Breit Hamiltonian and QED Effects for Spinless Particles

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Abstract We describe a simplified derivation for the relativistic corrections of order $\alpha^4$ for a bound system consisting of two spinless particles. We devote special attention to pionium, the bound system of two oppositely charged pions. The leading quantum electrodynamics (QED) correction to the energy levels is of the order of $\alpha^3$ and due to electronic vacuum polarization. We analyze further corrections due to the self-energy of the pions, and due to recoil effects, and we give a complete result for the scalar-QED leading logarithmic corrections which are due to virtual loops involving only the scalar constituent particles (the pions); these corrections are of order $\alpha^5 \ln \alpha$ for S states.

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1 Introduction

Exotic bound systems like pionium \cite{1,2} (the bound system of two oppositely charged pions) offer interesting possibilities for studies of fundamental properties of quantum mechanical bound states: the interplay between strong-interaction corrections and quantum electrodynamic corrections is of prime interest, and the small length scales characteristic of the heavy particles make it possible to explore effects of the virtual excitations of the quantum fields in previously unexplored kinematical regimes \cite{3–5}. We do not wish to hide the fact that any potential high-precision experiments in this area are faced with various experimental difficulties. Our calculations address QED corrections to the spectrum of bound systems whose constituent particles are spinless; relativistic corrections to the decay lifetime of pionium have recently been discussed in \cite{6} in the context of the D"{I}RAC experiment at CERN.

Here, we report on results regarding the spectrum of a bound system consisting of two spinless particles. We apply the simplified calculational scheme employed in \cite{7} for the relativistic and recoil corrections to a bound systems of two “non-Dirac” particles to the case of two interacting spinless particles (see Sec. 2). We then recall known results on leading-order vacuum polarization corrections in Sec. 3 and clarify the relative order-of-magnitude of the one- and two-loop electronic vacuum polarization, the relativistic and recoil corrections and the self-energy effects in pionium (also in Sec. 3). We then provide an estimate for the self-energy effect in Sec. 4, and we analyze the leading recoil correction of order $\alpha^5$ (the Salpeter correction) which leads us to complete results for the scalar-QED logarithmic corrections of order $\alpha^5 \ln \alpha$.

2 Breit Hamiltonian for Spinless Particles

We start from the Lagrangian for a charged spinless field coupled to the electromagnetic field [see equations (6-50) – (6-51b) of \cite{8}],

$$\mathcal{L}(x) = [(\partial_\mu - i e A_\mu) \phi^*(x)] (\partial^\mu + i e A^\mu) \phi(x) - m^2 \phi^*(x) \phi(x) - \frac{1}{4} F_{\mu \nu}(x) F^{\mu \nu}(x),$$

(1)

where the field strength tensor $F_{\mu \nu}$ reads $F_{\mu \nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$. We use natural Gaussian units with $\hbar = c = \epsilon_0 = 1$. The transition current for a free spinless particle ($A^\mu = 0$) can be inferred from (1); it reads in momentum space

$$j^\mu(p', p) = \phi^*(p') (p'^\mu + p^\mu) \phi(p).$$

(2)

This current now has to be expressed in terms of nonrelativistic wave functions. Specifically, the $j^0$-component has to reproduce the normalization of the nonrelativistic (Schrödinger) wave function. By contrast, according to Eq. (3) the zero-component of the current reads $2m \phi^* \phi$ in the nonrelativistic limit $p^0 \to m, p^0 \to m$. The nonrelativistic wave functions are normalized according to

$$\int d^3x \phi_S^*(x) \phi_S(x) = 1.$$ 

(3)

It is therefore evident that we cannot simply associate the relativistic wave function $\phi$ with $\phi_S$; rather, we should define according to Eqs. (13) – (14) of \cite{7}

$$\phi(p) = \frac{\phi_S(p^0, p)}{\sqrt{2p^0}},$$

(4)

where $p^0 = \sqrt{p^2 + m^2} \approx m$ is the energy of the free nonrelativistic particle (in deriving low-energy effective interactions, one always expands about free-particle amplitudes; all interactions
are treated as perturbations; note the analogy to nonrelativistic QED – NRQED – for spinor particles [2]. The Klein–Gordon current, in the presence of external fields, reads in contrast to (2)

\[ j^\mu(p', p) = \phi^*(p') \left( p'^\mu + p^\mu - 2 e A^\mu \right) \phi(p). \]

(5)

The zero-component of this current can be interpreted as a charge density, which is not necessarily positive definite. Questions related to the normalization of the Klein–Gordon wave functions in this case are discussed in detail in [10–13].

In terms of the Schrödinger wave function, the current is given as

\[ j^0(p', p) = \phi^*_S(p') \phi_S(p), \]

(6)

\[ j^i(p', p) = \phi^*_S(p') \frac{p'^i + p^i}{2m} \phi_S(p), \]

(7)

where \( m \) is the mass of the particle. The atomic momenta \( p^i \) and \( p'^i \) in Eq. (7) are of order \( Z\alpha \). As shown below, interactions involving the spatial components \( j^i \) of the transition current give rise to relativistic contributions of order \((Z\alpha)^4\) to the spectrum. This is exactly the order of magnitude that is the subject of the current investigation. Therefore, although Eq. (6) is only valid up to corrections of relative order \((Z\alpha)^2\), these can be neglected because the further corrections contribute to the energy levels at the order of \((Z\alpha)^6\). Specifically, we can expect corrections proportional to \((p^i p'^i)\) to the current \(j^i\) when a systematic expansion of the nonrelativistic current is performed; these terms are analogous to those obtained for relativistic corrections to the current of spinor particles which can be obtained via a Foldy–Wouthuysen transformation [14, 15].

In the following, the index \( S \) on the wave function will be dropped, and the nonrelativistic amplitudes describing the two interacting particles (with electric charges \( e_1 \) and \( e_2 \)) will be denoted as \( \phi_1 \) and \( \phi_2 \), respectively. Following [3], the Breit Hamiltonian \( U(p_1, p_2, q) \) in momentum space is related to the invariant scattering amplitude \( M \) and to the photon propagator \( D_{\mu\nu}(q) \) in the following way [see also equation (83,8) in [16]]:

\[ M = e_1 e_2 j^\mu_1(p'_1, p_1) D_{\mu\nu}(q) j^\nu_2(p'_2, p_2) \]

\[ = -\phi^*_1(p'_1) \phi^*_2(p'_2) \left[ \frac{e_1 e_2}{q^2} + U(p_1, p_2, q) \right] \phi_1(p_1) \phi_2(p_2) \]

(8)

where \( q = p'_2 - p_2 = -(p_1 - p'_1) \). We employ a Coulomb-gauge photon propagator,

\[ D_{00}(q) = -\frac{1}{q^2}, \quad D_{ij}(q) = -\frac{1}{q^2 - \omega^2} \left[ \delta^{ij} - \frac{q^i q^j}{q^2} \right], \]

(9)

where we can neglect the energy of the virtual photon for the derivation of next-to-leading order relativistic corrections,

\[ D_{ij}(q) \approx -\frac{1}{q^2} \left[ \delta^{ij} - \frac{q^i q^j}{q^2} \right]. \]

(10)

The invariant scattering amplitude \( M \) then reads

\[ \frac{M}{e_1 e_2} = -\phi^*_1(p'_1) \phi^*_2(p'_2) \frac{1}{q^2} \phi_1(p_1) \phi_2(p_2) \]

\[ + \phi^*_1(p'_1) \phi^*_2(p'_2) \left[ \frac{p'^1_1 + p'^1_2 + p'^2_2}{2 m_1} + \frac{p^1_1 + p^1_2 + p^2_2}{2 m_2} \frac{1}{q^2} \left[ \delta^{ij} - \frac{q^i q^j}{q^2} \right] \right] \phi_1(p_1) \phi_2(p_2). \]

(11)
We therefore identify

\[ U(p_1, p_2, q) = - \frac{e_1 e_2}{4m_1 m_2} \left( \frac{(2p_1 - q) \cdot (2p_2 + q)}{q^2} \right) \left[ \delta^i_j - \frac{q^i q^j}{q^2} \right] \]

\[ = - \frac{e_1 e_2}{4m_1 m_2} \left\{ \frac{(2p_1 - q) \cdot (2p_2 + q)}{q^2} - \frac{(2p_1 \cdot q - q^2)(2p_2 \cdot q + q^2)}{q^4} \right\}. \]

(12)

We now transform to the center-of-mass frame in which \( p_1 = -p_2 = p \), so that the expression for \( U(p_1, p_2, q) \) becomes even simpler,

\[ U(p, -p, q) = \frac{e_1 e_2}{m_1 m_2} \left[ \frac{p^2}{q^2} - \frac{(p \cdot q)^2}{q^4} \right]. \]

(13)

The formula (83,13) of [16] can now be employed in evaluating the Fourier transform,

\[ \int \frac{d^3q}{(2\pi)^3} \exp(iq \cdot r) \frac{4\pi(a \cdot q)(b \cdot q)}{q^4} = \frac{1}{2r} \left[ a \cdot b - \frac{(a \cdot r)(b \cdot r)}{r^2} \right]. \]

(14)

The Breit Hamiltonian, which we would like to denote by \( H_B \), is obtained by adding to the Fourier transform of \( \hat{H}_B \) the relativistic correction to the kinetic energy. Denoting with \( \hat{p} = -i \partial/\partial x \) the momentum operator in the coordinate-space representation, we obtain

\[ H_B(p, \hat{p}) = -\frac{\hat{p}^4}{8m_1} - \frac{\hat{p}^4}{8m_2} + \frac{e_1 e_2}{8\pi r m_1 m_2} \hat{p}^2 + \frac{e_1 e_2}{8\pi r^3} r \cdot (r \cdot \hat{p}) \hat{p}. \]

(15)

In the order of \((Z\alpha)^4\), there is no contribution due to virtual annihilation for spinless particles; corrections of this type would enter only for positronium and dimuonium because they are caused by the spin-dependent part of the transition current [see Eqs. (83,20) and (82,22) of [16]], which is absent for spinless particles. For S states, virtual annihilation is altogether prohibited by angular momentum conservation.

The matrix elements of the Breit Hamiltonian (15) for spinless particles can be evaluated on nonrelativistic bound states via computational techniques outlined in Sec. A3 of Ch. 1 of [17]. For \( m_1 = m_2 = m \) and \( e_1 e_2 = -4\pi Z\alpha \), we obtain

\[ E_B = -\frac{(Z\alpha)^2 m}{4n^2} - \frac{(Z\alpha)^4 m}{2n^2} \left[ \frac{1}{2l + 1} - \frac{1}{4} \delta_{l0} - \frac{11}{32n} \right]. \]

(16)

as the Breit energy for the energy levels of the bound system of two spinless particles, including relativistic corrections of order \((Z\alpha)^4\). Here, we keep \( Z \) as a parameter which denotes the nuclear charge number in a bound system. Of course, for two particles each carrying an elementary charge, \( Z \) has to be set to unity. The fine-structure constant is denoted by \( \alpha \). The result (16) agrees with previous calculations [17],[18],[22], notably with Eq. (38) of [22].

It is instructive to compare the result (16) with the known result for a single-particle system of mass \( m/2 \) satisfying the Klein–Gordon equation, bound to a nucleus with charge \( Ze \). According to Eq. (2-86) of [8], we obtain the “Klein–Gordon energy” (KG)

\[ E_{KG} = -\frac{(Z\alpha)^2 m}{4n^2} - \frac{(Z\alpha)^4 m}{2n^2} \left[ \frac{1}{2l + 1} - \frac{3}{8n} \right]. \]

(17)
The two results \((16)\) and \((17)\) are manifestly different in the order of \((Z\alpha)^4\).

From \((15)\) we conclude that the zitterbewegung term is absent for spinless particles. However, this statement is in need of further explanation because a considerable variety of physical interpretations exists in the literature with regard to the zitterbewegung term. We briefly expand: The Dirac \(\alpha\)-matrices fulfill \(\alpha = i[H_D, x]\) \((H_D\) is the Dirac hamiltonian\) as the relativistic generalization of the velocity operator. By contrast, in the nonrelativistic formalism, we have the analogous relation \(p/m = i[H_S, x]\) where \(H_S\) is the Schrödinger Hamiltonian. Since the \(\alpha\)-matrices have eigenvalues \(\pm 1\), the magnitude of the velocity of the electron – at face value – is equal to the velocity of light at any given instant. On p. 106 of [23], it is argued that “the explanation for this fact is that the electron carries out a fast irregular motion (“zitterbewegung”) – which is responsible for the spin – whereas the mean velocity is given by the momentum \(p/m\).

Note that the introduction of the Dirac matrix formalism is necessitated by the need to describe the internal degrees of freedom of the particle – the spin. On p. 71 of [8], it is shown that the zitterbewegung term can be traced to the positional fluctuations \(\langle \delta r^2 \rangle \sim 1/m^2\) of the electron, and a connection is drawn to the Darwin term which results naturally in the context of the Foldy–Wouthuysen transformed Dirac hamiltonian. On pp. 117–118 of [24] and p. 62 of [8], it is argued that the momentum \(p\) of a Dirac wave packet can be associated in a natural way with the group velocity, but that in addition to the group velocity term, there exist highly oscillatory terms which represent the zitterbewegung. Similarly, on pp. 139–140 of [24], it is shown that the zitterbewegung term can also be interpreted as arising from the interaction of the atomic electron with virtual electron-positron pairs created in the Coulomb field of the nucleus. This virtual electron-positron pair-creation is subject to the uncertainty principle and can occur only for time intervals of the order of \(\Delta t \sim \hbar/(2mc^2)\) \((\text{where we temporarily restore the factor } \hbar)\).

At the time the original atomic electron fills up the vacated negative-energy state \((\text{the bound-electron wave-function has negative-energy components})\), the escalated electron \((\text{which forms part of the virtual pair})\) is at most a distance \(c\Delta t \sim 1/m\) away from the original electron. This distance is precisely of the order of magnitude of the fluctuations of the electron coordinate and consistent with the discussion on p. 71 of [8]. All these interpretations elucidate different aspects of the same problem.

In the context of the Breit hamiltonian, we would like to adhere to the definition that the zitterbewegung term is the term of order \((Z\alpha)^4\) in the Breit Hamiltonian generated by a contribution which is manifestly proportional \(\delta(r)\) in coordinate space \((\text{or a constant in momentum space})\).

Such a term is absent in the result \((15)\). For spin-1/2 particles, such a term is generated by the multiplication of the photon propagator \((\text{proportional to } 1/q^2)\) with the zero-component of the transition current which is given for a spin-1/2 particle as \([\text{see equation (4) of [7]}]\)

\[
\bar{u}' \gamma^0 u = w^* \left( 1 - \frac{q^2}{8m^2} + \frac{i \sigma \cdot p' \times p}{4m^2} \right) w.
\]

Here, \(u\) is the bispinor amplitude for the bound particle, and \(w\) is the bound-state Schrödinger wave function related by

\[
u = \left( \begin{array}{c} 1 - \frac{p^2}{8m^2} \\sigma \cdot p \end{array} \right) \frac{w}{2m \sqrt{w}} \]

according to equation (3) of [7]. One might wonder why a term proportional to \(\delta_{00}\) \((\text{apparently generated by a } \delta\text{-function in coordinate space})\) prevails in the Breit energy \((16)\). This term arises naturally when evaluating a matrix element of the structure \(\langle \phi_S | (r \cdot (r \cdot \hat{p})) \hat{p} / r^3 | \phi_S \rangle\) \((\text{last term of equation (15)})\) on the nonrelativistic wave function \(\phi_S\) and should not be associated with the zitterbewegung.
Vacuum polarization corrections and self-energy effects, as well as corrections due to the strong interaction, are not included in (16). These corrections will be discussed in the two following sections.

3 Vacuum Polarization Effects

As pointed out by various authors (e.g. [25-31]), the electronic vacuum polarization enters already at the order of $\alpha^3$ [more precisely, $\alpha (Z\alpha)^2$] in bound systems with spinless particles, because the spinless particles are much heavier than the electron, which means that the Bohr radius of the bound system is roughly of the same order of magnitude as the Compton wavelength of the electron. The Compton wavelength of the electron, however, is the fundamental length scale at which the charge of any bound particle is screened by the electronic vacuum polarization.

The vacuum polarization (VP) correction to energy levels has been evaluated [5,25,27-30] with nonrelativistic wave functions. We recall that the leading-order VP correction (due to the Uehling potential) can be expressed as

$$\Delta E = \langle \psi | V_U | \psi \rangle = \frac{\alpha}{\pi} C_E E_\psi,$$

where

$$E_\psi = -\frac{(Z\alpha)^2 m}{4n^2}$$

is the Schrödinger binding energy for a two-body system with two particles each of mass $m$ [first term on the right-hand side of (16)]. For the $C_E$ coefficients, we recall the following known results [5,27,28]:

$$C_E(1S) = 0.22, \quad C_E(2S) = 0.10.$$  (22)

As an alternative to the nonrelativistic treatment, the Uehling correction could also be evaluated with relativistic bound-state Klein–Gordon wave functions (although these do not describe the two-body system accurately, as shown in Sec. 2). The difference could be interpreted as a rough estimate of further relativistic corrections not taken into account in the nonrelativistic treatment of the vacuum polarization. The result obtained by numerically solving the Klein–Gordon equation and numerically evaluating the Uehling correction is shown in the sixth row of Tab. 1. This relativistic result is in very good agreement with Eq. (22). As pointed out in [32], the strong interaction correction is also an $\alpha^3$ effect (like the vacuum polarization) and enters at a relative order of $\alpha$, i.e. on the level of about 1% in pionium.

We recall here also the known results for the vacuum polarization correction to the charge density at the origin [23,31]:

$$\left[ \frac{\Delta |\psi_{1S}(0)|^2}{|\psi_{1S}(0)|^2} \right]_{\pi^+\pi^-} = \frac{\alpha}{\pi} 1.36 \quad \text{and} \quad \left[ \frac{\Delta |\psi_{2S}(0)|^2}{|\psi_{2S}(0)|^2} \right]_{\pi^+\pi^-} = \frac{\alpha}{\pi} 1.14.$$  (23)

Two-loop vacuum polarization effects enter at a relative order $\alpha^2$ in pionium and are therefore of the same order of magnitude as the relativistic corrections mediated by the Breit interaction (discussed in Sec. 2 and termed “two-body correction” in Tab. 1). The self-energy correction which is discussed in the following section is even smaller, but of considerable theoretical interest.

4 Effects due to Scalar QED

As shown in [33], the leading logarithmic correction to the self energy can be obtained, in nonrelativistic approximation, from second-order perturbation theory based on nonrelativistic
quantum electrodynamics [33] (see also [34]). We will investigate here, in a systematic way, the leading logarithms generated for S states by self-energy and relativistic-recoil effects (the so-called Salpeter correction), and show that these are spin-independent.

The quantized electromagnetic field is [see Eq. (5) of [33]],

\[
A(r) = \sum_{\lambda=1,2} \int \frac{d^3k}{\sqrt{(2\pi)^3 2k}} \epsilon_\lambda(k) \left[ a_{k,\lambda}^+ \exp(-i k \cdot r) + a_{k,\lambda} \exp(i k \cdot r) \right],
\]

(24)

and the nonrelativistic interaction Hamiltonian for an atomic system with two spinless particles (charges \(e_1\) and \(e_2\) and masses \(m_1\) and \(m_2\)) reads

\[
H_1 = -\frac{e_1}{m_1} p_1 \cdot A(r_1) + \frac{e_2}{2m_1} A(r_1)^2 - \frac{e_2}{m_2} p_2 \cdot A(r_2) + \frac{e_2^2}{2m_2} A(r_2)^2.
\]

(25)

For two spin-1/2 particles, the terms

\[
-\frac{e_1}{m_1} \sigma_1 \cdot B(r_1) - \frac{e_2}{m_2} \sigma_2 \cdot B(r_2)
\]

(26)

have to be added to \(H_1\) [see Eq. (7) of [33]]. We will carry out the calculations for the general case of one particle of charge \(e_1 = e\) and the other having a charge \(e_2 = -Ze\) (we follow the convention of [8] that for hydrogen, \(e\) is the physical charge of the electron, i.e. \(e = -|e|\)). The unperturbed Hamiltonian of the system of the two particles and the electromagnetic field reads [see e.g. Eq. (6) of [33]],

\[
H_0 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{Z\alpha}{r} + \sum_{\lambda=1,2} \int d^3k k a_{k,\lambda}^+ a_{k,\lambda}.
\]

(27)

where \(r = r_1 - r_2\). The eigenstates of the “atomic part” \(H_0^A\) of this Hamiltonian in the center-of-mass system \(p_1 + p_2 = 0\) are the nonrelativistic Schrödinger–Coulomb wave functions for a reduced mass \(m_r = m_1 m_2/(m_1 + m_2)\) [here, the “atomic part” \(H_0^A\) excludes the photon field, i.e. the last term of (27)]. We denote \(p \equiv p_1 = -p_2\).

Given that the first-order perturbation \(\langle \phi_S | H_1 | \phi_S \rangle\) vanishes, the second-order perturbation yields the dominant nonvanishing perturbation. When evaluated on an atomic state, it is given by

\[
\delta E_{SE} = \langle \phi_S | H_1 \frac{1}{H_0 - E_S} H_1 | \phi_S \rangle.
\]

(28)

The interaction Hamiltonian (23) gives rise to QED corrections that involve both particles (in the current context, these are recoil corrections involving the product \(e_1 e_2\), and also to terms which involve only a single particle and are proportional to \(e_1^2\) or \(e_2^2\). The latter effects correspond to the self-energies of the two particles.

The low-energy part of the self-energy in leading order [33] can be inferred directly from (28), and it can be seen that the spin-dependent parts from (26) vanish in the leading order in the \((Z\alpha)\)-expansion [33]:

\[
E_L = -\frac{e_1^2}{6\pi^2} \int_0^\infty dk \, k \left( \langle \phi | \frac{p}{m_1} \frac{1}{H_0^A - (E_S - k)} \frac{p}{m_1} | \phi \rangle + (e_1 \leftrightarrow e_2, m_1 \leftrightarrow m_2) \right).
\]

(29)

where

\[
E_S = -\frac{(Z\alpha)^2 m_r}{2n^2}
\]

(30)
is the Schrödinger energy \((m_r\ln n^3)\) in the reduced mass of the atomic system under investigation. Starting from the \textit{spin-independent} expression \((29)\), it is now relatively straightforward to show that the leading “self-energy logarithm” for S states is given by

\[
\delta E_{SE} \approx \frac{4 \ln(Z\alpha)^{-2}}{3 \pi n^3} \delta_{10} \left[ \alpha (Z\alpha)^4 \frac{m_1^3}{m_1^2} + Z (Z\alpha)^5 \frac{m_2^3}{m_2^2} \right]. \tag{31}
\]

This result is by consequence spin-independent. The derivation is simplified when using the \(\epsilon\)-method developed and used in various bound-state calculations \([14, 35, 36]\). The two terms in square brackets in \((31)\) correspond to the two self-energies of the two constituent particles with charges \(e_1 = e\) and \(e_2 = -Ze\) and masses \(m_1\) and \(m_2\), respectively. It has been pointed out \([17]\) that in contrast to the self-energy corrections, the vacuum polarization corrections given in Eq. \((23)\) must not be double-counted. The “double-counting” of self-energy corrections (and lack of it in the vacuum-polarization case) finds a natural explanation in our formalism: whereas the vacuum-polarization correction mainly leads to a modification of the \(1/r\)-type Coulomb attraction in \((27)\) within a nonrelativistic effective theory, the structure of the interaction Hamiltonian \((25)\) implies the existence of the two self-energies of the two constituent particles of the atomic system.

It might be instructive to point out that the formula \((31)\) is consistent with Welton’s argument for estimating the self-energy effect on a bound particle which is based on analyzing the influence of the fluctuating electromagnetic field [a detailed discussion is given on pp. 80–82 of \([8]\)]. For a system with two particles of equal mass \(m_1 = m_2 = m\), we have \(m_r = m/2\).

The leading-order recoil correction (Salpeter correction) can also be inferred from the interaction Hamiltonian \((25)\) via second-order perturbation theory, by “picking up” terms that involve products \(e_1 e_2\). It has been shown in \([38]\) that the leading \textit{logarithm} (for S states) of the Salpeter correction is spin-independent (just like the leading logarithm of the self-energy correction). The Salpeter correction is usually referred to as a relativistic recoil (RR) correction. By following \([33]\), we obtain for the leading logarithm of this effect

\[
\delta E_{RR} \approx \frac{2 (Z\alpha)^5}{3 \pi n^3} \delta_{10} \ln \left( \frac{1}{Z\alpha} \right) \frac{m_1^3}{m_1 m_2}. \tag{32}
\]

This correction involves only the products \(e_1 e_2 = -4\pi Z\alpha\) and can therefore be written as a function of \(Z\alpha\) alone.

For pionium, we have \(Z = 1\), \(m_1 = m_2 = m = m_\pi\), \(m_r = m_\pi/2\). The leading logarithmic correction from scalar QED for pionium in the order of \(\alpha^5 \ln\alpha\) is obtained by adding the corrections \((31)\) and \((32)\),

\[
\delta E_{log} = \delta E_{SE} + \delta E_{RR} = \frac{3}{4} \frac{\alpha^5}{\pi n^3} \ln \left( \frac{1}{\alpha} \right) m_\pi. \tag{33}
\]

The non-logarithmic term of order \(\alpha^5\) is spin-dependent, and its evaluation requires a relativistic treatment of the self-energy effect of a bound spinless particle; such a calculation would be of considerable theoretical interest, but the size of the effect for pionium, which is roughly two orders of \(\alpha\) smaller than the leading vacuum polarization correction, precludes experimental verification in the near future. However, we would like to point out here that a fully relativistic treatment of this problem, including a detailed discussion of the renormalization of the self energy of the spinless particle, has not yet been accomplished. Scalar QED is a renormalizable theory \([8]\).

The dominance of vacuum polarization over self-energy effects in pionium is expressed, in particular, by the fact that even \textit{two-loop} vacuum polarization of order \(\alpha^4\) has a stronger effect on the spectrum of pionium than the leading logarithm from Eq. \((33)\), and that the strong-interaction
correction of order $\alpha^3$ [32] has to be well understood before any experimental verification of (33) appears feasible. Finally, we remark that for a manifestly non-elementary particle like the pion which has a finite charge radius, form-factor corrections have to be taken into account.

Table 1: QED contributions to the 1S level of pionium in eV. For a further discussion of the corrections see the text.

| Contribution                                    | Energy (eV) |
|------------------------------------------------|-------------|
| One–body Klein–Gordon [Eq. (17)]               | -1858.19895|
| Higher Order Klein–Gordon [Exact – Eq. (17)]   | -0.00001    |
| Form factor correction to Klein–Gordon (0.61 fm)| 0.01308     |
| Two-body correction (Breit) [Eq. (16) – Eq. (17)]| 0.04329     |
| Uehling (with a relativistic wave function)     | -0.94228    |
| Vac. Pol. (Wichmann-Kroll)                      | 0.00001     |
| Vac. Pol. (Källén-Sabry)                        | -0.00729    |
| Vac. Pol. (iterated Uehling)                    | -0.00113    |
| Self–Energy [Eq. (31)]                          | 0.00302     |
| Salpeter correction [Eq. (33)]                  | 0.00038     |
| **Total**                                       | **-1859.08986** |

5 Numerical evaluation of QED corrections

In order to provide a more complete picture of pionium we have evaluated numerically a number of QED corrections to the 1S level of pionium. We explicitly exclude QCD corrections whose evaluation represents a difficult separate problem [32]. We proceed as follows (see the sequence of the rows of Tab. [1]):

- We start from the one-body Klein–Gordon energy for a particle of (reduced) mass $m/2$, given in Eq. (17), including relativistic corrections of order $(Z\alpha)^4$.

- We use a Klein-Gordon equation numerical solver developed for pionic atoms which was developed originally for the precise evaluation of vacuum polarization corrections [35,36] in order to supplement the (almost negligible) difference between the exact one-body relativistic Klein–Gordon energy and the $(Z\alpha)^4$-result from Eq. (17), thereby confirming the expression (17) for the relativistic correction.

- The effect due to the pion Coulomb form factor is also included in an approximative framework by replacing the numerical solution of the Klein–Gordon equation the Coulomb potential by the interaction potential of two uniformly charged spheres of mean-square radius $R_{\text{rms}} = 0.61 \text{ fm}$. For the calculation, we employ the radius and the pion mass from the particle data group [40] (all other physical constants used in the evaluations come from the 1998 adjustment [41]).

- We add as a “two-body correction” the difference of the results from Eq. (16) for the energy of the relativistic two-body system and the one-body result given in Eq. (17).
• The Uehling potential is evaluated in a relativistic framework, using the relativistic numerical equation solvers \cite{38,39}. The result is in very good agreement with the nonrelativistic treatment discussed in Sec. 3 (this is natural because $Z = 1$).

• The higher-order VP corrections attributed to Wichmann–Kroll \cite{42} and Källén–Sabry \cite{43} are supplemented, as well as an evaluation of the iterated loop-after-loop Uehling contribution to all orders in $\alpha$. The Wichmann–Kroll correction \cite{42} is here negligible because $Z = 1$. VP potentials given in Ref. \cite{44} and analytic expressions from \cite{45} are used. More details about the numerical procedure to evaluate these corrections can be found in Ref. \cite{46}.

• Finally, the effects due to the scalar self-energy given in Eq. (31) and the relativistic recoil (Salpeter) correction listed in Eq. (33) are added.

The main observation one can draw from Tab. 1 is that the vacuum-polarization is 300 times larger than the self-energy. Even the Källén and Sabry correction, while of order $\alpha^2$ is about twice as large as the self-energy correction. The iterated Uehling correction, while dominated by terms of order $\alpha^2$ is dominated in turn by the scalar self-energy. Nevertheless, we stress here that the numbers contained in Tab. 1 will be modified when the Klein–Gordon equation is solved with the strong interaction potential incorporated directly into the equation solver. Analogously, one cannot avoid having to solve the Dirac equation exactly for high-$Z$ systems where the electron wave function rests significantly inside the nucleus \cite{47}. At present, in view of the formidable experimental difficulties associated with a study of the atomic spectrum of pionium, we give the numbers in Tab. 1 as an indication of the relative size and order-of-magnitude of the specific QED corrections.

6 Conclusion

We have presented in Sec. 2 a simplified derivation for the relativistic and recoil corrections of order $\alpha^4$ to a bound state of two spinless particles. The results agree with previous calculations \cite{20}. As evident from equation (15) and discussed in further detail in Sec. 3, the zitterbewegung term is absent in a bound system of two spinless particles.

The self-energy effect is suppressed in systems with spinless particles in comparison to the vacuum polarization effect as discussed in Secs. 3 and 4, because the lightest known spinless particle is much heavier than the electron, which implies that the electronic vacuum polarization effect is larger by two orders of $\alpha$ than the self-energy effect in bound systems of spinless particles.

We provide a complete result for the leading scalar-QED correction in pionium of order $\alpha^5 \ln \alpha$ in Sec. 4. A list of further QED corrections to the 1S level of pionium is presented in Tab. 1.

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