The Generalized Peierls Bracket

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Abstract

We first extend the Peierls algebra of gauge invariant functions from the space \( S \) of classical solutions to the space \( H \) of histories used in path integration and some studies of decoherence. We then show that it may be generalized in a number of ways to act on gauge dependent functions on \( H \). These generalizations (referred to as class I) depend on the choice of an “invariance breaking term,” which must be chosen carefully so that the gauge dependent algebra is a Lie algebra. Another class of invariance breaking terms is also found that leads to an algebra of gauge dependent functions, but only on the space \( S \) of solutions. By the proper choice of invariance breaking term, we can construct a generalized Peierls algebra that agrees with any gauge dependent algebra constructed through canonical or gauge fixing methods, as well as Feynman and Landau “gauge.” Thus, generalized Peierls algebras present a unified description of these techniques. We study the properties of generalized Peierls algebras and their pull backs to spaces of partial solutions and find that they may posses constraints similar to the canonical case. Such constraints are always first class, and quantization may proceed accordingly.
I. INTRODUCTION

In [1], we began an investigation of classical (commuting) *-Lie algebras $\mathcal{A}(\mathcal{H})$ on spaces $\mathcal{H}$ of histories. These algebras contained smooth complex functions on $\mathcal{H}$ with the usual operations of multiplication, addition, and complex conjugation (*) as well as a Lie bracket operation defined by extending the Poisson bracket to $\mathcal{H}$ from the phase space $\Gamma$ or reduced phase space $\Gamma_r$. This Lie bracket was often assembled from pieces defined locally.

Our motivation here as in [1] is to define algebras on $\mathcal{H}$, the space $\mathcal{S}$ of solutions, and the space $\mathcal{E}$ of evolutions (introduced in [1]) from which quantum theories can be derived. The hope is that such algebras will lead to a better understanding of the relationship between path integral and algebraic methods, provide a unified description of conventional gauge dependent algebras, and suggest new avenues for quantization. This approach is complementary to that of [2], [3], [4] and others which define a presymplectic structure on $\mathcal{H}$ and $\mathcal{S}$ since this presymplectic form and (often) our Lie algebras are degenerate so that neither can be inverted to obtain the other.

The close relationship between this extension and the usual Poisson bracket allowed us to compare the extended Poisson algebra $\mathcal{A}_H(\mathcal{H})$ with the familiar Poisson algebras $\mathcal{A}_H(\Gamma)$ and $\mathcal{A}_H(\Gamma_r)$ on the phase space $\Gamma$ and reduced phase space $\Gamma_r$ with little effort and to study quantization of $\mathcal{A}_H(\mathcal{H})$, $\mathcal{A}_H(\mathcal{E})$, and $\mathcal{A}_H(\mathcal{S})$ where $\mathcal{A}_H(\mathcal{E})$, and $\mathcal{A}_H(\mathcal{S})$ are pull backs of $\mathcal{A}_H(\mathcal{H})$ to $\mathcal{E}$ and $\mathcal{S}$. We saw that each of these algebras leads to a Heisenberg picture quantization resembling Dirac’s constraint quantization [5] but with certain differences. We also saw how canonical and gauge fixed Poisson algebras are both examples of “gauge breaking” schemes.

Here, we develop a more general construction of classical *-Lie algebras on $\mathcal{S}$ and $\mathcal{H}$ by extending and generalizing the Peierls bracket [6]. This construction, introduced in [7], uses more machinery than that of [1] but provides a unified perspective and its covariance is manifest. The work in [6] is an important link between the material presented here and more familiar techniques and it is strongly recommended that [6] be read before studying what
follows. The extended Poisson bracket will be shown to be a special case of the generalized Peierls bracket so that a comparison of these general methods with $\mathcal{A}_H(\Gamma)$, $\mathcal{A}_H(\Gamma_r)$, and the usual quantization techniques follows from [1].

We begin in section I with an introduction of notation as well as a brief review of the Peierls algebra $\mathcal{A}_L(S)$ and the associated techniques of [8] which may be unfamiliar. The extension $\mathcal{A}_L(H)$ of the Peierls bracket to $H$ for gauge-free systems is then direct. Appendix A shows that this extension is equivalent to the extended Dirac algebra of [1].

Using additional techniques of [8], section II discusses the case of gauge systems and describes the generalization of the Peierls bracket to act on gauge dependent functions. A subtle point concerning the spacetime support of these functions is discussed in appendix B. This generalization depends on the choice of an “invariance breaking term” which must be chosen carefully so that the generalized Peierls algebra is a Lie algebra. The difficult property to ensure is the Jacobi identity and Appendix C gives two examples for which this identity fails to hold. However, section II finds two general classes of invariance breaking terms guaranteed to produce Lie algebras, one of which defines a Lie bracket on $\mathcal{A}_L(H)$ and one of which defines such a bracket only on $\mathcal{A}_L(S)$.

Section IV explores the range of this generalization. With the help of Appendix A, it shows that the generalized Peierls algebra includes the extended Poisson algebras of [1]. In particular, both the canonical algebra of [3] and gauge fixed algebras can be derived from our procedure. Section IV also shows that generalized Peierls algebras include the Landau and Feynman “gauge” algebras, so that we have found a unified description of all of these techniques.

Our concentration on $H$ provides much of this unification since $S$, $E$, and other interesting spaces are contained in $H$. However, it is often desirable to deal with these subspaces directly. Because we consider $E$ and $S$ as subspaces of $H$ and not as projections as in [2], the natural notion is that of a pull back of $\mathcal{A}_L(H)$. Section V identifies subspaces to which such pull backs are well-defined and investigates their properties. The resulting algebras, as well as $\mathcal{A}_L(H)$ itself, posses “generalized constraints” which are analogous to the constraints of the
canonical theory but which are always first class. Appendix D shows how some pull backs may be obtained directly as generalized Peierls algebras on smaller spaces of histories. A summary discussion appears in section VI.

II. EXTENDING THE PEIERLS BRACKET

In this section, we describe an extension of the usual Peierls bracket to a space $\mathcal{H}$ of histories as preparation for our later generalization of the Peierls bracket to gauge dependent functions. This $\mathcal{H}$ is to be the domain of some action functional $S$ that is stationary on the space $S \subset \mathcal{H}$ of solutions. Typically, $\mathcal{H}$ will be the space of sufficiently regular fields on some differential manifold $M$. We note that the space $\mathcal{H}$ is not an inherent property of the system, but depends on our description through the choice of action $S$. For example, the space of histories for a scalar field depends on whether we use a canonical or covariant description.

The underlying manifold $M$ may or may not have some associated background structure such as a Lorentzian metric or causal structure. We will, however, make use of its differential structure and consider a number of distributions that are local or ultralocal on $M$. We also assume that variations $S_{,i}$ of $S$ with respect to coordinates $\phi^i$ for $i \in \mathcal{I}$ on $\mathcal{H}$ yield a set of differential equations on $M$ that has a causal structure which may be used to define Cauchy surfaces in $M$, though this structure may be dynamically determined through the fields $\phi^i$.

We will usually take the coordinates $\phi^i$ to refer to the values of fields at points in spacetime. Thus, the index $i$ contains a label for the field and for the spacetime point and $\phi^i$ is ultralocal in $M$. However, we assume only that such coordinates exist locally so that $\mathcal{H}$ is an infinite dimensional manifold. Certain steps may require that $\mathcal{H}$ be given further topological properties but we will not address this level of technicality. Note that we use the condensed notation of [8] as well as abstract index notation.

Our extension $(\cdot, \cdot)_\mathcal{H}$ is to be a Lie bracket of functions on $\mathcal{H}$. This means that it must be bilinear and antisymmetric and must satisfy the Jacobi identity and the derivation require-
ment:

\[(AB, C) = A(B, C) + (A, C)B\]  \hspace{1cm} (2.1)

It follows that the bracket is determined by the antisymmetric contravariant tensor field \(\tilde{G}^{ij} = (\phi^i, \phi^j)\) on \(\mathcal{H}\) through \(A, B) = A_i \tilde{G}^{ij} B_j\).

Because the Peierls bracket is unfamiliar to many researchers, subsection II A presents a brief review. Subsection II B then defines \((.,.)_\mathcal{H}\) for gauge-free systems using the techniques of [8].

A. The Peierls Bracket

In 1952, R.E. Peierls noticed that an algebraic structure equivalent to the Poisson bracket could be defined directly from any action principle without first performing a canonical decomposition into coordinates and momenta. His essential insight was to consider the advanced and retarded “effect of one quantity \((A)\) on another \((B)\).” Here, \(A\) and \(B\) are to be functions on \(\mathcal{H}\). They may be nonlocal in \(M\), but only in such a way that there exist cauchy surfaces both to the future and to the past of the support of \(A\) and \(B\). More general cases may be defined through limits when those limits converge.

The advanced \((D_A^+ B)\) and retarded \((D_A^- B)\) effects of \(A\) on \(B\) are then defined by comparing the original system with a new system defined by the action \(S_\epsilon = S + \epsilon A\) and the same space \(\mathcal{H}\) of histories. Under retarded (advanced) boundary conditions for which the solutions \(\phi^i \in \mathcal{S}\) and \(\phi^i_\epsilon \in \mathcal{S}_\epsilon\) coincide to the past (future) of the support of \(A\), the quantity \(B_0 = B(\phi^i)\) computed using \(\phi^i\) will in general differ from \(B_\epsilon = B(\phi^i_\epsilon)\) computed using \(\phi^i_\epsilon\).

For small epsilon, the difference between these quantities defines the retarded (advanced) effect of \(A\) on \(B\) through:

\[D_A^\pm B = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (B_\epsilon - B_0)\]  \hspace{1cm} (2.2)

which depends on the unperturbed solution \(\phi^i\).
The Peierls bracket is defined to be the difference of these two quantities:

\[(A, B) = D_A^+ B - D_B^- A = D_B^- A - D_A^+ B\] 

(2.3)

where the last equality follows from the fact that the dynamics is described by an action principle (see [6], but we will discuss a similar issue in section II B). The Peierls bracket is in fact a Lie bracket; the antisymmetry of this bracket is clear from the last line above and Peierls [6] shows that 2.3 satisfies the Jacobi identity and the derivation requirement.

Note that the Peierls bracket is defined on functions \(A\) and \(B\) of solutions; functions on the subspace \(S\) instead of all of \(H\). For a system without constraints, any map that evaluates the phase space coordinates at some time \(t\) takes \(S\) to the phase space \(\Gamma\) and Peierls’ original paper [6] shows that this map is a Lie bracket isomorphism between the Poisson and Peierls algebras.

Our main interest, however, will be in systems with gauge symmetries. For such a case, the Peierls bracket can be defined as above only when \(A\) and \(B\) are gauge invariant quantities. If, say, \(B\) is gauge dependent, then the value of \(B\) will not be uniquely determined by the initial data and \(D_B^- A\) will be ill-defined. Recall that it was necessary to define both \(D_A^+ B\) and \(D_B^- A\) to verify antisymmetry of the Peierls bracket using 2.3. The above argument breaks down if \(A\) or \(B\) is gauge dependent.

If we are to define a Peierls bracket of gauge dependent functions, it appears that we need a sort of gauge fixing for the disturbances \(\delta \phi^i = \phi^i_\epsilon - \phi^i\) which requires a local gauge fixing in \(S\). The structure we will introduce in section III is more general than this but is based on techniques of [8] that relate to such a gauge fixing. As such, the formalism and notation of [8] will be convenient. In II B, we introduce this formalism in the gauge-free case and find that it leads directly to the desired extension \((,)_H\).

B. Extensions to the Space of Histories

This section represents a brief digression from the main line of discussion. Instead of immediately generalizing the Peierls bracket to gauge dependent functions, we first extend it
to the space of histories $H$. This extension will prove useful in our study of the generalized Peierls algebra defined in [IIB] and provides the opportunities to introduce notation and techniques. We first define the advanced and retarded effects of $A$ and $B$ on each other as functions on $H$, from which the extended the Peierls bracket follows. This will be straightforward using the machinery of [8] and indeed, much of what follows is implicit in that treatment.

Following [8], recall that the undisturbed fields satisfy the equations of motion:

$$0 = S_{,i} (\phi^j)$$

(2.4)

while the disturbed fields satisfy:

$$0 = S_{,i} (\phi^j) = S_{,i} (\phi^j) + \epsilon A_{,i} (\phi^j)$$

(2.5)

To first order then, the perturbations $\delta \phi^j$ are governed by the equation:

$$- A_{,i} (\phi^k) = S_{,ij} (\phi^k) \delta \phi^j$$

(2.6)

and we see that both the boundary conditions (advanced or retarded) and any gauge fixing applies only to the inversion of the operator $S_{,ij} (\phi^k)$ in the above linear equation for $\delta \phi^j$ and not to the solution of (2.4) for $\phi^i$. In the case where there are no gauge symmetries, $S_{,ij}$ is invertible and has advanced and retarded Green’s functions $G^{\pm \ell j}$ that satisfy

$$S_{,ij} G^{\pm \ell j} = -\delta^\ell_i$$

(2.7)

so that the advanced and retarded solutions to the above equations are $\delta^{\pm \phi^j} = G^{\pm j\ell} A_{,i}$ where both $G^{\pm j\ell}$ and $A_{,i}$ depend on the unperturbed solution $\phi^i$. Since $\delta^{\pm B} = B_{,i} \delta^{\pm \phi^i}$, the Peierls bracket is just

$$(A, B) = A_{,i} \tilde{G}^{ij} B_{,j}$$

(2.8)

where

$$\tilde{G}^{ij} = G^{+ij} - G^{-ij}$$

(2.9)
As an aside, recall that when using the condensed notation it is important to note that contractions \( a^i b^j \) involve integrations over time (and space in a field theory) so that this operation may not be associative. Associativity \( (a^i b^j) c^j = a^i (b^j c^j) \) is guaranteed only when the various spacetime integrals converge appropriately. Nevertheless, because they involve only matrices of compact (usually local) spacetime support and Green’s functions that satisfy the proper (advanced or retarded) boundary conditions the expressions we consider do converge in the required fashion and the order of contractions will not be specified. This subtlety should nonetheless be kept in mind as convergence should be checked whenever an association is to be performed.

We may now take a different approach and define the Peierls bracket by Eq. 2.8 instead of Eq. 2.3. Since Eq. 2.9 is defined for all \( \phi^i \in \mathcal{H} \) for which \( S_{ij}(\phi) \) is invertible, this new “extended” Peierls bracket \( (\cdot,\cdot)_{\mathcal{H}} \) is defined on a much larger space. In practice, we will ignore any difference between this space and \( \mathcal{H} \) itself.

The term “extension” is appropriate because this large bracket has a well-defined pull back to \( \mathcal{S} \) where it coincides with the original Peierls bracket. To see this, we note that
\[
S_{ij} \tilde{G}^{jk} = 0 \quad \text{so that we have}
\]
\[
(A + a^i S_{i}, B + b^j S_{j})_{\mathcal{H}} = (A, B)_{\mathcal{H}} + c^k S_{,k} \tag{2.10}
\]
Thus, if \( i : \mathcal{S} \to \mathcal{H} \) is the inclusion map, \( F, G, J, \) and \( K \) are sufficiently smooth functions on \( \mathcal{H} \) such that \( F \circ i = J \circ i \) and \( G \circ i = K \circ i \), and \( S_{i} \) behave like coordinate functions near \( \mathcal{S} \) it follows that \( (F,G)_{\mathcal{H}} \circ i = (J,K)_{\mathcal{H}} \circ i \) and that we may consistently define a bracket \( (\cdot,\cdot)_{\mathcal{S}} \) on \( \mathcal{S} \) by
\[
(F \circ i, G \circ i)_{\mathcal{S}} \equiv (F,G)_{\mathcal{H}} \circ i \tag{2.11}
\]
This \( (\cdot,\cdot)_{\mathcal{S}} \) is just the original Peierls bracket.

As with \( (\cdot,\cdot)_{\mathcal{S}} \), the larger \( (\cdot,\cdot)_{\mathcal{H}} \) forms a Lie bracket. Antisymmetry is guaranteed since \( S_{ij} \) is symmetric and therefore \( G^{+ij} = G^{-ji} \). The derivation requirement is satisfied by construction and the Jacobi identity follows by a straightforward computation using the fact that \( \tilde{G}^{ij,k} = G^{+im} S_{mkn} G^{+nj} - G^{-im} S_{mkn} G^{-nj} \) and that \( S_{mnk} \) is symmetric.
Because this extension is defined by a matrix of second derivatives and not by a tensor field on $\mathcal{H}$, this bracket is invariant only under _linear_ changes of coordinates, just as was the extended Poisson bracket defined in [1]. The extension of the Peierls bracket is therefore not determined by $S$ and the manifold structure of $\mathcal{H}$ alone but requires a linearized structure. Such a structure was defined in [1] to be a set of coordinate patches that cover $\mathcal{H}$ and for which the transition functions between patches are linear. This structure is also known as a set of linearized coordinates and it defines a linearized manifold. Our algebra may be defined on any linearized manifold $\mathcal{H}$ by assembling algebras defined in patches in the manner described by appendix A of [1].

This coordinate dependence could be removed by introducing a covariant derivative operator ($\frac{\partial}{\partial}$) on $\mathcal{H}$. The condition that $S_{ij}$ be symmetric requires the associated connection to be torsion-free. We choose not to use this approach here for more convenient comparison with [1] in which a coordinate independent prescription is determined not by a covariant derivative but by a suitable set of functions. Such an introduction would not significantly change our discussion as it would only replace dependence on the linearized structure by dependence on the choice of covariant derivative.

In the following sections, we will consider generalizations of this extended Peierls bracket to gauge dependent quantities. Thus, we work with algebras on $\mathcal{H}$ and not just on $\mathcal{S}$. That our approach is more general than a strict gauge fixing of small disturbances around solutions will be evident from the fact that these algebras will in general not satisfy Eq. 2.10 or have well-defined pull back to $\mathcal{S}$.

Throughout our discussion we will draw parallels between such generalized Peierls brackets and the various extensions of the Poisson bracket defined in [1]. We thus make contact with more standard techniques through [1]. As a first comparison, Appendix A shows that the extended Peierls bracket for gauge-free systems is identical to the extended Poisson bracket defined by [1] on $\mathcal{H}$ when $S$ takes the canonical form $A_1$, $\mathcal{H}$ is the associated set of canonical histories, and both algebras are defined using the same linearized structure.
III. GENERALIZATIONS TO GAUGE DEPENDENT QUANTITIES

The extension of the Peierls bracket to $\mathcal{H}$ sets the stage for our generalization to gauge dependent functions. In III A, we introduce further methods of $\mathfrak{S}$ that can be used to define the Peierls algebra $\mathcal{A}_{GI}^G(S)$ of gauge invariant functions on $S$ using Green’s functions as was done in IIB for gauge-free systems. We then extend this algebra to $\mathcal{A}_{GI}^G(\mathcal{H})$ and generalize it to act on gauge dependent functions in III B.

A. Gauge Systems

We recall that DeWitt $\mathfrak{S}$ has shown how to calculate the Peierls Bracket of gauge invariants in the presence of a gauge symmetry in a manner similar to 2.8 above. Consider an action $S$ whose gauge invariances are generated by $Q^i_\alpha$ for $\alpha$ in some index set $\Lambda$ that includes (space)time labels. That is, $S$ is invariant under transformations of the form $\delta \phi^i = \epsilon^\alpha Q^i_\alpha$ for all $\epsilon^\alpha$ of compact support interior to the support of $S$. DeWitt then shows that the advanced and retarded effects of one invariant $A$ on another invariant $B$ can be written in the form

$$D_A^\pm B = B_{ij} G^{\pm ij} A_{ij}$$

(3.1)

where $G^{\pm ij}$ are the advanced and retarded Green’s functions that satisfy

$$F_{ij} G^{\pm jk} = -\delta^i_k$$

(3.2)

for any operator $F_{ij}$ of the form

$$F_{ij} = S_{ij} + P_{\alpha i} \eta^{\alpha \beta} P_{\beta j}$$

(3.3)

Here $\eta^{\alpha \beta}$ is an arbitrary symmetric invertible local continuous matrix (which we will in fact take to be ultralocal throughout our discussion) and $P_{\alpha i}$ are a set of one-forms on $\mathcal{H}$.

$^1$The set called $\Lambda$ here was referred to as $\mathcal{G}^I$ in $\mathfrak{I}$.
such that $\mathcal{F}_{\alpha\beta} = P_{\alpha i} Q^i_{\beta}$ is a non-singular differential operator. DeWitt shows in [8] that the operator $F_{ij}$ is always invertible when $\mathcal{F}_{\alpha\beta}$ is invertible and that when the source $\epsilon A$ is gauge invariant $3.1 - 3.3$ are equivalent to imposing the gauge fixing conditions $P_{\alpha i} \delta^\pm \phi^i = 0$ on the disturbances when calculating $\delta B$.

Greek indices will always take values in the set $\Lambda$ while Latin indices will take values in the set $I$. The combination $P_{\alpha i} \eta^{\alpha\beta} P_{\beta j}$ is referred to as the “invariance breaking term” and we will refer to $P_{\alpha i}$ and $\eta^{\alpha\beta}$ respectively as the “invariance breaking form” and the “invariance breaking metric” through this “metric” has nothing to do with any spacetime metric and may have arbitrary signature. Note that $\eta^{\alpha\beta}$ and $P_{\alpha i}$ need not be globally defined, nor even the invariance breaking term as a whole, so long as $F_{ij}$ is defined in patches that cover $S$ in such a way that the algebras in these patches are compatible in the sense of appendix A of [1]. This appendix describes the assembly of a globally defined algebra from algebras defined in patches so long as the Lie brackets of any two patches are identical when acting on functions with support in the intersection of those patches.

As with 2.8, 3.1 can be used to define a Lie bracket $(A, B)_S = A, \tilde{G}^{ij} B, j$ of gauge invariant functions (with spacetime support internal to that of $S$) on $S$. Again, antisymmetry follows because $F_{ij}$ is self-adjoint and again the derivation property is immediate. The Jacobi identity and the pull back property (Eq. 2.10) require more care but can be verified using the fact that the algebra is defined only on invariants and are derived in [8] by applying the useful fact:

$$Q^i_{\beta} G^{\pm \beta\alpha} = G^{\pm ij} P_{\beta j} \eta^{\beta\alpha},$$

which holds on $S$ and is derived in [8] by applying the symmetry generators to $F_{ij}$ and using the various definitions. Because this bracket is defined only on $S$, it does not depend on the choice of linearized structure.

That our algebra is independent of the choice of invariance breaking term can be seen by computing the change in the advanced and retarded Green’s functions induced by a change in the invariance breaking term:
\[
\delta G^{\pm ij} = -G^{\pm ik} \delta F_{kl} G^{\pm lj}
\]

\[
= -G^{\pm ik} \delta P_{\alpha k} \eta^{\alpha \beta} P_{\beta l} G^{\pm lj} - G^{\pm ik} P_{\alpha k} \delta \eta^{\alpha \beta} P_{\beta l} G^{\pm lj} - G^{\pm ik} P_{\alpha k} \eta^{\alpha \beta} \delta P_{\beta l} G^{\pm lj}
\]

(3.5)

and using (3.4). We find that \( \delta(A, B) = 0 \) under this variation if \( A \) and \( B \) are both gauge invariant. The resulting algebra may thus be called the extended Peierls algebra of gauge invariants. If the gauge generators \( Q^i_\beta \) are linear so that \( Q^i_\beta = 0 \) then Eq. 3.4 in fact holds on all of \( H \), from which it follows that the generalized Peierls bracket may be extended in this case to \( A_G^{GI}(H) \) as a Lie bracket that is independent of the choice of invariance breaking term. Appendix \[\[\]\] verifies that the restriction on the spacetime support of \( A \) and \( B \) is important.

**B. Lie Brackets of Gauge Dependent Functions**

We now generalize our bracket to act on gauge dependent functions. Recall that gauge invariance of \( A \) and \( B \) and properties of \( S \) were required only to derive the pull back property (Eq. 2.10) and the Jacobi identity. Therefore, if for some choice of \( F_{ij} \) the bracket

\[
(A, B)_H = A_{\alpha} G^{\alpha ij} B_{ij}
\]

(3.6)

satisfies the Jacobi identity even when \( A \) and \( B \) are gauge dependent, \( 3.6 \) defines a Lie bracket for all functions on the space of histories, although it may not have well-defined pull back to \( S \). Also, if \( 3.6 \) and 2.10 happen to hold on \( S \), then \((,)_S \) is a Lie bracket of functions on the space of solutions. Again, \( A \) and \( B \) should have spacetime support interior to that of \( S \), though more general cases may be defined by limits when such limits converge. Note that this bracket may depend on the choice of \( F_{ij} \) and may or may not satisfy the analogue of Eq. 2.10 when \( A \) and \( B \) are not invariants – these are issues to be explored in the coming sections. In this section we investigate the more fundamental issue of finding classes of invariance breaking terms for which the Jacobi identity is a consequence of \( 3.6 \).

We first observe that the Jacobi identity for systems without gauge symmetries followed only from the symmetry of \( S_{ijk} \). Similarly then, our generalized Peierls bracket is a Lie
bracket whenever \( F_{ij,k} \), or equivalently, \( (P_{ai}\eta^{\alpha\beta}P_{\beta j})_{,k} \), is symmetric. Such invariance breaking terms will be called “class I,” as will the associated algebras.

To find another condition under which 3.6 defines a Lie bracket (though only on \( S \)), we compute the antisymmetrized triple bracket: \( \epsilon_{\alpha\beta\gamma}(A^\alpha(A^\beta,A^\gamma)) \) for general \( A^1, A^2, A^3 \) where \( \epsilon_{\alpha\beta\gamma} \) is the completely antisymmetric symbol with three indices, \( \alpha, \beta, \gamma \in \{1,2,3\} \). Using \( G_{\pm ij,k} = G_{\pm im}S_{mnk}G_{\pm nj} \), and Eq. 3.4 arrives at

\[
\epsilon_{\alpha\beta\gamma}(A^\alpha,(A^\beta,A^\gamma)) = \epsilon_{\alpha\beta\gamma}A^\alpha_{,i}A^\beta_{,j}A^\gamma_{,k} \tilde{G}^{im}(G^{+jn}P_{\sigma n,m}G^{-\lambda\sigma}Q^k_\lambda \\
- Q^j_\sigma G^{+\lambda\eta_{\lambda\gamma}m}G^{-\delta\gamma}Q^k_\lambda + Q^j_\sigma G^{+\lambda\sigma}P_{\lambda n,m}G^{-kn} \\
- G^{-jn}P_{\sigma n,m}G^{+\lambda\eta_{\lambda\gamma}} + Q^j_\sigma G^{-\lambda\sigma}\eta_{\lambda\gamma}, mG^{+\delta\gamma}Q^k_\delta \\
- Q^j_\sigma G^{-\lambda\sigma}P_{\lambda n,m}G^{+kn})
\] (3.7)
on \( S \) (or when \( Q^j_{\beta,ij} = 0 \)) where \( \mathcal{G}^{\pm \alpha \beta} \) are the advanced and retarded right Green’s functions of \( \mathcal{F}_{\alpha \beta} \).

A common special case occurs when \( \mathcal{F}_{\alpha \beta} \) is invertible without imposing boundary conditions to the past or the future so that \( \mathcal{G}^{+\alpha \beta} = \mathcal{G}^{-\alpha \beta} = \mathcal{G}^{\alpha \beta} \). In this case 3.7 simplifies and vanishes when

\[
P_{\lambda n,m} = P_{\lambda m,n}
\] (3.8)

Thus, the second class of algebras (class II) that we will consider is defined by \( F_{ij} \) for which \( \mathcal{F}_{\alpha \beta} \) is invertible without past or future boundary conditions and \( P_{\lambda n,m} \) is symmetric in \( n \) and \( m \). Note that this condition does not involve the invariance breaking metric \( \eta^{\alpha \beta} \) but, as the Jacobi identity was verified using 3.4, it is guaranteed to form a Lie bracket only on \( S \) unless \( Q_{\beta,ij} = 0 \).

Finally, if this bracket is to be well defined on \( S \), we must show that \( (F \circ i,G \circ i)_S \), where \( F \) and \( G \) are two sufficiently smooth functions on \( \mathcal{H} \) and \( i : S \to \mathcal{H} \) is the natural embedding, does in fact depend only on \( F \circ i \) and \( G \circ i \). As before, we show this by computing (on \( S \))

\[
(S_{,i}^j A) = S_{,ij} \tilde{G}^{jk}A_{,k} = P_{ai}\eta^{\alpha\beta}P_{\beta j}(G^{+jk} - G^{-jk})A_{,k} \\
= -P_{ai}\tilde{G}^{\beta\alpha}Q^k_{\beta\sigma}A_{,k}
\] (3.9)
using Eq. 3.4. This vanishes for class II so that such generalized Peierls algebras are well defined.

We have given no proof and make no claim that one of these conditions is necessary for the Jacobi identity to hold. As a result, it is comforting to check that there do exist operators $F_{ij}$ such that the bracket defined through 3.6 does not satisfy the Jacobi identity. Such examples are discussed in appendix C and show that neither of the conditions that defines class II is, on its own, sufficient to guarantee that the Jacobi identity holds.

IV. SPECIAL CASES

Because a generalized Peierls algebra is determined by a choice of invariance breaking term, we have the potential for a variety of distinct generalizations. The purpose of this section is to exhibit this variety by investigating several invariance breaking terms and relating the results to familiar algebras of gauge dependent functions. In particular, IV A shows that generalized Peierls brackets can reproduce any gauge broken Poisson algebra defined in [1] and therefore both gauge fixed algebras and the canonical algebra of [5]. IV B shows that Feynman and Landau “gauge” algebras are also special cases of generalized Peierls brackets. We have thus found a unified description of these techniques.

A. Gauge Breaking

Gauge broken algebras on $\mathcal{H}$ were defined in [1] by introducing a set of locally defined functions $P^\alpha[\phi^i]$ that locally foliate $\mathcal{H}$ into slices $\mathcal{H}_{c^\alpha}$ on which $P^\alpha = c^\alpha$ and another set of locally defined functions $\phi^a$ such that $(\phi^a, P^\alpha)$ is a local product structure on $\mathcal{H}$. That is, $(\phi^a, P^\alpha)$ must form coordinates in patches such that the transitions functions preserve the product structure defined by the separation of coordinates into $\phi^a$ and $P^\alpha$. In addition, we ask that the $\phi^a$ be linearized and that $\phi^a$ and $P^\alpha$ be ultralocal on $M$.

If we now introduce the inclusion map $\Phi_{c^\alpha} : \mathcal{H}_{c^\alpha} \rightarrow \mathcal{H}$ with components $\Phi^i_{c^\alpha}$, where $\phi^i$ ranges over $\phi^a$ and $P^\alpha$, then the gauge broken Poisson bracket is defined by
\[ (A, B)_{\mathcal{H}}(p) = A_{i} \left( \phi^{i} \circ \Phi_{c^{a}}, \phi^{i} \circ \Phi_{c^{a}} \right)_{c^{a}}(x) B_{j} \]  

(4.1)

where \( p = (x, c^{a}) \in \mathcal{H}, x \in \mathcal{H}_{c^{a}}, \) and \( \phi^{i} \) ranges over \( \phi^{a} \) and \( P^{\alpha} \). The bracket \((\cdot)_{c^{a}}\) is the extended Poisson bracket defined by the linearized structure of the coordinates \( \psi^{a} = \phi^{a} \circ \Phi_{c^{a}} \) and the restriction \( S_{c^{a}} = S \circ \Phi \) (which has no gauge freedom) of the original action to \( \mathcal{H}_{c^{a}} \) when \( S_{c^{a}} \) takes the canonical form. When \( S_{c^{a}} \) does not take the canonical form, \((\cdot)_{c^{a}}\) is defined from an extended Poisson bracket \((\cdot)_{c^{a}}^{\prime}\) by the fact that it respects certain equations of motion that follow from \( S_{c^{a}} \) and these same equations are solved to construct the canonical action \( S_{c^{a}}^{\prime} \) that defines \((\cdot)_{c^{a}}^{\prime}\). By appendix A, \((\cdot)_{c^{a}}^{\prime}\) is just the extended Peierls algebra on \( \mathcal{H}_{c^{a}}^{\prime} \). Since the extended Peierls bracket on \( \mathcal{H}_{c^{a}} \) respects all equations of motion, it then follows from appendix D that, for a given linearized structure on \( \mathcal{H}_{c^{a}} \), \((\cdot)_{c^{a}}\) is the Peierls bracket on \( \mathcal{H}_{c^{a}} \) defined by \( S_{c^{a}} \). This Peierls bracket is built from the Green’s functions of

\[ S_{c^{a}; ab} = \Phi_{c^{a}; a}^{i} S_{ij} \Phi^{j}_{c^{a}; b} \]  

(4.2)

where the semicolon denotes a derivative with respect to the coordinates \( \psi^{a} \) on \( \mathcal{H}_{c^{a}} \). Our task is now to show how the proper choice of invariance breaking term can generate the operator 4.2.

To do so, we first consider a system described by an arbitrary action functional \( S[\phi^{i}] \) together with an arbitrary set of cotangent vectors \( P_{a i} \) locally defined on \( \mathcal{H} \) and ultralocal on \( M \). Note that the covectors \( P_{a i} \) form a one-form that takes values in the dual of the algebra of gauge generators. As such, it annihilates some section \( N \) of the tangent bundle \( T^{*}\mathcal{H} \). Since \( P_{a i} \) is ultralocal on \( M \), we may choose a set \( \{ \pi_{i}^{a} \} \) of basis vector fields (labelled by the index \( a \)) for \( N \) that are ultralocal on \( M \) and defined locally on \( \mathcal{H} \) such that, together with a set \( \{ \pi_{i}^{a} \} \) of vector fields also ultralocal on \( M \) and locally defined on \( \mathcal{H} \) but labelled by an index \( a \in \Lambda \) they form a basis \( \{ \pi_{i}^{j} \} \) of \( T^{p}\mathcal{H} \) at each \( p \in \mathcal{H} \). The matrix \( \pi^{-1}_{i j} \) is uniquely defined since the \( \pi_{i}^{j} \) are ultralocal in \( M \). We use latin indices \( (j) \) to run over both kinds of basis vectors and note that the \( \{ \pi_{i}^{a} \} \) are local components of a projection \( \pi : T_{*}\mathcal{H} \rightarrow N_{*} \) where \( N_{*} \) is the dual of \( N \).
What we would like to show is that in the limit of uniformly large eigenvalues of $\eta^{\alpha\beta}$, the algebra defined by $F_{ij}$ reduces to the algebra defined through $2.3$ by the operator $S_{ij}$ projected onto $N$ by $\pi_a^i$. That is, we would like to show that this limit is well defined if

$$S_{ab}^N \equiv \pi_a^i S_{ij} \pi_b^j$$  \hspace{1cm} (4.3)$$

is invertible and that the limit of $\tilde{G}^{ij}$ is given by the pull-back of $G^{N+ab} - G^{N-ab}$ as a bilinear form on $N_*$ to a bilinear form on $T^*_H$ through the projection $\pi$. Here, $G^{N\pm ab}$ are the advanced and retarded Green’s functions of $S_{ab}^N$.

We begin with the projection of the defining equation $3.2$ on the left:

$$\pi_a^i S_{ij} G^{\pm jk} = -\pi_a^k$$  \hspace{1cm} (4.4)$$

which, after multiplying by $G^{N\pm da}$ and expanding the identity operator on $T^*_H$ as

$$\delta_i^j = \pi_a^i \pi^{-1 k}_j = \pi_a^i \pi^{-1 a}_j + \pi_a^i \pi^{-1 a}_j$$

may be rewritten as

$$G^{N\pm ba} \pi_c^k - \pi^{-1 b}_j G^{\pm jk} + G^{N\pm ba} \pi_a^i S_{ij} \pi^{-1 a}_m G^{\pm mk} = 0$$  \hspace{1cm} (4.5)$$

Note that if we now contract this equation with $\pi^{-1 \beta}_k$ the first term is annihilated and the remaining terms give a linear relation between $\pi^{-1 b}_j G^{\pm jk} \pi^{-1 \beta}_k$ and $\pi^{-1 a}_j G^{\pm jk} \pi^{-1 \beta}_k$ with invertible coefficients that are independent of $\eta^{\alpha\beta}$. It follows that these two quantities must be of the same order in the scale $\eta$ of the eigenvalues of $\eta^{\alpha\beta}$ for large $\eta$.

If we project $3.2$ on both sides using $\pi_a^i$ on the left and $\pi^{-1 \beta}_k$ on the right, the result again involves only the two projections $\pi^{-1 b}_j G^{\pm jk} \pi^{-1 \beta}_k$ and $\pi^{-1 a}_j G^{\pm jk} \pi^{-1 \beta}_k$ of the Green’s functions $G^{\pm ij}$:

$$-\delta^\beta_\alpha = \pi_a^i S_{ij} \pi^a \pi^{-1 a}_m G^{\pm mk} \pi^{-1 \beta}_k + \pi_a^i S_{ij} \pi^j \pi^{-1 \gamma}_m G^{\pm mk} \pi^{-1 \beta}_k + \pi_a^i \eta^{\sigma \tau} P_{sj} \pi^j \pi^{-1 \gamma}_m G^{\pm mk} \pi^{-1 \beta}_k$$  \hspace{1cm} (4.6)$$

in which the last term is the largest for large $\eta$. Since $\pi^i_\alpha$ and $P_{\sigma i}$ are ultralocal in $M$ and $\pi^i_\alpha \notin N$, $\pi_a^i \pi^i_\alpha$ must be invertible. It follows that $\pi_a^i P_{\sigma i} \eta^{\sigma \tau} P_{sj} \pi^j_\gamma$ is also invertible and that the projected Green’s functions $\pi^{-1 \gamma}_m G^{\pm mk} \pi^{-1 \beta}_k$ are of order $1/\eta$ and vanish in the large
\[ \eta \text{ limit. By the discussion above, } \pi^{-1 \beta}_{m} G^{\pm mk} \pi^{-1 \alpha}_{k} \text{ must also vanish in this limit as must } \pi^{-1 \beta}_{m} G^{\pm mk} \pi^{-1 \alpha}_{k} \text{ since } F_{ij} \text{ is self-adjoint.} \]

Returning to 4.5, consider the projection through \( \pi^{-1 \alpha}_{k} \) on the right. All that remains in the large \( \eta \) limit is

\[ G^{N \pm ba} = \pi^{-1 b}_{j} G^{\pm j} \pi^{-1 a}_{k} \quad (4.7) \]

Together with our results about the other projections of \( G^{\pm ij} \), 4.7 implies that \( G^{\pm ij} \to \pi^{i}_{a} G^{N \pm ab} \pi^{j}_{b} \) so that in the large \( \eta \) limit the Green’s functions of \( F_{ij} \) become the pull-backs through the projection \( \pi \) of the Green’s functions of \( S_{ab}^{N} \) considered as a bilinear form on \( N_{a}^{*} \).

We have only to relate our basis \( \pi^{j}_{j} \) to the inclusion map \( \Phi_{c}^{\alpha} \) of 4.2 and verify 4.1 for an appropriate generalized Peierls algebra and our task will be complete. To do so, recall that local functions \( P^{\alpha} \) were introduced to define the gauge broken algebra and that the set \( (\phi^{\alpha}, P^{\alpha}) \) formed local coordinates on \( \mathcal{H} \). As a result, \( P^{\alpha}_{i,a} = 0 \). Now, since \( \Phi^{\beta}_{c} = c^{\beta} \), we have \( \Phi^{\beta}_{c,a} = 0 \) and \( P^{\alpha}_{i,a} \Phi^{i}_{c,a} = 0 \). Thus, if we choose the invariance breaking form to be \( P^{\alpha}_{i} = (\gamma_{\alpha \beta} P^{\beta})_{i} \) where \( \gamma_{\alpha \beta} \) is some locally defined nonsingular ultralocal matrix that does not depend on the \( \phi^{\alpha} \), then \( \pi^{i}_{a}(p) = \Phi^{i}_{c,a}(x) \) for \( p = (x, c_{a}) \), \( x \in \mathcal{H}_{c} \) form an appropriate basis for \( N \). We note that such \( \gamma_{\alpha \beta} \) exist whenever \( (\phi^{\alpha}, P^{\alpha}) \) form the required local product structure. The algebra defined by \( (A, B)_{\mathcal{H},c} = A_{a} \tilde{G}^{N ab} B_{b} \) is then identical to the extended Poisson bracket \( (,)^{c} \). Equation 4.1 follows since \( \pi^{-1 b}_{m} G^{\pm mk} \pi^{-1 a}_{k} \) and \( \pi^{-1 a}_{m} G^{\pm mk} \pi^{-1 \alpha}_{k} \) all vanish in the large \( \eta \) limit.

We have now shown that the invariance breaking term may be chosen such that the generalized Peierls bracket coincides with any gauge broken Poisson bracket, at least in the limit of large \( \eta \). That this may be done within class I follows from the fact that \( P^{\alpha}_{i,j} = 0 \) in the coordinates chosen. Thus, setting \( \eta^{\alpha \beta} = \eta \gamma^{\alpha \beta} \) where \( \eta \) is some constant that we take to infinity and \( \gamma^{\alpha \beta} \) is the inverse of \( \gamma_{\beta \delta} \), we find that \( P^{\alpha}_{i} \eta^{\alpha \beta} P^{\beta}_{j} = P^{\alpha}_{i} \eta \gamma_{\alpha \beta} P^{\beta}_{j} \) so that \( (P^{\alpha}_{i} \eta^{\alpha \beta} P^{\beta}_{j})_{k} = 0 \) and the invariance breaking term is class I. By the discussion of [1], this means that the generalized Peierls bracket can reproduce both gauge fixed algebras and the canonical algebra of [1] on \( \mathcal{H} \).
We note also that gauge broken algebras based on canonical gauge fixing are type II since 
\( P_{\alpha i} = P_{\alpha j} \) and \( \gamma_{\alpha \beta} P^\beta \) and \( F_{\alpha \beta} = Q^i_{\beta} (\gamma_{\alpha \sigma} P^\sigma)_{ij} \) are ultralocal. However, the canonical algebra of \([5]\) is only of type I, since it is reproduced by \( \gamma_{\alpha \beta} = \delta_{\alpha \beta}, P^\alpha = \lambda^\alpha \), and \( \eta^{\alpha \beta} = \delta^{\alpha \beta} \) and since \( Q^i_{\beta} (\delta_{\alpha \sigma} \lambda^\sigma)_{ij} \) is not ultralocal. Finally, we note that the choice of \( P_{\alpha i} \) ultralocal in \( M \) was not strictly necessary in the argument above but served to simplify the discussion.

If the above invariance breaking term is in class II and we are working on \( S \) or with linear gauge transformations, we need not even take the large \( \eta \) limit to attain this result. To see this, consider the change induced in the generalized Peierls algebra by a change \( \delta \eta^{\alpha \beta} \) of this metric. In particular, we evaluate the derivative:

\[
\frac{\delta G^{\pm ij}}{\delta \eta_{\alpha \beta}} = \eta_{\alpha \gamma} G^{\pm \gamma \sigma} Q^i_{\sigma} Q^j_{\lambda} \eta^{\pm \kappa} \eta_{\beta \kappa}
\]

using \( 3.3 \) and \( 3.4 \). Thus, \( \frac{\delta \tilde{G}^{ij}}{\delta \eta^{\alpha \beta}} \) vanishes on \( S \) for class II invariance breaking terms when \( 3.4 \) holds since \( G^{+ \alpha \beta} = G^{- \alpha \beta} \) and the limiting algebra is given by any finite \( \eta^{\alpha \beta} \).

**B. Feynman and Landau Gauge**

Another common technique \([9] \) in quantum field theory is to remove the gauge invariance of some action \( S \) by adding to it a term \( \Delta S \) which is a local quadratic form in the fundamental fields \( \phi^i \). One such term is added for each gauge invariance so that the full modification may be written

\[
\Delta S = \gamma P_{\alpha i} \phi^i \eta^{\alpha \beta} P_{\beta i} \phi^j
\]

where \( P_{\alpha i} \) and \( \eta^{\alpha \beta} \) are field independent. With \( \gamma = 1 \), this is a generalization of “Feynman gauge” while a generalized “Landau gauge” arises in the limit \( \gamma \to \infty \). Note that, by appendix \( \text{A} \), the extended Poisson algebra that follows from \( S + \Delta S \) is the generalized Peierls algebra of type I for the original action \( S \) and the invariance breaking term \( \Delta S_{ij} \). Interestingly, section \( \text{I} \) then guarantees that the algebra of gauge invariants defined by \( S \) is identical to the algebra of those same functions defined by \( S + \Delta S \).
V. PULL BACKS OF \((, )_H\)

As in [1], it is of interest to consider pull backs of \((, )_H\) to spaces of partial solutions. Such pull backs have a larger coordinate invariance than \((, )_H\) and are interesting for quantization both because, since the equations of motion hold, they lead naturally to Heisenberg picture formulations and because representations of a commutator algebra based on \((, )_H\) tend to be reducible to representations based on such a pull back (see [1]). In Section A we identify subspaces \(A \subset H\) to which this pull back is well-defined and in Section B we study the properties of the pulled back algebras.

A. Allowed Spaces

We would like to address the question: “To which spaces \(A \subset H\) of partial solutions does \((, )_H\) have a well-defined pull back?” For such a space, the addition of arbitrary quantities to \(A\) and \(B\) that vanish on \(A\) modifies \((A, B)_H\) only by a term that also vanishes on \(A\). Thus, \(A\) is characterized by a set \(\{ c^i_d S_{,i} = 0 \}\) of combinations of the equations of motion \(\{ S_{,i} = 0 \}\) for which \((A + a^d c^i_d S_{,i}, B + b^d c^i_d S_{,i})_H = (A, B)_H + q^d c^i_d S_{,i}\) for all \(A\), \(B\), \(a^d\), and \(b^d\) and some \(q^d\) that depends on \(A\), \(B\), \(a^d\), and \(b^d\). In particular, \((A, c^i_d S_{,i}) = q^f c^i_j S_{,i}\).

Evaluating the bracket of an equation of motion, we find

\[
(A, S_{,k})_H = A \tilde{G}^{jk} S_{,jk} = -A \tilde{G}^{jk} P_{\alpha k} \eta^{\alpha \beta} P_{\beta k} \tag{5.1}
\]

In general, \(5.1\) will vanish for some equations of motion \(S_{,i}\) but not for others. For example, we have \((A, S_{,k} a^k)_H = A_{,i} \tilde{G}^{ij} a^j S_{,k}\) when \(a^k P_{\alpha k} = 0\). Any algebra may thus be consistently pulled back to any subspace defined by \(c^i_d S_{,i} = 0\) for some \(\{ c^i_d \}\) such that \(c^i_d P_{\alpha i} = 0\) and \(c^i_d S_{,i} = K^b_{dj} c^i_d\) where \(K^b_{dj}\) and \(c^i_d\) have compact support.

Let us also recall Eq. \(3.9\):

\[
(A, S_{,k})_H = -A \tilde{Q}^i_\alpha \tilde{G}^{\alpha \beta} P_{\beta k} \tag{5.2}
\]
which holds on \( S \) or when \( Q^j_{\beta j} = 0 \) and which vanishes when \( \tilde{G}^{\alpha \beta} = 0 \). This shows that any algebra that is both class I and class II has a well-defined pull back to \( S \). In particular, this is true of any gauge broken algebra based on canonical gauge fixing.

B. Properties of Pulled Back Algebras

As hinted above, the properties of a pull back of a class I generalized Peierls algebra depend on whether or not the algebra is also in class II. When it is, the algebra may be pulled back to \( S \) where it depends only on the invariance breaking term and the manifold structure of \( \mathcal{H} \). Algebras not of class II can only be pulled back to a larger space of partial solutions, which we will call \( \mathcal{E} \).

On \( S \) class II algebras have locally nontrivial centers. That is, the patches on \( S \) may be chosen such that the local algebra of each patch has nontrivial center. This follows since \( P_{\alpha i,j} = P_{\alpha j,i} \) so that locally \( P_{\alpha i} = P_{\alpha i} \) for some \( P_{\alpha} \). We then have \( (A, P_{\alpha}) = A_{i} \tilde{G}^{ij} P_{\alpha j} = 0 \) as a result of 3.4 since \( G^{+\alpha \beta} = G^{-\alpha \beta} \). By IV, the same is true on \( \mathcal{H} \) for any algebra defined by a large \( \eta \) limit. Class II algebras thus have many of the same properties as gauge broken algebras based on canonical gauge fixing and quantization may proceed by imposing all of the equations of motion as operator equations with the results similar to V B of [1].

On the other hand, the situation for algebras not of type II is similar to the canonical case of [5]. This is no surprise after IV, where we saw that \( A_{\mathcal{H}}(\Gamma) \) is a class I generalized Peierls algebra that is not in class II.

We now work in the space \( \mathcal{E} \) containing \( S \) and on which those equations of motions that \( (,)_H \) respects vanish identically. In the Peierls equivalent of the canonical approach (in which \( P_{\alpha i} \) vanishes when the index \( i \) does not correspond to a Lagrange multiplier) these are the equations of motion generated by the Hamiltonian and the remaining equations of motion are the constraints. In general then, we refer to those equations of motions not respected by \( (,)_E \) as “generalized constraints.”

For quantization, the constraints should form a first class set. A set \( \{\phi_\alpha\} \) of constraints is
called first class when i) the bracket of two constraints is a linear combination of constraints:
\[(\phi_{\alpha}, \phi_{\beta}) = C_{\alpha\beta}^{\gamma} \phi_{\gamma}\] and ii) when the set of constraints does not generate further constraints by evolution. Here \(C_{\alpha\beta}^{\gamma}\) are functions on \(E\).

Because our constraints carry an index that contains a (space)time label, the set \(\{\phi_{\alpha}\}\) already contains the evolved versions of any constraint. The second condition is then trivially satisfied. While it is quite another issue whether or not the set of generalized constraints is equivalent to a subset for which the index \(\alpha\) lies in some single Cauchy surface, we have no need for such an assumption.

To show that the first condition is always satisfied as well, we note that since \((S_{i}, Q_{\alpha}^{j}), j = 0, (S_{i}, S_{k})_{H} = -P_{\alpha i} \eta^{\alpha \beta} P_{\beta j} \tilde{G}^{im} S_{km}\]
\[- P_{\alpha i} \tilde{G}^{\alpha \beta} Q_{\beta m} S_{km} - G^{+ \alpha \beta} Q_{\beta n} S_{ij} G^{+ nm} S_{km} + G^{- \alpha \beta} Q_{\beta n} S_{ij} G^{- nm} S_{km}\]
\[= P_{\alpha i} \tilde{G}^{\alpha \beta} Q_{\beta m} S_{km} - G^{+ \alpha \beta} Q_{\beta n} S_{ij} G^{+ nm} S_{km} + G^{- \alpha \beta} Q_{\beta n} S_{ij} G^{- nm} S_{km}\]
(5.3)
and the algebra of equations of motion on \(H\) is first class. Since all equations of motion that are not constraints vanish on \(E\), the algebra of generalized constraints is the pull back to \(E\) of this algebra on \(H\). It follows that the constraint algebra is first class as well. This is related to the fact derived in appendix A that when all constraints are second class the extended Peierls algebra produces not the extended Poisson algebra, but the extended Dirac algebra. Thus, any extended Peierls bracket can be used to define a quantum theory by following Dirac’s procedure [5] in which the constraints are imposed as conditions to select “physical” states.

Another similarity with [5] is that when \(S_{k}\) is a constraint, (5.2) shows that it generates the transformation \(\delta \phi^{i} = Q_{\alpha}^{i} \xi_{k}^{\alpha}\) where \(\xi_{k}^{\alpha} = \tilde{G}^{\alpha \beta} P_{\beta k}\) on \(S\) (or on \(E\), up to terms proportional to the constraints). This is not a gauge transformation, however, since \(\xi_{k}^{\alpha}\) does not have compact support. Instead, when \(P_{\alpha i} = P_{\alpha i}, i\), since \(P_{\alpha i} Q_{\beta k}^{i} = F_{\alpha \beta} \tilde{G}^{\beta \gamma} P_{\gamma k} = 0\) this transformation corresponds to one of the symmetries that would remain if the gauge conditions \(P_{\alpha} (\phi^{i}) = 0\) were imposed. This is just what was found for the extended canonical Poisson bracket in
VI. DISCUSSION

We have seen that the machinery of [8] can be used to first extend the Peierls bracket from the space of solutions to the space of histories and then to generalize it to act on gauge dependent functions. This generalization depends on the choice of an “invariance breaking term” which must be chosen in the proper way so that the generalized Peierls algebra is a Lie algebra. Two interesting classes of invariance breaking terms were identified, one that defines a Lie algebra on $A_L(\mathcal{H})$ and one that is guaranteed to be a Lie algebra only on $A_L(\mathcal{S})$.

Our generalized Peierls bracket includes the extended Poisson bracket of [1] as a special case so that [1] provides a comparison with more familiar methods. Because it includes Landau and Feynman gauge as well, the generalized Peierls algebra provides a unified descriptions of the conventional algebras of gauge dependent functions. Generalized Peierls algebras resemble the constrained canonical algebras of [5] and lead to a set of “generalized constraints.” Because these are always first class, quantization may proceed by analogy with [5].

Thus, we have described a large class of classical (commuting) *-Lie algebras of complex functions on $\mathcal{H}$, $\mathcal{S}$, and $\mathcal{E}$. They are constructed without performing a 3+1 decomposition so that their covariance is manifest and depends only on the covariance of the invariance breaking term. No gauge fixing or reduction is needed and, because the algebra is defined on functions of histories, a Heisenberg picture is the natural quantization.

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APPENDIX A: EQUIVALENCE OF PEIERLS AND DIRAC BRACKETS ON THE SPACE OF HISTORIES

Peierls original paper [6] shows that, for an unconstrained system, the Peierls bracket is mapped to the Poisson bracket under any map \( e_t : S \to \Gamma \) that evaluates the canonical fields at some time \( t \). In fact, when there are no constraints or gauge symmetries, the extended Peierls bracket is identical to the extended Poisson bracket and when only second class constraints are present, it yields the extended Dirac bracket of [1].

Recall that for a system described over a time interval \( I \) by the action

\[
S = \int_I dt \left( \frac{1}{2} \Omega_{AB} z^A(t) \dot{z}^B(t) - H(t) - \lambda^a \xi_a(t) \right) \tag{A1}
\]

for some field and time independent invertible antisymmetric matrix \( \Omega^{AB} \), \( H(t) = H(z^A(t), t) \), \( \xi_a(t) = \xi(z^A(t), t) \) for \( a \) in some index set \( T \), and some choice \( z^A(t) \) of linearized coordinates on a phase space \( \Gamma \), the extension of the Poisson bracket in [1] to \( H \subset \Gamma^I \times L^I \), where \( L \) is the range of \( \lambda^a \), is uniquely determined by the fact that it is a Lie bracket, that it respects the equations of motion \( \{ S, (A, t) \} = 0 \), and that

\[
(z^A(t), z^B(t))_H = \Sigma^{AB} \equiv \Omega^{AB} - \Omega^{AB} \xi_a|C(\Delta^{-1})^{ab} \xi_b|D \Omega^{DB} .
\]

The matrix \( \Delta_{ab} = \xi_a|A \Omega^{AB} \xi_b|B \) is invertible since the constraints are second class. Here, \( |A \) denotes a derivative with respect to the canonical coordinate \( z^A \) on \( \Gamma \) and we have separated the time label \( t \) from the other labels \( A \) and \( a \) so that our usual condensed index is \( i = (A, t) \) or \( i = (a, t) \).

We have already seen that the extended Peierls bracket is a Lie bracket and that it respects the same equations of motion when built using the same linearized structure. Thus, we now compute \((z^A(t), z^B(t))_H \) by inverting the operator \( S_{ij} \):

\[
S_{(a, t_1);(b, t_2)} = 0 \tag{A2a}
\]
\[ S_{i(a,t_1)(B,t_2)} = \xi_a|_B(t_1)\delta(t_1 - t_2) \]  
\[ (A2b) \]

\[ S_{i(A,t_1)(b,t_2)} = \xi_b|_A(t_1)\delta(t_1 - t_2) \]  
\[ (A2c) \]

\[ S_{i(A,t_1)(B,t_2)} = \Omega_{AB}^{-1}\delta'(t_1 - t_2) - H_{\xi|AB}(t_1)\delta(t_1 - t_2) \]  
\[ (A2d) \]

where the \( \prime \) denotes a derivative with respect to its argument and \( H_\xi(t) = H(t) - \lambda^a(t)\xi_a(t) \).

From the two equations
\[ S_{i(a,t_1)iG_{i(B,t_2)}} = 0 \quad \text{and} \quad S_{i(A,t_1)iG_{i(B,t_2)}} = -\delta_B^A\delta(t_1 - t_2) \]  
\[ (A3) \]

it follows that the Green’s functions \( G^{(A,t_1)(B,t_2)} \) are:
\[ G^{(A,t_1)(B,t_2)} = \pm \Sigma^{AC}P\exp\left[\int_{t_2}^{t_1} dt \, Q_{BC}[\theta(t_1 - t_2)]\right] \]  
\[ (A4) \]

where \( \theta \) is the usual step-function, \( P \) denotes path ordering, and
\[ Q_{BC}^B = \Delta^{-1}b\frac{\partial}{\partial t}(\xi_a|_B) \]  
\[ (A5) \]

Finally, we have \((z^A(t), z^B(t)) = \tilde{G}^{(A,t)(B,t)} = \Sigma^{AB} \) and it follows from [1] that the extended Dirac and Peierls brackets agree on \( \mathcal{H} \).

As discussed in section IV of [1], spaces \( \mathcal{L} \) of Lagrangian histories can typically be embedded as spaces of partial solutions in spaces of canonical histories. The Dirac bracket on \( \mathcal{L} \) is defined in [1] by pull back through this embedding map, while appendix D shows that the same pull back takes the extended Peierls bracket on \( \mathcal{H} \) to the extended Peierls bracket on \( \mathcal{L} \). It follows that the extended Dirac bracket and extended Peierls brackets are also identical on typical spaces of Lagrangian histories.

**APPENDIX B: GAUGE INVARIANTS WITH WIDER SUPPORT**

In this appendix we verify that the restriction of the Peierls algebra to gauge invariant functions with compact spacetime support *interior* to that of \( S \) is essential. Specifically, we
give an example for which the generalized Peierls bracket of gauge invariants whose support is not interior to that of \( S \) depends on the invariance breaking term in B.3.

Consider a free relativistic particle described by the action:

\[
S = \frac{1}{2} \int_{t_1}^{t_2} \left( \frac{\dot{x}^2}{N} - Nm^2 \right) \tag{B1}
\]

and two invariance breaking terms: one that leads to the canonical algebra:

\[
(x^\mu(t), x^\nu(t))_c = 0, \quad (x^\mu(t), \dot{x}^\nu(t))_c = N(t), \quad (\dot{x}^i(t), \dot{x}^j(t))_c = 0, \quad (A, N(t))_c = 0 \tag{B2}
\]

for any function \( A \) on \( \mathcal{E} \), and one that leads to the deparameterized algebra:

\[
(x^0(t), A)_d = 0, \quad (N(t), A)_d = (\frac{\sqrt{-\dot{x}^2(t)}}{m}, A)_d, \quad (x^i(t), \dot{x}^j(t))_d = \delta^{ij}, \quad (\dot{x}^i(t), \dot{x}^j(t))_d = 0 \tag{B3}
\]

for any function \( A \) on \( \mathcal{S} \). Here, \( i, j \in \{1, 2, 3\} \) and \( \mu, \nu \in \{0, 1, 2, 3\} \). That such terms exist are guaranteed by section IV and Appendix D.

We now note that any quantity of the form \( \int_{t_1}^{t_2} A(t) dt \) where \( A(t) \) is a scalar under time reparameterizations is gauge invariant since gauge transformations must vanish on the boundary. We then compute the bracket of

\[
T = \int_{t_1}^{t_2} N(t) dt \quad \text{and} \quad X^\mu = \int_{t_1}^{t_2} N(t)x^\mu(t) dt \tag{B4}
\]

in each algebra. Clearly, \( (T, X^\mu)_c = 0 \) and \( (T, X^0)_d = 0 \). However,

\[
(T, X^i)_d = T \int_{t_1}^{t_2} \left( \frac{\dot{x}^0}{\sqrt{m^2 + p^2(t_1)}}, x^i(t) \right)_d = TX^0 \frac{p^i}{(m^2 + p^2)^{3/2}} \neq 0 \tag{B5}
\]

where \( p^i(t) = \dot{x}^i(t)/\sqrt{-\dot{x}^2(t)} \) and the algebras disagree.

A more careful treatment of \( F_{ij} \) and \( G^{ij} \) near the past and future boundaries than that of [8] would clarify the standing of such invariants on. However, such a treatment is
complicated by the fact that, for most systems, gauge transformations must vanish on the boundary, even when the proper boundary terms are included in the action. This means that gauge transformations near the boundary cannot be described by generators $Q^i_\alpha$ in quite the same way as in the interior. The status of $Q^i_\alpha$ when $i$ lies on a boundary is unclear, so that the methods of [8] cannot be used directly.

Note that such difficulties never arise when comparing algebras on $S$. This is because, using the equations of motion, any gauge invariant can be written in terms of initial data on a single Cauchy surface. Its algebraic properties are then determined by a gauge invariant function with compact spacetime support interior to the support of $S$. It is only on $E$ and $H$ that each instant of time introduces genuinely new operators, such as $N(t)$ in this example. These lead to gauge invariants on $E$ like $T$ and $X^\mu$ that are not determined by the initial data.

APPENDIX C: FAILURES OF THE JACOBI IDENTITY

This appendix contains two examples of operators $F_{ij}$ such that the brackets they would define though Eq. 3.6 do not satisfy the Jacobi identity. We will of course choose these $F_{ij}$ carefully so that they do not fall into either of the two classes for which the Jacobi identity is guaranteed. In particular, for neither case is $F_{ij,k}$ symmetric and we will choose one $F_{ij}$ that cannot be written in the form 3.3 with a $P_{ai}$ that satisfies Eq. 3.8 but such that $F_{\alpha\beta}$ is invertible without past or future boundary conditions and one $F_{ij}$ for which $P_{ai}$ does satisfy 3.8 but for which the advanced and retarded Green’s functions of $F_{\alpha\beta}$ are distinct. Thus, we show that neither condition that defines class II is sufficient by itself to derive the Jacobi identity.

For the first example, consider a nonrelativistic particle in $\mathcal{R}^3$ with one component of its momentum constrained to vanish and a corresponding translational gauge symmetry. In an appropriate coordinate system, the canonical action for this system takes the form:

$$S = \int dt\left[p_i\dot{x}^i - \dot{p}^i p_i/2 - \lambda(p_1 + p_2 + p_3)\right] \quad (C1)$$
where \( \lambda \) is a Lagrange multiplier and \( i \in \{1, 2, 3\} \). Since there is only one gauge symmetry, the invariance parameter \( \alpha \) is a time parameter. The symmetry generators are

\[
Q^x(t_1) = -\delta(t_1 - t_2), \quad Q^p(t_1) = 0, \quad \text{and} \quad Q^\lambda(t_1) = -\delta'(t_1 - t_2)
\]

(C2)

where \( ' \) denotes a derivative with respect to the argument. If we choose \( \eta^x(t_2) = \gamma \delta(t_1 - t_2) \), \( P_{\lambda(t_1)t_2} = 0 = P_{p(t_1)t_2} \), and

\[
P_{x^1(t_1)t_2} = x^2(t_1)\delta(t_1 - t_2), \quad P_{x^2(t_1)t_2} = x^3(t_1)\delta(t_1 - t_2), \quad P_{x^3(t_1)t_2} = x^1(t_1)\delta(t_1 - t_2)
\]

(C3)

then \( F_{t_1t_2} = -(x^1 + x^2 + x^3)^2\delta(t_1 - t_2) \) is invertible without past or future boundary conditions.

A short calculation shows that the corresponding operator \( F_{ij} \) cannot be written in the form \[3.3\] with a \( P_{\alpha i} \) that satisfies \[3.8\]. A rather long calculation shows that \((x^1 + x^2 + x^3)^2\epsilon_{ijk}(x^i(t_1), x^j(t_2), x^k(t_3))\) when evaluated at \( \dot{x}^i = p^i = 0 \), is a polynomial in \( x^1, x^2, x^3, t_1, t_2, t_3 \) with coefficient \(-3\) for the \((t_1)^2(x^2)^4\) term so that \( \epsilon_{ijk}(x^i(t_1), x^j(t_2), x^k(t_3)) \neq 0 \) and this bracket does not satisfy the Jacobi identity on either \( \mathcal{S} \) or \( \mathcal{H} \).

For the second example, we describe the relativistic free particle by the action \( S = \int dt \sqrt{(-\ddot{x})} \) so that the invariance generators are \( Q^{t_1}_{t_2} = -\dot{x}^\mu \delta(t_1 - t_2) \), which generate time reparameterizations. If we choose \( \eta^t(t_2) = \frac{m_0}{\sqrt{-x^2}} \delta(t_1 - t_2) \) and

\[
P_{t_2(t_1)} = -(\sqrt{-\dot{x}(t_2)^2})_{x^\nu(t_1)} = -\frac{\dot{x}^\mu}{\sqrt{-x^2}} \delta'(t_1 - t_2)
\]

(C4)

then \( F_{t_1t_2} = -\frac{\partial}{\partial t_1}(\delta(t_1 - t_2) \sqrt{-\dot{x}(t_2)^2}) \) is invertible.

A much shorter computation than for the first example shows that

\[
((x^\mu(t), x^\nu(t')), x^\lambda(t'')) = \int_{t_1}^{t_2} ds [\dot{x}^\lambda(s) \eta^\mu\nu + \frac{\gamma + 1}{\gamma} (\eta^\mu\lambda \dot{x}^\nu(s) + \eta^\nu\lambda \dot{x}^\mu(s)) - \frac{2\gamma^2 + \gamma + 1}{\gamma^2} \dot{x}^\lambda(s) \dot{x}^\mu(s) \dot{x}^\nu(s)]
\]

(C5)

so that when \( \dot{x}^1 = 1, \dot{x}^2 = 1 \) we have

\[
\sum_{i,j,k \in \{1,2,3\}} \epsilon^{ijk}((x^\mu(t_i), x^\nu(t_j)), x^\lambda(t_k)) = (t_1 - t_2) \neq 0
\]

(C6)

for \( \mu_1 = \mu_2 = 1, \mu_3 = 2 \) and this bracket does not satisfy the Jacobi identity either.
APPENDIX D: SOLVING EQUATIONS OF MOTION

In this appendix we show that, modulo one assumption, the result of a well-defined pull back of a generalized Peierls algebra to a space $\hat{\mathcal{S}}$ of partial solutions is another generalized Peierls algebra. Specifically, consider such a space $\hat{\mathcal{S}}$ and the inclusion map $I : \hat{\mathcal{S}} \to \mathcal{H}$. We again assume that we have a local product structure such that the coordinates $\phi^i$ on $\mathcal{H}$ separate into two classes: $\{\phi^a\}$ and $\{\phi^z\}$ such that the pull backs $\phi^a \circ I$ to $\hat{\mathcal{S}}$ form coordinates on $\hat{\mathcal{S}}$ while the equations of motion $S_z = 0$ that follow from the second set are identically satisfied on $\hat{\mathcal{S}}$: $S_z \circ I \equiv 0$. We will label these classes with indices from opposite ends of the alphabet.

Let $\hat{\mathcal{S}}$ be the pull back $S \circ I$. The variations of $\hat{\mathcal{S}}$ and $S$ are related by: $\hat{S}_{;ab} = (S_{;ij} \circ I) I_{;ia} I_{;jb}$ where the semicolons denote derivatives with respect to $\phi^a \circ I$ on $\hat{\mathcal{S}}$. The second derivatives are related by:

$$\hat{F}_{ab} = (F_{ij} \circ I) I_{;ia} I_{;jb}$$

(D1)

since $I_{;ab} = \delta_a^b$ so that $I_{;bc} = 0 = I_{;bz}$ and since $S_z \circ I = 0$.

Now, suppose that some invariance breaking form $P_{\alpha i}$ was introduced on $\mathcal{H}$. We define an invariance breaking form on $\hat{\mathcal{S}}$ by pull back: $\hat{P}_{\alpha i} \equiv (P_{\alpha i} \circ I) I_{;ia}$ and similarly $\eta^\alpha \beta \equiv \eta^{\alpha \beta} \circ I$.

We now have an operator $\hat{F}_{ab}$ that is the pull back of our operator $F_{ij}$ on $\mathcal{H}$:

$$\hat{F}_{ab} = (F_{ij} \circ I) I_{;ia} I_{;jb}$$

(D2)

Since $I$ is an embedding, we can choose some matrix $I_{;ia}'$ on $\hat{\mathcal{S}}$ such that $I_{;ia}' I_{;jb} = \delta^a_b$. We then note that $\hat{G}^{\pm ab} = I_{;i} G^{\pm ij} I_{;j}'$ are Green’s functions of $F_{ab}$ and that the Peierls algebra they define is the pull back of $(,)_\mathcal{H}$ since $(A, B)_\mathcal{H} \circ I = (A_{;i} \hat{G}^{ij} B_{;j}) \circ I = (A_{;i} \circ I) I_{;ia} (\hat{G}^{+ab} - \hat{G}^{-ab}) I_{;ib} (B_{;j} \circ I) = (A \circ I)_{;ia} (\hat{G}^{+ab} - \hat{G}^{-ab}) (B \circ I)_{;ib}$.

It follows that the pull back of the generalized Peierls algebra to $\hat{\mathcal{S}}$ is another generalized Peierls algebra. In particular, it is the one that results from pulling back the action and the invariance breaking form to $\hat{\mathcal{S}}$. 

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