Asymmetric quantum multicast network coding: asymmetric optimal cloning over quantum networks

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In this study, we consider a quantum version of multicast network coding as a multicast protocol for sending universal quantum clones (UQCs) from a source node to the target nodes on a quantum network. By extending Owari et al.’s previous results for symmetric UQCs, we derive a protocol for multicasting $1 \rightarrow 2$ ($1 \rightarrow 3$) asymmetric UQCs of a $q$-dimensional state to two (three) target nodes. Our protocol works under the condition that each edge on a quantum network represented by an undirected graph $G$ transmits a $q$-dimensional state. There exists a classical solvable linear multicast network code with a source rate of $r$ on a classical network $G'$, where $G$ is an undirected underlying graph of an acyclic directed graph $G'$. We also assume free classical communication over a quantum network.

I. INTRODUCTION

The throughput of a network can be improved by applying non-trivial operations to the bitstream at intermediate nodes when there is a bottleneck on a network [1, 2]. This protocol can be referred to as network coding. The network coding research was started in classical information theory [3]. The network coding for a quantum network is called “quantum network coding” [4]. There has been a considerable amount of research on quantum network coding, which can improve the throughput of a quantum network in various situations [5–14]. Recently, it has been presented that quantum network coding can improve the security of a quantum network [13–18]. Further, it is useful for quantum repeater networks [19, 20] as well as for distributed quantum computation [21]. Although many studies have considered network coding on noisy classical networks, almost all the studies of quantum network coding consider noise-free quantum networks. This is because quantum network coding is regarded as a protocol implemented on a layer after the errors have already corrected. Hence, in this study, we consider noise-free quantum networks.

In classical network coding, majority of the studies have focused on multicast communication, where a single source node transmits the same information to multiple target nodes on a network [1, 2]. Figure 1 shows the network coding for the butterfly network. This is one of the simplest examples of classical multicast network coding. Another type of network coding is called multiple-unicast network coding. Here, there are $k$ pairs of source and target nodes $(s_0, t_0), \ldots, (s_{k-1}, t_{k-1})$ on the network, and each source node $s_i$ independently transmits a message to the corresponding target node $t_i$ for all $i$ [22]. The modified version of the butterfly network in Figure 2 is one of the simplest examples of classical multiple-unicast network coding.

Most of the research on quantum network coding considered multiple-unicast communication, i.e., multiple-unicast quantum network coding. This differs from classical network coding because each source node transmits a quantum state (instead of a classical message) to the corresponding target node [5–13]. The most important results are those of Kobayashi et al. If classical information (or measurement results) can be freely sent among the nodes on a quantum network, Kobayashi et al. gave a canonical procedure for constructing a quantum multiple-unicast network code from a classical multiple-unicast network code. The quantum network for the quantum code and the classical network for the classical code must represented by the same graph [7, 10].

Unlike quantum multiple-unicast network coding, there has been less research on quantum network coding focusing on multicast communication [6, 8, 11–13]. This is because in quantum information theory, no-cloning theory prohibits perfect multicast communication [23], and, thus, it is not straightforward to construct a multicast quantum network coding pro-
protocol as an extension of a classical multicast network coding protocol.

Shi et al.'s paper is the first to treat quantum multicast network coding \[8\]. They consider the problem of distributing $N$-identical copies of a state $\ket{\psi}$ from a single source node to $N$ target nodes. Since the number of copies of $\ket{\psi}$ is equal to the number of nodes, $\ket{\psi}$ can be distributed without cloning the quantum states. Shi et al. showed that coding on intermediate nodes can increase the throughput of the quantum network. The second work treating this topic is Kobayashi et al.'s paper \[8\]. In this paper, a single copy of a state $\ket{\psi} = \sum_{i=1}^{d} \alpha_i \ket{i}$ is given on the source node and the aim is to share a Greenberger-Horne-Zeilinger (GHZ-)type state $\sum_{i=1}^{d} \alpha_i \ket{i}_1 \otimes \cdots \otimes \ket{i}_N$ among target nodes, where the $i$th local system is on the $i$th target node. From this GHZ-type state between the target nodes, the input state $\ket{\Psi}$ can be reconstructed at any target node by local operations and classical communication (LOCC). Based on classical multicast network coding, Kobayashi et al. developed a quantum protocol to achieve the above task under the assumption of free classical communication among nodes on the quantum network.

Although Shi et al.'s protocol and Kobayashi et al.'s protocol can be considered as generalizations of classical multicast network coding to quantum networks, rigorously speaking, the goal of their protocols is not exactly to achieve a multicast of a quantum state. Since an optimal multicast quantum channel is nothing but an optimal cloning \[24, 26\], a protocol to share an optimal clone of an input state among target nodes of a quantum network can be considered as one of the most natural quantum extensions of a multicast classical network coding protocol. Based on this idea, Owari et al. constructed a protocol to share a symmetric optimal universal clone of an input state on the target nodes under the conditions that classical information can be sent freely among nodes on a quantum network and that a small amount of entanglement is shared on target nodes at the beginning of the protocol \[11, 13\].

In this paper, we focus on extending Owari et al.'s results to asymmetric optimal universal cloning \[27, 30\], which is a generalization of symmetric optimal universal cloning. Thus, we construct a protocol to efficiently multicast an asymmetric optimal clone of a $q^r$-dimensional input quantum state from one source node to two (three) target nodes, where $q$ is assumed to be a prime power. In this protocol, the following five conditions are assumed:

- The noise-free quantum network can be described by an undirected graph $G$ with one source node and two (three) target nodes.
- Each quantum channel on the quantum network can transmit one $q$-dimensional quantum system in a single session.
- There exists a classical solvable linear multicast network code with source rate $r$ for a noise-free classical network described by an acyclic directed graph $G'$, where $G$ is an undirected underlying graph of $G'$.
- Measurement results (or classical information) can be sent freely from one node to another node on the quantum network.
- A small amount of entanglement which does not scale with $q$, is shared among the target nodes. The amount of entanglement is at most 2 ebit for two target nodes, and at most $\left(2 + 4 \log_2 3\right)$ ebit for the case of tree target nodes.

Using the max-flow and min-cut theorem of multicast network coding \[11, 12\], for sufficiently large $q$, the assumption for the existence of a classical network code on $G'$ can be replaced by the condition that the minimum-cut between the source node $s$ and a target node $t_i$ is no less than $r$ for all $i$.

An outline of our protocol is as follows:

- We create two (three) asymmetric optimal clones of an input state with an ancilla system at a source node.
- We measure the ancilla system, and send the measurement outcomes to the target nodes.
- We compress the whole $d^r$ ($d^3$)-dimensional system into a $d$-dimensional system.
- We transmit the resulting state to two (three) target nodes using Kobayashi et al.'s multicast quantum network coding \[8\]. As a result, a GHZ-type state is shared among target nodes.

Figure 2: Multiple-unicast classical network coding on the butterfly network. A source node $s_0$ sends message $b_0 \in F_p$ to target node $t_0$, and source node $s_1$ sends message $b_1 \in F_p$ to target node $t_1$. 

\[ \sum_{i=1}^{d} \alpha_i \ket{i} = \sum_{i=1}^{d} \alpha_i \ket{i}_1 \otimes \cdots \otimes \ket{i}_N \]
We reconstruct the asymmetric clones of the input state from the GHZ-type state using LOCC with a small amount of entanglement among the target nodes and the measurement outcomes sent from the source node.

Using the above protocol, we can multicast asymmetric optimal clones from one source node to two (three) target nodes (Figure 3).

![Figure 3: Schematic diagram of a protocol for multicasting asymmetric optimal clones from one source node to two target nodes. The asymmetric optimal cloning protocol for the input state $|\psi\rangle$ is implemented at the source node. The result state is compressed into a network coding protocol in Section II. We present a protocol for asymmetric cloning and Kobayashi et al.’s quantum multicast network coding protocol. Finally, the asymmetric clones of the input state are reconstructed by LOCC on target nodes with the help of a small amount of entanglement.]

The rest of the paper is organized as follows: We explain the asymmetric cloning and Kobayashi et al.’s quantum multicast network coding protocol in Section II. We present a protocol for multicasting 1 → 2 asymmetric optimal clones in Section III. We also present a protocol for multicasting 1 → 3 asymmetric optimal clones in Section IV. Finally, we give an conclusion in Section V.

II. PRELIMINARIES

Optimal asymmetric universal quantum cloning, classical linear multicast network coding and Kobayashi et al.’s quantum multicast network coding protocol are all important in our protocol. In this section, we explain optimal asymmetric universal quantum cloning in Section II.A. Then, classical linear multicast network coding and the Kobayashi et al.’s multicast quantum network coding protocol are presented in Sections II.B and II.C respectively.

A. Optimal asymmetric quantum universal cloning machine

No-cloning theorem states that quantum mechanics prohibits a quantum operation that makes perfect copies of an unknown quantum state [23]. In other words, a perfect multicast of an unknown quantum state is impossible. On the other hand, quantum mechanics does not completely prohibit the approximate cloning of a quantum state. Hence, many studies have focused on quantum protocols to make an approximate copy of unknown states (so-called quantum cloning machines) [24, 25, 29, 30].

A quantum cloning machine (QCM) that produces $N$ approximate clones based on $M$ copies of a given quantum state $|\psi\rangle \in \mathcal{H}$ is a quantum channel (or a completely positive and trace preserving map) $\varepsilon$ from $\mathcal{B}(\mathcal{H}^\otimes M)$ to $\mathcal{B}(\mathcal{H}^\otimes N)$, where $\mathcal{B}(\mathcal{H})$ is a space of all linear operators on the Hilbert space $\mathcal{H}$. Suppose $\rho_i$ is a reduced density matrix of the output state on the $i$th subsystem: $\rho_i = \text{Tr}_{-i} \varepsilon \left( |\psi\rangle \langle \psi| \otimes I^\otimes (m-i) \right)$, where $\text{Tr}_{-i}$ is a partial trace of all subsystems except the $i$th subsystem. Since the purpose of a QCM is to make $\rho_i$ as closed as the input state $|\psi\rangle \langle \psi|$, the performance of a QCM can be described by the output fidelity $F_i$ between $\rho_i$ and $|\psi\rangle \langle \psi|$: 

$$F_i = \langle \psi | \rho_i | \psi \rangle, \quad (i = 1, \ldots, M).$$

A QCM is called universal, if $F_i$ does not depend on the input state $|\psi\rangle$. Further, a universal QCM (UQCM) is called symmetric, if the all clones are the same: $\rho_i = \rho_j$ for all $i$ and $j$. A UQCM that is not symmetric is asymmetric. An asymmetric UQCM whose output fidelities $F_i$ are optimal is called an optimal asymmetric UQCM. Since the output states of an asymmetric UQCM satisfy $\rho_i = \rho_j$, the output fidelities $F_i$ also depend on $i$. Hence, an optimal asymmetric UQCM in general depends on parameters that represent a bias among the output fidelities $\{F_i\}_{i=1}^N$.

Here, we give an optimal asymmetric UQCM with $M = 1$ and $N = 2$ (we call this protocol a 1 → 2 optimal asymmetric UQCM). This protocol uses three systems $A$, $B$, and $M$ whose Hilbert spaces are $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}_M$, respectively. Here $\mathcal{H}_A$ works as an input system and a first output system, $\mathcal{H}_B$ is a second output system, and $\mathcal{H}_M$ is an ancilla system. The dimensions of all three systems are the same, and we denote this dimension as $d$; that is, $\dim \mathcal{H}_A = \dim \mathcal{H}_B = \dim \mathcal{H}_M = d$. Then, for an input state $|\psi\rangle$ on system $A$, a 1 → 2 optimal asymmetric UQCM is given by an isometry $U'_{1\rightarrow 2}^{(a,b)}$ from $\mathcal{H}_A$ to $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_M$ satisfying [29]:

$$U'_{1\rightarrow 2}^{(a,b)}|\psi\rangle_A = a|\psi\rangle_A|\Phi_d^+\rangle_B + b|\psi\rangle_B|\Phi_d^+\rangle_A,$$

where $|\cdot\rangle$ is defined by $|+\rangle := \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle_M$, $|\Phi_d^+\rangle$ is a standard $d$-dimensional maximally entangled state:

$$|\Phi_d^+\rangle := \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle |k\rangle,$$

and $a$ and $b$ are real parameters satisfying:

$$a^2 + b^2 + \frac{2ab}{d} = 1.$$
The fidelity of the reduced density matrices, which have been proved to be optimum \cite{29}, are given by

\begin{equation}
F_A := \langle \psi | \text{Tr}_B \left( e_{1-2}^{(a,b)} | \psi \rangle \langle \psi | \right) | \psi \rangle = 1 - b^2 \frac{d-1}{d},
\end{equation}

\begin{equation}
F_B := \langle \psi | \text{Tr}_A \left( e_{1-2}^{(a,b)} | \psi \rangle \langle \psi | \right) | \psi \rangle = 1 - a^2 \frac{d-1}{d}. 
\end{equation}

Next, we give an optimal asymmetric UQCM with $M = 1$ and $N = 3$ (we call this protocol the $1 \rightarrow 3$ optimal asymmetric UQCM). This protocol use five systems $A, B, C, R,$ and $S$ whose Hilbert spaces are $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C, \mathcal{H}_R,$ and $\mathcal{H}_S,$ respectively. Here, $\mathcal{H}_A$ is an input system that is also the first output system. $\mathcal{H}_B$ and $\mathcal{H}_C$ are the second and third output systems, respectively. $\mathcal{H}_R$ and $\mathcal{H}_S$ are ancilla systems. The dimensions of all systems are the same, which we denote as $d$. For an input state $| \psi \rangle$ on system $A$, $1 \rightarrow 3$ optimal asymmetric UQCM is given by an isometry $U_{\text{ABCRS}}$ from $\mathcal{H}_A$ to $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_R \otimes \mathcal{H}_S$ satisfying the following equation:

\begin{equation}
U_{1-3}^{(a,b)} \left( | \psi \rangle \right) = \sqrt{\frac{d}{2d+2}} \left( \alpha | \psi \rangle_{A} (\Phi^{+})_{BR} (\Phi^{+})_{CS} + (\Phi^{+})_{BS} (\Phi^{+})_{CR} \right) + \beta \psi_B (\Phi^{+})_{AR} (\Phi^{+})_{CS} + (\Phi^{+})_{AS} (\Phi^{+})_{CR}) \right) + \gamma | \psi \rangle_C (\Phi^{+})_{AR} (\Phi^{+})_{BS} + (\Phi^{+})_{AS} (\Phi^{+})_{BR}),
\end{equation}

where $\alpha, \beta, \gamma$ are non-negative real parameters satisfying the following constraint \cite{26, 30}:

\begin{equation}
\alpha^2 + \beta^2 + \gamma^2 + \frac{2}{d} (\alpha \beta + \beta \gamma + \alpha \gamma) = 1. \tag{8}
\end{equation}

In terms of $U_{\text{ABCRS}}$, a $1 \rightarrow 3$ optimal asymmetric UQCM $e_{1-3}^{(a,b)}$ can be written as:

\begin{equation}
e_{1-3}^{(a,b,\gamma)} \left( | \psi \rangle \langle \psi | \right) := \text{Tr}_{RS} \left( U_{1-3}^{(a,b,\gamma)} \left( | \psi \rangle \langle \psi | \right) U_{1-3}^{(a,b,\gamma)^{-1}} \right). \tag{9}
\end{equation}

The fidelities between an input state and each reduced density matrix, which were proved to be optimum \cite{26, 30}, is given as follows:

\begin{equation}
F_A = 1 - \frac{d-1}{d} \left( \beta^2 + \gamma^2 + \frac{2 \beta \gamma}{d+1} \right), \tag{10}
\end{equation}

\begin{equation}
F_B = 1 - \frac{d-1}{d} \left( \alpha^2 + \gamma^2 + \frac{2 \alpha \gamma}{d+1} \right), \tag{10}
\end{equation}

\begin{equation}
F_A = 1 - \frac{d-1}{d} \left( \alpha^2 + \beta^2 + \frac{2 \alpha \beta}{d+1} \right). \tag{10}
\end{equation}

**B. Classical multicast network coding**

Since our protocol uses Kobayashi et al.’s protocol as a subroutine and since Kobayashi et al.’s protocol is based on a classical linear multicast network code, we introduce classical linear multicast network coding in this section. A detail description of classical linear multicast network coding can be found in standard text books of network coding like \cite{1, 2}.

A classical network is represented by a directed graph $G' = (V, E')$, where a vertex $v \in V$ represents a node of the network and an edge $e \in E'$ represents a noiseless classical channel. In this paper, we assume that $G'$ is acyclic. There exist a source node $s \in V,$ and $N$ target nodes $t_1, \ldots, t_N \in V$ on the network. A node that is neither a source nor a target node is called an intermediate node. In a single session of a classical multicast network coding, an alphabet on the finite field $\mathbb{F}_q$ is sent from node $u$ to node $v$ if $(u, v) \in E'$, where the order of $\mathbb{F}_q$ is a prime power $q$. Since $G'$ is an acyclic directed graph, a natural partial ordering can be defined on $E'$; that is, when $(u, v), (v, w) \in E'$, we define $(u, v) \prec (v, w)$. The order of transmissions of classical information can be determined by this partial ordering. That is, an edge $e \in E'$ transmits an alphabet after all edges $e' \in E'$ satisfying $e' \prec e$ have transmitted alphabets. We assume that there is no incoming edge to the source node $s$, and that there is no outgoing edge from any target node. Hence, all edges whose tail node is the source node $s$ are a local minimum, and all edges whose head node is a target node are a local maximum under the partial ordering. We further assume that all edges whose tail node is not the source node $s$ are not a local minimum and that all edges whose head node is not a target node are not a local maximum.

A classical linear multicast network code over $\mathbb{F}_q$ on $G'$ consists of a set of linear maps $\{f_e \mid e \in E'\}$. At the beginning of a session, an input message $\vec{x} := (x_1, \ldots, x_r) \in \mathbb{F}_q^r$ is chosen on the source node $s$, where $r$ is the source rate of the classical multicast network code. Suppose $e$ is an outgoing edge of $v$. At the first step of the network coding, an alphabet $y_e$ transmitted through the edge $e$ is chosen as a linear combination of $x_1, \ldots, x_r$. In other words, in terms of a linear function $f_e : \mathbb{F}_q^r \rightarrow \mathbb{F}_q, y_e$ can be written as

\begin{equation}
y_e := f_e(\vec{x}) = f_e(x_1, \ldots, x_r). \tag{11}
\end{equation}

After calculating $y_e$, $y_e$ is transmitted through $e$. After all edges outgoing from the source node $s$ transmitted an alphabet, all intermediate nodes transmit alphabet in the order determined by the partial ordering as follows: Suppose an intermediate node $v$ on the network has $m$ incoming edges and $e$ is an outgoing edge from $v$. After all transmissions of $m$ incoming edges to $v$ have finished, the node $v$ has $m$-alphabets $y_j \in \mathbb{F}_q$ ($j = 1, \ldots, m$), where $y_j$ is an alphabet sent through the $j$th incoming edge. Then, an alphabet $y_e$ transmitted through the edge $e$ is chosen as a linear combination of $y_1, \ldots, y_m$. In other words, there exists a linear function $f_e : \mathbb{F}_q^m \rightarrow \mathbb{F}_q$ such that

\begin{equation}
y_e := f_e(y_1, \ldots, y_m). \tag{12}
\end{equation}

After the calculation, $y_e$ is transmitted through $e$.

Suppose a target node $t_i$ has $m_i$ incoming edges. Then, after all edges have transmitted an alphabet, the target node $t_i$ has $m_i$-alphabets $y_{i,j}^{(0)} \in \mathbb{F}_q$ ($j = 1, \ldots, m_i$), where $y_{i,j}^{(0)}$ is an alphabet sent through the $j$th incoming edge to $t_i$. A classical linear multicast network code $\{f_e \mid e \in E'\}$ is called solvable if there exists a set of decoding operations \{$g_i^{(0)}\}_{i=1}^N$ such that $g_i : \mathbb{F}_q^{m_i} \rightarrow \mathbb{F}_q$ satisfies the following equation for all $i$:

\begin{equation}
\vec{x} = g_i(y_1^{(0)}, \ldots, y_m^{(0)}), \tag{13}
\end{equation}

where $y_e$ is the transmitted alphabet through the edge $e$. This equation means that each target node is able to recover the input message $\vec{x}$ using $m_i$ received messages $y_{i,j}^{(0)}$.
where $x \in F_q^r$ is the input message. If a classical linear multicast network code is solvable, any decoding operation $g_i$ can be chosen as a linear map.

There is a necessary and sufficient condition for the existence of a classical linear multicast network code \cite{1,2}. Suppose $C_r$ is the size of the minimum cut between $s$ and $t_i$. Then, there exists a classical linear multicast code with source rate $r$ on $G'$ over a sufficiently large field $F_q$, if and only if $C_r \geq r$ for all $i$.

## C. Quantum multicast network coding

In this section, we review Kobayashi et al.’s protocol \cite{8}. First, we give a problem setting for multicast quantum network coding that is common between our protocol and Kobayashi et al.’s protocol. A quantum network is described by an undirected graph $G = (V, E)$, where $V$ represents a set of nodes and $E$ represents a set of quantum channels. There exist a source node $s \in V$, and $N$ target nodes $t_1, \ldots, t_N \in V$ on the network. In a single session, any quantum channel $(u, v) \in E$ can send a $q$-dimensional quantum system $\mathcal{H}_e$ just once either from $u$ to $v$, or from $v$ to $u$, where $q$ is assumed to be a prime power. Further, any quantum operations can be implemented on any node $v \in V$, and measurement outcomes (or classical information) can be freely sent among nodes. At the beginning of a session, a single copy of input state $|\psi\rangle$ is given on the source node $s$. Here, the reason a quantum channel is represented by an undirected edge is that the direction of a quantum channel can be effectively reversed by quantum teleportation under the condition of free classical communication \cite{9}.

The purpose of both protocols is to multicast an input state $|\psi\rangle$ from the source node to all target nodes in a single session. Here, we should note that the meaning of “multicast” in Kobayashi et al.’s protocol is different from that in our protocol. As we have explained in the introduction, the purpose of our protocol is to construct optimal asymmetric universal clones among target nodes for a given $q'$-dimensional input state $|\psi\rangle = \sum_{j=0}^{q'-1} \alpha_j |j\rangle \in \mathcal{H}_s$ on a source node, where $\mathcal{H}_s$ is a $q'$-dimensional input space. In other words, we consider multicast quantum network coding with source rate $r$. On the other hand, the purpose of Kobayashi et al.’s protocol is to construct a GHZ-type state $\sum_{j=0}^{q'-1} \alpha_j |1\rangle \otimes \cdots \otimes |j\rangle_{N_y}$ among target nodes, where the $i$th local system is on the $i$th target node.

Both Kobayashi et al.’s protocol and our protocol are constructed under the assumption that there exists a solvable classical linear multicast network code $\{f_e\}_{e \in E'}$ with source rate $r$ on an acyclic directed graph $G' = (V, E')$ over a finite field $F_q$, where $G$ is an undirected underlying graph of $G'$. In other words, $G$ can be derived by replacing all directed edges on $G'$ by undirected edges. Using this replacement, a directed edge $e' \in E'$ is naturally mapped to an undirected edge $e \in E$, and this map is a bijection. Hence, in the following part of this section, we will not distinguish $e'$ from $e$, and write $e'$ as $e$.

Kobayashi et al.’s protocol imitates a classical linear multicast network code $\{f_e\}_{e \in E'}$ and corresponding decoding operations $\{g_i\}_{i=1}^N$ by unitary operators. Each linear map $f_e$ is imitated by a unitary operator $U_e$, and each recovery operation $g_i$ is imitated by a unitary operator $V_i$, where $U_e$ and $V_i$ are defined as follows: Since $\dim \mathcal{H}_e = q'$, due to the bijection between $[0, 1 \ldots, q'-1]$ and $F_q$, an input state $|\psi\rangle \in \mathcal{H}_e$ can be written as $|\psi\rangle = \sum_{e \in E} \alpha_e |\chi_e\rangle$. For an outgoing edge $e$ from the source node $s$, a unitary operator $U_e$ on $\mathcal{H}_t \otimes \mathcal{H}_s$ is defined by means of $f_e : F_q^r \rightarrow F_q$ as

$$U_e := \sum_{\bar{x} \in \mathbb{F}_q, x \in \mathbb{F}_q} |\bar{x}\rangle \langle x| \otimes |e + f_e (x)\rangle \langle e|_e ,$$

where $\mathcal{H}_t$ is a Hilbert space transmitted through $e$. Suppose $In(e)$ is a set of all incoming edges of $v$, where $v$ is a tail node of $e$, and suppose $\mathcal{H}_{In(e)} := \bigotimes_{e \in In(e)} \mathcal{H}_e$. Then, for an outgoing edge $e$ from an intermediate node $v$, a unitary operator $U_e$ on $\mathcal{H}_{In(e)} \otimes \mathcal{H}_s$ is defined by means of $f_e : \mathbb{F}_q^{In(e)} \rightarrow \mathbb{F}_q$ as

$$U_e := \sum_{\bar{y} \in \mathbb{F}_q^{In(e)} \otimes \mathbb{F}_q} |\bar{y}\rangle \langle y| \otimes |e + f_e (y)\rangle \langle e|_e .$$

Suppose $\mathcal{V}_t$ is a $q'$-dimensional output Hilbert space on a target node $t_i$. A unitary operator $V_i$ on $\mathcal{H}_{In(t_i)} \otimes \mathcal{V}_t$ is defined by means of the decoding operation $g_i : \mathbb{F}_q^{In(t_i)} \rightarrow \mathbb{F}_q$ as

$$V_i := \sum_{\bar{y} \in \mathbb{F}_q^{In(t_i)} \otimes \mathbb{F}_q} |\bar{y}\rangle \langle y| \otimes |e + g_i (\bar{y})\rangle \langle e|_{t_i} .$$

Kobayashi et al.’s quantum multicast network coding protocol is shown as protocol 1.

### Protocol 1 Kobayashi et al.’s quantum multicast network coding protocol

#### Step 1: Initialization

The source node $s$ prepares an initial state $|\psi\rangle$ on $\mathcal{H}_s$. Each node $v \in V$ prepares $|0\rangle$ on $\mathcal{H}_v$ for an edge $e \in E$ whose tail node is $v$. For all $i$ satisfying $1 \leq i \leq m$, a target node $t_i$ prepares $|0\rangle$ on $\mathcal{V}_{t_i}$.

#### Step 2: Transmission

First, for all edges $e \in E'$ whose tail node is the source node, the source node operates the unitary operator $U_e$ on $\mathcal{H}_s \otimes \mathcal{H}_e$ and sends $\mathcal{H}_e$ to the head node of $e$. Second, all intermediate nodes behave in the order defined by the natural partial ordering on $E'$ as follows: After an intermediate node $v$ has received Hilbert spaces from all edges whose head node is $v$, for all edges $e \in E'$ whose tail node is $v$, node $v$ operates the unitary operator $U_e$ on $\mathcal{H}_{In(e)} \otimes \mathcal{H}_e$ and sends $\mathcal{H}_e$ to the head node of $e$. Finally, after all edges have transmitted Hilbert spaces, for all $i$ satisfying $1 \leq i \leq m$, target node $t_i$ operates the unitary operator $V_i$ on $\mathcal{H}_{In(t_i)} \otimes \mathcal{V}_{t_i}$.

#### Step 3: Measurement on Fourier-basis

The source node $s$ measures the Hilbert space $\mathcal{H}_t$ in the Fourier basis, and sends the measurement outcome to all the terminal nodes $t_i$. For all edges $e \in E'$, the head node of $e$ measures the Hilbert space $\mathcal{H}_e$ in the Fourier basis, and sends the measurement outcome to all terminal nodes $t_i$.

#### Step 4: Recovery

All terminal nodes $t_i$ operate $Z(c_i) \otimes \cdots \otimes Z(c_i)$ on $\mathcal{V}_{t_i}$. Here, $[c_i]_{i=1}^N$ is a natural number that can be determined from the measurement outcomes received in step 3, the classical linear multicast network code $\{f_e\}_{e \in E'}$, and the decoding operators $\{g_i\}_{i=1}^N$ \cite{8}.
In step 3 of protocol 1, the Fourier basis of \( \{|z\rangle \}_{z \in \mathbb{F}_q} \subset \mathcal{H}_z \) of the computational basis \( \{|x\rangle \}_{x \in \mathbb{F}_q} \subset \mathcal{H}_x \) is defined as
\[
|z\rangle := \sum_{x \in \mathbb{F}_q} \omega^{Tr z x} |x\rangle,
\]
where \( \omega := \exp(-2\pi i / p) \). Here, \( Tr_z \) represents the element \( Tr_{M_z} \in \mathbb{F}_p \), where \( M_z \) is the matrix representation of the multiplication map \( x \mapsto zx \). Here, we note that the finite field \( \mathbb{F}_q \) can be identified with the vector space \( \mathbb{F}_p^t \), where \( t \) is the degree of the algebraic extension of \( \mathbb{F}_q \). For further details, see [31, Section 8.1.2]. We also define the generalized Pauli operators \( Z(t) \) as \( Z(t) := \sum_{x \in \mathbb{F}_q} \omega^{Tr-t} |x\rangle \langle x| \).

### III. \( 1 \rightarrow 2 \) ASYMMETRIC UQC MULTICAST PROTOCOL

In this section, we present a new protocol that multicasts optimal asymmetric UQCs from the source node \( s \) to two target nodes \( t_1 \) and \( t_2 \) on a quantum network. We present the protocol in Section III.A and prove that it creates optimal asymmetric UQCs in the subsection III.B.

#### A. \( 1 \rightarrow 2 \) quantum multicast protocol

In this section, we present the protocol for multicasting \( 1 \rightarrow 2 \) optimal asymmetric UQCs of an input quantum state from the source node \( s \) to two target nodes \( t_1 \) and \( t_2 \).

As we have explained in Section II.C, the problem settings for Kobayashi et al.’s protocol and our protocol are essentially the same, and the only their purposes are different. Here, we summarize the problem setting of our quantum multicast network coding: A quantum network is described by an undirected graph \( G = (V, E) \). There exist a source node \( s \in V \), and \( N \) target nodes \( t_1, \ldots, t_N \in V \) on the network. In this section, since we consider multicasting \( 1 \rightarrow 2 \) asymmetric UQCs, we set \( N = 2 \). In a single session, any quantum channel \( (u, v) \in E \) can send a \( q \)-dimensional quantum system \( \mathcal{H}_u \) just once, either from \( u \) to \( v \) or from \( v \) to \( u \), where \( q \) is assumed to be a prime power. Further, any quantum operations can be implemented on any node \( v \in V \), and measurement outcomes can be freely sent among nodes. At the beginning of a session, a single copy of input state \( |\psi\rangle \) is given on the source node \( s \).

Under these problem settings, the purpose of our protocol is to construct optimal asymmetric universal clones given by Eq. (5) between target nodes \( t_1 \) and \( t_2 \) for a given \( d \)-dimensional input state \( |\psi\rangle = \sum_{j=0}^{d-1} \alpha_j |j\rangle \in \mathcal{H}_s \) on a source node, where \( \mathcal{H}_t \) is a \( d \)-dimensional input space. We assume \( d = q^r \). In other words, we consider multicast quantum network coding with source rate \( r \). Here, note that since we assumed \( q \) is a prime power, \( d \) is also a prime power.

For this purpose, we use two additional assumptions: The first assumption is the same assumption that Kobayashi et al. used. That is, we assume that there exists a solvable classical linear multicast network code \( \{f_{r e} \}_{e \in E} \) with source rate \( r \) on an acyclic directed graph \( G' = (V, E') \) over a finite field \( \mathbb{F}_q \), where \( G \) is an undirected underlying graph of \( G' \). Hence, we can use Kobayashi et al.’s quantum multicast network coding protocol with source rate \( r \) on this quantum network \( G \). We further assume that at most 2 ebits of entanglement resource are shared between target node \( t_1 \) and \( t_2 \). Hence, the amount of this entanglement resource is constant with respect to the dimension \( d \) of the input state, and is negligible for large \( d \) in comparison to \( d \).

Before we present the protocol, we define the unitary operators used in it. Pauli operators \( X_d \) and \( Z_d \) are defined as
\[
X_d := \sum_{k=0}^{d-1} |k \oplus 1\rangle \langle k|, \quad Z_d := \sum_{k=0}^{d-1} \omega^k |k\rangle \langle k|, \quad (17)
\]
where \( \omega := e^{2\pi i / d} \). In the following part of the paper, unitary operators defined on \( \mathbb{C}^d \otimes \mathbb{C}^d \) and \( \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \) are called bipartite and tripartite unitary operators, respectively. \( \Upsilon(r) \) is defined as a bipartite unitary operator satisfying
\[
\Upsilon(r)(\cos \eta |jr\rangle + \sin \eta |jr\rangle) = |jr\rangle, \quad \Upsilon(r)(|rr\rangle) = |rr\rangle
\]
for all \( j \in \{0, \ldots, d-1\} \) satisfying \( j \neq r \), and
\[
\Upsilon(r)(|ij\rangle) = |ij\rangle \quad (19)
\]
for all \( i, j \in \{0, \ldots, d-1\} \) satisfying \( i, j \neq r \), where \( \eta \) is defined by
\[
\cos \eta = \frac{a}{\sqrt{1 - \frac{2ab}{d}}} \quad \text{and} \quad \sin \eta = \frac{b}{\sqrt{1 - \frac{2ab}{d}}} \quad (20)
\]
The bipartite unitary operator \( V(r) \) is defined by
\[
V(r) := \sum_{j \neq r} |j\rangle \langle j| \otimes U_{r,j} + |r\rangle \langle r| \otimes I \quad (21)
\]
where the unitary operator \( U_{r,j} \) is defined by
\[
U_{r,j} = I - |j\rangle \langle j| - |r-1\rangle \langle r-1| - |j\rangle \langle r-1| - |r-1\rangle \langle j|.
\]
The bipartite unitary operator \( \Delta(r) \) is defined by
\[
\Delta(r) := |r\rangle \langle r| \otimes \mathbb{1}^{2(r-1)} + \sum_{j \neq r} |j\rangle \langle j| \otimes I \quad (22)
\]
The tripartite unitary operator \( \Gamma(r) \) is defined by
\[
\Gamma(r) := |r\rangle \langle r| \otimes \text{swap} + \sum_{j \neq r} |j\rangle \langle j| \otimes I \quad (23)
\]
where \( \text{swap} \) is a unitary operator on \( \mathbb{C}^d \otimes \mathbb{C}^2 \) defined by
\[
\text{swap} := \sum_{i=2}^{d-1} |i\rangle \langle j| + \sum_{i=2}^{d-1} |i\rangle \langle j|.
\]
The unitary operator \( \Theta \) on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) is defined by
\[
\Theta(\cos \eta |00\rangle + \sin \eta |10\rangle) = |10\rangle \quad (24)
\]
\[
\Theta(\sin \eta |01\rangle - \cos \eta |10\rangle) = |01\rangle
\]
The bipartite unitary operator $A^{(r)}$ is defined by
\[ A^{(r)} := \sum_{j \neq r} |j\rangle\langle j| \otimes I + |r\rangle\langle r| \otimes X^r \]  
\(25\)

Before starting the protocol, we prepare three $d$-dimensional systems $A$, $B$ and $M$ at the source node $s$, $d$-dimensional systems $C$, $E$ and 2-dimensional systems $G$, $T_1$ at the target node $t_1$. Similarly, we prepare $d$-dimensional systems $D$, $F$ and 2-dimensional systems $H$, $T_2$ at $t_2$. The entanglement resource cos $\eta|1\rangle_E|1\rangle_E + \sin \eta|1\rangle_E|0\rangle_E$ is shared between $E$ and $F$, and the Bell state \(\frac{1}{\sqrt{2}}(|00\rangle_{T_1T_2} + |11\rangle_{T_1T_2})\) is shared between $T_1$ and $T_2$. Thus, the amount of entanglement resources is at most 2 ebits.

The protocol for $1 \rightarrow 2$ is shown as protocol2. Using the protocol, $1 \rightarrow 2$ asymmetric UQCs given by Eq. (5) are created in systems $EF$, where $E$ and $F$ are on the target nodes $t_1$ and $t_2$, respectively. Note that as we explained in the previous section, asymmetric UQCs depend on the parameters $a$ and $b$ in Eq. (2). We can set these parameters in step 2 of the protocol, when we apply $U^{(a,b)}_{1\rightarrow 2}$.

**B. Proof of $1 \rightarrow 2$ quantum multicast protocol**

In this section, we present the proof that protocol 2 creates $1 \rightarrow 2$ asymmetric UQCs given by Eq. (5) in system $EF$.

As we explained in the previous section, an input state at the source node $s$ can be written as
\[ |\psi\rangle = \sum_{j=0}^{d-1} \alpha_j |j\rangle \in \mathcal{H}_s. \]

Then, from Eq. (2), the state on system $ABM$ after step 1 can be written as:
\[ a|\psi\rangle_A|0^+\rangle_{BM} + b|\psi\rangle_B|1^+\rangle_{AM} \]
\(26\)

The unnormalized state $|\Psi^{(r)}_2\rangle_{AB}$ on system $AB$ after deriving measurement outcome $r$ in step 2 can be written as:
\[ |\Psi^{(r)}_2\rangle_{AB} := \beta_r |rr\rangle_{AB} + \sum_{j \neq r} \beta_j (\cos \eta |jr\rangle_{AB} + \sin \eta |jr\rangle_{AB}) \]
\(27\)

where $\eta$ is defined by Eq. (20), and $\{\beta_j\}_{j=0}^{d-1}$ is defined by
\[ \beta_r = \frac{\alpha_r}{\sqrt{d}} (a + b) \]
\[ \beta_j = \frac{\alpha_j}{\sqrt{d}} \sqrt{1 - \frac{2ab}{d}} \quad (\forall j \neq r). \]
\(28\)

Here, $\| |\Psi^{(r)}_2\rangle_{AB}\|^2 = \sum_j |\beta_j|^2$ is a probability in which outcome $r$ is derived in step 2. Since measuring system $M$ without seeing the outcome is mathematically equivalent to tracing out system $M$, $|\Psi^{(r)}_2\rangle_{AB}$ satisfies
\[ \varepsilon_{1\rightarrow 2}(|\psi\rangle \langle \psi|) = \sum_{r=0}^{d-1} |\Psi^{(r)}_2\rangle_{AB} \langle \Psi^{(r)}_2|, \]
\(29\)

**Protocol 2 1 $\rightarrow$ 2 quantum multicast network coding protocol**

**Step 1:** The source node $s$ prepares an input quantum state $|\psi\rangle_A$ on system $A$ and makes $1 \rightarrow 2$ asymmetric universal clones by applying an isometry $U^{(a,b)}_{1\rightarrow 2}$ defined by Eq. (2) from the system $A$ to the system $ABM$.

**Step 2:** The source node $s$ measures system $M$ in the computational basis, and sends the measurement outcome $r$ to the two target nodes $t_1$ and $t_2$.

**Step 3:** The source node $s$ applies the unitary operator $\Gamma_{AB}$ defined by Eqs. (18) and (19) to the systems $AB$, then discards the system $B$.

**Step 4:** The state on system $A$ is multicast to the target nodes $t_1$ and $t_2$ over the quantum network $G$ using Kobayashi et al.’s protocol. The target nodes $t_1$ and $t_2$ put the output GHZ-type state of Kobayashi et al.’s protocol on system $CD$.

**Step 5:** The target nodes $t_1$ and $t_2$ apply $X^{r_1}_{C} \otimes X^{r_1}_{C}$ to system $EF$ using the measurement outcome $r$ sent from the source node $s$.

**Step 6:** The target node $t_1$ applies $V_{CE}^{(r)}$ defined by Eq. (21) to system $CE$, and the target node $t_2$ applies $V_{DF}^{(r)}$, to system $DF$.

**Step 7:** Then, the target node $t_1$ applies $\Gamma^{CE}_{E,G}$ defined by Eq. (22) to system $CE$, and the target node $t_2$ applies $\Gamma^{DF}_{G,F}$ to system $DF$.

**Step 8:** The target node $t_2$ sends the state on system $H$ to system $T_1$ at the target node $t_1$ using the Bell state on system $T_1T_2$ by the quantum teleportation.

**Step 9:** The target node $t_1$ applies $\Theta_{Z_2}$ defined by Eq. (23) to systems $G$ and $T_1$, and discards $T_1$.

**Step 10:** The target node $t_1$ measures system $G$ in
\[ \{ |0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, |1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \} \]

and derives the measurement outcome $k$. Then, $t_1$ performs $Z_k$ on system $G$.

**Step 11:** The target node $t_1$ applies $\Lambda^{CE}_{E}$ defined by Eq. (25) on the system $CE$, and the target node $t_2$ applies $\Lambda^{DF}_{F}$ the system $DF$.

**Step 12:** The target nodes $t_1$ and $t_2$ measure system $C$ and $D$ in the Fourier basis
\[ \left\{ \sum_{\lambda=0}^{d-1} \lambda \langle \lambda | \right\}_{D_{CE}}, \]

respectively, and derive the measurement outcomes $p_1$ and $p_2$, respectively. Then, $t_1$ applies $Z^{p_1}$ to system $E$, and $t_2$ applies $Z^{p_2}$ to system $F$.

where $\varepsilon_{1\rightarrow 2}$ is a $1 \rightarrow 2$ optimal asymmetric UQCM defined by Eq. (5). Hence, the purpose of the remaining part of the protocol is to transfer $|\Psi_2\rangle$ to the target nodes. However, in our problem settings, the throughput of the quantum network is too small to send $|\Psi_2\rangle$ directly to the target nodes. Hence, first, we compress the state on the $d$-dimensional system in step 3. Then, the unnormalized state of system $AB$ after step 3 can be written as
\[ |\Psi_3\rangle_A = \sum_{j=0}^{d-1} \beta_j |j\rangle_A. \]
\(30\)

In step 4, Kobayashi et al.’s protocol successfully works
Then, the unnormalized state on \( C D F \) is
\[
\sum_{j=0}^{d-1} \beta_j |jj\rangle_{CD} \otimes (\cos \eta |0\rangle_E |1\rangle_F + \sin \eta |1\rangle_E |0\rangle_F).
\]
(31)

Then, the unnormalized state on \( C D F \) after step 5 can be written as
\[
\sum_{j=0}^{d-1} \beta_j |jj\rangle_{CD} \otimes (\cos \eta r - 1) |r\rangle_F + \sin \eta |r\rangle_E |r - 1\rangle_F.
\]
(32)

The unnormalized state on \( C D F \) after step 6 is
\[
\sum_{j\neq r} \beta_j |jj\rangle_{CD} \otimes (\cos \eta |j\rangle_E |r\rangle_F + \sin \eta |r\rangle_E |j\rangle_F) + \beta_r |rr\rangle_{CD} \otimes (\cos \eta |0\rangle_E |1\rangle_F + \sin \eta |1\rangle_E |0\rangle_F).
\]
(33)

Then, the unnormalized state on \( C D F G \) after step 7 can be written as
\[
\sum_{j\neq r} \beta_j |jj\rangle_{CD} \otimes (\cos \eta |j\rangle_E |r\rangle_F + \sin \eta |r\rangle_E |j\rangle_F) \otimes |00\rangle_{GH} + \beta_r |rr\rangle_{CD} \otimes |00\rangle_{EF} \otimes (\cos \eta |0\rangle_G |1\rangle_H + \sin \eta |1\rangle_G |0\rangle_H).
\]
(34)

Next, in step 8, the state on the system \( H \) is transferred to system \( T_1 \) by quantum teleportation. Thus, the unnormalized state on \( C D F G \) after step 9 can be written as
\[
\sum_{j\neq r} \beta_j |jj\rangle_{CD} \otimes (\cos \eta |j\rangle_E |r\rangle_F + \sin \eta |r\rangle_E |j\rangle_F) \otimes |00\rangle_{GH} + \beta_r |rr\rangle_{CD} \otimes |00\rangle_{EF} \otimes |1\rangle_G.
\]
(35)

Since system \( G \) is effectively removed in step 10, the unnormalized state on system \( C D F \) after step 10 can be written as
\[
\sum_{j\neq r} \beta_j |jj\rangle_{CD} \otimes (\cos \eta |j\rangle_E |r\rangle_F + \sin \eta |r\rangle_E |j\rangle_F) + \beta_r |rr\rangle_{CD} \otimes |00\rangle_{EF}.
\]
(36)

Then, the unnormalized state on \( C D F \) after step 11 can be written as
\[
\sum_{j\neq r} \beta_j |jj\rangle_{CD} \otimes (\cos \eta |j\rangle_E |r\rangle_F + \sin \eta |r\rangle_E |j\rangle_F) + \beta_r |rr\rangle_{CD} \otimes |rr\rangle_{EF}.
\]
(37)

In step 12, after system \( CD \) is measured in the Fourier basis \( \{d^{-1/2} \sum_{x=0}^{d-1} \omega^{|x|} |x\rangle\}_{|x| \in \mathbb{Z}_d} \) and is discarded, the unnormalized state on \( EF \) for the measurement outcomes \( p_1 \) and \( p_2 \) can be written as
\[
\sum_{j\neq r} \beta_j \omega^{-jp_1(p_1+1)}(\cos \eta |j\rangle_E |r\rangle_F + \sin \eta |r\rangle_E |j\rangle_F) + \beta_r \omega^{-r(p_1+p_2)} |rr\rangle_{EF}.
\]
(38)

Hence, after applying \( Z^{p_1} \otimes Z^{p_2} \) on system \( EF \), the unnormalized state on \( EF \) becomes
\[
\sum_{j=0}^{d-1} \beta_j |jj\rangle_{CD} \otimes (\cos \eta |j\rangle_E |r\rangle_F + \sin \eta |r\rangle_E |j\rangle_F) + \beta_r |rr\rangle_{EF}.
\]
(39)

This state is the state \( |\psi_2^{(r)}\rangle \) defined by Eq. (39). Since Eq. (39) is the unnormalized state corresponding to the outcome \( r \) in step 2, the final state of this protocol can be written as \( \sum_r |\psi_2^{(r)}\rangle |\psi_2^{(r)}\rangle \). Hence, by Eq. (29), the final states of protocol 2 on the target nodes \( t_1 \) and \( t_2 \) are \( 1 \rightarrow 2 \) optimal asymmetric UQCs of the input state \( |\psi\rangle \).

IV. 1 → 3 OPTIMAL ASYMMETRIC QUANTUM UNIVERSAL CLONES MULTICAST PROTOCOL

In this section, we present a protocol that multicasts optimal asymmetric UQCs from the source node \( s \) to three target nodes \( t_1, t_2, \) and \( t_3 \) on a quantum network. We present the protocol in Section IV A and in Section IV B we prove that creates optimal asymmetric UQCs.

A. 1 → 3 quantum multicast protocol

In this section, we present a protocol that multicasts \( 1 \rightarrow 3 \) optimal asymmetric UQCs of an input quantum state from the source node \( s \) to two target nodes \( t_1, t_2, \) and \( t_3 \).

The problem setting for the \( 1 \rightarrow 3 \) quantum multicast protocol is almost the same as that of the \( 1 \rightarrow 2 \) protocol given in the last section. Hence, we consider only the difference between these two problem settings. First, the number of target nodes is different. That is, in this section, a quantum network \( G \) has three target nodes \( t_1, t_2, \) and \( t_3 \). The purpose of the protocol is to construct \( 1 \rightarrow 3 \) optimal asymmetric universal clones given by Eq. (9) among target nodes \( t_1, t_2, \) and \( t_3 \) for a given \( d \)-dimensional input state \( |\psi\rangle = \sum_{j=0}^{d-1} \alpha_j |j\rangle \in \mathcal{H}_s \) on a source node, where \( \mathcal{H}_s \) is a \( d \)-dimensional input space. We again assume \( d = d' \). In other words, we consider a mulcast quantum network code with source rate \( r \). The assumption for the existence of a classical linear multicast network code is also similar. That is, a classical linear multicast network code is a code on \( \mathbb{F}_q \) used to multicast from the node \( s \) to the nodes \( t_1, t_2, t_3 \) on \( G' \) with source rate \( r \). The amount of entanglement shared among the target nodes is also different. In \( 1 \rightarrow 3 \) case, we assume that at most \( 2 + 4 \log_2 3 \) ebits are shared among the target nodes \( t_1, t_2, \) and \( t_3 \). Hence, the amount of this entanglement resource is constant with respect to the dimension \( d \) of the input state.
Before we present the protocol, we define the unitary operators used in the protocol. \( U^{(r,s)}_2 \) is a tripartite unitary operator satisfying the following conditions:

\[
U^{(r,s)}_2 = \frac{\alpha(jrs) + \beta(rjs) + \gamma(rsj) + \alpha(jsr) + \beta(sjr) + \gamma(srj)}{\sqrt{2\alpha^2 + 2\beta^2 + 2\gamma^2}} = |j00\rangle, \quad (\forall j \neq r, s)
\]

\[
U^{(r,s)}_2 = \frac{(\alpha + \beta)(rrs) + (\beta + \gamma)(srj) + (\gamma + \alpha)(rjr)}{\sqrt{(\alpha + \beta)^2 + (\beta + \gamma)^2 + (\gamma + \alpha)^2}} = |r00\rangle,
\]

\[
U^{(r,s)}_2 = \frac{(\alpha + \beta)(ssr) + (\beta + \gamma)(rrs) + (\gamma + \alpha)(srj)}{\sqrt{(\alpha + \beta)^2 + (\beta + \gamma)^2 + (\gamma + \alpha)^2}} = |s00\rangle.
\]

\[U^{(r,s)}_5 \] is a bipartite unitary operator defined by

\[
U^{(r,s)}_5 := |r\rangle\langle r| \otimes I + s\langle r| \otimes I + \sum_{j \neq r, s} |j\rangle\langle j| \otimes |s\rangle\langle s|, \quad (41)
\]

where \( \pi_{jrs} \) is a permutation satisfying the following conditions:

\[
\pi_{jrs}(0) = j, \quad \pi_{jrs}(1) = r, \quad \pi_{jrs}(2) = s \quad (42)
\]

\( U^{(r,s)}_6 \) is a tripartite unitary operator defined by

\[
U^{(r,s)}_6 := |r\rangle\langle r| \otimes \text{swap} + |s\rangle\langle s| \otimes \text{swap} + \sum_{j \neq r, s} |j\rangle\langle j| \otimes I \otimes I, \quad (43)
\]

where \( \text{swap} \) is a swap operator defined by \( \text{swap} := \sum_{j \neq j'} |i\rangle\langle j| \otimes |j\rangle\langle i| \). \( U^{(r,s)}_7 \) is a bipartite unitary operator defined by

\[
U^{(r,s)}_7 := \sum_{i \neq j} |i\rangle\langle i| \otimes |j\rangle\langle j|, \quad (44)
\]

where \( \pi'_{rs}(0) = x \) and \( \pi'_{rs}(1) = y \). \( U_8 \) is a unitary operator on \( \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \) satisfying

\[
U_8(\alpha'_{011} + \beta'_{010} + \gamma'_{010}) = |000\rangle,
\]

\[
U_8(\alpha'_{012} + |021\rangle + \beta'_{12}(|102\rangle + |201\rangle) + \gamma'_{12}(|120\rangle + |210\rangle)) = |100\rangle,
\]

where \( \alpha' \), \( \beta' \), \( \gamma' \), \( \alpha'' \), \( \beta'' \), and \( \gamma'' \) are defined by

\[
\alpha' = \frac{\alpha}{\sqrt{2\alpha^2 + 2\beta^2 + 2\gamma^2}}, \quad \beta' = \frac{\beta}{\sqrt{2\alpha^2 + 2\beta^2 + 2\gamma^2}},
\]

\[
\gamma' = \frac{\gamma}{\sqrt{2\alpha^2 + 2\beta^2 + 2\gamma^2}}.
\]

\[
\alpha'' = \frac{\alpha + \beta}{\sqrt{(\alpha + \beta)^2 + (\beta + \gamma)^2 + (\gamma + \alpha)^2}},
\]

\[
\beta'' = \frac{\beta + \gamma}{\sqrt{(\alpha + \beta)^2 + (\beta + \gamma)^2 + (\gamma + \alpha)^2}},
\]

\[
\gamma'' = \frac{\gamma + \alpha}{\sqrt{(\alpha + \beta)^2 + (\beta + \gamma)^2 + (\gamma + \alpha)^2}}.
\]

\( U^{(r,s,k)}_9 \) is a unitary operator on \( \mathbb{C}^d \) defined by

\[
U^{(r,s,k)}_9 := \sum_{j \neq r, s} |j\rangle\langle j| + (-1)^{k} |r\rangle\langle r| + (-1)^{k} |s\rangle\langle s|. \quad (47)
\]

\( U^{(r,s)}_2 \) is a tripartite unitary operator satisfying

\[
U^{(r,s)}_2 = \frac{2\alpha |rrj\rangle + 2\beta |rjr\rangle + 2\gamma |rrj\rangle}{\sqrt{(2\alpha)^2 + (2\beta)^2 + (2\gamma)^2}} = |j00\rangle, \quad (\forall j \neq r)
\]

\[
U^{(r,s)}_2 = \frac{2\alpha |rjr\rangle + 2\beta |rjr\rangle + 2\gamma |rrj\rangle}{\sqrt{(2\alpha)^2 + (2\beta)^2 + (2\gamma)^2}} = |r00\rangle, \quad (\forall j \neq r)
\]

\( U^{(r,s)}_5 \) is a bipartite unitary operator defined by

\[
U^{(r,s)}_5 := |r\rangle\langle r| \otimes I + \sum_{j \neq r} |j\rangle\langle j| \otimes \left( \sum_{x=0}^{d-1} |\pi_{jr}(x)\rangle \langle x| \right), \quad (49)
\]

where \( \pi_{jr} \) is a permutation satisfying

\[
\pi_{jr}(1) = r, \quad \pi_{jr}(0) = j \quad (50)
\]

\( U^{(r,s)}_6 \) is a tripartite unitary operator defined by

\[
U^{(r,s)}_6 := |r\rangle\langle r| \otimes \text{swap} + \sum_{j \neq r} |j\rangle\langle j| \otimes I \otimes I, \quad (51)
\]

where swap is an operator defined by \( \text{swap} := \sum_{i \neq j} |i\rangle\langle i| \otimes |j\rangle\langle j| \). \( U^{(r,s)}_7 \) is a tripartite unitary operator defined by

\[
U^{(r,s)}_7 := \sum_{i \neq j} |i\rangle\langle i| \otimes |j\rangle\langle j| + |r\rangle\langle r| \otimes \text{swap}, \quad (52)
\]

where \( X \) is the Pauli \( X \) operator defined by \( X := \sum_{i=0}^{d-1} |x\rangle\langle x+1| \). \( U_8 \) is a unitary operator on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) defined by

\[
U_8(\alpha'_{011} + \beta'_{010} + \gamma'_{010}) = |000\rangle
\]

\[
U_8(\alpha'_{012} + |021\rangle + \beta'_{12}(|102\rangle + |201\rangle) + \gamma'_{12}(|120\rangle + |210\rangle)) = |100\rangle,
\]

Finally, \( U^{(r,s,k)}_9 \) is a unitary operator on \( \mathbb{C}^d \) defined by

\[
U^{(r,s,k)}_9 := \sum_{j \neq r} |j\rangle\langle j| + (-1)^k |r\rangle\langle r|. \quad (54)
\]

We will also use in the protocol the projective measurement \( |P_k\rangle\langle P_k| \) defined by the following equations:

\[
P_0 := |0\rangle\langle 0|, \quad P_1 := |1\rangle\langle 1|, \quad P_2 := I - |0\rangle\langle 0| - |1\rangle\langle 1|, \quad (55)
\]

where \( |\tilde{0}\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}} \).

At the beginning of the protocol, the source node \( s \) has five \( d \)-dimensional systems \( A, B, C, R \) and \( S \). The target node \( t_1 \) has three \( d \)-dimensional systems \( D, M_1 \), and \( N_1 \). The target node \( t_2 \) has three \( d \)-dimensional systems \( E, M_2 \), and \( N_2 \). The target node \( t_3 \) has three \( d \)-dimensional systems \( F, M_3 \), and \( N_3 \). Further, the target nodes \( t_1 \) and \( t_2 \) share \( 1 + 2 \log_2 3 \) bits of entanglement, and the target nodes \( t_1 \) and \( t_2 \) share \( 1 + 2 \log_2 3 \) bits of entanglement. Hence, the amount of entanglement resources are \( 2 + 4 \log_2 3 \) bits in total.
### Protocol 3 1 → 3 quantum multicast network coding protocol (beginning)

**Step 1:** The source node \( s \) prepares an input quantum state \( |\psi\rangle_A \) on system \( A \), and makes \( 1 \rightarrow 3 \) asymmetric universal clones by applying an isometry \( U^{(\alpha,\beta,\gamma)}_{1 \rightarrow 3} \) defined by Eq. (4) from system \( A \) to system \( ABCRS \).

**Step 2:** The source node \( s \) measures the systems \( R \) and \( S \) in the computational basis, where the measurement outcomes of \( R \) and \( S \) are called \( r \) and \( s \), respectively. The source node \( s \) sends the measurement outcomes \( r \) and \( s \) to the target nodes \( t_1, t_2, \) and \( t_3 \). The following steps of the protocol depend on whether \( r \neq s \) or \( r = s \).

The beginning of the protocol for \( 1 \rightarrow 3 \) is given as protocol 3. In step 2 of protocol 3, the systems \( R \) and \( S \) are measured and the measurement outcomes \( r \) and \( s \) are derived. The continuation of the protocol branches depending on whether \( r \neq s \) or \( r = s \). The continuation for \( r \neq s \) is given as protocol 4, and for \( r = s \) is given as protocol 5. Using the protocol, \( 1 \rightarrow 3 \) asymmetric UQCs given by Eq. (9) are created system \( M_1 M_2 M_3 \), where \( M_1, M_2, \) and \( M_3 \) are on the target nodes \( t_1, t_2, \) and \( t_3 \), respectively. Note that as we explained in the previous section, asymmetric UQCs depends on the parameters \( \alpha, \beta, \) and \( \gamma \) in Eq. (7). We can set these parameters in step 1 of the protocol, when we apply \( U^{(\alpha,\beta,\gamma)}_{1 \rightarrow 3} \).

### B. Proof of 1 → 3 quantum multicast protocol

In this section, we prove that protocols 3, 4, and 5 create \( 1 \rightarrow 3 \) asymmetric UQCs given by Eq. (9) in system \( M_1 M_2 M_3 \).

Let the input state at the source node be \( |\psi\rangle = \sum_{f=0}^{d-1} c_f |f\rangle \). Then, from Eq. (7), the state on system \( ABCRA \) can be written as

\[
\sqrt{\frac{d}{2d+2}} [\alpha |\psi\rangle_A(|\Phi^+\rangle_{BR}|\Phi^+\rangle_{CS} + |\Phi^+\rangle_{BS}|\Phi^+\rangle_{CR}) \\
+ \beta |\psi\rangle_B(|\Phi^+\rangle_{AR}|\Phi^+\rangle_{CS} + |\Phi^+\rangle_{AS}|\Phi^+\rangle_{CR}) \\
+ \gamma |\psi\rangle_C(|\Phi^+\rangle_{AR}|\Phi^+\rangle_{BS} + |\Phi^+\rangle_{AS}|\Phi^+\rangle_{BR})]
\]

(58)

After step 2, the protocol branches depending on whether \( r \neq s \) or \( r = s \), where \( r \) and \( s \) are the measurement outcomes of system \( R \) and \( S \), respectively.

The unnormalized state \( |\psi^{(r,s)}_3\rangle \) after step 2 for \( r \neq s \) can be
Protocol 5 Continuation of protocol 3 for $1 \rightarrow 3$ quantum multicast network coding (for $r = s$)

$r = s$

Step 3: The source node $s$ applies unitary operator $U_{ss}^{(3)}$ defined by Eq. (55) to system $ABC$, and then, discards the systems $B$ and $C$.

Step 4: The state on system $A$ is multicast to the target nodes $t_1$, $t_2$, and $t_3$ over the quantum network $G$ using Kobayashi et al.’s protocol. The target nodes $t_1$, $t_2$, and $t_3$ put the output of Kobayashi et al.’s protocol on system DEF. Then, using 2 ebits of entanglement, the target nodes share the state on system $M_1M_2M_3$:

$$\sqrt{\alpha_1^2 + \beta_1^2} |111 \rangle_{M_1M_2M_3}$$

where $\alpha_1 = \sqrt{2\alpha \beta / \sqrt{(2\alpha^2 + \beta^2)^2}}$ and $\beta_1 = \sqrt{\beta^2 / \sqrt{(2\alpha^2 + \beta^2)^2}}$.

Further, they initialize all the systems $N_1$, $N_2$, and $N_3$ in $|0\rangle$.

Step 5: The target nodes apply $U_{ss}^{(3)} \otimes U_{ss}^{(3)} \otimes U_{ss}^{(3)}$ to system $DM_1EM_2FM_3$, where $U_{ss}^{(3)}$ is defined by Eq. (40).

Step 6: The target nodes apply $U_{ss}^{(3)} \otimes U_{ss}^{(3)} \otimes U_{ss}^{(3)}$ to system $DM_1N_2EM_2N_3FM_3$, where $U_{ss}^{(3)}$ is defined by Eq. (51).

Step 7: The target nodes apply $U_{ss}^{(3)} \otimes U_{ss}^{(3)} \otimes U_{ss}^{(3)}$ to system $DM_1EM_2FM_3$, where $U_{ss}^{(3)}$ is defined by Eq. (52).

Step 8: By using 2 ebits of entanglement resource, subspaces spanned by $|0\rangle, |1\rangle$ of the systems $N_2$ and $N_3$ are sent from the target nodes $t_2$ and $t_3$ to the target node $t_1$, respectively. The target node $t_1$ applies $U_{ss}^{(3)} \otimes U_{ss}^{(3)} \otimes U_{ss}^{(3)}$ to system $N_1N_2N_3$ and discards system $N_1$.

Step 9: The target node $t_1$ applies the projective measurement $\{P_i\}_{i=0}^{d-1}$ defined by Eq. (53) on system $N_1$ in the basis and discards the quantum system $N_1$. Then, depending on the measurement outcome $k$, the target node $t_1$ applies $U_{ss}^{(3)}$ defined by Eq. (53) on the system $D$.

Step 10: The target nodes $t_1$, $t_2$, and $t_3$ measure system $D$, $E$, and $F$ in the Fourier basis $d^{-1/2} \sum_{j=0}^{d-1} e^{i\omega j \lambda_j} |j\rangle$ respectively. Then, they apply $Z^{p_1(p_2+p_3)} \otimes Z^{p_2(p_1+p_3)} \otimes Z^{p_3(p_1+p_2)}$ to system $M_1M_2M_3$, where $p_1$, $p_2$, and $p_3$ are the measurement outcomes on the target nodes $t_1$, $t_2$, and $t_3$, respectively.

The unnormalized state $|\Psi^{(rs)}_2\rangle$ after step 2 for $r = s$ can be written as

$$|\Psi^{(rs)}_2\rangle = \sqrt{\frac{2}{d(d+1)}} [\alpha |\psi\rangle \otimes |r\rangle + \beta |\phi\rangle \otimes |r\rangle + \gamma |\delta\rangle \otimes |r\rangle]$$

As for the $1 \rightarrow 2$ quantum multicast network coding protocol, $|\psi^{(rs)}_{2}\rangle_{r,s=0}^{d-1}$ satisfies

$$\epsilon^{r \delta \gamma}_{j=0} = \sum_{j=0}^{d-1} |\psi^{(rs)}_{j}\rangle \langle \psi^{(rs)}_{j}|$$

where $\epsilon^{r \delta \gamma}$ is a 1 $\rightarrow$ 3 optimal asymmetric UQCM defined by Eq. (43). Hence, the purpose of the remaining part of the protocol is to transfer $|\Psi^{(rs)}_2\rangle$ to the target nodes.

First we give the continuation of the proof for $r \neq s$ (protocol 4). We compress the state on a $d$-dimensional system on step 3. The unnormalized state on system $A$ after step 3 can be written as

$$|\Psi^{(rs)}_2\rangle = \sum_{j=0}^{d-1} |\psi^{(rs)}_{j}\rangle$$

where $|\psi^{(rs)}_{j}\rangle_{j=0}^{d-1}$ is defined as

$$|\psi^{(rs)}_{j}\rangle = \sqrt{\frac{d}{2d+2}} |\psi\rangle \otimes |j\rangle$$

where $|\psi\rangle$ is a $1 \rightarrow 3$ optimal asymmetric UQCM defined by Eq. (43). Hence, the purpose of the remaining part of the protocol is to transfer $|\Psi^{(rs)}_2\rangle$ to the target nodes.

In step 4, Kobayashi et al.’s protocol successfully works based on the assumption for the existence of a classical linear multicast network code. The unnormalized state on system $D$ at the target node $t_1$, the system $E$ at the target node $t_2$, and on system $F$ at the target node $t_3$ can be written as

$$\sum_{j=0}^{d-1} |\psi^{(rs)}_{j}\rangle$$

Hence, the unnormalized state after step 4 can be written as

$$\sum_{j=0}^{d-1} |\psi^{(rs)}_{j}\rangle$$

\begin{align*}
&= \sum_{j=0}^{d-1} |\psi^{(rs)}_{j}\rangle_{DEF} \otimes (\alpha_{1}^{012} + \beta_{1}^{012} + \gamma_{1}^{120}) \\
&\quad + \alpha_{1}^{021} + \beta_{1}^{021} + \gamma_{1}^{120})_{M_1M_2M_3} \\
&\quad \otimes (\alpha_{1}^{001} + \beta_{1}^{010} + \gamma_{1}^{010})_{N_1N_2N_3}
\end{align*}

(64)
The purpose of the remaining part of the protocol is to reconstruct $|\Psi_{2}^{(r,s)}\rangle$ from the above state. The unnormalized state after step 5 can be written as

$$\left\{ \sum_{j_{r},s_{r}}^{n-1} \kappa_{j} (j_{r}j_{s_{r}})_{DEF} \otimes (\alpha_{r}^{1}|j_{r}s_{r}\rangle + \beta_{r}^{1}|r_{s_{r}}\rangle + \gamma_{r}^{1}|r_{j_{r}}s_{r}\rangle + \alpha_{r}^{1}|s_{r}j_{r}\rangle + \beta_{r}^{1}|s_{r}j_{r}\rangle + \gamma_{r}^{1}|s_{r}j_{r}\rangle)_{M_{r}M_{s_{r}}M_{r}} \right\} \otimes (\alpha_{r}^{1}(001) + \beta_{r}^{1}(100) + \gamma_{r}^{1}(010))_{N_{m}N_{m_{s_{r}}}}$$

(65)

The unnormalized state after step 6 can be written as

$$\sum_{j_{r},s_{r}}^{n-1} \kappa_{j} (j_{r}j_{s_{r}})_{DEF} \otimes (\alpha_{r}^{1}|j_{r}s_{r}\rangle + \beta_{r}^{1}|r_{s_{r}}\rangle + \gamma_{r}^{1}|r_{j_{r}}s_{r}\rangle + \alpha_{r}^{1}|s_{r}j_{r}\rangle + \beta_{r}^{1}|s_{r}j_{r}\rangle + \gamma_{r}^{1}|s_{r}j_{r}\rangle)_{M_{r}M_{s_{r}}M_{r}} \otimes (\alpha_{r}^{1}(001) + \beta_{r}^{1}(100) + \gamma_{r}^{1}(010))_{N_{m}N_{m_{s_{r}}}}$$

$$+ \left( \kappa_{s} \{|rr\rangle_{DEF} \otimes (\alpha_{r}^{1}(012) + \beta_{r}^{1}(102) + \gamma_{r}^{1}(120) + \alpha_{r}^{1}(021) + \beta_{r}^{1}(201) + \gamma_{r}^{1}(210))_{M_{r}M_{s_{r}}M_{r}} \right)$$

$$\otimes (\alpha_{r}^{1}(001) + \beta_{r}^{1}(100) + \gamma_{r}^{1}(010))_{N_{m}N_{m_{s_{r}}}} \right\}$$

(66)

Then, the unnormalized state after step 7 can be written as

$$\sum_{j_{r},s_{r}}^{n-1} \kappa_{j} (j_{r}j_{s_{r}})_{DEF} \otimes (\alpha_{r}^{1}|j_{r}s_{r}\rangle + \beta_{r}^{1}|r_{s_{r}}\rangle + \gamma_{r}^{1}|r_{j_{r}}s_{r}\rangle + \alpha_{r}^{1}|s_{r}j_{r}\rangle + \beta_{r}^{1}|s_{r}j_{r}\rangle + \gamma_{r}^{1}|s_{r}j_{r}\rangle)_{M_{r}M_{s_{r}}M_{r}} \otimes (\alpha_{r}^{1}(001) + \beta_{r}^{1}(100) + \gamma_{r}^{1}(010))_{N_{m}N_{m_{s_{r}}}}$$

$$+ \left( \kappa_{s} \{|rr\rangle_{DEF} \otimes (\alpha_{r}^{1}(012) + \beta_{r}^{1}(102) + \gamma_{r}^{1}(120) + \alpha_{r}^{1}(021) + \beta_{r}^{1}(201) + \gamma_{r}^{1}(210))_{M_{r}M_{s_{r}}M_{r}} \right)$$

$$\otimes (\alpha_{r}^{1}(001) + \beta_{r}^{1}(100) + \gamma_{r}^{1}(010))_{N_{m}N_{m_{s_{r}}}} \right\}$$

(67)

The unnormalized state after step 8 can be written as

$$\sum_{j_{r},s_{r}}^{n-1} \kappa_{j} (j_{r}j_{s_{r}})_{DEF} \otimes (\alpha_{r}^{1}|j_{r}s_{r}\rangle + \beta_{r}^{1}|r_{s_{r}}\rangle + \gamma_{r}^{1}|r_{j_{r}}s_{r}\rangle + \alpha_{r}^{1}|s_{r}j_{r}\rangle + \beta_{r}^{1}|s_{r}j_{r}\rangle + \gamma_{r}^{1}|s_{r}j_{r}\rangle)_{M_{r}M_{s_{r}}M_{r}} \otimes |0\rangle_{N_{s_{r}}}$$

$$+ \left( \kappa_{s} \{|rr\rangle_{DEF} \otimes (\alpha_{r}^{1}(012) + \beta_{r}^{1}(102) + \gamma_{r}^{1}(120) + \alpha_{r}^{1}(021) + \beta_{r}^{1}(201) + \gamma_{r}^{1}(210))_{M_{r}M_{s_{r}}M_{r}} \right)$$

$$\otimes (\alpha_{r}^{1}(001) + \beta_{r}^{1}(100) + \gamma_{r}^{1}(010))_{N_{m}N_{m_{s_{r}}}} \right\}$$

(68)

The unnormalized state after step 9 can be written as

$$\sum_{j_{r},s_{r}}^{n-1} \kappa_{j} (j_{r}j_{s_{r}})_{DEF} \otimes (\alpha_{r}^{1}|j_{r}s_{r}\rangle + \beta_{r}^{1}|r_{s_{r}}\rangle + \gamma_{r}^{1}|r_{j_{r}}s_{r}\rangle + \alpha_{r}^{1}|s_{r}j_{r}\rangle + \beta_{r}^{1}|s_{r}j_{r}\rangle + \gamma_{r}^{1}|s_{r}j_{r}\rangle)_{M_{r}M_{s_{r}}M_{r}}$$

$$+ \left( \kappa_{s} \{|rr\rangle_{DEF} \otimes (\alpha_{r}^{1}(012) + \beta_{r}^{1}(102) + \gamma_{r}^{1}(120) + \alpha_{r}^{1}(021) + \beta_{r}^{1}(201) + \gamma_{r}^{1}(210))_{M_{r}M_{s_{r}}M_{r}} \right)$$

$$\otimes (\alpha_{r}^{1}(001) + \beta_{r}^{1}(100) + \gamma_{r}^{1}(010))_{N_{m}N_{m_{s_{r}}}} \right\}$$

(69)

The unnormalized state after step 10 can be written as

$$\omega_{r+s}(p_{1},p_{2},p_{3}) \left\{ \sum_{j_{r},s_{r}}^{n-1} \kappa_{j} (j_{r}j_{s_{r}})_{DEF} \otimes (\alpha_{r}^{1}|j_{r}s_{r}\rangle + \beta_{r}^{1}|r_{s_{r}}\rangle + \gamma_{r}^{1}|r_{j_{r}}s_{r}\rangle + \alpha_{r}^{1}|s_{r}j_{r}\rangle + \beta_{r}^{1}|s_{r}j_{r}\rangle + \gamma_{r}^{1}|s_{r}j_{r}\rangle)_{M_{r}M_{s_{r}}M_{r}}$$

$$+ \left( \kappa_{s} \{|rr\rangle_{DEF} \otimes (\alpha_{r}^{1}(012) + \beta_{r}^{1}(102) + \gamma_{r}^{1}(120) + \alpha_{r}^{1}(021) + \beta_{r}^{1}(201) + \gamma_{r}^{1}(210))_{M_{r}M_{s_{r}}M_{r}} \right)$$

$$\otimes (\alpha_{r}^{1}(001) + \beta_{r}^{1}(100) + \gamma_{r}^{1}(010))_{N_{m}N_{m_{s_{r}}}} \right\} \right\}$$

(70)

We can easily see that the above state is equivalent to $|\Psi_{2}^{(r,s)}\rangle$ as defined by Eq. (65) except for a global phase. Hence, the proof is complete for $r \neq s$.

Next, we give the continuation of the proof for $r = s$ (protocol 5). The unnormalized state on system A after step 3 can be written as

$$\sum_{j=0}^{n-1} \kappa_{j} (j),$$

(71)
where \(|\kappa\rangle_{j=0}^{d-1}\) is defined as
\[
k_j' = \frac{2}{d(d+1)} \delta_j \sqrt{\alpha^2 + \beta^2 + \gamma^2} \quad (j \neq r),
\]
\[
k_r' = \frac{2}{d(d+1)} \delta_r (\alpha + \beta + \gamma). \tag{72}
\]
In step 4, Kobayashi et al.’s protocol successfully works, and the unnormalized state after step 9 can be written as
\[
\sum_{j=0}^{d-1} \kappa_j' |jj\rangle_{DEF} \otimes (\alpha'_2 |011\rangle + \beta'_2 |101\rangle + \gamma'_2 |110\rangle)_{M_iM_jM_k} \otimes |000\rangle_{N_iN_jN_k} \tag{73}
\]
Then, the unnormalized state after step 5 can be written as
\[
\sum_{j \neq r} \kappa_j' |jj\rangle_{DEF} \otimes (\alpha'_2 |rr\rangle + \beta'_2 |rr\rangle + \gamma'_2 |rr\rangle)_{M_iM_jM_k} \otimes |000\rangle_{N_iN_jN_k} + \kappa_r' |rr\rangle_{DEF} \otimes (\alpha'_2 |011\rangle + \beta'_2 |101\rangle + \gamma'_2 |110\rangle)_{M_iM_jM_k} \otimes |000\rangle_{N_iN_jN_k} \tag{74}
\]
The unnormalized state after step 6 can be written as
\[
\sum_{j \neq r} \kappa_j' |jj\rangle_{DEF} \otimes (\alpha'_2 |rr\rangle + \beta'_2 |rr\rangle + \gamma'_2 |rr\rangle)_{M_iM_jM_k} \otimes |000\rangle_{N_iN_jN_k} + \kappa_r' |rr\rangle_{DEF} \otimes |000\rangle_{M_iM_jM_k} (\alpha'_2 |011\rangle + \beta'_2 |101\rangle + \gamma'_2 |110\rangle)_{N_iN_jN_k} \tag{75}
\]
The unnormalized state after step 7 can be written as
\[
\sum_{j \neq r} \kappa_j' |jj\rangle_{DEF} \otimes (\alpha'_2 |rr\rangle + \beta'_2 |rr\rangle + \gamma'_2 |rr\rangle)_{M_iM_jM_k} \otimes |000\rangle_{N_iN_jN_k} + \kappa_r' |rr\rangle_{DEF} \otimes |rr\rangle_{M_iM_jM_k} (\alpha'_2 |011\rangle + \beta'_2 |101\rangle + \gamma'_2 |110\rangle)_{N_iN_jN_k} \tag{76}
\]
Then, the unnormalized state after step 8 can be written as
\[
\sum_{j \neq r} \kappa_j' |jj\rangle_{DEF} \otimes (\alpha'_2 |rr\rangle + \beta'_2 |rr\rangle + \gamma'_2 |rr\rangle)_{M_iM_jM_k} \otimes |0\rangle_{N_i} + \kappa_r' |rr\rangle_{DEF} \otimes |rr\rangle_{M_iM_jM_k} |1\rangle_{N_i} \tag{77}
\]
Finally, the unnormalized state after step 10 can be written as
\[
\omega^{p_i+p_i+r_i} \sum_{j=0}^{d-1} \kappa_j' |jj\rangle_{DEF} \otimes (\alpha'_2 |rr\rangle + \beta'_2 |rr\rangle + \gamma'_2 |rr\rangle)_{M_iM_jM_k} + \kappa_r' |rr\rangle_{DEF} \otimes |rr\rangle_{M_iM_jM_k} \tag{78}
\]
We can easily see that the above state is equivalent to \(|\Psi_2^{(r)}\rangle\) as defined by Eq. (60) except for global phase. Hence, the proof is complete for \(r = s\). Thus, we have achieved multicasting of asymmetric optimal clones for the systems \(M_1, M_2\) and \(M_3\) to the three target nodes.

V. CONCLUSION

In this paper, we considered quantum multicast network coding as the multicasting of optimal UQCs over a quantum network. By extending Owari et al.’s results [11,13] for multiset of symmetric optimal UQCs, we developed a protocol to multicast asymmetric optimal UQCs over a quantum network. Our results can be summarized as follows. Suppose a quantum network is described by an undirected graph G with one source node and two (three) target nodes, and each quantum channel on the quantum network G can transmit one \(q\)-dimensional quantum system in a single session. Further, suppose there exists a classical solvable multicast network code with source rate \(r\) for a classical network described by an acyclic directed graph \(G'\), where \(G\) is an undirected underlying graph of \(G'\). We showed that under the above assumptions, our protocol can multicast \(1 \to 2 \) (\(1 \to 3\)) asymmetric optimal UQCs of a \(q\)-dimensional state from the source node to the target nodes by consuming a small amount of entanglement that does not scale with \(q\), which is shared among the target nodes.

The extension of our protocol for \(1 \to n\) asymmetric optimal UQCs for \(n \leq 4\) is not so straightforward. Hence, we leave this study as our future work.

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