Uniform approximation of paraxial flat-topped beams

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A uniform asymptotic theory of the free-space paraxial propagation of coherent flattened Gaussian beams is proposed in the limit of nonsmall Fresnel numbers. The pivotal role played by the error function in the mathematical description of the related wavefield is stressed.

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1. Introduction

Several analytical models aimed at describing the propagation of coherent wavefields having an “initial” flat-topped profile were proposed in the past. Among them the supergaussian and the Fermi-Dirac profiles present a very simple mathematical structure but their free-space propagation cannot be described without resorting to numerical integration, even within paraxial approximation. To overcome such difficulties, in 1994 Gori introduced the first model, called flattened Gaussian (FG henceforth) [1], able to describe the paraxial propagation of coherent flat-topped beams in exact terms [2]. Differently from the supergaussian or the Fermi-Dirac, FG profiles are expressed through a finite sum in the following way:

\[
\text{FG}_N(\xi) = \exp(-\xi^2) \sum_{m=0}^{N} \frac{1}{m!} \xi^{2m},
\]

where \(N\) denotes the so-called order of the FG profile. Equation (1) was originally derived by starting from the identity

\[
1 = \exp(-\xi^2) \exp(\xi^2),
\]

and on truncating the Taylor expansion of the second exponential up to \(N\). On further rescaling the argument \(\xi\) by a factor \(\sqrt{N + 1}\), the FG profile assumes the characteristic flat-topped shape which for \(N = 0\) reduces to a Gaussian distribution whereas for \(N \to \infty\) tends to coincide with the characteristic function of the unitary disk. Since their introduction FG beams have proved to be a good model for describing the paraxial propagation of flat-topped beams, since the initial profile in Eq. (1) can be recast in terms of a finite number of Laguerre Gauss distributions, both of standard and “elegant” kind [3]. Shortly after its introduction, it was also proved that the initial distribution in Eq. (1) is practically indistinguishable from a supergaussian one with a suitable and simple adjustment of their respective parameters [4]. In 2002, Li [5] gave a more rigorous mathematical ground to the Gori’s trick used in Eq. (2), by finding that, for a typical axisymmetric profile, say \(F_N(\xi)\), to display a flat-topped shape it is mandatory to have the first \(N\) even \(\xi\)-derivatives to be vanishing at \(\xi = 0\). Then, on using such condition, which is certainly satisfied by Eq. (1), Li introduced a new class of flat-topped profiles which, differently from the Gori’s ones, are easily expressable via the superposition of \(N\) fundamental Gaussian functions having different widths, suitably chosen. Both Gori’s and Li’s approaches allow the paraxial propagation problem of flat-topped beams to be solved without approximations, as the
resulting fields turns out to be described in terms of finite sums and, at present, Refs. [1] and [5] have inspired hundreds of papers [6]. At the same time, however, although no computational difficulties are encountered in the numerical evaluation of propagated FG beams field, the behavior of those sums is hard to grasp, especially from an intuitive point of view and this led several authors to think that the FG model has a demanding analytical structure. The limiting cases \( N = 0 \) and \( N = \infty \) correspond to initial field distributions that are in some respect antithetic as far as their features are concerned: the former is the simpler example of tranverse coherent shape-invariant field distribution, whereas in the latter the presence of the field discontinuity leads to a complex and mathematically untractable (at least from an analytical viewpoint) behavior of the related paraxial free propagation features [7]. In between, the FG mathematics, whose exploration is the topic of the present work.

In particular, our aim is to find a uniform [8], with respect to the transverse radial position, analytical approximation of the free-space propagated field of a \( N \)th-order FG beams. The idea for the present work starts just from the limit \( N = \infty \) which, as said above, corresponds to the Fresnel diffraction by a circular hole [7] and from the, apparently not so known, asymptotic approximation of the related propagated field found in 1898 by Karl Schwarzschild [9], whose derivation can be found in Sec. 4.3 of [10], which is a much more accessible paper than the Schwarzschild’s original one. In particular, within such approximation a pivotal role is played, especially for large values of the Fresnel number [7], by the error function which rules the mathematical behaviour of the overall diffracted field. Certainly the presence of the error function in the uniform asymptotics of diffracted wavefields by hard-edge apertures is not surprising, as pointed out for instance by Stamnes thirty years ago [8], as it makes the field transition across the hole geometric shadow smooth. However, although for finite \( N \) the FG initial field distribution is continuous, our analysis will show that, even for moderately small values of \( N \) (of the order of ten), the propagated field is, in the limit of nonsmall values of the Fresnel number (roughly greater than two), still well represented by an error function. In particular across the initial plane (corresponding to an infinites Fresnel number) our analysis provides an interesting and, at least up to our knowledge, unsuspected direct connection between error function and FG profiles in Eq. (1), whose mathematical justification will be done \textit{a posteriori} by using the above quoted Li’s flatness prescription.
2. Theoretical Analysis

We start our analysis from the field distribution of an axisymmetric FG beam of order $N$ at the plane $z = 0$ of a cylindrical reference frame $(r, z)$, given by

$$U_N(r; 0) = \text{FG}_N \left( \frac{r}{w_0} \sqrt{N + 1} \right),$$

where an unessential amplitude constant has been set to one for simplicity and the symbol $w_0$ denotes the spot-size of the FG beam. To express the field, say $U_N(r; z)$, propagated in the free space at the transverse plane $z > 0$ we express the FG profile in Eq. (1) as follows [3]:

$$\text{FG}_N(\xi) = \sum_{m=0}^{N} (-1)^m \binom{N + 1}{m + 1} L_m(\xi^2) \exp(-\xi^2),$$

where $L_m(\cdot)$ denotes the $m$th-order Laguerre polynomial [11]. The decomposition in Eq. (4) has a clear physical interpretation once inserted into Eq. (3): the field of a $N$th-order FG beam is thought of as the superposition of elegant Laguerre-Gaussian modes [12], whose functional form is essentially the product of a Laguerre polynomial times a Gaussian function with the same complex argument. In this way it can be possible to express the propagated transverse field distribution $U_N(r; z)$ as follows [3]:

$$U_N(r; z) = \exp(\text{i}kz) \frac{\exp\left(\frac{N + 1}{\pi N_F} \left( \frac{r}{w_0} \right)^2 \right)}{1 + \text{i}N + 1}{\pi N_F} - \frac{N + 1}{1 + \text{i}N + 1}{\pi N_F} \left( \frac{r}{w_0} \right)^2 \right] \times \mathcal{G}_N \left[ \frac{1}{1 + \text{i}N + 1}{\pi N_F} , \frac{N + 1}{1 + \text{i}N + 1}{\pi N_F} \left( \frac{r}{w_0} \right)^2 \right],$$

where $N_F = w_0^2/\lambda z$ denotes the Fresnel number [7] and where the function $\mathcal{G}_N(\cdot, \cdot)$ is defined as

$$\mathcal{G}_N(t, s) = \sum_{n=0}^{N} (-1)^n \binom{N + 1}{n + 1} t^n L_n(s).$$

Note, in particular, that in the last equation the variables $s$ and $t$ are non independent, since

$$s = \frac{r^2}{w_0^2} (N + 1) t.$$

Equations (5) and (6) provide the exact expression of the propagated FG beam; moreover, the numerical evaluation of the $\mathcal{G}_N$ can be efficiently implemented, for large values of $N$, via a simple recursive computational scheme [3].
To derive a uniform asymptotic approximation of $U_N(r; z)$ with respect to $r$ we start from the following integral representation of Laguerre polynomials [13, Eq. (5.4.1)]:

$$L_n(s) = \frac{\exp(s)}{n!} \int_0^\infty d\xi \exp(-\xi) \xi^n J_0 \left(2 \sqrt{s \xi}\right),$$  \hspace{1cm} (8)

where $J_n(\cdot)$ denotes the $n$th-order Bessel function of the first kind [11]. On substituting from Eq. (8) into Eq. (6) and on taking into account that

$$\sum_{n=0}^N \frac{(-1)^n}{n!} \binom{N+1}{n+1} x^n = L_N^{(1)}(x),$$  \hspace{1cm} (9)

with $L_n^{(\alpha)}(\cdot)$ denoting the generalized Laguerre polynomial of order $n$ and degree $\alpha$ [11], the following integral representation of the function $G_N(t, s)$ is obtained:

$$\exp(-s) G_N(t, s) =$$ \hspace{1cm} (10)

$$= \int_0^\infty d\xi \exp(-\xi) J_0 \left(2 \sqrt{s \xi}\right) L_N^{(1)}(\xi t),$$

which will be estimated in the limit of non nonsmall values of $N$. To this end the Laguerre polynomial $L_N^{(1)}$ is first approximated by using Eq. (8.22.4) of [13] which gives

$$\exp \left(\frac{-\xi t}{2}\right) \frac{L_N^{(1)}(\xi t)}{N+1} \approx \frac{2 J_1 \left[2 \sqrt{(N+1) \xi t}\right]}{2 \sqrt{(N+1) \xi t}},$$  \hspace{1cm} (11)

that, once substituted into Eq. (10), on taking Eq. (7) into account, and after letting $\eta = (N+1) \xi t$, leads to

$$\exp(-s) G_N(t, s) \approx$$ \hspace{1cm} (12)

$$\approx \frac{2}{\sqrt{t}} \int_0^\infty d\eta \exp(-p\eta^2) J_0 \left(2 \frac{r}{w_0} \eta \sqrt{t}\right) J_1 \left(2 \eta \sqrt{t}\right),$$

where

$$p = \frac{1}{N+1} \left(1 - \frac{t}{2}\right).$$  \hspace{1cm} (13)

The integral in Eq. (12) can be evaluated on using formula 2.12.39.1 of [14] which, after long but straightforward algebra, allows Eq. (5) to be recast as follows:

$$U_N(r; z) \approx \exp(ikz) \mathcal{J} \left[\frac{2(N+1)}{1 + 2i \frac{N+1}{\pi N_F} \frac{r}{w_0}}, \frac{r}{w_0}\right],$$  \hspace{1cm} (14)
where the function $J(u, \rho)$ is defined by

$$J(u, \rho) = 1 - \exp(-u\rho^2) \int_u^\infty d\xi \exp(-\xi) I_0(2\rho \sqrt{\xi} u), \quad (15)$$

with $I_0(\cdot)$ denoting the zeroth-order modified Bessel function of the first kind [11].

Differently from the integral representation in Eq. (23), the function $J(u, \rho)$ is expressed in a mathematical form suitable to extract a uniform approximation with respect to $\rho$. To this end we make use of the, above quoted, Schwarzschild’s approach originally employed to study the Fresnel diffraction from a circular hole [10]. First of all we replace the modified Bessel function by its asymptotic expansion, namely

$$I_0(x) \sim \frac{\exp(x)}{\sqrt{2\pi x}}, \quad |x| \to \infty, \quad (16)$$

which, once substituted into Eq. (15) and after changing the integration variable $\xi$ into $u \xi^2$, gives, for $|u| \gg 1$,

$$J(u, \rho) \sim 1 - \sqrt{\frac{u}{\pi \rho}} \int_1^\infty \sqrt{\xi} d\xi \exp[-u(\xi - \rho)^2]. \quad (17)$$

The last integral can now be estimated by replacing the factor $\sqrt{\xi}$ by $(\sqrt{\xi} - \sqrt{\rho}) + \sqrt{\rho}$, so that

$$J(u, \rho) \sim 1 - \sqrt{\frac{u}{\pi \rho}} \int_1^\infty d\xi \exp[-u(\xi - \rho)^2]$$

$$- \sqrt{\frac{u}{\pi \rho}} \int_1^\infty d\xi (\sqrt{\xi} - \sqrt{\rho}) \exp[-u(\xi - \rho)^2]. \quad (18)$$

The first integral can be evaluated exactly by using formula 2.3.15.4 of [14], which gives

$$\sqrt{\frac{u}{\pi}} \int_1^\infty d\xi \exp[-u(\xi - \rho)^2] = \frac{1}{2} \text{erfc}[\sqrt{u}(1 - \rho)], \quad (19)$$

where $\text{erfc}(\cdot)$ denotes the complementary error function [11]. As far as the second integral in Eq. (18) is concerned, a first partial integration gives at once

$$- \sqrt{\frac{u}{\pi \rho}} \int_1^\infty d\xi (\sqrt{\xi} - \sqrt{\rho}) \exp[-u(\xi - \rho)^2] =$$

$$- \frac{1}{2\sqrt{u\pi \rho}} \exp[-u(1 - \rho)^2]$$

$$+ \frac{1}{4\sqrt{u\pi \rho}} \int_1^\infty d\xi \frac{\exp[-u(\xi - \rho)^2]}{\sqrt{\xi}(\sqrt{\xi} + \sqrt{\rho})^2}. \quad (20)$$
while further partial integrations would give rise to an asymptotic expansion in negative powers of \( u \) whose single terms can easily find by using standard techniques \[15\]. In Appendix \[A\] it is shown that

\[
\frac{1}{4\sqrt{u\pi \rho}} \int_{1}^{\infty} \frac{d\xi}{\sqrt{\xi}} \exp[-u(\xi - \rho)^2] \sim \frac{1}{32 u\rho^2} \text{erfc}[\sqrt{u}(1 - \rho)]
\]

and in the same Appendix it is also given to the interested reader an idea about how to derive the complete asymptotic series in such a way that the function \( \mathcal{J} \) can be written as follows:

\[
\mathcal{J}(u, \rho) = 1 - \frac{1}{2} \text{erfc}[\sqrt{u}(1 - \rho)] \sum_{m=0}^{\infty} \frac{a_m(\rho)}{u^m} - \frac{1}{2\sqrt{\pi}} \frac{\exp[-u(1 - \rho)^2]}{1 + \sqrt{\rho}} \sum_{m=0}^{\infty} \frac{b_m(\rho)}{u^{m+1/2}}, \quad |u| \gg 1,
\]

where

\[
a_0 = 1, \quad a_1 = -\frac{1}{16\rho^2}, \quad a_2 = -\frac{5}{512\rho^4}, \ldots,
\]

\[
b_0 = \frac{1}{\sqrt{\rho}}, \quad b_1 = \frac{1}{16\rho^2} \frac{4\rho + 3\sqrt{\rho} + 1}{(1 + \sqrt{\rho})^2}, \ldots,
\]

Equations (22) and (23) constitutes the main results of the present Paper and suggest an intriguing and, at least up to our knowledge, unexpected mathematical connection between FG profiles and the error function.

3. A new analytical form for flat-topped profiles

First of all we note that in the limit \( N_F \to \infty \) the propagated field \( U_N(r; z) \) must coincide with the initial field \( U_N(r; 0) \). From Eq. (14) we have

\[
U_N(r; 0) \simeq \mathcal{J} \left[ 2(N + 1) \frac{r}{w_0} \right],
\]

which, once compared with Eq. (3), leads to the following asymptotic relation:

\[
\text{FG}_N(\xi) \simeq 1 - \frac{1}{2} \text{erfc} \left[ \sqrt{2}(\sqrt{N+1} - \xi) \right],
\]
and where only the leading term in the expansion of Eq. (22) has been kept. Figure 1 shows a visual comparison of FG profiles obtained through the definition in Eq. (1) (solid curve) and through the asymptotic estimate in Eq. (25) (dashed curves).

It is surprising how, except for the case $N = 1$, the agreement between the two curves is remarkably good also for reasonably small values of $N$ (of the order of ten), the two curves being practically undistinguishable, at least at a visual level, for $N = 30$. To give an explanation of such behavior it is sufficient to prove that an initial erfc-based profile in the r.h.s. of Eq. (25), when is evaluated at the scaled variable $\xi/\sqrt{N + 1}$, satisfies the Li's flatness condition which, i.e.,

$$\left. \frac{d^{2n}}{d\xi^{2n}} F G_N \left( \frac{\xi}{\sqrt{N + 1}} \right) \right|_{\xi=0} = 0, \quad n = 1, 2, \ldots, N.$$  

In Appendix B it is shown that, when the FG profile is approximated via Eq. (25), it turns out to be

$$\left. \frac{d^{2n}}{d\xi^{2n}} F G_N \left( \frac{\xi}{\sqrt{N + 1}} \right) \right|_{\xi=0} \simeq$$

$$\simeq - \frac{1}{\sqrt{\pi}} \left( \frac{2}{N + 1} \right)^n \exp[-2(N + 1)] H_{2n-1}(\sqrt{2}\sqrt{N + 1}),$$

where $H_n(\cdot)$ denotes the $n$th-order Hermite polynomial [11]. Accordingly, we have to prove that the r.h.s. of Eq. (27) is negligible for $n < N$ with respect to the values assumed for $n > N$. A possibility is again offered by asymptotics: in Appendix C is it shown that, in the limit of nonsmall values of $n$, Eq. (27) can be replaced by

$$\left. \frac{d^{2n}}{d\xi^{2n}} F G_N \left( \frac{\xi}{\sqrt{N + 1}} \right) \right|_{\xi=0} \sim \frac{(n - 1)!}{2\pi} \left( - \frac{8}{N + 1} \right)^n$$

$$\times \exp[-(N + 1)] \sin[\sqrt{2}\sqrt{N + 1}\sqrt{4n - 1}],$$

which, on taking the absolute value of both members and on replacing the modulus of the sinusoidal function by its unitary upper bound, leads to the inequality

$$\left| \left. \frac{d^{2n}}{d\xi^{2n}} F G_N \left( \frac{\xi}{\sqrt{N + 1}} \right) \right|_{\xi=0} \right| \leq$$

$$\leq \frac{(n - 1)!}{2\pi} \left( \frac{8}{N + 1} \right)^n \exp[-(N + 1)].$$
Figure 2(a) shows a visual comparison between the behaviors of the modulus of the 2nth-order $\xi$-derivative in Eq. (27) (open circles), together with the asymptotic upper bound given in Eq. (29) (solid curve) for a typical value of the FG order $N$ (in this case $N = 30$). In order to better appreciate, from a quantitative point of view, the agreement between the two behaviors, the same plot is drawn in Fig. 2(b) but on a vertical logarithmic scale. Inequality (29) is the key to grasp why the erfc-based approximation in Eq. (25) satisfies the Li’s flatten condition: it predicts a “power times factorial” law which is known to displays the characteristic threshold-like behavior of Fig. 2(a) [16]. In particular, the terms of the sequence corresponding to the r.h.s. of Eq. (29) turns out to be exponentially small for $n < N$, whereas they grow according to a factorial divergence law for $n > N$. In Appendix D a nonrigorous proof that the threshold is around $n \sim N$ is given to the interested reader.

4. Some numerical results on free-space propagation

The present section is aimed at exploring, from a numerical point of view, the free-space propagation of FG beams by presenting a comparison between the exact values, obtained through Eqs. (5) and (6), and those provided by the uniform approximation in Eqs. (14), (22), and (23) for a FG profile of order $N = 30$. In Fig. 3 the amplitude (a) and the phase (b) of the FG beam propagated at the Fresnel number $N_F = 10$ are shown as functions of the normalized transverse radial distance $r/w_0$ (the dotted curve is the initial FG profile). Open circles are the exact values provided by Eq. (5), whereas the solid curve is the asymptotic estimate obtained by keeping only the leading term in the asymptotics expansion of Eq. (22). The dotted curve represents the (normalized) initial FG profile. As we can see, the agreement is really good. On reducing the Fresnel number to $N_F = 5$ the amplitude and phase distributions are those reported in Fig. 4(a) and Fig. 4(b), respectively. Also in this case it is possible to appreciate a reasonably acceptable agreement, especially far from the beam axis. On further reducing the $N_F$ the agreement continues to degrade: the situation at $N_F = 2$ is depicted in Fig. 5 where it is clear that the sole leading erfc-based asymptotics is no longer able to guarantee an acceptable, even at a visual level, agreement for the transverse beam amplitude. To try to improve such an agreement we add to the leading term of the asymptotics expansions in Eq. (22) that corresponding to $b_0$, thus obtaining the plots of Fig. 5 which now displays a much better agreement, especially close to the beam axis.

To conclude the present section we want to present a simulation concerning with the
case $N_F = 1$, which is customarily assumed as the boundary between the near-zone and the far-zone \[1\]. Figure 7 shows the amplitude (a) and phase (b) distributions obtained by keeping only the leading erfc-based term (dotted curve), by including the term $b_0$ (dashed curve), and by adding also $a_1$ and $a_2$ (solid curve). It must be appreciated that, although our analysis has been formally developed in the limit $N_F \gg 1$, the asymptotic series found above are able, with a modest increase of the computational effort, to provide a reasonably good description of the propagated field also beyond the analysis limits.

5. Conclusions

Still nowadays the supergaussian seems to be, especially among experimentalists, the most known and used model for dealing with flat-top beams \[17\]–\[21\], although it is well known that their propagation features are untractable in analytical terms. On the other hand, the “exact” models introduced by Gori and later by Li suffered by a (only apparent) formal complication of the propagated field expression which led several authors to still prefer the older and obsolete models. In the present paper we have shown that some interesting general features of the free-space propagation of flattened Gaussian beams can be grasped via uniform asymptotics. On using the FG beam representation in terms of elegant Laguerre-Gauss modes together with the Schwarzschild’s approach to derive uniform approximations of integrals we have found that the propagated field is well approximated, for large Fresnel numbers, by a simple error function. In particular, starting from the initial plane, the functional form of the propagated field remains the same and the effect of the propagation is taken into account by a simple (complex) scaling of the error function argument. On further reducing the Fresnel number this sort of “functional invariance” is progressively lost and it is necessary to include higher-order terms of the asymptotic series to achieve acceptable agreements between the exact and the approximated field. Since it has been proved in the past the practical indistinguishability of FG and supergaussian profiles, the results obtained in the present paper apply, upon evaluation of the related parameters, also to the latter. Moreover, a byproduct of our analysis is a new, at least up to our knowledge, analytical expression for flat-topped profiles, whose related flatness conditions have been quantitatively verified according to the Li’s prescriptions.

Finally, while the role of the error function in the mathematical treatment of diffraction by hard-edge apertures is well known, our results seem to suggest its involvement also
when dealing with smoother initial field distributions; moreover, since the representation
of FG beams via elegant Laguerre-Gauss modes is not limited to the case of the free-space
propagation it should be possible, in principle, to extend the analysis presented here to the
propagation problem through a typical paraxial ABCD optical system.

Appendix A: Proof of Eq. (21)

The Schwarzschild’s trick can be thought of as the first step of a systematic procedure to
extract uniform approximations of integrals, as pointed out for instance by Temme [15]. To
show this consider first the evaluation of integrals of the type
\[
\int_1^\infty d\xi f(\xi) \exp[-u(\xi - \rho)^2],
\]
where \( f(\cdot) \) denotes a function sufficiently regular in the interval \([1, \infty]\). To obtain a uniform
approximation with respect to \( \rho \) we have to replace the factor \( f(\xi) \) with \( f(\rho) + [f(\xi) - f(\rho)] \)
in such a way that Eq. (A1) becomes
\[
\begin{align*}
&f(\rho) \int_1^\infty d\xi \exp[-u(\xi - \rho)^2] \\
&\quad + \int_1^\infty d\xi [f(\xi) - f(\rho)] \exp[-u(\xi - \rho)^2].
\end{align*}
\]
The first integral can be evaluated again via Eq. (19), while the second integral can be
rearranged for a further partial integration, so that
\[
\int_1^\infty d\xi f(\xi) \exp[-u(\xi - \rho)^2] =
\]
\[
f(\rho) \sqrt{\frac{1}{2\pi u}} \text{erfc}[\sqrt{u}(1 - \rho)]
\]
\[
- \frac{1}{2u} \int_1^\infty \frac{f(\xi) - f(\rho)}{\xi - \rho} \exp[-u(\xi - \rho)^2] d\{\exp[-u(\xi - \rho)^2]\} =
\]
\[
f(\rho) \sqrt{\frac{1}{2\pi u}} \text{erfc}[\sqrt{u}(1 - \rho)]
\]
\[
+ \frac{1}{2u} \frac{f(1) - f(\rho)}{1 - \rho} \exp[-u(1 - \rho)^2]
\]
\[
+ \frac{1}{2u} \int_1^\infty \frac{d}{d\xi} \left[ \frac{f(\xi) - f(\rho)}{\xi - \rho} \right] \exp[-u(\xi - \rho)^2] d\xi,
\]
where the last integral gives a contribution of order $1/u^2$. To prove Eq. (21) it is sufficient to substitute into Eq. (A3) the function $f(\xi)$ given by

$$f(\xi) = \frac{1}{\sqrt{\xi} (\sqrt{\xi} + \sqrt{\rho})^2}, \quad (A4)$$

thus obtaining

$$\frac{1}{4\sqrt{u\pi\rho}} \int_1^\infty \frac{d\xi}{\sqrt{\xi}} \exp[-u(\xi - \rho)^2] \frac{1}{(\sqrt{\xi} + \sqrt{\rho})^2} \approx \frac{1}{32\ u^2 \rho} \text{erfc}[\sqrt{u}(1 - \rho)] \quad (A5)$$

$$+ \frac{\exp[-u(1 - \rho)^2]}{8u\sqrt{u\pi\rho}} \frac{4\rho\sqrt{\rho} - \rho - 2\sqrt{\rho} - 1}{4\rho\sqrt{\rho}(1 + \sqrt{\rho})^2(1 - \rho)},$$

where the last term can be further simplified on taking into account that

$$\frac{4\rho\sqrt{\rho} - \rho - 2\sqrt{\rho} - 1}{4\rho\sqrt{\rho}(1 + \sqrt{\rho})^2(1 - \rho)} =$$

$$= \frac{1}{4\rho\sqrt{\rho}(1 + \sqrt{\rho})^3} \frac{4\rho\sqrt{\rho} - \rho - 2\sqrt{\rho} - 1}{1 - \sqrt{\rho}} =$$

$$- \frac{4\rho + 3\sqrt{\rho} + 1}{4\rho\sqrt{\rho}(1 + \sqrt{\rho})^3},$$

so that, after rearranging, Eq. (A5) becomes

$$\frac{1}{4\sqrt{u\pi\rho}} \int_1^\infty \frac{d\xi}{\sqrt{\xi}} \exp[-u(\xi - \rho)^2] \frac{1}{(\sqrt{\xi} + \sqrt{\rho})^2} \approx \frac{1}{32\ u^2 \rho} \text{erfc}[\sqrt{u}(1 - \rho)]$$

$$- \frac{1}{2\sqrt{u\pi\rho}} \exp[-u(1 - \rho)^2] \frac{1}{1 + \sqrt{\rho}} \frac{\sqrt{\rho}(4\rho + 3\sqrt{\rho} + 1)}{16u^2 \rho} \frac{1}{(1 + \sqrt{\rho})^2}$$

$$+ \frac{1}{32\sqrt{\pi u^2 \rho^3}} \int_1^\infty d\xi \frac{\sqrt{\xi} + 4\sqrt{\rho}}{\sqrt{\xi} (\sqrt{\xi} + \sqrt{\rho})^4} \exp[-u(\xi - \rho)^2],$$

which, on neglecting the last integral, coincides with Eq. (21).
Appendix B: Proof of Eq. (27)

On substituting from Eq. (25) into Eq. (26) and on letting $\zeta = \xi / \sqrt{N+1}$, we have

$$\left[ \frac{d^{2n}}{d\xi^{2n}} F G_N \left( \frac{\xi}{\sqrt{N+1}} \right) \right]_{\xi=0} \approx$$

$$\approx - \frac{1}{2(N+1)^n} \left\{ \frac{d^{2n}}{d\zeta^{2n}} \text{erfc}[\sqrt{2}(\sqrt{N+1} - \zeta)] \right\}_{\zeta=0} = \quad (B1)$$

$$= \frac{1}{2(N+1)^n} \left[ \frac{d^{2n}}{d\zeta^{2n}} \text{erf}(\sqrt{2}\zeta) \right]_{\zeta=-\sqrt{N+1}},$$

and, on using formula 1.5.1.1 of [22], Eq. (27) follows.

Appendix C: Proof of Eq. (28)

We start from the following asymptotics of Hermite polynomials [see formula 8.22.7 of [13]]:

$$\exp \left( -\frac{x^2}{2} \right) H_{2n-1}(x) \sim -\frac{(-1)^n}{\sqrt{4n-1}} \frac{(2n)!}{n!} \sin(x\sqrt{4n-1}) , \quad (C1)$$

valid for nonsmall values of $n$. On using Stirling’s formula to estimate the factorials, Eq. (C1) becomes

$$\exp \left( -\frac{x^2}{2} \right) H_{2n-1}(x) \sim -\frac{(-1)^n}{\sqrt{4n-1}} \frac{2^{2n}n!}{\sqrt{\pi n}} \sin(x\sqrt{4n-1}) , \quad (C2)$$

and, on replacing $\sqrt{4n-1}$ by $2\sqrt{n}$, we obtain

$$\exp \left( -\frac{x^2}{2} \right) H_{2n-1}(x) \sim -\frac{(-4)^n}{2\sqrt{\pi}} (n-1)! \sin(x\sqrt{4n-1}) , \quad (C3)$$

which, once substituted into Eq. (27), after some algebra leads to Eq. (28).

Appendix D: Proving that the threshold is at $n \sim N$

We could conventionally set the treshold as the value of $n$ at which the modulus of the r.h.s. of Eq. (29) is of the order of the unity. Taking the logarithm this implies that, neglecting the factor $1/(2\pi)$,

$$\log \Gamma(n) - n \log \frac{N+1}{8} - (N+1) \sim 0 , \quad n \gg 1 , \quad (D1)$$

where $\Gamma(\cdot)$ denotes the gamma function [11]. For large values of $n$ the following asymptotics holds:

$$\log \Gamma(n) \sim n \log n - n , \quad n \gg 1 , \quad (D2)$$
which, after substituted into Eq. (D1) and after taking into account that \( \log 8 \simeq 2 \), gives

\[
n \log n + n - n \log(N + 1) - (N + 1) \sim 0, \quad n \gg 1,
\]

which, in the limit of nonsmall \( N \)’s, gives \( n \sim N \).
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List of figure captions

Fig. 1. Flattened Gaussian profiles $FG_N(\xi)$ evaluated, for different values of $N$, through Eq. (1) (solid curves) and through the asymptotic estimate in Eq. (25) (dashed curves).

Fig. 2. (a): behavior of the modulus of the $2n$th-order $\xi$-derivative in Eq. (27) (open circles), together with the asymptotic estimate in Eq. (29) (solid curve). $N = 30$. (b): the same plot as in figure (a), but on a vertical logarithmic scale.

Fig. 3. Behavior of the amplitude of a FG beam of order $N = 30$ propagated at the Fresnel number $N_F = 10$. Open circles: exact values provided by Eq. (5). Solid curve: erfc-based asymptotic estimate by keeping only the leading term in the asymptotics expansion in Eq. (22). The dotted curve represents the (normalized) initial FG profile. The phase distribution is wrapped.

Fig. 4. The same as in Fig. 3 but for $N_F = 5$.

Fig. 5. The same as in Fig. 3 but for $N_F = 2$.

Fig. 6. The same as in Fig. 5 but now including the term $b_0$ in the asymptotic expansion of Eq. (22).

Fig. 7. Amplitude (a) and phase (b) distributions obtained by keeping only the leading erfc-based term (dotted curve), by including the term $b_0$ (dashed curve), and by adding also $a_1$ and $a_2$ (solid curve). $N_F = 1$. 

17
Figure 1 - Riccardo Borghi
Figure 3 - Riccardo Borghi
Figure 4 - Riccardo Borghi
Figure 5 - Riccardo Borghi
Figure 6 - Riccardo Borghi
Figure 7 - Riccardo Borghi