Trace operator on von Koch’s snowflake

Krystian Kazaniecki\textsuperscript{1} and Michał Wojciechowski\textsuperscript{2}

\textsuperscript{1}Institute of Mathematics, University of Warsaw
\textsuperscript{2}Institute of Mathematics, Polish Academy of Sciences

Abstract

We study properties of the boundary trace operator on the Sobolev space $W^{1,1}(\Omega)$. Using density result by Koskela, Zhang \cite{8} we define a surjective operator $\text{Tr} : W^{1,1}(\Omega_K) \to X(\Omega_K)$, where $\Omega_K$ is a von Koch snowflake and $X(\Omega_K)$ is a trace space with the quotient norm. Main result of this paper is the existence of a right inverse to $\text{Tr}$, i.e. a linear operator $S : X(\Omega_K) \to W^{1,1}(\Omega_K)$ such that $\text{Tr} \circ S = \text{Id}_{X(\Omega_K)}$. To do this we define an extension of the trace operator to the space of functions of bounded variation which seems to be new for domains with fractal boundary. Moreover we identify the isomorphism class of the trace space. On the other hand, for $\Omega$ with regular boundary we provide a simple proof of the Peetre’s theorem \cite{10} about non-existence of the right inverse.

It was shown by Gagliardo \cite{6} that the trace operator transforms space $W^{1,1}(\Omega)$ onto $L^1(\partial \Omega)$ for domains with regular boundary. From this theorem immediately arises a question whether there exists a right inverse operator to the trace, i.e. a continuous, linear operator $S : L^1(\partial \Omega) \to W^{1,1}(\Omega)$ s.t. $\text{Tr} \circ S = \text{Id}_{L^1(\partial \Omega)}$. It turns out that in general such operator does not exist. This was proved by Peetre \cite{10}. In his paper he has shown the non-existence of right inverse to the trace for a half plane. From that by straightening the boundary one can deduce non-existence for $\Omega$ with a smooth boundary. More recent proofs can be found in \cite{11}, \cite{1}. In this article we present an extraordinary simple proof based on geometry of a Whitney covering and basic properties of classical Banach spaces.

**Theorem 1.** Let $\Omega$ be an open domain with Lipschitz boundary and $\partial \Omega$ be a Jordan curve. Let $\text{Tr} : W^{1,1}(\Omega) \to L^1(\partial \Omega)$ be a trace operator. Then there is no continuous, linear operator $S : L^1(\partial \Omega) \to W^{1,1}(\Omega)$ s.t. $T \circ S = \text{Id}_{L^1(\partial \Omega)}$.

In \cite{7} Hajlasz and Martio studied the existence of a right inverse to trace operator in the case of Sobolev spaces $W_p^1(\Omega)$ for $p > 1$. They characterize trace space as a generalized Sobolev space. However this characterization does not work for $p=1$. The behavior of the trace space changes dramatically for the domains with fractal boundary. In the third section we use the structure of a specific Whitney covering of $\Omega_K$ - domain bounded by the von Koch’s curve, we show that in this case the trace space of $W^{1,1}(\Omega_K)$ is isomorphic to Arens-Eels space with a suitable metric. Surprisingly, based on this observation we are able to construct a right inverse operator to the trace operator. The theorem below is the main result of this article.

**Theorem 2.** Let $\text{Tr} : W^{1,1}(\Omega_K) \to X(\Omega_K)$ be a trace operator, where $X(\Omega_K)$ is a trace space \cite{2}. There exists a continuous, linear operator $S : X(\Omega_K) \to W^{1,1}(\Omega_K)$ s.t. $\text{Tr} \circ S = \text{Id}_{X(\Omega_K)}$.

In the following section we define the trace operator, trace space and auxiliary properties $\text{BV}(\Omega)$ needed in the proof.
1 Properties of $BV(\Omega)$ and trace operator

From now on we assume that $\Omega \subset \mathbb{R}^2$, $\partial \Omega$ is a Jordan curve. Our approach to Theorem 1 up to technical differences works in higher dimensions. However in the proof of the Theorem 2 the properties of two dimensional euclidean space are crucial. We define the trace operator and the trace space for $W^{1,1}(\Omega)$. Let us recall a notion of (slightly generalized) Whitney covering of $\Omega$.

**Definition 3.** We call the family of polygons $A$ a Whitney decomposition of an open set $\Omega \subset \mathbb{R}^2$ if it satisfies:

1. For $A \in A$ the boundaries $\partial A$ are uniformly bi-lipschitz.
2. $\bigcup_{Q \in A} Q = \Omega$ and elements of $A$ have pairwise disjoint interiors.
3. $C^{-1} \text{vol}_2 A \leq \text{dist}(A, \partial \Omega)^n \leq C \text{vol}_2(A)$
4. If $\partial A \cap \partial B$ has a positive one dimensional Hausdorff measure then
   
   (a) $C^{-1} \leq \frac{\text{vol}_2(A)}{\text{vol}_2(B)} \leq C$

   (b) $C^{-1} \leq \frac{l(\partial A)}{l(\partial B)} \leq C$

   (c) $C^{-1} l(\partial A) \leq l(\partial A \cap \partial B) \leq C^{-1} l(\partial A)$,

   where $l(\cdot)$ denotes length of a curve, and $\text{vol}_2$ denotes the area of the polygon.
5. For a given polygon $A \in A$ there exists at most $N$ polygons $B \in A$ s.t. $\partial A \cap \partial B \neq \emptyset$.

For the purpose of this article we will also assume that polygons of $A$ are uniformly star shaped in the following sense

6. For every $A \in A$ there exists a point $x \in A$ and positive numbers $\lambda, \tau$ s.t. $B(x, \lambda) \subset A \subset B(x, \tau)$, where $\frac{\lambda}{\tau}$ is fixed and the polygon $A$ is star shaped with respect to $x$. We call such point a center of $A$.

Let $A$ be such covering then we can define a graph describing it’s geometry.

**Definition 4.** Let $A$ be a Whitney decomposition. We call a graph $G := G(A) = (V(A), E(A)) =: (V, E)$ a graph of $A$ if $V := A$ and $\{A, B\} \in E$ only if boundaries of $A$ and $B$ have intersection of positive one dimensional Hausdorff measure.

Then we introduce a special subspace of $BV(\Omega)$.

**Definition 5.** Let $A$ be a Whitney decomposition of $\Omega$. We define the following subspaces of $BV(\Omega)$

$$BV_{A,0} = \{ F \in BV(\Omega) : \forall A \in A \int_A F(x) dx = 0 \}$$

and

$$BV_G = \{ f \in BV(\Omega) : \forall A \in A \quad f|_A = f_A \in \mathbb{R} \}$$

It is a known fact that for a given Whitney decomposition the space $BV_{A,0}$ is a complemented subspace of $BV(\Omega)$. A proof of this fact can be found in ([12], [4]).
Lemma 6. For any domain $\Omega$:

$$BV(\Omega) = BV_{A,0} \oplus BV_G.$$ 

Let us observe that we can easily calculate the norm of function $f \in BV_G$.

$$\|f\|_{BV_G} := \|f\|_{BV(\Omega)} \simeq \sum_{A \in V} |f_A| \, \text{vol}_2(A) + \sum_{\{A,B\} \in E} |f_A - f_B| \, l(\partial A \cap \partial B)$$

In their unpublished preprint Derezinski, Nazarov, Wojciechowski [5] have proven that there is a spanning tree of the graph $G(A)$ with a desirable properties i.e.

Lemma 7. If $\Omega$ is simply connected planar domain and $A$ is its Whitney decomposition. Then there exists spanning tree $T = (V_T, E_T)$ of the graph $G(A)$ s.t.

1. for every $f \in BV_G(\Omega)$

$$\|f\|_{BV_G} \simeq \|f\|_{BV_T} := \sum_{A \in V_T} |f_A| \, \text{vol}_2(A) + \sum_{\{A,B\} \in E_T} |f_A - f_B| \, l(\partial A \cap \partial B) \quad (1)$$

2. for every point $x$ on the boundary there is an infinite branch $\text{br}(x)$ of $T$ s.t. $\text{br}(x) \cong \mathbb{Z}_+$ and $\text{dist}(A_n, x) \to 0$ as $n \to \infty$, where $A_n \in \text{br}(x)$. For a sequence of real numbers $\{a_{A_n}\}$ we call a limit $\lim_{n \to \infty} a_{A_n}$ a limit along the branch $\text{br}(x)$.

We will call such tree a Whitney tree of $A$.

It follows immediately that $BV_G \cong BV_T$, where $BV_T$ is a set $BV_G$ with the norm $\|\cdot\|_{BV_T}$. Using the above notation we define trace of $f \in W^1_1(\Omega)$. Since $\Omega$ is a domain with a Jordan curve as boundary it follows from Koskela, Zhang theorem ([8]) that restrictions of Lipschitz function $\text{Lip}(\mathbb{R}^2)$ are dense in $W^1_1(\Omega)$. For $f \in C(\overline{\Omega}) \cap W^1_1(\Omega)$ we define the trace operator as a restriction of $f$ to the boundary. We define a trace space $X(\Omega)$ as completion of a space $Tr(C(\overline{\Omega}) \cap W^1_1(\Omega))$ with respect to the norm $\|\cdot\|_X$, where

$$\|g\|_X(\Omega) := \inf \{ \|f\|_{W^1_1(\Omega)} : Tr f = g \text{ and } f \in C(\overline{\Omega}) \cap W^1_1(\Omega) \}. \quad (2)$$

Since Lipschitz functions on $\Omega$ are dense in $W^1_1(\Omega)$ we can define trace operator on a whole space $W^1_1(\Omega)$. It is obvious that $Tr : W^1_1(\Omega) \to X(\Omega)$ is a continuous linear operator and it is surjective. We want to extend the trace operator to the $BV(\Omega)$.

Lemma 8. There exists a continuous, linear operator $\Phi : BV_G \to W^1_1(\Omega)$ s.t. for every $A \in A$

$$f_A = \int_A f(y)dy = \int_A \Phi(f)(y)dy + o(\text{dist}(A, \partial \Omega)). \quad (3)$$

Proof. Let $\phi$ be a mollifier, i.e. $\phi \in C^\infty(\mathbb{R}^2, \mathbb{R}_+)$, supp $\phi \subset B(0, 1)$ and $\int_{B(0,1)} \phi = 1$. We define an operator $\Phi$ with the formula

$$\Phi(f)(x) = \int_\Omega f(x-t)\phi \left( \frac{t}{c \text{dist}^2(x, \partial \Omega)} \right) \frac{1}{c^2 \text{dist}^4(x, \partial \Omega)} dt$$

This formula defines a continuous operator from $BV_G$ to $W^1_1(\Omega)$. Let us observe that by the definition of $\Phi$, $\Phi(f)(x) = f_A$ for every $x \in A$ s.t dist$(x, \partial A) \geq c \text{dist}(x, \partial \Omega)^2$ which implies [3]. \qed
Let \( P : BV(\Omega) \to BV_G \) be a projection from \( BV(\Omega) \) onto \( BV_G \). We define \( \widetilde{Tr} : BV(\Omega) \to X(\Omega) \) by the formula

\[
\widetilde{Tr} f = Tr \Phi (Pf) \quad \forall f \in BV(\Omega).
\]

If \( f \in C(\overline{\Omega}) \cap W^1_1(\Omega) \) then the function \( \Phi (Pf) \) is continuous on \( \overline{\Omega} \). Therefore its trace is a restriction of \( \Phi (Pf) \) to the boundary. However the value of the restriction at point \( x \in \partial \Omega \) for the function from \( C(\overline{\Omega}) \) is equal to the limit of \( \int_A \Phi(Pf(y)) \, dy \) along the branch \( \text{br}(x) \). From (3) and the definition of the space \( BV_T \)

\[
\Phi(P(f))(x) = f(x) \quad \forall x \in \partial \Omega.
\]

Since \( f \in C(\overline{\Omega}) \cap W^1_1(\Omega) \) are dense in \( W^1_1(\Omega) \) and \( \widetilde{Tr} f = Tr f \) the operator \( \widetilde{Tr} \) is an extension of the trace operator to \( BV(\Omega) \). We will abuse the notation and from now on we will denote \( \widetilde{Tr} \) by \( Tr \). From the definition of trace it follows that

\[
Tr f = 0 \quad \forall f \in BV_{A,0} \tag{4}
\]

2 Proof of Peetre’s theorem

In this section we will give a proof of Theorem 1.

Proof. Since \( \Omega \) has Lipschitz boundary by theorem of Gagliardo \( X(\Omega) \cong L^1(\partial \Omega) \) - space of functions integrable with respect to the 1-dimensional Hausdorff measure. Let us denote by \( P : BV(\Omega) \to BV_G \) the projection onto \( BV_G \). Assume there exist \( S : L^1(\partial \Omega) \to W^1_1(\Omega) \subset BV(\Omega) \) s.t. \( Tr \circ S = Id_{L^1(\partial \Omega)} \). Then the following diagram is commutative

\[
\begin{array}{ccc}
L^1(\partial \Omega) & \xrightarrow{S} & BV(\Omega) & \xrightarrow{Tr} & L^1(\partial \Omega) \\
| & & \downarrow P & & \\
BV_G & & & \xrightarrow{Tr} & \end{array}
\]

From (4) and Gagliardo theorem we conclude that \( Tr_{|BV_A(\Omega)} \) is onto \( L^1(\partial \Omega) \). On the other hand, \( Tr \circ P \circ S = Id_{L^1(\partial \Omega)} \). Hence \( L^1(\partial \Omega) \) is isomorphic to a subspace of \( BV_G \). The definition of \( BV_G \) implies that \( BV_G \) is isomorphic to a subspace of \( \ell^1(V) \oplus \ell^1(E) \cong \ell^1 \). Since the measure on the boundary is non atomic, \( L^1(\partial \Omega) \cong L^1(\mathbb{T}) \). However, it is well known that \( L^1 \) could not be embedded in \( \ell^1 \). (To see this, note that by Khintchine inequality, Radamacher functions span \( \ell^2 \) in \( L^1 \) space. The space \( \ell^2 \) could not be embedded in \( \ell^1 \) because, every subspace of \( \ell^1 \) contains a copy \( \ell^1 \) (\cite{9}, Proposition 1.a.11).

3 Trace operator on von Koch’s snowflake

Let \( \Omega_K \) be a domain bounded by von Koch’s curve. Since \( \Omega_K \) is simply connected and von Koch’s curve is a Jordan curve, we can use all the properties from the first section. It is enough to show that there exists a right inverse \( S : X(\Omega_K) \to BV_G \) to the trace on \( BV_G \) because then \( \Phi \circ S : X(\Omega_K) \to \)
\(W^1(\Omega_K)\) and \(Tr \circ \Phi \circ S = Id_{X(\Omega_K)}\), where \(\Phi\) is an operator from Lemma 8. It is a well known fact that \(\Omega_K\) satisfies Poincare inequality (eg. [2]). Therefore

\[
\left\| f - \int_{\Omega_K} f(y)dy \right\|_{L^1(\Omega_K)} \leq |\nabla f|_{\Omega_K},
\]

where \(|\mu|_{\Omega_K}\) is a total variation of a measure \(\mu\) on \(\Omega_K\). This inequality implies

\[\text{BV}_G \cong \text{BV}_T \oplus \mathbb{R},\]

where

\[\text{BV}_T = \{ f \in \text{BV}_G : \int_{\Omega_K} f(y)dy = 0 \}\]

with the total variation of gradient as the norm. In this case the norm is equal to

\[
\| f \|_{\text{BV}_T} = \sum_{\{A,B\} \in E_T} |f_A - f_B| l(\partial A \cap \partial B).
\]

Similarly \(X(\Omega_K) = \mathbb{R} \oplus \hat{X}(\Omega_K)\) for a quotient space \(\hat{X}(\Omega) = X/P_0\), where \(P_0\) is the space of constant functions on \(\Omega\). We reduce the problem to finding the right inverse operator to the trace \(Tr : \text{BV}_T \rightarrow \hat{X}(\Omega_K)\). We know that for all \(g \in \hat{X}(\Omega)\),

\[
\| g \|_{\hat{X}(\Omega)} = \inf \{ \| f \|_{\text{BV}_T} : Tr f = g \}
\]

We will show that for a carefully chosen Whitney covering. We introduce the following notation

**Definition 9.** For a given tree \(T\) by \(R := R(T)\) we will denote the root of \(T\). For a vertex \(A \in V_T\) by \(D_n(A)\) we denote descendants of \(A\) of order exactly \(n\) and we put \(D_n = D_n(R)\). For a vertex \(A \in V_T\) by \(A \downarrow\) we denote its unique father. We will denote by \(D \uparrow (A)\) the set of all descendants of \(A\) i.e. \(D \uparrow (A) = \bigcup_n D_n(A)\).

We take a covering \(A_K\) as shown on the Figure 1. This covering of von Koch’s snowflake is easy to describe if we look at its Whitney tree \(T_K\). The root of \(T_K\) is a six pointed star with six “pants” shaped descendants. We denote it by \(R\). In this tree there are three types of polygons/vertices. The aforementioned root, “pants” shaped polygons and “palace” shaped polygons. The type of a vertex describes direct descendants of this vertex (Figure 2). Polygons in \(D_{n+1}\) are similar to polygons from \(D_n\) with a scale \(\frac{1}{3}\). The tree \(T_K\) is the tree from Lemma 7. Hence for such Whitney covering the norm of \(\text{BV}_{T_K}\) satisfies

\[
\| f \|_{\text{BV}_{T_K}} \simeq \sum_{n=1}^{\infty} |f_A - f_{A_k}| 3^{-n}
\]

Further we will use above formula as a norm on \(\text{BV}_{T_K}\). We want to study the norm on \(\hat{X}(\Omega_K)\). To be precise, we want to define and calculate the norm of \(\sum_j a_j \mathbb{I}_{[x_j,y_j]}\) in \(\hat{X}(\Omega_K)\).

**Definition 10.** Let us denote by \(D_\infty(A)\) a cylinder of \(A\), i.e. \(D_\infty(A) = \{ x \in \partial \Omega : A \in \text{br}(x) \}\). We call an arc rational if there exists a finite sequence \(A_1, ..., A_k \in V_{T_K}\) s.t. \([x,y] := \cup_{n=1}^{\infty} D_\infty(A_n)\) and we say that points \(x,y\) are rational points.
Figure 1: Self similar Whitney decomposition of von Koch’s snowflake

Figure 2: On the left "pants" shaped polygon and its descendants, on the right "palace" shaped polygon and its descendants
For a given arc \([x, y]\) there exist a sequence of vertices \(A_k \in V_{TK}\) s.t. \([x, y] = \bigcup_k D_\infty(A_k)\) and sets \(D_\infty(A_k)\) are pairwise disjoint. Moreover this sequence can be taken maximal in the sense that if vertex \(A\) is in the sequence then there exists \(z \in D_\infty(A \downarrow)\), which is not in \([x, y]\). Such sequence is unique for \([x, y]\). Let \(n(k)\) be a natural number such that \(A_k \in D_{n(k)}\). Let

\[
d([x, y]) = \sum_k 3^{-n(k)}.
\]

We introduce an auxiliary metric on the boundary \(\partial \Omega_K\)

\[
d_K(x, y) = \min\{d([x, y]), d([y, x])\}.
\]

It is easy to check that \(d_K(x, y)\) is a metric greater than two dimensional euclidean metric. We prefer this metric over euclidean metric because it is a monotone function on an arc \([x, y]\) with respect to the natural order on the arc. In the lemma below we show that for every rational arc and every monotone right continuous function on this arc there exists a "good" extension of this function to \(BV_{TK}\). We will say that a function is monotone on an arc if it is monotone with respect to the natural order on the arc.

**Lemma 11.** Let \(x, y \in \partial \Omega_K\) and \([x, y]\) be a short arc. Let function \(F : \partial \Omega_K \to \mathbb{R}\) be a monotone and continuous function on the arc \([x, y]\) and \(\text{supp}(F) \subseteq [x, y]\). There exists \(h \in BV_{TK}\) such that

1. \(\|h\|_{BV(\Omega_K)} \lesssim (|F(x)| + |F(x) - F(y)|)d_K(x, y)\),

2. \(F(z) = \lim_{A \to z} \int_{A_{\text{br}(z)}} h(y)dy \quad \forall z \in \partial \Omega\).

**Proof.** First we prove the existence of \(s h\) the good extension for characteristics functions on arcs \([s, y]\) \(\subset [x, y]\). Since arc \([x, y]\) is a short arc then an arc \([s, y]\) is a short arc and can be written as a countable sum \(\bigcup_{k=1}^M D_\infty(A_k)\) in a unique way mentioned in the definition of \(d_K\). From this assumption it is clear that \(#\{A_k : A_k \in D_n\} \leq 10\). Let us put

\[
s h_A = \sum_k 1_{D_\infty(A_k)}(A)
\]

Clearly along every infinite branch \(\text{br}(z)\) the limit of \(\lim_{A \to z} s h_A\) exists and it is equal to \(1_{[s, y]}(z)\).

We need to estimate the total variation of \(s h\). From the definition of \(A_k\) it follows that

\[
\|s h\|_{BV(\Omega_K)} = \sum_k 3^{-n(k)} = d_K(x, s) \leq |1_{[s, y]}(y) - 1_{[s, y]}(x)|d_K(x, y) = d_K(x, y)
\]

Let us assume that \(F\) is an increasing function. For \(F\) let \(\mu\) be its Lebesgue-Stieltjes measure \(\mu\) i.e. \(\mu((a, b]) = F(b) - F(a)\). From the definition of the measure \(\mu\) and the assumptions on \(F\) we get

\[
F(t) = F(x) + \int_x^t 1\ d\mu(s) = F(x)1_{[x, y]}(t) + \int_x^y 1_{[s, y]}(t)d\mu(s)
\]

We define \(h\) by the formula

\[
h_A = F(x)h_A + \int_x^y s h_A d\mu(s)
\]
It follows from the definition of $h$ that

$$\|h\|_{BV_T} = |F(x)|yh\|_{BV_T} + \int_x^y \|h\|_{BV_T} d\mu(s) \leq |F(x)|d_K(x,y) + \int_x^y d_K(x,y) d\mu(s)$$

$$= d_K(x,y)(|F(x) - F(y)| + |F(x)|)$$

Since $s h_A \leq 1$ it follows from Lebesgue dominated convergence theorem that

$$\lim_{A \in br(z)} h_A = F(x) + \lim_{A \in br(z)} \int_x^y s h_A d\mu(s)$$

$$= F(x) + \int_x^y \lim_{A \in br(z)} s h_A d\mu(s) = F(x) + \int_x^y \mathbb{1}_{[s,y]}(z) d\mu(s) = F(z)$$

**Lemma 12.** For every $x, y \in \partial\Omega_K$ and $a, b \in \mathbb{R}$ there exist monotone Lipschitz function $f$, with respect to the euclidean metric, on the arc $[x, y]$ such that $f(x) = a$ and $f(y) = b$.

**Proof.** In order to construct such function we proceed inductively. If we have defined values in the arc $[z, t]$ only at points $z$ and $t$ we choose point $s \in [z, t]$ such that $|s - z| \geq |z - t|/2$ and $|s - t| \geq |z - t|/2$ since arc $[z, t]$ is a one dimensional curve there exists such a point. We put $f(s) := f(z) + (f(t) - f(z))/2$. We continue the procedure until $f$ is defined on a dense subset. We extend $f$ to the whole arc. Function $f$ has desired properties.

In the lemma below we prove the existence of a class of functions in $BV$, which have desirable properties and every function from this class provides a good approximation of the norm of its trace on the boundary.

**Lemma 13.** Let $x, y \in \partial\Omega_K$. There are sequences of functions $f_n \in BV(\Omega_K), g_n \in C(\overline{\Omega_K}) \cap BV(\Omega_K)$, and $h_n \in BV(\Omega_K)$ s.t.

1. $f_n = h_n + g_n$,
2. For every $z \in \partial\Omega_K$, $\mathbb{1}_{[x,y]}(z) = \lim_{A \in br(z)} f_A f_n(y) dy$,
3. $\|g_n\|_{BV(\Omega_K)} \leq (1 + \frac{1}{n^2})\|T g_n\|_{\dot{X}(\Omega_K)}$,
4. $T g_n$ is a Cauchy sequence in $\dot{X}(\Omega_K)$
5. $\|h_n\|_{BV(\Omega_K)} \leq \frac{1}{n^2}$.
6. $\|f_n\|_{BV(\Omega_K)} \leq (1 + \frac{1}{n^2})\|T g_n\|_{\dot{X}(\Omega_K)} + \frac{1}{n^2}$.

**Proof.** We use Lemma [12] For every $\varepsilon$ and every rational arc $[x, y]$, the characteristic function of $[x, y]$ can be written as sum of a Lipschitz function $g$ and a two monotone Lipschitz functions $p_1, p_2$, with supports in arcs $[t_1, x], [y, t_2]$ respectively. Moreover $t_1, t_2$ are rational, $|t_1 - x| + |t_2 - y| \leq \varepsilon$ and the monotone functions $p_i$ are bounded uniformly by one. Hence from the Lemma [11] for every function $p_i$ there exists a function $f_i$ s.t $\|f_i\|_{BV(\Omega_K)} \leq C\varepsilon$ and for every $z \in \partial\Omega_K$

$$\lim_{A \in br(z)} f_A^i = p_i(z).$$
Any Lipschitz extension of \( g \) to \( \Omega_K \) is in \( W^1_1(\Omega_K) \). Hence \( g \) is in the trace space. From the definition of the trace space there exists a \( g_c \in C(\bar{\Omega}_K) \cap BV(\Omega_K) \) such that
\[
\|g_c\|_{BV(\Omega_K)} \leq (1 + \epsilon)\|g\|_{\dot{X}(\Omega_K)},
\]
\( Tr g_c = g \).

Since \( g_c \) is in \( C(\bar{\Omega}_K) \) we have
\[
\lim_{A \in \text{br}(x)} \int_A g_c(y) \, dy = g(x).
\]
Therefore the function \( f = g_c + f^1 + f^2 = g_c + h \) has desired properties. The limits along \( \text{br}(z) \) of \( \int_A g_c(y) \, dy \) exist and are equal to \( 1_{[x,y]}(z) \) for every \( z \in \partial \Omega \) and
\[
\|f\|_{BV} \leq (1 + \epsilon)\|g\|_{\dot{X}(\Omega_K)} + C\epsilon,
\]
where the term \( C\epsilon \) is the estimate on the norms of the functions \( f^i \). For every \( n \) we choose suitable \( \epsilon \) and we get desired properties. The sequence \( Tr g_n \) is Cauchy sequence. Indeed for a given function \( g_n \) and \( m > n \) there exists a continuous piecewise monotone function \( q \) with support on a small set on the boundary s.t.
\[
q + Tr g_n = Tr g_m,
\]
From Lemma [11] there exists a function \( \tilde{q} \in BV(\Omega) \) with a small norm such that
\[
Tr (g_n + \tilde{q}) = Tr g_m
\]
The size of the support of \( q \) depends only on \( g_n \). Therefore
\[
\|Tr g_n - Tr g_m\|_{\dot{X}(\Omega_K)} \leq \epsilon
\]
for sufficiently large \( n, m \).

The Cauchy sequence \( \{g_n\} \) defines an element in \( g \in \dot{X}(\Omega) \). From the analogous argument as in the above Lemma if \( f \in BV(\Omega) \) satisfies \( 1_{[x,y]}(z) = \lim_{A \in \text{br}(x)} \int_A f(y) \, dy \) for every \( z \) on the boundary then \( Tr f = g \). To simplify notation we denote \( g = 1_{[x,y]} \). From the point 6. of the Lemma [13] it follows
\[
\|g\|_{\dot{X}(\Omega_K)} = \lim_{n \to \infty} \|Tr g_n\|_{\dot{X}(\Omega_K)} = \lim_{n \to \infty} \|f_n\|_{BV(\Omega_K)}
\]
Since the projection from \( BV \) onto \( BV_{T_K} \) preserves the trace, we may assume that functions \( f_n \) are from \( BV_{T_K} \). Therefore the function \( g = \sum_j a_j 1_{[x_j, y_j]} \), whose arcs \( [x_j, y_j] \) are rational, satisfies
\[
\|g\|_{\dot{X}(\Omega_K)} \approx \inf \{\|f\|_{BV_{T_K}} : f \in L \text{ and } Tr f = g\},
\]
where \( L \subset BV_{T_K} \) consists of such \( f \) that the limit \( \lim_{A \in \text{br}(x)} f_A \) exist for every \( x \in \partial \Omega \) and it is equal to \( Tr f(x) \).

**Remark 14.** In the above lemmas we abuse the notation a bit. For rational points \( x \) there are two branches \( \text{br}(x) \). If we look at a finite linear combination of characteristic functions of arcs, the are finitely many points (endpoints of segments) on which the limits over this two the branches are different. However they are equal to the value of the trace either on left or right side of that endpoint. Further in the article we are only interested in branches which contain some specific vertex \( A \). Hence we are interested only in one of the problematic branches and it is clear what we mean by the limit.
We want to characterize the space $\mathcal{X}(\Omega_K)$. We introduce, a metric on von Koch’s curve by formula
\[
\tilde{d}(x, y) := \|1_{[x, y]}\|_{\mathcal{X}(\Omega_K)},
\]
where $1_{[x, y]}$ is a characteristic function of an arc on the von Koch’s curve which connects $x$ and $y$. It does not matter which one of the two arcs we take because the difference between their characteristic functions is a constant. Further in the proof it will be clear which one of arcs is considered. Since $\|\cdot\|_{\mathcal{X}(\Omega_K)}$ is a norm, $\tilde{d}$ is a metric on the boundary. For a given metric space $(Y, d_Y)$ we define the Arens-Eels space $\mathcal{X}$.

**Definition 15.** Let $(Y, d_Y)$ be a metric space. We call a function $f : Y \to \mathbb{R}$ a molecule if it has finite support and $\sum_{y \in Y} f(y) = 0$. Let $x, y \in Y$. We define special type of a molecule - an atom: $m_{xy} = 1_x - 1_y$, where $1_a$ is a characteristic of a set $a$. Let $m$ be a molecule, i.e. $m = \sum_{j=1}^{M} a_j m_{x_j y_j}$, then the Arens-Eels norm of $m$ is
\[
\|m\|_{AE(d_Y)} = \inf \left\{ \sum_j |a_j| d_Y(x_j, y_j) : m = \sum_j a_j m_{x_j y_j} \right\},
\]
where the infimum is taken over all possible representations of $m$ as a sum of $m_{pq}$. The Arens-Eels space is the completion of molecules with respect to the norm $\|\cdot\|_{AE}$.

We want to show that $\mathcal{X}(\Omega_K)$ is isomorphic to the Arens-Eels space with the metric $\tilde{d}$. We will denote by $M(\tilde{d})$ linear space of molecules. It is a non complete norm space. By the definition its dense it is dense in $AE(\tilde{d})$. We define the candidate for the isomorphism on the a linearly dense subsets of both spaces. We set $\Psi : AE(\tilde{d}) \to \mathcal{X}(\Omega_K)$ by the formula
\[
\Psi(m_{xy}) = 1_{[x, y]} \quad \forall x, y \in \partial\Omega_K.
\]  (5)

**Lemma 16.** $\Psi : AE(\tilde{d}) \to \mathcal{X}(\Omega_K)$ is an isomorphism between Banach spaces.

**Proof.** By triangle inequality and the definitions of $\tilde{d}(x, y)$ and Arens-Eels space, it follows that $\Psi$ is continuous
\[
\|\Psi(f)\|_{\mathcal{X}(\Omega_K)} \leq \|f\|_{AE(\tilde{d})}.
\]  (6)
In the trace space we have following density result.

**Lemma 17.** $\Phi(M(\tilde{d}))$ is dense in $\mathcal{X}(\Omega_K)$.

**Proof.** From $[8]$ we know that restrictions of Lipschitz functions on $\mathbb{R}^2$ are dense in $W^1_2(\Omega_K)$. Therefore Lipschitz functions are dense in $\mathcal{X}(\Omega_K)$. Hence for any $f \in \mathcal{X}(\Omega_K)$ there exist a sequence of Lipschitz functions $f_n$ s.t.
\[
\lim_{n \to \infty} \|f - f_n\|_{\mathcal{X}(\Omega_K)} = 0.
\]
So it is enough to approximate Lipschitz functions with piecewise constant functions. Let $f$ be a Lipschitz function. We define function $g_k = \sum \min \{f(x) : x \in [x_j, x_{j+1}]\} 1_{[x_j, x_{j+1}]}$, where $x_j$ are rational points of order $k$ i.e. $\exists A \in D_k$ s.t. $[x_j, x_{j+1}] = D_\infty(A)$. We define function
\[
h_A = \inf \{f(z) - g_k(z) : z \in D_\infty(A)\}$

10
The function $f$ is also Lipschitz with respect to the metric $d_K$. Let $K$ be Lipschitz constant of $f$ with respect to $d_K$. Observe that due to Lipschitz continuity of the function $f$ there are positive numbers $(b_i)_{i=1}^5, (c_i)_{i=1}^3$ s.t. for every pants shaped polygon $A \in D_n$ and $n \geq k$ we have
\[
\frac{1}{3^n} \sum_{Q \in D_1(A)} |h(A) - h(Q)| = \frac{1}{3^n} \sum_{i=1}^5 \frac{b_i}{3^n} \leq K \max_i b_i \frac{\#D_1(A)}{9^n}. 
\]
Similarly for palace shaped polygon $B$
\[
\frac{1}{3^n} \sum_{Q \in D_1(B)} |h(B) - h(Q)| = \frac{1}{3^n} \sum_{i=1}^3 \frac{c_i}{3^n} \leq K \max_i c_i \frac{\#D_1(B)}{9^n}. 
\]
Let $\rho := \max\{b_1, \ldots, b_5, c_1, c_2, c_3\}$. We can prove inductively that $\#D_j(A) \lesssim 4^j$. Let $A \in D_k$ we have following estimate on the variation on the subtree $D^\uparrow(A)$, starting with $A \downarrow$
\[
\frac{1}{3^{k-1}} |h_{A_k} - 0| + \sum_{i=1}^{\infty} \sum_{Q \in D_i(A)} |h_B - h_{B_k}| \frac{1}{3^{k-1+i}} \leq K \rho \sum_{j=k}^{\infty} \frac{\#D_{j-k}(A)}{9^j} 
\leq K \sum_{j=k}^{\infty} \frac{4^{j-k}}{9^j} \lesssim K \frac{4^k}{9^k}, 
\]
We sum above inequalities over all $A \in D_k$ and we get
\[
\|h\|_{BV_{TK}} \lesssim K \frac{4^k}{3^{2k}}. 
\]
Left hand side tens to zero with $k \to \infty$. Hence $\Psi(M(\hat{d}))$ is dense in $\hat{X}(\Omega_K)$.

To show that $\Psi$ is an isomorphism we need to prove the estimate from below on the norm of $\Psi(m)$. The next auxiliary lemma reduces our problem to a finite tree.

**Lemma 18.** Let $f \in L$ and $\text{Tr} f(z) = c$ for every $z \in [x, y]$. Function $\tilde{f} \in L$ given by the formula
\[
\tilde{f}_A = \begin{cases} 
  c & D_\infty(A) \subset [x, y], \\
  f_A & \text{in a opposite case}, 
\end{cases} 
\]
satisfies
\[
\|\tilde{f}\|_{BV_{TK}} \leq \|f\|_{BV_{TK}}. 
\]
**Proof.** Fix $A_0 \in V_T$ such that $D_\infty(A_0) \subset [x, y]$. Without loss of generality we assume that $f_{A_0} = 0$ and $c = 1$. If $B$ is a descendant of $A_0$ it follows from the definition that $D_\infty(B) \subset [x, y]$.

We can assume that for $B \in D^\uparrow(A_0)$ the value $f_B$ does not exceed one. Indeed if $B$ is such that $f_{B_\downarrow} \leq 1$ and $f_B > 1$ then we define an auxiliary function $h$
\[
h_Q = \begin{cases} 
  1 & Q = B \text{ or } Q \in D^\uparrow(B) \subset [x, y], \\
  f_Q & \text{in a opposite case}, 
\end{cases} 
\]
Function $h$ has the same trace as $f$ and differs from $f$ only on $D^\uparrow(B)$. Since
\[
|f_{B_\downarrow} - f_B| > |f_{B_\downarrow} - 1| 
\]
and $h$ is constant on $D \uparrow (B)$ it follows that

$$\|h\|_{BV_{TK}} < \|f\|_{BV_{TK}}.$$  

We can assume that $f$ is monotone (non-decreasing) on $D \uparrow (A_0)$ with respect to descendancy relation i.e. if $B \in D \uparrow (A_0)$ and $C$ is a descendant of $B$ then $f_B \leq f_C$. Indeed suppose that $f_C < f_B < 1$ for some $C \in D_1 (B)$. Since for functions in $L$ the value of trace $Tr f(x)$ is defined as limit along $br(x)$, but for $x \in D_\infty (A)$ the limit is one. Therefore on every branch $br(x)$ s.t. $x \in D_\infty (C)$ there exists a vertex $Q$ such that $f_Q \geq f_B$ and $f_Q \downarrow < f_B$. We denote by $\omega (C)$ the set of all such vertices. Let $T(C)$ be a tree with a root $C$ and set of leaves equal to $\{ Q \downarrow : Q \in \omega (C) \}$. We define auxiliary function $p$ by the formula

$$p_Q = \begin{cases} f_B & Q \in V_{T(C)}, \\ f_Q & \text{in a opposite case}, \end{cases}$$

On the tree $T(C)$ the variation of $p$ is equal to the weighted sum of differences on leaves. However for every $Q \in \omega (C)$

$$|p_Q - p_Q| = |f_Q - f_B| \geq |f_Q - f_Q|.$$  

Therefore

$$\|p\|_{BV_{TK}} < \|f\|_{BV_{TK}}.$$  

We have reduced our problem to the set of functions $Y (f) \subset L$ s.t. $h \in Y (f)$ iff it is a non-decreasing function on $D \uparrow (A_0)$ with respect to descendancy relation, $h_B = f_B$ for every $B \in V_{TK} \setminus D \uparrow (A_0)$ and $Tr h(x) = 1$ for $x \in D_\infty (A_0)$. We introduce a partial order on $Y (f)$. For $h, z \in Y (f)$

$$h \preceq z \iff \forall A \in V_{TK} \quad h_A \leq z_A \quad \text{and} \quad \|z\|_{BV_{TK}} \leq \|h\|_{BV_{TK}}.$$  

If $C \subset Y (f)$ is a chain with respect to the relation $\preceq$ then it has an upper bound in $Y (f)$. Indeed the function $z \in Y (f)$ defined by the formula

$$z_A = \sup_{u \in C} u_A$$

is an upper bound. Function $z$ is a supremum of non-decreasing functions hence it is non-decreasing. If every non-decreasing sequence $b^k_\alpha$ is convergent to one as $k \to \infty$ then $\sup_\alpha b^k_\alpha$ converges to one. Therefore $z$ has the same trace as functions in $Y (f)$. In particular $Tr h := 1$ for $x \in D_\infty (A_0)$. By the definition if $u \preceq v$ then $u_Q \leq v_Q$ for every $Q \in V_{TK}$ and the total variation $\|v\|_{BV_{TK}} \leq \|u\|_{BV_{TK}}$.

Therefore for every $n$ we can choose a sequence $f^k \in Y (f)$ s.t.

$$\lim_{k \to \infty} \|f^k\|_{BV_{TK}} = \inf_{u \in C} \|u\|_{BV_{TK}}.$$  

and $f_Q^k = z_Q$ for every $Q \in \bigcup_{j=1}^n D_j$. Therefore following estimate is satisfied

$$\sum_{j=1}^n \sum_{Q \in D_j} \frac{1}{3^j} |z_Q - z_Q| \leq \inf_{u \in C} \|u\|_{BV_{TK}}.$$  

Taking limit with $n \to \infty$ we get

$$\|z\|_{BV_{TK}} < \inf_{u \in C} \|u\|_{BV_{TK}}.$$
Since every chain in $Y(f)$ has an upper bound in $Y(f)$ by the Kuratowski-Zorn Lemma, there exists element of $Y(f)$ maximal with respect to $\preceq$. Let $w \in Y(f)$ be a maximal element. By the monotonicity of $w$, it follows that $w_{Q_x} \leq w_Q$ for every $Q \in D^\uparrow(A_0)$. Since for every $Q \in V_{T_0}$ the set of direct descendants $D_1(Q)$ has at least three elements,

$$|w_{Q_x} - w_Q| + \sum_{B \in D_1(Q)} \frac{1}{3}|w_B - w_Q| = w_Q - w_{Q_x} + \sum_{B \in D_1(Q)} \frac{1}{3}w_B - w_Q$$

$$= (1 - \frac{\#D_1(Q)}{3})w_Q - w_{Q_x} + \sum_{B \in D_1(Q)} \frac{1}{3}w_B$$

$$\geq (1 - \frac{\#D_1(Q)}{3}) \min_{B \in D_1(Q)} (w_B) - w_{Q_x} + \sum_{B \in D_1(Q)} \frac{1}{3}w_B.$$ 

Function $w$ is maximal with respect to $\preceq$, hence $w_Q = \min_{B \in D_1(Q)} w_B$ for every $Q \in D^\uparrow(A_0)$. Therefore there is an infinite branch $\text{br}(x)$ s.t. $x \in D_\infty(A_0)$ and $w$ is constant on $\text{br}(x) \cap D^\uparrow(A_0)$. However for $x \in D_\infty(A_0)$ the limit over any branch $\text{br}(x)$ is equal to one. Hence $h_B = 1$ for every $B \in D^\uparrow(A_0)$. We have proven that changing the values of $f$ to one on the descendants of $A_0$ does not increase the total variation. It remains to consider the value at the point $A_0$. By the triangle inequality and the fact that for every vertex $Q$, $\#D_1(Q) \geq 3$ we have

$$|f_{A_0} - f_{A_0}| + \sum_{B \in D_1(A_0)} \frac{1}{3}|1 - f_{A_0}| \leq |f_{A_0} - 1|.$$ 

Therefore changing the value of $f$ on $A_0$ and its descendants to one, will not increase the total variation. Since only assumption on $A_0$ was that $D_\infty(A_0) \subseteq [x,y]$ we have desired estimate

$$\|\bar{f}\|_{BV_{T_K}} \leq \|f\|_{BV_{T_K}}.$$ 

Lemma 19. Let $A_0 \in D_n$ and $[x, y] = D_\infty(A_0)$ then

$$\tilde{d}(x, y) = 3^{-n}.$$ 

Proof. For any $f \in BV_{T_K}$ s.t. $Tr f = 1_{[x,y]}$ we define

$$f_A = \begin{cases} 1 & D_\infty(A) \subseteq [x,y], \\ f_A & A \in D_k, k \leq n, \\ 0 & \text{in a opposite case.} \end{cases}$$

From the Lemma 18 it follows that

$$\|\tilde{f}\|_{BV_{T_K}} \leq \|f\|_{BV_{T_K}}$$

However

$$\|\tilde{f}\|_{BV_{T_K}} \geq \frac{1}{3^n} \sum_{B \in D_1(A_0)} |f_{A_0} - f_B| \geq \frac{1}{3^n} (|f_{A_0} - 1| + |f_{A_0}|) \geq \frac{1}{3^n}.$$ 

13
The right hand side of the inequality is a total variation of a function $p$

$$p_A = \begin{cases} 1 & \text{if } D_\infty(A) \subset [x, y], \\ 0 & \text{in a opposite case.} \end{cases}$$

Let us observe that the set of functions $\sum_j a_j 1_{[x_j, y_j]}$, where $x_j, y_j$ are rational, is dense in $\hat{X}(\Omega_K)$. Indeed for every irrational arc $[x, y]$ there exist a sequence of points $t_n, z_n$ s.t.

$$\|1_{[x, y]} - 1_{[t_n, z_n]}\|_{\hat{X}(\Omega_K)} \lesssim \frac{1}{3^n}.$$  

Similarly we observe that molecules $\sum_j a_j m_{x_jy_j}$, where $x_j, y_j$ are rational, are dense in Arens-Eels space.

We fix $g = \sum_j a_j 1_{[x_j, y_j]}$, where arcs $[x_j, y_j]$ are rational and pairwise disjoint. Let $f \in L$ be any function such that $\text{Tr} f = g$. There exists $n_0 = n_0(g)$ such that for $A \in D_{n_0}$ either there exist an arc $[x_j, y_j]$ s.t. $D_\infty(A) \subset [x_j, y_j]$ or $D_\infty(A)$ and $\bigcup_j [x_j, y_j]$ are disjoint. We define function $W f \in L$ by

$$W f_A = \begin{cases} a_j & D_\infty(A) \subset [x_j, y_j] \\ 0 & D_\infty(A) \cap \bigcup_j [x_j, y_j] = \emptyset, \end{cases} f_A \text{ in other cases.}$$

It is easy to observe that $\text{Tr} f = \text{Tr} W f$. Moreover from Lemma 18 it follows that

$$\|W f\|_{\hat{BV} T_K} \leq \|f\|_{\hat{BV} T_K}.$$  

Therefore

$$\inf \{\|f\|_{\hat{BV} T_K} : \text{Tr} f = g\} = \inf \{\|f\|_{\hat{BV} T_K} : \text{Tr} f = g \text{ and } f = W f\}.$$  

Since we minimize the total variation over the set $\{\text{Tr} f = g \text{ and } f = W f\}$, the values $f_A$ are fixed for $A \in D_{n_0}, k > n_0$. Therefore the total variation on this set is a function of finitely many variables. Moreover it is a piecewise linear function with finitely many pieces. Therefore the minimum is attained. We denote the total variation minimizer by $\psi$. We define by $\gamma^A \in \hat{BV} T_K$

$$\gamma^A_B = \begin{cases} 1 & B \in D \uparrow (A), \\ 0 & \text{in other cases.} \end{cases}$$

Therefore from Abel summation formula

$$\psi = \psi_R + \sum_{j=1}^{n_0} \sum_{A \in D_{n_0}} (\psi_A - \psi_A_{A_j}) \gamma^A.$$  

A simple calculation gives us

$$\|\psi\|_{\hat{BV} T_K} = \sum_{j=1}^{n_0} \sum_{A \in D_{n_0}} \|\psi_A - \psi_A_{A_j}\|_{\gamma^A} \|\gamma^A\|_{\hat{BV} T_K}. \tag{8}$$  

The function $\|\psi\|_{\hat{BV} T_K}$ minimize the variation for a given trace, hence

$$\|\text{Tr} f\|_{\hat{X}(\Omega_K)} = \|\psi\|_{\hat{BV} T_K}.$$  

14
Therefore from (8), (7)

\[ \|Tr \psi\|_{X(\Omega_K)} \simeq \sum_{j=1}^{n_0} \sum_{A \in \mathcal{D}_n} |\psi_A - \psi_{A_1}| d(x(A), y(A)) \]

\[ \geq \| \sum_j a_j m_{x_j, y_j} \|_{AE(\tilde{d})} \]

Therefore \( \Psi \) is an isomorphism of Banach spaces. \( \square \)

We have proven that the trace space is isomorphic to the Arens-Eels space. We will characterize \( AE(\tilde{d}) \) further.

**Lemma 20.** \( AE(\tilde{d}) \) is isomorphic to \( \ell^1 \)

**Proof.** In order to characterize \( AE(\tilde{d}) \) we introduce another metric on the von Koch’s curve. The von Koch’s curve is constructed inductively. The induction starts with a triangle and every segment of the triangle is replaced with a piecewise linear curve \( w \). This curve is made of 4 segments. In the next step every old segment is replaced with a rescaled copy of \( w \). Every segment is indexed in the following way. The segment \( S_x \) is replaced with segments \( S_{x_0}, S_{x_1}, S_{x_2}, S_{x_3} \).

\[ I = \{ x = (x_1, x_2, \ldots) : x_1 \in \{0, 1, 2\}, x_i \in \{0, 1, 2, 3\} \text{ for } i > 1 \} \] is a set of all infinite indices of segments in the von Koch’s curve construction. For every point \( x \in \partial \Omega_K \) there is a corresponding index \( i(x) \in I \) such that segments \( S_{i(x)_1}, \ldots, i(x)_k \to x \) as \( k \to \infty \). We define a bijection between set of indices and a one dimensional Torus with the euclidean metric

\[ T = \{ y : y = \frac{i(x)_1}{3} + \sum_{j=2}^{\infty} \frac{i(x)_j}{4^j} \mid i(x) \in I \}. \]

Every \( x \in \partial \Omega_K \) has a unique index in \( T \). Abusing notation we denote it by \( i(x) \). We can define a metric on \( \partial \Omega_K \) by

\[ d(x, y) := d_T(i(x), i(y)). \]

As is easily on a Figure 1 if \( A \in \mathcal{D}_n \) is a "pants" shaped polygon then \( D_\infty(A) = [x, y] \), where \( d(i(y), i(x)) = \frac{2}{3^n} \). It is so because its descendants cover two segments of \( n \)-th generation. Similarly if \( A \) is a "palace" shaped polygon, \( d(i(y), i(x)) = \frac{1}{4^n} \). In any of the above cases we have

\[ \tilde{d}(x, y) \simeq \frac{1}{3^n} = \frac{1}{4^n \log_4(3)} \simeq d(x, y)^{\log_4(3)}. \]

For rational points \( x, y \) we define

\[ f_{[x,y]}^A := \begin{cases} 1 & D_\infty(A) \subset [x, y], \\ 0 & \text{otherwise}. \end{cases} \]
Obviously $Tr f^{[x,y]} := 1_{[x,y]}$. Since $x, y$ are rational, there exists unique finite sequence of 
$\{A_k\}_{k \in I} \subset V_T$, such that $f^{[x,y]} = \sum_k \gamma^{A_k}$. Let $m = \min \{ n : \exists k \ A_k \in D_n \}$. From the definition of $f^{[x,y]}$ we deduce that $\gamma^{A_k}$ have disjoint support, and for every $n$ there are at most 10 polygons in $\{A_k\}_{k \in I \cap D_n}$. Therefore

$$d(x, y) = \sum_k d(x(A_k), y(A_k)) \leq 10 \sum_{i=m}^{4^i} \frac{1}{4^i} \approx \frac{1}{4^m},$$

and we have analogous estimate for $\tilde{d}$. Hence

$$\tilde{d}(x, y) \approx \frac{1}{3^m} = \frac{1}{4^m \log_4(3)} \approx d(x, y)^{\log_4(3)}. $$

Therefore $AE(\tilde{d}) \cong AE(d^{\log_4(3)})$. Since $0 < \log_4(3) < 1$ the claim of the lemma follows from the theorem below,

**Theorem 21.** Let $N \in \mathbb{N}$ and $X$ is isometric to infinite compact subset of $\mathbb{R}^N$. If $d, \tilde{d}$ are metrics on $X$ s.t $\tilde{d} \simeq d^\alpha$ for $0 < \alpha < 1$ then the space $AE(\tilde{d})$ is isomorphic to $\ell^1$.

The case $N=1$ was proven by Z. Ciesielki [3] and for $N > 1$ the above Theorem follows from Theorem 3.5.5 and Theorem 3.3.3 in [13].

Therefore $\tilde{X}(\Omega_K)$ is isomorphic to $\ell^1$. Let $\tilde{X}(\Omega_K) = \text{span}\{e_i\}$. From the definition of the trace space for every $e_i$ there exists $f_i \in BV_{T_K}$ such that $\|f_i\|_{BV_{T_K}} \leq 2\|e_i\|_{\tilde{X}(\Omega_K)}$ and $Tr f_i = e_i$. Hence the $S$ given by the formula

$$S \left( \sum_i a_i e_i \right) = \sum_i a_i f_i$$

is the desired right inverse operator with $\|S\| = 2$. Indeed

$$Tr \left( S \left( \sum_i a_i e_i \right) \right) = Tr \left( \sum_i a_i f_i \right) = \sum_i a_i e_i.$$ 

This concludes the proof of Theorem 2.
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Krystian Kazaniecki
krystian.kazaniecki@mimuw.edu.pl
Michał Wojciechowski
miwoj@impan.pl