GENERALIZED FORMS OF AN OVERCONSTRAINED SLIDING MECHANISM CONSISTING OF TWO EQUAL TETRAHEDRA

Endre Makai, Jr.*, Tibor Tarnai**

Abstract. We investigate motions of a bar structure consisting of two congruent tetrahedra, whose vertices in their basic position form the vertices of a rectangular parallelepiped. The constraint of the motion is that the originally intersecting edges should remain coplanar. We determine all finite motions of our bar structure. This generalizes our earlier work, where we did the same for the case that the rectangular parallelepiped was a cube.

1. Introduction

Drawing all the diagonals of all faces of a cube, we obtain the edges of two congruent regular tetrahedra. This position of these tetrahedra will be called the basic position. Keeping one of the tetrahedra fixed, we move the other under the condition that each edge pair of the two tetrahedra, which were originally diagonals of some face of the cube, should remain coplanar.

The structure of the above described two tetrahedra has been invented in 1982 by L. Tompos Jr., then an undergraduate of the Hungarian Academy of Craft and Design. He has built a physical model of the bar structure of these tetrahedra, such that the bars of one of the tetrahedra touched those of the other tetrahedron from inside (Fig. 1). He observed that this structure admits continuous motions. We note that [Fuller 1975] contains a figure of these tetrahedra, but their mobility is not investigated there.

By motion (sometimes we will say finite motion) we will not mean a continuous motion from the basic position, but any position of our structure that satisfies the constraints (possibly this position is not the result of a continuous motion). We mean by this that one of the tetrahedra is fixed, and the other is obtained from the basic position of itself by the application of an isometry (=congruence) of the space of determinant 1, such that the coplanarity conditions are satisfied. An isometry of determinant 1 will be written in the form \( \Phi(x) = Ax + b \), where \( A \) is a \( 3 \times 3 \) orthogonal matrix of determinant 1, and \( b \) is a vector in \( \mathbb{R}^3 \). Geometrically, \( A \) is a rotation about some straight line containing the origin.

2000 Mathematics Subject Classification 51M99.
Key words and phrases. sliding mechanism, tetrahedra.

*Research (partially) supported by Hungarian National Foundation for Scientific Research, grant nos. K68398, T046846.
**Research (partially) supported by Hungarian National Foundation for Scientific Research, grant no. T046846.
[Stachel 1988] and [Tarnai and Makai 1989a] determined all motions of this pair of tetrahedra. We give a brief description of them, using the names used for them in [Tarnai and Makai 1989a]. We note that because the bars have non-zero width, not all motions can be realized by a physical model, only such ones for which each respective pair of edges (which have to be coplanar) actually has a common point. A discussion of which motions are physically admissible, i.e., satisfy this more restrictive condition, is contained in [Chen 1991], Ch. 4. Further in this paper we will make no distinction between physically admissible and inadmissible motions.

We suppose that the vertices of the tetrahedra in the basic position are the points \((\pm 1, \pm 1, \pm 1)\). More exactly, the vertices of the fixed tetrahedron are \(P_0^1(1, -1, -1)\), \(P_0^2(-1, 1, -1)\), \(P_0^3(-1, -1, 1)\), \(P_0^4(1, 1, 1)\), and the vertices of the moving tetrahedron are denoted by \(Q_0^1, \ldots, Q_0^4\), where in the basic position \(Q_0^i\) is the mirror image of \(P_0^i\) w.r.t. the origin (= the centre of the cube), see Fig. 2.

There exist motions \(\Phi(x) = Ax + b\) of the Tompos’ tetrahedra, where \(A\) is a rotation about an axis \(0e_i, 0(e_i \pm e_j)\) or \(0(e_i \pm e_j \pm e_k)\) respectively, where \(e_1, e_2, e_3\) are the basic unit vectors in the space, and \(i, j, k\) are different. These motions \(\Phi(x)\) are called motions of the first, second and third kind, respectively (in the first case, for angle of rotation 180°, we count only the case \(b = 0\) to the motion of first kind). The angle of rotation of \(A\) in the first case is arbitrary, in the second case it is arbitrary, except 180°, in the third case it is arbitrary, except \(\pm 90°\). (The angle of rotation is positive if, looking from the axis vector backwards, it is positive.) It turned out that to each above rotation \(A\), except in the first case the rotation through 180°, there existed a unique translation \(b\) such that \(\Phi(x) = Ax + b\) is a motion of the Tompos’ tetrahedra. In the first case, for an angle of rotation different from 180°, we have \(b = 0\). In the first case, for a rotation through 180°, \(b\) is not unique (but only the case \(b = 0\) is counted to the motion of first kind).

There also exist motions \(\Phi(x) = Ax + b\) of the Tompos’ tetrahedra, where \(A\) is a rotation about an axis \(C_1e_i + C_2e_j\), where \(i \neq j\), \(C_1, C_2\) are real, not both 0. If \(C_1C_2 \neq 0\), and thus this is not a motion of the first kind, then the angle of the rotation \(A\) is arbitrary, except 180°, moreover for each such rotation \(A\) the translation \(b\) is uniquely determined. These motions, for \(C_1C_2 \neq 0\), together with the motions of the first kind (where \(C_1C_2 = 0\)) with angle of rotation \(\neq 180°\), are called motions of the intermediate kind. These motions contain the motions of the second kind as a special case.

Now let \(A\) be a rotation about an axis \(0e_i\) through 180°, and then let \(b\) be any vector of the form \(Ce_j\), or \(C_1e_j + C_2e_k\), respectively, where \(C, C_1, C_2\) are real and \(i, j, k\) are different. Then \(\Phi(x) = Ax + b\) is a motion of the Tompos’ tetrahedra, which is called motion of the fourth or fifth kind, respectively. The motions of the fifth kind contain the motions of the fourth kind as a special case. The motions of the first, second, third, fourth and fifth kinds are drawn in Fig. 3.

It turned out that the motions of each kind constituted a smooth manifold.
(those of the third kind a manifold of two connected components, cf. [Tarnai–Makai 1989b], p. 141) in the six-dimensional manifold of all motions of the space, of dimensions 1, 1, 1, 2, 1, 2, respectively (in the order as they have been enumerated above). These manifolds show certain bifurcation phenomena, which have been analyzed in [Tarnai–Makai 1989a]. Moreover, as shown by [Stachel 1988] and [Tarnai–Makai 1989a], the above enumerated motions are the only motions of the Tompos’ tetrahedra. We note that [Stachel 1988] also described the trajectories of the vertices during the physically admissible motions. Moreover, as shown by [Stachel 1988] and [Tarnai–Makai 1989a], the above enumerated motions are the only motions of the Tompos’ tetrahedra. We note that [Stachel 1988] also described the trajectories of the vertices during the physically admissible motions. Moreover, in [Hyder–Zsombor-Murray 1989, 1990, 1992] (of which only [Hyder–Zsombor-Murray 1992] has been available to the authors) and [Chen 1991] the motions of the Tompos tetrahedra have been further investigated, and some possible mechanical engineering applications have been pointed out.

In our paper we generalize the above investigations in two directions. First we start instead of a cube with a rectangular parallelepiped. All diagonals of all of its faces constitute the edges of two congruent tetrahedra. This position of the two tetrahedra is the basic position. Keeping one of the tetrahedra fixed we move the other one (i.e., apply to it an isometry of the space, with determinant 1) under the condition that

- each pair of edges of the two tetrahedra that were originally diagonals of some face of the rectangular parallelepiped should remain coplanar. (A)

First we give the description of all motions, and then we give a mathematical proof that this list of motions is complete. The results are rather analogous to the case of the cube, only in a special case there is a sixth kind of motion, which constitutes a one-dimensional manifold.

Secondly, we consider two congruent right pyramids with regular $n$-gonal bases ($n \geq 3$), with coincident axes of rotation, the basic edges of one pyramid intersecting the lateral edges of the other and conversely, both having a basic edge enclosing an angle $180^\circ / n$. This position will be called the basic position of these bar structures. Consider these pyramids as bar structures only, and move each vertex of this bar structure under the condition that

- one triangular face of one pyramid remains fixed, each bar retains its length, and each pair of originally intersecting bars (edges), one from each pyramid, remains coplanar. (B)

We make local investigations. As the motions of the Tompos tetrahedra have already been described, we further exclude the case that $n = 3$ and the two tetrahedra are regular. We have a one-dimensional manifold of finite motions (which are conjectured to be the only finite motions), where one pyramid remains fixed, the other undergoes a rigid motion (an isometry of the space with determinant 1) so that its
axis of rotation remains fixed. We investigate at the basic position of this motion the infinitesimal degree of freedom of our pyramids, for $3 \leq n \leq 7$, and for several values of the quotient lateral edge / basic edge. We find that this infinitesimal degree of freedom is, except the case of the Tompos tetrahedra, in all the investigated cases equal to 1.

2. The motions of the two tetrahedra derived from a rectangular parallelepiped

2.1. We will use analogous notations as in the case of a cube. Let the rectangular parallelepiped have the vertices $(\pm d_1, \pm d_2, \pm d_3)$, where $d_1, d_2, d_3 > 0$. The fixed vertices are $P_1(d_1, -d_2, -d_3), P_2(-d_1, d_2, -d_3), P_3(-d_1, -d_2, d_3), P_4(d_1, d_2, d_3)$, and the moving vertices are $Q_1, Q_2, Q_3, Q_4$, where in the basic position $Q_i$ is the mirror image of $P_i$ w.r.t. the origin. Thus $P_1P_2P_3P_4$ is the fixed tetrahedron, $Q_1Q_2Q_3Q_4$ is the moving tetrahedron, and we move it under condition (A), which is the same as in the case of a cube. First we describe the motions $\Phi(x) = Ax + b$ of this moving tetrahedron.

First of all, the motions of fourth and fifth kinds are defined word for word as in the case of a cube, and they evidently exist (i.e., the originally intersecting edges remain coplanar). The motion of first kind, also defined in the same way as for the cube, exists for any value of the rotation angle. Moreover, we have for this motion $b = 0$. These are shown in the same way as in [Tarnai–Makai 1989a], p. 428. The motion of intermediate kind (now we do not need to treat separately the motion of second kind), also defined the same way as for the cube, exists for any value of the rotation angle, except for $180^\circ$. For this kind of motion, for any given rotation $A$ the translation $b$ is uniquely determined (but possibly for such an $A$ there exist several other $b$'s yielding another kind of motion, see the next paragraph). These are also shown in the same way as in [Tarnai–Makai 1989a], p. 429.

We continue with describing the novel sixth kind of motion. Let us suppose our rectangular parallelepiped satisfies $d_k = d_i d_j / \sqrt{d_i^2 + d_j^2}$ (which is half the length of the altitude belonging to the hypotenuse of the right triangle bounded by two sides and a diagonal of the face perpendicular to $e_k$), where $i, j, k$ are different. Let e.g. $d_3 = d_1 d_2 / \sqrt{d_1^2 + d_2^2}$. Let us take an axis of rotation passing through 0 and parallel to one of the diagonals of a horizontal face, say, to $P_1P_2$, and let us consider a rotation about this axis through $90^\circ$, in positive or negative sense. For convenience, instead of taking one tetrahedron as fixed, the other as moving, we will rotate both tetrahedra about this axis, through $45^\circ$, in a way symmetric w.r.t. the $xy$ plane. Then the rotated tetrahedra will remain symmetric to the $xy$ plane, and the pairs of edges of the two tetrahedra, which originally were intersecting and lay
on a vertical face of the parallelepiped, will remain symmetric to the $xy$ plane, hence remain coplanar. This symmetry, and consequently coplanarity property remains unchanged, if we still translate vertically these tetrahedra, in a way symmetric w.r.t. the $xy$ plane. By a suitable vertical translation we can achieve that also the pairs of edges of the tetrahedra that originally lay on horizontal faces of the parallelepiped, will be, simultaneously, coplanar, thus obtaining a position corresponding to a motion of the intermediate kind (cf. [Tarnai–Makai 1989a], p. 429).

See Fig. 4, where the rotated and not yet translated tetrahedra are denoted by $P_1', P_2', P_3', P_4'$ and $Q_1', Q_2', Q_3', Q_4'$, with corresponding notations of the vertices. We assume that the sense of rotation is as drawn in Fig. 4 (the other case is similar, only the role of indices is changed).

The figure shows the orthogonal projection of the parallelepiped along the axis of rotation. By $d_3 = d_1d_2/\sqrt{d_1^2 + d_2^2}$ this projection is a rectangle whose horizontal side is twice as long as its vertical side. Note that this implies that the edges $P_1', P_2', P_3', P_4'$, and their mirror images $Q_2', Q_4'$, $Q_1', Q_3'$ are horizontal, and hence, respectively parallel. Therefore, their any translated copies remain respectively coplanar. Further the edges $P_1', P_4'$, $P_2', P_4'$, and their mirror images $Q_2', Q_3'$, $Q_1', Q_3'$ lie in a vertical plane (whose projection is a vertical line in the figure). Hence any translation in this vertical plane will leave them coplanar. Lastly, the edges $P_1', P_2'$ (whose projection in the figure is a point) and $Q_3', Q_4'$ will intersect, thus be coplanar, after some symmetric vertical translations of the two tetrahedra (and then by symmetry also $Q_2', Q_1'$ and $P_4', P_3'$ will intersect). However, then any translation in the direction of $P_1', P_2' (= the direction of $Q_3', Q_4'$ = the direction of the rotation axis) leaves them coplanar. Summing up: each pair of originally intersecting edges remains coplanar if, after a symmetric rotation about the above described axis through 45° and then a symmetric vertical translation, making $P_1', P_2'$ and $Q_3', Q_4'$ intersecting, we translate one of the tetrahedra in the direction of the rotation axis, through an arbitrary distance. Hence, this is a one-dimensional manifold of solutions, which we call, for any choice of the permutation $(i, j, k)$ (for which $d_k = d_id_j/\sqrt{d_i^2 + d_j^2}$ holds) a motion of sixth kind.

Lastly, we turn to the motion of third kind, which we are able to give in analytic form only. Let $D_i = d_i^{-2}$ and let $\Phi(x) = Ax + b$ be a motion, where $A$ is a rotation about an axis $0u$ – where $u = [u_1\ u_2\ u_3]$ and $u_1^2 + u_2^2 + u_3^2 = 1$ – through an angle $\varphi$. Note that $(u, \varphi)$ and $(-u, -\varphi)$ represent the same rotation. Suppose $0 < \varphi < 2\pi$, $\varphi \neq \frac{1}{2}\pi, \frac{3}{2}\pi$, and let $s = \cot\left(\frac{1}{2}\varphi\right) (\neq \pm 1)$. Then let

$$u_i^2 = \frac{(s^4 + 3s^2 + 1) - (3s^4 + 7s^2 + 1)}{2(s^2 + 1)} \frac{D_i}{D_1 + D_2 + D_3},$$
provided

\[
\frac{s^4 + 3s^2 + 1}{3s^4 + 7s^2 + 1} > \frac{\max D_i}{D_1 + D_2 + D_3}
\]

(then we have \(u_i^2 > 0\)). This determines the rotation part \(\mathbf{A}\) of the motion \(\Phi(x)\), and the translation part \(\mathbf{b}\) is then uniquely determined among all admitted motions.

This is a one-dimensional manifold of solutions (actually there are several branches according to the signs of the \(u_i\)'s). Moreover, if we restrict our attention to the case \(u_1, u_2, u_3 > 0\), then for \(\max D_i/(D_1 + D_2 + D_3) \geq 5/11\) the solution manifold is connected, while for \(\max D_i/(D_1 + D_2 + D_3) < 5/11\) the solution manifold consists of three connected components, one for \(s < -1\), one for \(-1 < s < 1\), one for \(s > 1\). For \(\max D_i/(D_1 + D_2 + D_3) > 5/11\) this solution manifold ends at two points satisfying \(u_i = 0\) (for \(D_i = \max(D_1, D_2, D_3)\)) and \(s \neq 0, \pm 1\) – these points not lying on this solution manifold – thus at a motion of the intermediate kind.

For \(\max D_i/(D_1 + D_2 + D_3) < 5/11\) the component for \(s < -1\) begins (for \(s\) minimal) and the component for \(s > 1\) ends (for \(s\) maximal) at a point satisfying \(u_i = 0\) (for \(D_i = \max(D_1, D_2, D_3)\)) and \(s \neq 0, \pm 1\) – this point not lying on the respective component – thus at a motion of the intermediate kind; moreover there are no end-points at \(s = \pm 1\) (the manifold components go to infinity there). For \(D_i = D_j = \max(D_1, D_2, D_3)\) the above mentioned end-points satisfy \(u_i = u_j = 0, s \neq 0\), thus are motions of the first kind. This third kind of motion never passes through the basic position, unless \(d_1 = d_2 = d_3\), when it does. The fact that these formulas in fact describe a motion of our tetrahedra, and the mentioned properties of this motion will follow from the proof of Theorem 1.

**Theorem 1.** Consider the two tetrahedra \(P_1P_2P_3P_4\) and \(Q_1Q_2Q_3Q_4\) derived above from a rectangular parallelepiped of vertices \((\pm d_1, \pm d_2, \pm d_3)\). The only finite motions admitted by these tetrahedron – i.e., all positions of the moving tetrahedron, satisfying (A) – are those of the first, intermediate, third, fifth kinds and, provided \(d_k = d_i d_j/\sqrt{d_i^2 + d_j^2}\) for some permutation \((i, j, k)\) of \((1, 2, 3)\), of the sixth type, described above.

**Proof.** 1) Let \(\Phi(x) = \mathbf{A}x + \mathbf{b}\) be a finite motion admitted by our tetrahedra, i.e., \(\mathbf{A} = [a_{ij}]\) is an orthogonal \(3 \times 3\) matrix with determinant +1, \(\mathbf{b} = [b_1 \ b_2 \ b_3]^T\) a vector in \(\mathbb{R}^3\), and condition (A) is satisfied. \(\mathbf{A}\) is a rotation about some axis \(\mathbf{0u}\), \(\mathbf{u} = [u_1 \ u_2 \ u_3]^T\), where \(u_1^2 + u_2^2 + u_3^2 = 1\), through an angle \(\varphi\) (with sense of rotation as described in §1).
We have, using the coordinates of the basic position of \(Q_i\)'s from the beginning of 2.1,

\[
Q_1(x_1, y_1, z_1) = A[-d_1 \quad d_2 \quad d_3]^T + b, \\
Q_2(x_2, y_2, z_2) = A[d_1 \quad -d_2 \quad d_3]^T + b, \\
Q_3(x_3, y_3, z_3) = A[d_1 \quad d_2 \quad -d_3]^T + b, \\
Q_4(x_4, y_4, z_4) = A[-d_1 \quad -d_2 \quad -d_3]^T + b.
\]

Let us denote

\[
D = \begin{bmatrix} d_1 & 0 & 0 \\
0 & d_2 & 0 \\
0 & 0 & d_3 \end{bmatrix}.
\]

Using the notations \(P_i^0, Q_i^0\) from §1 and \(P_i\) from the beginning of 2.1, we have \(P_i = DP_i^0\). Further we have \(Q_i = \Phi(Q_i)\), where \(Q_i\), or \(Q_i^0\), is the basic position of \(Q_i\), or \(Q_i^0\), respectively. These satisfy \(Q_i = DQ_i^0\). The coplanarity e.g. of the fixed vertices \(P_1 = DP_1^0, P_2 = DP_2^0\) and the moving vertices \(Q_3 = \Phi(Q_3) = \Phi(DQ_3^0), Q_4 = \Phi(Q_4) = \Phi(DQ_4^0)\) is equivalent to the coplanarity of the points \(P_1^0, P_2^0, D^{-1}(\Phi(D(Q_3^0))) = D^{-1}AD(Q_3^0) + D^{-1}b\) and \(D^{-1}(\Phi(D(Q_4^0))) = D^{-1}AD(Q_4^0) + D^{-1}b\). We have similar equivalent conditions for the coplanarity of other quadruples of vertices to be considered, so \(\Phi(x) = Ax + b\) represents a motion of our tetrahedra if and only if the transformation \(D^{-1}AD(x) + D^{-1}b\) of the vertices \(Q_1^0, \ldots, Q_4^0\) preserves coplanarity of the four vertices of any face of the cube with vertices \((\pm 1, \pm 1, \pm 1)\). We have \(D^{-1}AD = [a_{ij}^0] = [d_i^{-1}a_{ij}d_j]\), and \(D^{-1}b = [b_1^0 \quad b_2^0 \quad b_3^0]^T = [d_1^{-1}b_1 \quad d_2^{-1}b_2 \quad d_3^{-1}b_3]^T\). Thus, like in [Tarnai–Makai 1989a], p. 435, we have

\[
-(a_{02}^0 + a_{03}^0)b_1^0 + a_{12}^0b_2^0 + a_{13}^0b_3^0 = a_{21}^0a_{13}^0 + a_{31}^0a_{12}^0 + (a_{23}^0 + a_{32}^0)(1 - a_{11}^0),
\]

(I/1)

\[
-(a_{23}^0 + a_{32}^0)b_1^0 + a_{13}^0b_2^0 + a_{12}^0b_3^0 = a_{12}^0a_{21}^0 + a_{31}^0a_{13}^0 + (a_{22}^0 + a_{33}^0)(1 - a_{11}^0).
\]

(II/1)

There hold the analogous equations obtained from these ones by the permutation of indices \(1 \rightarrow 2 \rightarrow 3 \rightarrow 1\); these equations will be denoted by (I/2), (II/2). Similarly, using the permutation \(1 \rightarrow 3 \rightarrow 2 \rightarrow 1\) we get the equations (I/3), (II/3). Thus we have a system of six linear equations for \(b_1, b_2, b_3\), that expresses the coplanarity of the respective quadruples from the points \(P_i, Q_i\).
2) In our equations we replace \( a_{ij}^0 \) by \( d_i^ia_{ij}d_j \), and \( b_i^0 \) by \( d_i^{-1}b_i \). Then we express \( a_{ij} \) by \( u_1, u_2, u_3 \) and \( \phi \), by the well-known formula (cf. e.g. [Tarnai–Makai 1989a], p. 436). Let us then consider the \( 6 \times 4 \) matrix formed by the coefficients of \( b_1, b_2, b_3 \) and the right-hand sides of these equations. We call its rows I/\( i \), II/\( i \), according to the equation they correspond to. We multiply rows \( I/i \), II/\( i \) by \( d_i \), then divide the fourth column by \( 2d_i d_2 d_3 \), and then divide row II/\( i \) by \( d_1 d_2 d_3 d_i^{-1} \). Like above, denote \( D_i = d_i^{-2} \). Note that for \( \phi = 0 \), thus \( A = I \) we have the unique motion with \( b = 0 \). Henceforward we will assume \( 0 < \phi < 2\pi \). Letting \( s = \cot(\frac{1}{2}\phi) \), as above, we multiply each entry of the last obtained matrix by \( \frac{1}{2}(s^2 + 1) \). Thus our matrix becomes

\[
\begin{bmatrix}
  u_1^2 - s^2 & u_1 u_2 + su_3 & u_3 u_1 - su_2 & u_2 u_3 (D_2 + D_3) \\
  u_1 u_2 - su_3 & u_2^2 - s^2 & u_2 u_3 + su_1 & u_3 u_1 (D_3 + D_1) \\
  u_3 u_1 + su_2 & u_2 u_3 - su_1 & u_2^2 - s^2 & u_1 u_2 (D_1 + D_2) \\
  -u_2 u_3 (D_2 + D_3) + su_1 (D_3 - D_2) & (u_3 u_1 - su_2) D_2 & (u_1 u_2 + su_3) D_3 & 0 \\
  +su_1 (D_3 - D_2) & (u_3 u_1 - su_2) D_2 & (u_1 u_2 + su_3) D_3 & 0 \\
  (u_2 u_3 + su_1) D_1 & -u_3 u_1 (D_3 + D_1) + su_2 (D_1 - D_3) & (u_1 u_2 - su_3) D_3 & 0 \\
  (u_2 u_3 - su_1) D_1 & (u_3 u_1 + su_2) D_2 & -u_1 u_2 (D_1 + D_2) + su_3 (D_2 - D_1) & 0 \\
\end{bmatrix}
\]

(Note that for \( D_1 = D_2 = D_3 = 1 \) this reduces to (B) in [Tarnai–Makai 1989a], p. 437, up to a constant factor in the fourth column.) The rows of matrix (C) corresponding to the equations (I/i), (II/i) will be called rows I/i, II/i of (C). The solvability of the system of equations corresponding to this new matrix is equivalent to the solvability of our original system, and also the dimensions of the solution manifolds (if they are not empty) are the same, moreover they are parallel.

3) The upper left \( 3 \times 3 \) submatrix of (C) is independent of \( D_i \), hence it is singular in the same case as it is singular for \( D_1 = D_2 = D_3 = 1 \). Recall that the left-hand side of equation (I/1) equals \( d_i^{-1}[-(a_{22} + a_{33})b_1 + a_{12}b_2 + a_{13}b_3] \), and similarly for (I/2), (I/3). Hence the determinant of the considered \( 3 \times 3 \) submatrix is a non-zero constant times the determinant \(|a_{ij} - \delta_{ij}(a_{11} + a_{22} + a_{33})| \). By [Tarnai–Makai 1988], p. 270 or [Tarnai–Makai 1989a], p. 438 this determinant is 0 if and only if \( \phi = \pm \frac{1}{2}\pi \) or \( \phi = \pi \). (We do not distinguish between angles differing by multiples of \( 2\pi \).) Hence our equations can have a non-unique solution for \( b \) only for \( \phi = \pm \frac{1}{2}\pi \) and \( \phi = \pi \), i.e., for \( s = \pm 1 \) and \( s = 0 \).

The lower three rows of matrix (C) are linearly dependent: multiplying row II/i by \( D_i \) and summing them we obtain a zero row.

Multiplying row I/i of (C) by \( u_i \) and summing them we obtain
a straightforward but somewhat lengthy calculation

\[ [u_i(1-s^2) \ u_2(1-s^2) \ u_3(1-s^2) \ u_1u_2u_3(2D_1 + D_2 + D_3)]. \]

The corresponding equation having a solution implies that for \( s^2 = 1 \), i.e., \( \varphi = \pm \frac{1}{2} \pi \), we have \( u_1u_2u_3 = 0 \).

4) The determinant of the submatrix of (C) formed by the rows I/1, I/2, I/3, II/1, which is a homogeneous eighth degree polynomial of \( u_1, u_2, u_3, s \), turns out by a straightforward but somewhat lengthy calculation

\[
\begin{align*}
&su_1u_2u_3(D_1 + D_2 + D_3)\left\{ (u_1^2 + u_2^2 + u_3^2) : \\
&\cdot [u_1^2(D_3 - D_2) + u_2^2(-D_2 - D_3) + u_3^2(D_2 + D_3)] + \\
&\cdot s^2[u_1^2(3D_3 - 3D_2) + u_2^2(-3D_2 + D_3) + u_3^2(-D_2 + 3D_3)] + s^4(D_3 - D_2) \right\}.
\end{align*}
\]

(D)

If now \( su_1u_2u_3 \neq 0 \), then the factor in braces is 0, and two more analogous expressions are equal to 0, which are obtained from this expression by cyclic permutations of the indices (these arise analogously from the determinants of the submatrices formed by the rows I/1, I/2, I/3, II/2, and I/1, I/2, I/3, II/3 of (C)). These three equations are homogeneous linear in \( D_2, D_3, D_1 \), in \( D_3, D_1, D_2 \), respectively, and can be written as

\[
\begin{align*}
&\frac{(u_1^2 + u_2^2 + u_3^2)(-u_1^2 + u_2^2 + u_3^2) + s^2(u_1^2 + 3u_2^2 + 3u_3^2) + s^4}{D_1} = \\
&\frac{(u_1^2 + u_2^2 + u_3^2)(u_1^2 - u_2^2 + u_3^2) + s^2(3u_1^2 + u_2^2 + 3u_3^2) + s^4}{D_2} = \\
&\frac{(u_1^2 + u_2^2 + u_3^2)(u_1^2 + u_2^2 - u_3^2) + s^2(3u_1^2 + 3u_2^2 + u_3^2) + s^4}{D_3},
\end{align*}
\]

(E)

(thus they are actually only two equations).

5) First we discuss the case when the first factor of (D), i.e., \( s \), equals 0. Consider the 4 \times 4 matrix formed by the rows I/1, I/2, II/1, and the sum of \( (D_1 + D_2) \) times row II/2 and \( D_3 \) times row II/3 of our matrix (C). Its determinant is

\[
-u_1u_2u_3^2D_1D_3(D_1 + D_2 + D_3)\left[ u_2^2(D_2 + D_3) - u_3^2(D_3 + D_1) \right] \cdot (u_1^2 + u_2^2 + u_3^2),
\]

which equals 0. By cyclic permutation of rows I/i and II/i we get similar equations (the expressions obtained from the last expression by cyclic permutations of indices.
are equal to 0), and these three equations together imply \( u_1 u_2 u_3 = 0 \) or \( u_1^2 : u_2^2 : u_3^2 = (D_2 + D_3) : (D_3 + D_1) : (D_1 + D_2) \).

Let us first suppose e.g. \( u_1 = 0 \). Then the equation corresponding to row I/1 becomes 0 = \( u_2 u_3 (D_2 + D_3) \), thus \( u_2 u_3 = 0 \). Let e.g. \( u_1 = u_2 = 0 \), and \( s = 0 \), i.e., \( \varphi = \pi \). Then the pairs of edges of the two tetrahedra, originally lying on some vertical face of the parallelepiped, are parallel, thus remain coplanar after any translation, while those originally lying on some horizontal face are intersecting, thus remain coplanar after a translation through a vector \([b_1 \ b_2 \ b_3]^T\) exactly when \( b_3 = 0 \). Therefore the set of solution vectors \([b_1 \ b_2 \ b_3]^T\) is given by all vectors with \( b_3 = 0 \).

Let us secondly suppose \( u_1^2 : u_2^2 : u_3^2 = (D_2 + D_3) : (D_3 + D_1) : (D_1 + D_2) \); then by \( u_1^2 + u_2^2 + u_3^2 = 1 \) we have \( u_1 u_2 u_3 \neq 0 \). Observe that in this case the double equality (E) is satisfied. We will investigate this case further together with the investigation of (E), in 10) (when both cases \( s = 0 \) and \( s \neq 0 \) will be allowed). We only show now that in this case there is a unique solution of our equations for \( b \). (The existence of the solution will be showed in 10).) For this consider the \( 4 \times 4 \) matrix from the beginning of 5), and take its \( 3 \times 3 \) submatrix consisting of the first three elements of its first, third and fourth rows. Its determinant is a homogeneous polynomial of third degree in the \( D_i \)'s; to test if it is zero or not it suffices to substitute \( D_2 + D_3 = u_1^2 \), \( D_3 + D_1 = u_2^2 \), \( D_1 + D_2 = u_3^2 \). Thus this determinant becomes

\[
\frac{1}{8} u_1^2 u_2 u_3 (u_1^2 + u_2^2 + u_3^2)^2 (u_1^2 + u_2^2 - u_3^2)(-u_1^2 + u_2^2 + u_3^2).
\]

Here \( u_1^2 u_2 u_3 \neq 0 \), as shown above, and each other factor is positive – e.g. \( (u_1^2 + u_2^2 - u_3^2)/(u_1^2 + u_2^2 + u_3^2) = D_3/(D_1 + D_2 + D_3) > 0 \). Hence the considered determinant is non-zero, showing unicity of the solution for \( b \).

6) Secondly, we discuss the case when the factor \( u_1 u_2 u_3 \) of (D) equals 0. Let e.g. \( u_1 = 0 \). In this case we have the motions of the intermediate, fifth and sixth kinds. We have to show for \( u_2 u_3 \neq 0 \) that if the motion of intermediate kind does not exist – that is \( \varphi = \pi \), i.e., \( s = 0 \) – then we do not have any solution. (Recall that the motion of intermediate kind exists only for \( \varphi \neq \pi \), the motion of fifth kind – for \( u_1 = 0 \) – exists only for \( u_2 u_3 = 0 \), and the motion of sixth kind exists only for \( \varphi = \frac{1}{2} \pi \).) However, in 5) it has been shown that \( s = 0 \), \( u_1 = 0 \) imply \( u_2 u_3 = 0 \).

There remains the question of unicity of the translation part \( b \) of the motion. However, in 3) it has been shown that our equations can have a non-unique solution for \( b \) only for \( s = 0 \) and \( s = \pm 1 \). The case \( s = 0 \) has been dealt with in 5) (the question of unicity has been completely settled there), and we will deal with \( s = \pm 1 \) in 7).
7) We turn to discuss unicity of \( b \) for \( s = \pm 1 \), i.e., \( \varphi = \pm \frac{1}{2} \pi \). Replacing \( [u_1 \ u_2 \ u_3]^T \) by \( [-u_1 \ -u_2 \ -u_3]^T \) if necessary, we may assume \( s = 1 \). In 3) it has been shown that \( s^2 = 1 \) implies \( u_1u_2u_3 = 0 \). Let e.g. \( u_3 = 0 \). Then our matrix (C) becomes a function only of \( u_1, u_2, D_1, D_2, D_3 \). Loosing homogeneity, we will use \( u_1^2 + u_2^2 = 1 \). Thus we see rows I/1 and I/2 are proportional, they are \( -u_2, \) and \( u_1 \) times \( [u_2 \ -u_1 \ 1 \ 0] \). Since \( u_1, u_2 \) are not both zero, we may replace rows I/1, I/2 by one row \( [u_2 \ -u_1 \ 2 \ 0] \). As mentioned in 3), rows II/1, II/2, II/3 are linearly dependent, with non-zero coefficients, hence we may omit row II/3. Thus we obtain a \( 4 \times 4 \) matrix, and the question of the dimension of the solution manifold (for \( b \)) of the corresponding equations (at the considered rotation part of the motion) is equivalent to the same question regarding the matrix (C). By \( u_3 = 0 \) and \( \pm \frac{1}{2} \pi = \varphi \neq \pi \) one solution always exists, namely a motion of the intermediate kind. Thus the dimension of the solution manifold is \( 3 - r \), where \( r \) is the rank of the matrix

\[
M = \begin{bmatrix}
  u_2 & -u_1 & 1 \\
  u_2 & -u_1 & 0 \\
  u_1(D_3 - D_2) & -u_2D_2 & u_1u_2D_3 \\
  u_1D_1 & u_2(D_1 - D_3) & u_1u_2D_3
\end{bmatrix},
\]

obtained by omitting the last column from the above considered \( 4 \times 4 \) matrix.

Subtracting the second row from the first one, the first row becomes \( [0 \ 0 \ 2] \), and thus \( r = 1 + r' \) where \( r' \) is the rank of the \( 3 \times 2 \) submatrix \( M' \) of \( M \) at the lower left corner. If \( u_1 \) or \( u_2 \) is 0, we have \( r' = 2 \), thus \( r = 3 \), and we have a unique solution for \( b \). Let now \( u_1u_2 \neq 0 \). The determinants of the \( 2 \times 2 \) submatrices of \( M' \) obtained by omitting its first, second or third row, respectively, are \( u_1u_2D_3(D_1 + D_2 - D_3) \), \( u_2(D_1 - D_3) + u_1^2 D_1 \), and \( -u_2D_2 + u_1^2(D_3 - D_2) \). If any of these expressions is not 0, we have \( r' = 2 \), thus \( r = 3 \), and then there is a unique solution of our equations for \( b \). If all these above expressions are equal to 0, we have (equivalently) \( D_3 = D_1 + D_2 \), \( u_1^2D_1 = u_2^2D_2 \), hence \( u_1^2 = D_2/((D_1 + D_2)) \), \( u_2^2 = D_1/((D_1 + D_2)) \). In this case \( r' = 1 \), thus \( r = 2 \), and then the dimension of the solution manifold (for \( b \)) is 1 (at this rotation part of the motion).

It remains to show that geometrically this is the sixth kind of motion. Since \( D_i = d_i^{-2} \), thus \( D_3 = D_1 + D_2 \) means \( d_3 = d_1d_2/\sqrt{d_1^2 + d_2^2} \). We have \( [u_1 \ u_2 \ u_3]^T = [\pm d_1/\sqrt{d_1^2 + d_2^2} \ \pm d_2/\sqrt{d_1^2 + d_2^2} \ 0]^T \), hence the axis of rotation of the rotation part \( A \) of the motion is parallel to a diagonal of a horizontal face of our rectangular parallelepiped in its basic position. Further the angle of rotation is \( \pm \frac{1}{2} \pi \). This is just the rotation part of the sixth motion (for \( k = 3 \)). At describing the sixth motion we have exhibited a one-dimensional (linear) manifold of solutions for \( b \) at the above \( A \), which is therefore a subset of the entire solution manifold at this \( A \).
Now we have shown that the entire solution manifold at this \( A \) is exactly one-dimensional. Hence the entire solution manifold we just have found equals the manifold of solutions exhibited formerly, i.e., that of the motion of sixth kind.

8) Recall that non-unique solution for \( b \) is possible only for \( s = 0, \pm 1 \) (cf. 3)), and these have been discussed in 5) and 7), respectively.

For the existence of solutions we have derived in 4) equation (D) and some of its consequences. The case when the first factor in (D), i.e., \( s \), equals 0, has been settled in 5), except the case when \( u_1u_2u_3 \neq 0 \) and (E) is satisfied. The case when the factor \( u_1u_2u_3 \) of (D) equals 0, has been settled in 6), except the case of unicity at \( s = \pm 1 \) which in turn has been settled in 7). If \( su_1u_2u_3 \neq 0 \), we have derived in 4) equations (E).

Therefore all that remains is to solve equations (E), where \( s \) can be 0 or any non-zero number, and \( u_1u_2u_3 \neq 0 \), and to verify if they are solutions of the original system of equations. Recall that by 3) \( s^2 = 1 \) implies \( u_1u_2u_3 = 0 \), hence we will suppose \( s \neq \pm 1 \).

9) Now we show that for \( u_1u_2u_3 \neq 0 \) and \( s \neq \pm 1 \) any solution of equations (E) is a solution of our problem. This is necessary since equations (E) which are consequences of our original equations, have not been gained from our original system of equations by equivalent transformations, and also we do not have a geometrical description of the motion of third kind making its existence evident.

First we show that for \( u_1u_2u_3 \neq 0 \), \( s \neq 0, \pm 1 \) any solution of equations (E) is a solution of our problem. In fact, equations (E) have been derived for \( su_1u_2u_3 \neq 0 \) from the equation that expression (D) equals 0, and two other analogous equations, which respectively express linear dependence of rows I/1, I/2, I/3, II/i \((i = 1, 2, 3)\) of matrix (C). For \( s \neq 0, \pm 1 \) the determinant of the matrix formed by the first three elements of rows I/1, I/2, I/3 is not 0, cf. 3). Hence (E) expresses linear dependence of rows II/1, II/2, II/3 on the linearly independent rows I/1, I/2, I/3 (observe that already their first three elements form linearly independent vectors). Hence (E) implies that the rank of (C) is \( \leq 3 \), thus that the four column vectors of (C) are linearly dependent. However, at this linear dependence the fourth column vector must have a non-zero coefficient, since the first three column vectors are linearly independent (their first three elements already forming linearly independent vectors). This just means solvability of the equations represented by (C), thus of our original equations. (Recall that the equations represented by matrix (C) have been obtained from our original equations (I/1), \ldots , (II/3) by multiplying each equation by some non-zero factor and then multiplying the right-hand sides simultaneously by some non-zero factor, which is an equivalent transformation.)

Let now \( u_1u_2u_3 \neq 0, s = 0 \). (Recall that \( s = \pm 1 \) has been excluded.) We show that also now any solution of equations (E) is a solution of our problem. For \( s = 0 \) (E) gives \(-u_1^2 + u_2^2 + u_3^2)/D_1 = (u_1^2 - u_2^2 + u_3^2)/D_2 = (u_1^2 + u_2^2 - u_3^2)/D_3 \); say, their value is \( \lambda \). Then \( u_1^2 = (\lambda D_2 + \lambda D_3)/2 \), etc. (hence \( \lambda \neq 0 \) by \( u_1^2 + u_2^2 + u_3^2 = 1 \), thus
$u_1^2 : u_2^2 : u_3^2 = (D_2 + D_3) : (D_3 + D_1) : (D_1 + D_2)$. Then the rows I/1, I/2, I/3 of matrix $(C)$ are all proportional to $[u_1 \ u_2 \ u_3 \ 2u_1u_2u_3/\lambda]$, and rows II/1, II/2, II/3 are linearly dependent by 3). Hence (E) implies that the rank of $(C)$ is $\leq 3$. Since after some row manipulations some $3 \times 3$ submatrix, contained in the first three columns, has a non-zero determinant by 5), we have like at the case $s \neq 0$, $\pm 1$ that also in this case our equations have a solution.

10) It remained to solve equations (E) for $u_1^2u_2u_3 \neq 0$, where $s$ can be any real number different from $\pm 1$.

Using $u_1^2 + u_2^2 + u_3^2 = 1$, equations (E) become

$$\frac{[(1 - 2u_1^2) + s^2(3 - 2u_1^2) + s^4]}{D_i} = \lambda \quad \text{(F)}$$

where $\lambda$ is independent of $i \ (i = 1, 2, 3)$. Solving this for $u_i^2$ we obtain

$$u_i^2 = \frac{1}{2} \left( s^2 + 2 - \frac{\lambda D_i + 1}{s^2 + 1} \right) \quad \text{(G)}$$

Summing these for $i = 1, 2, 3$ we obtain

$$1 = \frac{1}{2} \left( 3s^2 + 6 - \frac{\lambda(D_1 + D_2 + D_3) + 3}{s^2 + 1} \right),$$

from which we express $\lambda$ and put it into (G). Thus we obtain

$$u_i^2 = \frac{(s^4 + 3s^2 + 1) - (3s^4 + 7s^2 + 1) \frac{D_i}{D_1 + D_2 + D_3}}{2(s^2 + 1)} \quad \text{(H)}$$

provided of course that all these expressions are non-negative. Actually by $u_1u_2u_3 \neq 0$ all these expressions have to be positive. It is easily seen that these expressions actually satisfy (F) and have sum 1, thus we have made equivalent transformations.

Using (H), the condition $\min u_i^2 > 0$ is equivalent to $f(s^2) = (s^4 + 3s^2 + 1)/(3s^4 + 7s^2 + 1) > \max D_i/(D_1 + D_2 + D_3)$. Here $f(s^2)$ strictly decreases in $[0, \infty)$, from 1 to $\frac{1}{3}$. Hence, except the case $D_1 = D_2 = D_3$ when this inequality is satisfied for all $s$, we have that this inequality is satisfied for $s^2 < f^{-1}[\max D_i/(D_1 + D_2 + D_3)] < \infty$ (and thus in this case this solution set is far from the basic position, which is characterized by $s^2 = \infty$). Here $f^{-1}$, defined on $\left( \frac{1}{3}, 1 \right]$, and strictly decreasing there from $\infty$ to 0, is the inverse of $f$, defined on $[0, \infty)$. In
particular $f(1) = \frac{5}{11}$ gives $f^{-1}\left(\frac{5}{11}\right) = 1$. Excluding further the case $D_1 = D_2 = D_3$ (which has been completely settled by [Stachel 1988] and [Tarnai–Makai 1989a]), for $f^{-1}\left[\frac{\text{max} \, D_i/(D_1 + D_2 + D_3)}{\text{max} \, D_i/(D_1 + D_2 + D_3)}\right] \leq 1$, i.e., $\text{max} \, D_i/(D_1 + D_2 + D_3) \geq 5/11$ we have for $u_1, u_2, u_3 > 0$ a connected manifold of solutions. Let now $f^{-1}\left[\frac{\text{max} \, D_i/(D_1 + D_2 + D_3)}{\text{max} \, D_i/(D_1 + D_2 + D_3)}\right] > 1$, i.e., $\text{max} \, D_i/(D_1 + D_2 + D_3) < 5/11$. For $s = \pm 1$ this would imply by (H) that $\min u_i^2 > 0$, thus $u_1 u_2 u_3 \neq 0$, and by 3) for $s = \pm 1$ and $u_1 u_2 u_3 \neq 0$ our equations do not have a solution (the corresponding b tends to infinity for $s \to \pm 1$). Hence now for $u_1, u_2, u_3 > 0$ we have three connected component of the solution manifold, one for $-\sqrt{f^{-1}\left[\frac{\text{max} \, D_i/(D_1 + D_2 + D_3)}{\text{max} \, D_i/(D_1 + D_2 + D_3)}\right]} < s < -1$, one for $-1 < s < 1$, one for $1 < s < \sqrt{f^{-1}\left[\frac{\text{max} \, D_i/(D_1 + D_2 + D_3)}{\text{max} \, D_i/(D_1 + D_2 + D_3)}\right]}$. The statements about the end-points of this solution manifold (for $\text{max} \, D_i/(D_1 + D_2 + D_3) \neq 5/11$) follow from the fact that by $s \neq 0, \pm 1$, for the motions of intermediate kind corresponding to these end-points the system of equations corresponding to matrix (C) is not singular, hence we have continuous dependence of the solution vector on the coefficients. □

2.2. In [Tarnai–Makai 1989a], pp. 438–440 a slight generalization of the question of Tompos’ tetrahedra has also been considered. We now present the corresponding question for the tetrahedra $P_1 P_2 P_3 P_4$, $Q_1 Q_2 Q_3 Q_4$ derived from a rectangular parallelepiped. In the physical model of these tetrahedra, the bars (edges) of one tetrahedral frame (of the fixed tetrahedron $P_1 P_2 P_3 P_4$, say) touch the corresponding bars (edges) of the other tetrahedral frame (of the moving tetrahedron $Q_1 Q_2 Q_3 Q_4$, say) from inside. Thus the actual physical constraint is only that each edge $P_i P_j$ lies “inside $Q_j Q_k$” ((i, j, k, l) any permutation of (1, 2, 3, 4)). This can be defined mathematically as follows (cf. [Tarnai–Makai 1989a], p. 439).

For any permutation (i, j, k, l) of (1, 2, 3, 4), the signed volume of the tetrahedron $P_i P_j Q_k Q_l$ is either 0 or has the opposite sign as that of the tetrahedron $P_i P_j R_k R_l$, where $R_k R_l$ is such a translate of the segment $Q_k Q_l$ in the basic position (i.e., of the segment $Q_k Q_l = (-P_k)(-P_l)$) that the midpoint of $R_k R_l$ is the centre of the rectangular parallelepiped in the basic position.

(I)

We take (I) as the definition of a generalized motion of our moving tetrahedron $Q_1 Q_2 Q_3 Q_4$ (while $P_1 P_2 P_3 P_4$ is fixed) and prove

**Theorem 2.** For the pair of tetrahedra considered in Theorem 1 the generalized admitted finite motions are identical with the finite motions admitted by them (described in Theorem 1).

**Proof** is analogous to that of [Tarnai–Makai 1989a], Theorem 2, pp. 439–440; details cf. there, we only indicate the differences.
Let $\Phi x = Ax + b$ be a generalized admitted finite motion. For $A = I$ we have $b = 0$. From now on we suppose $0 < \varphi < 2\pi$. Observe that now the constraints are expressed by six inequalities (corresponding to the equalities (1g)--(1l) in [Tarnai–Makai 1989a], p. 423), namely three expressions (the left-hand sides of (1g), (1h), (1i)) are non-negative three (the left-hand sides of (1j), (1k), (1l)) are non-positive (the difference from [Tarnai–Makai 1989a] is that in these formulas the moving vertices are not obtained by the motion $Ax + b$, but by the transformation $D^{-1}AD(x) + D^{-1}b$, like in our Theorem 1).

Subtracting from a non-negative above expression a non-positive above expression (corresponding to pairs of edges that were originally diagonals of opposite faces of the rectangular, parallelepiped), like in [Tarnai–Makai 1989a] p. 439, we obtain that instead of our equalities (II/i) in the proof of Theorem 1 we will have inequalities, the left-hand sides of (II/i) being not less than their right-hand sides. Thus the three equations corresponding to rows II/1, II/2, II/3 of matrix (C) become inequalities, the left-hand sides not less than the right-hand sides, which are equal to 0. However, by 3) of the proof of Theorem 1 a positive linear combination of these rows is 0. Hence like in [Tarnai–Makai 1989a] in each of the inequalities corresponding to rows II/1, II/2, II/3 we have equalities. Thus in all the six original constraint inequalities we have equalities, i.e., each pair of edges $P_iP_j$, $Q_kQ_l$ ($(i, j, k, l)$ any permutation of $(1, 2, 3, 4)$) are coplanar. Thus $\Phi$ is a motion admitted by our pair of tetrahedra. □

2.3. Let us depart instead of a rectangular parallelepiped from a general parallelepiped, and define the admitted motions as in (A). That means, with the notations from 1) in the proof of Theorem 1, that $P_i = DP_i^0$, $Q_i = DQ_i^0$, where now $D = [d_{ij}]$ is a general non-singular linear transformation. (We note that, by eventual change of the notation of the vertices, we may suppose det $D > 0$.) In what follows, we show that in certain cases the analogues of the motions for the case of the rectangular parallelepipeds exist. Further we prove the generalization of Theorem 2 to the case of general parallelepipeds, and investigate unicity of solutions of our equations for $b$.

Also now we have for $D^{-1}AD = [a_{ij}^0]$ and $D^{-1}b = [b_1^0 b_2^0 b_3^0]^T$ equations (I/i), (II/i), $i = 1, 2, 3$. Evidently the left-hand sides of (II/1), (II/2), (II/3) have sum 0. Their right-hand sides have sum $2Tr(D^{-1}AD) - 2m_2(D^{-1}AD)$, where for any $3 \times 3$ matrix $B$ $m_2(B)$ is the sum of the symmetric $2 \times 2$ subdeterminants of $B$. We have $Tr(D^{-1}AD) = TrA$. We also have $m_2(D^{-1}AD) = m_2(A)$, since these two numbers are the coefficients of $-\lambda$ in the characteristic polynomial of $D^{-1}AD$, and $A$, respectively, which polynomials however coincide. Hence the sum
of the right-hand sides of equations (II/1), (II/2), (II/3) is the same as for the case $D = I$, i.e. 0 (cf. [Tarnai–Makai 1989a], (II/1)). Thus the sum of equations (II/1), (II/2), (II/3) is the equation $0 = 0$, hence among our equations there are at most five independent ones.

Defining also for the case of general parallelepipeds the generalized admitted finite motions by (I), we have

**Theorem 2'**. For the pair of tetrahedra derived above from a general parallelepiped, the generalized admitted finite motions are identical with the finite motions admitted by them.

**Proof** is the same (for det $D > 0$) as for Theorem 2, using the above dependence among equations (II/1), (II/2), (II/3). □

Similarly like in 3) of the proof of Theorem 1, a non-unique solution of equations (I/i), (II/i), $i = 1, 2, 3$ for $b$ can occur only if $Tr(D^{-1}AD)$ is an eigenvalue of $D^{-1}AD$, i.e., $TrA$ is an eigenvalue of $A$, i.e., $\varphi = \pm 90^\circ$ or $\varphi = 180^\circ$. Further, like in 9) of the proof of Theorem 1, for $\varphi \neq \pm 90^\circ$, $180^\circ$, the vanishing of the determinants of equations (I/1), (I/2), (I/3) and (II/1), and of the equations (I/1), (I/2), (I/3) and (II/2) is also a sufficient condition for the solvability of our equations for $b$ (recall also the linear dependence among equations (II/1), (II/2), (II/3)).

Because of the dependence among the equations it is to be expected that there is a one-parameter set of solutions, an analogue of the motion of the third kind. This exists if the parallelepiped has a threefold rotational symmetry about a spatial diagonal, cf. §3. For the general case, if the parallelepiped is nearly a cube ($D$ is near to 1), in a neighbourhood of the solution manifold of the motions of third kind, for $0 < \varphi < 2\pi$, $s \neq 0, 1$, and, say, $u_1 = u_2 = u_3 > 0$, there is locally a one-parameter set of solutions. Namely, our problem is now equivalent to solving the system of two determinantal equations mentioned in the last paragraph, for $u_1, u_2, u_3, s$. For the case of the cube these equations say that non-zero multiples of $u_2 - u_3$, and of $u_3 - u_1$, are 0, cf. [Tarnai–Makai 1989a], p. 437, 3. Thus both these expressions change their signs at the curves (on the surface of the unit sphere) $u_2 = u_3$, and $u_3 = u_1$, respectively. Hence after a small perturbation of the equations this change of sign will take place near the above curves. Therefore, locally, on each surface $s = \text{const} (\neq 0, \pm 1)$ we have a solution, with $u_1, u_2, u_3$ nearly equal. (A strict proof of this directly follows from [Hurewicz–Wallman], p. 40, Proposition D.)

If the mid-plane between two parallel faces is a plane of symmetry of the parallelepiped, we have the motions of intermediate kind (for any value of $\varphi$, except $180^\circ$), and fifth kind, with axis of rotation parallel (and beside this of arbitrary direction), and perpendicular to this plane of symmetry, respectively. In this
symmetric case, if moreover the projection of the parallelepiped along a diagonal of one of the mentioned parallel faces is a rectangle of side ratio $2 : 1$ (the other diagonal of this face having a larger projection than the altitude belonging to this face) then we have motion of the sixth kind.

Further, of course, the basic position is a solution as well, but we do not know if it lies on a solution manifold of positive dimension (experiences with models seem not to exclude this possibility). By the results of §3 (about parallelepipeds having threefold rotational symmetry about a spatial diagonal) it does not lie in general on a solution manifold, smoothly embedded in the manifold of all motions of $\mathbb{R}^3$, of dimension greater than 1.

2.4. Another generalization of the tetrahedra derived from the cube is the following. Let us replace each edge of both tetrahedra by congruent circular arcs, with the same endpoints as the respective edges, each lying in the plane spanned by the respective edge and the centre of the cube. Let further either each arc lie in the inner half-plane of the mentioned plane, or each arc lie in the outer half-plane of the mentioned plane, bounded by the straight-line spanned by the edge (Fig. 5). (The inner half-plane is the one containing the centre of the cube, the outer half-plane is the other one.) Thus both tetrahedra become tetrahedron-like frames. Fixing one of these frames we move the other so that the pairs of circles containing the pairs of circular arcs corresponding to originally intersecting edges of the two tetrahedra have a common point. For the physically non-admissible case only a common point with complex coordinates, and in the projective sense, is required, since for this the condition is to be awaited simpler – using discriminants?? of polynomials, rather than determinants like in [Tarnai–Makai 1989a].

If the frames both lie on the surface of the circumsphere of the cube, then an arbitrary rotation about the centre of the cube, with translation part $b = 0$, is an admitted motion, so we have an at least 3-parameter set of motions. Maybe these are the only admitted motions in this case.

In the general case, at least in a neighbourhood of the basic position, the motions of intermediate and third kinds exist, as well as the motion of first kind with angle of rotation $180^\circ$, by the proofs of [Tarnai–Makai 1989a]. Moreover, there is an analogue of the motion of the fifth kind. At this motion the moving tetrahedron undergoes from the last mentioned position of first kind (obtained by rotation through $180^\circ$ about the $z$-axis, say) only a translation. This happens in such a way that an arbitrarily fixed point of the circle containing the circular arc corresponding to the edge $Q_1^0Q_2^0$, in its rotated position, will coincide after translation with an arbitrarily fixed point of the circle containing the circular arc corresponding to the edge $P_3^0P_4^0$ (cf. Fig. 2). This is a two-parameter motion, and at this motion the circles containing the arcs corresponding to the edges $Q_1^0Q_4^0$ (in its rotated position) and $P_1^0P_2^0$ also intersect, by a simple argument using central symmetry. All other pairs of respective circles (that should have common points) lie in respectively
parallel or coincident planes, and this guarantees that these pairs of circles in fact have common points, in the complex projective sense.

References

[1] Chen, H.-W., *Kinematics and introduction to dynamics of a movable pair of tetrahedra*, M. Eng. Thesis, Dept. Mech. Engng., McGill University, Montreal, Canada, 1991.

[2] Fuller, R. B., *Synergetics. Exploration in the geometry of thinking*, p. 7, Macmillan, New York, 1975.

[3] Hurewicz, W., Wallman, H., *Dimension theory*, Princeton Math. Series, vol. 4, Princeton Univ. Press, Princeton, N. J., 1941. MR 3; 312

[4] Hyder, A., Zsombor-Murray, P. J., Design, mobility analysis and animation of a double equilateral tetrahedral mechanism, *CIM-89-15 MeRCIM* Internal Report, McGill University, Montreal, Canada, 1989.

[5] Hyder, A., Zsombor-Murray, P. J., Design, mobility analysis and animation of a double equilateral tetrahedral mechanism, *Proc. Internat. Symp. on Robotics and Manufacturing*, ASME Press series, Vol. 3, ISSN 1052–4150, 49–56, 1990.

[6] Hyder, A., Zsombor-Murray, P. J., *Equilateral tetrahedral mechanism*, J. Robotics and Autonomous Systems 9 (1992), 227–236.

[7] Kötter, E., Über die Möglichkeit, \( n \) Punkte in der Ebene oder im Raume durch weniger als \( 2n – 3 \) oder \( 3n – 6 \) Stäbe von ganz unveränderlicher Länge unverschieblich miteinander zu verbinden, *Festschrift Heinrich Müller–Breslau Proc. Internat. Symp. on Robotics and Manufacturing*, Kröner, Leipzig, Forschr. Math. 43.945, 1912, 61–80.

[8] Stachel, H., *Ein bewegliches Tetraederpaar*, Elemente der Mathematik 43 (1988), 65–75. MR 89i:51029

[9] Tarnai T., Makai, E., Physically inadmissible motions of a movable pair of tetrahedra, *Proc. Third Internat. Conf. on Engineering Graphics and Descriptive Geometry* (eds. S. M. Slaby and H. Stachel), Vol. 2, 264–271, Technical Univ., Vienna, 1988.

[10] Tarnai T., Makai, E., *A movable pair of tetrahedra*, Proc. Royal Soc. London A 423 (1989a), 419–442. MR 90m: 52010

[11] Tarnai T., Makai, E., *Kinematical indeterminacy of a pair of tetrahedral frames*, Acta Techn. Acad. Sci. Hungar. 102 (1–2) (1989b), 123–145.

Endre Makai, Jr.,
Alfréd Rényi Mathematical Institute,
Hungarian Academy of Sciences,
H-1364 Budapest, P.O. Box 127,
HUNGARY
E-mail address: makai@renyi.hu, http://www.renyi.hu/~makai

Tibor Tarnai,
University of Technology and Economy, Budapest,
Department of Structural Mechanics,
H-1521 Budapest, Műegyetem rkp. 3,
HUNGARY
E-mail address: tarnai@ep-mech.me.bme.hu