Theory of Subradiant States of a One-Dimensional Two-Level Atom Chain

Yu-Xiang Zhang and Klaus Mølmer
Department of Physics and Astronomy, Aarhus University, 8000 Aarhus C, Denmark
(Dated: December 27, 2018)

Recently, the subradiant states of one-dimensional two-level atom chains coupled to light modes were found to have decay rates obeying a universal scaling, and an unexpected fermionic character of the multiply-excited subradiant states was discovered. In this Letter, we theoretically obtain the singly-excited subradiant states, and by eliminating the superradiant modes, we demonstrate a relation between the multiply-excited subradiant states and the Tonks-Girardeau limit of the Lieb-Liniger model which explains the fermionic behavior. In addition, we identify a new family of states with correlations different from the fermionic ansatz.

To achieve controllable and deterministic photon-atom interfaces for applications in quantum information processing and quantum sensing, large atom ensembles are usually used to enhance the coupling to photons [1]. The photons induce both coherent and dissipative atom-atom interactions that yield collective phenomena of super- or sub-radiance [2], wherein a collective excitation of the atom ensemble decays faster or slower than individual atomic excitations. While superradiance has been extensively studied since the seminal work of Dicke [3], subradiance of a large ensemble was observed only very recently in cold atom clouds [4] and metamaterial arrays [5]. Comprehensive theoretical tools for the subradiance are still elusive due to the complicated long-range interactions and many-body features of the atomic ensembles [2]. A one-dimensional (1D) chain of equally spaced two-level atoms offers the simplest geometry to gain insight in the collective decay mechanisms, and implementation of such chains coupled to nanofibers [6], 1D waveguides [7] and to the full vacuum electromagnetic field in 3D free space [11] has attracted considerable attention. Super- and subradiance phenomena are in these systems supplemented by further interesting properties and applications such as atomic mirrors [12], photon Fock state synthesis [13], enhancement of cooperativity [14] and applications in quantum computation [15].

Recently, the subradiant states of such 1D chains of N qubits in 3D free space [16] and coupled to 1D waveguide [17] were numerically found to have a series of seemingly universal properties: In the one-excitation sector where only one of the N atoms is excited, if we sort all eigenstates (to be elaborated) by increasing decay rates with integer labels from $\xi = 1$ to $\xi = N$, the most subradiant states ($\xi \ll N$) have decay rates $\gamma_\xi \propto \xi^2/N^3$. In the multi-excitation sectors, the most subradiant states have a fermionic character, e.g., a most subradiant state with two excitations is given by $|F_{1,2}\rangle \propto \sum_{i<j} (c_{i1}c_{j2} - c_{i2}c_{j1})|e_i,e_j\rangle$, built from subradiant states $|\psi_1(2)\rangle = \sum_{i} c_{i1}(2)|e_i\rangle$ in the one-excitation sector, where $|e_i\rangle$ ($|e_i,e_j\rangle$) represents the state with the $i$th ($i$th and the $j$th) atom excited to $|e\rangle$ while all other atoms are in the ground state $|g\rangle$. The decay rate of $|F_{1,2}\rangle$ is the sum of the decay rates of $|\psi_1\rangle$ and $|\psi_2\rangle$. The infidelity of the fermionic ansatz $|F_{1,2}\rangle$ to exact numerical results scale as $N^{-2}$ and $N^{-1}$ for two different classes of states.

It is intriguing how these properties arise and why they are universally seen in systems with different atom-atom couplings. A thorough theoretical understanding is needed to guide further experimental studies and applications of subradiance. In this Letter, we provide such understanding based on the approach summarized in Fig. 1.

**Spin Models**- For light-matter interactions where the Markovian approximation is applicable, we can eliminate the light modes to obtain a master equation describing only the atoms [18]. The master equation is equivalent to a Monte Carlo wave function formalism [19], where the atomic state evolves stochastically under quantum jumps and deterministically under $H = H_0 + H_{\text{eff}}$, where $H_0$ is the bare Hamiltonian of the atoms and the non-Hermitian $H_{\text{eff}}$ describes both coherent and dissipative atom-atom interactions mediated by the vacuum field. The right eigenstates of $H$, or equivalently of $H_{\text{eff}}$, in each manifold of states with any given number of atomic excitations have decay rates that are twice the imaginary

![Figure 1. Eigenstates of Im $H_{\text{eff}}$ span the superradiant and dark spaces (left panel), while Re $H_{\text{eff}}$ couples the two spaces perturbatively and produces the subradiant states. In the multi-excitation sector (right panel), the Holstein-Primakoff transformation bosonizes the super- and subradiant modes and introduces a coupling between them. The coupling effectively yields a strong interaction $V_{\text{sub}}$ among the bosonic subradiant modes leading to the formation of states with fermionic character.](image-url)
part of the corresponding right eigenvalues. We focus our analysis on the qubit chain coupled to a 1D waveguide, but our treatment provides sufficient insight to also account for the case of coupling to 3D vacuum field (see the Supplemental Material). For an atom chain coupled to a 1D waveguide, we have \[12\,17\]:

\[ H_{\text{eff}} = -\frac{i}{2} \Gamma_D \sum_{m,n=1}^{N} e^{i k_D |z_m-z_n|} \sigma_m^\dagger \sigma_n, \tag{1} \]

where \( \Gamma_D \) is the decay rate of a single atom coupled to the waveguide \([12]\), \( k_D \) is the wavenumber of the waveguide mode resonant with the atomic transition, and \( \sigma_m = |g\rangle_m \langle e| \), the Pauli lowering operator acting on the \( m \)th atom. We assume the atoms are equidistantly spaced by \( d \) so that the coordinate of the \( m \)th atom is \( z_m = md \). For convenience, we shall denote \( H_{\text{eff}} = \text{Re} H_{\text{eff}} - i \text{Im} H_{\text{eff}} \).

One-Excitation Sector- Given a large \( N \), the eigenstates of \( \text{Im} H_{\text{eff}} \) divide the one-excitation sector into a two-dimensional superradiant subspace (SRS) spanned by states \(|\pm k_D\rangle = N^{-1/2} \sum_{i=1}^{N} e^{ik_Dz_i} |e_i\rangle \) with eigenvalue \( N\Gamma_D/4 \) (assuming that \( \langle k_D| - k_D \rangle = 0 \)) and an \((N-2)\)-dimensional dark space (DS) with eigenvalue 0, see Fig. [1]. The dark states acquire weak decay rates because of their admixture of superradiant states induced by the perturbation from \( \text{Re} H_{\text{eff}} \). For the decay rate (expectation value of \( \text{Im} H_{\text{eff}} \)) to scale as \( N^{-3} \), the SRS state admixture must scale as \( N^{-2} \).

The Bloch states \(|k\rangle = N^{-1/2} \sum_j e^{ikz_j} |e_j\rangle \) with \( |k| \leq \pi/d \) have the overlap \( \langle \pm k|k\rangle = \frac{1}{N} e^{i(\pi k D/2)N/d} \) with the superradiant states which has an \( N^{-1} \) rather than the desired \( N^{-2} \) scaling. However, a proper superposition of the degenerate states \(|k\rangle \) and \(|-k\rangle\) of the 1D system may have a destructively interfering overlap with the SRS, and thus much smaller decay rates. We note the exact result

\[ H_{\text{eff}} |k\rangle = w_k |k\rangle - i \frac{N\Gamma_D}{2} (g_{k,+} |k\rangle - h_{k,-} |k\rangle) + \text{higher order terms}, \tag{2} \]

where \( w_k = \frac{\Gamma_D}{4} \sum_{\epsilon = \pm} \epsilon \cot(\frac{k_D + \epsilon k}{2} d) \) and the “tails” \( g_{k,\pm} = \frac{1}{N} e^{-i(\pi k D/2)N/d} \) and \( h_{k,\pm} = \frac{1}{N} e^{-i(\pi k D/2)N/d} \). It follows that a superposition of \(|k\rangle \) and \(|-k\rangle\) is an eigenstate of \( H_{\text{eff}} \) with eigenvalue \( w_k \) and no tails if \( k \) is a solution to the equation \( g_{k,+} h_{k,-} = g_{k,-} h_{k,+} \). This equation has solutions for complex values of \( k \) as long as \( k \neq \pm k_D \). In the regimes \( k \approx 0 \) or \( \pm \pi/d \), supposing \( k_{\xi} = 0 + \delta_{k,\xi} \) and \( k_{\xi} = -\pi/d + \delta_{k,\xi} \) respectively, we find to order \( N^{-2} \),

\[ \delta_{k,\xi} = \frac{\xi \pi}{N d} \times \begin{cases} 1 - i \frac{1}{2N} \cot(\frac{k_D d}{2}), & k \approx 0 \\ 1 + i \frac{1}{2N} \tan(\frac{k_D d}{2}), & k \approx \pm \pi/d \end{cases} \tag{3} \]

with \( \xi = 1, 2, 3 \cdots, \xi \ll N \). Note that the imaginary parts of \( \delta_{k,\xi} \) are \( 1/N \)-order corrections to the leading order solutions \( \delta_{k,0} = \xi \pi/(Nd) \), which are real and independent of \( k_D \).

Substitution of Eq. (3) into the expression for \( \omega_k \), which is parabolic near \( k \approx 0 \) and \( \pm \pi/d \), yields the corresponding decay rates:

\[ \gamma_{\xi} = \Gamma_D \frac{8}{\pi^2} \frac{\xi^2}{N^3} \times \begin{cases} \cot^2(\frac{k_D d}{2})/\sin\frac{k_D d}{2}, & k \approx 0 \\ \tan^2(\frac{k_D d}{2})/\cos\frac{k_D d}{2}, & k \approx \pm \pi/d. \end{cases} \tag{4} \]

The above derivation explicitly leads to the \( \xi^2/N^3 \)-scaling found by numerical diagonalization of \( H_{\text{eff}} \). For completeness and later convenience, we give the expression of the eigenstates,

\[ |\phi_{k,\xi} \rangle \propto g_{-k,\xi,0} |k,\xi \rangle - g_{k,\xi,0} |+k,\xi \rangle = \frac{1}{\sqrt{2}} \left( |k(0) \rangle - |0 \rangle \right) + O\left( \frac{\xi}{N} \right), \tag{5} \]

where \( k(0) \) equals \( k(0) \) or \( -\pi/d + k(0) \) and is also independent of \( k_D \).

Universal-We highlight three features of the above solutions that are relevant for the applicability of the \( \xi^2/N^3 \)-scaling in more complex systems:

1. The leading order solutions of \( k_D \) and the eigenstates \( |\phi_{k,\xi} \rangle \) are independent of the values of \( k_D \);

2. The imaginary corrections to \( k_D \) are of higher order in \( 1/N \) and preserve the linear dependence that \( \delta_{k,\xi} \propto \xi \), for all values of \( k_D \), see Eq. (3);

3. The dispersion relation \( \omega_k \) is parabolic at \( k \approx 0 \) and \( \pm \pi/d \).

If an effective Hamiltonian contains a sum of terms in the form of Eq. (1) but with different values of \( k_D \), Feature 1 implies that these terms share the leading order solutions of \( k_D \) and \( |\phi_{k,\xi} \rangle \) that are identical to what we obtained above. The imaginary corrections depend on \( k_D \), but Feature 2 implies that they are always \( 1/N \) smaller and the linearity \( \delta_{k,\xi} \propto \xi \) holds. Thus, Feature 3 directly leads to the \( \xi^2/N^3 \)-scaling decay rates.

The universal behaviour, e.g., applies to the atom chain coupled to 3D free space modes. Therein, we must sum over terms with \( \pm k_D \) in the interval \([-k_0, k_0]\) where \( k_0 \) is the resonant wavenumber. Only excited states with values of \( k \) outside this interval are dark states, given that \( k_0 \ll \pi/d \). The effective Hamiltonian also contains Hermitian terms similar to Eq. (1) but with \( k_D \rightarrow ik_D \). The above three features still apply for such imaginary \( k_D \). Therefore subradiant states with the \( \xi^2/N^3 \)-scaling decay rates are also obtained there (see the Supplemental Material).

Subradiant multiply-excited states- When the number of spin wave excitations \( n_e \ll N \), the Holstein-Primakoff (HP) transformation \[20\] applies and one may replace \( \sigma_m^\dagger \sigma_n \) with the bosonic operators \( b_m^\dagger b_n \) and obtain a quadratic bosonic Hamiltonian \( H_{\text{eff}} \). It reproduces correctly the superradiant modes with wavenumber \( \pm k_D \). But
for the subradiant multiple-excited states, the predicted states, i.e., bosonic exchange symmetry combinations of subradiant one-excitation states, have decay rates scaling as $N^{-1}$, i.e., much larger than the numerically observed $N^{-3}$-scaling \[16, 17\]. Instead, fermionic exchange anti-symmetric combinations were found to match the numerical results well \[16, 17\].

To see why the fermionic ansatz has the $N^{-3}$ decay rate, we introduce the two-excitation state $|k_1, k_2\rangle = \sum_{m<n} e^{i k_1 z_m + i k_2 z_n} |e_m, e_n\rangle$, and evaluate

$$\text{Im } H_{\text{eff}} |k_1, k_2\rangle = \frac{N \Gamma_D}{4} \sum_{\pm} \left[ g_{k_1, \epsilon} b_{k_2, \epsilon k_D}^\dagger + c_{k_1, k_2, \epsilon} b_{k_1 + k_2 - \epsilon k_D, \epsilon k_D} \right],$$

(6)

where the factors $g_{k, \pm}$ and $h_{k, \pm}$ are given after Eq. (2), $c_{k_1, k_2, \epsilon} = g_{k_1, \epsilon} + g_{k_2, \epsilon}$, and $(|k, k'|) = (|k', k\rangle + |k, k'|)$. All these factors scale as $N^{-1}$ rather than the desired $N^{-2}$-scaling. To reduce them to $N^{-2}$, as in the one-excitation sector, we may exploit the superspositions of $|k_1, k_2\rangle$, $|-k_1, k_2\rangle$, $|k_1, -k_2\rangle$ and $|-k_1, -k_2\rangle$ to construct state $|\phi_{k_1}, \phi_{k_2}\rangle = \sum_{m<n} \phi_{k_1}(z_m) \phi_{k_2}(z_n) |e_m, e_n\rangle$, where $\phi_{k}(z_m) = \langle e_m | \phi_k \rangle$. Then in the expression of $\text{Im } H_{\text{eff}} |\phi_{k_1}, \phi_{k_2}\rangle$, the state amplitudes on $|b_{k_1(2), \epsilon k_D}\rangle$, but not on $|b_{k_1 + k_2 - \epsilon k_D, \epsilon k_D}\rangle$, are successfully reduced to $N^{-2}$-scaling. To also suppress the latter, we form the superposition with the permuted state $|\phi_{k_2}, \phi_{k_1}\rangle$. The suitable superposition turns out to be “fermionic”, i.e., $|F_{k_1, k_2}\rangle \propto |\phi_{k_1}, \phi_{k_2}\rangle - |\phi_{k_2}, \phi_{k_1}\rangle$. The fermionic behavior also applies in regimes with more than two excitations \[16, 17\].

The $|b_{k_1+k_2-\epsilon k_D, \epsilon k_D}\rangle$ state components appear due to “scattering” with the superradiant modes with wavenumber $\epsilon k_D$, and are essential to understand the role played by the fermionic states. The scattering terms appear in the Holstein-Primakoff transformation only if we include the second order corrections due to saturation, i.e., $\sigma_m = (1 - b_m^\dagger b_m/2) b_m$ so that $\text{Im } H_{\text{eff}} = H_{SR} + Q + Q^\dagger$, where $H_{SR} = N \Gamma_D/4 \sum_{\epsilon = \pm} b_{\epsilon k_D}^\dagger b_{\epsilon k_D}$. Now

$$Q = -\frac{\Gamma_D}{8} \sum_{\epsilon = \pm} b_{\epsilon k_D}^\dagger b_{\epsilon p + q - \epsilon k_D} b_{\epsilon p} b_{\epsilon q}.$$ 

(7)

Here, $b_k^\dagger = N^{-1/2} \sum_m e^{i k z_m} b_m^\dagger$ and the summation over wavenumbers is taken over an orthonormal basis $|k\rangle_k$ containing $\pm k_D$. Note that the action of the quartic $Q$ operator is to map the state $b_{k_1}^\dagger b_{k_2}^\dagger |\varnothing\rangle$ to $b_{k_1+k_2-\epsilon k_D}^\dagger b_{\epsilon k_D}^\dagger |\varnothing\rangle$ as in Eq. (6). Thus the quartic terms $Q$ and $Q^\dagger$ must be taken into account in order to understand the fermionic behavior. In contrast to its role in many other applications, the saturation of the individual atomic excitation plays a significant role even in the low saturation regime ($n_e \ll N$).

The quadratic $H_{SR}$ defines the (bosonic) DS spanned by states $|\psi\rangle$ satisfying $b_{p-k_D}^\dagger |\psi\rangle = 0$, and the complementary (bosonic) SRS. As shown in the right panel of Fig. 1 terms $Q$ and $Q^\dagger$ couple the two spaces. Therefore, the effect of $Q$ and $Q^\dagger$ on the most subradiant states can be distilled by eliminating the degrees of freedom of the SRS, in a manner similar to the adiabatic elimination of excited state manifolds to yield the effective dynamics of quantum systems within their ground state manifold \[21\]. The bosonic SRS with only a single excitation of the superradiant modes has the strongest coupling to the DS. We hence omit other SRS and the effective coupling among subradiant states reduces to $V_{\text{sub}} = \frac{4}{N \Gamma_D} P_{DSQ^1} P_{SRSQ^1} P_{DS}$, with projection operators $P_{DS(S_{\text{RS}})}$ on the DS and SRS, respectively. To evaluate this expression we use the operator relations (valid on the DS), $b_{k_D} b_{k_D}^\dagger = \delta_{\epsilon, \epsilon'}$ and $b_{p+q-\epsilon'} b_{p+q-\epsilon'}^\dagger = \delta_{\epsilon, \epsilon'} \delta_{\epsilon, \epsilon'}$. The first equality is trivial, and the second equality is reasonable as the most subradiant states with $i$ indices $p, q \approx 0$ or $\pm \pi/d$, have values of $p+q-\epsilon k_D$ close to $\pm k_D$ in the periodic Brillouin zone, so that $b_{p+q-\epsilon k_D}^\dagger |\varnothing\rangle$ has a large decay rate when the full $H_{\text{eff}}$ is considered.

Finally, we obtain

$$V_{\text{sub}} = \frac{1}{8N} \Gamma_D \sum_{p,q,k} b_{p+q-kD}^\dagger b_{p} b_{q} b_{k}$$

$$= \frac{1}{8N} \Gamma_D \sum_{m=1}^{N} (b_{m}^\dagger)^2 b_{m}^2.$$ 

(8)

That is, $V_{\text{sub}}$ imposes a large decay rate $O(\Gamma_D)$ upon states having more than a single HP boson excitation at the same site.

As illustrated in Fig. 1, for the most subradiant states, the quadratic terms of the HP transformation of $H_{\text{eff}}$ are $U_{\text{sub}} = \sum_{\xi} (\omega_{\xi} - i \gamma_{\xi}) b_{\xi}^\dagger b_{\xi}$, where $\omega_{\xi}$ is given in the Supplemental Material and $b_{\xi}^\dagger = \sum_{m} \phi_{\xi}(z_m) b_{m}^\dagger$. Disregarding quartic contributions to $\text{Re } H_{\text{eff}}$, the effective theory for the manifold of the most subradiant states is therefore governed by $H_{\text{sub}} = U_{\text{sub}} - i \nu_{\text{sub}} = \sum_{\xi} \omega_{\xi} b_{\xi}^\dagger b_{\xi} - i \sum_{\gamma \in f} \gamma_{\xi} b_{\xi}^\dagger b_{\xi} + V_{\text{sub}}$. In the Supplemental Material we show that the continuous limit of $N \text{Im } H_{\text{sub}}$ is the second-quantized form of the Hamiltonian,

$$\mathcal{H} = \sum_{i=1}^{n} \left[ -\frac{\partial^2}{2 m_s} + H'(x_i) \right] + c_{\text{LUV}} \sum_{i<j=1}^{n} \delta(x_i - x_j),$$

(9)

where $c_{\text{LUV}} = N d \Gamma_D/8$ and $H'(x_i)$ is a single-particle potential that traps the bosons in the interval $[0, N d]$ and produces the one-excitation states $|\phi_{k_D}\rangle$ given by Eq. (5). The effective mass $m_s = \xi^2 \pi^2 / (2N^3 d^2 \gamma_{\xi})$ depends on $k_D$ and $d$ but not on $N$. The observation behind Eq. (9) is that $N \gamma_{\xi}$ has the form of the kinetic energy, $N \gamma_{\xi} = k_D^2 / (2 m_s)$ for $k_D \approx \xi / (N d)$; or the kinetic energy in a gauge field $N \gamma_{\xi} = (k_D + \pi/d)^2 / (2 m_s)$ when $k_D \approx -\pi/d + \xi / (N d)$.

As $c_{\text{LUV}} \propto N$ diverges with $N$, Eq. (9) yields the Tonks-Girardeau, hard-core boson limit \[22, 23\] of the Lieb-Liniger model \[24\]. In this limit, the bosonic eigenstates of $\mathcal{H}$ can be obtained via a fermion-boson mapping.
We transform $\sum \xi \gamma \xi b^\dagger \xi b^\dagger \xi$ to a sum $\sum \xi \gamma \xi f^\dagger \xi f^\dagger \xi$ of fermionic operators $f^\dagger \xi = \sum m \phi^\dagger \xi (z_m) f^\dagger m$, with $f^\dagger m f_m = \delta_{m,n}$. Taking fermionic (e.g., two-fermion) number states $f^\dagger m f^\dagger n \langle \phi \rangle = \sum \xi \xi \phi^\dagger \xi \phi \xi (z_m) \phi^\dagger \xi \phi \xi (z_n) f^\dagger m f^\dagger n \langle \phi \rangle$ and replacing the fermionic with bosonic operators, i.e., $f^\dagger m f^\dagger n \rightarrow \text{sign}(n - m) h^\dagger m h^\dagger n$, where sign$(j - i)$ ensures the consistency with the fermionic commutation relation, this yields $H = \sum m < n \phi^\dagger \xi \phi \xi (z_m) \phi^\dagger \xi \phi \xi (z_n) - e^{i \varphi} \phi^\dagger \xi \phi \xi (z_m) b^\dagger m b^\dagger n \langle \phi \rangle \langle e^{i \varphi} = 1 \text{ is introduced for later convenience}$. Finally the reversed HP transformation $h^\dagger m h^\dagger n \langle \phi \rangle \rightarrow |e_m, e_n \rangle$ leads exactly to the fermionic ansatz of the two-excitation sector \([16, 17]\). The above mapping also applies to more excitations \([23, 25]\), and the states obtained are also eigenstates of the single-particle part of Eq. \([9]\), and thus of $\text{Re} H_{\text{sub}}$ and, consequently, of $H_{\text{sub}}$. The Hamiltonian eigenvalues are not changed by the fermion-boson mapping, and the decay rates of the most subradiant multiply-excited states are merely the sum of the decay rates of its one-excitation constituents, (a direct implication of the representation as non-interacting fermions). However, as we shall see below, the fermionic states do not cover all subradiant eigenstates.

The effective coupling $V_{\text{sub}}$ also applies to atom chains in 3D free space where the $\xi^2/\!N^3$-scaling holds in the one-excitation sector. Now, we have to eliminate the SRS modes in the interval $[-k_0, k_0]$. As the coefficients of $V_{\text{sub}}$ depend only on the sum of the decay rates of the atoms, $c_{\text{LL}} \propto N$ and the fermionic behavior still holds as confirmed by numerical calculations \([16]\).

**Beyond Fermionic Ansatz-** For a finite atom chain, the fermionic ansatz $|F_{\xi_1, \xi_2}\rangle$ slightly deviates from the fermionic eigenstates $|\psi_{\text{num}}\rangle$. The deviation quantified by the infidelity $1 - |\langle F_{\xi_1, \xi_2}|\psi_{\text{num}}\rangle|^2$ is numerically found to scale as $N^{-2}$ when both components, $|\phi_{\xi_1}\rangle$ and $|\phi_{\xi_2}\rangle$, come from the same branch of the one-excitation subradiant states, i.e., $k_{\xi_1} - k_{\xi_2} \approx \pm \pi / d$ (or both $\approx 0$); otherwise, the infidelity scales as $N^{-1}$ (when $k_{\xi_1} \approx \pm \pi / d$ and $k_{\xi_2} \approx 0$) \([16, 17]\). These behaviours can also be explained from the Lieb-Liniger model of Eq. \([9]\). The fermion-boson mapping is not exact since $c_{\text{LL}}$ is always finite, and the phase factor $e^{i \varphi}$ mentioned above deviates from unity by a factor in the form of $(k_{\xi_1} - k_{\xi_2})/c_{\text{LL}}$ \([24]\). Since $k_{\xi_1} - k_{\xi_2}$ is about $O(N^{-1})$ or $\pi / d$ in the two cases considered, while $c_{\text{LL}}$ scales as $N$, their ratio scales exactly in the same manner as the numerically observed infidelities. Also, larger discrepancies with the fermionic ansatz are detectable when the decay rates increase.

The fermionic ansatz does not exhaust all the most subradiant eigenstates. For a medium-size ensemble of $N = 20$ atoms, we numerically obtained all the one- and two-excitation eigenstates of $H_{\text{eff}}$, and evaluated the maximal fidelity a fermionic ansatz from one-excitation states can achieve for every two-excitation eigenstate in Fig. \([2]\). It confirms that the fermionic ansatz is a good approximation for a broad range of the most subradiant states, but also highlights a few exceptions by the dips in the fidelity bar-chart. These exceptions were not noticed in previous studies, and we observe that the states are composed of components $|k_1, k_2\rangle$ with definite values of $k_1 + k_2$, as illustrated in Fig. \([2c]\). Further discussion of these states is out of the scope here and will be deferred to future works.

**Conclusions-** In this Letter, we have theoretically explained the $\xi^2/\!N^3$-scaled decay rates found numerically in \([16, 17]\) for the most subradiant states of atom chains coupled to 1D waveguide and 3D free-space modes. We obtained an effective coupling between the subradiant states by eliminating the superradiant modes from the theory. In the multi-excitation sector, this coupling engages higher order terms of the Holstein-Primakoff transformation and maps the spin model to the Lieb-Liniger model \([24]\) of a 1D hard core boson gas in the Tonks-Girardeau limit \([22, 23]\). It is the fermion-boson mapping of that system \([23, 25]\) that explains the fermionic behavior of the subradiant multiply-excited states \([16, 17]\), and, conversely offers interesting prospects for studying quantum fluctuations \([20, 28]\) in the Tonk-Girardeau limit of the Lieb-Liniger gas theory by detection of the excited state correlations among the atoms in a subradiant chain. We also noticed the existence of subradiant state beyond the fermionic ansatz. These results reveal the richness

![Figure 2. Two-excitation eigenstates of a system with $k_D = 0.2\pi/d$ and $N = 20$. (a) All the eigenstates (totally 190) are sorted by increasing energy level. Two regimes of most subradiant states are marked. (b) The 190 eigenstates are sorted by increasing decay rates. The bars show the maximal fidelity that a fermionic ansatz can achieve for each eigenstate. Dips for the states ranked, e.g., 7, 16 and 17, in the regime of the most subradiant states imply a failure of the fermionic ansatz. These states belong to Regime B of panel (a). (c) The values of $|\langle \psi_{16,17}|k_1, k_2\rangle|^2$, with normalized $|k_1, k_2\rangle$ where $k_{1,2}$ take values from $k_m = -\pi/d + (2m + 1)\pi/(Nd)$ with integer $m = 0, 1, 2, \cdots 19$.](image-url)
of subradiance physics and may facilitate experimental applications of subradiance and inspire further study of subradiance in light-matter interactions of more complex geometries, e.g., with chiral waveguides that break the parity symmetry and with setups with topological effects.

Acknowledgments—This work was supported by the Villum Foundation and by the European Unions Horizon 2020 research and innovation program (Grant No. 712721, NanoQtech).

[1] K. Hammerer, A. S. Sørensen, and E. S. Polzik, Rev. Mod. Phys. 82, 1041 (2010).
[2] A. F. Van Loo, A. Fedorov, K. Lahumière, B. C. Sanders, A. Blais, and A. Wallraff, Science 342, 1494 (2013).
[3] R. H. Dicke, Phys. Rev. 93, 99 (1954).
[4] W. Guerin, M. O. Araújo, and R. Kaiser, Phys. Rev. Lett. 116, 083601 (2016).
[5] P. Weiss, M. O. Araújo, R. Kaiser, and W. Guerin, New J. Phys. 20, 063024 (2018).
[6] S. D. Jenkins, J. Ruostekoski, N. Papasimakis, S. Savo, and N. I. Zheludev, Phys. Rev. Lett. 119, 053601 (2017).
[7] P. Solano, P. Barberis-Blostein, F. K. Fatemi, L. A. Orozco, and S. L. Rolston, Nature Communications 8, 1857 (2017).
[8] C. Noh and D. G. Angelakis, Rep. Prog. Phys. 80, 016401 (2016).
[9] D. F. Kornovan, A. S. Sheremet, and M. I. Petrov, Phys. Rev. B 94, 245416 (2016).
[10] H. R. Haakh, S. Faez, and V. Sandoghdar, Phys. Rev. A 94, 053840 (2016).
[11] R. T. Sutherland and F. Robicheaux, Phys. Rev. A 94, 013847 (2016).
[12] D. E. Chang, L. Jiang, A. Gorshkov, and H. Kimble, New J. Phys. 14, 063003 (2012).
[13] A. González-Tudela, V. Paulisch, H. J. Kimble, and J. I. Cirac, Phys. Rev. Lett. 118, 213601 (2017).
[14] D. Plankensteiner, C. Sommer, H. Ritsch, and C. Genes, Phys. Rev. Lett. 119, 093601 (2017).
[15] V. Paulisch, H. Kimble, and A. González-Tudela, New J. Phys. 18, 043041 (2016).
[16] A. Asenjo-Garcia, M. Moreno-Cardoner, A. Albrecht, H. J. Kimble, and D. E. Chang, Phys. Rev. X 7, 031024 (2017).
[17] A. Albrecht, L. Henriët, A. Asenjo-Garcia, P. B. Dieterle, O. Painter, and D. E. Chang, arXiv:1803.02115.
[18] H. T. Dung, L. Knöll, and D.-G. Welsch, Phys. Rev. A 66, 063810 (2002).
[19] K. Mölmer, Y. Castin, and J. Dalibard, J. Opt. Soc. Am. B 10, 524 (1993).
[20] T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940).
[21] F. Reiter and A. S. Sørensen, Phys. Rev. A 85, 032111 (2012).
[22] L. Tonks, Phys. Rev. 50, 955 (1936).
[23] M. Girardeau, Journal of Mathematical Physics 1, 516 (1960).
[24] E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).
[25] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, Rev. Mod. Phys. 83, 1405 (2011).
[26] B. Paredes, A. Widera, V. Murg, O. Mandel, S. Fölling, I. Cirac, G. V. Shlyapnikov, T. W. Hänsch, and I. Bloch, Nature 429, 277 EP (2004).
[27] T. Jacqmin, J. Armijo, T. Berrada, K. V. Kheruntsyan, and I. Bouchoule, Phys. Rev. Lett. 106, 230405 (2011).
[28] M. Budde and K. Mølmer, Phys. Rev. A 70, 053618 (2004).
[29] T. Ramos, H. Pichler, A. J. Daley, and P. Zoller, Phys. Rev. Lett. 113, 237203 (2014).
[30] J. Perczel, J. Borregaard, D. E. Chang, H. Pichler, S. F. Yelin, P. Zoller, and M. D. Lukin, Phys. Rev. Lett. 119, 023603 (2017).
[31] T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, and I. Carusotto, arXiv:1802.04173.
SUPPLEMENTAL MATERIAL

A. $\xi^2/N^2$-scaling in the energy shifts

In the main text we give only $\gamma_\xi$. Here we give the expressions of the energy shifts $\omega_\xi$. We have

$$\omega_\xi = \frac{\Gamma_D}{2} \cot\left(\frac{k_D}{2}d\right) + \Gamma_D^{\mathrm{cay}} \cot\left(\frac{k_D d/2}{\sin^2(k_D d/2)}\frac{\xi \pi}{2N}\right)^2 \quad (10)$$

for $k_\xi \approx 0$, and

$$\omega_\xi = -\frac{\Gamma_D}{2} \tan\left(\frac{k_D}{2}d\right) - \Gamma_D^{\mathrm{cay}} \tan\left(\frac{k_D d/2}{\cos^2(k_D d/2)}\frac{\xi \pi}{2N}\right)^2 \quad (11)$$

for $k_\xi \approx \pm \pi/d$. Hence we see an $\xi^2/N^2$-scaling. The constant parts are not important.

B. $\xi^2/N^3$-scaling for 1D atom chain in free space

Suppose the on resonant wavenumber is $k_0$ and the decay rate of a single atom in free space is $\gamma_0$. The effective Hamiltonian can be written as $H_{\mathrm{eff}} = H_{\mathrm{res}} + H_{\mathrm{eva}}$, where the non-Hermitian $H_{\mathrm{res}}$ has the same form as the case of waveguide, and $H_{\mathrm{eva}}$ is Hermitian.

Suppose the atoms are polarized in the same direction. If the polarization is parallel to the chain, we have

$$H_{\mathrm{res}} = -i 3\gamma_0 \sum_{m,n=1}^{N} \int_0^{k_0} \frac{dk_z}{4k_0} (1 - \frac{k_z^2}{k_0^2}) e^{i k_z |z_m - z_n|} \sigma_m^+ \sigma_n,$$

$$H_{\mathrm{eva}} = -3\gamma_0 \sum_{m,n=1}^{N} \int_0^{\infty} \frac{dk_z}{4k_0} (1 + \frac{k_z^2}{k_0^2}) e^{-i k_z |z_m - z_n|} \sigma_m^+ \sigma_n.$$  \quad (12)

If the polarization is transverse to the chain,

$$H_{\mathrm{res}} = -i 3\gamma_0 \sum_{m,n=1}^{N} \int_0^{k_0} \frac{dk_z}{8k_0} (1 - \frac{k_z^2}{k_0^2}) e^{i k_z |z_m - z_n|} \sigma_m^+ \sigma_n,$$

$$H_{\mathrm{eva}} = -3\gamma_0 \sum_{m,n=1}^{N} \int_0^{\infty} \frac{dk_z}{8k_0} (1 + \frac{k_z^2}{k_0^2}) e^{-i k_z |z_m - z_n|} \sigma_m^+ \sigma_n.$$  \quad (13)

As we discussed in the main text, the eigenstates of $H_{\mathrm{res}}$ will be similar to that of 1D waveguide, i.e., the deviation from $k = 0$ or $k = \pm \pi/d$ is

$$\delta_\xi = \frac{\xi \pi}{Nd} [1 + \frac{1}{N} f(k_0, d) + O(\frac{1}{N^2})],$$

where $\xi \ll N$ and $f(k_0, d)$ is a real-valued function of $k_0$ and $d$, accounting for all terms with $k_z \in [0, k_0]$.

The Hermitian Hamiltonian $H_{\mathrm{eva}}$ has the same form as for the waveguide, but with $k_D \rightarrow i k_D$. The analysis is similar to the case of real-valued $k_D$, and the eigenstates of $H_{\mathrm{eva}}$ are also

$$|\phi_\xi\rangle = \frac{1}{\sqrt{2}} (|k_\xi\rangle - |k_\xi\rangle) + O(\xi/N),$$

where the zeroth order solutions to $k_\xi$ are also given by $\xi \pi/Nd$. The linearity of $\xi$ at the first order correction is also preserved as above.

Since the eigenstates of $H_{\mathrm{res}}$ and $H_{\mathrm{eva}}$ are almost identical, the eigenstates of the total Hamiltonian $H_{\mathrm{eff}}$ will be indexed by

$$\delta_\xi = \frac{\xi \pi}{Nd} [1 + \frac{1}{N} g(k_0, d) + i \frac{1}{N} f(k_0, d) + O(\frac{1}{N^2})],$$

with some function $g(k_0, d)$. The $\xi^2/N^3$-scaling of the decay rates of 1D atom chains in free space are thus obtained, given the parabolic dispersion relation at $k \approx 0$ or $\pm \pi/d$.

C. Transformation to continuous form

Equation (9) of the main text is written in discrete notations. The continuous expression can be obtained from the discrete notations by the mapping

$$\sum_{i=1}^{N} \rightarrow \frac{1}{d} \int_0^{Nd} dx, \quad b_i \rightarrow \sqrt{d} b_x.$$  \quad (17)

The bosonic commuting relation changes from $[b_i, b_j^\dagger] = \delta_{i,j}$ to $[b_x, b_y^\dagger] = \delta(x - y)$. 
