Uniform Rates for Kernel Estimators of Weakly Dependent Data

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Abstract

This paper provides new uniform rate results for kernel estimators of absolutely regular stationary processes that are uniform in the bandwidth and in infinite-dimensional classes of dependent variables and regressors. Our results are useful for establishing asymptotic theory for two-step semiparametric estimators in time series models. We apply our results to obtain nonparametric estimates and their rates for Expected Shortfall processes.

Keywords: Uniform-in-bandwidth; Kernel estimation; Empirical process theory; Mixing.

JEL classification: C14; C22.

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1 Introduction

Kernel estimators were first introduced by Rosenblatt (1956) for density estimation and by Nadaraya (1964) and Watson (1964) for regression estimation. Uniform convergence for kernel estimators of weakly dependent stationary data has been considered in a number of papers, including Bierens (1983), Liero (1989), Roussas (1990), Peligrad (1991), Andrews (1995), Liebscher (1996), Masry (1996), Bosq (1998), Fan and Yao (2003), Ango Nze and Doukhan (2004), Hansen (2008), Kristenssen (2009), and Kong, Linton, and Xia (2010), among others. In this paper we provide a general uniform rate result for kernel estimators of absolutely regular stationary processes, where the uniformity is in the bandwidth and over possibly infinite-dimensional classes of dependent variables and regressors. Our results are useful for establishing asymptotic theory for two-step semiparametric estimators in time series models.

We generalize a number of uniform-in-bandwidth results that were obtained for independent and identically distributed observations by Einmahl and Mason (2005) and Escanciano, Jacho-Chavez and Lewbel (2014) to the weakly dependent stationary case. Our results complement related results given in Andrews (1995) and Kristenssen (2009). These authors permit more heterogeneity and different dependence concepts than ours. In contrast, we deal with unbounded dependent variables (unlike Andrews (1995)), infinite-dimensional classes of regressors and dependent variables, and provide uniform-in-bandwidth results (unlike Kristenssen (2009)). We provide primitive conditions for some of the equicontinuity assumptions required in Andrews (1995). Our conditions for infinite-dimensional classes are relatively easy to check.

We apply empirical processes tools developed in Doukhan, Massart and Rio (1995) to deal with the uniformity in the stochastic part of kernel estimators, replacing the use of the celebrated Talagrand’s inequality (see Talagrand, 1994) in the work of Einmahl and Mason (2005) and Escanciano, Jacho-Chavez and Lewbel (2014). This method of proof requires establishing some preliminary entropy bounds for classes indexed by the bandwidth, as in Einmahl and Mason (2005), but also over classes of dependent variables and regressors. The entropy bounds are for a special norm introduced in Doukhan et al. (1995), which accommodates the weak dependence structure.

We introduce notation from empirical processes theory that will be used throughout. For a class of measurable functions $\mathcal{G}$ from $\mathbb{R}^p$ to $\mathbb{R}$, let $\| \cdot \|$ be a generic pseudo-norm on $\mathcal{G}$, defined as a norm except for the property that $\|f\| = 0$ does not necessarily imply that $f \equiv 0$. Given two functions $l, u$, a bracket $[l, u]$ is the set of functions $f \in \mathcal{G}$ such that $l \leq f \leq u$. An $\epsilon$-bracket with respect to $\|\cdot\|$ is a bracket $[l, u]$ with $\|l - u\| \leq \epsilon$, $\|l\| < \infty$ and $\|u\| < \infty$ (note that $u$ and $l$ not need to be in $\mathcal{G}$). The covering number with bracketing $N_{\|\cdot\|}(\epsilon, \mathcal{G}, \|\cdot\|)$ is the minimal number of $\epsilon$-brackets with respect to $\|\cdot\|$ needed to cover $\mathcal{G}$. These definitions are extended to classes taking values in $\mathbb{R}^d$, with $d > 1$, by taking the maximum of the bracketing numbers of the coordinate classes. Let $\|\cdot\|_{L^2}$ be the $L_2(\mathbb{P})$ norm, i.e. $\|f\|_{L^2}^2 = \int f^2 d\mathbb{P}$. When $\mathbb{P}$ is clear from
the context, we simply write \(\|\cdot\|_2 \equiv \|\cdot\|_{2,p}\). Let \(\cdot\) denote the Euclidean norm, i.e. \(|A|^2 = A^T A\) \((A^T \) denotes the transpose of \(A\)). Define for any vector \(a\) of \(p\) integers the differential operator \(\partial_x^a := \partial^{a_1}/\partial x_1^{a_1} \ldots \partial x_p^{a_p}\), where \(|a|_1 := \sum_{t=1}^{p} a_t\). Let \(S\) be a convex set of \(\mathbb{R}^p\), with non-empty interior. For any smooth function \(h : S \subseteq \mathbb{R}^p \to \mathbb{R}\) and some \(\eta > 0\), let \(\eta\) be the largest integer strictly smaller than \(\eta\), and

\[
\|h\|_{\infty,\eta} := \max_{|a|_1 \leq 2} \sup_{x \in X} |\partial_x^a h(x)| + \max_{|a|_1 = \eta} \sup_{x \neq x'} \frac{|\partial_x^a h(x) - \partial_x^a h(x')|}{|x - x'|^{\eta-2}}.
\]

Further, let \(C^n_M(S)\) be the set of all continuous functions \(h : S \subseteq \mathbb{R}^p \to \mathbb{R}\) with \(\|h\|_{\infty,\eta} \leq M\). The sup norm is \(\|h\|_{\infty} := \sup_{x \in S} |h(x)|\). Finally, throughout \(C\) denotes a positive constant that may change from expression to expression. Henceforth, we abstract from measurability issues that may arise (see van der Vaart and Wellner (1996) for ways to deal with lack of measurability).

2 Uniform Rate Results

Let \(Z_n := \{Y_t, X_t\}_{t=1}^n\) represent a sample of size \(n\) from a sequence of stationary and \(\beta\)-mixing process \(Z_t = (Y_t, X_t)\), where \(Y_t\) takes values in \(S_Y \subseteq \mathbb{R}^q\) and \(X_t\) takes values in \(S_X \subseteq \mathbb{R}^p\). Recall the definition of a \(\beta\)-mixing process. Let \(\mathcal{F}^t_s \equiv \mathcal{F}^t_s(Z_t)\) denote the \(\sigma\)-algebra generated by \(\{Z_j, j = s, \ldots, t\}\), \(s \leq t, s, t \in \mathbb{Z}\). Define the \(\beta\)-mixing coefficients as (see, e.g., Doukhan (1994))

\[
\beta_j = \sup_{m \in \mathbb{Z}} \sup_{A \in \mathcal{F}^j_{\infty}} \mathbb{E} \left[ \mathbb{P}(A|\mathcal{F}^m_{\infty}) - \mathbb{P}(A) \right].
\]

Let \(\Upsilon\) be a class of measurable real-valued functions of \(Z_t\) and let \(\mathcal{W}\) be a class of measurable functions of \(X_t\) with values in \(\mathbb{R}^d\), \(d \leq q\). Define \(S_\Upsilon := \{W(x) \in \mathbb{R}^d : W \in \mathcal{W}, x \in X_X\}\). We denote by \(\psi := (\varphi, W)\) a generic element of the set \(\Psi := \Upsilon \times \mathcal{W}\). Let \(f_W(w)\) denote the Lebesgue density of \(W(X_t)\) evaluated at \(w\). Define the regression function \(m_\psi(w) := \mathbb{E}[\varphi(Z_t)|W(X_t) = w]\). Henceforth, we use the convention that a function evaluated outside its support is zero. Then, an estimator for \(T_\psi(w) := m_\psi(w)f_W(w)\) is given by

\[
\hat{T}_{\psi,h}(w) = \frac{1}{nh^d} \sum_{t=1}^{n} \varphi(Z_t) K \left( \frac{w - W(X_t)}{h} \right),
\]

where \(K(w) = \prod_{l=1}^{d} k(w_l)\), \(k(\cdot)\) is a kernel function, \(h := h_\eta > 0\) is a bandwidth and \(w = (w_1, \ldots, w_d)^T\). We consider the following regularity conditions on the data generating process, kernel, bandwidth and classes of functions.

**Assumption 1** \(\{Z_t\}_{t \in \mathbb{Z}}\) is a strictly stationary and absolutely regular (\(\beta\)-mixing), with mixing coefficients of order \(O(j^{-b})\), for some \(b\) such that \(b > \delta/(\delta - 2)\), where \(2 < \delta < \infty\).
Assumption 2 For $\delta > 2$ as in Assumption 1 and each $1 > \varepsilon > 0$: (i) the class $\mathcal{Y}$ satisfies $\log N(\varepsilon, \mathcal{Y}, \|\cdot\|_2) \leq C\varepsilon^{-v_\varphi}$, for some $v_\varphi < 2$, with an envelope $G(Z_t)$ such that $\mathbb{E}[G(Z_t)^h] < \infty$ and $\sup_{w \in \mathcal{S}_W} \mathbb{E}[G(Z_t)^2|W(X_t) = w] < C$; (ii) the class $\mathcal{W}$ is such that (a) $\log N(\varepsilon, \mathcal{W}, \|\cdot\|_\infty) \leq C\varepsilon^{-v_w}$, for some $v_w < 1$, or (b) $\log N(\varepsilon, \mathcal{W}, \|\cdot\|_2) \leq C\varepsilon^{-v_w}$, for some $v_w < 1/2$.

Assumption 3 $T_\psi \in C_M^r(\mathcal{S}_W)$, where $r$ is as in Assumption 4 below, and $f_{\mathcal{W}}(w)$ is uniformly bounded.

Assumption 4 The kernel function $k(t): \mathbb{R} \to \mathbb{R}$ is bounded, symmetric and satisfies the following conditions: $\int k(t)\,dt = 1$, $\int tk(t)\,dt = 0$ for $0 < l < r$, and $\int |t^r k(t)|\,dt < \infty$, for some $r \geq 2$. Moreover, either $k$ is Lipschitz and has a truncated support or $k$ is differentiable and satisfies $|\partial k(t)/\partial t| \leq C$ and for some $v > 1$, $|\partial k(t)/\partial t| \leq C|t|^{-v}$ for $|t| > L$, $0 < L < \infty$.

Assumption 5 The possibly data-dependent bandwidth $h$ satisfies $\mathbb{P}(a_n \leq h \leq b_n) \to 1$ as $n \to \infty$, for deterministic sequences of positive numbers $a_n$ and $b_n$ such that $b_n \to 0$ and $na_n^d \to \infty$.

Assumption 1 requires that observations are strictly stationary and $\beta$-mixing, as in Doukhan, Massart and Rio (1995). As usual, there is a tradeoff between the moments and the dependence allowed. Assumption 2 restricts the “size” of the classes $\mathcal{Y}$ and $\mathcal{W}$. There are numerous examples of classes satisfying Assumption 2, see, e.g., van der Vaart and Wellner (1996) and Nickl and Pötscher (2007). Note we do not require $\mathcal{S}_X$ nor $\mathcal{S}_W$ to be bounded. Assumption 3 is a standard assumption used for controlling the bias uniformly. Assumption 4 is taken from Hansen (2008), while Assumption 5 permits data dependent bandwidths, as in, e.g., Andrews (1995). In particular, our theory allows for plug-in bandwidths of the form $\hat{h}_n = \bar{c}h_n$ with $\bar{c}$ stochastic and $h_n$ a suitable deterministic sequence converging to zero as $n \to \infty$. Andrews (1995) points out that this condition holds in many common data dependent bandwidth selection procedures, such as cross-validation and generalized cross-validation.

Define the rate

$$d_n := \sqrt{\frac{1}{na_n^d}} + b_n^r.$$  

Theorem 2.1 Let Assumptions 1–5 hold. Then, we have

$$\sup_{a_n \leq h \leq b_n} \sup_{\psi \in \mathcal{Y}} \sup_{w \in \mathcal{S}_W} |\hat{T}_{\psi,h}(w) - T_\psi(w)| = O_p(d_n).$$  

We apply the previous result to obtain rates for Nadaraya-Watson kernel estimators. Define the kernel estimators

$$\hat{m}_{\psi,h}(w) := \frac{\hat{T}_{\psi,h}(w)}{\hat{f}_{W,h}(w)}, \text{ where }$$

$$\hat{f}_{W,h}(w) := \frac{1}{nh^{d}} \sum_{t=1}^{n} K\left(\frac{w - W(X_t)}{h}\right).$$
For a positive sequence $c_n$ define also

$$
\tau_n = \inf_{|w| \leq c_n, W \in W} f_W(w) > 0.
$$

**Corollary 2.1** Let Assumptions 1-5 and $\tau_n^{-1}d_n = o(1)$ hold. Then, we have

$$
\sup_{t_n \leq h \leq u_n, \psi \in \Psi} \sup_{|w| \leq c_n} |\hat{m}_{\psi,h}(w) - m_{\psi}(w)| = O_p(\tau_n^{-1}d_n).
$$

### 3 Application to Conditional Expected Shortfall Processes

There is an extensive literature on semiparametric and nonparametric estimation of Expected Shortfall (ES). Escanciano and Mayoral (2008) review the literature on parametric and semiparametric estimation of ES and provide a unified approach; see also Nadarajah, Zhang and Chan (2014). Nonparametric estimation of Conditional ES (CES) has been studied by Scaillet (2004). He proposed a kernel estimator for the quantity

$$
CES_{a,p} := \mathbb{E}[ -a^\top Y_t - a^\top Y_t > VaR(a,p)],
$$

where the vector $a$ are portfolio weights, $a \in \mathcal{A} \subseteq \{a \in \mathbb{R}^q : |a| = 1\}$, and $VaR(a,p)$ is the $p$th Value-at-Risk (VaR), $p \in (0,1)$, defined as

$$
\mathbb{P}( -a^\top Y_t > VaR(a,p)) = p.
$$

We introduce covariates and study nonparametric estimation of

$$
CES_{a,b,c,w}(w) := \mathbb{E}[ -a^\top Y_t - a^\top Y_t > c(X), b^\top X_t = w],
$$

as a process in $(a,b,c,w)$. Portfolio weights are often estimated. The motivation to consider $b^\top X_t$ is to reduce the dimensionality of the conditioning set. The motivation to consider a function $c(X)$ is to be able to obtain rates when a plugging estimator for the conditional VaR is considered. Fully nonparametric estimators for ES with covariates are proposed in Scaillet (2005), Cai and Wang (2008), and Linton and Xiao (2013). An application of the smoothed ES estimator of Scaillet (2004) with generated variables is given in Brownlees and Engle (2016).

To study $CES_{a,b,c}$, we use that

$$
CES_{a,b,c}(w) = \frac{\mathbb{E}[\varphi_1(Z_t) | W(X_t) = w]}{\mathbb{E}[\varphi_2(Z_t) | W(X_t) = w]},
$$

where $\varphi_1 \in \mathcal{F}_1$, $\varphi_2 \in \mathcal{F}_2$ and $W \in \mathcal{W}$, with

\[
\begin{align*}
\mathcal{F}_1 &= \{(y,x) \to -a^\top y 1(-a^\top y > c(x)) : a \in \mathcal{A}, c \in \mathcal{C}\} \\
\mathcal{F}_2 &= \{(y,x) \to 1(-a^\top y > c(x)) : a \in \mathcal{A}, c \in \mathcal{C}\} \\
\mathcal{W} &= \{x \to b^\top x : b \in \mathcal{B} \subseteq \mathbb{R}^p\}.
\end{align*}
\]
Here $1(E)$ is the indicator function of the event $E$, which equals one if $E$ is true and zero otherwise. A kernel estimator for $CES_{a,b,c}$ is then

$$
\widehat{CES}_{a,b,c}(w) = \frac{1}{nh} \sum_{i=1}^n \varphi_1(Z_i) K \left( \frac{w-W(X_i)}{h} \right),
$$

To apply our previous results, write $CES_{a,b,c}(w)$ and its estimator as indexed by $\psi := (\varphi_1, \varphi_2, W) \in \Psi := \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{W}$. Thus, we write $CES_{a,b,c}(w) = CES_\psi(w)$. Define the functions $m_{\psi_j}(w) := \mathbb{E}[\varphi_j(Z_i)|W(X_t) = w]$ and $T_{\psi_j}(w) := m_{\psi_j}(w) f_W(w)$ for $j = 1, 2$. Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalue for a positive definite symmetric matrix $A$. Then, consider the following assumptions:

**Assumption 6** (i) $\mathbb{E}[|Y_t|^2] < \infty$ and uniformly in $b \in \mathbb{R}^p : 0 < \lambda_{\min}(\mathbb{E}[Y_t Y_t^T | b^T X_t]) \leq \lambda_{\max}(\mathbb{E}[Y_t Y_t^T | b^T X_t]) < C$ a.s.; (ii) the class $\mathcal{C}$ is such that $\log N(\varepsilon, \mathcal{C}, \|\cdot\|_\infty) \leq C\varepsilon^{-v_c}$, for some $v_c < 1$; (iii) $\mathcal{B}$ is compact and $\mathbb{E}[|X_t|^2] < \infty$.

**Assumption 7** $T_{\psi_j} \in C^r_M(\mathcal{S}_W)$, where $r$ is as in Assumption 4, and the conditional and marginal densities of $a^T Y_t$ given $b^T X_t$ and $b^T X_t$, respectively, are uniformly bounded (in $a \in \mathcal{A}$ and $b \in \mathcal{B}$).

Define the rate

$$
d_n := \sqrt{\frac{1}{na_n} + b^r_n}.
$$

For a positive sequence $c_n$ define also

$$
\tau_n = \inf_{|w| \leq c_n, \psi \in \mathcal{F}_2} T_{\psi_2}(w) > 0.
$$

**Theorem 3.1** Let Assumptions 1, 4, 5, 6, 7 and $\tau_n^{-1}d_n = o(1)$ hold. Then, we have

$$
\sup_{t_n \leq h \leq u_n} \sup_{\psi \in \Psi} |CES_\psi(w) - CES_{\psi}(w)| = O_p(\tau_n^{-1}d_n).
$$

## 4 Proofs

**Proof of Theorem 2.1:** Write

$$
\sup |\hat{T}_{\psi,h}(w) - T_\psi(w)| \leq \sup |\hat{T}_{\psi,h}(w) - \mathbb{E} \left[ \hat{T}_{\psi,h}(w) \right]| + \sup |\mathbb{E} \left[ \hat{T}_{\psi,h}(w) \right] - T_\psi(w)|
$$

$$
\equiv S_n + B_n,
$$

where henceforth the sup is over the set in the left hand side of (1). We start investigating the stochastic part $S_n$. Define the product class of functions $\mathcal{G}_0 := \mathcal{K}_0 \cdot \mathcal{Y}$, where

$$
\mathcal{K}_0 = \left\{ x \to K \left( \frac{w - W(x)}{h} \right) : w \in \mathcal{S}_W, W \in \mathcal{W}, h \in (0, 1] \right\}.
$$
From the boundedness of the kernel, and the squared integrable envelope in Assumption 2 it is straightforward to prove that, for some positive constant $C$,

$$N_{[;]}(\varepsilon, G_0, \| \cdot \|_2) \leq N_{[;]}(C\varepsilon, K_0, \| \cdot \|_2) \times N_{[;]}(C\varepsilon, \Upsilon, \| \cdot \|_2).$$

(2)

By Lemma B.3 in Escanciano, Jacho-Chávez and Lewbel (2014) $K_0$ satisfies

$$N_{[;]}(C\varepsilon, K_0, \| \cdot \|_2) \leq C\varepsilon^{-\alpha_K} N(\varepsilon^2, W, \| \cdot \|_\infty),$$

for some $\alpha_K \geq 1$. An by Lemma A1 in Escanciano and Zhu (2015)

$$N_{[;]}(C\varepsilon, K_0, \| \cdot \|_2) \leq C\varepsilon^{-\alpha_K} N(\varepsilon^4, W, \| \cdot \|_2).$$

An inspection of the proof of these two Lemmas reveals that $S_W$ could be unbounded. Hence, by our assumptions on the classes $\Upsilon$ and $W$, we obtain that $\log N_{[;]}(\varepsilon, G_0, \| \cdot \|_2) \leq C\varepsilon^{-v}$, for some $v < 2$. Define the norm

$$\| f \|_{2,\beta}^2 = \int_0^1 \beta^{-1}(u) Q_f^2(u) du,$$

where $\beta^{-1}$ is the inverse cadlag of the decreasing function $u \to \beta_{[u]}$ ($[u]$ being the integer part of $u$, and $\beta_t$ being the mixing coefficient) and $Q_f$ is the inverse cadlag of the tail function $u \to \mathbb{P}(|f| > u)$ (see Doukhan, Massart and Rio 1995). Note that

$$\mathbb{P}(|f - g| > z) \leq \frac{\mathbb{E}[|f - g|^2]}{z^2}$$

and hence, for an $\sqrt{b\varepsilon}$-bracket $[f, g]$ wrt $\| \cdot \|_2$

$$\| f - g \|_{2,\beta}^2 \leq \int_0^1 \beta^{-1}(u) \frac{b\varepsilon^2}{u} du \leq b\varepsilon^2 \int_0^1 u^{b-1} du = \varepsilon^2.$$

Therefore,

$$\log N_{[;]}(\varepsilon, G_0, \| \cdot \|_{2,\beta}) \leq \log N_{[;]}(\sqrt{b\varepsilon}, G_0, \| \cdot \|_2) \leq C\varepsilon^{-v}.$$

Theorem 3 in Doukhan, Massart and Rio (1995) applied to the class $G_0$ then implies

$$\sup_{t_n \leq h \leq u_n} \mathbb{E} [\hat{T}_h(\psi) - \mathbb{E} [\hat{T}_h(\psi)]] = O_p \left( \frac{1}{n\alpha_n^d} \right),$$

provided $\| f \|^2_{2,\beta} \leq C h^{d/2}$ for all $f \in G_0$. But by Assumption 4 and Pollard (1984, pg. 36)

$$\mathbb{P}(|f| > z) \leq \frac{\mathbb{E}[|f|^2]}{z^2} \leq \frac{Ch^d}{z^2},$$
where have used sup_{w \in S_W} E[G(Z_t)^2|W(X_t) = w] < C and the bounded density and kernel assumption. Hence,
\[ \| f \|_{2, \beta}^2 \leq \int_0^1 \beta^{-1}(u) \frac{Ch^d}{u} du \leq Ch^d \int_0^1 u^{b-1} du = \frac{Ch^d}{b}, \]
where the latter inequality follows from Assumption 1.

We now study the bias part \( B_n \). By a multivariate Taylor expansion
\[ T_{\psi}(w + uh) = \sum_{|\alpha| < r-1} \frac{\partial^u_{\alpha} T_{\psi}(w)}{\alpha!} (uh)^\alpha + \sum_{|\alpha| = r-1} \frac{R_{\alpha}(w + uh)}{\alpha!} (uh)^\alpha, \]
where the remainder satisfies
\[ R_{\alpha}(w + uh) = (r-1) \int_0^1 (1 - \tau)^{r-2} \partial_{\alpha}^u T_{\psi}(w + \tau uh) d\tau. \]
Since \( T_{\psi} \in C^r_M(S_W) \),
\[ |R_{\alpha}(w + uh) - \partial_{\alpha}^u T_{\psi}(w)| \leq (r-1) \int_0^1 (1 - \tau)^{r-2} [\partial_{\alpha}^u T_{\psi}(w + \tau uh) - \partial_{\alpha}^u T_{\psi}(w)] d\tau \]
\[ \leq (r-1) M \int_0^1 (1 - \tau)^{r-2} |\tau uh| d\tau \]
\[ \leq M |uh|. \]
Thus, by a standard change of variables and Assumption 4
\[ \mathbb{E} \left[ \hat{T}_{\psi,h}(w) - T_{\psi}(w) \right] = \left| \int [T_{\psi}(w + uh) - T_{\psi}(w)] K(u) du \right| \]
\[ = \left| \int \sum_{|\alpha| = r-1} \frac{1}{\alpha!} [R_{\alpha}(w + uh) - \partial_{\alpha}^u T_{\psi}(w)] (uh)^\alpha K(u) du \right| \]
\[ \leq h^r \sum_{|\alpha| = r-1} \frac{M}{\alpha!} \int |u \cdot K(u)|^r du. \]
Hence,
\[ \sup \mathbb{E} \left[ \hat{T}_{\psi,h}(w) - T_{\psi}(w) \right] = O \left( b_n^r \right). \]

**Proof of Corollary 2.1:** From Theorem 2.1
\[ \sup_{l_n \leq h \leq u_n} \sup_{\psi \in \Psi} |\hat{T}_{\psi,h}(w) - T_{\psi}(w)| = O_{\mathbb{P}}(d_n) \]
and
\[ \sup_{l_n \leq h \leq u_n} \sup_{W \in W} \sup_{|w| \leq c_n} |\hat{f}_{W,h}(w) - f_W(w)| = O_{\mathbb{P}}(d_n). \]
Therefore
\[
\sup_{l_n \leq h \leq u_n} \sup_{W \in \mathcal{W}} \sup_{|w| \leq c_n} \left| \frac{\tilde{f}_{W,h}(w) - f_W(w)}{f_W(w)} \right| = O_P(\tau_n^{-1}d_n)
\]
and
\[
\sup_{l_n \leq h \leq u_n} \sup_{W \in \mathcal{W}} \sup_{|w| \leq c_n} \left| \frac{\tilde{T}_{\psi,h}(w) - T_{\psi}(w)}{f_W(w)} \right| = O_P(\tau_n^{-1}d_n).
\]
Thus, uniformly in \(l_n \leq h \leq u_n\), \(\psi \in \Psi\) and \(|w| \leq c_n\)
\[
\hat{m}_{\psi,h}(w) = \frac{\hat{T}_{\psi,h}(w)}{f_{W,h}(w)} = \frac{m_{\psi}(w) + O_P(\tau_n^{-1}d_n)}{1 + O_P(\tau_n^{-1}d_n)} = m_{\psi}(w) + O_P(\tau_n^{-1}d_n).
\]
Q.E.D.

The following result is well-known in empirical processes theory. Define the generic class of measurable functions 
\(\mathcal{F} := \{x \mapsto m(x, \theta, h) : \theta \in \Theta, h \in \mathcal{H}\}\), where \(\Theta\) and \(\mathcal{H}\) are endowed with the pseudo-norms \(|\cdot|_\Theta\) and \(|\cdot|_\mathcal{H}\), respectively.

**Lemma 4.1** (Pollard; Chen, Linton and Van Keilegom) Assume that for all \((\theta_0, h_0) \in \Theta \times \mathcal{H}\), \(m(z, \theta, h)\) is locally uniformly \(\| \cdot \|_2\) continuous, in the sense that
\[
\mathbb{E} \left[ \sup_{|\theta - \theta_0|_\Theta < \delta, |h - h_0|_\mathcal{H} < \delta} |m(Z, \theta, h) - m(Z, \theta_0, h_0)|^2 \right] \leq C\delta^s,
\]
for all sufficiently small \(\delta > 0\), some constant \(s \in (0, 2]\) and \(C > 0\). Then,
\[
N(\varepsilon, \mathcal{F}, \| \cdot \|_2) \leq N \left( \left( \frac{\varepsilon}{2C} \right)^{2/s}, \Theta, |\cdot|_\Theta \right) \times N \left( \left( \frac{\varepsilon}{2C} \right)^{2/s}, \mathcal{H}, |\cdot|_\mathcal{H} \right).
\]

**Proof of Theorem 3.1:** The proof proceeds as in Corollary 2.1 after checking the conditions of Theorem 2.1 to obtain, for \(j = 1, 2\),
\[
\sup_{l_n \leq h \leq u_n} \sup_{\psi_j \in \Psi} \sup_{|w| \leq c_n} |\hat{T}_{\psi_j,h}(w) - T_{\psi_j}(w)| = O_P(d_n),
\]
where
\[
\hat{T}_{\psi_j,h}(w) = \frac{1}{nh} \sum_{t=1}^n \varphi_j(Z_t) K \left( \frac{w - W(X_t)}{h} \right).
\]
To verify Assumption 3 with \(\Upsilon = \mathcal{F}_1\) we apply Lemma 4.1 with \(z = (y, x)\),
\[
m(z, \theta, h) = -\theta^\top y 1(-\theta^\top y > h(x))
\]
\( \Theta = A \) and \( \mathcal{H} = C \) with \( | \cdot |_H = \| \cdot \|_\infty \). We then obtain by triangle inequality

\[
\mathbb{E} \left[ \sup_{\theta:|\theta - \theta_0|, h: |h_0 - h| < \delta} |m(Z, \theta, h) - m(Z, \theta_0, h_0)|^2 \right]
\leq 2\mathbb{E} \left[ \sup_{\theta:|\theta - \theta_0|, h: |h_0 - h| < \delta} |m(Z, \theta, h) - m(Z, \theta_0, h)|^2 \right]
+ 2\mathbb{E} \left[ \sup_{\theta:|\theta - \theta_0|, h: |h_0 - h| < \delta} |m(Z, \theta_0, h) - m(Z, \theta_0, h_0)|^2 \right]
\leq 2\delta^2 \mathbb{E} \left[ |Y_i|^2 \right] + C\delta.
\]

where the last inequality uses that \( |m(z, \theta, h) - m(z, \theta_0, h)| \leq |\theta - \theta_0| |y| \) and

\[
\mathbb{E} \left[ \sup_{\theta:|\theta - \theta_0|, h: |h_0 - h| < \delta} |m(Z, \theta_0, h) - m(Z, \theta_0, h_0)|^2 \right]
\leq \mathbb{E} \left[ \left( \theta_0^\top Y_i \right)^2 1(h_0(X_t) - \delta < -\theta_0^\top Y_i + h_0(X_t) + \delta) \right]
\leq C\delta
\]

by Assumption 7. Then, Lemma 4.1 implies

\[
N_1(\varepsilon, \Gamma, \| \cdot \|_2) \leq N \left( \left( \frac{\varepsilon}{2C} \right)^2, \Theta, | |_\Theta \right) \times N \left( \left( \frac{\varepsilon}{2C} \right)^2, C, | |_\infty \right)
\leq C\varepsilon^{-v_\varphi},
\]

with \( v_\varphi = 2v_c < 2 \). The entropy condition on \( \mathcal{W} \) in Assumption 3(ii-b) follows from the compactness of \( \mathcal{B} \) and \( \mathbb{E} \left[ |X_t|^2 \right] < \infty \). This concludes the verification of Assumption 3. The same arguments apply to \( \bar{\mathcal{Y}} = \mathcal{F}_2 \). Conclude as in Corollary 2.1. Q.E.D.

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