CHARACTERIZING ALGEBRAIC STACKS

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Abstract. We extend the notion of algebraic stack to an arbitrary subcanonical site $C$. If the topology on $C$ is local on the target and satisfies descent for morphisms, we show that algebraic stacks are precisely those which are weakly equivalent to representable presheaves of groupoids whose domain map is a cover. This leads naturally to a definition of algebraic $n$-stacks. We also compare different sites naturally associated to a stack.

1. Introduction

Stacks arise naturally in the study of moduli problems in geometry. They were introduced by Giraud [Gi] and Grothendieck, and were used by Deligne and Mumford [DM] to study the moduli spaces of curves. They have recently become important also in differential geometry [Bry] and homotopy theory [G]. Higher order generalizations of stacks are also receiving much attention from algebraic geometers and homotopy theorists.

In this paper, we continue the study of stacks from the point of view of homotopy theory started in [H, H2]. The aim of these papers is to show that many properties of stacks and classical constructions with stacks are homotopy theoretic in nature. This homotopy theoretical understanding gives rise to a simpler and more powerful general theory. In [H] we introduced model category structures on different ambient categories in which stacks are the fibrant objects, and showed that they are all Quillen equivalent. In this paper we work with the simplest such model: the local model structure on presheaves of groupoids on site $C$, which we denote by $P(C, Grpd)_L$.

Deligne and Mumford introduced the notion of an algebraic stack in [DM, Definition 4.6]. This definition generalizes easily to an arbitrary site $C$ and our main result is a characterization of these (generalized) algebraic stacks on sites satisfying certain mild hypotheses.

A key observation is that the (2-category) fiber product (see Definition 2.1) is a model for the homotopy pullback in the model category $P(C, Grpd)_L$ and this allows us to rewrite the definition of representable morphism in the following homotopy invariant fashion:

$$f : M \to N \in P(C, Grpd)$$

is representable if for each $X \to N$ with $X \in C$, the homotopy pullback $M \times^h_N X$ is weakly equivalent to a representable.

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Generalizing [DM, Definition 4.4], we say that a presheaf of groupoids $\mathcal{M}$ on $\mathcal{C}$ is \textit{algebraic} if the diagonal $\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is a representable morphism and there exists a \textit{cover} $X \to \mathcal{M}$ with $X \in \mathcal{C}$. By cover we mean a representable morphism such that for all $Y \in \mathcal{C}$, the homotopy pullback

$$X \times^h_M Y \to Y$$

is weakly equivalent to a cover in $\mathcal{C}$.

We say that the (basis for the) topology on $\mathcal{C}$ is \textit{local} if the notion of cover is local on the target (Definition 4.2). This condition is satisfied by virtually all the topologies in use in algebraic geometry and one can always saturate a basis for a topology so that this condition is satisfied. A topology on $\mathcal{C}$ satisfies \textit{descent for morphisms} if the contravariant assignment $X \mapsto \text{Iso}(\mathcal{C}/X)$ is a stack (Definition 4.1). Our main result is then the following.

\textbf{Theorem 1.1} (Theorem 4.3). Let $\mathcal{C}$ be a Grothendieck topology which is local on the target and satisfies descent for morphisms. $\mathcal{M} \in P(\mathcal{C}, \text{Grpd})$ is algebraic if and only if $\mathcal{M}$ is weakly equivalent in $P(\mathcal{C}, \text{Grpd})_L$ to a representable presheaf of groupoids $(X_0, X_m)$ with the domain map $X_m \to X_0$ a cover in $\mathcal{C}$.

In particular, if $\mathcal{C} = \text{Aff}_{\text{flat}}$ (fpqc for affine schemes), this theorem characterizes algebraic stacks as those weakly equivalent to flat Hopf algebroids.

This result leads naturally to a definition of \textit{algebraic $\infty$-stacks} ($n$-stacks); they are those presheaves of simplicial sets on $\mathcal{C}$ which are weakly equivalent in $P(\mathcal{C}, sS\text{et})_L$ (see [DHI]) to (the $n$-coskeleton of) a simplicial object in $\mathcal{C}$ where all the boundary maps are covers.

In the appendix we consider several natural sites associated to a stack and compare them. The first is the classical site $\mathcal{C}/\mathcal{M}$ (see [DM]). In this topology, the objects are maps $X \to \mathcal{M}$ with $X \in \mathcal{C}$. It is natural to ask for a topology on the over category of $\mathcal{M}$ (in which $\mathcal{M}$ itself is an object). We use the notion of representability to construct a larger site $\text{Rep}/\mathcal{M}$ and prove that for $\mathcal{M}$ algebraic the sheaves on the two sites agree.

We also construct a topology on $P(\mathcal{C}, \text{Grpd})/\mathcal{M}$ where the covers are collections of fibrations $U_i \to \mathcal{M}$ such that the canonical map from the realization of the nerve $|U_i| \to \mathcal{M}$ is a weak equivalence. We characterize these covers as those sets of maps whose image is locally a covering sieve for the topology on $\mathcal{C}$ (see Proposition A.9).

\textbf{1.1. Relation to other work.} In his 2004 Northwestern thesis E. Pribble [P] constructs an equivalence of 2-categories between flat Hopf algebroids and rigidified algebraic stacks. This is essentially equivalent to Theorem 4.3 in case $\mathcal{C}$ is affine schemes in the flat topology.

\textbf{1.2. Acknowledgments.} I would like to thank G. Granja for helpful comments.

\textbf{1.3. Notation and conventions.} We assume that our fixed base site $\mathcal{C}$ is small and closed under finite products and pullbacks. By topology we mean what is usually called a basis for a topology [MM, Definition III.2.2]. We assume the topology is subcanonical, i.e. that the representable functors are sheaves, and identify the objects in $\mathcal{C}$ with the sheaves they represent.

We write $P(\mathcal{C}, \text{Grpd})$ for the category of presheaves of groupoids on $\mathcal{C}$. If $\{U_i \to X\}$ is a cover, we write $U = \coprod_i U_i$ for the coproduct of the sheaves and $U_* = \text{nerve of the cover}$ which is the simplicial object obtained by taking iterated fiber
products over $X$. We will sometimes abuse notation and write a cover as $U \to X$. $|U_{\bullet}|$ will denote the geometric realization of the simplicial object in $P(\mathcal{C}, \text{Grpd})$.

Recall that the geometric realization of a simplicial diagram $F_{\bullet}$ in $P(\mathcal{C}, \text{Grpd})$ is defined by $|F_{\bullet}(X)| = |F_{\bullet}(X)|$ (see [H] Section 2.2).

We will write $P(\mathcal{C}, \text{Grpd})$ for the category of presheaves of groupoids with the levelwise model structure where a map $F \to F'$ is a fibration (weak equivalence) if and only if $F(X) \to F'(X)$ is a fibration (weak equivalence) in $\text{Grpd}$ for all $X \in \mathcal{C}$. We will write $P(\mathcal{C}, \text{Grpd})_L$ for the local model structure which is the localization of $P(\mathcal{C}, \text{Grpd})$ with respect to the maps $|U_{\bullet}| \to X$ where $U \to X$ is a cover (see [H]). The local model structure $P(\mathcal{C}, \text{Grpd})_L$ is our default.

We will use repeatedly the basic result [H Theorem 5.7] which characterizes the weak equivalences in $P(\mathcal{C}, \text{Grpd})_L$ as those satisfying the local lifting conditions.

**Definition 1.2.** [H Definition 5.6] A map $F \to G \in P(\mathcal{C}, \text{Grpd})$ satisfies the local lifting conditions if

1. Given a commutative square

\[
\begin{array}{ccc}
\emptyset & \to & F(X) \\
\downarrow & & \downarrow \\
* & \to & G(X)
\end{array}
\Rightarrow \exists \text{ cover } U \to X,
\]

\[
\begin{array}{ccc}
\emptyset & \to & F(U) \\
\downarrow & & \downarrow \\
* & \to & G(U)
\end{array}
\]

1. For $A \to B$, one of the generating cofibrations $\partial \Delta^1 \to \Delta^1$, $B\mathbb{Z} \to \ast$, given a commutative square

\[
\begin{array}{ccc}
A & \to & F(X) \\
\downarrow & & \downarrow \\
B & \to & G(X)
\end{array}
\Rightarrow \exists \text{ cover } U \to X,
\]

\[
\begin{array}{ccc}
A & \to & F(U) \\
\downarrow & & \downarrow \\
B & \to & G(U)
\end{array}
\]

2. Fiber Product

In this section we will review the fiber product of stacks [LM-B Definition 2.2.2] from our homotopy theoretic point of view.

**Definition 2.1.** Let

\[M_1 \xrightarrow{i} N \xleftarrow{j} M_2\]

be a diagram in $P(\mathcal{C}, \text{Grpd})$. The homotopy fiber product $M_1 \times^h_{N} M_2$, is the presheaf of groupoids defined as follows:

1. the objects of $(M_1 \times^h_N M_2)(X)$ are triples $(a, b, \phi)$ with $a \in M_1(X), b \in M_2(X)$ and an isomorphism $\phi : i(a) \xrightarrow{\sim} j(b)$, and
2. morphisms of $(M_1 \times^h_N M_2)(X)$ from $(a, b, \phi)$ to $(a', b', \phi')$ are pairs $(\alpha, \beta)$ where $\alpha : a \cong a'$ and $\beta : b \cong b'$, such that $\phi' \circ i(\alpha) = j(\beta) \circ \phi$.

There are natural projections $p_i : M_1 \times^h_N M_2 \to M_i$ and natural homotopy $i \circ p_1 \to j \circ p_2$ which are universal in the following sense. To give a map $f : M \to M_1 \times^h_N M_2$ is the same as to give a maps $f_i : M \to M_i$ and a levelwise homotopy $i \circ f_1 \to j \circ f_2$.

The homotopy fiber product defined above is obviously the homotopy limit of the pullback diagram in the category $P(\mathcal{C}, \text{Grpd})$ with the levelwise model structure. In
fact, it also provides a model for the homotopy pullback in the local model structure as we now see.

**Lemma 2.2.** The homotopy fiber product of Definition 2.1 is a model for the homotopy pullback in $P(\mathcal{C}, \mathcal{G}_{rpd})_L$.

**Proof.** Consider the pullback diagram in Definition 2.1. Since $P(\mathcal{C}, \mathcal{G}_{rpd})_L$ is right proper [H Corollary 5.8] (and $P(\mathcal{C}, \mathcal{G}_{rpd})$ is obviously right proper), the homotopy fiber product in both of these model categories is obtained by replacing the map $M_2 \to N$ by a fibration and taking the pullback.

Factor $M_2 \to N$ into a trivial cofibration followed by a fibration $M_2 \simto M' \to N$ in $P(\mathcal{C}, \mathcal{G}_{rpd})_L$. Further factor $M_2 \simto M'$ into a levelwise trivial cofibration and a levelwise fibration $M_2 \simto M'' \to M'$. Then we have a levelwise weak equivalence $M_1 \times_N hM_2 \simeq M_1 \times_N M''$.

The map $M_1 \times_N M'' \to M_1 \times_N M'$ is the pullback of a levelwise fibration and weak equivalence and hence, by [H Corollary 5.8], it is itself a weak equivalence. □

**Remark 2.3.** Since homotopy limits commute with each other, if $M_1, M_2$ and $N$ in Definition 2.1 are stacks, (presheaves of groupoids satisfying the homotopy sheaf condition, see [H Definition 1.3]) the homotopy fiber product $M_1 \times_N hM_2$ is also a stack and agrees with what is usually called the fiber product of stacks [LM-B 2.2.2].

Given a groupoid object $(X_0, X_1)$ in $\mathcal{C}$ we abuse notation and let $(X_0, X_1)$ denote the presheaf of groupoids of which $X_0$ represents the objects and $X_m$ represents the morphisms. We let $M_{(X_0, X_1)}$ denote the fibrant replacement in $P(\mathcal{C}, \mathcal{G}_{rpd})_L$ of $(X_0, X_1)$, that is its stackification.

**Lemma 2.4.** Let $M$ be a presheaf of groupoids, $(X_0, X_1)$ be a groupoid object in $\mathcal{C}$, and $(X_0, X_1) \to M$ a weak equivalence in $P(\mathcal{C}, \mathcal{G}_{rpd})_L$. The map

$$X_1 \longrightarrow X_0 \times_M X_0$$

induced by the domain and range is a weak equivalence. If $M$ is a stack, it is a levelwise weak equivalence.

**Proof.** First we prove that the map is a weak equivalence for $M = M_{(X_0, X_1)}$. By Lemma 2.2 we need to verify the local lifting conditions of Definition 1.2 for the map $X_1 \to X_0 \times_M X_0$. By definition the map $(X_0, X_1) \to M$ is a weak equivalence and so by (1.2) given two objects $a, b \in X_0(Y)$ and an isomorphism between their images in $M(Y)$ there exists a cover $U \to Y \in \mathcal{C}$ such that this isomorphism lifts to $X_1(U)$. This implies that condition (1.2) holds for the map $X_1 \to X_0 \times_M X_0$.

Similarly, given $\phi_1, \phi_2 \in X_1(X)$, an isomorphism between their images in $(X_0 \times_M X_0)(X)$ is necessarily trivial (as $X_0(X)$ is discrete) and so the images of $\phi_1$ and $\phi_2$ in $M(X)$ are the same.

The fact that $(X_0, X_1) \to M$ satisfies condition (1.2) for the cofibration $BZ \to *$ guarantees the existence of the cover $U$ of $X$ such that $\phi_1$ and $\phi_2$ in $X_1(U)$. This proves one half of (1.2) for the map $X_0 \to X_0 \times_M X_0$ and the other half is automatic as $X_1(X)$ is discrete.

For general $M$, the fact that the map is a weak equivalence follows from the homotopy invariance of the homotopy fiber product (Lemma 2.2).
Since weak equivalences between fibrant objects in $P(\mathcal{C}, \mathfrak{g}p)_{L}$ are levelwise weak equivalences, if $M$ is a stack, the map is a levelwise weak equivalence.

**Remark 2.5.** If $M$ is a stack, the statement that $X_1 \to X_0 \times^h_M X_0$ is a levelwise weak equivalence means that evaluating at each $X \in \mathcal{C}$

$$(X_0, X_1)(X) \to M(X)$$

is bijective on Hom sets and that two objects with the same image in $M(X)$ are already isomorphic on $(X_0, X_1)(X)$. Thus this map is equivalent to the inclusion of a full subcategory of $M(X)$ for each $X \in \mathcal{C}$.

If $M$ is not a stack, composing the map with a fibrant replacement for $M$ shows that $(X_0, X_1)(X) \to M(X)$ is injective on morphisms and isomorphism classes.

3. **Representable Morphisms**

We begin by giving a definition of representable morphism in $P(\mathcal{C}, \mathfrak{g}p)$ generalizing the one for stacks in [DM, Definition 4.2]. Classically the definition of representable morphism applies only to maps between stacks, for which the following two notions agree (by Lemma 2.2).

**Definition 3.1.** A morphism $M \to N \in P(\mathcal{C}, \mathfrak{g}p)$ is called

- **strongly representable** if for each $X \in \mathcal{C}$ and each map $X \to N$, the homotopy fiber product $X \times^h_N M$ is levelwise weakly equivalent to a representable presheaf.
- **representable** if for each $X \in \mathcal{C}$ and each map $X \to N$, the homotopy fiber product $X \times^h_N M$ is weakly equivalent to a representable presheaf.

**Note 3.2.** Note the following easy consequence of the homotopy invariance of the homotopy pullback: If $f$ and $g$ are weakly equivalent morphisms in $P(\mathcal{C}, \mathfrak{g}p)_{L}$, (i.e. there exist $\alpha, \beta$ weak equivalences such that $\alpha \circ f = g \circ \beta$) then $f$ is representable if and only if $g$ is.

Representability allows one to extend certain properties of morphisms in $\mathcal{C}$ to arbitrary presheaves of groupoids.

**Definition 3.3.** Let $P$ be a property of morphisms in $\mathcal{C}$. We say $f : M \to N$ satisfies property $P$ if for all maps $X \to N$ with $X \in \mathcal{C}$, the map $X \times^h_N M \to X$ is weakly equivalent to a map in $\mathcal{C}$ which satisfies property $P$.

Similarly, a collection $\{U_i \to N\}$ is a cover if for each $X \to M$ with $X \in \mathcal{C}$, $\{U_i \times^h_N X \to X\}$ is weakly equivalent to a cover in $\mathcal{C}$.

Notice that if $f$ satisfies property $P$ as above then it is necessarily representable.

Given a presheaf of groupoids $F$ recall that $\pi_0 F$ is the presheaf of groupoids defined by $(\pi_0 F)(X) = \pi_0(F(X))$.

**Proposition 3.4.** A map $f : M \to N \in P(\mathcal{C}, \mathfrak{g}p)$ is representable iff any fibration $p : M' \to N'$ weakly equivalent to $f$ is strongly representable. In that case, for each map $X \to M$, (the sheaf) $\pi_0(X \times_M M')$ is isomorphic to a representable.

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1. In [LM-B, 3.9] for $\mathcal{C} = \text{Aff}_{\text{etale}}$ such morphisms are called schematic.
2. In the usual definition it is also required that $P$ is local on the target and stable under pullback (as in Definition 4.2). If $P$ is not stable under pullback then property $P$ for representable functors will be a stabilized version of the original property. For our purposes neither of these extra requirement makes a difference.
Proof. If \( p \) is strongly representable, \( f \) is obviously representable. For the converse note that since \( p \) is a fibration, given \( X \to \mathcal{N}' \), \( X \times_{\mathcal{N}'} \mathcal{M}' \) is levelwise weakly equivalent to the homotopy fiber product \( X \times_{\mathcal{N}'} \mathcal{M}' \), which is by assumption weakly equivalent to a representable. Now \( X \times_{\mathcal{N}'} \mathcal{M}' \to X \) is a fibration, representables are fibrant and weak equivalences between fibrant objects are levelwise weak equivalences. Hence \( X \times_{\mathcal{N}'} \mathcal{M} \) is levelwise weakly equivalent to a representable.

The second statement is clear. \( \square \)

Note 3.5. The previous lemma shows that a map \( f : \mathcal{M} \to \mathcal{N} \) is representable iff the associated map of stacks is strongly representable.

3.1. Generalized algebraic stacks. In this section we define the concept of a generalized algebraic presheaf of groupoids. We first recall the definition of algebraic stack which appears in [DM, 4.6]. This is usually called a Deligne-Mumford stack and we follow suite. We also recall the weakening of this which is usually called an algebraic stack [LM-B, 4.1].

Definition 3.6. Let \( S \) be a scheme and let \( \mathcal{C} \) be the category of \( S \)-schemes in the étale topology. A stack \( \mathcal{M} \) is called a Deligne-Mumford (resp. algebraic) stack if the diagonal \( \mathcal{M} \to \mathcal{M} \times \mathcal{M} \) is representable, separated and quasi-compact and if it admits an étale (resp. smooth) cover \( X \to \mathcal{M} \) with \( X \in \mathcal{C} \).

Definition 3.7. Let \( \mathcal{C} \) be a site. We say that \( \mathcal{M} \in P(\mathcal{C}, \text{Grpd}) \) is generalized algebraic if its diagonal is representable and there is a cover \( X \to \mathcal{M} \) with \( X \in \mathcal{C} \).

Note 3.8. The condition that the diagonal of \( \mathcal{M} \) be representable is equivalent to the requirement that for all \( X \to \mathcal{M}, Y \to \mathcal{M} \), with \( X, Y \in \mathcal{C} \) the product \( X \times_{\mathcal{M}} Y \) is weakly equivalent to a representable.

Lemma 3.9. The definition of generalized algebraic is invariant under weak equivalence. Thus a presheaf of groupoids is generalized algebraic if and only if its stackification is generalized algebraic.

Proof. If \( X \to \mathcal{M} \) is a cover and \( \mathcal{N} \to \mathcal{M} \) is a weak equivalence, the local lifting conditions [12] provide a cover of \( \mathcal{N} \). \( \square \)

4. Characterization of the generalized algebraic stacks

In this section we give a homotopy theoretic characterization of generalized algebraic stacks (Theorem [L3]). For this we will need the following definition which generalizes faithfully flat descent of morphisms [SGA, Theorem VIII.2.1].

Definition 4.1. Given a site \( \mathcal{C} \) consider the presheaf of groupoids on \( \mathcal{C} \) defined on objects by \( X \mapsto \text{iso}(\mathcal{C}/X) \) and on morphisms via pullback. We say that the site \( \mathcal{C} \) satisfies descent for morphisms if this is a stack.

Definition 4.2. We say that a topology on \( \mathcal{C} \) is local if the notion of cover is local on the target. This means that if \( \{ U_i \to X \} \) is a cover and \( \{ V_j \to X \} \) is a collection of morphisms such that \( \{ V_j \times_X U_i \to U_i \} \) is a cover for each \( i \) then \( \{ V_k \to X \} \) is also a cover.

\textsuperscript{3}Note that the definitions in [LM-B] use a weakened form of representability which only requires that the pullback be an algebraic space.
Theorem 4.3. Let $\mathcal{C}$ be a site which is local and satisfies descent for morphisms. Then $\mathcal{M}$ is a generalized algebraic presheaf of groupoids if and only if $\mathcal{M}$ is weakly equivalent in $P(\mathcal{C}, \mathcal{G}rpdl)$ to a groupoid object $(X_o, X_m)$ in $\mathcal{C}$, for which the domain map $X_m \to X_o$ is a cover.

The proof is broken down into the following two propositions.

Proposition 4.4. Let $\mathcal{M}$ be a generalized algebraic presheaf of groupoids and $X \to \mathcal{M}$ be a cover (in the sense of Definition 4.3) with $X \in \mathcal{C}$. Let $X_m$ denote the representable weakly equivalent to $X \times_h^M X$. Then the pair $(X, X_m)$ is a groupoid object in $\mathcal{C}$ and the natural map $(X, X_m) \to \mathcal{M}$ is a weak equivalence.

Proof. It suffices to prove this for $\mathcal{M}$ a stack. Given a generalized algebraic presheaf of groupoids there exists a representable morphism $X \to \mathcal{M}$ which is a cover. Let $X \sim \to \tilde{X} \to \mathcal{M}$ be a factorization as a trivial cofibration followed by a fibration, and let $\tilde{X}_\bullet$ denote the nerve of this cover.

$\tilde{X} \times_M X$ is levelwise weakly equivalent to a representable $X_m$ and $X_m \to X$ is a cover. Since $(\tilde{X}, \tilde{X} \times_M \tilde{X})$ is a groupoid object in the homotopy category so is $(X, X_m)$, and as $X$ and $X_m$ are both fibrant, cofibrant, and discrete $(X, X_m)$ is also a groupoid object in $P(\mathcal{C}, \mathcal{G}rpdl)$ and in $\mathcal{C}$.

Next we show that the map $(X, X_m) = |(X, X_m)_\bullet| \to \mathcal{M}$ is a weak equivalence by verifying that it satisfies the local lifting conditions. The first of the local lifting conditions follows from the fact that $X \to \mathcal{M}$ is a cover. By Remark 2.5 the map $(X, X_m) \to \mathcal{M}$ is levelwise equivalent to the inclusion of a full subcategory and so the second of the local lifting conditions is also satisfied (even not locally).

Proposition 4.5. Let $\mathcal{C}$ be a site which is local and satisfies descent for morphisms. If $(X_o, X_m)$ is a groupoid object in $\mathcal{C}$, with $X_m \to X_o$ a cover then the associated stack $\mathcal{M}(X_o, X_m)$ is generalized algebraic.

Proof. First we will show that under these hypothesis $X_o \to \mathcal{M} = \mathcal{M}(X_o, X_m)$ is representable.

Let $Y \in \mathcal{C}$ and $Y \to \mathcal{M}$ be a map in $P(\mathcal{C}, \mathcal{G}rpdl)$. Since $X_o \to \mathcal{M}$ is locally surjective there is a cover $U \to Y$ for which we have the following factorization

\[
\begin{array}{ccc}
U \times Y & \longrightarrow & U \\
\downarrow & & \downarrow \\
\tilde{X}_m & \longrightarrow & \tilde{X}_o \to \mathcal{M}
\end{array}
\]

By construction of the homotopy fiber product we obtain a simplicial diagram of fibrations $U \times Y \to \mathcal{M}$ augmented by $Y \times_M^h X_o \to Y$. Notice that besides $Y \times_M^h X_o$ all of the fiber products $(U \times Y U \cdots \times Y_U) \times_M^h X_o$ are levelwise weakly equivalent to representables, for example

$U \times_M^h X_o = U \times_{X_o} X_o \times_M^h X_o \sim U \times_{X_o} X_m$.

Fix $V \in \mathcal{C}$ and $\alpha : U \times_M^h X_o \sim V$, and let $\tilde{\alpha}$ be the induced isomorphism $\pi_0(U \times_M^h X_o) \sim \to V$. Let $pr_1, pr_2$ denote the projections $U \times Y U \to U$. It follows that we have weak equivalences

$\pi_0(U \times Y U \times_M^h X_o) \xrightarrow{pr_1^* \tilde{\alpha}} pr_1^* V$. 

and so we obtain an isomorphism over $U \times_Y U$,

$$[(pr_2^* \tilde{\alpha})^{-1} \circ pr_1^* \tilde{\alpha}] : pr_1^* V \to pr_2^* V$$

The simplicial identities imply that this isomorphism satisfies the hypothesis for descent for morphisms, which implies that there exists $V' \to Y \in \mathcal{C}$ together with an isomorphism $V \cong V' \times_Y U$ making the following diagram commute

$$U \times_Y U \times_Y h_M X_o \xrightarrow{\sim} U \times_Y U \times_Y h_M X_o \xrightarrow{\sim} U \times_Y U \times_Y h_M X_o \xrightarrow{\sim} U \times_Y V' \xrightarrow{\sim} V' \xrightarrow{\sim} U \times_Y U \xrightarrow{\sim} U \xrightarrow{\sim} Y$$

It follows that $V'$ is weakly equivalent to $|U_\bullet \times_Y V'| \xrightarrow{\sim} |U_\bullet \times_M X_o| \xrightarrow{\sim} Y \times_M h_M X_o$.

Essentially the same argument implies that the diagonal $\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable.

To see that $X_o \to \mathcal{M}$ is a cover, consider the pullback square

$$U \times_Y V' \cong U \times_{X_o} X_m \xrightarrow{\sim} V' \cong Y \times_M h_M X_o$$

The bottom horizontal arrow and left vertical one are covers since $X_m \to X_o$ is a cover. Since the topology on $\mathcal{C}$ is local $V' \to Y$ is a cover. □

By [SGA Exposé IX], we have the following corollary.

**Corollary 4.6.** Let $\mathcal{C} = \text{Schemes in the étale topology}$. Then $\mathcal{M}$ is a Deligne-Mumford stack in the sense of [DM, Definition 4.6] if and only if it is weakly equivalent to a groupoid object $(X_o, X_m)$ in $\mathcal{C}$, with $X_m \to X_o$ a cover.

The flat topology on affine schemes satisfies descent for morphisms by [SGA, Theorem VIII.2.1] so we have the following corollary.

**Corollary 4.7.** Let $\mathcal{C} = \text{Affine schemes in the flat topology}$. The generalized algebraic stacks are those stacks which are weakly equivalent to flat Hopf algebroids.

**Appendix A. Topologies on a stack**

We define two new sites $Rep/\mathcal{M}$ and $P(\mathcal{C}, \mathcal{G}_{rpd})/\mathcal{M}$ associated to a presheaf of groupoids $\mathcal{M}$. We show that if $\mathcal{M}$ is a generalized algebraic stack then the category of sheaves $Sh(Rep/\mathcal{M})$ agrees with the usual category of sheaves on $\mathcal{M}$. In addition we prove a comparison theorem explaining the relation between $Rep/\mathcal{M}$ and $P(\mathcal{C}, \mathcal{G}_{rpd})/\mathcal{M}$.

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4In geometric situations it is usually also required that the diagonal of $\mathcal{M}$ be quasi-compact and separated.
A.1. The site $\mathcal{C}/\mathcal{M}$. In this section we recall a site canonically associated to a presheaf of groupoids $\mathcal{M}$ first considered in [DM] Definition 4.10.

**Definition A.1.** Let $\mathcal{M}$ be in $P(\mathcal{C}, \mathcal{Grpd})$ and let $\mathcal{C}/\mathcal{M}$ denote the site whose

- **objects are pairs** $(X, f)$, where $X \in \mathcal{C}$ and $X \to \mathcal{M}$,
- **morphisms from** $X \to \mathcal{M}$ to $X' \to \mathcal{M}$ **are pairs** $(h, \alpha)$ where $X \to X'$ and $\alpha$ is a homotopy $f \to g \circ h$,
- **covers are collections of morphisms** which forget to covers in $\mathcal{C}$.

For a proof that this defines a Grothendieck topology see [H2] Section 2.1.

**Remark A.2.** Given maps $f, f' : X \to \mathcal{M}$ a homotopy $\alpha : f \to f'$ determines an isomorphism in $\mathcal{C}/\mathcal{M}$ between the objects $f$ and $f'$. So a presheaf $F$ on $\mathcal{C}/\mathcal{M}$ will satisfy $F(X, f) \cong F(X, f')$. The category $\mathcal{C}/\mathcal{M}$ is just the Grothendieck construction on the functor $\mathcal{M}$.

**Remark A.3.** Definition [A.1] generalizes the étale site [DM] 4.10 of a Deligne-Mumford stack which is the site defined above for $\mathcal{C}$ the category of schemes and étale maps in the étale topology. However, there is no site $\mathcal{C}$ which gives rise via [A.1] to the smooth-étale site [LM-B] 12.1 of an algebraic stack $\mathcal{M}$. For example, if we take $\mathcal{M}$ to be a scheme, the smooth-étale site is not the over category of $\mathcal{M}$ in some category of schemes which are the only kind of sites which arise via [A.1].

A.2. The site $\text{Rep}/\mathcal{M}$. The concept of representable morphism allows us to extend in a natural way the notion of cover to presheaves of groupoids and so gives rise to the following site.

**Definition A.4.** For $\mathcal{M}$ in $P(\mathcal{C}, \mathcal{Grpd})$ the site $\text{Rep}/\mathcal{M}$ has

- **Objects:** strongly representable morphisms $N \to \mathcal{M}$,
- **Morphisms from** $N_1 \to \mathcal{M}$ to $N_2 \to \mathcal{M}$ consist of pairs $(g, \alpha)$ with $g : N_1 \to N_2$ and $\alpha$ a homotopy $f_1 \to f_2 \circ g$.
- **Covers:** collections of morphisms $\{N_i \to \mathcal{N}\}$ such that the $u_i$ are strongly representable and for each $X \to \mathcal{N}$, $X \in \mathcal{C}$ the collection $\{X \times^h_{\mathcal{N}} N_i \to X\}$ is weakly equivalent to a cover in $\mathcal{C}$.

Note that the pullback in $\text{Rep}/\mathcal{M}$ is exactly the homotopy fiber product of Definition 2.1. It is also true that homotopy equivalences are isomorphisms in $\text{Rep}/\mathcal{M}$. The proof that $\text{Rep}/\mathcal{M}$ is a site is parallel to that for $\mathcal{C}/\mathcal{M}$.

**Proposition A.5.** Let $\mathcal{M}$ be a generalized algebraic stack then the category of sheaves on $\text{Rep}/\mathcal{M}$ is equivalent to the category of sheaves on the site $\mathcal{C}/\mathcal{M}$.

**Proof.** Since $\mathcal{M}$ is a generalized algebraic stack, $\mathcal{C}/\mathcal{M}$ embeds in $\text{Rep}/\mathcal{M}$ as a full subcategory. By [Ta] Proposition 3.9.1 it is enough to see that any object in $\text{Rep}/\mathcal{M}$ is covered by an object in $\mathcal{C}/\mathcal{M}$.

Given an object $f : N \to \mathcal{M}$ in $\text{Rep}/\mathcal{M}$, and a cover $X \to \mathcal{M}$ with $X \in \mathcal{C}$, $X \times^h_{\mathcal{M}} N$ is levelwise weakly equivalent to a representable $Z \cong \pi_0(X \times^h_{\mathcal{M}} N)$. The quotient map $p : X \times^h_{\mathcal{M}} N \to Z$ is a trivial fibration and $Z$ is cofibrant so $p$ is a homotopy equivalence and hence an isomorphism in $\text{Rep}/\mathcal{M}$. It follows that $Z \to X \times^h_{\mathcal{M}} N \to N$ is a cover in $\text{Rep}/\mathcal{M}$. □

**Note A.6.** One can make a definition analogous to Definition [A.4] using the concept of representable instead of strongly representable morphism, but then the result of
the previous proposition would not hold as local weak equivalences would not be isomorphisms in the category.

**Remark A.7.** Let \( \{ f_i : U_i \to N \} \) be a collection of representable morphisms and \( \{ f_i : U_i \to N \} \) be the family of fibrations obtained by functorial factorization in \( P(\mathcal{C}, \text{Grpd})_L \). Then the following are equivalent:

(i) The collection \( \{ f_i : U_i \to N \} \) is a cover in the sense of Definition 3.3.
(ii) The collection \( \{ f_i : U_i \to N \} \) is a cover in \( \text{Rep}/M \).

**A.3. The site \( P(\mathcal{C}, \text{Grpd})/M \).** We now define a site associated to \( M \) which is very natural from the point of view of homotopy theory and compare it to the ones discussed above.

**Theorem A.8.** Let \( \mathcal{C} \) be a site and \( M \in P(\mathcal{C}, \text{Grpd})_L \). Then there is a Grothendieck topology on \( P(\mathcal{C}, \text{Grpd})_L/M \) in which the covers are the sets of morphisms \( \{ U_i \to N \} \) which satisfy:

- \( U_i \to N \) are fibrations,
- \( |U_i| \to N \) is a weak equivalence.

**Proof.** First we prove that pullbacks of covers are covers. Let \( \coprod U_i \to N \) be a cover and \( M \to N \) a morphism. The morphism \( \coprod U_i \to N \) is an objectwise fibration and so the induced map \( |U_i| \to N \) is also an objectwise fibration. As geometric realization commutes with fiber products, \( |U_i| \times_N M \cong |U_i| \times_N M \) and so we have a pullback square

\[
\begin{array}{ccc}
|U_i \times_N M| & \longrightarrow & |U_i| \\
\downarrow & & \downarrow \\
\coprod U_i & \longrightarrow & N
\end{array}
\]

where the right vertical map is an objectwise fibration and a weak equivalence. By [H] Corollary 5.8 the pullback of a weak equivalence which is an objectwise fibration is a weak equivalence.

To see that covers compose, let \( \{ V_{ij} \to U_i \} \) be covers of each \( U_i \). The iterated fiber products of the covers \( \{ V_{ij} \to U_i \} \) form a bisimplicial object \( V_{\bullet, \bullet} \) augmented over \( U_{\bullet} \). The columns \( V_{n, \bullet} \) are iterated fiber products of the nerves of \( \{ V_{ij} \to U_i \} \) and therefore the map induced by the augmentation

\[
|V_{\bullet, \bullet}| \to |U_{\bullet}| \to N
\]

is a weak equivalence. The geometric realization of the bisimplicial object is equivalent to the geometric realization of its diagonal, so \( |\text{diag} V_{\bullet, \bullet}| \to N \) is a weak equivalence.

The nerve of the cover \( \{ V_{ij} \to N \} \) is the 0-th row \( V_{0, \bullet} \). Since \( V_{0, \bullet} \) is a 0-coskeleton over \( N \), there is a retraction to the canonical map \( V_{0, \bullet} \to \text{diag} V_{\bullet, \bullet} \) over \( N \) (see [DHI] Proposition A.4) and therefore \( V_{0, \bullet} \to N \) is a weak equivalence.

**Proposition A.9.** Given a collection \( \{ f_i : U_i \to N \} \) in \( P(\mathcal{C}, \text{Grpd})/M \) the following are equivalent:

(i) The collection \( \{ f_i : U_i \to N \} \) is a cover in \( P(\mathcal{C}, \text{Grpd})/M \).
(ii) For each \( X \to N \) the collection \( \{ U_i \times_N X \to X \} \) is a cover in \( P(\mathcal{C}, \text{Grpd})/M \).
(iii) For each \( X \to N \) the union of the images of \( U_i \times_N X \to X \) is a covering sieve of \( X \) in \( \mathcal{C} \).
Proof. The fact that (i) implies (ii) is a part of the axioms for a topology. First we prove that that (ii) implies (i). Given $X \in \mathcal{C}$, and $X \to N$, let $W$ denote $|U_\bullet|$, then the projection map $W \times_N X \to X$ is a weak equivalence since
\[ W \times_N X \cong |U_\bullet \times_N X| \cong |(U \times_N X)_\bullet|, \]
so this map is the induced map to $X$ from the nerve of the cover $U_i \times_N X \to X$.

Similarly, given any map $X \otimes \Delta^1 \to N$ the pullback $(X \otimes \Delta^1) \times_N W \to X \otimes \Delta^1$ is a weak equivalence since in the diagram
\[
\begin{array}{c}
W \times_N X \to W \times_N (X \otimes \Delta^1) \to W \\
\downarrow \sim \downarrow \downarrow \downarrow \\
X \sim \to X \otimes \Delta^1 \to N
\end{array}
\]
the top left map is a levelwise weak equivalence (because $W \to N$ is a levelwise fibration and $\mathcal{G}rpdl$ is right proper). It is now straightforward to check that $W \to N$ is a weak equivalence using the local lifting conditions [12].

To see that (ii) implies (iii) let $\{U_i \to X\}$ be a cover in $P(\mathcal{C}, \mathcal{G}rpdl)/\mathcal{M}$ and let $F$ be a sheaf on $\mathcal{C}$. $F$ is a discrete stack and so
\[ Map(X, F) \to \lim Map(U_\bullet, F) \cong \lim Map(U_\bullet, F) \cong Map(\text{colim } \pi_0 U_\bullet, F) \]
which shows that $X$ is the coequalizer of the sheafification $(\coprod \pi_0 U_{ij} \cong \coprod \pi_0 U_{ij})$, so by [MM Corollary III.7.7] the union of the images of $\pi_0 U_i \to X$ is a covering sieve in $\mathcal{C}$.

Conversely suppose that $\{U_i \to X\}$ generates a covering sieve. This means that $sh(\coprod \pi_0 U_i) \to X$ is a surjection of sheaves, from which it follows that $\text{colim } \pi_0 U_\bullet \to X$ is a weak equivalence. Since $U_\bullet$ is a 0-coskeleton in simplicial objects over $X$ the projection $|U_\bullet| \to \text{colim } \pi_0 U_\bullet$ is a levelwise weak equivalence. It follows that $\{U_i \to X\}$ is a cover in $P(\mathcal{C}, \mathcal{G}rpdl)/\mathcal{M}$.

Here is the relation between the notion of cover on $P(\mathcal{C}, \mathcal{G}rpdl)/\mathcal{M}$ just defined with the ones defined previously.

Corollary A.10. Let $\{f_i : U_i \to N\}$ be a collection of representable morphisms and $\{\tilde{f}_i : U_i \to N\}$ be fibrations obtained by functorial factorization in $P(\mathcal{C}, \mathcal{G}rpdl)_L$.

1. If $\{f_i : U_i \to N\}$ is a cover in the sense of Definition X.3 then $\{\tilde{f}_i : U_i \to N\}$ is a cover in $P(\mathcal{C}, \mathcal{G}rpdl)/\mathcal{M}$.

2. Conversely, if $\{\tilde{f}_i : U_i \to N\}$ is a cover in $P(\mathcal{C}, \mathcal{G}rpdl)/\mathcal{M}$ then for each $X \to N$ the collection $\{U_i \times_N X \to X\}$ determine a covering sieve in $\mathcal{C}$.

Let $\text{Sh}(\mathcal{M})$ be the category of sheaves on $P(\mathcal{C}, \mathcal{G}rpdl)_L/\mathcal{M}$ which take weak equivalences to isomorphisms. The above Corollary implies that we have a surjective restriction functor $\text{Sh}(\mathcal{M}) \to \text{Sh}(\mathcal{C}/\mathcal{M})$.

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