Almost everywhere convergence on the solution of Schrödinger equation is an important problem raised by Carleson in harmonic analysis. In recent years, this problem was essentially solved by building the sharp $L^p$-estimate of Schrödinger maximal function. Du-Guth-Li in [9] proved the sharp $L^p$-estimates for all $p \geq 2$ in $\mathbb{R}^{2+1}$. Du-Zhang in [13] proved the sharp $L^2$-estimate in $\mathbb{R}^{n+1}$ with $n \geq 3$, but for $p > 2$ the sharp $L^p$-estimate of Schrödinger maximal function is still unknown. In this paper, we obtain partial results on this problem by using polynomial partitioning.

1. INTRODUCTION

We consider the free Schrödinger equation:

\[
\begin{cases}
  iu_t - \Delta u = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\
  u(x, 0) = f(x), & x \in \mathbb{R}^n.
\end{cases}
\]

(1.1)

Its solution is given by

\[e^{it\Delta} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + |\xi|^2)\frac{t}{2}} \hat{f}(\xi) d\xi,\]

where $\hat{f}$ denotes the Fourier transform of the function $f$.

One of the fundamental problems in this setting is that of determining the optimal $s$ such that

\[\lim_{t \to 0} e^{it\Delta} f(x) = f(x), \quad \text{for a.e. } x \in \mathbb{R}^n\]

(1.2)

for all $f \in H^s(\mathbb{R}^n)$. In 1979, Carleson [7] first purposed this problem and proved that the almost everywhere convergence (1.2) holds for any $f \in H^{1/4}(\mathbb{R})$ by making use of the stationary phase method. A year later, Dahberg and Kenig in [7] were able to prove the condition $s \geq \frac{1}{4}$ given by Carleson is sharp, showing that the existence of functions in $H^s(\mathbb{R})$ with $s < \frac{1}{4}$ for which the convergence fails. For the situation in higher dimensions, many authors such as Carbery in [4] and Cowling in [6] studied this problem, and in 1987 Sjölin in [22] and Vega in [24] proved independently that (1.2) holds if $s > \frac{1}{2}$ no matter the dimension. After that some important positive results in higher dimensions have been obtained by many authors [1, 8, 10, 19, 20, 21]. More recently, Bourgain in [2] gave counterexamples showing that (1.2) can fail if $s < \frac{n}{2(n+1)}$. Du-Guth-Li in [9] and
Du-Zhang in [13] improved the sufficient condition to the almost sharp range \( s > \frac{n}{2(n+1)} \) when \( n = 2 \) and \( n \geq 3 \), respectively. Hence, the Carleson problem was essentially solved except the endpoint.

Now, we recall the main ideas in [9, 13]. Suppose that \( f \) satisfies \( \text{supp} \hat{f} \subset B^n(0, 1) \). When \( n = 2 \), Du-Guth-Li in [9] proved the sharp \( L^3 \)-estimate of Schrödinger maximal function:

\[
\left\| \sup_{0 < t \leq R} |e^{it\Delta} f| \right\|_{L^3(B^2(0, R))} \lesssim R^\epsilon \|f\|_{L^2(\mathbb{R}^2)}, \quad \forall \epsilon > 0.
\]  
(1.3)

By Littlewood-Paley decomposition and parabolic rescaling, they applied (1.3) to derive that (1.2) holds for \( s > 1/3 \). On the other hand, using Hölder’s inequality, one immediately obtains whole sharp \( L^p \)-estimates:

\[
\left\| \sup_{0 < t \leq R} |e^{it\Delta} f| \right\|_{L^p(B^2(0, R))} \lesssim \begin{cases} R^{\frac{2}{3} - \frac{\epsilon}{2}} \|f\|_{L^2(\mathbb{R}^2)}, & 2 \leq p < 3, \\ R^{\epsilon} \|f\|_{L^2(\mathbb{R}^2)}, & p \geq 3. \end{cases}
\]  
(1.4)

This fact means that \( p = 3 \) is the minimal number such that

\[
\left\| \sup_{0 < t \leq R} |e^{it\Delta} f| \right\|_{L^p(B^2(0, R))} \lesssim R^\epsilon \|f\|_{L^2(\mathbb{R}^2)}, \quad \forall \epsilon > 0.
\]  
(1.5)

When \( n \geq 3 \), Du-Zhang in [13] showed (1.2) holds for \( s > \frac{n}{2(n+1)} \) by establishing the sharp \( L^2 \)-estimate

\[
\left\| \sup_{0 < t \leq R} |e^{it\Delta} f| \right\|_{L^2(B^n(0, R))} \lesssim R^{\frac{n}{2(n+1)} + \epsilon} \|f\|_{L^2(\mathbb{R}^n)}, \quad \forall \epsilon > 0.
\]  
(1.6)

We observe that (1.6) can recover the case of \( p = 2 \) in (1.4) when \( n = 2 \). The sharp \( L^p \)-estimate of Schrödinger maximal function implies the sharp \( L^2 \)-estimate by Hölder’s inequality, but the reverse is invalid. So Du-Zhang in [13] further raised the problem of identifying the sharp exponent \( p \) such that

\[
\left\| \sup_{0 < t \leq R} |e^{it\Delta} f| \right\|_{L^p(B^n(0, R))} \lesssim R^\epsilon \|f\|_{L^2(\mathbb{R}^n)}, \quad \forall \epsilon > 0.
\]  
(1.7)

From (1.3), we see that

\[
\left\| \sup_{0 < t \leq R} |e^{it\Delta} f| \right\|_{L^{\frac{2(n+1)}{n}}(B^n(0, R))} \lesssim R^{\epsilon} \|f\|_{L^2(\mathbb{R}^n)}, \quad \forall \epsilon > 0
\]  
(1.8)

holds when \( n = 1, 2 \), see [9, 13]. It would be an amazing result if (1.8) holds for higher dimensions, since \( p = \frac{2(n+1)}{n} \) corresponds to the endpoint exponent of \((n+1)\)-dimensional restriction conjecture. Unfortunately, Du-Kim-Wang-Zhang in [11] proved (1.8) fails when \( n \geq 3 \). More precisely, by researching Bourgain’s counterexample in every intermediate dimension in [2], they proved that (1.7) fails if

\[
p < p_0 := \max_{1 \leq m \leq n} \frac{4}{2 + \frac{4}{n - 1 + m + \frac{n}{m}}}. \tag{1.9}
\]

In particular, (1.8) fails for \( n \geq 3 \) due to \( \frac{2(n+1)}{n} < p_0 \). Recently, Wu in [26] proved that (1.7) holds for

\[
p \geq 2 + \frac{4}{n + 2 - \frac{1}{n}}. \tag{1.10}
\]
Our main result in this paper is the following:

**Theorem 1.1.** Let $n \geq 3$. Suppose that $p$ satisfies

\[ p \geq 2 + \frac{4}{n + 1 + \frac{1}{2} + \ldots + \frac{1}{n}}. \]  \hspace{1cm} (1.11)

For any $0 < \epsilon \ll 1$, there exists a constant $C_\epsilon$ such that

\[ \sup_{0 < t \leq R} \left\| e^{it\Delta} f \right\|_{L^p(B^n(0,R))} \leq C_\epsilon R^p \| f \|_{L^2} \]  \hspace{1cm} (1.12)

for all $f$ with $\text{supp} \hat{f} \subset B^n(0,1)$.

We firstly recall the $k$-broad norm $BL^p_{k,A}$ initially defined by Guth in [15]. $BL^p_{k,A}$-norm has two advantages compared with classical $L^p$-norm. The first is that $BL^p_{k,A}$-norm can lead to the $k$-linear reduction. Secondly, $\| e^{it\Delta} f \|_{BL^p_{k,A}(B_R)}$ is negligible whenever most of the mass of $e^{it\Delta} f$ is concentrated on an algebraic surface with the dimension less than $k$, named the vanishing property. Guth applied these properties to improve the bound of restriction conjecture in higher dimensions. Motivated by Guth’s idea, Du-Li in [12] extended $BL^p_{k,A}$-norm to a mixed version $BL^{p,\infty}_{k,A}$-norm for studying $L^p$-estimate of Schrödinger maximal function. The definition we present below is slightly different from that of their article. They introduced a new parameter $M$ in the definition which can help them to achieve the bilinear reduction in the main proof [12, Section 2]. However this makes that the vanishing property may invalidate when $M$ is large. And it will lead to an obstacle in our following argument. For reserving this important property, we give the following definition. On the bilinear reduction, we are going to take a different way to achieve it.

Let $f$ satisfy $\text{supp} \hat{f} \subset B^n(0,1)$. Decomposing $B^n(0,1)$ into balls $\tau$ of radius $K^{-1}$, where $K$ is a large constant, we have $f = \sum \tau \ f_\tau$ such that each $f_\tau$ is Fourier supported on $\tau$. Denote

\[ G(\tau) = \{ G(\xi) : \xi \in \tau \}, \text{ where } G(\xi) = \frac{(-2^{k,1})}{(\xi - 2^{k,1})}. \]

If $V \subset \mathbb{R}^n \times \mathbb{R}$ is a subspace, we use $\angle(G(\tau), V)$ to be the smallest angle between any non-zero vectors $v \in V$ and $v' \in G(\tau)$.

Next we decompose $B^n(0,R)$ into balls $B_K$ of radius $K$, and decompose $[0,R]$ into intervals $I_K$ of length $K$. For a given positive integer $A$, we define

\[ \mu_{e^{it\Delta} f}(B_K \times I_K) := \min_{V_1, \ldots, V_A} \left( \max_{\tau \notin V_a} \int_{B_K \times I_K} |e^{it\Delta} f_\tau(x)|^p dx dt \right), \]  \hspace{1cm} (1.13)

where $V_1, \ldots, V_A$ are $(k-1)$-subspaces of $\mathbb{R}^{n+1}$, and $\tau \notin V_a$ means that $\angle(G(\tau), V_a) > K^{-1}$.

For any subset $U \subset B_R := B^n(0,R) \times [0,R]$, define

\[ \| e^{it\Delta} f \|_{BL^{p,\infty}_{k,A}(U)} := \sum_{B_K \subset B(0,R)} \max_{I_K \subset [0,R]} \frac{|U \cap (B_K \times I_K)|}{|B_K \times I_K|} \mu_{e^{it\Delta} f}(B_K \times I_K). \]  \hspace{1cm} (1.14)

We reduce Theorem 1.1 to the following Theorem 1.2, and we will explain how Theorem 1.2 implies Theorem 1.1 in Section 2.
Theorem 1.2. Let $p$ satisfy (1.11). For any $0 < \epsilon \ll 1$, there exist a positive integer $A$ and a constant $C(\epsilon, K)$ such that
\begin{equation}
\|e^{it\Delta}f\|_{BL^{p,\infty}_{k,A}(B^n_R)} \leq C(\epsilon, K)R^\epsilon\|f\|_{L^2}
\end{equation}
for all $R \geq 1$ and all $f$ with supp$\hat{f} \subset B^n(0,1)$.

Since $BL^{p,\infty}_{k,A}$-norm is not continuous, it is not convenient to use polynomial partitioning. For this purpose, we substitute $BL^{p,\infty}_{k,A}$-norm with $BL^{p,q}_{k,A}$-norm, defined as
\begin{equation}
\|e^{it\Delta}f\|_{BL^{p,q}_{k,A}(U)} := \sum_{B_R \subset B(0,R)} \left[ \sum_{I_K \subset [0,R]} \left( \left| U \cap (B_K \times I_K) \right| / |B_K \times I_K| \mu_{e^{it\Delta}f}(B_K \times I_K) \right)^{\frac{q}{p}} \right]^{\frac{p}{q}}.
\end{equation}
Since
\begin{equation}
\|e^{it\Delta}f\|_{BL^{p,\infty}_{k,A}(B^n_R)} = \lim_{q \to \infty} \|e^{it\Delta}f\|_{BL^{p,q}_{k,A}(B^n_R)},
\end{equation}
it suffices to show the following result.

Theorem 1.3. Let $p$ satisfy (1.11). For any $0 < \epsilon \ll 1$, let $0 < \delta \ll \epsilon$, there exist a positive integer $A$ and a constant $C(\epsilon, K)$ such that for any $q > 1/\delta$,
\begin{equation}
\|e^{it\Delta}f\|_{BL^{p,q}_{k,A}(B^n_R)} \leq C(\epsilon, K)R^\epsilon\|f\|_{L^2}
\end{equation}
for all $R \geq 1$ and all $f$ with supp$\hat{f} \subset B^n(0,1)$.

Let us say a few words on the proof scheme of Theorem 1.3. The basic tool is polynomial partitioning introduced by Guth in [15], which helps us to make the dimensional reduction. We will adopt the algorithm, repeated application of polynomial partitioning, which has similar spirit as Hickman-Rogers in [17]. Hickman-Rogers built the algorithm to estimate $\|Ef\|_{BL^{p,q}_{k,A}(B^n_R)}$, where $Ef$ represents the extension operator associated with restriction problem. By building polynomial partitioning on $BL^{p,q}_{k,A}$-norm, we show the algorithm also works to estimate $\|e^{it\Delta}f\|_{BL^{p,q}_{k,A}(B^n)}$.

This paper is organized as follows. In Section 2 we explain how Theorem 1.2 implies Theorem 1.1. In Section 3 we introduce some notations and standard results. The proof of Theorem 1.3 is in Section 4 and Section 5.

To end up this section, we will outline some notations used throughout the paper. If $X$ is a finite set, we use $|X|$ to denote its cardinality. If $X$ is a measurable set, we use $|X|$ to denote its Lebesgue measure. If the function $f$ has compact support, we use supp$f$ to denote the support of $f$. $C_\epsilon$ denotes a constant which depends on $\epsilon$. Write $A \lesssim B$ or $A = O(B)$ to mean that there exists a constant $C$ such that $A \leq CB$. We write $A \lesssim \epsilon$ to denote $A \leq \epsilon R^\epsilon B$.

2. How maximal broad $L^p$-norm implies maximal $L^p$-norm

In this section, we prove that Theorem 1.2 implies Theorem 1.3 by broad-narrow argument. Broad-narrow argument was developed by Bourgain and Guth [3] to study restriction conjecture for the first time. This method can help us to reduce original linear...
estimate to $k$-linear estimate. Though the broad norm that we consider in the paper is slightly weaker than classical $k$-linear norm, the method and reduction still work.

For the convenience of the discussion, we first introduce an equivalent definition of $BL^{p,q}_{k,A}$-norm. Denote
\[ B^K_{r,k,A}(e^{it\Delta}f)(x) := \min_{V_1, \ldots, V_A \cap \mathbb{R}^d} \max_{\tau \in \mathbb{R}} \max_{a} |e^{it\Delta} f_\tau(x)|, \]
where $V_1, \ldots, V_A$ are defined as in Section 1. Then we have
\[ \|e^{it\Delta}f\|_{BL^{p,q}_{k,A}(B^n)} \sim \left\| B^K_{r,k,A}(e^{it\Delta}f)(x) \right\|_{L_p^L(B^n)}. \] (2.1)

In fact, note that $\tau$ is a $K^{-1}$-ball, we can view $|e^{it\Delta} f_\tau|$ as constant on $B_K \times I$. So (1.13) becomes
\[ \mu e^{it\Delta}f(B \times I_K) \sim \int_{B \times I_K} \min_{V_1, \ldots, V_A \cap \mathbb{R}^d} \max_{\tau \in \mathbb{R}} \max_{a} |e^{it\Delta} f_\tau(x)|^p dxdt. \]

Summing all $B_K \subset B(0,R)$ and all $I_K \subset [0,R]$ in (1.13), we can get (2.1) immediately. Here we have used one important property named locally constant property. We will introduce it more precisely in Section 3.

**Proposition 2.1.** For every $0 < \epsilon \ll 1$ and every sufficiently large $R$, there exist
\[ 1 \ll A \ll K_1 \ll \cdots \ll K_{n-1} \ll K_n \ll R', \]
such that the following holds. Suppose that $f$ is Fourier supported on $B^n(0,1)$, then
\[ |e^{it\Delta} f(x)| \lesssim K_n^{2n} \sum_{\alpha_1, \ldots, \alpha_{n+1} \text{transverse}} \left( \prod_{j=1}^{n+1} |e^{it\Delta} f_{\alpha_j}(x)| \right)^{\frac{1}{n+1}} + K_n^{2(n-1)} \max_{V^n} B_{r,n,A}(e^{it\Delta} f_{V^n})(x) + AK_n^{2(n-2)} \max_{V^n=1} B_{r,n-2,A}(e^{it\Delta} f_{V^{n-1}})(x) + \cdots + A^{n-2} K_1^2 \max_{V_1^2} B_{2,1}(e^{it\Delta} f_{V_1})(x) + A^{n-1} \max_{\gamma; K_1^{-1}-\text{balls}} |e^{it\Delta} f_\gamma(x)|. \] (2.2)

Here each $\alpha_i$ denotes the ball with radius $K_n^{-1}$, and each $V^l$ denotes one $l$-dimensional subspace. The definition of transverse see (2.4). And for each $1 \leq l \leq n$, we have
\[ |e^{it\Delta} f(x)| \lesssim K_l^{2(l-1)} \max_{V^l} B_{l,A}(e^{it\Delta} f_{V^l})(x) + AK_l^{2(l-2)} \max_{V^{l-1}} B_{l-1,A}(e^{it\Delta} f_{V^{l-1}})(x) + \cdots + A^{n-2} K_1^2 \max_{V_1^2} B_{2,1}(e^{it\Delta} f_{V_2})(x) + A^{n-1} \max_{\gamma; K_1^{-1}-\text{balls}} |e^{it\Delta} f_\gamma(x)|. \] (2.3)

**Proof.** We only prove (2.2) since the proof of (2.3) is the same. We first fix $x$ and $t$. Decomposing the frequency support $B^n(0,1)$ to balls $\alpha$ with radius $K_n^{-1}$, then
\[ e^{it\Delta} f(x) = \sum_{\alpha} e^{it\Delta} f_{\alpha}(x), \]
where each $f_{\alpha}$ is Fourier supported in $\alpha$. Define
\[ S(x,t) := \left\{ \alpha : |e^{it\Delta} f_{\alpha}(x)| \geq \frac{1}{K_n} \max_{\alpha'} |e^{it\Delta} f_{\alpha'}(x)| \right\}. \]

There are two cases that may occur now:
There exist \( \alpha_1, ..., \alpha_{n+1} \in S(x, t) \) which are \( (n+1) \)-transverse: for any \( v_j \in G(\alpha_j) \),
\[
|v_1 \wedge v_2 \wedge ... \wedge v_{n+1}| \geq K_n^{-n}.
\]  
(2.4)

There exists an \( n \)-dimensional subspace \( V^n \) such that for each \( \alpha \in S(x, t) \),
\[
\angle(G(\alpha), V^n) \leq \frac{1}{K_n}.
\]

If we are in the first case,
\[
|e^{it\Delta} f(x)| \leq K_n^n \max_{\alpha'} |e^{it\Delta} f_{\alpha'}(x)| \leq K_n^{2n} \left( \prod_{j=1}^{n+1} |e^{it\Delta} f_{\alpha_j}(x)| \right)^{\frac{1}{n+1}}.
\]

If we are in the second case,
\[
|e^{it\Delta} f(x)| = \left| \sum_{\alpha \in S(x, t)} e^{it\Delta} f_{\alpha}(x) + \sum_{\alpha \notin S(x, t)} e^{it\Delta} f_{\alpha}(x) \right|
\leq \left| \sum_{\alpha \in S(x, t)} e^{it\Delta} f_{\alpha}(x) \right| + \max_{\alpha'} |e^{it\Delta} f_{\alpha'}(x)|
\leq \left| \sum_{\alpha : \angle(G(\alpha), V^n) \leq K_n^{-1}} e^{it\Delta} f_{\alpha}(x) \right| + \max_{\alpha'} |e^{it\Delta} f_{\alpha'}(x)|.
\]

Therefore we conclude
\[
|e^{it\Delta} f(x)| \lesssim K_n^n \left( \prod_{j=1}^{n+1} |e^{it\Delta} f_{\alpha_j}(x)| \right)^{\frac{1}{n+1}} + \sum_{\alpha : \angle(G(\alpha), V^n) \leq K_n^{-1}} e^{it\Delta} f_{\alpha}(x) \right| + \max_{\alpha'} |e^{it\Delta} f_{\alpha'}(x)|.
\]

Denote
\[
e^{it\Delta} f_{V^n}(x) := \sum_{\alpha : \angle(G(\alpha), V^n) \leq K_n^{-1}} e^{it\Delta} f_{\alpha}(x).
\]

Decomposing the frequency support \( N_{K_n^{-1}}(V^n) \) to balls \( \beta \) with radius \( K_n^{-1} \), then
\[
e^{it\Delta} f_{V^n}(x) = \sum_{G(\beta) \cap N_{K_n^{-1}}(V^n) \neq \emptyset} e^{it\Delta} f_{V^n, \beta}(x).
\]

Define
\[
S_{V^n}(x, t) := \left\{ \beta : |e^{it\Delta} f_{V^n, \beta}(x)| \geq \frac{1}{K_n^{-1}} \max_{\beta'} |e^{it\Delta} f_{V^n, \beta'}(x)|, G(\beta) \cap N_{K_n^{-1}}(V^n) \neq \emptyset \right\}.
\]

We choose \( (n-1) \)-dimensional subspaces \( V_1^{n-1}, ..., V_A^{n-1} \subset V^n \) to achieve the minimum of
\[
\max_{\beta : \angle(G(\beta), V_{A}^{n-1}) > K_n^{-1}, \forall \alpha} |e^{it\Delta} f_{V^n, \beta}(x)|.
\]

There are two cases that may occur now:

- There exists \( \beta \in S_{V^n}(x, t) \) such that for each \( a, \angle(G(\beta), V_a^{n-1}) > K_n^{-1} \).
- For each \( \beta \in S_{V^n}(x, t) \), there exists \( a \) such that \( \angle(G(\beta), V_a^{n-1}) \leq K_n^{-1} \).
If the first case happens,

\[
|e^{it\Delta}f_{V^n}(x)| \leq K_{n-1}^{n-1} \max_{\beta'} |e^{it\Delta}f_{V^n,\beta'}(x)| \\
\leq K_{n-1}^{2(n-1)} \min_{\beta \in S_{V^n}(x,t)} |e^{it\Delta}f_{V^n,\beta}(x)| \\
\leq K_{n-1}^{2(n-1)} \max_{\beta, \alpha} \min_{\ell(G(\beta),V^n)} \max_{\gamma} |e^{it\Delta}f_{V^n,\beta}\gamma(x)| \\
= K_{n-1}^{2(n-1)} B_{\ell_{n,\alpha}}^{K_{n-1}} (e^{it\Delta}f_{V^n})(x).
\]

If the second case happens,

\[
|e^{it\Delta}f_{V^n}(x)| = \left| \sum_{\beta \in S_{V^n}(x,t)} e^{it\Delta}f_{V^n,\beta}(x) + \sum_{\beta \notin S_{V^n}(x,t)} e^{it\Delta}f_{V^n,\beta}(x) \right| \\
\leq \left| \sum_{\beta \in S_{V^n}(x,t)} e^{it\Delta}f_{V^n,\beta}(x) \right| + \max_{\beta'} |e^{it\Delta}f_{V^n,\beta'}(x)| \\
\leq \sum_{a=1}^{A} \left| \sum_{\beta \in S_{V^n}(x,t)} e^{it\Delta}f_{V^n,\beta}(x) \right| + \max_{\beta'} |e^{it\Delta}f_{V^n,\beta'}(x)| \\
+ \sum_{a=1}^{A} \left| \sum_{\beta \in S_{V^n}(x,t)} e^{it\Delta}f_{\beta}(x) \right| \\
\leq \sum_{a=1}^{A} \left| \sum_{\beta \in S_{V^n}(x,t)} e^{it\Delta}f_{V^n,\beta}(x) \right| + \max_{\beta'} |e^{it\Delta}f_{V^n,\beta'}(x)| \\
+ \sum_{a=1}^{A} \left| \sum_{\beta \in S_{V^n}(x,t)} e^{it\Delta}f_{\beta}(x) \right| \\
+ \sum_{a=1}^{A} \left| \sum_{\alpha \in \beta} e^{it\Delta}f_{\alpha}(x) \right| + \max_{\beta'} |e^{it\Delta}f_{V^n,\beta'}(x)| \\
\leq \sum_{a=1}^{A} \left| \sum_{\beta \in S_{V^n}(x,t)} e^{it\Delta}f_{\beta}(x) \right| + A \sum_{\alpha \notin S(x,t)} |e^{it\Delta}f_{\alpha}(x)| + \max_{\beta'} |e^{it\Delta}f_{V^n,\beta'}(x)| \\
+ \sum_{a=1}^{A} \left| \sum_{\beta \in S_{V^n}(x,t)} e^{it\Delta}f_{\beta}(x) \right| + A \max_{\alpha} |e^{it\Delta}f_{\alpha}(x)| + \max_{\beta'} |e^{it\Delta}f_{V^n,\beta'}(x)|.
\]
Thus
\[
|e^{it\Delta}f_{V^n}(x)| \lesssim K_{n-1}^{2(n-1)} Br_{n,A} K_{n-1}^{n-1} (e^{it\Delta}f_{V^n})(x) + |e^{it\Delta}f_{\alpha}(x)| + A \max_{\alpha'} |e^{it\Delta}f_{\alpha'}(x)|.
\]
Combining the results of the first two steps, we obtain
\[
|e^{it\Delta}f(x)| \lesssim K_n^{2n+1} \left( \prod_{j=1}^{n+1} |e^{it\Delta}f_{\alpha_j}(x)| \right) + K_{n-1}^{2(n-1)} Br_{n,A} K_{n-1}^{n-1} (e^{it\Delta}f_{V^n})(x)
\]
\[
+ AK_{n-2} Br_{n-1,A} (e^{it\Delta}f_{V^n})(x) + \ldots + A^{n-2} K_1^2 Br_{V^n,A} (e^{it\Delta}f_{V^n})(x)
\]
\[
+ A \max_{\gamma;K_1^{-1}-balls} |e^{it\Delta}f_{\gamma}(x)|.
\]
Iterating on this formula, we can get
\[
|e^{it\Delta}f(x)| \lesssim K_n^{2n} \left( \prod_{j=1}^{n+1} |e^{it\Delta}f_{\alpha_j}(x)| \right) + K_{n-1}^{2(n-1)} Br_{n,A} K_{n-1}^{n-1} (e^{it\Delta}f_{V^n})(x)
\]
\[
+ AK_{n-2} Br_{n-1,A} (e^{it\Delta}f_{V^n})(x) + \ldots + A^{n-2} K_1^2 Br_{V^n,A} (e^{it\Delta}f_{V^n})(x)
\]
\[
+ A \max_{\gamma;K_1^{-1}-balls} |e^{it\Delta}f_{\gamma}(x)|.
\]
Finally, since the choices of $\alpha_j$, $V^2$, ..., $V^n$ depend on $x$ and $t$, we take the maximum at both ends of above inequality simultaneously, and then (2.2) follows.

Now we show that Theorem 1.2 implies Theorem 1.1. Take $l = 2$ in (2.3), then
\[
|e^{it\Delta}f(x)| \lesssim A^{n-2} K_1^2 \max_{V^2} Br_{V^2,A} (e^{it\Delta}f_{V^2})(x) + A^{n-1} \max_{\gamma;K_1^{-1}-balls} |e^{it\Delta}f_{\gamma}(x)|.
\]
We integrate over $B_R^*$ to obtain
\[
\left\| \sup_{0 < t \leq R} |e^{it\Delta}f| \right\|_{L^p(B^n(0, R))} \lesssim A^{n-2} K_1^2 \max_{V^2} Br_{V^2,A} (e^{it\Delta}f_{V^2})(x) \left\| \sup_{0 < t \leq R} |e^{it\Delta}f_{\gamma}| \right\|_{L^p(B^n(0, R))}^{1/p} + A^{n-1} \sum_{\gamma} \left( \left\| \sup_{0 < t \leq R} |e^{it\Delta}f_{\gamma}| \right\|_{L^p(B^n(0, R))}^{1/p} \right)^{1/p}.
\]
For the first term, we have
\[
A^{n-2} K_1^2 \max_{V^2} Br_{V^2,A} (e^{it\Delta}f_{V^2})(x) \left\| \sup_{0 < t \leq R} |e^{it\Delta}f_{\gamma}| \right\|_{L^p(B^n(0, R))} \lesssim A^{n-2} K_1^2 \sum_{V^2} \left\| Br_{V^2,A} (e^{it\Delta}f_{V^2})(x) \right\|_{L^p(B^n(0, R))}.
\]
Note that the choice of $V^2$ is allowed a $K_2^{-1}$-scale of perturbation, so we can only consider $K_2^n$ many $V^2$. Then using (2.1) and Theorem 1.2, it is further bounded by
\[
A^{n-2} K_2^n \sum_{V^2} \left\| e^{it\Delta}f_{V^2} \right\|_{BL^p(L^q(B^n_R))} \lesssim A^{n-2} K_1^{O(1)} K_2^n \left\| f \right\|_{L^2} \lesssim R' \left\| f \right\|_{L^2}.
\]
For the second term, by parabolic rescaling and induction on scale, one gets
\[ A^{n-1} \sum_{\gamma} \left( \sup_{0 < t \leq R} \|e^{it\Delta} f_\gamma\|_{L^p(B^n(0,R))}^p \right)^\frac{1}{p} \leq A^{n-1} K_1^{-\frac{n}{p} + \frac{n}{2} - \epsilon} R^c \left( \sum_{\gamma} \|f_\gamma\|_{L^2}^p \right)^\frac{1}{p} . \]

Since \( p \geq 2 \), we apply Minkowski’s inequality to obtain
\[ A^{n-1} \sum_{\gamma} \left( \sup_{0 < t \leq R} \|e^{it\Delta} f_\gamma\|_{L^p(B^n(0,R))}^p \right)^\frac{1}{p} \leq A^{n-1} K_1^{-\frac{n}{p} + \frac{n}{2} - \epsilon} R^c \|f\|_{L^2} . \]

Finally, we can choose the parameters \( A \) and \( K_1 \) satisfying \( A^{n-1} \ll K_1 \), and then the induction is close.

Since we will mainly consider \( BL^{p,q}_{2,q} \)-norm in the following discussion, from now on, we use \( BL^p_{2,q} \) to represent \( BL^{p,q}_{2,q} \).

3. SOME PREPARATIONS

The main ingredient of the proof of Theorem 1.3 is the algorithm which leads to different scales \( r \) satisfying \( 1 \ll r \leq R \), not just the scale \( R \). In this section, we introduce some notations and standard results with respect to the general scale \( r \).

1. Locally constant property

Locally constant property says that if a function \( f \) has compact Fourier support \( \Theta \), then we can view \( |f| \) essentially as constant on dual \( \Theta^* \).

**Lemma 3.1.** \([16, \text{Lemma 6.1}]\) Let \( \Theta \) be a compact symmetric convex set centered at \( C_\Theta \in \mathbb{R}^n \). If \( \widehat{g}_\Theta \) is supported in \( \Theta \) and \( T_\Theta \) is the dual convex \( \Theta^* := \{ y : |y \cdot x - C_\Theta| \leq 1, \forall x \in \Theta \} \), then there exists a positive function \( \eta_{T_\Theta} \) satisfying the following properties:

1. \( \eta_{T_\Theta} \) is essentially supported on \( 10T_\Theta \) and rapidly decaying away from it: for any integer \( N > 0 \), there exists a constant \( C_N \) such that \( \eta_{T_\Theta}(x) \leq C_N(1+n(x,10T_\Theta))^{-N} \) where \( n(x,10T_\Theta) \) is the smallest positive integer \( n \) such that \( x \in 10nT_\Theta \),
2. \( \|\eta_{T_\Theta}\|_{L^1} \lesssim 1 \),
3. \( |g_\Theta| \leq \sum_{T \neq T_\Theta} c_T \chi_T \leq |g_\Theta| \ast \eta_{T_\Theta} \), \hspace{1cm} (3.1)

where \( c_T := \max_{x \in T} |g_\Theta|(x) \) and the sum \( \sum_{T \neq T_\Theta} \) is over a finitely overlapping cover \( \{T\} \) of \( \mathbb{R}^n \) with each \( T \parallel T_\Theta \). Here \( T \parallel T_\Theta \) means that \( T \) is a translated copy of \( T_\Theta \).

In particular, for each ball \( \tau \) of radius \( K^{-1} \), note \( e^{it\Delta} f_\tau \) is Fourier supported on \( \{(w,|w|^2) : w \in \tau\} \) in a distributional sense, which is contained in a rectangular box of radius \( K^{-1} \) and thickness \( K^{-2} \). Therefore \( e^{it\Delta} f_\tau \) is locally constant on each tube of radius \( K \) and length \( K^2 \).

2. Wave packet decomposition on scale \( r \)
Let \( \varphi \) be a Schwartz function satisfying \( \text{supp} \hat{\varphi} \subset B(0, 3/2) \), and
\[
\sum_{k \in \mathbb{Z}^n} \hat{\varphi}(\xi - k) = 1, \quad \text{for all } \xi \in \mathbb{R}^n.
\]
Define
\[
\left\{ \begin{array}{l}
\hat{\varphi}_\theta(\xi) := r^\frac{n}{2} \hat{\varphi}(r^\frac{n}{2}(\xi - c(\theta))) ,
\hat{\varphi}_{\theta, \nu}(\xi) := e^{-i c(\nu) \cdot \xi} \hat{\varphi}_\theta(\xi), \\
c(\theta) \in r^{-1/2} \mathbb{Z}^n , \quad c(\nu) \in r^{1/2} \mathbb{Z}^n,
\end{array} \right.
\]
where \( \theta \) denote \( r^{-\frac{1}{2}} \)-balls in frequency space and \( \nu \) denote \( r^{\frac{1}{2}} \)-balls in physical space.

For each Schwartz function \( f \) with \( \text{supp} \hat{f} \subset B^n(0, 1) \), we have the following decomposition
\[
f = \sum_{\theta, \nu} c_{\theta, \nu} \varphi_{\theta, \nu}, \quad \theta \cap B^n(0, 1) \neq \emptyset,
\]
where each \( \varphi_{\theta, \nu} \) is Fourier supported in \( \theta \) and has physical support essentially in \( \nu \). A basic property is the functions \( \varphi_{\theta, \nu} \) are approximately orthogonal, i.e.
\[
\sum_{\theta, \nu} |c_{\theta, \nu}|^2 \sim \| f \|^2_{L^2}. \quad (3.3)
\]

Next we consider the decomposition on \( e^{it\Delta} f \) associated with \( (3.2) \)
\[
e^{it\Delta} f = \sum_{\theta, \nu} c_{\theta, \nu} e^{it\Delta} \varphi_{\theta, \nu}. \quad (3.4)
\]

By the stationary phase method, we obtain
\[
|e^{it\Delta} \varphi_{\theta, \nu}(x)| \leq r^{-\frac{n}{4}} \chi_{T_{\theta, \nu}}(x, t) + O(r^{-N}), \quad \forall \, N > 0. \quad (3.5)
\]
Here \( T_{\theta, \nu} \) is defined by
\[
T_{\theta, \nu} := \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq r, \ |x + 2tc(\theta) - c(\nu)| \leq r^{\frac{1}{2} + \delta} \right\},
\]
where \( \delta \) is a small positive parameter satisfying \( \delta \ll \epsilon \). \( T_{\theta, \nu} \) is a tube of length \( r \), of radius \( r^{1/2 + \delta} \) and in the direction \((-2c(\theta), 1)\). For each \( \nu \), \( e^{it\Delta} \varphi_{\theta, \nu} \) is Fourier supported in \( \xi(\theta) = \{(\xi, \xi^2) : \xi \in \theta \} \) in a distributional sense. For detailed proofs of wave packet decomposition one can see [9, 23].

3. The properties of \( BL_{A/2}^{p,q} \)-norm

Firstly, we recall the basic properties of \( BL_{A/2}^{p,q} \)-norm.

**Proposition 3.2** ([12, 13]). Let \( p \leq q \). The following properties hold:

(a) (Finite subadditivity) Let \( A \geq 1, U_1, U_2 \subset \mathbb{R}^{n+1} \), then
\[
\| e^{it\Delta} f \|_{BL_{A/2}^{p,q}(U_1)}^p \leq \| e^{it\Delta} f \|_{BL_{A}^{p,q}(U_1)}^p + \| e^{it\Delta} f \|_{BL_{A}^{p,q}(U_2)}^p.
\]

(b) (Triangle inequality) Let \( A \geq 2, U \subset \mathbb{R}^{n+1} \), then
\[
\| e^{it\Delta} (f + g) \|_{BL_{A/2}^{p,q}(U)}^p \leq \| e^{it\Delta} f \|_{BL_{A/2}^{p,q}(U)}^p + \| e^{it\Delta} g \|_{BL_{A/2}^{p,q}(U)}^p.
\]
(c) (Logarithmic convexity) Let $A \geq 2$, $U \subset \mathbb{R}^{n+1}$. Suppose that $1 \leq p, p_1, p_2 < \infty$ and $0 \leq \alpha \leq 1$ obey
\[ \frac{1}{p} = \frac{1 - \alpha}{p_1} + \frac{\alpha}{p_2}, \]
then
\[ \|e^{it\Delta}f\|_{BL_p^q(U)} \leq \|e^{it\Delta}f\|_{BL_{p_1,q}^{1-\alpha}(U)}^{1-\alpha} \|e^{it\Delta}f\|_{BL_{p_2,q}^\alpha(U)}^\alpha. \]

The proof of Proposition 3.2 is elementary, one can refer to [12, 15]. If $A = 1$, we can’t use (b) and (c) in Proposition 3.2 since $BL_p^q$-norm only makes sense for the positive integer $A$. But if we start with a large $A$, and use (b) and (c) only $O_e(1)$ times, then the choice of $A$ won’t influence our argument, see [17, 25]. On the other hand, we always choose the parameter $A$ satisfying $A \ll K$, which is an unimportant parameter. From now on, we write $BL_p^q$ to represent $BL_{A,p}^q$.

Now we consider the relationship between $BL_p^\infty$-norm and maximal $L^p$-norm, bilinear maximal $L^p$-norm.

**Proposition 3.3.** Let $p \geq 2$. Suppose that $f$ has Fourier support in $B^n(0,1)$. For any $\epsilon > 0$, if
\[ \left\| \sup_{0 < t \leq r} |e^{it\Delta}f| \right\|_{L^p(B_r)} \lesssim r^\epsilon \|f\|_{L^2}, \]
then
\[ \|e^{it\Delta}f\|_{BL_p^\infty(B_r^\epsilon)} \lesssim K^{O(1)}r^\epsilon \|f\|_{L^2}. \]

**Proof.** By the definition of $BL_p^\infty$-norm, we obtain
\[
\|e^{it\Delta}f\|_{BL_p^\infty(B_r^\epsilon)} = \left( \sum_{B_k \subset B(0,r)} \max_{I_k \subset [0,r]} \mu e^{it\Delta}f(B_k \times I_k) \right)^{\frac{1}{p}} \\
\leq \left[ \sum_{B_k \subset B(0,r)} \max_{I_k \subset [0,r]} \left( \sum_{\tau} \|e^{it\Delta}f\|_{L^p(B_k \times I_k)}^p \right)^{\frac{1}{p}} \right]^{\frac{1}{p}} \\
\leq \left( \sum_{\tau} \sum_{B_k \subset B(0,r)} \max_{I_k \subset [0,r]} \|e^{it\Delta}f\|_{L^p(B_k \times I_k)}^p \right)^{\frac{1}{p}} \\
\lesssim K^{O(1)} \left( \sum_{\tau} \|e^{it\Delta}f\|_{L^p(B_k \times I_k)}^p \right)^{\frac{1}{p}} \\
\lesssim K^{O(1)}r^\epsilon \|f\|_{L^2}. \]

□

**Proposition 3.4.** Let $p \geq 2$. Suppose that $f$ has Fourier support in $B^n(0,1)$. Let $f = \sum_{\tau} f_\tau$, where each $f_\tau$ is Fourier supported on $\tau$. For any $\epsilon > 0$, if
\[
\left\| \sup_{0 < t_1 \leq t} |e^{it_1\Delta}f_{t_1} e^{it_2\Delta}f_{t_2}| \right\|_{L^p(B_r)} \lesssim r^\epsilon \|f\|_{L^2}
\]
holds for all $\tau_1, \tau_2$ satisfying that the Fourier supports of $f_{\tau_1}$ and $f_{\tau_2}$ are separated by at least $1/K$, then
\[
\|e^{it\Delta}f\|_{BL_p^\infty(B_r^\epsilon)} \lesssim K^{O(1)}r^\epsilon \|f\|_{L^2}. \]
**Proof.** We recall the definition of \( \mu_{e^{it\Delta}} \):

\[
\mu_{e^{it\Delta}}(B_K \times I_K) = \min_{V_1, \ldots, V_A} \left( \max_{\tau \notin V_0} \int_{B_K \times I_K} |e^{it\Delta} f_\tau|^p dx dt \right).
\]

For each \( \tau \), by Lemma 3.1 there exists a positive function \( \eta_K \) which is essentially supported on \( B_K \times I_K \) satisfying \( \|\eta_K\|_{L^1} \lesssim 1 \) and

\[
|e^{it\Delta} f_\tau| \leq |e^{it\Delta} f_\tau| * \eta_K.
\]

Fix \( B_K \times I_K \), then there exists a \( \tau_0 \) such that

\[
\max_{\tau} \int_{B_K \times I_K} |e^{it\Delta} f_\tau|^p = \int_{B_K \times I_K} |e^{it\Delta} f_{\tau_0}|^p dx dt.
\]

We choose \( V_1', \ldots, V_A' \) such that \( \tau_0 \in V_a' \) for some \( a \in \{1, 2, \ldots, A\} \), then by (3.1) and Hölder inequality,

\[
\mu_{e^{it\Delta}}(B_K \times I_K) \leq \max_{\tau \notin V_0} \int_{B_K \times I_K} |e^{it\Delta} f_\tau|^p \leq \max_{\tau \notin V_a'} \left( \int_{B_K \times I_K} |e^{it\Delta} f_\tau|^p \right)^{\frac{1}{2}} \left( \int_{B_K \times I_K} |e^{it\Delta} f_{\tau_0}|^p \right)^{\frac{1}{2}} \lesssim \max_{\tau \notin V_a'} \left( |e^{it\Delta} f_\tau| * \eta_K \right) \left( |e^{it\Delta} f_{\tau_0}| * \eta_K \right) \leq \max_{\tau \notin V_a'} \left( \int_{B_K \times I_K} |e^{it\Delta} f_\tau|^p * \eta_K \right) \left( \int_{B_K \times I_K} |e^{it\Delta} f_{\tau_0}|^p * \eta_K \right) = \sum_{d(\tau_1, \tau_2) \geq (KM)^{-1}} \int_{B_K \times I_K} \left( \int_{B_K \times I_K} |e^{i(t-\xi_1)\Delta} f_{\tau_1}(x - x_1) e^{i(t-\xi_2)\Delta} f_{\tau_2}(x - x_2)|^p dx \right) \eta_K(x_1, t_1) \eta_K(x_2, t_2) dx_1 dx_2 dt_1 dt_2.
\]

Define \( f_{\tau_j, x_j, t_j} \) by

\[
\widehat{f_{\tau_j, x_j, t_j}} = e^{-i(t_j |x|^2 + x_j \cdot \xi)} \widehat{f_{\tau_j}},
\]

then

\[
e^{it\Delta} f_{\tau_j, x_j, t_j} = e^{i(t-\xi_j)\Delta} f_{\tau_j}(x - x_j), \quad j = 1, 2.
\]
We repeat the argument as in the proof of Proposition $3.3$ to obtain
\[
\left\|e^{it\Delta}f\right\|_{BL_p,\infty(B_r^*)} = \left( \sum_{B_K \subset B(0,r)} \max_{I_K \subset [0,r]} \left( \int \left\| e^{it\Delta}f_{t_1,x_1,t_2} e^{it\Delta}f_{t_2,x_2,t_2} \right\|_{L^p(B_K \times I_K)} \right)^{1/p} \right)^{1/p}.
\]

\[
\leq \left( \sum_{B_K \subset B(0,r)} \max_{I_K \subset [0,r]} \left( \int \left\| e^{it\Delta}f_{t_1,x_1,t_2} e^{it\Delta}f_{t_2,x_2,t_2} \right\|_{L^p(B_K \times I_K)} \right)^{1/p} \right)^{1/p}.
\]

\[
\sum_{d(\tau_1,\tau_2) \geq (KM)^{-1}} \int \left( \sum_{B_K \subset B(0,r)} \max_{I_K \subset [0,r]} \left( \int \left\| e^{it\Delta}f_{t_1,x_1,t_2} e^{it\Delta}f_{t_2,x_2,t_2} \right\|_{L^p(B_K \times I_K)} \right)^{1/p} \right)^{1/p}.
\]

\[
\leq K^{O(1)} \left( \sum_{d(\tau_1,\tau_2) \geq (KM)^{-1}} \int \left\| e^{it\Delta}f_{t_1,x_1,t_2} e^{it\Delta}f_{t_2,x_2,t_2} \right\|_{L^p(B_r^*)} \right)^{1/p}.
\]

\[
\lesssim K^{O(1)} r^\varepsilon \|f\|_{L^2}.
\]

\[
\square
\]

Remark 3.5. For $q > 1/\delta$, we have
\[
\left\| e^{it\Delta}f \right\|_{L^p_t L^q_x(B_r^*)} \lesssim r^\delta \left\| e^{it\Delta}f \right\|_{L^p_t L^p_x(B_r^*)}
\]

and
\[
\left\| e^{it\Delta}f \right\|_{BL_p^q(B_r^*)} \lesssim r^\delta \left\| e^{it\Delta}f \right\|_{BL_p B_\infty^q(B_r^*)}.
\]

So Proposition $3.3$ and $3.4$ still hold if we replace $BL_p B_\infty^q$-norm with $BL_p^q$-norm.

4. Polynomial partitioning

Polynomial partitioning was firstly introduced by Guth [14, 15] to improve the bound of restriction conjecture in three dimension. The main idea of polynomial partitioning is using a polynomial to cut the whole space to several parts, and each part has the same contribution.

Theorem 3.6. [14, Theorem 1.4] Suppose that $F$ is a non-negative $L^1$ function on $\mathbb{R}^n$. For each degree $d$, there exists a nontrivial polynomial $P$ of degree at most $d$ such that $\mathbb{R}^n \setminus Z(P)$ is a union of $\sim_n d^n$ pairwise disjoint open sets $O'_i$ (cells) and for each $i$ we have
\[
\int_{O'_i} F(x) dx \sim_n d^{-n} \int F(x) dx.
\]

Let $Z(P_1, \ldots, P_{n+1-m})$ be the set of common zeros of polynomials $P_1, \ldots, P_{n+1-m}$. We call the variety $Z(P_1, \ldots, P_{n+1-m})$ to be a transverse complete intersection if
\[
\nabla P_1(x) \wedge \ldots \wedge \nabla P_{n+1-m}(x) \neq 0, \quad \text{for all} \ x \in Z(P_1, \ldots, P_{n+1-m}).
\]
If $Z = Z(P_1, ..., P_{n+1-m})$ satisfies $\deg P_i \leq d$ for each $i = 1, 2, ..., n+1 - m$, we say the degree of $Z$ is at most $d$. Du-Li in [12] gave the following polynomial partitioning theorem on the broad norm version.

**Theorem 3.7.** [12] Theorem 4.5] Suppose that $f$ is a function with $\text{supp}\hat{f} \subset B(0, 1) \subset \mathbb{R}^n$, $U$ is a subset of $B^*_R$, and $1 \leq p, \lambda < \infty$. Then for each $d$, there exists a nontrivial polynomial $P$ of degree at most $O(d)$ such that $(\mathbb{R}^n \times \mathbb{R}) \setminus Z(P)$ is a union of $\sim d^{n+1}$ disjoint open sets $O'_i$ and for each $i$ we have

$$\|e^{it\Delta}f\|^p_{BL^{p,\lambda}(U)} \leq C_n d^{n+1}\|e^{it\Delta}f\|^p_{BL^{p,\lambda}(U\setminus O'_i)}.$$  

Moreover, $Z(P)$ is a finite union of transverse complete intersections.

Theorem 3.7 induces one spatial decomposition $B^*_R = (\bigcup_i O'_i) \cup Z(P)$. If we consider wave packet decomposition of $f$ on scale $R$: $f = \sum_{\theta, \nu} c_{\theta, \nu}\varphi_{\theta, \nu}$, then most of the mass of each $e^{it\Delta}\varphi_{\theta, \nu}$ is concentrated on $T_{\theta, \nu}$ in physical space. An example of $f$ says that there exists a related tube $T_{\theta, \nu}$ such that it can enter into all cells $O'_i$, which is not convenient to carry on this divide and conquer approach. Roughly speaking, this bad situation can happen because the diameter of each cell $O'_i$ is too large. So we shrunken the scale of cells $O'_i$ to make that each tube can only enter into a small part of the shrunken cells.

Using Theorem 3.7 with $\lambda = q$, we conclude that there exists a nontrivial polynomial $P$ of degree at most $O(d)$ such that $(\mathbb{R}^n \times \mathbb{R}) \setminus Z(P)$ is a union of $\sim d^{n+1}$ disjoint open sets $O'_i$ and for each $i$ we have

$$\|e^{it\Delta}f\|^p_{BL^{p,q}(B^*_R)} \leq C_n d^{n+1}\|e^{it\Delta}f\|^p_{BL^{p,q}(O'_i)}.$$  

(3.6)

Now we define

$$W := N_{R^{1+\delta}} Z(P) \cap B^*_R, \quad O_i := [O'_i \cap B^*_R] \setminus W,$$

(3.7)

where $N_{R^{1+\delta}} Z(P)$ denotes the $R^{1+\delta}$-neighborhood of the variety $Z$. By Proposition 3.2, one obtains

$$\|e^{it\Delta}f\|^p_{BL^{p,q}(B^*_R)} \leq \sum_i \|e^{it\Delta}f\|^p_{BL^{p,q}(O_i)} + \|e^{it\Delta}f\|^p_{BL^{p,q}(W)}.$$  

(3.8)

By the pigeonhole principle, we have at least one of the following cases holds:

**Cellular case:** there exist $O(d^{n+1})$ many cells $O_i$ such that for each $i$,  

$$\|e^{it\Delta}f\|^p_{BL^{p,q}(B^*_R)} \lesssim d^{n+1}\|e^{it\Delta}f\|^p_{BL^{p,q}(O_i)}.$$  

**Algebraic case:**  

$$\|e^{it\Delta}f\|^p_{BL^{p,q}(B^*_R)} \lesssim \|e^{it\Delta}f\|^p_{BL^{p,q}(W)}.$$  

Therefore we can attribute the contribution on the whole region to cellular contribution and algebraic contribution. Hickman-Rogers [17] also introduced the polynomial partitioning theorem in lower dimensions. To build the algorithm on $BL^{p,q}$-norm in the next section, we prove relevant theorem on the version of $BL^{p,q}$-norm.

We firstly recall the definition of tangential tube introduced by Guth in [14] [15].
Definition 3.8. Suppose that $Z = Z(P_1, ..., P_{n+1-m})$ is a transverse complete intersection in $\mathbb{R}^n \times \mathbb{R}$. We say that $T_{\theta, \nu}$ is concentrated on scale $r$ wave packets which are $r^{-\frac{1}{2}+\delta_m}$ tangent to $Z$ in $B_r$, if the following two conditions hold:

(i) Distance condition:

$$T_{\theta, \nu} \subset N_{\frac{1}{2}+\delta_m}, Z \cap B_r.$$ 

(ii) Angle condition: for any $x \in T_{\theta, \nu}$ and $z \in Z \cap B_r$ with $|z - x| \lesssim r^{1/2+\delta_m}$, one has

$$\angle(G(\theta), T_z) \lesssim r^{-\frac{1}{2}+\delta_m}.$$ 

Let

$$T_Z[r] := \{ (\theta, \nu) \mid T_{\theta, \nu} \text{ is concentrated on scale } r \text{ wave packets which are } r^{-\frac{1}{2}+\delta_m} \text{ tangent to } Z \}.$$ 

We say that $f$ is concentrated in wave packets from $T_Z[r]$ if

$$\sum_{(\theta, \nu) \notin T_Z[r]} \|f_{\theta, \nu}\|_{L^2} \leq O(r^{-N}) \|f\|_{L^2}, \quad \forall \, N > 0.$$

Theorem 3.9. Fix $r \gg 1$, $d \in \mathbb{N}$ and $1 \leq p, \lambda < \infty$. Suppose that $Z$ is an $m$-dimensional transverse complete intersection of degree at most $d$. Suppose that $f$ satisfies $\text{supp} \hat{f} \subset B(0, 1) \subset \mathbb{R}^n$, and $f$ is concentrated in wave packets from $T_Z[r]$. Then there exists a constant $D = D_Z = D(\varepsilon, d)$ such that at least one of the following cases holds:

Cellular case: there exist $O(D^m)$ shrunken cells $O_i$ in $\mathbb{R}^n \times \mathbb{R}$ such that for each $i$,

$$\|e^{it\Delta}f\|_{B^{p, \lambda}_{L^p}(O_i)} \lesssim D_m^m \|e^{it\Delta}f\|_{B^{p, \lambda}_{L^p}(X_i)}.$$ 

Algebraic case: there exists an $(m-1)$-dimensional transverse complete intersection $Y$ of degree at most $O(D)$ such that

$$\|e^{it\Delta}f\|_{B^{p, \lambda}_{L^p}(B_r \cap N_{1/2+\delta_m} Z)} \lesssim \|e^{it\Delta}f\|_{B^{p, \lambda}_{L^p}(B_r \cap N_{1/2+\delta_m} Y)}.$$

Proof. Suppose that the algebraic case does not occur. We say that a ball $B = B(x_0, r^{1/2+\delta_m}) \subset N_{1/2+\delta_m} Z \cap B_r$ is regular if, on each connected component of $Z \cap B(x_0, r^{1/2+\delta_m})$, the tangent space $TZ$ is constant up to angle 1/100. By the hypothesis and Guth’s argument such as in [15, Section 8], we have the regular balls contain most of the mass of $e^{it\Delta}f$, i.e.

$$\|e^{it\Delta}f\|_{B^{p, \lambda}_{L^p}(B_r)} \lesssim \|e^{it\Delta}f\|_{B^{p, \lambda}_{L^p}(\cup \{ \text{regular } B \})}.$$ 

For each regular ball $B$, we pick a point $z \in Z \cap B_{r^{1/2+\delta_m}}$ and define $V_B$ to be the $m$-dimensional tangent plane $T_z Z$. For each $m$-dimensional tangent plane $V$, we use $\mathcal{B}_V$ to denote the set of regular balls so that $\angle(V, V) \leq 1/100$. By the pigeonhole principle, there exists a plane $V$ such that

$$\|e^{it\Delta}f\|_{B^{p, \lambda}_{L^p}(B_r)} \lesssim \|e^{it\Delta}f\|_{B^{p, \lambda}_{L^p}(\cup \{ \text{regular } BB \})}.$$ 

Set $N_1 = \cup_{B \in \mathcal{B}_V} B$. Therefore, it suffices to consider the contribution on $N_1$. 


Since all wave packets of \( e^{it\Delta} f \) on scale \( r \) are contained in the \( \approx r^{1/2} \)-neighborhood of \( V \), \( |e^{it\Delta} f(x)| \) is essentially constant along a certain direction which is roughly normal to \( V \). Let \( P_V : V \to \mathbb{R} \) denote a polynomial defined on \( V \), and \( \pi : \mathbb{R}^{n+1} \to V \) be the orthogonal projection. Then we can extend \( V \) to a polynomial \( P \in \mathbb{P}_V \) of degree at most \( D = D(\epsilon, d) \) such that \( N_1 = (\cup_i O'_i) \cup Z(P) \) satisfying \( \#O'_i \sim D^m \), and for each \( i \),

\[
\|e^{it\Delta} f\|_{p,\lambda(B_r)}^p \sim \|e^{it\Delta} f\|_{BP^{\lambda}(O_i)}^p \lesssim D^m\|e^{it\Delta} f\|_{BP^{\lambda}(O'_i)}^p.
\]

Define

\[
W := N_{r^{-1/2}+3\delta} Z(P) \cap B_r, \quad O_i := [O'_i \cap B_r] \setminus W.
\]

By Proposition 3.2 we obtain

\[
\|e^{it\Delta} f\|_{BP^{\lambda}(B_r)}^p \lesssim \sum_i \|e^{it\Delta} f\|_{BP^{\lambda}(O_i)}^p + \|e^{it\Delta} f\|_{BP^{\lambda}(W)}^p.
\]

Since the algebraic case does not occur, the contribution from \( W \) is negligible. By the pigeonhole principle and (3.9), there exist \( O(D^m) \) cells \( O_i \) such that for each \( i \),

\[
\|e^{it\Delta} f\|_{BP^{\lambda}(B_r)}^p \lesssim D^m\|e^{it\Delta} f\|_{BP^{\lambda}(O_i)}^p.
\]

\[
\square
\]

5. The vanishing property of the broad norm

In the final part of this section, we prove one important property of the broad norm named the vanishing property. For the \( BL^{p,q}_{k,A} \)-norm, Guth [15] has proved this property. Now we show this property also holds for general \( BL^{p,q}_{k,A} \)-norm.

**Theorem 3.10.** Let \( r \gg 1 \), \( \delta \ll \epsilon \) and \( 1 \leq m < k \leq n \), and let \( Z \) be an \( m \)-dimensional transverse complete intersection. Suppose that \( f \) is concentrated on scale \( r \) wave packets which are \( r^{-1/2+\delta} \)-tangent to \( Z \) on \( B_r \), then

\[
\|e^{it\Delta} f\|_{BL^{p,q}_{k,A}(B_r)} = O(r^{-N})\|f\|_{L^2}
\]

for any \( N > 0 \).

**Proof.** We consider wave packet decomposition of \( f \) on scale \( r \): \( f = \sum_{\theta,\nu} c_{\theta,\nu} \varphi_{\theta,\nu} \). Since \( f \) is concentrated in wave packets from \( T_Z[r] \), it suffices to consider \((\theta, \nu) \in T_Z[r] \). For each \((\theta, \nu) \in T_Z[r] \), we can choose \( z_0 \in Z \cap B_r \cap N_{O(\epsilon^{1/2+\delta})} T_{\theta,\nu} \) such that

\[
\angle(G(\theta), T_z Z) \lesssim r^{-1/2+\delta}.
\]

This fact implies for any \( \tau \supset \theta \),

\[
\angle(G(\tau), T_z Z) \leq K^{-1}.
\]

Since \( T_z Z \) is \( m \)-dimensional, note \( m < k \), we have \( T_z Z \) is also a \((k-1)\)-dimensional subspace. So such \( \tau \)'s do not contribute to \( \mu_{e^{it\Delta} f}(B_K \times I_K) \), and then

\[
\|e^{it\Delta} f\|_{BL^{p,q}_{k,A}(B_r)} = O(r^{-N})\|f\|_{L^2}
\]

for any \( N > 0 \).

\[
\square
\]
4. Two algorithms

From this section, we start to prove Theorem 1.3. We will adopt two algorithms introduced by Hickman-Rogers in [17]. The method developed by Guth [15] in the study of restriction conjecture relies on polynomial partitioning and scale induction. Hickman-Rogers used the algorithm instead of scale induction to give a new proof of Guth’s result. The main advantage of the algorithm is to lead to a more detailed geometry analysis on different scales.

Now we introduce the first algorithm. It is a dimensional reduction, essentially passing from an $m$-dimensional to an $(m-1)$-dimensional situation. More precisely, we start with a function $f$ which is concentrated in wave packets from $T_Z[r]$, and $Z$ is an $m$-dimensional transverse complete intersection. Using Theorem 3.9, we can divide $\|e^{it\Delta}f\|_{BL^{p,q}(B_r)}$ into the cellular and algebraic case. The algebraic case can further separated into the transverse and tangential case according to the angle condition. If it’s the tangential case, we already pass from an $m$-dimensional to an $(m-1)$-dimensional situation, then the algorithm stops. Otherwise, we repeat the above steps to each cells produced by the cellular and transverse cases.

[Alg-1]-The first algorithm: Let $p \geq 2$, $0 < \epsilon \ll 1$, and $\delta, \delta_n, ..., \delta_1, \delta_0$ satisfy

$$\epsilon^C \leq \delta \ll \delta_n \ll \delta_{n-1} \ll ... \ll \delta_1 \ll \delta_0 \ll \epsilon$$

for some constant $C$.

**Input.**

- A ball $B_r \subset \mathbb{R}^{n+1}$ with radius $r \gg 1$.
- A transverse complete intersection $Z$ of dimension $m \geq 2$.
- A function $f$ satisfying $\text{supp} \hat{f} \subset B^n(0,1)$, and $f$ is concentrated on scale $r$ wave packets which are $r^{-1/2+\delta_m}$-tangent to $Z$ in $B_r$.

**Output.** The $j$-th step of recursion will produce:

- A word $\mathcal{E}_j$ of length $j$ in the alphabet $\{a,c\}$. Here $a$ is an abbreviation of “algebraic” and $c$ “cellular”. The algorithm is realized by repeated application of Theorem 3.9. More precisely, suppose that the algorithm has ran through $(j-1)$ steps, and produced a word $\mathcal{E}_{j-1}$ of length $j-1$ and a family of subsets $\mathcal{O}_{j-1}$ (see its definition below), the we again use Theorem 3.9 on each $\mathcal{O}_{j-1} \in \mathcal{O}_{j-1}$. If more than $1/2$ fraction of the $\mathcal{O}_{j-1}$ belong to the cellular case, then we define $\mathcal{E}_j$ as a word of length $j$, where the first $(j-1)$ letters are the same as $\mathcal{E}_{j-1}$ and the $j$-th letter is $c$. Otherwise, we defined the $j$-th letter of $\mathcal{E}_j$ to be $a$.
- A spatial scale $\rho_j \geq 1$. Let $\tilde{\delta}_{m-1}$ satisfy

$$\left(1 - \tilde{\delta}_{m-1}\right) \left(\frac{1}{2} + \delta_{m-1}\right) = \frac{1}{2} + \delta_m,$$
and $\sigma_k : [0, \infty) \to [0, \infty)$ satisfy
\[
\sigma_k(\rho) := \begin{cases} 
\frac{\rho}{2}, & \text{if the } k\text{-th letter of } E_j \text{ is } c, \\
\rho^{1-\delta_{m-1}}, & \text{if the } k\text{-th letter of } E_j \text{ is } a,
\end{cases}
\]
for each $1 \leq k \leq j$. With these definitions, take
\[
\rho_j := \sigma_j \circ \ldots \circ \sigma_1(r).
\]
Then $\rho_j$ satisfies:
\[
\rho_j \leq r^{(1-\delta_{m-1})\#a(j)} \quad \text{and} \quad \rho_j \leq \frac{r}{2^{\#c(j)}},
\]
where $\#a(j)$ and $\#c(j)$ denote the number of occurrences of $a$ and $c$ in the $E_j$.

- A family of subsets $O_j$ of $\mathbb{R}^{n+1}$ which will be referred to as cells. Each cell $O_j \in O_j$ has diameter at most $\rho_j$.
- A collection of functions $(f_{O_j})_{O_j \in O_j}$. Each $f_{O_j}$ is concentrated on scale $\rho_j$ wave packets which are $\rho_j^{-1/2+\delta_m}$-tangent to some translation of $Z$ on (a ball of radius $\rho_j$ containing) $O_j$.
- A large integer $d \in \mathbb{N}$ which depends only on the admissible parameters and $\deg Z$.

Then the following properties hold:

**Property I.** Most of the mass of $\|e^{it\Delta}f\|_{BL^p(B_r)}^p$ is concentrated on the $O_j \in O_j$:
\[
\|e^{it\Delta}f\|_{BL^p(B_r)}^p \leq C_j^I(d, r) \sum_{O_j \in O_j} \|e^{it\Delta}f_{O_j}\|_{BL^p(O_j)}^p + O(r^{-N})\|f\|_{L^2}. \tag{4.1}
\]

**Property II.** The functions $f_{O_j}$ satisfy
\[
\sum_{O_j \in O_j} \|f_{O_j}\|_{L^2}^2 \leq C_j^{II}(d, r) d^{\#c(j)}\|f\|_{L^2}^2. \tag{4.2}
\]

**Property III.** Each $f_{O_j}$ satisfies
\[
\|f_{O_j}\|_{L^2}^2 \leq C_j^{III}(d, r) \left(\frac{r}{\rho_j}\right)^{-\frac{n+1-m}{2}} d^{-\#c(j)(m-1)}\|f\|_{L^2}^2. \tag{4.3}
\]

Here
\[
\begin{align*}
C_j^{I}(d, r) & := d^{\#c(j)}(\log r)^{2\#a(j)(1+\delta)}, \\
C_j^{II}(d, r) & := d^{\#c(j)}\text{Poly}(d)^{\#a(j)(1+\delta)}, \\
C_j^{III}(d, r) & := d^{\delta_{m-1}} r^{\delta_{m}} d^{\#c(j)},
\end{align*}
\]

where $\text{Poly}(d) := d^{n+1}$ and $\overline{C}$ is some suitably chosen large constant. One easily verifies that
\[
C_j^{I}(d, r), \quad C_j^{II}(d, r), \quad C_j^{III}(d, r) \lesssim_{d, \delta} r^{\delta_{m}} d^{\#c(j)} \delta.
\]

**The first step.** The algorithm shall start by taking:
- $E_0 := \emptyset$ to be the empty word;
- $\rho_0 := r$;
- $O_0 := \{O_0\}$, where $O_0 := N_{\rho^{1/2+\delta_m}} Z \cap B_r$;
• $f_{O_0} := f$.
• A large integer $d \in \mathbb{N}$ to be determined later.

Under this setting, one easily verifies the validity of Property I, II, and III. In fact, Property I holds due to the hypothesis of $f$. Property II and III hold trivially.

**The $(j + 1)$-th step.** Let $j \geq 1$. We assume that the recursive algorithm has ran through $j$ steps. Since each function $f_{O_j}$ is concentrated on scale $\rho_j$ wave packets which are $\rho_j^{-1/2 + \delta_m}$-tangent to some translation of $Z$ in $B_{\rho_j}$, we again apply Theorem 3.9 to bound $\|e^{it\Delta} f_{O_j}\|^p_{BL^p,q(O_j)}$ for each $O_j \in \mathcal{O}_j$. One of two things can happen: either [Alg-1] terminates if the diameter of each cell is very small or it terminates if most of the mass of $\|e^{it\Delta} f_{O_j}\|^p_{BL^p,q(O_j)}$ is concentrated on some $(m-1)$-dimensional transverse complete intersections. We give the exact stopping conditions as follows.

**Stopping conditions.** The algorithm will be stopped if one of both conditions occurs:

**[tiny]** The algorithm terminates if $\rho_j \leq r^{\delta_{m-1}}$.

**[tang]** The algorithm terminates if

$$\sum_{O_j \in \mathcal{O}_j} \|e^{it\Delta} f_{O_j}\|^p_{BL^p,q(O_j)} \leq C_{tang} \sum_{S \in \mathcal{S}} \|e^{it\Delta} f_S\|^p_{BL^p,q(B_{\bar{\rho}})}$$

(4.5)

and

$$\sum_{S \in \mathcal{S}} \|f_S\|^2_{L^2} \leq C_{tang} r^{(n+1)\delta_{m-1}} \sum_{O_j \in \mathcal{O}_j} \|f_{O_j}\|^2_{L^2}$$

(4.6)

hold for $\bar{\rho} := \rho_j^{-1/2 + \delta_m}$ and for some constant $C_{tang}$. Here $\mathcal{S}$ is a collection of $(m-1)$-dimensional transverse complete intersections in $\mathbb{R}^{n+1}$ all of degree $O(d)$, and $f_S$ is a function to each $S \in \mathcal{S}$ which is $\bar{\rho}^{-1/2 + \delta_m}$-tangent to $S$ on $B_{\bar{\rho}}$.

**Remark 4.1.** (a) We can use Theorem 3.9 and scale induction to reformulate this algorithm. For details one can refer to Hickman-Rogers [17, Section 9].

(b) There exists a additional local estimate in Property III in [Alg-1] of Hickman-Rogers. The main effect of such estimate is to establish $L^\infty$-estimate from $L^2$-estimate at the smallest possible cost due to the hypothesis of restriction conjecture. In this paper, we always assure $f \in L^2$, so the local estimate is not essential in our argument though it also holds in this case.

Next we introduce the second algorithm. It consists of repeated application of the first algorithm to reduce to an appropriate dimensional case.

**[Alg-2]** The second algorithm: Let $\{p_\ell\}_{\ell=2}^{n+1}$ denote Lebesgue exponents such that

$$p_2 \geq p_3 \geq \ldots \geq p_{n+1} := p \geq 2.$$  

Let $0 \leq \alpha_\ell, \beta_\ell \leq 1$ be the numbers defined in terms of $p_\ell$ by

$$\frac{1}{p_\ell} := \frac{1 - \alpha_{\ell-1}}{2} + \frac{\alpha_{\ell-1}}{p_{\ell-1}}, \quad \beta_\ell := \prod_{i=\ell}^{n} \alpha_i \quad \text{for} \quad 3 \leq \ell \leq n,$$

and $\alpha_{n+1} := \beta_{n+1} := 1$. 

Input. Fix $R \gg 1$. Let $f$ satisfy $\text{supp} \hat{f} \subset B^n(0, 1)$ and the non-degeneracy hypothesis
\[ \|e^{it\Delta} f\|_{BL^{p,q}(B_R^n)} \geq C R^\ell \| f \|_{L^2}. \] (4.7)

Output. The $(n + 2 - \ell)$-th step of the recursion will produce:

- Scales $\vec{r}_1 = (r_{n+1}, \ldots, r_\ell)$, satisfying $R = r_{n+1} > r_n > \ldots > r_\ell$. Large and non-admissible parameters $\vec{D}_\ell = (D_{n+1}, \ldots, D_\ell)$. Here $D_i, \ell \leq i \leq n + 1$, are defined by the upper bound of (4.4).
- A family transverse complete intersections $\mathcal{S}_\ell$. Each $S_i \in \mathcal{S}_\ell, \ell \leq i \leq n + 1$, satisfies $\deg S_i = i$.
- A function $f_{\mathcal{S}_\ell}$ associated with each $S_i \in \mathcal{S}_\ell$ is concentrated on scale $r_\ell$ wave packets which are $r_\ell^{-1/2+\delta_\ell}$-tangent to $S_\ell$ in $B_{r_\ell}$.

Then the following properties hold:

Property 1.
\[ \|e^{it\Delta} f\|_{BL^{p,q}(B_R^n)} \lesssim M(\vec{r}_\ell, \vec{D}_{\ell}) \| f \|_{L^2}^{1-\beta_\ell} \left( \sum_{S_i \in \mathcal{S}_\ell} \| e^{it\Delta} f_{S_i} \|_{BL^{p,q}(B_{r_i})} \right)^{\beta_\ell \vec{D}_{\ell}}, \] (4.8)

where
\[ M(\vec{r}_\ell, \vec{D}_{\ell}) := \left( \prod_{i=1}^{n} D_i \right)^{(n+1-\ell)\delta} \left( \prod_{i=1}^{n} r_i^{\frac{1}{n+1-\beta_i}} \right) D_\ell^{\frac{1}{\beta_\ell}}. \]

Property 2. For $\ell \leq n$,
\[ \sum_{S_i \in \mathcal{S}_\ell} \| f_{S_i} \|_{L^2}^2 \lesssim D_\ell^{1+\delta} \sum_{S_{i+1} \in \mathcal{S}_{i+1}} \| f_{S_{i+1}} \|_{L^2}^2. \] (4.9)

Property 3. For $\ell \leq n$,
\[ \max_{S_i \in \mathcal{S}_\ell} \| f_{S_i} \|_{L^2}^2 \lesssim \left( \frac{r_{\ell+1}}{r_\ell} \right)^{\frac{n+1}{2}} D_\ell^{-\ell+\delta} \max_{S_{i+1} \in \mathcal{S}_{i+1}} \| f_{S_{i+1}} \|_{L^2}^2. \] (4.10)

The first step. The algorithm shall start by taking:

- $r_{n+1} := R$ and $D_{n+1} := 1$;
- $\mathcal{S}_{n+1} := \{ S_{n+1} \}$, where $S_{n+1} := \mathbb{R}^{n+1}$;
- $\mathcal{O}_0 := \{ O_0 \}$, where $O_0 := N_{r_{n+1}^{1/2+\delta_{n+1}}} Z \cap B_R$;
- $f_{S_{n+1}} := f$ is a function satisfying the non-degeneracy hypothesis, and $f$ is concentrated on scale $R$ wave packets which are $R^{-1/2+\delta_{n+1}}$-tangent to the $(n+1)$-dimensional variety $\mathbb{R}^{n+1}$ in $B_R^n$.

With these definitions, Property 1, 2 and 3 hold trivially.

The $(n + 3 - \ell)$-th step. Let $\ell \geq 1$. We assume that the algorithm has ran through $(n + 2 - \ell)$ steps. Since each function $f_{S_i}$ is concentrated on scale $r_\ell$ wave packets which are $r_\ell^{-1/2+\delta_\ell}$-tangent to $S_\ell$ in $B_{r_\ell}$, we repeat to apply [Alg-1] to bound $\|e^{it\Delta} f_{S_i} \|_{BL^{p,q}(B_{r_i})}$ for each $S_i \in \mathcal{S}_\ell$. One of two things can happen: either [Alg-2] terminates if most of $S_i \in \mathcal{S}_\ell$ ended with the stopping condition [tiny] after using [Alg-1] or it terminates due
to the stopping condition \[\text{[tang]}\]. This recursive process terminates if the contributions from terms of the former type dominate. We give the exact stopping conditions as follows.

**Stopping conditions.** The algorithm will be stopped if the following condition occurs: \[\text{[tiny-dom]}\] Suppose that the inequality
\[
\sum_{S_t \in S_t} \| e^{it\Delta} f_{S_t} \|_{BL^{p_\ell,q}(B_{r_t})}^{p_\ell} \leq 2 \sum_{S_t \in S_t, \text{tang}} \| e^{it\Delta} f_{S_t} \|_{BL^{p_\ell,q}(B_{r_t})}^{p_\ell}
\] (4.11)
holds, where the right-hand summation is restricted to those \( S_t \in S_t \) for which \[\text{[Alg-1]}\] terminates owing to the stopping condition \[\text{[tiny]}\]. Then \[\text{[Alg-2]}\] terminates.

**Property 2** and **3** of \[\text{[Alg-2]}\] can be obtained by repeated applications of **Property II** and **III** in \[\text{[Alg-1]}\]. We give the proof of **Property 1**, which relies on the following result. This result can be proved by the same argument as in \[\text{[10]}\], so here we omit the proof.

**Theorem 4.2.** Let \( n \geq 2, \, \delta \ll \epsilon, \) and \( k \) be a dimension in the range \( 2 \leq k \leq n \). Suppose that \( Z = Z(P_1, \ldots, P_{n+1-k}) \) is a transverse complete intersection where \( \deg(P_i) \leq D_Z = R^{\delta_{\text{deg}}} \). Here \( \delta_{\text{deg}} \ll \delta \) is a small parameter. Suppose that \( f \) has Fourier support in \( B^n(0,1) \), and \( f \) is concentrated on scale \( R \) wave packets which are \( R^{-\frac{k}{2}+\delta} \)-tangent to \( Z \) on \( B_R \). Then we have
\[
\left\| \sup_{0 < t \leq R} | e^{it\Delta} f | \right\|_{L^2(B^n(0,R))} \lesssim R^{\frac{k}{2}+\delta} \| f \|_{L^2}
\] (4.12)
for all \( R \geq 1 \).

**The proof of Property 1.** If the stopping condition \[\text{[tiny-dom]}\] not happened, then
\[
\sum_{S_t \in S_t} \| e^{it\Delta} f_{S_t} \|_{BL^{p_\ell,q}(B_{r_t})}^{p_\ell} \leq 2 \sum_{S_t \in S_t, \text{tang}} \| e^{it\Delta} f_{S_t} \|_{BL^{p_\ell,q}(B_{r_t})}^{p_\ell}
\] (4.13)
holds, where the right-hand summation is restricted to those \( S_t \in S_t \) for which \[\text{[Alg-1]}\] terminates owing to the stopping condition \[\text{[tang]}\]. For each \( S_t \in S_t, \text{tang} \), let \( S_{\ell-1}[S_t] \) denote a collection of transverse complete intersections in \( \mathbb{R}^{n+1} \) of dimension \( \ell - 1 \) which is produced by the stopping condition \[\text{[tang]}\] in \[\text{[Alg-1]}\] on \( \| e^{it\Delta} f_{S_t} \|_{BL^{p_\ell,q}(B_{r_t})}^{p_\ell} \); then by **Proposition 1** in \[\text{[Alg-1]}\],
\[
\| e^{it\Delta} f_{S_t} \|_{BL^{p_\ell,q}(B_{r_t})}^{p_\ell} \lesssim D_\ell^{\delta} \sum_{S_{\ell-1} \in S_{\ell-1}[S_t]} \| e^{it\Delta} f_{S_{\ell-1}} \|_{BL^{p_\ell,q}(B_{r_{\ell-1}})}^{p_\ell}.
\]
Define
\[
S_{\ell-1} := \{ S_{\ell-1} : S_t \in S_t, \text{tang} \} \quad \text{and} \quad S_{\ell-1} := S_{\ell-1}[S_t].
\]
By induction on \( \ell \), we have
\[
\| e^{it\Delta} f \|_{BL^{p_\ell,q}(B_{r_t})}^{p_\ell} \lesssim D_\ell^{\delta} M(\overline{r_t}, \overline{D_t}) \| f \|_{L^2}^{1-\beta_\ell} \left( \sum_{S_{\ell-1} \in S_{\ell-1}} \| e^{it\Delta} f_{S_{\ell-1}} \|_{BL^{p_\ell,q}(B_{r_{\ell-1}})}^{p_\ell} \right)^{\frac{\beta_\ell}{p_\ell}}.
\]
On the other hand, by Proposition 3.2 and Hölder’s inequality, we have
\[
\left( \sum_{S_{t-1} \in S_{t-1}} \| e^{it\Delta} f_{S_{t-1}} \|_{BL^{p,q}(B_{r_{t-1}})}^{p} \right)^{\frac{1}{p}} = \left\| e^{it\Delta} f_{S_{t-1}} \right\|_{BL^{p,q}(B_{r_{t-1}})}^{\frac{1}{p}} \leq \left\| e^{it\Delta} f_{S_{t-1}} \right\|_{BL^{p,q}(B_{r_{t-1}})}^{\frac{1}{p}} \leq \left\| e^{it\Delta} f_{S_{t-1}} \right\|_{BL^{p,q}(B_{r_{t-1}})}^{\frac{1}{p}}.
\]
From Theorem 4.2 it follows
\[
\left\| e^{it\Delta} f_{S_{t-1}} \right\|_{L_{r_{t-1}}^{2}(B_{r_{t-1}})} \leq \frac{t^{\frac{1}{2}}}{r_{t-1}} \left\| f_{S_{t-1}} \right\|_{L_{r_{t-1}}^{2}},
\]
and so we have by Proposition 3.3 and Remark 3.5
\[
\left\| e^{it\Delta} f_{S_{t-1}} \right\|_{BL^{2,q}(B_{r_{t-1}})} \lesssim \left\| e^{it\Delta} f_{S_{t-1}} \right\|_{BL^{2,\infty}(B_{r_{t-1}})} \lesssim \frac{t^{\frac{1}{2}}}{r_{t-1}} \left\| f_{S_{t-1}} \right\|_{L_{r_{t-1}}^{2}}.
\]
This inequality implies
\[
\left\| e^{it\Delta} f_{S_{t-1}} \|_{BL^{2,q}(B_{r_{t-1}})} \right\|_{L_{r_{t-1}}^{2}(S_{t-1})} \lesssim \frac{t^{\frac{1}{2}}}{r_{t-1}} \left( \prod_{i=1}^{n} D_{l+\delta}^{1} \right) \left\| f \right\|_{L_{r_{t-1}}^{2}}.
\]
Combining all estimates, one concludes
\[
\left\| e^{it\Delta} f \right\|_{BL^{p,q}(B_{r})} \lesssim M(r_{t-1}, D_{l+\delta}) \left\| f \right\|_{L_{r_{t-1}}^{2}} \left( \sum_{S_{t-1} \in S_{t-1}} \left\| e^{it\Delta} f_{S_{t-1}} \|_{BL^{p,q}(B_{r_{t-1}})} \right\|_{L_{r_{t-1}}^{2}(S_{t-1})} \right)^{\frac{1}{p}}.
\]

\[\square\]

5. The final stage

In this section, we apply the algorithms in the previous section to prove Theorem 3.13. We assume that \([\text{Alg-2}]\) terminates the step \(m\), then \(m \geq 2\). In fact, if the recursive process terminates the step \(m = 1\), then each function \(f_{S_{1}}\) is concentrated on wave packets which are tangent to some transverse complete intersection of dimension 1. By Theorem 3.10 we have
\[
\left\| e^{it\Delta} f_{S_{1}} \right\|_{BL^{p,q}(B_{r_{1}})} = O(r_{1}^{-N}) \left\| f \right\|_{L_{r_{1}}^{2}},
\]
for any \(N > 0\). Then it easily follows that
\[
\left\| e^{it\Delta} f \right\|_{BL^{p,q}(B_{r})} = O(R^{-N}) \left\| f \right\|_{L_{r_{1}}^{2}},
\]
which contradicts the non-degeneracy hypothesis (4.7).

Recall that \([\text{Alg-2}]\) terminates at the stopping condition \([\text{tiny-dom}]\), i.e.
\[
\sum_{S_{m} \in S_{m}} \left\| e^{it\Delta} f_{S_{m}} \|_{BL^{p,q}(B_{r_{m}})} \right\|_{L_{r_{m}}^{2}} \lesssim 2 \sum_{S_{m} \in S_{m,\text{tiny}}} \left\| e^{it\Delta} f_{S_{m}} \right\|_{BL^{p,q}(B_{r_{m}})}.
\]
For each \(S_{m} \in S_{m,\text{tiny}}\), let \(O[S_{m}]\) denote the final collection of cells output by \([\text{Alg-1}]\). By Properties I, II and III of \([\text{Alg-1}]\), we obtain
\[
\left\| e^{it\Delta} f_{S_{m}} \right\|_{BL^{p,q}(B_{r_{m}})} \lesssim D_{m}^{\delta} \sum_{O \in O[S_{m}]} \left\| e^{it\Delta} f_{O} \right\|_{BL^{p,q}(O)},
\]
\[
\text{(5.2)}
\]
In order to prove Theorem 1.3, we need to verify the definition of $\beta$

Let $O$ where

Hence (5.5) becomes

Since each $O \in \mathcal{O}$ has diameter at most $R^k$ by the stopping condition [tiny], one has

Hence (5.5) becomes

The definition of $\beta_m$ leads to

and so we obtain

On the other hand, using (5.3) and Property 2 from [Alg-2], one has

By (5.4) and repeated application of Property 3 from [Alg-2], it implies that

where $r_{m-1} := 1$. Combining (5.7), (5.8) and (5.9), one concludes

where

In order to prove Theorem 1.3, we need to verify $X_i \leq 0, \ m \leq i \leq n$ and $Y_i \leq 0, \ m - 1 \leq i \leq n$. 

One easily verifies that
\[ X_i \leq 0 \iff \left( \frac{1}{2} - \frac{1}{1} \right)^{-1} \left( \frac{1}{2} - \frac{1}{p_i} \right)^{-1} \leq \frac{i + 1}{i} \] (5.11)
and
\[ Y_{i-1} \leq 0 \iff \left( \frac{1}{2} - \frac{1}{p_i} \right)^{-1} - 2i \leq 0. \] (5.12)

We firstly choose \( p_m = \frac{2m}{m-1} \), then
\[ \left( \frac{1}{2} - \frac{1}{p_m} \right)^{-1} = 2m. \]

Let \( i = m \) in (5.11) and \( i = m + 1 \) in (5.12), then
\[ \left( \frac{1}{2} - \frac{1}{p_{m+1}} \right)^{-1} \leq \min \{ 2(m + 1), 2m + \frac{m + 1}{m} \} = 2m + \frac{m + 1}{m}. \]

We choose \( p_{m+1} \) such that
\[ \left( \frac{1}{2} - \frac{1}{p_{m+1}} \right)^{-1} = 2m + \frac{m + 1}{m}. \]

Repeating the above argument, we obtain the following formula
\[ \left( \frac{1}{2} - \frac{1}{p_i} \right)^{-1} = 2m + \frac{m + 1}{m} + \ldots + \frac{i}{i-1}, \quad i \geq m + 1. \]

In particular, let \( i = n + 1 \),
\[ \left( \frac{1}{2} - \frac{1}{p_{n+1}} \right)^{-1} = 2m + \frac{m + 1}{m} + \ldots + \frac{n + 1}{n}, \]
and so we have
\[ p_{n+1} = 2 + \frac{4}{n + m - 1 + \frac{1}{m} + \ldots + \frac{1}{n}}. \]

The worst case happens when \( m = 2 \), i.e.
\[ p_{n+1} = 2 + \frac{4}{n + 1 + \frac{1}{2} + \ldots + \frac{1}{n}}. \]

Therefore, we have proved Theorem 1.3.

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