Diffusion in Flashing Periodic Potentials

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Abstract

The one-dimensional overdamped Brownian motion in a symmetric periodic potential modulated by external time-reversible noise is analyzed. The calculation of the effective diffusion coefficient is reduced to the mean first passage time problem. We derive general equations to calculate the effective diffusion coefficient of Brownian particles moving in arbitrary supersymmetric potential modulated: (i) by external white Gaussian noise and (ii) by Markovian dichotomous noise. For both cases the exact expressions for the effective diffusion coefficient are derived. We obtain acceleration of diffusion in comparison with the free diffusion case for fast fluctuating potentials with arbitrary profile and for sawtooth potential in case (ii). In this case the parameter region where this effect can be observed is given. We obtain also a finite net diffusion in the absence of thermal noise. For rectangular potential the diffusion slows down in comparison with the case when particles diffuse freely, for all parameters of noise and of potential.

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I. INTRODUCTION

Recently a considerable amount of analysis has been devoted to investigating transport of Brownian particles in spatially periodic stochastic structures, such as Josephson junctions\(^1\), Brownian motors\(^2\) and molecular motors\(^3\). Specifically there has been great interest in studying influences of symmetric forces on transport properties, and in calculating the effective diffusion coefficient in the overdamped limit in particular\(^2, 4, 6, 7\). Analytical results were obtained in arbitrary fixed periodic potential, tilted periodic potentials, symmetric periodic potentials modulated by white Gaussian noise, and in supersymmetric potentials\(^4, 6, 7, 8, 10, 11\). The acceleration of diffusion in comparison with the free diffusion was obtained in Refs.\(^4, 6, 7, 11\). At thermal equilibrium there is not net transport of Brownian particles, while away from equilibrium, the occurrence of a current (ratchet effect) is observed generically. Therefore, the absence rather the presence of net flow of particles in spite of a broken spatial symmetry is the very surprising situation away from thermal equilibrium, as stated in Refs.\(^2, 5\). Moreover, the problem of sorting of Brownian particles by enhancement of their effective diffusion coefficient has been increasingly investigated in the last years both from experimental\(^12, 13, 14\) and theoretical point of view\(^4, 15\). Specifically the enhancement of diffusion in symmetric potentials was investigated in Refs.\(^4, 14\).

Motivated by these studies and by the problem of dopant diffusion acceleration in semiconductors physics\(^16\), we try to understand how nonequilibrium symmetrical correlated forces influence thermal systems when potentials are symmetric, and if there are new features which characterize the relaxation process in symmetric potentials. This is done by using a fluctuating periodic potential satisfying the supersymmetry criterion\(^5\), and with a different approach with respect to previous theoretical investigations (see review of P. Reimann in Ref.\(^2\)). Using the analogy between a continuous Brownian diffusion at large times and the "jump diffusion" model\(^10, 11\), we reduce the calculation of effective diffusion coefficient \(D_{eff}\) to the first passage time problem. We consider potentials modulated by external white Gaussian noise and by Markovian dichotomous noise. For the first case we derive the exact formula of \(D_{eff}\) for arbitrary potential profile. The general equations obtained for randomly switching potential are solved for the sawtooth and rectangular periodic potential, and the exact expression of \(D_{eff}\) is derived without any assumptions on the intensity of driving white
Gaussian noise and switchings mean rate of the potential.

II. FAST FLUCTUATING PERIODIC POTENTIAL

The effective diffusion coefficient in fast fluctuating sawtooth potential was first investigated and derived in Ref. [6]. In papers [7] we generalized this result to the case of arbitrary potential profiles. We consider the following Langevin equation

\[ \frac{dx}{dt} = -\frac{dU(x)}{dx} \cdot \eta(t) + \xi(t), \]

(1)

where \( x(t) \) is the displacement in time \( t \), \( \xi(t) \) and \( \eta(t) \) are statistically independent Gaussian white noises with zero means and intensities \( 2D \) and \( 2D_\eta \), respectively. Further we assume that the potential \( U(x) \) satisfies the supersymmetry criterion [5]

\[ E - U(x) = U\left(x - \frac{L}{2}\right), \]

(2)

where \( L \) is the spatial period of the potential (see Fig. 1). Following Ref. [8] and because we have \( \langle x(t) \rangle = 0 \), we determine the effective diffusion coefficient as the limit

\[ D_{\text{eff}} = \lim_{t \to \infty} \frac{\langle x^2(t) \rangle}{2t}. \]

(3)

To calculate the effective diffusion constant we use the "jump diffusion" model [10, 11]

\[ \tilde{x}(t) = \sum_{i=1}^{n(0,t)} q_i, \]

(4)

where \( q_i \) are random increments of jumps with values \( \pm L \) and \( n(0, t) \) denotes the total number of jumps in the time interval \( (0, t) \). In the asymptotic limit \( t \to \infty \) the "fine structure" of
a diffusion is unimportant, and the random processes \( x(t) \) and \( \tilde{x}(t) \) become statistically equivalent, therefore \( \langle x^2(t) \rangle \simeq \langle \tilde{x}^2(t) \rangle \). Because of the supersymmetry of potential \( U(x) \) the probability density reads \( P(q) = \frac{\delta(q-L) + \delta(q+L)}{2} \). From Eq. (1) we arrive at

\[
D_{\text{eff}} = \frac{L^2}{2\tau},
\]

where \( \tau = \langle \tau_j \rangle \) is the mean first-passage time (MFPT) for Brownian particle with initial position \( x = 0 \) and absorbing boundaries at \( x = \pm L \). In fluctuating periodic potentials therefore the calculation of \( D_{\text{eff}} \) reduces to the MFPT problem. Solving the equation for the MFPT of Markovian process \( x(t) \) we obtain the exact formula for \( D_{\text{eff}} \)

\[
D_{\text{eff}} = D \left[ \frac{1}{L} \int_0^L \frac{dx}{\sqrt{1 + D_\eta[U'(x)]^2/D}} \right]^{-2}.
\]

From Eq. (6), \( D_{\text{eff}} > D \) for an arbitrary potential profile \( U(x) \), therefore we have always the enhancement of diffusion in comparison with the case \( U(x) = 0 \). We emphasize that the value of diffusion constant does not depend on the height of potential barriers, as for fixed potential [8], but it depends on its gradient \( U'(x) \). The dependencies of effective diffusion constant \( D_{\text{eff}} \) on the intensity \( D_\eta \) of the modulating white noise are plotted in Fig. 2 for sawtooth, sinusoidal and piece-wise parabolic potential profiles.

![Fig. 2: Enhancement of diffusion in fast fluctuating periodic potential.](image)

**III. RANDOMLY SWITCHING PERIODIC POTENTIAL PROFILE**

Now we consider Eq. (1) where \( \eta(t) \) is a Markovian dichotomous noise, which takes the values \( \pm 1 \) with switchings mean rate \( \nu \). Thus, we investigate the Brownian diffusion in a
supersymmetric periodic potential flipping between two configurations $U(x)$ and $-U(x)$. In the "overturned" configuration the maxima of the potential become the minima and vice versa. In accordance with Eq. (2) we can rewrite Eq. (1) as

$$\frac{dx}{dt} = -\frac{\partial}{\partial x} U \left( x + \frac{L}{4} [\eta(t) - 1] \right) + \xi(t),$$

and the non-Markovian process $x(t)$ has Markovian dynamics between flippings. Because of supersymmetric potential and time-reversible Markovian dichotomous noise the ratchet effect is absent: $\langle \dot{x} \rangle = 0$. All Brownian particles are at the origin at $t = 0$ and the "jump diffusion" model (4) and (5) is used. The probability density of random increments $q_i$ is the same of previous case and the distribution of waiting times $t_j$ reads

$$w(t) = \frac{w_+(t) + w_-(t)}{2},$$

where $w_+(t)$ and $w_-(t)$ are the first passage time distributions for the configuration of the potential with $\eta(0) = +1$ and $\eta(0) = -1$ respectively. In accordance with Eq. (8), $\tau$ is the semi-sum of the MFPTs $\tau_+$ and $\tau_-$ corresponding to the probability distributions $w_+(\tau)$ and $w_-(\tau)$. The exact equations for the MFPTs of Brownian diffusion in randomly switching potentials, derived from the backward Fokker-Planck equation, are

$$D\tau''_+ - U''(x) \tau'_+ + \nu (\tau_+ - \tau_-) = -1,$$

$$D\tau''_- + U''(x) \tau'_- + \nu (\tau_+ - \tau_-) = -1,$$

where $\tau_+(x)$ and $\tau_-(x)$ are the MFPTs for initial values $\eta(0) = +1$ and $\eta(0) = -1$ respectively, with the starting position at the point $x$. We consider the initial position at $x = 0$ and solve Eqs. (2) with the absorbing boundaries conditions $\tau_\pm (\pm L) = 0$. Finally we obtain the general equations to calculate the effective diffusion coefficient

$$\theta'' - f(x) \int_0^x f(y) \theta'(y) dy - \frac{2\nu}{D} \theta = \frac{xf(x)}{D},$$

$$\frac{D_{\text{eff}}}{D} = \left[ 1 + \frac{2D}{L} \int_0^L \left( 1 - \frac{x}{L} \right) f(x) \theta'(x) dx \right]^{-1},$$

where $\theta(x) = [\tau_+(x) - \tau_-(x)]/2$. Equations (10) and (11) solve formally the problem.
A. Switching sawtooth periodic potential

In such a case (see Fig. 3) from Eqs. (10) and (11), after algebraic rearrangements, we obtain the following exact result

\[
\frac{D_{\text{eff}}}{D} = \frac{2\alpha^2 (1 + \mu) (A_\mu + \mu + \mu^2 \cosh 2\alpha)}{2\alpha^2 \mu^2 (1 + \mu) + 2\mu (A_{\mu 1}) \sinh^2 \alpha + 4\alpha \mu (A_\mu) \sinh \alpha + 8 (A_{\mu 2}) \sinh^2 (\alpha/2)},
\]

where \(A_\mu = 1 - 3\mu + 4\mu \cosh \alpha\), \(A_{\mu 1} = 7 - \mu + 2\alpha^2 \mu^2\), and \(A_{\mu 2} = 1 - 6\mu + \mu^2\). Here \(\alpha = \sqrt{(E/D)^2 + \nu L^2/(2D)}\) and \(\mu = \nu L^2 D/(2E^2)\) are dimensionless parameters, \(E\) is the potential barrier height. The Eq. (12) was derived without any assumptions on the intensity of white Gaussian noise, the mean rate of switchings and the values of the potential profile parameters. We introduce two new dimensionless parameters with a clear physical meaning: \(\beta = E/D\), and \(\omega = \nu L^2/(2D)\), which is the ratio between the free diffusion time through the distance \(L\) and the mean time interval between switchings. The parameters \(\alpha\) and \(\mu\) can be expressed in terms of \(\beta\) and \(\omega\) as

\[
\alpha = \sqrt{\beta^2 + \omega}, \quad \mu = \omega/\beta^2.
\]

Let us analyze the limiting cases. At very rare flippings \((\omega \to 0)\) we have \(\alpha \approx \beta, \mu \to 0\) and Eq. (12) gives

\[
\frac{D_{\text{eff}}}{D} \approx \frac{\beta^2}{4 \sinh^2 (\beta/2)},
\]

which coincides with the result obtained for the fixed periodic potential. For very fast switchings \((\omega \to \infty)\) the Brownian particles “see” the average potential, i.e. 
\([U(x) + (-U(x))]/2 = 0\), and we obtain diffusion in the absence of potential. If we put in Eq. (12) \(\alpha \approx \sqrt{\omega [1 + \beta^2/(2\omega)]} \to \infty\) and \(\mu = \omega/\beta^2 \to \infty\), we find

\[
\frac{D_{\text{eff}}}{D} \approx 1 + \frac{\beta^2}{\omega}.
\]

FIG. 3: Switching sawtooth periodic potential.
The normalized effective diffusion coefficient $D_{\text{eff}}/D$ as a function of the dimensionless mean rate of potential switching $\omega$, for different values of the dimensionless height of potential barriers $\beta$, is shown in Fig. 4. A non-monotonic behavior for all values of $\beta$ is observed.

![Graph showing normalized effective diffusion coefficient versus the dimensionless switchings mean rate of potential $\omega = \nu L^2/(2D)$ for different values of the dimensionless height of the potential barrier.](image)

FIG. 4: The normalized effective diffusion coefficient versus the dimensionless switchings mean rate of potential $\omega = \nu L^2/(2D)$ for different values of the dimensionless height of the potential barrier. Namely $\beta = 3, 7, 9$, for the curves $a$, $b$, and $c$ respectively.

$D_{\text{eff}}/D > 1$ for different values above of $\omega$. This value decreases with increasing height of the potential barrier. In the limiting case of $\beta \ll 1$, we find from Eq. (12)

$$D_{\text{eff}} \approx 1 + \frac{\beta^2 \cdot [(1 + 2\omega) \cosh 2\sqrt{\omega} - (4 \cosh \sqrt{\omega} - 3) (1 + 4 \sqrt{\omega} \sinh \sqrt{\omega} - 2\omega)]}{2\omega^2 \cosh 2\sqrt{\omega}},$$

(15)

and for low barriers we obtain the enhancement of diffusion at relatively fast switchings: $\omega > 9.195$. For very high potential barriers ($\beta \to \infty$) and fixed mean rate of switchings $\nu$, we have $\alpha \approx \beta \to \infty$, $\mu \to 0$, and $\alpha^2 \mu \to \omega$. As a result, we find from Eq. (12)

$$D_{\text{eff}} = \frac{\nu L^2}{\mu}.$$  

(16)

We obtained the interesting result: a diffusion at super-high potential barriers (or at very deep potential wells) is due to the switchings of the potential only. According to Eq. (16) the effective diffusion coefficient depends on the mean rate of flippings and the spatial period of potential profile only, and does not depend on $D$. The area of diffusion acceleration, obtained by Eq. (12), is shown on the plane $(\beta, \omega)$ in Fig. 5 as shaded area. This area lies inside the rectangle region defined by $\beta > 0$ and $\omega > 3.5$. 

7
FIG. 5: Shaded area is the parameter region on the plane $(\beta, \omega)$ where the diffusion acceleration compared with a free diffusion case can be observed.

B. Switching rectangular periodic potential

For switching rectangular periodic potential represented in Fig. 6 the main integro-

\[ D_{\text{eff}} = 1 - \frac{\tanh^2(\beta/2)}{\cosh(2\sqrt{\omega})}. \]  

(17)

FIG. 6: Switching rectangular periodic potential.
We have slowing down of diffusion for all values of the parameters $\beta$ and $\omega$. This is because in rectangular periodic potential the Brownian particles can only move by thermal force, crossing randomly the potential barriers as in fixed potential. The behavior of the normalized effective diffusion coefficient $D_{eff}/D$ as a function of the dimensionless height of the potential barrier $\beta$ for different values of the dimensionless mean rate of switchings $\omega$ is shown in Fig. 7. The dependence of $D_{eff}/D$ versus $\omega$ for different values of $\beta$ is shown in Fig. 8. For very rare

\[ \frac{D_{eff}}{D} \approx \frac{1}{\cosh^2(\beta/2)}. \]  

(18)
In the case of very fast flippings the effective diffusion coefficient, as for sawtooth potential (see Eq. (14)), is practically equal to the free diffusion one

$$\frac{D_{eff}}{D} \simeq 1 - 2e^{-2\sqrt{\omega}} \tanh^2 (\beta/2).$$

(19)

For relatively low potential barriers we get from Eq. (17)

$$\frac{D_{eff}}{D} \simeq 1 - \frac{\beta^2}{4 \cosh (2\sqrt{\omega})}.$$  

(20)

Finally for very high potential barriers $D_{eff}$ depends on the white noise intensity $D$

$$D_{eff} \simeq \frac{2D}{1 + \coth^2 \sqrt{\omega}}.$$  

(21)

IV. CONCLUSIONS

We studied the overdamped Brownian motion in fluctuating supersymmetric periodic potentials. We reduced the problem to the mean first passage time problem and derived the general equations to calculate the effective diffusion coefficient $D_{eff}$. We obtain the exact formula for $D_{eff}$ in periodic potentials modulated by white Gaussian noise. For switching sawtooth periodic potential the exact formula obtained for $D_{eff}$ is valid for arbitrary intensity of white Gaussian noise, arbitrary parameters of the external dichotomous noise and of potential. We derived the area on the parameter plane ($\beta, \omega$) where the enhancement of diffusion can be observed. We analyzed in detail the limiting cases of very high and very low potential barriers, very rare and very fast switchings. A diffusion process is obtained in the absence of thermal noise. For switching rectangular periodic potential the diffusion process slows down for all values of dimensionless parameters of the potential and the external noise.

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