Gap Statistics of the Sequence \{\alpha\sqrt{n}\}

CHRISTOPHER LUTSKO*

April 17, 2020

Abstract

The gaps in the sequence \{\sqrt{n}\} were shown by Elkies-McMullen (2004) to have a limiting distribution which is not the exponential distribution. However it is conjectured that the distribution of gaps in the sequence \{\alpha\sqrt{n}\} is exponential, provided \alpha^2 is irrational. For almost all values of \alpha, we prove an important step in this direction. In particular, we show that all the correlations are Poissonian along a subsequence.

MSC2020: 11K06; 11K60; 11L07; 37A44; 37A44

Key words and phrases: Local Statistics; Gap Distribution; Exponential Sums.

1 Introduction

Consider the sequence \{x_n\}_{n=1}^N where

\[ x_n := \alpha\sqrt{n} \mod 1. \]

It is easy to show that the sequence \{x_n\}_{n=1}^N is uniformly distributed on [0,1]. However, the higher order local statistics for this sequence are more nuanced.

In a beautiful paper, Elkies and McMullen [EM04] showed that, if \alpha = 1, the limiting gap distribution is given by an explicit function. In particular, the gap statistics do not converge to an exponential distribution as they do for uniformly distributed random variables on [0,1]. Elkies and McMullen’s result can be extended to all \alpha such that \alpha^2 \in \mathbb{Q}. In contrast if \alpha^2 is irrational it is conjectured that the associated gap statistics do converge to the exponential distribution – see [Mar07]. The proof of this conjecture remains a challenging open problem.

In this paper, we consider the sequence \{x_n\}_{n=1}^N for \alpha^2 irrational. We show that, for almost all \alpha \in \mathbb{R}, the m-level correlations converge to the Poissonian limit along a particular subsequence. In particular we show that if \alpha is well approximated by prime numbers then the sequence \{x_n\}_{n=1}^N is ‘Poissonian along a subsequence’ (see Theorem 1).

1.1 Setup and Main Theorem

Given our sequence \{x_n\}_{n \in \mathbb{N} \subset [0,1]}, and a vector \mathbf{j} \in \mathbb{Z}^m, let \Delta(\mathbf{j}) \in \mathbb{R}^{m-1} denote the difference vector:

\[ \Delta(\mathbf{j}) := (x_{j_1} - x_{j_2}, x_{j_2} - x_{j_3}, \ldots, x_{j_{m-1}} - x_{j_m}). \]

Then for \( f \in C_c^\infty(\mathbb{R}^{m-1}) \) we define the m-level correlation to be:

*University of Bristol
Email: chris@lutsko.com
\[ R^{(m)}(N, \alpha, f) := \frac{1}{N} \sum_{j \in \{1, \ldots, N\}^m} f(N(\Delta(j))), \]

where the notation \( \sum^* \) indicates that all entries of the vector \( j \) are distinct. We say the sequence \( \{\alpha \sqrt{n}\} \) is **Poissonian** if

\[ R^{(m)}(N, \alpha, f) \to \int_{\mathbb{R}^{m-1}} f(x)dx \quad (1.1) \]

in the limit as \( N \to \infty \), for every \( m \geq 2 \). That is, we say a sequence is Poissonian if, for every \( m \), the \( m \)-level correlation converges to the same limit as for a collection of uniformly distributed random variables.

It is important to note that if a sequence is Poissonian, then by the method of moments one can show that the spacing measures are also Poissonian. Therefore, if we could show that \( \{\alpha \sqrt{n}\} \) is Poissonian, this would imply the conjecture about the gap statistics of the sequence. Our main result states that the convergence in \((1.1)\) does occur for \( \{\alpha \sqrt{n}\} \), along certain subsequences. In particular, using the same standard argument, this implies that the gap distribution converges to the exponential distribution along this subsequence.

**Theorem 1.** Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) be such that, there exist infinitely many \( q_j \) prime, and \( b_j \) coprime to \( q_j \), for which

\[ \left| \alpha - \frac{b_j}{q_j} \right| < \psi(q_j), \quad (1.2) \]

where \( \psi(q_j) = o\left(\frac{1}{q_j}\right) \). Then there exists a subsequence \( N_j \to \infty \) such that \((1.1)\) holds along this subsequence for every \( m \geq 2 \). The sequence \( N_j \) satisfies \( \frac{\log(N_j)}{\log(q_j)} \to 0 \).

**Remark:** By Khintchin’s Theorem, the set of \( \alpha \) satisfying \((1.2)\) with \( \psi(x) := \frac{1}{x^2 \log^3(x)} \) has full measure.

**Idea of proof of Theorem 1:** To prove Theorem 1 we begin by following the methodology used by Rudnick-Sarnak-Zaharescu [RSZ01] who prove a similar statement about the sequence \( \{n^2 \alpha\} \). That is, in Section 2 we use their comparison lemma to show that we can replace the sequence \( \{\alpha \sqrt{n}\} \) by a family of sequences \( \{\frac{b_j}{q_j} \sqrt{\pi}\} \) of rational approximations. We show that Theorem 1 is implied by a similar convergence result for the approximating sequence (see Theorem 2).

To prove that the correlations for the rational approximations converge to the Poissonian limit, Rudnick-Sarnak-Zaharescu relied on a Fourier decomposition and then counting integer points on curves defined by Diophantine equations. In our case the underlying equation is not polynomial, but radical. Hence, while Rudnick-Sarnak-Zaharescu could use the Lipschitz principle and Riemann hypothesis for curves over finite fields, we will use Fourier analysis to reduce the statement about convergence to bounds on exponential sums. To achieve these bounds we then use Van der Corput’s B-process [Mon94, Chapter 3] – i.e stationary phase integrals – and some classical analytic tools.

**1.2 Motivation**

Dating back to the 1970s, mathematicians have observed a deep connection between the dynamics on a given surface, and the statistical distribution of the quantum spectrum on said surface. In particular the well-known Berry-Tabor conjecture [BT77] states that, for generic surfaces, if the Hamiltonian
dynamics on the surface are integrable, then the gap distribution for the spectrum has a well-defined limit, given by the exponential distribution. There exist counter-examples to this statement which can be explained away (thus the word generic), however in general this relationship has been confirmed by experiment. For more information see the review [Mar00].

While this conjecture is backed by experimental evidence and intuitive arguments, proving that a given sequence has Poissonian gap distribution appears to be a very hard problem. For example there has been a plethora of research concerning two rather simple sequences: \( \{abled\} \) [KR99, RSZ01, HB10] and the sequence with which we are concerned, \( \{\sqrt{a}\} \) [EM04, Sin12, EBMV15, BV16]. However, in both cases, proving that the gap statistics indeed converge to the exponential distribution (for a certain class of \( a \)) remains beyond modern techniques.

However, with regards \( \{\sqrt{a}\} \), we are slowly understanding more about the sequence. Elkies and McMullen [EM04] showed that for \( a = 1 \), one can use equidistribution of expanding horospheres to prove the existence and deduce the explicit form of the limiting distribution for the spacings (which is not exponential). This result was later made effective by Browning and Vinogradov [BV16]. Sinai [Sin12] later gave an alternative proof using Birkhoff sums involving rotations that there exists a limiting gap distribution (although his method does not give the explicit limit). Then El-Baz, Marklof, and Vinogradov [EBMV15] showed that for \( a = 1 \), the pair correlation does indeed converge to the Poissonian limit.

Put into this context, our result has a number of implications. First of all, for those \( a \) satisfying (1.2) our result suggests that not only are the spacings Poissonian, but the correlations are Poissonian, which is a stronger statement. From a technical point of view, the methodology is very general. After comparing the sequence \( \{\sqrt{a}\} \) to the Diophantine approximants, the only place where we use the exact form of the sequence is in bounding the exponential sums which arise. As such, the counting methods of Rudnick-Sarnak-Zaharescu, can be generalised to the non-Diophantine setting. For example, the methods used in this paper could be used to consider the problem \( \{\alpha n^\beta\} \).

Moreover, Sinai [Sin12] used Birkhoff sums for circle rotations to prove that the gap distribution for \( \{\sqrt{a}\} \) had a limiting distribution. Using Theorem 1, if one could prove the same statement as Sinai did for \( \{\sqrt{a}\} \) then one could prove the conjecture for the spacings. As such, Theorem 1 has reduced the complexity of this conjecture in the case that the Diophantine condition (1.2) holds.

**Notation:** To avoid confusion, fix the following notation, let \( f(n) = O(g(n)) \) if \( \lim_{n \to \infty} |f(n)/g(n)| < \infty \), and \( f(n) = o(g(n)) \) if \( \lim_{n \to \infty} |f(n)/g(n)| = 0 \). Equivalently let \( f(n) \ll g(n) \) denote \( f(n) = O(g(n)) \).

For \( x \in \mathbb{R} \) let \( \{x\} \) denote the fractional part of \( x \). Moreover we write \( a \equiv b \mod q \) and the \( q \) is obvious (in what follows \( \equiv \) will always be referring to modulo the variable \( q \)). Given a set \( \mathscr{A} \) we denote \( \mathscr{A}^* := \mathscr{A} \setminus \{0\} \). Finally, as usual \( e(z) := e^{2\pi iz} \).

## 2 Comparison

The first step in the proof is to reduce Theorem 1 to the following

**Theorem 2.** Fix \( m \geq 2 \), a function \( f \in C_c^\infty(\mathbb{R}^{m-1}) \), and a constant \( \delta > 0 \). Then as \( q \to \infty \) with \( q \) prime,

\[
R^{(m)}(N, \frac{b}{q}, f) \to \int_{\mathbb{R}^{m-1}} f(x)dx
\]

(2.1)

uniformly for all \( (b, q) = 1 \) and \( q^{\frac{1}{\log q} - \delta} \leq N \leq \frac{q}{\log q} \).

Consider two families of sequences.
\[ N := \{ x_N(n) : n \leq N \} \]
\[ N' := \{ x'_N(n) : n \leq N \}. \]

Define the separation between these two sequences to be
\[ \epsilon(N, N') := N \max_{n \geq N} |x_N(n) - x'_N(n)| \]
and let \( R^{(m)}(N, f) \) be the \( m \)-level correlation associated to the sequence \( N \). The following 'Comparison lemma' from [RSZ01] is crucial in showing that Theorem 2 implies Theorem 1:

**Lemma 3 ([RSZ01, Lemma 5]).** Assume \( N \) and \( N' \) are two families of sequences such that \( \epsilon(N, N') \to 0 \) as \( N \to \infty \). Then for all smooth test functions

\[
\left| R^{(m)}(N, f) - R^{(m)}(N', f) \right| \leq R^{(m)}(N, f_+) \epsilon(N, N'), \quad (2.2)
\]

for all \( N \) large enough and where \( f_+ \geq 0 \) is an admissible function depending only on \( f \).

**Proof of Theorem 1.** Assuming Theorem 2, let \( b_j \) and \( q_j \) be the convergents from the Diophantine condition (1.2) in Theorem 1. Let \( N_j := \left\{ \frac{q_j}{b_j} \sqrt{n} \equiv \alpha \mod 1 \right\} \).

By the comparison lemma, with
\[ N'_j := \{ \alpha \sqrt{n} (\mod 1) \}_{n \leq N_j} \]
\[ N_j := \{ \frac{b_j}{q_j} \sqrt{n} (\mod 1) \}_{n \leq N_j}. \]

We have that the difference
\[
\left| R^{(m)}(N_j, \alpha, f) - R^{(m)}(N_j, \frac{b_j}{q_j}, f) \right| \leq R^{(m)}(N_j, f_+) \epsilon(N_j, N'_j). \quad (2.3)
\]

Now note that unless \( n = 0 \) then, for \( n < N_j \), we have \( \min \left\{ \left| \frac{1}{q_j} \sqrt{n} \right| \right\} > \frac{r}{q_j}. \) That is, write
\[
\frac{b_j}{q_j} \sqrt{n} = \frac{b_j}{q_j} \sqrt{\left( \frac{q_j b_j}{b_j} \right)^2 + r b_j^2}
\]
where \( r, k \in \mathbb{Z} \) and \( r < q_j^2 \) and \( k < \frac{b_j}{q_j} \sqrt{n} \). Therefore, Taylor expanding the square-root gives:
\[
\frac{b_j}{q_j} \sqrt{n} = k + \frac{r}{2q_j^2 k} - \frac{3r^2}{4q_j^4 k^3} + ...
\]
from which it follows that \( \{ b_j \sqrt{n}/q_j \} > \frac{r}{q_j^2} \), one can perform a similar calculation to show 1 - \( \left\{ b_j \sqrt{n}/q_j \right\} > \frac{r}{q_j^2} \).

Therefore we can use (1.2) to write
\[
\left| \{ \alpha \sqrt{n} \} - \left\{ \frac{b_j}{q_j} \sqrt{n} \right\} \right| \leq \psi(q_j) \sqrt{n} \leq \psi(q_j) \sqrt{N_j},
\]

Therefore \( \epsilon(N_j, N'_j) \leq \psi(q_j) N_j^{3/2} \to 0 \). Hence Theorem 1 follows from Theorem 2 and (2.3).

\[ \square \]
In the remainder of the paper we prove Theorem 2.

3 Fourier Decomposition

The first step of the proof is to decompose \( R^{(m)}(N, \frac{b}{q}, f) \) into Fourier components and isolate the main term, let

\[
g(x) := \sum_{k \in \mathbb{Z}^{m-1}} f(N\left(\frac{b}{q}\sqrt{x_1} - \frac{b}{q}\sqrt{x_2} + k_1\right), \ldots, N\left(\frac{b}{q}\sqrt{x_{m-1}} - \frac{b}{q}\sqrt{x_m} + k_{m-1}\right)).
\]  

(3.1)

Then

\[
R^{(m)}(N, \frac{b}{q}, f) = \frac{1}{N} \sum_{x \in \{1, \ldots, N\}^m} \sum_{r \pmod{q}} \hat{g}(r) e\left(\frac{x \cdot r}{q}\right)
\]  

(3.2)

where the \( * \) over the sum signifies that the components \( x_i \) are distinct, and

\[
\hat{g}(r) := \frac{1}{q^m} \sum_{y \in \{1, \ldots, q\}^m} g(y) e\left(-\frac{r \cdot y}{q}\right),
\]

\[F_N(r) := \sum_{x \in \{1, \ldots, N\}^m} e\left(\frac{r \cdot x}{q}\right).
\]  

(3.3)

With that we separate \( R^{(m)} \) into two components: a main term with \( r = 0 \) and a remainder term for all other \( r \). That is, we write

\[
R^{(m)}(N, \frac{b}{q}, f) = \mathcal{M} + \mathcal{E},
\]  

(3.4)

where

\[
\mathcal{M} := N^{m-1} \frac{1}{q^m} \sum_{y \in \{1, \ldots, q\}^m} g(y),
\]

\[
\mathcal{E} := \frac{1}{N} \sum_{0 \neq r \pmod{q}} \hat{g}(r) F_N(r).
\]  

(3.5)

4 Exponential Sums

To control both \( \mathcal{M} \) and \( \mathcal{E} \) we rely on bounds for the following exponential sums:

\[
E_{k,r} := \left| \sum_{y \in \{1, \ldots, q\}} e\left(\frac{kb}{q}\sqrt{y} - \frac{r_1 y}{q}\right) \right|.
\]  

(4.1)

These bounds are slightly different depending on \( r_1 \). We collect all the necessary bounds in the following theorem:

**Theorem 4.** Let \( \frac{b}{q} \to \alpha \) as \( q \to \infty \), let \( 0 < |k| \leq Cq \) for any fixed constant \( C < \infty \) independent of \( q \) and let \( r_1 \in \{1, \ldots, q-1\} \). Then as \( q \to \infty \):
• If \( kb/2 \equiv r_1 \mod q \):
\[
E_{k,r_1} \ll \frac{q^{3/4}}{|k|^{1/2}}. \quad (4.2)
\]

• If not:
\[
E_{k,r_1} \ll \log(q) + \frac{q^{1/4}}{|k|^{1/2}} + \frac{q^{1/2}}{|k|}. \quad (4.3)
\]

For \( r = 0 \), for any fixed \( \epsilon > 0 \), (4.2) and (4.3) hold for \( |k| > q^{1/2+\epsilon} \), however if \( 0 < |k| < q^{1/2+\epsilon} \) then for any \( k \):
\[
E_{k,0} \ll \frac{q^{3/4}}{|k|^{1/2}}. \quad (4.4)
\]

The rest of this section is devoted to the proof of Theorem 4, which is based on Van der Corput’s B-Process – see [Mon94, Chapter 3] for an excellent reference. However, for our problem, the domain of the sum in (4.1) is very large. As a consequence we do not have good control on the derivatives of the exponent. Hence we need to prove the stationary phase integrals rather than cite a known reference. Then we will use some classical analytic tools to control the various terms which arise. For simplicity of notation we assume \( k > 0 \). Then to achieve the same bounds for negative \( k \) take complex conjugates.

### 4.1 Analytic Tools

For a real-valued function \( g(x) \) defined on \([A,B]\) we define:
\[
V(g) := \max_{(A,B)} |g(x)| + t.v(g(x)), \quad (4.5)
\]
where \( t.v(g(x)) := \int_A^B |g'(x)| \, dx \). For a proof of the following classical lemma see [Hux96, Lemma 5.1.4]

**Lemma 5** (Van der Corput’s \( k^{th} \) Derivative Test). Let \( f(x) \) be real and \( k \) times differentiable on \((A,B)\) with \( f^{(r)}(x) \geq \mu > 0 \) on \((A,B)\). Let \( g(x) \) be a real valued function. Then:
\[
\left| \int_A^B g(x) e(f(x)) \, dx \right| \ll \frac{V(g)}{\mu^{1/r}}. \quad (4.6)
\]

#### 4.1.1 Stationary Phase Integrals

If \( \frac{kb}{2\sqrt{q}} \leq r \leq \frac{kb}{2q} \), then the function \( h_{k,r}(x) = \frac{kb}{q} \sqrt{\tau - \frac{k}{4}} \) has a stationary point, \( \gamma_{k,r} = \left(\frac{q}{kb}\right)^2 \in (1,q) \) where \( h_{k,r}'(\gamma_{k,r}) = 0 \). In that case the following is an application of classical stationary phase integrals (see for example [Hux96, Lemma 5.5.2]):

**Proposition 6** (Stationary Phase Integral). Let \( r \in \mathbb{Z} \) be an integer such that \( \frac{kb}{2\sqrt{q}} \leq r < \frac{kb}{2} \), let \( k \in \{1,2,\ldots,q\} \), and let \( 0 < c_1 < 1 \) be any fixed constant. Then in the limit as \( q \to \infty \):
\[
\int_1^q e(h_{k,r}(x)) \, dx = e\left(\frac{(kb)^2}{4qr} + \frac{1}{8} \right) \frac{2\pi}{k^{1/2}} h_{k,r}'(1) + O\left(E_{k,r_1} + q^{1/2} \left(\frac{kb}{2q} - \frac{r}{q}\right)^{-2} \left(\frac{q}{\frac{kb}{2}}\right)^{c_1}\right). \quad (4.7)
\]
Proof. For this proof we will suppress the subscripts \( k, r \) whenever they appear. To prove (4.7) we fix two constants \( 0 < c_1 < 1 < c_2 < \infty \) independent of \( q \) and fix \( u := \max\{1, c_1 \gamma \} \) and \( v := \min\{q, c_2 \gamma \} \). Now we consider the three integrals on \((1, u), (u, v), \) and \((v, q)\) separately.

**Step 1:** To address the integral on \((1, u)\), we may assume \( u > 1 \) (and thus \( r < \frac{kq - r}{q} \)). Applying the first integral test gives:

\[
\int_1^u e(h(x))dx = \left[ \frac{e(h(x))}{2\pi i h'(x)} \right]_{x=1}^u + \frac{1}{2\pi i} \int_1^u \frac{h''(x)e(h(x))}{h'(x)^2} dx
\]

\[
= \left[ \frac{e(h(x))}{2\pi i h'(x)} \right]_{x=1}^u + \mathcal{O}\left(V \left( \frac{h''(x)}{h'(x)^2} \right) (\min |h'(x)|)^{-1} \right)
\]

Now note that on \((1, u)\), the derivative \( |h'(x)| \geq C \frac{k}{\gamma/2} \) where the constant \( C = \frac{1}{q} \left( \frac{1}{v_{c_1}} - 1 \right) \). Thus:

\[
\int_1^u e(h(x))dx = \left[ \frac{e(h(x))}{2\pi i h'(x)} \right]_{x=1}^u + \mathcal{O}\left( \frac{k^{1/2}}{k} \left( \frac{k b}{q} - \frac{r}{q} \right)^{2} + \frac{1}{k^{1/2}} \right)
\]

\[\text{(4.8)}\]

**Step 2:** The proof for \((v, q)\) is similar: First we use the first derivative test (Lemma 5) to show:

\[
\int_v^q e(h(x))dx = \left[ \frac{e(h(x))}{2\pi i h'(x)} \right]_{x=v}^q + \frac{1}{2\pi i} \int_v^q \frac{h''(x)e(h(x))}{h'(x)^2} dx
\]

\[
= \left[ \frac{e(h(x))}{2\pi i h'(x)} \right]_{x=v}^q + \mathcal{O}\left(V \left( \frac{h''(x)/h'(x)^2}{\min |h'(x)|} \right) \right)
\]

Moreover, evaluating \( V \) gives:

\[
V \left( \frac{h''(x)/h'(x)^2}{\min |h'(x)|} \right) = \mathcal{O}\left( \frac{1}{k^2} \right) .
\]

\[\text{(4.10)}\]

Therefore:

\[
\int_v^q e(h(x))dx = \left[ \frac{e(h(x))}{2\pi i h'(x)} \right]_{x=v}^q + \mathcal{O}\left( \frac{1}{k^2} \right) .
\]

\[\text{(4.11)}\]

**Step 3:** Working in \([u, v]\), set \( T = \frac{kq - r}{q} \) and \( M = \gamma \). Then for \( j = 2, 3, 4 \), there exist constants \( 0 < C_j < \infty \) such that

\[
|h^{(j)}(x)| \leq C_j \frac{T}{M^j} , \quad h''(x) \geq h''(v) \geq C_2^{-1} \frac{T}{M^2}
\]

Thus we can apply a classical stationary phase integral [Hux96, Lemma 5.5.2] to say:
\[ \int_v^u h(x) \, dx = e^{h(\gamma) + 1/8} + e^{h(u)} - e^{h(u)} + O\left( \frac{M^2}{T^{3/2}} \frac{M^4}{T^2} \left( \frac{h''(\gamma)}{k\eta} \right)^3 \right) \]

\[ = e^{h(\gamma) + 1/8} + e^{h(u)} + e^{h(u)} + O\left( \frac{\gamma^{3/4}}{k^{3/4}} \frac{\gamma^3}{k^2} \left( \frac{h''(\gamma)}{k\eta} \right)^3 \right) \]

(4.12)

(4.7), now follows directly from this bound together with (4.9) and (4.11).

4.2 Proof of Theorem 4

We write \( E := E_{k,r} \).

Proof of (4.3):

Our goal is to estimate

\[ E := \left| \sum_{y \in \{1, \ldots, q\}} e\left( \frac{kb}{q} \sqrt{y} - \frac{r_1 y}{q} \right) \right|. \]

Step 1: First, apply the truncated Poisson summation formula [Hux96, Lemma 5.4.3]: For \( A := \frac{kb}{2\sqrt{q}} \) and \( B := \frac{kb}{2q} \),

\[ E = \sum_{q(A - \frac{1}{2}) < r < q(B + \frac{1}{2})} \int_1^q e\left( f_k(x) - \frac{r x}{q} \right) \, dx + O(\log(q)). \]

Now we apply Proposition 6, the stationary phase integral:

\[ E = \sum_{q(A - \frac{1}{2}) < r < q(B + \frac{1}{2})} \left( e\left( \frac{(kb)^2}{4qr} + \frac{1}{8} \right) \frac{\gamma_{k,r}^{3/4}}{k^{1/2}} - \frac{e(h_{k,r}(1))}{2\pi i h_{k,r}(1)} + \frac{e(h_{k,r}(q))}{2\pi i h_{k,r}(q)} \right) + O\left( q^{1/4} \left( \frac{q^{1/2} + \log(q)}{k^{1/2}} \right) \right). \]

Note that we have used the fact that there are order \( k \) terms in the sum over \( r \) to pull one of the error terms out of the sum. Moreover the other error term inside the sum satisfies:

\[ q^{1/2} \sum_{q(A - \frac{1}{2}) < r < q(B + \frac{1}{2})} \left( \frac{1}{q} \right) \leq q^{1/2} \sum_{A - \frac{1}{2} < r < B} \left( \frac{1}{q} \right) \leq q^{1/2} \frac{1}{k}. \]

Therefore

\[ E = \sum_{q(A - \frac{1}{2}) < r < q(B + \frac{1}{2})} \left( e\left( \frac{(kb)^2}{4qr} + \frac{1}{8} \right) \frac{\gamma_{k,r}^{3/4}}{k^{1/2}} - \frac{e(h_{k,r}(1))}{2\pi i h_{k,r}(1)} + \frac{e(h_{k,r}(q))}{2\pi i h_{k,r}(q)} \right) + O\left( q^{1/4} \left( \frac{1}{k^{1/2}} + \frac{1}{k} + \log(q) \right) \right). \]
**Step 2:** The latter two terms in the sum can both be controlled in the same way (here we use that $kb/2 \neq r_1 \pmod q$):

\[
\sum_{q(A-\frac{1}{4}) \leq r < q(B+\frac{1}{4})} e(h_{k,r}(1)) \ll \sum_{A-\frac{1}{4} < r + q < B+\frac{1}{4}} \frac{1}{2\pi h_{k,r}(1)} \ll \log(q).
\] (4.13)

The same bound holds for the last term in the sum. Thus:

\[
E = \sum_{q(A-\frac{1}{4}) < r < q(B+\frac{1}{4})} e\left(\frac{(kb)^2}{4qr} + \frac{1}{8} \frac{\gamma_{k,r}}{k^{1/2}} + \mathcal{O}\left(\frac{q^{1/4}}{k^{1/2}} + q^{1/2} + \log(q)\right)\right).
\]

**Step 3:** Inserting the definition of $\gamma_{k,r}$, one can write the remaining term as:

\[
\bar{E} := \frac{e(1/8)kb^{3/2}}{2^{3/4}} \sum_{q(A-\frac{1}{4}) < r_1 + qr < q(B+\frac{1}{4})} e\left(\frac{(kb)^2}{4q(r_1 + qr)}\right) \frac{1}{(r_1 + qr)^{3/2}}.
\]

Now taking absolute values (and letting $\bar{A} := \min\{r : r_1 + qr > q(A - \frac{1}{4})\} \geq 0$ and $\bar{B} := \max\{r : r_1 + qr < q(B + \frac{1}{4})\}$)

\[
\bar{E} \ll kq^{3/2} \sum_{r = \bar{A}}^{\bar{B}} e\left(\frac{(kb)^2}{4q(r_1 + qr)}\right) \frac{1}{(r_1 + qr)^{3/2}}.
\]

Applying the Poisson summation formula then gives:

\[
\bar{E} \ll kq^{3/2} \sum_{l \in \mathbb{Z}} \int_{\bar{A}}^{\bar{B}} e\left(\frac{(kb)^2}{4q(r_1 + qr)}\right) \frac{1}{(r_1 + qr)^{3/2}} e(lx)dx.
\]

**Step 4:** First we separate out the $l = 0$ and change variables

\[
\bar{E} \ll kq^{1/2} \int_{q(A+r_1)}^{q(B+r_1)} \frac{(kb)^2}{x^{3/2}} dx + kq^{3/2} \sum_{l \in \mathbb{Z}} \int_{\bar{A}}^{\bar{B}} e\left(\frac{(kb)^2}{4q(r_1 + qr)}\right) \frac{1}{(r_1 + qr)^{3/2}} e(lx)dx.
\]

(if we were in the case $r_1 = 0$ for $k < q^{0.5+\epsilon}$, then the lower bound on the integral would be 0 which would cause a problem). Now we apply integration by parts to the term for $l = 0$ to get

\[
\bar{E} \ll k^{1/2} + kq^{3/2} \sum_{l \in \mathbb{Z}} \int_{\bar{A}}^{\bar{B}} e\left(\frac{(kb)^2}{4q(r_1 + qr)}\right) \frac{1}{(r_1 + qr)^{3/2}} e(lx)dx.
\]

Finally, to address the $|l| \geq 1$ term, if $|l| \geq q$ we can use the decay of Fourier coefficients for smooth functions to bound the terms in the sum. For $|l| \leq q$ we integrate by parts and note that the boundary term for $l$ will cancel with the boundary term for $-l$ (since $A, B \in \mathbb{Z}$). Thus:
4.2 to get that term is bounded by

$$E \lesssim \frac{1}{k^{1/2}} + \left| \sum_{l \in [-q,q]} \int_{\mathcal{A}} \left( e(lx) \frac{x}{l} \left( \frac{kb}{2q(r_1 + qx)} \right)^2 \frac{2(r_1 + qx)^{3/2}}{(r_1 + qx)^2} \right) dx \right|. $$

Now set $\theta(x) := \frac{(kb)^2}{4q(r_1 + qx)}$ and $\varphi(x) = \frac{(kb)^2}{2(r_1 + qx)^{3/2}} + \frac{q}{(r_1 + qx)^{5/2}}$, and integrate by parts a second time to give:

$$\tilde{E} \lesssim \frac{1}{k^{1/2}} + \left| \sum_{l \in [-q,q]} \int_{\mathcal{A}} \frac{d}{dx} \left( e(lx) \frac{x}{l} \theta(x) \right) \frac{x}{l} \left( \varphi(x) \theta'(x) \right) dx \right|$$

$$\lesssim \frac{1}{k^{1/2}} + \left| \sum_{l \in [-q,q]} \left\{ \left( \frac{\varphi(x)}{l^2 \theta'(x)} \right) + \frac{\varphi'(x)}{l \theta'(x)} + \left( \frac{\varphi(x) \theta''(x)}{l^2 \theta'(x)^2} \right) \right\} \right|,$$

where we have used the following bounds:

$$\varphi(\tilde{A}) \ll \frac{1}{k^{3/2}}, \quad \varphi(\tilde{B}) \ll \frac{1}{(kb)^{3/2}}, \quad \theta'(\tilde{A}) \ll q, \quad \theta'(\tilde{B}) \ll 1,$$

$$\varphi'(\tilde{A}) \ll \frac{q^{3/4}}{k^{3/2}}, \quad \varphi'(\tilde{B}) \ll \frac{1}{k^{5/2} q^{3/2}}, \quad \theta''(\tilde{A}) \ll \frac{q^{3/2}}{k}, \quad \theta''(\tilde{B}) \ll \frac{1}{k}.$$

**Proof of (4.2):** The proof of (4.2) follows similar lines to the proof of (4.3) with one exception. The reason the same proof does not work is in step 2 the sum in (4.13) blows up. To address this we start the estimate in the same way, by applying the Poisson summation formula to get:

$$E = \sum_{q(A - \frac{1}{2}) < r < q(B + \frac{1}{2})} \int_{1}^{q} e \left( f_k(x) - \frac{rx}{q} \right) dx + O(\log(q))$$

however, now we remove the term $r = \frac{kb}{2q} - \frac{1}{q}$ from the sum, and to that term we apply the second derivative test: Lemma 5 to get that that term is bounded by $\frac{q^{3/4}}{k^{3/2}}$. The rest of the proof follows exactly the same lines.

**Proof of (4.4):** The proof of (4.4) is the same as the proof of (4.2). That is, the problem with applying the original proof of (4.3) to (4.4) is, after the truncated Poisson summation formula, when $r = 0$. Therefore we separate that term from the sum and apply Lemma 5 to get the bound $\frac{q^{3/4}}{k^{3/2}}$. All other terms in the sum can be handled using the original method.
5 The Main Term

Recall the main term

\[ M = \frac{N^{m-1}}{q^m} \sum_{y \in \{1\ldots q\}^m} \sum_{k \in \mathbb{Z}^{m-1}} f(N(\beta(y) + k)). \]  

(5.1)

where \( \beta(y) := (\beta_1(y), \ldots, \beta_{m-1}(y)) \) and \( \beta_i(y) = \frac{b}{q} (\sqrt{y_i} - \sqrt{y_{i-1}}) \).

5.1 The Main Term for \( m = 2 \)

First apply Poisson summation to the sum over \( k \) to get

\[ M = \frac{N}{q^2} \sum_{y \in \{1\ldots q\}^2} \sum_{k \in [-q,q]^*} e(k\beta(y)) \int_{\mathbb{R}} f(Nx)e(xk)dx + O\left( \frac{N}{q^2} \sum_{y \in \{1\ldots q\}^2} \sum_{k \in \mathbb{Z} \setminus |k| > q} e^{-k/N} \right) \]

\[ = \frac{N}{q^2} \sum_{y \in \{1\ldots q\}^2} \sum_{k \in [-q,q]^*} e(k\beta(y)) \int_{\mathbb{R}} f(Nx)e(xk)dx + O\left( Ne^{-q/N} \right) \]

Note that since \( f \) is \( C_c^\infty \), the Fourier coefficients decay exponentially, which gives the error term (which is \( o(1) \)). Now split \( M \) into two parts with \( k = 0 \) and \( |k| > 0 \):

\[ M = \int_{\mathbb{R}} f(x)dx + \frac{N}{q^2} \sum_{y \in \{1\ldots q\}^2} \int_{\mathbb{R}} f(Nx)e(xk)dx \sum_{y \in \{1\ldots q\}^2} e(k\beta(y)) + o(1) \]

\[ = \int_{\mathbb{R}} f(x)dx + O\left( \frac{1}{q^2} \sum_{k \in [-q,q]^*} e\left( \frac{kh}{q}\sqrt{y_1} \right) \right) + o(1) \]  

(5.2)

Now we can apply Theorem 4 to bound the exponential sum:

\[ M = \int_{\mathbb{R}} f(x)dx + O\left( \frac{1}{q^2} \sum_{k \in [-q,q]^*} E_{k,0}^2 \right) + o(1) \]

\[ = \int_{\mathbb{R}} f(x)dx + O\left( \frac{\log(q)}{q^{7/2}} \right) + o(1) \]  

(5.3)

Therefore the main term converges to exactly the limit we require.

5.2 The Main Term for \( m \geq 2 \)

We want to bound

\[ M = \frac{N^{m-1}}{q^m} \sum_{y \in \{1\ldots q\}^m} \sum_{k \in \mathbb{Z}^{m-1}} f(N(\beta(y) + k)). \]  

(5.4)

Once again the strategy is to apply Poisson summation to each sum over \( k \):

\[ M = \frac{N^{m-1}}{q^m} \sum_{y \in \{1\ldots q\}^m} \sum_{k \in \mathbb{Z}^{m-1}} e(k \cdot \beta(y)) \int_{\mathbb{R}^{m-1}} f(Nx)e(k \cdot x)dx. \]  

(5.5)
Since \( f \in C_c^\infty \) we can again apply the decay of Fourier coefficients to bound the range of \( k_i \), and separate the \( k = 0 \) term:

\[
\mathcal{M} = \int_{\mathbb{R}^{m-1}} f(x)dx + \frac{N^{m-1}}{q^m} \sum_{y \in \{1, \ldots, q\}^m} \sum_{0 \neq k \in \mathbb{Z}^{m-1}} |k_i| \leq q \ e(k \cdot \beta(y)) \int_{\mathbb{R}^{m-1}} f(Nx)e(k \cdot x)dx + o(1). \tag{5.6}
\]

For the sake of notation, denote:

\[
\mathcal{M}_0 := \frac{N^{m-1}}{q^m} \sum_{y \in \{1, \ldots, q\}^m} \sum_{0 \neq k \in \mathbb{Z}^{m-1}} |k_i| \leq q \ e(k \cdot \beta(y)) \int_{\mathbb{R}^{m-1}} f(Nx)e(k \cdot x)dx
\]

\[
\ll \frac{1}{q^m} \sum_{0 \neq k \in \mathbb{Z}^{m-1}} |k_i| \leq q \ \left| \sum_{y \in \{1, \ldots, q\}^m} e(k \cdot \beta(y)) \right| = \frac{1}{q^m} \sum_{0 \neq k \in \mathbb{Z}^{m-1}} \prod_{i=1}^m \left| \sum_{y \in \{1, \ldots, q\}} e\left( \frac{(k_i - k_{i-1})b}{q} \sqrt{y_i} \right) \right| = \frac{1}{q^m} \sum_{0 \neq k \in \mathbb{Z}^{m-1}} \prod_{i=1}^m E(k_i - k_{i-1}), \tag{5.7}
\]

where we write \( k_0 = k_m = 0 \). Now write \( d(k) := k_{m-1} - k_{m-2} + \cdots \pm k_1 \) and change variables to give:

\[
\mathcal{M}_0 \ll \frac{1}{q^m} \sum_{0 \neq k \in \mathbb{Z}^{m-1}} E_{k_1,0} E_{k_2,0} \cdots E_{k_{m-1},0} E_{d(k),0}. \tag{5.7}
\]

Now we estimate the final term \( E_{d(k),0} \) using Theorem 4 and the trivial bound \( E_{0,0} = q \):

\[
\mathcal{M}_0 \ll \frac{q^{3/4}}{q^m} \sum_{0 \neq k \in \mathbb{Z}^{m-1}} E_{k_1,0} E_{k_2,0} \cdots E_{k_{m-1},0} 1(d(k) \neq 0) + \frac{1}{q^{m-1}} \sum_{0 \neq k \in \mathbb{Z}^{m-1}} E_{k_1,0} E_{k_2,0} \cdots E_{k_{m-1},0} 1(d(k) = 0)
\]

To this we can apply Theorem 4 (and the trivial bound \( E_{0,0} = q \), giving:

\[
\mathcal{M}_0 \ll \frac{q^{3/4}}{q^m} \prod_{i=1}^{m-1} (q + q \log(q)) + \frac{1}{q^{m-1}} q^{3/4} \prod_{i=1}^{m-2} (q + q \log(q)) = O\left( \frac{(\log(q))^m}{q^{1/4}} \right).
\]

That is, in both terms all of the factors of \( q + q \log(q) \) come from a direct application of Theorem 4 with the \( q \) coming from the small \( k \) contribution and the \( q \log(q) \) coming from the large \( k \) contribution. Moreover, the \( q^{3/4} \) in the second term comes from the condition \( d(k) = 0 \) and \( k \neq 0 \). Therefore, we conclude:

\[
|\mathcal{M}_0| = o(1). \tag{5.8}
\]

Therefore, inserting (5.8) into (5.6) gives:

\[
\mathcal{M} = \int_{\mathbb{R}^{m-1}} f(x)dx + o(1) \tag{5.9}
\]

and all that remains to prove Theorem 2 is to show that \( |E| \to 0 \).
6 The Remainder

Recall the remainder term from (3.5),

\[ E = \frac{1}{Nq^m} \sum_{0 \neq r \pmod{q}} F_N(r) \sum_{y \in \{1, \ldots, q\}^m} g(y)e\left(-\frac{r \cdot y}{q}\right). \]  

(6.1)

6.1 The Remainder Term \((m = 2)\)

First we consider the error for the two point correlation:

\[ E = \frac{1}{Nq} \sum_{0 \neq r \pmod{q}} F_N(r) \sum_{y \in \{1, \ldots, q\}} \sum_{k \in \mathbb{Z}} f\left(N(\beta(y) + k) - \frac{r \cdot y}{q}\right). \]  

(6.2)

Inserting the definition of \(g(y)\):

\[ E = \frac{1}{Nq} \sum_{0 \neq r \pmod{q}} F_N(r) \sum_{y \in \{1, \ldots, q\}} \sum_{k \in \mathbb{Z}} f\left(N(\beta(y) + k) - \frac{r \cdot y}{q}\right) \int_{\mathbb{R}} f(Nx)e(xk)dx + O\left(q^{-1}\right). \]

Applying the Poisson summation formula to the sum over \(k\)

\[ E = \frac{1}{Nq^2} \sum_{0 \neq r \pmod{q}} F_N(r) \left( \sum_{y \in \{1, \ldots, q\}} \sum_{k \in \mathbb{Z}} e\left(k\beta(y) - \frac{r \cdot y}{q}\right) \int_{\mathbb{R}} f(Nx)e(xk)dx \right) + O\left(q^{-1}\right) \]

Note that if \(k = 0\) then, since \(r \neq 0\), the sum over \(y\) is 0:

\[ \sum_{y \in \{1, \ldots, q\}^2} e\left(-\frac{r \cdot y}{q}\right) \int_{\mathbb{R}} f(Nx)dx = 0. \]

Therefore

\[ |E| \leq \frac{1}{Nq^2} \sum_{0 \neq r \pmod{q}} |F_N(r)| \left( \frac{1}{N} \sum_{y \in \{1, \ldots, q\}^2} \sum_{k \in \mathbb{Z}} e\left(k\beta(y) - \frac{r \cdot y}{q}\right) \right) + O\left(q^{-1}\right) \]

\[ = \frac{1}{N^2q^2} \sum_{0 \neq r \pmod{q}} |F_N(r)| \left( \sum_{k \in \mathbb{Z}^*} E_{k,r_1}E_{k,r_2} + O\left(\frac{1}{qN}\right) \right). \]

Summing over \(k\): First if \(r_1 = 0\) (and by symmetry if \(r_2 = 0\)) then we can apply (4.4) to the sum over \(r_1\):

\[ \sum_{k \in \mathbb{Z}^*} E_{k,0}E_{k,r_2} \lesssim \sum_{k \in \mathbb{Z}^*} E_{k,0}E_{r_2} + \sum_{k \in \mathbb{Z}^*} E_{k,0}E_{k,r_2} \lesssim \sum_{k \in \mathbb{Z}^*} E_{k,0}E_{r_2} \lesssim q^{5/4}. \]

Now for \(r_1, r_2 > 0\) we simply apply (4.2) and (4.3) to the sum to get:
\[
\sum_{k \in \mathbb{Z}^*} E_{k,r_1} E_{k,r_2} \ll q^{3/2}.
\]

**Final bound:** Now recall the definition of \( F_N(r) \):

\[
\sum_{0 \neq r \pmod{q}} |F_N(r)| = \sum_{0 \neq r \pmod{q}} \left| \sum_{x \in \{1, \ldots, N\}^2} e\left(\frac{r \cdot x}{q}\right) \right|
\leq \sum_{0 \neq r \pmod{q}} \left| \sum_{x \in \{1, \ldots, N\}^2} e\left(\frac{r \cdot x}{q}\right) \right| + \sum_{0 \neq r \pmod{q}} \left| \sum_{x \in \{1, \ldots, N\}} e\left(\frac{(r_1 + r_2) x}{q}\right) \right| 
\leq \sum_{0 \neq r \pmod{q}} \prod_{i=1}^2 \min \left\{N, \frac{q}{r_i}\right\} + \sum_{0 \neq r \pmod{q}} \min \left\{N, \frac{q}{r_1 + r_2}\right\}
\leq q^2 \log(q)^2.
\]

Therefore, we conclude:

\[
|\mathcal{E}| \ll \frac{q^{3/2} \log(q)^2}{N^2} \to 0.
\]

### 6.2 The Remainder Term \((m \geq 2)\)

As we did for the main term, applying the Poisson summation formula to sum over \( k \) in the remainder gives

\[
\mathcal{E} = \frac{1}{N^m q^m} \sum_{0 \neq r \pmod{q}} F_N(r) \sum_{y \in \{1, \ldots, q\}^m} e\left(\frac{-r \cdot y}{q}\right) \sum_{k \in \mathbb{Z}^{m-1}} e(k \cdot \beta(y)) \int_{\mathbb{R}^{m-1}} f(Nx)e(k \cdot x)dx.
\]

Once more we can use the decay of Fourier coefficients to assume \( k_i < q \). Moreover if \( k = 0 \) then the sum over \( y \) will give 0. Thus

\[
|\mathcal{E}| \ll o(1) + \frac{1}{N^m q^m} \sum_{0 \neq r \pmod{q}} |F_N(r)| \sum_{0 \neq k \in \mathbb{Z}^{m-1}, y \in \{1, \ldots, q\}^m} e\left(\frac{-r \cdot y}{q}\right) e(k \cdot \beta(y))
\leq o(1) + \frac{1}{N^m q^m} \sum_{0 \neq r \pmod{q}} |F_N(r)| \sum_{0 \neq k \in \mathbb{Z}^{m-1}} \prod_{i=1}^m \left| \sum_{y \in \{1, \ldots, q\}} e\left(\frac{(k_i - k_{i-1}) b}{q} \sqrt{y_i} - \frac{r_i y_i}{q}\right) \right|
\leq o(1) + \frac{1}{N^m q^m} \sum_{0 \neq r \pmod{q}} |F_N(r)| \sum_{0 \neq k \in \mathbb{Z}^{m-1}} E_{k_1, r_1} \cdots E_{k_{m-1}, r_{m-1}} E_{d(k), r_m},
\]

where we recall that \( d(k) = k_{m-1} - k_{m-2} + \cdots + k_1 \).

**Bounding** \( |F_N(r)| \): Turning now to the term \( F_N(r) \) we note that by the inclusion-exclusion principle:

\[
|F_N(r)| = \left| \sum_{x \in \{1, \ldots, N\}^m} e\left(\frac{r \cdot x}{q}\right) \right|
\leq \left| \sum_{x \in \{1, \ldots, N\}^m} e\left(\frac{r \cdot x}{q}\right) \right| + \left| \sum_{x \in \{1, \ldots, N\}^m} e\left(\frac{r \cdot x}{q}\right) \right| + \cdots + \left| \sum_{x \in \{1, \ldots, N\}^m} e\left(\frac{r \cdot x}{q}\right) \right| (6.6)
\]
where the notation \( \sum_{j} \) imposes the condition that at least \( j \) terms in the sum must be equal. Now we apply the bound \( \left| \sum_{x \in \{1, \ldots, N\}} e\left(\frac{ax}{N}\right) \right| \leq \prod_{i=1}^{m} \min\{N, \frac{q_i}{r_i}\} \) and similar for the other sums. Therefore, for any \( 0 \leq k < m \) we have

\[
\sum_{0 \neq r \pmod{q}} |F_N(r)| \cdot 1(r_1 = \cdots = r_k) = q^{m-k} \log(q)^{m-k} N^k. \tag{6.7}
\]

Which will prove to be exactly what we need.

**If** \( d(k) = 0 \): If \( d(k) = 0 \) then \( E_{0,r_m} = 0 \) unless \( r_m = 0 \). Thus, setting \( r' = (r_1, \ldots, r_{m-1}) \):

\[
\frac{1}{N^m q^m} \sum_{0 \neq r' \pmod{q}} |F_N(r')| \sum_{0 \neq k \in \mathbb{Z}_{m-1}^m \mid |k_i| < 2q} E_{k_1, r_1} \cdots E_{k_{m-1}, r_{m-1}} E_{0,0} 3(d(k) = 0)
= \frac{1}{N^m q^m} \sum_{0 \neq r' \pmod{q}} |F_N(r')| \sum_{0 \neq k \in \mathbb{Z}_{m-1}^m \mid |k_i| < 2q} E_{k_1, r_1} \cdots E_{k_{m-1}, r_{m-1}} E_{0,0} 3(d(k) = 0).
\]

To this we can apply Theorem 4 and (6.7) to get

\[
= \frac{1}{N^m q^m} (N^q/4) \prod_{i=1}^{m-2} \left( N^{q^{1/2} + 2} + q^2 \log(q)^2 + qN^2 \right),
\]

where the factor \( N^q/4 \) comes from the condition that \( d(k) = 0 \) which restricts one of the sums over \( k_i \), and Theorem 4. While the term \( N^{q^{1/2} + 2} \) comes from \( r_i = 0 \) and \( k_i \neq 0 \), the term \( q^2 \log(q)^2 \) comes from \( r_i > 0 \) and \( k_i \neq 0 \) and the final term, \( qN^2 \) comes from \( k_i = r_i = 0 \). Therefore, choosing \( \epsilon \) appropriately gives:

\[
\frac{1}{N^m q^m} \sum_{0 \neq r' \pmod{q}} |F_N(r')| \sum_{0 \neq k \in \mathbb{Z}_{m-1}^m \mid |k_i| < 2q} E_{k_1, r_1} \cdots E_{k_{m-1}, r_{m-1}} E_{0,0} 3(d(k) = 0) \to 0 \tag{6.8}
\]

**If** \( d(k) > 0 \): if \( d(k) > 0 \) then we can apply Theorem 4 to say that \( E_{d(k),r_m} \ll q^{1/2} \) except for a bounded number of \( r_m \) where \( E_{d(k),r_m} \ll q^{3/4} \). Thus (inserting (6.8))

\[
\mathcal{E} \ll o(1) + \left( \frac{q^{1/2} \log(q)}{N^m q^{m-1}} + \frac{q^{3/4}}{N^m q^m} \right) \sum_{0 \neq r' \pmod{q}} |F_N(r')| \sum_{0 \neq k \in \mathbb{Z}_{m-1}^m \mid |k_i| < 2q} E_{k_1, r_1} \cdots E_{k_{m-1}, r_{m-1}}. \tag{6.9}
\]

Now we can apply Theorem 4, (6.7), and the trivial bound \( E_{0,0} \) to conclude that

\[
\mathcal{E} \ll o(1) + \left( \frac{q^{1/2} \log(q)}{N^m q^{m-1}} + \frac{q^{3/4}}{N^m q^m} \right) \prod_{i=1}^{m-1} \left( N^{q^{1/2} + 2} + q^2 \log(q)^2 + qN \right) \tag{6.10}
\]

where in the final bracket, the first term comes from \( r_i = 0 \) and \( k_i \neq 0 \), the second term comes from the sum for \( r_i > 0 \) and \( k_i \neq 0 \) and the final term comes from \( k_i = r_i = 0 \). Thus for any \( \epsilon > 0 \)

\[
|\mathcal{E}| \ll \frac{q^{m-\frac{1}{2}+\epsilon}}{N^m} + o(1). \tag{6.11}
\]

The largest contribution comes when \( r' = 0 \) Thus, choosing \( \epsilon < \delta/m \) implies \( |\mathcal{E}| \to 0 \). Which completes
the proof of Theorem 2.

Acknowledgements

For part of this work, the author was supported by EPSRC Studentship EP/N509619/1 1793795. Furthermore, the author would like to thank Jens Marklof for several useful discussions, Nick Rome for proof-reading an earlier draft, and Zeev Rudnick for spotting an error in an earlier draft.

References

[BT77] M. Berry and M. Tabor. Level clustering in the regular spectrum. Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, 356(1686):375–394, 1977.

[BV16] T. Browning and I. Vinogradov. Effective Ratner theorem for $SL(2, \mathbb{R}) \ltimes \mathbb{R}^2$ and gaps in $\sqrt{n}$ modulo 1. Journal of the London Mathematical Society. Second Series, 94(1):61–84, 2016.

[EBMV15] D. El-Baz, J. Marklof, and I. Vinogradov. The two-point correlation function of the fractional parts of $\sqrt{n}$ is Poisson. Proceedings of the American Mathematical Society, 143(7):2815–2828, 2015.

[EM04] N. Elkies and C. McMullen. Gaps in $\sqrt{n}$ mod 1 and ergodic theory. Duke Mathematical Journal, 123(1):95–139, 2004.

[HB10] D. R. Heath-Brown. Pair correlation for fractional parts of $\alpha n^2$. Mathematical Proceedings of the Cambridge Philosophical Society, 148(3):385–407, 2010.

[Hux96] M. N. Huxley. Area, lattice points, and exponential sums, volume 13 of London Mathematical Society Monographs. New Series. The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.

[KR99] P. Kurlberg and Z. Rudnick. The distribution of spacings between quadratic residues. Duke Mathematical Journal, 100(2):211–242, 1999.

[Mar00] J. Marklof. The Berry-Tabor conjecture. In Proceedings of the 3rd European Congress of Mathematics, volume 202, pages 421–427, Barcelona, 2000. Birkhäuser, Basel.

[Mar07] J. Marklof. Distribution modulo one and Ratner’s theorem. In A. Granville and Z. Rudnick, editors, Equidistribution in Number Theory, An Introduction, pages 217–244, Dordrecht, 2007. Springer Netherlands.

[Mon94] H. Montgomery. Ten lectures on the interface between analytic number theory and harmonic analysis, volume 84 of CBMS Regional Conference Series in Mathematics. Published for the CBMS, Washington, DC; by the AMS, Providence, RI, 1994.

[RSZ01] Z. Rudnick, P. Sarnak, and A. Zaharescu. The distribution of spacings between the fractional parts of $n^2 \alpha$. Inventiones Mathematicae, 145(1):37–57, 2001.

[Sin12] Ya. G. Sinai. Statistics of gaps in the sequence \{$\sqrt{n}$\}. In Dynamical systems and group actions, volume 567 of Contemp. Math., pages 185–189. Amer. Math. Soc., Providence, RI, 2012.