STRONG MINIMALITY OF THE $j$-FUNCTION

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ABSTRACT. We show that the order three algebraic differential equation over $\mathbb{Q}$ satisfied by the analytic $j$-function defines a non-$\aleph_0$-categorical strongly minimal set with trivial forking geometry relative to the theory of differentially closed fields of characteristic zero answering a long-standing open problem about the existence of such sets. The theorem follows from Pila’s modular Ax-Lindemann-Weierstrass with derivatives theorem using Seidenberg’s embedding theorem and a theorem of Nishioka on the differential equations satisfied by automorphic functions. As a by product of this analysis, we obtain a more general version of the modular Ax-Lindemann-Weierstrass theorem, which, in particular, applies to automorphic functions for arbitrary arithmetic subgroups of $SL_2(\mathbb{Z})$.

1. INTRODUCTION

According to Sacks, “[t]he least misleading example of a totally transcendental theory is the theory of differentially closed fields of characteristic 0 (DCF₀)” [22]. This observation has been borne out through the discoveries that a prime differential field need not be minimal [21], the theory DCF₀ has the ENI-DOP property [12], and Morley rank and Lascar rank differ in differentially closed fields [7], amongst others. However, the theory of differentially closed fields of characteristic zero does enjoy some properties not shared by all totally transcendental theories, most notably the Zilber trichotomy holds for its minimal types [8] and there are infinite definable families of strongly minimal sets for which the induced structure on each such definable set is $\aleph_0$-categorical and orthogonality between the fibres is definable [5,6]. Early in the study of the model theory of differential fields, Lascar asked whether it might be the case that the induced structure on every strongly minimal set orthogonal to the constants must be $\aleph_0$-categorical [9]. From the existence of Manin kernels, one knows that there are strongly minimal sets relative to DCF₀ which are not $\aleph_0$-categorical [4], but the question of whether there are non-$\aleph_0$-categorical strongly minimal sets with trivial forking geometry has remained open [20]. We exhibit an explicit equation defining a set with such properties.

The analytic $j$-function, $j : h \to \mathbb{C}$, which has been known to mathematicians for quite some time, appearing implicitly in the work of Gauss already in the late Eighteenth Century [3], satisfies a differential equation over $\mathbb{Q}$ which when evaluated in a differentially closed field defines a non-$\aleph_0$-categorical strongly minimal set with trivial forking geometry.

This paper is organized as follows. In Section 2 we recall some of the basic theory of the $j$-function, including the theory of the Schwarzian derivative and the differential equation satisfied by the $j$-function.

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equation satisfied by \( j \). In Section 3 we recount a theorem of Nishioka showing that the \( j \)-function does not satisfy any nonzero algebraic differential equations over \( \mathbb{C} \) of order two or less. With Section 4 we complete the proof of our main theorem and draw some corollaries. The main ingredients of the proof, in addition to Nishioka’s theorem, are Seidenberg’s embedding theorem and a Pila’s modular Ax-Lindemann-Weierstrass with derivatives theorem.

2. BASIC THEORY OF THE \( j \)-FUNCTION

In this section we summarize some of the basic theory of the \( j \)-function. We denote the upper half-plane by

\[ \mathfrak{h} := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} . \]

The \( j \)-function is an analytic function on \( \mathfrak{h} \) whose Fourier expansion begins with

\[ j(\tau) = \exp(-2\pi i \tau) + 744 + 196884 \exp(2\pi i \tau) + 21493760 \exp(4\pi i \tau) + \cdots \]

The algebraic group \( \text{SL}_2(\mathbb{C}) \) acts on the projective line via linear fractional transformations and the restriction of this action to \( \text{SL}_2(\mathbb{R}) \) induces an action of \( \text{SL}_2(\mathbb{R}) \) on \( \mathfrak{h} \). The \( j \)-function is a modular function for \( \text{SL}_2(\mathbb{Z}) \) in the sense that \( j(\gamma \cdot \tau) = j(\tau) \) for each \( \gamma \in \text{SL}_2(\mathbb{Z}) \). Indeed, more is true: \( j(\tau) = j(\sigma) \) if and only if there is some \( \gamma \in \text{SL}_2(\mathbb{Z}) \) with \( \gamma \cdot \tau = \sigma \).

The differential equation satisfied by \( j \) is best expressed using the Schwarzian derivative. Writing \( x' \) for \( \frac{\partial x}{\partial t} \), we define the Schwarzian by

\[ S_{\frac{\partial}{\partial t}}(x) = \left( \frac{x''}{x'} \right)' - \frac{1}{2} \left( \frac{x''}{x'} \right)^2 . \]

The Schwarzian satisfies a chain rule:

\[ S_{\frac{\partial}{\partial t}}(f \circ g) = \left( \frac{\partial g}{\partial t} \right)^2 S_{\frac{\partial}{\partial t}}(f) \circ g + S_{\frac{\partial}{\partial t}}(g) . \]

A characteristic feature of the Schwarzian is that if \( (K, \partial) \) is a differential field of characteristic zero with field of constants \( C = \{ x \in K : \partial(x) = 0 \} \) and \( z \in K \) satisfies \( \partial(z) = 1 \), then for \( y \in K \) one has \( S_\partial(y) = 0 \) if and only if \( y = \frac{az + b}{cz + d} \) for some constants \( a, b, c \) and \( d \).

The following is an order three algebraic differential equation satisfied by \( j \).

\[ E : S_{\frac{\partial}{\partial \tau}}(y) + \frac{y^2 - y + 1}{(4y^2 - 1)(y)^2} \left( \frac{\partial}{\partial \tau} (y) \right)^2 = 0 \]

For the remainder of this paper, when we speak of the differential equation satisfied by \( j \), we mean \( E \).

3. NISHIOKA’S METHOD AND AUTOMORPHIC FUNCTIONS

Nishioka [15] proved a conjecture of Mahler [11] regarding automorphic functions and the differential equations that they might satisfy. In this section, we explain Nishioka’s result and review its proof here, noting certain necessary uniformities.

Let us recall the notion of an automorphic function, specifically adapted to Nishioka’s method. In our application, we take \( G = \text{SL}_2(\mathbb{Z}) \), \( D = \mathfrak{h} \) and \( f = j : \mathfrak{h} \to \mathbb{C} \).
**Definition 3.1.** Let $G \leq \text{SL}_2(C)$ be a subgroup. A function $f(z)$ which is analytic on some domain $D \subset \mathbb{P}^1$ is called automorphic if it satisfies the following properties:

1. $Tz \in D$ for any $z \in D$ and any $T \in G$.
2. $f(Tz) = f(z)$ for $z \in D$.

The main theorem of [15] is the following.

**Theorem 3.2.** Let $G$ be a Zariski dense subgroup of $\text{SL}_2(C)$. Then any nontrivial automorphic function of $G$ satisfies no algebraic differential equation of order two or less over $C$.

**Remark 3.3.** In [15] the hypothesis in Theorem 3.2 is that $G$ have at least three limit points rather than that it be Zariski dense in $\text{SL}_2(C)$, but these conditions are equivalent as is noted in [15].

**Remark 3.4.** The conclusion of Theorem 3.2 as stated in [15] is ostensibly stronger in that $f$ does not satisfy low order differential equations even over $C(z, \exp(2\pi iz))$. The inclusion of these additional parameters is a red herring. We explain towards the end of this note how such independence follows from the minimality of $tp(j/C)$.

There are some minor errors in Nishioka’s proof of Theorem 3.2. The first is very slight: there is a misplaced reference to Lemma 4 of his paper (which should be to Lemma 3). Somewhat more seriously, the necessary uniformity of the algebraic dependence in the first half of his proof is not mentioned at all. For the sake of completeness, we reproduce his proof with these defects remedied.

**Proof.** Let $f(z)$ be automorphic for $G$. Assume that $z, f(z), \frac{df}{dz}z, \frac{d^2f}{dz^2}z$ are algebraically dependent over $C$; specifically, there is some nonzero polynomial $F$ with constant coefficients for which

$$F(z, f(z), \frac{df}{dz}z, \frac{d^2f}{dz^2}z) = 0.$$ 

Then for each $T \in G$, the same is true when we substitute the variable $Tz$ for $z$. Now our four functions have become:

1. $Tz$
2. $f(Tz) = f(z)$
3. $\frac{df}{dz}Tz = (\frac{df}{dz}z)^{-1} \frac{df}{dz}z$
4. $(\frac{df}{dz})^2 f(Tz) = \left(\frac{df}{dz}z\right)^{-2} \frac{d^2f}{dz^2}z - \left(\frac{df}{dz}z\right)^{-3} \left(\frac{d^2f}{dz^2}Tz\right) \frac{df}{dz}z$.

So, by our earlier remarks, we have that

$$F(Tz, f(z), (\frac{df}{dz}z)^{-1} \frac{df}{dz}z, (\frac{df}{dz}z)^{-2} \frac{d^2f}{dz^2}z - (\frac{df}{dz}z)^{-3} (\frac{d^2f}{dz^2}Tz) \frac{df}{dz}z) = 0.$$ 

By clearing denominators, we obtain a nonzero polynomial over $C(z)$ which vanishes on the triple $(Tz, \frac{df}{dz}z, \frac{d^2f}{dz^2}z)$ for all $T \in G$. This violates Lemma 3 of [15].

□

4. **Minimality and the j-function**

In this section, we deduce our main theorem on the strong minimality of the set defined by $E$. Model theoretic notation is standard and generally follows that of [19]. We regard
Theorem 4.4. 

Proof. We need to check that any forking extension of \( \tau \) over any superset of \( C \) is algebraic. If \( B \supseteq C \) is any superset of \( C \) in our differentially closed field for which \( \text{tp}(j/B) \) forks over \( C \), then by the Shelah reflection principle mentioned above, we may find a finite Morley sequence \( \{d_i, \ldots, d_n\} \) in \( \text{tp}(j/B) \) which is not independent over \( C \). By Lemma 4.2, we may realize \( d_1, \ldots, d_n \) as \( j(g_1 \tau), \ldots, j(g_n \tau) \) for some \( g_i \in GL_2(C) \). The modular Ax-Lindemann-Weierstrass with derivatives theorem of [17] asserts that if \( g_1, \ldots, g_n \) are in distinct cosets
of $GL_2(\mathbb{Q})$, then $j(g_1\tau), \ldots, j(g_n\tau)$ are independent over $\mathbb{C}$. However, if $g_i$ and $g_j$ are in the same coset of $GL_2(\mathbb{Q})$, then $j(g_i\tau)$ and $j(g_j\tau)$ are interalgebraic over $\mathbb{Q}$ as witnessed by an appropriate modular polynomial $F_N(x, y)$ [14, see pages 183-186]; Pila [17, 16] calls these modular relations. The only way that the elements of a Morley sequence may be interalgebraic is if the type itself is algebraic. Hence, from the dependence of the Morley sequence we deduce that $tp(j/B)$ is algebraic, as required. □

Using Nishioka’s theorem, we strengthen Theorem 4.4 to the conclusion that $E$ defines a strongly minimal set.

**Theorem 4.5.** The set $X$ defined by the differential equation $E$ is strongly minimal.

**Proof.** As the equation $E$ has degree one in order three, it suffices to show that any differential specialization of $j(\tau)$ over $\mathbb{C}$ satisfies no lower order differential equation. By the proof of Lemma 4.2 and Theorem 4.1 and the fact that any differential specialization $f$ satisfies the equation

$$S_2(f) + \frac{f^2 - f + 1}{4((f)^2 - 1)(f^2)} \cdot \frac{\partial}{\partial t} f^2 = 0$$

one can assume that $f = j(g\tau)$ for some $g \in GL_2(\mathbb{C})$. Now $f$ satisfies the hypotheses of Theorem 3.2 (with $G = SL_2(\mathbb{Z})$) and so it satisfies no nontrivial order two or less equation over $\mathbb{C}$. □

**Remark 4.6.** Recall that we remarked in Remark 5.4 that Nishioka proved a slightly more general statement than that an automorphic form $f(t)$ satisfies no order two differential equation over $\mathbb{C}$. In fact, he proves that the conclusion holds over $\mathbb{C}(t, e^{2\pi i t})$. For the $j$-function, this conclusion follows from minimality of the type $tp(j/C)$. Indeed, this depends very little on the functions $t$ and $e^{2\pi i t}$. The same conclusion holds for any function (or collection of functions) $f(t)$ so that $f(t)$ satisfies an order two (or lower) differential equation over $\mathbb{C}$.

Similar remarks apply to Pila’s theorems [17]. Algebraic equations potentially satisfied by $\{j(g_i t), j'(g_i t), j''(g_i t)\}_{i=1}^n$ where $g_i \in SL_2(\mathbb{C})$ are considered over function fields which include exponential and Weierstrass $\wp$-functions. These satisfy first order differential equations, so the remarks from the previous paragraph apply.

The main theorem of Hrushovski’s manuscript [5] is that if the definable set $X$ is defined by an order one differential equation over the constants and is orthogonal to the constants, then the induced structure on $X$ over any finite set of parameters over which it is defined in $\aleph_0$-categorical. In [20], Rosen extended this theorem dropping the hypothesis that $X$ is defined over the constants, but added certain technical assumptions. It has been known since the identification of Manin kernels that not every strongly minimal which is orthogonal to the constants must have $\aleph_0$-categorical induced structure, but the question of whether a strongly minimal set with trivial forking geometry must have $\aleph_0$-categorical induced structure (raised explicitly by Lascar over thirty years ago [10]) has remained open until now.

**Theorem 4.7.** The set $X$ defined by the differential equation $E$ is strongly minimal with trivial forking geometry but does not have $\aleph_0$-categorical induced structure over any base set.
Proof. Our main theorem, Theorem 4.5, asserts that $X$ is strongly minimal. Triviality of the forking geometry of the generic type of $X$ (and, hence, of $X$ itself) is an immediate consequence of Pila’s modular Ax-Lindemann-Weierstrass theorem with derivatives. Indeed, suppose that $a_1, \ldots, a_n \in X$ are $n$ pairwise independent realizations of the generic type of $X$. We shall check that they are independent as a set. By Theorem 4.2 we may realize these points as meromorphic functions of the form $j(\gamma \cdot t)$ where $\gamma \in \mathrm{GL}_2(\mathbb{C})$. By Pila’s theorem, provided that $\gamma_1, \ldots, \gamma_n$ lie in distinct cosets of $\mathrm{GL}_2(\mathbb{Q})$, the differential field $\mathbb{C}\langle a_1, \ldots, a_n \rangle$ has transcendence degree $3n$ over $\mathbb{C}$, which is the dimension of the field generated by a Morley sequence of length $n$. On the other hand, if $\gamma_i = \delta \gamma_j$ for some $\delta \in \mathrm{GL}_2(\mathbb{Q})$, then $a_i$ and $a_j$ are in a modular relation and, hence, interalgebraic contradicting our assumption that the $a_i$’s are be pairwise independent.

On the other hand, the modular relations show that $X$ is not $\aleph_0$-categorical. All of the functions $j(\gamma \cdot \tau)$ for $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ lie in $X$ and each is algebraic over $j(\tau)$ as witnessed by a modular relation. As $\mathrm{GL}_2(\mathbb{Q})/\mathrm{GL}_2(\mathbb{Z})$ is infinite, we see that there are infinitely many elements of $X$ algebraic over the single element $j(\tau)$. Hence, $X$ does not have $\aleph_0$-categorical induced structure. \qed

Remark 4.8. Suppose that $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ is an arithmetic subgroup. One might inquire about the differential algebraic properties of $j_\Gamma(t)$, where $j_\Gamma$ is the analytic function expressing $\Gamma \backslash \mathfrak{H}$ as an algebraic curve. First, since $j_\Gamma(t)$ is not algebraic and is interalgebraic with $j$ over $\mathbb{Q}$, we can see that the type is strongly minimal. In differential algebra, generally, this would not be enough to conclude that the locus of the type is strongly minimal. However, Nishioka’s theorem applies to automorphic functions in this setting, and so one sees that there are no order two or lower solutions to the differential equation satisfied by $j_\Gamma$. Further, for $g \in \mathrm{SL}_2(\mathbb{C})$, we have the following diagram:

\[
\begin{array}{ccc}
\downarrow & & \\
j_\Gamma(t) & \rightarrow & j_\Gamma(gt) \\
\downarrow & & \\
j(t) & \longleftarrow & j(gt)
\end{array}
\]

The solid vertical lines indicate interalgebraicity. The relationship of $j(t)$ and $j(gt)$ is completely controlled by the Pila’s modular Ax-Lindemann-Weierstrass theorem [16] (and by the stronger theorem, to the derivatives as well [17]). It follows from the interalgebraicity of $j(gt)$ and $j_\Gamma(gt)$, that if $g_1, \ldots, g_n$ lie in distinct cosets of $\mathrm{SL}_2(\mathbb{Q})$, then the functions $j_\Gamma(g_1 t), \ldots, j_\Gamma(g_n t)$ are differentially algebraically independent. That is, the relationship (in terms of algebraic closure in the sense of differential fields) indicated by the top (curly) line is completely controlled by the modular Ax-Lindemann-Weierstrass theorem and the results of this paper; naturally, one obtains as a by-product the Ax-Lindemann-Weierstrass theorem with derivatives for $j_\Gamma(t)$.

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