Chapter 1

DOUBLE-MODE STELLAR PULSATIONS

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Abstract  The status of the hydrodynamical modelling of nonlinear multi-mode stellar pulsations is discussed. The hydrodynamical modelling of steady double-mode (DM) pulsations has been a long-standing quest that is finally being concluded. Recent progress has been made thanks to the introduction of turbulent convection in the numerical hydrodynamical codes which provide detailed results for individual models. An overview of the modal selection problem in the HR diagram can be obtained in the form of bifurcation diagrams with the help of simple nonresonant amplitude equations that capture the DM phenomenon.

Keywords: Nonlinear pulsations, Variable stars, Cepheids, RR Lyrae, Turbulence, Convection, Beat Cepheids, Double-mode pulsations.

1. INTRODUCTION

Self-excited multi-mode pulsations are quite common among the less evolved luminous stars, such as the delta Scuti stars, but they mostly involve nonradial modes of oscillation. In contrast, the classical variable stars, viz. the Cepheids and RR Lyrae stars, are believed to be radially pulsating, and two independent frequencies are
typically identified in the Fourier spectrum. These pulsating stars are referred to as double-mode even though, in principle, there could be more than two modes involved with locked frequencies. (We should add that there is evidence that suggests the presence for nonradial modes as well (Kovács et al., Moskalik, Olech et al. in Szabados & Kurtz 2000).

In the Galaxy, the Beat Cepheids, also called double-mode (DM) Cepheids are relatively rare. Only about a good dozen are known. On the other hand, the recent observations of the Magellanic Clouds (Beaulieu et al. 1995, 1997, Welch et al. 1995, Udalski et al. 1999) have shown that in these galaxies Beat Cepheids are quite common. Beat Cepheid pulsations occur either in the fundamental/first overtone (F/O\(_1\)) modes, or in the first/second overtone (O\(_1\)/O\(_2\)) modes. In the Galaxy there is only one O\(_1\)/O\(_2\) known DM pulsator (CO Aur), but in the SMC Udalski et al. 1999 claim 70 candidates out of 93 DM Cepheids.

The occurrence of DM RR Lyrae stars (RRd stars), depends on the type of cluster. For example, not a single RRd star is observed in \(\omega\) Centauri (cf. Kovács in this Volume) while in M68, 9 RRd stars are known out of 37 RR Lyrae stars.

The DM pulsators are important for pulsation theory in that they impose very stringent requirements on the numerical modelling. Double periodicity and indeed the overall view of the modal selection picture provide many more observational constraints than their single-mode siblings.

Despite the frequent occurrence of DM pulsations in nature, the numerical modelling of this type of pulsation had remained a serious challenge until recently. In fact, it had become abundantly clear that purely radiative models, i.e. models that disregarded convective transport, were not capable of yielding DM pulsations, except on a purely transient basis, i.e. they were switching from one mode to another, but too fast to account for the observed fairly steady nature of DM pulsators. Actually, some purely radiative models of RRd stars had been found already by Kovács and Buchler (1993), but those results were not satisfactory. However, recently, DM behavior has been found simultaneously, and fully independently, in RR Lyrae models by Feuchttinger (1998) and in Cepheid models by Kolláth et al. (1998), this with different numerical methods, viz. the
Vienna and the Florida codes. The breakthrough came as a result of the inclusion of time-dependent turbulent convection in the models.

2. AMPLITUDE EQUATIONS – SIMPLE MODELS FOR DM PULSATIONS

Numerical hydrodynamics may provide us with an accurate description of the pulsations of an individual stellar model, but it does not give us an overview of the behavior when model parameters are changed. On the sole basis of numerical hydrodynamical modelling one often has no clue why neighboring models may exhibit very different pulsational behavior. Examples of such sensitive behavior will be discussed below.

Therefore a global way had to be devised for gaining a view of the global modal selection picture (or the bifurcation diagram as it would be called more generally). The simplest global way to describe the interaction of pulsation modes, including DM behavior, is through the use of amplitude equations, which are also known as normal forms. A derivation of the amplitude equation formalism applied to stellar pulsation can be found elsewhere (Buchler & Goupil 1984, Buchler & Kovács 1986, cf. also Dziembowski & Kovács (1984), Takeuti (1985), Goupil & Buchler 1994 and for a review Buchler 1993). This formalism is based on dynamical systems theory, and it is very general and fundamental. The amplitude equations give the temporal behavior of the amplitudes and the phases of the modes that are involved in the nonlinear pulsation. They are of fundamental importance for understanding the changes in the morphology of the Fourier decomposition coefficients of the light and radial velocity curves as a function of pulsation period. For example, they explain the nature of the well-known Hertzsprung progression of the classical Cepheids. The amplitude equations show unambiguously how these phenomena are related to and caused by internal resonances of the pulsating mode with an overtone (for a review cf. Buchler 1993).

We consider here the situation in which only two modes are involved in the pulsation and there exists no resonance condition between them nor with the other modes of the star. This is in contrast to earlier purely radial modelling which had required the presence of resonances (Kovács & Buchler 1993), but which turned out to be unsatisfactory.
We denote the linear eigenvalues by $\sigma_j$, for an assumed $\exp(\sigma t)$ dependence,

$$
\sigma_j = \kappa_j + i\omega_j \quad (1.1)
$$

$$
P_j = 2\pi/\omega_j \quad (1.2)
$$

$$
\eta_j = 2\kappa_j P_j \quad (1.3)
$$

The absence of resonance means that there is no relation of the form $n_1\omega_0 + n_2\omega_1 \approx 0$ or $n_1\omega_0 + n_2\omega_1 + n_2\omega_3 \approx 0$, where $n_1, n_2, n_3$ are small positive or negative integers.

The generic amplitude equations, appropriate to the nonresonant situation, when truncated at 5th order read \textit{(e.g. Buchler 1993)}:

$$
\dot{a}_0 = a_0(\sigma_0 + Q_{00}|a_0|^2 + Q_{01}|a_1|^2 + S_1|a_0|^2|a_1|^2 \\
+ R_{00}|a_0|^4 + R_{01}|a_1|^4) \quad (1.4)
$$

$$
\dot{a}_1 = a_1(\sigma_1 + Q_{10}|a_0|^2 + Q_{11}|a_1|^2 + S_1|a_0|^2|a_1|^2 \\
+ R_{10}|a_0|^4 + R_{11}|a_1|^4) \quad (1.5)
$$

The $a_j$’s are the complex amplitudes of the two excited modes, and $Q_{jk}, S_j$ and $R_{jk}$ are the complex nonlinear coupling constants.

Note that only selected powers of the amplitudes appear in Eqs. (1.4,1.5); for example there are no quadratic terms. Normal form theory \textit{(e.g. Coullet & Spiegel 1983)} shows (a) that these equations are generic, \textit{i.e.} they apply to any system that is characterized by having two excited nonresonant modes, be it a pulsating star, a biological system, or any other system, in which the relative growth-rates of the modes are small ($\eta_j \ll 1$); and (b) that the omitted terms are not essential for the modal selection (bifurcation diagram), \textit{i.e.} that the \textit{nature} of the dynamical behavior does not depend on them.

Usually it is more convenient to use real amplitudes $A_j(t)$ and phases $\varphi_j(t)$, defined by $a_j(t) = A_j(t) \exp i\varphi_j(t)$. Inserting this relation into the above equations one gets:

$$
\dot{A}_j = A_j (\kappa_j + q_{j0}A_0^2 + q_{j1}A_1^2 + s_jA_0^2A_1^2 + r_{j0}A_0^4 + r_{j1}A_1^4) \quad (1.6)
$$

and

$$
\dot{\varphi}_j = \omega_j + \dot{q}_{j0}A_0^2 + \dot{q}_{j1}A_1^2 + \dot{s}_jA_0^2A_1^2 + \dot{r}_{j0}A_0^4 + \dot{r}_{j1}A_1^4, \quad (1.7)
$$
where we have introduced the real constants by the relations:

\[ Q_{jk} = q_{jk} + i\hat{q}_{jk} \]  
\[ S_{j} = s_{j} + i\hat{s}_{j} \]  
\[ R_{jk} = r_{jk} + i\hat{r}_{jk} \]

One notes that the introduction of the real amplitudes \( A_{j} \) in Eqs. (1.4, 1.5) produces a complete decoupling of the amplitudes from the phases.

In principle, the constants \( Q, S, \) and \( R \) can be derived from the stellar structure equations and the linear eigenvectors as was shown in Buchler & Goupil (1984). In practice, however this is a daunting task that has only been attempted in some very specific cases (Takeuti & Aikawa 1981), Klapp et al. 1985, Dziembowski & Krollikowska 1985). An alternative approach that works quite well is to derive them from numerical hydrodynamical studies (e.g. Buchler & Kovács 1987, and below).

In the earliest use of amplitude equations for describing DM pulsations (Buchler & Kovács 1986) the amplitude equations were truncated at the lowest nontrivial, \( i.e. \) cubic, terms. It was shown that in this order the single-mode fixed points and steady DM pulsations cannot simultaneously be stable for the same stellar model. Subsequently, the behavior of the hydrodynamical models and the observational constraints imposed by the Beat Cepheids and RR Lyrae stars have forced us to consider also the next order, quintic terms (Buchler, Yecko, Kolláth & Goupil 1999, hereafter BYKG). The properties of these equations and their applicability to the DM stellar pulsators were discussed in that paper. Here we present a slightly different discussion and approximation. While, for simplicity, BYKG retained only the \( r_{00} \) and \( r_{11} \) cubic terms, instead, we keep the \( s_{0} \) and \( s_{1} \) cubic terms because they are sufficient to give a good, and in fact a better fit to the results of the hydrodynamical calculations.

**Time-Independent Amplitude Equations**

If our primary interest is the study of steady pulsations, \( i.e. \) oscillations with constant amplitudes, then we need to consider only the fixed points (FP's) of the amplitude equations Eq. (1.6), defined by \( \dot{A}_{0} = \dot{A}_{1} = 0 \). When only one FP amplitude is nonzero (a single-
mode FP) the corresponding full-amplitude (nonlinear) pulsations of the star have constant amplitudes and they are mono-periodic. These pulsations are then called limit-cycles, when in addition they are stable.

Eqs. (1.7) show that for FP’s the phases become linearly increasing functions of time (because the $A_j$’s are constant) and they thus yield the nonlinear modal frequencies $\bar{\omega}_j = \dot{\phi}_j$.

The possible FP’s are therefore obtained by looking for the solutions of the 4 combinations of the equation pairs:

\[
\begin{align*}
\kappa_0 + q_{00}A_0^2 + q_{01}A_1^2 + s_0A_0^2A_1^2 + r_{00}A_0^4 + r_{01}A_1^4 &= 0 \\
A_1 &= 0
\end{align*}
\]

\[
\begin{align*}
\kappa_1 + q_{10}A_0^2 + q_{11}A_1^2 + s_1A_0^2A_1^2 + r_{10}A_0^4 + r_{11}A_1^4 &= 0 \\
A_0 &= 0
\end{align*}
\]

Without loss of generality, we label the two modes 0 and 1, because most of the time we will be talking about the fundamental and first overtone modes.

Eqs. (1.11 – 1.14) are a function of the squared amplitudes. It is therefore sometimes advantageous to work in $(A_0^2, A_1^2)$ space. Thus in Figure 1.1 we have plotted the loci defined by the first of each of Eqs. (1.11, 1.13) as solid and dotted lines, respectively, for an RR Lyrae model that will be discussed below. The remaining two loci are the positive x and y axes. The solid and open circles represent stable and unstable FPs which are the intersections of these loci. Note that the situation is much simpler when all the quintic terms are disregarded as in Buchler & Kovács (1986) because then the loci degenerate into a set of straight lines, and clearly there can be at most one DM solution in that case. Here we consider the more general case.

The trivial FP of the amplitude equations is the static stellar model, i.e. with both $A_0 = 0$ and $A_1 = 0$. If both growth-rates are negative ($\kappa_0, \kappa_1 < 0$) the FP is stable, and so is the star. The next simplest case is where only one $\kappa$ is positive in which case the only possible pulsation state of the star is a limit-cycle in that mode. For the existence of a DM solution both growth-rates need to be positive, but as we will see it is not a sufficient criterion for steady
Table 1.1  Possible fixed point structures of the nonresonant amplitude equations (in addition to the unstable trivial FP, $A_0 = A_1 = 0$), assuming $\kappa_0, \kappa_1 > 0$. S: stable, U: unstable.

| Number of FP’s | F  | O₁  | DM1 | DM2 |
|---------------|----|-----|-----|-----|
| 2             | S  | U   | –   | –   |
| 2             | U  | S   | –   | –   |
| 3             | S  | S   | U   | –   |
| 3             | U  | U   | S   | –   |
| 4             | S  | U   | S   | U   |
| 4             | U  | S   | S   | U   |

DM oscillations. With the nonresonant amplitude equations only a limited number of possible limit-cycle combinations are possible, and they are listed in Table 1.1.

Figure 1.1  DM solutions of the amplitude equations in the $(A_0^2, A_1^2)$ plane for an RR Lyrae model. Solid circles are stable and open circles are unstable fixed-points.
In order to make the problem tractable we now keep the $s$ terms, but omit the $r$ quintic terms in Eq. (1.11,1.13). It is simple to find the fundamental and first overtone limit-cycle amplitudes

$$A_{0,lc}^2 = \frac{-\kappa_0}{q_{00}}, \quad A_{1,lc} = 0$$  \hspace{1cm} (1.15)

$$A_{1,lc}^2 = \frac{-\kappa_1}{q_{11}}, \quad A_{0,lc} = 0. \hspace{1cm} (1.16)$$

Inserting those values into the amplitude equations corresponding to the other mode, we can easily obtain the conditions for the linear stability of the fundamental and overtone limit-cycles, respectively:

$$\bar{\kappa}_1 \equiv \kappa_1 + q_{10} A_{0,lc}^2 = \kappa_1 - \kappa_0 \frac{q_{10}}{q_{00}} < 0 \hspace{1cm} (1.17)$$

$$\bar{\kappa}_0 \equiv \kappa_0 + q_{01} A_{1,lc}^2 = \kappa_0 - \kappa_1 \frac{q_{01}}{q_{11}} < 0 \hspace{1cm} (1.18)$$

We illustrate the modal behavior with an RR Lyrae model for which we have computed the nonlinear coupling coefficients (see Table 1.2). The numbers have been rounded and the $q$’s and $s$’s
have been normalized to set the limit-cycle amplitudes to the unity (e.g. \( q_{00} = -\kappa_0 \) and \( q_{11} = -\kappa_1 \)).

In order to see the behavior of the solutions along a sequence of RR Lyrae models, we fix all these parameters but let \( \kappa_0 \) vary. This sequence of solutions, obtained from Eqs. (1.11 – 1.14), has already been depicted in Fig. 1.1.

From the stability of the limit-cycles one further finds that the range of stable fundamental mode pulsations is \( \kappa_0 > 0.007 \). The range for the stable overtone is \( \kappa_0 < 0.01 \). This indicates that both limit-cycles are possible (the ‘either-or’ regime in the pulsation jargon) and these multiple solutions exist in the \( 0.007 < \kappa_0 < 0.01 \) interval, and necessarily there is a DM fixed point for that regime which is furthermore unstable.

In Figure 1.1 one easily misses some important and interesting solutions that occur in an extremely narrow range of \( \kappa_0 = 0.0100 \) to 0.0101, located between the last two subfigures. In fact for the parameters of Table 1.3 the upper, solid curve undergoes two intersections with the dotted curve which corresponds to two additional DMs.

In order to exhibit this behavior in a more apparent way we consider the quantities \( F_j \), derived from Eqs. (1.11, 1.13).

\[
F_j = A_1^2 = -\frac{\kappa_j + q_{j0}A_0^2}{q_{j1} + s_jA_0^2} \quad j = 0, 1. \tag{1.19}
\]

Plots of \( \delta F = F_1 - F_0 \) as a function of \( A_0^2 \) give the amplitudes of the DMs solution. It can be seen than the fixed point is stable if the slope of \( F_0 \) is steeper than that of \( F_1 \), i.e. \( d\delta F/dA_0^2 > 0 \). Otherwise no stationary DM pulsation is possible.

A stable DM solution exists and both limit-cycles lose their stability if and only if \( \delta F \) is an increasing function of \( A_0^2 \) in the neighborhood of the fixed point. With the opposite slope both the fundamental and the first overtone limit-cycles are stable, giving the “either-or” region of pulsation (Buchler & Kovács 1986).

Figure 1.2 shows how the \( \delta F = F_1 - F_0 \) function varies as \( \kappa_0 \) is changed. Only a very narrow range exists where the nearly parabolic function has two intersections with the zero line.

Both Figs. 1.1 and 1.2 show that the DM amplitudes are always smaller than the corresponding SM limit-cycle amplitudes.
When the DM occurs near the $A_1$ axis, then $A_{1,\text{dm}} \leq A_{1,\text{lc}}$, but $A_{0,\text{dm}} \ll A_{0,\text{lc}}$. With the given parameters the maximum fundamental amplitude is only 30% of the limit-cycle amplitude. For most of the RRd stars the fundamental mode amplitude is indeed smaller than that of the first overtone (see Kovács in this Volume), indicating that the above RR Lyrae model is on the right track.

### Time-Dependent Amplitude Equations

The time-dependent solutions of the amplitude equations become essential when we interpret the numerical hydrodynamical integrations or explain the changes due to e.g. evolutionary changes of the star. As we see above, very small changes in $\kappa$'s can induce significant changes in the possible pulsation state of the star. In our test problem if the fundamental mode growth-rate is slowly increasing, at $\kappa_0 = 0.01$ the overtone mode loses its stability. Then the model evolves smoothly to the DM state, if the system has been pulsating in the overtone mode before. With a further increase of $\kappa_0$, at $\kappa_0 \approx 0.01007$, the fundamental mode pulsation remains the only stable state. Then the amplitude equations can be used to estimate how the stars switches from the DM state to the fundamental limit-cycle.

In Fig. 1.3 the temporal variation of the amplitudes are shown during the simulated mode switching. The integration of the amplitude equations was initiated (at $t=0$) at the stable DM fixed point, and $\kappa_0$ was continuously increased at a $10^{-7}$ years$^{-1}$ rate. The DM oscillations became unstable during the first 10 years, but it took more than 150 years to evolve the star to the fundamental mode pulsations.

Right at the bifurcation point where the stable DM pulsation trades stability with the unstable fundamental SM limit-cycle the growth-rate of the DM pulsation is infinitely small and it takes evolution for the star to move into the regime where this growth-rate
achieves sufficiently large values for the switch to the fundamental limit-cycle to occur. It is therefore clear that the mode switching time-scale depends both on the evolution time-scale and on the sensitivity of the modal growth-rate to the evolution time-scale.

Figure 1.3  Simulated mode switching as predicted by the amplitude equations, see text; solid: $A_0$, dotted: $A_1$.

In section 4. we discuss how the hydrodynamical calculations can be used to reconstruct the amplitude equations.

3. HYDRODYNAMICAL MODELLING OF DOUBLE-MODE PULSATION

The Hydrodynamical Code

The classical variable stars that we consider in this review are radial pulsators. The global motions thus can be modelled by 1D hydrodynamics (see however Kovács in this Volume for indication of nonradial modes in RR Lyrae stars). Since real 3D modelling remains a dream in nonlinear pulsation calculations, the local nonradial flows i.e. turbulence have to be treated in some one dimensional approximation. Recipes and pulsation codes for 1D turbulent convection (TC) have been developed by different authors (e.g. Stellingwerf 1982, Kuhfuß 1986, Gehmeyr & Winkler 1992, Feuchttinger, M. U. 1998, Yecko, Kolláth, & Buchler 1998).

The fluid dynamics part of the model calculations are given by the following equations:

$$\frac{du}{dt} = -\frac{1}{\rho} \frac{\partial}{\partial r} (p + p_t + p_v) - \frac{GM_r}{r^2} \quad (1.20)$$
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\[ \frac{de}{dt} + p \frac{dv}{dt} = -\frac{1}{\rho r^2} \frac{\partial}{\partial r} \left[ r^2 (F_r + F_c) \right] + C \]  (1.21)

The turbulent motion of the gas and the convection interacts with the hydrodynamics of the radial motion through the convective flux \( F_c \), the viscous eddy pressure \( p_v \), the turbulent pressure \( p_t \), and finally, through an energy coupling term \( C \). Those terms can be derived by averaging the 3D hydrodynamics equations. This procedure introduces a sequence of moment equations (see e.g. Canuto 1998; Canuto & Dubikov 1998). In the simplest recipe we truncate all this equations, but leave the time-dependent diffusion equation for the turbulent energy \( e_t \):

\[ \frac{de_t}{dt} + (p_t + p_v) \frac{dv}{dt} = -\frac{1}{\rho r^2} \frac{\partial}{\partial r} \left( r^2 F_t \right) - C \]  (1.22)

The coupling term that connects the gas and the turbulent energy equations is given by:

\[ C = -\frac{e_t^{1/2}}{\Lambda} \alpha_d \left( e_t - S_t \right), \]  (1.23)

where \( \Lambda = \alpha_A H_p, H_p = p r^2/(\rho GM) \) is the pressure scale height \( \alpha_A \) is the mixing length parameter and \( \alpha_d \) is a dimensionless parameter.

Both the convective flux and the source term of the turbulent energy \( (S_t) \) depend on the entropy gradient:

\[ Y = -\frac{H_p}{c_p} \frac{\partial s}{\partial r} \]  (1.24)

\[ S_t = (\alpha_s \alpha_A)^2 \frac{P}{\rho} \beta T Y f_{pec}, \]  (1.25)

\[ F_c = \alpha_c \alpha_A \rho e_t^{1/2} c_p Y f_{pec}, \]  (1.26)

where \( \beta \) is the thermal expansion coefficient \( \alpha_s, \alpha_c \) are parameters and other symbols have their usual meanings. The Péclet correction \( f_{pec} \) accounts for the decrease of convective efficiency when radiative losses are important. We approximate this correction factor by
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\[ f_{pec} = \frac{1}{1 + \alpha_e Pe^{-1}} \]  
\[ Pe = \frac{D_c}{D_r} \]  
\[ D_r = \frac{4 a c T^3}{3 \kappa \rho^2 c_p} \]  
\[ D_c = \Lambda e_i^{1/2} \]

The remaining quantities are defined as

\[ p_t = \alpha_p \rho e_t \]  
\[ p_\nu = -\frac{4}{3} \alpha_\nu \rho \Lambda e_i^{1/2} r \frac{\partial u}{\partial r} \]  
\[ F_t = -\alpha_t \rho \Lambda e_i^{1/2} \frac{\partial e_t}{\partial r} \]

The TC model equations are based on a very simple physical picture of turbulent energy generation and diffusion, and convective energy transport. The various expressions can essentially be derived from dimensional arguments (even though numerous attempts have been made to derive them from more detailed analyses - for a dated, but still excellent review, see Baker 1987). The above definitions of the turbulent energy source and of the convective flux are not unique, and slightly different expressions have been used by different authors. The formulation adopted is of that of Gehmeyr & Winkler (1992). For a comparison of the different recipes used for stellar pulsation calculations see Buchler et al. (1999).

The dimensionless \( \alpha \) parameters of our turbulent convection model equations are of order unity, but theory provides no guidance as to their numerical values. We will resort to a comparison of numerical results with empirical data, such as the position and the width of the instability strip, to calibrate those parameters.

We have not yet performed a full calibration, which is a giant task because of the number of parameters, and in fact it is not even clear whether all observational constraints can be fulfilled with the use of such a simple model for turbulent convection. In the meantime we have chosen several sets of \( \alpha \) parameters that satisfy some of the constraints, and we have explored the consequences of turbulent convection for the linear and nonlinear properties of pulsation.
It has been comforting that, for the first time, we have been able to calculate DM pulsation with these parameters for a wide set of (physical) stellar parameters. However, the results still miss some of the observational constraints, e.g. the model amplitudes are small compared to the observed ones. The modelling of 1st overtone/2nd overtone DM pulsations is still not quite satisfactory. At this time no detailed or comprehensive comparison has been made between the observations and the DM modelling. Thus, this paper is a status report on modelling rather than a closed and self-contained theory of DM pulsations. However, the tendencies, e.g. along a sequence with different temperatures, can be used as primary tests for our models.

The first step in the numerical modelling is the construction of the static envelope model, and its linear stability analysis which yields the possible pulsation periods and the corresponding excitation-rates. These modal growth-rates tell us only the rate of increase or decrease of the oscillation for tiny amplitudes. Furthermore, if more than one mode is excited we cannot infer from this information which of the modes will grow into full-amplitude nonlinear oscillations. This 'modal selection' problem has already been discussed in section 2, where it was shown how it can be attacked with the amplitude equation formalism.

There exists no really simple way for computing the limit-cycles or the final DM or multi-mode pulsation state of the stellar models. The hydrodynamical modelling is an initial value problem, and we need to evolve the model directly with the hydrodynamics code, or through a relaxation method to obtain the stable oscillations. Furthermore, the model can evolve to different final stationary states depending on the initial kick of the static envelope. Fortunately, the investigated models have only one or two possible stable pulsation states, since for radially pulsating envelopes only a few modes are generally linearly unstable (at least for the Cepheids and RR Lyrae). In the case of nonradial pulsations, where dozens of modes are linearly unstable, the number of stable NL oscillations can be higher.

A great advantage of the Florida code is that it can relax to the periodic solutions by iteration and it provides the stability (Floquet) analysis of the limit-cycles. When both the fundamental mode and the first overtone are linearly unstable but the other
modes are stable, then the Floquet exponents of their limit-cycles provide the primary information on modal selection.

**DM Solutions in the Numerical Calculations**

The computation of the possible full-amplitude (or nonlinear) periodic pulsation states of a stellar model is essentially routine work with the relaxation code. However there exists no similar method for finding DM or other multi-mode full amplitude pulsations. Even when the fundamental and the first overtones are both linearly unstable, nothing guarantees the existence of steady DM pulsations. If both periodic limit-cycles are unstable against perturbations, then the model should oscillate in a DM state only. However, as the solutions of the amplitude equations indicate (see Section 2.) stable DM solutions can exist even when one of the single-mode limit-cycles is stable. In such a case the Floquet analysis is insufficient for identifying DM pulsations.

The most efficient way of kicking the initial static model with the velocity eigenvectors from the linear stability analysis. In order to obtain different transient paths we thus use linear combinations of the fundamental and the overtone mode eigenvectors with a surface amplitude of $\approx 10$ km/s.

![Figure 1.4](image.png)

*Figure 1.4*  The initial transient in RR Lyrae model pulsations

In Figure 1.4 the radius variation of an RR Lyrae model is shown for the initial 100 days of model integration. The multi-component nature of the signal can be seen from an inspection of the curve – the beats are clearly visible. The amplitudes of the modes are seemingly saturated, but further integration of the model shows that this is not the case. At that stage of the hydrodynamical evolution one cannot predict the final pulsation state: it can be a
limit-cycle with any of the unstable modes, or a DM pulsation. The system generally has to be integrated for a very long time (usually for thousands of cycles) to determine whether the amplitude of the modal components saturates at some finite value.

To emphasize the importance of long term integration we have calculated the amplitudes from an extended continuation of the transient shown in Fig. 1.4. The variation of the amplitudes is displayed in Fig. 1.5. Note the different time-scales on the two figures: the curve shown in Fig. 1.4 corresponds to the very beginning of the amplitude evolution on Fig. 1.5.

![Figure 1.5](image)

*Figure 1.5* The evolution of the amplitudes after the transient shown in Fig. 1.4. Solid line: fundamental mode, dotted line: first overtone.

Claiming DM behavior is therefore not an easy task. Very long transients with mixed-mode behavior exist in the hydrodynamical calculations, even when the model finally converges to a SM limit-cycle. This means that the knowledge of the Fourier spectra and of the velocity and light-curves from a single hydrodynamical calculation of a model cannot provide a sufficient proof of steady DM behavior. A more elaborate set of calculations is necessary. This is the topic of the next sections.

**Time-Dependent Amplitudes and Phases**

Frequently a visual inspection of the time-series that is obtained from the hydrodynamical calculations may indicate that the pulsations have reached a steady state. However these indications can be quite treacherous. Even the Fourier analysis of the light, velocity or radius variation can be misleading because of the temporal changes of amplitudes and periods in the transitory stage of model calculations. In order to obtain reliable information from the numerical
results it is necessary to apply a well suited time-frequency analysis and derive instantaneous amplitudes and periods.

A variety of such methods are available. For example, Kovács, Buchler & Davis (1987) used a time-dependent Fourier method. They performed linear least-squares fits with sine functions to very small successive portions of the data. The bases of the successive fits were shifted by small portions of the fit, and the results were averaged to get smooth results. The disadvantage of this method is that short bases provide large errors on the amplitudes, while longer bases introduce temporal averaging destroying the instantaneous nature of the amplitudes and phases.

Instead of the time-dependent Fourier analysis one can follow Gábor (1946) to reconstruct the time-dependent amplitudes and frequencies of the pulsation modes. Let \( s(t) \) represent the real part of an assumed complex analytical function \( a(t) \). The imaginary part \( \tilde{s}(t) \) of \( a(t) \) can then be obtained via a Cauchy integral, which through contour deformation becomes a Hilbert transform. Physicists are generally familiar with this analytic signal concept through the Kramers–Kronig dispersion relations (Jackson 1975).

\[
a(t) = s(t) + i\tilde{s}(t) = s(t) + \frac{i}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{s(t')}{t-t'} dt' \quad (1.34)
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} s(t')e^{i\omega(t-t')} dt' d\omega \quad (1.35)
\]

\[
\equiv A(t)e^{i\varphi(t)} \quad (1.36)
\]

It is easy to verify for example that with \( s(t) = A \cos \omega t \) one finds \( \tilde{s}(t) = A \sin \omega t \). The Gábor construction makes it possible to unambiguously define the phase \( \varphi(t) \) and the amplitude \( A(t) \) of a signal. Consequently it allows one to extract the instantaneous frequency (Cohen 1994).

It is possible to extend the method to multi-component signals. To obtain the instantaneous amplitudes and periods of the pulsation modes separately one first filters the signal to eliminate the power from the other frequency components. It is convenient to make the filtering in Fourier space, and combine it with the definition of the analytic signal.
\[ Z_k(t) = a_k(t)e^{i\varphi_k(t)} = \frac{1}{\pi} \int_0^\infty H(\omega - \omega_k) \int_{-\infty}^\infty s(t')e^{i\omega(t-t')} dt' d\omega, \]

where \( H(\omega - \omega_k) \) is the window of the filtering, centered on \( \omega_k \). A Gaussian window with a half width of 0.2 c/d gives a satisfactory result for RR Lyrae models. The resulting amplitude and phase gives the temporal evolution of the given mode. The least-squares fitting of a Fourier sum to the radius variation gives essentially the same results. However the analytic signal gives better resolution and it is not necessary to average the amplitudes in time to get smooth results. Thus it is more appropriate for further analysis. A great advantage of the analytic signal is that the resulting amplitudes can be directly used to fit the coefficients of the amplitude equations in a differential form – a simple linear least-squares method. The amplitudes calculated by the method of KBD87 are not suitable for a differential fit, but only for a complicated nonlinear least-squares fit to the integral curves.

**Modal Selection Problem**

Armed with this tool we can now analyze the modal selection problem in the hydrodynamical models. A time consuming, but very reliable method is to perform a number of hydrodynamical integrations with different initial perturbations of the same static model, and then to extract the slowly varying amplitudes with the help of a time-frequency analysis. The resulting phase-portrait \((A_1(t) \text{ vs. } A_0(t))\) gives a clear global overview of the nonlinear behavior of the model, and thus permits one to see immediately if stable DMs exist.

In Fig. 1.6 we show a representative set of such hydrodynamical tracks for a Cepheid model. The initial perturbations have been consisted of a mixture of the fundamental and the first overtone eigenvectors. Fig. 1.6 provides conclusive proof of the presence of stable beat pulsation with amplitudes \( A_0 = 0.016 \) and \( A_1 = 0.028 \) that coexists with the stable fundamental mode with amplitude \( A_0 = 0.049 \). In this case there is hysteresis and the actual state of the star depends on its former evolutionary path. From Fig. 1.6 one can also infer that two additional nontrivial fixed points exist for the given model in the phase-space of the amplitudes, namely a
The flow in the \((A_0, A_1)\) phase-space. The large dots represent the hydrodynamical results. The short lines represent the flow in the phase-space. The two separatrices are clearly visible: the first runs from the origin upward between the fourth and fifth evolutionary tracks, and the second one is the arc connecting the two single-mode fixed points.

First overtone limit-cycle and a DM saddle-point, but these states must be unstable. The observed behavior of the model is in a good agreement with the results found with the amplitude equations. The amplitude equations can be used to find all the fixed points, even if the hydrodynamical trackss do not actually reach them, either because of insufficient integration time, or because they are unstable and thus simply unreachable this way. The flow field that is obtained from the amplitude equations is represented by the short thin lines in Fig. 1.6.

The procedure that we have described, namely numerical hydrodynamics integrations with judiciously chosen different initial
conditions, combined with a time-frequency analysis and with a Floquet analysis provide a complete picture of the possible pulsation states of a given model and of their stability. By repeating the calculations with different effective temperatures and luminosities one can thus map the whole picture of modal selection on the HR diagram.

4. RESULTS

We have already mentioned that the calibration of the turbulent parameters is a big task that has not been finished yet. The results that we present here can show the general tendencies that exist in the models, but we cannot pretend that the models can be used to infer for example physical parameters from individual variable star observations.

The Coupling Coefficients

The time-dependent amplitude equations describe the evolution of the amplitudes of the excited modes. From the knowledge of the transient hydrodynamical tracks for a given model it is thus possible to evaluate the a priori unknown nonlinear coefficients of the amplitude equations.

Of course, in order to extract reliable nonlinear coupling coefficients it is necessary to integrate the models from a sufficient number of initial conditions so that the ‘trajectories’ sample well the phase-phase \((A_0,A_1)\). Numerical simulations have show that multiple DM solutions, a stable fixed point and a saddle point can co-exist in the model pulsations. From the analysis of the fixed-points of the amplitude equations one can conclude that it is necessary to include at least one quintic term into (see Section 2.). According to our experiments the mixed quintic coefficients \((s_j)\) are usually sufficient to give a good fit. The inclusion of the other quintic terms \((r_{ij})\) results only in a minor improvement of the fit. We have to note however that fits with the \(r_{ii}\) terms only are also capable of reproducing the main features of the amplitude evolution.

In Table 1.3 we present the coefficients of the amplitude equations for a sequence of RR Lyrae models where a DM solution exists in the 6500\(K < T_{eff} < 6515K\) temperature range. There are no
significant variations with $T_{eff}$ for most of the coefficients, but the growth-rate $\kappa_0$ has a decreasing trend.

Table 1.3 Coupling coefficients in a sequence

| $T_{eff}$ | $\kappa_0$ | $\kappa_1$ | $q_{00}$ | $q_{01}$ | $q_{10}$ | $q_{11}$ | $s_0$  | $s_1$ |
|-----------|------------|------------|----------|----------|----------|----------|--------|--------|
| 6495      | 0.009397   | 0.03975    | -3.08    | -20.18   | -10.16   | -44.20   | -2789. | -729.  |
| 6500      | 0.009319   | 0.03976    | -3.07    | -20.29   | -10.13   | -44.12   | -2788. | -583.  |
| 6505      | 0.009235   | 0.03967    | -3.07    | -20.35   | -10.13   | -43.94   | -2750. | -500.  |
| 6515      | 0.009095   | 0.03976    | -3.08    | -20.62   | -10.08   | -43.90   | -2833. | -384.  |
| 6525      | 0.008896   | 0.03971    | -3.07    | -20.75   | -9.99    | -43.72   | -2792. | -360.  |

We have fitted the coefficients with a quadratic function of temperature (e.g. $s_j = C_0 + C_1 T_{eff} + C_2 T_{eff}^2$), and we have used the fit to map the solution of the amplitude equations as a function of temperature. Fig 1.7 shows the variation of the function $\delta F$ as the temperature is decreased. The displayed behavior is very similar to the one displayed in Fig 1.2, where only one of the parameters, the fundamental mode growth-rate was changed. If we fix all the parameters but decrease $\kappa_0$ according to the temperature fit, the basic tendencies on the figures remain the same i.e. for the given example the variation of the fundamental growth-rates plays the most important role.

The Fourier representation of steady DM pulsations

The primary information one can derive from observational data is the spectral content of the light variations, the amplitudes and periods presented in the Fourier spectra. (Examples of RR Lyrae frequency spectra can be found in Kovács’s review in this Volume.) In addition, because of the nonlinearity of pulsation, linear combinations of the primary frequencies also appears in the spectra. A good example for the existence of those terms is the analysis of the Cepheid TU Cassiopeia (Szabados 1993), where 29 frequencies have been identified. Even terms like $4f_0+3f_1$ or $5f_0+2f_1$ are inferred from the Fourier transform of the light-curve. In the RR Lyrae spectra typically only the $f_0+f_1$ and $f_0–f_1$ terms can be found in addition to $f_0$ and $f_1$. 
In order to relate the observed frequency spectra with those calculated from model pulsations, we have calculated synthetic data with the observed periodicities but with the same sampling and length as the hydrodynamical outputs (typically 1000 cycle). In Figure 1.8 we compare the spectra of TU Cas and of a MACHO DM star presented in Figure 2 of Kovács in this Volume with typical hydrodynamical results. We note that the amplitudes are given in bolometric magnitudes for the models and in $'V'$ (TU Cas) and in instrumental red (MACHO RR Lyr). The agreement is satisfactory between the RR Lyrae models and observations. TU Cas has a large amplitude variation – it is the high nonlinearity that causes the large power at the harmonics of the fundamental mode. Our model has a much lower amplitude and consequently a weaker harmonic power, and it thus misses this behavior.

When the harmonics of the modal frequencies can be observed, the corresponding Fourier parameters provide useful information. It has been observed that the Fourier parameters, $\varphi_{21}$ and $\varphi_{31}$, of both modes in DM Cepheids are very similar to those of the corresponding modes in mono-periodic stars with similar periods.
Figure 1.8 Fourier spectrum of the light variation: From top to bottom: TU Cassiopeia, MACHO RRd variable, Cepheid model, RR Lyrae model.

(e.g. Udalski et al. 1999, Beaulieu, private communication). Since the models relax very slowly on the separatrix connecting the F mode to the O\(_1\) through the DM solution, one can calculate the Fourier parameters along the separatrix. This test provides a clue how the Fourier phases depend on the modal amplitudes (Fig. 1.9). We have selected a model from the middle of the DM region for the plot, but we checked the Fourier parameters for the whole range and obtained very similar results. The Fourier parameters were calculated from the hydrodynamical model light-curves, and the amplitudes are in magnitudes. To indicate the position of the separatrix, we also show the \(A_1\) vs. \(A_0\) plot of the models in the top right box. As expected the \(R_{21}\) amplitude ratios have a strong dependence on the corresponding amplitudes. For the fundamental mode this dependence is almost exactly linear. The plots of the \(\varphi_{21}\)
Fourier phases are very flat because there are no resonances. The difference between the $\varphi_{21}$ value of the DM solution and that of the SM limit-cycle is less than 0.2. The $\varphi_{31}$ parameters have the same tendencies but with a little bit stronger amplitude dependence for the F mode.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_graphs}
\caption{The dependence of the Fourier parameters on the fundamental mode amplitude along the separatrix. Filled hexagons: fundamental mode, open squares: first overtone. The top right box shows the models on the $(A_0, A_1)$ phase-space.}
\end{figure}

**The Period ratios**

The period ratios offer an additional constraint for determining the physical parameters of DM Cepheids and RR Lyrae stars (e.g. Kovács in this Volume). These investigations make use of the linear periods and the latter are usually based on radiative pulsation models. Thus it is very important to check how turbulent convection and nonlinear effects affect the period ratios.
Turbulent convection

The dominant source of period shifts between the radiative and turbulent convective models is the convection induced change in the structure. In the radiative models there is a huge temperature gradient around the hydrogen partial ionization zone, because radiation is not sufficiently effective in transporting energy. In the turbulent convective models the convective flux carries some energy and thus reduces the temperature gradient. This change in the stellar structure causes a decrease in the period ratios. The shift depends on the efficiency of the convection, and thus is larger for lower temperature models. Since the convective flux parameter ($\alpha_c$) has not been satisfactorily calibrated so far, it is not possible to state definitive values for the shifts in the period ratios. Depending on the parameters of the models one finds $(P_1/P_0)_{tc} - (P_1/P_0)_{rad} \approx -0.0005$ to $-0.002$. Turbulent viscosity causes an additional shift of the periods, but that is small compared to the effect of the convective flux.

Nonlinearity

The magnitude of the nonlinear period shifts has remained unknown due to the lack of nonlinear DM pulsation models. In order to isolate the period shifts from the larger period variations with $T_{\text{eff}}$ we have used the time-frequency analysis of the model output together with the amplitude-equation formalism, instead of checking only the periods in the stationary DM oscillations.

The phase that is calculated from the analytic signal (Eq. 1.37) is sufficiently smooth to yield the instantaneous frequencies by a numerical time derivative of $\varphi(t)$. Hydrodynamical integrations with a set of different initial perturbations then can be used to fit the coefficients in the phase equation (Eq. 1.7). Since the nonlinear correction of the frequencies are small compared to $\omega_j$, the period ratios can be fitted directly with monomials

$$\frac{(P_1/P_0)_{nl} - (P_1/P_0)_{lin}}{(P_1/P_0)_{lin}} = \sum C_{k,l} a_0^k a_1^l.$$ (1.38)

Fig. 1.10 and 1.11 shows examples of the hydrodynamical tracks that have been used to extract the nonlinear instantaneous period shift as a function of the amplitudes in the transient states. Fig 1.10
refers to an RR Lyrae model \((M=0.77M_\odot, L=50L_\odot)\) and Fig 1.11 to a Cepheid model \((M=4.0M_\odot, L=1100L_\odot)\). Although this fit gives an interpolation formula for a specific model (with given mass, luminosity, and effective temperature) only, the fit is found to vary only weakly inside the DM region.

The 'equi-period-ratio' curves in the phase-space of the amplitudes are also shown in Fig 1.10 and 1.11. The relative changes of the nonlinear (TC) period ratios compared to the linear ones are indicated in the figures. For the Cepheid model the period ratio is always smaller than the linear one. In contrast, for RR Lyrae models the shift can be both positive and negative. These trends in the nonlinear shifts of the period-ratios have been found to persist in our model calculations throughout a wide range of model param-
Since the vast majority of the RR Lyrae stars pulsate with higher amplitude in the first overtone than the fundamental mode, the observed period ratios are likely to be smaller than the linear values. Combining the two results we can conclude that the nonlinear turbulent convective period ratios are thus generally, but not always, smaller than the linear radiative ones, and that the relative difference is of the order of several tenths of a percent.

**Modal selection – Bifurcation Diagram**

The model calculations with different initial perturbations of the static model give the possible fixed point solutions of the star. The behavior of the individual models can be classified into five groups:
(1) first overtone only (O),
(2) fundamental mode only (F),
(3) either fundamental or first overtone (E),
(4) DM only (D), and
(5) either fundamental or DM (H).

We have not encountered any model where the first overtone and the DM fixed points are simultaneously stable, but in principle such a situation is possible. For models with ‘E’ or ‘H’ characteristics there is hysteresis and one should examine a sequence along the possible evolutionary path of the star to predict its pulsation state. Sequences with different temperatures, but with a fixed luminosity, give good estimates for that purpose.

Figure 1.12  Modal selection on the \( (T_{\text{eff}}, \alpha_v) \) plane for a Cepheid model. The symbols of the modal states are described at the top of the figure.

In Fig. 1.12 the dependence of the modal selection on the eddy viscosity is shown for a Cepheid model sequence with SMC composition. The Galactic Cepheids and the RR Lyrae models show very
similar behavior (for similar $\alpha$ parameters). The consistency of this similarity is somewhat astonishing in light of the serious discrepancies that one finds between the the observations and the calculations of low metallicity single-mode Cepheids (to be discussed elsewhere).

![Figure 1.13](image)

Figure 1.13 Modal selection of galactic Cepheid model sequences on the HR diagram. See Fig. 1.12 for the notation.

Turbulent viscosity has two dominant effects. The first one is to shift the transition region to higher temperatures (and to shorter periods) at higher $\alpha_\nu$ values. The second one is to changes the characteristics of modal selection. For high viscosity parameters a DM-only solution exists sandwiched between the O and F state (we denote this transition by O-D-F). As the eddy viscosity decreased an 'either F or DM' (H) state appears at temperatures between the D and F solutions (O-D-H-F transition). For low $\alpha_\nu$ values the characteristics of the transition changes to O-E-H-F i.e. the DM only region is changed to an 'either-or' one.

For different stellar parameters (see Fig. 1.13) in addition to the transition types seen in Fig. 1.12 we could identify the following variations of modal behavior: O-E-F and O-D-H-E-F sequence.

For the O-D-F type sequence DM behavior occurs independently of the direction of evolution. However, for the O-E-H-F case DM
pulsation can be observed only if the star evolves from the blue side of the instability strip to the red. In that case the star evolves to the DM state, when the $O_1$ mode loses its stability, while on the opposite evolutionary path the star moves to the overtone directly from the F mode at the blue edge of the fundamental mode instability strip through a transient DM state.

The modal selection is illustrated in Figures 1.13 and 1.14 both for a Galactic Cepheid model and for an RR Lyrae model in an HR diagram.

We have calculated the Cepheid models with a mass–luminosity relation (the masses of the models are indicated at the right end of the gray stripes). One notes a very interesting feature, namely the disappearance of the DM solution at higher luminosities, even where stable first overtone pulsation exists. This result is in good agreement with the observations by Udalski et al. (1999).

Despite this very encouraging agreement we recall however that these bifurcation diagrams remain tentative until a full calibration of the $\alpha$ parameters has been made.
5. THE FUTURE

Thanks to the incorporation of turbulent convection in the hydrodynamical codes it is finally possible to model steady DM pulsations both in Cepheids and RR Lyrae stars. Some problems persist though. The large-scale surveys have found second overtone \((O_2/O_1)\) DM pulsators at similar population as \(O_1/F\) beat Cepheids. The models should reproduce this constraint as well. With the recent parameter setting we could find some \(O_2/O_1\) models, but in a very narrow range of temperature and turbulence parameters. To model these DM variables satisfactorily, we have to explore the \(\alpha\) parameter space in a wider range and perhaps improve the description of turbulent convection.

This effort is important because a consistent modelling of DM oscillations in both Cepheids and RR Lyrae stars and with different chemical compositions is necessary for understanding the physical mechanisms inside the stars, on the one hand, and for enhancing our confidence that we can use these stars reliably as standard candles, on the other hand.

Acknowledgments

This work has been supported by the Hungarian OTKA (T-026031) grant and by the National Science Foundation (AST9819608). It is a great pleasure to acknowledge the collaboration of Jean-Philippe Beaulieu, Zoltán Csubry, Róbert Szabó, and Phil Yecko on this project. We wish to thank Michael Feuchtinger for a thorough reading of the manuscript.

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