GRADED BETTI NUMBERS OF POWERS OF IDEALS

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Abstract. In this work, by using the concept of vector partition functions, we investigate the asymptotic behavior of Betti numbers of powers of $\mathbb{Z}$-homogeneous ideals in the polynomial ring with its usual grading. In fact, we show that the Hilbert function of non-standard graded polynomial rings is quasi-polynomial. Applying this result, we prove our main result that states the Betti numbers of powers of homogeneous ideals have a quasi-polynomial behavior when the power gets large enough which generalizes the result of Kodiyalam on this issue.

More precisely, for the couple $(\mu, t) \in \mathbb{Z}^2$ with $\dim_k (\text{Tor}^S_i (I^t, k)_\mu) \neq 0$, $\mathbb{Z}^2$ can be splitted into a finite number of regions such that in each of them $\dim_k (\text{Tor}^S_i (I^t, k)_\mu)$ is a quasi-polynomial in $\mu, t$ for $t$ large enough.

1. Introduction

Homological invariants of powers of homogeneous ideals in a Noetherian standard $\mathbb{N}$-graded algebra was a very active area in last two decades. One of the most important results in this area is about asymptotic linearity of Castelnuovo-Mumford regularity which is studied by many people such as Kodiyalam [19], Cutkosky, Herzog and Trung [11] and Trung and Wang [26] independently. Their result states that if $I$ is a homogeneous ideal in a Noetherian standard $\mathbb{N}$-graded algebra and $M$ is a finitely generated $\mathbb{Z}$-graded $S$-module, therefore regularity $\text{reg}(I^t M)$ is asymptotically a linear function for $t$ large enough. The proof of Cutkosky, Herzog and Trung is based on concluding the linearity of $\text{end} (\text{Tor}^S_i (I^t, k))$.

On the other hand, looking at $\text{Tor}^S_i (I^t, k)$ as a finitely generated $\mathbb{Z}$-graded module, makes natural to ask about the asymptotic behavior of Betti numbers of $I^t$. Northcott and Rees have done the first work in this direction. They investigated the asymptotic behavior of the first number of $I^t$, $\beta^S_1 (I^t)$, see [22]. Then, in 1993, using the Hilbert-Serre theorem, Kodiyalam [18, Theorem 1] proved that for any non-negative integer $i$ and sufficiently large $t$, the $i$-th Betti number, $\beta^S_i (I^t)$, is a polynomial in $t$. Also he showed that the degree of this polynomial is bounded by constants independent of $i$, and is of degree at most $l(I) - 1$ for $t$ large enough. Here $l(I)$ denotes the analytic spread of $I$.

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Recently, in [4], the authors obtained more general results. In fact, they proved the asymptotic linearity of multigraded Betti numbers when the polynomial ring \( S = A[x_1, \ldots, x_n] \) is graded over a finitely generated abelian group \( G \) and Noetherian ring \( A \), see [4, Theorem 4.6]. Also, in the case that the ideal is equi-generated in \( \gamma \) (all of the generators of the ideal are of the same degree \( \gamma \)) and there is a ring homomorphism \( A \to k \), they proved that for any \( \delta \) and any \( j \), the functions

\[
\dim_k \Tor^A_j(\Tor^S_i(M^t, A)_{\delta + t\gamma}, k)
\]

and

\[
\dim_k \Tor^S_i(M^t, k)_{\delta + t\gamma}
\]

are polynomials in the \( t \) for \( t \gg 0 \), See [4, Theorem 3.3].

To prove the second part, they used the Rees algebra \( \mathcal{R}_I \) of the ideal \( I \) shifted by \( \gamma \) and then, they considered this Rees ring as a finitely generated \( G \times \mathbb{Z} \)-graded module over \( R = S[T_1, \ldots, T_r] \) such that for all \( 1 \leq i \leq n \), \( \deg_{G \times \mathbb{Z}}(x_i) = (\deg_G(x_i), 0) \in G \times 0 \) and for all \( 1 \leq j \leq r \), \( \deg_{G \times \mathbb{Z}}(T_j) = (0, 1) \in 0 \times \mathbb{Z} \). Then, in this way after reducing to degree \( (\ast, t) \), the problem changes to the standard graded ring \( A[T_1, \ldots, T_r] \) that has a polynomial as its Hilbert function.

This technique was not applicable to the general case when the ideal \( I \) is an arbitrary homogeneous ideal, because if \( I \) is generated in not necessarily equal degrees \( d_1, \ldots, d_r \), although \( \mathcal{R}_I \) is again a finitely generated module over \( R = S[T_1, \ldots, T_r] \), we have not the condition \( \deg_{G \times \mathbb{Z}}(T_i) \in 0 \times \mathbb{Z} \). In other words, after reducing to degree \( (\ast, t) \), the ring \( A[T_1, \ldots, T_r] \) is not a standard graded ring.

In this paper, we are interested in the behavior of \( \dim_k \Tor^S_i(M^t, k)_{\alpha} \) in its general case when \( I \) is an arbitrary ideal and \( S = k[x_1, \ldots, x_n] \) is a \( \mathbb{Z} \)-graded polynomial ring over a field \( k \), with \( \deg(x_i) > 0 \) for all \( i \). In fact we prove that \( \beta_{ij}(I^t) \) is a quasi-polynomial when \( t \) gets large enough.

The theorem below is our main result in this case.

**Theorem 1.1.** (See Theorem 4.5). Let \( S = k[x_1, \ldots, x_n] \) be a positively graded polynomial ring over a field \( k \) and let \( I \) be a homogeneous ideal in \( S \). There exist, \( t_0, m, \Delta \in \mathbb{Z} \), linear functions \( L_i(t) = a_i t + b_i \), for \( i = 0, \ldots, m \), with \( a_i \) among the degrees of the minimal generators of \( I \) and \( b_i \in \mathbb{Z} \), and polynomials \( Q_{i,j} \in \mathbb{Q}[x, y] \) for \( i = 1, \ldots, m \) and \( j = 1, \ldots, \Delta \), such that, for \( t \geq t_0 \),

(i) \( L_i(t) < L_j(t) \iff i < j \),

(ii) \( \Tor^S_i(M^t, k)_\mu = 0 \) if \( \mu < L_0(t) \) or \( \mu > L_m(t) \),

(iii) \( \dim_k \Tor^S_i(M^t, k)_\mu = Q_{i,j}(\mu, t) \) if \( L_{i-1}(t) \leq \mu \leq L_i(t) \) and \( a_i t - \mu \equiv j \mod(\Delta) \).

Similar to other people who investigated the asymptotic behavior of powers of ideals, we will also use the Rees algebra with respect to \( I \) with a \( \mathbb{Z} \times \mathbb{Z} \)-grading structure over the ring \( R = S[T_1, \ldots, T_r] \) where for all \( 1 \leq i \leq n \), \( \deg_{\mathbb{Z} \times \mathbb{Z}}(x_i) = (1, 0) \) and for all \( 1 \leq j \leq r \),
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deg_{\mathbb{Z} \times \mathbb{Z}}(T_i) = (d_i, 1). Also, we will use a combinatorial concept, called vector partition function, to obtain the (quasi-)polynomial behavior. We will relate the dimension of Tor^S(I^t, k)\alpha in a certain degree to the \mathbb{N}-solutions of linear equations. More precisely, we will transform the problem of computing the dimension of Tor^S(I^t, k)\alpha to a combinatorial concept that computes the number of solutions of a certain system(s) of linear equations.

This paper, is organized as follows. In the next section, we provide some definitions and terminology that we will need. In section 3, we will discuss about Hilbert functions of non-standard graded rings and in the last section, we prove the main theorem. Moreover, we provide an example and we compute the dimension of Tor^S(I^t, k) in certain degree.

2. Preliminaries

In this section, we are going to collect some necessary notations and terminology used in the paper. For basic facts in commutative algebra, we refer the reader to [12, 20].

2.1. Hilbert series. Let S = k[x_1, \ldots, x_n] be a polynomial ring over field k. We first make clear our definition of grading.

Definition 2.1. Let G be an abelian group. A G-grading of S is a morphism deg : \mathbb{Z}^n \rightarrow G and deg(x^u) := deg(u) for a monomial x^u = x_1^{u_1} \cdots x_n^{u_n} \in S. An element \sum c_u x^u \in S is homogeneous of degree \mu \in G if deg(u) = \mu whenever c_u \neq 0. If G is torsion-free and S_0 = k, the grading is positive.

Criterions of positivity are given in [21, 8.6]. One defines graded ideals and modules similarly to the classical \mathbb{Z}-graded case. When G = \mathbb{Z}^d and the grading is positive, (generalized) Laurent series are associated to finitely generated graded modules:

Definition 2.2. The Hilbert function of a finitely generated module M over a positively graded polynomial ring is the map:

HF(M; -) : \mathbb{Z}^d \rightarrow \mathbb{N} \quad \mu \mapsto \dim_k(M_\mu).

The Hilbert series of M is the Laurent series

H(M; t) = \sum_{\mu \in \mathbb{Z}^d} \dim_k(M_\mu) t^\mu.

Remark 2.3. By [21, 8.8], if S is positively graded by \mathbb{Z}^d, then the semigroup Q = deg(\mathbb{N}^n) can be embedded in \mathbb{N}^d. Hence, after such a change of embedding, the above Hilbert series are Laurent series in the usual sense.

Let M be a finitely generated \mathbb{Z}^d-graded S-module. It admits a finite minimal graded free S-resolution

\mathbb{F}_* : 0 \rightarrow F_u \rightarrow \ldots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0.
Writing
\[ F_i = \oplus \mu S(-\mu)^{\beta_{i,\mu}(M)}, \]
the minimality shows that \( \beta_{i,\mu}(M) = \dim_k (\text{Tor}^S_\mu(M, k))_{\mu} \); as the maps of \( \mathbb{F}_\bullet \otimes_S k \) are zero. We also recall that the support of a \( \mathbb{Z}^d \)-graded module \( N \) is
\[ \text{Supp}_{\mathbb{Z}^d}(N) := \{ \mu \in \mathbb{Z}^d | N_\mu \neq 0 \}, \]
and use the abbreviated notations \( \mathbb{Z}[t] := \mathbb{Z}[t_1, \ldots, t_d] \) for \( t = (t_1, \ldots, t_d) \) and \( t^\mu := t_1^{\mu_1} \cdots t_d^{\mu_d} \) for \( \mu = (\mu_1, \ldots, \mu_d) \in \mathbb{Z}^d \).

**Proposition 2.4.** Let \( S = k[x_1, \ldots, x_n] \) be a positively graded \( \mathbb{Z}^d \)-graded polynomial ring over the field \( k \). Then the following hold.

1. The Hilbert series of \( S \) is the development in Laurent series of the rational function
\[ H(S(-\mu); t) = \frac{t^\mu}{\prod_{i=1}^n (1 - t_{\mu_i})}, \]
where \( \mu_i = \deg(x_i) \).

2. If \( M \) is a finitely generated graded \( S \)-module, setting \( \Sigma_M := \cup_\ell \text{Supp}_{\mathbb{Z}^d}(\text{Tor}_\ell^R(B, k)) \) and
\[ \kappa_M(t) := \sum_{a \in \Sigma_M} \left( \sum_\ell (-1)^\ell \dim_k (\text{Tor}_\ell^R(B, k))_{a} \right) t^a, \]
one has \( H(M; t) = \kappa_M(t) H(S; t) \).

### 2.2. Vector partition function

We first recall the definition of quasi-polynomials. Let \( d \geq 1 \) and \( \Lambda \) be a lattice in \( \mathbb{Z}^d \).

**Definition 2.5.** \cite{24} A function \( f : \mathbb{Z}^d \to \mathbb{Q} \) is a quasi-polynomial with respect to \( \Lambda \) if there exists a list of polynomials \( Q_i \in \mathbb{Q}[T] \) for \( i \in \mathbb{Z}^d/\Lambda \) such that \( f(s) = Q_i(s) \) if \( s \equiv i \mod \Lambda \).

Notice that \( \mathbb{Z}^d/\Lambda \) has \( |\det(\Lambda)| \) elements, and that when \( d = 1, \Lambda = q\mathbb{Z} \) for some \( q > 0 \), in which case \( f \) is also called a quasi-polynomial of period \( q \).

Now assume that a positive grading of \( S \) by \( \mathbb{Z}^d \) with \( Q := \deg(\mathbb{N}^n) \subseteq \mathbb{N}^d \) is given and that \( Q \) spans a subgroup of rank \( d \) in \( \mathbb{Z}^d \). In other words, the matrix \( A = (a_{i,j}) \) representing \( \deg : \mathbb{Z}^n \to \mathbb{Z}^d \) is a \( d \times n \)-matrix of rank \( d \) with entries in \( \mathbb{N} \). Let \( a_j := (a_{1,j}, \ldots, a_{d,j}) \) and
\[ \varphi_A : \mathbb{N}^d \to \mathbb{N} \]
\[ u \to \# \{ \lambda \in \mathbb{N}^n | A.\lambda = u \}. \]

Equivalently, \( \varphi_A(u) \) is the coefficient of \( t^u \) in the formal power series \( \prod_{i=1}^n \frac{1}{(1-t^{\lambda_i})} \).

Notice that \( \varphi_A \) vanishes outside of \( \text{Pos}(A) := \{ \sum \lambda_i a_i \in \mathbb{R}^n | \lambda_i \geq 0, 1 \leq i \leq n \} \).

Blakley showed in \cite{6} that \( \mathbb{N}^n \) can be decomposed into a finite number of parts, called chambers, in such a way that \( \varphi_A \) is a quasi-polynomial of degree \( n - d \) in each chamber.
Later, Sturmfels in [25] investigated these decompositions and the differences of polynomials from one piece to another.

Here we briefly introduce the basic facts and necessary terminology of vector partition functions, specially the chambers and the polynomials (quasi-polynomials) obtained from vector partition functions corresponding to a matrix $A$. For more details about the vector partition function, we refer the reader to [6, 7, 25].

If $\sigma \subseteq \{1, \ldots, n\}$ is such that the $a_i$’s for $i \in \sigma$ are linearly independent, we will say that $\sigma$ is independent. We set $A_\sigma := (a_{i,j})_{1 \leq i \leq d, j \in \sigma}$ and denote by $\Lambda_\sigma$ the $\mathbb{Z}$-module with base the columns of $A_\sigma$. When $\sigma$ has $d$ elements (i.e. is a maximal independent set), $\Lambda_\sigma$ is a sublattice of $\mathbb{Z}^d$.

To such a $\sigma$, a chamber complex is attached whose maximal chamber is the interior of $\text{Pos}(A_\sigma)$, chambers of dimension one less are interiors of $\text{Pos}(A_{\sigma-\{i\}})$ for $i \in \sigma$, etc.

The simplicial complexes attached to $A_\sigma$ with $\sigma$ in the set of independent subsets of $\{1, \ldots, n\}$ admit a common refinement into a chamber complex whose cells are interior of simplicial convex cones. This common refinement is called the chamber complex associated to $A$.

If $C$ is a chamber of maximal dimension in the chamber complex associated to $A$, we set $\Delta(C) := \{\sigma \subseteq \{1, \ldots, n\} \mid C \subseteq \text{Pos}(A_\sigma)\}$ and say that $\sigma \in \Delta(C)$ is non-trivial if $G_\sigma := \mathbb{Z}^d/\Lambda_\sigma \neq 0$, equivalently if $\det(\Lambda_\sigma) \neq \pm 1$ ($G_\sigma$ is finite because $C \subseteq \text{Pos}(A_\sigma)$).

Now, we are ready to state the vector partition function theorem, which relies on the chamber decomposition of $\text{Pos}(A) \subseteq \mathbb{N}^d$.

**Theorem 2.6.** (See [25, Theorem 1]) For each chamber $C$ of maximal dimension in the chamber complex of $A$, there exist a polynomial $P$ of degree $n-d$, a collection of polynomials $Q_\sigma$ and functions $\Omega_\sigma : G_\sigma \setminus \{0\} \to \mathbb{Q}$ indexed by non-trivial $\sigma \in \Delta(C)$ such that, if $u \in \mathbb{N}A \cap C$,

$$
\varphi_A(u) = P(u) + \sum\{\Omega_\sigma([u]_\sigma).Q_\sigma(u) : \sigma \in \Delta(C), [u]_\sigma \neq 0\}
$$

where $[u]_\sigma$ denotes the image of $u$ in $G_\sigma$. Furthermore, $\deg(Q_\sigma) = \#\sigma - d$.

Notice that setting $\Lambda$ for the intersection of the lattices $\Lambda_\sigma$ with $\sigma$ maximal, one has the following corollary:

**Corollary 2.7.** For each chamber $C$ of maximal dimension in the chamber complex of $A$, there exists a collection of polynomials $Q_\tau$ for $\tau \in \mathbb{Z}^d/\Lambda$ such that

$$
\varphi_A(u) = Q_\tau(u), \text{ if } u \in \mathbb{N}A \cap C \text{ and } u - \tau \in \Lambda.
$$

**Proof.** The class $\tau$ of $u$ modulo $\Lambda$ determines $[u]_\sigma$ in $G_\sigma = \mathbb{Z}^d/\Lambda_\sigma$. The term of the right-hand side of the equations in the above theorem is a polynomial determined by $[u]_\sigma$, hence by $\tau$. \qed
In the next section, we will use this theorem to show that the Hilbert function of a non-standard graded ring is a quasi-polynomial.

3. Hilbert functions of non-standard bigraded rings

Let \( R = k[y_1, \ldots, y_m] \) be a \( \mathbb{Z}^{d-1} \)-graded polynomial ring over a field and \( I = (f_1, \ldots, f_n) \) a graded ideal, with \( f_i \) homogeneous of degree \( d_i \). To get information about the behavior of \( i \)-syzygy module of \( I \) as \( t \) varies, we pass to Rees algebra \( R_I = \bigoplus_{t \geq 0} I^t \) which is a \( (\mathbb{Z}^{d-1} \times \mathbb{Z}) \)-graded algebra such that \((R_I)_{(\mu,n)} = (I^n)_\mu\).

Recall that \( R_I \) is a graded quotient of \( B := R[x_1, \ldots, x_n] \) with grading extended from the one of \( R \) by setting \( \deg(a) := (\deg(a), 0) \) for \( a \in R \) and \( \deg(x_j) := (d_j, 1) \) for all \( j \). As noticed in [4], if \( G_\bullet \) is a \( \mathbb{Z}^d \)-graded free \( B \)-resolution of \( R_I \), then, setting \( S := k[x_1, \ldots, x_n] = B/(y_1, \ldots, y_m) \),

\[
\text{Tor}_i^S(I^t, A)_\mu = H_i(G_\bullet \otimes_B S)_{(\mu,t)}.
\]

The complex \( G_\bullet \otimes_B S \) is a \( \mathbb{Z}^d \)-graded complex of free \( S \)-modules. Its homology modules are therefore finitely generated \( \mathbb{Z}^d \)-graded \( S \)-modules, on which we will apply results derived from the ones on vector partition functions describing the Hilbert series of \( S \).

We begin with a lemma that describes the chamber complex associated to the matrix corresponding to the degrees \((d_i, 1)\), when \( d = 1 \) (i.e. \( d_i \in \mathbb{N} \)).

**Lemma 3.1.** Let

\[
A = \begin{pmatrix}
  d_1 & \cdots & d_n \\
  1 & \cdots & 1
\end{pmatrix}
\]

be a \( 2 \times n \)-matrix with entries in \( \mathbb{N} \) such that \( d_1 \leq \ldots \leq d_n \). Then the chambers corresponding to \( \text{Pos}(A) \) are positive polyhedral cones \((\Delta)\) where \( \Delta \) is generated by \( \{(d_i, 1), (d_{i+1}, 1)\} \) for all \( d_i \neq d_{i+1} \) where \( i \) runs over \( \{1, \ldots, n\} \).

**Proof.** Let \( \xi = \{\delta_1, \ldots, \delta_r\} \) be the set of column vectors in the matrix \( A \). In other words, we collect the columns without multiplicities in \( \xi \). Then we denote polyhedral subdivision of linear combinations of elements in \( \xi \) by \( C(\xi) \) where it is the common refinement of simplicial cones \( C(\sigma) \) which \( \sigma \) is the basis for \( \mathbb{R}_r \). Since any arbitrary pair \( \{(d_i, 1), (d_j, 1)\} \) makes an independent set whenever \( d_i \neq d_j \), therefore the common refinement consists of disjoint union of open convex polyhedral cones generated by \( \{(d_i, 1), (d_{i+1}, 1)\} \) for all \( i = 1, \ldots, n \).

Notice that the chambers are generated by distinct consecutive columns of matrix. In the case that we have some equal columns, we will use one of them and we ignore the others because they generate the same chambers.

We recalled the definition of univariate quasi-polynomial. In more general case, if the function \( f \) is defined on \( \mathbb{Z}^n \), the congruence comes from the number of integer points in
a parallelepiped of a certain lattice. Here we give one of the definitions of multivariate quasi-polynomials due to Barvinok [3].

**Definition 3.2.** Let $L$ be a lattice in $\mathbb{Z}^d$. A function $f : \mathbb{Z}^d \to \mathbb{Q}$ is a multivariate quasi-polynomial of period $L$ if there exists a list of $Q_i \in \mathbb{Q}[T_1, \ldots, T_d]$ for $i \in \mathbb{Z}^d/L$ such that $f(s) = Q_i(s)$ if $s \equiv i \mod L$.

Now we are ready to prove the main result of this section.

**Theorem 3.3.** Let $B = k[T_1, \ldots, T_n]$ be a bigraded polynomial ring over field $k$ with $\deg(T_i) = (d_i, 1)$. Then there exist a finite index sublattice $L$ of $\mathbb{Z}^2$ and collections of polynomials $P_i$ and $Q_{ij}$ for $1 \leq i \leq r - 1$ and $1 \leq j \leq s$ such that for any $(\mu, \nu) \in \mathbb{Z}^2 \cap R_i$ and $(\mu, \nu) \equiv g_j \mod \mathbb{Z}^2/L := \{g_1, \ldots, g_s\}$,

$$HF(B, (\mu, \nu)) = P_i(\mu, \nu) + Q_{ij}(\mu, \nu)$$

where $R_i$ is the convex polyhedral cone generated by linearly independent vectors $\{(d_i, 1), (d_{i+1}, 1)\}$ and $r$ is the number of distinct $d_i$’s.

Furthermore, $Q_{ij}(\mu, \nu) = Q_{ij}(\mu', \nu')$ if $\mu - \nu d_i \equiv \mu' - \nu' d_i \mod (\det(L))$.

**Proof.** Let

$$A = \begin{pmatrix} d_1 & \cdots & d_n \\ 1 & \cdots & 1 \end{pmatrix}$$

be a $2 \times n$-matrix corresponding to degrees of $T_i$.

The Hilbert function in degree $u = (\mu, \nu)$ is the number of monomials $T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ such that $\alpha_1(d_1, 1) + \ldots + \alpha_n(d_n, 1) = (\mu, \nu)$. This equation is equivalent to the system of linear equations

$$A. \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = (\mu \nu).$$

In this sense $HF(B, (\mu, \nu))$ will be the value of vector partition function at $(\mu, \nu)$. Assume that $(\mu, \nu)$ belongs to the chamber $R_i$ which is the convex polyhedral cone generated by $\{(d_i, 1), (d_{i+1}, 1)\}$. By 2.6 and 2.7, we know that

$$\varphi_A((\mu, \nu)) = P((\mu, \nu)) + Q_{ij}(\mu, \nu).$$

Notice that in the theorem 3.3, if moreover we suppose that $d_i \neq d_j$ for all $i \neq j$, then the Hilbert function in degree $(\mu, \nu)$ will also be the number of possible ways to reach from $(0, 0)$ to $(\mu, \nu)$ in $\mathbb{Z}^2$ but it is not necessarily correct when we have equalities between some of degrees. For example if one has $d_i = d_{i+1} < d_{i+2}$, so the independent sets of vectors $\{(d_i, 1), (d_{i+2}, 1)\}$ and $\{(d_{i+1}, 1), (d_{i+2}, 1)\}$ generate the same chamber and the number of possible ways to reach from $(0, 0)$ to $(\mu, \nu)$ is less than $HF(B, (\mu, \nu))$. 


In the following example, we are going to give a formula for Hilbert function of a non-standard graded polynomial ring in the case of three indeterminates which is a special case of formula done by Xu in [28].

**Example 3.4.** Let $B = k[T_1, T_2, T_3]$ be a polynomial ring over field $k$ and $\deg T_i = (d_i, 1)$ for $1 \leq i \leq 3$ such that $d_i - d_{i+1} \geq 0$ for $i = 1, 2$. Set $Y_{ij} = d_i - d_j$ and suppose that $\gcd(Y_{12}, Y_{13}, Y_{23}) = 1$. Then there exist $f_{ij}, g_{ij}$ such that

\[
\begin{align*}
f_{12}Y_{23} + g_{12}Y_{23} &= \gcd(Y_{23}, Y_{13}) \gcd(f_{12}Y_{13} + g_{12}Y_{23}, Y_{12}) = 1, \\
f_{13}Y_{12} + g_{13}Y_{23} &= \gcd(Y_{12}, Y_{23}) \gcd(f_{13}Y_{12} + g_{13}Y_{23}, Y_{13}) = 1, \\
f_{23}Y_{13} + g_{23}Y_{12} &= \gcd(Y_{13}, Y_{12}) \gcd(f_{23}Y_{13} + g_{23}Y_{12}, Y_{23}) = 1,
\end{align*}
\]

with

\[
\begin{align*}
(f_{12}Y_{13} + g_{12}Y_{23})^{-1} (f_{12}Y_{13} + g_{12}Y_{23}) &\equiv 1 \mod Y_{12} \\
(f_{13}Y_{12} + g_{13}Y_{23})^{-1} (f_{13}Y_{12} + g_{13}Y_{23}) &\equiv 1 \mod Y_{13} \\
(f_{23}Y_{13} + g_{23}Y_{12})^{-1} (f_{23}Y_{13} + g_{23}Y_{12}) &\equiv 1 \mod Y_{23}.
\end{align*}
\]

and $f_{12}, g_{12}, f_{13}, g_{13}, f_{23}$ and $g_{23}$ can be calculated by an Euclidean algorithm.

Our chambers are regions

\[
\Omega_i = \{(\mu, \nu) \mid \nu d_i > \mu > \nu d_{i+1}\}
\]

for $i = 1, 2$.

Then for $(n_1, n_2)$ belonging to the positive cone generated by vectors $\{(d_1, 1), (d_2, 1), (d_3, 1)\}$, when $(n_1, n_2)^t \in \Omega_1 \cap \mathbb{Z}^2$, it is proved in [28, Theorem 4.3] that

\[
\text{HF} (B, (n_1, n_2)) = \frac{n_2 d_1 - n_1}{Y_{12} Y_{13}} + 1
\]

\[
- \left\{ \frac{(f_{12}Y_{13} + g_{12}Y_{23})^{-1}(n_2(f_{12}d_1 + g_{12}d_2) - n_1(f_{12} + g_{12}))}{Y_{12}} \right\}
\]

\[
- \left\{ \frac{(f_{13}Y_{12} + g_{13}Y_{23})^{-1}(n_2(f_{13}d_1 + g_{13}d_3) - n_1(f_{13} + g_{13}))}{Y_{13}} \right\}.
\]

**4. Main Result**

We now turn to the main result on Betti numbers of powers of ideals. We can treat without any further effort the case of a collection of ideals and include a graded module $M$, namely to study the behaviour of $\dim_k \text{Tor}_i^R(M I_1^{t_1} \cdots I_s^{t_s}, k)_\mu$ for $\mu \in \mathbb{Z}^d$ and $t \gg 0$. To this aim we first use the important fact that the module

\[
B_i := \bigoplus_{t_1, \ldots, t_s} \text{Tor}_i^R(M I_1^{t_1} \cdots I_s^{t_s}, k)
\]

is a finitely generated $(\mathbb{Z}^p \times \mathbb{Z}^s)$-graded ring, over $k[T_{i,j}]$ setting $\deg(T_{i,j}) = (\deg(f_{i,j}), e_i)$ with $e_i$ the $i$-th canonical generator of $\mathbb{Z}^s$ and $f_{i,j}$'s the minimal generators of $I_i$. Hence $\text{Tor}_i^R(M I_1^{t_1} \cdots I_s^{t_s}, k)_\mu = (B_i)_{\mu, t_1 e_1 + \cdots + t_s e_s}$.
The following result applied to $B_i$ will then give the asymptotic behaviour of Betti numbers. In the particular case of one $\mathbb{Z}$-graded ideal, we will use it to give a simple description of this eventual behaviour.

Let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^d$ with $\varphi(N^n) \subseteq \mathbb{N}^d$ be a positive $\mathbb{Z}^d$-grading of $R := k[T_{i,j}]$. Set $\mathbb{Z}^n := \sum_{i=1}^n \mathbb{Z}e_i$, let $E$ be the set of $d$-tuples $e = (e_{i_1}, \ldots, e_{i_d})$ such that $(\varphi(e_{i_1}), \ldots, \varphi(e_{i_d}))$ generates a lattice $\Lambda_e$ in $\mathbb{Z}^d$, and set

$$\Lambda := \cap_{e \in E} \Lambda_e, \quad \mathcal{S}_\Lambda : \mathbb{Z}^d \xrightarrow{\text{can}} \mathbb{Z}^d / \Lambda. $$

Denote by $C_i, i \in F$, the maximal cells in the chamber complex associated to $\varphi$. One has $C_i = \{\xi \mid H_{i,j}(\xi) \geq 0, 1 \leq j \leq d\}$ where $H_{i,j}$ is a linear form in $\xi \in \mathbb{Z}^d$.

**Proposition 4.1.** With notations as above, let $B$ be a finitely generated $\mathbb{Z}^d$-graded $R$-module. There exist convex sets of dimension $d$ in $\mathbb{R}^d$ of the form

$$\Delta_u = \{x \mid H_{i,j}(x) \geq a_{u,i,j}, \forall (i,j) \in G_u\} \subseteq \mathbb{R}^d$$

for $u \in U, U$ finite, with $a_{u,i,j} = H_{i,j}(a)$ for $a \in \cup_\ell \text{Supp}_{\mathbb{Z}^d}(\text{Tor}^R_\ell (B, k)), G_u \subseteq F \times \{1, \ldots, d\}$ and polynomials $P_{u,\tau}$ for $u \in U$ and $\tau \in \mathbb{Z}^d / \Lambda$ such that:

$$\dim_k(B_\xi) = P_{u,s_\Lambda(\xi)}(\xi), \quad \forall \xi \in \Delta_u,$$

and $\dim_k(B_\xi) = 0$ if $\xi \notin \cup_{u \in U} \Delta_u$.

**Proof.** By Proposition 2.4, there exists a polynomial $\kappa_B(t_1, \ldots, t_d)$ with integral coefficients such that

$$H(B; t) = \kappa_B(t)H(R; t)$$

and $\kappa_B(t) = \sum_{a \in A} c_at^a$ with $A \subset \cup_\ell \text{Supp}_{\mathbb{Z}^d}(\text{Tor}^R_\ell (B, k))$. Let $D_i := \cup_\ell \{x \mid H_{i,j}(x) = 0\}$ be the minimal union of hyperplanes containing the border of $C_i$. The union $C$ of the convex sets $\overline{C}_i + a$ can be decomposed into a finite union of convex sets $\Delta_u$, each $u \in U$ corresponding to one connected component of $C \setminus \bigcup_{i,a} (D_i + a)$. (Notice that $\mathbb{R}^d \setminus \bigcup_{i,a} (D_i + a)$ has finitely many connected components, which are convex sets of the form of $\Delta_u$, and that we may drop the ones not contained in $C$ as the dimension of $B_\xi$ is zero for $\xi$ not contained in any $\overline{C}_i + a$.) We define $u$ as the set of pairs $(i,a)$ such that $(C_i + a) \cap \Delta_u \neq \emptyset$, and remark that if $(i,a) \in u$ then $(j,a) \notin u$ for $j \neq i$.

If $\xi \notin \bigcup_i \overline{C}_i + a$, then $\dim_k R_{\xi-a} = 0$, while if $\xi \in \overline{C}_i + a$ then it follows from Corollary 2.7 that there exist polynomials $Q_{i,\tau}$ such that

$$\dim_k R_{\xi-a} = Q_{i,\tau}(\xi-a) \quad \text{if} \quad \xi-a \equiv \tau \mod (\Lambda).$$

Hence, setting $Q'_{i,a,\tau}(\xi) := Q_{i,\tau+a}(\xi-a)$, one gets the conclusion with

$$P_{u,\tau} = \sum_{(i,a) \in u} c_{a}Q'_{i,a,\tau}.$$
Remark 4.2. The above proof shows that if one has a finite collection of modules $B_i$, setting $A := \cup_{i, \ell} \text{Supp}_{Z^d} \text{Tor}_{\ell}^R(B_i, k)$, there exist convex polyhedral cones $\Delta_u$ as above on which any $B_i$ has its Hilbert function given by a quasi-polynomial with respect to the lattice $\Lambda$.

We now turn to the main result of this article. The more simple, but important, case of powers of an ideal in a positively $\mathbb{Z}$-graded module over a field will be detailed just after.

Let $S = k[y_1, \ldots, y_m]$ be a $\mathbb{Z}^p$-graded polynomial ring over a field. Assume that $\deg(y_j) \in \mathbb{N}^p$ for any $j$, and let $I_i = (f_{i,1}, \ldots, f_{i,r_i})$ be ideals, with $f_{i,j}$ homogeneous of degree $d_{i,j}$.

Consider $R := k[T_{i,j}]_{1 \leq s, 1 \leq j \leq r_i}$, set $\deg(T_{i,j}) = (\deg(f_{i,j}), e_i)$, with $e_i$ the $i$-th canonical generator of $\mathbb{Z}^s$ and the induced grading $\varphi : \mathbb{Z}^{r_1+\cdots+r_s} \to \mathbb{Z}^d := \mathbb{Z}^p \times \mathbb{Z}^s$ of $R$.

Denote as above by $\Lambda$ the lattice in $\mathbb{Z}^d$ associated to $\varphi$, by $s_\Lambda : \mathbb{Z}^d \to \mathbb{Z}^d/\Lambda$ the canonical morphism and by $C_i$, for $i \in F$, the maximal cells in the the chamber complex associated to $\varphi$. One has $C_i = \{(\mu, t) \mid H_{i,j}(\mu, t) \geq 0, 1 \leq j \leq d\}$ where $H_{i,j}$ is a linear form in $(\mu, t) \in \mathbb{Z}^p \times \mathbb{Z}^s = \mathbb{Z}^d$.

**Theorem 4.3.** In the situation above, there exist a finite number of polyhedral convex cones

$$\Delta_u = \{(\mu, t) | H_{i,j}(\mu, t) \geq a_{u,i,j}, (i, j) \in G_u\} \subseteq \mathbb{R}^d,$$

polynomials $P_{\ell,u,\tau}$ for $u \in U$ and $\tau \in \mathbb{Z}^d/\Lambda$ such that, for any $\ell$,

$$\dim_k(\text{Tor}_{\ell}^R(MI_1^{r_1} \cdots I_m^{r_m}, k)_\mu) = P_{\ell,u,s_\Lambda}(\mu, t), \quad \forall (\mu, t) \in \Delta_u,$$

and $\dim_k(\text{Tor}_e^R(MI_1^{s_1} \cdots I_m^{s_m}, k)_\mu) = 0$ if $(\mu, t) \not\in \cup_{u \in U} \Delta_u$.

Furthermore, for any $(u, i, j)$, $a_{u,i,j} = H_{i,j}(b)$, for some

$$b \in \cup_{i,\ell} \text{Supp}_{Z^d} \text{Tor}_{\ell}^R(\text{Tor}_e^R(MR_{I_1, \ldots, I_s}, R), k).$$

**Proof.** We know from [4] that $B_i := \oplus_{t_{i,1}, \ldots, t_{i,s}} \text{Tor}_e^R(MI_1^{t_{i,1}} \cdots I_m^{t_{i,m}}, k)$ is a finitely generated $\mathbb{Z}^d$-graded module over $R$. As $B_i \neq 0$ for only finitely many $i$, the conclusion follows from Proposition 4.1 and Remark 4.2. \hfill \Box

The above results tell us that $\mathbb{R}^d$ could be decomposed in a finite union of convex polyhedral cones $\Delta_u$ on which, for any $\ell$, the dimension of $\text{Tor}_{\ell}^R(MI_1^{s_1} \cdots I_m^{s_m}, k)_\mu$, as a function of $(\mu, t) \in \mathbb{Z}^{p+s}$ is a quasi-polynomial with respect to a lattice determined by the degrees of the generators of the ideals $I_1, \ldots, I_s$.

This general finiteness statement may lead to pretty complex decompositions in general, that depend on the number of ideals and on arithmetic properties of the sets of degrees of generators. This complexity is reflected both by the covolume of $\Lambda$ as defined above and by the number of simplicial chambers in the chamber complex associated to $\varphi$.

We now detail an important special case: one ideal in a positively $\mathbb{Z}$-graded polynomial ring over a field.
We will use the following elementary lemma.

**Lemma 4.4.** Let $L_i^j$ be the half-line parallel to the vector $(d_i, 1)$ and passing through the point $(\beta_i^1, \beta_i^2)$ for $1 \leq s, t \leq r$ and $1 \leq i, j \leq N$. Then there exist permutations $\sigma_i$ in $S_N$ and a positive integer $n_0$ such that for all $n \geq n_0$, we have the following properties:

1. $L_{i}^{\sigma_i(1)}(n) \leq L_{i}^{\sigma_i(2)}(n) \leq \cdots \leq L_{i}^{\sigma_i(N)}(n)$ for $1 \leq i \leq r$,
2. $L_{i}^{\sigma_i(N)} \leq L_{i+1}^{\sigma_i(1)}$.

Moreover $n_0$ can be considered as the biggest width of intersection points.

**Proof.** If two arbitrary half-lines $L_i^j$ and $L_u^v$ meet together at the point $A(x_A, y_A)$, therefore it is easy to conclude that $y_A = \frac{\det \begin{pmatrix} \beta_1^v & d_u \\ \beta_2^v & 1 \end{pmatrix} - \det \begin{pmatrix} \beta_1^j & d_i \\ \beta_2^j & 1 \end{pmatrix}}{d_i - d_u}$. It is easily seen that $n_0$ can be considered as a max of intersection points. \qed

**Figure 1.** 3-Shifts.
Now we are ready to prove the main result of this paper. Let \( E := \{e_1, \ldots, e_s\} \) with \( e_1 < \cdots < e_s \) be a set of positive integers. For \( \ell \) from 1 to \( s - 1 \), let

\[
\Omega_\ell := \left\{ a \binom{e_\ell}{1} + b \binom{e_{\ell+1}}{1}, \ (a, b) \in \mathbb{R}_\geq 0^2 \right\}
\]

be the closed cone spanned by \( \binom{e_\ell}{1} \) and \( \binom{e_{\ell+1}}{1} \). For integers \( i \neq j \), let \( \Lambda_{i,j} \) be the lattice spanned by \( \binom{e_i}{1} \) and \( \binom{e_j}{1} \) and

\[
\Lambda_\ell := \bigcap_{i \leq \ell < j} \Lambda_{i,j}.
\]

In the case \( E := \{d_1, \ldots, d_r\}, e_1 = d_1 \) and \( e_s = d_r \), and, if \( s \geq 2 \), it follows from Theorem 2.6 that

(i) \( \dim_k B_{\mu,t} = 0 \) if \( (\mu, t) \notin \bigcup_\ell \Omega_\ell \),

(ii) \( \dim_k B_{\mu,t} \) is a quasi-polynomial with respect to the lattice \( \Lambda_\ell \) for \( (\mu, t) \in \Omega_\ell \).

Notice further that \( \Lambda := \bigcap_{i < j} \Lambda_{i,j} \) is a sublattice of \( \Lambda_\ell \) for any \( \ell \).

**Theorem 4.5.** In the above situation, set \( \Lambda_0 := \Lambda_s := \mathbb{Z}^2 \). If \( M \) is a finitely generated graded \( B \)-module, there exist \( t_0, N \) and \( L_i(t) := a_i t + b_i \) for \( i = 1, \ldots, N \) with \( b_i \in \mathbb{Z} \) and \( \{a_1, \ldots, a_N\} = E \) such that for \( t \geq t_0 \):

(i) \( L_i(t) < L_j(t) \iff i < j \),

(ii) \( M_{\mu,t} = 0 \) if \( \mu < L_1(t) \) or \( \mu > L_N(t) \),

(iii) For \( t \geq t_0 \) and \( L_i(t) \leq \mu \leq L_{i+1}(t) \), \( \dim_k(M_{\mu,t}) \) is a quasi-polynomial \( Q_i(\mu, t) \) with respect to the lattice \( \Lambda_\ell \) if \( e_\ell = a_i < a_{i+1} = e_{\ell+1} \) and with respect to the lattice \( \Lambda_\ell \cap \Lambda_{\ell-1} \) if \( e_\ell = a_i = a_{i+1} \).

**Proof.** (i) and (ii) are clear from 4.4. To prove (iii), since \( M \) is a finitely generated bigraded module, therefore it has a finite free resolution, say \( F_* \), as follows:

\[
0 \to F_t = \bigoplus_{j=1}^{m_t} B(-a_{tj}, -b_{tj})^{\beta_{tj}} \to \cdots \to F_1 = \bigoplus_{j=1}^{m_1} B(-a_{1j}, -b_{1j})^{\beta_{1j}} \to M \to 0.
\]

If we consider \( m = \max_{1 \leq i \leq t} \{m_i\} \), then we define the set of shifts as a subset of power set of \( \{1, \ldots, t\} \times \{1, \ldots, m\} \). More precisely, we set

\[
\Gamma \subseteq \mathcal{P}(\{1, \ldots, t\} \times \{1, \ldots, m\})
\]
such that for all $\sigma \in \Gamma$ and for all $(i, j), (i', j') \in \sigma$ we have $(-a_{ij}, -b_{ij}) = (-a_{ij'}, -b_{ij'})$. Now we can write down the Hilbert function of $M$ as the below explicit way:

$$HF(M, (\alpha, n)) = \sum_{i=1}^{t} (-1)^{i+1} HF(F_i, (\alpha, n))$$

$$= \sum_{i=1}^{t} (-1)^{i+1} \left( \sum_{j=1}^{m_i} B(-a_{ij}, -b_{ij})^{\beta_{ij}}, (\alpha, n) \right)$$

$$= \sum_{i=1}^{t} (-1)^{i+1} \left[ \sum_{j=1}^{m_i} HF(B(-a_{ij}, -b_{ij})^{\beta_{ij}}, (\alpha, n)) \right]$$

$$= \sum_{\sigma \in \Gamma} \left( \sum_{(i, j) \in \sigma} (-1)^{i+1} \beta_{ij} \right) HF(B(-a_{\sigma}, -b_{\sigma}), (\alpha, n))$$

$$= \sum_{\sigma \in \Gamma} \left( \sum_{(i, j) \in \sigma} (-1)^{i+1} \beta_{ij} \right) HF(B, (\alpha - a_{\sigma}, n - b_{\sigma})).$$

Since $\Gamma$ is a finite set, we set $\Gamma$ as

$$\Gamma = \{(\beta_1^1, \beta_1^2), \ldots, (\beta_1^N, \beta_1^N)\}.$$

Without loss of generality, we assume that $\beta_1^1 \leq \beta_1^2 \leq \ldots \leq \beta_1^N$. For any $j$, we define the non-negative integer

$$L_i^{\beta_j} = \begin{cases} 
(\alpha - \beta_1^1) - (n - \beta_2^j)d_i & \alpha \geq \beta_1^1 + (n - \beta_2^j)d_i \\
0 & \text{otherwise}
\end{cases}$$

which is the distance between $\alpha$ and half-lines $L_i^j$. Therefore for any $1 \leq i \leq r$, there exists a permutation $\sigma_i$ in $S_N$ such that

$$(\alpha - \beta_1^{\sigma_1(h)}) - (n - \beta_2^{\sigma_1(h)})d_i \leq (\alpha - \beta_1^{\sigma_{i+1}(h)}) - (n - \beta_2^{\sigma_{i+1}(h)})d_i.$$
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References

[1] T. V. Alekseevskaya, I. M. Gel’fand and A. V. Zelevinskii. An arrangement of real hyperplanes and the partition function connected with it. Soviet Math. Dokl. 36 (1988), 589-593.
[2] K. Baclawski and A.M. Garsia. Combinatorial decompositions of a class of rings. Adv. in Math. 39 (1981), 155-184.
[3] A. Barvinok. A course in convexity, American Mathematical Society. 2002, 366 pages.
[4] A. Bagheri, M. Chardin and H.T. Hà. The eventual shape of the Betti tables of powers of ideals. To appear in Math. Research Letters.
[5] D. Berlekamp. Regularity defect stabilization of powers of an ideal. Math. Res. Lett. 19 (2012), no. 1, 109-119.
[6] S. Blakley. Combinatorial remarks on partitions of a multipartite number. Duke Math. J. 31 (1964), 335-340.
[7] M. Brion and M. Vergne. Residue formulae, vector partition functions and lattice points in rational polytopes. J. Amer. Math. Soc.. 10(1997), 797-833.
[8] W. Bruns, C. Krattenthaler, and J. Uliczka. Stanley decompositions and Hilbert depth in the Koszul complex. J. Commutative Algebra, 2 (2010), no. 3, 327-357.
[9] W. Bruns and J. Herzog. Cohen-Macaulay rings. Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993.
[10] M. Chardin. Powers of ideals and the cohomology of stalks and fibers of morphisms. Preprint. arXiv:1009.1271.
[11] S.D. Cutkosky, J. Herzog and N.V. Trung. Asymptotic behaviour of the Castelnuovo-Mumford regularity. Compositio Mathematica, 118 (1999), 243-261.
[12] D. Eisenbud. Commutative Algebra: with a View Toward Algebraic Geometry. Springer-Verlag, New York, 1995.
[13] D. Eisenbud and J. Harris. Powers of ideals and fibers of morphisms. Math. Res. Lett. 17 (2010), no. 2, 267-273.
[14] D. Eisenbud and B. Ulrich. Stabilization of the regularity of powers of an ideal. Preprint. arXiv:1012.0951.
[15] W. Fulton and B. Sturmfels, Intersection Theory On Toric Varieties. Topology, 36, (1997) no. 2, 335-353.
[16] H.T. Hà. Asymptotic linearity of regularity and $a^*$-invariant of powers of ideals. Math. Res. Lett. 18 (2011), no. 1, 1-9.
[17] J. Herzog and D. Popescu. Finite filtrations of modules and shellable multicomplexes. Manuscripta Math., 121 (2006), no. 3, 385-410.
[18] V. Kodiyalam. Homological invariants of powers of an ideal. Proceedings of the American Mathematical Society, 118, no. 3, (1993), 757-764.
[19] V. Kodiyalam. *Asymptotic behaviour of Castelnuovo-Mumford regularity*. Proceedings of the American Mathematical Society, 128, no. 2, (1999), 407-411.

[20] H. Matsumura. *Commutative Ring Theory*. Cambridge, 1986.

[21] E. Miller and B. Sturmfels. *Combinatorial commutative algebra*. Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005

[22] D. G. Northcott and D. Rees. *Reductions of ideals in local rings*. Proc. Cambridge Philos. Soc. 50 (1954). 145-158.

[23] R. Stanley. *Combinatorics and commutative algebra*. Birkhäuser, Boston, 1983.

[24] R. Stanley. *Enumerative Combinatorics*. Vol 1, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997, With a foreward by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[25] B. Sturmfels. *On vector partition functions*. J. Combinatorial Theory, Series A 72(1995), 302-309.

[26] N.V. Trung and H. Wang. *On the asymptotic behavior of Castelnuovo-Mumford regularity*. J. Pure Appl. Algebra, 201 (2005), no. 1-3, 42-48.

[27] G. Whieldon. *Stabilization of Betti tables*. Preprint. arXiv:1106.2355.

[28] Z. Xu, *An explicit formulation for two dimensional vector partition functions*, *Integer points in polyhedra-geometry, number theory, representation theory, algebra, optimization, statistics*. Contemporary Mathematics, 452(2008), 163-178.

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