ASYMPTOTIC BEHAVIOR OF A HIERARCHICAL SIZE-STRUCTURED POPULATION MODEL

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Abstract. We study in this paper a hierarchical size-structured population dynamics model with environment feedback and delayed birth process. We are concerned with the asymptotic behavior, particularly on the effects of hierarchical structure and time lag on the long-time dynamics of the considered system. We formally linearize the system around a steady state and study the linearized system by $C_0$-semigroup framework and spectral analysis methods. Then we use the analytical results to establish the linearized stability, instability and asynchronous exponential growth conclusions under some conditions. Finally, some examples are presented and simulated to illustrate the obtained results.

1. Introduction. In this paper we study the following system of a hierarchical size structured population model

\[
\begin{aligned}
\frac{\partial}{\partial t} u(s,t) + \frac{\partial}{\partial s} \left( \gamma(s) u(s,t) \right) &= -\mu(s) u(s,t), \quad s \in [0,m], \ t > 0, \\
u(0,t) &= \int_0^m \int_{-\tau}^0 \beta(s,\sigma,Q(s,t+\sigma))u(s,t+\sigma)d\sigma ds, \quad t > 0, \\
u(s,t) &= u^0(s,t), \quad s \in [0,m], \ t \in [-\tau,0],
\end{aligned}
\]

(1.1)

where the function $u = u(s,t)$ denotes the density of individuals of size $s \in [0,m]$ at time $t \in [0,\infty)$. The function $\gamma = \gamma(s)$ represents the growth rate of an individual of size $s$, while $\mu = \mu(s)$ is the mortality rate of an individual of size $s$. $\beta \in C([0,m] \times [-\tau,0] \times C^1[-\tau,\infty))$ should be understood as the probability that an individual of size $s$ reproduces after a time lag $-\tau$ starting from conception, here $\tau > 0$ is a constant denoting the maximal time delay the birth process takes. Such a form of the delayed birth process was initiated by Pizzera [30] by taking into account the time which the birth process may take, and it has been adopted in many papers, such as in papers [18, 19, 37, 38], to discuss the long-time behavior of the corresponding nonlinear systems and explore how the time lag influences the asymptotic behaviors of the considered systems.

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On the other hand, the vital rates of structured population models, including the birth rate $\beta$, frequently depend on the environment. In (1.1), it is assumed that the birth rate $\beta$ of individuals depend on the environment feedback at time $t$ which is given by (see [6, 14])

$$Q(s, t) = \alpha \int_0^s w(\eta)u(\eta, t)d\eta + \int_s^m w(\eta)u(\eta, t)d\eta, \quad s \in [0, m), \ t > 0. \quad (1.2)$$

Here the function $w$ represents a positive weight (population measure). For example, $w(\eta) = 1$ means the total number of individuals in the population while $w(\eta) = \eta$ means the total biomass (see [1]). The constant $\alpha$ is a parameter in $[0,1]$ measuring the degree of hierarchy in the population structure. More precisely, $\alpha$ is the weight of the lower ranks in the competition for resources. So, $\alpha = 0$ (which represents contest or asymmetric competition) means an absolute hierarchical social structure, whereas values of $\alpha$ tending to 1 mean that the effect of higher ranks (in the competition for resources) tends to be similar to the effect of lower ranks. The limit case $\alpha = 1$ (which represents scramble competition) corresponds to an environment given by the total population at time $t$, i.e., without any kind of social structure, and it has been treated in details in [15].

This kind of systems (whose rates depend on environment $Q$) are used to describe many structured population models. A simple example is given by a forest consisting of tree individuals in which the height of a tree determines its rank in the population (see [24]). Taller individuals have higher efficiency when competing for resources such as light, while individuals of lower rank cannot affect the vital rates of individuals of higher rank. Also in a food chain the size of the individual sometimes determines the size of its prey as well as that of its predator (see [2]). Recent efforts have been directed to the study of hierarchical structured models which take into account the efficiency of an individual in the competition amongst others (e.g. [1, 2, 6, 7, 8]). In [1, 6, 7, 8] the authors have investigated the existence-uniqueness of continuous solutions and the long time dynamics for various hierarchical structured models by using transforming and approximation approaches. In [2] Ackleh and Ito studied the existence of measure-valued solutions for a hierarchically size-structured population model with growth, mortality and reproduction rates depending on a function of the population density (environment). They discussed the problem by applying the vanishing viscosity method.

Particularly, in [14, 16] Farkas and Hagen studied some systems involving competitions among individuals which are based upon some hierarchy or cannibalism in the population relating to the size of individuals. They applied linear semigroup methods to formulate biologically interpretable conditions for the linear stability/instability of equilibria of such hierarchy or cannibalism size-structured population models.

In this work, we study the asymptotic behavior for the hierarchical size-structured population model (1.1) with delayed birth process. Our main purpose is to extend the work in [30, 31] by considering the hierarchical structure in the birth function $\beta$ and investigating the effect of the environment $Q$ in $\beta$ on the long-time behavior for System (1.1). It is seen that, other than in [30, 31], (1.1) is actually no longer a standard linear system due to the hierarchical structure in the birth function, and as a result, it may admit two equilibriums (the null solution and a positive stationary solution which is more biologically meaningful) and bring about more interesting dynamical behavior. Diekmann et al. have developed a general mathematical
framework to study analytical questions for structured populations (see [9, 10, 11]),
including those pertaining to linear/nonlinear stability of population equilibria. In
this context it was proven for large classes of structured population models, formulated
as integral (or delay) equations, that the nonlinear dynamics of a population
equilibrium, such as stability and bifurcations, is completely determined by its linearized
equation, which is commonly referred to as the “Principle of Linearization”,
see Theorem 2.6 and Section 4 in [10] and Theorem 1.3 in [23]. By virtue of this
principle, we investigate in this paper the dynamics of the linearized equation of the
system (1.1). Precisely, we shall explore the linearized stability and instability of
stationary solutions of the system (1.1) by using semigroup techniques and spectral
methods based on the characteristic equation, and meanwhile, the asynchronous
exponential growth property (AEG, for short, it will be interpreted in details in
Section 6 below) of solutions will be studied based on the spectral analysis as well.
We first present the necessary conditions of the existence of positive equilibrium and
linearize the considered system (1.1) around the equilibriums. We then employ the
Perron-Frobenius techniques as in [20, 30, 31] to carry on our discussion and some
results on linearized stability, instability and AEG for the linearized equations are
obtained under proper conditions. Particularly, we find interestingly that the AEG
phenomena may occur around both the equilibriums under the given conditions.
The examples and the simulation illustrate well the applications of the obtained
results.

Clearly, the system discussed here is biologically and mathematically more gen-
eral than that in [30, 31] because the discussed equations there are merely linear
systems. As described above, the nonlinearity of the boundary condition in (1.1)
requires us study the problems by using the linearization arguments, and corre-
spondingly, the obtained results in this article are richer than those in [30, 31] (note
that there is only the trivial stationary) and improve their main theorems directly.
On the other hand, it can be seen that, the stability/instability results obtained
here also extend and develop the corresponding ones in [1, 2, 14, 15, 16] as the
effects of time lag in birth process on asymptotic behavior are studied in this paper.
Moreover, the techniques employed here differs greatly from these papers due to the
extra time delay. We also would like to point out that, it should be more reasonable
or practical if the rates $\gamma(\cdot)$ and $\mu(\cdot)$ in (1.1) depend on the environment $Q$
like in [1, 2, 6, 7]. However it is very difficult to obtain the explicit expression of the
characteristic equation and so we only discuss here the simpler case for which the
rates $\gamma(\cdot)$ and $\mu(\cdot)$ are independent of $Q$.

The organization of this paper is as follows. In Section 2, we collect some nota-
tions and results on theory of $C_0$-semigroup which will be used in the later sections,
then we linearize the system (1.1). In Section 3, we set the linear system in the
framework of semigroup theory, and prove existence and uniqueness of the solutions
for the simplified system by showing that the related abstract Cauchy problem gives
rise to a strongly continuous semigroup via Hille-Yosida operator theory. In Section
4, some regularity properties are derived for the linearized system, following that we
also deduce the characteristic equation in this section. Then we discuss in Section
5 the linearized stability and instability of the stationary solution. And in Section
6 we devote to discussing the AEG property for the linearized system under some
conditions. Finally, in Section 7, we provide some simple examples to illustrate the
obtained results through numerical simulations.
2. Preliminaries and the linearized system. In this section we recall some preliminaries on linear operators and \( C_0 \)-semigroup which will be used later. We will also linearize the system (1.1) around a stationary solution so that we may construct the \( C_0 \)-semigroup framework for our discussion.

Let \( L : D(L) \subset X \to X \) be a linear operator on a Banach space \( X \). Denote by \( \rho(L) \) the resolvent set of \( L \). The spectrum of \( L \) is \( \sigma(L) = \mathbb{C} \setminus \rho(L) \). The point spectrum of \( L \) is given by the set
\[
\sigma_p(L) := \{ \lambda \in \mathbb{C} : N(\lambda I - L) \neq \{0\} \}.
\]

**Definition 2.1.** Let \( L : D(L) \subseteq X \rightarrow X \) be a linear operator. Assume that there exist real constants \( M \geq 1 \), and \( \omega \in \mathbb{R} \), such that \( (\omega, +\infty) \subseteq \rho(L) \), and
\[
\| (\lambda - L)^{-n} \| \leq \frac{M}{(\lambda - \omega)^n}, \text{ for } n \in \mathbb{N}_+, \text{ and all } \lambda > \omega.
\]

Then the linear operator \((L, D(L))\) will be called a Hille-Yosida operator.

If \((L, D(L))\) is a Hille-Yosida operator on the Banach space \( X \) and set
\[
X_0 := \{(D(L), \| \cdot \|)\},
\]
\[
D(L_0) := \{x \in D(L) : Lx \in X_0\},
\]
\[
L_0x := Lx \text{ for } x \in D(L_0).
\]

Then the operator \((L_0, D(L_0))\) is called the part of \( L \) in \( X_0 \) and one has that.

**Lemma 2.2** (see [13] and [29]). If \((L, D(L))\) is a Hille-Yosida operator, then its part \((L_0, D(L_0))\) generates a strongly continuous semigroup \((T_0(t))_{t \geq 0}\) on \( X_0 \).

If \( L \) is the generator of a strongly continuous semigroup \((T(t))_{t \geq 0}\), we present briefly the concepts of the extrapolation space and Favard class here, see [27] for more details.

**Definition 2.3.** On Banach space \( X \), a new norm associated to linear operator \( L \) can be defined by
\[
\|x\|_{-1} := \| (\lambda I - L)^{-1} x \|, \text{ for } x \in X \text{ and } \lambda > \omega.
\]

Then the completion of the space \((X, \| \cdot \|_{-1})\) will be called the extrapolation space of \( X_0 \) associated to \( L \) and will be denoted by \( X_{L-1} \) (or simply \( X_{-1} \)).

The (unique) extension of \( T(t) \) to \( X_{-1} \) is denoted by \( T_{-1}(t) \), it is a semigroup on \( X_{-1} \) and its generator is denoted as \( A_{-1} \).

**Definition 2.4.** The Favard class of the generator \( L \) is the space
\[
F = F^L := \{ x \in X : \lim_{t \to 0} \sup_{t} \frac{1}{t} \| T(t)x - x \| < +\infty \}
\]
equipped with the norm
\[
\| x \|_{F^L} := \lim_{t \to 0} \sup_{t} \frac{1}{t} \| T(t)x - x \| \text{ for } x \in F^L_0.
\]

In addition, a crucial quantity associated to it is the growth bound \( \omega_0(L) \), defined as
\[
\omega_0(L) := \lim_{t \to +\infty} \frac{\log \| T(t) \|}{t}.
\]
The essential growth bound \( \omega_{\text{ess}}(L) \) is defined by
\[
\omega_{\text{ess}}(L) := \lim_{t \to +\infty} t^{-1} \log(\alpha(T(t))),
\]
where \( \alpha \) is the measure of noncompactness defined as
\[
\alpha[T] := \inf \left\{ \| T - K \| : K \text{ is a compact operator on } X \right\}
\] for \( T \in \mathcal{L}(X) \).

Then there holds
\[
\omega_0(L) = \max \{ \omega_{\text{ess}}(L), s(L) \},
\]
where \( s(L) \) is the spectral bound of \( L \), i.e. \( s(L) = \sup \{ \Re \lambda : \lambda \in \sigma(L) \} \). The essential spectral radius of \( T \in \mathcal{L}(X) \) is given by
\[
r_{\text{ess}}(T) := \lim_{t \to +\infty} (\alpha[T^n])^{\frac{1}{n}}.
\]

With Definition 2.3, Definition 2.4 and these notations, we have the following perturbation result to be used to prove Theorem 6.4.

**Lemma 2.5.** (see [32], Corollary 2.2 or [35], Theorem 2.4) Let \((L, D(L))\) be a Hille-Yosida operator on a Banach space \( X \) and let \( C \) be a bounded operator from \( X_0 = \overline{D(L)} \) to \( F^{L_{-1}} \), the Favard class of \( L \) (see Definition 2.4). Let \((T_0(t))_{t \geq 0}\) and \((T_0(t))_{t \geq 0}\) be the \( C_0 \)-semigroup on \( X_0 \) generated by the parts of \( L \) and \( L_{-1} + C \) in \( X_0 \), respectively. If \( CT_0(t) \) is a compact operator from \( X_0 \) to \( F^{L_{-1}} \) for all \( t > 0 \), then
\[
r_{\text{ess}}(T_0(t)) = r_{\text{ess}}(T_0(t)).
\]

Next we set about to linearize the system (1.1). Obviously Eqs. (1.1) admit the trivial solution \( u_* \equiv 0 \). Realistically we also expect some additional positive (continuously differentiable) stationary solutions \( u_*>0 \) for (1.1). In the following proposition we formulate a necessary condition for the existence of a positive equilibrium solution of problem (1.1).

**Proposition 1.** For the given vital rate functions \( \beta, \mu, \gamma \), if \( u_* \) is a positive stationary solution of problem (1.1), then the function \( Q_* \) defined by
\[
Q_*(s) = \alpha \int_0^s w(\eta)u_*(\eta)d\eta + \int_s^m w(\eta)u_*(\eta)d\eta,
\]
(2.1)
satisfies
\[
1 = \int_0^m \frac{\gamma(0)}{\gamma(s)} e^{-\int_0^s \frac{\mu(y)}{\gamma(y)}dy} \int_0^0 \beta(s, \sigma, Q_*(s))d\sigma ds,
\]
(2.2)
or equivalently \( R(Q_*) = 1 \) where \( R : C([0, m]) \to \mathbb{R} \) is the inherent net reproduction rate
\[
R(Q) = \int_0^m \pi(s) \int_0^s \beta(s, \sigma, Q(s, t + \sigma))d\sigma ds
\]
(2.3)
and the operator \( \pi \) is given for \( 0 \leq s \leq m \) by
\[
\pi(s) = \frac{\gamma(0)}{\gamma(s)} e^{-\int_0^s \frac{\mu(y)}{\gamma(y)}dy}.
\]
(2.4)
In this case, the unique positive stationary solution \( u_* \) of problem (1.1) is given by
\[
u_* = \frac{U_* e^{-\int_0^s \frac{\mu(y)+\gamma(y)}{\gamma(y)}dy}}{\int_0^m e^{-\int_0^s \frac{\mu(y)+\gamma(y)}{\gamma(y)}dy}ds},
\]
(2.5)
that is
\[
u_* = \frac{U_* \pi(s)}{\int_0^m \pi(s)ds},
\]
(2.6)
where \( U_* = \int_0^m u_*(s)ds \) represents the positive population quantity.
Observing that the functional dependence of the fertility rate on $Q$ rather than on $u$ requires the linearization about $Q_*$. Thus by the approximation

$$\beta(s,\sigma,Q(s,t+\sigma)) = \beta(s,\sigma,Q_*(s)) + \beta_Q(s,\sigma,Q_*)(Q(s,t+\sigma) - Q_*(s)) + \text{h.o.t.},$$

in system (2.12) and dropping all the nonlinear terms, we arrive at the linearized problem

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} v(s,t) + \gamma(s) \frac{\partial}{\partial s} v(s,t) + \rho(s) v(s,t) = 0, \quad t > 0, \quad s \in [0,m], \\
v(0,t) = \int_0^m \int_{-\tau}^0 \beta(s,\sigma,Q_*(s)) v(s,t+\sigma) d\sigma ds \\
\quad + \int_0^m \int_{-\tau}^0 \beta_Q(s,\sigma,Q_*(s)) V(s,t+\sigma) d\sigma ds, \quad t > 0, \\
v(s,t) = v^0(s,t), \quad t \in [-\tau,0], \quad s \in [0,m],
\end{array} \right. \quad (2.13)$$

where we have set

$$\rho(s) = \gamma'(s) + \mu(s). \quad (2.14)$$
\[ V(s,t) = Q(s,t) - Q_*(s) = \alpha \int_0^s w(\eta)v(\eta,t)d\eta + \int_s^m w(\eta)v(\eta,t)d\eta. \] (2.15)

3. \textbf{C₀-semigroup for linear system.} To analysis the asymptotic behavior for linearized system (2.13), we establish in this section the \( C₀ \)-semigroup framework for this system and through which rewrite it into an abstract evolution equation.

Suppose \( u_\ast \) is any positive stationary solution of problem (1.1). We denote the Banach space \( X = L^1(0,m) \) with the usual norm \( \| \cdot \| \) and on this space we introduce the following operator

\[ (A_m f)(s) = -\gamma(s)f'(s) - \rho(s)f(s) \] for a.e. \( s \in [0,m] \),

with domain

\[ D(A_m) = W^{1,1}(0,m). \]

Here the subscript \('m'\) reminds that the operator is defined on its maximal domain and \( \rho(s) \) is given by (2.14). Moreover, we call the map \( P : D(A_m) \to X \) defined as

\[ Pf := f(0) \]

the boundary operator, which is used to express the boundary condition (see [21, 30]).

Now look at the Banach space

\[ E := L^1([-\tau,0],X) \cong L^1((0,m) \times [-\tau,0]). \]

On this space we introduce the operator \( \Phi \in \mathcal{L}(E,X) \), called delay operator, by

\[ \Phi g = \int_0^m \int_{-\tau}^0 \beta(s,\sigma, Q_*(s))g(s,\sigma)d\sigma ds + \int_0^m u_\ast(s) \int_{-\tau}^0 \beta Q(s,\sigma, Q_*(s)) \left( \alpha \int_0^s w(\eta)g(\eta,\sigma)d\eta + \int_s^m w(\eta)g(\eta,\sigma)d\eta \right) d\sigma ds. \]

Then with these operators the linearized system (2.13) can be cast in the form of an abstract boundary delay problem:

\[
\begin{cases}
\frac{d}{dt}v(t) = A_m v(t), & t \geq 0, \\
P v(t) = \Phi v_t, & t \geq 0, \\
v_0(t) = v^0(t), & t \in [-\tau,0],
\end{cases}
\] (3.1)

where \( v^0(t) := v^0(\cdot,t), v : [0, +\infty) \to X \) is defined as \( v(t) := v(\cdot,t) \) and \( v_t : [-\tau,0] \to X \) is the history segment defined in the usual way as

\[ v_t(\sigma) := v(t + \sigma), \quad \sigma \in [-\tau,0]. \]

In order to apply the \( C₀ \)-semigroup theory, we rewrite (3.1) as an abstract Cauchy problem. For this, as in [30, 31], on the space \( E \) we consider the differential operator

\[ (G_m g)(\sigma) := \frac{d}{d\sigma} g(\sigma), \]

with domain

\[ D(G_m) = W^{1,1}([-\tau,0],X). \]

In addition, we introduce another boundary operator \( Q : D(G_m) \to X \) defined as

\[ Qg := g(0). \]
Finally, we consider the product space
\[ X := E \times X, \]
on which we define the operator matrix
\[ A := \begin{pmatrix} G_m & 0 & 0 \\ 0 & A_m \end{pmatrix}, \tag{3.2} \]
with domain
\[ D(A) = \{ (g, f) \in D(G_m) \times D(A_m) : Qg = f, Pf = \Phi g \}. \]

With these notations, we obtain the following abstract Cauchy problem
\[ \begin{cases} \mathcal{V}'(t) = A\mathcal{V}(t), & t \geq 0, \\
\mathcal{V}(0) = \mathcal{V}_0, \end{cases} \tag{3.3} \]
associated to the operator \((A, D(A))\) on the space \(X\). Here the function \(\mathcal{V} : [0, +\infty) \to \mathcal{X}\) is given by
\[ \mathcal{V}(t) = \begin{pmatrix} v_t \\ v(t) \end{pmatrix}. \]

To obtain the well-posedness of solutions for the abstract Cauchy problem (3.3), in the sequel we will verify that \((A, D(A))\) generates a \(C_0\)-semigroup on \(X\).

In the first step, we consider the Banach space
\[ \mathcal{X} := E \times X \times X \times \mathbb{C} \]
and the matrix operator
\[ \mathcal{A} := \begin{pmatrix} G_m & 0 & 0 & 0 \\ -Q & 0 & Id & 0 \\ 0 & 0 & A_m & 0 \\ \Phi & 0 & -P & 0 \end{pmatrix}, \]
with domain
\[ D(\mathcal{A}) = D(G_m) \times \{0\} \times D(A_m) \times \{0\}. \]

We shall use the following well-known perturbation result to show the operator \((\mathcal{A}, D(\mathcal{A}))\) to be a Hille-Yosida operator:

**Lemma 3.1** (see [13, 29]). Let \((A, D(A))\) be a Hille-Yosida operator on a Banach space \(X\) and \(B\) be a bounded linear operator on \(X\), then the sum \(C = A + B\) is also a Hille-Yosida operator.

By this lemma we have that

**Proposition 2.** The operator \((\mathcal{A}, D(\mathcal{A}))\) is a Hille-Yosida operator on the Banach space \(\mathcal{X}\).

**Proof.** The operator \(\mathcal{A}\) can be written as the sum of two operators on \(\mathcal{X}\) as
\[ \mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2, \]
where
\[ \mathcal{A}_1 = \begin{pmatrix} G_m & 0 & 0 & 0 \\ -Q & 0 & 0 & 0 \\ 0 & 0 & A_m & 0 \\ 0 & 0 & -P & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & Id & 0 \\ 0 & 0 & 0 & 0 \\ \Phi & 0 & 0 & 0 \end{pmatrix}. \]
with $D(\mathcal{A}_1) = D(\mathcal{A})$ and $D(\mathcal{A}_2) = \mathcal{X}_\gamma$.

The restriction $(G_0, D(G_0))$ of $G_m$ to the kernel of $Q$ generates the nilpotent left shift semigroup $(S_0(t))_{t \geq 0}$ on $E$ given by the formula

$$(S_0(t)g)(s, \sigma) = \begin{cases} g(s, t + \sigma), & \text{if } t + \sigma \leq 0, \\ 0, & \text{if } t + \sigma \geq 0. \end{cases}$$

Similarly, the restriction $(A_0, D(A_0))$ of $A_m$ to the kernel of $\mathcal{P}$ generates a strongly continuous positive semigroup $(T_0(t))_{t \geq 0}$ on $X$ given by

$$((T_0(t)f)(s) = \begin{cases} e^{-\int_{\Gamma(s)}^{\Gamma(s)-t} \frac{\sigma}{\gamma(y)} dy} f(\Gamma^{-1}(\Gamma(s) - t)), & \text{if } \Gamma(s) \geq t, \\ 0, & \text{if } \Gamma(s) \leq t, \end{cases}$$

where

$$\Gamma(s) = \int_0^s \frac{1}{\gamma(y)} dy. \quad (3.4)$$

We claim that $\mathcal{A}_1$ is a Hille-Yosida operator. In fact, for any $\lambda \in \mathbb{C}$ and $f_1 \neq 0$, let

$$(\lambda I - A_0)f = f_1,$$

we get

$$f(s) = e^{-\int_0^s \frac{\lambda + \gamma(y)}{\gamma(y)} dy} \int_0^s f_1(r) e^{\int_r^s \frac{\lambda + \gamma(y)}{\gamma(y)} dy} dr,$$

which shows $\sigma(A_0) = \emptyset$ as $\gamma(s) > 0$. And similarly, $\sigma(G_0) = \emptyset$. So for every $\lambda \in \mathbb{C}$, its resolvent is given by

$$R(\lambda, \mathcal{A}_1) = \begin{pmatrix} R(\lambda, G_0) & \epsilon_{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & R(\lambda, A_0) & \varphi_{\lambda} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\epsilon_{\lambda}(\sigma) = e^{\lambda \sigma}, \sigma \in [-\tau, 0], \text{ and } \varphi_{\lambda}(s) = e^{-\int_0^s \frac{\lambda + \gamma(y)}{\gamma(y)} dy}, s \in [0, m]. \quad (3.5)$$

Moreover,

$$\ker(\lambda - G_m) = \{ f : \epsilon_{\lambda} : f \in X \},$$

$$\ker(\lambda - A_m) = \langle \varphi_{\lambda} >.$$

Let $(g \ f_1 \ f_2 \ x)^T \in \mathcal{X}$ and $\lambda > 0$. It is true that

$$\| R(\lambda, \mathcal{A}_1)(g \ f_1 \ f_2 \ x)^T \| = \| R(\lambda, G_0)g + \epsilon_{\lambda} f_1 \|_E + \| R(\lambda, A_0) f_2 + x \varphi_{\lambda} \|_X$$

$$\leq \| R(\lambda, G_0) f_1 \|_E + \| \epsilon_{\lambda} g \|_E + \| R(\lambda, A_0) f_2 \|_X + \| x \varphi_{\lambda} \|_X$$

$$\leq \int_{-\tau}^0 \frac{1}{\lambda} \| g(\sigma) \|_X d\sigma + \frac{1}{\lambda} \| f_1 \|_X + \frac{1}{\lambda} \| f_2 \|_X + \frac{1}{\lambda} \| x \|_X$$

$$= \frac{1}{\lambda}(\| g \|_E + \| f_1 \|_X + \| f_2 \|_X + \| x \|_X).$$

Therefore, we have

$$\| \lambda R(\lambda, \mathcal{A}_1) \| \leq 1,$$

and $\mathcal{A}_1$ is a Hille-Yosida operator.

Since the perturbation operator $\mathcal{A}_2$ is clearly bounded, $\mathcal{A}$ is a Hille-Yosida operator as well by Lemma 3.1.
According to Lemma 2.2, we have actually obtained by Proposition 2 that the
operator \((\mathcal{A}_0, D(\mathcal{A}_0))\), the part of the operator \((\mathcal{A}, D(\mathcal{A}))\) in the closure of its
domain, generates a \(C_0\)-semigroup on the space \(E \times \{0\} \times X \times \{0\}\). Now we show
that the operator \((\mathcal{A}, D(\mathcal{A}))\) generates a strongly continuous semigroup on \(X\) by
the following theorem.

**Theorem 3.2.** The operator \((\mathcal{A}, D(\mathcal{A}))\) is isomorphic to the part \((\mathcal{A}_0, D(\mathcal{A}_0))\)
of the operator \((\mathcal{A}, D(\mathcal{A}))\) in the closure of its domain \(\overline{D(\mathcal{A})}\).

In particular, \((\mathcal{A}, D(\mathcal{A}))\) generates a \(C_0\)-semigroup \(\{T(t)\}_{t \geq 0}\) of bounded linear
operators on \(X\).

**Proof.** From Lemma 2.2, we know that the part \((\mathcal{A}_0, D(\mathcal{A}_0))\) of \((\mathcal{A}, D(\mathcal{A}))\) in the

\[
D(\mathcal{A}_0) = \left\{ x \in D(\mathcal{A}) : \mathcal{A}x \in \overline{D(\mathcal{A})} \right\}
= \left\{ \begin{pmatrix} g \\ 0 \\ f \\ 0 \end{pmatrix} : g \in D(G_m), f \in D(A_m), \mathcal{A} \begin{pmatrix} g \\ 0 \\ f \\ 0 \end{pmatrix} \in \mathfrak{K}_0 \right\}
\]

Therefore, the operator \((\mathcal{A}, D(\mathcal{A}))\) is isomorphic to \((\mathcal{A}_0, D(\mathcal{A}_0))\) and thus generates
a strongly continuous semigroup on the space \(X\). \(\square\)

The following well-posedness result for (3.1) is then a direct consequence of Theorem 3.2.

**Corollary 1.** For initial data \(\varphi_0 \in E\) the linear boundary delayed problem (3.1)
has a unique solution \(v\) in \(C([-\tau, +\infty), X)\), given by \(v(s, t) = \varphi_0(s, t)\) for \(t \in \lbrack -\tau, 0 \rbrack\)
and
\[
v(s, t) = \Pi_2 \left( T(t) \begin{pmatrix} v_0(s, 0) \\ v_0(s) \end{pmatrix} \right), \quad \text{for } t > 0,
\]
where \(\Pi_2\) is the projection operator of \(T(t)\) on the space \(X\).

4. **Spectral analysis, regularities and characteristic equation.** In this section, we prove two regularity results about the \(C_0\)-semigroup generated by \((\mathcal{A}, D(\mathcal{A}))\)
which imply that the spectrally determined growth property holds true and that
the linearized stability of the steady-state solution is governed by the location of
the leading eigenvalue. Then we drive the characteristic equation for the linearized
system.

The first main result of this section is that

**Theorem 4.1.** The semigroup \((T(t))_{t \geq 0}\) generated by the operator \((\mathcal{A}, D(\mathcal{A}))\) is
eventually compact. Particularly, the semigroup operator \(T(t)\) is compact for all
\(t > 2(\Gamma(m) + \tau)\).
Proof. We observe the abstract Cauchy problem (3.3) with $A$ becomes

$$\begin{align*}
\frac{d}{dt} \begin{pmatrix} v_t \\ v(t) \end{pmatrix} &= A \begin{pmatrix} v_t \\ v(t) \end{pmatrix}, \quad t \geq 0,
\mathcal{P}v(t) &= \Phi v_t.
\end{align*}$$

With the definition of $A$, we have

$$\begin{align*}
\frac{d}{dt} v_t(s, \sigma) &= G_m v_t(s, \sigma) = \frac{d}{d\sigma} v_t(s, \sigma), \\
\frac{d}{dt} v(t) &= A_m v(t) = -\gamma(s) \frac{\partial}{\partial s} v(s, t) - \rho(s) v(s, t),
\end{align*}$$

subject to the boundary condition in the system (1.1).

From the equation (4.1), we get the solution of $v_t$ that

$$v_t(s, \sigma) = v(s, t + \sigma), \quad \sigma \in [-\tau, 0].$$

To obtain the solution of $v(s, t)$, let us introduce, for $t_0 > 0$,

$$n(s) = v(s, t(s)),$$

where $t(s) = t_0 + \Gamma(s)$ with $\Gamma$ defined in (3.4). Then

$$\frac{d}{ds} n(s) = \frac{\partial}{\partial s} v(s, t(s)) + \frac{\partial}{\partial t} v(s, t(s)) \Gamma'(s)$$

$$= \frac{\partial}{\partial s} v(s, t) + \frac{1}{\gamma(s)} \frac{\partial}{\partial t} v(s, t(s)).$$

With the equation (4.2), $n$ satisfies

$$\frac{d}{ds} n(s) + \frac{\rho(s)}{\gamma(s)} n(s) = 0,$$

which yields that

$$n(s) = n(0) e^{-\int_0^s \frac{\rho(\eta)}{\gamma(\eta)} d\eta},$$

where $n(0) = v(0, t_0) = \Phi(v_{t_0})$. By the formula of $\Phi$, for $t_0 = t - \Gamma(s) > 0$, one has that

$$v(s, t) = e^{-\int_0^s \frac{\rho(\eta)}{\gamma(\eta)} d\eta} \left[ \int_0^m \int_{-\tau}^0 \beta(s, \sigma, Q_\ast(s)) v_{t-\Gamma(s)}(s, \sigma) d\sigma ds \\
+ \int_0^m \int_0^s u_\ast(s) \beta_Q(s, \sigma, Q_\ast(s)) \alpha \int_0^\eta w(\eta) v_{t-\Gamma(s)}(\eta, \sigma) d\eta d\sigma ds \\
+ \int_0^m u_\ast(s) \int_{-\tau}^0 \beta_Q(s, \sigma, Q_\ast(s)) \int_s^m w(\eta) v_{t-\Gamma(s)}(\eta, \sigma) d\eta d\sigma ds \right].$$

Therefore, if $t > \Gamma(m) + \tau$, $v$ is continuous in $s$ and $t$. Consequently, Eq. (4.2) implies that $v$ is continuous differentiable if $t > 2(\Gamma(m) + \tau)$. Hence the semigroup generated by $A$ is differentiable for $t > 2(\Gamma(m) + \tau)$. Since $W^{1,1}(0, m)$ is compactly imbedded in $X$, the claim follows.

Theorem 4.1 has the following immediate and noteworthy consequence (see [13, 20]).

**Corollary 2.** The spectrum of the semigroup generator $(A, D(A))$ consists of isolated eigenvalues of finite multiplicity only and the Spectral Mapping Theorem holds true, i.e.,

$$\sigma(T(t)) = \{0\} \cup e^{\sigma(A)}, \quad t > 0.$$
Moreover, the semigroup is spectrally determined, i.e. the growth rate \( \omega(T(t)) \) of the \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) and the spectral bound \( s(A) \) of its generator coincide.

Because of Corollary 2 the linear stability of the steady-state solution is spectrally determined (see [13, 29], Theorem VI.1.15). Hence in the sequel it suffices to investigate the location of the leading eigenvalue of the generator of the \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \).

In order to state and prove the second main theorem of this section, we need to introduce two operators as follows. For \( \lambda \in \rho(G_0) \cap \rho(A_0) \), we define

\[
K_\lambda : X \to E \text{ by } K_\lambda := 1 \circ \epsilon_\lambda, \\
L_\lambda : E \to X \text{ by } L_\lambda := (1 \circ \varphi_\lambda) \Phi,
\]

or more precisely,

\[
K_\lambda (f) = f \cdot \epsilon_\lambda, \text{ for } f \in X, \\
L_\lambda (g) = \Phi (g) \varphi_\lambda, \text{ for } g \in E,
\]

where \( \epsilon_\lambda \) and \( \varphi_\lambda \) are from (3.5). The next result was formulated in [26].

**Lemma 4.2** (see [26], Lemma 2.5). \( K_\lambda \in \mathcal{L}(X, E) \) and \( L_\lambda \in \mathcal{L}(E, X) \), moreover, \( Q(K_\lambda (f)) = f, \ P(L_\lambda (g)) = \Phi (g) \) for all \( g \in D(G_m), f \in D(A_m), \) with \( \epsilon_\lambda \) and \( \varphi_\lambda \) given in (3.5).

The decomposition of the operator \( \lambda - A \) below has also been proved in ([26], Lemma 2.6).

**Lemma 4.3.** For \( \lambda \in \rho(G_0) \cap \rho(A_0) \), we have

\[
(\lambda - A) = \begin{pmatrix} \lambda - G_0 & 0 \\ 0 & \lambda - A_0 \end{pmatrix} B_\lambda,
\]

where \( B_\lambda := \begin{pmatrix} Id & -K_\lambda \\ -L_\lambda & Id \end{pmatrix} \) is a bounded linear operator matrix on \( D(G_m) \times D(A_m) \) and the matrix \( \begin{pmatrix} \lambda - G_0 & 0 \\ 0 & \lambda - A_0 \end{pmatrix} \) has domain \( D(G_0) \times D(A_0) \).

The next results are due to Nagel ([26], Theorem 2.7) and Engel ([12], Theorem 2.8).

**Theorem 4.4.** Let \( \lambda \in \rho(G_0) \cap \rho(A_0) \), for the following statements

(a) \( \lambda \in \rho(A) \), 
(b) \( 1 \in \rho(K_\lambda L_\lambda) \) for the operator \( K_\lambda L_\lambda \in \mathcal{L}(E) \), 
(c) \( 1 \in \rho(L_\lambda K_\lambda) \) for the operator \( L_\lambda K_\lambda \in \mathcal{L}(X) \).

Then one has the implications (a) \( \iff \) (b) \( \iff \) (c).

If, in particular, \( K_\lambda \) and \( L_\lambda \) are compact operators, then the statements (a), (b) and (c) are equivalent.

Since the operator \( L_\lambda \) here has one-dimensional range, it is compact, and hence \( K_\lambda L_\lambda \) and \( L_\lambda K_\lambda \) are compact too. Then, from Theorem 4.4 we obtain immediately that

**Theorem 4.5.** For the operator \( (A, D(A)) \), there hold that

(1) \( \lambda \in \sigma(A) \iff 1 \in \sigma(L_\lambda K_\lambda) \iff 1 \in \sigma_F(L_\lambda K_\lambda) \iff \lambda \in \sigma_F(A) \);
(2) Furthermore, if \( \lambda \in \rho(A) \) (equivalently \( 1 \in \rho(L_\lambda K_\lambda) \)), then

\[
R(\lambda, A) = \begin{pmatrix}
(1 - K_\lambda L_\lambda)^{-1}R(\lambda, A_0) & (1 - K_\lambda L_\lambda)^{-1}K_\lambda R(\lambda, A_0) \\
(1 - L_\lambda K_\lambda)^{-1}L_\lambda R(\lambda, A_0) & (1 - L_\lambda K_\lambda)^{-1}R(\lambda, A_0)
\end{pmatrix}.
\]

(4.4)

Proof. We only need to verify (4.4). By the equation of (4.3) in Lemma 4.3, the inverse of \((\lambda - A)\) is

\[
R(\lambda, A) = \mathcal{B}_\lambda^{-1} \begin{pmatrix}
R(\lambda, A_0) & 0 \\
0 & R(\lambda, A_0)
\end{pmatrix}.
\]

By the definition of \(\mathcal{B}_\lambda\), we have

\[
\mathcal{B}_\lambda^{-1} = \begin{pmatrix}
(1 - K_\lambda L_\lambda)^{-1} & (1 - K_\lambda L_\lambda)^{-1}K_\lambda \\
(1 - L_\lambda K_\lambda)^{-1}L_\lambda & (1 - L_\lambda K_\lambda)^{-1}
\end{pmatrix}.
\]

Then the expression (4.4) follows. \(\square\)

For positivity of a \(C_0\)-semigroup one has that

**Lemma 4.6** (see [13], Theorem VI.1.8). A strongly continuous semigroup \((T(t))_{t \geq 0}\) on a Banach lattice \(X\) is positive if and only if the resolvent \(R(\lambda, A)\) of its generator \(A\) is positive for all sufficiently large \(\lambda\).

Now, using this lemma, we can conclude this section by formulating conditions for the positivity of the obtained semigroup \((T(t))_{t \geq 0}\).

**Theorem 4.7.** Suppose that

\[
\int_{-\tau}^{0} \beta(\cdot, \sigma, Q_\lambda(\cdot))d\sigma + w(\cdot) \left( \int_{-\tau}^{0} \int_{-\tau}^{0} \beta_Q(\eta, \sigma, Q_\lambda(\eta))u_+(\eta)d\sigma d\eta \right) + a \int_{-\tau}^{m} \int_{-\tau}^{0} \beta_Q(\eta, \sigma, Q_\lambda(\eta))u_+(\eta)d\sigma d\eta \geq 0,
\]

(4.5)

holds. Then the semigroup \((T(t))_{t \geq 0}\) generated by the operator \((\mathcal{A}, D(\mathcal{A}))\) is positive.

Proof. We consider the operator \(K_\lambda L_\lambda\). By the definitions of \(K_\lambda\) and \(L_\lambda\) in the above of Lemma 4.2, it is easy to see that

\[
\lim_{Re\lambda \to +\infty} \|K_\lambda L_\lambda\| = 0.
\]

Therefore \(\|K_\lambda L_\lambda\| < 1\) for \(Re\lambda\) sufficiently large. Thus the operator \((1 - K_\lambda L_\lambda)\) is invertible and its inverse \((1 - K_\lambda L_\lambda)^{-1}\) is given by the Neumann series. Obviously \(K_\lambda L_\lambda\) is a positive operator by the condition (4.5), and hence \((1 - K_\lambda L_\lambda)^{-1}\) is positive as well for \(Re\lambda\) big enough. With the resolvent representation of \(\mathcal{A}\) in (4.4), \(R(\lambda, A)\) is nonnegative for such \(\lambda\). Thus, using Lemma 4.6 above, we infer that the operator \((\mathcal{A}, D(\mathcal{A}))\) generates a positive semigroup on the Banach lattice \(E \times X\). Then we get the assertion. \(\square\)

The positivity and eventual compactness of the \(C_0\)-semigroup \((T(t))_{t \geq 0}\) enable us to draw the following important conclusion.

**Corollary 3.** Suppose that the condition (4.5) holds true, then the spectral bound \(s(\mathcal{A}) := \sup \{Re\lambda \mid \lambda \in \sigma(\mathcal{A})\}\) belongs to the spectrum \(\sigma(\mathcal{A})\).
Now we turn to address the characteristic equation. From Corollary 2, we know that the linearized stability of stationary solutions of the system (1.1) is entirely determined by the eigenvalues of the semigroup generator \((A,D(A))\). Hence, it is essential to determine the eigenvalues of \(A\). The eigenvalue equation

\[
(\lambda I - A)v = 0
\]  

for \(\lambda \in \mathbb{C}\) and nontrivial function \(v\), is equivalent to the system

\[
\begin{align*}
\lambda v_t(\sigma) - \frac{d}{d\sigma} v_t(\sigma) &= 0, \\
\gamma(s)v'(s) + (\lambda + \rho(s))v(s) &= 0, \\
v(0) &= \int_{0}^{m} \int_{0}^{-\tau} e^{\lambda \sigma} \left( \beta(s,\sigma,Q_*(s))v(s) + \beta_Q(s,\sigma,Q_*(s))u_*(s)V(s) \right) d\sigma ds,
\end{align*}
\]  

where

\[
V(s) = \alpha \int_{0}^{s} w(\eta)v(\eta)d\eta + \int_{s}^{m} w(\eta)v(\eta)d\eta \\
= (\alpha - 1) \int_{0}^{s} w(\eta)v(\eta)d\eta + \int_{m}^{0} w(\eta)v(\eta)d\eta.
\]

For the remainder of this section let us assume that \(\alpha \in [0,1)\). From (4.10) we obtain

\[
V'(s) = (\alpha - 1)w(s)v(s) \quad \text{and} \quad V''(s) = (\alpha - 1)(w'(s)v(s) + w(s)v'(s)).
\]

Using the relations (4.11) we can rewrite system (4.7)-(4.9) in terms of \(V\) and its derivatives as follows

\[
V''(s) + \left( \frac{\lambda + \rho(s)}{\gamma(s)} - \frac{w'(s)}{w(s)} \right) V'(s) = 0.
\]

Eq.(4.12) is accompanied by boundary conditions of the form

\[
aV(0) = V(m),
\]

\[
V'(0) = w(0) \int_{0}^{m} \int_{-\tau}^{0} e^{\lambda \sigma} \frac{\beta(s,\sigma,Q_*(s))}{w(s)} V'(s)d\sigma ds + (\alpha - 1)w(0) \int_{m}^{0} \int_{-\tau}^{0} e^{\lambda \sigma} \beta_Q(s,\sigma,Q_*(s))u_*(s)V(s)d\sigma ds.
\]

Then the general solution of (4.12) can be formulated as

\[
V(s) = V(0) + V'(0) \int_{0}^{s} \frac{w(\eta)}{w(0)} \Pi(\lambda,\eta)d\eta,
\]

where

\[
\Pi(\lambda,\eta) = \frac{\gamma(0)}{\gamma(\eta)} e^{-\int_{0}^{\eta} \frac{\lambda + \rho(r)}{\gamma(r)} dr}.
\]

By the boundary condition (4.13) and the solution (4.15), we get

\[
(1 - \alpha)V(0) + V'(0) \int_{m}^{0} \frac{w(\eta)}{w(0)} \Pi(\lambda,\eta)d\eta = 0,
\]
while the boundary condition (4.14) and (4.15) imply that

\[
V(0) \left( w(0)(1 - \alpha) \int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \beta_Q(s, \sigma, Q_*(s)) u_*(s) d\sigma ds \right) 
+ V'(0) \left\{ 1 - \int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \beta(s, \sigma, Q_*(s)) \Pi(\lambda, s) d\sigma ds \right. 
+ (1 - \alpha) \int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \beta_Q(s, \sigma, Q_*(s)) u_*(s) \int_0^s w(y) \Pi(\lambda, y) dy d\sigma ds \} = 0. 
\]

The linear system (4.17)-(4.18) has a nontrivial solution \((V(0), V'(0))\) if and only if \(\lambda\) satisfies

\[
\Delta(\lambda) := \int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \beta(s, \sigma, Q_*(s)) \Pi(\lambda, s) d\sigma ds 
+ (\alpha - 1) \int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \beta_Q(s, \sigma, Q_*(s)) u_*(s) \int_0^s w(y) \Pi(\lambda, y) dy d\sigma ds 
+ \int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \beta_Q(s, \sigma, Q_*(s)) u_*(s) d\sigma ds \int_0^m w(y) \Pi(\lambda, y) dy - 1 = 0. 
\]

(4.19)

For the case \(\alpha = 1\), by the definition (4.10) of \(V\), we have that \(V\) is a constant. When we solve the problem (4.8)-(4.9) directly, we obtain again the condition

\[
\Delta(\lambda) = 0, 
\]

(4.20)

where \(\Delta(\lambda)\) is given by (4.19) with \(\alpha = 1\).

Hence (4.19) is the characteristic equation for all \(0 \leq \alpha \leq 1\).

5. Linear stability results. Based on the discussion in the previous sections we can no explore the asymptotic behaviors of solutions for (1.1). Firstly, in this part, we will formulate sufficient conditions about the stability and instability of stationary solutions for the equation (1.1). First we can establish the stability result for the null solution as follows.

**Theorem 5.1.** The trivial stationary solution \(u_* \equiv 0\) is linearly asymptotically stable if \(R(0) < 1\) and linearly unstable if \(R(0) > 1\).

**Proof.** For \(u_* \equiv 0\), in the characteristic equation (4.19), we observe that the second and the third terms are 0, then (4.19) becomes

\[
\Delta(\lambda) = \int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \beta(s, \sigma, Q_*(s)) \Pi(\lambda, s) d\sigma ds - 1, 
\]

following that we have

\[
\Delta(0) = R(0) - 1. 
\]

Clearly the condition (4.5) of Theorem 4.7 is fulfilled when the stationary solution \(u_*(s) \equiv 0\). Then we can restrict ourselves to \(\lambda \in \mathbb{R}\). Observe that

\[
\lim_{\lambda \to -\infty} \Delta(\lambda) = +\infty, \quad \lim_{\lambda \to +\infty} \Delta(\lambda) = -1. 
\]

(5.1)

and

\[
\Delta'(\lambda) = \int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \Pi(\lambda, s) \left( \sigma - \int_0^s \frac{1}{\gamma(y)} dy \right) d\sigma ds \leq 0. 
\]

(5.2)

Therefore, if \(R(0) < 1\) holds, \(\Delta(\lambda)\) decreases monotonically by the inequality (5.2), then the characteristic function cannot have nonnegative roots as \(\Delta(0) < 0\). If,
however, \( R(0) > 1 \) holds, then the Intermediate Value Theorem gives a positive root since \( \Delta(0) > 0 \) and (5.1) hold.

Next we focus on the linearized stability/instability of the positive stationary solutions. For this purpose, in the light of the arguments in the above sections, we deduce the following theorem.

**Theorem 5.2.** If the positive condition (4.5) is true, then following equivalence holds:

\[
s(A) = \omega_0(A) \leq 0 \Leftrightarrow \beta_Q(s, \sigma, Q_\ast) \leq 0.
\] (5.3)

**Proof.** It follows from ([13], Theorem VI.1.15) that \( s(A) = \omega_0(A) \), which is the dominate eigenvalue of the equation (4.19). Through the positivity condition (4.5) we deduce that

\[
\Delta'(\lambda) = \int_0^m \int_{-\tau}^0 e^{\lambda \sigma} \Pi(\lambda, s) \left( \sigma - \int_0^s \frac{1}{\gamma(y)} dy \right) \left\{ \beta(s, \sigma, Q_\ast(s)) + w(s) \left( \int_s^m \beta_Q(y, \sigma, Q_\ast(y)) u_\ast(y) dy + \alpha \int_s^m \beta_Q(y, \sigma, Q_\ast(y)) u_\ast(y) dy \right) \right\} d\sigma ds 
\]

Clearly, if restricted to \( \mathbb{R} \), \( \Delta(\lambda) \) is a continuous, decreasing, real function having

\[
\lim_{\lambda \to -\infty} \Delta(\lambda) = +\infty, \quad \lim_{\lambda \to +\infty} \Delta(\lambda) = -1.
\]

Therefore, \( \Delta_0 \) has a unique real zero \( \lambda_0 \) which is the spectral bound \( s(A) \) (see also Reference [13] Lemma VI.1.18). Moreover, we observe that

\[
\Delta(0) = R(Q_\ast) + \int_0^m \int_{-\tau}^0 \beta_Q(s, \sigma, Q_\ast(s)) u_\ast(s) \left( \alpha \int_s^m w(y) \pi(y) dy + \int_s^m w(y) \pi(y) dy \right) d\sigma ds - 1.
\] (5.5)

From Proposition 2.1 we have got \( R(Q_\ast) = 1 \), so

\[
\Delta(0) = \int_0^m \int_{-\tau}^0 \beta_Q(s, \sigma, Q_\ast(s)) u_\ast(s) \left( \alpha \int_s^m w(y) \pi(y) dy + \int_s^m w(y) \pi(y) dy \right) d\sigma ds. \quad (5.6)
\]

where \( \pi(y) \) is defined as (2.4). By the above considerations we conclude easily that

\[
s(A) = \omega_0(A) \leq 0 \Leftrightarrow \beta_Q(s, \sigma, Q_\ast) \leq 0,
\]

which gives (5.3).

From this theorem we get immediately the following results on the stability/instability of stationary solutions for the system (1.1).

**Theorem 5.3.** Let \( u_\ast \) be any positive stationary solution of Eqs. (1.1) and the positivity condition (4.5) hold true. Then the stationary solution is linearly asymptotically stable if

\[
\beta_Q(s, \sigma, Q_\ast) < 0.
\]

If, however,

\[
\beta_Q(s, \sigma, Q_\ast) \geq 0 \quad \text{and} \quad \beta_Q(s, \sigma, Q_\ast) \neq 0,
\]

then \( u_\ast \) is linearly unstable.
6. **Asynchronous exponential growth.** The purpose of this section is to gain a deeper insight into asymptotic properties of solutions of the linearized system (2.13). That is, we will use semigroup techniques and spectral analysis methods to obtain the property of asynchronous exponential growth (AEG for short) for (2.13) which is defined in the framework of semigroup theory as below.

**Definition 6.1.** A linear $C_0$-semigroup $(T(t))_{t \geq 0}$ on Banach space $X$ is said to exhibit an AEG if there exist a real number $\lambda_0 \geq 0$ (called Malthusian parameter) and a rank one projection $P$ on $X$ such that

$$
\lim_{t \to +\infty} e^{-\lambda_0 t}T(t)x = Px, \quad \text{for all } x \in X.
$$

The existence of such a $\lambda_0$ is related to the existence of a nonnegative strictly dominant eigenvalue in the spectrum of the generator of the semigroup.

The phenomenon of AEG appears frequently in age/size-structured populations models (see [5, 23, 36]). It describes the situation when the population grows exponentially in time but the proportion of individuals within any range of age/size compared to the total population tends, as time tends to infinity, to a limit which just depends on the chosen range. This is an important characteristic of solutions of population equations both from the theoretical and application point of view. In the past years there is much work on this topic, see for instance [5, 31, 37, 38].

For positive semigroups there exists the well-known characterization of AEG (see [28], Theorem C-IV.2.1). As in Ref. [31], our analytical approach will be guided toward the result as follows.

**Theorem 6.2.** Let $(T(t))_{t \geq 0}$ be a positive, strongly continuous semigroup generated by $(A, D(A))$ on a Banach lattice $X$ satisfying $\omega_{ess}(A) < \omega_0(A)$. Then the following hold:

(a) $\lambda_0 := s(A) = \omega_0(A)$ is a pole of the resolvent $R(\lambda, A)$ and a strictly dominant eigenvalue of the generator $A$, i.e. there exists $\epsilon > 0$ such that $\Re \lambda \leq \lambda_0 - \epsilon$ for every $\lambda \in \sigma(A)$, $\lambda \neq \lambda_0$. Moreover, if $k$ is the order of the pole $\lambda_0$ and $P$ denotes the related spectral projection, then $\text{rg} P = \text{Ker}(\lambda_0 - A)^k$.

(b) If the order of the pole $\lambda_0$ is equal to 1, there exist constants $M, \delta > 0$ such that

$$
\|e^{-t\lambda_0}T(t) - P\| \leq Me^{-\delta t}, \quad \text{for all } t \geq 0,
$$

where $P$ denotes the spectral projection corresponding to the eigenvalue $\lambda_0$ having

$$
\text{rg} P = \text{Ker}(\lambda_0 - A).
$$

**Remark 1.** It is easy to see that if (6.1) holds and $\text{Ker}(\lambda_0 - A)$ has dimension 1, then the semigroup $(T(t))_{t \geq 0}$ has AEG.

Before presenting the main result of this section, we need to do some preparation. Consider the following decomposition for the operator $\mathcal{A}$

$$
\mathcal{A} = \mathcal{A} + \mathcal{C},
$$

where

$$
\mathcal{A} := \begin{pmatrix}
G_m & 0 & 0 & 0 \\
-Q & \text{Id} & 0 \\
0 & 0 & A_m & 0 \\
0 & 0 & -\Phi & 0
\end{pmatrix}
$$

and

$$
\mathcal{C} := \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\Phi & 0 & 0 & 0
\end{pmatrix}
$$

(6.2)
with domains $D(\tilde{A}) := D(\mathcal{A})$ and $D(\mathcal{C}) := \mathcal{X}$.

Clearly $\mathcal{C}$ is a bounded operator on $\mathcal{X}$ and it is compact, while for operator $(\tilde{A}, D(\tilde{A}))$ we have the following result.

**Lemma 6.3.** The operator $(\tilde{A}, D(\tilde{A}))$ is a Hille-Yosida operator.

**Proof.** It is sufficient to apply Lemma 3.1 to $\tilde{A} = \mathcal{A}_1 + \mathcal{F}$ with

$$
\mathcal{F} := \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \text{Id} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

and $\mathcal{A}_1$ as in the proof of Proposition 2. \qed

**Proposition 3.** The part $(\tilde{A}_0, D(\tilde{A}_0))$ of $(\tilde{A}, D(\tilde{A}))$ in $\mathcal{X}_0$ generates a positive strongly continuous semigroup on $\mathcal{X}_0$ which is isomorphic to the semigroup

$$
T_0(t) := \left( \begin{array}{cc}
S_0(t) & T_t \\
0 & T_0(t)
\end{array} \right)
$$

on $\mathcal{X} := E \times X$, where $T_t : X \to E$ is linear operator defined as

$$(T_t f)(\sigma) := \begin{cases}
T_0(t+\sigma)f, & \text{if } t+\sigma > 0, \\
0, & \text{if } t+\sigma \leq 0.
\end{cases}$$

$(S_0(t))_{t \geq 0}$ and $(T_0(t))_{t \geq 0}$ are defined as in Proposition 2. The generator of $(T_0(t))_{t \geq 0}$ is

$$
\left( \begin{array}{cc}
G_m & 0 \\
0 & A_0
\end{array} \right)
$$

with domain

$$
\left\{ \left( \begin{array}{c}
g \\
f
\end{array} \right) \in D(G_m) \times D(A_0) : g(0) = f \right\}.
$$

**Proof.** The first part of the assertion is a consequence of Lemma 6.3 and Lemma 2.2. Moreover,

$$
D(\tilde{A}_0) = \left\{ \left( \begin{array}{c}
g \\
\varphi \\
f \\
x
\end{array} \right) \in D(\tilde{A}) : \tilde{A} \left( \begin{array}{c}
g \\
\varphi \\
f \\
x
\end{array} \right) \in \mathcal{X}_0 \right\}
$$

$$
= \left\{ \left( \begin{array}{c}
g \\
0 \\
f \\
o
\end{array} \right) : g \in D(G_m), f \in D(A_m), -Pf = 0, -Qg + f = 0 \right\}
$$

$$
= \left\{ \left( \begin{array}{c}
g \\
0 \\
f \\
o
\end{array} \right) \in D(G_m) \times \{0\} \times D(A_0) \times \{0\} : g(0) = f \right\}.
$$

Therefore, the operator $(\tilde{A}_0, D(\tilde{A}_0))$ is isomorphic to the operator

$$
\left( \begin{array}{cc}
G_m & 0 \\
0 & A_0
\end{array} \right)
$$

with domain

$$
\left\{ \left( \begin{array}{c}
g \\
f
\end{array} \right) \in D(G_m) \times D(A_0) : g(0) = f \right\},
$$
which generates the semigroup \((\mathcal{T}_0(t))_{t \geq 0}\) given in (6.3) (see [3], Proposition 3.1 for the proof) on the Banach space \(E \times X\). Thus, the semigroup generated by \((\tilde{\mathcal{A}}_0, D(\tilde{\mathcal{A}}_0))\) is isomorphic to \((\mathcal{T}_0(t))_{t \geq 0}\).  

Based on the \(C_0\)-semigroup and the spectral analysis, we are on the position now to show the system (2.13) has AEG under some conditions, namely,

**Theorem 6.4.** Let \((e^{tA})_{t \geq 0}\) denote the semigroup generated by the operator \((\mathcal{A}, D(\mathcal{A}))\). If the positive condition (4.5) and \(\beta_Q(s, \sigma, Q_*) \geq 0\) hold, then \(\lambda_0 := \omega_0(\mathcal{A}) = s(\mathcal{A}) \geq 0\) is an eigenvalue of \(\mathcal{A}\) with one dimensional spectral projection \(P\) and there exist constants \(\omega, \omega_0, C, \delta > 0\) such that
\[
\|e^{-t\lambda_0} e^{tA} - P\| \leq Me^{-\delta t}, \quad \text{for all } t \geq 0.
\]

In particular, \((e^{tA})_{t \geq 0}\) has AEG.

**Proof.** We apply Theorem 6.2 to show this result. From the positive condition (4.5) we obtain that the semigroup \((e^{t\mathcal{A}})_{t \geq 0}\) is positive, so we just have to verify that \(\lambda_0\) is a first order pole of \(R(\lambda_0, \mathcal{A})\) with one dimensional eigenspace, and \(\omega_{\text{ess}}(\mathcal{A}) < \omega_0(\mathcal{A})\).

From Theorem 5.2, we can obtain that \(\beta_Q(s, \sigma, Q_*) \geq 0\) means exactly \(\omega_0(\mathcal{A}) \geq 0\). Moreover, since \(\omega_0(\mathcal{A}) > -\infty\), we have \(\omega_0(\mathcal{A}) \in \sigma(\mathcal{A})\) (see [13], Theorem VI.1.10). We show now that \(\omega_{\text{ess}}(\mathcal{A}) < \omega_0(\mathcal{A})\). Since, by Proposition 3, the semigroup \((\mathcal{T}_0(t))_{t \geq 0}\), which is isomorphic to the semigroup generated by \((\tilde{\mathcal{A}}_0, D(\tilde{\mathcal{A}}))\), is positive on the Lebesgue space \(X = E \times X\), we get \(\omega_0(\tilde{\mathcal{A}}) = s(\tilde{\mathcal{A}})\) (see ([13], Theorem VI.1.15)). For the spectral bound of the generator of \((\mathcal{T}_0(t))_{t \geq 0}\), and hence for \(s(\tilde{\mathcal{A}}_0)\), we have
\[
s(\tilde{\mathcal{A}}_0) = s(A_0) = -\infty < 0,
\]
since \(\sigma(A_0) = \emptyset\). Therefore, we obtain
\[
\omega_0(\tilde{\mathcal{A}}_0) = s(\tilde{\mathcal{A}}_0) = s(A_0) < 0.
\]
In particular,
\[
\omega_{\text{ess}}(\tilde{\mathcal{A}}_0) < 0.
\]

Now we consider the operator \(\mathcal{C}\) in (6.2). \(\mathcal{C}\) can be restricted to \(\mathcal{X}_0\). Moreover, the space \(\mathcal{X}\) is continuously embedded in the extrapolated Favard class \(F(\tilde{\mathcal{A}})_1\) related to \((\tilde{\mathcal{A}}_0, D(\tilde{\mathcal{A}}))\). Therefore \(\mathcal{C}\) is compact from \(\mathcal{X}_0\) to \(F(\tilde{\mathcal{A}})_1\). Applying Lemma 2.5 to our situation we get
\[
r_{\text{ess}}(e^{tA_0}) = r_{\text{ess}}(e^{t\tilde{A}_0}).
\]
Since it is known (see [13], Proposition IV.2.10) that for a strongly continuous semigroup \(e^{tA}\)
\[
r_{\text{ess}}(e^{tA}) = e^{t\omega_{\text{ess}}(A)},
\]
it follows that
\[
\omega_{\text{ess}}(\tilde{\mathcal{A}}_0) = \omega_{\text{ess}}(\tilde{\mathcal{A}}_0) < 0,
\]
i.e.
\[
\omega_{\text{ess}}(\mathcal{A}) < 0.
\]
Thus by assumption \(\omega_0(\mathcal{A}) \geq 0\), we deduce that \(\omega_{\text{ess}}(\mathcal{A}) < \omega_0(\mathcal{A})\).

Finally, we show that \(\lambda_0 := \omega_0(\mathcal{A})\) has pole order 1 and that \(\text{Ker}(\lambda_0 - \mathcal{A}_0)\) is one dimensional, so that the dimension of the range of the corresponding spectral projection \(P\) is also equal to 1. Similar to [31], we apply the technique developed
by Greiner (see [20], Proof of Corollary 1.6). To achieve this assertion, first we note that \( \text{Ker}(\lambda_0 - \mathcal{A}_0) \) has dimension 1 because all eigenvectors are multiples of

\[
\psi_{\lambda_0} := \begin{pmatrix}
\varphi_{\lambda_0} \\
0 \\
\varphi_{\lambda_0} \\
0
\end{pmatrix},
\]

where \( \varepsilon_{\lambda_0}(\cdot) \) and \( \varphi_{\lambda_0}(\cdot) \) are given by (3.5).

Assume now that the pole \( \lambda_0 \) has order \( n > 1 \). Let

\[
U := \lim_{\lambda \to \lambda_0^+} (\lambda - \lambda_0)^n R(\lambda, \mathcal{A}_0),
\]

then it is clearly a positive operator, and \( Uh = 0 \) for every eigenvector \( h \) of \( \mathcal{A}_0 \), in particular, \( U\psi_{\lambda_0} = 0 \).

In addition, if \( |\phi| \leq \psi_{\lambda_0} \), then \( |U\phi| \leq U|\phi| \leq U\psi_{\lambda_0} = 0 \). Therefore, \( U \) vanishes on the set \( \{ \phi \in \mathcal{D}_0 : |\phi| \leq \psi_{\lambda_0} \} \) which is total in \( \mathcal{D}_0 \) (i.e. the set of all linear combinations \( \text{lin}\{ \phi \in \mathcal{D}_0 : |\phi| \leq \psi_{\lambda_0} \} \) is dense in \( \mathcal{D}_0 \)) because \( \psi_{\lambda_0} \) is strictly positive. Hence \( U = 0 \), i.e. \( \lambda_0 \) is a pole of order less than \( n \) which is a contradiction.

Then, from Theorem 6.2, the claim follows.

\( \square \)

7. Examples and simulations. In this section, we present some concrete data for the parameters in the system (1.1) and do the corresponding numerical simulations to illustrate the stability/instability and AEG results obtained in Theorems 5.1, 5.3 and 6.4 respectively.

(a) If the vital rates and the parameters of (1.1) are taken as:

\[
\gamma(s) = 1 + s, \quad \mu = 1, \quad \tau = 1, \quad \alpha = 0.8, \quad w = 2,
\]

\[
\beta(s, \sigma, Q(s, t + \sigma)) = \frac{1}{2} e^{\sigma}(1 + s)(1 + Q).
\]

Then, clearly, \( Q_* \equiv 0 \) and the positive condition holds true. From (2.3) we get \( R(0) \approx 0.755 < 1 \), according to Theorem 5.1 the trivial solution \( u_* \equiv 0 \) of System (1.1) is (locally) asymptotically stable, see Figure 1. While if we choose the vital rates and the parameters as

\[
\gamma(s) = 2(1 + s), \quad \mu = 2, \quad \tau = 1, \quad \alpha = 0.5, \quad w = 1,
\]

\[
\beta(s, \sigma, Q(s, t + \sigma)) = e^{\sigma}(1 + s)(1 + Q),
\]

then we derive that \( R(0) \approx 1.52 > 1 \), so \( u_* \equiv 0 \) is unstable, see Figure 2.

![Figure 1](image1.png)

**Figure 1.** Choosing the parameters with \( m = 10, \ T = 120 \), “a” represents \( u_* \) and \( U_* \), the initial conditions corresponding to curves \( b \) and \( c \) are: (b) \( u_0(s) = \frac{0.1}{1 + s^2} \); (c) \( u_0(s) = \frac{0.09}{1 + s^2} + e^{-3s} \).
Figure 2. Choosing the parameters with \( m = 10, \ T = 160, \) “a” represents \( u_* \) and \( U_* \), the initial conditions corresponding to curves \( b \) and \( c \) are: (b) \( u_0(s) = \frac{0.14}{1+s^2} \), (c) \( u_0(s) = \frac{0.09}{1+s^2} + e^{-3s} \).

(b) We take that:
\[
\gamma(s) = 1 - 0.5s, \ \mu = 1, \ \tau = 1, \ \alpha = 0.5, \ w = 1,
\]
\[
\beta(s, \sigma, Q(s, t + \sigma)) = \frac{480}{997} e^\sigma (1 + s)(3 - 2Q), \ 0 \leq Q \leq 1.5.
\]

Then it easy to compute that
\[
\beta_Q(s, \sigma, Q_*(s)) = -\frac{960}{997} e^\sigma (1 + s) < 0,
\]
\[
\int_{-\tau}^{0} \beta(\cdot, \sigma, Q_*(\cdot))d\sigma + w(\cdot) \left( \int_{0}^{0} \int_{-\tau}^{0} \beta_Q(\eta, \sigma, Q_*(\eta))u_*(\eta)d\sigma d\eta \right) + \alpha \int_{m}^{0} \int_{-\tau}^{0} \beta_Q(\eta, \sigma, Q_*(\eta))u_*(\eta)d\sigma d\eta \geq 0,
\]

Hence, in the light of Theorem 5.3, the positive stationary solution \( u_* \) of System (1.1) is locally asymptotically stable, see Figure 3 below.

If, however,
\[
\beta(s, \sigma, Q(s, t + \sigma)) = e^\sigma (1 + s)Q, \ Q \geq 0,
\]
then \( \beta_Q(s, \sigma, Q_*(s)) > 0 \) and all the conditions of Theorem 5.3 are satisfied, therefore \( u_* \) is unstable, see Figure 4.

Figure 3. Choosing the parameter \( m = 100, \ T = 1050, \) “a” represents \( u_* \) and \( U_* \), the initial conditions corresponding to curves \( b \) and \( c \) are: (b) \( u_0(s) = \frac{0.01}{1+s^2} + 0.0215 \), (c) \( u_0(s) = \frac{0.05}{2+s^2} + 0.0365 \).
(c) We now put in the system (1.1) that
\[ \gamma(s) = 1 + s, \quad \mu = 0.5 + 2s, \quad \tau = 1, \quad \alpha = 0.45, \quad w = 1.6, \]
\[ \beta(s, \sigma, Q(s, t + \sigma)) = \frac{3}{4} e^\sigma (2 + s)(3 + 2Q). \]
Obviously, \( \beta_0(s, \sigma, Q_0(s)) > 0 \) and the positive condition is satisfied. Based on Theorem 6.4 the solutions of System (1.1) have AEG, see Figure 5.

From Figure 5. we can see that, the total population grows exponentially as time evolves. On the other hand, the initial distribution of the time \( t \) from \(-1\) to \( 0 \) concentrates at \( s \in (5, 12) \) and \( s \in (25, 35) \), the population with size ranges \((5, 12)\) and \((25, 35)\) still dominate the main part of the total population as time increases and goes to infinity. This is just the expected phenomenon of AEG shown in Theorem 6.4.

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