Space-time non-commutativity tends to create bound states

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We study the spectrum of fluctuations about static solutions in 1+1 dimensional non-commutative scalar field models. In the case of soliton solutions non-commutativity leads to creation of new bound states. In the case of static singular solutions an infinite tower of bound states is produced whose spectrum has a striking similarity to the spectrum of confined quark states.

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I. INTRODUCTION

Over the recent years non-commutative field theory has developed into a mature discipline (see reviews [1]). It was argued (cf., e.g., [2]; more references can be found in [3]) that because of the presence of an infinite number of time derivatives space-time non-commutative theories cannot be quantized properly. However, the situation does not look hopeless. Perturbative unitarity can be successfully maintained [3] if one takes care of explicit Hermiticity of the Lagrangian. Even a canonical formalism can be developed at the expense of introducing an additional space-time dimension [4]. We don’t have much to add to this discussion. Moreover, our analysis will be essentially classical. We like to mention only that space-time non-commutative theories are not excluded, and that one can expect many non-standard features from these theories.

In this paper we consider some qualitative features of space-time non-commutative theories. Namely, we study fluctuations around static classical solutions in 1 + 1-dimensional non-commutative models with a real scalar field. Note, that the solutions themselves look exactly as in the commutative models. Therefore, non-commutativity can be seen only through the fluctuation spectra or through the scattering amplitudes [5]. We find that the frequency-dependent potential which appears in the equation for fluctuation has typically an “effective width” proportional to the frequency and to the non-commutativity parameter. This phenomenon is somewhat similar to delocalization of states discussed in [6] in a different context. In our case, this distortion of the potential leads to creation of new bound states (soliton backgrounds) or even to infinite families of new bound states (soliton backgrounds).

This paper is organized as follows. In the next section we fix our notations and conventions. Section II is devoted to fluctuations about solitonic solutions in the $\phi^4$ and in sine-Gordon models. Singular solutions are discussed in section III. Some concluding remarks are given in section IV.

II. NOTATIONS AND CONVENTIONS

Let us consider the non-commutative plane with a coordinate $\sigma = (t, x)$. The Groenewold-Moyal product is defined by the equation

$$f(\sigma) \star g(\sigma) = \left[ \exp \left( \frac{i}{2} \theta^{mn} \frac{\partial}{\partial \sigma^m} \frac{\partial}{\partial \sigma^n} \right) f(\sigma) g(\sigma') \right]_{\sigma' = \sigma},$$

(1)

where $\theta^{mn} = -\theta^{nm}$ is a constant antisymmetric $2 \times 2$ matrix which can be chosen as $\theta^{mn} = \theta e^{mn}$ with $e^{mn} = -e^{mn}$ and $e^{01} = 1$. This product is associative but non-commutative. A historical overview can be found in [7].

The following relations will be useful throughout this paper:

$$f(x) \star e^{i\omega t} = e^{i\omega t} f(x + \theta \omega), \quad e^{i\omega t} \star f(x) = e^{i\omega t} f(x - \theta \omega).$$

(2)

We shall study non-commutative deformations of the action

$$S = \int d^2 \sigma \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right]$$

(3)

for a real one-component field $\phi$ with some potential $V$. Non-commutative deformations of $V$ are constructed (as usual) in the following way. Let

$$V(\phi) = \sum_{p \geq 0} c_p \phi^p.$$

(4)

Then a non-commutative counterpart of $V$ is defined as

$$V_*(\phi) = \sum_{p \geq 0} c_p \phi \star \phi \cdots \star \phi,$$

(5)

where

$$\theta \theta = 0,$$

$$\theta = (i\hbar)^{-1} \left[ \begin{array}{cc} 0 & \sigma \\ \bar{\sigma} & 0 \end{array} \right],$$

$$\sigma = (\partial_1, \partial_2), \quad \bar{\sigma} = (\sigma_1^*, \sigma_2^*).$$

This matrix commutes with Hamiltonian $H$.
where the $p$th term contains $p$th star-power of $\phi$. We restrict ourselves to polynomial or exponential potential only, so that there is no problem with the convergence of $\Phi$. Convergence of $\Phi$ is a more subtle question, but we shall actually work with non-commutativity only in the perturbative regime.

Our primary example will be the $\phi^4$ model

$$V^{[4]} = -\frac{1}{2}m^2 \phi^2 + \lambda \frac{\phi^4}{4}.$$  \hfill (6)

We shall also consider the Liouville model

$$V^{[L]} = \gamma e^{\beta \phi},$$  \hfill (7)

the sine-Gordon model

$$V^{[sG]} = \frac{m^4}{6\lambda} \cos \left( \frac{\sqrt{6\lambda}}{m} \phi \right),$$  \hfill (8)

and the sinh-Gordon model

$$V^{[shG]} = \frac{m^2}{2} \cosh(2\phi).$$  \hfill (9)

The equation of motion following from the non-commutative deformation

$$S_* = \int d^2 \sigma \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V_*(\phi) \right]$$  \hfill (10)

of (6) reads:

$$\partial_\mu^2 \phi - \partial_\phi^2 \phi + [\partial_\phi V]_* = 0.$$  \hfill (11)

Obviously, star-product of functions depending on $x$ only coincides with the ordinary product. Therefore, all static solutions of a commutative model will also solve the non-commutative equation of motion (11). Note, that there could be of course non-static localized solutions in space-time non-commutative theories (cf. 3).

III. SOLITONS: NEW BOUND STATES

A. The $\phi^4$ model

In this section we consider the spectrum of fluctuations about static solitonic solutions (i.e. about localized solutions with finite energy). For the $\phi^4$ model (6) this is the kink solution:

$$\Phi(x) = \frac{m}{\sqrt{\lambda}} \tanh \left( \frac{m x}{\sqrt{2}} \right)$$  \hfill (12)

As we have already mentioned above, this is also a soliton in the non-commutative $\phi^4$.

Let us consider small fluctuations about the kink background, $\phi := \Phi + \delta \phi$. The equation of motion for the fluctuations reads

$$\delta \ddot{\phi} - \delta \phi'' - m^2 \delta \phi + \lambda (\delta \phi \Phi^2 + \Phi^2 \delta \Phi + \Phi \delta \Phi \Phi) = 0.$$

We shall look for the solutions in the form

$$\delta \phi = e^{i\omega t} \eta(x),$$  \hfill (14)

Then, by virtue of (2), one obtains

$$-\eta'' + \lambda (\Phi^2_+ + \Phi_+ \Phi_+ + \Phi^2_+) \eta = (\omega^2 + m^2) \eta,$$  \hfill (15)

where $\Phi_\pm \equiv \Phi(x \pm \theta \omega)$. This problem has a natural scale $\mu = m/\sqrt{2}$. It is convenient to introduce rescaled dimensionless variables: $\hat{x} = \mu x$, $\hat{\omega} = \omega/\mu$, $\hat{\theta} = \mu^2 \theta$. In terms of these variables eq. 15 reads

$$-\hat{\eta}''(\hat{x}) + U(\hat{x}) \eta(\hat{x}) = (\hat{\omega}^2 + 2) \eta$$  \hfill (16)

where prime denotes the differentiation with respect to $\hat{x}$, and

$$U = 2 \left( \tanh^2(\hat{x}_+) + \tanh(\hat{x}_+) \tanh(\hat{x}_-) + \tanh^2(\hat{x}_-) \right).$$  \hfill (17)

From now on we drop hats which is equivalent to setting $\mu = 1$.

In the commutative case, $\theta = 0$, $x_\pm = x$,

$$U = U_0 = 6 - \frac{6}{\cosh^2 x}.$$  \hfill (18)

There are two bound states for this potential with $\omega_1 = 0$ and $\omega_2 = \sqrt{3}$ with the wave functions:

$$\eta_1 = \frac{\sqrt{3}}{2 \cosh^2 x}, \quad \eta_2 = \frac{\sqrt{3}}{2} \frac{\sinh x}{\cosh^2 x}.$$  \hfill (19)

For $\theta \neq 0$ we have a complicated problem with a frequency dependent potential. Of course, in the generic case no exact solution for the eigenstates is available. For small $\theta$ and relatively low eigenfrequencies, $\theta \omega \ll 1$, we can use ordinary perturbation theory to estimate shift of the eigenvalues. An example of the potential $U(x)$ is shown on Fig. 1.

We write to the leading order in $\theta \omega$:

$$-\eta'' - \left( \frac{6}{\cosh^4 x} - U^{(1)} \right) \eta = (\omega^2 - 4) \eta,$$  \hfill (20)

where

$$U^{(1)} = 2(\theta \omega)^2 \left( \frac{7}{\cosh^4 x} - 6 \right).$$  \hfill (21)

We assume $\eta = \eta_0 + \delta \eta$, where $\eta_0$ satisfies 16 with $U = U_0$ given by 18. Remember that hats have been dropped. Then one immediately obtains that the bound

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1 To avoid confusions we have to mention that sometimes the Liouville model includes also an interaction with two-dimensional metric.
state frequency $\omega_1 = 0$ is not shifted to this order, while $\omega_2$ receives a negative correction:

$$\omega_2^2 = \omega_2^2|_{\theta=0} + \int dx \eta_2^2 U^{(1)} = 3 \left( 1 - \frac{8}{5} \theta^2 \right)$$

or

$$\omega_2^2 = (1 - 0.8\theta^2)\omega_2|_{\theta=0}. \quad (23)$$

 Corrections to this formula are of order $\theta^4$. We see, that in this regime the eigenfrequency of one of the bound states is being shifted. Let us remind that the kink solution itself does not depend on $\theta$. Therefore, shift of the eigenfrequencies may be a measurable manifestation of non-commutativity for small $\theta$ and low frequencies.

We have shown that $\omega$ decreases due to the non-commutativity. A natural question to ask is whether new bound states can appear. The answer is positive. The following analysis will be made in the large-$\theta$ limit.

It is easy to demonstrate that for large $\theta\omega$ the potential $U(x)$ (cf. 10) can be approximated by a square well potential $\tilde{U}$ (cf. Fig. 2) such that

$$\tilde{U}(x) = 6 \quad \text{for} \quad |x| > \theta\omega,$$

$$\tilde{U}(x) = 2 \quad \text{for} \quad |x| < \theta\omega. \quad (24)$$

Therefore, we replace (10) by

$$-\eta'' + \tilde{U}\eta = (\omega^2 + 2)\eta. \quad (25)$$

Note, that the characteristic width of $\tilde{U}$ depends on $\omega$. Clearly, bound states can only appear for $0 < \omega < 2$. By using the standard methods which can be found in any quantum mechanics textbook we obtain that eigenfrequencies of the bound states should satisfy one of the equations:

$$\tan(\theta\omega^2) = \sqrt{4 - \omega^2}, \quad \cot(\theta\omega^2) = -\sqrt{4 - \omega^2}, \quad (26)$$

where the first equation gives eigenfrequencies of the states with a symmetric wave function, while the second

\begin{align*}
\text{equation describes the states with antisymmetric wave functions.}
\end{align*}

In our approximation it is essential that $\omega \theta$ is large. Therefore, we shall consider (26) for $\omega$ near the upper limit $\omega = 2$. In a small interval near $\omega^2 = 4$ the functions on the right hand sides of the equations (26) are bounded and continuous, while $\tan(\theta\omega^2)$ and $\cot(\theta\omega^2)$ change from $-\infty$ to $+\infty$ when $\omega^2$ changes from $\pi(n - \frac{1}{2})/\theta$ to $\pi(n + \frac{1}{2})/\theta$ or from $\pi n/\theta$ to $\pi(n-1)/\theta$ respectively. This means if $\theta \gg \pi/8$ there is always at least one solution for each of the equations (26) near $\omega = 2$. We can even estimate roughly the number of the solutions in the upper half of the allowed interval (i.e. for $\omega^2 \in [2, 4]$) to be about $8\theta/\pi$.

We conclude that for a large non-commutativity parameter $\theta$ there are many new bound states for the fluctuations about the kink soliton as compared to the commutative case.

\section{B. The sine-Gordon model}

To make sure that the phenomenon of creation of new bound states due to the non-commutativity is present not only in the $\phi^4$ model, let us consider the sine-Gordon model $\Phi^4$. Static solutions in this model in both commutative and non-commutative regimes should satisfy the equation:

$$-\phi'' + \frac{m^3}{\sqrt{6}\lambda} \sin \left( \frac{\sqrt{6}\lambda}{m} \phi \right) = 0. \quad (27)$$

There is a one-soliton solution of (27) which reads

$$\Phi(x) = \frac{4m}{\sqrt{6}\lambda} \arctan (e^{mx}). \quad (28)$$

To obtain an equation for fluctuations we have to expand a non-commutative exponential. This can be done with the help of the equation:

$$e_{\ast}^{A+B} = e_{\ast}^{A} + \int_{0}^{1} d\sigma \, e_{\ast}^{A} \ast B \ast e_{\ast}^{(1-\sigma)A} + O(B^2), \quad (29)$$
which has a purely combinatorial origin and is true regardless of the choice of associative product involved (this could be the ordinary operator or matrix product, for example, or the Groenewold-Moyal star as in our case). The formulae (29), (14) and (2) yield

\[ -\eta'' + m^3 \left( \sin \frac{\sqrt{6} \lambda}{m} \Phi_+ - \sin \frac{\sqrt{6} \lambda}{m} \Phi_- \right) \eta = \omega^2 \eta. \]  

(30)

For large \( \theta \omega \) the effective potential \( U(x) \) behaves similarly to that for the \( \phi^4 \) model (see Fig. 3). Namely, the potential \( U(x) \) can be approximated by a square well potential with the width \( 2\hat{\theta}\hat{\omega} \). All arguments of the previous subsection apply for this case almost without modification. We conclude, that for large non-commutativity we have new bound states. This seems to be a generic feature of the fluctuation equation on the background of a static solitonic solution in a two-dimensional non-commutative space-time.

IV. SINGULAR SOLUTIONS AND CONFINING POTENTIALS

A. Massless \( \phi^4 \) model

The \( \phi^4 \) model with \( m = 0 \) admits a singular static solution which reads:

\[ \Phi(x) = \frac{\sqrt{2}}{x\sqrt{\lambda}}. \]  

(31)

Then, by acting exactly as in the previous section we obtain the following equation for the fluctuations:

\[ -\eta'' + \frac{2(3x^2 + \theta^2\omega^2)}{(x^2 - \theta^2\omega^2)^2} \eta = \omega^2 \eta. \]  

(32)

Note, that this equation is scale-invariant, i.e., if we rescale \( x \to x\mu, \omega \to \omega/\mu, \theta \to \theta\mu^2 \) the scaling parameter cancels out. As a consequence, we can assume that we are working in dimensionless variable, so that \( \theta \omega = 3 \) on Fig. 4 indeed makes sense.

Obviously, for \( \theta = 0 \) there are no bound states. If \( \theta \neq 0 \) the situation changes drastically (cf. Fig. 4). In this case we have two infinitely high potential barriers located at \( x = \pm \omega\theta \). Physics between these two barriers can be approximated by an infinitely deep well of width \( 2\theta\omega \). This approximation is good for high frequencies. In this case the equation for small fluctuations can be easily solved yielding

\[ \omega_N \simeq \sqrt{\frac{\pi N}{2\theta}}, \]  

(33)

where \( N \in \mathbb{N} \). Accuracy of this formula increases for large \( N \) and/or large \( \theta \).

The spectrum obtained in this simple model has a striking similarity to the spectrum of hadrons. Indeed, we observe an infinite number of bound states with a linear dependence of \( \omega^2 := M^2 \) on an integer spectral parameter. Since we do not have something like angular momentum in two dimensions we cannot push these arguments further.

B. Exponential interactions

Again, we would like to test the observation made for the \( \phi^4 \) model by considering other models admitting similar types of the classical solution. We start with the Liouville model (7). Classical equation of motion for this model reads

\[ \ddot{\phi} - \phi'' + \alpha e^{\beta\phi} = 0, \]  

(34)

where \( \alpha = \beta \gamma \). In the commutative case \( \theta = 0 \) there is a general solution to this equation:

\[ \phi = \frac{1}{\beta} \log \left( -\frac{2G(p)F'(q)}{\alpha \beta (G(p) + F(q))^2} \right), \]  

(35)
with \( p = (t + x)/2 \), \( q = (t - x)/2 \). \( G \) and \( F \) are arbitrary functions. The simplest static solution (which again is common for the commutative and non-commutative models) is obtained by setting \( G = p, \ F = -q \):

\[
\Phi(x) = -\frac{1}{\beta} \log \left( \frac{\alpha \beta x^2}{2} \right), \quad \alpha \beta > 0. \tag{36}
\]

Next we use again the expansion \( \varphi \), the ansatz \( \eta \), and the property \( \Phi \) to write down the equation for fluctuations:

\[
-\eta'' + \alpha \left( \frac{e^{\beta \Phi_+} - e^{\beta \Phi_-}}{\Phi_+ - \Phi_-} \right) \eta = \omega^2 \eta. \tag{37}
\]

The substitution of \( \Phi \) in \( \eta \) yields the following effective potential

\[
U = \frac{8 \theta \omega x}{(x_+ x_-)^2 \log \left( \frac{x_+}{x_-} \right)}. \tag{38}
\]

This potential is similar to the one appearing in the \( \phi^4 \) model (cf. \( \Phi \)). Again, we have two infinitely high potential barriers with a “confinement” region between them. Effective width of this region is \( 2 \theta \omega \). Therefore, the spectrum of higher excited states is again given by \( \Phi \).

A very similar behaviour can be found also in the sinh-Gordon model \( \eta \) near the static solution

\[
\Phi(x) = \frac{1}{2} \log \tanh^2 \left( \frac{m x}{2} \right). \tag{39}
\]

We leave this case for the reader as an exercise.

We like to stress that in each case the universal formula \( \Phi \) appears.

V. CONCLUSIONS

Our main result is that in the presence of the space-time non-commutativity the effective potential describing fluctuations on a static background becomes delocalized with the effective width \( \sim \theta \omega \). As a consequence, in the case of large non-commutativity \( \theta \) we have much more bound states on solitonic background than in corresponding commutative theories. On singular static backgrounds the picture is even more interesting. Non-commutativity produces an infinite tower of bound states with linear dependence of \( \omega^2 \) on an integer quantum number (for large frequencies). This behaviour is universal, the large frequency spectrum depends only on \( \theta \), but not on the details of the models. This result suggests that the space-time non-commutativity may have some relation to the problem of quark confinement. Although we have not presented any general proof, the number of examples considered seems to justify our conclusion that creation of new bound states is a generic feature of space-time non-commutative theories in \( 1 + 1 \) dimensions.

In principle, the frequency spectrum can be used to calculate quantum corrections to the mass of the solitons in non-commutative theories. In \( \Phi \) it was argued that the \( \zeta \)-function or other heat kernel based methods may be a suitable instrument (although, it is not clear whether the fluctuation operator involved can indeed be reduced to \( \Phi \)-Laplacians considered in \( \Phi \)). As an intermediate step one has to put the system in a large box so that the spectrum becomes discrete. At the end of the calculation the boundary is moved to the infinity and, if necessary, the boundary contribution to the vacuum energy is subtracted from the total energy of the system (cf. \( \Phi \) where this procedure is applied to the kink). However, in the present case no fixed boundary can be far away enough since the “effective width” of the potential is proportional to the frequency. This seems to be another manifestation of the mixing between ultraviolet and infra red scales in non-commutative theories \( \Phi \). Consequences of this mixing for quantum corrections to (space) non-commutative solitons in \( 2 + 1 \) dimensions were considered recently in \( \Phi \).

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