DUALS OF NON-ZERO SQUARE

HANNAH R. SCHWARTZ

Abstract. In this short note, for each non-zero integer \( n \), we construct a 4-manifold containing a smoothly concordant pair of spheres with a common dual of square \( n \) but no automorphism carrying one sphere to the other. Our examples, besides showing that the square zero assumption on the dual is necessary in Gabai’s and Schneiderman-Teichner’s versions of the 4D Light Bulb Theorem, have the interesting feature that both the Freedman-Quinn and Kervaire-Milnor invariant of the pair of spheres vanishes. The proof gives a surprising application of results due to Akbulut-Matveyev and Auckly-Kim-Melvin-Ruberman pertaining to the well-known Mazur cork.

0. Introduction and Motivation

We work throughout in the smooth, oriented category. Begin by considering a pair of homotopic 2-spheres \( S \) and \( T \) embedded in a smooth 4-manifold \( X \), with an embedded 2-sphere \( G \subset X \) intersecting both \( S \) and \( T \) transversally in a single point. Such a sphere is called a common dual of \( S \) and \( T \). Recent work of Gabai [9] and Schneiderman-Teichner [17] has completely characterized the conditions under which the spheres \( S \) and \( T \) are isotopic, so long as their common dual \( G \) has square zero, i.e. a trivial normal bundle, in the 4-manifold \( X \). We call such a dual standard, and non-standard otherwise. The objective of this note is to show that the assumption of a standard dual is necessary one in [9] and [17].

Main Theorem. For each \( n \neq 0 \), there exists a 4-manifold \( X_n \) containing smoothly concordant embedded spheres \( S_n \) and \( T_n \) with a common dual of square \( n \) such that there is no automorphism of \( X_n \) carrying one sphere to the other.

The proof of our Main Theorem gives a surprising application of well-studied 4-dimensional objects called corks: compact contractible 4-manifolds \( C \) equipped with an orientation preserving diffeomorphism \( h: \partial C \to \partial C \). The study of corks was initially motivated by the fact that the cork twist \( X_{C,h} = (X - \operatorname{int}(C)) \cup_h C \) of an embedded cork \( C \subset X \) is homeomorphic to \( X \) by Freedman [7], but need not be diffeomorphic to \( X \) by Akbulut [1]. Such an embedding of a cork is called non-trivial. Our construction builds upon examples given by Akbulut and Matveyev [3] of non-trivial embeddings of corks.

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1. Warm-up

The first example of a cork with a non-trivial embedding was produced by Akbulut in [1]. Now ubiquitous, the “Akbulut-Mazur cork” \( (W, \tau) \) consists of the Mazur manifold \( W \) shown in Figure 1 and \( \tau \) the involution on its boundary induced by a rotation of \( \pi \) around the indicated axis of symmetry. Many 4-manifolds are now known to admit non-trivial embeddings of the Mazur cork; we outline one such embedding due to Akbulut and Matveyev [3] as a warm-up to the proof of the Main Theorem.

Let \( X \) denote the compact 4-manifold shown on the left in Figure 2 built from the Mazur manifold \( W \) by adding a single 2-handle. Note that \( X \) has a handlebody decomposition consisting

* Mazur’s [15] contractible 4-manifolds are each built with a single 0,1, and 2-handle. They are not homeomorphic to the 4-ball, but their products with the interval give the standard 5-ball.
of a single 1-handle, and two 2-handles each attached along knots in $S^1 \times S^2$ with framings less than their maximum Thurston-Bennequin numbers, as illustrated on the bottom right of Figure 3. Therefore $X$ is a compact Stein domain\footnote{For a precise definition of what we mean by “compact Stein domain”, see \cite{Eliashberg2}.} by a result of Eliashberg \cite{Eliashberg1}; see also \cite{Eliashberg3} for more exposition.

On the other hand, the cork twist $X_{W,\tau}$ contains an embedded 2-sphere of square $-1$, seen in the diagram for $X_{W,\tau}$ in Figure 2 as the union of the shaded disk $D$ and the core of the 2-handle attached along $\partial D$. Therefore $X_{W,\tau}$ must not be a compact Stein domain. This follows from a result due to Lisca and Matić \cite{Lisca1} that compact Stein domains embed in minimal, closed Kähler surfaces, which contain no smoothly embedded 2-spheres of square $-1$. Therefore, $X$ and $X_{W,\tau}$ are not diffeomorphic.

\section{Main theorem}

To contextualize our main result, we outline the previous results about common duals referred to in Section 1. By Gabai \cite{Gabai1} and Schneiderman-Teichner \cite{Schneiderman1}, the existence of a common standard dual for homotopic spheres $S, T \subset X$ guarantees a smooth isotopy between $S$ and $T$ whenever the \textbf{Freedman-Quinn invariant}, a concordance invariant defined in \cite{Freedman1}, of the pair $(S, T)$ vanishes. Recent work of Gabai \cite{Gabai2} shows an analogous result holds for certain properly embedded disks with a common standard dual and vanishing \textbf{Dax invariant}, an isotopy invariant of properly embedded disks recently formulated by Gabai in \cite{Gabai2} using homotopy theoretic work of Dax \cite{Dax1}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The Akbulut-Mazur cork $(W, \tau)$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The manifold $X$ (left) and the cork twist $X_{W,\tau}$ (right)}
\end{figure}
from the 70’s. To guarantee even a smoothly embedded concordance between $S$ and $T$ when their common dual is non-standard, it is also required that their Kervaire-Milnor invariant, defined by Stong in [20], vanishes.

2.1. Remark. Recently, Klug-Miller [13, Example 7.2] showed that it is necessary that the dual have square zero for Gabai [9] and Schneiderman-Teichner [17] to achieve an isotopy, by presenting a pair of spheres whose common dual of square +1, with vanishing Freedman-Quinn invariant but non-vanishing Kervaire-Milnor invariant. On the other hand, for each $n \neq 0$, the Main Theorem gives examples of pairs of spheres with dual of square $n$ whose Freedman-Quinn invariant and Kervaire-Milnor invariants vanish, but that are not related by any automorphism of the ambient 4-manifold. Such an automorphism always exists for spheres when the common dual is standard by [18, Lemma 2.3], since in this case the common dual can be surgered (Gabai remarks after [10, Theorem 0.8] that by a similar proof, this also holds for properly embedded disks with a common standard dual).

Proof of Main Theorem. For $n \leq -1$, consider the 4-manifold $X_n$ pictured in Figure 4. Since $X_n$ is simply-connected, the spheres $S_n$ and $T_n$ are not only homologous, but also homotopic. It is also immediate that the both the Freedman-Quinn and Kervaire-Milnor invariants of the pair $(S_n, T_n)$ vanish, since these invariants are elements of $H_1(X_n; \mathbb{Z}_2)$ and a quotient of $\mathbb{Z}[\pi_1(X)]$ respectively, which are both trivial in this case. Let $R_n$ denote the sphere of square $n$ gotten by capping off the red disk in Figure 1 with the core of the 2-handle attached with framing $n$ along its boundary. The sphere $R_n$ is dual to both $S_n$ and $T_n$, since $S_n$ and $T_n$ each pass once (geometrically) over the 2-handle with framing 1 in the topmost diagram of Figure 4. Therefore, by [8] and [20], the spheres $S_n$ and $T_n$ are smoothly concordant in $X_n \times I$.

The manifold $X_n$ contains Akbulut and Matveyev’s manifold $X$ [4] discussed in Section 1. To show that there is no automorphism of $X_n$ carrying $S_n$ to $T_n$, we use an argument similar to one of Auckly-Kim-Melvin-Ruberman [4, Theorem A]; see in particular Figure 18 of their paper. For, blowing down $S_n$ gives the bottom left manifold of Figure 5, which is not Stein since it contains an embedded sphere of square $-1$, as in the argument from Section 1. On the other hand, blowing down $T_n$ gives the bottom right manifold of Figure 5, which is Stein whenever $n \leq -1$ by [6], since all 2-handles are attached along Legendrian knots whose framings are strictly less than their Thurston-Bennequin numbers.

Figure 3. Identical handlebody structures for $X$, drawn with (left and middle) and without (right) the dotted circle notation for 1-handles from [12, Chapter I.2]. The Thurston-Bennequin framing of the attaching circle of each 2-handle is computed from the rightmost diagram using the usual formula (writhe) - (number of right cusps).
As the manifolds that result from blowing down $S_n$ and $T_n$ are not diffeomorphic, there can be no automorphism of $X_n$ carrying one sphere to the other when $n \leq -1$. The result therefore also holds for $n \geq 1$, setting $X_n = -X_{-n}$ and considering the spheres $S_n, T_n \subset X_n$ that are the images of the spheres $S_{-n}, T_{-n} \subset X_{-n}$ under the (orientation reversing) identity map.

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Figure 5. Blowing down the spheres $S_n$ and $T_n$

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Princeton University, Princeton, NJ 08544

Email address: hs25@princeton.edu