Special contact Wilson loops

Andrei Mikhailov

Kavli Institute for Theoretical Physics, University of California
Santa Barbara, CA 93106, USA

and

Institute for Theoretical and Experimental Physics,
117259, Bol. Cheremushkinskaya, 25, Moscow, Russia

Abstract

Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory correspond at strong coupling to extremal surfaces in $AdS_5$. We study a class of extremal surfaces known as special Legendrian submanifolds. The "hemisphere" corresponding to the circular Wilson loop is an example of a special Legendrian submanifold, and we give another example. We formulate the necessary conditions for the contour on the boundary of $AdS_5$ to be the boundary of the special Legendrian submanifold and conjecture that these conditions are in fact sufficient. We call the solutions of these conditions "special contact Wilson loops". The first order equations for the special Legendrian submanifold impose a constraint on the functional derivatives of the Wilson loop at the special contact contour which should be satisfied in the Yang-Mills theory at strong coupling.

\footnote{e-mail: andrei@kitp.ucsb.edu}
1 Introduction.

Our understanding of the correspondence between gauge fields and strings improved recently due to the development of the idea of the AdS/CFT correspondence [1, 2, 3]. The most remarkable achievement was the comparison of the superstring and field theory computations of the quantities which are not protected by the symmetries. An important example is the study of the Wilson loop functional [4, 5, 6, 7]. For the loop of the circular shape it was computed to all orders in the perturbation theory in the large $N$ Yang-Mills [9, 10]. The result analytically continued to the strong coupling limit was found to be in agreement with the string theory computation. In another very recent development, the anomalous dimension of the twist two operators in the $\mathcal{N} = 4$ Yang-Mills theory was computed from the properties of the extremal surfaces [11, 12]. Already these two examples show the importance of the study of the extremal surfaces in AdS space for understanding the relation between gauge fields and strings.

In our paper we will study a class of the Wilson loops which correspond to a special class of the extremal surfaces in AdS — special Legendrian submanifolds. This is a large class of extremal surfaces which are easier to study than the generic external surface. The reason for simplifications is that the special Legendrian submanifolds satisfy the first order differential equations while the generic minimal surfaces satisfy the second order differential equations. We were not able to find explicitly the generic special Legendrian surface in AdS space and we think that it is actually not possible. But nontrivial explicit examples of the special Legendrian manifolds are known for $S^5$ [13, 14, 15, 16] and presumably can be constructed by similar methods in $AdS_5$. In this paper we will consider only a simplest nontrivial example and will mostly concentrate on general aspects of the special Legendrian manifolds in $AdS_5$.

We will give the definition of the special Legendrian manifold in Section 2. The surface ending on the circular contour on the boundary which was found in [1, 5] is a special case of a special Legendrian manifold. We will give another example in Section 2. In Section 3 we will find the necessary conditions for the contour in $\mathbb{R} \times S^3$ to be the boundary of the special Legendrian manifold in $AdS_5$ and find all solutions to these conditions in terms of a real function of one real variable. We conjecture that at least for the contours which are close to the circular contour these conditions are in fact necessary and sufficient. We call such contours the "special contact Wilson loops".
In Section 4 we will consider the infinitesimal deformations of the special Legendrian manifold ending on the circular Wilson loop. We will confirm by the explicit calculation that the infinitesimal deformations preserve the differential conditions on the special contact Wilson loop. In Section 5 we will show that the special contact Wilson loop does not in general preserve any supersymmetry. In Section 6 we will study the behavior of a special Legendrian manifold near the boundary of the AdS space. In Section 7 we study the regularized area of the special Legendrian manifold. We did not succeed in calculating the regularized area for the general special contact boundary. This would presumably require the knowledge of the actual special Legendrian surface. But we do know something about the infinitesimal variation of the regularized area under the variation of the contour (not necessarily preserving the special contact condition). There is a special vector field $\lambda^\mu$ in $\mathbb{R} \times S^3$ which enters into the definition of the special Legendrian manifold. We show that the variation of the regularized area functional under the infinitesimal deformation of the special contact contour is zero provided that the normal vector describing the variation is orthogonal to $\lambda^\mu$ pointwise on the contour. (The deformed contour does not have to be special contact.) In other words $\frac{\delta}{\delta C}[W[C]] \sim \lambda_\mu$ for the special contact $C$. It would be interesting to see whether this is true only at strong coupling.

An interesting feature of the special Legendrian surfaces is that the worldsheet coordinates satisfy the first order differential equations. This is usually associated with the supersymmetry. However the string wrapped on the special Legendrian manifold does not in fact preserve any supersymmetry. A special case of the special contact Wilson loop is the circular Wilson loop. On the field theory side the circular Wilson loop classically preserves half of the supersymmetry. But quantum mechanically all the supersymmetry is presumably broken because of the problems with the regularization \cite{17}. On the string theory side the corresponding string worldsheet is not supersymmetric even on the classical level. And for the special contact Wilson loops which are not circular we show in Section 6 that the supersymmetry is completely broken already in the field theory even classically.
2 Special Legendrian manifolds.

2.1 Definitions.

We consider $\text{AdS}_5$ embedded into $\mathbb{R}^{2+4}$ as the hyperboloid

$$y_{-1}^2 + y_0^2 = 1 + y_1^2 + y_2^2 + y_3^2 + y_4^2$$  \hspace{1cm} (1)

The boundary of $\text{AdS}_5$ is the projectivization of the lightcone $C$:

$$C: \quad y_{-1}^2 + y_0^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2$$  \hspace{1cm} (2)

The lightcone separates the future $\mathbb{R}^{2+4}_+$ from the past $\mathbb{R}^{2+4}_-$.

We will introduce in $\mathbb{R}^{2+4}$ the complex coordinates $z_0 = y_{-1} + iy_0$, $z_1 = y_1 + iy_2$, $z_2 = y_3 + iy_4$. The metric and the complex structure define the Kahler form

$$\omega = dz_0 \wedge d\bar{z}_0 - dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2$$  \hspace{1cm} (3)

We will also need the holomorphic 3-form

$$\Omega = dz_0 \wedge dz_1 \wedge dz_2$$  \hspace{1cm} (4)

**Lagrangian submanifolds.** The submanifold $L \subset \mathbb{C}^{1+2}$ is called Lagrangian if $\omega|_{TL} = 0$. One can see that for the Lagrangian manifold

$$\Omega|_{TL} = e^{i\phi(x)} \text{vol}_L$$  \hspace{1cm} (5)

**Special Lagrangian submanifolds.** The Lagrangian manifold is called special Lagrangian \cite{18} if the phase $\phi(x)$ is constant (does not depend on $x$). In flat space it is enough to consider the case when $\phi = 0$ (because the cases with $\phi \neq 0$ are related to the cases with $\phi = 0$ by a symmetry):

$$\text{Im} \ \Omega|_{TL} = 0$$  \hspace{1cm} (6)

The special Lagrangian manifolds are extremal in the sense that when we deform them the variation of the volume is of the second order in the deformation. We will review the proof of this fact in Appendix.

The Euler vector field $E = x^\mu \frac{\partial}{\partial x^\mu}$ is orthogonal to $\text{AdS}_5$. The complex structure in $\mathbb{R}^{2+4}$ defines the one-form $\lambda$ in the AdS space:

$$\lambda = i_E \omega$$  \hspace{1cm} (7)
The corresponding vector field $\lambda^\mu$ can be restricted to the boundary; the restriction is also denoted $\lambda^\mu$. The restriction of the Kahler form $\omega$ to the AdS space is $\omega|_{AdS} = \frac{1}{2}d\lambda$.

**Contact submanifolds.** A submanifold $X$ of $AdS_5$ is called contact if the restriction of $\lambda$ on $X$ is zero. This is equivalent to the cone over $X$ being Lagrangian. The maximal dimension of a contact submanifold $X$ in $AdS_5$ is two.

**Special Legendrian submanifolds.** A special Lagrangian manifold is called a special Lagrangian cone if it is invariant under the rescalings generated by $E$. The intersection of a special Lagrangian cone with $AdS_5$ is called a special Legendrian submanifold $\Sigma$. A special Legendrian submanifold is contact and the restriction of $\epsilon^{\lambda\mu\nu}z_\lambda dz_\mu \wedge dz_\nu$ to its tangent space is real. It is an extremum of the area functional.

**The "hemisphere".** An example of the special Lagrangian cone is the plane given by the equations

$$y_0 = y_1 = y_4 = 0$$

The corresponding special Legendrian manifold is $AdS_2 \subset AdS_5$. Its boundary is the circular Wilson loop. Historically it was one of the first Wilson loops computed in the strong coupling limit by the AdS/CFT correspondence [7, 10]. The extremal surface looks like a hemisphere in the Penrose coordinates.

The infinitesimal deformations of the Lagrangian submanifold $\Sigma$ are in one to one correspondence with the generating functions $H$; the corresponding normal vector field $\xi_H$ is given by the equation

$$dH(\eta) = \omega(\xi_H, \eta)$$

for any vector $\eta$ tangent to $\Sigma$. If we require $\Sigma$ to be a cone $H$ has to satisfy $H(ty) = t^2 H(y)$ — a homogeneous function of the degree 2. If we require both $\Sigma$ and its deformation to be special Lagrangian then $H$ has to be harmonic:

$$\frac{\partial^2 H}{\partial y^\mu \partial y_\mu} = 0$$

Therefore the deformations of the special Lagrangian cones are parametrized by a single homogeneous harmonic function of the degree two. In some sense there are as many special Lagrangian cones as there are harmonic functions of
the degree two on a three-dimensional space. Notice that the deformations of the general three-dimensional extremal cone in $\mathbb{R}^{2+4}$ are parametrized roughly speaking by three homogeneous harmonic functions of the degree two (the worldsheet coordinates). The general extremal cone is parametrized by three harmonic functions while the special Lagrangian cone is parametrized by one harmonic function.

### 2.2 Special Legendrian manifolds are described by the first order differential equations.

The special Legendrian manifolds have a property which resembles the principle of analytic continuation for the complex curves. It turns out that the special Legendrian manifold is completely determined by any one-dimensional contour belonging to it $\Sigma$. Consider the contour $z_\mu(\sigma)$, $\sigma \in \mathbb{R}$ inside the special Legendrian manifold $\Sigma$. At any given point on the contour $z_\mu(\sigma_0)$ the tangent space to $\Sigma$ is generated by two vectors one of which is $\partial_\sigma z_\mu(\sigma_0)$. The other one may be chosen to be orthogonal to $\partial_\sigma z_\mu$. Let us call it $\xi$. It turns out that $\xi$ is completely determined up to the multiplication by a real number by the condition that $\Sigma$ is special Legendrian. Indeed, $\xi$ should satisfy the equations:

\[
\begin{cases}
  z_\mu^* \xi^\mu = 0 \\
  \partial_\sigma z_\mu^* \xi^\mu = 0 \\
  \epsilon_{\mu\nu\lambda} z^\mu \partial_\sigma z^\nu \xi^\lambda \in \mathbb{R}
\end{cases}
\]  

(11)

The first of these equations says that $\xi^\mu$ belongs to $AdS_5$ (real part) and to the kernel of $\lambda$ (imaginary part). The second equation says that $\xi^\mu$ is orthogonal to $\partial_\sigma z$ (real part) and $\omega(\partial_\sigma z, \xi) = 0$ (imaginary part). These are five real equations on six real components of $\xi^\mu$ therefore the direction of $\xi^\mu$ is determined:

\[
\xi^\mu(\sigma) \sim \epsilon^\mu_{\nu\lambda} z^\nu \partial_\sigma z^\lambda
\]  

(12)

We can deform the contour $z^\mu(\sigma) \rightarrow z^\mu(\sigma) + \epsilon \xi^\mu(\sigma)$. The deformed contour still belongs to $\Sigma$. Therefore we get a family of contours sweeping $\Sigma$. This family can be parametrized by a real parameter $\tau$:

\[
\begin{align*}
  \partial_\tau z_0 &= \frac{\partial}{\partial_{\sigma}} z_2 \\
  \partial_\tau z_1 &= -\frac{\partial}{\partial_{\sigma}} z_0 \\
  \partial_\tau z_2 &= \frac{\partial}{\partial_{\sigma}} z_0
\end{align*}
\]  

(13)
This is a system of the first order equations on the worldsheet coordinates. By this construction any contact one-dimensional contour in AdS$_5$ will give a special Legendrian manifold. Contact one-dimensional contours in AdS$_5$ depend on three real functions of a real variable. Two one-dimensional contours give the same special Legendrian manifold if they are related by the deformation (12). Therefore a special Legendrian submanifold in AdS$_5$ is parametrized by two real functions of a real variable. This probably suggests that a special Legendrian manifold will generally have at least two boundaries. Indeed, as we have seen these manifolds are parametrized roughly speaking by a harmonic function; but a harmonic function is defined by its boundary values which gives one real function of a real variable per boundary.

2.3 Example.

In this section we will repeat in AdS$_5$ the construction of the special Legendrian manifold in $S^5$ suggested in [16]. We consider the following surface parametrized by the two real parameters $\sigma$ and $\tau$:

$$z_\mu(\tau, \sigma) = ig_\mu(\tau)e^{i\alpha_\mu\sigma}$$  \hspace{1cm} (14)

where $g_\mu(\tau)$ is real. This is a special Legendrian manifold if

$$\alpha_0 + \alpha_1 + \alpha_2 = 0$$  \hspace{1cm} (15)

and $g_0(\tau)$, $g(\tau)$, $g_2(\tau)$ satisfy the algebraic equations:

$$g_0(\tau)^2 - g_1(\tau)^2 - g_2(\tau)^2 = 1$$
$$\alpha_0 g_0(\tau)^2 - \alpha_1 g_1(\tau)^2 - \alpha_2 g_2(\tau)^2 = 0$$  \hspace{1cm} (16)

Indeed $z_\mu(\tau, \sigma)$ satisfies (13) if we choose $\tau$ so that $g_\mu(\tau)$ satisfy the system of differential equations:

$$\dot{g}_0 = -(\alpha_1 - \alpha_2)g_1g_2$$
$$\dot{g}_1 = (\alpha_2 - \alpha_0)g_2g_0$$
$$\dot{g}_2 = -(\alpha_1 - \alpha_0)g_1g_0$$  \hspace{1cm} (17)

for which (16) are integrals of motion. Equations (16) have real solutions only if $\alpha_1$ and $\alpha_2$ have different sign. Also notice that $\alpha_\mu \mapsto -\alpha_\mu$ is a symmetry. This means that without any loss of generality we may assume

$$-\alpha_2 > \alpha_1 > 0$$  \hspace{1cm} (18)
With this choice of $\alpha_\mu$ the solution is:

\[ z_0 = i \left( (\alpha_1 - \alpha_2)T + \frac{1}{3} \right)^{1/2} e^{-i(\alpha_1+\alpha_2)\sigma} \]
\[ z_1 = i \left( (-\alpha_1 - 2\alpha_2)T - \frac{1}{3} \right)^{1/2} e^{i\alpha_1\sigma} \]
\[ z_2 = \pm i \left( 2\alpha_1 + \alpha_2 \right)T - \frac{1}{3} \right)^{1/2} e^{i\alpha_2\sigma} \]

(19)

where $T \in \left[ \frac{1}{3(2\alpha_1+\alpha_2)}, +\infty \right]$. The induced metric on the worldsheet is:

\[ ds^2 = - \left[ T(\alpha_1 - \alpha_2)(-\alpha_1 - 2\alpha_2)(2\alpha_1 + \alpha_2) - \frac{2}{3}(\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2) \right] \times \]
\[ \times \left\{ d\sigma^2 + \left( \left[ (\alpha_1 - \alpha_2)T + \frac{1}{3} \right] \left[ (-\alpha_1 - 2\alpha_2)T - \frac{1}{3} \right] \left[ (2\alpha_1 + \alpha_2)T - \frac{1}{3} \right] \right)^{-1} \frac{dT^2}{4} \right\} \]

(20)

This metric is negative definite and becomes asymptotically the metric of the $AdS_2$ when $T \to +\infty$. The Laplacian of the worldsheet coordinates in the induced metric is:

\[ \frac{1}{4} \left[ (\alpha_1 - \alpha_2)T + \frac{1}{3} \right]^{-1} \left[ (-\alpha_1 - 2\alpha_2)T - \frac{1}{3} \right]^{-1} \left[ (2\alpha_1 + \alpha_2)T - \frac{1}{3} \right]^{-1} \frac{\partial^2 z_\mu}{\partial \sigma^2} + \]
\[ + \frac{9}{T} \left[ (\alpha_1 - \alpha_2)T + \frac{1}{3} \right]^{-1} \left[ (-\alpha_1 - 2\alpha_2)T - \frac{1}{3} \right]^{-1} \left[ (2\alpha_1 + \alpha_2)T - \frac{1}{3} \right]^{-1} \frac{\partial^2 z_\mu}{\partial T} = \]
\[ = \frac{1}{2} \frac{T(\alpha_1 - \alpha_2)(-\alpha_1 - 2\alpha_2)(2\alpha_1 + \alpha_2) - \frac{2}{3}(\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2)}{\sqrt{[(\alpha_1 - \alpha_2)T + \frac{1}{3}][(-\alpha_1 - 2\alpha_2)T - \frac{1}{3}][(2\alpha_1 + \alpha_2)T - \frac{1}{3}]}} z_\mu \]

(21)

which explicitly shows that the surface is extremal. The boundary is at $T = +\infty$. It can be parametrized by $\sigma$:

\[ z_0 = i\sqrt{\alpha_1 - \alpha_2} e^{-i(\alpha_1+\alpha_2)\sigma} \]
\[ z_1 = i\sqrt{-\alpha_1 - 2\alpha_2} e^{i\alpha_1\sigma} \]
\[ z_2 = \pm i\sqrt{\alpha_2 + 2\alpha_1} e^{i\alpha_2\sigma} \]

(22)

It is spacelike and consists of two components corresponding to the choice of the sign in the formula for $z_2$. It is interesting to consider the limit $\alpha_2 = -2\alpha_1 + a$ where $a$ is positive and small. The boundary consists of two spirals with the common central line. It is spacelike becoming lightlike when $a = 0$. The distance between the spirals is comparable to the length of the period of each spiral.

An interesting property of this example is that the double spiral extremizes the Wilson loop functional. We will prove it at the end of Section 7.
3 Special contact Wilson loops.

We want to characterize the one-dimensional contours in the boundary of \( AdS_5 \) which are the boundaries of the special Legendrian manifolds in \( AdS_5 \).

We will first describe some necessary conditions for such a contour. The first condition is that the contour should be contact. This means that the one form on the boundary of the AdS space which defines the contact structure should be zero on a tangent vector to the contour. This condition by itself is not enough. To formulate the second condition we define another one-form on the boundary. More precisely, this one-form is defined only on those vectors which are tangent to the contact structure. We prove that if the contour is the boundary of the special Lagrangian cone then this second one-form should be also zero on the contour.

We then conjecture that these two necessary conditions are sufficient at least for those contours which are small deformations of a circular contour. We explicitly describe the solutions to these conditions.

3.1 Two necessary conditions.

The boundary of \( AdS_5 \) is conformally \( S^1 \times S^3 \). It is the projectivization of the lightcone \( C \subset \mathbb{R}^{2+4} \). Instead of considering the one-dimensional curves in \( S^1 \times S^3 \) we will consider the two-dimensional subcones of \( C \). Suppose that \( X \subset C \) is a two-dimensional subcone. When \( X \) is the boundary of the special Lagrangian cone \( \mathcal{L} \subset \mathbb{R}^{2+4}_+ \)? The obvious necessary condition is that the restriction of \( \omega \) on \( X \) is zero:

\[
\iota_E \omega = z^* \frac{\partial}{\partial z^\mu} z^\mu = 0
\]  

(23)

To formulate the second necessary condition we will need to define a complex one-form \( \Lambda \). Let \( TX \) be the tangent bundle to \( X \). Let \( S \subset TX \) be the subspace tangent to the contact structure:

\[
S = \{ \xi \in TX | \omega(E, \xi) = 0 \}
\]  

(24)

Consider the following one-form on \( S \):

\[
\Lambda = \frac{z_1dz_2 - z_2dz_1}{z_0^*} |_S
\]  

(25)
Notice that $\Lambda$ is $SU(1, 2)$ invariant. Indeed one can see that

$$
\frac{z_1 dz_2 - z_2 dz_1}{z_0^*} \Bigg|_S = \frac{z_0 dz_2 - z_2 dz_0}{z_1^*} \Bigg|_S = \frac{z_1 dz_0 - z_0 dz_1}{z_2^*} \Bigg|_S
$$

These three forms of rewriting $\Lambda$ prove the invariance of $\Lambda$ under $SU(0, 2)$ which stabilizes $z_0$ and two $SU(1, 1)$ which stabilize $z_1$ and $z_2$. These three groups generate $SU(1, 2)$.

For $X$ to be the boundary of the special Lagrangian $L$ it is necessary that

$$\text{Im } \Lambda|_{T_X} = 0 \quad (27)$$

Let us prove that this is a necessary condition.

Let us fix a point $l \in X$. The tangent space $T_l X$ is generated by two linearly independent vectors $l$ and $\eta$, where $\eta$ should be orthogonal to $l$. To generate the tangent space to $L$ we need to add a third vector $\nu$ which is orthogonal to $\eta$ and leads out of the light cone. By an $SU(1, 2)$ transformation we can bring $\eta$ to the form

$$
\eta = \begin{bmatrix} 0 \\ 0 \\ \eta_2 \end{bmatrix}, \quad \eta_2 \in \mathbb{C}
$$

We know that $l$ is orthogonal to $\eta$ and $I.\eta$. Therefore $l$ should be of the form:

$$
l = \begin{bmatrix} q_0 \\ q_1 \\ 0 \end{bmatrix}
$$

Consider the subgroup $U(1, 1) \subset SU(1, 2)$ which rotates $\eta$ by a phase: $U(1, 1) = \{ g \in SU(1, 2) \mid g.\eta = e^{i\phi} \eta \}$. We can use this subgroup to make $q_0$ and $q_1$ real. Notice that $q_0^2 - q_1^2 = 0$. What can we say about $\nu$? Since $L$ is special Lagrangian $\nu$ should be orthogonal to both $I.\eta$ and $I.l$. Also remember that we have chosen $\nu$ to be orthogonal to $\eta$. This means that $\nu$ is of the form:

$$
\nu = \begin{bmatrix} r_0 \\ r_1 \\ 0 \end{bmatrix} + a I.l
$$

where $r_0, r_1$ and $a$ are real numbers. Now we have $\Omega(\nu, \eta, l) = (r_0 q_1 - r_1 q_0) \eta_2$ and $\Lambda(\eta) = \frac{q_1 \eta_2}{q_0}$. Therefore $\Omega(\nu, \eta, l) \in \mathbb{R}$ implies $\Lambda(\eta) \in \mathbb{R}$. This proves that (27) is a necessary condition.
We conjecture that at least for the contours sufficiently close to the circle $(23)$ and $(27)$ are actually sufficient. Our motivation for this conjecture is the counting of the parameters. Consider the "hemisphere" $(8)$. The small deformations of this hemisphere correspond to the degree two harmonic functions on the corresponding cone. These harmonic functions should be determined by their boundary values (we will explain the details in Section 4.) The boundary value of the function is a real function on a circle. We conjecture that these deformations are unobstructed. This was proven in [19] for compact special Lagrangian manifolds, but we are dealing with non-compact cases. If it is true that the deformations corresponding to harmonic functions are unobstructed then the special Legendrian manifolds close to the hemisphere should be parametrized by a real function on a circle. In the next subsection we will see that the contours satisfying $(23)$ and $(27)$ are also parametrized by a real function on a circle. This suggests that there is a one to one correspondence between the contours satisfying $(23)$ and $(27)$ and the special Legendrian manifolds, at least in the vicinity of the circular contour.

3.2 Special contact loops.

We will call special contact loops the solutions to the necessary conditions $(23)$ and $(27)$:

\begin{align*}
\iota_E \omega(\eta) & = 0 \\
\text{Im } \Lambda(\eta) & = 0 \tag{31}
\end{align*}

These special contact loops can be described very explicitly. Consider a closed path $u(\sigma)$ in $S^2 = CP^1$ which restricts the domain of zero area (area is counted with the orientation; $\infty$ is an example of a path which restricts zero area.) This path can be lifted to the horizontal curve in $S^3 \xrightarrow{\iota} CP^1$ which satisfies

\begin{equation}
\gamma_1^* \hat{\partial}_\sigma y_1 + \gamma_2^* \hat{\partial}_\sigma y_2 = 0, \quad y_2(\sigma)/y_1(\sigma) = u(\sigma) \tag{32}
\end{equation}

This gives a solution to $(31)$:

\begin{align*}
z_1(\sigma) & = e^{i\psi(\sigma)} y_1(\sigma) \tag{33} \\
z_2(\sigma) & = e^{i\psi(\sigma)} y_2(\sigma) \tag{34} \\
z_0(\sigma) & = e^{i\psi(\sigma)} \sqrt{|y_1(\sigma)|^2 + |y_2(\sigma)|^2} \tag{35}
\end{align*}
where
\[ e^{i\psi(\sigma)} = \left( \frac{y_1^* (\sigma) \bar{\partial}_\sigma y_2 (\sigma)}{y_1 (\sigma) \bar{\partial}_\sigma y_2 (\sigma)} \right)^{\frac{1}{2}} \] (36)

Therefore the special contact Wilson loops correspond to the closed contours in \( CP^1 \) restricting a domain of zero area.

### 3.3 In Poincare coordinates.

Let us introduce the Poincare coordinates:
\[ (z_0; z_1, z_2) = \left( \frac{1 - x_\mu^2 + h^2}{2h} + i \frac{x_0}{h}; \frac{x_1 + ix_2}{h}, \frac{x_3 + \frac{1 + x_\mu^2 - h^2}{2h}}{h} \right) \] (37)

Let us write the conditions for the contour in these coordinates. It is convenient to introduce:
\[ x_\pm = x_0 \pm x_3 \] (38)

The contact condition \( \lambda = 0 \) reads:
\[ dx_+ + x_- \bar{d} x_\mu^2 - 2x_1 \bar{d} x_2 = 0 \] (39)

We find it more convenient to consider special Lagrangian manifolds with \( i\Omega \in \mathbb{R} \) (rather than \( \Omega \in \mathbb{R} \)) when working in Poincare coordinates. The special condition \( \text{Re}(z_0 (\bar{d} z_2)) = 0 \) becomes:
\[ \text{Re} \left[ (x_1 + ix_2) \left( \frac{1}{2} d x_+ - \frac{1}{2} x_- \bar{d} x_\mu^2 + ix_0 \bar{d} x_3 + \frac{i}{2} \bar{d} x_\mu^2 \right) \right] = 0 \] (40)

The solutions to these conditions are parametrized by a complex valued function \( y(\sigma) \):
\[ x_1 + ix_2 = y(\partial_\sigma \bar{y})^{1/3}/\text{Re} (\partial_\sigma y)^{1/3} \]
\[ x_- = -\text{Im} (\partial_\sigma y)^{1/3}/\text{Re} (\partial_\sigma y)^{1/3} \]
\[ x_+ = x_- |y|^2 + i \int d\sigma \left( y \partial_\sigma \bar{y} - \bar{y} \partial_\sigma y \right) + C \] (41)

where \( C \) is a constant. Adding constant to \( x_+ \) corresponds to the \( su(1,2) \) transformation \( \delta(z_0; z_1, z_2) = (iz_0 + z_2; 0, z_0 - iz_2) \).
4 Infinitesimal deformations of the special Lagrangian plane.

As we have explained in Section 2 the deformations of the special Lagrangian cone correspond to the harmonic functions, homogeneous of the degree two. If the cone is a plane we can describe such functions and the corresponding deformations rather explicitly.

An infinitesimal deformation of the plane $\mathbb{R}^{1+2} \subset \mathbb{R}^{2+4}$ is described by a vector field $\xi(v)$, $v \in \mathbb{R}^{1+2}$ which is orthogonal to $\mathbb{R}^{1+2}$. The Lagrangian deformations correspond to functions $H$ on $\mathbb{R}^{1+2}$ in the following way:

$$\xi_H(v) = I.\nabla H(v)$$

where $I$ is the complex structure in $\mathbb{R}^{2+4}$ (multiplication by $i$ in $\mathbb{C}^{1+2}$.) We want the deformed submanifold to be a special Lagrangian cone. This leads us to considering the harmonic functions $H$ which are homogeneous, $H(tv) = t^2 H(v)$. Such a function $H(v)$ can be reconstructed from its values on the lightcone. Let us choose a vector $v_0 = [1, 0, 0]$ in $\mathbb{R}^{1+2}$ and consider the circle $S(v_0)$ — the set of points $l$ on the lightcone $(l, l) = 0$ satisfying $(l, v) = 1$. According to the Asgeirsson theorem about the mean value of the harmonic function\footnote{For the explanation of the Asgeirsson theorem see for example \cite{20}.}

$$H(v) = \frac{1}{3\pi} \int_{S(v_0)} \frac{dH(l)v^5}{(v \cdot l)^3} \quad (42)$$

In fact we could have replaced $S(v_0)$ by any closed path on the lightcone; this integral with the naturally defined measure $dl$ does not depend on the choice of the path. The limiting value near the intersection with the light cone $\xi(v)|_{v \to l}$ depends on $H(l)$, $\frac{d}{d\sigma} H(l)$ and $\frac{d^2}{d\sigma^2} H(l)$ where $\sigma$ is the angular coordinate on $S(v_0)$. The direct computation expressing $\xi(v)|_{v \to l}$ through $H(l)$ and its first and second derivative should be rather cumbersome. We will use a trick. First let us evaluate a particular integral:

$$I[Q](v) = \frac{1}{3\pi} \int \frac{v^5}{(v \cdot l)^3} (l, Q.l)[dl] \quad (43)$$

where $Q$ is a constant $6 \times 6$ matrix. Because of the $SO(2, 4)$ invariance

$$I[Q](v) = Av^2 \text{tr} Q + B(v, Q.v) \quad (44)$$
From $I[Q = 1](v) = 0$ we get

$$A = -\frac{1}{3}B \quad (45)$$

From the theorem about the mean value, $B = 1$. Therefore

$$I[Q](v) = -\frac{1}{3}v^2 \text{tr} Q + (v, Q,v) \quad (46)$$

Let us take

$$H_Q(l) = (l, Q.l) \quad \text{with } Q = \begin{bmatrix} Q_{00} & Q_{01} & Q_{02} \\ -Q_{01} & Q_{11} & Q_{12} \\ -Q_{02} & Q_{12} & Q_{22} \end{bmatrix} \quad (47)$$

Take $l(\sigma) = (1, \cos \sigma, \sin \sigma)$. We will write $H(\sigma)$ instead of $H(l(\sigma))$. At $\sigma = 0$ we have $H_Q(0) = Q_{00} + 2Q_{01} - Q_{11}$, $H'_Q(0) = 2(Q_{02} - Q_{12})$, $H''_Q(0) = 2(Q_{11} - Q_{01} - Q_{22})$. At the same time

$$\nabla I[Q] = -\frac{2}{3}v \text{ tr} Q + 2Q.v =
$$

$$= -\frac{2}{3} \times \begin{bmatrix} Q_{00} + Q_{11} + Q_{22} - 3(Q_{00} + Q_{01}) \\ Q_{00} + Q_{11} + Q_{22} - 3(-Q_{01} + Q_{11}) \\ 3(Q_{02} - Q_{12}) \end{bmatrix} = \frac{1}{3} \times \begin{bmatrix} H''_Q(0) + 4H_Q(0) \\ -2H_Q(0) + H'_Q(0) \\ -3H'_Q(0) \end{bmatrix} \quad (48)$$

If the boundary data has $H(0) = H_Q(0)$, $H'(0) = H'_Q(0)$ and $H''(0) = H''_Q(0)$ then $\nabla H(0) = \nabla H_Q(0)$. Therefore:

$$\nabla H(\tau) = \frac{1}{3} \times \begin{bmatrix} H''(\tau) + 4H(\tau) \\ (H''(\tau) - 2H(\tau)) \cos \tau + 3H'(\tau) \sin \tau \\ -3H'(\tau) \cos \tau + (H''(\tau) - 2H(\tau)) \sin \tau \end{bmatrix} \quad (49)$$

One can verify that this deformation preserves both $\bar{z}_\mu \overset{d}{\rightarrow} z^\mu = 0$ and $\text{Im}[z_0(z_1 \overset{d}{\rightarrow} z_2)] = 0$. Indeed the unperturbed contour is $(z_0; z_1, z_2) = (1; \cos \tau, \sin \tau)$. The contact form:

$$(I.(l + I.\nabla H), \partial_\tau l + I\partial_\tau \nabla H) = -(\nabla H, \partial_\tau z) + (z, \partial_\tau \nabla H) = 0 \quad (50)$$
The special condition:
\[ \xi_0 (\cos \tau \, \partial_{\tau} \sin \tau) + (\xi_1 \, \partial_{\tau} \sin \tau) + (\cos \tau \, \partial_{\tau} \, \xi_2) = 0 \quad (51) \]

In this example we see that the deformation of the special contact Wilson loop is described in terms of a single function \( H(\sigma) \). The formula for the deformation is rather complicated involving up to two derivatives of \( H \). The deformation preserves the special contact conditions (23) and (27).

5 No supersymmetry.

The extremal surface in \( AdS_5 \) with the boundary on the contour \( C \) corresponds on the field theory side to the insertion of the Wilson loop functional:

\[ W[C] = \frac{1}{N} \text{tr} \, P \exp \int d\sigma (i A_\mu \partial_\sigma x^\mu + \Phi_1 |\partial_\sigma x|) \quad (52) \]

This functional is invariant under the superconformal transformations which are generated by the conformal Killing spinor \( \psi(x) \) satisfying the constraint:

\[ \gamma_\mu \partial_\sigma x^\mu \psi(x) = i |\partial_\sigma x| \Gamma_1 \psi(x) \quad (53) \]

Here \( \gamma_\mu \) are the space-time gamma matrices and \( \Gamma_1 \) is first of the six gamma-matrices generating the Clifford algebra of \( R^6 \). We should have \( \{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \) and \( \{\Gamma_i, \Gamma_j\} = 2 \delta_{ij} \) and \( [\gamma_\mu, \Gamma_i] = 0 \). The conformal Killing spinors are of the form

\[ \psi(x) = \psi_0 + \gamma_\mu x_\mu \psi_1 \quad (54) \]

where both \( \psi_0 \) and \( \psi_1 \) are constant spinors\(^3\). The condition (53) is satisfied for the circular Wilson loop. Indeed, consider the circular Wilson loop in the plane \( (x_1, x_2) \) given by the equation:

\[ x_1^2 + x_2^2 = 1 \quad (55) \]

The condition (53) is satisfied for the following conformal Killing spinor:

\[ \psi(x) = \chi_0 - i(x_1 \gamma_1 + x_2 \gamma_2)\gamma_1\Gamma_1 \chi_0 \quad (56) \]

where \( \chi_0 \) is an arbitrary constant spinor. But the generic special contact Wilson loop does not preserve any supersymmetry. For example let us consider the Wilson loop corresponding to the contour \( y(\sigma) \) shown on the picture:

\(^3\)The conformal Killing spinors on flat \( R^4 \) should satisfy \( \partial_\mu \psi = \gamma_\mu \psi_1 \). It follows that \( \psi_1 \) should be constant. Indeed, \( \psi_1 \) would satisfy \( \gamma_\mu \partial_\nu \psi_1 = \gamma_\nu \partial_\mu \psi_1 \) which implies that \( \psi_1 \) is a constant.
Suppose that this contour preserves the conformal Killing spinor $\psi_0 + \gamma_\mu x_\mu \psi_1$. Let us consider three intervals $(A, B)$, $(C, D)$, and $(E, \infty)$. For the interval $(A, B)$:

$$\gamma_1(\psi_0 + x_1 \gamma_1 \psi_1) = i \Gamma_1 (\psi_0 + x_1 \gamma_1 \psi_1)$$

which implies that $\gamma_1 \psi_0 = i \Gamma_1 \psi_0$ and $\gamma_1 \psi_1 = i \Gamma_1 \psi_1$. For the interval $(C, D)$:

$$\gamma_1(\psi_0 + (x_1 \gamma_1 + x_+ \gamma_-) \psi_1) = i \Gamma_1 (\psi_0 + (x_1 \gamma_1 + x_+ \gamma_-) \psi_1)$$

therefore $\gamma_- \psi_1 = 0$. Finally, for the interval $(E, \infty)$

$$\frac{1}{\sqrt{x_1^2 + x_2^2}} (\dot{x}_1 \gamma_1 + \dot{x}_2 \gamma_2 + \dot{x}_+ \gamma_-)(\psi_0 + (x_1 \gamma_1 + x_2 \gamma_2 + x_- \gamma_+) \psi_1) = i \Gamma_1 (\psi_0 + (x_1 \gamma_1 + x_2 \gamma_2 + x_- \gamma_+) \psi_1)$$

In the limit $|x| \to \infty$:

$$\frac{1}{\sqrt{x_1^2 + x_2^2}} \dot{x}_+ x_- \psi_1 = i \Gamma_1 (x_1 \gamma_1 + x_2 \gamma_2) \psi_1$$

This would imply that $\gamma_2 \psi_1 = \pm i \Gamma_1 \psi_1$ which contradicts $\gamma_1 \psi_1 = i \Gamma_1 \psi_1$. Therefore the special contact loops are generally not supersymmetric.

### 6 Behavior near the boundary.

In this section we will study the special Legendrian submanifold $X$ in the vicinity of a point on its boundary. The main result is the following. Consider
a curve inside $X$ which originates from the point $l_0$ on the boundary of $X$ and is orthogonal to the boundary at this point. Consider the acceleration of this curve and take the component of the acceleration normal to $X$. It turns out that the normal component of the acceleration is directed along $l_0$. We will first prove it in the simpler case when the contour is locally exactly a straight line, and then give a general proof.

**Special choice of the Poincare coordinates.** Fix a point $l_0$ on the boundary of our special Legendrian submanifold. We will use the Poincare coordinates:

$$(z_0; z_1, z_2) = \left( \frac{1 - x_\mu^2 + h^2}{2h}, \frac{i x_0}{h}, \frac{x_1 + i x_2}{h}, x_3 + \frac{i}{2h} \right) \quad (61)$$

The point $l_0$ corresponds to $(1; 0, i)$. We will choose the Poincare coordinates in such a way that the Wilson loop near the point $l_0$ is nearly a straight line:

$$x_0 = \beta_0 x_1^3 + \ldots, \quad x_2 = \beta_1 x_1^3 + \ldots, \quad x_3 = \beta_2 x_1^3 + \ldots \quad (62)$$

In other words the curvature of the Wilson loop at the point $l_0$ is zero. Notice that we can always choose such coordinates. Indeed the Wilson loop in the vicinity of $l_0$ is nearly a circle. This circle is the boundary of the intersection of the AdS space with some plane. Because the Wilson loop is special contact the plane is special Lagrangian. Let us choose another lightlike vector $\tilde{l}_0$ on the plane and a space like vector $e$ on the plane orthogonal to both $l_0$ and $\tilde{l}_0$, $(e, e) = -1$. The Poincare coordinates are: $h = (v, \tilde{l}_0)^{-1}$, $x_1 = (v, e)$, $x_2 = (v, I.e)$, $x_0 + x_3 = (v, I.l_0)$ and $x_0 - x_3 = (v, I.\tilde{l}_0)$.

**General form of the extremal surface near the point of the boundary.** We assume that the coordinates of the extremal surface have a series expansion in $x_1$, $h$ near the point of the boundary $x_\mu = h = 0$. The equation for the extremal surface is to the lowest order in $h, x_1$:

$$(\partial_{x_1}^2 + \partial_h^2)y - \frac{2}{h}\partial_h y = 0 \quad (63)$$

where $y = x_0, x_2, x_3$. If the contour near $x_\mu = h = 0$ is a straight line plus corrections of the order $x_1^3$ then the extremal surface is given near the point $x_\mu = h = 0$ by the expression cubic in $x_1$ and $h$ plus higher orders. There are four cubic monomials $x_1^3, x_1^2 h, x_1 h^2, h^3$ and the equation (63) leaves two combinations:

$$y = a h^3 + \beta(x_1^3 + 3h^2 x_1) + \ldots \quad (64)$$
where dots denote terms of the degree higher than 3 (the degree of the monomial $h^a x_1^b$ is $a + b$.) This is the general form of the extremal surface near the point of the boundary where the curvature of the Wilson loop is zero. The role of the coefficients $\alpha$ and $\beta$ is very different. The coefficients $\beta$ are determined from the local behavior of the contour near the point $l_0$ (they measure the cubic deviation of the contour from the straight line.) But the coefficients $\alpha$ of $h^3$ depend globally on the contour. They may be defined as the acceleration of the geodesic on the extremal surface starting from the point $l_0$. To determine $\alpha$ for the general contour we have to actually know the extremal surface. But for the special contact contour we can determine the direction of $\alpha$ without actually knowing the extremal surface.

**The direction of $\alpha$.** In the case of the special Legendrian surfaces, we want to prove that $\alpha$ is directed along $l_0$. Let us first consider the case when the contour is locally a straight line. Near $x_1 = 0$, $h = 0$ the surface should have the form:

$$
(z_0; z_1, z_2) = \frac{1}{h} \left( \frac{1 + x_1^2 + x_1^3}{2} + i \alpha_0(x_1) h^3; x_1 + i \alpha_1(x_1) h^3, \frac{i - ix_1^2 - ih^2}{2} + \alpha_2(x_1) h^3 \right)
$$

Let us understand when this surface is Legendrian:

$$
\text{Im} \left[ z_0^* \partial_{x_1} z_0 - z_1^* \partial_{x_1} z_1 - z_2^* \partial_{x_1} z_2 \right] = \\
= \frac{3}{2} \left[ (1 + x_1^3) \partial_{x_1} \alpha_0 - 2 x_1 \partial_{x_1} \alpha_1 + (1 - x_1^2) \partial_{x_1} \alpha_2 - 2 x_1 \alpha_0 + 2 \alpha_1 + 2 x_1 \alpha_2 \right] h + o(h) = 0
$$

$$
\text{Im} \left[ z_0^* \partial_h z_0 - z_1^* \partial_h z_1 - z_2^* \partial_h z_2 \right] = \\
= \frac{3}{2} \left[ (1 + x_1^3) \alpha_0 - 2 x_1 \alpha_1 + (1 - x_1^2) \alpha_2 \right] + o(1) = 0, \quad h \to 0
$$

The condition for being special coincides with the second of these equations:

$$
\text{Re} \, \iota_E \Omega = \frac{3}{2} \left( (1 + x_1^3) \alpha_0 - 2 x_1 \alpha_1 + (1 - x_1^2) \alpha_2 \right) \frac{dx_1 \wedge dh}{h} + \ldots = 0
$$

because all the extremal Legendrian manifolds are special Legendrian. From (66) we find

$$
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2
\end{bmatrix}
= a(x_1)
\begin{bmatrix}
1 + x_1^2 \\
2x_1 \\
-1 + x_1^2
\end{bmatrix}
$$

(68)
This means that $\alpha$ is directed along $I.l$ which is what we wanted to prove.

Now suppose that the contour has a general cubic shape (62) rather than being a straight line. The extremal surface has the following shape:

$$(z_0; z_1, z_2) = \frac{1}{h} \left( \frac{1+x_1^2+h^2-\Phi_{0}}{2} + i\Phi_0(x_1, h); x_1 + i\Phi_1(x_1, h), \frac{1-x_1^2-h^2+\Phi_2^2}{2} + \Phi_2(x_1, h) \right)$$

(69)

where $\Phi_{0}^2 = \Phi_{0}^2 - \Phi_{1}^2 - \Phi_{2}^2$. We assume that $\Phi(x_1, h)$ has a series expansion in $x_1$ and $h$ starting with cubic terms:

$$\Phi_{\mu}(x_1, h) = \alpha_{\mu} h^3 + \beta_{\mu} x_1^3 + \gamma_{\mu} x_1 h^2 + \delta_{\mu} x_1^2 h + \ldots$$

In fact we know that $\gamma_{\mu} = 3\beta_{\mu}$ and $\delta_{\mu} = 0$. The condition for being Legendrian is:

$$(1 + x_1^2 + h^2 - \Phi_{0}^2 + 2x_1 \partial_{\Phi_0} - 2x_1 \partial_{\Phi_1} + (1 - x_1^2 - h^2 + \Phi_{2}^2) \partial_{\Phi_2} = 0$$

(70)

The term on the left hand side of the lowest degree in $x_1$ and $h$ comes from $d(\Phi_0 + \Phi_2)$ and has degree three. (We count $x_1, h, dx_1, dh$ as having the degree one.) This implies that $\Phi_0 + \Phi_2$ does not have degree three terms (starts with the degree four.) Therefore $\alpha_0 + \alpha_2 = 0$. The next term is quartic and it comes from $d(\Phi_0 + \Phi_2) - 2x_1 \partial_{\Phi_1}$. This implies that the quartic term in $dx_1 \wedge d\Phi_1$ is zero. Therefore $\alpha_1 = \gamma_1 = \delta_1 = 0$. But $\alpha_1 = \alpha_0 + \alpha_2 = 0$ means that $\alpha$ is directed along $I.l_0$.

7 Variation of the regularized area.

The Wilson loop functional in the strong coupling regime is the exponential of the regularized area of the extremal surface (63). We should place the boundary of the extremal surface at small constant $h = h_0$ rather than $h = 0$. The regularized area of the surface is

$$A_{\text{reg}} = A - \frac{1}{h_0} L[C]$$

(71)

where $L[C]$ is the length of the contour $C$ in the metric $dx_{\mu}^2$. The variation of the loop can be described by the displacement vector $\delta x^\mu$. The variation of the area defines the vector field $p_{\mu}(\sigma)$:

$$\delta A_{\text{reg}}[C] = \int_C dl \ p_{\mu}(\sigma) \ \delta x^\mu(\sigma)$$

(72)
This vector field $p_\mu$ can be found from the shape of the extremal surface near the boundary. Consider the geodesic on the extremal surface originating from the point on the boundary. The acceleration of this geodesic (orthogonal to the extremal surface) is $h^3 p_\mu + o(h^3)$. The leading term does not depend on the choice of the geodesic and gives a geometrical definition of $p_\mu$. In particular, for the circular Wilson loop $p_\mu = 0$ — the circular Wilson loop is an extremum of the functional $W[C]$. In the Poincare coordinates near the point $l_0$ of the boundary:

$$p_\mu = 3\alpha_\mu$$

(73)

where $\alpha^\mu$ is determined by (64).

Let us explain why the variation of the regularized area is related to the acceleration of a geodesic near the boundary. Consider the variation of the extremal surface corresponding to the variation of the contour on the boundary. We can describe the variation of the extremal surface by the normal vector $\xi(\sigma_1, \sigma_2)$. The vector describing the variation of the surface is fixed only up to vectors parallel to the surface; but it is essential for our argument that we choose $\xi(\sigma_1, \sigma_2)$ to be normal to the surface. We need to regularize the area by choosing a boundary of the surface, for example by cutting the surface at $h = h_0$ for small enough $h_0$. Since it is a good regularization it does not matter how precisely we choose the boundary. It is very natural to define the boundary of the deformed surface to be the displacement of the boundary of the original surface by the vector field $\xi$. Then the variation of the area $A$ of the surface with the boundary will be zero. Indeed, the variation of the area of the surface is the integral over the surface of the trace of the second fundamental form contracted with $\xi$. But for the extremal surface the trace of the second fundamental form is zero. The only reason why $A_{\text{reg}}$ changes is the variation of the length of the boundary which we subtract. The variation of the length of the contour is the integral over the contour of its acceleration contracted with $\xi$. But again, let us take into account that the trace of the second fundamental form of the extremal surface is zero. This means that the normal component of the acceleration of the boundary is minus the normal component of the acceleration of a curve orthogonal to the boundary, which we can choose to be a geodesic on the surface.

Calculation of $p_\mu(\sigma)$ or equivalently the acceleration of the geodesic starting from the boundary requires the full knowledge of the extremal surface. There is no simple general formula expressing the $p_\mu$ in terms of the contour.
However in the special case of the special contact Wilson loop we know from the previous section that the acceleration is directed along the lightlike vector $I.l$. In Poincare coordinates:

$$\frac{\delta}{\delta c_{\mu}(\sigma)} W[C] = p^\mu(\sigma) = (p^0, p^1, p^2, p^3) = c(\sigma) \lambda^\mu(x(\sigma)) =$$

$$= c(\sigma) \left( \frac{1 + x_2^2 + x_3^2 + x_1^2}{2}, -x_2 + x_1, x_1 + x_2, -\frac{1 + x_2^2 - x_1^2 - x_3^2}{2} \right)$$

(74)

All the infinitesimal variations with $\delta x^\mu(\sigma)$ orthogonal for any $\sigma$ to $I.l(\sigma)$ will not change the regularized area.

For the example considered in Section 2.3 it turns out that $p^\mu = 0$, just as for the circular Wilson loop. Indeed let us compute $p_\mu(\sigma = 0)$. Let us choose the Poincare coordinates with the origin at $\sigma = 0$. Notice that $z_0, z_1, z_2$ are odd functions of $\sqrt{T}$ when $T$ is large and $\sigma = 0$. This means that $h$ is an odd function of $\sqrt{T}$ and $x_0, \ldots, x_3$ are even functions of $\sqrt{T}$. This means that $x_\mu$ is an even function of $h$ and therefore it cannot have an $h^3$ term, which is just what we wanted to prove. It would be interesting to understand in general which contours $C$ extremise $W[C]$.

**Acknowledgements.**

I want to thank Yu. Makeenko and K. Zarembo for discussions on the supersymmetry of the circular Wilson loop. This work was supported in part by the National Science Foundation under Grant No. PHY99-07949 and in part by the RFBR Grant No. 00-02-116477 and in part by the Russian Grant for the support of the scientific schools No. 00-15-96557.

**A Special Lagrangian manifolds are extrema of the area functional.**

Consider a three dimensional submanifold $X \subset \mathbb{R}^{2+4}$. Let us introduce on $X$ the coordinates $\sigma_\mu$, $\mu = 0, 1, 2$. The volume of $X$ is

$$\text{vol } X = \int_X d\sigma_0 \wedge d\sigma_1 \wedge d\sigma_2 \sqrt{(w, w)}$$

(75)
where \( w = \frac{\partial}{\partial \sigma_0} \wedge \frac{\partial}{\partial \sigma_1} \wedge \frac{\partial}{\partial \sigma_2} \) is the three-vector tangent to the surface. Suppose that \( \mathbb{R}^{2+4} \) has a complex structure which makes it \( \mathbb{C}^{1+2} \). Let us define \( w_{3,0} \):

\[
w_{3,0} = \left( \frac{1 + iI}{2} \otimes \frac{1 + iI}{2} \otimes \frac{1 + iI}{2} \right) \cdot w
\]  

(76)

The key point is that the space of \((3,0)\) forms has complex dimension one. For that reason, there exists a \((3,0)\) form \( \Omega \) such that 

\[
(w_{3,0}, w_{3,0}) = |\Omega(w_{3,0})|^2 = |\Omega(w)|^2.
\]  

On the other hand

\[
(w_{3,0}, w_{3,0}) = (w, w) - 3(w, (I \otimes I \otimes 1) \cdot w)
\]  

(77)

Therefore the volume is:

\[
\text{vol } X = \int_X d\sigma_0 \wedge d\sigma_1 \wedge d\sigma_2 = \\
= \int d\sigma_0 d\sigma_1 d\sigma_2 \sqrt{(w_{3,0}, w_{3,0}) + 3(w, (I \otimes I \otimes 1) \cdot w)} = \\
= \int d\sigma_0 d\sigma_1 d\sigma_2 \sqrt{(\text{Re } \Omega(w))^2 + (\text{Im } \Omega(w))^2 + 3(w, (I \otimes I \otimes 1) \cdot w)} = \\
= \int \text{Re } \Omega + \ldots
\]  

(78)

where dots denote the terms which are second order in the variation of \( X \) when \( X \) is special Lagrangian. (Notice that \((w, (I \otimes I \otimes 1) \cdot w)\) is of the second order in the deformation because \( \left( \frac{\partial}{\partial \sigma^\mu}, I \cdot \frac{\partial}{\partial \sigma^\nu} \right) \) is of the first order in the deformation.) Since \( \text{Re } \Omega \) is a closed form, this proves that \( X \) extremises the volume functional.

References

[1] J.M. Maldacena, “The Large N Limit of Superconformal Field Theories and Supergravity”, Adv.Theor.Math.Phys. 2 (1998) 231-252; Int.J.Theor.Phys. 38 (1999) 1113-1133; hep-th/9711200.

[2] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, “Gauge Theory Correlators from Non-Critical String Theory”, Phys.Lett. B428 (1998) 105-114, hep-th/9802109.

[3] E. Witten, “Anti De Sitter Space And Holography”, Adv.Theor.Math.Phys. 2 (1998) 253-291, hep-th/9802150.
[4] S.J. Rey, J.T. Yee, ”Macroscopic strings as heavy quarks: Large-N gauge theory and anti-de Sitter supergravity”, Eur.Phys.J. C22 (2001) 379-394, hep-th/9803001.

[5] J.M. Maldacena, ”Wilson loops in large N field theories”, Phys.Rev.Lett. 80 (1998) 4859-4862, hep-th/9803002.

[6] S.J. Rey, S. Theisen, J.T. Yee, ”Wilson-Polyakov Loop at Finite Temperature in Large N Gauge Theory and Anti-de Sitter Supergravity”, Nucl.Phys. B527 (1998) 171-186, hep-th/9803135.

[7] D. Berenstein, R. Corrado, W. Fischler, J. Maldacena, ”The Operator Product Expansion for Wilson Loops and Surfaces in the Large N Limit”, Phys.Rev. D59 (1999) 105023, hep-th/9809188

[8] N. Drukker, D.J. Gross, H. Ooguri, ”Wilson Loops and Minimal Surfaces”, Phys.Rev. D60 (1999) 125006, hep-th/9904191.

[9] J.K. Erickson, G.W. Semenoff, K. Zarembo, ”Wilson Loops in N=4 Supersymmetric Yang–Mills Theory”, Nucl.Phys. B582 (2000) 155-175, hep-th/0003053.

[10] N. Drukker, D.J. Gross, ”An Exact Prediction of N=4 SUSYM Theory for String Theory”, J.Math.Phys. 42 (2001) 2896-2914, hep-th/0010274.

[11] M. Kruczenski, ”A note on twist two operators in N=4 SYM and Wilson loops in Minkowski signature”, hep-th/0210115.

[12] Yu. Makeenko, ”Light-Cone Wilson Loops and the String/Gauge Correspondence”, hep-th/0210256.

[13] I. Castro and F. Urbano, ”New examples of minimal Lagrangian tori in the complex projective plane”, Manuscripta Math. 85 (1994) 265-281; ”On a minimal Lagrangian submanifold of $\mathbb{C}^n$ foliated by spheres”, Michigan Math. J. 46 (1999) 71-82.

[14] M. Haskins, ”Special Lagrangian Cones”, math.DG/0005164.

[15] M. Gross, ”Examples of Special Lagrangian Fibrations”, Symplectic geometry and mirror symmetry (Seoul, 2000), 81–109, World Sci. Publishing, River Edge, NJ, 2001, math.AG/0012002.
[16] D. Joyce, "Special Lagrangian \(m\)-folds in \(\mathbb{C}^m\) with symmetries", math.DG/0008021; "Constructing special Lagrangian \(m\)-folds in \(\mathbb{C}^m\) by evolving quadrics", Math. Ann. 320 (2001), no. 4, 757–797, math.DG/0008153; "Evolution equations for special Lagrangian 3-folds in \(\mathbb{C}^3\)”, Ann. Global Anal. Geom. 20 (2001), no. 4, 345–403, math.DG/0010030.

[17] M. Bianchi, M.B. Green, S. Kovacs, "Instanton corrections to circular Wilson loops in N=4 Supersymmetric Yang-Mills”, JHEP 0204 (2002) 040, hep-th/0202003.

[18] R. Harvey and H.B. Lawson, Jr., "Calibrated geometries”, Acta Math. 148 (1982) 47-157.

[19] R.C. McLean, "Deformations of calibrated submanifolds”, Comm. Anal. Geom. 6 (1998) no.4, 705-747.

[20] R. Courant, D. Hilbert, "Partial Differential Equations”, 1962