CANONICAL BILINEAR FORM AND EULER CHARACTERS

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ABSTRACT. An explicit formula for the canonical bilinear form on the Grothendieck ring of the Lie supergroup $GL(n, m)$ is given. As an application we get an algorithm for the decomposition Euler characters in terms of characters of irreducible modules in the category of partially polynomial modules.

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1. INTRODUCTION

Let $\mathcal{F}$ be the category of finite dimensional modules over complex algebraic group $GL(n)$ and $K(\mathcal{F})$ be its Grothendieck ring. There exists a natural pairing on the ring $K(\mathcal{F})$

$$([U], [V]) = \dim \text{Hom}_{GL(n)}(U, V)$$

Let us identify the ring $K(\mathcal{F})$ with the ring of symmetric Laurent polynomials $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$ then the above pairing in terms of characters can be expressed in the following form

$$([U], [V]) = \left[(ch U)^* ch V \prod_{i \neq j} \left(1 - \frac{x_i}{x_j}\right)\right]_0$$

where $[f]_0$ means the constant term of Laurent polynomial and $f^* (x_1, \ldots, x_n) = f(x_1^{-1}, \ldots, x_n^{-1})$ (see [10, 11]). This formula is very interesting from many points of view. On one side it allows to connect problems in representation
theory to some problems with symmetric functions and their generalisations including Jack and Macdonald polynomials. On the other side the above formula can be extended to the root system of any semisimple Lie algebra and the corresponding analogues of symmetric polynomials. The main goal of this paper is to prove the same kind of formula for complex algebraic Lie supergroup $GL(m,n)$ and to illustrate some of its applications. The category of finite dimensional representations of $GL(m,n)$ is not semisimple. Therefore in this case we have the natural pairing only between projective modules $P(\mathcal{F})$ and finite dimensional modules $K(\mathcal{F})$ (see [1], [7])

$$P(\mathcal{F}) \times K(\mathcal{F}) \rightarrow \mathbb{Z}, \quad ([U],[V]) = \dim \text{Hom}_{GL(m,n)}(U,V)$$

where $[L]$ means the class of module $L$ in appropriate version of Grothendieck ring. We also should mention that the category of partially polynomials modules defined below is a convenient object from the point of view of our bilinear form.

2. Preliminaries

Instead of general linear complex algebraic supergroup $GL(m,n)$ it is more convenient to deal with the complex Lie superalgebra $gl(m,n)$. So we will consider only finite dimensional representations of $gl(n,m)$ such that every one of them can be lifted to the representation of $GL(m,n)$.

Let us remind that the Lie superalgebra $gl(m,n)$ is the Lie superalgebra of the linear transformations of a $\mathbb{Z}_2$ graded vector space $V = V_0 \oplus V_1$ ($V$ is also called the standard representation of $g$). We have

$$g_0 = gl(m) \oplus gl(n), \quad g_1 = V_0 \otimes V_1^* \oplus V_1 \otimes V_0^*$$

We also have $\mathbb{Z}$ graded decomposition $g = g_- \oplus g_0 \oplus g_1$ where

$$g_- = V_1 \otimes V_0^*, \quad g_1 = V_0 \otimes V_1^*,$$

Let us fix bases in $V_0 =$ $< e_1, \ldots, e_m >$ and $V_1 =$ $< f_1, \ldots, f_n >$ respectively. Let $b$ be the subalgebra of upper triangular matrix in $gl(m,n)$ and $k$ be the subalgebra of diagonal matrix in $gl(m,n)$ in the above basis. By $\varepsilon_1, \ldots , \varepsilon_m, \delta_1, \ldots , \delta_n$ we will denote the weights of standard representation with respect to $f$. The corresponding system of positive roots $R^+ = R^+_0 \cup R^+_1$ of $gl(m,n)$ can be described in the following way

$$R^+_0 = \{ \varepsilon_i - \varepsilon_j : 1 \leq i < j \leq m : \delta_k - \delta_l, 1 \leq k < l \leq n \}$$

$$R^+_1 = \{ \varepsilon_i - \delta_k, 1 \leq i \leq m, 1 \leq k \leq n \}$$

Let also

$$P = \{ \chi = \lambda_1 \varepsilon_1 + \cdots + \lambda_m \varepsilon_m + \mu_1 \delta_1 + \cdots + \mu_n \delta_n, \ | \ n_i, m_j \in \mathbb{Z} \}$$

be the weight lattice and

$$P^+ = \{ \chi \in P \ | \ \lambda_i - \lambda_j \geq 0, i < j : \mu_k - \mu_l \geq 0, k < l \}$$

be the set of highest weights.
We will use the following parity on the weight lattice due to C. Gruson and V. Serganova [5] and Brundan and Stroppel [3] by saying that $\varepsilon_i$ (resp. $\delta_j$) is even (resp. odd). It is easy to check that every finite dimensional module $L$ can be represented in the form

$$L = L^+ \oplus L^-$$

where $L^+$ is the submodule of $L$ in which weight space has the same parity as the corresponding weight and $L^-$ is the submodule in which the parities differ. We should note that this construction is a particular case of Deligne construction category $Rep(G, z)$ from the paper [4] for $G = GL(m, n)$ and $z = diag(1, \ldots, 1, -1, \ldots, -1)$.

Let us denote by $\mathcal{F}$ the category of finite dimensional modules over $\mathfrak{gl}(n, m)$ such that every module in $\mathcal{F}$ is semisimple over Cartan subalgebra $\mathfrak{k}$ and all its weights are in $P$. By $K(\mathcal{F})$ we will denote the quotient of the Grothendieck ring of $\mathcal{F}$ by the relation $[L] - [\Pi(L)] = 0$ where $\Pi(L)$ is the module with the shifted parity $\Pi(L)_0 = L_1, \Pi(L)_1 = L_0$ and $x*y = (-1)^{p(x)} xv, x \in \mathfrak{gl}(m, n)$. For every $L \in \mathcal{F}$ we can define

$$\text{ch} L = \sum_{\chi} \dim L_{\chi} e^{\chi}$$

where the sum is taken over all weights of $L$. It is easy to see that $\text{ch} L$ is well defined function on $K(\mathcal{F})$.

The ring $K(\mathcal{F})$ can be describe explicitly in the following way. Let

$$P_{m,n} = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$$

be the ring of Laurent polynomials in variables $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$.

If we set $x_i = e^{\varepsilon_i}$, $y_j = e^{\delta_j}$ then we get a character map

$$ch : K(\mathcal{F}) \rightarrow P_{m,n}$$

Let also

$$\Lambda_{m,n}^\pm = \{ f \in P_{m,n}^{S_m \times S_n} \ | \ x_i \frac{\partial f}{\partial x_i} + y_j \frac{\partial f}{\partial y_j} \in (x_i + y_j) \}$$

be the subring of $P_{m,n}$ of supersymmetric Laurent polynomials.

**Theorem 2.1.** [15] The ring $K(\mathcal{F})$ is isomorphic to the ring $\Lambda_{m,n}^\pm$ under the character map.

**Remark 2.2.** Actually in the paper [15] slightly different versions of Grothendieck ring and the algebra $\Lambda_{m,n}^\pm$ were considered. But it is easy to check that they are isomorphic to our ones. We prefer to use characters instead of supercharacters in this paper in order to avoid some unnecessary signs.

It will be needed later an explicit description of the projective covers of the irreducible finite dimensional modules due to Brundan [1]. We give the description here in a slightly different way.
First let us for any $\chi \in P^+$ define a pair of sets
\[ A = \{(\chi + \rho, \varepsilon_1), \ldots, (\chi + \rho, \varepsilon_m)\}, \quad B = \{(\chi + \rho, \delta_1), \ldots, (\chi + \rho, \delta_n)\} \]
where
\[ \rho = \frac{1}{2} \sum_{\alpha \in R_0^+} \alpha - \frac{1}{2} \sum_{\alpha \in R_1^+} \alpha + \frac{1}{2}(n-m+1)(\sum_{i=1}^m \varepsilon_i - \sum_{j=1}^n \delta_j) \]
\[ = \sum_{i=1}^m (1-i)\varepsilon_i + \sum_{j=1}^n (m-j)\delta_j \]

Our $\rho$ is slightly different from the standard one but it is more convenient since the elements of $A$ and $B$ are integers. So instead of highest weights we will use the set of pairs $(A, B)$ such that $A, B \subset \mathbb{Z}$ and $|A| = m, |B| = n$. We will also use the language of diagrams which is due to Brundun and Stroppel [3] but we will use it here in a form due to I. Musson and V. Serganova [9].

**Definition 2.3.** Let $(A, B)$ be a pair of subsets in $\mathbb{Z}$ such that $|A| = m, |B| = n$. Then the corresponding diagram is the following function on $\mathbb{Z}$
\[
f(x) = \begin{cases} 
\times, & x \in A \cap B \\
\circ, & x \in A' \cup B' \\
> , & x \in A \setminus B \\
< , & x \in B \setminus A 
\end{cases}
\]

Let us also set
\[ \varphi(\times) = 1, \quad \varphi(\circ) = -1, \quad \varphi(>) = \varphi(<) = 0 \]
\[ [a,b] = \{c \in \mathbb{Z} \mid a \leq c \leq b\}, \quad \{a,b\} = \{c \in \mathbb{Z} \mid a \leq c < b\}, \]
\[ (a,b) = \{c \in \mathbb{Z} \mid a < c < b\} \]
and for integers $a < b$ let us define a transposition
\[ \pi^b_a : \mathbb{Z} \rightarrow \mathbb{Z}, \quad \pi^b_a(x) = \begin{cases} 
x, & x \neq a, b \\
b, & x = a \\
a, & x = b 
\end{cases} \]

**Definition 2.4.** We will call a transposition $\pi^b_a$ an admissible for $f$ if $a \in f^{-1}(\times), b \in f^{-1}(\circ)$ and the following conditions are fulfilled
\[ b > a, \quad \sum_{i \in [a,b]} (\varphi \circ f)(i) = 0, \quad \sum_{i \in [a,c]} (\varphi \circ f)(i) > 0, \text{ for any } c \in [a,b). \]

Since $b$ is uniquely defined by $f$ and $a$ we sometimes will omit $b$.

The following Lemma easily follows from the definition above.
Lemma 2.5. The following statements hold true

1) If \( \pi_a^b, \pi_c^d \) are two admissible transpositions for \( f \) then one of the following conditions is fulfilled
\[
[a, b] \cap [c, d] = \emptyset, \ [a, b] \subset (c, d), \ [c, d] \subset (a, b)
\] (1)

2) Let \( \pi_a^b \) be an admissible transposition for \( f \) and \( d \in [a, b] \) be such that \( f(d) = \circ \). Then there exist an admissible transposition for \( f \) of the form \( \pi_c^d \).

Corollary 2.6. Admissible transpositions pairwise commute.

Proof. It easily follows from Lemma 2.5. \( \square \)

Now let us define for a diagram \( f \) and any \( C \subset f^{-1}(\times) \) the permutation of \( \mathbb{Z} \) by the formula
\[
\pi_C = \prod_{c \in C} \pi_c \quad (2)
\]

We should mention that the above product is well defined since admissible transpositions commute with each other.

Definition 2.7. Let \( P(f) \) be the projective cover of irreducible module \( L(f) \). We will denote by \( P(f) \) the set of \( g \) such that \( K(g) \) is a subquotient of \( P(f) \).

Now we can formulate the main result of Brundan [1].

Theorem 2.8. \( P(f) \) has a multiplicity free Kac flag and
\[
P(f) = \{ g \mid g = \pi_C(f), C \subset f^{-1}(\times) \}
\]

In order to get an algorithm for the decomposition Kac modules into the sum of irreducible modules we need the following combinatorial Lemma.

Lemma 2.9. Let \( f, g \) be such diagrams that
\[
g = \tau_r \circ \tau_2 \circ \cdots \circ \tau_1(f)
\]
where \( \tau_i = \pi_{a_i}^{b_i} \) is a transposition and \( f_i = \tau_{i-1} \circ \cdots \circ \tau_1(f) \), \( i = 1, \ldots, r \).

Suppose also that for any pair of \( i > j \) we have
\[
[a_i, b_i] \cap [a_j, b_j] = \emptyset, \text{ or } [a_i, b_i] \subset (a_j, b_j)
\] (3)

Then for any \( i = 1, \ldots, r \) the transposition \( \tau_i \) is admissible for \( f \) if and only if \( \tau_i \) is admissible for \( f_i, i = 1, \ldots, r \).

Proof. Let us prove first that functions \( f_i \) and \( f \) coincide on the segment \( [a_i, b_i] \) for \( 1 \leq i \leq r \). The following equalities are easy to check
\[
\varphi \circ f_i = \varphi \circ f + 2 \sum_{j=1}^{i-1} (\delta_{b_j} - \delta_{a_j}), \quad i = 1, \ldots, r
\]
\[
f_i^{-1}(\times) = (f^{-1}(\times) \setminus \{a_1, \ldots, a_{i-1}\}) \cup \{b_1, \ldots, b_{i-1}\}
\]

If \( t \in [a_i, b_i] \) then from the conditions of the Lemma it follows that \( \delta_{a_j}(t) = \delta_{b_j}(t) = 0 \) for any \( 1 \leq j < i \). Therefore \( \varphi \circ f_i(t) = \varphi \circ f(t) \) on the segment \( [a_i, b_i] \). Now Lemma follows from Definition 2.3. \( \square \)
We will use induction on \( f \).

Let us prove by induction that conditions of Lemma 2.9 are fulfilled.

The last condition is impossible since \( a_r = a_i \). Therefore by Lemma 2.9 transposition \( \pi_{a_r}^c \) is admissible for \( f \). Therefore \( \pi_{a_r}^b \pi_{a_r}^c = \pi_{a_r}^c \) is admissible for \( f \).

**Corollary 2.10.** We will keep the notations from Lemma 2.9. Suppose that \( \tau_1 = \pi_{a_1}^{b_1}, \ldots, \tau_r = \pi_{a_r}^{b_r} \) is a set of transpositions such that \( a_1 < a_2 < \cdots < a_r \) and \( a_i \neq b_j, 1 \leq i, j \leq r \). Suppose also that for any \( i = 1, \ldots, r \) transposition \( \tau_i \) is admissible for \( f_i \). Then all transpositions \( \tau_1, \ldots, \tau_r \) are admissible for \( f \).

**Proof.** Let us prove by induction that conditions of Lemma 2.9 are fulfilled. We will use induction on \( r \). If \( r = 1 \) then the statement of the Lemma is trivial. Let \( r > 1 \). By inductive assumption transpositions \( \tau_1, \ldots, \tau_{r-1} \) are admissible for \( f \). Therefore

\[
f_r^{-1}(x) = (f^{-1}(x) \setminus \{a_1, \ldots, a_{r-1}\}) \cup \{b_1, \ldots, b_{r-1}\}
\]

Since \( \tau_r \) is admissible for \( f \) we have \( a_r \in f_r^{-1}(x) \). By assumptions of the Lemma \( a_r \neq b_1, \ldots, b_{r-1} \) therefore \( a_r \in f^{-1}(x) \). Let \( \pi_{a_r}^c \) be the corresponding admissible transposition for \( f \). Then for any \( i \leq r \) one of the following conditions holds true

\[
[a_i, b_i] \cap [a_r, c] = \emptyset, \ [a_r, c] \subset (a_i, b_i), \ [a_i, b_i] \subset (a_r, c)
\]

The last condition is impossible since \( a_r > a_i \). Therefore by Lemma 2.9 transposition \( \pi_{a_r}^c \) is admissible for \( f \). Therefore \( \pi_{a_r}^b \pi_{a_r}^c = \pi_{a_r}^c \) is admissible for \( f \).

**Corollary 2.11.** Suppose that \( \tau_1 = \pi_{a_1}^{b_1}, \ldots, \tau_r = \pi_{a_r}^{b_r} \) is a set of transpositions such that \( b_1 > b_2 > \cdots > b_r \) and \( a_i \neq b_j, 1 \leq i, j \leq r \). Suppose also that \( \tau_i \) is admissible for \( f_i, i = 1, \ldots, n \). Then \( \tau_i, i = 1, \ldots, r \) is admissible for \( f \).

**Proof.** Let us prove by induction that conditions of [3] are fulfilled. We will use induction on \( r \). If \( r = 1 \) then the statement of the Lemma is trivial. Let \( r > 1 \). By inductive assumption the conditions of [3] are fulfilled therefore by Lemma 2.9 transpositions \( \tau_1, \ldots, \tau_{r-1} \) are admissible for \( f \). Therefore

\[
f_r^{-1}(x) = (f^{-1}(x) \setminus \{a_1, \ldots, a_{r-1}\}) \cup \{b_1, \ldots, b_{r-1}\}
\]

Since \( \tau_r \) is admissible for \( f \) we have \( a_r \in f_r^{-1}(x) \). By our assumptions \( a_r \neq b_1, \ldots, b_{r-1} \) therefore \( a_r \in f^{-1}(x) \). Besides since \( b_r \neq a_1, \ldots, a_{r-1} \) we have \( b_r \in f^{-1}(c) \). Let \( \pi_{a_r}^c \) be the corresponding admissible transposition for \( f \). Suppose that \( b_r < c \), then \( b_r \in [a_r, c] \). Therefore by Lemma 2.5 there exist an admissible for \( f \) transposition \( \pi_{a_r}^{b_r} \). Then for any \( i < r \) one of the following conditions holds true

\[
[a_i, b_i] \cap [a_r, b_r] = \emptyset, \ [a_r, b_r] \subset (a_i, b_i), \ [a_i, b_i] \subset (a_r, b_r)
\]

The last condition is impossible since \( b_r < b_i \). Therefore by Lemma 2.9 transposition \( \pi_{a_r}^{b_r} \) is admissible for \( f \). Therefore \( \pi_{a_r}^{b_r} \pi_{a_r}^c = \pi_{a_r}^c \) is admissible for \( f \). If \( b_r \geq c \) then again condition \( [a_i, b_i] \subset (a_r, c) \) is impossible and \( \pi_{a_r}^{b_r} = \pi_{a_r}^c \) is admissible for \( f \). \( \square \)
Corollary 2.12. Irreducible module \( L(f) \) is a subquotient of Kac module \( K(g) \) if and only if there exist a sequence of transpositions
\[
\sigma_1 = \pi_{c_1}^{d_1}, \ldots, \sigma_r = \pi_{c_r}^{d_r}
\]
where \( c_i < d_i, i = 1, \ldots, r \) such that
1) \( \sigma_i \) is admissible for \( \sigma_i \circ \cdots \circ \sigma_1(g) \), \( i = 1, \ldots, r \) and \( \sigma_r \circ \cdots \circ \sigma_1(g) = f \)
2) \( c_1 > c_2 > \cdots > c_r \)
3) \( c_i \neq d_j, 1 \leq i, j \leq r \)

Proof. Suppose that all conditions of the Corollary are fulfilled. Then
\[
g = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_r (f)
\]
If we set \( \tau_i = \sigma_{r+1-i}, a_i = c_{r-i+1}, b_i = d_{r-i+1} \) where \( i = 1, \ldots, r \) then it is easy to see that all conditions of Corollary 2.10 are fulfilled. Therefore \( K(g) \) is a subquotient of \( P(f) \). Therefore by BGG reciprocity \( L(f) \) is a subquotient of \( K(g) \).

Now let us suppose that \( L(f) \) is a subquotient of Kac module \( K(g) \). Then again by BGG reciprocity \( K(g) \) is a sub quotient of \( P(f) \). Therefore by Theorem 2.8 \( g = \pi_A(f), A \subset f^{-1}(\times) \). Let \( A = \{a_1, a_2, \ldots, a_r\} \) where \( a_1 < a_2 < \cdots < a_r \). Since admissible transpositions pairwise commute we have
\[
g = \tau_r \circ \tau_2 \circ \cdots \circ \tau_1 (f)
\]
where \( \tau_i = \pi_{a_i}^{b_i} \). Let us check that conditions 3 are fulfilled. It is enough to verify that inclusion \([a_j, b_j] \subset (a_i, b_i)\) is impossible if \( i > j \). Indeed if it is so then \( a_j > a_i \) and we get a contradiction. Therefore by Lemma 2.9 \( \tau_i \) is admissible for \( f_i = \tau_{i-1} \circ \cdots \circ \tau_1(f) \) and we can set
\[
\sigma_i = \tau_{r-i+1}, c_i = a_{r-i+1}, d_i = b_{r-i+1} \quad i = 1, \ldots, r.
\]

\[\square\]

In the same way we can prove the following Corollary.

Corollary 2.13. Irreducible module \( L(f) \) is a subquotient of Kac module \( K(g) \) if and only if there exist a sequence of transpositions
\[
\sigma_1 = \pi_{c_1}^{d_1}, \ldots, \sigma_r = \pi_{c_r}^{d_r}
\]
where \( c_i < d_i, i = 1, \ldots, r \) such that
1) \( \sigma_i \) is admissible for \( \sigma_i \circ \cdots \circ \sigma_1(g) \), \( i = 1, \ldots, r \) and \( \sigma_r \circ \cdots \circ \sigma_1(g) = f \)
2) \( d_1 < d_2 < \cdots < d_r \)
3) \( c_i \neq d_j, 1 \leq i, j \leq r \)

The above corollaries can be used to calculate the irreducible subquotients of Kac modules.

Definition 2.14. \( K(g) = \{ f \mid \text{Hom}_g(P(f), K(g)) \neq 0 \} \)
Example 2.15. Let \( m = n = 2 \) and \( g^{-1}(\times) = \{2, 3\} \). We are going to describe the set \( \mathcal{K}(g) \).

As the first step we are going to find a transpositions \( \pi^b_a \) such that \( b \in g^{-1}(\times) \), and \( \pi^b_a \) is admissible for \( \pi^b_a(g) \). And it is easy to see that there exists only one such transposition \( \pi_1^2 \).

The next step is to find a transposition \( \pi^b_a \) such that \( b \in \pi_1^2(g)^{-1}(\times) \), \( \pi^b_a \) is admissible for \( \pi_0^a \circ \pi_1^2(g) \) and \( a < 1 \). It is easy to check that there exists only one such transposition \( \pi_3^0 \). So we have

\[
\mathcal{K}(g) = \{g, \pi_1^2(g), \pi_0^3 \circ \pi_1^2(g)\}
\]

Remark 2.16. We should mention that our algorithm is essentially the same as in the paper [9]. Legal move of weight zero \( g \xrightarrow{[b,a]} f \), \( a < b \) in the sense of [9] is the same as \( \sigma = \pi^b_a \) is an admissible transposition for \( f = \sigma(g) \). And a regular increasing pass from \( g \) to \( f \) is the same as the sequence of transpositions

\[
\sigma_1 = \pi^b_{a_1}, \ldots, \sigma_r = \pi^b_{a_r}, \quad a_1 < b_1, \ldots, a_r < b_r
\]

such that

1) \( \sigma_i \) is admissible for \( \sigma_i \circ \cdots \circ \sigma_1(g) \), \( i = 1, \ldots, r \) and \( \sigma_r \circ \cdots \circ \sigma_1(g) = f \)
2) \( b_1 < b_2 < \cdots < b_r \)
3) \( a_i \neq b_j, 1 \leq i, j \leq r \).

3. A bilinear form on the ring \( P_{m,n} \)

In this section we are going to define a bilinear form on the ring of Laurent polynomials \( P_{m,n} \) and connect this bilinear form with the canonical bilinear form on the Grothendieck ring of Lie superalgebra \( \mathfrak{g} = \mathfrak{gl}(m,n) \). Let \( p \rightarrow p^* \) be the following automorphism of \( P_{m,n} \)

\[
x_i^* = x_i^{-1}, \quad i = 1, \ldots, m, \quad y_j^* = y_j^{-1}, \quad j = 1, \ldots, n
\]

Definition 3.1. Let us set

\[
\Delta(x) = \prod_{i > j} \left(1 - \frac{x_i}{x_j} \right), \quad \Delta(y) = \prod_{i > j} \left(1 - \frac{y_i}{y_j} \right), \quad \Delta(x, y) = \prod_{i,j} \left(1 + \frac{y_j}{x_i} \right)
\]

and for \( p, q \in P_{m,n} \) let us define

\[
(p, q) = \frac{1}{m! n!} \left[ p^* q \frac{\Delta(x) \Delta(x)^* \Delta(y) \Delta(y)^*}{\Delta(x, y) \Delta(x, y)^*} \right]_0
\]

where \( [\cdot, \cdot]_0 \) means the constant term and \( (\Delta(x, y) \Delta(x, y)^*)^{-1} \) should be understood as

\[
(\Delta(x, y) \Delta(x, y)^*)^{-1} = \frac{(y_1 \cdots y_n)^m}{(x_1 \cdots x_m)^n} \prod_{i,j} \left(1 + \frac{y_j}{x_i} \right)^{-2}
\]
Theorem 3.2. The following equality hold true
\[ \dim \text{Hom}_g(P, L) = (ch P, ch L) \]
where \( P \) is a finite dimensional projective module, \( L \) is any finite dimensional module.

Proof. We are going to prove the Theorem in several steps. First we are going to prove that characters of Kac modules are pairwise orthogonal with respect to the pairing \(( , )\). Let \( K(f), K(g) \) be two Kac modules and \( \chi = (\lambda, \mu) \) and \( \tilde{\chi} = (\nu, \tau) \) are the corresponding highest weights, where \( \lambda, \nu \) are highest weights of \( \mathfrak{gl}(m) \) and \( \mu, \tau \) are highest weights of \( \mathfrak{gl}(n) \). Then we have
\[ chK(f) = \Delta(x, y)s_\lambda(x)s_\mu(y), \quad chK(g) = \Delta(x, y)s_\nu(x)s_\tau(y), \]
where \( s_\lambda, s_\mu, s_\nu, s_\tau \) are Schur functions. Therefore we have
\[ (f, g) = \frac{1}{m!} \left[ s_\lambda^t s_\mu^* \Delta(x)^* \Delta(y)^* s_\nu s_\tau \Delta(x) \Delta(y) \right]_0 = \delta_{\lambda, \nu} \delta_{\mu, \tau} \]
according to the orthogonality of Schur polynomials.

Now let \( P(f) \) be the projective cover of the irreducible module \( L(f) \) and \( K(g) \) be a Kac module. Then we are going to prove that
\[ \dim \text{Hom}_g(P, K) = (ch P, ch K) \] (5)
We can suppose that \( P^+(f) = P(f), K^+(g) = K(g) \). In other words the parity of every weight vector coincides with the parity of the weight. We have
\[ \dim \text{Hom}_g(P(f), K(g)) = n_{g,f} \]
where \( n_{g,f} \) is the multiplicity of irreducible module \( L(f) \) in the Jordan - Helder series of the module \( K(g) \). On the other hand from the orthonormality of Kac modules it follows that \( (P(f), K(g)) = m_{f,g} \), where \( m_{f,g} \) is the multiplicity of Kac module \( K(g) \) in the Kac flag of the module \( P(f) \). But by BGG reciprocity \( m_{f,g} = n_{g,f} \) and we proved equality (5).

To complete the proof, it just remains to show that the following equality
\[ \dim \text{Hom}_g(P(f), L) = (ch P(f), ch L) \]
is true for any finite dimensional module \( L \). For this, we give two different arguments, the first based on a fact proved by Serganova in [13] and the second using instead completion in the spirit of Brundan ([2] §, 4c).

Now let \( L \) be a module which has a Kac flag. Then
\[ \dim \text{Hom}_g(P(f), L) = \sum_g \dim \text{Hom}_g(P(f), K(g)) = \sum_g (ch P(f), ch K(g)) = (ch P(f), ch L) \]
where \( K(g) \) runs over all subquotients of \( L \).

Now let \( L \) be any finite dimensional module and \( P \) be a projective module. By Serganova [13] there exist a resolvent of \( L \)
\[ \cdots \to K_i \to K_{i-1} \to \cdots \to K_1 \to L \to 0 \] (6)
where every $K_i$ has a flag of Kac modules. Therefore we have an exact sequence of vector spaces

$$\cdots \to \text{Hom}_q(P, K_i) \to \cdots \to \text{Hom}_q(P, K_1) \to \text{Hom}_q(P, L) \to 0$$

For any finite dimensional module $V$ let us denote by $\text{wt}(V)$ the set of the weights of the module $V$. Let $N$ be such that for any $i > N$ we have $\text{wt}(P) \cap \text{wt}(K_i) = \emptyset$. Then for any $i > N$ we have $\text{Hom}_q(P, K_i) = 0$ and

$$\dim(P, L) = \dim(P, K_1) - \dim(P, K_2) + \cdots + (-1)^{i+1} \dim(P, K_i) \quad (7)$$

On the other hand from equality (6) we have

$$\text{sch}_L - \text{sch}_K + \cdots + (-1)^i \text{sch}_{K_i} + \cdots = 0$$

The above sum makes sense since every weight entries the sum with finite multiplicity.

Now let us calculate $(\text{ch}_P, \text{ch}_K_i)$. We have by definition

$$(\text{ch}_P, \text{ch}_K_i) = \frac{1}{m! n!} \left[ (\text{ch}_P)^* \text{ch}_K_i \Delta^*(x) \Delta(y)^* \Delta(x) \Delta(y) \right]_0^\infty = \frac{1}{m! n!} \left[ (\text{ch}_P)^* \text{ch}_K_i \Delta^*(x) \Delta(y)^* \Delta(x) \Delta(y) \sum \prod \left( \frac{y}{x_i} \right)^{n_{ij}} \right]_0^\infty$$

Now let us take $M$ such that for any $i > M$ all monomials of the polynomial $(\text{ch}_P)^* \text{ch}_K_i \Delta^*(x) \Delta(y)^* \Delta(x) \Delta(y)$ were negative degree with respect to $x_1, \ldots, x_m$. Therefore all monomials in the above expansion have negative degree with respect to $x_1, \ldots, x_m$. Therefore $(\text{ch}_P, \text{ch}_K_i) = 0$. Therefore for $i > M$ we have

$$(\text{ch}_P, \text{ch}_L) - (\text{ch}_P, \text{ch}_K_1) + \cdots + (-1)^i (\text{ch}_P, \text{ch}_K_i) = 0 \quad (8)$$

Therefore if we take $i > \max\{N, M\}$ then from the equalities (7), (8) we have $\dim \text{Hom}_q(P, L) = (\text{ch}_P, \text{ch}_L)$ and Theorem 3.2 is proved.

Now let us use a completion. For $\chi = \lambda_1 \varepsilon_1 + \cdots + \lambda_m \varepsilon_m + \mu_1 \delta_1 + \cdots + \mu_n \delta_n$ let us set $m(\chi) = \mu_1 + \cdots + \mu_n$. Let $K(F)_d$ be the subgroup of the Grothendieck group $K(F)$ generated by $\{L(\chi)\}$ for $\chi \in P^+$ with $m(\chi) \geq d$.

We know that $[K(\chi)]$ is the finite linear combination of $L(\chi)$ and $[L(\bar{\chi})]$ where $\bar{\chi} < \chi$. Therefore we can find the sequence $\{A_i\}_{i \geq 1}$ of finite subsets in $P^+$ such that $A_i \subset A_{i+1}$ and

$$[L(\chi)] = \sum_{\bar{\chi} \in A_i} c_{\bar{\chi}} [K(\bar{\chi})] \in K(F)_{d_i}$$

where $d_1 < d_2 < d_3 \ldots$. It is easy to see that for given projective module $P$ there exists $N_1$ such that for any $d \geq N_1$ we have $\dim_q(P, L) = 0$ for any irreducible module $L \in K_d$. And it follows from formula (4) that there exists $N_2$ such than for any $d \geq N_2$ we have $(\text{ch}_P, \text{ch}_L) = 0$ for any irreducible module $L \in K_d$. Therefore for $d_i \geq \max\{N_1, N_2\}$ we have

$$\dim \text{Hom}(P, L(\chi)) = \sum_{\bar{\chi} \in A_i} c_{\bar{\chi}} \dim \text{Hom}(P, K(\bar{\chi})) =$$
and we proved the Theorem in this way. □

**Corollary 3.3.**

\[ \mathcal{P}(f) = \{ g \mid (\text{ch } P(f), \text{ch } K(g)) \neq 0 \} \]

4. **Kac modules and Euler characters**

Now we are going to calculate the number \((\text{ch } K(f), \text{ch } E(g))\) where \(K(f)\) is a Kac module and \(E(g)\) is an Euler virtual module. General formula for Euler characters was given by V. Serganova in [13]. For any parabolic subalgebra \(p \subset g\) and any finite dimensional module \(M\) of \(p\) by a super version of Borel - Weil - Bott construction one can define the virtual Euler module \(E^p(M)\). According to the general formula due to Serganova [13]

\[
\text{ch} E^p(M) = \sum_{w \in W} w \left( \frac{D e^\rho_{\text{ch} M}}{\prod_{\alpha \in R_p \cap R_1^+} (1 - e^\alpha)} \right)
\]

with

\[
D = \frac{\prod_{\alpha \in R_1^+} (e^{\alpha/2} - e^{-\alpha/2})}{\prod_{\alpha \in R_0^+} (e^{\alpha/2} - e^{-\alpha/2})}
\]

Here \(\rho\) is the half-sum of the even positive roots minus the half-sum odd positive roots, \(R_p\) is the set of roots \(\alpha\) such that \(g_{\pm \alpha} \subset p\). Consider now \(g = \mathfrak{gl}(m, n)\) and let \((r, s)\) be a pair of integers such that \(0 \leq r \leq m, 0 \leq s \leq n, r - s = m - n\). We will denote the set of such pairs as \(P(m, n)\). Next let us choose for \((r, s) \in P(m, n)\) the following system of simple roots

\[
\{ \varepsilon_{i+1} - \varepsilon_i, \delta_{p+1} - \delta_p, \varepsilon_{r+1} - \varepsilon_r, \delta_{s+1} - \delta_s, \varepsilon_{m+1} - \varepsilon_m, \varepsilon_{n+1} - \varepsilon_n \}, i \in [1, m] \setminus \{r\}, j \in [1, n] \setminus \{s\}.
\]

So we have the corresponding set of positive even and odd roots. Consider now the parabolic subalgebra \(p\) with

\[
R_p = \{ \varepsilon_{i+1} - \varepsilon_i, \delta_{p+1} - \delta_p, \pm (\varepsilon_i - \delta_j) \},
\]

where \(r + 1 \leq i, j \leq m, i \neq j\) and \(s + 1 \leq p, q \leq n, p \neq q\).

If we set

\[
\chi_{r,s} = \sum_{i=1}^{r} \tau_i \varepsilon_i + \sum_{j=1}^{s} \nu_j \delta_j
\]

where

\[
\tau = (\tau_1, \ldots, \tau_r), \nu = (\nu_1, \ldots, \nu_s)
\]

are non increasing sequences of integers then \(\chi\) defines one dimensional representation of \(p\). For any function \(f(x_1, \ldots, x_m, y_1, \ldots, y_n)\) let us define the following alternation operation

\[
\{ f(x, y) \} = \sum_{w \in S_m \times S_n} \varepsilon(w) w(f(x, y)).
\]
Then it is easy to check that Euler character is given by the following formula

\[
ch E(\chi_{r,s}) \Delta(x)x^{\rho_m} \Delta(y)y^{\rho_n} = \left\{ \prod_{(ij) \in D_+} \left(1 + \frac{y_j}{x_i}\right) \prod_{(ij) \in D_-} \left(1 + \frac{x_i}{y_j}\right) x^{\tau} y^{\nu} x^{\rho_m} y^{\rho_n} \right\}
\]

(9)

where

\[D_+ = [1, r] \times [1, n], \quad D_- = [r + 1, m] \times [1, s].\]

**Remark 4.1.** If we apply to the formula (9) the automorphism \(\omega\) which acts identically on \(x_1, \ldots, x_n\) and acts multiplication by \(-1\) on \(y_1, \ldots, y_m\) then we get the Euler supercharacter (see [14] Proposition 5.10). And it was proved in [14] that Euler supercharacters \(\omega(E(\chi_{r,s}))\) where \((r, s) \in P(m, n)\) form a basis in the ring of supercharacters. Therefore\(ch E(\chi_{r,s})\) where \((r, s) \in P(m, n)\) form a basis in the ring \(K(F)\).

As before we can use diagram \(g = (A, B)\) where

\[A = \{\tau_1, \tau_2 - 1, \ldots, \tau_r + 1 - r\},\]

\[B = \{s - r - \nu_s, s - r - \nu_{s-1} - 1, \ldots, -r - \nu_1\}\]

As a particular case we have the formula for character of Kac module \(K(\tilde{\chi})\) where \(\tilde{\chi} = (\lambda, \mu)\) and \(\lambda = (\lambda_1, \ldots, \lambda_m), \mu = (\mu_1, \ldots, \mu_n)\), are non increasing sequences of integers. In this case we have

\[ch K(\tilde{\chi}) = \Delta(x, y)s_{\lambda}(x)s_{\mu}(y)\]

and the corresponding diagram \(f = (\tilde{A}, \tilde{B})\) where

\[\tilde{A} = \{\lambda_1, \lambda_2 - 1, \ldots, \lambda_m + 1 - m\},\]

\[\tilde{B} = \{n - m - \mu_n, n - m - \mu_{n-1} - 1, \ldots, -m - \mu_1\}\]

**Definition 4.2.** Let \(X, Y\) be two sets of integers such that \(X \cap Y = \emptyset\). Let \(x_1 > x_2 > \ldots, x_m\) be the elements of \(X\) in decreasing order and \(y_1 > y_2 > \cdots > y_n\) be the elements of \(Y\) in decreasing and \(z_1 > x_2 > \ldots, z_{m+n}\) be the elements of \(Z = X \cup Y\) in decreasing order. The sign of a permutation \(\sigma\) such that

\[\sigma(x_1, \ldots, x_m, y_1, \ldots, y_n) = (z_1, \ldots, z_{m+n})\]

will be denoted by \(\varepsilon(X, Y)\).

Let us keep the notation of the above definition. Then the following Lemma can be easily proved.

**Lemma 4.3.** Let us set

\[a_i = |X \cap (-\infty, x_i)|, \quad b_j = |Y \cap (y_j, +\infty)|, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.\]

where \(|A|\) means the cardinality of \(A\). Then the following equalities hold true

\[\varepsilon(X, Y) = (-1)^{a_1 + \cdots + a_m} = (-1)^{b_1 + \cdots + b_n}\]
Definition 4.4. Let \( h \) be a diagram and \( C \subset h^{-1}(\circ) \). Then by \( h * C \) we will denote the following diagram

\[
(h * C)^{-1}(x) = \begin{cases} h^{-1}(x), x = \langle, > \\ h^{-1}(x) \cup C, x = \times \\ h^{-1}(x) \setminus C, x = \circ \end{cases}
\]

Now we can formulate the main result of this section.

Theorem 4.5. The following statement holds true:

\[
(ch K(f), ch E(g)) = \begin{cases} \varepsilon(f, g), f = g * C, C \subset g^{-1}(\circ) \cap \mathbb{Z}_{\leq n-m} \\ 0, \text{ otherwise} \end{cases}
\]  \hspace{1cm} (10)

where

\[
\varepsilon(f, g) = (-1)^{\frac{t(r-1)}{2} + \frac{m(m-1)}{2} + (m-r) + S(C)} \varepsilon(A, C) \varepsilon(C, B)
\]

and \( S(C) \) is equal to the sum of the elements of \( C \).

Proof. We have from the definition of Kac module that

\[
\frac{ch K(f)}{\Delta(x, y)} = s_\lambda(x)s_{\mu}(y)
\]

and from the definition of Euler character

\[
\frac{\Delta(x, y) y^m \Delta(y) y^n}{\Delta(x, y)} = \left\{ \prod_{(i, j) \in D_{r,s}} \left(1 + \frac{y_i}{x_j}\right)^{-1} x^r y^r, x^m y^m \right\}
\]

where \( D_{r,s} = [r + 1, m] \times [s + 1, n] \). Therefore

\[
\frac{ch E(g) \Delta(x) x^m \Delta(y) y^n}{\Delta(x, y)} = \sum_{a_1 \geq a_2 \geq \ldots \geq a_{m-r} \geq 0} (-1)^{|a|} \left\{ \left( \frac{x_{r+1} \cdots x_m}{y_1 \cdots y_r} \right)^{s} s_a(y_{s+1}, \ldots, y_n) s_a(x_{r+1}, \ldots, x_m) \right\}
\]

where \( |a| = a_1 + \cdots + a_{m-r} \). Further we have

\[
\left\{ x_{r+1} \cdots x_m s_a(x_{r+1}, \ldots, x_m) x_{1} \cdots x_{r} x^m \right\} =
\left\{ x_{1}^{r} \cdots x_{r} x_{r+1}^{s-a_{m-r}} \cdots x_{m}^{s-a_1} x^m \right\} = s_{\nu_1} \cdots s_{\nu_r}, s_{a_{m-r}}, \ldots, s_{a_1} \Delta(x) x^m.
\]

In the same way it is easy to see that

\[
\left\{ y_{1}^{r-m} \cdots y_{r}^{m} s_a(y_{s+1}, \ldots, y_n) y_{1}^{\nu_1} \cdots y_{s}^{\nu_s} y^m \right\} = s_{\nu_1} + s_{-n}, \ldots, s_{\nu_3} + s_{-n}, a_1, \ldots, a_{m-r} \Delta(y) y^m
\]

Therefore

\[
(ch K(f), ch E(g)) =
\]
\[
\sum_a (-1)^{|a|} (s_{\lambda}, s_{r, s-a_{m-r}, \ldots, s-a_1}) (s_{\mu}, s_{\nu_1+r-m, \ldots, \nu_s+r-m, a})
\]

It is easy to check that for given \( \lambda \) there exists a unique sequence \( a \) and a permutation \( \sigma \in S_m \) such that
\[
(\lambda_1, \ldots, \lambda_m) + \rho_m = \sigma((\tau_1, \ldots, \tau_r, s-a_{m-r}, \ldots, s-a_1) + \rho_m)
\]
or in an equivalent form \( \tilde{A} = \sigma(A, C) \) where
\[
C = \{s-r-a_{m-r}, \ldots, s+1-m-a_1\}.
\]
In the same way there exists a permutation \( \tau \in S_n \) such that
\[
(\mu_1, \ldots, \mu_n) + \rho_n = \tau((\nu_1 + r - m, \ldots, \nu_s + r - m, a_1, \ldots, a_{n-s}) + \rho_n)
\]
or in the equivalent form \( \tilde{B} = w_n \circ \tau \circ w_n(CB) \), where \( w_n(i) = n-i+1, i = 1, \ldots, n \). Therefore
\[
(K(f), E(g)) = (-1)^{|a|} \text{sign}(\sigma) \text{sign}(\tau).
\]
But
\[
S(C) = s(m-r) + \frac{1}{2}r(r-1) - \frac{1}{2}m(m-1) - |a|
\]
and the Theorem is proved. \( \square \)

**Corollary 4.6.** Let \( f, g \) be such diagrams that \((K(f), E(g)) \neq 0 \) and \( g = (A, B) \). Let us also suppose that for transposition \( \tau = \pi_a^b \) we have \( \tau(g) = g, a \in f^{-1}(x) \) and \( a, b \leq n - m \). Then
\[
(ch K(\tau(f)), ch E(g)) = (-1)^{n_{ab}+m_{ab}+a-b} (ch K(f), ch E(g))
\]
where \( n_{a,b} = |A \cap (a, b)|, m_{a,b} = |B \cap (a, b)| \).

**Proof.** By Theorem 4.5 \( f = g \circ C \) where \( C \subset g^{-1}(\varnothing) \cap G_{\leq n-m} \). Since \( \tau(g) = g \) we have \( \tau(f) = g \circ \tau(C) \) and by Theorem 4.5 we have
\[
(K(f), E(g)) = (-1)^{\frac{1}{2}r(r-1)+\frac{1}{2}m(m-1)+s(m-r)+S(C) \varepsilon(A, C) \varepsilon(C, B)}
\]
\[
(\tau(f), E(g)) = (-1)^{\frac{1}{2}r(r-1)+\frac{1}{2}m(m-1)+s(m-r)+S(\tau(C)) \varepsilon(A, \tau(C)) \varepsilon(\tau(C), B)}.
\]
Further we have the following equalities in \( \mathbb{Z}_2 \): \( S(C) - S(\tau(C)) = a - b \) and by Lemma 4.4
\[
\varepsilon(A, C) - \varepsilon(A, \tau(C)) = |A \cap (a, b)|, \varepsilon(C, B) - \varepsilon(\tau(C), B) = |B \cap (a, b)|.
\]
Lemma is proved. \( \square \)

**Corollary 4.7.** Let \( f, g, h \) be such diagrams that
\[
(ch P(f), ch K(g)) \neq 0, \quad (ch K(g), ch E(h)) \neq 0
\]
and \( \tau = \pi_a^b, a, b \leq n - m \) be an admissible transposition for \( f \) such that \( a, b \notin h^{-1}(\varnothing) \). Then
\[
(ch K(\tau(g)), ch E(h)) + (ch K(g), ch E(h)) = 0
\]
Proof. By Corollary 4.6 it is enough to prove that \( n_{ab} + m_{ab} + a - b \) is an odd number. Let us denote by \((a, b)_x = g^{-1}(x) \cap (a, b)\). Then we have
\[
(a, b) = (a, b)_> \cup (a, b)_< \cup (a, b)_x \cup (a, b)_<
\]
Therefore
\[
b - a - 1 = |(a, b)_> | + |(a, b)_< | + |(a, b)_x | + |(a, b)_< |.
\]
where \(|A|\) means the cardinality of the set \(A\). But
\[
n_{a,b} = |(a, b)_> | + |(a, b)_x |, m_{a,b} = |(a, b)_< | + |(a, b)_< |,
\]
Therefore it is enough to prove that \(|(a, b)_x | + |(a, b)_< |\) is an even number. Let \(C = \{c_1, \ldots, c_r\}\). We have
\[
\varphi \circ g = \varphi \circ f + 2 \sum_{i=1}^{r} (\delta_{d_i} - \delta_{c_i})
\]
and by definition admissible transposition we have \(\sum_{i \in (a, b)} \varphi \circ f(i) = 0\). Therefore
\[
|(a, b)_x | - |(a, b)_< | = \sum_{i \in (a, b)} \varphi \circ g(i) = 2 \sum_{i=1}^{r} (\delta_{d_i} - \delta_{c_i}).
\]
Corollary is proved.

5. Partially polynomial representations

Definition 5.1. A weight \(\chi \in P\)
\[
\chi = \lambda_1 \varepsilon_1 + \cdots + \lambda_m \varepsilon_m + \mu_1 \delta_1 + \cdots + \mu_n \delta_n
\]
is called partially polynomial (in \(y_1, \ldots, y_m\)) if \(\mu_1, \ldots, \mu_m \in \mathbb{Z}_{\geq 0}\).

Corollary 5.2. A diagram \(f = (A, B)\) corresponds to the partially polynomial highest weight if and only if all elements of \(B\) are not greater than \(n - m\).

Definition 5.3. A representation \(V\) of \(\mathfrak{gl}(m, n)\) is called partially polynomial (in \(y_1, \ldots, y_n\)) if all its weights are partially polynomials or its character is a polynomial in \(y_1, \ldots, y_n\).

We should note that there is no loss of generality in selecting our attention to partially polynomial representations, since an arbitrary finite dimensional irreducible representation of \(\mathfrak{gl}(m, n)\) can be obtained from some partially polynomial representation by tensoring with a one dimensional representation.

Example 5.4. Standard representation with the character \(x_1 + \cdots + x_n + y_1 + \cdots + y_m\) is partially polynomial. One dimensional representation with character \(\frac{y_1 \cdots y_m}{x_1 \cdots x_m}\) is also partially polynomial representation.

The subcategory of the modules with partially polynomials weights will be denote by \(\mathcal{F}^+\).
Definition 5.5. For any $M \in \mathcal{F}$ let us denote by $M^-$ the submodule generated by all weight vectors with non partially polynomials weights. Let us also define a functor $F^+ : \mathcal{F} \to \mathcal{F}^+$ by the following formula

$$F^+(M) = M/M^-$$

Lemma 5.6. 1) Functor $F^+$ is right exact.

2) Functor $F^+$ maps projective objects in $\mathcal{F}$ to projective objects in $\mathcal{F}^+$.

Proof. 1) By definition of $M^-$ we have the following equality for all $N \in \mathcal{F}^+$

$$\text{Hom}_\mathcal{F}(M, N) = \text{Hom}_\mathcal{F}(F(M), N)$$

Consider a functor $G : \mathcal{F}^+ \to \mathcal{F}$ such that $G(N) = N$. Then the above equality means that $G$ is right adjoint to $F$. Therefore $F$ is right exact.

2) follows from 1). \qed

Lemma 5.7. The following statements hold true

1) Let $\chi \in \mathcal{P}$ and $K(\chi)$ be the corresponding Kac module, then $F^+(K(\chi)) = K(\chi)$ if $\lambda$ is a partially polynomial weight and 0 otherwise.

2) Let $L(\chi)$ be the irreducible finite dimensional module corresponding to the weight $\chi$. Then $F^+(L(\chi)) = L(\chi)$ if $\chi$ is a partially polynomial weight and 0 otherwise.

Proof. 1) Let $\chi$ be a partially polynomial highest weight. Therefore $\chi - \alpha$ is also partially polynomial for any positive root $\alpha$. Therefore all weights of the module $K(\chi)$ are partially polynomials, so $K(\chi)^- = 0$ and $F^+(K(\chi)) = K(\chi)$. If $\chi$ is not partially polynomial then $K(\chi)^- = K(\chi)$ since it is generated by the vector of the weight $\chi$. Therefore $F^+(K(\chi)) = 0$.

2) Let $\chi$ be a partially polynomial weight. Since $L(\chi)$ is a quotient of $K(\chi)$ then by the first statement we have $F^+(L(\chi)) = L(\chi)$. If $\chi$ is not partially polynomial then by the first statement of the Lemma and by Lemma 5.6 $F^+(L(\lambda)) = 0$. \qed

Corollary 5.8. Let $M \in \mathcal{F}$ and suppose that it has composition series of Kac modules and in the Grothendieck group of $\mathcal{F}$ we have

$$[M] = \sum_{\chi \in I} m_{\chi}[K(\chi)]$$

Then in the Grothendieck group of $\mathcal{F}^+$ we have

$$[F^+([M])] = \sum_{\chi \in I_{\text{pol}}} m_{\chi}[K(\chi)]$$

where $I_{\text{pol}}$ is a subset of partially polynomial weights of $I$. 

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6. Projective covers and Euler characters

In this section we are going to give an algorithm to represent Euler characters as the sum of characters of irreducible modules. For any finite dimensional module \( V \) we have the following formula in the ring \( \Lambda_{m,n}^\pm \)

\[
ch V = \sum_P (ch P, ch V) ch L_P
\]

where sum is taken over all projective covers and \( L_P \) is the irreducible module corresponding to \( P \).

So if \( V = E(h) \) is an Euler character then we need to calculate \((ch P(f), ch E(h))\) for fixed \( h \) and all \( f \). In order to do so we calculate first \((ch P(f), ch E(h))\) for fixed \( f \) and all \( h \).

**Definition 6.1.** Let \( f \) be a diagram. Let us set

\[
f_0^{-1}(x) = \{ a \in f^{-1}(x) \mid \pi_0^b \text{ is admissible and } b \leq n - m \},
\]

\[
f_1^{-1}(x) = \{ a \in f^{-1}(x) \mid \pi_0^b \text{ is admissible and } b > n - m \},
\]

**Definition 6.2.** Let us set

\[
E(f) = \{ h \mid (ch P(f), ch E(h)) \neq 0 \}
\]

**Definition 6.3.** Let \( f \) be a diagram and \( B \subset f^{-1}(x) \). We will denote by \( f_B \) the following diagram

\[
f_B^{-1}(x) = \begin{cases} f^{-1}(x), x = <, > \\ f^{-1}(x) \setminus B \\ f^{-1}(\circ) \cup B \end{cases}
\]

And for a diagram \( f \) we will denote by \( f_{>d} \) the diagram \( f_B \) in the case when \( B = f^{-1}(x) \cap \mathbb{Z}_{>d} \). In other words \( f_{>d} \) is the diagram which can be obtained from \( f \) by deleting from \( f^{-1}(x) \) all the numbers which are strictly grater that \( d \) and adding them to \( f^{-1}(\circ) \).

The following Theorem describes the pairing between projective covers and Euler characters.

**Theorem 6.4.** The following equalities hold true

1) \( E(f) = \{ h \mid h * A \in \mathcal{P}(f), A \subset f_1^{-1}(x) \cap \mathbb{Z}_{\leq n-m} \} \) (11)

2) If \( h \in E(f) \) then

\[
(ch P(f), ch E(h)) = (ch K(h * A), ch E(h))
\]

**Proof.** Let us prove the first statement. We will denote by \( Q \) the write hand side of the equality (11). Let \( h \in E(f) \). We are going to prove that \( h \in Q \). Let us denote by \( A(h) \) the set of all \( g \) such that

\[
(ch P(f), ch K(g)) \neq 0, \quad (ch K(g), ch E(h)) \neq 0
\]
Then we have

\[(ch P(f), ch E(h)) = \sum_{g \in A(h)} (ch K(g), ch E(h))\]

Since \(h \in \mathcal{E}(f)\) the set \(A(h)\) is not empty and there exists \(g \in A(h)\). By Theorem 4.5 we have

\[g = h \ast A, A \subset \mathbb{Z}_{\leq n-m}\]

So we only need to prove that \(A \subset f_{1}^{-1}(\times)\). If \(A = \emptyset\) then \(A \subset f_{1}^{-1}(\times)\). Let \(A \neq \emptyset\) and \(a \in A\). There are two cases \(a \notin f^{-1}(\times)\) and \(a \in f^{-1}(\times)\).

Let us consider the first case. Let \(\tau = \pi_{a}^{b}\) be the corresponding admissible transposition then \(c \notin g^{-1}(\times)\) and therefore \(a \notin h^{-1}(\times)\). By Corollary 4.7 we prove that \(f \ast (A(h))\) is not empty and there exists \(\tilde{g} \in A(h)\) we have

\[(ch K(\tilde{g}), ch E(h)) + (ch K(\tau(\tilde{g})), ch E(h)) = 0\]

Therefore \((ch P(f), ch E(h)) = 0\) and the first case is impossible.

Consider the second case \(a \in f^{-1}(\times)\). Let \(\tau = \pi_{a}^{b}\) be the corresponding admissible transposition then \(b \notin g^{-1}(\times)\) and therefore \(b \notin h^{-1}(\times)\). We have two possibilities \(b \leq n - m\) and \(b > n - m\). If \(b \leq n - m\) then in the same way as above we can prove that \((ch P(f), ch E(h)) = 0\). So the only possibility left \(b > n - m\). Therefore \(A \subset f_{1}^{-1}(\times)\) and \(\mathcal{E}(f) \subset \mathcal{Q}\).

Now let us prove the opposite inclusion. Let \(h \ast A = \mathcal{P}(f)\) and \(A \subset f^{-1}(\times) \cap \mathbb{Z}_{\leq n-m}\). Suppose that \(h \ast A \in \mathcal{P}(f)\). Clearly \((K(h \ast A), E(h)) \neq 0\). Let \(g \in \mathcal{P}(f)\) such that \((ch K(g), ch E(h)) \neq 0\). Then by Theorem 4.5 we have \(g = h \ast \tilde{A}, \tilde{A} \subset \mathbb{Z}_{m-m}\). Let \(a \in A\) and \(\pi_{a}^{b}\) be the corresponding admissible transposition. Then one of the elements \(a, b\) belongs to \(\tilde{A}\). Suppose that \(a \notin \tilde{A}\). Therefore \(b \in \tilde{A}\) but it is impossible since \(b > n - m\). So \(a \in \tilde{A}\). Therefore \(A = \tilde{A}\). So we see that

\[(ch P(f), ch E(h)) = (ch K(h \ast A), ch E(h)) \neq 0\]

So we proved the inclusion \(\mathcal{E}(f) \subset \mathcal{Q}\) and the second statement. The Theorem is proved.

\[\square\]

Now we are going to investigate the case of partially polynomial representations in more details.

**Definition 6.5.** Let us denote by \(\mathcal{E}^{+}(f)\) the set of partially polynomial diagrams \(h\) such that \((ch P(f), ch E(h)) \neq 0\).

**Corollary 6.6.** Let \(f\) be a partially polynomial diagram then the following equality holds true

\[\mathcal{E}^{+}(f) = \{ h | h = \pi_{C}(f)_{> n-m}, C \subset f_{1}^{-1}(\times)\}\]

*Proof.* Let us denote the right hand side the above equality by \(\mathcal{R}\) and by \(\mathcal{Q}^{+}\). we will denote the set of partially polynomial diagrams in \(\mathcal{Q}\), where \(\mathcal{Q}\) is the same as in the proof of Theorem 4.3. By definition we have

\[\mathcal{E}^{+}(f) = \mathcal{Q}^{+}\]
and we need to prove that $\mathcal{R} = \mathbb{Q}^+$. Let $h \in \mathbb{Q}^+$ then by Theorem 6.4 we have $g = h * A \in P(f)$, $A \subset f_1^{-1}(x) \cap \mathbb{Z} \leq n - m$ and since $h, f$ are partially polynomial diagrams then $g$ is a partially polynomial diagram too. Futher we have

$$h = g_A = (\pi_A(g))_{> n - m}$$

Besides since $g \in P(f)$ we have $g = \pi_B(f)$, $B \subset f_0^{-1}(x)$. Therefore

$$h = (\pi_A \pi_B(f))_{> n - m} = (\pi_C(f))_{> n - m}, \quad C = A \cup B$$

So $h \in \mathcal{R}$. Now let us take $h \in \mathcal{R}$. Then by definition

$$h = \pi_C(f)_{> n - m}, C \subset f^{-1}(x)$$

Let us set $B = f_0^{-1}(x) \cap C, A = f_1^{-1}(x) \cap C$. Then $h * A = \pi_B(f) \in P(f)$. Therefore $h \in \mathbb{Q}^+$ and we proved the Corollary. 

**Corollary 6.7.** Let $h$ be a partially polynomial diagram and

$$ch E(h) = \sum_f b_{f,h} ch L(f)$$

be the decomposition of Euler character $E(h)$ in terms of characters of irreducible modules. Then $b_{f,h} = 0, \pm 1$ and it is nonzero if and only if there exists the sequence of transpositions

$$\sigma_1 = \pi_{c_1}^{d_1}, \ldots, \sigma_s = \pi_{c_s}^{d_s}, \quad \sigma_{s+1} = \pi_{c_{s+1}}^{d_{s+1}}, \ldots, \sigma_r = \pi_{c_r}^{d_r}$$

such that

$$f = \sigma_r \circ \cdots \circ \sigma_{s+1} ((\sigma_s \circ \cdots \sigma_1(h)) * \{d_{s+1}, \ldots, d_r\})$$

and

1) $\sigma_i$ is admissible for $h_i = \sigma_i \circ \cdots \circ \sigma_1(h), \quad i = 1, \ldots, s$
2) $c_1 > c_2 > \cdots > c_s, \quad d_1, \ldots, d_s \leq n - m$ and $c_i \neq d_j, \quad 1 \leq i, j \leq s$
3) $c_{s+1} > \cdots > c_r, \quad d_{s+1}, \ldots, d_r > n - m$
4) $\sigma_i$ is admissible for $h_i = \sigma_i(h_{i-1} * \{d_i\}), \quad i = s + 1, \ldots, r$
5) $\{c_1, \ldots, c_s\} \cap \{c_{s+1}, \ldots, c_r\} = \emptyset, \quad \{c_1, \ldots, c_r\} \cap \{d_1, \ldots, d_s\} = \emptyset$

**Proof.** Suppose that all conditions of the Corollary are fulfilled. Then

$$f = \sigma_r \circ \cdots \circ \sigma_{s+1} ((\sigma_s \circ \cdots \sigma_1(h)) * \{d_{s+1}, \ldots, d_r\})$$

Since $d_1, \ldots, d_s \leq n - m$ and $d_{s+1}, \ldots, d_r > n - m$ we can rewrite the above equality in the form

$$f = \sigma_r \circ \cdots \circ \sigma_{s+1} \circ \sigma_s \circ \cdots \circ \sigma_1 (h * \{d_{s+1}, \ldots, d_r\})$$

Now we are going to prove that $\sigma_1, \ldots, \sigma_r$ are admissible for $f$. Let us set

$$\tau_i = \sigma_{r-i+1}, a_i = c_{r-i+1}, b_i = d_{r-i+1}, \quad 1 \leq i \leq r$$

Then we have

$$h * \{b_{r-s}, \ldots, b_1\} = (\tau_r \circ \cdots \circ \tau_{r-s+1} \circ \tau_{r-s} \circ \cdots \circ \tau_1(f))$$

Again from the conditions of the Lemma it follows that $\tau_i, \quad i = 1, \ldots, r$ is admissible for $f_i = \tau_{i-1} \circ \cdots \circ \tau_1(f)$. We have $a_{r-s+1} < a_{r-s+2} < \cdots <$
Let $A_τ \overset{2.10}{\rightarrow} \text{Corollary 2.10}$

Consider the following conditions pairwise commute we have

$b \overset{2.11}{\rightarrow} \text{Corollary 2.11}$

In this case there are two possibilities for the first step.

1) We are going to find a transpositions $\pi_a$ such that $b \in h^{-1}(\times)$, and $\pi_a$ is admissible for $\pi_a(h)$. And it is easy to see that there exists only one such transposition $\pi_a^{-1}$.  

2) We also need to find a transposition $\pi_b$ such that $\pi_b$ is admissible for $\pi_b(h) \ast \{b\}$ and $b > 0$. It is also easy to check that there exist two such transpositions $\pi_0, \pi_{-2}$.

In the second step there are also two possibilities.

1) We are going to find a transpositions $\pi_a$ such that $b \in (\pi_{-2}^{-1}h)^{-1}(\times)$ and $\pi_a$ is admissible for $\pi_a \circ \pi_{-2}^{-1}(h)$. And it is easy to see that there is no such a transposition which satisfies conditions 1), 2), 3) of Corollary 6.7.

2) We are going to find a transpositions $\pi_a$ such that $\pi_a$ is admissible for $\pi_a \circ \pi_{-2}^{-1}(h) \ast \{b\}$ and $b > 0$. And it is easy to see that there is only one such transposition $\pi_0$.  

So we have

$$E^+(h) = \{\pi_0,(\pi_{-2}^{-1}(h)) \ast \{1\}, \pi_0(h \ast 1), \pi_{-2}(h \ast 1)\}$$

And it is easy to see that

$$E(h) = -L(\pi_0((\pi_{-2}^{-1}(h)) \ast \{1\})) - L(\pi_0(h \ast 1),) - L(\pi_{-2}(h \ast 1))$$

Example 6.8. Let $n = m = 2$ and $h^{-1}(\times) = -1$, $h^{-1}(\ast) = h^{-1}(\ast) = \emptyset$.

We are going to describe the set $E^+(h)$.

In this case there are two possibilities for the first step.

1) We are going to find a transpositions $\pi_a$ such that $b \in h^{-1}(\times)$, and $\pi_a$ is admissible for $\pi_a(h)$. And it is easy to see that there exists only one such transposition $\pi_a^{-1}$.

2) We also need to find a transposition $\pi_b$ such that $\pi_b$ is admissible for $\pi_b(h) \ast \{b\}$ and $b > 0$. It is also easy to check that there exist two such transpositions $\pi_0, \pi_{-2}$.

In the second step there are also two possibilities.

1) We are going to find a transpositions $\pi_a$ such that $b \in (\pi_{-2}^{-1}h)^{-1}(\times)$ and $\pi_a$ is admissible for $\pi_a \circ \pi_{-2}^{-1}(h)$. And it is easy to see that there is no such a transposition which satisfies conditions 1), 2), 3) of Corollary 6.7.

2) We are going to find a transpositions $\pi_a$ such that $\pi_a$ is admissible for $\pi_a \circ \pi_{-2}^{-1}(h) \ast \{b\}$ and $b > 0$. And it is easy to see that there is only one such transposition $\pi_0$.  

So we have

$$E^+(h) = \{\pi_0((\pi_{-2}^{-1}(h)) \ast \{1\}), \pi_0(h \ast 1), \pi_{-2}(h \ast 1)\}$$

And it is easy to see that

$$E(h) = -L(\pi_0((\pi_{-2}^{-1}(h)) \ast \{1\})) - L(\pi_0(h \ast 1),) - L(\pi_{-2}(h \ast 1))$$
7. Some special classes of irreducible modules

In this section we will only consider diagrams of the form \( f = (A, A) \) where \( A \subseteq \mathbb{Z}_{\leq 0} \) and instead of \( E(f) \) we will write \( E(A) \) for Euler virtual module and \( L(A), P(A) \) for irreducible module and for projective indecomposable module correspondently. Below \( |A| \) means the number of elements in the set \( A \). Our aim in this section is to give an explicit formula for characters of irreducible modules for the most atypical block of Lie superalgebra \( \mathfrak{gl}(2, 2) \).

**Definition 7.1.** Let us set
\[
P_n^{(m)} = \{ A \subseteq \mathbb{Z}_{\leq 0}, |A| = n \mid \exists B \subseteq \mathbb{Z}_{\leq 0}, |B| \leq m, (ch P(A), ch E(B)) \neq 0 \}
\]
and denote by \( \omega \) the following shift
\[
\omega : \mathbb{Z} \rightarrow \mathbb{Z}, \; \omega(x) = x - 1
\]

In the following Lemma we give an inductive description of the set \( P_n^{(m)} \).

**Lemma 7.2.** The following formulae hold true for \( m < n \)
\[
P_n^{(m)} = \bigcup_{i=0}^{m} P_{n,i}^{(m)}, \quad P_{n,i}^{(m)} = \{ (-i, C) \mid C \in \omega^{i+1}(P_{n-1}^{(m-i)}) \}
\]

**Proof.** It is easy to see that
\[
P_n^{(m)} = \bigcup_{i=0}^{m} A_i, \quad A_i = \{ A \in P_n^{(m)} \mid \max_{a \in A} a = i \}
\]
and we only need to show that \( A_i = P_{n,i}^{(m)} \).

First let us note that \( A \in P_n^{(m)} \) if and only if there exist at least \( n - m \) elements from \( A \) such that every corresponding admissible transposition has one positive element. If in addition \( A \) contains 0 then transposition \( \pi_0^1 \) is admissible for \( A \). Therefore for the set \( A \setminus \{0\} \) there must be at least \( n - m - 1 \) admissible transpositions with one positive element. This proves the equality \( A_0 = P_{n,0}^{(m)} \). The same arguments work for any \( 0 < i \leq m \). Lemma is proved. \( \square \)

In the case \( m = 1 \) we can give an explicit description of the set \( P_n^{(m)} \).

**Lemma 7.3.** We have \( P_n^{(1)} = S_1 \cup S_2 \) where
\[
S_1 = \{ \{0, -1, -2, \ldots, 2 - n, a\} \mid a \leq 1 - n\},
\]
\[
S_2 = \{ \{0, -1, -2, \ldots, -n\} \setminus \{b\} \mid b = 0, -1, \ldots, 1 - n, -n\}
\]

**Proof.** Use of Lemma 7.2 and induction on \( n \). \( \square \)

The following Lemma gives the value of our bilinear form on some pairs of projective modules and Euler characters.
Lemma 7.4. Let \( A_0 = \{0, -1, -2, \ldots, 1-n, -n\} \). Then we have
\[
E^+(A_0 \setminus \{-n\}) \cap \mathcal{P}^{(1)}_n = \{E(\emptyset), E(0), \ldots, E(1-n)\}, \quad (1, 1, -1, \ldots, (-1)^{n-1})
\]
\[
E^+(A_0 \setminus \{b\}) \cap \mathcal{P}^{(1)}_n = \{E(b), E(b-1)\}, \quad ((-1)^{n-1}, (-1)^{n-1}), \ b = 0, \ldots, 1-n
\]
\[
E^+(0, -1, \ldots, 2-n, b) \cap \mathcal{P}^{(1)}_n = \{E(b+1), E(b)\}, \ ((-1)^{n-1}, (-1)^{n-1}), \ b \leq -n
\]
we also indicate on the right the corresponding value \((ch P(A), ch E(B))\).

Proof. It easily follows from Theorem [6.4] \(\square\)

Corollary 7.5. The following formulae hold true
\[
1) \quad ch L(A_0 \setminus \{-n\}) = ch E(\emptyset),
\]
\[
2) \quad \text{if } b \in [1-n, 0] \text{ then }
ch L(A_0 \setminus \{b\}) = (-1)^{n-1} [ch E(b) - ch E(b+1) + \cdots +
(-1)^b ch E(0) + (-1)^{b+1} (1-b) ch E(\emptyset)]
\]
\[
3) \quad \text{if } b \leq -n \text{ then }
ch L(0, -1, \ldots, 2-n, b) = (-1)^{n-1} [ch E(b+1) - \cdots +
(-1)^{b+1} ch E(0) + (-1)^b n ch E(\emptyset)]
\]

Proof. Let us prove the first statement. It is enough to check that
\[
(P(A_0 \setminus \{-n\}), E(\emptyset)) = 1,
\]
and
\[
(P(B), E(\emptyset)) = 0, \text{ for any } B, |B| = n, B \subset \mathbb{Z}_{\geq 0}, B \neq A.\hspace{1cm} (13)
\]

Equality (12) follows from Lemma [7.4] Besides if \( B \notin \mathcal{P}^{(1)}_n \) then equality (13) follows from the definition of the set \( \mathcal{P}^{(1)}_n \) and if \( B \in \mathcal{P}^{(1)}_n \) then this equality follows from Lemma[7.4] The other two statements can be proved in the same manner. \(\square\)

Definition 7.6. Let us define a linear operator on the characters of Kac modules by the formula
\[
T(ch K(A)) = ch K(\omega(A)), \quad \omega(a) = a - 1
\]

It is easy to see that in the category \( \mathcal{F}^+ \) the operator \( T \) corresponds to tensor multiplication on one dimensional module with the character \( \frac{y_1 \cdots y_n}{x_1 \cdots x_n} \).

Now we are going to describe the action of the linear operator \( T \) on the Euler characters.

Lemma 7.7. The following formulae hold true (we suppose that \( A, B \subset \mathbb{Z}_{\leq 0} \))
\[
T(ch E(B)) = (-1)^{n+p} [ch E(\omega(B)) - (-1)^p ch E(\emptyset \cup \omega(B))]
\]
where \( p = |B| \) and we suppose that \( E(B) = 0 \), if the number of elements in \( B \) is strictly greater than \( n \).
Therefore, have let us calculate separately the left hand side and the right hand side. We suppose that 0

Lemma is proved. □

ίrreducible characters in this bloc we need some special type of graphs. Let Definition 7.8.

But by Theorem 4.5 we have

So formulae (14) holds true in this case.

Now consider the case of the most atypical bloc for Lie superalgebra gl(2,2) in the category \( F^+ \). In order to give a reasonable description of the irreducible characters in this bloc we need some special type of graphs.

Definition 7.8. Let \( n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{\geq 1} \). Let us denote by \( \Gamma_{n,m} \) the graph with \( n + m \) vertices that are integers from the segment \( [1 - n - m, 0] \) such that:

1) there exists exactly one edge containing any two vertices from \([1 - n, 0]\); 
2) there exists exactly one edge joining every vertex from \([1 - n, 0]\) with every vertex from \([-n, 1 - n - m]\).
Definition 7.9. For every graph $\Gamma = \Gamma_{n,m}$ let us define the following element of the Grothendieck ring by the formula
\[
\chi(\Gamma) = ch\ E(\emptyset) - \sum_v \varepsilon(v) ch\ E(v) - \sum_e \varepsilon(e) ch\ E(e)
\]
where $\varepsilon(v) = (-1)^i$ if $v = \{i\}$ and $\varepsilon(e) = \varepsilon(v) \varepsilon(u)$ if $e$ contains $v, u$.

Remark 7.10. It is convenient to define $\chi(\Gamma_{n,m})$ in the case when $n = -1$. In such a case we set $\chi(\Gamma_{n,m}) = E(\emptyset)$.

Example 7.11.
\[
\chi(\Gamma_{-1,3}) = ch\ E(\emptyset)
\]
\[
\chi(\Gamma_{2,1}) = ch\ E(\emptyset) - ch\ E(0) + ch\ E(-1) - ch\ E(-2) +\]
\[
ch\ E(0,-1) - ch\ E(0,-2) + ch\ E(-1,-2)
\]
\[
\chi(\Gamma_{0,2}) = ch\ E(\emptyset) - ch\ E(0) + ch\ E(-1)
\]

Theorem 7.12. The following equalities hold true
\[
ch\ L(a,a - 1) = \chi(\Gamma_{a,-1,1}), \ a \leq 0
\]  \hfill (15)
\[
ch\ L(a,b) = (-1)^{a-b-1} \left[ \chi(\Gamma_{a,-1,1}) + \chi(\Gamma_{a,a-b}) \right], \ a - b \geq 2, \ a \leq 0
\]  \hfill (16)

Proof. We are going to use the functor $T$. In the case of $n = 2$ the functor acts by the following formulae
\[
T(ch\ E(\emptyset)) = ch\ E(\emptyset) - ch\ E(0),
\]
\[
T(ch\ E(a)) = -ch\ E(a - 1) - ch\ E(0,a - 1)
\]
and
\[
T(ch\ E(a,b)) = ch\ E(a - 1, b - 1)
\]
It is not difficult to verify the following equality
\[
T(\chi(\Gamma_{n,m})) = \chi(\Gamma_{n+1,m})
\]
Further we see that $ch\ L(a,a - 1) = T^{[a]}(ch\ E(\emptyset))$. We will prove equality \[15\] induction on $|a|$. If $a = 0$ the equality is trivial $ch\ L(0,-1) = ch\ E(\emptyset)$. Let $|a| > 0$ then we have
\[
T^{[a]}(ch\ E(\emptyset)) = T(T^{[a]-1}(ch\ E(\emptyset))) = T(\chi(\Gamma_{|a|,-2,1})) = \chi(\Gamma_{|a|,-1,1})
\]
Now let us prove equality \[16\] also induction on $|a|$. If $a = 0$ and $b \leq -2$ then by corollary \[15\] we have
\[
ch\ L(0,b) = (-1)^{b+1}(2ch\ E(\emptyset) - ch\ E(0) + \cdots +
\]
\[
(-1)^b ch\ E(b + 1)) = (-1)^{b+1}[ch\ E(\emptyset) + \chi(\Gamma_{0,|b|})]
\]
If we apply to both sides of the above formula functor $T^r$ then we get
\[
ch\ L(-r,b - r) = (-1)^{b+1}T^r[\chi\ E(\emptyset) + \chi(\Gamma_{0,|b|})] =
\]
\[
(-1)^{b+1}[\chi(\Gamma_{r-1,1}) + \chi(\Gamma_{r,|b|})]
\]
If we replace $r$ by $-a$ and $b$ by $b - a$ we get the statement. \hfill $\square$
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