RECONSTRUCTION OF THE TIME-DEPENDENT SOURCE TERM IN A STOCHASTIC FRACTIONAL DIFFUSION EQUATION

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Abstract. In this work, an inverse problem in the fractional diffusion equation with random source is considered. The measurements we use are the statistical moments of the realizations of single point observation \( u(x_0, t, \omega) \). We build a representation of the solution \( u \) in the integral sense, then prove some theoretical results like uniqueness and stability. After that, we establish a numerical algorithm to solve the unknowns, where a mollification method is used.

1. Introduction.

1.1. Mathematical statement. In this work, the following stochastic fractional diffusion equation is considered,

\[
\begin{cases}
\partial_t^\alpha u + Au = f(x)[g_1(t) + g_2(t)\hat{W}(t)], & (x, t) \in D \times (0, T], \quad \alpha \in (1/2, 1), \\
u(x, t) = 0, & (x, t) \in \partial D \times (0, T], \\
u(x, 0) = 0, & x \in D,
\end{cases}
\]

in which \( D \subset \mathbb{R}^d \), \( 1 \leq d \leq 3 \) is bounded with sufficiently smooth boundary, \( A \) is a positive symmetric elliptic operator, like \(-\Delta\), and \( \partial_t^\alpha \) means the Djrbashyan-Caputo fractional derivative,

\[
\partial_t^\alpha \psi = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \psi'(\tau) \, d\tau.
\]

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For the source term \( f(x)g(t, \omega) := f(x)[g_1(t) + g_2(t)\dot{\mathcal{W}}(t)] \), the spatial component \( f(x) \) is deterministic and known, and the unknown \( g(t, \omega) \) contains uncertainty, where \( \omega \) is the random variable. In this paper, we consider the case that \( g(t, \omega) = g_1(t) + g_2(t)\dot{\mathcal{W}}(t) \) is an Ito process, and the notation \( \dot{\mathcal{W}}(t) \) means the white noise derived from the time-dependent Brownian motion. For the derivation and properties of Ito process, see [45] for details. For the two unknowns \( g_1, g_2 \) in \( g(t, \omega) \), we try to use the statistical moments of the observation \( u(x_0, t, \omega) \) to recover them. However, from the properties of \( \mathcal{W} \), which are displayed in section 2, the sign of \( g_2 \) can not affect the stochastic process \( g_2(t)\dot{\mathcal{W}}(t) \). Sequentially, \( g_2^2 \) is concerned instead of \( g_2 \).

The precise mathematical description of this inverse problem is given as follows: using the statistical moments of the realizations of \( h(t, \omega) := u(x_0, t, \omega) \), \( x_0 \in D \) to recover \( g_1 \) and \( g_2^2 \) simultaneously.

1.2. Physical background. In microscopical level, the random motion of a single particle can be regarded as a diffusion process. Under the assumption that the mean squared displacement of jumps after a long time is proportional to time, i.e. \( (\Delta x)^2 \propto t, \ t \to \infty \), the classical diffusion equation to describe the motion of particles can be derived. However, recently, some anomalous diffusion phenomena have been found, accompanying with considerable physical evidences, [27, 8, 41]. In such anomalous diffusions, the assumption \( (\Delta x)^2 \propto t, \ t \to \infty \) may be violated, and it may possess the asymptotic behavior of \( t^\alpha \), i.e. \( (\Delta x)^2 \propto t^\alpha, \ \alpha \neq 1 \). The different rate leads to a reformulation of the diffusion equation, introducing the time fractional derivative in it, and the corresponding equations are called fractional differential equations (FDEs). We list some of the applications of FDEs, to name a few, the diffusion process in a medium with fractal geometry [43], non-Fickian transport in geological formations [9], mathematical finance [19], theory of viscoelasticity [3, 28] and so on.

If uncertainty is added in the right-hand side, then this FDE system will become more complicated and interesting. This means that the random property in the source term will lead to a solution expressed as a stochastic process. Considering this case is meaningful since it is common to meet a diffusion source defined as a stochastic process to describe the uncertain character imposed by nature. Hence, mathematically, it is worth to investigate the diffusion system with a random source.

1.3. Previous literature. Considerable researchers have made efforts in the investigation of fractional differential equations and some valuable work are produced. For a comprehensive understanding of fractional calculus and fractional differential equations, we refer to [26, 53, 4] and the references therein; for numerical approaches, see [54, 39, 21, 40, 59, 37].

In the field of inverse problems, for an extensive review, [24] is referred. For time-fractional inverse problems, see [33, 25, 2, 47, 57, 55, 58, 48, 22, 61, 36, 60]; for fractional Calderón problems, [18] is one of the first works, and see [49, 11, 29, 17, 13, 12, 20] for more details. Furthermore, if we extend the assumption \( (\Delta x)^2 \propto t^\alpha \) to a more general case \( (\Delta x)^2 \propto F(t) \), the multi-term fractional diffusion equations and even the distributed-order differential equations will be generated, see [50, 34, 14] for details.

The inverse problems of determining the uncertain unknowns also have drawn more and more attention from researchers. We refer to [16, 7, 31, 5, 32, 30, 6, 44] and the references therein.
1.4. Main results and outline. Throughout this paper, the following restrictions on \( f \) and the unknowns \( g_1, g_2 \) are assumed to be valid.

**Assumption 1.** Let the function \( f \) and the unknowns \( g_1, g_2 \) satisfy the following conditions.

- \( g_1, g_2 \in C[0, T] \).
- \( f \in D(A^2) \subset H^4(D) \) and \( f(x_0) \neq 0 \).

The definition of the space \( D(A^2) \) is stated in section 2. Also, we give the definitions of probability space and the statistical moments \( E, V \) below.

**Definition 1.1.** We call \((\Omega, \mathcal{F}, \mathbb{P})\) a probability space if \( \Omega \) denotes the nonempty sample space, \( \mathcal{F} \) is the \( \sigma \)-algebra of \( \Omega \) and \( \mathbb{P} : \mathcal{F} \to [0, 1] \) is the probability measure.

For a random variable \( X \) defined on \((\Omega, \mathcal{F}, \mathbb{P})\), the expectation \( E \) and variance \( V \) are defined as follows:

\[
E[X] = \int_\Omega X(\omega) \, d\mathbb{P}(\omega), \quad V[X] = E[(X - E[X])^2].
\]

Then we can state our main theorem—the stability theorem.

**Theorem 1.2** (Stability). Let \( h(t, \omega) := u(x_0, t, \omega) \) be the realizations of single point measurements. Then under Assumption 1, the following stability results for \( g_1 \) and \( g_2 \) hold:

\[
\begin{align*}
|g_1|_{H^{-1}(0, T)} &\leq C\|E[I_t^{1-\alpha}h(\cdot, \omega)]\|_{L^2(0, T)}, \\
|g_2^2|_{H^{-1}(0, T)} &\leq C\|V[I_t^{1-\alpha}h(\cdot, \omega)]\|_{L^2(0, T)}.
\end{align*}
\]

Here the constant \( C > 0 \) is independent of \( h \).

The notations \( |\cdot|_{H^{-1}(0, T)} \) and \( I_t^{1-\alpha} \) mean the seminorm of space \( H^{-1}(0, T) \) and the fractional integral operator, respectively. These knowledge can be found in section 2.

In Theorem 1.2, we estimate \( g_2^2 \) instead of \( g_2 \), which means the sign of \( g_2 \) can not be determined by the variance moment. This is caused by the Ito formula, stated in Lemma 2.1. See the further proof for details. With Theorem 1.2, the following proposition about uniqueness can be derived immediately.

**Proposition 1** (Uniqueness). Under Assumption 1, \( g_1 \) and \( g_2^2 \) can be uniquely determined by the moments

\[
E[I_t^{1-\alpha}h(t, \omega)], \quad V[I_t^{1-\alpha}h(t, \omega)], \quad t \in [0, T].
\]

More precisely, suppose \( g_1, g_2, \tilde{g}_1, \tilde{g}_2 \) satisfy Assumption 1, and denote the corresponding realizations as \( h, \tilde{h} \), respectively. Then

\[
\begin{align*}
E[I_t^{1-\alpha}h(t, \omega)] &= E[I_t^{1-\alpha}\tilde{h}(t, \omega)], \\
V[I_t^{1-\alpha}h(t, \omega)] &= V[I_t^{1-\alpha}\tilde{h}(t, \omega)], \quad t \in [0, T],
\end{align*}
\]

lead to \( g_1 = \tilde{g}_1, g_2^2 = \tilde{g}_2^2 \) on \([0, T]\).

The remaining part of this manuscript is structured as follows. Section 2 includes some preliminary knowledge, such as the Ito isometry formula and the maximum principles in fractional diffusion equations. Also, we give the definition of the stochastic weak solution \( u : (0, T] \times \Omega \to L^2(D) \). In section 3, the proofs for Theorem 1 and Proposition 1.2 are built. After the theoretical analysis, we consider the numerical reconstruction in section 4. A mollification method is established and some numerical results are displayed.
2. Preliminary setting.

2.1. Eigensystem of $A$ and the space $D(A^\gamma)$.

Since $A$ is a symmetric elliptic operator with domain $H^2(D) \cap H_0^1(D)$, its eigensystem \{\(\lambda_n, \varphi_n(x) : n \in \mathbb{N}^+\)\} (multiplicity counted) satisfies the following properties:

- \(0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots\) and \(\lambda_n \to \infty\) as \(n \to \infty\),
- \(\{\varphi_n(x) : n \in \mathbb{N}^+\} \subset H^2(D) \cap H_0^1(D)\) form an orthonormal basis in \(L^2(D)\).

Then given \(\gamma > 0\), we define the space \(D(A^\gamma)\) as

\[D(A^\gamma) := \left\{ \psi \in L^2(D) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma} \langle \psi(\cdot), \varphi_n(\cdot) \rangle_{L^2(D)}^2 < \infty \right\},\]

where \(\langle \cdot, \cdot \rangle_{L^2(D)}\) denotes the inner product in \(L^2(D)\). It holds that \(D(A^\gamma) \subset H^{2\gamma}(D)\) since \(D\) has smooth boundary, \([52]\).

Moreover, we give the definition of Sobolev space with negative integer order \(H^{-m}(0, T)\), which is used in Theorem 1.2. \(H^{-m}(0, T)\) is defined as the dual space of \(H^m(0, T)\), namely \(H^{-m}(0, T) = (H^m(0, T))'\), and from \([35, \text{Theorem 12.1}]\), we have

\[H^{-m}(0, T) = \left\{ \psi : \psi = \sum_{|j| \leq m} D^j \psi_j; \exists \text{ some } \psi_j \in L^2(0, T) \right\}.\]

Accordingly, define the \(H^{-m}\) semi-norm as

\[|\psi|_{H^{-m}(0, T)} = ||\psi_m||_{L^2(0, T)}.\]

2.2. Probability setting and Ito isometry formula.

For the time dependent Brownian motion, which is described as Wiener process \(\mathcal{W}(t)\), we list some of its properties.

**Remark 1.** The Wiener process \(\mathcal{W}(t)\) used in this article possesses the following properties:

- \(\mathcal{W}(0) = 0\),
- \(\mathcal{W}(t)\) is almost surely continuous,
- \(\mathcal{W}(t)\) has independent increments and satisfies

\[\mathcal{W}(t) - \mathcal{W}(s) \sim \mathcal{N}(0, t - s), \quad 0 \leq s \leq t,\]

where \(\mathcal{N}\) is the normal distribution.

Now we can state the Ito isometry formula, which plays a crucial role in our analysis. In the following lemma, the random measure derived from \(\mathcal{W}\) is denoted by \(d\mathcal{W}(t)\).

**Lemma 2.1.** ([45]). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \(\psi : [0, \infty) \times \Omega \to \mathbb{R}\) satisfy the following properties:

1. \((t, \omega) \mapsto \psi(t, \omega)\) is \(\mathcal{B} \times \mathcal{F}\)-measurable, where \(\mathcal{B}\) denotes the Borel \(\sigma\)-algebra on \([0, \infty)\),
2. \(\psi(t, \omega)\) is \(\mathcal{F}_t\)-adapted,
3. \(\mathbb{E} \left[ \int_0^S \psi^2(\tau, \omega) \ d\mathcal{W}(\tau) \right] < \infty\) for \(S > 0\).

Then the Ito integral \(\int_0^S \psi(\tau, \omega) \ d\mathcal{W}(\tau)\) is well defined and it follows that

\[\mathbb{E} \left[ \left( \int_0^S \psi(\tau, \omega) \ d\mathcal{W}(\tau) \right)^2 \right] = \mathbb{E} \left[ \int_0^S \psi^2(\tau, \omega) \ d\tau \right].\]
2.3. **Stochastic weak solution.** Since we can not make sure \( \mathbb{W}(t) \) is continuous or differentiable for a given \( \omega \in \Omega \), we need to consider the weak solution in the sense of Ito integral.

Firstly, we define the fractional integral operator as follows.

**Definition 2.2.** The fractional integral operator \( I_\alpha \) is given as

\[
I_\alpha \psi(t) = \Gamma(\alpha)^{-1} \int_0^t (t - \tau)^{-\alpha} \psi(\tau) \, d\tau, \quad \psi \in L^1_{\text{loc}}(0, \infty).
\]

The following remark explains why we need to set the restriction \( \alpha \in (1/2, 1) \) on the fractional order \( \alpha \).

**Remark 2.** With the above definition, we can define the Ito integral \( I_\alpha^\tau g(t, \omega) \) as

\[
I_\alpha^\tau g(t, \omega) = I_\alpha^0 g_1(t) + \Gamma(\alpha)^{-1} \int_0^t (t - \tau)^{-\alpha} g_2(\tau) \, d\mathbb{W}(\tau).
\]

From the conditions \( \alpha \in (1/2, 1) \) and \( g_2 \in C[0, T] \), we have \( (t - \tau)^{-\alpha} g_2(\tau) \) is square-integrable on \([0, T]\). Then the well-definedness of the Ito integral \( \int_0^T (t - \tau)^{-\alpha} g_2(\tau) \, d\mathbb{W}(\tau) \) is ensured by Lemma 2.1.

In addition, the direct calculation gives that

\[
I_\alpha^\tau \partial_t^\alpha \psi(t) = \psi(t) - \psi(0),
\]

which can help us build the definition of the weak solution for equation (1).

**Definition 2.3 (Stochastic weak solution).** We say the stochastic process \( u(\cdot, t, \omega) : (0, T] \times \Omega \to L^2(D) \) is a stochastic weak solution of equation (1) if for each \( \psi \in H^2(D) \cap H_0^1(D) \) and \( t, \omega \in (0, T] \times \Omega \), it holds that

\[
\langle u(\cdot, t, \omega), \psi(\cdot) \rangle_{L^2(D)} + I^\tau_\alpha \langle Au(\cdot, t, \omega), \psi(\cdot) \rangle_{L^2(D)} = I^\tau_\alpha g(t, \omega) \langle f(\cdot), \psi(\cdot) \rangle_{L^2(D)}.
\]

The reference [44] investigated the direct problem of equation (1) under Definition 2.3. For example, it proved that the expectation moment \( \mathbb{E}[\|u\|_{L^2(D \times (0, T))}^2] \) can be bounded by the source. See [44, Proposition 1.1] for details.

2.4. **Auxiliary lemmas.** The following lemmas will be used in the proof of stability.

**Lemma 2.4.** ([46, Theorem 1.6]) The Mittag-Leffler function \( E_{\alpha, \beta}(z) \) is defined as

\[
E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}.
\]

Let \( 0 < \alpha < 2 \) and \( \beta \in \mathbb{R} \). Then for \( \mu \) satisfying \( \pi \alpha/2 < \mu < \min\{\pi, \pi \alpha\} \), there exists a constant \( C = C(\alpha, \beta, \mu) > 0 \) such that

\[
|E_{\alpha, \beta}(z)| \leq \frac{C}{1 + |z|^{1/\beta}}, \quad \mu \leq |\arg(z)| \leq \pi.
\]

**Lemma 2.5.** (Maximum principle, [38, Theorem 2]). Fix \( T \in (0, \infty) \), let \( \psi \) satisfy the following fractional diffusion equation

\[
\partial_\tau^\alpha \psi + A\psi = F(x, t), \quad (x, t) \in D \times (0, T],
\]

and define \( \Lambda_T = \partial D \times [0, T] \cup \partial D \times \{0\} \). If \( F \leq 0 \), then

\[
\psi(x, t) \leq \max\{0, \max\{\psi(x, t) : (x, t) \in \Lambda_T\}\}, \quad (x, t) \in D \times (0, T].
\]
The next lemma contains an $L^2$-regularity.

**Lemma 2.6.** Let $v(x,t)$ be the solution of the following fractional diffusion equation,

\[
\begin{aligned}
\partial_t^\alpha v + Av &= 0, \quad (x,t) \in D \times (0,T], \\
v(x,t) &= 0, \quad (x,t) \in \partial D \times (0,T], \\
v(x,0) &= f(x), \quad x \in D,
\end{aligned}
\]

then there exists $C > 0$ depending on $f(x), \alpha, T$ such that $\|v_t(x, \cdot)\|_{L^2(0,T)} \leq C$.

**Proof.** From [52], we can give the representation of $v$ as

\[v(x,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha) f_n \varphi_n(x), \quad f_n = \langle f(\cdot), \varphi_n(\cdot) \rangle_{L^2(D)},\]

and the regularity result $v \in C([0,T]; H^2(D))$. Since $D \subset \mathbb{R}^d$, $1 \leq d \leq 3$, from the Sobolev inequalities, we have that $v(\cdot,t) \in C(D)$ and $\|v(\cdot,t)\|_{C(D)} \leq C \|v(\cdot,t)\|_{H^2(D)}$. By this continuous regularity, we can write $v(x_0,t)$ as the convergent series,

\[v(x_0,t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha) f_n \varphi_n(x_0)\]

and denote its partial sum as $v_N(x_0,t)$, i.e. $v_N(x_0,t) = \sum_{n=1}^{N} E_{\alpha,1}(-\lambda_n t^\alpha) f_n \varphi_n(x_0)$. Recall the formula

\[E_{\alpha,1}(-\lambda_n t^\alpha)' = -\lambda_n t^{\alpha-1} E_{\alpha,1}(-\lambda_n t^\alpha),\]

and define

\[V(t) = -\sum_{n=1}^{\infty} \lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) f_n \varphi_n(x_0),\]

\[V_N(t) = -\sum_{n=1}^{N} \lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) f_n \varphi_n(x_0).\]

Next, we will show that given $\epsilon > 0$, $v_t(x_0,t) = V(t)$ on $[\epsilon,T]$.

For $t \in [\epsilon,T]$, the following estimate holds by Lemma 2.4,

\[|\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)| \leq C \frac{\lambda_n t^{\alpha-1}}{1 + \lambda_n t^\alpha} \leq C \epsilon^{-1}.\]

Then with Sobolev inequalities, we have

\[|V(t) - V_N(t)|^2 \leq \left\| \sum_{n=N+1}^{\infty} \lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) f_n \varphi_n(\cdot) \right\|^2_{C(D)} \]

\[\leq C \left\| \sum_{n=N+1}^{\infty} \lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) f_n \varphi_n(\cdot) \right\|^2_{H^2(D)} \]

\[\leq C \sum_{n=N+1}^{\infty} |\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 \lambda_n^2 f_n^2 \]

\[\leq C \epsilon^{-2} \sum_{n=N+1}^{\infty} \lambda_n^2 f_n^2.\]

Since $\sum_{n=1}^{\infty} \lambda_n^2 f_n^2 = \|f\|^2_{L^2(D)} < \infty$, the upper bound $C \epsilon^{-2} \sum_{n=N+1}^{\infty} \lambda_n^2 f_n^2$ will converge to zero as $N \to \infty$, and note that it is independent of $t$. Then we can conclude.
that the series $V(t)$ is uniformly convergent on $[\epsilon, T]$. Realize that $V(t)$ is derived from $v(x_0, t)$ by termwise differentiation, then the uniform convergence of $V(t)$ and the convergence of $v(x_0, t)$ give that $V(t) = v_t(x_0, t)$ on $[\epsilon, T], \epsilon > 0$.

Now we can show $v_t(x_0, t) \in L^2(0, T)$. For $t \in [\epsilon, T]$,

$$|v_t(x_0, t)|^2 \leq \left\| \lim_{n \to \infty} \sum_{n=1}^{\infty} \lambda_n t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^\alpha) f_n \varphi_n(\cdot) \right\|_{H^2(D)}^2$$

$$\leq C\left\| \lim_{n \to \infty} \sum_{n=1}^{\infty} \lambda_n t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^\alpha) f_n \varphi_n(\cdot) \right\|_{H^2(D)}^2$$

$$\leq C t^{2\alpha-2} \sum_{n=1}^{\infty} |E_{\alpha, \alpha}(-\lambda_n t^\alpha)|^2 \lambda_n f_n^2$$

$$\leq C t^{2\alpha-2} \sum_{n=1}^{\infty} \lambda_n f_n^2 \leq C \|f\|_{L^2(A)}^2 t^{2\alpha-2}.$$

Consequently, recalling that $\alpha \in (1/2, 1)$,

$$\|v_t(x_0, \cdot)\|_{L^2(\epsilon, T)}^2 \leq C(f) \int_{\epsilon}^{T} t^{2\alpha-2} \, dt$$

$$= C(f, \alpha) (T^{2\alpha-1} - \epsilon^{2\alpha-1}) \leq C(f, \alpha, T) < \infty.$$ Since $C(f, \alpha, T)$ is independent of the choice of $\epsilon$, we have

$$\|v_t(x_0, \cdot)\|_{L^2(0, T)}^2 = \lim_{\epsilon \to 0^+} \|v_t(x_0, \cdot)\|_{L^2(\epsilon, T)}^2 \leq C(f, \alpha, T) < \infty,$$

which completes the proof. \qed

3. Main results. In this section we will give the proofs of Theorem 1.2 and Proposition 1.

3.1. The second kind Volterra equation. The next lemma yields the representation of the weak solution $u(x, t, \omega)$.

**Lemma 3.1.** Under Definition 2.3, the weak solution $u$ can be written as

$$u(x, t, \omega) = I_t^\alpha g(t, \omega) f(x) + \int_0^t I_t^\alpha g(\tau, \omega) v_t(x, t - \tau) \, d\tau,$$

where $v(x, t)$ satisfies equation (3).

**Proof.** Note that

$$\mathcal{A} v_t = (\mathcal{A} v)_t = -\partial(\partial_t^\alpha v) / \partial t,$$

then we have

$$I_t^\alpha \mathcal{A} \left( \int_0^t I_t^\alpha g(\tau, \omega) v_t(x, t - \tau) \, d\tau \right)$$

$$= -\Gamma(\alpha)^{-1} \int_0^t \int_0^t \Gamma(\alpha)^{-1} \int_0^t I_t^\alpha g(\tau, \omega) \frac{\partial(\partial_t^\alpha v)}{\partial t}(x, t - \tau) \, d\tau \, d\tau$$

$$= -\Gamma(\alpha)^{-1} \int_0^t \int_0^t \Gamma(\alpha)^{-1} \int_0^t I_t^\alpha g(\tau, \omega) \frac{\partial(\partial_t^\alpha v)}{\partial t}(x, t - \tau) \, d\tau \, d\tau$$

$$= -\Gamma(\alpha)^{-1} \int_0^t I_t^\alpha g(\tau, \omega) \int_0^t (t - s)^{\alpha-1} \frac{\partial(\partial_t^\alpha v)}{\partial t}(x, s - \tau) \, ds \, d\tau.$$
From direct calculation, we have
\[
\Gamma(\alpha)^{-1} \int_{\tau}^{t} (t-s)^{\alpha-1} \frac{\partial (\partial^\alpha_t v)}{\partial t}(x, s-\tau) \, ds = \partial^{1-\alpha}(\partial^\alpha_t v)(x, t-\tau)
\]
\[
= v_t(x, t-\tau) - \Gamma(\alpha)^{-1} (t-\tau)^{\alpha-1} \partial^\alpha_t v(x, 0)
\]
\[
= v_t(x, t-\tau) + \Gamma(\alpha)^{-1} (t-\tau)^{\alpha-1} Af(x).
\]
Hence,
\[
I^\alpha_t A \int_{0}^{t} I^\alpha_t g(\tau, \omega)v_t(x, t-\tau) \, d\tau
\]
\[
= - \int_{0}^{t} I^\alpha_t g(\tau, \omega)v_t(x, t-\tau) \, d\tau - \Gamma(\alpha)^{-1} Af(x) \int_{0}^{t} I^\alpha_t g(\tau, \omega)(t-\tau)^{\alpha-1} \, d\tau
\]
\[
= - \int_{0}^{t} I^\alpha_t g(\tau, \omega)v_t(x, t-\tau) \, d\tau - (I^\alpha_t)^2 g(t, \omega) Af(x),
\]
which leads to
\[
I^\alpha_t Au = I^\alpha_t A \int_{0}^{t} I^\alpha_t g(\tau, \omega)v_t(x, t-\tau) \, d\tau + (I^\alpha_t)^2 g(t, \omega) Af(x)
\]
\[
= - \int_{0}^{t} I^\alpha_t g(\tau, \omega)v_t(x, t-\tau) \, d\tau
\]
\[
= f I^\alpha_t g - u.
\]
Moreover, we can derive \(v_t(\cdot, t) \in L^2(D)\) from the analysis in [52] and the condition \(f \in D(A^2)\). Inserting these regularity estimates in (4) yields that \(u(\cdot, t, \omega) \in L^2(D)\).

Now \(u\) in (4) satisfies Definition 2.3 and the proof is complete. \(\square\)

With Lemmas 2.1 and 3.1, the lemma below follows, which includes the unknowns \(g_1, g_2\) and the statistical moments of \(h(t, \omega)\).

**Lemma 3.2.** Define

\[
G_1(t) = \int_{0}^{t} g_1(\tau) \, d\tau, \quad G_2(t) = \int_{0}^{t} g_2(\tau) \, d\tau,
\]

then it holds that for \(t \in (0, T]\),

\[
G_1(t) = f^{-1}(x_0)E[I^{1-\alpha}t h(t, \omega)] - f^{-1}(x_0) \int_{0}^{t} G_1(\tau)v_t(x_0, t-\tau) \, d\tau,
\]

\[
G_2(t) = f^{-2}(x_0)V[I^{1-\alpha}t h(t, \omega)] - 2f^{-2}(x_0) \int_{0}^{t} G_2(\tau)v_t(x_0, t-\tau)v_t(x_0, t-\tau) \, d\tau.
\]
Proof. From (4), we can obtain the following result

$$I_t^{1-\alpha}u(x,t,\omega) = \frac{f(x)}{\Gamma(\alpha)(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \int_0^\tau (\tau-s)^{\alpha-1} g(s,\omega) \, ds \, d\tau$$

$$+ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \int_0^\tau \mathcal{I}^\alpha g(\tau-s,\omega)v_t(x,s) \, ds \, d\tau$$

$$= \frac{f(x)}{\Gamma(\alpha)(1-\alpha)} \int_0^t g(s,\omega) \int_s^t (t-\tau)^{-\alpha}(\tau-s)^{\alpha-1} \, d\tau \, ds$$

$$+ \frac{1}{\Gamma(1-\alpha)} \int_0^t v_t(x,s) \int_s^t (t-\tau)^{-\alpha} \mathcal{I}^\alpha g(\tau-s,\omega) \, d\tau \, ds$$

$$= \frac{1}{\Gamma(\alpha)(1-\alpha)} \left[ f(x) \int_0^t g(s,\omega) \int_s^t (t-\tau)^{-\alpha}(\tau-s)^{\alpha-1} \, d\tau \, ds$$

$$+ \int_0^t v_t(x,s) \int_0^{t-s} g(\tau,\omega) \int_0^{t-s} (t-s-\tau)^{-\alpha}(\tau-\tau)^{\alpha-1} \, d\tau \, d\tau \right].$$

Due to

$$\int_s^t (t-\tau)^{-\alpha}(\tau-s)^{\alpha-1} \, d\tau = B(1-\alpha,\alpha) = \Gamma(1-\alpha)\Gamma(\alpha)/\Gamma(1),$$

where $B$ is the Beta function, we have

$$I_t^{1-\alpha}u(x,t,\omega) = f(x) \int_0^t g(s,\omega) \, ds + \int_0^t v_t(x,s) \int_0^{t-s} g(\tau,\omega) \, d\tau \, ds$$

$$= f(x) \int_0^t g(s,\omega) \, ds + \int_0^t g(\tau,\omega)[v(x,t-\tau) - v(x,0)] \, d\tau$$

$$= \int_0^t g(\tau,\omega)v(x,t-\tau) \, d\tau,$$

i.e.

$$I_t^{1-\alpha}u(x,t,\omega) = \int_0^t g_1(\tau)v(x,t-\tau) \, d\tau + \int_0^t g_2(\tau)v(x,t-\tau) \, dW(\tau).$$

Hence,

$$I_t^{1-\alpha}h(t,\omega) = \int_0^t g_1(\tau)v(x_0,t-\tau) \, d\tau + \int_0^t g_2(\tau)v(x_0,t-\tau) \, dW(\tau).$$

(7)  $$I_t^{1-\alpha}h(t,\omega) = \int_0^t g_1(\tau)v(x_0,t-\tau) \, d\tau + \int_0^t g_2(\tau)v(x_0,t-\tau) \, dW(\tau).$$

Applying Lemma 2.1 to (7), the following result can be derived,

$$\mathbb{E}[I_t^{1-\alpha}h(t,\omega)] = \int_0^t g_1(\tau)v(x_0,t-\tau) \, d\tau,$$

$$\mathbb{V}[I_t^{1-\alpha}h(t,\omega)] = \int_0^t g_2^2(\tau)[v(x_0,t-\tau)]^2 \, d\tau, \quad t \in (0,T].$$

(8)  $$\mathbb{E}[I_t^{1-\alpha}h(t,\omega)] = \int_0^t g_1(\tau)v(x_0,t-\tau) \, d\tau,$$

$$\mathbb{V}[I_t^{1-\alpha}h(t,\omega)] = \int_0^t g_2^2(\tau)[v(x_0,t-\tau)]^2 \, d\tau, \quad t \in (0,T].$$
The right hand sides of the above equations can be written as
\[
\int_0^t g_1(\tau) v(x_0, t - \tau) \, d\tau = \int_0^t v(x_0, t - \tau) \, d(G_1(\tau))
= f(x_0)G_1(t) + \int_0^t G_1(\tau) v_1(x_0, t - \tau) \, d\tau,
\]
and
\[
\int_0^t g_2^2(\tau) [v(x_0, t - \tau)]^2 \, d\tau = \int_0^t [v(x_0, t - \tau)]^2 \, d(G_2(\tau))
= f^2(x_0)G_2(t) + 2 \int_0^t G_2(\tau) v(x_0, t - \tau) v_1(x_0, t - \tau) \, d\tau,
\]
which together with (8) yield the desired result. \hfill \Box

3.2. Proofs of theorem 1.2 and proposition 1.

Lemma 3.3. \(G_1 \) and \(G_2\), defined in (5), satisfy the following estimates,
\[
\|G_1\|_{L^2(0,T)} \leq C \|\mathbb{E}[I_t^{1-\alpha} h(\cdot, \omega)]\|_{L^2(0,T)},
\]
\[
\|G_2\|_{L^2(0,T)} \leq C \|\mathbb{V}[I_t^{1-\alpha} h(\cdot, \omega)]\|_{L^2(0,T)},
\]
where the constant \(C > 0\) does not depend on \(h\).

Proof. From [10], equation (6) can be solved as
\[
G_1(t) = f^{-1}(x_0)\mathbb{E}[I_t^{1-\alpha} h(t, \omega)] + \int_0^t R_1(t - \tau) \mathbb{E}[I_t^{1-\alpha} h(\tau, \omega)] \, d\tau,
\]
\[
G_2(t) = f^{-2}(x_0)\mathbb{V}[I_t^{1-\alpha} h(t, \omega)] + \int_0^t R_2(t - \tau) \mathbb{V}[I_t^{1-\alpha} h(\tau, \omega)] \, d\tau,
\]
where \(R_1, R_2\) are the resolvent kernels depending on \(f(x_0), v(x_0, \cdot), v_1(x_0, \cdot)\). With Sobolev inequalities, the conditions \(f \in H^d(D)\) and \(D \subset \mathbb{R}^d\), \(1 \leq d \leq 3\) give that \(\|f\|_{C(D)} \leq C \|f\|_{H^d(D)} < \infty\). This and Lemma 2.5 ensure the boundedness of \(|v(x_0, t)|\) on \([0, T]\). Also, from Lemma 2.6, we have \(v_1(x_0, \cdot) \in L^2(0, T)\). Hence, by [10, Theorem 8.3.3], the resolvent kernels \(R_1, R_2\) both belong to the type \((L^2, T)\), namely,
\[
\int_0^T \int_0^t |R_j(t - s)|^2 \, ds \, dt = \int_0^T \|R_j\|_{L^2(0,t)}^2 \, dt < \infty, \quad j = 1, 2.
\]
Now let’s build the upper bounds of \(G_1, G_2\). Hölder inequality gives that
\[
\|G_1\|_{L^2(0,T)} \leq C(f) \|\mathbb{E}[I_t^{1-\alpha} h(\cdot, \omega)]\|_{L^2(0,T)} + \int_0^T \|R_1\|_{L^2(0,t)} \|\mathbb{E}[I_t^{1-\alpha} h(\cdot, \omega)]\|_{L^2(0,t)} \, dt
\leq \left( C(f) + \int_0^T \|R_1\|_{L^2(0,t)} \, dt \right) \|\mathbb{E}[I_t^{1-\alpha} h(\cdot, \omega)]\|_{L^2(0,T)}
\leq C\|\mathbb{E}[I_t^{1-\alpha} h(\cdot, \omega)]\|_{L^2(0,T)}.
\]
Analogously, we can prove that \(\|G_2\|_{L^2(0,T)} \leq C\|\mathbb{V}[I_t^{1-\alpha} h(\cdot, \omega)]\|_{L^2(0,T)}\). The proof is complete. \hfill \Box

Now the stability and uniqueness can be proved.

Proofs of Theorem 1.2 and Proposition 1. From Lemma 3.3 and the definition of \(|\cdot|_{H^{-1}}\), we have
\[
|g_1|_{H^{-1}(0,T)} \leq C \|\mathbb{E}[I_t^{1-\alpha} h(\cdot, \omega)]\|_{L^2(0,T)},
\]
\[
|g_2^2|_{H^{-1}(0,T)} \leq C \|\mathbb{V}[I_t^{1-\alpha} h(\cdot, \omega)]\|_{L^2(0,T)}.
\]
Also (8) and Lemma 3.3 give that
\[ |g_1 - \tilde{g}_1|_{H^{-1}(0,T)} = |g_2 - \tilde{g}_2|_{H^{-1}(0,T)} = 0, \]
which together with the continuities of \( g_j, \tilde{g}_j, j = 1, 2 \) leads to \( g_1 = \tilde{g}_1, g_2 = \tilde{g}_2 \). \( \Box \)

**Remark 3.** Provided a stronger regularity condition for the spatial term \( f(x) \) that
\[ f(x) \in H^\eta(D), \eta > 4, \]
the result in Theorem 1.2 can be extended to the higher dimensional case, i.e. \( d \geq 4 \).

To support the stability result, we need the time \( L^2 \)-regularity of \( v_t(x_0, \cdot) \), which is proved by Lemma 2.6. In the case of \( d \geq 4 \), a stronger condition than \( f \in H^4(D) \) can keep the Sobolev inequality used in the proof of Lemma 2.6 be valid, which ensures the result of Lemma 2.6. The choice of \( \eta \) will depend on the value of \( d \).

4. **Numerical reconstruction.** In this section, we illustrate the numerical reconstruction of the unknowns \( g_1, g_2^2 \) from equation (6). Firstly we set
\[ D \times [0,T] = [0,1]^2, \alpha = 0.8, \ x_0 = 1/2, \ \mathcal{A} = -\Delta, \ f(x) = \sin(\pi x), \]
then we consider the following experiments:

- (e1) : \( g_1(t) = t + \sin(2\pi t) + \sin(3\pi t), g_2(t) = \begin{cases} \sin(2\pi t) - 0.3, & t \in [0,1/2), \\ \sin(2\pi t) + 0.3, & t \in [1/2,1], \end{cases} \)
- (e2) : \( g_1(t) = t + \sin(2\pi t) + \sin(3\pi t), g_2(t) = \sin(\pi t), \)
- (e3) : \( g_1(t) = \begin{cases} \sin(2\pi t) - 0.3, & t \in [0,1/2), \\ \sin(2\pi t) + 0.3, & t \in [1/2,1], \end{cases} g_2(t) = \sin(\pi t), \)
- (e4) : \( g_1(t) = t + \sin(2\pi t) + \sin(3\pi t), g_2(t) = \begin{cases} 4, & t \in [0,0.3), \\ 2, & t \in [0.3,0.6), \\ 1, & t \in [0.6,1], \end{cases} \)
- (e5) : \( g_1(t) = \begin{cases} \sin(2\pi t) - 0.3, & t \in [0,1/2), \\ \sin(2\pi t) + 0.3, & t \in [1/2,1], \end{cases} g_2(t) = \begin{cases} 4, & t \in [0,0.3), \\ 2, & t \in [0.3,0.6), \\ 1, & t \in [0.6,1]. \end{cases} \)

We add various sizes of white noise to our observation \( h(t,\omega) \), i.e. \( h^\sigma(t,\omega) = h(t,\omega) + \xi(t,\omega) \), with Gaussian independent identical distribution \( \xi(t,\omega) \sim \sigma \mathcal{N}(0,1) \), i.e. \( \xi(t,\omega) \sim \mathcal{N}(0,\sigma^2) \). Then we average a fractional integral of the point observations \( h^\sigma(t,\omega) \) to obtain the perturbed moments \( \bar{E}^\sigma, \bar{V}^\sigma \), and display the numerical results under different \( \sigma \) in section 4.3.

4.1. **Direct solver.** To obtain the point observation \( h(t,\omega) = u(x_0,t,\omega) \), the fractional diffusion equation (1) needs to be solved, and to this end, the discretized scheme is constructed as follows.

For the spatial variable \( x \), we apply a finite element approach using piecewise linear bases \( \{\phi_j(x)\}_{j=1}^m \), namely \( \phi_j(x_k) = \delta_{jk} \), where \( \{x_j\}_{j=1}^m \) consists of a Delaunay triangulation of domain \( D \). Then we define the finite element space as \( \mathcal{V}_m = \text{Span}\{\phi_j(x) : j = 1, \cdots, m\} \). The projections of \( u \) and \( f \) in \( \mathcal{V}_m \) are defined as
\[ \tilde{u}(x,t,\omega) = \sum_{j=1}^m u(x_j,t,\omega)\phi_j(x), \quad \tilde{f}(x) = \sum_{j=1}^m f(x_j)\phi_j(x). \]
Then considering equation (1), we have
\[
\sum_{j=1}^{m} \partial_t^\alpha u(x_j, t, \omega)(\phi_j, \phi_i)_{L^2(D)} + \sum_{j=1}^{m} u(x_j, t, \omega)(A\phi_j, \phi_i)_{L^2(D)} = g(t, \omega) \sum_{j=1}^{m} f(x_j)(\phi_j, \phi_i)_{L^2(D)}.
\]

From the above equation, we define the mass matrix \( \bar{M} \) and stiff matrix \( \bar{S} \) w.r.t. basis \( \{\phi_j\}_{1}^{m} \) as
\[
\bar{M} = \left[ (\phi_j, \phi_i)_{L^2(D)} \right]_{i,j=1}^{m}, \quad \bar{S} = \left[ (A\phi_j, \phi_i)_{L^2(D)} \right]_{i,j=1}^{m},
\]
which will be used to construct the discretized scheme for equation (1).

For the discretization on time, the \( L_1 \)-stepping scheme is used, which can be seen in [23, 51]. Set the discrete time mesh \( 0 = t_0 < t_1 < \cdots < t_N = T \) and denote the time step size as \( \Delta t = T/N \). Accordingly, the fractional derivative is approximated by
\[
\partial_t^\alpha \psi(t_1) \approx b_{1,0} (\psi(t_1) - \psi(t_0)),
\]
\[
\partial_t^\alpha \psi(t_n) \approx \sum_{k=1}^{n-1} (b_{n,k-1} - b_{n,k}) \psi(t_k) + b_{n,n-1} \psi(t_n) - b_{n,0} \psi(t_0), \quad n = 2, \cdots, N,
\]
where parameters
\[
b_{n,k} = \Gamma(2-\alpha)^{-1} \Delta t^{-\alpha} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}], \quad k = 0, \cdots, n-1.
\]

For the random term \( g(t_n, \omega) = g_1(t_n) + g_2(t_n) \bar{W}(t_n) \), from the property \( \bar{W}(t) - \bar{W}(s) \sim \mathcal{N}(t-s) \), the following approximation is given
\[
\bar{W}(t_n) \approx [\bar{W}(t_n) - \bar{W}(t_{n-1})]/\Delta t \sim \mathcal{N}(0, 1).
\]

Therefore, the discretized scheme for solving equation (1) is given as: for \( \hat{u}_n \in V_m, \ n = 1, \cdots, N \), its vector form \( \bar{u}_n \) satisfies
\[
\begin{align*}
(b_{1,0} \bar{M} + \bar{S}) \bar{u}_1 &= \bar{M} \left( g_1(t_1) \bar{f} + g_2(t_1) \Delta t^{-1/2} \mathcal{N}(0, 1) \bar{f} + b_{1,0} \hat{u}_0 \right), \\
(b_{n,n-1} \bar{M} + \bar{S}) \bar{u}_n &= \bar{M} \left( g_1(t_n) \bar{f} + g_2(t_n) \Delta t^{-1/2} \mathcal{N}(0, 1) \bar{f} \right. \\
&\quad + \sum_{k=1}^{n-1} (b_{n,k} - b_{n,k-1}) \hat{u}_k + b_{n,0} \hat{u}_0),
\end{align*}
\]
noting that \( \hat{u}_0 = 0 \) due to the zero initial condition. To capture the randomness in system (1), we need to collect numerous realizations for the single point solution \( u(x_0, t, \omega) \). Furthermore, the solution \( v(x, t) \) defined in equation (3) can be numerically simulated analogously.

4.2. **Inverse problem and mollification.** Here we consider the inverse problem of solving the unknowns \( g_1, g_2^2 \) and denote the numerical approximations as \( \hat{g}_1, \hat{g}_2^2 \) respectively.

After measuring the noisy data \( h^\sigma(t, \omega) \) disturbed by white noise \( \sigma \) from \( h(t, \omega) = u(x_0, t, \omega) \), we obtain the mean and variance moments of the fractional integral

\[<t_0, \omega> = \mathcal{E} \left( h(t, \omega) - u(x_0, t, \omega) \right).\]
\[ H^\sigma := I_1^{1-\alpha} h^\sigma(t, \omega), \] denoting as \( E^\sigma(t) \) and \( V^\sigma(t) \), respectively. To reconstruct \( G_1 \) and \( G_2 \) from (6), we need to solve the second kind Volterra equation

\[ Y(t) = X(t) + \int_0^t X(\tau)K(t - \tau)d\tau =: X(t) + \mathcal{K}X(t), \quad t \in [0, T]. \]

One of the common choices is the iterative method with initial guess \( Y(t) \), namely

\[ X_{n+1}(t) = Y(t) - \mathcal{K}X_n(t), \quad X_0(t) = Y(t). \]

The convergence of (10) can be assured if \( Y, \mathcal{K} \in L^2[0, T], [56]. \)

Before using iteration (10), we introduce the mollification method here. As defined in [1, 42], a mollifier is a nonnegative, real-valued function in \( C_0^\infty(\mathbb{R}) \). For example we take

\[ J(t) = \begin{cases} c \cdot \exp[-1/(1 - |t|^2)], & |t| < 1, \\ 0, & |t| \geq 1, \end{cases} \]

where \( c > 0 \) is chosen so that \( \int_\mathbb{R} J(t)dt = 1 \). Then for any \( \epsilon > 0 \), \( J_\epsilon(t) = c^{-1}J(t/\epsilon) \)

is a mollifier with compact support belonging to \((-\epsilon, \epsilon)\). The convolution

\[ J_\epsilon \ast \phi(t) = \int_\mathbb{R} J_\epsilon(t - \tau)\phi(\tau)d\tau \]

is called a mollification or regularization of \( \phi \) if the right side converges. The convergence of \( J_\epsilon \ast \phi \) as \( \epsilon \to 0 \) is assured by the regularity of \( \phi \). To that moment \( \hat{E}^\sigma(t), \hat{V}^\sigma(t), \quad t \in [0, T], \) in order to apply the mollification, we extend the domain by symmetric \( 2T \)-periodic extension, and denote the extensions by \( \hat{E}^\sigma(t), \hat{V}^\sigma(t) \), i.e.

\[ \hat{E}^\sigma(t) = \begin{cases} E^\sigma(t - 2kT), & t \in [2kT, (2k + 1)T], \\ E^\sigma(2kT - t), & t \in [(2k - 1)T, 2kT], \end{cases} \quad k \in \mathbb{Z}, \]

\[ \hat{V}^\sigma(t) = \begin{cases} V^\sigma(t - 2kT), & t \in [2kT, (2k + 1)T], \\ V^\sigma(2kT - t), & t \in [(2k - 1)T, 2kT], \end{cases} \quad k \in \mathbb{Z}. \]

Therefore, we can solve (9) and (10) to get \( \int_0^t \hat{g}_1(\tau)d\tau \) and \( \int_0^t \hat{g}_2(\tau)d\tau \) by letting

\[ Y(t) = f^{-1}(x_0) J_\epsilon \ast \hat{E}^\sigma(t), \quad K(t) = f^{-1}(x_0)v_l(x_0, t), \]

\[ Y(t) = f^{-2}(x_0) J_\epsilon \ast \hat{V}^\sigma(t), \quad K(t) = 2f^{-2}(x_0)v(x_0, t)v_l(x_0, t), \]

respectively.

Next we’re going to state the meaning of using mollification. Before that, to measure the accuracy of the reconstructions, we give the \( L^2 \) error as

\[ er(g_1) = \| g_1 - \hat{g}_1 \|_{L^2[0, T]} \approx \left( \frac{\Delta t}{2} \sum_{i=0}^{N} p_i \left| g_1(t_i) - \hat{g}_1(t_i) \right|^2 \right)^{1/2}, \]

\[ er(g_2) = \| g_2 - |\hat{g}_2| \|_{L^2[0, T]} \approx \left( \frac{\Delta t}{2} \sum_{i=0}^{N} p_i \left| |g_2(t_i)| - |\hat{g}_2(t_i)| \right|^2 \right)^{1/2}, \]

where weights \( p_0 = p_N = 1, \quad p_i = 2, \quad i = 1, \cdots, N - 1. \) It is natural that the more realizations we use, the better the numerical results are. This is confirmed by Table 1. However, large amount of samples causes high cost, as a consequence, the mollification is applied. In Table 2, we can see the improvement caused by mollification. To sum up, mollification can help us generate satisfactory results.

\footnote{In this subsection, we set \( \sigma = 0 \), i.e. no observation noise.}
with less samples. Moreover, taking \((e1)\) as an example, Figures 1, 2, 3 and 4 provide visual evidences to support the necessity of mollification.

**Table 1.** Errors without mollification.

| \(\epsilon\) | 10³ samples | 10⁴ samples | 10⁵ samples |
|----------------|--------------|--------------|--------------|
| \(e1\)        | 0.299794     | 0.056932     | 0.028705     |
| \(e2\)        | 0.449422     | 0.084018     | 0.046238     |
| \(e3\)        | 0.447672     | 0.083309     | 0.081429     |
| \(e4\)        | 1.570322     | 0.288016     | 0.151976     |
| \(e5\)        | 1.542568     | 0.310195     | 0.171542     |

**Table 2.** Errors for different \(\epsilon\), 10³ realizations used.

| \(\epsilon\) | Without mollification |
|--------------|-----------------------|
| \(e1\)      | 0.299794 0.056932    |
| \(e2\)      | 0.449422 0.084018    |
| \(e3\)      | 0.447672 0.083309    |
| \(e4\)      | 1.570322 0.288016    |
| \(e5\)      | 1.542568 0.310195    |

Now, we are interested in the dependence of the reconstruction results on the mollification parameter \(\epsilon\). Still using \((e1)\), as is shown in Figure 11, as the mollification parameter increases, the error decreases at the beginning but then grows up. This is because large \(\epsilon\) will lead to oversmoothing, making the moments deviate from the true values substantially. More precisely, we choose different values marked on Figure 11 to illustrate that too small or too large values of \(\epsilon\) are not desirable, see Figure 5.

4.3. **Numerical experiments.** In this subsection, we will focus on the adaptation of our algorithm to different sizes of observation noise. Most of the observation noises are raised by observations, and obey a normal distribution, denoted as \(\xi \sim N(0, \sigma^2)\). As stated in [15], from the Bayesian standpoint, the probability of the random variable \(\xi \sim N(0, \sigma^2)\) belongs to the interval \([-2.58\sigma, 2.58\sigma]\) is 99%. Thus, we will use the bound 2.58\(\sigma\) to evaluate the size of observation noises.

Figures 6-10, 12-16 give the reconstructions for experiments \((ej)\), \(1 \leq j \leq 5\). We can see that our algorithm performs well even for discontinuous cases. This means the continuous restrictions on the unknowns \(g_1, g_2\) may be weakened in the numerical aspect. Note that the amount of samples is 10³, and we adopt \(\epsilon = 0.05\), which is chosen according to the time step size \(\Delta t = 10^{-3}\).

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Inverse random source problem

Figure 1. Experiment (e1), $\hat{E}$ (top) and $g_1, \hat{g}_1$ (bottom) for different amount of samples.

Figure 2. Experiment (e1), $\hat{E}, J_\epsilon \ast \hat{E}$ (top) and $g_1, \hat{g}_1$ (bottom) for different $\epsilon$, $10^3$ realizations.
Figure 3. Experiment (e1), $\hat{V}$ (top) and $|g_2|, |\hat{g}_2|$ (bottom) for different amount of samples.

Figure 4. Experiment (e1), $\hat{V}$, $J_\epsilon * \hat{V}$ (top) and $|g_2|, |\hat{g}_2|$ (bottom) for different $\epsilon$, $10^3$ realizations.
Inverse random source problem

Figure 5. Experiment (e1), $g_1, \hat{g}_1$ (top) and $|g_2|, |\hat{g}_2|$ (bottom) for different $\epsilon$.

Figure 6. Experiment (e1), $g_1, \hat{g}_1$ (top) and $|g_2|, |\hat{g}_2|$ (bottom) for different $\sigma$. 
Figure 7. Experiment (e2), $g_1, \hat{g}_1$ (top) and $|g_2|, |\hat{g}_2|$ (bottom) for different $\sigma$.

Figure 8. Experiment (e3), $g_1, \hat{g}_1$ (top) and $|g_2|, |\hat{g}_2|$ (bottom) for different $\sigma$. 
Inverse random source problem

Figure 9. Experiment (e4), $g_1, \hat{g}_1$ (top) and $|g_2|, |\hat{g}_2|$ (bottom) for different $\sigma$.

Figure 10. Experiment (e5), $g_1, \hat{g}_1$ (top) and $|g_2|, |\hat{g}_2|$ (bottom) for different $\sigma$. 
Figure 11. Experiment (e1), \( \epsilon r(g_1) \) and \( \epsilon r(|g_2|) \) under different \( \epsilon \).

Figure 12. Experiment (e1), \( \sigma r(g_1) \) (left) and \( \sigma r(|g_2|) \) (right) for different \( \sigma \).

Figure 13. Experiment (e2), \( \sigma r(g_1) \) (left) and \( \sigma r(|g_2|) \) (right) for different \( \sigma \).
Figure 14. Experiment (e3), $\epsilon r(g_1)$ (left) and $\epsilon r(|g_2|)$ (right) for different $\sigma$.

Figure 15. Experiment (e4), $\epsilon r(g_1)$ (left) and $\epsilon r(|g_2|)$ (right) for different $\sigma$.

Figure 16. Experiment (e5), $\epsilon r(g_1)$ (left) and $\epsilon r(|g_2|)$ (right) for different $\sigma$. 
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