BRUALDI-TYPE INEQUALITIES ON THE MINIMUM EIGENVALUE FOR THE FAN PRODUCT OF M-TENSORS

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Abstract. In this paper, we focus on some inequalities for the Fan product of M-tensors. Based on Brualdi-type eigenvalue inclusion sets of M-tensors and similarity transformation methods, we establish Brualdi-type inequalities on the minimum eigenvalue for the Fan product of two M-tensors. Furthermore, we discuss the advantages of different Brualdi-type inequalities. Numerical examples verify the validity of the conclusions.

1. Introduction. Let \( \mathbb{C}(\mathbb{R}) \) be the set of all complex (real) numbers, \( \mathbb{R}_+(\mathbb{R}_{++}) \) be the set of all nonnegative (positive) numbers, \( \mathbb{C}^n(\mathbb{R}^n) \) be the set of all dimension \( n \) complex (real) vectors, and \( \mathbb{R}_n^+(\mathbb{R}_{n+}^+) \) be the set of all dimension \( n \) nonnegative (positive) vectors. An \( m \)-order \( n \)-dimensional tensor \( A = (a_{i_1i_2\ldots i_m}) \) is a higher-order generalization of matrices, which consists of \( n^m \) entries:
\[
a_{i_1i_2\ldots i_m} \in \mathbb{R}, \quad i_k \in N = \{1, 2, \ldots, n\}, \quad k = 1, 2, \ldots, m.
\]
\( A \) is called nonnegative (positive) if \( a_{i_1i_2\ldots i_m} \in \mathbb{R}_+(a_{i_1i_2\ldots i_m} \in \mathbb{R}_{++}) \).

Tensor eigenvalue problems have attracted much attention in recent decades due to their wide applications in medical resonance imaging, higher-order Markov chains, positive definiteness of even-order multivariate forms in automatical control, blind source separation, see [1, 7, 8, 10, 14, 15]. Fan product and Hadamard product of tensors arises in a wide variety of ways, such as trigonometric equation kernel, the weak minimum principle in partial differential equations, and characteristic functions in probability theory, see [4, 9, 10]. For example, in the study of structured tensors, it is known that strong Hankel tensors, complete Hankel tensors, completely positive tensors and \( H \)-tensors are closed under Hadamard product and Fan product [17, 25, 31]. As a pioneer in the study of the Fan product, Horn et al. [9] proposed lower bounds on the minimum eigenvalue for the Fan product of two \( M \)-matrices. Improved results can be founded in [5, 12, 13, 29]. Recently, matrices with special structures such as \( M \)-matrices, \( Z \)-matrices and nonnegative matrices have been extended to higher order tensors and these are becoming the focus of tensor in recent research [6, 11, 19, 20, 23, 25, 26, 27, 28, 30]. Based on Gershgorin-type eigenvalue inclusion sets and Perron-Frobenius theorems for nonnegative tensors.

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Sun et al. [18] investigated some inequalities for the Hadamard product of tensors and obtained some bounds on the spectral radius, and used them to estimate the spectral radius of a directly weighted hypergraph. Wang et al. [21, 22] established lower bounds on the minimum eigenvalues for the Fan product of two $M$-tensors based on its algebra properties [28]. Meanwhile, it is noted that Brualdi-type inclusion set (Brauer-type inclusion set) is tighter than Gershgorin-type inclusion set [2, 3]. Motivated by these observations, we want to establish Brualdi-type (Brauer-type) inequalities for the Fan product of $M$-tensors, and further discuss comparisons among different Brualdi-type (Brauer-type) inequalities.

This paper is organized as follows. In Section 2, we introduce important notation and recall preliminary results on tensor analysis. In Section 3, we establish Brualdi-type inequalities on the minimum eigenvalues for the Fan product when $\Gamma(A \star B)$ is weakly connected. Furthermore, Brauer-type inequalities are proposed on the minimum eigenvalues for the Fan product when $\Gamma(A \star B)$ may be not weakly connected. Numerical examples verify the validity of the conclusions.

2. Notation and preliminaries. We start this section with some fundamental notion and properties related to eigenvalue of a tensor [11, 16], which are needed in the subsequent analysis.

**Definition 2.1.** Let $A$ be an $m$-order $n$-dimensional tensor. Assume that $Ax^{m-1}$ is not identical to 0. We say that $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an eigenvalue-eigenvector of $A$ if

$$Ax^{m-1} = \lambda x^{m-1},$$

where $(Ax^{m-1})_i = \sum_{i_2, \ldots, i_m=1}^n a_{i_1 \ldots i_m} x_{i_2} \ldots x_{i_m}$, $x^{[m-1]} = [x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}]^T$, and $(\lambda, x)$ is called an $H$-eigenpair if they are both real.

Friedland et al. [6] defined weakly irreducible polynomial maps and weakly primitive polynomial maps by the connectivity of a graph associated with a polynomial map.

Given a tensor $A = (a_{i_1 \ldots i_m})$, we associate $A$ with a digraph $\Gamma_A$ as follows. The vertex set of $\Gamma_A$ is $V(A) = \{1, \ldots, n\}$ and the arc set of $\Gamma_A$ is $E(A) = \{(i, j) : a_{i_1 \ldots i_m} \neq 0, j \in \{i_2, \ldots, i_m\} \neq \{i, \ldots, i\}\}$. A directed graph $\Gamma_A$ is called weakly connected if for each vertex $v_i \in V$, there exists a circuit such that $v_i$ belongs to the circuit. A directed graph $\Gamma_A$ is called strongly connected if for each ordered pair of distinct vertices $v_i$ and $v_j$, there is a path from $v_i$ to $v_j$. Further, the tensor $A$ is called weakly irreducible if the directed graph $\Gamma_A$ is strongly connected.

The Perron-Frobenius theorems for weakly irreducible nonnegative tensors have been established in [6].

**Lemma 2.2.** Let $A$ be a weakly irreducible nonnegative tensor of order $m$ and dimension $n$. Then, there exists a unique $x$ up to a multiplicative constant such that $(\rho(A), x)$ is a positive eigenpair.

The following specially structured tensors are extended from matrices [4, 28].

**Definition 2.3.** Let $A$ and $I$ be $m$-order $n$-dimensional tensors.

(i) We call $\sigma(A)$ as the set of all eigenvalues of $A$. Assume $\sigma(A) \neq \emptyset$. Then the minimum eigenvalue $A$ is denoted by

$$\tau(A) = \min\{\lambda : \lambda \in \sigma(A)\}.$$
We call $A$ as a $Z$-tensor if it can be written as $A = cI - B$, where $c > 0$, $I$ is a unit tensor with entries
\[ \delta_{i_1 i_2 \ldots i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \cdots = i_m \\ 0, & \text{otherwise} \end{cases} \]
and $B$ is a nonnegative tensor. Furthermore, if $c \geq \rho(B)$, then $A$ is said to be an $M$-tensor. $A$ is a weakly irreducible $Z$-tensor if $B$ is weakly irreducible.

Note that all the off-diagonal entries of a $Z$-tensor are non-positive and the (strong) $M$-tensor is closely linked with the diagonal dominance defined below [28].

**Definition 2.4.** For an $m$-order $n$-dimensional tensor $A$, it is called diagonally dominant if
\[ |a_{i_1 \ldots i_m}| \geq \sum_{\delta_{i_2 \ldots i_m} = 0} |a_{i_{i_2} \ldots i_{i_m}}|, \quad \forall i \in N. \]

Tensor $A$ is called strictly diagonally dominant if the strict inequalities hold for all $i \in N$.

Let $A$ be an $m$-order $n$-dimensional tensor and $D = \text{diag}(d_1, \ldots, d_n)$ be a positive diagonal matrix. Set
\[ A_D = A \cdot D^{-(m-1)} \overline{D} \cdots \overline{D} \]
with $(A_D)_{i_1 \ldots i_m} = a_{i_1 \ldots i_m} d_{i_1}^{-(m-1)} d_{i_2} \ldots d_{i_m}$.

**Lemma 2.5.** [31] Suppose $A$ is a $Z$-tensor and its all diagonal elements are non-negative (positive). Then, $A$ is an (strong) $M$-tensor if and only if there exists a positive diagonal matrix $D$ such that $B = A \cdot D^{-(m-1)} \overline{D} \cdots \overline{D}$ is (strictly) diagonally dominant.

**Lemma 2.6.** [27] Let $A$ and $B$ be order $m$ dimension $n$ tensors. If there is a diagonal nonsingular matrix $D$ such that $B = A \cdot D^{-(m-1)} \overline{D} \cdots \overline{D}$, then they have the same eigenvalues.

Bu et al. [2, 3] gave Brualdi-type eigenvalue inclusion sets and Brauer-type eigenvalue inclusion sets for weakly connected tensors and general tensors.

**Lemma 2.7.** (Theorem 3.1 of [2]) Let $A = (a_{i_1 \ldots i_m})$ be an $m$-order $n$-dimensional tensor such that $\Gamma_A$ is weakly connected. Then,
\[ \sigma(A) \subseteq \bigcup_{\gamma \in C(\Delta)} \{ z \in C : \prod_{i \in \gamma} |z - a_{i \ldots i}| \leq \prod_{i \in \gamma} r_i(A) \}. \]
where $r_i(A) = \sum_{\delta_{i_1 \ldots i_m} = 0} |a_{i_{i_1} \ldots i_m}|$.

**Lemma 2.8.** (Theorem 4.1 of [3]) Let $A = (a_{i_1 \ldots i_m})$ be an $m$-order $n$-dimensional tensor with $r_{i_j}(A) \neq 0$ for all $i \in N$. Then
\[ \sigma(A) \subseteq \bigcup_{(a_{i_1 \ldots i_m} \neq 0, \forall j \in \{1, \ldots, n\}) \neq (1, \ldots, 1)} \{ z \in C : \prod_{j=1}^{m} |z - a_{j \ldots j}| \leq \prod_{j=1}^{m} r_{i_j}(A) \}. \]

To end this section, we give the definition of the Fan product for tensors.
Definition 2.9. Let $\mathcal{A}$ and $\mathcal{B}$ be $m$-order $n$-dimensional tensors. Fan product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $\mathcal{A} \star \mathcal{B} = (c_{i_1 i_2 \ldots i_m})$, where
\[
c_{i_1 i_2 \ldots i_m} = \begin{cases} a_{i_1 \ldots i}, & \text{if } i_1 = i_2 = \ldots = i_m = i, \\ -|a_{i_1 i_2 \ldots i_m} b_{i_1 i_2 \ldots i_m}|, & \text{otherwise.} \end{cases}
\]

3. Inequalities for the Fan product of $M$-tensors. In this section, we shall establish Brualdi-type results on the minimum eigenvalue for Fan product of $M$-tensors. To this end, we present several important lemmas.

Lemma 3.1. Let $\mathcal{A} = (a_{i_1 i_2 i_3 \ldots i_m})$ be an $m$-order $n$-dimensional $M$-tensor such that $\Gamma_\mathcal{A}$ is weakly connected. Then, there exists a circuit $\gamma \in C(\mathcal{A})$ such that
\[
\prod_{i \in \gamma} (a_{i_1 \ldots i} - \tau(\mathcal{A})) \leq \prod_{i \in \gamma} \tilde{r}_i(\mathcal{A}),
\]
where $\tilde{r}_i(\mathcal{A}) = \sum_{\delta_{i_1 i_2 \ldots i_m} = 0} -a_{i_1 i_2 \ldots i_m}$.

Proof. Letting $\tau(\mathcal{A})$ be the minimum eigenvalue of $\mathcal{A}$, from Lemma 2.7, we obtain
\[
\prod_{i \in \gamma} |a_{i_1 \ldots i} - \tau(\mathcal{A})| \leq \prod_{i \in \gamma} r_i(\mathcal{A}),
\]
where $r_i(\mathcal{A}) = \sum_{\delta_{i_1 i_2 \ldots i_m} = 0} |a_{i_1 i_2 \ldots i_m}|$. Since $\mathcal{A}$ is a $Z$-tensor, from Lemma 4.1 of [18], one has
\[
\tau(\mathcal{A}) \leq \min_{i \in N} a_{i_1 \ldots i_1} \text{ and } \tilde{r}_i(\mathcal{A}) = r_i(\mathcal{A}).
\]
So, (2) is equivalent to
\[
\prod_{i \in \gamma} (a_{i_1 \ldots i} - \tau(\mathcal{A})) \leq \prod_{i \in \gamma} \tilde{r}_i(\mathcal{A}).
\]

Lemma 3.2. Let $\mathcal{A}$ be a weakly irreducible $M$-tensor of order $m$ dimension $n$. Then, there exists a positive vector $u$ such that
\[
\mathcal{A} u^{m-1} = \tau(\mathcal{A}) u^{[m-1]}. \]

Proof. Since $\mathcal{A}$ is a weakly irreducible $Z$-tensor, there exist a real number $s > 0$ and a weakly irreducible nonnegative tensor $Q$ such that
\[
\mathcal{A} = s\mathcal{I} - Q \text{ and } \rho(Q) = s - \tau(\mathcal{A}),
\]
where $s \geq \rho(Q)$. It follows from weak irreducibility of $Q$ and Lemma 2.2 that there exists a positive vector $u$ such that
\[
Q u^{m-1} = \rho(Q) u^{[m-1]} = (s - \tau(\mathcal{A})) u^{[m-1]}.
\]
Hence,
\[
(s\mathcal{I} - Q) u^{m-1} = \tau(\mathcal{A}) u^{[m-1]},
\]
which implies
\[
\mathcal{A} u^{m-1} = \tau(\mathcal{A}) u^{[m-1]}.
\]

Lemma 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two $M$-tensors of order $m$ dimension $n$. Then, $\mathcal{A} \star \mathcal{B}$ is an $M$-tensor. Furthermore, if either $\mathcal{A}$ or $\mathcal{B}$ is a strong $M$-tensor, then $\mathcal{A} \star \mathcal{B}$ is a strong $M$-tensor.
Proof. By the definition of $A \star B$, it holds that
$$A \star B = \begin{cases} a_{i_1 \ldots i_m}b_{i_1 \ldots i_m}, & \text{if } i_2 = i_3 = \ldots = i_m = i, \\ -|a_{i_1 \ldots i_m}b_{i_1 \ldots i_m}|, & \text{otherwise}. \end{cases}$$

Since $A$ and $B$ are $M$-tensors, by Lemma 2.6, there exist positive diagonal matrices $C, D$ such that
$$P = A \cdot C^{-(m-1)}a_1 \ldots a_m, \quad Q = B \cdot D^{-(m-1)}a_1 \ldots a_m$$
with
$$|p_{i_2 \ldots i_m}| = |a_{i_1 \ldots i_m}| \geq \sum_{i_{i_2 \ldots i_m}=0} |a_{i_1 \ldots i_m}| \geq \sum_{i_{i_2 \ldots i_m}=0} |a_{i_1 \ldots i_m}|C_{i_1 \ldots i_m},$$

$$|q_{i_2 \ldots i_m}| = |b_{i_1 \ldots i_m}| \geq \sum_{i_{i_2 \ldots i_m}=0} |b_{i_1 \ldots i_m}| = \sum_{i_{i_2 \ldots i_m}=0} |b_{i_1 \ldots i_m}|D_{i_1 \ldots i_m}.$$

Certainly,
$$|p_{i_1 \ldots i_m}q_{i_2 \ldots i_m}| = |a_{i_1 \ldots i_m}b_{i_1 \ldots i_m}| \geq \sum_{i_{i_2 \ldots i_m}=0} (|p_{i_1 \ldots i_m}|C_{i_1 \ldots i_m}) \sum_{i_{i_2 \ldots i_m}=0} (|q_{i_2 \ldots i_m}|D_{i_2 \ldots i_m}) \geq \sum_{i_{i_2 \ldots i_m}=0} (|p_{i_1 \ldots i_m}|C_{i_1 \ldots i_m}) \sum_{i_{i_2 \ldots i_m}=0} (|q_{i_2 \ldots i_m}|D_{i_2 \ldots i_m}) \geq \sum_{i_{i_2 \ldots i_m}=0} |p_{i_1 \ldots i_m}q_{i_2 \ldots i_m}|(c_{i_1 d_1} \ldots c_{i_m d_m}).$$

From (4), there exists a positive diagonal matrix $U = \text{diag}(c_1 d_1, c_2 d_2, \ldots, c_n d_n)$ such that
$$|p_{i_1 \ldots i_m}q_{i_2 \ldots i_m}| \geq \sum_{i_{i_2 \ldots i_m}=0} |p_{i_1 \ldots i_m}q_{i_2 \ldots i_m}|(u_i)^{(m-1)}u_{i_2} \ldots u_{i_m}.$$

It follows from Lemma 2.5 that $A \star B$ is an $M$-tensor. Similar to the argument for the first conclusion, we can obtain the second conclusion. \hfill \Box

3.1. $\Gamma(A \star B)$ is weakly connected. Based on the characterizations on the minimum eigenvalue of $M$-tensors, we propose lower bounds on minimum eigenvalue for the Fan product of two weakly connected $M$-tensors.

**Theorem 3.4.** Let $A$ and $B$ be two strong $M$-tensors of order $m$ and dimension $n$. Fan product $A \star B$ is an $M$-tensor such that $\Gamma(A \star B)$ is weakly connected. For any $\gamma \in C(A \star B)$, then
$$\tau(A \star B) \leq \min_{i \in N} a_{i_1 \ldots i_m},$$

$$\prod_{i \in \gamma}(a_{i_1 \ldots i_m} - \tau(A \star B)) \leq \max_{\gamma \in C(A \star B)} \prod_{i \in \gamma}(a_{i_1 \ldots i_m} - \tau(A))(b_{i_1 \ldots i_m} - \tau(B)),$$

where $C(A \star B)$ is the set of circuits in $\Gamma(A \star B)$.

Proof. From Lemma 4.1 of [18], (5) holds. Next, we focus on (6). For this purpose, we break the proof into two cases.

Case 1. $A$ and $B$ are weakly irreducible. From Lemma 3.2, there exist positive vector $u, v$ such that
$$a_{1 \ldots 1}v^{(m-1)} + \sum_{\delta_{i_2 \ldots i_m}=0} a_{i_1 \ldots i_m}u_{i_2} \ldots u_{i_m} = \tau(A)u_i^{(m-1)},$$
\[ b_{1 \ldots i_{m}}^{[m-1]} + \sum_{\delta_{i_{12} \ldots i_{m}} = 0} b_{i_{1}i_{2} \ldots i_{m}} v_{i_{2}} \ldots v_{i_{m}} = \tau(B)v_{i}^{[m-1]} \]. \tag{8} \]

Set \( D = \text{diag}(u_{1}v_{1}, \ldots, u_{n}v_{n}) \). It is obvious that \( D \) is a positive diagonal matrix. Thus, \( \sigma(A \ast B) = \sigma(D^{-m}(A \ast B)D) \). By Lemma 3.1, (7) and (8), there exists a circuit \( \gamma \in C(A \ast B) \) such that

\[
\prod_{i \in \gamma} (a_{i \ldots i} - \tau(A \ast B)) \leq \prod_{i \in \gamma} \sum_{\delta_{i_{12} \ldots i_{m}} = 0} \frac{a_{i_{1}i_{2} \ldots i_{m}} b_{i_{1}i_{2} \ldots i_{m}} u_{i_{2}} \ldots u_{i_{m}} v_{i_{2}} \ldots v_{i_{m}}}{u_{i}^{[m-1]} v_{i}^{[m-1]}} \\
\leq \prod_{i \in \gamma} \sum_{\delta_{i_{12} \ldots i_{m}} = 0} \frac{-a_{i_{1}i_{2} \ldots i_{m}} u_{i_{2}} \ldots u_{i_{m}}}{u_{i}^{[m-1]}} \sum_{\delta_{i_{12} \ldots i_{m}} = 0} \frac{-b_{i_{1}i_{2} \ldots i_{m}} v_{i_{2}} \ldots v_{i_{m}}}{v_{i}^{[m-1]}} \\
= \prod_{i \in \gamma} (a_{i \ldots i} - \tau(A))(b_{i \ldots i} - \tau(B)). \tag{9} \]

Due to the uncertainty of \( \gamma \in C(A \ast B) \), (6) is established.

Case 2. Either \( A \) or \( B \) is weakly reducible. Let \( S \) be an order \( m \) dimension \( n \) tensor with

\[ s_{i_{1}i_{2} \ldots i_{m}} = \begin{cases} 1, & \text{if } i_{2} = i_{3} = \cdots = i_{m} \neq i, \\ 0, & \text{otherwise}. \end{cases} \]

Then both \( A - \epsilon S \) and \( B - \epsilon S \) are weakly irreducible tensors for any \( \epsilon > 0 \). Now, we claim that \( A - \epsilon S \) and \( B - \epsilon S \) are both strong \( M \)-tensors when \( \epsilon > 0 \) is sufficiently small. Since \( A, B \) are strong \( M \)-tensors, there exist positive diagonal matrices \( C, D \) such that

\[ P = A \cdot C^{-(m-1)} \underbrace{C \ldots C}_{m-1}, \quad Q = B \cdot D^{-(m-1)} \underbrace{D \ldots D}_{m-1} \]

with

\[ |p_{i \ldots i}| = |a_{i \ldots i}| > \sum_{\delta_{i_{12} \ldots i_{m}} = 0} |a_{i_{1}i_{2} \ldots i_{m}}| = \sum_{\delta_{i_{12} \ldots i_{m}} = 0} |p_{i_{1}i_{2} \ldots i_{m}} c_{i_{1}}^{-(m-1)} c_{i_{2}} \ldots c_{i_{m}}|, \]

\[ |q_{i \ldots i}| = |b_{i \ldots i}| > \sum_{\delta_{i_{12} \ldots i_{m}} = 0} |b_{i_{1}i_{2} \ldots i_{m}}| = \sum_{\delta_{i_{12} \ldots i_{m}} = 0} |q_{i_{1}i_{2} \ldots i_{m}} d_{i_{1}}^{-(m-1)} d_{i_{2}} \ldots d_{i_{m}}|. \]

Set

\[ L = \max_{i \neq j} \left\{ \frac{|c_{i}^{[m-1]}|}{c_{i_{1}i_{2} \ldots i_{m}}}, \frac{|d_{i}^{[m-1]}|}{d_{i_{1}i_{2} \ldots i_{m}}} \right\} \]

and

\[ \epsilon_{0} = \min_{i \neq j} \left\{ \frac{|a_{i \ldots i}| - \sum_{\delta_{i_{12} \ldots i_{m}} = 0} |a_{i_{1}i_{2} \ldots i_{m}} c_{i_{1}}^{-(m-1)} c_{i_{2}} \ldots c_{i_{m}}|}{(n-1)L}, \frac{|b_{i \ldots i}| - \sum_{\delta_{i_{12} \ldots i_{m}} = 0} |b_{i_{1}i_{2} \ldots i_{m}} d_{i_{1}}^{-(m-1)} d_{i_{2}} \ldots d_{i_{m}}|}{(n-1)L} \right\}. \]

Then for any \( 0 < \epsilon < \epsilon_{0} \), it holds that \( A - \epsilon S \) and \( B - \epsilon S \) are two strong \( M \)-tensors. Noting that \( A \) and \( B \) are two strong \( M \)-tensors, for the circuit \( \gamma \in C(A \ast B) \), we get \( \gamma \in C((A - \epsilon S) \ast (B - \epsilon S)) \). Substituting \( A - \epsilon S \) and \( B - \epsilon S \) for \( A \) and \( B \) and letting \( \epsilon \rightarrow 0 \), we can obtain the desired results by the continuity of \( \tau(A - \epsilon S) \) and \( \tau(B - \epsilon S) \).

By making use of the information of the absolute maximum in the off-diagonal elements, we are at the position to establish the following theorem.
Theorem 3.5. Let $A$ and $B$ be two strong $M$-tensors of order $m$ and dimension $n$. Fan product $A \ast B$ is an $M$-tensor such that $\Gamma_{(A \ast B)}$ is weakly connected. For any $\gamma \in C(A \ast B)$, then

$$\tau(A \ast B) \leq \min_{i \in N} a_i b_i,$$

(10)

$$\prod_{i \in \gamma} (a_{i_1 \ldots i_m} - \pi(A \ast B)) \leq \max_{\gamma \in C(A \ast B)} \prod_{i \in \gamma} (\alpha_i \beta_i (a_{i_1 \ldots i} - \pi(A)) (b_{i_1 \ldots i} - \pi(B)))^{\frac{1}{2}},$$

where $\alpha_i = \max_{\delta_{i_1 \ldots i_m} = 0} -a_{i_1 \ldots i_m}$ and $\beta_i = \max_{\delta_{i_1 \ldots i_m} = 0} -b_{i_1 \ldots i_m}$.

Proof. The following argument is divided into two cases.

Case 1. $A$ and $B$ are weakly irreducible. From Lemma 3.2, there exist two positive eigenvectors $u = (u_i^T)$ corresponding to $\tau(A)$ and $\tau(B)$ such that

$$a_{i_1 \ldots i} u_i^{2[m-1]} + \sum_{\delta_{i_1 \ldots i_m} = 0} a_{i_1 \ldots i_m} u_{i_2}^2 \ldots u_{i_m}^2 = \tau(A) u_i^{2[m-1]},$$

(11)

$$b_{i_1 \ldots i} v_i^{2[m-1]} + \sum_{\delta_{i_1 \ldots i_m} = 0} b_{i_1 \ldots i_m} v_{i_2}^2 \ldots v_{i_m}^2 = \tau(B) v_i^{2[m-1]},$$

(12)

Without loss of generality, assume that $u, v \in \mathbb{R}^n_{++}$. Set $D = \text{diag}(u_1 v_1, \ldots, u_n v_n)$. Thus, $\sigma(A \ast B) = \sigma(D^{1-m}(A \ast B D))$. By Lemma 3.1, (11) and (12), there exists a circuit $\gamma \in C(A \ast B)$ such that

$$\prod_{i \in \gamma} (a_{i_1 \ldots i} - \pi(A \ast B)) \leq \prod_{i \in \gamma} \sum_{\delta_{i_1 \ldots i_m} = 0} \frac{a_{i_1 \ldots i_m} u_{i_2}^2 \ldots u_{i_m}^2}{u_i^{2[m-1]}} \frac{b_{i_1 \ldots i_m} v_{i_2}^2 \ldots v_{i_m}^2}{v_i^{2[m-1]}},$$

(13)

$$\leq \prod_{i \in \gamma} \left( \sum_{\delta_{i_1 \ldots i_m} = 0} \frac{-a_{i_1 \ldots i_m} u_{i_2}^2 \ldots u_{i_m}^2}{u_i^{2[m-1]}} \frac{-b_{i_1 \ldots i_m} v_{i_2}^2 \ldots v_{i_m}^2}{v_i^{2[m-1]}} \right) \prod_{i \in \gamma} \left( \sum_{\delta_{i_1 \ldots i_m} = 0} \frac{a_{i_1 \ldots i_m} u_{i_2}^2 \ldots u_{i_m}^2}{u_i^{2[m-1]}} \frac{b_{i_1 \ldots i_m} v_{i_2}^2 \ldots v_{i_m}^2}{v_i^{2[m-1]}} \right)^{\frac{1}{2}},$$

$$\leq \prod_{i \in \gamma} (\alpha_i \beta_i (a_{i_1 \ldots i} - \pi(A)) (b_{i_1 \ldots i} - \pi(B)))^{\frac{1}{2}},$$

where the third inequality uses the Cauchy-Schwarz inequality. From the uncertainty of $\gamma \in C(A \ast B)$, (10) holds.

Case 2. Either $A$ or $B$ is weakly reducible. Similar to the proof of Theorem 3.4, we obtain the desired result.

We next give a simple comparison of between Theorem 3.4 and Theorem 3.5.

Remark 1. Let $A$ and $B$ be strong $M$-tensors of order $m$ dimension $n$. Define $f(x) = \prod_{i \in \gamma} (a_{i_1 \ldots i} b_{i_1 \ldots i} - x)$. For $x \in D = (-\infty, \min_{i \in N} a_{i_1 \ldots i} b_{i_1 \ldots i})$ and $0 \leq k_1 \leq k_2$, we have

$$S_{k_1} = \{ x \in D : f(x) \leq k_1 \} \subseteq S_{k_2} = \{ x \in D : f(x) \leq k_2 \}.$$

Thus, the lower bound of $S_{k_1}$ is not less than that of $S_{k_2}$. Set

$$k_1 = \prod_{i \in \gamma} (a_{i_1 \ldots i} - \pi(A)) (b_{i_1 \ldots i} - \pi(B)), k_2 = \prod_{i \in \gamma} (\alpha_i \beta_i (a_{i_1 \ldots i} - \pi(A)) (b_{i_1 \ldots i} - \pi(B)))^{\frac{1}{2}}.$$
where $\beta$. 

**Proof.** By Lemma 3.1 and (7), there exists a circuit $\gamma \in D \in \gamma$, such that

$$\prod_{i \in \gamma} (a_i - \tau(A)) (b_{i...i} - \tau(B)) \geq \prod_{i \in \gamma} [\alpha_i \delta_i (a_{i...i} - \tau(A))((b_{i...i} - \tau(B))]^2.$$ 

Similarly, the lower bound in (6) is not less than that of (10).

**Theorem 3.6.** Let $A$ and $B$ be two strong $M$-tensors of order $m$ and dimension $n$. Fan product $A \ast B$ is an $M$-tensor such that $\Gamma_{(A \ast B)}$ is weakly connected. For any $\gamma \in C(A \ast B)$, then

$$\tau(A \ast B) \leq \min_{i \in N} a_i b_{i...i},$$

$$\prod_{i \in \gamma} (a_i b_{i...i} - \tau(A \ast B)) \leq \max_{\gamma \in C(A \ast B)} \beta_i (a_{i...i} - \tau(A)),$$ (14) where $\beta_i = \max_{\delta_{i...i}} = 0 - b_{i...i}.$

**Proof.** The following argument is divided into two cases.

Case 1. $A$ is weakly irreducible. From Lemma 3.2, there exists a positive vector $u$ such that (7) holds. Set $D = \text{diag}(u_1, \ldots, u_n)$. Thus, $\sigma(A \ast B) = \sigma(D^{1-m}(A \ast B)D).$ By Lemma 3.1 and (7), there exists a circuit $\gamma \in C(A \ast B)$ such that

$$\prod_{i \in \gamma} (a_i b_{i...i} - \tau(A \ast B)) \leq \prod_{i \in \gamma} \sum_{\delta_{i...i}} a_{i...i} b_{i...i} u_{i...i} \leq \prod_{i \in \gamma} \beta_i \sum_{\delta_{i...i}} -a_{i...i} u_{i...i} \leq \prod_{i \in \gamma} \beta_i (a_{i...i} - \tau(A)).$$ (15)

By the uncertainty of $\gamma \in C(A \ast B)$, (14) holds.

Case 2. $A$ is weakly reducible. Similar to the proof of Theorem 3.4, then $A - \epsilon S$ is a weakly irreducible $M$-tensor for sufficiently small positive real number $\epsilon$. Noting that $A$ and $B$ are two strong $M$-tensors, for the circuit $\gamma \in C(A \ast B)$, we get $\gamma \in C((A - \epsilon S) \ast B).$ Substituting $A - \epsilon S$ for $A$ and letting $\epsilon \to 0$ on (15), we can obtain the desired results by the continuity of $\tau(A - \epsilon S)$. \hfill $\Box$

Since Fan product is commutative, the inequality (14) remains correct if $A$ and $B$ are switched. Moreover, the following result can be immediately obtained.

**Theorem 3.7.** Let $A$ and $B$ be two strong $M$-tensors of order $m$ and dimension $n$. Fan product $A \ast B$ is an $M$-tensor such that $\Gamma_{(A \ast B)}$ is weakly connected. For any $\gamma \in C(A \ast B)$, then

$$\tau(A \ast B) \leq \min_{i \in N} a_i b_{i...i},$$

$$\prod_{i \in \gamma} (a_i b_{i...i} - \tau(A \ast B)) \leq \max_{\gamma \in C(A \ast B)} \alpha_i (b_{i...i} - \tau(B)),$$ (16) where $\alpha_i = \max_{\delta_{i...i}} = 0 - a_{i...i}.$

We next give comparisons among Theorems 3.4, 3.6 and 3.7.
Remark 2. Let $A$ and $B$ be strong $M$-tensors of order $m$ dimension $n$. 

(i) For $\gamma \in C(A \ast B)$ and $i \in \gamma$, if $b_{i...i} - \tau(B) \leq \beta_i$, then
\[
\prod_{i \in \gamma} (a_{i...i} - \tau(A))(b_{i...i} - \tau(B)) \leq \prod_{i \in \gamma} \beta_i(a_{i...i} - \tau(A)).
\]

Thus, the lower bound in (6) is not less than that of (14).

For $\gamma \in C(A \ast B)$ and $i \in \gamma$, if $b_{i...i} - \tau(B) \geq \beta_i$, then
\[
\prod_{i \in \gamma} (a_{i...i} - \tau(A))(b_{i...i} - \tau(B)) \geq \prod_{i \in \gamma} \beta_i(a_{i...i} - \tau(A)).
\]

Hence, the lower bound in (6) is not more than that of (14).

(ii) For $\gamma \in C(A \ast B)$ and $i \in \gamma$, if $a_{i...i} - \tau(A) \leq \alpha_i$, then
\[
\prod_{i \in \gamma} (a_{i...i} - \tau(A))(b_{i...i} - \tau(B)) \leq \prod_{i \in \gamma} \alpha_i(b_{i...i} - \tau(B)).
\]

Therefore, the lower bound in (6) is not less than that of (16).

For $\gamma \in C(A \ast B)$ and $i \in \gamma$, if $a_{i...i} - \tau(A) \geq \alpha_i$, then
\[
\prod_{i \in \gamma} (a_{i...i} - \tau(A))(b_{i...i} - \tau(B)) \geq \prod_{i \in \gamma} \alpha_i(b_{i...i} - \tau(B)).
\]

Consequently, the lower bound in (6) is not more than that of (16).

Similarly, we still establish similar comparisons of among Theorems 3.5-3.7. For the sake of simplicity, the details are omitted.

The following example exhibits efficiency of Theorems 3.4-3.7.

Example 3.1. Let $A = (a_{ijk}), B = (b_{ijk})$ be two strong $M$-tensors of order 3 dimension 3 with elements defined as follows:

\[A = [A(1, :, ::), A(2, :, ::), A(3, :, ::)], B = [B(1, :, ::), B(2, :, ::), B(3, :, ::)],\]

where
\[
A(1, :, ::) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix},
A(2, :, ::) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 4 & 0 \\ 0 & -2 & 0 \end{pmatrix},
A(3, :, ::) = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix},
\]
\[
B(1, :, ::) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix},
B(2, :, ::) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ -1 & -2 & 0 \end{pmatrix},
B(3, :, ::) = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}.
\]

By computations, we get $\tau(A) = 2.0000, \tau(B) = 1.3826, \alpha_1 = \alpha_2 = \alpha_3 = 2, \beta_1 = 1, \beta_2 = 2, \beta_3 = 3, \tau(A \ast B) = 13.8992$. Obviously, $\Gamma(A \ast B)$ is weakly connected and $\Gamma(A \ast B)$ has 3 circuits:

\[2 \rightarrow 2; 1 \rightarrow 2 \rightarrow 3 \rightarrow 1; 1 \rightarrow 3 \rightarrow 1.\]

By Theorem 3.4, for the circuit 2 \rightarrow 2, we have
\[\tau(A \ast B) \geq 10.7652;\]

for the circuit 1 \rightarrow 2 \rightarrow 3 \rightarrow 1, we deduce
\[\tau(A \ast B) \geq 10.9565;\]

for the circuit 1 \rightarrow 3 \rightarrow 1, we know
\[\tau(A \ast B) \geq 11.0714;\]
Thus, \[ \tau(A \ast B) \geq \min_{\gamma \in C(A \ast B)} \{10.7652, 10.9565, 11.0714\} = 10.7652. \]

Similarly, by Theorem 3.5, we obtain
\[ \tau(A \ast B) \geq \min_{\gamma \in C(A \ast B)} \{11.4241, 12.3464, 12.0722\} = 11.4241. \]

Similarly, it follows from Theorem 3.6 that
\[ \tau(A \ast B) \geq \min_{\gamma \in C(A \ast B)} \{12, 12, 12\} = 12. \]

Similarly, using Theorem 3.7, we know
\[ \tau(A \ast B) \geq \min_{\gamma \in C(A \ast B)} \{10.7652, 12.6588, 13.4023\} = 10.7652. \]

To sum up above theorems, we yield
\[ 12 \leq \tau(A \ast B) \leq 15. \]

3.2. \( \Gamma_{(A \ast B)} \) may not be connected. When \( \Gamma_{(A \ast B)} \) may not be connected, Brualdi-type inclusion sets can not be satisfied. To overcome the difficulties, Bu et al. [3] established Brauer-type eigenvalue inclusion sets for general tensors. Based on these results, we establish Brauer-type lower bounds for the Fan product of two \( M \)-tensors.

**Theorem 3.8.** Let \( A \) and \( B \) be two strong \( M \)-tensors of order \( m \) and dimension \( n \). Then,
\[
\tau(A \ast B) \leq \min_{\tau \in \mathbb{N}} \prod_{j=1}^{m} (a_{i_j...i_j} - \tau(A \ast B)) \leq \max_{a_{i_1...i_m} \neq 0} \prod_{j=1}^{m} \left[ (a_{i_j...i_j} - \tau(A)) (b_{i_j...i_j} - \tau(B)) \right].
\]

(17)

**Proof.** If \( A \) and \( B \) are both weakly irreducible, similar to the proof of Theorem 3.4, by Lemma 2.8, we obtain (17).

If either \( A \) or \( B \) is weakly reducible, then both \( A - \epsilon S \) and \( B - \epsilon S \) are weakly irreducible for any \( \epsilon > 0 \). Similar to the proof of Theorem 3.4, we claim that \( A - \epsilon S \) and \( B - \epsilon S \) are both strong \( M \)-tensors when \( \epsilon > 0 \) is sufficiently small. Observing that \( A \) and \( B \) are strong \( M \)-tensors, for \( -a_{i_1...i_m} b_{i_1...i_m} \neq 0 \) in \( A \ast B \), we get \( -(a_{i_1...i_m} - \epsilon)(b_{i_1...i_m} - \epsilon) \neq 0 \) in \( (A - \epsilon S) \ast (B - \epsilon S) \). Substituting \( A - \epsilon S \) and \( B - \epsilon S \) for \( A \) and \( B \) and letting \( \epsilon \to 0 \), we can obtain the desired results by the continuity of \( \tau(A - \epsilon S) \) and \( \tau(B - \epsilon S) \). So, (17) holds.

Based on Lemma 2.8 and Theorems 3.5-3.7, we propose Brauer-type lower bounds for the Fan product of two \( M \)-tensors.

**Theorem 3.9.** Let \( A \) and \( B \) be two strong \( M \)-tensors of order \( m \) and dimension \( n \). Then,
\[
\tau(A \ast B) \leq \min_{\tau \in \mathbb{N}} \prod_{j=1}^{m} (a_{i_j...i_j} - \tau(A \ast B)) \leq \max_{a_{i_1...i_m} \neq 0} \prod_{j=1}^{m} \left[ (a_{i_j...i_j} \beta_{i_j...i_j} - \tau(A))(b_{i_j...i_j} - \tau(B)) \right].
\]
Theorem 3.10. Let $\mathcal{A}$ and $\mathcal{B}$ be two strong $M$-tensors of order $m$ and dimension $n$. Then,

$$\tau(\mathcal{A} \star \mathcal{B}) \leq \min_{i \in \mathbb{N}} a_{i...i} b_{i...i},$$

$$\prod_{j=1}^{m} (a_{i...j} b_{j...i} - \tau(\mathcal{A} \star \mathcal{B})) \leq \max_{a_{i1...im} b_{i1...im} \neq 0} \prod_{j=1}^{m} \beta_{ij} (a_{i1...im} b_{i1...im} - \tau(\mathcal{A})).$$

Theorem 3.11. Let $\mathcal{A}$ and $\mathcal{B}$ be two strong $M$-tensors of order $m$ and dimension $n$. Then,

$$\tau(\mathcal{A} \star \mathcal{B}) \leq \min_{i \in \mathbb{N}} a_{i...i} b_{i...i},$$

$$\prod_{j=1}^{m} (a_{i...j} b_{j...i} - \tau(\mathcal{A} \star \mathcal{B})) \leq \max_{a_{ij...im} b_{ij...im} \neq 0} \prod_{j=1}^{m} \alpha_{ij} (b_{ij...im} - \tau(\mathcal{B})).$$

4. Conclusion. In this paper, we characterized some properties on $M$-tensors. Based on these properties, we established Brualdi-type (Brauer-type) inequalities on the minimum eigenvalue for the Fan product by similarity transformation methods, which are novel even for matrices. Finally, we discussed the advantages of different Brualdi-type (Brauer-type) inequalities.

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