OSCILLATION CRITERIA FOR SECOND-ORDER QUASI-LINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION

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Abstract. In this work, new sufficient conditions for oscillation of solution of second order neutral delay differential equation are established. One objective of our paper is to further simplify and complement some results which were published lately in the literature. In order to support our results, we introduce illustrating examples.

1. Introduction. In numerous applications in biology, electrical engineering and physiology, the dependence on the past appears naturally. Differential equations with delay appears in modeling of these natural phenomena, see [4, 15]. Whereas, delay differential equations or retarded equations, shape a class of mathematical models which let the systems rate of change to not depend only on its present state, but its past history also. The differential equation is called a neutral delay, If the delayed argument occurs in the highest derivative of the state variable. In recent decades, there has been great interest in the study of neutral differential equations. The significant practical importance and theoretical interest for qualitative study of the neutral differential equations is due to the appearance of differential equations with neutral delay appear in mechanics in problem of oscillating masses connected to an elastic bar, in solution of the problems of variations with delay in the time and the problems of electric networks which containing transmission lines without loss, as an example, lines of transmission without loss are used to interconnect switching circuits in computers with high speed, see [7]. Lately, there were a many articles dedicated to the oscillation of solutions of the differential equations with delay or neutral delay, see [1, 2, 3, 5, 6, 8, 10, 11, 12, 14, 17, 19].

One of these equations has got a lot of attention [9, 14, 20], the quasi-linear neutral delay equation

\[
\left( r(t) \left[ (x(t) + p(t) x(\tau(t)))' \right]^\alpha \right)' + f(t, x(\sigma(t))) = 0,
\]

where \( t \geq t_0 \) and provided that the following are satisfied:

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(h₁) r and p are positive functions, p(t) ∈ [0,1), α is a quotient of odd positive integers and

\[ \eta(t₀) := \int_{t₀}^{\infty} r^{-1/\alpha}(v) \, dv < \infty; \]  

(2)

(\eta t₀) \tau, \sigma \in C([t₀, \infty), \mathbb{R}), \sigma'(t) > 0, \tau(t) \leq t, \lim_{t \to \infty} \tau(t) = \infty and \lim_{t \to \infty} \sigma(t) = \infty;

(\eta t₁) f(t,x) \in C([t₀, \infty) \times \mathbb{R}, \mathbb{R}), xf(t,x) > 0 for all x ≠ 0, there exists a positive continuous function q(t) such that |f(t,x)| ≥ q(t)|x|^\beta and \beta is a quotient of odd positive integers;

(h₂) p(t) < \eta(t)/\eta(\tau(t)).

Now, we let z(t) = x(t) + p(t)x(\tau(t)). A non-trivial real function x(t) ∈ C([tₓ, \infty)), tₓ ≥ t₀, is said to be a solution of Eq. (1) if x(t) satisfies (1) on [tₓ, \infty), and has the properties z(t) and r(t)[z'(t)]^\alpha are continuously differentiable for all t ∈ [tₓ, \infty). In this study, we take into account only the solutions of (1) which satisfy \sup\{x(t) : t ≥ t₀\} > 0 for any t ≥ tₓ. If x(t) has arbitrary large zeros, then x(t) is called oscillatory, otherwise it is called non-oscillatory.

Actually, under the assumption \eta(t₀) = \infty, there are many studies in the literature that have been concerned with the oscillation and nonoscillation criteria of solutions of Eq. (1), see for example [2, 10, 17, 19].

For the assumption \eta(t₀) < \infty, Liu et al. [9] obtained some sufficient conditions which guarantee that all solution of Eq. (1) is either oscillatory or tends to zero.

**Theorem 1.1. [9]** Suppose that (h₁) - (h₃) hold,

\[ p'(t) ≥ 0, r'(t) ≥ 0 \quad and \quad \lim_{t \to \infty} p(t) = C. \]  

(3)

If there exist \rho, \varphi ∈ C([t₀, \infty), [0, \infty)) and \rho > 0, such that

\[ \int_{t₀}^{\infty} \left( \rho(s) q(s) (1 - p(\sigma(s)))^{\beta} - \frac{2^{\beta} (\rho'(s))^{2} r^{\beta/\alpha}(\sigma(s))}{8 \beta M^{\frac{\beta}{\alpha}} \sigma^{-\beta - 1}(s) \sigma'(s) \rho(s)} \right) \, ds = \infty, \]  

(4)

for any positive constant M, and

\[ \int_{t₀}^{\infty} \left( \frac{1}{r(t) \varphi(t)} \int_{t₀}^{t} \varphi(s) q(s) \, ds \right)^{1/\alpha} \, dt = \infty, \]  

(5)

then every solution x(t) of Eq. (1) oscillates or lim_{t \to \infty} x(t) = 0.

Moreover, in [20], Wu et al. established some criteria of oscillation for the neutral equation

\[ \left( r(t) |z'(t)|^{\alpha - 1} z'(t) \right)' + q(t) |x(\sigma(t))|^{\beta - 1} x(\sigma(t)) = 0. \]  

(6)

**Theorem 1.2. [20]** Assume that (h₁), (h₂) hold,

\[ p'(t) ≥ 0, r'(t) ≥ 0, r'(t) ≥ 0 \quad and \quad \sigma(t) ≤ \tau(t). \]  

(7)

Every solution of Eq. (1) oscillates if there exist a function \rho ∈ C([t₀, \infty), (0, \infty)) and a constant K > 0 such that

\[ \int_{t₀}^{\infty} \left( \rho(s) q(s) (1 - p(\sigma(s)))^{\beta} - \frac{(\rho'(s))^{\lambda + 1} r(\lambda(s))}{(\lambda + 1)^{\lambda + 1} (\lambda \rho(s))^{\sigma'(s)}} \right) \, ds = \infty \]  

(8)

and

\[ \int_{t₀}^{\infty} \left( \eta^{\mu}(s) q(s) (1 - p(s))^{\beta} - \frac{K r^{\mu + 1} (\nu(s))^{\lambda}}{\eta(s)} \right) \, ds = \infty, \]  

(9)
where $\lambda = \min \{\alpha, \beta\}$, $\mu = \max \{\alpha, \beta\}$,

$$
\lambda(t) = \begin{cases} 
\sigma(t) & \text{if } \beta \geq \alpha \\
 t & \text{if } \beta < \alpha 
\end{cases}
$$
and $m = \begin{cases} 
1 & \alpha = \beta \\
0 < m \leq 1 & \alpha \neq \beta 
\end{cases}$.

In the following theorem, Saker [14] obtained a sufficient condition for the oscillation of Eq. (1).

**Theorem 1.3.** [14] Let $(h_1) - (h_3)$ hold, 

$$p'(t) \geq 0 \quad \text{and} \quad \tau'(t) \geq 0.$$ 
Moreover, suppose that there exists $t \in [t_0, \infty)$ such that

$$
\int_t^\infty \mu(s) q(s) \left(1 - p(\sigma(s))\right) \frac{\int_t^{\sigma(t)} r^{-1/\alpha}(u) \, du}{\int_t^{\sigma(t)} r^{-1/\alpha}(u) \, du} \, ds = \infty \tag{10}
$$

and

$$
\int_t^\infty \left(\frac{1}{r(s)} \int_t^s \eta^\beta(u) q(u) (1 - p(u))^\beta \, du \right)^{1/\alpha} \, ds = \infty, \tag{11}
$$

where

$$
\mu(t) = \begin{cases} 
1 & \text{if } \beta = \alpha \\
c_2 & \text{if } \beta < \alpha \\
c_1 & \text{if } \beta > \alpha
\end{cases}
$$

and $c_1, c_2$ are any positive constants. Then every solution of (1) oscillates.

In the previous results, we note the following:

- Liu’s conditions in Theorem 1.1 guarantee that all solution of Eq. (1) is either oscillatory or tends to zero;
- To apply the Theorems 1.1, 1.2 and 1.3, we must ensure that the functions $\rho(t)$ and $\tau(t)$ nondecreasing;
- Theorems 1.1, 1.2 and 1.3 includes extra conditions (4), (8) and (10).

In this paper, we establish a new criteria of oscillation for the quasi-linear neutral Eq. (1). One objective of our paper is to further simplify, improve and complement Theorems 1.1, 1.2 and 1.3. Firstly, we improve Theorem 1.1 so that the condition guarantee that all solution of Eq. (1) is oscillatory, and without imposing restrictions on the derivatives of $\rho(t)$ and $\tau(t)$. As well, we simplify Theorems 1.2 and 1.3 by obtaining a new criteria ensure oscillation of (1) without check the excess conditions.

2. **Main results.** Let us show the following notations:

$$G(t) = q(t) \left(1 - p(\sigma(t))\frac{\eta(\tau(\sigma(t)))}{\eta(\sigma(t))}\right)^\beta$$

and

$$\tilde{G}(t) = \left(\frac{1}{r(t)} \int_t^t G(s) \, ds \right)^{1/\alpha}.$$ 

**Theorem 2.1.** If

$$
\int_{t_0}^\infty \left(\frac{1}{r(u)} \int_{t_0}^u \eta^\beta(s) G(s) \, ds \right)^{1/\alpha} \, du = \infty, \tag{12}
$$
then (1) is oscillatory.
Proof. Contrary to what is required, we will assume that Eq. (1) has non-oscillatory solution. Moreover, we suppose that \( x(t) \) is a positive solution of (1) (the case \( x(t) < 0 \) is taught by the same way). Thus, there exists a \( t_1 \geq t_0 \) such that \( x(\tau(t)) > 0 \) and \( x(\sigma(t)) > 0 \) for all \( t \geq t_1 \). Since \( x(t) > 0 \), we have that \( z(t) > 0 \) and

\[
(r(t)[z'(t)]^\alpha)' = -f(t, x(\sigma(t))) \leq -q(t) x^\beta (\sigma(t)) < 0. \tag{13}
\]

Then the function \( r(t)[z'(t)]^\alpha \) is strictly decreasing on \([t_0, \infty)\) and of one sign. Assume first that \( z'(t) < 0 \) for \( t \geq t_1 \). Hence,

\[
z(t) \geq -\int_{t}^{\infty} \frac{1}{r^{1/\alpha}(s)} \left(r^{1/\alpha}(s) z'(s)\right) ds \geq -r^{1/\alpha}(t) z'(t) \eta(t).
\]

Since \( (r(t)[z'(t)]^\alpha)' < 0 \), we get

\[
r(t)[z'(t)]^\alpha \leq r(t_1)[z'(t_1)]^\alpha := -K < 0,
\]

for all \( t \geq t_1 \) and thus

\[
z(t) \geq K^{1/\alpha} \eta(t) \quad \text{for all } t \geq t_1. \tag{14}
\]

From (14), we obtain

\[
\frac{d}{dt} \left(\frac{z(t)}{\eta(t)}\right) \geq 0.
\]

We can also note that

\[
x(t) = z(t) - p(t)x(\tau(t))
\]

and

\[
x(t) \geq z(t) - p(t)z(\tau(t)).
\]

Then,

\[
x(t) \geq z(t) \left(1 - p(t) \frac{\eta(\tau(t))}{\eta(t)}\right).
\]

From (13), we have

\[
(r(t)[z'(t)]^\alpha)' \leq -q(t) z^\beta (\sigma(t)) \left(1 - p(\sigma(t)) \frac{\eta(\tau(\sigma(t)))}{\eta(\sigma(t))}\right)^\beta. \tag{15}
\]

Combining (14) with (15) yields

\[
(r(t)[z'(t)]^\alpha)' \leq -K^{\beta/\alpha} \eta^\beta (\sigma(t)) G(t). \tag{16}
\]

By integrating (16) from \( t_1 \) to \( t \), we obtain

\[
r(t)[z'(t)]^\alpha \leq -K^{\beta/\alpha} \int_{t_1}^{t} \eta^\beta (\sigma(s)) G(s) ds. \tag{17}
\]

Integrating (17) from \( t_1 \) to \( t \), we get

\[
z(t) \leq z(t_1) - K^{\beta/\alpha^2} \int_{t_1}^{t} \left(\frac{1}{r(u)} \int_{t_1}^{u} \eta^\beta (\sigma(s)) G(s) ds\right)^{1/\alpha} du. \tag{18}
\]

From (12) and (18), we obtain \( z(t) \to -\infty \) as \( t \to \infty \), and this contradicts to the positivity of \( z(t) \).

Next, we assume that \( z'(t) > 0 \). Thus, we get that \( z(t) > z(\tau(t)) > x(\tau(t)) \) and hence

\[
x(t) = (1 - p(t))z(t).
\]
From (13), we obtain
\[
(r(t) [z'(t)]^\alpha)' \leq -q(t) (1 - p(\sigma(t)))^\beta z^\beta (\sigma(t)).
\] (19)

Since \( \eta'(t) < 0 \), we get that \( \eta(\tau(\sigma(t))) \geq \eta(\sigma(t)) \) for \( t \geq t_2 \geq t_1 \). Then, and from (19), we find
\[
(r(t) [z'(t)]^\alpha)' \leq -q(t) (1 - p(\sigma(t)))^\beta z^\beta (\sigma(t)).
\] (20)

Integrating (20) from \( t_1 \) to \( t \), we find
\[
r(t) [z'(t)]^\alpha \leq r(t_2) [z'(t_2)]^\alpha - \int_{t_2}^{t} q(s) (1 - p(\sigma(s)))^\beta z^\beta (\sigma(s)) ds
\]
\[
\leq r(t_2) [z'(t_2)]^\alpha - \int_{t_2}^{t} q(s) (1 - p(\sigma(s)))^\beta ds
\]
\[
\leq r(t_2) [z'(t_2)]^\alpha - \int_{t_2}^{t} q(s) (1 - p(\sigma(s))) \frac{\eta(\tau(\sigma(s)))}{\eta(\sigma(s))}^\beta ds
\]
\[
\leq r(t_2) [z'(t_2)]^\alpha - \int_{t_2}^{t} G(s) ds.
\] (22)

Since \( \eta'(t) < 0 \), we get
\[
\int_{t_2}^{t} \eta^\beta (\sigma(s)) G(s) ds \leq \eta^\beta (\sigma(t_2)) \int_{t_2}^{t} G(s) ds.
\] (23)

It follows from (12) and (h1) that \( \int_{t_2}^{t} \eta^\beta (\sigma(s)) G(s) ds \) must be unbounded. Hence, from (23), we find
\[
\int_{t_2}^{t} G(s) ds \to \infty \text{ as } t \to \infty.
\] (24)

Thus, and from (22), we obtain \( z'(t) \to -\infty \) as \( t \to \infty \), and this contradicts to \( z'(t) > 0 \). Then, the proof is completed. \( \square \)

**Theorem 2.2.** Assume that the delay equation
\[
z'(t) + \tilde{G}(t) z^{\beta/\alpha} (\sigma(t)) = 0
\] (25)

is oscillatory and
\[
\int_{t_0}^{\infty} G(s) ds = \infty.
\] (26)

Then (1) is oscillatory.

**Proof:** We can proceed exactly as in the proof of the Theorem 2.1. Then, \( z'(t) \) is of one sign eventually. Now, we let \( z'(t) < 0 \) for all \( t \geq t_1 \). Integrating (15) from \( t_1 \) to \( t \), we get
\[
r(t) [z'(t)]^\alpha \leq r(t_1) [z'(t_1)]^\alpha - \int_{t_1}^{t} z^\beta (\sigma(s)) G(s) ds.
\]

Since \( \sigma'(t) > 0 \), we obtain
\[
r(t) [z'(t)]^\alpha \leq -z^\beta (\sigma(t)) \int_{t_1}^{t} G(s) ds
\]
and so
\[ z' (t) \leq -z^{\beta/\alpha} (\sigma (t)) \left( \frac{1}{r (t)} \int_{t_1}^{t} G (s) \, ds \right)^{1/\alpha}. \]

Hence, we have that \( z \) is a positive solution of the inequality
\[ z' (t) + \tilde{G} (t) z^{\beta/\alpha} (\sigma (t)) \leq 0. \] (27)

In view of [18, Lemma 1], we see that the first-order delay differential Eq. (25) has a positive solution. This means that there is no positive solution of Eq. (1), a contradiction.

Next, we assume that \( z' (t) > 0 \) for all \( t \geq t_1 \). Therefore, we get that (26) lead to (24), then the remaining part of this proof is similar to that of proof of Theorem 2.1. Thus, the proof is completed.

The oscillatory behavior of Eq. (25) has been discussed in the literature. Erbe et al [6] and Ladde et al [8] shown that every solution of Eq. (25) when \( \beta/\alpha \in (0, 1) \), is oscillatory if and only if
\[ \int_{t_0}^{t} \tilde{G} (s) \, ds = \infty. \] (28)

On the other hand, in [16], it was pointed out that the Eq. (25) when \( \beta/\alpha > 1 \), still may have a nonoscillatory solution even though
\[ \lim_{t \to \infty} \int_{\sigma (t)}^{t} \tilde{G} (s) \, ds = \infty. \]

Tang [18] have studied the oscillation behavior of solutions of Eq. (25) when \( \beta/\alpha > 1 \). In the following, by using the results of [6] and [18], we will obtain a new criteria for oscillation of solutions of (1).

**Corollary 1.** Assume that \( 0 < \beta < \alpha \). If (28) holds, then (1) is oscillatory.

**Proof.** From [6] and [8], it is well-known that condition (28) implies oscillation of (25) when \( \beta/\alpha < 1 \). Also, we note that (26) is necessary for the validity of (28). Therefore, the proof is completed.

**Corollary 2.** Assume that \( \beta > \alpha \) and (26) holds. Let there exists a function \( \theta (t) \in C^1 ([t_0, \infty), \mathbb{R}) \) such that \( \theta' (t) > 0 \), \( \lim_{t \to \infty} \theta (t) = \infty \),
\[ \limsup_{t \to \infty} \frac{\beta \theta' (\sigma (t)) \sigma' (t)}{\alpha \theta' (t)} < 1 \]
and
\[ \liminf_{t \to \infty} \left( \frac{1}{\theta' (t)} e^{-\theta (t)} \tilde{G} (t) \right) > 0. \] (29)

Then (1) is oscillatory.

**Proof.** From [18, Theorem 1], we see that condition (29) implies oscillation of (25) when \( \beta/\alpha > 1 \). Therefore, the proof is completed.

**Corollary 3.** Assume that \( \beta > \alpha \) and (26) holds.
(a) If \( \sigma (t) = t - \sigma_0, \sigma_0 > 0 \) and there exists \( \delta > \sigma_0 \ln (\beta/\alpha) \), where
\[ \liminf_{t \to \infty} \left( \tilde{G} (t) \exp (e^{-\delta t}) \right) > 0, \]
then (1) is oscillatory.
(b) If \( \sigma(t) = \theta t, \theta \in (0, 1) \) and there exists \( \upsilon > -\ln(\beta/\alpha)/\ln \theta \), where

\[
\liminf_{t \to \infty} \left( \frac{G(t) \exp(-t^\upsilon)}{t} \right) > 0, \quad (31)
\]

then (1) is oscillatory.

**Proof.** From Theorems 3 and 4 in [18], we see that condition (30) implies oscillation of Eq. (25).

**Example 1.** Consider the neutral equation

\[
\left( e^t (x(t) + e^{-t} x(\lambda t))^\gamma \right)' + e^{\alpha t} x^\beta (\mu t) = 0, \quad t > 0, \quad (32)
\]

where \( \lambda, \mu \in (0, 1) \) and \( \beta > 1 \). Here, \( \alpha = 1, r(t) = e^t, p(t) = e^{-t}, q(t) = e^{\alpha t}, \tau(t) = \lambda t \) and \( \sigma(t) = \mu t \). This gives that \( \eta(t) = e^{-t} \) and

\[
G(t) = e^{\alpha t} (1 - e^{-\lambda \mu t})^\beta.
\]

It is easy to verify the conditions \((h_1) - (h_4)\). Moreover, we find

\[
\begin{align*}
\int_{t_0}^\infty \left( \frac{1}{r(u)} \right)^{1/\alpha} \int_{t_0}^u \eta^\beta (s) G(s) \, ds \, du &= \int_{t_0}^\infty \left( e^{-u} \int_{t_0}^u e^{\alpha t} \left( e^{-\lambda \mu t} (1 - e^{-\lambda \mu t})^\beta \right) \, ds \right) \, du = \infty,
\end{align*}
\]

if \( \kappa > \beta \mu + 1 \). Thus, by Theorem 2.1, every solution of (32) oscillates if \( \kappa > \beta \mu + 1 \).

**Example 2.** Consider the delay equation

\[
\left( t^2 \left( x(t) + \frac{\gamma}{t} x(t - \lambda) \right)^\gamma \right)' + \kappa(t - \mu) x(t - \mu) = 0, \quad (33)
\]

where \( t \geq t_0, \gamma, \lambda, \kappa \) and \( \mu \) are positive real numbers and \( t_0 = \max \{ \gamma + \lambda, \mu + \lambda \} \).

We see that \( \alpha = \beta = 1, r(t) = t^2, p(t) = \gamma/t, \tau(t) = t - \lambda, q(t) = \kappa(t - \mu) \) and \( \sigma(t) = t - \mu \). This gives that \( \eta(t) = t^{-1} \) and

\[
G(t) = \kappa(t - \mu) \left( 1 - \frac{\gamma}{t - \mu - \lambda} \right).
\]

It is easy to verify the conditions \((h_1) - (h_3)\). Since \( t > \gamma + \lambda \), we have that \( \gamma t < (t - \lambda)/t \), and hence \( p(t) < \eta(t)/\eta(\tau(t)) \). Note that,

\[
\int_{t_0}^\infty \left( \frac{1}{u^2} \int_{t_0}^u \left( 1 - \frac{\gamma}{t - \mu - \lambda} \right) \, ds \right) \, du = \infty.
\]

By Theorem 2.1, we have that all solution of (33) is oscillatory.

**Example 3.** Let the equation

\[
\left( t^{2 \alpha} \left[ \left( x(t) + \frac{1}{3} x \left( \frac{t}{2} \right) \right)^{1/\alpha} \right]' \right)' + \delta \sigma^\beta (t) x^\beta (\sigma(t)) = 0, \quad (34)
\]

where \( t \geq 1, \delta \) is a positive real number and \( \alpha \in (0, 1) \). We note that \( \tau(t) = t/2, p(t) = 1/3, q(t) = \delta \sigma^\beta (t) \) and \( r(t) = t^{2 \alpha} \). This gives that \( \eta(t) = t^{-1} \) and

\[
G(t) = \frac{\delta}{3^3} \sigma^\beta (t).
\]
It is easy to verify the conditions \((h_1) - (h_1)\). Moreover, we have
\[
\int_{t_0}^{\infty} \left( \frac{1}{u^{2\alpha}} \int_{t_0}^{u} \delta^3 \, ds \right)^{1/\alpha} \, du = \infty,
\]
for \(\alpha < 1\). Using Theorem 2.1, we get that every solution of (34) oscillates.

**Remark 1.** In Example 1, we note that \(p'(t) < 0\), so the results of [9], [14] and [20] cannot be applied in Eq. (32).

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