Symmetric Functions and Caps

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Abstract
Given a finite subset \( S \subset \mathbb{F}_d^p \), let \( a(S) \) be the number of distinct \( r \)-tuples \( \alpha_1, \ldots, \alpha_r \in S \) such that \( \alpha_1 + \cdots + \alpha_r = 0 \). We consider the “moments” \( F(m, n) = \sum_{|S|=n} a(S)^m \). Specifically, we present an explicit formula for \( F(m, n) \) as a product of two matrices, ultimately yielding a polynomial in \( q = p^d \). The first matrix is independent of \( n \) while the second makes no mention of finite fields. However, the complexity of calculating each grows with \( m \). The main tools here are the Schur-Weyl duality theorem, and some elementary properties of symmetric functions. This problem is closely to the study of maximal caps.

1 Introduction

Given a finite subset \( S \subset \mathbb{F}_d^p \), let \( a(S) \) be the number of distinct \( r \)-tuples \( \alpha_1, \ldots, \alpha_r \in S \) such that \( \alpha_1 + \cdots + \alpha_r = 0 \). We consider the “zero-sum” problem \([4]\) is determining how large \( |S| \) can be subject to the constraint \( a(S) = 0 \). In this paper, we consider the functions

\[
F(m, n) = \sum_{|S|=n} a(S)^m,
\]

where \( S \subset \mathbb{F}_d^p \), \( p \) is a prime. \( F(m, n)/F(0, n) \) is the \( m \)th moment of the push-forward of the uniform distribution through \( a \). Since this distribution is compactly supported, the values \( F(m, n) \) for \( 0 \leq m \leq \max_{|S|=n} a(S) \) determine the entire distribution.

The strategy of this paper is to describe \( F(m, n) \) as the trace of an operator that does not depend on \( n \). Sections \([2]\) and \([3]\) describe the operator, and
the decomposition mentioned in the abstract. Section 4 is the main theorem about the decomposition, while sections 5 and 6 give explicit formulas for the matrix elements. Finally, section 7 covers the example \(m = 2, n = 10, p = 3, \) \(d\) arbitrary.

2 The Moment Function as a Trace

Let \(V = \mathbb{C} \cdot \mathbb{F}^d_p\), the \(\mathbb{C}\)-vector space with basis \(v_\alpha, \alpha \in \mathbb{F}^d_p\), \(g = \text{End}(V)\), the Lie algebra of \(GL(V)\), and

\[
T(g) = \bigoplus_{k \geq 0} T_k = \bigoplus_{k \geq 0} g^{\otimes k},
\]

the tensor algebra of \(g\). For any representation \(\rho : GL(V) \to GL(\rho(V))\) we have a map

\[
\varphi = \varphi_\rho : T(g) \to \mathcal{U}(g) \to \text{End}(\rho(V))
\]

given by

\[
\varphi(A_1 \otimes \cdots \otimes A_k) = \rho'(A_1) \circ \cdots \circ \rho'(A_k),
\]

where \(\rho'\) is the derivative of \(\rho\).

When \(\rho = \wedge^n\), \(\rho(V)\) is the vector space with basis \(v_S, \#S = n\). Our aim is to represent the operator

\[
v_S \mapsto a(S)v_S
\]
as \(\varphi(B)\) for some \(B \in T(g)\). Then \(F(m, n)\) takes the form

\[
F(m, n) = h(B^{\otimes m}), \quad h = \text{Tr} \circ \varphi_{\wedge^n}.
\]

It turns out that one can choose \(B\) so that \(\deg B \leq p\), and so that \(B\) is generated by simultaneously commuting elements of \(g\). One can therefore take \(F(m, n) = h(B^m)\), where \(B^m = \Phi(B^{\otimes m})\), and \(\Phi = \Phi_k\) is the symmetrization map

\[
\Phi(A_1 \otimes \cdots \otimes A_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \cdot A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(k)}.
\]

We capitalize on this element using a factorization of the composed map \(f = h \circ \Phi\): \(f\) is a \(GL(V) \times S_k\)-invariant map to \(\mathbb{C}\), so factors through

\[
f : g^{\otimes k} \xrightarrow{g} (g^{\otimes k})_{\text{trivial}} \xrightarrow{h} \mathbb{C},
\]
where the middle term is the trivial subspace with respect to the $GL(V) \times S_k$ action.

Our theorem is a calculation of the matrix elements of $g$, $h$ in a canonical basis of $(g^\otimes k)_{\text{trivial}}$. The use of this decomposition is can be seen from a complexity point of view: the matrix elements of $g$ depend on $\mathbb{F}^d$, and $m$, but do not contain an $n$ term. The complexity of computing $h$ depends on $m$ and $n$, but not on $B$.

### 3 The Tensor Algebra Element

To find $B$, we use a fact used by Bierbrauer and Edel in [1]: if $\alpha \in \mathbb{F}_p^d$, then

$$
\frac{1}{q} \sum_{\beta \in \mathbb{F}_p^d} \zeta^{\alpha \cdot \beta} = \delta_{\alpha,0}, \quad \zeta = e^{2\pi i/p}.
$$

If $S = \{\alpha_1, ..., \alpha_n\}$, then

$$
a(S) = \frac{1}{q} \sum_{\beta} e_r(\zeta^{\alpha_1 \cdot \beta}, ..., \zeta^{\alpha_n \cdot \beta})
$$

where $e_r$ is the $r$th elementary symmetric polynomial. $e_r$ has a useful expression in the power-sum basis

$$
e_r = \sum_\rho \langle e_r, p_\rho \rangle \delta(\rho) p_\rho,
$$

where $\rho \in \Lambda(r)$, the set of partitions of $r$, $p_\rho = \prod_{j=1}^{\ell(\rho)} p_{\rho_j}$, $p_k = \sum_i a_i^k$, and

$$
\langle p_\rho, p_{\rho'} \rangle = \delta_{\rho,\rho'} \delta(\rho), \quad \delta(\rho) = |\text{Aut}(\rho)| \prod_j \rho_j,
$$

the standard inner product on symmetric polynomials.

We can now describe $B$. Let

$$
A_\beta : V \rightarrow V, \quad A_\beta \cdot v_\alpha = \zeta^{\alpha \cdot \beta}v_\alpha,
$$

and

$$
B = \frac{1}{q} \sum_{\beta, \rho} \frac{\langle e_r, p_\rho \rangle}{\delta(\rho)} A_{\rho_1 \beta} \cdots A_{\rho_l \beta}
$$

in $T_{k \leq p}$. The product here is the symmetric product, i.e. the symmetrization of the tensor product. By (1) and (2), $B$ has the desired property $f(B^m) = F(m, n)$. 

3
4 Schur-Weyl Duality and Theorem

The canonical basis of $\left( g^\otimes k \right)_{\text{trivial}}$ is better said under the isomorphism

$$\text{End}(V)^{\otimes k} \cong V^{\otimes k} \otimes V^{\ast \otimes k} \cong \text{End}(V^{\otimes k}).$$

(4)

The decomposition is now Schur’s lemma and Schur-Weyl duality:

$$\left( g^\otimes k \right)_{\text{trivial}} \cong \bigoplus_{|\mu|=k} \mathbb{C} \cdot P_\mu.$$

Here $\mu$ is a partition of $k$, and $P_\mu \in \text{End}(V^{\otimes k})$ is the projection onto $S_\mu(V) \otimes S_\mu$ in the Schur-Weyl decomposition

$$V^{\otimes k} \cong \bigoplus_\mu S_\mu(V) \otimes S_\mu$$

into irreducible representations of $GL(V) \times S_k$. $S_\mu$ is Schur functor, and $S_\mu$ is the irreducible Specht module [6] associated to $\mu$. The fact that the multiplicity of each irreducible is a most one means that $P_\mu$ is a canonical basis. We calculate the matrix elements of $g, h$ in this basis.

**Theorem 1.** The moment function is given by

$$F(m, n) = \sum_{|\mu|=k} \frac{G(m, \mu) \cdot H(\mu, n)}{\dim S_\mu(V) \cdot \dim S_\mu},$$

where

$$G(m, \mu) = \text{Tr}_{V^{\otimes k}} B^m \circ P_\mu, \quad H(\mu, n) = h(P_\mu).$$

Furthermore, combinatorial formulas for $G, H, F$ are given in [7], [14] and [15], and the answer is a polynomial of degree $\leq n - 1$ in $q$.

**Proof.** This just expresses $f(B^m) = h \cdot g(B^m)$ in the basis $P_\mu$. The explicit formulas for $G, H,$ and $F$ are given in sections [5] [6] and [7] The fact that the answer is a polynomial in $q$ of degree $n - 1$ follows from (15), and the fact that $m_\nu(\lambda) = 0$ for $\ell(\lambda) < \ell(\nu)$. The bound for the degree also follows from the simple fact that

$$\lim_{q \to \infty} \frac{F(m, n)}{F(0, n)} = 0,$$

so that $\deg(F(m, n)) < \deg(F(0, n)) = n.$

\qed
5 Decomposition of the Algebra Element

This section calculates $G(m, \mu)$, the coordinate of $g(B_m)$. First, we expand $B^m$:

$$
B^m = \sum_{|m|=m} \prod_{\rho} \left( \frac{(e_{r, p\rho})}{\lambda(\rho)} \right)^{m(\rho)} \cdot \frac{m!}{\prod_{\rho} m(\rho)!} \cdot \Phi(B_m),
$$

Where the sum is over functions $m : \Lambda(r) \rightarrow \mathbb{N}$ with $|m| = \sum_{\rho} m(\rho) = m$, $k = \sum_{\rho} m(\rho)\ell(\rho)$, and

$$
B_m = \frac{1}{q^m} \otimes_{\rho} \left( \sum_{\beta} A_{\rho_1\beta} \otimes \cdots \otimes A_{\rho_\ell\beta} \right)^{\otimes m(\rho)} \in T_k,
$$

with the tensor taken in the lexicographic order on $\Lambda(r)$. Let $G(m, \mu)$ be the coordinate of each component $g(B_m)$,

$$
G(m, \mu) = \text{Tr}_{V^\otimes k} B_m \circ P_\mu.
$$

Given $|\mu| = |\nu| = k$, let $\chi_\mu^\nu$ denote the character of the irreducible representation of $S_k$ corresponding to $\mu$ on $\sigma \in S_k$ of cycle-type $\nu$. By Schur-Weyl duality, this is the same as $(s_\mu, p_\nu)$, where $s_\mu$ is the Schur polynomial. Using [7], theorem 8(ii),

$$
G(m, \mu) = \sum_{\nu} \frac{n_\mu \chi_\mu^\nu}{k!} G_0(m, \nu), \quad G_0(m, \nu) = \sum_{\sigma \sim \nu} \text{Tr} \otimes_{\rho} B_m \circ \sigma, \quad n_\mu = \dim S_\mu.
$$

Now let $\pi$ denote a function

$$
\{1, \ldots, k\} \rightarrow \{1, \ldots, \ell\}, \quad \ell = \ell(\nu),
$$

and $[\pi]$ the corresponding partition of $\{1, \ldots, k\}$ into the level sets of $\pi$. Associated to each $\sigma$ we have a partition $[\pi](\sigma)$ which groups numbers if they are in the same cycle in $\sigma$. Also set $[\pi_m]$ denote the set partition corresponding to $m$:

$$
\{1, \ldots, k\} = \bigsqcup_{\rho} \bigsqcup_{1 \leq a \leq m(\rho)} C_{\rho,a},
$$

with $C_{\rho,a}$ a block of size $\ell(\rho)$, and where the disjoint union is with respect to the lexicographic ordering of $\Lambda(r)$. This associates to each $1 \leq c \leq k$ a 3-tuple

$$
c \leftrightarrow (\rho, a, b),
$$

(6)
being the index within $C_{\rho,a}$.

The summand in $G_0$ only depends on $[\pi](\sigma)$:

$$G_0(m, \nu) = \sum_{[\pi] \sim \nu} \# \{ \sigma : [\pi](\sigma) = [\pi] \}$$

$$\sum_{\alpha_1, \ldots, \alpha_k} \delta(\alpha_c \sim \alpha_{c'} \text{ if } c \sim c' \text{ in } [\pi]) \delta(\sum_{c \in C_{\rho,a}} \rho_b \alpha_c \cong 0, \text{ for each } \rho, a).$$

The sum over $\alpha_j$ is just the dimension of the kernel over $\mathbb{F}_p$ of the $m$ by $\ell$ matrix $X = X(m, \pi)$, where

$$X_{i,j} = \sum_{c \in \pi^{-1}(i) \cap \pi^{-1}(j)} \rho_b,$$

with $\rho, a, b$ associated to $c$ as in (6).

The summand depends only on the equivalence class $[X]_{m,\ell}$, where

$$[X]_m = \prod_{\rho} S_{m(\rho)} \cdot X, \quad [X]_\ell = X \cdot S_\ell, \quad [X]_{m,\ell} = [[X]_m]_\ell.$$

Now $G_0$ becomes

$$G_0(m, \nu) = \sum_{[X]_{m,\ell}} G_1(m, [X]_{m,\ell}, \nu) q^{\dim \ker X \pmod{p}}.$$

$$G_1(m, [X]_{m,\ell}, \nu) = \# \{ \sigma : \sigma \sim \nu, [X]_{m,\ell}(\sigma) = [X]_{m,\ell} \}.$$

It is helpful to think of $[X]_{m,\ell}$ as an equivalence class of the bipartite graph with adjacency matrix $X$.

Since every map is onto with fibers of the same size in

$$\begin{array}{ccc}
\pi & \xrightarrow{g_2} & [X]_\ell \\
\downarrow g_3 & & \downarrow g_4 \\
\sigma \sim \nu & \xrightarrow{g_1} & [\pi] \xrightarrow{g_5} [X]_{m,\ell}
\end{array}$$

we get

$$G_1(m, [X], \nu) = \frac{|g_1^{-1}([\pi])| \cdot |g_2^{-1}([X]_\ell)| \cdot |g_3^{-1}([X]_{m,\ell})|}{|g_3^{-1}([\pi])|}.$$
\[
\frac{\prod_j (\nu_j - 1)! \#[X]_{m,\ell}}{\ell!}[g_2^{-1}([X]_{\ell})],
\]
\[
|g_2^{-1}([X]_{\ell})| = \sum_{x \in [X]_{\ell}} \#\{\pi \sim \nu | \pi \mapsto X\} = \frac{\ell!}{|\text{Aut}(\nu)|} [m_{\nu}] \Phi_{\ell}(J(X)),
\]
where \([\mu_{\nu}]\) is the coefficient of \(m_{\nu},\)

\[
J(X)(x_1, x_2, ...) = \prod_i \sum_{\pi_0} \delta \left( X_{ij} = \sum_{b \in \pi_0^{-1}(j)} \rho_b, \text{ each } j \right) \prod_b x_{\pi_0(b)};
\]
\(\rho\) is associated to \(1 \leq i \leq m\) under \(\pi_m\), and the sum is over \(\pi_0 : \{1, ..., \ell(\rho)\} \to \{1, ..., \ell\}\). The final formula is

\[
G(m, \mu) = \sum_{[X]_{m,\ell}} \#[X]_{m,\ell} \cdot q^{\dim \ker X \mod p} \cdot \sum_{\nu} \left( \frac{n_{\mu} \chi_{\nu}^\mu}{k! \hat{\delta}(\nu)} \prod_j \nu_j! \right) \cdot [m_{\nu}] \Phi_{\ell}(J(X)).
\]

### 6 Trace of the Schur Projections

Finally, we calculate the matrix element

\[
H(\mu, n) = \text{Tr} \varphi_{\wedge^n}(P_\mu) = \sum_{\nu} \frac{n_{\mu} \chi_{\nu}^\mu}{k!} H_0(\nu, n), \quad H_0(\nu, n) = \sum_{\sigma \sim \nu} \text{Tr} \varphi_{\wedge^n}(\sigma),
\]
applying \(\varphi\) using the identification \((4)\).

Proceeding directly leads to difficult combinatorics, but the computation can be much simplified using the identification of symmetric polynomials with (virtual) representations of \(GL(V)\). Specifically, one identifies

\[
\wedge^n(V) \leftrightarrow e_n, \quad e_n(x_1, ..., x_q) \text{ being the character of } \wedge^n(\text{diag}(x_1, ..., x_q)).
\]

By \((2)\),

\[
H_0(\nu, n) = \sum_{\lambda} \langle e_n, p_{\lambda} \rangle \delta(\lambda) H_1(\nu, \lambda), \quad H_1(\nu, \lambda) = \sum_{\sigma \sim \nu} \text{Tr} \varphi_{\rho_{\lambda}}(\sigma),
\]

### 7
where $\rho_\lambda(V)$ is the vector space with basis

$$v_I = v_{i_1} \otimes \cdots \otimes v_{i_\ell}, \quad \ell = \ell(\lambda),$$

$v_i$ are basis vectors of $V$, and

$$\rho_\lambda(x) v_I = \left( x^{\lambda_1} v_{i_1} \right) \otimes \cdots \otimes \left( x^{\lambda_\ell} v_{i_\ell} \right).$$

$\rho_\lambda$ is not representation, but

$$\rho_\lambda' (\xi) v_I = \sum_k v_{i_1} \otimes \cdots \otimes (\lambda_k \xi \cdot v_{i_k}) \otimes \cdots \otimes v_{i_\ell}$$

is a linear map $\text{End}(V) \to \text{End}(\rho_\lambda(V))$ so $\varphi_{\rho_\lambda}$ is well-defined. $H_1$ turns out to be the better-behaved expression.

Continuing,

$$H_1(\nu, \lambda) = \sum_{\sigma \sim \nu} \text{Tr} \sum_J \varphi_{\rho_\lambda}(E_{\sigma(j_1), j_1} \otimes \cdots \otimes E_{\sigma(j_k), j_k}) =$$

$$\sum_{\sigma \sim \nu} \sum_{J, K} \langle \rho_\lambda' (E_{\sigma(j_1), j_1}) \circ \cdots \circ \rho_\lambda' (E_{\sigma(j_k), j_k}) v_K, v_K \rangle, \quad \langle v_I, v_J \rangle = \delta_{I, J}.$$  

We calculate each summand,

$$\langle \rho_\lambda' (E_{i_1, j_1}) \circ \cdots \circ \rho_\lambda' (E_{i_k, j_k}) v_K, v_K \rangle =$$

$$\sum_{\pi} \prod_{1 \leq b \leq \ell} \lambda_b^\xi (E_{i_{a_1}, j_{a_1}} \circ \cdots \circ E_{i_{a_c}, j_{a_c}} v_{k_b}, v_{k_b}) =$$

$$\sum_{\pi} \prod_{1 \leq b \leq \ell} \lambda_b^\xi \delta(k_b - i_{a_1}) \delta(j_{a_1} - i_{a_2}) \cdots \delta(j_{a_c} - k_b), \quad \delta(k) = \delta_{k, 0}. \quad (11)$$

The sum is over $\pi : \{1, \ldots, k\} \to \{1, \ldots, \ell\}$ which keeps track of where each $E_{i_{a_c}, j_{a_c}}$ is inserted as in (10). Also, we have set $\{a_1, \ldots, a_c\} = \pi^{-1}(b)$, with the ordering $a_1 < \cdots < a_c$.

Associated to each $\pi$ there is a unique permutation $\sigma'(\pi) \in S_k$ which is “sorted,” and such that $[\pi](\sigma') = [\pi]$. Sorted here means that each cycle of
\(\sigma'\) can be written as \((a_1, \ldots, a_c)\) with \(a_1 < \cdots < a_c\). The summand depends only on \(\sigma'(\pi)\). Summing over \(J, K\) and setting \(\sigma' \sim \nu', \sigma^{-1} \cdot \sigma' \sim \nu''\) gives

\[
H_1(\nu, \lambda) = \sum_{\sigma \sim \nu, \sigma'-\text{sorted}} q^{\ell(\lambda) - \ell(\nu')} \left( \sum_{\pi, [\pi ] = [\pi ](\sigma'')} \prod_{b} \lambda^c_b \right) \sum_{j_1, \ldots, j_k} \prod_{k} \delta(j \sigma'(k) - j \sigma(k)).
\]

Since the summand is invariant under simultaneously conjugating \((\sigma, \sigma') \mapsto (\tau \sigma \tau^{-1}, \tau \sigma' \tau^{-1})\), we may exchange the “sorted” condition for a factor of \(\prod_{j} \nu_j'!\):

\[
\sum_{\sigma \sim \nu, \sigma'} \frac{|\text{Aut}(\nu')|}{\prod_{j} (\nu_j' - 1)!} q^{\ell(\lambda) - \ell(\nu') + \ell(\nu'')} m_{\nu'}(\lambda) = \sum_{\nu', \nu''} \frac{3(\nu')}{\prod_{j} \nu_j''!} q^{\ell(\lambda) - \ell(\nu') + \ell(\nu'')} H_2(\nu, \nu', \nu'')(\lambda),
\]

\[
H_2(\nu, \nu', \nu'') = \# \{ \sigma \sim \nu, \sigma' \sim \nu'| \sigma \cdot \sigma' \sim \nu'' \},
\]

and \(m_{\nu'}\) is the monomial symmetric function.

This expression can be simplified: first, \(H_2\) is the same as the coefficient of \(\sum_{\sigma'' \sim \nu''} \sigma''\) in

\[
\left( \sum_{\sigma \sim \nu} \sigma \right) \cdot \left( \sum_{\sigma' \sim \nu'} \sigma' \right)
\]

in the center of the group-ring \(\mathbb{C}[S_k]\). Passing to the idempotent basis, one easily deduces that

\[
H_2(\nu, \nu', \nu'') = \frac{(k!)^2}{3(\nu)3(\nu')3(\nu'')} \sum_{\rho} \frac{\chi^\rho_{\nu} \chi^\rho_{\nu'} \chi^\rho_{\nu''}}{n_{\rho}}, \quad \text{(12)}
\]

Summing over \(\nu''\) gives

\[
H_1(\nu, \lambda) = \frac{(k!)^2}{\prod_{j} \nu_j''!} \sum_{\nu'} \frac{1}{\prod_{j} \nu_j''!} \sum_{\rho} \dim S_{\rho} \cdot \frac{\chi^\rho_{\nu} \chi^\rho_{\nu'}}{n_{\rho}} m_{\nu'}(\lambda), \quad \text{(13)}
\]

where we have used (12), and

\[
\sum_{\nu'} \frac{q^{\ell(\nu')}}{\prod_{j} \nu_j''!} \chi^\rho_{\nu'} = \text{eval}(s_{\rho}, p_k \mapsto q) = \dim S_{\rho}(V).
\]
Finally, summing over $\nu$ gives

$$H(\mu, n) = k! \dim \mathbb{S}_\mu(V) \sum_{\nu', \lambda} q^{\ell(\lambda) - \ell(\nu')} \frac{\chi^{(1^n)}_\lambda \chi^\mu_{\nu'}}{\delta(\lambda) \prod_j \nu'_j!} m_{\nu'}(\lambda),$$

(14)

using (13), (8), (9), and orthogonality of characters.

7 An Example

Combining equations (5), (7), (14), and the formula

$$\sum_{\rho} \chi^\mu_{\nu'} \chi^\mu_{\nu} = \delta_{\nu, \nu'} \delta(\nu),$$

we arrive at a formula for $F$:

$$F(m, n) = \sum_{\lambda} \left( \frac{q^{\ell(\lambda)} \chi^{(1^n)}_\lambda}{\delta(\lambda)} \right) F(m)(\lambda),$$

where

$$F(m) = \sum_{[m]=m} \frac{m!}{\prod_{\rho} m(\rho)!} \prod_{\rho} \left( \frac{\chi^{(1^n)}_{\rho}}{\delta(\rho)} \right)^{m(\rho)} F(m),$$

$$F(m)(x_1, x_2, ...) = \sum_{[X]m, \ell} q^{-rk_{p, \ell}} X_{[X]m, \ell} \cdot \Phi_{q}(J(X)).$$

(15)

$F(m)$ is a symmetric polynomial of degree $k$, independent of $n$.

As an example, we present the second moment for $n = 10$ points in $\mathbb{F}_3^d$. As the formula shows, this requires a sum over partitions of size $k \leq 6$, equivalence classes of bipartite graphs with $\leq 6$ edges, and over the character table of $S_{10}$. The answer is

$$\frac{F(2, 10)}{F(0, 10)} = \frac{120(q^2 + 89q - 540)}{(q - 5)(q - 4)(q - 2)}.$$

Notice that $F(2, 10)$ vanishes at $q = 1, 3, 9$, since there are no subsets of size 10 when $d = 0, 1, 2$.  

10
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