

Cinderella, Quadrilaterals and Conics

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Abstract

We study quadrilaterals inscribed and circumscribed about conics. Our research is guided by experiments in software Cinderella. We extend the known results in projective geometry of conics and show how modern mathematical software brings new ideas in pure and applied mathematics. Poncelet theorem for quadrilaterals is proved by elementary means together with Poncelet’s grid property.

1 Introduction

This paper is one in the serial of our forthcoming papers in geometry of curves, combinatorics and dynamic systems. The use of software Cinderella is common for all of them and our aim is to show that the good software is more than a box with nice examples and calculations. The smart use could lead us not only to discovering new results, but it gives the complete and correct proofs! In this sense Cinderella could go beyond the limits of geometry of conics and mechanical experiments, even to the curves of higher degree and abstract combinatorics, geometry and topology.

The positive experience with Cinderella in the paper Illumination of Pascal’s Hexagrammum and Octagrammum Mysticum by Baralić and Spasojević, [1] encouraged us to continue the research. The problems we study are strongly influenced by very inspirational paper Curves in Cages: an Algebro-geometric Zoo of Gabriel Katz printed in American Mathematical Monthly, [10]. Many important questions in dynamical systems and combinatorics have their equivalents in the terms of algebraic curves. Richard Schwartz and Serge Tabachnikov in [21] asked for the proof of Theorem 4.c. They found the theorem studying the pentagram maps, introduced in [19]. This is still open hypothesis and could be reformulated in the question about curves.

We have not found the proof for Schwartz and Tabachnikov Theorem 4.c but during recent work we discovered new interesting facts about quadrilaterals inscribed and quadrilaterals circumscribed about conic. Theorems about quadrilaterals and conics are usually known like degenerate cases of Pascal and Brianchon theorems. In [1] Baralić and Spasojević proved some new results about two quadrilaterals inscribed

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in a conic. However, in this paper we study more complicated structures involving both tangents at the vertices and the side lines of quadrilateral. We start from the degenerate form of Pascal and Brianchon theorems for the quadrilateral and then we discover new interesting points, conics and loci.

The objects are studied by elementary means. Some of the results are in particular the corollary of Great Poncelet Theorem for the case when \( n \)-gon is quadrilateral. Here we give the short proof for this case. Some special facts about this special case are explained as well.

Finally, we compare two theorems - Mystic Octagon theorem for the case of two quadrilaterals and Poncelet Theorem for the quadrilaterals. Both of them have in common that they state that certain 8 points coming from two quadrilaterals inscribed in a conic lie on the same conic. While the first one is pure algebro-geometric fact, the latter involves much deeper structure of the space and can not be seen naturally as the special case of the first. Thus, we could not find 'Theorem of all theorems for conics in projective geometry' and elementary surprises in projective geometry like those in [21] could come as the special case of different general statements.

2 From Pascal to Brocard Theorem

In this section we show how Pascal theorem for hexagon (1639) inscribed in a conic degenerates to Brocard theorem for the quadrilateral inscribed in a circle. All results here are well known and are part of the standard olympiad problem solving curriculum, but our aim is to illustrate the power of degeneracy tool and prepare the background for the next sections.

**Lemma 2.1.** Let \( ABCD \) be a quadrilateral inscribed in a conic \( C \) and let \( M \) be the intersection point of the lines \( AD \) and \( BC \), \( N \) be the intersection point of the lines \( AB \) and \( CD \), \( P \) be the intersection point of the tangents to \( C \) at \( A \) and \( C \), and \( Q \) be the intersection point of the tangents to \( C \) at \( B \) and \( D \). Then, the points \( M, N, P \) and \( Q \) are collinear (see Figure 1).

![Figure 1: Lemma 2.1](image)

**Proof:** Apply Pascal theorem to degenerate hexagon \( AABCCD \) and we get the points \( M, N \) and \( P \) are collinear. Apply Pascal theorem to degenerate hexagon \( ABBCD \) and we get the points \( M, N \) and \( Q \) are collinear. \( \square \)

Dual statement to Lemma 2.1 is the following:
Lemma 2.2. Let conic $C$ touch the sides $AB$, $BC$, $CD$ and $DA$ of a quadrilateral $ABCD$ in the points $M$, $N$, $P$ and $Q$, respectively. Then the lines $AC$, $BD$, $MP$ and $NQ$ pass through the same point $O$ (see Figure 2).

The Lemmas 2.1 and 2.2 will be used to prove the other interesting relations among the lines and points that naturally occur in a quadrilateral inscribed in conics configurations. Many points are going to be introduced so we are going to organize labels of the points.

Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a conic $C$ and let $M_1$ be the intersection point of the lines $A_1A_2$ and $A_3A_4$, $M_2$ of $A_2A_3$ and $A_4A_1$ and $M_3$ of $A_3A_1$ and $A_2A_4$. Let $N_3$ be the intersection point of the tangent lines to the conic at $A_1$ and $A_3$, $P_3$ of the tangents at $A_2$ and $A_4$, $N_2$ of the tangents at $A_1$ and $A_4$, $P_2$ of the tangents at $A_2$ and $A_3$, $N_1$ of the tangents at $A_1$ and $A_2$ and $P_1$ of the tangents at $A_3$ and $A_4$. Let $U_1$ and $U_2$ be the points where tangents from $M_1$ touch $C$, and analogously $V_1$, $V_2$ and $W_1$, $W_2$ for the points $M_2$ and $M_3$ respectively.

Lemma 2.1 states that the points $M_1$, $M_2$, $N_3$, $P_3$ are collinear, and also the points $M_2$, $M_3$, $N_1$, $P_1$ and $M_3$, $M_1$, $N_2$ and $P_2$. Denote this three lines by $m_3$, $m_1$ and $m_2$, respectively. We are going to prove that $U_1$ and $U_2$ lie on the line $m_3$, $V_1$ and $V_2$ on the line $m_2$ and $W_1$ and $W_2$ on $m_1$ - so that $m_1$, $m_2$ and $m_3$ are the polar lines of the points $M_1$, $M_2$ and $M_3$ with respect to $C$.

Lemma 2.3. The points $U_1$, $M_2$, $U_2$ and $M_3$ are collinear.

Proof: There is a projective transformation $\varphi$ that maps the points $A_1$, $A_2$, $A_3$ and $A_4$ onto the vertices of a square. Thus $\varphi(M_3)$ is the center of a square with vertices...
ϕ(A₁), ϕ(A₂), ϕ(A₃) and ϕ(A₄). The points ϕ(M₁) and ϕ(M₂) are at infinity. There is a unique way to inscribe the square into the conic, and the lines ϕ(A₁)ϕ(A₂) and ϕ(A₁)ϕ(A₄) are parallel to the axes of the conic ϕ(C). The points ϕ(U₁) and ϕ(U₂) must be mapped onto the axis parallel to the line ϕ(A₁)ϕ(A₄).

Now the points ϕ(U₁), ϕ(U₂), ϕ(M₂) and ϕ(M₃) lie on the axis of conic ϕ(C). Consequently, the points U₁, M₂, U₂ and M₃ then lie at the same line.

Figure 4: Quadrilateral inscribed in a conic

Lemma 2.3 clearly implies the analogous statement for the lines m₂ and m₃. This is the classical theorem of the projective geometry and a very useful tool. Some other facts about a quadrilateral inscribed in conics are going to be proved in the next sections.

Figure 5: Brocard theorem

In the end, we treat one very special case - when the conic C is a circle. Projective geometry gives us the plenty of techniques. For example, in the proof of Lemma 2.3 we used the projective transformation. We have already described degeneracy tool when we take some limit cases of polygons inscribed (or circumscribed) in a conic.
It is good to keep in mind that conic could degenerate itself for example to the two lines. This is a way to get interesting configurations of points and lines.

The configuration in the case of a circle has nice a property which is known as the Brocard theorem. Let $O$ be the center of a circle $C$. Then the quadrilateral $M_1U_1OU_2$ is deltoid and we get $M_1O \perp m_1$. Similarly, $M_2O \perp m_2$. Thus:

**Theorem 2.1 (Brocard theorem).** Let $O$ be the center of circumscribed circle of a cyclic quadrilateral $A_1A_2A_3A_4$. Then $O$ is the orthocenter of triangle $\triangle M_1M_2M_3$.

## 3 More lines and pencils of lines

We continue in the same manner. The lines and the pencils of lines we study came from various degenerations of the vertices of hexagon inscribed in a conic. Let us remind that configuration associated with 60 Pascal lines has been described in [13], [23] and [1]. All results from this section could be obtained as the certain degenerate case. But we are going to treat them by elementary means.

Let $T_1$ be the point of intersection of the line $A_3A_4$ and tangent at $A_1$ to $C$, $T_2$ of $A_2A_3$ and tangent at $A_2$, $T_3$ of $A_1A_2$ and tangent at $A_3$ and $T_4$ of $A_2A_3$ and tangent at $A_4$. Let $X_1$ be the point of intersection of the line $A_2A_3$ and tangent at $A_1$, $X_2$ of $A_3A_4$ and tangent at $A_2$, $X_3$ of $A_4A_1$ and tangent at $A_3$ and $X_4$ of $A_1A_2$ and tangent at $A_4$. Let $Y_1$ be the point of intersection of the line $A_2A_3$ and tangent at $A_1$, $Y_2$ of $A_1A_4$ and tangent at $A_2$, $Y_3$ of $A_1A_4$ and tangent at $A_3$ and $Y_4$ of $A_2A_3$ and tangent at $A_4$.

**Proposition 3.1.** The following 16 triples of points are collinear: $(M_1,Y_1,V_2)$, $(M_1,Y_3,V_4)$, $(M_1,X_3,T_4)$, $(M_1,X_1,T_2)$, $(M_2,Y_1,V_4)$, $(M_2,Y_2,V_3)$, $(M_2,X_4,T_1)$, $(M_2,X_2,T_3)$, $(M_3,T_1,T_3)$, $(M_3,X_2,X_4)$, $(M_3,X_1,X_3)$, $(M_3,T_2,T_4)$, $(X_2,Y_3,T_4)$, $(X_1,Y_4,T_1)$, $(X_4,Y_1,T_2)$.

**Proof:** The collinearity of the points $M_1$, $X_3$ and $T_4$ follows from the Pascal theorem for degenerate hexagon $A_1A_4A_3A_4A_2$, the collinearity of the points $M_1$, $X_3$ and $Y_4$ from degenerate hexagon $A_1A_3A_4A_3A_2$ and the collinearity of the points $X_2$, $X_3$ and $T_4$ from degenerate hexagon $A_2A_3A_4A_3A_4A_2$. The proof for the rest is analogous. □

**Proposition 3.2.** The following 6 triples of lines are concurrent: $(M_2M_3,X_2Y_3,X_3Y_4)$, $(M_1M_3,X_1Y_2,X_2Y_3)$, $(M_1M_2,X_1Y_2,X_3Y_4)$, $(M_2M_3,X_1Y_2,X_4Y_1)$, $(M_1M_3,X_4Y_1,X_3Y_4)$, $(M_1M_2,X_4Y_1,X_2Y_3)$.

**Proof:** By Lemma 4.1 (to be proved in the next section) the points $X_1$, $X_2$, $X_3$, $X_4$, $T_1$, $T_2$, $T_3$ and $T_4$ lie on the same conic. From Pascal theorem for the hexagon $T_1X_3X_1T_2X_4X_2$ we get that lines $M_1M_3$, $X_3Y_1$ and $X_3Y_4$ are concurrent. Analogously for other triples. □

Define the points as the intersections of the lines: $B_1 = l(A_2V_1) \cap l(A_1V_2)$, $C_1 = l(A_1V_1) \cap l(A_2V_2)$, $D_1 = l(A_3V_1) \cap l(A_4V_2)$, $E_1 = l(A_4V_1) \cap l(A_3V_2)$, $B_2 = l(A_4V_1) \cap l(A_3V_2)$, $C_2 = l(A_4V_2) \cap l(A_3V_1)$, $D_2 = l(A_3V_1) \cap l(A_4U_2)$, $E_2 = l(A_4U_1) \cap l(A_4U_2)$, $B_3 = l(A_3U_1) \cap l(A_2U_2)$, $C_3 = l(A_3U_2) \cap l(A_2U_1)$, $D_3 = l(A_2U_1) \cap l(A_3U_2)$, $E_3 = l(A_2U_1) \cap l(A_3U_2)$, $B_4 = l(A_1U_1) \cap l(A_1U_2)$, $C_4 = l(A_1U_2) \cap l(A_1U_1)$, $D_4 = l(A_1U_2) \cap l(A_1U_2)$, $E_4 = l(A_1U_2) \cap l(A_1U_2)$, $B_5 = l(A_1U_2) \cap l(A_1U_2)$, $C_5 = l(A_1U_2) \cap l(A_1U_2)$, $D_5 = l(A_1U_2) \cap l(A_1U_2)$, $E_5 = l(A_1U_2) \cap l(A_1U_2)$, $B_6 = l(A_1U_2) \cap l(A_1U_2)$, $C_6 = l(A_1U_2) \cap l(A_1U_2)$, $D_6 = l(A_1U_2) \cap l(A_1U_2)$, $E_6 = l(A_1U_2) \cap l(A_1U_2)$.
Proposition 3.3. The points $B_1, C_1, D_1, E_1, F_1, G_1, H_1, I_1$ lie on the line $M_2M_3$. Similarly, the points $B_2, C_2, D_2, E_2, F_2, G_2, H_2, I_2$ lie on the line $M_3M_1$ and the points $B_3, C_3, D_3, E_3, F_3, G_3, H_3, I_3$ lie on the line $M_1M_2$.

Proof: Consider the quadrilateral formed by the tangent lines to the conic $C$ at the points $A_4, A_2, V_1$ and $V_2$. Applying Lemma 2.2 we get that the point $B_3$ lies on the line $M_1M_2$. Analogously for other points. \qed
4 Surprising conics

In the upper sections many points were introduced. We have showed some of them are collinear while some are the intersections of certain lines. But some of them lie on the curves of degree two!

**Lemma 4.1.** The points $X_1$, $X_2$, $X_3$, $X_4$, $T_1$, $T_2$, $T_3$, and $T_4$ lie on the same conic $C_1$; $Y_1$, $Y_2$, $Y_3$, $Y_4$, $X_1$, $X_3$, $T_2$, and $T_4$ lie on the same conic $C_2$; $T_3$, $X_2$, $X_4$, $Y_1$, $Y_2$, $Y_3$ and $Y_4$ lie on the same conic $C_3$ (see Figure 8).

**Proof:** This statement is the special case of the Mystic Octagon theorem, the first time formulated by Wilkinson in [24]. The first conic appears when we consider degenerate octagon $A_1A_2A_3A_4A_5$, the second for $A_1A_3A_5A_2A_4A_1$, and the third for $A_1A_3A_5A_4A_2A_2A_1$. □

Let $J_{2i−1}$ be the intersection points of the tangents at $X_{i−2}$ and $T_i$ on the conic $C_1$, and $J_{2i}$ the intersection points of the tangents at $X_{i−1}$ and $T_i$ (modulo 4), for $i = 1, 2, 3, 4$. Then the following claim is true:

**Theorem 4.1.**

- The lines $J_iJ_{i+4}$, for $i = 1, 2, 3, 4$ intersect at the point $M_3$.
- The lines $J_1J_7$, $J_2J_6$ and $J_3J_5$ intersect at $M_1$ and the lines $J_1J_3$, $J_4J_8$ and $J_5J_7$ intersect at $M_2$.
- The lines $J_1J_4$ and $J_2J_5$ intersect at $A_1$, the lines $J_4J_7$ and $J_3J_6$ at $A_2$, the lines $J_6J_1$ and $J_5J_8$ at $A_3$ and the lines $J_3J_8$ and $J_2J_7$ at $A_4$.
- The intersection points $l(J_2J_4) \cap l(J_6J_8)$, $l(J_2J_8) \cap l(J_4J_6)$, $l(J_3J_6) \cap l(J_2J_7)$, $l(J_5J_8) \cap l(J_1J_4)$, $l(J_3J_8) \cap l(J_4J_7)$, $l(J_2J_5) \cap l(J_1J_6)$ and $l(J_iJ_{i+1}) \cap l(J_{i+4}J_{i+5})$ for $i = 1, 2, 3, 4$ lie on the same line $M_1M_2$.
- The intersection points $l(J_4J_3) \cap l(J_7J_8)$ and $l(J_3J_4) \cap l(J_1J_8)$ lie on the same line $M_1M_3$, the intersection points $l(J_2J_3) \cap l(J_5J_6)$ and $l(J_1J_2) \cap l(J_6J_7)$ lie on the same line $M_2M_3$.
- The point $P_3$ lies on the line $J_3J_7$ and the point $N_3$ on the line $J_1J_5$. 

Figure 8: Propositions 4.1
• Three lines \( J_{2i-2} J_{2i+2}, J_{2i+1} J_{2i-2} \) and \( J_{2i-1} J_{2i+2} \) (modulo 8) are concurrent for \( i = 1, 2, 3, 4 \).

**Proof:** Consider the quadrilateral formed by tangents to \( C_1 \) at \( J_2 \) and \( J_6 \). By Lemma 2.2 and Proposition 3.1 the points \( M_3 \) and \( M_2 \) lie on the line \( J_2 J_6 \) (we could take the order of points differently). Analogously, the lines \( J_1 J_5 \), \( J_3 J_7 \) and \( J_4 J_8 \) pass through the point \( M_3 \). In similar fashion we prove other statements for the points \( M_1 \) and \( M_2 \) as well as the points \( N_3 \) and \( P_3 \).

Lemma 2.2 applied to the quadrilateral formed by tangents to \( C_1 \) at \( J_2 \) and \( J_5 \) proves that line \( J_2 J_5 \) pass through \( A_1 \). Similarly, \( A_1 \) belongs to the line \( J_1 J_4 \). Analogously, we prove the corresponding statements for the points \( A_2, A_3 \) and \( A_4 \).

From Lemma 2.3 applied on the quadrilateral \( T_2 X_1 T_4 X_3 \) and Proposition 3.1 it follows that the intersection point of the lines \( J_3 J_4 \) and \( J_7 J_8 \) and the intersection point of the lines \( J_4 J_5 \) and \( J_8 J_1 \) lie on the line \( M_1 M_2 \). Then by Brianchon Theorem for the hexagon formed by the tangents to \( C_1 \) at \( T_2, X_1, T_3, T_1, X_3 \) and \( T_4 \) the intersection point of the line \( J_1 J_4 \) and \( J_5 J_8 \) lie on the line \( M_1 M_2 \). Analogously for the others.

Brianchon Theorem for the hexagon formed by the tangents to \( C_1 \) at \( T_2, X_1, X_4, T_1, X_3 \) and \( T_4 \) applies the concurrency of the lines \( J_2 J_6, J_1 J_4 \) and \( J_5 J_8 \). We use the similar argument for the rest of the proof.

Let \( K_i \) be the intersection points of lines \( J_i J_{i+1} \) and \( J_{i+2} J_{i+3} \) (modulo 8) for \( i = 1, \ldots, 8 \).

**Theorem 4.2.** The points \( K_i \) lie on the same conic \( D_1 \).

**Proof:** It is not hard to prove that the lines \( K_1 K_5, K_2 K_6, K_3 K_7 \) and \( K_4 K_8 \) pass through the point \( M_3 \), the lines \( K_2 K_3, K_1 K_4, K_5 K_8 \) and \( K_6 K_7 \) pass through the point \( M_1 \) and the lines \( K_2 K_7, K_1 K_8, K_3 K_6 \) and \( K_4 K_5 \) pass through the point \( M_2 \). From the collinearity of the points \( M_1, J_2 \) and \( l(J_4 J_5) \cap l(J_7 J_8) \) the points \( K_1, K_2, K_4, K_5, K_7 \) and \( K_8 \) lie on the same conic. Using the similar argument we show that \( K_2, K_4, K_3, K_6, K_7 \) and \( K_8 \) lie on the same conic. Because there is a unique conic determined by its 5 points then all the points \( K_1, K_2, K_4, K_5, K_6, K_7 \) and \( K_8 \) are on the same conic. Then it is easy to prove that \( K_3 \) also lies on the conic. \( \square \)
Let \( Z_1 = l(M_1U_1) \cap l(M_2V_1) \), \( Z_2 = l(M_1U_1) \cap l(M_2V_2) \), \( Z_3 = l(M_1U_2) \cap l(M_2V_2) \) and \( Z_4 = l(M_1U_2) \cap l(M_2V_1) \).

**Theorem 4.3.** The points \( N_1, N_2, P_1, P_2, Z_1, Z_2, Z_3 \) and \( Z_4 \) lie on the same conic.

**Proof:** There exists projective transformation \( \varphi \) that maps vertices \( A_1, A_2, A_3 \) and \( A_4 \) onto the vertices of a square. Then point \( \varphi(M_3) \) is mapped onto the center of conic \( \varphi(C) \) and the lines \( \varphi(N_1) \varphi(P_1) \) and \( \varphi(N_2) \varphi(P_2) \) are the axes. The points \( \varphi(U_1), \varphi(U_2), \varphi(V_1) \) and \( \varphi(V_2) \) also lie on the axes. As we could see in Figure 11 everything is symmetric and it is easy to conclude that there is a conic through \( \varphi(Z_1), \varphi(Z_2), \varphi(Z_3), \varphi(Z_4), \varphi(N_1), \varphi(N_2), \varphi(P_1) \) and \( \varphi(P_2) \).

Theorems 4.1, 4.2 and 4.3 associate new conics to the quadrilateral inscribed in a conic. They have interesting properties which will be explained in the next section.

## 5 Poncelet’s quadrilateral porism

Jean-Victor Poncelet’s famous *Closure theorem* states that if there exists one \( n \)-gon inscribed in conic \( C \) and circumscribed about conic \( D \) then any point on \( C \) is the
vertex of some \( n \)-gon inscribed in conic \( C \) and circumscribed about conic \( D \). Poncelet published his theorem in \([17]\). However, this result influenced mathematics until nowadays. In recent book \([7]\) by Dragovic and Radnovic there are several proofs of Closure theorem, it’s generalizations as well as it’s relations with elliptic functions theory. The proof is not elementary for general \( n \), although in the case \( n = 3 \) elegant proof could be found in almost every monograph in projective geometry, see \([16]\).

Theorems 4.2 and 4.3 are the special cases of Poncelet theorem for \( n = 4 \). Actually, quadrilaterals and conics in them have poristic property. We kept the spirit of elementarity through our paper and our agenda was: Firstlu, we experiment in Cinderella, after that the proof is recovered by elementary tools (again directly guided by Cinderella’s tools). In the same style we continue and offer direct analytic proof of Poncelet theorem for quadrilaterals without using differentials and elliptic functions.

**Lemma 5.1.** Let \( \lambda, \mu \) be such that conics \( C : \lambda x^2 + (1 - \lambda)y^2 - 1 = 0 \) and \( D : x^2 + \mu xy + y^2 + \frac{n^2 - 1}{4} = 0 \) are non-degenerate. Let \( A \) be a point on \( C \) and \( B \) and \( B' \) be the intersections of the tangent lines from \( A \) to \( D \) with conic \( C \). Then the points \( B \) and \( B' \) are symmetric with respect to the origin.

**Proof:** Let a line \( t : y = kx + n \) be a tangent line to conic \( D \). The condition of tangency between \( t \) and \( D \) is

\[
n^2 = k^2 + mk + 1.
\]  

The coordinates of the intersection points of \( t \) and \( C \) are

\[
(x_1, y_1) = \left( \frac{-2(1 - \lambda)kn - \sqrt{D}}{2(\lambda + (1 - \lambda)k^2)}, k \cdot \left( \frac{-2(1 - \lambda)kn - \sqrt{D}}{2(\lambda + (1 - \lambda)k^2)} \right) + n \right)
\]
and

\[(x_2, y_2) = \left( \frac{-2(1 - \lambda)kn + \sqrt{D}}{2(\lambda + (1 - \lambda)k^2)}, k \cdot \left( \frac{-2(1 - \lambda)kn + \sqrt{D}}{2(\lambda + (1 - \lambda)k^2)} \right) + n \right),\]

where \(D = 4(\lambda - \lambda(1 - \lambda)n^2 + (1 - \lambda)k^2)\). It is necessary and enough to prove that the line through the points \((-x_1, -y_1)\) and \((x_2, y_2)\) is tangent to \(D\). This line has the equation \(y = \tilde{k}x + \tilde{n}\) where \(\tilde{k}\) and \(\tilde{n}\) could be calculated as

\[\tilde{k} = \frac{-\lambda}{(1 - \lambda)k} \quad \text{and} \quad \tilde{n} = \frac{\sqrt{D}}{2k(1 - \lambda)}.\]

(2)

We need to check if

\[\tilde{n}^2 = \tilde{k}^2 + m\tilde{k} + 1.\]

It is directly verified that condition (1) multiplied by \(\frac{\lambda(1 - \lambda)}{\lambda^2(1 - \lambda)^2}\) finishes our proof. \(\square\)

**Theorem 5.1.** Let \(C\) and \(D\) be conics such that there exists one quadrilateral inscribed in a conic \(C\) and circumscribed about a conic \(D\). Then any point on \(C\) is the vertex of some quadrilateral inscribed in conic \(C\) and circumscribed about conic \(D\).

**Proof:** There exists a projective transformation that maps the vertices of the quadrilateral inscribed in conic \(C\) and circumscribed about conic \(D\) onto the points \((1, 1), (1, -1), (-1, -1)\) and \((-1, 1)\) (in the standard chart). Thus, conics \(C\) and \(D\) are transformed in those with the equations as in Lemma 5.1. Now the claim follows. \(\square\)

In fact, we proved more. All quadrilaterals with poristic property with respect to \(C\) and \(D\) have the common point of the intersection of diagonals (lines joining opposite vertices) and the common line passing through the intersections of opposite side lines. Our work in previous section, now could be reviewed in the new light.

![Figure 13: The first five conics in the sequence](image-url)
again we come to similar conclusions. Thus, by repeating this procedure, we obtain the infinite sequence of conics, see Figure 13. Every two consecutive conics in this sequence are Poncelet 4-connected.

Our theorems resemble Darboux’s theorem, see [6]. They could be seen as a very special case of Dragović-Radnović theorem 8.38, [7]. Such constructions are also studied in the paper of Schwartz, see [20]. The following result further explains their connection, but first we define 16 points of the intersections $R_1 = l(Z_1Z_2) \cap l(N_1N_2)$, $R_2 = l(Z_2Z_2) \cap l(N_1P_2)$, $R_3 = l(Z_2Z_3) \cap l(N_1P_2)$, $R_4 = l(Z_2Z_3) \cap l(P_1P_2)$, $R_5 = l(Z_3Z_4) \cap l(P_1P_2)$, $R_6 = l(Z_3Z_4) \cap l(N_1N_2)$, $R_7 = l(Z_1Z_4) \cap l(N_1N_2)$, $R_8 = l(Z_1Z_4) \cap l(N_1N_2)$, $R_9 = l(Z_1Z_2) \cap l(P_1P_2)$, $R_{10} = l(Z_3Z_4) \cap l(N_1P_2)$, $R_{11} = l(Z_2Z_3) \cap l(P_1N_2)$, $R_{12} = l(Z_2Z_4) \cap l(P_1P_2)$, $R_{13} = l(Z_3Z_4) \cap l(N_1N_2)$, $R_{14} = l(Z_1Z_2) \cap l(P_1P_2)$, $R_{15} = l(Z_1Z_4) \cap l(N_1P_2)$ and $R_{16} = l(Z_2Z_3) \cap l(N_1N_2)$, see Figure 13.

**Theorem 5.2.** The next groups of 8 points lie on the same conic:
- $\{R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8\}$, $\{R_9, R_{10}, R_{11}, R_{12}, R_{13}, R_{14}, R_{15}, R_{16}\}$,
- $\{R_1, R_2, R_3, R_6, R_{11}, R_{12}, R_{15}, R_{16}\}$, $\{R_3, R_4, R_7, R_8, R_9, R_{10}, R_{13}, R_{14}\}$,
- $\{R_1, R_5, R_7, R_9, R_{11}, R_{13}, R_{15}\}$ and $\{R_2, R_3, R_4, R_6, R_8, R_{10}, R_{12}, R_{14}, R_{16}\}$.

The proof of Theorem 5.2 uses the same arguments we used in the previous proofs so we omit it.

If we look at the conic $C$ and the conic $\mathcal{F}$ through the points $\{R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8\}$ we see they are Poncelet 8-connected and appropriate conics from Theorem 5.2 conic from Theorem 143 with the line $M_1M_2$ form Poncelet-Darboux grid. Two conics $\{R_2, R_3, R_4, R_6, R_8, R_{10}, R_{12}, R_{14}, R_{16}\}$ and $\{R_1, R_5, R_7, R_9, R_{11}, R_{13}, R_{15}\}$ are not coming from Poncelet-Darboux grid, but they could be directly obtained from Dragović-Radnović theorem 8.38, [7]. This result improves the result of Schwartz [20] in a particular case.

![Figure 14: Theorem 5.2](image)

### 6 Few words about Tabachnikov-Schwartz Theorem 4c

In the end we think it is suitable to say something about already mentioned Theorem 4c stated in [21]. Tabachnikov and Schwartz asked us for the proof. For this occasion
we reformulate it in the following manner:

**Theorem 6.1** (Tabachnikov-Schwartz Theorem 4c). Let $A_1A_2 \ldots A_{12}$ be a 12-gon inscribed in a conic $C$. Let $\pi$ maps 12-gon $X_1X_2 \ldots X_{12}$ onto a new 12-gon according to the rule $\pi(X_i) = l(X_iX_{i+4}) \cap l(X_{i+1}X_{i+5})$. Then, 12-gon $A_1A_2 \ldots A_{12}$ is mapped with $\pi^{(3)} = \pi \circ \pi \circ \pi$ onto a 12-gon inscribed in a conic.

This theorem was the starting point of our research. It seemed that this theorem is a perfect candidate to use the technique illustrated in [1], although in an unpublished paper of Tabachnikov [22] one can find nice proofs for the theorems from [21]. Encouraged by our previous success, we tried to prove Theorem 4c. We used Cinderella again to test the result and to obtain a nice picture. But at the beginning we present the problem. We will explain Figure 15 carefully. We start with a 12-gon $A_1A_2 \ldots A_{12}$ (the green points lying on the violet conic) inscribed in a conic and define the (yellow) points obtained by $\pi$, (blue and violet lines), $\pi^{(2)}$ the red points (green and orange lines) and $\pi^{(3)}$ the violet points (black and yellow lines). It looks like that at the every step we have a $6 \times 6$ cage of curves, see [10]. But instead of dealing with 24 points at the second step we take only 12 of them. It is not possible to catch the curves we want in the cage. By Mystic Octagon theorem we could catch three interesting conics and one quartic in the blue-violet cage. What to do with curves at other steps. Definitely we should try to add some new points and then apply Bézout’s theorem or a similar statement. But what are that points and how to find them? If we look more carefully, three quadrilaterals can be noticed ($A_1A_4A_7A_{10}$, $A_2A_5A_8A_{11}$ and $A_3A_6A_9A_{12}$) inscribed in a conic and usually the steps are always defined as the certain intersection points of the side lines of quadrilaterals. Thus, we thought if we want to overcome the problems we faced, it is good to understand the quadrilaterals in a conic better.

We have not succeeded in proving the Theorem 4c. But we conducted some experiments in Cinderella that we think are important. Firstly, usually algebro-geometric facts give us some freedom (for example, a product of $n$ lines could be often generalized to a curve of degree $n$, see [1], etc.) but here we have not found any such generalizations. Also, the technique in [1] usually does not differ order of points, that means that certain permutations lead to new objects of the same type (for example Pascal lines). Due to the difference of three quadrilaterals we did not
find new conic at the third step. After all these experiments we believe Tabachnikov
and Schwartz Theorem 4c is more surprising and deeper fact then it looks at the first
glance!

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References

[1] D. Baralić and I. Spasojević, Illumination of Pascal’s Hexagrammum and Octa-
grammum Mysticum, arXiv:1209.4795v2, sent.

[2] D. Baralić and I. Spasojević, New Theorems about Pascal’s Hexagram, to appear
in PROCEEDINGS OF THE SECOND MATHEMATICAL CONFERENCE OF
REPUBLIC OF SRPSKA

[3] Marcel Berger, Geometry Revealed - A Jacob’s Ladder to Modern Higher Geom-

etry, Springer-Verlag Heidelberg 2010

[4] R. Bix, Conics and Cubics - A Concrete Introduction to Algebraic Curves,
Springer, 2006.

[5] T. W. Chaundy, Poncelet poristic polygons, Proceedings of the London Mathe-
matical Society, Vol. 22, (1924), 104-123.

[6] G. Darboux, Lecons sur la theorie generale des surfaces et les applications ge-
ometriques du calcul infinitesimal, Vol. 2 and 3, Gauthier-Villars, Paris, 1914

[7] V. Dragovic and M. Radnović, Poncelet Porisms and Beyond, Integrable Billi-
lards, Hyperelliptic Jacobians and Pencil of Quadrics, Springer Basel (2011).

[8] V. Dragovic and M. Radnović, Hyperelliptic Jacobians as billiard algebra of
pencils of quadrics: beyond Poncelet porisms. Adv. Math. 219 (2008), no. 5,
1577-1607.

[9] V. Dragovic, Poncelet-Darboux curves, their complete decomposition and Marden
theorem, Int. Math. Res. Not. IMRN 2011, no. 15, 3502-3523.

[10] G. Katz, Curves in Cages: an Algebra-geometric Zoo, Amer. Math. Monthly,
113(9), 2006, 777-791.

[11] H. P. M. van Kempen, On Some Theorems of Poncelet and Carnot, Forum Ge-
ometricorum, Volume 6 (2006), 229-234.

[12] F. Kirwan, Complex Algebraic Curves, Cambridge University Press, 1992.

[13] C. Ladd, The Pascal Hexagram, American Journal of Mathematics, Volume 2,
1879, 1-12.

[14] V. Ovsienko, R Schwartz and S. Tabachnikov, The Pentagram Map, A Discrete
Integrable System, Comm. Math. Phys. 299 (2010), 409-446.
[15] B. Pascal, *Essai por les coniques*, Eubres, Brunschvigg et Boutroux, Paris, Volume I, 1908, 245.

[16] V. V. Prasolov and V. M. Tikhomirov, *Geometry*, American Mathematical Society, 2001.

[17] J. V. Poncelet, *Traité des propriétés projectives des figures*, Bachelier, Paris 1822.

[18] J. Richter-Gebert, *Perspectives on Projective Geometry*, Springer-Verlag Berlin Heidelberg, 2011.

[19] Richard Evan Schwartz, *The Pentagram Map*, Journal of Experimental Math 1 (1992), 90-95.

[20] Richard Evan Schwartz, *Poncelet grid*, Advances in Geometry 7 (2007), 157-175.

[21] R. Schwartz and S. Tabachnikov, *Elementary surprises in projective geometry*, Math. Intelligencer, 32 (2010), 31-34.

[22] R. Schwartz and S. Tabachnikov, *Letter to prof. Deligne about 'Elementary surprises in projective geometry';* unpublished letter.

[23] G. Veronese, *Nuovi Teoremi sull’ Hexagrammum Misticum*, Memorie dei Reale Accademi dei Lincei, Volume I, 1877, 649-703.

[24] T. T. Wilkinson, Mathematical questions with their solutions, 2015, Educational Times XVII (1872) p.72.