Supplementary Information of “The scaled-invariant Planckian metal and quantum criticality in Ce$_{1-x}$Nd$_x$Coln$_5$”

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ABSTRACT

In this Supplementary Information, we provide additional details and derivations that are not included in the main text.

Supplementary Note 1: Specific heat coefficient and its scaling for zero Nd doping ($x = 0$)

Supplementary Figure 1 shows the specific heat coefficient and its scaling at zero Nd doping ($x = 0$) under different magnetic fields. A $T \sim (B)$-power-law scaling behavior within the quantum-critical regime, $\gamma(T/B) \sim (T/B)^{-m}$, with exponent $m = 0.51$, closed to the exponents for $x = 0.02$ and $x = 0.05$, is also found here (Supplementary Figure 1b).

Supplementary Note 2: Estimation of carrier concentration $n$ and $\alpha$ coefficient

In this section, we provide derivation of the relevant equations for carrier concentration $n$ based on the quantum oscillation measurements. We will further use those equations to reproduce $n$ and the Planckian coefficients $\alpha$ shown in Table 1 of the main text.

We start from the formula of quantum oscillation frequency $F$, given by

$$F = \frac{(h/2\pi)S_F}{2\pi e}$$  \hspace{1cm} (S1)

with $S_F$ being the extremal cross-sectional area of the Fermi surface. For simplicity, we assume a circular cross-section of Fermi surface here, hence $S_F = \pi k_F^2$ with $k_F$ being the “averaged” Fermi wave vector of the circular Fermi surface. This links the dHvA frequency and the “average” Fermi wave vector of the Fermi surface by $F = \frac{(h/2\pi)k_F^2}{2\pi e}$. Here, we can make a link of $F$ and the carrier concentration $n$ through $k_F$. While considering the effective dimensionality of critical modes, the carrier concentration takes the following form$^1$:

$$n = \frac{2k_F}{\pi d_b d_c} = \frac{2}{\pi d_b d_c} \sqrt{\frac{2eF}{h/2\pi}} \quad \text{(for 1d)},$$

$$n = \frac{k_F^2}{2\pi d_c} = \frac{1}{2\pi d_c} \left( \frac{2eF}{h/2\pi} \right) \quad \text{(for 2d)},$$

$$n = \frac{k_F^3}{3\pi^2} = \frac{1}{3\pi^2} \left( \frac{2eF}{h/2\pi} \right)^{3/2} \quad \text{(for 3d)},$$  \hspace{1cm} (S2)

where $d_b$ and $d_c$ are the lattice constants of unit cell along the $b$ and $c$ axes. Note that, for the 2d case of Eq. (S2), we assume the system has a strong anisotropy along the $c$ direction while it remains isotropic in the $a$-$b$ plane.

As an example, we provide detailed derivation of carrier concentration for the 2d case shown in Eq. (S2) and then generalize this derivation to the case with fractional quasi-2d dimension.

Assume the critical modes occur on the isotropic ab-plane. The total number of states can be expressed as

$$N = N_{ab}N_c = 2 \times \frac{\pi k_F^2}{\Delta V_k} \times \frac{L_c}{d_c} = 2 \times \frac{A\pi k_F^2}{4\pi^2} \frac{L_c}{d_c}$$

$$\rightarrow n = \frac{N}{V'} = \frac{N}{AL_c} = \frac{k_F^2}{2\pi d_c},$$  \hspace{1cm} (S3)

where $N_{ab} = \frac{\pi k_F^2}{2\Delta V_k}$ is the number of states on the ab plane per spin while $N_c = L_c/d_c$ for that along the c-axis. Here, $\Delta V_k$ represents the unit volume in $k$ space occupied by a state, $L_c$ denotes the sample site along c, $A = L^2$ is the area of the sample on the ab plane. This indicates the total volume of the sample $V' = AL_c$.

The above approach of deriving the carrier concentration for the effective 2d critical modes embedded in a 3d lattice can
**Supplementary Figure 1.** Specific heat coefficient \( C/T \) and its scaling for zero Nd doping \((x = 0)\). a shows the electronic specific heat coefficient \( C/T \) with different fields \( B \parallel c \) for zero Nd doping \((x = 0)\) while b displays the power-law \( T/B \) scaling of a. 

The total number of states per spin for an isotropic \( d \)-dimensional system is given by

\[
N = \sum_{|k| \leq k_F} \Theta(k) = \int_{0}^{k_F} d^d k \frac{\Delta V_k}{\Delta V_k} = \frac{V_d(k_F)}{V_d}, \tag{S4}
\]

where \( \Delta V_k = (2\pi/L)^d = (2\pi)^d/V_d \) with \( V_d \equiv L^d \) and \( V_d(k_F) \) is the volume of a \( d \)-dimensional sphere with radius \( k_F \).

\[
V_d(k_F) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} k_F^d. \tag{S5}
\]

In the above equation, \( \Gamma(x) \) denotes the Gamma function. For a general \( d \)-dimensional critical modes embedded in a \( 3d \) lattice, the total number of states reads

\[
N = 2 \times N_{ab}^{(d)} \times N_c^{(3-d)}, \tag{S6}
\]

where the prefactor 2 comes from the spin degrees of freedom. Here, we assume that the quasi-\( 2d \) critical modes mostly arise from the \( ab \)-plane.

From Eq. (S4), we have

\[
N_{ab}^{(d)} = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \frac{k_F^d}{V_d} = \frac{\pi^{d/2} V_d k_F^d}{(2\pi)^d \times \Gamma(d/2 + 1)}, \tag{S7}
\]

while

\[
N_c^{(3-d)} = \left( \frac{L_c}{d_c} \right)^{3-d} \tag{S8}
\]

The total number of states is then given by

\[
N = \frac{\gamma k_F^d}{2^{d-1} \pi^{d/2} d_c^{3-d} \Gamma(d/2 + 1)}, \tag{S9}
\]

giving rise to the carrier concentration

\[
n = \frac{N}{\gamma} = \frac{k_F^d}{2^{d-1} \pi^{d/2} d_c^{3-d} \Gamma(d/2 + 1)}. \tag{S10}
\]
Using the relation of $k_F$ and $F$, we obtain the expression of carrier concentration for arbitrary $d$-dimensional critical modes,

$$
n = \frac{1}{2^{d-1} \pi^{d/2} d! \Gamma\left(\frac{d}{2} + 1\right)} \left(\frac{2eF}{\hbar/2\pi}\right)^d. \tag{S11}\n$$

When taking $d = 2$, the above expression of $n$ goes back the 2$d$ case in Eq. (S2).

**Supplementary Note 3: Estimating the Planckian coefficients $\alpha$ for Ce$_{1-x}$Nd$_x$CoIn$_5$**

Below, we estimate the carrier concentration $n$ and Planckian coefficients $\alpha$ shown in Table 1 of the main text for Ce$_{1-x}$Nd$_x$CoIn$_5$ with $x = 0, 0.02, 0.05$, and 0.1 using the dHvA frequency $F$ and effective mass $m^*$ in Ref. 2 (for $\alpha$-band) and in Refs. 1,3 (for $\beta$-band).

- **For $x = 0$.** The average dHvA frequency for the $\alpha$-band is $F = 4.9$KT while $m^* = 11.7m_0$ is its average effective mass. Since Fermi surface of pure CeCoIn$_5$ has been shown to be 2$d$-like, we thus use the equation of the 2$d$ version in Eq. (S2) to estimate the carrier density, given by

$$
n = \frac{(2 \times 1.6 \times 10^{-19}) \times (4.9 \times 10^3)}{(7.549 \times 10^{-10}) \times (6.6 \times 10^{-34})} \approx 0.31 \times 10^{28} \text{m}^{-3} \quad \text{(for $\alpha$-band).} \tag{S12}\n$$

For the $\beta$-band of pure CeCoIn$_5$, we adopt its Fermi surface parameters: $F = 9.75$KT and $m^* = 100m_0$. Following the similar approach, the carrier concentration for $\beta$-band is

$$
n = 0.63 \times 10^{28} \text{m}^{-3} \quad \text{(for $\beta$-band).} \tag{S13}\n$$

The Planckian coefficient from the $\alpha$-band can be straightforwardly obtained via $\alpha = \frac{e^2(\hbar/2\pi)}{k_BT} A_1 \frac{n}{m^*}$, suggesting

$$
\alpha = \frac{(1.6 \times 10^{-19})^2 \times (9.98 \times 10^{-8}) \times (0.31 \times 10^{28})}{(1.38 \times 10^{-23}) \times (11.7 \times 9.11 \times 10^{-34})} \times (1.05 \times 10^{-34}) = 0.56 \quad \text{(for $\alpha$-band),} \tag{S14}\n$$

and, for the $\beta$-band, we have

$$
\alpha = 0.13 \quad \text{(for $\beta$-band).} \tag{S15}\n$$

These two contributions give $\alpha \approx 0.7$ for pure CeCoIn$_5$. The $A_1$ coefficient for pure CeCoIn$_5$ at zero field, $A_1 = 0.98 \mu\Omega \cdot \text{cm/K}$, is used for the estimation of the $\alpha$ coefficients shown above.

- **For $x = 0.02$.** The average dHvA frequency for the $\alpha$-band is $F = 4.89$KT while $m^* = 11.7m_0$ is its average effective mass. Angular dependence of the dHvA frequencies indicates a 2$d$ Fermi surface of the $\alpha$-band for $x = 0.02$. Following the similar approach, the carrier concentration of the $\alpha$-band is calculated as

$$
n = 0.32 \times 10^{28} \text{m}^{-3} \quad \text{(for $\alpha$-band).} \tag{S16}\n$$

Here, we assume that the carrier density and the relevant band parameters as well as the effective dimension of the $\beta$-band do not significantly altered while doping 2% of Nd, indicating that $n = 0.63 \times 10^{28} \text{m}^{-3}$ and $m^* = 100m_0$ for the $\beta$-band here. The $\alpha$-coefficients for the $\alpha$- and $\beta$-band are thus estimated

$$
\alpha = 0.58 \quad \text{(for $\alpha$-band),} \quad \alpha = 0.14 \quad \text{(for $\beta$-band),} \tag{S17}\n$$

giving the total Planckian coefficient $\alpha = 0.72$ for $x = 0.02$. The gradient of the linear-$T$ resistivity $A_1 = 1.0\mu\Omega \cdot \text{cm/K}$ is used in this case.

- **For $x = 0.05$.** The fundamental band parameters of the $\alpha$-band for $x = 0.05$ is $F = 4.88$KT and $m^* = 9.15m_0$. Angular dependence of the dHvA frequencies indicates a 2$d$-to-3$d$ dimensional crossover of Fermi surface of the $\alpha$-band at $x = 0.05$. Accompanying with the prediction of a QCP at $x_c = 0.03$ and the theoretical studies on that QCP, we treat the dimensionality for $x = 0.05$ to be $d = 2.45$. Using Eq. (S11), the carrier concentration of $\alpha$-band is estimated as

$$
n = 0.26 \times 10^{28} \text{m}^{-3} \quad \text{(for $\alpha$-band).} \tag{S18}\n$$

Likewise, we assume the band parameters of the $\beta$-band also remains the same for $x = 0.05$, thus $m^* = 100m_0$ and $F = 9.75$KT. The carrier concentration of the $\beta$-band with $d = 2.45$ is found to be

$$
n = 0.6 \times 10^{28} \text{m}^{-3} \quad \text{(for $\beta$-band).} \tag{S19}\n$$

Using $A_1 = 1.17\mu\Omega \cdot \text{cm/K}$ for $x = 0.05$, the $\alpha$-coefficients for the $\alpha$- and $\beta$-band can be straightforwardly calculated as

$$
\alpha = 0.7 \quad \text{(for $\alpha$-band),} \quad \alpha = 0.15 \quad \text{(for $\beta$-band),} \tag{S20}\n$$

giving the total Planckian coefficient $\alpha = 0.85$.

- **For $x = 0.1$.** The fundamental band parameters of the $\alpha$-band for $x = 0.1$ is $F = 4.41$KT and $m^* = 7m_0$. We treat the effective dimensionality of Fermi surface of the $\alpha$-band to be three-dimensional as this compound at $x = 0.1$ is deep inside the AF state. Using Eq. (S11) and
\( d = 3 \), the carrier concentration of \( \alpha \)-band is estimated as

\[ n = 0.17 \times 10^{28} \text{ m}^{-3} \quad \text{(for \( \alpha \)-band).} \quad \text{(S21)} \]

Similarly, we assume that \( m^* = 100m_0 \) and \( F = 9.75kT \)
are also applicable for the \( \beta \)-band for \( x = 0.1 \) here. The carrier concentration of the \( \beta \)-band with \( d = 3 \) is found to be

\[ n = 0.55 \times 10^{28} \text{ m}^{-3} \quad \text{(for \( \beta \)-band).} \quad \text{(S22)} \]

Using \( A_1 = 1.49\mu\Omega \cdot \text{cm/K} \) for \( x = 0.1 \), the \( \alpha \)-coefficients for the \( \alpha \)- and \( \beta \)-band can be straightforwardly estimated as

\[ \alpha = 0.77 \quad \text{(for \( \alpha \)-band),} \]
\[ \alpha = 0.19 \quad \text{(for \( \beta \)-band),} \quad \text{(S23)} \]

giving the total Planckian coefficient \( \alpha = 0.96 \).

**Supplementary References**

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