Mod $p$ Homology of Unordered Configuration Spaces of Surfaces

MATTHEW CHEN

MENTOR: ADELA ZHANG
MIT MATHEMATICS DEPARTMENT
Abstract

We prove that dimensions of the mod $p$ homology groups of the unordered configuration space $B_k(T)$ of $k$ points in a torus are equivalent to its Betti numbers for $p > 2$ and $k \leq p$. Hence, the integral homology has no $p$-power torsion. I use the same argument to explicitly compute the dimensions of the mod 2 homology groups of the unordered configuration space $B_2(\mathbb{RP}^{2n+1})$ of two points in any odd dimensional real projective space. The same argument works for the punctured genus $g$ surface with $g > 0$, thereby recovering a result of Brantner-Hahn-Knudsen via Lubin-Tate theory.

Summary

Given two surfaces, it is a classical interest of topologists to determine whether you can continuously deform one surface into the other, as if the two surfaces were like play-doh. One approach is to look for algebraic structures that remain unchanged regardless of how you deform either surface. Then, if the two surfaces have different such structures, we can conclude they cannot be deformed into one another. In this project, we compute a special algebraic structure for two classes of topological objects by breaking down the structure into different dimensional components. By computing the sizes of these different dimensional components, we can also better understand how complex these topological objects are—the larger the size, the more complex.
1 Introduction

Configuration spaces are important mathematical objects with various applications, such as motion planning robotics in Farber [1]. For a closed manifold $M$, the configuration space $\text{Conf}_k(M)$ is defined as the space of $k$-tuples of distinct points of $M$. The $k$th symmetric group acts freely on $\text{Conf}_k(M)$ by permuting the distinct points, so the quotient of the action can be used to obtain $B_k(M)$ as the unordered configuration space of $k$ points in $M$.

One can associate a topological space with its homology, which is a collection of algebraic structures called homology groups. Having larger and more complicated homology groups indicates a more complex space. Moreover, the homology of a space is a homotopy invariant; that is, if two topological spaces have different homology groups, there exists no continuous deformation transforming one space to the other, allowing topologists to better classify and distinguish spaces.

It has been a classical challenge to compute the homology of configuration spaces over various coefficient rings for different possible manifolds. Research on configuration spaces dates back to the 1960s, with generalizations of $B_k(M)$ to labeled configuration spaces beginning in 1973 with Segal [2] and McDuff [3]. A recent development in the computation of the homology of configuration spaces by Knudsen [4] showed that when $M$ is a framed manifold, the homology with coefficients in a ring $R$ of labeled configuration spaces can be computed using a series of homological approximations called a spectral sequence. When $R = \mathbb{Q}$, Knudsen [5] showed that the spectral sequence could be computed from only its first approximation. Building on the work of Knudsen, Drummond-Cole and Knudsen [6] computed the dimensions of the rational homology groups of unordered configuration spaces of surfaces.

This paper is primarily interested in mod $p$ homology; that is, homology over finite field $\mathbb{F}_p$. However, the mod $p$ homology of unordered configuration spaces are difficult to compute. The only cases known classically are $M = \mathbb{R}^\infty$ by May [7] in 1972, $M = \mathbb{R}^n$ by Cohen [8] in 1986, and any odd-dimensional $M$ by Bödigheimer et al. [9] in 1988. Using Knudsen’s spectral sequence for the homology of labeled configuration spaces with coefficients in Lubin-Tate $E$-theory, Brantner et al. [10] computed the mod $p$ homology of $B_p(\Sigma_{g,1})$ in 2019, where $\Sigma_{g,1}$ is the genus $g$ surface with a point removed and $p > 2$. However, their methods do not extend easily to genus $g$ surfaces. In 2021, Zhang [11] showed that for $k = 2$ or 3, Knudsen’s spectral sequence for the mod $p$ homology of $B_k(M)$ remains relatively computable for any framed manifold $M$, giving an algorithm to compute an explicit basis for the mod $p$ homology of $B_k(M)$.

We build off the spectral sequence from Zhang [11] to compute the mod $p$ homology of $B_k(T)$ for $p \geq 5$ and larger $k$, where $T$ denotes the torus, the only closed and parallelizable genus $g$ surface. In Section 2 I define the homology and introduce preliminaries. We outline the general method in Section 3 which will be used for all later sections for given $M$ and $k$. Specifically, in Section 4 we prove that when $M = T$ and $p \geq 5$, the dimensions of the mod $p$ homology groups of $B_k(T)$ agree with the Betti numbers obtained by Drummond-Cole and Knudsen [6] for $k \leq p$. In Section 5 I
compute the mod 2 homology when $M = \mathbb{R}P^{2n+1}$ for $k = 2$ and $n \geq 0$.

## 2 Preliminaries

A chain complex $C_\bullet$ over a ring $R$ is a sequence of $R$-modules

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$
equipped with $R$-module homomorphisms $\partial_n : C_n \to C_{n-1}$ such that $\partial_n \partial_{n+1} = 0$ for all $n$. For a given topological space $X$, one can associate a chain complex $C_\bullet(X; R)$ with $C_n(X; R)$ generated by the $n$-dimensional simplices of $X$. The homology of $X$ with coefficients in $R$, denoted by $H_n(X; R)$, is a sequence of $R$-modules $H_0(X; R), H_1(X; R), H_2(X; R), \ldots$, where the $n$th homology group is the quotient

$$H_n(X; R) := \ker \partial_n / \text{im} \partial_{n+1}.$$

The intuition behind this definition is that, over a rational coefficient ring, the rank of the basis of $H_n$ is the number of $n$-dimensional holes of $X$. Indeed, $\ker \partial_n$ is the set of $n$-dimensional cycles, and $\text{im} \partial_{n+1}$ is the set of $n$-dimensional boundaries of $(n + 1)$-dimensional elements, so the quotient leaves non-boundary cycles—or in other words, holes. We are interested in the dimension of the homology groups over $\mathbb{F}_p$, namely $H_i(X; \mathbb{F}_p) \cong \mathbb{F}_p^{\dim H_i(X; \mathbb{F}_p)}$.

**Example 2.1** (Hatcher [12]). For an $n$-sphere $S^n$ and any ring $R$, the 0th homology group is $H_0(S^n; R) \cong R$, the 0th homology group is $H_n(S^n; R) \cong R$, and 0 for all other groups.

**Example 2.2** (Hatcher [12]). For any genus $g$ surface $\Sigma_g$ and ring $R$, the 0th homology group is $H_0(\Sigma_g; R) \cong R$, the 1st homology group is $H_1(\Sigma_g; R) \cong R^{2g}$, the 2nd homology group is $H_2(\Sigma_g; R) \cong R$, and 0 for all other groups.

Henceforth, denote the generators of $H_*(\Sigma_g; R)$ with $c$ as the generator of $H_2(\Sigma_g; R)$, generators $a_1, \ldots, a_g, b_1, \ldots, b_g$ for $H_1(\Sigma_g; R)$, and $d$ as the generator of $H_0(\Sigma_g; R)$.

**Example 2.3.** For odd dimensional real projective space $\mathbb{R}P^{2n+1}$, the $i$th mod 2 homology group is $H_i(\mathbb{R}P^{2n+1}; \mathbb{F}_2) \cong \mathbb{F}_2$ for $0 \leq i \leq 2n + 1$. Equivalently, one can let $x^k$ denote the generator for $H_k$, so the homology is

$$H_i(\mathbb{R}P^{2n+1}; \mathbb{F}_2) = \mathbb{F}_2[x] / x^{2n+2}.$$

For closed and oriented manifolds $M^n$, there is a dual notion to homology called cohomology. The $k$th cohomology group, $H^k(M^n)$, was shown by Poincaré duality to satisfy

$$H^k(M^n) \cong H_{n-k}(M^n).$$

For Examples 2.1, 2.2 and 2.3, $H^* \cong H_*$. There also exists a bilinear operation

$$\cup : H^p(M^n; R) \times H^q(M^n; R) \to H^{p+q}(M^n; R).$$
on the cohomology called the \textit{cup product}. For classes $a \in H^k(M^n; R)$ and $b \in H^l(M^n; R)$, $a \cup b$ is a class in $H^{k+l}(M^n; R)$. Although there exists a formula for the cup product in Appendix \[A\], the cup products we compute lie in cohomology groups with $\dim H^{k+l} = 0$ or 1 for the surfaces we encounter, making it trivial to compute $a \cup b$.

The chain complexes we work with are defined in terms of free shifted Lie algebras, free \(\overline{R}\) algebras, or as free \(s\text{Lie}_\overline{R}\) algebras. We define all three algebras.

A \textit{shifted Lie algebra} \(L\) over a ring \(R\) is a graded \(R\)-module equipped with a \textit{shifted Lie bracket} \([-,-]: L_i \otimes L_j \to L_{i+j-1}\), where \(L_k\) denotes the elements of \(L\) with \textit{internal degree} \(k\). \(L\) is graded by internal degree, and for \(x \in L\), we denote the internal degree of \(x\) as \(|x|\). For brevity, we refer to the shifted Lie bracket as simply a Lie bracket henceforth. The Lie bracket obeys the following properties.

- \textbf{Bilinearity:} \([ax + by, z] = a[x, z] + b[y, z]\) and \([z, ax + by] = a[z, x] + b[z, y]\),
- \textbf{Graded anti-commutativity:} \([x, y] = (-1)^{|x||y|}[y, x]\),
- \textbf{Jacobi Identity:} \((−1)^{|z||y|[x, [y, z]]} + (−1)^{|y||z|[y, [z, x]]} + (−1)^{|x||z|[z, [x, y]]} = 0\).

The free shifted Lie algebra on an \(R\)-module \(M\) also admits a \textit{weight grading}, making it a bigraded module. Explicitly, every element of \(M\) has weight 1, and the Lie bracket \([x, y]\) adds the weights of \(x, y\). For a chain complex \(C\), let\(\text{wt}_k(C)\) denote the weight \(k\) part of \(C\).

There is a free functor \(\text{Free}^{s\text{Lie}}\) sending an \(R\)-module \(M\) to the free shifted Lie algebra on \(M\), which can be thought of as applying iterations of Lie brackets on elements in \(M\) but not evaluating the Lie bracket. One can apply \(\text{Free}^{s\text{Lie}}\) to the underlying \(R\)-module of a shifted Lie algebra \(L\) over \(R\); note that this is not the same as the Lie bracket being in \(L\). This gives rise to the shifted Lie algebra structure map \(d : \text{Free}^{s\text{Lie}}(L) \to L\) of \(L\). For instance, suppose \(x\) and \(y\) are elements of \(L\). Then

\[ [x, y] \xrightarrow{d} [x, y], \]

where the different colored Lie brackets indicate that the free functor is applying the red Lie bracket to act on \(x\) and \(y\), while \(d\) maps to the element \([x, y]\) in \(L\). The free functor and structure maps can naturally extend into a longer chain complex. For example, let \(x, y, z\) be generators of \(L\) and \(d_1 : \text{Free}^{s\text{Lie}} \circ \text{Free}^{s\text{Lie}}(L) \to \text{Free}^{s\text{Lie}}(L)\) and \(d_2 : \text{Free}^{s\text{Lie}}(L) \to L\). The color of the functor corresponds to which color Lie bracket it is applying. Then

\[
\begin{align*}
[[x, y], z] & \xrightarrow{d_1} [[x, y], z] \\
 & \xrightarrow{d_2} [x, y], z.
\end{align*}
\]

In short, each of the Lie brackets decrease in level after each structure map, represented by a change to the next color.

On the other hand, the \textit{free} \(\overline{R}\)-\textit{algebra} is bigraded by internal degree and weight and generated by a set of unary operations \(\{\beta^eQ^i \ | \ i \in \mathbb{Z}, \ e = 0, 1\}\). For a free module \(M\) over \(\overline{R}\), these unary operations satisfy the following properties.
Definition 2.1 (Kjaer [13]). Let $p > 2$ be a prime. Suppose that $x$ is an element in a module $M$ over the algebra $\bar{R}$. Then the following hold:

- $\beta^i \bar{Q}^i x \neq 0$ for $\epsilon = 0, 1$ if and only if $i \geq |x|/2$,
- The internal degree of $\beta^i \bar{Q}^i x$ is $|\beta^i \bar{Q}^i x| = 2i(p-1) + |x| - 1 - \epsilon$,
- The weight of $\beta^i \bar{Q}^i x$ is $p$ times the weight of $x$.

There is a free functor $\text{Free}^\bar{R}$ sending an $\mathbb{F}_p$-module $M$ to the free $\bar{R}$-module over $M$, behaving analogously to $\text{Free}^{\text{sLie}}$ but applying $\beta^i \bar{Q}^i$ as opposed to a Lie bracket.

The free $\text{sLie}_{\bar{R}}$ algebra is equipped with both unary operations and the Lie bracket. The two structures interact in the following way: for any module $M$ over $\text{sLie}_{\bar{R}}$, Kjaer [13] proved that for all $x$ and $y$ in $M$, $[x, \beta^i \bar{Q}^i y] = 0$. There also exists a free functor for the $\text{sLie}_{\bar{R}}$-algebra, with underlying $\mathbb{F}_p$-module given by

$$\text{Free}^{\text{sLie}}_{\bar{R}}(M) = \text{Free}^\bar{R} \circ \text{Free}^{\text{sLie}}(M).$$

For a shifted Lie algebra $L$, it is known that there exists an isomorphism between the shifted Lie algebra homology of $L$ and the homology of another chain complex called the Chevalley-Eilenberg complex [14] of $L$. We define both homologies.

Definition 2.2. The shifted Lie algebra homology of $L$ is the homology of the chain complex $D_{\bullet}$ with $D_i = (\text{Free}^{\text{sLie}})^{\circ i}(L)$ and boundary maps $\partial_i : D_i \to D_{i-1}$ given by

$$\partial_i = \sum_{j=0}^{i} (-1)^j d_j,$$

where $d_j$ is induced by the structure map $\text{Free}^{\text{sLie}} \circ (\text{Free}^{\text{sLie}})^{\circ (i-j)}(L) \to (\text{Free}^{\text{sLie}})^{\circ (i-j)}(L)$.

Computing the homology of $D_{\bullet}$ is equivalent to computing the homology of the Chevalley-Eilenberg complex over rationals. The Chevalley-Eilenberg complex is expressed in terms of exterior and divided power algebras. The divided power algebra $\Gamma$ is a commutative ring $A$ equipped with ideal $I$ and divided power operations $\gamma_n$ for $n \geq 1$ satisfying a set of conditions available in Appendix A. Of most relevance to us is if $px \neq 0$ for all $x \in A$ and $x \neq 0$. In such instance, we say $A$ is $p$-torsion free, and $n! \gamma_n(x) = x^n$ for all $x \in I$, $n \geq 0$. Over an ideal $I$ where $x^n$ is an $n!$-multiple, such as in $\mathbb{Q}$, then $\gamma_n(x) = x^n/n!$.

For a vector space $V$ over a field, the exterior algebra $\Lambda V$ is an algebra satisfying associativity and unitality. It is equipped with the binary operation wedge product $\wedge$ with the identity that $x \wedge x = 0$ for all $x \in V$. For generators $x_1, x_2, \ldots x_n$, denote for brevity’s sake,

$$x_1 \wedge x_2 \wedge \cdots \wedge x_n = \langle x_1, x_2, \ldots, x_n \rangle.$$
Definition 2.3 (Chevalley-Eilenberg [14]). For a shifted Lie algebra \( L \) over a field, let \( L_{\text{even}} \) and \( L_{\text{odd}} \) denote the elements in \( L \) with even and odd degree, respectively. The Chevalley-Eilenberg complex of \( L \) over any coefficient ring \( R \) is the chain complex

\[
CE(L) = (\Gamma(L_{\text{even}}) \otimes \Lambda(L_{\text{odd}}), \partial),
\]

where the value of differential \( \partial \) on a general element

\[
\gamma_{k_1}(x_1)\gamma_{k_2}(x_2)\cdots\gamma_{k_m}(x_m)(y_1, y_2, \ldots, y_n) \in \Gamma(L_{\text{even}}) \otimes \Lambda(L_{\text{odd}})
\]

is the expression

\[
\begin{align*}
&\sum_{1 \leq i < j \leq m} \gamma_{k_1}(x_1)\cdots\gamma_{k_{i-1}}(x_i)\cdots\gamma_{k_{j-1}}(x_j)\cdots\gamma_{k_m}(x_m)\langle[x_i, x_j], y_1, \ldots, y_n \rangle \\
&+ \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} \gamma_{k_1}(x_1)\cdots\gamma_{k_m}(x_m)\langle[y_i, y_j], y_1, \ldots, \hat{y}_i, \ldots, \hat{y}_j, \ldots, y_n \rangle \\
&+ \frac{1}{2} \sum_{i=1}^m \gamma_{k_1}(x_1)\cdots\gamma_{k_{i-2}}(x_i)\cdots\gamma_{k_m}(x_m)\langle[x_i, x_i], y_1, \ldots, y_n \rangle \\
&+ \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j-1} \gamma_1([x_i, y_j])\gamma_{k_1}(x_1)\cdots\gamma_{k_{i-1}}(x_i)\cdots\gamma_{k_m}(x_m)\langle y_1, \ldots, \hat{y}_j, \ldots, y_n \rangle.
\end{align*}
\]

where \( \hat{y} \) denotes the omission of \( y \) from the expression.

All elements in the chain complex are bigraded by simplicial degree and internal degree; the simplicial degree is the number of Lie brackets in an element and determines the chain group the element is in. For a chain complex of sLie\(_R\) algebras, applying a unary operation also adds one to the simplicial degree. The simplicial degree is commonly denoted by \( s \) and internal degree by \( t \). Together, \( s + t \) is the total degree of the element.

The dimension of the \( i \)th homology group over \( \mathbb{Q} \) is the \( i \)th Betti number \( \beta_i \). Drummond-Cole and Knudsen [6] computed the Betti numbers of \( B_k(\Sigma_g) \) for \( g > 0 \).

Theorem 2.1 (Drummond-Cole, Knudsen [6]). The Betti numbers of \( B_k(\Sigma_g) \) are equivalent to the Betti numbers of \( \text{wt}_k(CE(g')) \), where \( g' \) is the shifted Lie algebra

\[
g' = H^*(\Sigma_g; \mathbb{Q}) \otimes \text{FreeLie}_\mathbb{Q}(\mathbb{Q}\{x_2\}).
\]

The basis elements of \( g' \) with weight at most \( p \) are

\[
\begin{align*}
\{c \otimes x_2, a_i \otimes x_2, b_i \otimes x_2, d \otimes x_2\} & \quad 1 \leq i \leq g, \\
\{c \otimes [x_2, x_2], a_i \otimes [x_2, x_2], b_i \otimes [x_2, x_2], d \otimes [x_2, x_2]\} & \quad 1 \leq i \leq g.
\end{align*}
\]

Drummond-Cole and Knudsen [6] gave explicit formulas for the Betti numbers of \( B_k(\Sigma_g) \) for all \( k \) and \( g \). These formulas and tables for small values can be found in Appendix A.

There is the following well established inequality in mathematical folklore regarding mod \( p \) and rational homology groups of a finite type topological space \( X \).
Theorem 2.2. For any finite type topological space $X$, the dimension of $H_n(X; \mathbb{F}_p)$ is at most the $n$th Betti number of $X$ for all $n$.

3 General Method of Computing the $\mathbb{F}_p$ Homology

We present a general outline of the method we use to compute the mod $p$ homology of an arbitrary framed closed manifold $M^n$ of dimension $n$, building on Sections 6 and 7 in Zhang [11]. In subsequent sections, we implement this general algorithm for specific values of $k$ with respect to $p$ and framed manifolds $M^n$.

We compute the $\mathbb{F}_p$ homology of unordered configuration spaces using a spectral sequence. In general, a spectral sequence consists of bigraded objects $E^r_{s,t}$, called the $E^r$-pages, for $r \geq 2$ and differentials $d_r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ such that $E^{r+1}_{s,t} = H_s(E^r_{s,t}, d_r)$. For certain sequences, the spectral sequence converges to the $E^\infty$-page

$$E^2_{s,t} \to E^3_{s,t} \to \cdots \to E^\infty.$$ 

If $d_r = 0$, then $E^{r+1}_{s,t} \cong E^r_{s,t}$. We say that the spectral sequence collapses on the $E^r$-page when $d_i = 0$ for all $i \geq r$.

Denote by $\mathfrak{g}$ the $s\text{Lie}_{\mathbb{R}}$-algebra

$$\mathfrak{g} = H^s(M^n; \mathbb{F}_p) \otimes \text{Free}^{s\text{Lie}_{\mathbb{R}}}((\mathbb{F}_p \{x_n\})),$$

where the $\mathbb{F}_p$-module generator $x_n$ has internal degree $n$ and tensoring with an element of $H^i(M^n; \mathbb{F}_p)$ decreases the internal degree by $i$. Define a chain complex $C_\bullet(\mathfrak{g})$ as follows: $C_\bullet$ consists of chain groups $C_s = (\text{Free}^{s\text{Lie}_{\mathbb{R}}})^{s}(\mathfrak{g})$ and boundary maps $\partial_s : C_s \to C_{s-1}$ such that

$$\partial_s = \sum_{i=0}^{s} (-1)^i d_i,$$

where $d_i$ is the structure map

$$d_i : \text{Free}^{s\text{Lie}_{\mathbb{R}}} \circ (\text{Free}^{s\text{Lie}_{\mathbb{R}}})^{(s-i)}(\mathfrak{g}) \to (\text{Free}^{s\text{Lie}_{\mathbb{R}}})^{(s-i)}(\mathfrak{g}).$$

In particular, the differentials in the chain complex preserve the weight grading. Let $C^k_\bullet$ denote the weight $k \geq 1$ part of $C_\bullet$ and $D^k_\bullet$ the weight $k$ part of the shifted Lie algebra chain complex $D_\bullet$, defined in Definition 2.2. Then

$$H_s(C_\bullet) = \bigoplus_{k \geq 1} \text{wt}_k H_s(C^k_\bullet) = \bigoplus_{k \geq 1} H_s(C^k).$$

The $E^2$-page of the spectral sequence is $H_s(C_\bullet)$, and it converges to $\bigoplus_{k \geq 1} H_s(B_k(M^n); \mathbb{F}_p)$. More specifically, because $E^2$ and $E^\infty$ are both graded by weight, then we consider the spectral sequence in weight $k$ to compute $H_s(B_k(M^n); \mathbb{F}_p)$.

This gives a general approach for computing the mod $p$ homology for the configuration space of a genus $g$ surface: compute the homology of the weight $k$ part of $C_\bullet$ and find the nonzero differentials
in the spectral sequence to compute the $E^\infty$-page.

## 4 Mod $p$ Homology of Configuration Spaces of $M^n = T$

In this section, we compute the mod $p$ homology of the configuration spaces of $k \leq p$ points in a torus. Depending on the size of $k$ with respect to $p$, the operations in the weight $k$ part of $C_\ast$ differ, and we compute these cases separately. For this section, $M^n = T$ is a 2-dimensional manifold of genus 1, so in the chain complex $C_\ast(g)$,

$$g = H^*(T; \mathbb{F}_p) \otimes \text{Free}^\text{Lie}(\mathbb{F}_p\{x_2\}).$$

The basis elements of $g$ with weight at most $p$ are the following:

$$\begin{cases} 
\{c \otimes x_2, a_1 \otimes x_2, b_1 \otimes x_2, d \otimes x_2\} \\
\{c \otimes [x_2, x_2], a_1 \otimes [x_2, x_2], b_1 \otimes [x_2, x_2], d \otimes [x_2, x_2]\} \\
\{d \otimes \bar{\beta}^j Q^i x_2, a_1 \otimes \bar{\beta}^j Q^i x_2, b_1 \otimes \bar{\beta}^j Q^i x_2\} & j \geq 1, \epsilon \in \{0, 1\}, \\
\{c \otimes \bar{\beta}^j Q^i x_2\} & j \geq 0, \epsilon \in \{0, 1\}.
\end{cases}$$

The Lie bracket $[p \otimes x_2, q \otimes x_2]$ in $g$ for $p, q \in H^*(T; \mathbb{F}_p)$ is given by

$$[p \otimes x_2, q \otimes x_2] = (p \cup q)[x_2, x_2].$$

A unary operation in the $R$ algebra acts on $g$ via

$$\beta^j Q^i (p \otimes x_2) = p \otimes \beta^j Q^i x_2.$$

### 4.1 Homology for $k < p$ and $p \geq 3$

For $k < p$, all elements of weight $k$ must be the iteration of Lie brackets, for applying a unary operation via the functor $\text{Free}^\text{Lie}(R)$ to any element in $C_\ast$ results in weight $p > k$. As a result, $C^k \cong wt_k(CE(g; \mathbb{F}_p)) = D^k$, implying $E^2_{s,t} = H_*(C^k) \cong H_*(D^k)$. We show that the spectral sequence collapses on the $E^2$-page and compute the homology of the Chevalley-Eilenberg complex over $\mathbb{F}_p$.

**Theorem 4.1.** For $k < p$, the dimensions of the mod $p$ homology groups of the Chevalley-Eilenberg complex of $g$ as $\mathbb{F}_p$-modules equal the Betti numbers of the Chevalley-Eilenberg complex of $g'$ in Theorem 2.1.

**Proof.** For a given basis element $e$ with weight $k$ of the form $x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} (y_1, y_2, \ldots, y_n)$ in $CE(g'; \mathbb{Q})$, define the map $\phi$ mapping the basis of the weight $k < p$ part of $CE(g'; \mathbb{Q})$ to the basis of the weight $k < p$ part of $CE(g; \mathbb{F}_p)$ such that

$$\phi(e) = \phi(x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} (y_1, y_2, \ldots, \cdot)) = x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} (y_1, y_2, \ldots, y_n).$$

\[ (*) \]
Because $k < p$, then $k_i < p$ for all $i$, so the weight $k$ part of $CE(g'; \mathbb{F}_p)$ is $p$-torsion free and

$$\phi(e) = \gamma_{k_1}(x_1)\gamma_{k_2}(x_2) \cdots \gamma_{k_m}(x_m)(y_1, y_2, \ldots, y_n).$$

The basis elements only differ by a unit, so $\phi$ is a surjection mapping a basis in $CE(g'; \mathbb{Q})$ to a basis in $CE(g; \mathbb{F}_p)$. Moreover, two distinct basis elements cannot be mapped to the same element in $CE(g; \mathbb{F}_p)$ lest $x_i^{k_i}$ is the same for all $i$, so $\phi$ is also injective and bijective. Because $e$ and $\phi(e)$ differ only by a unit, then this bijection is also preserved under the differential. Therefore, when $e$ is in the rational homology of $CE(g'; \mathbb{Q})$, $\phi(e)$ is in the mod $p$ homology of $CE(g; \mathbb{F}_p)$.

We claim the spectral sequence over $\mathbb{F}_p$ collapses on the $E^2$-page. Suppose for the sake of contradiction there existed a nonzero differential $d_r$ in the spectral sequence with $r \geq 2$. Then, in the $E^{r+1}$-page, some elements vanish from the kernel $d_r$ and by extension from the $E^r$-page.

On the other hand, no elements vanish from the $E^r$-page in the rational homology from Knudsen [5]. Therefore, dim $H_*(E^r, d_r)$ over $\mathbb{F}_p$ is strictly less than dim $H_*(E^r, d_r)$ over $\mathbb{Q}$, contradicting Theorem 2.2 instead, it must be that the $E^2$-page over $\mathbb{F}_p$ collapses as well, so the dimensions of the mod $p$ homology of $CE(g; \mathbb{F}_p)$ agree with the Betti numbers of $CE(g'; \mathbb{Q})$.

Using the formula from Theorem 2.1 we can compute the Betti numbers of the rational homology case and subsequently the dimensions of the mod $p$ homology groups of $B_k(T)$ for $k < p$ by Theorem 4.1.

4.2 Homology for $k = p$ and $p \geq 5$

For $k = p$, all elements of weight $p$ must be either iterations of Lie brackets or a single unary operation applied to a weight one element $x \in g$. In other words,

$$C^p = D^p \oplus \{\beta^i \bar{Q}^i x \mid x \in wt_1(g)\}.$$

Hence, the homology of $C^p$ is

$$H_*(C^p) \cong H_*(D^p) \oplus \{\beta^i \bar{Q}^i x \in \ker d_1\},$$

where $\partial_1 : C_1 \to C_0$ is the boundary map. We first show that $H_*(D^p)$ is computable. Then, we find all $\beta^i \bar{Q}^i x \in \ker \partial_1$ and all nonzero differentials in the spectral sequences.

Lemma 4.2. In weight $p$, the dimensions of the mod $p$ homology groups of the Chevalley-Eilenberg complex of $g$ as $\mathbb{F}_p$-modules equal the Betti numbers of the Chevalley Eilenberg complex of $g'$.

Proof. For all basis elements with $k_i < p$ for all $i$, the same mapping as $(\ast)$ from the proof of Theorem 4.1 produces a valid bijection. For a weight one generator $x$, we map the remaining element $x^p$ over $\mathbb{Q}$ to the remaining basis element $\gamma_p(x)$ over $\mathbb{F}_p$. The basis element $x^p$ in $CE(g'; \mathbb{Q})$ corresponds to $p! \gamma_p(x) = 0$ in $CE(g; \mathbb{F}_p)$, so $CE(g; \mathbb{F}_p)$ is no longer $p$-torsion free.

It suffices to show that the differentials, which we denote by $\partial_\mathbb{Q}$ and $\partial_\mathbb{F}_p$, in each complex are equivalent for their respective elements. Because we are interested in the homology, it suffices to
show that \(x^p \in \ker \partial_Q\) if and only if \(\gamma_p(x) \in \ker \partial_{\Sigma_p}\). Indeed, applying the formula for the differential from Definition 2.3,

\[
\partial_Q(x^p) = \frac{1}{2} x^{p-2}[x, x],
\]

while over \(\mathbb{F}_p\),

\[
\partial_{\Sigma_p}(\gamma_p(x)) = \frac{1}{2} \gamma_{p-2}[x, x] = \frac{1}{2} \frac{x^{p-2}}{(p-2)!}[x, x].
\]

Because \((p-1)! \equiv -1 \pmod{p}\) from Wilson’s Theorem,

\[
\partial_{\Sigma_p}(\gamma_p(x)) = \frac{1}{2} \frac{x^{p-2}}{(p-2)!}[x, x] = \frac{1}{2} (1-p)x^{p-2}[x, x] = \frac{1}{2} x^{p-2}[x, x].
\]

Because the differentials are equivalent on \(x^p\) in \(CE(\mathfrak{g}; \mathbb{Q})\) and \(\gamma_p(x)\) in \(CE(\mathfrak{g}; \mathbb{F}_p)\), we conclude the bijection is indeed preserved under the differential. Consequently, the dimensions of the mod \(p\) homology groups of \(CE(\mathfrak{g}; \mathbb{F}_p)\) agree with the Betti numbers of \(CE(\mathfrak{g}; \mathbb{Q})\).

We can thus compute \(H_*(D^p)\) using the Betti number formulas in Theorem 2.1. To compute elements in the kernel of \(\partial_1\) with a unary operation, we claim that \(\beta^i \bar{Q}^i(q \otimes x_2) \in \ker \partial_1\) if and only if \(|q \otimes x_2|/2 < i < 1\) for generator \(q \in H^*(T)\). The lower bound is necessary for \(\beta^i \bar{Q}^i(a \otimes x_2)\) to be nonzero. For the upper bound, if \(i < 1 = |x_2|/2\), then

\[
\beta^i \bar{Q}^i(a \otimes x_2) = a \otimes \beta^i \bar{Q}^i x_2 = a \otimes 0 = 0.
\]

If \(i \geq |x_2|/2\), then \(\beta^i \bar{Q}^i x_2 \neq 0\) and \(\beta^i \bar{Q}^i(a \otimes x_2)\) would not be in the kernel.

Therefore, the only elements in \(\{\beta^i \bar{Q}^i x \in \ker \partial_1\}\) are \(\bar{Q}^0(c \otimes x_2)\) and \(\beta \bar{Q}^0(c \otimes x_2)\). We have computed \(H_*(C^p)\) and, equivalently, the weight \(p\) part of the \(E_{s+t}^2\)-page. It remains to find the nonzero differentials in the spectral sequence in order to compute \(E^\infty\). We begin with the following lemma proving there exists a unique element with minimal total degree.

Lemma 4.3. In the homology of the weight \(p\) part of the Chevalley-Eilenberg complex, \(\gamma_p(c \otimes x_2)\) is the unique element with minimal dimension \(s + t = 0\).

Proof. From Definition 2.1 given an element \(e = \gamma_{k_1}(x_1) \cdots \gamma_{k_i}(x_i)(y_1, \ldots, y_j)\) in \(D^p\) with simplicial degree \(s\), the total degree of \(e\) is

\[
|e| + s = ((k_1|x_1| + \cdots + k_i|x_i|) + (|y_1| + \cdots + |y_j|) - s) + s,
\]

\[
= (k_1|x_1| + \cdots + k_i|x_i|) + (|y_1| + \cdots + |y_j|),
\]

\[
\geq (k_1 + \cdots + k_i) \cdot \min_{1 \leq t \leq i} |x_t| + j \cdot \min_{1 \leq t \leq j} |y_t|,
\]

\[
\geq (k_1 + \cdots + k_i + j) \cdot \min_{x \in \mathfrak{g}} |x|,
\]

\[
\geq p \cdot \min_{x \in \mathfrak{g}} |x|.
\]
because \( k_1 + \cdots + k_i + j = p \) in weight \( p \). Because the unique minimal degree element in \( g \) is \( c \otimes x_2 \), then \( \gamma_p(c \otimes x_2) \) is the unique minimal degree element with total degree 0. Moreover, under the differential in the Chevalley-Eilenberg complex from Definition 2.3, it is evident that

\[
\partial_{E_p}(\gamma_p(c \otimes x_2)) = \frac{1}{2}\gamma_{p-2}(c \otimes x_2)[c \otimes x_2, c \otimes x_2],
\]

\[
= \frac{1}{2}\gamma_{p-2}(c \otimes x_2) \cdot (c \cup c)[x_2, x_2],
\]

\[
= 0.
\]

because \( c \cup c = 0 \). Therefore \( \gamma_p(c \otimes x_2) \) is in the kernel of \( \partial_{E_p} \), so it is in the homology of the Chevalley Eilenberg complex. \( \square \)

Equipped with Theorem 2.2 and Lemmas 4.2 and 4.3, we are prepared to state the result for the \( k = p \) case.

**Theorem 4.4.** The dimensions of the mod \( p \) homology \( H_*(B_p(T); \mathbb{F}_p) \) agree with the Betti numbers of \( B_p(T) \).

**Proof.** Using Definition 2.1, \( \check{Q}^0(c \otimes x_2) \) and \( \beta \check{Q}^0(c \otimes x_2) \) have internal degree \(-1\) and \(-2\), respectively. Under a nonzero differential \( d_r : E^r_{s,t} \to E^r_{s-r,t+r-1} \), the total degree of a given element in the \( E^r \) page decreases by 1. In particular, \( \dim \gamma_p(c \otimes x_2) = 0 \), so \( \dim d_r(\gamma_p(c \otimes x_2)) = -1 \).

We claim that \( d_r(\gamma_p(c \otimes x_2)) = \beta \check{Q}^0(c \otimes x_2) \). By Lemma 4.3, all elements apart from \( \gamma_p(c \otimes x_2) \) have dimension strictly greater than 0 while \( \dim \beta \check{Q}^0(c \otimes x_2) = -1 \). It follows that only \( d_r(\gamma_p(c \otimes x_2)) = \beta \check{Q}^0(c \otimes x_2) \). If not, then \( \beta \check{Q}^0(c \otimes x_2) \) survives to the \( E^\infty \)-page and becomes a generator in the \( H_{-1} \) homology group, contradicting the fact that \( B_p(T) \) is a space and only has nonnegative dimension homology groups.

The simplicial degrees of \( \gamma_p(c \otimes x_2) \) and \( \beta \check{Q}^0(c \otimes x_2) \) are \( p - 1 \) and 1, respectively, so the only possible nonzero differential is \( d_{p-2} \). As a result, because \( d_r = 0 \) for \( 2 \leq r < p - 2 \), then

\[
E^2_{s,t} \cong E^3_{s,t} \cong \cdots \cong E^{p-2}_{s,t}.
\]

It remains to show that one can uniquely determine the \( E^{p-1}_{s,t} \)-page from \( d_{p-2} \). Because \( \check{Q}^0(c \otimes x_2) \) already has lowest simplicial degree, it cannot be that \( \check{Q}^0(c \otimes x_2) \) maps to a nonzero element. It therefore suffices to verify that no elements in \( H_*(D^p) \) map to \( \check{Q}^0(c \otimes x_2) \).

Suppose for the sake of contradiction there exists \( x \in H_*(D^p) \) such that \( d_{p-2}(x) = \check{Q}^0(c \otimes x_2) \). Then, \( \check{Q}^0(c \otimes x_2) \) is quotiented out by the image of \( d_{p-2} \), so it vanishes from the homology of \( E^{p-2} \). Likewise, \( \gamma_p(c \otimes x_2) \) is no longer in the kernel of \( d_{p-2} \), so it too vanishes from the homology of \( E^{p-2} \). Then \( E^{p-1} \cong E^{\infty} \) is isomorphic to \( H_*(D^p) \) excluding \( \gamma_p(c \otimes x_2) \), implying

\[
\dim H_*(B_p(T); \mathbb{F}_p) < \dim H_*(D^p) = \dim H_*(B_p(T); \mathbb{Q}),
\]

contradicting Theorem 2.2. We thus conclude the only elements which are removed from the \( E^{p-2}_{s,t} \)-page after \( d_{p-2} \) are \( \gamma_p(c \otimes x_2) \) and \( \beta \check{Q}^0(c \otimes x_2) \).
Hence, we have determined that $d_{p-2}$ is the only nonzero differential, the two elements removed from the $E^2 \cong E^{p-2}$-page are $\beta \tilde{Q}^0(c \otimes x_2)$ and $\gamma_p(c \otimes x_2)$, and $\tilde{Q}^0(c \otimes x_2)$ is the only element added. Moreover, because $\dim \gamma_p(c \otimes x_2) = \dim \tilde{Q}^0(c \otimes x_2)$, the dimensions of the mod $p$ homology groups of $B_p(T)$ remain unchanged from $CE(g; \mathbb{F}_p)$, which agree with the Betti numbers of $B_p(T)$ by Lemma 4.2.

**Remark 4.5.** When $p = 3$, $\gamma_3(c \otimes x_2)$ has internal degree $-2$ and simplicial degree 2, whereas $\beta \bar{Q}^0(c \otimes x_2)$ has internal degree $-2$ and simplicial degree 1. For all differentials $d_r$ with $r \geq 2$ in the spectral sequence, the simplicial degree of $d_r(c \otimes x_2)$ is less than the simplicial degree of $\beta \bar{Q}^0(c \otimes x_2)$. This implies there must exist a generator in the $H_{-1}$ homology group, contradiction. Instead, we suspect there exists a missing equivalence relation

$$\beta \bar{Q}^0(c \otimes x_2) \sim \gamma_3(c \otimes x_2)$$

in Definition 2.1 undiscovered by Kjaer [13]. For this reason, we restrict our results to $p \geq 5$ for now.

**Remark 4.6.** Because they are framed, the exact same argument also works for punctured genus $g$ surfaces $\Sigma_{g,1}$ by comparing

$$g = H^*(\Sigma_{g,1}; \mathbb{F}_p) \otimes \text{Free}^{s\text{Lie}_\bar{R}}(\mathbb{F}_p\{x_2\})$$

and

$$g' = H^*(\Sigma_{g,1}; \mathbb{Q}) \otimes \text{Free}^{s\text{Lie}_\bar{Q}}(\mathbb{Q}\{x_2\})$$

to show that the dimensions of the $\mathbb{F}_p$ homology groups equal the Betti numbers of $B_k(\Sigma_{g,1})$ for $k \leq p$. Consequently, this provides an elementary proof for Corollary 1.11 of Brantner-Hahn-Knudsen [10].

## 5 Mod 2 Homology of Configuration Spaces of $\mathbb{R}P^{2n+1}$

The method in Section 3 applies to other framed manifolds, so long as the Chevalley-Eilenberg complex can be computed over $\mathbb{F}_p$. It is therefore natural to compute the mod $p$ homology of configuration spaces for other well known framed manifolds, not only the torus.

When the manifold is an odd-dimensional real projective space $\mathbb{R}P^{2n+1}$, its mod $p$ homology groups are trivially 0 for all $p > 2$. Therefore, I compute the nontrivial mod 2 homology of the configuration space for 2 points on $\mathbb{R}P^{2n+1}$. When $p = 2$, the Lie bracket and unary operation are different from those in the odd primary cases.

**Definition 5.1** (Antolín-Camarena [15]). Suppose that $M$ is an $\mathbb{F}_2$-module over the $s\text{Lie}_\bar{R}$ algebra. The action of the Lie bracket and the unary operations $\tilde{Q}^i$ on $M$ satisfy the following.

- Applying $\tilde{Q}^i$ to $x \in M$ multiplies the weight of $x$ by 2 instead of by $p$,
- The condition $\tilde{Q}^i x \neq 0$ occurs when $i \geq |x|$ instead of $i \geq |x|/2$, 


• There is an identity $\bar{Q}^{|x|}x = [x, x]$, 

• The Lie bracket is now commutative, implying $[y_{2n+1}, y_{2n+1}] \neq 0$.

To compute $H_\ast(B_k(\mathbb{RP}^{2n+1}); \mathbb{F}_2)$, it still suffices to compute the homology of the weight $k$ part of $C_\ast(g)$, where for $\mathbb{RP}^{2n+1}$,

$$g = H^\ast(\mathbb{RP}^{2n+1}) \otimes \text{Free}^{\text{Lie}}(\mathbb{F}_2\{y_{2n+1}\})$$

**Theorem 5.1.** The dimension of the $m$th homology group of $H_\ast(B_2(\mathbb{RP}^{2n+1}); \mathbb{F}_2)$ for $n \geq 1$ is

$$\begin{cases}
\left\lfloor \frac{\min(m,4n-m)}{2} \right\rfloor + \frac{-1}{2}m + \left\lfloor \frac{m+1}{2} \right\rfloor & 0 \leq m \leq 2n, \\
\left\lfloor \frac{\min(m,4n-m)}{2} \right\rfloor + 1 + \left\lfloor \frac{4n-3-m}{2} \right\rfloor & 2n + 1 \leq m \leq 4n.
\end{cases}$$

**Proof.** For a weight 1 generator $x$ in $g$, any weight 2 element must be either of the form $\bar{Q}^i x$ or $[x, x]$, both of which have simplicial degree 1. As a result, there are no nonzero differentials in the spectral sequence, meaning the $E^\infty$-page is the kernel of $\partial_1$. There are three possible cases for basis elements in $\ker \partial_1$:

1. $[x^a \otimes y_{2n+1}, x^b \otimes y_{2n+1}] = x^{a+b} \otimes [y_{2n+1}, y_{2n+1}]$ for unordered pairs $(a, b)$ where $a + b \geq 2n + 2$ and $a, b \leq 2n + 1$,

2. $\bar{Q}^i(x^k \otimes y_{2n+1})$ with $2n + 1 - k < i < 2n + 1$ for $k \geq n + 1$, where the lower bound is strict because $i = 2n + 1 - k$ is counted in case 1 from the identity $\bar{Q}^{|x|}x = [x, x]$, and $2n + 1 - k \leq i < 2n + 1$ for $k \leq n$.

3. $[x^a \otimes y_{2n+1}, x^b \otimes y_{2n+1}] - [x^c \otimes y_{2n+1}, x^d \otimes y_{2n+1}] = (x^{a+b} - x^{c+d}) \otimes [y_{2n+1}, y_{2n+1}]$ for unordered quadruples $(a, b, c, d)$ where $a + b = c + d < 2n + 2$ and $(a, b) \neq (c, d)$.

**Case 1:** For a given $2n + 2 \leq a + b \leq 4n + 2$, the total degree of $x^{a+b} \otimes [y_{2n+1}, y_{2n+1}]$ is $4n + 2 - (a + b)$. Therefore, for $0 \leq m \leq 2n$, the ordered pairs $(a, b)$ with total degree $m$ are

$$(2n + 1 - m, 2n + 1), (2n + 2 - m, 2n + 1), \ldots, (2n + 1, 2n + 1 - m),$$

giving $\left\lfloor \frac{m+1}{2} \right\rfloor$ many unordered pairs and, accordingly, kernel basis elements with dimension $m$ in $\ker \partial_1$.

**Case 2:** The total degree of $\bar{Q}^i(x^k \otimes y_{2n+1})$ for given pair $(i, k)$ is $m = 2n + 1 + i - k$. Note that the number of elements in $H_m$ are the number of pairs $(i, k)$ with $i - k$ constant for $1 \leq k \leq 2n + 1$ and $2n + 1 - k \leq i \leq 2n$. Plotting all such pairs $(i, k)$ in Table 1 with their total degrees, pairs lying on the same diagonal have equal total degrees. Note that for $k \geq n + 1$, if $i + k = 2n + 1$, then from the identity $\bar{Q}^{|x|}x = [x, x]$,

$$\bar{Q}^{2n+1-k}(x^k \otimes y_{2n+1}) = [x^k \otimes y_{2n+1}, x^k \otimes y_{2n+1}] = x^{2k} \otimes [y_{2n+1}, y_{2n+1}] = 0.$$
However, elements of this form were counted in Case 1, so I exclude \(\tilde{Q}^{2n+1-k}(x^k \otimes y_{2n+1})\) for \(n+1 \leq k \leq 2n+1\). In Table 1 I denote excluded elements with a \(\emptyset\). From here, it is evident that in general, the number of \(m\)'s in the table for \(0 \leq m \leq 2n\) is

\[
\left\lfloor \frac{\min(m, 4n - m)}{2} \right\rfloor + \frac{1 - (-1)^m}{2},
\]

and for \(2n + 1 \leq m \leq 4n\), there are

\[
\left\lfloor \frac{\min(m, 4n - m)}{2} \right\rfloor + 1.
\]

| \(k\) \(i\) | 0 | 1 | 2 | \cdots | \(n-1\) | \(n\) | \(n+1\) | \cdots | \(2n-2\) | \(2n-1\) | \(2n\) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | \(\emptyset\) | \(\emptyset\) | \(\emptyset\) | \(\cdots\) | \(\emptyset\) | \(\emptyset\) | \(\emptyset\) | \(\cdots\) | \(\emptyset\) | \(\emptyset\) | \(4n\) |
| 2 | \(\emptyset\) | \(\emptyset\) | \(\emptyset\) | \(\cdots\) | \(\emptyset\) | \(\emptyset\) | \(\emptyset\) | \(\cdots\) | \(\emptyset\) | \(4n-2\) | \(4n-1\) |
| 3 | \(\emptyset\) | \(\emptyset\) | \(\emptyset\) | \(\cdots\) | \(\emptyset\) | \(\emptyset\) | \(\emptyset\) | \(\cdots\) | \(4n-4\) | \(4n-3\) | \(4n-2\) |
| \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) |
| \(n\) | \(\emptyset\) | \(\emptyset\) | \(\emptyset\) | \(\cdots\) | \(\emptyset\) | \(\emptyset\) | \(2n+2\) | \(\cdots\) | \(3n-1\) | \(3n\) | \(3n+1\) |
| \(n+1\) | \(\emptyset\) | \(\emptyset\) | \(\emptyset\) | \(\cdots\) | \(\emptyset\) | \(\emptyset\) | \(2n+1\) | \(\cdots\) | \(3n-2\) | \(3n-1\) | \(3n\) |
| \(n+2\) | \(\emptyset\) | \(\emptyset\) | \(\emptyset\) | \(\cdots\) | \(\emptyset\) | \(2n-1\) | \(2n\) | \(\cdots\) | \(3n-3\) | \(3n-2\) | \(3n-1\) |
| \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) |
| \(2n-1\) | \(\emptyset\) | \(\emptyset\) | \(\emptyset\) | \(\cdots\) | \(n+1\) | \(n+2\) | \(n+3\) | \(\cdots\) | \(2n\) | \(2n+1\) | \(2n+2\) |
| \(2n\) | \(\emptyset\) | \(\emptyset\) | \(3\) | \(\cdots\) | \(n\) | \(n+1\) | \(n+2\) | \(\cdots\) | \(2n-1\) | \(2n\) | \(2n+1\) |
| \(2n+1\) | \(1\) | \(2\) | \(\cdots\) | \(n-1\) | \(n\) | \(n+1\) | \(\cdots\) | \(2n-2\) | \(2n-1\) | \(2n\) |

Table 1: Total degrees for all unary elements in the mod 2 homology of \(B_2(\mathbb{RP}^{2n+1})\), where \(\emptyset\) denotes an excluded unary element equivalent to a previously counted Lie bracket from the identity \(\tilde{Q}^{|x|x} = [x, x]\). Note that all pairs \(i, k\) with total degree \(m\) lie on the same diagonals.

**Case 3:** Suppose \(0 \leq a + b = c + d = k \leq 2n+1\). For a given \(k\), the total degree of \((x^a + b - x^c + d) \otimes [y_{2n+1}, y_{2n+1}]\) is \(4n - 2 - k\), implying that for \(2n+1 \leq m = 4n + 2 - k \leq 4n + 2\), the possible ordered pairs with total degree \(m\) are \((0, k), (1, k-1), \ldots, (k, 0)\), giving \([k+1] = \left\lfloor \frac{4n+3-m}{2} \right\rfloor\) unordered pairs with total degree \(m\). Therefore, there are \(\left\lfloor \frac{4n+3-m}{2} \right\rfloor\) distinct kernel basis elements of the form \([x^a \otimes y_{2n+1}, x^b \otimes y_{2n+1}] - [x^c \otimes y_{2n+1}, x^d \otimes y_{2n+1}]\). Note that there are 0 such elements for \(m = 4n + 1, 4n + 2\), so \(2n + 1 \leq m \leq 4n\).

Summing over all three cases then gives the desired formula.
References

[1] M. Farber. Topological complexity of motion planning. *Discrete Comput. Geom.*, 29:211–221, 2003.

[2] D. McDuff. Configuration-spaces and iterated loop-spaces. *Invent. Math.*, 21(3):213–221, 1973.

[3] D. McDuff. Configuration spaces of positive and negative particles. *J. Topol.*, 14(1):91–107, 1972.

[4] B. Knudsen. Higher enveloping algebras. *Geom. Topol.*, 22(7):4013–4066, 2017.

[5] B. Knudsen. Betti numbers and stability for configuration spaces via factorization homology. *Algebraic Geom. Topol.*, 17(5):3137–3187, 2017.

[6] G. Drummond-Cole and B. Knudsen. Betti numbers of configuration spaces of surfaces. *arXiv preprint arXiv:1608.07490*, 2016.

[7] J. May. The geometry of iterated loop spaces. *Lect. Notes Math.*, 271, 1972.

[8] F. Cohen, T. Lada, and P. May. The homology of iterated loop spaces. *Lect. Notes Math.*, 533, 2007.

[9] C. Bödigheimer, F. Cohen, and L. Taylor. On the homology of configuration spaces. *J. Topol.*, 28(1):111–123, 1989.

[10] L. Brantner, J. Hahn, and B. Knudsen. The Lubin-Tate theory of configuration spaces: I. *arXiv preprint arXiv:1908.11321*, 2019.

[11] A. Zhang. André-quillen homology of spectral lie algebras with application to mod p homology of labeled configuration spaces. *arXiv preprint arXiv:2110.08428*, 2021.

[12] A. Hatcher. *Algebraic Topology*. Cambridge Univ. Press, 2000.

[13] J. J. Kjaer. On the odd primary homology of free algebras over the spectral lie operad. *arXiv preprint arXiv:1608.06605*, 2016.

[14] C. Chevalley and S. Eilenberg. Cohomology theory of lie groups and lie algebras. *Trans. Am. Math. Soc.*, 63:85–124, 1948.

[15] O. A. Camarena. The mod 2 homology of free spectral lie algebras. *arXiv preprint arXiv:1611.08771*, 2016.
Appendix

A Extra Preliminaries Definitions and Formulas

Properties of $\gamma$ Operations in a Divided Power Algebra

The $\gamma_n : I \rightarrow I$ operations over an ideal $I$ in commutative ring $A$ satisfy:

1. For any $x \in I$, define $\gamma_0(x) = 1$ and $\gamma_1(x) = x$.

2. For $x, y \in I$, $\gamma_n(x + y) = \sum_{i=0}^{n} \gamma_{n-i}(x)\gamma_i(y)$.

3. For any $c \in A$ and $x \in I$, $\gamma_n(cx) = c^n\gamma_n(x)$.

4. For every $x \in I$, $\gamma_{m+n}(x) = \frac{m!\gamma_m(x) \cdot n!\gamma_n(x)}{(m+n)!}$.

5. For $x \in I$, $\gamma_m(\gamma_n(x)) = \frac{(nm)!}{(n!)^mm!}\gamma_{mn}(x)$.

Cup Product Formula

**Definition A.1** (Hatcher [12]). Denote $X$ a simplicial complex with cohomology $C^*(X; R)$ over coefficient ring $R$. For cochains $a \in C^p(X; R)$ and $b \in C^q(X; R)$, the cup product $a \smile b \in C^{p+q}(X; R)$ is the cochain whose value on a singular simplex $\sigma : \Delta^{p+q} \rightarrow X$ is given by the formula

$$(a \smile b)(\sigma) = a|_{[v_0, \ldots, v_p]}b|_{[v_{k+1}, \ldots, v_{p+q}]}$$

where $\sigma|_{[v_i, \ldots, v_j]}$ denotes the simplex $\sigma$ restricted to a face generated by vertices $v_i, \ldots, v_j$.

**Betti Number Formulas for $B_k(\Sigma_g)$**

Formulas for the Betti numbers $\beta_i(B_k(\Sigma_g))$ were explicitly computed in general for all $i \geq 0$ and $g \geq 0$ by Drummond-Cole and Knudsen [6]. They showed that $\beta_i(B_k(\Sigma_g)) = 0$ for all $i > k + 1$ and $\beta_i(B_k(\Sigma_g))$ is independent of $k$ for all $k > i$. It therefore suffices to consider when $i = k + 1$, $i = k$, and $i < k$.

**Theorem A.1** (Drummond-Cole, Knudsen [6]). For $i = k + 1 \geq 5$ and $g \geq 0$, the Betti number $\beta_i(B_{i-1}(\Sigma_g))$ is

$$\beta_i(B_{i-1}(\Sigma_g)) = \sum_{j=0}^{g-1} \sum_{m=0}^{j} (-1)^{g+j+1} \frac{2j - 2m + 2}{2j - m + 2} \binom{6j+2i+2g-2m-5-3(-1)^{i+j+g+m}}{m, 2j - m + 1}.$$
For $i < 5$, the Betti numbers are

$$
\beta_i(B_{i-1}(\Sigma_g)) = \begin{cases}
0 & i = 1 \\
1 & i = 2 \\
0 & i = 3 \\
2g & i = 4
\end{cases}
$$

**Theorem A.2** (Drummond-Cole, Knudsen [6]). For $i = k \geq 5$ and $g \geq 0$, the Betti number $\beta_i(B_i(\Sigma_g))$ is

$$
\beta_i(B_i(\Sigma_g)) = \left(\frac{2g + i - 4}{2g - 2}\right) + \sum_{j=0}^{g-1} \sum_{m=0}^{j} (-1)^{g+j+1} \frac{2j - 2m + 2}{2j - m + 2} \left[ \left(\frac{6j + 2i + 2g - 2m + 1 + 3(-1)^{i+j+g+m}}{m, 2j - m + 1}\right) + \left(\frac{6j + 2i + 2g - 2m - 3 + 3(-1)^{i+j+g+m}}{m, 2j - m + 1}\right) + \left(\frac{6j + 2i + 2g - 2m - 5 - 3(-1)^{i+j+g+m}}{m, 2j - m + 1}\right) \right].
$$

For $i < 5$, the Betti numbers are

$$
\beta_i(B_i(\Sigma_g)) = \begin{cases}
1 & i = 0 \\
2g & i = 1 \\
2g^2 - g & i = 2 \\
4 & i = 3, g = 1 \\
(4g^3 - g + 3)/3 & i = 3, g \neq 1 \\
0 & i = 4, g = 0 \\
4 & i = 4, g = 1 \\
24 & i = 4, g = 2 \\
(4g^4 + 4g^3 - g^2 + 11g)/6 & i = 4, g > 2
\end{cases}
$$

**Theorem A.3** (Drummond-Cole, Knudsen [6]). For $5 \leq i < k$ and $g \geq 0$, the Betti number is

$$
\beta_i(B_k(\Sigma_g)) = \left(\frac{2g + i - 1}{2g - 2}\right) - \left(\frac{2g + i - 4}{2g - 2}\right) + \sum_{j=0}^{g-1} \sum_{m=0}^{j} (-1)^{g+j+1} \frac{2j - 2m + 2}{2j - m + 2} \left[ \left(\frac{6j + 2i + 2g - 2m + 1 + 3(-1)^{i+j+g+m}}{m, 2j - m + 1}\right) + \left(\frac{6j + 2i + 2g - 2m + 1 + 3(-1)^{i+j+g+m}}{m, 2j - m + 1}\right) + \left(\frac{6j + 2i + 2g - 2m - 3 + 3(-1)^{i+j+g+m}}{m, 2j - m + 1}\right) + \left(\frac{6j + 2i + 2g - 2m - 5 - 3(-1)^{i+j+g+m}}{m, 2j - m + 1}\right) \right].
$$
For \( i < 5 \), the Betti numbers are

\[
\beta_i(B_k(\Sigma_g)) = \begin{cases} 
1 & i = 0 \\
2g & i = 1 \\
0 & i = 2, g = 0 \\
3 & i = 2, g = 1 \\
2g^2 - g & i = 2, g > 1 \\
1 & i = 3, g = 0 \\
5 & i = 3, g = 1 \\
16 & i = 3, g = 2 \\
(4g^3 - g + 3)/3 & i = 3, g > 2 \\
0 & i = 4, g = 0 \\
7 & i = 4, g = 1 \\
28 & i = 4, g = 2 \\
90 & i = 4, g = 3 \\
(4g^4 + 4g^3 - g^2 + 11g)/6 & i = 4, g > 3.
\end{cases}
\]

The dimensions of the first few Betti numbers are explicitly computed in Tables 2, 3, and 4 out of convenience for future reference. For \( k \leq p \), by Theorem 4.4, the values in the column \( g = 1 \) give the dimensions of the \( \mathbb{F}_p \) homology groups of \( B_k(T) \), and all values listed in the tables correspond to the dimensions of the \( \mathbb{F}_p \) homology groups of \( B_k(\Sigma_{g,1}) \).
| $k \setminus g$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---------|---|---|---|---|---|---|---|
| 1       | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2       | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3       | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4       | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
| 5       | 0 | 1 | 6 | 15 | 28 | 45 | 66 |
| 6       | 0 | 3 | 10 | 35 | 84 | 165 | 286 |
| 7       | 0 | 2 | 20 | 70 | 210 | 495 | 1001 |
| 8       | 0 | 4 | 25 | 140 | 462 | 1287 | 3003 |
| 9       | 0 | 3 | 45 | 224 | 966 | 3003 | 8008 |
| 10      | 0 | 5 | 51 | 406 | 1764 | 6567 | 19448 |
| 11      | 0 | 4 | 84 | 574 | 3318 | 13035 | 44187 |
| 12      | 0 | 6 | 91 | 966 | 5370 | 25575 | 92950 |
| 13      | 0 | 5 | 140 | 1266 | 9372 | 45342 | 189761 |
| 14      | 0 | 7 | 148 | 2010 | 13959 | 82368 | 359359 |
| 15      | 0 | 6 | 216 | 2505 | 23001 | 134420 | 678678 |
| 16      | 0 | 8 | 225 | 3795 | 32263 | 230230 | 1186185 |
| 17      | 0 | 7 | 315 | 4565 | 50765 | 352781 | 2109536 |
| 18      | 0 | 9 | 325 | 6655 | 68068 | 576433 | 3461536 |
| 19      | 0 | 8 | 440 | 7799 | 103103 | 840762 | 5865782 |
| 20      | 0 | 10 | 451 | 11011 | 133497 | 1321398 | 9148074 |
| 21      | 0 | 9 | 594 | 12649 | 195832 | 1852630 | 14894516 |
| 22      | 0 | 11 | 606 | 17381 | 246610 | 2817750 | 22279180 |
| 23      | 0 | 10 | 780 | 19656 | 351988 | 3824898 | 35068110 |
| 24      | 0 | 12 | 793 | 26390 | 433356 | 5656002 | 50658674 |
| 25      | 0 | 11 | 1001 | 29470 | 604044 | 7474050 | 77452816 |
| 26      | 0 | 13 | 1015 | 38780 | 729912 | 10784970 | 108636528 |
| 27      | 0 | 12 | 1260 | 42860 | 996540 | 13932282 | 161938634 |
| 28      | 0 | 14 | 1275 | 55420 | 1185444 | 19676514 | 221476255 |
| 29      | 0 | 13 | 1560 | 60724 | 1589160 | 24931401 | 322850451 |
| 30      | 0 | 15 | 1576 | 77316 | 1865325 | 34545819 | 432010733 |
| 31      | 0 | 14 | 1904 | 84099 | 2460291 | 43046685 | 617367817 |
| 32      | 0 | 16 | 1921 | 105621 | 2854845 | 58640175 | 810508270 |
| 33      | 0 | 15 | 2295 | 114171 | 3711099 | 72014294 | 1137836128 |
| 34      | 0 | 17 | 2313 | 141645 | 4263448 | 96611086 | 1468946336 |
| 35      | 0 | 16 | 2736 | 152285 | 5470157 | 117137735 | 2029362335 |

Table 2: Chart of values for the $k$th Betti number of the $(k-1)$th unordered configuration space of a genus $g$ surface for $1 \leq k \leq 35$ and $0 \leq g \leq 6$. By Theorem 4.4, if $k \leq p + 1$, these are also the dimensions of the $k$th $\mathbb{F}_p$ homology group of $B_{k-1}(\Sigma_{g,1})$, and the $g = 1$ column lists the dimensions of the $k$th $\mathbb{F}_p$ homology group of $B_{k-1}(T)$.
Table 3: Chart of values for the $k$th Betti number of the $k$th unordered configuration space of a genus $g$ surface for $0 \leq k \leq 35$ and $0 \leq g \leq 6$. By Theorem 4.4, if $k \leq p$, these are also the dimensions of the $k$th $\mathbb{F}_p$ homology group of $B_k(\Sigma_{g,1})$, and the $g = 1$ column lists the dimensions of the $k$th $\mathbb{F}_p$ homology group of $B_k(T)$.
Table 4: Chart of values for the $i$th Betti number of the $k$th unordered configuration space of a genus $g$ surface for all $i < k$, $0 \leq i \leq 35$ and $0 \leq g \leq 6$. By Theorem 4.4, if $k \leq p$, these are also the dimensions of the $i$th $\mathbb{F}_p$ homology group of $B_k(\Sigma_{g,1})$, and the $g = 1$ column lists the dimensions of the $i$th $\mathbb{F}_p$ homology group of $B_k(T)$. 

| $i \setminus g$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----------------|---|---|---|---|---|---|---|
| 0               | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1               | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
| 2               | 0 | 3 | 6 | 15 | 28 | 45 | 66 |
| 3               | 1 | 5 | 16 | 36 | 85 | 166 | 287 |
| 4               | 0 | 7 | 28 | 90 | 218 | 505 | 1013 |
| 5               | 0 | 9 | 48 | 169 | 532 | 1332 | 3069 |
| 6               | 0 | 11 | 75 | 335 | 1098 | 3300 | 8294 |
| 7               | 0 | 13 | 114 | 569 | 2289 | 7227 | 20878 |
| 8               | 0 | 15 | 162 | 979 | 4187 | 15587 | 47762 |
| 9               | 0 | 17 | 225 | 1531 | 7748 | 30294 | 105963 |
| 10              | 0 | 19 | 300 | 2396 | 13034 | 58860 | 216281 |
| 11              | 0 | 21 | 393 | 3520 | 22079 | 105118 | 436150 |
| 12              | 0 | 23 | 501 | 5151 | 34866 | 188319 | 818752 |
| 13              | 0 | 25 | 630 | 7211 | 55223 | 315369 | 1530869 |
| 14              | 0 | 27 | 777 | 10039 | 82965 | 30294 | 206281 |
| 15              | 0 | 29 | 948 | 13529 | 124690 | 842884 | 473680 |
| 16              | 0 | 31 | 1140 | 18125 | 179921 | 1343826 | 7912036 |
| 17              | 0 | 33 | 1359 | 23689 | 259302 | 2050653 | 13221792 |
| 18              | 0 | 35 | 1602 | 30784 | 361900 | 3132029 | 21159269 |
| 19              | 0 | 37 | 1875 | 39236 | 504021 | 4615128 | 33879846 |
| 20              | 0 | 39 | 2175 | 49741 | 684067 | 6800508 | 52294099 |
| 21              | 0 | 41 | 2508 | 62085 | 926002 | 9727432 | 80742936 |
| 22              | 0 | 43 | 2871 | 77111 | 1227304 | 13904838 | 120830579 |
| 23              | 0 | 45 | 3270 | 94561 | 1622011 | 19387707 | 180821641 |
| 24              | 0 | 47 | 3702 | 115439 | 2106363 | 27001767 | 263434743 |
| 25              | 0 | 49 | 4173 | 139439 | 2727348 | 36822006 | 383668154 |
| 26              | 0 | 51 | 4680 | 167740 | 3479594 | 50140352 | 545978070 |
| 27              | 0 | 53 | 5229 | 199984 | 4426415 | 67056804 | 776480287 |
| 28              | 0 | 55 | 5817 | 237539 | 5560388 | 89530551 | 1082270541 |
| 29              | 0 | 57 | 6450 | 279991 | 6965069 | 117692377 | 1507214918 |
| 30              | 0 | 59 | 7125 | 328911 | 8630475 | 154433796 | 2062327850 |
| 31              | 0 | 61 | 7848 | 383825 | 10664900 | 199929976 | 2818996389 |
| 32              | 0 | 63 | 8616 | 446521 | 13055217 | 258327002 | 3793935067 |
| 33              | 0 | 65 | 9435 | 516461 | 15939574 | 329858905 | 5100110873 |
| 34              | 0 | 67 | 10302 | 595664 | 19301036 | 420398901 | 6762379052 |
| 35              | 0 | 69 | 11223 | 683524 | 23313381 | 530213298 | 8955099361 |