A non-existence result for a nonlinear Neumann problem

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Abstract. In this note we consider a semilinear elliptic equation in $B_R$ with the nonlinear boundary condition, where $B_R$ is a ball of radius $R$. Under certain conditions, we establish a sufficient condition on the non-existence of solutions provided that $R$ is sufficiently large. The main argument is based on applying the asymptotic analysis to the equation with respect to $R \gg 1$.

Keywords. Elliptic equation, Nonlinear Neumann boundary condition, Non-existence, Radially symmetric solutions

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1. Introduction

Let $R > 0$ and $N \geq 2$. We consider a semilinear elliptic equation

$$-u''(r) - \frac{N-1}{r}u'(r) + f(u(r)) = 0, \quad r \in (0, R),$$

with the nonlinear Neumann boundary condition

$$u'(0) = 0, \quad u'(R) = g(u(R)).$$

(1.1)–(1.2) is the $N$-dimensional radial version of the nonlinear Neumann problem (cf. [9, 10]):

$$
\begin{cases}
-\Delta u + f(u) = 0 & \text{in } B_R, \\
\frac{\partial u}{\partial \vec{v}} = g(u) & \text{on } \partial B_R,
\end{cases}
$$

where $\Delta$ stands for the Laplacian operator, $B_R$ is a ball of radius $R$ centered at the origin in $\mathbb{R}^N$, and $\frac{\partial}{\partial \vec{v}}$ is the normal derivative with respect to the unit outward normal vector $\vec{v}$ to $\partial B_R$. We refer the reader to [2, 5] for the physical background of nonlinear boundary conditions and references therein.

The associated energy functional of (1.3) is defined by

$$
\mathcal{E}[u] = \int_{B_R} \left( \frac{|\nabla u|^2}{2} + F(u) \right) \, dx - \int_{\partial B_R} \int_0^u g(t) \, dt \, d\sigma_x, \quad u \in H^1(B_R),
$$

where $F$ is a primitive of $f$:

$$
F(t) = \int_0^t f(s) \, ds.
$$

Under assumptions with physical meanings that $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and $g : \mathbb{R} \to \mathbb{R}$ is monotonically decreasing and non-negative, the author in his previous work [9] applied the standard direct method to $\mathcal{E}$ and established the existence of weak solutions to (1.3). Then, following the standard argument consisting of the maximum principle and the elliptic regularity theorem (cf. [6]), (1.3) he obtained the uniqueness of solutions to (1.3). Furthermore, by the uniqueness this solution is radially symmetric in $B_R$, and satisfies (1.1)–(1.2).

To the best of our knowledge, however, when $g$ is not necessary a decreasing function, the issue about the existence result of equation (1.3) and its radial version (1.1)–(1.2) remains to be open. Based on [9], in this note we shall focus on the radial version (1.1)–(1.2) and assume that

$$
f \in C(\mathbb{R}; \mathbb{R}) \text{ is strictly increasing}, \quad f(0) = 0 \text{ and } \liminf_{t \to 0} \frac{f(t)}{t} > 0.
$$

(1.5)

Moreover, by (1.5), $F \in C^1(\mathbb{R}; \mathbb{R})$ is strictly convex and has the minimum value $F(0) = 0$ in $\mathbb{R}$. We further make an assumption for $F$: there exists $\theta_0 > 1$ such that

$$
tf(t) \geq \theta_0 F(t) \quad \text{for } |t| \gg 1.
$$

(1.6)
Note that, by (1.6), there holds \( \lim_{|t| \to \infty} F(t) = \infty \). When \( \theta_0 > 2 \), (1.6) particularly implies that \( f \) and \( F \) are superlinear at infinity. Such an assumption was introduced by Ambrosetti and Rabinowitz [1]. Besides, an application of (1.3)–(1.6) is \( f(u) = \sinh u \) which appears in the so-called Poisson–Boltzmann equation \([3, 8, 10]\) and sinh–Gordon equation \([7]\).

With these properties of \( f \) and \( F \), we propose a condition of \( g \) for the non-existence of (1.1)–(1.2) as \( R > 0 \) is sufficiently large, which is stated as follows.

**Theorem 1.1.** Under (1.5)–(1.6), if \( g \in C(\mathbb{R}; \mathbb{R}) \) satisfies

\[
g^2(t) \neq 2F(t), \ \forall t \in \mathbb{R}, \text{ and } \lim_{|t| \to \infty} \frac{g^2(t)}{2F(t)} \neq 1,
\]

then there exists \( R^* = R^*(f, g) > 0 \) depending on \( f \) and \( g \) such that when \( R > R^* \), equation (1.1)–(1.2) has no solution.

An example of (1.7) is \( f(t) = \sinh t \) and \( g(t) = \pm \left(1 + 4 \sinh \frac{|t|}{2}\right) \). In Section 2 we will state the proof of Theorem 1.1.

**Remark 1.2.** It should be stressed that \( g \) satisfying (1.7) is not a decreasing function. Note that by (1.4) and (1.7) we have \( g(0) \neq 0 \). Without loss of generality, we may assume \( g(0) < 0 = F(0) \). Suppose on the contrary that \( g \) is decreasing on \( \mathbb{R} \). Then \( g^2 \) is increasing on \( (t_-, 0] \), where \( t_- \) is finite such that \( g(t_-) = 0 \) or \( t_- = -\infty \) if \( g < 0 \) on \((-\infty, 0)\). Since \( 2F \) is strictly decreasing on \((-\infty, 0)\) with \( \lim_{t \to -\infty} F(t) = \infty \) and \( g^2(0) > 2F(0) \), by the intermediate value theorem there exists \( t_0 \in (t_-, 0) \) such that \( g^2(t_0) = 2F(t_0) \). This contradicts to (1.7).

**Remark 1.3.** As an application of Theorem 1.1 we shall point out that if \( g \) satisfies one of the following statements (i) and (ii), then as \( R > 0 \) is sufficiently large, equation (1.1)–(1.2) has no solution:

(i) \( g(0) > 0 \) and \( \inf_{t \in \mathbb{R} \setminus \{0\}} \frac{g(t)}{\sqrt{2F(t)}} > 1 \);

(ii) \( g(0) < 0 \) and \( \sup_{t \in \mathbb{R} \setminus \{0\}} \frac{g(t)}{\sqrt{2F(t)}} < -1 \).

Theorem 1.1 also shows that under (1.5)–(1.6), if \( g \in C(\mathbb{R}; \mathbb{R}) \) satisfies (1.7), then as \( R > R^* \), equation (1.3) has no radially symmetric solution. Accordingly, we shall state a problem which, to the best of our knowledge, is unsolved.

**Open problem.** Assume that \( f \) and \( F \) satisfy (1.5)–(1.6) and \( g \) satisfies (1.7). Does there exist \( R_\ast > 0 \) such that for each \( R > R_\ast \), equation (1.3) has a solution which is not radially symmetric in \( B_R \)?

2. **Proof of Theorem 1.1**

We first consider a change of variables

\[
U(x) = u(r) \text{ with } x = \frac{r}{R} \text{ and } \varepsilon = \frac{1}{R} > 0.
\]

Then (1.1)–(1.2) is equivalent to the equation

\[
\begin{align*}
-\varepsilon^2 \left(U''(x) + \frac{N-1}{x} U'(x) \right) + f(U(x)) &= 0, \quad x \in (0, 1), \\
U'(0) &= 0, \quad \varepsilon U'(1) = g(U(1)).
\end{align*}
\]

Moreover, from (2.3) let us set

\[
U(1) = \lambda_\varepsilon, \quad \varepsilon U'(1) = g(\lambda_\varepsilon).
\]
Although the solution $U$ depends on the parameter $\varepsilon$ and should be denoted by $U_\varepsilon$, without the confusion we omit its subscript for a sake of simplicity.

Equation (2.2) with the boundary condition (2.1) is not an overdetermined problem since $\lambda_\varepsilon$ will be determined. Note also that assumption (1.5) implies $f(U(x)) = C(x)U(x)$ for some function $C(x) > 0$. Thus, for each $\varepsilon > 0$ and $\lambda_\varepsilon \in \mathbb{R}$, equation (2.2) with the boundary conditions $(U'(0), U(1)) = (0, \lambda_\varepsilon)$ satisfies the maximum principle and has a unique classical solution (see, e.g., [4] and [9, Section 2]).

To be specific we shall study the asymptotics (with respect to $\varepsilon \downarrow 0$) of solutions to equation (2.2) with boundary conditions $(U'(0), U(1)) = (0, \lambda_\varepsilon)$. In doing so, it is expected to obtain the refined asymptotics of $U'(1)$ so that we can further investigate $\lambda_\varepsilon$ via $\varepsilon U'(1) = g(\lambda_\varepsilon)$ with respect to $\varepsilon \downarrow 0$.

**Lemma 2.1.** Let $U$ be the unique classical solution of (2.2) with the boundary conditions $(U'(0), U(1)) = (0, \lambda_\varepsilon)$. Then, we have
\[
\min\{0, \lambda_\varepsilon\} \leq U \leq \max\{0, \lambda_\varepsilon\} \quad \text{and} \quad \lambda_\varepsilon U' \geq 0 \quad \text{on} \ [0, 1]. \tag{2.5}
\]

**Proof of Lemma 2.1.** We first assume $\lambda_\varepsilon \geq 0$. Suppose $U(0) < 0$. Then there exists $\delta > 0$ such that $U < 0$ on $[0, \delta)$. Along with (2.2), one may employ (1.5) to obtain $\varepsilon^2(x^{N-1}U'(x))' = x^{N-1}f(U(x)) < 0$ on $(0, \delta)$. In particular, $U' < 0$ on $(0, \delta)$. As a consequence, $U$ arrives at its minimum value at an interior point $x_0 \in (0, 1)$, and by (2.2) we get $f(U(x_0)) \geq 0$. This leads a contradiction since $U(x_0) \geq 0 > U(0)$. Hence, there holds $U(0) \geq 0$. Applying the maximum principle to (2.2), one arrives at $0 \leq U(0) \leq U(x) \leq U(1) = \lambda_\varepsilon$. Thus,
\[
\varepsilon^2(x^{N-1}U'(x))' = x^{N-1}f(U(x)) \geq 0 \quad \text{on} \ (0, 1),
\]
and we further obtain $U' \geq 0$ on $[0, 1]$.

Similarly, for the case $\lambda_\varepsilon < 0$ we have $\lambda_\varepsilon = U(1) \leq U(x) \leq U(0) \leq 0$ and $U' \leq 0$ on $[0, 1]$. This completes the proof of (2.5). □

When $\lambda_\varepsilon = 0$, (2.5) implies that equation (2.2) with the boundary condition $(U'(0), U(1)) = (0, \lambda_\varepsilon)$ only has a trivial solution $U \equiv 0$, together with (2.1) we obtain $g(0) = 0$. This is impossible due to (1.7). In what follows, without loss of generality, it suffices to consider the case $\lambda_\varepsilon > 0$. Hence, we have $U \geq 0$, $f(U) \geq 0$ and $U' \geq 0$ on $[0, 1]$. Moreover, (2.6) holds, and we have the following estimates.

**Lemma 2.2.** Under the same assumptions as in Lemma 2.1 we assume $\lambda_\varepsilon > 0$. Then, there exists a positive constant $M$ independent of $\varepsilon$ such that as $\varepsilon \in (0, \frac{M}{\sqrt{2(N-1)}})$, we have
\[
0 \leq U(x) \leq 2\lambda_\varepsilon \exp\left(-\frac{M}{\varepsilon}(1 - x)\right), \quad x \in [0, 1], \tag{2.7}
\]
and that:

(i) If $\limsup_{\varepsilon \downarrow 0} \lambda_\varepsilon < \infty$, then
\[
\lim_{\varepsilon \downarrow 0} \frac{g^2(\lambda_\varepsilon)}{2} - F(\lambda_\varepsilon) = 0. \tag{2.8}
\]

(ii) If $\lambda_\varepsilon \nrightarrow \infty$, then
\[
\lim_{\varepsilon \downarrow 0} \frac{g^2(\lambda_\varepsilon)}{2F(\lambda_\varepsilon)} = 1. \tag{2.9}
\]

**Proof of Lemma 2.2.** Multiplying (2.2) by $U$ and using (1.5) and (1.6), one may check that
\[
\frac{\varepsilon^2}{2} (U^2(x))'' = \varepsilon^2 \left( (U'(x))^2 - \frac{N - 1}{x} U(x)U'(x) \right) + U(x)f(U(x)) \geq \left( M^2 - \varepsilon^2 \frac{(N-1)^2}{4x^2} \right) U^2(x).
\]
Here we have used (1.5) and (1.6) to verify a positive constant $M$ independent of $\varepsilon$ such that $tf(t) \geq Mt^2$ for all $t \in \mathbb{R}$. As a consequence, for $0 < \varepsilon^* < \frac{\sqrt{M}}{N-1}$, we have
\[
\varepsilon^2 (U^2(x))^\nu \geq M^2 U^2(x), \quad x \in \left[ \frac{N-1}{\sqrt{2M}}, 1 \right) \text{ and } \varepsilon \in (0, \varepsilon^*).
\]
Along with (2.5) for $\lambda_\varepsilon > 0$, we follow the comparison theorem to obtain
\[
0 \leq U(x) \leq \lambda_\varepsilon \left( \exp \left( -\frac{M(x - \frac{N-1}{\sqrt{2M}})}{\varepsilon^*} \right) + \exp \left( -\frac{M(1 - x)}{\varepsilon} \right) \right), \quad (2.10)
\]
for $x \in \left[ \frac{N-1}{\sqrt{2M}}, 1 \right)$ and $\varepsilon \in (0, \varepsilon^*)$. In particular, for $x \in [0, \frac{1}{2} \left( \frac{N-1}{\sqrt{2M}} + 1 \right)]$ with $\varepsilon^* = \frac{M}{\sqrt{2(N-1)}}$, (2.5) and (2.10) imply
\[
0 \leq U(x) \leq U \left( \frac{1}{2} \left( \frac{N-1}{\sqrt{2M}} + 1 \right) \right) \leq 2\lambda_\varepsilon \exp \left( -\frac{M}{2\varepsilon} \left( 1 - \frac{N-1}{\sqrt{2M}} \varepsilon^* \right) \right) = 2\lambda_\varepsilon \exp \left( -\frac{M}{4\varepsilon} \right) \quad (2.11)
\]
for $x \in [0, \frac{3}{4}]$ and $\varepsilon \in (0, \frac{M}{\sqrt{2(N-1)}})$.

On the other hand, by (2.10) with $\varepsilon^* = \frac{M}{\sqrt{2(N-1)}}$, we have
\[
0 \leq U(x) \leq \lambda_\varepsilon \left( \exp \left( -\frac{M(x - \frac{1}{2})}{\varepsilon} \right) + \exp \left( -\frac{M(1 - x)}{\varepsilon} \right) \right) \leq 2\lambda_\varepsilon \exp \left( -\frac{M}{\varepsilon} \left( 1 - x \right) \right), \quad \text{for } x \in \left[ \frac{3}{4}, 1 \right) \text{ and } \varepsilon \in (0, \frac{M}{\sqrt{2(N-1)}}).
\]
Therefore, (2.7) follows from (2.11) and (2.12).

It remains to prove (2.8) – (2.9). Multiplying (2.6) by $x^{N-1}U'(x)$, one may check via simple calculations that
\[
\left( \frac{\varepsilon^2}{2} x^{2N-2} U'^2(x) - x^{2N-2} F(U(x)) \right)' = -(2N-2)x^{2N-3} F(U(x)) \quad (2.13)
\]
Integrating (2.13) over the interval $(0,1)$ and using (2.4), we have
\[
\frac{g^2(\lambda_\varepsilon)}{2} - F(\lambda_\varepsilon) = -(2N-2) \int_0^1 x^{2N-3} F(U(x)) dx \quad (2.14)
\]
and the following two cases for the estimate of $\int_0^1 x^{2N-3} F(U(x)) dx$:

**Case 1.** When $\limsup_{\varepsilon \downarrow 0} \lambda_\varepsilon < \infty$, we assume $0 < \lambda_\varepsilon \leq L^*$ as $0 < \varepsilon \ll 1$, where $L^* > 0$ is independent of $\varepsilon$. Then, by (1.5) and (2.7),
\[
0 \leq \int_0^1 x^{2N-3} F(U(x)) dx \leq f(L^*) \int_0^1 U(x) dx \leq \frac{2L^* f(L^*)}{M} \varepsilon \xrightarrow{\varepsilon \downarrow 0} 0. \quad (2.15)
\]

**Case 2.** When $\lambda_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \infty$, we notice that, by (1.4) and (1.6), \( (\frac{F(t)}{t})' = \frac{tf(t) - F(t)}{t^2} \geq \frac{\theta_0 - 1}{t^2} F(t) > 0 \) for $t \gg 1$. Hence, (2.5) gives $\sup_{[0,1]} \frac{F(U)}{U} = \frac{F(U(1))}{U(1)} \leq \frac{F(\lambda_\varepsilon)}{\lambda_\varepsilon}$, and
\[
0 \leq \int_0^1 x^{2N-3} F(U(x)) dx \leq \left( \sup_{[0,1]} \frac{F(U)}{U} \right) \int_0^1 U(x) dx 
\leq \left( \frac{F(\lambda_\varepsilon)}{\lambda_\varepsilon} \right) \frac{2\lambda_\varepsilon \varepsilon}{M} = \frac{2F(\lambda_\varepsilon)}{M} \varepsilon. \quad (2.16)
\]
As a consequence, by (2.14) and (2.15), we prove (2.8); by (2.14) and (2.16), we prove (2.9). Thus, the proof of Lemma 2.2 is completed.

Having Lemma 2.2 in hands, we state the proof of Theorem 1.1 as follows.

Proof of Theorem 1.1. Suppose on the contrary that there exists a strictly increasing sequence $R_i \to \infty$ such that for each equation (1.1)–(1.2) corresponding to $R = R_i$ has a classical solution $u_i$. Then by (2.1) we set $\varepsilon_i = \frac{1}{R_i} \to 0$ and $\lambda_{\varepsilon_i} = U_i(1) = u_i(R_i)$. Note that the sequence $\{\lambda_{\varepsilon_i}\}_{i \in \mathbb{N}}$ contains infinitely many members of non-negative numbers or non-positive numbers. Hence, without loss of generality, we may assume $\lambda_{\varepsilon_i} > 0$, $\forall i \in \mathbb{N}$. (As mentioned previously, if $\lambda_{\varepsilon_i} = 0$, then $U_i \equiv 0$ on $(0, 1)$ and $g(0) = 0$ which is impossible!)

We now consider two situations for $\{\lambda_{\varepsilon_i}\}_{i \in \mathbb{N}}$. If $\limsup_{i \to \infty} \lambda_{\varepsilon_i} = \lambda^* < \infty$, then there exists a subsequence $\{\lambda_{\varepsilon_{n_i}}\}$ such that $\lim_{n_i \to \infty} \lambda_{\varepsilon_{n_i}} = \lambda^*$. Since both $g$ and $F$ are continuous on $\mathbb{R}$, by Lemma 2.2(i) we obtain $g^2(\lambda^*) = 2F(\lambda^*)$ which contradicts to (1.7). On the other hand, if $\lim_{i \to \infty} \lambda_{\varepsilon_i} = \infty$, then by Lemma 2.2(ii) we have $\lim_{i \to \infty} \frac{g^2(\lambda_{\varepsilon_i})}{2F(\lambda_{\varepsilon_i})} = 1$ which still contradicts to (1.7). Therefore, there exists $R^* = R^*(f, g) > 0$ depending on $f$ and $g$ such that when $R > R^*$, equation (1.1)–(1.2) has no solution. We thus complete the proof of Theorem 1.1.

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