Sum of fractional series through extended $q$-difference operators

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Abstract
In this paper, we define the extended $q$-difference operator, $q$-polynomial factorial and inverse of the extended $q$-difference operator and obtain the relation between shift operator and extended $q$-polynomial factorials. Also, we obtain the formula for some fractional series of arithmetic and geometric progressions in the field of Numerical Methods using the inverse of extended $q$-difference operator. Suitable examples are provided to illustrate the main results.

Keywords
Extended q-difference operator, Finite Series, Infinite Series, Polynomial factorial.

AMS Subject Classification
39A12, 39A70, 47B39, 39B60.

1. Introduction
The theory of difference equations is based on the difference operator $\Delta$ defined as

$$\Delta u(k) = u(k+1) - u(k), k \in N = \{0, 1, 2, \cdots\}. \quad (1.1)$$

Also, many authors [1, 4, 5, 11] have suggested the definition of difference operator $\Delta_\ell$ as

$$\Delta_\ell u(k) = u(k+\ell) - u(k), k \in [0, \infty), \ell \in (0, \infty), \quad (1.2)$$

and no significant results developed in the field of numerical methods. In 2006, by taking the definition of $\Delta$ as given in (1.2) and the theory of difference equations was developed in a different direction and many interesting results were obtained in the field of Numerical Methods [6]-[10].

In the field of approximation theory, the applications of $q$-calculus are new area in last 30 years. The first $q$-analogue of the well-known Bernstein polynomials was introduced by Lupas in the year 1987. In 1997, Phillips considered another $q$-analogue of the classical Bernstein polynomials. Later several other researchers have proposed the $q$-extension of the well-known exponential-type operators which includes Baskakov operators, Szasz-Mirakyan operators, Meyer-Konig-Zeller operators, Bleiman, Butzer and Hahn operators, Picard operators, and Weierstrass operators. Also, the $q$-analogue of some standard integral operators of Kantorovich and Durrmeyer type was introduced, and their approximation properties were discussed [2].

In [12], while discussing the definition of $q-$derivative operator $\Delta_q$ as

$$\Delta_q u(k) = \frac{u(kq) - u(k)}{(q-1)k}, q \in (0, \infty),$$

and they didn’t developed any significant results in Numerical Methods. But recently, V.Chandrasekar and K.Suresh have generalized the definition of $\Delta_q$ by $\Delta_{q,\ell}$ as

$$\Delta_{q,\ell} u(k) = \frac{u(kq) - \ell u(k)}{(q-\ell)k}.$$
for the real valued function $u(k)$ and $\ell \in (0, \infty)$ and also obtained the several types of arithmetic-geometric progressions in the field of Numerical methods [3].

With this background, in this paper, we define the extended $q-$difference operator and derive the formula for fractional series in the field of Numerical Analysis using its inverse operator.

## 2. Preliminaries

In this section, we present some basic definitions and preliminary results for further subsequent discussions.

**Definition 2.1.** If $u(k)$ is real valued function, then we define the extended $q-$difference operator $\Delta_q^\ell (k)$ as

$$\Delta_q^\ell (u(k)) = u((k + \ell)q) - u(k), \quad q, \ell \in (0, \infty).$$

**Lemma 2.2.** The relation between $\Delta_q^\ell (1)$ and $E_q^\ell (1)$ is

$$E_q^\ell (1) = \Delta_q^\ell (1) + 1.$$  

**Proof.** The shift operator $E_q^\ell (1)$ is defined by

$$E_q^\ell (1) u(k) = u((k + \ell)q), \quad k \in [0, \infty).$$

The proof follows from (2.1) and (2.3).

**Lemma 2.3.** If $q, \ell \in N(1) = \{1, 2, \cdots \}$, then

$$1 + \Delta_q^\ell = (1 + \Delta_q^\ell)^q.$$  

**Lemma 2.4.** If $q$ and $\ell$ are positive reals and $n$ is positive integer, then

$$E_q^n = \sum_{r=0}^n nCr \Delta_q^{\ell r}.$$  

**Proof.** Equation (2.5) follows by (2.2).

The following two Lemma’s are easily deductions from $\Delta_q^\ell (1)$.

**Lemma 2.5.** Let $u(k)$ and $v(k) \neq 0$ be any two real valued functions. Then

$$\Delta_q^\ell [u(k)v(k)] = v((k + \ell)q) \Delta_q^\ell u(k) + u(k) \Delta_q^\ell v(k).$$  

**Lemma 2.6.** If $u(k)$ and $v(k) \neq 0$ are any two real valued functions, then

$$\Delta_q^\ell \left[ \frac{u(k)}{v(k)} \right] = \frac{v((k + \ell)q) \Delta_q^\ell u(k) - u(k) \Delta_q^\ell v(k)}{v(k)v((k + \ell)q)}.$$  

The following is the binomial theorem according to $\Delta_q^\ell (1)$.

**Theorem 2.7.** If $m$ and $n$ are any two positive integers, then

$$[(k + \ell)]^m = \frac{1}{q^m} \left[ \sum_{r=0}^m nCr \Delta_q^{\ell r} \right].$$

**Proof.** The proof follows by operating both sides on $u(k) = k^m$ in (2.5).

**Example 2.8.** If $\theta$ is in degrees taking only integer values in the anticlockwise direction then

$$[\sin (k + \theta)] = \frac{1}{q^m} \left[ \sum_{r=0}^m nCr \Delta_q^{\ell r}(k^m) \right].$$

**Proof.** The proof follows by taking $\ell = \theta$ and operating on $u(k) = \sin (k)$ in (2.5).

## 3. Extended $q-$Polynomial Factorial

In this section, we define the extended $q-$polynomial factorial, relation between $q-$polynomial factorial and $q-$difference operator according to $\Delta_q^\ell (1)$.

**Definition 3.1.** If $n$ is positive integer, then we define the extended $q-$polynomial factorial is denoted by $k_q^{(n)}$ is defined as

$$k_q^{(n)} = k \left( \frac{k - \ell}{q} \right) \left( \frac{k - 2\ell}{q^2} \right) \cdots \left( \frac{k - (n - 1)\ell}{q^n} \right).$$

**Lemma 3.2.** If $q$ and $\ell$ are positive reals and $n$ is a positive integer, then

$$\Delta_q^\ell k_q^{(n)} = k_q^{(n-1)} \left[ \frac{q^n - 1}{q^n - 1} k + C_q^{(n)} \right],$$

where $C_q^{(n)} = \frac{(q^n + (n-1)\ell)}{q^n - 1}$.

**Proof.** The proof follows from (2.1) and (3.1).

**Theorem 3.3.** If $k_q^{(n)}$ is extended $q-$polynomial factorial and $m,n$ are the any two positive integers then

$$\Delta_q^\ell k_q^{(n)} = \frac{(q^n - 1)}{q^n - 1} \Delta_q^{m-1} k_q^{(n,1)} \left[ k_q^{(n,1)} \right] + \frac{(q^n + (n-1)\ell)\ell}{q^n - 1} \Delta_q^{m-1} k_q^{(n-1)}.$$  

**Proof.** The proof follows by induction method on $m$ and $n$.

**Theorem 3.4.** If $q$ and $\ell$ are positive reals and $n$ is a negative integer, then

$$\Delta_q^\ell \left[ \frac{k + n\ell}{k + (n-1)\ell} \right] = \frac{(1 - q)\ell k - q\ell^2}{k + (n-1)\ell \left[ (k + \ell) q + (n-1)\ell \right]}.$$  

**Proof.** (3.4) follows from (2.1) and using lemma 2.6.
4. Inverse of Extended q-difference Operator

In this section, we define the inverse of extended q-difference operator and derived some interesting results using its inverse.

Definition 4.1. The inverse of extended q-difference operator denoted by $\Delta_{q(\ell)}^{-1}$ is defined as

$$\Delta_{q(\ell)} v(k) = u(k) \text{ then } v(k) = \Delta_{q(\ell)}^{-1} u(k) + c_j$$

(4.1)

and the $n$th order inverse operator denoted by $\Delta_{q(\ell)}^{-n}$ is defined as

$$\text{if } \Delta_{q(\ell)} v(k) = u(k) \text{ then } v(k) = \Delta_{q(\ell)}^{-n} u(k) + c_j,$$

where $c_j$ is a constant, depends upon $k \in N_{\ell}(j), j = k - \left[ \frac{k}{\ell} \right] \ell$.

Lemma 4.2. If $u(k)$ and $v(k) \neq 0$ are any two real valued functions, then

$$\Delta_{q(\ell)}^{-1} [u(k) v(k)] = u(k) \Delta_{q(\ell)}^{-1} v(k) - \Delta_{q(\ell)}^{-1} \left[ \Delta_{q(\ell)}^{-1} v((k + \ell)q) \Delta_{q(\ell)} u(k) \right].$$

Proof. The proof follows from (2.6) and Definition 4.1.

Theorem 4.3. If $k, \ell$ and $q$ are positive real values, then

$$\sum_{t=1}^{\frac{1}{\ell}} u \left( k - \ell \sum_{t=1}^{r} q^t \right) \Delta_{q(\ell)}^{-1} u(k) = \Delta_{q(\ell)}^{-1} \left( j_{q(\ell)} \right),$$

(4.2)

where $j_{q(\ell)} = \frac{k-\ell \left[ \frac{k}{q} \right]}{q^r}$.

Proof. The proof follows from (4.1) and the relation

$$\Delta_{q(\ell)} \left[ \sum_{t=0}^{\frac{1}{\ell}} u \left( k - \ell \sum_{t=1}^{r} q^t \right) \right] = u(k).$$

Lemma 4.4. For $\lambda \neq 1, k \geq 2q\ell$ and $P(k)$ is any function of $k$ then

$$\sum_{t=1}^{\frac{1}{\ell}} \lambda \left( \frac{k-\ell \sum_{t=1}^{r} q^t}{q^r} \right) p \left( k - \ell \sum_{t=1}^{r} q^t \right) = \frac{\lambda^{kq}}{\lambda^k - 1} - 1 - \frac{\lambda^{\Delta_{q(\ell)} u(k)}}{\Delta_{q(\ell)} u(k) - 1} + \frac{\lambda^{2\Delta_{q(\ell)} u(k) \Delta_{q(\ell)}}}{\Delta_{q(\ell)} u(k) - 1} + \cdots P(k) + c_j.$$

Proof. If $F(k)$ is any function of $k$ then

$$\Delta_{q(\ell)} \lambda^k F(k) = \lambda^{(k+\ell)q} F((k+\ell)q) - \lambda^k F(k)$$

$$= \lambda^{kq} \left[ \lambda^{qE_{q(\ell)}} - \lambda^{(1-q)k} \right] F(k)$$

$$= \lambda^{kq} P(k)$$

where

$$P(k) = \left[ \lambda^{qE_{q(\ell)}} - \lambda^{(1-q)k} \right] F(k) \text{ (or)}$$

$$\left( \lambda^{(1-q)k} \right)^{-1} \left[ \frac{\lambda^{qE_{q(\ell)}}}{\lambda^{(1-q)k}} - 1 \right]^{-1} P(k) = F(k).$$

Operating $\Delta_{q(\ell)}^{-1}$ on both sides of the equation

$$\Delta_{q(\ell)} \lambda^k F(k) = \lambda^{kq} P(k),$$

we get

$$\Delta_{q(\ell)}^{-1} \lambda^{kq} P(k) = \lambda^k F(k) + c$$

$$= \lambda^k \left( \lambda^{(1-q)k} \right)^{-1} \left[ \frac{\lambda^{qE_{q(\ell)}}}{\lambda^{(1-q)k}} - 1 \right]^{-1} P(k) + c_j.$$

The proof follows by (2.2), (4.1) and the Binomial theorem.

5. Applications in Numerical Methods

In this section, we derived some fractional series using the inverse of extended q-difference operators with suitable examples are provided.

Theorem 5.1. If $q$ and $\ell$ are positive reals, then

$$\sum_{t=1}^{\frac{1}{\ell}} q^{2t} \left[ \left( \frac{k-\ell \sum_{t=1}^{r} q^t}{q^r} \right)^2 + \ell^2 \left( 2q^2 - 2q + 1 \right) \left( \frac{k-\ell \sum_{t=1}^{r} q^t}{q^r} \right) \right] = k_{q(\ell)}^h.$$

Proof. (5.1) follows from (2.1) and lemma 4.2.

Example 5.2. Consider the fractional series

$$F = \frac{79002}{(27)(57)} + \frac{70686}{(729)(17)} + \frac{(1284822)(9)}{(16683)(103)} + \cdots + \frac{(1.32350526 \times 10^{11})(2187)}{(2.82495365 \times 10^{11})(43663)}$$

Solution: Taking $k = 65, \ell = 8, q = 3$ in (5.1), we get

$$F = 65 - \frac{65 - 8 \sum_{t=1}^{\frac{1}{8}} 3^t}{3^8}$$

$$= 65 + 11.98826398 = 76.98826398.$$
Theorem 5.3. If $k \in [0, \infty)$ and $n$ is a negative integer, then

$$\sum_{r=1}^{\lfloor \ell \rfloor} \left( 1 - q \right)^{\ell} \left( \frac{k - \ell \sum_{j=1}^{r} q^{j}}{q^{j}} \right) - q^{\ell^{2}}$$

Proof. From (2.1) and (4.1), we have

$$\Delta_{q^{\ell}}^{-1} \left[ \frac{1 - q^{\ell}}{(k + \ell)q + (n - 1)\ell} \right] = \left[ \frac{k + n\ell}{k + (n - 1)\ell} \right].$$

The proof follows from (4.2) and (5.3).

Example 5.4. Consider the fractional series

$$F = \frac{-78}{(-13)(13)} + \frac{-39}{(-26)(-13)} + \frac{-39}{(-39)(4096)} + \cdots + \frac{(376)(19683)}{-319475(159731)}$$

Solution: Substituting $n = -10, k = 46, \ell = 3,$ and $q = 2$ in (5.2), we get

$$F = \frac{16}{13} \left[ \frac{46 - 3 \sum_{j=1}^{15} 2^{j}}{2^{15}} + (-30) \right]$$

$$= 1.230769231 - 0.923073792$$

$$= 0.307695439.$$

Theorem 5.5. If $q$ and $\ell$ are positive reals and $n$ is a positive integer, then

$$\Delta_{q^{\ell}} \left[ \frac{k + n\ell}{k + (n + 1)\ell} \right] = \frac{(q - 1)\ell^{2}k + q^{\ell^{2}}}{(k + (n + 1)\ell)(k + (n + 1)\ell)}$$

Proof. By using (2.1) and lemma 2.6, we get (5.4).

Theorem 5.6. If $k \in [0, \infty)$ and $n \in N(1),$ then

$$\sum_{r=1}^{\lfloor \ell \rfloor} \left( q - 1 \right)^{\ell} \left( \frac{k - \ell \sum_{j=1}^{r} q^{j}}{q^{j}} \right) + q^{\ell^{2}}$$

$$= \frac{k + n\ell}{k + (n + 1)\ell^{1/\ell(\ell)}}.$$

Proof. From (2.1) and (4.1), we have

$$\Delta_{q^{\ell}}^{-1} \left[ (q - 1)\ell^{2}k + q^{\ell^{2}} \right] = \left[ \frac{k + n\ell}{k + (n + 1)\ell} \right].$$

The proof follows from (4.2) and the above relation.

Example 5.7. Consider the fractional series

$$F = \frac{376}{9125} + \frac{(376)(3)}{35125} + \frac{(376)(9)}{210469} + \cdots + \frac{(376)(19683)}{(7.85789644 \times 10^{1})}$$

Solution: Substituting $n = 7, k = 41, \ell = 4,$ and $q = 3$ in (5.5), we get

$$F = \frac{69}{73} - \left[ \frac{41 - 4 \sum_{j=1}^{10} y^{j}}{3^{60}} + 28 \right]$$

$$= 0.945205479 - 0.846158555$$

$$= 0.099046924.$$

6. Conclusion

In this paper, an advance has been developed for some results on the solutions of extended $q$-difference equations governed by (4.2) along with the function $u(k)$ in the field of Numerical analysis. Also, by selecting large value for $k$ and small positive value for $q$ and $\ell$ one can find the sum of several fractional series easily using the Theorem mentioned above.

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