A CLOSED FORMULA FOR THE GENERATING FUNCTION OF $p$-BERNOULLI NUMBERS

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ABSTRACT. In this paper, using geometric polynomials, we obtain a generating function of $p$-Bernoulli numbers in terms of harmonic numbers. As consequences of this generating function, we derive closed formulas for the finite summation of Bernoulli and harmonic numbers involving Stirling numbers of the second kind. We also give a relationship between the $p$-Bernoulli numbers and the generalized Bernoulli polynomials.

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1. Introduction. The Bernoulli numbers $B_n$ are defined by

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} \quad (|t| < 2\pi)$$

and the recurrence formula for $B_n$ has the form

$$B_0 = 1 \text{ and } \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \text{ for } n \geq 1.$$ 

In order to compute the $B_n$ faster, Kaneko [11] (also see [1]) established the following recurrence relation

$$\sum_{k=0}^{n+1} \binom{n+1}{k} \tilde{B}_{n+k} = 0,$$

where $\tilde{B}_n = (n+1) B_n$.

More recently, Rahmani [15] introduced the $p$-Bernoulli numbers $B_{n,p}$ by constructing an infinite matrix as follows: the first row of the matrix $B_{0,p} = 1$, the
first column of the matrix $B_{n,0} = B_n$ and each entry $B_{n,p}$ is given recursively by

$$B_{n+1,p} = pB_{n,p} - \frac{(p+1)^2}{p+2} B_{n,p+1}, \text{ for } n, p \geq 0.$$ 

These numbers have an explicit formula involving the Stirling numbers of the second kind

$$(1.1) \quad B_{n,p} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k+p+1}{k}^{-1} k!,$$

and are closely related to Bernoulli numbers by the following formula

$$(1.2) \quad B_{n,p} = \frac{p+1}{p!} \sum_{j=0}^{p} (-1)^j \left[ \begin{array}{c} p \\ j \end{array} \right] B_{n+j},$$

where $\left[ \begin{array}{c} n \\ k \end{array} \right]$ is the Stirling number of the first kind [10]. Moreover, for every integer $p \geq -1$, we have the following generating function

$$(1.3) \quad \sum_{n \geq 0} B_{n,p} \frac{t^n}{n!} = {}_2F_1 \left( 1, 1; p+2; 1-e^t \right),$$

where ${}_2F_1 \left( a, b; c; z \right)$ denotes the Gaussian hypergeometric function defined by

$${}_2F_1 \left( a, b; c; z \right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!},$$

The first few generating functions for $B_{n,p}$ ($p = 1, 2$ and $3$ in (1.3)) are

$$\sum_{n=0}^{\infty} B_{n,1} \frac{t^n}{n!} = \frac{2(t-1)e^t + 1}{(e^t - 1)^2},$$

$$\sum_{n=0}^{\infty} B_{n,2} \frac{t^n}{n!} = \frac{3(2t-3)e^{2t} + 4e^t - 1}{2(e^t - 1)^3},$$

$$\sum_{n=0}^{\infty} B_{n,3} \frac{t^n}{n!} = \frac{2(6t-11)e^{3t} + 18e^{2t} - 9e^t + 2}{3(e^t - 1)^4}.$$

The main purpose of this study is to give a close form of the above results as

$$(1.4) \quad \sum_{n=0}^{\infty} B_{n,p} \frac{t^n}{n!} = \frac{(p+1)(t-H_p)e^{pt}}{(e^t - 1)^{p+1}} + (p+1) \sum_{k=1}^{p} \binom{p}{k} \frac{H_k}{(e^t - 1)^{k+1}},$$

where $H_n$ is the harmonic numbers, defined by [10, p. 258]

$$H_n = \sum_{j=1}^{n} \frac{1}{j}.$$
As a consequence of (1.4), we have closed formulas for the finite summation of Bernoulli and harmonic numbers.

For the proof of (1.4), we use some properties of geometric polynomials. The geometric polynomials \( w_n(x) \) are defined by means of the following generating function [17]

\[
\frac{1}{1 - x (e^t - 1)} = \sum_{n=0}^{\infty} w_n(x) \frac{t^n}{n!},
\]

and have the explicit formula

\[
w_n(x) = \sum_{k=0}^{n} \binom{n}{k} k! x^k,
\]

where \( \binom{n}{k} \) is the Stirling number of the second kind [10, 14]. The Stirling numbers of the second kind are defined by means of the following generating function

\[
\sum_{n=0}^{\infty} \binom{n}{k} \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!}.
\]

Some other properties of geometric polynomials can be found in [3, 4, 5, 6, 8, 12].

2. **A new generating function for \( p \)-Bernoulli numbers.** In this section, the main theorem and its applications are given. Employing the geometric polynomials, we derive the following result concerning the generating function of \( p \)-Bernoulli numbers.

**Theorem 2.1.** For \( p \geq 0 \), the following generating function holds true:

\[
\sum_{n=0}^{\infty} B_{n,p} \frac{t^n}{n!} = (p + 1) \frac{t - H_p}{(e^t - 1)^{p+1}} + (p + 1) \sum_{k=1}^{p} \binom{p}{k} \frac{H_k}{(e^t - 1)^{k+1}}.
\]

For the proof of main theorem, we first need the following proposition.

**Proposition 2.2.** For \( n > p \geq 0 \), we have

\[
\frac{(-1)^p}{(p + 1)!} B_{n,p} = \int_{0}^{x_3} \cdots \int_{0}^{x_2} \int_{0}^{x_1} w_n(x_1) dx_1 dx_2 \cdots dx_{p-1} dx_p.
\]

**Proof.** If we integrate both sides of (1.6) with respect to \( x_1 \) from 0 to \( x_2 \), we have

\[
\int_{0}^{x_2} w_n(x_1) dx_1 = \sum_{k=0}^{n} \binom{n}{k} k! \frac{x_2^{k+1}}{k + 1}.
\]
Integrating both sides of the above equation with respect to $x_2$ from 0 to $x_3$, we obtain
\[
\int_0^{x_3} \int_0^{x_2} w_n(x_1) \, dx_1 \, dx_2 = \sum_{k=0}^{n} \binom{n}{k} k! \frac{x_3^{k+2}}{(k+1)(k+2)}.
\]

Applying the same procedure for $p - 1$ times yields
\[
\int_0^{x_p} \ldots \int_0^{x_2} \int_0^{x_1} w_n(x_1) \, dx_1 \, dx_2 \ldots \, dx_{p-1} = \sum_{k=0}^{n} \binom{n}{k} k! \frac{x_{p+k}}{(k+1) \ldots (k+p)}.
\]

Finally, integrating both sides of the above equation with respect to $x_p$ from $-1$ to 0 and using (1.1) gives the desired equation.

We note that taking $p = 0$ in (2.2) gives [13, Theorem 1.2]. Now, we are ready to give the proof of the main theorem.

**Proof.** (Proof of Theorem 2.1) Multiplying both sides of (2.2) with $\frac{t^n}{n!}$ and summing over $n$ from 0 to infinity, we have
\[
\frac{(-1)^p}{(p+1)!} \sum_{n=0}^{\infty} B_{n,p} \frac{t^n}{n!} = \int_{-1}^{0} \int_0^{x_p} \ldots \int_0^{x_2} \left( \sum_{n=0}^{\infty} \int_0^{x_1} w_n(x_1) \frac{t^n}{n!} \, dx_1 \right) \, dx_2 \ldots \, dx_{p-1} \, dx_p.
\]

If we evaluate the innermost integral, we obtain
\[
\int_0^{x_2} \frac{1}{1 - x_1 (e^t - 1)} \, dx_1 = \frac{-1}{e^t - 1} \ln \left(1 - x_2 (e^t - 1)\right).
\]

We integrate both sides of this equation with respect to $x_2$ from 0 to $x_3$
\[
\frac{-1}{e^t - 1} \int_0^{x_3} \ln \left(1 - x_2 (e^t - 1)\right) \, dx_2
\]
\[
= \frac{1}{(e^t - 1)^2} \left[ (1 - x_3 (e^t - 1)) \ln \left(1 - x_3 (e^t - 1)\right) - (1 - x_3 (e^t - 1)) + 1 \right].
\]
For induction on \( p \), let us assume that the following equation holds

\[
(2.3) \quad \int_0^{x_p} \ldots \int_0^{x_3} \int_0^{x_2} w_n(x_1) \, dx_1 \, dx_2 \ldots dx_{p-1} \\
= \frac{(-1)^p}{(e^t - 1)^p (p-1)!} \left( 1 - x_p (e^t - 1) \right)^{p-1} \ln \left( 1 - x_p (e^t - 1) \right) \\
- \frac{(-1)^p}{(e^t - 1)^p (p-1)!} \left[ H_{p-1} \left( 1 - x_p (e^t - 1) \right)^{p-1} + H_{p-1} \right] \\
+ \sum_{k=1}^{p-2} \frac{(-1)^{p-k} H_{p-1-k} x_k^k}{(e^t - 1)^p (p-1-k)! k!}.
\]

Now, we want to prove that (2.3) implies the case \( p+1 \). Let us integrate both sides of (2.3) with respect to \( x_p \) from 0 to \( y \). Then we have

\[
\int_0^y \int_0^{x_p} \ldots \int_0^{x_3} \int_0^{x_2} w_n(x_1) \, dx_1 \, dx_2 \ldots dx_{p-1} \, dx_p \\
= \frac{(-1)^p}{(e^t - 1)^p (p-1)!} \int_0^y \left( 1 - x_p (e^t - 1) \right)^{p-1} \ln \left( 1 - x_p (e^t - 1) \right) \, dx_p \\
- \frac{(-1)^p H_{p-1}}{(e^t - 1)^p (p-1)!} \int_0^y \left( 1 - x_p (e^t - 1) \right)^{p-1} \, dx_p \\
+ \frac{(-1)^p H_{p-1}}{(e^t - 1)^p (p-1)!} \int_0^y dx_p + \sum_{k=1}^{p-2} \frac{(-1)^{p-k} H_{p-1-k}}{(e^t - 1)^p (p-1-k)! k!} \int_0^y x_k^k \, dx_p.
\]

The first integral in the right hand-side equals

\[
(2.4) \quad \frac{(-1)^p}{(e^t - 1)^p (p-1)!} \int_0^y \left( 1 - x_p (e^t - 1) \right)^{p-1} \ln \left( 1 - x_p (e^t - 1) \right) \, dx_p \\
= \frac{(-1)^{p+1}}{(e^t - 1)^{p+1} p!} \left[ (1 - y (e^t - 1))^p \ln \left( 1 - y (e^t - 1) \right) - \frac{(1 - y (e^t - 1))^p}{p} + \frac{1}{p} \right].
\]

For the second integral in the right hand-side, we obtain

\[
(2.5) \quad \frac{(-1)^p H_{p-1}}{(e^t - 1)^p (p-1)!} \int_0^y \left( 1 - x_p (e^t - 1) \right)^{p-1} \, dx_p \\
= \frac{(-1)^{p+1} H_{p-1}}{(e^t - 1)^{p+1} p!} \left( 1 - y (e^t - 1) \right)^p - \frac{(-1)^{p+1} H_{p-1}}{(e^t - 1)^{p+1} p!}.
\]
For the third and fourth integrals, we find
\begin{equation}
(2.6) \quad \frac{(-1)^p H_{p-1}}{(e^t - 1)^p (p - 1)!} \int_0^y \frac{d x_p}{0} = \frac{(-1)^{p+1} H_{p-1}}{(e^t - 1)^p (p - 1)!} y
\end{equation}
and
\begin{equation}
(2.7) \quad \sum_{k=1}^{p-2} \frac{(-1)^{p-k} H_{p-1-k}}{(e^t - 1)^{p-k} (p - 1 - k)! k!} \int_0^y \frac{x_k^k d x_p}{0} = \sum_{k=2}^{p-1} \frac{(-1)^{p+1-k} H_{p-k} y^k}{(e^t - 1)^{p+1-k} (p - k)! k!},
\end{equation}
respectively. Combining (2.4), (2.5), (2.6) and (2.7), we achieve that
\begin{equation}
\int_0^y \int_0^{x_2} \cdots \int_0^{x_2} \int_0 \cdots \int_0 w_n(x_1) d x_1 d x_2 \cdots d x_p d x_p = \frac{(-1)^{p+1}}{(e^t - 1)^{p+1} p!} (1 - y (e^t - 1))^p \ln (1 - y (e^t - 1))
\end{equation}
\begin{equation}
- \frac{(-1)^{p+1}}{(e^t - 1)^{p+1} p!} [H_p (1 - y (e^t - 1))^p + H_p] + \sum_{k=1}^{p-1} \frac{(-1)^{p+1-k} H_{p-k} y^k}{(e^t - 1)^{p+1-k} (p - k)! k!}.
\end{equation}
Finally, setting $y = -1$ in the above equation and using (2.2), we arrive at the desired equation. \hfill \square

As an application of Theorem 2.1, we give the following theorem.

**Theorem 2.3.** For $n > p \geq 0$, we have
\begin{equation}
(2.8) \quad \sum_{k=p+1}^{n} \left( \begin{array}{c} n \\ k \\ p+1 \end{array} \right) B_{n-k,p} = \frac{p^{n-1} (n - p H_p)}{p!} + \sum_{j=1}^{p} \left( \begin{array}{c} n \\ p-j \end{array} \right) H_j.
\end{equation}

**Proof.** Multiplying both sides of (2.1) with $\frac{(e^t - 1)^{p+1} (p+1)!}{(p+1)!}$ and using (1.7), the left hand side of (2.1) becomes
\begin{equation}
(2.9) \quad \frac{(e^t - 1)^{p+1}}{(p+1)!} \sum_{n=0}^{\infty} B_{n,p} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \begin{array}{c} k \\ p+1 \end{array} \right) B_{n,p} \frac{t^{n+k}}{k! n!}
\end{equation}
\begin{equation}
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \\ p+1 \end{array} \right) B_{n-k,p} \right) \frac{t^n}{n!}.
\end{equation}
From the right hand side of (2.1), we obtain
\begin{equation}
(2.10) \quad \frac{t e^t}{p!} - \frac{H_p e^t}{p!} + \sum_{k=1}^{p} \frac{H_k (e^t - 1)^{p-k}}{k! (p-k)!}
\end{equation}
\begin{equation}
= \sum_{n=1}^{\infty} \left( \frac{p^{n-1} (n - p H_p)}{p!} + \sum_{k=1}^{p} \frac{H_k}{k!} \left( \begin{array}{c} n \\ p-k \end{array} \right) \right) \frac{t^n}{n!}.
\end{equation}
Finally, comparing the coefficients of $\frac{t^n}{n!}$ in (2.9) and (2.10) completes the proof. \hfill \Box

Using (1.2) in Theorem 2.3 gives the following corollary.

**Corollary 2.4.** For $n > p \geq 0$,

$$\sum_{k=p+1}^{n} \sum_{j=0}^{p} \binom{n}{k} \binom{k}{p+1} \binom{p}{j} (-1)^j B_{n-j-k}$$

$$= \frac{p^{n-1} (n - pH_p)}{p + 1} + \frac{p!}{p + 1} \sum_{j=1}^{p} \binom{n}{p-j} \frac{H_j}{j!}.$$

As a consequence of Corollary 2.4, the following sums are obtained $(p = 1, 2)$:

$$\sum_{k=2}^{n} \binom{n}{k} \binom{k}{2} B_{n+1-k} = -\frac{1}{2} (n - 1),$$

$$\sum_{k=3}^{n} \binom{n}{k} \binom{k}{3} (B_{n+2-k} - B_{n+1-k}) = \frac{2^{n-1} (n - 3) + 2}{3}.$$

Setting $n = p + 1$ in Theorem 2.3 and using $B_{0,p} = 1$, we arrive at the following corollary.

**Corollary 2.5.** For $p \geq 1$, we obtain a new closed formula for the finite summation of harmonic numbers

$$\sum_{j=1}^{p} \binom{p+1}{p-j} \frac{H_j}{j!} = 1 - \frac{p^p (p(1 - H_p) + 1)}{p!}.$$

Our final proposition will show the relationship between the $p$-Bernoulli numbers and the generalized Bernoulli polynomials. Recall that the generalized Bernoulli polynomials $B^{(\alpha)}_n(x)$ of degree $n$ in $x$ are defined by the exponential generating function

$$\left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B^{(\alpha)}_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi; \ 1^\alpha := 1)$$

for arbitrary parameter $\alpha$. In particular $B^{(1)}_n(0) := B^{(\alpha)}_n$ denotes the generalized Bernoulli numbers of order $\alpha$, and $B^{(1)}_n(x)$ denotes the classical Bernoulli polynomials. For a recent treatment see [2, 7, 9, 14, 16].

**Proposition 2.6.** For $p \geq 0$, we have

$$B_{n,p} = \frac{p+1}{p!} \frac{B^{(p+1)}_{n+p}}{n+p} - \frac{H_p}{p!} \frac{B^{(p+1)}_{n+p+1}}{n+p+1} + \sum_{k=0}^{p} \frac{\binom{p+1}{k+1} H_k}{k+1} \frac{B^{(k+1)}_{n+k+1}}{n+k+1}.$$
Proof. By (2.1) and (2.11) we have
\[
\sum_{n=0}^{\infty} B_{n,p} \frac{t^n}{n!} = (p+1) \left( \sum_{n=0}^{\infty} B_{n}^{(p+1)} (p) \frac{t^{n-p}}{n!} - H_p \sum_{n=0}^{\infty} B_{n}^{(p+1)} (p) \frac{t^{n-p-1}}{n!} \right) + (p+1) \sum_{k=1}^{p} \binom{p}{k} H_k \sum_{n=0}^{\infty} B_{n}^{(k+1)} \frac{t^{n-k-1}}{n!}.
\]
This simplifies to
\[
\sum_{n=0}^{\infty} B_{n,p} \frac{t^n}{n!} = (p+1) \sum_{n=0}^{\infty} \left( \frac{n!B_{n+p}^{(p+1)} (p)}{(n+p)!} - H_p \frac{n!B_{n+p+1}^{(p+1)} (p)}{(n+p+1)!} \right) \frac{t^n}{n!} + (p+1) \sum_{n=0}^{\infty} \sum_{k=1}^{p} \binom{p}{k} H_k \frac{n!B_{n+k+1}^{(k+1)} (p)}{(n+k+1)!} \frac{t^n}{n!}.
\]
Equating the coefficient of \( \frac{t^n}{n!} \), we get the result. □

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