Distance functions with dense singular sets

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Abstract

We characterize the denseness of the singular set of the distance function from a $C^1$-hypersurface in terms of an inner ball condition and we address the problem of the existence of viscosity solutions of the Eikonal equation whose singular set (i.e. set of non-differentiability points) is not no-where dense.

1 Introduction

The distance function $\delta_K$ from a closed subset $K \subseteq \mathbb{R}^n$ is a viscosity solution of the Eikonal equation $|\nabla u|^2 = 1$ on $\mathbb{R}^n \sim K$ and it plays a central role in the theory of Hamilton-Jacobi equations. The function $\delta_K$ is locally semiconcave on $\mathbb{R}^n \sim K$ and it is continuously differentiable on $\mathbb{R}^n \sim (K \cup \Sigma(K))$ with a locally Lipschitz gradient, where $\Sigma(K)$ is the set of non-differentiability points of $\delta_K$. In view of these facts the topological and measure-theoretic properties of the sets $\Sigma(K)$ and $\overline{\Sigma(K)}$ have always been a central theme of research (see [IT01, MM03, LN05, CM07, ACNS13]). The set $\Sigma(K)$ can be covered, outside a set of $H^{n-1}$ measure zero, by the union of countably many $C^2$ hypersurfaces (see [Zaj79]). Assuming at least that $K$ is a closed $C^2$ hypersurface, the Lebesgue measure of $\Sigma(K)$ is zero and upper bounds on the Hausdorff dimension of the set $\overline{\Sigma(K)}$ are known ([IT01, MM03, LN05, CM07]); see also [Min16] for the case of $C^{1,1}$ hypersurfaces that are almost $C^2$. On the other hand a well known example of Mantegazza and Mennucci in [MM03, pag. 10] describes a convex body $C$ with $C^{1,1}$-boundary such that $\overline{\Sigma(\partial C)}$ is a no-where dense subset of $C$ with positive Lebesgue measure. This example raises the natural question to understand if (and under which hypothesis) the set $\overline{\Sigma(K)}$ can have interior points. It is particularly interesting the case $K = \partial C$, where $C$ is a convex body with $C^{1,1}$ boundary, since if this example exists then one can construct by a well known procedure a viscosity solution of the Eikonal equation on all of $\mathbb{R}^n$ whose singular set is not no-where dense. This question was addressed in [Rif08, Theorem 1, footnote pag. 520], which contains the assertion that every viscosity solution of the Eikonal equation on an open subset of $\mathbb{R}^n$ must be differentiable outside a no-where dense set. Unfortunately the proof of this statement is invalid (see [Rif20]) and, as we show in this paper, it turns out that the statement is actually not true.
In this note we aim to establish the existence of a counterexample to the aforementioned assertion in [Rif08, Theorem 1] and to provide geometric conditions on $C^1$-hypersurfaces $K$ that ensures that $\Sigma(K)$ has non-empty interior. Specifically we prove the following facts:

1. If $\Omega$ is an open subset with $C^1$ boundary then $\Omega \sim \Sigma(\partial \Omega) \neq \emptyset$ if and only if $\Omega$ satisfies an inner uniform ball condition on some open subset of $\partial \Omega$ (see 2.7-2.8).

2. If $K$ is a closed and connected $C^1$ hypersurface that is $C^2$-unrectifiable, then $\Sigma(K) = \mathbb{R}^n$ (see 2.9).

3. For most of the convex bodies $C$ with $C^1$ boundary (in the sense of Baire Category) the set $\Sigma(\partial C)$ is dense in $C$ (see 3.1).

4. There exists a convex body $C$ with $C^{1,1}$ boundary such that $\Sigma(\partial C)$ has interior points (see 3.3).

5. There exists viscosity solutions of the Eikonal equation $|\nabla u|^2 = 1$ on all of $\mathbb{R}^n$ that are not differentiable on a set that is not nowhere dense (see 3.3).

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2 Inner ball condition and dense singular sets

In this section for an open set $\Omega$ with $C^1$ boundary $K$ we characterize the denseness of the set of non differentiability points of $\delta_K$ in $\Omega$ in terms of an inner ball condition (see 2.7-2.8). We use then this result to show that closed $C^1$-hypersurfaces that are $C^2$-unrectifiable have a singular set dense in all of $\mathbb{R}^n$ (see 2.9).

2.1 Definition. Let $k \geq 1$ be an integer, $0 \leq \alpha \leq 1$ and $M \subseteq \mathbb{R}^n$. We say that $M$ is a $C^{k,\alpha}$-hypersurface if and only if for every $a \in M$ there exists an open subset $U$ of $\mathbb{R}^n$, an $n-1$ dimensional subspace $Z$ of $\mathbb{R}^n$ and a $C^{k,\alpha}$-diffeomorphism $\sigma : U \to \mathbb{R}^n$ such that

$$\sigma(U \cap M) = Z \cap \sigma(U).$$

2.2 Definition. Let $k \geq 1$ be an integer. A $k$-manifold is an Hausdorff space which is locally homeomorphic to an open subset of $\mathbb{R}^k$.

Let $K \subseteq \mathbb{R}^n$ be a closed set and let $\xi_K : \mathbb{R}^n \to 2^K$ be the nearest point projection onto $K$:

$$\xi_K(x) = K \cap \{ a : |x - a| = \delta_K(x) \}$$
for every $x \in \mathbb{R}^n$. The singular set of $\delta_K$ is defined as
\[
\Sigma(K) = (\mathbb{R}^n \sim K) \cap \{ x : \delta_K \text{ is not differentiable at } x \}.
\]

2.3 Remark. The reader might wonder what are the points in $K$ where $\delta_K$ is not differentiable. In this regard one observes that if $x \in K$ and $\delta_K$ is differentiable at $x$ then $\nabla \delta_K(x) = 0$. It follows that if $x \in K$ and the tangent cone (see [Fed59, 4.3]) of $K$ at $x$ is not equal to $\mathbb{R}^n$ then $\delta_K$ is not differentiable at $x$. In particular if $K$ is $C^1$ hypersurface [resp. $K$ is a convex body] $\delta_K$ is not differentiable at all points of $K$ [resp. all points of $\partial K$].

2.4 Remark. It is well known that $\delta_K$ is locally semiconcave in $\mathbb{R}^n \sim K$, see [CS04, 2.2.2]. As a consequence of general structural results on the singular sets of convex functions (see [Zaj79]) we deduce that $\Sigma(K)$ can be covered, outside a set of $H^{n-1}$ measure zero, by the union of countably many $C^2$-hypersurfaces.

We recall from [CS04, 3.4.5] a well known characterization of $\Sigma(K)$.

2.5 Lemma. Suppose $K \subseteq \mathbb{R}^n$ is closed and $x \notin K$.
Then $x \notin \Sigma(K)$ if and only if $\xi_K(x)$ is a singleton and
\[
\nabla \delta_K(x) = \frac{x - \xi_K(x)}{\delta_K(x)}.
\]

2.6 Definition. For $x \in \mathbb{R}^n$ and $r > 0$ we define
\[
U(x, r) = \mathbb{R}^n \cap \{ y : |y - x| < r \}.
\]

2.7 Definition. Suppose $\Omega$ is an open subset of $\mathbb{R}^n$ and $S \subseteq \partial \Omega$. We say that $\Omega$ satisfies an inner uniform ball condition on $S$ if and only if there exists $\rho > 0$ such that each $x \in S$ belongs to the boundary of an open ball $B$ of radius $\rho$ which is contained in $\Omega$.

2.8 Theorem. Let $K \subseteq \mathbb{R}^n$ be a closed $C^1$-hypersurface and let $\Omega$ be an open subset of $\mathbb{R}^n$ such that $\partial \Omega = K$.
Then $\Omega \sim \overline{\Sigma(K)} \neq \emptyset$ if and only if $\Omega$ satisfies an inner uniform ball condition on a non-empty open subset of $K$.

Proof. Suppose $\Omega \sim \overline{\Sigma(K)} \neq \emptyset$. Choose $w \in \Omega \sim \overline{\Sigma(K)}$ and $0 < \epsilon < \delta_K(w)$ such that $U(w, \epsilon) \subseteq \Omega \sim \overline{\Sigma(K)}$. Then define
\[
S = U(w, \epsilon) \cap \{ x : \delta_K(x) = \delta_K(w) \}.
\]
Since, by [Fed59] the Lipschitz function $\delta_K$ is differentiable at each $x \in U(w, \epsilon)$ and $|\nabla \delta_K(x)| = 1$, we apply the implicit function theorem of Clarke [Class3 7.11] to conclude that $S$ is an $(n-1)$-manifold. Moreover $\xi_K|S$ is continuous by [Fed59 4.8(4)]. We prove that $\xi_K|S$ is an injective map. Suppose $x, y \in S$ such that $\xi_K(x) = \xi_K(y)$. Then
\[
x - \xi_K(x) \in \text{Nor}(K, \xi_K(x)), \quad y - \xi_K(x) \in \text{Nor}(K, \xi_K(x))
\]
and $|x - \xi_K(x)| = |y - \xi_K(x)| = \delta_K(w)$. Since $\dim \text{Nor}(K, \xi_K(x)) = 1$, it follows that either $x - \xi_K(x) = y - \xi_K(x)$ or $x - \xi_K(x) = \xi_K(x) - y$. The latter would imply that

$$|x - y| = |x - \xi_K(x) + \xi_K(x) - y| = 2|x - \xi_K(x)| = 2\delta_K(w)$$

which is clearly impossible, since $|x - y| < \epsilon < 2\delta_K(w)$. Henceforth $x = y$ and $\xi_K|S$ is injective. Since $S$ and $\partial \Omega$ are $(n-1)$-manifolds, we apply Brouwer’s theorem on invariance of domain (see [Do172 IV, 7.4]) to conclude that $\xi_K(S)$ is open in $K$. Noting that

$$U(x, \delta_K(w)) \subseteq \Omega \quad \text{and} \quad \xi_K(x) \in \partial U(x, \delta_K(w)),$$

for every $x \in S$, we conclude that $\Omega$ satisfies an inner uniform ball condition on $\xi_K(S)$.

Suppose $S \subseteq K$ is open in $K$ and $\Omega$ satisfies an inner uniform ball condition on $S$. Let $\nu : K \to S^{n-1}$ be the inner unit normal of $\Omega$. Our hypothesis implies that there exists $\rho > 0$ such that

$$U(a + \rho \nu(a), \rho) \subseteq \Omega \quad \text{for every } a \in S.$$

Define $\phi : S \times (0, \rho) \to \mathbb{R}^n$ by $\phi(a, t) = a + t\nu(a)$ for $(a, t) \in S \times (0, \rho)$. Then $\phi[S \times (0, \rho)]$ is open in $\mathbb{R}^n$ and $\xi_K(x)$ is a singleton for every $x \in \phi[S \times (0, \rho)]$ then it is clear by [2.5] that $\phi[S \times (0, \rho)]$ does not intersect $\Sigma(K)$ and $\Omega \sim \Sigma(K) \neq \emptyset$. To prove the two assertions above we first show that $\phi$ is injective. Let $(a, t), (b, s) \in S \times (0, \rho)$ such that $a + t\nu(a) = b + s\nu(b)$. We notice that

$$t = \delta_K(a + t\nu(a)) = \delta_K(b + s\nu(b)) = s,$$

$$|a + t\nu(a) - b| = t.$$

If $a \neq b$ then $|a + \rho \nu(a) - b| < \rho$ and $b \in \Omega$, which is a contradiction. Henceforth $a = b$ and $\phi$ is injective. If $b \in \xi_K(a + t\nu(a))$ for some $(a, t) \in S \times (0, \rho)$ then we notice that $\nu(b) = t^{-1}[(a + t\nu(a) - b)$ and $a = b$ by the injectivity of $\phi$. Therefore $\xi_K(x)$ is a singleton for every $x \in \phi[S \times (0, \rho)]$. Moreover, since $S \times (0, \rho)$ is an $n$-manifold, we conclude that $\phi[S \times (0, \rho)]$ is an open subset of $\mathbb{R}^n$ by [Do172 IV, 7.4].

This corollary shows that every $C^1$-hypersurface that is $C^2$-unrectifiable generates a dense singular set.

2.9 Corollary. Suppose $K$ is a closed and connected $C^1$ hypersurface such that $\mathcal{H}^{n-1}(K \cap M) = 0$ whenever $M$ is a $C^2$-hypersurface of $\mathbb{R}^n$.

Then $\Sigma(K) = \mathbb{R}^n$.

Proof. Let $U$ and $V$ the two connected open subsets of $\mathbb{R}^n$ such that $\partial U = \partial V = K$, $U \cap V = \emptyset$ and $U \cup V = \mathbb{R}^n$. It follows from [MS19] that if $S$ is a subset of $K$ such that either $U$ or $V$ satisfies an inner uniform ball condition on $S$ then $\mathcal{H}^{n-1}(S) = 0$. In particular neither $U$ nor $V$ can satisfy an inner uniform ball condition on some non empty open subset of $K$. Therefore we conclude from [2.5] that $\Sigma(K) = \mathbb{R}^n$. 

□
2.10 Remark. Let $0 < \alpha < 1$. It follows from \[Koh77\] that there exists a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ whose graph $K$ is a closed $C^{1,\alpha}$-hypersurface such that $\mathcal{H}^{n-1}(K \cap M) = 0$ for every $C^2$-hypersurface $M \subseteq \mathbb{R}^n$.

2.11 Remark. It follows from the theory of sets of positive reach (see \[Fed59, \S 4\]) that if $K$ is a closed $C^{1,1}$-hypersurface then there exists an open neighbourhood $U$ of $K$ such that $\Sigma(K) \cap U = \emptyset$.

3 Convex sets

In this section we show that there exist many $C^{1,1}$ convex hypersurfaces $K$ such that $\Sigma(K)$ has non empty interior. Consequently there exist many viscosity solutions of the Eikonal equation on $\mathbb{R}^n$ such that the singular set is nowhere dense.

Let $\mathcal{K}_n^r$ be the space of all compact convex subsets in $\mathbb{R}^n$ with non empty interior such that $\partial C$ is a $C^{1,1}$ hypersurface. We equip $\mathcal{K}_n^r$ with the Hausdorff metric and we recall (see \[Sch14, 2.7.1\]) that it is a Baire space (i.e. countable intersections of dense open subsets are dense). A subset of a metric space is called meager if and only if it is countable union of nowhere-dense sets and it is called comeager if and only if it is the complementary of a meager set. It is customary to call typical the elements of a comeager subset of a Baire space.

The next statement contains the observation that for a typical convex body $C \in \mathcal{K}_n^r$ the distance function from the boundary $\partial C$ is not differentiable on a dense subset of $C$. This statement easily follows combining Theorem 2.8 with well known properties of the curvature of a typical convex body.

3.1 Theorem. For all $C$ in $\mathcal{K}_n^r$, except those belonging to a meager subset of $\mathcal{K}_n^r$,

$$C = \Sigma(\partial C).$$

Proof. By \[Sch14, 2.7.4\] there exists a comeager $T$ of $\mathcal{K}_n^r$ such that if $C \in T$ then $\text{Int}(C)$ does not satisfy an inner uniform ball condition on a comeager subset of $\partial C$. It follows from 2.8 that $C \subseteq \Sigma(\partial C)$ for every $C \in T$. On the other hand it is well known that $\delta_{\partial A} \in C^{1,1}_{\text{loc}}(\mathbb{R}^n \sim A)$ for every convex body $A$ (see for instance \[Fed59, 4.8\]) and the conclusion follows.

3.2 Lemma. If $C$ is a convex body then for every $\epsilon > 0$ the set

$$C_\epsilon = \mathbb{R}^n \cap \{x : \delta_C(x) \leq \epsilon\}$$

is convex, $\partial C_\epsilon$ is a $C^{1,1}$-hypersurface and $\Sigma(\partial C) \subseteq \Sigma(\partial C_\epsilon)$.

Proof. Evidently $C_\epsilon$ is a convex body and is well known that $\partial C_\epsilon$ is a $C^{1,1}$ hypersurface (see \[Fed59, 4.8\]). We observe that

$$\delta_{\partial C_\epsilon}(x) = \epsilon + \delta_{\partial C}(x) \quad \text{for } x \in C,$$

and we conclude that $\Sigma(\partial C) \subseteq \Sigma(\partial C_\epsilon)$.

\[In fact $\mathcal{K}_n^r$ is a comeager of the space of all convex bodies (with non empty interior) equipped with the Hausdorff metric.\]
3.3 Theorem. There exists $C \in K_n^r$ such that $\partial C$ is a $C^{1,1}$-hypersurface and $\Sigma(\partial C)$ has non empty interior. Moreover the function $u : \mathbb{R}^n \to \mathbb{R}$ defined by

$$u(x) = \delta_{\partial C}(x) \quad \text{for } x \in C \quad \text{and} \quad u(x) = -\delta_{\partial C}(x) \quad \text{for } x \in \mathbb{R}^n \sim C$$

is a viscosity solution of the Eikonal equation on $\mathbb{R}^n$ and the closure of the set of points where $u$ is not differentiable has non empty interior.

Proof. The existence of a convex body $C$ such that $\partial C$ is a $C^{1,1}$ hypersurface and $\Sigma(\partial C)$ has non empty interior directly follows from 3.1 and 3.2.

It follows from [Fed59, 4.20] that $\partial C$ has positive reach. Therefore one infers from [KPS1, Theorem 2] that there exists an open neighborhood $U$ of $\partial C$ such that $u$ is continuously differentiable on $U$. Since it is clear that $u$ is continuously differentiable on $\mathbb{R}^n \sim C$, we conclude by 2.5 that

$$|\nabla u(x)|^2 = 1 \quad \text{for every } x \in (\mathbb{R}^n \sim C) \cup U.$$ 

Moreover $u$ is locally semiconcave on the interior $\text{Int}(C)$ of $C$ and $|\nabla u(x)|^2 = 1$ for $\mathbb{L}^n$ a.e. $x \in \text{Int}(C)$. Henceforth, it follows from [CS04, 5.3.1] that $|\nabla u|^2 = 1$ in the viscosity sense in $\text{Int}(C)$. It is now evident that $|\nabla u|^2 = 1$ in the viscosity sense in $\mathbb{R}^n$. \qed

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