NEW ANALYTIC RUNNING COUPLING IN QCD:
HIGHER LOOP LEVELS

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The properties of the new analytic running coupling are investigated at the higher loop levels. The expression for this invariant charge, independent of the normalization point, is obtained by invoking the asymptotic freedom condition. It is shown that at any loop level the relevant β function has the universal behaviors at small and large values of the invariant charge. Due to this feature the new analytic running coupling possesses the universal asymptotics both in the ultraviolet and infrared regions irrespective of the loop level. The consistency of the model considered with the general definition of the QCD invariant charge is shown.

1. Introduction

The description of hadron interaction in the infrared region remains an actual problem of Quantum Chromodynamics (QCD). Since the standard perturbation theory provides no definite answer on this problem, a variety of nonperturbative methods is usually invoked for the comprehensive investigation of this matter. The point is that the renormalization group (RG) summation leads to the violation of the proper analytic properties of the relevant physical quantities, that contradicts the general principles of the theory. For instance, the ghost pole appears in the expression for the running coupling at the one-loop level. Therefore, in order to improve this situation one has to involve into consideration the condition of analyticity.

Being based on the first principles of the local Quantum Field Theory, the analytic approach seeks to recover the violated after RG summation proper analytic properties of the relevant physical quantities. Its original ideas were formulated in the framework of Quantum Electrodynamics (QED) in the late 1950's. The basic idea of this approach is an explicit imposition of the causality condition which implies the requirement of the analyticity in the \( q^2 \) variable for the relevant physical quantities. The analytic approach has recently been extended to QCD and applied to the ‘analytization’ of the perturbative series for the QCD observables.

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The term ‘analytization’ means the recovering of the proper analytic properties in the $q^2$ variable by making use of the Källén–Lehmann spectral representation

$$\{A(q^2)\}_\text{an} \equiv \int_0^{\infty} \frac{\varrho(\sigma)}{\sigma + q^2} d\sigma$$

(1)

with the spectral density $\varrho(\sigma)$ determined by the initial (perturbative) expression for a quantity $A(q^2)$:

$$\varrho(\sigma) \equiv \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \left[ A(-\sigma - i\varepsilon) - A(-\sigma + i\varepsilon) \right], \quad \sigma \geq 0.$$  

(2)

The analytic approach has recently been applied to the ‘improvement’ of the $\beta$ function perturbative expansion. In accordance with the model proposed in Refs. 5, 6, the new analytic running coupling (NARC) is the solution of the renormalization group equation for the invariant charge with the analytized $\beta$ function. This running coupling possesses a number of appealing features. Its most important advantages are the following. First of all, the new analytic running coupling has no unphysical singularities. Further, the NARC explicitly involves the ultraviolet (UV) asymptotic freedom with the infrared (IR) enhancement in a single expression. It is worth noting here that there is a number of evidences for such a behavior of the QCD invariant charge. In particular, the recent lattice simulations as well as the solution of the Schwinger–Dyson equations testify to the IR enhancement of the QCD running coupling. Furthermore, it was demonstrated recently that the NARC provides the quark confinement, the one-gluon exchange model being employed. Remarkably, the new model for the analytic running coupling has no additional parameters, i.e., similarly to the perturbative approach, $\Lambda_{QCD}$ remains the basic characterizing parameter of the theory.

In this letter the investigation of the new analytic running coupling is continued. The primary objective is to study the properties of the NARC at the higher loop levels. Since we have only the integral representation for the NARC here, its straightforward investigation turns out to be rather complicated. Nevertheless, by examining the properties of the $\beta$ function corresponding to the NARC one succeeded in the description of the asymptotic behavior of the new analytic running coupling at the higher loop levels.

The layout of the letter is as follows. In Sec. 2 the new analytic running coupling is briefly discussed. By invoking the asymptotic freedom condition the expression for the NARC, independent of the normalization point, is obtained. Further, the conclusions concerning the higher loop and scheme stability of the current approach are drawn. In Sec. 3 the $\beta$ function corresponding to the new analytic running coupling at the higher loop levels is investigated. It is shown that this $\beta$ function has universal behaviors both at the small and large values of the invariant charge, irrespective of the loop level considered. This results in the universal asymptotics of the new analytic running coupling both in the UV and IR regions at any loop level. In Sec. 4 the compatibility of the new model for the analytic running coupling
with the general definition of the QCD invariant charge is shown. The ways of the analyticity requirement implementation are discussed. In Sec. 5 the obtained results are summarized.

2. New Analytic Running Coupling in QCD

The complementation of the $\beta$ function perturbative expansion with the analyticity requirement leads to the analytized RG equation for the new analytic running coupling (see Refs. 5, 7 for the details). At the $\ell$-loop level this equation acquires the form

$$\frac{d}{d \ln \mu^2} \ln \begin{bmatrix} N_{\tilde{\alpha}}^{\text{an}}(\mu^2) \end{bmatrix} = - \left\{ \sum_{j=0}^{\ell-1} B_j \begin{bmatrix} \tilde{\alpha}_s^{\text{an}}(\mu^2) \end{bmatrix}^{j+1} \right\}_{\text{an}}, \quad B_j = \frac{\beta_j}{\beta_0^{j+1}}. \tag{3}$$

Here $N_{\alpha_{\text{an}}}^{\text{an}}(\mu^2)$ is the new analytic running coupling at the $\ell$-loop level, $\alpha_s^{\text{an}}(\mu^2)$ denotes the $\ell$-loop perturbative running coupling, $\tilde{\alpha}(\mu^2) = \alpha(\mu^2) \beta_0/(4\pi)$, $\beta_0 = 11 - 2n_f/3$ and $\beta_1 = 102 - 38n_f/3$ are the standard coefficients for the $\beta$ function expansion, and $n_f$ is the number of active quarks. At the one-loop level Eq. (3) can be integrated explicitly with the result

$$N_{\alpha_{\text{an}}}^{\text{an}}(q^2) = \frac{4\pi}{\beta_0} \frac{z - 1}{z \ln z}, \quad z = \frac{q^2}{\Lambda^2}. \tag{4}$$

The properties of the one-loop NARC (4) have been studied in details in Ref. 8. It is worth to emphasize here that the new analytic running coupling (4) possesses the following significant feature

$$N_{\alpha_{\text{an}}}^{\text{an}}(q^2) = \frac{A^2}{q^2} N_{\tilde{\alpha}_{\text{an}}}^{\text{an}} \left( \frac{A^4}{q^2} \right). \tag{5}$$

Recently it has been shown that this symmetry of the QCD invariant charge precisely corresponds to the conformal inversion symmetry of the instanton size distribution. In particular, this implies that one can also derive the expression (4) for the QCD running coupling proceeding from the entirely different motivations. It is interesting to note also that the relation

$$\frac{N_{\alpha_{\text{an}}}^{\text{an}}(q^2)}{N_{\tilde{\alpha}_{\text{an}}}^{\text{an}}(q^2)} + \frac{N_{\beta_{\text{an}}}^{\text{an}}(\Lambda^4/q^2)}{N_{\alpha_{\text{an}}}^{\text{an}}(\Lambda^4/q^2)} = -1 \tag{6}$$

holds for all $q^2 > 0$ in the framework of the model proposed. Here $N_{\alpha_{\text{an}}}^{\text{an}}(q^2)$ is the invariant charge and $N_{\beta_{\text{an}}}^{\text{an}}$ denotes the relevant $\beta$ function (see Eq. (4) further).

Let us turn to the higher loop levels. From the very beginning one should note that the solution of Eq. (3) is determined up to a constant factor due to the logarithmic derivative on its left-hand side. In previous studies\(^a\) this problem has

\(^a\) In Ref. 8 the factor $\sigma_0/z_0$ has been taken out from the exponent of Eq. (7) due to the property $\int_0^\infty R^{\ell}(\sigma) \sigma^{-1} d\sigma = 1$ for all $\ell$.
been eliminated by normalization of the solution of Eq. (3) to its value at a point \( q_0^2 \):

\[
\frac{\tilde{\alpha}_{\text{an}}^{(\ell)}(q^2)}{\tilde{\alpha}_{\text{an}}^{(1)}(q^2)} = \frac{\alpha_{\text{an}}^{(\ell)}(q_0^2)}{\alpha_{\text{an}}^{(1)}(q_0^2)} \exp\left[\int_0^\infty \Delta R^{(\ell)}(\sigma) \ln\left(\frac{1 + \sigma/z}{1 + \sigma/z_0}\right) \frac{d\sigma}{\sigma}\right],
\]

(7)

where

\[
\Delta R^{(\ell)}(\sigma) = \lim_{\epsilon \to +0} \frac{\ell-1}{2\pi i} \sum_{j=0}^{\ell-1} B_j \left\{ \left[ \tilde{\alpha}_{\text{an}}^{(\ell)}(-\sigma - i\epsilon) \right]^{j+1} - \left[ \tilde{\alpha}_{\text{an}}^{(\ell)}(-\sigma + i\epsilon) \right]^{j+1} \right\}.
\]

(8)

However, in some cases it turns out to be more convenient to deal with the explicit expression for the running coupling independent of the normalization point.

In this letter we propose a simple physical method for removing the ambiguity mentioned above. Indeed, this can easily be achieved by involving the condition of the asymptotic freedom, namely \( \tilde{\alpha}_{\text{an}}^{(\ell)}(q^2) \to \tilde{\alpha}_{\text{an}}^{(1)}(q^2) \) when \( q^2 \to \infty \) (in fact, this has already been used in Eq. (3)). It is worth noting that this method does not violate the renormalization invariance of Eq. (7). In general, one becomes able to derive the integral representation for the \( \ell \)-loop new analytic running coupling, independent of the normalization point, by invoking the similar condition \( \tilde{\alpha}_{\text{an}}^{(\ell)}(q^2) \to \tilde{\alpha}_{\text{an}}^{(1)}(q^2) \) when \( q^2 \to \infty \). For this purpose let us consider the ratio of the \( \ell \)-loop new analytic running coupling (1) normalized at a point \( q_0^2 \) to the one-loop NARC written in the form (1) and normalized at the same point \( q_0^2 \):

\[
\frac{\tilde{\alpha}_{\text{an}}^{(\ell)}(q^2)}{\tilde{\alpha}_{\text{an}}^{(1)}(q^2)} = \frac{\alpha_{\text{an}}^{(\ell)}(q_0^2)}{\alpha_{\text{an}}^{(1)}(q_0^2)} \exp\left[\int_0^\infty \Delta R^{(\ell)}(\sigma) \ln\left(\frac{1 + \sigma/z}{1 + \sigma/z_0}\right) \frac{d\sigma}{\sigma}\right],
\]

(9)

where

\[
\Delta R^{(\ell)}(\sigma) = R^{(\ell)}(\sigma) - R^{(1)}(\sigma).
\]

Proceeding to the limit \( q_0^2 \to \infty \), we arrive at the expression for the \( \ell \)-loop new analytic running coupling

\[
\alpha_{\text{an}}^{(\ell)}(q^2) = \frac{4\pi}{\beta_0} \frac{z - 1}{z \ln z} \exp\left[\int_0^\infty \Delta R^{(\ell)}(\sigma) \ln\left(1 + \frac{\sigma}{z}\right) \frac{d\sigma}{\sigma}\right].
\]

(10)

It is worth noting that there is the integral representation of the Källén–Lehmann type for the new analytic running coupling

\[
\tilde{\alpha}_{\text{an}}^{(\ell)}(q^2) = \int_0^\infty \frac{\tilde{\alpha}^{(\ell)}(\sigma)}{\sigma + z} d\sigma,
\]

(11)

where

\[
\tilde{\alpha}^{(\ell)}(\sigma) = \tilde{\alpha}^{(1)}(\sigma) \exp\left[\int_0^\infty \Delta R^{(\ell)}(\zeta) \ln\left|1 - \frac{\zeta}{\sigma}\right| d\zeta\right] \times \left[\cos \psi^{(\ell)}(\sigma) + \frac{\ln \sigma}{\pi} \sin \psi^{(\ell)}(\sigma)\right],
\]

(12)

\[b\] In particular, this was used in Ref. 6 when evaluating the normalization coefficients.
is the $\ell$-loop spectral density,

$$N^{(1)}(\sigma) = \left(1 + \frac{1}{\sigma}\right) \frac{1}{\ln^2 \sigma + \pi^2}$$

(13)

is the one-loop spectral density, and

$$\psi^{(\ell)}(\sigma) = \pi \int_\sigma^\infty \Delta R^{(\ell)}(\zeta) \frac{d\zeta}{\zeta}.$$  

(14)

In the exponent of Eq. (12) the principle value of the integral is assumed. However, in the practical use the expression (10) turns out to be more convenient (at least, beyond the one-loop level).

One of the important features of the new analytic running coupling is the absence of unphysical singularities at any loop level. Furthermore, it involves both the asymptotic freedom behavior and the IR enhancement in a single expression. This feature enables one to describe a broad range of physical phenomena including both the standard perturbative and the intrinsically nonperturbative ones (see Ref. 6 for the details). In particular, it has been shown 5, 12 that in the framework of the one-gluon exchange model the new analytic running coupling (4) explicitly leads to the rising at large distances static quark-antiquark potential. Let us emphasize here that the additional parameters are not introduced in the theory. The detailed description of the properties and advantages of the NARC (4) is given in the papers.

![Fig. 1. The new analytic running coupling at different loop levels, $z = q^2/\Lambda^2$.](image)

The Figure 1 shows the new analytic running coupling (10) at the one-, two- and three-loop levels. It is obvious that NARC possesses the higher loop stability. Thus, the curves corresponding to the two-, and three-loop levels are practically...
indistinguishable. Proceeding from this one can also draw the conclusion concerning the scheme stability of the current approach. In particular, in Fig. 1 the curve corresponding to the three-loop new analytic running coupling \( \tilde{\alpha}_3^{(3)} \) is plotted by making use of the coefficient \( \beta_2 = \frac{2857}{2} - \frac{5033}{18} + \frac{325}{54} n_f \) computed in the \( \overline{\text{MS}} \) scheme. The account of the third term on the right-hand side of Eq. (3) does not lead to a valuable quantitative variation of its solution in comparison with the two-loop approximation. Therefore it is clear that using the coefficient \( \beta_2 \) computed in another subtraction scheme does not lead to significant variation of the solution to Eq. (3) in comparison with the considered case of the \( \overline{\text{MS}} \) scheme. This statement follows also from the fact that the contribution of every subsequent term on the right-hand side of Eq. (3) is substantially suppressed by the contributions of the preceding ones.

Since we have only the integral representation for the new analytic running coupling at the higher loop levels, its straightforward investigation becomes rather complicated. Nevertheless, the examining of the \( \beta \) function corresponding to the NARC enables one to elucidate a number of important questions, in particular, the asymptotic behavior of the new analytic running coupling.

3. The \( \beta \) Function: Higher Loop Levels

In the previous letter the \( \beta \) function

\[
\beta(a) = \frac{d \ln \alpha(\mu^2)}{d \ln \mu^2} \tag{15}
\]

corresponding to the one-loop new analytic running coupling \( \tilde{\alpha}_1^{(1)} \) has been derived and studied in details. Due to the explicit expression for the NARC at the one-loop level, one succeeded in performing the relevant investigation manifestly. Thus, the \( \beta \) function corresponding to the one-loop NARC \( \tilde{\alpha}_1^{(1)} \) acquires the form

\[
\tilde{\alpha}_1^{(1)}(a) = 1 - N(a)/a, \quad a(\mu) \equiv \tilde{\alpha}_1^{(1)}(\mu^2), \tag{16}
\]

where the function \( N(a) \)

\[
N(a) = \begin{cases} 
N_0(a), & 0 < a \leq 1, \\
N_{-1}(a), & 1 < a,
\end{cases}
\]

\[
N_k(a) = -a W_k \left[ -\frac{1}{a} \exp \left( -\frac{1}{a} \right) \right] \tag{17}
\]

is defined in terms of the many-valued Lambert \( W \) function (see Ref. 8 for the details). Figure 2 presents the \( \beta \) function and the result corresponding to the one-loop perturbative running coupling \( \alpha_s^{(1)}(q^2) = 1/\ln z \), namely \( \beta_s^{(1)}(a) = -a \).

\( ^c \) At least, in schemes that do not have unnaturally large expansion coefficients (see Ref. 15 and references therein for the detailed discussion of this matter).

\( ^d \) In Ref. 8 the definition \( \beta(a) = d \alpha(\mu^2)/d \ln \mu^2 \) was used.

\( ^e \) In definitions (14) and (15) \( k \) denotes the branch index of the Lambert \( W \) function.
The investigation of the properties of the function $N(a)$ defined in Eq. (17) enables one to find the explicit asymptotic behavior of the $\beta$ function (16). Thus, for small values of the running coupling $\beta_{\text{an}}^{(1)}$ Eq. (16) coincides with the well-known perturbative result

$$\beta_{\text{an}}^{(1)}(a) = -a + O \left[ a^{-1} \exp \left( -\frac{1}{a} \right) \right], \quad a \to 0_+.$$  

(19)

It is worth noting here that the second term in Eq. (19) points to the intrinsically nonperturbative nature of the $\beta$ function (16). For large values of $a$ one has

$$\beta_{\text{an}}^{(1)}(a) = -1 + O \left[ \frac{\ln(\ln a)}{\ln^2 a} \right], \quad a \to \infty.$$  

(20)

Such a behavior of the $\beta$ function provides the IR enhancement of the invariant charge, namely (up to a logarithmic factor) $\alpha(q^2) \sim 1/z$, when $z \to 0$. In turn, this leads in a straightforward way to the confining static quark–antiquark potential (see Refs. 5, 12 for the details). Therefore, at the one-loop level the $\beta$ function (16) explicitly incorporates the UV asymptotic freedom and the IR enhancement of the running coupling in a single expression.

Let us proceed to the higher loop levels. As it has been mentioned in the previous section, here we have only the integral representation for the new analytic running coupling (10). This fact significantly complicates the investigation and leads to the necessity of applying the numerical methods. It is worth to mention here again that we originate in the standard perturbative expansion for the $\beta$ function and complement it with the analyticity requirement. The results of the computations corresponding to the two-, and three-loop levels are shown in Figs. 3 and 4, respectively. Figure 3 presents the $\beta$ function $\beta_{\text{an}}^{(2)}(a)$ corresponding to the two-loop new
analytic running coupling $\beta_{\text{an}}^{(2)}(a)$ together with the respective perturbative result $\beta_{\text{s}}^{(2)}(a) = -a - B_1 a^2$. Figure 3 shows the analogous functions at the three-loop level, namely $\beta_{\text{an}}^{(3)}(a)$ and $\beta_{\text{s}}^{(3)}(a) = -a - B_1 a^2 - B_2 a^3$. It is clear from the figures 2, 3 and 4 that the $\beta$ function corresponding to the new analytic running coupling coincides with its perturbative analog in the region of small values of the invariant charge at any loop level. In other words, in the framework of the model in hand the complete recovering of the perturbative limit in the UV region takes place.

Let us turn now to the asymptotics of the $\beta$ function $\beta_{\text{an}}^{(\ell)}(a)$ at large values of the running coupling. Figure 4 presents the $\beta$ functions $\beta_{\text{an}}^{(\ell)}(a)$, $\ell = 1, 2, 3$.
Fig. 5. The $\beta$ function corresponding to the new analytic running coupling at different loop levels (solid curves). The one-loop perturbative result is shown as the dot-dashed curve.

and the one-loop perturbative result $\beta^{(1)}_a(a) = -a$. This figure clearly shows the perturbative limit at small $a$, as well as the universal asymptotic behavior of the $\beta$ function corresponding to the NARC $\beta^{(N)}_a(a) \to -1$, when $a \to \infty$.

The latter statement can also be proved in an independent way. Indeed, due to the IR enhancement of the new analytic running coupling, the value of the right-hand side of Eq. (3) (it is nothing but the $\beta$ function at the relevant loop level) when $\mu^2 \to 0$ corresponds to the limit $\beta^{(N)}_a(a) \to \infty$. One can show that

$$\lim_{q^2 \to 0} \left\{ \beta^{(N)}_a(q^2) \right\} = 1,$$

and

$$\lim_{q^2 \to 0} \left\{ \left[ \beta^{(N)}_a(q^2) \right]^{j+1} \right\} = 0 \quad (j \geq 1 \text{ is a positive integer number}),$$

irrespective of the loop level. Therefore, we infer that at any loop level the $\beta$ function corresponding to the new analytic running coupling tends to the universal limit

$$\lim_{a \to \infty} \beta^{(N)}_a(a) = -1. \quad (21)$$

As it has been mentioned above, such a behavior of the $\beta$ function leads to the IR enhancement of the invariant charge, namely $\alpha(q^2) \sim 1/z$, when $z \to 0$.

Thus, the $\beta$ function corresponding to the new analytic running coupling has the universal asymptotic behaviors both at the small ($\beta^{(\infty)}_a(a) \simeq -a$) and large ($\beta^{(\infty)}_a(a) \simeq -1$) values of the invariant charge irrespective of the loop level. Therefore, the new analytic running coupling possesses the universal asymptotics both in the UV ($\beta^{(\infty)}_a(q^2) \sim 1/\ln z$) and IR ($\beta^{(\infty)}_a(q^2) \sim 1/z$) regions at any loop level. In particular, this implies that using the new analytic running coupling (10) at the higher loop levels will also results in the confining quark-antiquark potential.
4. Discussion

In general, the QCD invariant charge is defined as the product of the corresponding Green functions and vertexes. For example, in the transverse gauge the following definition takes place, where \( \alpha(q^2) \) is the QCD running coupling, \( \alpha(\mu^2) \) denotes its value at a normalization point \( \mu^2 \), \( G(q^2) \) and \( g(q^2) \) are dimensionless gluon and ghost propagators, respectively. It is essential to note here that the Green functions possess the proper analytic properties in the \( q^2 \) variable (namely, there is the only left cut \( q^2 \leq 0 \)) before the RG summation. Indeed, for the considering in this letter massless case a Green function can be represented as the perturbative power series in \( \alpha(\mu^2) \) with the coefficients being a polynomial in \( \ln(q^2/\mu^2) \). Obviously, in this case the integral representation of the Källén–Lehmann type must hold for any finite order of the perturbative expansion for the Green function. The RG summation leads to violation of such a properties of the Green functions (e.g., in the simplest one-loop case the ghost pole appears). As it has already been mentioned above, the analytic approach to QCD seeks to recover the proper analytic properties of the relevant quantities. However, for the consistency of involving the analyticity condition with the definition of the invariant charge (22), one has to apply the analytization procedure to the logarithm of the Green function. Really, if the spectral function of the Källén–Lehmann representation for the Green function \( G \) is of a fixed sign (that is true for the leading orders of perturbation theory), then \( G \) has no nulls in the complex \( q^2 \) plane. Hence, its logarithm, \( \ln G \), can also be represented as the spectral integral of the Källén–Lehmann type.

In the framework of perturbative approach the considered above Green functions have the following form: \( g(q^2) = (1/\ln z)^{d_g} \) and \( G(q^2) = (1/\ln z)^{d_G} \), where \( z = q^2/\Lambda^2 \), \( d_g \) and \( d_G \) denote the corresponding anomalous dimensions. For the latter the relationship \( 2d_g + d_G = 1 \) holds, that plays the crucial role in the definition (22). Applying the analytization procedure to these functions in the described above way, one arrives at the following result: \( g_{an}(q^2) = [(z - 1)/(z \ln z)]^{d_g} \) and \( G_{an}(q^2) = [(z - 1)/(z \ln z)]^{d_G} \). Therefore, in this case the invoking the analyticity condition does not affect the definition of the invariant charge (22). It is interesting to note here that the similar situation takes place in the nonperturbative \( a \)-expansion method also. Thus, we infer that the NARC (4) is consistent with the general definition of the QCD invariant charge.

It is worth to mention that there are different methods of removing of the unphysical singularities from the running coupling in the framework of the analytic approach to QCD. In original work the analytization procedure was applied to the perturbative invariant charge in a straightforward way (see Ref. 18 also). The model in hand is based on the complementation of the \( \beta \) function perturbative

\footnote{One should note that, unlike the case of the QED photon propagator, the relevant spectral function here can’t be directly identified with the spectral density.}
expansion with the analyticity condition. In both cases the additional parameters are not introduced into the theory, the models being the ‘minimal’ ones in this sense. Of course, in the UV region these models have identical behavior determined by the asymptotic freedom. However, there is a qualitative distinction between them in the IR region (see discussion in Ref. 7 also).

Let us note here that in equation (3) for the new analytic running coupling the analytization of its right-hand side as a whole is assumed. In turn, this results in the representation of the corresponding $\beta$ function as a non-power series. As it has already been shown,\textsuperscript{19,20} it is this way of the analyticity requirement involving that leads to the stability of obtaining results in a broad range of energies. In particular, such an implementation of the analytization procedure ultimately leads to the universal IR behavior of the NARC.

As it was mentioned above, the NARC incorporates the IR enhancement with the UV asymptotic freedom in a single expression. The recent results of the nonperturbative studies (namely, lattice simulations\textsuperscript{9,10} and Schwinger–Dyson equations\textsuperscript{11}) testify to such a behavior of the QCD invariant charge. For the completeness of the pattern let us also note here that these nonperturbative methods, being the matter of contemporary comprehensive investigations, provide no unique point of view on the IR behavior of the QCD invariant charge.

5. Conclusions

The properties of the new analytic running coupling are studied at the higher loop levels. By making use of the asymptotic freedom condition the expression for the NARC, independent of the normalization point, is obtained. The conclusions about the loop and scheme stability of the current approach are drawn. The $\beta$ function corresponding to the NARC is constructed and examined. It is shown that the behaviors of this $\beta$ function at both its key asymptotics (namely, when $\alpha \to 0$ and when $\alpha \to \infty$) are the same at any loop level. This results, irrespective of the loop level, in an universal asymptotics of the new analytic running coupling. Namely, its UV behavior ($\bar{\alpha}_{an}(\ell, q^2) \sim 1/\ln z$, when $q^2 \to \infty$) is determined by the asymptotic freedom, and there is the IR enhancement of the new analytic running coupling ($\bar{\alpha}_{an}(\ell, q^2) \sim 1/z$, when $q^2 \to 0$). The consistency of the model considered with the general definition of the QCD invariant charge is shown.

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