Local stability conditions for asymmetric Fermi superfluid

S.-K. Yip

Institute of Physics, Academia Sinica, Nankang, Taipei 115, Taiwan

(Dated: September 30, 2018)

Abstract

Two different local stability conditions for an asymmetric superfluid has been discussed in the literature. We here consider the relations between them.

PACS numbers: 03.75.Ss, 05.30.Fk, 34.90.+q
In our recent paper [1], we considered a two component Fermi gas with unequal populations under a short-ranged attractive interaction. We constructed the phase diagram by first finding uniform state solutions to the number and gap equations. We then impose the local stability condition that the susceptibility matrix has to have only positive eigenvalues. In particular then, in terms of the differences in populations and chemical potentials $n_d$ and $h$, local stability/instability is decided via

$$\left( \frac{\partial n_d}{\partial h} \right)_\mu > 0$$  \hspace{1cm} (1)

Here $\mu$ is the average chemical potential. There we stated, without providing details, that a negative susceptibility implies instability towards phase separation. We thus then exclude all states with negative susceptibilities and identified the corresponding point in the phase diagram as not having a stable uniform state.

The phase diagram and local stability condition have been discussed in a different manner [2, 3, 6, 7, 8, 9, 10]. These authors considered the free energy of the system $\tilde{\Omega}(\Delta; \mu, h)$ as a function of the order parameter $\Delta$ for fixed $\mu$ and $h$ [11]. The locally stable states are selected according to the condition that $\tilde{\Omega}$ must be a relative minimum of $\tilde{\Omega}$ as a function of $\Delta$. Thus, at the extremum points $\left( \frac{\partial \tilde{\Omega}}{\partial \Delta} \right)_{\mu, h} = 0$, they apply the condition

$$\left( \frac{\partial^2 \tilde{\Omega}}{\partial \Delta^2} \right)_{\mu, h} > 0$$  \hspace{1cm} (2)

to distinguish locally stable and unstable states.

Since our condition eq (1) was stated without providing detailed justifications, there are some confusions as to how the conditions (1) and (2) are related. Sheehy and Radzihovsky [7, 8] suggest recently that states with positive susceptibilities can still correspond to a relative maximum of the free energy $\tilde{\Omega}(\Delta; \mu, h)$. Other groups [12, 13, 14] have set out to show the relation between the conditions (1) and (2). These proofs, while correct, involve algebra which motivation may be a priori somewhat mysterious. The purpose of this note is to provide perhaps a physically more transparent picture on the relations between the conditions (1) and (2).

We start by considering the locus of the relative extremum of $\tilde{\Omega}(\Delta, \mu, h)$, that is, the solution to the gap equation $\left( \frac{\partial \tilde{\Omega}}{\partial \Delta} \right)_{\mu, h} = 0$. This gives $\Delta$ as a function of $\mu$ and $h$. For definiteness [3], we consider a "cut" of this function at some given $\mu$. First, let us note
that the normal state $\Delta = 0$ is always a solution to this gap equation, and hence always an extremum of $\tilde{\Omega}$ (when considered a function of $\Delta$ at fixed $\mu$ and $h$). For some region of parameters, there may be in addition non-trivial ($\Delta \neq 0$) solution to the gap equation. An example deep in the BEC (strong attractive interaction) limit is as sketched in the inset of Fig 1. The non-trivial solution to the gap equation gives a $\Delta$ that is monotonically decreasing with $h$. (This non-trivial solution for $h < h_1$ represents a gapless superfluid [1]).

This function however is not necessarily single valued. An example is as shown in the main part of Fig 1. This happens in particular in the weak interaction limit, as was first shown by Sarma [4] (and continues to hold until the attractive interaction is sufficiently strong). There, the non-trivial solution for $\Delta$, instead of decreasing monotonically with $h$, "bends around" at the point B. For $h < h_A$, there is one non-trivial solution (labelled by S-A), whereas there are two non-trivial solutions for $h_A < h < h_B$ (labelled by A-B and B-C).

First we note a simple but useful fact. The free energy $\tilde{\Omega}$ must continue to increase as $\Delta \to \infty$, as $\Delta = \infty$ is not a solution to the gap equation, and $\tilde{\Omega}$ cannot decrease indefinitely. Since the solutions to the gap equation already give all the extrema, it follows that the solution to the gap equation with the largest $\Delta$ always corresponds to a relative minimum of $\tilde{\Omega}$. In fact, we have the following statements:

1. If $\Delta = 0$ is the only solution to the gap equation, then it is the free energy minimum.
2. If there is only one non-trivial solution $\Delta = \Delta_1$, then it is the free energy minimum, and the normal state is a relative maximum. (In this case, starting from $\Delta = 0$, $\tilde{\Omega}$ first decreases, then reaches its minimum at $\Delta_1$, then increases for all $\Delta > \Delta_1$.) This applies to the inset for $h < h_1$, and $h < h_A$ for the main Fig 1.
3. If there are two non-trivial solutions $\Delta_1 < \Delta_2$, then $\Delta_1$ must be a relative maximum, whereas both $\Delta_2$ and the normal state are relative minima. (In this case, starting from $\Delta = 0$, $\tilde{\Omega}$ first increases, then reaches a relative maximum at $\Delta_1$, decreases till $\Delta_2$, then increases again for all $\Delta > \Delta_2$.

Therefore, for the non-trivial solutions in Fig 1, the branch S-A is the absolute minimum for $\tilde{\Omega}$, A-B is a relative minimum, B-C must be a relative maximum. For the trivial solution $\Delta = 0$, it corresponds to a relative maximum for $h < h_C = h_A$ and a relative minimum for $h > h_C$. (see also the last paragraph of this note).

Now we consider $n_d$ as a function of $h$. For the situation in the inset of Fig 1 there is one branch of $n_d$ for the normal state, and one for the superfluid state. A sketch is as
shown in the inset of Fig 2, where these two branches are denoted by the dashed and full lines respectively. Note that the full line must have positive slope as it is stable. Therefore there is one possible $n_d$ for each $h$ for $h > h_1$, but two for each $h < h_1$. Now consider decreasing the attractive interaction (or increasing $\mu$) so that the $\Delta$-$h$ plot evolves from the form in the inset of Fig 1 to that in the main part of that figure. Consider now what must happen to the $n_d$-$h$ plot. There continues to be one branch (dashed line in main part of Fig 2) which corresponds to the normal state. For the non-trivial solution to the gap equation (S-A-B-C in Fig 1), there is an S-A-B part in Fig 2 with positive slope corresponding to the locally stable S-A-B part in Fig 1. However, since there is one $n_d$ for each solution to the gap equation, there must be two possible $n_d$’s corresponding to the non-trivial solutions in the range $h_C = h_A < h < h_B = h_D$. Thus if we follow the path S-A-B-C in Fig 1 the corresponding path in the $n_d$ versus $h$ plot must also ”turn around” at point B, where the $\Delta$ versus $h$ curve turns around. At this point, \( \left( \frac{\partial n_d}{\partial h} \right)_\mu \) necessarily diverges and changes sign at B, As a result, the branch BC in Fig 2 has \( \left( \frac{\partial n_d}{\partial h} \right)_\mu < 0 \), corresponding to \( \left( \frac{\partial^2 \tilde{\Omega}}{\partial \Delta^2} \right)_{\mu, h} < 0 \) in Fig 1.

At this point let us make connection with the arguments given in [12, 13, 14, 15]. In essence, these arguments are based on the following equation or its variation:

\[
\left( \frac{\partial n_d}{\partial h} \right)_\mu = \left( \frac{\partial \tilde{n}_d}{\partial h} \right)_{\Delta, \mu} + \frac{\left( \frac{\partial \tilde{n}_d}{\partial \Delta} \right)^2_{\mu, h}}{\left( \frac{\partial^2 \tilde{\Omega}}{\partial \Delta^2} \right)_{\mu, h}}.
\]

(3)

Here $\tilde{n}_d$ is a function of $\Delta$, $\mu$, $h$ defined by the relation $\tilde{n}_d \equiv -\left( \frac{\partial \tilde{\Omega}}{\partial h} \right)_{\Delta, \mu}$. Note that, at the solution to the gap equation, $n_d(\mu, h) = \tilde{n}_d(\Delta, \mu, h)$ since $n_d = -\left( \frac{\partial \tilde{\Omega}}{\partial h} \right)_\mu = -\left( \frac{\partial \tilde{\Omega}}{\partial h} \right)_{\Delta, \mu}$ since \( \left( \frac{\partial \tilde{\Omega}}{\partial \Delta} \right)_{\mu, h} = 0 \). Suppose we follow the line S-A-B-C in Fig 1. At B, one changes from a branch where \( \left( \frac{\partial^2 \tilde{\Omega}}{\partial \Delta^2} \right)_{\mu, h} > 0 \) to < 0. Since \( \left( \frac{\partial^2 \tilde{\Omega}}{\partial \Delta^2} \right)_{\mu, h} \) appears in the denominator of eq (3), \( \left( \frac{\partial n_d}{\partial h} \right)_\mu \) must changes from positive to negative through $\pm \infty$ near B [17]. As we have already seen above, this behavior is a natural consequence of the fact that the curve $n_d$ versus $h$ must turn around at point B in Fig 2.

As seen from eq (3) however, since \( \left( \frac{\partial n_d}{\partial h} \right)_{\Delta, \mu} > 0 \) always (c.f. [8]), there seems no general principle requiring that \( \left( \frac{\partial n_d}{\partial h} \right)_\mu < 0 \) whenever \( \left( \frac{\partial^2 \tilde{\Omega}}{\partial \Delta^2} \right)_{\mu, h} < 0 \). Similarly, in principle with further change of interaction parameters, some part of the BC curve in Fig 2 may evolve further so that it can acquire a positive slope. However, for this to happen, it can easily be
seen that one must have some region of $n_d$ which corresponds to at least three possible $h$’s, and with at least one point where $(\frac{\partial n_d}{\partial h})_{\mu}$ vanishes. This does not seem to happen for the current system of asymmetric Fermi gas in mean-field theory. Moreover, there is no reason why the change to positive slope would occur at the same point in parameter space as where $\Delta$ versus $h$ turns around in Fig 1. Thus evaluating the susceptibility should still indicate when one goes from a local minimum to maximum of $\tilde{\Omega}$ as a function of $\Delta$.

Finally, we notice here also that Fig 2 is similar to the one in theory of liquid gas transition such as the Van der Waals equation of state for a non-ideal gas. $h$ and $-n_d$ here plays the role of the pressure $P$ and volume $V$ respectively. (A sign change in $n_d$ is necessary since here $d\Omega = -n_d dh$, whereas for the gas, the differential of Gibbs free energy is $dG = VdP$). Indeed, Fig 2 has precisely that form if we make the above identification (and ignore the branch NC, which is never stable). The branch BC there is locally unstable since the compressibility is negative. By again the analogy to the Van der Waals gas, the absolutely stable branch can be determined by Maxwell construction. These branches are S-A-E$_s$ for $h < h_E \equiv h_{Es} = h_{En}$ and E$_n$-D-N’ if $h > h_E$, with the condition that the areas E$_s$-B-M and M-C-E$_n$ are equal. At $h = h_E$, the system phase separates into a mixture with part of the volume in state E$_s$ and part in E$_n$.

This research was supported by the NSC of Taiwan under grant number NSC94-2112-M-001-002.

[1] C.-H. Pao, S.-T. Wu and S.-K. Yip, Phys. Rev. B 73, 132506 (2006) (see also Ref [15])

[2] We shall first discuss the local stability condition. After the local minima are found, the state that is globally stable can be found by comparing the free energies of these local minima (see below)

[3] Since we have a two component gas, we have two concentrations $n_{\sigma}$ and two chemical potentials $\mu_{\sigma}$ where $\sigma = \uparrow, \downarrow$ for the two species. In general we have to consider the susceptibility matrix $\mathcal{M}_{\sigma,\sigma'} \equiv \left(\frac{\partial n_{\sigma}}{\partial \mu_{\sigma'}}\right)$. It is clear that the susceptibility in eq (1) is just the ”expectation value” of $\mathcal{M}$ taken along a ”vector” in the $(1, -1)$ direction in the $(\mu_{\uparrow}, \mu_{\downarrow})$ plane. It can be easily seen that the same argument in text applies for any such directions, and hence the set of conditions such as eq (1) is equivalent to positive versus negative eigenvalues of the matrix $\mathcal{M}$. 


[4] G. Sarma, Phys. Chem. Solids 24, 1029 (1963)

[5] P. F. Bedaque, H. Caldas and G. Rupak, Phys. Rev. Lett. 91, 247002 (2003)

[6] D. E. Sheehy and L. Radzihovsky, Phys. Rev. Lett. 96, 060401 (2006)

[7] D. E. Sheehy and L. Radzihovsky, cond-mat/0607803

[8] D. E. Sheehy and L. Radzihovsky, cond-mat/0608172

[9] Z.-C. Gu, G. Warner and F. Zhou, cond-mat/0603091

[10] M. M. Parish, F. M. Marchetti, A. Lamacraft and B. D. Simons, cond-mat/0605744

[11] We use a tilde to denote the free energy as a function of the variable $\Delta$ before it has been eliminated. The true free energy $\Omega(\mu, h)$ is obtained by evaluating this function $\tilde{\Omega}$ at its minimum.

[12] L. Y. He, M. Jin and P. F. Zhuang, cond-mat/0606322

[13] C.-C. Chien, Q. Chen, Y. He and K. Levin, cond-mat/0605039

[14] Q. Chen, Y. He, C.-C. Chien and K. Levin, cond-mat/0608454

[15] C. H. Pao, S. T. Wu and S.-K. Yip, cond-mat/0608501

[16] At zero temperature, part or all of the branch S-A-B corresponding to the superfluid in in Figs 1 and 2 have zero slope. Moreover, strictly speaking the dashed line in Fig 2 is not a simple straight line. We ignore these complications since they have no effect on our arguments below.

[17] For this argument to hold, the quantity $\left(\frac{\partial n_d}{\partial \Delta}\right)^2_{\mu, h}$ must not vanish. In particular it does not apply to the branch NC. On approaching C, one can show by a Ginzburg-Landau type of argument that the second term in eq 3 remains finite.
FIG. 1: $\Delta$ versus $h$ at a given chemical potential.

FIG. 2: $n_d$ versus $h$ at a given chemical potential.