First Degree Cohomology of Specht Modules for Two Part Partitions and $p$-ary Designs

Liam Jolliffe

Abstract

Using purely combinatorial methods we calculate the first degree cohomology of Specht modules indexed by two part partitions over fields of characteristic $p \geq 3$. These combinatorial methods give an explicit description of all of the non-split extensions of the Specht module, $S^\lambda$, by the trivial module and is related to the construction of universal $p$-ary designs of constant block size. We also solve the problem of the existence of such designs and show their uniqueness, up to similarity, unless $\lambda$ is pointed, in which case we decompose the design into two parts, which are unique up to similarity.

1 Introduction

We shall briefly review some concepts from the representation theory of the symmetric group in this section, but we refer the reader to James’ book [9], from which our notation is taken, for more detail. Let $\lambda \vdash n$ be a partition and let $S^\lambda$ be the corresponding Specht module for the symmetric group $S_n$. The Specht module is a submodule of the transitive permutation module $M^\lambda$. The cohomology $H^i(S_n, S^\lambda) = \text{Ext}_{S_n}^i(k, S^\lambda)$ is known for $i \leq 1$. Indeed, $H^0(S_n, S^\lambda) = \text{Hom}_{kS_n}(k, S^\lambda)$ and is determined module $M^\lambda$, as $\text{Hom}_{kS_n}(k, S^\lambda)$ is contained in the one dimensional $\text{Hom}_{kS_n}(k, M^\lambda)$. This means that calculating $H^0(S_n, S^\lambda)$ is equivalent to determining if the trivial submodule of $M^\lambda$ is also contained in $S^\lambda$, which is an entirely combinatorial task, via James’ Kernel Intersection Theorem [Theorem 11].

Hemmer [7] had suggested an alternative approach for calculating $H^1(S_n, S^\lambda)$ similar to the approach of James [8] in calculating $H^0(S_n, S^\lambda)$, which is the one we shall take in this paper. This approach is based on the observation that, like
\(H^0\), the first cohomology \(H^1\) is also determined by \(M^\lambda\) when the field \(k\) has odd characteristic, as in this case any non-split extension of \(S^\lambda\) by the trivial module, \(k\), is contained in \(M^\lambda\). This approach is entirely combinatorial and remains within the setting of the representation theory of the symmetric group. The other benefit of this approach is that it also gives an explicit description of the non-split extensions of \(S^\lambda\) by \(k\). We will complete the calculation of \(H^1(S_n, S^\lambda)\) in the case that \(\lambda\) is a two part partition, recovering the result of Donkin and Geranios in this case \([2]\). We remark that this Hemmer had also calculated \(H^1(S_n, S^\lambda)\) for two part partitions \([6]\), although not via the combinatorial approach he suggested which motivated this paper; instead he shows that the calculation follows from work of Erdmann \([3]\) on the cohomology of \(SL_2\), and so the proof does not remain in the setting of the symmetric group.

Recall that given a partition \(\lambda \vdash n\), a \(\lambda\)-tableau is a bijection from \([n] := \{1, \ldots, n\}\) to \([\lambda]\), the Young diagram of shape \(\lambda\), and a \(\lambda\)-tabloid is an equivalence class of \(\lambda\)-tableaux under the relation of row equivalence: \(t \sim_R s\) if the entries in each row of \(t\) are the same as the entries in the corresponding row of \(s\). These equivalence classes will be denoted by writing the name of the tableau in braces, \(\{t\}\). There is an obvious action of the symmetric group \(S_n\) on the set of \(\lambda\)-tabloids, by permuting the entries. We extend this set to a vector space over a field \(k\) by taking formal sums of \(\lambda\)-tabloids, and we call the resulting permutation module \(M^\lambda\). In fact, we may extend this and define \(M^\lambda\) in the case where \(\lambda\) is a composition of \(n\) rather than a partition; that is \(\sum_{i=0}^r \lambda_i = n\), but we do not require that the \(\lambda_i\) are non-increasing.

Let \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n\) and let \(i, v \in \mathbb{N}\) be such that \(i < r\) and \(v \leq \lambda_i\). Let \(\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + \lambda_{i+1} - v, v, \lambda_{i+2}, \ldots, \lambda_r)\). Define the homomorphism \(\psi_{i, v} : M^\lambda \rightarrow M^{\lambda'}\) by

\[
\psi_{i, v}(\{t\}) = \sum_{t' \in T_{i,v}} \{t'\},
\]

where the sum is over all those \(\{t'\}\) who agree with \(\{t\}\) on all rows other than rows \(i\) and \(i + 1\), and whose \(i + 1\)th row is a subset of the \(i + 1\)th row of \(\{t\}\). The Specht module \(S^\lambda\) can be described as the span of set of distinguished elements of \(M^\lambda\), the polytabloids, or equivalently can be characterised as follows:

**Theorem 1** (Kernel Intersection Theorem). \([3]\)

\[
S^\lambda = \bigcap_{i=1}^{r-1} \bigcap_{v=0}^{\lambda_i-1} \text{Ker}(\psi_{i, v}) \subseteq M^\lambda.
\]

Denote the the sum of all \(\lambda\)-tabloids by \(f_\lambda\) and observe that \(f_\lambda\) is fixed by the action of \(S_n\). Clearly \(H^0(S_n, S^\lambda)\) is one dimensional if \(f_\lambda \in S^\lambda\) and is 0 otherwise, which allows us to prove the following:

**Theorem 2.** \(H^0(S_n, S^\lambda)\) is one dimensional if \(\binom{\lambda_i + j}{j} \equiv 0 \pmod{p}\) for \(1 \leq j \leq \lambda_{i+1}\) for all \(i < r\) and is 0 otherwise.

We call a partition satisfying the condition above a James partition. We will provide equivalent characterisations of James partitions in Lemma \([6]\) Hemmer proved a similar result for the first cohomology \([7]\):
Theorem 3. Let \( p \geq 3 \) and \( \lambda = (\lambda_1, \ldots, \lambda_r) \vdash n \), then \( \text{Ext}^1(k, S^\lambda) \neq 0 \) if and only if there is an element \( u \in M^\lambda \) with the following properties:

1. For each \( 1 \leq i < r \) and \( 0 \leq v < \lambda_i \), \( \psi_{i,v}(u) \in M^{\lambda'} \) is a multiple of \( f_{\lambda'} \), at least one of which is a non-zero multiple.

2. There does not exist a scalar \( c \in k \) such that all the \( \psi_{i,v}(c \cdot f_{\lambda} - u) = 0 \).

If such a \( u \) exists then the subspace \( \langle S^\lambda, u \rangle \subseteq M^\lambda \) spanned by \( S^\lambda \) and \( u \) is a non-split extension of \( u \).

We will call an element \( u \) satisfying the above conditions Hemmer. Over fields of characteristic \( p \geq 3 \), any non-split extension of \( S^\lambda \) by \( k \) is contained in \( M^\lambda \), and so for any non split extension we have such a \( u \). The second condition ensures that \( \langle S^\lambda, u \rangle \) is not the direct sum of \( S^\lambda \) and a trivial module, and is automatic when \( H^0(S_n, S^\lambda) \neq 0 \). In [12], Weber uses this method to give a far reaching combinatorial condition which sufficient for first degree cohomology to be trivial. We shall use Hemmer’s method to calculate the first cohomology of in the case where \( \lambda \) is a two part partition, \( \lambda = (a, b) \). In section 2 we construct a Hemmer element in \( M^{(a,b)} \) in the special case that \( b \) is a \( p \)-power, generalising the example given by Nguyen [11], which was in turn a generalisation of the example in Hemmer’s original paper [7]. We shall then follow Nguyen and draw connections between this Hemmer element and the theory of combinatorial designs, and describe how to construct a Hemmer element when \( \lambda = (a, b) \) is James. In section 4 we shall go further and construct a Hemmer element \( u \in M^\lambda \) for a larger class of two part partitions, namely pointed partitions, and study designs over fields of positive characteristic, \( p \)-ary designs. Careful analysis of these designs will then reveal, in section 5, that Hemmer elements, \( u \in M^{(a,b)} \), do not exist unless \( (a, b) \) is either James or pointed, and that our choice of extension \( \langle S^\lambda, u \rangle \) is unique up to isomorphism. Of course, the element \( u \) is not unique as if \( u \) is Hemmer, then so is \( u - v \) for any \( v \in S^\lambda \). We conclude by summarising the results of this paper in two different ways: first in the language of the theory of \( p \)-ary designs, and then in terms of the cohomology of Specht Modules, which was the motivation for this paper.

We shall now state some well known results on the divisibility of binomial coefficients, as many of the results in the theory of \( p \)-ary designs involve determining whether certain binomial coefficients are \( 0 \ (\text{mod } p) \) or not.

Let \( a = \sum_{i=0}^{\alpha} a_i p^i \) be the base \( p \) expansion of \( a \); that is \( 0 \leq a_i \leq p - 1 \) and \( a_\alpha \neq 0 \). The \( p \)-adic valuation \( \text{val}_p(a) \) is the least \( i \) such that \( a_i \) is non-zero, we call \( \alpha \) the \( p \)-adic length of \( a \) and write \( l_p(a) = \alpha \).

Lemma 4. Let \( p \) be a prime and \( a, b \in [N] \), then \( \text{val}_p\left(\binom{a+b}{b}\right) \), the highest power of \( p \) that divides \( \binom{a+b}{b} \), is the number of carries that occurs when \( a \) and \( b \) are added in their base \( p \) expansions.

Lemma 5. Let \( a = \sum_{i=0}^{r} a_i p^i \) and \( b = \sum_{i=0}^{r} b_i p^i \), with \( 0 \leq a_i, b_i \leq p - 1 \). Then

\[
\binom{a}{b} \equiv \binom{a_0}{b_0} \binom{a_1}{b_1} \cdots \binom{a_r}{b_r} \pmod{p}.
\]
In particular, \( \binom{a}{b} \equiv 0 \pmod{p} \) if and only if some \( a_i < b_i \).

**Lemma 6.** \([9]\) Let \( a, b \in \mathbb{N} \). The binomial coefficients \( \binom{a+1}{1}, \binom{a+2}{2}, \ldots, \binom{a+b}{b} \) are all divisible by \( p \) if and only if \( a \equiv 1 \pmod{p^{b/p(b)}} \).

**Remark.** This gives an alternative characterisation of a James partition, in particular \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) is James if and only if \( \lambda_i \equiv 1 \pmod{p^{b/p(\lambda_i+1)}} \) for all \( i < r \), or equivalently \( l_p(\lambda_i) < \text{val}_p(\lambda_i+1) \) for all \( i < r \).

### 2 A special case

In this section we shall construct a Hemmer element in \( M^\lambda \), when \( \lambda = (a, b) \) and \( b \) is a \( p \) power and \( a \not\equiv 1 \pmod{p^{b/p(b)-1}} \), extending a result of Nguyen \([11]\), who solved the case when \( (a, b) = (rp^\beta, p^\beta) \) for \( r \leq p - 1 \). We remark here that Nguyen constructs a candidate for \( u \) in the case that \( \lambda = (a, p^\beta) \) and \( a \equiv 1 \pmod{p^{\beta+1}} \), however this element is not Hemmer as it does not satisfy the second condition of Theorem 3. We first give an small example, in order to introduce some notation and to illustrate the more general example constructed in the proof of Theorem 3.

When \( \lambda = (a, b) \) is a two part partition, there is a natural bijection between \( \lambda \)-tabloids and subsets of \([a+b] \) of size \( b \). We shall make use of this throughout and identify the tabloid whose second row contains the elements \( x_1, \ldots, x_b \) with the set \( \{x_1, \ldots, x_b\} \). Let \( p = 3 \) and let \( \lambda = (3, 3) \vdash 6 \). Define \( u \in M^{(3,3)} \) by \( u = \sum \{t\} \), where the sum is over all \( \{t\} \) with 1 appearing in the top row. That is:

\[
\begin{align*}
u &= \{2, 3, 4\} + \{2, 3, 5\} + \{2, 4, 5\} + \{3, 4, 5\} + \{2, 3, 6\} \\
&\quad + \{2, 4, 6\} + \{3, 4, 6\} + \{2, 5, 6\} + \{3, 5, 6\} + \{4, 5, 6\}.
\end{align*}
\]

Observe that:

\[
\begin{align*}
\psi_{1,2}(u) &= 3 \cdot (\{2, 3\} + \{2, 4\} + \{3, 4\} + \{2, 5\} + \{3, 5\} \\
&\quad + \{4, 5\} + \{2, 6\} + \{3, 6\} + \{4, 6\} + \{5, 6\}) \\
&= 0 \\
\psi_{1,1}(u) &= 6 \cdot (\{2\} + \{3\} + \{4\} + \{5\} + \{6\}) \\
&= 0 \\
\psi_{1,0}(u) &= 10 \cdot \emptyset \\
&= -f_6.
\end{align*}
\]
While

\[\psi_{1,2}(f_{(3,3)}) = \binom{4}{1} f_{(4,2)} = f_{4,2}\]
\[\psi_{1,1}(f_{(3,3)}) = \binom{5}{2} f_{(5,1)} = f_{5,1}\]
\[\psi_{1,0}(u) = \binom{6}{3} f_{(6)} = -f_{(6)}\]

Clearly there is no scalar \(c \in k\) such that all the \(\psi_{i,v}(c \cdot f_{(3,3)} - u) = 0\) and thus \(u\) is Hemmer. We have shown:

**Proposition 7.** In characteristic 3, \(H^1(S_6, S^{(3,3)}) \neq 0\). Moreover, the module spanned by \(S^{(3,3)}\) and the element \(u\) above is a non-split extension of \(S^{(3,3)}\).

**Remark.** This is a different example to the Hemmer element constructed in [7], but the difference \(u - v\), where \(v\) is the Hemmer element from [7], is similar to \(f_{\lambda}\) in the sense of Definition 16.

A similar construction can be used whenever \(b = p^\beta\). If we take \(u\) to be the sum of all tabloids which have the entries 1, 2, ..., \(m\) all appearing in the top row then the coefficient of a set \(X\) of size \(v\) in \(\psi_{1,v}(u)\) is 0 if \(X \cap [m] \neq \emptyset\) and is \((a+b-m-v)^{(a+m)}\) otherwise. If we can find \(m\) such that this coefficient is zero except when \(v = 0\), then \(u\) will satisfy condition 1 of Theorem 3. In particular, we seek an \(m\) such that the binomial coefficients \(\binom{a-m+1}{1}, \binom{a-m+2}{2}, \ldots, \binom{a+m+b-1}{b-1}\) are all divisible by \(p\), which, by Lemma 4 occurs if and only if \(a - m \equiv -1 \ (mod\ p^{l+1})\). If \(a \equiv -1 \ (mod\ p^{l+1})\), then \(u\) will not satisfy condition 2 of Theorem 3 but otherwise we may set \(m = a - b + 1\) and \(u\) will be Hemmer.

**Theorem 8.** Let \(k\) have characteristic \(p \geq 3\), and let \(\lambda = (a,b) \vdash n\) be such that \(b = p^\beta\) and \(p^{\text{val}_p(a+1)} < b\). Then \(H^1(S_n, S^\lambda) \neq 0\).

**Proof.** The above discussion shows that if \(u\) is the sum of all tabloids whose top row contains the entries 1, 2, ..., \(a - b + 1\) then \(u\) is Hemmer. The result then follows from Theorem 3.

We shall now draw a connection between Hemmer elements and designs, allowing us to construct Hemmer elements for a larger class of partitions.

### 3 Designs

In this section we shall introduce the theory of designs, first over the integers, and then over fields of positive characteristic. We shall see that a Hemmer
element corresponds to a design which is not similar to the trivial design, and thus the combinatorial theory of designs will lead to an understanding of Specht module cohomology.

3.1 Integral Designs

We provide a brief overview of the theory of integral designs here, more detail can be found in [1].

**Definition 9.** Let \( v, b, \mu_1, \mu_2, \ldots, \mu_t \) be integers with \( 0 \leq t < b \leq v \) and each \( \mu_i \geq 0 \). Let \( V = [v] \), \( V = \{ X \mid X \subseteq V \} \) and \( V_b = \{ X \in V \mid |X| = b \} \). An integral \((v, \mu_1, \mu_2, \ldots, \mu_t)\)-design of block size \( b \) is a function \( c : V_b \to \mathbb{Z} \) such that

\[
\hat{c}(Y) := \sum_{X \supseteq Y} c(X) = \mu_s, \quad \text{if } |Y| = s \leq t.
\]

We call the integers \( \mu_i \) the coefficients of the design, and if all the coefficients are 0 then we call \( c \) a null design. Following [3], we denote the \( \mathbb{Z} \)-module of null \((v, \mu_1, \mu_2, \ldots, \mu_t)\)-designs of block size \( b \) by \( N_{v,b} \).

**Example 10.** Let \( \alpha \in \mathbb{Z} \), then the constant design \( c(X) = \alpha \) for all \( X \in V_b \) is an integral \((v, \mu_1, \mu_2, \ldots, \mu_t)\)-design of block size \( b \) with \( \mu_i = \alpha^{(\mu_i)} \).

Observe that if \( c \) is a \((a+b, \mu_1, \mu_2, \ldots, \mu_{b-1})\)-design of block size \( b \) and we define \( u = \sum_{X \subseteq [a+b]} c(X) X \in M^{(a,b)} \), then \( u \) satisfies condition 1 of Theorem 3.

Using this correspondence, we shall refer to such an element \( u \in M^{(a,b)} \) as a design. Graver and Jurkat [4] developed a method for constructing such a design, if one exists, and we outline their construction below.

**Theorem 11.** [3] Let \( v, b, \mu_1, \mu_2, \ldots, \mu_t \) be integers where \( v \geq 1 \) and \( 0 \leq t < b \leq v \). There exists an integral \((v, \mu_1, \mu_2, \ldots, \mu_t)\)-design of block size \( b \) if and only if \( \mu_{s+1} = \frac{b-s}{s-\mu_s} \) for \( 0 \leq s < t \).

The inclusion matrix, \( A^{b}_i(v) \), where \( i \leq b \leq v \), is the \((v)_i \times (v)_b \) matrix whose rows are indexed by subsets of \([v]\) of size \( i \) and whose columns are indexed by subsets of \([v]\) of size \( b \). The entry corresponding to position \( X, Y \) is 1 if \( X \subseteq Y \) and 0 otherwise. Gottlieb showed matrix is of full rank over characteristic 0 [5]. If integral \((v, \mu_1, \mu_2, \ldots, \mu_t)\) design of block size \( b \), then considering \( c \) as a vector of length \((v)_b \), we see that

\[
A^{b}_i(v)c = \mu_i 1_i,
\]

where \( 1_i \) is the vector of length \((v)_i \) consisting of 1’s. It is clear that

\[
A^{b}_j(v)A^{b}_i(v) = \left(\begin{array}{c} b-j \\ i-j \end{array}\right) A^{b}_j(v),
\]

and thus

\[
\left(\begin{array}{c} v-i \\ i-j \end{array}\right) \mu_i = \left(\begin{array}{c} b-j \\ i-j \end{array}\right) \mu_j,
\]
proving the necessity of the conditions in Theorem 11. To prove the sufficiency we need the following result:

**Theorem 12.** Let \( 0 \leq t < b \leq v - t \). Then \( \mathcal{A}_{t+1}(N_{t,b}) = N_{t,t+1} \).

**Proof of Theorem 12.** We have already seen that the conditions are necessary. We shall prove sufficiency of the conditions by induction on \( t \), noting that if \( t = 0 \) then the design which assigns \( \mu_0 \) to the set \([b]\) and 0 to all other sets of size \( b \) is of the form we seek. Now assume that these conditions are sufficient for \( t \geq 0 \), and that \( \mu_1, \mu_2, \ldots, \mu_{t+1} \) satisfy these conditions. Then there is some \((v, \mu_1, \mu_2, \ldots, \mu_t)\)-design, \( c' \), of block size \( b \). We shall construct \( c \), a \((v, \mu_1, \mu_2, \ldots, \mu_{t+1})\)-design, of block size \( b \). If \( b \geq v - t \) then \( \mathcal{A}_{t} \) is of full column rank and thus the only designs are multiples of the constant design. In this case \( c' = \alpha \mathbf{1}_b \) is also a \((v, \mu_1, \mu_2, \ldots, \mu_{t}, \mu'_{t+1})\)-design. The relationship between the coefficients of the design established previously ensure that \( \mu_{t+1} = \mu'_{t+1} \).

We now consider the case where \( b < v - t \). Observe, \( \mathcal{A}_{t+1}^b = (l - t)\mathcal{A}_{t+1}^c - \mu_{t+1} \mathbf{1}_{t+1} \). Setting \( c = c' - d \) we see that

\[
\mathcal{A}_{t+1}^l c = \mathcal{A}_{t+1}^l c' - \mathcal{A}_{t+1}^l d = \mu_{t+1} \mathbf{1}_{t+1},
\]

and the relationship between coefficients ensures this is a \((v, \mu_1, \mu_2, \ldots, \mu_{t}, \mu'_{t+1})\)-design, as required. \( \square \)

An element \( u \in \mathcal{M}^{(a,b)} \) which is an \((a + b, \mu_1, \mu_2, \ldots, \mu_{b-1})\)-design of block size \( b \) satisfies the first condition of Theorem 13 as long as one of the \( \mu_i \) is non-zero (in \( k \)). We have to take care that when we construct such an element that \( u \) also satisfies the second condition of Theorem 13.

**Theorem 13.** Let \( \lambda = (a, b) \), then there exists an integral design which corresponds to a Hemmer element if and only if \( \lambda \) is James.

**Proof.** Any integral design must have coefficients satisfying the conditions of Theorem 11 \( \mu_{s+1} = \frac{b-s}{a+b-s} \mu_s \) for \( 0 \leq s < t \). This means that

\[
\mu_s = \left( \frac{a+b-s}{a} \right)^{s} \mu_0.
\]

To ensure that some \( \mu_i \neq 0 \) (mod \( p \)) we must take \( \mu_s = c \left( \frac{a+b-s}{a} \right)^{s} \mu_0 \) where \( c \in k \) is non-zero and \( d \) is the least power of \( p \) dividing some \( \left( \frac{a+b-s}{a} \right)^{s} \) for \( s \in \{0, 1, \ldots, b-1\} \).
1). That is, \( d = \min_{s \in b} \{ \text{val}_p(a + b - s) \} \). Observe that

\[
\psi_{1,j}(f(a,b)) = \left( \frac{a + b - j}{b - j} \right) f(a + b - j, j)
\]

\[
= c^{-1} p^d \mu_j f(a + b - j, j),
\]

and so if \( p^d \) is a unit in \( k \), that is if \( d = 0 \), then

\[
\psi_{1,j}(c \cdot f(a,b) - u) = 0,
\]

and \( u \) is not Hemmer. This means \( u \) is Hemmer if and only if \( p \mid (a + b - j) \) for all \( j \in \{0,1,\ldots,b-1\} \), which by Lemma 4 is if and only if \( \lambda \) is James. \( \Box \)

Of course, not being able to construct an integral design for a partition \((a, b)\) does not mean that we cannot find a Hemmer element in \( M(a, b) \). If \( u \) is a Hemmer element, then the \( j \) sets occurring in \( \psi_{1,j}(u) \) need only to have the same coefficient in \( k \), not over \( \mathbb{Z} \). To solve this problem in a more general setting, we need to investigate designs over fields of positive characteristic.

### 3.2 \( p \)-ary designs

Designs over positive characteristic do not behave in the same way as designs over the integers, so we have to modify the definition slightly.

**Definition 14.** Let \( v, b, t \) be integers with \( 0 \leq t < b \leq v \), and \( k \) be a field of positive characteristic, \( \text{char}(k) = p \). Let \( V = [v] \), \( \mathcal{V} = \{ X \mid X \subseteq V \} \) and \( \mathcal{V}_b = \{ X \in \mathcal{V} \mid |X| = b \} \). A \( p \)-ary \( t \)-design of block size \( b \) is a function \( c : \mathcal{V}_b \to k \) such that

\[
\hat{c}(Y) := \sum_{X \supseteq Y} c(X) = \alpha, \quad \text{if} \quad |Y| = t.
\]

If the coefficient \( \alpha = 0 \) then we call \( c \) a null \( t \)-design.

Observe here that we only require \( \hat{c} \) be constant on sets of size \( t \). Recall, over the integers the relationship between the coefficients of designs ensured that \( \hat{c} \) was constant on the sets of size \( j \) for any \( j \leq t \). We have a similar relationship here:

**Proposition 15.** Let \( c : \mathcal{V}_b \to k \) be a \( p \)-ary \( t \)-design of block size \( b \) on a set of size \( v \) with coefficient \( \mu_t \). Let \( j \leq b \) be such that \( \binom{b - j}{t - j} \not\equiv 0 \pmod{p} \), then \( c \) is also a \( j \)-design, with coefficient

\[
\mu_j = \left( \frac{v - j}{b - j} \right) \mu_t.
\]
Proof.

\[
\binom{b-j}{t-j} A_j^b(v)c = A_j^t(v)A_t^b(v)c \\
= A_j^t(v)\mu_t 1_t \\
= \binom{v-j}{t-j} \mu_t 1_t.
\]

\[\square\]

Remark. Wilson [13] showed that there are examples of \(t\)-designs which are not \(j\) designs whenever \(\binom{b-j}{t-j} \equiv 0 \pmod{p}\), which is very different to the behaviour of integral designs.

We shall call a design of block size \(b\) universal if it is simultaneously a \(t\)-design for all \(t < b\). Clearly a Hemmer element corresponds to a universal design, however, not all universal designs are Hemmer.

**Definition 16.** Let \(u\) and \(v\) be universal designs of block size \(b\) with coefficients \(\mu_1, \ldots, \mu_{b-1}\) and \(\mu'_1, \ldots, \mu'_{b-1}\) respectively. Then we say \(u\) and \(v\) are similar if there is some constant \(\alpha \in k\) such that \(\mu_i = \alpha \mu'_i\) for all \(i < b\).

It is clear that the constant design \(c(X) = 1\) for all \(X \in \mathcal{V}_b\) corresponds to the element \(f_{(a,b)} \in M^{(a,b)}\), and that the second condition of Theorem 3 is equivalent to requiring \(u\) is not similar to this constant design. The existence of \(t\)-designs was solved by Wilson [13], whose result we state below. In the following section we shall solve the problem of the existence of universal designs which are not similar to the constant design; that is the problem of the existence of Hemmer elements in \(M^{(a,b)}\).

**Theorem 17.** Let \(t \leq b \leq v - t\). Then there is a non-null \(p\)-ary \(t\)-design of block size \(b\) if and only if

\[
\binom{b-i}{t-i} \equiv 0 \pmod{p} \quad \text{implies} \quad \binom{v-i}{t-i} \equiv 0 \pmod{p}
\]

for all \(i \leq t\).

4 A more general case

In section 2 we constructed a Hemmer element for the partition \((a, b)\) when \(b\) was a \(p\)-power and \(a\) is such that \(\text{val}_p(a+1) < b\). We will now generalise this example, using Wilson’s work, to the case where \(b = p^\beta + \hat{b}\), where \(\text{val}_p(a+1) < p^\beta\) as before, and \(\hat{b} < \text{val}_p(a+1)\). Following Donkin and Geranios [2], we call such a partition **pointed**. Let \(u\) be the Hemmer element for \((a, p^\beta)\) constructed in section 2. Recall that \(\psi_{1,j}(u) = 0\) except for when \(j = 0\). We shall modify \(u\) to create a Hemmer element for \((a, p^\beta + \hat{b})\).
The Hemmer element can be thought of as a sum subsets of \([a + p^β]\) of size \(p^β\), say \(u = \sum_{X \subseteq u} X\). Let \(Y = \{a + p^β + 1, \ldots, a + b\}\), then \(Y\) is a set of size \(\hat{b}\). Let \(u_Y \sum_{X \subseteq u} X \cup Y\) be the element in \(M(a,p^β)\) obtained by adjoining \(Y\) to the bottom row of all tabloids appearing in \(u\). Similarly \(u'^Y\) is obtained by adjoining \(Y\) to the top row of all tabloids appearing in \(u\). Consider \(ψ_{1,j}(u_Y)\), which is a formal sum of sets of size \(j\), by grouping terms by the size of their intersection with \(Y\). First, consider the case where \(\hat{b} < j < b\):

\[
ψ_{1,j}(u_Y) = ψ_{1,j-\hat{b}}(u) + \sum_{y \in Y} ψ_{1,j-\hat{b}+1}(u)_{Y \cap \{y\}} + \cdots + ψ_{1,j}(u)^Y.
\]

Each of these terms is 0, by our choice of \(u\), so \(ψ_{1,j}(u_Y) = 0\). Similarly for \(j ≤ \hat{b}\)

\[
ψ_{1,j}(u_Y) = \sum_{i=0}^{j} \sum_{|Y \cap Y'| = i} ψ_{1,j-i}(u)'\]

where \(\mu_0 \neq 0\) is the coefficient of \(u\) as a 0-design. Observe that if \(Y\) is any subset of \([a + b]\), then we may define \(u_Y\) similarly, by relabeling \(u\) so that it has entries in \([a + b] \setminus Y\).

Let \(X \subseteq [a + b]\) of size \(b - 1 = p^β + \hat{b} - 1\). Define \(u_X := \sum_{Y \subseteq X} u_Y\). Then

\[
ψ_{1,j}(u_X) = \sum_{Y \subseteq X} ψ_{1,j}(u_Y),
\]

which is 0 if \(\hat{b} < j < b\). When \(j ≤ \hat{b}\),

\[
ψ_{1,j}(u_X) = \sum_{Y \subseteq X} ψ_{1,j}(u_Y)
\]

\[
= \sum_{Y \subseteq X} \sum_{Y' \subseteq Y} \mu_0 Y'
\]

\[
= \left(p^β - 1 + \hat{b} - j\right) \mu_0 \sum_{Y' \subseteq X} Y',
\]

which is 0 if \(j \neq \hat{b}\). So

\[
ψ_{1,\hat{b}}(u_X) = \mu_0 \sum_{Y' \subseteq X} Y',
\]

where the sum is over all subsets \(Y' \subseteq X\) of size \(\hat{b}\). If \(U\) is a non null \(p\)-ary \(\hat{b}\)-design of block size \(b - 1\) and coefficient \(α\) then setting

\[
u_U := \sum_{X} U(X) u_X,
\]

10
where the sum is over all sets $X$ of size $b-1$ and $U(X)$ is the coefficient of $X$ in the $b$-design $U$, we see

$$\psi_{1,b}(u_U) = \sum_X U(X) \psi_{1,b} u_X$$

$$= \sum_X U(X) \mu_0 \sum_{Y' \subseteq X} Y'$$

$$= \alpha \mu_0 \sum_{Y' \subseteq X} Y',$$

and of course

$$\psi_{1,j}(u_U) = 0$$

for all other $j$.

**Theorem 18.** Let $k$ have characteristic $p \geq 3$, and let $\lambda = (a,b) \vdash n$ be such that $b = p^\beta + \hat{b}$ and $\hat{b} < p^{val_p(a+1)} < b$. Then $H^1(S_n, S^\lambda) \neq 0$

**Proof.** By Theorem 3 it suffices to find a Hemmer element in $M^\lambda$. Observe that

$$\psi_{1,j}(f_\lambda) = \left(\begin{array}{c} a + b - j \\ b - j \end{array}\right) f_{(a+b-j,j)}.$$

As $\hat{b} < p^{val_p(a+1)} < b$ there is some digit which occurs before the position corresponding to $p^\beta$ in the $p$-ary expansion of $a$ which is not $p-1$. Let this digit be in the position corresponding to $p^\alpha$. If $j$ is chosen such that $b - j = p^\alpha$, then $(a+b-j) \neq 0$, by Lemma 4. This means an element of the form $u_U$ as described above is Hemmer, as $\psi_{1,j}(u_U) = 0$.

It remains to prove such an element exists, that is that there is a non null $p$-ary $b$-design of block size $b-1$. By Theorem 17 we may construct such a design if (and only if) $(a+b-1-i) \equiv 0 \pmod{p}$ whenever $(b-1-i) \equiv 0 \pmod{p}$. Of course $(b-1-i) = (p^\beta + b-1-i) \equiv 0 \pmod{p}$ for all $i < \hat{b}$, so it remains to see that $(a+b-1-i) \equiv 0 \pmod{p}$ for all $i < b$; that is, that $(a+p^\beta-j) \equiv 0 \pmod{p}$ for all $j < b$. This follows from Lemma 5 as $a + p^\beta \equiv -1 \pmod{p^\beta}$.

We have so far seen how to construct Hemmer elements for two part partitions $\lambda$ when $\lambda$ is either pointed or James. In the next section we shall see that these are indeed the only cases where Hemmer elements for two part partitions exist.

## 5 Existence of Hemmer elements

We shall see in this section that there are no Hemmer elements in $M^\lambda$ if $\lambda = (a, b)$ is not James or pointed. We shall do this by investigating what universal designs can exist and whether they are Hemmer. We will call a $p$-ary $t$-design of block size $b$ on a set of size $a+b$ a $t$-design for the partition $\lambda = (a, b)$, as it corresponds to an element in $M^\lambda$ satisfying the first condition of Theorem 3.
Proposition 19. Let $\lambda = (a,b)$. A design for $\lambda$ is universal if and only if it is a $(b-p)$-design for all $l \leq l_p(b)$.

Proof. Of course a universal design is a $(b-p)$-design. A $(b-p)$-design, is also a $j$ design for all $j < b - p$ with $(b_{-p-j}) \neq 0$; that is, for any $j$ such that the sum $(b - j - p)$ has no carries in p-ary notation, by Lemma 4. This is precisely those $j$ for which the coefficient of $p^j$ in the p-ary expansion of $b - j$, which we shall denote $(b - j)_i$, is non zero. If $j < b$, then some $(b-j)_i \neq 0$, and as $u$ is a $(b-p)$-design $u$ is also a $j$-design by Proposition 15. 

Proposition 20. There are non-null p-ary $(b-p)$-designs for $(a,b)$ if and only if $a_l \neq -1 \ (mod \ p)$ or $b \leq p^{l+1}$.

Proof. By Theorem 17 a non-null $(b-p)$-design exists if

$$\binom{b-j}{p^l} \equiv 0 \ (mod \ p) \ \text{imply} \ \binom{a+b-j}{a+p^l} \equiv 0 \ (mod \ p).$$

If $a_l \equiv -1 \ (mod \ p)$ and $b > p^{l+1}$ then setting $j = b - p$ we see that non-null designs can not exist. On the other hand if $b \leq p^{l+1}$ then $(b_{-p-j}) \neq 0 \ (mod \ p)$ for all $j < b - p$ so there are non-null $(b-p)$-designs. Finally, if $a \neq -1 \ (mod \ p)$ then $(b_{-p-j}) \equiv 0 \ (mod \ p)$ whenever $(b-j)_i = 0$. If $(b-j)_i = 0$ then the sum $(a+p^j) + (b-j-p)$ necessarily has a carry in p-ary notation, so $(a+b-j) \equiv 0 \ (mod \ p)$ by Lemma 4. 

Combining this with the relationship between coefficients, established in Proposition 15 we obtain more integers $j$ for which a universal design for $(a,b)$ is null.

Proposition 21. If a universal design, $u$, for $(a,b)$ is non-null as a $j$-design, then $(b-j)_m + a_m < p$ for all $m < l_p(b)$.

Proof. Suppose $u$ is non-null as a $j$-design with coefficient $\mu_j$, and let $m < l+p(b)$ be such that $(b-j)_m \neq 0$. As $u$ is non-null for $j$, we must have $u$ is non-null for $b-p^m$, by Proposition 15 as

$$\mu_j = \binom{a+b-j}{b-p-m-j} \binom{b-p-m-j}{b-j} \mu_{b-p^m}.$$ 

For $u$ to be non-null as a $j$-design, we must have $(\binom{a+b-j}{b-p-m-j}) \neq 0$. Proposition 20 ensures that $a_m \neq -1 \ (mod \ p)$ and thus $(a+p^j) + (b-j-p)$ having no carries is equivalent to $(a) + (b-j)$ having no carries. Using Lemma 4 we see that if $u$ is non-null then $(a) + (b-j)$ has no carries, and therefore $(b-j)_m + a_m < p$ for all $m < l_p(b)$.

Our next goal is to determine what the relationship is between the non-zero coefficients of a universal design. Let $u$ be a universal design for $(a,b)$, and
let $X$ be the set of all $j$ with $(b-j)_m + a_m < p$ for all $m < l_p(b)$. Observe if $j \notin X$ then $u$ must be a null $j$-design, and so $X$ contains all $j$ such that $u$ is a non-null $j$-design. We shall define a partial ordering on $X$ by setting $i \geq_X j$ if $i > j$ and $b_{i-j} \neq 0 \pmod{p}$. If $i \geq_X j$ and $\mu_i$ and $\mu_j$ are the coefficients of $u$ corresponding to $i$ and $j$ respectively, then $\mu_j = \frac{(i+j)}{(i-j)} \mu_i$, so we have a relationship between the coefficients appearing in the same connected component of $X$.

**Proposition 22.** If $\lambda = (a,b)$ is James, then $X$ has a single connected component.

**Proof.** Recall if $\lambda$ is James then $b < p^{\text{val}_p(a+1)}$, and $a_m \equiv -1 \pmod{p}$ for all $m < l_p(b)$. Write $b = \alpha p^\beta + \hat{b}$, where $\beta = l_p(b)$ and $b < p^\beta$, and observe, by Proposition 21 that $X = \{\hat{b}, \alpha p^\beta + \hat{b}, \ldots, (\alpha - 1)p^\beta + \hat{b}\}$, which, by Lemma 5 is a single connected component. \qed

**Proposition 23.** If $\lambda = (a,b)$ is not James, and $b = \alpha p^\beta + \hat{b}$ then $X$ has a single connected component, unless $\lambda$ is pointed, in which case $X$ has two connected components, one of which consists only of the element $\hat{b}$.

**Proof.** Observe that $i,j \in X$ are comparable if and only if $(b-i)_m \leq (b-j)_m$ for all $m \leq l_p(b)$, or $(b-j)_m \leq (b-i)_m$ for all $m \leq l_p(b)$. Observe also that $(b-i)_m = 0$ for all $m < l_p(b)$ for which $a_m \equiv -1 \pmod{p}$. The join of $i,j \in X$, if it exists is the element $i \vee j = x$ such that $(b-x)_m = \max\{(b-i)_m, (b-j)_m\}$, the meet, $y = i \wedge j$, is the element $y$ such that $(b-y)_m = \min\{(b-i)_m, (b-j)_m\}$. These may fail to be in $X$ as it may be that $b - x > b$ or $b - y = 0$, but $i \wedge j \in X$ if $(b-i)_m$ and $(b-j)_m$ are both non-zero for some $m$.

Let $x$ be such that $(b-x)_m = p - 1 - a_m$ for $m < \beta$ and $(b-x)_\beta = \alpha - 1$, and observe that $x \in X$ by Proposition 21. Clearly $j \in X$ with $j \gg \hat{b}$ is comparable to $x$. If $j < \hat{b} \in X$, or if $j = \hat{b}$ and $\alpha \neq 1$ then $x \wedge j \in X$.

It only remains to consider the case where $j = \hat{b}$ and $\alpha = 1$, which, if $\hat{b} > p^{\text{val}_p(a+1)}$ is clearly comparable to $\hat{b} - p^{\text{val}_p(a+1)}$, which is in the same component as $x$. It follows that if $\lambda$ is not pointed then there is only one connected component of $X$.

On the other hand, when $\lambda$ is pointed $\hat{b}$ is not comparable to any other element and thus is in a connected component of its own. This is as no $j < \hat{b}$ is in $X$ as no $j < \hat{b}$ has $(b-j)_m = 0$ for all $m < l_p(b)$ where $a_m \equiv -1 \pmod{p}$. Similarly no $j > \hat{b}$ has $(b-j)_\beta \geq 1$, so $j$ and $\hat{b}$ are incomparable. \qed

If $u$ is a universal design for $(a,b)$, then its coefficients are entirely determined by the connected components of $X$, thus an understanding of this poset allows us to determine the possible coefficients of designs. We shall conclude by proving the main results of this paper, on the existence and uniqueness of universal $p$-ary designs and a calculation of the first degree cohomology of the Specht module.
6 Main theorems

The correspondence between designs and Hemmer elements allow the main result of this paper can be stated in two ways which we state in this section as we believe both formulations are of interest. The first is a purely combinatorial result on the existence of universal $p$-ary designs, while the second is a result on the first degree cohomology of Specht modules.

**Theorem 24.** Let $a, b \in \mathbb{N}$, with $a \geq b$ and let $u$ be a non-null universal $p$-ary design for $(a,b)$. If $(a,b)$ is neither pointed or James, then $u$ is similar to the constant design. If $(a,b)$ is James then $u$ is unique up to similarity, while if $(a,b)$ is pointed then $u = u' + c$ where $u'$ is non-null only as a $b$-design, while $c$ is similar to the constant design.

**Proof.** The coefficients of $u$ are determined by the coefficients on the connected components of the poset $X$. If $(a,b)$ is not pointed then $X$ has a single connected component, by Propositions 22 and 23, and thus all non-null designs are similar. If $(a,b)$ is not James, then the constant design is non-null and thus all non-null designs are similar to the constant design. If $(a,b)$ is not James, then the design constructed in section 2 is non-null, and all other non-null designs are similar.

On the other hand, if $(a,b)$ is pointed, then $X$ has two connected components. Observe that the constant design is non-null on the largest of these two components, and possibly the connected component containing only $\{b\}$. Thus we can write $u = u' + c$ where $u'$ is a multiple of the design constructed in Theorem 18, which is non-null only as a $b$-design, and $c$ is similar to the constant design.

Equivalently:

**Theorem 25.** Let $\lambda = (a,b) \vdash n$, and $p \geq 3$ then

$$\dim(H^1(S_n, S^\lambda)) = \begin{cases} 1 & \text{if $\lambda$ is James or pointed,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover an element $u \in M^\lambda$ such that $\langle S^\lambda, u \rangle$ is a non-split extension of $S^\lambda$ is constructed in Theorem 18 if $\lambda$ is James, and in the discussion preceding Theorem 18 if $\lambda$ is pointed.

**Proof.** By Theorem 8 a non-split extension of $S^\lambda$ corresponds to a Hemmer element in $M^\lambda$. If $u$ and $v$ are similar Hemmer elements, then there is some $\alpha$ such that $\psi_{i,j}(u - \alpha v) = 0$ for all $j$. Then $u - \alpha v \in S^\lambda$ by Theorem 4 and thus the extensions they define are the same and $\dim(H^1(S_n, S^\lambda)) = 1$. Similarily, in the case where $\lambda$ is pointed, we may have Hemmer elements $u$ and $v$, which are not similar. Without loss of generality we may assume that $u = v + f_\lambda$ by subtracting off some $v' \in S^\lambda$, in which case the extensions $\langle S^\lambda, u \rangle$ and $\langle S^\lambda, v \rangle$ are equivalent and $\dim(H^1(S_n, S^\lambda)) = 1$. If $\lambda$ is neither pointed or James, then there are no Hemmer elements in $M^\lambda$, by Theorem 24 and thus $\dim(H^1(S_n, S^\lambda)) = 0$. 

\[14\]
Observe that the above result recovers the results of Hemmer [5], and Donkin and Geranios for the case of two part partitions [2], however our proof is entirely in the setting of the symmetric group. This result goes further by describing how to construct a Hemmer element, \( u \) such that the extension, \( \langle S^\lambda, u \rangle \) is non-split.

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L. J. Jolliffe, Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WB, United Kingdom

E-mail address: ljj33@cam.ac.uk