Universal complementarity between coherence and intrinsic concurrence for two-qubit states

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Abstract

Entanglement and coherence are two essential quantum resources for quantum information processing. A natural question arises of whether there is a direct link between them. In this work, we propose a definition of intrinsic concurrence for two-qubit states. Although the intrinsic concurrence is not a measure of entanglement, it embodies the concurrence of four pure states which are members of a special pure state ensemble for an arbitrary two-qubit state. And we show that intrinsic concurrence is always complementary to first-order coherence. In fact, this relation is an extension of the complementary relation satisfied by two-qubit pure states. Interestingly, we apply the complementary relation in some composite systems composed by a single-qubit state coupling with four typical noise channels respectively, and discover their mutual transformation relation between concurrence and first-order coherence. This universal complementarity provides reliable theoretical basis for the interconversion of the two important quantum resources.

1. Introduction

Entanglement and coherence represent two crucial natural properties which are widely applied to quantum information processing and computation [1]. For a physical system, commonly used entanglement measures mainly consider correlations between their subsystems, whereas we usually think the physical system as a whole in the research of coherence omitting its structure [2]. Entanglement as one of earlier resources is a crucial ingredient for various quantum information processing protocols [3], such as remote state preparation [4, 5], quantum teleportation [6], super-dense coding [7] and so on. With the development of the entanglement measures, entanglement of formation [8], concurrence [9], relative entropy of entanglement [10] and negativity [11] have been proposed. Although entanglement can be measured by a variety of methods, there exist intrinsic relations between them. For instance, a functional relation between the entanglement of formation and concurrence has been put forward by [9].

On the other hand, coherence is a consequence of the superposition of quantum states, which can be used to characterize the interference capability of interaction fields. But in quantum physics, coherence can be known as the entanglement or correlation when it is further broadened to that between two or more subsystems [3]. Svozik et al [12] investigated relation between first-order coherence and Clauser–Horne–Shimony–Holt Bell’s-like inequality in 2015. They showed that the classical coherence of a given subsystem can be converted to the quantum correlations between subsystems for multipartite quantum systems. Cernoch et al experimentally verified the conservation of the maximally accessible first-order coherence while it migrated between classical coherence and quantum correlations [13].

Now that both entanglement and coherence are characterized by the resource theory, the understanding of common evolution of coherence and entanglement will be crucial. In particular, the researches of the intrinsic relations hidden in these resources have been made in recent years [14, 15]. Since the chosen types of resources and measure approaches are various, there exist distinct differences for these intrinsic relations among the quantum states. Thus, the main goal of our research is how to obtain a universal intrinsic relation. In this paper,
we accomplish two main tasks: we put forward a definition of intrinsic concurrence for a general two-qubit state; its intrinsic concurrence can be complementary to its coherence.

The remainder of this paper is organized as follows. In section 2, we review the quantification of first-order coherence and concurrence. In section 3, we provide detailed proofs of the existence about a special pure state decomposition and put forward a definition of intrinsic concurrence. In section 4, we give detailed proofs of the complementarity and describe the mutual transformation of coherence and intrinsic concurrence. In section 5, we give out the unified complementary relation of single-qubit state in open systems. In final, we end up our paper with a brief conclusion.

2. Preliminaries

Concurrence is usually used as a measure for entanglement of two-qubit states [3, 16]. For a two-qubit pure state $|\psi\rangle$, its spin-flipped state is defined as $|\tilde{\psi}\rangle = (\sigma_2 \otimes \sigma_2)|\psi^\text{flip}\rangle$, where $|\psi^\text{flip}\rangle$ is the complex conjugate of $|\psi\rangle$ and $\sigma_2$ is the Pauli matrix

$$
\begin{pmatrix}
0 & -i \\
-i & 0
\end{pmatrix}.
$$

The concurrence is defined as [9]

$$
C(|\psi\rangle) = \langle \psi|\tilde{\psi}\rangle.
$$

For a general two-qubit state $\rho$, its spin-flipped density matrix $\tilde{\rho}$ can be expressed as

$$
\tilde{\rho} = (\sigma_2 \otimes \sigma_2)\rho^\text{flip}(\sigma_2 \otimes \sigma_2).
$$

The concurrence is defined by the convex-roof [17, 18] as follows

$$
C(\rho) = \min_{\{x_n\}} \sum_n x_n C(|\phi_n\rangle).
$$

The minimization is taken over all possible decompositions $\rho$ into pure states. An analytic solution of concurrence can be calculated [9]

$$
C(\rho) = \max \{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\},
$$

where $\lambda_n (n \in \{1, 2, 3, 4\})$ are the eigenvalues, in decreasing order, of the non-Hermitian matrix $\rho\tilde{\rho}$. The definition of concurrence is based on the convex-roof construction, and it is suitable for use in both pure states and mixed states [19–21].

A widely used measure of coherence in optical systems is the first-order coherence [22], which is similar with the degree of polarization coherence [23]. We consider a two-qubit state $\rho = \sum_{n=1}^4 p_n |\psi_n\rangle \langle \psi_n|$, composed of subsystems $A$ and $B$, where $p_1 \geq p_2 \geq p_3 \geq p_4 \geq 0$ and $\sum_{n=1}^4 p_n = 1$. This quantum state $\rho$ can be obtained by applying a unitary operation $V$ to the non-entangled state $\rho_A$. Here, the unitary operation $V$ contains the corresponding eigenvectors $|\psi_n\rangle$ and the state $\rho_A$ is a diagonal matrix with the eigenvalues $p_n$. In other words, the state $\rho$ can be generally written (spectral decomposition) as [1]

$$
\rho = V\rho_A V^\dagger.
$$

Each subsystem of the state $\rho$ is characterized by the reduced density matrix $\rho_A = \text{Tr}_B(\rho)$ and $\rho_B = \text{Tr}_A(\rho)$. The degree of first-order coherence of each subsystem can be given by [22]

$$
D_{A,B} = \sqrt{2\text{Tr}(\rho^2_{A,B}) - 1}.
$$

Therefore, a measure of coherence for both subsystems, when they are considered independently, has the following form [12]

$$
D = \frac{D_A^2 + D_B^2}{2}.
$$

When both subsystems are coherent, that is, $0 < D \leq 1$, while only if both subsystems show no coherence, $D = 0$.

3. Intrinsic concurrence

For a general two-qubit pure state $|\psi\rangle$, its concurrence is defined as [24]

$$
C(|\psi\rangle) = \sqrt{2[1 - \text{Tr}(\rho_2^B)]} = \sqrt{2[1 - \text{Tr}(\rho_1^B)]},
$$

where $\rho_A$ and $\rho_B$ are the reduced density matrix of the pure state $|\psi\rangle$. Combining the definition of the first-order coherence with equation (8), it is obvious that the complementary relation of the pure state $|\psi\rangle$ can be written as
\[ C^2(\langle \psi \rangle) + D^2(\langle \psi \rangle) = 1. \]  

(9)

However, for a general two-qubit mixed state, the square sum of these two quantities is not any longer a conserved quantity. It is well-known that a general two-qubit state \( \rho \) has many pure state ensembles and the minimization of equation (3) is taken over all possible decompositions \( \rho \) into pure states. In order to generalize the complementary relation from the pure state to the mixed state, we try to look for a pure state ensemble \( \{ q_n | \varphi_n \} \) of the state \( \rho \), i.e. \( \rho = \sum_n q_n | \varphi_n \rangle \langle \varphi_n | \), where the concurrence \( C(\langle \varphi_n \rangle) \) of pure states \( | \varphi_n \rangle \) and corresponding probabilities \( q_n \) satisfy the relation that \( D^2(\rho) + \sum_n q_n^2 C^2(\langle \varphi_n \rangle) \) is a conserved quantity.

Interestingly, we find that when the pure states \( | \varphi_n \rangle \) are the eigenvectors of the non-Hermitian matrix \( \rho \), the quantity \( D^2(\rho) + \sum_n q_n^2 C^2(\langle \varphi_n \rangle) \) is indeed a conserved quantity. We will give a detailed proof of this conservation relation in section 4.

Similar to equation (2), we consider the spin-flipped operator \( \tilde{F} \) corresponding to a Hermitian operator \( F \), whose order is 2n, defined as

\[ \tilde{F} = \sigma_2^\otimes n F^* \sigma_2^\otimes n, \]  

(10)

where \( F^* \) is the complex conjugate of \( F \).

First of all, we introduce some peculiarities about spin-flipped operators. These properties provide a solid basis for the proof of two theories that we are going to talk about. Obviously, the spin-flipped operator \( \tilde{F} \) satisfies the Hermitian property. If the Hermitian operator \( F \) can be written as the form \( F = F_1F_2 \), then the corresponding spin-flipped operator \( \tilde{F} \) has a similar form

\[ \tilde{F} = \sigma_2^\otimes n F_1^* F_2^* \sigma_2^\otimes n = (\sigma_2^\otimes n F_1^* \sigma_2^\otimes n)(\sigma_2^\otimes n F_2^* \sigma_2^\otimes n) = \tilde{F}_1\tilde{F}_2. \]  

(11)

For the Pauli operators \( \sigma_n \), one obtains some special properties

\[ \sigma_i = \sigma_2^\otimes_n \sigma_2 = - \sigma_n, \]  

(12)

where \( i \in \{ 1, \ 2, \ 3 \} \). This indicates that the spin-flipped state \( \tilde{\rho}_n \), related to the single qubit state \( \rho_n \), is reversed with the state \( \rho_n \) in the Bloch sphere space.

And then, in order to prove the fact that there exists the pure state ensemble \( \{ q_n | \varphi_n \} \) for the state \( \rho \), whose pure states \( | \varphi_n \rangle \) are the eigenvectors of the non-Hermitian matrix \( \rho \), we put forward two theorems.

**Theorem 1.** If a two-qubit state \( \rho \) has a pure state decomposition \( \rho = \sum_{n=1}^4 q_n | \varphi_n \rangle \langle \varphi_n | \) and these pure states \( | \varphi_n \rangle \) satisfy the tilde orthogonal relation \( \langle \varphi_m | \varphi_n \rangle = \delta_{mn} \langle \varphi_m | \varphi_n \rangle \), then the eigenvectors of the non-Hermitian matrix \( \rho \) will be \( | \varphi_n \rangle \) and the corresponding eigenvalues will be expressed as \( \lambda_n = q_n^2 \ C^2(\langle \varphi_n \rangle) \).

**Proof of theorem 1.** Assuming that the two-qubit state \( \rho \) has a pure state decomposition

\[ \rho = \sum_{n=1}^4 q_n | \varphi_n \rangle \langle \varphi_n | \]  

(13)

where these pure states \( | \varphi_n \rangle \) satisfy the tilde orthogonal relation

\[ \langle \varphi_m | \varphi_n \rangle = \delta_{mn} \langle \varphi_m | \varphi_n \rangle. \]  

(14)

Then, combining equation (2), we obtain that the non-Hermitian matrix \( \rho \) can be given by

\[ \rho \rho^* = \sum_{m=1}^4 q_m | \varphi_m \rangle \langle \varphi_m | \sum_{n=1}^4 q_n | \varphi_n \rangle \langle \varphi_n | \]  

(15)

Hence, one obtain that the eigenvalue-eigenvector equation of the non-Hermitian matrix \( \rho \) has the form

\[ \rho^* | \varphi_n \rangle = \sum_{m=1}^4 q_n^2 \langle \varphi_m | \varphi_n \rangle (\varphi_m | \varphi_n \rangle) = q_n^2 \delta_{nm} | \varphi_n \rangle \]  

(16)

**Theorem 2.** For a general two-qubit state \( \rho \), if the eigenvalue-eigenvector equation of the non-Hermitian matrix \( \rho^* \) has the form \( \rho^* | \varphi_n \rangle = \lambda_n | \varphi_n \rangle \), then these eigenvectors \( | \varphi_n \rangle \) will satisfy the tilde orthogonal relation \( \langle \varphi_m | \varphi_n \rangle = \delta_{nm} \langle \varphi_m | \varphi_n \rangle \) and the state \( \rho \) will have a pure state decomposition \( \rho = \sum_{n=1}^4 q_n | \varphi_n \rangle \langle \varphi_n | \) where \( q_n \) satisfy the relation \( \lambda_n = q_n^2 \ C^2(\langle \varphi_n \rangle) \).

**Proof of theorem 2.** The eigenvalue-eigenvector equation of the non-Hermitian matrix \( \rho \) about the state \( \rho \) can be expressed as

\[ \rho^* | \varphi_n \rangle = \lambda_n | \varphi_n \rangle. \]  

(17)
Then let us apply spin-flip to both sides of the equation (16), one can obtain
\[
\tilde{\rho} \tilde{\varphi}_n = \lambda_n \varphi_n.
\] (18)

Obviously, the non-Hermitian matrix $\tilde{\rho}$ is the Hermitian conjugate of the non-Hermitian matrix $\rho \tilde{\rho}$, because of
\[
(\rho \tilde{\rho})^\dagger = \rho^\dagger \tilde{\rho} = \rho \tilde{\rho}.
\] (19)

Therefore, we obtain that the eigenvalue spectral decomposition of the non-Hermitian matrix $\rho \tilde{\rho}$ can be expressed as [25]
\[
\rho \tilde{\rho} = \sum_{n=1}^{4} \lambda_n \varphi_n \langle \varphi_n | \varphi_n \rangle,
\] (20)

and these eigenvectors $| \varphi_n \rangle$ satisfy the tilde orthogonal relation $\langle \varphi_m | \varphi_n \rangle = \delta_{mn}$. And then there is a special decomposition of the state $\rho$
\[
\rho = \sum_{n=1}^{4} q_n | \varphi_n \rangle \langle \varphi_n |,
\] (21)

where $q_n$ satisfy the relation
\[
\lambda_n = q_n^2 C^2(| \varphi_n \rangle).
\] (22)

The theorems 1 and 2 reveal the fact that there exists the pure state ensemble $\{ | q_n \rangle, | \varphi_n \rangle \}$ for the state $\rho$, whose pure states $| \varphi_n \rangle$ are the eigenvectors of the non-Hermitian matrix $\rho \tilde{\rho}$. And this two theorems provide a special pure state decomposition $\rho = \sum_{n=1}^{4} q_n | \varphi_n \rangle \langle \varphi_n |$ method for a two-qubit state $\rho$, which provides a convenience for us to solve the pure state decomposition.

And next, we give the definition of the intrinsic concurrence. For a general two-qubit pure state $| \psi \rangle$, its intrinsic concurrence is defined as
\[
C_I (| \psi \rangle) = \frac{1}{4} \sqrt{\sum_{n=1}^{4} q_n^2 C^2(| \varphi_n \rangle)},
\] (23)

where these pure states $| \varphi_n \rangle$ are the eigenvectors of the non-Hermitian matrix $\rho \tilde{\rho}$ and $q_n$ are the corresponding probabilities.

Finally, we obtain three properties about the intrinsic concurrence, which indicate the rough relation between the concurrence and the intrinsic concurrence.

**Property 1.** For an arbitrary two-qubit pure state $| \psi \rangle$, there is an equivalence relation between its concurrence and intrinsic concurrence. The relation can be expressed as
\[
C_I (| \psi \rangle) = C (| \psi \rangle).
\] (24)

**Proof.** For an arbitrary two-qubit pure state $| \psi \rangle$, its pure state decomposition is just itself. Therefore $C_I (| \psi \rangle) = C (| \psi \rangle)$.

**Property 2.** For an arbitrary two-qubit state $\rho$, there is a relation between its intrinsic concurrence and the quantity $\text{Tr} (\rho \tilde{\rho})$. The formula can be expressed as
\[
C_I^2 (\rho) = \text{Tr} (\rho \tilde{\rho}).
\] (25)

**Proof.** For an arbitrary two-qubit state $\rho$, its quantity $\text{Tr} (\rho \tilde{\rho}) = \sum_{n=1}^{4} \lambda_n$, where $\lambda_n (n \in \{ 1, 2, 3, 4 \})$ are the eigenvalues, in decreasing order, of the non-Hermitian matrix $\rho \tilde{\rho}$. Combining equations (22) and (23), one obtain $\text{Tr} (\rho \tilde{\rho}) = C_I^2 (\rho)$.

**Property 3.** For a general two-qubit state $\rho$, there is a lower bound of its intrinsic concurrence. And the lower bound is its concurrence. The inequality about its concurrence and intrinsic concurrence can be written as
\[
C_I (\rho) \geq C (\rho).
\] (26)

**Proof.** Combining equations (4) and (25), one obtain
\[
C_I (\rho) = \sqrt{\text{Tr} (\rho \tilde{\rho})} = \sqrt{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4} \geq \sqrt{\lambda_1} \geq \max \{ 0, \sqrt{\lambda_4} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \} = C (\rho).
\] (27)

According to the *proof of the property 3*, one obtain an inference that the necessary and sufficient condition of $C_I (\rho) = C (\rho)$ is $R (\rho \tilde{\rho}) \in \{ 0, 1 \}$, where $R (\rho \tilde{\rho})$ is the rank of the non-Hermitian matrix $\rho \tilde{\rho}$. 


4. Complementary relation between intrinsic concurrence and first-order coherence

In this section, we introduce the complementary relation between the intrinsic concurrence and the first-order coherence.

**Theorem 3.** For a general two-qubit state $\rho$, there is a complementary relation

$$C_I^2(\rho) + D^2(\rho) = \text{Tr}(\rho^2).$$

**Proof of theorem 3.** In general, a two-qubit state $\rho$ is denoted as

$$\rho = \frac{1}{4}[\mathbb{I} \otimes \mathbb{I} + (\vec{A} \cdot \vec{\sigma}) \otimes \mathbb{I} + \mathbb{I} \otimes (\vec{B} \cdot \vec{\sigma}) + \sum_{m,n=1}^{3} T_{mn} \sigma_m \otimes \sigma_n],$$

where $\mathbb{I}$ stands for identity operator of single qubit, $\sigma_n$ stand for three Pauli operators, $\vec{A} = (a_1, a_2, a_3)$ and $\vec{B} = (b_1, b_2, b_3)$ are vectors in $\mathbb{R}^3$ and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. Just to make it easy to calculate, we rewrite the state $\rho$ as

$$\rho = \rho_A \otimes \rho_B + \frac{1}{4} \sum_{m,n=1}^{3} (T_{mn} - a_m b_n) \sigma_m \otimes \sigma_n,$$

where $\rho_A = \text{Tr}_B(\rho)$ and $\rho_B = \text{Tr}_A(\rho)$. According to equation (6), we obtain that the coherence of each subsystem has the following form

$$D_K = \sqrt{2 \text{Tr}(\rho_K^2) - 1} = |\vec{K}|,$$

where $K \in \{A, B\}$. Therefore, combining equations (7) and (31), one obtains that its first-order coherence can be given by the following formula

$$D(\rho) = \sqrt{\frac{|\vec{A}|^2 + |\vec{B}|^2}{2}}.$$

It is obvious that the spin-flipped operator $\tilde{\rho}$ corresponding to the state $\rho$ can be expressed as

$$\tilde{\rho} = \tilde{\rho}_A \otimes \tilde{\rho}_B + \frac{1}{4} \sum_{m,n=1}^{3} (T_{mn} - a_m b_n) \sigma_m \otimes \sigma_n,$$

where $\tilde{\rho}_A = \sigma_1 \rho_A \sigma_2 = \frac{1}{2}(1 - \vec{A} \cdot \vec{\sigma})$ and $\tilde{\rho}_B = \sigma_2 \rho_B \sigma_3 = \frac{1}{2}(1 - \vec{B} \cdot \vec{\sigma})$. Therefore, for each subsystem, the definition of the first-order coherence can be rewritten as

$$D_K = |\vec{K}| = \sqrt{\text{Tr}[\rho(\vec{K} \cdot \vec{\sigma})]} = \sqrt{\text{Tr}[\rho_K (\vec{K} \cdot \vec{\sigma})]}.$$

Similarly, for the whole system, the definition of the first-order coherence can be rewritten as

$$D(\rho) = \sqrt{\frac{|\vec{A}|^2 + |\vec{B}|^2}{2}} = \sqrt{\text{Tr}[\rho(\vec{A} \cdot \vec{\sigma} \otimes \mathbb{I})] + \text{Tr}[\rho(\mathbb{I} \otimes \vec{B} \cdot \vec{\sigma})]} = \sqrt{\text{Tr}[\rho(\rho_A \otimes \rho_B - \tilde{\rho}_A \otimes \tilde{\rho}_B + \tilde{\rho})]}.\quad (35)$$

Therefore, the square sum of its first-order coherence and intrinsic concurrence is a conserved quantity. The derivative process can be described as follows

$$C_I^2(\rho) + D^2(\rho) = \text{Tr}(\rho \tilde{\rho}) + \text{Tr}[\rho(\rho_A \otimes \rho_B - \tilde{\rho}_A \otimes \tilde{\rho}_B)] = \text{Tr}[\rho(\rho_A \otimes \rho_B - \tilde{\rho}_A \otimes \tilde{\rho}_B + \tilde{\rho})] = \text{Tr}(\rho^2).\quad (36)$$

Note that, the theorem 3 reveals that for a general two-qubit state $\rho$, its first-order coherence $D(\rho)$ and intrinsic concurrence $C_I(\rho)$ are a pair of complementary quantities. And it shows that if the evolution of the whole system is unitary, there is a mutual transformation relationship between its first-order coherence and intrinsic concurrence. Combining with equation (23), we can rewrite the complementarity as

$$D^2(\rho) + \sum_{n=1}^{4} q_n^2 C^2(\rho_n) = \text{Tr}(\rho^2).\quad (37)$$

According to the property 3, one obtain an inference: if the state $\rho$ satisfies condition $R(\rho \tilde{\rho}) \in \{0, 1\}$, then this complementarity can be rewritten as

$$D^2(\rho) + C^2(\rho) = \text{Tr}(\rho^2).\quad (38)$$

For a general two-qubit pure state, its coherence and concurrence satisfy the complementarity ($q_1 = 1$, $q_2 = q_3 = q_4 = 0$) for equation (37), i.e. that of equation (9). If the mixed state $\rho$ satisfies condition $R(\rho \tilde{\rho}) = 1$, then its coherence and concurrence satisfy the complementarity that of equation (38). For a general two-qubit
mixed state \( \rho \), the coherence of the state \( \rho \) and the concurrence of the four pure states \( |\varphi_i\rangle \) satisfy the complementarity.

### 5. The complementary relation of quantum state in open systems

In fact, a quantum system is inevitably coupled with the surrounding environment, and quantum resources are constantly exchanged between the quantum system and the environment. In this section, we consider only the interactions between a single qubit state \( \rho_A = \frac{1}{2}(1 + \tilde{A} \cdot \varnothing) \) and the environment, and the dimension of the environment is the same as the dimension of the particle \( A \). Here, \( \tilde{A} = (a_1, a_2, a_3) \) and \( \varnothing = (\sigma_1, \sigma_2, \sigma_3) \). In order to obtain the complementary relation between coherence and concurrence, we assume that the initial state of the environment is \( |\varphi_0\rangle \langle \varphi_0| \). Then the state of the whole quantum system at the initial time (IT) can be described as \( \rho_{IT} = \rho_A \otimes |\varphi_0\rangle \langle \varphi_0| \). And its evolution state \( \rho \) can be described by unitary evolution operator \( U\hat{1} U^\dagger \). Thus, the state \( \varepsilon(\rho_A) \) of the particle \( A \) can be rewritten as

\[
\varepsilon(\rho_A) = \sum_{k=0}^{l} |e_k| \langle e_k| e_k \rangle = \sum_{k=0}^{l} E_k \rho_A E_k^\dagger,
\]

where \( E_k = |e_k\rangle \langle e_k| \) \((k \in \{0, 1\})\) are the Kraus operators of the environment.

We will discuss only a few common channels, namely amplitude damped (AD) channel, bit flip (BF) channel, bit-phase flip (BPF) channel, and phase flip (PF) channel. The Kraus operators \( E_k \) and unitary evolution operator \( U \) corresponding to these channels are given in the table 1, respectively. Here, the parameter \( p = 1 - e^{-\gamma t} \) is the strength of the noise, where \( \gamma \) is the decay factor and \( t \) represents the time the particle \( A \) is in the environment. Therefore one can obtain the dynamics of coherence, concurrence and intrinsic concurrence for the evolution states \( \rho \) of these quantum systems (see the figure 1). From these figures, we find that the concurrence \( C(\rho) \) of the composite system \( \rho \) will be increased from the decrease of its coherence \( D(\rho) \).

For the state \( \rho \), we obtain that the ratio \( S(\rho) = \frac{C(\rho)}{\sqrt{\rho}} \) is independent of the parameter \( p \). And the rank \( R(\rho \varnothing) \) of the non-Hermitian matrix \( \rho \varnothing \) and the ratio \( S(\rho) \) are given in the table 2. Therefore, the complementary relation of the whole system \( \rho \), can be rewritten as

\[
D^2(\rho) + S^2(\rho) C^2(\rho) = \frac{1 + |\tilde{A}|^2}{2}.
\]

The above equation (40) expresses that when the state \( \rho_A \) is coupled with these channels (AD, BF, BPF and PF) respectively, the decrease of the first-order coherence function \( D^2(\rho) \) can be converted to the concurrence function \( C^2(\rho) \) with a conversion efficiency \( \frac{1}{S(\rho)} \).
In this paper, we have completed three tasks about the complementary relation between intrinsic concurrence and first-order coherence. At first, we show that there is a special pure state ensemble \( \{ \rho_i, |\phi_i^> \} \) for an arbitrary two-qubit mixed state \( \rho \), and put forward the definition of intrinsic concurrence. Next, we establish a universal relation that intrinsic concurrence can be complementary to the first-order coherence. This complementarity reveals the transformation between coherence of the state \( \rho \) and concurrence of the pure state ensemble \( \{ \rho_i, |\phi_i^> \} \), i.e. equation (37). Finally, as an application, we consider the interactions between an arbitrary single qubit state and environment, and give out the unified complementary relation between coherence and entanglement for the compound system which is formed by coupling the single-qubit state and the environment. Quantum entanglement of the compound system will be increased at the cost of coherence. In the other words, a single-qubit system with coherence under noisy channels can generate entanglement of total qubit state and environment, and give out the uni

| Channel type          | AD channel | BF channel | BPF channel | PF channel |
|-----------------------|------------|------------|-------------|------------|
| Rank \( R(\rho) \)    | \( R(\rho) = 1 \) | \( R(\rho) = 2 \) | \( R(\rho) = 2 \) | \( R(\rho) = 2 \) |
| Ratio \( S(\rho) \)    | \( S(\rho) = 1 \) | \( S(\rho) = 2 \) | \( S(\rho) = 2 \) | \( S(\rho) = 2 \) |

Figure 1. The coherence \( D(\rho) \), concurrence \( C(\rho) \) and intrinsic concurrence \( C_i(\rho) \) versus the parameter \( p \) for the composite system \( \rho \), which is formed by coupling the single qubit state \( \rho_s = \frac{1}{2}(1 + \hat{A} \cdot \hat{\sigma}) \) with the amplitude damped (AD) channel, bit flip (BF) channel, bit–phase flip (BPF) channel and phase flip (PF) channel respectively, where \( \hat{A} = (0.50, \ 0.61, \ 0.16) \).

6. Conclusions

In this paper, we have completed three tasks about the complementary relation between intrinsic concurrence and first-order coherence. At first, we show that there is a special pure state ensemble \( \{ \rho_i, |\phi_i^> \} \) for an arbitrary two-qubit mixed state \( \rho \), and put forward the definition of intrinsic concurrence. Next, we establish a universal relation that intrinsic concurrence can be complementary to the first-order coherence. This complementarity reveals the transformation between coherence of the state \( \rho \) and concurrence of the pure state ensemble \( \{ \rho_i, |\phi_i^> \} \), i.e. equation (37). Finally, as an application, we consider the interactions between an arbitrary single qubit state and environment, and give out the unified complementary relation between coherence and entanglement for the compound system which is formed by coupling the single-qubit state and the environment. Quantum entanglement of the compound system will be increased at the cost of coherence. In the other words, a single-qubit system with coherence under noisy channels can generate entanglement of total system. We hope that these results will find some interesting applications in controlling interconversion of coherence and entanglement for a two-qubit system.

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