Non-Abelian hierarchies of compatible maps, associated integrable difference systems and Yang-Baxter maps

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Abstract. We present two non-equivalent families of hierarchies of non-Abelian compatible maps and we provide their Lax pair formulation. These maps are associated with families of hierarchies of non-Abelian Yang-Baxter maps, which we provide explicitly. In addition, these hierarchies correspond to integrable difference systems with variables defined on edges of an elementary cell of the \(\mathbb{Z}^2\) graph, that in turn lead to hierarchies of difference systems with variables defined on vertices of the same cell. In that respect we obtain the non-Abelian lattice-modified Gel’fand-Dikii hierarchy, together with the explicit form of a non-Abelian hierarchy that we refer to as the lattice-NQC (or lattice-(\(Q3_0\)) Gel’fand-Dikii hierarchy.

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1. Introduction

Unlike the scalar case, contributions on integrable multi-component and integrable difference systems defined on higher order stencils, are rather sparse \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\]. Furthermore, unlike the continuous setting, contributions on non-Abelian generalizations and extensions of integrable difference systems are rather scattered in the literature \[17, 18, 19, 20, 21, 22\] and moreover quite rare when they concern hierarchies \[23, 24, 25, 26, 27\]. The results of this paper serve in the renewed and growing interest in deriving and extending integrable difference systems and structures to the non-Abelian domain \[23, 28, 29, 30, 31, 32, 33, 34, 35\]. The term non-Abelian, refers to the requirement that the multiplication operator is no longer Abelian. In that respect, the variables that participate on difference systems do not a-priori mutually commute.
Specifically, in this article we introduce non-Abelian hierarchies of integrable difference systems in edge and in vertex variables. In detail, we derive two $3D-$compatible non-Abelian hierarchies of maps that we refer to as the $\mathcal{K}^{(i)}$, $i = 0, 1$ hierarchies. We prove their multidimensional compatibility and we implicitly provide the companion maps that constitute hierarchies of Yang-Baxter maps. These companion hierarchies are the ones associated with the so-called $\mathcal{K}_I$, and $\Lambda_I$ Yang-Baxter maps introduced in [33]. In the Abelian setting, $\mathcal{K}_I$ and $\Lambda_I$ maps are equivalent to the so-called Harrison map, a.k.a $H_I$ in [38], so in that respect and in the Abelian restriction, our results provide the hierarchy of the $H_I$ Yang-Baxter map. In addition, we show that both $\mathcal{K}^{(i)}$ hierarchies arise through Lax pair formulation by deforming known Lax matrices. Moreover, these hierarchies serve as deformations of the edge-variable avatars of the non-Abelian lattice Gel’fand-Dikii hierarchies introduced in [23]. In the Abelian case lattice Gel’fand-Dikii hierarchies in vertex and edge form were introduced [39, 40], furthermore, modifications and extensions of their lower order members as well as the hierarchies themselves can be found in [3, 4, 5, 6, 11, 12, 13, 14, 15]. On top of that, it is shown here that both $\mathcal{K}^{(i)}$ hierarchies could also be obtained through periodic reductions of deformed versions of the non-Abelian Hirota-Miwa system. Note that the non-Abelian Hirota-Miwa system was firstly introduced in [22], in the Abelian setting it was earlier introduced in [45].

Each one of the $\mathcal{K}^{(i)}$ hierarchies gives rise to two hierarchies of integrable vertex systems i.e. integrable difference systems with variables defined of the vertices of an elementary cell of the $\mathbb{Z}^2$ graph. It turns out that the vertex systems associated with $\mathcal{K}^{(0)}$ are point equivalent to the ones associated with the $\mathcal{K}^{(1)}$ hierarchy. So from $\mathcal{K}^{(0)}$ we obtain the non-Abelian lattice-modified Gel’fand-Dikii hierarchy as well as the explicit form of a hierarchy that we refer to as the non-Abelian lattice-NQC (or lattice-$(Q3)_0$) Gel’fand-Dikii hierarchy. The non-Abelian lattice-modified Gel’fand-Dikii hierarchy was introduced in [23] whereas in the Abelian case it was firstly implicitly provided in [39] and explicitly in [42]. The first member of the non-Abelian lattice-NQC Gel’fand-Dikii hierarchy is the so-called NQC integrable lattice equation that was firstly introduced in [46], cf. also [47]. Note that the NQC integrable lattice equation is gauge equivalent to the lattice equation that is referred as $(Q3)_0$ in [48]. The whole hierarchy in the Abelian case, was implicitly provided in [41], whereas its second member i.e. the Boussinesq analogue of $(Q3)_0$, was explicitly derived in [43].

We start this manuscript with a brief introduction. We continue in Section 2, where we present the basic notions and definitions used throughout this paper. In addition, we recall from the literature two Lax matrices that play a crucial role to this work. In Section 3, we deform these Lax matrices and we obtain two non-Abelian integrable hierarchies of $3D-$compatible maps, the $\mathcal{K}^{(i)}$, $i = 0, 1$ hierarchies. Furthermore, we prove their multidimensional compatibility and we provide implicitly their corresponding

‡ The Harrison map was derived in [36] and serves as the nonlinear superposition formula for the Bäcklund transformation of the Ernst equation [37] in general relativity.
Yang-Baxter maps. These hierarchies of maps, are naturally associated with non-Abelian hierarchies of integrable difference systems with variables defined on the edges of an elementary cell of the $\mathbb{Z}^2$ graph. We proceed with Section 4 where we associate with the $\mathcal{K}(0)$ hierarchy, two integrable hierarchies of non-Abelian difference systems defined on the vertices of an elementary cell of the $\mathbb{Z}^2$ graph. Thus we obtain explicitly the non-Abelian lattice-modified and the lattice-$(Q3)_0$ Gel'fand-Dikii hierarchies. Finally in Section 5 we present some ideas on further research. We conclude this article with Appendix A where we present non-Abelian forms of the lattice-potential KdV equation.

2. Notation, definitions and the Lax matrices $L^{(N,j)}$ and $M^{(N,j)},$

$j = 1, \ldots, N - 1, N \geq 2 \in \mathbb{N}$

Here we present the basic objects and definitions that will be considered in this paper. Firstly, let $S$ be any set. We proceed with the following definitions.

**Definition 1** The maps $R : S \times S \to S \times S$ and $\hat{R} : S \times S \to S \times S$ will be called equivalent if there exists a bijection $\kappa : S \to S$ such that $(\kappa \times \kappa) \circ R = \hat{R} \circ (\kappa \times \kappa)$.

**Definition 2** (3D-compatible/consistent maps [49]) Let $Q : S \times S \ni (x, y) \mapsto (u, v) = (f(x, y), g(x, y)) \in S \times S$, be a map and $Q_{ij} \ i \neq j \in \{1, 2, 3\}$, be the maps that act as $Q$ on the $i$–th and $j$–th factor of $S \times S \times S$ and as identity to the remaining factor. In detail we have

\[
Q_{12} : (x, y, z) \mapsto (\tilde{x}, \tilde{y}, z) = (f(x, y), g(x, y), z), \\
Q_{13} : (x, y, z) \mapsto (x, \tilde{y}, \tilde{z}) = (f(x, z), y, g(x, z)), \\
Q_{23} : (x, y, z) \mapsto (\tilde{x}, \tilde{y}, z) = (x, f(y, z), g(y, z)).
\]

The map $Q : S \times S \to S \times S$ will be called 3D-compatible or 3D-consistent map if it holds $\tilde{x} = \hat{x}, \tilde{y} = \hat{y}, \tilde{z} = \hat{z}$, that is

\[
f(\tilde{x}, \tilde{y}) = f(\hat{x}, \hat{y}), \quad \text{or} \quad f(f(x, y), f(y, z)) = f(f(x, y), g(y, z)), \quad (1)\\
g(\hat{x}, \hat{y}) = g(\tilde{y}, \tilde{z}), \quad \text{or} \quad g(f(x, z), f(y, z)) = g(f(x, y), g(x, z)), \quad (2)\\
g(\tilde{x}, \tilde{z}) = g(\hat{y}, \hat{z}), \quad \text{or} \quad g(f(x, y), g(y, z)) = g(g(x, y), g(x, z)). \quad (3)
\]

Relations (1)-(3) serve as the compatibility (consistency) relations of the map $Q$ on the cube (see Figure 1). Consequently, they imply compatibility on any 3 dimensional face of the $n$ dimensional cube $Q_n$. We refer to this property as multidimensional compatibility or equivalently as multidimensional consistency property.

**Definition 3** (Quadrirational maps and their companion maps [50, 49]) A map $R : S \times S \ni (x, y) \mapsto (u, v) \in S \times S$ will be called quadrirational, if both the map $R$ and the so-called companion map $cR : S \times S \ni (x, v) \mapsto (u, y) \in S \times S$, are birational maps.
An alternative notion that incorporates the 3D-compatibility property of a map, is the so-called Yang-Baxter property. The maps that satisfy the Yang-Baxter property will be called Yang-Baxter maps. Note that if a 3D-compatible map is quadrirational, its companion map is a Yang-Baxter map.

**Definition 4 (Yang-Baxter maps [51, 52])** A map \( R : S \times S \ni (x, y) \mapsto (u, v) = (s(x, y), t(x, y)) \in S \times S \), will be called a Yang-Baxter map if it satisfies

\[
R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12},
\]

where \( R_{ij} i \neq j \in \{1, 2, 3\} \), denotes the maps that act as \( R \) on the \( i \)-th and the \( j \)-th factor of \( S \times S \times S \), and as identity to the remaining factor.

Yang-Baxter maps serve as set-theoretical-solutions of the functional Yang-Baxter equation (4) and the first instances of such solutions appeared in [51, 52]. Note that the term Yang-Baxter maps was introduced in [53, 54].

Yang-Baxter property, as a compatibility property, uses another set of initial data on the cube than the 3D-compatibility property (see Figure 1). In that respect, when the maps \( R, Q \) are quadrirational and \( R = cQ \), the Yang-Baxter property is equivalent to the 3D-compatibility property. So, as soon as a Yang-Baxter map is quadrirational, its companion is 3D-compatible and vice versa, irrespectively of the underlying sets on which the map acts. Note that in general the companion map \( cQ \) of a 3D-compatible map \( Q \) has different functional form than \( Q \), so it cannot be a Yang-Baxter map at the same time. There exist though cases where the companion map \( cQ \) of a 3D-compatible map \( Q \) coincides (has the same functional form) with \( Q \), then the map \( Q \) shares both the Yang-Baxter and the 3D-compatibility property [49].

![Figure 1](image-url)

**Figure 1:** (a): Left hand side of the 3D-compatibility formulas (1), (3), that is

\[(x, y, z) \xrightarrow{Q_{12}} (\hat{x}, \hat{y}, z), (x, \hat{y}, z) \xrightarrow{Q_{21}} (x, \hat{y}, \hat{z}), (\hat{x}, y, \hat{z}) \xrightarrow{Q_{13}} (\hat{x}, y, \hat{z}).\]

(b): Right hand side of the Yang-Baxter equation, that is

\[(x, y, z) \xrightarrow{R_{23}} (x, y, \hat{z}), (x, \hat{y}, z) \xrightarrow{R_{13}} (x, \hat{y}, \hat{z}), (\hat{x}, \hat{y}, \hat{z}) \xrightarrow{R_{12}} (\hat{x}, \hat{y}, \hat{z}).\]

**Definition 5** A bijection \( \phi : S \to S \) will be called a symmetry of the map \( R : S \times S \to S \times S \), if \((\phi \times \phi) \circ R = R \circ (\phi \times \phi)\).
Definition 6 ([55, 56]) The matrix $L(x; \lambda)$ will be called the Lax matrix of the $3D$-compatible map $H : (x, y) \mapsto (u, v)$, if the relation

$$L(u; \lambda)L(y; \lambda) = L(v; \lambda)L(x; \lambda)$$

implies for all $\lambda$ that $H(x, y) = (u, v)$. $L(x; \lambda)$ will be called a strong Lax matrix, if mapping $H$ is implied uniquely from $\lambda$.

Next, we define the order $N$ lower and upper-triangular nilpotent matrices.

Definition 7 With $\nabla^k$, and $\Delta^k$, $k = 1, 2, \ldots, N - 1$, we respectively define the order $N$ lower-triangular and upper-triangular nilpotent matrices i.e.

$$(\nabla^k)_{ij} := \begin{cases} 0, & i \leq j \\ \delta_{i,j+k}, & i > j \end{cases} \quad (\Delta^k)_{ij} := \begin{cases} \delta_{i+N-k,j}, & i < j \\ 0, & i \geq j \end{cases}$$

Finally, we define the notion of Deformation matrices.

Definition 8 (Deformation matrix) Let $L(x; \lambda)$ be an order $N$ strong Lax matrix of the $3D$-compatible map $H : (x, y) \mapsto (u, v)$. Fix $k \in \mathbb{N}$ with $0 \leq k < N$. The matrix $D^k(x, \alpha)$, where $\alpha$ a collection of constants, will be called deformation matrix and the constants $\alpha$ deformation constants, if it holds:

(i) $D^k(x, 0) = \Delta^k + \nabla^k$, where $\Delta^k$, and $\nabla^k$ the order $N$ nilpotent matrices of Definition 7, and $0$ a collection of zeros,

(ii) $\hat{L}(x; \lambda) := D^k(x, \alpha)L(x; \lambda)$ serves as a strong Lax matrix for a family of maps $\hat{H}(\alpha) : (x, y) \mapsto (\hat{u}, \hat{v})$.

The Lax matrix $\hat{L}(x; \lambda)$ will be called deformed Lax matrix.

Remark 2.1 When $k = 0$, for the deformation matrix $D^0(x, \alpha)$, it holds that $D^0(x, 0) = I$, where $I$ the order $N$ identity matrix, and clearly there follows $\hat{H}(0) \equiv H$. The deformation matrix $D^0(x, \alpha)$ will be called diagonal deformation matrix. Note that a diagonal deformation matrix is not necessarily a diagonal matrix.

There is a natural correspondence of a map with a difference system defined on the edges of an elementary quad of the $\mathbb{Z}^2$ graph. Specifically, a map $R : S \times S \ni (x, y) \mapsto (u, v) \in S \times S$, can be considered as a difference system defined on the edges of an elementary quadrilateral of the $\mathbb{Z}^2$ graph by making the following identifications

$$x \equiv x_{m+1/2,n}, \quad y \equiv y_{m,n+1/2}, \quad u \equiv x_{m+1/2,n+1}, \quad v \equiv y_{m+1,n+1/2}, \quad m, n \in \mathbb{Z}. \quad (6)$$

Moreover, we can adopt the compendious notation (see Figure 2)

$$x_1 := x_{m+3/2,n}, \quad y_1 := y_{m+1,n+1/2}, \quad x_2 := x_{m+1/2,n+1} \equiv u, \quad y_2 := y_{m,n+3/2}, \quad \text{etc.}, \quad m, n \in \mathbb{Z}. $$
Note that $\mathbf{x}$ could be a collection of variables, so $\mathbf{x} = (x^{(1)}, \ldots, x^{(N)})$, $N \in \mathbb{N}$. In this case with $\mathbf{x}_2$ we denote $\mathbf{x}_2 = (x_2^{(1)}, \ldots, x_2^{(N)})$, that is all variables from the collection shifted in the second direction and similarly for $\mathbf{y}_1$. So, unless otherwise stated and for the rest of this article, we represent the components of a vector with superscripts inside parentheses. When the superscripts denote powers, we do not use parentheses.

Let $A$ be an associative algebra over a field $\mathbb{F}$, with multiplicative identity that we denote with $1$. Throughout this paper we consider $\mathcal{S} = A^\times \times \cdots \times A^\times$, $N \in \mathbb{N}$, where $A^\times$ denotes the subgroup of elements $w \in A$ having multiplicative inverse $w^{-1} \in A$, s.t. $ww^{-1} = w^{-1}w = 1$. In addition, with $C(A^\times)$ we denote the center of algebra $A^\times$ i.e. a commutative subalgebra of $A^\times$ consisting of invertible elements.

In this general setting, $A^\times$ could be a division ring for instance bi-quaternions. More generally, $A^\times$ could stand for the subgroup of invertible matrices of the algebra $A$ of $n \times n$ matrices.

2.1. The Lax matrices $L^{(N,j)}$, $M^{(N,j)}$, $j = 1, \ldots, N - 1$ \( N \geq 2 \in \mathbb{N} \) and integrable hierarchies of difference systems

The following Lax matrices of order $N$, have appeared in various occasions and in different context inside the theory of integrable systems, see for instance [57, 58, 1, 59]. One of these Lax matrices, in particular the Lax matrix that in what follows we denote as $L^{(N,1)}$, was firstly considered in [39] in connection with the lattice Gel’fand-Dikii hierarchy in the Abelian setting. After this seminal work, modifications of this Lax matrix led to various hierarchies of integrable difference systems in the Abelian [42, 44] and recently in the non-Abelian domain [23, 26]. The Lax matrix $L^{(N,1)}$ explicitly reads
and serves as a specific member (j=1) of the family of Lax matrices \( L^{(N,j)}(x; \lambda) := I_N + \nabla^{(j)}X + \lambda \Delta^{(j)}X \), \( j = 1, \ldots, N - 1 \), where \( I_N \) the order \( N \) identity matrix and \( X \) an order \( N \) diagonal matrix with entries \((X)_{i,i} = x^{(i)}\).

The compatibility conditions \( L^{(N,1)}(u; \lambda)L^{(N,1)}(y; \lambda) = L^{(N,1)}(v; \lambda)L^{(N,1)}(x; \lambda) \), read

\[
u^{(i)}y^{(i-1)} = v^{(i)}x^{(i-1)}, \quad u^{(i)} + y^{(i)} = v^{(i)} + x^{(i)}, \quad i = 1, 2, \ldots, N,
\]

and they assure that the the Lax matrix \( L^{(N,1)} \) serves as a strong Lax matrix for the hierarchy of maps

\[
\mathcal{G} : (x^{(1)}, \ldots, x^{(N)}, u^{(1)}, \ldots, u^{(N)}) \mapsto (u^{(1)}, \ldots, u^{(N)}, v^{(1)}, \ldots, v^{(N)}), \quad (8)
\]

\[
u^{(i)} = (x^{(i)} - y^{(i)})x^{(i-1)}(x^{(i-1)} - y^{(i-1)})^{-1}, \quad \nu^{(i)} = (y^{(i)} - x^{(i)})y^{(i-1)}(y^{(i-1)} - x^{(i-1)})^{-1}, \quad i = 1, 2, \ldots, N.
\]

This hierarchy in the non-Abelian setting was firstly considered in [26], and in vertex variables depending on a two-fold choice of potential functions, serves as the lattice-modified or the lattice-Schwarzian Gel’fand-Dikii hierarchy. Moreover its companion hierarchy of maps, defines the hierarchy of \( H^{(N)}_{1,1} \) Yang-Baxter maps. It is easy to show that \( \mathcal{G} \), has as symmetry the bijection

\[
\psi : (x^{(1)}, x^{(2)}, \ldots, x^{(N)}) \mapsto (x^{(N)}, x^{(1)}, \ldots, x^{(N-1)}).
\]

The family of Lax matrices \( M^{(N,j)}(x; \lambda) := X + \nabla^{(j)} + \lambda \Delta^{(j)} \), \( j = 1, \ldots, N - 1 \), can be considered as a dual family to the family of Lax matrices \( L^{(N,j)} \). For the detailed study, in the Abelian case, of such family of Lax matrices we refer to [44]. Here we consider one member of this family, namely, \( M^{(N,1)} \) that explicitly reads:

\[
M^{(N,1)}(x; \lambda) := X + \nabla^{(1)} + \lambda \Delta^{(1)} = \begin{pmatrix}
x^{(N)} & 0 & \cdots & 0 & \lambda \\
1 & x^{(1)} & 0 & \cdots & 0 \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & 0 & 1 & \cdots & x^{(N-1)}
\end{pmatrix}. \quad (9)
\]

The compatibility conditions \( M^{(N,1)}(u; \lambda)M^{(N,1)}(y; \lambda) = M^{(N,1)}(v; \lambda)M^{(N,1)}(x; \lambda) \), read

\[
u^{(i)}y^{(i)} = v^{(i)}x^{(i)}, \quad u^{(i)} + y^{(i-1)} = v^{(i)} + x^{(i-1)}, \quad i = 1, 2, \ldots, N,
\]
and they assure that the Lax matrix $M^{(N,1)}$ serves as a strong Lax matrix for the hierarchy of maps

$$D : (x^{(1)}, \ldots, x^{(N)}, y^{(1)}, \ldots, y^{(N)}) \mapsto (u^{(1)}, \ldots, u^{(N)}, v^{(1)}, \ldots, v^{(N)}),$$

(10)

$$u^{(i)} = (x^{(i-1)} - y^{(i-1)}) x^{(i)} (x^{(i)} - y^{(i)})^{-1},$$

$$v^{(i)} = (y^{(i-1)} - x^{(i-1)}) y^{(i)} (y^{(i)} - x^{(i)})^{-1}, \quad i = 1, 2, \ldots, N.$$ 

The non-Abelian hierarchy of maps $D$ was firstly considered in [23, 24], where it was also considered as a periodic reduction of the non-Abelian Hirota-Miwa system [22]. Furthermore, in [24] it was shown that the hierarchy of companion maps of $D$, defines the hierarchy of $H^{B}_{III}$ Yang-Baxter maps. In vertex variables, $D$ defines either the lattice-modified or the lattice-Schwarzian Gel’fand-Dikii hierarchy.

3. Deformed Lax matrices and integrable hierarchies of difference systems

Following Definition 8 and in particular Remark 2.1 here we are searching for families of diagonal deformation matrices for the Lax matrices $L^{(N,1)}$ and $M^{(N,1)}$, of the previous Section. In detail, we search for diagonal deformation matrices $D^{(0)}(x, \alpha, \beta)$, under the additional requirements, first, they are diagonal matrices of order $N$, second, their entries read

$$(D^{(0)}(x, \alpha, \beta))_{i,i} = (\alpha^{(i-1)} - \beta^{(i-1)} x^{(i-1)})^{-1}, \quad i = 1, \ldots, N, \quad \alpha^{(i)}, \beta^{(i)} \in C(\mathbb{A}^\times).$$

(11)

In order for $D^{(0)}(x, \alpha, \beta)$ to be a deformation matrix for the instance of the strong Lax matrix $M^{(N,1)}$, a necessary condition (see Definition 8) is that the deformed Lax matrix $\tilde{M}^{(N,1)}(x; \lambda) := D^{(0)}(x, \alpha, \beta) M^{(N,1)}(x; \lambda)$ remains a strong Lax matrix, that is it defines uniquely a family of maps. Clearly that is a strong requirement that will put some conditions on the the deformation parameters. Indeed, the Lax equation

$$\tilde{M}^{(N,1)}(u; \lambda) \tilde{M}^{(N,1)}(y; \lambda) = \tilde{M}^{(N,1)}(v; \lambda) \tilde{M}^{(N,1)}(x; \lambda),$$

provides the following three sets of equations, where each set consists of $N$ equations,

$$(\alpha^{(i)} - \beta^{(i)} u^{(i)})^{-1} (\alpha^{(i-1)} - \beta^{(i-1)} y^{(i-1)})^{-1} = (\alpha^{(i)} - \beta^{(i)} v^{(i)})^{-1} (\alpha^{(i-1)} - \beta^{(i-1)} x^{(i-1)})^{-1},$$

$$(\alpha^{(i)} - \beta^{(i)} u^{(i)})^{-1} u^{(i)} (\alpha^{(i)} - \beta^{(i)} y^{(i)})^{-1} y^{(i)} = (\alpha^{(i)} - \beta^{(i)} v^{(i)})^{-1} v^{(i)} (\alpha^{(i)} - \beta^{(i)} x^{(i)})^{-1} x^{(i)},$$

$$(\alpha^{(i)} - \beta^{(i)} u^{(i)})^{-1} (\alpha^{(i-1)} - \beta^{(i-1)} y^{(i-1)})^{-1} y^{(i-1)} + (\alpha^{(i)} - \beta^{(i)} y^{(i)})^{-1} u^{(i)} (\alpha^{(i)} - \beta^{(i)} x^{(i)})^{-1} x^{(i)} -$$

$$= (\alpha^{(i)} - \beta^{(i)} v^{(i)})^{-1} (\alpha^{(i-1)} - \beta^{(i-1)} x^{(i-1)})^{-1} x^{(i-1)} + (\alpha^{(i)} - \beta^{(i)} v^{(i)})^{-1} v^{(i)} (\alpha^{(i)} - \beta^{(i)} x^{(i)})^{-1},$$

and the superscript $i = 1, \ldots, N$, is considered modulo $N$. If we demand that the three sets of equations above are linearly dependent, we obtain that $\beta^{(1)} = \ldots = \beta^{(N)} = \beta$, so the diagonal matrices $D^{(0)}(x, \alpha, \beta)$, with entries $(D^{(0)}(x, \alpha, \beta))_{i,i} = (\alpha^{(i)} - \beta x^{(i-1)})^{-1}$ serves as a family of diagonal deformation matrices for the strong Lax matrix $M^{(N,1)}$ and
the deformed Lax equation implies as unique solution the following family of hierarchies of maps
\[
(x^{(1)}, \ldots, x^{(N)}, y^{(1)}, \ldots, y^{(N)}) \mapsto (u^{(1)}, \ldots, u^{(N)}, v^{(1)}, \ldots, v^{(N)}),
\]
where:
\[
u^{(i)} = (\alpha^{(i-1)} - \beta^{(i)} y^{(i-1)})^{-1} y^{(i-1)} x^{(i-1)} (x^{(i-1)} - y^{(i-1)})^{-1} (\alpha^{(i)} - \beta^{(i)} y^{(i)}) ,
\]
\[
u^{(i)} = (\alpha^{(i-1)} - \beta^{(i)} x^{(i-1)})^{-1} x^{(i-1)} (y^{(i-1)} - x^{(i-1)})^{-1} (\alpha^{(i)} - \beta^{(i)} x^{(i)}) ,
\]
\[i = 1, 2, \ldots, N.\]

Note that w.l.o.g. we could set \(\alpha^{(i)}, \beta \in \{0, 1\}, i = 1, \ldots, N\). The generic case corresponds to \(\alpha^{(1)} = \ldots = \alpha^{(N)} = \beta = 1\), while degenerate cases arise when we equalize to zero some of the \(\alpha^{(i)}\). Note that in our setting, the most degenerate case corresponds to set \(\beta = 0\), and then the family of hierarchies above, coincides with (10).

Working similarly, we find that the diagonal matrices \(\widetilde{D}^{(0)}(x, \alpha, \beta)\) with entries
\[
\left(\frac{\widetilde{D}^{(0)}(x, \alpha, \beta)}{\alpha - \beta^{(i-1)} x^{(i-1)}}\right)_{i,i} = (\alpha - \beta^{(i-1)} x^{(i-1)})^{-1},
\]
serves as a family of diagonal deformation matrices for the strong Lax matrix \(L^{(N,1)}\). We denote the deformed Lax matrix as \(\tilde{L}^{(N,1)}(x; \lambda) := \tilde{D}^{(0)}(x, \alpha, \beta)L^{(N,1)}(x; \lambda)\) and the deformed Lax equation implies as unique solution the following family of hierarchies of maps
\[
(x^{(1)}, \ldots, x^{(N)}, y^{(1)}, \ldots, y^{(N)}) \mapsto (u^{(1)}, \ldots, u^{(N)}, v^{(1)}, \ldots, v^{(N)}),
\]
where:
\[
u^{(i)} = (\alpha - \beta^{(i)} y^{(i)})^{-1} y^{(i)} x^{(i-1)} (x^{(i-1)} - y^{(i-1)})^{-1} (\alpha - \beta^{(i)} y^{(i-1)}),
\]
\[
u^{(i)} = (\alpha - \beta^{(i)} x^{(i)})^{-1} x^{(i)} (y^{(i-1)} - x^{(i-1)})^{-1} (\alpha - \beta^{(i)} x^{(i-1)}),
\]
\[i = 1, 2, \ldots, N.\]

The generic case corresponds to \(\beta^{(1)} = \ldots = \beta^{(N)} = \alpha = 1\), while the most degenerate case corresponds to set \(\beta^{(1)} = \ldots = \beta^{(N)} = 0\), and then the family of hierarchies above, coincides with (8).

Note that for the generic members of both families of hierarchies presented above the deformation matrices coincide with the following matrix that we denote as \(D(x)\)
\[
D(x) := \begin{pmatrix}
(1 - x^{(N)})^{-1} & 0 & \cdots & 0 & 0 \\
0 & (1 - x^{(1)})^{-1} & 0 & \cdots & 0 \\
0 & 0 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & (1 - x^{(N-2)})^{-1} & 0 \\
0 & 0 & 0 & 0 & (1 - x^{(N-1)})^{-1}
\end{pmatrix} .
\]

A detailed analysis on the generic members of both families of hierarchies presented above, is presented in the forthcoming Section.
3.1. Two hierarchies of 3D-compatible maps

In the following propositions, we respectively present in detail the generic members of both families of hierarchies of 3D-compatible maps that correspond to the deformed Lax matrices \( \widehat{M}^{(N,1)} \) and \( \widehat{L}^{(N,1)} \), of the previous Section.

Under identification (9), 3D-compatible maps correspond to integrable difference systems with variables defined on the edges of an elementary cell of the \( \mathbb{Z}^2 \) graph. In that respect, in this Section although we refer to hierarchies of 3D-compatible maps, at the same time we refer to hierarchies of integrable difference systems with edge variables.

**Proposition 3.1** The hierarchy of maps

\[ \mathcal{K}^{(0)} : (x^{(1)}, \ldots, x^{(N)}, y^{(1)}, \ldots, y^{(N)}) \mapsto (u^{(1)}, \ldots, u^{(N)}, v^{(1)}, \ldots, v^{(N)}), \]

where

\[
\begin{align*}
u^{(i)} &= (1 - y^{(i-1)})^{-1} \left( y^{(i-1)} - (v^{(i)})^{-1} y^{(i)} \right) (x^{(i)} - y^{(i)})^{-1} \left( 1 - y^{(i)} \right), \\
u^{(i)} &= (1 - x^{(i-1)})^{-1} \left( y^{(i-1)} - x^{(i)} \right) y^{(i)} (y^{(i)} - x^{(i)})^{-1} \left( 1 - x^{(i)} \right),
\end{align*}
\]

\( i = 1, 2, \ldots, N, \)

(i) has as strong Lax matrix the matrix \( \widehat{M}^{(N,1)}(x; \lambda) := D(x) M^{(N,1)}(x; \lambda), \)

where \( M^{(N,1)}(x; \lambda) \) the Lax matrix (9) and \( D(x) \) the deformation matrix (14):

(ii) has as symmetry the bijection

\[ \psi : (x^{(1)}, x^{(2)}, \ldots, x^{(N)}) \mapsto (x^{(N)}, x^{(1)}, \ldots, x^{(N-1)}). \]

**Proof:** Explicitly the Lax matrix \( \widehat{M}^{(N,1)}(x; \lambda) \) reads

\[
\widehat{M}^{(N,1)}(x; \lambda) := \begin{pmatrix}
(1 - x(N))^{-1} x(N) & 0 & \cdots & 0 & \lambda (1 - x(N))^{-1} \\
(1 - x(1))^{-1} & 0 & \cdots & 0 & 0 \\
0 & (1 - x(2))^{-1} & \cdots & 0 & 0 \\
0 & \cdots & (1 - x(N-2))^{-1} & 0 & 0 \\
0 & \cdots & \cdots & (1 - x(N-1))^{-1} & 0 \\
0 & \cdots & \cdots & \cdots & (1 - x(N-1))^{-1} x(N-1)
\end{pmatrix},
\]

The compatibility conditions \( \widehat{M}^{(N,1)}(u; \lambda) \widehat{M}^{(N,1)}(y; \lambda) = \widehat{M}^{(N,1)}(v; \lambda) \widehat{M}^{(N,1)}(x; \lambda) \) read:

\[
\begin{align*}
(1 - u^{(i)})^{-1} (1 - y^{(i-1)})^{-1} &= (1 - v^{(i)})^{-1} (1 - x^{(i-1)})^{-1}, \\
(1 - u^{(i)})^{-1} u^{(i)} (1 - y^{(i)})^{-1} y^{(i)} &= (1 - v^{(i)})^{-1} v^{(i)} (1 - x^{(i)})^{-1} x^{(i)}, \\
(1 - u^{(i)})^{-1} (1 - y^{(i-1)})^{-1} y^{(i-1)} + (1 - u^{(i)})^{-1} u^{(i)} (1 - y^{(i)})^{-1} y^{(i)} &= (1 - v^{(i)})^{-1} (1 - x^{(i-1)})^{-1} x^{(i-1)} + (1 - v^{(i)})^{-1} v^{(i)} (1 - x^{(i)})^{-1} x^{(i)},
\end{align*}
\]

where the superscript \( i = 1, \ldots, N, \) is considered modulo \( N. \) The three sets of compatibility conditions presented above are not functionally independent since it can be easily shown that equations (15) can be obtained by adding (16) and (17). Then it is easy to verify that the solved form of (16), (17), is exactly (15). In addition, the compatibility conditions (16)-(18) are clearly invariant under the map \( \phi := \psi \times \psi, \) that proves that \( \psi \) is a symmetry of \( \mathcal{K}^{(0)} \).

\[ \square \]
Proposition 3.2 The hierarchy of maps

\[ \mathcal{K}^{(1)} : (x^{(1)}, \ldots, x^{(N)}, y^{(1)}, \ldots, y^{(N)}) \mapsto (u^{(1)}, \ldots, u^{(N)}, v^{(1)}, \ldots, v^{(N)}), \]

where

\[ u^{(i)} = (1 - y^{(i)})^{-1} (x^{(i)} - y^{(i)}) x^{(i-1)} (x^{(i-1)} - y^{(i-1)})^{-1} (1 - y^{(i-1)}), \]
\[ v^{(i)} = (1 - x^{(i)})^{-1} (y^{(i)} - x^{(i)}) y^{(i-1)} (y^{(i-1)} - x^{(i-1)})^{-1} (1 - x^{(i-1)}), \]

\[ i = 1, 2, \ldots, N, \]

(i) has as strong Lax matrix the matrix \( \tilde{L}^{(N,1)}(x; \lambda) := D(x) L^{(N,1)}(x; \lambda), \) where \( L^{(N,1)}(x; \lambda) \) the Lax matrix \([7]\) and \( D(x) \) the deformation matrix \([14]\);

(ii) has as symmetry the bijection

\[ \psi : (x^{(1)}, x^{(2)}, \ldots, x^{(N)}) \mapsto (x^{(N)}, x^{(1)}, \ldots, x^{(N-1)}). \]

(iii) The hierarchies \( \mathcal{K}^{(0)} \) and \( \mathcal{K}^{(1)} \) are not equivalent.

Proof: The compatibility conditions \( \tilde{L}^{N,1}(u; \lambda) \tilde{L}^{N,1}(y; \lambda) = \tilde{L}^{N,1}(v; \lambda), \) explicitly read:

\[ (1 - u^{(i)})^{-1} (1 - y^{(i)})^{-1} = (1 - v^{(i)})^{-1} (1 - x^{(i)})^{-1}, \]
\[ (1 - u^{(i)})^{-1} u^{(i)} (1 - y^{(i-1)})^{-1} y^{(i)} = (1 - v^{(i)})^{-1} v^{(i)} (1 - x^{(i-1)})^{-1} x^{(i)} - 1, \]
\[ (1 - u^{(i)})^{-1} (1 - y^{(i)})^{-1} y^{(i)} + (1 - u^{(i)})^{-1} u^{(i)} (1 - y^{(i-1)})^{-1} \]
\[ = (1 - v^{(i)})^{-1} (1 - x^{(i)})^{-1} x^{(i)} + (1 - v^{(i)})^{-1} v^{(i)} (1 - x^{(i-1)})^{-1} \]

where the superscript \( i = 1, \ldots, N \), is considered modulo \( N \). Similarly with the Proposition 3.1 the three sets of compatibility conditions presented above are not functionally independent since it can be easily shown that equations (22) can be obtained by adding (20) and (21). Then it is easy to verify that the solved form of (20), (21), is exactly (19). Furthermore, the compatibility conditions (20)-(22) are clearly invariant under the map \( \phi := \psi \times \psi \), that proves that \( \psi \) is a symmetry of \( \mathcal{K}^{(1)} \). Finally, the hierarchies \( \mathcal{K}^{(1)} \) and \( \mathcal{K}^{(0)} \) are related via the change of variables \( (u^{(i)}, x^{(i)}, v^{(i)}, y^{(i)}) \mapsto (u^{(i)}, x^{(i+1)}, v^{(i)}, y^{(i+1)}), \forall i \in \{1, \ldots, N\} \). Indeed, under this change of variables, \( \mathcal{K}^{(1)} \) reads

\[ u^{(i)} = (1 - y^{(i+1)})^{-1} (x^{(i+1)} - y^{(i+1)}) x^{(i)} (x^{(i)} - y^{(i)})^{-1} (1 - y^{(i)}), \]
\[ v^{(i)} = (1 - x^{(i+1)})^{-1} (y^{(i+1)} - x^{(i+1)}) y^{(i)} (y^{(i)} - x^{(i)})^{-1} (1 - x^{(i)}), \]

\[ i = 1, 2, \ldots, N, \]

that coincides with the opposite of the inverse hierarchy of maps \( \mathcal{K}^{(0)} \). Note that the opposite \( H^{opp} \) of a non-Abelian map \( H \), is defined as the map that is obtained from the map \( H \) when all multiplications are taken in opposite order.

Note also that there does not exists any bijection \( \kappa \) such that \( \mathcal{K}^{(1)} = (\kappa^{-1} \times \kappa^{-1}) \circ \mathcal{K}^{(0)} \circ (\kappa \times \kappa) \), so \( \mathcal{K}^{(0)} \) and \( \mathcal{K}^{(1)} \) are not equivalent up to the equivalence relation of Definition 1.
3.1.1. Multidimensional compatibility of the hierarchies $K^{(i)}, i = 0, 1$  
In this Section we prove that both hierarchies $K^{(i)}, i = 0, 1$ are multidimensional compatible. Actually we prove the multidimensional compatibility of the $K^{(1)}$ hierarchy by providing explicitly its multidimensional compatibility formula. The multidimensional compatibility of $K^{(0)}$ can be proven in an exactly similar manner.

By performing the identifications
\[
X_{i,a}^{i} := x^{(i)}, \quad X_{b}^{i,a} := u^{(i)}, \quad X_{i}^{i,b} := y^{(i)}, \quad X_{a}^{i,b} := v^{(i)}, \quad (23)
\]
the hierarchy of maps (19) obtains the compact form
\[
X_{b}^{i,a} = (1 - X_{i,b}^{i})^{-1} \left(X_{i,a}^{i} - X_{i,b}^{i}\right) X_{i-1,a}^{i} \left(X_{i-1,a}^{i} - X_{i-1,b}^{i}\right)^{-1} \left(1 - X_{i-1,b}^{i}\right), \quad (24)
\]
i = 1, \ldots, N, a \neq b \in \{1,2\}. At the same time, (24) serves as a difference system with variables assigned on the edges of an elementary cell of the $\mathbb{Z}^2$ graph, where the subscripts denote appropriate discrete shifts (see Figure 2 under the identification (23)).

**Lemma 3.3** It holds
\[
1 - X_{c}^{i,b} = (1 - X_{i,c}^{i})^{-1} \left(X_{i,b}^{i} - X_{i,c}^{i}\right) \Gamma^{(i)}(b,c), \quad (25)
\]
\[
X_{c}^{i,a} - X_{c}^{i,b} = (1 - X_{i,c}^{i})^{-1} \left(X_{i,b}^{i} - X_{i,c}^{i}\right) \Delta^{(i)}(a,b,c) \left(X_{i-1,a}^{i} - X_{i-1,c}^{i}\right)^{-1} \left(1 - X_{i-1,c}^{i}\right), \quad (26)
\]
i = 1, \ldots, N, a \neq b \neq c \neq a \in \{1, \ldots, n\}, where
\[
\Gamma^{(i)}(b,c) := (X_{i,b}^{i} - X_{i,c}^{i})^{-1} \left(1 - X_{i,c}^{i}\right) - X_{i-1,b}^{i} \left(X_{i-1,b}^{i} - X_{i-1,c}^{i}\right)^{-1} \left(1 - X_{i-1,c}^{i}\right),
\]
\[
\Delta^{(i)}(a,b,c) := (X_{i,b}^{i} - X_{i,c}^{i})^{-1} \left(X_{i,a}^{i} - X_{i,c}^{i}\right) X_{i-1,a}^{i} - X_{i-1,b}^{i} \left(X_{i-1,b}^{i} - X_{i-1,c}^{i}\right)^{-1} \left(X_{i-1,a}^{i} - X_{i-1,c}^{i}\right).
\]
The functions $\Gamma^{(i)}(b,c)$ and $\Delta^{(i)}(a,b,c)$ are antisymmetric under the interchange $b \leftrightarrow c$ i.e. $\Gamma^{(i)}(b,c) + \Gamma^{(i)}(c,b) = 0$, $\Delta^{(i)}(a,b,c) + \Delta^{(i)}(a,c,b) = 0$.

**Proof:** Substituting from (24) the expressions of $X_{c}^{i,a}$ and $X_{c}^{i,b}$ into the lhs of (25) and (26), upon expansion, recollection of terms we verify these formulae.

Using the identity
\[
(1 - AB^{-1})^{-1} + (1 - BA^{-1})^{-1} = 1,
\]
where $A, B$ non-commuting symbols, we can show that $\Gamma^{(i)}(b,c) + \Gamma^{(i)}(c,b) = 0$, $\Delta^{(i)}(a,b,c) + \Delta^{(i)}(a,c,b) = 0$ and that proves the fact that the functions $\Gamma^{(i)}(b,c)$ and $\Delta^{(i)}(a,b,c)$, are antisymmetric under the interchange $b \leftrightarrow c$ of the discrete shifts. □

**Proposition 3.4** The hierarchy of difference equations (24) can be extended in a compatible way to $n-$dimensions as follows
\[
X_{b}^{i,a} = (1 - X_{i,b}^{i})^{-1} \left(X_{i,a}^{i} - X_{i,b}^{i}\right) X_{i-1,a}^{i} \left(X_{i-1,a}^{i} - X_{i-1,b}^{i}\right)^{-1} \left(1 - X_{i-1,b}^{i}\right), \quad (27)
\]
with $i = 1, \ldots, N$, $a \neq b \in \{1, \ldots, n\}$. The compatibility conditions
\[
X_{i,a}^b = X_{c}^i, \quad i = 1, \ldots, N, \quad a \neq b \neq c \neq a \in \{1, \ldots, n\}
\]
hold.

Proof: Shifting relations (27) at the $c-$direction, we obtain
\[
X_{i,a}^b = (1 - X_{i}^{1,b})^{-1} \left( X_{i}^{1,a} - X_{i}^{1,b} \right) X_{i}^{1,a} \left( X_{i}^{1,a} - X_{i}^{1,b} \right)^{-1} (1 - X_{i}^{1,b})
\]
Substituting from (27) the expression of $X_{i}^{1,a}$, $X_{i}^{1,b}$, $X_{i}^{a}$ and $X_{i}^{b}$ to the relations above and by making use of Lemma (3.3) we obtain the following multidimensional compatibility formula
\[
X_{i,a}^b = (\Gamma^{(i)}(b,c))^{-1} \Delta^{(i)}(a,b,c)X_{i}^{1,a} \left( \Delta^{(i)}(a,b,c) \right)^{-1} \Gamma^{(i-1)}(b,c),
\]
that is clearly symmetric under the interchange $b$ to $c$, since it consists of the product of an even number of antisymmetric functions (under the interchange $b \leftrightarrow c$) and that completes the proof. $\square$

Remark 3.5 In exactly similar manner we can prove that the hierarchy of difference equations (15) can be extended in a compatible way to $n-$dimensions as follows
\[
X_{i,a}^b = (1 - X_{i}^{1,b})^{-1} \left( X_{i}^{1,a} - X_{i}^{1,b} \right) X_{i}^{1,a} \left( X_{i}^{1,a} - X_{i}^{1,b} \right)^{-1} (1 - X_{i}^{1,b}),
\]
with $i = 1, \ldots, N$, $a \neq b \in \{1, \ldots, n\}$. If we consider $i \in \mathbb{Z}$, (28) remains multidimensional compatible and moreover it serves as a deformation of the Hirota-Miwa system. Indeed by setting $X_{i}^{k,d} = \epsilon X_{i}^{k,d}$, $k \in \mathbb{Z}$, $d \in \{1, \ldots, n\}$, $\epsilon \in C(\mathbb{R}^+)$. and by taking the limit $\epsilon \to 0$, we obtain exactly the non-Abelian discrete KP hierarchy, which was derived in [24], from the non-Abelian Hirota-Miwa system [22]. In that respect and for $i \in \mathbb{Z}$, (28) serves as an integrable deformation of the discrete KP hierarchy.

3.1.2. Quadrirationality of the hierarchies $\mathcal{K}^{(i)}$ $i = 0, 1$ In what follows, we consider an analysis of proving quadrirationality of the $\mathcal{K}^{(1)}$ hierarchy under some commutativity assumptions that are referred to as the centrality assumptions.

Remark 3.6 The products $\mathcal{P}^{(N,k)}(x) := \prod_{i=1}^{N} x^{(N+k-1)}$, $k = 1, 2, \ldots, N$ satisfy the relations:
\[
\mathcal{P}^{(N,k)}(u) = (1 - y^{(N+k-1)})^{-1} \left( x^{(N+k-1)} - y^{(N+k-1)} \right) \mathcal{P}^{N,k-1}(x) \quad (29)
\]
\[
\mathcal{P}^{(N,k)}(v) = (1 - x^{(N+k-1)})^{-1} \left( y^{(N+k-1)} - x^{(N+k-1)} \right) \mathcal{P}^{N,k-1}(y) \quad (30)
\]
k = 1, \ldots, N, and $u, v$ stand for the defining relations of the $\mathcal{K}^{(1)}$ hierarchy (19). Note that the direct substitution of (19) into (24), (30) validates the formulas above. Also note that we consider the products presented above in ascending order of the superscripts f.i.
\[
\prod_{i=1}^{n} x^{(i)} := x^{(1)} x^{(2)} \ldots x^{(m)}.
\]
Clearly in the Abelian case the products $P^{(N,k)}$ are independent of $k$ i.e. all products $P^{(N,k)}$ coincide with the product $P^{(N)}(x) = \prod_{i=1}^{N} x^{(i)}$. Furthermore, there is $P^{(N)}(u) = P^{(N)}(x)$, and $P^{(N)}(v) = P^{(N)}(y)$, hence the products $P^{(N)}(x)$ and $P^{(N)}(y)$ serve as invariants of the map $K^{(1)}$. In the non-Abelian case, these facts mentioned earlier are no longer true. Due to Remark 3.6, the products $P^{(N,k)}$ are no longer independent of $k$ and moreover are not invariants of the map. Nevertheless, if we assume that the products $P^{(N,k)}$ belong to the center of the algebra $A^{\infty}$, from Remark 3.6 we obtain the invariant relations $P^{(N,k)}(u) = P^{(N,k-1)}(x)$, and $P^{(N,k)}(v) = P^{(N,k-1)}(y)$, $k = 1, \ldots, N$, hence the functions $P^{(N,k)}(x), P^{(N,k)}(y)$ are covariants of the map. We denote the functions $P^{(N,k)}(x), P^{(N,k)}(y)$, respectively as $p^{(k-1)}$ and $q^{(k-1)}$. To recapitulate we have

$$p^{(k-1)} := P^{(N,k)}(x), \quad q^{(k-1)} := P^{(N,k)}(y), \quad i = 1, \ldots, N. \quad (31)$$

In addition, as a consequence of (31) it holds that $p^{(i)} = p^{(j)}$, and $q^{(i)} = q^{(j)}$, $\forall i, j \in \{0, 1, \ldots, N-1\}$.

From further on, when we refer to the centrality assumption, we refer to the formulas:

$$x^{(N)} \cdot x^{(2)} x^{(1)} = p \in C(A^{\infty}), \quad y^{(N)} \cdot y^{(2)} y^{(1)} = q \in C(A^{\infty}), \quad (32)$$

where with $C(A^{\infty})$ we denote the center of the algebra $A^{\infty}$ and we also have denoted $p := p^{(0)}$, $q := q^{(0)}$. The centrality assumptions were first introduced in [23, 24] for the so-called $N-$ periodic reduction of the non-Abelian Hirota-Miwa system (KP-map). Centrality assumptions play a crucial role to the quadrirationality of the hierarchy of maps (19), as it is shown in the Proposition that follows.

**Proposition 3.7** The hierarchy of maps (19) is birational. When the centrality assumptions (32) are imposed the hierarchy of maps is quadrirational.

**Proof:** First we prove that (19) is birational. The compatibility conditions (20), (21), can be solved rationally for $x^{(i)}, y^{(i)}$ in terms of $u^{(i)}, v^{(i)}$ and that proves birationality of the system. Specifically from (20), (21) we obtain:

$$x^{(i)} = (1 - v^{(i+1)}) \left( u^{(i+1)} - v^{(i+1)} \right)^{-1} u^{(i+1)} (u^{(i)} - v^{(i)}) \left( 1 - v^{(i)} \right)^{-1},$$

$$y^{(i)} = (1 - u^{(i+1)}) \left( v^{(i+1)} - u^{(i+1)} \right)^{-1} v^{(i+1)} (v^{(i)} - u^{(i)}) \left( 1 - u^{(i)} \right)^{-1}, \quad (33)$$

$$i = 1, 2, \ldots, N.$$ 

From (33) it is easy to validate the following formulæ

$$P^{(N,k)}(x) = (1 - v^{(N+k)})^{-1} (u^{(N+k)} - v^{(N+k)}) P^{(N,k+1)}(u) \left( u^{(N+k)} - v^{(N+k)} \right)^{-1} \left( 1 - v^{(N+k)} \right),$$

$$P^{(N,k)}(y) = (1 - u^{(N+k)})^{-1} (v^{(N+k)} - u^{(N+k)}) P^{(N,k+1)}(v) \left( v^{(N+k)} - u^{(N+k)} \right)^{-1} \left( 1 - u^{(N+k)} \right),$$

$$k = 1, \ldots, N,$$

where the expressions $P^{(N,k)}$ are defined in Remark 3.6.
To prove that (19) is quadrirational it suffices to solve (19) rationally for \( u^{(i)}, y^{(i)} \) in terms of \( x^{(i)}, v^{(i)} \) and show that the resulting map is birational.

From (20) and (21) we obtain respectively

\[
(1 - u^{(i)})^{-1} = (1 - v^{(i)})^{-1} (1 - x^{(i)})^{-1} (1 - y^{(i)}) ,
\]

and

\[
(1 - u^{(i)})^{-1} u^{(i)} = (1 - v^{(i)})^{-1} v^{(i)} (1 - x^{(i-1)})^{-1} x^{(i-1)} (y^{(i-1)})^{-1} (1 - y^{(i-1)}) .
\]

Substituting these expressions into the lhs of (22) we get

\[
y^{(i)} y^{(i-1)} - (x^{(i)} + A^{(i)}) y^{(i-1)} + A^{(i)} x^{(i-1)} = 0 ,
\]

where

\[
A^{(i)} := (1 - x^{(i)}) v^{(i)} (1 - x^{(i-1)})^{-1} .
\]

Multiplying from the right equation (36) respectively with

\[
y^{(i-2)}, y^{(i-2)} y^{(i-3)}, \ldots, \prod_{l=2}^{m} y^{(i-l)} ,
\]

and by substituting at each step from (36) the expressions \( y^{(k)} y^{(k-1)} \) we obtain:

\[
\prod_{l=0}^{m} y^{(i-l)} - f_{m}^{(i)} y^{(i-m)} + f_{m-1}^{(i)} A^{(i-m+1)} x^{(i-m)} = 0 , \quad m \geq 2 ,
\]

where \( f^{(i)} \) satisfies the recurrences

\[
f^{(i)}_{n+2} = f^{(i)}_{n+1} (x^{(i-n-1)} + A^{(i-n-1)}) - f^{(i)}_{n} A^{(i-n)} x^{(i-n-1)} , \quad n \in \mathbb{Z} ,
\]

with

\[
f^{(i)}_{0} = 1 , \quad f^{(i)}_{1} = x^{(i)} + A^{(i)} , \quad i = 1, \ldots, N .
\]

Setting \( m = N - 1 \) and by assuming the centrality assumptions (32), equations (37) read

\[
q - f^{(i)}_{N-1} y^{(i-N+1)} + f^{(i)}_{N-2} A^{(i-N+2)} x^{(i-N+1)} = 0 ,
\]

where \( f^{(i)}_{N-1}, f^{(i)}_{N-2} \) are determined by the recurrences (38), (39). So from (40) we have obtained \( y^{(i)} \) as a function of \( x^{(i)}, v^{(i)} \), \( i = 1, \ldots, N \), and together with (34) we finally obtain \( y^{(i)}, u^{(i)} \) in terms of \( v^{(i)}, x^{(i)} \), i.e. the companion hierarchy of maps of the hierarchy of maps \( K^{(i)} \). In exactly similar manner we can express rationally \( x^{(i)}, v^{(i)} \) in terms of \( u^{(i)}, y^{(i)} \), that proves birationality of the companion hierarchy of maps, and that completes the proof. \( \square \)
Proposition 3.8 Let the expressions $f_{N-1}^{(i)}, f_{N-2}^{(i)}; i = 1, \ldots, N,$ be determined by the recurrences, (38), (39), and the expressions $g_{N-1}^{(i)}, g_{N-2}^{(i)}; i = 1, \ldots, N,$ be determined by the recurrences

$$g_{n+2}^{(i)} = (v^{(i+n)} + B^{(i+n+1)}) g_{n+1}^{(i)} - v^{(i+n)} B^{(i+n)} g_n^{(i)}, \quad n \in \mathbb{Z},$$

with

$$g_0^{(i)} = 1, \quad g_1^{(i)} = v^{(i-1)} + B^{(i)}, \quad B^{(i)} := (1 - v^{(i)})^{-1} x^{(i-1)} (1 - v^{(i-1)}), \quad i = 1, \ldots, N.$$ (42)

Assuming the centrality assumptions (32) that results quadrirationality, the companion hierarchy $c\mathcal{K}^{(i)}$ of the hierarchy of maps $\mathcal{K}^{(i)}$ explicitly reads

$$c\mathcal{K}^{(i)} : (x^{(1)}, \ldots, x^{(N)}, v^{(1)}, \ldots, v^{(N)}) \mapsto (u^{(1)}, \ldots, u^{(N)}, y^{(1)}, \ldots, y^{(N)}),$$

where

$$u^{(i)} = \left( p + v^{(i)} B^{(i)} g_{N-2}^{(i-1)} \right) \left( g_{N-1}^{(i-1)} \right)^{-1},$$

$$y^{(i)} = \left( f_{N-1}^{(i+N-1)} \right)^{-1} \left( q + f_{N-2}^{(i+N-1)} A^{(i+1)} x^{(i)} \right), \quad i = 1, \ldots, N,$$ (43)

and it serves as a hierarchy of Yang-Baxter maps.

Proof: The formulas of $y^{(i)}$ in (43) are just the solved for $y^{(i)}$ form of (40). To obtain the formulas for $u^{(i)}$ of (43), so that we obtain the explicit formulas for the hierarchy $c\mathcal{K}^{(i)}$, we have to substitute the formulas of $y^{(i)}$ to (44) and solve for $u^i$. Equivalently, from the defining relations of $\mathcal{K}^{(i)}$ (20) and (21), if we eliminate $y^{(i)}$ we will obtain $u^{(i)}$ as functions of $v^{(i)}$ and $x^{(i)}$, hence the first part of the formulas (43). This is what we do for the rest of the proof.

From (20) and (21) we obtain

$$u^{(i+1)} u^{(i)} - u^{(i+1)} (v^{(i)} + B^{(i+1)}) + v^{(i+1)} B^{(i+1)} = 0,$$ (44)

where

$$B^{(i)} := (1 - v^{(i)})^{-1} x^{(i-1)} (1 - v^{(i-1)}).$$

Multiplying from the left equation (44) respectively with

$$u^{(i+2)}, \quad u^{(i+3)} y^{(i+2)}, \quad \ldots, \quad \prod_{l=0}^{m-2} u^{(i+m-l)}$$

and by substituting at each step from (44) the expressions $u^{(k+1)} u^{(k)}$ we obtain:

$$\prod_{l=0}^{m} u^{(i+m-l)} - u^{(i+m)} g_m^{(i)} + v^{(i+m)} B^{(i+m)} g_{m-1}^{(i)} = 0, \quad m \geq 2,$$ (45)
where \( g^{(i)} \) satisfies the recurrences
\[
\begin{align*}
g^{(i)}_{n+2} &= \left( v^{(i+n+1)} + B^{(i+n+2)} \right) g^{(i)}_{n+1} - v^{(i+n+1)} B^{(i+n+1)} g^{(i)}_n, \\
\quad n \in \mathbb{Z},
\end{align*}
\]
with
\[
\begin{align*}
g^{(i)}_0 &= 1, \\
g^{(i)}_1 &= v^{(i-1)} + B^{(i)}, \\
\quad i = 1, \ldots, N.
\end{align*}
\] (47)

Setting \( m = N - 1 \) and by assuming the centrality assumption (32), equations (45) read
\[
\begin{align*}
p - u^{(i+N-1)} g^{(i)}_{N-1} + v^{(i+N-1)} B^{(i+N-1)} g^{(i)}_{N-2} &= 0,
\end{align*}
\] (48)
where \( g^{(i)}_{N-1}, g^{(i)}_{N-2} \) are determined by the recurrences (46), (47). Solving (48) for \( u^{(i)} \), we obtain the first expression of (43). Since \( cK^{(1)} \) serves as the companion hierarchy of a 3D–compatible hierarchy, \( cK^{(1)} \) is hierarchy of Yang-Baxter maps (see Section 2) and that completes the proof.

\[ \square \]

Remark 3.9 By following an exactly similar analysis as in this Section, we can show the quadrirationality of the hierarchy \( \mathcal{K}^{(0)} \) and provide its companion that serves as a hierarchy of Yang-Baxter maps.

4. Hierarchies of integrable difference systems in vertex variables

The hierarchies of integrable difference systems in edge variables associated with the 3D-compatible maps \( \mathcal{K}^{(0)} \) and \( \mathcal{K}^{(1)} \), can be also rewritten in terms of vertex variables. The procedure that incorporates the transition from edge (or even face) to vertex variables in integrable difference systems, was introduced in [60]. This procedure is widely applied nowadays [61, 62, 63, 64], under the attributed name potentialisation.

In what follows we apply the potentialization procedure to the integrable hierarchy \( \mathcal{K}^{(0)} \), to obtain integrable hierarchies of difference systems in vertex variables. Applying the potentialization procedure to the hierarchy \( \mathcal{K}^{(1)} \), leads to point equivalent integrable hierarchies of difference systems with the hierarchy \( \mathcal{K}^{(0)} \).

From the defining relations of the 3D–compatible map \( \mathcal{K}^{(0)} \), or equivalently from the compatibility conditions (20), (21), it is guaranteed the existence of two sets of potential functions that allow us to rewrite the 3D–compatible map as two (related by a Bäcklund transformation) integrable hierarchies of difference systems with dynamical variables defined on the vertices of the \( \mathbb{Z}^2 \) graph.

4.1. The first set of potential functions and the lattice-modified Gel’fand-Dikii hierarchy

The integrable hierarchy \( \mathcal{K}^{(0)} \), in its polynomial form consists of the sets of equations (16) and (17). The set of equations (16), guarantees the existence of potential functions
\( \phi^{(i)}, i = 1, \ldots, N, \) such that
\[
(1 - x^{(i)})^{-1} = \phi_1^{(i)} \left( \phi^{(i-1)} \right)^{-1},
(1 - y^{(i)})^{-1} = \phi_2^{(i)} \left( \phi^{(i-1)} \right)^{-1},
(1 - u^{(i)})^{-1} = \phi_{12}^{(i)} \left( \phi_2^{(i-1)} \right)^{-1},
(1 - v^{(i)})^{-1} = \phi_{12}^{(i)} \left( \phi_1^{(i-1)} \right)^{-1},
\]
i = 1, \ldots, N. \quad (49)

In terms of the potential functions \( \phi^j \), the set of equations \((16)\) is identically satisfied, while the set of equations \((17)\) becomes
\[
\left( \phi_{12}^{(i)} \left( \phi_2^{(i-1)} \right)^{-1} - 1 \right) \left( \phi_2^{(i)} \left( \phi^{(i-1)} \right)^{-1} - 1 \right) = \left( \phi_{12}^{(i)} \left( \phi_1^{(i-1)} \right)^{-1} - 1 \right) \left( \phi_1^{(i)} \left( \phi^{(i-1)} \right)^{-1} - 1 \right),
\]
i = 1, \ldots, N and constitute a hierarchy of difference systems in vertex variables.

**Proposition 4.1** For the hierarchy of difference systems in vertex variables \((50)\) we have:

(i) it arises as the compatibility conditions of the Lax equation
\[
L(\phi_{12}, \phi_2; \lambda)L(\phi_2, \phi; \lambda) = L(\phi_{12}, \phi_1; \lambda)L(\phi_1, \phi; \lambda),
\]
associated with the strong Lax matrix
\[
L = \begin{pmatrix}
\phi_1^{(N)} \left( \phi^{(N-1)} \right)^{-1} - 1 & 0 & \cdots & 0 & \lambda \phi_1^{(N)} \left( \phi^{(N-1)} \right)^{-1} \\
0 & \phi_1^{(1)} \left( \phi^{(N)} \right)^{-1} - 1 & 0 & \cdots & 0 \\
0 & \phi_1^{(2)} \left( \phi^{(1)} \right)^{-1} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \phi_1^{(N-2)} \left( \phi^{(N-1)} \right)^{-1} - 1 & 0 \\
0 & 0 & \cdots & \phi_1^{(N-1)} \left( \phi^{(N-2)} \right)^{-1} - 1 & \phi_1^{(N-1)} \left( \phi^{(N-2)} \right)^{-1} - 1
\end{pmatrix};
\]

(ii) it is multidimensional consistent;

(iii) it respects the rhombic symmetry
\[
\tau : (\phi, \phi_1, \phi_2, \phi_{12}) \mapsto (\phi, \phi_2, \phi_1, \phi_{12}), \quad \sigma : (\phi, \phi_1, \phi_2, \phi_{12}) \mapsto (\phi_{12}, \phi_2, \phi_1, \phi);
\]

(iv) it is an integrable hierarchy of difference systems in vertex variables.

**Proof:** Let us prove the statements of this Proposition.

(i) We substitute the expressions of the potential functions \((49)\) into the Lax matrix \(\widehat{M}^{(N,1)}(x; \lambda)\) of Proposition \((3.1)\) to obtain the Lax matrix presented here.

(ii) The multidimensional consistency of \((50)\) is a direct consequence of the multidimensional compatibility of the underlying difference system in edge variables \((15)\). Indeed, for the system \((15)\) that its multidimensional extension reads
\[
X_b^{i,a} = (1 - X^{i-1,b})^{-1} (X^{i-1,a} - X^{i-1,b}) X^{i,a} (X^{i,a} - X^{i,b})^{-1} (1 - X^{i,b}), \quad (51)
\]
\( i = 1, \ldots, N, \ a \neq b \neq c \neq a \in \{1, \ldots, n\}, \) it holds the multidimensional compatibility formula

\[
X_{bc}^{i,a} = X_{cb}^{i,a}, \quad i = 1, \ldots, N, \ a \neq b \neq c \neq a \in \{1, \ldots, n\}. \tag{52}
\]

Also for the potential functions \( \phi^{(i)} \) we have

\[
(1 - X_{b}^{i,a})^{-1} = \phi_{a}^{(i)}(\phi^{(i-1)})^{-1}, \quad i = 1, \ldots, N, \ a \in \{1, \ldots, n\}. \tag{53}
\]

From (52) and (53) we obtain

\[
\phi_{abc}^{(i)} \left( \phi_{bc}^{(i-1)} \right)^{-1} = \phi_{acb}^{(i)} \left( \phi_{cb}^{(i-1)} \right)^{-1}.
\]

So in order to prove the multidimensional consistency of (50) that is

\[
\phi_{abc}^{(i)} = \phi_{acb}^{(i)}, \quad i = 1, \ldots, N, \ a \neq b \neq c \neq a \in \{1, \ldots, n\},
\]

it suffices to prove that

\[
\phi_{ab}^{(i)} = \phi_{ba}^{(i)}, \quad a \neq b \in \{1, \ldots, n\}. \tag{54}
\]

Indeed by shifting appropriately (53) we obtain

\[
(1 - X_{b}^{i,a})^{-1} \phi_{b}^{(i-1)} = \phi_{ab}^{(i)}, \quad (1 - X_{a}^{i,b})^{-1} \phi_{a}^{(i-1)} = \phi_{ba}^{(i)}
\]

and by using again (53) to eliminate \( \phi_{b}^{(i-1)} \) and \( \phi_{a}^{(i-1)} \), we arrive to

\[
(1 - X_{b}^{i,a})^{-1} (1 - X_{b}^{i-1,a})^{-1} \phi_{b}^{(i-2)} = \phi_{ab}^{(i)}, \quad (1 - X_{a}^{i,b})^{-1} (1 - X_{a}^{i-1,b})^{-1} \phi_{a}^{(i-2)} = \phi_{ba}^{(i)}.
\tag{55}
\]

From (51) we get

\[
1 - X_{b}^{i,a} = (1 - X_{b}^{i-1,a})^{-1} (X_{b}^{i-1,a} - X_{b}^{i-1,b}) \Gamma_{b,a}^{(i)},
\]

\[
1 - X_{a}^{i,b} = (1 - X_{a}^{i-1,a})^{-1} (X_{a}^{i-1,b} - X_{a}^{i-1,a}) \Gamma_{a,b}^{(i)},
\tag{56}
\]

where the expressions \( \Gamma^{(i)}(a, b) \) are antisymmetric under the interchange \( a \leftrightarrow b \), and they are of similar form to the ones of Lemma 3.3. Using (56) to eliminate \( 1 - X_{b}^{i,a} \) and \( 1 - X_{a}^{i,b} \) from (55), together with the fact that \( \Gamma^{(i)}(a, b) + \Gamma^{(i)}(b, a) = 0 \), we obtain exactly (54) and that proves the multidimensional consistency of the hierarchy of difference systems in vertex variables (50). (iii) It can be easily verified that (50) is invariant under \( \tau \), while acting with \( \sigma \) on (50) we get

\[
\left( \phi_{1}^{(i)} \left( \phi_{1}^{(i-1)} \right)^{-1} - 1 \right) \left( \phi_{1}^{(i)} \left( \phi_{12}^{(i-1)} \right)^{-1} - 1 \right),
\]

\[
\tag{57}
\left( \phi_{2}^{(i)} \left( \phi_{2}^{(i-1)} \right)^{-1} - 1 \right) \left( \phi_{2}^{(i)} \left( \phi_{12}^{(i-1)} \right)^{-1} - 1 \right).
\]
which is (50) in disguise. Indeed, by acting on (50) with $T_{-1}T_{-2}$ we obtain
\[
\left(\phi^{(i)}\left(\phi_{-1}^{(i-1)}\right)^{-1} - 1\right)\left(\phi_{-1}^{(i)}\left(\phi_{-1-2}^{(i-1)}\right)^{-1} - 1\right)
\]
\[
= \left(\phi^{(i)}\left(\phi_{-2}^{(i-1)}\right)^{-1} - 1\right)\left(\phi_{-2}^{(i)}\left(\phi_{-1-2}^{(i-1)}\right)^{-1} - 1\right),
\]
and if we perform the change of the dependent variables $\phi_{m,n} = \Phi_{m,-n}$, followed by the change of independent variables $m' = -m, n' = -n$, we obtain that $\Phi_{m',n'}$ satisfies (57). Here with $T_j$ we denote the forward shift operator in the $j$–th direction while with $T_{-k}$ we denote the backward shift operator in the $k$–th direction i.e.
\[
T_1 : \phi^{(i)} \mapsto \phi_1^{(i)}, \quad T_{-1} : \phi^{(i)} \mapsto \phi_{-1}^{(i)}, \quad T_{-2} : \phi^{(i)} \mapsto \phi_{-2}^{(i)}, \quad \text{etc.}
\]
(iv) Due to statements (1) – (3), (50) constitutes an integrable hierarchy of difference systems in vertex variables defined on the black-white (chessboard) lattice [65, 66, 67].

Note that the hierarchy of difference systems (50), can be solved rationally only for the sets of variables $\Phi$ and $\phi_{12}$. If the centrality assumptions (32) are imposed, that in terms of the potential functions $\phi$ read
\[
\prod_{l=1}^{N} \left(1 - \phi^{(N-l)}\left(\phi_1^{(N+1-l)}\right)^{-1}\right) = p \in C(\mathbb{A}^\times), \quad (58)
\]
\[
\prod_{l=1}^{N} \left(1 - \phi^{(N-l)}\left(\phi_2^{(N+1-l)}\right)^{-1}\right) = q \in C(\mathbb{A}^\times), \quad (59)
\]
then (50) can be solved for any corner variable set, mimicking the Abelian case.

To recapitulate, hierarchy (50) serves as an integrable hierarchy in vertex variables that respects the rhombic symmetry. If in addition the centrality assumptions (58), (59) are imposed, then one potential function can be eliminated and the resulting hierarchy consists of $N - 1$ equations in $N - 1$ potential functions. The resulting hierarchy serves as the non-Abelian lattice-modified Gel’fand-Dikii hierarchy, since its lowest member turns out to be the lattice-potential-modified KdV equation (in the abelian case derived in [68, 69, 46] and referred to as $(H3)_0$ in [48]), as the example that follows suggest. Remember that the lattice-modified Gel’fand-Dikii hierarchy in the Abelian case was firstly considered implicitly in [39] and explicitly in [42], whereas in the non-Abelian setting in [23].
Example 4.2 \((N = 2)\) The first member of the hierarchy \(\mathcal{M}\) \((N = 2)\) reads:

\[
\begin{align*}
(\phi_{12}^{(1)} (\phi_2^{(2)})^{-1} - 1) (\phi_{12}^{(1)} (\phi_2^{(2)})^{-1} - 1) &= (\phi_{12}^{(1)} (\phi_1^{(2)})^{-1} - 1) \\
(\phi_{12}^{(2)} (\phi_2^{(1)})^{-1} - 1) (\phi_{12}^{(2)} (\phi_2^{(1)})^{-1} - 1) &= (\phi_{12}^{(2)} (\phi_1^{(1)})^{-1} - 1)
\end{align*}
\]

(60)

(61)

If we impose the centrality assumptions \((58),(59)\) that now read

\[
\begin{align*}
1 - \phi^{(1)} (\phi_2^{(2)})^{-1} = p, \\
1 - \phi^{(2)} (\phi_2^{(1)})^{-1} = q,
\end{align*}
\]

we can eliminate f.i. \(\phi^2\) and its shifts from \((60)\) to obtain

\[
(1 - q - \phi \phi_2^{-1}) \phi_2 = (1 - p - \phi \phi_1^{-1}) \phi_1,
\]

where for simplicity we have denoted \(\phi := \phi^{(1)}, \phi_2 := \phi_2^{(2)}, \text{ etc.} \) Under the re--parametrization \(p \mapsto 1 - 1/p^2, q \mapsto 1 - 1/q^2, \) followed by the point transformation \(\phi_{m+m_1,n+n_1} \mapsto p^{m_1} q^n \phi_{m+m_1,n+n_1}, \) the equation above takes the form

\[
\left(\frac{1 - 1/p}{q - \phi \phi_2^{-1}}\right) \phi_2 = \left(\frac{1 - 1/q}{p - \phi \phi_1^{-1}}\right) \phi_1,
\]

(62)

that is exactly \((H3)_0\) but extended in the non-Abelian domain.

4.2. The second set of potential functions and the lattice-\((Q3)_0\) Gel’fand-Dikii hierarchy

As we mentioned earlier, the polynomial form of the integrable hierarchy \(\mathcal{K}^{(0)}\), consists of the sets of equations \((16)\) and \((17)\). The set of equations \((17)\), guarantees the existence of potential functions \(\psi^i, i = 1, \ldots, N, \) such that

\[
\begin{align*}
(1 - x^{(i)})^{-1} x^{(i)} &= \psi_1^{(i)} (\psi^{(i)})^{-1},
(1 - y^{(i)})^{-1} y^{(i)} &= \psi_2^{(i)} (\psi^{(i)})^{-1}, \\
(1 - u^{(i)})^{-1} u^{(i)} &= \psi_{12}^{(i)} (\psi_2^{(i)})^{-1},
(1 - v^{(i)})^{-1} v^{(i)} &= \psi_{12}^{(i)} (\psi_1^{(i)})^{-1},
\end{align*}
\]

(63)

In terms of the potential functions \(\psi^{(i)}\), the set of equations \((17)\) is identically satisfied, while the set of equations \((16)\) becomes

\[
\begin{align*}
\left(1 + \psi_1^{(i)} (\psi_2^{(i)})^{-1}\right) \left(1 + \psi_2^{(i-1)} (\psi_1^{(i-1)})^{-1}\right) &= \left(1 + \psi_1^{(i)} (\psi_1^{(i)})^{-1}\right) \\
\left(1 + \psi_1^{(i-1)} (\psi_1^{(i)})^{-1}\right) &\left(1 + \psi_1^{(i-1)} (\psi_1^{(i)})^{-1}\right),
\end{align*}
\]

(64)

\(i = 1, \ldots, N\) and constitute a hierarchy of difference systems in vertex variables.
Proposition 4.3 For the hierarchy of difference systems in vertex variables (64) we have:

(i) it arises as the compatibility conditions of the Lax equation

\[ L(\psi_{12}, \psi_2; \lambda) L(\psi_2, \psi; \lambda) = L(\psi_{12}, \psi_1; \lambda) L(\psi_1, \psi; \lambda), \]

associated with the strong Lax matrix

\[
L = \begin{pmatrix}
\psi_1^{(N)} (\psi^{(N)})^{-1} & 0 & \cdots & 0 & \lambda (1 + \psi_1^{(N)} (\psi^{(N)})^{-1}) \\
1 + \psi_1^{(1)} (\psi^{(1)})^{-1} & \psi_1^{(1)} (\psi^{(1)})^{-1} & 0 & \cdots & 0 \\
0 & 1 + \psi_1^{(2)} (\psi^{(2)})^{-1} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 + \psi_1^{(N-1)} (\psi^{(N-1)})^{-1} & \psi_1^{(N-1)} (\psi^{(N-1)})^{-1}
\end{pmatrix};
\]

(ii) it is multidimensional consistent;

(iii) it respects the rhombic symmetry

\[
\tau : (\psi, \psi_1, \psi_2, \psi_{12} \mapsto (\psi, \psi_2, \psi_1, \psi_{12}), \quad \sigma : (\psi, \psi_1, \psi_2, \psi_{12} \mapsto (\psi_{12}, \psi_2, \psi_1, \psi);
\]

(iv) it is an integrable hierarchy of difference systems in vertex variables.

Proof: The proof of this Proposition follows from the proof of the Proposition 4.1. □

Similarly as with the hierarchy of difference systems (50), the hierarchy of difference systems (64), can be solved rationally only for the sets of variables \( \psi \) and \( \psi_{12} \). If the centrality assumptions (65) are imposed, that in terms of the potential functions \( \psi \) read

\[
\prod_{l=1}^{N} \left( 1 + \psi^{(N-l+1)} (\psi_1^{(N-l+1)})^{-1} \right)^{-1} = p \in C(\mathbb{A}^{\times}), \quad (65)
\]

\[
\prod_{l=1}^{N} \left( 1 + \psi^{(N-l+1)} (\psi_2^{(N-l+1)})^{-1} \right)^{-1} = q \in C(\mathbb{A}^{\times}), \quad (66)
\]

then (64) can be solved for any corner variable set, mimicking the Abelian case.

To recapitulate, hierarchy (64) serves as an integrable hierarchy in vertex variables that respects the rhombic symmetry. If in addition the centrality assumptions (65), (66) are imposed, then one potential function can be eliminated and the resulting hierarchy consists of \( N - 1 \) equations in \( N - 1 \) potential functions. The resulting hierarchy serves as the non-Abelian lattice-(Q3)\(_0\) Gel’fand-Dikii hierarchy, since its lowest member turns out to be (Q3)\(_0\) (in the Abelian case goes back to [47] and referred to as (Q3)\(_0\) in [48]), as the example that follows suggest. Note that the lattice-(Q3)\(_0\) Gel’fand-Dikii hierarchy in the Abelian case was firstly considered implicitly in [41] and in [43] it was explicitly derived just the second member of this hierarchy i.e. the Boussinesq analogue of (Q3)\(_0\).
Example 4.4 \((N = 2)\) The first member of the hierarchy \((N = 2)\) reads:

\[
\left( \psi^{(1)}_{12} \left( \psi^{(1)}_{2} \right)^{-1} + 1 \right) \left( \psi^{(2)}_{2} \left( \psi^{(2)}_{1} \right)^{-1} + 1 \right) = \left( \psi^{(1)}_{12} \left( \psi^{(1)}_{1} \right)^{-1} + 1 \right) \left( \psi^{(2)}_{1} \left( \psi^{(2)}_{2} \right)^{-1} + 1 \right),
\]

\[(67)\]

If we impose the centrality assumptions \((65),(66)\) that now read

\[
\left( 1 + \psi^{(1)} \left( \psi^{(1)} \right)^{-1} \right)^{-1} \left( 1 + \psi \left( \psi \right)^{-1} \right)^{-1} = p,
\]

\[
\left( 1 + \psi^{(2)} \left( \psi^{(2)} \right)^{-1} \right)^{-1} \left( 1 + \psi \left( \psi \right)^{-1} \right)^{-1} = q,
\]

we can eliminate f.i. \(\psi^{(2)}\) and its shifts from the first equation of \((67)\) to obtain

\[
\left( 1 + \psi_{12} \psi^{(2)} \left( \psi^{(2)} \right)^{-1} \right) \left( 1 - q \left( 1 + \psi^{(2)} \psi^{-1} \right) \right) = \left( 1 + \psi_{12} \psi^{(1)} \right) \left( 1 - p \left( 1 + \psi \psi^{-1} \right) \right),
\]

where for simplicity we have denoted \(\psi := \psi^{(1)}\), \(\psi_{2} := \psi^{(1)}_{2}\), etc. Under the re-parametrization \(p \mapsto \frac{p}{p^2 - 1}\), \(q \mapsto \frac{q^2}{q^2 - 1}\) followed by the point transformation

\[
\psi_{m+m_{1},n+n_{1}} \mapsto (-1)^{m_{1}+n_{1}} p^{m_{1}} q^{n_{1}} \psi_{m+m_{1},n+n_{1}},
\]

the equation above takes the form

\[
\left( \psi_{2} - p \psi_{12} \right) \left( \psi_{2} - \frac{q}{q^2 - 1} \left( q \psi_{2} - \psi \right) \right)^{-1} = \left( \psi_{1} - q \psi_{12} \right) \left( \psi_{1} - \frac{p}{p^2 - 1} \left( p \psi_{1} - \psi \right) \right)^{-1},
\]

\[(68)\]

that is exactly \((Q3)_{0}\) but extended in the non-Abelian domain.

Example 4.5 \((N > 2)\) For \(N > 2\) we have the lattice-(Q3)\(_{0}\) Gel’fand-Dikii hierarchy i.e. the set of equations \((64)\) that as we showed, respect the rhombic symmetry.

From the centrality assumptions \((65),(66)\), we obtain

\[
1 + \psi^{(N)}_{1} \left( \psi^{(N)} \right)^{-1} = \left( 1 - p \prod_{l=1}^{N-1} \left( 1 + \psi^{(l)} \left( \psi^{(l)} \right)^{-1} \right) \right)^{-1},
\]

\[
1 + \psi^{(N)}_{2} \left( \psi^{(N)} \right)^{-1} = \left( 1 - q \prod_{l=1}^{N-1} \left( 1 + \psi^{(l)} \left( \psi^{(l)} \right)^{-1} \right) \right)^{-1}.
\]

Using the relations above, we can eliminate the potential \(\psi^{N}\) and its shifts from \((64)\), to obtain the following form of the lattice-(Q3)\(_{0}\) Gel’fand-Dikii hierarchy.
\[
\left(1 + \psi_{12}^{(1)} \left(\psi_{2}^{(1)}\right)^{-1}\right) \left(1 - q \prod_{l=1}^{N-1} \left(1 + \psi_{l}^{(l)} \left(\psi_{2}^{(l)}\right)^{-1}\right)\right)^{-1}
\]
\[
= \left(1 + \psi_{12}^{(1)} \left(\psi_{1}^{(1)}\right)^{-1}\right) \left(1 - p \prod_{l=1}^{N-1} \left(1 + \psi_{l}^{(l)} \left(\psi_{1}^{(l)}\right)^{-1}\right)\right)^{-1},
\]
\[
\left(1 + \psi_{12}^{(i)} \left(\psi_{2}^{(i)}\right)^{-1}\right) \left(1 + \psi_{2}^{(i-1)} \left(\psi_{1}^{(i-1)}\right)^{-1}\right) = \left(1 + \psi_{12}^{(i)} \left(\psi_{1}^{(i)}\right)^{-1}\right) \left(1 + \psi_{1}^{(i-1)} \left(\psi_{1}^{(i-1)}\right)^{-1}\right),
\]
\[
\left(1 - p \prod_{l=1}^{N-1} \left(1 + \psi_{2}^{(l)} \left(\psi_{12}^{(l)}\right)^{-1}\right)\right)^{-1} \left(1 + \psi_{2}^{(N-1)} \left(\psi_{(N-1)}\right)^{-1}\right)
\]
\[
= \left(1 - q \prod_{l=1}^{N-1} \left(1 + \psi_{1}^{(l)} \left(\psi_{12}^{(l)}\right)^{-1}\right)\right)^{-1} \left(1 + \psi_{1}^{(N-1)} \left(\psi_{(N-1)}\right)^{-1}\right),
\]
i = 2, \ldots, N - 1.

5. Conclusions

In this article we introduced two families of hierarchies of non-Abelian compatible maps. We have explicitly provided their associated Lax matrices, we have shown that they serves as deformations of known hierarchies of maps [23, 24] and we have proven their compatibility by explicitly providing their multidimensional compatibility formula. Furthermore, by imposing to these hierarchies certain compatible commutativity assumptions, we prove the quadrirationality of the latter and we provide explicitly the associated families of hierarchies of Yang-Baxter maps. Finally, we derived the corresponding integrable hierarchies of difference systems in non-commuting edge variables and the associated integrable difference hierarchies in vertex variables together with their Lax matrices. The lattice-modified Gel’fand-Dikii hierarchy and the lattice-NQC Gel’fand-Dikii hierarchy, both in non-commuting variables, together with the underlying integrable difference system in edge variables, were obtained.

The results of this article can be extended and/or generalized in various ways. We anticipate that by following [70, 71] we can obtain entwining hierarchies of maps associated with the ones presented here. Moreover, by switching on the deformation parameters (see the beginning of Section 3), we can obtain degeneracies of the hierarchies \(K^{(i)}, i = 1, 2\), that in turn will lead to degeneracies of the Gel’fand-Dikii hierarchies presented here, as well as their corresponding Yang-Baxter maps which are expected to be related with the non-Abelian extension of the results in [72].

Furthermore \(K^{(1)}\), serves as a member of a family of hierarchies which correspond to the following order \(N \in \mathbb{N}\) Lax matrices
\[
\hat{L}^{(N,\kappa_1,\kappa_2)} := DL^{(N,\kappa_1,\kappa_2)},
\]
where $D_L$ the diagonal deformation matrix with entries $(D_L)_{i,i} = (\alpha^{(i-\kappa_1)} - \beta x^{(i-\kappa_1)})^{-1}$, \(\beta, \alpha \in C(A^\times)\) and

\[
L^{(N,\kappa_1,\kappa_2)} := \nabla^{(\kappa_2)} + \Delta^{(\kappa_2)} + \nabla^{(\kappa_1)} \mathbf{X} + \lambda \Delta^{(\kappa_1)} \mathbf{X},
\]

with the matrices $\nabla, \Delta$, and $\mathbf{X}$ given as in Section 2 and $\kappa_1 \neq \kappa_2 \in \{1, \ldots, N\}$. In this setting, $\mathcal{K}^{(1)}$ corresponds to the Lax matrix $\hat{L}^{(N,1,0)}$, ($\nabla^0 + \Delta^0$ denotes the order $N$ identity matrix $I_N$). Similarly $\mathcal{K}^{(0)}$, serves as a member of a dual family of hierarchies which correspond to the following order $N \in \mathbb{N}$ Lax matrices

\[
\hat{M}^{(N,\kappa_1,\kappa_2)} := D_M M^{(N,\kappa_1,\kappa_2)},
\]

where $D_M$ the diagonal deformation matrix with entries $(D_M)_{i,i} = (\alpha - \beta x^{(i)})^{-1}$, $\alpha, \beta \in C(A^\times)$ and

\[
M^{(N,\kappa_1,\kappa_2)} := \nabla^{(\kappa_2)} \mathbf{X} + \Delta^{(\kappa_2)} \mathbf{X} + \nabla^{(\kappa_1)} + \lambda \Delta^{(\kappa_1)},
\]

where again $\kappa_1 \neq \kappa_2 \in \{1, \ldots, N\}$, and in this setting, $\mathcal{K}^{(0)}$ corresponds to the Lax matrix $\hat{M}^{(N,1,0)}$. We postpone the study of the discrete spectral problems associated with the mentioned Lax matrices for a future contribution.

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Appendix A. Non-Abelian lattice-potential KdV equations

A non-Abelian form of the lattice-potential KdV equation was firstly derived [21]. Here we propose an alternative $D_4$ symmetric form of the non-Abelian lattice-potential KdV equation, as well as a non-Abelian form that respects the rhombic symmetry.

The following Lax matrix

\[
L(x; \lambda) := \begin{pmatrix}
 x^{(1)} & (x^{(1)} + x^{(2)}) x^{(1)} - \lambda \\
 1 & x^{(1)}
\end{pmatrix},
\]

was introduced in [33] and corresponds to a 3D—compatible map that in the Abelian setting reduces to the companion map of the Adler map [73]. The compatibility
conditions \( L(u; \lambda)L(y; \lambda) = L(v; \lambda)L(x; \lambda) \), are equivalent to the following set of equations

\[
\begin{align*}
  u^{(1)} + y^{(1)} &= v^{(1)} + x^{(1)}, \\
  (u^{(1)} + u^{(2)}) u^{(1)} - (y^{(1)} + y^{(2)}) y^{(1)} &= (u^{(1)} + v^{(2)}) v^{(1)} - (x^{(1)} + x^{(2)}) x^{(1)}, \\
  (u^{(1)} + u^{(2)}) u^{(1)} + u^{(1)} y^{(1)} &= (u^{(1)} + v^{(2)}) v^{(1)} + v^{(1)} x^{(1)}, \\
  (u^{(1)} + u^{(2)}) (u^{(1)} y^{(1)} + u^{(1)} (y^{(1)} + y^{(2)}) y^{(1)} &= (v^{(1)} + v^{(2)}) v^{(1)} x^{(1)} + v^{(1)} (x^{(1)} + x^{(2)}) x^{(1)},
\end{align*}
\]

that according to identification (6), serve as a difference system in edge variables. In that respect, equations (A.1) and (A.2), guarantee the existence of potential functions \( \phi, \psi \) such that:

\[
\begin{align*}
  x^{(1)} &= \phi_1 - \phi, \\
  y^{(1)} &= \phi_2 - \phi, \\
  (x^{(1)} + x^{(2)}) x^{(1)} &= \psi_1 + \psi, \\
  (y^{(1)} + y^{(2)}) y^{(1)} &= \psi_2 + \psi.
\end{align*}
\]

In terms of these potential functions, (A.1) and (A.2) are identically satisfied while equations (A.3) and (A.4) respectively read

\[
\begin{align*}
  (\phi_{12} - \phi_2)(\phi_2 - \phi) - (\phi_{12} - \phi_1)(\phi_1 - \phi) &= \psi_1 - \psi_2, \\
  (\psi_2 + \psi_1)(\phi_2 - \phi) + (\phi_{12} - \phi_2)(\psi_2 + \psi) &= (\psi_1 + \psi_1)(\phi_1 - \phi) + (\phi_{12} - \phi_1)(\psi_1 + \psi),
\end{align*}
\]

and constitute a non-Abelian form of lattice-potential KdV equation. Note that this form of the lattice-potential KdV equation respects the rhombic symmetry, hence it is defined on a black-white lattice.

Furthermore, if the centrality assumptions \( x^{(2)} x^{(1)} = -p, y^{(2)} y^{(1)} = -q \) are assumed, relations (A.5) give

\[
\begin{align*}
  (\phi_1 - \phi)(\phi_1 - \phi) - p &= \psi_1 + \psi, \\
  (\phi_2 - \phi)(\phi_2 - \phi) - q &= \psi_2 + \psi,
\end{align*}
\]

that serve as the Bäcklund transformation between the non-Abelian multiquadratic relation§ that the potential \( \psi \) satisfies and the \( D_4 \) symmetric form of the non-Abelian lattice-potential KdV equation that the potential function \( \phi \) satisfies i.e.

\[
(\phi_1 - \phi_2)(\phi_{12} - \phi) + (\phi_{12} - \phi)(\phi_1 - \phi_2) = 2(p - q).
\]

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§ In the Abelian case this multiquadratic relation was first obtained in [74] and serves as the nonlinear superposition principle of the Bäcklund transformation of the KdV equation.
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