Quantum Secure Communication via W States

Jaewoo Joo, Jinhyoung Lee, Jingak Jang, and Young-Jai Park

1Department of Physics, Sogang University, CPO Box 1142, Seoul 100-611, Korea
2Institute of Quantum Information Processing and Systems University of Seoul, Seoul, Korea
3School of Mathematics and Physics, The Queen’s University, Belfast BT7 1NN, United Kingdom
4National Security Research Institute, Taegon, Korea
5Department of Physics and Basic Research Institute, Sogang University, Seoul 121-742, Korea

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W states of multipartite systems are pair-wisely entangled, belonging to a different class from Greenberger, Horne, and Zeilinger states. Based on W states, we propose three variant protocols for quantum secure communication, i.e., quantum key distribution, partial quantum secret sharing, and their synthesis. By the synthesis we mean that both quantum key distribution and partial quantum secret sharing are executed in a single protocol. For these protocols it is discussed how authorized communicators detect individual attacks by an eavesdropper.

I. INTRODUCTION

Quantum entanglement is at the heart of quantum information processes. In particular bipartite entanglement has been extensively studied for quantum key distribution, teleportation, conclusive teleportation, entanglement swapping, and quantum computation. On the other hand multipartite entanglement has been relatively less studied in terms of information-theoretic aspects.

Recently the classification on the states of three qubits has been proposed. A typical class includes Greenberger-Horne-Zeilinger (GHZ) states which exhibit nonlocality among distant local observables and have nonvanishing tangle. W states are pair-wisely entangled with no tangle and they belong to the different class from GHZ states in the sense that they cannot be transformed to any GHZ states under local operation and classical communication (LOCC). These facts suggest that W states have the physical properties considerably different from GHZ states. W states have been rarely considered for quantum information protocols, whereas GHZ states have been employed for quantum cryptography among several distant parts.

The nonlocality of W states were studied by explicitly considering non-commuting observables. The generation of W states has been investigated with the physical models based on linear optical elements, cavity QED, and Heisenberg XY model. In this paper we propose three variant protocols by using W states of three qubits which enable the quantum secure communication among three distant persons. The protocols include quantum key distribution (QKD) in the pair-wise way and partial quantum secret sharing (PQSS) among the three persons. In addition we propose a synthesis protocol by which both pair-wise QKD and PQSS are executed simultaneously when it is necessary. It is discussed how authorized communicators detect individual attacks by an eavesdropper.

II. CHARACTERISTICS OF W STATES

We consider briefly the known characteristics on W states of three qubits and compare them with those of GHZ states. W states can not be transformed to the GHZ state under LOCC, implying they belong to a different class from the GHZ states.

A pure triseparable state of three qubits is defined by Schmidt-like decomposition as

\[ |\phi_3\rangle = \sum_{i=1}^{2} \lambda_i |\alpha_i, \beta_i, \gamma_i\rangle \]

where \(|\alpha_i\rangle\), \(|\beta_i\rangle\) and \(|\gamma_i\rangle\) are orthonormal basis sets on the Hilbert spaces for the three qubits, respectively, and \(\lambda_i\) are positive. The triseparability is closely related to the tangle \(T = 2\lambda_1\lambda_2\). When \(\lambda_1 = 1/\sqrt{2}\), the triseparable state becomes a maximal GHZ state with \(T = 1\). When the partial trace is performed over one qubit, the reduced density operator of the other qubits can be represented by a convex sum of product states. It implies that the reduced density operator is separable. On the other hand, it can be shown by generalized Schmidt decomposition scheme that W states are not triseparable since they can not be represented in the form of Eq. (1). Thus W states have no tangle, \(T = 0\). But the reduced density operator for each pair among the three qubits is inseparable because its partial transposition has at least one negative eigenvalue. Thus, the W state is pair-wisely entangled with no tangle.

The nonlocality of W states can be revealed by three local observers. Let \(z^q\) and \(x^q\) be the outcomes (+ or −) in measuring \(\hat{\sigma}_z\) and \(\hat{\sigma}_x\) on qubit \(q\) (\(q = A, B, C\)). For the W state the local observables \(z^q\) and \(x^q\) satisfy

\[ x^a = x^b = x^c = 0, z^a = z^b = z^c = 0. \]
Einstein-Podolsky-Rosen (EPR) criterion of elements of reality and should be predetermined before any measurement \[21\]. However, such a predetermination is impossible according to quantum mechanics. Recently, Cabello \[10\] has shown that quantum mechanics refutes EPR’s elements of reality by the proof similar to the GHZ’s. It is notable that the refutation results from the inferences by only two of three qubits while the inferences by all three qubits are necessary for GHZ’s.

Bell’s theorem for W states of three qubits can be investigated also by considering Clauser-Horne-Bell (CH-Bell) inequality \[14\].

\[-1 \leq A_{11} - A_{12} - A_{21} - A_{22} \leq 0, \tag{2}\]

where \(A_{11} = P(z^i = z_+, z^j = z_+)\) is the probability that two qubits \(i\) and \(j\) among the three raise the outcomes of \(z_+\) as they are measured by \(\hat{\sigma}_z\), and \(A_{12} = P(z^i = z_+, x^j \neq x^k)\) is the probability that one qubit \(i\) measured by \(\hat{\sigma}_z\) raises the outcome of \(z_+\) and the others \(j\) and \(k\) measured in \(\hat{\sigma}_z\) raise the outcomes opposite to each other. Similarly, \(A_{21} = P(x^i \neq x^k, z^j = z_+),\) and \(A_{22} = P(x^i = x^j = x^k)\).

In particular consider a symmetric W state \[11\],

\[|W\rangle = \frac{1}{\sqrt{3}} (|z_- z_+ z_+ \rangle + |z_+ z_- z_+ \rangle + |z_+ z_+ z_- \rangle), \tag{3}\]

where \([z_+, z_-]\) is the set of the eigen states for \(\hat{\sigma}_z\) and the symbol “\(\otimes\)” for the direct product is omitted unless any confusions arise. For this symmetric W state \[3\],

\[
\begin{align*}
P(z^i = z_+, z^j = z_+) &= 1, \tag{4} \\
P(z^i = z_+, x^j \neq x^k) &= 0, \tag{5} \\
P(x^i \neq x^k, z^j = z_) &= 0, \tag{6} \\
P(x^i = x^j = x^k) &= 3/4. \tag{7}
\end{align*}
\]

The middle term in Eq. (2) is 1/4, implying the violation of the CH-Bell inequality \[14\].

### III. QUANTUM SECURE COMMUNICATION VIA W STATES

#### A. Pair-wise quantum key distribution

Quantum key distribution (QKD) is a secure communication scheme by which two distant persons have in common a secret key message via quantum channels and classical communication. A QKD protocol includes three basic steps: a) Encoding a key message on a quantum state, b) transmitting the quantum system in the encoded state, and c) decoding the key message from the state. Protocols for QKD may be divided into two sets: protocols assisted by an entangled quantum channel and the rest of protocols. The protocol suggested by Ekert (E91) \[1\] and that by Bennett and Brassard (BB84) \[24\] are representative of the two sets respectively.

In BB84, a sender encodes a key message on non-orthogonal states of a quantum system (e.g., photon polarization) which is directly transferred to an authorized person, a receiver. The receiver can retrieve the key message by a measurement on the state if the measurement is approved through classical communication with the sender. Security of BB84 has been investigated based on no cloning theorem \[22, 23\]. Modified protocols have been proposed \[25\].

On the other hand, E91 utilizes an entangled EPR pair as a quantum channel. Two distant persons share an EPR pair. Each person measures an observable randomly chosen among three non-commuting observables. By doing so, the two persons can have in common a key message if both measurements are approved through classical communication. The procedures for encoding and decoding are executed at the same time, contrary to BB84. Security of E91 is based on Bell’s theorem \[27\]. As the quantum channel is in an EPR state, the authorized persons can see a violation of Bell’s inequality from measurement outcomes. Provided an eavesdropper enforces an intercept-resend strategy to extract the key message \[29\], the attempt breaks the entanglement of the quantum channel and thus the Bell’s inequality is not violated.

The link between security of QKD and Bell’s inequality has been intensively discussed against individual attacks by an eavesdropper \[13\]. Bennett et al. \[27\] suggested a slightly variant protocol (BBM92) from E91 with a simplified set of observables and showed that BBM92 is actually equivalent to BB84 although they have different characteristics on quantum channels. They investigated the security by comparing a priori probabilities with posteriori ones of outcomes instead of Bell’s inequality.

We propose a pair-wise QKD protocol via W states of three qubits, which is similar to BBM92 using EPR state of two qubits. Suppose that three authorized persons, Alice, Bob, and Charlie, would like to perform a secure communication in the pair-wise way such that every pair among the three persons tries to have a key message and in particular the members in the pair have in common the key message.

Consider a composite system of three qubits which is

| Alice | Bob | Charlie | decider |
|-------|-----|---------|---------|
| \(z_+\) | \(x_+\) | \(x_+\) | Alice |
| \(z_+\) | \(x_-\) | \(x_-\) | |
| \(z_-\) | \(x_+\) or \(x_-\) | \(x_+\) or \(x_-\) | |
| \(x_+\) | \(z_+\) | \(x_+\) | Bob |
| \(x_-\) | \(z_+\) | \(x_-\) | |
| \(x_+\) or \(x_-\) | \(z_-\) | \(x_+\) or \(x_-\) | |
| \(x_+\) | \(x_+\) | \(z_+\) | Charlie |
| \(x_-\) | \(x_-\) | \(z_+\) | |
| \(x_+\) or \(x_-\) | \(x_+\) or \(x_-\) | \(z_-\) | |

TABLE I: Deciders and mutual relations among the outcomes in local measurements for pair-wise quantum key distribution
in a symmetric W state in Eq. (3). The three authorized persons share the three qubits, one qubit for each person. As in E91 or BBM92 they acquire key messages by performing local measurements. Each person randomly chooses an observable out of \( \hat{\sigma}_x \) and \( \hat{\sigma}_z \). After his measurement, he announces the axis but not the outcome. When Alice measures along \( d_1 \) axis, Bob along \( d_2 \), and Charlie along \( d_3 \), the set of the axes is denoted by \( d_1-d_2-d_3 \). We employ particular sets of axes, i.e., \( x-x-z \), \( x-z-x \), and \( z-x-x \) for the present pair-wise QKD. The probability to choose one of the sets is 3/8.

In Table I, we present mutual relations among outcomes when the authorized persons perform local measurements along a particular set of axes. One who measured along \( z \)-axis decides whether it is possible for the others to have in common a key message. We call him a decider. For example, consider the case of \( x-x-z \). If Charlie obtains an outcome \( z_- \) in his local measurement along the \( z \)-axis, then Alice and Bob have their pair in the product state \( |z_+z_+\rangle \) and further they obtain outcomes randomly out of \( x_+ \) and \( x_- \) in their local measurements along the \( x \)-axis. The outcomes are useless for the present protocol and discarded as they exhibit no correlation. However, if Charlie obtains \( z_+ \), the pair that Alice and Bob share comes to be in the maximally entangled state \( (|z_-z_+\rangle + |z_+z_-\rangle)/\sqrt{2} \). In the measurements along \( x \)-axis, Alice and Bob obtain the same outcome of \( x_+ \) or \( x_- \). They may now regard the outcome as a key bit. Similar processes are applied to the rest of \( x-z-x \) and \( z-x-x \). The success probability in distributing a key bit is \( 2/3 \) once a particular set of measurement axes is chosen among \( x-x-z \), \( x-z-x \), and \( z-x-x \).

The present pair-wise QKD protocol via W states is summarized as following

K.1 Each of Alice, Bob, and Charlie chooses, at random, the axis of measuring instrument out of \( x \) and \( z \)-axes.

K.2 Each person announces a bit information on the axis of his local measurement but not the outcome.

K.3 For the purpose of security one requests to announce their outcomes at random in the trials of distributing key messages.

K.4a All keep their outcomes if the set of the measurement axes is \( x-x-z \), \( x-z-x \), or \( z-x-x \). Otherwise restart the protocol.

K.4b A decider who measured along \( z \)-axis tells the rest to regard their outcomes as a key bit if his outcome is \( z_+ \). Otherwise restart the protocol.

K.5 Repeat the protocol until they have key bits as many as they want.

K.6 By obtaining frequencies of security-check events over the outcomes which were announced at the step K.3, verify the security of quantum channel against attacks by an eavesdropper (See Sec. [V] for details). If the errors are larger than permitted, throw away the key bits which have been obtained so far.

Overall success probability \( P_s \) in obtaining a key bit is given by multiplying together the probability of choosing a particular set of measurement axes and the success probability of distributing a key bit for a given particular set of axes. The number of distributed key bits \( K_t \) is given as

\[
K_t = P_s N_e \tag{8}
\]

where \( P_s \) is an overall success probability and \( N_e \) an effective number of trials. In the pair-wise QKD \( P_s = 1/4 \) and \( N_e = N - M \) where \( N \) is a number of total trials in the protocol and \( M \) a number of trials for the purpose of security. For a given \( K_t \), the protocol requires \( n_q \) qubits such that

\[
n_q = \frac{3K_t}{P_s(1 + M/N)} \tag{9}
\]

In the limit of \( N \to \infty \), \( M/N \to 0 \) and \( n_q \to 3K_t/P_s \). Thus the protocol requires 12 qubits per a key bit at average. On the other hand, the protocol of E91 has the overall success probability \( P_s = 2/9 \) and it requires 9 qubits per a key bit.

### B. Partial quantum secret sharing

Quantum secret sharing is a key distribution protocol among \( N \) persons in such a way that one’s key message can be retrieved by the others only if they cooperate all together. Variant protocols have been proposed by using multipartite GHZ states \([13, 15]\) and bipartite EPR states \([28, 29]\).

Here we consider a partial quantum secret sharing protocol (PQSS) using W states of three qubits. The procedure for PQSS among three persons is similar to the pair-wise QKD. The difference is the set of measurement axes, i.e., \( z-z-z \) is employed.

Suppose Alice, Bob, and Charlie would like to perform QSS by sharing three qubits in a symmetric W state \([4]\). Each person randomly chooses an observable out of \( \hat{\sigma}_x \) and \( \hat{\sigma}_z \) as done in the pair-wise QKD. After his measurement, he announces the axis but not the outcome. In the case of \( z-z-z \), he may regard the outcome as a key bit. More explicitly, consider a case that Bob and Charlie are expected to retrieve Alice’s key message in their cooperation. If Alice has the outcome \( z_+ \), then Bob and Charlie have opposite outcomes out of \( z_+ \) and \( z_- \). Otherwise both have the same outcome \( z_+ \). When Bob and Charlie cooperate so as to collect their outcomes, they can correctly deduct Alice’s key bit. We note that, if Bob or Charlie obtains \( z_- \), he will realize that the other and Alice have the same outcome \( z_+ \). In the case he can deduct Alice’s key bit without help of the other. It will be done in the probability of \( 1/3 \). The fact implies that
Bob and Charlie may have partial information on Alice’s key bit. However, Bob and Charlie must still cooperate to retrieve completely Alice’s key bits. In the sense we call the present protocol a PQSS.

Although the actual key sharing is performed in the case of z-z-z, the authorized persons must choose $\hat{\sigma}_z$ for their observables as well for the purpose of security. If they chose only z-axis for PQSS, an eavesdropper could extract all information by measuring along z-axis.

The protocol of PQSS has steps in common with the pair-wise QKD protocol: Most steps in the pair-wise QKD protocol are applicable to PQSS but the step K.4 is modified as

S.4 All keep their outcomes if the set of the measurement axis is z-z-z. Otherwise restart the protocol.

In PQSS we have the overall success probability $P_s = 1/8$ which is determined by the probability of choosing z-z-z. Due to the arguments below Eq. (8), 24 qubits are necessary to share a key bit at average in the limit of $N \to \infty$. On the other hand, HBB99 with $P_s = 1/2$ needs 6 qubits per a key bit.

\[\hat{U}_{CE}|z+z\rangle_{CE} = |z+z\rangle_{CE}\]
\[\hat{U}_{CE}|z-z\rangle_{CE} = \cos \phi |z+z\rangle_{CE} + \sin \phi |z-\rangle_{CE}\]

\[P = (1 - \cos \phi)(5 + \cos \phi) / 18\]

which is obtained by averaging the probabilities over Charlie’s two cases. Keeping it in mind that such an event never happens if there is no attack by Eve, the authorized persons can detect Eve’s eavesdropping by checking whether such an event occurs. We call, by
a security-check event, an event which can be used to detect Eve’s eavesdropping. For the security of QKD, PQSS, and the synthesis, we employ the event that one person has the outcome $z_i = z_+$ and the others have the opposite outcomes $x_j \neq x_k$. This security event has an advantage over others such that they never occur if no attack by Eve and their occurrence indicates directly Eve’s attack.

Subensemble of local-measurement outcomes suffices to examine the security of the proposed protocols. One of three persons may request at random to announce their outcomes by $M$ times among $N$ trials for distributing key messages. We call $M$ a number of security trials. Based on $M$ sets of outcomes, they obtain the frequency of the security-check event and verify whether there is an attack by Eve. This procedure is appended to the protocols as steps K.3 and K.6.

V. REMARKS

We have proposed three variant protocols of quantum key distribution, partial quantum secret sharing, and their synthesis based on $W$ states, which have the different nonlocal characteristics from GHZ states. We have shown that these protocols are secure against the simple individual attacks by an eavesdropper.

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