On phase deficit of the super-twisting second-order sliding mode control algorithm

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Summary
The notion of phase deficit was introduced in [12], which allows one to determine if a nonlinear system can reveal finite-time or asymptotic convergence. Twisting and suboptimal second-order sliding mode control algorithms and a conventional relay feedback system were analyzed from the perspective of the phase deficit. However, phase deficit for the super-twisting algorithm was not determined at that time due to the complexity of the problem. Therefore, although it is known that the super-twisting algorithm reveals finite-time convergence (when no parasitic dynamics are present), it was not supported by the criterion based on the phase deficit. In the current article this problem has been solved through an open-loop interpretation of the phase deficit, which is proposed in the article. Another problem addressed in the article is the relationship between the frequency characteristics of the system and the type of convergence (finite-time or asymptotic). The mechanism of convergence is analyzed through considering time-varying frequency of self-excited oscillations and phase lag of the linear part of the system.

KEYWORDS
Lure system, second-order sliding mode, sliding mode, super-twisting algorithm

1 | INTRODUCTION
The problems of finite-time or infinite-time (asymptotic) convergence in the analysis of conventional sliding mode (SM), second-order sliding mode (SOSM) and other nonlinear control systems are of high importance. Usually great efforts are needed to prove finite-time convergence in a system. In fact, a mode and a respective algorithm cannot be legitimately called SM or SOSM without proof of the finite-time convergence. In particular the super-twisting algorithm[1] attracted a lot of attention from researchers, who proposed various Lyapunov functions to prove finite-time convergence of the algorithm,[2–6] analyzed chattering and designed control with specification on chattering.[7,8] Lately finite-time stability was established in a number of publications mentioned above through the homogeneity technique in continuous[9] and discontinuous systems.[10] Another opportunity for establishing finite-time stability is provided by fractional differential inequalities.[11]

The notion of phase deficit introduced in Reference 12 offers a simple and universal (not algorithm-specific) necessary condition for finite-time convergence. However, it is based on the approximate describing function (DF) method and requires finding a limit of the DF at amplitude \( a \rightarrow 0 \) and frequency \( \omega \rightarrow \infty \). This limit is not easy to find due to the two...
variables approaching their limiting values simultaneously. The phase deficit was previously found for the conventional SM control,\textsuperscript{13,14} the twisting algorithm,\textsuperscript{15} and the suboptimal algorithm,\textsuperscript{16,17} but was not found for the super-twisting algorithm,\textsuperscript{1} terminal SM,\textsuperscript{18,19} and some other SOSM algorithms.

In the present article the issue of finding limits of the DF is resolved through a transformation of the structure of a SOSM algorithm. With this transformation an input-output formulation of the phase deficit is proposed. Although it is illustrated only on the super-twisting algorithm the methodology of this transformation and the subsequent analysis can be applied to any known SOSM algorithm. Another result of the present research is some input-output frequency properties of the super-twisting algorithm that are noted and proved. And the third new result is an input-output interpretation (or a model) of the mechanism of the finite-time convergence. It further details the previously proposed criterion of the phase deficit and explains its mechanism.

\section{Problem Formulation}

Let us consider a nonlinear system that has a loop connection of a linear plant given by

\begin{equation}
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx,
\end{aligned}
\end{equation}

where \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times 1}, C \in \mathbb{R}^{1 \times n}\), \(x \in \mathbb{R}^n\) is a state vector, \(u \in \mathbb{R}\) control \(y \in \mathbb{R}\) output, and a nonlinear controller

\begin{equation}
u = f(y, \dot{y}).\end{equation}

If the controller is presented by a nonlinearity only then the system is a Lure system. We shall consider the systems that satisfy the following conditions:

(i) the magnitude of the controller DF given by

\begin{equation}
N(a, \omega) = \frac{\omega}{\pi a} \int_0^{2\pi/\omega} u(t) \sin \omega t \, dt + \frac{\omega}{\pi a} \int_0^{2\pi/\omega} u(t) \cos \omega t \, dt,
\end{equation}

where \(u(t)\) is the controller reaction to the harmonic excitation, \(a\) amplitude, and \(\omega\) frequency, tends to infinity when the amplitude tends to zero:

\begin{equation}
\lim_{a \to 0} |N(a, \omega)|_{\omega=\omega^*} = \infty, \quad \forall \omega^* \in (0, \infty)
\end{equation}

and (ii) the transfer function of the linear plant is proper rational, has relative degree of one or two and is given by

\begin{equation}
G(s) = \frac{B_{n-1}(s)}{A_n(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_1s + b_0}{a_ns^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0},
\end{equation}

where \(b_{n-1} = 0\) for relative degree two. The notion of the phase deficit was introduced in Reference 12 to classify the types of steady and transient oscillations that may occur in the system (1), (2). The phase deficit is defined with involvement of the DF analysis\textsuperscript{20} for the situation of negative reciprocal of the DF of the controller having a point of intersection with the Nyquist plot of the plant only at the origin. Phase deficit is defined as the angle between the high-frequency asymptote of the Nyquist plant of the linear plant and the low-amplitude asymptote of the negative reciprocal of the DF of the nonlinear controller:

\begin{equation}
\phi_d = \pi + \lim_{a \to 0} \arg N(a) + \lim_{\omega \to \infty} \arg G(j\omega).
\end{equation}

It was shown in Reference 12 that if the phase deficit is a positive quantity then the phase balance cannot occur and finite-time convergence occurs in the system, if the phase deficit is zero then asymptotic convergence occurs, and if the phase deficit is a negative quantity (which means that the phase balance occurs) then finite-frequency oscillations occur. The validity of the criterion based on the phase deficit was demonstrated for the twisting and the suboptimal second-order SM algorithms in References 12 and 21, respectively, as well as relay feedback systems with the linear plant having relative degree one (finite-time convergence), two (asymptotic convergence), and higher than two
The super-twisting algorithm was not analyzed using this approach, as the value of the phase deficit is hard to determine for this system due to the simultaneous dependence of the DF of the algorithm on the frequency and the amplitude. The notion of the phase deficit is revisited below, which allows us to extend the respective analysis to a system with the super-twisting algorithm. It also involves a new approach to analysis of convergence.

Another possibly even more important problem that is being solved in the article is analysis and explanation of the mechanism of convergence (finite-time or asymptotic). This analysis is done below through the consideration of time-varying frequency of the self-excited oscillations and the phase lag introduced by the linear dynamics. The relationships have to be established for a relay feedback system first, after which the same approach is applied to the super-twisting algorithm.

3 | CONVERGENCE IN RELAY FEEDBACK SYSTEMS

Let us consider a relay feedback system with a controller being a relay nonlinearity and a plant being linear dynamics (Figure 1).

Let us assume that the relay has hysteresis value of \( b \geq 0 \). If \( b > 0 \), oscillations of finite frequency are excited in the loop\(^*\), that according to the DF method are given by \( y(t) = a_y \sin(\Omega t + \phi_y) \). The frequency \( \Omega \) and the amplitude \( a_y \) of these oscillations can be found through the DF method from the following equations:

\[
\text{Im}G(j\Omega) = \frac{\pi b}{4c}, \tag{7}
\]

\[
a_y = \frac{4c}{\pi} |G(j\Omega)|, \tag{8}
\]

where \( b \) and \( c \) are hysteresis value and the amplitude of the relay. The output of the relay is symmetric square pulses of frequency \( \Omega \) and amplitude \( c \): \( u(t) = c \text{sign} \left[ \sin(\omega t) \right] \). If, however, the relay is ideal \( (b = 0) \) and the plant has first or second relative degree then the phase balance cannot occur and periodic motions cannot be excited in the loop. What occurs in this case is the process involving vanishing oscillations.

Let us consider, as per the DF method, only the first harmonic in the output of the relay and find the response of linear dynamics to a harmonic input, with frequency changing on each half-period. To find this response, we first find the reaction to a harmonic input \( u(t) = \sin(\omega t) \). For a relaxed system the reaction can be found as a convolution of the input signal and the impulse response \( g(t) \), which yields:

\[
y(t) = \int_0^t g(\tau) \cdot \sin \omega(t - \tau) d\tau
= \sin \omega t \int_0^t g(\tau) \cdot \cos \omega \tau \, d\tau - \cos \omega t \int_0^t g(\tau) \cdot \sin \omega \tau \, d\tau. \tag{9}
\]

One can notice that \( \lim_{t \to \infty} \int_0^t g(\tau) \cdot \cos \omega \tau \, d\tau = \text{Re}G(j\omega) \) and \( \lim_{t \to \infty} \int_0^t g(\tau) \cdot \sin \omega \tau \, d\tau = -\text{Im}G(j\omega) \), where \( G(s) \) is the transfer function of the linear dynamics. Therefore, the response (9) can be presented as the sum of the periodic component \( y_p(t) \)

\[
y_p(t) = \sin \omega t \text{Re}G(j\omega) + \cos \omega t \text{Im}G(j\omega) \tag{10}
\]

\(^*\)The conditions of excitation of self-sustained oscillations are considered in details in Reference 22.
and of the vanishing component \( y_v(t) \) of the output signal:

\[
y_v(t) = -\sin \omega t \int_t^\infty g(r) \cdot \cos \omega r \, dr + \cos \omega t \int_t^\infty g(r) \cdot \sin \omega r \, dr.
\]  

(11)

The reaction of first-order linear dynamics with the transfer function \( G(s) = 1/(s+1) \) is illustrated by Figure 2, which gives the input signal, the periodic component of the output signal, the vanishing component, and the output signal. One can see that at \( t = 0 \), \( y_v(0) = -y_p(0) \), and therefore \( y(0) = 0 \). Moreover, \( y_v(t) \) is, indeed, vanishing because \( g(r) \) is absolutely integrable, with \( \lim_{t \to \infty} g(t) = 0 \) [these are the conditions for bounded-input-bounded-output (BIBO) stability] and with the increase of \( t \) the interval of integration becomes smaller until it vanishes at \( t \to \infty \). Therefore, for a BIBO system \( \text{Re}(j\omega) \) and \( \text{Im}(j\omega) \) can be considered approximations for \( \int_0^t g(r) \cdot \cos \omega r \, dr \) and \( \int_0^t g(r) \cdot \sin \omega r \, dr \), respectively.

Consider now specifically the second-order dynamics because it can produce the phase lag from the range \([-180^\circ, -90^\circ]\), which is of interest in our analysis because with the phase lag from the given range the harmonic balance condition cannot be met in the relay feedback system (having an ideal relay). Formulate the following theorem.

**Theorem 1.** The response \( y(t) \) of second-order and relative degree two linear dynamics \((n = 2 \text{ and } b_{n-1} = b_1 = 0 \text{ in formula (5)})\) to a sinusoidal input \( u(t) = \sin(\omega t) \) of frequency \( \omega \) can be represented as a sum of two components: vanishing \( y_v(t) \) and periodic \( y_p(t) \), where \( y_p(t) = |G(j\omega)| \cdot \sin(\omega t + \arg G(j\omega)) \). If the frequency \( \omega \) is high enough, so that the phase response of the linear dynamics is \( \arg G(j\omega) \in [-180^\circ, -90^\circ] \), then at the time \( t = \frac{\pi}{\omega} \) the response of the linear dynamics is \( y(\frac{\pi}{\omega}) < y_p(\frac{\pi}{\omega}) \).

**Proof.** As an auxiliary step, find both components of the response \( y(t) \) for the first-order dynamics (see Figure 2). If \( G(s) = \frac{1}{s+1} \) then the frequency response is: \( G(j\omega) = \frac{1-j\omega T}{1+\omega^2 T^2} \), with \( \text{Re}(G(j\omega)) = \frac{1}{1+\omega^2 T^2} \), and \( -\text{Im}(G(j\omega)) = \frac{-\omega T}{1+\omega^2 T^2} \). The impulse response for this transfer function is:

\[
g(t) = \frac{1}{T} e^{-\frac{t}{T}}.
\]

(12)

With the impulse response given by (12) the first integral in (9) becomes

\[
\int_0^t g(r) \cdot \cos \omega r \, dr = \frac{1}{1 + \omega^2 T^2} \left[ e^{-\frac{t}{T}} \left( -\cos \omega t + T\omega \sin \omega t \right) + 1 \right].
\]

At the time \( t = 0 \) this integral is zero and at \( t = \frac{\pi}{\omega} \) (reaction at the time equal to one half-period) this integral is

\[
\int_0^{\frac{\pi}{\omega}} g(r) \cdot \cos \omega r \, dr = \frac{1}{1 + \omega^2 T^2} \left[ e^{-\frac{\pi}{2 T}} + 1 \right].
\]
Therefore,
\[
\int_0^\frac{\pi}{\omega} g(\tau) \cdot \cos \omega \tau \, d\tau = \text{Re} \, G(j\omega) \cdot \left( e^{-\frac{\omega}{T}} + 1 \right). \tag{13}
\]

For the second integral from (9), we can write:
\[
\int_0^t g(\tau) \cdot \sin \omega \tau \, d\tau = \frac{1}{1 + \omega^2 T^2} \left[ e^{\frac{\omega}{T}} (-\sin \omega t - \omega T \cos \omega t) + \omega T \right].
\]

At the time \( t = 0 \) this integral is zero and at \( t = \frac{\pi}{\omega} \) (reaction at the time equal to one half-period) it is
\[
\int_0^\frac{\pi}{\omega} g(\tau) \cdot \sin \omega \tau \, d\tau = \frac{\omega T}{1 + \omega^2 T^2} \left[ e^{-\frac{\omega}{T}} + 1 \right].
\]

Therefore,
\[
\int_0^\frac{\pi}{\omega} g(\tau) \cdot \sin \omega \tau \, d\tau = -\text{Im} \, G(j\omega) \cdot \left( e^{-\frac{\omega}{T}} + 1 \right). \tag{14}
\]

With formulas (13) and (14) available we can now find the value of \( y(t) \) at \( t = \frac{x}{\omega} \) through the substitution of (13), (14) into (9):
\[
y\left( \frac{x}{\omega} \right) = -\text{Im} \, G(j\omega) \cdot \left( e^{-\frac{x}{T}} + 1 \right).
\]

Because specifically we consider linear parts that can introduce the phase lag within the range of \([-180^\circ, -90^\circ]\), which cannot be provided by the first-order dynamics, we have to consider the second-order dynamics. Let us assume that the transfer function of the second-order dynamics is given by the transfer function \( G_{SO}(s) = \frac{1}{(T_1 s + 1)(T_2 s + 1)} \), where \( T_1 < T_2 \). This transfer function would serve as a certain representative dynamics, which satisfies the noted phase lag property. The results obtained for the first-order dynamics can be used in analysis of response of \( G_{SO}(s) \) if the latter is expanded into partial fractions:
\[
G_{SO}(s) = -\frac{T_1}{T_2 - T_1} G_1(s) + \frac{T_2}{T_2 - T_1} G_2(s),
\]

where \( G_1(s) = \frac{1}{T_1 s + 1} \), \( G_2(s) = \frac{1}{T_2 s + 1} \). The response of \( G_{SO}(s) \) to the sinusoidal input \( u(t) = \sin(\omega t) \) at time \( t = \frac{x}{\omega} \) (reaction at the time equal to one half-period) is
\[
y\left( \frac{x}{\omega} \right) = \frac{-1}{\omega (T_2 - T_1)} \left[ \frac{\omega^2 T_1^2}{1 + \omega^2 T_1^2} \left( e^{-\frac{x}{T_1}} + 1 \right) - \frac{\omega^2 T_2^2}{1 + \omega^2 T_2^2} \left( e^{-\frac{x}{T_2}} + 1 \right) \right].
\]

The periodic component of the response is given by
\[
y_p\left( \frac{x}{\omega} \right) = \frac{1}{\omega (T_2 - T_1)} \left[ \frac{\omega^2 T_2^2}{1 + \omega^2 T_2^2} - \frac{\omega^2 T_1^2}{1 + \omega^2 T_1^2} \right]. \tag{15}
\]

And the vanishing component is
\[
y_v\left( \frac{x}{\omega} \right) = \frac{1}{\omega (T_2 - T_1)} \left[ \frac{\omega^2 T_2^2}{1 + \omega^2 T_2^2} e^{-\frac{x}{T_2}} - \frac{\omega^2 T_1^2}{1 + \omega^2 T_1^2} e^{-\frac{x}{T_1}} \right]. \tag{16}
\]

One can see that with time the output signal approaches the periodic component, so that using \( y_p(t) \) as an approximation for \( y(t) \) becomes possible. It is also possible to show that specifically at time \( t = \frac{x}{\omega} \), which corresponds to the half-period, the value of the vanishing component for a system having relative degree two is positive. For example, for the transfer function \( G(s) = \frac{1}{(T_1 s + 1)(T_2 s + 1)} \), \( y_v(\pi/\omega) \) is positive.
One can notice that the expression in brackets is a difference of two values of the same function \( \Theta = \frac{\eta^2}{1+\eta^4} e^{-\eta} \), with \( \eta = \omega T_1 \) in one case and \( \eta = \omega T_2 \) in the other. Function \( \Theta \) is monotone increasing for positive \( \eta \) because both factors in \( \Theta \) are monotone increasing functions of \( \eta \). As was assumed above, \( T_1 < T_2 \); therefore, \( y_v \left( \frac{\pi}{\omega} \right) > 0 \) according to (16). At the same time, as per (15), one can see that \( y_p \left( \frac{\pi}{\omega} \right) > 0 \).

Therefore, the following conclusions can be made:

- Approximately the response of the linear dynamics \( G_{SO}(s) \) to the sinusoidal input \( u(t) = \sin(\omega t) \) can be evaluated using the asymptotic relationship. The larger the time the more precise this approximation is. This is due to the vanishing character of \( y_v(t) \). Particularly at the time \( t = \frac{\pi}{\omega} \) the value of \( y_v \left( \frac{\pi}{\omega} \right) \) can be found using the asymptotic relationship (15).

- The error caused by disregarding \( y_v(t) \) is such that the use of the estimate (15) would correspond to the delayed switching because at the time \( t = \frac{\pi}{\omega}, y_p \left( \frac{\pi}{\omega} \right) > 0 \) and \( y_p(t) \) is growing (this follows from the fact that \( y_p(t) \) is a sinusoid lagged with respect to the input by the phase from the range \([-180^\circ, -90^\circ]\)), and the addition of \( y_v \left( \frac{\pi}{\omega} \right) > 0 \) would correspond to an earlier crossing of zero level by \( y(t) \).

The same conclusions can be made for other types of second-order dynamics: under-damped and integrating, and under certain limiting conditions (the frequency must be high enough) be extended to higher-order dynamics with relative degree two that can produce the phase within \([-180^\circ, -90^\circ]\). The conclusions for the general case are supported by formulas (9) to (11).

Because of the positiveness of \( y_v(\pi/\omega) \) and given the fact that the phase lag is within \([-180^\circ, -90^\circ]\), the use of \( y_p(t) \) as an approximation for \( y(t) \) would correspond to a delayed crossing of zero level by \( y_p(t) \) in comparison to \( y(t) \). Therefore, the following approximate model of the frequency change would provide conservative estimates of switching instants and convergence time, so that the actual convergence time will be shorter than the estimated one provided by this model.

Let us assume the following model relating the frequency (variable) applied to the linear plant, the reaction of the linear plant, and the change of the frequency that this entails. According to this model we shall consider that the phase shift \( \phi_y \), between the control and the output signals is due to the phase lag introduced by the plant, which is

\[
\phi_y = \arg(G(j\Omega)).
\] (17)

Let us also consider the first harmonic analysis only because of the low-pass filtering properties of the plant, which is in line with the DF analysis of the system. Assuming that the control signal produced by the relay is approximately equal:

\[
\begin{align*}
u(t) & \approx \frac{4C}{\pi} \sin(\Omega t) = a_u \sin(\Omega t),
\end{align*}
\] (18)

where \( a_u = \frac{4C}{\pi} \) is the amplitude of the first harmonic, and disregarding all higher harmonics we can find the response of the linear plant to this control signal.

**Assumption 1.** The plant is BIBO stable. It is known that if the plant is BIBO stable then with input being \( u(t) = a_u \sin(\omega t) \), where \( \omega \) is arbitrary frequency, the output of the plant at \( t \to \infty \) is a harmonic signal of the same frequency and has the amplitude of \( a_v = a_u |G(j\omega)| \) and phase \( \phi_y \) is given by \( \phi_y = \arg(G(j\omega)) \).

**Assumption 2.** The phase response of the plant is a monotounous function of frequency \( \omega \). It is shown below that this assumption is not a limitation of the approach but allows us to eliminate some complex scenarios, when a few different modes of oscillations may be available in a system.

Under Assumption 2 the phase shift provided by the plant always belongs to the interval \([\frac{-\pi}{2}d, 0]\), where \( d \) is relative degree of the plant.

Consider now how the phase lag is produced in system Figure 1. For producing self-sustained oscillations of constant frequency \( \Omega \), it is necessary that the phase balance condition should be satisfied:

\[
\phi_y + \phi_r = -\pi,
\] (19)
where $\phi_r$ is a phase lag introduced by the relay: $\phi_r = -\arcsin \frac{b}{a}$.  

If the phase lag cannot reach $-\pi$ at any frequency, what is termed by the phase deficit in Reference 12, the switching of the relay still occurs but with shortening of the period of the oscillation at every switch. This may happen if hysteresis value of the relay equals zero (eg, is set to zero from a nonzero initial value when self-excited periodic motion were taking place). This would result in the phase lag of the relay $\phi_r = 0$ and the necessity for the frequency to grow to the value providing the phase balance (normally $\omega \to \infty$). During this process period shortening happens because the phase balance cannot be provided and the switching condition ($y = 0$) is produced before the end of the period of the oscillation of the input signal (Figure 3). In accordance with this consideration, the phase deficit can be viewed in relay systems as the following limit:

$$\phi_d = \pi + \lim_{\omega \to \infty} \phi_y(\omega).$$

(20)

Let us consider the following model of excitation of vanishing transient oscillations, when the series of switches of the relay occur as a result of the oscillations of the output signal (Figure 3): $i$th switch of the relay causes an oscillation of the output signal of a certain frequency $\omega_i$; the output oscillation is considered as a propagation of the control signal oscillation; it is shifted with respect to the control signal (switch of the relay) by the phase angle $\phi_{yi} = \arg G(j\omega_i)$ or by time $\Delta T_i = (\pi + \phi_{yi})/\omega_i$; and the crossing of zero level by the output oscillation initiates the next switch of the relay.

It is easy to understand now why for the finite-time convergence of the process it is necessary that the phase deficit must be a positive quantity. If the phase lag (absolute value) of the plant at the current frequency is smaller than $\pi$ then every next period of the oscillation is shortened in accordance with the phase shift at the current frequency, which approximately can be described as:

$$T_{i+1} = T_i - \frac{\pi + \phi_{yi}}{\omega_i} = T_i - \Delta T_i,$$

(21)

where $T_i$ is the $i$th interval between two consecutive switches of the relay $\omega_i = \frac{\pi}{T_i}$, $\phi_{yi} = \arg G(j\omega_i)$.

Let us estimate the infinite sum $\sum_{i=1}^{\infty} T_i$, which gives the convergence time. In accordance with Assumption 2, $\phi_{yi} \geq \lim_{\omega \to \infty} \phi_y(\omega)$, and, therefore, as per phase deficit definition $\phi_{yi} \geq \phi_d - \pi$. Therefore,

$$\frac{\Delta T_i}{T_i} = \frac{\pi + \phi_{yi}}{\pi} \geq \frac{\phi_d}{\pi}$$

(22)

It follows from (22) that $\Delta T_i \geq \frac{\phi_d}{\pi} T_i$. And therefore, (21) leads to the following inequality:

$$T_{i+1} \leq T_i - \frac{\phi_d}{\pi} T_i = \left(1 - \frac{\phi_d}{\pi}\right) T_i,$$

(23)
Considering the sequence $T_1, T_2, T_3 \ldots$ as a geometric series with common ratio not exceeding $r = 1 - \frac{\phi_d}{\pi} < 1$, we find an estimate for the infinite sum:

$$\sum_{i=1}^{\infty} T_i \leq \frac{1}{1-r} T_1 = \frac{\pi}{\phi_d} T_1.$$  \hfill (24)

Formula (24) gives an estimate of the time required for the oscillatory process to end. Therefore, it gives an estimate of the convergence time and shows that with $\phi_d > 0$ convergence is finite-time. If $\phi_d = 0$ then finite-time convergence cannot be ensured. In fact, it is proved in other research (see eg, 12) that convergence is asymptotic in this case, or takes infinite-time, which fully agrees with (24).

With the notion of phase deficit available, convergence type: whether it is finite-time or asymptotic, can be easily determined through a relatively simple input-output analysis of the plant or of the open-loop system. An example of analysis of a system with the super-twisting algorithm is given in the following section.

### 4 ANALYSIS OF PHASE DEFICIT OF THE SUPER-TWISTING CONTROLLER

The super-twisting controller was proposed in Reference 15 as follows:

$$\dot{u}_1 = c \, \text{sign}(\sigma),$$  \hfill (25)

$$u_2 = \lambda |\sigma|^{q/p} \, \text{sign}(\sigma),$$  \hfill (26)

$$u = u_1 + u_2,$$  \hfill (27)

where $c$ is the amplitude of the relay, $\lambda > 0$ and $q < p$, $q$ and $p$ integers, $\sigma = -y$ is the controller input, $u$ controller output.

With the approach presented above let us find the phase deficit in the system with the super-twisting controller. In the beginning let us assume that the linear plant is an integrator, which satisfies the requirement for the plant to be relative degree one dynamics.\textsuperscript{15} We consider the system as an open-loop relay system. To be able to do this we transpose the nonlinearity as shown in Figure 4.

With this representation (Figure 4) it becomes possible to find the reaction of the output $y(t)$ to the periodic input $\sigma(t) = a \sin(\omega t)$. Let us use the DF method to find propagation of the first harmonic. Because the amplitude of the first harmonic of the relay output is $a_r = \frac{4c}{\pi}$, the amplitude of the integrator output signal $u_1(t)$ is $a_1 = \frac{4c}{\pi \omega}$. To find the amplitude of $y(t)$, let us close the loop through $f(y)$ replacing the nonlinearity with its DF:\textsuperscript{20}

$$N_f(a) = \frac{2\lambda a^{\frac{q}{p} - 1} \Gamma \left( \frac{a}{2p} + 1 \right)}{\sqrt{\pi} \Gamma \left( \frac{a}{2p} + 1.5 \right)},$$  \hfill (28)

where $\Gamma$ is gamma-function.\textsuperscript{23} Rewrite (28) as

$$N_f(a) = \beta a^\gamma,$$  \hfill (29)
where \( \beta = \frac{2\sqrt{\frac{q^2}{p^2} + 1}}{\sqrt{\pi}} \), \( \gamma = \frac{q}{p} - 1 \).

The transfer function of the loop consisting of the integrator and the nonlinear function \( f(y) \) (see Figure 4) can be written as:

\[
G_f(s) = \frac{1}{s + N_f(ay)}.
\]

where \( ay \) is the amplitude of the oscillations of \( y(t) \), \( N_f(ay) \) is given by (29). Amplitude \( ay \) is the result of propagation of the oscillation having amplitude \( a_1 \) and frequency \( \omega \) through \( G_f \), which yields the following equation for \( ay \): \( ay = a_1 |G_f(j\omega)| \),

which results in:

\[
ay = a_1 ||G_f(j\omega)|| = \left| a_1 \right| |\frac{a_1}{\sqrt{\frac{\pi}{2}}\beta a_1^2 \gamma} + j\omega|,
\]

or

\[
\omega^2 a_y^2 + \beta^2 a_y^{2r+2} - a_1^2 = 0,
\] (31)

Formula (31) can be considered an equation for \( ay \) that can be found for every value of the frequency of the oscillations (amplitude \( a_1 \) is a function of the frequency too). It does not have analytic solution in a general case. However, it may have solution for some values of \( \gamma \). From the definition of \( \gamma = \frac{q}{p} - 1 \), it follows that \(-1 < \gamma < 0\), and Equation (31) has analytic solution for \( \gamma = -0.5 \), for example. This value of \( \gamma \) corresponds to the most popular fractional power of the nonlinearity used in the super-twisting controller: \( \frac{q}{p} = 0.5 \). Let us consider this case in detail.

If \( \gamma = -0.5 \) Equation (31) is a quadratic equation:

\[
\omega^2 a_y^2 + \beta^2 a_y^2 - a_1^2 = 0,
\] (32)

that has the following solution:

\[
a_y = -\frac{\beta^2 + \sqrt{\beta^4 + 4\omega^2 a_1^2}}{2\omega^2},
\] (33)

The other root of (32) is negative and not considered as a physically meaningful solution.

We can find the phase deficit of the open-loop system in Figure 4 as per (20) by obtaining first the phase lag at a finite frequency \( \omega \) and then considering the limit at \( \omega \to \infty \).

\[
\pi + \phi_y(\omega) = \frac{\pi}{2} - \arctan \left( \frac{1}{\sqrt{2}} \sqrt{-1 + \sqrt{1 + \frac{64c^2}{\pi^2 \beta^4}}} \right)
\] (34)

One can see from (34) that the phase lag given by the open-loop system in Figure 4 does not depend on the frequency of the oscillations, which is a remarkable property of the super-twisting algorithm. The phase deficit is always positive, which leads to a finite-time convergence.

\[
\phi_d = \frac{\pi}{2} - \arctan \left( \frac{1}{\sqrt{2}} \sqrt{-1 + \sqrt{1 + \frac{64c^2}{\pi^2 \beta^4}}} \right) > 0.
\] (35)

In accordance with the criterion proposed in Reference 12 and the analysis given above, the closed-loop system with the super-twisting algorithm and the plant given by the integrator would exhibit finite-time convergence.

For the higher order dynamics of the plant having relative degree one, the constancy of the phase lag cannot be ensured. But the analysis of the phase deficit can be found in the similar way. This leads to the following formula for the phase deficit:

\[
\phi_d = \frac{\pi}{2} - \arctan \left( \frac{1}{\sqrt{2}} \sqrt{-1 + \sqrt{1 + \frac{64c^2 a_n^2}{\pi^2 \beta^4 b_{n-1}^2}}} \right) > 0.
\] (36)
Therefore, the closed-loop system with the super-twisting algorithm and the plant having relative degree one would exhibit finite-time convergence.

5 | EXAMPLES

Example 1. In Figure 3 and Table 1 an example of transient oscillations for the following system is given: the controller is an ideal relay and the plant provides a fixed value of the phase lag of $-173.69^\circ$ at every frequency. One can see that distribution of value of $T_i$, both actual and predicted by (21), are geometric series. However, the common ratio for the actual distribution of half-periods (0.88) is smaller than the predicted one (0.96). As a result, the convergence time predicted by formula (24) is 807.3, which is higher than the actual one (235.8). One can see that even relatively small differences in the common ratio value may lead to large differences in the convergence time. However, as noted above, the developed model provides conservative estimates.

The conclusions that can be drawn from the considered example are:

- Formula (24), indeed, gives a conservative estimate of the convergence time.
- Despite this inaccuracy, a clear distinction between finite-time convergence and infinite-time (asymptotic) convergence is possible through analysis of the phase deficit.

Example 2. This example demonstrates the constancy of the phase lag of the open-loop system under the super-twisting algorithm, regardless of the input frequency. Consider an open-loop system having the super-twisting controller (25) to (27) and an integrator being the plant (Figure 4, with the dashed line disregarded), and with the following parameters: $q_p = 0.5$, $c = 0.8$, $\lambda = 0.6$. According to the DF formula for the nonlinearity, $\beta = \frac{2.651(1.25)}{\sqrt{2}(1.75)} = 0.6677$. And as per (35) the phase deficit is

$$\phi_d = \frac{\pi}{2} - \arctan\left(\frac{1}{\sqrt{2}}\sqrt{-1 + \sqrt{1 + \frac{64-0.8^2}{x^20.6677}}} \right) = 0.6354 \text{ rad} = 36.41^\circ.$$  

Results of the simulations and analysis through the formulas presented above are summarized in Table 2. One can see a good match between the analytical solutions and the results of the simulations, and a support to the conclusion that the phase lag in the system with the super-twisting algorithm does not depend on the frequency.

6 | CONCLUSIONS

A frequency-domain interpretation of the finite-time convergence phenomenon is proposed. It is based on the consideration of the phase lag in an open-loop system that can be realized under the harmonic excitation of frequency varying from zero to infinity. It is shown that if the phase lag is smaller than $\pi$ (phase response is above $-180^\circ$), which constitutes a positive phase deficit, then the system with the controller being an ideal relay shows finite-time convergence. It is difficult at this time to provide examples in which other methods fail and the developed approach based on the phase deficit

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The method of realization of the plant having a fixed phase lag at any frequency is based on the property of the super-twisting algorithm given by formula (34).

| $i$ | Actual $T_i$ | Predicted $T_i$ |
|-----|--------------|-----------------|
| 1   | 28.3         | 28.3            |
| 2   | 25.0         | 27.3            |
| 3   | 22.0         | 26.4            |
| 4   | 19.4         | 25.6            |
| 5   | 17.2         | 25.0            |
TABLE 2 Frequency response of super-twisting algorithm

| Frequency $\omega_{in}$ (rad/s) | 10    | 40    | 120   |
|----------------------------------|-------|-------|-------|
| Amplitude $a_y$ (simulations)    | $8.25\times10^{-3}$ | $5.15\times10^{-4}$ | $5.71\times10^{-5}$ |
| Amplitude $a_y$ (DF analysis)    | $8.20\times10^{-3}$ | $5.12\times10^{-4}$ | $5.89\times10^{-5}$ |
| $\varphi(\omega_{in})$ (°)      | $-143.15$ | $-143.35$ | $-143.26$ | simulations |
| $\varphi(\omega_{in})$ (°)      | $-143.59$ | $-143.59$ | $-143.59$ | DF analysis |

analysis might succeed. This is because the algorithms known from publications are already accompanied by finite-time stability proofs. However, in the future the situation may change: a developed algorithm can be first tested by the phase deficit criterion and after that, if necessary, efforts can be put into finding a Lyapunov function.

The undertaken analysis of a system with the super-twisting algorithm shows that the phase lag may be independent of the frequency of excitation. This remarkable property can possibly be utilized in some applications.

The provided analysis and the developed approach shed light on the complex relationship between the conventional (classical or first-order) SM control, second-order SM control, and higher order SM control. According to the presented approach, the difference between the first-order SM and the second-order SM is in the value of the phase deficit: it is 90° for the first-order SM and smaller than 90° for the second-order SM. It is a negative value for systems having nonvanishing self-sustained oscillations. In this respect, it is currently unclear how the third- and higher-order SM fits into this classification and whether the higher-order SM is a sufficiently distinct type of SM or through some transformations it can be presented as one of the former types of SM. In either case, we believe that the presented approach contributes to the fuller understanding of the complex phenomenon of finite-time convergence and SM control.

In the future research the accuracy can possibly be improved through the dynamic harmonic balance,21,24 which can provide a more accurate model of the convergence transient process.

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REFERENCES

1. Levant A. Universal SISO sliding-mode controllers with finite-time convergence. *IEEE Trans Automat Control*. 2001;46(9):1447-1451.
2. Moreno JA, Osorio M. Strict Lyapunov functions for the super-twisting algorithm. *IEEE Trans Automat Control*. 2012;57(4):1035-1040.
3. Polyakov A, Poznyak A. Reaching time estimation for “super-twisting” second order sliding mode controller via Lyapunov function designing. *IEEE Trans Automat Control*. 2009;54(8):1951-1955.
4. Orlov Y, Aoustin Y, Chevallereau C. Finite time stabilization of a perturbed double integrator - Part i: Continuous sliding modebased output feedback synthesis. *IEEE Trans Automat Control*. 2010;56(3):1035-1040.
5. Seeber R, Horn M, Fridman L. A novel method to estimate the reaching time of the super-twisting algorithm. *IEEE Trans Automat Control*. 2018;63(12):4301-4308. https://doi.org/10.1109/TAC.2018.2812789.
6. Utkin V. On convergence time and disturbance rejection of super-twisting control. *IEEE Trans Automat Control*. 2013;58(8):2013-2017.
7. Rosales A, Shtessel Y, Fridman L, Panathula CB. Chattering analysis of HOSM controlled systems: frequency domain approach. *IEEE Trans Automat Control*. 2017;62(8):4109-4115.
8. Shtessel Y, Fridman L, Rosales A, Panathula CB. Practical stability phase and gain margins concept. *Advances in Variable Structure Systems and Sliding Mode Control Theory and Applications*. New York, NY: Springer; 2018:101-132.
9. Bhat SP, Bernstein DS. Finite-time stability of homogeneous systems. Paper presented at: Proceedings of the 1997 American Control Conference (ACC); 1997:2513-2514; Albuquerque, New Mexico.
10. Orlov Y. Finite time stability and robust control synthesis of uncertain switched systems. *SIAM J Control Optim*. 2004;43(4):1253-1271.
11. Haddad WM, L’Afflitto A. Finite-time partial stability theory and fractional Lyapunov differential inequalities. Paper presented at: Proceedings of the 2015 American Control Conference (ACC); 2015:5347-5352; Chicago, IL.
12. Boiko I. On frequency-domain criterion of finite-time convergence of second-order sliding mode control algorithms. *Automatica*. 2011;47(9):1969-1973.
13. Utkin V. Sliding Modes in Control and Optimization. Berlin, Germany: Springer-Verlag; 1992.
14. Utkin V, Guldner J, Shi J. Sliding Mode Control in Electromechanical Systems. London, UK: Taylor and Francis; 1999.
15. Levantovsky LV. Sliding order and sliding accuracy in sliding mode control. *Int J Control*. 1993;58(6):1247-1263.
16. Bartolini G, Ferrara A, Usai E. Chattering avoidance by second-order sliding mode control. *IEEE Trans Automat Control*. 1998;43(2):241-246.
17. Bartolini G, Ferrara A, Levant A, Usai E. On second order sliding mode controllers. In: Young KD, Ozguner U, eds. Variable Structure Systems, Sliding Mode and Nonlinear (Lecture Notes in Control and Information Science). Vol 247. London, UK: Springer-Verlag; 1999:329-350.
18. Man Z, Poplinsky AP, Wu HR. A robust MIMO terminal sliding mode control for rigid robotic manipulators. *IEEE Trans Automat Control*. 1994;39(12):2464-2468.
19. Yu X, Wu Y, Man Z. On global stabilization of nonlinear dynamical systems. In: Young KD, Ozguner U, eds. Variable Structure Systems, Sliding Mode and Nonlinear Control. London, UK: Springer; 1999:109-122.
20. Atherton DP. Nonlinear Control Engineering - Describing Function Analysis and Design. Van Nostrand Company Limited: Workingham, Berks, UK; 1975.
21. Boiko I. Non-parametric Tuning of PID Controllers: Modified Relay Feedback Test Approach. London, UK: Springer; 2013.
22. Boiko I. Discontinuous Control Systems: Frequency-Domain Analysis and Design. Boston, MA: Birkhauser; 2009.
23. Burlington RS, Handbook of Mathematical Tables and Formulas. New York, NY: McGraw-Hill; 1973.
24. Boiko I. Dynamic harmonic balance principle and analysis of rocking block motions. *J Frankl Inst*. 2012;349(3):1198-1212.

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