Re-Recounting Dyons in N=4 String Theory

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Abstract

The purpose of this brief note is to understand the reason for the appearance of a genus two Riemann surface in the expression for the microscopic degeneracy of 1/4 BPS dyons in N=4 String Theory.
\[ N = 4 \text{ string theory has a duality group} \]
\[ SL(2, Z) \times O(22, 6, Z) \quad (1) \]

The \( U(1) \) charges of the theory form a \( SL(2, Z) \) doublet \( (q_e, q_m) \) of \( O(22, 6, Z) \) vectors. Objects with \( q_e = 0 \) or \( q_m = 0 \) are half BPS, while \( 1/4 \) BPS dyons may have generic charges. The degeneracy of dyonic states depends only on the \( O(22, 6, Z) \) invariants

\[ N_e = \frac{q_e^2}{2} \quad N_{em} = q_e \cdot q_m \quad N_m = \frac{q_m^2}{2} \quad (2) \]

The generating function for the degeneracies is conjectured to be

\[ \sum_{N_e, N_{em}, N_m} e^{2\pi i (N_e \rho + N_{em} v + N_m \sigma)} d(N_e, N_{em}, N_m) = \Phi \left[\begin{array}{c} \rho \\ v \\ \sigma \end{array}\right] \quad (3) \]

Here \( \Phi[\Omega] \) is the partition function of 24 chiral bosons on a genus two Riemann surface of period matrix \( \Omega \). This is also the inverse of the unique automorphic form of weight 10. The conjecture has been verified in various ways (for example [6]), here we want to understand more directly why a genus two Riemann surface would appear in computing the partition function for \( 1/4 \) BPS states.

In the following we will use the realization of \( N = 4 \) string theory as IIB superstring theory compactified on \( K3 \times T^2 \). 26 of the possible \( U(1) \) charges correspond to various branes and strings wrapped along one of the \( S^1 \) and on cycles of the \( K3 \). Two more electric charges correspond to momentum along the other circle and one KK monopole charge. We will not turn on these two charges, although it is possible to do so and repeat the following construction with minor modifications.

If the \( K3 \) is much smaller than the \( T^2 \), a \( 1/2 \) BPS state with electric charges appears as a black string wrapped on one cycle of the torus. This string is a bound state of \( D5, NS5 \) branes wrapped on \( K3 \), \( D3 \) branes wrapped on the 22 2-cycles of \( K3 \), \( D1,F1 \) strings.

A black string wrapped on the other cycle of the torus corresponds instead to a \( 1/2 \) BPS magnetic object. The partition function for such \( 1/2 \) BPS objects is known to be

\[ \sum d(N_e) e^{2\pi i N_e \tau} = \frac{1}{\eta(\tau)^{24}} \quad (4) \]

This is the torus partition function of 24 chiral bosons. Indeed in the low energy limit, the CFT living on the black string is the same as the one for a fundamental heterotic string compactified on a \( T^6 \) [2,3]. The electric charges form a Narain lattice of signature \( (22, 6) \). A supersymmetric ground state for the string has level 0 for the right movers, level matching requires a level \( N_e \) for the left mover oscillators. The partition function is then the partition function for the 24 left moving bosons.

An \( 1/4 \) BPS dyon state is generically a bound state of an electric string and a magnetic string with charges respectively \( q_e \) and \( q_m \), wrapped on the two cycles of the torus.
The naive configuration of the two strings wrapping the two cycles and intersecting at a four-pronged intersection is not supersymmetric. We want to argue that it can reach a lower energy, 1/4 BPS ground state by relaxing the intersection and splitting it up, to form two three-pronged string junctions joined by a stub of charges $q_e + q_m$ or $q_e - q_m$ (See Figure 1).

In other words the dyon can be realized as a network or web of black strings wrapping the $T^2$, made out of three strands of different charges joined at two supersymmetric junctions.

It is well known that a junction of three BPS strings can preserve 1/4 of the supersymmetries. As long as the strings are in a plane, charge is conserved at the junction and the relative angles are fixed by mechanical equilibrium [4] the BPS bound is saturated. This is true in particular of these black strings from IIB compactified on $K3$. Periodic, honeycomb-like networks of strings of charges $q_e,q_m,q_e + q_m$ can be built out of those intersections (See for example [5]). Quotient by the periodicity of the lattice gives the 1/4 BPS network wrapping a $T^2$, with two three-pronged intersections, that corresponds to a BPS dyon in four dimensions. (See Figure 2)

The actual angles between the strands depend on the background values of the scalar moduli: the $K3$ moduli fix the tensions of the three strands and hence the relative angles at the junction, while the shape of the torus fixes the position and orientation of the junction. It should be possible to make use of this construction to understand the attractor equations geometrically.

The topology of the string network wrapping the torus is the same as a genus two Riemann surface with very thin handles. This can be made precise: to count the (index) number of BPS microstates of such a string network, one computes the partition function with euclidean time compactified on a supersymmetric circle.

By taking the limit in which the supersymmetric thermal circle is very small we can dualize IIB theory to M-theory on a torus. The partition function is then computed in M-theory...
Figure 2: An honeycomb network can be quotiented to give the network on the torus compactified on $K3 \times T^4$.

The duality relates a network of black strings wrapped on the torus to an $M5$ brane wrapping $K3$ and a holomorphic curve in the $T^4$. The topology of the string network with two three-pronged junctions implies that the curve is of genus two. There is a natural holomorphic map from a genus two Riemann surface into a $T^4$, that is the map between the complex curve and its Jacobian. This map is unique.

The partition function for the $M5$ brane wrapping the $K3$ and a genus two Riemann surface is easily computed. For example by a second duality from M-theory on $K3$ to heterotic string on $T^3$, this is the partition function for a fundamental heterotic string wrapping the genus two Riemann surface.

As the fermions are periodic around each circle, the partition function is computed with Ramond boundary conditions around each cycle of the curve, that projects the right-moving sector onto the Ramond ground state. The electric and magnetic charges along each strand of the network fix the value of the Narain momenta running across the various cycles of the Riemann surface.

Level matching identifies then $N_e$ with the left oscillator level propagating along one handle of the surface, $N_m$ along the other handle, $N_e + N_m + N_{em}$ along the stub. The number of states contributing to the partition function is computed from the Fourier coefficients of the genus two partition function for 24 chiral bosons: $\Phi((\rho, \nu, \sigma)) = \sum d(N_e, N_{em}, N_m) e^{2\pi i (N_e \rho + N_{em} \nu + N_m \sigma)}$.
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