SURFACES WITH LIGHT-LIKE POINTS IN
LORENTZ-MINKOWSKI 3-SPACE WITH APPLICATIONS

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ABSTRACT. With several concrete examples of zero mean curvature surfaces in $\mathbb{R}^3_1$ containing a light-like line recently having been found, here we construct all real analytic germs of zero mean curvature surfaces by applying the Cauchy-Kovalevski theorem for partial differential equations. A point where the first fundamental form of a surface degenerates is said to be light-like. We also show a theorem on a property of light-like points of a surface in $\mathbb{R}^3_1$ whose mean curvature vector is smoothly extendable. This explains why such surfaces will contain a light-like line when they do not change causal types. Moreover, several applications of these two results are given.

INTRODUCTION

In this paper, we denote by $\mathbb{R}^3_1$ the Lorentz-Minkowski 3-space with inner product $\langle \cdot , \rangle$ with signature $(-++),$ and write the canonical coordinate system of $\mathbb{R}^3_1$ as $(t,x,y).$

Klyachin [13] showed that a zero mean curvature $C^3$-immersion $F: U \to \mathbb{R}^3_1$ of a domain $U \subset \mathbb{R}^2$ in the Lorentz-Minkowski 3-space $\mathbb{R}^3_1$ containing a light-like point $o$ satisfies one of the following two conditions:

(a) There exists a null curve $\sigma$ (i.e., a regular curve in $\mathbb{R}^3_1$ whose velocity vector field is light-like) on the image of $F$ passing through $F(o)$ which is non-degenerate (i.e. its projection into the $xy$-plane is a locally convex plane curve, cf. Definition 3.1). Moreover, the causal type of the surface changes from time-like to space-like across the curve.

(b) There exists a light-like line segment passing through $F(o)$ consisting of the light-like points of $F.$ Zero mean curvature surfaces which change type across a light-like line belong to this class.

The case (a) is now well understood (cf. [9], [13] and [12]). In fact, under the assumption that $F$ is real analytic, the surface in the class (a) can be reconstructed from the null curve $\sigma$ as follows:

\[
F(u,v) := \begin{cases}
    \frac{\sigma(u + i\sqrt{v}) + \sigma(u - i\sqrt{v})}{2} & (v \geq 0), \\
    \frac{\sigma(u + \sqrt{|v|}) + \sigma(u - \sqrt{|v|})}{2} & (v < 0),
\end{cases}
\]

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where $i = \sqrt{-1}$, and we extend the real analytic curve $\sigma$ as a complex analytic map into $C^3$.

We call the point $o$ as in the case (a) a non-degenerate light-like point of $F$. A typical example of such a surface is obtained by a null curve $\gamma(u) = (u, \cos u, \sin u)$ and the resulting surface is a helicoid, which is a zero-mean curvature surface (i.e. ZMC-surface) in $R^3_1$ as well as in the Euclidean 3-space. The reference [6] is an expository article of this subject. Moreover, an interesting connection between type change of ZMC-surfaces and 2-dimensional fluid mechanics was also given in [6]. The existence and properties of entire ZMC-graphs in $R^3_1$ with non-degenerate light-like points are discussed in [3]. Embedded ZMC-surfaces with non-degenerate light-like points with many symmetries are given in [8] and [4].

On the other hand, several important ZMC-surfaces satisfying (b) are given in [5] and [1]. In contrast to the case (a), these examples of surfaces do not change causal types across the light-like line $1$. A family of surfaces constructed in [7] satisfies (b) and also changes its causal type. In spite of this progress, there was still no machinery available to find surfaces of type (b) and no simple explanation for why only two cases occur at light-like points.

In this paper, we clarify such phenomena as follows: We denote by $Y^r$ ($r \geq 3$) the set of germs of $C^r$-differentiable immersions in $R^3_1$ whose mean curvature vector field can be smoothly extended at a light-like point. We prove a property of regular surfaces in the class $Y^3$, which contains the above Klyachin’s result as a special case. Our approach is different from that of [13]: we use the uniqueness of ordinary differential equations to prove the assertion. We also show a general existence of real analytic ZMC-surfaces and surfaces in $Y^\omega$ using the Cauchy-Kovalevski theorem for partial differential equations. As a consequence, new examples of ZMC-surfaces which change type along a given light-like line are obtained.

1. Preliminaries

We denote by $0 := (0, 0, 0)$ (resp. $o := (0, 0)$) the origin of the Lorentz-Minkowski 3-space $R^3_1$ of signature $(- + +)$ (resp. the plane $R^2$) and denote by $(t, x, y)$ the canonical coordinate system of $R^3_1$. An immersion $F : U \to R^3_1$ of a domain $U \subset R^3$ into $R^3_1$ is said to be space-like (resp. time-like, light-like) at $p$ if the tangent plane of the image $F(U)$ at $F(p)$ is space-like (resp. time-like, light-like), that is, the restriction of the metric $(\ , \ )$ to the tangent plane is positive definite (resp. indefinite, degenerate). We denote by $\tilde{I}^r$ ($r \geq 2$) the set of germs of $C^r$-immersions into $R^3_1$ which map the origin $o$ in the $uv$-plane to the origin $0$ in $R^3_1$. (F $\in \tilde{I}^\omega$ means that $F$ is real analytic.) Let $F : (U, o) \to R^3_1$ be an immersion in the class $\tilde{I}^r$. We denote by $U_+$ (resp. $U_-$) the set of space-like (resp. time-like) points, and set

$$U_* := U_+ \cup U_-.$$ 

A point $p \in U$ is light-like if $p \notin U_*$. We denote by $\tilde{I}^r_L(\subset \tilde{I}^r)$ the set of germs of $C^r$-immersion such that $o$ is a light-like point.

1 In this paper, we say that a surface changes its causal types across the light-like line if the causal type of one-side of the line is space-like and the other-side is time-like. If the causal type of the both sides of the line coincides, we say that the surface does not change its causal type across the light-like line.
If $F \in \tilde{I}_L$, the tangent plane of the image of $F$ at $o$ contains a light-like vector and does not contain time-like vectors. Thus, we can express the surface as a graph

\[ F = (f(x, y), x, y), \]

where $f(x, y)$ is a $C^r$-function defined on a certain neighborhood of the origin of the $xy$-plane. Let

\[ B_F := 1 - f_x^2 - f_y^2 \quad \left( f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y} \right). \]

Then the point of the graph (1.1) is space-like (resp. time-like) if and only if $B_F > 0$ (resp. $B_F < 0$) at the point. Since $F \in \tilde{I}_L$, the origin $o = (0, 0)$ is light-like, that is, $B_F(0, 0) = 0$. Hence there exists $\theta \in [0, 2\pi)$ such that

\[ f_x(0, 0) = \sin \theta, \quad f_y(0, 0) = \cos \theta. \]

So by a rotation about the $t$-axis, we may assume

\[ f_x(0, 0) = 0, \quad f_y(0, 0) = 1 \]

without loss of generality.

We denote by $I_L^r (\subset \tilde{I}_L)$ the set of germs of $C^r$-immersion $F$ with properties (1.1) and (1.3). Then

\[ \iota : I_L^r \ni F \mapsto f = \iota_F \in \{ f \in C^r_o(\mathbb{R}^2) ; f(0, 0) = f_x(0, 0) = 0, f_y(0, 0) = 1 \}, \]

which maps $F$ to the function $f$ as in (1.1), is a bijection, where $C^r_o(\mathbb{R}^2)$ is the set of $C^r$-function germs on a neighborhood of $o \in \mathbb{R}^2$.

For $F \in I_L^r$,

\[ U_+ = \{ p \in U ; B_F(p) > 0 \}, \quad U_- = \{ p \in U ; B_F(p) < 0 \} \]

hold, where $B_F$ is the function as in (1.2). We let

\[ A_F := (1 - f_x^2)f_yy + 2f_x f_y f_{xy} + (1 - f_y^2)f_{xx}. \]

Then the mean curvature function of $F$

\[ H_F := \frac{A_F}{2|B_F|^{3/2}} \]

is defined on $U_+$ (cf. [10, Lemma 2.1]). We first remark the following:

**Proposition 1.1** (cf. Klyachin [13, Example 4]). If $B_F$ vanishes identically, then so does $A_F$.

**Proof.** Since $B_F = 0$, we have $1 - f_x^2 = f_y^2$. By differentiating this, we get $f_x f_{xy} = -f_y f_{yy}$, and

\[ (1 - f_x^2) f_{yy} = f_y (f_y f_{yy}) = -f_x f_y f_{xy}. \]

Similarly, we have

\[ (1 - f_y^2) f_{xx} = f_x (f_x f_{xx}) = -f_x f_y f_{xy}. \]

By (1.8) and (1.9), we get the identity $A_F = 0$. \qed

We denote by $\Lambda^r$ the set of germs of immersions $F \in I_L^r$ with identically vanishing $B_F$, that is, $\Lambda^r$ is the set of germs of light-like immersions. We denote by $C^\omega_o(\mathbb{R}, 0_2)$ the set of real analytic functions $\varphi$ satisfying $\varphi(0) = d\varphi(0)/dx = 0$. Then the following assertion holds:
Proposition 1.2. The map
\[ \lambda : \Lambda \ni F \mapsto (\lambda_F :=) f(x,0) \in C^o_\omega(R, 0_2) \]
is bijective, where \( f = \iota_F \) (cf. (1.4)).

Proof. Since \( B_F \) vanishes identically, taking into account of (1.3), we can write
\[ f_y = \sqrt{1 - f_x^2} \]
This can be considered as a normal form of a partial differential equation under the initial condition (1.10)
\[ f(x, 0) = \psi(x) \quad (\psi \in C^o_\omega(R, 0_2)) \]
because of the condition \( f_x(0, 0) = \psi'(0) = 0 \). So we can apply the Cauchy-Kovalevski theorem (cf. [15]) and show the uniqueness and existence of the solution \( f \) satisfying (1.10).

Remark 1.3. The above proof of the existence of a light-like surface \( F(x, y) \) satisfying (1.10) is local, that is, it is defined only for small \(|y|\). Later, we will show that \( F \) is a ruled surface and has an explicit expression, see Corollary 3.7 and (3.8).

Example 1.4. The light-like plane \( F(x, y) = (y, x, y) \) belongs to the class \( \Lambda \omega \) such that \( \lambda_F = 0 \).

Example 1.5. The light-cone \( F(x, y) = (\sqrt{x^2 + (1 + y)^2} - 1, x, y) \) is a light-like surface satisfying \( \lambda_F = \sqrt{1 + x^2} - 1 \).

2. Surfaces with smooth mean curvature vector field.

Let \( F : (U, o) \to (R^3_1, 0) \) be an immersion of class \( \mathcal{I}_L^r \) \((r \geq 3)\) such that \( U \) is open and dense in \( U \), and fix a \( C^{r-2} \)-function \( \varphi \). We say that \( F \) is \( \varphi \)-admissible if
\[ A_F - \varphi B_F^2 = 0 \]
holds, where \( A_F \) and \( B_F \) are as in (1.6) and (1.2), respectively. We denote
\[ \mathcal{Y}_\varphi^r := \{ F \in \mathcal{I}_L^r : F \text{ is } \varphi\text{-admissible} \} \]
An immersion germ \( F \in \mathcal{I}_L^r \) is called admissible if it is \( \varphi \)-admissible for a certain \( \varphi \in C^{r-2}_o (R^2) \). The set
\[ \mathcal{Y}_r := \bigcup_{\varphi \in C^{r-2}_o (R^2)} \mathcal{Y}_\varphi^r \]
consists of all germs of \( \varphi \)-admissible immersions. The following assertion explains why the class \( \mathcal{Y}_r \) is important.

Proposition 2.1 (10). Let \( F : (U, o) \to (R^3_1, 0) \) be an immersion in the class \( \mathcal{I}_L^r \) for \( r \geq 3 \). Then the mean curvature vector field \( H_F \) can be \( C^{r-2} \)-differentially extended on a neighborhood of \( o \) if and only if \( F \) belongs to the class \( \mathcal{Y}_r \).

Proof. This assertion follows from the fact that
\[ H_F = \frac{A_F}{2(B_F)^2}(F_x \times F_y) \]
on \( U^* \), where \( \times \) denotes the vector product in \( R^3_1 \).
Proof. Suppose that $f = F$ like point of $\mathcal{I}(\mathbb{R})$ at the light-like point $o$. By definition, we have

$$
\mathcal{I}^r := \{ F \in \mathcal{I}_r(\mathbb{R}) : A_F = 0 \} = \mathcal{Y}_0^r
$$

is the set of germs of zero mean curvature immersions in $\mathbb{R}^3$ at the light-like point $o$. By definition, we have

$$
\mathcal{I}^r \subseteq \mathcal{Z}^r \subseteq \mathcal{Y}^r \quad (r \geq 3).
$$

Surfaces in the class $\mathcal{Y}^r$ are investigated in [10], and an entire graph in $\mathcal{Y}^r$ which is not a ZMC-surface was given. In this section, we shall show a general existence result of surfaces in the class $\mathcal{Y}^\omega$. We fix a germ of a real analytic function $\varphi \in C^\omega_\omega(\mathbb{R}^2)$, and take an immersion $F \in \mathcal{Y}_\varphi^\omega$.

**Definition 2.2.** Let $f := \iota_F$ be the function associated with $F \in \mathcal{Y}_\varphi^\omega$ (cf. (1.4)). Then

$$
\gamma_F(x) := (f(x, 0), f_y(x, 0))
$$

is a real analytic plane curve, which we call the *initial curve* associated with $F$.

We denote by $C^\omega_\omega(\mathbb{R}, \mathbb{R}^2)$ the set of germs of $C^\omega$-maps $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^2, (0, 1))$. By definition (cf. (1.3)), $\gamma_F(x) \in C^\omega_\omega(\mathbb{R}, \mathbb{R}^2)$ holds. We prove the following assertion:

**Theorem 2.3.** For $\varphi \in C^\omega(\mathbb{R}^2)$, the set $\mathcal{Y}_\varphi^\omega$ is non-empty. More precisely, the map

$$
\mathcal{Y}_\varphi^\omega \ni F \mapsto \gamma_F \in C^\omega_\omega(\mathbb{R}, \mathbb{R}^2)
$$

is bijective. Moreover, the base point $o$ is a non-degenerate (resp. degenerate) light-like point of $F$ if $\dot{\gamma}_F(0) \neq (0, 0)$ (resp. $\dot{\gamma}_F(0) = (0, 0)$), where “dot” denotes $d/dx$.

**Proof.** Suppose that $F \in \mathcal{Y}_\varphi^\omega$. Since $A_F - \varphi B^2_F$ vanishes identically (cf. (2.1)), $f = \iota_F$ satisfies

$$
\begin{align*}
  f_y &= g, \\
  g_y &= \frac{2f_x g g_x + (1 - g^2)f_{xx} - (1 - f^2_x - g^2)^2 \varphi}{1 - f_x^2},
\end{align*}
$$

which is the normal form for partial differential equations. So we can apply the Cauchy-Kovalevski theorem (cf. [15]) for a given initial data

$$(f(x, 0), g(x, 0)) := \gamma(x) \quad (\gamma \in C^\omega_\omega(\mathbb{R}, \mathbb{R}^2)).$$

Then the solution $(f, g)$ of (2.6) is uniquely determined. Obviously, the resulting immersion $F_\gamma := (f(x, y), x, y)$ gives a surface in $\mathcal{Y}_\varphi^\omega$ whose initial curve is $\gamma$. The second assertion follows from the fact that $\dot{\gamma}(0) = (0, 0)$ if and only if $\nabla B_F(0, 0) = 0$, where $\nabla B_F := ((B_F)_x, (B_F)_y)$.

When $\varphi = 0$, we get the following:

**Corollary 2.4.** The map $\mathcal{Z}^\omega \ni F \mapsto \gamma_F \in C^\omega_\omega(\mathbb{R}, \mathbb{R}^2)$ is bijective.

The following is a direct consequence of this corollary and Theorem 2.3.

**Corollary 2.5.** In the above correspondence, it holds that

$$
\Lambda^\omega = \left\{ F \in \mathcal{Z}^\omega ; \gamma_F = \left( \psi, \sqrt{1 - \psi^2} \right), \psi \in C^\omega_\omega(\mathbb{R}, 0_2) \right\}.
$$
3. A PROPERTY OF LIGHT-LIKE POINTS.

**Definition 3.1.** Let $I$ be an open interval, and $\sigma : I \to \mathbb{R}^3$ a regular curve of class $C^r$ ($r \geq 3$). The space curve $\sigma$ is called **null** if $\sigma'(t) = d\sigma/dt$ is light-like. Moreover $\sigma$ is called **non-degenerate** if $\sigma''(t)$ is not proportional to $\sigma'(t)$ for each $t \in I$.

The orthogonal projection of a non-degenerate null curve into the $xy$-plane is a locally convex plane curve. The following assertion is a generalization of Klyachin’s result in the introduction, since ZMC-surfaces are elements of $\mathcal{Y}^3$.

**Theorem 3.2.** Let $F : (U, o) \to \mathbb{R}^3$ be an immersion of class $\mathcal{Y}^3$. Then, one of the following two cases occurs:

(a) $\nabla B_F$ does not vanish at $o$, and the image of the level set $F(\{B_F = 0\})$ consists of a non-degenerate null regular curve in $\mathbb{R}^3$, where $f = \iota_F$.

(b) $\nabla B_F$ vanishes at $o$, and the image of the level set $F(\{B_F = 0\})$ contains a light-like line segment in $\mathbb{R}^3$ passing through $F(o)$.

**Proof.** The first assertion (a) was proved in [10, Proposition 3.5]. So it is sufficient to prove (b). We may assume that $F \in \mathcal{Y}^3$ and $\varphi \in C^1_0(\mathbb{R}^2)$. Let $f := \iota_F$. Since $f$ is of class $C^3$, applying the division lemma (Lemma A.1 in Appendix A) for $g(x, y) := 2(f(x, y) - f(0, y) - x f_x(0, y))$, there exists a $C^1$-function $h$ such that

$$ (3.1) \quad f(x, y) = a_0(y) + a_1(y)x + \frac{h(x, y)}{2}x^2 \quad (a_0(y) := f(0, y), \quad a_1(y) := f_x(0, y)). $$

By (1.3), it holds that

$$ (3.2) \quad a_0(0) = 0, \quad a_0'(0) = 1, \quad a_1(0) = 0. $$

Moreover, since $\nabla B_F$ vanishes at $o$, we have

$$ (3.3) \quad a_1'(0) = 0. $$

We set (cf. (1.6) and (2.1))

$$ \tilde{A} := A_F - \varphi B_F^2. $$

Since $F \in \mathcal{Y}^3$, $\tilde{A}$ vanishes identically. We set

$$ h_0(y) := h(0, y), \quad h_1(y) := h_x(0, y), \quad h_2(y) := h_y(0, y), $$

$$ \varphi_0(y) := \varphi(0, y), \quad \varphi_1(y) := \varphi_x(0, y), $$

Then we have

$$ (3.4) \quad 0 = \tilde{A}_{|x=0} = \varphi_0((a_0')^2 + a_1^2 - 1)^2 + h_0 \left(1 - (a_0')^2\right) + (1 - a_1^2) a_0'' + 2a_1 a_0 a_1', $$

$$ (3.5) \quad 0 = \tilde{A}_{|x=0} = -2a_1 h_0 a_0'' + 4\varphi_0 \left(1 - (a_0')^2 - a_1^2\right) \left(a_1 h_0 + a_0 a_1'\right) + 2a_1 h_2 a_0'' - \varphi_1 \left(-(a_0')^2 - a_1^2 + 1\right)^2 $$

$$ - h_1 \left((a_0')^2 - 1\right) - (a_1^2 - 1)a_0'' + 2a_1 (a_1')^2. $$

These two identities (3.4) and (3.5) can be rewritten in the form

$$ (1 - a_1^2) a_0'' = \Psi_1(x, y, a_0, a_0', a_1, a_1'), $$

$$ -2a_1 h_0 a_0'' + (a_1^2 - 1)a_0'' = \Psi_2(x, y, a_0, a_0', a_1, a_1'), $$

where $\Psi_1$ and $\Psi_2$ are continuous functions of five variables. Since $1 - a_1^2(0) = 1$, this gives a normal form of a system of ordinary differential equations with unknown
functions $a_0$ and $a_1$. Moreover, this system of differential equations satisfies the local Lipschitz condition, since $\Psi_1$ and $\Psi_2$ are polynomials in $a_0, a_0', a_1$ and $a_1'$. Here,

\begin{equation}
(a_0, a_1) = (y, 0)
\end{equation}

gives a solution of this system of equations. Then the uniqueness of the solution with the initial conditions (3.2) and (3.3) implies that (3.6) holds for $F$. As a consequence, we have $F(0, y) = (y, 0, y)$, proving the assertion.

\textbf{Remark 3.3.} If $F$ is of non-zero constant mean curvature $H$, then $A_F - 2H|B_F|^3/2$ vanishes identically. In this case, we also get the relations $A_F|_{x=0} = (A_F)|_{x=0} = 0$, which can be considered as a system of ordinary equations like as in the above proof. However, this does not satisfy the local Lipschitz condition, and the above proof does not work in this case. Fortunately, an analogue of Theorem 3.2 can be proved for surfaces with non-zero constant mean curvature using a different approach (see [16] for details).

As a corollary, we immediately get the following:

\textbf{Corollary 3.4.} Any light-like points on a surface in the class $\mathcal{Y}^3$ are not isolated.

If a light-like point $o$ is non-degenerate, then $B_F$ changes sign, that is, $F$ changes causal type. So the following corollary is also obtained.

\textbf{Corollary 3.5.} If an immersion $F \in \mathcal{Y}^3$ does not change its causal type at $o$, then there exists a light-like line $L$ passing through $f(o)$ such that the set of the light-like points of $F$ is a regular curve $\gamma$ and the image of $F \circ \gamma$ lies on the line $L$.

We next give an application of Theorem 3.2 for light-like surfaces:

\textbf{Definition 3.6.} Let $\sigma(t)$ ($t \in I$) be a space-like $C^\infty$-regular curve defined on an interval $I$. Since the orthogonal complement of $\sigma'(t)$ is Lorentzian, there exists a non-vanishing vector field $\xi(t)$ along $\sigma$ such that $\xi(t)$ points the light-like direction which is orthogonal to $\sigma'(t) = d\sigma(t)/dt$, that is, it holds that

$$\langle \xi(t), \xi(t) \rangle = \langle \xi(t), \sigma'(t) \rangle = 0$$

where $\langle , \rangle$ means the canonical Lorentzian inner product in $\mathbb{R}^3_1$. The possibility of such vector fields $\xi(t)$ are essentially two up to a multiplication of non-vanishing smooth functions. Then the map

\begin{equation}
F(t, s) := \sigma(t) + s \xi(t) \quad (t \in I, |s| < \epsilon)
\end{equation}

gives a light-like immersion if $\epsilon > 0$ is sufficiently small (This representation formula was given in Izumiya-Sato [11]). We call such a $F$ a light-like ruled surface associated to the space-like curve $\sigma$).

For example, consider an ellipse $\sigma(t) = (0, a \cos t, \sin t)$ on the $xy$-plane in $\mathbb{R}^3_1$, where $a > 0$ is a constant. Then an associated light-like surface is given by (3.7) by setting

$$\xi(t) := \left( \sqrt{a^2 \sin^2 t + \cos^2 t}, \cos t, a \sin t \right).$$

Figure 1 presents the resulting light-like surface for $a = 2$.

The following corollary asserts that light-like regular surfaces are locally regarded as ruled surfaces.
Corollary 3.7. A light-like surface germ $F \in \Lambda^\infty$ can be parametrized by a light-like ruled surface along a certain space-like regular curve.

Proof. Let $F$ be a light-like surface such that $\iota_F(x, 0) = \psi(x)$ as in (1.10), where $\psi(0) = \dot{\psi}(0) = 0$ ($\dot{\cdot} = d/dx$). Then it holds that $\sigma(x) := F(x, 0) = (\psi(x), x, 0)$ is a space-like curve for sufficiently small $x$. There are $\pm$-ambiguity of light-like vector fields

$$\xi^\pm(x) := \left(1, \psi(x), \pm \sqrt{1 - \dot{\psi}(x)^2}\right)$$

along the curve $\sigma(x)$ perpendicular to $\dot{\sigma}(x)$. By Corollary 3.5, $F$ must be a ruled surface foliated by light-like lines. Since the light-like line in the image of $F$ passing through the origin is $y \mapsto (y, 0, y)$, the light-like ruled surface

$$(t, s) \mapsto \sigma(t) + s \xi^+(t)$$

(3.8)

gives a new parametrization of the surface $F$, that proves the assertion. $\square$

It should be remarked that the property of light-like points (cf. Theorem 3.2) can be generalized for surfaces in arbitrarily given Lorentzian 3-manifolds, see [10]. It is well-known that space-like ZMC-surfaces have non-negative Gaussian curvature. Regarding this fact, we prove the following assertion on the sign of Gaussian curvature at non-degenerate light-like points:

Proposition 3.8. Let $F$ be an immersion in the class $\mathcal{Y}^r$ ($r \geq 3$). Suppose that $o$ is a non-degenerate light-like point. Then the Gaussian curvature function $K$ diverges to $\infty$ at $o$.

When $F$ is of ZMC, the assertion was proved in Akamine [2].

Proof. Let $f = \iota_F$ is the function associated with $F \in \mathcal{Y}^r$ (cf. (1.4)) with (3.1) and set

$$C_F := f_{xx}f_{yy} - (f_{xy})^2.$$ 

Then the Gaussian curvature of $F$ is given by

$$(3.9) \quad K := -\frac{C_F}{(B_F)^2},$$
where $B_F$ is the function as in (1.2). Since the function $A_F$ as in (1.6) satisfies $A_F(0) = 0$, we have
\[
C_F(o) = h(o) - a'_1(0)^2 = -a'_1(0)^2.
\]
Here $a'_1(0) \neq 0$ since $o$ is a non-degenerate light-like point. Thus $C_F(o) \neq 0$ holds, and we get the conclusion because of the fact $B_F(o) = 0$. \hfill \Box

4. Properties of surfaces in $\mathcal{Y}_o^\omega$

We denote by $\mathcal{Y}_o^\omega$ (resp. $\mathcal{Y}_b^\omega$) the subset of $\mathcal{Y}$ consisting of surfaces such that the origin $o$ is a non-degenerate (resp. degenerate) light-like point. Then
\[
\mathcal{Y}_o^\omega := \mathcal{Y}_a^\omega \cup \mathcal{Y}_b^\omega
\]
holds. We next define two subsets of $\mathcal{Z}$ (cf. (2.3)) as
\[
\mathcal{Z}_a := \{ F \in \mathcal{Z}; \gamma F(0) \neq (0,0) \} = \mathcal{Y}_a \cap \mathcal{Z},
\]
\[
\mathcal{Z}_b := \{ F \in \mathcal{Z}; \gamma F(0) = (0,0) \} = \mathcal{Y}_b \cap \mathcal{Z},
\]
where $B_F$ is the function as in (1.2).

As explained in the introduction, surfaces in $\mathcal{Z}_o^\omega$ can be constructed using the formula as in (1.1). On the other hand, to get ZMC-surfaces in $\mathcal{Z}_b^\omega$, we can apply Corollary 2.4. However, it is only a general existence result, and is not useful if one would like to know the precise behavior of the surfaces along the degenerate light-like lines. Here, we focus to the class $\mathcal{Z}_o^\omega$. We first show that $\mathcal{Y}_o^\omega$ and will sow that surfaces in the class $\mathcal{Y}_b^\omega$ have quite similar properties as zero-mean curvature surfaces in the class $\mathcal{Z}_o^\omega$: Each surface $F \in \mathcal{Y}_o^\omega$ is expressed as the graph of a function
\[
f(x,y) := y + \frac{\alpha_F(y)}{2} x^2 + \frac{\beta_F(y)}{3} x^3 + h(x,y)x^4,
\]
where $\alpha_F$, $\beta_F$ and $h$ are certain real analytic functions. In fact, as seen in the proof of Theorem 3.2, $a_1(y) = 0$ holds when $o$ is a degenerate light-like point, where $a_1(y) := f_3(0,y)$. We call $\alpha_F(y)$ and $\beta_F(y)$ in (4.1) the second approximation function and the third approximation function of $F$, respectively. These functions give the following approximation of $F$:
\[
f(x,y) \approx y + \frac{\alpha_F(y)}{2} x^2 + \frac{\beta_F(y)}{3} x^3.
\]

Proposition 4.1. For $F \in \mathcal{Y}_o^\omega$, there exists a real number $\mu_F$ (called the characteristic of $F$) such that $\alpha_F$ and $\beta_F$ satisfy
\[
\alpha_F' + \alpha_F^2 + \mu_F = 0,
\]
\[
\beta_F' + 4\alpha_F\beta_F = 0.
\]
Moreover, if $\mu_F > 0$ (resp. $\mu_F < 0$) then $F$ has no time-like points (resp. no space-like points). In particular, if $F$ changes causal type, the $\mu_F = 0$.

When $F \in \mathcal{Z}_o^\omega$, this assertion for $\alpha := \alpha_F$ was proved in [5]. So the above assertion is its generalization.

Proof. Since $F$ in $\mathcal{Y}_o^\omega$, there exists a function $\varphi \in C_o^\omega(R^2)$ such that $F \in \mathcal{Y}_o^\omega$. Let $f := f_F$ and set $A := A_F$ and $B := B_F$ as in (1.2) and (1.6), respectively. Then by (4.1) implies that
\[
B(0,y) = B_x(0,y) = 0.
\]
Since \( \varphi = A/B^2 \) is a smooth function, the L'Hospital rule yields that

\[
\varphi(0, y) = \lim_{x \to 0} \frac{A(x, y)}{B(x, y)^2} = \lim_{x \to 0} \frac{A_x(x, y)}{2Bz(x, y)B(x, y)} = \lim_{x \to 0} \frac{A_{xx}(x, y)}{2B_{xx}(x, y)B(x, y) + 2B_x(x, y)B_x(x, y)}.
\]

Thus, (4.5) yields that we have

\[
0 = A(0, y) = A_x(0, y) = A_{xx}(0, y) = A_{xxx}(0, y).
\]

Here

\[
A \bigg|_{x=0} = A \bigg|_{x=0} = 0
\]

does not produce any restrictions for \( \alpha := \alpha_F \) and \( \beta := \beta_F \). On the other hand, we have

\[
0 = A_{xx} \bigg|_{x=0} = 2\alpha \alpha' + \alpha'',
\]

where the prime means the derivative with respect to \( y \). Hence \( \alpha' + \alpha^2 \) is a constant function, and get the first relation. We then get

\[
0 = A_{xxx} \bigg|_{x=0} = 4\alpha \beta' + \beta'',
\]

which yields the second assertion. The last assertion follows from the fact that

\[
B_{xx}(0, y) = -2(\alpha' + \alpha^2) = 2\mu F.
\]

We can find the solution \( \alpha = \alpha_F \) of the ordinary differential equation (4.3) under the conditions

\[
\alpha(0) = \ddot{u}(0), \quad \mu_F = -\left(\ddot{u}(0)^2 + \ddot{v}(0)\right)
\]

for a given initial curve \( \gamma(x) = (u(x), v(x)) \). By a homothetic change

\[
\tilde{F}(x, y) := (\tilde{f}(x, y), x, y), \quad \tilde{f}(x, y) := \frac{1}{m}f(mx, my), \quad m > 0,
\]

one can normalize the characteristic \( \mu_F \) to be \(-1, 0\) or \(1\). In fact, as shown in [5],

\[
\alpha^+ := -\tan(y + c) \quad (|c| < \pi/2)
\]

is a general solution of (4.3) for \( \mu_F = 1 \),

\[
\alpha^0_I := 0 \quad \text{and} \quad \alpha^0_H := \frac{1}{y + c} \quad (c \in \mathbb{R} \setminus \{0\})
\]

are the solutions for \( \mu_F = 0 \), and

\[
\alpha^-_I := \tanh(y + c) \quad (c \in \mathbb{R}),
\]

\[
\alpha^-_H := \coth(y + c) \quad (c \in \mathbb{R} \setminus \{0\}),
\]

\[
\alpha^-_{III} := \pm 1
\]

are the solutions for \( \mu_F = -1 \). Thus, as pointed out in [5], \( \mathcal{Y}_b^\omega \) consists of the following six subclasses:

\[
\mathcal{Y}^+, \quad \mathcal{Y}^0_I, \quad \mathcal{Y}^0_H, \quad \mathcal{Y}^-_I, \quad \mathcal{Y}^-_H, \quad \mathcal{Y}^-_{III}.
\]

Remark 4.2. To find surfaces in the class \( \mathcal{Y}^-_{III} \), we may set \( \alpha^-_H := 1 \) without loss of generality. In fact, if \( F \in \mathcal{Y}_b^\omega \) satisfies \( \alpha_F = -1 \), then we can write

\[
F = (f(x, y), x, y), \quad f(x, y) = y - \frac{x^2}{2} + k(x, y)x^3.
\]
where $k(x, y)$ is a $C^\omega$-function defined on a neighborhood of the origin. If we set $X := -x, Y := y$ and set $F := -F$, then we have
\[ \tilde{F}(X, Y) = \left( Y + \frac{X^2}{2} + k(-X, Y)X^3, X, Y \right). \]
Thus $\alpha_{\tilde{F}} = 1$.

Like as in the case of $\alpha = \alpha_F$, we can find the solution $\beta := \beta_F$ of the ordinary differential equation (4.4) with the given initial condition $2(\beta(0), \beta'(0)) = \gamma_F(0)$. In fact, $\beta$ can be written explicitly for each $\alpha = \alpha^+, \alpha_I^0, \alpha^-, \alpha_H^+, \alpha_H^−, \alpha_{III}^-$ as follows:

\begin{align*}
\beta^+ &= c_1 \left( 2 + \sec^2(y + c) \right) \tan(y + c) + c_2, \\
\beta_I^0 &= c_1 y + c_2, \\
\beta_H^0 &= \frac{c_1}{(y + c)^3} + c_2, \\
\beta_I^- &= c_1 \left( 2 + \text{sech}^2(y + c) \right) \tanh(y + c) + c_2, \\
\beta_H^- &= c_1 \left( 2 - \text{csch}^2(y + c) \right) \coth(y + c) + c_2, \\
\beta_{III}^- &= c_1 e^{\pm 4y} + c_2.
\end{align*}

In particular, we get the following assertion:

**Proposition 4.3.** The second and the third approximation functions of each $F \in Y^\omega_\beta$ can be written in terms of elementary functions.

The following assertion implies that our approximation for $F \in Y^\omega$ itself is an element of $Y^\omega$:

**Proposition 4.4.** Let $\alpha$ and $\beta$ be analytic functions satisfying
\[ \alpha'' + 2\alpha \alpha' = 0, \quad \beta'' + 4\alpha \beta' = 0. \]
Then the immersion
\[ F_{\alpha, \beta}(x, y) := \left( y + \frac{\alpha(y)}{2} x^2 + \frac{\beta(y)}{3} x^3, x, y \right) \]
belongs to the class $Y^\omega$.

This assertion follows from the proof of Proposition 4.1 immediately. Moreover, the following assertion holds:

**Proposition 4.5.** We fix an analytic function $\varphi \in C^\omega_0(R^2)$. Let $\alpha, \beta$ be analytic functions satisfying (4.8). Then there exists an immersion $F \in Y^\omega_\varphi$ such that $\alpha = \alpha_F$ and $\beta = \alpha_F$. In the case of $\varphi = 0$, this implies the existence of $F_{\alpha, \beta} \in Z^\omega_0$.

**Proof.** We set
\[ \gamma(x) = (1, 0) + \frac{x^2}{2} (\alpha(0), \beta(0)) + \frac{x^3}{3} (\alpha'(0), \beta'(0)). \]
By Theorem 2.3 there exists a unique immersion $F \in Y^\omega_\varphi \cap Y^\omega_0$ such that $\gamma_F = \gamma$. Then we have $\alpha_F = \alpha$ and $\beta_F = \beta$, proving the assertion. \[\Box\]

Finally, we prove the following assertion for the sign of the Gaussian curvature near a degenerate light-like point. (cf. Proposition 3.8 for the non-degenerate case.)
Proposition 4.6. Let \( F \) be an immersion in the class \( \mathcal{Y}_b^\omega \). Then the Gaussian curvature function \( K \) diverges to \( \infty \) at a degenerate light-like point if \( \mu_F > 0 \). On the other hand, if \( \mu_F = 0 \) and \( \delta_F \neq 0 \), then \( K(x, y) \) diverges to \( +\infty \) (resp. \( -\infty \)) on the domain of \( B_F(x, y) > 0 \) (resp. \( B_F(x, y) < 0 \)) as \( (x, y) \to (0, 0) \), where
\[
\delta_F := \beta'(0) + 3\alpha(0)\beta(0)
\]
and \( \alpha = \alpha_F, \beta = \beta_F \).

Proof. Recall that the Gaussian curvature \( K \) is expressed as (cf. (3.9))
\[
K = -\frac{C_F}{(B_F)^2} \quad (C_F = f_{xx}f_{yy} - f_{xy}^2),
\]
where \( f = \iota_F \) and \( B_F \) is the function defined as in (1.2). Since \( F \in \mathcal{Y}_b^\omega \), can be expanded as (4.1). Then we have
\[
C_F = f_{xx}f_{yy} - (f_{xy})^2 = \left( \frac{1}{2} \alpha'' - (\alpha')^2 \right) x^2 + \left( \beta \alpha'' - 2\alpha'\beta' + \frac{1}{3} \alpha\beta'' \right) x^3 + \text{(higher order terms)}.
\]
Since \( F \in \mathcal{Y}_b^\om \), the relations (4.3) and (4.4) hold, and then we have
\[
C_F = \alpha' \mu_F x^2 - \frac{2}{3} \left( 3\alpha'\beta' + 3\alpha\alpha' + 2\alpha^2 \beta' \right) x^3 + \text{(higher order terms)}.
\]
If \( \mu_F > 0 \), then \( \alpha'(0) < 0 \) by (4.3), so we get the conclusion. We next assume \( \mu_F = 0 \), then \( \alpha' = -\alpha^2 \) and we have
\[
C_F(x, 0) = \frac{2\alpha(0)^2\delta_F}{3} x^3 + \text{(higher order terms)}.
\]
In this situation, it holds that
\[
B_F(x, 0) = -\frac{2\delta_F}{3} x^3 + \text{(higher order terms)}.
\]
Thus the sign of the Gaussian curvature \( K(x, 0) \) coincides with that of \( B_F(x, 0) \), proving the assertion. \( \square \)

5. Examples

In this section, we give several examples of zero mean curvature surfaces: We now give here a recipe to give more refined approximate solutions as follows: For \( F \in \mathcal{Y}_b^\omega \), we can expand the function \( f = \iota_F \) as
\[
f(x, y) = y + \sum_{k=2}^{\infty} \frac{a_k(y)}{k} x^k.
\]
We call each function \( a_k(y) \) as the \( k \)-th approximation function of \( F \). Remark that \( a_2 \) and \( a_3 \) coincide with \( \alpha_F \) and \( \beta_F \) in (4.1), respectively:
\[
\alpha_F = a_2, \quad \beta_F = a_3.
\]

We give here several examples:

Example 5.1. The light-like plane (cf. Example 1.4) \( F(x, y) = (y, x, y) \) belongs to \( \Lambda^\omega \cap \mathcal{Z}_1^I \) such that \( \gamma_F = (0, 1) \) and \( \alpha_F = \beta_F = 0 \).
Example 5.2. The light-cone (cf. Example 1.5)

\[ F(x, y) = (\sqrt{x^2 + (1 + y)^2} - 1, x, y) \]

also belongs to \( \Lambda^\omega \cap \mathcal{Z}_1^H \) such that

\[ \gamma_F = (\sqrt{1 + x^2} - 1, 1/\sqrt{1 + x^2}) \]

and \( \alpha_F = 1/(1 + y) \), \( \beta_F = 0 \).

Example 5.3. The surface \( F(x, y) = (y + x^2/2, x, y) \) is a zero-mean curvature surface in \( \mathcal{Z}_1^H \), which satisfies \( \gamma_F = (x^2/2, 0) \) and

\[ \alpha_F = 1, \quad \beta_F = 0. \]

Example 5.4. Recall the space-like Scherk surface \( \{ (t, x, y) \in \mathbb{R}^3_1; \cos t = \cos x \cos y \} \) (cf. [5, Example 3]). We replace \( (t, x, y) \) by \( (t + (\pi/2), x, (\pi/2) - y) \), we have the expression

\[ F(x, y) = \left( -\arccos(\cos x\sin y - \frac{\pi}{2}), x, y \right) \]

satisfying (1.3). This is a zero-mean curvature surface in \( \mathcal{Z}_1^+ \), which satisfies \( \gamma_F = (-\pi, \cos x) \) and

\[ \alpha_F = -\tan y, \quad \beta_F = 0. \]

Example 5.5. The time-like Scherk surface of the first kind (cf. [5, Example 4]) can be normalized as

\[ F(x, y) = \left( \arccosh(\cosh x \cosh(y + 1)) - 1, x, y \right), \]

which is a zero-mean curvature surface in \( \mathcal{Z}_1^- \). This satisfies

\[ \gamma_F = \left( -1 + \arccosh(\cosh x \cosh 1), \frac{\sinh 1 \cosh x}{\sqrt{\cosh 1 \cosh x^2 - 1}} \right) \]

and

\[ \alpha_F = \coth y, \quad \beta_F = 0. \]

Example 5.6. The time-like Scherk surface of the second kind (cf. [5, Example 5])

\[ F(x, y) = \left( \arcsinh(\cosh x \sinh y), x, y \right) \]

is a zero-mean curvature surface in \( \mathcal{Z}_1^- \), which satisfies \( \gamma_F = (0, \cosh x) \) and

\[ \alpha_F = \tanh y, \quad \beta_F = 0 \]

hold.

For \( F \in \mathcal{Z}_b^\omega \), it holds for \( k \geq 4 \) that

\[ \frac{d^k A}{dx^k} \bigg|_{x=0} = 0 \quad (A := A_F), \]

which can be considered as an ordinary differential equation of the \( k \)-th approximation function \( a_k \) as in (5.1). As shown in [7], (5.3) is equivalent to

\[ a_k'' + 2(k - 1)a_2a_k' + k(3 - k)a_2a_k + k(P_k + Q_k - R_k) = 0, \]
where $P_k, Q_k, R_k$ are terms written using $\{a_s\}_{s<k}$ as follows:

\[
\begin{align*}
P_k &= \sum_{m=3}^{k-1} \frac{2(k - 2m + 3)}{k - m + 2} a_m a'_{k-m+2}, \\
Q_k &= \sum_{m=2}^{k-2} \sum_{n=2}^{k-m} \frac{3n - k + m - 1}{mn} a'_m a'_n a_{k-m-n+2}, \\
R_k &= \sum_{m=2}^{k-2} \sum_{n=2}^{k-m} a_m a_n a''_{k-m-n+2}.
\end{align*}
\]

When $k = 4$, (5.4) reduces to

\[
a''_4 + 6a_2 a'_4 - 4a'_2 a_4 + 3a_2 (a'_2)^2 - 2(a_2)^2 a''_2 + \frac{8}{3}a_3 a'_3 = 0.
\]

Using (5.4) and (5.1), one can get an appropriate approximation for $F$.

Finally, we remark on existence results of zero mean curvature surface using Corollary 2.4: For $F \in \mathcal{Z}_{\omega}$, we set

\[
\gamma_F = (0,1) + (0,v_1) x + \sum_{n=2}^4 \left( u_n, v_n \right) \frac{x^n}{n} + \text{(higher order terms)}.
\]

Then $F \in \mathcal{Z}_{\omega}$ if and only if $v_1 = 0$. Under the assumption $v_1 = 0$, the characteristic (cf. (4.3)) $\mu_F$ and the constant $\delta_F$ in (4.9) satisfy

\[
\mu_F = -(u_2^2 + v_2) \quad \text{and} \quad \delta_F = 3u_2 u_3 + v_3.
\]

We set

\[
\Delta_F := 4u_3^2 + 8u_2 u_4 + v_2^2 + 2v_4.
\]

**Proposition 5.7.** A surface $F \in \mathcal{Z}_{\omega}$ belongs to $\mathcal{Z}_0^I$ (resp. $\mathcal{Z}_0^I$) if $\mu_F = 0$ and $u_2 = 0$ (resp. $\mu_F = 0$ and $u_2 \neq 0$). Suppose that $\mu_F = 0$. Then

1. $F$ changes causal type if $\delta_F \neq 0$, and
2. $F$ has no time-like (resp. space-like) part if $\delta_F = 0$ and $\Delta_F < 0$ (resp. $\Delta_F > 0$).

**Proof.** The causal type of $F$ depends on the sign of $B := B_F$. As shown in [5], $B|_{x=0} = B_2|_{x=0} = 0$. Moreover, one can easily see that

\[
(5.5) \quad B(x,0) = \mu_F x^2 - \frac{2\delta_F}{3} x^3 - \frac{\Delta_F}{4} x^4 + \text{(higher order terms)}.
\]

So we get the conclusion. \qed

**Example 5.8.** In [7], $F \in \mathcal{Z}_{\omega}$ satisfying

\[
\gamma_F(x) := (0,1 + 3cx^3)
\]

is constructed, which belongs to the class $\mathcal{Z}_{\omega}^r$ and changes its causal type. Although the existence of this $F$ is obtained by applying Corollary 2.4, the advantage of the method in [7] is that we can get the explicit approximation for $F$ at the same time.

Until now, the existence of zero mean curvature surfaces (i.e. ZMC-surfaces) in the following three cases was unknown (cf. the footnote of [6 Page 194]):

1. ZMC-surfaces in $\mathcal{Z}_0^I$ without space-like part,
2. ZMC-surfaces in $\mathcal{Z}_0^I$ without time-like part,
3. ZMC-surfaces in $\mathcal{Z}_0^I$ which changes causal type.
We can show the existence of the above remaining cases:

Corollary 5.9. There exist ZMC-immersions satisfying (i) (ii) and (iii) respectively.

Proof. We set \( \mu_F = 0 \). If \( u_2 \neq 0 \) and \( \delta_F \neq 0 \), then \( F \in Z_0^0 \) which changes causal type (i.e. it gives the case (iii)). On the other hand, if \( u_2 = \delta_F = 0 \) and \( \Delta F < 0 \) (resp. \( \Delta F > 0 \)), then it gives the case (ii) (resp. (i)). \( \square \)

Appendix A. Division Lemma

Lemma A.1. Let \( g \) be a \( C^r \)-function (\( r \geq 1 \)) defined on a convex domain \( U \) of the \( xy \)-plane including the origin \( o \), satisfying

\[
(A.1) \quad g(0, y) = \frac{\partial g}{\partial x}(0, y) = \frac{\partial^2 g}{\partial x^2}(0, y) = \cdots = \frac{\partial^k g}{\partial x^k}(0, y) = 0 \quad ((0, y) \in U)
\]

for a non-negative integer \( k < r \). Then there exists a \( C^{r-k-1} \)-function \( h \) defined on \( U \) such that

\[
(A.2) \quad g(x, y) = x^{k+1}h(x, y) \quad ((x, y) \in U).
\]

Proof. We shall prove by an induction in \( k \). Since

\[
g(x, y) = \int_0^1 \frac{dg(tx, y)}{dt} dt = \int_0^1 xg_x(tx, y) dt = x \int_0^1 g_x(tx, y) dt,
\]

the conclusion follows for \( k = 0 \), by setting

\[
h(x, y) := \int_0^1 g_x(tx, y) dt.
\]

Assume that the statement holds for \( k - 1 \). If \( g \) satisfies (A.1), there exists a \( C^{r-k} \)-function \( \varphi(x, y) \) defined on \( U \) such that

\[
(A.3) \quad g(x, y) = x^k \varphi(x, y) \quad ((x, y) \in U).
\]

Differentiating this \( k \)-times in \( x \), we have

\[
0 = \frac{\partial^k g}{\partial x^k}(0, y) = k! \varphi(0, y)
\]

because of (A.1). Hence, by the case \( k = 0 \) of this lemma, there exists \( C^{k-r-1} \)-function \( h(x, y) \) defined on \( U \) such that \( \varphi(x, y) = xh(x, y) \). The function \( h \) is the desired one. \( \square \)

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