About electric field lines of an arbitrarily moving point charge

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We present an elementary derivation of a parametric equation describing electric field lines of an arbitrarily moving point charge. The found equation describes electric field lines as parametrized lines and occurs to be much simpler than the equation for the electric field which requires the solution of the implicit retarded-time equation. The found equation is generally valid and can be applied to arbitrarily complex and relativistic motions of a point charge. We exemplify this equation by considering four particular examples: an uniformly accelerated charge, an uniformly decelerated charge, a charge moving in a free electron laser, and a charge moving in a synchrotron.

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I. INTRODUCTION

Electric and magnetic fields of arbitrarily moving point charges are a fascinating topic where relativistic physics meets classical electrodynamics. In particular, accelerated point charges are the generators for almost all electromagnetic radiation, such as emitted by oscillating electric dipoles, synchrotrons, or free electron lasers. As is met classical electrodynamics. In particular, accelerated charges are a fascinating topic where relativistic physics is the particles velocity \( v \) divided by the speed of light \( c \), \( \gamma = 1/\sqrt{1 - \beta^2} \), and a dot denotes differentiation after time. Here and below, a hat above a vector always symbolizes normalization (unit vector). For finding the electric field at a given position \( r \) and time \( t \), we have first to solve the retarded time equation (1), and then evaluates the right hand side of (1) at time \( t' \), which is typically a numerically demanding task. Another way of visualizing an electric field is to use electric field lines, continuous lines of force tangential to the electric field. Visualization of field lines can help in better understanding complex field configurations generated by non-trivial particle trajectories, and knowledge of field lines could also be used to find the electric field strength, due to the interconnection between local field line density and field strength as embodied by the zero divergence of the electric field in source-free space.

\[
E(r, t) = q \left\{ \frac{\hat{R} - \beta}{\gamma^2 R^2 \left( 1 - \hat{R} \cdot \beta \right)^3} + \frac{\hat{R} \times \left[ (\hat{R} - \beta) \times \beta \right]}{c R \left( 1 - \hat{R} \cdot \beta \right)^3} \right\} \cdot \lambda n',
\]

where the vector \( R \) is the spatial part of the four-dimensional null-vector

\[
\{ c(t - t'), r - r_0(t') \}
\]

which defines the retarded time \( t' < t \) via

\[
t - t' = \frac{R}{c} = \frac{|r - r_0(t')|}{c}
\]

at which the right hand side of equation (1) has to be evaluated. Here, \( r_0(t) \) is the particle’s trajectory as a function of time \( t \). Furthermore, the symbol \( \beta = v/c \) is the particles velocity \( v \) divided by the speed of light \( c \), \( \gamma \) is the usual Lorentz factor \( \gamma = 1/\sqrt{1 - \beta^2} \), and a dot denotes differentiation after time. Here and below, a hat above a vector always symbolizes normalization (unit vector). For finding the electric field at a given position \( r \) and time \( t \), we have first to solve the retarded time equation (1), and then evaluates the right hand side of (1) at time \( t' \), which is typically a numerically demanding task. Another way of visualizing an electric field is to use electric field lines, continuous lines of force tangential to the electric field. Visualization of field lines can help in better understanding complex field configurations generated by non-trivial particle trajectories, and knowledge of field lines could also be used to find the electric field strength, due to the interconnection between local field line density and field strength as embodied by the zero divergence of the electric field in source-free space.

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\end{align*}\]
II. THEORY

Let us describe a field line at time $t$ by the parametric three-dimensional curve $\mathbf{p}(t, \tau)$ which is parametrized by the variable $\tau$. At all positions, it will be parallel to the electric field vector, which means that it has to obey the differential equation

$$\frac{d\mathbf{p}(t, \tau)}{d\tau} \propto E[p(t, \tau)]. \quad (4)$$

Taking into account the non-trivial form of the electric field as given by eq. (1), finding analytic solutions to this equation is a formidable task. Nonetheless, this was successfully achieved by Arutyunyan in the eighties of the last century, see refs. [8,9]. Here, we will present a much simpler approach starting for the electric field of a non-moving point charge. Consider the situation depicted in Fig. 1 and consider the electric field on the sphere $|\mathbf{r} - \mathbf{r}_0(t')| = c(t - t')$ at time $t$ which is generated by the charge at the position $\mathbf{r}_0(t')$ at retarded time $t'$. Let us first look at the situation from the rest frame of the point charge if it would travel uniformly from its position $\mathbf{r}_0(t')$ at time $t'$. In this rest frame, field lines point radially away from particle’s position, and on the sphere they are positioned at $\mathbf{r}_0(t') + c(t - t')\hat{n}$, where a tilde above a variable symbolizes that it refers to the particle’s rest frame, and $\hat{n}$ is a unit vector defining the starting direction of a field line in the particles rest frame.

The important idea is to recognize that the vector $c(t - t')\hat{n}$ can be considered as the spatial part of the null-vector $\{c(t - t'), c(t - t')\hat{n}\}$. To find the field line position in the observer’s lab frame, we have to boost this vector back by $-\hat{\beta} = -\mathbf{r}_0(t')/c$. For the transformed null vector, we then find

$$\{c(t - t'), \beta c(t - t') + \lambda \hat{n}'\} = \gamma \{c(\hat{t} - \hat{t'})/(1 + \mathbf{\beta} \cdot \hat{n}), c(\hat{t} - \hat{t'})\hat{n} + \hat{\beta}\} \quad (5)$$

where we have split the spatial part of the transformed vector into one part describing the anticipated position of the point charge if it would travel uniformly from its position $\mathbf{r}_0(t')$ during time $c(t - t')$ and a second part describing a vector $\lambda \hat{n}'$ point from this position to $\mathbf{r}$, see Fig. 1. From this equation, we obtain

$$c(\hat{t} - \hat{t'}) = \frac{c(t - t')}{\gamma(1 + \mathbf{\beta} \cdot \hat{n})} \quad (6)$$

and

$$\beta c(t - t') + \lambda \hat{n}' = c(t - t')\frac{\hat{n} + \hat{\beta}}{\gamma(1 + \mathbf{\beta} \cdot \hat{n})} \quad (7)$$

which then leads, after some algebraic transformations, to the final result

$$\mathbf{p}(t, t' | \hat{n}) = \mathbf{r}_0(t') + c(t - t')\left[\mathbf{\beta} + \frac{(\gamma^{-1} - 1)(\mathbf{\hat{n}} \cdot \hat{\beta}) \hat{\beta} + \hat{n}}{\gamma(1 + \mathbf{\hat{n}} \cdot \mathbf{\beta})}\right] \text{\hat{t}' \quad (8)}$$

Here, $\mathbf{\beta}$ and $\gamma$ in the square bracket are evaluated at time $t'$ and, as before, a hat over a vector symbolizes its normalization (unit vector).

The found expression for $\mathbf{p}(t, t' | \hat{n})$, Eq. (8), gives an explicit parametric representation of a field line, which is fully determined by its “starting” direction $\hat{n}$ in the particle’s rest frame, and which is parametrized by the retarded time variable $t'$. For finding a particular field line, one first defines $\hat{n}$ and then traces the line for decreasing values of $t'$ starting from $t' = t$. Thus, $t'$ plays the role of a line parameter and does not have to be found a priori from an implicit retarded time equation such as eq. (7), as has to be done when calculating the electric field. The derived representation of eq. (8) is generally valid for an arbitrarily moving point charge, and it is identical to the result of Arutyunyan found via direct integration of eq. (4), see e.g. eq. (1) in Ref. [4].

Let us check eq. (8) against the well-known case of a point charge uniformly moving with velocity $\mathbf{r}_0(t')$. Then, the electric field reads

$$\mathbf{E}(\mathbf{r}, t) = \frac{q\gamma\Delta \mathbf{r}}{\left\{\gamma^2(\Delta \mathbf{r} \cdot \mathbf{\beta})^2 + \left[\Delta \mathbf{r} - (\Delta \mathbf{r} \cdot \mathbf{\beta})\mathbf{\beta}\right]^2\right\}^{3/2}} \quad (9)$$

where we have used the abbreviation $\Delta \mathbf{r} = \mathbf{r} - \mathbf{r}_0(t') - c\mathbf{\beta}(t - t')$, and $\mathbf{\beta}$ is a unit vector along $\mathbf{\beta}$. This describes an isotropic electric field which is “squeezed” by a factor $\gamma^{-1}$ along the direction of motion. Thus, if a field line is directed along unit vector $\hat{n}$ in the particle’s rest frame, it will point along direction

$$\hat{n}' = \frac{(\gamma^{-1} - 1)(\mathbf{\hat{n}} \cdot \hat{\beta}) \hat{\beta} + \mathbf{\hat{n}}}{\sqrt{1 - (\mathbf{\hat{n}} \cdot \mathbf{\beta})^2}} \quad (10)$$

in the observers’ lab frame. Comparing eq. (10) with eq. (8) shows that eq. (8) indeed describes straight field lines along directions $\hat{n}'$ starting from the momentous position $\mathbf{r}_0(t') + c\beta(t - t')$ of the uniformly moving charge. This nicely confirms the validity of eq. (8) for a uniformly moving charge.

Remarkably, eq. (8) depends only on the particle’s trajectory and velocity, but not on its acceleration, in contrast to the electric field where the term containing the acceleration accounts for the part of the field which is connected with electromagnetic radiation. However, if $\mathbf{p}(t, t' | \hat{n})$ would contain terms including the acceleration $\mathbf{r}_0(t')$, then its differentiation after $t'$ would lead to the occurrence of third order time derivatives of $\mathbf{r}_0(t')$, which are absent in the electric field and would thus violate
eq. (4). This gives an alternative heuristic justification of eq. (8): Because it holds for a uniformly moving charge, and because the general expression, as just said, can only depend on position and velocity of the particle, eq. (8) must be generally true for an arbitrarily moving charge.

III. APPLICATIONS

All figures for the examples presented in this section were calculated with MATHEMATICA, directly using eqs. (1) and (8). As a first application of eq. (8) we consider the well-known classical example of a uniformly accelerated charge which is at rest at time zero, then (relativistically) accelerates along the x-direction with constant acceleration within one unit of time to the speed $c/\sqrt{2}$, and then continues to move uniformly with that constant velocity. For such a motion, the particle’s x-position as a function of time is given by

$$x_0(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ c \left(1 + t^2 - \sqrt{1 + t^2}\right), & \text{if } 0 < t \leq 1 \\ c \left(\sqrt{2} - 1 + (t - 1)/\sqrt{2}\right), & \text{if } t > 1 \end{cases}$$ (11)

The resulting field lines and electric field for $t = 4$ are presented in Fig. 2. Although the motion of the charge is a simple time-reversal of the first example, the field lines and electric field look significantly different, which is, of course, a direct consequence of the retarded time effect.

The second example considers the opposite situation: A uniformly moving charge (uniform speed $c/\sqrt{2}$) starts to decelerate at time zero with constant deceleration so that it stops moving at time one. Now, its position is given by

$$x_0(t) = \begin{cases} ct/\sqrt{2}, & \text{if } t \leq 0 \\ c \left(1 + t/\sqrt{2} - \sqrt{1 + t^2}\right), & \text{if } 0 < t \leq 1 \\ c \left(1 + 1/\sqrt{2} - \sqrt{2}\right), & \text{if } t > 1 \end{cases}$$ (12)

The third and more interesting example refers to the motion of a point charge in a free electron laser: A point charge moves with constant speed $c/\sqrt{2}$ along the x-direction and oscillates (wiggles) along the orthogonal...
The resulting field lines and electric field are shown in Fig. 4. One nicely sees that regions of strong transversal field-line orientation (with respect to the line of sight from the particle) and thus field line density correspond to regions of large electric field strength, as it should be.

![Electric field lines and electric field amplitude for a circularly moving point charge.](image)

**FIG. 5.** Electric field lines and electric field amplitude for a circularly moving point charge (indicated by green line). Here, the unit of length is 1 km, and the radius of the circular motion is 0.1 km. High field intensities coincide with strong bunching of electric field lines, demonstrating nicely the tight connection between field intensity and field line density.

The last considered example refers to the motion of a point charge in a synchrotron: a motion with uniform speed around a circle with radius $a$ and angular frequency $\omega$ as described by

$$\mathbf{r}_0(t) = a \left( \cos \omega t \hat{x} + \sin \omega t \hat{y} \right)$$

(14)

For the calculation, we have chosen a radius $a$ of 100 m and a travel speed of 0.99c, which corresponds to a circular frequency of $\omega = 0.99 \, c/a \sim 2.97 \cdot 10^6 \, \text{s}^{-1}$, or an oscillation period of $\sim 2.1 \, \mu\text{s}$. The resulting field line structure and electric field are presented in Fig. 5.

One can use this field line structure also for finding the direction of the electromagnetic energy flux at a given position: Recalling the the magnetic field is given by $\mathbf{B}(\mathbf{r},t) = \hat{s}(t') \times \mathbf{E}(\mathbf{r},t)$ [2], where $\hat{s}(t')$ is the unit vector pointing from the charge’s position at retarded time $t'$ to the position $\mathbf{r}$ where the field is calculated, one finds that the direction of the Poynting vector at position $\mathbf{r}$ and time $t$ is along direction $\hat{n} \times \partial (\mathbf{p}(t,t'|\hat{n})/dt + \mathbf{E}(\mathbf{r},t)) / dt'$. Thus, for a motion confined to a plane, as is the case for the synchrotron example, the Poynting vector is everywhere perpendicular to the field lines and, outside the trajectory’s circle, directed outwards. Thus, Fig. 5 shows for example that the light emission (‘shock waves’ of field lines) of a synchrotron is directed along lines tangential to the circle of the particle’s rotation.

**IV. CONCLUSION**

We have presented an elementary derivation of a parametric equation describing electric field lines for an arbitrarily moving charge. It is explicit and easy to evaluate, thus providing a powerful tool for visualizing the electric field structure generated by a point charge moving along arbitrarily complex and relativistic trajectories.

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[1] L. Landau and E. Lifshitz, *The classical theory of fields*, ch. 3, pp. 1–10. Pergamon, 1971.
[2] J. Jackson, *Classical electrodynamics*, ch. 14, pp. 661–707. John Wiley & Sons, 1998.
[3] S. Arutyunyan, “Force lines of electric and magnetic fields of an arbitrarily moving charge,” *Radiophysics and Quantum Electronics*, vol. 28, no. 7, pp. 619–624, 1985.
[4] S. Arutyunyan, “Electromagnetic field lines of a point charge moving arbitrarily in vacuum,” *Physics-Uspekhi*, vol. 29, no. 11, pp. 1053–1057, 1986.
[5] S. Arutyunyan, “Equivalent representation of orthogonal-field electrodynamics by system of covariant lines of force,” *Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika*, vol. 35, no. 3, pp. 313–323, 1992.