The Domination Polynomials of Cubic Graphs of Order 10

Saieed Akbari\textsuperscript{a}, Saeid Alikhani\textsuperscript{b,d}, Yee-hock Peng\textsuperscript{c,d}

\textsuperscript{a}Department of Mathematical Sciences, Sharif University of Technology
11365-9415 Tehran, Iran
\textsuperscript{b}Department of Mathematics, Yazd University
89195-741, Yazd, Iran
\textsuperscript{c}Department of Mathematics, and
\textsuperscript{d}Institute for Mathematical Research, University Putra Malaysia
43400 UPM Serdang, Malaysia

ABSTRACT

Let $G$ be a simple graph of order $n$. The domination polynomial of $G$ is the polynomial $D(G, x) = \sum_{i=\gamma(G)}^{n} d(G, i)x^i$, where $d(G, i)$ is the number of dominating sets of $G$ of size $i$, and $\gamma(G)$ is the domination number of $G$. In this paper we study the domination polynomials of cubic graphs of order 10. As a consequence, we show that the Petersen graph is determined uniquely by its domination polynomial.

Mathematics Subject Classification: 05C60.

Keywords: Domination polynomial; Equivalence class; Petersen graph; Cubic graphs.

1 Introduction

Let $G = (V, E)$ be a simple graph. The order of $G$ is the number of vertices of $G$. For any vertex $v \in V$, the open neighborhood of $v$ is the set $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of $S$ is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of $S$ is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set if $N[S] = V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. A dominating set with cardinality $\gamma(G)$ is called a $\gamma$-set. The family of all $\gamma$-sets of a graph $G$ is denoted by $\Gamma(G)$. For a detailed treatment of these parameters, the reader is referred to \cite{2}. The
An \textit{i}-subset of $V(G)$ is a subset of $V(G)$ of size $i$. Let $D(G, i)$ be the family of dominating sets of a graph $G$ with cardinality $i$ and let $d(G, i) = |D(G, i)|$. The \textit{domination polynomial} $D(G, x)$ of $G$ is defined as $D(G, x) = \sum_{i=\gamma(G)}^{\vert V(G) \vert} d(G, i)x^i$, where $\gamma(G)$ is the domination number of $G$ (see [1]).

We denote the family of all dominating sets of $G$ with cardinality $i$ and contain a vertex $v$ by $D_v(G, i)$, and $d_v(G, i) = |D_v(G, i)|$. Two graphs $G$ and $H$ are said to be \textit{dominating equivalence}, or simply $D$-equivalent, written $G \sim H$, if $D(G, x) = D(H, x)$. It is evident that the relation $\sim$ of being $D$-equivalence is an equivalence relation on the family $\mathcal{G}$ of graphs, and thus $\mathcal{G}$ is partitioned into equivalence classes, called the $D$-equivalence classes. Given $G \in \mathcal{G}$, let $[G] = \{H \in \mathcal{G} : H \sim G\}$.

We call $[G]$ the equivalence class determined by $G$. A graph $G$ is said to be \textit{dominating unique}, or simply $D$-unique, if $[G] = \{G\}$.

The minimum degree of $G$ is denoted by $\delta(G)$. A graph $G$ is called \textit{k-regular} if all vertices have the same degree $k$. A \textit{vertex-transitive graph} is a graph $G$ such that for every pair of vertices $v$ and $w$ of $G$, there exists an automorphism $\theta$ such that $\theta(v) = w$. One of the famous graphs is the Petersen graph which is a symmetric non-planar cubic graph. In the study of domination polynomials, it is interesting to investigate the dominating sets and domination polynomial of this graph. We denote the Petersen graph by $P$.

In this paper, we study the dominating sets and domination polynomials of cubic graphs of order 10. As a consequence, we show that the Petersen graph is determined uniquely by its domination polynomial. In the next section, we obtain domination polynomial of the Petersen graph. In Section 3, we list all $\gamma$-sets of connected cubic graphs of order 10. This list will be used to study the $D$-equivalence of these graphs in the last section. In Section 4, we prove that the Petersen graph is $D$-unique. In the last section, we study the $D$-equivalence classes of some cubic graphs of order 10.
2 Domination Polynomial of the Petersen Graph

In this section we shall investigate the domination polynomial of the Petersen graph. First, we state and prove the following lemma:

**Lemma 1.** Let $G$ be a vertex transitive graph of order $n$ and $v \in V(G)$. For any $1 \leq i \leq n$, $d(G, i) = \frac{n}{i}d_v(G, i)$.

**Proof.** Clearly, if $D$ is a dominating set of $G$ with size $i$, and $\theta \in Aut(G)$, then $\theta(D)$ is also a dominating set of $G$ with size $i$. Since $G$ is a vertex transitive graph, then for every two vertices $v$ and $w$, $d_v(G, i) = d_w(G, i)$. If $D$ is a dominating set of size $i$, then there are exactly $i$ vertices $v_{j_1}, \ldots, v_{j_i}$ such that $D$ counted in $d_{v_{j_r}}(G, i)$, for each $1 \leq r \leq i$. Hence $d(G, i) = \frac{n}{i}d_v(G, i)$, and the proof is complete. \qed

**Theorem 1.** ([2], p.48) If $G$ is a connected graph of order $n$ with $\delta(G) \geq 3$, then $\gamma(G) \leq \frac{3n}{8}$.

We need the following theorem for finding the domination polynomial of the Petersen graph.

**Theorem 2.** Let $G$ be a graph of order $n$ with domination polynomial $D(G, x) = \sum_{i=1}^{n} d(G, i)x^i$. If $d(G, j) = \binom{n}{j}$ for some $j$, then $\delta(G) \geq n - j$. More precisely, $\delta(G) = n - l$, where $l = \min \left\{ j | d(G, j) = \binom{n}{j} \right\}$, and there are at least $\binom{n}{l-1} - d(G, l-1)$ vertices of degree $\delta(G)$ in $G$. Furthermore, if for every two vertices of degree $\delta(G)$, say $u$ and $v$ we have $N[u] \neq N[v]$, then there are exactly $\binom{n}{l-1} - d(G, l-1)$ vertices of degree $\delta(G)$.

**Proof.** Since $d(G, j) = \binom{n}{j}$ for any $r \geq j$, every $r$-subset of the vertices of $G$ forms a dominating set for $G$. Suppose that there is a vertex $v \in V(G)$ such that $\deg(v) < n - j$. Consider $V(G) \setminus N[v]$. Clearly, $\left| V(G) \setminus N[v] \right| \geq j$, and $V(G) \setminus N[v]$ is not a dominating set for $G$, a contradiction. Now, assume that $S$ is a $(l-1)$-subset of $V(G)$ which is not a dominating set. Thus there is a vertex $u \in V(G) \setminus S$ which is not covered by $S$. Since $\delta(G) \geq n - l$, we have $\deg(u) = n - l$. Let $S' \neq S$ be a $(l-1)$-subset which is not a dominating set. Thus there exists
a vertex $u' \in V(G) \setminus S'$ which is not covered by $S'$. As we did before $\deg(u') = n - 1$. We claim that $u \neq u'$. Since $S \neq S'$, there exists a vertex $x \in S \cap (V(G) \setminus S')$. We know that $u'x \in E(G)$. If $u = u'$, then $ux \in E(G)$, a contradiction. Thus $u \neq u'$ and the claim is proved. If $u$ and $v$ are two vertices of degree $\delta(G)$ and $N[u] = N[v]$, then $V(G) \setminus N[u] = V(G) \setminus N[v]$ is an $(l - 1)$-subset which is not a dominating set for $G$. Thus we have at least $(\binom{n}{l-1} - d(G, l - 1))$ vertices of degree $\delta(G)$ in $G$. The last part of theorem is obvious. 

Figure 1: Cubic graphs of order 10.
Indeed, by Theorem 2, we have the following theorem which relates the domination polynomial and the regularity of a graph $G$.

**Theorem 3.** Let $H$ be a $k$-regular graph, where for every two vertices $u, v \in V(H)$, $N[u] \neq N[v]$. If $D(G, x) = D(H, x)$, then $G$ is also a $k$-regular graph.

There are exactly 21 cubic graphs of order 10 given in Figure 1 (see [3]). Using Theorem 1, the domination number of a connected cubic graph of order 10 is 3. There are just two non-connected cubic graphs of order 10. Clearly, for these graphs, the domination number is also 3. Note that the graph $G_{17}$ is the Petersen graph. For the labeled graph $G_{17}$ in Figure 1, we obtain all dominating sets of size 3 and 4 in the following lemma.

**Lemma 2.** For the Petersen graph $P$, $d(P, 3) = 10$ and $d(P, 4) = 75$.

**Proof.** First, we list all dominating sets of $P$ of cardinality 3, which are the $\gamma$-sets of the labeled Petersen graph (graph $G_{17}$) given in Figure 1:

$D(P, 3) = \{\{1, 3, 7\}, \{1, 4, 10\}, \{1, 8, 9\}, \{2, 4, 8\}, \{2, 5, 6\}, \{2, 9, 10\}, \{3, 5, 9\}, \{3, 6, 10\}, \{4, 6, 7\}, \{5, 7, 8\}\}$. Now, we shall compute $d(P, 4)$. By Lemma 1, it suffices to obtain the dominating sets of cardinality 4 containing one vertex, say the vertex labeled 1. These dominating sets are listed below:

$D_1(P, 4) = \{\{1, 2, 3, 7\}, \{1, 2, 4, 8\}, \{1, 2, 4, 10\}, \{1, 2, 5, 6\}, \{1, 2, 8, 9\}, \{1, 2, 9, 10\}, \{1, 3, 4, 7\}, \{1, 3, 4, 10\}, \{1, 3, 5, 7\}, \{1, 3, 5, 9\}, \{1, 3, 6, 7\}, \{1, 3, 6, 10\}, \{1, 3, 7, 8\}, \{1, 3, 7, 9\}, \{1, 3, 7, 10\}, \{1, 3, 8, 9\}, \{1, 3, 9, 10\}, \{1, 4, 5, 10\}, \{1, 4, 6, 7\}, \{1, 4, 6, 10\}, \{1, 4, 7, 8\}, \{1, 4, 7, 10\}, \{1, 4, 8, 9\}, \{1, 4, 8, 10\}, \{1, 4, 9, 10\}, \{1, 5, 8, 9\}, \{1, 6, 8, 9\}, \{1, 7, 8, 9\}, \{1, 8, 9, 10\}\}$. Therefore by Lemma 1, $d(P, 4) = \frac{10 \times 30}{4} = 75$.

We need the following lemma:

**Lemma 3.** Let $G$ be a cubic graph of order 10. Then the following hold:

(i) $d(G, i) = \binom{10}{i}$, for $i = 7, 8, 9, 10$. 

5
(ii) if $t$ and $s$ are the number of subgraphs isomorphic to $K_4\{e\}$ ($e$ is an edge) and $K_4$ in $G$, respectively, then $d(G, 6) = \binom{10}{6} - (10 - t - 3s)$.

(iii) if $G$ has no subgraph isomorphic to graphs given in Figure 2, then $d(G, 5) = \binom{10}{5} - 60$.

![Figure 2: Graphs illustrated in Lemma 3.](image-url)

**Proof.** (i) It follows from Theorem 2.

(ii) If $G$ is a cubic graph of order 10, then for every $v \in V(G)$, $V(G)\setminus N[v]$ is not a dominating set. Also, if $S \subset V(G)$, $|S| = 6$ and $S$ is not a dominating set, then $S = V(G)\setminus N[v]$, for some $v \in V(G)$. Note that if $G$ has $K_4\{e\}$ as a subgraph, then there are two vertices $u_1$ and $u_2$ such that $G\setminus N[u_1] = G\setminus N[u_2]$. Also if $G$ has $K_4$ as its subgraph, then there are four vertices $u_i$, $1 \leq i \leq 4$ such that $G\setminus N[u_i] = G\setminus N[u_j]$, for $1 \leq i \neq j \leq 4$. Hence we have $d(G, 6) = \binom{10}{6} - (10 - t - 3s)$.

(iii) It suffices to determine the number of 5-subsets which are not dominating set. Suppose that $S \subset V(G)$, $|S| = 5$, and $S$ is not a dominating set for $G$. Thus there exists $v \in V(G)$ such that $N[v]\cap S = \emptyset$. Now, note that for every $x \in V(G)$, $V(G)\setminus (N[x] \cup \{y\})$, where $y \in V(G)\setminus N[x]$ is a 5-subset which is not a dominating set for $G$. Also since none of the graphs given in Figure 2 is a subgraph of $G$, for every two distinct vertices $x$ and $x'$ and any two arbitrary vertices $y \in V(G)\setminus N[x]$ and $y' \in V(G)\setminus N[x']$, we have $V(G)\setminus (N[x] \cup \{y\}) \neq V(G)\setminus (N[x'] \cup \{y'\})$. This implies that the number of 5-subsets of $V(G)$ which are not dominating sets is $10 \times 6 = 60$. So we have $d(G, 5) = \binom{10}{5} - 60$. □

**Corollary 1.** For cubic graphs of order 10, the following hold:
Therefore, if \( G \in \{G_{20}, G_{21}\} \), then \( d(G, 6) = \binom{10}{6} - 7 \).

(ii) If \( G \in \{G_1, G_{18}\} \), then \( d(G, 6) = \binom{10}{6} - 8 \).

(iii) If \( G \in \{G_3, G_5\} \), then \( d(G, 6) = \binom{10}{6} - 9 \).

(iv) For each \( i, 1 \leq i \leq 21 \), if \( i \notin \{1, 3, 5, 18, 20, 21\} \), then \( d(G_i, 6) = \binom{10}{6} - 10 \).

(v) If \( G \in \{G_6, G_7, G_8, G_{10}, G_{17}\} \), then \( d(G, 5) = \binom{10}{5} - 60 \).

**Theorem 4.** The domination polynomial of the Petersen graph \( P \) is:

\[
D(P, x) = x^{10} + \binom{10}{9}x^9 + \binom{10}{8}x^8 + \binom{10}{7}x^7 + \left[\binom{10}{6} - 10\right]x^6 + \left[\binom{10}{5} - 60\right]x^5 + 75x^4 + 10x^3.
\]

**Proof.** The result follows from Lemma 2 and Corollary 1. □

### 3 \( \gamma \)-Sets of cubic graphs of order 10.

In this section, we present all \( \gamma \)-sets of connected cubic graphs \( G_1, G_2, \ldots, G_{18}, G_{19} \) shown in Figure 11. The results here will be useful in studying the \( D \)-equivalence of these graphs in the last section.

\[ 
\Gamma(G_1) = \left\{ \{1, 3, 7\}, \{1, 3, 8\}, \{1, 3, 9\}, \{1, 4, 7\}, \{1, 4, 8\}, \{1, 4, 9\}, \{1, 5, 7\}, \{1, 5, 8\}, \{1, 5, 9\}, \{2, 5, 8\}, \{2, 5, 9\}, \{2, 6, 8\}, \{2, 6, 9\}, \{2, 6, 10\}, \{3, 6, 8\}, \{3, 6, 9\}, \{3, 6, 10\}, \{3, 7, 10\}, \{4, 6, 8\}, \{4, 6, 9\}, \{4, 6, 10\}, \{4, 7, 10\} \right\}. \]

Therefore, \( d(G_1, 3) = |\Gamma(G_1)| = 22 \).

\[ 
\Gamma(G_2) = \left\{ \{1, 3, 8\}, \{1, 4, 8\}, \{1, 4, 9\}, \{1, 4, 10\}, \{1, 5, 8\}, \{2, 5, 8\}, \{3, 6, 8\}, \{3, 6, 9\}, \{3, 6, 10\}, \{3, 7, 10\}, \{4, 6, 9\}, \{5, 6, 9\} \right\}. \]

Therefore, \( d(G_2, 3) = |\Gamma(G_2)| = 12 \).

\[ 
\Gamma(G_3) = \left\{ \{1, 3, 6\}, \{1, 3, 7\}, \{1, 4, 8\}, \{1, 5, 9\}, \{1, 5, 9\}, \{1, 7, 9\}, \{2, 3, 6\}, \{2, 3, 7\}, \{2, 4, 8\}, \{2, 5, 8\}, \{2, 5, 9\}, \{3, 6, 10\}, \{3, 7, 10\}, \{4, 6, 10\}, \{4, 7, 10\}, \{4, 8, 10\}, \{5, 9, 10\} \right\}. \]

Therefore, \( d(G_3, 3) = |\Gamma(G_3)| = 17 \).

\[ 
\Gamma(G_4) = \left\{ \{1, 5, 6\}, \{1, 5, 8\}, \{1, 6, 7\}, \{1, 7, 8\}, \{2, 4, 9\}, \{2, 5, 6\}, \{2, 5, 8\}, \{2, 6, 7\}, \{2, 7, 8\}, \{3, 4, 9\}, \{3, 6, 9\}, \{3, 8, 9\}, \{4, 5, 10\}, \{4, 7, 10\}, \{4, 9, 10\} \right\}. \]

Therefore, \( d(G_4, 3) = |\Gamma(G_4)| = 15 \).
\[ \Gamma(G_5) = \{\{1, 4, 7\}, \{1, 4, 8\}, \{1, 4, 9\}, \{1, 5, 7\}, \{1, 5, 8\}, \{1, 5, 9\}, \{1, 6, 7\}, \{1, 6, 8\}, \{1, 6, 9\}, \\
\{2, 4, 7\}, \{2, 4, 8\}, \{2, 4, 9\}, \{2, 5, 7\}, \{2, 5, 8\}, \{2, 5, 9\}, \{2, 6, 7\}, \{2, 6, 8\}, \{2, 6, 9\}, \{3, 4, 9\}, \\
\{3, 5, 9\}, \{3, 6, 9\}, \{4, 7, 10\}, \{4, 8, 10\}, \{4, 9, 10\}\}. \] Therefore \(d(G_5, 3) = |\Gamma(G_5)| = 24.\]

\[ \Gamma(G_6) = \{\{1, 4, 7\}, \{1, 4, 8\}, \{1, 5, 8\}, \{2, 5, 8\}, \{2, 5, 9\}, \{2, 6, 9\}, \{3, 6, 9\}, \{3, 6, 10\}, \{3, 7, 10\}, \\
\{4, 7, 10\}\}. \] Therefore \(d(G_6, 3) = |\Gamma(G_6)| = 10.\]

\[ \Gamma(G_7) = \{\{1, 4, 8\}, \{2, 5, 8\}, \{2, 5, 9\}, \{3, 6, 9\}, \{3, 7, 10\}, \{4, 7, 10\}\}. \]

Therefore \(d(G_7, 3) = |\Gamma(G_7)| = 6.\]

\[ \Gamma(G_8) = \{\{1, 3, 9\}, \{1, 4, 8\}, \{1, 5, 7\}, \{2, 6, 10\}, \{3, 6, 9\}, \{4, 6, 8\}\}. \]

Therefore \(d(G_8, 3) = |\Gamma(G_8)| = 6.\]

\[ \Gamma(G_9) = \{\{1, 3, 9\}, \{1, 4, 8\}, \{1, 5, 7\}, \{2, 5, 7\}, \{2, 5, 8\}, \{2, 6, 8\}, \{3, 6, 9\}, \{4, 6, 10\}, \{4, 7, 10\}, \\
\{5, 7, 10\}\}. \] Therefore \(d(G_9, 3) = |\Gamma(G_9)| = 10.\]

\[ \Gamma(G_{10}) = \{\{1, 2, 9\}, \{1, 5, 8\}, \{1, 8, 9\}, \{2, 3, 10\}, \{2, 9, 10\}, \{3, 4, 6\}, \{3, 6, 10\}, \{4, 5, 7\}, \{4, 5, 7\}, \\
\{4, 6, 7\}\}. \] Therefore \(d(G_{10}, 3) = |\Gamma(G_{10})| = 10.\]

\[ \Gamma(G_{11}) = \{\{1, 3, 7\}, \{1, 5, 9\}, \{2, 3, 7\}, \{2, 5, 8\}, \{2, 5, 9\}, \{2, 6, 8\}, \{2, 7, 8\}, \{3, 6, 9\}, \{3, 7, 10\}, \\
\{4, 5, 10\}, \{4, 6, 10\}, \{4, 7, 10\}\}. \] Therefore \(d(G_{11}, 3) = |\Gamma(G_{11})| = 12.\]

\[ \Gamma(G_{12}) = \{\{1, 3, 9\}, \{1, 4, 8\}, \{1, 5, 7\}, \{2, 5, 8\}, \{2, 6, 8\}, \{2, 6, 9\}, \{2, 6, 10\}, \{2, 7, 9\}, \{3, 5, 10\}, \\
\{3, 6, 8\}, \{3, 6, 9\}, \{4, 6, 8\}, \{4, 6, 9\}, \{4, 6, 10\}, \{4, 7, 10\}\}. \] Therefore \(d(G_{12}, 3) = |\Gamma(G_{12})| = 15.\]

\[ \Gamma(G_{13}) = \{\{1, 3, 8\}, \{1, 4, 9\}, \{2, 5, 8\}, \{2, 7, 8\}, \{3, 6, 9\}, \{4, 5, 10\}, \{4, 7, 10\}\}. \] Therefore \(d(G_{13}, 3) = |\Gamma(G_{13})| = 8.\]

\[ \Gamma(G_{14}) = \{\{1, 3, 7\}, \{1, 3, 8\}, \{1, 3, 9\}, \{1, 4, 7\}, \{1, 4, 8\}, \{1, 4, 9\}, \{1, 5, 7\}, \{1, 5, 8\}, \{1, 5, 9\}, \\
\{2, 3, 7\}, \{2, 4, 7\}, \{2, 5, 7\}, \{2, 5, 8\}, \{2, 5, 9\}, \{2, 6, 8\}, \{3, 6, 9\}, \{3, 7, 10\}, \{4, 6, 10\}, \\
\{4, 7, 10\}, \{5, 7, 10\}, \{5, 8, 10\}, \{5, 9, 10\}\}. \] Therefore \(d(G_{14}, 3) = |\Gamma(G_{14})| = 22.\]

\[ \Gamma(G_{15}) = \{\{1, 2, 8\}, \{1, 3, 8\}, \{1, 4, 8\}, \{2, 5, 10\}, \{2, 6, 9\}, \{2, 7, 8\}, \{3, 5, 10\}, \{3, 6, 7\}, \{3, 6, 9\}, \} \]
\{4, 5, 10\}, \{4, 6, 9\}, \{4, 7, 10\}\}. Therefore \(|\Gamma(G_{15})| = 12\).

\(\Gamma(G_{16}) = \left\{1, 3, 8\right\}, \{1, 4, 8\}, \{1, 4, 9\}, \{3, 6, 8\}, \{3, 6, 9\}, \{4, 6, 9\}\}. Therefore \(d(G_{16}, 3) = 6\).

\(\Gamma(G_{17}) = \left\{1, 3, 7\right\}, \{1, 4, 10\}, \{1, 8, 9\}, \{2, 4, 8\}, \{2, 5, 6\}, \{2, 9, 10\}, \{3, 5, 9\}, \{3, 6, 10\}, \{4, 6, 7\}, \{5, 7, 8\}\}. Therefore, \(d(G_{17}, 3) = 10\).

\(\Gamma(G_{18}) = \left\{1, 5, 8\right\}, \{1, 5, 9\}, \{2, 5, 8\}, \{2, 5, 9\}, \{2, 6, 7\}, \{2, 6, 8\}, \{2, 6, 9\}, \{2, 6, 10\}, \{3, 5, 8\}, \{3, 5, 9\}, \{3, 6, 7\}, \{3, 6, 8\}, \{3, 6, 9\}, \{3, 6, 10\}, \{4, 5, 8\}, \{4, 5, 9\}\}. Therefore, \(d(G_{18}, 3) = 16\).

\(\Gamma(G_{19}) = \left\{1, 4, 8\right\}, \{1, 5, 8\}, \{2, 4, 7\}, \{2, 5, 8\}, \{2, 5, 9\}, \{2, 6, 9\}, \{2, 7, 9\}, \{3, 5, 10\}, \{3, 6, 9\}, \{3, 6, 10\}, \{3, 7, 10\}, \{4, 7, 10\}, \{5, 8, 10\}\}. Therefore \(|\Gamma(G_{19})| = 13\).

4 \ \textbf{\(\mathcal{D}\)-Equivalence class of the Petersen Graph}

In this section we show that the Petersen graph is \(\mathcal{D}\)-unique.

\textbf{Theorem 5.} The Petersen graph \(P\) is \(\mathcal{D}\)-unique.

\textbf{Proof.} Assume that \(G\) is a graph such that \(D(G, x) = D(P, x)\). Since for every two vertices \(x, y \in V(P)\), \(N[x] \neq N[y]\), by Theorem 3 \(G\) is a 3-regular graph of order 10. Using the \(|\Gamma(G_i)|\) for \(i = 1, \ldots, 21\) in Section 3 we reject some graphs from \([P]\). Since \(d(G_{6}, 3) = d(G_{9}, 3) = d(G_{10}, 3) = d(G_{17}, 3) = 10\), we compare the cardinality of the families of dominating sets of these four graphs of size 4.

\(\mathcal{D}_1(G_6, 4) = \left\{1, 2, 3, 4\right\}, \{1, 2, 3, 10\}, \{1, 2, 4, 7\}, \{1, 2, 4, 8\}, \{1, 2, 4, 9\}, \{1, 2, 5, 8\}, \{1, 2, 5, 9\}, \{1, 2, 6, 9\}, \{1, 2, 9, 10\}, \{1, 3, 4, 6\}, \{1, 3, 4, 7\}, \{1, 3, 4, 8\}, \{1, 3, 5, 8\}, \{1, 3, 6, 8\}, \{1, 3, 6, 9\}, \{1, 3, 6, 10\}, \{1, 3, 7, 10\}, \{1, 3, 8, 10\}, \{1, 4, 5, 7\}, \{1, 4, 5, 8\}, \{1, 4, 6, 7\}, \{1, 4, 6, 8\}, \{1, 4, 6, 9\}, \{1, 4, 7, 8\}, \{1, 4, 7, 9\}, \{1, 4, 7, 10\}, \{1, 4, 8, 9\}, \{1, 4, 8, 10\}, \{1, 5, 6, 8\}, \{1, 5, 6, 9\}, \{1, 5, 7, 8\}, \{1, 5, 8, 9\}, \{1, 5, 8, 10\}, \{1, 6, 8, 9\}, \{1, 8, 9, 10\}\).

Therefore, by Lemma \(1\) \(d(G_6, 4) = \frac{34 \times 10}{4} = 85 > d(P, 4) = 75\).
Therefore, by Lemma 1, \( d(G_{10}, 4) = \frac{34 \times 10}{4} = 85 > d(P, 4) = 75. \)

Now, we obtain the family of all dominating sets of \( G_9 \) of size 4.
\[
\mathcal{D}(G_9, 4) = \bigg\{ \{1, 2, 3, 9\}, \{1, 2, 4, 8\}, \{1, 2, 5, 7\}, \{1, 2, 6, 8\}, \{1, 2, 8, 9\}, \{1, 3, 4, 5\}, \{1, 3, 4, 8\}, \{1, 3, 4, 9\}, \{1, 3, 5, 7\}, \{1, 3, 5, 8\}, \{1, 3, 5, 9\}, \{1, 3, 6, 8\}, \{1, 3, 6, 9\}, \{1, 3, 7, 9\}, \{1, 3, 8, 9\}, \{1, 3, 9, 10\}, \{1, 4, 5, 6\}, \{1, 4, 5, 7\}, \{1, 4, 5, 8\}, \{1, 4, 6, 9\}, \{1, 4, 6, 10\}, \{1, 4, 7, 8\}, \{1, 4, 7, 9\}, \{1, 4, 7, 10\}, \{1, 4, 8, 9\}, \{1, 4, 8, 10\}, \{1, 5, 6, 7\}, \{1, 5, 6, 8\}, \{1, 5, 7, 8\}, \{1, 5, 7, 9\}, \{1, 5, 7, 10\}, \{1, 5, 8, 10\}, \{1, 5, 8, 9\}, \{1, 6, 8, 9\}, \{1, 7, 8, 9\}, \{1, 8, 9, 10\} \bigg\}.
\]

Therefore, \( d(G_9, 4) = 91 > d(P, 4). \)

Hence \( [P] = \{P\}, \) and so the Petersen graph is \( \mathcal{D} \)-unique. \( \square \)

By the arguments in the proof of Theorem 5, we have the following corollary.

**Corollary 2.** (i) The graph \( G_9 \) is \( \mathcal{D} \)-unique,

(ii) \( [G_9] = \{G_6, G_{10}\} \) with the following domination polynomial:
\[
x^{10} + \binom{10}{9}x^9 + \binom{10}{8}x^8 + \binom{10}{7}x^7 + \left(\binom{10}{6} - 10\right)x^6 + \left(\binom{10}{5} - 60\right)x^5 + 85x^4 + 10x^3.
\]
5 \( \mathcal{D} \)-equivalence class of cubic graphs of order 10

In this section, we shall study the \( \mathcal{D} \)-equivalence classes of other cubic graphs of order 10.

We need the following theorem:

**Theorem 6.** (\( \square \)) If a graph \( G \) has \( m \) components \( G_1, \ldots, G_m \), then \( D(G, x) = D(G_1, x) \cdots D(G_m, x) \). \( \square \)

**Corollary 3.** Two graphs \( G_{20} \) and \( G_{21} \) are \( \mathcal{D} \)-equivalence, with the following domination polynomial:

\[
D(G_{20}, x) = D(G_{21}, x) = x^{10} + 10x^9 + 45x^8 + 120x^7 + 203x^6 + 216x^5 + 134x^4 + 36x^3.
\]

**Proof.** Two graphs \( G_{20} \) and \( G_{21} \) are disconnected with two components. In other words \( G_{20} = H \cup K_4 \) and \( G_{21} = H' \cup K_4 \), where \( H \) and \( H' \) are graphs with 6 vertices. It is not hard to see that

\[
D(H, x) = D(H', x) = x^6 + 6x^5 + 15x^4 + 20x^3 + 9x^2.
\]

On the other hand, \( D(K_4, x) = x^4 + 4x^3 + 6x^2 + 4x \). By Theorem 6 we have the result. \( \square \)

**Theorem 7.** The graphs \( G_{12}, G_{13}, G_{14}, G_{16}, \) and \( G_{19} \) in Figure 7 are \( \mathcal{D} \)-unique.

**Proof.** Using \( \gamma \)-sets in Section 3, \( |\Gamma(G_{12})| = 15, |\Gamma(G_{13})| = 7, |\Gamma(G_{14})| = 22, \) and \( |\Gamma(G_{19})| = 13. \)

By comparing these numbers with the cardinality of \( \gamma \)-sets of other 3-regular graphs, we have the result. Now, we consider graph \( G_{16} \). Since \( d(G_{7}, 3) = d(G_{8}, 3) = d(G_{16}, 3) = 6 \), we shall obtain \( d(G_i, 4) \) for \( i = 7, 8, 16 \).

\[
\mathcal{D}_1(G_7, 4) = \left\{ \{1, 2, 3, 4\}, \{1, 2, 4, 8\}, \{1, 2, 4, 9\}, \{1, 2, 4, 10\}, \{1, 2, 5, 8\}, \{1, 2, 5, 9\}, \{1, 2, 6, 8\}, \{1, 2, 6, 10\}, \{1, 3, 4, 6\}, \{1, 3, 4, 7\}, \{1, 3, 4, 8\}, \{1, 3, 5, 7\}, \{1, 3, 5, 8\}, \{1, 3, 6, 7\}, \{1, 3, 6, 8\}, \{1, 3, 6, 9\}, \{1, 3, 7, 10\}, \{1, 3, 8, 10\}, \{1, 4, 5, 8\}, \{1, 4, 6, 8\}, \{1, 4, 6, 9\}, \{1, 4, 6, 10\}, \{1, 4, 7, 8\}, \{1, 4, 7, 9\}, \{1, 4, 7, 10\}, \{1, 4, 8, 9\}, \{1, 4, 8, 10\}, \{1, 5, 6, 9\}, \{1, 5, 7, 9\}, \{1, 5, 8, 9\}, \{1, 6, 8, 9\}, \{1, 8, 9, 10\} \right\}.
\]

Therefore, by Lemma 1 \( d(G_7, 4) = \frac{32 \times 5}{2} = 80. \)
Figure 3: Cubic graphs of order 10 with identical domination polynomial.

$D_1(G_8, 4) = \left\{ \{1, 2, 3, 5\}, \{1, 2, 3, 9\}, \{1, 2, 4, 8\}, \{1, 2, 5, 7\}, \{1, 2, 5, 8\}, \{1, 2, 6, 10\}, \{1, 3, 4, 8\}, \{1, 3, 4, 9\}, \{1, 3, 5, 7\}, \{1, 3, 5, 8\}, \{1, 3, 5, 9\}, \{1, 3, 6, 8\}, \{1, 3, 6, 9\}, \{1, 3, 7, 9\}, \{1, 3, 8, 9\}, \{1, 3, 9, 10\}, \{1, 4, 5, 7\}, \{1, 4, 5, 8\}, \{1, 4, 6, 8\}, \{1, 4, 6, 9\}, \{1, 4, 7, 8\}, \{1, 4, 7, 9\}, \{1, 4, 7, 10\}, \{1, 4, 8, 9\}, \{1, 4, 8, 10\}, \{1, 5, 6, 7\}, \{1, 5, 7, 8\}, \{1, 5, 7, 9\}, \{1, 5, 7, 10\}, \{1, 6, 7, 10\}, \{1, 7, 9, 10\} \right\}$. Therefore, by Lemma [1] $d(G_8, 4) = \frac{32 \times 5}{2} = 80$.

Now, we obtain $d(G_{16}, 4)$.

$D_1(G_{16}, 4) = \left\{ \{1, 2, 3, 8\}, \{1, 2, 4, 5\}, \{1, 2, 4, 7\}, \{1, 2, 4, 8\}, \{1, 2, 4, 9\}, \{1, 2, 5, 6\}, \{1, 2, 5, 7\}, \{1, 2, 5, 8\}, \{1, 2, 6, 7\}, \{1, 2, 7, 8\}, \{1, 2, 7, 9\}, \{1, 2, 8, 10\}, \{1, 3, 4, 8\}, \{1, 3, 4, 9\}, \{1, 3, 5, 8\}, \{1, 3, 5, 9\}, \{1, 3, 6, 8\}, \{1, 3, 6, 9\}, \{1, 3, 7, 8\}, \{1, 3, 8, 9\}, \{1, 3, 8, 10\}, \{1, 4, 5, 8\}, \{1, 4, 5, 9\}, \{1, 4, 5, 10\}, \{1, 4, 6, 8\}, \{1, 4, 6, 9\}, \{1, 4, 6, 10\}, \{1, 4, 7, 9\}, \{1, 4, 7, 10\}, \{1, 4, 8, 9\}, \{1, 4, 8, 10\}, \{1, 4, 9, 10\}, \{1, 5, 6, 10\}, \{1, 5, 7, 10\} \right\}$. 

12
Therefore, by Lemma 1, \(d(G_{16}, 4) = \frac{34 \times 5}{2} = 85\). Hence \([G_{16}] = \{G_{16}\}\).

By the arguments in the proof of Theorem 7, we have the following corollary.

**Corollary 4.** Two graphs \(G_7\) and \(G_8\) are \(D\)-equivalence.

In summary, in this paper we showed that the Petersen graph is \(D\)-unique. Also, we proved that the graphs \(G_2, G_9, G_{11}, G_{12}, G_{13}, G_{14}, G_{15}, G_{16}, G_{17}\), and \(G_{19}\) are \(D\)-unique, and \([G_6] = \{G_6, G_{10}\}\), \([G_7] = \{G_7, G_8\}\), \([G_{20}] = \{G_{20}, G_{21}\}\) (see Figure 3). We are not able to determine the \(D\)-equivalence of \(G_1, G_3, G_4, G_5,\) and \(G_{18}\), but we think that they are \(D\)-unique.

**Acknowledgement.** The first author is indebted to the Institute for Mathematical Research (INSPEM) at University Putra Malaysia (UPM) for the partial support and hospitality during his visit.

**References**

[1] S. Alikhani, Y. H. Peng, Introduction to Domination polynomial of a graphs, Ars Combinatoria, to appear.

[2] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.

[3] G. B. Khosrovshahi, Ch. Maysoori, Tayfeh-Rezaie, A Note on 3-Factorizations of \(K_{10}\), J. Combin. Designs 9 (2001), 379-383.