A comparative study of system size dependence of the effect of non-unitary channels on different classes of quantum states

Geetu Narang · Shruti Dogra · Arvind

Received: 8 May 2020 / Accepted: 20 October 2020 / Published online: 4 November 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract
We investigate the effect of different types of non-unitary quantum channels on multiqubit quantum systems. For an \(n\)-qubit system and a particular channel, in order to draw unbiased conclusions about the system as a whole as opposed to specific states, we evolve a large number of randomly generated states under the given channel. We increase the number of qubits and study the effect of system size on the decoherence processes. The entire scheme is repeated for various types of environments which include dephasing channel, depolarizing channel, collective dephasing channel and zero temperature bath. Non-unitary channels representing the environments are modeled via their Kraus operator decomposition or master equation approach. Further, for a given \(n\) we restrict ourselves to the study of particular subclasses of entangled states, namely the GHZ-type and W-type states. We generate random states within these classes and study the class behaviors under different quantum channels for various values of \(n\).

Keywords Quantum channels · Quantum decoherence · Quantum dissipation

Arvind
arvind@iisermohali.ac.in

Geetu Narang
geet29@gmail.com

Shruti Dogra
shrutidogra@iisermohali.ac.in

1 Department of Applied Sciences, U.I.E.T, Panjab University, Chandigarh 160025, India
2 Department of Physical Sciences, Indian Institute of Science Education and Research, Mohali, India
3 QTF Centre of Excellence, Department of Applied Physics, Aalto University School of Science, AALTO, P.O. Box 15100, 00076 Espoo, Finland
1 Introduction

Inability to overcome the effects of decoherence is the most crucial hurdle in quantum information processing [1,2]. Hence, one of the fundamental requirements to build a quantum computer is to understand and control the process of decoherence [3–7]. Several platforms have been proposed for scalable implementation of quantum computers on the basis of superconductors [8], semiconductors [9], ion traps [10], spins in solids [11,12] and spins of molecules using NMR techniques [13,14]. There have also been several attempts to characterize decoherence processes in different experimental scenarios. Study of the dynamics of entanglement in trapped ions [15], study of the dephasing channel using photonic qubits [16], decoherence in superconducting qubits [17,18], modeling decoherence [19] and the protection of tripartite entanglement [20] and quantum discord [21] in NMR are some of the examples.

In all the physical contexts, decoherence timescales are typically estimated for individual qubits, whereas practical implementation of a quantum computer requires the use of many qubits. For multiqubit systems, correlations between qubits can arise and Hilbert space dimension grows exponentially with number of qubits. Estimating the decoherence costs and effect of decohering environments on quantum discord and entanglement for multiqubit systems has also been investigated by several authors [22–26]. Since new ways in which decoherence can affect the system can emerge for multiqubit systems, investigation of the behavior of the system with increased number of qubits is important.

The non-unitary environmental effects can be classified as dissipation and dephasing. While dissipation involves energy exchange and is possible at the classical level too [27,28], dephasing is a purely quantum mechanical phenomenon [29]. In any case, both lead to information loss and state degradation. If we consider system and the environment as a whole, their dynamics is unitary. The environment by its very nature is inaccessible, and to obtain the dynamics of the system alone, we can trace over the environment. This may lead to a non-unitary evolution of the system. At a fundamental level, environment-induced non-unitary processes are completely positive maps and such maps allow a representation via Kraus operators as follows [30–32]:

\[ \rho^{\text{out}} = E(\rho) = \sum_{\nu=1}^{N} K_{\nu} \rho K_{\nu}^{\dagger} \]

where \( K_{\nu} \) are the Kraus operators and they satisfy the condition \( \sum_{\nu} K_{\nu}^{\dagger} K_{\nu} = 1 \) for trace preserving maps [1]. This evolution is in general non-unitary leading to decoherence; however, the unitary quantum evolution is included and corresponds to a situation when only one of the Kraus operators is nonzero. Depending upon the kinds of Kraus operators involved, the channels are classified. In our analysis, depolarizing channel and collective dephasing channels will be described through their Kraus operators.

Another approach to model the evolution of environment-induced non-unitary dynamics of a system involves an explicit description of the environment Hamiltonian \( H_{E} \), as well as its interaction with the system via interaction Hamiltonian \( V \).
Total Hamiltonian has a general form:

\[ H = H_S + H_E + V \]  

(2)

where \( H_S \) is the system Hamiltonian. We assume a large number of environment degrees of freedom and a weak system–environment interaction. To begin with, state of the system \( \rho_S \) and environment state \( \rho_E \) are assumed to be in a tensor product of the form \( \rho_S \otimes \rho_E \), which eventually gets entangled due to the system–environment interaction. Under such evolution, when the environment is traced out, we obtain a non-unitary channel. Further, under the Born–Markov approximation, this channel is represented by Lindblad master equation whose solution provides us with time evolution of the system \([6,33,34]\). For a particular kind of environment, beginning with the total Hamiltonian of the system we obtain the equation governing the dynamics of the system density operator alone called the master equation \([35,36]\). We will employ this method while dealing with the channel termed “zero temperature bath” and dephasing channel.

Our aim in this work is to study the behavior of quantum systems under different non-unitary processes with a focus on its dependence on the system size. Under a given non-unitary channel, different states of a system behave differently. In order to draw conclusion about the system as a whole, we generate a large number of random states and average our results over this sample set. Assuming that we have a large enough sample set and the sampling of the state space is uniform, our conclusions pertain to the system as a whole and are state independent. For an \( n \)-qubit system, we take a particular channel and see its effect as we change the values of \( n \). Then, we repeat the exercise for another channel. This allows us to analyze the size dependences of the effect of these non-unitary channels and make comparisons of these effects across different channels in a state-independent manner. Channels that we consider include zero temperature bath, dephasing channel, collective dephasing channel and depolarizing channel.

In a similar vein for the \( n \)-qubit Hilbert space (for \( n > 1 \)), we consider entangled states and study their decoherence properties under four decoherening channels that we considered for the earlier study. Motivated by the structure of different inequivalent maximally entangled states for three qubits, namely the GHZ and W states, we define “GHZ-type” and “W-type” states for systems with \( n > 1 \). We study these families separately and make comparisons about their decoherence under various channels. We find very interesting comparisons and contrasts in the behavior. Throughout, while studying a particular class of states we generate a large number of samples in that particular class and average the behavior over these samples to obtain state-independent results as was done for the full \( n \)-qubit state space.

The effect of decoherence can be estimated by computing the change in the state that takes place due to the environmental factors. For the case where we start with an initial pure state, a good measure of deviation is fidelity defined in terms of the overlap of the initial pure state \( |\psi\rangle \) and the final mixed or pure state \( \rho^{\text{out}} \):

\[ F = \langle \psi | \rho^{\text{out}} | \psi \rangle. \]  

(3)
Fig. 1 Variation of average fidelity for $n = 1$ to 6 qubits for zero temperature bath model. Input states are $n$-qubit pure states, GHZ-type states and W-type-states. The value of $\gamma_1 t = 1$. As can be seen, the W-type states degenerate much faster than the GHZ-type states and general pure states.

Fidelity can take values between 0 and 1, and the deviation from 1 indicates the amount of degradation or change.

The computations involve a mix of analytical and numerical tools. The general forms of output states are computed analytically, and then, numerical simulations are carried out on randomly generated states from the family of states under consideration. The uniform distribution is achieved by the appropriate use of pseudo-random function of Mathematica. We observe that, in the case of zero temperature bath channel, degradation rate with respect to number of qubits is maximum in case of W-type states and minimum in case of GHZ-type states. The rate of degeneration in case of dephasing channel is minimum for GHZ-type states and maximum for general pure states. Depolarizing channel destroys all the three sets of states in a similar way. In the case of collective dephasing channel, degeneration rate of the state with respect to system size is negligible for GHZ-type states, whereas it is very similar for general pure states and W-type states. We have also computed and displayed the fidelity distributions for different classes of states, under different channels and their variation with number qubits.

The paper is organized as follows: In Sect. 2, we define the three classes of states, namely the $n$-qubit general pure states, the GHZ-type states and the W-type states. We then define and discuss the four non-unitary channels: the zero temperature bath, the dephasing channel, the collective dephasing channel and the depolarizing channel and the evolution of the family of states under these channels. The results are shown as average fidelities as a function of number of qubits for different state classes for a given channel. We also display the fidelity distributions. In Sect. 3, we compare the effects of all four channels on each set of state classes. Here, the graphs of average fidelity as a function of the number of qubits are shown for a given class of states for all four non-unitary channels. Section 4 offers some concluding remarks.
2 Classes of states and their evolution under different channels

In this section, we describe our main results where we study certain families of states of an $n$-qubit system under different non-unitary channels. For $n = 1$, we have only one class of states which are the most general pure states of the system. For an $n > 1$, we consider three types of states, namely the $n$-qubit pure states (or general pure states), GHZ-type states and W-type states. The latter two types are motivated by the structure of superpositions involved in the two inequivalent classes of maximally entangled states for three qubits. These families are defined as follows:

(a) \textit{n-qubit pure states}: for an $n$-qubit system, the most general pure state can be expressed as a linear combination of all the computational basis states as follows:

$$| \psi_{\text{General}} \rangle = \alpha_0 |000...0\rangle + \alpha_1 |000...1\rangle + \alpha_2 |000...10\rangle + ....... + \alpha_{2^n-1} |111......1\rangle$$  \quad (4)

where $\alpha_0, \alpha_1, ..., \alpha_{2^n-1}$ are complex numbers satisfying $\sum_{j=0}^{2^n-1} |\alpha_j|^2 = 1$

(b) \textit{GHZ-type states}: a GHZ-type state for an $n$-qubit system is defined as follows:

$$| \psi_{\text{GHZ}} \rangle = \alpha |000....0\rangle + \beta |111......1\rangle$$ \quad (5)

where $\alpha$ and $\beta$ can have any complex numbers with $|\alpha|^2 + |\beta|^2 = 1$.

(c) \textit{W-type states}: a W-type state is defined as follows:

$$| \psi_{\text{W}} \rangle = \beta_1 |000.....001\rangle + \beta_2 |000.....010\rangle + \beta_3 |000...0100\rangle + ....... + \beta_n |1000.....000\rangle$$ \quad (6)

where $\beta_1, \beta_2, ..., \beta_n$ are again complex numbers with $\sum_{j=1}^{n} |\beta_j|^2 = 1$. 

Fig. 2 Variation of average fidelity as a function of system size for the dephasing channel. Fidelity is calculated for general pure, GHZ-type and W-type states. The value of $\gamma_2 t = 2.48$.
We are now ready to study the effect of different non-unitary channels on the classes of states defined above. We will start with a single qubit and try to go up to eight qubits.

2.1 Channel with zero temperature bath as environment

For the non-unitary process where we have a zero temperature bath of qubits in the environment, we assume that each qubit interacts with the bath qubits independently. We consider the Lindblad master equation for the evolution of the system density operator $\rho$ to model the interaction of the system with the bath, given as [35]:

$$\frac{d\rho}{dt} = \sum_{k=1}^{n} (I \otimes \ldots \otimes L_k \otimes \ldots \otimes I)\rho.$$  \hspace{1cm} (7)
Here, $L_k$ is a single qubit operator and is defined by its action on the $k$th qubit in terms of Pauli operators $\sigma_\pm = \sigma_1 \pm i \sigma_2$ as:

$$L_k \rho_k = \gamma_1 \left( 2 \sigma_- \rho_k \sigma_+ - \sigma_+ \rho_k \sigma_- - \rho_k \sigma_+ \sigma_- \right)$$

(8)

The parameter $\gamma_1$ depends upon the strength of the system–bath interaction.

For the $n = 1$ case, let the initial state of the system be $\rho = |\psi_0\rangle \langle \psi_0|$. Then, the final state obtained at time $t$ by solving Eq. (7) is given as

$$\rho_{\text{out}} = \left( \begin{array}{cc} e^{-t \gamma_1/2} \rho_{11} & e^{-t \gamma_1/2} \rho_{12} \\ e^{-t \gamma_1/2} \rho_{21} & (1 - e^{-t \gamma_1}) \rho_{11} + \rho_{22} \end{array} \right)$$

(9)

where $\rho_{ij} = \langle i | \rho | j \rangle$ is the $i$, $j$th element of the initial state $\rho$ in the computational basis. Since both diagonal and off-diagonal terms are being affected by the channel, it is clear that the interaction of the system with the bath results in both dissipation and decoherence of the state. Similarly, final states for the systems up to six qubits can be calculated analytically. The expressions are long and therefore are not being displayed. As $t \to \infty$, the decohered state in Eq. (9) approaches the lower energy state with $\rho_{22}^{\text{out}} = \rho_{11} + \rho_{22} = 1$. This happens for higher number of qubits too and is a reflection of the fact that we are working with a zero temperature bath.

Once we have the final state, the fidelity can be calculated using Eq. (3). We generate 1,00,000 random states numerically and compute the fidelity and the average fidelity. The process is repeated for up to six qubits. Next for $n > 1$, we restrict ourselves to GHZ-type and W-type states and again generate random states and compute average fidelity. The average fidelities are shown in Fig. 1. All fidelities are computed for time, $t = 1/\gamma_1$, which is a typical timescale to observe significant decoherence. The histograms of fidelities are shown in Figs. 5, 6 and 7 where the first column in each figure corresponds to the zero temperature bath channel.

The difference in degeneration properties of the three classes of states is clearly visible in Fig. 1. While the W-type states degenerate more rapidly compared to general pure states, GHZ-type states are more robust. Another interesting result obtained while calculating the fidelities for W-type states is that all states in W-type family have same fidelity given by:

$$F = e^{-(n-1)\gamma_1 t},$$

(10)

where $n$ is the number of qubits. Clearly, the fidelity in this case is independent of the randomly chosen coefficients ($\beta_i$s). The derivation of this result is provided in Appendix A. It implies that in case system is interacting with a zero temperature bath, all the $n$-qubit states in W-type state space degenerate in exactly the same way. This is clearly seen from the first column of Figure 7. The first columns of Figs. 5, 6 and 7 show how the fidelities are distributed, for $n$-qubit pure, GHZ-type and W-type states, respectively. The fidelity distribution is most broad for the GHZ-type states, and for $n$-qubit pure states, the distribution tends to become narrow as the number of qubits increases.
2.2 Dephasing channel

Dephasing channel destroys the off-diagonal elements of a density matrix which correspond to coherences among the computational basis states [37,38]. In this case like the zero temperature bath, the qubits interact with the environment individually and we use the master equation model described in Eq. (7). Lindblad operator in this case is again given through its action on single qubit density operator in terms of Pauli matrices as:

$$L_k \rho_k = \frac{\gamma_2}{2} (2 \sigma_- \sigma_+ \rho_k \sigma_- \sigma_+ - \sigma_- \sigma_+ \sigma_- \sigma_+ \rho_k - \rho_k \sigma_- \sigma_+ \sigma_- \sigma_+)$$  \hspace{1cm} (11)

Here, $\gamma_2$ depends upon the interaction strength between the system and the bath. Using the similar procedure as used in the zero temperature bath case, we obtain the final density matrix for the state under dephasing channel for a single qubit

$$\rho_{\text{out}} = \begin{pmatrix} \rho_{11} e^{-\gamma_2 t / 2} & e^{-\gamma_2 t / 2} \rho_{12} \\ e^{-\gamma_2 t / 2} \rho_{21} & \rho_{22} \end{pmatrix}$$  \hspace{1cm} (12)

It is clear from the RHS of the above equation that the channel affects only the off-diagonal terms, whereas the diagonal terms remain unaffected. The final state for $n > 1$ can be calculated and have similar structure; however, we are not displaying the long expression. The analytical expressions for fidelity up to eight qubits can be obtained using the final density matrices. Generating a large number of random states, as in case of zero temperature bath, we obtain the fidelity distributions for all three classes of states for $n = 1$ to $n = 8$. The average fidelities are shown in Fig. 2, while the histograms of fidelities are shown in the second columns of Figs. 5, 6 and 7. Comparison of the average fidelities of three set of states is shown in Fig. 2 revealing how classes are affected by the channel. Decoherence of GHZ-type states is minimum, and the decoherence of general pure states is maximum. This is quite different from the zero temperature bath.

We can attribute the slow decoherence of GHZ-type states and W-type states in comparison with $n$-qubit pure states to the number of phases involved in both the GHZ- and W-type states. Number of relative phases in case of GHZ-type states is just one; therefore, it has only one way to degrade. In case of W-type states, more relative phases are involved; therefore, W-type states degenerate relatively more in comparison with GHZ-type states. Number of relative phases goes up with number of qubits in case of $n$-qubit pure states; therefore, degeneration is drastic. An important observation about the GHZ-type states is that their fidelity converges to 0.66 as the number of qubits increases.

Looking at the second columns of Figs. 5, 6 and 7, we can see how the fidelities are distributed. Again the fidelity distributions are very different for the three classes of states and very different from the zero temperature bath.
The variance of fidelity distribution is maximum in case of zero temperature bath and is minimum in case of depolarizing channel. The fidelity distributions of dephasing and collective dephasing channel show similar behavior with increasing number of qubits.

**Fig. 5** Fidelity distribution of uniformly distributed random $n$-qubit pure states as a function of system size.
| n=2 | Zero temp. bath | Dephasing Channel | Collective Dephasing Channel | Depolarising Channel |
|-----|-----------------|-------------------|-------------------------------|----------------------|
| 0.005 | 0 | 0 | 0.05 | 0 | 0.04 |
| 0.0025 | 0 | 0 | 0.025 | 0 | 0.02 |

| n=3 | Zero temp. bath | Dephasing Channel | Collective Dephasing Channel | Depolarising Channel |
|-----|-----------------|-------------------|-------------------------------|----------------------|
| 0.005 | 0 | 0 | 0.05 | 0 | 0.04 |
| 0.0025 | 0 | 0 | 0.025 | 0 | 0.02 |

| n=4 | Zero temp. bath | Dephasing Channel | Collective Dephasing Channel | Depolarising Channel |
|-----|-----------------|-------------------|-------------------------------|----------------------|
| 0.005 | 0 | 0 | 0.05 | 0 | 0.04 |
| 0.0025 | 0 | 0 | 0.025 | 0 | 0.02 |

| n=5 | Zero temp. bath | Dephasing Channel | Collective Dephasing Channel | Depolarising Channel |
|-----|-----------------|-------------------|-------------------------------|----------------------|
| 0.005 | 0 | 0 | 0.05 | 0 | 0.04 |
| 0.0025 | 0 | 0 | 0.025 | 0 | 0.02 |

| n=6 | Zero temp. bath | Dephasing Channel | Collective Dephasing Channel | Depolarising Channel |
|-----|-----------------|-------------------|-------------------------------|----------------------|
| 0.0075 | 0 | 0 | 0.075 | 0.05 | 0.04 |
| 0.0025 | 0 | 0 | 0.025 | 0 | 0.02 |

| n=7 | Zero temp. bath | Dephasing Channel | Collective Dephasing Channel | Depolarising Channel |
|-----|-----------------|-------------------|-------------------------------|----------------------|
| 0.005 | 0 | 0 | 0.05 | 0 | 0.04 |
| 0.0025 | 0 | 0 | 0.025 | 0 | 0.02 |

| n=8 | Zero temp. bath | Dephasing Channel | Collective Dephasing Channel | Depolarising Channel |
|-----|-----------------|-------------------|-------------------------------|----------------------|
| 0.005 | 0 | 0 | 0.05 | 0 | 0.04 |
| 0.0025 | 0 | 0 | 0.025 | 0 | 0.02 |

**Fig. 6** Fidelity distribution of uniformly distributed random GHZ-type states as a function of system size. The variance of fidelity distribution is maximum in case of zero temperature bath and is minimum in case of depolarizing channel. The fidelity distributions of dephasing and collective dephasing channel show similar behavior with increasing number of qubits.
Fig. 7 Fidelity distribution of uniformly distributed random generalized W-type states as a function of system size. The variance of fidelity distribution is zero in case of zero temperature bath model as well as depolarizing channel. Fidelity distribution of W-type state for zero temperature bath model shows that it is affecting all states equally, which is in accordance with Eq. (10)
2.3 Collective dephasing channel

The collective dephasing channel is similar to dephasing channel. However, for its action we need at least two qubits which are collectively coupled to an environment. This channel can be described using the Kraus operators as follows [39]:

\[
K_1 = \begin{pmatrix}
\gamma_3(t) & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \gamma_3(t)
\end{pmatrix}, \quad \gamma_3(t) = e^{-\frac{t}{T}}
\]

\[
K_2 = \begin{pmatrix}
\omega_1(t) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega_2(t) \\
0 & 0 & 0 & \omega_3(t)
\end{pmatrix}, \quad \omega_1(t) = \sqrt{1 - e^{-\frac{t}{T}}} \\
\omega_2(t) = -e^{-t/T} \sqrt{1 - e^{-\frac{t}{T}}}
\]

\[
K_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega_3(t)
\end{pmatrix}, \quad \omega_3(t) = \sqrt{(1 - e^{-\frac{t}{T}})(1 - e^{-\frac{2t}{T}})}.
\]

The phase relaxation time \( T \) due to the collective interaction of the system with the bath which is the inverse of the damping rate \( \Gamma \) of the system is the single parameter characterizing the channel. The action of the channel on a general two qubit quantum state \( \rho \) is given as:

\[
\rho_{\text{out}} = \sum_{j=1}^{3} K_j \rho K_j^\dagger.
\] (14)

Since collective dephasing channel acts on two qubits at a time, we have considered even number of qubits, namely 2, 4, 6 and 8, in our analysis. Once again, beginning with a general pure state of two qubits we let it evolve under the channel defined in Eq. (14) and evaluate the output state, which turns out to be:

\[
\rho_{\text{out}} = \begin{pmatrix}
(\gamma_3^2 + \omega_1^2)\rho_{11} & \gamma_3\rho_{12} & \gamma_3\rho_{13} & (\gamma_3^2 + \omega_1\omega_2)\rho_{14} \\
\gamma_3\rho_{21} & \rho_{22} & \rho_{23} & \gamma_3\rho_{24} \\
\gamma_3\rho_{31} & \rho_{32} & \rho_{33} & \gamma_3\rho_{34} \\
(\gamma_3^2 + \omega_1\omega_2)\rho_{41} & \gamma_3\rho_{42} & \gamma_3\rho_{43} & (\gamma_3^2 + \omega_2^2 + \omega_3^2)\rho_{44}
\end{pmatrix}
\] (15)

The expression of the final state shows that the 2-3 subspace corresponding the “zero quantum” is not effected at all. This interesting feature is reflected in Fig. 3, where, for \( n = 2 \), the average fidelity in the case of W-type state is 1, while it has much smaller values for GHZ-type and general pure states. The final states corresponding to four, six and eight qubits can be calculated in a similar way, and from the final state, we can calculate the fidelity. Again we generate a large number random states within the same three classes and compute the distribution of fidelities. The average fidelities
are shown in Fig. 3, while the fidelity distributions are shown in the third columns of Figs. 5, 6 and 7.

The average fidelity degeneration behaviors with changing $n$ shown in Fig. 3 are similar to that of dephasing channel. Decay of GHZ-type states in comparison with $n$-qubit pure states and W-type states is minimum. The change in the degeneration rate of GHZ-type states is very small as we increase number of qubits. General pure states are more fragile compared to the other two families, and increase in their degeneration rate is fastest with respect to the number of qubits. As was explained in the case of dephasing channel, states degeneration depends on the number of relative phases contained in them. GHZ-type states contain minimum number of phases; therefore, degeneration is minimum. General pure states contain maximum number of phases; therefore, degeneration is maximum in their case. The general pattern of fidelity distributions shown in the third columns of Figs. 5, 6 and 7 shows that overall the behaviors is similar to the dephasing channel.

2.4 Depolarizing channel

Depolarizing channel describes the system–environment interaction in the large temperature regime. There are ways to obtain this channel from explicit models of such interactions in the high temperature limit; however, we directly use the model of this channel using the Kraus operators. For a single qubit, a general Pauli channel has its Kraus operators represented by Pauli matrices $\sigma_j$, $j = 1, 2, 3$ as follows:

$$\epsilon(\rho) = p_0\rho + \sum_{i=1}^3 p_i\sigma_i\rho\sigma_i$$  \hspace{1cm} (16)

where $p_i \geq 0$, $p_0 + p_1 + p_2 + p_3 = 1$. When $p_1 = p_2 = p_3$, the above channel corresponds to the depolarizing channel. The depolarizing channel, therefore, can be represented by a single parameter $p$ as follows:

$$\rho^{\text{out}} = E(\rho) = (1 - p)\rho + \frac{p}{3}(\sigma_1\rho\sigma_1 + \sigma_2\rho\sigma_2 + \sigma_3\rho\sigma_3).$$  \hspace{1cm} (17)

For single qubit state, the action of the depolarizing channel can be computed using Eq. (17) resulting in the transformation of input $\rho$ to the output state $\rho^{\text{out}}$ with

$$\rho^{\text{out}} = \begin{pmatrix}
-\frac{1}{2}(-2 + p)\rho_{11} & -(1 + p)\rho_{12} + \frac{1}{2}p\rho_{21} \\
-(1 + p)\rho_{21} + \frac{1}{2}p\rho_{12} & -\frac{1}{2}(-2 + p)\rho_{22}
\end{pmatrix}$$  \hspace{1cm} (18)

Since the depolarizing channel affects both the diagonal and off-diagonal terms of the state, it results in both decoherence and dissipation of the system. The effect of a depolarizing channel on states for $n > 1$ can be computed by using Eq. (17). We generate random states within the three families of states under consideration and pass them through the depolarizing channel. The fidelities are computed, and the average fidelity and fidelity distributions are plotted. The average fidelity for different types of
Variation of average fidelity of \( n \)-qubit pure states as a function of system size for different channels. As we can see, the degeneration dependence on \( n \) is same for zero temperature bath, dephasing channel and collective dephasing channel. It is only the depolarizing channel that effects the states differently.

Depolarizing channel is supposed to be most unbiased way of carrying out state degradation. In agreement with that view, the average fidelity for the three classes of states is same and shows the same behaviors with number of qubit as is evident from Fig. 4. The fidelity distribution of each set of states with increasing number of qubits also shows the same pattern as can be seen from the fourth columns of Figs. 5, 6 and 7. Furthermore, there is no variation in fidelity as is expected from the depolarizing channel.

### 3 Comparison of fidelity for different channels

In the previous section, a comparison was drawn between the degeneration rates of three set of states with respect to number of qubits for all the four channels. Here, we compare the degeneration behavior of each family of states under the effect of all four channels. We have used the same data that were used in the previous section to draw conclusion in this section. As was mentioned on each graph in the previous section that we used specific parameter values, we used \( \gamma_1 t = 1 \) for zero temperature bath, \( \gamma_2 t = 2.48 \), and for dephasing channel, \( \Gamma t = 5 \) for the collective dephasing channel and \( p = 0.8348 \) for the depolarizing channel. These values appear arbitrary, and similar behaviors will be seen for other values. The reason behind this choice is that the starting fidelities for all the channels should be same for \( n \)-qubit pure states, which for zero temperature bath, dephasing channel and depolarizing channel is \( n = 1 \) and for collective dephasing channel is \( n = 2 \).
3.1 Variation of average fidelity for $n$-qubit pure states

We consider $n$-qubit pure states and plot the average fidelity as a function of number of qubits corresponding to the different channels. The results are shown in Fig. 8. It is clear that the dependence of degeneration on number of qubits is same for zero temperature bath, dephasing and collective dephasing channels. The states degenerate differently under the depolarization channel where the degeneration grows slower with number of qubits compared to the other cases.

3.2 Variation of average fidelity for GHZ-type states

Next, we take GHZ states and plot their average fidelity as a function of number of qubits for different channels. The results are shown in Fig. 9. The relative behavior of GHZ-type states under the effect of four channels is quite different. Figure 9 shows that degeneration of the states is least in the case of collective dephasing channel followed by dephasing channel. The graph obtained in this case is quite different from that for $n$-qubit pure states. The reason can be attributed to very small number of relative phases involved in the state. The action of zero temperature bath and depolarizing channel is similar to that of $n$-qubit pure states.

3.3 Variation of average fidelity for W-type states

In this case, we consider the average fidelity as a function of number of qubits for W-type states for different channels. The results are shown in Fig. 10. It is clear from the figure that the dependence of decoherence of W-type states the number qubits is similar in the case of zero temperature bath and depolarizing channel and for dephasing and collective dephasing it is similar. The decoherence effects increase more rapidly.
for the first two cases compared to the last two cases. This behavior is quite different from the GHZ-type states as well as from the general pure states.

4 Conclusions

We studied one to eight qubit quantum systems under zero temperature bath, dephasing channel, collective dephasing channel and depolarizing channel. The main aim was to study the dependence of degradation rates on the system size which in this case was quantified by the number qubits. For each case (\(n > 1\)), we considered three family of states, namely the \(n\)-qubit pure states, GHZ-type states and W-type states, and studied them for their behaviors under different environments listed above. For \(n = 1\), we studied only general pure states. In order to draw state-independent conclusions, we averaged the fidelities over the family of states that we considered. We also studied the fidelity distributions.

While the average fidelity was observed to drop with increasing number of qubits, the three classes of states behaved differently. In case of zero temperature bath channel, degeneration rate with respect to number of qubits is maximum in the case of W-type states and minimum in the case of GHZ-type states. On the other hand, the degeneration rate for dephasing channel is minimum for GHZ-type states and maximum for general pure states. Depolarizing channel degrades all the three sets of states in a similar way. In case of collective dephasing channel, degeneration rate of the state with respect to system size is negligible for GHZ-type state, whereas it is very similar for general pure states and W-type states.

We would like to clarify that we have defined GHZ-type and W-type states in a certain way. This definition is not same as GHZ-class and W-class states which are the two inequivalent classes of maximally entangled states for three qubits. Our definition is motivated by the structure of superpositions involved in the original definition of GHZ and W states. For example, for two qubits for us the states \(\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)\) will
be GHZ-type, while \( \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle) \) will be W-type, although they are all equivalent to each other under local transformations.

We would like to stress that we have obtained state-independent conclusions by generating a large number of unbiased random state for each class of states that we studied. We hope that this study will help in the direction of understanding effect of non-unitary channels on different classes of states and their relative fragility for different numbers of qubits.

### Appendix A: Fidelity of W-type states under zero temperature bath

We provide here the derivation of the analytical expression for fidelity given in Eq. (10) for W-type states under the zero temperature bath. Let us consider orthogonal ket vectors with \( m \) qubits in state \( |1\rangle \) and \( n - m \) qubits in state \( |0\rangle \). This set of vectors forms a basis for the \( n \) qubit states if we considered all values of \( m \). For a given value of \( m \), we have \( \mathcal{P}^{(m)} = n!/(m!(n - m)!) \) permutations of \( m \) \( |1\rangle \)s and \( (n - m) \) \( |0\rangle \)s. W-type states for \( n \)-qubits defined in Eqn. (6) can be expanded in terms of subset of \( n \) vectors from this set with \( m = 1 \) as

\[
|\psi_W\rangle = \sum_{i=1}^{\mathcal{P}} \beta_i |0\rangle \otimes (n-1) \langle 1 | \mathcal{P}^{(1)}_i \tag{A1}
\]

where \( \beta_i \)s are the complex coefficients and \( \mathcal{P}^{(1)}_i \) represent the \( i \)th permutation. For instance, in a three-qubit W-type state, permutations \( \mathcal{P}^{(1)}_1 \), \( \mathcal{P}^{(1)}_2 \) and \( \mathcal{P}^{(1)}_3 \) correspond to basis vectors \( |001\rangle \), \( |010\rangle \) and \( |100\rangle \), respectively. The corresponding density matrix is given by

\[
\rho_W = \sum_{i,j=1}^{\mathcal{P}} \beta_i \beta_j^* |0\rangle \otimes (n-1) \langle 1 | \mathcal{P}^{(1)}_i \otimes (n-1) \langle 1 | \mathcal{P}^{(1)}_j \tag{A2}
\]

Under the influence of zero temperature bath, final density matrix can be expressed in the following form in the basis considered above:

\[
\rho_{W\text{out}} = e^{-(n-1)\gamma t} \sum_{m=1}^{n} (e^{\gamma t} - 1)^{(m-1)} \times \sum_{i,j=1}^{\mathcal{P}^{(m)}} C_{i,j} |0\rangle \otimes (n-m) \langle 1 | \mathcal{P}^{(m)}_i \otimes (n-m) \langle 1 | \mathcal{P}^{(m)}_j \tag{A3}
\]

where \( C_{i,j} \) are the complex coefficients. The final state expression contains clearly structured \( n \) blocks (labeled by \( m \)) of the basis vector with no cross-terms. With a view to calculate overlap of the final state with the initial W-type state, we explicitly
segregate the terms keeping \( m = 1 \) case separate as follows:

\[
\rho_{\text{out}}^{\text{W}} = e^{-(n-1)\gamma t} \sum_{k,l=1}^{\mathcal{P}(1)} \beta_k \beta_l^* |0\otimes(n-1)1\rangle \mathcal{P}_k^{(1)} \langle 0\otimes(n-1)1| \mathcal{P}_l^{(1)} + e^{-(n-1)\gamma t} \sum_{m=2}^{n} (e^{\gamma t} - 1)^{(m-1)} \\
\times \sum_{i,j=1}^{\mathcal{P}(m)} C_{i,j} |0\otimes(n-m)1\otimes(m)\rangle \mathcal{P}_i^{(m)} \langle 0\otimes(n-m)1\otimes(m)| \mathcal{P}_j^{(m)}.
\]

(A4)

Given this expression where the first term is W-type state, the fidelity of final state \((\rho_{\text{out}}^{\text{W}})\) with respect to the initial pure state \(|\psi_{\text{W}}\rangle\) can be calculated readily as

\[
F = \langle \psi_{\text{W}} | \rho_{\text{out}}^{\text{W}} | \psi_{\text{W}} \rangle = e^{-(n-1)\gamma t} \sum_{i,j=1}^{n} |\beta_i \beta_j^*|^2.
\]

(A5)

Since \( \sum_{i,j=1}^{n} |\beta_i \beta_j^*|^2 = Tr(\rho_{\text{W}}^2) = 1 \), we have

\[
F = e^{-(n-1)\gamma t}
\]

(A6)

Thus, the fidelity of W-type states as they evolve in an environment comprising of a zero temperature bath turns out to be independent of the initial state and depends only on the number of qubits involved.

References

1. Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Quantum Information, Quantum Computation, Cryptography. Cambridge University Press, Cambridge (2000)
2. Zeilinger, A., Bouwmeester, D.: Ekert, Artur: The Physics of Quantum Information. Springer, Berlin (2000)
3. Redfield, A.G.: On the theory of relaxation processes. IBM J. Res. Dev. 1, 19–31 (1957)
4. Feynman, R.P., Vernon, F.L.: The theory of a general quantum system interacting with a linear dissipative system. Ann. Phys. 24, 118–173 (1963)
5. Caldeira, A.O., Leggett, A.J.: Path integral approach to quantum Brownian motion. Phys. A: Stat. Mech. Appl. 121, 587–616 (1983)
6. Gardiner, C.W., Zoller, P.: Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics, Springer Series in Synergetics, 3rd edn. Springer, Berlin, Heidelberg (2004)
7. Breuer, H.-P., Petruccione, F.: The Theory of Open Quantum Systems. Oxford University Press, Oxford (2007)
8. Mooij, J.E., Orlando, T.P., Levitov, L., Tian, L., van der Wal, C.H., Lloyd, S.: Josephson persistent-current qubit. Science 285, 1036–1039 (1999)
9. Loss, D., DiVincenzo, D.P.: Quantum computation with quantum dots. Phys. Rev. A 57, 120–126 (1998)
10. Cirac, J.I., Zoller, P.: Quantum computation with cold trapped ions. Phys. Rev. Lett. 74, 4091–4094 (1995)
11. Kane, B.: A silicon-based nuclear spin computer. Nature (London) 393, 133 (1998)
12. Recher, P., Trauzettel, B.: Quantum dots and spin qubits in graphene. Nanotechnology 21, 302001 (2010). https://doi.org/10.1088/0957-4484/21/30/302001
13. Jones, J.A.: Quantum computing with NMR. Prog. Nucl. Magn. Reson. Spectrosc. 59, 91–120 (2011)
14. Ladd, T.D., Jelezko, F., Laflamme, R., Nakamura, Y., Monroe, C., O'Brien, J.L.: Quantum computers. Nature 464, 45 (2010)
15. Barreiro, J.T., Schindler, P., Ghne, O., Monz, T., Chwalla, M., Roos, C.F., Henrich, M., Blatt, R.: Experimental multiparticle entanglement dynamics induced by decoherence. Nat. Phys. 6, 943–946 (2010)
16. Liu, Z., Lyyra, H., Sun, Y., et al.: Experimental implementation of fully controlled dephasing dynamics and synthetic spectral densities. Nat. Commun. 9, 3453 (2018)
17. Burnett, J.J., Bengtsson, A., Scigliuzzo, M., Niepce, D., Kudra, M., Delsing, P., Bylander, J.: Decoherence benchmarking of superconducting qubits. NPJ Quant. Inf. 5, 54 (2019)
18. Ficheux, Q., Jezouin, S., Leghtas, Z., Huard, B.: Dynamics of a qubit while simultaneously monitoring its relaxation and dephasing. Nat. Commun. 9, 1926 (2018)
19. Singh, H., Arvind, Dorai, K.: Using a lindbladian approach to model decoherence in two coupled nuclear spins via correlated phase-damping and amplitude damping noise channels (2020). arXiv:2007.12972
20. Singh, H., Arvind, Dorai, K.: Evolution of tripartite entangled states in a decohering environment and their experimental protection using dynamical decoupling. Phys. Rev. A 97, 022302 (2018)
21. Singh, H., Arvind, Dorai, K.: Experimentally freezing quantum discord in a dephasing environment using dynamical decoupling. EPL (Europhys. Lett.) 118, 50001 (2017)
22. Gühne, O., Bodoky, F., Blaauuber, M.: Multiparticle entanglement under the influence of decoherence. Phys. Rev. A 78, 060301 (2008)
23. Borras, A., Majtey, A.P., Plastino, A.R., Casas, M., Plastino, A.: Robustness of highly entangled multiqubit states under decoherence. Phys. Rev. A 79, 022108 (2009)
24. Aolita, L., Cavalcanti, D., Acín, A., Salles, A., Tiersch, M., Buchleitner, A., de Melo, F.: Scalability of Greenberger–Horne–Zeilinger and random-state entanglement in the presence of decoherence. Phys. Rev. A 79, 032322 (2009)
25. Berrada, K., Eleuch, H., Hassouni, Y.: Asymptotic dynamics of quantum discord in open quantum systems. J. Phys. B: Atom., Mol. Opt. Phys. 44, 145503 (2011)
26. Ali, M., Gühne, O.: Robustness of multiparticle entanglement: specific entanglement classes and random states. J. Phys. B 47, 055503 (2014)
27. Földi, P., Benedict, M.G., Czirják, A., Molnár, B.: Decoherence of wave packets in an anharmonic oscillator. Phys. Rev. A 67, 032104 (2003)
28. Kaur, M., Arora, B., Arvind, A.: Effect of dissipative environment on collapses and revivals of a non-linear quantum oscillator. Eur. Phys. J. D 72, 136 (2018)
29. Joos, E., Zeh, H.D., Kiefer, C., Giuliini, D.J.W., Kupsch, J., Stamatescu, I.-O.: Decoherence and the Appearance of a Classical World in Quantum Theory, 2nd edn. Springer, Berlin, Heidelberg (2003)
30. Kraus, K.: General state changes in quantum theory. Ann. Phys. 64, 311–335 (1971)
31. Kraus, K.: States, Effects and Operations: Fundamental Notions of Quantum Theory. Lecture Notes in Physics, Quantum Information, Quantum Computation, Cryptography, vol. 190. Springer, New York (1983)
32. Arsenijevi, M., Jekni-Dugi, J., Dugi, M.: Kraus operators for a pair of interacting qubits: a case study. Braz. J. Phys. 48, 242–248 (2018)
33. Lindblad, G.: On the generators of quantum dynamical semigroups. Commun. Math. Phys. 48, 119–130 (1976)
34. Gorini, V., Kossakowski, A., Sudarshan, E.C.G.: Completely positive dynamical semigroups of N-level systems. J. Math. Phys. 17, 821–825 (1976)
35. Carvalho, A.R.R., Mintert, F., Buchleitner, A.: Decoherence and multipartite entanglement. Phys. Rev. Lett. 93, 230501 (2004)
36. Mintert, F., Carvalho, A.R.R., Kus, M., Buchleitner, A.: Measures and dynamics of entangled states. Phys. Rep. 415, 207–259 (2005)
37. Roszak, K., Machnikowski, P.: Complete disentanglement by partial pure dephasing. Phys. Rev. A 73, 022313–022319 (2006). arXiv:quant-ph/0507027

38. Ann, K., Jaeger, G.: Disentanglement and decoherence in two-spin and three-spin systems under dephasing. Phys. Rev. B 75, 115307 (2007)

39. Cai, J.-M., Zhou, Z.-W., Guo, G.-C.: Stability of pairwise entanglement in a decoherent environment. Phys. Rev. A 72, 022312–0221318 (2005). arXiv:quant-ph/05040221

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.