Separability of qubit-qudit quantum states with strong positive partial transposes

Kil-Chan Ha¹

¹Faculty of Mathematics and Statistics, Sejong University, Seoul 143-747, Korea
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We show that all 2 ⊗ 4 states with strong positive partial transposes (SPPT) are separable. We also construct a family of 2 ⊗ 5 entangled SPPT states, so the conjecture on the separability of SPPT states is completely settled. In addition, we clarify the relation between the set of all 2 ⊗ d separable states and the set of all 2 ⊗ d SPPT states for the case of d = 3, 4.

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The notion of quantum entanglement plays a key role in the current study of quantum information and quantum computation theory. One of the central problems in the theory of quantum entanglement is to check whether a given density matrix representing a quantum state of composite system is separable or entangled. Let us recall that a state \( \rho \) acting on the Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \) is called separable if it is a convex combination of product states, that is, \( \rho = \sum_k \rho_k \otimes \tilde{\rho}_k \), where \( \rho_k \) and \( \tilde{\rho}_k \) are states acting on \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively [1].

There are two main criteria for separability: One criterion was given by Horodecki et al. [2] using positive linear maps between matrix algebras, and this was formulated as the notion of entanglement witnesses [3]. Another criterion, the so called partial positive transpose (PPT) criterion [4] tells us that if a state \( \rho \) is separable then its partial transposition \( \rho^{T_A} = (T \otimes 1) \rho \) is positive. It was shown by Horodecki et al. [5] that the PPT criterion is also a sufficient condition for separability in the system 2 ⊗ 2 and 2 ⊗ 3. For higher dimensions, this is not the case by a work of Woronowicz [6] who gave an example of a 2 ⊗ 4 PPT entangled state, see also Ref. [7] for the 3 ⊗ 3 cases, in the early eighties. See also Ref. [8] for the 2 ⊗ 4 case. Therefore, it is important to understand which PPT states are separable and which are entangled.

In this context, a subclass of PPT states, whose PPT properties are ensured by the canonical construction using Cholesky decomposition, were considered in Ref. [9]. Such states are called strong positive partial transpose (SPPT) states. Based on several examples of SPPT states, it was conjectured in [10] that all SPPT states are separable. Unfortunately this is not true in general, where it was shown that all SPPT states are separable independently of the rank of \( X_1 \) such that \( r(X_1) = d \). In this case, \( X_1 \) is nonsingular, and so \( S \) is normal matrix by the condition (3). Thus we have the spectral decomposition for \( S \)

\[
S = \sum_{i=1}^{d} \lambda_i P_i,
\]

where \( P_i \)’s are rank one projections with \( \sum_{i=1}^{d} P_i = 1 \).

Then we can write

\[
\begin{pmatrix}
1 & S \\
S^\dagger & S^\dagger S
\end{pmatrix} = \sum_{i=1}^{d} \sigma_i \otimes P_i \quad \text{with} \quad \sigma_i = \left( \frac{1}{\lambda_i^2} \right)
\]

and we see that

\[
\begin{pmatrix}
X_1^\dagger & 0 \\
0 & X_1^\dagger
\end{pmatrix} \begin{pmatrix}
1 & S \\
S^\dagger & S^\dagger S
\end{pmatrix} \begin{pmatrix}
X_1 & 0 \\
0 & X_1
\end{pmatrix} = \sum_{i=1}^{d} \sigma_i \otimes X_1^\dagger P_i X_1
\]

is an unnormalized separable state. Therefore, we can conclude that

\[
\rho = \begin{pmatrix}
X_1^\dagger & 0 \\
0 & X_1^\dagger
\end{pmatrix} \begin{pmatrix}
1 & S \\
S^\dagger & S^\dagger S
\end{pmatrix} \begin{pmatrix}
X_1 & 0 \\
0 & X_1
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
0 & X_2^\dagger X_2
\end{pmatrix}
\]

result displays the difference between the 2 ⊗ d case and the general m ⊗ n one (m, n > 2), even difference between 2 ⊗ 4 and 2 ⊗ 5 cases. We also clarify the relation between the notion of separability and the SPPT property.
is separable since the second matrix in the righthand side is \( |1\rangle |1\rangle \otimes X_1 \otimes X_2 \), where \( |1\rangle = (0, 1)^t \). Consequently we have the following.

**Proposition 1** Let \( \rho \) be a \( 2 \otimes d \) SPPT state of the form (2) with \( r(X_1 \otimes X_1) = d \). Then \( \rho \) is separable.

From the above Proposition, we obtain a sufficient condition for separability of \( 2 \otimes d \) PPT states \( \rho \) with \( r(0\rho|0\rho) = d \). To see this, we observe the condition when such a PPT state is SPPT. Let \( \rho \) be a \( 2 \otimes d \) PPT state of the form

\[
\rho = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}
\]
with \( A = d \) and \( r(\rho) = d \). Now, we write \( \Sigma \) and \( \tilde{\Sigma} \) as block matrices

\[
\Sigma = \begin{pmatrix} D_k & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\Sigma} = \begin{pmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{pmatrix},
\]

where \( D_k \) and \( \tilde{S}_{11} \) are \( k \times k \) matrices. Then we have

\[
\tilde{\rho} := \begin{pmatrix} \Sigma^2 & \Sigma \tilde{S} \Sigma \\ \Sigma \tilde{S}^\dagger \Sigma & \Sigma \tilde{S}^\dagger \tilde{S} \Sigma \end{pmatrix} = \begin{pmatrix} D_k^2 & 0 & D_k \tilde{S}_{11} D_k \\ 0 & 0 & \tilde{S}_{12} \\ D_k \tilde{S}_{11} D_k & \tilde{S}_{21} & \tilde{S}_{22} \end{pmatrix} D_k
\]

with \( D_k(\tilde{S}_{11} \tilde{S}_{11} + \tilde{S}_{21} \tilde{S}_{21})D_k = D_k(\tilde{S}_{11} \tilde{S}_{11} + \tilde{S}_{12} \tilde{S}_{12})D_k \) by the condition (3). It is easy to see that \( \tilde{\rho} \) is unnormalized separable state if and only if the following reduced unnormalized \( 2 \otimes k \) state

\[
\begin{pmatrix} D_k^2 \\ D_k \tilde{S}_{11} D_k \\ D_k(\tilde{S}_{11} \tilde{S}_{11} + \tilde{S}_{21} \tilde{S}_{21})D_k \end{pmatrix}
\]

is separable. We note that the above \( 2 \otimes k \) state is a PPT state, although it may not be SPPT. Therefore, if \( k \leq 3 \) then we see that \( \tilde{\rho} \) is separable. In this case, we can conclude that \( \rho \) is separable in (5). Consequently, we have the following.

**Proposition 3** Let \( \rho \) be a \( 2 \otimes d \) SPPT state of the form (2) with \( r(X_1 \otimes X_1) \leq 3 \). Then \( \rho \) is separable.

To answer the conjecture asked in 10, we will show the following.

**Theorem 4** All \( 2 \otimes d \) states with strong positive partial transpose are separable if and only if \( d \leq 4 \).

**Proof.** By combining the Proposition 1 and 2, we see that all \( 2 \otimes 4 \) states are separable. To complete the proof, we construct a \( 2 \otimes 5 \) state which is not separable. Define \( 5 \times 5 \) matrices \( X_1 \) and \( S \) by

\[
X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & 0 & 0 & \beta_1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

where \( \beta_1 = [(1 - b)/(2b)]^{1/2} \) and \( \beta_2 = [(1 + b)/(2b)]^{1/2} \) with \( 0 < b < 1 \). We also put \( X_2 \) by \( 5 \times 5 \) zero matrix. Then we define \( \rho_0 \) by Eqs. (1) and (2):

\[
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & \beta_1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

where \( \gamma_1 = (b + 1)/(2b) \), \( \gamma_2 = \sqrt{b^2 - 1}/(2b) \), and \( \dot{\cdot} \) denotes zero. We note that the corresponding reduced \( 2 \otimes 4 \) state as in (5) is the PPT entangled state given by Horodecki [9]. Therefore, we conclude that \( \rho_0 \) is an entangled state with strong positive partial transpose. This completes the proof.
\[
\begin{pmatrix}
\rho_{11} & \rho_{12} \\
\rho_{12}^\dagger & \rho_{22}
\end{pmatrix}
\]
derived by

\[
\rho_{11} = \begin{pmatrix} 3 & \cdot & \cdot \\
\cdot & 4 & 2 \\
\cdot & 2 & 3
\end{pmatrix},
\rho_{12} = \begin{pmatrix} \cdot & \cdot & 1 \\
\cdot & 1 & -1 \\
1 & -1 & \cdot
\end{pmatrix},
\rho_{22} = \begin{pmatrix} 2 & 1 & -1 \\
1 & 6 & 1 \\
-1 & 1 & 3
\end{pmatrix}.
\]

Then, we can easily check that four \(3 \times 3\) matrices \(\rho_{11}, \rho_{22}, \rho_{22} - \rho_{12}^\dagger \rho_{11} \rho_{12} - \rho_{12} \rho_{11}^\dagger \rho_{12}^\dagger\) are all positive definite matrices. Therefore, we see that \(\rho_1\) is a \(2 \otimes 3\) PPT state, and so \(\rho_1\) is separable. On the other hand, we see that

\[
\rho_{12}^\dagger \rho_{11} \rho_{12} - \rho_{12} \rho_{11}^\dagger \rho_{12}^\dagger = \frac{1}{12}
\begin{pmatrix}
6 & -6 & -3 \\
-6 & 0 & 0 \\
-3 & 0 & -4
\end{pmatrix}.
\]

Therefore, \(\rho_1\) is not SPPT by the Corollary\(^\text{[2]}\) Now, we define \(2 \otimes 4\) state \(\rho_2\) using the above \(2 \otimes 3\) state \(\rho_1\) as follows:

\[
\rho_2 = \begin{pmatrix}
\rho_{11} & 0 & \rho_{12} & 0 \\
0 & 1 & 0 & 0 \\
\rho_{12}^\dagger & 0 & \rho_{22} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Therefore, \(\rho_2\) is separable. We can also show that \(\rho_2\) is not SPPT by the Corollary\(^\text{[2]}\). This completes the proof of claim.

In conclusion, we showed that all \(2 \otimes 4\) SPPT states are separable. We also constructed a family of \(2 \otimes 5\) SPPT entangled states using Horodecki’s \(2 \otimes 4\) PPT entangled states. So the conjecture\(^\text{[10]}\) on the separability of SPPT states is completely settled. We also clarify the relation between the set of all \(2 \otimes d\) separable states and the set of all \(2 \otimes d\) SPPT states for the case of \(d = 3, 4\).

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