A note on the relationship between the Szlenk and $w^*$-dentability indices of arbitrary $w^*$-compact sets

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Abstract We prove the optimal estimate between the Szlenk and $w^*$-dentability indices of an arbitrary $w^*$-compact subset of the dual of a Banach space. For a given $w^*$-compact, convex subset $K$ of the dual of a Banach space, we introduce a two player game the winning strategies of which determine the Szlenk index of $K$. We give applications to the $w^*$-dentability index of a Banach space and of an operator.

Keywords Asplund space · Szlenk index · Dentability index

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1 Introduction

Since its inception in [14], the Szlenk index has been an important tool in renorming theory [8, 10, 13]. In [7], the notion of $\xi$-asymptotically uniformly smooth operators was given, with the 0-asymptotically uniformly smooth notion generalizing the notion of an asymptotically uniformly smooth Banach space. It was shown in [7] that an operator $A : X \to Y$ has Szlenk index not exceeding $\omega^{\xi+1}$ if and only if there exists an equivalent norm $|\cdot|$ on $Y$ making $A : X \to (Y, |\cdot|)$ $\xi$-asymptotically uniformly smooth. Applying this to the identity of a Banach space, we deduce that a Banach space $X$ has Szlenk index not exceeding $\omega^{\xi+1}$ if and only if there exists an equivalent norm $|\cdot|$ on $X$ such that $(X, |\cdot|)$ is $\xi$-asymptotically uniformly smooth.

Another index has been used to study the class of Asplund spaces, the $w^*$-dentability index. The $w^*$-dentability index is distinct from the Szlenk index, but each char-
characterizes \( w^* - \text{fragmentability} \) of a \( w^* - \text{compact} \) set. Since both indices characterize \( w^* - \text{fragmentability} \), it is natural to ask what relationship must exist between the indices. It follows immediately from the definitions that the Szlenk index of a set cannot exceed its \( w^* - \text{dentability} \) index. We discuss in the next section the different results obtained in the literature regarding the relationship between the \( w^* - \text{dentability} \) and Szlenk indices.

In what follows, \( S_z(K) \) (resp. \( D_z(K) \)) will denote the Szlenk (resp. \( w^* - \text{dentability} \) index) of the set \( K \).

**Theorem 1** Let \( X \) be a Banach space, let \( K \subset X^* \) be \( w^* - \text{compact} \), and let \( \xi \) be an ordinal.

1. If \( S_z(K) \leq \omega^\xi \), then \( D_z(K) \leq \omega^{1+\xi} \).
2. Suppose that \( K \) is convex. Then \( D_z(K) \leq \omega S_z(K) \), and if \( S_z(K) \geq \omega^\omega \), \( D_z(K) = S_z(K) \).

As was discussed in [9], for every \( n \in \mathbb{N} \cup \{0\} \), there exist Banach spaces \( X_n, Y_n \) such that \( S_z(X_n) = S_z(Y_n) = \omega^n \) while \( D_z(X_n) = \omega^{n+1} \) and \( D_z(Y_n) = \omega^n \). These examples show the sharpness of Theorem 1.

In [1], it was shown that one can compute the Szlenk index of a separable Banach space containing no isomorphism of \( \ell_1 \) by considering convex combinations of the branches of trees of vectors satisfying a certain weak nullity condition. We also recall a particular two-player game played on a Banach space. For \( \varepsilon > 0 \) and every \( n \in \mathbb{N} \), Player I chooses a subspace \( Z_1^n \) of \( X \) such that \( \dim(X/Z_1^n) < \infty \), Player II chooses a vector \( x_1^n \in B_{Z_1^n}, \ldots, \) Player I chooses a subspace \( Z_n^n \) of \( X \) such that \( \dim(X/Z_n^n) < \infty \), and Player II chooses a vector \( x_n \in B_{Z_n^n} \). We say that Player II wins the game if for every \( n \in \mathbb{N} \), \( \| n^{-1} \sum_{i=1}^n x_i^n \| > \varepsilon \), and Player I wins otherwise. Then if \( X \) is a separable Banach space not containing \( \ell_1 \), the results of [1] combined with the results of [10] imply that \( S_z(X) \leq \omega \) if and only if for every \( \varepsilon > 0 \), Player I has a winning strategy in this game. Since this game is determined, \( S_z(X) > \omega \) if and only if for some \( \varepsilon > 0 \), Player II has a winning strategy in this game. Note that we require a certain “smallness” condition on a specific convex combination \( n^{-1} \sum_{i=1}^n x_i^n \) of \( (x_i^n)_{i=1}^n \).

In [4], the results of [1] were extended to allow one to compute the Szlenk index of an arbitrary \( w^* - \text{compact} \) subset of the dual of an arbitrary Banach space. In analogy to the game defined above, we wish to define for a given ordinal \( \xi \) a certain game the winning strategies of which determine whether the Szlenk index of an arbitrary \( w^* - \text{compact} \) set exceeds \( \omega^\xi \). Given a Banach space \( X \), let \( D \) denote the subspaces of \( X \) having finite codimension in \( X \), and let \( K \) denote the norm-compact subsets of \( X \). Let \( K \subset X^* \) be \( w^* - \text{compact} \). Suppose that \( \Lambda \) is a set, \( T \) is a non-empty collection of non-empty sequences in \( \Lambda \) such that there does not exist an infinite sequence \( (\xi_i)_{i=1}^\infty \subset \Lambda \) all the finite initial segments of which lie in \( T \) (such a collection \( T \) is called a non-empty, well-founded B-tree). Assume also that \( \mathbb{P} : T \rightarrow \mathbb{R} \) is a fixed function. For \( \varepsilon > 0 \), we let Player I choose \( Z_1 \in D \) and \( \xi_1 \in \Lambda \) such that \( (\xi_1) \in T \). Player I then chooses \( C_1 \in K \). Next, assuming \( (\xi_i)_{i=1}^n \in T \), \( Z_1, \ldots, Z_n \in D \), and \( C_1, \ldots, C_n \in K \) have been chosen, if \( (\xi_i)_{i=1}^n \) has no proper extensions in \( T \), the game terminates. Otherwise Player I chooses \( Z_{n+1} \in D \) such that \( (\xi_i)_{i=1}^{n+1} \in T \) and \( Z_{n+1} \in D \). Player II chooses \( C_{n+1} \in K \). Our assumptions on \( T \) yield that this game must terminate after finitely many turns. Let us assume the game terminates with the
choices \((\zeta_i)_{i=1}^n, (Z_i)_{i=1}^n, (C_i)_{i=1}^n\). We say that Player II wins the game if there exist a sequence \((x_i)_{i=1}^n \in \prod_{i=1}^n (B_X \cap Z_i \cap C_i)\) and \(x^* \in K\) such that

\[
Re \ x^* \left( \sum_{i=1}^n \mathbb{P}((\zeta_j)_{j=1}^i)x_i \right) \geq \varepsilon,
\]

and let us say Player I wins otherwise. Let us refer to this as the \((\varepsilon, K, \mathbb{P})\) game on \(T.D.K\). Our main result in this direction is the following.

\textbf{Theorem 2} For every ordinal \(\xi\), there exists a non-empty, well-founded \(B\)-tree \(\Gamma_{\xi}\) on \([0, \omega^\xi]\) and a function \(\mathbb{P}_\xi : \Gamma_{\xi} \to \mathbb{R}\) such that for any Banach space \(X\) and any \(w^*-\)compact \(K \subset X^*\), \(SZ(K) > \omega^\xi\) if and only if there exists \(\varepsilon > 0\) such that Player II has a winning strategy in the \((\varepsilon, K, \mathbb{P}_\xi)\)-game on \(\Gamma_{\xi}.D.K\), and \(SZ(K) \leq \omega^\xi\) if and only if for every \(\varepsilon > 0\), Player I has a winning strategy in the \((\varepsilon, K, \mathbb{P}_\xi)\) game on \(\Gamma_{\xi}.D.K\).

\section*{2 Definitions}

\subsection*{2.1 Definition of the indices}

Let \(X\) be a Banach space and let \(K \subset X^*\). For \(\varepsilon > 0\), we let \(s_\varepsilon(K)\) denote those \(x^* \in K\) such that for every \(w^*-\)neighborhood \(V\) of \(x^*\), \(diam(V \cap K) > \varepsilon\). We let \(d_\varepsilon(K)\) denote those \(x^* \in K\) such that for every \(w^*-\)open slice \(S\) containing \(x^*\), \(diam(S \cap K) > \varepsilon\).

Recall that a \(w^*-\)open slice is a subset of \(X^*\) of the form \(\{y^* : Re y^*(x) > a\}\) for some \(x \in X\) and \(a \in \mathbb{R}\). We then define \(s_\varepsilon^0(K) = K, s_\varepsilon^{\xi+1}(K) = s_\varepsilon(s_\varepsilon^\xi(K))\), and \(s_\varepsilon^\xi(K) = \cap_{\xi < \eta} s_\varepsilon^\eta(K)\) when \(\xi\) is a limit ordinal. We set \(SZ(K, \varepsilon) = \min\{\xi : s_\varepsilon^\xi(K) = \emptyset\}\) if this class of ordinals is non-empty, and we set \(SZ(K, \varepsilon) = \infty\) otherwise. We let \(SZ(K) = \sup_{\varepsilon > 0} SZ(K, \varepsilon)\), where we agree that \(\xi < \infty\) for all ordinals \(\xi\). If \(X\) is a Banach space, we let \(SZ(X, \varepsilon) = SZ(B_{X^*}, \varepsilon)\) and \(SZ(X) = SZ(B_{X^*})\). If \(A : X \to Y\) is an operator, we let \(SZ(A, \varepsilon) = SZ(A^*B_Y, \varepsilon)\) and \(SZ(A) = SZ(A^*B_Y)\). We define \(d_\varepsilon^\xi(K), DZ(K, \varepsilon), DZ(K),\) etc., similarly. It is quite clear that \(SZ(K) \leq DZ(K)\).

We recall that \(K\) is said to be \(w^*-\)fragmentable provided that for every non-empty subset \(L \subset K\) and every \(\epsilon > 0\), there exists a \(w^*-\)open subset \(U\) of \(X^*\) such that \(U \cap L \neq \emptyset\) and \(\text{diam}(U \cap L) < \epsilon\). We say that \(K\) is \(w^*-\)dentable if for any non-empty subset \(L \subset K\) and every \(\epsilon > 0\), there exists a \(w^*-\)open slice \(S \subset X^*\) such that \(S \cap L \neq \emptyset\) and \(\text{diam}(S \cap L) < \epsilon\). It is clear that \(K\) is \(w^*-\)fragmentable (resp. \(w^*-\)dentable) if and only if \(SZ(K)\) (resp. \(DZ(K)\)) is an ordinal. Moreover, \(w^*-\)fragmentability and \(w^*-\)dentability are equivalent, which is a consequence of Theorem 1. Since these properties are equivalent, it is natural to consider the relationship between \(SZ(K)\) and \(DZ(K)\). Lancien [11] proved using descriptive set theoretic techniques that there exists a function \(\phi : [0, \omega_1] \to [0, \omega_1]\) such that if \(\xi < \omega_1\) and if \(X\) is a Banach space with \(SZ(X) < \xi\), \(DZ(X) < \phi(\xi)\). Raja [13] proved that for any Banach space (without assumption of countability of \(SZ(X)\)) that \(DZ(X) \leq \omega SZ(X)\). Hájek and Schlumprecht [9] showed that if \(SZ(X)\) is countable, \(DZ(X) \leq \omega SZ(X)\). The content of Theorem 1
extends this result of Hájek and Schlumprecht to the general case of an arbitrary $w^*$-compact, convex set $K$ as opposed to the case $K = B_{X^*}$, and removes the hypothesis of countability of $Sz(K)$.

We note that the most interesting case, of course, is the case $K = B_{X^*}$. However, the case $K = A^*B_{Y^*}$ for an operator $A : X \to Y$ is also of interest. We refer the reader to [2,6,7] for results concerning the Szlenk index of an operator, including renorming theorems for asymptotically uniformly smooth operators. However, to our knowledge, the $w^*$-dentability index of an operator has not been investigated.

2.2 $B$-trees

Given a set $\Lambda$, we let $\Lambda^{<\mathbb{N}}$ denote the finite sequences in $\Lambda$, including the empty sequence, $\emptyset$. We write $s \leq t$ if $s$ is an initial segment of $t$. If $t \in \Lambda^{<\mathbb{N}}$, we let $|t|$ denote the length of $t$ and for $0 \leq i \leq |t|$, $t_i$ is the initial segment of $t$ having length $i$. If $\emptyset \neq t$, we let $t^- = t_{|t|-1}$. We let $s^{-}\cup t$ denote the concatenation of $s$ and $t$. A subset $T$ of $\Lambda^{<\mathbb{N}}$ is called a tree if for all $t \in T$ and $s \leq t$, $s \in T$. A subset $T$ of $\Lambda^{<\mathbb{N}} \setminus \{\emptyset\}$ will be called a $B$-tree provided that for any $t \in T$ and any $\emptyset \prec s \leq t$, $s \in T$. We let $MAX(T)$ denote the members of $T$ which are $<\Lambda$-maximal and $T' = T \setminus MAX(T)$.

We define $T^0 = T$, $T^k+1 = (T^k)'$, and $T^k = \cap_{\xi<\mathbb{N}} T^\xi$ when $\xi$ is a limit ordinal. We say $T$ is well-founded if there exists an ordinal $\xi$ such that $T^\xi = \emptyset$, and we let $o(T)$ denote the smallest such $\xi$. If no such $\xi$ exists, we say $T$ is ill-founded and write $o(T) = \infty$. Note that $o(T) = \infty$ if and only if there exists an infinite sequence $(\xi_i)_{i=1}^\infty \subset \Lambda$ such that $(\xi_i)_{i=1}^n \in T$ for all $n \in \mathbb{N}$. Given a tree or $B$-tree $T$, we let $\Pi(T) = \{(s,t) \in T \times T : s \leq t, t \in MAX(T)\}$.

Recall that for any $B$-trees $S$, $T$, a function $\theta : S \to T$ is called monotone provided that for any $\emptyset \prec s \prec s_1 \in S$, $\theta(s) \prec \theta(s_1)$.

Given non-empty sets $\Lambda_1$, $\ldots$, $\Lambda_k$, we identify the set $(\prod_{i=1}^k \Lambda_i)^{<\mathbb{N}}$ with the set $\{(t_i)_{i=1}^k \in \prod_{i=1}^k \Lambda_i^{<\mathbb{N}} : |t_1| = \cdots = |t_k|\}$. The identification is obtained by identifying $\emptyset$ with $(\emptyset, \ldots, \emptyset)$ and, for $n > 0$,

$$\left((a_{1i}, \ldots, a_{ki})\right)_{i=1}^n \leftrightarrow \left((a_{1i})_{i=1}^n, (a_{2i})_{i=1}^n, \ldots, (a_{ki})_{i=1}^n\right).$$

Let $X$ be a Banach space and let $T$ be a $B$-tree. Let us say that a collection $(x_t)_{t \in T} \subset X$ is weakly null provided that for every ordinal $\xi$, every $t \in (T \cup \{\emptyset\})^\xi+1$, and every $Z \leq X$ with $\dim(X/Z) < \infty$, there exists $s \in T^\xi$ with $s^- = t$ such that $x_s \in Z$.

We last define some $B$-trees which will be important for us. If $(\xi_i)_{i=1}^n$ is a sequence of ordinals and $\xi$ is an ordinal, we let $\xi + (\xi_i)_{i=1}^n = (\xi + \xi_i)_{i=1}^n$. If $G$ is a collection of non-empty sequences of ordinals and $\xi$ is an ordinal, we let $\xi + G = \{\xi + t : t \in G\}$. We let

$$\mathcal{T}_0 = \emptyset,$$

$$\mathcal{T}_{\xi+1} = \{(\xi + 1)\setminus t : t \in [\emptyset] \cup \mathcal{T}_\xi\},$$
and if \( \xi \) is a limit ordinal, we let

\[
T_{\xi} = \bigcup_{\zeta < \xi} T_{\zeta + 1}.
\]

Note that this union is a totally incomparable union. For each ordinal \( \xi \), \( T_{\xi} \) is a \( B \)-tree on \([0, \xi]\) with \( o(T_{\xi}) = \xi \).

Next, let

\[
\Gamma_0 = \{(1)\},
\]

\[
\Gamma_{\xi+1} = \left\{(\omega^\xi(n - 1) + t_1) \cap \ldots \cap (\omega^\xi(n - m) + t_m) : n \in \mathbb{N}, 1 \leq m \leq n, t_i \in \Gamma_\xi, t_i \in MAX(\Gamma_\xi) \text{ for each } 1 \leq i < m\right\}.
\]

and when \( \xi \) is a limit ordinal,

\[
\Gamma_{\xi} = \bigcup_{\zeta < \xi} (\omega^\xi + \Gamma_{\xi+1}).
\]

For each ordinal \( \xi \), \( \Gamma_{\xi} \) is a \( B \)-tree on \([1, \omega^\xi]\) with \( o(\Gamma_{\xi}) = \omega^\xi \). We define \( P_{\xi} : \Gamma_{\xi} \to [0, 1] \) by letting \( P_{\xi}(1) = 1 \),

\[
P_{\xi+1}((\omega^\xi(n - 1) + t_1) \cap \ldots \cap (\omega^\xi(n - m) + t_m)) = P_{\xi}(t_m)/n,
\]

and

\[
P_{\xi}(\omega^\xi + t) = P_{\xi+1}(t), \quad t \in \Gamma_{\xi+1}.
\]

We refer the reader to [5] for a discussion that these functions are well-defined and for every ordinal \( \xi \) and every \( t \in MAX(\Gamma_{\xi}) \), \( \sum_{s \preceq t} P_{\xi}(s) = 1 \).

### 3 Games on well-founded \( B \)-trees

Given a non-empty, well-founded \( B \)-tree \( T \) on the set \( \Lambda \), let \( R_T = \{\zeta \in \Lambda : (\zeta) \in T\} \).

Given a non-empty, well-founded \( B \)-tree \( T \) and two non-empty sets \( \mathcal{D}, \mathcal{K} \), we let \( T.\mathcal{D}.\mathcal{K} \) denote the sequences \((\zeta_i, Z_i, C_i)_{i=1}^n\) such that \( Z_i \in \mathcal{D}, C_i \in \mathcal{K}, (\zeta_i)_{i=1}^n \in T \). Let \( T.\mathcal{D} = \{(\zeta_i, Z_i)_{i=1}^n : Z_i \in \mathcal{D}, (\zeta_i)_{i=1}^n \in T\} \). Note that \( T.\mathcal{D}.\mathcal{K} \) and \( T.\mathcal{D} \) are non-empty, well-founded \( B \)-trees with the same order as \( T \). Given a subset \( \mathcal{E} \subseteq MAX(T.\mathcal{D}.\mathcal{K}) \), we define the \( \mathcal{E} \)-game on \( T.\mathcal{D}.\mathcal{K} \) as follows: Player I chooses \( Z_1 \in \mathcal{D} \) and \( \zeta_1 \in R_T \). Player II chooses \( C_1 \in \mathcal{K} \). Next, assuming that \( Z_1, \ldots, Z_n \in \mathcal{D}, C_1, \ldots, C_n \in \mathcal{K}, \) and \( \zeta_1, \ldots, \zeta_n \in \Lambda \) have been chosen such that \((\zeta_i)_{i=1}^n \in T \), if \((\zeta_i)_{i=1}^n \in MAX(T)\), the game terminates. Otherwise Player I chooses \( Z_{n+1} \in \mathcal{D} \) and \( \zeta_{n+1} \in \Lambda \) such that \((\zeta_i)_{i=1}^{n+1} \in T \) and player II chooses \( C_{n+1} \in \mathcal{K} \). Since \( T \) is well-founded, the game terminates after some finite number of steps. Suppose that the game
terminates after the choices $C_1, \ldots, C_n \in \mathcal{K}$, $Z_1, \ldots, Z_n \in \mathcal{D}$, and $\zeta_1, \ldots, \zeta_n \in \Lambda$. Then Player I wins provided $(\zeta_i, Z_i, C_i)_{i=1}^n \in MAX(T, D, \mathcal{K}) \setminus \mathcal{E}$, and Player II wins if $(\zeta_i, Z_i, C_i)_{i=1}^n \in \mathcal{E}$. We call such a game a game on a non-empty, well-founded $B$-tree.

A strategy for Player I is a function $\varphi : T'.D.\mathcal{K} \cup \{\varnothing\} \rightarrow \Lambda \times D$ such that if $\varphi(\varnothing) = (\zeta, Z)$, $\zeta \in R_T$, and if $\varphi((\zeta_j, Z_j, C_j)_{j=1}^n) = ((\zeta_{n+1}, Z_{n+1}), (\zeta_i)_{i=1}^{n+1}) \in T$, a sequence $(\zeta_i, Z_i, C_i)_{i=1}^n \in MAX(T, D, \mathcal{K})$ is $\varphi$-admissible if $((\zeta_j, Z_j) = \varphi((\zeta_i, Z_i, C_i)_{i=1}^n))$ for each $1 \leq j \leq n$. A strategy for Player I $\varphi$ is called a winning strategy for the $\mathcal{E}$ game on $T.D.\mathcal{K}$ provided that every $\varphi$-admissible sequence lies in $MAX(T, D, \mathcal{K}) \setminus \mathcal{E}$. A winning strategy for Player I for the $\mathcal{E}$ game on $T.D.\mathcal{K}$ is a subset $S$ of $T'.D.\mathcal{K} \cup \{\varnothing\}$ containing $\varnothing$ and a function $\phi : S \rightarrow \Lambda \times D$ such that, if $(\zeta, Z) = \phi(\varnothing)$,

(i) $S = \{\varnothing\} \cup \{t \in T'.D.\mathcal{K} : (\exists C \in \mathcal{K})((\zeta, Z, C) \leq t))$,

(ii) $\xi \in R_T$,

(iii) if $t = (\zeta_i, Z_i, C_i)_{i=1}^n \in S$ and $(\zeta_{n+1}, Z_{n+1}) = \phi(t)$, then $(\zeta_i)_{i=1}^n \in T$,

(iv) if $(\zeta_i, Z_i, C_i)_{i=1}^n \in MAX(T, D, \mathcal{K})$, $(\zeta, Z_1) = (\zeta, Z)$, and $(\zeta_j, Z_j) = \phi((\zeta_i, Z_i, C_i)_{i=1}^n)$ for each $1 \leq j \leq n$, then $t \notin \mathcal{E}$.

Note that if Player I has a winning strategy for the $\mathcal{E}$ game on $T.D.\mathcal{K}$, then Player I has a winning strategy. Indeed, given a winning strategy $\phi : S \rightarrow \Lambda \times D$, we fix any $Z' \in \mathcal{D}$ and define a strategy $\varphi : T'.D.\mathcal{K} \rightarrow \Lambda \times D$ by letting $\varphi|S = \phi$ and, if $t = (\zeta_i, Z_i, C_i)_{i=1}^n \in T'.D.\mathcal{K} \setminus S$, we let $\varphi(t) = (\zeta_{n+1}, Z')$ for any $\zeta_{n+1} \in \Lambda$ such that $(\zeta_i)_{i=1}^{n+1} \in T$. Such a $\zeta_{n+1}$ exists since $(\zeta_i)_{i=1}^n \in T'$. Let $(\zeta, Z) = \phi(\varnothing)$. It is straightforward to verify that this is a strategy for Player I. Since any $\varphi$-admissible sequence $(\zeta_i, Z_i, C_i)_{i=1}^n$ satisfies $(\zeta_1, Z_1) = (\zeta, Z)$, property (iv) of winning strategy guarantees that $(\zeta_i, Z_i, C_i)_{i=1}^n \in MAX(T, D, \mathcal{K}) \setminus \mathcal{E}$.

A strategy for Player II is a $\mathcal{K}$-valued function $\psi$ on the set of all pairs $(t, (\zeta_{n+1}, Z_{n+1}))$ such that $t \in \{\varnothing\} \cup T.D.\mathcal{K}$, $(\zeta_{n+1}, Z_{n+1}) \in \Lambda \times D$, and if $t = (\zeta_i, Z_i, C_i)_{i=1}^n \in T$. A sequence $(\zeta_i, Z_i, C_i)_{i=1}^n \in MAX(T, D, \mathcal{K})$ is $\psi$-admissible provided that for every $1 \leq k \leq n$, $\psi((\zeta_i, Z_i, C_i)_{i=1}^k, (\zeta_k, Z_k)) = C_k$. A strategy for player II $\psi$ is called a winning strategy for the $\mathcal{E}$ game on $T.D.\mathcal{K}$ provided that every $\psi$-admissible sequence lies in $\mathcal{E}$. Obviously for a given subset $\mathcal{E}$ of $MAX(T, D, \mathcal{K})$, Player I and Player II cannot both have a winning strategy.

**Proposition 3.1** Every game on a non-empty, well-founded $B$-tree is determined. That is, exactly one of Player I and Player II has a winning strategy.

**Proof** We prove by induction on $\xi \geq 1$ that if $T$ is a non-empty, well-founded $B$-tree with $o(T) \leq \xi$, then for any $\mathcal{E} \subset MAX(T, D, \mathcal{K})$, either Player I has a winning strategy or Player II has a winning strategy. Assume that for some ordinal $\xi$ and every $1 \leq \gamma < \xi$, the hypothesis is true for $\gamma$. Let $T$ be a non-empty, well-founded $B$-tree with $o(T) = \xi$. For every $\zeta \in R_T$, let $T(\zeta)$ denote those non-empty sequences $t$ such that $(\zeta) \triangleright t \in T$. Note that $T(\zeta)$ is a $B$-tree with $o(T(\zeta)) < \xi$, and $T(\zeta) = \varnothing$ if and only if $\zeta \in MAX(T)$. Given $\zeta \in R_T$, $Z \in D$, and $C \in \mathcal{K}$, let $\mathcal{E}(\zeta, Z, C)$ denote those non-empty sequences $(\zeta, Z_i, C_i)_{i=1}^n$ such that $(\zeta, Z, C) \triangleright (\zeta_i, Z_i, C_i)_{i=1}^n \in \mathcal{E}$. Let $W$ denote the set of those $(\zeta, Z) \in T.D$ such that either

(i) $\zeta \in MAX(T)$ and for every $C \in \mathcal{K}$, $(\zeta, Z, C) \in MAX(T.D.\mathcal{K}) \setminus \mathcal{E}$, or
(ii) $\zeta \notin MAX(T)$ and for every $C \in \mathcal{K}$, Player I has a winning strategy in the $\mathcal{E}(\zeta, Z, K)$ game on $T(\zeta).D.K$.

By the inductive hypothesis, if $(\zeta, Z) \in T.D\setminus W$, then either

(i) $\zeta \in MAX(T)$ and there exists $C \in \mathcal{K}$ such that $(\zeta, Z, C) \in \mathcal{E}$, or

(ii) $\zeta \notin MAX(T)$ and there exists $C \in \mathcal{K}$ such that Player II has a winning strategy in the $\mathcal{E}(\zeta, Z, C)$ game on $T(\zeta).D.K$.

It is obvious that Player I has a winning strategy in the $\mathcal{E}$ game on $T.D.K$ if $W \neq \emptyset$, and Player II has a winning strategy in the $\mathcal{E}$ game on $T.D.K$ if $W = \emptyset$. For completeness, we define the strategies in each case.

Suppose $W \neq \emptyset$. Fix $(\zeta, Z) \in W$ and let $S = \{\emptyset\} \cup \{t \in T^'.D.K : (\exists C \in \mathcal{K})((\zeta, Z, C) \preceq t)\}$. If $\zeta \in MAX(T)$, then we define $\phi(\emptyset) = (\zeta, Z)$. Next, suppose $\zeta \notin MAX(T)$. For each $C \in \mathcal{K}$, fix a winning strategy $\psi_C : T(\zeta)^'.D.K \to \Lambda \times D$ in the $\mathcal{E}(\zeta, Z, K)$ game on $T(\zeta).D.K$. Let $\phi(\emptyset) = (\zeta, Z)$ and for each $C \in \mathcal{K}$ and each extension $s = (\zeta, Z, C)\preceq t \in T^'.D.K$ of $(\zeta, Z, C)$, let $\phi(s) = \psi_C(t)$. In either case, we have produced a winning substrategy, which we may extend to a winning strategy by the remarks preceding the proposition.

Next, suppose $W = \emptyset$. Fix $C' \in \mathcal{K}$. Fix $(\zeta, Z) \in R_T \times D$. If $(\zeta) \in MAX(T)$, fix $C_{\zeta, Z} \in K$ such that $(\zeta, Z, C_{\zeta, Z}) \in \mathcal{E}$ and let $\psi(\emptyset, (\zeta, Z)) = C_{\zeta, Z}$. If $(\zeta) \in T'$, let $\psi(\emptyset, (\zeta, Z)) = C_{\zeta, Z}$, where $C_{\zeta, Z} \in \mathcal{K}$ is such that Player II has a winning strategy in the $\mathcal{E}(\zeta, Z, C_{\zeta, Z})$ game on $T(\zeta).D.K$, and let $\psi_{\zeta, Z}$ be a winning strategy on the appropriate domain. For $s = (\zeta, Z, C)\preceq (\zeta_i, Z_i, C_i)_{i=1}^{n+1}$ and $(\zeta_{n+1}, Z_{n+1}) \in \Lambda \times D$ such that $(\zeta, \zeta_1, \ldots, \zeta_{n+1}) \in T$, let $\psi(s, (\zeta_{n+1}, Z_{n+1})) = C'$ if $C \neq C_{\zeta, Z}$ and $\psi(s, (\zeta_{n+1}, Z_{n+1})) = \psi_{\zeta, Z}(\zeta_i, Z_i, C_i)_{i=1}^{n+1}$ if $C = C_{\zeta, Z}$. This defines a winning strategy for Player II.

\section{Szlenk games}

In Sects. 3, 4 and 5, $X$ will be a fixed Banach space, $D$ will the subspaces of $X$ having finite codimension in $X$, and $\mathcal{K}$ will denote set of norm compact subsets of $X$. Given a non-empty, well-founded $B$-tree $T$ and a collection $(x(s, t))_{s \in \Pi(T, D)} \subset X$, we say the collection is \textit{normally weakly null} provided that for any $s = (\zeta_i, Z_i)_{i=1}^{n+1} \in T.D$ and any $t$ such that $(s, t) \in \Pi(T, D)$, $x(s, t) \in B_{Z_n}$. We will also use normally weakly null to describe a collection $(x_s)_{s \in T.D}$ such that if $s = (\zeta_i, Z_i)_{i=1}^{n+1} \in T.D$, $x_s \in Z_i$. This is a special case of the previous definition in which the collection $(x(s, t))_{s \in \Pi(T, D)}$ is such that $x(s, t)$ is independent of $t$.

\subsection{Determination of Szlenk index by games}

Given $K \subset X^*$, $\varepsilon \in \mathbb{R}$, a $B$-tree $T$, and a function $\mathbb{P} : T \to \mathbb{R}$, we let $\mathcal{E}_{K, \varepsilon}(T.D.K, \mathbb{P})$ denote those $(\zeta_i, Z_i, C_i)_{i=1}^{n+1} \in MAX(T.D.K)$ such that there exist $x^* \in K$ and $(x_i)_{i=1}^{n} \in \prod_{i=1}^{n} (B_X \cap Z_i \cap C_i)$ such that

\[ \text{Re } x^* \left( \sum_{i=1}^{n} \mathbb{P}((\zeta_i)_{j=1}^{j=i})x_i \right) \geq \varepsilon. \]
Given a function \( \mathbb{P} : T \rightarrow \mathbb{R} \), we will consider the function \( \mathbb{P} \) to be also defined on \( T.\mathcal{D} \) by \( \mathbb{P}((\xi_i, Z_i)^{n}_{i=1}) = \mathbb{P}((\xi_i)^{n}_{i=1}) \).

**Lemma 4.1** Fix a non-empty, well-founded \( B \)-tree \( T \), a function \( \mathbb{P} : T \rightarrow \mathbb{R}, \varepsilon \in \mathbb{R} \), and a subset \( K \) of \( X^* \). If Player II has a winning strategy in the \( \mathcal{E}_{K,\varepsilon}(T.\mathcal{D}.\mathcal{K}, \mathbb{P}) \) game, then there exist a normally weakly null collection \((x(s,t))_{(s,t)\in\Pi(T.\mathcal{D})} \subset B_X\), a collection \( (x^*_i)_{i\in\text{MAX}(T.\mathcal{D})} \subset K \), and a collection \((C_s)_{s\in T.\mathcal{D}} \subset \mathcal{K} \) such that

(i) for every \( t \in \text{MAX}(T.\mathcal{D}) \),

\[
\text{Re} \; x^*_t \left( \sum_{s \leq t} \mathbb{P}(s)x(s,t) \right) \geq \varepsilon,
\]

and

(ii) for every \( s \in T.\mathcal{D} \) and any maximal extension \( t \in T.\mathcal{D} \) of \( s \), \( x(s,t) \in C_s \).

**Proof** Fix a winning strategy \( \psi \) for Player II in the \( \mathcal{E}_{K,\varepsilon}(T.\mathcal{D}.\mathcal{K}, \mathbb{P}) \) game. We first define \( C_s \in \mathcal{K} \) for \( s \in T.\mathcal{D} \) by induction on \( |s| \). If \( |s| = 1 \), write \( s = (\xi, Z) \) and let \( C_s = \psi(\xi, (\xi, Z)) \). Next, suppose that for some \( j \in \mathbb{N} \) and some sequence \( s = (\xi_i, Z_i)^{j+1}_{i=1} \in T.\mathcal{D} \), \( C_{s|i} \) has been defined for each \( 1 \leq i \leq j \). Let \( C_s = \psi((\xi_i, Z_i, C_{s|i})^{j}_{i=1}, (\xi_{j+1}, Z_{j+1})) \). This completes the definition of \((C_s)_{s\in T.\mathcal{D}}\). Note that with this definition, for every \( t = (\xi_i, Z_i)^{n}_{i=1} \in \text{MAX}(T.\mathcal{D}) \), the sequence \((\xi_i, Z_i, C_{t|i})^{n}_{i=1}\) is \( \psi \)-admissible and therefore lies in \( \mathcal{E}_{K,\varepsilon}(T.\mathcal{D}.\mathcal{K}, \mathbb{P}) \). Thus there exists \( x^*_i \in K \) and a sequence \((x^*_i)^{|r|}_{i=1} \in \prod_{i=1}^{|r|} (B_X \cap Z_i \cap C_{t|i}) \) such that

\[
\text{Re} \; x^*_t \left( \sum_{s \leq t} \mathbb{P}(s)x(s,t) \right) \geq \varepsilon.
\]

Letting \( x(s,t) = x^*_i |_{s|} \) finishes the proof. \( \square \)

Given \( B \)-trees \( S, T \), we say a pair of functions \( \theta : S.\mathcal{D} \rightarrow T.\mathcal{D}, e : \text{MAX}(S) \rightarrow \text{MAX}(T) \) is an extended pruning provided it is monotone, if \( s = (\xi, Z)^{m}_{i=1} \) and \( \theta(s) = (\mu_i, W_i)^{n}_{i=1} \), \( W_n \subset Z_m \), and for any \( (s, t) \in \Pi(S.\mathcal{D}), \theta(s) \leq e(t) \). We will write \((\theta, e) : S.\mathcal{D} \rightarrow T.\mathcal{D} \) to denote an extended pruning.

**Lemma 4.2** For any ordinal \( \gamma > 0 \) and any finite cover \( P_1, \ldots, P_n \) of \( \text{MAX}(T_\gamma.\mathcal{D}) \), there exist an extended pruning \((\theta, e) : T_\gamma.\mathcal{D} \rightarrow T_\gamma.\mathcal{D} \) and \( 1 \leq i \leq n \) such that \( e(\text{MAX}(T_\gamma.\mathcal{D})) \subset P_i \).

**Proof** It was shown in [3] that for any \( 0 < \xi \leq \gamma \), there exists a function \( \phi : T_\xi \rightarrow T_\gamma \) such that for any \( \emptyset \prec s \preceq s_1 \in T_\xi, \phi(s) \prec \phi(s_1) \). From this we easily deduce that for any \( 0 < \xi \leq \gamma \), there exists an extended pruning \((\theta, e) : T_\xi.\mathcal{D} \rightarrow T_\gamma.\mathcal{D} \). Indeed, we first note that the function \( \varphi : T_\xi \rightarrow T_\gamma \) given by \( \varphi(s) = \phi(s)|_{s} \) is well-defined and still has the property that for any \( \emptyset \prec s \preceq s_1 \in T_\xi, \varphi(s) \prec \varphi(s_1) \), and \( \varphi \) preserves lengths. We may then define \( \theta((\xi_i, Z_i)^{n}_{i=1}) = (\mu_i, Z_i)^{n}_{i=1} \), where \( \varphi((\xi_i)^{n}_{i=1}) = (\mu_i)^{n}_{i=1} \). Then for every \( t \in \text{MAX}(T_\xi) \), let \( e(t) \) be any maximal extension of \( \theta(t) \), at least one of which exists by well-foundedness.
Recall that \( T_1.D = \{(1, Z) : Z \in D\} \). There exists \( 1 \leq i \leq n \) such that the set \( M = \{Z : (1, Z) \in P_i\} \) is cofinal in \( D \). This means that for any \( Z \in D \), there exists \( W_Z \in M \) such that \( W_Z \leq Z \) and we may let \( \theta((1, Z)) = e((1, Z)) = (1, W_Z) \). Then \( e(MAX(T_1.D)) \subseteq P_i \).

Next, suppose \( \gamma \) is a limit ordinal and the result holds for all \( \xi < \gamma \). Recall that \( T_\gamma.D = \cup_{\xi<\gamma} T_{\xi+1}.D \) and this is a disjoint union. For every \( \xi < \gamma \), there exist an extended pruning \( (\theta_\xi, e_\xi) : T_{\xi+1}.D \to T_{\xi+1}.D \) and \( 1 \leq i_\xi \leq n \) such that \( e_\xi(MAX(T_{\xi+1}.D)) \subseteq P_{i_\xi} \). There exists \( 1 \leq i \leq n \) such that \( M = \{\xi < \gamma : i_\xi = i\} \) has supremum \( \gamma \). For every \( \xi < \gamma \), fix \( \eta_\xi \in M \) with \( \xi < \eta_\xi \) and an extended pruning \( (\theta'_\xi, e'_\xi) : T_{\xi+1}.D \to T_{\xi+1}.D \), as we may by the first paragraph of the proof. Let 
\[
\theta|_{T_{\xi+1}.D} = \theta_{\eta_\xi+1} \circ \theta'_\xi \quad \text{and} \quad e|_{MAX(T_{\xi+1}.D)} = e_{\eta_\xi+1} \circ e'_\xi. \quad \text{Then } e(MAX(T_\gamma.D)) \subseteq P_i.
\]

Next, assume the result holds for an ordinal \( \xi > 0 \) and \( \gamma = \xi + 1 \). For \( \gamma \in D \), identifying \( \{(\gamma, Z)^t : t \in T_\gamma.D\} \) with \( T_\gamma.D \), we may find an extended pruning \( (\theta_Z, e_Z) : T_\gamma.D \to T_\gamma.D \) and \( 1 \leq i_Z \leq n \) such that \( \{(\gamma, Z)^\circ e_Z(t) : t \in MAX(T_\gamma.D)\} \subseteq P_{i_Z} \). There exists \( 1 \leq i \leq n \) such that \( M = \{Z \in D : i_Z = i\} \) is cofinal in \( D \). For \( Z \in D \), fix \( W_Z \in M \) such that \( W_Z \leq Z \) and define
\[
\theta((\gamma, Z)) = (\gamma, W_Z),
\theta((\gamma, Z)^\circ t) = (\gamma, W_Z)^\circ \theta_{W_Z}(t), \quad t \in T_\gamma.D,
\]
\[
e((\gamma, Z)^\circ t) = (\gamma, W_Z)^\circ e_{W_Z}(t), \quad t \in MAX(T_\gamma.D).
\]

This is an extended pruning with \( e(MAX(T_\gamma.D)) \subseteq P_i \). ☐

**Lemma 4.3** Fix an ordinal \( \xi > 0 \). Suppose that \( T \) is a well-founded, non-empty \( B \)-tree with \( o(T) \geq \xi \) and \( (x_{(s,t)})_{(s,t)\in \Pi(T,D)} \subseteq B_X \) is normally weakly null. Suppose also that for every \( s \in T.D, C_s \) is a norm compact subset of \( X \) such that for every maximal extension \( t \) of \( s, x_{(s,t)} \in C_s \). Then for any \( \delta > 0 \), there exists a collection \( (x^t_i)_{t\in T_\xi.D} \subseteq B_X \) which is normally weakly null and an extended pruning \( (\theta, e) : T_\xi.D \to T.D \) such that for every \( (s, t) \in \Pi(T_\xi.D), \|x^t_i - x_{(s,t)}\| < \delta \).

**Proof** We induct on \( \xi \). First suppose \( \xi = 1 \). Recall that \( T_1.D = \{(1, Z) : Z \in D\} \), so that \( \Pi(T_1.D) = \{(1, Z), (1, Z) : Z \in D\} \). Fix any \( \xi \in R_T \), as we may, since \( o(T) \geq 1 \). For every \( Z \in D \), fix a maximal extension \( t_Z \) of \( (\xi, Z) \). Let \( \theta((1, Z)) = (\xi, Z), \) \( e((1, Z)) = t_Z \), and let \( x^t_i(Z) = x_{(1, Z)}(\xi, Z)^\circ t \). The conclusions are easily seen to be satisfied in this case with \( \delta = 0 \).

The limit ordinal case is trivial, since \( T_\xi.D = \cup_{\xi<\xi} T_{\xi+1}.D \) is an incomparable union.

Assume \( \gamma > 0 \), the statement holds for \( \gamma \), and \( \xi = \gamma + 1 \). Fix any \( \xi \) such that \( (\xi) \in T' \). Let \( S \) denote those non-empty sequences \( u \) such that \( (\xi)^\circ s \in T \). Fix \( Z \in D \). Since \( (\xi, Z) \in T' \), \( o(S.D) \geq \gamma \) and \( (x_{((\xi, Z)^\circ s, (\xi, Z)^\circ t)})_{(s,t)\in \Pi(S.D)} \) is normally weakly null. Applying the inductive hypothesis to this collection and the sets \( (C_{((\xi, Z)^\circ s, (\xi, Z)^\circ t)})_{(s,t)\in \Pi(S.D)} \), we dederive the existence of a normally weakly null collection \( (x^t_i)_{(s,t)\in \Pi(T_\gamma.D)} \subseteq B_X \) and an extended pruning \( (\theta_Z, e_Z) : T_\gamma.D \to S.D \) such that for every \( (s, t) \in \Pi(T_\gamma.D), \)
\[
\|x^t_i - x_{((\xi, Z)^\circ t)}\| < \delta.
\]
Next, let \((v_i)_{i=1}^n\) be a finite \(\delta/2\)-net of \(C(\zeta, Z)\). Then if
\[
P_i = \{ t \in MAX(T_\gamma, D) : \| v_i - x_{((\zeta, Z), (\zeta, Z)^{-}e_Z(t))} \| < \delta/2 \},
\]
by Lemma 4.2, there exists an extended pruning \((\theta_\gamma, e_\gamma) : T_\gamma, D \to T_\gamma, D\) and \(1 \leq i \leq n\) such that \(e_\gamma (MAX(T_\gamma, D)) \subset P_{i_\gamma}\). Fix \(i_0 \in MAX(T_\gamma, D)\) and let
\[
x^\gamma((\xi, Z)) = x_{((\zeta, Z), (\zeta, Z)^{-}e_Z(o_\gamma^\ast(t)))}, \quad x^\gamma((\xi, Z), (\zeta, Z)^{-}e_Z(o_\gamma^\ast(t))),
\]
\[
\theta((\xi, Z)) = (\zeta, Z), \quad \theta((\xi, Z)^{-}e) = (\zeta, Z)^{-}e_Z \circ \theta^\gamma((\xi, Z)^{-}e),
\]
\[
e((\xi, Z)^{-}e) = e_{Z} \circ e_{Z}^\gamma(t).
\]

\[\square\]

**Remark 4.4** Let \(N\) denote any weak neighborhood basis at 0 in \(X\). Given a nonempty \(B\)-tree \(T\), let us say that \((x_i)_{i=1}^n \subset B_X\) is usually weakly null if for every \(t = (\zeta, U_i)_{i=1}^n \in T_N, x_i \in U_i\). Note that for any \(\delta > 0\), there exist functions \(\rho : D \to N\) and \(\rho : N \to D\) such that for any \(Z \in D\) and \(U \in N\), \(B_Z \subset \rho(Z) \cap B_X\) and for any \(x \in U \cap B_X\), there exist \(y \in \rho(U)\) with \(\| x - y \| < \delta\). For \(\varepsilon > 0\) and \(\emptyset \neq K \subset X^\ast\), let \(H^K\) denote the empty sequence together with those sequences \((x_i)_{i=1}^n \subset B_X^\ast\) such that there exists \(x^\ast \in K\) such that for every \(1 \leq i \leq n\), \(R(x^\ast) \geq \varepsilon\). The main theorem of [4] is the existence of a constant \(c > 0\) such that

(i) if there exists a usually weakly null \((x_i)_{i=1}^n \subset B_X\) such that for every \(t \in T_{\rho\alpha}, N, (x_i)_{i=1}^n \subset H^K\), then \(S_Z(K, \varepsilon_1) > \omega^k\) for every \(0 < \varepsilon_1 < \varepsilon\), and

(ii) if \(S_Z(K, c\varepsilon) > \omega^k\), there exists a usually weakly null \((x_i)_{i=1}^n \subset B_X\) such that for every \(t \in T_{\rho\alpha}, N, (x_i)_{i=1}^n \subset H^K\).

This combined with the existence of the functions \(\rho, \rho\alpha\) above, we deduce that

(i) if there exists a normally weakly null \((x_i)_{i=1}^n \subset B_X\) such that for every \(t \in T_{\rho\alpha}, D, (x_i)_{i=1}^n \subset H^K\), then \(S_Z(K, \varepsilon_1) > \omega^k\) for every \(0 < \varepsilon_1 < \varepsilon\), and

(ii) for any \(c' > c\), if \(S_Z(K, c'e) > \omega^k\), then there exists a normally weakly null \((x_i)_{i=1}^n \subset B_X\) such that for every \(t \in T_{\rho\alpha}, D, (x_i)_{i=1}^n \subset H^K\).

From this, it follows that \(S_Z(K) > \omega^k\), then there exists \(c > 0\) such that Player II has a winning strategy in the \(E_{K, \varepsilon}(\Gamma_\xi, D, \mathcal{K}, P_\xi)\) game. Indeed, there exists \(c > 0\) such that \(S_Z(K, 2c\varepsilon) > \omega^k\), and a normally weak null \((x_i)_{i=1}^n \subset B_X\) such that for every \(t \in T_{\rho\alpha}, D, (x_i)_{i=1}^n \subset H^K\). Since there exists a length-preserving, monotone \(\theta : \Gamma_\xi \to T_{\rho\alpha}, \theta(t) = \phi((\zeta, Z_i)_{i=1}^n) = (\mu_i, Z_i^n, (\mu_i)_{i=1}^n = \phi((\zeta_i)_{i=1}^n)\). By relabeling, we may assume we have a normally weak null \((x_i)_{i=1}^n \subset E_{K, \varepsilon}(\Gamma_\xi, D, \mathcal{K}, P_\xi)\) game. Let \(\psi(\emptyset, (\zeta, Z)) = (x_{(\xi, Z)})\) and \(\psi((\zeta, Z, C_i)_{i=1}^n = (\zeta_{n+1}, Z_{n+1})) = (\{x_{(\xi, Z, i)_{i=1}^n}\})\) for any compact sets \(C_1, \ldots, C_n\). Fix \(t = (\zeta_i, Z_i, C_i)_{i=1}^n \in MAX(\Gamma_\xi, D, \mathcal{K})\) which is \(\psi\)-admissible, let \(s = (\zeta_i, Z_i)_{i=1}^n\), and note that for each \(1 \leq i \leq n\), \(C_i = \{x_{(i)}\}\). Then
Corollary 4.5 Suppose that $K \subseteq X^*$ is $w^*$-compact, $\varepsilon > 0$, and $\xi$ is an ordinal such that Player II has a winning strategy in the $E_{K, \varepsilon}(\Gamma_\xi, D, K, \mathbb{P}_\xi)$ game. Then for any $0 < \varepsilon_1 < \varepsilon$, $S\Sigma(K, \varepsilon_1) > \omega_\xi$.

Proof Fix $\varepsilon_1 < \varepsilon' < \varepsilon$. By Lemma 4.1, we may fix a normally weakly null $(x(s,t))_{s,t} \in \Pi(\Gamma_\xi, D) \subseteq B_X$, $(x^*_t)_{t} \in \text{MAX}(\Gamma_\xi, D) \subseteq K$, and $(C_s)_{s} \in \Gamma_\xi, D \subseteq K$ such that for every $t \in \text{MAX}(\Gamma_\xi, D)$,

$$\text{Re } x^*_t \left( \sum_{s \leq t} \mathbb{P}_\xi(s)x(s,t) \right) > \varepsilon$$

and for every $s \in \Gamma_\xi, D$ and every maximal extension $t$ of $s$, $x(s,t) \in C_s$. Fix $R > 0$ such that $K \subseteq RB_{X^*}$ and define the function

$$f : \Pi(\Gamma_\xi, D) \rightarrow [-R, R]$$

by $f(s, t) = \text{Re } x^*_t(x(s,t))$. For every $t \in \text{MAX}(\Gamma_\xi, D)$,

$$\sum_{s \leq t} \mathbb{P}_\xi(s)f(s, t) = \text{Re } x^*_t \left( \sum_{s \leq t} \mathbb{P}_\xi(s)x(s,t) \right) > \varepsilon.$$

By [5, Theorem 4.3], there exists an extended pruning $(\theta, e) : \Gamma_\xi, D \rightarrow \Gamma_\xi, D$ such that for every $(s, t) \in \Pi(\Gamma_\xi, D)$, $\text{Re } x^*_{e(t)}(x(\theta(s), e(t))) = f(\theta(s), e(t)) > \varepsilon$. Fix $\delta > 0$ such that $R\delta < \varepsilon' - \varepsilon_1$. We may apply Lemma 4.3 with this $\delta$ to the collection $(x(\theta(s), e(t)))_{s,t} \in \Pi(\Gamma_\xi, D)$ and $(C_{\theta(s)})_{s} \in \Gamma_\xi, D$ to obtain another extended pruning $(\theta', e') : T_{0^\delta, D} \rightarrow \Gamma_\xi, D$ and a normally weakly null collection $(x'_t)_{t} \in T_{0^\delta, D} \subseteq B_X$ such that for every $s \in T_{0^\delta, D}$ and every maximal extension $t$ of $s$,

$$\|x_s - x_{\theta_0\theta'_0(s), e_{e'_0}(t)}\| < \delta.$$

Fix any maximal $t \in T_{0^\delta, D}$ and note that $x^*_{e_{e'_0}(t)} \in K \subseteq RB_{X^*}$. For any $1 \leq i \leq |t|$,

$$\text{Re } x^*_{e_{e'_0}(t)} (x'_{t^i}) \geq \text{Re } x^*_{e_{e_{e'_0}(t)}} (x_{\theta_0\theta'(t^i), e_{e'_0}(t)}) - R\|x'_{t^i} - x_{\theta_0\theta'(t^i), e_{e'_0}(t)}\| \geq \varepsilon' - R\delta.$$

Since $\varepsilon' - R\delta > \varepsilon_1$, Remark 4.4 guarantees that $S\Sigma(K, \varepsilon_1) > \omega_\xi$. \hfill \Box

Corollary 4.6 Given an ordinal $\xi$ and a $w^*$-compact set $K \subseteq X^*$, $S\Sigma(K) > \omega_\xi$ if and only if there exists $\varepsilon > 0$ such that Player II has a winning strategy in the $E_{K, \varepsilon}(\Gamma_\xi, D, K, \mathbb{P}_\xi)$ game.
4.2 Applications to essentially bounded trees in $L_p(X)$

The results in this subsection are important for the proof of Theorem 1. We recall the following special case of the main theorem of [1].

**Theorem 4.7** ([1]) If $X$ is a separable Banach space not containing $\ell_1$, then $Sz(X) > \omega$ if and only if there exists a $B$-tree $B$ with $o(B) = \omega$ and a weakly null collection $(f_t)_{t \in B} \subset B_X$ such that for every $t \in B$ and $f \in co(f_s : s \leq t)$, $\|f\| \geq \varepsilon$.

It is easy to see that $Sz(X) = 1$ if and only if $X$ has finite dimension. It was shown in [10] that any asymptotically uniformly smooth Banach space has Szlenk index not exceeding $\omega$, whence for any $1 < p < \infty$, $Sz(L_p) = \omega$. It is also easy to see that the Szlenk index is an isomorphic invariant, so that any Banach space isomorphic to $L_p$ has Szlenk index $\omega$.

Recall that for $1 < p < \infty$, $L_p(X)$ denotes the Banach space of (equivalence classes of) Bochner integrable functions $f : [0, 1] \to X$ such that $\int \|f\|^p < \infty$, where $[0, 1]$ is endowed with Lebesgue measure. We let $L_{\infty}(X)$ denote the $X$-valued strongly measurable functions which are essentially bounded. It is well known and easy to see that for any subspace $Z$ of $X$, $L_p(X)/L_p(Z)$ is canonically isometrically isomorphic to $L_p(X/Z)$ by the operator $\Phi$ such that for each $\sigma \in [0, 1]$, $\Phi(f + L_p(Z))(\sigma) = f(\sigma) + Z$. Moreover, if $\dim X/Z < \infty$, $L_p(X/Z)$ is either the zero vector space or isomorphic to $L_p$ and therefore has Szlenk index not exceeding $\omega$.

This means that for any $B$-tree $T$ with $o(T) \geq \omega$ and any weakly null collection $(\overline{f}_t)_{t \in T} \subset B_{L_p(X/Z)}$ and any $\delta > 0$, there exists $t \in T$ and a convex combination $\overline{f}$ of $(\overline{f}_s : s \leq t)$ such that $\|f\| < \delta$. This means that if $T$ is a $B$-tree with $o(T) \geq \omega$, $\dim X/Z < \infty$, $\delta > 0$, and if $(f_t)_{t \in T} \subset B_{L_p(X)}$ is a weakly null collection, there exists $t \in T$ and $f \in co(f_s : s \leq t)$ such that $\|f\|_{L_p(X)/L_p(Z)} < \delta$. Indeed, we simply let $\overline{f}_t = f_t + L_p(Z)$ and use the previous fact, noting that $(\overline{f}_t)_{t \in T}$ is still weakly null and contained in $B_{L_p(X)/L_p(Z)}$ and using the isometric identification of $L_p(X)/L_p(Z)$ and $L_p(X/Z) \cong L_p$. Finally, if $f \in CB_{L_{\infty}(X)}$ and $\|f\|_{L_p(X)/L_p(Z)} < \delta$, then there exists a simple function $g \in 2CB_{L_{\infty}(X)}$ such that $\|f - g\|_{L_p(X)} < \delta$. Indeed, we may first fix $h \in L_p(Z)$ such that $\|f - h\|_{L_p(X)} < \delta$ and, by density of simple functions in $L_p(Z)$, assume $h$ is simple. Next, let $E = \{\sigma : \|h(\sigma)\| > 2C\}$. Note that there exists a subset $N$ of $E$ having measure zero such that for all $\sigma \in E \setminus N$,

$$\|f(\sigma)\| \leq C \leq \|h(\sigma)\| - \|f(\sigma)\| \leq \|h(\sigma) - f(\sigma)\|.$$ 

Thus we deduce that

$$\|f - 1_{EC}h\|^p = \int_E \|f\|^p + \int_{EC} \|f - h\|^p \leq \int_E \|f\|^p + \int_{EC} \|f - h\|^p < \delta^p.$$ 

Thus $g = 1_{EC}h$ is the simple function we seek.

Fix $1 < p < \infty$ and let $q$ be the conjugate exponent to $p$. For a fixed $K \subset X^*$, let $M$ denote the $K$-valued, measurable simple functions in $L_q(X^*)$. Recall that $L_q(X^*)$ is canonically isometrically included in $L_p(X)^*$ via the action $g(f) = \int g(\sigma)(f(\sigma))d\sigma$. 


Theorem 4.8  With $K$, $M$ as above, if $Sz(K) \leq \omega^k$, then for any $B$-tree $S$ with $o(S) \geq \omega^{1+k}$, any weakly null in $(L_p(X))$ collection of simple functions $(f_t)_{t \in S} \subset \frac{1}{2}BL_\infty(X)$, and any $\varepsilon > 0$, there exist $t \in S$ and $f \in co(f_s : s \leq t)$ such that $\sup_{n \in M} Re \int h f \leq \varepsilon$.

Proof  Fix $R > 0$ such that $K \subset RB_X$. By Proposition 3.1, the $E = E_{K,\varepsilon/2}(\Gamma_k, D, K, M)$ game on $\Gamma_\varepsilon D$ is determined. Since $Sz(K) \leq \omega^k$, Corollary 4.6 implies that Player II cannot have a winning strategy, and therefore Player I has a winning strategy. Fix a winning strategy $\psi$ for Player I. Define $m : \Gamma_k D \to [0, \omega^k]$ by letting $m(t) = \max \{ \gamma \omega^k : t \in (\Gamma_k D)^\gamma \}$.

We next define several sequences recursively. Let $\varphi(\emptyset) = (\zeta_1, Z_1)$. Let $\gamma_1 = m(\zeta_1, Z_1)$. Note that since $\gamma_1 < \omega^k$ and $o(S) \geq \omega^{1+k}$, $o(S^{\omega \gamma_1}) \geq \omega$ and $(f_t)_{t \in S^{\omega \gamma_1}} \subset \frac{1}{2}BL_\infty(X)$ is normally weakly null. By the remarks in the paragraphs preceding the statement of the theorem, there exist $s_1 \in S^{\omega \gamma_1}$, a convex combination $f_1$ of $(f_t : t \leq s_1)$, and a simple function $g_1 \in BL_\infty(Z_1)$ such that $\| f_1 - g_1 \|_{L_p(X)} < \varepsilon / 2 R$. By redefining $g_1$ on a set of measure zero, we may assume $range(g_1) \subset B_{Z_1}$ is finite. Let $C_1 = range(g_1) \subset B_X$.

Next, suppose that for each $1 \leq i \leq n, \zeta_i, Z_i, C_i, s_i, \gamma_i, f_i, g_i$ have been defined to have the following properties:

1. $\varphi((\zeta_j, Z_j, C_j)_{j=1}^{i-1}) = (\zeta_i, Z_i)$,
2. $\| f_i - g_i \|_{L_p(X)} < \varepsilon / 2 R$,
3. $C_i = range(g_i) \subset B_{Z_i}$ is finite,
4. $\gamma_i = m((\zeta_j, Z_j)_{j=1}^{i-1})$,
5. $f_i \in co(f_s : s_{i-1} < s \leq s_i)$, (where $s_0 = \emptyset$).

If $(\zeta_i, Z_i)_{i=1}^{n_1}$ is maximal in $\Gamma_k D$, we have completed the recursive construction. Suppose that $(\zeta_i, Z_i)_{i=1}^{n_1}$ is not maximal in $\Gamma_k D$. Let $\varphi((\zeta_i, Z_i, C_i)_{i=1}^{n}) = (\zeta_{n+1}, Z_{n+1})$. Let $\gamma_{n+1} = m((\zeta_i, Z_i)_{i=1}^{n+1})$. Let $U$ denote those non-empty sequences $s$ such that $s_n s \in S^{\omega \gamma_{n+1}}$. Applying the remarks in the paragraphs preceding the proof to the collection $(f_s)_{s \in U}$, we deduce the existence of $s_{n+1} \in S^{\omega \gamma_{n+1}}$, $f_{n+1} \in co(f_s : s_n < s \leq s_{n+1})$, and $g_{n+1} \in BL_\infty(Z_{n+1})$ such that $\| f_{n+1} - g_{n+1} \|_{L_p(X)} < \varepsilon / 2 R$.

Here we have used that since $s_n \in S^{\omega \gamma_n}$ and $\gamma_{n+1} < \gamma_n, o(U) \geq \omega$. By redefining $g_{n+1}$ on a set of measure zero, we may assume $range(g_{n+1}) \subset B_{Z_{n+1}}$ is finite. Let $C_{n+1} = range(g_{n+1})$.

Since $\Gamma_k D$ is well-founded, this process must eventually terminate. Assume that the process terminates with the sequence $(\zeta_i, Z_i)_{i=1}^{n} \in MAX(\Gamma_k D)$, the sequences $s_i$, and the functions $f_i, g_i$. By our choices, $(\zeta_i, Z_i, C_i)_{i=1}^{n} = \varphi$-admissible, and therefore not a member of $E_{K,\varepsilon/2}(\Gamma_k D, M, M)$. This means that for any $(x_i)_{i=1}^{n} \in \prod_{i=1}^{n} (B_X \cap Z_i \cap C_i)$ and for all $x^* \in K$, $Re x^* (\sum_{i=1}^{n} \mathbb{P}_k(t_i) x_i) < \varepsilon / 2$. But for any $\sigma \in [0, 1], (g_i(\sigma))_{i=1}^{n} \in \prod_{i=1}^{n} (B_X \cap Z_i \cap C_i)$, whence for any $x^* \in K$, $Re x^* (\sum_{i=1}^{n} \mathbb{P}_k(t_i) g_i(\sigma)) < \varepsilon / 2$. Then with $g = \sum_{i=1}^{n} \mathbb{P}_k(t_i) g_i$ and $h \in M$, $Re \int h g \leq \varepsilon / 2$, whence

$$\sup_{h \in M} Re \int h g \leq \varepsilon / 2.$$
Let \( f = \sum_{i=1}^{n} \| \xi(t_i) f_i \| \leq \sum_{i=1}^{n} \| \xi(t_i) f_i \| < \varepsilon / 2R. \)

Since \( M \subset RB_{L_p(X^*)} \), it follows that

\[
\sup_{h \in M} \text{Re} \int h f \leq \sup_{h \in M} \text{Re} \int h g + R\| f - g \| \leq \varepsilon.
\]

\[\square\]

5 The \( w^* \)-dentability index and a result of Lan"{c}ien

In this section, we prove further results which will be used in the proof of Theorem 1. We again fix \( 1 < p < \infty \) and let \( q \) be the conjugate exponent to \( p \). Let \( \mathcal{W} \) be a \( w^* \)-neighborhood basis at 0 in \( L_p(X^*) \). The following was shown in [4] in the case that \( L \) is \( w^* \)-compact. However, the proof given there does not depend upon the \( w^* \)-compactness of \( L \). For the remainder of the section, \( K \subset X^* \) will be a fixed \( w^* \)-compact, non-empty set and \( M \) will denote the subset of \( L_q(X^*) \subset L_p(X)^* \) consisting of all \( K \)-valued, measurable simple functions.

**Proposition 5.1** For an ordinal \( \xi \), if \( h \in s_{2\xi}^g (L) \), there exists a collection \( (h_t)_{t \in (T_\xi)} \cup (\emptyset), \mathcal{W} \subset L \) such that \( h_\emptyset = h \) and for every \( t \in T_\xi, \mathcal{W} \), if \( t = (\xi_i, V_i)_{i=1}^{\infty}, \| h_t - h_{t^-} \| > \varepsilon \) and \( h_t - h_{t^-} \in V_n \).

A collection \( (h_t)_{t \in (T_\xi)} \cup (\emptyset), \mathcal{W} \) satisfying the condition that for any \( t \in T_\xi, \mathcal{W} \), if \( t = (\xi_i, V_i)_{i=1}^{\infty} \), then \( h_t - h_{t^-} \in V_n \) will be called normally \( w^* \)-closed. A collection such that for any \( t \in T_\xi, \mathcal{W} \), \( \| h_t - h_{t^-} \| > \varepsilon \) will be called \( \varepsilon \)-separated.

Although it was not stated in this way, the following theorem was shown in [12]. Since the statement of this theorem differs significantly from the statement in [12], we will sketch the proof here for completeness.

**Theorem 5.2** [12, Lemmal1] Suppose that \( K \) is convex. If \( n \in \mathbb{N} \) and \( x_1^*, \ldots, x_n^* \in d_{2\xi}^g (K) \), then \( \sum_{i=1}^{n} x_i^* \frac{1}{i^\frac{1}{2}} \in s_{\xi}^g (M) \).

**Sketch** Given \( K_0 \subset X^* \) and \( L_0 \subset L_p(X)^* \), let us say the pair \((K_0, L_0)\) is nice provided that \( K_0 \) is \( w^* \)-compact, convex, and symmetric, and for any \( n \in \mathbb{N} \) and \( x_1^*, \ldots, x_n^* \in K_0; \sum_{i=1}^{n} x_i^* \frac{1}{i^\frac{1}{2}} \in L_0 \). Of course, if \((K_0, L_0)\) is nice and \( K_0 \neq \emptyset \), then \( L_0 \neq \emptyset \).

We claim that if \((K_0, L_0)\) is nice, then for any \( \varepsilon > 0 \), the pair \((d_{2\xi}^g (K_0), s_{\xi}^g (L_0))\) is nice. An easy induction then yields that for any ordinal \( \xi \), the pair \((d_{2\xi}^g (K_0), s_{\xi}^g (L_0))\) is nice, whence \( DZ(K_0, 2\varepsilon) \leq S_Z(L_0, \varepsilon) \). We obtain Theorem 5.2 by noting that \((K, M)\) is nice.

We prove the claim that \((d_{2\xi}^g (K_0), s_{\xi}^g (L_0))\) is nice, assuming \((K_0, L_0)\) is nice. Of course, \( d_{2\xi}^g (K_0) \) is \( w^* \)-compact, convex, and symmetric. Fix \( n \in \mathbb{N} \) and \( x_1^*, \ldots, x_n^* \in
$d_2(\mathcal{K}_0)$. Let $f = \sum_{i=1}^{n} x_i^* 1_{(\frac{i-1}{n}, \frac{i}{n})} \in L_0$. Let $V$ be a $w^*$-open neighborhood of $f$. It follows by the Hahn-Banach theorem that for each $1 \leq i \leq n$, $x_i^*$ lies in the $w^*$-closed, convex hull of $\mathcal{K}_0 \setminus (x_i^* + \varepsilon \mathcal{B}_{\mathcal{X}}^*)$. There exists $k \in \mathbb{N}$ such that for each $1 \leq i \leq n$, there exist $(x_{ij}^*)_{j=1}^{k} \subset \mathcal{K}_0 \setminus (x_i^* + \varepsilon \mathcal{B}_{\mathcal{X}}^*)$ such that

$$g = \sum_{i=1}^{n} \left( \frac{k}{k-1} \sum_{j=1}^{k} x_{ij}^* \right) 1_{(\frac{i-1}{n}, \frac{i}{n})} \in V.$$  

For each $l \in \mathbb{N}$, let

$$\psi_l = \sum_{i=1}^{n} \sum_{j=1}^{k} \sum_{m=1}^{l} x_{ij}^* 1_{ijm},$$

where

$$I_{ijm} = \left[ \frac{i - 1}{n} + \frac{m - 1}{nl} + \frac{j - 1}{nl}, \frac{i - 1}{n} + \frac{m - 1}{nl} + \frac{j}{nlk} \right].$$

Note that $\psi_l \to g$, whence $\psi_l \in V$ for sufficiently large $l \in \mathbb{N}$. Since $(\mathcal{K}_0, L_0)$ is nice, $\psi_l \in L_0$ for all $l \in \mathbb{N}$. Also, for any $\omega \in [0, 1]$, $\| f(\omega) - \psi_l(\omega) \| > \varepsilon$, whence $\| f - \psi_l \|_{L_p(X)^*} > \varepsilon$. This shows that $f \in s_\varepsilon(L_0)$. \hfill \Box

We remark that if $h \in L_{q}(X^*)$ is a simple function such that $\| h \|_{L_q(X^*)} > \varepsilon > 0$ and $\| h \|_{L_\infty(X^*)} \leq C$, there exists a simple function $f \in B_{L_p(X)}$ with $\| f \|_{L_\infty(X)} \leq C^{q-1}/\varepsilon^{q-1}$ and $\int \! hf > \varepsilon$. Indeed, write $h = \sum_{i=1}^{n} x_i^* 1_{F_i}$ with $F_i$ pairwise disjoint and measurable. Fix $0 < \rho < 1$ such that $\rho \| h \|_{L_q(X^*)} > \varepsilon$. For each $1 \leq i \leq n$, fix $x_i \in S_X$ such that $x_i^* (x_i) > \rho \| x_i^* \|$. Then $f = \| h \|_{L_q(X^*)}^{1-q} \sum_{i=1}^{n} x_i^* \| x_i^* \|^q x_i 1_{F_i}$ has the indicated properties by familiar computations.

**Lemma 5.3** Suppose that $R \geq 1$ is such that $\mathcal{K} \subset \mathcal{RB}_{\mathcal{X}}$. Assume $(h_t)_{t \in (T_\emptyset \cup \{\emptyset\}), \mathcal{W}} \subset M$ is normally $w^*$-closed and $\varepsilon$-separated for some $\varepsilon \in (0, 1)$. Then there exist a function $\theta : T_\emptyset \mathcal{N} \to T_\emptyset \mathcal{W}$ and a weakly null (in $L_p(X)$) collection $(f_t)_{t \in T_\emptyset \mathcal{N}} \subset \frac{1}{2} B_{L_\infty(X)}$ such that for any $\emptyset < s \leq t \in T_\emptyset \mathcal{N},$

$$\text{Re} \int \! h_{\theta(t)} f_s \geq \frac{\varepsilon^q}{3 \cdot 2^q R^{q-1}}.$$  

Here, $\mathcal{N}$ denotes the directed set of convex, weakly open neighborhoods of $0$ in $L_p(X)$.

**Proof** We will need the following claim.

**Claim 5.4** If $(h_t)_{t \in (T_\emptyset \cup \{\emptyset\}), \mathcal{W}} \subset M$ is normally $w^*$-closed and $\varepsilon$-separated, then for any sequence $(\varepsilon_n)_{n=0}^{\infty}$ of positive numbers, there exists a monotone, length-preserving
function $\theta : T_\xi N \to T_\xi \mathcal{W}$ and a collection $(g_t)_{t \in T_\xi N} \subseteq \frac{2^{q-1} R^{q-1}}{q-1} B_{L^\infty(X)}$ such that for every $s \in T_\xi N$, $\text{Re} \int h_{\theta(t)} g_s > \varepsilon/2 - \varepsilon_0$ and such that for any $\emptyset < s < t$, $|\int (h_{\theta(t)} - h_{\theta(s)}) g_s| < \varepsilon_t$, and if $s = (\xi_i, U_i)_{i=1}^n$, $g_s \in U_n$.

We first assume the claim and finish the proof. We apply the claim with some sequence $(\varepsilon_n)_{n=0}^\infty$ such that $\varepsilon/2 - \sum_{n=0}^\infty \varepsilon_n > \varepsilon/3$. Fix any $t \in T_\xi N$ and let $\emptyset < s \leq t$. Then

$$\text{Re} \int h_{\theta(t)} g_s \geq \text{Re} \int h_{\theta(s)} g_s - \sum_{s < u \leq t} \left| \int (h_{\theta(u)} - h_{\theta(u')}) g_s \right| > \varepsilon/2 - \varepsilon_0 - \sum_{n=|s|+1}^{\infty} \varepsilon_n > \varepsilon/3.$$

From this it follows that for any convex combination $g$ of $(g_s : s \leq t)$, $\text{Re} \int h_{\theta(t)} g > \varepsilon/3$. Letting $f_t = \frac{2^{q-1}}{2 q^{q-1}} g_t$ gives the desired collection. Since $\varepsilon/2 - \sum_{n=0}^\infty \varepsilon_n > \varepsilon/3$. Fixing some $(\xi_i, U_i)_{i=1}^n$ for some $V_1, \ldots, V_n \in \mathcal{W}$. Note that these properties together imply that for all $s = (\xi_i, U_i)_{i=1}^n$ and $1 \leq m \leq n$, $\theta(s|m) = (\xi_i, V_i)_{i=1}^m$.

Suppose $(h_t)_{t \in (T_\xi \cup \emptyset)} \subseteq \mathcal{M}$ as in the claim. Fix some $(\xi, U) \in T_\xi N$. For every $V \in \mathcal{W}$, $\|h_{(\xi, V)} - h_\emptyset\|_{L^q(X^*)} > \varepsilon$ and $h_{(\xi, V)} - h_\emptyset$ is a simple function with $\|h_{(\xi, V)} - h_\emptyset\|_{L^\infty(X^*)} \leq 2R$. By the remarks preceding the lemma, there exists a simple function $j_V \in B_{L_p(X)} \cap \frac{2^{q-1} R^{q-1}}{q-1} \mathcal{W}$ such that $\text{Re} \int h_{(\xi, V)} - h_\emptyset j_V > \varepsilon$. By [7, Lemma 3.3], for each $U \in N$, there exist $V_1^U, V_2^U \in \mathcal{W}$ such that

$$\text{Re} \int h_{(\xi, V_2^U)} \left( \frac{j_{V_2^U} - j_{V_1^U}}{2} \right) > \varepsilon/2 - \varepsilon_0$$

and

$$\frac{j_{V_2^U} - j_{V_1^U}}{2} \in U.$$ 

We let $g_{(\xi, U)} = \frac{j_{V_2^U} - j_{V_1^U}}{2}$ and $\theta((\xi, U)) = (\xi, V_2^U)$.

Now suppose that for some $s = s_1^\emptyset (\eta, W) \in T_\xi N$ with $s_1 \neq \emptyset$, and for every $\emptyset < u \leq s_1$, $g_u$ and $\theta(u)$ have been defined to have the indicated properties. Let $t = \theta(s_1)$. For every $V \in \mathcal{W}$, $\|h_{(t, \emptyset)} - h_t\|_{L^q(X^*)} > \varepsilon$, $\|h_{(t, \emptyset)} - h_t\|_{L^\infty(X^*)} \leq 2R$, and the function $h_{(t, \emptyset)} - h_t$ is simple, whence there exists a simple function $i_V \in B_{L_p(X)}$ with $\|i_V\|_{L^\infty(X)} \leq 2^{q-1} R^{q-1} / q-1$ such that $\text{Re} \int (h_{(t, \emptyset)} - h_t)i_V > \varepsilon$. Again using [7, Lemma 3.3], there exist $V_1^W, V_2^W \in \mathcal{W}$ such that
\[ \text{Re} \int h_{(\eta, V^W_2)} \left( \frac{i V^W_2 - i V^W_1}{2} \right) > \varepsilon / 2 - \varepsilon_0, \]
\[ \frac{i V^W_2 - i V^W_1}{2} \in W, \]

and

\[ V^W_2 \subset \{ h \in L_p(X)^* : (\forall \emptyset < u \leq s_1)(|h(g_u)| < \varepsilon_{\{s\}}) \}. \]

We let \( g_s = \frac{i V^W_2 - i V^W_1}{2} \) and \( \theta(s) = t \leftarrow (\eta, V^W_2) \). This finishes the construction, and the conclusions of the claim are easily verified. \( \Box \)

**Corollary 5.5** If \( K \) is convex and \( D_K > \omega^{1+\xi} \), then there exists a constant \( \varepsilon' > 0 \) and a weakly null collection \( (f_t)_{t \in T_{\omega^{1+\xi}}} \mathcal{N} \subset \frac{1}{2} BL_\infty(X) \) such that for every \( t \in T_{\omega^{1+\xi}} \mathcal{N} \) and every convex combination \( f \) of \( (f_s) : s \leq t \),

\[ \sup_{h \in M} \text{Re} \int h f \geq \varepsilon'. \]

**Proof** Suppose \( D_K > \omega^{1+\xi} \). Fix \( \varepsilon > 0 \) such that \( d_{4s}^{\omega^{1+\xi}}(K) \neq \emptyset \) and fix \( x^* \in d_{4s}^{\omega^{1+\xi}}(K) \). Then by Theorem 5.2, \( x^* 1_{[0,1)} \in s_{\omega}^{\omega^{1+\xi}}(M) \). By Proposition 5.1, there exists \( (h_t)_{t \in T_{\omega^{1+\xi}}} \mathcal{W} \subset M \) which is normally \( w^* \)-closed and \( \varepsilon \)-separated. By Lemma 5.3, there exist a function \( \theta : T_{\omega^{1+\xi}} \mathcal{N} \rightarrow T_{\omega^{1+\xi}} \mathcal{W} \) and a weakly null (in \( L_p(X) \)) collection \( (f_t)_{t \in T_{\omega^{1+\xi}}} \mathcal{N} \subset \frac{1}{2} BL_\infty(X) \) such that for any \( \emptyset < s \leq t \),

\[ \text{Re} \int h_\theta(t) f_s \geq \frac{\varepsilon}{3 \cdot 2^q R^{q-1}}, \]

where \( R > 0 \) is such that \( K \subset RB_{X^*} \). Then \( (f_t)_{t \in T_{\omega^{1+\xi}}} \mathcal{N} \) satisfies the conclusion with \( \varepsilon' = \frac{\varepsilon}{3 \cdot 2^q R^{q-1}} \). Indeed, for any \( t \in T_{\omega^{1+\xi}} \) and any \( f = \sum_{\emptyset < s \leq t} a_s f_s \in \mathrm{co}(f_s : \emptyset < s \leq t) \),

\[ \text{Re} \int h_\theta(t) f = \sum_{\emptyset < s \leq t} a_s \text{Re} \int h_\theta(t) f_s \geq \varepsilon'. \]

\( \Box \)

**6 Proof of the main theorem**

**Proof of Theorem 1** First assume \( K \) is convex and symmetric. If \( D_K > \omega^{1+\xi} \), there exists a constant \( \varepsilon' > 0 \) and a normally weakly null collection \( (f_t)_{t \in T_{\omega^{1+\xi}}} \mathcal{N} \subset \frac{1}{2} BL_\infty(X) \) as in the conclusion of Corollary 5.5. By Theorem 4.8, the existence of such a collection implies that \( S_K > \omega^{\xi} \). It follows that \( D_K \leq \omega^{1+\xi} \) if \( S_K \leq \omega^{\xi} \).
This proves (i) in the case that $K$ is convex. Now for a general $K$, let $K_0$ be the $w^*$-closed, convex, symmetrized hull of $K$. By [5, Theorem 1.1], $Sz(K_0) \leq \omega^\xi$ if $Sz(K) \leq \omega$. Thus by the convex case, $Dz(K_0) \leq \omega^{1+\xi}$. Since $Dz(K) \leq Dz(K_0)$, this gives (i) in the general $K$.

For (ii), note that if $K$ is convex, $Sz(K) = \omega^\xi$ for some ordinal $\xi$, or $Sz(K) = \infty$ if $K$ is not $w^*$-fragmentable. In the first case, by (i), we deduce that $Dz(K) \leq \omega^{1+\xi} = \omega Sz(K)$. If $K$ is not $w^*$-fragmentable, it is not $w^*$-dentable, and $Dz(K) = \infty = \omega \infty = \omega Sz(K)$ by convention. If $Sz(K) \geq \omega^0$, then either $Sz(K) = Dz(K) = \infty$ or $Sz(K) = \omega^\xi$ and $Dz(K) \leq \omega^{1+\xi}$ for an ordinal $\xi \geq \omega$. But since $\xi \geq \omega$, $1 + \xi = \xi$, and $Dz(K) \leq \omega^{1+\xi} = \omega^\xi = Sz(K)$. □

As we have already mentioned, for every $n \in \mathbb{N} \cup \{0\}$, there exist a pair of Banach spaces $X_n, Y_n$ such that $Sz(X_n) = Dz(X_n) = \omega^n$ and $Dz(Y_n) = \omega Sz(Y_n) = \omega^{n+1}$, so that Theorem 1 is sharp.

If $A : X \to Y$ is an operator, for any $1 < p < \infty$, $A$ induces an operator $A_p : L_p(X) \to L_p(Y)$ such that for any $\sigma \in [0, 1]$, $(A_p f)(\sigma) = A(f(\sigma))$. Since $(A^* B_{Y^*}, (A_p)^* B_{L_p(Y)^*})$ is nice, Theorem 5.2 yields that $Dz(A) \leq Sz(A_p)$. Thus a positive solution to the next question implies Theorem 1.

**Question 6.1** For any operator $A : X \to Y$ and $1 < p < \infty$, is it true that $Sz(A_p) \leq \omega Sz(A)$?

By [9], Question 6.1 has a positive answer when $A$ is an identity operator and $Sz(A)$ is countable. It is possible to deduce using arguments similar to those in [9] that if $Sz(A)$ is countable, Question 6.1 has a positive answer.

A positive solution to the following question would imply a positive solution to Question 6.1.

**Question 6.2** For any operator $A : X \to Y$ and $1 < p < \infty$, is it true that $Dz(A) = Sz(A_p)$?

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