World-Sheet Geometry and Baby Universes in 2-D Quantum Gravity

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We show that the surface roughness for $c < 1$ matter theories coupled to 2D quantum gravity is described by a self-similar structure of baby universes. There exist baby universes whose neck thickness is of the order of the ultraviolet cutoff, the largest of these having a macroscopic area $\sim A^{\frac{1}{1-\gamma}}$, where $A$ is the total area and $\gamma$ the string susceptibility exponent.
Sums over random surfaces have appeared in a number of contexts, for example, the study of real membranes made up of molecules, superstring theory, graph theory. In the context of membranes they have been used to describe experimentally observed phase transitions in which the geometric nature of the surface changes from smooth to rough with various kinds of roughness: crumpled, spongy, layered, etc. In this letter we are interested in the same geometric question – how rough is the surface and how does one characterize its roughness – but in the context of $c \leq 1$ conformal matter theories coupled to 2-dimensional quantum gravity. For quantum gravity theories this question concerns the short-distance quantum nature of spacetime. Geometric properties of dominant surfaces in a sum are of physical interest also for QCD and the 3-d Ising model. We hope that extracting these properties for $c \leq 1$ models will be of use in other areas also, particularly if they result in a geometric understanding of the $c = 1$ barrier and help to go beyond it.

The internal fractal dimension of the surface has been discussed recently from the dynamical triangulation \[1\][2][3] and Liouville \[4][5] viewpoints. While there is general agreement that the typical surface is quite rough there is yet no agreement on the value of the fractal dimension. At present we also lack a picture of what the typical surface looks like at various length scales and how this picture depends upon the matter living on the surface. From their numerical studies of geodesic observables on the surface Agishtein and Migdal \[2\] conclude that the typical surface for the $c = 0$ system (pure gravity) is highly branched. David \[5\] provides evidence from Liouville computations that a typical geodesic passes through many regions where the Liouville field is highly negative (the local scale factor is very small).

In this letter we characterize the roughness of the typical surface in terms of the distribution of baby universes on it. We estimate the average number of baby universes of different areas and neck sizes, and study the variation of these distributions with the central charge of the matter that lives on the surface. The results are obtained analytically by estimating the product of entropies of the baby and parent together with the number of ways of joining the two.

**Definition of baby universes.** A baby universe is defined as a simply connected region of the surface whose boundary length is much smaller than the square-root of its area. This definition is meant to capture, in purely intrinsic terms, the intuitive picture of a baby universe as an inflated balloon like region of the surface attached to it by a small neck. The thickness of a neck will be defined as the length of the loop located at the thinnest point of the neck. Thus, to determine the thickness we need to identify, given a closed loop on
the surface encircling the neck of a baby universe, where the length of this loop becomes a minimum as we slide it along the neck. It is not enough to have a ‘local’ minimum, for then it is possible that a short distance away along the neck there is another local minimum of even smaller length, in which case the latter would be a preferable location of the beginning of the baby universe. It seems reasonable that in order to check for a minimum we probe an area that is some fraction of the area of the baby universe itself on either side of a candidate loop, and as a working definition we choose this fraction to be a half. To be precise, a simply connected region of area $B$ with a boundary $C$ of length $l$ will be called a baby universe of size $B$ that begins at $C$ and has a neck thickness $l$ provided $l$ is the minimum length of all loops to which $C$ can be continuously deformed and which lie in any annular region of area $B/2$ on either side of $C$. Further, $l < f \sqrt{4\pi B}$, where $f \ll 1$, and $B < A/2$, where $A$ is the area of the whole surface. The last two conditions are needed to justify the nomenclature ‘baby universe’.

The purpose of this definition is that we characterize a baby universe like region of the surface by an essentially unique choice of boundary curve $C$. (The degeneracy due to loops of exactly the same size close by on the same neck is not expected to be significant for our results.) Hence determining the distribution of baby universes on a surface just involves counting loops on it with the above mentioned properties. A surface can have baby universes of various sizes living on it and baby universes can grow upon other baby universes. By definition the area of a ‘daughter baby universe’ is smaller than its ‘parent baby universe’ since the latter includes the former, but both are counted as distinct baby universes.

Baby universes with the thinnest possible neck will play an important role. Thus, we picture the surface as having some ultraviolet cutoff, e.g., a triangulated surface covered with equilateral triangles. We will always think of a surface as defined by such a triangulation; distinct surfaces are specified by distinct triangulations. All lengths are in units of the edge size of each elementary triangle, all areas are in units of the elementary area (a surface of area $A$ has $A$ triangles), and matter lives on the vertices of triangles. Thus we can imagine the thinnest neck as having a circumference of 3; such a neck is obtained by removing one triangle each from two closed surfaces and joining them together at the boundary.

A baby universe whose neck thickness equals this minimum possible circumference on the surface will be called a ‘minimum neck baby universe’ (abbreviated ‘minbu’). The
smallest possible area of a minbu is 3 (for a 3-plaquette tetrahedral ‘blip’) and the largest
minbu can have an area of $A/2$.

1. Results
Suppose we choose a range $B < B' < 2B$ for the area of baby universes. We first mark
out all baby universes within this area range and with neck thickness 3. Next we mark
out baby universes (in the same size range) and neck thickness 4, and so on. We ask the
question: What fraction $F(f)$ of the surface is covered by baby universes in this size range,
with neck thicknesses of order $f \sqrt{B'}$, $(0 < f < 1)$? If the surface was smoother at larger
scales than at smaller scales, then as we increased $B$, $F(f)$ would decrease for small $f$
and increase for larger $f$. If on the other hand the surface was self-similar through all
scales, then $F(f)$ would be a function independent of $B$. We find that the surface is in
fact self-similar, if we use in our power law ansatz for the entropy of ‘spheres with a neck’
an exponent indicated by the known entropy of closed surfaces of higher genera.

The entropy of higher genus surfaces is used in the following way. A genus one surface
can be obtained from a genus zero surface with two equal holes by identifying the holes.
Thus with an ansatz for the entropy of genus zero surfaces with holes we can estimate the
entropy of higher genus surfaces in terms of the power law exponents in the ansatz. A
simple argument gives that the string susceptibility $\gamma(g)$ increases linearly with genus $g$.
Identifying the slope in the linear growth with the known value we deduce the value of
the ‘boundary length’ exponent. This value of the exponent gives rise to the self similar
world-sheet structure mentioned above.

We now describe the results in more detail. $\bar{n}_A(B, l)$ denotes the average number of
baby universes of fixed area $B$ and neck thickness $l$ on a typical closed surface of area $A$.
$n_A(B_1 \rightarrow B_2, l)$ denotes the average number of baby universes of area in the range $B_1$
to $B_2$ and fixed neck thickness $l$. The average is always taken in the ensemble of closed
surfaces of fixed area $A$. $n_A(B, 3) \equiv \bar{n}_A(B)$ refers to minbu’s.

A. Distribution of minimum neck baby universes. We find that the average number of
minbu’s of area $B$ on a closed genus $g$ surface of area $A$ is given by

$$\bar{n}_A(B) \simeq k A^{3-\gamma(g)} (A - B)^{\gamma(g)-2} B^{g-3}.$$  (1)

Here $\gamma(g) = \gamma + g(2 - \gamma)$, where $\gamma$ is the string susceptibility exponent, given by $\gamma = \frac{1}{12} [c - 1 - \sqrt{(25 - c)(1 - c)}]$ for $c \leq 1$ matter theories coupled to gravity. $k$ is a
constant independent of $A$ and $B$ and of order unity for $c < 1$. (Henceforth we will
not display such constants.) As will be seen in section 2, the only assumption needed to prove (1) is that the partition function for closed genus zero surfaces of area $B$ is given by $Z(B) \sim e^{\mu B} B^{\gamma - 3}$, and the partition function for genus $g$ closed surfaces of area $A$ is given by $Z(A) \sim e^{\mu A} A^{\gamma(g) - 3}$. Thus (1) holds whenever $A$ and $B$ are large enough for these formulae to be valid.

**Corollaries:** For $B \ll A$, (1) reduces to $\bar{n}_A(B) \sim A B^{\gamma - 2}$, in keeping with the expectation that the local structure of a sufficiently large surface should be independent of the genus. The average number of minbu’s of size between $B$ and $2B$ is

$$\bar{n}_A(B \to 2B) = \sum_{B' = B}^{2B} \bar{n}_A(B') \to \int_B^{2B} dB' \bar{n}_A(B') \sim A B^{\gamma - 1}.$$  \hspace{1cm} (2)

This implies that the surface always has small minbu’s, the average number decreasing with increasing size of the minbu. Strictly speaking (2) is derived for large $B$, but if we extrapolate it down to $B$ of order 3 (i.e., blips), it would imply that the average number of blips is of the order $A$, i.e., a significant fraction of the surface is covered with blips. In the large $B$ domain $\bar{n}_A(B \to 2B)$ decreases to $O(1)$ for $B \sim A^{1/\gamma}$, implying that the largest minbu to be seen on a surface of sufficiently large area $A$ is typically of size

$$B_{max} \sim A^{1/\gamma}.$$  \hspace{1cm} (3)

Thus for $\gamma < 0$ (i.e., for $c < 1$) the size of the largest minbu on the surface is always less than $O(A)$ for sufficiently large $A$. The average area residing in minbu’s of size $B$, i.e., the sum of the areas of all minbu’s of size $B$ is $B \bar{n}_A(B)$, hence the average area residing in minbu’s of size between $B$ and $2B$ is given by $\bar{a}_A(B \to 2B) = \int_B^{2B} dB' B' \bar{n}_A(B') \sim A \frac{1}{\gamma}[(2B)^\gamma - B^\gamma].$

For large $B$ this decreases monotonically as $B$ increases, like $A B^\gamma$.

Note that the $B$ dependence in (2) changes character at the scale $B \sim A_0 \equiv \exp(\frac{1}{|\gamma|})$, which is $O(1)$ at $c = 0$ and infinite at $c = 1$. For $c$ close to 1, $B^\gamma \simeq 1$ in the domain $1 \ll B \ll A_0$, and if (2) were valid at these scales, it would imply $\bar{n}_A(B \to 2B) \sim A/B$ and $\bar{a}_A(B \to 2B) \sim A$. This would mean that minbu’s in every area range $B$ to $2B$ capture the same total area, which is $O(A)$ and independent of $B$.

**B. Baby universes with arbitrary neck size.** We estimate the average number of baby universes of area $B$ and neck size $l$ on a genus $g$ surface of area $A$ to be given by

$$\bar{n}_A(B, l) \sim A^{3-\gamma(g)} (A - B)^{\gamma(g) - 2} B^{\gamma - 2} l^{-(1+2\gamma)}.$$  \hspace{1cm} (4)
The derivation of this formula uses an ansatz (Eq. (9)) discussed in section 2. (4) is valid only when $B$ is sufficiently large and for $l$ less than some fraction of $\sqrt{B}$.

**Corollaries:** Again for $B \ll A$, $\bar{n}_A(B, l) \sim A B^{\gamma - 2} l^{-(1+2\gamma)}$, independent of genus. The number of baby universes of area between $B$ and $2B$ and neck thickness between $l$ and $2l$ is the sum $\sum_{B'=B}^{2B} \sum_{l'=l}^{2l} \bar{n}_A(B', l')$ which is given by

$$\bar{n}_A(B \to 2B, l \to 2l) \sim A B^{\gamma - 1} l^{-2\gamma}.$$  \hspace{1cm} (5)

The same result (upto a numerical factor of order unity) is obtained by including baby universes of neck size smaller than $l$ since the sum over $l'$ is dominated by the larger necks for $\gamma < 0$. The area carried by this set of baby universes is $\bar{a}_A(B \to 2B, l \to 2l) \sim A B^\gamma l^{-2\gamma}$. If we set $l$ to be a fixed fraction $f$ of $\sqrt{B}$, we get a result that is independent of the scale $B$:

$$\bar{a}_A(B \to 2B, f\sqrt{B} \to 2f\sqrt{B}) \sim A f^{-2\gamma}.$$ \hspace{1cm} (6)

Thus the typical surface is *self-similar* at sufficiently large length scales, since, given a scale $B$, the fraction of the total area captured by baby universes defined by that scale and $f$ (namely, those whose area is of order $B$ and whose neck thickness is of the order of $f\sqrt{B}$) is independent of the scale $B$ and depends only on $f$. (4) implies that $F(f) \sim f^{-2\gamma}$. The $f$ dependence of this quantity involves $\gamma$ and can be used to distinguish theories with different central charges. As $c$ increases it implies a drift towards narrower necks, and hence rougher surfaces. The total area captured by baby universes defined by the scale $B$ is of the order of the area of the whole surface $A$ for any $B$, which means that small baby universes must live on larger ones in a self similar way. Note that this self similarity is a consequence of the very specific power law $l^{-(1+2\gamma)}$ in (4).

**C. Baby universes at $c = 1$.** As $c$ approaches 1 from below, $\gamma$ approaches zero from below and $A_0$ diverges. Thus if (4) were valid at scales $1 \ll B \ll A_0$ for $c \leq 1$, the baby universe distributions discussed above for $B \ll A_0$ would be valid for the whole surface at $c = 1$. i.e., the surface is bubbly with minbu’s, with minbu’s upon minbu’s from the smallest to the largest size in such a way that minbu’s in *every* area range $B \to 2B$ capture the *same* total area of the order of the area of the whole surface, $A$, leaving little space for baby universes on the surface with neck sizes much larger than the minimum. This would be quite interesting since it would mean that the picture of the surface at $c = 1$ is already visible at $c < 1$ provided we look at baby universes of area less than $A_0$. This would suggest that $c = 1$ is some kind of a phase transition point and $A_0(c)$ defines a ‘correlation
length’ on the surface that diverges as the critical point is approached. (Note that this divergence has the form $\exp(const./\sqrt{1-c})$ rather than a power law in $1-c$.)

However, at $c = 1$, logarithmic scaling violations [8] are believed to modify the fixed area partition function to $Z(A) \sim e^{\mu A} A^{-1} (A \ln A)^{2(9-1)}$. That in turn modifies (4) to $\tilde{n}_A(B) \sim A (B \ln B)^{-2}$ (instead of just $A B^{-2}$ for $\gamma = 0$). Then minbu’s at all length scales do not capture the same area $A$, but instead $\tilde{a}_A(B \to 2B) \sim A (\ln B)^{-2}$, i.e., larger minbu’s are suppressed by the logarithmic factor. A speculation on the distribution of baby universes with larger neck sizes is discussed at the end of the next section.

This raises the question as to whether logarithmic scaling violations are visible even for $c < 1$ theories at scales less than $A_0$. It is possible that subleading corrections to the asymptotic formula for $Z(A)$ at fixed genus could lead to such a behaviour (e.g., corrections suppressed by factors of $A^7$) [8] If so, that would still mean that $c = 1$ behaviour is captured in $c < 1$ theories at scales less than $A_0(c)$.

2. Proofs

Let us define some notation: The fixed area partition function is $Z(A) \equiv \sum_S W(S)$, where the sum is over all closed surfaces (distinct triangulations) of area $A$. $W(S)$ is the weight factor to be attached to the surface $S$ which includes the integral over matter fields. Similarly, the partition function for surfaces with one boundary is $Z(A, l) \equiv \sum_{S_1} W(S)$, where the sum is over all surfaces of area $A$ and one boundary of length $l$. For pure gravity $W(S) = 1$, and then $Z(A)$ is just the number of closed surfaces with area $A$, and $Z(A, l)$ the number of surfaces having one boundary and with area $A$ and boundary length $l$. To make the combinatoric argument transparent we first restrict to the pure gravity case, and further, consider only surfaces with no handles. The generalizations to include matter and surfaces with handles will be discussed subsequently.

If we join two surfaces of area $B$ and $A - B$ (with $B < A - B$), each having a single boundary of length $l$, along their boundaries, we obtain a closed surface of area $A$ with a marked loop of length $l$ partitioning it into parts of area $B$ and $A - B$. Since the boundary has $l$ links there are generically $l$ ways of joining the two surfaces to obtain distinct final surfaces. (If the initial surfaces were highly symmetric, some of these $l$ ways of joining them would not be distinct, but for large $B$ and $A - B$ this would happen for relatively very few surfaces.) It is obvious that any surface of area $A$ and a marked loop of length

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1 We thank F. David for suggesting this possibility.
that partitions it into sizes $B$ and $A - B$ can be represented as a join of the aforesaid two surfaces, and for every distinct choice of any of the original two surfaces a distinct final marked surface is obtained. Thus the number, denoted $G(A, B, l)$, of closed surfaces of area $A$ with a marked loop of length $l$ that partitions the surface into parts of area $B$ and $A - B$ is given by $G(A, B, l) \simeq l \ Z(B, l) \ Z(A - B, l)$.

Consider now the quantity $n_S(B, l)$, which is defined to be the number of ways of marking a nonintersecting loop of length $l$ on a surface $S$ such that an area $B$ is enclosed by the loop. Then by definition, since $G(A, B, l)$ is the total number of surfaces so marked, one has $G(A, B, l) = \sum_S n_S(B, l)$. From this it follows that the average value of $n_S(B, l)$ in the ensemble of closed surfaces of area $A$ equals $G(A, B, l)/Z(A)$ and hence is given by

$$\langle n_S(B, l) \rangle_A \simeq \frac{1}{Z(A)} l \ Z(B, l) \ Z(A - B, l). \quad (7)$$

Case A: $l = 3$. In this case the region enclosed by every such loop is a minbu of size $B$ on the surface. A slight reflection will convince the reader that since $l = 3$ is the smallest loop possible, $n_S(B, l)$ also equals the number of minbu’s of area $B$ on $S$. (If $l > 3$, $n_S(B, l)$ in general overcounts the number of baby universes of area $B$, since now one can possibly slide the loop along the neck keeping the area $B$ enclosed the same, thereby obtaining another entry in $n_S(B, l)$ for the same baby universe. For $l = 3$, the minimum loop length, no sliding that preserves $B$ is possible.) Thus

$$\bar{n}_A(B) \simeq \frac{1}{Z(A)} 3 \ Z(B, 3) \ Z(A - B, 3). \quad (8)$$

We need to estimate $Z(B, 3)$, the number of surfaces of area $B$ and one boundary of length 3. Since this boundary is created by removing one triangle from a closed surface with $B + 1$ triangles, and since this triangle can be chosen in $B + 1$ ways to give, generically, a different final surface, it follows that $Z(B, 3) \sim (B + 1) \ Z(B + 1)$. Using this in (8) and the fact that $Z(A) \sim e^{\mu A} \ A^{\gamma - 3}$ we obtain (1) for $g = 0$.

Case B: $l > 3$. From the set of $n_S(B, l)$ loops on the surface we need to select the subset that are true boundaries of baby universes in the sense described in the introduction. Imagine cutting the surface into two parts along such a ‘true’ neck. We would like to estimate the entropy of each of the two parts. The relevant quantities are not $Z(B, l)$ and $Z(A - B, l)$, because these just count the number of surfaces with given area and boundary length, with no reference to the fact that the cut was along a ‘minimal’ loop. We need instead the quantity $Z_1(B, l)$ which we define as follows. $Z_1(B, l)$ is the sum over surfaces
of area $B$, boundary $l$, with the further property that this boundary cannot be deformed along the surface to a smaller curve if area less than $B/2$ is swept out in the deformation. Thus $Z_1(B,l)$ counts a subset of the surfaces included in $Z(B,l)$; it excludes those surfaces of area $A$ and boundary length $l$ for which the boundary is not a minimal loop in the above sense. Our ansatz for $Z_1$ is

$$Z_1(B,l) \sim e^{\mu B} B^{\gamma-2} l^{-(1+\gamma)}. \quad (9)$$

Substituting this in the expression $\bar{n}_A(B,l) \simeq \frac{1}{Z(A)} l Z_1(B,l) Z_1(A-B,l)$ which is analogous to (7) but counts only true necks, we get (4).

Discussion of the ansatz (9). As a justification of (9) we start from $Z_1(B,l) \sim e^{\mu A} A^{\gamma-2} l^{-(1+\alpha)}$, where the area exponent follows from the entropy $\sim A Z(A)$ for locating the centre of a loop on a closed surface, and the boundary length exponent $\alpha$ is left undetermined. This suggests a corresponding ansatz for genus zero surfaces of area $A$ with two holes of length $l$ each:

$$Z_1(A,l,l) \sim Z(A) [A l^{-(1+\alpha)}]^2, \quad (10)$$

where following the spirit of the definition of $Z_1(B,l)$ it is assumed that the boundaries cannot be deformed to smaller loops anywhere along the surface. Each factor of $A l^{-(1+\alpha)}$ corresponds to having one loop on the surface. Identifying the two holes in the $l$ possible ways we get for the number of surfaces of genus one

$$Z^{g=1}(A) \sim \int_3^{\sqrt{A}} dl \ l Z_1(A,l,l) \sim e^{\mu A} A^{-1}, \quad (11)$$

which agrees with the known result. The difference between using $Z$ and $Z_1$ is important since the former would give genus one surfaces with an arbitrary marked loop while the latter gives just genus one surfaces (since it marks a unique loop, the smallest). The above argument also extends to give the known linear increase of the string susceptibility exponent with genus, $\gamma(g) = \gamma + g(2 - \gamma)$. The extra power of $A^{2-\gamma}$ for every handle has the simple interpretation: $A^2$ for locating two holes, $A^{-\gamma}$ for integrating over their boundary lengths.

The same power laws in $B$ and $l$ as in (9) have appeared in [10] in the context of $Z(B,l)$, and are derived by completely different methods. Therefore one wonders about the relationship between $Z(B,l)$ and $Z_1(B,l)$. For example, one might ask: can the expression on the r.h.s. of (9) correspond to $Z(B,l)$? In fact it cannot correspond to $Z(B,l)$ since
it can be seen combinatorially that \( Z(B, l) \) increases with \( l \) for small \( l \) while (9) decreases with \( l \). \( Z(B, l) \) is expected to have an increasing factor like \( e^{\rho l} (\rho > 0) \) which cannot be present in \( Z_1(B, l) \); in the latter it would lead to the contradiction that non-overlapping baby universes cover more than the entire area of a typical surface. It is interesting that the two physically distinct quantities \( Z \) and \( Z_1 \) differ by a large non-universal factor, but their subleading behaviour given by the power laws seems to be the same. At large \( l \) \((l \geq \sqrt{A})\), we expect \( Z_1 \) to be exponentially damped by factors of the type \( e^{-\frac{l^2}{2\pi}} \), like \( Z \).

The derivation of baby universe distributions presented above generalizes easily to higher genus surfaces. The baby universe remains a sphere with one boundary but the parent is a higher genus surface whose entropy is appropriately modified. Accordingly the average number of baby universes is given by

\[
\bar{n}_A(B, l) \simeq \frac{1}{Z(g)(A)} \ l \ Z_1^{(0)}(B, l) \ Z_1^{(g)}(A - B, l),
\]

and one uses \( Z(g)(A) \sim e^{\mu A} A^{\gamma(g)-3} \), \( Z_1^{(g)}(A - B, l) \sim e^{\mu(A-B)} (A - B)^{\gamma(g)-2} l^{-(1+\gamma)} \). This completes the proof of (4) and (4) for pure gravity. Of course this assumes that the total number of plaquettes is much larger than the genus.

**Generalization of proof for unitary \( c < 1 \) matter.** One might wonder if the non-local contribution to the weight factor \( W(S) \) arising from the integration over matter permits the simple product relation \( G(A, B, l) \simeq l \ Z(B, l) \ Z(A - B, l) \) which was essential in the above analysis. Consider the state created at the boundary of a baby universe on a fixed surface by integrating matter over the baby universe. Expanding the matter integral in a complete set of states at the neck, we find that only the Virasoro tower above the identity can contribute because the other primaries have vanishing one-point functions on the sphere. Further, for narrow necks \((f \ll 1)\) the identity itself gives the predominant contribution, as it is separated by an energy gap from the next highest state in its Virasoro tower. From this it follows that the above factorised form for \( G(A, B, l) \) is a good estimate, and our analysis extends to include unitary matter living on the triangulated surfaces. We are unable to make any statement for nonunitary matter theories where the lowest dimension operator is not the identity, because for these theories the entropy of fixed area surfaces is not known.

**Speculation for \( c = 1 \).** The appearance of logs in \( Z(A) \) for \( c = 1 \) implies that one must depart from the power law ansatz (9) for \( Z_1 \). We discuss here a possible modification.
We ask: what $Z_1(A,l,l)$ needs to be substituted in (11) to get the genus dependence $Z^{(g)}(A) \sim e^{\mu A} A^{-1} (A \ln A)^{(g-1)}$ for the closed surface partition function? It is easy to check that $Z_1^{(g)}(A,l,l) \sim Z^{(g)}(A) [A l^{-1} (\ln l)^{1/2}]^2$ substituted in (11) gives the correct result for $Z^{(g+1)}(A)$. This suggests that for $c = 1$ the analogue of (11) might be $Z_1^{(g)}(B,l) \sim Z^{(g)}(B) B l^{-1} (\ln l)^{1/2}$. Substituting this ansatz in (12), we get $\bar{n}_A(B,l) \sim A (B \ln B)^{-2} l^{-1} \ln l$ for $B \ll A$. This in turn implies $\bar{a}_A(B \to 2B, f\sqrt{B} \to 2f\sqrt{B}) \sim A (\ln B)^{-1} (1 + \frac{\ln f}{\ln \sqrt{B}})$, and $\bar{a}_A(B \to 2B, 3 \to fB^\alpha) \sim A \alpha^2 (1 + \frac{2\ln f}{\ln B^\alpha})$. This would mean that baby universes whose neck thicknesses are much less than $O(\sqrt{B})$ capture a significant fraction of the total area of the surface, which is in keeping with the drift towards narrower necks with increasing $c$ mentioned in the previous section. We would like to emphasize, however, that due to subtleties of subleading effects at $c = 1$ which are related to the appearance of the logs, these expressions are at the moment on a weaker footing than the $c < 1$ results. It is necessary to check our speculated ansatz for $Z_1$ by other methods.

3. Discussion

We have analysed the distribution of baby universes on randomly triangulated surfaces, taking into account the entropies of the baby and the parent. It would be interesting to analyse along these lines the random ‘triangulations’ of 3-dimensional and higher dimensional manifolds.

We have obtained a simple argument for the linear increase of $\gamma(g)$ with genus, given the ansatz (9). This linear increase is crucial to the existence of the double scaling limit in the corresponding string theories. The correct slope of $\gamma(g)$ is given for the same power laws in $Z_1(B,l)$ that give self similarity of the surface geometry. This suggests that there could be a connection between the geometrical structure on the world-sheet and target space properties like the Virasoro symmetry observed in matrix models.

We find that the typical surface does have minimum-sized necks. The surface ‘pinches’ to the ultraviolet cutoff at these necks. From a continuum viewpoint, all surfaces with minbu’s are at the boundary of the space of surfaces. A simple ‘block-renormalization’ of a discretely triangulated surface would cause the minbu’s to get disconnected from the parent leaving a puncture on both pieces. This is likely to be a generic feature of quantum gravity theories. One would therefore have to keep track of how the punctures evolve under the renormalization group. Alternatively, one could possibly use a kind of ‘nested’ renormalization group, wherein, upon encountering a small neck, one first integrates over the baby universe and smoothes out the neck, and then proceeds to integrate over the
parent. We believe that the development of the RG approach for quantum gravity theories is an important problem, relevant for understanding observed phenomena far above the scale of fluctuations of spacetime.

It is interesting that our formulae for the number distribution of baby universes also arise in a semiclassical calculation of the Liouville path integral. The Liouville action for a spherical baby universe of area $B$ and neck $l$ is $\sim (1 - \gamma) \ln(B/l^2)$ \[^1\]. Taking the entropy arising from different possible placings of the baby on the parent to be $\ln(A/l^2)$, one gets the contribution of this configuration to the path integral to be $e^{\ln(A/l^2) - (1 - \gamma) \ln(B/l^2)} = AB^{\gamma - 1} l^{-2\gamma}$. We observe that this expression is the same as our eq. (5) for the average number of baby universes of area $\sim B$ and neck $\sim l$. In particular, setting the free energy $\ln(A/l^2) - (1 - \gamma) \ln(B/l^2)$ to be unity for $l$ equalling the cutoff gives (3) for the largest minbu, which, as a special case implies that for $c < 1$ there are no minbu’s of area $\sim A$ \[^1\]. Note that if we assume the expression for $\bar{n}_A(B \to 2B, l \to 2l)$ is independent of the cutoff then self-similarity follows immediately; $B$ and $l$ would enter only in the combination $B/l^2$ (at least for $B \ll A$).

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\[^2\] It seems that one should take the entropy to be $\ln(A/l^2)$ instead of the $\ln A$ expected from a Kosterlitz-Thouless type argument. This could be because the Liouville ansatz is to be interpreted as an effective theory, in which, for configurations such as the above, gravitational fluctuations smaller than the neck size which do not destroy the essential configuration, are already integrated over.
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