Generalized Nonequilibrium Quantum Transport of Spin and Pseudospins: Entanglements and Topological Phases

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Abstract

General nonequilibrium quantum transport equations are derived for a coupled system of charge carriers, Dirac spin, isospin (or valley spin), and pseudospin, such as either one of the band, layer, impurity, and boundary pseudospins. Limiting cases are obtained for one, two or three different kinds of spin occurring in a system. We show that a characteristic integer number $N_s$ determines the formal form of spin quantum transport equations, irrespective of the type of spins or pseudospins, as well as the maximal entanglement entropy. The results may shed a new perspective on the mechanism leading to zero modes and chiral/helical edge states in topological insulators, integer quantum Hall effect topological insulator (QHE-TI), quantum spin Hall effect topological insulator (QSHE-TI) and Kondo topological insulator (Kondo-TI). It also shed new light in the observed competing weak localization and antilocalization in spin-dependent quantum transport measurements. In particular, a novel mechanism of localization and delocalization, as well as the new mechanism leading to the onset of superconductivity in bilayer systems seems to emerge naturally from torque entanglements in nonequilibrium quantum transport equations of spin and pseudospins. Moreover, the general results may serve as a foundation for engineering approximations of the quantum transport simulations of spintronic devices based on graphene and other 2-D materials such as the transition metal dichalcogenides (TMDs), as well as based on topological materials exhibiting quantum spin Hall effects. The extension of the formalism to spin caloritronics and pseudo-spin caloritronics is straightforward.

Keywords: magnetization quantum transport, spin entanglements, topological insulators, spintronics, pseudospintronics.
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I. Introduction

The recent history of condensed matter physics has shown that the study of vortices, cyclotron orbits, spinors, Berry connections (in older form as Peierls phase factor), Berry curvatures, and Chern numbers have ushered the incarnation of topology in quantum physics. Indeed, the importance of several discrete-two degrees of freedom i.e., spinors and their entanglements has emerge as ubiquitous in the physics of new and low dimensional materials, such as graphene, TMDs and topological systems. For example, Pauli-Dirac spin, valley spin (isospin), pseudospin due to low-energy electron-hole symmtery at Dirac points (electron and hole have the same pseudospin), and bilayer pseudospin have gained importance in consideration of the spin quantum transport of two-dimenional materials.

The effect of atomic-layer pseudospin degrees of freedom in bilayer graphene and TMD materials has been vigorously pursued both theoretically and experimentally due to exotic properties, namely, exciton condensation and a new mechanism for the onset of superconductivity. Entangled electron-hole pairs have also been proposed for monolayer graphene. Indeed, in bilayer materials a new mechanism for the onset of superconductivity has been an intriguing discovery. This is related to the onset of exciton condensate due to
the entanglement of layer pseudospin with \( \nu = 1 \) filling of the lowest Landau levels (LLL), i.e., lowest Landau orbit (LLO) in each of the layers, with intralayer quantum Hall effect at \( \nu = 1 \) of LL0. In topological insulators, the Anderson type of localization of metallic edge states due to interaction with magnetic impurities causing successive spin flips has also been proposed (in view of our recent findings, this kind of edge-states localization maybe attributed to the entanglement with impurity spin). Spin Kondo effect at the metallic edge states has also been treated. In other words, the edge states have been studied and subjected to the usual treatment of 1-D conductors, mostly based on the usual Hamiltonian and exchange interaction context.

In this paper, we propose a new perspective on the physics of topological insulators, QHE-TI, QSHE-TI, and Kondo-TI based on quantum nonlocality as a result of the entanglement of torques induced by the various spin degrees of freedom. This is inferred from our results of the generalized nonequilibrium quantum transport equations of spin and pseudospins, and their entanglements. The present proposal implies a new mechanism for localization and delocalization based on the series of spin and pseudospin torque entanglements. The sort of localization being referred to here is typefied by cyclotron-orbit current localized around orbit center due to either external magnetic field or intrinsic Berry curvature in strongly spin-orbit coupled materials. On the other hand, the sort of delocalization referred to is represented by the metallic edge states in topological insulators as a result of the quantum nonlocality brought about by spin and pseudospin entanglements, the type of delocalization responsible for the resonant tunneling transport phenomena in resonant tunneling diodes.

Here, the point of view in all of these is that of torque entanglement in materials with multi-spin degrees of freedom. Localization and delocalization of vortices have also found some treatments in the literature that could be interpreted as due to torque interactions. In fact, these sort of localization and quantum-nonlocality delocalization effects also shed light on the experimentally observed and ubiquitous phenomena in spin-dependent quantum transport physics. These are the so-called weak localization (WL) and weak antilocalization, (WAL) often referred to in the literature. By virtue of their competing effects in nonequilibrium quantum transport, the crossover from WL to WAL has also been experimentally observed.

On the other hand, without treating the Cooper pairing it is not clear what information, if any, concerning gapless majorana edge states in \( p_x + ip_y \)-superconductors, the so-called spin-
triplet superconductor or where topological band theory claimed that majorana excitation is trap in the vortex core, due to single Cooper-unpaired state that appears exactly at zero energy. A one-dimensional topological superconductor, namely the spinless \( p \)-wave superconductor, bears similarities with polyacetylene \( [C_2H_2]_n \), which is a prototype of a one-dimensional topological insulator.

In general, Cooper pairings, i.e., the anomalous Green’s functions, correspond trivially to paired majorana fermions in non-topological superconductors, but a nontrivial pairing of \( m_i = (\Psi^\dagger_i + \Psi_i)/2 \) and \( m_j = (\Psi^\dagger_j - \Psi_j)/2i \), with \( \langle m_im_j \rangle \neq 0 \) as condensed pair of majorana fermions at neighboring sites, i.e., \( i \neq j \), describes a spinless or ferromagnetic one-dimensional topological superconductor. This combination is nontrivial precisely because when \( \langle m_i \rangle = 0 \), then \( \langle m_j \rangle \neq 0 \) and vice versa, which indicates the presence of unpaired majorana at site \( j \) or \( i \) as the case maybe. This can be induced in a Kitaev wire\(^{19}\) by proximity effect with \( s \)-type superconductor.

Whether through the use of anomalous Green’s function and ordinary Green’s function, say \( \frac{1}{4i} \left[ \langle \Psi^\dagger_i \Psi^\dagger_j \rangle - \langle \Psi_i \Psi_j \rangle - 2 \langle \Psi^\dagger_i \Psi^\dagger_j \Psi_i \Psi_j \rangle \right] \), to simulate \( \langle m_im_j \rangle \) in our nonequilibrium quantum transport equations, would have something to say about bound majorana fermion at the ends of a topological wire is yet unclear. This kind of treatment may require more detailed information of the Hamiltonian and the nature of boundary scattering, either leading to majorana chiral modes at the edges of 2-D systems or bound/unpaired majorana at the ends of a Kitaev wire, where a 1-D and 2-D lattice physics seem to give a clearer picture\(^{20}\). In principle, the zeros of either \( \frac{1}{4} \left[ 1 - \langle \Psi^\dagger_i \Psi^\dagger_j \rangle + \langle \Psi_i \Psi_j \rangle \right] \) or \( \frac{1}{4} \left[ 1 + \langle \Psi^\dagger_i \Psi^\dagger_j \rangle + \langle \Psi_i \Psi_j \rangle \right] \) would then constitute the presence of unpaired majorana, i.e., the probability finding ‘only-half’ \( (\frac{1}{2}) \) fermion. In numerical computer simulation, in a lattice physics methodology, one can look for the big imbalance between these two quantities to determine the presence of unpaired-majorana density.

What is not clear in the context of quantum transport is the sort of ‘discontinuity/boundary’ pseudospin or vortex entanglements\(^{14}\) in 2-D at the boundary that would break the majorana pairs and thus produce unpaired majoranas as zero mode excitations. The edge excitations, would therefore correspond to delocalized chiral majorana fermions in 2-D or bound zero modes in 1-D. However, Cooper pairings are all beyond the scope of the present treatment.

The general outline of this paper is that first we give all the nonequilibrium quantum
transport equations for various number of spin degrees of freedom in a system. This is followed by a discussion and concluding remarks in Sec. XII. Some of the detailed calculations are relegated to the Appendix.

II. Quantum Transport Equations

Our starting point is the general quantum transport expressions for fermions, where nonequilibrium quantum (super)fields and their correlations are the basic variables. These are obtained from the real-time nonequilibrium quantum superfield theoretical transport formulation of Buot

\[ \frac{i \hbar}{\partial t} \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^{\bar{z} z} = \left[ \mathcal{H} G^{\bar{z} z} - G^{\bar{z} z} \mathcal{H} \right] \]

+ \left[ \Sigma^r G^{\bar{z} z} \right] \]

+ \left[ \Sigma^a G^{\bar{z} z} \right] \]

+ \left[ \Delta_{hh} g_{ee}^{\bar{z} z} - g_{hh}^{\bar{z} z} \Delta_{ee}^a \right] \]

+ \left[ \Delta_{hh} g_{ee}^{\bar{z} z} - g_{hh}^{\bar{z} z} \Delta_{ee}^a \right]. \tag{1} \]

The last two brackets account for the Cooper pairings between fermions of the same specie. These do not concern us in the present paper (their corresponding transport equations are important in nonequilibrium superconductivity, where we also need to solve for the nonequilibrium anomalous Green functions). In what follows, we will drop these last two square brackets of the RHS of Eq. (1).

III. ‘Cube’ Matrix Quantum Transport Equations

In the absence of Cooper pairing between fermions of the same specie Eq. (1) becomes, by separately writing the discrete spin quantum-label arguments, essentially as \(8 \times 8\) matrix equations on the spin and pseudospin indices,

\[ \frac{i \hbar}{\partial t_1} \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^{\bar{z} z}_{kk' \ell \ell' mm'} (12) = \]

\[ \left[ \mathcal{H}_{kj} \gamma_m a \right] G^{\bar{z} z}_{jk' \gamma' l' a m'} (22) - G^{\bar{z} z}_{kj} \gamma_m \left(12\right) \mathcal{H}_{jk' \gamma' l' a m'} \delta_{22} \]

+ \left[ \Sigma^r_{kj} \gamma_m \left(12\right) G^{\bar{z} z}_{jk' \gamma' l' a m'} (22) - G^{\bar{z} z}_{kj} \gamma_m \left(12\right) \Sigma^a_{jk' \gamma' l' a m'} (22) \right] \]

+ \left[ \Sigma^a_{kj} \gamma_m \left(12\right) G^{\bar{z} z}_{jk' \gamma' l' a m'} (22) - G^{\bar{z} z}_{kj} \gamma_m \left(12\right) \Sigma^a_{jk' \gamma' l' a m'} (22) \right], \tag{2} \]

where the Greek subscript indices correspond to discrete degrees of freedom, namely, Pauli-Dirac spin \(\{k, k' = \{\downarrow, \uparrow\}\}\), valley indices \(\{l, l' = \{K, K'\}\}\), and layer or band indices as the case may be \(\{e.g., m, m' = \{t, b\}\}\). The numeral indices correspond to the two-point space-time arguments. In what follows, we will treat either the two-layer model of chiral Dirac
fermions \((m, m')\), e.g. the bottom and top layer of graphene and TMDs in strong magnetic field or other 2-D materials with strong spin-orbit coupling but without magnetic fields\(^{24}\).

In the absence of Cooper pairing of superconductivity, we may also write the general transport equation for \(\geq\)-quantities as [here \(\{A, B\}\) and \([A, B]\) means anticommutator and commutator, respectively, of operators \(A\) and \(B\)], by dropping all arguments and discrete indices, as

\[
i\hbar\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right) G^\geq =
\left[\tilde{\mathcal{H}}, G^\geq\right] + \left[\Sigma^\geq, \text{Re} G^r\right] - \frac{i}{2} \left\{\Gamma, G^\geq\right\} + \frac{i}{2} \left\{\Sigma^\geq, A\right\}, \tag{3}\]

where \(\Gamma = -\frac{1}{\hbar} \text{Im} \Sigma^r\), is the scattering-out matrix, \(A = -\frac{1}{\hbar} \text{Im} G^r\) is the spectral function, and the single-particle Hamiltonian is given by,

\[
\tilde{\mathcal{H}} = \mathcal{H}_0 + \Sigma^d + \text{Re} \Sigma^r, \tag{4}\]

where \(\Sigma^d\) and \(\text{Re} \Sigma^r\) correspond to the renormalization of the bands or mass terms induced by the self-energy\(^{21,22}\). The appearance of the imaginary \(i\) in Eq. (3) in the anticommutator is commensurate when viewed in their gradient expansions in terms of Poisson bracket differential operator wherein a commutator of Eq. (3) has a factor \(i\) whereas the anticommutator does not have. As is well-known, in the classical limit, the gradient expansion leads to the Boltzmann kinetic transport equation\(^{23}\).

IV. Canonical Spinor Form of a 'Cube' Matrix

Consider \(G^{<}_{k,k',l,l',m,m'}\), where the indices denote the elements of the 8 \(\times\) 8 matrix. Here \(k, k'\) denote the spin indices, \(l, l'\) denote the isospin or valley indices, and \(m, m'\) denote the pseudospin of either band or bilayer indices. We will now demonstrate how to reduce the 64 variables of the 8 \(\times\) 8 matrix to just 8 tensor variables, which include pure scalars or total charge. This is reminiscent to a decomposition, in the absence of dissipation, of the \(SU(8)\) into a direct product of \(SU(2) \otimes SU(2) \otimes SU(2)\), i.e., in order to keep track of the spin-like degrees of freedom. In what follows, we make constant use of the theorem of Sec. A in the Appendix.

A: First Stage: reduction of Pauli-Dirac spin indices to scalar and vector or Pauli-Dirac spin components, we have, where \(x, y, z\) denote components of vector quantities,

\[
G^{<}_{k,k',l,l',m,m'} = \frac{1}{2} \begin{pmatrix}
G^{<}_{o,l,l',m,m'} + G^{<}_{z,l,l',m,m'} & G^{<}_{x,l,l',m,m'} - iG^{<}_{y,l,l',m,m'} \\
G^{<}_{x,l,l',m,m'} + iG^{<}_{y,l,l',m,m'} & G^{<}_{o,l,l',m,m'} - G^{<}_{z,l,l',m,m'}
\end{pmatrix}
\]
B: Second stage: reduction of the remaining valley indices to scalar and vector or isospin components.

\[
\frac{1}{2} G^<_{o,l,m,m'} = \frac{1}{4} \left( G^<_{o,o,m,m'} + G^<_{o,z,m,m'} G^<_{o,x,m,m'} - iG^<_{o,y,m,m'} \right)
\]

\[
\frac{1}{2} G^<_{k,l,m,m'} = \frac{1}{4} \left( G^<_{k,o,m,m'} + G^<_{k,z,m,m'} G^<_{k,x,m,m'} - iG^<_{k,y,m,m'} \right)
\]

C: Third and final stage: reduction of the last band or layer indices to scalar and vector or pseudospin components.

\[
\frac{1}{4} G^<_{o,o,m,m'} = \left( \frac{1}{2} \right)^3 \left( G^<_{o,o,o} + G^<_{o,o,z} G^<_{o,o,x} - iG^<_{o,o,y} \right)
\]

\[
\frac{1}{4} G^<_{o,l,m,m'} = \left( \frac{1}{2} \right)^3 \left( G^<_{o,l,o} + G^<_{o,l,z} G^<_{o,l,x} - iG^<_{o,l,y} \right)
\]

\[
\frac{1}{4} G^<_{k,o,m,m'} = \left( \frac{1}{2} \right)^3 \left( G^<_{k,o,o} + G^<_{k,o,z} G^<_{k,o,x} - iG^<_{k,o,y} \right)
\]

\[
\frac{1}{4} G^<_{k,l,m,m'} = \left( \frac{1}{2} \right)^3 \left( G^<_{k,l,o} + G^<_{k,l,z} G^<_{k,l,x} - iG^<_{k,l,y} \right)
\]

Collecting results of the third and final reduction stage, we finally transformed \( G^<_{k,k',l',m,m'} \) into its spinor form as

\[
(2)^3 G^<_{k,k',l',m,m'} \]

\[
\begin{pmatrix}
G^<_{o,o,o} + G^<_{o,o,z} G^<_{o,o,x} - iG^<_{o,o,y} \\
G^<_{o,o,x} + iG^<_{o,o,y} G^<_{o,o,o} - G^<_{o,o,z}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
G^<_{o,l,o} + G^<_{o,l,z} G^<_{o,l,x} - iG^<_{o,l,y} \\
G^<_{o,l,x} + iG^<_{o,l,y} G^<_{o,l,o} - G^<_{o,l,z}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
G^<_{k,o,o} + G^<_{k,o,z} G^<_{k,o,x} - iG^<_{k,o,y} \\
G^<_{k,o,x} + iG^<_{k,o,y} G^<_{k,o,o} - G^<_{k,o,z}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
G^<_{k,l,o} + G^<_{k,l,z} G^<_{k,l,x} - iG^<_{k,l,y} \\
G^<_{k,l,x} + iG^<_{k,l,y} G^<_{k,l,o} - G^<_{k,l,z}
\end{pmatrix}
\]
Thus, from Eq. (5), we end up with the eight tensor transport variables, similar to the number of distinct configurations of three spin-qubits, which include joint distributions or spin-torque entanglements, defined in the Table 1.

TABLE I: The Eight Transport Variables

| $G_{0,0,0}^<$ | complete scalar, i.e., particle number density or total charge density |
|--------------|---------------------------------------------------------------------|
| $G_{k,0,0}^<$ | Pauli-Dirac spin magnetization density                               |
| $G_{0,l,0}^<$ | isospin magnetization density                                        |
| $G_{0,0,m}^<$ | pseudospin magnetization density                                     |
| $G_{k,l,0}^<$ | entangled or joint Pauli-Dirac spin and isospin magnetization density |
| $G_{k,0,m}^<$ | entangled or joint Pauli-Dirac spin and pseudospin magnetization density |
| $G_{0,l,m}^<$ | entangled or joint isospin-pseudospin magnetization density          |
| $G_{k,l,m}^<$ | entangled or joint Pauli-Dirac spin, isospin, pseudospin magnetization density |

Note that in Table 1 the 8 tensor variables, including the total charge, are mapped to the configurations of three qubits as exhibited by the subscripts of the nonequilibrium tensorial Green’s function, $G^<$. 

V. Quantum Transport Equations for a 'Cube' Matrix $G^\otimes$, $N_s = 3$

By treating all $SU(2)$ indices on equal footing, we evaluate the canonical form of every matrix involved in all matrix products as a series of product of $2 \times 2$ matrices: $(kj) (jk')$, $(l\gamma) (\gamma l')$, $(ma) (am')$, as was done in Sec IV. We finally end up with the eight coupled quantum transport equations for $G_{o,o,o}^<$, $G_{k,o,o}^<$, $G_{o,l,o}^<$, $G_{o,o,m}^<$, $G_{k,l,o}^<$, $G_{k,o,m}^<$, $G_{o,l,m}^<$, and $G_{k,l,m}^<$ defined in Table 1. The technique followed in the derivation is explained in the Sec. A[C]8 of the Appendix. Note that in line with the four terms in the right-hand-side (RHS) of Eq. (3), the RHS of each of the eight coupled transport equations correspondingly consist of four groups. We have generalized the Hamiltonian spinor to account for either of the magnetic field and/or Dresselhaus and/or Rashba spin-orbit coupling.

We have for the total scalar,

$$2^{N_s}i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{o,o,o}^\otimes =$$

$$= \left[ \tilde{H}, G^\otimes \right]_{o,o,o} + \left[ \Sigma^\otimes, \text{Re}G^r \right]_{o,o,o} - i \left\{ \Gamma, G^\otimes \right\}_{o,o,o} + \frac{i}{2} \left\{ \Sigma^\otimes, A \right\}_{o,o,o} \quad \text{(6)}$$
where

\[
\left[ \tilde{\mathcal{H}}, G^\Sigma_{o,o,o} \right] =
\]

\[
= \left[ H_{o.o.o}, G^\Sigma_{o.o.o} \right] + \left[ H_{o.l.o}, G^\Sigma_{o.l.o} \right] + \left[ H_{o.l.m}, G^\Sigma_{o.l.m} \right]
\]

\[
+ \left[ H_{k,o.o}, G^\Sigma_{k,o.o} \right] + \left[ H_{k,o.m}, G^\Sigma_{k,o.m} \right] + \left[ H_{k,l,o}, G^\Sigma_{k,l,o} \right] + \left[ H_{k,l,m}, G^\Sigma_{k,l,m} \right]
\]

(7)

\[
\left[ \Sigma^\Sigma, \Re G^\nu \right]_{o,o,o} =
\]

\[
= \left[ \Sigma^\Sigma_{o.o.o}, \Re G^\nu_{o.o.o} \right] + \left[ \Sigma^\Sigma_{o.o.m}, \Re G^\nu_{o.o.m} \right] + \left[ \Sigma^\Sigma_{o.l.o}, \Re G^\nu_{o.l.o} \right] + \left[ \Sigma^\Sigma_{o.l.m}, \Re G^\nu_{o.l.m} \right]
\]

\[
+ \left[ \Sigma^\Sigma_{k,o.o}, \Re G^\nu_{k,o.o} \right] + \left[ \Sigma^\Sigma_{k,o.m}, \Re G^\nu_{k,o.m} \right] + \left[ \Sigma^\Sigma_{k,l.o}, \Re G^\nu_{k,l.o} \right] + \left[ \Sigma^\Sigma_{k,l,m}, \Re G^\nu_{k,l,m} \right]
\]

(8)

\[
\left\{ \Gamma, G^\Sigma \right\}_{o,o,o} =
\]

\[
= \left\{ \Gamma_{o.o.o}, G^\Sigma_{o.o.o} \right\} + \left\{ \Gamma_{o.o.m}, G^\Sigma_{o.o.m} \right\} + \left\{ \Gamma_{o.l.o}, G^\Sigma_{o.l.o} \right\} + \left\{ \Gamma_{o.l.m}, G^\Sigma_{o.l.m} \right\}
\]

\[
+ \left\{ \Gamma_{k,o.o}, G^\Sigma_{k,o.o} \right\} + \left\{ \Gamma_{k,o.m}, G^\Sigma_{k,o.m} \right\} + \left\{ \Gamma_{k,l.o}, G^\Sigma_{k,l.o} \right\} + \left\{ \Gamma_{k,l,m}, G^\Sigma_{k,l.m} \right\}
\]

(9)

\[
\left\{ \Sigma^\Sigma, A \right\}_{o,o,o} =
\]

\[
= \left\{ \Sigma^\Sigma_{o,o.o}, A_{o.o.o} \right\} + \left\{ \Sigma^\Sigma_{o.o.m}, A_{o.o.m} \right\} + \left\{ \Sigma^\Sigma_{o.l.o}, A_{o.l.o} \right\} + \left\{ \Sigma^\Sigma_{o.l.m}, A_{o.l.m} \right\}
\]

\[
+ \left\{ \Sigma^\Sigma_{k,o.o}, A_{k,o.o} \right\} + \left\{ \Sigma^\Sigma_{k,o.m}, A_{k,o.m} \right\} + \left\{ \Sigma^\Sigma_{k,l.o}, A_{k,l.o} \right\} + \left\{ \Sigma^\Sigma_{k,l.m}, A_{k,l.m} \right\}
\]

(10)

where repeated vector-component indices, corresponding to dot products, are summed over following the Einstein summation convention, and \( N_s \) is the integer number of spin-like degrees of freedom. We have Pauli-Dirac spin, valley spin (or isospin) and pseudospin degrees of freedom acting on the systems. Here in Eqs. (6) - (11), we have \( N_s = 3 \).

We have for the Pauli-Dirac spin,

\[
2^{N_s} \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^\Sigma_{k,o,o} =
\]

\[
= \left[ \tilde{\mathcal{H}}, G^\Sigma_{k,o,o} \right] + \left[ \Sigma^\Sigma, \Re G^\nu \right]_{k,o,o} - \frac{i}{2} \left\{ \Gamma, G^\Sigma \right\}_{k,o,o} + \frac{i}{2} \left\{ \Sigma^\Sigma, A \right\}_{k,o,o}
\]

(11)
where,

\[
\begin{align*}
\left[ \hat{\mathcal{H}}, G^\Sigma \right]_{k,o,o} &= \left[ H_{o,o,o}, G^\Sigma_{k,o,o} \right] + \left[ H_{o,o,m}, G^\Sigma_{k,o,m} \right] + i \epsilon_{k,k_2} \left\{ H_{k_1,o,o}, G^\Sigma_{k_2,o,o} \right\} \\
&+ \left[ H_{o,l,o}, G^\Sigma_{k,l,o} \right] + \left[ H_{o,l,m}, G^\Sigma_{k,l,m} \right] + i \epsilon_{k,k_2} \left\{ H_{k_1,l,o}, G^\Sigma_{k_2,l,o} \right\} \\
&+ \left[ H_{k,o,o}, G^\Sigma_{o,o,o} \right] + \left[ H_{k,o,m}, G^\Sigma_{o,o,m} \right] + i \epsilon_{k,k_2} \left\{ H_{k_1,o,m}, G^\Sigma_{k_2,o,m} \right\} \\
&+ \left[ H_{k,l,o}, G^\Sigma_{o,l,o} \right] + \left[ H_{k,l,m}, G^\Sigma_{o,l,m} \right] + i \epsilon_{k,k_2} \left\{ H_{k_1,l,m}, G^\Sigma_{k_2,l,m} \right\} \\
\end{align*}
\]

\[
\left[ \Sigma^\Sigma, \text{Re} G^r \right]_{k,o,o} = \\
\left[ \Sigma^\Sigma_{o,o,o}, \text{Re} G^r_{k,o,o} \right] + \left[ \Sigma^\Sigma_{o,o,m}, \text{Re} G^r_{k,o,m} \right] + i \epsilon_{k,k_2} \left\{ \Sigma^\Sigma_{k_1,o,o}, \text{Re} G^r_{k_2,o,o} \right\} \\
+ \left[ \Sigma^\Sigma_{o,l,o}, \text{Re} G^r_{k,l,o} \right] + \left[ \Sigma^\Sigma_{o,l,m}, \text{Re} G^r_{k,l,m} \right] + i \epsilon_{k,k_2} \left\{ \Sigma^\Sigma_{k_1,l,o}, \text{Re} G^r_{k_2,l,o} \right\} \\
+ \left[ \Sigma^\Sigma_{k,o,o}, \text{Re} G^r_{o,o,o} \right] + \left[ \Sigma^\Sigma_{k,o,m}, \text{Re} G^r_{o,o,m} \right] + i \epsilon_{k,k_2} \left\{ \Sigma^\Sigma_{k_1,o,m}, \text{Re} G^r_{k_2,o,m} \right\} \\
+ \left[ \Sigma^\Sigma_{k,l,o}, \text{Re} G^r_{o,l,o} \right] + \left[ \Sigma^\Sigma_{k,l,m}, \text{Re} G^r_{o,l,m} \right] + i \epsilon_{k,k_2} \left\{ \Sigma^\Sigma_{k_1,l,m}, \text{Re} G^r_{k_2,l,m} \right\}
\]

\[
\left\{ \Gamma, G^\Sigma \right\}_{k,o,o} = \\
\left\{ \Gamma_{o,o,o}, G^\Sigma_{k,o,o} \right\} + \left\{ \Gamma_{o,o,m}, G^\Sigma_{k,o,m} \right\} + i \epsilon_{k,k_2} \left[ \Gamma_{k_1,o,o}, G^\Sigma_{k_2,o,o} \right] \\
+ \left\{ \Gamma_{o,l,o}, G^\Sigma_{k,l,o} \right\} + \left\{ \Gamma_{o,l,m}, G^\Sigma_{k,l,m} \right\} + i \epsilon_{k,k_2} \left[ \Gamma_{k_1,l,o}, G^\Sigma_{k_2,l,o} \right] \\
+ \left\{ \Gamma_{k,o,o}, G^\Sigma_{o,o,o} \right\} + \left\{ \Gamma_{k,o,m}, G^\Sigma_{o,o,m} \right\} + i \epsilon_{k,k_2} \left[ \Gamma_{k_1,o,m}, G^\Sigma_{k_2,o,m} \right] \\
+ \left\{ \Gamma_{k,l,o}, G^\Sigma_{o,l,o} \right\} + \left\{ \Gamma_{k,l,m}, G^\Sigma_{o,l,m} \right\} + i \epsilon_{k,k_2} \left[ \Gamma_{k_1,l,m}, G^\Sigma_{k_2,l,m} \right]
\]

\[
\left\{ \Sigma^\Sigma, A \right\}_{k,o,o} = \\
\left\{ \Sigma^\Sigma_{o,o,o}, A_{k,o,o} \right\} + \left\{ \Sigma^\Sigma_{o,o,m}, A_{k,o,m} \right\} + i \epsilon_{k,k_2} \left[ \Sigma^\Sigma_{k_1,o,o}, A_{k_2,o,o} \right] \\
+ \left\{ \Sigma^\Sigma_{o,l,o}, A_{k,l,o} \right\} + \left\{ \Sigma^\Sigma_{o,l,m}, A_{k,l,m} \right\} + i \epsilon_{k,k_2} \left[ \Sigma^\Sigma_{k_1,l,o}, A_{k_2,l,o} \right] \\
+ \left\{ \Sigma^\Sigma_{k,o,o}, A_{o,o,o} \right\} + \left\{ \Sigma^\Sigma_{k,o,m}, A_{o,o,m} \right\} + i \epsilon_{k,k_2} \left[ \Sigma^\Sigma_{k_1,o,m}, A_{k_2,o,m} \right] \\
+ \left\{ \Sigma^\Sigma_{k,l,o}, A_{o,l,o} \right\} + \left\{ \Sigma^\Sigma_{k,l,m}, A_{o,l,m} \right\} + i \epsilon_{k,k_2} \left[ \Sigma^\Sigma_{k_1,l,m}, A_{k_2,l,m} \right]
\]

We have for the valley spin,

\[
2^{N_s} i h \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^\Sigma_{o,l,o} = \\
\left[ \tilde{\mathcal{H}}, G^\Sigma \right]_{o,l,o} + \left[ \Sigma^\Sigma, \text{Re} G^r \right]_{o,l,o} - \frac{i}{2} \left\{ \Gamma, G^\Sigma \right\}_{o,l,o} + \frac{i}{2} \left\{ \Sigma^\Sigma, A \right\}_{o,l,o}
\]
where

\[
\left[ \mathcal{H}, G^{\Sigma} \right]_{o,l,o} = \left[ H_{o,o,o}, G^{\Sigma}_{o,l,o} \right] + \left[ H_{o,o,m}, G^{\Sigma}_{o,l,m} \right] + i \epsilon_{l_1l_2} \left\{ H_{o,l_1,o}, G^{\Sigma}_{o,l_2,o} \right\} = + \left[ H_{o,l,o}, G^{\Sigma}_{o,o,o} \right] + \left[ H_{o,l,m}, G^{\Sigma}_{o,l,m} \right] + i \epsilon_{l_1l_2} \left\{ H_{o,l_1,m}, G^{\Sigma}_{o,l_2,m} \right\} + \left[ H_{k,o,o}, G^{\Sigma}_{k,l,o} \right] + \left[ H_{k,o,m}, G^{\Sigma}_{k,l,m} \right] + i \epsilon_{l_1l_2} \left\{ H_{k,l_1,o}, G^{\Sigma}_{k,l_2,o} \right\} + \left[ H_{k,l,o}, G^{\Sigma}_{k,o,o} \right] + \left[ H_{k,l,m}, G^{\Sigma}_{k,o,m} \right] + i \epsilon_{l_1l_2} \left\{ H_{k,l_1,m}, G^{\Sigma}_{k,l_2,m} \right\} \tag{17}\]

\[
\left[ \Sigma^{\Sigma}, \text{Re} G^{\sigma} \right]_{o,l,o} = \left[ \Sigma^{\Sigma}_{o,o,o}, \text{Re} G^{\sigma}_{o,l,o} \right] + \left[ \Sigma^{\Sigma}_{o,o,m}, \text{Re} G^{\sigma}_{o,l,m} \right] + i \epsilon_{l_1l_2} \left\{ \Sigma^{\Sigma}_{o,l_1,o}, \text{Re} G^{\sigma}_{o,l_2,o} \right\} = + \left[ \Sigma^{\Sigma}_{o,l,o}, \text{Re} G^{\sigma}_{o,o,o} \right] + \left[ \Sigma^{\Sigma}_{o,l,m}, \text{Re} G^{\sigma}_{o,l,m} \right] + i \epsilon_{l_1l_2} \left\{ \Sigma^{\Sigma}_{o,l_1,m}, \text{Re} G^{\sigma}_{o,l_2,m} \right\} + \left[ \Sigma^{\Sigma}_{k,o,o}, \text{Re} G^{\sigma}_{k,l,o} \right] + \left[ \Sigma^{\Sigma}_{k,o,m}, \text{Re} G^{\sigma}_{k,l,m} \right] + i \epsilon_{l_1l_2} \left\{ \Sigma^{\Sigma}_{k,l_1,o}, \text{Re} G^{\sigma}_{k,l_2,o} \right\} + \left[ \Sigma^{\Sigma}_{k,l,o}, \text{Re} G^{\sigma}_{k,o,o} \right] + \left[ \Sigma^{\Sigma}_{k,l,m}, \text{Re} G^{\sigma}_{k,o,m} \right] + i \epsilon_{l_1l_2} \left\{ \Sigma^{\Sigma}_{k,l_1,m}, \text{Re} G^{\sigma}_{k,l_2,m} \right\} \tag{18}\]

\[
\left\{ \Gamma, G^{\Sigma} \right\}_{o,l,o} = \left\{ \Gamma_{o,o,o}, G^{\Sigma}_{o,l,o} \right\} + \left\{ \Gamma_{o,o,m}, G^{\Sigma}_{o,l,m} \right\} + i \epsilon_{l_1l_2} \left\{ \Gamma_{o,l_1,o}, G^{\Sigma}_{o,l_2,o} \right\} = + \left\{ \Gamma_{o,l,o}, G^{\Sigma}_{o,o,o} \right\} + \left\{ \Gamma_{o,l,m}, G^{\Sigma}_{o,l,m} \right\} + i \epsilon_{l_1l_2} \left\{ \Gamma_{o,l_1,m}, G^{\Sigma}_{o,l_2,m} \right\} + \left\{ \Gamma_{k,o,o}, G^{\Sigma}_{k,l,o} \right\} + \left\{ \Gamma_{k,o,m}, G^{\Sigma}_{k,l,m} \right\} + i \epsilon_{l_1l_2} \left\{ \Gamma_{k,l_1,o}, G^{\Sigma}_{k,l_2,o} \right\} + \left\{ \Gamma_{k,l,o}, G^{\Sigma}_{k,o,o} \right\} + \left\{ \Gamma_{k,l,m}, G^{\Sigma}_{k,o,m} \right\} + i \epsilon_{l_1l_2} \left\{ \Gamma_{k,l_1,m}, G^{\Sigma}_{k,l_2,m} \right\} \tag{19}\]

\[
\left\{ \Sigma^{\Sigma}, A \right\}_{o,l,o} = \left\{ \Sigma^{\Sigma}_{o,o,o}, A_{o,l,o} \right\} + \left\{ \Sigma^{\Sigma}_{o,o,m}, A_{o,l,m} \right\} + i \epsilon_{l_1l_2} \left\{ \Sigma^{\Sigma}_{o,l_1,o}, A_{o,l_2,o} \right\} = + \left\{ \Sigma^{\Sigma}_{o,l,o}, A_{o,o,o} \right\} + \left\{ \Sigma^{\Sigma}_{o,l,m}, A_{o,l,m} \right\} + i \epsilon_{l_1l_2} \left\{ \Sigma^{\Sigma}_{o,l_1,m}, A_{o,l_2,m} \right\} + \left\{ \Sigma^{\Sigma}_{k,o,o}, A_{k,l,o} \right\} + \left\{ \Sigma^{\Sigma}_{k,o,m}, A_{k,l,m} \right\} + i \epsilon_{l_1l_2} \left\{ \Sigma^{\Sigma}_{k,l_1,o}, A_{k,l_2,o} \right\} + \left\{ \Sigma^{\Sigma}_{k,l,o}, A_{k,o,o} \right\} + \left\{ \Sigma^{\Sigma}_{k,l,m}, A_{k,o,m} \right\} + i \epsilon_{l_1l_2} \left\{ \Sigma^{\Sigma}_{k,l_1,m}, A_{k,l_2,m} \right\} \tag{20}\]

We have for the pseudospin,

\[
2^{N_r} i \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^{\Sigma}_{o,o,m} = \left[ \mathcal{H}, G^{\Sigma} \right]_{o,o,m} + \left[ \Sigma^{\Sigma}, \text{Re} G^{\sigma} \right]_{o,o,m} - \frac{i}{2} \left\{ \Gamma, G^{\Sigma} \right\}_{o,o,m} + \frac{i}{2} \left\{ \Sigma^{\Sigma}, A \right\}_{o,o,m} \tag{21}\]
where,

$$
\begin{align*}
\left[ \mathcal{H}, G^z \right]_{o.o.m} &= \\
&= [H_{o.o,o}, G^z_{o.o,m}] + [H_{o.o,m}, G^z_{o.o,o}] + i\epsilon_{m_1m_2} \{H_{o.o,m_1}, G^z_{o.o,m_2} \} \\
&+ [H_{o,l,o}, G^z_{o,l,m}] + [H_{o,l,m}, G^z_{o,l,o}] + i\epsilon_{m_1m_2} \{H_{o,l,m_1}, G^z_{o,l,m_2} \} \\
&+ [H_{k,o,o}, G^z_{k,o,m}] + [H_{k,o,m}, G^z_{k,o,o}] + i\epsilon_{m_1m_2} \{H_{k,o,m_1}, G^z_{k,o,m_2} \} \\
&+ [H_{k,l,o}, G^z_{k,l,m}] + [H_{k,l,m}, G^z_{k,l,o}] + i\epsilon_{m_1m_2} \{H_{k,l,m_1}, G^z_{k,l,m_2} \}
\end{align*}
$$

(22)

$$
\begin{align*}
\left[ \Sigma^z, \text{Re} G^r \right]_{o,o.m} &= \\
&= [\Sigma^z_{o.o,o}, \text{Re} G^r_{o,o,m}] + [\Sigma^z_{o.o,m}, \text{Re} G^r_{o.o,o}] + i\epsilon_{m_1m_2} \{\Sigma^z_{o.o,m_1}, \text{Re} G^r_{o,o,m_2} \} \\
&+ [\Sigma^z_{o,l,o}, \text{Re} G^r_{o,l,m}] + [\Sigma^z_{o,l,m}, \text{Re} G^r_{o,l,o}] + i\epsilon_{m_1m_2} \{\Sigma^z_{o,l,m_1}, \text{Re} G^r_{o,l,m_2} \} \\
&+ [\Sigma^z_{k,o,o}, \text{Re} G^r_{k,o,m}] + [\Sigma^z_{k,o,m}, \text{Re} G^r_{k,o,o}] + i\epsilon_{m_1m_2} \{\Sigma^z_{k,o,m_1}, \text{Re} G^r_{k,o,m_2} \} \\
&+ [\Sigma^z_{k,l,o}, \text{Re} G^r_{k,l,m}] + [\Sigma^z_{k,l,m}, \text{Re} G^r_{k,l,o}] + i\epsilon_{m_1m_2} \{\Sigma^z_{k,l,m_1}, \text{Re} G^r_{k,l,m_2} \}
\end{align*}
$$

(23)

$$
\begin{align*}
\{ \Gamma, G^z \}_{o,o.m} &= \\
&= \{\Gamma_{o,o,o}, G^z_{o,o,m}\} + \{\Gamma_{o,o,m}, G^z_{o,o,o}\} + i\epsilon_{m_1m_2} \{\Gamma_{o,o,m_1}, G^z_{o,o,m_2} \} \\
&+ \{\Gamma_{o,l,o}, G^z_{o,l,m}\} + \{\Gamma_{o,l,m}, G^z_{o,l,o}\} + i\epsilon_{m_1m_2} \{\Gamma_{o,l,m_1}, G^z_{o,l,m_2} \} \\
&+ \{\Gamma_{k,o,o}, G^z_{k,o,m}\} + \{\Gamma_{k,o,m}, G^z_{k,o,o}\} + i\epsilon_{m_1m_2} \{\Gamma_{k,o,m_1}, G^z_{k,o,m_2} \} \\
&+ \{\Gamma_{k,l,o}, G^z_{k,l,m}\} + \{\Gamma_{k,l,m}, G^z_{k,l,o}\} + i\epsilon_{m_1m_2} \{\Gamma_{k,l,m_1}, G^z_{k,l,m_2} \}
\end{align*}
$$

(24)

$$
\begin{align*}
\{ \Sigma^z, A \}_{o,o.m} &= \\
&= \{\Sigma^z_{o.o,o}, A_{o.o,m}\} + \{\Sigma^z_{o.o,m}, A_{o.o,o}\} + i\epsilon_{m_1m_2} \{\Sigma^z_{o.o,m_1}, A_{o.o,m_2} \} \\
&+ \{\Sigma^z_{o,l,o}, A_{o,l,m}\} + \{\Sigma^z_{o,l,m}, A_{o,l,o}\} + i\epsilon_{m_1m_2} \{\Sigma^z_{o,l,m_1}, A_{o,l,m_2} \} \\
&+ \{\Sigma^z_{k,o,o}, A_{k,o,m}\} + \{\Sigma^z_{k,o,m}, A_{k,o,o}\} + i\epsilon_{m_1m_2} \{\Sigma^z_{k,o,m_1}, A_{k,o,m_2} \} \\
&+ \{\Sigma^z_{k,l,o}, A_{k,l,m}\} + \{\Sigma^z_{k,l,m}, A_{k,l,o}\} + i\epsilon_{m_1m_2} \{\Sigma^z_{k,l,m_1}, A_{k,l,m_2} \}
\end{align*}
$$

(25)

We have for the entangled Dirac spin and valley spin,

$$
2^{N_i} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^z_{k,l,o} = \\
= \left[ \mathcal{H}, G^z \right]_{k,l,o} + [\Sigma^z, \text{Re} G^r]_{k,l,o} - \frac{i}{2} \{\Gamma, G^z\}_{k,l,o} + \frac{i}{2} \{\Sigma^z, A\}_{k,l,o}
$$

(26)
where,

$$\left[ \mathcal{H}, G^\Sigma \right]_{k,l,o} =$$

$$\left[ H_{o,o,o}, G^\Sigma_{k,l,o} \right] + \left[ H_{o,o,m}, G^\Sigma_{k,l,m} \right] + i\epsilon_{kk_1k_2} \left\{ H_{k_1,o,o}, G^\Sigma_{k_2,l,o} \right\}$$

$$+ i\epsilon_{ll_1l_2} \left\{ H_{o,l,o}, G^\Sigma_{k,l,o} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \left[ H_{k_1,l,o}, G^\Sigma_{k_2,o,l} \right]$$

$$+ \left[ H_{o,l,o}, G^\Sigma_{k,o,o} \right] + \left[ H_{o,l,m}, G^\Sigma_{k,o,m} \right] + i\epsilon_{kk_1k_2} \left\{ H_{k_1,l,o}, G^\Sigma_{k_2,o,o} \right\}$$

$$+ i\epsilon_{ll_1l_2} \left\{ H_{k_1,l,o} G^\Sigma_{o,l_2,o} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \left[ H_{k_1,l,o}, G^\Sigma_{k_2,o,o} \right]$$

$$+ \left[ H_{k_1,o,o}, G^\Sigma_{o,o,o} \right] + \left[ H_{k_1,o,m}, G^\Sigma_{o,o,m} \right] + i\epsilon_{kk_1k_2} \left\{ H_{k_1,l,m}, G^\Sigma_{k_2,o,m} \right\}$$

$$+ i\epsilon_{ll_1l_2} \left\{ H_{k_1,l,m}, G^\Sigma_{o,l_2,m} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \left[ H_{k_1,l,m}, G^\Sigma_{k_2,o,m} \right]$$

$$+ \left[ H_{k_1,o,o}, G^\Sigma_{o,o,o} \right] + \left[ H_{k_1,o,m}, G^\Sigma_{o,o,m} \right] + i\epsilon_{kk_1k_2} \left\{ H_{k_1,l,m}, G^\Sigma_{k_2,l,m} \right\}$$

$$+ i\epsilon_{ll_1l_2} \left\{ H_{o,l,m}, G^\Sigma_{k_2,l,m} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \left[ H_{k_1,l,m}, G^\Sigma_{k_2,l,m} \right]$$

$$\left(27\right)$$

$$\left[ \Sigma^\Sigma, \text{Re}G^\nu \right]_{k,l,o} =$$

$$\left[ \Sigma^\Sigma_{o,o,o}, \text{Re}G^\nu_{k,l,o} \right] + \left[ \Sigma^\Sigma_{o,o,m}, \text{Re}G^\nu_{k,l,m} \right] + i\epsilon_{kk_1k_2} \left\{ \Sigma^\Sigma_{k_1,o,o}, \text{Re}G^\nu_{k_2,l,o} \right\}$$

$$+ i\epsilon_{ll_1l_2} \left\{ \Sigma^\Sigma_{o,l_1,o}, \text{Re}G^\nu_{k,l,o} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \left[ \Sigma^\Sigma_{k_1,l,o}, \text{Re}G^\nu_{k_2,l,o} \right]$$

$$+ \left[ \Sigma^\Sigma_{o,o,o}, \text{Re}G^\nu_{k,o,o} \right] + \left[ \Sigma^\Sigma_{o,o,m}, \text{Re}G^\nu_{k,o,m} \right] + i\epsilon_{kk_1k_2} \left\{ \Sigma^\Sigma_{k_1,l,o}, \text{Re}G^\nu_{k_2,o,o} \right\}$$

$$+ i\epsilon_{ll_1l_2} \left\{ \Sigma^\Sigma_{o,l_1,o}, \text{Re}G^\nu_{k,o,o} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \left[ \Sigma^\Sigma_{k_1,l,o}, \text{Re}G^\nu_{k_2,o,o} \right]$$

$$+ \left[ \Sigma^\Sigma_{k,l,o}, \text{Re}G^\nu_{o,o,o} \right] + \left[ \Sigma^\Sigma_{k,l,m}, \text{Re}G^\nu_{o,o,m} \right] + i\epsilon_{kk_1k_2} \left\{ \Sigma^\Sigma_{k_1,l,m}, \text{Re}G^\nu_{k_2,o,o} \right\}$$

$$+ i\epsilon_{ll_1l_2} \left\{ \Sigma^\Sigma_{o,l_1,m}, \text{Re}G^\nu_{k_2,l,o} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \left[ \Sigma^\Sigma_{k_1,l,m}, \text{Re}G^\nu_{k_2,l,o} \right]$$

$$\left(28\right)$$

$$\left\{ \Gamma, G^\Sigma \right\}_{k,l,o} =$$

$$\left\{ \Gamma_{o,o,o}, G^\Sigma_{k,l,o} \right\} + \left\{ \Gamma_{o,o,m}, G^\Sigma_{k,l,m} \right\} + i\epsilon_{kk_1k_2} \left\{ \Gamma_{k_1,o,o}, G^\Sigma_{k_2,l,o} \right\}$$

$$+ i\epsilon_{ll_1l_2} \left\{ \Gamma_{o,l_1,o}, G^\Sigma_{k,l,o} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \left\{ \Gamma_{k_1,l,o}, G^\Sigma_{k_2,o,l} \right\}$$

$$+ \left\{ \Gamma_{o,l,o}, G^\Sigma_{k,o,o} \right\} + \left\{ \Gamma_{o,l,m}, G^\Sigma_{k,o,m} \right\} + i\epsilon_{kk_1k_2} \left\{ \Gamma_{k_1,l,o}, G^\Sigma_{k_2,o,o} \right\}$$

$$+ i\epsilon_{ll_1l_2} \left\{ \Gamma_{k_1,l,o} G^\Sigma_{o,l_2,o} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \left\{ \Gamma_{k_1,l,o}, G^\Sigma_{k_2,o,o} \right\}$$

$$+ \left\{ \Gamma_{k,l,o}, G^\Sigma_{o,o,o} \right\} + \left\{ \Gamma_{k,l,m}, G^\Sigma_{o,o,m} \right\} + i\epsilon_{kk_1k_2} \left\{ \Gamma_{k_1,l,m}, G^\Sigma_{k_2,o,m} \right\}$$

$$+ i\epsilon_{ll_1l_2} \left\{ \Gamma_{k,l,m}, G^\Sigma_{o,l_2,m} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \left\{ \Gamma_{k_1,l,m}, G^\Sigma_{k_2,o,m} \right\}$$

$$+ \left\{ \Gamma_{k,o,o}, G^\Sigma_{o,o,o} \right\} + \left\{ \Gamma_{k,o,m}, G^\Sigma_{o,o,m} \right\} + i\epsilon_{kk_1k_2} \left\{ \Gamma_{k_1,o,m}, G^\Sigma_{k_2,o,m} \right\}$$

$$+ i\epsilon_{ll_1l_2} \left\{ \Gamma_{o,l,m}, G^\Sigma_{k_2,o,m} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \left\{ \Gamma_{k_1,o,m}, G^\Sigma_{k_2,o,m} \right\}$$

$$\left(29\right)$$
\begin{align*}
\{ \Sigma^S, A \}_{k,l,o} = & \{ \Sigma^S_{o,o,o}, A_{k,l,o} \} + \{ \Sigma^S_{o,o,m}, A_{k,l,m} \} + i\epsilon_{kk_1k_2} \{ \Sigma^S_{k_1,o,o}, A_{k_2,l,o} \} \\
& + i\epsilon_{ll_1l_2} \{ \Sigma^S_{o,l,o}, A_{k,l,o} \} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \{ \Sigma^S_{k_1,l,o}, A_{k_2,l,o} \} \\
& + \{ \Sigma^S_{o,l,m}, A_{k,o,m} \} + i\epsilon_{kk_1k_2} \{ \Sigma^S_{k_1,l,m}, A_{k_2,o,m} \} \\
& + \{ \Sigma^S_{o,l,m}, A_{k,o,m} \} + i\epsilon_{kk_1k_2} \{ \Sigma^S_{k_1,l,m}, A_{k_2,l,m} \} \\
& + \{ \Sigma^S_{o,l,m}, A_{k,o,m} \} + i\epsilon_{kk_1k_2} \{ \Sigma^S_{k_1,l,m}, A_{k_2,l,m} \} \\
& + \{ \Sigma^S_{o,l,m}, A_{k,o,m} \} + i\epsilon_{kk_1k_2} \{ \Sigma^S_{k_1,l,m}, A_{k_2,l,m} \} \\
& = \{ \Sigma^S_{k,l,o}, A_{o,o,o} \} + \{ \Sigma^S_{k,l,m}, A_{o,o,m} \} + i\epsilon_{kk_1k_2} \{ \Sigma^S_{k_1,l,m}, A_{k_2,o,m} \} \\
& + \{ \Sigma^S_{k,l,m}, A_{o,l,m} \} + \{ \Sigma^S_{k,l,m}, A_{o,l,m} \} + i\epsilon_{kk_1k_2} \{ \Sigma^S_{k_1,l,m}, A_{k_2,l,m} \} \\
& + \{ \Sigma^S_{k,l,m}, A_{o,l,m} \} + \{ \Sigma^S_{k,l,m}, A_{o,l,m} \} + i\epsilon_{kk_1k_2} \{ \Sigma^S_{k_1,l,m}, A_{k_2,l,m} \} \\
& + \{ \Sigma^S_{k,l,m}, A_{o,l,m} \} + \{ \Sigma^S_{k,l,m}, A_{o,l,m} \} + i\epsilon_{kk_1k_2} \{ \Sigma^S_{k_1,l,m}, A_{k_2,l,m} \} \\
& + \{ \Sigma^S_{k,l,m}, A_{o,l,m} \} + \{ \Sigma^S_{k,l,m}, A_{o,l,m} \} + i\epsilon_{kk_1k_2} \{ \Sigma^S_{k_1,l,m}, A_{k_2,l,m} \} \\
& + \{ \Sigma^S_{k,l,m}, A_{o,l,m} \} + \{ \Sigma^S_{k,l,m}, A_{o,l,m} \} + i\epsilon_{kk_1k_2} \{ \Sigma^S_{k_1,l,m}, A_{k_2,l,m} \} \\
& = 2^{N_s} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^S_{k,o,m} = \left[ \bar{H}, G^S \right]_{k,o,m} + \left[ \Sigma^S, \text{Re} G^S \right]_{k,o,m} - \frac{i}{2} \{ \Gamma, G^S \}_{k,o,m} + \frac{i}{2} \{ \Sigma^S, A \}_{k,o,m} \tag{31} \end{align*}

where,

\begin{align*}
\left[ \bar{H}, G^S \right]_{k,o,m} = & \left[ H_{o,o,o}, G^S_{k,o,m} \right] + \left[ H_{o,o,m}, G^S_{k,o,m} \right] + i\epsilon_{kk_1k_2} \left\{ H_{k_1,o,o}, G^S_{k_2,o,m} \right\} \\
& + i\epsilon_{mm_1m_2} \left\{ H_{o,o,m_1}, G^S_{k,o,m_2} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{mm_1m_2} \left[ H_{k_1,o,m_1} G^S_{k_2,o,m_2} \right] \\
& + \left[ H_{o,l,o}, G^S_{k,l,m} \right] + \left[ H_{o,l,m_1}, G^S_{k,l,m} \right] + i\epsilon_{kk_1k_2} \left\{ H_{k_1,l,o}, G^S_{k_2,l,m} \right\} \\
& + i\epsilon_{mm_1m_2} \left\{ H_{o,l,m_1}, G^S_{k,l,m_2} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{mm_1m_2} \left[ H_{k_1,l,m_1} G^S_{k_2,l,m_2} \right] \\
& + \left[ H_{k,o,o}, G^S_{o,o,m} \right] + \left[ H_{k,o,m_1}, G^S_{o,o,m} \right] + i\epsilon_{kk_1k_2} \left\{ H_{k_1,o,m_1}, G^S_{k_2,o,o} \right\} \\
& + i\epsilon_{mm_1m_2} \left\{ H_{k,o,m_1}, G^S_{o,o,m_2} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{mm_1m_2} \left[ H_{k_1,o,m_1} G^S_{k_2,o,o} \right] \\
& + \left[ H_{k,l,o}, G^S_{o,l,m} \right] + \left[ H_{k,l,m_1}, G^S_{o,l,m} \right] + i\epsilon_{kk_1k_2} \left\{ H_{k_1,l,o}, G^S_{k_2,l,m} \right\} \\
& + i\epsilon_{mm_1m_2} \left\{ H_{k,l,m_1}, G^S_{o,l,m_2} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{mm_1m_2} \left[ H_{k_1,l,m_1} G^S_{k_2,l,m} \right] \tag{32} \end{align*}
\[
[\Sigma^\subseteq, \text{Re} \mathcal{G}^r]_{k,o,m} = \\
[\Sigma^\subseteq_{o,o,m}, \text{Re} \mathcal{G}^r_{k,o,m}] + [\Sigma^\subseteq_{o,o,m}, \text{Re} \mathcal{G}^r_{k,o,m}] + i\epsilon_{kk}k_2 \left\{ \Sigma^\subseteq_{k_1,o,o}, \text{Re} \mathcal{G}^r_{k_2,o,m} \right\} \\
+ i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{o,m,m_1}, \text{Re} \mathcal{G}^r_{k,m,m_2} \right\} + \frac{1}{2} i\epsilon_{kk}k_2 i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{k_1,o,m_1}, \text{Re} \mathcal{G}^r_{k_2,o,m_2} \right\} \\
+ [\Sigma^\subseteq_{o,l,o}, \text{Re} \mathcal{G}^r_{k,l,m}] + [\Sigma^\subseteq_{o,l,o}, \text{Re} \mathcal{G}^r_{k,l,m}] + i\epsilon_{kk}k_2 \left\{ \Sigma^\subseteq_{k_1,l,o}, \text{Re} \mathcal{G}^r_{k_2,l,m} \right\} \\
+ i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{o,l,m_1}, \text{Re} \mathcal{G}^r_{k,l,m_2} \right\} + \frac{1}{2} i\epsilon_{kk}k_2 i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{k_1,l,m_1}, \text{Re} \mathcal{G}^r_{k_2,l,m_2} \right\} \\
+ [\Sigma^\subseteq_{k,o,o}, \text{Re} \mathcal{G}^r_{o,o,m}] + [\Sigma^\subseteq_{k,o,o}, \text{Re} \mathcal{G}^r_{o,o,m}] + i\epsilon_{kk}k_2 \left\{ \Sigma^\subseteq_{k_1,o,m}, \text{Re} \mathcal{G}^r_{k_2,o,m} \right\} \\
+ i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{k,o,m_1}, \text{Re} \mathcal{G}^r_{o,o,m_2} \right\} + \frac{1}{2} i\epsilon_{kk}k_2 i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{k_1,o,m_1}, \text{Re} \mathcal{G}^r_{k_2,o,m_2} \right\}
\]

(33)

\[
\{\Gamma, \mathcal{G}^\subseteq\}_{k,o,m} = \\
\{\Gamma_{o,o,o}, \mathcal{G}^\subseteq_{k,o,m}\} + \{\Gamma_{o,o,m}, \mathcal{G}^\subseteq_{k,o,m}\} + i\epsilon_{kk}k_2 \left\{ \Gamma_{k_1,o,o}, \mathcal{G}^\subseteq_{k_2,o,m} \right\} \\
+ i\epsilon_{mm}m_2 \left\{ \Gamma_{o,o,m_1}, \mathcal{G}^\subseteq_{k_2,o,m_2} \right\} + \frac{1}{2} i\epsilon_{kk}k_2 i\epsilon_{mm}m_2 \left\{ \Gamma_{k_1,o,m_1}, \mathcal{G}^\subseteq_{k_2,o,m_2} \right\} \\
+ \{\Gamma_{o,l,o}, \mathcal{G}^\subseteq_{k,l,m}\} + \{\Gamma_{o,l,m}, \mathcal{G}^\subseteq_{k,l,m}\} + i\epsilon_{kk}k_2 \left\{ \Gamma_{k_1,l,o}, \mathcal{G}^\subseteq_{k_2,l,m} \right\} \\
+ i\epsilon_{mm}m_2 \left\{ \Gamma_{o,l,m_1}, \mathcal{G}^\subseteq_{k,l,m_2} \right\} + \frac{1}{2} i\epsilon_{kk}k_2 i\epsilon_{mm}m_2 \left\{ \Gamma_{k_1,l,m_1}, \mathcal{G}^\subseteq_{k_2,l,m_2} \right\} \\
+ \{\Gamma_{k,o,o}, \mathcal{G}^\subseteq_{o,o,m}\} + \{\Gamma_{k,o,m}, \mathcal{G}^\subseteq_{o,o,m}\} + i\epsilon_{kk}k_2 \left\{ \Gamma_{k_1,o,m}, \mathcal{G}^\subseteq_{k_2,o,m} \right\} \\
+ i\epsilon_{mm}m_2 \left\{ \Gamma_{k,o,m_1}, \mathcal{G}^\subseteq_{o,o,m_2} \right\} + \frac{1}{2} i\epsilon_{kk}k_2 i\epsilon_{mm}m_2 \left\{ \Gamma_{k_1,o,m_1}, \mathcal{G}^\subseteq_{k_2,o,m_2} \right\}
\]

(34)

\[
\{\Sigma^\subseteq, A\}_{k,o,m} = \\
\{\Sigma^\subseteq_{o,o,o}, A_{k,o,m}\} + \{\Sigma^\subseteq_{o,o,m}, A_{k,o,m}\} + i\epsilon_{kk}k_2 \left\{ \Sigma^\subseteq_{k_1,o,o}, A_{k_2,o,m} \right\} \\
+ i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{o,o,m_1}, A_{k,o,m_2} \right\} + \frac{1}{2} i\epsilon_{kk}k_2 i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{k_1,o,m_1}, A_{k_2,o,m_2} \right\} \\
+ \{\Sigma^\subseteq_{o,l,o}, A_{k,l,m}\} + \{\Sigma^\subseteq_{o,l,m}, A_{k,l,m}\} + i\epsilon_{kk}k_2 \left\{ \Sigma^\subseteq_{k_1,l,o}, A_{k_2,l,m} \right\} \\
+ i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{o,l,m_1}, A_{k,l,m_2} \right\} + \frac{1}{2} i\epsilon_{kk}k_2 i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{k_1,l,m_1}, A_{k_2,l,m_2} \right\} \\
+ \{\Sigma^\subseteq_{k,o,o}, A_{o,o,m}\} + \{\Sigma^\subseteq_{k,o,m}, A_{o,o,m}\} + i\epsilon_{kk}k_2 \left\{ \Sigma^\subseteq_{k_1,o,m}, A_{k_2,o,o} \right\} \\
+ i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{k,o,m_1}, A_{o,o,m_2} \right\} + \frac{1}{2} i\epsilon_{kk}k_2 i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{k_1,o,m_1}, A_{k_2,o,m_2} \right\} \\
+ \{\Sigma^\subseteq_{k,l,o}, A_{l,o,m}\} + \{\Sigma^\subseteq_{k,l,m}, A_{l,o,m}\} + i\epsilon_{kk}k_2 \left\{ \Sigma^\subseteq_{k_1,l,m}, A_{k_2,l,o} \right\} \\
+ i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{k,l,m_1}, A_{l,o,m_2} \right\} + \frac{1}{2} i\epsilon_{kk}k_2 i\epsilon_{mm}m_2 \left\{ \Sigma^\subseteq_{k_1,l,m_1}, A_{k_2,l,o} \right\}
\]

(35)
We have for the entangled valley spin and pseudospin,

\[
2^{N_s} i \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{o,l,m}^\Sigma =
\]

\[
= \left[ \hat{\mathcal{H}}, G_{o,l,m}^\Sigma \right] + \left[ \Sigma^\Sigma, \text{Re}G^r \right]_{o,l,m} - \frac{i}{2} \left\{ \Gamma, G^\Sigma \right\}_{o,l,m} + \frac{i}{2} \left\{ \Sigma^\Sigma, \mathcal{A} \right\}_{o,l,m} \tag{36}
\]

where,

\[
\left[ \hat{\mathcal{H}}, G_{o,l,m}^\Sigma \right] =
\]

\[
= \left[ H_{o,o,o} G_{o,l,m}^\Sigma + \left[ H_{o,o,m} G_{o,l,o}^\Sigma + i \epsilon_{m_1 m_2} \left\{ H_{o,o,m_1}, G_{o,l,m_2}^\Sigma \right\} \right] + \frac{i}{2} \epsilon_{l_1 l_2} i \epsilon_{m_1 m_2} \right\} H_{o,l_1,m_1,1} G_{o,l_2,m_2}^\Sigma
\]

\[
+ \left[ H_{o,l,o} G_{o,o,m}^\Sigma + \left[ H_{o,l,m} G_{o,o,o}^\Sigma + i \epsilon_{m_1 m_2} \right\} H_{o,l,m_1,1} G_{o,o,m_2}^\Sigma \right] + \frac{i}{2} \epsilon_{l_1 l_2} i \epsilon_{m_1 m_2} \right\} H_{o,l_1,m_1,1} G_{o,o,m_2}^\Sigma
\]

\[
= \left[ H_{k,l,o} G_{k,o,o}^\Sigma + \left[ H_{k,l,m} G_{k,o,o}^\Sigma + i \epsilon_{m_1 m_2} \right\} H_{k,l,m_1,1} G_{k,o,m_2}^\Sigma \right] + \frac{i}{2} \epsilon_{l_1 l_2} i \epsilon_{m_1 m_2} \right\} H_{k,l_1,m_1,1} G_{k,o,m_2}^\Sigma
\]

\[
+ \left[ H_{k,o,o} G_{k,l,o}^\Sigma + \left[ H_{k,o,m} G_{k,l,o}^\Sigma + i \epsilon_{m_1 m_2} \right\} H_{k,o,m_1,1} G_{k,l,m_2}^\Sigma \right] + \frac{i}{2} \epsilon_{l_1 l_2} i \epsilon_{m_1 m_2} \right\} H_{k,o,l_1,m_1,1} G_{k,l,m_2}^\Sigma
\]

\[
\left[ \Sigma^\Sigma, \text{Re}G^r \right]_{o,l,m} =
\]

\[
= \left[ \Sigma_{o,o,o}^\Sigma, \text{Re}G_{o,l,m}^r \right] + \left[ \Sigma_{o,o,m}^\Sigma, \text{Re}G_{o,l,o}^r \right] + i \epsilon_{m_1 m_2} \left\{ \Sigma_{o,o,m_1}^\Sigma, \text{Re}G_{o,l,m_2}^r \right\} \right\} \Sigma_{o,l_1,m_1,1} \text{Re}G_{o,l_2,m_2}^r
\]

\[
+ \left[ \Sigma_{o,l,o}^\Sigma, \text{Re}G_{o,o,m}^r \right] + \left[ \Sigma_{o,l,m}^\Sigma, \text{Re}G_{o,o,o}^r \right] + i \epsilon_{m_1 m_2} \left\{ \Sigma_{o,l,m_1}^\Sigma, \text{Re}G_{o,o,m_2}^r \right\} \right\} \Sigma_{o,l_1,m_1,1} \text{Re}G_{o,o,m_2}^r
\]

\[
= \left[ \Sigma_{k,l,o}^\Sigma, \text{Re}G_{k,o,o}^r \right] + \left[ \Sigma_{k,l,m}^\Sigma, \text{Re}G_{k,o,o}^r \right] + i \epsilon_{m_1 m_2} \left\{ \Sigma_{k,l,m_1}^\Sigma, \text{Re}G_{k,o,m_2}^r \right\} \right\} \Sigma_{k,l_1,m_1,1} \text{Re}G_{k,l_2,o}^r
\]

\[
+ \left[ \Sigma_{k,o,o}^\Sigma, \text{Re}G_{k,l,o}^r \right] + \left[ \Sigma_{k,o,m}^\Sigma, \text{Re}G_{k,l,o}^r \right] + i \epsilon_{m_1 m_2} \left\{ \Sigma_{k,o,m_1}^\Sigma, \text{Re}G_{k,l,m_2}^r \right\} \right\} \Sigma_{k,l_1,m_1,1} \text{Re}G_{k,l_2,o}^r
\]

\[
\left(38\right)
\]
\[ \{ \Gamma, G^s \}_{o,l,m} = \{ \Gamma_{o,o,o}, G^s_{o,l,m} \} + \{ \Gamma_{o,o,m}, G^s_{o,l,o} \} + i\epsilon_{m,m_1} [\Gamma_{o,o,m_1}, G^s_{o,l,m_2}] \\
+ i\epsilon_{l_1 l_2} \{ \Gamma_{o,l_1,o}, G^s_{o,l_2,m} \} + \frac{1}{2} i\epsilon_{l_1 l_2} i\epsilon_{m,m_1} [\Gamma_{o,l_1,m_1}, G^s_{o,l_2,m_2}] \\
+ \{ \Gamma_{o,l,m}, G^s_{o,o,o} \} + i\epsilon_{m_1,m_2} [\Gamma_{o,l,m_1}, G^s_{o,o,m_2}] \\
+ i\epsilon_{l_1 l_2} \{ \Gamma_{o,l_1,m_1}, G^s_{o,l_2,o} \} + \frac{1}{2} i\epsilon_{l_1 l_2} i\epsilon_{m,m_1} [\Gamma_{o,l_1,m_1}, G^s_{o,l_2,m_2}] \\
+ \{ \Gamma_{k,l,o}, G^s_{k,o,m} \} + \{ \Gamma_{k,l,m}, G^s_{k,o,o} \} + i\epsilon_{m_1,m_2} [\Gamma_{k,l,m_1}, G^s_{k,o,m_2}] \\
+ i\epsilon_{l_1 l_2} \{ \Gamma_{k,l_1,m_1}, G^s_{k,l_2,o} \} + \frac{1}{2} i\epsilon_{l_1 l_2} i\epsilon_{m,m_1} [\Gamma_{k,l_1,m_1}, G^s_{k,l_2,m_2}] \\
+ \{ \Gamma_{k,o,o}, G^s_{k,l,m} \} + \{ \Gamma_{k,o,m}, G^s_{k,l,o} \} + i\epsilon_{m_1,m_2} [\Gamma_{k,o,m_1}, G^s_{k,l,m_2}] \\
+ i\epsilon_{l_1 l_2} \{ \Gamma_{k,l_1,o}, G^s_{k,l_2,m} \} + \frac{1}{2} i\epsilon_{l_1 l_2} i\epsilon_{m,m_1} [\Gamma_{k,l_1,m_1}, G^s_{k,l_2,m_2}] \] (39)

\[ \{ \Sigma^s, A \}_{o,l,m} = \{ \Sigma^s_{o,o,o}, A_{o,l,m} \} + \{ \Sigma^s_{o,o,m}, A_{o,l,o} \} + i\epsilon_{m,m_1} [\Sigma^s_{o,o,m_1}, A_{o,l,m_2}] \\
+ i\epsilon_{l_1 l_2} \{ \Sigma^s_{o,l_1,o}, A_{o,l_2,m} \} + \frac{1}{2} i\epsilon_{l_1 l_2} i\epsilon_{m,m_1} [\Sigma^s_{o,l_1,m_1}, A_{o,l_2,m_2}] \\
+ \{ \Sigma^s_{o,l,m}, A_{o,o,o} \} + [\Sigma^s_{o,l,m_1}, A_{o,o,m_2}] + i\epsilon_{m_1,m_2} [\Sigma^s_{o,l,m_1}, A_{o,o,m_2}] \\
+ i\epsilon_{l_1 l_2} \{ \Sigma^s_{o,l_1,m_1}, A_{o,l_2,o} \} + \frac{1}{2} i\epsilon_{l_1 l_2} i\epsilon_{m,m_1} [\Sigma^s_{o,l_1,m_1}, A_{o,l_2,o}] \\
+ \{ \Sigma^s_{k,l,o}, A_{k,o,m} \} + \{ \Sigma^s_{k,l,m}, A_{k,o,o} \} + i\epsilon_{m_1,m_2} [\Sigma^s_{k,l,m_1}, A_{k,o,m_2}] \\
+ i\epsilon_{l_1 l_2} \{ \Sigma^s_{k,l_1,m_1}, A_{k,l_2,o} \} + \frac{1}{2} i\epsilon_{l_1 l_2} i\epsilon_{m,m_1} [\Sigma^s_{k,l_1,m_1}, A_{k,l_2,o}] \\
+ \{ \Sigma^s_{k,o,o}, A_{k,l,m} \} + \{ \Sigma^s_{k,o,m}, A_{k,l,o} \} + i\epsilon_{m_1,m_2} [\Sigma^s_{k,o,m_1}, A_{k,l,m_2}] \\
+ i\epsilon_{l_1 l_2} \{ \Sigma^s_{k,l_1,o}, A_{k,l_2,m} \} + \frac{1}{2} i\epsilon_{l_1 l_2} i\epsilon_{m,m_1} [\Sigma^s_{k,l_1,m_1}, A_{k,l_2,m_2}] \] (40)

We have for the entangled Dirac spin, valley spin, and pseudospin,

\[ 2^N \hbar \left( \frac{\partial}{\partial l_1} + \frac{\partial}{\partial l_2} \right) G^s_{k,l,m} = \] 

\[ = \left[ \tilde{H}, G^s \right]_{k,l,m} + [\Sigma^s, \text{Re} G^r]_{k,l,m} - \frac{i}{2} \{ \Gamma, G^s \}_{k,l,m} + \frac{i}{2} \{ \Sigma^s, A \}_{k,l,m} \] (41)
where,

\[
\left[ \mathcal{H}, G^\Sigma \right]_{k,l,m} = 
\]

\[= +i\epsilon_{kk_1k_2} i\epsilon_{ll_1l_2} \left[ \Sigma^\Sigma_{k,l,m_1}, A_{k_2,l_2} \right] + \left\{ \Sigma^\Sigma_{k_2,l_2,m_2} \right\}
\]

**VI. Limiting Case of Valley Spin and Pseudospin, \( N_s = 2 \)**

\[
2^{N_s} \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^\Sigma_{o,o} = 
\]

\[= \left[ \mathcal{H}, G^\Sigma \right]_{o,o} + \left[ \Sigma^\Sigma, \text{Re} G^r \right]_{o,o} - \frac{i}{2} \left\{ \Gamma, G^\Sigma \right\}_{o,o} + \frac{i}{2} \left\{ \Sigma^\Sigma, A \right\}_{o,o}
\]
where,

\[
\begin{align*}
\left[ \mathcal{H}, G^\le_\circ, \circ \right] & = \\
& = \left[ H_{\circ,\circ}, G^\le_{\circ,\circ} \right] + \left[ H_{\circ,\bar{m}}, G^\le_{\circ,\bar{m}} \right] + \left[ H_{\bar{m},\circ}, G^\le_{\bar{m},\circ} \right] + \left[ H_{\bar{m},\bar{m}}, G^\le_{\bar{m},\bar{m}} \right] 
\end{align*}
\] (47)

\[
\begin{align*}
\left[ \Sigma^\le, \mathrm{Re} G^r \right] & = \\
& = \left[ \Sigma^\le_{\circ,\circ}, \mathrm{Re} G^r_{\circ,\circ} \right] + \left[ \Sigma^\le_{\circ,\bar{m}}, \mathrm{Re} G^r_{\circ,\bar{m}} \right] + \left[ \Sigma^\le_{\bar{m},\circ}, \mathrm{Re} G^r_{\bar{m},\circ} \right] + \left[ \Sigma^\le_{\bar{m},\bar{m}}, \mathrm{Re} G^r_{\bar{m},\bar{m}} \right] 
\end{align*}
\] (48)

\[
\begin{align*}
\left\{ \Gamma, G^\le \right\} & = \\
& = \left\{ \Gamma_{\circ,\circ}, G^\le_{\circ,\circ} \right\} + \left\{ \Gamma_{\circ,\bar{m}}, G^\le_{\circ,\bar{m}} \right\} + \left\{ \Gamma_{\bar{m},\circ}, G^\le_{\bar{m},\circ} \right\} + \left\{ \Gamma_{\bar{m},\bar{m}}, G^\le_{\bar{m},\bar{m}} \right\} 
\end{align*}
\] (49)

\[
\begin{align*}
\left\{ \Sigma^\le, A \right\} & = \\
& = \left\{ \Sigma^\le_{\circ,\circ}, A_{\circ,\circ} \right\} + \left\{ \Sigma^\le_{\circ,\bar{m}}, A_{\circ,\bar{m}} \right\} + \left\{ \Sigma^\le_{\bar{m},\circ}, A_{\bar{m},\circ} \right\} + \left\{ \Sigma^\le_{\bar{m},\bar{m}}, A_{\bar{m},\bar{m}} \right\} 
\end{align*}
\] (50)

\[
\begin{align*}
2^{N_s} \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^\le_{l,\circ} = \\
& = \left[ \mathcal{H}, G^\le \right]_{l,\circ} + \left[ \Sigma^\le, \mathrm{Re} G^r \right]_{l,\circ} - \frac{i}{2} \left\{ \Gamma, G^\le \right\}_{l,\circ} + \frac{i}{2} \left\{ \Sigma^\le, A \right\}_{l,\circ} 
\end{align*}
\] (51)

where,

\[
\begin{align*}
\left[ \mathcal{H}, G^\le \right] & = \\
& = \left[ H_{\circ,\circ}, G^\le_{\circ,\circ} \right] + \left[ H_{\circ,\bar{m}}, G^\le_{\circ,\bar{m}} \right] + \left[ H_{\bar{m},\circ}, G^\le_{\bar{m},\circ} \right] + \left[ H_{\bar{m},\bar{m}}, G^\le_{\bar{m},\bar{m}} \right] 
\end{align*}
\] (52)

\[
\begin{align*}
\left[ \Sigma^\le, \mathrm{Re} G^r \right] & = \\
& = \left[ \Sigma^\le_{\circ,\circ}, \mathrm{Re} G^r_{\circ,\circ} \right] + \left[ \Sigma^\le_{\circ,\bar{m}}, \mathrm{Re} G^r_{\circ,\bar{m}} \right] + \left[ \Sigma^\le_{\bar{m},\circ}, \mathrm{Re} G^r_{\bar{m},\circ} \right] + \left[ \Sigma^\le_{\bar{m},\bar{m}}, \mathrm{Re} G^r_{\bar{m},\bar{m}} \right] 
\end{align*}
\] (53)

\[
\begin{align*}
\left\{ \Gamma, G^\le \right\} & = \\
& = \left\{ \Gamma_{\circ,\circ}, G^\le_{\circ,\circ} \right\} + \left\{ \Gamma_{\circ,\bar{m}}, G^\le_{\circ,\bar{m}} \right\} + \left\{ \Gamma_{\bar{m},\circ}, G^\le_{\bar{m},\circ} \right\} + \left\{ \Gamma_{\bar{m},\bar{m}}, G^\le_{\bar{m},\bar{m}} \right\} 
\end{align*}
\] (54)
\[
\{ \Sigma^{\subseteq}, A \}_{l,o} = \\
= \{ \Sigma_{o,o}^{\subseteq}, A_{l,o} \} + \{ \Sigma_{o,m}^{\subseteq}, A_{l,m} \} + i \epsilon_{l_1 l_2} [\Sigma_{l_1 o, A_{l_2 o}}^{\subseteq}] \\
+ \{ \Sigma_{l,o}^{\subseteq}, A_{o,o} \} + \{ \Sigma_{l,m}^{\subseteq}, A_{o,m} \} + i \epsilon_{l_1 l_2} [\Sigma_{l_{1,m} o, A_{l_{2,m}}}] 
\]

\begin{align*}
2^{N_s} i \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{o,m}^{\subseteq} &= \\
= \left[ \mathcal{H}, G^{\subseteq} \right]_{o,m} + [\Sigma^{\subseteq}, \text{Re} G^\gamma]_{o,m} - \frac{i}{2} \{ \Gamma, G^{\subseteq} \}_{o,m} + \frac{i}{2} \{ \Sigma^{\subseteq}, A \}_{o,m}
\end{align*}

where,

\[
\left[ \mathcal{H}, G^{\subseteq} \right]_{o,m} = \\
= \left[ H_{o,o}, G_{o,m}^{\subseteq} \right] + \left[ H_{o,m}, G_{o,o}^{\subseteq} \right] + i \epsilon_{m m_1 m_2} \left\{ H_{o,m_1}, G_{o,m_2}^{\subseteq} \right\} \\
+ \left[ H_{l,o}, G_{l,m}^{\subseteq} \right] + \left[ H_{l,m}, G_{l,o}^{\subseteq} \right] + i \epsilon_{m m_1 m_2} \left\{ H_{l,m_1}, G_{l,m_2}^{\subseteq} \right\}
\]

\[
\left[ \Sigma^{\subseteq}, \text{Re} G^\gamma \right]_{o,m} = \\
= \left[ \Sigma_{o,o}^{\subseteq}, \text{Re} G_{o,m}^\gamma \right] + \left[ \Sigma_{o,m}^{\subseteq}, \text{Re} G_{o,o}^\gamma \right] + i \epsilon_{m m_1 m_2} \left\{ \Sigma_{o,m_1}^{\subseteq}, \text{Re} G_{o,m_2}^\gamma \right\} \\
+ \left[ \Sigma_{l,o}^{\subseteq}, \text{Re} G_{l,m}^\gamma \right] + \left[ \Sigma_{l,m}^{\subseteq}, \text{Re} G_{l,o}^\gamma \right] + i \epsilon_{m m_1 m_2} \left\{ \Sigma_{l,m_1}^{\subseteq}, \text{Re} G_{l,m_2}^\gamma \right\}
\]

\[
\left\{ \Gamma, G^{\subseteq} \right\}_{o,m} = \\
= \left\{ \Gamma_{o,o}, G_{o,o,m}^{\subseteq} \right\} + \left\{ \Gamma_{o,m}, G_{o,o}^{\subseteq} \right\} + i \epsilon_{m m_1 m_2} \left[ \Gamma_{o,m_1}, G_{o,m_2}^{\subseteq} \right] \\
+ \left\{ \Gamma_{l,o}, G_{l,m}^{\subseteq} \right\} + \left\{ \Gamma_{l,m}, G_{l,o}^{\subseteq} \right\} + i \epsilon_{m m_1 m_2} \left[ \Gamma_{l,m_1}, G_{l,m_2}^{\subseteq} \right]
\]

\[
\left\{ \Sigma^{\subseteq}, A \right\}_{o,m} = \\
= \left\{ \Sigma_{o,o}^{\subseteq}, A_{o,o,m} \right\} + \left\{ \Sigma_{o,m}^{\subseteq}, A_{o,o} \right\} + i \epsilon_{m m_1 m_2} \left[ \Sigma_{o,m_1}^{\subseteq}, A_{o,m_2} \right] \\
+ \left\{ \Sigma_{l,o}^{\subseteq}, A_{l,m} \right\} + \left\{ \Sigma_{l,m}^{\subseteq}, A_{l,o} \right\} + i \epsilon_{m m_1 m_2} \left[ \Sigma_{l,m_1}^{\subseteq}, A_{l,m_2} \right]
\]

\begin{align*}
2^{N_s} i \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{l,m}^{\subseteq} &= \\
= \left[ \mathcal{H}, G^{\subseteq} \right]_{l,m} + [\Sigma^{\subseteq}, \text{Re} G^\gamma]_{l,m} - \frac{i}{2} \{ \Gamma, G^{\subseteq} \}_{l,m} + \frac{i}{2} \{ \Sigma^{\subseteq}, A \}_{l,m}
\end{align*}
where,

\[
\left[ \tilde{H}, G^S \right]_{l,m} = \\
\left[ H_{o,o}, G^S_{l,m} \right] + \left[ H_{l,o}, G^S_{l,o} \right] + \left[ H_{l,m}, G^S_{o,m} \right] + \left[ H_{l,m}, G^S_{o,o} \right] + \frac{i\epsilon_{\mu\mu\nu\nu}}{2} \left\{ H_{o,m} G^S_{l,m}, G^S_{o,m} \right\} + \frac{i\epsilon_{\nu\nu\mu\mu}}{2} \left\{ H_{l,m} G^S_{l,m}, G^S_{l,o} \right\} + \frac{i\epsilon_{\mu\nu\nu\nu}}{2} \left\{ H_{l,m} G^S_{l,m}, G^S_{l,m} \right\} 
\]

(62)

\[
\left[ \Sigma^S, \text{Re}G^r \right]_{l,m} = \\
\left[ \Sigma^S_{o,o}, \text{Re}G^r_{l,m} \right] + \left[ \Sigma^S_{o,m}, \text{Re}G^r_{l,o} \right] + \left[ \Sigma^S_{l,o}, \text{Re}G^r_{o,m} \right] + \left[ \Sigma^S_{l,m}, \text{Re}G^r_{o,o} \right] + \frac{i\epsilon_{\mu\mu\nu\nu}}{2} \left\{ \Sigma^S_{o,m}, \text{Re}G^r_{l,m}, \text{Re}G^r_{o,m} \right\} + \frac{i\epsilon_{\nu\nu\mu\mu}}{2} \left\{ \Sigma^S_{l,o}, \text{Re}G^r_{l,m}, \text{Re}G^r_{l,o} \right\} + \frac{i\epsilon_{\mu\nu\nu\nu}}{2} \left\{ \Sigma^S_{l,m}, \text{Re}G^r_{l,m}, \text{Re}G^r_{l,m} \right\} 
\]

(63)

\[
\left\{ \Gamma, G^S \right\}_{l,m} = \\
\left\{ \Gamma_{o,o}, G^S_{l,m} \right\} + \left\{ \Gamma_{o,m}, G^S_{l,o} \right\} + \left\{ \Gamma_{l,o}, G^S_{o,m} \right\} + \left\{ \Gamma_{l,m}, G^S_{o,o} \right\} + \frac{i\epsilon_{\mu\mu\nu\nu}}{2} \left\{ \Gamma_{o,m}, G^S_{l,m}, G^S_{o,m} \right\} + \frac{i\epsilon_{\nu\nu\mu\mu}}{2} \left\{ \Gamma_{l,o}, G^S_{l,m}, G^S_{l,o} \right\} + \frac{i\epsilon_{\mu\nu\nu\nu}}{2} \left\{ \Gamma_{l,m}, G^S_{l,m}, G^S_{l,m} \right\} 
\]

(64)

\[
\left\{ \Sigma^S, A \right\}_{l,m} = \\
\left\{ \Sigma^S_{o,o}, A_{l,m} \right\} + \left\{ \Sigma^S_{o,m}, A_{l,o} \right\} + \left\{ \Sigma^S_{l,o}, A_{o,m} \right\} + \left\{ \Sigma^S_{l,m}, A_{o,o} \right\} + \frac{i\epsilon_{\mu\mu\nu\nu}}{2} \left\{ \Sigma^S_{o,m}, A_{l,m}, A_{o,m} \right\} + \frac{i\epsilon_{\nu\nu\mu\mu}}{2} \left\{ \Sigma^S_{l,o}, A_{l,m}, A_{l,o} \right\} + \frac{i\epsilon_{\mu\nu\nu\nu}}{2} \left\{ \Sigma^S_{l,m}, A_{l,m}, A_{l,m} \right\} 
\]

(65)

VII. Limiting Case of Pauli-Dirac Spin and Valley Spin, \( N_s = 2 \)

\[
2^{N_s} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^S_{o,o} = \\
= \left[ \tilde{H}, G^S \right]_{o,o} + \left[ \Sigma^S, \text{Re}G^r \right]_{o,o} - \frac{i}{2} \left\{ \Gamma, G^S \right\}_{o,o} + \frac{i}{2} \left\{ \Sigma^S, A \right\}_{o,o} 
\]

(66)
where,

$$\left[ \tilde{\mathcal{H}}, G^\lessdot_{k,o} \right] =$$

$$= \left[ H_{o,o}^\lessdot, G^\lessdot_{o,o,o} \right] + \left[ H_{o,l}^\lessdot, G^\lessdot_{o,o,l} \right] + \left[ H_{k,o}^\lessdot, G^\lessdot_{k,o,o} \right] + \left[ H_{k,l}^\lessdot, G^\lessdot_{k,l} \right]$$

$$\quad (67)$$

$$\left[ \Sigma^\lessdot, \text{Re} G^r \right]_{o,o} =$$

$$= \left[ \Sigma^\lessdot_{o,o}, \text{Re} G^r_{o,o} \right] + \left[ \Sigma^\lessdot_{o,l}, \text{Re} G^r_{o,l} \right] + \left[ \Sigma^\lessdot_{k,o}, \text{Re} G^r_{k,o} \right] + \left[ \Sigma^\lessdot_{k,l}, \text{Re} G^r_{k,l} \right]$$

$$\quad (68)$$

$$\{ \Gamma, G^\lessdot \}_{o,o} =$$

$$= \{ \Gamma_{o,o}^\lessdot, G^\lessdot_{o,o,o} \} + \{ \Gamma_{o,l}^\lessdot, G^\lessdot_{o,o,l} \} + \{ \Gamma_{k,o}^\lessdot, G^\lessdot_{k,o,o} \} + \{ \Gamma_{k,l}^\lessdot, G^\lessdot_{k,l,o} \}$$

$$\quad (69)$$

$$\{ \Sigma^\lessdot, A \}_{o,o} =$$

$$= \{ \Sigma^\lessdot_{o,o}, A_{o,o} \} + \{ \Sigma^\lessdot_{o,l}, A_{o,l} \} + \{ \Sigma^\lessdot_{k,o}, A_{k,o} \} + \{ \Sigma^\lessdot_{k,l}, A_{k,l} \}$$

$$\quad (70)$$

$$2^{N_s}i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^\lessdot_{k,o} =$$

$$= \left[ \tilde{\mathcal{H}}, G^\lessdot \right]_{k,o} + \left[ \Sigma^\lessdot, \text{Re} G^r \right]_{k,o} - \frac{i}{2} \{ \Gamma, G^\lessdot \}_{k,o} + \frac{i}{2} \{ \Sigma^\lessdot, A \}_{k,o}$$

$$\quad (71)$$

where,

$$\left[ \tilde{\mathcal{H}}, G^\lessdot \right]_{k,o} =$$

$$= \left[ H_{o,o}^\lessdot, G^\lessdot_{k,o} \right] + \left[ H_{k,o}^\lessdot, G^\lessdot_{k,o,o} \right] + \left[ H_{k,l}^\lessdot, G^\lessdot_{k,l} \right] + \left[ H_{o,l}^\lessdot, G^\lessdot_{k,l} \right]$$

$$\quad (72)$$

$$\left[ \Sigma^\lessdot, \text{Re} G^r \right]_{k,o} =$$

$$= \left[ \Sigma^\lessdot_{o,o}, \text{Re} G^r_{k,o} \right] + \left[ \Sigma^\lessdot_{o,l}, \text{Re} G^r_{o,l} \right] + \left[ \Sigma^\lessdot_{k,l}, \text{Re} G^r_{k,o} \right] + \left[ \Sigma^\lessdot_{k,l}, \text{Re} G^r_{k,l} \right]$$

$$\quad (73)$$

$$\{ \Gamma, G^\lessdot \}_{k,o} =$$

$$= \{ \Gamma_{k,o}^\lessdot, G^\lessdot_{k,o,o} \} + \{ \Gamma_{k,l}^\lessdot, G^\lessdot_{k,l,o} \} + \{ \Gamma_{o,l}^\lessdot, G^\lessdot_{k,l} \}$$

$$\quad (74)$$
\[
\{ \Sigma^S, A \}_{k,o} = \\
= \{ \Sigma^S_{o,o}, A_{k,o} \} + \{ \Sigma^S_{k,o}, A_{o,o} \} + \{ \Sigma^S_{k,l}, A_{o,l} \} + \{ \Sigma^S_{o,l}, A_{k,l} \} + \{ \Sigma^S_{k,1}, A_{k,2} \} \\
+ i \epsilon_{kk_1k_2} \{ \Sigma^S_{kk_1,o}, A_{k_2,o} \} + i \epsilon_{kk_1k_2} \{ \Sigma^S_{kk_1,l}, A_{k_2,l} \}
\]

(75)

\[
2^{N_s}ih \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^S_{o,l} = \\
= \left[ \tilde{H}, G^S \right]_{o,l} + \{ \Sigma^S, \text{Re} G^r \}_{o,l} - \frac{i}{2} \{ \Gamma, G^S \}_{o,l} + \frac{i}{2} \{ \Sigma^S, A \}_{o,l}
\]

(76)

where,

\[
\left[ \tilde{H}, G^S \right]_{o,l} = \\
= \left[ H_{o,o}, G^S_{o,l} \right] + \left[ H_{o,l}, G^S_{o,o} \right] + \left[ H_{k,o}, G^S_{k,l} \right] + \left[ H_{k,l}, G^S_{k,o} \right] \\
+ i \epsilon_{ll_1l_2} \left\{ H_{o,l_1}, G^S_{o,l_2} \right\} + i \epsilon_{ll_1l_2} \left\{ H_{k,l_1}, G^S_{k,l_2} \right\}
\]

(77)

\[
\{ \Sigma^S, \text{Re} G^r \}_{o,l} = \\
= \left[ \Sigma^S_{o,o}, \text{Re} G^r_{o,l} \right] + \left[ \Sigma^S_{o,l}, \text{Re} G^r_{o,o} \right] + \left[ \Sigma^S_{k,o}, \text{Re} G^r_{k,l} \right] + \left[ \Sigma^S_{k,l}, \text{Re} G^r_{k,o} \right] \\
+ i \epsilon_{ll_1l_2} \left\{ \Sigma^S_{o,l_1}, \text{Re} G^r_{o,l_2} \right\} + i \epsilon_{ll_1l_2} \left\{ \Sigma^S_{k,l_1}, \text{Re} G^r_{k,l_2} \right\}
\]

(78)

\[
\{ \Gamma, G^S \}_{o,l} = \\
= \left\{ \Gamma_{o,o}, G^S_{o,l} \right\} + \left\{ \Gamma_{o,l}, G^S_{o,o} \right\} + \left\{ \Gamma_{k,o}, G^S_{k,l} \right\} + \left\{ \Gamma_{k,l}, G^S_{k,o} \right\} \\
+ i \epsilon_{ll_1l_2} \left\{ \Gamma_{o,l_1}, G^S_{o,l_2} \right\} + i \epsilon_{ll_1l_2} \left\{ \Gamma_{k,l_1}, G^S_{k,l_2} \right\}
\]

(79)

\[
\{ \Sigma^S, A \}_{o,l} = \\
= \left\{ \Sigma^S_{o,o}, A_{o,l} \right\} + \left\{ \Sigma^S_{o,l}, A_{o,o} \right\} + \left\{ \Sigma^S_{k,o}, A_{k,l} \right\} + \left\{ \Sigma^S_{k,l}, A_{k,o} \right\} \\
+ i \epsilon_{ll_1l_2} \left\{ \Sigma^S_{o,l_1}, A_{o,l_2} \right\} + i \epsilon_{ll_1l_2} \left\{ \Sigma^S_{k,l_1}, A_{k,l_2} \right\}
\]

(80)

\[
2^{N_s}ih \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^S_{k,l} = \\
= \left[ \tilde{H}, G^S \right]_{k,l} + \{ \Sigma^S, \text{Re} G^r \}_{k,l} - \frac{i}{2} \{ \Gamma, G^S \}_{k,l} + \frac{i}{2} \{ \Sigma^S, A \}_{k,l}
\]

(81)

where,

\[
\left[ \tilde{H}, G^S \right]_{k,l} = \\
= \left[ H_{o,o}, G^S_{k,l} \right] + \left[ H_{k,l}, G^S_{o,o} \right] + \left[ H_{o,l}, G^S_{k,o} \right] + \left[ H_{k,o}, G^S_{o,l} \right] \\
+ i \epsilon_{kk_1k_2} \left\{ H_{k_1,o}, G^S_{k_2,l} \right\} + i \epsilon_{kk_1k_2} \left\{ H_{k_1,l}, G^S_{k_2,o} \right\} \\
+ i \epsilon_{ll_1l_2} \left\{ H_{o,l_1}, G^S_{o,l_2} \right\} + i \epsilon_{ll_1l_2} \left\{ H_{k,l_1}, G^S_{k,l_2} \right\} \\
+ \frac{i}{2} i \epsilon_{kk_1k_2} i \epsilon_{ll_1l_2} \left[ H_{k_1,l_1}, G^S_{k_2,l_2} \right] + \frac{i}{2} i \epsilon_{kk_1k_2} i \epsilon_{ll_1l_2} \left[ H_{k_1,l_1}, G^S_{k_2,l_2} \right]
\]

(82)
\[
\left[ \Sigma^\parallel, \text{Re} G^r \right]_{k,l} = \\
\left[ \Sigma^\parallel_{o,o}, \text{Re} G^r_{k,l} \right] + \left[ \Sigma^\parallel_{k,l}, \text{Re} G^r_{o,o} \right] + \left[ \Sigma^\parallel_{k,o}, \text{Re} G^r_{o,l} \right] + \left[ \Sigma^\parallel_{o,l}, \text{Re} G^r_{k,o} \right] \\
+ i \epsilon_{kk_{1}k_{2}} \left\{ \Sigma^\parallel_{k_{1},o}, \text{Re} G^r_{k_{2},l} \right\} + i \epsilon_{kk_{1}k_{2}} \left\{ \Sigma^\parallel_{k_{1},l}, \text{Re} G^r_{k_{2},o} \right\} \\
+ i \epsilon_{ll_{1}l_{2}} \left\{ \Sigma^\parallel_{o,o,l}, \text{Re} G^r_{k_{2},l} \right\} + i \epsilon_{ll_{1}l_{2}} \left\{ \Sigma^\parallel_{k_{2},l}, \text{Re} G^r_{o,o} \right\} \\
+ \frac{1}{2} i \epsilon_{kk_{1}k_{2}} i \epsilon_{ll_{1}l_{2}} \left\{ \Sigma^\parallel_{k_{1},l}, \text{Re} G^r_{k_{2},l} \right\} + \frac{1}{2} i \epsilon_{kk_{1}k_{2}} i \epsilon_{ll_{1}l_{2}} \left\{ \Sigma^\parallel_{k_{1},l}, \text{Re} G^r_{k_{2},l} \right\}
\]

\[
\left\{ \Gamma, G^\parallel \right\}_{k,l} = \\
\left\{ \Gamma_{o,o}, G^\parallel_{k,l} \right\} + \left\{ \Gamma_{k,l}, G^\parallel_{o,o} \right\} + \left\{ \Gamma_{k,o}, G^\parallel_{o,l} \right\} + \left\{ \Gamma_{o,l}, G^\parallel_{k,o} \right\} \\
+ i \epsilon_{kk_{1}k_{2}} \left\{ \Gamma_{k_{1},o}, G^\parallel_{k_{2},l} \right\} + i \epsilon_{kk_{1}k_{2}} \left\{ \Gamma_{k_{1},l}, G^\parallel_{k_{2},o} \right\} \\
+ i \epsilon_{ll_{1}l_{2}} \left\{ \Gamma_{o,o,l}, G^\parallel_{k_{2},l} \right\} + i \epsilon_{ll_{1}l_{2}} \left\{ \Gamma_{k_{2},l}, G^\parallel_{o,o} \right\} \\
+ \frac{1}{2} i \epsilon_{kk_{1}k_{2}} i \epsilon_{ll_{1}l_{2}} \left\{ \Gamma_{k_{1},l}, G^\parallel_{k_{2},l} \right\} + \frac{1}{2} i \epsilon_{kk_{1}k_{2}} i \epsilon_{ll_{1}l_{2}} \left\{ \Gamma_{k_{1},l}, G^\parallel_{k_{2},l} \right\}
\]

\[
\left\{ \Sigma^\parallel, A \right\}_{k,l} = \\
\left\{ \Sigma^\parallel_{o,o}, A_{k,l} \right\} + \left\{ \Sigma^\parallel_{k,l}, A_{o,o} \right\} + \left\{ \Sigma^\parallel_{k,o}, A_{o,l} \right\} + \left\{ \Sigma^\parallel_{o,l}, A_{k,o} \right\} \\
+ i \epsilon_{kk_{1}k_{2}} \left\{ \Sigma^\parallel_{k_{1},o}, A_{k_{2},l} \right\} + i \epsilon_{kk_{1}k_{2}} \left\{ \Sigma^\parallel_{k_{1},l}, A_{k_{2},o} \right\} \\
+ i \epsilon_{ll_{1}l_{2}} \left\{ \Sigma^\parallel_{o,o,l}, A_{k_{2},l} \right\} + i \epsilon_{ll_{1}l_{2}} \left\{ \Sigma^\parallel_{k_{2},l}, A_{o,o} \right\} \\
+ \frac{1}{2} i \epsilon_{kk_{1}k_{2}} i \epsilon_{ll_{1}l_{2}} \left\{ \Sigma^\parallel_{k_{1},l}, A_{k_{2},l} \right\} + \frac{1}{2} i \epsilon_{kk_{1}k_{2}} i \epsilon_{ll_{1}l_{2}} \left\{ \Sigma^\parallel_{k_{1},l}, A_{k_{2},l} \right\}
\]

VIII. Limiting Case of Pauli-Dirac Spin and Pseudospin, \( N_s = 2 \)

This case has been treated in the paper by Buot et al.\(^{27}\) There the Dirac spin semiconductor Bloch equations (DSSBEs) are treated in the electron-hole picture, although the corresponding DSSBEs for holes was not treated in that paper. When we cast the DSSBEs in the electron picture, either by virtue the symmetry at low energies of the conduction-valence bands of Dirac points, or on account of layer pseudospin in bilayer systems, the resulting equations for Pauli-Dirac spin and pseudospin, using the method of using appropriate combinations of the DSSBEs employed there, exactly reproduce the results as given in the present limiting case. We have,

\[
2^{N_s} i \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^\parallel_{o,o} = \\
= \left[ \tilde{H}, G^\parallel \right]_{o,o} + \left[ \Sigma^\parallel, \text{Re} G^r \right]_{o,o} - \frac{i}{2} \left\{ \Gamma, G^\parallel \right\}_{o,o} + \frac{i}{2} \left\{ \Sigma^\parallel, A \right\}_{o,o}
\]
where,

\[
\begin{align*}
\left[ \mathcal{H}, G^< \right]_{o,o} &= \left[ H_{o,o}, G_{o,o}^< \right] + \left[ H_{o,m}, G_{o,m}^< \right] + \left[ H_{k,o}, G_{k,o}^< \right] + \left[ H_{k,m}, G_{k,m}^< \right] \\
[\Sigma^<, \text{Re} G^r]_{o,o} &= \left[ \Sigma_{o,o}^<, \text{Re} G_{o,o}^r \right] + \left[ \Sigma_{o,m}^<, \text{Re} G_{o,m}^r \right] + \left[ \Sigma_{k,o}^<, \text{Re} G_{k,o}^r \right] + \left[ \Sigma_{k,m}^<, \text{Re} G_{k,m}^r \right]
\end{align*}
\] (87)

\[
\left\{ \Gamma, G^< \right\}_{o,o} = \left\{ \Gamma_{o,o}, G_{o,o}^< \right\} + \left\{ \Gamma_{o,m}, G_{o,m}^< \right\} + \left\{ \Gamma_{k,o}, G_{k,o}^< \right\} + \left\{ \Gamma_{k,m}, G_{k,m}^< \right\}
\] (88)

\[
\left\{ \Sigma^<, A \right\}_{o,o} = \left\{ \Sigma_{o,o}^<, A_{o,o} \right\} + \left\{ \Sigma_{o,m}^<, A_{o,m} \right\} + \left\{ \Sigma_{k,o}^<, A_{k,o} \right\} + \left\{ \Sigma_{k,m}^<, A_{k,m} \right\}
\] (89)

\[
2^{N_s} i \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \frac{\partial}{\partial t_2} \ G_{k,o}^< =
\left[ \mathcal{H}, G^< \right]_{k,o} + \left[ \Sigma^<, \text{Re} G^r \right]_{k,o} - \frac{i}{2} \left\{ \Gamma, G^< \right\}_{k,o} + \frac{i}{2} \left\{ \Sigma^<, A \right\}_{k,o}
\] (91)

where,

\[
\begin{align*}
\left[ \mathcal{H}, G^< \right]_{k,o} &= \left[ H_{o,o}, G_{k,o}^< \right] + \left[ H_{o,m}, G_{k,m}^< \right] + i \epsilon_{k1k2} \left\{ H_{k1,o}, G_{k2,o}^< \right\} \\
+ \left[ H_{k,o}, G_{o,o}^< \right] + \left[ H_{k,m}, G_{o,m}^< \right] + i \epsilon_{k1k2} \left\{ H_{k1,m}, G_{k2,m}^< \right\}
\end{align*}
\] (92)

\[
\begin{align*}
\left[ \Sigma^<, \text{Re} G^r \right]_{k,o} &= \left[ \Sigma_{o,o}^<, \text{Re} G_{k,o}^r \right] + \left[ \Sigma_{o,m}^<, \text{Re} G_{k,m}^r \right] + i \epsilon_{k1k2} \left\{ \Sigma_{k1,o}^<, \text{Re} G_{k2,o}^r \right\} \\
+ \left[ \Sigma_{k,o}^<, \text{Re} G_{o,o}^r \right] + \left[ \Sigma_{k,m}^<, \text{Re} G_{o,m}^r \right] + i \epsilon_{k1k2} \left\{ \Sigma_{k1,m}^<, \text{Re} G_{k2,m}^r \right\}
\end{align*}
\] (93)

\[
\left\{ \Gamma, G^< \right\}_{k,o} = \left\{ \Gamma_{o,o}, G_{k,o}^< \right\} + \left\{ \Gamma_{o,m}, G_{k,m}^< \right\} + i \epsilon_{k1k2} \left\{ \Gamma_{k1,o}, G_{k2,o}^< \right\} \\
+ \left\{ \Gamma_{k,o}, G_{o,o}^< \right\} + \left\{ \Gamma_{k,m}, G_{o,m}^< \right\} + i \epsilon_{k1k2} \left\{ \Gamma_{k1,m}, G_{k2,m}^< \right\}
\] (94)
\begin{align*}
\{ \Sigma^\xi, A \}_{k,o} &= \left\{ \begin{array}{l}
\{ \Sigma^\xi_{o,o}, A_{k,o} \} + \{ \Sigma^\xi_{o,m}, A_{k,m} \} + i \epsilon_{k k_1 k_2} \left[ \Sigma^\xi_{k_1, o}, A_{k_2, o} \right] \\
+ \{ \Sigma^\xi_{k,o}, A_{o,o} \} + \{ \Sigma^\xi_{k,m}, A_{o,m} \} + i \epsilon_{k k_1 k_2} \left[ \Sigma^\xi_{k_1, m}, A_{k_2, m} \right]
\end{array} \right\} \\
&= 2^{N_s} i \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^\xi_{o,m} = \\
&= \left[ \mathcal{H}, G^\xi \right]_{o,m} + [\Sigma^\xi, \text{Re} G^\sigma]_{o,m} - i \frac{1}{2} \left\{ \Gamma, G^\xi \right\}_{o,m} + i \frac{1}{2} \left\{ \Sigma^\xi, A \right\}_{o,m} \number{95}
\end{align*}

where,
\begin{align*}
\left[ \mathcal{H}, G^\xi \right]_{o,m} &= \left[ H_{o,o}, G^\xi_{o,m} \right] + \left[ H_{o,m}, G^\xi_{o,o} \right] + i \epsilon_{m m_1 m_2} \left\{ H_{o,m_1}, G^\xi_{o,m_2} \right\} \\
&+ \left[ H_{k,o}, G^\xi_{k,m} \right] + \left[ H_{k,m}, G^\xi_{k,o} \right] + i \epsilon_{m m_1 m_2} \left\{ H_{k,m_1}, G^\xi_{k,m_2} \right\} \number{97}
\end{align*}

\begin{align*}
[\Sigma^\xi, \text{Re} G^\sigma]_{o,m} &= \left[ \Sigma^\xi_{o,o}, \text{Re} G^\sigma_{o,o} \right] + \left[ \Sigma^\xi_{o,m}, \text{Re} G^\sigma_{o,m} \right] + i \epsilon_{m m_1 m_2} \left\{ \Sigma^\xi_{o,m_1}, \text{Re} G^\sigma_{o,m_2} \right\} \\
&+ \left[ \Sigma^\xi_{k,o}, \text{Re} G^\sigma_{k,o} \right] + \left[ \Sigma^\xi_{k,m}, \text{Re} G^\sigma_{k,m} \right] + i \epsilon_{m m_1 m_2} \left\{ \Sigma^\xi_{k,m_1}, \text{Re} G^\sigma_{k,m_2} \right\} \number{98}
\end{align*}

\begin{align*}
\left\{ \Gamma, G^\xi \right\}_{o,m} &= \left\{ \Gamma_{o,o}, G^\xi_{o,m} \right\} + \left\{ \Gamma_{o,m}, G^\xi_{o,o} \right\} + i \epsilon_{m m_1 m_2} \left[ \Gamma_{o,m_1}, G^\xi_{o,m_2} \right] \\
&+ \left\{ \Gamma_{k,o}, G^\xi_{k,m} \right\} + \left\{ \Gamma_{k,m}, G^\xi_{k,o} \right\} + i \epsilon_{m m_1 m_2} \left[ \Gamma_{k,m_1}, G^\xi_{k,m_2} \right] \number{99}
\end{align*}

\begin{align*}
\left\{ \Sigma^\xi, A \right\}_{o,m} &= \left\{ \Sigma^\xi_{o,o}, A_{o,m} \right\} + \left\{ \Sigma^\xi_{o,m}, A_{o,o} \right\} + i \epsilon_{m m_1 m_2} \left[ \Sigma^\xi_{o,m_1}, A_{o,m_2} \right] \\
&+ \left\{ \Sigma^\xi_{k,o}, A_{k,m} \right\} + \left\{ \Sigma^\xi_{k,m}, A_{k,o} \right\} + i \epsilon_{m m_1 m_2} \left[ \Sigma^\xi_{k,m_1}, A_{k,m_2} \right] \number{100}
\end{align*}

\begin{align*}
2^{N_s} i \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^\xi_{k,m} &= \\
&= \left[ \mathcal{H}, G^\xi \right]_{k,m} + [\Sigma^\xi, \text{Re} G^\sigma]_{k,m} - i \frac{1}{2} \left\{ \Gamma, G^\xi \right\}_{k,m} + i \frac{1}{2} \left\{ \Sigma^\xi, A \right\}_{k,m} \number{101}
\end{align*}

where,
\begin{align*}
\left[ \mathcal{H}, G^\xi \right]_{k,m} &= \left[ H_{o,o}, G^\xi_{k,m} \right] + \left[ H_{o,m}, G^\xi_{k,o} \right] + i \epsilon_{k k_1 k_2} \left[ H_{k_1, o}, G^\xi_{k_2, m} \right] \\
&+ i \epsilon_{m m_1 m_2} \left[ H_{o,m_1}, G^\xi_{k,m_2} \right] + \frac{1}{2} i \epsilon_{k k_1 k_2} i \epsilon_{m m_1 m_2} \left[ H_{k_1, m_1}, G^\xi_{k_2, m_2} \right] \\
&+ \left[ H_{k,o}, G^\xi_{o,m} \right] + \left[ H_{k,m}, G^\xi_{o,o} \right] + i \epsilon_{k k_1 k_2} \left[ H_{k_1, m_1}, G^\xi_{k_2, o} \right] \\
&+ i \epsilon_{m m_1 m_2} \left[ H_{k,m_1}, G^\xi_{o,m_2} \right] + \frac{1}{2} i \epsilon_{k k_1 k_2} i \epsilon_{m m_1 m_2} \left[ H_{k_1, m_1}, G^\xi_{k_2, m_2} \right] \number{102}
\end{align*}
\[
\left[ \Sigma^\varphi, \text{Re}G^\varphi \right]_{k,m} = \\
= \left[ \Sigma^\varphi_{o,o}, \text{Re}G^\varphi_{k,m} \right] + \left[ \Sigma^\varphi_{o,m}, \text{Re}G^\varphi_{k,o} \right] + i\epsilon_{kk_1k_2} \left\{ \Sigma^\varphi_{k_1,o} \cdot \text{Re}G^\varphi_{k_2,m} \right\} \\
+ i\epsilon_{mm_1m_2} \left\{ \Sigma^\varphi_{k_1,m} \cdot \text{Re}G^\varphi_{k_2,m} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{mm_1m_2} \left[ \Sigma^\varphi_{k_1,m_1} \cdot \text{Re}G^\varphi_{k_2,m_2} \right]
\] (103)

\[
\{ \Gamma, G^\varphi \}_{k,m} = \\
\left\{ \Gamma_{o,o}, G^\varphi_{k,m} \right\} + \left\{ \Gamma_{o,m}, G^\varphi_{k,o} \right\} + i\epsilon_{kk_1k_2} \left[ \Gamma_{k_1,o}, G^\varphi_{k_2,m} \right] \\
+ i\epsilon_{mm_1m_2} \left[ \Gamma_{k_1,m} \cdot G^\varphi_{k_2,m} \right] + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{mm_1m_2} \left\{ \Gamma_{k_1,m_1} \cdot G^\varphi_{k_2,m_2} \right\} \\
+ i\epsilon_{mm_1m_2} \left[ \Gamma_{k_1,m_1} \cdot G^\varphi_{k_2,m_2} \right] + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{mm_1m_2} \left\{ \Gamma_{k_1,m_1} \cdot G^\varphi_{k_2,m_2} \right\}
\] (104)

\[
\{ \Sigma^\varphi, A \}_{k,m} = \\
\left\{ \Sigma^\varphi_{o,o}, A_{k,m} \right\} + \left\{ \Sigma^\varphi_{o,m}, A_{k,o} \right\} + i\epsilon_{kk_1k_2} \left[ \Sigma^\varphi_{k_1,o} \cdot A_{k_2,m} \right] \\
+ i\epsilon_{mm_1m_2} \left\{ \Sigma^\varphi_{k_1,m} \cdot A_{k_2,m} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{mm_1m_2} \left\{ \Sigma^\varphi_{k_1,m_1} \cdot A_{k_2,m_2} \right\} \\
+ i\epsilon_{mm_1m_2} \left\{ \Sigma^\varphi_{k_1,m_1} \cdot A_{k_2,m_2} \right\} + \frac{1}{2} i\epsilon_{kk_1k_2} i\epsilon_{mm_1m_2} \left\{ \Sigma^\varphi_{k_1,m_1} \cdot A_{k_2,m_2} \right\}
\] (105)

The explicit form of the expressions containing the product of two Levi Civita symbols can be obtained by the method of Ref.\textsuperscript{22} using the DSSBEs derived there but cast in the electron picture. The typical result containing two Levi Civita symbols, for example, the \(x\)-component of pseudospin in our present notation, is

\[
2^{N_s} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \tilde{G}^\varphi_x
\]

\[
\rightarrow \left[ \frac{i}{2} \left\{ \left( \tilde{G}^\varphi_x \times \tilde{G}^\varphi_z \right) - \tilde{G}^\varphi_z \times \tilde{G}^\varphi_x \right\} + \left( \tilde{G}^\varphi_y \times \tilde{G}^\varphi_z \right) - \tilde{G}^\varphi_z \times \tilde{G}^\varphi_y \right\} \right] (106)
\]

where the vector denotes the Pauli-Dirac spin. This translates to our result in term of tensor components, for arbitrary vector components for both Dirac spin and pseudospin, as

\[
2^{N_s} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^\varphi_{k,m}
\]

\[
\rightarrow + \left[ \frac{i}{2} \epsilon_{kk_1k_2} \epsilon_{mm_1m_2} \left\{ \Gamma_{k_1,m_1} \cdot G^\varphi_{k_2,m_2} \right\} \right] (107)
\]
Equations (106) - (107) give a clear meaning of the terms containing two Levi-Civita symbols in the transport equations. Indeed for the scattering terms, it restores the locality character as contained in the original equations, Eq. (3) [refer to the discussion in Sec. 3 of the Appendix].

A. Comparison with the expression in Ref[27]

In the light of the present notations used, the corresponding quantum transport equation for Pauli-Dirac spin and pseudospin with different notations employed by Buot et al.[27] is shown in Sec. 4 of the Appendix to be identical to the present results.

IX. Limiting Case of Pseudospin, \( N_s = 1 \)

We have the equation for particle number density as,

\[
2^{N_s} i \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_\sigma^\infty = \\
[H_\sigma, G_\sigma^\infty] + [H_m, G_m^\infty] \]

\[
+ \left[ \Sigma_\sigma^\infty, \text{Re} G_o^\infty \right] + \left[ \Sigma_m^\infty, \text{Re} G_m^\infty \right] \]

\[
- \frac{i}{2} \left\{ \{ \Gamma_\sigma, G_\sigma^\infty \} + \{ \Gamma_m, G_m^\infty \} \right\} \]

\[
+ \frac{i}{2} \left\{ \{ \Sigma_\sigma^\sigma, A_o \} + \{ \Sigma_m^\sigma, A_m \} \right\} \quad (108)
\]

where \( N_s = 1 \). The pseudospin magnetization density is given by,

\[
2^{N_s} i \hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_m^\infty = \\
[H_\sigma, G_m^\sigma] + [H_m, G_o^\infty] + i \epsilon_{mm_1m_2} \left\{ H_{m_1}, G_{m_2}^\sigma \right\} \]

\[
+ \left[ \Sigma_\sigma^\infty, \text{Re} G_o^\sigma \right] + \left[ \Sigma_m^\infty, \text{Re} G_m^\sigma \right] + i \epsilon_{mm_1m_2} \left\{ \Sigma_{m_1}^\infty, \text{Re} G_{m_2}^\sigma \right\} \]

\[
- \frac{i}{2} \left\{ \{ \Gamma_\sigma, G_m^\sigma \} + \{ \Gamma_m, G_o^\sigma \} + i \epsilon_{mm_1m_2} \left[ \Gamma_{m_1}, G_{m_2}^\sigma \right] \right\} \]

\[
+ \frac{i}{2} \left\{ \{ \Sigma_\sigma^\sigma, A_m \} + \{ \Sigma_m^\sigma, A_o \} + i \epsilon_{mm_1m_2} \left[ \Sigma_{m_1}^\sigma, A_{m_2} \right] \right\} \quad (109)
\]
X. Limiting Case of Valley Spin, $N_s = 1$

\[ 2^{N_s} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^z_o = \]
\[ = [H_o, G^z_o] + [H_t, G^z_t] \]
\[ + [\Sigma^z_o, \text{Re} G^x_o] + [\Sigma^z_t, \text{Re} G^x_t] \]
\[ - \frac{i}{2} \{ \{ \Gamma_o, G^z_o \} + \{ \Gamma_t, G^z_t \} \} \]
\[ + \frac{i}{2} \{ \{ \Sigma^z_o, A_o \} + \{ \Sigma^z_t, A_t \} \} \] (110)

\[ 2^{N_s} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^z_t = \]
\[ = [H_o, G^z_o] + [H_t, G^z_t] + i\epsilon_{ll_1 l_2} \{ H_{l_1}, G^z_{l_2} \} \]
\[ + [\Sigma^z_o, \text{Re} G^x_o] + [\Sigma^z_t, \text{Re} G^x_t] + i\epsilon_{ll_1 l_2} \{ \Sigma^z_{l_1}, \text{Re} G^x_{l_2} \} \]
\[ - \frac{i}{2} \{ \{ \Gamma_o, G^z_o \} + \{ \Gamma_t, G^z_t \} + i\epsilon_{ll_1 l_2} [\Gamma_{l_1}, G^z_{l_2}] \} \]
\[ + \frac{i}{2} \{ \{ \Sigma^z_o, A_o \} + \{ \Sigma^z_t, A_t \} \} + i\epsilon_{ll_1 l_2} \{ \Sigma^z_{l_1}, A_{l_2} \} \] (111)

XI. Limiting Case of Pauli-Dirac Spin, $N_s = 1$

\[ 2^{N_s} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^z_o = \]
\[ = [H_o, G^z_o] + [H_k, G^z_k] \]
\[ + [\Sigma^z_o, \text{Re} G^x_o] + [\Sigma^z_k, \text{Re} G^x_k] \]
\[ - \frac{i}{2} \{ \{ \Gamma_o, G^z_o \} + \{ \Gamma_k, G^z_k \} \} \]
\[ + \frac{i}{2} \{ \{ \Sigma^z_o, A_o \} + \{ \Sigma^z_k, A_k \} \} \] (112)

\[ 2^{N_s} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^z_k = \]
\[ = [H_o, G^z_o] + [H_k, G^z_k] + i\epsilon_{kk_1 k_2} \{ H_{k_1}, G^z_{k_2} \} \]
\[ + [\Sigma^z_o, \text{Re} G^x_o] + [\Sigma^z_k, \text{Re} G^x_k] + i\epsilon_{kk_1 k_2} \{ \Sigma^z_{k_1}, \text{Re} G^x_{k_2} \} \]
\[ - \frac{i}{2} \{ \{ \Gamma_o, G^z_o \} + \{ \Gamma_k, G^z_k \} + i\epsilon_{kk_1 k_2} [\Gamma_{k_1}, G^z_{k_2}] \} \]
\[ + \frac{i}{2} \{ \{ \Sigma^z_o, A_o \} + \{ \Sigma^z_k, A_k \} \} + i\epsilon_{kk_1 k_2} \{ \Sigma^z_{k_1}, A_{k_2} \} \] (113)
A. Comparison with the expression given in Ref.\textsuperscript{23}

Equation (113) is identical to that obtained by Buot et al.,\textsuperscript{23} which is rearranged as follows,

$$2i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}^< =$$

$$= \left\{ \right.$$

\begin{align*}
&+ \left[ (\vec{B} + \text{Re} \vec{\Xi}^r), S^<_o \right] + i \left[ (\vec{B} + \text{Re} \vec{\Xi}^r) \times \vec{S}^<- \vec{S}^< \times \left( \vec{B} + \text{Re} \vec{\Xi}^r \right) \right] \\
&+ \left[ \Sigma^<, \text{Re} \vec{S}^r \right] + \left[ \vec{\Xi}^<, \text{Re} S^r_o \right] + i \left[ \vec{\Xi}^< \times \text{Re} \vec{S}^r - \text{Re} \vec{S}^r \times \vec{\Xi}^< \right] \\
&- \frac{i}{2} \left\{ \right. \\
&\left. \{ \Gamma, \vec{S}^<_o \} - \frac{i}{2} \{ \vec{\gamma}, S^<_o \} + \frac{i}{2} \left\{ \vec{\gamma} \times \vec{S}^< + \vec{S}^< \times \vec{\gamma} \right\} \right. \\
&\left. + \frac{i}{2} \left\{ \right. \\
&\left. \{ \Sigma^<, A \} + \frac{i}{2} \{ \vec{\Xi}^<, A_o \} - \frac{i}{2} \{ \vec{\Xi}^< \times A + A \times \vec{\Xi}^< \} \right. \right. \right. \\
&\right\} \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
and diffusive processes in phase space or drift-diffusion transport in position space. Indeed, to leading order in gradient expansion, the commutator represents the inherent nonlocality in quantum mechanics and describes diffusive motion in phase space of transport kinetics, whereas the anticommutator describes local or nondiffusive events in \((\vec{p}, \vec{q})\) space typefied either by cyclotron-orbit current tied to orbit center or by decay and growth of a phase-space distribution function tied to a point \((\vec{p}, \vec{q})\) in phase space. For example, the nondiffusive character of the scattering anticommutator has been simply approximated by a relaxation-time approximation in Refs.\textsuperscript{10,11}, whereas the commutator is entirely responsible for bringing in the nonlocality of quantum transport physics, specifically the tunneling transport in resonant tunneling structures\textsuperscript{10,11}.

For convenience (see Appendix Sec. F for more details), we give here the following lattice-Weyl (LW) transform to phase space of a commutator \([A, B]\) and an anticommutator \(\{A, B\}\) in terms of Poisson bracket differential operator, \(\Lambda\), as

\[
[A, B] (p, q) = \cos \Lambda [a (p, q) b (p, q) - b (p, q) a (p, q)]
- i \sin \Lambda \{a (p, q) b (p, q) + b (p, q) a (p, q)\},
\]

(115)

\[
\{A, B\} (p, q) = \cos \Lambda \{a (p, q) b (p, q) + b (p, q) a (p, q)\}
- i \sin \Lambda [a (p, q) b (p, q) - b (p, q) a (p, q)],
\]

(116)

where \(\Lambda = \hbar \left( \frac{\partial^{(a)}}{\partial p} \cdot \frac{\partial^{(b)}}{\partial q} - \frac{\partial^{(a)}}{\partial q} \cdot \frac{\partial^{(b)}}{\partial p} \right)\). Thus if the LW transforms \(a (p, q)\) and \(b (p, q)\) are not matrices, then to lowest order, we have

\[
[A, B] (p, q) = -i \frac{\hbar}{2} \left( \frac{\partial a (p, q)}{\partial p} \cdot \frac{\partial b (p, q)}{\partial q} - \frac{\partial a (p, q)}{\partial q} \cdot \frac{\partial b (p, q)}{\partial p} \right),
\]

(117)

\[
\{A, B\} (p, q) = \{a (p, q) b (p, q) + b (p, q) a (p, q)\} = 2 a (p, q) b (p, q).
\]

(118)

clearly showing the local and diffusive properties in phase space of anticommutator and commutator, respectively. Likewise, in its integral representation [Sec. F in the Appendix] which is suitable for carrying numerical simulations, Eq. (117) has a nonlocal kernel, whereas Eq. (118) has a local kernel of integration. It is precisely the nonlocal character of the kernel given in Eq. (F11) of the Appendix that is responsible for quantum tunneling in resonant tunneling diodes\textsuperscript{10,11}.

One observes that by virtue of Eqs. (A1) - (A2) in the Appendix, the entangling with a second pseudospin can transform local terms in the transport equation into diffusive or
mobile terms, a sort of delocalization. This type of delocalization is not surprising as it is exhibited even in the classical case, often used to explain the absence of diamagnetism in classical free electron gas. In the classical case, when localized currents or cyclotron orbits interacts with a constriction or opposite boundaries (providing torque interaction), delocalized currents in opposite directions reside in opposite boundaries, respectively.

Thus, the quantum edge states in integer quantum Hall effect (precursor to topological insulators) are often portrayed semiclassically as counterpropagating “skipping orbits” that propagate (delocalize) along the boundary of the system in the opposite sense to that of the Landau orbits, when a particle in a Landau orbit interacts with the boundary, and specularly bounces off it (see Fig. 1).

We are lead to conclude that, quantum mechanically speaking, in the specular reflection at the boundary, the incoming and outgoing states present the boundary pseudospin resulting in the boundary-pseudospin dependent interaction of the Landau orbits. This boundary-pseudospin dependent interaction results in the delocalization of the electron current along the boundary. Since the interaction is elastic and does not cost energy the delocalized current excitations do not have mass or are zero-excitation modes resulting in the chiral/helical dispersion relations or Dirac point of the excitations as in the case of integer quantum Hall effect. This is the mechanism leading to QHE-TI.

![Fig. 1](image)

**FIG. 1:** Classical skipping orbits in space, showing the holographic nature of topological entanglement, i.e., bulk-boundary correspondence. [Reproduced from Ref. 2].

Therefore, starting from the commutator \([A, B]\) of Eq. (3), the dependence in spinor pseudospin degrees of freedom results in a series of alternating commutator and anti-commutator of the spin tensor components as the number of entangled spin degrees of freedom, \(N_s\), grows, with commutator for even number of \(i\epsilon_{ijk}\) and anti-commutator for odd number of \(i\epsilon_{ijk}\). Had we started with \(\{A, B\}\), then there will be a series of alternating anti-commutator and commutator instead. In other words, entanglement of spins will result in transformations from
local to diffusive motion in phase space, and vice versa. This novel delocalization mechanism seems to have direct relevance to topological insulators where torques, due either to spin-orbit coupling (Berry curvature)\textsuperscript{24,28} or magnetic fields resulting in localization or a gapped spectrum in the bulk, akin to integer quantum Hall effect, are then entangled with pseudospin torque at the boundary consistent with Dirac points, akin to classical specular reflection of cyclotron orbits at boundaries, resulting in delocalization at the edges (conductive edge quantum states) of 2-D systems. We therefore expect that localization of the edge states can occur if further entanglement with spin-dependent (spinor) impurities (resulting in spin flips)\textsuperscript{8} or when layer pseudospin becomes important in bilayer topological insulators, i.e., when three spin degrees of freedom become entangled at the edge. There are actually some evidence on this aspect of edge-state localization in bilayer topological quantum Hall 2-D systems where edge states are not even mentioned in the discussion of transport\textsuperscript{29}. On the other hand, the intralayer gapped bulk states of the LLO in bilayer systems become mobile due to interaction with layer pseudospin, in fact in pseudospin ferromagnetic state this becomes superconducting mediated by the condensation of bilayer excitons\textsuperscript{7}.

To illustrate the point in the above discussion, consider for instance the first line of Eq. (92), specifically we are interested in the term, $i\epsilon_{kk_1k_2} \{ H_{k_1,o}, G_{k_2,o}^\pm \}$, which is the term coming from the Hamiltonian [our spinor Hamiltonian is generalized to account for the spinor Re$\Sigma$\textsuperscript{30,31} (which is included in $H$, Eq. (4) as a spinor mass term] and/or magnetic field and/or Dresselhaus and/or Rashba spin-orbit coupling] that incorporates the Landau orbit and is thus a local term by virtue of the anticommutator. However, when the dependence on pseudospin, $m$, comes into play this local term is transformed into a nonlocal term in Eq. (102), specifically through the term, $i\epsilon_{kk_1k_2}i\epsilon_{mm_1m_2} \left[ H_{k_1,m_1} G_{k_2,m_2}^\pm \right]$ which is now a nonlocal term, i.e., mobile term in terms of transport point of view, as expressed by the commutator. If the pseudospin $m$ is the boundary pseudospin, then the delocalization will result in the so-called edge metallic states. Similarly, in bilayer system with each layer in their LLO localized state, the entangling with the layer pseudospin will delocalized the LLO states into nonlocal conducting states. In fact in the layer-pseudospin ferromagnetic state, experiments has shown that the electron-hole condensate leads to a new mechanism for the onset of superconductivity.

Thus, in Eq. (113) we expect localization from Dresselhaus and Rashba spin-orbit coupling as incorporated in the Hamiltonian, $H$, yielding the orbit-term $i\epsilon_{kk_1k_2} \{ H_{k_1}, G_{k_2}^\pm \}$,
whereas delocalization contribution is expected from the Dyakonov-Perel and Elliott-Yafet mechanism as incorporated in the scattering-out $\Gamma$-term, and scattering-in $A$-term. These are the commutator terms in Eq. (113) containing one Levi-Civita symbol, which describe a complex motion similar to the motion and deformation of Landau-orbits, induced by strong magnetic fields, due to the torque exerted by the electric field\textsuperscript{32} resulting in the build-up of Hall voltage. There are indeed theoretical and experimental works which support this assertion, the effect is often referred to in the literature as weak localization (WL) due to Dresselhaus and Rashba spin-orbit coupling\textsuperscript{15} and weak antilocalization (WAL) due to the Dyakonov-Perel and Elliott-Yafet scattering mechanisms\textsuperscript{16,17}. These contrasting and competing effects as well as their respective dependence on different parameters can result in an observed crossover between WL and WAL\textsuperscript{18}. The terminology 'weak' as used in the experimental observations is due to the fact that in Eqs. (112) - (113), there are other important terms that contribute to quantum nonlocality and hence conductivity in transport measurements. Delocalization or nonlocal scattering terms may also occur where the scattering centers have a pseudo-spin character, such as the two-level dependence of the scattering in Wigner-function quantum transport formalism studied by Rossi et al.\textsuperscript{33,34}.

Remark 1. We remark on quantum spin Hall effect topological insulator (QSHE-TI) as it relates to the above discussion on QHE-TI. In time-reversal symmetric systems with strong spin-orbit coupling, Kramer’s degenerate states play an important role. Since spin-orbit coupling does not break time reversal symmetry, the Kramer’s degenerate pair remains intact and moves in opposite directions with opposite spins. Thus, the effective spin-orbit magnetic field rotates the Kramer’s degenerate pair in opposite sense, say clockwise and counterclockwise rotating pair. When this pair interacts with the boundary pseudospin, in the same sense as discussed above, this oppositely moving Kramer’s Landau-orbit pair forms metallic edge states of oppositely moving currents having opposite spins, i.e., helical edge states. This is the mechanism leading to QSHE-TI.

Remark 2. It is worthwhile to add comments on topological Kondo insulator\textsuperscript{35}, which represents the intersection of topology and strongly correlated and heavy fermion systems. Within the quantum transport perspective, the hybridization of localized $f$ and mobile $d$-band, which gives rise to a hybridization gap insulator, is within the domain of quantum transport localization term expressed by the ‘orbit-term’ $i\epsilon_{kk_1k_2} \{ H_{k_1}, G_{k_2}^2 \}$ driven by strong spin-orbit
interaction of the f-electron\textsuperscript{35} as discussed above. Likewise, the metallic edge states are described by the delocalization term $i\epsilon_{kk_1}i\epsilon_{mm_1,m_2}[H_{k_1,m_1}G_{k_2,m_2}^\Sigma]$, similar to the mechanism leading to QSHE-TI with helical edge or surface states. It should be pointed out that since the orbit centers are the atoms themselves the delocalization at the surface could result in the restructuring of the atomic surface atoms resulting in twisting or warping, as exhibited by some Kondo insulator materials, such as CeNiSn possessing nonsymmorphic symmetries, \textit{i.e.}, containing three glide reflections, three screw rotations and an inversion symmetry\textsuperscript{36}. There, nonsymmorphic symmetries give rise to a momentum-dependent twist that enables the surface states to be detached from the bulk on the glide plane leading to the so-called Möbius-twisted surface states\textsuperscript{36}.

In summary, it is worth noting that although topological phases are often discussed in the Hamiltonian context, it has also been shown that associated topological protection and phenomena can also emerge in open quantum systems with engineered dissipation\textsuperscript{37}. In this paper, we have shown that a rounded picture and intuitive aspect of topological phases arise from the entanglement of different pseudospins in highly nonlinear and nonequilibrium quantum transport equations of spin and pseudospins. These mechanisms also shed light in the so-called weak localization (WL) and delocalization (WAL) as observed experimentally\textsuperscript{18}. Whereas two torques or two spin degrees of freedom, one in the bulk and one at the boundary, are needed to characterize topological insulators, only one torque or spin degrees of freedom is needed to characterize WL and WAL.

It was mentioned in the \textbf{Introduction}, that a parallel treatment might be possible for treating majorana edge states in topological superconductors. In superconductors the anomalous Green’s function or field correlations, $\langle \Psi^\dagger_\alpha \Psi^\dagger_{\alpha'} \rangle \neq 0$ and $\langle \Psi_\alpha \Psi_{\alpha'} \rangle \neq 0$. This also implies that paired majorana correlation, namely, $\frac{1}{4} \langle (\Psi_\alpha + \Psi^\dagger_\alpha) (\Psi_{\alpha'} + \Psi^\dagger_{\alpha'}) \rangle \neq 0$. However, neither this combination nor the combination $\frac{-1}{4} \langle (\Psi^\dagger_\alpha - \Psi_\alpha) (\Psi^\dagger_{\alpha'} - \Psi_{\alpha'}) \rangle \neq 0$ changes the physics of superconductors, thus, these are trivial pairings. However, the nontrivial pairing of $m_i = (\Psi^\dagger_i + \Psi_i)/2$ and $m_j = (\Psi^\dagger_j - \Psi_j)/2i$, at neighboring sites for spinless fermions, and hence the pairing $\langle m_i m_j \rangle \neq 0$ completely changes the physics of superconductor to that of the physics of paired majorana fermions, \textit{i.e.}, to the physics of topological superconductor. In this sense, a Dirac fermion is a paired majorana fermions. Indeed, $\langle m_i \rangle = 0$, indicates $\langle m_j \rangle \neq 0$ and \textit{vice versa}. In general, the different sites for
spinless fermions could be replaced by different discrete indices, such as different spins, different wavevectors, different valleys, and different bands, etc..

Ideally, the zeros of either \( \frac{1}{4} \left[ 1 - \langle \Psi^\dagger_i \Psi^\dagger_j \rangle + \langle \Psi_i \Psi_j \rangle \right] \) or \( \frac{1}{4} \left[ 1 + \langle \Psi^\dagger_i \Psi^\dagger_j \rangle + \langle \Psi_i \Psi_j \rangle \right] \) would then constitute the probability of finding only 'one-half \( \frac{1}{2} \) fermion' or unpaired majorana at the edges of 2-D topological superconductor and at both ends of Kitaev wire\(^\text{19}\). The entanglement entropy of a majorana pair is of course determined by this probability, and is given by \( S = \ln 2 \). In numerical computer simulation, one can look for the big imbalance between the two quantities to signal the presence of unpaired-majorana, i.e., *one-half* fermion density.

The \( \langle m_i m_j \rangle \neq 0 \) is indeed realized with spinless fermions in the so-called 1-D topological superconductor, with unpaired majorana, i.e., \( \langle m_i \rangle \neq 0 \) and \( \langle m_j \rangle = 0 \) or vice versa at the ends\(^\text{19}\). Thus, \( \frac{1}{4\pi} \left[ \langle \Psi^\dagger_i \Psi^\dagger_j \rangle - \langle \Psi_i \Psi_j \rangle - 2 \langle \Psi^\dagger_i \Psi_j \rangle \right] \) seems an appropriate starting point for a parallel treatment of topological superconductors that would describe the nonequilibrium quantum transport equations of majorana fermions. It seems likely that the corresponding combined gap functions is compatible with \( (p_x + ip_y) \)-gap function of topological superconductors. This would be an interesting topic for further research, probably as a nonequilibrium quantum superfield theory in a lattice-space framework\(^\text{20}\).

The nonequilibrium quantum transport results of this paper also serve as springboard to applications in the emerging field of spincaloritronics and pseudo-spincaloritronics\(^\text{38,39}\). The formalism employed here can readily be extended to the inclusion of the spinor form of the electron-phonon/plasmon and spinon/magnon self-energies, in accounting for the coupling of the equations obtained here to the spin-dependent nonequilibrium quantum transport of phonons and plasmons\(^\text{22}\). This will be discussed in another communication concerning heat flow, spincaloritronics and pseudo-spincaloritronics.

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A. Product of $2 \times 2$ matrices

**Theorem 3.** Any product of $2 \times 2$ matrices given by

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

can be expressed in terms of $2 \times 2$ identity and Pauli matrices as

$$AB = \frac{1}{4} \left\{ S_{o}(ab) I + V(ab) \cdot \vec{\sigma} \right\}$$

where $S_{o}(ab)$ is the scalar-transforming operator on product $AB$ and $V(ab)$ is the vector-transforming operator on $AB$. Using the Einstein summation convention for the repeated indices, we have

$$S_{o}(ab) = [\bar{a}\bar{b} + a_{k}b_{k}]$$
$$V_{k}(ab) = [\bar{a}b_{k} + a_{k}\bar{b} + i\epsilon_{klm} a_{l}b_{m}]$$.

Here, the scalar and the vector components of $\vec{a}$ and $\vec{b}$ are defined by,

$$\bar{a} = a_{11} + a_{22}, \quad \bar{b} = b_{11} + b_{22}$$
$$a_{x} = a_{12} + a_{21}, \quad b_{x} = b_{12} + b_{21}$$
$$ia_{y} = a_{12} - a_{21}, \quad ib_{y} = b_{12} - b_{21}$$
$$a_{z} = a_{11} - a_{22}, \quad b_{z} = b_{11} - b_{22}$$

**Proof.** Expressing each matrix in terms of the identity and Pauli matrices, we have

$$AB = \frac{1}{2} \begin{pmatrix} \bar{a} + a_{z} & a_{x} - ia_{y} \\ a_{x} + ia_{y} & \bar{a} - a_{z} \end{pmatrix} \frac{1}{2} \begin{pmatrix} \bar{b} + b_{z} & b_{x} - ib_{y} \\ b_{x} + ib_{y} & \bar{b} - b_{z} \end{pmatrix}$$

$$= \frac{1}{4} \left\{ (\bar{a} + a_{z})(\bar{b} + b_{z}) + (a_{x} - ia_{y})(b_{x} + ib_{y}) (\bar{a} + a_{z})(b_{x} - ib_{y}) + (a_{x} - ia_{y})(\bar{b} - b_{z}) \\ (a_{x} + ia_{y})(\bar{b} + b_{z}) + (\bar{a} - a_{z})(b_{x} + ib_{y}) (a_{x} + ia_{y})(b_{x} - ib_{y}) + (\bar{a} - a_{z})(\bar{b} - b_{z}) \right\}$$

Upon evaluating the last line, we end up with

$$AB = \frac{1}{4} \left\{ \left[ \bar{a}b + \bar{a} \cdot \bar{b} \right] I + \left[ \bar{a}b + \bar{a} \cdot \bar{b} + i \left( \bar{a} \times \bar{b} \right) \right]_{z} \sigma_{z} + \left[ \left( \bar{a} \bar{b} + \bar{a} b \right) + i \left( \bar{a} \times b \right) \right]_{x} \sigma_{x} + \left[ \left( \bar{a} \bar{b} + \bar{a} b \right) + i \left( \bar{a} \times b \right) \right]_{y} \sigma_{y} \right\}$$

\[ \Box \]
Lemma 4. The theorem can be generalized to any binary operation of spinor matrices.

\[ S_o (a \otimes b) = \left[ \bar{a} \otimes \bar{b} + \bar{a} \otimes \bar{b} \right] \]
\[ = \frac{1}{4} \left[ \bar{a} \otimes \bar{b} + a_k \otimes b_k \right] \]
\[ \tilde{S} (a \otimes b) = \frac{1}{4} \left[ a \otimes b + \bar{a} \otimes \bar{b} + i \left( \bar{a} \times b \right) \otimes \bar{b} \right] \]
\[ S_k (ab) = \frac{1}{4} \left[ a \otimes b_k + a_k \otimes b + i \epsilon_{klm} a_l \otimes b_m \right]. \]

where \( \otimes \) will generally be a different binary operation from that of \( \otimes \). This will become clear in the example below.

1. Examples

The Pauli-Dirac spinor form of the commutator of \( A \) and \( B \) is given by

\[ [A, B] = \frac{1}{4} \left( [\bar{a}, b] + [\bar{a}, b] + i [\bar{a} \times b - b \times \bar{a}] \right) \]
\[ = \frac{1}{4} \left( [\bar{a}, b] + [a_k, b] + i \epsilon_{kk_1k_2} \{a_{k_1}, b_{k_2}\} \right) \]  (A1)

where the bar indicates total independence with respect to the spin degrees of freedom, \( \{a_{k_1}, b_{k_2}\} \) is the anticommutation of \( a_{k_1} \) and \( b_{k_2} \). Note the change from commutator to anticommutator for the term containing one Levi-Civita symbol. To include dependence on another spin degrees of freedom, it is convenient to arrange the component of the next spin as the succeeding subscripts, such as dyadic tensors \( a_{k,l} \) and \( b_{k,l} \) where the index \( l \) coresponds to the next spin variable, such as valley isospin, for example. Instead of using the bar, quantities independent of two spins will be designated by \( a_{o,o} \) and \( b_{o,o} \). In all instances, the theorem on binary product of \( 2 \times 2 \) matrices can be sucessively applied. Thus, the fully entangled ‘torque’ [coming from the last term of Eq. (A1)] now becomes,

\[ i \epsilon_{ll_1l_2} i \epsilon_{kk_1k_2} \{a_{k_1,l_1}, b_{k_2,l_2}\} = - \epsilon_{ll_1l_2} \epsilon_{kk_1k_2} \{a_{k_1,l_1}, b_{k_2,l_2}\} \]  (A2)

Upon adding dependence on one more pseudospin degrees of freedom, such the pseudospin arising either from the symmetry at low energies of the band indices at Dirac points in graphene or from layer pseudospin in bilayer graphene, the entanglement of the torques becomes

\[ - i \epsilon_{mm_1m_2} \epsilon_{ll_1l_2} \epsilon_{kk_1k_2} \{a_{k_1,l_1,m_1}, b_{k_2,l_2,m_2}\} \]  (A3)
which is a third rank tensor. Assuming we have considered only the band pseudospin in Eq. (A3), if one further consider the pseudospin from layer degrees of freedom in bilayer graphene, for example, then the fully entangled ‘torque’ becomes

\[ \epsilon_{nn_1} \epsilon_{mm_1} \epsilon_{ll_1} \epsilon_{kk_1} \left[ a_{k_1,l_1,m_1,n_1} b_{k_2,l_2,m_2,n_2} \right] \]

Note that starting from the commutation of \([A, B]\), the dependence in spin degrees of freedom results in the alternating commutator and anti-commutator of the spin tensor components in the expression for entangled torques, with commutator for even number of \(i\epsilon_{ijk}\) (Levi Civita symbols) and anti-commutator for odd number of \(i\epsilon_{ijk}\). Had we started with \(\{A, B\}\), then there will be alternating anti-commutator and commutator instead as the number of spin degrees of freedom grows.

These considerations have important significance in formulating the nonequilibrium spin quantum transport equations. However, for economy of space, we will only consider the Pauli-Dirac spin, valley spin or isospin, and pseudospins either due to symmetry of band indices at low energies at the Dirac points or due to bilayer pseudospin. Here we only have eight coupled transport equations to consider. On the other hand, by including any other fourth pseudospin degrees of freedom we will double the number of coupled quantum transport equations to sixteen. Although sixteen coupled spin quantum transport equations are not treated here, the derivation is straightforward following the procedures exemplified in this paper. From the general nonequilibrium coupled spin quantum transport equations various limiting cases appropriate to the problem at hand can be obtained as given in the text.

**B. Local and Diffusive Terms**

To leading order in gradient expansion, the commutator describes diffusive motion in phase space of transport kinetics, whereas the anticommutator describes local or nondiffusive events in \((\vec{p}, \vec{q})\) space typefied by cyclotron-orbit current tied to orbit center or decay and growth of a phase-space distribution function tied to a point \((\vec{p}, \vec{q})\) in phase space. Moreover, the integral form of the commutator and anticommutator exhibits nonlocal kernel and local kernel of integration, respectively. See also the discussion in Sec. F in this Appendix.

However, note that by Eqs. (A1) - (A2), the entangling with a second pseudospin can transform local terms in the transport equation into diffusive or mobile term, a sort of delocalization. This type of delocalization is not surprising as it is exhibited even in the
classical case. In the classical case, when localized currents or cyclotron orbits (magnetic torque) interacts with a constriction or opposite boundaries (providing torque interaction), delocalized currents in opposite directions resides in opposite boundaries, respectively. 

Thus, the quantum edge states in integer quantum Hall effect (precursor to topological insulators) are often portrayed semiclassically as counterpropagating “skipping orbits” that propagate (delocalize) along the boundary of the system in the opposite sense to that of the Landau orbits, when a particle in a Landau orbit interacts with the boundary, and specularly bounces off it.

Therefore, starting from the commutation of $[A, B]$, the dependence in spin or pseudospin degrees of freedom results in the alternating commutator and anti-commutator of the spin tensor components, with commutator for even number of $i\epsilon_{ijk}$ (Levi Civita symbols) and anti-commutator for odd number of $i\epsilon_{ijk}$. Had we started with $\{A, B\}$, then there will be alternating anti-commutator and commutator instead, as the number of entangled spin degrees of freedom grows. In other words, entanglement of spins will result in transformations from local to diffusive motion in phase space, and vice versa. This novel delocalization mechanism seems to have direct relevance to topological insulators where torques, either due spin-orbit coupling (Berry curvature) or magnetic fields resulting in localization or a gapped spectrum in the bulk, akin to integer quantum Hall effect, are then entangled with pseudospin torque at the boundary consistent with Dirac points, akin to classical specular reflection of cyclotron orbits at boundaries, resulting in delocalization at the edges (conductive edge quantum states) of 2-D systems. We therefore expect that localization of the edge states can occur if further entanglement with spin-dependent (spinor) impurities (resulting in spin flips) or when layer pseudospin becomes important in bilayer topological insulators, i.e., when three spin degrees of freedom become entangled at the edge. There are actually some evidence on this aspect of edge-state localization in bilayer topological 2-D systems. On the other hand, the intralayer gapped bulk states in bilayer systems become mobile states due to interaction with layer pseudospin, in fact in some cases becomes superconducting mediated by the condensation of bilayer excitons.

C. Binary Product of 'Cube Matrices'

The result of any binary product of matrices characterized by Pauli-Dirac spin indices, valley or isospin indices, and energy bands or pseudospin indices can be classified into eight tensorial groups, obtained by successive iteration of the spinor transformation of $2 \times 2$
matrices given in Sec. A. We obtained the following expressions, where as before repeated indices are summed.

1. **Total scalars or tensors of rank zero**

   Here we have \( \{ k = 0 \text{ or } \sum_k, \ l = 0 \text{ or } \sum_l, \ m = 0 \text{ or } \sum_m \} \), yielding

   \[
   S_{o,o,o} = \left( \frac{1}{4} \right)^3 \begin{cases}
   [\bar{a}_{o,o,o}b_{o,o,o} + a_{k,o,o}b_{k,o,o} + \bar{a}_{o,o,m}b_{o,o,m}] \\
   +\bar{a}_{o,l,o}b_{o,l,o} + a_{k,l,o}b_{k,l,o} + \bar{a}_{o,l,m}b_{o,l,m} \\
   +a_{k,o,m}b_{k,o,m} + a_{k,l,m}b_{k,l,m}
   \end{cases}
   \]

2. **Dirac-spin vectors and torques \([l = 0 \text{ or summed, } m = 0 \text{ or summed}]\)

   \[
   V_{k,o,o} = \left( \frac{1}{4} \right)^3 \begin{cases}
   \bar{a}_{o,o,o}b_{k,o,o} + a_{k,o,o}b_{o,o,o} + i\epsilon_{kk1}k_2 (a_{k_1,o,o}b_{k_2,o,o}) \\
   +\bar{a}_{o,l,o}b_{k,l,o} + a_{k,l,o}b_{o,l,o} + i\epsilon_{kk1}k_2 (a_{k_1,l,o}b_{k_2,l,o}) \\
   +\bar{a}_{o,o,m}b_{k,o,m} + a_{k,o,m}b_{o,o,m} + i\epsilon_{kk1}k_2 (a_{k_1,o,m}b_{k_2,o,m}) \\
   +\bar{a}_{o,l,m}b_{k,l,m} + a_{k,l,m}b_{o,l,m} + i\epsilon_{kk1}k_2 (a_{k_1,l,m}b_{k_2,l,m})
   \end{cases}
   \] \quad (C1)

3. **Isospin vectors and torques \([k = 0 \text{ or summed, } m = 0 \text{ or summed}]\)

   \[
   V_{o,l,o} = \left( \frac{1}{4} \right)^3 \begin{cases}
   [\bar{a}_{o,o,o}b_{o,l,o} + \bar{a}_{o,l,o}b_{o,o,o} + i\epsilon_{ll1}l_2 (a_{o,l_1,o}b_{o,l_2,o})] \\
   +\bar{a}_{o,o,m}b_{o,l,m} + a_{o,l,m}b_{o,o,m} + i\epsilon_{ll1}l_2 (a_{o,l_1,m}b_{o,l_2,m}) \\
   + [\bar{a}_{k,o,o}b_{k,l,o} + a_{k,l,o}b_{k,o,o} + i\epsilon_{ll1}l_2 (a_{k,l_1,o}b_{k,l_2,o})] \\
   +\bar{a}_{k,o,m}b_{k,l,m} + a_{k,l,m}b_{k,o,m} + i\epsilon_{ll1}l_2 (a_{k,l_1,m}b_{k,l_2,m})
   \end{cases}
   \] \quad (C2)

4. **Pseudospin vectors and torques \([k = 0 \text{ or summed, } l = 0 \text{ or summed}]\)

   \[
   V_{o,o,m} = \left( \frac{1}{4} \right)^3 \begin{cases}
   [\bar{a}_{o,o,o}b_{o,o,o} + \bar{a}_{o,o,m}b_{o,o,o} + i\epsilon_{mm1}m_2 (a_{o,o,m_1}b_{o,o,m_2})] \\
   +[a_{k,o,o}b_{k,o,o} + a_{k,o,m}b_{k,o,o} + i\epsilon_{mm1}m_2 (a_{k,o,m_1}b_{k,o,m_2})] \\
   + [\bar{a}_{o,l,o}b_{o,l,o} + a_{o,l,o}b_{o,l,o} + i\epsilon_{mm1}m_2 (a_{o,l_1,o}b_{o,l_2,o})] \\
   +[a_{k,l,o}b_{k,l,o} + a_{k,l,m}b_{k,l,o} + i\epsilon_{mm1}m_2 (a_{k,l_1,m}b_{k,l_2,m})]
   \end{cases}
   \]
5. Dirac spin-isospin dyadics and torques \([m = 0 \text{ or summed}]\)

\[
T_{k,l} = \left( \frac{1}{4} \right)^3 \left\{ \begin{array}{l}
\bar{a}_{o.o,o}b_{k.l,o} + a_{k,l,o}\bar{b}_{o.o,o} + \bar{a}_{o.l,o}b_{k.o.o} + a_{k,o.o}\bar{b}_{o.l,o} \\
+i\epsilon_{kk_1k_2}(a_{k_1.o,o}b_{k_2.l,o}) + i\epsilon_{kk_1k_2}(a_{k_1,l,o}b_{k_2.o,o}) \\
+ [i\epsilon_{ll_1l_2}(\bar{a}_{o,l_1,o}b_{k,l_2,o}) + i\epsilon_{ll_1l_2}(a_{k,l_1,o}\bar{b}_{o,l_2,o})] \\
+ \frac{1}{2}[i\epsilon_{kk_1k_2}i\epsilon_{ll_1l_2}(a_{k_1,l_1,o}b_{k_2,l_2,o})] + \frac{1}{2}[i\epsilon_{kk_1k_2}i\epsilon_{ll_1l_2}(a_{k_1,l_1,m}b_{k_2,l_2,m})]
\end{array} \right. 
\]

6. Dirac spin-pseudospin dyadics and torques \([l = 0 \text{ or summed}]\)

\[
T_{k,m} = \left( \frac{1}{4} \right)^3 \left\{ \begin{array}{l}
\bar{a}_{o.o,o}b_{k.o,m} + a_{k.o,m}\bar{b}_{o.o,o} + \bar{a}_{o.o,m}b_{k.o.o} + a_{k.o,o}\bar{b}_{o.o,m} \\
+i\epsilon_{kk_1k_2}(a_{k_1.o,o}b_{k_2.o,m}) + i\epsilon_{kk_1k_2}(a_{k_1,o,m}b_{k_2.o,o}) \\
+i\epsilon_{mm_1m_2}(\bar{a}_{o,o,m_1}b_{k.o,m_2}) + i\epsilon_{mm_1m_2}(a_{k.o,m_1}\bar{b}_{o.o,m_2}) \\
+ \frac{1}{2}i\epsilon_{kk_1k_2}i\epsilon_{mm_1m_2}(a_{k_1,o,m_1}b_{k_2,o,m_2}) \\
+ \frac{1}{2}i\epsilon_{kk_1k_2}i\epsilon_{mm_1m_2}(a_{k_1,o,m_1}b_{k_2,o,m_2}) \\
\end{array} \right. 
\]
7. Isospin-pseudospin dyadics and torques \([k = 0 \text{ or summed}]\)

\[
T_{l,m} = \left( \frac{1}{4} \right)^3 \begin{bmatrix}
\bar{a}_{o,o,o} b_{o,l,m} + \bar{a}_{o,l,o} b_{o,o,o} + \bar{a}_{o,o,m} b_{o,l,o} \\
+ i \epsilon_{mm,m_2} \left( a_{o,o,m_1} b_{o,l,m_2} \right) + i \epsilon_{mm,m_2} \left( a_{o,l,m_1} b_{o,o,m_2} \right) \\
+ i \epsilon_{l_1l_2} \left( \bar{a}_{o,l_1,o} b_{o,l_2,m} \right) + i \epsilon_{l_1l_2} \left( \bar{a}_{o,l_1,m} b_{o,l_2,o} \right) \\
+ \frac{1}{2} i \epsilon_{l_1l_2} i \epsilon_{mm,m_2} \left( a_{o,l_1,m_1} b_{o,l_2,m_2} \right) + \frac{1}{2} i \epsilon_{l_1l_2} i \epsilon_{mm,m_2} \left( a_{o,l_1,m_1} b_{o,l_2,m_2} \right)
\end{bmatrix}
\]

\[
+ \left( \frac{1}{4} \right)^3 \begin{bmatrix}
\bar{a}_{k,l,o} b_{k,o,o} + a_{k,o,m} b_{k,l,o} + a_{k,l,m} b_{k,o,o} + a_{k,o,o} b_{k,l,m} \\
+ i \epsilon_{mm,m_2} \left( a_{k,l,m_1} b_{k,o,m_2} \right) + i \epsilon_{mm,m_2} \left( a_{k,o,m_1} b_{k,l,m_2} \right) \\
+ i \epsilon_{l_1l_2} \left( a_{k,l_1,m_1} b_{k,l_2,o} \right) + i \epsilon_{l_1l_2} \left( a_{k,l_1,o} b_{k,l_2,m} \right) \\
+ \frac{1}{2} i \epsilon_{l_1l_2} i \epsilon_{mm,m_2} \left( a_{k,l_1,m_1} b_{k,l_2,m_2} \right) + \frac{1}{2} i \epsilon_{l_1l_2} i \epsilon_{mm,m_2} \left( a_{k,l_1,m_1} b_{k,l_2,m_2} \right)
\end{bmatrix}
\]

8. Tensors of third rank:

This consists of 2 entangled torques and 3 entangled torques as follows,

\[
T_{k,l,m} = \left( \frac{1}{4} \right)^3 \begin{bmatrix}
i \epsilon_{kk_1k_2} i \epsilon_{l_1l_2} \left( a_{k_1,l_1,o} b_{k_2,l_2,m} + a_{k_1,l_1,m} b_{k_2,l_2,o} \right) \\
i \epsilon_{kk_1k_2} i \epsilon_{mm,m_2} \left( a_{k_1,o,m_1} b_{k_2,l_2,m_2} + a_{k_1,l_1,m} b_{k_2,o,m_2} \right) \\
i \epsilon_{l_1l_2} i \epsilon_{mm,m_2} \left( a_{o,l_1,m_1} b_{k_2,l_2,m_2} + a_{k_1,l_1,m} b_{o,l_2,m_2} \right) \\
i \epsilon_{kk_1k_2} i \epsilon_{l_1l_2} i \epsilon_{mm,m_2} \left( a_{k_1,l_1,m_1} b_{k_2,l_2,m_2} \right)
\end{bmatrix}
\]

D. Comparison with Multiband Quantum Transport Equations

Here we make a comparison for \(N_s = 2\) with multi-band quantum spin transport equations of Buot et al.\(^{24}\) Note that Ref.\(^{24}\) essentially calculates Dirac spin semiconductor Bloch equations (DSSBEs) of the spin magnetization quantum transport equations (SMQTEs) in the electron-hole picture. We immediately see that the expression of \(G_{k,o}^\leq\), i.e., pseudospin independent Pauli Dirac spin transport equations, is identical to the expression of the transport equation of a conduction-band spin transport of \(\bar{S}_{cc}\) of Ref.\(^{24}\), upon using the correspondence of the symbols used in Ref.\(^{24}\) to the symbols used in this paper, i.e., \(\Xi_{\gamma\gamma} \mapsto \Sigma_{\gamma\gamma}, \delta_{\alpha\beta}^\leq \mapsto \Sigma_{\alpha\beta}^\leq, \text{ and } S^\geq \mapsto G^\leq\).

By transforming the DSSBEs to the electron picture using Tables 2 - 4, and using the following relations,

\[
i A (1, 2) = -2 i \text{Im} G^r = - \left( G^> (1, 2) - G^< (1, 2) \right),
\]

\[
i \Gamma (1, 2) = -2 i \text{Im} \Sigma^r = - \left( \Sigma^> (1, 2) - \Sigma^< (1, 2) \right)
\]
in combining the equations to obtain the equations for the components of the magnetization density tensors, the resulting equations exactly reproduce the equations for $N_s = 2$ of this paper. The virtue of doing the calculations from the DSSBEs is that it give us an explicit expression for the sort of terms containing two Levi-Civita tensors. The calculation shows the equivalence, for $m = x$-component of pseudospin with $k$ an arbitrary component of Dirac spin, of the expression,

$$
\left( \vec{\Sigma}_y \times \vec{A}_z - \vec{\Sigma}_z \times \vec{A}_y \right) + \left( \vec{A}_y \times \vec{\Sigma}_z - \vec{A}_z \times \vec{\Sigma}_y \right)
$$

$$
= \left\{ \begin{array}{l}
\epsilon_{kk_1k_2}\epsilon_{mm_1m_2}\vec{\Sigma}_{k_1,m_1}\vec{A}_{k_2m_2} \\
+\epsilon_{kk_1k_2}\epsilon_{mm_1m_2}\vec{A}_{k_1,m_1}\vec{\Sigma}_{k_2m_2}
\end{array} \right\}
= \epsilon_{kk_1k_2}\epsilon_{mm_1m_2}\left\{ \vec{\Sigma}_{k_1,m_1}\vec{A}_{k_2m_2} \right\}
$$

(D1)

Note that the $\epsilon_{kk_1k_2}\epsilon_{mm_1m_2}\left\{ \vec{\Sigma}_{k_1,m_1}\vec{A}_{k_2m_2} \right\}$ and similar other terms are completely symmetric in the simultaneous exchange of $k_1$ and $k_2$ and $m_1$ and $m_2$, respectively. The last line of Eq. (D1) holds for arbitrary components, $k$ and $m$. Note that in this paper, we have generalized the spinor Hamiltonian, Eq. (4), to account for the spinor Re$\Sigma_{30,31}$ and/or magnetic field and/or Dresselhaus and/or Rashba spin-orbit coupling.

E. Conversion Map from Electron-Hole Picture to Electron Picture

We have the mapping between electron picture and electron-hole picture using the following table:\textsuperscript{21}

| electron picture | $<e - \text{field}>$ | $<e - h \text{field}>$ | $e - h$ picture |
|-----------------|----------------|-----------------|----------------|
| $-i\hbar G_{vv}^{<}$ (12) | $\langle \psi^\dagger_v (2) \psi_v (1) \rangle$ | $\langle \phi_v (2) \phi^\dagger_v (1) \rangle$ | $i\hbar G_{vv}^{h,>T}$ (12) |
| $-i\hbar G_{vc}^{<}$ (12) | $\langle \psi^\dagger_v (2) \psi_c (1) \rangle$ | $\langle \phi_v (2) \phi_c (1) \rangle$ | $-i\hbar g_{vc}^{e-h,\text{<}}$ (12) |
| $-i\hbar G_{cc}^{<}$ (12) | $\langle \psi^\dagger_c (2) \psi_c (1) \rangle$ | $\langle \phi_c (2) \phi_c (1) \rangle$ | $-i\hbar g_{cc}^{e-h,\text{<}}$ (12) |
| $i\hbar G_{vv}^{>T}$ (12) | $\langle \psi_v (2) \phi^\dagger_v (1) \rangle$ | $\langle \phi_v (2) \phi_v (1) \rangle$ | $-i\hbar G_{vv}^{h,\text{<}}$ (12) |
| $i\hbar G_{vc}^{>T}$ (12) | $\langle \psi_v (1) \phi^\dagger_v (2) \rangle$ | $\langle \phi_v (1) \phi_v (2) \rangle$ | $-i\hbar G_{vc}^{h,\text{<}T}$ (12) |

We also have the following Tables, which can also be similarly applied to the self-energies,
Table 3.

| electron picture | e – h picture |
|------------------|--------------|
| $G_{vv}^r(12)$   | $-G_{ev}^{e-h,aT}(12)$ |
| $G_{vc}^r(12)$   | $g_{ee,vc}^{e-h,r}(12)$ |
| $G_{cv}^r(12)$   | $g_{hh,cv}^{e-h,r}(12)$ |

Table 4.

| electron picture | e – h picture |
|------------------|--------------|
| $G_{vv}^a(12)$   | $-G_{ev}^{e-h,rT}(12)$ |
| $G_{vc}^a(12)$   | $g_{ee,vc}^{e-h,a}(12)$ |
| $G_{cv}^a(12)$   | $g_{hh,cv}^{e-h,a}(12)$ |

F. Transport Equations in Phase Space

The transport equations in phase space are obtained by applying the "lattice" Weyl transformation (although continuum approximation is often employed in this paper, this is not essential and we use the word "lattice" when referring to solid-state problems) of the propagator equations for the respective excitations by using the following identities:

1. "Lattice" Weyl transform of differential operators

   \[ i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) Y(t_1, t_2) \Leftrightarrow i\hbar \frac{\partial}{\partial t} Y(E, t) \]  
   \[ \left( \frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} \right) Y(t_1, t_2) \Leftrightarrow \frac{2i}{\hbar} E \frac{\partial}{\partial t} Y(E, t) \]  
   \[ (\nabla_1^2 - \nabla_2^2) Y(r_1, r_2) \Leftrightarrow \frac{2i}{\hbar} p \cdot \nabla q Y(p, q) \]

2. Lattice Weyl transform of product of arbitrary operators

   a. Poisson bracket expansion

   In terms of differential "Poisson bracket" operator,

   \[ AB(p, q) = \exp \frac{\hbar}{i} \left( \frac{\partial^{(a)}}{\partial p} \cdot \frac{\partial^{(b)}}{\partial q} - \frac{\partial^{(a)}}{\partial q} \cdot \frac{\partial^{(b)}}{\partial p} \right) a(p, q) b(p, q) \]  

   Thus, we are lead to the lattice Weyl transform of a commutator, \([A, B](p, q)\), and an anticommutator, \{A, B\}(p, q) as

   \[ [A, B](p, q) = \cos \cos \Lambda [a(p, q) b(p, q) - b(p, q) a(p, q)] - i \sin \Lambda [a(p, q) b(p, q) + b(p, q) a(p, q)] \]

   \[ \{A, B\}(p, q) = \cos \cos \Lambda [a(p, q) b(p, q) + b(p, q) a(p, q)] - i \sin \Lambda [a(p, q) b(p, q) - b(p, q) a(p, q)] \]

   where \(\Lambda = \frac{\hbar}{2} \left( \frac{\partial^{(a)}}{\partial p} \cdot \frac{\partial^{(b)}}{\partial q} - \frac{\partial^{(a)}}{\partial q} \cdot \frac{\partial^{(b)}}{\partial p} \right)\).
### b. Integral representation

In terms of integral operators, we have,

\[
AB(p,q) = \frac{1}{(\hbar^4)^2} \int dp' \ dq' \ K_A^+(p,q; p' q') \ b(p', q')
\]

\[
= \frac{1}{(\hbar^4)^2} \int dp' \ dq' \ a(p', q') \ K_B^-(p,q; p'q')
\]  

(F7)

where integral kernels are defined by

\[
K_Y^\pm (p,q; p' q') = \int dudv \ exp \left\{ \frac{i}{\hbar} [(p - p') \cdot v + (q - q') \cdot u]\right\} y \left( p \pm \frac{u}{2}, q \mp \frac{v}{2}\right).
\]  

(F8)

Therefore, in terms of integral operators,

\[
[A,B] (p,q) = \frac{1}{(\hbar^4)^2} \int dp' \ dq' \ K_A^+(p,q; p' q') b(p', q') - b(p', q') K_B^- (p,q; p'q') \]  

(F9)

\[
\{A,B\} (p,q) = \frac{1}{(\hbar^4)^2} \int dp' \ dq' \ K_A^+(p,q; p' q') b(p', q') + b(p', q') K_B^- (p,q; p'q') \]  

(F10)

c. Local and nonlocal integral kernels

The above expressions simplify considerably when the lattice Weyl transform \(a(p,q)\) and \(b(p,q)\) are scalar functions. We have for the integral representations,

\[
[A,B] (p,q) = \frac{1}{(\hbar^4)^2} \int dp' \ dq' dudv \ exp \left\{ \frac{i}{\hbar} [(p - p') \cdot v + (q - q') \cdot u]\right\}
\]

\[
\times \left[ a \left( p + \frac{u}{2}, q - \frac{v}{2} \right) - a \left( p - \frac{u}{2}, q + \frac{v}{2} \right) \right] b(p', q')
\]  

(F11)

\[
\{A,B\} (p,q) = \frac{1}{(\hbar^4)^2} \int dp' \ dq' dudv \ exp \left\{ \frac{i}{\hbar} [(p - p') \cdot v + (q - q') \cdot u]\right\}
\]

\[
\times \left[ a \left( p + \frac{u}{2}, q - \frac{v}{2} \right) + a \left( p - \frac{u}{2}, q + \frac{v}{2} \right) \right] b(p', q')
\]  

(F12)

Thus, in its integral forms, which are better suited for carrying numerical simulations, Eq. (F11) exhibits a nonlocal kernel for the commutator, \([A,B] (p,q)\), whereas Eq. (F12) shows an ‘averaging’ type of kernel tied at point \((p,q)\) or local kernel for the anticommutator \(\{A,B\} (p,q)\) to a leading order. Indeed, it is precisely the nonlocal character of the kernel in
Eq. (F11) that is entirely responsible for quantum-tunneling transport in resonant tunneling diodes [10].

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