Finite-time attitude stabilization of an output-constrained rigid spacecraft

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Abstract
This article concentrates on the study of finite-time attitude stabilization for a rigid spacecraft with output constraints. The dynamics of the spacecraft are expressed by modified Rodrigues parameters, so that the singularity of covariance matrix owing to quaternion’s redundancy is eliminated. Based on the backstepping technique in combination with adding a power integrator technique, the attitude stabilization control law is constructed. Rigorous mathematical proof shows that the closed-loop system is finite time stable and all the outputs remain bounded by using a barrier Lyapunov function technique. The effectiveness of the proposed finite-time stabilization scheme is verified by a simulation example.

Keywords
Finite-time stabilization, spacecraft, output constraint, modified Rodrigues parameters, barrier Lyapunov function, backstepping

Introduction
As is well-known, all practical systems are subject to physical constraints. Ignoring these constraints’ effects may result in undesirable performance degradation, safety matters, and even the instability of the system. How to effectively compensate the effect of the constraints in the control design has always been a practically important research task. For these reasons, the study of the control problem for systems subject to constraints has received more and more attention by researchers.1–7

Control problem of the spacecrafts with input constraints has gained increasing attention8–12 in the last decades. Considering the actuator saturation constraints, an adaptive dynamic inverse control was developed to handle the spacecrafts maneuver tracking problem in the study by Tandale and Valasek.13 Ruiter14 presented a nonlinear adaptive controller for spacecraft attitude tracking under actuator saturation. Bustan et al.15 designed a fault-tolerant attitude controller based on variable structure control theory for spacecraft with input saturation and actuator failure. The work within the study by Wang et al.16 described a discrete gain scheduled controller to handle the robust control problem for circular orbital spacecraft rendezvous system with consideration of the input saturation.

Although various schemes have been studied for the constrained control problem, such as neural networks (NNs) control17–19 and adaptive fuzzy control20. However, there have been relatively few works on finite-time attitude stabilization for spacecrafts with output constraints in the literature. In fact, it is ordinary that system outputs have to be bounded within certain ranges for physical application. The violation of the output constraints during operation...
may lead to undesirable performance or system damage. Thus, we should attach great importance to the output constraints from a practical viewpoint, and it is significant to handle the control design problems for the output-constrained systems. To make the outputs remain a bounded range, the barrier Lyapunov functions (BLFs) have been widely proposed to guarantee that constraints are not violated; Tee et al.\textsuperscript{21,22} studied two adaptive controls for nonlinear systems with output constraint. And various types of BLF were appeared, for example, the log-type BLF by Tee et al.,\textsuperscript{21} the integral-type BLF by Tee and Ge,\textsuperscript{23} the tan-type BLF by Jin,\textsuperscript{24} and so on. The work within the study by Ren et al.\textsuperscript{25} described an output feedback-based adaptive neural control algorithm for a class of nonlinear systems with output constraint which can ensure that the closed-loop system is uniformly bounded. In the study by Li and Li,\textsuperscript{26} an adaptive NN controller is proposed for a class of continuous stirred tank reactor with output constraint and uncertainties by using BLF to satisfy the constraints. Tension control as well as vibration control of moving string with consideration of the output constraints is concerned by using BLF technique by He et al.\textsuperscript{27}

In recent years, finite-time control has received much interests from researchers since it can provide more advantages, such as faster convergence rate, higher performance of precision control, and better property of disturbance rejection.\textsuperscript{27–34} The major difference between a finite-time controller and an asymptotical controller is that the former can guarantee the convergence of system states to its equilibrium point in a finite time.\textsuperscript{35} Finite-time control is meaningful to spacecrafts and it leads to enhanced application efficiency.\textsuperscript{36–43} Within the work by Du et al.,\textsuperscript{44} finite-time attitude tracking control of a single spacecraft as well as the finite-time attitude synchronization of a group spacecraft have been concerned by using continuous finite-time control technique to design the control law. In the study by Lu and Xia,\textsuperscript{45} adaptive terminal sliding mode control is utilized to design the finite-time attitude tracking control for rigid spacecraft. In the study by Song et al.,\textsuperscript{46} spacecraft attitude control problem is studied by using a fast terminal sliding mode control technique with double closed loops. Two nonlinear proportional–derivative (PD) controllers were presented by Su and Zheng\textsuperscript{47} to design the global finite-time stabilization law for a rigid spacecraft.

Finite-time stabilization by considering states constraints can improve the control performance. In the study by Xia et al.,\textsuperscript{48} an adaptive fuzzy controller is designed to address the finite-time tracking control problem for a class of strict feedback nonlinear systems with full state constraints. Based on adding a barrier integrator technique, adaptive fuzzy finite-time control strategy is proposed for a class of nonlinear switched systems by Zheng and Li.\textsuperscript{49} In the study by Du et al.,\textsuperscript{50} by using NNS, finite-time theory, and BLFs, an adaptive finite-time control approach is constructed to three degrees of freedom active suspension systems. In this article, the problem of finite-time attitude stabilization of a rigid spacecraft in the face of output constraints is investigated inspired by above discussion. The main work and contributions of the article lie in the following:

1. This work studies the problem of attitude stabilization with output constraints for a rigid spacecraft system. By using BLFs, the system outputs are constrained in a reasonable range, and it further enhances the system safety.
2. The output-constrained problems are considered in the finite-time control design process, which is more general for applications in practical engineering. The desired performance of the closed-loop system can be guaranteed by appropriately selecting the design parameters.
3. The proposed scheme effectively limited the transient overshoot of the system outputs as well as the property of finite-time convergent is achieved, and the finite-time stable of the resulting closed-loop system with output constraints is rigorously proved.

The rest of the article is organized as follows. The second section provides a brief overview of the problem formulation and preliminaries. In the third section, we present the basic idea of the detailed implementation of an adaptive finite-time control for a spacecraft with output constraints. In order to verify the effectiveness of the proposed approach, simulation experiments are performed and analyzed in the fourth section. Finally, concluding remarks are given in the last section.

**Problem statement**

**Kinematics and attitude dynamics**

For the parameterizations of three-dimensional coordinate expression, modified Rodrigues parameters (MRPs)\textsuperscript{51,52} are used in this article to describe the attitude of spacecrafts with respect to an inertial coordinate. It is denoted by \( \sigma = \eta \tan(\frac{\theta}{2}) \in \mathbb{R}^3 (0 \leq \theta \leq 2\pi) \), where \( \eta \) and \( \theta \) denote the Euler eigenaxis and eigenangle, respectively. The attitude kinematics and dynamics for the described rigid spacecraft can be expressed as

\[
\dot{\sigma} = G(\sigma)\omega \tag{1}
\]

\[
J\dot{\omega} + S[\omega]J\omega = u \tag{2}
\]

with

\[
G(\sigma) = \frac{1}{2} \left[ \frac{1 - \sigma^T \sigma}{2} I_3 + S[\sigma] + \sigma \sigma^T \right] \tag{3}
\]

where \( \omega = [\omega_1, \omega_2, \omega_3]^T \in \mathbb{R}^3 \) is the angular velocity vector expressed in the body coordinate. \( J \in \mathbb{R}^{3 \times 3} \) is the constant positive definite symmetric inertia matrix. Let \( S[a] = [0, -a_3, a_2; a_3, 0, -a_1; -a_2, a_1, 0] \) be the skew-symmetric matrix of vector \( a = [a_1, a_2, a_3]^T \). We use \( I_3 \) to
denote the $3 \times 3$ identity matrix and the control torque is represented by $u = [u_1, u_2, u_3]^T \in \mathbb{R}^3$.

**Remark 1.** When $\theta$ approaches $2\pi$, geometric singularity will be occurred by using MRPs representation. Therefore, in order to avoid the kinematic singularity associated with the MRPs, we should choose the desired Euler eigenangle $y_d \leq \sigma_M < 2\pi$, where $\sigma_M$ is a positive constant.

**Remark 2.** Simple algebraic manipulation shows that $G(\sigma)$ in equation (3) has the following properties which will be used later

$$
\sigma^T G(\sigma) \omega = \frac{1}{4} (1 + \sigma^T \sigma) \sigma^T \omega
$$

(4)

$$
G(\sigma) G(\sigma)^T = \frac{1}{4} (1 + \sigma^T \sigma) I_3
$$

(5)

Consider $F(\sigma) = G(\sigma)^{-1}$, we have

$$
\dot{F}(\sigma) = -G(\sigma)^{-1} \dot{G}(\sigma) G(\sigma)^{-1}
$$

(6)

where

$$
\dot{G}(\sigma) = \frac{1}{2} \sigma^T \dot{\sigma} I_3 - \frac{1}{2} S[\dot{\sigma}] + \frac{1}{2} \sigma \sigma^T + \frac{1}{2} \sigma \dot{\sigma}^T
$$

(7)

By defining $x_1 = \sigma$ and $x_2 = \dot{\sigma}$, we can transform the dynamics of spacecraft (1) and (2) into the following form

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -G(x_1) \dot{F}(x_1) x_2 - G(x_1) J^{-1} S [F(x_1) x_2] J F(x_1) x_2 + G(x_1) J^{-1} u \\
y &= x_1
\end{aligned}
$$

(8)

Define

$$
f(\bar{x}_2) = -G(x_1) \dot{F}(x_1) x_2 - G(x_1) J^{-1} S [F(x_1) x_2] J F(x_1) x_2
$$

(9)

$$
g(x_1) = G(x_1) J^{-1}
$$

(10)

where $\bar{x}_2 = [x_1, x_2]^T$, and by substituting equations (6) and (7) into equation (9), one can obtain

$$
f(\bar{x}_2) = -0.5 x_2^T x_2 F(x_1) x_2 - 0.5 S[x_2] F(x_1) x_2
$$

$$
+ 0.5 x_2 x_1^T F(x_1) x_2 + 0.5 x_2 x_1^T F(x_1) x_2
$$

$$
- G(x_1) J^{-1} S [F(x_1) x_2] J F(x_1) x_2
$$

(11)

From the practical point of view, the output $y = [y_1, y_2, y_3]^T \in \mathbb{R}^3$ should be bounded in $|y| \leq k_b$ (output constraint) with positive constant vector $k_b = [k_{b_1}, k_{b_2}, k_{b_3}]^T, \ \forall t \geq 0$.

The main objective of this work is to design a finite-time attitude stabilizer for an output-constrained spacecraft described by systems (1) and (2). The designed finite-time attitude stabilization law can ensure the global finite-time stability of the closed-loop system in the presence of output constraints.

**Preliminaries**

A BLF defined as follows is employed to prevent the outputs of system from violating their constraints.

**Definition 1.** A BLF defined with respect to the system $\dot{x} = f(x)$ on an open region $D$ containing the origin is a scalar function $V(x)$. It is continuous and positive definite and has continuous first-order partial derivatives at every point of $D$. A BLF has the property $V(x) \to +\infty$ as $x$ approaches the boundary of $D$. For $x(t)$ in $D$ and some positive constant $b$, it satisfies $V(x(t)) \leq b \dot{t} \geq 0$ along the solution of $\dot{x} = f(x)$.

**Remark 3.** BLFs are a kind of control Lyapunov functions (CLFs) whose values approach infinity if the state variable or the output tends to the constraint boundary. The logarithmic and tangent BLFs are widely used for constrained control design.\(^5\)

**Lemma 1.** For any constant $0 < k_{b_1}$, $Z_1 := \{z_1 \in R : k_{b_1} > |z_1| \} \subset R$ and $N := R^+ \times Z_1 \subset R^{n+1}$ are open sets.\(^2\) Consider the system, $\xi = H(t, \zeta)$, with the state $\zeta := [w, z_1]^T \in N$ and the function $H : R^+ \times N \to R^{n+1}$ is piecewise continuous in $t$ and locally Lipschitz in $z_1$, uniformly in $t$, on $R_+ \times N$. If there exist positive definite and continuously differentiable functions $U : R^1 \to R_+$ and $V_1 : Z_1 \to R_+$, such that

$$
V_1(z_1) \to \infty as |z_1| \to k_{b_1}
$$

(12)

$$
\gamma_2(||w||) \geq U(w) \geq \gamma_1(||w||)
$$

(13)

where $\gamma_1$ and $\gamma_2$ are class $K_\infty$ functions. Let $V(\eta) := V_1(z_1) + U(w)$, and $z_1(0) \in Z_1$. Suppose that the following inequality holds

$$
\dot{V} = \frac{\partial V}{\partial \zeta} H \leq -\gamma V + \nu
$$

(14)

in the set $\zeta \in N$, and $\gamma$ and $\nu$ are positive constants, then $w$ remains bounded and $z_1(t) \in Z_1 \forall t \in [0, \infty)$.

**Lemma 2.** Considering the nonlinear system in the form of\(^30,31\)

$$
\dot{x} = f(x), f(0) = 0, x \in \mathbb{R}^n
$$

(15)

where $f : \mathbb{R}_0 \to \mathbb{R}^n$ is continuous on an open neighborhood $\mathbb{U}_0$ and the origin is 0, and supposing that there is a $C^1$ function $V(x)$ defined in neighborhood $\mathbb{U} \subset \mathbb{R}^n$ of the origin, and that real numbers $c > 0$ and $0 < \mu < 1$, such that

(i) $V(x)$ is positive definite in $\mathbb{U}$.

(ii) $\dot{V}(x) + cV^\mu(x) \leq 0, \forall x \in \mathbb{U}$.\(^3\)
Thus, the origin of system (15) is locally finite-time stable. The settling time, which depends on the initial state \( x(0) = x_0 \), satisfies
\[
T_s(x_0) \leq \frac{V(x_0)^{1-\mu}}{c(1-\mu)}
\]
for all \( x_0 \) in some open neighborhoods of the origin. If \( \dot{U} = \mathbb{R}^n \) and if \( V(x) \) is also radially unbounded (i.e. \( V(x) \to +\infty \) as \( ||x|| \to +\infty \)), the origin of system
\[
\dot{x} = f(x,u), f(0,0) = 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m
\]
is globally finite-time stable.

**Lemma 3.** If continuous function \( f: [a,b] \to \mathbb{R} (b > a) \) is a monotone and satisfies \( f(a) = 0 \), then \( |f(b)| \cdot |b-a| \geq \int_a^b |f(x)|dx \).

**Lemma 4.** Consider \( x_i \in \mathbb{R}, i = 1,2,\cdots, n \), if \( 1 \geq a \geq 0 \), then \( (x_1^a + x_2^a + \cdots + x_n^a)^b \leq (x_1^a + x_2^a + \cdots + x_n^a)^b \) and \( (|x_1| + |x_2| + \cdots + |x_n|)^a \leq (|x_1| + |x_2| + \cdots + |x_n|)^b \).

Consider odd integers \( 0 < c \) and \( 0 < d \), let \( 1 \geq a = \frac{2}{p} \), then \( 2^{1-a}|x-y|^a \geq |x-y|^a \).

**Lemma 5.** Consider \( a, b, c \in \mathbb{R} \), if \( a \geq b \geq c > 0 \), then \( |x|^a + |x|^c = |x|^a(1 + |x|^{-c-a}) \geq |x|^b \).

**Lemma 6.** For the given positive real numbers \( m_1, m_2 \) and a function \( \Phi(x,y) \), there holds
\[
\left| \frac{d}{dx} \left( \frac{m_1}{m_2} \Phi(x,y) \right) \right| \leq \Theta(x,y) |\frac{d}{dx} \left( \frac{m_1}{m_2} \Phi(x,y) \right) |
\]
\[
\text{where } \Theta(x,y) > 0, \text{ for any } x \in \mathbb{R} \text{ and } y \in \mathbb{R}.
\]

**Lemma 7.** Consider \( W = \int_a^b \left( \beta - a \right)^{1-p} d\beta \), where \( \beta \) is the integration variable, if \( \rho > 0 \), then \( W \) is positive definite.

**Proof 1.** By lemma 4, we have
\[
|\beta - a| = \left| (\beta - a)^p - (a - a)^p \right| \leq |2^{1-p}|\beta - a|^p
\]
then
\[
|\beta - a| \geq (2^{p-1}|\beta - a|^p)^\frac{1}{p}
\]
If \( b \geq a \), by equation (20), we have
\[
W = \int_a^b \left( \beta - a \right)^{2-p} d\beta
\]
\[
\geq \int_a^b (2^{p-1}|\beta - a|^{2-p}) \frac{2}{p} \, d\beta
\]
\[
= p \times 2^{2-p-2p} (b - a)^2/p
\]
In the case of \( b < a \), the proof is similar and equation (21) is still true, then we conclude that \( W \) is positive definite.

**Control design**

The aim of the proposed control is to ensure that the output constraints are not violated with the property of finite-time convergence. The backstepping technique is used in the process of deriving control. The following steps describe the details of design procedure. Define the error \( e_1 = x_1 - x_{1id} \), for the stabilization problem in this work, we have \( x_{id} = [0,0]^T \).

**Assumption 1.** The initial value of \( x_{1i} \), satisfies \( |x_{1i}(0)| < h_{bi} \).

**Step 1.** Consider a BLF candidate: \( V_1 = \sum_{i=1}^3 V_{1i} \), and
\[
V_{1i} = \frac{1}{2} \log \frac{k_{bi}^2}{k_{bi}^2 - x_{1i}^2}
\]
Then, we can obtain
\[
\dot{V}_{1i} = \frac{x_{1i}}{k_{bi}^2 - x_{1i}^2} [x_{2i}^2 + \alpha_i - \alpha_i]
\]
The following virtual control law is chosen in this article
\[
\alpha_i = -x_{1i}^2 \left[ 2(k_{bi}^2 - x_{1i}^2) + \left( \frac{1}{2} - \frac{1}{p} \right) (k_{bi}^2 - x_{1i}^2) \right]
\]
Considering equation (24) and substituting it into equation (23), one obtains
\[
\dot{V}_{1i} = \frac{x_{1i}}{k_{bi}^2 - x_{1i}^2} [x_{2i}^2 - \alpha_i] \leq -x_{1i}^2 \left[ 2 + 2(k_{bi}^2 - x_{1i}^2) \right] + |x_{1i}| \left( |x_{2i} - \alpha_i| \right)
\]
It follows lemma 4 that
\[
|x_{2i} - \alpha_i| \leq 2^{1-p} \left( \frac{1}{2} - \frac{1}{p} \right) \left( x_{2i}^2 - \alpha_i^2 \right)^{1/p}
\]
Substituting equation (26) into equation (25), one obtains
where

\[ \mu_{1i} = \frac{p}{1 + p} \left( \frac{2p}{1 + p} \right) \frac{1 + \rho}{r} \]

Substituting equations (28) and (29) into equation (27), one obtains

\[ \dot{V}_{1i} \leq -x_{1i}^{1 + \rho} \left[ 2 + (2(k_{bi}^2 - x_{1i}^2)) \right] \]

\[ + \frac{1}{2} |x_{1i}|^{1 + \rho} + \mu_{1i} |x_{2i} - \alpha_i|^{1 + \rho} \]

Step 2. Consider Lyapunov function candidate \( V_2 = \sum_{i=1}^{3} V_{2i} \) with

\[ V_{2i} = \int_{0}^{\alpha_{2i}} \left( \frac{1}{\beta_i} - \alpha_i \right)^{2 - \rho} d\beta \]

According to lemma 7.32, we know that \( V_{2i}, i = 1, 2, 3 \) are proper and differentiable, positive definite. And the time derivative of \( V_{2i} \) will be

\[ \dot{V}_{2i} = \frac{\partial V_{2i}}{\partial x_{1i}} \dot{x}_{1i} + \frac{\partial V_{2i}}{\partial x_{2i}} \dot{x}_{2i} \]

Denote \( \dot{x}_{1i} = x_{1i} \) and \( \dot{x}_{2i} = x_{2i} - \alpha_i \), according to lemma 5 and equation (24), one obtains

\[ |x_{2i}| = \left( |\dot{x}_{2i} + \alpha_i| \right)^{\rho} \]

\[ \leq |\dot{x}_{2i}|^{\rho} + |\alpha_i| \]

\[ \leq |\dot{x}_{2i}|^{\rho} + |\dot{x}_{2i}|^{\rho} \cdot |\phi_i(\cdot)| \]

where \( \phi_i(\cdot) = 2(k_{bi}^2 - x_{1i}^2) + 2 (k_{bi}^2 - x_{1i}^2) \). Considering equation (24), one obtains

\[ \psi_i(x_{1i}) = \frac{\partial \alpha_{1i}}{\partial x_{1i}} = 2 \frac{1 - p}{p} \frac{\phi_i(x_{1i}^2 - x_{1i}^2)}{x_{1i}} \]

\[ + \frac{4}{p} \frac{1}{\phi_i} x_{2i}^{\rho} - \phi_i^{\rho} \]

Therefore, we have

\[ \left| \frac{\partial x_{1i}}{\partial x_{1i}} \right| \leq \left| \frac{\partial x_{1i}}{\partial x_{1i}} \right| |x_{1i}| \leq |\psi_i(\cdot)| \cdot (|\dot{x}_{1i}|^{\rho} + |x_{2i}|^{\rho} \cdot |\phi_i(\cdot)|) \]

\[ \leq (|\dot{x}_{1i}|^{\rho} + |x_{2i}|^{\rho}) \cdot \Phi_i \]
where $\Phi_i \geq \max\{|\psi_i(\cdot)|, |\psi_i(\cdot)| \cdot |\phi_i(\cdot)|\}$ is a $C^1$ function and it is nonnegative.

Considering lemma 3, one can obtain

$$\int_{\alpha_i}^{\beta_i} \left( \frac{1}{\beta_i - \alpha_i} \right)^{1-p} \, d\beta \leq 2|\dot{x}_{2i}|$$  \hspace{1cm} (36)

Then, one obtains

$$\frac{\partial V_{2i}}{\partial x_{1i}} \cdot \dot{x}_{1i} = \left( 2-p \right) \frac{\partial \alpha_i}{\partial x_{1i}} \int_{\alpha_i}^{\beta_i} \left( \frac{1}{\beta_i - \alpha_i} \right)^{1-p} \, d\beta \cdot \dot{x}_{1i}$$

\leq 2(2-p)|\dot{x}_{2i}| \cdot (|\lambda_{1i}|^p + |\lambda_{2i}|^p) \cdot \Phi_i$$

\leq \frac{1}{2} |\dot{x}_{2i}|^{1+p} + \mu_{2i} |\dot{x}_{2i}|^{1+p}$$

where $\mu_{2i}(\cdot)$ is also a nonnegative $C^1$ function with

$$\mu_{2i} = \frac{1}{1+p} \left( \frac{1}{1+p} \right)^p \left[ 2(2-p)\phi_i \right]^{1+p} + 2(2-p)\phi_i$$

(37)

(38)

Considering $\frac{\partial V_{2i}}{\partial x_{2i}} = x_{2i}^{\lambda_{1i}^p}$ and substituting equation (37) into equation (32) yields

$$\dot{V}_{2i} \leq \frac{1}{2} |\lambda_{1i}|^{1+p} + \mu_{2i} |\lambda_{2i}|^{1+p} + x_{2i}^{2-p} \cdot \dot{x}_{2i}$$

(39)

Considering equations (30) and (39), denoting $\tau_i = \dot{x}_{2i}$, we have

$$\dot{V} = \sum_{i=1}^{3} V_{1i} + \sum_{i=1}^{3} V_{2i} \leq - \sum_{i=1}^{3} \left[ 2(k_{2i}^2 - \lambda_{1i}^2) \right] \frac{1}{\lambda_{1i}^p} |\dot{x}_{2i}|^{1+p}$$

\leq - \sum_{i=1}^{3} |\dot{x}_{2i}|^{1+p} + \sum_{i=1}^{3} (\mu_{1i} + \mu_{2i}) |\dot{x}_{2i}|^{1+p} + \sum_{i=1}^{3} \mu_{2i} \cdot \tau_i$$

(40)

If the input torque is designed as

$$u = -JF(x_1) \Gamma(x_1, x_2) [x_2^T - \alpha_s^T]^{2p-1}$$

$$+ G(x_1) \Gamma^{-1} S[F(x_1)x_2] JF(x_1)x_2$$

$$+ 0.5x_2^T x_2 F(x_1)x_2 + 0.5S[x_2] F(x_1)x_2$$

$$- 0.5x_2^T F(x_1)x_2 - 0.5x_1 x_2^T F(x_1)x_2$$

where $\dot{x}_2 = [x_{21}, x_{22}, x_{23}]^T$, $\alpha_s^T = [\alpha_1^T, \alpha_2^T, \alpha_3^T]^T$, and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{(a to c) Angular velocity $\omega$ by using PD method, FPD method, and proposed method. PD: proportional–derivative; FPD: finite-time PD.}
\end{figure}
Then, we have

\[
\tau_i = -(1 + \mu_{1i} + \mu_{2i})\hat{\lambda}_{2i}^{2p-1}
\]  

(43)

Substituting this expression into equation (40), we have

\[
\dot{V} \leq -\sum_{i=1}^{3} \left[ 2(k_{bi}^2 - \hat{\lambda}_{1i}^2) \right] - \frac{1}{2} \hat{\lambda}_{1i}^{1+p} - \sum_{i=1}^{3} |\hat{\lambda}_{1i}|^{1+p} - \sum_{i=1}^{3} |\hat{\lambda}_{2i}|^{1+p}
\]  

(44)

Remark 4. Lyapunov function \( V_{2i} \) contains a power integrator, which will make the whole design procedure singular free and finally result in a continuous controller.

Remark 5. From the above design procedure, we know that the constraint is applied to the virtual control signal \( \alpha_i \). In this way, as long as \( e_{1i} \) is bounded within the certain compact, \( x_{1i} \) will be constrained to the predefined range. Furthermore, from the view of computational complexity, the proposed controller structure is very simple and can be determined explicitly. There is no need to determine some complex functions which come from repeated derivations of virtual controller.

To this end, the main result of above design can be stated as follows.

Theorem 1. Consider the rigid spacecraft system model defined by equations (1) and (2), the proposed state feedback control law (41) can ensure the global finite-time stabilization of the closed-loop system at the origin and without violation of the output constraints, that is, \( |y| < k_b, \forall t > 0 \).

Proof 2. Consider lemma 3, we have

Figure 3. (a to c) Control inputs \( u \) by using PD method. PD: proportional–derivative.
Let \( r = (1 + \rho)/2 \), we have

\[
|V_{2i}| = \left| \int_{\alpha_i}^{x_{2i}} \left( \frac{1}{\beta_i - \alpha_i} \right)^{2-\rho} \, d\beta \right| \leq \left| \frac{1}{\beta_i - \alpha_i} \right|^{2-\rho} |x_{2i} - \alpha_i| \leq 2|x_{2i}|^2 \tag{45}
\]

Substituting equation (47) into equation (46), we have

\[
V^\rho \leq 2 \left[ 2(k_{bi}^2 - \lambda_{1i}^2) \right]^{-\rho} \lambda_{2i}^{2\rho} + 2 \sum_{i=1}^{3} \lambda_{2i}^{2\rho} \tag{48}
\]

Consider equations (41) and (48), we can obtain

\[
V + \frac{1}{2} V^\rho \leq -3 \sum_{i=1}^{3} \lambda_{1i}^{2\rho} \tag{49}
\]

By using lemma 2, the closed-loop system will converge to origin within time \( T \) with

\[
\text{Figure 4. (a to c) Control inputs } u \text{ by using FPD method. FPD: finite-time proportional–derivative.}
\]
Therefore, the closed-loop system (1) and (2) with equation (41) is finite time stable within \( |y| < k_b, \forall t > 0 \).

**Remark 6.** The proposed control strategy not only can achieve finite-time stabilization but also can guarantee the output constraints. This is achieved by using the BLF. If the fraction power is chosen as \( p = 1 \), the finite-time control law (41) will result in a kind of traditional backstepping control law considering output constraints. As a result, the desired attitude will be asymptotically stabilized under the traditional backstepping control law and the exponential convergence rate will be achieved.

**Remark 7.** The proposed method can also use the stabilization problem with consideration of output constraints for other motion systems, such as robot systems and vehicle systems.

**Simulation**

To show the effectiveness of the proposed scheme, it is applied to a rigid spacecraft model with output constraints, and the detailed responses are numerically simulated and analyzed. The parameters are set as inertia moments 
\[ J = \begin{bmatrix} 200 & 030 & 004 \end{bmatrix} \text{kg m}^2 \]  
and subject to the initial conditions: \( \sigma(0) = [0.3; 0; -0.3], \omega(0) = [0.4; 0; -0.4] \text{ rad/s} \). The bounds of control inputs are all set as \( \pm 40 \text{ Nm} \). The objective of this simulation is to make the attitude orientation of the spacecraft converge to a small set around the origin, while the output constraints \( |y| < [0.3; 0.31; 0.31] \) are not violated.

For the purposes of comparison, the developed controller is compared with the finite-time controller in the study by Su and Zheng\(^{47} \) and PD control with the same initial conditions. The controller parameters were chosen by trial and error until the excellent performance was achieved in our simulation. The design parameter of proposed output-constrained finite-time stabilizer (41) is \( p = 11/13 \), and \( k_b \) is set as \( [0.5, 0.5, 0.5]^T \). The parameters of traditional PD controller are selected as \( K_P = \text{diag}(30, 30, 30) \) and \( K_D = \text{diag}(15, 15, 15) \). And the finite-time PD (FPD) controller developed by Su and Zheng\(^{47} \) is

\[ u_{FPD} = -G^T(\sigma)K_1\text{Sig}^{\alpha_1}(\sigma) - K_2\text{Sig}^{\alpha_2}(\omega) \]

where \( \alpha_1 = 0.5 \) and \( \alpha_2 = 0.4 \). Figure 5. (a to c) Control inputs \( u \) by using proposed method.
where the vector $\text{Sig}^\alpha(a) \in \mathbb{R}^3$ is defined as $\text{Sig}^\alpha(a) = [|a_1| \text{sign}(a_1), |a_2| \text{sign}(a_2), |a_3| \text{sign}(a_3)]^T$ and $\text{sign}(\cdot)$ denotes standard sign function. The gains for the FPD controller are chosen as $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $K_1 = \text{diag}(30, 30, 30)$, and $K_2 = \text{diag}(15, 15, 15)$.

The comparative simulations are conducted, and the results are shown in Figures 1 to 5. Figure 1 shows the comparison results of the output responses by using PD method, FPD method, and the proposed method. It can be concluded that the proposed controller (41) guarantees that the origin of the closed-loop system is finite time stable quickly and the output constraints of system are not violated, while the system performance is significantly degraded using the PD controller. Figure 2 shows that the angle velocity responses are all converge to stable. Figures 3 to 5 show the actual control torques. It can be observed that all signals are bounded, and the control torques are all within the maximum allowable limits, that is, 40 Nm.

To investigate the performance of the closed-loop system under different selections of $k_b$, the simulation experiments have been carried out under $k_b = [0.45; 0.45; 0.45]$, $k_b = [0.75; 0.75; 0.75]$, and $k_b = [1; 1; 1]$. The results are shown in Figures 6 to 9. Figure 6 shows the Rodrigues parameters $\sigma$ under different values of $k_b$. Figure 7 shows the angular velocity $\omega$ under different values of $k_b$. Figure 8 shows the virtual control $\alpha$ under different values of $k_b$. Figure 9 shows the control inputs $u$ under different values of $k_b$. From Figures 6 to 9, we can get that with the increase
of parameter $k_b$, the convergence speed is faster, but the chattering of the control inputs is increased.

Summarizing all the simulation results, it is noted that the proposed scheme can successfully accomplish attitude stabilization with high attitude pointing accuracy and stability in the presence of output constraints in both theory and simulations. In addition, comparison simulations were also performed to traditional PD control and simple FPD method by Su and Zheng\(^47\) without considering the output constraints. These results show that the closed-loop system attitude stabilization is accomplished while meeting the constraints on the output.

**Conclusion**

This article has investigated the finite-time stabilization problem for an output-constrained rigid spacecraft based on finite-time control theory and the BLF technique. Under the proposed control, the closed-loop system is shown to be global finite time stable, and all the closed-loop signals are bounded. The results of simulation have verified that the proposed finite-time controller is feasible and effective in stabilizing the demanded attitude orientation for a rigid spacecraft with output constraints.

**Declaration of conflicting interests**

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