Rigidity of noncompact complete manifolds with harmonic curvature

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Abstract

Let $(M, g)$ be a noncompact complete $n$-manifold with harmonic curvature and positive Sobolev constant. Assume that $L^2$ norms of Weyl curvature and traceless Ricci curvature are finite. We prove that $(M, g)$ is Einstein if $n \geq 5$ and $L^{n/2}$ norms of Weyl curvature and traceless Ricci curvature are small enough.

1 Introduction

One of the important problems in Riemannian geometry is to understand metrics near the Einstein space or the constant curvature space. The metric with harmonic curvature is a natural candidate for this study. Since the curvature tensor is decomposed into orthogonal parts, we compare this metric with the Einstein metric and the constant curvature space, when Weyl curvature and traceless Ricci curvature are small in $L^{n/2}$ sense. By this comparison, we obtain some $L^{n/2}$-type rigidity of noncompact complete metrics with harmonic curvature in Theorem 1.5.

A Riemannian $n$-manifold $(M, g)$ is a manifold with harmonic curvature if the divergence of curvature tensor $\text{Riem}$ vanishes, i.e.,

$$0 = (\delta \text{Riem})_{jkl} = \nabla^i R_{ijkl}.$$  

(1)
which is equivalent to

\[ 0 = \nabla_k R_{jl} - \nabla_l R_{jk}. \]  

(2)

By the Bianchi identity, scalar curvature is constant for \( n \geq 3 \). An important example of metrics with harmonic curvature is the Einstein space. The parallel Ricci curvature space is another example of the manifold with harmonic curvature, but the converse does not hold (see Derdzinski [3]). In the space of connections, the Yang-Mills connection corresponds to a metric with harmonic curvature (see Bourguignon [6]). Important properties of the manifold with harmonic curvature were surveyed by Besse [3]. Einstein manifolds and manifolds with harmonic curvature share many important properties. A natural question is when a manifold of harmonic curvature must be Einstein or flat. Hebey and Vaugon proved \( L_{n/2} \)-type rigidity for compact manifolds with harmonic curvature and positive scalar curvature [7]. In this paper, we provide some rigidity properties for noncompact complete manifold with harmonic curvature. For the rigidity proof, we use an elliptic estimation for the Laplacian of curvature tensors.

## 2 Manifolds with harmonic curvature

Let \((M, g)\) be a noncompact complete Riemannian manifold of dimension \( n \geq 3 \) with scalar curvature \( R \). The Sobolev constant \( Q(M, g) \) is defined by

\[
Q(M, g) \equiv \inf_{0 \neq u \in C_0^\infty(M)} \frac{\int_M \nabla u^2 + (\frac{n-2}{4(n-1)}) R_g u^2 dV_g}{(\int_M |u|^{2n/(n-2)} dV_g)^{(n-2)/n}}.
\]

There are noncompact complete Riemannian manifolds of negative scalar curvature with positive Sobolev constant. For example, any simply connected locally conformally flat manifold has positive Sobolev constant [12]. Contrary to noncompact case, the Sobolev constant of a given compact manifold is determined by the sign of scalar curvature (see [11]). It is also known that \( Q(M, g) \geq 0 \) if a noncompact complete manifold \((M, g)\) has zero scalar curvature [4]. In this paper, we show that the rigidity of noncompact complete manifolds with harmonic curvature holds not only for nonnegative scalar curvature metrics but also for negative scalar curvature metrics with positive Sobolev constant and the dimension \( n \geq 5 \).

Let \( E_{ij} \) be the traceless Ricci tensor, i.e., \( E_{ij} = R_{ij} - \frac{1}{4} R g_{ij}, |E| = |E_{ij}|, W \) the Weyl
curvature. In this paper, we use $c$ and $c'$ to denote some positive constant, which can be varied.

**Theorem 1** Let $(M, g)$ be a noncompact complete Riemannian $n$-manifold with harmonic curvature and $Q(M, g) > 0$. Assume that $\int_M |W|^2 + |E|^2 \, dV_g$ is finite, and (A) or (B) holds:

(A) Scalar curvature $R \geq 0$.

(B) Scalar curvature $R < 0$ and $n \geq 5$.

Then there exists a small number $c_0$ such that if $\int_M |W|^{n/2} + |E|^{n/2} \, dV_g \leq c_0$, then $(M, g)$ is Einstein.

**Proof.** We need to prove that $|E_{ij}| = 0$. To simplify notations, we will work in an orthonormal frame. The Laplacian of traceless Ricci tensor is:

$$\triangle E_{ij} = \nabla_k \nabla_k E_{ij}$$

$$= \nabla_k \nabla_k E_{kj} \quad (3)$$

$$= \nabla_i \nabla_k E_{kj} + R_{kikm} E_{mj} + R_{kijm} E_{km} \quad (4)$$

$$= R_{im} E_{mj} + R_{kijm} E_{km}, \quad (5)$$

where (2) is used in (3) and $\nabla_k E_{kj} = 0$ is used in (5). Since

$$R_{kijm} = W_{kijm} + \frac{R}{n(n-1)} (g_{kj} g_{im} - g_{km} g_{ij})$$

$$+ \frac{1}{n-2} (E_{kj} g_{im} + E_{im} g_{kj} - E_{km} g_{ij} - E_{ij} g_{km}), \quad (6)$$

$$E_{ij} \triangle E_{ij} = W_{kijm} E_{km} E_{ij} + \frac{R}{n(n-1)} g_{kj} g_{im} E_{km} E_{ij}$$

$$+ \frac{1}{n-2} (E_{kj} g_{im} + E_{im} g_{kj}) E_{km} E_{ij} + R_{im} E_{mj} E_{ij} \quad (7)$$

$$= W_{kijm} E_{km} E_{ij} + \frac{R}{n(n-1)} |E|^2 + \frac{n}{n-2} \text{tr} E^3 + \frac{R}{n} \text{tr} E^2 \quad (8)$$

$$= W_{kijm} E_{km} E_{ij} + \frac{R}{(n-1)} |E|^2 + \frac{n}{n-2} \text{tr} E^3. \quad (9)$$
Note that

\[ W_{kijm}E_{km}E_{ij} \leq \sqrt{\frac{n-2}{2(n-1)}} |W||E|^2, \]  
\[ \text{tr}E^3 \leq \frac{n-2}{\sqrt{n(n-1)}} |E|^3 \]  
and

\[ |\nabla|E||^2 \leq \frac{n}{n+2} |\nabla E|^2 \]  

because \( E_{ij} \) is a traceless Codazzi tensor (see [7]). Inequalities (10) and (11) are proved by Huisken [8]. From (9),

\[ |E|\Delta|E| = |\nabla E|^2 - |\nabla|E||^2 + E_{ij}\Delta E_{ij} \geq \frac{2}{n} |\nabla|E||^2 - \sqrt{\frac{n-2}{2(n-1)}} |W||E|^2 - \sqrt{\frac{n}{n-1}} |E|^3 + \frac{R|E|^2}{n-1}, \]  

where (10–12) is used in (13). Let \( u = |E| \). Multiplying a smooth compact supported function \( \phi^2 \) to (13) and integrating on \( M \), for any positive constant \( \epsilon_1 \),

\[ \int_M (\epsilon_1 + 1 + 2/n)\phi^2 |\nabla u|^2 - \epsilon_1^{-1} |u\nabla \phi|^2 dV_g \]  
\[ \leq \int_M (1 + 2/n)\phi^2 |\nabla u|^2 + 2\phi u \nabla \phi \cdot \nabla u dV_g \]  
\[ \leq \int_M \phi^2 \left[ \sqrt{\frac{n-2}{2(n-1)}} |W|u^2 + \sqrt{\frac{n}{n-1}} u^3 - \frac{R}{n-1} u^2 \right] dV_g. \]  

Using (16), for any positive constant \( \epsilon_2 \),

\[ \int_M |\nabla(\phi u)|^2 + \frac{n-2}{4(n-1)} Ru^2 \phi^2 dV_g \]  
\[ \leq \int_M (1 + \epsilon_2)\phi^2 |\nabla u|^2 + (1 + \epsilon_2^{-1})u^2 |\nabla \phi|^2 + \frac{n-2}{4(n-1)} Ru^2 \phi^2 dV_g \]  
\[ \leq \int_M c_1 u^2 |\nabla \phi|^2 + c_2 Ru^2 \phi^2 + A dV_g, \]  

(19)
where
\[ c_1 = (1 + \epsilon_2)(-\epsilon_1 + 1 + 2/n)^{-1} \epsilon_1^{-1} + \epsilon_2^{-1} + 1, \quad (20) \]
\[ c_2 = \left( (n - 2)/4 - (1 + \epsilon_2)(-\epsilon_1 + 1 + 2/n)^{-1} \right)/(n - 1), \quad (21) \]
\[ c_3 = (1 + \epsilon_2)(-\epsilon_1 + 1 + 2/n)^{-1}, \quad (22) \]
\[ A = c_3 \left( \sqrt{\frac{n - 2}{2(n - 1)}} |W| u^2 + \sqrt{\frac{n}{n - 1}} u^3 \right) \phi^2. \quad (23) \]

Note that \( c_2 \) is positive when \( n \geq 5 \) and \( \epsilon_1 \) and \( \epsilon_2 \) are sufficiently small. For any \( n \), \( c_2 \) is negative when \( \epsilon_1 \) is sufficiently close to \( 1 + 2/n \). Therefore we can choose \( \epsilon_1 \) and \( \epsilon_2 \) so that \( c_2 \) is positive if \( n \geq 5 \) and \( R < 0 \), and \( c_2 \) is negative if \( R > 0 \) for any given \( n \), which makes the second term of (19) non-positive. Using the Sobolev constant \( \Lambda_0 \equiv Q(M, g) \) and (19),

\[ A_0 \left( \int_M (\phi u)^{2n/(n-2)} dV_g \right)^{n/(n-2)} \leq \int_M |\nabla (\phi u)|^2 + \frac{n - 2}{4(n - 1)} Ru^2 \phi^2 dV_g \]
\[ \leq \int_M c_1 u^2 |\nabla \phi|^2 + A dV_g. \quad (24) \]

Note that

\[ \int_M A \ dV_g \]
\[ \leq c_3 \sqrt{\frac{n - 2}{2(n - 1)}} \left( \int_M |W|^{n/2} dV_g \right)^{2/n} \left( \int_M (u\phi)^{2n/(n-2)} dV_g \right)^{n/(n-2)} \]
\[ + c_3 \sqrt{\frac{n}{n - 1}} \left( \int_M u^{n/2} dV_g \right)^{2/n} \left( \int_M (u\phi)^{2n/(n-2)} dV_g \right)^{(n-2)/n}. \quad (25) \]

Since \( \int_M |E|^{n/2} + |W|^{n/2} dV_g \) is sufficiently small, \( \int_M A \ dV_g \) can be absorbed into left hand side of (24). Therefore, there exists a constant \( c' \) such that

\[ c' \left( \int_M (\phi u)^{2n/(n-2)} dV_g \right)^{(n-2)/n} \leq \int_M |\nabla \phi|^2 u^2 dV_g. \quad (26) \]

Let \( B_t = \{ x \in M \mid d(x, x_0) \leq t \} \) for some fixed \( x_0 \in M \) and choose \( \phi \) as

\[ \phi = \begin{cases} 
1 & \text{on } B_t, \\
0 & \text{on } M - B_{2t}, \\
|\nabla \phi| \leq \frac{2}{t} & \text{on } B_{2t} - B_t,
\end{cases} \quad (27) \]
and $0 \leq \phi \leq 1$. From (26)
\[
c' \left( \int_M (u\phi)^{2n/(n-2)} dV_g \right)^{n/(n-2)} \leq \frac{4}{t^2} \int_{B(t)} u^2 dV_g.
\] (28)

By taking $t \to \infty$, we have $u = 0$ since $\int_M |E|^2 dV_g$ is finite. Therefore $(M, g)$ is Einstein.

Since there is no noncompact complete Einstein metric with positive scalar curvature, Theorem 1 implies that there should be a lower bound for $\int_M |W|^{n/2} + |E|^{n/2} dV_g$ if $\int_M |W|^2 + |E|^2 dV_g$ is finite.

**Theorem 2** Let $(M, g)$ be a noncompact complete Riemannian $n$-manifold with harmonic curvature and positive Sobolev constant. Assume that scalar curvature $R$ is positive and $\int_M |W|^2 + |E|^2 dV_g$ is finite. Then there exists a positive constant $c$ such that $\int_M |W|^{n/2} + |E|^{n/2} dV_g \geq c$.

Next we get a rigidity result without $L^2$ finiteness of Weyl curvature and traceless Ricci curvature.

**Theorem 3** Let $(M, g)$ be a noncompact complete Riemannian $n$-manifold with harmonic curvature and $Q(M, g) > 0$. Assume that $n \geq 4$ and (A’) or (B’) holds:

(A’) Scalar curvature $R \geq 0$.

(B’) Scalar curvature $R < 0$ and $n \geq 6$.

Then there exists a small number $c_0$ such that if $\int_M |W|^{n/2} + |E|^{n/2} dV_g \leq c_0$, then $(M, g)$ is Einstein.

**Proof.** We need to prove that $|E_{ij}| = 0$. Let $u = |E|$. Multiplying a smooth compact supported function $\phi^2 u^{-2 + n/2}$ to (13) and integrating on $M$, for any positive constant $\epsilon_1$,

\[
16 \left( \frac{n}{2} + \frac{2}{n} - 1 - \epsilon_1 \right) n^2 \int_M \phi^2 |\nabla u^{n/4}|^2 dV_g \leq \int_M \epsilon_1^{-1} |u^{n/2} \nabla \phi|^2 + \phi^2 \left[ \sqrt{\frac{n-2}{2(n-1)}} |W| u^{n/2} \right. \\
+ \sqrt{\frac{n}{n-1}} u^{1+n/2} - \frac{R}{n-1} u^{n/2} \phi^2 \left. \right] dV_g
\] (29)

\[
+ \sqrt{\frac{n}{n-1}} u^{1+n/2} - \frac{R}{n-1} u^{n/2} \phi^2 \right] dV_g
\] (30)
Using (31), for any positive constant $\epsilon_2$,
\[
\int_M |\nabla (\phi u^{n/4})|^2 + \frac{n-2}{4(n-1)} Ru^{n/2} \phi^2 dV_g \leq \int_M (1 + \epsilon_2) \phi^2 |\nabla u^{n/4}|^2 + (1 + \epsilon_2^{-1}) u^{n/2} |\nabla \phi|^2 + \frac{n-2}{4(n-1)} Ru^{n/2} \phi^2 dV_g \tag{32}
\]

\[
\leq \int_M c'_1 u^{n/2} |\nabla \phi|^2 + c'_2 Ru^{n/2} \phi^2 + \Lambda' \ dV_g, \tag{33}
\]

where
\[
c'_1 = (1 + \epsilon_2) n^2 [16(n/2 + 2/n - 1 - \epsilon_1)]^{-1} \epsilon_1^{-1} + \epsilon_2^{-1} + 1, \tag{34}
\]
\[
c'_2 = \left( (n-2)/4 - (1 + \epsilon_2) n^2 [16(n/2 + 2/n - 1 - \epsilon_1)]^{-1} \right) / (n-1), \tag{35}
\]
\[
c'_3 = (1 + \epsilon_2) n^2 [16(n/2 + 2/n - 1 - \epsilon_1)]^{-1}, \tag{36}
\]
\[
\Lambda' = c'_3 \left( \sqrt{\frac{n-2}{2(n-1)}} |W| u^{n/2} + \sqrt{n-1} u^{1+n/2} \right)^2. \tag{37}
\]

Note that $c'_2$ is positive when $n \geq 6$ and $\epsilon_1$ and $\epsilon_2$ are sufficiently small. For any $n$, $c'_2$ is negative when $\epsilon_1$ is sufficiently close to $-1 + 2/n + n/2$. Therefore we can choose $\epsilon_1$ and $\epsilon_2$ so that $c'_2$ is positive if $n \geq 6$ and $R < 0$, and $c'_2$ is negative if $R > 0$ for any given $n$, which makes the second term of (33) non-positive. Using the Sobolev constant and (33),
\[
\Lambda_0 \left( \int_M (\phi u^{n/4})^{2n/(n-2)} dV_g \right)^{n/(n-2)} \leq \int_M |\nabla (\phi u^{n/4})|^2 + \frac{n-2}{4(n-1)} Ru^{n/2} \phi^2 dV_g 
\]
\[
\leq \int_M c'_1 u^{n/2} |\nabla \phi|^2 + \Lambda' \ dV_g, \tag{38}
\]

Note that
\[
\int_M \Lambda' \ dV_g 
\]
\[
\leq c'_3 \sqrt{\frac{n-2}{2(n-1)}} \left( \int_M |W|^{n/2} dV_g \right)^{2/n} \left( \int_M (u^{n/4} \phi)^{2n/(n-2)} dV_g \right)^{n/(n-2)} \tag{39}
\]
\[
+ c'_3 \sqrt{\frac{n}{n-1}} \left( \int_M u^{n/2} dV_g \right)^{2/n} \left( \int_M (u^{n/4} \phi)^{2n/(n-2)} dV_g \right)^{(n-2)/n}.
\]

Since $\int_M |E|^{n/2} + |W|^{n/2} dV_g$ is sufficiently small, $\int_M \Lambda' dV_g$ can be absorbed into left hand side of (35). Therefore, there exists a constant $c'$ such that
\[
c' \left( \int_M (\phi u^{n/4})^{2n/(n-2)} dV_g \right)^{(n-2)/n} \leq \int_M |\nabla \phi|^2 u^{n/2} dV_g. \tag{40}
\]
Choosing a compact supported function similar to (27), we can easily show that $u^{n/4} = 0$ on $M$. We conclude that $(M, g)$ is a Einstein space.

3 Rigidity of Einstein spaces

In this section, we provide an $L_{n/2}$-type rigidity for noncompact complete Einstein manifolds. Previously, this type of rigidity was proved for compact Einstein manifolds with positive scalar curvature [10, 11, 7, 9]. Throughout this section $(M, g)$ denotes a noncompact complete Einstein manifold of dimension $n \geq 4$.

**Theorem 4** Let $(M, g)$ be a noncompact complete Einstein $n$-manifold with $Q(M, g) > 0$ and finite $\int_M |W|^2 \, dV_g$. Assume that $(M, g)$ satisfies (C) or (D):

(C) $(M, g)$ is Ricci-flat.

(D) Scalar curvature $R$ is negative and $n \geq 8$.

Then there exists a small constant $c$ such that if $\int_M |W|^{n/2} \, dV_g \leq c$, then $(M, g)$ is a constant curvature space.

**Proof.** We need to prove that $|W| = 0$. For an Einstein metric, the Laplacian of Weyl tensor is

$$\Delta W_{ijkl} = \nabla_t \nabla_t W_{ijkl} = \frac{2}{n} RW_{ijkl} + W \ast W,$$

where $W \ast W$ denotes quadratic terms of Weyl tensor (see Singer [11]). Multiplying $W_{ijkl}$ on (42),

$$W_{ijkl} \Delta W_{ijkl} = \frac{2}{n} R |W_{ijkl}|^2 + W \ast W \ast W.$$  

In above, $W \ast W \ast W$ denotes cubic terms of Weyl tensor, which is bounded by $c_4 |W|^3$ for a positive constant $c_4$. Multiplying a smooth compact supported function $\phi$ to (43) and integrating on $M$,

$$\int_M (1 - \epsilon_1)\phi^2 |\nabla W|^2 dV_g \leq \int_M -\frac{2}{n} R^2 |W|^2 \phi^2 + c_4 \phi^2 |W|^3 + \frac{1}{\epsilon_1} |\nabla \phi|^2 |W|^2 dV_g.$$  

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for any positive constant $\epsilon_1$. The following refined Kato inequality was proved for Weyl tensor of the Einstein manifold,

$$|\nabla W|^2 \leq \frac{n-1}{n+1}|\nabla W|^2,$$

(45)

when $|W| \neq 0$ (see Bando et al. [2]). Let $f = |W|$. For any positive constant $\epsilon_2$,

$$|\nabla (\phi f)|^2 \leq (1 + \frac{1}{\epsilon_2})|f \nabla \phi|^2 + (1 + \epsilon_2)|\phi \nabla f|^2$$

(46)

$$\leq (1 + \frac{1}{\epsilon_2})|f \nabla \phi|^2 + (1 + \epsilon_2)\frac{n-1}{n+1}|\phi \nabla W|^2.$$  

(47)

From (44),

$$\Lambda_0 \left( \int_M (\phi f)^{2n/(n-2)} dV_g \right)^{\frac{n-2}{n}} \leq \int_M |\nabla (\phi f)|^2 + \frac{n-2}{4(n-1)} R f^2 \phi^2 dV_g$$

(48)

$$\leq \int_M c_5 R f^2 \phi^2 + c_6 |\nabla \phi|^2 f^2 + c_7 \phi^2 f^3 dV_g,$$

(49)

where

$$c_5 = \frac{2(1 + \epsilon_2)(n-1)}{n(n+1)(1-\epsilon_1)} + \frac{n-2}{4(n-1)},$$

(50)

$$c_6 = 1 + \epsilon_2^{-1} + \frac{(1 + \epsilon_2)(n-1)}{(n+1)(1-\epsilon_1)\epsilon_1},$$

(51)

$$c_7 = \frac{(1 + \epsilon_2)(n-1)}{(n+1)(1-\epsilon_1)c_4}.$$  

(52)

Note that $c_5$ is positive when $n \geq 8$ and $\epsilon_1$ and $\epsilon_2$ are sufficiently small. $c_5$ is negative when $\epsilon_1$ is sufficiently close to 1 for any given $n$. Therefore we can choose $\epsilon_1$ and $\epsilon_2$ so that $c_5$ is positive when $R < 0$ and $n \geq 8$, which makes the first term of (49) non-positive. The last term of (49) is bounded by

$$c_7 \int_M \phi^2 f^3 dV_g \leq c_7 \left( \int_M f^{n/2} dV_g \right)^{2/n} \left( \int_M (f \phi)^{2n/(n-2)} dV_g \right)^{n/(n-2)}. $$

(53)

Since $\int_M |W|^{n/2} dV_g$ is sufficiently small, (53) can be absorbed into left hand side of (48). Therefore, there exists a constant $c'$ such that

$$c' \left( \int_M (\phi f)^{2n/(n-2)} dV_g \right)^{(n-2)/n} \leq \int_M |\nabla \phi|^2 f^2 dV_g.$$  

(54)

Choosing a compact supported function similar to (27), we can easily show that $f = 0$ on $M$. We conclude that $(M, g)$ is a constant curvature space.

From Theorem 1 and Theorem 4.
Theorem 5  Let \((M, g)\) be a noncompact complete Riemannian \(n\)-manifold with harmonic curvature and \(Q(M, g) > 0\). Assume that \(\int_M |W|^2 + |E|^2 \, dV_g\) is finite, and (E) or (F) holds:

(E) Scalar curvature \(R = 0\).

(F) Scalar curvature \(R\) is negative and \(n \geq 8\).

Then there exists a small number \(c\) such that if \(\int_M |W|^n/2 + |E|^{n/2} \, dV_g \leq c\), then \((M, g)\) is a constant curvature space.

Remarks. The constants \(c_0\) and \(c\) in Theorem 1-5 depend on the Sobolev constant. For the compact manifold, these constants depend only on the dimension of a given manifold, because of the finiteness of volume [11, 7, 9]. The applicability of Theorem 1-5 to lower dimensional manifolds needs further study.

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