A Note on the Statistics of Hardcore Fermions

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Abstract. It is shown that the statistics of the hardcore fermions is A–superstatistics of order one [see T.D.P. J. Math. Phys. 21 1293 (1980)]. The Pauli principle for these particles is formulated. The Hubbard operators, which constitute a basis in the Lie superalgebra $gl(1|n)$, are expressed via the creation and the annihilation operators of the hardcore fermions.

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The aim of the present note is to show that the so-called hard core fermions can be viewed as particles obeying the $A$-superstatistics introduced in [1] (see also the review paper [2]).

Hardcore fermions appear implicitly in various Hubbard lattice models [3] of strongly correlated electron systems and in particular in models of high temperature superconductivity. By definition the particles are hardcore if they obey the hardcore restriction (HC restriction): each site of the lattice cannot accommodate more than one particle.

The HC restriction is stronger than the Pauli principle. Indeed, if the number of the orbitals at site $i$ is $n_i$, then the Pauli principle for fermions asserts that the site $i$ can accommodate up to $n_i$ particles, whereas the HC property admits at most one particle.

Physically, the HC property stems from the repulsion between the electrons at low energies. Mathematically, this property is described by projecting down the entire fermionic Fock space $W$ onto the subspace $W^{(1)}$ of states with at most one particle per site. The latter can be achieved in different ways. Some of the pioneering papers to mention are the Gutzwiller variational method [4], further developed by Kotliar and Ruckenstein [5], the slave-boson approach of Barnes [6], extended by Coleman [7], Read and Newns [8], but in general the literature on the subject is vast.

We proceed to show that the statistics of hardcore particles is $A$--superstatistics of order 1 [1]. Consider a lattice with $N$ sites. Let

$$F(f_{11}^\pm, f_{12}^\pm, ..., f_{1n}^\pm; f_{21}^\pm, f_{22}^\pm, ..., f_{2n}^\pm; ..., f_{N1}^\pm, f_{N2}^\pm, ..., f_{Nn}^\pm)$$

be any polynomial of $Nn$ pairs of Fermi creation and annihilation operators (CAO’s), where $f_{i\alpha}^+$ creates/annihilates a fermion at the site $i = 1, 2, ..., N$, with, say, a flavor index (including spin and other internal characteristics) $\alpha = 1, 2, ..., n$. Denote by $\mathcal{P}$ a projection operator from the entire fermionic state space $W$ onto the subspace $W^{(1)}$ defined above. Then

$$\mathcal{P} F(f_{11}^\pm, f_{12}^\pm, ..., f_{1n}^\pm; f_{21}^\pm, f_{22}^\pm, ..., f_{2n}^\pm; ..., f_{N1}^\pm, f_{N2}^\pm, ..., f_{Nn}^\pm) = F(a_{11}^\pm, a_{12}^\pm, ..., a_{1n}^\pm; a_{21}^\pm, a_{22}^\pm, ..., a_{2n}^\pm; ..., a_{N1}^\pm, a_{N2}^\pm, ..., a_{Nn}^\pm),$$

where

$$a_{i\alpha}^+ = \mathcal{P} f_{i\alpha}^+ \mathcal{P}, \quad a_{i\alpha}^- = \mathcal{P} f_{i\alpha}^- \mathcal{P}, \quad i = 1, 2, ..., N, \quad \alpha = 1, 2, ..., n$$

For instance, if

$$H = -t \sum_{ij} \sum_\alpha (f_{i\alpha}^+ f_{j\alpha}^- + f_{j\alpha}^+ f_{i\alpha}^-) + U \sum_i \sum_{\alpha \neq \beta} f_{i\alpha}^+ f_{i\beta}^- f_{i\beta}^+ f_{i\alpha}^-,$$
then
\[ \mathcal{H} = \mathcal{P} \mathcal{H} \mathcal{P} = -t \sum_{ij} \sum_{\alpha}(a_{i\alpha}^+ a_{j\alpha}^- + a_{j\alpha}^+ a_{i\alpha}^-) + U \sum_{\alpha} \sum_{i \neq \beta} a_{i\alpha}^+ a_{i\beta}^- a_{i\alpha}^+ a_{i\beta}^- . \] (5)

(some of the \( \mathcal{P} \)'s in (3) can be skipped, but we keep them for symmetry).

By a straightforward computation, one verifies that at each site \( i = 1, 2, ..., N \), the above operators satisfy the following relations in \( \mathcal{W}(1) \):

\[
\begin{align*}
[a_{i\alpha}^+ + i\alpha, a_{i\gamma}^-] &= \delta_{\beta \gamma} a_{i\alpha}^+ - \delta_{\alpha \beta} a_{i\gamma}^-, \\
[a_{i\alpha}^+ - i\alpha, a_{i\gamma}^-] &= -\delta_{\alpha \gamma} a_{i\beta}^- + \delta_{\alpha \beta} a_{i\gamma}^-, \\
\{a_{i\alpha}^+, a_{i\beta}^+\} &= \{a_{i\alpha}^-, a_{i\beta}^-\} = 0 .
\end{align*}
\]
(6)

The triple relations (6) are defining relations for the creation and annihilation operators of \( A \)-superstatistics at site \( i \) [1].

At different sites, the operators anticommute,

\[
\{a_{i\alpha}^+, a_{j\beta}^+\} = \{a_{i\alpha}^+, a_{j\beta}^+\} = \{a_{i\alpha}^-, a_{j\beta}^-\} = 0, \quad i \neq j = 1, ..., N .
\] (7)

From (6) one concludes that if the creation and the annihilation operators \( a_{i\alpha}^\pm \) are postulated to be odd elements, then the linear span

\[
lin.\,span.\{a_{i\alpha}^\pm, a_{i\beta}^\pm|\alpha, \beta, \gamma = 1, ..., n\}
\] (8)
is a Lie superalgebra (LS) with an even subalgebra specified by

\[
lin.\,span.\{a_{i\beta}^\pm, a_{i\gamma}^\pm|\alpha = 1, ..., n\}.
\]

A more detailed analysis [1] shows that at each site \( i \), the operators \( a_{i\alpha}^\pm \) generate the Lie superalgebra \( sl(1|n)^{(i)} \). Then (7) implies that the operators \( a_{i\alpha}^\pm, \alpha = 1, 2, ..., n, \ i = 1, 2, ..., N \), generate a Lie superalgebra which is a direct sum of \( N \) identical copies of \( sl(1|n) \),

\[
A(N, n) = sl(1|n)^{(1)} \oplus sl(1|n)^{(2)} \oplus ... \oplus sl(1|n)^{(N)} .
\] (9)

The circumstance that the hardcore CAO’s generate a Lie superalgebra carries important information. The immediate conclusion is that the hardcore operators give one particular solution of the relations (6) or - with another words - these CAO’s give one particular representation of each Lie superalgebra \( sl(1|n)^{(i)} \) and together with (7) - a representation of the LS (9). Next, it is known that the LS \( sl(1|n)^{(i)} \) has several other
representations, i.e., several other solutions of (6) and in view of (7) also the algebra $A(N, n)$ has different solutions. Are these new representations of any interest? What is their physical interpretation, if any? These are the questions we will address next.

As usually we shall write $sl(1|n)^{(i)}$ in a basis of the general linear Lie superalgebra $gl(1|n)^{(i)}$. Such an extension is convenient, since the $gl(1|n)^{(i)}$ basis is simpler. Moreover it does not change anything related to $sl(1|n)^{(i)}$ since the representation space remains the same (every irreducible $sl(1|n)^{(i)}$ module can be extended to an irreducible $gl(1|n)^{(i)}$ module). But as we shall see almost immediately, the extension is more than simply convenient.

As a basis in $gl(1|n)^{(i)}$ we choose $(n + 1)^2$ generators $X_{AB}^{(i)}$ with $A, B = 0, 1, ..., n$. The odd generators are $X_{0\alpha}^{(i)}$ and $X_{\alpha 0}^{(i)}$, $\alpha = 1, 2, .., n$. All other generators are even. The $X$ operators satisfy the supercommutation relations:

$$[X_{AB}^{(i)}, X_{CD}^{(i)}]_{\pm} = \delta_{BC}X_{AD}^{(i)} \pm \delta_{AD}X_{CB}^{(i)},$$

whereas at different sites

$$[X_{AB}^{(i)}, X_{CD}^{(j)}]_{\pm} = 0, \ i \neq j. \quad (11)$$

In the above $A, B, C, D = 0, 1, ..., n$, and the upper sign (+) stands for the case when both generators in the LHS are odd, otherwise the lower sign (−) should be adopted.

The supercommutation relations (10) determine completely the LS $gl(1|n)^{(i)}$. Here they are written in somewhat unusual for this LS form. We have adopted such notation because the $X$ operators (10), called Hubbard operators, play an important role in condensed matter physics as an alternative way for description of strongly correlated electron systems.

The Hubbard operators yield one possible basis of the LS $gl(1|n)^{(i)}$. In fact these operators define a particular representation, the fundamental $(n + 1)$—dimensional representation of each $gl(1|n)^{(i)}$. In a matrix form the Hubbard operators are nothing but the $(n + 1)$—dimensional matrix units ($X_{AB}^{(i)}$ has 1 at position (A, B) and 0 elsewhere).

Locally, at each site $i$, the creation and annihilation operators $a_{i\alpha}^{\pm}$ together with $X_{00}^{(i)}$ generate $gl(1|n)^{(i)}$. Therefore the Hubbard operators can be expressed via the hardcore creation and annihilation operators and $X_{00}^{(i)}$:

$$X_{0\alpha}^{(i)} = a_{i\alpha}^{-}, \quad X_{\alpha 0}^{(i)} = a_{i\alpha}^{+}, \quad \alpha = 1, ..., n. \quad (12)$$

Observe that the CAO’s coincide with the odd $X$—operators. Then

$$X_{\alpha\beta}^{(i)} = \{a_{i\alpha}^{+}, a_{i\beta}^{-}\}, \quad \alpha \neq \beta = 1, ..., n, \quad (13a)$$
\[
X^{(i)}_{\alpha\alpha} = \{a^+_{i\alpha}, a^-_{i\alpha} \} - X^{(i)}_{00}, \quad \alpha = 1, ..., n. \tag{13b}
\]

Below, see (21), we express also \(X^{(i)}_{00}\) via the CAO’s, so that all Hubbard generators become functions of only CAO’s.

In [1] we have introduced a concept of Fock representations of a simple Lie (super)algebra. The \(sl(1|n)\) Fock modules, considered here, are finite-dimensional and irreducible. They are labelled by all positive integers \(p = 1, 2, ...,\) the order of statistics. As in parastatistics the representation space \(W(n, p, i)\) at site \(i\) and with order of statistics \(p\) is reconstructed from the relations

\[
a^-_{i\alpha} a^+_{i\beta} |0\rangle = \delta_{\alpha\beta} p |0\rangle, \quad a^-_{i\alpha} |0\rangle = 0.
\] (14)

Without loss of generality for \(sl(1|n)\), we extend \(W(n, p, i)\) to a \(gl(1|n)\) module setting

\[
X^{(i)}_{00} |0\rangle = p |0\rangle. \tag{15}
\]

The representations corresponding to different orders of statistics \(p\) are inequivalent finite-dimensional irreducible representations. At each site \(i\) all states

\[
|p; \theta_{i1}, \theta_{i2}, ..., \theta_{in}\rangle = \sqrt{\frac{(p - \sum_{\alpha} \theta_{i\alpha})!}{p!}} (a^+_{i1})^{\theta_{i1}} ... (a^+_{in})^{\theta_{in}} |0\rangle, \quad \sum_{\alpha=1}^{n} \theta_{i\alpha} \leq \min(n, p) \tag{16}
\]

with \(\theta_{i1}, ..., \theta_{in} = 0, 1,\) constitute an orthogonal basis in \(W(n, p, i)\).

The transformations of the basis under the action of the odd generators read [1]:

\[
a^-_{i\alpha} |..., \theta_{i\alpha}, ..\rangle = \theta_{i\alpha} (-1)^{\theta_{i1} + ... + \theta_{i,\alpha-1}} \sqrt{p - \sum_{\beta} \theta_{i\beta}} + 1 |..., \theta_{i\alpha} - 1, ..\rangle \tag{17a}
\]

\[
a^+_{i\alpha} |..., \theta_{i\alpha}, ..\rangle = (1 - \theta_{i\alpha}) (-1)^{\theta_{i1} + ... + \theta_{i,\alpha-1}} \sqrt{p - \sum_{\beta} \theta_{i\beta}} |..., \theta_{i\alpha} + 1, ..\rangle \tag{17b}
\]

Moreover

\[
X^{(i)}_{00} |p; \theta_{i1}, \theta_{i2}, ..., \theta_{in}\rangle = (p - \sum_{\beta} \theta_{i\beta}) |p; \theta_{i1}, \theta_{i2}, ..., \theta_{in}\rangle. \tag{18}
\]

From (17) one can compute the action of all the even generators. In particular the number operator \(N_{i\alpha}\) for particles of flavor \(\alpha\) on the site \(i\) is (see (13a))

\[
N_{i\alpha} = \{a^+_{i\alpha}, a^-_{i\alpha} \} - X^{(i)}_{00} = X^{(i)}_{\alpha\alpha}, \tag{19}
\]
namely,
\[ N_{i\alpha}|.., \theta_{i\alpha}, ..\rangle = \theta_{i\alpha}|.., \theta_{i\alpha}, ..\rangle. \]  

(20)

The operators \( X_{00}^{(i)} \) do not belong to \( sl(1|n) \). Nevertheless within each irreducible module \( W(n, p, i) \) these operators can be expressed via the \( sl(1|n) \) generators:

\[ X_{00}^{(i)} = \frac{1}{n-1} \left( \sum_{\alpha=1}^{n} \{ a_{\alpha i}^+, a_{\alpha i}^- \} - p \right). \]  

(21)

Then

\[ X_{\alpha\alpha}^{(i)} = \{ a_{\alpha i}^+, a_{\alpha i}^- \} - X_{00}^{(i)}, \quad \alpha = 1, ..., n. \]  

(22)

From (16) there emerges an important conclusion, which is in fact

The Pauli principle for \( A- \)superstatistics (at site \( i \)): if the order of statistics is \( p \) then each \( \theta_{i\alpha} = 0, 1 \) (fermionic like property: on each orbital there can be no more than one particle), but in addition each site \( i \) can accommodate up to \( \min(p, n) \) particles.

The case \( p = 1 \) corresponds to hardcore fermions. The particles, corresponding to an arbitrary \( p \) could be called hardcore fermions of order \( p \).

From a mathematical point of view we have constructed explicitly (back in 1980 [1]) a class of representations, the Fock representations, of each local \( gl(1|n) \), labelled by \( p = 1, 2, ... \). The construction is relatively easy and this is due to the fact that the creation operators anticommute. The similar problem for parastatistics turned to be very difficult. It was solved very recently, see [10] [11], more than 50 years after the discovery of Green’s parastatistics.

We should mention that apart from in [1] the transformation relations of the state space under the actions of the Hubbard generators were written down also in [12]. The results coincide. There is only a difference in the notation. In particular our \( p \) corresponds to the biggest amount of particles to be accommodated on a site \( i \), whereas \( n_0 \) in [12], (5) and (6) is \( n_0 = p - \sum_{b=1}^{N} \theta_{\beta} \). An interesting speculation would be to interpret \( n_0 \) as a number of particles in a reservoir and to study the related thermodynamics.

In conclusion, we have shown that locally (at each site) the statistics of hardcore fermions is \( p = 1 \) \( A- \)superstatistics and more precisely \( sl(1|n) - \)statistics of order one. Similar as for parastatistics the related CAO’s obey triple relations (6). The \( A- \)superstatistics admits also other configurations. In particular if the order of statistics is \( p \), then each site can accommodate no more than \( p \) particles.
We have indicated that the Hubbard operators (taken at each site) are generators of the LS $gl(1|n)$ in the usual sense: they constitute a basis in $gl(1|n)$ considered as a linear space. Based on the circumstance that within each irreducible module the CAO’s generate $gl(1|n)$, we have expressed the Hubbard operators via the creation and annihilation operators within any Fock space (for any $p$).

Finally we mention that the Lie superalgebra $gl(1|n)$ has many other irreducible representations. They are labelled by $n + 1$ numbers [13], [14], [15]. Hence the same holds for the hardcore fermions and for the Hubbard operators. Are these generalized hardcore fermions of interest? Do they carry new physical information? These are questions still to be answered and a motivation for further investigations.

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