GEOMETRIC DECOMPOSITIONS OF DIV-CONFORMING FINITE ELEMENT TENSORS

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ABSTRACT. A unified construction of div-conforming finite element tensors, including vector div element, symmetric div matrix element, traceless div matrix element, and in general tensors with constraints, is developed in this work. It is based on the geometric decomposition of Lagrange elements into bubble functions on each sub-simplex. Then the tensor at each sub-simplex is decomposed into the tangential and the normal component. The tangential component forms the bubble function space and the normal component characterizes the trace. A deep exploration on boundary degrees of freedom is presented for discovering various finite elements. The developed finite element spaces are div conforming and satisfy the discrete inf-sup condition. An explicit basis of the constraint tensor space is also established.

1. INTRODUCTION

Hilbert complexes play a fundamental role in the theoretical analysis and the design of stable numerical methods for partial differential equations [2, 4, 1, 7]. Recently in [5] Arnold and Hu have developed a systematical approach to derive new complexes from well-understood differential complexes such as the de Rham complex. On the right end of the Bernstein-Gelfand-Gelfand (BGG) diagram in [5] is the generalized div tensor spaces, cf. Fig. 1, where $\Lambda^{n-1,k} := \Lambda^{n-1} \otimes \Lambda^k$ is a tensor product of differential forms and $\kappa_k$ is the Koszul operator for de Rham complex. Noticeable example are $H(\text{div}; S)$ with the symmetric matrix $S$, which plays an important role in the discretization of elasticity equation in the mixed form, and $H(\text{div}; T)$ with the trace-free matrix $T$. In general, the tensor space from this diagram is defined as

$$\mathbb{X} := \ker(s^{n-1,k}) \cap \Lambda^{n-1,k},$$

where $s^{n-1,k} = \text{div} \kappa_k - \kappa_k \text{div} : \Lambda^{n-1,k} \rightarrow \Lambda^n, k-1$ is the operator on the diagonal. A proxy of the tensor $\Lambda^{n-1,k}$ is a matrix $A = (a_{i,\sigma}) \in \mathbb{R}^{n \times (k)}$ and $\mathbb{X}$ is then a subset of matrices satisfying certain constraints.

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The purpose of this paper is to develop a systematical construction of finite elements for \( H(\text{div}; \mathbb{R}) \) space, which is the first step towards a unified construction of finite element complexes for the whole BGG diagram unifying the recent progress on finite element complexes \([11, 9, 8, 19, 20, 21]\). The approach we shall use is the geometric decomposition, which leads to explicit bases for finite elements. The geometric decomposition of standard finite elements for de Rham complex is well-studied in \([3, 2, 17]\), and in \([13]\) for nodal finite element de Rham complexes. The geometric decomposition is an important tool for the finite element analysis. For example, it is used in \([16]\) to construct a local and bounded co-chain projection to the discrete de Rham complexes. The finite element system in \([14]\) also originates from the geometric decomposition.

Let \( E^n \) be the \( n \)-dimensional linear vector space and \( T \) be an \( n \)-dimensional simplex. The set of \( \ell \)-dimensional sub-simplex of \( T \) is denoted by \( \Delta_\ell(T) \). We start from the Bernstein decomposition of vector Lagrange element \([3]\)

\[
P_r(T) \otimes E^n = \bigoplus_{r=0}^n \bigoplus_{f \in \Delta_\ell(T)} b_f \mathbb{P}_{r-\ell+1}(f) \otimes E^n,
\]

where \( \mathbb{P}_k(f) \) is the set of real valued polynomials defined on \( f \) of degree no more than \( k \), which can be extended to \( T \) by barycentric coordinates, and \( b_f \) is the bubble function on \( f \). That is for each \( f \), there is a vector polynomial with vanishing trace on \( \partial f \). The vector polynomial \( \mathbb{P}_r(T) \otimes E^n \) is used to develop div-conforming finite elements for the differential form \( \Lambda^{n-1}(T) \), which, however, is an intrinsic quantity in the sense that the form itself is coordinate-free.

Based on this observation, we shall choose different and in general non-orthonormal basis adapted to each \( f \in \Delta_\ell(T) \): choose \( \ell \) tangential vectors \( t^f_1, \ldots, t^f_\ell \) of \( f \), which span the tangent plane \( T_f \), and \( n - \ell \) normal vectors \( n^f_1, \ldots, n^f_{n-\ell} \) for the normal plane \( N_f \). Then \( b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes E^n \) can be decomposed into the tangential component and the normal component

\[
B^f_r(T) = b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes T_f, \quad N^f_r(T) = b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes N_f.
\]

The tangential component will form the bubble space for the div operator. For \( r \geq 1 \), it holds that

\[
\mathbb{B}_r(\text{div}, T) := \ker(\text{tr}^{\text{div}}) \cap (\mathbb{P}_r(T) \otimes E^n) = \bigoplus_{r=1}^n \bigoplus_{f \in \Delta_\ell(T)} B^f_r(T).
\]

The normal component will contribute to the trace of div operator and thus is crucial for the div-conformity. Let \( \mathcal{T}_h \) be a conforming triangulation of a domain \( \Omega \). A set of degrees of freedom (DoFs) is

1. \( \nu(\nu) \quad \forall \nu \in \Delta_0(\mathcal{T}_h) \),
2. \( \int_f \nu \cdot n^f_i \, p \, ds, \quad f \in \Delta_\ell(\mathcal{T}_h), p \in \mathbb{P}_{r-(\ell+1)}(f), \ell = 1, \ldots, n-1, i = 1, \ldots, n-\ell \),
3. \( \int_T \nu \cdot p \, dx, \quad p \in \mathbb{B}_r(\text{div}, T), T \in \mathcal{T}_h \),

which is known as the Stenberg element \([25, 13]\). The tangential component is merged to \( \mathbb{B}_r(\text{div}, T) \) in (3) and thus is considered as local. The normal basis \( \{n^f_i\} \) is global in the sense that it only depends on \( f \) not on the element \( T \) containing \( f \). Therefore \( \nu \cdot n^f \) is uniquely determined and ensures the div-conformity. Furthermore, DoFs (1)-(2) can be redistributed facewisely, by choosing the normal basis as \( \{n_F, f \subset F \in \partial T\} \), which leads to Brezzi-Douglas-Marini (BDM) element \([6]\).
In order to construct div-conforming finite elements for $H(\text{div}; \mathbb{X})$, the crucial step is to get a similar $T - \text{N}$ decomposition of $\mathbb{X}$, i.e.,

$$\mathbb{X} = T^f(\mathbb{X}) \oplus N^f(\mathbb{X}),$$

where $T^f(\mathbb{X}) = (T^f \otimes \Lambda^k) \cap \ker(s^{n-1,k})$ and $N^f(\mathbb{X}) = \pi_X(N^f \otimes \Lambda^k)$ with an oblique (non-orthogonal) projection operator $\pi_X : N^f \otimes \Lambda^k \to \mathbb{X}$. This induces the geometric decomposition

$$P_r(T; \mathbb{X}) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_\ell(T)} [B^f_r(T; \mathbb{X}) \oplus N^f_r(T; \mathbb{X})],$$

where $B^f_r(T; \mathbb{X}) = b_f P_r(T; \mathbb{X}) \otimes \mathbb{X}$ and $N^f_r(T; \mathbb{X}) = b_f P_r(T; \mathbb{X}) \otimes N^f_r(T; \mathbb{X})$. As a direct result of decomposition (4), the following DoFs

(5) $\omega(v_i) \quad \forall i = 0, \ldots, n,$

(6) $(\omega, \eta)_f \quad \forall \eta \in P_{r-(\ell+1)}(f) \otimes \{} * dx_i^f \otimes dx_\sigma^f \mid (i, \sigma) \text{ is free}, \quad f \in \Delta_\ell(T), \ell = 0, \ldots, n-1,$

(7) $(\omega, \eta)_{\Omega} \quad \forall \eta \in \mathbb{P}_r(\text{div}; T; \mathbb{X}) = \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(T)} B^f_r(T; \mathbb{X}), T \in T_h,$

will determine a space $V_h \subset H(\text{div}; \Omega; \mathbb{X})$. Discrete inf-sup condition will be established with requirement $r \geq n + 1$ and with modification of DoFs for $r \geq k + 1$ for $1 \leq k \leq n - 2$. Variants can be constructed by further tuning the DoFs (5)-(7) which will recover the existing construction of $H(\text{div}; \mathbb{S})$ elements [10, 18, 22] and $H(\text{div}; T)$ elements [8, 19].

Another result which is of its own interest is a basis of $\mathbb{X}$

$$\{ d\lambda_{\sigma(0),\omega(i)} \otimes d\lambda_{\omega(i)} \}_{\sigma \in \Sigma(0; n-k; 0:n)}.$$

Indeed for any $t - n$ basis $\{x_i^f, i = 1, \ldots, n\}$ of $f \in \Delta_n-k(T)$,

$$\{ * dx_i^f \otimes (dx_{n-k+1}^f \land \cdots \land dx_n^f) \}_{i=1, \ldots, n-k}^{f \in \Delta_n-k(T)}$$

is a basis of $\mathbb{X}$. It generalizes the result: $\{ t_r \otimes t_r \} r \in \Delta_n(T)$ is a basis of $\mathbb{S}$, which is crucial in designing the $H(\text{div}; \mathbb{S})$ element [18, 22] and useful in the Regge calculus [12].

As basis functions are well documented for the Lagrange element, our finite elements for $H(\text{div}; \mathbb{X})$ can be efficiently implemented by appropriate tensor product of Lagrange basis functions. One can introduce data structure to manage DoFs for each sub-simplex $f$ and treat some components local and some global.

The rest of this paper is organized as follows. Notation and background are given in Section 2. Section 3 reviews the geometric decomposition of Lagrange elements. Geometric decomposition of div elements for vectors is shown in Section 4. A constraint div tensor space is discussed in Section 5. And geometric decomposition of div tensors and div-conforming finite elements for constraint tensors are developed in Section 6.

2. Notation and Background

We mainly follow the notation set in [3]. We summarize the most important notation and integers in the beginning:

- $\mathbb{R}^n : n$ is the dimension of the ambient Euclidean space;
- $P_r : r$ is the degree of the polynomial;
- $\Lambda^k : k$ is the order of the differential form;
- $\Delta_\ell(T) : \ell$ is the dimension of a sub-simplex $f \in \Delta_\ell(T)$.
2.1. Increasing sequence. For non-negative integers $a, b, l, m$, with $0 \leq b - a \leq m - l$, define
\[
\Sigma(a : b, l : m) := \{\sigma : \{a, \ldots, b\} \rightarrow \{l, \ldots, m\} \mid \sigma(a) < \sigma(a + 1) < \cdots < \sigma(b)\}.
\]
The range of $\sigma$ is denoted by $[\sigma]$, i.e., for $\sigma \in \Sigma(a : b, l : m)$, $[\sigma] = \{\sigma(i) \mid i = a, \ldots, b\}$.
The set $\Sigma(0 : k, 0 : n)$ will be used for the description of sub-simplexes, and $\Sigma(1 : k, 1 : n)$ for $k$ differential forms in $\mathbb{R}^n$.

For $\sigma \in \Sigma(0 : k, 0 : n)$, denote by $\sigma^* \in \Sigma(1 : n - k, 0 : n)$ the complementary map characterized by
\[
[\sigma] \cup [\sigma^*] = \{0, 1, \ldots, n\}.
\]
For $\sigma \in \Sigma(1 : k, 1 : n)$, its complementary map $\sigma^c \in \Sigma(1 : n - k, 1 : n)$ satisfies
\[
[\sigma] \cup [\sigma^c] = \{1, \ldots, n\}.
\]
For an increasing sequence $\sigma$ with length 1, we use the value for more intuitive indexing. In particular, for $i = 1, \ldots, n$, $i^* \in \Sigma(1 : n - 1, 1 : n)$ s.t. $\{i\} \cup \{i^*\} = \{1, \ldots, n\}$ and $i^* \in \Sigma(1 : n, 0 : n)$ s.t. $\{i\} \cup \{i^*\} = \{0, 1, \ldots, n\}$. For the unique element in $\Sigma(a : b, a : b)$, we simply write it as $[a : b]$.

We follow [23] to introduce notation on the addition and subtraction of increasing sequences. For non-negative integers $a, b, l, m$, with $0 \leq b - a \leq m - l$, let $\sigma \in \Sigma(a : b, l : m)$. If $q \in [l : m]\backslash[\sigma]$, then we write $\sigma + q = q + \sigma$ for the unique element of $\Sigma(a : b + 1, l : m)$ with image $[\sigma] \cup \{q\}$. In that case, we also write $\epsilon(q, \sigma)$ for the signum of the permutation that orders the sequence $q, \sigma(a), \ldots, \sigma(b)$ in the ascending order. For $q \in [\sigma]$, $\sigma - q$ is the unique element in $\Sigma(a : b - 1, l : m)$ s.t. $(\sigma - q) + q = \sigma$.

2.2. Simplex. Let $T \in \mathbb{R}^n$ be an $n$-dimensional simplex with vertices $v_0, v_1, \ldots, v_n$ in general position. We let $\Delta(T)$ denote all the subsimplices of $T$, and $\Delta_\ell(T)$ denotes the set of subsimplices of dimension $\ell$, for $0 \leq \ell \leq n$. The cardinality of $\Delta_\ell(T)$ is $\binom{n + 1}{\ell + 1}$.

There is a one-to-one correspondence between $\Delta_\ell(T)$ and $\Sigma(0 : \ell, 0 : n)$. Let
\[
f_\sigma = [v_{\sigma(0)}, \ldots, v_{\sigma(\ell)}] \in \Delta_\ell(T), \quad \sigma \in \Sigma(0 : \ell, 0 : n)
\]
be the closed convex hull of the vertices $v_{\sigma(0)}, \ldots, v_{\sigma(\ell)}$. The face $f_\sigma$ is uniquely determined by $[\sigma]$, while the ascendent ordering in $\sigma$ gives an orientation of $f_\sigma$.

If $f \in \Delta_\ell(T)$, then $f^* \in \Delta_{n-\ell-1}(T)$ will denote the subsimplex of $T$ opposite to $f$, i.e., the subsimplex whose index set is the complement of $[\sigma]$ in $\{0, 1, \ldots, n\}$. If $f = f_\sigma$ with $\sigma \in \Sigma(0 : \ell, 0 : n)$, then $f^* = f_{\sigma^*}$. Again for an increasing sequence $\sigma$ with length 1, we use its value and skip $\sigma$. For example, $f_i$ is the $i$-th vertex $v_i$ and $F_{\ell}$ is the $(n - 1)$-dimensional face opposite to $v_\ell$. Here capital $F$ is reserved for an $(n - 1)$-dimensional face of $T$ and the set of $F$ is $\partial T := \Delta_{n-1}(T)$. For lower dimensional sub-simplexes, we will use more conventional notation. For example, the vertex will be denoted by $v_i$ and the edge formed by $v_i$ and $v_j$ will be denoted by $e_{ij}$.

For each $f \in \Delta_\ell(T)$, there are $n - \ell$ faces $F \in \Delta_{n-1}(T)$ containing $f$. Indeed if $f = f_\sigma$, then $f \subseteq F_i$ for $i \in [\sigma^*]$ and thus the normal vector $n_{F_i}$ of $F_i$ is orthogonal to $f$. Then these normal vectors $\{n_{F_i}, i \in [\sigma^*]\}$ span an $(n - \ell)$-dimensional hyperplane called the normal plane $\mathbb{N}^f$ of $f$. The $\ell$-dimensional hyperplane containing $f$ is called the tangent plane of $f$ and denoted by $T^f$.
2.3. Barycentric coordinates and polynomials. If $\Omega \subseteq \mathbb{R}^n$ and integer $r \geq 0$, then $\mathbb{P}_r(\Omega)$ denotes the set of real valued polynomials defined on $\Omega$ of degree less than or equal to $r$ and $\mathbb{H}_r(\Omega)$ is the homogenous polynomial with degree $r$. For simplicity, we let $\mathbb{P}_r = \mathbb{P}_r(\mathbb{R}^n)$. Hence, if $n$-dimensional domain $\Omega$ has nonempty interior, then $\dim \mathbb{P}_r(\Omega) = \dim \mathbb{P}_r = \binom{r+n}{n}$.

When $\Omega$ is a point, $\mathbb{P}_r(\Omega) = \mathbb{R}$ for all $r \geq 0$. And we set $\mathbb{P}_r(\Omega) = \{0\}$ when $r < 0$.

For $n$-dimensional simplex $T$, we denote by $\lambda_0, \lambda_1, \ldots, \lambda_n$ the barycentric coordinate functions with respect to $T$. That is $\lambda_i \in \mathbb{P}_1(T)$ and $\lambda_i(v_j) = \delta_{ij}$, $0 \leq i, j \leq n$, where $\delta_{ij}$ is the Kronecker delta function. The functions $\{\lambda_i, i = 0, 1, \ldots, n\}$ form a basis for $\mathbb{P}_1(T)$. $\sum_{i=0}^{n} \lambda_i(x) = 1$, and $0 \leq \lambda_i(x) \leq 1$, $i = 0, 1, \ldots, n$, for $x \in T$. The subsimplices of $T$ correspond to the zero sets of the barycentric coordinates, i.e., if $f = f_\sigma$ for $\sigma \in \Sigma(0 : \ell, 0 : n)$, then $f$ is characterized by $f = \{x \in T \mid \lambda_i(x) = 0, i \in [\sigma^*]\}$.

We will use the multi-index notation $\alpha \in \mathbb{N}_0^n$, meaning $\alpha = (\alpha_1, \ldots, \alpha_n)$ with integer $\alpha_i \geq 0$. We define $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $|\alpha| := \sum_{i=1}^{n} \alpha_i$. We will also use the set $\mathbb{N}_0^n$ of multi-indices $\alpha = (\alpha_0, \ldots, \alpha_n)$, with $x^\alpha := x_0^{\alpha_0} \cdots x_n^{\alpha_n}$.

The Bernstein representation of polynomial of degree $r$ on a simplex $T$ is

$$\mathbb{P}_r(T) := \{\lambda^\alpha_{[0:n]} = \lambda_0^{(0)} \lambda_1^{(1)} \cdots \lambda_n^{(n)}, |\alpha| = r, \alpha \in \mathbb{N}_0^n\}.$$  

In the Bernstein form, for an $f_\sigma \in \Delta_\ell(T)$,

$$\mathbb{P}_r(f_\sigma) = \{\lambda^\alpha_{[\ell]}, |\alpha| = r, \alpha \in \mathbb{N}_0^\ell\}.$$  

Through the natural extension defined by the barycentric coordinate, $\mathbb{P}_r(f_\sigma) \subseteq \mathbb{P}_r(T)$.

The bubble polynomial of $f_\sigma$ is a polynomial of degree $\ell + 1$:

$$b_f := \lambda_\sigma = \lambda_{\sigma(0)} \lambda_{\sigma(1)} \cdots \lambda_{\sigma(\ell)}.$$  

Then $b_f|_{f'} = 0$ for other $f' \in \Delta_\ell(T), f' \neq f$ and $b_f|_e = 0$ for all $e \in \Delta_m(T), m < \ell$. And $b_f|_e = 0$ for faces $F, f \not\subset F$. On the other hand, for a polynomial $u \in \mathbb{P}_r(f)$, if $u|_{\partial f} = 0$, then $u = b_f q$ for some $q \in \mathbb{P}_{r-(\ell+1)}(f)$.

2.4. Differential forms. We consider an $n$-dimensional domain $\Omega \subset \mathbb{R}^n$. Usually we choose a Cartesian coordinate and describe a point $x = (x_1, \ldots, x_n) \in \Omega$ in this coordinate. To distinguish a set of points with a linear space of vectors, we will use $\mathbb{E}^n$ to denote the $n$-dimensional linear vector space which can be identified with $\mathbb{R}^n$ by identifying a point $x$ with the vector $x = \alpha \vec{e}$. We use $\partial x_i$ as the unit vector from the origin $o$ to point $(0, \ldots, 1, \ldots)$, which is considered as an element in the tangent space $T_o \Omega$. The dual basis of $(\mathbb{E}^n)^*$ is denoted by $\{dx_i, i = 1, 2, \ldots, n\}$, i.e., $(dx_i, \partial x_j) = \delta_{ij}$. Moreover, we assign the standard inner product of vectors relative to this Cartesian basis to make $\mathbb{E}^n$ a Hilbert space.

For a vector space $V$, we define the space of exterior $k$-forms as the skew symmetric multilinear functional space on $V^k := V \times \cdots \times V$ and denote it by $\Lambda^k(V)$ or simply $\Lambda^k$ if $V$ is clear in the context. We define $\Lambda(V) := \bigoplus_{k=0}^{\infty} \Lambda^k(V)$. By definition, $\Lambda^k \subset (V^k)^*$. The best way to study a $k$-form is through the action on $k$ vectors in $V$.

Beside the trivial addition, we now define a product on $\Lambda$ to make it an algebra. Let $\omega \in \Lambda^p$ and $\eta \in \Lambda^q$, we define the wedge product $\omega \wedge \eta \in \Lambda^{p+q}$:

$$\omega \wedge \eta = \sum_{\sigma} \text{sign}(\sigma) \omega(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \eta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}),$$
where the sum is over all permutations \( \sigma \) of \( \{1, \ldots, p+q\} \), for which \( \sigma(1) < \sigma(2) < \cdots < \sigma(p) \) and \( \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q) \). We have the determinant formulae on the wedge product. For \( \omega_i \in \Lambda^1, v_i \in V, i = 1, \ldots, p, \)

\[
(\omega_1 \wedge \cdots \wedge \omega_p)(v_1, \ldots, v_p) = \det (\omega_i(v_j))_{i,j=1,\ldots,p}.
\]

For a smooth manifold \( \Omega \), a \( k \)-th order differential form is a section of the tangent bundle \( \bigcup_{x \in \Omega} \Lambda^k(T_x \Omega) \). The linear space formed by all \( k \)-th differential forms is denoted by \( \Lambda^k(\Omega) \). As \( \Omega \) is a domain in \( \mathbb{R}^n \), given any point \( x \) in the interior of \( \Omega \), the tangent space \( T_x \Omega \) is isomorphism to \( T_x \Omega \) by shifting the origin to \( x \). That is we can use one basis \( \{ dx_i \} \) for all \( \Lambda^1(T_x \Omega) \). An element \( \omega \in \Lambda^k(\Omega) \) thus has a representation

\[
\omega = \sum_{\sigma \in \Sigma(1:k,1:n)} a_\sigma(x) \, dx_\sigma, \quad x \in \Omega,
\]

where we extend the multi-index notation to the wedge product

\[
dx_\sigma := dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(k)}.
\]

This representation enables us to identify a vector function with a differential form:

\[
\omega \leftrightarrow (a_\sigma)_{\sigma \in \Sigma(1:k,1:n)},
\]

which is usually called a vector proxy of \( \omega \). Be aware that, by the definition, the differential form is coordinate independent. Also note that, for a general manifold, the basis \( \{ dx_i \} \) is defined on the local chart while in (8) a global coordinate, which will be called an ambient coordinate, is used.

Using (8), we define the exterior derivative \( d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega) \) as: for \( \omega = \sum_{\sigma \in \Sigma(1:k,1:n)} a_\sigma(x) \, dx_\sigma, \) define \( d \omega \in \Lambda^{k+1}(\Omega) \) by

\[
d \omega = \sum_{\sigma \in \Sigma(1:k,1:n)} \sum_{i \notin \sigma} \partial_i a_\sigma \, dx_i \wedge dx_\sigma = \sum_{\sigma \in \Sigma(1:k,1:n)} \sum_{i \notin \sigma} \partial_i a_\sigma(i, \sigma) \, dx_{i+\sigma}.
\]

It can be verified that this definition of \( d \omega \) is independent of the choice of bases.

As \( \sum_{i=0}^n \lambda_i = 1, \sum_{i=0}^n \partial_i \lambda_i = 0 \) and \( \{ d\lambda_0, \ldots, d\lambda_n \} \) is not a basis of \( \Lambda^1 \). Set a vertex as the origin, without loss of generality, say \( v_0, \) then \( \{ d\lambda_1, \ldots, d\lambda_n \} \) is a basis of \( \Lambda^1 \). In general, through the index \( \sigma \), there is one-to-one correspondence between \( \Delta_{k-1}(F_0^r) \) and \( \Lambda^k(T) \). Namely for \( \sigma \in \Sigma(1:k,1:n) \), \( f_\sigma \) is a \( (k-1) \)-dimensional simplex contained in \( F_0^r \) and \( d\lambda_{\sigma} = (d\lambda_{\sigma(1)} \wedge \cdots \wedge d\lambda_{\sigma(k)}) \in \Lambda^k \).

The 1-form \( d\lambda_i \) has a vector representation \( \nabla \lambda_i \), which is orthogonal to face \( F_i^r \). Therefore for a simplex \( f_\sigma \in \Delta_k(T), \{ \nabla \lambda_i, i \in [\sigma^*] \} \) are \( n-\ell \) normal vectors of \( f_\sigma \) and can span the normal plane of \( f_\sigma \). The vector representations of \( (n-1) \)-forms, for \( i = 1, \ldots, \ell \),

\[
d\lambda_{[\sigma(0),\sigma(i)]}^* = d\lambda_{[0:n]-\sigma(0)-\sigma(i)} := d\lambda_0 \wedge \cdots \wedge d\lambda_{\sigma(0)} \wedge \cdots \wedge d\lambda_{\sigma(i)} \wedge \cdots \wedge d\lambda_n,
\]

are \( \ell \) tangential vectors of \( f_\sigma \). This is illustrated in Fig. 2.

3. Geometric Decomposition of Lagrange Elements

We begin with a geometric decomposition of the Lagrange element. Let \( T \) be an \( n \)-dimensional simplex and \( f = f_\sigma \in \Delta_k(T) \) be an \( \ell \)-dimensional face of \( T \). Recall that \( b_f := \lambda_\alpha \in \mathbb{P}_{\ell+1}(f_\sigma) \) is the bubble polynomial of \( f \) and \( \mathbb{P}_r(f_\sigma) = \{ \lambda_\alpha^*, |\alpha| = r, \alpha \in \mathbb{N}_0^o, \ell \} \subseteq \mathbb{P}_r(T) \).
Consider a Hilbert space $V$ with the inner product $(\cdot, \cdot)$. Define $\mathcal{N} : V \rightarrow V^*$ as: for any $p \in V$, $\mathcal{N}(p) \in V^*$ is given by $\langle \mathcal{N}(p), \cdot \rangle = (\cdot, p)$. When restricted to face $f$, $\mathbb{P}_r(f)$ is a subspace of $L^2(f)$ and thus $\mathcal{N}(\mathbb{P}_r(f))$ consists of the degrees of freedom (DoFs):

$$\langle \mathcal{N}(p), \cdot \rangle = \int_f p \, ds, \quad p \in \mathbb{P}_r(f).$$

For the 0-dimensional face, i.e., a vertex $v$, we understand that $\langle \mathcal{N}(v), u \rangle = u(v)$.

**Lemma 3.1** (Bernstein decomposition of Lagrange element, (2.6) in [3]). For the polynomial space $\mathbb{P}_r(T)$ with $r \geq 1$ on an $n$-dimensional simplex $T$, we have the following decomposition

$$\mathbb{P}_r(T) = \bigoplus_{\ell=0}^{n} \bigoplus_{f \in \Delta_\ell(T)} b_f \mathbb{P}_{r-(\ell+1)}(f).$$

And the function $u \in \mathbb{P}_r(T)$ is uniquely determined by DoFs

$$\int_f u p \, ds \quad \forall \ p \in \mathbb{P}_{r-(\ell+1)}(f), f \in \Delta_\ell(T), \ell = 0, 1, \ldots, n,$$

which can be written as

$$\mathbb{P}_r^*(T) = \bigoplus_{\ell=0}^{n} \bigoplus_{f \in \Delta_\ell(T)} \mathcal{N}(\mathbb{P}_{r-(\ell+1)}(f)).$$

**Proof.** We first prove the decomposition (9). Each component $b_f \mathbb{P}_{r-(\ell+1)}(f) \subset \mathbb{P}_r(T)$ and the sum is direct due to the property of $b_f$ that $b_f|_{f'} = 0$ for other $f' \in \Delta_\ell(T)$, $f' \neq f$ and $b_f|_e = 0$ for all $e \in \Delta_m(T)$ with $m < \ell$. Then count the dimension to finish the proof.

To prove the unisolvence, we choose a basis $\{\phi_i\}$ of $\mathbb{P}_r(T)$ by the decomposition (9) and denote DoFs (10) as $\{N_i\}$. By construction, the dimension of $\{\phi_i\}$ matches the number of DoFs $\{N_i\}$. The square matrix $(N_i(\phi_j))$ is block lower triangular in the sense that for $\phi_f \in b_f \mathbb{P}_{r-(\ell+1)}(f)$,

$$\int_e \phi_f p \, ds = 0, \quad \forall e \in \Delta(T) \text{ with } \dim e < \dim f, p \in \mathbb{P}_{r-\dim e+1}(e)$$

as $b_f|_e = 0$ for all $e \in \Delta_m(T)$ with $m < \ell$. The diagonal block is invertible as $b_f : \mathbb{P}_{r-(\ell+1)}(f) \rightarrow b_f \mathbb{P}_{r-(\ell+1)}(f)$ is an isomorphism.

So the unisolvence follows from the invertibility of this lower triangular matrix. □

We refer to [3, Fig. 2.1] for an illustration of this geometric decomposition.

Let $\{\mathcal{T}_h\}$ be a family of partitions of $\Omega$ into nonoverlapping simplexes with $h_K := \text{diam}(K)$ and $h := \max_{K \in \mathcal{T}_h} h_K$. Let $\Delta_\ell(\mathcal{T}_h)$ be the set of all $\ell$-dimensional faces of
the partition \( \mathcal{T}_h \) for \( \ell = 0, 1, \cdots, n - 1 \). The mesh \( \mathcal{T}_h \) is conforming in the sense that the intersection of any two simplex is either empty or a common lower sub-simplex. The global Lagrange finite element space \( S_h^\ell := \{ v \in C(\Omega) : v|_T \in \mathbb{P}_r(T), \forall T \in \mathcal{T}_h \} \) has the geometric decomposition
\[
S_h^\ell = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_r(\mathcal{T}_h)} b_f \mathbb{P}_{r-(\ell+1)}(f).
\]
Here we extend the polynomial \( b_f \mathbb{P}_{r-(\ell+1)}(f) \) to each element \( T \) containing \( f \) by the Bernstein form in the barycentric coordinate and thus it is a piecewise polynomial function and continuous in \( \Omega \). Consequently the dimension of \( S_h^\ell \) is
\[
\dim S_h^\ell = \sum_{\ell=0}^n |\Delta_r(\mathcal{T}_h)| \binom{r-1}{\ell},
\]
where \( |\Delta_r(\mathcal{T}_h)| \) is the cardinality, i.e., the number of \( \ell \)-dimensional simplexes in \( \mathcal{T}_h \).

4. GEOMETRIC DECOMPOSITION OF DIV ELEMENTS

Recall that \( \Lambda^k = \Lambda^k(\mathbb{R}^n) \) is the space of \( k \)-forms with constant coefficients, \( \Lambda^k(\Omega) \) is the space with coefficients being functions on \( \Omega \), \( \mathbb{P}_r \Lambda^k(\Omega) \) is the space with polynomial coefficients, and \( L^2 \Lambda^k(\Omega) \) is the space with square-integrable coefficient functions. The space \( H\Lambda^k(\Omega) := \{ \omega \in \Lambda^k(\Omega) : \omega \in L^2 \Lambda^k(\Omega), d\omega \in L^2 \Lambda^{k+1}(\Omega) \} \). In this section, we consider \( k = n - 1 \) and \( H\Lambda^{n-1}(\Omega) \) is isomorphism to \( H(\text{div}, \Omega) := \{ v \in L^2(\Omega; \mathbb{R}^n) : \text{div} v \in L^2(\Omega) \} \).

4.1. Differential form and vector proxy. Let \( T \) be an \( n \)-dimensional simplex. We consider finite elements with shape function space \( \mathbb{P}_r \Lambda^{n-1}(T) \). Recall that, for \( i = 1, \ldots, n \), \( i^r \in \Sigma(1 : n-1, 1 : n) \) s.t. \( \{i\} \cup \llbracket i^r \rrbracket = \{1, \ldots, n\} \). We choose an orthonormal basis of \( \Lambda^1(\mathbb{R}^n) \) as \( \{dx_i, i = 1, \ldots, n\} \) and \( dx_{i^r} := dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_n \in \Lambda^{n-1} \). The Hodge star for an orthonormal basis is defined as
\[
\star dx_i = (-1)^{i-1} dx_{i^r}, \quad \star dx_{i^r} = (-1)^{n-i} dx_i,
\]
which satisfies
\[
dx_i \wedge \star dx_i = dx, \quad dx_{i^r} \wedge \star dx_{i^r} = dx,
\]
with the volume \( dx := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \). Then
\[
\Lambda^{n-1}(\mathbb{R}^n) = \text{span}\{\star dx_i, i = 1, \ldots, n\},
\]
For \( \omega \in \Lambda^{n-1} \), we can then write
\[
\omega = \sum_{i=1}^n v_i \star dx_i, \quad \star \omega = (-1)^{n-1} \sum_{i=1}^n v_i dx_i,
\]
which induces an isomorphism
\[
\text{Prox}_{n-1} : \omega \rightarrow v = (v_1, v_2, \ldots, v_n)^T,
\]
and \( v \) is called a proxy vector of \( \omega \). Here boldface letters are used to denote the vector proxy in a fixed orthonormal basis of the ambient space \( \mathbb{R}^n \).

Consequently the differential form
\[
\mathbb{P}_r \Lambda^{n-1}(T) \cong \mathbb{P}_r(T; \mathbb{R}^n) := \mathbb{P}_r(T) \otimes \mathbb{R}^n.
\]
That is we can treat a polynomial differential form in \( \mathbb{P}_r \Lambda^{n-1}(T) \) as a polynomial vector function. The representation of \( d\omega \) is \( \text{div} v \), i.e.,
\[
d\omega = (\text{div} v) dx.
We will mix the usage of differential forms and vector functions and provide results in both versions. Note that a differential form is coordinate-free while a proxy vector representation depends on the choice of the basis.

For a 1-form \( \eta = \sum_{i=1}^{n} u_i \, dx_i \in \Lambda^1 \), define

\[
\text{Prox}_1(\eta) = u = (u_1, u_2, \ldots, u_n)^\top.
\]

For a vector \( t \) represented in the same orthonormal coordinate, \( t = (t_1, t_2, \ldots, t_n) \) representing the tangential vector \( \sum_{i=1}^{n} t_i \partial x_i \), then

\[
\eta(t) = \sum_{i,j=1}^{n} u_i t_j \langle dx_i, \partial x_j \rangle = u \cdot t.
\]

For \( F \in \Delta_{n-1}(T) \), let the trace operator \( \text{tr}_{F}^{n-1} : \Lambda^{n-1}(T) \to \Lambda^{n-1}(F) \) be the pullback of the inclusion \( F \hookrightarrow T \). That is for any tangent vectors \( v_1, \ldots, v_{n-1} \) of \( F \), we have

\[
\text{tr}_{F}^{n-1} \omega(v_1, \ldots, v_{n-1}) = \omega(v_1, \ldots, v_{n-1}), \quad \omega \in \Lambda^{n-1}(T).
\]

We denoted by \( \text{tr}^{n-1} : \Lambda^{n-1}(T) \to \bigcup_{F \in \partial T} \Lambda^{n-1}(F) \) as \( \text{tr}^{n-1} \omega|_F = \text{tr}_{F}^{n-1} \omega \).

In the ambient basis, \( \omega \) has a vector proxy \( v \) and \( \mathbf{n}_F \) is the normal vector of \( F \) so that the orientation of \( F \), which is given by the volume \( dx_F \in \Lambda^{n-1}(F) \), and \( \mathbf{n}_F \) form a consistent orientation of the ambient orthonormal basis. Then

\[
(11) \quad \text{tr}_F \omega = \mathbf{n}_F \cdot v \, dx_F.
\]

Based on (11), we can discuss the trace operator in the more familiar vector function setting. The trace operator for space \( H(\text{div}, T) \)

\[
\text{tr}^{\text{div}} : H(\text{div}, T) \to H^{1/2}(\partial T)
\]

is a continuous extension of \( \text{tr}^{\text{div}} v = \mathbf{n} \cdot v|_{\partial T} \) defined on smooth functions. Denoted by \( \mathbb{P}_r(\partial T) := \{ q \in L^2(\partial T) : q|_F \in \mathbb{P}_r(F) \text{ for each } F \in \Delta^{n-1}(T) \} \), which is a Hilbert space with inner product \( \sum_{F \in \Delta^{n-1}(T)} \langle \cdot, \cdot \rangle_F \). Obviously

\[
\text{tr}^{\text{div}}(\mathbb{P}_r(T; \mathbb{E}^n)) \subseteq \mathbb{P}_r(\partial T).
\]

It can be shown that \( \text{tr}^{\text{div}} : \mathbb{P}_r(T; \mathbb{E}^n) \to \mathbb{P}_r(\partial T) \) is onto; see [10, Lemma 3.2]. Define the polynomial bubble space

\[
\mathbb{B}_r(\text{div}, T) := \ker(\text{tr}^{\text{div}}) \cap \mathbb{P}_r(T; \mathbb{E}^n).
\]

It is obvious that for \( k = 0, \mathbb{B}_0(\text{div}, T) = \{0\} \). With a slight abuse of notation, \( \mathbb{B}_r(\text{div}, T) = \ker(\text{tr}^{n-1}) \cap \mathbb{P}_r(\Delta^{n-1}(T)) \) is also defined for differential \( (n-1) \)-forms.

4.2. Geometric decomposition of vector Lagrange elements. The geometric decomposition of the vector Lagrange elements is a straightforward generalization of Lemma 3.1.

The shape function space and DoFs are

\[
(12) \quad \mathbb{P}_r(T; \mathbb{E}^n) = \bigoplus_{t=0}^{n} \bigoplus_{f \in \Delta_t(T)} \left[ b_f \mathbb{P}_r(\partial t+1)(f) \otimes \mathbb{E}^n \right],
\]

\[
(13) \quad \mathbb{P}_r^+(T; \mathbb{E}^n) = \bigoplus_{t=0}^{n} \bigoplus_{f \in \Delta_t(T)} \left[ \mathcal{N}(\mathbb{P}_r(\partial t+1)(f)) \otimes \mathbb{E}^n \right].
\]

In (12)-(13), a fixed orthonormal basis is implicitly assumed in which the vector is expanded. It is usually the Cartesian coordinate describing the domain \( \Omega \).

The differential form \( \Lambda^{n-1}(T) \), however, is an intrinsic quantity in the sense that the form itself is coordinate-free although vector representations could vary. We shall choose different and in general non-orthonormal basis adapted to \( f \) to modify the geometric decomposition (12). We will use vector functions to present the main idea first and then write in the language of differential forms.
For a simplex $f \in \Delta_{\ell}(T)$, choose $\ell$ linearly independent tangential vectors $t^f_1, \ldots, t^f_\ell$ of $f$ and $n - \ell$ linearly independent normal vectors $n^f_1, \ldots, n^f_\ell$ of $f$. Then the set of $n$ vectors $\{t^f_1, \ldots, t^f_\ell, n^f_1, \ldots, n^f_\ell\}$ is a basis of $\mathbb{E}^n$ and each vector $u \in \mathbb{E}^n$ can be written uniquely as

\begin{equation}
\label{eq:14}
u = \sum_{i=1}^{\ell} u^f_i \cdot t^f_i + \sum_{i=\ell+1}^{n} u^f_i \cdot n^f_i, \quad u^f_i \in \mathbb{R}, \ i = 1, \ldots, n.
\end{equation}

Notice that for $\ell = 0$, no tangential vectors and for $\ell = n$, no normal vectors. Also note that $\{t^f_i\}$ or $\{n^f_i\}$ may not be orthonormal. But $t^f_i \cdot n^f_j = 0$ for $i = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots, n - \ell$. Define the tangent plane and normal plane of $f$ as

\begin{align*}
\mathbb{T}^f := \left\{ \sum_{i=1}^{\ell} c_i t^f_i : c_i \in \mathbb{R} \right\}, \quad \mathbb{N}^f := \left\{ \sum_{i=1}^{n-\ell} c_i n^f_i : c_i \in \mathbb{R} \right\}.
\end{align*}

For $r \geq 1$, define

\begin{align*}
\mathbb{B}_{r}^f(T) = b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathbb{T}^f, \quad \mathbb{N}_{r}^f(T) = b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathbb{N}^f.
\end{align*}

Note that $\mathbb{B}_{r}^f(T), \mathbb{N}_{r}^f(T) \subseteq \mathbb{P}_r(T; \mathbb{E}^n)$ for $r \geq 1$. Here we use letter $\mathbb{B}$ instead of $\mathbb{T}$ for the tangential component as it corresponds to the bubble polynomial functions of div-conforming elements.

**Lemma 4.1.** Let $r \geq 1$ be an integer. The shape function space $\mathbb{P}_r(T; \mathbb{E}^n)$ has a geometric decomposition

\begin{equation}
\label{eq:15}\mathbb{P}_r(T; \mathbb{E}^n) = \bigoplus_{\ell=0}^{n} \bigoplus_{f \in \Delta_{\ell}(T)} \left[ \mathbb{B}_{r}^f(T) \oplus \mathbb{N}_{r}^f(T) \right].
\end{equation}

A function $\mathbf{u} \in \mathbb{P}_r(T; \mathbb{E}^n)$ is uniquely determined by the DoFs

\begin{align*}
\int_f (\mathbf{u} \cdot t^f_i) \ p \, ds, & \quad i = 1, \ldots, \ell, \quad \text{(16)} \\
\int_f (\mathbf{u} \cdot n^f_j) \ p \, ds, & \quad j = 1, \ldots, n - \ell, \quad \text{(17)}
\end{align*}

for all $p \in \mathbb{P}_{r-(\ell+1)}(f), f \in \Delta_{\ell}(T), \ell = 0, 1, \ldots, n$, which can be written as

\begin{equation}
\label{eq:16}\mathbb{P}_r(T; \mathbb{E}^n) = \bigoplus_{\ell=0}^{n} \bigoplus_{f \in \Delta_{\ell}(T)} \left[ \mathbb{N}(\mathbb{P}_{r-(\ell+1)}(f) \otimes \mathbb{T}^f) \oplus \mathbb{N}(\mathbb{P}_{r-(\ell+1)}(f) \otimes \mathbb{N}^f) \right].
\end{equation}

**Proof.** Since $\{t^f_1, \ldots, t^f_\ell, n^f_1, \ldots, n^f_{n-\ell}\}$ forms a basis of $\mathbb{E}^n$, DoFs (16)-(17) are equivalent to

\[\int_f \mathbf{u} \cdot \mathbf{p} \, ds \quad \forall \mathbf{p} \in \mathbb{P}_{r-(\ell+1)}(f; \mathbb{E}^n).\]

Then the unisolvence follows from the unisolvency of the Lagrange element. \hfill \Box

### 4.3. Geometric decomposition of div element

Next we use $\mathbb{B}_{r}^f(T), \mathbb{N}_{r}^f(T)$ to characterize the kernel or range of the trace operator, respectively.

**Lemma 4.2.** For $r \geq 1$, it holds that

\begin{equation}
\label{eq:18}\mathbb{B}_r(\text{div}, T) = \bigoplus_{\ell=1}^{n} \bigoplus_{f \in \Delta_{\ell}(T)} \mathbb{B}_{r}^f(T),
\end{equation}

and

\begin{equation}
\label{eq:19}\text{tr} : \bigoplus_{\ell=0}^{n-1} \bigoplus_{f \in \Delta_{\ell}(T)} \mathbb{N}_{r}^f(T) \rightarrow \text{tr}\mathbb{P}_r(T; \mathbb{E}^n)
\end{equation}

is a bijection.
Proof. Verification of
\[ \mathbb{B}_v^f(T) \subseteq \mathbb{B}_r(\text{div}, T) \]
is straightforward. For face \( F \) not containing \( f \), \( b_f|_F = 0 \). For face \( F \) containing \( f \),
\[ \text{tr}_F u = n_F \cdot u = 0 \text{ as } n_F \cdot t_i^f = 0. \] Therefore \( \bigoplus_{\ell=1}^{n} \bigoplus_{f \in \Delta_{\ell}(T)} \mathbb{B}_v^f(T) \subseteq \mathbb{B}_r(\text{div}, T). \)

Then apply the trace operator to the decomposition (15) and use \( \text{tr}(\mathbb{B}_v^f(T)) = 0 \) to obtain \( \text{tr} \left( \bigoplus_{\ell=0}^{n-1} \bigoplus_{f \in \Delta_{\ell}(T)} N_v^f(T) \right) = \text{tr}\mathbb{P}_r(T; \mathbb{E}^n) \). So the map \( \text{tr} \) in (19) is onto.

Now we prove it is also injective. Take a function \( u = \sum_{\ell=0}^{n-1} \sum_{f \in \Delta_{\ell}(T)} u_{f}^\ell \) with \( u_{f}^\ell \in N_v^f(T) \) and \( \text{tr} \ u = 0 \). We will prove \( u = 0 \) by induction. Since \( u(v_i) = u_{f}^\ell(v_i) \) for \( i = 0, \ldots, n \), we get from \( \text{tr} \ u = 0 \) that \( u_{f}^{\ell_0} = 0 \). Assume \( u_{f}^\ell = 0 \) for all \( f \in \Delta(T) \) whose dimension is less than \( j \) with \( 1 \leq j < n \). Then
\[ u = \sum_{\ell=j}^{n-1} \sum_{f \in \Delta_{\ell}(T)} u_{f}^\ell. \]
Take \( f \in \Delta_j(T) \). We have \( u_{f}^\ell \in N_v^f(T) \). Hence \( (u_{f}^\ell \cdot n)|_f = (u \cdot n)|_f = 0 \) for any \( n \in N_v^f(T) \), which results in \( u_{f}^\ell = 0 \). Therefore \( u = 0 \).

Once we have proved the map \( \text{tr} \) in (19) is bijective, we conclude (18) from the decomposition (15).

Notice that the index of bubble polynomials is start from 1 and no bubble functions associated to the vertices, which implies \( B_1(\text{div}, T) = \{ 0 \} \).

4.4. Geometric decomposition of differential forms. We present the geometric decomposition (15) in term of differential forms. Given a vector \( u = (u_1, \ldots, u_n)^T \in \mathbb{E}^n \) represented in the coordinate \( x_i \), let \( \omega = \sum_{i=1}^{n} u_i \, dx_i \in \Lambda^1 \) and \( *\omega = \sum_{i=1}^{n} u_i (-1)^{i-1} \, dx_i \in \Lambda^{n-1} \). Then \( \text{Prox}_{1}(\omega) = u \) or \( \omega = \text{Prox}_{1}^{-1}(u) \). To resemble the notation of differential forms, we introduce \( d \, u := \omega \cdot x = \sum_{i=1}^{n} u_i \, dx_i \). Here in \( d \, u \), \( d \) is understood as a dual operator mapping a tangent vector \( u \) to a co-tangent vector \( du \in \Lambda^1(\mathbb{E}^n) \), and the symbol \( d \) is not associated to any differentiation. The proxy vector of \( d \, u \) and \( *\, d \, u \) is simply \( u \). A textbook notation of \( d \, u \) is usually \( \star u \) and \( *\, d \, u \) is \( \lambda_{n-1} u \).

So given the set of the \( n \)-vectors \( \{ t_1^f, \ldots, t_n^f, n_1^f, \ldots, n_{n-1}^f \} \) which is a basis of \( \mathbb{E}^n \), the set of the dual vectors \( \{ dt_1^f, \ldots, dt_n^f, dn_1^f, \ldots, dn_{n-1}^f \} \) is a basis of \( \Lambda^1(\mathbb{E}^n) \). Through the Hodge star, \( \{ \star dt_1^f, \ldots, \star dt_n^f, \star dn_1^f, \ldots, \star dn_{n-1}^f \} \) is a basis of \( \Lambda^{n-1}(\mathbb{E}^n) \). In this notation, the proxy vector representation of \( \star dt_i^f \) and \( \star \, dt_i^f \) is \( t_i^f \) and that of \( \star \, dn_i^f \) and \( \star \, dn_i^f \) is \( n_i^f \). Symbolically one can switch between vector functions and differential forms easily.

One particular choice of a \( t - n \) basis is as follows. Take a face \( f \sigma \) with \( \sigma \in \Sigma(0 : \ell, 0 : n) \). Without loss of generality, set \( \sigma(0) \) as the origin. The tangential vectors can be chosen as \( \{ t_i, i = 1 : \ell \} \), where \( t_i \) is a vector of edge \( e_{\sigma(0), \sigma(i)} \). The normal vectors can be chosen as \( \{ \nabla \lambda_i, i \in [\sigma^*] \} \), which is a basis of the normal plane of \( f \). In this \( t - n \) basis, \( \star \, d n_i^f = \lambda_i, i \in [\sigma^*] \), and \( \star \, dt_i^f \) is proportional to \( d \lambda_{[\sigma(0), \sigma(i)]^*} \); see Fig. 2.

We expand \( \omega \in b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \Lambda^{n-1}(T) \) in the \( t - n \) basis as
\[ \omega = \sum_{i=1}^{\ell} u_i^f \ast dt_i^f + \sum_{i=\ell+1}^{n} u_i^f \ast dn_i^f, \quad u_i^f \in b_f \mathbb{P}_{r-(\ell+1)}(f), i = 1, \ldots, n, \]
and its vector proxy is exactly (14).
Sometimes it is more convenient to write as one coordinate \( \{ x_i^f \} \) of \( \mathbb{R}^n \) adapted to \( f \in \Delta^v(T) \). We assume this coordinate satisfies the following requirements

1. \( \{ \partial x_i^f, i = 1, 2, \ldots, \ell \} \) is a basis of the tangent plane of \( f \);
2. \( \{ \partial x_i^f, i = \ell + 1, 2, \ldots, n \} \) is a basis of the normal plane of \( f \).

Then \( \{ dx_i^f, i = 1, 2, \ldots, n \} \) is a basis of \( \Lambda^1 \) and \( \{ \star dx_i^f, i = 1, 2, \ldots, n \} \) is a basis of \( \Lambda^{n-1} \).

Note that \( \partial x_i^f, i = 1, 2, \ldots, n \) may not be orthonormal but block-wisely orthogonal, i.e., \( (\partial x_i^f, \partial x_j^f) = 0 \) for \( i = 1, 2, \ldots, \ell \) and \( j = \ell + 1, 2, \ldots, n \). Then we can write (20) as

\[
\omega = \sum_{i=1}^n u_i^f \star dx_i^f.
\]

When a \( t-n \) basis \( \{ t^f_1, \ldots, t^f_\ell, n^f_1, \ldots, n^f_{n-\ell} \} \) is chosen, we can define the coordinate \( \{ x^f_i, i = 1, 2, \ldots, n \} \) as

\[
x^f_i := \begin{cases} t^f_i \cdot x & \text{for } i = 1, \ldots, \ell, \\ n^f_{i-\ell} \cdot x & \text{for } i = \ell + 1, \ldots, n. 
\end{cases}
\]

But the definition of \( \{ x^f_i \} \) may not require the tangential and normal vectors. We present the differential form version of Lemma 4.1.

**Lemma 4.3.** The shape function space \( \mathbb{P}_r \Lambda^{n-1}(T) \), for \( r \geq 1 \), has a geometric decomposition

\[
\mathbb{P}_r \Lambda^{n-1}(T) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta^v(T)} \left[ \mathbb{B}^f_\ell(T) \oplus \mathbb{N}^f_\ell(T) \right],
\]

where \( \{ t^f_1, \ldots, t^f_\ell, n^f_1, \ldots, n^f_{n-\ell} \} \) is a \( t-n \) basis at \( f \) and

\[
\mathbb{B}^f_\ell(T) := \left\{ \sum_{i=1}^\ell u^f_i \star dt^f_i, u^f_i \in b_f \mathbb{P}_{r-\ell}(f) \right\},
\]

\[
\mathbb{N}^f_\ell(T) := \left\{ \sum_{i=1}^{n-\ell} u^f_i \star dn^f_i, u^f_i \in b_f \mathbb{P}_{r-\ell}(f) \right\}.
\]

A function \( \omega \in \mathbb{P}_r \Lambda^{n-1}(T) \) is uniquely determined by the DoFs

\[
\int_f \star \omega(t^f_i) \ p \ ds, \quad i = 1, \ldots, \ell,
\]

\[
\int_f \star \omega(n^f_j) \ p \ ds, \quad j = 1, \ldots, n-\ell,
\]

for all \( p \in \mathbb{P}_{r-\ell+1}(f), f \in \Delta^v(T), \ell = 0, 1, \ldots, n \).

**Remark 4.4.** Geometric decomposition (21) can be derived from more general form in terms of \( d\lambda \) (cf. [3, Theorem 6.1]) but ours seems simpler.

### 4.5. Discrete inf-sup condition

In the unisolvence of vector Lagrange element, cf. the proof of Lemma 4.1, any basis of \( \mathbb{P}^n \) at \( f \) is allowed. The \( t-n \) basis is used for two purpose: 1. a div-conforming element; 2. the discrete inf-sup condition.

It is well known that for a piecewise smooth function \( v \) on \( \Omega \), i.e., \( v|_T \in C^1(T), T \in \mathcal{T}_h \), to be in \( H^{1}(\text{div}, \Omega) \) if and only if \( v \cdot n_F \) is continuous across all \( (n-1) \)-dimensional faces of \( \mathcal{T}_h \). For each \( f \in \Delta^v(T) \), we choose a \( t-n \) coordinate \( \{ t^f_1, \ldots, t^f_\ell, n^f_1, \ldots, n^f_{n-\ell} \} \).

If a basis vector \( t^f_1 \) or \( n^f_1 \) depends only on \( f \) not on element \( T \) containing \( f \), we call it global and otherwise \( t^f_1(T) \) or \( n^f_1(T) \) is local and the corresponding DoFs are different for different \( T \) containing \( f \). For a global basis vector, the corresponding DoF (16) or (17)
only depends on $f$ and thus impose continuity in that direction. With the global $t^f_i$ or $n^f_j$, (16) or (17) are called the global DoFs. With the local $t^f_i(T)$ or $n^f_j(T)$, we can define local DoFs

$$\int_f (u \cdot t^f_i(T)) \, p \, ds, \quad i = 1, \ldots, \ell; \quad \int_f (u \cdot n^f_j(T)) \, p \, ds, \quad j = 1, \ldots, n - \ell,$$

which are also element-dependent.

In the $t - n$ basis, we can choose some components to be global and some are local. For example, all $\{t^f_i\}$ are local and all $\{n^f_j\}$ are global. Then we obtain an element with normal continuity on all lower dimensional sub-simplexes. In the extreme case, if all $t - n$ basis vectors are global, we then obtain the Lagrange element. Later on we will present an example even a local basis for the normal plane is chosen, the div-conforming can be still ensured by assign DoFs facewisely.

If the div-conformity is the only concern, we can simply choose the Lagrange element. There is another consideration from the inf-sup condition. In the continuous level, we have the inf-sup condition that $\text{div} : H(\text{div}, \Omega) \to L^2(\Omega)$ is surjective and has a continuous right inverse. A regular potential also exists.

By the Euler’s formula for homogenous degree polynomials $H_{r-1}(T)$, i.e. $\text{div}(xq) = (r - 1 + n)q$ for any $q \in H_{r-1}(T)$, clearly we have $\text{div} \mathbb{P}_r(T; E^n) = \mathbb{P}_{r-1}(T)$. Hence the discrete inf-sup condition in one element always holds. We now discuss the global version. Let

$$V_h := \{v_h \in H(\text{div}, \Omega) : v_h|_T \in \mathbb{P}_r(T; E^n) \text{ for each } T \in \mathcal{T}_h\},$$

$$Q_h := \{q_h \in L^2(\Omega) : q_h|_T \in \mathbb{P}_{r-1}(T) \text{ for each } T \in \mathcal{T}_h\}.\tag{22}$$

The discrete inf-sup condition is $\text{div} V_h = Q_h$ and the operator $\text{div}$ has a continuous right inverse.

The following characterization of the range of the bubble polynomial space implies that it is better to let tangential vectors be local.

**Lemma 4.5 (Lemma 3.6 in [10]).** It holds

$$\text{div} \mathbb{B}_r(\text{div}, T) = \mathbb{P}_{r-1,0}^\perp(T),$$

where $\mathbb{P}_{r-1,0}^\perp(T) := \mathbb{P}_{r-1}(T) \cap L^2(\Omega)$.

**Proof.** The inclusion $\text{div}(\mathbb{B}_r(\text{div}, T)) \subseteq \mathbb{P}_{r-1,0}^\perp(T)$ is proved through integration by parts

$$(\text{div} v, p)_T = - (v, \text{grad} p)_T = 0 \quad \forall \, p \in \ker(\text{grad}) = \mathbb{R}.$$

If $\text{div}(\mathbb{B}_r(\text{div}, T)) \neq \mathbb{P}_{r-1,0}^\perp(T)$, there exists a $p \in \mathbb{P}_{r-1,0}^\perp(T)$ and $p \perp \text{div}(\mathbb{B}_r(\text{div}, T))$, which is equivalent to $\text{grad} p \perp \mathbb{B}_r(\text{div}, T)$. Expand the vector $\text{grad} p$ in the basis $\{n_i, i = 1, \ldots, n\}$ as $\text{grad} p = \sum_{i=1}^n q_i n_i$ with $q_i \in \mathbb{P}_{r-2}(T)$. Then set $v_p = \sum_{i=1}^n |\nabla \lambda_i| q_i \lambda_0 \lambda_i t_{i,0} \in \mathbb{B}_r(\text{div}, T)$, where $t_{i,0} := v_0 - v_i$. We have

$$(\text{grad} p, v_p)_T = \sum_{i=1}^n \int_T q_i^2 \lambda_0 \lambda_i \, dx = 0,$$

which implies $q_i = 0$ for $i = 1, 2, \ldots, n$, i.e., $\text{grad} p = 0$ and $p = 0$ as $p \in \mathbb{P}_{r-1,0}^\perp(T)$. \qed

4.6. Brezzi-Douglas-Marini element. We shall derive the classical Brezzi-Douglas-Marini (BDM) element [6] from a special \( t - n \) basis. Given an \( f \in \Delta_\ell(T) \), we choose \( \{n_F, f \subseteq F \in \partial T\} \) as the basis for its normal plane and an arbitrary basis for the tangent plane.

**Lemma 4.6 (Local BDM element).** The shape function space \( P_r(T; E_n) \) is uniquely determined by the DoFs

\[
\int_f (v \cdot n_F) p \, ds, \quad f \in \Delta_\ell(T), F \in \Delta_{\ell-1}(T), f \subseteq F, \quad p \in \mathbb{P}_{\ell-(\ell+1)}(f), \ell = 0, 1, \ldots, n-1,
\]

\[
\int_T v \cdot p \, dx \quad p \in \mathbb{B}_r(\text{div}, T).
\]

**Proof.** First of all, by the geometric decomposition of \( \mathbb{P}_r(T; E_n) \) (15) and Theorem 4.2, the number of DoFs is equal to the dimension of the shape function space.

DoFs on the normal space can be redistributed according to the following combinatorial identity

\[
(n - \ell) \binom{n+1}{\ell+1} = (n+1) \binom{n}{\ell+1}.
\]

On the left, there are \( (\ell+1) \) \( \ell \)-dimensional sub-simplex \( f \) in \( \Delta_\ell(T) \) and each \( f \) has \( n - \ell \) normal vectors. On the right, there are \( n+1 \) faces \( F \in \Delta_{n-1}(T) \) and each \( F \) will have \( (\ell+1) \) \( \ell \)-dimensional sub-simplex \( f \). We can distribute \( n - \ell \) copies of DoFs on an \( \ell \)-dimensional \( f \) to each face \( F \) to determine the trace \( n_F \cdot v \in \mathbb{P}_r(F) \).

So we can rearrange (23) as: for each face \( F \in \Delta_{n-1}(T) \),

\[
\int_f v \cdot n_F p \, ds, \quad p \in \mathbb{P}_{\ell-(\ell+1)}(f), f \subseteq F, \quad f \in \Delta_\ell(F), \ell = 0, 1, \ldots, n-1.
\]

Based on the geometric decomposition of \( \mathbb{P}_r(F) = \bigoplus_{\ell=0}^{n-1} \bigoplus_{f \in \Delta_\ell(F)} b_f \mathbb{P}_{\ell-(\ell+1)}(f) \), (25) will determine the trace \( v \cdot n_F \in \mathbb{P}_r(F) \).

Therefore given a \( v \in \mathbb{P}_r(T; E_n) \), if (23) vanishes, then \( \text{tr} v = 0 \) and consequently \( v \in \mathbb{B}_r(\text{div}, T) \). Finally the vanishing DoF (24) implies \( v = 0 \). \( \square \)

Again by the geometric decomposition of Lagrange element, DoF (23) can be merged into

\[
\int_F v \cdot n_F p \, ds, \quad p \in \mathbb{P}_r(F), F \in \Delta_{n-1}(T),
\]

which is exactly the boundary DoFs for BDM element.

The geometric decomposition and the facewise redistribution of DoFs of BDM element is illustrated in Fig. 3.

To glue local finite elements to form a div-conforming finite element of \( H(\text{div}, \Omega) \), we need to enforce continuity of \( v \cdot n_F \) by DoFs. The face moment (23) is presented for each \( f \). To make \( v \cdot n_F \) continuous, we switch the view point and ask \( \int_f v \cdot n_F p \, ds \) depended only on \( F \) not element \( T \).
Lemma 4.7 (BDM space). For each \( F \in \Delta_{n-1}(T_h) \), choose a normal \( \mathbf{n}_F \). For the shape function space \( \mathbb{P}_r(T; \mathbb{E}^n) \), the following DoFs

\[
\int_T \mathbf{v} \cdot \mathbf{n}_F \mathbf{p} \, dx, \quad F \in \Delta_{n-1}(T_h), \quad f \in \Delta_{\ell}(F), \quad p \in \mathbb{P}_{r-(\ell+1)}(f), \quad \ell = 0, \ldots, n-1
\]

\[
\int_T \mathbf{v} \cdot \mathbf{p} \, dx \quad T \in \mathcal{T}_h, \quad \mathbf{p} \in \mathbb{B}_r(\text{div},T),
\]

defines a div-conforming space \( V_{\text{BDM}} = \{ \mathbf{v}_h \in H(\text{div},\Omega) : \mathbf{v}_h |_T \in \mathbb{P}_r(T; \mathbb{E}^n), \forall T \in \mathcal{T}_h \} \). 

Proof. On each element \( T \), DoFs (27)-(28) will determine a function in \( \mathbb{P}_r(T; \mathbb{E}^n) \) by Lemma 4.6. DoF (27) will determine the trace \( \mathbf{v} \cdot \mathbf{n}_F \) on \( F \) independent of the element containing \( F \) and thus the function is \( H(\text{div},\Omega) \)-conforming. In view of (26), the obtained space \( V_h \) is the BDM space. \( \square \)

Notice that given an \( f \) with \( \dim f < n-1 \), the basis of its normal plane is not global. The div-conformity is obtained by asking DoF (27) depends only on \( \mathbf{n}_F \) and regroup facewisely, i.e., given \( F \in \Delta_{n-1}(T_h) \), run over all \( f \in \Delta_{\ell}(F) \). While when defining the local element, DoF (23) is given \( f \in \Delta_{\ell}(T) \), run over all \( F \in \Delta_{n-1}(T), f \subseteq F \). We shall call (27) facewise DoF and the change from (23) to (27) the facewise redistribution of normal DoFs. One benefit from the geometric decomposition of BDM element is that the well documented Lagrange basis functions can be used in the implementation.

Lemma 4.8. Let \( r \geq 1 \) and \( V_{\text{BDM}} \) be the BDM space defined in Lemma 4.7 and \( Q_h \) defined by (22). It holds the discrete inf-sup condition

\[
\| q_h \|_0 \lesssim \sup_{\mathbf{v}_h \in V_{\text{BDM}}} \frac{(\text{div} \mathbf{v}_h, q_h)}{\| \mathbf{v}_h \|_0 + \| \text{div} \mathbf{v}_h \|_0} \quad \forall q_h \in Q_h.
\]

Proof. By Theorem 1.1 in [15], there exists \( \mathbf{v} \in H^1(\Omega; \mathbb{R}^n) \) such that

\[
\text{div} \mathbf{v} = q_h, \quad \| \mathbf{v} \|_1 \lesssim \| q_h \|_0.
\]
Let $\tilde{v}_h \in V_{\text{BDM}}$ such that
\[
\int_F \tilde{v}_h \cdot \mathbf{n}_F \, p \, ds = \int_F \mathbf{v} \cdot \mathbf{n}_F \, p \, ds \quad \forall \, p \in \mathbb{P}_r(F), \, F \in \Delta_{n-1}(T_h),
\]
\[
\int_T \tilde{v}_h \cdot \mathbf{p} \, dx = \int_T \mathbf{v} \cdot \mathbf{p} \, dx \quad \forall \, \mathbf{p} \in \mathbb{P}_r(div, T), \, T \in T_h.
\]
By the scaling argument,
\[
||\tilde{v}_h||_0 + ||\text{div}\, \tilde{v}_h||_0 \lesssim ||\mathbf{v}||_1 \lesssim ||q_h||_0.
\]
Clearly we have $\text{div}(\tilde{v}_h - \mathbf{v})|_T \in \mathbb{P}_{r-1}^+(T)$ for each $T \in T_h$. By Lemma 4.5, there exists $b_h \in L^2(\Omega; \mathbb{R}^n)$ such that $b_h|_T \in \mathbb{P}_r(\text{div}, T)$ for each $T \in T_h$, and
\[
\text{div} \, b_h = \text{div}(\mathbf{v} - \tilde{v}_h), \quad ||b_h||_{0,T} \lesssim h_T \|\text{div} (\tilde{v}_h - \mathbf{v})\|_{0,T}.
\]
Take $\mathbf{v}_h = \tilde{v}_h + b_h \in V_{\text{BDM}}$. By (30) and (32), it holds
\[
\text{div} \, \mathbf{v}_h = \text{div} \, \tilde{v}_h + \text{div} \, b_h = \text{div} \, \mathbf{v} = q_h.
\]
It follows from (31) and (32) that
\[
||\mathbf{v}_h||_0 + ||\text{div} \, \mathbf{v}_h||_0 = ||\mathbf{v}_h||_0 + ||q_h||_0 \leq ||\tilde{v}_h||_0 + ||b_h||_0 + ||q_h||_0 \lesssim ||\tilde{v}_h||_0 + h ||\text{div} \, \tilde{v}_h||_0 + ||q_h||_0 \lesssim ||q_h||_0.
\]
Combining (33)-(34) yields (29).\hfill\square

We have the geometric decomposition of the global BMD element space
\[
V_{\text{BDM}} = \bigoplus_{F \in \Delta_{n-1}(T_h)} \bigoplus_{\ell=0}^{n-1} \bigoplus_{f \in \Delta_\ell(F)} N^f_\ell(F, \Omega)
\]
\[
\bigoplus_{T \in T_h} \bigoplus_{\ell=1}^n \bigoplus_{F \in \Delta_\ell(T)} B^f_\ell(T, \Omega)
\]
and
\[
\dim V_{\text{BDM}} = |\Delta_{n-1}(T_h)| \binom{n + r - 1}{r} + |\Delta_n(T_h)| \sum_{\ell=1}^n \binom{n + 1}{\ell} \binom{r - 1}{\ell}.
\]
Here
\[
N^f_\ell(F, \Omega) := \{ \mathbf{v}_h \in H(\text{div}, \Omega) : \mathbf{v}_h|_T \in N^f_\ell(T) \text{ for } T \in T_h, \, F \subseteq T, \}
\]
\[
\mathbf{v}_h|_{T'} = 0 \text{ for } T' \in T_h, \, F \not\subseteq T',
\]
\[
B^f_\ell(T, \Omega) := \{ \mathbf{v}_h \in H(\text{div}, \Omega) : \mathbf{v}_h|_T \in B^f_\ell(T), \, \mathbf{v}_h|_{T'} = 0 \text{ for } T' \in T_h \setminus \{T\} \}.
\]

4.7. **Stenberg’s type element.** We can obtain an $H(\text{div})$-conforming element satisfying the discrete inf-sup condition with more continuity as follows. We let all $\{t^f_\ell\}$ local and all $\{n^f_\ell\}$ global. That is the basis of the normal plane of $f$ is global, while the tangential basis is still local and thus contributes to the bubble functions. The obtained finite element space is continuous on the normal plane of $f$. In particular, it is continuous at vertices. Variants of Stenberg’s element can be obtained by chosen a global normal basis for some $f$ and a face-wise normal plane for others.
Lemma 4.9 (Stenberg type element). Let \(-1 \leq k \leq n-2\). For each \(f \in \Delta_{\ell} (T_h)\) with \(\ell \leq k\), we choose \(n - \ell\) normal vectors \(\{n^f_1, \ldots, n^f_{n-\ell}\}\). Then the DoFs

\[
\int_f \mathbf{v} \cdot \mathbf{n}_i^f \, p \, ds, \quad p \in \mathbb{P}_{r-(\ell+1)}(f), \, f \in \Delta_{\ell} (T_h),
\]

\(i = 1, \ldots, n - \ell, \, \ell = 0, \ldots, k,\)

\[
\int_f (\mathbf{v} \cdot \mathbf{n}_F) \, p \, ds, \quad p \in \mathbb{P}_{r-(\ell+1)}(f), \, f \in \Delta_{\ell}(F),
\]

\(F \in \Delta_{n-1}(T_h), \, \ell = k + 1, \ldots, n - 1,\)

\[
\int_T \mathbf{v} \cdot \mathbf{p} \, dx, \quad \mathbf{p} \in \mathcal{B}_r(\text{div}, T), \, T \in T_h,
\]

will determine a space \(V^r_{\text{Stenberg}} \subset H(\text{div}, \Omega)\).

Proof. For \(T \in T_h\) and \(f \in \Delta_\ell(T)\), both \(\{n^f_1, \ldots, n^f_{n-\ell}\}\) and \(\{n_F, F \in \Delta_{n-1}(T), f \subset F\}\) are basis of the normal plane \(N^f\). Then the number of DoFs (35)-(37) restricted to \(T\) equals to the number of DoFs (23)-(24). DoF (35) determines DoF (27) for \(\ell = 0, \ldots, k\). Then we conclude the result from Lemma 4.7. \(\Box\)

When \(k = 0\), it is the original Stenberg’s element [25], i.e., only continuous at vertices. When \(k = n - 2\), it is the generalization of \(H(\text{div})\) element constructed in [13]. We allow \(k = -1\) to include the BDM element.

The continuity at normal planes introduces some constraint and make the discrete inf-sup condition harder. As all bubble functions are treated local, \(\text{div} \mathcal{B}_r(\text{div}, T) = \mathbb{P}_{r-1,0}(T)\) still holds locally. We only need to show \(\cup_{T \in T_h} \mathbb{P}_0(T)\) is in the range of \(\text{div} V_h\) which requires the face moment and condition \(r \geq n\).

Lemma 4.10. Let \(r \geq n\) and \(V^r_{\text{Stenberg}}\) be the Stenberg’s element defined in Lemma 4.9. The following discrete inf-sup condition holds with a constant independent of \(h\)

\[
\|q_h\| \lesssim \sup_{\mathbf{v}_h \in V^r_{\text{Stenberg}}} \frac{\|\text{div} \mathbf{v}_h, q_h\|}{\|\mathbf{v}_h\| + \|\text{div} \mathbf{v}_h\|} \quad \forall \, q_h \in Q_h.
\]

Proof. By Theorem 1.1 in [15], there exists \(v \in H^1(\Omega; \mathbb{R}^n)\) such that

\[
\text{div} \mathbf{v} = q_h, \quad \|\mathbf{v}\|_1 \lesssim \|q_h\|_0.
\]

Let \(v_1\) be the Scott-Zhang interpolation of \(v\) in the vector Lagrange element space \(S^r \otimes \mathbb{R}^n\) [24] with \(S^r_h\) being the Lagrange element space of degree \(r\), then

\[
\|v_1\|_1 \lesssim \|v\|_1 \lesssim \|q_h\|_0.
\]

Let \(\tilde{v}_h \in V^r_{\text{Stenberg}}\) such that

\[
\int_f (\tilde{v}_h - v_1) \cdot n^f_i \, p \, ds = 0 \quad \forall \, f \in \Delta_{\ell}(T_h), \, p \in \mathbb{P}_{r-(\ell+1)}(f), \, 1 \leq i \leq n - \ell, \, 0 \leq \ell \leq k,
\]

\[
\int_f (\tilde{v}_h - v_1) \cdot n_F \, p \, ds = 0 \quad \forall \, F \in \Delta_{n-1}(T_h), \, f \in \Delta_{\ell}(F), \, p \in \mathbb{P}_{r-(\ell+1)}(f), \, \ell = k + 1, \ldots, n - 2,
\]

\[
\int_F \tilde{v}_h \cdot n_F \, p \, ds = \int_F \mathbf{v} \cdot n_F \, p \, ds \quad \forall \, F \in \Delta_{n-1}(T_h), \, p \in \mathbb{P}_{r-n}(F),
\]

\[
\int_T \tilde{v}_h \cdot \mathbf{p} \, dx = \int_T \mathbf{v} \cdot \mathbf{p} \, dx \quad \forall \, \mathbf{p} \in \mathcal{B}_r(\text{div}, T), \, T \in T_h.
\]
Since \( r \geq n \), the face moment implies \( \int_T \text{div}(\nabla u - \nu) \, dx = 0 \), i.e., \( \text{div}(\nabla u - \nu)|_T \in \mathbb{P}_{r-1,0}^+(T) \) for each \( T \in \mathcal{T}_h \). Therefore we can apply the same argument as Lemma 4.8 to conclude (38).

We have the geometric decomposition of the global Stenberg element space
\[
V^r_{\text{Stenberg}} = \bigoplus_{\ell=0}^k \bigoplus_{f \in \Delta_\ell(\mathcal{T}_h)} N_f^\ell(\Omega) + \bigoplus_{F \in \Delta_{n-1}(\mathcal{T}_h)} \bigoplus_{\ell=k+1}^{n-1} \bigoplus_{f \in \Delta_\ell(F)} N_f^\ell(F,\Omega)
\]
and
\[
\dim V^r_{\text{Stenberg}} = \sum_{\ell=0}^k |\Delta_\ell(\mathcal{T}_h)| (n-\ell) \binom{r-1}{\ell} + |\Delta_{n-1}(\mathcal{T}_h)| \sum_{\ell=k+1}^{n-1} \binom{n}{\ell+1} \binom{r-1}{\ell} + |\Delta_n(\mathcal{T}_h)| \sum_{\ell=1}^n \binom{n+1}{\ell+1} \ell \binom{r-1}{\ell}.
\]

Here
\[
N_f^\ell(\Omega) := \{ \nu_h \in H(\text{div},\Omega) : \nu_h|_T \in N_f^\ell(T) \text{ for } T \in \mathcal{T}_h, f \subseteq T, \nu_h|_{T'} = 0 \text{ for } T' \in \mathcal{T}_h, f \not\subseteq T' \}.
\]
Clearly \( V^r_{\text{Stenberg}} \subseteq V_{\text{BDM}} \) and \( \dim V^r_{\text{Stenberg}} < \dim V_{\text{BDM}} \).

5. Div Conforming Tensor Spaces

In this section we shall introduce a constraint tensor space \( \mathcal{X} \subset \Lambda^{i,k}(\mathbb{E}^n) \) as a kernel space of operator \( s^{n-1,k} \), and discover a basis and its dual basis of \( \mathcal{X} \).

5.1. An algebraic operator. Define \( \Lambda^{i,k}(\mathbb{E}^n) = \Lambda^i(\mathbb{E}^n) \otimes \Lambda^k(\mathbb{E}^n) \) for \( i, k = 0, 1, \ldots, n \). In particular \( \Lambda^{n-1,k}(\mathbb{E}^n) = \Lambda^{n-1}(\mathbb{E}^n) \otimes \Lambda^k(\mathbb{E}^n) \cong \mathbb{E}^n \otimes \Lambda^k(\mathbb{E}^n) \). Thus, \( \dim \Lambda^{n-1,k} = n \binom{n}{k} \). To simplify the notation, the linear space \( \mathbb{E}^n \) will be skipped. In [5], the algebraic operator \( s^{n-1,k} : \Lambda^{n-1,k} \to \Lambda^{n,k-1} \) is defined as
\[
s^{n-1,k}\omega(v_1, \ldots, v_n)(w_1, \ldots, w_{k-1}) := \sum_{i=1}^n (-1)^{i-1} \omega(v_1, \ldots, \overset{\hat{i}}{v_i}, \ldots, v_n)(v_i, w_1, \ldots, w_{k-1})
\]
\[\forall v_1, \ldots, v_n, w_1, \ldots, w_{k-1} \in \mathbb{E}^n.\]

Recall that we have reserved \( \{ dx_i \} \) for a fixed orthonormal basis of \( \Lambda^1(\mathbb{E}^n) \). We are going to derive more concrete forms of operator \( s^{n-1,k} \) in a generic basis \( \{ dy_i \} \), which may not be orthonormal. In the next section, \( \{ dx'_i \} \) will be \( \{ dx_i \} \) a basis adapted to the sub-simplex \( f \).

We expand \( \omega \in \Lambda^{n-1,k} \) in this basis as
\[
\omega = \sum_{i=1}^n \sum_{\sigma \in \Sigma(1:k,1:n)} a_{i,\sigma} * dy_i \otimes dy_\sigma,
\]
where \( * dy_i := (-1)^{i-1} dy_{i'} \) satisfying \( dy_1 \wedge * dy_i = dy_i \). Note that the Hodge \( * \) is defined only for an orthonormal basis and when \( dy_i \) is orthonormal, \( * dy_i = * dy_i \). Then
an element in \( \Lambda^{n-1,k} \) can be identified as a matrix \( A = (a_{i,\sigma}) \) of size \( n \times \binom{n}{k} \) indexed by \((i, \sigma)\) for \( i = 1, 2, \ldots, n \) and \( \sigma \in \Sigma(1 : k, 1 : n) \). The mapping will be denoted by
\[
\operatorname{Prox}_{n-1,k} : \Lambda^{n-1,k} \to \mathbb{R}^{n \times \binom{n}{k}}.
\]
A labeling of \( \Sigma(1 : k, 1 : n) \) can be further introduced so that a single column index can be used, i.e., \((a_{i,j})\). Different labeling will lead to different matrix representations (up to permutation of columns) and some will have better structure. Note that a proxy matrix is basis dependent. In particular, the proxy matrix in the ambient basis \( \{ dx_i \} \) will be denoted by \( A \).

**Lemma 5.1.** For \( \omega = \sum_{i=1}^{n} \sum_{\sigma \in \Sigma(1 : k, 1 : n)} a_{i,\sigma} \delta y_i \otimes \delta y_{\sigma} \), we have
\[
\begin{align*}
\omega(\sigma) &= \sum_{\tau \in \Sigma(1 : k-1, 1 : n)} \epsilon(i, \tau) a_{i,\tau} \delta y_i \otimes \delta y_{\tau}.
\end{align*}
\]

**Proof.** Let \( \{ \delta y_i \} \) be tangent vectors dual to \( \{ dy_i \} \). That is \( dy_i(\partial y_j) = \delta_{i,j} \) for \( 1 \leq i, j \leq n \). Given \( \sigma \in \Sigma(1 : k, 1 : n) \), we use the notation \( \partial y_{\sigma} \) for \( k \) vectors \( (\partial y_{\sigma(1)}, \ldots, \partial y_{\sigma(k)}) \). Then \( \partial y_i(\partial y_{\sigma'}) = \delta_{\sigma,\sigma'}, \sigma, \sigma' \in \Sigma(1 : k, 1 : n) \). To get the coefficient of \( \partial y_i \otimes \partial y_{\tau} \), we apply to
\[
\omega(\sigma) = \sum_{\tau \in \Sigma(1 : k-1, 1 : n)} \epsilon(i, \tau) a_{i,\tau} \delta y_i \otimes \delta y_{\tau}.
\]

If \( i \in [\tau] \), then vectors \( \partial y_i, \partial y_{\tau(1)}, \ldots, \partial y_{\tau(k-1)} \) are linear dependent and thus the term vanishes. So only \( i \notin [\tau] \) are left in the summation. \( \square \)

If we identify the \((k-1)\)-form \( \delta y_i \) with the sub-simplex \( f_\tau \), then \( \{ f_{i+\tau} \}_{i \in [\tau]} \) corresponds to all \((k-1)\)-dimensional sub-simplex except \( f_{0+\tau} \) using \( \tau \) as a boundary face. On the other hand, for each \( \sigma \), there are \( n \) components in the first index \( i = 1, \ldots, n \). An index \((i, \sigma)\) will appear in the formulae (39) if and only if \( i \in [\sigma] \).

\begin{figure}[h]
\centering
\begin{subfigure}[b]{0.3\textwidth}
\centering
\includegraphics[width=\textwidth]{fig4a.png}
\caption{s^{4,1} for \( \tau = \emptyset \)}
\end{subfigure} \hspace{1cm}
\begin{subfigure}[b]{0.3\textwidth}
\centering
\includegraphics[width=\textwidth]{fig4b.png}
\caption{s^{4,3} for \( \tau = (1, 2) \)}
\end{subfigure} \hspace{1cm}
\begin{subfigure}[b]{0.3\textwidth}
\centering
\includegraphics[width=\textwidth]{fig4c.png}
\caption{s^{4,4} for \( \tau = (1, 2, 3) \)}
\end{subfigure}
\caption{Illustration of constraint sequences and operator \( s^{n-1,k} \) for \( k = 1, 3 \) and \( k = n-1 \) for a simplex in \( \mathbb{R}^5 \).}
\end{figure}
For a given $\tau \in \Sigma(1 : k − 1, 1 : n)$, we call the sequence of index $(i_m^\tau, \sigma_m^\tau), i_m^\tau \in [r^\tau]$ and $\sigma_m^\tau = i_m^\tau + \tau$, for $m = 1, 2, \ldots, n − k + 1$, the constraint sequence of $\tau$, which will be abbreviated as $(i_m, \sigma_m)$ when $\tau$ is clear from the context. We collect the coefficients of a constraint sequence and denoted by $a_\tau = (a_{i_m, \sigma_m})_{m=1,2,\ldots,n-k+1}$. Let $\epsilon_\tau = (\epsilon(i_m, \tau))_{m=1,2,\ldots,n-k+1}$ be the sign vector. When $k = 1$, $\tau$ is empty, which is represented by vertex $v_0$ in Fig. 4, and the length of the constraint sequence is $n$. When $k = n$, the length is 1. For two indices $(i, i + \tau)$ and $(j, j + \tau')$ with $\tau \neq \tau'$, then $(i, i + \tau) \neq (j, j + \tau')$. That is indices in different constraint sequences are different.

We can identify $\Lambda_{n,k} - 1$ as a vector in $\mathbb{R}^{\dim \Lambda_{n,k} - 1}$. With the proxy representations, the $s_{n,k} - 1$ operator induces an operator from matrix $X$ to a vector in $\mathbb{R}^{\dim \Lambda_{n,k} - 1}$ and will be denoted by $S^n_{n,k}$. Using these notations we can write the operator as

$$(S^n_{n,k} A)_\tau = a_\tau \cdot \epsilon_\tau.$$

**Lemma 5.2.** For $k = 1, \ldots, n - 1$, the operator $s_{n,k} - 1 \colon \Lambda_{n,k} - 1 \rightarrow \Lambda_{n,k} - 1$ is onto. And $s_{n,k} - 1 \colon \Lambda_{n,k} - 1 \rightarrow \Lambda_{n,k} - 1$ is a bijection and its proxy $S^n_{n,k}$ is an identity operator.

**Proof.** First consider $k = 1, \ldots, n - 1$. By the linearity, it suffices to prove for $\tau = [1, \ldots, k - 1]$, there exists $\omega = \sum_{i=1}^n \sum_{\sigma \in \Sigma(1:k,1:n)} a_{i,\sigma} \cdot \omega y_i \otimes \omega y_\sigma$ such that $s_{n,k} - 1 \cdot \omega = \omega y_\tau \otimes \omega y_\tau$. Take $a_{k,[1,\ldots,k]} = \epsilon(k, [1:k-1]) = (-1)^{k-1}$, and $a_{i,\sigma} = 0$ for the rest. Then

$$s_{n,k} - 1 \cdot \omega = \sum_{\tau \in \Sigma(1:k-1,1:n)} \left( \sum_{i \in [r^\tau]} \epsilon(i, \tau) a_{i,i+\tau} \right) \omega y_i \otimes \omega y_\tau = \omega y_\tau \otimes \omega y_\tau.$$

Next consider $k = n$. For $\omega = \sum_{i=1}^n a_i \cdot \omega y_i \otimes \omega y_i$, we have

$$s_{n,n} - 1 \cdot \omega = \sum_{\tau \in \Sigma(1:n-1,1:n)} \left( \sum_{i \in [r^\tau]} \epsilon(i, \tau) a_i \right) \omega y_i \otimes \omega y_\tau$$

$$= \sum_{i=1}^n a_i \omega y_i \otimes \omega y_i.$$

Namely $S^n_{n,n}$ is the identity operator. \hfill \Box

The above proof will be trivial if thinking geometrically. For a given $\tau$, we just pick up one $\sigma = i + \tau$ from its constraint sequence and set the coefficient be $\epsilon(i, \tau)$.

### 5.2. Constraint tensor space.

Now we are ready to introduce the tensor space

$$\mathbb{X} := \ker(s^n_{n,k}) \cap \Lambda_{n,k} = \{ \omega \in \Lambda_{n,k} \mid s^n_{n,k} \omega = 0 \}.$$

For a given basis $\{ \omega y_i \}$, it will be more convenient to work on the matrix representation

$$\mathbb{X} := \left\{ \omega = \sum_{i=1}^n \sum_{\sigma \in \Sigma(1:k,1:n)} a_{i,\sigma} \cdot \omega y_i \otimes \omega y_\sigma \mid A = (a_{i,\sigma}) \in \mathbb{R}^{n \times (\binom{n}{k})} : \right. \sum_{i \in [r^\tau]} \epsilon(i, \tau) a_{i,i+\tau} = 0 \ \forall \tau \in \Sigma(1 : k - 1, 1 : n) \left\}. \right.$$

As $s^n_{n,k}$ is surjective, the $\left( \begin{array}{c} n \\ k - 1 \end{array} \right)$ constraints

$$\sum_{i \in [r^\tau]} \epsilon(i, \tau) a_{i,i+\tau} = 0 \ \forall \tau \in \Sigma(1 : k - 1, 1 : n)$$


are linear independent, and consequently

\[
\dim \mathcal{X} = \dim \Lambda^{n-1,k} - \dim \Lambda^{n,k-1} = n \binom{n}{k} - \binom{n}{k-1} = (n-k) \binom{n+1}{k}.
\]

Smooth function space will be obtained if the coefficients \(a_{i,o}\) is a smooth function. In particular, for the orthonormal basis \( \{dx_i\} \), we introduce

\[
H(\text{div},T;\mathcal{X}) := \left\{ A = (a_{i,o}) \in L^2(T,\mathcal{X}) : \nabla \cdot A \in L^2(T,\mathbb{R}^n) \right\}
\]

with \( \nabla \cdot A := (\sum_{i=1}^n \partial_i a_{i,o}) \), i.e., the divergence operator is applied column-wise. And its differential form version is

\[
H(\text{d}_n-1,T;\mathcal{X}) := \left\{ \omega \in L^2(\Lambda^{n-1,k}(T)) : s_n^{n-1,k}\omega = 0, \text{d}_n-1\omega \in L^2(\Lambda^{n,k}(T)) \right\},
\]

where the exterior derivative \(\text{d}_n-1\) is applied to the component \(\Lambda^{n-1}\) in \(\Lambda^{n-1,k}\).

In the next section we will construct div-conforming finite elements with shape function space

\[
\mathbb{P}_r(T;\mathcal{X}) := \mathbb{P}_r(T) \otimes \mathcal{X}.
\]

For \(\omega \in \mathcal{X}\), the trace \(\text{tr}_{n-1}\) is applied to the component \(\Lambda^{n-1}\) in \(\Lambda^{n-1,k}\). In view of matrix proxy, the trace on face \(F\)

\[
\text{tr}_F A = n_F^T A = n_F \cdot A
\]

is a row vector of length \(\binom{n}{k}\) and should be continuous across simplexes. Here again, boldface letter \(A\) is used for the special proxy matrix in the orthonormal basis \(\{dx_i\}\).

**Example 5.3.** Consider \(k = n-1\). For \(\omega = \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \ast dy_i \otimes \ast dy_j\), we have

\[
s_n^{n-1,n-1}\omega = \sum_{\tau \in \Sigma(1:n-2:1:n)} \left( \sum_{i \in [\tau]} (-1)^{1:1:n} e(i,\tau) a_{i,1:1:n} \right) dy_i \otimes dy_j,
\]

where \(dy_{(i,j)} := dy_1 \wedge \cdots \wedge \hat{dy}_i \wedge \cdots \wedge \hat{dy}_j \wedge \cdots \wedge dy_n\). In terms of the matrix proxy, it holds

\[
S_n^{n-1,n-1}(A) = 2\text{vskw}(A) \quad \text{with} \quad A = (a_{i,j})_{n \times n} \in \mathbb{R}^{n \times n},
\]

where operator \(\text{vskw} : \mathbb{R}^{n \times n} \to \mathbb{R}^{(n-1)/2}\) is defined by

\[
(\text{vskw}(A))_{[i,j]} = \frac{1}{2} (-1)^{i+j} (a_{i,j} - a_{j,i}) \quad \text{with} \quad [i,j] \in \Sigma(1:2,1:n).
\]

And thus \(\mathcal{X} = \mathbb{S}\) consists of all symmetric matrices.

**Example 5.4.** Consider \(k = 1\). For \(\omega = \sum_{i=1}^n \sum_{j=1}^n a_{i,j} \ast dy_i \otimes dy_j\), we have

\[
s_n^{n-1,1}\omega = \sum_{i=1}^n a_{i,i} dy_i.
\]

In terms of the matrix proxy,

\[
S_n^{n-1,1}(A) = \text{trace}(A).
\]

Thus \(\mathcal{X} = \mathbb{T}\), which is the traceless matrix space.
5.3. A basis of the constraint tensor space. From (39), we have
\begin{equation}
\left( y_{\sigma}^{i} \right)_{n-1,k} = 0 \quad \text{if and only if} \quad i \in [\sigma^\top].
\end{equation}

It will be more convenient to write the index \((i, \sigma)\) instead of the basis. So we call index \((i, \sigma)\) in \(\text{ker}(s^{n-1,k})\) or in \(X\) if \(s^{n-1,k}(d_{y_{\sigma}^{i} \otimes dy_{\sigma}^{i}}) = 0\).

If we identify the \(k\)-form \(dy_{\sigma}\) with the sub-simplex \(f_{\sigma}\) and the \((n-1)\)-form \(*dy_{i}\) with vertex \(v_{i}\), then the index \((i, \sigma)\) is in \(\text{ker}(s^{n-1,k})\) if and only if \(v_{i}\) is not a vertex of \(f_{\sigma}\) or equivalently \(v_{i}\) is a vertex of \(f_{\sigma^\top}\). Here \(i = 1, 2, \ldots, n\) and \(\sigma \in \Sigma(1 : k, 1 : n)\) as the vertex \(v_{0}\) is treated as the origin. Of course we can set other vertices as the origin, which motivates the following basis of \(X\) in terms of the wedge product of \(d\lambda_{i}, i = 0, 1, \ldots, n\).

**Lemma 5.5.** For any \(\sigma \in \Sigma(0 : n-k, 0 : n)\) and \(i = 1, \ldots, n-k\), it holds
\[
d\lambda_{\sigma(0), \sigma(i)}^\top \otimes d\lambda_{\sigma^\top} \in X,
\]
where
\[
d\lambda_{\sigma(0), \sigma(i)}^\top = d\lambda_{\sigma(0)-\sigma(i)} = d\lambda_{0} \wedge \cdots \wedge d\lambda_{\sigma(i)-\sigma(0)} \wedge \cdots \wedge d\lambda_{n}.
\]

**Proof.** We treat \(\sigma(0)\) as the origin. Let \(y_{i} = \lambda_{\sigma(i)}\) for \(i = 1, \ldots, n-k\), and \(y_{n-k+i} = \lambda_{\sigma^\top(i)}\) for \(i = 1, \ldots, k\). Then \(d\lambda_{\sigma(0), \sigma(i)}^\top \otimes d\lambda_{\sigma^\top} = (-1)^{i-1} \otimes dy_{i} \wedge (dy_{n-k+1} \wedge \cdots \wedge dy_{n}) \in X\) for \(i = 1, \ldots, n-k\) from (41).

Recall that the inner product of \(k\)-form \((\omega, \eta) := \omega \wedge \star \eta, \omega, \eta \in \Lambda^{k}\), which can be extend to \(\Lambda^{n-1,k}\) by the tensor product. In view of the matrix proxy, it is the Frobenius product of matrices. Define \(P_{X}\) as the orthogonal projection from \(\Lambda^{n-1,k}\) to \(X\) in the inner product \(\langle \cdot, \cdot \rangle\).

**Lemma 5.6.** The set
\[
\left\{ d\lambda_{\sigma(0), \sigma(i)}^\top \otimes d\lambda_{\sigma^\top} \right\}_{\sigma \in \Sigma(0 : n-k, 0 : n)}
\]
in \(X\) is dual to the set
\[
\left\{ P_{X} \left( * d\lambda_{\sigma(i)} \otimes * d\lambda_{\sigma - \sigma(i)} \right) \right\}_{\sigma \in \Sigma(0 : n-k, 0 : n)}
\]

**Proof.** Since \(d\lambda_{\sigma(0), \sigma(i)}^\top \otimes d\lambda_{\sigma^\top} \in X\), we only need to prove that for any \(\sigma, \tau \in \Sigma(0 : n-k, 0 : n)\), and \(i, j = 1, \ldots, n-k\),
\[
( d\lambda_{\sigma(0), \sigma(i)}^\top \otimes d\lambda_{\sigma^\top} ) \wedge ( d\lambda_{\tau(j)} \otimes d\lambda_{\tau - \tau(j)} ) \neq 0
\]
if and only if
\[
\sigma = \tau \text{ and } i = j.
\]
By definition,
\[
( d\lambda_{\sigma(0), \sigma(i)}^\top \otimes d\lambda_{\sigma^\top} ) \wedge ( d\lambda_{\tau(j)} \otimes d\lambda_{\tau - \tau(j)} )
= ( d\lambda_{\sigma(0), \sigma(i)}^\top \wedge d\lambda_{\tau(j)} ) \otimes ( d\lambda_{\sigma^\top} \wedge d\lambda_{\tau - \tau(j)} ).
\]
Then \(( d\lambda_{\sigma(0), \sigma(i)}^\top \otimes d\lambda_{\sigma^\top} ) \wedge ( d\lambda_{\tau(j)} \otimes d\lambda_{\tau - \tau(j)} ) \neq 0\) is equivalent to
\[
\tau(j) \in \{ \sigma(0), \sigma(i) \} \quad \text{and} \quad [\tau - \tau(j)] \subset [\sigma].
\]
This indicates \([\tau] \subseteq [\sigma]\). We finish the proof by the fact \(\tau\) and \(\sigma\) have the same length. \(\square\)
Theorem 5.7. The set
\[ \{ d\lambda_{[\sigma(0),\sigma(i)]^*} \otimes d\lambda_{\sigma^*} \}_{\sigma \in \Sigma(0;n-k,0:n)} \]

is a basis of $\mathcal{X}$. Its dual basis is
\[ (42) \quad \{ P_\mathcal{X}(\ast d\lambda_{\sigma(i)} \otimes \ast d\lambda_{\sigma(0)}) \}_{\sigma \in \Sigma(0:n-k,0:n)} \, . \]

For any $t - n$ basis $\{ x_i^T, i = 1, \ldots, n \}$ of $f \in \Delta_{n-k}(T)$,
\[ \{ \ast d x_i^T \otimes (d x_{n-k+1}^f \wedge \cdots \wedge d x_n^f) \}_{i = 1, \ldots, n-k} \]

is a basis of $\mathcal{X}$.

Proof. Both the vector proxy of $\{ \ast (d\lambda_{[\sigma(0),\sigma(i)]}) \}_{i = 1, \ldots, n-k}$ and $\{ \ast (d x_i^f) \}_{i = 1, \ldots, n-k}$ are bases of the tangent space of $f$. And $d\lambda_{\sigma^*}$ equals to $d x_{n-k+1}^f \wedge \cdots \wedge d x_n^f$ multiplied by a non-zero constant. Hence $\{ d\lambda_{[\sigma(0),\sigma(i)]} \otimes d\lambda_{\sigma^*} \}_{\sigma \in \Sigma(0:n-k,0:n)}$ and $\{ \ast d x_i^T \otimes (d x_{n-k+1}^f \wedge \cdots \wedge d x_n^f) \}_{i = 1, \ldots, n-k}$ can be linearly expressed by each other. As a result we only need to prove the first part of this lemma.

The number of the set $\{ d\lambda_{[\sigma(0),\sigma(i)]} \otimes d\lambda_{\sigma^*} \}_{\sigma \in \Sigma(0:n-k,0:n)}$ is $(n-k) \binom{n+1}{k}$, which equals to $\dim \mathcal{X}$, cf. (40). Hence it suffices to prove that these $k$-forms are linearly independent. Assume there exist $c_{i,\sigma} \in \mathbb{R}$ for each $\sigma \in \Sigma(0:n-k,0:n)$ and $i = 1, \ldots, n-k$ such that
\[ \sum_{\sigma \in \Sigma(0:n-k,0:n)} \sum_{i = 1}^{n-k} c_{i,\sigma} d\lambda_{[\sigma(0),\sigma(i)]^*} \otimes d\lambda_{\sigma^*} = 0. \]

Then apply wedge product with $d\lambda_{\tau(j)} \otimes d\lambda_{\tau(0) - \tau(j)}$ for $\tau \in \Sigma(0:n-k,0:n)$ and $0 \leq j \leq n-k$, due to Lemma 5.6, we obtain $c_{j,\tau} = 0$. As $(j, \tau)$ runs over the whole set $\{1, \ldots, n-k\} \times \Sigma(0:n-k,0:n)$, we conclude all $c_{j,\tau}$ vanishes. \hfill $\square$

The vector proxy of $d\lambda_{[\sigma(0),\sigma(i)]^*}$ is a scaling of the edge vector $t_{\sigma(0)\sigma(i)}$ which is on the tangent plane of $f$. The $k$-form $d\lambda_{\tau(0)\tau(i)}$ is the volume of the normal plane of $f = f_{\sigma} \in \Delta_{n-k}(T)$; see Fig. 2.

When $k = n - 1$, the vector proxy of $d\lambda_{\tau(0)\tau(i)}$ is also a scaling of $t_{\tau(0)\tau(i)}$. A basis of $\mathcal{X}$ is thus given by $\{ d\lambda_{\sigma^*} \otimes d\lambda_{\sigma^*} \}_{\sigma \in \Sigma(0,1,0:n)}$. Equivalently, in the matrix proxy, a basis of $\mathcal{X}$ is $\{ t_e \otimes t_e \}_{e \in \Delta(T)}$, which is crucial in designing the $H(\text{div}; \mathcal{X})$ element [18, 22] and useful in the Regge calculus [12].

When $k = 1$, the vector proxy of $d\lambda_{\tau(0)\tau(i)}$ is $n_F$. In the matrix proxy, a basis of $\mathcal{T}$ is $\{ t_{F,i} \otimes n_F \}_{i = 1, \ldots, n-1}$, which is included in [19].

5.4. Dual operator and integration by parts. For a smooth tensor $u = (u_\sigma)$ with index $\sigma \in \Sigma(1:k,1:n)$, let $\nabla u$ be a tensor with size $\mathbb{R}^{n \times \binom{n}{1}}$ give by

\[ (\nabla u)_{i,\sigma} := \partial_{x_i} u_\sigma. \]

Lemma 5.8. Let $\mathcal{X}^\perp$ be the orthogonal complement of $\mathcal{X}$ in the Frobenius inner product. It holds that

\[ \ker(P_\mathcal{X} \nabla) = \mathbb{P}_0(T; \mathbb{R}^{\binom{n}{2}}) + x^T \mathcal{X}^\perp. \]
Proof. Noting that \( \text{grad} (P_0(T; \mathbb{R}^3)) + x^T X^\perp = X^\perp \), hence
\[
P_0(T; \mathbb{R}^3) + x^T X^\perp \subseteq \ker(P_X \nabla).
\]
By (31) in [5], \( \ker(P_X \nabla) \subseteq P_1(T; \mathbb{R}^3) \). Take \( c + x^T A \in \ker(P_X \nabla) \). By \( \text{grad} (c + x^T A) = A \), we have \( P_X A = 0 \), i.e. \( A \in X^\perp \). Therefore \( \ker(P_X \nabla) \subseteq P_0(T; \mathbb{R}^3) + x^T X^\perp \).

We introduce notation \( R_X := \ker(P_X \nabla) \). Then \( R_X = RM \) is the rigid motion for \( X = S \), and \( RX = RT \) for \( X = T \), where \( RT := P_0(T; \mathbb{R}^n) + xP_0(T) \) is the lowest Ravirant-Thomas element. In general, \( RX \) is the Whitney form \( P_0 \Lambda^k + \kappa_{k+1} P_0 \Lambda^{k+1} \), which is another characterization of \( \ker(P_X \nabla) \).

Clearly \( P_X \nabla \) is the proxy of \( P_X d := (-1)^{n-1} \star P_X \star d : \Lambda^{0,k} \rightarrow \Lambda^{1,k} \). Indeed \( \int_T \omega \wedge P_X d\eta = (-1)^{n-1-1} \int_T \omega \wedge \star d\eta = \int_T \omega \wedge d\eta \), then the integration by parts holds
\[
\int_T d\omega \wedge \eta = (-1)^{n-2} \int_T \omega \wedge P_X d\eta + \int_{\partial T} \text{tr}_{n-1} \omega \wedge \text{tr}_0 \eta
\]
for any \( \omega \in X \subseteq \Lambda^{n-1,k} \) and \( \eta \in \Lambda^{0,k} \). In the matrix and vector proxy, we have
\[
\int_T (\nabla \cdot A) \cdot u \, dx = \int_T A : P_X \nabla u \, dx + \int_{\partial T} n^T A \cdot u \, dS.
\]

5.5. Formula on the projection. We present an explicit formulae on \( P_X \). Recall that \( S^{n-1,k} A = (a_\tau \cdot \epsilon_\tau) \in \mathbb{R}^{n-1} \) for \( A = (a_{i,\sigma}) \in \mathbb{R}^{n \times (n)} \). We embed \( \epsilon_\tau \) into \( \Lambda^{n-1,k} \) as follows:
\[
\epsilon_\tau^A := \sum_{i \in \tau^{-1}} \epsilon(i, \tau) \star dy_i \otimes dy_{i+\tau}.
\]
The proxy of \( \epsilon_\tau^A \) is a matrix with value \( \epsilon(i, \tau) \) at the location \( (i, i+\tau) \).

Lemma 5.9. It holds
\[
X^\perp = \text{span}\{ \epsilon_\tau^A, \tau \in \Sigma(1 : k-1, 1 : n) \}.
\]
And for \( \omega \in \Lambda^{n-1,k} \) with proxy \( A = (a_{i,\sigma}) \in \mathbb{R}^{n \times (n)} \), it holds
\[
P_X \omega = \sum_{\tau \in \Sigma(1; k-1, 1; n)} \frac{a_\tau \cdot \epsilon_\tau}{n-k+1} \epsilon_\tau^A.
\]
Consequently for \( \sigma \in \Sigma(1 : k, 1 : n) \),
\[
(P_X \omega)_{i,\sigma} = \begin{cases}
a_{i,\sigma} & i \in [\sigma^c], \\
a_{i,\sigma} - \frac{a_\tau \cdot \epsilon_\tau}{n-k+1} & i \in [\sigma] \text{ with } \tau = \sigma - i.
\end{cases}
\]

Proof. Let \( \omega = \sum_{i=1}^n a_{i,\sigma} \star dy_i \otimes dy_{\sigma} \in X \). Then
\[
\langle \sum_{i \in \tau^{-1}} \epsilon(i, \tau) \star dy_i \otimes dy_{i+\tau}, \omega \rangle = \sum_{i \in \tau^{-1}} \epsilon(i, \tau) a_{i, i+\tau} = 0 \quad \forall \tau \in \Sigma(1 : k-1, 1 : n).
\]
Notice that for two indices \( (i, i+\tau) \) and \( (j, j+\tau') \) with \( \tau \neq \tau' \), then \( (i, i+\tau) \neq (j, j+\tau') \). So \( \{ \epsilon_\tau^A, \tau \in \Sigma(1 : k-1, 1 : n) \} \) is orthogonal and (44) follows from the dimensions match.

Then equation (45) holds from (44) and
\[
\langle \omega, \epsilon_\tau^A \rangle = a_\tau \cdot \epsilon_\tau, \quad \langle \epsilon_\tau^A, \epsilon_\tau^A \rangle = n - k + 1.
\]
Combining (45) and $P_X = I - P_{X^\perp}$ gives (46).

6. Geometric Decomposition of Div Tensors

In this section, we generalize the geometric decomposition of $H(div)$-element to the tensor $H(div; X)$-conforming element. We first work on one simplex $T$ and give a decomposition of $X$ into a direct sum of the tangential bubble subspace and a normal subspace. Then we present DoFs and show the div-conformity and the discrete inf-sup condition.

6.1. Decomposition of the constraint tensor space. As before, for a face $f \in \Delta_f(T)$, we choose a $t - n$ basis $\{t_1^f, \ldots, t_{\ell}^f, n_1^f, \ldots, n_{n-\ell}^f\}$, where the set of $\ell$ tangential vectors $\{t_1^f, \ldots, t_{\ell}^f\}$ is basis of the tangent plane $T^f$ of $f$ and the set of $n - \ell$ normal vectors $\{n_1^f, \ldots, n_{n-\ell}^f\}$ forms a basis of the normal plane $N^f$ of $f$. We also introduce the $t - n$ coordinate $\{x_i^f\}$

$$x_i^f := \begin{cases}
t_i^f \cdot x & \text{for } i = 1, \ldots, \ell, \\
n_i^f \cdot x & \text{for } i = \ell + 1, \ldots, n.
\end{cases}$$

The matrix proxy of $\omega \in \Lambda^{n-1,k}$ in this basis will be denoted by $A^f$ to emphasize the dependence of $f$. Define

$$(47) \quad T^f(X) = (T^f \otimes \Lambda^k) \cap \ker(s^{n-1,k}).$$

In view of matrix representations,

$$T^f(X) = \left\{ A^f = (a_{i,\sigma}^f) \in \mathbb{R}^{n \times (k)} : a_{i,\sigma}^f = 0 \ \forall \ i > \ell, S^{n-1,k}(A^f) = 0 \right\}.$$ 

We will define $N^f(X)$ so that $X = T^f(X) \oplus N^f(X)$.

Recall that $\Lambda^{n-1} = T^f \oplus N^f$ and thus

$$\Lambda^{n-1,k} = \Lambda^{n-1} \otimes \Lambda^k = (T^f \otimes \Lambda^k) \oplus (N^f \otimes \Lambda^k).$$

A naive definition of $N^f(X)$ similar to (47) would be $(N^f \otimes \Lambda^k) \cap \ker(s^{n-1,k})$. Unfortunately for three subspaces:

$$(A \cap C) \oplus (B \cap C) \not\subseteq (A \oplus B) \cap C.$$ 

And the equality may not hold. In view of the constraints defining $X$, we need to make sure one constraint is used only once either in $T^f(X)$ or $N^f(X)$. To this end, we introduce the concepts: an active constraint and an inactive constraint.

Let $(l, m), l, m \in [n^\tau]$ and $l, m = i_m + \tau$, for $m = 1, 2, \ldots, n - k + 1$, be the constraint sequence of $\tau$. In the sequel, we consider the non-trivial case: $1 \leq k < n$ and thus the length of the constraint sequence $n - k + 1$ is greater or equal to 2. We call the constraint $a_i^\tau \cdot e_* = 0$ is active in $N^f \otimes \Lambda^k$ if $l_m \geq \ell + 1$ for all $m = 1, 2, \ldots, n - k + 1$. That is $[\tau^\sigma] \subset [\ell + 1 : n]$. Otherwise it is called inactive in $N^f \otimes \Lambda^k$. For active constraints, it will be imposed inside the normal component. The number of the active constraints is

$$\binom{n - \ell}{n - k + 1} = \binom{n - \ell}{k - 1 - \ell}$$

and thus when $\ell \geq k$, there is no active constraint.

We introduce an oblique (non-orthogonal) projection operator $\pi_X : N^f \otimes \Lambda^k \rightarrow X$ as follows: take a basis function $* dx_i^f \otimes dx_j^f$ ($i > \ell$) in $N^f \otimes \Lambda^k$, if $s^{n-1,k}(\star dx_i^f \otimes dx_j^f) = 0$, which is the case if $i \in [\sigma^\tau]$, we keep it unchanged. If $i \in [\sigma]$ and $(i, \sigma)$ belongs to an active constraint sequence of $\tau = \sigma - i$, we can find another index $(j, j + \tau)$ in the same constraint sequence, and modify the basis to

$$\pi_X(\star dx_i^f \otimes dx_j^f) := \star dx_i^f \otimes dx_j^f - \epsilon(i, \tau)\epsilon(j, \tau) \star dx_j^f \otimes dx_{j+\tau}^f,$$
then \( \pi_X(*dx_i^f \otimes dx_j^f) \in \mathbb{X} \). In terms of the coefficient vector \( a_r, \pi_X \) will map the vector \((0, \ldots, 1, \ldots, 0)\) to \((0, \ldots, 1, \ldots, -\epsilon(i, \tau) \epsilon(j, \tau), \ldots)\) so that the constraint \( a_r \cdot e_r = 0 \) is satisfied. If \((i, \sigma)\) belongs to an inactive constraint sequence, we further require the pair index \((j, \sigma_j)\) satisfying \( j \leq \ell \), i.e., it is in the tangential component. After that \( \pi_X \) is defined on \( \mathbb{N}^f \otimes \Lambda^k \) by linear combination of the basis \( \{*dx_i^f \otimes dx_j^f\} \). One can thought \( \pi_X \) as a generalization of the \( \text{sym} \) operator for a square matrix to a rectangular matrix. In \( \text{sym} \), the pair of indices is uniquely determined as the length of the constraint is 2. In general, the length is \( n - k + 1 \) and choice of the pair index in \( \pi_X \) is not unique.

We define
\[
\mathbb{N}^f(\mathbb{X}) = \text{span}\{\pi_X(*dx_i^f \otimes dx_j^f) \mid *dx_i^f \otimes dx_j^f \in \mathbb{N}^f \otimes \Lambda^k\} = \pi_X(\mathbb{N}^f \otimes \Lambda^k).
\]

From the generating set \( \{ \pi_X(*dx_i^f \otimes dx_j^f) \mid (i, \sigma) \in \mathbb{N}^f \otimes \Lambda^k \} \), we can find a basis and call the corresponding index \((i, \sigma)\) is free (in \( \mathbb{N}^f \otimes \Lambda^k \)), i.e., \( \{ \pi_X(*dx_i^f \otimes dx_j^f) \mid (i, \sigma)\) is free in \( \mathbb{N}^f \otimes \Lambda^k \} \) is a basis of \( \mathbb{N}^f(\mathbb{X}) \). One possible choice of a set of free indices is as follows. For \((i, \sigma)\) not in the constraint sequence, it is free. If \((i, \sigma)\) is in some inactive constraint sequence, it is also free as in \( \pi_X(*dx_i^f \otimes dx_j^f) \), the pair index is in \( \mathbb{T}^f \otimes \Lambda^k \). For each active constraint, we randomly choose one index in the constraint sequence, denoted by \((i_a, \sigma_a)\), and leave others free. In the definition of \( \pi_X \), the fixed index \((i_a, \sigma_a)\) will be used as the pair index for each free index in the sequence. See Fig. 6.3 for an illustration of the concepts of the pair index and free indices etc. By construction, one active constraint will reduce the dimension of \( \mathbb{N}^f \otimes \Lambda^k \) by one. Especially if there is no active constraint, which is the case if \( \ell \geq k \), as the length of the constraint \( n - k + 1 \) will be greater than the dimension of the normal plane \( n - \ell \), then
\[
\mathbb{N}^f(\mathbb{X}) \cong \mathbb{N}^f \otimes \Lambda^k.
\]

Namely all indices are free and \( \mathbb{N}^f(\mathbb{X}) \) has a tensor product structure when \( \ell \geq k \).

**Lemma 6.1.** Given a \( t - n \) coordinate \( \{x_i^f\} \) of a face \( f \in \Delta_s(T) \), we have the following decomposition
\[
\mathbb{X} = \mathbb{T}^f(\mathbb{X}) \oplus \mathbb{N}^f(\mathbb{X}),
\]
where \( \mathbb{T}^f(\mathbb{X}) = (\mathbb{T}^f \otimes \Lambda^k) \cap \text{ker}(s^{n-1,k}) \) and \( \mathbb{N}^f(\mathbb{X}) = \pi_X(\mathbb{N}^f \otimes \Lambda^k) \). Their dimensions are
\[
dim \mathbb{T}^f(\mathbb{X}) = \ell \left( \begin{array}{c} n \\ k \end{array} \right) + \left( \begin{array}{c} n - \ell \\ n - k + 1 \end{array} \right) - \left( \begin{array}{c} n \\ k - 1 \end{array} \right),
\]
\[
dim \mathbb{N}^f(\mathbb{X}) = (n - \ell) \left( \begin{array}{c} n \\ k \end{array} \right) - \left( \begin{array}{c} n - \ell \\ n - k + 1 \end{array} \right).
\]

**Proof.** By construction, the sum is direct. It suffices to count the dimension. The number of constraints is \( \dim \Lambda^{k-1} \). By the proof of the surjectivity of \( s^{n-1,k} \), all constraints are linearly independent and all active constraints in \( \mathbb{N}^f \otimes \Lambda^k \) are also linearly independent.

Therefore the dimension of
\[
\mathbb{X} = \dim \Lambda^{n-1} \dim \Lambda^k - \dim \Lambda^{k-1} = n \dim \Lambda^k - \dim \Lambda^{k-1}.
\]

For each active constraint in \( \mathbb{N}^f \otimes \Lambda^k \), it will reduce the dimension by one and thus
\[
\dim \mathbb{N}^f(\mathbb{X}) = (n - \ell) \times \dim \Lambda^k - \# \text{ active constraint}.
\]

If a constraint is inactive in \( \mathbb{N}^f \otimes \Lambda^k \), then it will be used in \( \mathbb{T}^f \) to reduce the dimension by one and therefore
\[
\dim \mathbb{T}^f(\mathbb{X}) = \ell \times \dim \Lambda^k - \# \text{ inactive constraint}.
\]
Sum these two and use the fact

\[ \# \text{ active constraint} + \# \text{ inactive constraint} = \# \text{ all constraints} = \dim \Lambda^{k-1} \]

to get the dimensions matched.

To count the dimension of \( \mathcal{T}^f(\mathcal{X}) \) and \( \mathcal{N}^f(\mathcal{X}) \), we notice that the number of the active constraints is \( \binom{n-\ell}{n-k+1} \) and thus the number of the inactive constraints is \( \binom{n}{k-1} - \binom{n-\ell}{n-k+1} \).

\[ \square \]

We can also use the orthogonal projection \( \mathbb{P}_\mathcal{X} \) to define \( \mathcal{N}^f(\mathcal{X}) \) and obtain the following decomposition

\[ (48) \quad \mathcal{X} = \mathcal{T}^f(\mathcal{X}) \oplus \mathcal{N}^f(\mathcal{X}), \]

where \( \mathcal{T}^f(\mathcal{X}) = (\mathcal{T}^f \otimes \Lambda^k) \cap \ker(s^{n-1,k}) \) and \( \mathcal{N}^f(\mathcal{X}) = \mathbb{P}_\mathcal{X}(\mathcal{N}^f \otimes \Lambda^k) \).

### 6.2. Geometric decomposition of polynomial tensors.

We start from tensor product with Lagrange elements

\[ \mathbb{P}_r(T; \mathcal{X}) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_\ell(T)} [b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathcal{X}], \]

\[ \mathbb{P}_r^+(T; \mathcal{X}) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_\ell(T)} [\mathcal{N}(\mathbb{P}_{r-(\ell+1)}(f)) \otimes \mathcal{X}]. \]

For an \( \ell \)-dimensional face \( f \in \Delta_\ell(T) \), there is a polynomial matrix function \( A^f \in \mathbb{R}^{n \times \binom{n}{\ell}} \) satisfying the constraint \( s^{n-1,k}(A^f) = 0 \). The \( H(\text{div}) \) element is \( k = 0 \) for which the matrix \( A \) is degenerated to a column vector of length \( n \) and no constraint is imposed. For \( k = 1 \) or \( n - 1 \), the matrix \( A^f \in \mathbb{R}^{n \times n} \) is squared and satisfies constraint \( \text{trace}(A^f) = 0 \) or \( \text{vskw}(A^f) = 0 \), respectively. It is the constraint \( s^{n-1,k}(A^f) = 0 \) that makes the construction difficult.

Define the bubble polynomial space

\[ \mathbb{B}_r(\text{div}, T; \mathcal{X}) := \mathbb{P}_r(T; \mathcal{X}) \cap \ker(\text{tr}). \]

Notice that the trace operator is applied to the column of the matrix proxy or the first component in the tensor product defining \( \Lambda^{n-1,k} \). For each \( f \in \Delta_\ell(T) \), choose a \( t - n \) basis, and define

\[ \mathbb{B}_r^f(T; \mathcal{X}) := b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathcal{T}^f(\mathcal{X}), \]

\[ \mathbb{N}_r^f(T; \mathcal{X}) := b_f \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathcal{N}^f(\mathcal{X}). \]

In view of matrix proxy

\[ \mathbb{B}_r^f(T; \mathcal{X}) = \left\{ A^f = \left( a_{i,\sigma}^f \right): a_{i,\sigma}^f \in b_f \mathbb{P}_{r-(\ell+1)}(f), a_{i,\sigma}^f = 0 \forall i > \ell, s^{n-1,k}(A^f) = 0 \right\}, \]

\[ \mathbb{N}_r^f(T; \mathcal{X}) = \left\{ \pi_\mathcal{X}(A^f) : A^f = \left( a_{i,\sigma}^f \right) \text{ with } a_{i,\sigma}^f \in b_f \mathbb{P}_{r-(\ell+1)}(f), a_{i,\sigma}^f = 0 \forall i \leq \ell \right\}. \]

In the definition of \( \mathbb{N}_r^f(T; \mathcal{X}) \), the operator \( \pi_\mathcal{X} \) can be replaced by \( \mathbb{P}_\mathcal{X} \).

**Lemma 6.2.** Let \( r \geq 1 \) be an integer. The shape function space \( \mathbb{P}_r(T; \mathcal{X}) \) has a geometric decomposition

\[ \mathbb{P}_r(T; \mathcal{X}) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_\ell(T)} [\mathbb{B}_r^f(T; \mathcal{X}) \oplus \mathbb{N}_r^f(T; \mathcal{X})]. \]
A function \( \omega \in \mathbb{P}_r(T; \mathbb{X}) \) is uniquely determined by the DoFs

\[ (\omega, \eta)_f \quad \forall \eta \in \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathbb{T}^f(\mathbb{X}), \quad f \in \Delta_\ell(T), \ell = 1, \ldots, n, \]

which can be written as

\[ \mathbb{P}_r^\ast(T; \mathbb{X}) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_\ell(T)} \left[ \mathcal{N}(\mathbb{P}_{r-(\ell+1)}(f) \otimes \mathbb{T}^f(\mathbb{X})) \oplus \mathcal{N}(\mathbb{P}_{r-(\ell+1)}(f) \otimes \mathbb{N}^f(\mathbb{X})) \right]. \]

**Proof.** Since \( \mathbb{X} = \mathbb{T}^f(\mathbb{X}) \oplus \mathbb{N}^f(\mathbb{X}) \), DoFs (49)-(50) are equivalent to

\[ (\omega, \eta)_f \quad \forall \eta \in \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathbb{X}. \]

Then the unisolvence follows from the unisolvence of the Lagrange element. \( \square \)

By construction \( \mathbb{B}_r^f(T; \mathbb{X}) \subseteq \mathbb{B}_r^f(T) \otimes \Lambda^k(T) \subseteq \mathbb{B}_r(\text{div}, T) \otimes \Lambda^k(T) \) and thus

\[ \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_r^f(T; \mathbb{X}) \subseteq \mathbb{B}_r(\text{div}, T; \mathbb{X}). \]

Thanks to (48) and the same proof as Lemma 4.2, we then obtain the following result.

**Theorem 6.3.** It holds

\[ \mathbb{B}_r(\text{div}, T; \mathbb{X}) = \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_r^f(T; \mathbb{X}), \]

and

\[ \text{tr} : \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_\ell(T)} \mathbb{N}_r^f(T; \mathbb{X}) \to \text{tr}\mathbb{P}_r(T; \mathbb{X}) \]

is a bijection.

**Example 6.4.** Consider \( k = n - 1, \mathbb{X} = \mathbb{S}, \) and face \( f \in \Delta_\ell(T) \). By Example 5.3, for

\[ \omega = \sum_{i=1}^n \sum_{j=1}^n a_{ij} * dy_i^f \otimes dy_j^f, \]

we have

\[ S_{n-1,n-1}A = 2\text{vskw}(A) \quad \text{with} \quad A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n}. \]

Thus

\[ \mathbb{B}_r^f(T; \mathbb{S}) = \{ A = (a_{ij})_{n \times n} : a_{ij} = a_{ji} \in b_j^r \mathbb{P}_{r-(\ell+1)}(f) \text{ for } i, j = 1, \ldots, n, \]

\[ \text{and } a_{ij} = 0 \text{ for } i = \ell + 1, \ldots, n \}. \]

This implies

\[ \mathbb{B}_r(T; \mathbb{S}) = \sum_{1 \leq i \leq j \leq \ell} b_j^r \mathbb{P}_{r-(\ell+1)}(f) \text{ sym}(t_i^f \otimes t_j^f), \]

and

\[ \mathbb{N}_r^f(T; \mathbb{S}) = \sum_{i=1}^{\ell} \sum_{j=1}^{n-\ell} b_j^r \mathbb{P}_{r-(\ell+1)}(f) \text{ sym}(n_i^f \otimes t_j^f) \]

\[ + \sum_{1 \leq i \leq j \leq n-\ell} b_j^r \mathbb{P}_{r-(\ell+1)}(f) \text{ sym}(n_i^f \otimes n_j^f). \]

Due to Lemma 5.7, it follows from the dimension identity

\[ \sum_{\ell=1}^{r-1} \binom{n+1}{\ell+1} \binom{\ell+1}{2} \frac{r-1}{r-\ell-1} = \binom{n+1}{2} \sum_{\ell=1}^{r-1} \binom{n-1}{\ell-1} \frac{r-1}{r-\ell-1} = \binom{n+1}{2} \frac{n+r-2}{r-2} \]

\[ \binom{n+1}{2} \frac{n+r-2}{r-2} \]
Corollary 6.6. That

\[ \mathbb{B}_r(\text{div}, T; \mathbb{S}) = \bigoplus_{e \in \Delta_\ell(T)} b_e \mathbb{P}_{r-2}(T)(\mathbf{t}_e \otimes \mathbf{t}_e), \]

which is the characterization given in [18, 22]. Hence \( \mathbb{B}_r(\text{div}, T; \mathbb{S}) \cong \mathbb{P}_{r-2}(T) \otimes \mathbb{S} \).

Example 6.5. Consider \( k = 1, \mathbb{X} = \mathbb{T} \), and face \( f \in \Delta_\ell(T) \). By Example 5.4, for

\[ \omega = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \ast dy_i \otimes dy_j, \]

we have

\[ S^{n-1} \mathbf{A} = \text{trace}(\mathbf{A}) \quad \text{with} \quad \mathbf{A} = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n}. \]

Thus

\[ \mathbb{B}_r^f(T; \mathbb{T}) = \{ \mathbf{A} = (a_{ij})_{n \times n} : a_{ij} \in b_j \mathbb{P}_{r-\ell+1}(f) \text{ for } i, j = 1, \ldots, n, \]

\[ \text{trace}(\mathbf{A}) = 0 \text{ and } a_{ij} = 0 \text{ for } i = \ell + 1, \ldots, n \}. \]

This implies

\[ \mathbb{B}_r^f(T; \mathbb{T}) = \sum_{i=1}^{\ell} \sum_{j=1}^{n-\ell} b_j \mathbb{P}_{r-\ell+1}(f) \mathbf{t}_i^f \otimes \mathbf{n}_j^f + \sum_{1 \leq i < j \leq \ell} b_j \mathbb{P}_{r-\ell+1}(f) \mathbf{t}_i^f \otimes \mathbf{t}_j^f \]

\[ + \sum_{i=2}^{\ell} b_j \mathbb{P}_{r-\ell+1}(f) (\mathbf{t}_i^f \otimes \mathbf{t}_j^f - \mathbf{t}_i^f \otimes \mathbf{t}_j^f), \]

and

\[ \mathbb{N}_r^f(T; \mathbb{T}) = \sum_{i=1}^{\ell} \sum_{j=1}^{n-\ell} b_j \mathbb{P}_{r-\ell+1}(f) \mathbf{n}_j^f \otimes \mathbf{t}_i^f + \sum_{1 \leq i < j \leq n-\ell} b_j \mathbb{P}_{r-\ell+1}(f) \mathbf{n}_j^f \otimes \mathbf{n}_j^f \]

\[ + \sum_{j=1}^{n-\ell} b_j \mathbb{P}_{r-\ell+1}(f) (\mathbf{n}_j^f \otimes \mathbf{n}_j^f - \mathbf{t}_j^f \otimes \mathbf{t}_j^f). \]

If \( P_k \) is used to define \( \mathbb{N}_r^f(T; \mathbb{T}) \), the last component is replaced by \( \mathbf{n}_j^f \otimes \mathbf{n}_j^f - \frac{1}{n} \mathbf{I} \).

We can simplify the DoFs as follows.

Corollary 6.6. Let \( r \geq 1 \) be an integer. The shape function space \( \mathbb{P}_{r}(T; \mathbb{X}) \) is uniquely determined by the DoFs

\[ (\omega, \eta)_f \quad \forall \eta \in \mathbb{P}_{r-\ell+1}(f) \otimes \{ * \, dx_i^f \otimes dx_j^f | (i, \sigma) \text{ is free} \}, \]

\[ f \in \Delta_\ell(T), \ell = 0, \ldots, n - 1, \]

\[ (\omega, \eta)_T \quad \forall \eta \in \mathbb{B}_r(\text{div}, T; \mathbb{X}) = \bigoplus_{\ell=1}^{n} \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}_r^f(T; \mathbb{X}). \]

Proof. First of all, in the Frobenius product,

\[ (< * \, dx_i^f \otimes dx_j^f, * \, dx_j^f \otimes dx_j^f>) = \delta_{i,j} \delta_{\sigma, \tilde{\sigma}} \]

for all \( 1 \leq i, j \leq n, \) and \( \sigma, \tilde{\sigma} \in \Sigma(1 : k, 1 : n) \). In \( \pi_\Sigma(* \, dx_i^f \otimes dx_j^f) \), a pair index will be involved and may destroy the duality. In the following, we shall show that the pair index is not a free index and thus the duality still holds inside the free index set.

Recall that \( \{ \pi_\Sigma(* \, dx_i^f \otimes dx_j^f) | (i, \sigma) \text{ is free in } \mathbb{N}_r^f \otimes \Lambda_k \} \) is a basis of \( \mathbb{N}_r^f(\mathbb{X}) \). For \( (i, \sigma) \) in an inactive constraint, the pair index is out of \( \mathbb{N}_r^f \otimes \Lambda_k \) and cannot be free. For \( (i, \sigma) \) in an active constraint sequence, the modification is \( \pi_\Sigma(* \, dx_i^f \otimes dx_j^f) = * \, dx_i^f \otimes dx_j^f \).
\[ dx^f_i - \epsilon(i, \sigma - i)(i_*, \sigma_* - i_*) \ast dx^f_i \otimes dx^f_\sigma, \]

where \((i_*, \sigma_*)\) is not in the free index set. Therefore for two free indices \((i, \sigma)\) and \((j, \tilde{\sigma})\), the Frobenius inner product
\[ \langle \pi_X(\ast dx^f_i \otimes dx^f_\sigma), \ast dx^f_j \otimes dx^f_\tilde{\sigma} \rangle = \delta_{ij} \delta_{\sigma, \tilde{\sigma}}. \]

Consequently the DoF (50) can be changed to (51). If (51) vanishes, so is (50). By Theorem 6.3, \(\omega \in \mathbb{B}_r((\text{div}; T; \mathbb{X})\) and thus DoF (49) can be changed to (52).

DoF (50) is like the Galerkin method, where both trial and test functions are in the same space. While DoF (51) is Petrov-Galerkin method, where the test function space is different with the trial space.

6.3. Div-conforming finite element spaces. Now we glue local finite element spaces to form a div-conforming subspace of \(H(\text{div}; \Omega; \mathbb{X})\). For each \(f \in \Delta_\ell(T_h)\), we choose a global \(t - n\) basis \(\{t^f_1, \ldots, t^f_\ell, n^f_1, \ldots, n^f_{n-\ell}\}\) and use the following DoFs to define a div-conforming tensor space with normal continuity.

Lemma 6.7. For each \(f \in \Delta_\ell(T_h), \ell = 1, \ldots, n - 1\), we choose a global \(t - n\) basis \(\{t^f_1, \ldots, t^f_\ell, n^f_1, \ldots, n^f_{n-\ell}\}\). The following DoFs
\[
\begin{align*}
(\omega, \eta)_{f_1} & \quad \forall \eta \in \mathbb{P}_{r-(\ell+1)}(f) \otimes \{\ast dx^f_i \otimes dx^f_\sigma \mid \text{(i, \sigma) is free}\}, \\
(\omega, \eta)_{f_2} & \quad \forall \eta \in \Delta_\ell(T_h), \ell = 0, \ldots, n - 1, \\
(\omega, \eta)_T & \quad \forall \eta \in \mathbb{B}_r(\text{div}; T; \mathbb{X}) = \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(T)} \mathbb{B}^f_\ell(T; \mathbb{X}), T \in T_h,
\end{align*}
\]
will determine a space \(V_h \subset H(\text{div}; \Omega; \mathbb{X})\).

Proof. By Corollary 6.6, on each simplex, DoFs (53)-(55) will define a function \(\omega \in \mathbb{P}_r(T; \mathbb{X})\). We only need to verify the trace is uniquely determined. The DoFs (53)-(54) on \(F\) will imply
\[
(\omega, \eta)_f = 0 \quad \forall \eta \in \mathbb{P}_{r-(\ell+1)}(f) \otimes (N^f \otimes \Lambda^k), f \in \Delta_\ell(F), \ell = 0, \ldots, n - 1.
\]
Let \(\omega \in \mathbb{P}_r(T; \mathbb{X})\) such that the DoFs (56) vanish. Write
\[ N^f \otimes \Lambda^k = \mathbb{P}_X(N^f \otimes \Lambda^k) \oplus \mathbb{P}_X^\perp (N^f \otimes \Lambda^k). \]
Then as \(\omega \in \mathbb{X}\), we have
\[ (\omega, \eta)_f = 0 \quad \forall \eta \in \mathbb{P}_{r-(\ell+1)}(f) \otimes (N^f \otimes \Lambda^k), f \in \Delta_\ell(F), \ell = 0, \ldots, n - 1. \]

Namely the test function \(\eta\) can be changed to the whole subspace \(N^f \otimes \Lambda^k\).

The tensor product space \(N^f \otimes \Lambda^k\) is independent of the choice of the coordinate. For a given simplex \(T\), we switch the basis of \(N^f\) to \(\{n_F, f \subseteq F, F \in \Delta_{n-1}(T)\}\) and write a basis of \(N^f \otimes \Lambda^k\) as \(\{\ast d\eta_F \otimes dx^f_\sigma\}\). We have the formula
\[
(\omega, \ast d\eta_F \otimes dx^f_\sigma) = (\text{tr}_F \omega, dx^f_\sigma).
\]
It follows that
\[ (\text{tr}_F \omega, \eta)_f = 0 \quad \forall \eta \in \mathbb{P}_{r-(\ell+1)}(f) \otimes \Lambda^k, f \in \Delta_\ell(F), \ell = 0, \ldots, n - 1. \]
Then \(\text{tr}_F \omega = 0\) follows from the uni-solvence of the Lagrange element as \(\text{tr}_F \omega \in \mathbb{P}_r(F) \otimes \Lambda^k\).
Notice that when $\ell = 0$, DoF (54) becomes (53). We single out the vertex DoF to emphasize the continuity at vertices. Take an vertex in $\Delta_0(T)$, for example $v_0$. Then $(n^T_{F_v} A)(v_0)$ is determined by $(n^T_{F_v} A)|_{F_v}$ for $i = 1, \ldots, n$, where $F_v \in \Delta_{n-1}(T)$ is opposite to $v_i$. The number of elements in $(n^T_{F_v} A)(v_0)$ is $\dim \Lambda^k$ for each face. Running $i$ from 1 to $n$, $A(v_0)$ is determined by $n \dim \Lambda^k$ conditions, which is more than $\dim \mathcal{X}$, hence DoF (53) is necessary. In other words, the constraint makes the tensor product of vector DoFs fails and introduce additional smoothness.

The basis $\{t^1, \ldots, t^\ell, n^1, \ldots, n^\ell\}$ depends on $f$ not the element containing $f$ and thus (54) implies the certain continuity on the normal component $\mathbb{N}^f \otimes \Lambda^k$. When $\mathbb{N}^f(\mathcal{X}) \cong \mathbb{N}^f \otimes \Lambda^k$, i.e., all indices $(i, \sigma)$ in $\mathbb{N}^f \otimes \Lambda^k$ are free, we can redistribute the DoFs facewisely and relax the normal continuity. Then the normal continuity is only imposed on sub-simplexes with dimension $\leq k - 1$. The key formulae is (57), where we can switch the basis for $\mathbb{N}^f(\mathcal{X})$ while keep the basis for $\Lambda^k$ at $f$ is global.

**Lemma 6.8.** For each $f \in \Delta_\ell(T_h)$, $\ell = 1, \ldots, k-1$, we choose a global $t - n$ basis $\{t^1, \ldots, t^\ell, n^1, \ldots, n^{\ell+1}\}$. The following DoFs

\begin{align}
\omega(v_i) & \quad \forall i = 0, \ldots, n, \\
(\omega, \eta)_f & \quad \forall \eta \in \mathbb{P}_{-l+1}(f) \otimes \{\star \, dx^T_i \otimes dx^f_\sigma \mid (i, \sigma) \text{ is free}\}, \\
 & \quad f \in \Delta_\ell(T_h), \ell = 1, \ldots, k-1, \\
(tr_F \omega, \eta)_f & \quad \forall \eta \in \mathbb{P}_{-l+1}(f) \otimes \Lambda^k, F \in \Delta_{n-1}(T_h), \\
 & \quad f \in \Delta_\ell(F), \ell = k, \ldots, n-1,
\end{align}

will determine a space $V_h \subset H(\text{div}; \mathcal{X})$.

**Proof.** By the definition of trace operator $tr_F$ and $\omega|_f \in L^2(f; \mathcal{X})$, DoF (60) can be rewritten as

\[(tr_F \omega, \eta)_f = (\omega, \star \, d n_F \otimes \eta)_f = (\omega, P_\Sigma(\star \, d n_F \otimes \eta))_f.\]

And notice that $\{n^f\}_{f \in \Delta_\ell(T)}$ is a basis of $\mathbb{N}^f_0$, then DoF (60) is equivalent to $\omega(v_i)$.

Hence we conclude the result by applying the proof of Lemma 6.7. □

In Lemma 6.8, we split the tensor $\mathcal{X}$ horizontally such that the lower part has a tensor product structure and can redistribute DoFs facewisely. We can also obtain a tensor product structure vertically. To this end, we introduce the concept of a free column. The indices $(i, \sigma)$ of $\mathbb{N}^f \otimes \Lambda^k$ form a matrix of size $(n - \ell) \times \dim \Lambda^k$. Take a $\sigma \in \Sigma(1 : k, 1 : n)$, the column $(i, \sigma), i = \ell + 1 : n$ in $\mathbb{N}^f \otimes \Lambda^k$ is called free if no index is in the active constraints. All free columns will form a sub-matrix and called the free block. The rest is called the (active) constraint block. Notice that a free column consists of free indices but a column with all free indices may not be a free column since a free index can be in an active constraint sequence. For each active constraint sequence, we only remove one index and all others are free. But those free indices are in the active constraint and thus will eliminate the corresponding column from the free block. For a free column, we have a vector in $\mathbb{R}^n$ consists of $a_1, a_\ell, a_{\ell+1}, \ldots, a_n, a_{n+1}$. The first $\ell$ components will be in the tangential component and is determined locally in each simplex. The part $a_{n+1}, \ldots, a_n$ is in the normal component and the corresponding DoFs can be redistributed facewisely. In short, a free column is just like a vector div element.
The constraint block disappears when $\ell \geq k$ as the length of the constraint $n - k + 1$ will be greater than the dimension of the normal plane $n - \ell$. Consider the case $\ell < k$. If index $\sigma \in \Sigma(1 : k, 1 : n)$ is in the constraint block, there exists some $i > \ell$ such that $\tau = \sigma - i \in \Sigma(1 : k - 1, 1 : n)$ satisfies $[\tau^\sigma] \subseteq [\ell + 1 : n]$, which is equivalent to $[\sigma^\tau] \subseteq [\ell + 1 : n]$. Hence the number of columns in the constraint block is $\binom{n - \ell}{n - k}$: among all $n - \ell$ index of the normal plane, choose $n - k$ indices to form $\sigma^\tau$. When $\ell = 0$, $\binom{n}{n - k} = \binom{n}{k}$, i.e., all columns belong to the constraint block and consequently the tensor should be continuous at vertices.

In general, on the active constraints block, the normal continuity is still imposed which can be thought of as the super-smoothness induced by the constraint of $X$. For example, if $X = S$, then $A$ is also symmetric on the normal plane of an edge and the projection of $A$ to the normal plane should depend on $f$ only. We are allowed to modify the free columns as in the definition of $\pi_X$, the pair index is in the tangential component which is local. While for an active constraint, the pair index is still in the normal component and changing the basis of the normal plane will make $\pi_X$ element-dependent and destroy the conformity. We refer to Fig. 6.3 for an illustration.

**Lemma 6.9.** For each $f \in \Delta_\ell(T_h)$, $\ell = 1, \ldots, k - 1$, we choose a global $t - n$ basis $\{t^f_1, \ldots, t^f_\ell, n^f_1, \ldots, n^f_{n-\ell}\}$. The following DoFs

$$\omega(v_i) \quad \forall i = 0, \ldots, n,$$

$$\omega, \eta \quad \forall \eta \in P_{r-(\ell+1)}(f) \otimes \{\star dx_i^f \otimes dx^f_\sigma | \sigma \text{ is in the constraint block}\},$$

$$f \in \Delta_\ell(T_h), \ell = 1, \ldots, n - 1,$$

$$\tau_F \omega, \eta \quad \forall \eta \in P_{r-(\ell+1)}(f) \otimes \{dx^f_\sigma | \sigma \text{ is in the free block}\},$$

$$F \in \Delta_{n-1}(T_h), f \in \Delta_\ell(F), \ell = 1, \ldots, n - 1,$$

$$\omega, \eta \quad \forall \eta \in \mathbb{B}_r(div; T; X) = \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(T)} B^f_\ell(T; X), T \in \mathcal{T}_h,$$

will determine a space $V_h \subset H(div; \Omega; X)$.

**Proof.** We write DoF (63) as

$$\omega, \eta \quad \forall \eta \in \mathbb{B}_r(div; T; X) = \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(T)} B^f_\ell(T; X), T \in \mathcal{T}_h.$$

As $\{n_F, f \subseteq F, F \in \Delta_{n-1}(T)\}$ is a basis of $\mathbb{B}_r$, DoF (65) will also determine $\omega, \eta \in \mathbb{B}_r(div; T; X)$ for $i = \ell + 1, \ldots, n$. Together with (62), we obtain (54). \hfill\square

In Lemma 6.8, we relax the normal continuity when $\ell \geq k$ as the whole index set is free. In Lemma 6.9, it says even in the lower dimensional sub-simplex, DoFs of the free block can be still facewisely defined.

**Example 6.10.** Consider $k = n - 1, X = S$. The DoFs are

$$\omega(v_i) \quad \forall v_i \in \Delta_0(T_h), i = 1, \ldots, n + 1,$$

$$\omega, \eta \quad \forall \eta \in \mathbb{P}_{r-(\ell+1)}(f) \otimes \{\star dx^f_i \otimes dx^f_\sigma | \sigma \text{ is in the constraint block}\},$$

$$f \in \Delta_\ell(T_h), \ell = 1, \ldots, n - 1, 1 \leq i \leq j \leq n - \ell,$$

$$\tau_F \omega, \eta \quad \forall \eta \in \mathbb{P}_{r-(\ell+1)}(f) \otimes \{dx^f_\sigma | \sigma \text{ is in the free block}\},$$

$$F \in \Delta_{n-1}(T_h), f \in \Delta_\ell(F), \ell = 1, \ldots, n - 1, j = 1, \ldots, \ell,$$

$$\omega, \eta \quad \forall \eta \in \mathbb{P}_{r-2}(T; S), T \in \mathcal{T}_h.$$
The constraint is imposed on the normal plane \((n^f_i)^\top A n^f_j\). Define the global finite element space \([10]\)

\[
V_h := \{\omega_h \in L^2(\Omega; \mathbb{S}) : \omega_h|_T \in \mathbb{P}_r(T; \mathbb{S}) \quad \forall \, T \in \mathcal{T}_h, \quad \text{the DoF (66) is single-valued across } f \in \Delta_\ell(\mathcal{T}_h) \text{ for } \ell = 0, \ldots, n-1, \quad \text{the DoF (67) is single-valued across } F \in \Delta_{n-1}(\mathcal{T}_h)\}. \]

By Lemma 6.9, \(V_h \subset H(\text{div}; \Omega; \mathbb{S})\). Only the normal-normal part of \(A\) is required to be continuous in the global finite element space \(V_h\) since the normal-tangential DoF (67) corresponds to the free block, it can be face-wise. This is the finite element for symmetric tensors constructed in [10]. If (67) is changed to \(((n^f_i)^\top A t^f, q) f\), i.e., it is also continuous across \(f\) rather than \(F\), we get the Hu-Zhang finite element in [18, 22].

**Example 6.11.** Consider \(k = 1, \mathbb{X} = \mathbb{T}\). The DoFs are

\[
\begin{align*}
A(\nu_i) & \quad \forall \, \nu_i \in \Delta_0(T), i = 0, \ldots, n, \\
(n^T_F A, q)_f & \quad \forall \, q \in \mathbb{P}_{r-(\ell+1)}(f) \otimes \mathbb{E}^n, f \in \Delta_\ell(F), F \in \Delta_{n-1}(T), \ell = 1, \ldots, n-1, \\
(A, q)_T & \quad \forall \, q \in \mathbb{B}_r(\text{div}, T; \mathbb{S}).
\end{align*}
\]

As there is no active constraint for \(\ell \geq 1\), the DoFs are almost the tensor product of BDM element except the continuity at vertices.

**6.4. Discrete inf-sup condition.** For simplicity of notation, consider the finite elements defined in Lemma 6.8. Proofs of the discrete inf-sup condition can be easily adapted to
other cases. Define the global finite element space
\[ V_h := \{ \omega_h \in L^2(\Omega; \mathbb{X}) : \omega_h|_T \in \mathbb{P}_r(T; \mathbb{X}) \quad \forall \, T \in \mathcal{T}_h, \] 
the DoF (58) is single-valued across \( \Delta_0(\mathcal{T}_h) \),
the DoF (59) is single-valued across \( f \in \Delta_e(\mathcal{T}_h) \) for \( \ell = 0, \ldots, k-1 \),
the DoF (60) is single-valued across \( F \in \Delta_{n-1}(\mathcal{T}_h) \),
\[ Q_h := \{ q_h \in L^2(\Omega; \Lambda^k) : q_h|_T \in \mathbb{P}_{r-1}(T; \Lambda^k) \quad \forall \, T \in \mathcal{T}_h \}. \]

Thanks to Lemma 6.8, \( V_h \subset H(\operatorname{div}; \Omega; \mathbb{X}) \). We are going to verify the discrete inf-sup condition \( \operatorname{div} V_h = Q_h \). Recall that \( \operatorname{RX} = \ker(P_\mathbb{X} \nabla) \). The following characterization of the range of the bubble polynomials is a generalization of a result in [18, 22]; see Lemma 4.5 for the \( H(\operatorname{div}) \) case.

**Lemma 6.12.** For each \( T \in \mathcal{T}_h \), it holds
\[ \operatorname{div} \mathbb{B}_r(\operatorname{div}; T; \mathbb{X}) = \mathbb{P}_{r-1}^\perp(\operatorname{div}; T). \]

**Proof.** Apply the integration by parts (43) to get \( \operatorname{div} \mathbb{B}_r(\operatorname{div}; T; \mathbb{X}) \subset \mathbb{P}_{r-1}^\perp(\operatorname{div}; T) \). Next we focus on the proof of the other side \( \mathbb{P}_{r-1}^\perp(\operatorname{div}; T) \subset \mathbb{B}_r(\operatorname{div}; T; \mathbb{X}) \). For simplicity, write \( d\lambda_{\sigma(0),\sigma(i)} \otimes d\lambda_{\tau} \) as \( \phi_{i,\sigma} \) for each \( \sigma \in \Sigma(0 : n-k, 0 : n) \) and \( i = 1, \ldots, n-k \). By Lemma 5.7, \( \{ \phi_{i,\sigma} \}_{i=1}^{n-k} \) is a basis of \( \mathbb{X} \), whose dual basis (appropriate rescaling of (42)) is denoted by \( \{ \psi_{j,\tau} \}_{j=1}^{n-k} \), that is \( \psi_{j,\tau} \in \mathbb{X} \), \( \langle \phi_{i,\sigma}, \psi_{j,\tau} \rangle = 1 \) for \( \sigma = \tau \) and \( i = j \), otherwise it vanishes.

Consider the edge \( e = e_{\sigma(0),\sigma(i)} \). The vector proxy of \( d\lambda_{\sigma(0),\sigma(i)}^* \) is proportional to \( t_e \). Couple with the edge bubble function \( b_e = \lambda_{\sigma(0),\sigma(i)} \), the vector function \( b_e t_e \) satisfies
\[ n_F \cdot b_e t_e|_F = 0, \quad \forall \, F \in \Delta_{n-1}(T), \]
as if the edge \( e \not\subseteq F \), then \( b_e|_F = 0 \); otherwise \( n_F \cdot t_e = 0 \). Therefore \( \lambda_{\sigma(0),\sigma(i)} \) satisfies
\[ \lambda_{\sigma(0),\sigma(i)} \in \mathbb{B}_2(\operatorname{div}; T; \mathbb{X}). \]
If \( \mathbb{P}_{r-1}^\perp(\operatorname{div}; T) \not\subseteq \mathbb{B}_r(\operatorname{div}; T; \mathbb{X}) \), then there exists \( u \in \mathbb{P}_{r-1}^\perp(\operatorname{div}; T) \) satisfying \( (u, \operatorname{div} \omega)|_T = 0 \) for any \( \omega \in \mathbb{B}_r(\operatorname{div}; T; \mathbb{X}) \). Equivalently
\[ (P_\mathbb{X} \nabla u, \omega)|_T = 0 \quad \forall \, \omega \in \mathbb{B}_r(\operatorname{div}; T; \mathbb{X}). \]

By expressing \( P_\mathbb{X} \nabla u = \sum_{\sigma \in \Sigma(0:n-k,0:n)} \sum_{i=1}^{n-k} q^{i,\sigma} \psi_{i,\sigma} \) with \( q^{i,\sigma} \in \mathbb{P}_{r-2}(T) \), we choose
\[ \omega = \sum_{\sigma \in \Sigma(0:n-k,0:n)} \sum_{i=1}^{n-k} \lambda_{\sigma(0)} \lambda_{\sigma(i)} q^{i,\sigma} \phi_{i,\sigma} \in \mathbb{B}_r(\operatorname{div}; T; \mathbb{X}). \]
Then we have
\[ \sum_{\sigma \in \Sigma(0:n-k,0:n)} \sum_{i=1}^{n-k} (\lambda_{\sigma(0)} \lambda_{\sigma(i)} q^{i,\sigma}, q^{i,\sigma}) = 0. \]
Therefore \( q^{i,\sigma} = 0 \) for all \( i \) and \( \sigma \) and thus \( u = 0 \).

Employing the same argument as the proof of (38), the discrete inf-sup condition follows from (68).

**Lemma 6.13.** Let \( r \geq n+1 \). It holds
\[ \operatorname{div} V_h = Q_h. \]
Proof. Take $q_h \in Q_h$. By (37) in [5], $\text{div} \ H^1(\Omega; \mathbb{X}) = L^2(\Omega; \mathbb{L}^k)$. Then there exists $\omega \in H^1(\Omega; \mathbb{X})$ satisfying $\text{div} \omega = q_h$. Let $\omega_1$ be the Scott-Zhang interpolation of $\omega$ in the vector $r$th order Lagrange element space $S^+_h \otimes \mathbb{X}$ [24] such that

$$(\text{tr}_F(\omega - \omega_1), \eta)_F = 0 \quad \forall \eta \in \mathbb{P}_{r-n}(F) \otimes \mathbb{L}^k, F \in \Delta_{n-1}(T_h).$$

Since $r \geq n + 1$, $\mathbb{R}X \subset \mathbb{P}_1(T) \subseteq \mathbb{P}_{r-n}(T)$, the last equation together with the integration by parts implies $\text{div}(\omega - \omega_1)|_T \in \mathbb{P}_{r-1,\mathbb{R}X}(T)$. By (68), there exists $\omega_2 \in V_h$ such that $\omega_2|_T \in \mathbb{B}_r(\text{div}; T; \mathbb{X})$ for each $T \in T_h$ and $\text{div} \omega_2 = \text{div}(\omega - \omega_1)$. Finally take $\omega_h = \omega_1 + \omega_2 \in V_h$ to get $\text{div} \omega_h = q_h$. \hfill \Box

For $\ell \geq k$, all DoFs are redistributed facewisely as no active constraint exists. We can further modify the DoFs to show that the discrete inf-sup condition $\text{div} V_h = Q_h$ holds for $r \geq k + 1$ when $k = 1, \ldots, n - 2$. To this end, for $F \in \Delta_{n-1}(T)$, let

$$\mathbb{P}^0_{k+1}(F) := \{ v \in \mathbb{P}_{k+1}(F) : (v, q)_F = 0 \text{ for all } q \in \mathbb{P}_1(F),$$

$v|_f = 0$ for all $f \in \Delta_{\ell}(F), \ell = 0, \ldots, k-1 \}$.

Lemma 6.14. Let $1 \leq k \leq n-2$ and $F \in \Delta_{n-1}(T)$. The degrees of freedom

$$(69) \quad (v, q)_F \quad \forall \ q \in \mathbb{P}_{k-\ell}(F), \ f \in \Delta_{\ell}(F), \ell = 0, \ldots, k-1,$$

$$(70) \quad (v, q)_F \quad \forall \ q \in \mathbb{P}_1(F),$$

$$(v, q)_F \quad \forall \ q \in \mathbb{P}^0_{k+1}(F)$$

are uni-solvent for $\mathbb{P}^0_{k+1}(F)$.

Proof. By the geometric decomposition of $\mathbb{P}^0_{k+1}(F)$, a function $v \in \mathbb{P}^0_{k+1}(F)$ will be determined by (69) if the index $\ell = 0, \ldots, k-1, k$. We are going to move $\text{dim} \mathbb{P}^0_{k+1}(F) - \#(69)$ DoFs to face $F$. DoF (70) is introduced to preserve the face moment up to $\mathbb{P}_1(F)$ and by the definition of $\mathbb{P}^0_{k+1}(F)$, it suffices to prove the DoFs (69) and (70) in the dual space of $\mathbb{P}^0_{k+1}(F)$ are linearly independent. Without loss of generality, we assume the vertices of $F$ are $v_1, \ldots, v_n$. Assume there exist constants $c^\alpha_f, c_i \in \mathbb{R}$ such that

$$(71) \quad \sum_{\ell=0}^{k-1} \sum_{\tau \in \Sigma(0; \ell, 1: n)} \sum_{\alpha \in [0]_{k-\ell}^d} c^\alpha_f (v, \lambda^\alpha \tau)_f + \sum_{i=1}^n c_i (v, \lambda_i)_f = 0 \quad \forall \ v \in \mathbb{P}^0_{k+1}(F).$$

Taking $v = \lambda_{\sigma}$ for any $\sigma \in \Sigma(0: k, 1: n)$ in (71), we get

$$\sum_{i=0}^k c_{\sigma(i)} + \sum_{i=1}^n c_i = 0 \quad \forall \ \sigma \in \Sigma(0: k, 1: n).$$

This implies all $\sum_{i=0}^k c_{\sigma(i)}$ are equal for $\sigma \in \Sigma(0: k, 1: n)$, and then $c_1 = \cdots = c_n = 0$. Then (71) reduces to

$$\sum_{\ell=0}^{k-1} \sum_{\tau \in \Sigma(0; \ell, 1: n)} \sum_{\alpha \in [0]_{k-\ell}^d} c^\alpha_f (v, \lambda^\alpha \tau)_f = 0 \quad \forall \ v \in \mathbb{P}^0_{k+1}(F).$$

Thus $c^\alpha_f = 0$ follows from the uni-solvence of Lagrange element. \hfill \Box

The requirement $k \leq n-2$ is to ensure the number of DoFs (69) and (70) is not greater than $\text{dim} \mathbb{P}^0_{k+1}(F)$. Indeed, we have

$$\text{dim} \mathbb{P}^0_{k+1}(F) = \#(69) = \binom{n}{k+1} \geq n = \#(70) \quad \text{if } k \leq n-2.$$
To extend the degree to $r$, define
\[ P_{r,k+1}^i(F) := \{ v \in P_r(F) : (v, q)_F = 0 \text{ for all } q \in P^i_{k+1}(F) \}, \]
and $P_{r,k+1}^i(F; \Lambda^k) := P_{r,k+1}^i(F) \otimes \Lambda^k$, and $\mathbb{P}_{r,k+1}^0(F) := \mathbb{P}_{r,k+1}^0(F) \otimes \Lambda^k$.

Combining Lemma 6.9 and Lemma 6.14, we get another DoFs for $P_r(T; \Xi)$.

**Theorem 6.15.** Let $1 \leq k \leq n - 2$ and $r \geq k + 1$. The DoFs
\[ \omega(v_i) \quad \forall i = 0, \ldots, n, \]
\[ (\omega, \eta)_f \quad \forall \eta \in P_{r-\ell+1}(f) \otimes \{ \star dx^f_1 \otimes dx^f_2 \mid \sigma \text{ is in the constraint block} \}, \]
\[ f \in \Delta_\ell(T_h), \ell = 1, \ldots, k - 1, \]
\[ (\text{tr}_F \omega, \eta)_f \quad \forall \eta \in P_{r-\ell+1}(f) \otimes \{ dx^f_\ell \mid \sigma \text{ is in the free block} \}, \]
\[ f \in \Delta_{n-1}(T_h), \ell = 1, \ldots, k - 1, \]
\[ (72) \quad (\text{tr}_F \omega, \eta)_F \quad \forall \eta \in P_1(F; \Lambda^k) \oplus P_{k+1}^0(F; \Lambda^k) \oplus P_{r,k+1}^0(F; \Lambda^k), F \in \Delta_{n-1}(T_h), \]
\[ (\omega, \eta)_T \quad \forall \eta \in B_{r+1}(\text{div}, T; \Xi), T \in T_h \]
will determine a space $V_h \subset H(\text{div}, \Omega; \Xi)$.

Thanks to the first part of DoF (72), we acquire the following discrete inf-sup condition by applying the same argument as the proof of Lemma 6.13.

**Corollary 6.16.** Let $1 \leq k \leq n - 2$ and $r \geq k + 1$. Let $V_h \subset H(\text{div}, \Omega; \Xi)$ be the finite element space defined in Theorem 6.15 and $Q_h$ be the space defined in (22). It holds
\[ \text{div } V_h = Q_h. \]

**Remark 6.17.** An incomplete polynomial finite element space can be chosen as
\[ (73) \quad P_{r+1}^{-}(T; \Xi) := B_{r+1}(\text{div}, T; \Xi) \oplus N_{r+1}(\text{div}, T; \Xi), \]
where $N_{r+1}(\text{div}, T; \Xi) = \bigoplus_{\ell=0}^{n-1} \bigoplus_{f \in \Delta_\ell(T_h)} N_{r+1}^f(T; \Xi)$. As the trace remains unchanged, we can keep the DoFs (61)-(63) and simply replace (64) by
\[ (74) \quad (\omega, \eta)_T \quad \forall \eta \in B_{r+1}(\text{div}, T; \Xi), T \in T_h. \]

Namely (61), (62), (63), and (74) will determine a space $V_{r+1,h}^{-} \subset H(\text{div}, \Omega; \Xi)$ with inf-sup condition $\text{div } V_{r+1,h}^{-} = Q_{r+1,h}$. The bubble functions can be further reduced by removing the ker(\text{div}) \cap B_{r+1}(\text{div}, T; \Xi) but (73) is more friendly to the implementation as a basis of $B_{r+1}(\text{div}, T; \Xi)$ can be easily constructed from the Lagrange basis.

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