ABOUT THE SEMIAMPLE CONE OF THE SYMMETRIC PRODUCT OF A CURVE

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ABSTRACT. Let $C$ be a smooth curve which is complete intersection of a quadric and a degree $k > 2$ surface in $\mathbb{P}^3$ and let $C^{(2)}$ be its second symmetric power. In this paper we study the finite generation of the extended canonical ring $R(\Delta, K) := \bigoplus_{(a,b) \in \mathbb{Z}^2} H^0(C^{(2)}, a\Delta + bK)$, where $\Delta$ is the image of the diagonal and $K$ is the canonical divisor. We first show that $R(\Delta, K)$ is finitely generated if and only if the difference of the two $g_1^k$ on $C$ is torsion non-trivial and then show that this holds on an analytically dense locus of the moduli space of such curves.

INTRODUCTION

Let $C$ be a smooth complex curve of genus $g > 1$, denote by $C^2$ the cartesian product of $C$ with itself, by $C^{(2)}$ the second symmetric product of $C$ and by $\pi : C^2 \to C^{(2)}$ the double cover $(p, q) \mapsto p + q$. Let $\Delta$ be the image of the diagonal via $\pi$ and $K$ be a canonical divisor of $C^{(2)}$. In this paper we are interested in the finite generation of the extended canonical ring:

$$R(\Delta, K) := \bigoplus_{(a,b) \in \mathbb{Z}^2} H^0(C^{(2)}, a\Delta + bK).$$

It is not difficult to show that finite generation is equivalent to ask that the two-dimensional cone $\text{Nef}(\Delta, K)$ consisting of classes of nef divisors within the rational vector space spanned by the classes of $\Delta$ and $K$ must be generated by two semiample classes (see proof of Theorem 1). The Kouvidakis conjecture [5] states that, for the very general curve, if $\text{Nef}(\Delta, K)$ is closed then $g$ must be a square, i.e. $g = (k-1)^2$ for $k > 1$ integer. We thus focus our attention to the case of square genus and assume $C$ to be a complete intersection of a quadric $Q$ with a degree $k > 2$ hypersurface. We denote by $\eta_C$ the class of the difference of the two $g_1^k$ of $C$ induced by the two rulings of $Q$, which is trivial if $Q$ is a cone. Our first result is the following.

**Theorem 1.** Let $C \subseteq \mathbb{P}^3$ be a smooth complete intersection of a quadric with a degree $k > 2$ surface. Then the following are equivalent.

(i) $R(\Delta, K)$ is finitely generated.

(ii) $\eta_C$ is torsion non-trivial.

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Moreover if both the above conditions are satisfied then $\eta_C$ has order at least $k$.

Let $\mathcal{F}_k$ be the open subset of the Hilbert scheme of curves of bi-degree $(k, k)$ of $\mathbb{P}^1 \times \mathbb{P}^1$ consisting of smooth curves and let $\mathcal{F}^{\text{tor}}_k \subseteq \mathcal{F}_k$ be the subset consisting of curves $C$ such that the class $\eta_C$ of the difference between the two $g_k$ is torsion. Our second theorem is the following.

**Theorem 2.** The locus $\mathcal{F}^{\text{tor}}_k$ is a countable union of subvarieties of complex dimension $\geq 4k - 1$ and the set of subvarieties of dimension $4k - 1$ is dense in $\mathcal{F}_k$ in the analytic topology.

The paper is organized as follows. In Section 1, after recalling some basic facts about the symmetric product of a curve, we prove Theorem 1. In Section 2 we introduce the grid family consisting of curves of bi-degree $(k, k)$ on a smooth quadric which pass through a complete intersection of type $(k, 0), (0, k)$. We show that the grid family is exactly the subvariety of $\mathcal{F}^{\text{tor}}_k$ corresponding to torsion of order $k$ and has the expected dimension $4k - 1$. Section 3 is devoted to the proof of Theorem 2. In Section 4 we prove a density theorem for hyperelliptic curves, providing an alternative proof for Theorem 2 in case $g = 4$ (see Corollary 4.3). This result has an independent interest and is proved in the spirit of Griffiths computations of the infinitesimal invariant [9]. Finally, in Section 5 we consider examples of curves $C$ with $\eta_C$ torsion.

In all the paper we work over the field of complex numbers except for Section 1 (see Remark 1.8).

## 1. The second symmetric product

Let $C$ be a smooth projective curve of genus $g > 1$ defined over an algebraically closed field $K$ of characteristic 0.

**Proposition 1.1.** The diagonal embedding $\iota : C \to C^{(2)}$ induces an isomorphism $\iota^* : \text{Pic}^0(C^{(2)}) \to \text{Pic}^0(C)$ of abelian varieties.

**Proof.** To prove the statement we explicitly construct the inverse map of $\iota^*$. Given a point $p \in C$ let $H_p$ be the curve of $C^{(2)}$ which is the image of $\{p\} \times C$ via $\pi$. Define the map $\text{Div}(C) \to \text{Div}(C^{(2)})$ by $\sum n_p H_p \mapsto \sum n_p H_p$ and observe that it maps principal divisors to principal divisors. The induced map of Picard groups restricts to a homomorphism $\text{Pic}^0(C) \to \text{Pic}^0(C^{(2)})$ which is easily seen to be a right inverse of $\iota^*$. Since the two abelian varieties $\text{Pic}^0(C)$ and $\text{Pic}^0(C^{(2)})$ have the same dimension we conclude that $\iota^*$ is an isomorphism. \qed

Observe that $\Delta$ is the branch divisor of the double cover $\pi$ and thus its class is divisible by 2 in $\text{Pic}(C^{(2)})$. Moreover the following linear equivalences

$$\Delta|_\Delta \sim -2K_\Delta \quad K|_\Delta \sim 3K_\Delta$$

can be proved by passing to $C^2$ and calculating the restriction of $K_{C^2}$ to the diagonal. By the Riemann-Hurwitz formula we get the equalities $2(2g - 2)^2 = K_{C^2}^2 = 2(1 + \frac{1}{2})^2$ from which we deduce the following

$$K^2 = (g - 1)(4g - 9) \quad K \cdot \Delta = 6(g - 1) \quad \Delta^2 = -4(g - 1).$$

In particular the classes of $\Delta$ and $K$ are independent in the Néron-Severi group of $C^{(2)}$. We let $\langle \Delta, K \rangle$ be the rational vector subspace of $\text{Pic}(C^{(2)}) \otimes \mathbb{Q}$ generated
by the classes of $\Delta$ and $K$ and form the following cone

$$\text{Nef}(\Delta, K) = \{D \in \langle \Delta, K \rangle : D \text{ is nef} \}.$$ 

This cone is related to the Kouvidakis conjecture which predicts which ones are the extremal rays of $\text{Nef}(\Delta, K)$. In case the genus is a square, i.e. $g = (k - 1)^2$, the conjecture is known to be true [5] for a very general curve $C$ and it holds as well if $C$ has an irreducible $g_k^1$, that is the curve $(1.2)$ defined below is irreducible. In this case the extremal rays of $\text{Nef}(\Delta, K)$ are spanned by the classes of $2K + 3\Delta$ and $2K + (5 - 2k)\Delta$.

**Proposition 1.2.** Let $C$ be a smooth curve of genus at least two. Then the divisor $2K + 3\Delta$ of $C^{(2)}$ is semiample.

**Proof.** Observe that the divisor $2K + 3\Delta$ is big since $K$ is ample and $\Delta$ is effective. Moreover it is nef since $(2K + 3\Delta) \cdot \Delta = 0$. We have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{C^{(2)}}(2K + 2\Delta) \longrightarrow \mathcal{O}_{C^{(2)}}(2K + 3\Delta) \longrightarrow \mathcal{O}_\Delta \longrightarrow 0.$$ 

Since $2K + 2\Delta = N + K + \frac{1}{2}\Delta$, with $N = \frac{1}{2}(2K + 3\Delta)$ nef and big, then by the Kawamata-Viehweg vanishing theorem and the long exact sequence in cohomology of the above sequence we conclude that $\Delta$ is not contained in the base locus of $|2K + 3\Delta|$. The statement follows by the ampleness of $K$ and the Zariski-Fujita theorem [13, Remark 2.1.32].

Assume now that $C$ is a smooth curve of genus $g = (k - 1)^2 > 1$ which admits a $g_k^1$ and define the following curve of $C^{(2)}$:

$$(1.2) \quad \Gamma := \{p + q : g_k^1 - p - q \geq 0\}.$$ 

It can be easily proved that $\Gamma$ is irreducible if the $g_k^1$ does not contain a $g_k^1$ with $r < k$. In particular this holds if the $g_k^1$ is simple. Observe that the $g_k^1$ defines a morphism $\Gamma \to \mathbb{P}^1$ of degree $\frac{1}{2}k(k - 1)$ whose branch points are exactly those of the $g_k^1$. Thus if the $g_k^1$ is simple we deduce that

$$(1.3) \quad 2g(\Gamma) - 2 = -k(k - 1) + 2(k - 2)(k - 1)k,$$

where $g(\Gamma)$ is the genus of $\Gamma$. By continuity the genus of $\Gamma$ stays the same even if the $g_k^1$ is not simple.

**Lemma 1.3.** If $C$ admits a $g_k^1$ then the divisor $2K + (5 - 2k)\Delta$ is numerically equivalent to $(4k - 8)\Gamma$. Moreover if $C \in F_k$, the curves $\Gamma$ and $\Gamma'$ corresponding to the two distinct $g_k^1$ on $C$ are disjoint.

**Proof.** Let $H$ be the curve of $C^{(2)}$ defined by $\{p + q : q \in C\}$. By the Riemann-Hurwitz formula one has the numerical equivalence

$$K \equiv (2g - 2)H - \frac{1}{2}\Delta.$$ 

Thus by the genus formula $2g(\Gamma) - 2 = \Gamma^2 + (2g - 2)\Gamma \cdot H - \frac{1}{2}\Delta \cdot \Gamma$, by Equation (1.3) and the equalities $\Gamma \cdot H = k - 1$, $\Delta \cdot \Gamma = 2g - 2 + 2k$ we deduce $\Gamma^2 = 0$. Since the intersection matrix of the divisors $\Gamma, K, \Delta$

$$\begin{pmatrix}
0 & (2k - 5)(k - 1)k & 2(k - 1)k \\
(2k - 5)(k - 1)k & (2k - 5)(k - 2)(2k + 1) & 6(k - 2)k \\
2(k - 1)k & 6(k - 2)k & -4(k - 2)k
\end{pmatrix}$$


has rank two, by the Hodge Index theorem $\Gamma$ is numerically equivalent to a rational linear combination of $K$ and $\Delta$. Moreover, being the kernel of the above matrix generated by the vector $(4k - 8, -2, 2k - 5)$, we get the first statement. Observe that $\Gamma$ and $\Gamma'$ have no common component since otherwise the map from $C$ to the quadric given by the two $g^1_k$ would not be an embedding. Thus the second statement immediately follows from the equality $\Gamma \cdot \Gamma' = \Gamma^2 = 0$. \hfill $\square$

Lemma 1.4. If $f : C^{(2)} \to Y$ is a fibration with connected fibers onto a smooth curve, then $Y$ has genus at most one. If moreover a fiber of $f$ is numerically equivalent to a rational linear combination of $\Delta$ and $K$ then $Y$ is rational.

Proof. To prove the first statement observe that if $Y$ had genus $\geq 2$, then there would be two linearly independent holomorphic 1-forms $w_1, w_2$ of $C^{(2)}$, obtained by pull-back of 1-forms of $Y$, such that $w_1 \wedge w_2 = 0$. On the other hand we have a commutative diagram of isomorphisms

$$
\begin{array}{ccc}
\Lambda^2 H^0(C^{(2)}, \Omega^1) & \xrightarrow{w_1 \wedge w_2 \mapsto w_1 \wedge w_2} & H^0(C^{(2)}, \Omega^2) \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\Lambda^2 H^0(C, \Omega^1) & & \\
\end{array}
$$

where $\alpha$ is induced by pull-back via the diagonal embedding $C \to C^{(2)}$ and $\beta$ is defined by $w_1 \wedge w_2 \mapsto \frac{1}{4}(\pi_1^* w_1 \wedge \pi_2^* w_2 + \pi_2^* w_1 \wedge \pi_1^* w_2)$, with $\pi_i : C^2 \to C$ the projections onto the two factors. Thus such 1-forms cannot exist and $Y$ must have genus at most one.

To prove the second statement, assume that a fiber $F$ of $f$ is numerically equivalent to a rational linear combination of $\Delta$ and $K$. If $Y$ has genus one then we have the following commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\iota} & C^{(2)} \\
\downarrow{\alpha_C} & & \downarrow{\alpha_{C^{(2)}}} \\
\Alb(C) & \xrightarrow{\alpha_\iota} & \Alb(C^{(2)}) \\
\end{array} \cong \begin{array}{c}
\xrightarrow{\alpha_Y} \\
\Alb(Y) \\
\end{array}
$$

where $\alpha_\iota$ and $\alpha_f$ are induced by the universal property of the Albanese morphism and $\alpha_\iota$ is an isomorphism by Proposition 1.1. Let $\Theta$ be the theta divisor of $\Alb(C^{(2)})$. Recall that both $\alpha_{C^{(2)}}(\Delta)$ and $\alpha_{C^{(2)}}(K)$ are numerically equivalent to a multiple of $\Theta^{q-1}$. By our assumption on $F$ we deduce that $\alpha_{C^{(2)}}(F) \equiv c \Theta^{q-1}$. Moreover $c$ is non-zero since $F$ cannot be contracted by $\alpha_{C^{(2)}}$. But this gives a contradiction, since $\alpha_f \circ \alpha_{C^{(2)}}$ contracts $F$, while $\alpha_f(\Theta^{q-1})$ cannot be a point, being the image of $\alpha_Y \circ f \circ \iota$. \hfill $\square$

For a proof of the following lemma see also [3, Lemma III.8.3].

Lemma 1.5. Let $f : S \to \mathbb{C}$ be a proper morphism and let $nF$ be a multiple fiber of $f$ with multiplicity $n > 1$. Then $\mathcal{O}_F(F)$ is torsion non-trivial.

Proof. We first show that $\mathcal{L} = \mathcal{O}_S(F)$ is not trivial. Consider the closure of the graph of $f$ in $S \times \mathbb{P}^1$, let $\bar{S}$ be its minimal resolution and $\bar{f} : \bar{S} \to \mathbb{P}^1$ be the fibration given by the projection to the second factor. Assume by contradiction that $F \sim 0$ in $S$. Thus $F \sim \alpha F'$ in $\bar{S}$, where $F'$ is a divisor with support contained
in $S - S = \tilde{f}^{-1}(\infty)$ and $\alpha$ is a positive integer. Since $h^0(S, nF) = 2$ with $n > 1$, then $h^0(S, F) = 1$, giving a contradiction.

The line bundle $L$ thus defines a non-trivial étale cyclic covering $\eta: S' \to S$. By taking the Stein factorization of $f \circ \eta$ we get a commutative diagram

\[
\begin{array}{ccc}
S' & \xrightarrow{\eta} & S \\
| \downarrow f' & & \downarrow f \\
B & \xrightarrow{\nu} & C
\end{array}
\]

where $f'$ is a morphism with connected fibers and $\nu$ is a finite map. If $L|_B = O_F(F)$ is trivial, then $\nu$ is an étale covering of $C$, since the restriction of $L$ to any fiber of $f$ is trivial. Thus $\nu$ is the trivial covering and $B$ has $n$ connected components, a contradiction since $\eta$ is non-trivial.

Since $L^\otimes n$ is trivial, then clearly its restriction to $F$ is trivial. This concludes the proof. \hfill \Box

**Lemma 1.6.** Let $C$ be a non-hyperelliptic curve of genus $g = (k - 1)^2 > 1$ which is complete intersection of a quadric cone with a degree $k$ surface of $\mathbb{P}^3$. Then the divisor $\Gamma$ of $C^{(2)}$, corresponding to the $g_k^1$ of $C$ defined by the ruling of the cone, is not semiample.

**Proof.** We first show that the line bundle $O_{\Gamma}(\Gamma)$ is trivial. Indeed let $Q_t \subseteq \mathbb{P}^3 \times \mathbb{A}^1$ be a family of quadrics whose central fiber $Q_0$ is the cone containing $C$ and whose general fiber is a smooth quadric. Let $D$ be a divisor of $\mathbb{P}^3 \times \mathbb{A}^1$ which cuts out on the general fiber $Q_t$ a smooth curve $C_t$ of type $(k, k)$ with two simple $g_k^1$ and $C$ on $Q_0$. The family $C \to \mathbb{A}^1$ of curves $C_t$ gives a family $C^{(2)} \to \mathbb{A}^1$ whose general fiber is $C^{(2)}_t$. On any such fiber there are two curves $\Gamma_t, \Gamma'_t$ corresponding to the two $g_k^1$ on $C_t$. The line bundle $O_{\Gamma_t}(\Gamma'_t)$ is trivial, by Lemma 1.3, and its limit is $O_{\Gamma}(\Gamma)$, which proves the claim.

Assume now, by contradiction, that $\Gamma$ is semiample. Since $\Gamma^2 = 0$, a multiple $n\Gamma$ defines a morphism $f: C^{(2)} \to B$, where $B$ is a curve. Moreover, after possibly normalizing, we can assume $B$ to be smooth. Now, let $f = \nu \circ \varphi$ be the Stein factorization of $f$, where $\varphi: C^{(2)} \to Y$ is a morphism with connected fibers. By Lemma 1.5 and the fact that $O_{\Gamma}(\Gamma)$ is trivial we deduce that $\Gamma$ is a union of fibers of $\varphi$. Moreover both the hypotheses of Lemma 1.4 are satisfied, thus $Y$ must be a smooth rational curve. Let $H$ be the curve of $C^{(2)}$ which is the image of the curve $\{p\} \times C$ via $\pi$. The equality $\Gamma : H = k - 1$ implies that $\varphi|_H$ is a covering of $Y$ whose degree $d$ divides $k - 1$. Thus $C \cong H$ would admit two maps to $\mathbb{P}^1$ of degrees $k$ and $d$, respectively. Being the degrees coprime, the curve $C$ would be birational to a curve of bi-degree $(k, d)$ of $\mathbb{P}^1 \times \mathbb{P}^1$, whose genus is smaller than $g$, a contradiction. \hfill \Box

**Remark 1.7.** If $D$ is a prime divisor on a projective surface $X$ such that $|D|$ has dimension 0 then $D$ is semiample if and only if $\dim|nD| > 0$ for some $n > 1$. Indeed the “only if” part is obvious, while the other implication follows from the fact that the fixed divisor of $|nD|$ is $mD$ for some $m < n$ and thus the base locus of $|(n - m)D|$ is at most zero-dimensional and one concludes by Zariski-Fujita theorem [13, Remark 2.1.32]
Proof of Theorem 1. We show that \((i) \Rightarrow (ii)\) holds. Assume that the extended canonical ring \(R := R(\Delta, K)\) is finitely generated. We begin to show that any divisor whose class is in \(\text{Nef}(\Delta, K)\) is semiample. By Proposition 1.2 it suffices to show that \(2K + (5 - 2k)\Delta\) is semiample, or equivalently that the divisor \(\Gamma\) defined by a \(g^1_k\) of \(C\) is semiample, by Lemma 1.3. Let \(f_1, \ldots, f_r\) be a minimal system of homogeneous generators of \(R\) with respect to the \(\mathbb{Z}^2\)-grading and let \(w_i := \deg(f_i) \in \mathbb{Z}^2\) be the degree of \(f_i\) for any \(i\). Let \(f_1 \in R\) be a generator whose degree belongs to the ray generated by \([\Gamma]\) and let \(D = a\Gamma + bK\), with \(a, b > 0\) whose class \(w\) lies in the interior of the cone

\[
\bigcap_{i=2}^{r} \text{cone}(w_1, w_i) \cap \text{cone}(w_1, [K])
\]

If \(\Gamma\) is not semiample, then any section of \(R_w = H^0(C^{(2)}, D)\) is divisible by \(f_1\), a contradiction. Thus we showed that \(\Gamma\) is semiample and as a consequence of Lemma 1.6 the curve \(C\) has two \(g^1_k\). We denote the corresponding curves of \(C^{(2)}\) by \(\Gamma\) and \(\Gamma'\). A multiple of \(\Gamma\) defines a morphism \(f: C^{(2)} \to B\) onto a smooth curve \(B\)

Two fibers of \(\varphi\) are \(n\Gamma\) and \(m\Gamma'\) for some positive rational numbers \(n, m\). Since \(\Gamma\) is numerically equivalent to \(\Gamma'\) by Lemma 1.3, then \(n = m\). Moreover by Lemma 1.4 the curve \(Y\) is rational, so that \(n\Gamma\) is linearly equivalent to \(n\Gamma'\). This implies that the class of \(\Gamma - \Gamma'\) is torsion non-trivial in \(\text{Pic}^0(C^{(2)})\) and thus the same holds for

\[
2\eta_C = \text{ramification of } g^1_k - \text{ramification of } g^1_k' = \iota^*(\Gamma - \Gamma').
\]

We now show that \((ii) \Rightarrow (i)\) holds. First of all observe that if \(\eta_C\) is torsion non-trivial, then the same holds for \(\Gamma - \Gamma'\) by the above equalities and the fact that \(\iota^*\) is an isomorphism. In this case \(n\Gamma \sim n\Gamma'\) for some positive integer \(n\) and this implies that \([n\Gamma]\) is base point free, being \(\Gamma^2 = 0\). Thus \(\Gamma\) is semiample. Hence, if \(L \subseteq \mathbb{Z}^2\) is the submonoid generated by the integer points of the cone \(\text{cone}(2K + 3\Delta, 2K + (5 - 2k)\Delta)\), the following subalgebra

\[
S := \bigoplus_{(a,b) \in L} H^0(C^{(2)}, a\Delta + bK)
\]

of \(R\) is finitely generated by \([2, \text{Lemma 4.3.3.4}]\). The homogeneous elements of \(R\) which do not belong to \(S\) are sections of Riemann-Roch spaces \(H^0(C^{(2)}, D)\) with \(D \cdot \Delta < 0\). Hence any such section is divisible by the generator \(f_\Delta\) of \(R\) which is a defining section for \(\Delta\). Thus we conclude that \(R\) is generated by \(S\) and \(f_\Delta\) and the statement follows. \(\square\)

Remark 1.8. Over an algebraically closed field \(K\) of positive characteristic the statement of Theorem 1 must be modified as follows: the algebra \(R(\Delta, K)\) is finitely generated.
generated if and only if $\eta_C$ is torsion. Indeed if $\eta_C$ is trivial, then $\mathcal{O}_T(\Gamma)$ is trivial as well as shown in the proof of Lemma 1.6, thus $\Gamma$ is semiample by [11, Theorem 0.2]. Moreover, if $K$ is the algebraic closure of a finite field, then $\eta_C$ is always torsion and thus $R(\Delta, K)$ is always finitely generated.

The conclusion of Lemma 1.5 is no longer true in positive characteristic. Indeed in this case the algebraic fundamental group of the affine line $K$ is not trivial and thus there is no contradiction. For example, in characteristic $p > 0$, if $C$ is the curve of $K^2$ defined by the equation $x_1^p - x_1 = x_2$, then the projection onto the second factor defines a non-trivial étale covering of $K$.

Remark 1.9. We observe that the locus of smooth curves of genus $(k - 1)^2 > 1$ which admit two $g^1_k$ has one component of maximal dimension which consists of curves of type $(k, k)$ on a smooth quadric $Q$. Indeed by [1] the only other component which could have bigger dimension would consist of curves $C$ admitting an involution. A parameter count shows that for our curves this component has smaller dimension.

Moreover any smooth curve $C$ of type $(k, k)$ on $Q$ admits exactly two $g^1_k$. Indeed let $S = \{p_1, \ldots, p_k\}$ be a set of $k$ distinct points with $p_1 + \cdots + p_k \in g^1_k$. By the Riemann-Roch theorem $S$ is in Cayley-Bacharach configuration with respect to the curves of type $(k - 2, k - 2)$. It follows that all the points of $S$ are collinear. Indeed, let $\ell$ be the line through the first two points $p_1, p_2$, let $H$ be a general hyperplane which contains $\ell$ and let $q \in S \setminus \{p_1, p_2\}$. Take a union $\Lambda$ of $k - 3$ hyperplanes through all the points of $S \setminus \{p_1, p_2, q\}$ and such that $q \notin \Lambda$. Then $H \cup \Lambda$ cuts out on $Q$ a curve of type $(k - 2, k - 2)$ which, by the Cayley-Bacharach configuration, must pass through $q$. Hence $q \in H$ and by the generality assumption on $H$ we deduce $q \in \ell$.

2. The grid family

In this section we study families of curves of type $(k, k)$ on a smooth quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ such that the class of the difference of the two $g^1_k$ induced by the two rulings is $n$-torsion. We prove that $n \geq k$ and construct the family with $n = k$.

Definition 2.1. Given two effective divisors $L_1$ and $L_2$ of $Q$ of type $(k, 0)$ and $(0, k)$ respectively, the grid linear system defined by $L_1$ and $L_2$ is the linear system of curves of $Q$ of bi-degree $(k, k)$ which pass through the complete intersection $L_1 \cap L_2$. The grid family

$$G_k \subseteq F_k$$

is the family of all curves of type $(k, k)$ which belong to some grid linear system.

Observe that if $C$ is a smooth curve in $G_k$ and $\eta_C \in \text{Pic}^0(C)$ is the class of the difference of the two $g^1_k$ cut out by the two rulings then $\eta_C^{\otimes k}$ is trivial. This justifies the inclusion $G_k \subseteq F_k^{\text{tor}}$. Any curve $C$ in $G_k$ admits an equation of the form

$$f_1(x_0, x_1)g_2(y_0, y_1) + g_1(x_0, x_1)f_2(y_0, y_1) = 0,$$

where $f_1, f_2, g_1$ and $g_2$ are homogeneous polynomials of degree $k$. Indeed it is enough to prove the claim for a curve $C$ in a grid family where both $L_1$ and $L_2$ consist of distinct lines and then conclude by specialization that the same holds for any curve $C$ of $G_k$. Let $h = 0$ be an equation for $C$ and let $f_i = 0$ be an equation for $L_i$, for $i = 1, 2$. By the equality

$$V(h, f_1) = V(f_1, f_2),$$
the fact that the two ideals \((h, f_1)\) and \((f_1, f_2)\) are both radical and saturated with respect to the irrelevant ideal \((x_0, x_1) \cap (y_0, y_1)\) of \(Q\), we deduce that the equality \((h, f_1) = (f_1, f_2)\) holds. The claim follows since \(h\) has bi-degree \((k, k)\). Observe that the general element of \(G_k\) is irreducible and smooth.

**Proposition 2.2.** The grid family \(G_k\) has dimension \(4k - 1\).

**Proof.** The projectivization of the set of homogeneous polynomials of bidegree \((k, k)\) of type \(f(x_0, x_1)g(y_0, y_1)\) can be identified with the image \(S\) of the Segre embedding of \(\mathbb{P}^k \times \mathbb{P}^k \to \mathbb{P}^N\), where \(N = k^2 + 2k\). Thus, a curve in \(G_k\) can be identified with a point of the 1-secant variety \(Sec(S)\) of \(S\). By [8, Theorem 1.4] the dimension of \(Sec(S)\) is \(4k - 1\) and the statement follows. \(\square\)

**Lemma 2.3.** Let \(C \in F_k\) and let \(D\) be a divisor of \(C\) cut out by one of the two rulings. Then, for any \(1 \leq n \leq k - 1\), the linear system \(|nD|\) is composed with the pencil \(|D|\).

**Proof.** To prove the statement it is enough to show that \(h^0(C, nD) = n + 1\) for \(1 \leq n \leq k - 1\). Let \(Q = \mathbb{P}^1 \times \mathbb{P}^1\). Without loss of generality we can assume that \(D\) is cut out by the first ruling of \(Q\), so that we have the following exact sequence of sheaves

\[
0 \longrightarrow \mathcal{O}_Q(-k + n, -k) \longrightarrow \mathcal{O}_Q(n, 0) \longrightarrow \mathcal{O}_C(nD) \longrightarrow 0.
\]

Taking cohomology, using the vanishing of the higher cohomology groups of the middle sheaf, the Serre’s duality theorem and the hypothesis on \(n\) we deduce the following equalities:

\[
h^1(C, nD) = h^0(Q, \mathcal{O}_Q(k - n - 2, k - 2)) = (k - n - 1)(k - 1).
\]

By the adjunction formula \(C\) has genus \((k - 1)^2\). Thus by the above and the Riemann-Roch formula we conclude

\[
h^0(C, nD) = nk + 1 - (k - 1)^2 + (k - n - 1)(k - 1) = n + 1
\]

and the statement follows. \(\square\)

**Proposition 2.4.** Let \(C\) be a smooth curve of type \((k, k)\) on \(Q\) and let \(\eta_C\) be the class of the difference of the two \(g^1_k\). Then \(\eta_C\) has order \(\geq k\) and the following are equivalent:

(i) \(\eta_C\) has order \(k\);

(ii) the curve \(C\) belongs to the grid family \(G_k\).

**Proof.** We first show that \(\eta_C\) has order \(\geq k\). Let \(D_1\) be a divisor in the first \(g^1_k\) and let \(D_2\) be a divisor in the second \(g^1_k\). Let \(n < k\) be a positive integer. By Lemma 2.3 the linear system \(|nD_i|\) is composed with \(|D_i|\) for any \(i\). Thus the linear equivalence \(nD_1 \sim nD_2\) would imply that for any set of \(n\) lines of the first ruling there are \(n\) lines of the second ruling which cut out the same set of points on \(C\). This cannot be since such set of \(nk\) points would lie on a grid with \(n^2\) points. This proves the claim.

We now show that (i) \(\Rightarrow\) (ii) holds. By the exact sequence of sheaves

\[
0 \longrightarrow \mathcal{O}_Q(0, -k) \longrightarrow \mathcal{O}_Q(k, 0) \longrightarrow \mathcal{O}_C(kD_1) \longrightarrow 0
\]

the fact that \(h^0 = 0, h^1 = k - 1\) for the first sheaf and \(h^0 = k + 1, h^1 = 0\) for the second sheaf we get the equality \(h^0(C, \mathcal{O}_C(kD_1)) = 2k\). By hypothesis
\(|kD_1| = |kD_2|\) holds and by the above calculation the linear system is a projective space of dimension \(2k - 1\). Let \(H_1 \subseteq |kD_1| \cong \mathbb{P}^{2k-1}\) be the projectivization of the \(k\)-th symmetric power of the vector space \(H^0(C, D_1)\). Since both \(H_1\) and \(H_2\) have dimension \(k\), their intersection \(H_1 \cap H_2\) is at least one-dimensional. A point of this intersection corresponds to a divisor \(D\) which is cut out on \(C\) by \(k\) lines of the first ruling and by \(k\) lines of the second ruling. This proves the claim.

The implication \((ii) \Rightarrow (i)\) is obvious.

\section{Density of the torsion locus}

We define \(F_k\) to be the open subset of the Hilbert scheme of the smooth curves of type \((k, k)\) of \(\mathbb{P}^1 \times \mathbb{P}^1\). Let \(\pi: C \to F_k\) be the universal family, let \(H = R^1\pi_*\mathbb{C}\) be the associated Hodge bundle whose fiber over a point \(C \in F_k\) is the cohomology group \(H^1(C, \mathbb{C})\) and let \(H^{1,0}\) be the subbundle of \(H\) whose fiber over \(C\) is \(H^{1,0}(C)\). We recall that the jacobian family \(J \to F_k\) can be defined as

\[
J = \frac{H}{H^{1,0} + R^1\pi_*\mathbb{Z}}.
\]

Given a point \(t \in F_k\) we denote by \(C_t\) the fiber of \(\pi\) over \(t\) and by \(\nu_t = L_1 \otimes L_2^{-1} \in J(C_t)\) the class of the difference of the two \(g_k^1\) cut out by the two rulings. This gives a normal function \(\nu: F_k \to J\) defined by \(t \mapsto \nu_t\). We now consider a \(C^\infty\) trivialization \(\varphi\) of the jacobian family over an open, simply connected subset \(U\) of \(F_k\):

\[
\varphi: J \longrightarrow \mathbb{C} \times \frac{R^1\pi_*\mathbb{R}}{R^1\pi_*\mathbb{Z}} = \mathbb{C} \times \mathbb{T},
\]

where \(\mathbb{T}\) is the real torus \(\mathbb{R}^2 / \mathbb{Z}^2\). We let \(\text{pr}_2: \mathbb{C} \times \mathbb{T} \to \mathbb{T}\) be the projection on the second factor and let \(\eta: U \to \mathbb{T}, \ t \mapsto \text{pr}_2(\varphi(\nu(t)))\).

We will show that, for any \(k \geq 3\), the image of \(\eta\) contains a non-empty open subset. Our strategy will be to prove that the fibers of \(\eta\) are complex subvarieties of \(U\), that \(\eta\) has a fiber of the expected real dimension \(8k - 2 = \dim F_k - \dim \mathbb{T}\) and we conclude by Proposition 3.1 (see also the argument in the proof of [7, Proposition 3.4]).

\textbf{Proposition 3.1.} \textit{Let} \(V \subseteq X\) \textit{be an inclusion of complex manifolds,} \(Y\) \textit{be a real manifold and} \(\eta: X \to Y\) \textit{be a} \(C^\infty\) \textit{map whose fibers are complex subvarieties of} \(X\). \textit{Assume that there is a point} \(p \in V\) \textit{such that the following hold:}

\begin{enumerate}
  \item[(i)] \textit{the differential} \(d\eta_p\) \textit{is surjective,}
  \item[(ii)] \textit{the fiber of the restriction} \(\eta|_V\) \textit{over} \(\eta(p)\) \textit{has dimension} \(\dim V - \dim Y \geq 0\).
\end{enumerate}

\textit{Then} \(\eta(V)\) \textit{contains a non-empty open subset of} \(Y\).

\textbf{Proof.} \textit{Let} \(F\) \textit{be the fiber of} \(\eta\) \textit{over} \(\eta(p)\) \textit{and let} \(U\) \textit{be a coordinate neighbourhood of} \(p\) \textit{such that the restriction} \(\eta|_U\) \textit{is a submersion. Let} \(d = \dim V - \dim Y\). \textit{After possibly intersecting} \(V\) \textit{with} \(d\) \textit{general smooth complex hypersurfaces passing through} \(p\) \textit{we can reduce to the case} \(d = 0\). \textit{Thus, after possibly shrinking} \(U\) \textit{we can assume} \(F \cap V \cap U = \{p\}\). \textit{For} \(q\) \textit{in a sufficiently small neighbourhood} \(W\) \textit{of} \(\eta(p)\) \textit{the intersection number} \(\eta^{-1}(q) \cap V\) \textit{does not change by} [10, Pag.664], \textit{so that the intersection is non-empty. Then} \(\eta(V)\) \textit{contains} \(W\) \textit{and the statement follows.}\n
\textbf{Proposition 3.2.} \textit{The fibers of the map} \(\eta\) \textit{are complex varieties.}
Proof. We restrict the Hodge and the jacobian bundle to the simply connected open subset $U$ of $\mathcal{F}_k$, where the family is topologically trivial. Let $t_0 \in U$ be a distinguished point and let $C = C_{t_0}$ be the corresponding curve. We construct the following commutative diagram

\begin{equation}
\begin{array}{cccccc}
\mathcal{U} & \xrightarrow{\nu} & \mathcal{H} & \xrightarrow{=} & \mathcal{U} \times H^1(C, \mathbb{C}) & \xrightarrow{=} & H^1(C, \mathbb{C}) \\
\downarrow{\nu|_U} & & \downarrow{\varphi} & & \downarrow{\varphi} & & \downarrow{\varphi} \\
\mathcal{U} & \xrightarrow{\nu|_U} & \mathcal{J} & \xrightarrow{\varphi} & \mathcal{U} \times \mathbb{T} & \xrightarrow{} & \mathbb{T}
\end{array}
\end{equation}

where $\tilde{\nu}$ is a lifting of the restriction $\nu|_U$ and the maps in the top row are complex analytic morphisms. Observe that, given $x \in \mathbb{T}$, we have

$$\eta^{-1}(x) = \{ t \in \mathcal{U} : \tilde{\nu}(t) - \tilde{x} + c \in H^{1,0} \},$$

where the projection of $\tilde{x}$ to $\mathcal{J}$ is $x$ and $c \in H^1(C, \mathbb{Z})$. This implies that $\eta^{-1}(x)$ is a complex subvariety of $\mathcal{U}$ since $H^{1,0}$ is a complex subbundle of $\mathcal{H}$. □

Proof of Theorem 2. We will start proving that the image of the map $\eta$ contains a non-empty open subset of $\mathbb{T}$. We consider the map

$$\phi: \mathcal{J}_U \to \mathbb{T}$$

which is composition of the trivialization $\varphi$ with the projection onto the second factor. Observe that the differential of $\phi$ has maximal rank at any point. By applying Proposition 3.1 to the map $\phi$, the subvariety $V = \nu|_U(\mathcal{U})$ of $\mathcal{J}$ and the fiber $G_k$ of the restriction $\phi|_V$ we deduce that $\eta(V)$ contains a non-empty open subset. Observe that the trivialization $\varphi$ can be taken real analytic, since it is obtained by taking the real part of the complex analytic isomorphism coming from the trivialization of the Hodge bundle. By Sard theorem the set of non-critical values of $\phi$ is an open subset in the analytic topology. Since $\phi$ admits an analytic continuation on any simple arc in $\mathcal{J}$, its set of non-critical points is dense in the analytic topology. We conclude by observing that the set of torsion points of $\mathbb{T}$ is dense and $\mathcal{F}_k^{\text{tor}}$ is the preimage of this set via $\eta = \phi \circ \nu$. □

4. Hyperelliptic curves

In this section we prove a density theorem for hyperelliptic curves. This result has an independent interest and is proved in the spirit of Griffiths computations of the infinitesimal invariant [9]. As an application to our problem, this result provides an alternative proof for Theorem 2 in case $g = 4$.

Let $\pi: C \to \mathcal{U}$ be a versal family of hyperelliptic curves of genus $g > 1$, where $\mathcal{U}$ is simply connected of dimension $2g - 1$. We denote by $j \in \text{Aut}(C)$ the hyperelliptic involution and by $\mathcal{J} \to \mathcal{U}$ the jacobian family. We consider the Abel-Jacobi map

$$\nu: C \to \mathcal{J}, \quad x \mapsto \int_{j(x)}^x.$$

If we take $\pi^* \mathcal{J} \to \mathcal{C}$ the pull-back of the Jacobian family on $\mathcal{C}$, we may consider $\nu$ as a normal function. As in Section 3 we consider a $C^\infty$ trivialization of the jacobian family $\mathcal{J} \cong \mathcal{U} \times \mathbb{T}$, where $\mathbb{T} \cong J(C)$, to construct a map

$$\gamma: \mathcal{C} \to \mathbb{T}.$$
Theorem 4.1. If \( p \in \mathcal{C} \) is not a Weierstrass point, then the differential of \( \gamma \) at \( p \) is surjective.

Our strategy is as follows. For any holomorphic form \( \omega \in \mathcal{H}^{0}(C, \Omega_{\mathcal{C}}) \) we show that there is a curve \( r(t) \) in \( \mathcal{C} \) such that \( r(0) = p \) and \( d\gamma_{p}(r'(0)) \cdot \omega \) is non-zero. To this aim we produce \( r(t) \) accordingly to the order \( n \) of vanishing of \( w \) at \( p \). Since the divisor \( \text{div}(w) \) is \( j \)-invariant, then it is natural to consider \( D = p + j(p) \). We thus have a filtration

\[
(4.1) \quad \mathcal{H}^{0}(C, \Omega_{\mathcal{C}}) = L^{0} \supseteq L^{1} \supseteq \ldots \supseteq L^{g-1} \supseteq L^{g} = 0,
\]

where \( L^{k} \) is the Riemann-Roch space \( \mathcal{H}^{0}(C, \Omega_{\mathcal{C}}(-kD)) \). Given \( \omega \in L^{k} \setminus L^{k+1} \), with \( k > 0 \), we construct \( \zeta = \partial(f) \in \mathcal{H}^{1}(C, T_{C}) \) as in Subsection 4.1, where \( f \in \mathcal{H}^{0}(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \). The one-dimensional family

\[
\mathcal{C}_{\mathcal{Z}} \rightarrow \Delta
\]
defined by \( \zeta \), plus a choice of a smooth section through the point \( p \) in the family, defines a curve \( r(t) \) in \( \mathcal{C} \). We show that the family is equipped with a \( C^{\infty} \) 1-form \( \Theta \) such that the restriction of the \((1,0)\)-part \( \Theta_{\mathcal{Z}} \) to the central fiber \( C \) admits a local expansion at \( p \) of the form \( w + f(z)dz + o(t) \), where \( z \) is a coordinate in \( C \), \( f|_{\mathcal{Z}} = f \) and \( t \in \Delta \). We finally prove

\[
d\gamma_{p}(r'(0)) \cdot \omega = \lim_{t \to 0} \frac{1}{t} \left( \int_{\Gamma_{t}} \Theta_{t} - \int_{\Gamma_{0}} \omega \right) = 2f(p) \neq 0,
\]

where \( \Gamma_{t} \) is a path between \( r(t) \) and \( j(r(t)) \). When \( k = 0 \) we choose \( r(t) \) to be a path within the central fiber \( C \), we write \( \omega \) locally as \( h(z)dz \) and prove the equality

\[
d\gamma_{p}(r'(0)) \cdot \omega = \lim_{t \to 0} \frac{1}{t} \int_{p}^{r(t)} \omega = h(0) \neq 0.
\]

4.1. Deformation of curves. We recall first a result on deformation and on extension of line bundles which will be applied to hyperelliptic curves (see also [6] and [14]). Let \( C \in \mathbb{C} \) be a smooth curve of genus \( g > 1 \) and let \( T_{C} \) and \( \Omega_{C} \) be respectively the holomorphic tangent bundle and the canonical line bundle of \( C \). Fix a non trivial \( \omega \in \mathcal{H}^{0}(C, \Omega_{C}) \) and let \( Z \) be the canonical divisor associated to \( \omega \). The form \( \omega \) defines the following exact sequence

\[
0 \longrightarrow T_{C} \overset{\omega}{\longrightarrow} \mathcal{O}_{C} \longrightarrow \mathcal{O}_{Z} \longrightarrow 0.
\]

Passing to the long exact sequence in cohomology we obtain

\[
0 \longrightarrow \mathbb{C} \cong \mathcal{H}^{0}(C, \mathcal{O}_{C}) \longrightarrow \mathcal{H}^{0}(Z, \mathcal{O}_{Z}) \overset{\partial}{\longrightarrow} \mathcal{H}^{1}(C, T_{C}) \overset{\omega}{\longrightarrow} \mathcal{H}^{1}(C, \mathcal{O}_{C}).
\]

Given an element \( f \in \mathcal{H}^{0}(Z, \mathcal{O}_{Z}) \) its image \( \zeta = \partial(f) \) defines an extension of \( \mathcal{O}_{C} \) by \( T_{C} \) via the isomorphism \( \mathcal{H}^{1}(C, T_{C}) \cong \text{Ext}^{1}(\mathcal{O}_{C}, T_{C}) \)

\[
0 \longrightarrow T_{C} \longrightarrow E_{\zeta} \overset{\omega}{\longrightarrow} \mathcal{O}_{C} \longrightarrow 0.
\]

Taking tensor product with \( \Omega_{C} \) and recalling that \( T_{C} \) is dual with \( \mathcal{O}_{C} \) we get the following exact sequence

\[
(4.2) \quad 0 \longrightarrow \mathcal{O}_{C} \overset{\omega}{\longrightarrow} E_{\zeta} \otimes \Omega_{C} \longrightarrow \mathcal{O}_{C} \longrightarrow 0,
\]
which passing to the long exact sequence in cohomology gives the following sequence whose coboundary is the cup product with \( \zeta \):

\[
\begin{array}{ccccccc}
0 & \rightarrow & C \cong H^0(C, \mathcal{O}_C) & \rightarrow & H^0(C, E_\zeta \otimes \Omega_C) & \rightarrow & H^1(C, \mathcal{O}_C) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^0(C, \mathcal{O}_C) & \rightarrow & H^0(Z, \mathcal{O}_Z) & \rightarrow & H^1(C, T_C) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^0(C, E_\zeta \otimes \Omega_C) & \rightarrow & H^0(Z, E_\zeta \otimes \Omega_C|_Z) & \rightarrow & H^1(C, E_\zeta) \\
\downarrow k & & \downarrow r & & \downarrow & & \downarrow \\
H^0(C, \mathcal{O}_C) & \rightarrow & H^0(C, \mathcal{O}_C) & \rightarrow & H^0(Z, \Omega_C|_Z) & \rightarrow & H^1(C, \mathcal{O}_C)
\end{array}
\]

Since \( \zeta \in \ker(\omega) \), or equivalently the cup product \( \zeta \cdot \omega \) vanishes, there exists an element \( \Omega \in H^0(C, E_\zeta \otimes \Omega_C) \) such that \( k(\Omega) = \omega \). Now we consider the commutative diagram

The restriction of the lifting \( \Omega \) to \( Z \) gives an element \( r(\Omega) \in H^0(Z, E_\zeta \otimes \Omega_C|_Z) \) that by construction is in the image of the map \( \rho \), that is \( \Omega = \rho(g) \) for some \( g \in H^0(Z, \mathcal{O}_Z) \). A diagram chase proves indeed that \( \partial(g) = \zeta \) holds. Since \( \ker(\partial) \) is isomorphic to \( C \), we conclude that \( f \) equals \( g \) up to a constant. This means that we can realize the function \( f \) by a unique suitable lifting \( \Omega \) of \( \omega \). We collect the discussion in the following lemma.

**Proposition 4.2.** Let \( \omega \) be a non-zero element in \( H^0(C, \mathcal{O}_C) \) and let \( Z \) be the divisor of \( \omega \). Then for any \( f \in H^0(Z, \mathcal{O}_Z) \) there is a unique \( \Omega \in H^0(C, E_\zeta \otimes \Omega_C) \) such that \( r(\Omega) = \rho(f) \).

When we interpret \( H^1(C, T_C) \) as the space of first order deformations of \( C \), so that \( \zeta \) corresponds to a family

\[
\mathcal{C}_\zeta \rightarrow \text{Spec} \mathcal{C}[\varepsilon],
\]

the isomorphism \( H^1(C, T_C) \rightarrow \text{Ext}^1(\mathcal{O}_C, \Omega_C) \) gives the identifications \( E_\zeta \cong T_C|_C \) and \( E_\zeta \otimes \Omega_C \cong \Omega_C|_C \). Therefore the sequence (4.2) is the cotangent sequence of the first order deformation. In coordinates we may write

\[
(4.3) \quad \omega = h(z)dz, \quad \Omega = h(z)dz + \tilde{f}(z)dt,
\]

where \( dt \) is the global section of the cotangent \( \Omega_C|_C \) and \( f = \tilde{f}|_Z \) is a section of \( H^0(Z, \mathcal{O}_Z) \) such that \( r(\Omega) = \rho(f) \).

**4.2. The normal function.** In this subsection we will specialize the previous construction to hyperelliptic families. First of all, given a hyperelliptic curve \( C \) of genus \( g > 1 \), we consider the \( j \)-invariant subspace

\[
H^1(C, T_C)^j \subseteq H^1(C, T_C),
\]

which corresponds to the directions that are preserved by the hyperelliptic involution \( j \). Since \( j \) acts on \( H^0(Z, \mathcal{O}_Z) \) as \( f \mapsto f \circ j \) and it acts as \(-1\) on \( H^0(C, \mathcal{O}_C) \), then \( \partial(f) = \zeta \) is \( j \)-invariant if and only if \( f \circ j \) equals \(-f \) up to a constant.
Let \( p \) be a non-Weierstrass point of \( C \) and \( \omega \in H^0(C, \Omega_C) \) be a holomorphic form which vanishes with order \( k > 0 \) at \( p \), that is \( \omega \in L^k \setminus L^{k+1} \). We now take \( f_\omega \in H^0(Z, \mathcal{O}_Z) \), where \( Z = k(p + j(p)) + Z' = \text{div}(\omega) \), such that

\[
f_\omega(p) = 1, \quad f_\omega(j(p)) = -1, \quad f_\omega(p') = 0 \quad \text{for} \quad p' \in Z'.
\]

Given \( \zeta_\omega = \partial(f_\omega) \), by the previous remark we have \( j(\zeta_\omega) = \zeta_\omega \). Thus there exists a smooth family of hyperelliptic curves

\[
\pi: C_\omega \to \Delta,
\]

where \( \Delta \subset \mathcal{U} \) is a disk, such that \( \pi^{-1}(0) = C \) and such that the Kodaira-Spencer class of the family is \( \zeta_\omega \). Let \( \Omega_{\omega} \) be the section of \( E_\zeta \otimes \Omega_C \) associated to \( f_\omega \) as in Lemma 4.2. By means of a trivialization of the family we can construct a closed differential 1-form \( \Theta \) on \( C_\omega \) which is invariant with respect to the involution \( j \) and such that the restriction of the \((1, 0)\)-part \( \Theta(1, 0) \) to the central fiber equals \( \Omega_\omega \). In local coordinates we can write

\[
\Theta(z, t)^{(1, 0)} = \Omega_{\omega} + o(t) = \omega + f_\omega(z) dt + o(t).
\]

We also assume to have a holomorphic section \( r \) of \( \pi_{\omega} \) such that \( r(0) = p \) and define \( r' = j(r) \). Moreover, we fix a differentiable map

\[
\Gamma(t, s): \Delta \times [0, 1] \to C_\omega
\]

such that \( \Gamma(t, s) \in \pi^{-1}(t) \), \( \Gamma(t, 0) = r'(t) \) and \( \Gamma(t, 1) = r(t) \). Observe that \( \Gamma \) is a family of sections connecting \( r' \) and \( r \). We define the function

\[
g: \Delta \to \mathbb{C}, \quad t \mapsto \int_{\Gamma_t} \Theta_t.
\]

Following Griffiths [9, (6.6)] and using the fact that the Gauss-Manin connection vanishes on \( \Theta \), we have that the derivative of \( g \) at 0 equals \( d\gamma_p(r'(0)) \cdot \omega \).

**Proof of Theorem 4.1.** With the previous notation, given any \( \omega \in H^0(C, \Omega_C) \) vanishing of order \( k > 0 \) at \( p \), we consider the family \( \pi_{\omega} \), with its sections \( r = r_{\omega} \) and \( r' = j(r) \), and \( \Theta \) the corresponding differential form on \( C_\omega \). By the previous remark we have that

\[
d\gamma_p(r'(0)) \cdot \omega = g'(0).
\]

We now compute the latter term:

\[
g(t) - g(0) = \int_{\Gamma_t} \Theta_t - \int_{\Gamma_0} \Theta_0 = \int_{\Gamma_t} \Theta_t - \int_{\Gamma_0} \omega.
\]

We call \( r_t \) and \( r'_t \) the arcs \( r([0, t]) \) and \( r'([0, t]) \) respectively. Since \( \Theta \) is closed, then \( \Gamma^*(\Theta) \) is exact and we have \( 0 = \int_{r'_t} \Theta_t + \int_{r_t} \Theta_t - \int_{r_t} \Theta_t - \int_{r_0} \Theta_t \), hence

\[
g(t) - g(0) = \int_{r_t} \Theta_t - \int_{r'_t} \Theta_t = 2 \int_{r_t} \Theta_t,
\]

where the last equality is due to the fact that \( j^*(\Theta) = -\Theta \). Finally, since \( \Theta = \omega + f_\omega(z) dt + o(t) \), by the fundamental theorem of calculus we get

\[
\lim_{t \to 0} \frac{1}{t} \int_{r_t} \Theta_t = \lim_{t \to 0} \frac{1}{t} \int_{r_t} \Theta_t^{(1, 0)} = f_\omega(p) \neq 0.
\]
Thus \( g'(0) \neq 0 \). If \( k = 0 \), that is \( \omega \) does not vanish at \( p \), we will choose a loop \( r(t) \) in \( C \) with \( r(0) = p \) and we will compute the derivative of the Abel-Jacobi map \( AJ \) on \( C \). First note that if we take a Weierstrass point \( q \) of \( C \) we have

\[
AJ(r(t) - j(r(t))) = 2AJ(r(t) - q).
\]

Take a coordinate \( z \) centered at \( p \) such that \( \omega(z) = h(z)dz \) with \( h(0) \neq 0 \). Fix a loop \( r(t) \) such that \( z(r(t)) = t \), then

\[
\lim_{t \to 0} \frac{1}{t} \int_{r(t)}^q \omega = \lim_{t \to 0} \frac{1}{t} \int_0^t h(z)dz = h(0)
\]

and we complete our result.

**Corollary 4.3.** The locus of curves \( C \) in \( \mathcal{M}_4 \) such that \( \eta_C \) is a non-trivial torsion point is a countable union of subvarieties of complex dimension \( \geq 5 \) and the set of subvarieties of dimension 5 is dense in \( \mathcal{M}_4 \) in the analytic topology.

**Proof.** Let \( \mathcal{U} \) be an open subset of \( \mathcal{M}_4 \) which intersects the hyperelliptic locus and let \( \tilde{\mathcal{U}} \) be the moduli space of pairs \( (C, g^1_3) \), where \( [C] \in \mathcal{U} \). Let \( \iota : \mathcal{H} \to \tilde{\mathcal{U}} \) be the subvariety containing pairs where \( C \) is hyperelliptic. Given a universal family \( \pi : \tilde{\mathcal{U}} \to \mathcal{U} \), we construct the following commutative diagram

\[
\begin{array}{c}
\tilde{\mathcal{U}} \times \mathbb{T} \\
\downarrow \varphi \\
\mathcal{U} \\
\downarrow \pi \\
\end{array}
\begin{array}{c}
\mathcal{J} \\
\downarrow \iota \ast C \\
C \\
\end{array}
\]

where \( \eta_C = g^1_3 - g^1_1 = K_C - 2g^1_3 \) and \( \varphi \) is a \( C^\infty \)-trivialization (defined after possibly shrinking \( \mathcal{U} \)). The commutativity of the top square comes from the fact that on a hyperelliptic curve \( C \) we have \( g^1_3 = p + g^1_2 \) and \( K_C - 2g^1_3 = j(p) - p \), where \( j \in \text{Aut}(C) \) is the hyperelliptic involution. By Theorem 4.1 the differential of the map \( \gamma : \iota \ast C \to \mathbb{T} \) obtained composing the maps in the diagram is surjective at any point \( p \) which is not Weierstrass. This implies that the map \( \eta : \mathcal{U} \to \mathbb{T} \) is locally a submersion at any point corresponding to a hyperelliptic curve, in particular its image contains an open subset of \( \mathbb{T} \). We thus conclude as in the last part of the proof of Theorem 2 given in section 3. \( \square \)

5. **Examples**

In this section we will provide further examples of curves having two \( g^1_k \)'s whose difference is a torsion element in the Jacobian. In particular we will show how to use automorphism groups to construct new examples (see Example 5.3).

**Proposition 5.1.** Let \( C \) be a curve in \( \mathcal{F}_k \). If \( G \) is an automorphism group of \( C \) of order \( n \) which preserves each \( g^1_k \) of \( C \) and such that \( C/G \) has genus zero, then the order of \( \eta_C \) divides \( n \).
Proof. Let \( \pi : C \to C/G \cong \mathbb{P}^1 \) be the quotient morphism, let \( D = p_1 + p_2 + \cdots + p_k \) be an element of the first \( g^1_k \) and let \( g_i = \pi(p_i) \). Then the following linear equivalences hold
\[
nD \sim \sum_{\sigma \in G} \sigma^*(D) = \pi^*(q_1) + \pi^*(q_2) + \cdots + \pi^*(q_k) \sim kF,
\]
where \( F \) is a fiber of \( \pi \) and the first equivalence is due to the fact that \( G \) preserves the linear series \( g^1_k \). Since the same property holds for an element \( D' \) of the second \( g^1_k \), the linear equivalence \( nD \sim nD' \) follows. \( \square \)

Example 5.2. Let \( \sigma \) be the order \( k \) automorphism of \( \mathbb{P}^1 \times \mathbb{P}^1 \) defined by
\[
\sigma(x_0, x_1, y_0, y_1) = (\zeta_k x_0, x_1, y_0, y_1),
\]
where \( \zeta_k \) is a primitive \( k \)-th root of unity. We now show that a curve \( C \in \mathcal{F}_k \) which is \( \sigma \)-invariant admits an equation of the form
\[
x_0^k g_2(y_0, y_1) + x_1^k f_2(y_0, y_1) = 0,
\]
where \( f_2, g_2 \) are homogeneous of degree \( k \) in \( y_0, y_1 \). In particular the quotient \( C/\langle \sigma \rangle \) has genus zero. Thus \( C \in \mathcal{F}_k^{\text{tor}} \) by either Proposition 5.1 or Proposition 2.4. The automorphism \( \sigma \) preserves each ruling of the quadric and acts identically on one of the two rulings. Consider a point in \( \mathcal{H}_1 \cap \mathcal{H}_2 \), with the notation in the proof of Proposition 2.4, which corresponds to a \( \sigma \)-invariant grid. The lines of the grid which belong to the first ruling are defined by either \( x_0^k - x_1^k = 0 \) or \( x_0^k = 0 \). An equation of \( C \) in such coordinates is then of the form
\[
(x_0^k - \mu x_1^k) h_1 + (x_0^k + \lambda x_1^k) h_2 = x_0^k (h_1 + h_2) + x_1^k (\lambda h_2 - \mu h_1) = 0,
\]
for \( \mu \in \{0, 1\} \), \( \lambda \in \mathbb{C} \) and \( h_1, h_2 \) homogeneous of degree three in \( y_0, y_1 \).

Example 5.3. The moduli space of non-hyperelliptic curves \( C \) of genus four having an order five automorphism \( \sigma \) such that \( C/\langle \sigma \rangle \) has genus zero is a 1-dimensional subvariety of \( \mathcal{F}_5^{\text{tor}} \). Moreover, any such \( C \) is isomorphic to a curve in the following family
\[
x_0 x_1^2 y_1^3 + \alpha x_0^2 x_1 y_0 y_1 + \beta x_0^3 y_0 y_1 + \gamma x_1^3 y_0 y_1 = 0,
\]
where \( \sigma(x_0, x_1, y_0, y_1) = (\zeta_5 x_0, x_1, \zeta_5^3 y_0, y_1) \) and \( \zeta_5 \) is a primitive fifth root of unity. The family contains curves which pass through the points of a grid of type \( (5, 5) \), for example the curve with
\[
\alpha = -\zeta_5, \quad \beta = \zeta_5^3 + \zeta_5^2 + \zeta_5, \quad \gamma = \zeta_5^2 + \zeta_5.
\]
However, the general element of the family is not of grilled type. This means that if \( D_i \) is a divisor of the \( i \)-th \( g^1_5 \) and \( \mathcal{H}_i \subseteq |5D_1| \cong \mathbb{P}^1 \) is the projectivization of the fifth symmetric power of \( H^0(C, D_i) \), then the intersection \( \mathcal{H}_1 \cap \mathcal{H}_2 \) is empty. For example this holds for the curve with \( \alpha = -1, \beta = \gamma = 1 \). The statements for both curves can be checked by means of the Magma [4] program available here http://www2.udec.cl/~alaface/software/semiample/aut.

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