NEW STATISTICAL SYMMETRIES OF THE MULTI-POINT EQUATIONS AND ITS IMPORTANCE FOR TURBULENT SCALING LAWS

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ABSTRACT. We presently show that the infinite set of multi-point correlation equations, which are direct statistical consequences of the Navier-Stokes equations, admit a rather large set of Lie symmetry groups. This set is considerably extended compared to the set of groups which are implied from the original set of equations of fluid mechanics. Specifically a new scaling group and translational groups of the correlation vectors and all independent variables have been discovered. These new statistical groups have important consequences on our understanding of turbulent scaling laws to be exemplarily revealed by two examples. Firstly, one of the key foundations of statistical turbulence theory is the universal law of the wall with its essential ingredient is the logarithmic law. We demonstrate that the log-law fundamentally relies on one of the new translational groups. Second, we demonstrate that the recently discovered exponential decay law of isotropic turbulence generated by fractal grids is only admissible with the new statistical scaling symmetry. It may not be borne from the two classical scaling groups implied by the fundamental equations of fluid motion and which has dictated our understanding of turbulence decay since the early thirties implicated by the von-Kármán-Howarth equation.

1. Introduction. Although there has been considerable progress in theory as well as in experiments, turbulence still is one of the most important and most challenging unresolved problems in classic mechanics. It has indirectly been chosen as one of the seven Millennium problems of Mathematics (with six still unresolved) since existence and smoothness of Navier-Stokes equations have been posed as one of them.

The special importance of turbulence is determined by its presence in numerous natural and technical systems. Examples for natural turbulent flows are atmospheric flow and oceanic current which to calculate is a crucial point in climate research. Only with the advent of super computers it became apparent that the Navier-Stokes equations provide a very good continuum mechanical model for turbulent flows. Still, the application of Navier-Stokes equations to practical flow
problems at high Reynolds number without invoking any extra assumptions is still several decades away.

However, in most applications it is not at all necessary to know all the detailed fluctuating of velocity and pressure present in a turbulent flows but for the most part statistical measures would be sufficient.

This was in fact the key idea of O. Reynolds who was the first to suggest a statistical description of turbulence. The Navier-Stokes equations, however, constitute a non-linear and, due to the pressure Poisson equation, a non-local set of equations. As an immediate consequence of this the equations for the mean values leads to an infinite set of statistical equations, or, if truncated at some level of statistics, an un-closed system is generated.

In order to obtain at least some insight into the statistical behavior of turbulence we presently apply Lie group theory to the full infinite set of statistical equations investigating two canonical turbulent flow situations.

2. Equations of statistical turbulence theory.

2.1. Navier-Stokes equations. The initial point of the entire analysis to follow is based on the three dimensional Navier-Stokes equations for an incompressible fluid under the assumption of a Newtonian material with constant density and viscosity. In Cartesian tensor notation we have the continuity equation

$$\frac{\partial U_k}{\partial x_k} = 0 \quad (1)$$

and the momentum equation writes

$$\frac{DU_i}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k}, \quad i = 1, 2, 3, \quad (2)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U_k \frac{\partial}{\partial x_k} \quad (3)$$

denotes the material derivative. \( t \in \mathbb{R}^+ \), \( x \in \mathbb{R}^3 \), \( U = U(x, t) \) and \( P = P(x, t) \) represent time, position vector, instantaneous velocity vector and pressure. The density \( \rho \) and the viscosity \( \nu \) are positive constants. Furthermore pressure can be normalized with the constant density. The new pressure term reduces to

$$P^* = \frac{P}{\rho}, \quad (4)$$

which, inserted into (2), leads to the modified momentum equation and the asterisk is omitted from here on

$$\mathcal{M}_i(x) = \frac{DU_i}{Dt} + \frac{\partial P}{\partial x_i} - \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} = 0, \quad i = 1, 2, 3, \quad (5)$$

where all terms have been collected on one side to further down deal with the \( \mathcal{M} \) notation.

2.2. Statistical averaging. In the following we define the classical Reynolds averaging. Though definitely mathematical more sound it is not intended to define the necessary multi-point probability density function to derive the multi-point equations in the subsequent section.

The quantity \( Z \) represents an arbitrary statistical variable, i.e. \( U \) and \( P \), which in the following we also denote as instantaneous value. According to the classic
definition by Reynolds all instantaneous quantities are decomposed into their mean and their fluctuation value

\[ Z = \bar{Z} + z. \]  

(6)

The overbar always marks a statistically averaged quantity whereas the lower-case \( z \) denotes the fluctuation value of \( Z \). The most general definition of statistically averaged quantity is given by an ensemble average operator \( \mathcal{K} \)

\[ Z = \bar{Z}(x, t) = \mathcal{K}[Z(x, t)] = \lim_{N \to \infty} \left( \frac{1}{N} \sum_{n=1}^{N} Z_n(x, t) \right). \]  

(7)

The definition of the mean value leads, according to the Reynolds decomposition, to the fluctuation value of \( Z \)

\[ z = Z - \bar{Z}. \]  

(8)

From the averaging operator above, several calculation rules may be derived e.g.

\[ \bar{z} = 0, \]
\[ \bar{Z} = Z, \]
\[ Z_1 + Z_2 = \bar{Z_1} + \bar{Z_2}, \]
\[ \frac{\partial \bar{Z}}{\partial s} = \frac{\partial Z}{\partial s}, \]
\[ \bar{Z}_1 \ldots \bar{Z}_m \bar{z}_n = 0, \]
\[ \bar{z}_1 \bar{z}_2 \ldots \bar{z}_k \neq 0 \text{ with } k > 1, \]  

(9)

which in the following sub-sections will be employed to derive the multi-point equations.

2.3. Reynolds averaged transport equations. After \( U \) and \( P \) are decomposed according to (6), we may apply \( \mathcal{K} \) to the continuity equation (1) and the momentum equation (5) leading to its averaged versions i.e. continuity equation

\[ \frac{\partial \bar{U}_k}{\partial x_k} = 0, \]  

(10)

and momentum equation

\[ \frac{\bar{D}U_i}{Dt} = -\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 \bar{U}_i}{\partial x_k \partial x_k} - \frac{\partial \bar{U}_i}{\partial x_k} \frac{\partial \bar{U}_k}{\partial x_k}, i = 1, 2, 3, \]  

(11)

where

\[ \frac{\bar{D}}{Dt} = \frac{\partial}{\partial t} + \bar{U}_k \frac{\partial}{\partial x_k} \]  

(12)

denotes the mean material derivative.

At this point we observe the well-known closure problem of turbulence since, compared to the original set of equations, the unknown Reynolds stress tensor \( \bar{u}_i u_k \) has appeared. However, rather different from the classical approach we will not proceed with deriving the Reynolds stress tensor transport equation which contains additional four unclosed tensors. Instead the multi-point correlation approach is put forward the reason being twofold.
First, if the infinite set of correlation equations is considered the closure problem is somewhat bypassed. Second, the multi-point correlation delivers additional information on the turbulence statistics such as length scale information which may not be gained from the Reynolds stress tensor, which is a single-point approach.

For this we need the equations for the fluctuating quantities \( u \) and \( p \) which are derived by taking the differences between the averaged and the non-averaged equations. The resulting fluctuation equations read

\[
\frac{\partial u_k}{\partial x_k} = 0 ,
\]

(13)

and

\[
N_i(x) = \frac{D u_i}{Dt} + u_k \frac{\partial \tilde{U}_i}{\partial x_k} + \frac{\partial u_i u_k}{\partial x_k} + \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} \quad i = 1, 2, 3 \quad .
\]

(14)

2.4. Multi-point correlation equations. The idea of two- and multi-point correlation equations in turbulence was presumably first established by Friedmann and Keller ([8]). In the beginning it was assumed that all correlation equations of orders higher than two may be neglected. Theoretical considerations led to the result that all higher correlations have to be taken into account. Consequently, all multi-point correlation equations have to be considered in the symmetry analysis to follow.

Two different sets of multi-point correlation (MPC) equations will be derived below. The first is based on the instantaneous values of \( \mathbf{U} \) and \( P \) while the second one is in accordance with the classical notation based on the fluctuating quantities \( u \) and \( p \).

2.4.1. MPC equations: instantaneous approach. In order to write the MPC equations in a very compact form, we introduce the following notation. The multi-point velocity correlation tensor of order \( n + 1 \) is defined as follows:

\[
H_{i(n+1)} = H_{i(0)\cdots i(n)} = \frac{\partial \tilde{U}_i}{\partial x_k} \cdots \frac{\partial \tilde{U}_i}{\partial x_k} U_{i(0)}(x_{(0)}) \cdots U_{i(n)}(x_{(n)}) ,
\]

(15)

where the first index of the \( \mathbf{H} \) tensor defines the tensor character of the term and the second index in braces denotes the order of the tensor. The curly brackets point out that not an index of a tensor but an enumeration is meant. It is important to mention that the indices start with 0 which is an advantage when introducing a new coordinate system based on the Euclidean distance of two space points. The value in curly brackets is the actual order of the tensor and takes into account that counting starts at zero. Apparently we have the connection to the mean velocity according to \( H_{i(1)} = H_{i(0)} = \bar{U}_i \).

In some cases the list of indices is interrupted by one or more other indices which is pointed out by attaching the replaced value in brackets to the index

\[
H_{i(n+1)}[i(0)\cdots k(l)] = \frac{\partial \tilde{U}_i}{\partial x_k} \cdots \frac{\partial \tilde{U}_i}{\partial x_k} U_{i(0)}(x_{(0)}) \cdots U_{i(l-1)}(x_{(l-1)}) U_{k(l)}(x_{(l)}) U_{i(l+1)}(x_{(l+1)}) \cdots U_{i(n)}(x_{(n)}) .
\]

(16)

This is further extended by

\[
H_{i(n+2)}[i(n+1)\cdots k(l)] = \frac{\partial \tilde{U}_i}{\partial x_k} \cdots \frac{\partial \tilde{U}_i}{\partial x_k} U_{i(0)}(x_{(0)}) \cdots U_{i(n)}(x_{(n)}) U_{k(l)}(x_{(l)}) ,
\]

(17)

where not only that index \( i(n+1) \) is replaced by \( k(l) \), but also that the independent variable \( x_{(n+1)} \) is replaced by \( x_{(l)} \). If indices are missing e.g. between \( i(l-1) \) and \( i(l+1) \) we define

\[
H_{i(n)}[i(l)\cdots] = \frac{\partial \tilde{U}_i}{\partial x_k} \cdots \frac{\partial \tilde{U}_i}{\partial x_k} U_{i(l-1)}(x_{(l-1)}) U_{i(l+1)}(x_{(l+1)}) \cdots U_{i(n)}(x_{(n)}) .
\]

(18)
Finally, if the pressure is involved we write
\[ I_{i(n+1)} = \overline{U_{i(0)}(x(0))} \cdots \overline{U_{i(n-1)}(x(1))} \overline{P(x(1))} \overline{U_{i(n+1)}(x(2))} \cdots \overline{U_{i(n)}}(x(n)) \]
(19)
which is, considering all the above definitions, sufficient to derive the MPC equations from the equations of instantaneous velocity and pressure i.e. equation (1) and (5).

Applying the Reynolds averaging operator (7) according to the sum below
\[ S_{i(n+1)}(x(0), \ldots, x(n)) = \mathcal{M}_{i(0)}(x(0))U_{i(1)}(x(1)) \cdots \overline{U_{i(n)}}(x(n)) \]
+ \[ \overline{U_{i(0)}(x(0))}\mathcal{M}_{i(1)}(x(1))U_{i(2)}(x(2)) \cdots \overline{U_{i(n)}}(x(n)) \]
+ \ldots
+ \[ \overline{U_{i(0)}(x(0))} \cdots \overline{U_{i(n-2)}(x(n-2))} \mathcal{M}_{i(n-1)}(x(n-1)) \overline{U_{i(n)}}(x(n)) \]
(20)
we obtain the \( S \)-equation which writes
\[ S_{i(n+1)}^l = \frac{\partial H_{i(n+1)}}{\partial t} + \sum_{l=0}^{n} \left[ \frac{\partial H_{i(n+2)}[t_{i(n+1)} \rightarrow k_{i(l)}]}{\partial x_k} \right] \frac{x(n+1) \rightarrow x(l)}{x_{k(l)}} \]
+ \[ \frac{\partial I_{i(n+1)}}{\partial x_k} \frac{\partial^2 H_{i(n+1)}}{\partial x_k \partial x_k} = 0 \]
(21)
for \( n = 1, \ldots, \infty \).

Loosely speaking equation (21) implies the statistical information of the Navier-Stokes equations at the expense to deal with an infinite dimensional chain of differential equations starting with order 2 i.e. \( n = 1 \). The rather remarkable consequence of the derivation is that (21) is a linear equation which considerably simplifies the finding of Lie symmetries to be pointed out below.

From equation (1) a continuity equation for \( H_{i(n+1)} \) and \( I_{i(n+1)} \) can be derived. This leads to
\[ \frac{\partial H_{i(n+1)}[t_{i(n+1)} \rightarrow k_{i(l)}]}{\partial x_k} = 0 \quad \text{for} \quad l = 0, \ldots, n \]
(22)
and
\[ \frac{\partial I_{i(n)}[t_{i(n)} \rightarrow m_{i(l)}]}{\partial x_k} = 0 \quad \text{for} \quad k, l = 0, \ldots, n \quad \text{and} \quad k \neq l . \]
(23)

At this point we adopt the classic notation of distance vectors. Accordingly the usual position vector \( x \) is employed and the remaining independent spatial variables are expressed as the difference of two position vectors \( x(l) \) and \( x(0) \). The coordinate transformation and the corresponding rules of derivation are
\[ x = x(0), \quad r(l) = x(l) - x(0) \quad \text{with} \quad l = 1, \ldots, n \]
(24)
and
\[ \frac{\partial}{\partial x_{k(0)}} = \frac{\partial}{\partial x_k} - \sum_{l=1}^{n} \frac{\partial}{\partial r_{k(l)}} , \quad \frac{\partial}{\partial x_{k(l)}} = \frac{\partial}{\partial r_{k(l)}} \quad \text{for} \quad l \geq 1 . \]
(25)

For consistency reasons the first index \( i(0) \) is replaced by \( i \). Thus the indices of the tensor \( H_{i(n+1)} \) are \( H_{i(1) \ldots i(n)} \). Using the rules of transformation (24) and (25) the \( S \)-equation leads to
\[ S_{i(n+1)} = \frac{\partial H_{i(n+1)}}{\partial t} + \frac{\partial H_{i(n+2)}[t_{i(n+1)} \rightarrow k]}{\partial x_k} \]
\[ \frac{x(n+1) \rightarrow x}{x} \]
are defined accordingly

\[ \frac{\partial H_{i(n+2)}[i(n+1)\rightarrow k(i)]}{\partial r_{k(i)}}[x_{n+1} \rightarrow r_{(i)}] \]

\[ + \frac{\partial I_{i(n)}[0]}{\partial x_{i}} + \sum_{l=1}^{n} \left( \frac{\partial I_{i(n)}[0]}{\partial m_{(i)}[m_{(i)}-i]} + \frac{\partial I_{i(n)}[0]}{\partial r_{k(i)}} \right) - \nu \left[ \frac{\partial^2 H_{i(n+1)}}{\partial x_{k} \partial x_{k}} \right] = 0 \]

for \( n = 1, \ldots, \infty \),

and the two continuity equations become

\[ \frac{\partial H_{i(n+1)}[i(0)\rightarrow k]}{\partial x_{k}} - \sum_{j=1}^{n} \frac{\partial H_{i(n+1)}[i(0)\rightarrow k(j)]}{\partial r_{k(j)}} = 0 \>,

\[ \frac{\partial H_{i(n+1)}[k(i)\rightarrow k]}{\partial r_{k(i)}} = 0 \quad \text{for} \quad l = 1, \ldots, n \]

and

\[ \frac{\partial I_{i(n)}[k][i\rightarrow m]}{\partial x_{m}} - \sum_{j=1}^{n} \frac{\partial I_{i(n)}[k][i\rightarrow m]}{\partial r_{m(j)}} = 0 \quad \text{for} \quad k = 1, \ldots, n \>,

\[ \frac{\partial I_{i(n)}[k][i\rightarrow m]}{\partial r_{m(j)}} = 0 \quad \text{for} \quad k = 0, \ldots, n \>, \quad l = 1, \ldots, n \>, \quad k \neq l \>.

2.4.2. MPC equations: fluctuation approach. In the present subsection we adopt the classical approach i.e. all correlation functions are based on the fluctuating quantities \( u \) and \( p \) as introduced by Reynolds and not on the full instantaneous quantities as in the previous sub-section. Hence, similar to (15) we have the multipoint correlation for the fluctuation velocity

\[ R_{i(n+1)} = R_{i(0)i(1)\cdots i(n)} = u_{i(0)}(x_{(0)}) \cdot \cdots \cdot u_{i(n)}(x_{(n)}) \]  

Further, all other correlations defined in sub-section 2.4.1 are defined accordingly i.e. equivalent to the definitions (16)-(19) we respectively define \( R_{i(n+1)[i(l)\rightarrow k]} \), \( R_{i(n+2)[i(n+1)\rightarrow k(i)j]}[x_{n+1} \rightarrow x_{(i)}] \), \( R_{i(n)[i(l)\rightarrow 0]} \) and \( P_{i(n)}[l] \).

Finally, we define the correlation equation in analogy to (20) where \( M_{i} \) is replaced by the equation for the fluctuations (14) denoted by \( \mathcal{N}_{i} \) and \( U_{i} \) and \( P \) are substituted by \( u_{i} \) and \( p \). The resulting equation is denoted by \( T_{i(n+1)} \)

\[ T_{i(n+1)} = \frac{\partial R_{i(n+1)}}{\partial t} + \sum_{l=0}^{n} \left[ \mathcal{U}_{k(l)}(x_{(i)}) \frac{\partial R_{i(n+1)}}{\partial x_{k(l)}} + R_{i(n+1)[i(l)\rightarrow k]} \frac{\partial U_{i(l)}(x_{(i)})}{\partial x_{k(l)}} \right] 

+ \frac{\partial P_{i(n)}[l]}{\partial x_{(i)}} - \nu \left[ \frac{\partial^2 R_{i(n+1)}}{\partial x_{k(l) \partial x_{k(l)}}} - R_{i(n)[i(l)\rightarrow 0]} \frac{\partial m_{(i)}[k(l)](x_{(i)})}{\partial x_{k(l)}} \right] 

+ \frac{\partial R_{i(n+2)[i(n+1)\rightarrow k]}[x_{n+1} \rightarrow x_{(i)}]}{\partial x_{k(l)}} = 0 \]

for \( n = 1, \ldots, \infty \).
The first tensor equation of this infinite chain propagates $R_{i(2)}$, which has a close link to the Reynolds stress tensor

$$\lim_{x(k) \to x(l)} R_{i(2)} = \lim_{x(k) \to x(l)} R_{i(0) i(1)} = \overline{u_{i(0)} u_{i(1)}}(x(l)) \quad \text{mit} \quad k \neq l, \quad (31)$$

which is the key unclosed quantity in the Reynolds stress transport equation (11). Here $x(k)$ and $x(l)$ can be arbitrary vectors out of $x(0), \ldots, x(n)$.

Also equation (30) implies all statistical information of the Navier-Stokes equations. However, apart from the latter simple relation to the Reynolds stress tensor it possesses the key disadvantage of being a non-linear infinite dimensional system of differential equations which make the extraction of Lie symmetries from this equation rather cumbersome. There are two essential sources of non-linearity in these equations. One is the known convection non-linearity which links the mean velocity to all correlation equations. The second source of non-linearity originates from the second row of equation (30). It is based on the fact that the gradient of the Reynolds stress tensor is contained in the equations of fluctuation. Hence, considering the following identity, this term is not equal to zero for turbulent flows and for multi-point correlation tensors of order higher than two

$$R_{i(1)[k(n)] \to 0} = 0. \quad (32)$$

As a direct consequence all multi-point correlation equations of order $n > 1$ are coupled to the two-point correlation equation.

From equation (13) a continuity equation for $R_{i(n+1)}$ and $P_{i(n)}[l]$ can be derived. They have identical form to (22) and (23) for $H_{i(n+1)}$ and $I_{i(n)}$ and hence will be omitted for brevity.

As above in 2.4.1 the classic distance vector notation is adopted which leads to

$$T_{i(n+1)} = \frac{\partial R_{i(n+1)}}{\partial t} + \sum_{l=1}^{n} \left[ \overline{U_{i(l)}}(x + r(l)) - \overline{U_{i(l)}}(x) \right] \frac{\partial R_{i(n+1)}}{\partial r_{i(l)}}$$

$$+ R_{i(n+1)[k \to k]} \frac{\partial U_{i(l)}}{\partial x_k} + \sum_{l=1}^{n} R_{i(n+1)[i(l) \to k(l)]} \frac{\partial U_{i(l)}}{\partial x_k}$$

$$+ \frac{\partial P_{i(n)}[0]}{\partial x_i} + \sum_{l=1}^{n} \left( \left[ \frac{\partial P_{i(n)}[0]}{\partial r_{m(l)}} \right]_{[m(l) \to i]} + \frac{\partial P_{i(n)}[l]}{\partial r_{i(l)}} \right)$$

$$- \nu \left[ \frac{\partial^2 R_{i(n+1)}}{\partial x_k \partial x_k} + \sum_{l=1}^{n} \left( \frac{2}{\partial x_k \partial r_{k(l)}} \sum_{m=1}^{n} \frac{\partial^2 R_{i(n+1)}}{\partial r_{m(l)} \partial r_{k(l)}} + \frac{\partial^2 R_{i(n+1)}}{\partial r_{k(l)} \partial r_{k(l)}} \right) \right]$$

$$- R_{i(n)[i \to 0]} \frac{\partial \overline{n_i u_i}}{\partial x_k} - \sum_{l=1}^{n} R_{i(n)[i(l) \to 0]} \frac{\partial \overline{n_i u_{i(l)}}(x(l))}{\partial x_k (l)} \frac{x(l) \to x}{x(l) \to x + r(l)}$$

$$+ \frac{\partial R_{i(n+2)}[i(n+1) \to k]}{\partial x_k} \frac{x(n+1) \to x}{x(n+1) \to x} - \sum_{l=1}^{n} \frac{\partial R_{i(n+2)}[i(n+1) \to k(l)]}{\partial r_{k(l)}} \frac{x(n+1) \to r(l)}{x(n+1) \to r_(l)} = 0, \quad (33)$$

for $n = 1, \ldots, \infty$, and the two continuity equations transform alike.

From (6)-(9) it is apparent that there is a unique relation between the instantaneous and the fluctuation approach though the actual crossover is somewhat cumbersome in particular with increasing tensor order because they may only be given
in recursive form. Since needed later we give the first relations

\[ H_{i(0)} = \bar{U}_{i(0)} \]  
\[ H_{i(0)i(1)} = \bar{U}_{i(0)} \bar{U}_{i(1)} + R_{i(0)i(1)} \]  
\[ H_{i(0)i(1)i(2)} = \bar{U}_{i(0)} \bar{U}_{i(1)} \bar{U}_{i(2)} + R_{i(0)i(1)i(2)} + R_{i(1)i(2)i(0)} \] \[ \vdots \] \[ \vdots \]

where the indices also refer to the spatial points as indicated.

As a special case of the equations (34) we consider \( n = 1 \), thus we derive the equation for the two-point correlation tensor. To abbreviate the notation we introduce the following nomenclature:

\[ R_{ij} = R_{ij} \] \[ \{2\} \]

In this case equation (26) reduces to

\[ T_{ij} = \frac{D R_{ij}}{Dt} + R_{kj} \frac{\partial \bar{U}_i(x,t)}{\partial x_k} + R_{ik} \frac{\partial \bar{U}_j(x,t)}{\partial x_k} \bigg|_{x \rightarrow x+r} \]

\[ + \left[ \bar{U}_k(x+r,t) - \bar{U}_k(x,t) \right] \frac{\partial R_{ij}}{\partial x_k} + \frac{\partial \overline{m_{ij}}}{\partial x_i} - \frac{\partial \overline{u_{ij}}}{\partial r_i} + \frac{\partial \overline{u_{ij}}}{\partial r_j} \]

\[ - \nu \left[ \frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} - 2 \frac{\partial^2 R_{ij}}{\partial x_k \partial r_k} + 2 \frac{\partial^2 R_{ij}}{\partial r_k \partial r_k} \right] \]

\[ + \frac{\partial R_{ikj}}{\partial x_k} - \frac{\partial}{\partial r_k} \left[ R_{ikj} - R_{ikj} \right] = 0 \] \[ \{38\} \]

The vectors \( \overline{m_{ij}} \) and \( \overline{u_{ij}} \) are special cases of \( P_{i\{n\}[k]} \) and defined as

\[ \overline{m_{ij}}(x,r,t) = \overline{m}(x(0),t) u_j(x(1),t) \text{ and } \overline{u_{ij}}(x,r,t) = u_i(x(0),t) \overline{u}(x(1),t) \] \[ \{39\} \]

For the two-point case the continuity equations take the form

\[ \frac{\partial R_{ij}}{\partial x_i} - \frac{\partial R_{ij}}{\partial r_i} = 0 \text{, } \frac{\partial R_{ij}}{\partial r_j} = 0 \] \[ \{40\} \]

and

\[ \frac{\partial \overline{m}}{\partial r_i} = 0 \text{, } \frac{\partial \overline{u_{ij}}}{\partial x_j} - \frac{\partial \overline{u_{ij}}}{\partial r_j} = 0. \] \[ \{41\} \]

The non-locality of the two- and multi-point correlation equations is most obvious when we use the commutation of the two-point correlation tensor. Given \( u_i(x(0)) u_j(x(1)) = u_j(x(1)) u_i(x(0)) \) with equation (24) leads to the functional relations

\[ R_{ij}(x,r,t) = R_{ji}(x+r,-r,t) \] \[ \{42\} \]

\[ \overline{m_{ij}}(x,r,t) = \overline{m_{ij}}(x+r,-r,t) \] \[ \{43\} \]

Analogous identities can be derived for all other two- and multi-point correlation tensors.
2.4.3. Small scale asymptotics. The following singular asymptotic behavior in the correlation space \( r \) is fundamental for understanding the laws of turbulence developed in section 3. They are based on Kolmogorov’s idea of a cascade process [10, 11] in which the energy of the turbulent eddies is transferred to the eddy next in size. The smallest characteristic length scale in the flow is given by the Kolmogorov length

\[
\eta_K = \left( \frac{\nu^3}{\epsilon} \right)^{\frac{1}{4}},
\]

where \( \epsilon \) denotes the rate of turbulent energy dissipation. Besides this measure two other length scales may be defined and their magnitudes may be distinguish against each other as follows:

\[
\eta_K \ll \lambda \ll \ell_t.
\]

\( \lambda \) denotes the Taylor length and is defined as

\[
\lambda = \sqrt{\frac{\nu k}{\epsilon}}
\]

and further we have the integral length scale \( \ell_t \) and it characterizes the magnitude of most energetic turbulent eddies

\[
\ell_t = \frac{1}{\overline{u_m u_m}} \int R_{kk} d\Omega dr,
\]

where \( \Omega \) denotes the angle in space. Both are related by the turbulent Reynolds number

\[
\frac{\eta_K}{\ell_t} = Re_t^{-\frac{3}{4}}, \quad \frac{\lambda}{\ell_t} = Re_t^{-\frac{1}{2}} \quad \text{where} \quad Re_t = \frac{\ell_t \sqrt{k}}{\nu},
\]

and \( k \) represents the turbulent energy \( k = \frac{1}{2} \overline{u_k u_k} \). Mean velocity as well as correlation, e.g. \( R_{ij} \), are mostly determined by the high-energy eddies of magnitude \( \ell_t \) and \( k \). Eddies of magnitude \( \eta_K \) merely induce the necessary turbulent energy dissipation.

In the language of two- and multi-point correlation this refers to the fact that there has to be made a distinction between the cases \( |r| \gg \eta_K \) and \( |r| = O(\eta_K) \).

For the first case i.e. the large scale limit we make all variables dimensionless with \( \ell_t \) and \( k \). Implemented into the two-point correlation equations and taking the high Reynolds number limit i.e. \( Re_t \to \infty \) this leads to a reduced form of equation (38):

\[
T_{(2)}^> = \frac{\partial}{\partial t} \frac{\partial u_j^>(x,t)}{\partial x_k^>} + R_{jk}^> \frac{\partial}{\partial x_k^>} \frac{\partial u_j^>(x,t)}{\partial x_k^>} \bigg|_{x \cdot r} + [\bar{u}_k^>(x+r,t) - \bar{u}_k^>(x,t)] \left( \frac{\partial R_{ij}^>}{\partial r_k^>} + \frac{\partial u_j^>}{\partial x_i^>} - \frac{\partial u_i^>}{\partial x_j^>} \right) \frac{\partial R_{ik}^>}{\partial r_j^>} + \frac{\partial R_{(kj)}^>}{\partial x_k^>} - \frac{\partial}{\partial r_k^>} \left( R_{(ik)}^> - R_{(jk)}^> \right) = 0,
\]

where “\( > \)” stands for \( |r| \gg \eta_K \). Analogously, all multi-point correlation equations without viscosity terms can be derived. Obviously, for \( |r| \gg \eta_K \) mean velocity and correlation functions are determined by the frictionless part of the correlation equations which is a long standing folk wisdom in the turbulence community.
In the second case of \( r \) being in the order of the Kolmogorov length scale, the dissipation term, which is the last term in the fourth row in equation (38), must not be omitted.

The limit demands \( r \) to be made dimensionless with \( \lambda \) and the odd moments of velocity and the pressure-velocity correlation with \( k^2 \lambda / \ell \). All remaining variables are still rescaled by \( \ell \) and \( k \). In the leading order for \( \text{Re}_{\ell} \to \infty \) we obtain

\[
\mathcal{T}^<_{i(2)} = \frac{\mathcal{D}R^<_{ij}}{\mathcal{D}t} + R^<_{kj} \frac{\partial \bar{u}^<_{\ell}(x,t)}{\partial x^<_{k}} + R^<_{ik} \frac{\partial \bar{u}^<_{\ell}(x,t)}{\partial x^<_{k}} + \bar{u}^<_{i} \frac{\partial R^<_{ij}}{\partial r^<_{i}} - \frac{\partial \bar{p}u^<_{i}}{\partial r^<_{i}} \quad (50)
\]

Analogous equations can be obtained for higher correlations. The essential difference to equation (49) is that the second derivative of \( R_{ij} \) with respect to \( r \) remains present. In the limit of \( r = 0 \) the latter expression represents the rate of dissipation.

The above splitting of the characteristic length scales allows to separate large scales from energy dissipation in their influence on the velocity.

The remaining of the paper deals with the large-scale quantities i.e. especially the mean velocity and the correlation tensors.

Due to length and complexity of the full multi-point tensor equations (21) and (30) mathematical details are primarily focused on the two-point correlations and the related equations though can be extended straightforward for all higher order correlations.

3. Symmetries of statistical transport equations. In the present section we first revisit the Lie symmetries of the Euler and Navier-Stokes equations. In turn they will all be transferred to its corresponding ones for the MPC equations. In the second part we show that the MPC equations admit even more Lie symmetries which are not reflected in the original Euler and Navier-Stokes equations.

Both sets of symmetries will finally be employed in section 4 to show that classical and new scaling laws may not be from the classical symmetries alone but essentially rely on the new symmetries which we will call statistical symmetries.

The Lie groups will either be presented in global or in infinitesimal form [3].

3.1. Symmetries of the Euler and Navier-Stokes equations. The Euler equations, i.e. equation (1) and (5) with \( \nu = 0 \) admit a ten-parameter symmetry group, here in infinitesimal from,

\[
X_1 = \frac{\partial}{\partial t},
\]

\[
X_2 = x_i \frac{\partial}{\partial x_i} + U_j \frac{\partial}{\partial U_j} + 2P \frac{\partial}{\partial P},
\]

\[
X_3 = t \frac{\partial}{\partial t} - U_i \frac{\partial}{\partial U_i} - 2P \frac{\partial}{\partial P},
\]

\[
X_4 = -x_2 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_2} - U_2 \frac{\partial}{\partial U_1} + U_1 \frac{\partial}{\partial U_2},
\]

\[
X_5 = -x_3 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_3} - U_3 \frac{\partial}{\partial U_1} + U_2 \frac{\partial}{\partial U_3},
\]

\[
X_6 = -x_3 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_3} - U_3 \frac{\partial}{\partial U_1} + U_1 \frac{\partial}{\partial U_3},
\]
we primarily limit ourselves to the global form of the symmetries.

For brevity and ease of reading the analysis to follow in this and the next subsection 3.3 we primarily limit ourselves to the global form of the symmetries.

The entire set of Lie symmetries span a linear vector space or may be interpreted as a multi-parameter group of the form

\[ X = \sum_{i=1}^{10} c_i X_i \tag{52} \]

with \( c_i \) being arbitrary constants.

The symmetries \( X_7-X_9 \) comprise translational invariance in space for constant \( f_1-f_3 \) as well as the classical Galilei group if \( f_1-f_3 \) are linear in time. In its rather general form \( X_7-X_9 \) and \( X_{10} \) are direct consequences of an incompressible flow and do not have a counterpart in the case of compressible flows. The complete record of all point-symmetries (51) was first published by Pukhnachev [18].

Making the formal transfer from Euler to the Navier-Stokes equations symmetry properties change and a recombination of the two scaling symmetries \( X_2 \) and \( X_3 \) is observed

\[ X_{ScaleNS} = 2t \frac{\partial}{\partial t} + x_i \frac{\partial}{\partial x_i} - U_j \frac{\partial}{\partial U_j} - 2P \frac{\partial}{\partial P}, \tag{53} \]

while the remaining groups stay unaltered.

It should be noted that further symmetries exist for dimensional restricted cases such as plane or axisymmetric flows (see e.g. [1, 4]).

Lie’s key theorem (see e.g. [3]) states the full equivalence between the infinitesimal and the global form of a Lie group and hence we may give the symmetries (51) also in global form

\[ T_1 : t^* = t + a_1, \quad x^* = x, \quad U^* = U, \quad P^* = P, \]
\[ T_2 : t^* = t, \quad x^* = e^{a_2}x, \quad U^* = e^{a_2}U, \quad P^* = e^{2a_2}P, \]
\[ T_3 : t^* = e^{a_3}t, \quad x^* = x, \quad U^* = e^{-a_3}U, \quad P^* = e^{-2a_3}P, \]
\[ T_4 - T_6 : t^* = t, \quad x^* = a \cdot x, \quad U^* = a \cdot U, \quad P^* = P, \]
\[ T_7 - T_9 : t^* = t, \quad x^* = x + f(t), \quad U^* = U + \frac{df}{dt}, \quad P^* = P - x \cdot \frac{d^2f}{dt^2}, \]
\[ T_{10} : t^* = t, \quad x^* = x, \quad U^* = U, \quad P^* = P + f'_4(t), \tag{54} \]

where \( a_1-a_3 \) are independent group-parameters, \( a \) denotes a constant rotation matrix with the properties \( a a^T = a^T a = I \) and \( |a| = 1 \), and \( f(t) = (f_1(t), f_2(t), f_3(t))^T \) respectively \( f'_4(t) \) satisfies the conditions mentioned after (51).

3.2. Symmetries of the MPC implied by Euler and Navier-Stokes symmetries. For brevity and ease of reading the analysis to follow in this and the next subsection 3.3 we primarily limit ourselves to the global form of the symmetries.
Adopting the classical Reynolds notation first where the instantaneous quantities are split into mean and fluctuating values we may directly derive from (6)-(9) and (54)

\[
\begin{align*}
\bar{T}_1 : & \quad t^* = t + a_1, \quad x^* = x, \quad r^*_p(t) = r(t), \quad \bar{U}^* = \bar{U}, \quad \bar{P}^* = \bar{P}, \\
\bar{R}^*_n(t) = \bar{R}(n), \quad \bar{P}^*_n(t) = \bar{P}(n), \\
\bar{T}_2 : & \quad t^* = t, \quad x^* = e^{a_2}x, \quad r^*_p(t) = e^{a_2}r(t), \quad \bar{U}^* = e^{a_2}\bar{U}, \quad \bar{P}^* = e^{2a_2}\bar{P}, \\
\bar{R}^*_n(t) = e^{a_2}\bar{R}(n), \quad \bar{P}^*_n(t) = e^{(n+2)a_2}\bar{P}(n), \\
\bar{T}_3 : & \quad t^* = e^{a_3}t, \quad x^* = x, \quad r^*_p(t) = r(t), \quad \bar{U}^* = e^{-a_3}\bar{U}, \quad \bar{P}^* = e^{-2a_3}\bar{P}, \\
\bar{R}^*_n(t) = e^{-na_3}\bar{R}(n), \quad \bar{P}^*_n(t) = e^{-(n+2)a_3}\bar{P}(n), \\
\bar{T}_4 - \bar{T}_9 : & \quad t^* = t, \quad x^* = a \cdot x, \quad r^*_p(t) = r(t), \quad \bar{U}^* = a \cdot \bar{U}, \quad \bar{P}^* = \bar{P}, \\
\bar{R}^*_n(t) = A_n \otimes \bar{R}(n), \quad \bar{P}^*_n(t) = A_n \otimes \bar{P}(n), \\
\bar{T}_{10} : & \quad t^* = t, \quad x^* = x, \quad r^*_p(t) = r(t), \quad \bar{U}^* = \bar{U} + \frac{df}{dt}, \quad \bar{P}^* = \bar{P} - x \cdot \frac{df}{dt^2}, \\
\bar{R}^*_n(t) = \bar{R}(n), \quad \bar{P}^*_n(t) = \bar{P}(n), \\
\end{align*}
\]

where all function and parameter definitions are adopted from 3.1 and \(A\) is a concatenation of rotation matrices as \(A_{i(0),j(0),l(1),l(1),\ldots,i(n),j(n)} = a_{i(0),j(0)a_{i(1),j(1)} \ldots a_{i(n),j(n)}}\).

The latter symmetries may also be transformed into the \(H\)-notation. This will be omitted for briefness and also because scaling laws have never been considered in this notation.

3.3. Statistical symmetries of the MPC equations. First hints towards a considerably extended set of symmetries for the MPC equation may e.g. be taken from [14, 9]. Its importance were not observed therein - rather it was stated that they may be mathematical artifacts of the averaging process and probably physically irrelevant.

The actual finding of symmetries of the non-rotating MPC is rather difficult since an infinite system of equations has to be analyzed. For this task, however, it is considerably easier to investigate the linear \(H-I\)-system (26)-(28) rather than the non-linear \(R-P\)-system (33) extended by its corresponding continuity equations. Since the latter formulation, however, is more common symmetries will finally be re-written in this notation.

Overall three sets of new symmetries have been identified. The first one may be written as

\[
\begin{align*}
\bar{T}_1' : & \quad t^* = t, \quad x^* = x, \quad r^*_p(t) = r(t) + a(t), \quad H^*_n(t) = H(n), \quad I^*_n(t) = I(n), \\
\end{align*}
\]

where \(a(t)\) represents the related set of group parameters. Note that this group is not related to the classical translation group in usual \(x\)-space (here \(T_2 - T_3\) in equation (54) with \(f = \text{const.}\). In classical notation we have

\[
\begin{align*}
\bar{T}_1 : & \quad t^* = t, \quad x^* = x, \quad r^*_p(t) = r(t) + a(t), \quad \bar{U}^* = \bar{U}, \quad \bar{P}^* = \bar{P}, \\
\bar{R}^*_n(t) = \bar{R}(n), \quad \bar{P}^*_n(t) = \bar{P}(n). \\
\end{align*}
\]
The second set of statistical symmetries was in fact already partially identified in [15] however, falsely taken for the Galilean group. In its general form it reads

\[ \tilde{T}'_2(n) : \quad t^* = t, \quad x^* = x, \quad r^*_l = r(l), \quad H^*_i = H_i + C_i, \quad l^*_n = l + D_n, \]  
(58)

where \( C_{i(n)} \) and \( D_{i(n)} \) refer to group parameters independent for each of the according tensor orders. To clarify this we single out the groups for the first two tensor order i.e.

\[ \tilde{T}'_2(1) : \quad t^* = t, \quad x^* = x, \quad r^*_l = r(l), \quad H^*_i = H_i + C_i, \quad H^*_{i(1)i(1)} = H_{i(1)i(1)}, \ldots \]  
(59)

and

\[ \tilde{T}'_2(2) : \quad t^* = t, \quad x^* = x, \quad r^*_l = r(l), \quad H^*_i = H_i + C_i, \quad H^*_{i(1)i(1)} = H_{i(1)i(1)} + C_{i(1)i(1)}, \ldots . \]  
(60)

Note, that even (59) and (60) refer to multi-parameter Lie groups since the constants \( C_{i(0)} \) and \( C_{i(1)i(1)} \) denote three and nine independent group parameters respectively. This may be further extended by also varying the parameter for \( D_{i(n)} \). Apparently the above notation implies a very large number of groups since each tensor has its own independent corresponding group parameter.

Though each of these groups appear to be almost trivial since they are simple translational groups in the dependent coordinates they exhibit an increasingly complexity with increasing tensor order if written in the \( \tilde{U}-R \) formulation. Here, we use the above two examples and rewrite them using (34) and (35) yielding

\[ \tilde{T}'_2(1) : \quad t^* = t, \quad x^* = x, \quad r^*_l = r(l), \quad \tilde{U}^*_i = \tilde{U}_i + C_i, \quad R^*_{i(1)i(1)} = R_{i(1)i(1)} + \tilde{U}_i + C_i, \]  
(61)

and

\[ \tilde{T}'_2(2) : \quad t^* = t, \quad x^* = x, \quad r^*_l = r(l), \quad \tilde{U}^*_i = \tilde{U}_i, \quad R^*_{i(1)i(1)} = R_{i(1)i(1)} + C_{i(1)i(1)}. \]  
(62)

Finally, the third statistical group that has been identified denotes simple scaling of all MPC tensors as may be directly read off from equation (21)

\[ \tilde{T}'_3 : \quad t^* = t, \quad x^* = x, \quad r^*_l = r(l), \quad H^*_i = e^{\alpha_i} H_i, \quad l^*_n = e^{\alpha_n} l_n, \ldots \]  
(63)

Again, employing (34) and (35) we obtain this group in classical notation though only a recursive form may be given

\[ \tilde{T}'_3 : \quad t^* = t, \quad x^* = x, \quad r^*_l = r(l), \]  

\[ \tilde{U}^*_i = e^{\alpha_i} \tilde{U}_i + e^{\alpha_i} \tilde{U}_i, \quad R^*_{i(1)i(1)} = e^{\alpha_i} \left[ R_{i(1)i(1)} + \left( 1 - e^{\alpha_i} \right) \tilde{U}_i \right], \]  
(64)

It should be finally added that due to the linearity of the MPC equation (21) another rather generic symmetry is admitted. This is in fact featured by all linear differential equations (see e.g. [3]). It merely reflects the super-position property of linear differential equations though usually cannot directly be adopted for the practical derivation of group invariant solutions.
4. Turbulent scaling laws. The rather old idea of a turbulent scaling law (see e.g. [17]) usually refers to two distinct facts:

- Introducing a certain set of parameters to non-dimensionalize statistical turbulence variables such as the mean velocity leads to a collapse of data if one external parameter is varied such as the Reynolds number.
- An explicit mathematical function is given for statistical turbulence variables such as the mean velocity, Reynolds stresses, etc..

Presently we only contemplate with the second definition. In order to rigorously derive such laws directly from the MPC equations we need to introduce the definition of a group invariant solution (see e.g. [3]).

Suppose \( F = 0 \) is the set of differential equations under investigation, \( X \) denotes the multi-parameter Lie symmetry group taken from the condition \( \left[ \left. X F \right|_{F=0} \right] = 0 \) and, generically as above, \( x \) and \( y \) refer to the entire set of independent and dependent variables respectively. We define the group invariant solution \( y = \Theta(x) \) comprising the two conditions

(i) \( y - \Theta(x) \) is invariant under \( X \) and
(ii) \( y = \Theta(x) \) is a solution of the differential equation \( F=0 \).

With this we obtain

\[
X [y - \Theta(x)] = 0 \quad \text{where} \quad y = \Theta(x).
\]

Differentiating this out we obtain the hyperbolic system

\[
\xi_k (x, \Theta) \frac{\partial \Theta_l}{\partial x_k} = \eta_l (x, \Theta)
\]

(66)

to be solved by the method of characteristics which finally ends up with the characteristic condition

\[
\frac{dx_1}{\xi_1(x, y)} = \frac{dx_2}{\xi_2(x, y)} = \cdots = \frac{dx_n}{\xi_n(x, y)} = \frac{dy_1}{\eta_1(x, y)} = \frac{dy_2}{\eta_2(x, y)} = \cdots = \frac{dy_n}{\eta_n(x, y)} ,
\]

(67)

where \( \Theta \) has been replaced by \( y \). The latter is usually referred to invariant surface condition.

In the remaining two subsections we adopt the latter condition for the derivation of the accordant invariant solution alternatively later also named scaling laws, which is the usual phrase in the turbulence literature.

4.1. Stationary wall-bounded turbulent shear flows. Due to its eminent practical importance wall-bounded shear flow is by far the most intensively investigated turbulent flows thereby employing a vast number of numerical, experimental and modeling approaches and this, in fact, for more than a century.

From all the theoretical approaches the universal law of the wall is the most widely cited and also accepted approach with its essential ingredient the logarithmic law. Though a variety of different approaches have been put forward for its derivation neither of them employ the multi-point equation, which is the basis for statistical turbulence, nor do they solve an equation that is related to the Navier-Stokes equations.

In the following we demonstrate that the log-law is an invariant solution of the infinite set of multi-point equations and further it is shown that it essentially relies on one of the new translational groups.
Already in [13] it was observed that in the limit of high Reynolds numbers and $|r| \gg \eta K$ the logarithmic wall law allows for a self-similar solution of the two-point correlation equation (38). This is rather remarkable since for inhomogeneous flows equation (38) is not a partial differential equation in the classic sense but a non-local differential equation. Non-locality is denoted by the fact that for a given point $x$ and $r$ not only the dependent variables and derivatives are connected but also terms “at the point” $x + r$ contribute to the equation. In equation (38) this is given by the last term in the first row and the first term in the second row.

Indeed the terms mentioned above were the major cause for the limitation of the two-point correlation equation to homogeneous flows i.e. flows where the mean velocity gradient is a constant, or even more simple the mean velocity itself is a constant. It is important to note that this limitation is not needed for the present group theoretical approach.

Within this subsection we exclusively examine wall-parallel turbulent flows only depending on the wall-normal coordinate $x_2$. Further, we only explicitly write the two-point correlation $R_{ij}$ though all results are also valid for all higher order correlations. This finally yields

$$\bar{U}_1 = \bar{U}_1(x_2), \quad R_{ij} = R_{ij}(x_2, r), \quad \ldots . \quad (68)$$

With these geometrical assumptions we identify a reduced set of groups. The two scaling groups in (55) may in infinitesimal form be written as

$$\bar{X}_2 = x_2 \frac{\partial}{\partial x_2} + \bar{U}_1 \frac{\partial}{\partial \bar{U}_1} + r_i \frac{\partial}{\partial r_i} + 2R_{ij} \frac{\partial}{\partial R_{ij}} + \ldots \quad (69)$$

and

$$\bar{X}_3 = -\bar{U}_1 \frac{\partial}{\partial \bar{U}_1} - 2R_{ij} \frac{\partial}{\partial R_{ij}} + \ldots , \quad (70)$$

where in the latter the term $\partial/\partial t$ has been omitted since only statistically steady flows according to (68) are considered. The translation invariance in $x_2$-direction has the form

$$X_{x_2} = \frac{\partial}{\partial x_2} \quad (71)$$

and all previous three symmetries are implied by the Navier-Stokes equations. Further we have the statistical groups i.e. the translational group in correlation space (57)

$$\bar{X}_{r_i} = \frac{\partial}{\partial r_i} \quad (72)$$

and the translational group (61) for $\bar{U}_1$

$$\bar{X}_{\bar{U}_1} = \frac{\partial}{\partial \bar{U}_1} - (\bar{U}_1(x_2) + \bar{U}_1(x_2 + r_2)) \frac{\partial}{\partial \bar{R}_{11}} + \ldots \quad (73)$$

and (62) for $R_{ij}$

$$\bar{X}_{R_{ij}} = \frac{\partial}{\partial R_{ij}} + \ldots \quad (74)$$

and finally the scaling group (64)

$$\bar{X}_s = \bar{U}_1 \frac{\partial}{\partial \bar{U}_1} + (R_{ij} - \bar{U}_1(x_2)\bar{U}_1(x_2 + r_2)\delta_{i1}\delta_{j1}) \frac{\partial}{\partial R_{ij}} + \ldots . \quad (75)$$

In some earlier work on group invariance of two-point correlations [14, 15] (73) was not complete and was falsely interpreted as Galilean group. The symmetries (69)-(75) are mutually independent transformations of the two- and, if fully written
out, of the MPC equations under the restriction (68). Hence the linear combination of the above symmetries form a multi-parameter group

$$X = k_2 X_2 + k_3 X_3 + k_{x_2} X_{x_2} + k_r X_r + k_{U_1} X_{U_1} + k_{R_{ij}} X_{R_{ij}} + k_s X_s$$  (76)

and hence constitute a new symmetry of the entire set of correlation equations.

Using the latter multi-parameter group in the equation for invariant solutions (67) we obtain the invariance condition

$$\frac{dx_2}{k_2 x_2 + k_{x_2}} = \frac{dr_{[k]}}{k_2 r_{[k]} + k_{r_{[k]}}} = \frac{dU_1}{(k_2 - k_3 + k_s) U_1 + k_{U_1}} = \frac{dR_{ij}}{ξ_{R_{ij}}} = \cdots \text{ with}$$

$$ξ_{R_{ij}} = (2k_2 - 2k_3 + k_s) R_{ij} - (k_s U_1(x_2) U_1(x_2 + r_2) + k_{U_1} (U_1(x_2) + U_1(x_2 + r_2))) δ_{i1} δ_{j1} + k_{R_{ij}}$$  (77)

where no summation is implied by the indices in square brackets and instead a concatenation is implied where the indices are consecutively assigned its values. For brevity explicit dependencies on the independent variables are only given where there is an unambiguity.

In particular with a distinct combinations of parameters $k_2$, $k_3$ and $k_s$ a multitude of flows may be described where here we only focus on the log-law. We may keep in mind that $\bar{U}_1$ exclusively depends on $x_2$ and not on $r$.

The meaning of the scale symmetries (69), (70) and (75) can be discussed most efficiently by investigating its concatenated global transformations $T_2$ and $T_3$ in (55) and (64) for the mean velocity and the space coordinate

$$x_2^* = e^{k_2 x_2} \quad \text{and} \quad \bar{U}_1^* = e^{k_2 - k_3 + k_s} \bar{U}_1 .$$  (78)

For the present case to examine the classic case of the logarithmic wall law its basic idea of a symmetry breaking in (78) can be best seen by revisiting the key idea of von Kármán. He assumed that close to the wall the wall-friction velocity $u_\tau$ is the only parameter determining the flow. He defined the wall-friction velocity by investigating the balance of momentum (11) in $x_1$-direction close to the wall assuming a parallel shear flow without any pressure gradient. Only the viscous and turbulent shear stresses remain and the resulting equation may be integrated once with respect to $x_2$ leading to

$$\nu \frac{∂\bar{U}_1}{∂x_2} - \frac{u_1 u_2}{ρ} = \frac{τ_w}{ρ} = u_τ^2 ,$$  (79)

where $τ_w$ is the wall shear stress.

Since $u_τ$ is a given external parameter specifying the velocity scale, scaling of $\bar{U}_1$ and thereby an arbitrary choice of the parameters $k_2$, $k_3$ and $k_s$ in (78) is not admissible. Thus we find

$$k_2 - k_3 + k_s = 0 ,$$  (80)

since $u_τ$ causes a symmetry breaking in the scalability of $\bar{U}_1$.

Under this assumption (77) leads to the classical functional form of the mean velocity

$$\bar{U}_1 = \frac{k_{U_1}}{k_2} \ln \left( x_2 + \frac{k_{x_2}}{k_2} \right) + C_{log}$$  (81)
and the invariant correlations read
\[ \tilde{r}_k = \frac{r_k}{x_2 + \frac{k_s}{k_2} x_2}, \quad R_{ij} = \left( x_2 + \frac{k_{x_2}}{k_2} \right)^{-\frac{k_{x_2}}{k_2}} \tilde{R}_{ij}(\tilde{r}) + \frac{k_{R_{ij}}}{k_s} \] for \( ij \neq 11 \)
\[ R_{11} = \tilde{R}_{11}(\tilde{r}) \left( x_2 + \frac{k_{x_2}}{k_2} \right)^{-\frac{k_{x_2}}{k_2}} \left( \frac{C_{\log} k_{U_1}}{k_2} + \ln(\tilde{r}_2 + 1) + \frac{k_s}{k_2} \right) \left( x_2 + \frac{k_{x_2}}{k_2} \right) \] (82)
\[ \cdots \]
where \( C_{\log} \) in (81) and the variables marked with "\( \tilde{\} \)" in (82) are constants of integration. These are the new invariant coordinates solely depending on \( \tilde{r} \). \( C_{\log} \) is an exception since according to (68) \( \bar{U}_1 \) depends on \( x_2 \) alone.

Obviously equation (81) is a slightly generalized form of the classic logarithmic wall law since by the term \( \frac{k_{x_2}}{k_2} \) a displacement of the origin is admitted. In its classical dimensionless form it reads
\[ u^+ = \frac{1}{k} \ln(x_2^+ + A^+) + C. \] (83)

As betoken above the non-locality of the two- and multi-point correlation equations appears not to hinder the analysis and, of course, a reduced form of the equation may be given for any multi-point correlation tensor equation.

An experimental validation of the logarithmic wall law is not given here since it has been verified in numerous publications.

The invariant variable \( \tilde{r} \) in (82) for the logarithmic wall law was already given by Hunt et al. [5] though in a less general form and has been numerically verified by DNS data of a turbulent channel flow. Especially for the two-point correlation tensor \( R_{22} \) this data matches very well. This is very remarkable since the DNS data is based on a low Reynolds number. The considerations taken by Hunt et al. are not based on an examination of the two-point correlation equations but merely on a dimensional analysis of the independent coordinate.

4.2. Turbulent decay scaling laws. Decaying turbulence is, if also homogeneity and isotropy is invoked, the most simple turbulent flow one can think of. In this case the two-point correlation equations reduce to the von Kármán-Howarth equation - a scalar transport equation for the correlation function [7]. Physically it refers to the fact that fluid in an infinite domain is stirred to generate a statistically homogeneous and isotropic flow field and then the stirring force is interrupted and turbulence decays freely.

Classical theories on decaying turbulence in the limit of large Reynolds number such as Birkhoff’s [2] and Loitsyansky’s [12] integral entirely rely on the groups \( \bar{T}_2 \) and \( \bar{T}_3 \) and, if also a time shift is considered, also on \( \bar{T}_1 \). These two scaling groups give rise to a one-parameter family of invariant solutions where e.g. the turbulent kinetic energy decays and the integral length scale increases according to a power law
\[ k \sim (t + t_0)^{-m}, \quad \ell_t \sim (t + t_0)^n \] (84)
where the exponent is usually settled by one of the above proposed conserved integrals (see e.g. [16]).
Presently using the above extended set of symmetry groups (56)-(64) restricted to the present flow case a much wider class of invariant solutions or turbulent scaling laws is admissible. The combined infinitesimals read

\[
\bar{X} = k_1 \bar{X}_1 + k_2 \bar{X}_2 + k_3 \bar{X}_3 + k_r \bar{X}_r + k_{R_{ij}} \bar{X}_{R_{ij}} + k_s \bar{X}_s .
\]

Employing the latter in (67) we obtain the invariant surface condition

\[
\frac{dt}{k_3 t + k_1} = \frac{dr_{[ik]}}{k_2 r_{[ik]} + k_{R_{ij}}} = \frac{dR_{[ij]}}{(2k_2 - 2k_3 + k_s)R_{[ij]} + k_{R_{ij}}} = \cdots
\]

with the notation identical to the one explained below (77).

It is important to note that any solution for an arbitrary set of group parameters of the latter invariant surface condition allows for an invariant solution of (33). Three fundamentally different cases may be distinguished depending on the choice of the scaling group parameters \(k_2, k_3\) and \(k_s\).

The classical cases may easily be extended if we assume \(k_2, k_3\) and \(k_s\) are distinct from each other and non-zero. With this we find the following invariants of the system (86)

\[
\hat{r}_k = \frac{r_k + \frac{k_r}{(t + t_0)^n}}{ \frac{k_{R_{ij}}}{R_{[ij]}} + \frac{k_{R_{ij}}}{R_{[ij]}}} , \quad r_{ij}(r, t) = (t + t_0)^{-m} \tilde{R}_{ij}(\tilde{r}) , \quad \ldots .
\]

Here \(m\) and \(n\) are indeed independent numbers simply related to the scaling group parameters. If, however, we limit ourselves to the classical case i.e. \(k_s = 0\) we obtain that \(n = k_2/k_3\) and \(m = 2(1 - k_2/k_3)\). The variables \(\hat{r}\) and \(\tilde{R}_{ij}\) are again the constants of integration of the invariant surface condition (86) or in other words the invariants of the system. They are to be taken as new independent and dependent variables of the system (33) leading to a similarity reduction depicting the classical power law behavior. Therein \(m = 6/5, n = 2/5\) and \(m = 10/7, n = 2/7\) respectively correspond to Birkhoff’s and Loitsiansky’s integrals. An overview on the classical cases and its variations is given in [16].

The most remarkable new scaling behavior of decaying turbulence, however, was observed by [6, 19]. Therein they reported on an experiment where they generated turbulence with a grid of fractal structure that is rather different to the classical grids which are usually manufactured from simple bars of a given thickness assembled in a rectangular manner at a given constant distant from each other. The fractal grids have different base forms e.g. one set is installed by squares made of bars. These squares have different sizes composing a geometrical series. On all of the four corners of the squares the next smaller square is mounted and this is iterated up to the fourth or fifth level. In the actual experiment air is blown through the fractal grid generating turbulence on very different length scales which is rather different from the classical square grids of a single size.

In order to interpret this flow from a symmetry point of view we rewrite the scaling groups in global form to obtain

\[
t^* = e^{k_1} t \quad \text{and} \quad R_{ij}^* = e^{2(k_2 - k_3) + k_s} R_{ij} ,
\]

where we should recall from above that \(k_2\) refers to the scaling group of space.

Interpreting the length of the different squares as symmetry breaking we immediately observe that \(k_2 = 0\). If we further take the ratios of the flow velocity at which the grid is flown at and the different square grids we obtain a multitude of time scales acting on the flows and hence we have a symmetry breaking of time meaning \(k_3 = 0\).
Employing the latter information into equation (86) we observe two important conclusions. First, due to \( k_3 = 0 \) and combining the first and the third term in (86) we have an exponential scaling of the two- and MPC with time. The second case i.e. the first two terms in (86) for \( t \) and \( r \) is somewhat more delicate because one might expect some traveling wave type of behavior due to the only non-zero group parameters \( k_1 \) and \( k_{r[i]} \). This however is physically not feasible and hence we have to further take \( k_{r[i]} = 0 \). With this we may deduce that \( r \) itself is an invariant since all infinitesimals in the denominator of the second term in (86) vanish.

Following the methodology above this leads to an invariant solution for the infinite set of MPC tensors (33) where the first term in the row, i.e. the two-point tensor, has the following form

\[
\tilde{r} = r, \quad R_{ij}(r, t) = e^{-t/t_0} \tilde{R}_{ij}(\tilde{r}), \quad \ldots,
\]

where \( \tilde{R}_{ij} \) is the variable of the reduced set of MPC equations only depending on the similarity \( \tilde{r} \).

In order to compare (89) to experimentally observable one-point quantities we recall the relation between two-point correlation and Reynolds stress tensor (31) which leads to the turbulent kinetic energy \( k = \frac{1}{2} u_k u_k \) and the integral length \( \ell_t \) scale defined in (47).

If we finally employ (89) into the latter definitions we obtain the new turbulent scaling law

\[
k \sim e^{-t/t_0}, \quad \ell_t \sim \text{const.}
\]

which is very different from the classical algebraic decay law.

It was in fact exactly this behavior that was first reported in [6] and fully consolidated in [19]. For certain cases they also find a constant integral length scale (47) (and also Taylor length scale) downstream of the fractal grid. A variety of different fractal grids were employed for the experiment such as cross-grids, square-grids and \( I \)-grids.

From the different sets of measurements they conducted it was noticed that the stronger the stretching of the fractal grid and bar spacing was the better the above scaling behavior (90) was observed.

The physical interpretation of the latter results is that a broad bandwidth of external scales have been imposed on the flow which are symmetry breaking.

Beside the classical algebraic scaling laws and the latter new exponential scaling law for decaying turbulence, we report another new turbulent scaling law which, to the best or the authors knowledge, has never been observed experimentally or reported from theoretical considerations.

For this we consider that there is a constant time scale, say \( \tau_0 \), acting on the flow i.e. the scaling of time is broken and hence \( k_3 = 0 \). Employing the latter into the invariant surface condition (86) and integrate it leads to the following invariant solution for the correlation tensors

\[
\tilde{r} = r e^{-t/\tau_0}, \quad R_{ij}(r, t) = e^{-\gamma t/\tau_0} \tilde{R}_{ij}(\tilde{r}), \ldots,
\]

where essentially only the scaling groups of space, the statistical scaling group and the translation group in time respectively corresponding to the parameters \( k_2, k_s \) and \( k_1 \) have been employed.

Translated into the language of one-point quantities we obtain

\[
k \sim e^{-\gamma t/\tau_0}, \quad \ell_t \sim e^{t/\tau_0},
\]
where in the latter two equations $\gamma$ emerged from the scaling group parameters $k_2$ and $k_s$. This result, however, is to be be validated in future experiments or numerical simulations.

5. **Summary and outlook.** Within the present paper it was shown that the admitted symmetry groups of the infinite set of multi-point correlation equations are considerable extended by three classes of groups compared to those originally stemming from the Euler and the Navier-Stokes equations. In fact, it was demonstrated that it is exactly these symmetries which are essentially needed to validate certain classical laws from first principles and to derive new scaling laws.

Still, even with the new symmetry groups at hand which give a much deeper understanding on turbulence statistics there are still some key open questions to be answered:

- From turbulence data it is apparent that the appearing group parameters do have certain decisive values which are to be determined. In some very rare cases such as the classical decaying turbulence case values such as the decay exponent can be determined. In others, such as the logarithmic law, the numerical value of the parameters may presently only be taken from experiments.
- Solutions such as the logarithmic law only explain certain regions of a turbulent flow and are usually embedded within other layers of turbulence. It is an open questions how turbulent scaling laws may be matched to each other.

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