The strong field gravitational lensing in the Schwarzschild black hole pierced by a cosmic string

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Abstract

In this work the gravitational lensing in the strong field limit around the Schwarzschild black hole pierced by a cosmic string is studied. We find the angular position and magnification of the relativistic images depending on the tension of cosmic string. It is interesting that the angular separation s increases and the relative magnification r decreases when the tension of string is greater. It is also interesting that the deflection angle is greater when the tension of cosmic string is stronger.

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1. Introduction

As an important application of general relativity the gravitational lensing is on the deflection of electromagnetic radiation in a gravitational field. Recently more efforts have been contributed to the gravitational lensing and the issue has been developed greatly [1-3]. In fact, it is difficult to investigate the relations between the deflection angle and the properties of gravitational source because the deflection angle of light passing close to a compact and massive source is shown in integral forms. The integral expressions can be dealt with in the limiting cases such as weak field approximation and strong field limit. When the light goes very close to a heavy compact body like a black hole, an infinite series of images generate, which provide more information about the nature of the black hole’s surrounding. In these cases the weak field approximation is not valid and the strong gravitational lensing can help us to explore the characteristics of the gravitational source. More topics within the region of the strong field have been discussed. We can list that the strong gravitational lensing was treated in a Schwarzschild black hole [4, 5], gravitational source with naked singularities [6], a Reissner-Nordström black hole [7], a GMGHS charged black hole [8], a spinning black hole [9, 10], a braneworld black hole [11, 12], an Einstein-Born-Infeld black hole [13], a black hole in Brans-Dicke theory [14], a black hole with Barriola-Vilenkin monopole [15, 16] and the deformed Horava-Lifshitz black hole [17], etc..

As a kind of topological defects formed during the phase transition in the early universe due to the Kibble mechanism, the cosmic strings have been a focus for many years [18]. According to the inflationary models resulting form string theory the cosmic string networks which consist of long strings and string loops appeared at the end of inflation [19, 20]. The evolution of cosmic string loops have been studied in various backgrounds [21-23]. The cosmic string networks evolve then the cosmic strings interacted with the other gravitational source or each other. The Schwarzschild black holes pierced by cosmic strings could have formed in the process of early phase transition and survive. It was shown that the Schwarzschild black hole pierced by a cosmic string with full $U(1)$ Abelian-Higgs model has long-range "hair", excluding the no hair conjecture [24, 25]. A cosmic string piercing a Schwarzschild black hole has also been considered in the limit of thin string and the corrections were brought about in the Schwarzschild solution [26]. In the new metric the angular variable $\phi$ is replaced by $\gamma \phi$, where the parameter $\gamma$ is associated with the deficit angle by $\Delta = 2\pi(1 - \gamma)$ due to the cosmic string. In order to explore the properties of the spacetime, the geodesic equation near this kind of gravitational source is investigated carefully [27].

It is interesting to probe the fancy massive source in different directions. The main purpose of this work is to investigate the gravitational lensing on the Schwarzschild black hole pierced by a cosmic string. We plan to study the deflection angle of electromagnetic radiation passing very close to this kind of massive source with cosmic string to show the angle deviation resulting from the cosmic string by means of Bozza’s device put forward in Ref. [28]. We plot the dependence of the observational gravitational lensing coefficients on the parameter subject to the cosmic string going through the black hole. Our results will be listed in the end.
2. The deflection angle of the Schwarzschild black hole pierced by a cosmic string

The explicit form of the metric that describing a Schwarzschild black hole pierced by a cosmic string is [26],

\[ ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \gamma^2 \sin^2 \theta d\varphi^2) \]  (1)

where

\[ A(r) = B^{-1}(r) = 1 - \frac{2GM}{r} \]  (2)

\[ \gamma = 1 - 4G\mu \]  (3)

and \( G \) is Newton’s constant. \( \mu \) is the tension of string.

Here we choose that both the observer and the gravitational source lie in the equatorial plane with condition \( \theta = \frac{\pi}{2} \) according to the methods of [1, 28]. The whole trajectory of the photon is subject to the same plane that is perpendicular to the long cosmic string. On the equatorial plane the metric reads,

\[ ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)d\varphi^2 \]  (4)

where

\[ C(r) = \gamma^2 r^2 \]  (5)

The deflection angle for the electromagnetic radiation emitted from the distant place depends on the closest distance between the ray and the source and can be expressed as,

\[ \alpha(r_0) = I(r_0) - \pi \]  (6)

where

\[ I(r_0) = 2 \int_{r_0}^{\infty} \frac{\sqrt{B(r)}}{\sqrt{C(r)} \sqrt{C(r)A(r_0) - C(r)A(r) - 1}} \, dr \]  (7)

and here \( r_0 \) represents the minimum distance from the photon trajectory to the gravitational source. If the denominator of expression (7) is equal to the zero, the deflection angle will be divergent, then the photons will move around the source forever instead of leaving. The null denominator in the integral expression leads the equation

\[ \frac{C'(r)}{C(r)} = \frac{A'(r)}{A(r)} \]  whose largest root is named as the radius of the photon sphere. Having solved the equation, we obtain the radius of the photon sphere in the metric of Schwarzschild black hole pierced by a long cosmic string (4) as follow,

\[ r_m = 3GM \]  (8)
It is interesting that the photon sphere radius is the same as that of Schwarzschild metric without cosmic string. The cosmic string does not change the event horizon either. Of course the radius of photon sphere is larger than the black hole radius. It is clear that the cosmic string does not modify the photon sphere radius and the event horizon within the plane that the string is perpendicular to. With $C(r) = \gamma^2 r^2$, we re-derive to find that the following expressions consisting of line elements of metric describe the deflection angle and coefficients in the strong field limit and have the same form as Bozza’s in Ref. [28]. We can make use of Bozza’s results and just let $C(r) = \gamma^2 r^2$ to explore our topic. Further we wonder how the parameter $\gamma$ produced by cosmic string brings about the influence on the gravitational lensing. According to what mentioned above, we follow the procedure by Bozza in Ref. [28] to express the deflection angle in the strongfield limit as follow,

$$\alpha(\theta) = -\tilde{a} \ln \left( \frac{\theta D_{OL}}{u_m} - 1 \right) + \tilde{b} + O(u - u_m)$$

(9)

where $D_{OL}$ means the distance between observer and gravitational lens. The impact parameter is

$$u = \sqrt{\frac{C(r_0)}{A(r_0)}}$$

(10)

Certainly,

$$u_m = u|_{r_0=r_m}$$

(11)

The strong field limit coefficients $\tilde{a}$ and $\tilde{b}$ are expressed as,

$$\tilde{a} = \frac{R(0, r_m)}{2\sqrt{d_m}}$$

(12)

$$\tilde{b} = -\pi + I_R(r_m) + \tilde{a} \ln \frac{2d_m}{A(r_m)}$$

(13)

where

$$d_m = d|_{r_0=r_m}$$

(14)

It is necessary to explain the variables $\tilde{a}$ and $\tilde{b}$. We define some functions as follow,

$$R(z, r_0) = \frac{2\sqrt{B(r)A(r)}}{C(r)A'(r)}(1 - A(r_0))\sqrt{C(r_0)}$$

(15)

$$f(z, r_0) = \frac{1}{\sqrt{A(r_0) - [(1 - A(r_0))z + A(r_0)]\frac{C(r_0)}{C(r)}}}$$

(16)

$$f_0(z, r_0) = \frac{1}{\sqrt{cz + dz^2}}$$

(17)

$$I_D(r_0) = \int_0^1 R(0, r_m)f_0(z, r_0)dz$$

(18)
\[ I_R(r_0) = \int_0^1 [R(z, r_0) f(z, r_0) - R(0, r_m) f_0(z, r_0)] dz \]  
\( (19) \)

where

\[ z = \frac{A(r) - A(r_0)}{1 - A(r_0)} \]  
\( (20) \)

\[ c = \frac{1 - A(r_0)}{C(r_0) A'(r_0)} (C'(r_0) A(r_0) - C(r_0) A'(r_0)) \]  
\( (21) \)

\[ d = \frac{(1 - A(r_0))^2}{2C^2(r_0) A^2(r_0)} [2C_0 C'_0 A_0'^2 + (C_0 C''_0 - 2C_0'^2) A_0 A'_0 - C_0 C'_0 A_0' A''_0] \]  
\( (22) \)

with \( A_0 = A(r_0), C_0 = C(r_0) \). Substituting the metric (4) into the equations above, we obtain that

\[ R(z, r_0) = \frac{2}{\beta} \]  
\( (23) \)

\[ f(z, r_0) = [-\frac{2GM}{r_0} z^3 - (1 - \frac{6GM}{r_0}) z^2 + 2(1 - \frac{3GM}{r_0}) z]^{-\frac{1}{2}} \]  
\( (24) \)

\[ f(0) z, r_0 = [2(1 - \frac{3GM}{r_0}) z - (1 - \frac{6GM}{r_0}) z^2]^{-\frac{1}{2}} \]  
\( (25) \)

According to the defined functions and equations above, the coefficients \( \bar{a}, \bar{b} \) and the minimum impact parameter of the deflection angle can be written as,

\[ \bar{a} = \frac{1}{\gamma} = \frac{1}{1 - 4G\mu} \]  
\( (26) \)

\[ \bar{b} = \frac{4}{1 - 4G\mu} \ln(3 - \sqrt{3}) + \frac{1}{1 - 4G\mu} \ln 6 - \pi \]  
\( (27) \)

\[ u_m = 3\sqrt{3}(1 - 4G\mu)GM \]  
\( (28) \)

The deflection angle of the Schwarzschild black hole pierced by a cosmic string is,

\[ \alpha(\theta) = -\frac{1}{1 - 4G\mu} \ln\left(\frac{\theta D_{OL}}{3\sqrt{3}(1 - 4G\mu)GM} - 1\right) + \frac{4}{1 - 4G\mu} \ln(3 - \sqrt{3}) + \frac{1}{1 - 4G\mu} \ln 6 - \pi \]  
\( (29) \)

It is clear that the tension of cosmic string modifies the deflection angle subject to the standard Schwarzschild black hole without cosmic string. The deviation from the cosmic string going through the massive source helps us to distinguish a pure Schwarzschild black hole from that one involving a cosmic string.

We relate the position and the magnification to the strong field limit coefficients. The cosmic string influence on the coefficients can appear in the observables. The lens equation in the strong field limit reads [29],
\[ \beta = \theta - \frac{D_{LS}}{D_{OL}} \Delta \alpha_n \]  

(30)

where \( \beta \) denotes the angular separation between the source and the lens. \( \theta \) is the angular separation between the lens and the images. \( D_{LS} \) represents the distance between the lens and the source. Once we subtract all of the loops made by the photon, the offset of the deflection angle is expressed as \( \Delta \alpha_n = \alpha(\theta) - 2n\pi \). Because of \( u_m \ll D_{OL} \) the position of the n-th relativistic image can be approximated as,

\[ \theta_n = \theta_0^n + \frac{u_m e_n (\beta - \theta_0^n D_{OS})}{\bar{a} D_{LS} D_{OL}} \]  

(31)

where

\[ D_{OS} = D_{OL} + D_{LS} \]  

(32)

\[ e_n = e_{\frac{\bar{b}}{\bar{a}}} - 2n\pi \bar{a} \]  

(33)

while the term in the right-hand side of Eq. (31) is much smaller than \( \theta_0^n \) and we choose \( \theta_0^n \) as \( \alpha(\theta_0^n) = 2n\pi \). The magnification of n-th relativistic image is the inverse of the Jacobian evaluated at the position of the image like

\[ \mu_n = \left. \frac{1}{\bar{a} D_{OS}} \frac{\partial \theta_0^n}{\partial \theta_0^n} \right|_{\theta_0^n} = \frac{u_m e_n (1 + e_n) D_{OS}}{\bar{a} \beta D_{LS} D_{OL}^2} \]  

(34)

As an observable the angular separation between the first image and the others is defined as,

\[ s = \theta_1 - \theta_\infty \]  

(35)

where \( \theta_1 \) means the outermost image in the situation that the outermost one is thought as a single image and all of the remaining ones are packed together at \( \theta_\infty \). As an another obsevable the ratio of the fluxes from the first image and all of the others images is,

\[ r = \frac{\mu_1}{\sum_{n=2}^{\infty} \mu_n} \]  

(36)

According to \( e_{\frac{2\pi}{a}} \gg 1, e_{\frac{\bar{b}}{\bar{a}}} \approx 1 \) and \( u_m = D_{OL} \theta_\infty \), the above two formulae can be simplified as,

\[ s = \theta_\infty e_{\frac{\bar{b}}{\bar{a}}} e_{\frac{2\pi}{a}} \]  

(37)

\[ r = e_{\frac{2\pi}{a}} \]  

(38)
It is significant that the strong field limit coefficients such as $\bar{a}$ and $\bar{b}$ containing the influence from cosmic string are directly connected to the observables like $r$ and $s$, which can help us to explore whether the Schwarzschild black hole has been pierced by a cosmic string.

It is necessary to estimate the numerical values of the coefficients of gravitational lensing in the strong field limit and the related observables to show the influence from the cosmic string piercing the gravitational source. The cosmic string tension $\mu$ appears in our results. Our results will recover to those of standard Schwarzschild black hole with $\mu = 0$. We calculate these coefficients and observables according to the equations above and plot their dependence on the cosmic string tension in the figures. From Fig. 1 we show that the two strong gravitational lensing coefficients $\bar{a}$ and $\bar{b}$ both become larger with the increase of tension $\mu$. They increase quickly as the tension is large enough. Fig. 2 exhibits that the angular separation $s$ is also an increasing function of string tension $\mu$. In Fig. 3 we find that the relative magnification $r$ decreases when the tension $\mu$ increases. According to Eq. (29), we also show the dependence of the deflection angle on the corrections from the cosmic string in Fig. 4. The curves of the dependence of deflection angle on $G\mu$ with various values of $\frac{\theta D_{O}}{r_{D}}$ respectively are similar. The deflection angle becomes larger as the tension of cosmic string increases. When the increasing tension of cosmic string is large enough, the image will deviate more greatly. It should also be emphasized that the influences from cosmic string on the deflection angle $\alpha(\theta)$, angular separation $s$ and the relative magnification $r$ are evident. The deviations of observables such as $\alpha(\theta)$, $s$ and $r$ generate as the gravitational source involves the cosmic string. We hope that the astronomical observations for the cosmic string influence on these observables will become a new way to explore the cosmic string in future.

3. Summary

In this work we research on the gravitational lensing in the strong field limit around the Schwarzschild black hole pierced by a cosmic string. It is interesting that the cosmic string leads the minimum impact parameter $u_m$ to decrease although the radius of photon sphere has nothing to do with the string. We reveal that the strong gravitational lensing coefficients increase as the tension of cosmic string is greater. The influence from the cosmic string is brought about to the observables such as deflection angle, angular separation and relative flux. We find that the stronger string tension increases the angular separation $s$ while decreases the relative flux $r$. We also indicate that the deflection angle increase with the increase in tension $\mu$ and the images deviate more when the increasing tension of cosmic string is sufficiently large. The deviation of the deflection angle is manifest and distinct. Observing these phenomena supplies us a new way to explore the cosmic string. The cosmic strings can produce some other distinct characteristics besides two images presented in Ref. [18]. The related topics need to be studied further in future.
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Figure 1: The solid and dot curves corresponds to the dependence of the coefficient of strong field limit $\tilde{a}$ and $\tilde{b}$ respectively on the parameter $x = G\mu$ representing the tension of cosmic string piercing the Schwarzschild black hole.
Figure 2: The dependence of the angular separation $s$ in the Schwarzschild black hole with global monopole on the parameter $x = G\mu$ in unit of $\theta_\infty$, the asymptotic position of approached by a set of images and $\mu$ representing the tension of cosmic string piercing the Schwarzschild black hole.
Figure 3: The dependence of the relative flux $r$ on the parameter $x = G\mu$ representing the tension of cosmic string piercing the Schwarzschild black hole.
Figure 4: The solid, dot and dashed curves correspond to the dependence of the deflection angle $\alpha$ on the parameter $x = G\mu$ representing the tension of cosmic string piercing the Schwarzschild black hole for $\frac{\theta_{D}G\mu}{GM} = 10^{10}, 10^{12}, 10^{14}$ respectively.