CHARACTER VARIETIES

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Abstract. Let $G$ be a complex reductive algebraic group and let $\Gamma$ be a finitely generated group. We study properties of irreducible and completely reducible representations $\rho: \Gamma \to G$ in the context of the geometric invariant theory of the $G$ action on the space of $G$-representations of $\Gamma$ by conjugation.

Let $X_G(\Gamma)$ be the $G$-character variety of $\Gamma$. We prove that if $\rho: \Gamma \to G$ is completely reducible and it represents a reduced point of the character variety then

$$T_{[\rho]} X_G(\Gamma) = T_0(H^1(\Gamma, \text{Ad}\rho)//C_G(\rho(\Gamma)))$$

where $H^1(\Gamma, \text{Ad}\rho)$ is the 1st cohomology group of $\Gamma$ with coefficients in the Lie algebra $\mathfrak{g}$ of $G$ twisted by the homomorphism

$$\Gamma \xrightarrow{\rho} G \xrightarrow{\text{Ad}} \text{GL}(\mathfrak{g})$$

and $C_G(\rho(\Gamma))$ is the centralizer of $\rho(\Gamma)$ in $G$.

Let $M$ be an orientable 3-manifold with a connected boundary $F$ of genus $g \geq 2$. Let $X_G^p(F)$ be the subset of the $G$-character variety of $\pi_1(F)$ composed of conjugacy classes of good representations, that is irreducible representations $\rho: \Gamma \to G$ such that the centralizer of $\rho(\Gamma)$ is the center of $G$. By a theorem of Goldman, $X_G^p(F)$ is a holomorphic symplectic manifold. The main goal of this paper is to prove that the set of good $G$-representations of $\pi_1(F)$ which extend to representations of $\pi_1(M)$ is a complex isotropic submanifold of $X_G^p(F)$. Furthermore, if these representations correspond to reduced points of the $G$-character variety of $M$, then this submanifold is Lagrangian. This result has important applications to Chern-Simons theory and quantum topology.

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1. **Summary of Results**

Let $G$ be a reductive complex algebraic group, for example a classical group of matrices, $GL(n, \mathbb{C}), SL(n, \mathbb{C}), O(n, \mathbb{C}), Sp(n, \mathbb{C})$ or one of their quotients. Let $\Gamma$ be a finitely generated group. We say that a representation $\rho : \Gamma \to G$ is **irreducible** if $\rho(\Gamma)$ is not contained in any proper parabolic subgroup of $G$. Additionally, we say that $\rho$ is **completely reducible** if for every parabolic subgroup $P \subset G$ containing $\rho(\Gamma)$, there is a Levi subgroup $L \subset P$ containing $\rho(\Gamma)$. In particular, $\rho : \Gamma \to GL(n, \mathbb{C})$ is irreducible if and only if $\mathbb{C}^n$ is a simple $\Gamma$-module (via $\rho$) and it is completely reducible if $\mathbb{C}^n$ is a semi-simple $\Gamma$-module. In Sections 3-4 we study properties of irreducible and completely reducible representations. In particular we prove the following statements:

**Theorem 1.** 1. $\rho : \Gamma \to G$ is completely reducible if and only if the algebraic closure of $\rho(\Gamma)$ in $G$ is a linearly reductive group.
2. A completely reducible representation $\rho : \Gamma \to G$ is irreducible if and only if the centralizer of $\rho(\Gamma)$ is a finite extension of $C(G)$.

The space, $\text{Hom}(\Gamma, G)$, of all group homomorphisms from $\Gamma$ to $G$ is an algebraic set on which $G$ acts by conjugation. We study properties of this action from the point of view of the Geometric Invariant Theory in Section 7. In particular we observe that $\rho$ is a poly-stable point under that action if and only if $\rho$ is completely reducible. If $\rho$ is irreducible then it is a stable point. Finally, $\rho$ is properly stable if and only if $\rho$ is irreducible and $C(G)$ is finite.

The categorical quotient $X_G(\Gamma) = \text{Hom}(\Gamma, G)/G$ is called the $G$-character variety of $\Gamma$, c.f. Section 11. Although it is a coarser quotient than the set theory one, it has the advantage of having a natural structure of an affine algebraic set. Every element of $X_G(\Gamma)$ is represented by a unique completely reducible representation.

Let $H^*(\Gamma, Ad, \rho)$ denote the group cohomology of $\Gamma$ with coefficients in the Lie algebra $g$ of $G$ twisted by the homomorphism $\rho : G \to G$.

$$\Gamma \xrightarrow{\rho} G \xrightarrow{Ad} \text{End}(g),$$

where $Ad$ is the adjoint representation, $Ad(g)(x) = gxg^{-1}$. Denote the stabilizer of $\rho$ under the $G$ action on $\text{Hom}(\Gamma, G)$ by $C_G(\rho(\Gamma))$. (It is the centralizer of $\rho(\Gamma)$ in $G$.) There is a natural action of $C_G(\rho(\Gamma))$ on $H^1(\Gamma, Ad \rho)$, c.f. Sec. 12.

We say that $\rho$ is **reduced** if it is a reduced point of the algebraic scheme of $G$-representations of $\Gamma$, c.f. Section 9.

**Theorem 2.** (Proof in Sec. 12) (1) For every reductive $G$ and a completely reducible $\rho$ there is a natural linear map

$$\phi : T_0 X_G(\Gamma) \to T_0 \left( H^1(\Gamma, Ad \rho)/C_G(\rho(\Gamma)) \right).$$

2. $\phi$ is an isomorphism if $\rho$ is reduced.
A version of this result stating that 

\[ T_{|\rho|} X_G(\Gamma) = H^1(\Gamma, \text{Ad} \rho) \]

belongs to folk knowledge, although it is often used without proper assumptions – in particular the requirement of \(\rho\) being reduced and \(C_G(\rho(\Gamma))\) acting trivially on \(H^1(\Gamma, \text{Ad} \rho)\). The quotient on the right side may be non-trivial even if \(\rho\) is irreducible. (In Section 4 we give examples of irreducible representations whose centralizers are proper extensions of \(C(G)\).)

It is easy to see that all representations of free groups are reduced. Furthermore, we prove:

**Theorem 3.** For every reductive \(G\) and for every orientable surface \(F\) of genus \(\geq 2\), all irreducible \(G\)-representations of \(\pi_1(F)\) are reduced.

Denote the set of irreducible \(G\)-representations of \(\Gamma\) by \(\text{Hom}^i(\Gamma, G)\). It is a Zariski open subset of \(\text{Hom}(\Gamma, G)\), c.f. Proposition 20. Since each equivalence class in \(\text{Hom}(\Gamma, G)\) contains a unique closed orbit and the orbit of every irreducible representation is closed (Proposition 20), the categorical quotient of \(\text{Hom}(\Gamma, G)\) by \(G\) restricted to \(\text{Hom}^i(\Gamma, G)\) is the set theoretic quotient. Denote

\[ \text{Hom}^i(\Gamma, G) \sqcap G = \text{Hom}^i(\Gamma, G) / G \]

by \(X^i_G(\Gamma)\).

**Proposition 4.** Let \(\Gamma\) be a free group or the fundamental group of a closed orientable surface of genus \(\geq 2\). Then

1. \(X^i_G(\Gamma)\) is an orbifold.
2. If \(G = \text{GL}(n, \mathbb{C})\) or \(\text{SL}(n, \mathbb{C})\) then \(X^1_G(\Gamma)\) is a manifold. (See also [FL2].)

We do not know if Proposition 4 holds for any reductive groups other than \(\text{GL}(n, \mathbb{C})\) and \(\text{SL}(n, \mathbb{C})\), c.f. Question 13 and Proposition 13.

A representation \(\rho: \Gamma \to G\) is good if and only if it is irreducible and \(C_G(\rho(\Gamma)) = C(G)\). (By Theorem 2, the condition of irreducibility can be relaxed to complete reducibility.)

Denote the set of good \(G\)-representations of \(\Gamma\) by \(\text{Hom}^g(\Gamma, G)\). It is a Zariski open subset of \(\text{Hom}(\Gamma, G)\), c.f. Proposition 22. Let

\[ X^g_G(\Gamma) = \text{Hom}^g(\Gamma, G) \sqcap G = \text{Hom}^g(\Gamma, G) / G \]

By the above discussion \(X^g_G(\Gamma)\) is an open subset of \(X^1_G(\Gamma)\) and a smooth manifold for free groups and surface groups \(\Gamma\).

For a topological space \(Y\), we abbreviate \(X_G(\pi_1(Y))\) by \(X_G(Y)\). Let \(F\) be a closed orientable surface of genus \(\geq 2\). Goldman proved that every non-degenerate symmetric bilinear \(\text{Ad}\)-invariant form \(B\) on the Lie algebra, \(g\), of \(G\) gives rise to a holomorphic symplectic 2-form \(\omega_B\) on \(X^g_G(F)\), [Go2], c.f. Section 14.

According to folk knowledge, if \(M\) is a compact orientable 3-manifold with a connected boundary \(F = \partial M\) such that \(\rho: \pi_1(F) \to \Gamma\) is a representation, then \(X^g_G(M)\) is a Lagrangian subspace of \(X^g_G(F)\). In Section 15, we formulate this claim precisely and prove it in detail. Let \(X_G(M)\) be the smooth part of \(X^g_G(F) \cap r_*(X_G(M))\).
Theorem 5. (1) $Y_G(M)$ is an isotropic submanifold of $X_G^0(F)$ with respect to $\omega_B$ for every symmetric non-degenerate bilinear Ad-invariant form $B$ on $\mathfrak{g}$. (In particular, every connected component of $Y_G(M)$ of dimension $\frac{1}{2}\dim X_G(F)$ is Lagrangian.)

(2) If a connected component of $Y_G(M)$ contains the conjugacy class of a reduced irreducible $G$-representation of $\pi_1(M)$ then it is a Lagrangian submanifold of $Y_G(M)$.

Denote the set of equivalence classes of reduced representations in $X_G(M)$ by $X_G^r(M)$.

Theorem 6. $X_G^0(F) \cap r_* (X_G^r(M))$ is an immersed Lagrangian submanifold of $X_G^0(F)$.

Note however that $r_* : X_G(M) \to X_G(F)$ does not have to be an immersion. We observe in Proposition 45 that for every $G$ there is no upper bound on $\dim X_G(M)$ over compact manifolds $M$ with connected boundary of fixed genus $g$. ($\dim X$ is the maximum of dimensions of irreducible components of $X$.)

Theorems 5 and 6 have applications to Chern-Simons theory, quantum topology, and, potentially, Casson-type 3-manifold invariants and Floer symplectic homology, c.f. comments in Section 15.

In the paper we assume familiarity with basic algebraic geometry and the theory of algebraic groups. The standard references for these topics are [Ha, Sh, Bo, Hu].

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2. Reductive Groups

Every algebraic group $G$ contains a unique maximal normal connected solvable subgroup, called its radical and denoted by $\text{Rad } G$. A connected group $G$ is semi-simple if and only if $\text{Rad } G$ is trivial. A connected group $G$ is reductive if and only if $\text{Rad } G$ is an algebraic torus, $(\mathbb{C}^*)^n$. In particular, $\mathbb{C}^*$ and all classical matrix groups, $\text{SL}(n, \mathbb{C})$, $\text{O}(n, \mathbb{C})$, $\text{Sp}(n, \mathbb{C})$, are reductive. Furthermore, Cartesian products, quotients and finite connected covers of reductive groups are reductive. In fact, all reductive groups can be obtained in this way from simple algebraic groups.

Denote the center of $G$ by $C(G)$ and the connected component of the identity in $C(G)$ by $C^0(G)$. For every reductive $G$, $C^0(G) = (\mathbb{C}^*)^n$. A reductive group $G$ is semi-simple if and only if $C^0(G)$ is trivial.

Let $[G, G]$ be the commutator of $G$. If $G$ is reductive then $[G, G]$ is semi-simple. Furthermore, by [Bo, Proposition IV.14.2], the epimorphism

$$\nu : C^0(G) \times [G, G] \to G, \quad \nu(g, h) = g \cdot h$$

has a finite kernel, isomorphic to $K = C^0(G) \cap [G, G]$.

Therefore, there is a finite quotient

$$\pi : G \to C^0(G)/K \times [G, G]/K.$$ 

An algebraic group $G$ is linearly reductive if its all $GL(n, \mathbb{C})$-representations are completely reducible. $G$ is linearly reductive if and only if the connected component, $G^0$, of its identity is reductive. (This property does not hold for groups over fields of non-zero characteristic.) Therefore, linearly reductive groups are “virtually reductive”.


A maximal connected solvable subgroup of $G$ is called a Borel subgroup. A closed subgroup $P \subset G$ is parabolic if one of the following equivalent conditions holds: (a) $G/P$ is a complete variety, (b) $G/P$ is a projective variety, (c) $P$ contains a Borel subgroup of $G$, c.f. [Bo].

A Levi subgroup of an algebraic group $H$ is a connected subgroup $L \subset H$ such that $H$ is a semi-direct product of $L$ and the unipotent radical of $H$. Since $L$ is isomorphic to the quotient of $H$ by its unipotent radical, it is always reductive. By a result of Mostow, every algebraic group contains a Levi subgroup, c.f. [Bo, IV.11.22]

3. Irreducible and Completely Reducible Subgroups

We say that a subgroup $H$ (closed or not) of $G$ is irreducible if it is not contained in any proper parabolic subgroup of $G$. We also say that $H$ is completely reducible if for every parabolic subgroup $P \subset G$ containing $H$, there is a Levi subgroup of $P$ containing $H$ as well, [Se, BMR]. In particular, every irreducible subgroup is completely reducible.

The following is an important characterization of completely reducible subgroups:

**Proposition 7.** For every reductive $G$, a subgroup $H \subset G$ is completely reducible if and only if the algebraic closure of $H$ in $G$ is a linearly reductive group.

**Proof.** $\Rightarrow$ (1) Assume first that $H$ is irreducible, i.e. not contained in any proper parabolic subgroup of $G$. Let $\overline{H}$ be the Zariski closure of $H$ in $G$ and $\text{Rad}_u(\overline{H})$ be the unipotent radical of $\overline{H}$. Let $P = P(\text{Rad}_u(\overline{H}))$ be the parabolic subgroup defined in [Hu, 30.3]. Then $\overline{H} \subset N_G(\text{Rad}_u(\overline{H}))$ and $N_G(\text{Rad}_u(\overline{H})) \subset P$ by [Hu, 30.3 Corollary A]. If $\overline{H}$ is not linearly reductive then $\text{Rad}_u(\overline{H})$ is non-trivial and $\text{Rad}_u(\overline{H}) \subset \text{Rad}_u(P)$ by [Hu, 30.3 Corollary A]. Therefore, $P$ is a proper subgroup of $G$.

(2) Now we carry the proof in full generality by induction with respect to $\dim G$:

If $\dim G = 1$ then $G = \mathbb{C}^*$ and the statement holds. Assume now that it holds for all reductive algebraic groups $G$ of dimension less than $n$. Let $\dim G = n$. If $H$ lies in a proper parabolic subgroup of $G$ then it also lies in a Levi subgroup of $P$ and the statement follows from inductive hypothesis. If $H$ does not lie in a proper parabolic subgroup of $G$ then $H$ is irreducible in $G$ and the statement follows from part (1).

$\Leftarrow$ Suppose $\overline{H}$ is linearly reductive and $H \subset P$. Since $P$ is closed, $\overline{H} \subset P$. Now the statement follows from the fact that every closed linearly reductive subgroup of $P$ lies in a Levi subgroup of $P$. Since we do not know a good reference to this fact, we enclose its proof here: There is an exact sequence

$$
\{e\} \rightarrow \text{Rad}_u P \rightarrow P \xrightarrow{\tau} L \rightarrow \{e\},
$$

where $\text{Rad}_u P$ is the unipotent radical of $P$. Since $\overline{H}$ is reductive, it has no connected unipotent subgroups and, therefore, $\tau$ is an embedding of $\overline{H}$ into $L$. Therefore, the kernel $K$ of $\tau$ restricted $\overline{H}$ is finite. By [Bo, Corollary 4.8], $\text{Rad}_u P$ is a subgroup of upper triangular matrices and, therefore, it has no elements of finite order. Hence, $K$ is trivial. $\square$
A representation \( \phi : \Gamma \rightarrow G \) is irreducible or completely reducible if \( \phi(\Gamma) \subset G \) is. In particular, a representation \( \rho : \Gamma \rightarrow GL(n, \mathbb{C}) \) is irreducible if and only if \( \mathbb{C}^n \) does not have any \( \Gamma \)-invariant subspaces other than \( \{0\} \) and \( \mathbb{C}^n \). Additionally, \( \rho : \Gamma \rightarrow GL(n, \mathbb{C}) \) is completely reducible if and only if \( \mathbb{C}^n \) decomposes into a sum of irreducible representations.

Since a quotient of a reductive group is reductive, Proposition 7 implies:

**Corollary 8.** For every homomorphism \( \phi : G_1 \rightarrow G_2 \) of reductive groups the image of completely reducible subgroup of \( G_1 \) is completely reducible in \( G_2 \).

Similarly, we have:

**Lemma 9.** For every epimorphism \( \phi : G_1 \rightarrow G_2 \) of reductive groups, the image of an irreducible subgroup of \( G_1 \) is irreducible in \( G_2 \).

**Proof.** Suppose that \( \phi(H) \) lies inside a proper parabolic subgroup \( P \subset G_2 \). Since \( \phi \) induces an isomorphism \( G_1 / \phi^{-1}(P) \rightarrow G_2/P \) and \( G_2/P \) is complete, \( G_1 / \phi^{-1}(P) \) is complete as well, implying that \( \phi^{-1}(P) \) is a proper parabolic subgroup of \( G_1 \) containing \( H \). \( \square \)

The following example shows that irreducibility of \( H \subset G_1 \) does not imply irreducibility of \( \phi(H) \subset G_2 \) if \( \phi : G_1 \rightarrow G_2 \) is not an epimorphism, even if \( \phi \) is irreducible itself.

**Proposition 10.** Let \( H = \{ A : A \cdot A^T = \pm I \} \subset SL(2, \mathbb{C}) \).

1. \( H \) is isomorphic to \( O(2, \mathbb{C}) \).
2. \( H \subset SL(2, \mathbb{C}) \) is irreducible.
3. The image of \( H \) under the adjoint representation \( Ad : SL(2, \mathbb{C}) \rightarrow SL(3, \mathbb{C}) \) is completely reducible but not irreducible in \( SL(3, \mathbb{C}) \).

**Proof.** (1) \( H \) is a non-abelian split \( \mathbb{Z}/2 \) extension of \( SO(2) \). Now the statement follows from the fact that \( O(2, \mathbb{C}) \) is the unique non-abelian split extension of \( SO(2, \mathbb{C}) \) by \( \mathbb{Z}/2 \).

(2) Since \( H \) is reductive, it is completely reducible in \( SL(2, \mathbb{C}) \) by Proposition 7. If it was reducible, it would be a subgroup of diagonal matrices, \( \mathbb{C}^* \). Since \( SO(2, \mathbb{C}) \simeq \mathbb{C}^* \) and \( H = \mathbb{C}^* \times \mathbb{Z}/2 \) does not embed into \( \mathbb{C}^* \), it is irreducible in \( SL(2, \mathbb{C}) \).

(3) Complete reducibility follows from Corollary 8. We claim that the group \( Ad(H) \) lies in the parabolic subgroup of \( SL(3, \mathbb{C}) \) composed of transformations of \( sl(2, \mathbb{C}) \simeq \mathbb{C}^3 \) which preserve \( \text{Span}(M) \subset sl(2, \mathbb{C}) \), where \( M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Indeed, since \( SL(2, \mathbb{C}) = Sp(2, \mathbb{C}) \), \( AMA^T = M \) for every \( A \in SL(2, \mathbb{C}) \). If \( A \in H \) then \( A^T = \pm A^{-1} \) and the claim follows. \( \square \)

We say that \( H \subset G \) is Ad-irreducible if \( Ad(H) \subset GL(\mathfrak{g}) \) is irreducible. The above \( H \subset SL(2, \mathbb{C}) \) is irreducible but not Ad-irreducible. By Proposition 7 every irreducible subgroup \( H \subset G \) is completely Ad-reducible, i.e. \( Ad(H) \subset GL(\mathfrak{g}) \) is completely reducible. We are going to show that Ad-irreducibility implies irreducibility.

**Lemma 11.** Let \( \phi : G \rightarrow GL(n, \mathbb{C}) \) be an irreducible representation of a reductive group \( G \). If \( H \) a subgroup of \( G \) such that \( \phi(H) \) is irreducible then either

(a) \( H \) is irreducible, or
(b) \( \ker \phi \) contains the unipotent radical (i.e. the maximal connected unipotent subgroup) of a Borel subgroup of \( G \).

**Proof.** Suppose that \( H \subset P \subsetneq G \). Then \( \phi \) restricted to \( P \) is irreducible as well. Let \( U \) be the unipotent radical of \( P \). Denote the space of vectors in \( V = \mathbb{C}^n \) invariant under the \( U \)-action by \( V^U \). Since \( P \) is a semi-direct product of \( U \) and its Levi subgroup \( L \), \( L \times U \), \( l^{-1}ul \in U \), for every \( u \in U \) and \( l \in L \), and

\[
u \cdot l \cdot v = l \cdot l^{-1}ul \cdot v = l \cdot v \quad \text{for every} \quad v \in V^U.
\]

Therefore \( \nu \cdot v \in V^U \) and, consequently, \( V^U \) is preserved by \( P \). Since \( \phi \) restricted to \( P \) is irreducible, by Shur’s Lemma \( V^U \) is either 0 or \( V \). However, \( U \) is a connected solvable group and, therefore, \( V^U \neq 0 \), by Lie-Kolchin theorem, \( \text{[Bo, Cor 10.5]} \). Hence \( V^U = V \) and, consequently, \( U \subset \ker \phi \). If \( B \) is a Borel subgroup of \( G \) contained in \( P \) then the unipotent radical of \( B \) is contained in \( U \). \( \square \)

Since the kernel of the adjoint representation is the center of \( G \), \( \text{[Bo I.3.15]} \), and its unipotent radical is trivial, Lemma \( \text{[11]} \) implies:

**Corollary 12.** Every \( Ad \)-irreducible subgroup of a reductive group is irreducible.

4. **Stabilizers of irreducible representations**

**Proposition 13.** The centralizer of an \( Ad \)-irreducible subgroup of a reductive group \( G \) is the center of \( G \).

**Proof.** Let \( H \subset G \) be \( Ad \)-irreducible. By Shur’s Lemma the centralizer of \( Ad(H) \) is the group of scalar matrices in \( GL(g) \). Hence,

\[
Ad(C_G(H)) \subset C_{GL(g)}(Ad(H)) \subset \{ c \cdot I : c \in \mathbb{C}^* \}.
\]

On the other hand, since the center of \( Ad(G) = G/C(G) \) is trivial, c.f. \( \text{[FH Thm 23.16]} \), \( Ad(G) \cap \{ c \cdot I : c \in \mathbb{C}^* \} = \{ I \} \). Hence, \( Ad(C_G(H)) \) is trivial, implying that \( C_G(H) \subset C(G) \). \( \square \)

**Proposition 14.** The centralizer of an irreducible subgroup of a reductive group \( G \) is a finite extension of the center of \( G \).

**Proof.** Suppose that the centralizer, \( C_G(H) \), of \( H \) is an infinite extension of the center. Let \( T \) be a maximal torus in \( C_G(H) \). Then \( \text{rank } T > \text{rank } C(G) \) and \( H \subset C_G(T) \). Recall that \( T \) is either regular, semi-regular, or singular, \( \text{[Bo §13.1]} \). If \( T \) is regular then \( C_G(T) \) is a maximal torus. If \( T \) is semi-regular, then \( C_G(T) \) is contained in a Borel subgroup, c.f. proof of \( \text{[Bo IV.13.1 Proposition]} \). In either case \( H \subset C_G(T) \) is reducible. Therefore, \( T \) is singular. In that case \( T \) is the connected component of identity in \( \bigcap_{\alpha \in I} \ker \alpha \), where the intersection is over a certain proper, non-empty subset \( I \) of positive roots. By \( \text{[Bo IV.14.17]} \), \( T \) lies inside of a proper parabolic subgroup of \( G \) (denoted by Borel by \( P_I \)). \( \square \)

The following lemma will be useful later.

**Lemma 15.** For every Levi subgroup \( L \) of every proper parabolic subgroup of a reductive group \( G \), \( \dim C(L) > \dim C(G) \).

**Proof.** Suppose that \( L \) is a Levi subgroup of a parabolic group \( P \) in \( G \). As before, let \( C^0(G) \) be the connected component of the identity in the center of \( G \). Then \( C^0(G) \subset L \) and \( L' = L/C^0(G) \) is a Levi subgroup of the parabolic subgroup
$P' = P/C^0(G)$ of $G' = G/C^0(G)$. Therefore, without loss of generality we can assume that $\dim C(G) = 0$.

By [Bo] Prop. 11.23, it is enough to prove that the radical of $P$, $\mathcal{R}P$, has a positive dimension. Fix a root system for $G$. We are going to use the notation of [Bo]. By classification of parabolic subgroups in [Bo §14.17], $P = P_I$ for some subset $I$ of positive roots $\Delta$ of $G$. Let $T_I$ be the identity component of $\cap_{\alpha \in I} \text{Ker} \alpha$. By [Bo] Prop. 14.18, $T_I \subset \mathcal{R}P$. Since $T_I$ is an algebraic torus of dimension $\text{rank } G - |I|$, $\dim \mathcal{R}P > 0$ unless $I = \Delta$. In this case, $P = G$. \qed

Proposition [14] and Lemma [15] imply:

**Corollary 16.** A completely reducible subgroup $H \subset G$ is irreducible if and only if $\dim C_G(H) = \dim C(G)$.

We will say that a reductive group $G$ has property CI if the centralizer of every irreducible subgroup of $G$ coincides with the center of $G$.

**Example 17.** $G = GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$ are CI. Indeed, $H \subset G$ is irreducible if and only if elements of $H$ linearly span $M(n, \mathbb{C})$. Consequently, the centralizer of every irreducible subgroup $H \subset G$ is the center of $G$.

**Question 18.** Are $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$ the only CI-groups?

**Example 19.** $PSL(2, \mathbb{C})$ is not CI. To see that consider the subgroup $H \subset PSL(2, \mathbb{C})$, generated by $\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $|H| = 4$, $H$ is the Klein group.

One can easily see that $H$ is its own centralizer in $PSL(2, \mathbb{C})$ (while the center of $PSL(2, \mathbb{C})$ is trivial). Being finite, $H$ is linearly reductive and completely reducible by Proposition [7] By Corollary [16] $H$ is irreducible in $PSL(2, \mathbb{C})$.

**Example 20.** $SO(n, \mathbb{C})$ is not CI: Let $DM_n$ be the group of diagonal matrices in $SO(n, \mathbb{C}) = \{ A : A \cdot A^T = I \} \subset SL(n, \mathbb{C})$. Then $DM_n \simeq (\mathbb{Z}/2)^{n-1}$ and it is easy to see that $DM_n$ is its own centralizer in $SO(n, \mathbb{C})$. Being finite, $DM_n$ is linearly reductive and completely reducible by Proposition [7] By Corollary [16] $DM_n$ is irreducible in $SO(n, \mathbb{C})$.

**Proposition 21.** $Sp(2n, \mathbb{C})$ is not CI.

**Proof.** (based on the idea of S. Lawton, c.f. [12]) Denote by $D(\alpha_1, ..., \alpha_n)$ the diagonal matrix with entries $\alpha_1, ..., \alpha_n$, and by $AD(\alpha_1, ..., \alpha_n)$ the anti-diagonal matrix

\[
\begin{pmatrix}
0 & 0 & 0 & \alpha_1 \\
0 & 0 & ... & 0 \\
0 & ... & 0 & 0 \\
\alpha_n & 0 & 0 & 0
\end{pmatrix}.
\]

The matrices $D(\alpha_1, ..., \alpha_n, \alpha_n^{-1}, ..., \alpha_1^{-1})$ and $AD(\beta_1, ..., \beta_n, -\beta_n^{-1}, ..., -\beta_1^{-1})$, for $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n \in \mathbb{C}^*$ form a subgroup of $Sp(2n, \mathbb{C}) = \{ A : AJA^T = J \}$, where $J = AD(1, ..., 1, -1, ..., -1)$. Denote that subgroup by $H_n$. An elementary computation shows that the center of $H_n$ is composed of matrices $D(\alpha_1, ..., \alpha_n, \alpha_n^{-1}, ..., \alpha_1^{-1})$, where $\alpha_1, ..., \alpha_n \in \{ \pm 1 \}$. Since $H_n$ is a finite extension of $(\mathbb{C}^*)^n$, it is linearly reductive and, hence, by Proposition [7] it is completely reducible in $Sp(2n, \mathbb{C})$. Since $C(\Gamma_n)$ is a finite extension of $C(Sp(2n, \mathbb{C})) = \{ \pm 1 \}$, $\Gamma_n$ is irreducible by Corollary [16] \qed
By the following result, $PSO(n, \mathbb{C}), PSp(2n, \mathbb{C})$ are not CI either.

**Proposition 22.** A quotient of a non-CI group by a finite subgroup is non-CI.

**Proof.** Let $\Gamma \subset G$ be irreducible and such that $C_G(\Gamma)$ is a proper extension of $C(G)$. If $\pi : G \to G'$ is a quotient with finite kernel then $\text{Ker } \pi \subset C(G)$ and, consequently, the centralizer of $\pi(\Gamma)$ in $G'$ is a proper extension of $C(G')$. Now the statement follows from Proposition 9. □

5. **Representation Varieties**

If $\Gamma$ is a finitely generated group and $G$ an affine complex algebraic group, then the space of all $G$-representations of $\Gamma$, $\text{Hom}(\Gamma, G)$, is an algebraic set.

**Example 23.**

(3) \[ \text{Hom}(\Gamma_1 * \Gamma_2, G) = \text{Hom}(\Gamma_1, G) \times \text{Hom}(\Gamma_2, G). \]

Hence, for the free group on $n$ generators, $\text{Hom}(F_n, G) = G^n$.

**Example 24.** Each point of $\text{Hom}(\mathbb{Z}^2, SL(2, \mathbb{C}))$ is represented by $\rho : \mathbb{Z}^2 \to SL(2, \mathbb{C})$ defined by

\[ \rho(1, 0) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \quad \rho(0, 1) = \begin{pmatrix} x_5 & x_6 \\ x_7 & x_8 \end{pmatrix}, \]

satisfying relations

\[ x_1 x_4 - x_2 x_3 - 1 = x_5 x_8 - x_6 x_7 - 1 = x_2 x_7 - x_3 x_6 = 0, \]
\[ -x_2 x_5 + x_1 x_6 - x_4 x_8 + x_2 x_8 = x_3 x_5 - x_1 x_7 + x_4 x_7 - x_3 x_8 = 0. \]

The algebraic set $\text{Hom}(\mathbb{Z}^2, SL(2, \mathbb{C})) \subset \mathbb{C}^8$ is irreducible by [RH Thm C].

For a more thorough study of representation varieties it is useful to associate with each $\Gamma$ and $G$ as above an affine algebraic scheme, also called the representation variety, whose set of close points coincides with $\text{Hom}(\Gamma, G)$. That scheme, containing sometimes more subtle information about $G$-representations of $\Gamma$ than $\text{Hom}(\Gamma, G)$, is constructed below.

If $G$ is an affine complex algebraic group, then $\mathbb{C}[G]$ is a Hopf algebra with the coproduct

\[ \Delta : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G] = \mathbb{C}[G \times G] \]

being the dual to the group product $G \times G \to G$ and the antipode

\[ S : \mathbb{C}[G] \to \mathbb{C}[G] \]

being the dual to the inverse map $g \to g^{-1}$. Consequently, for any commutative $\mathbb{C}$-algebra $A$ with product $m : A \times A \to A$, the space of algebra homomorphisms, $\text{Hom}(\mathbb{C}[G], A)$, is a group with the multiplication

\[ \text{Hom}(\mathbb{C}[G], A) \times \text{Hom}(\mathbb{C}[G], A) \ni (f, g) \mapsto m(f \otimes g)\Delta \in \text{Hom}(\mathbb{C}[G], A) \]

and the inverse

\[ \text{Hom}(\mathbb{C}[G], A) \ni f \mapsto fS \in \text{Hom}(\mathbb{C}[G], A). \]

We will denote $\text{Hom}(\mathbb{C}[G], A)$ with that group structure by $G(A)$. The functor $G(\cdot)$ is called an affine group scheme, [Va]. For example, $G(A) = SL(n, A)$ for $G = SL(n, \mathbb{C})$.

We say that a commutative $\mathbb{C}$-algebra $R(\Gamma, G)$ is a universal representation algebra of $\Gamma$ into $G$ and $\rho_{U} : \Gamma \to G(R(\Gamma, G))$ is a universal representation if
for every commutative \( \mathbb{C} \)-algebra \( A \) and every representation \( \rho : \Gamma \to G(A) \), there is a \( \mathbb{C} \)-algebra homomorphism \( f : R(\Gamma, G) \to A \) inducing a representation \( G(f) : G(R(\Gamma, G)) \to G(A) \) such that \( \rho = G(f)\rho_U \), [BH] [SH].

**Lemma 25.** For every \( \Gamma \) and every \( G \) as above,
1. \( R(\Gamma, G) \) and \( \rho_U \) exist.
2. \( R(\Gamma, G) \) is well defined up to an isomorphism of \( \mathbb{C} \)-algebras.
3. \( \rho_U : \Gamma \to G(R(\Gamma, G)) \) is unique up to a composition with \( G(f) \) where \( f \) is a \( \mathbb{C} \)-algebra automorphism of \( R(\Gamma, G) \).

**Proof.**
1. Since each affine algebraic group \( G \) is a closed subgroup of \( GL(n, \mathbb{C}) \), the coordinate ring of \( G \),

\[ \mathbb{C}[GL(n, \mathbb{C})] = \mathbb{C}[d, x_{ij}, 1 \leq i, j \leq n]/(d \cdot \det(x_{ij}) - 1). \]

Let

\[ \mathbb{C}[G] = \mathbb{C}[d, x_{ij}, 1 \leq i, j \leq n]/I_G, \]

for an appropriate ideal \( I_G \). For the free group, \( F_N = \langle \gamma_1, ..., \gamma_N \rangle \),

\[ R(F_N, G) = \mathbb{C}[d_1, x_{1ij}, 1 \leq i, j \leq n]/I_G \otimes \cdots \otimes \mathbb{C}[d_N, x_{Nij}, 1 \leq i, j \leq n]/I_G \]

and

\[ \rho_U(\gamma_t) = (x_{tij}) \in G(R(\Gamma, G)), \quad \text{for} \ t = 1, ..., N \]

satisfy the required universal properties.

If

\[ \Gamma = \langle \gamma_1, ..., \gamma_N \rangle / H, \]

where \( H \) is the group of relations between the generators \( \gamma_1, ..., \gamma_N \) then we define \( R(\Gamma, G) \) as the quotient of \( R(\langle \gamma_1, ..., \gamma_N \rangle, G) \) by an ideal \( I \) generated by all relations necessary for \( \rho_U \) to be a well defined group homomorphism. Therefore, each normal generator of \( H \lhd \langle \gamma_1, ..., \gamma_N \rangle \) introduces \( n^2 \) relations to \( I \) (although some of them may be redundant). It is easy to see that \( \rho_U \) descends to a universal representation \( \rho_U : \Gamma \to G(R(\Gamma, G)) \).

2. and 3. follow immediately from the definition. \[ \square \]

Every \( \rho \in Hom(\Gamma, G) \) defines a \( \mathbb{C} \)-algebra homomorphism \( h_\rho : R(\Gamma, G) \to \mathbb{C} \) (unique up to an automorphism of \( R(\Gamma, G) \)) such that

\[ \rho = G(h_\rho)\rho_U. \]

\( Ker h_\rho \) is a maximal ideal in \( R(\Gamma, G) \) and, hence, a closed point in the affine scheme \( Spec R(\Gamma, G) \). Conversely, every closed point in \( Spec R(\Gamma, G) \) defines a representation \( \rho : \Gamma \to G \). Therefore, \( Hom(\Gamma, G) \) is the set of closed points of \( Spec R(\Gamma, G) \) and

\[ R(\Gamma, G)/\sqrt{U} = \mathbb{C}[Hom(\Gamma, G)]. \]

By [KM] Thm 1.2, \( R(\Gamma, PSL(2, \mathbb{C})) \) contains non-zero nilpotent elements for some Artin groups \( \Gamma \). Furthermore, M. Kapovitch proves that \( R(\pi_1(M), SL(2, \mathbb{C})) \) and \( R(\pi_1(M), PSL(2, \mathbb{C})) \) contain non-zero nilpotents for some 3-dimensional manifolds \( M \), [Ka1] [Ka2]. See further comments in Sec. 13.
6. Spaces of irreducible representations

**Proposition 26.** For every $\Gamma$ and every reductive group $G$ the set of irreducible $G$-representations of $\Gamma$ is open in $\text{Hom}(\Gamma, G)$.

**Proof.** The proposition follows from Corollary 16 and from [Ne, Prop 3.8]. Since the proof of this referenced result is non-elementary, we enclose a complete simple proof here:

1. First, a simple proof for $G = \text{GL}(n, \mathbb{C})$ and $\text{SL}(n, \mathbb{C})$: If $\rho : \Gamma \to G$ is irreducible then, by Shur’s Lemma, the elements of $\rho(\Gamma)$ linearly span the space of $n \times n$ matrices, $M(n, \mathbb{C})$. Conversely, if $\rho(\Gamma)$ lies in a parabolic subgroup of $G$ then the elements of $\rho(\Gamma)$ do not span $M(n, \mathbb{C})$.

Enumerate all elements of $\Gamma$ in a sequence $\gamma_1, \gamma_2, \ldots$. Let $U_s$ be the space of all $\rho$’s such that $\rho(\gamma_1), \ldots, \rho(\gamma_s)$ span $M(n, \mathbb{C})$. Since the space of all irreducible $\rho$’s is the union of all $U_s$’s, it is enough to prove that each $U_s$ is open. This condition is equivalent to an existence of a sequence $i_1, \ldots, i_{n^2}$ such that the $n^2 \times n^2$ matrix whose columns are $\rho(\gamma_{i_1}), \ldots, \rho(\gamma_{i_{n^2}})$ considered as vectors in $M(n, \mathbb{C}) = \mathbb{C}^{n^2}$ has a non-zero determinant. This is an open condition.

2. Here is a fairly elementary proof for all $G$:

The set of irreducible representations $\Gamma \to G$ is the complement of

$$\bigcup_P \text{Hom}(\Gamma, P) \subset \text{Hom}(\Gamma, G)$$

where the union on the left is over all proper parabolic subgroups of $G$. By [Bo, Thm 14.18], there are only finitely many parabolic subgroups of $G$ up to conjugation. Therefore it is enough to prove that for a given $P$

$$X_P = \bigcup_{g \in G} \text{Hom}(\Gamma, gPg^{-1}) \subset \text{Hom}(\Gamma, G)$$

is closed. $X_P$ is the union of closed sets parameterized by a complete variety $G/P$. By Projective Extension Theorem, [?, Ch 8 §5 Thm 6], such union is closed. □

The adjoint representation induces a map $\text{Ad}_* : \text{Hom}(\Gamma, G) \to \text{Hom}(\Gamma, \text{GL}(g))$. $\rho : \Gamma \to G$ is $\text{Ad}_*$-irreducible if $\text{Ad}\rho$ is irreducible. Since the set of $\text{Ad}$-irreducible representations $\Gamma \to G$ is the $\text{Ad}_*$-preimage of the irreducible representations in $\text{Hom}(\Gamma, \text{GL}(g))$ we conclude with

**Corollary 27.** The set of $\text{Ad}$-irreducible representations is open in $\text{Hom}(\Gamma, G)$.

**Proposition 28.** Let $G$ be a reductive group.

1. For a free group, $F_N$, of rank $N \geq 2$, the irreducible representations form a dense subset of $\text{Hom}(F_N, G)$.

2. For every closed orientable surface $F$ of genus $g \geq 2$, the irreducible representations are dense in a non-empty set of irreducible components of $\text{Hom}(\pi_1(F), G)$.

**Proof.** (1) Since $\text{Hom}(F_N, G)$ is an irreducible algebraic set and the set of irreducibles is open in it, it is enough to show that the set of irreducibles is non-empty. Since every free group $F_N$ of rank $N \geq 2$ maps onto $F_2$, it is enough to prove that statement for $F_2$. The set of irreducible $G$-representations of $F_2$ is the complement of

$$\bigcup_P \text{Hom}(F_2, P) \subset \text{Hom}(F_2, G) = G \times G,$$
where the union of sets on the left is over all proper parabolic subgroups of \( G \). By \([153]\) Thm 14.18, there are only finitely many parabolic subgroups of \( G \) up to conjugation. Since for each \( P \)

\[
\bigcup_{g \in G} \text{Hom}(F_2, gPg^{-1})
\]

is the image of the \( G \) action on \( \text{Hom}(F_2, P) \) with stabilizer \( P \), its dimension is at most

\[
2 \cdot \dim P + \dim G - \dim P < 2 \cdot \dim G = \dim \text{Hom}(F_2, G)
\]

Therefore, there exists an irreducible representation.

(3) Again, it is enough to prove that an irreducible representation exists. This follows from the fact that \( \pi_1(F) \) maps onto the free group of rank 2.

\[\square\]

7. Stable and properly stable representations in the sense of GIT

Let \( O_\rho \) be the orbit of \( \rho \in \text{Hom}(\Gamma, G) \) under the \( G \) action on \( \text{Hom}(\Gamma, G) \) by conjugation. In the language of geometric invariant theory, \( \rho \) is poly-stable if \( O_\rho \) is closed.

**Theorem 29.** For any reductive algebraic group \( G \), \( O_\rho \subset \text{Hom}(\Gamma, G) \) is closed if and only if \( \rho \) is completely reducible.

**Proof.** (The proof for \( G = GL(n, \mathbb{C}) \), can be found in \([151]\) Thm 1.27])

\[\Rightarrow \] We follow an argument of the proof of \([151]\) Thm 1.1]: Assume that \( \rho \) is closed. If \( \rho \) is contained in a proper parabolic subgroup \( P \) then by conjugating \( \rho \) with a one parameter group in the center of a Levi subgroup \( L \) of \( P \) one can obtain a representation \( \rho' \in O_\rho \) whose image lies in \( L \). Since \( O_\rho \) is closed, \( \rho' = g^{-1}\rho g \), for some \( g \in P \). Hence \( \rho \) lies in the Levi subgroup \( gLg^{-1} \).

\[\Leftarrow \] Any finitely generated group \( \Gamma \) is a quotient of a free group \( F \) of finite rank. Denote the epimorphism \( F \to \Gamma \) by \( \pi \). Since \( \text{Hom}(\Gamma, G) \) is a closed subset of \( \text{Hom}(F, G) \) and \( O_\rho = O_\rho \cap \text{Hom}(\Gamma, G) \), it is enough to prove that \( O_\rho \subset \text{Hom}(F, G) \) is closed. This statement follows from \([152]\) Thm. 3.6].

According to the geometric invariant theory, a point \( x \) of an affine set \( X \) is stable with respect to a \( G \) action on \( X \) (and the trivial line bundle on \( X \)) if there is a Zariski open neighborhood of \( x \) preserved by \( G \) on which the \( G \) action is closed, \([153]\) [Do].

**Corollary 30.** (1) Every irreducible representation is a stable point of \( \text{Hom}(\Gamma, G) \) under the \( G \) action by conjugation.

(2) \( \rho \in \text{Hom}(F_n, G) \) is stable if and only if \( \rho \) is irreducible.

**Proof.** (1) Follows from Proposition 29

(2) Every stable \( \rho \) it is completely reducible by Theorem 29. Every completely reducible representation of a free group which is not irreducible can be deformed by an arbitrarily small deformation to a representation which is not completely reducible.

\[\square\]

A point \( x \in X \) is properly stable if it is stable and its stabilizer, \( S_G(x) \), is finite.
Corollary 31. For every reductive group \( G \),
(1) \( \rho \) is a properly stable point of \( \text{Hom}(\Gamma, G) \) under conjugation action of \( G \) if and only if \( \rho \) is irreducible and \( C(G) \) is finite.
(2) \( \rho \) is a properly stable point of \( \text{Hom}(\Gamma, G) \) under conjugation action of \( G/C(G) \) if and only if \( \rho \) is irreducible.

Proof. (1) \( \Rightarrow \): \( \rho \) is completely reducible by Theorem 29, and it is irreducible by Lemma 15.
\( \Leftarrow \): by Theorem 29 and Propositions 26 and 14.
The same argument shows (2) \( \square \)

\[ \text{JM} \] say that a representation \( \rho \) is good if \( O_\rho \) is closed and \( C_G(\rho(\Gamma)) \) is the center of \( G \). By Theorem 29 and Corollary 16, every good representation is irreducible. By Proposition 13, every \( \text{Ad} \)-irreducible representation is good.

Proposition 32. For every \( \Gamma \) the space of good \( G \)-representations is open in the space of all \( G \)-representations of \( \Gamma \).

Proof. By \[ \text{JM, Proposition 1.1} \], the \( G \) action on the space of all irreducible \( G \)-representations of \( \Gamma \) is proper. The good representations, if they exist, form a set which is the union of the principal orbits of that action. For every proper action, the union of principal orbits is an open subset, c.f. \[ \text{CO, Thm. 1.5} \]. \( \square \)

8. Tangent Spaces

Let \( A \) be a commutative \( \mathbb{C} \)-algebra, let \( m \) be a closed point of \( \text{Spec} A \), i.e. a maximal ideal \( m \trianglelefteq A \), and let \( r_m \) be the projection \( A \to A/m = \mathbb{C} \). The tangent space to \( \text{Spec} A \) at \( m \) is the dual vector space to \( m/m^2 \),
\[
T_m \text{Spec} A = (m/m^2)^*.
\]

Here is an equivalent definition of the tangent space which will be useful later: Let \( \pi : \mathbb{C}[\varepsilon]/(\varepsilon^2) \to \mathbb{C} \) be the homomorphism sending \( \varepsilon \) to 0 and let \( T_m \text{Spec} A \) be the complex vector space of \( \mathbb{C} \)-algebra homomorphisms \( A \to \mathbb{C}[\varepsilon]/(\varepsilon^2) \) which descend to \( r_m \) when composed with \( \pi \). Any such homomorphism is of the form \( r_m + \tau \varepsilon \), where \( \tau : A \to \mathbb{C} \) is a derivation,
\[
T_m \text{Spec} A = \{ \tau : A \to \mathbb{C} : \tau(ab) = r_m(a)\tau(b) + r_m(b)\tau(a) \}.
\]
A straightforward calculation shows that for every \( v \in T_m \text{Spec} A \),
\[
\lambda_v(a) = v(a - r_m(a))
\]
is a derivation in \( T_m \text{Spec} A \). A direct computation shows that
\[
\lambda : T_m \text{Spec} A \to T_m \text{Spec} A
\]
sending \( v \) to \( \lambda_v \) is an isomorphism of vector spaces, \[ \text{EH, VI.1.3} \]. From now on we will identify these two spaces and call them the Zariski tangent space to \( \text{Spec} A \) at \( m \).

The above discussion applies to \( A = \text{Spec} R(\Gamma, G) \). Each \( \rho \in \text{Hom}(\Gamma, G) \) defines a projection \( r_\rho : R(\Gamma, G) \to \mathbb{C} \) and a closed point \( m_\rho = \text{Ker} \rho \) in \( \text{Spec} R(\Gamma, G) \). We will abbreviate \( T_{m_\rho} \text{Spec} R(\Gamma, G) \) to \( T_\rho \text{Spec} R(\Gamma, G) \). Each tangent vector \( \tau \in T_\rho \text{Spec} R(\Gamma, G) \) defines a group homomorphism
\[
\Gamma \xrightarrow{\rho_\tau} G(R(\Gamma, G)) \xrightarrow{G(r_\rho + \tau \varepsilon)} G(\mathbb{C}[\varepsilon]/(\varepsilon^2)).
\]
By abuse of notation, we denote by $\pi$ the extension of the homomorphism
$$\pi : \mathbb{C}[\varepsilon]/(\varepsilon^2) \to \mathbb{C}, \quad \pi(\varepsilon) = 0,$$
to the induced group homomorphism
$$\pi : G(\mathbb{C}[\varepsilon]/(\varepsilon^2)) \to G(\mathbb{C}) = G.$$

**Proposition 33.** Consider a closed embedding $G \subset GL(n, \mathbb{C})$. (Such an embedding exists for every affine algebraic group.)

1. For every $g \in G(\mathbb{C}[\varepsilon]/(\varepsilon^2))$,
$$\sigma(g) = \frac{g \cdot \pi(g)^{-1} - I}{\varepsilon} \in M_n(\mathbb{C}[\varepsilon])$$
and $\sigma(g)$ belongs to the Lie algebra $\mathfrak{g} \subset M_n(\mathbb{C})$ of $G$.

2. For every $g_1, g_2 \in G(\mathbb{C}[\varepsilon]/(\varepsilon^2))$,
$$\sigma(g_1 g_2) = \sigma(g_1) + Ad \pi(g_1) \cdot \sigma(g_2).$$

where $Ad : G \to GL(\mathfrak{g})$ is (as before) the adjoint representation.

**Proof.** (1) If $h \in G(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ is such that $\pi(h) = I$ then $\frac{h-I}{\varepsilon}$ belongs to the Zariski tangent space to $G$ at the identity, that is the Lie algebra of $G$. Now the statement follows from substitution $h = g\pi(g)^{-1}$.

(2) follows by a direct computation. $\square$

For every $\tau \in T_\rho Spec R(\Gamma, G)$, $G(r_\rho + \tau\varepsilon)\rho\nu(\gamma) \in G(\mathbb{C}[\varepsilon]/(\varepsilon^2))$, c.f. 1. Therefore, by Proposition 33 we have a function $\Gamma \to \mathfrak{g}$

$$\gamma \to \left(\frac{G(r_\rho + \tau\varepsilon)\rho\nu(\gamma)}{\varepsilon}\right)^{-1} \mod \varepsilon$$
satisfying the cocycle condition for the first cohomology group of $\Gamma$ with coefficients in $\mathfrak{g}$ twisted by $Ad \rho$. Hence, 1 defines a map

$$\Psi_\rho : T_\rho Spec R(\Gamma, G) \to Z^1(\Gamma, Ad \rho)$$
sending $\tau$ to $\sigma$.

The adjoint action of the centralizer of $\rho(\Gamma)$, $C_G(\rho(\Gamma))$, on $\mathfrak{g}$ induces a $C_G(\rho(\Gamma))$-action on $Z^1(\Gamma, Ad \rho)$. Additionally, every $g \in C_G(\rho(\Gamma))$ acts on $T_\rho Spec R(\Gamma, G)$ by sending $\tau \in T_\rho Spec R(\Gamma, G)$ to $g\tau$ such that

$$G(r_\rho + g\tau\varepsilon) = gG(r_\rho + \tau\varepsilon)g^{-1}.$$

The homomorphism $\Psi_\rho$ is a $C_G(\rho(\Gamma))$-equivariant.

We are going to prove that $\Psi_\rho$ is an isomorphism by constructing its inverse. An easy calculation shows that for every $\sigma \in Z^1(\Gamma, Ad \rho)$,

$$\gamma \to (I + \sigma(\gamma)\varepsilon) \cdot \rho(\gamma)$$
is a group homomorphism from $\Gamma$ to $G(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ (c.f. [LM] Prop. 2.2 for $G = GL(n, \mathbb{C})$). Therefore, $\sigma$ defines a homomorphism $\Phi_\rho(\sigma) : R(\Gamma, G) \to \mathbb{C}[\varepsilon]/(\varepsilon^2)$ such that $\pi\Phi_\rho(\sigma) = r_\rho$. Hence, $\Phi_\rho(\sigma) \in T_\rho Spec R(\Gamma, G)$. In other words, we have defined the map

$$\Phi_\rho : Z^1(\Gamma, Ad \rho) \to T_\rho Spec R(\Gamma, G).$$

A straightforward computation shows (c.f. [LM] Lemma 2.2 and [LM] Prop 2.2 for $G = GL(n, \mathbb{C})$):

**Theorem 34.** $\Psi_\rho$ and $\Phi_\rho$ are inverses of each other, and therefore, they are $C_G(\rho(\Gamma))$-equivariant isomorphisms between $Z^1(\Gamma, Ad \rho)$ and $T_\rho Spec R(\Gamma, G)$. 
9. Reduced Representations

A closed point \( x \) of an algebraic scheme \( X \) is reduced if the local ring \( O_{X,x} \) has no non-zero nilpotent elements. By definition, reduced points form an open subset of \( X \).

A closed point \( x \) of an affine algebraic set or of an algebraic scheme \( X \) is simple if \( \dim T_x X \) coincides with the largest dimension of an irreducible component of \( X \) containing \( x \). (Simple points are also called non-singular.) Simple points form a complex manifold which is an open subset of \( X \). Every simple point is reduced.

We say that \( \rho : \Gamma \to G \) is reduced (respectively: simple) if \( \rho \) is a reduced (respectively: simple) point of \( \text{Spec} \, R(\Gamma, G) \).

**Corollary 35.** The set \( \text{Hom}^r(\Gamma, G) \) of reduced representations is an open subset of \( \text{Hom}(\Gamma, G) \).

**Proof.** \( \text{Hom}^r(\Gamma, G) \) is a preimage of the open set of reduced points in \( \text{Spec} \, R(\Gamma, G) \) under the map \( \text{Hom}(\Gamma, G) = \text{Spec} \, R(\Gamma, G)/\sqrt{0} \to \text{Spec} \, R(\Gamma, G) \). \( \square \)

For example all \( G \)-representations of a free group are reduced, since \( R(F_n, G) \) is the coordinate ring of the \( n \)-th cartesian power of \( G \), which is a smooth algebraic set.

**Proposition 36.** For every reductive group \( G \) and every closed orientable surface \( F \) of genus \( \geq 2 \), all irreducible representations \( \rho : \pi_1(F) \to G \) are simple and, hence, reduced.

**Proof.** By Proposition [35], the centralizer of \( \rho(\pi_1(F)) \) is a finite extension of the center of \( G \). Hence, by [Go1, Prop. 1.2],

\[
\dim Z^1(\pi_1(F), \text{Ad} \rho) = (2g-1)\dim G + \dim C(G).
\]

By Theorem [34],

\[
(10) \quad \dim T_{\rho} \text{Hom}(\pi_1(F), G) \leq \dim T_{\rho} \text{Spec} \, R(\pi_1(F), G) = \dim Z^1(\pi_1(F), \text{Ad} \rho) = (2g-1)\dim G + \dim C(G).
\]

(1) Assume first (for simplicity) that \( G \) is semi-simple. Then \( \dim C(G) = 0 \). Since \( \pi_1(F) \) has a presentation with \( 2g \) generators and one relation,

\[
\dim C \geq (2g-1)\dim G
\]

for all irreducible components \( C \subset \text{Hom}(\Gamma, G) \). Therefore, all quantities in (10) are equal, implying that \( \rho \) is a simple point of \( \text{Spec} \, R(\Gamma, G) \).

(2) For an arbitrary reductive group \( G \) consider epimorphism [11],

\[
\nu : C^0(G) \times [G, G] \to G.
\]

Since it has a finite kernel, the induced map

\[
\text{Hom}(\pi_1(F), C^0(G)) \times \text{Hom}(\pi_1(F), [G, G]) \to \text{Hom}(\pi_1(F), G)
\]

is finite. Since \( \text{Hom}(\pi_1(F), C^0(G)) = (C^0(G))^{2g} \) and, by (1), \( \text{Hom}(\pi_1(F), [G, G]) \) is composed of irreducible components of dimension at least \( (2g-1)\dim [G, G] \), the set \( \text{Hom}(\pi_1(F), G) \) is composed of irreducible components of dimension at least

\[
2g \cdot \dim C^0(G) + (2g-1) \dim [G, G] = (2g-1)\dim G + \dim C^0(G).
\]

Therefore, both sides of (10) are equal implying that \( \rho \) is a simple point of \( \text{Spec} \, R(\Gamma, G) \). \( \square \)
Corollary 37. For every reductive group \( G \) and every closed orientable surface \( F \) of genus \( \geq 2 \), every irreducible representation belongs to a unique irreducible component of \( \text{Hom}(\Gamma, G) \).

Proof. By \[\text{§II \§2 Thm 6}\] every simple point belongs to a unique irreducible component. \(\square\)

Proposition 36 does not hold for non-surface groups. In fact, there appears to be no easy characterization of simple points of \( \text{Hom}(\Gamma, G) \) in general.

Example 38. Let \( \rho_1 : \mathbb{Z}^2 \to SL(2, \mathbb{C}) \) be the trivial representation and let \( \rho_2 : \Gamma \to SL(2, \mathbb{C}) \) be an irreducible representation. These representations define a representation \( \rho_1 * \rho_2 : \mathbb{Z}^2 * \Gamma \to SL(2, \mathbb{C}) \) which is irreducible. On the other hand, the Jacobian matrix \( \partial \tau(x)/\partial x_j \) of the five relations in Example 24 has rank 2 at \( \rho_1 = (x_1, \ldots, x_8) = (1, 0, 0, 1, 1, 0, 0, 1) \). Therefore, \( \text{Hom}(\mathbb{Z}^2, SL(2, \mathbb{C})) \) is singular at \( \rho_1 \) and, by \[3\], \( \rho_1 * \rho_2 \) is a singular point of \( \text{Hom}(\mathbb{Z}^2 \ast \Gamma, G) \).

10. Orbits

As before, let \( O_{\rho} \) be the orbit of \( \rho \) in \( \text{Hom}(\Gamma, G) \) under the \( G \) action by conjugation. Since \( O_{\rho} \) is homogeneous and it has a simple point\(^1\), all its points are simple, i.e. \( O_{\rho} \) is a smooth algebraic set.

The map

\[
\text{Hom}(\Gamma, G) = \text{Spec } R(\Gamma, G)/\sqrt{0} \to \text{Spec } R(\Gamma, G)
\]

induces an embedding \( T_\rho \text{Hom}(\Gamma, G) \subset T_\rho \text{Spec } R(\Gamma, G) \).

The following theorem generalizes \[\text{§II \ Lemma 2.2} \] and \[\text{LM \ Cor 2.4}\].

Theorem 39. For every \( \rho \) the inclusion

\[
T_\rho O_{\rho} \subset T_\rho \text{Hom}(\Gamma, G) \subset T_\rho \text{Spec } R(\Gamma, G)
\]

corresponds to

\[
B^1(\Gamma, \text{Ad } \rho) \cap T_\rho \text{Hom}(\Gamma, G) \subset Z^1(\Gamma, \text{Ad } \rho)
\]

under the isomorphism \( \Psi_{\rho} \).

Proof. Since \( O_{\rho} \) is the image of the map \( f_\rho : G \to \text{Hom}(\Gamma, G) \), \( f_\rho(\gamma) = g g^{-1} \), \( O_{\rho} \) is the left quotient of \( G \) by the stabilizer of \( \rho \), \( C_G(\rho(\Gamma)) \), c.f. \[\text{Bo \ II.6.1}\]. Since quotient maps are surjections, the differential

\[
d(f_\rho) : T_g G \to T_{g g^{-1}} O_{\rho}
\]
is an epimorphism.

Consider a closed embedding \( G \leftarrow GL(n, \mathbb{C}) \). Every 1-coboundary \( \tau \in B^1(\Gamma, \text{Ad } \rho) \) is a function \( \Gamma \to \mathfrak{g} \) of the form

\[
\tau(\gamma) = A - \text{Ad } \rho(\gamma) \cdot A = A - \rho(\gamma) A \rho(\gamma)^{-1},
\]
for some \( A \in \mathfrak{g} \). By the discussion in Section \[\text{§I \ } \Phi_{\rho}(\tau) \in T_\rho \text{Spec } R(\Gamma, G) \]
corresponds to a homomorphism \( R(\Gamma, G) \to \mathbb{C}[[\varepsilon]]/(\varepsilon^2) \) yielding a representation

\[
\Gamma \to G(\mathbb{C}[[\varepsilon]]/(\varepsilon^2)), \quad \gamma \to (I + \tau \varepsilon) \rho(\gamma)
\]

which by \[\text{II \ } \text{is} \]

\[
\gamma \to (I + (A - \rho(\gamma) A \rho(\gamma)^{-1}) \varepsilon) \rho(\gamma) = (I + A \varepsilon) \rho(\gamma)(I - A \varepsilon).
\]

\(^1\)Every algebraic set has a simple point.
If \( \tau \) can be written as

\[
(I + A\varepsilon)^{-1} = I - A\varepsilon \mod \varepsilon^2,
\]

representation (12) can be written as

\[
\gamma \mapsto (I + A\varepsilon)\rho(\gamma)(I + A\varepsilon)^{-1}.
\]

Since \( \tau \in T_\rho \operatorname{Hom}(\Gamma, G) \) then this representation does represent an element \( d\rho(v) \in T_\rho O_\rho \) where

\[
v = I + A\varepsilon \in T_1 G.
\]

Conversely, since every tangent vector to \( O_\rho \) at \( \rho \) is of the form (13), the statement follows.

Observe that the stabilizer of \( \rho, C_G(\rho(\Gamma)) \), acts on \( O_\rho \) by conjugation and, hence, it acts on \( T_\rho O_\rho \) as well. Furthermore, \( B^1(\Gamma, \operatorname{Ad} \rho) \cap T_\rho \operatorname{Hom}(\Gamma, G) \) is preserved by the \( C_G(\rho(\Gamma)) \)-action on \( Z^1(\Gamma, G) \). Since \( \Psi_\rho \) is \( C_G(\rho(\Gamma)) \)-equivariant, we conclude:

**Corollary 40.** The isomorphism \( T_\rho O_\rho \to B^1(\Gamma, \operatorname{Ad} \rho) \cap T_\rho \operatorname{Hom}(\Gamma, G) \) of Theorem 39 is \( C_G(\rho(\Gamma)) \)-equivariant.

11. Character Varieties

The categorical quotient of \( \operatorname{Hom}(\Gamma, G) \) by the \( G \) action by conjugation,

\[
X_G(\Gamma) = \operatorname{Hom}(\Gamma, G) // G,
\]

is called the \( G \)-character variety of \( \Gamma \). By definition, it is an affine algebraic set together with the map \( \operatorname{Hom}(\Gamma, G) \to \operatorname{Hom}(\Gamma, G) // G \) which is constant on all \( G \)-orbits, with the property that every morphism from \( \operatorname{Hom}(\Gamma, G) \) into an affine algebraic set \( Y \) which is constant on all \( G \)-orbits factors through \( \operatorname{Hom}(\Gamma, G) \to \operatorname{Hom}(\Gamma, G) // G \), c.f. [Do] [Fp] [MFK]. If \( G \) is reductive then the categorical quotient exists. The reason for considering the categorical quotient rather than the set theory quotient is that the quotient topology on \( \operatorname{Hom}(\Gamma, G)/G \) is not a Zariski topology of any algebraic set. For example, often it contains points which are not closed. Character varieties are often reducible, despite the term "variety" in their name.

Every equivalence class in \( X_G(\Gamma) \) contains a unique closed orbit. Therefore, by Proposition 29 each element of the \( G \)-character variety of \( \Gamma \) is represented by a uniquely completely reducible representation.

**Example 41.** Let \( T \) be a maximal torus of \( G \). The map \( T \to \operatorname{Hom}(\mathbb{Z}, G) \) assigning to \( g \) the \( G \)-representation of \( \mathbb{Z} \) sending 1 to \( g \) factors to an isomorphism

\[
T/W \to X_G(\mathbb{Z}),
\]

where \( W \) is the Weyl group of \( G \), c.f. [SL 6.4].

**Example 42.** (1) The \( \operatorname{SL}(2, \mathbb{C}) \)-character variety of the free group, \( F_2 \), on two generators is isomorphic to \( \mathbb{C}^3 \).

(2) \( \operatorname{SL}(3, \mathbb{C}) \)-character variety of the free group on two generators is a hypersurface in 9-dimensional affine space, [Lo1 Thm 8], [Si1].

Denote the set of irreducible representations in \( \operatorname{Hom}(\Gamma, G) \) by \( \operatorname{Hom}^i(\Gamma, G) \). The \( G \) action by conjugation preserves \( \operatorname{Hom}^i(\Gamma, G) \). Since all orbits in \( \operatorname{Hom}^i(\Gamma, G) \) are closed (c.f. Proposition 29) and each equivalence class in a categorical quotient contains a unique closed orbit, the categorical quotient \( \operatorname{Hom}^i(\Gamma, G) // G \) is the set-theoretic quotient. Denote \( \operatorname{Hom}^i(\Gamma, G) // G = \operatorname{Hom}^i(\Gamma, G)/G \) by \( X_G^i(\Gamma) \). From
now on, the quotient of the complex topology on $Hom^i(\Gamma, G)$ will be the default topology on $X^i_G(\Gamma)$.

**Proposition 43.** Let $G$ be a reductive group.

1. For the free group $F_n$, $X^i_G(F_n)$ is a complex (Hausdorff) orbifold of dimension $(n - 1)\dim G + \dim C(G)$.

2. For the closed orientable surface $S_g$ of genus $g > 1$, $X^i_G(\pi_1(S_g))$ is a complex orbifold of dimension $(2g - 2)\dim G + 2\dim C(G)$.

3. If $G$ is CL (e.g. $G = GL(n, \mathbb{C}), SL(n, \mathbb{C})$) then $X^i_G(F_n)$ and $X^i_G(\pi_1(S_g))$ are manifolds for all $n, g > 1$. (See also [FL2].)

**Proof.** (1) $X^i_G(F_n) = Hom^i(F_n, G)/G$. Since $Hom^i(F_n, G)$ is an open subset of $G^n$, it is smooth. By Proposition 14, $G/C(G)$ acts on $Hom^i(F_n, G)$ with a finite centralizers.

(2) By Proposition 32, all irreducible representations of $\pi_1(S_g)$ are reduced. Therefore, by Theorem [FL] and by [Go2, Prop. 1.2], $Hom^i(\pi_1(S_g))$ is a complex manifold of dimension $(2g - 1)\dim G + \dim C(G)$. Since $G/C(G)$ acts on $Hom^i(F_n, G)$ with a finite centralizers, the statement follows.

(3) By definition of a CL group, the $G/C(G)$ action on $X^i_G(F_n)$ is free. By [JM, Thm 1.1], the action is also properly discontinuous. Since the quotient of a manifold by a free properly discontinuous group action is also proper, [?], Proposition 3.5.7, the statement follows. \qed

Recall that a representation $\rho : \Gamma \to G$ is good if and only if it is irreducible and the centralizer of $\rho(\Gamma)$ coincides with the center of $G$. By Proposition 32 the space of good representations $Hom^g(\Gamma, G)$ is an open subset of the irreducible ones. Since $G/C(G)$ acts freely on $Hom^g(\Gamma, G)$, we conclude with:

**Corollary 44.** For every reductive group $G$ and every surface group or a free group $\Gamma$, $X^i_G(\Gamma) = Hom^g(\Gamma, G)/G$ is an open subset of $X^i_G(\Gamma)$ and a smooth complex manifold. 

For a topological space $Y$, we will abbreviate $X_G(\pi_1(Y))$ by $X_G(Y)$.

**Proposition 45.** (1) If $M$ is a compact manifold with connected boundary of genus $g$ then

$$\dim X_G(M) \geq \dim G \cdot (g - 1).$$

(2) For a given non-abelian reductive group $G$ and a positive integer $g$ there is no upper bound on $\dim X_G(M)$ over compact manifolds $M$ with connected boundary of fixed genus $g$.

**Proof.** (1) If $M$ is a compact manifold with connected boundary of genus $g$ then $\pi_1(M)$ has a presentation with $n$ generators and $p$ relations such that

$$1 - n + p = \chi(M) = 1 - g.$$ 

Hence $\dim Hom(\pi_1(M), G) \geq \dim G \cdot (n - p) = \dim G \cdot g$.

(2) It is enough to prove that there is no upper bound on $\dim Hom(\pi_1(M), G)$, over compact manifolds $M$ with connected boundary of fixed genus $g$. Since every non-abelian reductive group contains either $SL(2, \mathbb{C})$ or $PSL(2, \mathbb{C})$, $\dim Hom(\pi_1(M), G) \geq$
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The inequality stems from the fact that this map does not have to be onto.) Therefore, it is enough to prove that there is no upper bound on \( \dim \text{Hom}(\pi_1(M), SL(2, \mathbb{C})) \).

Let \( K_n \) be the connected sum of \( n \) copies of a knot \( K \). Cooper and Long, \( \text{CL} \), proved that \( \dim \text{Hom}(\pi_1(S^3 \setminus K_n), SL(2, \mathbb{C})) \geq n + 3 \). (Although their argument is made for hyperbolic knots \( K \) only, it generalizes to all knots by the result of \( \text{KrM} \), c.f. \( \text{DC} \).) Let \( K_{n,g} \) be a graph obtained by connecting \( g \) unlinked copies of \( K_n \) in \( S^3 \) by \( g-1 \) tunnels and let \( M_{n,g} \) be the complement of an open tabular neighborhood of \( K_{n,g} \) in \( S^3 \). Then \( \pi_1(M_{n,g}) \) is the free product of \( g \) copies of \( \pi_1(S^3 \setminus K_n) \) and

\[
\dim \text{Hom}(\pi_1(M_{n,g}), SL(2, \mathbb{C})) \geq g \cdot (n + 3).
\]

Since for every \( n \), \( \partial M_{n,g} \) is a surface of genus \( g \), the statement follows. \( \square \)

In general, character varieties are very difficult to describe by explicit algebraic equations. For further information on character varieties, we refer the reader to [AP, BC, BH, BK1, BK2, BL1, BL2, CM, JM, H0, Lo3, Lo5, LM, LP, Na, PBK, Sa, S11, SL]. Character varieties of surface groups will be discussed in Section 14. See also [BKCh, Go1, Go2, Go3, Go4, Go5, Go6, Go7, Go8, Go9, Je, Li, LR1, LR2, PX, RBK, RBC, SS] and other papers of these authors. Applications of character varieties to low-dimensional topology and geometry are discussed in [BF, BB, BLZ, BN, BZ1, BZ2, BZ3, Bu, CCGLS, CS, CL, Cu, DDW, Du, FGL, Ga, Ge, GM1, GM2, Gu, HP1, HP2, HLM1, HLM2, HS, JM, ?, KM, Le1, Le2, LR1, LR2, MS, Mo, PS, Ra, S11, S12, T] and in other papers of these authors.

12. Tangent spaces to character varieties

For any \( \rho : \Gamma \to G \), the \( C_G(\rho(\Gamma)) \) action on \( H^1(\Gamma, G) \) descends to an action on \( H^1(\Gamma, G) \).

Theorem 46. For every reductive \( G \) and every completely reducible \( \rho \) there is a natural linear map

\[
\phi : T_{[\rho]} X_G(\Gamma) \to T_0 \left( H^1(\Gamma, \text{Ad } \rho) / \text{C}_G(\rho(\Gamma)) \right).
\]

If \( \rho \) is reduced then \( \phi \) is an isomorphism.

S. Lawton pointed out to us that this statement for \( G = PSL(2, \mathbb{C}) \) appears in [HP2 Prop. 5.2]. (Although the assumption on \( \rho \) being reduced does not appear explicitly in their assumptions, one can guess it from the context.)

A version of this result stating that

\[
T_{[\rho]} X_G(\Gamma) = H^1(\Gamma, \text{Ad } \rho)
\]

belongs to folk knowledge, although it is often used without proper assumptions – in particular the requirement of \( \rho \) being reduced and \( C_G(\rho(\Gamma)) \) acting trivially on \( H^1(\Gamma, \text{Ad } \rho) \).

Proof of Theorem 46. By Luna’s étale slice theorem, \( \text{Lu} \), c.f. \( \text{MFK} \), \( \text{PV} \) Thm 6.1, there exists an affine subset (the étale slice) \( S \) of \( Hom(\Gamma, G) \) containing \( \rho \) and an excellent \( G \)-equivariant morphism

\[
G \times C_G(\rho(\Gamma)) S \to Hom(\Gamma, G).
\]
By definition of being excellent, the induced quotient map $S//C_G(\rho(\Gamma)) \to \text{Hom}(\Gamma, G)//G$ is étale and, therefore, it induces an isomorphism

$$T_\rho \text{Hom}(\Gamma, G)//G \to T_\rho (S//C_G(\rho(\Gamma))).$$

Now, by [PV, Thm 6.4], $T_\rho (S//C_G(\rho(\Gamma))) = T_0 (T_\rho S//C_G(\rho(\Gamma)))$. Since $T_\rho S$ is an $C_G(\rho(\Gamma))$-equivariant complement of $T_\rho O_\rho$ in $T_\rho \text{Hom}(\Gamma, G)$,

$$T_\rho S = T_\rho \text{Hom}(\Gamma, G)/T_\rho O_\rho,$$

as $C_G(\rho(\Gamma))$-modules. By Theorems 33 and 39, the embedding

$$T_\rho \text{Hom}(\Gamma, G) \hookrightarrow Z^1(\Gamma, G)$$

factors to

$$T_\rho \text{Hom}(\Gamma, G)/T_\rho O_\rho \to Z^1(\Gamma, G)/B^1(\Gamma, G) = H^1(\Gamma, G).$$

If $\rho$ is reduced, then this map is an isomorphism.

13. Character Varieties as Algebraic Schemes

For every reductive $G$ and every finitely generated $\Gamma$, the invariant part of $R(\Gamma, G)$ under the $G$ action defines an algebraic scheme $X_G(\Gamma) = \text{Spec } R(\Gamma, G)^G$ which is a scheme “sibling” of $X_G(\Gamma)$. It is often also called the $G$-character variety of $\Gamma$. By the definition of the categorical quotient, $\mathbb{C}[X_G(\Gamma)] = \mathbb{C}[\text{Hom}(\Gamma, G)]^G$. The epimorphism $R(\Gamma, G) \to \mathbb{C}[\text{Hom}(\Gamma, G)]$ induces the epimorphism $R(\Gamma, G)^G \to \mathbb{C}[\text{Hom}(\Gamma, G)]^G$ and

$$R(\Gamma, G)^G/\sqrt{0} = \mathbb{C}[\text{Hom}(\Gamma, G)]^G = \mathbb{C}[X_G(\Gamma)].$$

In other words, there is a natural bijection between the closed points of $X_G(\Gamma)$ and points of $X_G(\Gamma)$.

In [SH], we have described $R(\Gamma, SL(n, \mathbb{C}))^{SL(n, \mathbb{C})}$ as a space of $n$-valent graphs reminiscent of Feynman diagrams in an arbitrary path connected topological space $X$ with $\pi_1(X) = \Gamma$.

Kapovich and Millson proved that for every affine (possibly unreduced) variety $X$ over $\mathbb{Q}$ there is an Artin group $\Gamma$ such that a Zariski open subset of $X_{PSL(2, \mathbb{C})}(\Gamma)$ is isomorphic to a Zariski open subset of $X$, [KM]. Additionally, for every $x \in X$ there is a representation $\rho$ of an Artin group $\Gamma$ into $PSL(2, \mathbb{C})$ such that the analytic germ of $X_{PSL(2, \mathbb{C})}(\Gamma)$ at $[\rho]$ coincides with the analytic germ of $X$ at $x$, [KM].

Kapovich proved that the same is true for 3-manifold groups. That is, for every $x \in X$ as above there is a closed 3-manifold $M$ and a representation $\rho : \Gamma = \pi_1(M) \to PSL(2, \mathbb{C})$ such that the analytic germ of $X_{PSL(2, \mathbb{C})}(\Gamma)$ at $[\rho]$ coincides with the analytic germ of $X$ at $x$, [Ka1] [Ka2]. In particular $X_{PSL(2, \mathbb{C})}(\Gamma)$ contains non-zero nilpotent elements for some Artin groups and some 3-manifold groups $\Gamma$.

**Question 47.** Under what conditions on $\rho : \Gamma \to G$

$$T_{[\rho]}X_G(\Gamma) = T_0 (H^1(\Gamma, \text{Ad } \rho)//C_G(\rho(\Gamma))) ?$$
14. Simplicity of Character Varieties of Surfaces

Let $G$ be a reductive group and let $\mathfrak{g}$ be its Lie algebra. A bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is $Ad$-invariant if $B(Ad(g)x, Ad(g)y) = B(x, y)$.

Let $F$ be a closed orientable surface. For every representation $\rho : \pi_1(F) \to G$ and every $Ad$-invariant bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, the cup product defines a pairing

$$(14) \quad \omega_B : H^1(F, \text{Ad} \rho) \times H^1(F, \text{Ad} \rho) \xrightarrow{\cup} H^2(F, \text{Ad} \rho \otimes \text{Ad} \rho) \xrightarrow{B} H^2(F, \mathbb{C}) = \mathbb{C}$$

which can be also identified with the pairing

$$(15) \quad H^1(F, \text{Ad} \rho) \times H_1(F, \text{Ad} \rho) \xrightarrow{\cap} H_0(F, \text{Ad} \rho \otimes \text{Ad} \rho) \xrightarrow{B} H_0(F, \mathbb{C}) = \mathbb{C}$$

via the Poincare duality with twisted coefficients, $[\mathbb{F}]$, $\cap [F] : H^n(F, \text{Ad} \rho) \to H_{2-n}(F, \text{Ad} \rho)$,

where $[F] \in H_2(F, \mathbb{C})$ is a fundamental class of $F$.

**Lemma 48.** (1) Let $\Gamma$ be a group and let $(C_*, \partial)$ be a chain complex of left $\mathbb{Z}\Gamma$-modules. Let $M_1, M_2$ be left $\mathbb{Z}\Gamma$-modules. If $B : M_1 \times M_2 \to \mathbb{C}$ is a $\mathbb{Z}\Gamma$-invariant pairing, i.e. $B(rm_1, rm_2) = B(m_1, m_2)$ for every $r \in \mathbb{Z}\Gamma$, $m_1 \in M_1$, $m_2 \in M_2$, then the cap product induces a pairing

$$(16) \quad H^n(\text{Hom}_{\mathbb{Z}\Gamma}(C_*, M_2), \partial) \times H_n(M_1 \otimes_{\mathbb{Z}\Gamma} C_*, \partial) \to M_1 \otimes_{\mathbb{Z}\Gamma} M_2 \xrightarrow{B} \mathbb{C}.$$  

In the above formula $M_1$ is considered as a right $\mathbb{Z}\Gamma$-module via $m \cdot \gamma = \gamma^{-1} \cdot m$.

(2) If $B$ is a duality pairing, i.e. if $B$ induces an isomorphism $M_1 \simeq \text{Hom}(M_2, \mathbb{C})$, then the pairing $(16)$ is non-degenerate.

**Proof.** (1) is well known, c.f. [Br] V §3].

(2) The cochain complex $(\text{Hom}_{\mathbb{Z}\Gamma}(C_*, M_2), \partial)$ can be written as

$$(\text{Hom}_{\mathbb{Z}\Gamma}(C_*, \text{Hom}(M_1, \mathbb{C})), \partial) = (\text{Hom}_{\mathbb{Z}\Gamma}(C_*, \mathbb{C} \otimes M_1, \mathbb{C}), \partial) =$$

$$(\text{Hom}(C_* \otimes_{\mathbb{Z}\Gamma} M_1, \mathbb{C}), \partial) = \text{Hom}(C_* \otimes_{\mathbb{Z}\Gamma} M_1, \partial, \mathbb{C}).$$

Hence

$$(17) \quad H_n(\text{Hom}_{\mathbb{Z}\Gamma}(C_*, M_2), \partial) = H_n(\text{Hom}(C_* \otimes_{\mathbb{Z}\Gamma} M_1, \partial, \mathbb{C})).$$

Since $\mathbb{C}$ is a divisible group, $\text{Hom}(\cdot, \mathbb{C})$ is an exact functor in the category of abelian groups. Hence, $(17)$ becomes

$$H_n(\text{Hom}_{\mathbb{Z}\Gamma}(C_*, M_2), \partial) = \text{Hom}(H_n(C_* \otimes_{\mathbb{Z}\Gamma} M_1, \partial, \mathbb{C})).$$

It is easy to verify that this isomorphism is induced by $(16)$. \hfill $\Box$

If $B$ is symmetric then $(14)$ is skew-symmetric. Therefore, Lemma 48 implies:

**Corollary 49.** (c.f. [Go]) If $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is symmetric, $Ad$-invariant, and non-degenerate, then $(14)$ is a symplectic form on $H^1(F, \text{Ad} \rho)$.

If $\mathfrak{g}$ is simple then the Killing form is unique among symmetric, $G$-invariant, non-degenerate forms on $\mathfrak{g}$, up to a constant multiple.

Let $F$ be a closed orientable surface of genus $\geq 2$ now. By Corollary 44, the space of $G$-conjugacy classes of good representations,

$$X^\circ_G(F) = \text{Hom}^g(\pi_1(F), G)/G = \text{Hom}^g(\pi_1(F), G)/G$$
is a complex manifold and, by Theorem 16
\[ T_{\pi_1} X^g_G(F) = H^1(\pi_1(F), \text{Ad} \rho). \]

Remark 50. \( \omega_B \) is an “algebraic” form on \( X^g_G(F) \), i.e. it is a global section of the second exterior power of the vector bundle of Kähler differentials on \( X^g_G(F) \). In particular, \( \omega_B \) is holomorphic.

Goldman proves by an argument from gauge theory that for every non-degenerate, symmetric, \( \text{Ad} \)-invariant \( B \), \( \omega_B \) is closed, [Go1]. Therefore, \( (X^g_G(F), \omega_B) \) is a holomorphic symplectic manifold.

15. 3-MANIFOLDS AND LAGRANGIAN SUBSPACES

Let \( M \) be an orientable compact 3-manifold with a connected boundary \( F \). The embedding \( \partial M \hookrightarrow M \) induces a homomorphism \( r : \pi_1(F) \to \pi_1(M) \) and a map \( r_* : X_G(M) \to X_G(F) \). Let \( Y_G(M) \) be the smooth part of \( r_*(X_G(M)) \) in \( X^g_G(F) \).

Our goal is to prove the following two theorems:

Theorem 51. (1) \( Y_G(M) \) is an isotropic submanifold of \( X^g_G(F) \) with respect to \( \omega_B \) for every symmetric non-degenerate bilinear \( \text{Ad} \)-invariant form \( B \) on \( \mathfrak{g} \).

(In particular, every connected component of \( Y_G(M) \) of dimension \( \frac{1}{2} \dim X_G(F) \) is Lagrangian.)

(2) If a connected component of \( Y_G(M) \) contains an equivalence class of a reduced \( G \)-representation of \( \pi_1(M) \) then it is a Lagrangian submanifold of \( Y_G(M) \).

Theorem 52. \( X^g_G(F) \cap r_*(X^g_G(M)) \) is an immersed Lagrangian submanifold of \( X^g_G(F) \).

Note that we do not claim that \( X^g_G(M) \cap r_*^{-1}(X^g_G(F)) \) is a manifold. Furthermore, even if it is, \( r_* \) does not have to be an immersion by Proposition 45(2).

Theorems 51 and 52 have important applications to Chern-Simons theory, [Fr], [Ba], as well as to quantum topology, c.f. for example [Gu, JW, We1, We2, Si3]. (In the scheme of geometric quantization, one associates Hilbert spaces \( H \) to symplectic manifolds \( X \) and vectors in \( H \) to Lagrangian subspaces of \( X \).)

C. Curtis defined an analog of the Casson-Walker 3-manifold invariant for \( SL(2, \mathbb{C}) \), [Cu]. Her construction is based on a Heegaard splitting of a 3-manifold into two handlebodies \( H_1 \) and \( H_2 \) and on counting the intersection points of the Lagrangian subspaces \( r_*X_{SL(2,\mathbb{C})}(H_1) \) and \( r_*X_{SL(2,\mathbb{C})}(H_2) \) inside \( X_G(F) \), where \( F = \partial H_1 = \partial H_2 \).

The above results suggest a possible generalization of her work to arbitrary splittings of closed 3-manifolds \( M = M_1 \cup_F M_2 \) along surfaces \( F \) of genus \( \geq 2 \), c.f. [BC].

Furthermore, if \( Y_G(M_1), Y_G(M_2) \) are Lagrangian submanifolds of \( X_G(F) \) then one may be tempted to build an algebraic version of Floer symplectic homology theory for such submanifolds.

For every representation \( \rho : \pi_1(M) \to G \) the homomorphism \( r : \pi_1(F) \to \pi_1(M) \) induces \( r^* : H^1(M, \text{Ad} \rho) \to H^1(F, \text{Ad} \rho r) \). The proofs of Theorems 51 and 52 are based on the following:

Theorem 53. For every \( \rho : \pi_1(M) \to G \), \( r^*H^1(M, \text{Ad} \rho) \) is a Lagrangian subspace of the symplectic space \( (H^1(F, \text{Ad} \rho r), \omega_B) \) with respect to every non-degenerate, \( \text{Ad} \)-invariant, symmetric, bilinear form \( B \) on \( \mathfrak{g} \).
In particular, for the trivial representation $\rho : \pi_1(M) \to \mathbb{C}^*$, Theorem 53 implies the following classical result:

**Corollary 54.** For every compact, orientable 3-manifold with a connected boundary $F$ the image of the map $r^* : H^1(M, \mathbb{C}) \to H^1(F, \mathbb{C})$ induced by the embedding $r : F \hookrightarrow M$ is a Lagrangian subspace of $H^1(F, \mathbb{C})$ with the symplectic form being the cup product.

**Proof of Theorem 53.** (1) We prove that
\[ \dim r^* H^1(M, Ad \rho) = \frac{1}{2} \dim H^1(F, Ad \rho r) \]
fist, by filling in the details of the approach of [17]. (This approach was communicated to us by Charlie Frohman.) By Poincare-Lefschetz duality we have
\[
\begin{align*}
H_2(M, F, Ad \rho) & \xrightarrow{\partial} H_1(F, Ad \rho r) \xrightarrow{r^*} H_1(M, Ad \rho) \\
H^1(M, Ad \rho) & \xrightarrow{r^*} H^1(F, Ad \rho r) \xrightarrow{\delta} H^2(M, F, Ad \rho),
\end{align*}
\]
where all vertical maps are isomorphisms induced by Poincare duality. By Corollary 49 the cap product
\[ H^1(F, Ad \rho r) \times H_1(F, Ad \rho r) \xrightarrow{\cap} H_0(F, Ad \rho r \otimes Ad \rho r) \xrightarrow{B} H_0(F, \mathbb{C}) = \mathbb{C}, \]
is non-degenerate. Similarly,
\[
\begin{align*}
H^1(M, Ad \rho) \times H_1(M, Ad \rho) & \xrightarrow{\cap} H_0(M, Ad \rho \otimes Ad \rho) \xrightarrow{B} \mathbb{C}, \\
H^2(M, F, Ad \rho) \times H_2(M, F, Ad \rho) & \xrightarrow{\cap} H_0(M, F, Ad \rho \otimes Ad \rho) \xrightarrow{B} \mathbb{C}
\end{align*}
\]
are non-degenerate by Lemma 48 (2). Consider the isomorphisms
\[ H^1(M, Ad \rho) \cong (H_1(M, Ad \rho))^*, \quad H^1(F, Ad \rho r) \cong (H_1(F, Ad \rho r))^*, \]
\[ H^2(M, F, Ad \rho) \cong (H_2(M, F, Ad \rho))^* \]
defined by these pairings. Under these identifications, $r_*$ and $r^*$ and $\partial$ and $\delta$ are the duals of each other. Hence
\[ \text{rank } r^* = \text{rank } r_* = \text{rank } \delta = \dim H^1(F, Ad \rho r) - \dim Ker \delta \]
\[ = \dim H^1(F, Ad \rho r) - \dim r^*. \]

(2) It remains to be proven that $r^* H^1(M, Ad \rho)$ is an isotropic subspace of $H^1(F, Ad \rho)$.

The pairing \[18\] identifies $H_1(F, Ad \rho r)$ with $H^1(F, Ad \rho r)^*$. The isomorphism $\eta^{-1}$ of \[18\] sends $\alpha \in H^1(F, Ad \rho r)$ to $\eta^{-1}(\alpha) \in H_1(F, Ad \rho r)$ which under the above identification is the functional $f_\alpha : H^1(F, Ad \rho r) \to \mathbb{C}$, $f_\alpha(\beta) = \omega_B(\alpha, \beta)$.

By Lemma 48 the pairing
\[ H^1(M, Ad \rho) \times H_1(M, Ad \rho) \xrightarrow{\cap} H_0(M, Ad \rho \otimes Ad \rho) \xrightarrow{B} H_0(M, \mathbb{C}) = \mathbb{C}, \]
is non-degenerate. If we use it to identify $H_1(M, Ad \rho)$ with $H^1(M, Ad \rho)^*$ then $r_*$ in \[18\] sends $f_\alpha$ to $f_{r_* \alpha} : H^1(M, Ad \rho) \to \mathbb{C}$. By commutativity and exactness of \[18\], $f_{r_* \alpha} = 0$ for every $\alpha \in r^*(H^1(M, Ad \rho))$. In other words, $f_\alpha(\beta) = 0$ for every $\alpha, \beta \in r^*(H^1(M, Ad \rho))$. \qed
Proposition 55. If $G$ is reductive and $\rho : \pi_1(M) \to G$ is such that $\rho r : \pi_1(F) \to G$ is good, then the following diagram commutes:

$$
\begin{array}{ccc}
T_{[\rho]}X_G(M) & \xrightarrow{dr^*} & T_{[\rho r]}X_G(F) \\
\downarrow \phi & & \downarrow \phi \\
H^1(M, \text{Ad } \rho) & \xrightarrow{r^*} & H^1(F, \text{Ad } \rho r),
\end{array}
$$

where $\phi$ is the morphism of Theorem 14 considered for $\Gamma = \pi_1(M)$ on the left and for $\Gamma = \pi_1(F)$ on the right.

**Proof.** Since $\rho r$ is good also $\rho$ is good. Hence

$$
T_0(H^1(\pi_1(M), \text{Ad } \rho)/\langle C_G(\rho(\Gamma)) \rangle) = H^1(\pi_1(M), \text{Ad } \rho)
$$

and

$$
T_0(H^1(\pi_1(F), \text{Ad } \rho r)/\langle C_G(\rho r \pi_1(F)) \rangle) = H^1(\pi_1(F), \text{Ad } \rho r).
$$

Now the statement follows directly from the relevant definitions. \qed

**Proof of Theorem 51:** (1) is a direct consequence of Theorems 46, 53 and Proposition 55.

(2) Let $\rho : \pi_1(M) \to G$ be a reduced, irreducible representation who conjugacy class belongs to $Y_G(M)$. By Theorem 14 both $\phi$'s in the diagram of Proposition 55 are isomorphisms. By Theorem 53 $\dim T_{[\rho]} C = \frac{1}{2} \dim X_G(F)$. Now the statement follows from (1). \qed

**Proof of Theorem 52** By Proposition 32, $X^\rho_G(F) \subset X_G(F)$ is open and, hence, $U = r_*^{-1}X^\rho_G(F)$ is an open subset of $X_G(M)$ and of $X^\rho_G(M)$. By Theorem 46 and Proposition 55,

$$
r_* : T_{[\rho]} U \to T_{[\rho]} X^\rho_G(F)
$$

has constant rank for all $[\rho]$ in $U$. By rank theorem for algebraic sets, [BCR, Thm 9.6.1], every $[\rho] \in U$ has a neighborhood $V$ such that $r_* (V)$ a submanifold of $X^\rho_G(F)$. Consequently, $r_*(U) = r_* (X_G(M)) \cap X^\rho_G(F)$ is an immersed submanifold of $X^\rho_G(F)$. \qed

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