Shortest Reconfiguration of Sliding Tokens on a Caterpillar

Takeshi Yamada and Ryuhei Uehara
School of Information Science, JAIST, Japan.
{tyama,uehara}@jaist.ac.jp

Abstract. Suppose that we are given two independent sets $I_b$ and $I_r$ of a graph such that $|I_b| = |I_r|$, and imagine that a token is placed on each vertex in $I_b$. Then, the SLIDING TOKEN problem is to determine whether there exists a sequence of independent sets which transforms $I_b$ into $I_r$ so that each independent set in the sequence results from the previous one by sliding exactly one token along an edge in the graph. The SLIDING TOKEN problem is one of the reconfiguration problems that attract the attention from the viewpoint of theoretical computer science. The reconfiguration problems tend to be PSPACE-complete in general, and some polynomial time algorithms are shown in restricted cases. Recently, the problems that aim at finding a shortest reconfiguration sequence are investigated. For the 3SAT problem, a trichotomy for the complexity of finding the shortest sequence has been shown; that is, it is in P, NP-complete, or PSPACE-complete in certain conditions. In general, even if it is polynomial time solvable to decide whether two instances are re-configured with each other, it can be NP-complete to find a shortest sequence between them. Namely, finding a shortest sequence between two independent sets can be more difficult than the decision problem of reconfigurability between them. In this paper, we show that the problem for finding a shortest sequence between two independent sets is polynomial time solvable for some graph classes which are subclasses of the class of interval graphs. More precisely, we can find a shortest sequence between two independent sets on a graph $G$ in polynomial time if either $G$ is a proper interval graph, a trivially perfect graph, or a caterpillar. As far as the authors know, this is the first polynomial time algorithm for the SHORTEST SLIDING TOKEN problem for a graph class that requires detours.

1 Introduction

Recently, the reconfiguration problems attract the attention from the viewpoint of theoretical computer science. The problem arises when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible and each step abides by a fixed reconfiguration rule, that is, an adjacency relation defined on feasible solutions of the original problem. The reconfiguration problems have been studied extensively for several
well-known problems, including INDEPENDENT SET \[12,13,14,16,20\], SATISFIABILITY \[11,18\], SET COVER, CLIQUE, MATCHING \[14\], VERTEX-COLORING \[2,3,6\], SHORTEST PATH \[15\], and so on.

The reconfiguration problem can be seen as a natural “puzzle” from the viewpoint of recreational mathematics. The 15 puzzle is one of the most famous classic puzzles, that had the greatest impact on American and European society of any mechanical puzzle the word has ever known in 1880 (see \[23\] for its rich history). It is well known that the 15 puzzle has a parity; for any two placements, we can decide whether two placements are reconfigurable or not by checking the parity. Therefore, we can solve the reconfiguration problem in linear time just by checking whether the parity of one placement coincides with the other or not. Moreover, we can say that the distance between any two reconfigurable placements is \(O(n^3)\), that is, we can reconfigure from one to the other in \(O(n^3)\) sliding pieces when the size of the board is \(n \times n\). However, surprisingly, for these two reconfigurable placements, finding a shortest path is NP-complete in general \[22\]. Namely, although we know that it is \(O(n^3)\), finding a shortest one is NP-complete. Another interesting property of the 15 puzzle is in another case of generalization. In the 15 puzzle, every piece has the same unit size of 1 \(\times\) 1. We have the other famous classic puzzles that can be seen as a generalization of this viewpoint. That is, when we allow to have rectangles, we have the other classic puzzles, called “Dad puzzle” and its variants (see Fig. 1). Gardner said that “These puzzles are very much in want of a theory” in 1964 \[10\], and Hearn and Demaine have gave the theory after 40 years \[12\]; they prove that these puzzles are PSPACE-complete in general using their nondeterministic constraint logic model \[13\]. That is, the reconfiguration of the sliding block puzzle is PSPACE-complete in general decision problem, and linear time solvable if every block is the unit square. However, finding a shortest reconfiguration for the latter easy case is NP-complete. In other words, we can characterize these three complexity classes using the model of sliding block puzzle.

From the viewpoint of theoretical computer science, one of the most important problems is the 3SAT problem. For this 3SAT problem, a similar trichotomy for the complexity of finding a shortest sequence has been shown recently; that is, for the reconfiguration problem of 3SAT, finding a shortest sequence between two satisfiable assignments is in P, NP-complete, or PSPACE-complete in certain
Fig. 2. A sequence $\langle I_1, I_2, \ldots, I_5 \rangle$ of independent sets of the same graph, where the vertices in independent sets are depicted by small black circles (tokens).

conditions [19]. In general, the reconfiguration problems tend to be PSPACE-complete, and some polynomial time algorithms are shown in restricted cases. In the reconfiguration problems, finding a shortest sequence can be a new trend in theoretical computer science because it has a great potential to characterize the class NP from a new viewpoint.

Beside the 3SAT problem, one of the most important problems in theoretical computer science is the independent set problem. Recall that an independent set of a graph $G$ is a vertex-subset of $G$ in which no two vertices are adjacent. (See Fig. 2 which depicts five different independent sets in the same graph.) For this notion, the natural reconfiguration problem is called the SLIDING TOKEN problem introduced by Hearn and Demaine [12]: Suppose that we are given two independent sets $I_b$ and $I_r$ of a graph $G = (V, E)$ such that $|I_b| = |I_r|$, and imagine that a token (coin) is placed on each vertex in $I_b$. Then, the SLIDING TOKEN problem is to determine whether there exists a sequence $\langle I_1, I_2, \ldots, I_\ell \rangle$ of independent sets of $G$ such that

- $(a)$ $I_1 = I_b$, $I_\ell = I_r$, and $|I_i| = |I_b|$ for all $i$, $1 \leq i \leq \ell$; and
- $(b)$ for each $i$, $2 \leq i \leq \ell$, there is an edge $\{u, v\}$ in $G$ such that $I_{i-1} \setminus I_i = \{u\}$ and $I_i \setminus I_{i-1} = \{v\}$, that is, $I_i$ can be obtained from $I_{i-1}$ by sliding exactly one token on a vertex $u \in I_{i-1}$ to its adjacent vertex $v$ along $\{u, v\} \in E$.

Figure 2 illustrates a sequence $\langle I_1, I_2, \ldots, I_5 \rangle$ of independent sets which transforms $I_b = I_1$ into $I_r = I_5$. Hearn and Demaine proved that the SLIDING TOKEN problem is PSPACE-complete for planar graphs.

(We note that the reconfiguration problem for INDEPENDENT SET have some variants. In [16], the reconfiguration problem for INDEPENDENT SET is studied under three reconfiguration rules called “token sliding,” “token jumping,” and “token addition and removal.” In this paper, we only consider token sliding model, and see [16] for the other models.)

For the SLIDING TOKEN problem, some polynomial time algorithms are investigated as follows: Linear time algorithms have been shown for cographs (also known as $P_4$-free graphs) [10] and trees [7]. Polynomial time algorithms are shown for bipartite permutation graphs [9], and claw-free graphs [4]. On the other hand, PSPACE-completeness is also shown for graphs of bounded treewidth [21], and planar graphs [13].
In this context, we investigate for finding a shortest sequence of the sliding token problem, which is called the shortest sliding token problem. That is, our problem is formalized as follows:

Input: A graph \( G = (V, E) \) and two independent sets \( I_b, I_r \) with \( |I_b| = |I_r| \).

Output: A shortest reconfiguration sequence \( I_b = I_1, I_2, \ldots, I_\ell = I_r \) such that \( I_i \) can be obtained from \( I_{i-1} \) by sliding exactly one token on a vertex \( u \in I_{i-1} \) to its adjacent vertex \( v \) along \( \{u, v\} \in E \) for each \( i \), \( 2 \leq i \leq \ell \).

We note that \( \ell \) is not necessarily in polynomial of \( |V| \); this is an issue how we formalize the problem, and if we do not know that \( \ell \) is in polynomial or not. If the length \( k \) is given as a part of input, we may be able to decide whether \( \ell \leq k \) in polynomial time even if \( \ell \) itself is not in polynomial. However, if we have to output the sequence itself, it cannot be solved in polynomial time if \( \ell \) is not in polynomial.

In this paper, we will show that the shortest sliding token problem is solvable in polynomial time for the following graph classes:

* **Proper interval graphs:** We first prove that any two independent sets of a proper interval graph can be transformed into each other. In other words, every proper interval graph with two independent sets \( I_b \) and \( I_r \) is a yes-instance of the problem if \( |I_b| = |I_r| \). Furthermore, we can find the ordering of tokens to be slid in a minimum-length sequence in \( O(n) \) time (implicitly), even though there exists an infinite family of independent sets on paths (and hence on proper interval graphs) for which any sequence requires \( \Omega(n^2) \) length.

* **Trivially perfect graphs:** We then give an \( O(n) \)-time algorithm for trivially perfect graphs which actually finds a shortest sequence if such a sequence exists. In contrast to proper interval graphs, any shortest sequence is of length \( O(n) \) for trivially perfect graphs. Note that trivially perfect graphs form a subclass of cographs, and hence its polynomial time solvability has been known [16].

* **Caterpillars:** We finally give an \( O(n^2) \)-time algorithm for caterpillars for the shortest sliding token problem. To make self-contained, we first show a linear time algorithm for decision problem that asks whether two independent sets can be transformed into each other. (We note that this problem can be solved in linear time for a tree [7].) For a yes-instance, we next show an algorithm that finds a shortest sequence of token sliding between two independent sets.

We here remark that, since the problem is PSPACE-complete in general, an instance of the sliding token problem may require the exponential number of independent sets to transform. In such a case, tokens should make detours to avoid violating to be independent (as shown in Fig. 2). As we will see, caterpillars certainly require to make detours to transform. Therefore, it is remarkable that any yes-instance on a caterpillar requires a sequence of token-slides of polynomial length. This is still open even for a tree. That is, in a tree, we can determine if two independent sets are reconfigurable in linear time due to [7], however, we do not know if the length of the sequence is in polynomial.
As far as the authors know, this is the first polynomial time algorithm for the shortest sliding token problem for a graph class that requires detours of tokens.

2 Preliminaries

In this section, we introduce some basic terms and notations. In the sliding token problem, we may assume without loss of generality that graphs are simple and connected. For a graph $G = (V, E)$, we let $n = |V|$ and $m = |E|$.

2.1 Sliding token

For two independent sets $I_i$ and $I_j$ of the same cardinality in a graph $G = (V, E)$, if there exists exactly one edge $\{u, v\}$ in $G$ such that $I_i \setminus I_j = \{u\}$ and $I_j \setminus I_i = \{v\}$, then we say that $I_j$ can be obtained from $I_i$ by sliding a token on the vertex $u \in I_i$ to its adjacent vertex $v$ along the edge $\{u, v\}$, and denote it by $I_i \triangleright I_j$. We remark that the tokens are unlabeled, while the vertices in a graph are labeled.

A reconfiguration sequence between two independent sets $I_1$ and $I_\ell$ of $G$ is a sequence $\langle I_1, I_2, \ldots, I_\ell \rangle$ of independent sets of $G$ such that $I_{i-1} \triangleleft I_i$ for $i = 2, 3, \ldots, \ell$. We denote by $I_1 \triangleright^\ast I_\ell$ if there exists a reconfiguration sequence between $I_1$ and $I_\ell$. We note that a reconfiguration sequence is reversible, that is, we have $I_1 \triangleright^\ast I_\ell$ if and only if $I_\ell \triangleright^\ast I_1$. Thus we say that two independent sets $I_1$ and $I_\ell$ are reconfigurable into each other if $I_1 \triangleright^\ast I_\ell$. The length of a reconfiguration sequence $S$ is defined as the number of independent sets contained in $S$. For example, the length of the reconfiguration sequence in Fig. 2 is 5.

The sliding token problem is to determine whether two given independent sets $I_b$ and $I_r$ of a graph $G$ are reconfigurable into each other. We may assume without loss of generality that $|I_b| = |I_r|$; otherwise the answer is clearly “no.” Note that the sliding token problem is a decision problem asking for the existence of a reconfiguration sequence between $I_b$ and $I_r$, and hence it does not ask an actual reconfiguration sequence. In this paper, we will consider the shortest sliding token problem that computes a shortest reconfiguration sequence between two independent sets. Note that the length of a reconfiguration sequence may not be in polynomial of the size of the graph since the sequence may contain detours of tokens.

We always denote by $I_b$ and $I_r$ the initial and target independent sets of $G$, respectively, as an instance of the (shortest) sliding token problem; we wish to slide tokens on the vertices in $I_b$ to the vertices in $I_r$. We sometimes call the vertices in $I_b$ blue, and the vertices in $I_r$ red; each vertex in $I_b \cap I_r$ is blue and red.

2.2 Target-assignment

We here give another notation of the sliding token problem, which is useful to explain our algorithm.
Let $I_b = \{b_1, b_2, \ldots, b_k\}$ be an initial independent set of a graph $G$. For the sake of convenience, we label the tokens on the vertices in $I_b$; let $t_i$ be the token placed on $b_i$ for each $i$, $1 \leq i \leq k$. Let $\mathcal{S}$ be a reconfiguration sequence between $I_b$ and an independent set $I$ of $G$, and hence $I_b \leftrightarrow I$. Then, for each token $t_i$, $1 \leq i \leq k$, we denote by $f_\mathcal{S}(t_i)$ the vertex in $I$ on which the token $t_i$ is placed via the reconfiguration sequence $\mathcal{S}$. Notice that $\{f_\mathcal{S}(t_i) \mid 1 \leq i \leq k\} = I$.

Let $I_c$ be a target independent set of $G$, which is not necessarily reconfigurable from $I_b$. Then, we call a mapping $g : I_b \rightarrow I_c$ a target-assignment between $I_b$ and $I_c$. The target-assignment $g$ is said to be proper if there exists a reconfiguration sequence $\mathcal{S}$ such that $f_\mathcal{S}(t_i) = g(b_i)$ for all $i$, $1 \leq i \leq k$. Note that there is no proper target-assignment between $I_b$ and $I_c$ if $I_b \not\leftrightarrow I_c$. Therefore, the sliding token problem can be seen as the problem of determining whether there exists at least one proper target-assignment between $I_b$ and $I_c$.

### 2.3 Interval graphs and subclasses

The neighborhood of a vertex $v$ in a graph $G = (V, E)$ is the set of all vertices adjacent to $v$, and we denote it by $N(v) = \{u \in V \mid \{u, v\} \in E\}$. Let $N[v] = N(v) \cup \{v\}$. For any graph $G = (V, E)$, two vertices $u$ and $v$ are called strong twins if $N[u] = N[v]$, and weak twins if $N(u) = N(v)$. In our problem, strong twins have no meaning: when $u$ and $v$ are strong twins, only one of them can be used by a token. Therefore, in this paper, we only consider the graphs without strong twins. That is, for any pair of vertices $u$ and $v$, we have $N[u] \neq N[v]$.

We have to take care about weak twins; see Section 5 for the details.

A graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ is an interval graph if there exists a set $\mathcal{I}$ of (closed) intervals $I_1, I_2, \ldots, I_n$ such that $\{v_i, v_j\} \in E$ if and only if $I_i \cap I_j \neq \emptyset$ for each $i$ and $j$ with $1 \leq i, j \leq n$. We call the set $\mathcal{I}$ of intervals an interval representation of the graph, and sometimes identify a vertex $v_i \in V$ with its corresponding interval $I_i \in \mathcal{I}$. We denote by $L(I)$ and $R(I)$ the left and right endpoints of an interval $I \in \mathcal{I}$, respectively. That is, we always have $L(I) \leq R(I)$ for any interval $I = [L(I), R(I)]$.

To specify the bottleneck of the running time of our algorithms, we suppose that an interval graph $G = (V, E)$ is given as an input by its interval representation using $O(n)$ space. (If necessary, an interval representation of $G$ can be found in $O(n + m)$ time [17].) More precisely, $G$ is given by a string of length $2n$ over alphabets $\{L(I_1), L(I_2), \ldots, L(I_n), R(I_1), R(I_2), \ldots, R(I_n)\}$. For example, a complete graph $K_3$ with three vertices can be given by an interval representation $L(I_1)L(I_2)L(I_3)R(I_1)R(I_2)R(I_3)$, and a path of length two is given by an interval representation $L(I_1)L(I_2)R(I_1)L(I_3)R(I_2)R(I_3)$.

An interval graph is proper if it has an interval representation such that no interval properly contains another. The class of proper interval graphs is also known as the class of unit interval graphs [2]: an interval graph is unit if it has an interval representation such that every interval has unit length. Hereafter, we

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1 In this paper, a bold $I$ denotes an “independent set,” an italic $I$ denotes an “interval,” and calligraphy $\mathcal{I}$ denotes “a set of intervals.”
assume that each proper interval graph is given in the interval representation of intervals of unit length. In the context of the interval representation, an interval graph is proper if and only if \( L(I_i) < L(I_j) \) if and only if \( R(I_i) < R(I_j) \).

An interval graph is \textit{trivially perfect} if it has an interval representation such that the relationship between any two intervals is either disjoint or inclusion. That is, for any two intervals \( I_i \) and \( I_j \) with \( L(I_i) < L(I_j) \), we have either \( L(I_i) < L(I_j) < R(I_j) < R(I_i) \) or \( L(I_i) < R(I_i) < R(I_j) < L(I_j) \).

A \textit{caterpillar} \( G = (V, E) \) is a tree (i.e., a connected acyclic graph) that consists of two subsets \( S \) and \( L \) of \( V \) as follows. The vertex set \( S \) induces a path \((s_1, \ldots, s_n')\) in \( G \), and each vertex \( v \) in \( L \) has degree 1, and its unique neighbor is in \( S \). We call the path \((s_1, \ldots, s_n')\) spine, and each vertex in \( L \) leaf. In this paper, without loss of generality, we assume that \( n' \geq 2 \), \( \deg(s_1) \geq 2 \), and \( \deg(s_n') \geq 2 \). That is, the endpoints \( s_1 \) and \( s_n' \) of the spine \((s_1, \ldots, s_n')\) should have at least one leaf. It is easy to see that the class of caterpillars is a proper subset of the class of interval graphs, and these three subclasses are incomparable with each other.

## 3 Proper Interval Graphs

We show the main theorem in this section for proper interval graphs, which first says that the answer of \textit{sliding token} is always “yes” for connected proper interval graphs. We give a constructive proof of the claim, and it certainly finds a shortest sequence in linear time.

**Theorem 1.** For a connected proper interval graph \( G = (V, E) \), any two independent sets \( I_b \) and \( I_r \) with \( |I_b| = |I_r| \) are reconfigurable into each other, that is, \( I_b \longleftrightarrow I_r \). Moreover, the shortest reconfiguration sequence can be found in polynomial time.

We give a constructive proof for Theorem 1, that is, we give an algorithm which actually finds a shortest reconfiguration sequence between any two independent sets \( I_b \) and \( I_r \) of a connected proper interval graph \( G \).

A connected proper interval graph \( G = (V, E) \) has a unique interval representation (up to reversal), and we can assume that each interval is of unit length in the representation \([8]\). Therefore, by renumbering the vertices, we can fix an interval representation \( \mathcal{I} = \{I_1, I_2, \ldots, I_n\} \) of \( G \) so that \( L(I_i) < L(I_{i+1}) \) (and \( R(I_i) < R(I_{i+1}) \)) for each \( i, 1 \leq i \leq n-1 \), and each interval \( I_i \in \mathcal{I} \) corresponds to the vertex \( v_i \in V \).

Let \( I_b = \{b_1, b_2, \ldots, b_k\} \) and \( I_r = \{r_1, r_2, \ldots, r_k\} \) be any given initial and target independent sets of \( G \), respectively. Without loss of generality, we assume that the blue vertices \( b_1, b_2, \ldots, b_k \) are labeled from left to right (according to the interval representation \( \mathcal{I} \) of \( G \)), that is, \( L(b_i) < L(b_{j}) \) if \( i < j \); similarly, we assume that the red vertices \( r_1, r_2, \ldots, r_k \) are labeled from left to right. Then, we define a target-assignment \( g : I_b \to I_r \), as follows: for each blue vertex \( b_i \in I_b \)

\[
g(b_i) = r_i. \tag{1}
\]
To prove Theorem 1, it suffices to show that \( g \) is proper, and each token takes no detours.

### 3.1 String representation

By traversing the interval representation \( I \) of a connected proper interval graph \( G \) from left to right, we can obtain a string \( S = s_1s_2 \cdots s_{2k} \) which is a superstring of both \( b_1b_2 \cdots b_k \) and \( r_1r_2 \cdots r_k \), that is, each letter \( s_i \) in \( S \) is one of the vertices in \( I_b \cup I_r \), and \( s_i \) appears in \( S \) before \( s_j \) if \( L(s_i) < L(s_j) \). We may assume without loss of generality that \( s_1 = b_1 \) since the reconfiguration rule is symmetric in SLIDING TOKEN. If a vertex is contained in both \( I_b \) and \( I_r \), as \( b_i \) and \( r_j \), then we assume that it appears as \( b_ir_j \) in \( S \), that is, the blue vertex \( b_i \) appears in \( S \) before the red vertex \( r_j \). Then, for each \( i, 1 \leq i \leq 2k \), we define the height \( h(i) \) at \( i \) by the number of blue vertices appeared in the substring \( s_1s_2 \cdots s_i \) minus the number of red vertices appeared in \( s_1s_2 \cdots s_i \). For the sake of notational convenience, we define \( h(0) = 0 \). Then \( h(i) \) can be recursively computed as follows:

\[
h(i) = \begin{cases} 
0 & \text{if } i = 0; \\
h(i-1) + 1 & \text{if } s_i \text{ is blue;} \\
h(i-1) - 1 & \text{if } s_i \text{ is red.}
\end{cases}
\]  

Note that \( h(2k) = 0 \) for any string \( S \) since \( |I_b| = |I_r| \).

Using the notion of height, we split the string \( S \) into substrings \( S_1, S_2, \ldots, S_h \) at every point of height 0, that is, in each substring \( S_j = s_{2p+1}s_{2p+2} \cdots s_{2q} \), we have \( h(2q) = 0 \) and \( h(i) \neq 0 \) for all \( i, 2p + 1 \leq i \leq 2q - 1 \). For example, a string \( S = b_1b_2b_1r_1r_2b_3r_3b_4r_4b_5r_5b_6r_6b_7r_7b_8b_9b_8b_7b_8 \) can be split into four substrings \( S_1 = b_1b_2r_1r_2, S_2 = b_3r_3, S_3 = r_4r_5b_4r_4b_5r_7b_6r_8b_7b_8 \) and \( S_4 = b_9r_9 \). Then, the substrings \( S_1, S_2, \ldots, S_h \) form a partition of \( S \), and each substring \( S_j \) contains the same number of blue and red tokens. We call such a partition the **partition of \( S \) at height 0**.

#### Lemma 1

Let \( S_j = s_{2p+1}s_{2p+2} \cdots s_{2q} \) be a substring in the partition of the string \( S \) at height 0. Then,

(a) the blue vertices \( b_{p+1}, b_{p+2}, \ldots, b_q \) appear in \( S_j \), and their corresponding red vertices \( r_{p+1}, r_{p+2}, \ldots, r_q \) appear in \( S_j \);

(b) if \( S_j \) starts with the blue vertex \( b_{p+1} \), then each blue vertex \( b_i, p+1 \leq i \leq q, \) appears in \( S_j \) before its corresponding red vertex \( r_i \); and

(c) if \( S_j \) starts with the red vertex \( r_{p+1} \), then each blue vertex \( b_i, p+1 \leq i \leq q, \) appears in \( S_j \) after its corresponding red vertex \( r_i \).

**Proof.** By the definitions, the claim (a) clearly holds. We thus show that the claim (b) holds. (The proof for the claim (c) is symmetric.)

Since \( h(2p) = 0 \) and \( S_j \) starts with a blue vertex, we have \( h(2p+1) = 1 > 0 \). We now suppose for a contradiction that there exists a blue vertex \( s_x = b_i \) which appears in \( S_j \) after its corresponding red vertex \( s_y = r_i \). Then, \( y < x \). We assume that \( y \) is the minimum index among such blue vertices in \( S_j \). Then, in the substring \( s_1s_2 \cdots s_y \) of \( S \), there are exactly \( i' \) red vertices. On the other
hand, since $y < x$, the substring $s_1 s_2 \cdots s_q$ contains at most $i' - 1$ blue vertices. Therefore, by the definition of height, we have $h(y) < 0$. Since $h(2p + 1) = 1 > 0$ and $h(y) < 0$, by Eq. 2 there must exist an index $z$ such that $h(z) = 0$ and $2p < z < y$. This contradicts the fact that $S_j$ is a substring in the partition of $S$ at height 0. □

3.2 Algorithm

Recall that we have fixed the unique interval representation $\mathcal{I} = \{I_1, I_2, \ldots, I_n\}$ of a connected proper interval graph $G$ so that $L(I_i) < L(I_{i+1})$ for each $i$, $1 \leq i < n$, and each interval $I_i \in \mathcal{I}$ corresponds to the vertex $v_i \in V$. Since all intervals in $\mathcal{I}$ have unit length, the following proposition clearly holds.

**Proposition 1.** For two vertices $v_i$ and $v_j$ in $G$ such that $i < j$, there is a path $P$ in $G$ which passes through only intervals (vertices) contained in $[L(I_i), R(I_j)]$. Furthermore, if $I_i \cap I_j = \emptyset$ for some index $i'$ with $i' < i$, no vertex in $v_{i_1}, v_{i_2}, \ldots, v_{i'}$ is adjacent to any vertex in $P$. If $I_j \cap I_j' = \emptyset$ for some index $j'$ with $j < j'$, no vertex in $v_{j'}, v_{j'+1}, \ldots, v_n$ is adjacent to any vertex in $P$.

Let $S$ be the string of length $2k$ obtained from two given independent sets $I_b$ and $I_r$ of a connected proper interval graph $G$, where $k = |I_b| = |I_r|$. Let $S_1, S_2, \ldots, S_h$ be the partition of $S$ at height 0. The following lemma shows that the tokens in each substring $S_j$ can always reach their corresponding red vertices. (Note that we sometimes denote simply by $S_j$ the set of all vertices appeared in the substring $S_j$, $1 \leq j \leq h$.)

**Lemma 2.** Let $S_j = s_{2p+1} s_{2p+2} \cdots s_{2q}$ be a substring in the partition of $S$ at height 0. Then, there exists a reconfiguration sequence between $I_b \cap S_j$ and $I_r \cap S_j$ such that tokens are slid along edges only in the subgraph of $G$ induced by the vertices contained in $[L(s_{2p+1}), R(s_{2q})]$.

**Proof.** We first consider the case where $S_j$ starts with the blue vertex $b_{p+1}$, that is, $s_{2p+1} = b_{p+1}$. Then, by Lemma 1(b) each blue vertex $b_i$, $p + 1 \leq i \leq q$, appears in $S_j$ before the corresponding red vertex $r_i$. Therefore, we know that $s_{2q} = r_q$, and hence it is red. Suppose that $s_x = b_q$, then all vertices appeared in $s_{x+1} s_{x+2} \cdots s_{2q}$ are red. Roughly speaking, we slide the tokens $t_q, t_{q-1}, \ldots, t_{p+1}$ from left to right in this order.

We first claim that the token $t_q$ can be slid from $b_q$ ($= s_x$) to $r_q$ ($= s_{2q}$). By Proposition 1 there is a path $P$ between $b_q$ and $r_q$ which passes through only intervals contained in $[L(b_q), R(r_q)]$. Since $I_b$ is an independent set of $G$, the vertex $b_q$ is not adjacent to any other vertices $b_{p+1}, b_{p+2}, \ldots, b_{q-1}$ in $I_b \cap S_j$. Since $L(b_{p+1}) < L(b_{p+2}) < \cdots < L(b_{q-1}) < L(b_q)$, by Proposition 1 all vertices in $P$ are not adjacent to any of tokens $t_{p+1}, t_{p+2}, \ldots, t_{q-1}$ that are now placed on $b_{p+1}, b_{p+2}, \ldots, b_{q-1}$, respectively. Therefore, we can slide the token $t_q$ from $b_q$ to $r_q$. We fix the token $t_q$ on $r_q = s_{2q}$, and will not slide it anymore.

We then slide the next token $t_{q-1}$ on $b_{q-1}$ to $r_{q-1}$ along a path $P'$ which passes through only intervals contained in $[L(b_{q-1}), R(r_{q-1})]$. Since $I_b$ is an independent set of $G$, the corresponding red vertex $r_{q-1}$ is not adjacent to $r_q$ on
which the token \( t_q \) is now placed. Recall that \( L(r_{q-1}) < L(r_q) \), and hence by Proposition 1, \( r_q \) is not adjacent to any vertex in \( P' \). Similarly as above, the tokens \( t_{p+1}, t_{p+2}, \ldots, t_{q-2} \) are not adjacent to any vertex in \( P' \). Therefore, we can slide the token \( t_{q-1} \) from \( b_{q-1} \) to \( r_{q-1} \).

Repeat this process until the token \( t_{p+1} \) on \( b_{p+1} \) is slid to \( r_{p+1} \). In this way, there is a reconfiguration sequence between \( I_b \cap S_j \) and \( I_r \cap S_j \) such that tokens are slid along edges only in the subgraph of \( G \) induced by the vertices contained in \([L(b_{p+1}), R(r_q)]\).

The symmetric arguments prove the case where \( S_j \) starts with the red vertex \( r_{p+1} \). Note that, in this case, we slide the tokens \( t_{p+1}, t_{p+2}, \ldots, t_q \) from right to left in this order.

**Proof of Theorem 1** We now give an algorithm which slides all tokens on the vertices in \( I_b \) to the vertices in \( I_r \). Recall that \( S_1, S_2, \ldots, S_h \) are the substrings in the partition of \( S \) at height 0. Intuitively, the algorithm repeatedly picks up one substring \( S_j \), and slides all tokens in \( I_b \cap S_j \) to \( I_r \cap S_j \). By Lemma 2, it works locally in each substring \( S_j \), but it should be noted that a token in \( S_j \) may be adjacent to another token in \( S_{j-1} \) or \( S_{j+1} \) at the boundary of the substrings. To avoid this, we define a partial order over the substrings \( S_1, S_2, \ldots, S_h \), as follows.

Consider any two consecutive substrings \( S_j \) and \( S_{j+1} \), and let \( S_j = s_{2p+1}s_{2p+2} \cdots s_{2q} \). Then, the first letter of \( S_{j+1} \) is \( s_{2q+1} \). We first consider the case where both \( s_{2q} \) and \( s_{2q+1} \) are the same color. Then, since \( s_{2q} \) and \( s_{2q+1} \) are both in the same independent set of \( G \), they are not adjacent. Therefore, by Proposition 1 and Lemma 2, we can deal with \( S_j \) and \( S_{j+1} \) independently. In this case, we thus do not define the ordering between \( S_j \) and \( S_{j+1} \). We then consider the case where \( s_{2q} \) and \( s_{2q+1} \) have different colors; in this case, we have to define their ordering. Suppose that \( s_{2q} \) is blue and \( s_{2q+1} \) is red; then we have \( s_{2q} = b_q \) and \( s_{2q+1} = r_q \). By Lemma 2, the token \( t_q \) on \( s_{2q} \) is slid to left, and the token \( t_{q+1} \) will reach \( r_{p+1} \) from right. Therefore, the algorithm has to deal with \( S_j \) before \( S_{j+1} \). Note that, after sliding all tokens \( t_{p+1}, t_{p+2}, \ldots, t_q \) in \( S_j \), they are on the red vertices \( r_{p+1}, r_{p+2}, \ldots, r_q \), respectively, and hence the tokens in \( S_{j+1} \) are not adjacent to any of them. By the symmetric argument, if \( s_{2q} \) is red and \( s_{2q+1} \) is blue, \( S_{j+1} \) should be dealt with before \( S_j \).

Notice that such an ordering is defined only for two consecutive substrings \( S_j \) and \( S_{j+1} \), \( 1 \leq j \leq h - 1 \). Therefore, the partial order over the substrings \( S_1, S_2, \ldots, S_h \) is acyclic, and hence there exists a total order which is consistent with the partial order defined above. The algorithm certainly slides all tokens from \( I_b \) to \( I_r \) according to the total order. Therefore, the target-assignment \( g \) defined in Eq. (1) is proper, and hence \( I_b \vdash^* I_r \).

Therefore, there always exists a reconfiguration sequence between two independent sets \( I_b \) and \( I_r \) of a connected proper interval graph \( G \). We now discuss the length of reconfiguration sequences between \( I_b \) and \( I_r \), together with the running time of our algorithm.

**Proposition 2.** For two given independent sets \( I_b \) and \( I_r \) of a connected proper interval graph \( G \) with \( n \) vertices,
(1) the ordering of tokens to be slid in a shortest reconfiguration sequence between them can be computed in \(O(n)\) time and \(O(n)\) space; and
(2) a shortest reconfiguration sequence between them can be output in \(O(n^2)\) time and \(O(n)\) space.

Proof. We first modify our algorithm so that it finds a shortest reconfiguration sequence between \(I_b\) and \(I_r\). To do that, it suffices to slide each token \(t_i\), \(1 \leq i \leq k\), from the blue vertex \(b_i\) to its corresponding red vertex \(r_i\) along the shortest path between \(b_i\) and \(r_i\). We may assume without loss of generality that \(L(b_i) < L(r_i)\), that is, the token \(t_i\) will be slid from left to right. Then, for the interval \(b_i\), we choose an interval \(I_j \in \mathcal{I}\) such that \(b_i \cap I_j \neq \emptyset\) and \(L(I_j)\) is the maximum among all \(I_j \in \mathcal{I}\). If \(L(r_i) \leq L(I_j)\), we can slide \(t_i\) from \(b_i\) to \(r_i\) directly; otherwise we slide \(t_i\) to the vertex \(I_j\), and repeat.

We then prove the claim (1). If we simply want to compute the ordering of tokens to be slid in a shortest reconfiguration sequence, it suffices to compute the partial order over the substrings \(S_1, S_2, \ldots, S_h\) in the partition of the string of length \(n\). It is not difficult to implement our algorithm in Section 3.2 to run in \(O(n)\) time and \(O(n)\) space. Therefore, the claim (1) holds.

We finally prove the claim (2). Remember that each token \(t_i\), \(1 \leq i \leq k\), is slid along the shortest path from \(b_i\) to \(r_i\). Furthermore, once the token \(t_i\) reaches \(r_i\), it is not slid anymore. Therefore, the length of a shortest reconfiguration sequence between \(I_b\) and \(I_r\) is given by the sum of all lengths of the shortest paths between \(b_i\) and \(r_i\). It is clear that this sum is \(O(kn) = O(n^2)\). We output only the shortest paths between \(b_i\) and \(r_i\), together with the ordering of the tokens to be slid. Therefore, the claim (2) holds.

This proposition also completes the proof of Theorem 1.

It is remarkable that there exists an infinite family of instances for which any reconfiguration sequence requires \(\Omega(n^2)\) length. To show this, we give an instance such that each shortest path between \(b_i\) and \(r_i\) is \(\Theta(n)\). Simple example is: \(G\) is a path \((v_1, v_2, \ldots, v_{8k})\) of length \(n = 8k\) for any positive integer \(k\), \(I_b = \{v_{1}, v_3, v_5, \ldots, v_{2k-1}\}\), and \(I_r = \{v_{3k+2}, v_{3k+4}, \ldots, v_{8k}\}\). In this instance, each token \(t_i\) must be slid \(\Theta(n)\) times, and hence it requires \(\Theta(n^2)\) time to output all of them. We note that a path is not only a proper interval graph, but also a caterpillar. Thus this simple example also works as a caterpillar.

4 Trivially perfect graphs

The main result of this section is the following theorem.

**Theorem 2.** The sliding token problem for a trivially perfect graph \(G = (V, E)\) can be solved in \(O(n)\) time and \(O(n)\) space. Furthermore, one can find a shortest reconfiguration sequence between two given independent sets \(I_b\) and \(I_r\) in \(O(n)\) time and \(O(n)\) space if there exists.
In this section, we explicitly give such an algorithm as a proof of Theorem 2. Note that there are no-instances for trivially perfect graphs. However, for trivially perfect graphs, we construct a proper target-assignment between $I_b$ and $I_r$ efficiently if it exists.

4.1 MPQ-tree for trivially perfect graphs

The MPQ-tree of an interval graph $G$ is a kind of decomposition tree, developed by Korte and Möhring [17], which represents the set of all feasible interval representations of $G$. For an interval graph $G$, although there are exponentially many interval representations for $G$, its corresponding MPQ-tree is unique up to isomorphism. For the notion of MPQ-trees, the following theorem is known:

**Theorem 3 ([17]).** For any interval graph $G = (V, E)$, its corresponding MPQ-tree can be constructed in $O(n + m)$ time.

Since it is involved to define MPQ-tree for general interval graphs, we here give a simplified definition of MPQ-tree only for the class of trivially perfect graphs. (See [17] for the detailed definition of the MPQ-tree for a general interval graph.) Let $G = (V, E)$ be a trivially perfect graph. Recall that a trivially perfect graph has an interval representation such that the relationship between any two intervals is either disjoint or inclusion. Then, the MPQ-tree $T$ of $G$ is a rooted tree such that each node, called a $P$-node, in $T$ is associated with a non-empty set of vertices in $G$ such that (a) each vertex $v \in V$ appears in exactly one $P$-node in $T$, and (b) if a vertex $v_i \in V$ is in an ancestor node of another node that contains $v_j \in V$, then $L(I_i) \leq L(I_j) < R(I_j) \leq R(I_i)$ in any interval representation of $G$, where $v_i$ and $v_j$ correspond to the intervals $I_i$ and $I_j$, respectively (see Fig. 3 as an example). By the property (b), the ancestor/descendant relationship on $T$ corresponds to the inclusion relationship in the interval representation of $G$. Thus, $N[v_j] \subseteq N[v_i]$ if $v_i$ is in an ancestor node of another node that contains $v_j$ in the MPQ-tree.

Let $T$ be the (unique) MPQ-tree of a connected trivially perfect graph $G = (V, E)$. For two vertices $u$ and $w$ in $G$, we denote by LCA$(u, w)$ the least common ancestor in $T$ for the nodes containing $u$ and $w$. By the property (a) the node LCA$(u, w)$ can be uniquely defined.

4.2 Basic properties and key lemma

Let $T$ be the (unique) MPQ-tree of a connected trivially perfect graph $G = (V, E)$. Recall that the interval representation of a trivially perfect graph has just disjoint or inclusion relationship. This fact implies the following observation.

**Observation 1** Every pair of vertices $u$ and $w$ in a connected trivially perfect graph $G$ has a path of length at most two via a vertex in LCA$(u, w)$.

**Proof.** We first observe that every $P$-node of $T$ is non-empty. If the root is empty, the graph is disconnected, which is a contradiction. If some non-root $P$-node $P$
is empty, joining all children of \( P \) to the parent of \( P \), we obtain a simpler MPQ-tree than \( T \), which contradicts the construction of the unique MPQ-tree in [17]. Thus every P-node is non-empty. Therefore, there is at least one vertex \( v \) in \( \text{LCA}(u, w) \). Then, the property (b) implies that \( N[u] \subseteq N[v] \) and \( N[w] \subseteq N[v] \), and hence \( \{u, v\} \) and \( \{w, v\} \) are both in \( E \). Therefore, there is a path \((u, v, w)\) of length at most two between \( u \) and \( w \) via \( v \). When we have \( v = u \) or \( v = w \), the path degenerates to the edge \( \{u, w\} \in E \).

\[ \square \]

Let \( \text{LCA}^*(u, w) \) be the set of vertices in \( V \) appearing in the P-nodes on the (unique) path from \( \text{LCA}(u, w) \) to the root of the MPQ-tree. By the definition of MPQ-tree, we clearly have the following observation. (Recall also that each token must be slid along an edge of \( G \).)

**Observation 2** Consider an arbitrary reconfiguration sequence \( S \) which slides a token \( t_i \) from \( b_i \in I_b \) to some vertex \( r_i \). Then, \( t_i \) must pass through at least one vertex in \( \text{LCA}^*(b_i, r_i) \), that is, there exists at least one independent set \( I' \) in \( S \) such that \( I' \cap \text{LCA}^*(b_i, r_i) \neq \emptyset \).

We are now ready to give the key lemma for trivially perfect graphs.

**Lemma 3.** Let \( g : I_b \rightarrow I_r \) be a target-assignment between \( I_b \) and \( I_r \). Then, \( g \) is proper if and only if the nodes \( \text{LCA}(b_i, g(b_i)) \) and \( \text{LCA}(b_j, g(b_j)) \) are not in the ancestor/descendant relationship on \( T \) for every pair of vertices \( b_i, b_j \in I_b \).

**Proof.** We first show the sufficiency. For a target-assignment \( g \) between \( I_b \) and \( I_r \), suppose that the nodes \( \text{LCA}(b_i, g(b_i)) \) and \( \text{LCA}(b_j, g(b_j)) \) are not in the ancestor/descendant relationship on \( T \) for every pair of vertices \( b_i, b_j \in I_b \). Then, we
can simply slide the tokens one by one in an arbitrary order; by Observation 1, each token $t_i$, $1 \leq i \leq k$, can be slid along a path from $b_i$ to $g(b_i)$ via a vertex $v_{j'}$ in $LCA(b_i, g(b_i))$. Note that there is no token $t_j$ adjacent to $v_{j'}$, because the nodes $LCA(b_i, g(b_i))$ and $LCA(b_j, g(b_j))$ are not in the ancestor/descendant relationship on $T$. Thus, there is a reconfiguration sequence between $I_b$ and $I_r$ according to $g$, and hence $g$ is proper.

We then show the necessity. Suppose that $g$ is proper, and suppose for a contradiction that there exists a pair of vertices $b_i, b_j \in I_b$ such that the nodes $LCA(b_i, g(b_i))$ and $LCA(b_j, g(b_j))$ are in the ancestor/descendant relationship on $T$; without loss of generality, we assume that $LCA(b_i, g(b_i))$ is an ancestor of $LCA(b_j, g(b_j))$. Since $g$ is proper, there exists a reconfiguration sequence $S$ between $I_b$ and $I_r$ which slides the token $t_i$ from $b_i$ to $g(b_i)$ and also slides the token $t_j$ from $b_j$ to $g(b_j)$. By Observation 2 there is at least one vertex $v_{j'}$ in $LCA'(b_i, g(b_i))$ which is passed through by $t_i$. Similarly, there is at least one vertex $v_{j'}$ in $LCA'(b_j, g(b_j))$ which is passed through by $t_j$. Let $P_i$ and $P_j$ be the P-nodes that contains $v_{j'}$ and $v_{j''}$, respectively. Since $LCA(b_i, g(b_i))$ and $LCA(b_j, g(b_j))$ are in the ancestor/descendant relationship on $T$, so are $P_i$ and $P_j$. First suppose $P_j$ is an ancestor of $P_i$. Then we have $N[b_j] \subseteq N[v_{j'}]$, $N[g(b_j)] \subseteq N[v_{j'}]$ and $N[v_{j''}] \subseteq N[v_{j'}]$. Therefore, if we slide $t_i$ via $v_{j'}$, then $t_i$ would be adjacent to the other token $t_j$ which is on one of the three vertices $b_j, g(b_j)$ and $v_{j''}$. Thus, the token $t_j$ should “escape” from $b_j$ before sliding $t_i$. However, we can establish the same argument for any descendant of $P_i$, and hence $t_j$ must escape to some vertex $u$ that is contained in an ancestor of $P_i$ at first. However, the vertex $u$ is adjacent to all of $b_i, g(b_i), v_{j'}$, and hence $t_j$ cannot escape before sliding $t_i$. This contradicts the assumption that $S$ slides the token $t_i$ from $b_i$ to $g(b_i)$ and also slides the token $t_j$ from $b_j$ to $g(b_j)$. The other case, $P_i$ is an ancestor of $P_j$, is symmetric.

\[ \square \]

4.3 Algorithm and its correctness

We now describe our linear-time algorithm for a trivially perfect graph. Let $T$ be the MPQ-tree of a connected trivially perfect graph $G = (V, E)$. Let $I_b = \{b_1, b_2, \ldots, b_k\}$ and $I_r = \{r_1, r_2, \ldots, r_k\}$ be given initial and target independent sets of $G$, respectively. Then, we determine whether $I_b \vdash^* I_r$ as follows:

(A) construct some particular target-assignment $g^*$ between $I_b$ and $I_r$; and

(B) check whether $g^*$ is proper or not, using Lemma 3.

We will show later in Lemma 5 that it suffices to check only $g^*$ in order to determine whether $I_b \vdash^* I_r$ or not. Indeed, our linear-time algorithm executes (A) and (B) above at the same time, in the bottom-up manner based on $T$.

Description of the algorithm Remember that the vertex-set associated to each P-node in $T$ induces a clique in $G$. Therefore, for any independent set $I$ of $G$, each P-node contains at most one vertex in $I$, and hence contains at most one token. We put a “blue token” for each P-node containing a blue vertex in $I_b$, and also put a “red token” for each P-node containing a red vertex in $I_r$. Note that a P-node may contain a pair of blue and red tokens. Our algorithm lifts up
the tokens from the leaves to the root of $T$, and if a blue token $b$ meets a red
token $r$ at their least common ancestor $\text{LCA}(b, r)$ in $T$, then we replace them by
a single “green token.” This corresponds to setting $g^*(b) = r$. More precisely, at
initialization step, the algorithm first collects all leaves of $T$ in a queue, which
is called $\text{frontier}$. The algorithm marks the nodes in the frontier, and lifts up
each token to its parent $P$-node. Each $P$-node $P$ is put into the frontier if its all
children are marked, and then, all children of $P$ are removed from the frontier
after the following procedure at $P$:

Case (1) $P$ contains at most one token: the algorithm has nothing to do.
Case (2) $P$ contains only one pair of blue token $b$ and red token $r$: the algorithm
replaces them by a single green token, and let $g^*(b) = r$.
Case (3) $P$ contains only green tokens: the algorithm replaces them by a single
green token.
Case (4) $P$ contains two or more blue tokens, or two or more red tokens: the
algorithm outputs “no” and halts (that is, $\mathbf{I}_b \not\vdash \mathbf{I}_r$ in this case).
Case (5) $P$ contains at least one green token and at least one blue or red token:
the algorithm outputs “no” and halts (that is, $\mathbf{I}_b \not\vdash \mathbf{I}_r$ in this case).

Repeating this process, and the algorithm outputs “yes” if and only when the
frontier contains only the root $P$-node $r$ of $T$ which is in one of Cases (1)–(3)
above.

Correctness of the algorithm
It is not difficult to implement our algorithm to
run in $O(n)$ time and $O(n)$ space. Therefore, we here prove the correctness of
the algorithm.

We first show that $\mathbf{I}_b \vdash^* \mathbf{I}_r$ if the algorithm outputs “yes.” In this case, the
algorithm is in Cases (1), (2), or (3) at each $P$-nodes in $T$ (including the root
$r$). Then, the target-assignment $g^*$ has been (completely) constructed: for each
blue vertex $b_i \in \mathbf{I}_b$, $g^*(b_i)$ is the red vertex in $\mathbf{I}_r$ such that $\text{LCA}(b_i, v_i')$ has the
minimum height in $T$ among all vertices $v_i' \in \mathbf{I}_r$. Then, we have the following
lemma.

Lemma 4. If the algorithm outputs “yes,” then $\mathbf{I}_b \vdash^* \mathbf{I}_r$.

Proof. By Lemma 3 it suffices to show that the target-assignment $g^*$ constructed
by the algorithm satisfies that the nodes $\text{LCA}(b_i, g^*(b_i))$ and $\text{LCA}(b_j, g^*(b_j))$ are not in the ancestor/descendant relationship on $T$ for every pair of vertices $b_i, b_j \in \mathbf{I}_b$.

We first consider the case where a $P$-node $P$ is in Case (2). Then, there is
exactly one pair of a blue token $b$ and a red token $r = g^*(b)$, and $P = \text{LCA}(b, r)$. Since $b$ and $r$ did not meet any other tokens before $P$, the subtree $T_P$ of $T$ contains only the two tokens $b$ and $r$. Therefore, the lemma clearly holds for $T_P$.

We then consider the case where a $P$-node $P$ is in Case (3). Then, two or more least common ancestors of pairs of blue and red tokens meet at this node $P$. Notice that the green tokens were placed on children’s node of $P$ in the
previous step of the algorithm, and hence they were sibling in $T$. Therefore, their corresponding least common ancestors are not in the ancestor/descendant relationship on $T$.  

\hfill $\Box$
The following lemma completes the correctness proof of our algorithm.

**Lemma 5.** If the algorithm outputs “no,” then \( I_b \not\models I_r \).

*Proof.* We assume that the algorithm outputs “no.” Then, by Lemma 3 it suffices to show that there is no target-assignment \( g \) between \( I_b \) and \( I_r \) such that \( \text{LCA}(b_i, g(b_i)) \) and \( \text{LCA}(b_j, g(b_j)) \) are not in the ancestor/descendant relationship on \( T \) for every pair of vertices \( b_i, b_j \in I_b \).

Suppose for a contradiction that the algorithm outputs “no,” but there exists a target-assignment \( g' \) between \( I_b \) and \( I_r \) such that \( \text{LCA}(b_i, g'(b_i)) \) and \( \text{LCA}(b_j, g'(b_j)) \) are not in the ancestor/descendant relationship on \( T \) for every pair of vertices \( b_i, b_j \in I_b \). Then, by Lemma 3 \( g' \) is proper and hence \( I_b \not\models I_r \).

Since the algorithm outputs “no,” there is a \( P \)-node \( P \) which is in either Case (4) or (5).

We first assume that the \( P \)-node \( P \) is in Case (4). Without loss of generality, at least two blue tokens \( b_1 \) and \( b_2 \) meet at this node \( P \). Then, the MPQ-tree \( T \) contains two red tokens \( r_1 \) and \( r_2 \) placed on \( g'(b_1) \) and \( g'(b_2) \), respectively.

Notice that, since \( b_1 \) and \( b_2 \) did not meet any red token before at the node \( P \), both \( r_1 \) and \( r_2 \) must be placed on either \( P \) or nodes in \( T \setminus T_P \). Then, the least common ancestor \( \text{LCA}(b_1, g'(b_1)) \) must be an ancestor of \( P \), and so is the least common ancestor \( \text{LCA}(b_2, g'(b_2)) \). Therefore, the nodes \( \text{LCA}(b_1, g'(b_1)) \) and \( \text{LCA}(b_2, g'(b_2)) \) are in the ancestor/descendant relationship, a contradiction.

Thus, the algorithm outputs “no” because the \( P \)-node \( P \) is in Case (5). In this case, without loss of generality, at least one blue token \( b_1 \) and at least one green token \( c_1 \) meet at this node \( P \). Then, the red token \( r_1 \) corresponding to \( g'(b_2) \) must be placed on either \( P \) or some node in \( T \setminus T_P \). Therefore, the least common ancestor \( \text{LCA}(b_1, g'(b_1)) \) is an ancestor of \( c_1 \). Note that \( c_1 \) corresponds to the least common ancestor of some pair of blue and red tokens, say \( b_1 \) and \( g'(b_1) \), and \( P \) is an ancestor of it. Therefore, the nodes \( \text{LCA}(b_1, g'(b_1)) \) and \( \text{LCA}(b_2, g'(b_2)) \) are in the ancestor/descendant relationship on \( T \), a contradiction. \( \square \)

### 4.4 Shortest reconfiguration sequence

To complete the proof of Theorem 2 we finally show that our algorithm in Section 4.3 can be modified so that it actually finds a shortest reconfiguration sequence between \( I_b \) and \( I_r \).

Once we know that \( I_b \models I_r \) holds by the \( O(n) \)-time algorithm in Section 4.3 we run it again with modification that “green” tokens are left at the corresponding least common ancestors. As in the proof of Lemma 3 we can now obtain a reconfiguration sequence \( S = \langle I_1, I_2, \ldots, I_\ell \rangle \) between \( I_b = I_1 \) and \( I_r = I_\ell \) such that each token \( t_i \), \( 1 \leq i \leq k \), is slid at most twice. It is sufficient to output \( I_{i+1} \setminus I_i \) and \( I_i \setminus I_{i+1} \), and hence the running time of the modified algorithm is proportional to \( \ell \), the number of independent sets in \( S \). Since \( k = |I_b| = O(n) \) and each token \( t_i \), \( 1 \leq i \leq k \), is slid at most twice in \( S \), we have \( \ell = O(n) \), that is, the length \( \ell \) of \( S \) is \( O(n) \). Therefore, the modified algorithm also runs in \( O(n) \) time and \( O(n) \) space. Notice that each token \( t_i \) is slid to its target vertex \( g^*(b_i) \)
along a shortest path (of length at most two) between $b_i$ and $g^*(b_i)$ without detour, and hence $S$ has the minimum length.

This completes the proof of Theorem 2.

5 Caterpillars

The main result of this section is the following theorem.

**Theorem 4.** The sliding token problem for a connected caterpillar $G = (V, E)$ and two independent sets $I_b$ and $I_r$ of $G$ can be solved in $O(n)$ time and $O(n)$ space. Moreover, for a yes-instance, a shortest reconfiguration sequence between them can be output in $O(n^2)$ time and $O(n)$ space.

Let $G = (S \cup L, E)$ be a caterpillar with spine $S$ which induces the path $(s_1, \ldots, s_{n'})$, and leaf set $L$. We assume that $n' \geq 2$, $\deg(s_1) \geq 2$, and $\deg(s_{n'}) \geq 2$. First we show that we can assume that each spine vertex has at most one leaf without loss of generality.

**Lemma 6.** For any given caterpillar $G = (S \cup L, E)$ and two independent sets $I_b$ and $I_r$ on $G$, there is a linear time reduction from them to another caterpillar $G' = (S' \cup L', E')$ and two independent sets $I'_b$ and $I'_r$ such that (1) $G$, $I_b$, and $I_r$ are a yes-instance of the sliding token problem if and only if $G'$, $I'_b$, and $I'_r$ are a yes-instance of the sliding token problem, (2) the maximum degree of $G'$ is at most 3, and (3) $\deg(s_1) = \deg(s_{n'}) = 2$, where $n' = |S'|$. In other words, the sliding token problem on a caterpillar is sufficient to consider only caterpillars of maximum degree 3.

**Proof.** On $G$, let $s_i$ be any vertex in $S$ with $\deg(s_i) > 3$. Then there exist at least two leaves $\ell_i$ and $\ell'_i$ attached to $s_i$ (note that they are weak twins). Now we consider the case that two tokens in $I_b$ are on $\ell_i$ and $\ell'_i$. Then, we cannot slide these two tokens at all, and any other token cannot pass through $s_i$ since it is blocked by them. If $I_r$ contains these two tokens also, we can split the problem into two subproblems by removing $s_i$ and its leaves from $G$, and solve it separately. Otherwise, the answer is “no” (remind that the problem is reversible; that is, if tokens cannot be slid, there are no other tokens which slide into the situation). Therefore, if at least two tokens are placed on the leaves of a vertex of the original graph, we can reduce the case in linear time. Thus we assume that every spine vertex with its leaves contains at most one token in $I_b$ and $I_r$, respectively. Then, by the same reason, we can remove all leaves but one of each spine vertex. More precisely, regardless whether $I_b \leftarrow^* I_r$ or $I_b \leftrightarrow^* I_r$, at most one leaf for each spine vertex is used for the transitions. Therefore, we can remove all other useless leaves but one from each spine vertex. Especially, removing all useless leaves, we have $\deg(s_1) = \deg(s_{n'}) = 2$. \hfill $\square$

Hereafter, we only consider the caterpillars stated in Lemma 6. That is, for any given caterpillar $G = (S \cup L, E)$ with spine $(s_1, \ldots, s_{n'})$, we assume that
deg(s_1) = deg(s_{n'}) = 2 and 2 ≤ deg(s_i) ≤ 3 for each 1 < i < n'. Then, we denote the unique leaf of s_i by ℓ_i if it exists.

We here introduce a key notion of the problem on these caterpillars that is named locked path. Let G and I be a caterpillar and an independent set of G, respectively. A path P = (p_1, p_2, ..., p_k) on G is locked by I if and only if

(a) k is odd and greater than 2,
(b) I ∩ P = {p_1, p_3, p_5, ..., p_k},
(c) deg(p_1) = deg(p_k) = 1 (in other words, they are leaves), and deg(p_3) = deg(p_5) = ⋅⋅⋅ = deg(p_{k-2}) = 2.

This notion is simplified version of a locked tree used in [7]. Using the discussion in [7], we obtain the condition for the immovable independent set on a caterpillar:

Theorem 5 ([7]). Let G and I be a caterpillar and an independent set of G, respectively. Then we cannot slide any token in I on G at all if and only if there exist a set of locked paths P_1, ..., P_h for some h such that I is a union of them.

The proof can be found in [7], and omitted here. Intuitively, for any caterpillar G and its independent set I, if I contains a locked path P, we cannot slide any token through the vertices in P. Therefore, P splits G into two subgraphs, and we obtain two completely separated subproblems. (We note that the endpoints of P are leaves with tokens, and their neighbors are spine vertices without tokens. This property admits us to cut the graph at the spine vertices on the locked path.) Therefore, we obtain the following lemma:

Lemma 7. For any given caterpillar G = (S ∪ L, E) and two independent sets I_b and I_r on G, there is a linear time reduction from them to another caterpillar G' = (S' ∪ L', E') and two independent sets I_{b}' and I_{r}' such that (1) G, I_b, and I_r are a yes-instance of the sliding token problem if and only if G', I_{b}', and I_{r}' are a yes-instance of the sliding token problem, and (2) both of I_{b}' and I_{r}' contain no locked path.

Proof. In G, when I_b contains a locked path P, it should be appear in I_r; otherwise, the answer is no. Therefore, we can remove all vertices in P and obtain the new graph G'' with two independent sets I_{b}'' := I_b \ P and I_{r}'' := I_r \ P such that G with I_b and I_r is a yes-instance if and only if G'' with I_{b}'' and I_{r}'' is a yes-instance. Repeating this process, we obtain disconnected caterpillar G and two independent sets I_{b} and I_{r} such that both of I_{b} and I_{r} contain no locked paths. On a disconnected graph, we can solve the problem separately for each connected component. Therefore, we can assume that the graph is connected, which completes the proof.

Hereafter, without loss of generality, we assume that the caterpillar G with two independent sets I_{b} and I_{r} satisfies the conditions in Lemmas 6 and 7. That is, each spine vertex s_i has at most one leaf ℓ_i, s_1 and s_{n'} have one leaf ℓ_1 and ℓ_{n'}, respectively, both of I_{b} and I_{r} contain no locked path, and |I_{b}| = |I_{r}|. By the result in [7], this is a yes-instance. Thus, it is sufficient to show an O(n^2) time algorithm that computes a shortest reconfiguration sequence between I_{b} and I_{r}.
Each pair \((s_i, \ell_i)\) can have at most one token. Therefore, without loss of generality, we can assume that the blue vertices \(b_1, b_2, \ldots, b_k\) in \(I_b\) are labeled from left to right (according to the order \((s_1, \ell_1), (s_2, \ell_2), \ldots, (s_n, \ell_n)\) of \(G\)), that is, \(L(b_i) < L(b_j)\) if \(i < j\); similarly, the red vertices \(r_1, r_2, \ldots, r_k\) are also labeled from left to right. Then, we define a target-assignment \(g: I_b \rightarrow I_r\), as follows: for each blue vertex \(b_i \in I_b\)

\[
g(b_i) = r_i. \tag{3}\]

To prove Theorem 4, it suffices to show that \(g\) is proper, and we can slide tokens with fewest detours. Here, any token cannot bypass the other token since each token is on a leaf or spine vertex. Thus, by the results in [7], it has been shown that \(g\) is proper. We show that we can compute a shortest reconfiguration in case analysis.

Now we introduce \textit{direction} of a token \(t\) denoted by \(\text{dir}(t)\) as follows: when \(t\) slides from \(v_i \in \{s_i, \ell_i\}\) in \(I_b\) to \(v_j \in \{s_j, \ell_j\}\) in \(I_r\) with \(i < j\), the direction of \(t\) is said to be \(R\) and denoted by \(\text{dir}(t) = R\). If \(i > j\), it is said to be \(L\) and denoted by \(\text{dir}(t) = L\). If \(i = j\), the direction of \(t\) is said to be \(C\) and denoted by \(\text{dir}(t) = C\).

We first consider a simple case: all directions are either \(R\) or \(L\). In this case, we can use the same idea appearing in the algorithm for a proper interval graph in Section 3. We can introduce a partial order over the tokens, and slide them straightforwardly using the same idea in Section 3.2. Intuitively, a sequence of \(R\) tokens are slid from left to right, and a sequence of \(L\) tokens are slid from right to left, and we can define a partial order over the sequences of different directions. The only additional considerable case is shown in Fig. 4. That is, when the token \(a\) slides to \(\ell_i\) from left and the other token \(c\) slides to \(s_{i+1}\) from right, \(a\) should precede \(c\). It is not difficult to see that this (and its symmetric case) is the only exception than the algorithm in Section 3.2 when all tokens slide to right or left. In other words, in this case, detour is required, and unavoidable.

We next suppose that \(I_b\) (and hence \(I_r\)) contains some token \(t\) with \(\text{dir}(t) = C\). In other words, \(t\) is put on \(s_i\) or \(\ell_i\) for some \(i\) in both of \(I_b\) and \(I_r\). We have five cases.

Case (1): \(t\) is put on \(\ell_i\) in \(I_b\) and \(I_r\). In this case, we have nothing to do; \(t\) does not need to be slid.
Case (2): $t$ is put on $s_i$ in $I_b$ and slid to $\ell_i$ in $I_r$. In this case, we first slide it from $s_i$ to $\ell_i$, and do nothing any more. Then no detour is needed for $t$.

Case (3): $t$ is put on $\ell_i$ in $I_b$ and slid to $s_i$ in $I_r$. In this case, we lastly slide it from $\ell_i$ to $s_i$, and no detour is needed for $t$ again.

Case (4): $t$ is put on $s_i$ in $I_b$ and $I_r$, and $\ell_i$ exists. Using a simple induction by the number of tokens, we can determine if $t$ should make a detour or not in linear time. If not, we never slide $t$. Otherwise, we first slide $t$ to $\ell_i$, and lastly slide back from $\ell_i$ to $s_i$. It is clear that the length of detour with respect to $t$ is as few as possible.

Case (5): $t$ is put on $s_i$ in $I_b$ and $I_r$, and $\ell_i$ does not exist. By assumption, $1 < s < n'$ (since $\ell_1$ and $\ell_{n'}$ exist). Without loss of generality, we suppose $t$ is the leftmost spine vertex having the condition. We first observe that $|I_b \cap \{s_{i-1}, \ell_{i-1}, s_{i+1}, \ell_{i+1}\}|$ is at most 1. Clearly, we have no token on $s_{i-1}$ and $s_{i+1}$. When we have two tokens on $\ell_{i-1}$ and $\ell_{i+1}$, the path $(\ell_{i-1}, s_{i-1}, s_i, s_{i+1}, \ell_{i+1})$ is a locked path, which contradicts the assumption. We also have $|I_r \cap \{s_{i-1}, \ell_{i-1}, s_{i+1}, \ell_{i+1}\}| \leq 1$ by the same argument. Now we consider the most serious case since the other cases are simpler and easier than this case. The most serious case is that a blue token on $\ell_{i-1}$ and a red token on $\ell_{i+1}$. Since any token cannot bypass the other, $I_b$ contains an L token on $\ell_{i-1}$, and $I_r$ contains an L token on $\ell_{i+1}$. In this case, by the L token on $\ell_{i-1}$, first, $t$ should make a detour to right, and by the L token in $I_r$, $t$ next should make a detour to left twice after the first detour. It is clear that this three slides should not be avoided, and this ordering of three slides cannot be violated. Therefore, $t$ itself should slide at least four times to return to the original position, and $t$ can done it in four slides. During this slides, since $t$ is the leftmost spine with this condition, the tokens on $s_1, \ell_1, s_2, \ell_2, \ldots, s_{i-1}, \ell_{i-1}$ do not make any detours. Thus we focus on the tokens on $s_{i+1}, \ell_{i+1}, \ldots$. Let $t'$ be the token that should be on $\ell_{i+1}$ in $I_r$. Since $t$ is on $s_i$, $t'$ is not on $\{s_{i+1}, \ell_{i+1}\}$. If $t'$ is on one of $\ell_{i+2}, s_{i+3}, \ell_{i+3}, s_{i+4}, \ldots$ in $I_b$, we have nothing to do; just make a detour for only $t$. The problem occurs when $t'$ is on $s_{i+2}$ in $I_b$. If there exists $\ell_{i+2}$, we first slide $t'$ to it, and this detour for $t'$ is unavoidable. If $\ell_{i+2}$ does not exist, we have to slide $t'$ to $s_{i+3}$ before slide of $t$. This can be done immediately except the only considerable case; when we have another L or S token $t''$ on $s_{i+3}$. We can repeat this process recursively and confirm that each detour is unavoidable. Since $G$ with $I_b$ and $I_r$ contains no locked path, this process will halts. (More precisely, this process will be stuck if and only if this sequence of tokens forms a locked path on $G$, which contradicts the assumption.) Therefore, traversing this process, we can construct the shortest reconfiguration sequence.

**Proof of Theorem 4.** For a given independent set $I_b$ on a caterpillar $G = (V, E)$, we can check if each vertex is a part of locked path as follows in $O(n)$ time (which is much simpler than the algorithm in [7]):

1. Initialize a state $S$ by “not locked path”.

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For \(i = 1, 2, \ldots, n', \) check \(s_i\) and \(\ell_i.\) We here denote their states by \((s_i, \ell_i) = (x, y),\) where \(x \in \{0, 1\}\) and \(y \in \{0, 1, -\}\) such that 1 means “token is placed on the vertex”, 0 means “no token is placed on the vertex”, and − means “the leaf does not exist.” In each case, update the state \(S\) as follows:

Case \((0, 1):\) If \(S\) is “not locked path,” set \(S\) by “locked path?,” and remember \(i\) as a potential left endpoint of a locked path. If \(S\) is “locked path,” \((s_i, \ell_i)\) is a part of locked path. Therefore, mark all vertices between the previously remembered left endpoint to this endpoint as “locked path”. After that, set \(S\) by “locked path?” again, and remember \(i\) as a potential left endpoint of the next locked path.

Case \((0, -\) and \((0, 0):\) If there is a token on \(s_{i-1},\) nothing to do. If there is no token on \(s_{i-1},\) reset \(S\) by “not locked path” (regardless of the previous state of \(S\)).

Case \((1, 0):\) reset \(S\) by “not locked path” (regardless of the previous state of \(S\)).

Case \((1, -):\) nothing to do.

Simple case analysis shows that after the procedure above, every vertex in a locked path is marked in \(O(n)\) time. Thus, we first run this procedure twice for \((G, I_b)\) and \((G, I_r)\) in \(O(n)\) time, and check whether the marked vertices coincide with each other. If not, the algorithm outputs “no”. Otherwise, the algorithm splits the caterpillar \(G\) into subgraphs \(G_1, G_2, \ldots, G_h\) induced by only unmarked vertices. Then we can solve the problem for each subgraph: we note that two endpoints of which tokens are placed of a locked path \(P\) are leaves. That is, for example, when a locked path \(P = (p_1, p_2, \ldots, p_k)\) splits \(G\) into \(G_1\) and \(G_2,\) the neighbors of \(G_1\) and \(G_2\) in \(P\) are \(p_2\) and \(p_{k-1},\) and there are no token on them. Thus, in the case, we can solve the problem on \(G_1\) and \(G_2\) separately, and we do not need to consider their neighbors.

For each subgraph \(G_1, \ldots, G_h,\) the algorithm next checks whether each subgraph contains the same number of blue and red tokens. If they do not coincide with each other, the algorithm output “no.” Otherwise, we have a yes-instance.

The correctness of the algorithm so far follows from Theorem 5 with results in [7] immediately. It is also easy to implement the algorithm to run in \(O(n)\) time and space.

It is not difficult to modify the algorithm to output the sequence itself based on the previous case analysis. For each token, the number of detours made by the token is bounded above by \(O(n)\), the number of slides of the token itself is also bounded above by \(O(n)\), and the computation for the token can be done in \(O(n)\) time. Therefore, the algorithm runs in \(O(n^2)\) time, and the length of the sequence is \(O(n^2).\) (As shown in the last paragraph in Section 8 there exist instances that require a shortest sequence of length \(\Theta(n^2).\)) \(\Box\)
6 Concluding Remarks

In this paper, we showed that the shortest sliding token problem can be solved in polynomial time for three subclasses of interval graphs. The computational complexity of the problem for chordal graphs, interval graphs, and trees are still open. Especially, tree seems to be the next target. We can decide if two independent sets are reconfigurable in linear time [7], then can we find a shortest sequence for a yes-instance in polynomial time? As in the 15-puzzle, finding a shortest sequence can be NP-hard. For a tree, we do not know that the length can be bounded by any polynomial or not. It is an interesting open question whether there is any instance on some graph classes whose reconfiguration sequence requires super-polynomial length.

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reachable?

Yes!