A note on the singular set of area-minimizing hypersurfaces

Nick Edelen

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Abstract
We prove an isoperimetric-type bound on the \((n-7)\)-dimensional measure of the singular set for a large class of area-minimizing \(n\)-dimensional hypersurfaces, in terms of the geometry of their boundary.

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Area-minimizing surfaces in general will not be smooth, and a basic question in minimal surface theory is to understand the size and nature of the singular region. The cumulative works of many (Federer, De Giorgi, Allard, Simons, to name only a few) prove that for absolutely-area-minimizing \(n\)-dimensional hypersurfaces in \(\mathbb{R}^{n+1}\) (“codimension-one area-minimizing integral currents”), the interior singular set is at most \((n-7)\)-dimensional. This dimension bound is sharp, and is directly tied to the existence of low-dimensional, non-flat minimizing cones.

Hardt and Simon [10] proved that for such codimension-one area-minimizers, if the boundary is known to be \(C^{1,\alpha}\) and multiplicity-one, then in fact no singularities lie within a neighborhood of the boundary. Combined with interior regularity, this theorem gives a very nice structure of these minimizing hypersurfaces.

Recently remarkable work of [13,14] quantified the interior partial regularity, by demonstrating effective local (interior) bounds on the \(\mathcal{H}^{n-7}\) measure of the singular set. Their methods also prove \((n-7)\)-rectifiability of the singular set, which was originally established through an entirely different approach by [17].

In this short note, we obtain a global, effective a priori estimate on the \((n-7)\)-Hausdorff measure of singular set for a large class of area-minimizing hypersurfaces in terms of the boundary geometry. Our results are loosely analogous to the a priori bounds of [3] (see also the recent works [11,12]).

We work in \(\mathbb{R}^{n+1}\), for \(n \geq 7\). Let us write \(\mathcal{I}_n(U)\) for the space of integral \(n\)-currents acting on forms supported in the open set \(U\). Given an \(n\)-dimensional, oriented manifold \(E\), write \([E]\) for the current induced by integration. Let \(\eta_\lambda(x) = \lambda x\), and \(\tau_y(x) = x + y\).
If $T \in \mathcal{I}_n(U)$, we say $T$ is area-minimizing if $||T||(W) \leq ||T + S||(W)$ for every open $W \subset U$, and every $S \in \mathcal{I}_n(U)$ satisfying $\partial S = 0$, $\text{spt}S \subset W$. The regular set $\text{reg}T$ is the (open) set of points where $\text{spt}T$ is locally the union of embedded $C^{1,\alpha}$ manifolds (for some $\alpha \in (0, 1)$). The singular set is $\text{sing}T = \text{spt}T \setminus \text{reg}T$. Write $||T||$ for the mass measure of $T$.

Given an $k$-manifold $S$, and $x \in S$, let $r_{1,\alpha}(S, x)$ be the largest radius $r$, so that $(S - x)/r$ is the graph of a $C^{1,\alpha}$ function $u$, with $|u|_{1,\alpha} \leq 1$. Define $r_{1,\alpha}(S) = \inf_{x \in S} r_{1,\alpha}(S, x)$.

Our main Theorem is the following.

**Theorem 0.1** For any $\alpha \in (0, 1)$, there is a constant $c = c(n, \alpha)$ so that the following holds. Let $T \in \mathcal{I}_n(\mathbb{R}^{n+1})$ be area-minimizing. Suppose $\partial T$ is a multiplicity-one, compact, oriented $C^{1,\alpha}$ manifold $S$, and assume that $S$ is contained in the boundary of some convex set. Then

$$\mathcal{H}^{n-7}(\text{sing}T) \leq c(n, \alpha) \frac{||T||(\mathbb{R}^{n+1})}{r_{1,\alpha}(S)^7}. \tag{1}$$

In particular, we have

$$\mathcal{H}^{n-7}(\text{sing}T) \leq c'(n, \alpha) \frac{\mathcal{H}^{n-1}(S)^{\frac{n}{n-1}}}{r_{1,\alpha}(S)^7}. \tag{2}$$

I believe Theorem 0.1 should hold for more general $S$, but there are subtleties even in the idealized case when $S$ is a line. See the discussion below. However, let us make some remarks.

**Remark 0.2** Convexity is not necessary. If there is a domain $\Omega$ with $S \subset \partial \Omega$, and one assumes $\text{spt}T \subset \overline{\Omega}$, it would suffice to know (e.g.): $r_{1,\alpha}(\partial \Omega) \geq r_0 > 0$, or $\Omega$ satisfies an exterior ball condition, of radius $\geq r_0 > 0$. In this case our proof gives

$$\mathcal{H}^{n-7}(\text{sing}T) \leq c(n, \alpha) \max\{r_0^{-7}, r_{1,\alpha}(S)^{-7}\} \mathcal{H}^{n-1}(S)^{\frac{n}{n-1}}.$$ 

In the statement of Theorem 0.1 we assume convexity of $\Omega$, because this becomes purely an assumption on the boundary curve $S$.

**Remark 0.3** The following variant of Theorem 0.1 holds for almost-area-minimizers. Suppose $T$ as in Theorem 0.1 is instead almost-area-minimizing, in the sense that

$$||T||(B_r(x)) \leq ||T + S||(B_r(x)) + c_0 r^{n+2\alpha},$$

for any $S \in \mathcal{I}_n(\mathbb{R}^{n+1})$, $\partial S = 0$, $\text{spt}S \subset B_r(x)$, and some fixed $c_0$. If there is a domain $\Omega$ as in Remark 0.2, then

$$\mathcal{H}^{n-7}(\text{sing}T) \leq c(n, \alpha) \max\{c_0^{\frac{\alpha}{\alpha - 1}}, r_0^{-7}, r_{1,\alpha}(S)^{-7}\} ||T||(\mathbb{R}^{n+1})^{\frac{n}{n-1}}.$$ 

The same proof works, using [4,5] in place of [1,10], and a minor modification of [13].

**Remark 0.4** [13] in fact prove a Minkowski estimate on the singular set, and correspondingly we have a variant of Theorem 0.1 for the Minkowski content: given $T$, $S$, $\alpha$ as in Theorem 0.1, then

$$r^{-8} \text{Vol}(B_r(\text{sing}T)) \leq c(n, \alpha) r_{1,\alpha}(S)^{-7} \mathcal{H}^{n-1}(S)^{\frac{n}{n-1}}$$

for every $0 < r < r_{1,\alpha}(S)$. Here $\text{Vol}(B_r(\text{sing}T))$ denotes the $(n + 1)$-volume of the $r$-tubular neighborhood of $\text{sing}T$. Again, the same proof works, except in Lemma 1.2 we use the Minkowski estimate of [13] instead of the Hausdorff estimate.
We also have a version of Theorem 0.1 in the case when $T$ has free-boundary. Given open sets $U$, $\Omega$, we say $T \in \mathcal{D}(U)$ is area-minimizing with free-boundary in $\Omega$ if: sp$T \subset \Omega$, and $||T||(W) \leq ||S + T||(W)$ for all $W \subset U$, and every $S \in \mathcal{D}(U)$ satisfying sp$S \subset \Omega \cap W$ and sp$(\partial S) \subset \partial \Omega$. Gruter [8] proved boundary singularities have dimension at most $n - 7$.

**Theorem 0.5** Let $\Omega$ be a domain with $C^2$-boundary, and $\infty > r_{1,1}(\partial \Omega) > 0$. Let $T$ be a compactly supported, area-minimizing current with free-boundary in $\Omega$, with $\partial T \cap \Omega = 0$. Then

$$\mathcal{H}^{n-7}(\text{sing} T) \leq c(n) \frac{||T||(\Omega)}{r_{1,1}^{7}(\partial \Omega)}.$$

The key to proving both Theorems is the observation that Naber-Valtorta’s technique gives the following linear interior bound on the singular set: if $T$ is area-minimizing in $U \subset \mathbb{R}^{n+1}$, with $\partial T \cap U = 0$, then for every $\epsilon > 0$, we have:

$$\mathcal{H}^{n-7}(\text{sing} T \cap U \setminus B_{\epsilon}(\partial U)) \leq c(n)\epsilon^{-7}||T||(U \setminus B_{\epsilon/2}(\partial U)).$$

For the Neumann problem (Theorem 0.5), we can adapt the techniques of [13] and the regularity of [9] to prove a similar linear estimate on the singular set in a neighborhood of the barrier. Unfortunately, it’s not clear that a good Dirichlet boundary version of [13] exists, in any more generality than is considered in Theorem 0.1.

There are two problems. First, both we and [13] require good control on the mass of $T$, either directly or indirectly by “nicely” splitting $T$ into a union of minimizers with good mass control. This holds classically for area-minimizers away from the boundary, and also near a Neumann boundary, but as illustrated in Example 0.6 can fail near a Dirichlet boundary.

Second, [13] is fundamentally a measure theoretic result, proving an estimate on sets determined by symmetries (the $(k, \epsilon)$-strata), and to relate this back to the singular set one needs some kind of regularity theorem which says that whenever $T$ looks “very close” to a plane (i.e. looks close to having maximal symmetry), then $T$ is nearby regular. It is possible a regularity theorem like this fails for the general Dirichlet problem: if there exists a singular, minimizing hypersurface with Euclidean area growth and linear boundary, then by [10] any tangent cone at infinity would be planar. It would be very interesting to construct such an example.

Instead, in the setup of Theorem 0.1, we can prove an effective version of [10], which says that the singular set is some uniform distance away from the boundary curve. It’s tempting to think an ineffective, quantitative version of [10] might hold for more general Dirichlet setups, but the problem seems to be the same.

The following examples illustrates some of the problems in extending our proof of Theorem 0.1 to more general settings.

**Example 0.6** Both the half-helicoid and half of Enneper’s surface [15,18] are area-minimizing 2-dimensional currents in $\mathbb{R}^3$. (For the half-helicoid, just observe that by rotating the half-helicoid about the $z$-axis, one obtains a smooth foliation of $\mathbb{R}^3 \setminus z$-axis by oriented minimal surfaces).

The half-helicoid structure could be seen locally for finite $S$, if one does not assume a priori area bounds on $S$. For example, one can imagine a connected boundary curve $S$, which is composed of line segment $L$, and a curve that wraps around $L$ many times. By taking the wrapping curve to go further and further out, one can arrange $S$ to satisfy $r_{1, \alpha}(S) \geq 1$, but take the separation along $L$ of the wrappings to zero. The minimizing integral current $T$ spanning $S$ will look very much like a compressed half-helicoid near the line segment. We cannot decompose this $T$ near $L$ into pieces of uniformly bounded area.
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1 Proof of Theorem 0.1

The following quantifies Hardt–Simon’s boundary regularity.

Lemma 1.1 There is a constant $\epsilon_1(n, \alpha)$, so that the following holds. Let $T$ and $S$ be as in Theorem 0.1. Then for all $x$ $\in$ $\text{sing}T$, we have

$$\inf_{y \in S} |x - y| \geq \epsilon_1(n, \alpha).$$

(2)

Proof Towards a contradiction, suppose we have minimizing currents $T_i$, with boundary curves $S_i$, each contained the boundary of the convex set $\Omega_i$, and $x_i \in \text{sing}T_i$, and $y_i \in S_i$, so that

$$\text{dist}(x_i, S_i) \leq r_{1, \alpha}(S, y_i)/i.$$  

By the maximum principle, $\text{spt}T_i \subset \overline{\Omega_i}$, and by [10], $\text{dist}(x_i, S_i) > 0$. Since $r_{1, \alpha}(S, y) \leq r_{1, \alpha}(S, y)/2$ for $y \in B_{r_{1, \alpha}(S, y)/2}(y)$, there is no loss in generality in assuming that $y_i$ realizes the distance in $S$ to $x_i$.

After a rotation, dilation, translation, we can assume $y_i = 0$, $r_{1, \alpha}(S_i, y_i) = 2$, and $e_1$ is a choice of vector so that $\Omega_i \subset \{ x : x \cdot e_1 < 0 \}$. Moreover, we can take $S_i \cap B_1$ to be the graph of a function $u_i$, defined on the line $L = \{ x_1 = x_{n+1} = 0 \}$, with $|u_i|_{1, \alpha} \leq 1$. Notice that $\text{dom}(u_i) \supset B_{1/2} \cap L$, and that $x_i \to 0$. Let us assume $x_i \in B_{1/2}$ for all $i$.

Let $h(t, x) : [-1, 1] \times (L \cap B_{1/2}) \to \mathbb{R}^{n+1}$ be defined as

$$h(t, x) = \begin{cases} x + tu_i(x) & t \geq 0 \\ x - t\sqrt{1 - |x|^2}e_1 & t \leq 0 \end{cases},$$

and let $R_i = (h_i)_c([-1, 1] \times [L \cap B_{1/2}])$. Then, as an element of $\mathcal{I}_n(B_{1/2})$, $\partial R_i = [S_i]_c B_{1/2}$. In particular, $T_i - R_i \in \mathcal{I}_n(B_{1/2})$ has no boundary. By standard decomposition of codimension-one currents [16, Sect. 37], we can find open sets $E_{i, j} \subset E_{i, j+1} \subset \ldots B_{1/2}$, so that $[E_{i, j}] \in \mathcal{I}_n(B_{1/2})$ satisfies:

$$T_i - R_i = \sum_j \partial[E_{i, j}].$$

(3)

I claim that in fact there is $j_i$, so that

$$||T_i|| = ||\partial[E_{i, j_i}] + R|| + \sum_{j \neq j_i} ||\partial[E_{i, j}]||.$$  

(4)

To see this, recall that $T_i \cap \Omega_i^c = 0$ and $R_i \cap \Omega_i^c = R_i$, and hence by (3) we have that $\text{spt}(\partial[E_{i, j_i}] \cap \Omega_i^c) \subset \text{spt} R_i$. $R_i$ is the current associated to an oriented Lipschitz manifold, and $\partial[E_{i, j_i}]$ have multiplicity one, and so by the constancy theorem ([16, Theorem 26.27]) we have $\partial[E_{i, j_i}] \cap \Omega_i^c = k_{i, j} R_i$, for some $k_{i, j} \in \{-1, 0, 1\}$. But now (3) implies $\sum_j k_{i, j} = 1$, giving that there must be $j_i$ for which

$$\partial[E_{i, j_i}] \cap \Omega_i^c = -R_i, \quad \partial[E_{i, j \neq j_i}] \cap \Omega_i^c = 0.$$  

From this (4) follows immediately.
Equation (4) implies $\partial[E_{i,j}] + R_i$ and each $\partial[E_{i,j} \neq j,i]$ are area-minimizing. By volume comparison agains balls, and the estimate $||\partial[E]||(B_r) \leq c(n)r^n$, we get that

$$||\partial[E_{i,j}]||(B_r) \leq c(n)r^n \quad \forall r < 1/4.$$  

After passing to a subsequence, we can break into two cases. First, assume that $x_i \in \text{spt}(\partial[E_{i,j}]) + R_i$ for all $i$. Let $\lambda_i = |x_i|^{-1}$, and consider the dilates $T_i' := \partial[(\eta_{\lambda_i})_2 E_{i,j}] + (\eta_{\lambda_i})_2 R_i$.

So that $T_i'$ has a singularity at distance 1 from $\lambda_i S_i$.

We can pass to a subsequence (also denoted $i$), so that $(\eta_{\lambda_i})_2[E_{i,j}] \rightarrow [E]$, for some open set $E$. Since $\lambda_i S_i \rightarrow L$ in $C^{1,\alpha}$, we have $(\eta_{\lambda_i})_2 R_i \rightarrow [H]$, where $H = \{x_{n+1} = 0, x_1 > 0\}$ and $[H]$ is endowed with the orientation so that $\partial[H] = [L]$.

In particular, we have $T_i' \rightarrow T = \partial[E] + [H]$, where $\partial T = [L]$. Since each $T_i'$ is minimizing, $T$ is minimizing also, and $T_i'$ converge as both currents and measures. By construction, $T$ has a singularity at distance 1 from $L$, $T$ has Euclidean volume growth, and $\text{spt} T \subset \{x : x \cdot e_1 \leq 0\}$.

Since $T$ is minimizing with Euclidean volume growth, we can take a tangent cone $C$ at infinity (as both currents and measures). $C$ satisfies $\partial C = [L]$, and so by [10, Step II] $C$ is planar. Since we can write $C = \partial[F] + [H]$ for some open set $F$, and $\text{spt} C \subset \{x : x \cdot e_1 \leq 0\}$, in fact $C$ must be a multiplicity-one half-plane. By monotonicity we must have that $T$ is a multiplicity-one half-plane also, and hence $T$ is regular. This is a contradiction.

We are left with the second case: for all $i, x_i \in \text{spt}(\partial[E_{i,j}])$ for some $j \neq j_i$. Write $E_i = E_{i,j}$ for the open set, for which $x_i \in \text{spt}(\partial E_i)$. Consider the dilates $E_i' = \lambda_i E_i$. Then we can pass to a subsequence, to get convergence as currents $[E_i'] \rightarrow [E]$, convergence as currents and measures $\partial[E_i'] \rightarrow \partial[E]$, for $0 \neq [E] \in \mathcal{I}_{n+1}(\mathbb{R}^{n+1})$ satisfying: a) $\partial[E]$ is minimizing; b) $\partial[E]$ has a singularity at distance 1 from the origin; and c) $E \subset \{x : x \cdot e_1 \leq 0\}$.

Properties a), c) imply that any tangent cone at infinity of $\partial[E]$ is a multiplicity-one plane, and hence $\partial[E]$ is a multiplicity-one plane. This contradicts property b), and therefore completes the proof of Lemma 1.1. 

**Lemma 1.2** Let $T \in \mathcal{I}_n(B_1)$ be area-minimizing, with $\partial T = 0$. Then we have

$$\mathcal{H}^{n-7}(\text{sing} T \cap B_{1/2}) \leq c(n)||T||(B_1).$$

**Proof.** We can decompose $T = \sum_i \partial[E_i]$, for $E_i \subset E_{i+1} \subset \ldots \subset B_1$, so that $||T|| = \sum_i ||\partial[E_i]||$, and hence each $\partial[E_i] \in \mathcal{I}_n(B_1)$ is minimizing in $B_1$.

Since $||\partial[E_i]||(B_1) \leq ||\partial[E_i \cup B_{3/4}]||(B_1)$, we have $||\partial[E_i]||(B_{3/4}) \leq c(n)$. On the other hand, by monotonicity, if $\text{spt}(\partial[E_i] \cap B_{1/2} \neq \emptyset$, then $||\partial[E_i]||(B_{1/2}) \geq 1/c(n)$. From the estimates of [13], we have

$$\mathcal{H}^{n-7}(\text{sing} \partial[E_i] \cap B_{1/2}) \leq c(n) \leq c(n)||\partial[E_i]||(B_1).$$

We can sum up contributions:

$$\mathcal{H}^{n-7}(\text{sing} T \cap B_{1/2}) \leq \sum_i \mathcal{H}^{n-7}(\text{sing} \partial[E_i] \cap B_{1/2}) \leq c(n) \sum_i ||\partial[E_i]||(B_1) = c(n)||T||(B_1).$$

□
Proof of Theorem 0.1. By scaling, there is no loss in assuming $r_{1, \alpha}(S) = 1$. Lemma 1.1 implies that $B_{\epsilon}(S) \cap \text{sing} T = \emptyset$, where $\epsilon = \epsilon_1(n, \alpha)$.

Let $\{x_j\}_j$ be a maximal $(\epsilon/4)$-net in $\text{spt} T \setminus B_{\epsilon}(S)$. Then the balls $\{B_{\epsilon/2}(x_j)\}_j$ cover $\text{spt} T \setminus B_{\epsilon}(S)$, and the balls $\{B_{\epsilon}(x_j)\}_j$ have overlap bounded by $c(n)$. For each $j$, $\partial T \cup B_{\epsilon}(x_j) = \emptyset$, and so by Lemma 1.2 we have

$$\mathcal{H}^{n-7}(\text{sing} T \cap B_{\epsilon/2}(x_j)) \leq \frac{c(n)}{\epsilon^7} \cdot ||T|| \cdot (B_{\epsilon}(x_j)).$$

Using bounded overlap of the $\{B_{\epsilon}(x_j)\}_j$, and the isoperimetric inequality for currents (see [16, Theorem 30.1] or [2]), we deduce that

$$\mathcal{H}^{n-7}(\text{sing} T) = \mathcal{H}^{n-7}(\text{sing} T \setminus B_{\epsilon}(S))$$
$$\leq \sum_j \mathcal{H}^{n-7}(\text{sing} T \cap B_{\epsilon/2}(x_j))$$
$$\leq c(n, \alpha) \sum_j ||T|| \cdot (B_{\epsilon}(x_j))$$
$$\leq c(n, \alpha)||T|| \cdot (\mathbb{R}^{n+1})$$
$$\leq c(n, \alpha)\mathcal{H}^{n-1}(S)^{n/(n-1)}.$$

\[\square\]

2 Proof of Theorem 0.5

We will show that the arguments of [8,13], and [9] prove the following: there is an $\epsilon = \epsilon(n)$, so that for $x \in \text{spt} T \cap \partial \Omega$, and $r = r_{1,1}(\partial \Omega)$, we have

$$\mathcal{H}^{n-7}(\text{sing} T \cap B_{\epsilon r/2}(x)) \leq c(n)||T|| \cdot (B_{\epsilon r}(x)) \quad (5)$$

Given this estimate, the bound of Theorem 0.5 follows by a straightforward covering argument as in the proof of Theorem 0.1.

By scaling, we can and shall assume that $r_{1,1}(\partial \Omega) = 1/ \Gamma$, for $\Gamma = \epsilon_2(n)$ chosen sufficiently small so that in $B_3(\partial \Omega)$ the nearest-point projection $\xi(x)$ to $\partial \Omega$ is well-defined and satisfies $|\xi|_{C^1} \leq 2$. Define the reflection function $\sigma(x) = 2\xi(x) - x$, and the linear reflection $i_x$ about $T_{\xi(x)} \partial \Omega$.

Take $T \in \mathcal{T}_n(B_2)$ area-minimizing with free-boundary in $\Omega$. Define $T' = T - \sigma_2 T$, so that $\partial T' = 0$. Then we can decompose $T'$ as

$$T' = \sum_i \partial [E_i], \quad ||T'|| = \sum_i ||\partial [E_i]|.$$

for nested open sets $E_i \subset E_{i+1}$. Moreover, since $T' \cup \Omega = T$ we can write

$$T = \sum_i \partial [E_i] \cup \Omega, \quad ||T|| = \sum_i ||\partial [E_i]| \cup \Omega||. \quad \text{(6)}$$

From (6), we get that each $\partial [E_i] \cup \Omega$ is area-minimizing, with free-boundary in $\Omega$. By comparison against $\partial [E_i \cup B_r(x)] \cup \Omega$, we have the a priori mass bounds

$$||\partial [E_i] \cup \Omega|| \cdot (B_r(x)) \leq c(n)r^n \quad \forall B_r(x) \subset \subset B_2.$$
Additionally, [8] showed $T'$ admits a certain almost-minimizing property, in the following sense:

$$||(T'||(B_r(x)) \leq ||T' + S||(B_r(x)) + c(n)\Gamma r||T'||(B_r(x)).$$  \hspace{1cm} (7)

for every $S \in \mathcal{I}_n(B_2)$ with $\partial S = 0$, $\text{spt} S \subset B_r(x)$, and every $B_r(x) \subset B_2$ with $x \in \partial \Omega$.

Gruter and Jost [9] define the following monotonicity. For $x \in B_1$, and $r < 1 - |x|$, let

$$\tilde{\theta}_T(x, r) = r^{-n}||T'||(B_r(x)) + r^{-n}||T'||\{y : |\sigma(y) - x| < r\}.\hspace{1cm} (\text{8})$$

Notice that when $\Omega$ is a half-space, then $\tilde{\theta}_T(x, r) = \theta_T(x, r)$, and in general we have $\tilde{\theta}_T(x, r) = \theta_T(x, r)$ when $r < \text{dist}(x, \partial \Omega)$. Here $\theta_T(x, r) = r^{-n}||T'||(B_r(x))$ for the usual Euclidean density ratio, and $\tilde{\theta}_T(x) = \lim_{r \to 0} \tilde{\theta}_T(x, r)$ whenever it exists.

For $0 < s < r < 1 - |x|$, [9] prove

$$\int_{B_r(x) \setminus B_s(x)} |y - x|^{-n-2} \left(|(y - x)^\perp|^2 + |i_y(\sigma(y) - x)^\perp|^2\right) d||T'||(y) \leq \tilde{\theta}_T(x, r) - \tilde{\theta}_T(x, s) + c(n)\Gamma r\tilde{\theta}_T(n, r).$$ \hspace{1cm} (8)

Monotonicity (8) implies that the density $\tilde{\theta}_T(x) = \lim_{r \to 0} \tilde{\theta}_T(x, r)$ is a well-defined, upper-semi-continuous function on $B_1$, which is $\geq 1$ on $\text{spt} T$.

The above discussion, and the works of [8,9], give:

**Lemma 2.1** Let $\Omega_i$ be a sequence of $C^2$ domains, with $r_{1,1}(\partial \Omega_i \cap B_2) \to \infty$, and $T_i \in \mathcal{I}_n(B_2)$ a sequence of area-minimizing currents with free-boundary in $\Omega_i$. Suppose $T_i \to T$. Then

1. $T$ is area-minimizing, with free-boundary in a half-space, and $||T_i|| \to ||T||$.
2. $T_i' \to T'$ as currents and measures, and $\tilde{\theta}_{T_i'}(x, r) \to \tilde{\theta}_{T'}(x, r)$ for all $x \in B_2$, and a.e. $0 < r < 2 - |x|$. Here $T_i' = T_i - \sigma_i\eta T_i$, where $\sigma_i$ is the reflection function associated to $\Omega_i$.
3. If $x_i \to x \in B_2$, and $r_i \to 0$, then $\lim \sup\tilde{\theta}_{T_i'}(x_i, r_i) \leq \tilde{\theta}_T(x) = \theta_{T'}(x)$.\hspace{1cm} (4)
4. If $T'$ is regular, then the $T_i' \cap B_1$ are regular for $i$ sufficiently large.

**Proof** Since $\sigma_i \to \sigma$ in $C^1$, we have $T_i' \to T'$. The convergence of measures $||T_i'|| \to ||T'||$ is a standard argument using the almost-minimizing property (8). Convergence $||T_i|| \to ||T||$ then follows from the fact that $T_i' \cap \Omega_i = T_i$.

Convergence of the $\tilde{\theta}_T$ follows because we can estimate

$$\left||T_i||\{y : |\sigma_i(y) - x| < r\}\right| - ||T_i||\{\sigma(B_r(x))\} \leq c||T_i||\{B(1 + \kappa_i)r(x) \setminus B(1 - \kappa_i)r(x)\},$$

where $\kappa_i \to 0$ as $i \to \infty$, and because $||T||(\partial B_r(x)) = 0$ for a.e. $r$. Upper-semi-continuity follows by convergence of $\tilde{\theta}_T(x, r)$, and monotonicity.

The last property (4) is a direct consequence of the decomposition (6) and the Allard-type regularity theory of [9].\hspace{1cm} $\square$

We show the following variant of [13] (recall that $r_{1,1}(\partial \Omega) = 1/\Gamma$).

**Theorem 2.2** (compare from [13]). There is an $\epsilon_3 = \epsilon_3(n, \Lambda)$, so that if $T \in \mathcal{I}_n(B_2)$ is area-minimizing, with free-boundary in $\Omega$, and $||T|| \leq \Lambda$, and $\Gamma \leq \epsilon_3$, then

$$\mathcal{H}^{n-7}(\text{sing}T \cap B_1) \leq c(n, \Lambda).$$
When $T = \partial [E]$, we get $\Lambda = c(n)$, and then using the decomposition (6) in an identical argument to Lemma 1.2, we deduce the required (5).

The argument of [13] requires only the monotonicity formula (8), and the following two theorems, which are essentially Lemmas 7.2, 7.3 and Theorem 6.1 in [13] (or Lemma 3.1, Theorem 5.1 in [6]). The rest of [13] is entirely general (see e.g. [7]).

**Theorem 2.3** There is an $\eta_0 = \eta_0(n, \alpha, \Lambda, \gamma, \rho)$, so that the following holds. Take $B_{9r}(x) \subset B_2$. Let $T \in \mathcal{I}_n(B_{9r}(x))$ be an area-minimizer with free-boundary in $\Omega$, and take $\eta \leq \eta_0$. Suppose

$$\tilde{\theta}_T(x, 6r) \leq \Lambda, \quad \Gamma \leq \eta, \quad \sup_{B_3r(x)} \tilde{\theta}_T(z, 3r) \leq E,$$

then at least one of the following occurs:

1. We have
   $$\text{sing} T \cap B_r(x) \subset \{z \in B_r(x) : \tilde{\theta}_T(z, \gamma r) \geq E - \gamma, \ \text{or} \}.$$
2. There is an affine $(n - 8)$-space $p + L^{n-8}$, so that
   $$\{z \in B_r(x) : \tilde{\theta}_T(z, 3\eta r) \geq E - \eta\} \subset B_{pr}(p + L).$$

**Theorem 2.4** There is a $\delta(n, \alpha, \Lambda)$ so that the following holds. Take $B_{10r}(x) \subset B_2$. Let $T \in \mathcal{I}_n(B_{10r}(x))$ be an area-minimizer with free-boundary in $\Omega$, and $\mu$ a finite Borel measure. Suppose that

$$\tilde{\theta}_T(x, 10r) \leq \Lambda, \quad \Gamma \leq \delta, \quad \tilde{\theta}_T(x, 8r) - \tilde{\theta}_T(x, \delta r) < \delta, \quad x \in \text{sing}(T).$$

Then we have

$$\inf_{p + L^{n-7}} \frac{1}{r^{n-7}} \int_{B_r(x)} \text{dist}(z, p + L)^2 d\mu(z) \leq c(n, \alpha, \Lambda) \int_{B_r(x)} \tilde{\theta}_T(z, 8r) - \tilde{\theta}_T(z, r) + c(n)\Gamma r d||T||_p(z),$$

where the infimum is over affine $(n - 7)$-planes $p + L^{n-7}$.

**Proof of Theorem 0.1.** The proof consists of two contradiction arguments, verbatim to Theorem 5.1 in [6]. In place of the $\epsilon$-strata, we use the following consequence of Lemma 2.1: Suppose $T_i \in \mathcal{I}_n(B_6)$ is a sequence of area-minimizers with free-boundary in $\Omega_i$, so that $r_{1,1}(\partial \Omega_i \cap B_6) \rightarrow \infty$ and $T_i \rightarrow T$. If $T'_\perp B_2$ coincides with a cone, that is invariant along an $(n - 6)$-space, then $T'_\perp B_1$ is regular for sufficiently large $i$. □

**Proof of Theorem 2.4.** The proof divides into two parts, which are verbatim to Lemma 6.2 and Proposition 6.6 in [13] (or Theorem 5.1 in [6]). The first part is a direct consequence of the monotonicity formula (8). The second part is a straightforward contradiction argument. The proof in [13] uses varifold convergence. For integral currents, one can use the fact for any $(n + 1)$-form $\omega$ and any vector $v$, we have

$$<\vec{T}, \omega \wedge v > = < v \wedge \vec{T}, \omega > \quad \text{and} \quad |v \wedge \vec{T}| = |\pi_{T\perp}(v)|. \quad \Box$$
References

1. Allard, W.: On the first variation of a varifold. Ann. Math. 2(95), 417–491 (1972)
2. Almgren Jr., F.J.: Optimal isoperimetric inequalities. Indiana Univ. Math. J. 35, 451–547 (1986)
3. Almgren, F.J., Lieb, Elliott H.: Singularities of energy minimizing maps from the ball to the sphere: examples, counterexamples, and bounds. Ann. Math. 128(3), 483–530 (1988)
4. Bombieri, E.: Regularity theory for almost minimal currents. Arch. Rational Mech. Anal. 78, 99–130 (1982)
5. Duzaar, F., Steffan, K.: Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals. J. Reine Angew. Math. 546, 73–138 (2002)
6. Edelen, N., Engelstein, M.: Quantitative stratification for some free-boundary problems. Trans. Am. Math. Soc. (2017)
7. Edelen, N.: Notes on a measure theoretic version of Naber–Valtorta’s rectifiability theorem. http://math.mit.edu/~edelen/general-nv.pdf. Accessed 10 Jan 2019
8. Gruter, M.: Optimal regularity for codimension one minimal surfaces with a free-boundary. Manuscr. Math. 58, 295–343 (1987)
9. Gruter, M., Jost, J.: Allard type regularity results for varifolds with free boundaries. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 13, 129–169 (1986)
10. Hardt, R., Simon, L.: Boundary regularity and embedded solutions for the oriented plateau problem. Ann. Math. 2(110), 439–486 (1979)
11. Mazowiecka, K., Miskiewicz, M., Schikorra, A.: On the size of the singular set of minimizing harmonic maps into the sphere in dimension three (2018). arXiv:1811.00515
12. Mazowiecka, K, Miskiewicz, M., Schikorra, A.: On the size of the singular set of minimizing harmonic maps into a 2 sphere in dimension four and higher (2018). arXiv:1902.03161
13. Naber, A., Valtorta, D.: The singular structure and regularity of stationary and minimizing varifolds (2015). arXiv:1505.03428
14. Naber, A., Valtorta, D.: Rectifiable-reifenberg and the regularity of stationary and minimizing harmonic maps. Ann. Math. 2(185), 131–227 (2017)
15. Perez, J.: Stable embedded minimal surfaces bounded by a straight line. Calc. Variations Partial Differ Equ 29, 267–279 (2007)
16. Simon, L.: Lectures on geometric measure theory. In: Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3. Australian National University, Centre for Mathematical Analysis, Canberra (1983)
17. Simon, L.: Rectifiability of the singular set of energy minimizing maps. Calc. Variations Partial Differ. Equ. 3, 1–65 (1995)
18. White, B.: Half of Enneper’s surface minimizes area. In: Jost, J. (ed.) Geometric Analysis and the Calculus of Variations for Stefan Hildebrandt, pp. 361–367. International Press, Somerville (1996)

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