Abstract

This paper defines an argumentation semantics for extended logic programming and shows its equivalence to the well-founded semantics with explicit negation. We set up a general framework in which we extensively compare this semantics to other argumentation semantics, including those of Dung, and Prakken and Sartor. We present a general dialectical proof theory for these argumentation semantics.

1 Introduction

Argumentation has attracted much interest in the area of AI. On the one hand, argumentation is an important way of human interaction and reasoning, and is therefore of interest for research into intelligent systems. Application areas include automated negotiation via argumentation [5, 4, 13] and legal reasoning [17]. On the other hand, argumentation provides a formal model for various assumption based (or non-monotonic, or default) reasoning formalisms [4, 11]. In particular, various argumentation based semantics have been proposed for logic programming with default negation [4, 11].

Argumentation semantics are elegant since they can be captured in an abstract framework [4, 11, 24, 12], for which an elegant theory of attack, defence, acceptability, and other notions can be developed, without recourse to the concrete instance of the reasoning formalism at hand. This framework can then be instantiated to various assumption based reasoning formalisms. Similarly, a dialectical proof theory, based on dialogue trees, can be defined for an abstract argumentation framework, and then applied to any instance of such a framework [4, 11].

In general, an argument \( A \) is a proof which may use a set of defeasible assumptions. Another argument \( B \) may have a conclusion which contradicts the assumptions or the conclusions of \( A \), and thereby \( B \) attacks \( A \). There are two fundamental notions of such attacks: undercut and rebut [17] or equivalently ground-attack and reductio-ad-absurdum attack [6]. We will use the terminology of undercuts and rebuts. Both attacks differ in that an undercut attacks a premise of an argument, while a rebut attacks a conclusion.

Given a logic program we can define an argumentation semantics by iteratively collecting those arguments which are acceptable to a proponent, i.e. they can be defended against all opponent attacks. In fact, such a notion of acceptability can be defined in a number of ways depending on which attacks we allow the proponent and opponent to use.

Normal logic programs do not have negative conclusions, which means that we cannot use rebuts. Thus both opponents can only launch undercuts on each other’s assumptions. Various argumentation semantics have been defined for normal logic programs [4, 11, 13], some of which are equivalent to existing semantics such as the stable model semantics [4] or the well-founded semantics [6].

Extended logic programs [10, 2, 21], on the other hand, introduce explicit negation, which states that a literal is explicitly false. As a result, both undercuts and rebuts are possible forms of attack; there are further variations depending on whether any kind of counter-attack is admitted. A variety of argumentation semantics arise if one allows one notion of attack as defence for the proponent, and another as attack for the opponent. Various argumentation semantics have been proposed for extended logic programs [6, 17]. Dung has shown that a certain argumentation semantics is equivalent to the answer set semantics [6], a generalisation of the stable model semantics [4]. To our knowl-
edge, no argumentation semantics has yet been found equivalent to the well-founded semantics for extended logic programs, WFSX.\cite{15,24}.

This paper makes the following contributions: we define a least fixpoint argumentation semantics for extended logic programs, and show its equivalence to the well-founded semantics with explicit negation.\cite{15,24,16}. In order to relate this semantics to other argumentation semantics, we set up a general framework to classify notions of justified arguments, and use it to compare our argumentation semantics to those of Dung\cite{6} and Prakken and Sartor\cite{17} among others. We develop a general dialectical proof theory for the notions of justified arguments we introduce.

The paper is organised as follows: First we define arguments and notions of attack and acceptability. Then we set up a framework for classifying different least fixpoint argumentation semantics, based on different notions of attack. In Section 2, we recall the definition of WFSX, and in Section 3, we prove the equivalence of an argumentation semantics and WFSX. A general dialectical proof theory for arguments is presented in Section 4, and its soundness and completeness is proven.

2 Extended Logic Programming and Argumentation

We summarise the definitions of arguments for extended logic programs, and define various notions of attack between arguments.

2.1 Arguments

Definition 1 An objective literal is an atom $A$ or its explicit negation $\neg A$. We define $\neg \neg L = L$. A default literal is of the form $L$ where $L$ is an objective literal. A literal is either an objective or a default literal.

An extended logic program is a (possibly infinite) set of rules of the form

$L_0 \leftarrow L_1, \ldots, L_m, \neg L_{m+1}, \ldots, \neg L_{m+n}$

where $m, n \geq 0$, and each $L_i$ is an objective literal ($0 \leq i \leq m + n$).

For such a rule $r$, we call $L_0$ the head of the rule, $head(r)$, and $L_1, \ldots, \neg L_{m+n}$ the body of the rule, $body(r)$.

Our definition of an argument for an extended logic program is based on\cite{16}. Essentially, an argument is a partial proof, resting on a number of assumptions, i.e. a set of default literals.\footnote{In\cite{16,24}, an argument is a set of assumptions; the two approaches are equivalent in that there is an argument with a conclusion $L$ if there is a set of assumptions from which $L$ can be inferred. See the discussion in\cite{17}.}

Definition 2 Let $P$ be an extended logic program. An argument for $P$ is a finite sequence $A = [r_1, \ldots, r_n]$ of rules $r_i \in P$ such that for every $1 \leq i \leq n$, for every objective literal $L_j$ in the body of $r_i$ there is a $k > i$ such that $head(r_k) = L_j$.

A subargument of $A$ is a subsequence of $A$ which is an argument. The head of a rule in $A$ is called a conclusion of $A$, and a default literal not $L$ in the body of a rule of $A$ is called an assumption of $A$. We write $assm(A)$ for the set of assumptions and $conc(A)$ for the set of conclusions of an argument $A$.

An argument $A$ with a conclusion $L$ is a minimal argument for $L$ if there is no subargument of $A$ with conclusion $L$. An argument is minimal if it minimal for some literal $L$. Given an extended logic program $P$, we denote the set of minimal arguments for $P$ by $Args_P$.

The restriction to minimal arguments is not essential, but convenient, since it rules out arguments constructed from several unrelated arguments. Generally, one is interested in the conclusions of an argument, and wants to avoid having rules in an argument which do not contribute to the desired conclusion.

2.2 Notions of Attack

There are two fundamental notions of attack: undercut, which invalidates an assumption of an argument, and rebut, which contradicts a conclusion of an argument.\cite{6,17}. From these, we may define further notions of attack, by allowing either of the two fundamental kinds of attack, and considering whether any kind of counter-attack is allowed or not. We will now formally define these notions of attacks.

Definition 3 Let $A_1$ and $A_2$ be arguments.

1. $A_1$ undercut $A_2$ if there is an objective literal $L$ such that $L$ is a conclusion of $A_1$ and $\neg L$ is an assumption of $A_2$.

2. $A_1$ rebuts $A_2$ if there is an objective literal $L$ such that $L$ is a conclusion of $A_1$ and $\neg L$ is a conclusion of $A_2$.

3. $A_1$ attacks $A_2$ if $A_1$ undercuts or rebuts $A_2$.

4. $A_1$ defeats $A_2$ if $A_1$ undercuts $A_2$, or ($A_1$ rebuts $A_2$ and $A_2$ does not undercut $A_1$).
5. $A_1$ strongly attacks $A_2$ if $A_1$ attacks $A_2$ and $A_2$ does not undercut $A_1$.

6. $A_1$ strongly undercuts $A_2$ if $A_1$ undercuts $A_2$ and $A_2$ does not undercut $A_1$.

The notions of undercut and rebut, and hence attack, are fundamental for extended logic programs [6, 17].

The notion of undercut instead of undercut. For this reason, we use the term strict undercut, i.e. an undercut that is not counter-defeated. For arguments without priorities, rebuts are these notions of attack. For arguments with priorities, rebutts are somewhat weaker than undercuts, because it is symmetric: a rebut is always counter-rebutted, while the same does not hold for undercuts.

2.3 Acceptability and Justified Arguments

Given the above notions of attack, we define acceptability of an argument. Basically, an argument is acceptable if it can be defended against any attack. Depending on which particular notion of attack we use as defence and which for the opponent’s attacks, we obtain a host of acceptability notions.

Acceptability forms the basis for our argumentation semantics, which is defined as the least fixpoint of a function, which collects all acceptable arguments. The least fixpoint is of particular interest [6], because it provides a canonical fixpoint semantics and it can be constructed inductively.

Definition 6 Let $x$ and $y$ be notions of attack. Let $A$ be an argument, and $S$ a set of arguments. Then $A$ is $x/y$-acceptable wrt. $S$ if for every argument $B$ such that $(B,A) \in x$ there exists an argument $C \in S$ such that $(C,B) \in y$.

Based on the notion of acceptability, we can then define a fixpoint semantics for arguments.

Definition 7 Let $x$ and $y$ be notions of attack, and $P$ an extended logic program. The operator $F_{P,x/y} : \mathcal{P}(\text{Args}_P) \rightarrow \mathcal{P}(\text{Args}_P)$ is defined as

$$F_{P,x/y}(S) = \{ A \mid A \text{ is } x/y\text{-acceptable wrt. } S \}$$

We denote the least fixpoint of $F_{P,x/y}$ by $J_{P,x/y}$. If the program $P$ is clear from the context, we omit the subscript $P$. An argument $A$ is called $x/y$-justified if $A \in J_{x/y}$; an argument is called $x/y$-overruled if it is attacked by an $x/y$-justified argument; and an argument is called $x/y$-defensible if it is neither $x/y$-justified nor $x/y$-overruled.

For any program $P$, the least fixpoint exists by the Knaster-Tarski fixpoint theorem [6, 7], because $F_{P,x/y}$ is monotone. It can be constructed by transfinite induction as follows:

$$J_{x/y}^0 = \emptyset \quad J_{x/y}^{\alpha+1} = F_{P,x/y}(J_{x/y}^\alpha) \quad J_{x/y}^\lambda = \bigcup_{\alpha<\lambda} J_{x/y}^\alpha$$

for $\alpha$ a successor ordinal and $\lambda$ a limit ordinal.

This diagram contains the notions of attack used in [6, 7], plus strongly attacks which seemed a natural intermediate notion between strongly undercuts and defeats. We have not included rebuts, because in the absence of priorities, rebuts is somewhat weaker than undercuts, because it is symmetric: a rebut is always counter-rebutted, while the same does not hold for undercuts.
Then there exists a least ordinal $\lambda_0$ such that $F_{x/y}(J_{x/y}^{\lambda_0}) = J_{x/y}^\lambda = J_{x/y}$.

## 3 Relationships of Notions of Justifiability

This section is devoted to an analysis of the relationship between the different notions of justifiability, leading to a hierarchy of notions of justifiability illustrated in Figure 2.

First of all, it is easy to see that the least fixpoint increases if we weaken the attacks, or strengthen the defence.

**Proposition 1** Let $x' \subseteq x$, $y \subseteq y'$ be notions of attack, then $J_{x/y} \subseteq J_{x'/y'}$.

Theorem 3 states that it does not make a difference if we allow only the strong version of the defence. This is because an argument need not defend itself on its own, but it may rely on other arguments to defend it.

We only give a formal proof for the first theorem; the proofs for the other theorems are similar, and we provide an intuitive informal explanation instead.

**Theorem 2** Let $x$ and $y$ be notions of attack such that $x \supseteq$ undercuts, and let $sy = y -$ undercuts. Then $J_{x/y} = J_{x/sy}$.

**Proof.** Informally, every $x$-attack $B$ to an $x/y$-justified argument $A$ is $y$-defended by some $x/sy$-justified argument $C$ (by induction). Now if $C$ was not a $sy$-attack, then it is undercut by $B$, and because $x \supseteq$ undercuts and $C$ is justified, there exists a strong defence for $C$ against $B$, which is also a defence of the original argument $A$ against $C$.

The formal proof is by transfinite induction. By Proposition 2, we have $J_{x/sy} \subseteq J_{x/y}$. By the inverse inclusion by showing that for all ordinals $\alpha$: $J_{x/y}^\alpha \subseteq J_{x/sy}^\alpha$, by transfinite induction on $\alpha$.

*Base case* $\alpha = 0$: $J_{x/y} = \emptyset = J_{x/sy}$.

*Successor ordinal* $\alpha \rightarrow \alpha + 1$: Let $A \in J_{x/y}^{\alpha+1}$, and $(B,A) \in x$. By definition, there exists $C \in J_{x/y}^\alpha$ such that $(C,B) \in y$. By induction hypothesis, $C \in J_{x/sy}^\alpha$. If $B$ does not undercut $C$, then we are done. If, however, $B$ undercut $C$, then because $C \in J_{x/sy}^\alpha$, and undercuts $x$, there exists $D \in J_{x/sy}^\alpha(\emptyset)(\alpha_0 < \alpha)$ such that $(D,B) \in y$. It follows that $A \in J_{x/sy}^{\alpha+1}$.

*Limit ordinal* $\lambda$: Assume $J_{x/y}^\alpha \subseteq J_{x/sy}^\alpha$ for all $\alpha < \lambda$. Then $J_{x/y}^\lambda = \bigcup_{\alpha < \lambda} J_{x/y}^\alpha \subseteq \bigcup_{\alpha < \lambda} J_{x/sy}^\alpha = J_{x/sy}^\lambda$.

In particular, the previous Theorem states that undercut and strong undercut are equivalent as a defence, as are attack and strong attack. This may be useful in an implementation, where we may use the stronger notion of defence without changing the semantics, thereby decreasing the number of arguments to be checked. The following Corollary shows that because defeat lies between attack and strong attack, it is equivalent to both as a defence.

**Corollary 3** Let $x$ be a notion of attack such that $x \supseteq$ undercuts. Then $J_{x/a} = J_{x/d} = J_{x/sa}$.

**Proof.** With Proposition 1 and Theorem 2, we have $J_{x/a} \subseteq J_{x/d} \subseteq J_{x/sa} = J_{x/a}$.

**Theorem 4** Let $x$ be a notion of attack such that $x \supseteq$ strongly attacks. Then $J_{x/a} = J_{x/d} = J_{x/a}$.

**Proof.** Every $x$-attack $B$ to a $x/a$-justified argument $a$ is attacked by some $x/y$-justified argument $C$ (by induction). If $C$ is a rebut, but not an undercut, then because $B$ strongly attacks $C$, and because $x \supseteq$ strongly attacks, there must have been an argument defending $C$ by undercutting $B$, thereby also defending $A$ against $B$.

The statement for defeats follows in a similar way to Corollary 2.

**Theorem 5** $J_{sa/su} = J_{sa/sa}$

The proof is similar to Theorem 2.

**Theorem 6** $J_{sa/a} = J_{sa/d}$

**Proof.** Every strong undercut $B$ to a $su/a$-justified argument $A$ is attacked by some $su/d$-justified argument $C$ (by induction). If $C$ does not defeat $A$, then there is some argument $D$ defending $C$ by defeating $B$, thereby also defending $A$ against $B$.

We will now present some example programs which distinguish various notions of justifiability.

**Example 1** Consider $P_1$ in Figure 1. For any notion of attack $x$, we have $J_{su/x} = J_{sa/x} = \{[p \leftarrow not q], [q \leftarrow not p]\}$, because there is no strong undercut or strong attack to any of the arguments. However, $J_{a/x} = J_{d/x} = J_{c/x} = \emptyset$, because every argument is undercut (and therefore defeated and attacked).

**Example 2** Consider $P_2$ in Figure 2. Let $x$ be a notion of attack. Then $J_{d/x} = J_{a/x} = \emptyset$, because every argument is defeated (hence attacked).
Example 3 Consider $P_3$ in Figure 1. Let $x$ be a notion of attack. Then $J_{s/a\to x} = \emptyset$, because every argument is strongly attacked.

$J_{s/u\to u} = J_{s/a\to sa} = \{[q \not \rightarrow p]\}$, because $[q \not \rightarrow p]$ is the only argument which is not strongly attacked, but it does not strongly attack any other argument. $J_{a/u} = J_{a/u} = \{[-p]\}$, because there is no undercut to $[-p]$, but $[-p]$ does not undercut any other argument. $J_{a/a} = \{[-p],[q \not \rightarrow p]\}$, because there is no undercut to $[-p]$, and the undercut $[p \not \rightarrow p]$ to $[q \not \rightarrow p]$ is attacked by $[-p]$. We also have $J_{a/a} = \{[-p],[q \not \rightarrow p]\}$, because $[q \not \rightarrow p]$ is not strongly attacked, and the strong attack $[p \not \rightarrow q]$ on $[\neg p]$ is undercut by $[q \not \rightarrow p]$.

Example 4 Consider $P_4$ in Figure 1. Let $x$ be a notion of attack. Then $J_{u/x} = J_{d/x} = J_{a/x} = \emptyset$, because every argument is under-attack. $J_{s/u\to u} = J_{s/a\to sa} = J_{a/a} = J_{s/a\to sa} = \{[p \not \rightarrow q],[q \not \rightarrow p]\}$. In this case, the strong attacks are precisely the strong undercuts; The argument $[r \not \rightarrow p]$ is not justified, because the strong undercut $[p \not \rightarrow q]$ is under-cut, but not strongly under-cut, by $[q \not \rightarrow p]$. $J_{s/u\to u} = J_{s/a\to sa} = J_{a/a} = J_{s/a\to sa} = \{[p \not \rightarrow q],[q \not \rightarrow p],[r \not \rightarrow p]\}$. Again, undercuts and attacks, and strong undercuts and strong attacks, coincide; but now $[r \not \rightarrow p]$ is justified, because non-strong undercuts are allowed as defence.

Example 5 Consider $P_5$ in Figure 1. Then $J_{a/x} = \emptyset$, because both arguments attack each other, while $J_{d/x} = \{[-p]\}$, because $[-p]$ defeats $[p \not \rightarrow \neg p]$, but not vice versa.

Example 6 Consider $P_6$ in Figure 1. Let $x$ be a notion of attack. Then $J_{s/a\to x} = J_{d/x} = J_{a/x} = \emptyset$, because every argument is strongly attacked (hence defeated and attacked), while $J_{u/x} = J_{a/x} = \{[p],[q]\}$.

Theorem 7 The notions of justifiability are ordered (by set inclusion) according to the Hasse diagram in Figure 2.

By definition, Dung’s grounded argumentation semantics is exactly $a/\text{d}-\text{justifiability}$, while Prakken and Sartor’s semantics, if we disregard priorities, amounts to $d/su$-justifiability. As corollaries to Theorem 4 we obtain relationships of these semantics to the other notions of justifiability.

Corollary 8 Let $J_{\text{Dung}}$ be the set of justified arguments according to Dung’s grounded argumentation semantics. Then $J_{\text{Dung}} = J_{a/\text{s/u}} = J_{a/\text{u}} = J_{a/\text{d}} = J_{a/\text{s/a}}$ and $J_{\text{Dung}} \subseteq J_{x/y}$ for all notions of attack $x$ and $y$.

Corollary 9 Let $J_{\text{PS}}$ be the set of justified arguments according to Prakken and Sartor’s argumentation semantics, where all arguments have the same priority. Then $J_{\text{PS}} = J_{\text{d/\text{s/u}}} = J_{\text{d/\text{u}}} = J_{\text{d/a}} = J_{\text{d/\text{d}}} = J_{\text{d/\text{s/a}}}$, $J_{\text{PS}} \subseteq J_{x/y}$ for all notions of attack $x \neq a$ and $y$, and $J_{\text{PS}} \supseteq J_{a/\text{u}}$ for all notions of attack $y$.

Remark 1 1. The notions of $a/x-$, $d/x-$ and $sa/x$-justifiability are very sceptical in that a fact $p$ may not be justified, if there is a rule $\neg p \rightarrow B$ (where not $p \not \in B$) that is not $x$-attacked. On the other hand this is useful in terms of avoiding inconsistency.

2. $sx/y$-justifiability is very credulous, because it does not take into account non-strong attacks, so e.g. the program $\{p \not \rightarrow q,q \not \rightarrow p\}$ has the justified arguments $[p \not \rightarrow q]$ and $[q \not \rightarrow p]$.
Remark 2 One might ask whether any of the semantics in Figure 2 are equivalent for non-contradictory programs, i.e. programs for which there is no literal \( L \) such that there exist justified arguments for both \( L \) and \( \neg L \). The answer to this question is no: all the examples above distinguishing different notions of justifiability involve only non-contradictory programs.

In particular, even for non-contradictory programs, Dung’s and Prakken and Sartor’s semantics differ, and both differ from \( u/a \)-justifiability, which will be shown equivalent to the well-founded semantics WFSX \((\mathbb{I}, \lambda)\) in the following section.

4 Well-founded semantics

We recollect the definition of the well-founded semantics for extended logic programs, WFSX. We use the definition of \((\mathbb{I}, \lambda)\), because it is closer to our definition of argumentation semantics than the original definition of \((\mathbb{I}, \mathbb{I})\).

Definition 8 The set of all objective literals of a program \( P \) is called the Herbrand base of \( P \) and denoted by \( \mathcal{H}(P) \). A pseudo-interpretation of a program \( P \) is a set \( T \cup \neg F \) where \( T \) and \( F \) are subsets of \( \mathcal{H}(P) \). An interpretation is a pseudo-interpretation where the sets \( T \) and \( F \) are disjoint. An interpretation is called two-valued if \( T \cup F = \mathcal{H}(P) \).

Definition 9 Let \( P \) be an extended logic program, \( I \) an interpretation, and let \( P' \) (resp. \( I' \)) be obtained from \( P \) (resp. \( I \)) by replacing every literal \( \neg A \) by a new atom, say \( A \). The GL-transformation \( P' \) is the program obtained from \( P' \) by removing all rules containing a default literal not \( A \) such that \( A \in P' \), and then removing all remaining default literals from \( P' \), obtaining a definite program \( P'' \). Let \( J \) be the least model of \( P'' \). \( \Gamma P \) is obtained from \( J \) by replacing the introduced atoms \( \neg A \) by \( \neg A \).

Definition 10 The semi-normal version of a program \( P \) is the program \( P_s \) obtained from \( P \) by replacing every rule \( L \leftarrow \text{Body} \) in \( P \) by the rule \( L \leftarrow \neg L, \text{Body} \).

If the program \( P \) is clear from the context, we write \( \Gamma P(I) \) for \( \Gamma P(I) \) and \( \Gamma P(I) \) for \( \Gamma P(I) \).

Definition 11 Let \( P \) be a program whose least fixpoint of \( \Gamma_{s} \) is \( T \). Then the paraconsistent well-founded model of \( P \) is the pseudo-interpretation \( \text{WFM}_{P}(P) = T \cup \neg (\mathcal{H}(P) - \Gamma_{s} T) \). If \( \text{WFM}_{P}(P) \) is an interpretation, then \( P \) is called non-contradictory, and \( \text{WFM}_{P}(P) \) is the well-founded model of \( P \), denoted by \( \text{WFM}(P) \).

The paraconsistent well-founded model can be defined iteratively by the transfinite sequence \( \{ I_{\alpha} \} \):

\[
\begin{align*}
I_{0} & = \emptyset \\
I_{\alpha+1} & = \Gamma_{s} I_{\alpha} \\
I_{\lambda} & = \bigcup_{\alpha<\lambda} I_{\alpha} \quad \text{for limit ordinal } \lambda
\end{align*}
\]

There exists a smallest ordinal \( \lambda_{0} \) such that \( I_{\lambda_{0}} \) is the least fixpoint of \( \Gamma_{s} \), and \( \text{WFM}_{P}(P) = I_{\lambda_{0}} \cup \neg \mathcal{H}(P) \).

5 Equivalence of argumentation semantics and WFSX

In this section, we will show that the argumentation semantics \( J_{u/a} \) and the well-founded model coincide. That is, the conclusions of justified arguments are exactly the objective literals which are true in the well-founded model; and those objective literals all of whose arguments are overruled are exactly the literals which are false in the well-founded model. The result holds also for contradictory programs under the paraconsistent well-founded semantics. This is important, because it shows that contradictions in the argumentation semantics are precisely the contradictions under the well-founded semantics, and allows the application of contradiction removal (or avoidance) methods to the argumentation semantics. Because for non-contradictory programs, the well-founded semantics coincides with the paraconsistent well-founded semantics, we obtain as a corollary that argumentation semantics and well-founded semantics coincide for non-contradictory programs.

In order to compare the argumentation semantics with the well-founded semantics, we define the set of literals which are a consequence of the argumentation semantics.

Definition 12 \( A(P) = T \cup \neg F \), where \( T = \{ L \mid \text{there is a justified argument for } L \} \) and \( F = \{ L \mid \text{all arguments for } L \text{ are overruled} \} \).

The following Proposition shows a precise connection between arguments and consequences of a program \( \mathcal{P} \).

Proposition 10 Let \( I \) be a two-valued interpretation.

1. \( L \in \Gamma(I) \) iff \( \exists \) argument \( A \) with conclusion \( L \) such that \( \text{assm}(A) \subseteq I \).
2. \( L \in \Gamma_{s}(I) \) iff \( \exists \) argument \( A \) with conclusion \( L \) such that \( \text{assm}(A) \subseteq I \) and \( \neg \text{conc}(A) \cap I = \emptyset \).
3. \( L \notin \Gamma(I) \) iff \( \forall \) arguments \( A \) with conclusion \( L \), \( \text{assm}(A) \cap I \neq \emptyset \).
4. \( L \notin \Gamma_s(I) \) iff \( \forall \) arguments \( A \) with conclusion \( L \),
\( \text{assm}(A) \cap I \neq \emptyset \) or \( \neg \text{conc}(A) \cap I \neq \emptyset \).

Proof. See Appendix. \( \square \)

Theorem 11 Let \( P \) be an extended logic program. Then \( \text{WFMP}(P) = A(P) \).

Proof. First, note that \( A \) undercuts \( B \) iff \( \exists \) \( L \) s.t.
\( \neg L \in \text{assm}(A) \) and \( L \in \text{conc}(B) \); and \( A \) rebuts \( B \) iff \( \exists \) \( L \in \text{conc}(A) \cap \neg \text{conc}(B) \).

We show that for all ordinals \( \alpha \), \( I_\alpha = A_\alpha \), by
transfinite induction on \( \alpha \).
Base case \( \alpha = 0 \): \( I_\alpha = \emptyset = A_\alpha \)
Successor ordinal \( \alpha \rightarrow \alpha + 1 \):
\( L \in I_{\alpha+1} \) iff (Def. of \( I_{\alpha+1} \))
\( L \in \Gamma_{s}I_\alpha \) iff (Prop. [10])
\( \exists \) argument \( A \) for \( L \) such that \( \text{assm}(A) \subseteq \Gamma_sI_\alpha \)
iff (Def. of \( \subseteq \), and \( \Gamma_sI_\alpha \) is two-valued)
\( \exists \) argument \( A \) for \( L \) such that
\( \forall \) not \( L \in \text{assm}(A) \), \( L \notin \Gamma_sI_\alpha \)
iff (Prop. [10])
\( \exists \) argument \( A \) for \( L \) such that \( \forall \) not \( L \in \text{assm}(A) \), for
any argument \( B \) for \( L \),
\( \exists \) not \( L' \in \text{assm}(B) \) s.t. \( L' \in I_\alpha \) or
\( \exists L'' \in \text{conc}(B) \) s.t. \( \neg L'' \in I_\alpha \)
iff (Induction hypothesis)
\( \exists \) argument \( A \) for \( L \) such that \( \forall \) not \( L \in \text{assm}(A) \), for
any argument \( B \) for \( L \), ( \( \exists \) not \( L' \in \text{assm}(B) \) s.t. \( \exists \)
argument \( C \in J_\alpha \) for \( L' \), or
\( \exists L'' \in \text{conc}(B) \) s.t. \( \exists \) argument \( C \in J_\alpha \) for \( \neg L'' \)
iff (Def. of undercut and rebut)
\( \exists \) argument \( A \) for \( L \) such that for any undercut \( B \) to
\( A \), ( \( \exists \) argument \( C \in J_\alpha \) s.t. \( C \) undercuts \( B \), or
\( \exists \) argument \( C \in J_\alpha \) s.t. \( C \) rebuts \( B \)
iff \( \exists \) argument \( A \) for \( L \) such that for any undercut \( B \) to
\( A \), \( \exists \) argument \( C \in J_\alpha \) s.t. \( C \) attacks \( B \)
iff (Def. of \( J_{\alpha+1} \))
\( \exists \) argument \( A \in J_{\alpha+1} \) for \( L \)
iff (Def. of \( A_{\alpha+1} \))
\( L \in A_{\alpha+1} \)

Limit ordinal \( \lambda \):
\( I_\lambda = \bigcup_{\alpha<\lambda} I_\alpha \) and \( A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha \), so by induction hypothesis (\( I_\alpha = A_\alpha \) for all \( \alpha < \lambda \)), \( I_\lambda = A_\lambda \).

Now, we show that a literal \( \text{not} \ L \) is in the
well-founded semantics iff every argument for \( L \) is
overruled.
\( \text{not} \ L \in \text{WFMP}(P) \)
iff (Def. of \( \text{WFMP}(P) \))
\( L \notin \Gamma_sI \)
iff (Prop. [10])
for all arguments \( A \) for \( L \),
\( ( \exists \) not \( L' \in \text{assm}(A) \) s.t. \( L' \in I \), or
\( \exists L'' \in \text{conc}(A) \) s.t. \( \neg L'' \in I \)
iff (I=A)
for all arguments \( A \) for \( L \),
\( ( \exists \) not \( L' \in \text{assm}(A) \) s.t. \( \exists \) argument \( B \in J \) for \( L' \),
or \( \exists L'' \in \text{conc}(A) \) s.t. \( \exists \) argument \( B \in J \) for \( \neg L'' \)
iff (Def. of undercut and rebut)
for all arguments \( A \) for \( L \), ( \( \exists \) argument \( B \in J \) s.t. \( B \)
undercuts \( A \), or \( \exists \) argument \( B \in J \) s.t. \( B \) rebuts \( A \)
iff
every argument for \( L \) is attacked by a justified
argument in \( J \)
iff (Def. of overruled)
every argument for \( L \) is overruled
iff (Def. of \( A(P) \))
\( \text{not} \ L \in A(P) \) \( \square \)

Corollary 12 Let \( P \) be a non-contradictory program. Then \( \text{WFMP}(P) = A(P) \).

Remark 3 In a similar way, one can show that the
\( \Gamma \) operator corresponds to undercuts, while the \( \Gamma_s \) operator corresponds to attacks, and so the least
fixpoints of the \( \Gamma \), \( \Gamma_s \), and \( \Gamma_s \) correspond to \( J_{u/w} \), \( J_{a/w} \), and
\( J_{a/a} \), respectively. In \([1] \), the least fixpoints of
these operators are shown to be ordered as \( \text{lfp}(\Gamma_s, \Gamma) \subseteq \text{lfp}(\Gamma, \Gamma_s) \subseteq \text{lfp}(\Gamma_s, \Gamma_s) \).
Because \( J_{u/w} = J_{a/a} \subseteq J_{u/w} \subseteq J_{u/a} \)
by Theorem [3], we can strengthen this statement to
\( \text{lfp}(\Gamma_s, \Gamma) = \text{lfp}(\Gamma_s, \Gamma_s) \subseteq \text{lfp}(\Gamma_s) \).

6 Proof theory

One of the benefits of relating the argumentation semantics \( J_{a/a} \) to WFSX is the existence of an efficient
top-down proof procedure for WFSX \([1] \), which we can
use to compute justified arguments in \( J_{a/a} \). On the other hand, dialectical proof theories, based on dialogue
trees, have been defined for a variety of argumentation semantics \([1] \). \([1] \). \([1] \). \([1] \). In this section, we
present a sound and complete dialectical proof theory
for the least fixpoint argumentation semantics \( J_{a/y} \) for
any notions of attack \( x \) and \( y \). Our presentation closely
follows \([17] \). As a further consequence, we obtain an
equivalence of the proof theory for WFSX and the
dialectical proof theory for arguments.

Definition 13 An \( x/y \)-dialogue is a finite nonempty
sequence of moves \( \text{move}_i = (\text{Player}_i, \text{Arg}_i)(i > 0) \),
such that
1. Player \( i = P \) iff \( i \) is odd; and Player \( i = O \) iff \( i \) is even.

2. If Player \( i = \text{Player}_j \) and \( i \neq j \), then \( \text{Arg}_i \neq \text{Arg}_j \).

3. If Player \( i = P \) and \( i > 1 \), then \( \text{Arg}_i \) is a minimal argument such that \( (\text{Arg}_i, \text{Arg}_{i-1}) \in y \).

4. If Player \( i = O \), then \( (\text{Arg}_i, \text{Arg}_{i-1}) \in x \).

The first condition states that the players \( P \) (Propponent) and \( O \) (Opponent) take turns, and \( P \) starts. The second condition prevents the proponent from repeating a move. The third and fourth conditions state that both players have to attack the other player's last move, where the opponent is allowed to use the notion of attack \( x \), while the proponent may use \( y \) to defend its arguments.

**Definition 14** An \( x/y \)-dialogue tree is a tree of moves such that every branch is a dialogue, and for all moves \( m_{\text{move}} = (P, \text{Arg}_i) \), the children of move \( m \) are all those moves \( (O, \text{Arg}_j) \) such that \( (\text{Arg}_j, \text{Arg}_i) \in x \).

**Definition 15** A player wins an \( x/y \)-dialogue iff the other player cannot move. A player wins an \( x/y \)-dialogue tree iff it wins all branches of the tree. An \( x/y \)-dialogue tree which is won by the proponent is called a winning \( x/y \)-dialogue tree. An argument \( A \) is provably \( x/y \)-justified iff there exists a \( x/y \)-tree with \( A \) as its root, and won by the proponent. A literal \( L \) is a provably justified conclusion iff it is a conclusion of a provably \( x/y \)-justified argument. The height of a dialogue tree is \( 0 \) if it consists only of the root, and otherwise \( \text{height}(t) = \bigcup \text{height}(t_i) + 1 \) where \( t_i \) are the trees rooted at the grandchildren of \( t \).

We show that the proof theory of \( x/y \)-dialogue trees is sound and complete for any notions of attack \( x \) and \( y \).

**Theorem 13** An argument is provably \( x/y \)-justified iff it is \( x/y \)-justified.

**Proof.** “If”-direction. We show by transfinite induction: If \( A \in J^\alpha_{x/y} \), then there exists a winning \( x/y \)-dialogue tree of height \( < \alpha \) for \( A \).

Base case \( \alpha = 0 \): Then there exists no argument \( B \) such that \( (B, A) \in x \), and so \( A \) is a winning \( x/y \)-dialogue tree for \( A \) of height \( 0 \).

Successor ordinal \( \alpha + 1 \): If \( A \in J^{\alpha+1}_{x/y} \), then for any \( B_i \) such that \( (B_i, A) \in x \) there exists a \( C_i \in J^\alpha_{x/y} \) such that \( (C_i, B_i) \in y \). By induction hypothesis, there exist winning \( x/y \)-dialogue trees for the \( C_i \). Thus, we have a winning tree rooted for \( A \), with children \( B_i \), whose children are the winning trees for \( C_i \).

Limit ordinal \( \lambda \): If \( A \in J^\lambda_{x/y} \), then there exists an \( \alpha < \lambda \) such that \( A \in J^\alpha_{x/y} \); by induction hypothesis, there exists a winning \( x/y \)-dialogue tree of height \( \alpha \) for \( A \).

“Only-if”-direction. We prove by transfinite induction: If there exists a winning tree of height \( \alpha \) for \( A \), then \( A \in J^\alpha_{x/y} \).

Base case \( \alpha = 0 \): Then there are no arguments \( B \) such that \( (B, A) \in x \), and so \( A \in J^0_{x/y} \).

Successor ordinal \( \alpha + 1 \): Let \( T \) be a tree with root \( A \), whose children are \( B_i \), and the children of \( B_i \) are winning trees rooted at \( C_i \). By induction hypothesis, \( C_i \in J^\alpha_{x/y} \). Because the \( B_i \) are all those arguments such that \( (B_i, A) \in x \), then \( A \) is defended against each \( B_i \) by \( C_i \), and so \( A \in J^{\alpha+1}_{x/y} \).

As a corollary, we can relate the proof theory of WFSX and the \( u/a \)-proof theory.

**Corollary 14** \( L \) is a provably \( u/a \)-justified conclusion iff there exists a successful \( T \)-tree \( \exists T \) for \( L \).

**Proof.** Follows from the fact that \( u/a \)-dialogue trees are sound and complete for \( u/a \)-justifiability (Theorem 13), that \( T \)-trees are sound and complete for WFSX \( \exists T \), and that \( u/a \)-justifiability and WFSX are equivalent (Theorem 11).

7 Conclusion and Further Work

We have identified various notions of attack for extended logic programs. Based on these notions of attack, we defined notions of acceptability and least fixpoint semantics. These fixpoint semantics were related by establishing a lattice of justified arguments, based on set inclusion. We identified an argumentation semantics \( J_{u/a} \) equal to the well-founded semantics for logic programs with explicit negation, WFSX \( \exists T \), and established that \( J_{\text{Dung}} \subseteq J_{PS} \subseteq J_{u/a} = \text{WFSX} \), where \( J_{\text{Dung}} \) and \( J_{PS} \) are the least fixpoint argumentation semantics of Dung \( \exists T \) and Prakken and Sartor \( \exists T \). We have defined a dialectical proof theory for argumentation. For all notions of justified arguments introduced, we prove that the proof theory is sound and complete wrt. the corresponding fixpoint argumentation semantics. In particular, we show the equivalence of successful \( T \)-trees \( \exists T \) in WFSX to provably \( u/a \) justified arguments.
Finally, it remains to be seen whether a variation in the notion of attack yields interesting variations of alternative argumentation semantics for extended logic programs such as preferred extensions or stable extensions [6]. It is also an open question how the hierarchy changes when priorities are added as defined in [17][20].

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Appendix

Proof of Proposition 10

1. “If”-direction: Induction on the length n of the derivation of \( L \in \Gamma(I) \).

   **Base case:** \( n = 1 \):
   Then there exists a rule \( L \leftarrow \neg L_1, \ldots, \neg L_n \) in \( P \) s.t. \( L_1, \ldots, L_n \not\in I \), and \( [L \leftarrow \neg L_1, \ldots, \neg L_n] \) is an argument for \( L \) whose assumptions are contained in \( I \).

   **Induction step:** \( n \sim n + 1 \):
   Let \( L \in \Gamma^{n+1}(I) \). Then there exists a rule \( r = L \leftarrow L_1, \ldots, L_n, \neg L'_1, \ldots, \neg L'_m \) in \( P \) s.t. \( L_i \in \Gamma^n(I) \), and \( r \not\in I \). By induction hypothesis, there exists arguments \( A_1, \ldots, A_n \) for \( L_1, \ldots, L_n \) with \( \text{assm}(A_i) \subseteq I \). Then \( A = [r] \cdot A_1 \cdots A_n \) is an argument for \( L \) such that \( \text{assm}(A) \subseteq I \).

   “Only-if” direction: Induction on the length of the argument.

   **Base case:** \( n = 1 \):
   Then \( A = [L \leftarrow \neg L_1, \ldots, \neg L_n] \), and \( \neg L, L_1, \ldots, L_n \not\in I \). Then \( L \leftarrow \neg L_1, \ldots, L_n \not\in I \). By induction hypothesis, \( L \in \Gamma^1(I) \). Hence \( L \in \Gamma(I) \).

   **Induction step:** \( n \sim n + 1 \):
   Let \( A = [L \leftarrow L_1, \ldots, L_n, \neg L'_1, \ldots, \neg L'_m; r_2, \ldots, r_n] \) be an argument s.t. \( \text{assm}(A) \subseteq I \), and \( \neg \text{conc}(A) \cap I = \emptyset \). \( A \) contains subarguments \( A_1, \ldots, A_n \) for \( L_1, \ldots, L_n \), with \( \text{assm}(A_i) \subseteq I \). Because \( L'_1, \ldots, L'_m \not\in I \), and \( \neg L \not\in I \), then \( L \leftarrow L_1, \ldots, L_n \not\in P \). By induction hypothesis, \( L_i \in \Gamma(I) \). So also \( L \in \Gamma(I) \).

3. and 4. follow immediately from 1. and 2., because \( I \) is two-valued.

“Only-if” direction: Induction on the length of the argument.

**Base case:** \( n = 1 \):
Then \( A = [L \leftarrow \neg L_1, \ldots, \neg L_n] \), and \( \neg L, L_1, \ldots, L_n \not\in I \). Then \( L \leftarrow \neg L_1, \ldots, L_n \not\in I \). By induction hypothesis, \( L \in \Gamma^1(I) \). Hence \( L \in \Gamma(I) \).

**Induction step:** \( n \sim n + 1 \):
Let \( A = [L \leftarrow L_1, \ldots, L_n, \neg L'_1, \ldots, \neg L'_m; r_2, \ldots, r_n] \) be an argument s.t. \( \text{assm}(A) \subseteq I \), and \( \neg \text{conc}(A) \cap I = \emptyset \). \( A \) contains subarguments \( A_1, \ldots, A_n \) for \( L_1, \ldots, L_n \), with \( \text{assm}(A_i) \subseteq I \). Because \( L'_1, \ldots, L'_m \not\in I \), and \( \neg L \not\in I \), then \( L \leftarrow L_1, \ldots, L_n \not\in P \). By induction hypothesis, \( L_i \in \Gamma(I) \). So also \( L \in \Gamma(I) \).

2. “If”-direction: Induction on the length n of the derivation of \( L \in \Gamma_s(I) \).

   **Base case:** \( n = 1 \):
   Then there exists a rule \( L \leftarrow \neg L_1, \ldots, \neg L_n \) in \( P \) s.t. \( \neg L, L_1, \ldots, L_n \not\in I \), and \( [L \leftarrow \neg L_1, \ldots, \neg L_n] \) is an argument for \( L \) whose assumptions are contained in \( I \), and \( \neg L \not\in I \).

   **Induction step:** \( n \sim n + 1 \):
   Let \( L \in \Gamma^{n+1}(I) \). Then there exists a rule \( r = L \leftarrow L_1, \ldots, L_n, \neg L'_1, \ldots, \neg L'_m \) in \( P \) s.t. \( L_i \in \Gamma^n(I) \), and \( \neg L \not\in I \). By induction hypothesis, there exists arguments \( A_1, \ldots, A_n \) for \( L_1, \ldots, L_n \) with \( \text{assm}(A_i) \subseteq I \) and \( \neg \text{conc}(A_i) \cap I = \emptyset \). Then \( A = [r] \cdot A_1 \cdots A_n \) is an argument for \( L \) such that \( \text{assm}(A) \subseteq I \), and \( \neg \text{conc}(A) \cap I = \emptyset \).