ELEMENTARY APPROACH TO HOMOGENEOUS C*-ALGEBRAS

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ABSTRACT. An elementary proof of Fell’s theorem on models of homogeneous C*-algebras is presented. A spectral theorem and a functional calculus for finite systems of elements which generate homogeneous C*-algebras are proposed.

1. Introduction. In 1961, Fell [9] introduced models for n-homogeneous C*-algebras in terms of certain fibre bundles. It is a natural generalization of the commutative Gelfand-Naimark theorem, which gives models for commutative C*-algebras. However, Fell’s proof involves the machinery of (general) operator fields and, as such, is more advanced than Gelfand’s theory of commutative Banach algebras. Tomiyama and Takesaki [28] gave another proof of Fell’s theorem, which involved techniques of von Neumann algebras. In this paper, we propose a new proof of this theorem (starting from the very beginning), which is elementary and resembles the standard proof of the commutative Gelfand-Naimark theorem. We avoid the abstract language of fibre bundles; instead of them we introduce n-spaces, which are counterparts of locally compact Hausdorff spaces in the commutative case. These are locally compact Hausdorff spaces endowed with a (continuous) free action of the group \( \mathcal{U}_n = \mathcal{U}_n / Z(\mathcal{U}_n) \) where \( \mathcal{U}_n \) is the unitary group of \( n \times n \) matrices and \( Z(\mathcal{U}_n) \) is its center.

Our approach to the subject mentioned above enables us to generalize the spectral theorem (for a normal Hilbert space operator) to the context of finite systems generating homogeneous C*-algebras. It also...
allows building so-called $n$-functional calculus for such systems. These and related topics are discussed in the present paper.

The paper is organized as follows. Section 2 is devoted to an operator-valued version of the Stone-Weierstrass theorem, which plays an important role in our proof of Fell’s theorem on homogeneous $C^*$-algebras (presented is Section 5). In Section 3, we define and establish basic properties of so-called $n$-spaces $(X,.)$ (which, in fact, are the same as Fell’s fibre bundles) and, corresponding to them, $C^*$-algebras $C^*(X,.)$. These investigations are continued in the next part where we define spectral $n$-measures and characterize by means of them all representations of $C^*(X,.)$ for any $n$-space $(X,.)$. In Section 5, we give a new proof of Fell’s characterization of homogeneous $C^*$-algebras. In Section 6, we formulate the spectral theorem for finite systems of elements which generate $n$-homogeneous $C^*$-algebras and build the $n$-functional calculus for them.

**Notation and terminology.** If a $C^*$-algebra $A$ has a unit $e$, the spectrum of $x$ is denoted by $\sigma(x)$, and it is the set of all $\lambda \in \mathbb{C}$ for which $x - \lambda e$ is noninvertible in $A$. For two self-adjoint elements $a$ and $b$ of $A$ we write $a \leq b$ provided $b - a$ is nonnegative. If $a \leq b$ and $b - a$ is invertible in $A$, we shall express this by writing $a < b$ or $b > a$. The $C^*$-algebra of all bounded operators on a (complex) Hilbert space $H$ is denoted by $B(H)$. Representations of unital $C^*$-algebras need not preserve unities and they are understood as $*$-homomorphisms into $B(H)$ for some Hilbert space $H$. A representation of a $C^*$-algebra is $n$-dimensional if it acts on an $n$-dimensional Hilbert space. A *map* is a continuous function.

**2. Operator-valued Stone-Weierstrass theorem.** The classical Stone-Weierstrass theorem finds many applications in functional analysis and approximation theory. It reached many generalizations as well, see e.g., [2, 10, 11, 14, 17, 21, 27] and the references therein (consult also [5, Corollary 11.5.3], [9, Theorem 1.4] and [23, subsection 4.7]). A first significant counterpart of it for general $C^*$-algebras was established by Glimm [11]. Much later, Longo [17] and Popa [21] proved independently a stronger version of Glimm’s result, solving a long-standing problem in theory of $C^*$-algebras. In comparison to the classical Stone-Weierstrass theorem or, for example, to its generaliza-
tion by Timofte [27], Glimm’s and Longo’s-Popa’s theorems are not settled in function spaces. In this section, we propose another version of the theorem under discussion which takes place in spaces of functions taking values in $C^*$-algebras. As such, it may be considered as its very natural generalization. Although the results of Glimm and Longo and Popa are stronger and more general than ours, they involve advanced machinery of $C^*$-algebras and advanced language of this theory, while our approach is very elementary and its proof is similar to Stone’s [24, 25]. To formulate our result, we need to introduce the following notion.

**Definition 2.1.** Let $X$ be a set, $x$ and $y$ distinct points of $X$, and let $\mathcal{A}$ be a unital $C^*$-algebra. A collection $\mathcal{F}$ of functions from $X$ to $\mathcal{A}$ spectrally separates points $x$ and $y$ if there is $f \in \mathcal{F}$ such that $f(x)$ and $f(y)$ are normal elements of $\mathcal{A}$ and their spectra are disjoint. If $\mathcal{F}$ spectrally separates any two distinct points of $X$, we say that $\mathcal{F}$ spectrally separates points of $X$.

The reader should notice that a collection of complex-valued functions spectrally separates two points if and only if it separates them.

Whenever $\mathcal{A}$ is a unital $C^*$-algebra and $a$ is a self-adjoint element of $\mathcal{A}$, let us denote by $M(a)$ the real number $\max \sigma(a)$. Further, if $X$ is a locally compact Hausdorff space and $f : X \to \mathcal{A}$ is a map, we say that $f$ vanishes at infinity if and only if for every $\varepsilon > 0$ there is a compact set $K \subset X$ such that $\|f(x)\| < \varepsilon$ for any $x \in X \setminus K$. The set of all $\mathcal{A}$-valued maps on $X$ vanishing at infinity is denoted by $C_0(X, \mathcal{A})$. Notice that $C_0(X, \mathcal{A})$ is a $C^*$-algebra when it is equipped with pointwise actions and the supremum norm induced by the norm of $\mathcal{A}$. Moreover, $C_0(X, \mathcal{A})$ is unital if and only if $X$ is compact (recall that we assume here that $\mathcal{A}$ is unital).

A full version of our Stone-Weierstrass type theorem has the following form.

**Theorem 2.2.** Let $X$ be a locally compact Hausdorff space, and let $\mathcal{A}$ be a unital $C^*$-algebra. Let $E$ be a $*$-subalgebra of $C_0(X, \mathcal{A})$ such that:

$$(AX0)$$ if $X$ is noncompact, then for each $z \in X$ either $f_0(z)$ is invertible in $\mathcal{A}$ for some $f_0 \in E$ or $f(z) = 0$ for any $f \in E$;
and for any two points \( x \) and \( y \) of \( X \) one of the following two conditions is fulfilled:

1. (AX1) either \( x \) and \( y \) are spectrally separated by \( \mathcal{E} \), or
2. (AX2) \( M(f(x)) = M(f(y)) \) for any self-adjoint \( f \in \mathcal{E} \).

Then the (uniform) closure of \( \mathcal{E} \) in \( C_0(X, \mathcal{A}) \) coincides with the *-algebra \( \Delta_2(\mathcal{E}) \) of all maps \( u \in C_0(X, \mathcal{A}) \) such that for any \( x, y \in X \) and each \( \varepsilon > 0 \) there exists \( v \in \mathcal{E} \) with \( \|v(z) - u(z)\| < \varepsilon \) for \( z \in \{x, y\} \).

As a consequence of the above result we obtain the following result, which is a special case of [5, Corollary 11.5.3].

**Proposition 2.3.** Let \( X \) be a locally compact Hausdorff space, and let \( \mathcal{A} \) be a unital \( C^* \)-algebra. A *-subalgebra \( \mathcal{E} \) of \( C_0(X, \mathcal{A}) \) is dense in \( C_0(X, \mathcal{A}) \) if and only if \( \mathcal{E} \) spectrally separates points of \( X \) and for every \( x \in X \) the set \( \mathcal{E}(x) := \{f(x) : f \in \mathcal{E}\} \) is dense in \( \mathcal{A} \).

It is worth noting that we know no characterization of dense *-subalgebras of \( C_0(X, \mathcal{A}) \) in case \( \mathcal{A} \) does not have a unit.

The proof of Theorem 2.2 is partially based on the original proof of the Stone-Weierstrass theorem given by Stone [24, 25]. However, the key tool in our proof is the so-called Loewner-Heinz inequality (for the discussion on this inequality, see [1, page 150]), first proved by Loewner [18]:

**Theorem 2.4.** Let \( a \) and \( b \) be two self-adjoint nonnegative elements in a \( C^* \)-algebra such that \( a \leq b \). Then, for every \( s \in (0, 1) \), \( a^s \leq b^s \).

The proof of Theorem 2.2 is preceded by several auxiliary results. For simplicity, the unit of \( \mathcal{A} \) will be denoted by 1 and the function from \( X \) to \( \mathcal{A} \) constantly equal to 1 will be denoted by \( 1_X \). We also preserve the notation of Theorem 2.2. Additionally, \( \overline{\mathcal{E}} \) stands for the (uniform) closure of \( \mathcal{E} \) in \( C_0(X, \mathcal{A}) \).

**Lemma 2.5.** Suppose \( X \) is compact. Then \( 1_X \in \overline{\mathcal{E}} \) if and only if for every \( x \in X \) there is \( f \in \mathcal{E} \) such that \( f(x) \) is invertible in \( \mathcal{A} \).
Lemma 2.7. Suppose $X$ is compact and $1_X \in \overline{E}$. Let $x \in X$ and $\delta > 0$ be arbitrary. For any self-adjoint $f \in \Delta_2(\mathcal{E})$, there are self-adjoint $g, h \in \overline{E}$ such that $g(x) = f(x) = h(x)$ and $g - \delta \cdot 1_X \leq f \leq h + \delta \cdot 1_X$.

Proof. It follows from the definition of $\Delta_2(\mathcal{E})$ (and the fact that $*$-homomorphisms between $C^*$-algebras have closed ranges) that for every $y \in X$ there is an $f_y \in \overline{E}$ with $f_y(z) = f(z)$ for $z \in \{x, y\}$. Replacing, if needed, $f_y$ by $(f_y + f_y^*)/2$, we may assume that $f_y$ is self-adjoint. Let $U_y \subset X$ consist of all $z \in X$ such that $\|f_y(z) - f(z)\| < \delta$. Take a finite number of points $x_1, \ldots, x_p$ for which $X = \bigcup_{j=1}^p U_{x_j}$. For simplicity, put $V_j = U_{x_j}$ and $g_j = f_{x_j}$ ($j = 1, \ldots, p$). Observe that $f - \delta \cdot 1_X \leq g_j$ on $V_j$ and $g_j(x) = f(x)$. We define, by induction, functions $h_1, \ldots, h_p \in \overline{E}$: $h_1 = g_1$ and $h_k = (h_{k-1} + g_k + |h_{k-1} - g_k|)/2$ for $k = 2, \ldots, p$ where $|u| = \sqrt{u^*u}$ for each $u \in \overline{E}$. Since $\overline{E}$ is a $C^*$-algebra, we clearly have $h_k \in \overline{E}$. Use induction to show that $h_j(x) = f(x)$ and $g_j \leq h_p$ for $j = 1, \ldots, p$. Then $h = h_p$ is the function we searched for. Indeed, $h(x) = f(x)$, and for any $y \in X$, there is $j \in \{1, \ldots, p\}$ such that $y \in V_j$, which implies that $f(y) - \delta \cdot 1 \leq g_j(y) \leq h(y)$.

Now if we apply the above argument to the function $-f$, we shall obtain a self-adjoint function $h' \in \overline{E}$ such that $-f(x) = h'(x)$ and $-f \leq h' + \delta \cdot 1_X$. Then put $g := -h'$ to complete the proof. \qed

Lemma 2.6. Suppose $X$ is compact and $1_X \in \overline{E}$. Let $x \in X$ and $\delta > 0$ be arbitrary. For any self-adjoint $f \in \Delta_2(\mathcal{E})$, there are self-adjoint $g, h \in \overline{E}$ such that $g(x) = f(x) = h(x)$ and $g - \delta \cdot 1_X \leq f \leq h + \delta \cdot 1_X$. 
self-adjoint elements of $A$ such that $0 \leq a_j \leq b$, $b a_j = a_j b$ ($j = 1, \ldots, k$) and $\|b\| \leq r$, then $a_s \leq (\sum_{j=1}^{k} a_j^n)^{1/n} \leq b + \varepsilon \cdot 1$ for any $s \in \{1, \ldots, k\}$ and $n \geq N$.

**Proof.** Let $N > 2$ be such that $\sqrt[n]{k} \leq 1 + \varepsilon/r$ for each $n \geq N$, and let $a_1, \ldots, a_k, b$ be as in the statement of the lemma. Then since $a_j^n \leq \sum_{j=1}^{k} a_j^n$, Theorem 2.4 yields $a_s \leq (\sum_{j=1}^{k} a_j^n)^{1/n}$. Further, since $b$ commutes with $a_j$, we get $a_j^n \leq b^n$, and consequently, $\sum_{j=1}^{k} a_j^n \leq k b^n$. So, another application of Theorem 2.4 gives us $(\sum_{j=1}^{k} a_j^n)^{1/n} \leq \sqrt[n]{k} b$. So, it suffices to have $\sqrt[n]{k} b \leq b + \varepsilon \cdot 1$ which is fulfilled for $n \geq N$ because $\|(\sqrt[n]{k} - 1)b\| \leq (\sqrt[n]{k} - 1)r \leq \varepsilon$. 

**Lemma 2.8.** Suppose $X$ is compact and $1_X \in \mathcal{E}$. If $f \in \Delta_2(\mathcal{E})$ commutes with every member of $\mathcal{E}$, then $f \in \mathcal{E}$.

**Proof.** Since $\Delta_2(\mathcal{E})$ is a $\ast$-algebra, we may assume that $f$ is self-adjoint. Fix $\delta > 0$. By Lemma 2.6, for every $x \in X$, there is an $f_x \in \mathcal{E}$ with $f_x(x) = f(x)$ and $f_x \leq f + \delta \cdot 1_X$. Let $U_x \subset X$ consist of all $y \in X$ such that $f_x(y) > f(y) - \delta \cdot 1$. We infer from the compactness of $X$ that $X = \bigcup_{j=1}^{k} U_{x_j}$ for some points $x_1, \ldots, x_k \in X$. For simplicity, we put $V_j = U_{x_j}$ and $g_j = f_{x_j}$. We then have

\[(2.1)\quad g_j(x) \geq f(x) - \delta \cdot 1 \quad \text{for any } x \in V_j\]

and

\[(2.2)\quad g_j(x) \leq f(x) + \delta \cdot 1 \quad \text{for any } x \in X.\]

It follows from the compactness of $X$ that there is a constant $c > 0$ such that $g_j + c \cdot 1_X \geq 0$ ($j = 1, \ldots, k$) and $f + (c - \delta) \cdot 1_X \geq 0$. Further, there is an $r > 0$ such that $f + (c + \delta) \cdot 1_X \leq r \cdot 1_X$. Now let $N = N(\delta, r, k)$ be as in Lemma 2.7. Since $f$ commutes with each member of $\mathcal{E}$, we conclude from that lemma and from (2.2) that $g_j(x) + c \cdot 1 \leq [\sum_{j=1}^{k} (g_j(x) + c \cdot 1)^n]^{1/n} \leq f(x) + (c + 2\delta) \cdot 1$ for any $x \in X$. Finally, since $1_X \in \mathcal{E}$, the function

$$ g := \left[ \sum_{j=1}^{k} (g_j + c \cdot 1_X)^n \right]^{1/n} - c \cdot 1_X $$
belongs to $\overline{E}$. What is more, $g < f + 2\delta \cdot 1_X$ and $g(x) \geq g_j(x) \geq f(x) - \delta \cdot 1$ for $x \in V_j$ (cf., (2.1)). This gives $f - \delta \cdot 1_X \leq g$ on the whole space $X$, and therefore $-\delta \cdot 1_X \leq g - f \leq 2\delta \cdot 1_X$, which is equivalent to $\|g - f\| \leq 2\delta$ and finishes the proof. \hfill $\square$

**Lemma 2.9.** Suppose $X$ is compact, $1_X \in \overline{E}$ and there exists an equivalence relation $\mathcal{R}$ on $X$ such that two points $x$ and $y$ are spectrally separated by $\mathcal{E}$ whenever $(x, y) \notin \mathcal{R}$. Then every map $g: X \to C\cdot 1 \subset A$ which is constant on each equivalence class with respect to $\mathcal{R}$ belongs to $\overline{E}$.

**Proof.** By Lemma 2.8, we only need to check that $g \in \Delta_2(\mathcal{E})$. We may assume that $g: X \to \mathbb{R} \cdot 1$. Let $x$ and $y$ be arbitrary. Write $g(x) = \alpha \cdot 1$ and $g(y) = \beta \cdot 1$. If $(x, y) \in \mathcal{R}$, then both $x$ and $y$ belong to the same equivalence class, and hence $\alpha = \beta$. Then $g(z) = (\alpha \cdot 1_X)(z)$ for $z \in \{x, y\}$ (and $\alpha \cdot 1_X \in \overline{E}$). Now assume that $(x, y) \notin \mathcal{R}$. Then, by assumption, there is an $f \in \mathcal{E}$ such that both $f(x)$ and $f(y)$ are normal and $\sigma(f(x)) \cap \sigma(f(y)) = \emptyset$. Let $\varphi: \mathbb{C} \to \mathbb{R}$ be a map such that $\varphi|_{\sigma(f(x))} \equiv \alpha$ and $\varphi|_{\sigma(f(y))} \equiv \beta$. There is a sequence of polynomials $p_1(z, z), p_2(z, z), \ldots$, which converge uniformly to $\varphi$ on $K := \sigma(f(x)) \cup \sigma(f(y))$. Then $p_n(f, f^*) \in \mathcal{E}$ and, for $w \in \{x, y\}$, $[p_n(f, f^*)](w) = p_n(f(w), [f(w)]^*)$. Since $f(w)$ is normal and its spectrum is contained in $K$, we see that

$$\lim_{n \to \infty} [p_n(f, f^*)](w) = \varphi(f(w)).$$

Now notice that $\varphi(f(x)) = \alpha \cdot 1 = g(x)$ and $\varphi(f(y)) = \beta \cdot 1 = g(y)$ finishes the proof. \hfill $\square$

We recall that, if $X$ is a compact Hausdorff space and $\mathcal{R}$ is a closed equivalence relation on $X$, then the quotient topological space $X/\mathcal{R}$ is Hausdorff as well.

**Lemma 2.10.** Suppose $X$ is compact and there is a closed equivalence relation $\mathcal{R}$ on $X$ such that $M(f(x)) = M(f(y))$ for each self-adjoint $f \in \mathcal{E}$ whenever $(x, y) \in \mathcal{R}$. Let $\pi: X \to X/\mathcal{R}$ denote the canonical projection, $f \in \overline{E}$ be self-adjoint, $a$ and $b$ two real numbers, and let $U = \{x \in X: a \cdot 1 < f(x) < b \cdot 1\}$. Then $\pi^{-1}(\pi(U)) = U$ and $\pi(U)$ is open in $X/\mathcal{R}$. 
Put $\alpha$ that the sets $\pi$ follow from the definition of $R$ be a partition of unity such that $\beta$ (which is compact and Hausdorff). Now let $\gamma$ and $\delta$ (2.4)

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there is an $x \in U$ such that $(x, y) \in R$. We then have $\alpha < f(x) < b \cdot 1$, $M(f(x)) = M(f(y))$ and $M(-f(x)) = M(-f(y))$ (the last two relations follow from the fact that $f \in \mathcal{E}$). The first of these relations says that $[-M(-f(x)), M(f(x))] \subset (a, b)$, from which we infer that $[-M(-f(y)), M(f(y))] \subset (a, b)$, and consequently $y \in U$. \hfill \Box

The following is a special case of Theorem 2.2.

**Lemma 2.11.** Suppose $X$ is compact, $1_X \in \mathcal{E}$ and, for any $x, y \in X$, one of conditions (AX1)–(AX2) is fulfilled. Then $\Delta_2(\mathcal{E}) = \overline{\mathcal{E}}$.

**Proof.** We only need to show that $\Delta_2(\mathcal{E})$ is contained in $\overline{\mathcal{E}}$. Let $f \in \Delta_2(\mathcal{E})$ be self-adjoint, and let $\delta > 0$. We shall construct $w \in \overline{\mathcal{E}}$ such that $\|w - f\| \leq 3\delta$. By Lemma 2.6, for each $x \in X$, there are functions $u_x, v_x \in \overline{\mathcal{E}}$ such that $u_x(x) = f(x) = v_x(x)$ and $u_x - \delta \cdot 1_X < f < v_x + \delta \cdot 1_X$. Let $G_x \subset X$ consist of all $y \in X$ such that $v_x(y) - \delta \cdot 1 < f(y) < u_x(y) + \delta \cdot 1$. Since $x \in G_x$ and $X$ is compact, there is a finite system $x_1, \ldots, x_k \in X$ for which $X = \bigcup_{j=1}^k G_{x_j}$. For simplicity, we put $W_j = G_{x_j}$, $p_j = u_{x_j}$ and $q_j = v_{x_j}$. Observe that then

\[(2.3) \quad p_j(x) - \delta \cdot 1 < f(x) < q_j(x) + \delta \cdot 1 \quad \text{for any } x \in X\]

and

\[(2.4) \quad q_j(x) - \delta \cdot 1 < f(x) < p_j(x) + \delta \cdot 1 \quad \text{for any } x \in W_j.\]

Let $D_j$ consist of all $x \in X$ such that $-2\delta \cdot 1 < p_j(x) - q_j(x) < 2\delta \cdot 1$. We infer from (2.3) and (2.4) that $W_j \subset D_j$, and thus $X = \bigcup_{j=1}^k D_j$. Further, let $R$ be an equivalence relation on $X$ given by the rule: $(x, y) \in R \iff M(u(x)) = M(u(y))$ for each self-adjoint $u \in \mathcal{E}$. It follows from the definition of $R$ that $R$ is closed in $X \times X$. Denote by $\pi: X \to X/R$ the canonical projection. We deduce from Lemma 2.10 that the sets $\pi(D_1), \ldots, \pi(D_k)$ form an open cover of the space $X/R$ (which is compact and Hausdorff). Now let $\beta_1, \ldots, \beta_k: X/R \to [0, 1]$ be a partition of unity such that $\beta_j^{-1}((0, 1]) \subset \pi(D_j)$ for $j = 1, \ldots, k$. Put $\alpha_j = (\beta_j \circ \pi) \cdot 1: X \to \mathbb{C} \cdot 1 \subset A$. Lemma 2.9 combined with
conditions (AX1)–(AX2) yields that \( \alpha_1, \ldots, \alpha_k \in \overline{\mathcal{E}} \). Define \( w \in \overline{\mathcal{E}} \) by

\[
w = \sum_{j=1}^{k} \alpha_j p_j.
\]

Since \( \sum_{j=1}^{k} \alpha_j = 1_X \), we conclude from (2.3) that \( w \leq f + \delta \cdot 1_X \). So, to end the proof, it is enough to check that \( f(x) \leq w(x) + 3\delta \cdot 1 \) for each \( x \in X \). This inequality will be satisfied, provided

\[
(2.5) \quad \alpha_j(x)(f(x) - 3\delta \cdot 1) \leq \alpha_j(x)p_j(x)
\]

for any \( j \). We consider two cases. If \( x \in D_j \), then \( p_j(x) > q_j(x) - 2\delta \cdot 1 > f(x) - 3\delta \cdot 1 \) (by (2.3)) and consequently (2.5) holds. Finally, if \( x \notin D_j \), then \( \pi(x) \notin \pi(D_j) \) (see Lemma 2.10) and therefore \( \alpha_j(x) = 0 \), which easily gives (2.5).

\[\square\]

**Proof of Theorem 2.2.** We only need to check that \( \Delta_2(\mathcal{E}) \subset \overline{\mathcal{E}} \). We consider two cases.

First assume \( X \) is compact. Let \( \mathcal{E}' = \mathcal{E} + \mathbb{C} \cdot 1_X \). Observe that \( \mathcal{E}' \) is a *-algebra and, for any two points \( x \) and \( y \), one of the conditions (AX1)–(AX2) is fulfilled with \( \mathcal{E} \) replaced by \( \mathcal{E}' \). Consequently, it follows from Lemma 2.11 that \( \overline{\mathcal{E}} = \Delta_2(\mathcal{E}') \). But \( \overline{\mathcal{E}} = \overline{\mathcal{E}} + \mathbb{C} \cdot 1_X \). So, for any \( g \in \Delta_2(\mathcal{E}) \), we clearly have \( g \in \Delta_2(\mathcal{E}') \), and hence \( g = f + \lambda \cdot 1_X \) for some \( f \in \overline{\mathcal{E}} \) and \( \lambda \in \mathbb{C} \). If \( \lambda = 0 \), then \( g = f \in \overline{\mathcal{E}} \), and we are done. Otherwise, \( 1_X = (g - f)/\lambda \in \Delta_2(\mathcal{E}) \), which implies that the assumptions of Lemma 2.5 are satisfied. We infer from that lemma that \( 1_X \in \overline{\mathcal{E}} \) and, therefore, \( g \in \overline{\mathcal{E}} \) as well.

Now assume that \( X \) is noncompact. Let \( \hat{X} = X \cup \{\infty\} \) be the one-point compactification of \( X \). Every function \( f \in C_0(X, \mathcal{A}) \) admits a unique continuous extension \( \hat{f}: \hat{X} \to \mathcal{A} \), given by \( \hat{f}(\infty) = 0 \). Denote by \( \hat{\mathcal{E}} \) the *-subalgebra of \( C(\hat{X}, \mathcal{A}) \) consisting of all extensions of (all) functions from \( \mathcal{E} \). We claim that, for any \( x, y \in \hat{X} \), one of the conditions (AX1)–(AX2) is fulfilled with \( \mathcal{E} \) replaced by \( \hat{\mathcal{E}} \). Indeed, if both \( x \) and \( y \) differ from \( \infty \), this follows from our assumptions about \( \mathcal{E} \). And if, for example, \( y = \infty \neq x \), condition (AX0) implies that either \( M(\hat{f}(x)) = M(\hat{f}(y)) \) for each \( f \in \mathcal{E} \) or \( \hat{u}(x) \) is invertible in \( \mathcal{A} \) for some \( u \in \mathcal{E} \). But then \( f = u^*u \in \mathcal{E} \) is normal and \( 0 \notin \sigma(\hat{f}(x)) \), while \( \sigma(\hat{f}(y)) = \{0\} \), which shows that \( x \) and \( y \) are spectrally separated by \( \hat{\mathcal{E}} \). So, it follows from the first part of the proof that the closure of \( \hat{\mathcal{E}} \) in \( C(\hat{X}, \mathcal{A}) \) coincides with \( \Delta_2(\hat{\mathcal{E}}) \). But the closure of \( \hat{\mathcal{E}} \) coincides with
\{ \hat{f} \colon f \in \mathcal{E} \} \text{ and } \Delta_2(\mathcal{E}) = \{ \hat{f} \colon f \in \Delta_2(\mathcal{E}) \}. \text{ We infer from these that } \Delta_2(\mathcal{E}) = \mathcal{E}, \text{ and the proof is complete.} \qed

Proof of Proposition 2.3. The necessity of the condition is clear (since, for any two distinct points } x \text{ and } y \text{ in } X \text{ and any elements } a \text{ and } b \text{ of } \mathcal{A}, \text{ there is a function } f \in \mathcal{C}_0(X, \mathcal{A}) \text{ such that } f(x) = a \text{ and } f(y) = b). \text{ To prove the sufficiency, assume } \mathcal{E} \text{ spectrally separates points of } X \text{ and, for each } x \in X, \text{ the set } \mathcal{E}(x) \text{ is dense in } \mathcal{A}. \text{ First notice that then for each } x \in X \text{ there is an } f \in \mathcal{E} \text{ such that } f(x) \text{ is invertible in } \mathcal{A}. \text{ This shows that all assumptions of Theorem 2.2 are satisfied. According to that result, we only need to show that, for any two distinct points } x \text{ and } y \text{ of } X, \text{ the set } L := \{(f(x), f(y)) \colon f \in \mathcal{E}\} \text{ is dense in } \mathcal{A} \times \mathcal{A}. \text{ Since } x \text{ and } y \text{ are spectrally separated by } \mathcal{E}, \text{ the proof of Lemma 2.9 shows that } (1,0), (0,1) \in \overline{\mathcal{E}}. \text{ Further, since both } \mathcal{E}(x) \text{ and } \mathcal{E}(y) \text{ are dense in } \mathcal{A}, \text{ we conclude that } \{ f(x) \colon f \in \mathcal{E} \} = \{ f(y) \colon f \in \mathcal{E} \} = \mathcal{A} \text{ and, therefore, for arbitrary two elements } a \text{ and } b \text{ of } \mathcal{A}, \text{ there are } u, v \in \mathcal{E} \text{ for which } u(x) = a \text{ and } v(y) = b. \text{ Then } (a,b) = (u(x), u(y)) \cdot (1,0) + (v(x), v(y)) \cdot (0,1) \in \overline{\mathcal{E}} \text{ (we use here the coordinatewise multiplication), and we are done.} \qed

3. Topological \textit{n}-spaces. In Fell’s characterization of homogeneous \textit{C*}-algebras \cite{9} (consult also \cite[Theorem IV.1.7.23]{3} and \cite{28}) special fibre bundles appear. To make our lecture as simple and elementary as possible, we avoid this language and, instead of using fibre bundles, we shall introduce so-called \textit{n}-spaces (see Definition 3.1 below). To this end, let } M_n \text{ be the } \textit{C*}-algebra of all complex } n \times n \text{-matrices. Let } \mathcal{U}_n \text{ be the unitary group of } M_n \text{ and } I \text{ its neutral element. Let } \mathbb{T} = \{ z \in \mathbb{C} \colon |z| = 1 \}. \text{ Let } \mathcal{U}_n \text{ denote the compact topological group } \mathcal{U}_n/(\mathbb{T} \cdot I), \text{ and let } \pi_n \colon \mathcal{U}_n \to \mathcal{U}_n \text{ be the canonical homomorphism. Members of } \mathcal{U}_n \text{ will be denoted by } u \text{ and } v. \text{ The (probabilistic) Haar measure on } \mathcal{U}_n \text{ will be denoted by } du. \text{ For any } A \in M_n \text{ and } u \in \mathcal{U}_n, \text{ let } u. A \text{ denote the matrix } U A U^{-1} \text{ where } U \in \mathcal{U}_n \text{ is such that } \pi_n(U) = u. \text{ It is easily seen that the function }

\mathcal{U}_n \times M_n \ni (u, A) \longmapsto u \cdot A \in M_n

\text{is a well-defined continuous action of } \mathcal{U}_n \text{ on } M_n \text{ (which means that } j.A = A \text{ where } j \text{ is the identity of } \mathcal{U}, \text{ and } u.(vA) = (uv).A \text{ for any } u, v \in \mathcal{U}_n \text{ and } A \in M_n). \text{ More generally, for any } \textit{C*}-algebra } \mathcal{A}, \text{ let } M_n(\mathcal{A}) \text{ be the algebra of all } n \times n \text{-matrices with entries in } \mathcal{A}. (M_n(\mathcal{A})
may naturally be identified with $A \otimes M_n$.) For any matrix $A \in M_n(A)$ and each $u \in \mathcal{U}_n$, $u.A$ is defined as $UAU^{-1}$ where $U \in \mathcal{U}_n$ is such that $\pi_n(U) = u$, and $UAU^{-1}$ is computed in a standard manner.

**Definition 3.1.** A pair $(X,\cdot)$ is said to be an $n$-space if $X$ is a locally compact Hausdorff space and $\mathcal{U}_n \times X \ni (u, x) \mapsto u.x \in X$ is a continuous free action of $\mathcal{U}_n$ on $X$. Recall that the action is free if and only if the equality $u.x = x$ (for some $x \in X$) implies that $u$ is the identity of $\mathcal{U}$.

Let $(X,\cdot)$ be an $n$-space. Let $C^*(X,\cdot)$ be the $*$-algebra of all maps $f \in C_0(X,M_n)$ such that $f(u.x) = u.f(x)$ for any $u \in \mathcal{U}_n$ and $x \in X$. $C^*(X,\cdot)$ is a $C^*$-subalgebra of $C_0(X,M_n)$.

By a morphism between two $n$-spaces $(X,\cdot)$ and $(Y,\ast)$, we mean any proper map $\psi : X \to Y$ such that $\psi(u.x) = u \ast \psi(x)$ for any $u \in \mathcal{U}_n$ and $x \in X$. (A map is proper if the inverse images of compact sets under this map are compact.) A morphism which is a homeomorphism is said to be an isomorphism. Two $n$-spaces are isomorphic if there exists an isomorphism between them.

The reader should notice that the (natural) action of $\mathcal{U}_n$ on $M_n$ is not free. However, one may check that the set $\mathcal{M}_n$ of all irreducible matrices $A \in M_n$ (that is, $A \in \mathcal{M}_n$ if and only if every matrix $X \in M_n$ which commutes with both $A$ and $A^\ast$ is of the form $\lambda I$ where $\lambda \in \mathbb{C}$) is open in $M_n$ (and, thus, $\mathcal{M}_n$ is locally compact) and the action $\mathcal{U}_n \times \mathcal{M}_n \ni (u, A) \mapsto u.A \in \mathcal{M}_n$ is free, which means that $(\mathcal{M}_n,\cdot)$ is an $n$-space.

In this section, we establish basic properties of $C^*$-algebras of the form $C^*(X,\cdot)$ where $(X,\cdot)$ is an $n$-space. To this end, recall that, whenever $(\Omega, \mathcal{M}, \mu)$ is a finite measure space and $f : \Omega \ni \omega \mapsto (f_1(\omega), \ldots, f_k(\omega)) \in \mathbb{C}^k$ is an $\mathcal{M}$-measurable (which means that $f^{-1}(U) \in \mathcal{M}$ for every open set $U \subset \mathbb{C}^k$) bounded function, then $\int_{\Omega} f(\omega) \, d\mu(\omega)$ is (well) defined as

$$\left( \int_{\Omega} f_1(\omega) \, d\mu(\omega), \ldots, \int_{\Omega} f_k(\omega) \, d\mu(\omega) \right).$$

If $\| \cdot \|$ is any norm on $\mathbb{C}^k$, then

$$\left\| \int_{\Omega} f(\omega) \, d\mu(\omega) \right\| \leq \int_{\Omega} \| f(\omega) \| \, d\mu(\omega).$$
In particular, the above rules apply to matrix-valued measurable functions.

From now on, \( n \geq 1 \) and an \( n \)-space \((X,.)\) are fixed. A set \( A \subset X \) is said to be invariant provided \( u.a \in A \) for any \( u \in \mathcal{U}_n \) and \( a \in A \). Observe that, if \( A \) is closed or open and \( A \) is invariant, then \( A \) is locally compact and consequently \((A,.)\) is an \( n \)-space (when the action of \( \mathcal{U}_n \) is restricted to \( A \)). We begin with:

**Lemma 3.2.** For each \( f \in C_0(X,M_n) \), let \( f^U : X \to M_n \) be given by:

\[
f^U(x) = \int_{\mathcal{U}_n} u^{-1}.f(u.x) \, du \quad (x \in X).
\]

(a) For any \( f \in C_0(X,M_n) \), \( f^U \in C^*(X,.) \).
(b) If \( f \in C_0(X,M_n) \) and \( x \in X \) are such that \( f(u.x) = u.f(x) \) for any \( u \in \mathcal{U}_n \), then \( f^U(x) = f(x) \).
(c) Let \( A \subset X \) be a closed invariant nonempty set. Every map \( g \in C^*(A,.) \) extends to a map \( \tilde{g} \in C^*(X,.) \) such that \( \sup_{a \in A} \|g(a)\| = \sup_{x \in X} \|\tilde{g}(x)\| \).
(d) For any \( x \in X \) and \( A \in M_n \), there is an \( f \in C^*(X,.) \) with \( f(x) = A \).
(e) Let \( x \) and \( y \) be two points of \( X \) such that there is no \( u \in \mathcal{U}_n \) for which \( u.x = y \). Then, for any \( A, B \in M_n \), there is an \( f \in C^*(X,.) \) such that \( f(x) = A \) and \( f(y) = B \).
(f) \( C^*(X,.) \) has a unit if and only if \( X \) is compact.

**Proof.** It is clear that \( f^U \) is continuous for every \( f \in C_0(X,M_n) \). Further, if \( K \subset X \) is a compact set such that \( \|f(x)\| \leq \varepsilon \) for each \( x \in X \setminus K \), then \( \|f^U(z)\| \leq \varepsilon \) for any \( z \in X \setminus \mathcal{U}_n.K \) where \( \mathcal{U}_n.K = \{u.x : u \in \mathcal{U}_n, \ x \in K\} \). The note that \( \mathcal{U}_n.K \) is compact leads to the conclusion that \( f^U \in C_0(X,M_n) \). Finally, for any \( v \in \mathcal{U}_n \), any representative \( V \in \mathcal{U}_n \) of \( v \) and each \( x \in X \), we have:

\[
f^U(v.x) = \int_{\mathcal{U}_n} u^{-1}.f(uv.x) \, du = \int_{\mathcal{U}_n} (uv^{-1})^{-1}.f(u.x) \, du
\]

\[
= \int_{\mathcal{U}_n} v.[u^{-1}.f(u.x)] \, du = \int_{\mathcal{U}_n} V[u^{-1}.f(u.x)]V^{-1} \, du
\]

\[
= V \cdot \left( \int_{\mathcal{U}_n} u^{-1}.f(u.x) \, du \right) \cdot V^{-1} = v.f^U(x),
\]
which proves (a). Point (b) is a simple consequence of the definition of $f^U$. Further, if $g$ is as in (c), it follows from Tietze’s type theorem that there is a $G \in C_0(X, M_n)$ which extends $g$ and satisfies $\sup_{a \in A} \|g(a)\| = \sup_{x \in X} \|G(x)\|$ (if $X$ is noncompact, consider the one-point compactification $\hat{X} = X \cup \{\infty\}$ of $X$, and note that then the set $\hat{A} = A \cup \{\infty\}$ is closed in $\hat{X}$ and $g$ extends continuously to $\hat{A}$). Then $\bar{g} = G^U$ is a member of $C^*(X,.)$ (by (a)) which we searched for (see (b)).

We turn to (d) and (e). Let $K = \mathfrak{U}_n, \{x\}$ and $f_0: K \to M_n$ be given by $f_0(u.x) = u.A$ ($u \in \mathfrak{U}_n$). Since the action of $\mathfrak{U}_n$ on $X$ is free, $f_0$ is a well-defined map. Since $K$ is compact, (c) yields the existence of $f \in C^*(X,.)$ which extends $f_0$. To prove (e), we argue similarly: put $L = \mathfrak{U}_n, \{x,y\}$, and let $g_0: L \to M_n$ be given by $g_0(u.x) = u.A$ and $g_0(u.y) = u.B$ ($u \in \mathfrak{U}_n$). We infer from the assumption of (e) that $g_0$ is a well defined map. Consequently, since $L$ is compact, there exists, by (c), a map $g \in C^*(X,.)$ which extends $g_0$. This finishes the proof of (e), while point (f) immediately follows from (d).

\[ \text{Proposition 3.3.} \]

(a) For every closed two-sided ideal $I$ in $C^*(X,.)$, there exists a (unique) closed invariant set $A \subset X$ such that $I$ coincides with the ideal $I_A$ of all functions $f \in C^*(X,.)$ which vanish on $A$. Moreover, $C^*(X,.)/I$ is “naturally” isomorphic to $C^*(A,.)$.

(b) Let $k \leq n$, and let $\pi: C^*(X,.) \to M_k$ be a nonzero representation. Then $k = n$, and there is a unique point $x \in X$ such that $\pi(f) = f(x)$ for $f \in C^*(X,.)$.

(c) Let $(Y,*)$ be an $n$-space. For every $*$-homomorphism $\Phi: (X,.) \to (Y,*)$, there is a unique pair $(U, \varphi)$ where $U$ is an open invariant subset of $Y$, $\varphi: (U,*) \to (X,.)$ is a morphism of $n$-spaces and

\[
[\Phi(f)](y) = \begin{cases} f(\varphi(y)) & \text{if } y \in U \\ 0 & \text{if } y \notin U. \end{cases}
\]

In particular, $C^*(X,.)$ and $C^*(Y,*)$ are isomorphic if and only if so are $(X,.)$ and $(Y,*)$.

\text{Proof.} The uniqueness of the set $A$ in (a) follows from point (e) of
Lemma 3.2. To show its existence, let $A$ consist of all $x \in X$ such that $f(x) = 0$ for any $f \in \mathcal{I}$. It is clear that $A$ is closed and invariant and that $\mathcal{I} \subset \mathcal{I}_A$. To prove the converse inclusion we shall involve Theorem 2.2 for $\mathcal{E} = \mathcal{I}$. First of all, it follows from Lemma 3.2 (d) that, for each $x \in X$, the set $\mathcal{I}(x) := \{f(x) : f \in \mathcal{I}\}$ is a two-sided ideal in $M_n$. Since $\{0\}$ is the only proper ideal of $M_n$, we conclude that $\mathcal{I}(x) = \{0\}$ for $x \in A$ and $\mathcal{I}(x) = M_n$ for $x \in X \setminus A$. This shows that condition (AX0) of Theorem 2.2 is satisfied. Further, if $x$ and $y$ are arbitrary points of $X$, then either:

- $x, y \in A$; in that case, (AX2) is fulfilled; or
- $x \in A$ and $y \notin A$ (or conversely); in that case, there is $f \in \mathcal{I}$ such that $f(y) = 1$, and $f(x) = 0$ (since $x \in A$)—this implies that $x$ and $y$ are spectrally separated by $\mathcal{I}$; or
- $x, y \notin A$ and $y = u.x$ for some $u \in \mathcal{U}_n$; in that case, (AX2) is fulfilled since, for any self-adjoint $f \in \mathcal{I}$, $f(y) = u.f(x)$ and consequently $\sigma(f(x)) = \sigma(f(y))$; or
- $x, y \notin A$ and $y \notin \mathcal{U}_n \{x\}$; in that case, there are $f_1 \in \mathcal{I}$ and $f_2 \in C^*(X, .)$ such that $f_1(x) = 1 = f_2(x)$ and $f_2(y) = 0$ (cf., Lemma 3.2 (e)), then $f = f_1f_2 \in \mathcal{I}$ is such that $f(x) = 1$ and $f(y) = 0$, and hence $x$ and $y$ are spectrally separated by $\mathcal{I}$.

Now, according to Theorem 2.2, it suffices to check that $\mathcal{I}_A \subset \Delta_2(\mathcal{I})$ (since $\mathcal{I}$ is closed). To this end, we fix $f \in \mathcal{I}_A$ and two arbitrary points $x$ and $y$ of $X$. We consider similar cases as above:

1°) If $x, y \in A$, we have nothing to do because then $f(x) = f(y) = 0$.
2°) If $x \in A$ and $y \notin A$ (or conversely), then there is $g \in \mathcal{I}$ such that $g(y) = f(y)$. But also $g(x) = 0 = f(x)$, and we are done.
3°) If $x, y \notin A$ and $y = u.x$ for some $u \in \mathcal{U}_n$, then there is a $g \in \mathcal{I}$ with $g(x) = f(x)$. Then also $g(y) = g(u.x) = u.g(x) = u.f(x) = f(y)$, and we are done.
4°) Finally, if $x, y \notin A$ and $y \notin \mathcal{U}_n \{x\}$, there are functions $g_1, g_2 \in \mathcal{I}$ and $h_1, h_2 \in C^*(X, .)$ such that $g_1(x) = f(x)$, $g_2(y) = f(y)$, $h_1(x) = 1 = h_2(y)$ and $h_1(y) = 0 = h_2(x)$. Then $g = g_1h_1 + g_2h_2 \in \mathcal{I}$ satisfies $g(z) = f(z)$ for $z \in \{x, y\}$.

The arguments (1°)–(4°) show that $f \in \Delta_2(\mathcal{I})$, and thus $\mathcal{I} = \mathcal{I}_A$. It follows from Lemma 3.2 (c) that the $*$-homomorphism $C^*(X, .) \ni f \mapsto f|_A \in C^*(A, .)$ is surjective. What is more, its kernel coincides with $\mathcal{I}_A = \mathcal{I}$ and therefore $C^*(X, .)/\mathcal{I}$ and $C^*(A, .)$ are isomorphic.
We now turn to (b). We infer from (a) that there is a closed invariant set $A \subset X$ such that $\ker(\pi) = J_A$. Since $\pi$ is nonzero, $A$ is nonempty. Further, $k^2 \geq \dim \pi(C^*(X, .)) = \dim(C^*(X, .)/\ker(\pi)) = \dim C^*(A, .) \geq n^2$ (by Lemma 3.2 (d) and by (a)), and thus $k = n$, $\dim C^*(A, .) = n^2$ and $\pi$ is surjective. Fix $a \in A$, and observe that $A = \mathfrak{U}_n\{a\}$ because otherwise $\dim C^*(A, .) > n^2$ (thanks to Lemma 3.2 (e)). Now define $\Phi: M_n \to M_n$ by the rule $\Phi(X) = f(a)$ where $\pi(f) = X$. It may easily be checked (using the fact that $\ker(\pi) = J_{\mathfrak{U}_n\{a\}}$) that $\Phi$ is a well defined one-to-one $\ast$-homomorphism of $M_n$. We conclude that there is a $u \in \mathfrak{U}_n$ for which $\Phi(X) = u.X$ (in the algebra of matrices this is quite an elementary fact; however, this follows also from [23, Corollary 2.9.32]). Put $x = u^{-1}a$ and note that then $f(a) = \Phi(\pi(f)) = u.\pi(f)$, and consequently $\pi(f) = u^{-1}.f(a) = f(x)$, for each $f \in C^*(X, .)$. The uniqueness of $x$ follows from Lemma 3.2 (d), (e).

We turn to (c). Let $\Phi: C^*(X, .) \to C^*(Y, \ast)$ be a $\ast$-homomorphism of $C^*$-algebras. Put

$$U = Y \setminus \{y \in Y : [\Phi(f)](y) = 0 \quad \text{for each } f \in C^*(X, .)\}.$$ 

It is clear that $U$ is invariant and open in $Y$. For any $y \in U$, the function $C^*(X, .) \ni f \mapsto [\Phi(f)](y) \in M_n$ is a nonzero representation and therefore, thanks to (b), there is a unique point $\varphi(y) \in X$ such that $[\Phi(f)](y) = f(\varphi(y))$ for each $f \in C^*(X, .)$. In this way, we have obtained a function $\varphi: U \to X$ for which (3.1) holds. By the uniqueness in (b), we see that $\varphi(u.y) = u.\varphi(y)$ for any $u \in \mathfrak{U}_n$ and $y \in U$. So, to prove that $\varphi$ is a morphism of $n$-spaces, it remains to check that $\varphi$ is a proper map. First we shall show that $\varphi$ is continuous. Suppose, to the contrary, that there is a set $D \subset U$ and a point $b \in U \cap D$ ($\overline{D}$ is the closure of $D$ in $Y$) such that $a := \varphi(b) \notin \varphi(\overline{D})$ (the closure taken in $X$). Let $V$ be an open neighborhood of $a$ whose closure is compact and disjoint from $F := \overline{\varphi(\overline{D})}$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $M_n$, that is, $\langle X,Y \rangle = \text{tr}(Y^*X)$ (‘tr’ is the trace) and let $\|X\|_2 := \sqrt{\text{tr}(X^*X)}$. Take an irreducible matrix $Q \in M_n$ with $\|Q\|_2 = 1$. For simplicity, put $B = \{X \in M_n : \|X\|_2 \leq 1\}$. Our aim is to construct $f \in C^*(X, .)$ such that $f(a) = Q$ and $f^{-1}(\{Q\}) \subset V$. Observe that there is a compact convex nonempty set $K$ such that

(3.2) $Q \notin K \subset B$ and $\{u.a : u \in \mathfrak{U}_n, u.Q \notin K\} \subset V$. 
implies that $Z \subset \mathbb{R}$ gives $F$ from (3.1) that holds. Since $g_0(W \setminus V) \subset K$ (by (3.2)) and the set $K$ (being compact, convex and nonempty) is a retract of $M_n$, there is a map $g_1 \in C_0(X \setminus V, M_n)$ such that $g_1(X \setminus V) \subset K$ and $g_1(x) = g_0(x)$ for $x \in W \setminus V$. Finally, there is a $g \in C_0(X, M_n)$ which extends both $g_0$ and $g_1$, and $g(X) \subset B$. Now put $g = g_{\Phi} \in C^*(X, \cdot)$, and notice that $f(a) = Q$ (by Lemma 3.2 (b)). We claim that

$$f^{-1}([Q]) \subset V.$$  

Let us prove the above relation. Let $x \in X \setminus V$. Then $g(x) = g_1(x) \in K$, and hence $g(x) \notin Q$ (see (3.2)). The set $\mathcal{G} := \{u \in \mathcal{U}_n : u^{-1}.g(u.x) \notin Q\}$ is open in $\mathcal{U}_n$ and nonempty, which implies that its Haar measure is positive. Further, $|\langle u^{-1}.g(u.x), Q\rangle| \leq 1$ for any $u \in \mathcal{U}_n$ and $\langle u^{-1}.g(u.x), Q\rangle \neq 1$ for $u \in \mathcal{G}$ (since $g(X) \subset B$). We infer from these remarks that $\int_{\mathcal{U}_n} \langle u^{-1}.g(u.x), Q\rangle \, du \neq 1$. Equivalently, $\langle f(x), Q\rangle \neq 1$, which implies that $f(x) \neq Q$ and finishes the proof of (3.3). For $m \geq 1$, let

$$C_m = \{y \in Y : ||\Phi(f)(y) - Q||_2 \leq 2^{-m}\}$$

and

$$F_m = \{x \in X : ||f(x) - Q||_2 \leq 2^{-m}\}.$$ 

Since $f \in C_0(X, M_n)$ and $\Phi(f) \in C_0(Y, M_n)$, $F_m$ is compact and $C_m$ is a compact neighborhood of $b$. Consequently, $C_m \cap D \neq \emptyset$. We infer from (3.1) that $\varphi(C_m \cap D) \subset F_m \cap F$. Now the compactness argument gives $F \cap \bigcap_{m=1}^{\infty} F_m \neq \emptyset$. Let $c$ belong to this intersection. Then $f(c) = Q$ and $c \notin V$, which contradicts (3.3) and finishes the proof of the continuity of $\varphi$.

To see that $\varphi$ is proper, take a compact set $K \subset X$ and note that $L = \mathcal{U}_n.K$ is compact as well. Let $G \subset X$ be an open neighborhood of $L$ with compact closure. Take a map $\beta \in C_0(X, M_n)$ such that $\beta(x) = I$ for $x \in L$ and $\beta$ vanishes off $G$. Let $f = \beta_{\Phi} \in C^*(X, \cdot)$ and observe that $f(x) = I$ for $x \in L$. Since $\Phi(f) \in C_0(Y, M_n)$, the set $Z := \{y \in Y : \Phi(f)(y) = I\}$ is a compact subset of $Y$. But (3.1) implies that $Z \subset U$ and $\varphi^{-1}(K) \subset Z$. This finishes the proof of the
fact that \( \varphi \) is a morphism. The uniqueness of the pair \((U, \varphi)\) follows from Lemma 3.2 and is left to the reader.

Now if \( \Phi \) is a \( \ast \)-isomorphism of \( C^\ast \)-algebras, then \( U = Y \) (by Lemma 3.2 (d)) and thus \( \Phi(f) = f \circ \varphi \). Similarly, \( \Phi^{-1} \) is of the form \( \Phi^{-1}(g) = g \circ \psi \) for some morphism \( \psi : (X,.) \rightarrow (Y,\ast) \). Then \( f = f \circ (\varphi \circ \psi) \) for each \( f \in C^\ast(X,.) \), and the uniqueness in (c) gives \( (\varphi \circ \psi)(x) = x \) for each \( x \in X \). Similarly, \( (\psi \circ \varphi)(y) = y \) for any \( y \in Y \), and consequently \( \varphi \) is an isomorphism of \( n \)-spaces. The proof is complete. \( \square \)

4. Representations of \( C^\ast(X,.) \). In this section, we will characterize all representations of \( C^\ast(X,.) \) for an arbitrary \( n \)-space \((X,.)\). But first we shall give a ‘canonical’ description of all continuous linear functionals on \( C^\ast(X,.) \). We underline here that we are not interested in the formula for the norm of a functional. The results of the section will be applied in the next two parts where we formulate our version of Fell’s characterization of homogeneous \( C^\ast \)-algebras (Section 5) and a counterpart of the spectral theorem for finite systems of operators which generate \( n \)-homogeneous \( C^\ast \)-algebras (Section 6).

**Definition 4.1.** Let \((X,.)\) be an \( n \)-space. Let \( \mathcal{B}(X) \) denote the \( \sigma \)-algebra of all Borel subsets of \( X \); that is, \( \mathcal{B}(X) \) is the smallest \( \sigma \)-algebra of subsets of \( X \) which contains all open sets. For any \( u \in \mathcal{U}_n \) and \( A \in \mathcal{B}(X) \), the set \( u.A := \{u.a: a \in A\} \) is Borel as well. We shall denote by \( \chi_A : X \rightarrow \{0,1\} \) the characteristic function of \( A \). Further, \( \mathcal{B}C^\ast(X,.) \) stands for the \( C^\ast \)-algebra of all bounded Borel (i.e., \( \mathcal{B}(X) \)-measurable) functions \( f : X \rightarrow M_n \) such that \( f(u.x) = u.f(x) \) for any \( u \in \mathcal{U}_n \) and \( x \in X \).

An \( n \)-measure on \((X,.)\) is an \( n \times n \)-matrix \( \mu = [\mu_{jk}] \) where \( \mu_{jk} : \mathcal{B}(X) \rightarrow \mathbb{C} \) is a regular (complex-valued) measure and \( \mu(u.A) = u.\mu(A) \) for any \( u \in \mathcal{U}_n \) and \( A \in \mathcal{B}(X) \) (here, of course, \( \mu(A) = [\mu_{jk}(A)] \in M_n \)). The set of all \( n \)-measures on \((X,.)\) is denoted by \( M(X,.) \).

For any bounded Borel function \( f : X \rightarrow M_n \) and an \( n \times n \)-matrix \( \mu = [\mu_{jk}] \) of complex-valued regular Borel measures we define the
integral $\int f \, d\mu$ as the complex number

$$\sum_{j,k} \int_X f_{jk} \, d\mu_{kj},$$

where $f(x) = [f_{jk}(x)]$ for $x \in X$. We emphasize that in the formula for $\int f \, d\mu$, $f_{jk}$ meets $\mu_{kj}$ (not $\mu_{jk}$ (!)).

The first purpose of this section is to prove the following

**Theorem 4.2.** For every continuous linear functional $\varphi: C^*(X,\cdot) \to \mathbb{C}$ there exists a unique $\mu \in \mathcal{M}(X,\cdot)$ such that $\varphi(f) = \int f \, d\mu$ for any $f \in C^*(X,\cdot)$.

The above result is a simple consequence of the next one.

**Proposition 4.3.** Let $\mu = [\mu_{jk}]$ be an $n \times n$-matrix of complex-valued regular Borel measures on $X$. Then $\mu \in \mathcal{M}(X,\cdot)$ if and only if, for every map $f \in C_0(X,M_n)$,

$$(4.1) \quad \int f \, d\mu = \int f^\mu \, d\mu.$$ 

**Proof.** For any $n \times n$-matrix $A$ we shall write $A_{jk}$ to denote the suitable entry of $A$. We adapt the same rule for functions $f \in C_0(X,M_n)$ and matrix-valued measures. Further, for two arbitrarily fixed indices $(j,k)$ and $(p,q)$, the function $\mathfrak{U}_n \ni u \mapsto u_{jk} \overline{u}_{pq} \in \mathbb{C}$ is well defined and continuous (although $'u_{jk}'$ is not well defined). Observe that for any $A \in M_n$, $u \in \mathfrak{U}_n$ and an index $(p,q)$ one has:

$$(u.A)_{p,q} = \sum_{j,k} u_{pj} \overline{u}_{qk} \cdot A_{jk}$$

and

$$(u^{-1}.A)_{p,q} = \sum_{j,k} u_{kq} \overline{u}_{jp} \cdot A_{jk}.$$ 

Further, for $u \in \mathfrak{U}_n$ and a complex-valued regular Borel measure $\nu$ on $X$, let $\nu^u$ be the (complex-valued regular Borel) measure on $X$ given
by $\nu^u(A) = \nu(u.A)$ ($A \in \mathfrak{B}(X)$). It follows from the transport measure theorem that, for any $g \in C_0(X, \mathbb{C})$,

$$\int_X g(u.x) \, d\nu^u(x) = \int_X g(x) \, d\nu(x).$$

We adapt the above notation also for $n \times n$-matrix $\mu$ of measures: $\mu^u(A) = \mu(u.A)$. Notice that $(\mu^u)_{jk} = (\mu_{jk})^u$.

Now assume that $\mu \in \mathcal{M}(X,\cdot)$. This means that, for any $u \in \mathfrak{U}_n$, $u.\mu = \mu^u$. For $f \in C_0(X,M_n)$ and $x \in X$, we have

$$(f^\mu)_{pq}(x) = \sum_{j,k} \int_{\mathfrak{U}_n} u_{kj} f_{jp}(u.x) \, du,$$

and therefore, by Fubini’s theorem,

$$\int f^\mu \, d\mu = \sum_{p,q} \int_X (f^\mu)_{p,q} \, d\mu_{qp}$$

$$= \sum_{p,q} \sum_{j,k} \int_X \int_{\mathfrak{U}_n} u_{kj} f_{jp}(u.x) \, du \, d\mu_{qp}(x)$$

$$= \sum_{j,k} \int_{\mathfrak{U}_n} \int_X f_{jk}(u.x) \, d(\sum_{p,q} u_{kj} f_{jp} \cdot \mu_{qp})(x) \, du$$

$$= \sum_{j,k} \int_{\mathfrak{U}_n} \int_X f_{jk}(u.x) \, d(u.\mu)_{kj}(x) \, du$$

$$= \sum_{j,k} \int_{\mathfrak{U}_n} \int_X f_{jk}(u.x) \, d(\mu_{kj})^u(x) \, du$$

$$= \sum_{j,k} \int_X f_{jk}(x) \, d\mu_{kj}(x)$$

$$= \int f \, d\mu,$$

which gives (4.1). Conversely, assume (4.1) if fulfilled for any $f \in C_0(X,M_n)$ and fix a compact $\mathcal{G}_\delta$ subset $K$ of $X$ and an index $(p,q)$. Let $g \in C_0(X,\mathbb{C})$ be arbitrary, and let $f \in C_0(X,M_n)$ be such that $f_{pq} = g$ and $f_{jk} = 0$ for $(j,k) \neq (p,q)$. Applying (4.1) for such an $f$,
we obtain

\[(4.2) \quad \int_X g \, d\mu_{qp} = \sum_{j,k} \int_{U_n} \int_X \mu_{qk} \|u\|_p \cdot g(u,x) \, du \, d\mu_{kj}(x).\]

Further, since $K$ is compact and $G$, there is a sequence $(g_k)_{k=1}^\infty \subset C_0(X,\mathbb{C})$ such that $g_k(x) \subset [0,1]$ and $\lim_{k \to \infty} g_k(x) = \chi_K(x)$ for any $x \in X$. Substituting $g = g_k$ in (4.2) and letting $k \to \infty$, we obtain (by Lebesgue’s dominated convergence theorem as well as Fubini’s):

\[
\mu_{q,p}(K) = \sum_{j,k} \int_{U_n} \int_X \mu_{qk} \|u\|_p \cdot \chi_K(u,x) \, du \, d\mu_{kj}(x).
\]

We infer from the arbitrariness of $(p,q)$ in the above formula that

\[
\mu(K) = \int_{U_n} \mu(u^{-1}.K) \, du.
\]

Now, if $v \in U_n$, the set $v.K$ is also compact and $G$, and therefore

\[
\mu(v.K) = \int_{U_n} \mu(u^{-1}.v.K) \, du
= \int_{U_n} v.\mu(u^{-1}.K) \, du
= v.\left(\int_{U_n} \mu(u^{-1}.K) \, du\right)
= v.\mu(K).
\]

Finally, since $\mu$ is regular, the relation $\mu(v.A) = v.\mu(A)$ holds for any $A \in \mathcal{B}(X)$, and we are done. \(\square\)

**Proof of Theorem 4.2.** Note that the function $P: C_0(X,M_n) \ni f \mapsto f^M \in C^*(X,\mathbb{C})$ is a continuous linear projection (that is, $P(f) = f$ for $f \in C^*(X,\mathbb{C})$). So, if $\varphi: C^*(X,\mathbb{C}) \to \mathbb{C}$ is a continuous linear functional, so is $\psi := \varphi \circ P: C_0(X,M_n) \to \mathbb{C}$. Since $C_0(X,M_n)$ is isomorphic, as a Banach space, to $[C_0(X,\mathbb{C})]^n$, the Riesz-type representation
theorem yields that there is a unique $n \times n$-matrix $\mu$ of complex-valued regular Borel measures such that $\psi(f) = \int f \, d\mu$. Observe that $\psi(f^u) = \psi(f)$ for any $f \in C_0(X, M_n)$, and hence $\mu \in M(X,.)$, thanks to Proposition 4.3. The uniqueness of $\mu$ follows from the above construction, Proposition 4.3 and the uniqueness in the Riesz-type representation theorem. \hfill $\Box$

Now we turn to representations of $C^*(X,.)$. To this end, we introduce

**Definition 4.4.** An operator-valued $n$-measure on the $n$-space $(X,.)$ is any function of the form $E: \mathcal{B}(X) \ni A \mapsto [E_{jk}(A)] \in M_n(\mathcal{B}(\mathcal{H}))$ (where $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a Hilbert space) such that:

(M1) for any $h, w \in \mathcal{H}$ and $j, k \in \{1, \ldots, n\}$, the function

$$E_{jk}^{(h,w)}: \mathcal{B}(X) \ni A \mapsto \langle E_{jk}(A)h, w \rangle \in \mathbb{C}$$

is a (complex-valued) measure,

(M2) for any $u \in \mathfrak{U}_n$ and $A \in \mathcal{B}(X)$, $E(u.A) = u.E(A)$.

In other words, an operator-valued $n$-measure is an $n \times n$-matrix of operator-valued measures which satisfies axiom (M2). The operator-valued $n$-measure $E$ is regular if and only if $E_{jk}^{(h,w)}$ is regular for any $h, w$ and $j, k$.

Recall that if $\mu: \mathcal{B}(X) \to \mathcal{B}(\mathcal{H})$ is an operator-valued measure and $f: X \to \mathbb{C}$ is a bounded Borel function, $\int_X f \, d\mu$ is a bounded linear operator on $\mathcal{H}$, defined by an implicit formula:

$$\left\langle \left( \int_X f \, d\mu \right)h, w \right\rangle = \int_X f \, d\mu^{(h,w)}, \quad (h, w \in \mathcal{H}),$$

where $\mu^{(h,w)}(A) = \langle \mu(A)h, w \rangle \quad (A \in \mathcal{B}(X))$. Now assume that $E = [E_{jk}]: \mathcal{B}(X) \to M_n(\mathcal{B}(\mathcal{H}))$ is an $n$-measure and $f = [f_{jk}]: X \to M_n$ is a bounded Borel function. We define $\int f \, dE$ as a bounded linear operator on $\mathcal{H}$ given by

$$\int f \, dE = \sum_{j,k} \int_X f_{jk} \, dE_{kj}.$$
We are now ready to introduce

**Definition 4.5.** A spectral $n$-measure is any operator-valued regular $n$-measure $E : \mathcal{B}(X) \to M_n(\mathcal{B}(\mathcal{H}))$ such that

\begin{equation}
\left( \int f \, dE \right)^* = \int f^* \, dE,
\end{equation}

\begin{equation}
\int f \cdot g \, dE = \int f \, dE \cdot \int g \, dE
\end{equation}

for any $f, g \in \mathcal{B}C^*(X,\cdot)$. (The product $f \cdot g$ is computed pointwise as the product of matrices.) In other words, a spectral $n$-measure is an operator-valued regular $n$-measure $E : \mathcal{B}(X) \to M_n(\mathcal{B}(\mathcal{H}))$ such that the operator

\begin{equation}
\mathcal{B}C^*(X,\cdot) \ni f \mapsto \int f \, dE \in \mathcal{B}(\mathcal{H})
\end{equation}

is a representation of a $C^*$-algebra $\mathcal{B}C^*(X,\cdot)$.

The main result of this section is the following.

**Theorem 4.6.** Let $(X,\cdot)$ be an $n$-space and $\pi : C^*(X,\cdot) \to \mathcal{B}(\mathcal{H})$ a representation. There is a unique spectral $n$-measure $E : \mathcal{B}(X) \to M_n(\mathcal{B}(\mathcal{H}))$ such that

\begin{equation}
\pi(f) = \int f \, dE \quad (f \in C^*(X,\cdot)).
\end{equation}

In particular, every representation of $C^*(X,\cdot)$ admits an extension to a representation of $\mathcal{B}C^*(X,\cdot)$.

In the proof of the above result we shall involve the following:

**Lemma 4.7.** Let $\mu : \mathcal{B}(X) \to \mathbb{R}_+$ be a regular measure. For any $f \in \mathcal{B}C^*(X,\cdot)$ and $\varepsilon > 0$, there exists $g \in C^*(X,\cdot)$ such that

\[ \sup_{x \in X} \|g(x)\| \leq \sup_{x \in X} \|f(x)\| \quad \text{and} \quad \int_X \|f(x) - g(x)\| \, d\mu(x) < \varepsilon. \]

**Proof.** Let $f = [f_{jk}] \in \mathcal{B}C^*(X,\cdot)$, and let $M > 0$ be such that

\[ \sup_{x \in X} \|f(x)\| \leq M. \]
It follows from the regularity of $\mu$ that, for each $(j,k)$, there is a compact set $L_{jk}$ such that $\mu(X \setminus L_{jk}) \leq \frac{\varepsilon}{2mn^2}$

and $f_{jk}|_{L_{jk}}$ is continuous. Put

$$L = \bigcap_{j,k} L_{jk}$$

and $K = \cup_n L$. Then $K$ is compact and invariant, and

$$\mu(X \setminus K) \leq \frac{\varepsilon}{2M}.$$ 

What is more, $f|_{K}$ is continuous (this follows from the facts that $f|_{L}$ is continuous and $f(u.x) = u.f(x)$). Now Lemma 3.2 (c) yields the existence of $g \in C^*(X,\cdot)$ such that $\sup_{x \in X} \|g(x)\| \leq \sup_{x \in X} \|f(x)\|$ and $g|_{K} = f|_{K}$. Then:

$$\int_X \|f(x) - g(x)\| \, d\mu(x) = \int_{X \setminus K} \|f(x) - g(x)\| \, d\mu$$

$$\leq 2M \cdot \mu(X \setminus K)$$

$$= \varepsilon,$$

and we are done. 

\[ \square \]

**Proposition 4.8.** Let $E = [E_{jk}]: \mathcal{B}(X) \to M_n(\mathcal{B}(\mathcal{H}))$ be a regular $n$-measure.

(a) $E$ satisfies (4.3) for any $f \in \mathcal{B}C^*(X,\cdot)$ if and only if (4.3) is fulfilled for any $f \in C^*(X,\cdot)$, if and only if $(E_{jk}(A))^* = E_{kj}(A)$ for each $A \in \mathcal{B}(X)$;

(b) $E$ is spectral if and only if (4.3) and (4.4) are satisfied for any $f, g \in C^*(X,\cdot)$.

**Proof.** For any complex-valued regular Borel measure $\nu$ on $X$ we shall denote by $|\nu|$ the variation of $\nu$. Recall that $|\nu|$ is a nonnegative finite regular Borel measure on $X$. Further, for any $h, w \in \mathcal{H}$ and $j, k \in \{1, \ldots, n\}$, let $E^{(h,w)}_{jk}$ be as in Definition 4.4. Finally, $\langle \cdot, \cdot \rangle$ stands for the scalar product of $\mathcal{H}$. 


We begin with (a). Fix \( h, w \in \mathcal{H} \) and \( j, k \in \{1, \ldots, n\} \). First assume that (4.3) is fulfilled for any \( f \in C^*(X, \cdot) \). Let \( E^{(h, w)} := [E_{jk}^{(h, w)}] \), and note that \( E^{(h, w)} \in \mathcal{M}(X, \cdot) \) since \( E_{pq}(u.A) = \sum_{j,k} u_{pj} \pi_{qk} \cdot E_{jk}(A) \).

Thus, \( E_{pq}^{(h, w)}(u.A) = \sum_{j,k} u_{pj} \pi_{qk} \cdot E_{jk}^{(h, w)}(A) = (u.E^{(h, w)}(A))_{pq} \).

Observe that \( (E^{(h, w)})^* \in \mathcal{M}(X, \cdot) \) as well where \( (E^{(h, w)})^*(A) = (E^{(h, w)}(A))^* \) (because \( (u.P)^* = u.P^* \) for any \( P \in M_n \)). Further, for each \( f \in C^*(X, \cdot) \), we have

\[
\int f^* \, dE^{(h, w)} = \sum_{j,k} \int_X (f^*)_{jk} \, dE_{kj}^{(h, w)} = \sum_{j,k} \int_X f_{kj} \, dE_{kj}^{(h, w)}
\]

and, on the other hand,

\[
\int f^* \, dE^{(h, w)} = \langle \left( \int f^* \, dE \right) h, w \rangle = \langle \left( \int f \, dE \right)^* h, w \rangle = \langle \left( \int f \, dE \right) w, h \rangle = \int f \, dE^{(w, h)}.
\]

The uniqueness in Theorem 4.2 implies that \( (E^{(h, w)})^* = E^{(w, h)} \), which means that, for each \( A \in \mathfrak{B}(X) \), \( \langle (E_{jk}(A))w, h \rangle = \langle (E_{kj}(A))^* w, h \rangle = \langle (E_{kj}(A))^* w, h \rangle \). We conclude that \( (E_{jk}(A))^* = E_{kj}(A) \). Finally, if the last relation holds for any \( j, k \in \{1, \ldots, n\} \), then for every \( f \in \mathfrak{B}C^*(X, \cdot) \) we get:

\[
\left( \int f \, dE \right)^* = \sum_{j,k} \left( \int_X f_{jk} \, dE_{kj} \right)^* = \sum_{j,k} \int_X \overline{f}_{jk} \, d(E_{kj})^*
\]

\[
= \sum_{j,k} \int_X (f^*)_{kj} \, dE_{jk} = \int f^* \, dE.
\]

This completes the proof of (a).

We now turn to (b). We assume that (4.3) and (4.4) are fulfilled for any \( f, g \in C^*(X, \cdot) \). We know from (a) that actually (4.3) is satisfied for any \( f \in \mathfrak{B}C^*(X, \cdot) \). The proof of (4.4) is divided into three steps, stated below.
Step 1. If $\xi \in \mathcal{BC}^*(X,.)$ is such that

\begin{equation}
\int g \cdot \xi \, dE = \int g \, dE \cdot \int \xi \, dE
\end{equation}

for any $g \in C^*(X,.)$, then

$$\int f \cdot \xi \, dE = \int f \, dE \cdot \int \xi \, dE \text{ for any } f \in \mathcal{BC}^*(X,.)$$

Proof of Step 1. Fix $f \in \mathcal{BC}^*(X,.)$, $h, w \in \mathcal{H}$ and $\varepsilon > 0$. Let $M \geq 1$ be such that $\sup_{x \in X} \|\xi(x)\| \leq M$. Put

$$v = \left( \int \xi \, dE \right) h$$

and

$$\mu = \sum_{j,k} (|E_{jk}^{(h,w)}| + |E_{jk}^{(v,w)}|).$$

Since $\mu$ is finite and regular, Lemma 4.7 gives us a map $g \in C^*(X,.)$ such that

$$\int_X \|f(x) - g(x)\| \, d\mu(x) \leq \frac{\varepsilon}{M}.$$

Then (4.6) holds and, therefore, (remember that $M \geq 1$):

$$\left| \left\langle \int f \cdot \xi \, dE - \int f \, dE \cdot \int \xi \, dE \right\rangle_{h,w} \right|$$

$$\leq \left| \left\langle \int f \cdot \xi \, dE - \int g \cdot \xi \, dE \right\rangle_{h,w} \right|$$

$$+ \left| \left\langle \int g \, dE \cdot \int \xi \, dE - \int f \, dE \cdot \int \xi \, dE \right\rangle_{h,w} \right|$$

$$= \left| \sum_{j,k} \int_X ((f - g)\xi)_{jk} \, dE_{kj}^{(h,w)} \right|$$

$$+ \left| \sum_{j,k} \int_X (g_{jk} - f_{jk}) \, dE_{kj}^{(v,w)} \right|$$

$$\leq \sum_{j,k} \int_X \|(f(x) - g(x))\xi(x)\| \, d|E_{kj}^{(h,w)}|(x)$$
\[ + \sum_{j,k} \int_X \|g(x) - f(x)\| \, d|E_{kj}^{(v,w)}|(x) \]
\[ \leq M \int_X \|f(x) - g(x)\| \, d\mu(x) \leq \varepsilon. \]

**Step 2.** For any \( f \in \mathfrak{B}C^*(X,.) \) and \( g \in C(X,.) \), (4.4) holds.

*Proof of Step 2.* It follows from Step 1 and our assumptions in (b) that
\[ \int g^* \cdot f^* \, dE = \int g^* \, dE \cdot \int f^* \, dE. \]
Now it suffices to apply (4.3):
\[ \int f \cdot g \, dE = \left( \int g^* \cdot f^* \, dE \right)^* = \left( \int g^* \, dE \cdot \int f^* \, dE \right)^* \]
\[ = \int f \, dE \cdot \int g \, dE. \]

**Step 3.** The condition (4.4) is satisfied for any \( f, g \in \mathfrak{B}C^*(X,.) \).

*Proof of Step 3.* Just apply Step 2 and then Step 1. \( \square \)

*Proof of Theorem 4.6.* According to Proposition 4.8 (b), it suffices to show that there exists a regular \( n \)-measure \( E : \mathfrak{B}(X) \to M_n(\mathcal{B}(\mathcal{H})) \) such that (4.5) holds and that such an \( E \) is unique. According to Theorem 4.2, for any \( h, w \in \mathcal{H} \) there is a unique \( \mu_{(h,w)}^{(h,w)} = [\mu_{jk}^{(h,w)}] \in \mathcal{M}(X,.) \) such that
\[ (4.7) \quad \langle \pi(f)h, w \rangle = \int f \, d\mu^{(h,w)} \]
for each \( f \in C^*(X,.) \) (\( \langle \cdot, - \rangle \) is the scalar product of \( \mathcal{H} \)). Now, for any \( j, k \in \{1, \ldots, n\} \) and each \( A \in \mathfrak{B}(X) \), there is a unique bounded operator on \( \mathcal{H} \), denoted by \( E_{jk}(A) \), for which \( \mu_{jk}^{(h,w)}(A) = \langle (E_{jk}(A))h, w \rangle \) \((h, w \in \mathcal{H})\). We put \( E(A) = [E_{jk}(A)] \in M_n(\mathcal{B}(\mathcal{H})). \)
We want to show that \( E(u.A) = u.E(A) \). Since \( \mu^{(h,w)} \in \mathcal{M}(X,.) \), we obtain:
\[ \langle (E_{pq}(u.A))h, w \rangle = (\mu^{(h,w)}(u.A))_{pq} = (u.\mu^{(h,w)}(A))_{pq} \]
\[ = \sum_{j,k} u_{pj} \, \pi_{qk} \cdot \mu_{jk}^{(h,w)}(A) = \sum_{j,k} u_{pj} \, \pi_{qk} \cdot \langle (E_{jk}(A))h, w \rangle \]
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\[ \langle (u.E(A))_{pq}h, w \rangle, \]

which shows that indeed \( E(u.A) = u.E(A) \). Further, observe that \( E^{(h,w)}_{jk} = \mu_{jk}^{(h,w)} \), and thus \( E \) is an operator-valued regular \( n \)-measure and

\[ \left\langle \left( \int f \, dE \right) h, w \right\rangle = \langle \pi(f)h, w \rangle \]

(thanks to (4.7)). Consequently,

\[ \int f \, dE = \pi(f), \]

and we are done.

The uniqueness of \( E \) follows from the above construction, and its proof is left to the reader. \( \square \)

**Example 4.9.** Let \((X,.)\) be an \( n \)-space, and let \( E = [E_{jk}] : \mathfrak{B}(X) \to M_n(\mathfrak{B}(\mathcal{H})) \) be a spectral \( n \)-measure. We denote by \( \mathfrak{B}_{inv}(X) \) the \( \sigma \)-algebra of all invariant Borel subsets of \( X \) (that is, \( A \in \mathfrak{B}(X) \) belongs to \( \mathfrak{B}_{inv}(X) \) if and only if \( u.A = A \) for any \( u \in \mathfrak{U}_n \)). Let

\[ F : \mathfrak{B}_{inv}(X) \ni A \mapsto \sum_j E_{jj}(A) \in \mathfrak{B}(\mathcal{H}). \]

Then, for every \( A \in \mathfrak{B}_{inv}(X) \), one has:

1. \( E_{jk}(A) = 0 \) whenever \( j \neq k \),
2. \( E_{11}(A) = \ldots = E_{nn}(A) = \frac{1}{n} F(A) \),

and \( F \) is a spectral measure (possibly with \( F(X) \neq I_H \) where \( I_H \) is the identity operator on \( \mathcal{H} \)). Let us briefly prove these claims. Since \( E(A) = E(u.A) = u.E(A) \) for any \( u \in \mathfrak{U}_n \), conditions (E1)–(E2) are fulfilled. Further, if \( j_A : X \to M_n \) is given by \( j_A(x) = \chi_A(x) \cdot I \) where \( I \in M_n \) is the unit matrix, then \( j_A \in \mathfrak{B} C^*(X,.) \) and, for \( B \in \mathfrak{B}_{inv}(X) \),

\[ F(A \cap B) = \int j_{A \cap B} \, dE = \int j_A \cdot j_B \, dE \]

\[ = \int j_A \, dE \cdot \int j_B \, dE = F(A)F(B). \]

What is more, Proposition 4.8 (a) implies that \( F(A) \) is self-adjoint, and hence \( F \) is indeed a spectral measure. One may also easily check
that \( F(X) = I_H \) if and only if the representation \( \pi_E: C^*(X,.) \ni f \mapsto \int f \, dE \in B(H) \) is nondegenerate.

The spectral measure \( F \) defined above corresponds to the representation of the center \( Z \) of \( C^*(X,.) \). It is a simple exercise that \( Z \) consists precisely of all \( f \in C_0(X, C \cdot I) \) which are constant on the sets of the form \( \mathcal{U}_n, \{x\} \ (x \in X) \). Thus, \( \mathfrak{B}_{inv}(X) \) may naturally be identified with the Borel \( \sigma \)-algebra of the spectrum of \( Z \), and consequently, \( F \) is the spectral measure induced by the representation \( \pi_E|_Z \) of \( Z \).

Conditions (E1)–(E2) show that a nonzero spectral \( n \)-measure \( E \) for \( n > 1 \) never satisfies the condition of a spectral measure—that \( E(A \cap B) = E(A)E(B) \). Indeed, \( E(X) \neq (E(X))^2 \).

The next result is well known. For the reader’s convenience, we give its short proof.

**Lemma 4.10.** Let \( \mathcal{A} \) be a \( C^* \)-algebra, and let \( \pi: \mathcal{A} \to M_n \) (where \( n \geq 1 \) is finite) be a nonzero irreducible representation of \( \mathcal{A} \). Then \( \pi \) is surjective.

**Proof.** Let \( J = \pi(\mathcal{A}) \). Since \( \pi \) is irreducible, \( J' = C \cdot I \), and consequently, \( J'' = M_n \). But it follows from von Neumann’s double commutant theorem that \( J'' = J + C \cdot I \) (here we use the fact that \( n \) is finite). So, the facts that \( J \) is a \( * \)-algebra and \( M_n = J + C \cdot I \) imply that \( J \) is a two-sided ideal in \( M_n \). Consequently, \( J = \{0\} \) or \( J = M_n \). But \( \pi \neq 0 \), and hence \( J = M_n \). \( \square \)

With the aid of the above lemma and Theorem 4.6, we shall now characterize all irreducible representations of \( C^*(X,.) \).

**Proposition 4.11.** Every nonzero irreducible representation \( \pi \) of \( C^*(X,.) \) (where \( (X,.) \) is an \( n \)-space) is \( n \)-dimensional and has the form \( \pi(f) = f(x) \) (for some \( x \in X \)).

**Proof.** Let \( \pi: C^*(X,.) \to B(H) \) be a nonzero irreducible representation. It follows from Theorem 4.6 that there is a spectral \( n \)-measure \( E: \mathfrak{B}(X) \to B(H) \) such that (4.5) holds. Let \( \mathfrak{B}_{inv}(X) \) and \( F: \mathfrak{B}_{inv}(X) \to B(H) \) be as in Example 4.9. Then \( F \) is a (regular)
spectral measure (with $F(X) = I_\mathcal{H}$ because $\pi$ is nondegenerate) and, for any $A \in \mathfrak{B}_{inv}(X)$ and $f \in C^*(X,.)$, we have

$$\int f \, dE \cdot \int \chi_A I \, dE = \int \chi_A I \, dE \cdot \int f \, dE$$

(where $I$ is the unit $n \times n$-matrix). Since $\pi$ is irreducible, we deduce that, for every $A \in \mathfrak{B}_{inv}(X)$, $\int \chi_A I \, dE$ is a scalar multiple of the identity operator on $\mathcal{H}$. This implies that we may think of $F$ as a complex-valued (spectral) measure. But $\mathfrak{B}_{inv}(X)$ is naturally ‘isomorphic’ to the $\sigma$-algebra of all Borel sets of $X/\mathfrak{U}_n$ (which is locally compact) and thus $F$ is supported on a set $S := \mathfrak{U}_n.a$ for some $a \in X$. But then

$$\int \chi_X I \, dE = \int \chi_S I \, dE,$$

and consequently, $\pi(f) = \int f \big|_S \, dE_S$, where $E_S$ is the restriction of $E$ to $\mathfrak{B}(S)$. Since the vector space $\{f \big|_S; \ f \in C^*(X,.)\}$ is finite dimensional (and its dimension is equal to $n^2$), we infer that $\mathcal{A} := \pi(C^*(X,.)$) is finite dimensional as well and $\dim \mathcal{A} \leq n^2$. So, the irreducibility of $\pi$ implies that $\mathcal{H}$ is finite dimensional, while Lemma 4.10 shows that $\dim \mathcal{H} \leq n$. Finally, Proposition 3.3 (b) completes the proof.

5. Homogeneous $C^*$-algebras.

Definition 5.1. A $C^*$-algebra is said to be $n$-homogeneous (where $n$ is finite) if and only if every nonzero irreducible representation of it is $n$-dimensional.

Our version of Fell’s characterization of $n$-homogeneous $C^*$-algebras [9] reads as follows.

Theorem 5.2. For a $C^*$-algebra $\mathcal{A}$ and finite $n \geq 1$, the following conditions are equivalent:

(i) $\mathcal{A}$ is an $n$-homogeneous $C^*$-algebra;
(ii) there is an $n$-space $(X,.)$ such that $\mathcal{A}$ is isomorphic (as a $C^*$-algebra) to $C^*(X,.)$.

What is more, if $\mathcal{A}$ is $n$-homogeneous, the $n$-space $(X,.)$ appearing in (ii) is unique up to isomorphism.
Proof of Theorem 5.2. We infer from Proposition 3.3 (c) that the \( n \)-space \((X,.)\) appearing in (ii) is unique up to isomorphism. In addition, it easily follows from Proposition 4.11 that \( C^*(X,.) \) is \( n \)-homogeneous for any \( n \)-space \((X,.)\). So, it remains to show that (i) implies (ii).

To this end, assume \( A \) is \( n \)-homogeneous, and let \( \mathcal{X} \) be the set of all representations (including the zero one) \( \pi: A \rightarrow M_n \), equipped with the topology of pointwise convergence. Since each representation is a bounded linear operator of norm not greater than 1, \( \mathcal{X} \) is compact. Consequently, \( X := \mathcal{X} \setminus \{0\} \) is locally compact. We define an action of \( \mathcal{U}_n \) on \( X \) by the formula:

\[(u.\pi)(a) = u.\pi(a) \quad (a \in A, \pi \in X, u \in \mathcal{U}_n).\]

It is easily seen that the action is continuous. What is more, Lemma 4.10 ensures us that it is free as well. So, \((X,.)\) is an \( n \)-space. The next step of construction is very common. For any \( a \in A \), let \( \tilde{a}: X \rightarrow M_n \) be given by \( \tilde{a}(\pi) = \pi(a) \). It is clear that \( \tilde{a} \in C_0(X,M_n) \) (indeed, if \( X \) is noncompact, then \( \mathcal{X} = X \cup \{0\} \) is a one-point compactification of \( X \) and \( \tilde{a} \) extends to a map on \( \mathcal{X} \) which vanishes at 0).

We also readily have \( \tilde{a}(u.\pi) = u.\tilde{a}(\pi) \) for any \( u \in \mathcal{U}_n \). So, we have obtained a \( * \)-homomorphism \( \Phi: A \ni a \mapsto \tilde{a} \in C^*(X,.) \). It follows from (i) (and the fact that all irreducible representations separate points of a \( C^* \)-algebra) that \( \Phi \) is one-to-one and, consequently, \( \Phi \) is isometric. So, to end the proof, it suffices to show that \( \mathcal{E} = \Phi(A) \) is dense in \( C^*(X,.) \).

To this end, we involve Theorem 2.2. It follows from Lemma 4.10 that condition (AX0) is fulfilled. Further, let \( \pi_1 \) and \( \pi_2 \) be arbitrary members of \( X \).

We consider two cases. First assume that \( \pi_2 = u.\pi_1 \) for some \( u \in \mathcal{U}_n \). Then \( \tilde{a}(\pi_2) = u.\tilde{a}(\pi_1) \), and consequently, \( \sigma(\tilde{a}(\pi_1)) = \sigma(\tilde{a}(\pi_2)) \) \((a \in A)\). So, in that case (AX2) holds. Now assume that there is no \( u \in \mathcal{U}_n \) for which \( \pi_2 = u.\pi_1 \). We shall show that, in that case:

\begin{equation}
\pi_1(a) = 0 \quad \text{and} \quad \pi_2(a) = I \quad \text{for some} \ a \in A.
\end{equation}

Let \( \mathcal{M} \subset M_{2n} \) consist of all matrices of the form

\[
\begin{pmatrix}
\pi_1(x) & 0 \\
0 & \pi_2(x)
\end{pmatrix}
\quad \text{with} \ x \in A.
\]

Since \( \mathcal{M} \) is a finite-dimensional \( C^* \)-algebra, it is singly generated (see, e.g., [22]) and unital (cf., [26, subsection 1.11]).
Lemma 4.10, $\mathcal{M}$ contains matrices of the form
\[
\begin{pmatrix}
I & 0 \\
0 & A
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
B & 0 \\
0 & I
\end{pmatrix}
\quad \text{for some } A, B \in M_n.
\]

We conclude that the unit of $\mathcal{M}$ coincides with the unit of $M_{2n}$. This, combined with the fact that $\mathcal{M}$ is singly generated, yields that there is $z \in \mathcal{A}$ such that, for $A_j = \pi_j(z)$ ($j = 1, 2$), we have
\[
\mathcal{M} = \left\{ \begin{pmatrix} p(A_1, A_1^*) & 0 \\
0 & p(A_2, A_2^*) \end{pmatrix} : p \in \mathcal{P} \right\}
\]

where $\mathcal{P}$ is the free algebra of all polynomials in two noncommuting variables. Observe that then $M_n = \pi_j(\mathcal{A}) = \{p(A_j, A_j^*) : p \in \mathcal{P}\}$ ($j = 1, 2$), which means that $A_1$ and $A_2$ are irreducible matrices. What is more, $A_1$ and $A_2$ are not unitarily equivalent, that is, there is no $u \in \mathbb{U}_n$ for which $A_2 = u A_1$ (indeed, if $A_2 = u A_1$, then $\pi_2 = u \pi_1$ since, for every $x \in \mathcal{A}$, there is a $p \in \mathcal{P}$ such that $\pi_j(x) = p(A_j, A_j^*)$). These two remarks imply that
\[
\begin{pmatrix}
0 & 0 \\
0 & I
\end{pmatrix}
\in \mathcal{M},
\]

because the $*$-commutant in $M_{2n}$ of the matrix
\[
\begin{pmatrix}
A_1 & 0 \\
0 & A_2
\end{pmatrix}
\]
consists of matrices of the form
\[
\begin{pmatrix}
X & 0 \\
0 & Y
\end{pmatrix}
\]
(this follows from the so-called Schur’s lemma on intertwining transformations; see [6, Theorem 1.5, Corollary 1.8]; cf., also [19, Proposition 5.2.1]), and consequently,
\[
\begin{pmatrix}
0 & 0 \\
0 & I
\end{pmatrix}
\in \mathcal{M}'' = \mathcal{M}.
\]

So, there is an $a \in \mathcal{A}$ such that
\[
\begin{pmatrix}
0 & 0 \\
0 & I
\end{pmatrix}
= \begin{pmatrix}
\pi_1(a) & 0 \\
0 & \pi_2(a)
\end{pmatrix},
\]
which gives (5.1). Replacing $a$ by $(a + a^*)/2$, we may assume that $a$ is self-adjoint. Then $f = \widehat{a} \in \mathcal{E}$ is self-adjoint (and hence normal) and
\[ \sigma(f(\pi_1)) \cap \sigma(f(\pi_2)) = \emptyset, \] which shows that \( \pi_1 \) and \( \pi_2 \) are spectrally separated by \( \mathcal{E} \). According to Theorem 2.2, it therefore suffices to check that each \( g \in C^*(X,.) \) belongs to \( \Delta_2(\mathcal{E}) \). To this end, we fix \( \pi_1, \pi_2 \in X \) and consider the same two cases as before. If \( \pi_2 = u.\pi_1 \), it follows from Lemma 4.10 that there is an \( x \in \mathcal{A} \) for which \( \pi_1(x) = g(\pi_1) \). Then \( \tilde{x}(\pi_1) = g(\pi_1) \) and \( \tilde{x}(\pi_2) = u.\tilde{x}(\pi_1) = u.g(\pi_1) = g(\pi_2) \), and we are done.

Finally, if \( \pi_2 \neq u.\pi_1 \) for any \( u \in \mathcal{U}_n \), (5.1) implies that there are points \( a_1, a_2 \in \mathcal{A} \) such that \( \pi_1(a_1) = I = \pi_2(a_2) \) and \( \pi_1(a_2) = 0 = \pi_2(a_1) \). Moreover, there are points \( x, y \in \mathcal{A} \) such that \( \pi_1(x) = g(\pi_1) \) and \( \pi_2(y) = g(\pi_2) \) (by Lemma 4.10). Put \( z = xa_1 + ya_2 \in \mathcal{A} \) and note that \( \tilde{x}(\pi_j) = g(\pi_j) \) for \( j = 1, 2 \), which means that \( g \in \Delta_2(\mathcal{E}) \). The whole proof is complete. \( \square \)

**Definition 5.3.** Let \( \mathcal{A} \) be an \( n \)-homogeneous \( C^* \)-algebra. By an \( n \)-spectrum of \( \mathcal{A} \) we mean any \( n \)-space \((X,.)\) such that \( \mathcal{A} \) is isomorphic to \( C^*(X,.) \). It follows from Theorem 5.2 that an \( n \)-spectrum of \( \mathcal{A} \) is unique up to isomorphism of \( n \)-spaces. By concrete \( n \)-spectrum of \( \mathcal{A} \) we mean the \( n \)-space of all nonzero representations \( \pi : \mathcal{A} \to M_n \) endowed with the pointwise convergence topology and the natural action of \( \mathcal{U}_n \).

The trivial algebra \( \{0\} \) is \( n \)-homogeneous and its \( n \)-spectrum is the empty \( n \)-space.

The reader interested in general ideas of operator spectra should consult [6, subsection 2.5]; [7, 8, 9]; [4] as well as [12, 13]; [15, 16]; [20].

Our approach to \( n \)-homogeneous \( C^* \)-algebras allows us to prove briefly the following

**Proposition 5.4.** Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be two \( n \)-homogeneous \( C^* \)-algebras such that \( \mathcal{A}_1 \subset \mathcal{A}_2 \).

(a) Every representation \( \pi_1 : \mathcal{A}_1 \to M_n \) is extendable to a representation \( \pi_2 : \mathcal{A}_2 \to M_n \).

(b) If every \( n \)-dimensional representation (including the zero one) of \( \mathcal{A}_1 \) has a unique extension to an \( n \)-dimensional representation of \( \mathcal{A}_2 \), then \( \mathcal{A}_1 = \mathcal{A}_2 \).
Proof. We begin with (a). We may and do assume that \( \pi_1 \) is nonzero. For \( j = 1, 2 \), let \((X_j,.)\) denote an \( n\)-spectrum of \( A_j \), and let \( \Psi_j: A_j \to C^*(X_j,.) \) be a *-isomorphism of \( C^*-\)algebras. Let \( j: A_1 \to A_2 \) be the inclusion map. Then \( \Phi := \Psi_2 \circ j \circ \Psi_1^{-1}: C^*(X_1,.) \to C^*(X_2,.) \) is a one-to-one *-homomorphism. We infer from Proposition 3.3 that there are an invariant open (in \( X_2 \)) set \( U \) and a morphism \( \varphi: (U,.) \to (X_1,.) \) such that (3.1) holds. We claim that

\[
(5.2) \quad \varphi(U) = X_1.
\]

Since \( \varphi \) is proper, the set \( F := \varphi(U) \) is closed in \( X_1 \). It is also invariant. So, if \( F \neq X_1 \), we may take \( b \in X_1 \setminus F \) and apply Lemma 3.2 (c) to get a function \( f \in C^*(X_1,.) \) such that \( f\big|_F \equiv 0 \) and \( f(b) = I \). Then \( \Phi(f) = 0 \), by (3.1), which contradicts the fact that \( \Phi \) is one-to-one. So, (5.2) is fulfilled.

Further, Proposition 3.3 yields that there is an \( x \in X_1 \) such that \( \pi_1(\Psi_1^{-1}(f)) = f(x) \) for any \( f \in C^*(X_1,.) \). It follows from (5.2) that we may find \( z \in U \) for which \( \varphi(z) = x \). Now define \( \pi_2: A_2 \to M_n \) by \( \pi_2(a) = [\Psi_2(a)](z) \) \((a \in A_2)\). It remains to check that \( \pi_2 \) extends \( \pi_1 \). To see this, for \( a \in A_1 \) put \( f = \Psi_1(a) \), and note that \( \pi_2(a) = [\Psi_2(a)](z) = [\Psi_2(\Psi_1^{-1}(f))](z) = [\Phi(f)](z) = f(\varphi(z)) = f(x) = \pi_1(a) \) (cf., (3.1)).

Now, if the assumption of (b) is satisfied, the above argument shows that \( \varphi \) is one-to-one (since different points of \( X_2 \) correspond to different \( n\)-dimensional representations of \( A_2 \)). It may also easily be checked that, for every \( z \in X_2 \setminus U \), the representation \( A_2 \ni a \mapsto [\Psi_2(a)](z) \in M_n \) vanishes on \( A_1 \) (use (3.1) and the definition of \( \Phi \)). So, we conclude from the uniqueness of the extension of the zero representation of \( A_1 \) that \( U = X_2 \), and hence, both \( \varphi \) and \( \Phi \) are isomorphisms. Consequently, \( A_1 = A_2 \), and we are done. \( \square \)

6. Spectral theorem and \( n\)-functional calculus. Whenever \( A \) is a unital \( C^*-\)algebra and \( x_1,\ldots,x_k \) are arbitrary elements of \( A \), let \( C^*(x_1,\ldots,x_k) \) denote the \( C^*-\)subalgebra of \( A \) generated by \( x_1,\ldots,x_k \), and let \( C^*_1(x_1,\ldots,x_k) \) be the smallest \( C^*-\)subalgebra of \( A \) which contains \( x_1,\ldots,x_k \) as well as the unit of \( A \) (so, \( C^*_1(x_1,\ldots,x_k) = C^*(x_1,\ldots,x_k) + \mathbb{C} \cdot 1 \) where \( 1 \) is the unit of \( A \)). We would like to distinguish those systems \((x_1,\ldots,x_k)\) for which one of these two \( C^*-\)algebras defined above is \( n\)-homogeneous. However, the property of
being $n$-homogeneous is not hereditary for $n > 1$. That is, when $n > 1$, every nonzero $n$-homogeneous $C^*$-algebra contains a $C^*$-subalgebra which is not $n$-homogeneous (namely, a nonzero commutative one). This results in the class of distinguished systems possibly depending on the choice of $C^*$-algebras related to them. Fortunately, this does not happen, which is explained in the following.

**Lemma 6.1.** Let $A$ be a unital $C^*$-algebra and $x_1, \ldots, x_k \in A$. If $C^*_1(x_1, \ldots, x_k)$ is $n$-homogeneous for some $n > 1$, then

$$C^*_1(x_1, \ldots, x_k) = C^*(x_1, \ldots, x_k).$$

**Proof.** Suppose, to the contrary, that the assertion is false. Observe that $\mathcal{I} := C^*(x_1, \ldots, x_k)$ is a two-sided ideal in $\mathcal{B} := C^*_1(x_1, \ldots, x_k)$, since

$$\mathcal{B} = \mathcal{I} + \mathbb{C} \cdot 1 \quad (1 = \text{the unit of } A).$$

Moreover, $\mathcal{B}/\mathcal{I}$ is isomorphic (as a $C^*$-algebra) to $\mathbb{C}$, which means that the canonical projection $\pi: \mathcal{B} \rightarrow \mathcal{B}/\mathcal{I}$ may be considered as a one-dimensional (nonzero) representation. It is obviously irreducible, which contradicts the fact that $\mathcal{B}$ is $n$-homogeneous (since $n > 1$). \quad \Box

Taking into account the above result, we may now introduce

**Definition 6.2.** A system $(x_1, \ldots, x_k)$ of elements of an (unnecessarily unital) $C^*$-algebra $A$ is said to be $n$-homogeneous (where $n \geq 1$ is finite) if the $C^*$-subalgebra $C^*(x_1, \ldots, x_k)$ of $A$ generated by $x_1, \ldots, x_k$ is $n$-homogeneous.

This part of the paper is devoted to studies of (finite) $n$-homogeneous systems. We begin with:

**Proposition 6.3.** Let $(x_1, \ldots, x_k)$ be an $n$-homogeneous system in a $C^*$-algebra $A$. Let $(\mathfrak{X}, \cdot)$ be the concrete $n$-spectrum of $C^*(x_1, \ldots, x_k)$, and let

$$(6.1) \quad \sigma_n(x_1, \ldots, x_k) := \{ (\pi(x_1), \ldots, \pi(x_k)) : \pi \in \mathfrak{X} \}$$

be equipped with the topology inherited from $(M_n)^k$ and with the action

$$u.(A_1, \ldots, A_k) := (u.A_1, \ldots, u.A_k)$$
(where \( u \in U_n \) and \( (A_1, \ldots, A_k) \in \sigma_n(x_1, \ldots, x_k) \)).

(Sp1) The pair \( (\sigma_n(x_1, \ldots, x_k), \cdot) \) is an \( n \)-space.

(Sp2) The function

\[
H: (\mathcal{X}, \cdot) \ni \pi \mapsto (\pi(x_1), \ldots, \pi(x_k)) \in (\sigma_n(x_1, \ldots, x_k), \cdot)
\]

is an isomorphism of \( n \)-spaces.

(Sp3) Every member of \( \sigma_n(x_1, \ldots, x_k) \) is irreducible; that is, if \( (A_1, \ldots, A_k) \in \sigma_n(x_1, \ldots, x_k) \) and \( T \in M_n \) commutes with each of \( A_1, A_1^*, \ldots, A_k, A_k^* \), then \( T \) is a scalar multiple of the unit matrix.

(Sp4) The set \( \sigma_n(x_1, \ldots, x_k) \) is either compact or its closure in \( (M_n)^k \) coincides with \( \sigma_n(x_1, \ldots, x_k) \cup \{0\} \).

**Proof.** Let \( \pi_0: A \to M_n \) be the zero representation, and let \( \Omega = \mathcal{X} \cup \{\pi_0\} \) be equipped with the pointwise convergence topology. Then \( \Omega \) is compact (cf., the proof of Theorem 5.2). If \( \pi_1, \pi_2 \in \Omega \), then the set \( \{x \in C^*(x_1, \ldots, x_k): \pi_1(x) = \pi_2(x)\} \) is a \( C^* \)-subalgebra of \( C^*(x_1, \ldots, x_k) \). This implies that the function \( \tilde{H}: \Omega \ni \pi \mapsto (\pi(x_1), \ldots, \pi(x_k)) \in \sigma_n(x_1, \ldots, x_k) \cup \{0\} \) is one-to-one. It is obviously seen that \( \tilde{H} \) is surjective and continuous. Consequently, \( \tilde{H} \) is a homeomorphism (since \( \Omega \) is compact). This proves (Sp4) and shows that \( \sigma_n(x_1, \ldots, x_k) \) is locally compact. It is also clear that \( H(u, \pi) = u.H(\pi) \), which is followed by (Sp1) and (Sp2). Finally, for any \( \pi \in \mathcal{X} \), \( C^*(\pi(x_1), \ldots, \pi(x_k)) = \pi(C^*(x_1, \ldots, x_k)) = M_n \) (see Lemma 4.10), which yields (Sp3) and completes the proof. \( \square \)

**Definition 6.4.** Let \( (x_1, \ldots, x_k) \) be an \( n \)-homogeneous system in a \( C^* \)-algebra. The \( n \)-space \( (\sigma_n(x_1, \ldots, x_k), \cdot) \) defined by (6.1) is said to be the \( n \)-spectrum of \( (x_1, \ldots, x_k) \). According to Proposition 6.3, the \( n \)-spectrum of \( (x_1, \ldots, x_k) \) is an \( n \)-spectrum of \( C^*(x_1, \ldots, x_k) \).

**Proposition 6.5.** Let \( x = (x_1, \ldots, x_k) \) be an \( n \)-homogeneous system in a \( C^* \)-algebra. There exists a unique \(*\)-homomorphism

\[
\Phi_x: C^*(\sigma_n(x), \cdot) \longrightarrow C^*(x)
\]

such that \( \Phi_x(p_j) = x_j \), where \( p_j: \sigma_n(x) \ni (A_1, \ldots, A_k) \mapsto A_j \in M_n \) \((j = 1, \ldots, k)\). Moreover, \( \Phi_x \) is a \(*\)-isomorphism of \( C^* \)-algebras.
Proof. Let \((X,.)\) be the concrete \(n\)-spectrum of \(C^*(x)\), and let \(H: (X,.) \to (\sigma_n(x),.)\) be the isomorphism as in point (Sp2) of Proposition 6.3. For \(x \in C^*(x)\) let \(\tilde{x} \in C(X,.)\) be given by \(\tilde{x}(\pi) = \pi(x)\). The proof of Theorem 5.2 shows that the function \(C^*(x) \ni x \mapsto \tilde{x} \in C^*(X,.)\) is a \(*\)-isomorphism of \(C^\ast\)-algebras. Consequently, \(\Psi: C^*(x) \ni x \mapsto \tilde{x} \circ H^{-1} \in C^*(\sigma_n(x),.)\) is a \(*\)-isomorphism as well. A direct calculation shows that \(\Psi(x_j) = p_j\) \((j = 1,\ldots,k)\). This implies that \(C^*(p_1,\ldots,p_k) = C^*(\sigma_n(x),.)\), from which we infer the uniqueness of \(\Phi_x\). To convince about its existence, just put \(\Phi_x = \Psi^{-1}\).

We are now ready to introduce the following:

**Definition 6.6.** Let \(x = (x_1,\ldots,x_k)\) be an \(n\)-homogeneous system, and let \(\Phi_x\) be as in Proposition 6.5. For every \(f \in C^*(\sigma_n(x_1,\ldots,x_k),.)\), we denote by \(f(x_1,\ldots,x_k)\) the element \(\Phi_x(f)\). The assignment \(f \mapsto f(x_1,\ldots,x_k)\) is called the \(n\)-functional calculus.

The reader familiar with functional calculus on normal operators (or normal elements in \(C^\ast\)-algebras) has to be careful with the \(n\)-functional calculus, because its main disadvantage is that its values are not \(n\)-homogeneous elements in general. Therefore, we cannot speak of the \(n\)-spectrum of \(f(x_1,\ldots,x_k)\) in general. What is more, it may happen that \(\sigma_n(x_1,\ldots,x_k)\) is compact, but \(j(x_1,\ldots,x_k)\), where \(j\) is constantly equal to the unit matrix, differs from the unit of the underlying \(C^\ast\)-algebra \(A\) from which \(x_1,\ldots,x_k\) were taken. This happens precisely when \(C^*(x_1,\ldots,x_k)\) has a unit, but this unit is not the unit of \(A\).

As a consequence of Theorem 4.6 and Proposition 6.5 we obtain the spectral theorem (for \(n\)-homogeneous systems) announced before.

**Theorem 6.7.** Let \(T = (T_1,\ldots,T_k)\) be an \(n\)-homogeneous system of bounded linear operators acting on a Hilbert space \(\mathcal{H}\). There exists a unique spectral \(n\)-measure \(E_T: \mathfrak{B}(\sigma_n(T)) \to M_n(\mathcal{B}(\mathcal{H}))\) such that

\[
\int p_j \, dE_T = T_j \quad (j = 1,\ldots,k)
\]

where \(p_j: \sigma_n(T) \ni (A_1,\ldots,A_k) \mapsto A_j \in M_n\).
**Definition 6.8.** Let $T = (T_1, \ldots, T_k)$ be an $n$-homogeneous system of bounded Hilbert space operators, and let $E_T$ be the spectral $n$-measure as in Theorem 6.7. $E_T$ is called the spectral $n$-measure of $T$, and the assignment

$$ \mathfrak{B} C^*(\sigma(T), .) \ni f \mapsto f(T_1, \ldots, T_n) := \int f \, dE_T \in \mathcal{B}(\mathcal{H}) $$

is called the extended $n$-functional calculus.

There is nothing surprising in the following

**Proposition 6.9.** Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and let $T = (T_1, \ldots, T_k)$ be an $n$-homogeneous system of operators belonging to $\mathcal{M}$. Let $X = \sigma_n(T)$.

(a) For any $f \in \mathfrak{B} C^*(X, .), f(T) \in \mathcal{M}$.

(b) If $f^{(1)}, f^{(2)}, \ldots \in \mathfrak{B} C^*(X, .)$ converge pointwise to $f: X \to M_n$ and

$$ \sup_{m \geq 1} \|f^{(m)}(x)\| < \infty, $$

then $f \in \mathfrak{B} C^*(X, .)$ and $\lim_{m \to \infty}(f^{(m)}(T))h = (f(T))h$ for each $h \in \mathcal{H}$.

**Proof.** We begin with (a). It is clear that $g(T) \in \mathcal{M}$ for $g \in C^*(X, .)$. Let $E_T = [E_{pq}]$. Denote by $\langle , \rangle$ the scalar product of $\mathcal{H}$, and fix $f = [f_{pq}] \in \mathfrak{B} C^*(X, .)$. We shall show that $f(T)$ belongs to the closure of $\{g(T): g \in C^*(X, .)\}$ in the weak operator topology of $\mathcal{B}(\mathcal{H})$, which will give (a). To this end, we fix $h_1, w_1, \ldots, h_r, w_r \in \mathcal{H}$ and $\varepsilon > 0$. Put $\mu = \sum_{s=1}^r \sum_{p,q}|E_{pq}^{(h_s, w_s)}|$. By Lemma 4.7, there is a $g = [g_{pq}] \in C^*(X, .)$ such that

$$ \int_X \|f(x) - g(x)\| \, d\mu(x) \leq \varepsilon. $$
But then, for each $s \in \{1, \ldots, r\}$,

\[
\left| \left\langle \left( \int f \, dE_T - \int g \, dE_T \right) h_s, w_s \right\rangle \right| \\
= \left| \sum_{p,q} \int_X (f_{pq} - g_{pq}) \, dE^{(h_s,w_s)}_{qp} \right| \\
\leq \sum_{p,q} \int_X |f_{pq} - g_{pq}| \, d|E^{(h_s,w_s)}_{qp}| \\
\leq \int_X \|f(x) - g(x)\| \, d\mu(x) \\
\leq \varepsilon,
\]

and we are done (since $f(T) = \int f \, dE_T$ and $g(T) = \int g \, dE_T$).

We turn to (b). It is clear that $f \in \mathfrak{B}C^*(X,\cdot)$. Replacing $f^{(m)}$ by $f^{(m)} - f$, we may assume $f = 0$. Observe that, then,

\[
\lim_{m \to \infty} \left( (f^{(m)})^* f^{(m)} \right)_{pq}(x) = 0
\]

for any $x \in X$ and $p, q \in \{1, \ldots, n\}$, and the functions $((f^{(1)})^* f^{(1)})_{pq}$, $((f^{(2)})^* f^{(2)})_{pq}, \ldots$ are uniformly bounded. Therefore (by Lebesgue’s dominated convergence theorem), for any $h \in H$,

\[
\|((f^{(m)}(T))h\|^2 = \left\langle (f^{(m)}(T))^* f^{(m)}(T) h, h \right\rangle \\
= \left\langle \left( \int (f^{(m)})^* f^{(m)} \, dE_T \right) h, h \right\rangle \\
= \sum_{p,q} \int_X ((f^{(m)})^* f^{(m)})_{pq} \, dE^{(h,h)}_{qp} \to 0 \quad (m \to \infty),
\]

which finishes the proof. \hfill \Box

We end the paper with the note that the above result enables defining the extended $n$-functional calculus for $n$-homogeneous systems in $W^*$-algebras.

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