Hochshild cohomology of the associative conformal algebra $Cend_{1,x}$

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It is established in this work that second Hochshild cohomology group of the associative conformal algebra $Cend_{1,x}$ is zero. As a corollary, this algebra split off in each extension with a nilpotent kernel.

Key words: associative conformal algebra, splitting off radical, Hochshild cohomology.

1 Introduction and preliminary information

1.1 Introduction

The axiomatic quantum field theory was originated in 50th of the last century in A. Whiteman’s articles etc. Fields with a conformal symmetry were considered in [5]. It can be represented as infinite in both side formal power series whose coefficients are linear operators. A key role in the conformal field theory is played by construction of an operator product expansion (OPE) an algebraical description of which lead us to a vertex operator algebra (VOA) concept. An axiomatic definition of vertex algebra was introduced by R. Borcherds in [6].

V.G. Kac established [see 9] that coefficients of a singular part of the OPE allow us to describe a commutator of formal power series responding for VOA multiplication operators. In case of an algebraical description of these structures there appears a concept of conformal Lie algebras. For these algebras the structure theory was constructed in [8] and the representations
theory respectively in [7]. During the process of studying representations of
the finite type there appear natural algebraic structures playing the same role
with respect to conformal Lie algebras as common associative algebras to Lie
algebras. These structures are named associative conformal algebras [see 7].

Let $H = \mathbb{F}[D]$ be a common polynomial algebra over a field $\mathbb{F}$ with a zero
char, $Z_+$ are non-negative integers, $D^{(n)}$ is for $\frac{1}{n!}D^n$, $n \in Z_+$ and $D^{(n)} = 0$, $n < 0$.

DEFINITION. A conformal algebra is a left $H$-module $A$ equipped with
a countable number of $\mathbb{F}$-linear maps $\circ_n : A \otimes A \to A$, $n \in Z_+$ satisfying axioms:

1. $a \circ_n b = 0$ for sufficiently large $n$,
2. $Da \circ_n b = -na \circ_{n-1} b$,
3. $a \circ_n Db = D(a \circ_n b) + na \circ_{n-1} b$,

where $a, b \in A, n \in Z_+$.

First axiom allows us to determine a locality function $N: A \times A \to Z_+$
by the rule: $N(a, b) = \min\{n \in Z_+ | a \circ_k b = 0 \ \forall k \geq n\}$, $a, b \in A$.

A conformal algebra is called associative if for every $a, b, c \in A$ holds

\[ (a \circ_n b) \circ_m c = \sum_{s \geq 0} (-1)^s \binom{n}{s} a \circ_{n-s} (b \circ_{m+s} c), \ m, n \in Z_+, \]

or equivalent condition

\[ a \circ_n (b \circ_m c) = \sum_{t \geq 0} \binom{n}{t} (a \circ_{n-t} b) \circ_{m+t} c, \ m, n \in Z_+. \]

Construction of simple and semisimple associative algebras of the finite
type was described in [8] (see also [12]). It is established in [13] that analogue
of the Wedderburn theorem about a splitting radical (in reference to Lie con-
formal algebras the same statement, which is analogue of the Levi - Maltsev
theorem for Lie algebras, is incorrect [4]) is fulfilled for associative conformal
algebras of the finite type.

Associative conformal algebras with a faithful representation of the finite
type form more wide class in comparison with algebras of the finite type. All
such algebras is subalgebras of $\text{Cend}_n$ (the algebra of conformal linear maps of free $n$-generated $\mathbb{F}[D]$-module)

$$\text{Cend}_n = M_n(\mathbb{F}[D, x]),$$

where $\circ_m : \text{Cend}_n \otimes \text{Cend}_n \to \text{Cend}_n$ is determined by a common matrix differentiation, i.e. $A(x) \circ_m B(x) = A(x)B^{(m)}(x)$.

Simple and semisimple subalgebras of $\text{Cend}_n$ were described in [10] and there also was proved that every such algebra $C$ has a nilpotent ideal (radical) defining a semisimple quotient with a faithful representation of the finite type.

The radical splitting problem of this class of algebras was considered in [11, see also 1, 2]. There is a definition of Hochschild cohomology for associative conformal algebras in [1, 2] and in a context of pseudoalgebras [3] (slightly different from the definition in [4]). Cohomology play a key role in the radical splitting problem. Specifically, to proof analogue of the Wedderburn radical splitting theorem for a conformal algebra $A = C/R$ it’s enough to show that for arbitrary conformal $A$-bimodule $M$ second Hochschild cohomology group $H^2(A, M)$ is zero.

It is shown in [1] that for $A = \text{Cend}_1$ and for every bimodule $M$ next condition $H^2(A, M) = 0$ is fulfilled. Contrariwise, it follows from the example [see 11] that there exist $A$-bimodule $M$ for $A = \text{Cend}_{1,x^2} = x^2\text{Cend}_1$ such that $H^2(A, M) \neq 0$.

In this paper we consider an intermediate case: $A = \text{Cend}_{1,x} = x\text{Cend}_1$. This associative conformal algebra is the universal enveloping algebra for the simple conformal Virasoro algebra with locality 2. The main result of this paper is next

**THEOREM.** Let $C = \text{Cend}_{1,x}$ and let $M$ be arbitrary conformal bimodule over $C$. Then $H^2(C, M) = 0$.

### 1.2 Modules over conformal algebras

**DEFINITION.** A left module over an associative conformal algebra $A$ is an $H$-module $V$, closed with respect to the left action by elements of $A$. Specifically,
there is determined such linear maps $\circ_n : A \otimes V \rightarrow V$, $n \in \mathbb{Z}_+$ that for every $a, b \in A$, $v \in V$, $m \in \mathbb{Z}_+$

(1) $a \circ_n v = 0$ for sufficiently large $n$,
(2) $Da \circ_n v = -na \circ_{n-1} v$,
(3) $a \circ_n Dv = D(a \circ_n v) + na \circ_{n-1} v$,
(4) $(a \circ_m b) \circ_n v = \sum_{s \geq 0} (-1)^s \binom{m}{s} a \circ_{m-s} (b \circ_{n+s} v)$.

Hereinafter a left module over an associative conformal algebra $A$ is a simply left $A$-module. A right $A$-module could be defined in the same way.

A bimodule is a simultaneously left and right $A$-module with one additional axiom:

$$(a \circ_m v) \circ_n b = \sum_{s \geq 0} (-1)^s \binom{m}{s} a \circ_{m-s} (v \circ_{n+s} b).$$

2 Main definitions and formulation of the problem

2.1 Connection with pseudoalgebras

An algebra $H = F[D]$ is Hopf’s algebra with a comultiplication $\Delta : H \rightarrow H \otimes H$, $\Delta(D) = D \otimes 1 + 1 \otimes D$, a counit $\varepsilon : H \rightarrow F$, $\varepsilon(D) = 0$ and an antipode $S : H \rightarrow H$, $S(D) = -D$ where $\Delta$, $\varepsilon$ and $S$ are homomorphisms of algebras.

Assuming $\Delta_1 = \Delta$ define by induction $\Delta_n = (\Delta \otimes id \otimes \cdots \otimes id)\Delta_{n-1}$, $n > 1$.

Because of conformal algebras are objects of the pseudotensor category of left $H$-modules, it could be constructed by the pseudotensor language.

One can define the right action $H$ on $H^\otimes n$ by the rule:

$$(h_1 \otimes h_2 \otimes \cdots \otimes h_n)h = (h_1 \otimes h_2 \otimes \cdots \otimes h_n)\Delta_{n-1}(h).$$

Then $H^\otimes n$ is a right $H$-module.

Let $V$ be a left unital $H$-module endowed with a bilinear operation $* : A \otimes A \rightarrow (H \otimes H) \otimes_H A$. Such $H$-module $A$ is called an $H$-pseudoalgebra and the operation $*$ is a pseudomultiplication.

Every conformal algebra $A$ is a $H$-pseudoalgebra and next equality

$$a * b = \sum_{n \geq 0} (-D)^{(n)} \otimes_H (a \circ_n b), \quad a, b \in A,$$
set up a correspondence between the pseudomultiplication $*$ and operations $\circ_n, n \in \mathbb{Z}_+$.  

The definition of a module (bimodule) over an associative conformal algebra along with the associativity property could be represented through pseudomultiplication-like operations, see details in [1, 3]. For research purposes we translate main definitions in [1](a cochain, cocycle and coboundary) from the pseudoalgebras language to the $\circ_n$-multiplications language, $n \in \mathbb{Z}_+$.

### 2.2 Definition of the Hochschild cohomology group

**DEFINITION.** An $n$-cochain with coefficients in $V$ is a $H$-polylinear map $\varphi : A \otimes^n \to (H \otimes^n) \otimes_H V$, i.e.

$$\varphi(h_1 a_1 \otimes \cdots \otimes h_n a_n) = (h_1 \otimes \cdots \otimes h_n \otimes_H 1)\varphi(a_1 \otimes \cdots \otimes a_n).$$

Denote by $C^n(A, V)$ a set of every $n$-cochains of the algebra $A$ and with coefficients in the bimodule $V$. In this paper only 1- and 2-cochains are interesting for us. Obviously, an 1-cochain $\tau$ is arbitrary $H$-polylinear map $\tau : A \to V$. For 2-cochains we use the natural representation:

$$\varphi(a, b) = \sum_{s \geq 0} ((-D)^{(s)} \otimes 1) \otimes_H \varphi_s(a, b), \text{ where } \varphi_s(a, b) \in V. \quad (1)$$

Thus a 2-cochain $\varphi$ is a collection of bilinear maps $\varphi_s : A \otimes A \to V, s \in \mathbb{Z}_+$, such that

1. $\varphi_s(Da, b) = -s\varphi_{s-1}(a, b),$
2. $\varphi_s(a, Db) = D\varphi_s(a, b) + s\varphi_{s-1}(a, b),$
3. $\varphi_s(a, b) = 0$ for sufficiently large $s$.

Similarly, a 3-cochain $\psi$ is a 2-parametric collection of linear maps $\psi_{mn} : A \otimes A \otimes A \to V, n, m \in \mathbb{Z}_+$ with the same properties.

**DEFINITION.** A differential is a map $\delta_n : C^n(A, V) \to C^{n+1}(A, V)$ given by the rule $(\delta_n \varphi)(a_1, \ldots, a_{n+1}) = a_1 \cdot \varphi(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(a_1, \ldots, a_i * a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} \varphi(a_1, \ldots, a_n) * a_{n+1}.$
Translating this definition to the language of \( \circ_n \)-operations, \( n \in \mathbb{Z}_+ \), we obtain the differential for the 1-cochain \( \tau \)

\[
(\delta_1 \tau)_n(a, b) = \tau(a) \circ_n b - \tau(a \circ_n b) + a \circ_n \tau(b),
\]

and for the 2-cochain \( \varphi \)

\[
(\delta_2 \varphi)_{mn}(a, b, c) = a \circ_m \varphi_n(b, c) + \varphi_m(a, b \circ_n c) - \sum_{s \geq 0} \binom{n}{s}(\varphi_{n+s}(a \circ_{m-s} b, c) + \varphi_{m-s}(a, b) \circ_{n+s} c), \quad a, b, c \in M, \ n, m \in \mathbb{Z}_+.
\]

Say an \( n \)-cochain \( \varphi \) is called an \( n \)-cocycle if \( \delta_n \varphi = 0 \). A cocycle \( \varphi \in Z^n(A, V) \) is called an \( n \)-coboundary if there exists an \( (n-1) \)-cochain \( \psi \) such that \( \varphi = \delta_{n-1} \psi \). Denote by \( Z^n(A, V) \) a set of every \( n \)-cocycles and by \( B^n(A, V) \) a set of every \( n \)-coboundaries.

DEFINITION. The Hochschild cohomology group of an associative conformal algebra \( A \) on an \( A \)-bimodule \( V \) is \( H^n(A, V) = Z^n(A, V)/B^n(A, V) \).

2.3 Extensions. Connection with second cohomology group

DEFINITION. An extension of a conformal algebra \( A \) is an ordered pair \((B, \sigma)\) where \( B \) is a conformal algebra and \( \sigma \) is a homomorphism \( B \) to \( A \).

Any extension can be identified with an exact sequence

\[
0 \to \text{Ker} \sigma \to B \to A \to 0.
\]

The conformal algebra \( A \) is called splitting off in the extension \((B, \sigma)\) if \( B = A' \otimes \text{Ker} \sigma \) where \( A' \subseteq B \) is a subalgebra isomorphic to \( A \).

The extension \( B \) called singular if \((\text{Ker} \sigma \circ_n \text{Ker} \sigma) = 0 \) for every \( n \in \mathbb{Z}_+ \).

Let \( M \) be a bimodule over the conformal algebra \( A \) and \( \varphi \in C^2(A, M) \). Construct a singular extension of the algebra \( A \) by the bimodule \( M \). Let \( B \) be equal to the \( H \)-modules direct sum \( A \oplus M \). The operation \(*\) can be defined by the rule

\[
(a_1 + u_1) * (a_2 + u_2) = a_1 * a_2 + a_1 * u_2 + u_1 * a_2 + \varphi(a_1, a_2),
\]

where \((a_1 + u_1), (a_2 + u_2) \in B\). Then \( B \) is a conformal algebra. Denote constructed extension by \((A; M, \varphi)\).
Next results are proved in [1]:

**LEMMA.** If $\varphi \in Z^2(A, M)$, then $B = (A; M, \varphi)$ is an associative conformal algebra.

**THEOREM.** A conformal algebra $A$ is split off in a singular extension $(B, \sigma)$ if and only if a cocycle $\varphi$ is trivial in $H^2(A, \text{Ker } \sigma)$.

### 2.4 Formulation of the problem

Earlier we determined the conformal algebra $Cend_n$, $n \geq 1$. Here we consider case $n = 1$. The conformal subalgebra $Cend_{1,x} = xCend_1 \subset Cend_1$ is a free $H$-module with a basis given by elements $x^k = x \circ_0 x \circ_0 \cdots \circ_0 x$ ($k$ times), $k \geq 1$.

A question has been investigated in the paper sounds in this way: do the conformal algebra $Cend_{1,x}$ split off in every singular extension? In other words, whether we can construct for every exact sequence

$$0 \rightarrow M \rightarrow C \rightarrow Cend_{1,x} \rightarrow 0,$$

(where $M^2 = 0$ in sense of $\circ$-multiplication) such cross-section $Cend_{1,x} \rightarrow C$ which is a conformal algebras homomorphism.

### 3 Main result

As it is shown in 2.3 the splitting off problem for the conformal algebra $Cend_{1,x}$ in a singular extension with the nilpotent kernel $M$ is in a tough connection with algebra’s second Hochschild cohomology group with coefficients in $M$.

The main result of the paper is next

**Theorem 1.** Let $Cend_{1,x}$ be the associative conformal algebra described back in the text, $M$ is a bimodule over it. Then $H^2(Cend_{1,x}, M) = 0$.

**Proof.** We prove the theorem in several steps.

**Lemma 1.** Arbitrary 2-cocycle $\varphi \in Z^2(Cend_{1,x}, M)$ is completely determined by elements $\varphi_t(x, x)$ and $\varphi_0(x, x^l)$, $t, l \geq 1$ laying down in the bimodule $M$. 
Proof. Use the 2-cocycle definition:

\[ x^k \circ_n \varphi_m(x^l, x^q) + \varphi_m(x^k, x^l \circ_m x^q) = \sum_{s \geq 0} \binom{n}{s} (\varphi_{m+s}(x^k \circ_{n-s} x^l, x^q) + \varphi_{n-s}(x^k, x^l) \circ_{m+s} x^q), \]

(2)

where \( n, m \in \mathbb{Z}_+, k, l, q \geq 1. \)

For \( n = 0, k = 1: \)

\[ \varphi_m(x^{l+1}, x^q) = x \circ_0 \varphi_m(x^l, x^q) + m! \left( \binom{q}{m} \varphi_0(x, x^{l+q-m}) - \varphi_0(x, x^l) \circ_m x^q. \right) \]

(3)

Use induction and express every values of the cocycle through \( \varphi_m(x, x^q), m \geq 0, q \geq 1. \)

Consider (2) for \( m = 0, k = l = 1: \)

\[ x^0 \circ_n \varphi_0(x, x^q) + \varphi_n(x, x^{q+1}) = \varphi_n(x^2, x^q) + n \varphi_{n-1}(x, x^q) - \sum_{s \geq 0} \binom{n}{s} \varphi_{n-s}(x, x) \circ_s x^q. \]

(4)

From (3) one can see that \( \varphi_n(x^2, x^q) = x \circ_0 \varphi_n(x, x^q) + n! \left( \binom{q}{m} \varphi_0(x, x^{q-n+1}) - \varphi_0(x, x) \circ_n x^q. \right) \). We substitute the result into expression (4):

\[ \varphi_n(x, x^{q+1}) = x \circ_0 \varphi_n(x, x^q) + n! \left( \binom{q}{m} \varphi_0(x, x^{q-n+1}) - \varphi_0(x, x) \circ_n x^q + n \varphi_{n-1}(x, x^q) - \sum_{s \geq 0} \binom{n}{s} \varphi_{n-s}(x, x) \circ_s x^q - x \circ_n \varphi_0(x, x^q). \]

Assuming \( n = 1 \) and using induction on \( q \) we obtain that \( \varphi_1(x, x^{q+1}) \) is expressed through \( \varphi_l(x, x) \) and \( \varphi_0(x, x^l) \) for \( q, l \geq 1 \). Applying sequently induction on \( n \) and on \( q \) we obtain the same expression for each \( \varphi_n(x, x^{q+1}), n, q \geq 1. \) As required.

Lemma 2. Let \( \varphi_1(x, x) = 0 \) and \( \varphi_0(x, x^l) = 0 \) for every \( l \geq 1. \) Then \( \varphi = 0. \)

Proof. Due to Lemma 1 it is enough to prove that \( \varphi_1(x, x) = 0 \) implies \( \varphi_l(x, x) = 0. \)

Hereinafter designate \( \varphi_m(x, x) = \varphi_m, (x \circ_m \cdot) = L_m \in \text{End}(M), (\cdot \circ_m x) = R_m \in \text{End}(M). \)
Applying associativity identities we’ve rewritten useful formulas in the new designation:

\[ L_n L_m(u) = \sum_{s=0}^{n} \binom{n}{s} ((x \circ_{n-s} x) \circ_{m+s} u) = L_0 L_{m+n}(u) + n L_{m+n-1}(u), \quad (5) \]

\[ R_n R_m(u) = \sum_{s=0}^{m} (-1)^s \binom{m}{s} (x \circ_{m-s} (x \circ_{n+s} x)) = \begin{cases} R_m(u), & n = 1, \\ 0, & n > 1, \end{cases} \quad (6) \]

\[ R_n L_m(u) = \sum_{s=0}^{m} (-1)^s \binom{m}{s} (x \circ_{m-s} (u \circ_{n+s} x)) = \sum_{s=0}^{m} (-1)^s \binom{m}{s} L_{m-s} R_{n+s}(u), \quad (7) \]

for \( u \in M \).

Using 2-cocycle identity \([2]\) obtain following relation:

\[ (x \circ_1 \varphi_m(x, x) - x \circ_0 \varphi_{m+1}(x, x)) - \varphi_m(x \circ_1 x, x) + (\varphi_1(x, x \circ_m x) - \varphi_0(x, x \circ_{m+1} x)) - \varphi_1(x, x) \circ_m x = 0 \]

Since the \( \circ \)-multiplication act on \( x^k \in Cend_{1,x} \) like a differentiation we obtain equalities:

\[ x \circ_0 \varphi_2 = x \circ_1 \varphi_1 - \varphi_1 \circ_1 x, \quad (8) \]

\[ x \circ_0 \varphi_{m+1} = x \circ_1 \varphi_m - \varphi_m - \varphi_1 \circ_m x, \quad m > 1. \quad (9) \]

An assumption \( \varphi_1 = 0 \) implies there are neither right part in \([8]\) nor last summand in \([7]\).

For \( n = 1, m = 0 \) formula \([5]\) gives \( L_1 L_0 = ((x \circ_1 x) \circ_0 \cdot) + ((x \circ_0 x) \circ_1 \cdot) = L_0 + L_0 L_1 \). Applying it to \([7]\) obtain next equality

\[ L_0^{m-2} \varphi_m = (L_1 - (m - 2)) \ldots (L_1 - 1) \varphi_2. \]

The locality axiom implies there is such \( N \) that \( \varphi_m = 0 \) for each \( m > N \).

Let \( N > 1 \). Then for \( m = N + 1 \): \( (L_1 - (N - 1)) \ldots (L_1 - 1) \varphi_2 = 0. \)

Denoting \( (L_1 - (N - 2)) \ldots (L_1 - 1) \varphi_2 = \prod_{l=1}^{N-2} (L_1 - l) \varphi_2 = \chi_1 \in M \) we get a new equalities system:

\[ \begin{align*}
L_1 \chi_1 &= x \circ_1 \chi_1 = (N - 1) \chi_1, \\
L_0 \chi_1 &= x \circ_0 \chi_1 = 0.
\end{align*} \quad (10) \]
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Second equality is obtained from (8) by the sequentially left multiplication on 
\((L_1 - 2), \ldots, (L_1 - (N - 1))\) and with the condition \((L_1 - 1)L_0 = L_0L_1\).

Show the system implies \(\chi_1 = 0\). Indeed, using (5), the associativity identity and system (10):

\[
0 = (x \circ_2 x) \circ_0 \chi_1 = L_2L_0\chi_1 - 2L_1L_1\chi_1 + L_0L_2\chi_1 = L_0L_2\chi_1 - 2L_1L_1\chi_1 \Rightarrow
L_0L_2\chi_1 = 2L_1L_1(N - 1)^2\chi_1 = L_1L_1\chi_1 = (x \circ_1 x) \circ_1 \chi_1 + (x \circ_0 x) \circ_2 \chi_1 = L_1\chi_1 + L_0L_2\chi_1 = (N - 1)\chi_1 + 2(N - 1)^2\chi_1 \Rightarrow 0 = ((N - 1) + (N - 1)^2)\chi_1 = N(N - 1)\chi_1.
\]

As long as \(N > 1\) last element in the chain of equalities is zero if only \(\chi_1 = 0\).

Let \(\chi_{k-1} = \prod_{l=1}^{N-k} (L_1 - l)\varphi_2 = (L_1 - (N - k))\chi_k, k = 2, \ldots, N \ (\chi_{N-1} = \varphi_2)\). Suppose reckoning above as the base of induction. Let \(\chi_{k-1} = 0\) for certain \(k, 1 < k < N\). Then

\[
\begin{align*}
L_1\chi_k &= (N - k)\chi_k, \\
L_0\chi_1 &= 0.
\end{align*}
\]

By similar actions one can obtain

\[
0 = (N - (k - 1))(N - k)\chi_k = (N - (k - 1))(N - k) \prod_{l=1}^{N-(k+1)} (L_1 - l)\varphi_2.
\]

hence \(\chi_k = 0\).

Therefore \(\chi_k = 0\) for \(k = 1, \ldots, N - 1\) and particularly \(\chi_{N-1} = \varphi_2 = 0\).

Now, with the condition \(\varphi_2 = 0\) we have (9) rewritten in next way

\[
L_0\varphi_3 = 0, \quad L_0\varphi_{m+1} = (L_1 - 1)\varphi_m
\]

and with the same reflections we get \(\varphi_3 = 0\) and so on until \(\varphi_{N+1} = 0\) by the locality axiom. This is it, \(\varphi_m = 0\) for every \(m \in \mathbb{Z}_+\).

\textbf{Lemma 3.} Consider a cocycle \(\varphi \in Z^2(Cend_1,x,M)\). Assume exists such \(\psi_1 \in M\) that

\[
\varphi_1(x,x) = x \circ_1 \psi_1 - \psi_1 \circ_1 x.
\]

Then \(\varphi \in B^2(Cend_1,x,M)\).
Proof. Define by induction $\psi_l, l \geq 2$

$$\psi_{l+1} = -\varphi_0(x, x^l) + x \circ_0 \psi_l + \psi_1 \circ_0 x^l, \quad l \geq 1. \quad (13)$$

Construct a 1-cochain with values $\psi_l(x^l) = \psi_l$. Consider $\delta \psi = \varphi' \in B^2(\text{Cend}_{1,x}, M)$. Then $[12]$ implies $\varphi'_1(x, x) = \varphi_1(x, x)$ and from $[13]$ follows $\varphi'_0(x, x^l) = \varphi_0(x, x^l)$.

Denote $\varphi - \varphi' = \pi \in Z^2(\text{Cend}_{1,x}, M)$. Then $\pi_1(x, x) = 0$ by the hypothesis, $\pi_0(x, x^l) = 0$ by the choice of $\psi_l$ and then regarding Lemma 2 we have $\pi = 0$, i.e. $\varphi = \delta \psi$ as required.

It is remained to show existence of the element $\psi_1$.

Firstly we derive a formula for $L_0^m \varphi_{m+1}, m \geq 1$.

Lemma 4. If $\varphi \in Z^2(\text{Cend}_{1,x}, M)$ then

$$L_0^m \varphi_{m+1} = (L_1 - m) \ldots (L_1 - 2)L_1 \varphi_1 - \sum_{s=1}^{m} (L_1 - m) \ldots (L_1 - (s+1))L_0^{s-1}R_s \varphi_1. \quad (14)$$

Proof. By induction on $m$. Using $[8]$ for $m = 1$ we obtain $L_0 \varphi_2 = (L_1 - R_1) \varphi_1$. Then it is multiplied on the left by $(L_1 - 2)$. Applying $[5]$:

$$(L_1 - 2)(L_1 - R_1) \varphi_1 = (L_1 - 2)L_0 \varphi_2 = L_0(L_1 - 1) \varphi_2.$$

Substitute the result in $[13]$:

$$L_0^3 \varphi_3 = (L_1 - 2)(L_1 - R_1) \varphi_1 - L_0 R_2 \varphi_1.$$

$m \to m + 1$: Let the formula be true for $L_0^{m-1} \varphi_m$. Then

$$L_0^m \varphi_{m+1} = L_0^{m-1}(L_0 \varphi_{m+1}) = L_0^{m-1}((L_1 - 1) \varphi_m - R_m \varphi_1)$$

$$= (L_1 - m)L_0^{m-1} \varphi_m - L_0^{m-1}R_m \varphi_1 = (L_1 - m)((L_1 - (m-1)) \ldots (L_1 - 2)L_1 \varphi_1$$

$$- \sum_{s=1}^{m-1} (L_1 - (m-1)) \ldots (L_1 - (s+1))L_0^{s-1}R_s \varphi_1) - L_0^{m-1}R_m \varphi_1$$

$$= (L_1 - m) \ldots (L_1 - 2)L_1 \varphi_1 - \sum_{s=1}^{m} (L_1 - m) \ldots (L_1 - (s+1))L_0^{s-1}R_s \varphi_1,$$

As required. \[\square\]
Consider in this step multiplications $L_{k-1}^k L_k$. By induction on $k$ we will show that the equality

$$L_{k-1}^k L_k = L_1(L_1 - 1) \ldots (L_1 - (k - 1))$$

(15)

is fulfilled.

Using (15) obtain $k = 2$:

$$L_0 L_2 = L_1^2 - L_1 = L_1(L_1 - 1);$$

$k \to k + 1$:

$$(L_1 - k) L_{k-1}^k L_k = L_{k-1}^k (L_1 - 1)L_k = -L_{k-1}^k L_k + L_{k-1}^k (L_k + L_0 L_{k+1}) = L_{k-1}^k L_{k+1}.$$

By the hypothesis the formula is correct for $L_{k-1}^k L_k$. Obviously, $(L_1 - k)$ commutates with every elements of the same form. So (15) is correct.

Now it’s easy to prove next result.

**Lemma 5.** Let $V \subset M$ be a finite-dimensional space invariant with respect to $L_1$. Then there exists a decomposition in the direct sum of $L_1$-invariant subspaces $V = V_0 \oplus V_1 \oplus \ldots \oplus V_k$ where $L_1 v = iv$ for $v \in V_i$.

**Proof.** Since $\dim V < \infty$ there exists only finite number of the basis elements $v_1, \ldots, v_n$. Formula (15) for each $v_i$ gives:

$$L_{k-1}^k L_{k+1} v_i = L_1(L_1 - 1) \ldots (L_1 - k)v_i.$$

The left part of the equality is zero for sufficiently large $k$ due to locality. Choose $k = \max(k_1, \ldots, k_n)$. Then $P_k = t(t-1) \ldots (t-k)$ is such polynomial that $P_k(L_1)V = 0$, i.e. $P_k(L_1)$ is the annihilator of the space $V$. This polynomial has no multiple roots thus the minimal polynomial of the operator $L_1$ on $V$ has no multiple roots likewise, i.e. $L_1$ is semisimple and every eigenvalues are laying down into $\mathbb{Z}_+$.

Then required decomposition $V = V_0 \oplus V_1 \oplus \ldots \oplus V_k$ is exists but generally speaking there might be zero spaces in the direct sum.

Consider the algebra of endomorphisms $\text{End } M$ and subalgebra generated by operators $L_1$ and $L_{0}^{m} R_{m+1}$, $m \geq 0$:

$$A = \text{sub} \langle L_1, L_{0}^{m} R_{m+1} \mid m \geq 0 \rangle.$$
Lemma 6. Let $W = Au = \{ au \mid a \in A \}$ be an $A$-module for arbitrary $u \in M$. Then $\dim W < \infty$. 

Proof. It’s easy to see that $W \subseteq W' = \text{span}(L_0^mL_1^nR_{n+s}, m, s \geq 0, n \geq 1)$. Indeed, it follows from (5), (6), (7) that $W'$ is closed with respect to the action of generators of $A$. Let us show that $\dim W' < \infty$.

Firstly notice that the locality axiom implies $\dim V' < \infty$ where $V' = \text{span}(R_{n+s}(u), s \geq 0, n > 0)$. After,

$$L_1^m = L_1 + F(L_0L_2, \ldots, L_0^{m-2}L_{m-1}) + L_0^{m-1}L_m,$$

where $F$ is a linear function. Actually, using (5) obtain $L_1^2 = L_1 + L_0L_2, L_1^3 = L_1(L_1^2) = L_1(L_1 + L_0L_2) = L_1^2 + L_1L_0L_2 = L_1 + L_0L_2 + L_0(L_1 + 1)L_2 = L_1 + 2L_0L_2 + L_0L_1L_2 = L_1 + 2L_0L_2 + L_0L_2 + L_0^2L_3 = L_1 + 3L_0L_2 + L_0^2L_3$ and so on. Hence elements $L_1V'$ generate the same linear subspace as $L_0L_{k+1}V'$ which are finite dimensional by virtue of the locality axiom.

For end of the proof consider the algebra $A \subseteq \text{End}(M)$ defined above and the $A$-module $V = A\varphi_1$. Let $\varphi_1 \in M$ be a value of a 2-cocycle $\varphi$ and it fulfill the formula (14). For sufficiently large $m$ it turns to

$$(L_1 - m) \ldots (L_1 - 2)L_1\varphi_1 = \sum_{s=1}^{m} (L_1 - m) \ldots (L_1 - (s + 1))L_0^{s-1}R_s\varphi_1 \quad (16)$$

As it shown back in the proof the operator $L_1$ divides the algebra $V$ on the finite number of invariant subspaces hence

$$\varphi_1 = v_0 + \cdots + v_m,$$

where $L_1v_i = iv_i$. Substitute that decomposition in (16). Since there is a collection of nilpotent operators in the left part of the formula in result only $v_1$ remains:

$$\mp(m - 1)!v_1 = \sum_{s=1}^{m} (L_1 - m) \ldots (L_1 - (s + 1))L_0^{s-2}R_{s-1}\varphi_1$$
Use the operator $R_k$, $k \geq 2$ to both part of the equality and obtain

$$R_k v_1 = \text{Const} \cdot R_k \left( \sum_{s=2}^{m} (L_1 - m) \cdots (L_1 - s) L_0^{s-2} R_{s-1} - L_0^{m-1} R_m \right) \varphi_1.$$  

It follows from (6) and (7) that $R_k v_1 = 0$, for $k \geq 2$.

Initially find such $z \in V$ that $(L_1 - 1)z = v_0 + v_2 + v_3 \cdots + v_m = \varphi_1 - v_1$. It exists because $(L_1 - 1)$ annihilate only eigenspace $V_1$ and is non-degenerate in other. Choose a 1-cochain $\xi$ such that its value in a bimodule $M$ is $z$, i.e. $\xi(x) = z$. Denote $\varphi' = \varphi - \delta \xi \in Z^2(\text{End}_{1,x}, M)$. Since (6), (7) the cocycle $\varphi'$ has a property:

$$R_k \varphi'_1 = R_k(\varphi_1 - (L_1 + R_1 - 1)z) = R_k(\varphi_1 - (L_1 - 1)z - R_1 z) = R_k(v_1 - R_1 z) = 0$$

for $k \geq 2$.

Without loss of generality one can assume

$$R_k \varphi_1 = 0, \ k \geq 2. \quad (17)$$

Applying (5), (6), (7) to a decomposition of $a \varphi_1 \in A \varphi_1$, $a \in A$ we gather all operators $R_k$ at the right. Obtain $V = A \varphi_1 \subseteq \mathbb{F}[L_1, L_0] R_1 \varphi_1$. It’s easy to check using (7), (17) that $R_k V = 0$ for each $k \geq 2$.

Relying on this result, show there is $\psi_1$ satisfying the equality $(L_1 + R_1 - 1)\psi_1 = \varphi_1$.

Notice the identity $[L_k, R_1] = 0$, $k \geq 0$ is fulfilled in $\text{End}(V)$ ($[\cdot , \cdot]$ is a common commutator). Indeed, for $k = 0$ endomorphisms commutate. For other $k$ it follows because of $R_k V = 0$:

$$[L_k, R_1]v = x \circ_k (v \circ_1 x) - (x \circ_k v) \circ_1 x$$

$$= x \circ_k (v \circ_1 x) - \sum_{l=0}^{k} (-1)^l \binom{k}{l} x \circ_{k-l} (v \circ_{1+l} x)$$

$$= x \circ_k (v \circ_1 x) - x \circ_k (v \circ_1 x) + \sum_{l=1}^{k} (-1)^{l-1} \binom{k}{l} L_{k-l} R_{1+l} v = 0. \quad (18)$$

Also (6) implies that $R_1$ acts on $M$ as the projection operator, i.e. $R_1^2 = R_1$. Hence the space $V$ is decomposed in the direct sum of the kernel $V^{(0)}$ and fixed points $V^{(1)}$. It follows from (18) that $V^{(0)}$ and $V^{(1)}$ are closed with respect to the action of $L_1$. 

\begin{align*}
[L_k, R_1]v &= x \circ_k (v \circ_1 x) - (x \circ_k v) \circ_1 x \\
&= x \circ_k (v \circ_1 x) - \sum_{l=0}^{k} (-1)^l \binom{k}{l} x \circ_{k-l} (v \circ_{1+l} x) \\
&= x \circ_k (v \circ_1 x) - x \circ_k (v \circ_1 x) + \sum_{l=1}^{k} (-1)^{l-1} \binom{k}{l} L_{k-l} R_{1+l} v = 0. \quad (18)
\end{align*}
Having the decomposition $\varphi_1 = \varphi_1^{(0)} + \varphi_1^{(1)}$ it’s enough to show there exists such $\psi^{(0)}$, $\psi^{(1)}$ that

\begin{align}
(L_1 + R_1 - 1)\psi^{(0)} &= (L_1 - 1)\varphi_1^{(0)}, \\
(L_1 + R_1 - 1)\psi^{(1)} &= L_1\varphi_1^{(1)} = \varphi_1^{(1)}.
\end{align}

Values of the cocycle $\varphi_1$ satisfies to (14) and (19) hence

\begin{align}
0 &= (L_1 - m)\ldots(L_1 - 2)(L_1 - R_1)\varphi_1 \\
&= (L_1 - m)\ldots(L_1 - 2)(L_1 - R_1)\varphi_1^{(0)} + (L_1 - m)\ldots(L_1 - 2)(L_1 - R_1)\varphi_1^{(1)} \\
&= (L_1 - m)\ldots(L_1 - 2)L_1\varphi_1^{(0)} + (L_1 - m)\ldots(L_1 - 2)(L_1 - 1)\varphi_1^{(1)}.
\end{align}

First summand is the element of $V^{(0)}$, second summand is the element of $V^{(1)}$ and the sum might be zero if and only if each summand is zero:

\begin{align}
(L_1 - m)\ldots(L_1 - 2)L_1\varphi_1^{(0)} &= 0, \\
(L_1 - m)\ldots(L_1 - 2)(L_1 - 1)\varphi_1^{(1)} &= 0.
\end{align}

Therefore $\varphi_1^{(0)} \in V_0 \oplus V_2 \oplus \cdots \oplus V_k$ and the operator $L_1 - 1$ is non-degenerate in this direct sum of spaces. It implies there exists such $\psi^{(0)} \in V$ that $(L_1 - 1)\psi^{(0)} = \varphi_1^{(0)}$.

By the same way, $\varphi_1^{(1)} \in V_1 \oplus \cdots \oplus V_k$ and $L_1$ is non-degenerate there and hence exists $\psi^{(1)} \in V$ such that $L_1\psi^{(1)} = \varphi_1^{(1)}$.

Hereby it is shown existence of the element $\psi_1 = \psi^{(0)} + \psi^{(1)}$ which is required for the Lemma [3] The theorem is proved.

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