Lack of Unique Factorization as a Tool in Block Cipher Cryptanalysis

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Abstract. Linear (or differential) cryptanalysis may seem dull topics for a mathematician: they are about super simple invariants characterised by say a word on $n = 64$ bits with very few bits at 1, the space of possible attacks is small, and basic principles are trivial. In contrast mathematics offers an infinitely rich world of possibilities. If so, why is that cryptographers have ever found so few attacks on block ciphers? In this paper we argue that black-box methods used so far to find attacks in symmetric cryptography are inadequate and we work with a more recent white-box algebraic methodology. Invariant attacks can be constructed explicitly through the study of roots of the so-called Fundamental Equation (FE). We also argue that certain properties of the ring of Boolean polynomials such as lack of unique factorization allow for a certain type of product construction attacks to flourish. As a proof of concept we show how to construct a complex and non-trivial attack where a polynomial of degree 7 is an invariant for any number of rounds for a complex block cipher.

Key Words: unique factorization, cyclotomic integers, algebraic number theory, multivariate polynomials, polynomial invariants, irreducible polynomials, prime numbers, block ciphers, Boolean functions, non-linearity, annihilator space, ANF, polynomial rings Feistel ciphers, weak keys, backdoors, T-310, Generalized Linear Cryptanalysis, algebraic cryptanalysis.

1 Introduction - Maths vs. Crypto Question

A major topic in number theory is to try to solve the usual problems by creating new abstract algebraic extensions (or new types of imaginary objects), which extend the usual possibilities of doing arithmetic. Solving diophantine equations and proving results about their solvability, is one of the biggest research topics of all times, and have benefited greatly from various algebraic innovations such as cyclotomic integers or various extension fields. In cryptography the problem is similar: breaking ciphers may be seen as solving a system of equations, and finding new non-trivial cryptographically significant properties of ciphers can, and should be formalised in this way. An interesting area is block cipher cryptanalysis. For decades the only attacks ever found were simple iterative attacks such as Linear Cryptanalysis where the set of possible attacks is very small. The space of possible attacks is tiny and it is hard to overlook the attacks if they exist. Cryptanalysis with multivariate polynomials was studied extensively too [9] but with very poor results. We need to ask the following question: why is it that cryptographers have ever found so few attacks. Have mathematics have
failed in its core mission of providing tools for understanding the world in which we live? We hypothesise that the fact why we have never found many powerful attacks on block ciphers has three core reasons:

1. One reason is combinatorial complexity: the number of all possible non-linear polynomial invariant attacks grows as $2^{2^n}$. Many authors stress that systematic exploration is not feasible [4]. For this reason, there are extremely few positive results on this topic [43, 13] and any method to approximate the solution is valuable. One new working example as in this paper will typically allow the researchers to find more similar examples.

2. A second answer lies in the limitations of block-box work methodology (e.g. in Linear Cryptanalysis): from basic properties on some black boxes we derive complex ones on larger boxes. In this paper we hypothesise that cryptographers have been missing a lot by using this methodology. We need a white-box or algebraic methodology for the study of block ciphers and once we have one, we are going to discover new attacks never seen before.

3. A third answer is about phase transitions and entry barriers. A very substantial effort to find some attacks with lower degree polynomials failed to produce anything more than just slightly better than best Matsui’s attack, cf. [15]. A radical improvement can be achieved by switching to higher degree polynomial attacks as shown in a recent paper [14]. At a higher degree we suddenly discover that invariants true with probability 1 may exist.

In this paper we construct just one new attack on one block cipher setup. Our invariant is a product of polynomials as in [14]. However the steps needed to show that the attack works are more complex than ever before and the attack does not resemble any previous attack. Before we study further details, we are now going to argue that there exist an interesting property of polynomial arithmetic which has been a source of serious trouble for mathematicians but it is extremely helpful\(^1\) for finding new attacks on block ciphers.

\subsection{Invention of Polynomial Rings vs. Unique Factorization}
As already explained mathematics and number theory have developed by study and creation of increasingly complex algebraic structures which were created as tools to solve important problems. Rings are also useful in cryptography: study of specific polynomial rings is at the heart of so called algebraic cryptanalysis. Mathematics have over the centuries seen the role of abstract algebra increase. The same is likely to happen if we want to make some progress in cryptanalysis.

In 1847 Gabriel Lamé has announced that he has proven the Fermat’s last theorem in the general case. Actually two mathematicians: Lamé and Cauchy have submitted so called “secret packets” recorded in the proceedings of the Paris Academy of Science [30] which contained similar ideas. The proof attempts were then reviewed and studied. The work used the so called cyclotomic integers in $\mathbb{Z}_n[\alpha]$ with a root of unity $\alpha$ with $\alpha^p = 1$ and with $p = 23$. However Kummer has proven few years earlier that for $p = 23$ ring $\mathbb{Z}_n[\alpha]$ does NOT have a unique factorization. Thus a major attempt to prove the Fermat last theorem has failed, and we had to wait for another 138 years for this question to be solved.

\(^1\) Our later attack is about equality of two products.
1.2 Lack of Unique Factorization

In this paper we show that lack of unique factorization for multivariate Boolean polynomials has important consequences in block cipher cryptanalysis. We claim that it helps the attacker in a very strong way. This is extremely clear here because we explore the so called “product construction” for polynomial invariant attacks on block ciphers proposed in [14]. Without unique factorization there are substantially fewer (and only trivial) ways to make two polynomials which are both products of linear factors equal to each other, which will be essentially permutations in the order of factors. In this paper however we show that there is another way to make them equal. In general we conjecture that potentially every non-trivial polynomial invariant attack will contain something which are “lack of unique factorization events” or annihilation events, absorption events etc, see Thm. 6.1 and page 19 in Section 6. An illuminating example of how this works can be found in page 19 where a polynomial of degree 5 can be expressed in 2 different ways as a multiple of two very different degree 3 polynomials where sets of variables are almost entirely disjoint (!), cf. page 19.

Recent papers show that with a lot of pain, one can construct some small degree invariant attacks on a block cipher with invariants being of low degree [13] and such attacks only work for extremely few degenerate and super weak Boolean functions. Then the situation improves dramatically in [14] at degree 8. We get a “phase transition” from hard to easy, or from a case where extremely few Boolean functions work, to where a very large proportion of Boolean functions will work with the attack. The product attack constructed in [14] is a degree 8 invariant and product of 8 linear factors. In this paper we argue that a strong phase transition happens somewhere near degree 7 or 8, at least here in this specific cipher setup. However finding one strong attack does not solve the general problem HOW to find further attacks on block ciphers (or attacks on stronger block ciphers). Here mathematics does not help a lot: mathematicians have rarely studied situations which are abundant in cryptography where polynomials have a large number of variables over a very small finite field. A huge problem for a cryptanalyst here is combinatorial explosion or vast complexity: the number of all possible polynomial invariant attacks on a cipher with n-bit blocks grows as $2^{2^n}$. In this paper we argue that this problem can be approached by explicit white-box constructions of polynomial invariant attacks where the attacker is able to make some immensely complex polynomial reduce to zero: i.e. equal to zero for any of $2^n$ inputs. It is also about tricks where the number of variables involved at some place can be reduced by some sort of miracle and very complex polynomials get simply cancelled. Only recently such attacks have been show to exist cf. [13, 43].

In this paper we construct a novel degree 7 invariant attack. We prove step by step that our attack works. This attack is irregular and has a very different structure than any previous attack ever seen in block cipher cryptanalysis. It does seem natural in any way that an attack with 7 well-chosen polynomials is at all possible for a highly regular Feistel cipher with 4 branches and 4 internal stages inside the round function. We believe that this paper is a first step towards building a theory of algebraic polynomial invariant attacks on block ciphers.
claim that possibly the only way to find some cryptographic attacks is to con- 
struct some attacks explicitly, using specific algebraic identities, even if this is 
done one weaker cipher or toy ciphers. The dominant black-box methodology of 
finding new attacks by composition should be considered as bankrupt, and for 
many decades cryptography researchers have never found any complex poly- 
nomial attacks, only very simple ones. This paper also suggest that the amount 
of mathematics and algebra in future works on block cipher cryptanalysis will 
need to increase very substantially.

Our methodology for discovering the attacks is based on the study of roots of 
the so-called Fundamental Equation (FE) which is a simple I/O sum of two poly-
nomials (an Input polynomial and an Output polynomial) and is formally defined 
much later in Section 5. Several methods for constructing polynomial invariant 
attacks and therefore also solving the $FE$ have been proposed recently. One is 
by study of certain type of “imperfect” cycles on monomials cf. [31, 13]. Another 
up by RowEchelon elimination in polynomial spaces [10, 13]. Many attacks have 
been constructed by paper and pencil methods [31, 13, 10, 14, 25]. Attacks can 
be transposed from one place to another [13, 10] or applied to a modifed cipher 
[25]. Finally we have a product construction in [14]. The attack is constructed 
by multiplying polynomials from simpler attacks where the equations have no 
roots (the do not correspond to any actual attack). Yet eventually and suddenly 
we find ourselves at another, brighter side: we construct an attack where the 
equation has roots [25] or even a large number of high quality roots [14].

2 Block Ciphers and Round Invariant Attacks

Block ciphers are in widespread use since the 1970s. Their iterated structure 
is prone to numerous round invariant attacks for example in Linear Cryptanal-
ysis (LC). The next step is to look at non-linear polynomial invariants with 
Generalised Linear Cryptanalysis (GLC) first proposed by Harpes, Kramer, and 
Massey cf. [32] (Eurocrypt’95).

2.1 Our Block Cipher

Our examples are constructed for T-310, and old Feistel cipher with 4 branches. 
This cipher offers great flexibility in the choice of the internal wiring. Most 
ciphers such as DES or AES also have this sort of flexibility in the choice of 
P-boxes, arbitrary invertible matrices inside the S-box, inside the mixing layers, 
however later these components are fixed. In T-310 this flexibility is “offically” 
supported: a large variety of possible choices of cipher wiring can be specified and 
used. Here if we find a weak setup, it will be entirely compatible with original 
historical hardware. The exact cipher wiring specification in T-310 is called LZS 
or Langzeitschlüssel cf. [20] and various key studied by researchers and various 
known complete specifications are denoted by 2 digit or 3 digit numbers such as 
LZS 31 or LZS 903, cf. [29, 20]. Our cipher uses Boolean functions on 6 variables 
which in our work will become a variable $Z$: initially we study degenerated cases, 
which eventually at the end become a scenario where this Boolean function is 
no longer chosen by the attacker, and a single attack works for a large number 
of possible Boolean functions.
2.2 Why Boolean Polynomials

An interesting question is why do we use Boolean polynomials in cryptanalysis. Why not for example using roots of unity with $p = 23$ cf. Section 1.1. Potentially it is a natural and the best choice, and potentially there is no reason to do so. The answer is that the arithmetic modulo 2 is a choice of the attacker, and potentially it is an arbitrary choice, and if it increases his chances of success, another choice could be made. A recent work done by a student [31] contains some highly detailed examples of how to construct a simple non-linear invariant attack on a block cipher in several elementary steps.

For example the polynomial $bd$ at the input is equal to the sum of term $Fc$ and polynomial $ce$ at the output. We keep this term $Fc$ on the side hoping it will be eliminated later and focus on transitions of type $bd \rightarrow ce$ for several steps. Then we form cycles where from $bd$ we come back to $bd$ as shown on Fig. 2 and in [31]. At the end we sum all the monomials in black (or green) in one or several cycles and hope that other terms such as $Fc$ in blue on Fig. 2 below will
appear an even number of times and will therefore eventually be cancelled when we sum all the non-linear I/O equations together. This is how we can show that the polynomial

$$P = a + b + c + ac + d + bd + e + ce + f + df + g + ag + eg + h + bh + fh$$

is an invariant for 1 round of encryption for a certain cipher setup 827, see Section 5.1 in [10] and [31] for a more detailed explanation.

**Attacks Modulo 3 or 4**

From here it is easy to see that there is no obstacle to construct an invariant attack modulo 3, we just need to make sure that a number of variables or products we do not want to see (like the in blue colour) in the final invariant, will be divisible by 3 not by 2 and cancelled likewise. This is maybe harder but not impossible.

Furthermore if we read a more recent paper [14] we are going to realize that in order to make this sort of attacks more successful we need to maximize the chances that certain polynomials can be annihilated. This suggests that maybe we need to look for attacks modulo 4, where $2 \cdot 2 = 0$ could create additional opportunities for annihilation. Furthermore, we might need to follow the example of mathematicians (who have eventually proven the Fermat’s last theorem) and consider invariant properties in other rings or fields. The choice of the arithmetic used in an attack ultimately lies with the attacker.

**2.3 Boolean Polynomials, Annihilators and Absorbers**

Let $B_n$ be the ring of Boolean polynomials in $n$ variables (polynomials in their ANF without powers or with $x^2 = x$ cancellations done when multiplying the polynomials). In this paper we do not use the annihilator method of [14] but we rather work on absorption properties, a polynomial $f$ absorbs $g$ if $fg = f$.

In theory both sorts of event are equivalent, absorption of $g$ is the same as annihilation with $f(g + 1) = 0$. However in practice in our attack we emphasise absorption and lack of unique factorization, as key mathematical events which occur in order to make our attack work.

**2.4 On Mathematical Theory of Invariants.**

There exists an extensive theory of multivariate polynomial algebraic invariants w.r.t linear transformations going back to 1845 [7, 27, 28]. This classical 19-th century invariant theory however deals with invariants in the situations where (simultaneously):

1) invariants are polynomials of small degree,

2) they have only 2 sometimes up to 5 variables,

3) polynomials are over large fields and rings, frequently algebraically closed or infinite (or both), or in fields with large characteristic,

4) invariants should not change when we operate a LINEAR input variable transformation $L$, a very important limitation,

3+4 makes that there is a scaling scalar or factor $\sigma$ in most invariants known in classical mathematics: a determinant of the linear transformation $L$,
5) these invariants are in general multivariate polynomials.

In modern invariant theory, however, there are of course more possibilities [28], and here is what the first author of this book has put in a preface of his another (unrelated) book, cf. slide 28 in [26]:

[...] Everybody in mathematics knows that going from one to several variables is an important jump that is accompanied by great difficulties and calls for completely new methods [...] 

In general however we are quite far from the traditional preoccupations of mathematicians. The common points are:
1) we study polynomial invariants $P$ of limited degree and
5) our invariants are multivariate polynomials over some fields,

However there are very substantial differences:
2') we work with many more variables for $P$, typically between 8 and 36 at a time.
3') we work on $GF(2)$ mainly,
4') and finally we are looking for invariants which remain the same after applying an extremely complex NON-LINEAR transformation called $\phi$, or any power of it $\phi^k$, which are no longer linear cf. point 4) above, very peculiar, and not of the sort the mathematicians would consider worth studying,
3+4 Here the scaling factor $\delta$ could only be equal to 1 and should be omitted.

2.5 Round Invariants for Block Ciphers
A major risk in mathematics is that mathematical theories operate mainly at a syntactic level and they could be dealing essentially with an empty set. This unless the objective is to prove the security by proving that the set is empty, see [5, 4]. Current research in application of polynomial invariants in symmetric cryptography has lacked substance or material to work in the form of real-life positive examples which work. Numerous results are about cipher components rather than full ciphers. For example for the AES-like S-box, it is possible to us the so called cross-ratio, (which is already an invariant in the more general non-linear case which is more rarely studied in mathematics). However this type of invariant is however still quite simple or we work with only one variable. In our research we study a substantially wider variety of multivariate invariants with increasing size and complexity.

In cryptographic invariants the main object to study are round invariants for one round. We would like to have $P$ (Inputs) = $P$ (Output ANF) where $P$ will be a Boolean function. We are looking for polynomials $P$ the value of which does not change after we apply a transformation called “a round” we call $\phi$.

This round function $\phi$ is typically a bijection and is like one round of encryption. In addition, typically it is NOT one fixed permutation but it has a parameter, a secret key and potentially additional parameters. The more parameters, the harder it becomes to find invariants. For example the T-310 cipher can be viewed as each round is applying one of the 8 possible permutations
\( \phi_0 : \{0,1\}^{36} \to \{0,1\}^{36} \) up to \( \phi_7 : \{0,1\}^{36} \to \{0,1\}^{36} \) and the choice which \( \phi \) is actually used depends on 2 bits of the secret key and 1 bit of the IV (which is public and known to the attacker), all using the original notations of [39].

Technically speaking, just finding such invariants is easy and they exist in vast numbers yet many are in some sense trivial or degenerated, cf. [13]. A key problem is to find simultaneous invariants to hold in all the eight cases, i.e. for all of \( \phi_0 \ldots \phi_7 \) simultaneously. This was a big problem in early research on this topic cf. [13] but it is not a problem in this paper (we construct a solution which works directly and there are many more variables which are eliminated also).

### 2.6 Group Theory vs. Invariants for Block Ciphers

We recall that for the finite field “inverse” S-box, and NOT for the actual AES S-box, cf. [16], it is possible to use the so called cross-ratio. Again this is already an invariant in our more general non-linear \( \phi \) case. We refer to Section 4 in [16] for more details and further references. The “Whitening Paradox” paradox is a proof of concept that a group-theoretic claims in cryptography [39–41, 2] can be highly misleading and can lead to ciphers where the group of transformations generated by the cipher is proven mathematically to be extremely large, and which are nevertheless insecure and can simultaneously broken for an exponentially large number of rounds. We refer to [16, 17] for more details.

There exists numerous modern works on the group of transformations generated by a block cipher [40, 38, 41, 2, 16] and primitive groups, [40, 38, 2, 8]. This research topic was very clearly was invented during the Cold War and was already studied very carefully in the 1970s with very specific security claims which are contained in [39], however these claims are not formulated as precise mathematical theorems in [39] and are therefore subject to interpretation.

### 2.7 Weak Ciphers and Backdoors

A first impression when reading this paper is that it is our paper is about how to “backdoor” a block cipher: how to make it weak on purpose [16, 17, 38, 18, 19, 8, 3, 1, 37]. This first impression is rather totally incorrect: what we do rather “proper” cryptanalysis of block ciphers. This is because absolutely every property\(^2\) we study about the cipher wiring happens with a relatively large probability. Overall we will get an attack which may happen accidentally, also when the whole cipher specification and the non-linear components are the strongest possible and were not chosen by the attacker. Moreover a wider variety of such attacks exist. If our polynomial invariant \( P \) of degree 7 does not work, another similar property could eventually work and the success probability is likely to be higher than it seems from the strict attack presented here below.

### 2.8 On Irreducible Polynomials

Recent papers show how to construct polynomial non-linear invariant attacks on some block ciphers. Some such attacks are clearly trivial, for example products

\(^2\) Such as assumptions on \( P \) and \( D \), or the fact that the Boolean function will have a cubic annihilator or absorber of some peculiar form.
of linear polynomials which are already invariants for the same block cipher\(^3\). No new attack is found in this case. For this reason some early papers on this topic emphasise invariants being irreducible polynomials [13, 10].

Irreducible polynomials turns out to be a false good idea. In general there is no problem whatsoever with products of linear polynomials: in general these linear factors will NOT correspond to any attack on the same cipher (and can only correspond to an attack a substantially weaker cipher). This is very clearly demonstrated in recent papers [13, 25] and the strongest round invariant attacks ever found are of this type [14].

2.9 On Sporadic Attacks vs. Product Construction

An interesting question is discovering some “sporadic” properties the existence of which we have not anticipated. The attack presented was actually first discovered accidentally, by trying the exact cipher setup 265 of [14] with various Boolean functions chosen at random. We discovered that they occur. Only later we developed a detailed mathematical explanation about WHY such an attack may work. At several we find it quite surprising that such an attack may at all be ever made to work on any block cipher. Indeed we make two extremely complex polynomials in 16 variables to be equal for any input on 36 bits, for any key, any IV (and any number of rounds in further cryptanalytic applications). Moreover the are 7 polynomials and there no trivial transitions of type 4+4 due to the Feistel structure with 4 branches which was a key feature in previous attacks [14, 31].

After discovering this new invariant we realized that the polynomial can be factored. We believe that this is NOT accidental. Not only a strong attack in [14] is a product of linear polynomials but also the new unexpected attack is such. This is really one of the main claims of this paper: the product construction of [14] and another new and (substantially more complex) product construction in this paper are not a good attacks by accident. They correspond to a powerful method of constructing attacks and there is some sort of phase transition where attacks become easier to construct as the degree of this product increases. This is because higher degree monomials can be obtained in several different ways, hence opportunities for cancelations, or obtaining the same polynomial in different ways. In general well-made polynomials which are products of many terms are MORE likely to lead to the sort of cancellations we need in order to make an attack work. All this is conjectured but not proven (better attacks could also exist elsewhere) and in our attack we need to make two polynomials being products of many factors equal, not just annihilate one such product, which however seems to be achievable roughly for the same reasons.

\(^3\) For example in appendix of [13] the author found an invariant with \(169 = 13^2\) terms which and it turns out to be equal to a product or two linear polynomials with 13 terms which are also round invariants. Such examples contain nothing new w.r.t. Matsui’s Linear Cryptanalysis.
3 Linear Cryptanalysis - New Possibilities

Linear cryptanalysis is a popular research topic since Matsui, [36]. However recent research shows that it is older than initially thought cf. [22]. There exist numerous papers on linear attacks and such attacks are a tough game. They require large quantities of data encrypted with a single key and properties are highly regular. For example the best attack on DES by Matsui depends on a property with a period of 14 rounds. For T-310 there exist numerous examples where the periodic property has say 6 or 8 or 12 rounds, and extremely few for say 13 rounds, cf. [20].

At the same time if we read the specification of T-310 cf. [42], we will see that state bits which are actually used for encryption will be extracted at the speed of 1 bit every 127 rounds, where 127 is a prime. We also learn that key bits are repeated every 120 rounds. Finally we learn that the IV bits are repeated every $2^{61} - 1$ bits which is also a prime, and this choice was clearly a deliberate choice by the designers, cf. [39]. So there is no hope that we could find any “relevant” periodic invariant attack, right, and for example it is unthinkable that an invariant with a period being a larger prime such as 127 would exist for a block cipher? In this paper we show that an invariant with a period of 127 may exist for a block cipher. Moreover such invariants do exist for some real-life keys. Such a degree of complexity was never seen before in Linear Cryptanalysis.

3.1 A New Discovery - a Large Prime Periodic Property

We found that when we set the Boolean function to zero (equal to zero for every input) the real-life setup LZS 31 and LZS 33 of cipher T-310 exhibits a complex linear property valid for 127 rounds. We should insist on the fact that 127 is a prime, and this is a solid irreducible complex invariant property which is most likely not a consequence or a combination of any simpler periodic properties. The number of active bits (out of 26) follows a complex pattern:

12->12->14->16->15->15->17->17->16->18->16->17->18->17->19->18->15->16->15->13->14->16->18->20->23->22->22->21->20->20->21->23->23->22->21->21->19->14->16->18->20->23->22->22->21->20->20->21->23->23->22->21->21->19-> and back to 12

Is there any hope that this property could be ever used in cryptanalysis?

3.2 How to Maybe Break T-310 in a Real-Life Historical Setting

In this section we sketch how this property can maybe be used to decrypt T-310 communications. The attack is hypothetical and is meant to highlight the role of algebra in constructing an attack of the sort never seen before. A possible attack will work as follows:

1. We consider one of the real-life cipher variants LZS 31 or 33 cf. [29, 20].
2. We express the problem of finding an invariant for 127 rounds as a system of algebraic equations where the unknowns are the coefficients of the Algebraic Normal Form (ANF) of the Boolean function $Z$. We call $FE$ this equation, cf. which is a simple I/O sum of two polynomials (an Input polynomial and an Output polynomial) cf. Section 5.
3. In general such equation has no solutions and it is very hard to know if it has any. However here we already have a case where $FE$ has a solution!
4. Now we multiply our 127 polynomials by a well-chosen polynomial at each step. We get a more complex $FE$ which is expected to have more than one solution (if we are lucky or if the polynomials were well chosen).
5. At the end if we do it well, this could work for a Boolean function not chosen by the attacker. The key property is that there are many ways to annihilate Boolean functions in this type of attacks which are sufficiently powerful in order to attempt to break ciphers with arbitrary Boolean functions, see [14] and this paper for specific examples.
6. At the end we might be able to construct an invariant attack with an invariant with a prime period of 127 or with another completely different prime period.
7. A period of 127, given that bits used for encryption also obey a period of 127, should lead to powerful attacks which allow to decrypt communications.

We have not yet presented any non-linear attack yet, however we have a powerful framework or methodology to construct one. From a trivial completely linear attack initially, or/and from set of polynomials which represent a hypothetical attack which does not work or where the Boolean function is degenerated, we can attempt to construct an attack which works for more complex random Boolean functions. If the reader doubts whether this methodology works we refer to [25] which contains an elaborate complete proof of concept where the cipher is modified several times simpler linear attacks are removed and complex high degree invariants only are kept, while the complexity of the invariants and the complexity of the Boolean function also increases.

**What is Next?** In what follows we will construct a new elaborate example of an invariant polynomial of degree 7. We will however construct an invariant for one round (instead of 127 rounds). In fact it is potentially extremely difficult to work with 127 rounds directly in the general non-linear case. The size of the polynomial equations to study and the size of $FE$ to solve would literally explode. Substantial simplifications are needed (or some way to limit the size of polynomials) and we are still learning how to build non-linear attacks on block cipher with our new white-box methodology. Our ultimate goal would be to obtain something which works for a non-negligible fraction such as say 1 % of all Boolean functions on 6 variables, cf. [14], which goal will be achieved later inside this paper. In all cases an important hint is that invariants where the polynomial is a product of several simpler polynomials are quite powerful cf. Section 2.8 above and [14].
4 Non-Linear Cryptanalysis through Formal ANF Coding

The concept of cryptanalysis with non-linear polynomials a.k.a. Generalized Linear Cryptanalysis (GLC) was introduced at Eurocrypt’95, cf. [32]. A key question is the existence of round-invariant I/O sums: when a value of a certain polynomial is preserved after 1 round. Many researchers have in the past failed to find any such properties, Bi-Linear and Multi-Linear attacks were introduced [15, 16] for Feistel ciphers branches specifically. In this paper and unlike in [43] we focus on invariants which work for 100 % of the keys and we focus on stronger invariants which hold with probability 1.

We call $P$ a polynomial invariant if the value of $P$ is preserved after one round of encryption, i.e. if $P($Inputs$) = P($Output ANF$)$. In this paper we work with one specific block cipher with 36-bit blocks. The main point is that any block cipher round translates into relatively simple Boolean polynomials, if we look at just one round. We follow the methodology of [13] in order to specify the exact mathematical constraint, known as the Fundamental Equation or $FE$, cf. Section 5, so that we could have a polynomial invariant attack on our cipher. Such an attack will propagate for any number of rounds (if independent of key and other bits). In addition it makes sense following [13] to consider that the Boolean function is an unknown. We denote this function by a special variable $Z$. We then see that our attack works if and only if $Z$ is a solution to a certain algebraic equation [with additional variables]. The main interest of making $Z$ a variable is to see that even if $Z$ is extremely strong, some advanced “product” attacks will work nevertheless.

4.1 Notation and Methodology

In this paper the sign $+$ denotes addition modulo 2, and frequently we omit the sign $*$ in products. For the sake of compact notation we frequently use short or single letter variable names. For example let $x_1, \ldots, x_{36}$ be inputs of a block cipher each being $\in \{0, 1\}$. We will avoid this notation and name them with small letters $a - z$ and letters $M - V$ when we run out of lowercase letters. We follow the backwards numbering convention of [13] with $a = x_{36}$ till $z = x_{11}$ and then we use specific capital letters $M = x_{10}$ till $V = x_1$. This avoids some “special” capital letters following notations used since the 1970s [24, 42, 39]. We consider that each round of encryption is identical except that they can differ only in some “public” bits called $F$, a round constant\(^4\), known to the attacker and some “secret” or key bits called $S1$ or $K$ and $S2 = L$. Even though these bits ARE different in different rounds we will omit to specify in which round we take them because our work is about constructing one round invariants (extending to any number of rounds). This framework covers most block ciphers ever made except that some ciphers would have more “secret” or “public” bits in one round. The capital letter $Z$ is a placeholder for substitution of the following kind

\(^4\) It is different in each round and it also known as IV bits which are derived from na LFSR with a very large period cf. [42].
where \( e_1 \ldots e_6 \) will be some 6 of the other variables. In practice, the \( e_i \) will represent a specific subset of variables of type \( a-z \), or other such as \( L \). At the end \( Z \) must be replaced by a formula like:

\[
Z \leftarrow Z00 + Z01 \cdot L + Z02 \cdot c + Z03 \cdot Lc + \ldots + Z62 \cdot c(kf)h + Z63 \cdot Lckf
\]

where \( Zij \) are coefficients of the Algebraic Normal Form (ANF).

4.2 Constructive Approach Given the Cipher Wiring

Our attack methodology starts from a given block cipher specified by its ANFs for one round. Specific examples will be shown for T-310. The block size is 36 bits and the key has 240 bits. The hardware encryption cost with T-310 is hundreds of times bigger than AES or 3DES, cf. [24]. Does it make this cipher very secure? Not quite, if we can construct algebraic invariants which work for any number of rounds.

4.3 ANF Coding of One Full Round

We number the cipher state bits from 1 to 36 where bits 1, 5, 9, \ldots, 33 are those freshly created in one round, cf. Fig 1. Let \( x_1, \ldots, x_{36} \) be the inputs and let \( y_1, \ldots, y_{36} \) be the outputs. One round of our cipher can be described as 36 Boolean polynomials out of which only 9 are non-trivial:

\[
\begin{align*}
    y_{33} &= F + x_{D(9)} \\
    Z1 &= Z(S2, x_{P(1)}, \ldots, x_{P(5)}) \\
    y_{29} &= F + Z1 + x_{D(8)} \\
    y_{25} &= F + Z1 + x_{P(6)} + x_{D(7)} \\
    Z2 &= Z(x_{P(7)}, \ldots, x_{P(12)}) \\
    y_{21} &= F + Z1 + x_{P(6)} + Z2 + x_{D(6)} \\
    y_{17} &= F + Z1 + x_{P(6)} + Z2 + x_{P(13)} + x_{D(5)} \\
    Z3 &= Z(x_{P(14)}, \ldots, x_{P(19)}) \\
    y_{13} &= F + Z1 + x_{P(6)} + Z2 + x_{P(13)} + S2 + Z3 + x_{D(4)} \\
    y_{9} &= F + Z1 + x_{P(6)} + Z2 + x_{P(13)} + S2 + Z3 + x_{P(20)} + x_{D(3)} \\
    Z4 &= Z(x_{P(21)}, \ldots, x_{P(26)}) \\
    y_{5} &= F + Z1 + x_{P(6)} + Z2 + x_{P(13)} + S2 + Z3 + x_{P(20)} + Z4 + x_{D(2)} \\
    y_{1} &= F + Z1 + x_{P(6)} + Z2 + x_{P(13)} + S2 + Z3 + x_{P(20)} + Z4 + x_{P(27)} + x_{D(1)} \\
    x_{0} &= S1 \\
    y_{i+1} &= x_{i} \text{ for all other } i \neq 4k \quad (\text{with } 1 \leq i \leq 36)
\end{align*}
\]
Two things remain unspecified: the $P$ and $D$ boxes or the internal wiring. In T-310 this specification is called an LZS or Langzeitsschlüssel which means a long-term key setup. We simply need to specify two functions $D : \{1 \ldots 9\} \rightarrow \{0 \ldots 36\}$, $P : \{1 \ldots 27\} \rightarrow \{1 \ldots 36\}$. For example $D(5) = 36$ will mean that input bit 36 is connected to the wire which becomes $U5 = y_{17}$ after XOR of Fig. 1. Then $P(1) = 25$ will mean that input 25 is connected as $v_1$ or the 2nd input of $Z_1$. We also apply a special convention where the bit $S_1$ is used instead of one of the $D(i)$ by specifying that $D(i) = 0$.

![Fig. 3. The internal structure of one round of T-310 block cipher.](image)

### 4.4 The Substitutions.
Overall one round can be described as 36 Boolean polynomials of degree 6; out of which only 9 are non-trivial. One round of encryption is viewed as a sequence of substitutions where an output variable is replaced by a polynomial algebraic expression in the input variables. Here is a (shortened) example following the cipher specification step-by-step for the long-term key 551 used in [13]:

$$\begin{align*}
a & \leftarrow b \\
b & \leftarrow c \\
c & \leftarrow d \\
d & \leftarrow F + i \\
\ldots & \\
\ldots & \\
V & \leftarrow F + Z1 + O + Z2 + q + L + Z3 + i + Z4 + k + K
\end{align*}$$

In order to have shorter expressions to manipulate we frequently replace $Z1 − Z4$ by shorter abbreviations $Z, Y, X, W$ respectively. We also replace $S2$ by a single letter $L$ (used at 2 places). The other key bits $S1 = K$ will only be used if some $D(i) = 0$. 
5 The Fundamental Equation

In order to break our cipher we need to find a polynomial expression $P$ say

$$P(a, b, c, d, e, f, g, h, \ldots) = abcdijkl + efg + efh + egh + fgh$$

using any number between 1 and 36 variables such that if we substitute in $P$ all the variables by the substitutions defined we would get exactly the same polynomial expression $P$, i.e. $P($Inputs$) = P($Output ANF$)$ are equal as multivariate polynomials. For example (or to start with) we assume that this polynomial $P$ is fixed. Then the attacker will write ONE SINGLE algebraic equation which he is going to solve to determine the unknown Boolean function $Z$, if it exists. There are several forms of his equation written with more less laconic vs. very precise notations.

5.1 Compact Notations - Basic High-Level Fundamental Equation

Definition 5.2 (Compact Uni/Quadri-variate FE). Our “Fundamental Equation (FE)” to solve is to make sure that sum of two polynomials like:

$$FE = P($Inputs$) + P($Output ANF$)$$

reduces to 0, or more precisely we are aiming at $FE = 0$ for any input, or in other words we want to achieve a formal equality of two Boolean polynomials like

$$P($Inputs$) + P($Transformed Outputs$) = 0$$

or even more precisely

$$P(a, b, c, d, e, f, g, h, \ldots) = P(b, c, d, F + i, f, g, h, F + Z1 + e, \ldots)$$

where $Z1 - Z4$ will be later replaced by Boolean functions $Z(), Y(), X(), W()$.

**Alternative Notation.** There is also another notation which is more like notations used in classical invariant theory. Instead of writing $FE = P($Inputs$) + P($Transformed Outputs$)$ we can also write:

$$FE = P + P^\phi$$

where

$$P^\phi \overset{\text{def}}{=} P($input \cdot \phi$) = P($Transformed Outputs$)$$

which is the same as above, and we can also write:

$$P^i \overset{\text{def}}{=} P$$

$$P^o \overset{\text{def}}{=} P($input \cdot \phi$)$$

where $\phi$ is the transformation induced by 1 round of encryption. This usage of exponents is similar as in the mathematical (Hilbertian) invariant theory. Our exponents can be simply interpreted as transformations on polynomials, or more precisely as operations belonging to a certain group of transformations acting
on a set of Boolean polynomials $P$ or $A$ or other say $(a_zM + b) \in B_{36}$ where $B_{36}$ is the precise ring of all Boolean polynomials in 36 variables named $a - z$ and $M - V$ as in this paper. The notation $P^\phi$ is very elegant and unhappily ambiguous in general, because in general $\phi$ depends also on $F$ and various key bits. Then it happens that $P^\phi$ is likely to be unique nevertheless: we are aiming at computing $P^\phi$ primarily and precisely in cases where the result, the transformed and substituted polynomial $P^\phi$ is such that the final result $P^\phi$ does NOT depend on $F, K, L$.

At this stage expressions of type $Z_1$ later renamed as $Z()$ or $Z_3$ which will be later replaced by a function $X()$ are placeholders for degree 6 polynomials yet to be specified fully.

### 5.3 Less Compact Notations - Expanding the Fundamental Equation

The main unknown in FE is a Boolean function $Z$ and in some very simple cases the FE can be of type $fZ = g$ where $f$ and $g$ are two polynomials. In general however this equation is very complex and maybe has no solutions $Z$ whatsoever. In order to study this question in detail, in the next step, $Z$ will be represented by its Algebraic Normal Form (ANF) with 64 binary variables which are the coefficients of the ANF of $Z$, and there will be several equations, and four instances $Z_1 - Z_4$ renamed as $Z, Y, X, W$ of the same $Z$:

**Definition 5.4 (A Multivariate FE).** At this step we will rewrite FE as follows. We will replace $Z_1$ by:

$$Z \leftarrow Z_00 + Z_01 \ast L + Z_02 \ast j + Z_03 \ast Lj + \ldots + Z_62 \ast jhfpd + Z_63 \ast Ljhfpd$$

Likewise we will also replace $Z_2$:

$$Y \leftarrow Z_00 + Z_01 \ast k + Z_02 \ast l + Z_03 \ast kl + \ldots + Z_62 \ast loent + Z_63 \ast kloent$$

and likewise for $X = Z_3$ and $W = Z_4$ and the coefficients $Z_00 \ldots Z_63$ will be the same inside $Z_1 - Z_4$, however the subsets of 6 variables chosen out of 36 will be different in $Z_1 - Z_4$. Moreover, some coefficients of $P$ may also be variable.

In all cases, all we need to do is to solve the equation above for $Z$, plus a variable amount of extra variables e.g. $Z_63$. This formal algebraic approach, if it has a solution, still called $Z$ for simplicity, or $(P, Z)$ will guarantee that our invariant $P$ holds for 1 round. This is, and in this paper we are quite lucky, IF this equation does not depend on three bits $F, K, L$. This is the discovery process of [13] which we do not use here. We rather work with basic paper and pencil maths and build our attack from scratch in stages.

---

5 Such equations have numerous non-trivial solutions, cf. [11].
6 A New Invariant Attack of Degree 7

Let
\[
\begin{align*}
A &\equiv (i + m) \quad \text{which is bits 24, 28} \\
B &\equiv (j + n) \quad \text{which is bits 23, 27} \\
C &\equiv (k + o) \quad \text{which is bits 22, 26} \\
D &\equiv (l + p) \quad \text{which is bits 21, 25} \\
E &\equiv (y + O) \quad \text{which is bits 8, 12} \\
F &\equiv (z + P) \quad \text{which is bits 7, 11} \\
G &\equiv (M + Q) \quad \text{which is bits 6, 10} \\
H &\equiv (N + R) \quad \text{which is bits 5, 9}
\end{align*}
\]

\[\text{Fig. 4. Variable naming conventions}\]

Theorem 6.1 (A Degree 7 Invariant Attack). Let
\[
P = (A + B) (C + D) (D + F)(B + F) (E + F)(G + F)(G + H)
\]
then \(P\) is a non-zero polynomial of degree 7. We also assume that
\[
\begin{align*}
\{D(2), D(3)\} &= \{6 \cdot 4, 7 \cdot 4\} \\
\{D(6), D(7)\} &= \{2 \cdot 4, 3 \cdot 4\}
\end{align*}
\]
and that inputs of \(Y\) are in order bits 27, 6, 10, 23, 21, 25 and inputs of \(W\) are in order bits 26, 9, 5, 22, 7, 11. We assume that the Boolean function used inside the cipher has after adding 1 TWO degree 3 annihilators as follows:
\[
\begin{align*}
(Z+1)\cdot(f+e)(d+a)(b+c)&=0 \\
(Z+1)\cdot(f+e+1)(d+a+1)(b+c+1)&=0
\end{align*}
\]
Then \(P\) is a round invariant for any key any IV and any number of rounds.

Remarks: We describe an attack initially designed for LZS 265 however it uses only few properties of 265, and any key which has properties listed above will also be broken. It may seem that Boolean functions which satisfy these properties are extremely rare. In reality annihilators of degree 3 are near-systematic cf. Thm. 6.0.1. in [11], and annihilators of degree 3 of a very special form required are also quite frequent, see [14]. Therefore this attack CAN happen when a Boolean function chosen at random (this is how it was found in the first place). One example of a Boolean function which works is \(b + ac + bc + abc + bd + abd + bcd + abcd + c + ce + ace + bce + af + bf + abf + bc + df + cdf + abd + ef + be + ef + ace + f + bce + ace + bdef + abedef + 1\). Moreover there exist further transposed variants of this attack where the inputs of \(Z\) are negated and \(P\)
need to be modified likewise. Moreover there exist numerous permutations of inputs of \( Y \) and \( W \) which will also work. For example \( b, c \) could be swapped or exchanged with the pair with \( e, f \) in any order. However even the basic attack is very complex to explain and it is hard to see if all the stages are correct. This is why we present only the basic attack (carefully verified step by step with computer algebra software). A full extended attack with additional cases will be published in the future.

**Proof of Thm. 6.1** We use the same notations as in [14] and we distinguish input and output-side variables and polynomials by \( A^o \) vs. \( A^i \). We recall our assumption:

\[
\begin{align*}
\{D(2), D(3)\} &= \{6 \cdot 4, 7 \cdot 4\} \\
\{D(6), D(7)\} &= \{2 \cdot 4, 3 \cdot 4\}
\end{align*}
\]

and following [14] or simply following step by step a walk from output 9 to output 5 on Fig. 3 above we see that:

\[
\begin{align*}
H^o &= y_9 + y_5 = x_{D(3)} + W(.) + x_{D(2)} = W(.) + A^i \\
D^o &= y_{25} + y_{21} = x_{D(7)} + Y(.) + x_{D(6)} = Y(.) + E^i
\end{align*}
\]

Then we remark that some transitions are trivial for example \( H^i = G^i \) and many other: \( H \rightarrow G \rightarrow F \rightarrow E \) and \( D \rightarrow C \rightarrow B \rightarrow A \). The output polynomial is then equal to:

\[
\begin{align*}
\left( A^o + B^o \right) &\left( C^o + D^o \right) \left( D^o + F^o \right) \left( E^o + F^o \right) \left( G^o + F^o \right) \left( G^o + H^o \right) \\
\left( B^i + C^i \right) &\left( D^i + Y(.) + E^i \right) \left( Y(.) + E^i + G^i \right) \left( C^i + G^i \right) \left( F^i + G^i \right) \left( H^i + W(.) + A^i \right)
\end{align*}
\]

at this moment we have only inputs left and we can use shorter notations:

\[
\begin{align*}
\left( B + C \right) &\left( D + Y + E \right) \left( Y + E + G \right) \left( C + G \right) \left( F + G \right) \left( H + G \right) \left( H + W + A \right)
\end{align*}
\]

Finally we add the last expression to the input polynomial \( (A + B)(C + D)(D + F)(B + F)(E + F)(G + F)(G + H) \) and obtain that the our invariant hold is and only our sum two polynomials is zero. In other terms our \( FE \) sum which we would like to be zero is exactly equal to:

\[
\frac{(G + H)(F + G) \cdot \left[ (A+B)(C+D)(D+F)(B+F)(E+F)+(B+C)(C+G)(D+Y+E)(Y+E+G)(H+W+A) \right]}{(A+B)(C+D)(D+F)(B+F)(E+F)}
\]

which is the same as:

\[
\begin{align*}
&\left( F + G \right) \left( G + H \right) \left( C + D \right) \left( B + C \right) \left( D + F \right) \\
&\left[ (A+B)(E+F)(B+F)+(A+H)(D+E)(B+F)+Y(G+D+1)(B+F)(H+F+1)(A+H)+ \\
&W(H+F+1)(G+D+1)(D+E)+YW \right]
\end{align*}
\]

This is more complex than previous results of this type. The next step will be to absorb \( Y \) and \( W \). We will need two trivial properties which come from \( (Z + 1) \ast (f + e)(d + a)(b + c) = 0 \) where the order of the 6 variables matters
and exact definitions of A-H matter, and this leads to the following 2 absorption properties:

\[ CHF \cdot W = CHF \quad \text{and} \quad BDG \cdot Y = BDG \quad (4) \]

Then we have two more trivial properties where the order of the 6 variables matters which come from the fact that \((Z+1)(f+e+1)(d+a+1)(b+c+1) = 0\). We get two more absorption properties:

\[ (C + 1)(H + 1)(F + 1) \cdot W = (C + 1)(H + 1)(F + 1) \quad (5) \]

and

\[ (B + 1)(D + 1)(G + 1) \cdot Y = (B + 1)(D + 1)(G + 1) \quad (6) \]

Now we are going to prove an intermediate result (lemma). Let

\[ \mu = (G + F)(G + H)(C + D)(B + C)(D + F) \quad (7) \]

then we have

\[ Y \mu = \mu \quad \text{and} \quad W \mu = \mu \quad (8) \]

An interesting question is how do we factor multivariate polynomials. Factorisation is not unique and not (or not yet) implemented in SAGE. However the computation of the annihilator space is implemented. Here is a code snippet:

```sage
R.<a,b,c,d,e,f> = BooleanPolynomialRing(6)
sage: F=BooleanFunction(mu)
sage: F.annihilator(1, dim=True)
```

With this ability to find individual factors, here each time we have some a linear factor \(f\), and we will select one such factor at random if there is more than one, we can then divide by \(f + 1\). We repeat this process at random (there are many possible factorizations typically). In this way we obtain the following two remarkable factorisations. Here exactly is where the lack of unique factorization plays an important role. We get both that:

\[ \mu = [H(B + 1)(D + 1)(G + 1) + (H + 1)BDG](C + H + 1)(C + F + 1)(F + H + 1) \quad (9) \]

\[ \mu = [G(C + 1)(F + 1)(H + 1) + (G + 1)CHF](B + D + 1)(D + G + 1)(B + G + 1) \quad (10) \]

these two facts imply that both \(Y\) and \(W\) can be absorbed by \(\mu\). More precisely in (9) we see that \(Y\) is absorbed by the first factor using (6) and (4) therefore \(Y \mu = \mu\). Similarly in (10) \(W\) is absorbed by the first factor using (5) and (4) therefore \(W \mu = \mu\). We have eventually proven the result claimed earlier:

\[ Y \mu = \mu \quad \text{and} \quad W \mu = \mu \quad (8) \]

In the previous formula we are now allowed to replace \(Y\) and \(W\) by 1 (and \(YW\) also by 1). This is because the whole expression has \(\mu\) as a factor, which term
absorbs both $Y$ and $W$. In other words, the previous (3) becomes (11) where we replaced $Y$ and $W$ by 1 (due to absorption by $\mu$ which is the first factor):

\[
\mu \cdot 
\]

\[
[(A+B)(E+F)(B+F)+(A+H)(D+E)(B+F)+(G+D+1)(B+F)(H+F+1)(A+H)+
(H + F + 1)(G + D + 1)(D + E) + 1] 
\]

(11)

At the end it becomes a purely syntaxic process: as $W$ and $Y$ are no longer present. It is now sufficient to verify with formal algebra software that this polynomial is zero. Moreover at this stage definitions of A-H do no longer matter and there result holds more generally. In SAGE maths we just need to type:

```sage
sage: B8.<A,B,C,D,E,F,G,H> = BooleanPolynomialRing()
sage: mu = (F + G)*(G + H)*(C + D)*(B + C)*(D + F)
sage: f = (A + B)*(E + F)*(B + F) + (A + H)*(D + E)*(B + F) +
(G + D + 1)*(B + F)*(H + F + 1)*(A + H) +
(H + F + 1)*(G + D + 1)*(D + E) + 1
sage: mu*f
0
```

This verification ends our proof. Formal annihilation of the XOR of the input and output polynomials means their formal equality for any input key and IV bit, and insures that we have found an invariant attack on our block cipher.
7 Conclusion

Most cryptographic attacks are extremely simple, regular and easily decomposed into simpler events. This paper is about the existence of more sporadic irregular attacks on block ciphers. For example we show a complex irregular linear property with a large prime period of 127 rounds cf. Section 3. The next step is the study of non-linear invariants which is substantially harder. We need to work in a white-box way with polynomial algebra. Can we find complex invariant properties which arise “ex-nihilo” and are not combinations of simpler invariants? We study conditions under which polynomial invariant attacks are possible: two polynomials must be equal. This might give an impression that we try to “backdoor” a block cipher: showing how to make it weak on purpose. In fact we do rather “proper” cryptanalysis of block ciphers. This is because absolutely every property\footnote{Such as assumptions on \( P \) and \( D \), or the fact that the Boolean function will have a cubic annihilator or absorber of some peculiar form.} we study about the cipher wiring happens with a relatively large probability. It may happen accidentally, also when the whole cipher specification and the non-linear components are the strongest possible and were not chosen by the attacker. Then if our polynomial invariant \( P \) of degree 7 does not work, another one could work. Furthermore, as the degree of \( P \) increases, the power of our attack increases, and the success probabilities go up, for example the previous attack of degree 8 described in \[14\] requires just one degree 3 special annihilator event on \( Z \), two of which are required in the new attack of degree 7 described in this paper. It remains an open problem which degree, maybe 12, is required to break some real-life versions of this cipher. Due to the phase transition law (observed empirically) we expect further favourable outcomes at higher degrees.

This paper shows that there exist complex periodic polynomial invariant attacks on block ciphers where the polynomial is a well chosen product of 7 linear polynomials. This sort of attacks were unthinkable in block cipher cryptanalysis even few months ago. Our new product construction is a unique attack with a product of 7 polynomials, unlike any other attack seen before and very surprising also when compared to the other recent product attack in \[14\] based on two cycles of length 4 closely following natural transitions in a Feistel cipher with 4 branches. Our attack seems to be of completely different nature. We believe that the primary reason why such attacks can be made to work shifts to deeper properties and the structure of the ring of multivariate polynomials \( B_n \) with lack of unique factorization. This ring offers numerous opportunities to make complex polynomials with a large number of variables (e.g. 16) disappear totally (annihilation events), or to obtain the same polynomials in different ways (absorption events, non-unique factorisation events). In our attack we have managed to make two different products of 7 factors equal (as polynomials). A non-trivial solution of this sort would be rather impossible with unique factorization. We conjecture that if we want to discover further new types of non-trivial invariant attacks on block ciphers the attention needs to be brought primarily to study of these types of polynomial cancelation/absorption/equality/factoring events.
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A More Invariant Attacks at Degree 7

This is not the only invariant attack of degree 7 we can construct by the methods of this paper. For example it is possible to see that for the same Lzs 265, and for the same definitions of $A, B$, etc, and for a different Boolean function

$$Z = b + ac + bc + b + ab + cd + e + ce + ac + abc + ab + abf + bcf + df + cdf + abcdf + ef + bef + acef + bce + bcdef + abce + df + + 1$$

which was also selected essentially at random with some amount of trial and error, we have the following invariant property:

$$P = (1 + A + H)(B + H)(1 + C + H)(D + H)(E + H)(1 + F + H)(G + H)$$

This is very similar to our attack of Thm. 6.1 where we had:

$$P = (A + B)(C + D)(D + F)(B + F)(E + F)(G + F)(G + H)$$

There are indeed many similarities and some important differences. In the first polynomial the letter $H$ appears in all terms. In the second one no letter appears everywhere. Therefore we do not expect the two attacks to be isomorphic: identical modulo some permutation of the inputs of $Z$ and negating some inputs. However we expect that they are similar at another level, and that the success probability when the Boolean function is chosen at random is more or less identical in both attacks due to similar types of degree 3 annihilators for $Z$.

If One Attack Fails...

More importantly we expect that the sets of Boolean functions for which each attack works are essentially disjoint, and different invariants will break them in different cases, opening the possibility of breaking this cipher also with the exact Boolean function which was specified in the 1970s and used in real-life encryption.

Phase Transitions: From Hard to Easy. We conjecture that for every Lzs wiring such as 265, there exist a certain degree $D \approx 8$ such that the proportion of Boolean functions for which the cipher is broken by some invariant polynomial of degree $D$ is not negligible. We expect that this proportion grows with $D$ and approaches 1 when $D \to 36$. Finally we expect that when maybe say when $S \geq 12$ the attacks get yet stronger, cf. the concept of (rapid) “Phase Transition” in Section 7 and in Section 2.4 in [14].