Extrapolation estimation for nonparametric regression with measurement error

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**Abstract**
For the nonparametric regression models with covariates contaminated with the normal measurement errors, this paper proposes an extrapolation algorithm to estimate the regression functions. By applying the conditional expectation directly to the kernel-weighted least squares of the deviations between the local linear approximation and the observed responses, the proposed algorithm successfully bypasses the simulation step in the classical simulation extrapolation, thus significantly reducing the computational time. It is noted that the proposed method also provides an exact form of the extrapolation function, although the extrapolation estimate generally cannot be obtained by simply setting the extrapolation variable to negative one in the fitted extrapolation function, if the bandwidth is less than the SD of the measurement error. Large sample properties of the proposed estimation procedure are discussed, as well as simulation studies and a real data example being conducted to illustrate its applications.

**KEYWORDS**
local linear smoothing, measurement error, nonparametric regression, simulation and extrapolation
1 INTRODUCTION

Due to its conceptual simplicity and the capability of harnessing the modern computational power, the simulation extrapolation (SIMEX) estimation procedure has been attracting significant attention from practical data analysts as well as theoretical researchers. The simplicity of the SIMEX lies in the fact that it allows us to directly use any standard estimates based on the known data as the building block, and its simulation nature makes the estimation process computer-dependent only. To be specific, suppose we want to estimate a parameter $\theta$, possibly multidimensional, in a statistical population $X$ of dimension $p \geq 1$. In certain situations where we cannot collect observations directly from $X$, what we observe is a surrogate value $Z$ of $X$. In measurement error literature, a classical assumption on the relationship between $X$ and $Z$ is $Z = X + U$, where $U$ is called the measurement error, which is often assumed to be independent of $X$, and has a normal distribution with mean 0 and known covariance matrix $\Sigma_u$. If there is an estimator $T(X)$ of $\theta$ when a sample $X = \{X_1,\ldots,X_n\}$ of size $n$ from $X$ is available, then when only $Z$ can be observed, the classical SIMEX procedure estimates $\theta$, using sample $Z = \{Z_1,\ldots,Z_n\}$ from $Z$, by going through the following three steps. First, we generate $n$ i.i.d. random vectors $V_i$’s from $N(0, \Sigma_u)$, select a nonnegative number $\lambda$, calculate $\tilde{Z}_i(\lambda) = Z_i + \sqrt{\lambda}V_i$ for $i = 1,2,\ldots,n$, and compute $T(\tilde{Z}(\lambda))$ based on $\tilde{Z}(\lambda) = \{\tilde{Z}_1(\lambda),\ldots,\tilde{Z}_n(\lambda)\}$. Second, we calculate the conditional expectation of $T(\tilde{Z}(\lambda))$ given $Z$. If the conditional expectation has a closed form, then it will be the estimate of $\theta$, otherwise, we repeat the previous step $B$ times to obtain $B$ values of $T_b(\tilde{Z}(\lambda))$, $b = 1,2,\ldots,B$, and the average $\bar{T}(\lambda)$ of these $B$ values of $T_b(\tilde{Z}(\lambda))$’s is computed. Finally, we repeat the first step and second step for a sequence of nonnegative $\lambda$ values, for example, $0 = \lambda_1 < \ldots < \lambda_K$ for some $K$. We denote these $K$ averages as $\bar{T}(\lambda_1),\ldots,\bar{T}(\lambda_K)$. To conclude, the trend of $\bar{T}(\lambda)$ with respect to $\lambda$ will be formulated as a function of $\lambda$, and the extrapolated value of this function at $\lambda = -1$ is the desired SIMEX estimate of $\theta$. In real applications, $K$ is suggested to be less than 20 and these $K$ $\lambda$-values are chosen equally spaced from $[0,2]$. The early development of the classical SIMEX estimation procedure can be found in Cook and Stefanski (1994), Stefanski and Cook (1995), and Carroll et al. (1996), with extensive applications in Mallick et al. (2002) for cox regression, Sevilmedu et al. (2019) for Log-logistic accelerated failure time models, Gould et al. (1999) for the catch-effort analysis, Hwang and Huang (2003), Stoklosa et al. (2016) for the capture-recapture models, Lin and Carroll (1999) for the analysis of the Framingham heart disease data using the logistic regression, Hardin et al. (2003) for generalized linear models, and Ponzi et al. (2019) for some applications in ecology and evolution, to name a few.

However, the discussion of the classical SIMEX estimation procedure in the nonparametric setup seems scant in the literature. Stefanski and Bay (1996) applied the SIMEX procedure to estimate the cumulative distribution function of a finite population based on the Horvitz–Thompson estimator. Since the conditional expectation of the Horvitz–Thompson estimator with the true variable replaced by the pseudo-data given the observable surrogates has an explicit form, the simulation step can be bypassed. Also, the quadratic function of $\lambda$ is shown to be a reasonable extrapolation function. Carroll et al. (1999) extended the classical SIMEX procedure to the nonparametric regression setup and it was implemented with the local linear estimator. In Carroll et al. (1999)’s work, the three steps in the classical SIMEX procedure are strictly followed. To estimate the unknown variance function in a general one-way analysis of variance model, Carroll and Wang (2008) proposed a permutation SIMEX estimation procedure to completely remove the bias after extrapolation. Wang et al. (2010) generalized Stefanski and Bay (1996)’s method to estimate the smooth distribution function in the presence of heteroscedastic normal measurement errors. Aiming at improving the SIMEX local linear estimator in Carroll et al. (1999), Staudenmayer
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and Ruppert (2004) introduced a new local polynomial estimator with the SIMEX algorithm. The improvement over the existing estimation procedure is made possible by using a bandwidth selection procedure. Again, Staudenmayer and Ruppert (2004)’s method still strictly followed the three-steps in the classical SIMEX.

Compared to various applications in both the parametric and nonparametric statistical models, the SIMEX procedure developed in Stefanski and Bay (1996) and Wang et al. (2010) successfully dodged the simulation step, which is the most time-consuming part in the classical SIMEX algorithm. The very reason why their methods work is that the averaged naive estimator from the pseudo-data, conditioning on the observed surrogates, has an explicit limit ready for extrapolation. Clearly, the strategy used in both references cannot be directly extended to other scenarios where such limits do not have user-friendly forms. In this paper, we will propose a new method, which in spirit is a variant of the classical SIMEX procedure, for estimating the nonparametric regression function but the new method successfully circumvents the simulation step.

2 | MOTIVATING EXAMPLES

In this section, we discuss two motivating examples which inspired our interest in searching for a more efficient bias reduction estimation procedure in the nonparametric setup. Our ambition is to keep the attractive feature of the extrapolation component in the classical SIMEX algorithm, while at the same time, significantly reduce the computational burden.

2.1 | Simple linear regression model

Let Y and X be two univariate random variables, which obey a simple linear relationship \( E(Y|X) = \alpha + \beta X \). Suppose we cannot observe X but we have data on \( Z = X + U \) and \( U \sim N(0, \sigma^2_u) \) with \( \sigma^2_u \) being known. As discussed in Carroll et al. (1999), for any fixed \( \lambda > 0 \), the average of the least squares slope estimates based on the pseudo-data consistently estimates \( g(\lambda) = \beta \sigma^2_X / (\sigma^2_X + (1 + \lambda)\sigma^2_u) \). Obviously, extrapolating \( \lambda \) to \(-1\), we have \( g(-1) = \beta \). This clearly shows that SIMEX works very well for linear regression model. In fact, in the seminal paper Cook and Stefanski (1994), the SIMEX estimators of \( \alpha \) and \( \beta \) can be derived without the simulation step. However, the derivation relies on a notion of NON-IID pseudo-errors. More details about the NON-IID pseudo-errors can be found in Cook and Stefanski (1994) and section 5.3.4.1 in Carroll et al. (2006). Here we would like to point out that the SIMEX estimators of \( \alpha \), \( \beta \) can be obtained from a different perspective.

Recall that the least squares (LS) estimator of \( \alpha \) and \( \beta \) can be obtained by minimizing the LS criterion \( \sum_{i=1}^{n}(Y_i - \alpha - \beta X_i)^2 \). Since \( X_i \) are not available, following the SIMEX idea, we generate the pseudo-data \( Z_i(\lambda) = Z_i + \sqrt{\lambda}V_i \), \( i = 1, 2, \ldots, n \). However, instead of following the classical SIMEX road map to minimize the LS target function \( \sum_{i=1}^{n}(Y_i - \alpha - \beta Z_i(\lambda))^2 \), we minimize the conditional expectation \( E\left[ \sum_{i=1}^{n}(Y_i - \alpha - \beta Z_i(\lambda))^2 \mid D \right] \), where \( D = (Y, Z) \), \( Y = (Y_1, \ldots, Y_n) \) and \( Z = (Z_1, \ldots, Z_n) \).

Since \( V_i \)'s are i.i.d. from \( N(0, \sigma^2_u) \) and independent of other random variables in the model, so this conditional expectation equals \( \sum_{i=1}^{n}(Y_i - \alpha - \beta Z_i)^2 + n\lambda \beta^2 \sigma^2_u \). The minimizer of the above expression is simply \( \hat{\beta}(\lambda) = (S_{ZZ} + \lambda \sigma^2_u)^{-1}S_{YZ} \) and \( \hat{\alpha}(\lambda) = \bar{Y} - \hat{\beta}(\lambda) \bar{X} \). By choosing \( \lambda = -1 \), we immediately have the commonly used bias-corrected estimators or the SIMEX estimators derived using NON-IID pseudo-errors. Note that here not only do we not need the simulation step, but also the extrapolation step is unnecessary.
2.2 Kernel density estimation

Suppose we want to estimate the density function \( f_X(x) \) of \( X \) in the measurement error model \( Z = X + U \). When observations can be made directly on \( X \), the kernel density estimation procedure is often called on for this purpose. Starting with the classical kernel estimator, Wang et al. (2009) followed the classical SIMEX algorithm, constructed an average of the kernel estimator \( \hat{f}_{B,n}(x) = B^{-1} \sum_{b=1}^{B} \left[ n^{-1} \sum_{i=1}^{n} K_h(x - Z_i - \sqrt{\lambda} V_{i,b}) \right] \) with \( B \) pseudo-datasets \( \{ Z_i + \sqrt{\lambda} V_{i,b} \}_{i=1}^{n}, b = 1, 2, \ldots, B \), where \( K_h(\cdot) = h^{-1} K(\cdot/h), K \) is a kernel density function, and \( h \) is a sequence of positive numbers often called bandwidths. By the law of large numbers, \( \hat{f}_{B,n}(x) \to n^{-1} \sum_{i=1}^{n} \int K_h(x - Z_i - \sqrt{\lambda} u) \phi(u) du = \tilde{f}_n(x) \) in probability as \( B \to \infty \). After some algebra, Wang et al. (2009) proposed to estimate \( f_X(x) \) using \( \hat{f}_n(x) = n^{-1} \sum_{i=1}^{n} (\sqrt{\lambda} \sigma_u)^{-1} \phi((x - Z_i)/\sqrt{\lambda} \sigma_u) \) which approximates the limit \( \tilde{f}_n(x) \) for sufficiently large \( n \). In fact, before initiating the simulation step, Cook and Stefanski (1994) suggested one should try to calculate the conditional expectation \( E[\tilde{f}_{B,n}(x)|Z] \) first. If this conditional expectation has a tractable form, then it will be chosen as the SIMEX estimator. Clearly, the conditional expectation is simply \( \tilde{f}_n(x) \). It is interesting to note that if we deliberately choose the kernel function \( K \) to be the standard norm density, we can show that \( \tilde{f}_n(x) = (n \sqrt{\lambda} \sigma_u^2 + h^2)^{-1} \sum_{i=1}^{n} \phi((x - Z_i)/\sqrt{\lambda} \sigma_u + h^2) \) which can also be directly used for extrapolation. Because there is no approximation done here, \( \tilde{f}_n(x) \) should potentially perform better than the estimator \( \tilde{f}_n(x) \) as proposed in Wang et al. (2009).

It is easy to see that the technique used in the kernel density estimation cannot be extended to the regression setup, since the commonly used kernel regression estimators, either the Nadaraya–Watson estimator, or the local linear estimator, often appear as a fraction of kernel components, which fails to provide a tractable conditional expectation for direct extrapolation. However, the observation of recovering the commonly used bias-corrected estimators or the SIMEX estimators derived using NON-IID pseudo-errors in the linear errors-in-variables regression indicates that we could have some interesting findings if we can apply the conditional expectation argument directly on the target functions, instead of computing the conditional expectation of the resulting naive estimator. In the next section, we will implement this idea via estimating the nonparametric regression function using a local linear smoothing procedure.

3 EXTRAPOLATION ESTIMATION PROCEDURE VIA LOCAL LINEAR SMOOTHER

For the sake of simplicity, we restrict ourselves to the univariate predictor cases. The proposed methodology can handle the multivariate predictor cases very well at the cost of introducing more complex notations. To be specific, suppose that the random pair \((X, Y)\) obeys the following nonparametric regression model

\[
Y = g(X) + \epsilon, \quad Z = X + U, \tag{1}
\]

with the common assumption on \( \epsilon, E(\epsilon|X) = 0 \) and \( 0 < \tau^2(X) = E(\epsilon^2|X) < \infty \). \( X \) and \( U \) are independent and \( U \) has a normal distribution \( N(0, \sigma_U^2) \) with known \( \sigma_U^2 \). If \((X, Y)\) are available, the local linear estimator for \( g(x) \) at a fixed \( x \)-value in the domain of \( X \) is defined as

\[
\hat{g}_n(x) = \frac{S_{2n}(x)T_{0n}(x) - S_{1n}(x)T_{1n}(x)}{S_{2n}(x)S_{0n}(x) - S_{1n}^2(x)},
\]
where
\[
S_{jn}(x) = n^{-1} \sum_{i=1}^{n} (X_i - x)^{j} K_h(X_i - x),
\]
and \( j = 0, 1, 2 \) for \( S_{jn}(x) \), \( j = 0, 1 \) for \( T_{jn}(x) \), \( K_h(\cdot) = h^{-1} K(\cdot/h) \). In the measurement error setup, a classical SIMEX estimator of \( g \) can be obtained through three steps: simulation, estimation, and extrapolation. For the sake of completeness, the following algorithm provides a detailed guideline for implementing the three steps in estimating \( g(x) \) from data on \( Y, Z \).

**SIMEX Algorithm of Local Linear Smoother**

1. Preselect a sequence of positive numbers \( \lambda = \lambda_1, \ldots, \lambda_K \).
2. For \( \lambda = \lambda_1 \), repeat the following steps \( B \) times. At the \( b \)-th repetition,
   (i) Generate \( n \) i.i.d. random observations \( V_{i,b} \)'s from \( N(0, \sigma_v^2) \), and calculate \( Z_{i,b}(\lambda) = Z_i + \sqrt{\lambda} V_{i,b} = X_i + U_i + \sqrt{\lambda} V_{i,b}, i = 1, 2, \ldots, n. \)
   (ii) Compute
   \[
   \hat{g}_{n,b}(x; \lambda_1) = \frac{S_{2nb}(x) T_{0nb}(x) - S_{1nb}(x) T_{1nb}(x)}{S_{2nb}(x) S_{0nb}(x) - S_{1nb}(x)^2},
   \]
   where
   \[
   S_{jnb}(x) = \frac{1}{n} \sum_{i=1}^{n} (Z_{i,b}(\lambda) - x)^{j} K_h(Z_{i,b}(\lambda) - x), \quad j = 0, 1, 2,
   \]
   \[
   T_{lnb}(x) = \frac{1}{n} \sum_{i=1}^{n} (Z_{i,b}(\lambda) - x)^{l} Y_i K_h(Z_{i,b}(\lambda) - x), \quad l = 0, 1.
   \]
3. Calculate \( \hat{g}_{n,b}(x; \lambda_1) = B^{-1} \sum_{b=1}^{B} \hat{g}_{n,b}(x; \lambda_1) \).
4. Repeat (2) and (3) for \( \lambda = \lambda_2, \ldots, \lambda_K \).
5. Identify a parametric trend of the pairs \( (\lambda_k, \hat{g}_{n,b}(x; \lambda_k)) \), \( k = 1, 2, \ldots, K \) and denote the trend as a function \( \Gamma(x; \lambda) \). The SIMEX estimator of \( g \) is defined as \( \hat{g}_{\text{SIMEX}}(x) = \Gamma(x; -1) \).

As a rough guideline, the \( \lambda \) values are often selected as a sequence of equally spaced grid points from \([0, 2], K \) is a positive integer as small as 5 or as large as 20, and \( B \) is often chosen to be 100 or above. With such choices, one can see the classical SIMEX procedure is computationally intensive.

We start with the commonly used local linear procedure. If \( X \) can be observed, then based on a sample \((X_i, Y_i), i = 1, 2, \ldots, n \) from model (1), the local linear estimator of the regression function \( g \), as well as its first-order derivative at \( x \), can be obtained by minimizing the following target function \( L(\beta_0, \beta_1) = \sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1 (X_i - x))^2 K_h(X_i - x) \) with respect to \( \beta_0 \) and \( \beta_1 \). In fact, the solution of \( \beta_0 \) is the local linear estimator of \( g(x) \) and \( \beta_1 \) is the local linear estimator of \( g'(x) \).

For a positive constant \( \lambda \), we replace \( X_i \) with the pseudo-data \( Z_i(\lambda) = Z_i + \sqrt{\lambda} V_i \) in the weighted least squares \( L(\beta_0, \beta_1) \), and calculate its conditional expectation given \((Z_i, Y_i), i =\)
1, 2, ..., n. A straightforward calculation shows that the minimizer of
\[ \sum_{i=1}^{n} E \left( [Y_i - \hat{\beta}_0 - \hat{\beta}_1 (Z_i(\lambda) - x)]^2 K_h(x - Z_i(\lambda))[(Y_i, Z_i)] \right), \]

with respect to \( \hat{\beta}_0, \hat{\beta}_1 \) is given by the solution of the following equations
\[ \begin{cases} \sum_{i=1}^{n} E \left( [Y_i - \hat{\beta}_0 - \hat{\beta}_1 (Z_i(\lambda) - x)] K_h(Z_i(\lambda) - x) [(Y_i, Z_i)] = 0, \\ \sum_{i=1}^{n} E \left( [Y_i - \hat{\beta}_0 - \hat{\beta}_1 (Z_i(\lambda) - x)] (Z_i(\lambda) - x) K_h(Z_i(\lambda) - x) [(Y_i, Z_i)] = 0. \end{cases} \tag{2} \]

The choice of kernel function \( K \) is not critical in theory, but for the ease of computation, choosing \( K \) to be standard normal can bring us extra benefits. In fact, with such a choice, together with the normality of the measurement error, the conditional expectations in (2) have explicit forms. Note that \( V_i \)'s are i.i.d. from \( N(0, \sigma^2_u) \) and independent of \((Z_i, Y_i)\), routine calculation (see Appendix) shows that
\[ E[K_h(\lambda Z_i - x) | Y, Z] = \phi(x; Z, h^2 + \lambda \sigma^2_u), \tag{3} \]
\[ E[(\lambda Z_i - x) K_h(\lambda Z_i - x) | Y, Z] = \frac{h^2}{h^2 + \lambda \sigma^2_u} (Z - x) \phi(x; Z, h^2 + \lambda \sigma^2_u), \tag{4} \]
\[ E[(\lambda Z_i - x)^2 K_h(\lambda Z_i - x) | Y, Z] = \frac{h^4}{(h^2 + \lambda \sigma^2_u)^2} (Z - x)^2 \phi(x; Z, h^2 + \lambda \sigma^2_u) + \frac{\lambda \sigma^2_u h^2}{h^2 + \lambda \sigma^2_u} \phi(x; Z, h^2 + \lambda \sigma^2_u), \tag{5} \]

here, also throughout this paper, \( \phi(x; \mu, \sigma^2_u) \) denotes the normal density function with mean \( \mu \) and variance \( \sigma^2_u \). Denote \( A_n(x) = n^{-1} \sum_{i=1}^{n} (Z_i - x)^j \phi(x; Z_i, h^2 + \lambda \sigma^2_u) \) for \( j = 0, 1, 2 \), and \( B_n(x) = n^{-1} \sum_{i=1}^{n} Y_i (Z_i - x)^j \phi(x; Z_i, h^2 + \lambda \sigma^2_u) \) for \( l = 0, 1 \). Then the solution of \((\hat{\beta}_0, \hat{\beta}_1)\) of Equation (2), or \((\hat{g}_n(x; \lambda), \hat{g}'_n(x; \lambda))\) is
\[ \begin{pmatrix} \hat{g}_n(x; \lambda) \\ \hat{g}'_n(x; \lambda) \end{pmatrix} = \begin{pmatrix} A_{n0}(x) & r(\lambda, h) A_{n1}(x) \\ r(\lambda, h) A_{n1}(x) & r(\lambda, h) [A_{n2}(x) + \lambda \sigma^2_u A_{n0}(x)] \end{pmatrix}^{-1} \begin{pmatrix} B_{n0}(x) \\ r(\lambda, h) B_{n1}(x) \end{pmatrix}, \tag{6} \]

where \( r(\lambda, h) = h^2 / (h^2 + \lambda \sigma^2_u) \).

Note that (6) itself can be used for extrapolation. However, unlike the estimator \( \hat{\lambda}(\lambda), \hat{\alpha}(\lambda) \) derived in the example of the linear regression, \( \lambda = -1 \) cannot be plugged directly into (6) to get the SIMEX estimator. In fact, when the sample size \( n \) gets bigger, the bandwidth \( h \) decreases to 0, resulting a negative \( h^2 + \lambda \sigma^2_u \) for \( \lambda = -1 \), which can not be the variance of a normal distribution. This implies the extrapolation step is necessary.

Therefore, we propose the following two-step extrapolation (EX) procedure to obtain an estimate of the regression function \( g \).

**EX Algorithm of The Local Linear Smoother**

1. For each \( \lambda \) from the preselected sequence \( \lambda = \lambda_1, \ldots, \lambda_K \), calculate \( \hat{g}_n(x; \lambda) \);
2. Identify a trend of the pairs \((\lambda_k, \hat{g}_n(x; \lambda_k)), k = 1, 2, \ldots, K \). Denote the trend as a function \( \Gamma(x; \lambda) \), respectively.

Then, the EX estimator of \( g \) is defined by \( \hat{g}_{EX}(x) = \Gamma(x; -1) \).
Obviously, the above EX algorithm is much more efficient than the classical three-step SIMEX algorithm. It is also easy to see that \( \hat{g}_n(x; \lambda) \) from the EX algorithm is not the limit of \( \hat{g}_{n,b}(x; \lambda) \) in the SIMEX algorithm as \( B \to \infty \). In fact, given the observed data \((Z_i, Y_i)_{i=1}^n\), by the law of large numbers, for a fixed \( \lambda \), as \( B \to \infty \), \( \hat{g}_{n,b}(x; \lambda) = B^{-1} \sum_{b=1}^B \hat{g}_{n,b}(x; \lambda) \) converges to \( \hat{g}_n(x; \lambda) \) in probability, where

\[
\hat{g}_n(x; \lambda) = \int \frac{S_{2n}(x, \nu)T_{0n}(x, \nu) - S_{1n}(x, \nu)T_{1n}(x, \nu)}{S_{2n}(x, \nu)S_{0n}(x, \nu) - S_{1n}^2(x, \nu)} \phi(\nu; 0, \lambda \sigma_n^2) d\nu,
\]

\( \nu = (v_1, \ldots, v_n)^T, \phi(\nu; 0, \lambda \sigma_n^2) = \prod_{i=1}^n \phi(v_i; 0, \lambda \sigma_n^2), \) and

\[
S_{jn}(x, \nu) = \frac{1}{n} \sum_{i=1}^n (Z_i + v_i - x)^j K_0(Z_i + v_i - x), \quad j = 0, 1, 2,
\]

\[
T_{ln}(x, \nu) = \frac{1}{n} \sum_{i=1}^n (Z_i + v_i - x)^l Y_i K_0(Z_i + v_i - x), \quad l = 0, 1.
\]

However, the estimator \( \tilde{g}_n(x; \lambda) \) defined in (6) has the form of

\[
\tilde{g}_n(x; \lambda) = \frac{\tilde{S}_{2n}(x)\tilde{T}_{0n}(x) - \tilde{S}_{1n}(x)\tilde{T}_{1n}(x)}{\tilde{S}_{2n}(x)\tilde{S}_{0n}(x) - \tilde{S}_{1n}^2(x)}
\]

\[
= \frac{A_{n2}(x)B_{n0}(x) + \lambda \sigma_n^2 A_{n0}(x)B_{n0}(x) - r(\lambda, h)A_{n1}(x)B_{n1}(x)}{A_{n2}(x)A_{n0}(x) + \lambda \sigma_n^2 A_{n0}^2(x) - r(\lambda, h)A_{n1}^2(x)},
\]

where

\[
\tilde{S}_{jn}(x) = \int S_{jn}(x, \nu) \phi(\nu; 0, \lambda \sigma_n^2) d\nu, \quad \tilde{T}_{ln}(x) = \int T_{ln}(x, \nu) \phi(\nu; 0, \lambda \sigma_n^2) d\nu,
\]

for \( j = 0, 1, 2 \) and \( l = 0, 1 \), respectively. Therefore, \( \tilde{g}_n(x; \lambda) \) is different from \( \hat{g}_n(x; \lambda) \), which indicates that \( \hat{g}_n(x; \lambda) \) from the SIMEX algorithm is not the limit of the EX algorithm as \( B \to \infty \). In fact, \( \hat{g}_n(x; \lambda) \) can be viewed as the limit of \( \hat{g}_{b,n}(x; \lambda) \) with \( S_{jn}(b, x) \) and \( T_{ln}(b, x) \) replaced by \( B^{-1} \sum_{b=1}^B S_{jn}(x, \nu) \) and \( B^{-1} \sum_{b=1}^B T_{ln}(x, \nu) \), \( j = 0, 1, 2, l = 0, 1 \), respectively, as \( B \to \infty \). We may extrapolate \( \tilde{g}_n(x; \lambda) \) back to \( \lambda = -1 \) to get the SIMEX estimator, but the proposed EX procedure is more computationally efficient than the classical SIMEX since certain numerical integrations are needed to obtain \( \hat{g}_n(x; \lambda) \).

### 4 | ASYMPTOTIC THEORY OF EX ALGORITHM

In this section, we shall investigate the large sample behaviours for the EX algorithm proposed in the previous section. We will show that as \( n \to \infty \), \( \hat{g}_n(x; \lambda) \) indeed converges to a function of both \( x \) and \( \lambda \), but the latter can approximate the true regression function \( g(x) \) as \( \lambda \to -1 \), thus justifying the effectiveness of extrapolation. The asymptotic joint distribution of \( \hat{g}_n(x; \lambda) \) at different \( \lambda \) values, including \( \lambda = 0 \) which corresponds to the naive estimator, will be also discussed.

The following is a list of regularity conditions needed to justify all the theoretical derivations.
C1. \( f_X(x), g(x) \) \( \tau^2(x) = \mathbb{E}(\varepsilon^2 | X = x), \mu(x) = \mathbb{E}|\varepsilon| \mathbb{E}|X = x) \) are twice continuously differentiable; also for each \( x \) in the support of \( X \), as a function of \( t \), \( \eta(t + x), \eta''(t + x) \in L_2(\Phi(t, 0, \sigma_n^2)) \), where \( \eta \) is a generic function denoting \( f_X, g, \tau^2, \mu \).

C2. The bandwidth \( h \) satisfies \( h \to 0, nh \to \infty \) as \( n \to \infty \).

For the sake of brevity, we write \( \tilde{S}_{nj}(x) \) and \( \tilde{T}_{nj}(x) \) as \( \tilde{S}_{nj} \) and \( \tilde{T}_{nj} \), respectively. Adding and subtracting the expectations from \( \tilde{S}_{nj} \) and \( \tilde{T}_{nj} \), \( \hat{g}_n(x; \lambda) \) can be decomposed into

\[
\hat{g}_n(x; \lambda) = \frac{E\tilde{S}_{n2} \tilde{T}_{n0} - E\tilde{S}_{n1} \tilde{T}_{n1}}{E\tilde{S}_{n2} \tilde{S}_{n0} - [E\tilde{S}_{n1}]^2} + \frac{1}{D_n} \left[ \sum_{j=0}^{2} C_{nj}(\tilde{S}_{nj} - E\tilde{S}_{nj}) + \sum_{l=0}^{2} D_{nl}(\tilde{T}_{nl} - E\tilde{T}_{nl}) \right],
\]

where

\[
D_n = (\tilde{S}_{n2} \tilde{S}_{n0} - \tilde{S}_{n1}^2)(E\tilde{S}_{n2} \tilde{S}_{n0} - (E\tilde{S}_{n1})^2), \quad C_{n0} = E\tilde{S}_{n2}[E\tilde{S}_{n1} \tilde{T}_{n1} - E\tilde{S}_{n2} \tilde{T}_{n0}], \quad C_{n1} = 2E\tilde{S}_{n1} \tilde{S}_{n2} \tilde{T}_{n0} - (E\tilde{S}_{n1})^2 \tilde{T}_{n1} - E\tilde{T}_{n1} \tilde{S}_{n2} \tilde{S}_{n0},
\]

\[
C_{n2} = E\tilde{S}_{n1}[E\tilde{S}_{n0} \tilde{T}_{n1} - E\tilde{T}_{n0} \tilde{S}_{n1}], \quad D_{n0} = E\tilde{S}_{n2}[E\tilde{S}_{n2} \tilde{S}_{n0} - (E\tilde{S}_{n1})^2], \quad D_{n1} = E\tilde{S}_{n1}[(E\tilde{S}_{n1})^2 - E\tilde{S}_{n2} \tilde{S}_{n0}].
\]

To proceed, for integers \( j \) we denote

\[
f_{j,\lambda}(x) = \int \phi(t; x, (\lambda + 1)\sigma_n^2)dt, \quad g_{j,\lambda}(x) = \int t^l g(t) \phi(t; x, (\lambda + 1)\sigma_n^2)dt,
\]

\[
G_{j,\lambda}(x) = \int t^l g^2(t) \phi(t; x, (1 + \lambda)\sigma_n^2)dt, \quad H_{j,\lambda}(x) = \int t^l \tau^2(t) f_X(t) \phi(t; x, (1 + \lambda)\sigma_n^2)dt.
\]

By a routine and tedious calculation, we can show that the first term in the decomposition of \( \hat{g}_n(x; \lambda) \) has the following asymptotic expansion.

**Theorem 1.** Under conditions C1 and C2, for each \( \lambda \geq 0 \), we have

\[
\frac{E\tilde{S}_{n2} \cdot E\tilde{T}_{n0} - E\tilde{S}_{n1} \cdot E\tilde{T}_{n1}}{E\tilde{S}_{n2} \cdot E\tilde{S}_{n0} - [E\tilde{S}_{n1}]^2} = \frac{g_{0,\lambda}(x)}{f_{0,\lambda}(x)} + h^2 B(x; \lambda) + o(h^2),
\]

as \( n \to \infty \), where \( B(x; \lambda) \) equals

\[
\frac{f_{0,\lambda}(x)(g''_{0,\lambda}(x) - f''_{0,\lambda}(x))g_{0,\lambda}(x)}{2f_{0,\lambda}^2(x)} + \frac{(f_{1,\lambda}(x) - xf_{0,\lambda}(x))(g_{0,\lambda}(x)f_{1,\lambda}(x) - f_{0,\lambda}(x)g_{1,\lambda}(x))}{(\lambda + 1)^2 \sigma_n^4 f_{0,\lambda}^2(x)},
\]

where \( \tilde{S}_{nj} \) and \( \tilde{T}_{nl} \) for \( j = 0, 1, 2 \) and \( l = 0, 1 \) are defined in (9).

Note that, as \( \lambda \to -1, g_{0,\lambda}(x) \to g(x)f_X(x), \) and \( f_{0,\lambda}(x) \to f_X(x) \). Thus,

\[
f_{1,\lambda}(x) - xf_{0,\lambda}(x) = \int (t - x)f_X(t)\phi(t; x, (\lambda + 1)\sigma_n^2)dt = (\lambda + 1)\sigma_n^2 f_X^2(x) + o((\lambda + 1)\sigma_n^2),
\]

and \( g_{1,\lambda}(x) - xg_{0,\lambda}(x) \) can be written as

\[
\int (t - x)g(t)f_X(t)\phi(t; x, (\lambda + 1)\sigma_n^2)dt = (\lambda + 1)\sigma_n^2 [g f_X]'(x) + o((\lambda + 1)\sigma_n^2).
\]
as \( \lambda \to -1 \). We can easily show that

\[
B(x; \lambda) = \frac{f_{0,\lambda}(x)g''_{0,\lambda}(x) - f'_{0,\lambda}(x)g_{0,\lambda}(x)}{2f^2_{0,\lambda}(x)} + \frac{g_{0,\lambda}(x)(f''_{x}(x))^2}{f^3_{0,\lambda}(x)} - \frac{f_x(x)[g_{x}(x)']^2}{f^2_{0,\lambda}(x)} + o(1),
\]

where \( o(1) \) denotes that the corresponding terms converge to 0 as \( \lambda \to -1 \). Therefore, we have \( \lim_{\lambda \to -1} B(x; \lambda) = g''(x)/2 \). Also, from Theorem 1, we can easily see that

\[
\lim_{\lambda \to -1} \frac{E\tilde{S}_{n2}(x) \cdot ET_{n0}(x) - E\tilde{S}_{n1}(x) \cdot ET_{n1}(x)}{E\tilde{S}_{n2}(x) \cdot E\tilde{S}_{n0}(x) - [E\tilde{S}_{n1}(x)]^2} = g(x).
\]

To investigate the asymptotic distribution of \( \hat{g}_n(x; \lambda) \), denote

\[
\begin{align*}
c_{0,\lambda}(x) &= -\frac{g_{0,\lambda}(x)}{f_{0,\lambda}(x)}, & c_{1,\lambda}(x) &= \frac{2[f_{1,\lambda}(x) - x f_{0,\lambda}(x)]g_{0,\lambda}(x) - [g_{1,\lambda}(x) - x g_{0,\lambda}(x)]f_{0,\lambda}(x)}{(\lambda + 1)\sigma^2_{0,\lambda}(x)}, \\
c_{2,\lambda}(x) &= \frac{[f_{1,\lambda}(x) - x f_{0,\lambda}(x)][g_{1,\lambda}(x) - x g_{0,\lambda}(x)]f_{0,\lambda}(x) - [f_{1,\lambda}(x) - x f_{0,\lambda}(x)]^2g_{0,\lambda}(x)}{(\lambda + 1)^2\sigma^4_{0,\lambda}(x)}, \\
d_{0,\lambda}(x) &= \frac{1}{f_{0,\lambda}(x)}, & d_{1,\lambda}(x) &= -\frac{f_{1,\lambda}(x) - x f_{0,\lambda}(x)}{(\lambda + 1)\sigma^2_{0,\lambda}(x)}.
\end{align*}
\]

Then, from Lemma 6 to Lemma 10 in Appendix, we can show that, for \( \lambda \geq 0, D^{-1}_{n}C_n = c_{j,\lambda}(x) + o_p(1), \) for \( j = 0, 1, 2 \) and \( D^{-1}_{n}D_{nj} = d_{j,\lambda}(x) + o_p(1) \) for \( j = 0, 1 \). Define

\[
\begin{align*}
\xi_{0,\lambda}(x) &= \frac{h^2}{h^2 + \lambda \sigma^2_x} \left[ (Z_i - x)\phi(x; Z_i, h^2 + \lambda \sigma^2_x) - E(Z - x)\phi(x; Z, h^2 + \lambda \sigma^2_x) \right], \\
\xi_{1,\lambda}(x) &= \frac{h^4}{(h^2 + \lambda \sigma^2_x)^2} \left[ (Z_i - x)^2\phi(x; Z_i, h^2 + \lambda \sigma^2_x) - E(Z - x)^2\phi(x; Z, h^2 + \lambda \sigma^2_x) \right] \\
&\quad + \frac{\lambda \sigma^2_x h^2}{h^2 + \lambda \sigma^2_x} \left[ \phi(x; Z_i, h^2 + \lambda \sigma^2_x) - E\phi(x; Z, h^2 + \lambda \sigma^2_x) \right], \\
\eta_{0,\lambda}(x) &= Y_i\phi(x; Z_i, h^2 + \lambda \sigma^2_x) - EY\phi(x; Z, h^2 + \lambda \sigma^2_x), \\
\eta_{1,\lambda}(x) &= \frac{h^2}{h^2 + \lambda \sigma^2_x} \left[ Y_i(Z_i - x)\phi(x; Z_i, h^2 + \lambda \sigma^2_x) - EY(Z - x)\phi(x; Z, h^2 + \lambda \sigma^2_x) \right].
\end{align*}
\]

Then from Theorem 1, we have

\[
\hat{g}_n(x; \lambda) = \frac{g_{0,\lambda}(x)}{f_{0,\lambda}(x)} + h^2 B(x; \lambda) + o(h^2) + \sum_{j=0}^{2} [c_{j,\lambda}(x) + o(1)](\tilde{S}_{nj} - E\tilde{S}_{nj}) + \sum_{k=0}^{1} [d_{k,\lambda}(x) + o(1)](\tilde{T}_{nk} - ET_{nk}).
\]

Since the terms \( o(1) \) in the above expression does not affect the asymptotic distribution of \( \hat{g}_n(x; \lambda) \), so we can safely neglect the \( o(1) \) terms from the sum, and therefore the two sums can be written.
as an i.i.d. average $n^{-1} \sum_{i=1}^{n} v_{i\lambda}(x)$, where $v_{i\lambda}(x)$ is defined by

$$c_{0\lambda}(x)\xi_{0\lambda,i}(x) + c_{1\lambda}(x)\xi_{1\lambda,i}(x) + c_{2\lambda}(x)\xi_{2\lambda,i}(x) + d_{0\lambda}(x)\eta_{0\lambda,i}(x) + d_{1\lambda}(x)\eta_{1\lambda,i}(x).$$

By verifying the Lyapunov condition, we can show that for each $\lambda > 0$, $n^{-1} \sum_{i=1}^{n} v_{i\lambda}(x)$ is asymptotically normal. This asymptotic normality is summarized in the following theorem.

**Theorem 2.** Under conditions C1 and C2, for each $\lambda > 0$,

$$\sqrt{n} \left\{ \hat{g}_n(x; \lambda) - \frac{g_{0\lambda}(x)}{f_{0\lambda}(x)} - h^2 B(x; \lambda) + o(h^2) \right\} \Rightarrow N(0, \Delta_{\lambda,\lambda}(x)),
$$

and for $\lambda = 0$,

$$\sqrt{n} h \left\{ \hat{g}_n(x; 0) - \frac{g_{0,0}(x)}{f_{0,0}(x)} - h^2 B(x; 0) + o(h^2) \right\} \Rightarrow N(0, \Delta_{0,0}(x)),
$$

where

$$\Delta_{\lambda,\lambda}(x) = c_{0\lambda}^2 \left[ \frac{f_{0\lambda/2}(x)}{2\sqrt{\pi \lambda \sigma_u^2}} - \frac{f_{0\lambda}(x)}{2\sqrt{\pi \lambda \sigma_u^2}} \right] + d_{0\lambda}^2 \left[ \frac{G_{0\lambda/2}(x) + H_{0\lambda/2}(x)}{2\sqrt{\pi \lambda \sigma_u^2}} - \frac{g_{0\lambda}(x)}{f_{0\lambda}(x)} \right]
+ 2c_{0\lambda}d_{0\lambda} \left[ \frac{g_{0\lambda/2}(x)}{2\sqrt{\pi \lambda \sigma_u^2}} - \frac{g_{0\lambda}(x)f_{0\lambda}(x)}{f_{0\lambda}(x)} \right],
$$

and

$$\Delta_{0,0}(x) = \frac{1}{2\sqrt{\pi}} \left[ \frac{G_{00}(x) + H_{00}(x)}{f_{00}(x)} - \frac{g_{00}^2(x)}{f_{00}^3(x)} \right].$$

When $\sigma_u^2 = 0$, that is, no measurement error in $X$, then one can easily see that $\Delta_{0,0}(x) = \tau^2(x)/(2\sqrt{\pi} f_X(x))$, which is exactly the asymptotic variance in local linear estimator of the regression function in the error-free cases. The theorem below states the asymptotic joint normality of $[\hat{g}_n(x; 0), \hat{g}_n(x; \lambda_1), \ldots, \hat{g}_n(x; \lambda_K)]'$.

**Theorem 3.** Under conditions C1 and C2, for $0 < \lambda_1 < \cdots < \lambda_K < \infty$,

$$\begin{pmatrix}
\sqrt{n} h & 0 & \cdots & 0 \\
0 & \sqrt{n} h & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \sqrt{n}
\end{pmatrix}
\begin{pmatrix}
\hat{g}_n(x; 0) - g_{0,0}(x)/f_{0,0}(x) - h^2 B(x; 0) + o(h^2) \\
\hat{g}_n(x; \lambda_1) - g_{0,\lambda_1}(x)/f_{0,\lambda_1}(x) - h^2 B(x; \lambda_1) + o(h^2) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{g}_n(x; \lambda_K) - g_{0,\lambda_K}(x)/f_{0,\lambda_K}(x) - h^2 B(x; \lambda_K) + o(h^2)
\end{pmatrix}
\Rightarrow N(0, \mathbf{\Delta}(x),$$
where $B(x; \lambda)$ is defined in (11),

$$\Delta(x) = \begin{pmatrix} \Delta_{0,0}(x) & 0 & 0 & \ldots & 0 \\ 0 & \Delta_{1,1}(x) & \Delta_{1,2}(x) & \ldots & \Delta_{1,K}(x) \\ 0 & \Delta_{2,1}(x) & \Delta_{2,2}(x) & \ldots & \Delta_{2,K}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \Delta_{K,1}(x) & \Delta_{K,2}(x) & \ldots & \Delta_{K,K}(x) \end{pmatrix},$$

and $\Delta_{i,j}(x), i = 0, 1, \ldots, K, j = 1, 2, \ldots, K,$ are given by

$$\frac{c_{0,i}(x)c_{0,j}(x)}{\sqrt{2\pi(\lambda_i + \lambda_j)\sigma_u^2}} \int \phi \left( t; x, \left( \frac{\lambda_i \lambda_j}{\lambda_i + \lambda_j} + 1 \right) \sigma_u^2 \right) f_X(t) dt - f_{0,i}(x)f_{0,j}(x)$$

$$+ \frac{c_{0,i}(x)d_{0,j}(x)}{\sqrt{2\pi(\lambda_i + \lambda_j)\sigma_u^2}} \int g(t)\phi \left( t; x, \left( \frac{\lambda_i \lambda_j}{\lambda_i + \lambda_j} + 1 \right) \sigma_u^2 \right) f_X(t) dt - f_{0,i}(x)g_{0,j}(x)$$

$$+ \frac{d_{0,i}(x)c_{0,j}(x)}{\sqrt{2\pi(\lambda_i + \lambda_j)\sigma_u^2}} \int g(t)\phi \left( t; x, \left( \frac{\lambda_i \lambda_j}{\lambda_i + \lambda_j} + 1 \right) \sigma_u^2 \right) f_X(t) dt - f_{0,i}(x)g_{0,j}(x)$$

$$+ \frac{d_{0,i}(x)d_{0,j}(x)}{\sqrt{2\pi(\lambda_i + \lambda_j)\sigma_u^2}} \int g^2(t)\phi \left( t; x, \left( \frac{\lambda_i \lambda_j}{\lambda_i + \lambda_j} + 1 \right) \sigma_u^2 \right) f_X(t) dt - g_{0,i}(x)g_{0,j}(x).$$

The proof of the joint normality is a straightforward application of the multivariate CLT on the following random vector

$$\begin{pmatrix} \sqrt{n}h \left[ g_n(x; 0) - g_{0,0}(x) / f_{0,0}(x) - h^2B(x; 0) + o(h^2) \right] \\ \sqrt{n} \left[ g_n(x; \lambda_1) - g_{0,\lambda_1}(x) / f_{0,\lambda_1}(x) - h^2B(x; \lambda_1) + o(h^2) \right] \\ \vdots \\ \sqrt{n} \left[ g_n(x; \lambda_K) - g_{0,\lambda_K}(x) / f_{0,\lambda_K}(x) - h^2B(x; \lambda_K) + o(h^2) \right] \end{pmatrix} \sim \frac{1}{\sqrt{n}} \begin{pmatrix} \sqrt{h}\sum_{i=1}^{n}v_{0,i}(x) \\ \sum_{i=1}^{n}v_{i,\lambda_1}(x) \\ \vdots \\ \sum_{i=1}^{n}v_{i,\lambda_K}(x) \end{pmatrix}.$$

For the sake of brevity, the proof will be omitted. In addition to the condition C2, if we further assume that $nh^4 \to 0,$ then the asymptotic bias can be removed.

## 5 | Extrapolation Function

Theorem 2 indicates that the extrapolation function can be defined as $\Gamma(\lambda) = g_{0,\lambda}(x)/f_{0,\lambda}(x).$ From the definitions of $g_{0,\lambda}$ and $f_{0,\lambda}(x),$ we know that

$$\Gamma(\lambda) = \frac{\int g(t)f_X(t)\phi(t; x, (\lambda + 1)\sigma_u^2)dt}{\int f_X(t)\phi(t; x, (\lambda + 1)\sigma_u^2)dt}.$$ (13)

As a function of $\lambda,$ $\Gamma(\lambda)$ does not have a tractable form, and some approximation is needed for extrapolating. By changing variable, we have

$$\int g(t)f_X(t)\phi(t; x, (\lambda + 1)\sigma_u^2)dt = \int g(x + \sqrt{\lambda + 1}\sigma_u v)f_X(x + \sqrt{\lambda + 1}\sigma_u v)\phi(v)dv.$$
Denote $\alpha = (\lambda + 1)\sigma_u^2$, and assume that $g$ and $f_X$ are four times continuously differentiable. We have

$$\int g(t)f_X(t)\phi(t; x, (\lambda + 1)\sigma_u^2)dt = g(x)f_X(x) + \frac{[f_X(x)g(x)]''}{2}\alpha + \frac{[f_X(x)g(x)]^{(4)}}{4!}\alpha^2 + o(\alpha^2),$$

where $o(\cdot)$ is understood as a negligible quantity when $\lambda \to -1$. Similarly, we have

$$\int f_X(t)\phi(x; t, (\lambda + 1)\sigma_u^2)dt = f_X(x) + \frac{f_X''(x)}{2}\alpha + \frac{f_X^{(4)}(x)}{4!}\alpha^2 + o(\alpha^2).$$

Therefore, after neglecting the $o(\lambda + 1)$ term, from (13), we obtain

$$\Gamma(\lambda) \approx \frac{g(x)f_X(x) + \sigma_u^2(\lambda + 1)[f_X(x)g(x)]''/2}{f_X(x) + \sigma_u^2(\lambda + 1)f_X''(x)/2}.$$

It is easy to see that the right-hand side approaches $g(x)$ as $\lambda \to -1$, and indeed, for fixed $x$-value, it has the form of $a + b/(c + \lambda)$, which is the nonlinear extrapolation function often used in the classical SIMEX estimation procedure. If we further apply the approximation

$$\frac{1}{f_X(x) + \sigma_u^2(\lambda + 1)f_X''(x)/2} = \frac{1}{f_X(x)} \left[1 - \frac{\sigma_u^2(\lambda + 1)f_X''(x)}{2f_X(x)} + o((\lambda + 1))\right],$$

or the approximation with higher-order expansions, we can obtain the quadratic extrapolation function $a + b\lambda + c\lambda^2$ and the polynomial extrapolation functions.

Almost all literature involving the classical SIMEX method, mostly in the parametric setups, assumes that the true extrapolation function has a known form when discussing the asymptotic distributions of the SIMEX estimators. However, this rarely happens in real applications. Based on the above discussion, the true extrapolation function of the EX procedure is never known either. To see this point clearly, we further assume that $X \sim N(0, \sigma_X^2)$. From (13), for any $x \in \mathbb{R}$,

$$\Gamma(\lambda) = \int g(t)\phi\left(t; \frac{x\sigma_X^2}{(\lambda + 1)\sigma_u^2 + \sigma_X^2}, \frac{\lambda + 1}{(\lambda + 1)\sigma_u^2 + \sigma_X^2}\right) dt.$$

Since the normal distribution family is complete, so the above expression implies that $\Gamma(\lambda)$ and $g(t)$ are uniquely determined by each other. Since $g$ is unknown, so neither is $\Gamma$. This discouraging finding really invalidates all the potential theoretical developments based on known extrapolation functions.

### 6 NUMERICAL STUDY

In this section, we conduct some simulation studies to evaluate the finite sample performance of the proposed SIMEX procedure. We also analyze a dataset from the National Health and Nutrition Examination Survey (NHANES) to illustrate the application of the proposed estimation procedure.
TABLE 1 $g(x) = x \sin(x); X \sim N(0, 1)$; Evaluation interval $[-3, 3]$.

| $\sigma_u^2$ | Method | $n = 100$ | $n = 200$ | $n = 500$ |
|-------------|--------|-----------|-----------|-----------|
|             | MSE    | Time(s)   | MSE      | Time(s)   | MSE      | Time(s)   |
| 0.1         | SIMEX  | 0.440     | 76.961   | 0.225     | 140.422  | 0.207     | 332.668  |
|             | $B = 50$ |          |           |           |           |           |           |
| 0.25        | EX     | 0.287     | 4.435    | 0.088     | 6.324    | 0.049     | 12.860   |
|             | $B = 100$ |        |           |           |           |           |           |
| 0.1         | Naive  | 0.197     | 0.141    | 0.101     | 0.259    | 0.066     | 0.620    |
|             | SIMEX  | 0.169     | 74.768   | 0.321     | 137.058  | 0.237     | 337.586  |
| 0.25        | EX     | 0.268     | 148.301  | 0.619     | 271.099  | 0.971     | 676.629  |
|             | $B = 100$ |        |           |           |           |           |           |
| 0.1         | Naive  | 0.221     | 0.132    | 0.139     | 0.263    | 0.119     | 0.649    |

Abbreviations: EX, extrapolation; MSE, mean squared error.

6.1 Simulation study

In this simulation study, the data are generated from the regression model $Y = g(X) + \epsilon, Z = X + U$, where $X \sim N(0, 1), U \sim N(0, \sigma_u^2)$. Three choices of the regression function $g(x)$ were considered, namely $g(x) = x^2, \exp(x)$ and $x \sin(x)$. To see the effect of $\sigma_u^2$ on the resulting estimate, we choose $\sigma_u^2 = 0.1$ and 0.25. The sample sizes are chosen to be $n = 100, 200, 500$. In each scenario, the estimate is calculated for 200 equally spaced $x$-values $x_j, j = 1, 2, \ldots, 200$, chosen from $[-3, 3]$. To implement the extrapolation step, the grid of $\lambda$ is taken from 0 to 2 separated by 0.2. The mean squared errors (MSE) is used to evaluate the finite sample performance of the proposed EX procedure. The bandwidth $h$ is chosen to be $n^{-1/5}$, a theoretical optimal order when estimating the regression function based on the error-free data. To get a stable result, all simulations were performed for 10 independent datasets, and the average of the 10 estimates was taken to be the final estimate at each of 200 $x$-values, and the MSE defined by $200^{-1} \sum_{j=1}^{200} [\hat{g}_n(x_j) - g(x_j)]^2$ is used for evaluate the finite sample performance of the proposed EX estimate. For comparison, we also apply the classical SIMEX algorithm ($B = 50, 100$) and the naive method to estimate these three regression functions. Besides the report on the MSEs from the three algorithms, we also record the computation time in seconds from each procedure to evaluate the algorithm efficiency. All the simulations were conducted on a desktop computer with Windows system running on Intel(R) Core(TM) i7-10700 CPU of 2.90GHz, and 16GB RAM. The simulation results are summarized in Tables 1–3. In all the tables, we use the EX to denote the proposed Extrapolation algorithm, SIMEX to denote the classical SIMEX method, and Naive for naive method.

In Tables 1, 2, and 3, the bold number indicates the smallest MSE values. The simulation results clearly show that the proposed EX algorithm is more efficient than the classical SIMEX method in terms of computational speed. The finite sample performance of both methods SIMEX and EX, as measured by the MSE, are comparable. When sample sizes get larger, and the measurement error variances get smaller, both procedures performs better, as expected. The advantage of using EX or SIMEX over the naive method may not be obvious when the sample size or the noise level $\sigma_u^2$ is small, but both methods outperform the naive method when either the sample size or the noise level increases. It is well known in measurement error literature that the performance of the estimation procedure heavily depends on the signal to noise ratio, or the ratio of $\sigma_x^2$ and $\sigma_u^2$. The signal to noise ratios in the previous simulation studies are 10 and 4.
TABLE 2 \( g(x) = x^2; X \sim N(0, 1); \) Evaluation interval \([-3, 3]\).

| \( \sigma^2_u \) | Method   | \( n = 100 \) | \( n = 200 \) | \( n = 500 \) |
|-------------|-----------|-------------|-------------|-------------|
|             | MSE       | Time (s)    | MSE         | Time (s)    | MSE         | Time (s)    |
| SIMEX       |           |             |             |             |             |             |
| B = 50      | 2.127     | 88.813      | 0.350       | 156.776     | 0.420       | 369.107     |
| 0.1         | 1.104     | 177.900     | 1.257       | 310.116     | 0.772       | 734.960     |
| EX          | 0.388     | 4.372       | 0.115       | 6.359       | 0.055       | 12.772      |
| Naive       | **0.318** | 0.133       | 0.450       | 0.247       | 0.385       | 0.615       |
| SIMEX       |           |             |             |             |             |             |
| B = 50      | 2.920     | 90.052      | 0.147       | 157.879     | 0.748       | 370.895     |
| 0.25        | 1.354     | 178.934     | 2.919       | 311.773     | 1.269       | 740.275     |
| EX          | **0.127** | 4.249       | **0.346**   | 5.989       | **0.025**   | 11.702      |
| Naive       | 1.820     | 0.134       | 2.282       | 0.259       | 1.456       | 0.610       |

Abbreviations: EX, extrapolation; MSE, mean squared error.

TABLE 3 \( g(x) = \exp(x); X \sim N(0, 1); \) Evaluation interval \([-3, 3]\).

| \( \sigma^2_u \) | Method   | \( n = 100 \) | \( n = 200 \) | \( n = 500 \) |
|-------------|-----------|-------------|-------------|-------------|
|             | MSE       | Time (s)    | MSE         | Time (s)    | MSE         | Time (s)    |
| SIMEX       |           |             |             |             |             |             |
| B = 50      | 0.832     | 75.432      | 1.384       | 138.462     | 0.919       | 338.790     |
| 0.1         | 2.512     | 148.797     | 6.176       | 273.750     | 0.820       | 675.488     |
| EX          | 0.684     | 4.372       | **0.122**   | 6.318       | **0.121**   | 12.626      |
| Naive       | **0.637** | 0.133       | 1.282       | 0.247       | 0.798       | 0.614       |
| SIMEX       |           |             |             |             |             |             |
| B = 50      | 4.174     | 75.571      | 0.873       | 139.348     | 2.771       | 342.374     |
| 0.25        | 1.795     | 149.799     | 4.620       | 275.701     | 1.203       | 785.072     |
| EX          | **0.425** | 4.148       | **0.484**   | 5.929       | **0.053**   | 11.756      |
| Naive       | 3.773     | 0.131       | 3.786       | 0.267       | 2.749       | 0.610       |

Abbreviations: EX, extrapolation; MSE, mean squared error.

It is well known that the local linear smoother is free of boundary effect in the classical nonparametric regression setup. However, from the simulation results, we can observe that both the SIMEX and EX methods perform poorly at the boundary or when the sample size is small, which can be attributed to the extrapolation nature of SIMEX and EX methods, as well as the sparsity of the sample data near the boundary. We also notice that the MSE values from the SIMEX method does not always decrease when \( B \) gets larger. This is indeed a perplexing phenomenon. Extra simulation studies on larger sample sizes, and narrower evaluation intervals (e.g., \([-2, 2]\)) have been conducted, and this pattern persists. A reviewer suggested us to try larger \( B \) values. Unfortunately, extra simulation studies show that larger \( B \) values does not help too much. We would like to argue that more pseudo-data might exacerbate the disadvantages of extrapolation nature of SIMEX, however, more convincing evidence should be investigated before making any formal judgements. This issue deserves an independent study in the future. This discovery indicates that the proposed EX procedure surely serves as a very competitive alternative to SIMEX.

For illustration purposes, we plot the fitted regression curves for \( n = 200, \sigma^2_u = 0.25, \) and \( B = 50 \) for SIMEX in Figures 1–3 from all three methods, with the true regression function as the
FIGURE 1  Naive, extrapolation, and SIMEX estimates of \( g(x) = x \sin(x) \).

reference. Clearly the SIMEX and the proposed EX estimators are closer to the true regression lines than the naive estimator when the predictor values not in the boundary.

Some simulation studies are also conducted for a simple linear, a logistic, and a bivariate regression models. Simulation results can be found in Data S1.

6.2 Real data application: NHANES dataset

To determine the relationship between the serum 25-hydroxyvitamin D (25(OH)D) and the long term vitamin D average intake, Curley (2017) analyzed a dataset from the National Health and Nutrition Examination Survey (NHANES), and used a nonlinear function for modeling the regression mean of 25(OH)D on the long-term vitamin D average intake. In this section, we apply the proposed estimation procedure on a subset of the 2009–2010 NHANES study. The selected dataset contains dietary records of 806 Mexican–American females. The long-term vitamin D average intake \( (X) \) is not measured directly. Instead, two independent daily observations of vitamin D intake are collected. Let \( W_{ji} \) be the vitamin D intake from the \( i \)th subject on the \( j \)th time, and we assume that the additive structures \( W_{ji} = X_i + U_{ji} \) hold for all \( i = 1, 2, \ldots, 806, j = 1, 2 \). We use \( W_i = (W_{1i} + W_{2i})/2 \) to represent the observed vitamin intake, and by assuming that \( U_{1i} \) and \( U_{2i} \) are independently and identically distributed, the estimate of the SD of the measurement error \( U \) using the sample SD of the differences \( (W_{1i} - W_{2i})/2, i = 1, 2, \ldots, 806 \), is 2.5737. As in Curley (2017), we also apply a square root transformation on the 25(OH)D which results in a more symmetric structure, but the Shapiro normal test reports a \( p \)-value of .04, indicating that the transformed 25(OH)D values, denoted as \( Y \), is still not quite normal.

We adopt the local linear estimator to fit the regression function of \( Y \) against \( X \) to capture the mean regression function using the Naive, SIMEX \( (B = 200) \) and the proposed EX methods. Three fitted regression functions with the bandwidth \( h = n^{-1/5} \), together with the scatter plots
FIGURE 2 Naive, extrapolation, and SIMEX Estimates of \( g(x) = x^2 \).

FIGURE 3 Naive, extrapolation, and SIMEX Estimates of \( g(x) = \exp(x) \).

of \( Y \) against \( W \), are plotted in Figure 4. In Figure 4, the solid line is the fitted EX regression function, the dashed line is the fitted regression function using the classical SIMEX, and the dotted line is the fitted regression curve using the naive method. Clearly the naive estimator captures the central structure of the raw data, as expected. The fitted regression function from the classical SIMEX nearly overlaps the proposed EX estimator. Compared with the naive regression, the SIMEX and the EX procedures provide relatively conservative fitted 25(OH)D values.
DISCUSSION

The limited simulation studies and real data application conducted in Section 6 indicate that the EX algorithm is more effective than the classical SIMEX in most simulation configurations, as evidenced by the smaller MSEs or less computation time. To show the variability of the EX estimate, we also add a pointwise bootstrap confidence band, shown as gray-colored curves, with confidence level 95% in Figure 4 based on 500 bootstrap samples. The wide confidence bounds at both ends clearly shows the boundary effects of the local smoother. Since the performance of the SIMEX estimate is almost same as the EX estimate, we did not report its bootstrap confidence band. We should note that it will take SIMEX algorithm over 29 h to obtain the 500 bootstrap estimates.

7 | DISCUSSION

Instead of taking the conditional expectation of the estimator based on the pseudo-data or following the three steps in the classical SIMEX algorithm, the proposed EX method applies the conditional expectation directly to the target function to be optimized based on the pseudo-data, thus successfully bypassing the simulation step. When the vitamin D intake values are small, which might be interpreted as an evidence of the subjects under-reporting their vitamin D intakes. Because fewer data points on the upper end, so we truncated the graph when the observed vitamin D intake is bigger than 15, therefore more caution should be paid when interpreting the trend on the right. More scientific explanations from the analysis need to consult with experts on nutrition studies. The computation times for each of the three methods are, 0.209 s for Naive, 0.839 s for the EX, and 209.58 s for the classical SIMEX. Again, one can see that the proposed EX method is more efficient than the classical SIMEX.

To show the variability of the EX estimate, we also add a pointwise bootstrap confidence band, shown as gray-colored curves, with confidence level 95% in Figure 4 based on 500 bootstrap samples. The wide confidence bounds at both ends clearly shows the boundary effects of the local smoother. Since the performance of the SIMEX estimate is almost same as the EX estimate, we did not report its bootstrap confidence band. We should note that it will take SIMEX algorithm over 29 h to obtain the 500 bootstrap estimates.

FIGURE 4  Naive, SIMEX, and extrapolation (EX) estimates. (Naive: dotted line; EX: solid line; SIMEX: dashed line; Confidence Band: gray line).
Let $s_j = S_jn(x; v)$, $	ilde{s_j} = \tilde{S}_jn(x)$ for $j = 0, 1, 2$, and $t_j = T_{jn}(x; v)$, $\tilde{t}_j = \tilde{T}_{jn}(x)$ for $j = 0, 1$. Then it is easy to see that $\tilde{g}_n(x; \lambda) = E[F(s)|D]$, and $\hat{g}_n(x; \lambda) = F(\tilde{s})$. Therefore, from the above Taylor expansion, we can see that $\tilde{g}_n(x; \lambda) - \hat{g}_n(x; \lambda)$ approximately equals

$$E\left((s - \tilde{s})^T \frac{\partial^2 F(s)}{\partial s \partial s^T} \bigg|_{s = \tilde{s}} (s - \tilde{s}) \right)D = \text{trace} \left( \frac{\partial^2 F(s)}{\partial s \partial s^T} \bigg|_{s = \tilde{s}} E\left((s - \tilde{s})(s - \tilde{s})^T|D\right) \right).$$

Note that the matrix $E\left((s - \tilde{s})(s - \tilde{s})^T|D\right)$ is nonnegative definite, so it is sufficient to consider the expectation of each entry only in the matrix to determine its order. We can show that all 25 terms are of the order $O(1/nh)$ for $\lambda \geq 0$. This implies that, for $\lambda > 0$, the estimators $\hat{g}_n(x; \lambda)$ and $\tilde{g}_n(x; \lambda)$ are equivalent, in the sense of having the same asymptotic distribution, if $nh^4 \to 0$, and for $\lambda = 0$, they are equivalent if $nh^5 \to 0$. All the necessary computations supporting these claims can be found in the Data S1.

### 7.1 On the extrapolation function

As we discussed in Section 5, theoretically it is impossible to specify the true form of the extrapolation function in the nonparametric regression setups. However, if the regression function $g$ has a parametric form, then according to the above discussion, we can indeed nail down the extrapolation function. To see this, consider the power function $x^p$ with some $p \geq 1$, with $X$ still assumed to be $N(0, \sigma_X^2)$. Then some algebra leads to

$$\int t^p \phi \left( t; \frac{x \sigma_X^2}{(\lambda + 1)\sigma_\alpha^2 + \sigma_X^2}, \frac{(\lambda + 1)\sigma_\alpha^2}{(\lambda + 1)\sigma_X^2 + \sigma_\alpha^2} \right) dt = \sum_{j=0}^{[p/2]} \binom{p/2}{2j} x^{p-2j} \sigma_X^2 \sigma_\alpha^2 (\lambda + 1)^{2j} \left[(\lambda + 1)\sigma_X^2 + \sigma_\alpha^2\right]^j.$$

This implies that for a polynomial regression function $g$ of order $p$, if $X$ is normal, then the extrapolation function can be taken as a polynomial function of order $p$. Without loss of generality, assume that $H(\lambda) = s^T(\lambda) \alpha$, where $s(\lambda) = (1, \lambda, \lambda^2, \ldots, \lambda^p)^T$. Then $\alpha$ can be estimated by the minimizer of $L(\alpha) = \sum_{j=0}^{K} [\tilde{g}_j(x) - s^T(\lambda_j) \alpha]^2$. In fact, the minimizer $\hat{\alpha} = \left[\sum_{j=0}^{K} s(\lambda_j) s^T(\lambda_j)^{-1} \right]^{-1} \sum_{j=0}^{K} [\tilde{g}_j(x) s(\lambda_j)]$. Similar to Carroll et al. (1999), we have $\sqrt{nh}(\hat{\alpha} - \alpha)$ is asymptotically normal with mean $\mu(x, h) = h^2 B(x, 0) \left[\sum_{j=0}^{K} s(\lambda_j) s^T(\lambda_j)^{-1} s(0)\right]$, and covariance matrix

$$\tau^2(x) = \Delta_0(x) \left[ \sum_{j=0}^{K} s(\lambda_j) s^T(\lambda_j)^{-1} s(0) s^T(0) \left[\sum_{j=0}^{K} s(\lambda_j) s^T(\lambda_j)\right]^{-1} \right],$$

as $nh^5 \to 0$. Thus, for the EX estimator $s^T(-1) \hat{\alpha}$, $\sqrt{nh}(s^T(-1) \hat{\alpha} - g(x) - s^T(-1) \mu(x, h))$ converges to $N(0, \tau^2(x))$ in distribution. Of course, the discussion only has some theoretical significance. If we knew in advance that $g$ has a parametric form, we would not estimate it using the nonparametric methods.

### 7.2 On the measurement error variance

In the previous discussion, we assume that the measurement error variance $\sigma_u^2$ is known. This assumption is rather restrictive, since in real applications, rarely do we know the value of $\sigma_u^2$. 


An estimate of $\sigma_u^2$ can be obtained if validation data or replications are available. For example, if we can observe multiple surrogate values $Z$ at each $X$, then we can estimate $\sigma_u^2$ via the sample variance of the replicated values from $Z$. To be specific, suppose we can observe $Z_{ij} = X_i + U_{ij}$, $j = 1, \ldots, n_i (> 1)$, $i = 1, 2, \ldots, n$, then an estimator of $\sigma_u^2$ can be defined as

$$\hat{\sigma_u^2} = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (Z_{ij} - \bar{Z}_i)^2 \right].$$

Then the SIMEX and EX estimation procedures can be implemented by taking $\hat{\sigma_u^2}$ as the measurement variance. A simulation study with estimated measurement variance based on replicated observation can be found in the Data S1.

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**REFERENCES**

Carroll, R., Rupper, D., Crainiceanu, C., & Stefanski, L. (2006). *Measurement error in nonlinear models: A modern perspective* (2nd ed.). Chapman and Hall/CRC.

Carroll, R. J., Kuchenhoff, H., Lombard, F., & Stefanski, L. A. (1996). Asymptotics for the simex estimator in nonlinear measurement error models. *Journal of the American Statistical Association, 91*(433), 9.

Carroll, R. J., Maca, J. D., & Ruppert, D. (1999). Nonparametric regression in the presence of measurement error. *Biometrika, 86*(3), 541–554.

Carroll, R. J., & Wang, Y. (2008). Nonparametric variance estimation in the analysis of microarray data: A measurement error approach. *Biometrika, 95*(2), 437–449.

Cook, J. R., & Stefanski, L. A. (1994). Simulation-extrapolation estimation in parametric measurement error models. *Journal of the American Statistical Association, 89*(428), 1314–1328.

Curley, B. (2017). *Nonlinear models with measurement error: Application to vitamin D* [PhD thesis]. Iowa State University.

Gould, W., Stefanski, L., & Pollock, K. (1999). Use of simulation–extrapolation estimation in catch–effort analyses. *Canadian Journal of Fisheries and Aquatic Sciences, 56*(7), 1234–1240.

Hardin, J. W., Schmiediche, H., & Carroll, R. J. (2003). The simulation extrapolation method for fitting generalized linear models with additive measurement error. *The Stata Journal, 3*(4), 373–385.

Hwang, W., & Huang, S. Y. (2003). Estimation in capture–recapture models when covariates are subject to measurement errors. *Biometrics, 59*(4), 1113–1122.

Lin, X., & Carroll, R. J. (1999). Simex variance component tests in generalized linear mixed measurement error models. *Biometrics, 55*(2), 613–619.

Mallick, R., Fung, K., & Krewski, D. (2002). Adjusting for measurement error in the cox proportional hazards regression model. *Journal of Cancer Epidemiology and Prevention, 7*(4), 155–164.

Ponzi, E., Keller, L. F., & Muff, S. (2019). The simulation extrapolation technique meets ecology and evolution: A general and intuitive method to account for measurement error. *Methods in Ecology and Evolution, 10*(10), 1734–1748.
Sevilimedu, V., Yu, L., Samawi, H., & Rochani, H. (2019). Application of the misclassification simulation extrapolation procedure to log-logistic accelerated failure time models in survival analysis. Journal of Statistical Theory and Practice, 13(1), 1–16.

Staudenmayer, J., & Ruppert, D. (2004). Local polynomial regression and simulation–extrapolation. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 66(1), 17–30.

Stefanski, L., & Bay, J. (1996). Simulation extrapolation deconvolution of finite population cumulative distribution function estimators. Biometrika, 83(2), 407–417.

Stefanski, L. A., & Cook, J. R. (1995). Simulation-extrapolation: The measurement error jackknife. Journal of the American Statistical Association, 90(432), 1247–1256.

Stoklosa, J., Dann, P., Huggins, R. M., & Hwang, W.-H. (2016). Estimation of survival and capture probabilities in open population capture–recapture models when covariates are subject to measurement error. Computational Statistics & Data Analysis, 96, 74–86.

Wang, X.-F., Fan, Z., & Wang, B. (2010). Estimating smooth distribution function in the presence of heteroscedastic measurement errors. Computational Statistics & Data Analysis, 54(1), 25–36.

Wang, X.-F., Sun, J., & Fan, Z. (2009). Deconvolution density estimation with heteroscedastic errors using simex. arXiv: Statistics Theory.

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APPENDIX A

To prove Theorem 1, we need find out the expectations and variances of each component appearing in $\tilde{g}_n(x; \lambda)$ defined in (8). The calculation is facilitated by the following lemmas.

**Lemma 4.** Let $a, c$ be any positive constants. Suppose that for any $x$ in the support of $f_X, m'(t + x), m''(t + x) \in L_1(\phi(\cdot; 0, c))$ and are continuous as functions of $t$. Then, as $h \to 0$, we have

$$\int \phi(t; x, ah^2 + c)m(t)dt = \int \phi(t; x, c)m(t)dt + \frac{ah^2}{2} \int m''(t)\phi(t; x, c)dt + o(h^2).$$

Furthermore, if

$$\frac{\partial^j}{\partial x^j} \int m(t + x)\phi(t; 0, c)dt = \int \frac{\partial^j m(t + x)}{\partial x^j} \phi(t; 0, c)dt, \quad j = 1, 2,$$

then we have

$$\int \phi(t; x, ah^2 + c)m(t)dt = \int \phi(t - x; 0, c)m(t)dt + \frac{ah^2}{2} \cdot \frac{\partial^2}{\partial x^2} \int m(t)\phi(t - x; 0, c)dt + o(h^2).$$

In Lemma 4, if we take $c = 0$, then $\int \phi(t; x, ah^2)m(t)dt = m(x) + \frac{ah^2}{2} m''(x) + o(h^2)$. 

SUPPORTING INFORMATION

Additional supporting information can be found online in the Supporting Information section at the end of this article.
The proof of Lemma 4. Note that
\[
\frac{\partial m(x + u\sqrt{ah^2 + c})}{\partial h} \bigg|_{h=0} = 0, \quad \frac{\partial^2 m(x + u\sqrt{ah^2 + c})}{\partial h^2} \bigg|_{h=0} = m'(x + u\sqrt{c})\frac{au}{\sqrt{c}}.
\]
Therefore, using Taylor expansion,
\[
\int \phi(t; x, ah^2 + c)m(t)dt = \int \phi(t - x; 0, c) \left[ m(t) + m'(t)\frac{ah^2(t - x)}{2c} \right] dt + o(h^2). \tag{A1}
\]
Note that under the condition of \(m'(t + x) \in L_1(\phi(\cdot; 0, c))\) for any \(x \in \mathbb{R}\), we can get \(c^{-1} \int \phi(t - x; 0, c)m'(t)(t - x)dt = \int m''(t)\phi(t - x; 0, c)dt\). This, together with (A1), implies the first expansion.

For the second expansion, notice that under the derivative-integration exchange-ability condition, we have
\[
\int m''(t)\phi(t - x; 0, c)dt = \int \frac{\partial^2 m(t + x)}{\partial t^2} \phi(t; 0, c)dt = \int \frac{\partial^2 m(t + x)}{\partial x^2} \phi(t; 0, c)dt.
\]
This concludes the proof. \[\blacklozenge\]

The following lemma lists some facts about normal density functions which are used often in the proofs of our main results. For the sake of brevity, the proofs of these facts are omitted since they can be found in standard statistics books.

**Lemma 5.** For normal density function \(\phi(u, \mu, \sigma^2)\) with mean \(\mu\) and variance \(\sigma^2\), we have
\[
\phi^2(u; \mu, \sigma^2) = \frac{1}{2\sqrt{\pi}\sigma^2} \phi\left(u; \mu, \frac{\sigma^2}{2}\right),
\]
\[
\phi^3(u; \mu, \sigma^2) = \frac{1}{2\sqrt{3}\pi}\sigma^2 \phi\left(u; \mu, \frac{\sigma^2}{3}\right),
\]
\[
\phi(u; \mu_1, \sigma_1^2)\phi(u; \mu_2, \sigma_2^2) = \phi(\mu_1 - \mu_2; 0, \sigma_1^2 + \sigma_2^2)\phi\left(u; \frac{\sigma_1^2\mu_2 + \sigma_2^2\mu_1}{\sigma_1^2 + \sigma_2^2}, \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right),
\]
\[
\int \phi(u; \mu_1, \sigma_1^2)\phi(u; \mu_2, \sigma_2^2)du = \phi(\mu_1 - \mu_2; 0, \sigma_1^2 + \sigma_2^2),
\]
\[
\int u\phi(u; \mu_1, \sigma_1^2)\phi(u; \mu_2, \sigma_2^2)du = \frac{\sigma_1^2\mu_2 + \sigma_2^2\mu_1}{\sigma_1^2 + \sigma_2^2} \phi(\mu_1 - \mu_2; 0, \sigma_1^2 + \sigma_2^2),
\]
\[
\int u^2\phi(u; \mu_1, \sigma_1^2)\phi(u; \mu_2, \sigma_2^2)du = \left[\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2} + \left(\frac{\sigma_1^2\mu_2 + \sigma_2^2\mu_1}{\sigma_1^2 + \sigma_2^2}\right)^2\right] \phi(\mu_1 - \mu_2; 0, \sigma_1^2 + \sigma_2^2).
\]

Then using the above two lemmas, for the components \(\hat{S}_{jn}(x)\) or \(\hat{s}_j\) for \(j = 0, 1, 2\), \(\hat{T}_{ln}(x)\) for \(l = 0, 1\), in the definition of \(\hat{g}_n(x)\) given in (8), we can get the following series of results on the asymptotic expansions of their expectations and variances. For brevity, in the proof, we denote \(\delta_j^2 = h^2 + (\lambda + j)\sigma_u^2\) for \(j = 0, 1, 2\).
Lemma 6. For \( \tilde{S}_{n0}(x) \), we have

\[
E(\tilde{S}_{n0}(x)) = f_{0,\lambda}(x) + h^2f_{0,\lambda}''(x)/2 + o(h^2), \quad \lambda \geq 0,
\]
\[
Var(\tilde{S}_{n0}(x)) = \begin{cases} 
\frac{f_{0,\lambda}(x)}{2n} - \frac{f_{0,\lambda}^2(x)}{n} + O\left(\frac{h^2}{n}\right), & \lambda > 0, \\
\frac{f_{0,\lambda}(x)}{2nh\sqrt{\pi}} - \frac{f_{0,\lambda}^2(x)}{n} + O\left(\frac{h}{n}\right), & \lambda = 0.
\end{cases}
\]

Proof of Lemma 6. By the independence of \( X \) and \( U \), and applying Lemma 4 with \( a = 1, c = (\lambda + 1)\sigma_u^2 \) and \( m(t) = f_X(t) \), for \( \tilde{S}_{n0} \), we have

\[
Ef(xZ, \delta_{0h}^2) = \int \int \phi(x; t + u, \delta_{0h}^2)\phi(u; 0, \sigma_u^2)f_X(t)dudt = f_{0,\lambda}(x) + h^2f_{0,\lambda}''(x)/2 + o(h^2).
\]

Also, applying Lemma (4) with \( a = 1/2, c = (\lambda + 2)\sigma_u^2/2 \), and \( m(t) = f_X(t) \), we have

\[
Ef^2(x; Z, \delta_{0h}^2) = \int \int \phi^2(x; t + u, \delta_{0h}^2)\phi(u; 0, \sigma_u^2)f_X(t)dudt
\]
\[
= \frac{1}{2\sqrt{\pi}\delta_{0h}^2} \int \phi\left(t; x, \frac{\delta_{0h}^2}{2}\right)f_X(t)dt
\]
\[
= \frac{f_{0,\lambda/2}(x)}{2\sqrt{\pi}\delta_{0h}^2} + \frac{h^2f_{0,\lambda/2}''(x)}{2\sqrt{\pi}\delta_{0h}^2} + O\left(\frac{h^2}{2\sqrt{\pi}\delta_{0h}^2}\right). \quad (A2)
\]

Therefore, we have

\[
\text{var}\left[\tilde{S}_{n0}(x)\right] = \frac{4f_{0,\lambda/2}(x) + h^2f_{0,\lambda/2}''(x)}{8n\sqrt{\pi}\delta_{0h}^2} - \frac{f_{0,\lambda}^2(x) + f_{0,\lambda}(x)f_{0,\lambda}''(x)h^2}{n} + o\left(\frac{h^2}{2n\sqrt{\pi}\delta_{0h}^2}\right).
\]

This concludes the proof of Lemma 6.

Lemma 7. For \( \tilde{S}_{n1}(x) \), we have

\[
E(\tilde{S}_{n1}(x)) = \frac{h^2}{(\lambda + 1)\sigma_u^2}f_{1,\lambda}(x) - \frac{xh^2}{(\lambda + 1)\sigma_u^2}f_{0,\lambda}(x) + o\left(h^2\right), \quad \lambda \geq 0,
\]
\[
Var(\tilde{S}_{n1}(x)) = \begin{cases} 
\frac{h^4}{2n(\lambda + 2)^2\sigma_u^2}\int f_{1,\lambda/2}(x) - xf_{1,\lambda/2}(x) + x^2f_{0,\lambda/2}(x)dx, & \lambda > 0, \\
\frac{h^4}{2n(\lambda + 2)^2\sigma_u^2}\int f_{0,\lambda/2}(x) - xf_{0,\lambda/2}(x) + x^2f_{0,\lambda/2}(x)dx, & \lambda = 0.
\end{cases}
\]

Proof of Lemma 7. Applying Lemma 4 with \( a = 1, c = (\lambda + 1)\sigma_u^2 \), \( m(t) = f_X(t) \) and \( tf_X(t) \), and Lemma 5, we have

\[
E(Z - x)\phi(x; Z, \delta_{0h}^2) = \int \int (t + u - x)\phi(x; t + u, \delta_{0h}^2)\phi(u; 0, \sigma_u^2)f_X(t)dudt
\]
Therefore, the variance of $\tilde{S}_{1h}(x)$ equals

\[
\frac{h^4}{2n\delta^2_{1h}\sqrt{\pi\delta^2_{0h}}} \left( \left[ f_{1.2/2}(x) + \frac{h^2}{4} f''_{1.2/2}(x) + o(h^2) \right] - x \left[ f_{1.2/2}(x) + \frac{h^2}{4} f''_{1.2/2}(x) + o(h^2) \right] \right)
+ \left( \frac{h^4 x^2}{2\delta^2_{2h} n \sqrt{\pi \delta^3_{0h}}} + \frac{\sigma_u^2 h^4}{2\delta^2_{2h} \sqrt{\pi \delta^3_{0h}}} \right) \left[ f_{0.2/2}(x) + \frac{h^2}{4} f''_{0.2/2}(x) + o(h^2) \right]
- \frac{1}{n} \left( \frac{h^2}{2} f_{1.2}(x) + \frac{h^2}{2} f''_{1.2}(x) \right) - \frac{h^2}{2 \delta^2_{1h}} \left[ f_{0.2}(x) + \frac{h^2}{2} f''_{0.2}(x) \right] + o \left( \frac{h^4}{n} \right).
\]

This concludes the proof of Lemma 7. \hfill \blacksquare

**Lemma 8.** For $\tilde{S}_{n2}(x)$, we have

\[
E(\tilde{S}_{n2}(x)) = h^2 f_{0.2}(x) + o(h^2), \quad \lambda \geq 0,
\]

\[
\text{Var}(\tilde{S}_{n2}(x)) = \begin{cases} 
\frac{h^4}{n} \left[ f_{0.2}(x) - \frac{1}{2} \frac{f_{0.2}(x)}{\sqrt{\pi \sigma_u^2}} \right] + o \left( \frac{h^4}{n} \right), & \lambda > 0, \\
\frac{3h^4 f_{0.2}(x)}{8n \sqrt{\pi}} + o \left( \frac{h^4}{n} \right), & \lambda = 0.
\end{cases}
\]

**Proof of Lemma 8.** Note that

\[
E(Z - x)^2 \phi(x; Z, \delta^2_{0h}) = \int \int (t + u - x)^2 \phi(x; t + u, \delta^2_{0h}) \phi(u; 0, \sigma_u^2 f_X(t)) dudt
\]
Therefore, we get

\[
E[S_{2n}(x)] = \frac{h^4}{\delta_{4h}^2} \left[ \frac{\delta_{2h}^2}{\delta_{2h}^2} \right]^2 \left[ f_{2,h}(x) + \frac{h^2}{2} f_{1,h}(x) + o(h^2) - 2x \left( f_{1,h}(x) + \frac{h^2}{2} f_{1,h}(x) + o(h^2) \right) \right.
+ x^2 \left( f_{0,h}(x) + \frac{h^2}{2} f_{0,h}(x) + o(h^2) \right) \left[ f_{0,h}(x) + \frac{h^2}{2} f_{0,h}(x) + o(h^2) \right] \right]
+ \lambda \sigma_0^2 h^2 \left[ f_{0,h}(x) + \frac{h^2}{2} f_{0,h}(x) + o(h^2) \right].
\]

Then we have to calculate \( E(Z - x)^4 \phi^2(x; Z, \delta_{4h}^2) \). Recall that for \( u \sim N(\mu, \sigma_u^2) \), we have \( Eu^3 = 3\mu\sigma_u^2 + \mu^3, Eu^4 = 3\sigma^4 + 6\mu^2\sigma_u^2 + \mu^4 \). So,

\[
E(Z - x)^4 \phi^2(x; Z, \delta_{4h}^2) = \int \int (t + u - x)^4 \phi^2(u; x - t, \delta_{4h}^2) \phi(u; 0, \sigma_u^2) f_X(t) dt du
\]

\[
= \frac{1}{2 \sqrt{\pi \delta_{4h}^2}} \int (t - x)^4 \phi \left( t ; x, \frac{\delta_{2h}^2}{2} \right) f_X(t) dt - \frac{4\sigma_u^2}{\delta_{4h}^2 \sqrt{\pi \delta_{4h}^2}} \int (t - x)^4 \phi \left( t ; x, \frac{\delta_{2h}^2}{2} \right) f_X(t) dt
+ \frac{6}{2 \sqrt{\pi \delta_{4h}^2}} \int (t - x)^2 \phi \left( t ; x, \frac{\delta_{2h}^2}{2} \right) \left[ \left( \frac{2\sigma_u^2}{\delta_{2h}^2} \right)^2 + \left( \frac{\sigma_u^2}{\delta_{2h}^2} \right)^2 \right] f_X(t) dt
+ \frac{4}{2 \sqrt{\pi \delta_{4h}^2}} \int (t - x) \phi \left( t ; x, \frac{\delta_{2h}^2}{2} \right) \left[ \left( \frac{3}{2} \frac{\sigma_u^2}{\delta_{2h}^2} \frac{2\sigma_u^2}{\delta_{2h}^2} \frac{2\sigma_u^2}{\delta_{2h}^2} \right) + \left( \frac{2\sigma_u^2}{\delta_{2h}^2} \right)^3 \right] f_X(t) dt
+ \frac{1}{2 \sqrt{\pi \delta_{4h}^2}} \int \phi \left( t ; x, \frac{\delta_{2h}^2}{2} \right) \left[ \left( \frac{3}{2} \frac{\sigma_u^2}{\delta_{2h}^2} \frac{2\sigma_u^2}{\delta_{2h}^2} \frac{2\sigma_u^2}{\delta_{2h}^2} \right) + \left( \frac{2\sigma_u^2}{\delta_{2h}^2} \right)^2 \right] - 2 \left( \frac{2\sigma_u^2}{\delta_{2h}^2} \right)^4 dt.
\]

It can be further written as

\[
\left\{ \frac{1}{2 \sqrt{\pi \delta_{4h}^2}} - \frac{4\sigma_u^2}{\delta_{4h}^2 \sqrt{\pi \delta_{4h}^2}} \right\} \int (t - x)^4 \phi \left( t ; x, \frac{\delta_{2h}^2}{2} \right) f_X(t) dt
+ \frac{6}{2 \sqrt{\pi \delta_{4h}^2}} \int (t - x)^2 \phi \left( t ; x, \frac{\delta_{2h}^2}{2} \right) \left[ \left( \frac{2\sigma_u^2}{\delta_{2h}^2} \right)^2 + \left( \frac{\sigma_u^2}{\delta_{2h}^2} \right)^2 \right] f_X(t) dt
\]
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E (Z − x)^4 \phi^2 (x; Z, \delta_{0h}^2) = A(h; \lambda) \int (t − x)^4 \phi \left( t; x, \frac{\delta_{2h}^2}{2} \right) f_x(t) dt \\
+ B(h; \lambda) \int (t − x)^2 \phi \left( t; x, \frac{\delta_{2h}^2}{2} \right) f_x(t) dt + C(h; \lambda) \int \phi \left( t; x, \frac{\delta_{2h}^2}{2} \right) f_x(t) dt \\
= A(h; \lambda) \left[ f_{4,\lambda/2}(x) + \frac{h^2}{4} f_{1,\lambda/2}(x) + o(h^2) \right] - 4x A(h; \lambda) \left[ f_{3,\lambda/2}(x) + \frac{h^2}{4} f_{2,\lambda/2}(x) + o(h^2) \right] \\
+ (6x^2 A(h; \lambda) + B(h; \lambda)) \left[ f_{2,\lambda/2}(x) + \frac{h^2}{4} f_{1,\lambda/2}(x) + o(h^2) \right] \\
- (4x^3 A(h; \lambda) + 2x B(h; \lambda)) \left[ f_{1,\lambda/2}(x) + \frac{h^2}{4} f_{0,\lambda/2}(x) + o(h^2) \right] \\
+ [x^4 A(h; \lambda) + x^2 B(h; \lambda) + C(h; \lambda)] \left[ f_{0,\lambda/2}(x) + \frac{h^2}{4} f_{0,\lambda/2}(x) + o(h^2) \right].

Summarizing above derivations eventually leads to

\text{var}[\tilde{S}_{2n}(x)] = \frac{h^8}{n \delta_{0h}^8} \left[ E(Z − x)^4 \phi^2 (x; Z, \delta_{0h}^2) - (E(Z − x)^2 \phi(x; Z, \delta_{0h}^2))^2 \right] \\
+ \frac{\lambda^2 \sigma_u^4 h^4}{n \delta_{0h}^4} \left[ E\phi^2 (x; Z, \delta_{0h}^2) - (E\phi(x; Z, \delta_{0h}^2))^2 \right] + \frac{2\lambda \sigma_u^2 h^6}{n \delta_{0h}^6} \left[ E(Z − x)^2 \phi^2 (x; Z, \delta_{0h}^2) - E(Z − x)^2 \phi(x; Z, \delta_{0h}^2) E\phi(x; Z, \delta_{0h}^2) \right].

This concludes the proof of Lemma 8.
Lemma 9. For $\tilde{T}_{n0}(x)$, we have

$$E(\tilde{T}_{n0}(x)) = g_{0,\lambda}(x) + \frac{h^2}{2} g_{0,\lambda}''(x) + o(h^2), \quad \lambda > 0,$$

$$\text{Var}(\tilde{T}_{n0}(x)) = \begin{cases} \frac{1}{n} \left[ \frac{1}{2\sqrt{2\pi \sigma^2_u}} [G_{0,\lambda/2}(x) + H_{0,\lambda/2}(x)] - g_{0,\lambda}^2(x) \right] + O \left( \frac{h^2}{n} \right), & \lambda > 0, \\
\frac{1}{2nh^2} \left[ G_{0,0}(x) + H_{0,0}(x) \right] + o \left( \frac{1}{nh} \right), & \lambda = 0. \end{cases}$$

Proof of Lemma 9. Note that

$$E[Y \phi(x; Z, \delta^2_{\text{oh}})] = E[(g(X) + \varepsilon) \phi(x; Z, \delta^2_{\text{oh}})]$$

$$= E \left( E[(g(X) + \varepsilon) \phi(x; Z, \delta^2_{\text{oh}})|X, U] \right) = E[g(X)\phi(x; Z, \delta^2_{\text{oh}})]$$

$$= \int g(t)f_X(t)(\phi(x-t;0,(\lambda + 1)\sigma^2)dt$$

$$+ \frac{h^2}{2} \frac{\partial^2}{\partial x^2} \int g(t)f_X(t)(\phi(x-t;0,(\lambda + 1)\sigma^2)dt + o(h^2).$$

Note that $\tau^2(X) = E(\varepsilon^2|X)$, we also have

$$E[Y^2 \phi^2(x; Z, \delta^2_{\text{oh}})] = E \left( E[(g(X) + \varepsilon)^2 \phi^2(x; Z, \delta^2_{\text{oh}})|X, U] \right)$$

$$= E[(g^2(X) + \tau^2(X))\phi^2(x; Z, \delta^2_{\text{oh}})] = \int [g^2(t) + \tau^2(t)]\phi^2(u-x-t,\delta^2_{\text{oh}})\phi(u; 0, \sigma^2_u)f_X(t)dt$$

$$= \frac{1}{2\sqrt{\pi \delta^2_{\text{oh}}}^3} \int [g^2(t) + \tau^2(t)]\phi(x-t;0,(\lambda + 2)\sigma^2_u/2)f_X(t)dt$$

$$+ \frac{h^2}{8\sqrt{\pi \delta^2_{\text{oh}}}^3} \int [g^2(t) + \tau^2(t)]\phi(x-t;0,(\lambda + 2)\sigma^2_u/2)f_X(t)dt + o \left( \frac{h^2}{2\sqrt{\pi \delta^2_{\text{oh}}}^3} \right).$$

Therefore,

$$\text{var} \left[ \tilde{T}_{n0}(x) \right] = \frac{1}{2n\sqrt{\pi \delta^2_{\text{oh}}}^3} \int [g^2(t) + \tau^2(t)]\phi(x-t;0,(\lambda + 2)\sigma^2_u/2)f_X(t)dt$$

$$+ \frac{h^2}{8n\sqrt{\pi \delta^2_{\text{oh}}}^3} \int [g^2(t) + \tau^2(t)]\phi(x-t;0,(\lambda + 2)\sigma^2_u/2)f_X(t)dt$$

$$+ o \left( \frac{h^2}{2n\sqrt{\pi \delta^2_{\text{oh}}}^3} \right) - \frac{1}{n} \left[ \int g(t)f_X(t)\phi(x-t;0,(\lambda + 1)\sigma^2_u)dt \right.$$

$$+ \left. \frac{h^2}{2} \frac{\partial^2}{\partial x^2} \int g(t)f_X(t)(\phi(x-t;0,(\lambda + 1)\sigma^2_u)dt + o(h^2) \right]^2.$$  

This implies the result in Lemma 9.
Lemma 10. For $\hat{\Theta}_{n1}(x)$, we have

$$E(\hat{\Theta}_{n1}(x)) = \frac{h^2}{(\lambda + 1)\sigma_u^2} g_{1,\lambda}(x) - \frac{hx^2}{(\lambda + 1)\sigma_u^2} g_{0,\lambda}(x) + o(h^2), \quad \lambda \geq 0,$$

$$\text{Var}(\hat{\Theta}_{n1}(x)) = \begin{cases} 
\frac{h^4}{n(\lambda + 2)^2\sqrt{\lambda \pi \sigma_u^4}} \left[ \frac{1}{2} [G_{2,\lambda/2}(x) + H_{2,\lambda/2}(x)] - x[G_{1,\lambda/2}(x) + H_{1,\lambda/2}(x)] \right] \\
+ \frac{h^4}{n(\lambda + 2)\sqrt{\lambda \pi \sigma_u^4}} \left[ \frac{x^2}{(\lambda + 2)\sigma_u^2} + \frac{1}{\lambda} \right] [G_{0,\lambda/2}(x) + H_{0,\lambda/2}(x)] \\
- \frac{h^4}{n(\lambda + 1)^2\sigma_u^4} [g_{1,\lambda}(x) - xg_{0,\lambda}(x)]^2, \quad \lambda > 0, \\
\frac{h}{4n\sqrt{\pi}} [G_{0,0}(x) + H_{0,0}(x)] + o\left(\frac{h}{n}\right), \quad \lambda = 0. 
\end{cases}$$

Proof of Lemma 10. Note that

$$E \left[ Y(Z - x)\phi(x; Z, \delta_{oh}^2) \right] = E \left[ (g(X) + \epsilon)(Z - x)\phi(x; Z, \delta_{oh}^2) \right] = E \left( E[(g(X) + \epsilon)(Z - x)\phi(x; Z, \delta_{oh}^2) \mid X, U] \right) = E \left[ g(X)(Z - x)\phi(x; Z, \delta_{oh}^2) \right]$$

$$= \int \int g(t)(t + u - x)\phi(x; t + u, \delta_{oh}^2)\phi(u; 0, \sigma_u^2) f_X(t)du dt$$

$$= \frac{\delta_{oh}^2}{\delta_{1h}^2} \left[ \int \int g(t)f_X(t)\phi(x; t, (\lambda + 1)\sigma_u^2)dt + \frac{h^2}{2} \frac{\partial^2}{\partial x^2} \int \int g(t)f_X(t)\phi(x; (\lambda + 1)\sigma_u^2)dt + o(h^2) \right]$$

$$- \frac{\delta_{oh}^2}{\delta_{1h}^2} \left[ \int \int g(t)f_X(t)\phi(x; (\lambda + 1)\sigma_u^2)dt + \frac{h^2}{2} \frac{\partial^2}{\partial x^2} \int \int g(t)f_X(t)\phi(x; (\lambda + 1)\sigma_u^2)dt + o(h^2) \right]$$

Next, we see that

$$E[Y^2(Z - x)^2\phi^2(x; Z, \delta_{oh}^2)] = E \left( E[(g(X) + \epsilon)^2(Z - x)^2\phi^2(x; Z, \delta_{oh}^2) \mid X, U] \right)$$

$$= \int \int [g^2(t) + (\tau^2(t))] \int (t + u - x)^2\phi^2(x; t + u, \delta_{oh}^2)\phi(u; 0, \sigma_u^2) f_X(t)du dt$$

$$= \frac{\delta_{oh}^3}{2\delta_{1h}^4} \sqrt{\pi} \left[ \int [g^2(t) + (\tau^2(t))] f_X(t)\phi(x - t; 0, (\lambda + 2)\sigma_u^2/2)dt + o(h^2) \right]$$

$$+ \frac{h^2\delta_{oh}^3}{8\delta_{1h}^4} \frac{\partial^2}{\partial x^2} \int [g^2(t) + (\tau^2(t))] f_X(t)\phi(x - t; 0, (\lambda + 2)\sigma_u^2/2)dt$$

$$- \frac{x\delta_{oh}^3}{\delta_{1h}^4} \sqrt{\pi} \left[ \int [g^2(t) + (\tau^2(t))] f_X(t)\phi(x - t; 0, (\lambda + 2)\sigma_u^2/2)dt + o(h^2) \right]$$

$$- \frac{hx^2\delta_{oh}^3}{4\delta_{1h}^4} \frac{\partial^2}{\partial x^2} \int [g^2(t) + (\tau^2(t))] f_X(t)\phi(x - t; 0, (\lambda + 2)\sigma_u^2/2)dt$$

$$+ \frac{x^2\delta_{oh}^3}{2\delta_{1h}^4} \frac{\partial^2}{\partial x^2} \int [g^2(t) + (\tau^2(t))] f_X(t)\phi(x - t; 0, (\lambda + 2)\sigma_u^2/2)dt$$

$$+ \frac{x^2h^2\delta_{oh}^3}{8\delta_{1h}^4} \frac{\partial^2}{\partial x^2} \int [g^2(t) + (\tau^2(t))] f_X(t)\phi(x - t; 0, (\lambda + 2)\sigma_u^2/2)dt$$
Therefore, we can show that the variance of $\hat{T}_{1n}(x)$ equals

\[
\frac{h^4}{2n\delta_{2h}^4} \int \left[ g^2(t) + \tau^2(t) \right] f_X(t) \phi(x - t; 0, (\lambda + 2)\sigma_u^2/2) dt + o(h^2) \\
+ \frac{h^6}{8n\delta_{2h}^4 \sqrt{\pi \delta_{0h}^2}} \int \left[ g^2(t) + \tau^2(t) \right] f_X(t) \phi(x - t; 0, (\lambda + 2)\sigma_u^2/2) dt \\
- \frac{xh^4}{n\delta_{2h}^4 \sqrt{\pi \delta_{0h}^2}} \int \left[ g^2(t) + \tau^2(t) \right] f_X(t) \phi(x - t; 0, (\lambda + 2)\sigma_u^2/2) dt \\
+ \frac{x^2h^4}{2n\delta_{2h}^4 \sqrt{\pi \delta_{0h}^2}} \int \left[ g^2(t) + \tau^2(t) \right] f_X(t) \phi(x - t; 0, (\lambda + 2)\sigma_u^2/2) dt \\
+ \frac{x^2h^6}{8n\delta_{2h}^4 \sqrt{\pi \delta_{0h}^2}} \int \left[ g^2(t) + \tau^2(t) \right] f_X(t) \phi(x - t; 0, (\lambda + 2)\sigma_u^2/2) dt \\
+ \frac{\sigma_u^2h^4}{2n\delta_{2h}^2 \delta_{0h}^3 \sqrt{\pi}} \int \left[ g^2(t) + \tau^2(t) \right] f_X(t) \phi(x - t; 0, (\lambda + 2)\sigma_u^2/2) dt \\
+ \frac{\sigma_u^2h^6}{8n\delta_{2h}^2 \delta_{0h}^3 \sqrt{\pi}} \int \left[ g^2(t) + \tau^2(t) \right] f_X(t) \phi(x - t; 0, (\lambda + 2)\sigma_u^2/2) dt \\
- \frac{h^4}{n\delta_{2h}^4} \int t g(t) f_X(t) \phi(x - t; 0, (\lambda + 1)\sigma_u^2) dt \\
+ \frac{h^2}{2} \int \frac{\partial^2}{\partial x^2} t g(t) f_X(t) \phi(x - t; 0, (\lambda + 1)\sigma_u^2) dt \\
- x \int g(t) f_X(t) \phi(t; x, (\lambda + 1)\sigma_u^2) dt - \frac{xh^2}{2} \int \frac{\partial^2}{\partial x^2} g(t) f_X(t) \phi(t; x, (\lambda + 1)\sigma_u^2) dt + o(h^2) \right] ^2.
\]

This concludes the proof of Lemma 10.

**Proof of Theorem 2.** To verify the Lyapunov condition, we have to find out the asymptotic expansions of $Ev^2(x)$, and an upper bound for $E|v^3(x)|$. Note that

\[
Ev^2(x) = c_{04}^2 E\xi_{04}^2 (x) + c_{14}^2 E\xi_{14}^2 (x) + c_{24}^2 E\xi_{24}^2 (x) + d_{04}^2 E\eta_{04}^2 (x) + d_{14}^2 E\eta_{14}^2 (x) \\
+ 2c_{04}c_{14}E[\xi_{04}(x)\xi_{14}(x)] + 2c_{04}c_{24}E[\xi_{04}(x)\xi_{24}(x)] + 2c_{14}d_{04}E[\xi_{04}(x)\eta_{04}(x)] \\
+ 2c_{04}d_{14}E[\xi_{04}(x)\eta_{14}(x)] + 2c_{14}c_{24}E[\xi_{14}(x)\xi_{24}(x)] + 2c_{14}d_{14}E[\xi_{14}(x)\eta_{14}(x)]
\]
+ 2c_{1,4}d_{1,4}E[\xi_1(x_1)\eta_1(x)] + 2c_{2,4}d_{0,4}E[\xi_2(x_1)\eta_1(x)] + 2c_{2,4}d_{1,4}E[\xi_2(x_1)\eta_1(x)] + 2d_{0,4}d_{1,4}E[\eta_0(x_1)\eta_1(x)].

Routine and tedious calculations show that when \( \lambda > 0 \), except for

\[
E[\xi_0(x)\eta_0(x)] = \frac{1}{2\sqrt{\pi} \lambda \sigma^2_u} g_{0,0}(x) - g_{0,0}(x)f_{0,0}(x) + O(h^2),
\]

all other expectations of the cross products are of the order \( O(h^2) \), which, together with the previous derivations with respect to \( E_{\zeta_{0}}(x), E_{\zeta_{1}}(x), E_{\zeta_{2}}(x), E_{\eta_{0}}(x), E_{\eta_{1}}(x) \), we can obtain

\[
E^{2}(x) = c^{2}_{0,4} \left[ \frac{1}{2\sqrt{\pi} \lambda \sigma^2_u} f_{0,0/2}(x) - f_{0,0}(x) \right] + d^{2}_{0,4} \left[ \frac{1}{2\sqrt{\pi} \lambda \sigma^2_u} \{ G_{0,0/2}(x) + H_{0,0/2}(x) \} - g_{0,0}(x) \right] + O(h^2).
\]

When \( \lambda = 0 \), except for \( E[\xi_{00}(x)\eta_{00}(x)] = \frac{1}{2\sqrt{\pi}} g_{0,0}(x) + O(h) \), all other expectations of the cross products are of the order \( O(h) \), which, together with the previous derivations with respect to \( E_{\zeta_{00}}(x), E_{\zeta_{10}}(x), E_{\zeta_{20}}(x), E_{\eta_{00}}(x), E_{\eta_{10}}(x) \), leads to

\[
E^{2}(x) = \frac{1}{2h\sqrt{\pi}} \left[ c^{2}_{0,00} f_{00}(x) + d^{2}_{0,0} \{ G_{00}(x) + H_{00}(x) \} + 2c_{00}d_{00}g_{00}(x) \right] + O(h)
\]

\[
= \frac{1}{2h\sqrt{\pi}} \left[ \frac{G_{00}(x) + H_{00}(x)}{f^{2}_{00}(x)} - \frac{g^{2}_{00}(x)}{f^{2}_{00}(x)} \right] + O(h).
\]

To find a proper order for \( E[\psi(x)]^3 \), we have to find the orders for the expectations

\[
E(\bar{Z}-x)\psi^3(x;Z,\delta^{2}_{0h}), \quad EY\psi^3(x;Z,\delta^{2}_{0h}), \quad EY(\bar{Z}-x)\psi^3(x;Z,\delta^{2}_{0h}),
\]

\[
EY(\bar{Z}-x)^2\psi^2(x;Z,\delta^{2}_{0h}), \quad EY(\bar{Z}-x)^2\psi^2(x;Z,\delta^{2}_{0h}), \quad EY(\bar{Z}-x)^2\psi^2(x;Z,\delta^{2}_{0h}),
\]

\[
E\phi^3(x;Z,\delta^{2}_{0h}), \quad E\bar{Z}-x|\phi^3(x;Z,\delta^{2}_{0h}), \quad E\bar{Z}-x|\phi^3(x;Z,\delta^{2}_{0h}),
\]

\[
E|\phi^3(x;Z,\delta^{2}_{0h}), \quad E|\phi^3(x;Z,\delta^{2}_{0h}).
\]

More complicated calculations show that

\[
E\phi^3(x;Z,\delta^{2}_{0h}) = \frac{1}{2\pi \delta^{2}_{0h} \sqrt{3}} \left[ f_{0,0/3}(x) + \frac{h^2}{6} f_{0,0/3}''(x) + o(h^2) \right],
\]

which is \( O(1) \) when \( \lambda > 0 \) and \( O(1/h^2) \) when \( \lambda = 0 \), and \( E|\bar{Z}-x|^3\phi^3(x;Z,\delta^{2}_{0h}) \) is bounded above by

\[
\frac{2}{\pi \sqrt{3\delta^{2}_{0h}}} \cdot \frac{8\sqrt{2}}{\sqrt{\pi}} \left( \frac{\sigma^2_u \delta^{2}_{0h}}{h^2 + (\lambda + 3)\sigma^2_u} \right)^{\frac{1}{2}} \int \phi \left( t; x, \frac{h^2 + (\lambda + 3)\sigma^2_u}{3} \right) f_X(t) dt
\]
\[
\begin{align*}
+ \frac{2}{\pi \sqrt{3\delta^2_{o_h}}} & \cdot 4 \left( \frac{3\sigma_u^2}{h^2 + (\lambda + 3)\sigma_u^2} \right)^3 \int |x - t|^3 \phi \left( t; x, \frac{h^2 + (\lambda + 3)\sigma_u^2}{3} \right) f_X(t) dt \\
+ \frac{2}{\pi \sqrt{3\delta^2_{o_h}}} & \int |x - t|^3 \phi \left( t; x, \frac{h^2 + (\lambda + 3)\sigma_u^2}{3} \right) f_X(t) dt,
\end{align*}
\]

which is \(O(1)\) when \(\lambda > 0\) and \(O(1/h^2)\) when \(\lambda = 0\). We also have

\[
E|Z - x|^6 \phi^3(x; Z, \delta_{o_h}^2) \leq \frac{7680\sigma_u^6}{\pi \sqrt{3\delta^2_{o_h}}} \left( \frac{\sigma_u^2 \delta_{o_h}^2}{h^2 + (\lambda + 3)\sigma_u^2} \right)^3 \times \int \phi \left( t; x, \frac{h^2 + (\lambda + 3)\sigma_u^2}{3} \right) f_X(t) dt \\
+ \frac{512}{\pi \sqrt{3\delta^2_{o_h}}} \left( \frac{3\sigma_u^2}{h^2 + (\lambda + 3)\sigma_u^2} \right)^6 \int |x - t|^6 \phi \left( t; x, \frac{h^2 + (\lambda + 3)\sigma_u^2}{3} \right) f_X(t) dt \\
+ \frac{16}{\pi \sqrt{3\delta^2_{o_h}}} \int |x - t|^6 \phi \left( t; x, \frac{h^2 + (\lambda + 3)\sigma_u^2}{3} \right) f_X(t) dt,
\]

which is \(O(1)\) when \(\lambda > 0\) and \(O(1/h^2)\) when \(\lambda = 0\). Denote \(\delta(X) = E \left( |e|^3 |X| \right)\), then

\[
E|Y|^3 \phi^3(x; Z, \delta_{o_h}^2) \leq 4E|g(X)|^3 \phi^3(x; Z, \delta_{o_h}^2) + 4E\delta(X) \phi^3(x; Z, \delta_{o_h}^2).
\]

Eventually, we can show that \(E|Y|^3 \phi^3(x; Z, \delta_{o_h}^2)\) is bounded above by

\[
\frac{2}{\pi \sqrt{3\delta^2_{o_h}}} \int \left[ |g(t)|^3 + \delta(t) \right] \phi \left( t; x, \frac{h^2 + (\lambda + 3)\sigma_u^2}{3} \right) f_X(t) dt,
\]

which is \(O(1)\) for \(\lambda > 0\) and \(O(1/h^2)\) for \(\lambda = 0\). Finally, for \(E|Y|^3 |Z - x|^3 \phi^3(x; Z, \delta_{o_h}^2)\), we can show it is bounded above by

\[
\frac{64\sqrt{2}}{\pi \sqrt{3\sigma_u^2 \delta_{o_h}^2}} \left( \frac{\sigma_u^2 \delta_{o_h}^2}{h^2 + (\lambda + 3)\sigma_u^2} \right)^{3/2} \int \left[ |g(t)|^3 + \delta(t) \right] \phi \left( t; x, \frac{h^2 + (\lambda + 3)\sigma_u^2}{3} \right) f_X(t) dt \\
+ \frac{32}{\pi \sqrt{3\delta^2_{o_h}}} \left( \frac{3\sigma_u^2}{h^2 + (\lambda + 3)\sigma_u^2} \right)^3 + \frac{8}{\pi \sqrt{3\delta^2_{o_h}}} \cdot \int \left[ |g(t)|^3 + \delta(t) \right] |x - t|^3 \phi \left( t; x, \frac{h^2 + (\lambda + 3)\sigma_u^2}{3} \right) f_X(t) dt,
\]

which is \(O(1)\) when \(\lambda > 0\) and \(O(1/h^2)\) when \(\lambda = 0\). Therefore, for \(\lambda \geq 0\), we can show that \(\sum_{i=1}^n E|V_i(x)|^3 \left( \sum_{i=1}^n E\nu^2_{i}\right)^{3/2} \rightarrow 0\). So, by Lyapunov central limit theorem, we proved Theorem 2.

**Proof of (3)–(5).** By the normality assumption of \(V\) and its independence from other random variables in the model, and the kernel function \(K\) being the standard normal
density, from Lemma 5, we have

\[
E[K_h(Z(\lambda) - x)|Y, Z] = \int \phi(v; x - Z, h^2)\phi(v; 0, \lambda \sigma_u^2)dv = \phi(x; Z, \delta_{0h}^2),
\]

which is (3). (4) can be derived from the following algebra,

\[
E[(Z(\lambda) - x)K_h(Z(\lambda) - x)|Y, Z] = (Z - x)\phi(x - Z; 0, \delta_{0h}^2)
+ \frac{\lambda \sigma_u^2(x - Z)}{\delta_{0h}^2} \phi(x - Z; 0, \delta_{0h}^2).
\]

Finally, note that \(E[(Z(\lambda) - x)^2K_h(Z(\lambda) - x)|Y, Z]\) can be written as

\[
(Z - x)^2\phi(x - Z; 0, \delta_{0h}^2) - \frac{2\lambda \sigma_u^2(Z - x)^2}{\delta_{0h}^2} \phi(x - Z; 0, \delta_{0h}^2)
+ \left[ \frac{\lambda \sigma_u^2h^2}{\lambda \sigma_u^2 + h^2} + \left( \frac{\lambda \sigma_u^2(x - Z)}{\lambda \sigma_u^2 + h^2} \right)^2 \right] \phi(x - Z; 0, \delta_{0h}^2),
\]

this is exactly (5).