Persistent Laplacians: properties, algorithms and implications

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Abstract

The combinatorial graph Laplace operator has been a fundamental object in the analysis of and optimization on graphs. Its spectral structure has been widely used in graph optimization problems (e.g., clustering), and it is also connected to network circuit theory. There is also a topological view of the graph Laplacian, which permits the extension of the graph Laplacian to the more general $q$-th combinatorial Laplace operator $\Delta^K_q$ for a simplicial complex $K$. In this way, the standard graph Laplace operator is simply the 0-th case (when $q = 0$) of this family of operators. Taking this topological view, Wang et al. introduced the so-called persistent Laplace operator $\Delta^K_{q,L}$, which is an extension of the combinatorial Laplace operator to a pair of simplicial complexes $K \hookrightarrow L$.

In this paper, we present a thorough study of properties and algorithms for the persistent Laplace operator. In particular, we first prove that the rank of the null space of the $q$-th persistent Laplacian $\Delta^K_{q,L}$ gives rise to the $q$-th persistent Betti number from $K$ to $L$, in a way analogous to the non-persistent version. We then present a first algorithm to compute the matrix representation of $\Delta^K_{q,L}$, which helps reveal insights into the meaning of persistent Laplacian. We next show a new relation between the persistent Laplacian and the so-called Schur complement of a matrix. This has several interesting implications. For example, in the graph case, this uncovers a relation with the notion of effective resistance, as well as a persistent version of the Cheeger inequality. This also gives a second, very simple algorithm to compute the $q$-th persistent Laplace operator. This in turn leads to a new algorithm to compute the $q$-th persistent Betti number for a pair of spaces in a fundamentally different manner from existing algorithms in the literature, and the new algorithm can be significantly more efficient than existing algorithms under some conditions. Finally, we also study persistent Laplacians for a filtration of simplicial complexes, and present interesting stability results for their eigenvalues.

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1 Introduction

The combinatorial graph Laplace operator has been a fundamental object in the analysis of and optimization on graphs. Its spectral structure has been widely used in graph optimization problems.
(e.g. spectral clustering [7, 21, 27, 33]), solving systems of equations e.g., [19, 23, 31, 32], and it is also connected to circuit network theory via the notion of effective resistance [1, 8, 24, 30]. There is also a topological view of the graph Laplacian by considering the boundary operators and specific inner product on the chain groups [22]. This allows the extension of graph Laplacian to the more general \( q \)-th combinatorial Laplace operator \( \Delta^K_q \) for a simplicial complex \( K \); e.g., [12, 16], to which the standard graph Laplace operator is simply the 0-th case (when \( q = 0 \)). It connects to the topology of the input complex via the Hodge Decomposition, which implies that the rank of the null space of the \( q \)-th combinatorial Laplacian equals to the rank of the \( q \)-th (co-)homology group of \( K \) under real coefficients. The high-dimensional combinatorial Laplacian (and variants) have received attention in recent years; see e.g. [12, 13, 14, 26]. The recent work of [18] also aims to extend the related concept, effective resistance from circuit theory, to the high dimensional case.

Adopting the topological view of the \( q \)-th combinatorial Laplace operator, Wang et al. [34] introduced the so-called persistent Laplace operator \( \Delta^K_{q,L} \), which is an extension of the combinatorial Laplace operator mentioned above to a pair simplicial complexes \( K \hookrightarrow L \). To our best knowledge, this paper [34] is the first to connect ideas behind persistent homology [10, 35], one of the most important developments in the field of applied and computational topology in the past two decades, with the Laplace operator, a common and fundamental object with a vast literature. It is thus natural and also highly desirable to achieve better understanding, as well as algorithmic developments, for this persistent Laplacian, which will also help broaden its potential applications. The present paper aims to close this gap.

**New work.** In this paper, we present a thorough study of the properties of and algorithms for the persistent Laplace operator. An overview of our results are summarized below:

- In Section 2, we present several results on properties of the \( q \)-th persistent Laplacian \( \Delta^K_{q,L} \), including Theorem 2.5, which proves that the nullity of \( \Delta^K_{q,L} \) yields the \( q \)-th persistent Betti number from \( K \) to \( L \), a result which is analogous to the one in non-persistent case.

- In Section 3, we give a first algorithm (Algorithm 1) to compute a matrix representation of \( \Delta^K_{q,L} \), which uses some reduction ideas from computing the persistent homology. This algorithm also reveals insights about the meaning of the persistent Laplace operator. \(^1\)

- In Section 4, we show a relation between the persistent Laplacian and the so-called Schur complement of a matrix (Theorem 4.6). This has several interesting implications. For example, in the graph case (and \( q = 0 \)), this leads to a persistent Cheeger-like inequality, as well as to revealing that the persistent Laplacian gives rise to the effective resistance of pairs of vertices in \( K \) w.r.t the larger graph \( L \). Furthermore, this connection gives a second, very simple algorithm (see Section 4.2) which can compute the persistent Laplace operator \( \Delta^K_{q,L} \) (for any \( q \)) efficiently, purely based on a linear algebraic formulation (Theorem 4.6). This in turn leads to a new algorithm to compute the \( q \)-th persistent Betti number for a pair of spaces in a fundamentally different manner from existing algorithms in the computational topology literature, and the new algorithm can be significantly more efficient than existing algorithms.

\(^1\)In [34] it is suggested that the \( q \)-th persistent Laplacian \( \Delta^K_{q,L} \) can be computed by (i) taking a certain submatrix of the boundary operator and then (ii) multiplying it by its transpose. However, simply following these two steps does not yield a correct algorithm. The calculation of the matrix form of the persistent Laplacian turned out to be rather subtle. See Remark 3.4 for details.
under some conditions. We think this new algorithm to compute persistent Betti numbers is of independent interest.

• Finally, in Section 5, we first discuss certain stability results for the persistent Laplace operator for a filtration of simplicial complexes (connected by inclusions). We then describe an efficient algorithm to iteratively compute the persistent Laplace operator between all pairs of complexes in a filtration; this is given in Appendix E.

All missing technical details can be found in the appendix.

2 The persistent Laplacian for simplicial pairs $K \hookrightarrow L$

In this section, after introducing some basic notions/definitions in Section 2.1, we formulate the persistent Laplacian for simplicial pairs in Section 2.2 and present some basic properties of persistent Laplacians in Section 2.3.

2.1 Basics

**Simplicial complexes.** An (abstract) simplicial complex $K$ over a finite ordered set $V$ is a collection of finite subsets of $V$ such that for any $\sigma \in K$, if $\tau \subseteq \sigma$, then $\tau \in K$. Denote by $\mathbb{N}$ the set of non-negative integers. For each $q \in \mathbb{N}$, an element $\sigma \in K$ is called a $q$-simplex if $|\sigma| = q + 1$, where we use $|A|$ to denote the cardinality of a set $A$. A 0-simplex, usually denoted by $v$, is also called a vertex. Denote by $S^K_q$ the set of $q$-simplices of $K$. Note that $S^K_0 \subseteq V$. The dimension of $K$, denoted by $\text{dim}(K)$, is the largest $q$ such that $S^K_q \neq \emptyset$. A 1-dim simplicial complex is also called a graph and we often use $K = (V_K, E_K)$ to represent a graph, where $V_K := S^K_0$ denotes the vertex set and $E_K := S^K_1$ denotes the edge set.

An oriented simplex, denoted by $[\sigma]$, is a simplex $\sigma \in K$ with the ordering on its vertices inherited from the ordering of $V$. Let $\bar{S}^K_q := \{[\sigma] : \sigma \in S^K_q\}$. The $q$-th chain group $C^K_q := C^q(K, \mathbb{R})$ of $K$ is the vector space over $\mathbb{R}$ with basis $\bar{S}^K_q$. Let $n^K_q := \dim C^K_q = |S^K_q|$. We define the boundary operator $\partial^K_q : C^K_q \to C^K_{q-1}$ by

$$\partial^K_q([v_0, \ldots, v_q]) := \sum_{i=0}^{q} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_q]$$

for each $\sigma = [v_0, \ldots, v_q] \in \bar{S}^K_q$, where $\hat{v}_i$ denotes the omission of the $i$-th vertex. The $q$-th homology group of $K$ is $H_q(K) = \frac{\ker(\partial^K_q)}{\text{im}(\partial^K_{q+1})}$ and $\beta^K_q := \text{rank } (H_q(K))$ is its $q$-th Betti number.

**Combinatorial Laplacian.** For each $q \in \mathbb{N}$ consider the inner product $\langle \cdot, \cdot \rangle_{C^K_q}$ such that $\bar{S}^K_q$ is an orthonormal basis (ONB) of $C^K_q$. We denote by $(\partial^K_q)^* : C^K_{q-1} \to C^K_q$ the adjoint of $\partial^K_q$ under these inner products. Then, we define the $q$-th combinatorial Laplace operator or just the $q$-th Laplacian $\Delta^K_q : C^K_q \to C^K_q$ as follows:

$$\Delta^K_q = \begin{cases} 
\Delta^K_{q+1} \circ (\partial^K_{q+1})^* + (\partial^K_q)^* \circ \Delta^K_q, & \text{if } q < n^K_q, \\
\Delta^K_{q, \text{up}} & \text{if } q = n^K_q, \\
\Delta^K_{q, \text{down}} & \text{if } q > n^K_q.
\end{cases}$$
Theorem 2.1 ([9]). For each \( q \in \mathbb{N} \), \( \beta_q^K = \text{nullity} (\Delta^K_q) \).

**Simplicial pairs and simplicial filtrations.** A simplicial pair, denoted \( K \hookrightarrow L \), consists of any pair \( K \) and \( L \) of finite simplicial complexes over the same finite ordered set \( V \) such that \( K \subseteq L \), i.e., \( S^K_q \subseteq S^L_q \) for all \( q \in \mathbb{N} \). A simplicial filtration \( \mathbf{K} = \{ K_t \}_{t \in T} \) is a set of simplicial complexes over the same finite ordered set \( V \) indexed by a subset \( T \subseteq \mathbb{R} \) such that for all \( s \leq t \in T \), \( K_s \hookrightarrow K_t \) is a simplicial pair. For an integer \( q \geq 0 \) and for each \( s \leq t \in T \), via functoriality of Homology [15] one obtains a map \( f^s_t : H_q(K_s) \rightarrow H_q(K_t) \) and the \( q \)-th persistent homology groups are defined as the images of these maps. The \( q \)-th persistent Betti numbers \( \beta_q^{s,t} \) of \( K \) are in turn defined as the ranks of these groups. Of course when one is just presented with a simplicial pair \( K \hookrightarrow L \), for each \( q \) one also obtains the analogously defined \( q \)-th persistent Betti number \( \beta_q^{K,L} \).

### 2.2 Definition of the persistent Laplacian

Suppose that we have a simplicial pair \( K \hookrightarrow L \) and that \( q \in \mathbb{N} \). Consider the subspace

\[
C^K_{q+1} := \{ c \in C^K_q : \partial^K_q(c) \in C^K_{q-1} \} \subseteq C^K_q
\]

consisting of those \( q \)-chains in \( C^K_q \) such that their images under the boundary operator \( \partial^K_q \) is in the subspace \( C^K_{q-1} \) of \( C^K_{q-1} \). Let \( n^K_{q,L} := \dim \left( C^K_{q,L} \right) \).

Now, for each \( q \) let \( \partial^K_{q,L} \) denote the restriction of \( \partial^K_q \) to \( C^K_{q,L} \) so that we obtain the “diagonal” operators \( \partial^K_{q,L} : C^K_{q,L} \rightarrow C^K_{q-1,L} \). As we mentioned earlier, for each \( q \) both \( C^K_q \) and \( C^K_q \) are endowed with inner products \( \langle \cdot , \cdot \rangle_{C^K_q} \) and \( \langle \cdot , \cdot \rangle_{C^K_q} \) so that we can consider the adjoints of \( \partial^K_{q+1} \) and \( \partial^K_q \). See the diagram below for the construction where the blue arrows signal the important part of the diagram:

![Diagram](image)

One can then define the \( q \)-th persistent Laplace operator [34] \( \Delta^K_{q,L} : C^K_q \rightarrow C^K_q \) by:

\[
\Delta^K_{q,L} := \partial^K_{q+1} \circ \left( \partial^K_{q+1} \right)^\ast + \left( \partial^K_q \right)^\ast \circ \partial^K_q
\]

where we have also defined the \( q \)-th up persistent Laplace operator \( \Delta^K_{q,L}^{\text{up}} \) with the same domain/codomain as \( \Delta^K_{q,L} \). When \( q = 0 \), since \( \partial^K_0 = 0 \), \( \Delta^K_{0,L}^{\text{up}} = \partial^K_{1,L} \circ \left( \partial^K_{1,L} \right)^* = \Delta^K_{0,L} \).
Example 2.2 (Trivial cases).
1. When \( C_{q+1}^{L,K} = \{0\} \), \( \partial_{q+1}^{L,K} = 0 \) and thus \( \Delta_{q,\text{up}}^{K,L} = 0 \).

2. When \( K = L \), then obviously \( \Delta_{q}^{K,L} = \Delta_{q}^{L} \), the usual Laplacian on \( L \).

3. If \( S_{q}^{K} = S_{q}^{L} \) (or equivalently, \( n_{q}^{K} = n_{q}^{L} \)), then \( \Delta_{q,\text{up}}^{K,L} = \Delta_{q,\text{up}}^{L} \). In particular, if \( S_{0}^{K} = S_{0}^{L} \), then \( \Delta_{0}^{K,L} = \Delta_{0}^{L} \). If furthermore \( S_{q-1}^{K} = S_{q-1}^{L} \), then \( \Delta_{q,\text{down}}^{K,L} = \Delta_{q,\text{down}}^{L} \) and thus \( \Delta_{q}^{K,L} = \Delta_{q}^{L} \).

Obviously, \( \Delta_{q}^{K,L} \) is a positive semi-definite operator on \( C_{q}^{K} \) and thus has non-negative eigenvalues. We denote by \( 0 \leq \lambda_{q,1}^{K,L} \leq \lambda_{q,2}^{K,L} \leq \ldots \leq \lambda_{q,n_{q}^{K}}^{K,L} \) the eigenvalues of \( \Delta_{q}^{K,L} \) sorted in increasing order, including repetitions.

Weighted case. For simplicity of presentation we have so far only defined the persistent Laplacian for unweighted simplicial pairs. See Appendix B.1 for the weighted case.

2.3 Basic properties of the persistent Laplacian

We now show some basic properties of \( \Delta_{q}^{K,L} \). All proofs are given in Appendix B.2.

Lemma 2.3. Suppose \( L \) has \( n \) connected components \( L_{1}, \ldots, L_{n} \). Suppose \( K \) only intersects with the first \( m \) connected components. Let \( K_{i} := K \cap L_{i} \) for each \( i = 1, \ldots, m \). Then, \( \Delta_{q}^{K,L} \) is the direct sum of persistent Laplace operators \( \Delta_{q}^{K_{i},L_{i}} \) on \( C_{q}^{K_{i}} \) for \( i = 1, \ldots, m \), i.e., \( \Delta_{q}^{K,L} = \bigoplus_{i=1}^{m} \Delta_{q}^{K_{i},L_{i}} \).

Given a graph \( K \), the multiplicity of the 0 eigenvalue of the graph Laplacian \( \Delta_{0}^{K} \) coincides with the number of components of \( K \) [25]. The following result is a persistent version of this.

Theorem 2.4. The eigenvalues of \( \Delta_{0}^{K,L} \) satisfy the following basic properties.

1. \( \lambda_{0,1}^{K,L} = 0 \); and if \( L \) is connected, then \( \lambda_{0,2}^{K,L} > 0 \).

2. Let \( m \) be the multiplicity of the 0 eigenvalue of \( \Delta_{0}^{K,L} \), then \( K \) intersects with exactly \( m \) connected components of \( L \).

The following result showing persistent Laplacians recover persistent Betti numbers was mentioned in passing and without proof in [34]. We give a full proof in Appendix B.2.

Theorem 2.5 (Persistent Betti numbers from persistent Laplacians). For each integer \( q \geq 0 \), we have that \( \beta_{q}^{K,L} = \text{nullity} \left( \Delta_{q}^{K,L} \right) \).

The following result describes the behavior of the up persistent Laplacian on interior simplices, where a \( q \)-simplex \( \sigma \in S_{q}^{K} \) is called an interior simplex if \( \sigma \) only shares co-faces with \( q \)-simplices in \( K \), i.e., \( \forall \sigma' \in S_{q}^{L_{i}} \), if \( \sigma \cup \sigma' \in S_{q+1}^{L_{i+1}} \), then \( \sigma' \in S_{q}^{K} \).

Theorem 2.6. Let \( c^{L} \in C_{q}^{L} \) and let \( c^{K} \) be the image of \( c^{L} \) under the orthogonal projection \( C_{q}^{L} \to C_{q}^{K} \). Then, for any interior simplex \( \sigma \in S_{q}^{K} \), we have that

\[
\langle \Delta_{q,\text{up}}^{L,K}, [\sigma] \rangle_{C_{q}^{L}} = \langle \Delta_{q,\text{up}}^{K,L}, [\sigma] \rangle_{C_{q}^{K}}.
\]
3 A first algorithm for computing a matrix representation of $\Delta^K_q$ 

In this section, we first provide a matrix representation $\Delta^K_q$ of $\Delta^K_q$ given the canonical basis $\tilde{S}^K_q$ of $C^K_q$ and then devise an algorithm computing $\Delta^K_q$. 

**Note:** For simplicity, given a simplicial pair $K \mapsto L$, for each $q \in \mathbb{N}$ we assume an ordering $\tilde{S}_q^L = \{ \{ \sigma_i \} \}_{i=1}^{n_q^L}$ on $\tilde{S}_q^L$ such that $\tilde{S}_q^K = \{ \{ \sigma_i \} \}_{i=1}^{n_q^K}$. Unless otherwise specified, matrix representations of operators between chain groups are always from such orderings on canonical bases $\tilde{S}_q^K$ and $\tilde{S}_q^L$ of $C^K_q$ and $C^L_q$, respectively. 

**Theorem 3.1** (Matrix representation of $\Delta^K_q$). Assume that $n_q^{L,K} := \dim \left( C^{L,K}_{q+1} \right) > 0$. Choose any basis of $C^{L,K}_{q+1} \subseteq C^L_{q+1}$ represented by a column matrix $Z \in \mathbb{R}^{n_{q+1}^L \times n_{q+1}^{L,K}}$. Let $B^K_q$ and $B^{L,K}_{q+1}$ be matrix representations of the boundary maps $\partial^K_q$ and $\partial^{L,K}_{q+1}$, respectively. Then, the matrix representation $\Delta^K_q$ of $\Delta^K_q$ is expressed as follows:

$$\Delta^K_q = B^{L,K}_{q+1} \left( Z^T Z \right)^{-1} \left( B^{L,K}_{q+1} \right)^T + \left( B^K_q \right)^T B^K_q.$$ (4)

Moreover, $\Delta^K_q$ is invariant under the choice of basis for $C^{L,K}_{q+1}$. 

To prove the theorem, we need the following result whose proof is in Appendix C.2. 

**Lemma 3.2.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let $f^* : \mathbb{R}^m \to \mathbb{R}^n$ be its adjoint with respect to the canonical inner products. Choose arbitrary bases for $\mathbb{R}^n$ and $\mathbb{R}^m$. Let $F \in \mathbb{R}^{m \times n}$ denote the corresponding matrix representation of $f$. Let $W_n \in \mathbb{R}^{n \times n}$ and $W_m \in \mathbb{R}^{m \times m}$ denote the inner product matrices corresponding to the chosen bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then, the matrix representation $F^*$ is $W_m^{-1} F^TW_n$. 

**Proof of Theorem 3.1.** With respect to our choice of bases for $C^{L,K}_{q+1}, C^K_q$ and $C^{K-1}_q$, the inner product matrices for them are $Z^T Z, I_{n^K_q}$ and $I_{n^{K-1}_q}$, respectively, where $I_d$ is the $d$-dim identity matrix. By Lemma 3.2, the matrix representation for $(\partial^K_q)^*$ is $\left( Z^T Z \right)^{-1} \left( B_{q+1}^{L,K} \right)^T$ and the matrix representation for $(\partial^K_q)^*$ is $\left( B^K_q \right)^T$. By Equation (3), we have that

$$\Delta^K_q = B^{L,K}_{q+1} \left( Z^T Z \right)^{-1} \left( B^{L,K}_{q+1} \right)^T + \left( B^K_q \right)^T B^K_q.$$ 

Since $(\partial^K_q)^*$ is a self-operator on $C^K_q$, its matrix representation $\Delta^K_q$ only depends on the choice of basis of $C^K_q$ and it is thus independent of the choice of basis of $C^{L,K}_{q+1}$. 

**An algorithm for computing the matrix representation of $\Delta^K_q$.** We use the symbol $[n]$ to denote the set $\{1, \ldots, n\}$ when $n$ is a positive integer. We first introduce a notation for representing submatrices. Let $M \in \mathbb{R}^{m \times n}$ be a real matrix and let $\emptyset \neq I \subseteq [m]$ and $\emptyset \neq J \subseteq [n]$. We denote by $M(I, J)$ the submatrix of $M$ consisting of those rows and columns indexed by $I$ and $J$, respectively. Moreover, we use $M(:, J)$ (or $M(I, :)$) to denote $M([m], J)$ (or $M(I, [n])$).
By Theorem 3.1, to compute a matrix representation of $\Delta_{q+1}^{K,L}$, the key is to produce a basis (i.e., Z) for $C_{q+1}^{L,K}$. Let $B_{q+1}^L \in \mathbb{R}^{n_{q+1}^L \times n_{q+1}^L}$ be the matrix representation of the boundary map $\partial_{q+1}^L$. We assume that $n_{q+1}^K < n_{q+1}^L$ since the case $n_{q+1}^K = n_{q+1}^L$ is trivial (cf. Example 2.2). Then, the following lemma (proof in Appendix C.2) suggests a way of constructing Z from $B_{q+1}^L$.

**Lemma 3.3.** Let $D_{q+1}^L := B_{q+1}^L ([n_q^L \setminus n_q^K], \cdot)$. Then, there exists a non-singular matrix $Y \in \mathbb{R}^{n_{q+1}^L \times n_{q+1}^L}$ such that $R_{q+1}^L := D_{q+1}^LY$ is column reduced. Moreover, let $I \subseteq [n_q^L]$ be the index set of 0 columns of $R_{q+1}^L$. The following hold:

1. If $I = \emptyset$, then $C_{q+1}^{L,K} = \{0\}$;
2. If $I \neq \emptyset$, let $Z := Y(:, I)$, then columns of $Z$ constitute a basis of $C_{q+1}^{L,K}$.

Moreover, if $I \neq \emptyset$, then $B_{q+1}^{L,K} = \left(B_{q+1}^L\right)([n_q^K, I])$ is the matrix representation of $\partial_{q+1}^{L,K}$.

We can apply a column reduction process (e.g., Gaussian elimination) to $D_{q+1}^L$ to obtain $Y \in \mathbb{R}^{n_{q+1}^L \times n_{q+1}^L}$ and $R_{q+1}^L := D_{q+1}^LY$ requested in Lemma 3.3. See Algorithm 1 for a pseudocode for computing $\Delta_{q+1}^{K,L}$.

**Remark 3.4.** It is suggested in [34] that $\Delta_{q+1}^{K,L}$ can be computed by (i) considering a certain submatrix of the boundary operator and then (ii) multiplying it by its transpose. However, simply combining these two steps does not produce a valid algorithm: see Example C.1 in Appendix C.1. As we show in Theorem 3.1 and Lemma 3.3, finding the matrix representations of both the new boundary operator and its dual is much more involved: the boundary matrix $B_{q+1}^L$ has to be reduced, and the matrix representation of the dual operator $\left(\partial_{q+1}^{L,K}\right)^*$ is not simply the transpose $(B_{q+1}^{L,K})^T$ but instead has the form $(Z^T Z)^{-1} (B_{q+1}^{L,K})^T$ (cf. lemma 3.3).

**Complexity analysis.** The computation of $\Delta_{q+1}^{K,L}$ takes time $O\left((n_q^K)^2\right)$ (See Appendix A for details). The size of $D_{q+1}^L$ is $(n_q^L - n_q^K) \times n_{q+1}^L$. Then, the column reduction process takes time $O\left((n_q^L - n_q^K)(n_{q+1}^L)^2\right)$. The computation of the product $B_{q+1}^LY$ takes time $O\left(n_q^L(n_{q+1}^L)^2\right)$. The size of $Z$ is $n_{q+1}^L \times \lvert I \rvert$, where $\lvert I \rvert \leq n_{q+1}^L$. Then, computing $(Z^T Z)^{-1}$ takes time at most $O\left((n_{q+1}^L)^3\right)$. The computation of the product $(B_{q+1}^L)^T (Z^T Z)^{-1} (B_{q+1}^L)^T$ takes time $O\left(n_q^K(n_{q+1}^L)^2\right)$. Hence Algorithm 1 takes $O\left(n_q^L(n_{q+1}^L)^2 + (n_{q+1}^L)^3 + (n_q^K)^2\right)$ total time. One can also improve this time complexity by using fast matrix-multiplication to both perform reduction and compute multiplication/inverse. We omit details.

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$^2$We say a matrix is column reduced, if for each two non-zero columns, their indices of lowest non-zero elements are different.
Algorithm 1 Persistent Laplacian: matrix representation

1: **Data:** $B^K_q$ and $B^L_{q+1}$
2: **Result:** $\Delta^K_{q,L}$
3: compute $\Delta^K_{q,\text{down}}$ from $B^K_q$
4: if $n^K_q = n^K_{q+1}$ then
5: compute $\Delta^K_{q,\text{up}}$ from $B^L_{q+1}$; return $\Delta^K_{q,\text{up}} + \Delta^K_{q,\text{down}}$
6: $D^K_{q+1} = B^K_{q+1}([n^K_q \setminus n^K_{q+1}], :)$
7: $(R^K_{q+1}, Y) = \text{ColumnReduction}(D^K_{q+1})$
8: $I \leftarrow$ index set corresponding to the all-zero columns of $R^K_{q+1}$
9: if $I == \emptyset$ then return $\Delta^K_{q,\text{down}}$
10: $Z = Y(:, I)$
11: $B^K_{q+1} = (B^K_{q+1} Y) ([n^K_q], I)$
12: return $B^K_{q+1} (Z^T Z)^{-1} (B^K_{q+1})^T + \Delta^K_{q,\text{down}}$

4 Schur complement, persistent Laplacian and implications

Let $M \in \mathbb{R}^{n \times n}$ be a block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $D \in \mathbb{R}^{d \times d}$ is a square matrix. Then, the (generalized) Schur complement of $D$ in $M$ [4], denoted by $M/D$, is $M/D := A - BD^\dagger C$, where $D^\dagger$ is the Moore-Penrose generalized inverse of $D$. Note that having $D$ to be the bottom right submatrix is only for notational simplicity. In fact, Schur complement is defined for any principal submatrix. More precisely, let $\emptyset \neq I \subset [n]$ be a proper subset. Then, we define the (generalized) Schur complement of $M(I, I)$ in $M$ by

$$M/M(I, I) := M([n] \setminus I, [n] \setminus I) - M([n] \setminus I, I) M(I, I)^\dagger M(I, [n] \setminus I).$$

(5)

After a simultaneous permutation of rows and columns of $M$, one can obtain $\hat{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $D = M(I, I)$. Then, it is clear that $\hat{M}/D = M/M(I, I)$.

Now we introduce some basic properties of the Schur complement.

**Definition 4.1** (Proper submatrices). *Consider the square block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where both $A$ and $D$ are square matrices. We say the submatrix $D$ is proper in $M$ if $\ker(D) \subseteq \ker(B)$ and $\ker(D^\text{T}) \subseteq \ker(C^\text{T})$.*

**Example 4.2** (Positive semi-definite matrices). *If $M$ is positive semi-definite, then $D$ is automatically proper. Indeed, since $M$ is positive semi-definite, there exists a square matrix $E \in \mathbb{R}^{n \times n}$ such that $M = EE^\text{T}$. Assume that $D$ has size $d \times d$. Let $E_1 := E([n-d], :)$ and let $E_2 := E([n] \setminus [n-d], :)$. Then, $B = E_1 E_2^\text{T}$ and $D = E_2 E_2^\text{T}$. Since $\ker(D) = \ker(E_2^\text{T})$, we have that $\ker(D) \subseteq \ker(B)$. Similarly, we have $\ker(D^\text{T}) \subseteq \ker(C^\text{T})$ and thus $D$ is proper.*

**Lemma 4.3** ([4, Theorem 1]). *Let $M$ be a square block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $A$ and $D$ are square matrices. Then, $\text{rank}(M) \geq \text{rank}(D) + \text{rank}(M/D).$*
Lemma 4.4 (Quotient Formula [4, Theorem 4]). Let $M, D$ and $H$ be square matrices with the following block structures: $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $D = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$. If $D$ is proper in $M$ and $H$ is proper in $D$, then $D/H$ is proper in $M/H$ and $M/D = (M/H)/(D/H)$.

Lemma 4.5 (Eigenvalue interlacing property). Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a $n \times n$ positive semi-definite matrix such that $D$ is a $d \times d$ proper square matrix. Then,

$$\lambda_k(M) \leq \lambda_k(M/D) \leq \lambda_k(A), \quad \forall 1 \leq k \leq n - d,$$

where $\lambda_k(A)$ denotes the $k$-th smallest eigenvalue of $A$ (counted with multiplicity).

Lemma 4.5 was proved in [11, Theorem 3.1] when $D$ is non-singular. See Appendix D.4 for a proof of Lemma 4.5 for the case when $D$ is proper in $M$.

4.1 Up-persistent Laplacian as a Schur complement

For a simplicial pair $K \hookrightarrow L$, recall from Section 3 that for each $q \in \mathbb{N}$ we assume an ordering $\bar{S}_q^L = \{[\sigma_i]\}_{i=1}^{n_q^L}$ on $\bar{S}_q^L$ such that $\bar{S}_q^K = \{[\sigma_i]\}_{i=1}^{n_q^K}$. Given such orderings on canonical bases of $C_q^K$ and $C_q^L$, the matrix representation $\Delta_{q,up}^{K,L}$ of $\Delta_{q,up}^{K,L} : C_q^K \rightarrow C_q^L$ is related to the matrix representation $\Delta_{q,up}^L$ of $\Delta_{q,up}^L (I_{I_K}^L, I_{I_K}^L)$ via the Schur complement as follows:

Theorem 4.6 (Up-persistent Laplacian as Schur complement). Let $K \hookrightarrow L$ be a simplicial pair. Assume that $n_q^K < n_q^L$ and let $I_{I_K}^L := [n_q^L]\backslash [n_q^K]$. Then,

$$\Delta_{q,up}^{K,L} = \Delta_{q,up}^L / \Delta_{q,up}^L (I_{I_K}^L, I_{I_K}^L).$$

To prove the above theorem, we first need the following lemma (whose proof is given in Appendix D.4) which relates Schur complements with a certain matrix operation.

Lemma 4.7. Let $B \in \mathbb{R}^{n \times m}$ be a block matrix $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, where $B_1 \in \mathbb{R}^{d \times m}$ for some $1 \leq d < n$. Let $M := BB^T$, which is a block matrix $M = \begin{pmatrix} B_1B_1^T & B_1B_2^T \\ B_2B_1^T & B_2B_2^T \end{pmatrix}$. Let $M_{ij} := B_iB_j^T$ for $i, j \in \{1, 2\}$. If $B_2$ has full column rank, then $M/M_{22} = 0$. Otherwise, for any non-singular block matrix $Y = \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} \in \mathbb{R}^{m \times m}$, if $B_2Y_1 = 0$ and $B_2Y_2$ has full column rank, then $M/M_{22} = B_1Y_1 (Y_1^TY_1)^{-1} (B_1Y_1)^T$.

Proof of Theorem 4.6. Let $B = B_{q+1}^L$ be the matrix representation of $\partial_{q+1}^L$. Let $B_1 := B([n_q^K], :)$ and let $B_2 := B ([n_q^L]\backslash [n_q^K], :)$. Then, $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$. Note that $B_2 = D_{q+1}^L$ using notations in Lemma 3.3. By Lemma 3.3, there exists a non-singular matrix $Y \in \mathbb{R}^{n_{q+1}^L \times n_{q+1}^L}$ such that $R_{q+1}^L := D_{q+1}^L Y$ is column reduced. Let $I \subseteq [n_q^L]$ be the index set of 0 columns of $R_{q+1}^L$. If $I = \emptyset$, then by Lemma 3.3 we have that $C_{q+1}^L = \{0\}$ and thus $\Delta_{q,up}^{K,L} = 0$. On the other hand, $I = \emptyset$ implies that $B_2$ has full column rank. Then by Lemma 4.7, we have that

$$\Delta_{q,up}^L / \Delta_{q,up}^L (I_{I_K}^L, I_{I_K}^L) = M/M_{22} = 0 = \Delta_{q,up}^{K,L}.$$
Now, we assume that \( I \neq \emptyset \). Without loss of generality, we assume that \( I = [n_{q+1}^L] \subseteq [n_{q+1}^L] \) (otherwise we multiply \( Y \) by a permutation matrix). Let \( Y_1 := Y (:, I) = Z \) and let \( Y_2 := Y (:, [n_{q+1}^L]\setminus I) \). Then, \( Y = (Y_1, Y_2) \) is a block matrix such that \( B_2Y_1 = 0 \) and that \( B_2Y_2 \) has full column rank. Note that \( M = BB^T = \Delta_{q,up}^L \) and \( M_{22} = \Delta_{q,up}^L (I_K, I_K^L) \). Then, by Lemma 4.7, we have that

\[
\Delta_{q,up}^L \frac{\Delta_{q,up}^L}{I_K^L, I_K^L} = M/M_{22} = B_1Y_1 (Y_1^TY_1)^{-1} (B_1Y_1)^T.
\]

Note also that \( B_{q+1}^{L,K} = B_1Y_1 \). Then, by Lemma 3.3 we have that

\[
\Delta_{q,up}^{K,L} = B_{q+1}^{L,K} (Z^T Z)^{-1} (B_{q+1}^{L,K})^T = B_1Y_1 (Y_1^TY_1)^{-1} (B_1Y_1)^T = \Delta_{q,up}^L \Delta_{q,up}^L (I_K^L, I_K^L).
\]

This finishes the proof of Theorem 4.6.

\[\Box\]

### 4.2 Fast computation of the matrix representation of \( \Delta_{q}^{K,L} \)

For a simplicial pair \( K \hookrightarrow L \), by Theorem 4.6, we now simply compute \( \Delta_{q,up}^{K,L} \) via Equation (7) using only Schur complement computations, which then give us \( \Delta_q^{K,L} = \Delta_{q,up}^{K,L} \Delta_{q,down}^{K,L} \). A pseudocode for this simple algorithm is given in Appendix D.1.

**Time complexity.** Computing \( \Delta_{q,up}^L \) takes time \( O(n_{q+1}^L) \) and computing \( \Delta_{q,down}^L \) takes \( O \left( (n_q^L)^2 \right) \) (see Appendix A for details). Computing the Schur complement \( \Delta_{q,up}^L \Delta_{q,up}^L \) takes time \( O \left( (n_q^L)^2 + (n_q^L - n_q^L)^3 + n_q^L \left( n_q^L - n_q^L \right)^2 \right) = O \left( (n_q^L)^3 \right) \). Therefore, the total time complexity of computing \( \Delta_{q,up}^{K,L} \) via Equation (7) is \( O \left( (n_q^L)^3 + n_{q+1}^L \right) \), which is more efficient than the complexity of Algorithm 1, \( O \left( n_q^L \left( n_{q+1}^L \right)^2 + (n_{q+1}^L)^3 + (n_q^L)^2 \right) \), when \( n_q^L = O(n_{q+1}^L) \). By using fast matrix multiplication algorithm (which takes \( O(r^\omega), \omega < 2.373 \), to multiply two \( r \times r \) matrices), this time complexity can be improved to \( O \left( (n_q^L)^{\omega} + n_{q+1}^L \right) \).

**Computation of persistent Betti numbers.** By Theorem 2.5, we can compute the persistent Betti number \( \beta_q^{K,L} \) in the following manner: we first compute \( \Delta_{q,up}^{K,L} \) and then compute \( \beta_q^{K,L} = \text{nullity} \left( \Delta_{q,up}^{K,L} \right) \). Since calculating the nullity of an \( n_q^L \times n_q^L \) square matrix can be done in time \( O \left( (n_q^L)^{\omega} \right) = O \left( (n_q^L)^{\omega} \right) \), we obtain a method for computing the persistent Betti number in time \( O \left( (n_q^L)^{\omega} + n_{q+1}^L \right) \) (which is \( O \left( (n_q^L)^{\omega} \right) \) if \( n_q^L = O(n_{q+1}^L) \)). Currently, the existing approach in the literature to compute the persistent Betti numbers is through computing the persistent homology of the pair \( K \hookrightarrow L \) using boundary matrices \( B_{q+1}^L \) and \( B_{q+1}^K \), which can be done in \( O \left( (n_q^L)^2 n_{q+1}^L + \left( n_q^L - n_{q+1}^L \right)^2 n_q^L \right) \) time or in \( O \left( (n_q^L)^{\omega-1} n_{q+1}^L + (n_q^L - n_{q+1}^L)^{\omega-1} n_q^L \right) \) (if we assume that \( n_q^L = O(n_{q+1}^L) \) and \( n_{q+1}^L = O(n_q^L) \)) using earliest basis (via fast matrix multiplication) approach [2]. Our new algebraic formulation of persistent Laplace operator (via Schur complement) thus also leads to a (potentially significantly) faster algorithm to compute the persistent Betti number for a pair of spaces if \( n_q^L = O(n_{q+1}^L) \). Note that the condition \( n_q^L = O(n_{q+1}^L) \) holds in many practical scenarios, especially for the popular Rips or Čech complexes and their variants. Given that this new algorithm is fundamentally different from existing ones (using only simple Schur complement computations), we believe that this is an independent contribution and of independent interest.
4.3 Persistent Cheeger inequality for graph pairs $K \hookrightarrow L$

The Cheeger constant $h^K$ of a graph $K = (V_K, E_K)$ is defined as follows [14]:

$$h^K := \min_{\emptyset \neq A \subseteq V_K} \frac{|V_K| |E_K(A, V_K \setminus A)|}{|A| |V_K \setminus A|},$$

where $E_K(A, B)$ denotes the set of edges $\{v, w\} \in E_K$ such that $v \in A$ and $w \in B$. The Cheeger constant $h^K$ measures the expansion of $K$ and it is related with the second smallest eigenvalue $\lambda_{0,2}^K$ of the graph Laplacian $\Delta_0^K$ as follows:

$$\lambda_{0,2}^K \leq h^K \leq \sqrt{8 \max_{\emptyset \neq A \subseteq V_K} \lambda_{0,2}^A}, \quad (8)$$

where $d_{\text{max}}$ is the maximal degree of vertices in the graph $K$. Equation (8) is called the discrete Cheeger inequality [6, 14], which is a discrete analogue to isoperimetric inequalities in Riemannian geometry [3, 5].

In this section, we define a persistent Cheeger constant for any graph pair $K \hookrightarrow L$ and establish a corresponding persistent Cheeger inequality.

A path in a given graph $K = (V_K, E_K)$ is a tuple $p = (v_0, \ldots, v_n)$ such that $v_i \in V_K$ for each $i = 0, \ldots, n$ and $\{v_i, v_{i+1}\} \in E_K$ for each $i = 0, \ldots, n-1$. The length of a path $p = (v_0, \ldots, v_n)$ is $n$, denoted by $\text{length}(p) = n$. For nonempty subsets $A, B \subseteq V_K$, we denote by $P_K(A, B)$ the set of all paths $p = (v_0, \ldots, v_n)$ in $K$ satisfying: (i) $v_0 \in A, v_n \in B$; (ii) $\{v_i, v_{i+1}\} \neq \{v_j, v_{j+1}\}$ for $i \neq j$. If $A = \{v\}$ and $B = \{w\}$ are one-point sets, then we also denote $P_K(v, w) := P_K(\{v\}, \{w\})$. We have the following obvious properties of $P_K(A, B)$.

**Lemma 4.8** (Basic properties). $P_K(A, B)$ satisfies:

1. $|P_K(A, B)| = |P_K(B, A)|$.
2. If $A = A_1 \bigcup A_2$, then $P_K(A, B) = P_K(A_1, B) \bigcup P_K(A_2, B)$.

We then define the persistent Cheeger constant $h^{K,L}$ for the graph pair $K \hookrightarrow L$ as follows:

$$h^{K,L} := \min_{\emptyset \neq A \subseteq V_K} \frac{|V_K| |P_L(A, V_K \setminus A)|}{|A| |V_K \setminus A|}.$$

Similarly to the case of a single graph, the persistent Cheeger constant for a graph pair $K \hookrightarrow L$ is related to the second smallest eigenvalue $\lambda_{0,2}^{K,L}$ of the persistent Laplacian $\Delta_0^{K,L}$.

**Theorem 4.9** (Persistent Cheeger inequality). Let $K \hookrightarrow L$ be a graph pair, then $\lambda_{0,2}^{K,L} \leq h^{K,L}$.

The proof is based on the following case when $V_K$ is a two-point subset of $V_L$; the proof of Lemma 4.10 is given in Appendix D.4.

**Lemma 4.10.** When $|V_K| = 2$, then $\lambda_{0,2}^{K,L} \leq h^{K,L}$.

**Proof of Theorem 4.9.** By Lemma 4.10, we only consider the case when $|V_K| \geq 3$. Choose arbitrarily two distinct points $v, w \in V_K$ and denote by $K'$ a subgraph of $K$ with vertex set $\{v, w\}$. We then assign an ordering $V_L = \{v_1, \ldots, v_{n_0}\}$ such that $V_K = \{v_1, \ldots, v_{n_0}\}$ and
\(V_K = \{v_1, v_2\}\). By Theorem 4.6, \(\Delta_0^{K,L} = \Delta_0^L / \Delta_0^L \left( I_{K}^L, I_{K}^L \right) \) and \(\Delta_0^{K',L} = \Delta_0^L / \Delta_0^L \left( I_{K'}^L, I_{K'}^L \right) \) where \(I_{K}^L = [n_0^L] \setminus [n_0^K] \) and \(I_{K'}^L = [n_0^L] \setminus [2]\). Since \(\Delta_0^L \) is positive semi-definite, both \(\Delta_0^L \left( I_{K}^L, I_{K}^L \right) \) and \(\Delta_0^L \left( I_{K'}^L, I_{K'}^L \right) \) are proper in \(\Delta_0^L \) (cf. Example 4.2). Similarly, \(\Delta_0^L \left( I_{K'}^L, I_{K'}^L \right) \) is proper in \(\Delta_0^L \left( I_{K}^L, I_{K}^L \right) \).

Then, by Lemma 4.4, \(\Delta_0^{K',L} \) is the Schur complement of some proper principal submatrix in \(\Delta_0^{K,L} \). Then, by Lemma 4.5, \(\lambda_{0,2}^{K,L} = \lambda_2 \left( \Delta_0^{K,L} \right) \leq \lambda_2 \left( \Delta_0^{K',L} \right) \). By Lemma 4.10, we have that \(\lambda_{0,2}^{K,L} \leq \lambda_2 \left( \Delta_0^{K',L} \right) \leq h^{K',L} = 2 |P_L(v,w)|.

Now, let \(\emptyset \neq A \subseteq V_K \) be such that \(h^{K,L} = \frac{|V_K| |P_L(A,V_K \setminus A)|}{|A||V_K \setminus A|} \). Without loss of generality, we assume that \(|V_K \setminus A| > 1\) (otherwise replace \(A\) with \(V_K \setminus A\)). Then, by Lemma 4.8 we have

\[
h_{K,L} = \frac{|V_K| |P_L(A,V_K \setminus A)|}{|A||V_K \setminus A|} \geq \frac{1}{|A|} \sum_{v \in A} |P_L(v,V_K \setminus A)| \geq |P_L(v_*, V_K \setminus A)| = \sum_{w \in V_K \setminus A} |P_L(v_*, w)| \geq |V_K \setminus A||P_L(v_*, w_*)| \geq 2 |P_L(v_*, w_*)| \geq \lambda_{0,2}^{K,L},
\]

where \(v_* \in A\) is such that \(|P_L(v_*, V_K \setminus A)| = \min_{v \in A} |P_L(v,V_K \setminus A)|\) and \(w_* \in V_K \setminus A\) is such that \(|P_L(v_*, w_*)| = \min_{w \in V_K \setminus A} |P_L(v_*, V_K \setminus A)|\).

**Remark 4.11.** When \(K = L\), \(h_{K,K} \) is usually much larger than \(h_K\) since \(|P_K(A,V_K \setminus A)| \geq |E_K(A,V_K \setminus A)|\) for any \(A \subset V_K\). This suggests us considering a smaller set of paths than \(P_I(A,B)\) and devising a smaller persistent Cheeger constant. See Appendix D.2 for a detailed discussion and a conjecture about a stronger persistent Cheeger inequality.

### 4.4 Relationship with the notion of effective resistance

Let \(K = (V_K, E_K)\) be a connected graph. For any two nodes \(v, w \in V_K\), we let \(\partial_{[v,w]} := [v] + [w] \in C_0^K\). Let \(D_{[v,w]}^K \in \mathbb{R}^{n_0^K}\) denote the vector representation of \(\partial_{[v,w]}\) in \(C_0^K\). We consider that each edge \(e \in E_K\) has a conductance 1. Then, the effective resistance \(R_{v,w}^K\) between \(v\) and \(w\) is defined by

\[
R_{v,w}^K := \left( D_{[v,w]}^K \right)^T \left( \Delta_0^K \right)^{+} D_{[v,w]}^K. \tag{9}
\]

Given a graph pair \(K \hookrightarrow L\), by Theorem 4.6 the persistent Laplacian \(\Delta_0^{K,L}\) turns out to be same as the Kron-reduced matrix from the Kron reduction of \(L\); see Appendix D.3 for details. The Kron reduction [20] has been used in network circuit theory and related applications. In particular, the Kron reduction preserves effective resistance (cf. [8, Theorem 3.8]). This in turn implies that the persistent Laplacian \(\Delta_0^{K,L}\) is enough to recover effective resistance \(R_{v,w}^L\) w.r.t. the larger graph \(L\) for all pairs of nodes \(v, w \in K\).

**Theorem 4.12.** Let \(K = (V_K, E_K) \hookrightarrow L = (V_L, E_L)\) be a graph pair where \(L\) is connected. For two distinct nodes \(v, w \in V_K\), note that \(\partial_{[v,w]} \in C_0^K \subseteq C_0^L\). Then,

\[
R_{v,w}^L = \left( D_{[v,w]}^L \right)^T \left( \Delta_0^L \right)^{+} D_{[v,w]}^L = \left( D_{[v,w]}^K \right)^T \left( \Delta_0^{K,L} \right)^{+} D_{[v,w]}^K,
\]

where \(D_{[v,w]}^L \in \mathbb{R}^{n_0^L}\) and \(D_{[v,w]}^K \in \mathbb{R}^{n_0^K}\) denote the vector representations of \(\partial_{[v,w]}\) in \(C_1^L\) and \(C_1^K\), respectively.
Proof. This follows directly from Theorem 4.6 and [8, Theorem 3.8].

Remark 4.13 (Higher dimension generalizations). Theorem 4.12 actually holds in the more general setting of weighted graphs. Moreover, the effective resistance has been generalized to the case of weighted simplicial complexes in [18]. In Appendix D.3, we show a higher-dimensional extension of Theorem 4.12.

In the case when $K$ consists of only two points in $L$, we have the following explicit relation between the persistent Laplacian on the simplicial pair $K \hookrightarrow L$ and the effective resistance.

Corollary 4.14. Let $L$ be a connected graph and let $K$ be a two-vertex subgraph with vertex set $V_K = \{v, w\}$. Then,

$$ \Delta_0^{K,L} = \begin{pmatrix} \frac{1}{R_{v,w}^L} & -\frac{1}{R_{v,w}^L} \\ -\frac{1}{R_{v,w}^L} & \frac{1}{R_{v,w}^L} \end{pmatrix}. $$

Combining with the persistent Cheeger inequality Equation (8), we obtain the following control of the effective resistance between a pair of nodes: the effective resistance between two points is lower bounded by the reciprocal of number of paths connecting the two points.

Corollary 4.15. Let $L$ be a connected graph. Let $v, w \in V_L$ be two distinct vertices; then we have

$$ R_{L,v,w} \geq \frac{1}{|P_L(v,w)|}. $$

See Appendix D.4 for proofs of the above two corollaries.

5 The persistent Laplacian for simplicial filtrations

We now extend the setting of Section 2 for simplicial pairs to a simplicial filtration.

5.1 Formulation

Let $K = \{K_t\}_{t \in T}$ be a simplicial filtration with an index set $T \subseteq \mathbb{R}$. For each $t \in T$ and integer $q \geq 0$ we let $C^t_q := C^{K_t}_q$ and $S^t_q := S^{K_t}_q$. For $s \leq t \in T$ we let

$$ C^{t,s}_q := \{ c \in C^t_q : \partial^t_q(c) \in C^s_q \} \subseteq C^t_q. $$

Let $\partial^t_q$ be the restriction of $\partial^t_q$ to $C^{t,s}_q$. Then, $\partial^t_q$ is a map from $C^{t,s}_q$ to $C^s_q$. Finally, we define the $q$-th persistent Laplacian $\Delta^{s,t}_q : C^t_q \rightarrow C^s_q$ by

$$ \Delta^{s,t}_q := \partial^{t,s}_q \circ (\partial^t_q)^* + (\partial^s_q)^* \circ \partial^t_q. $$

(10)

where we view $C^t_q$ for each $t \in T$ as a Hilbert space such that the collection $\tilde{S}^t_q := \{ [\sigma] : \sigma \in S^t_q \}$ of oriented simplices is an ONB ($C^{t,s}_q$ is just a Hilbert subspace of $C^t_q$) and $A^*$ means the adjoint of an operator $A$ under these inner products. We also let $\Delta^t_q$ denote the $q$-th Laplacian of $K_t$ for $t \in T$. Note that $\Delta^{s,t}_q = \Delta^t_q$ (cf. Example 2.2).
Computation. Given a filtration $K_1 \hookrightarrow K_2 \hookrightarrow \cdots \hookrightarrow K_m$, it turns out that we can compute all $\Delta^{i,j}_q$ for $1 \leq i \leq j \leq m$ faster than applying directly for all pairs $K_i \hookrightarrow K_j$ the algorithm in Section 4.2. See the algorithm given in Appendix E, which is an iterative procedure again leveraging the connection to the Schur complement.

5.2 Monotonicity and stability of persistent eigenvalues

Recall from Section 2.3 that for a simplicial pair $K \hookrightarrow L$, $\lambda^{K,L}_{q,k}$ denotes its $k$-th smallest eigenvalue of $\Delta^{K,L}_q$. Now, given a simplicial filtration $K = \{K_t\}_{t \in T}$, we define its $k$-th persistent eigenvalue $\lambda^{s,t}_{q,k}(K)$ for each $s \leq t \in T$ by $\lambda^{s,t}_{q,k}(K) := \lambda^{K_s,K_t}_{q,k}$. Similarly, we define the $k$-th up-persistent eigenvalue $\lambda^{s,t}_{q,\up}(K)$ for each $s \leq t \in T$ to be the $k$-th smallest eigenvalue of $\Delta^{K_s,K_t}_{q,\up}$. Whenever the underlying filtration $K$ is clear from the context, we let $\lambda^{s,t}_{q,k} := \lambda^{s,t}_{q,k}(K)$ and $\lambda^{s,t}_{q,\up} := \lambda^{s,t}_{q,\up}(K)$.

In [34] the authors suggest that invariants similar to persistent eigenvalues could be useful for shape classification applications. With that in mind, we now explore both their monotonicity and stability properties, concluding with theorem 5.8.

Theorem 5.1 (Monotonicity of eigenvalues of persistent up-Laplacian). Let $K = \{K_t\}_{t \in T}$ be a simplicial filtration and let $q \in \mathbb{N}$. Then, for any $t_1 \leq t_2 \leq t_3 \in T$, we have for each $k = 1, \ldots, n^t_q$ that $\lambda^{t_1,t_2}_{q,\up,k} \leq \lambda^{t_1,t_3}_{q,\up,k}$ and $\lambda^{t_2,t_3}_{q,\up,k} \leq \lambda^{t_1,t_3}_{q,\up,k}$.

The proof exploits the connection of the up-Laplacian with Schur complements (Theorem 4.6).

Proof. By the min-max theorem (see for example [16, Theorem 2.1]), we have for any $s \leq t \in T$ and for each $k = 1, \ldots, n^t_q$ that

$$\lambda^{s,t}_{q,\up,k} = \min_{V_k \subseteq C^s_q} \max_{g \in V_k} \frac{\langle \Delta^{s,t}_{q,\up,k} g, g \rangle_{C^s_q}}{\langle g, g \rangle_{C^s_q}},$$

where the minimum is taken over all $k$-dim subspaces $V_k$ of $C^s_q$. Then, in order to prove that $\lambda^{t_1,t_2}_{q,\up,k} \leq \lambda^{t_1,t_3}_{q,\up,k}$, we only need to verify that $\langle \Delta^{t_1,t_2}_{q,\up,k} g, g \rangle_{C^s_q} \leq \langle \Delta^{t_1,t_3}_{q,\up,k} g, g \rangle_{C^s_q}$ for any $k$-dim subspace $V_k \subseteq C^s_q$ and any $g \in V_k$.

Now, since $C^{t_2,t_1}_{q+1} \subseteq C^{t_3,t_1}_{q+1}$, we consider an orthogonal decomposition $C^{t_3,t_1}_{q+1} = C^{t_2,t_1}_{q+1} \oplus (C^{t_2,t_1}_{q+1})^\perp$. Then, we have the decomposition $\partial^{t_3,t_1}_{q+1} = \partial^{t_2,t_1}_{q+1} \oplus \partial^\perp$, where $\partial^\perp$ maps $(C^{t_2,t_1}_{q+1})^\perp$ into $C^{t_1}_{q+1}$. Therefore, we have that

$$\Delta^{t_1,t_3}_{q,\up,k} = \partial^{t_3,t_1}_{q+1} (\partial^{t_2,t_1}_{q+1})^* = \partial^{t_2,t_1}_{q+1} (\partial^{t_2,t_1}_{q+1})^* + \partial^\perp (\partial^\perp)^* = \Delta^{t_1,t_2}_{q,\up,k} + \partial^\perp (\partial^\perp)^*.$$

This implies the following and thus $\lambda^{t_1,t_2}_{q,\up,k} \leq \lambda^{t_1,t_3}_{q,\up,k}$:

$$\langle \Delta^{t_1,t_3}_{q,\up,k} g, g \rangle_{C^s_q} = \langle \Delta^{t_1,t_2}_{q,\up,k} g, g \rangle_{C^s_q} + \langle \partial^\perp (\partial^\perp)^* g, g \rangle_{C^s_q} = \langle \Delta^{t_1,t_2}_{q,\up,k} g, g \rangle_{C^s_q} + \langle (\partial^\perp)^* g, (\partial^\perp)^* g \rangle_{C^s_q} \geq \langle \Delta^{t_1,t_2}_{q,\up,k} g, g \rangle_{C^s_q}.$$
As for \( \lambda_{t_2, t_3, k} \leq \lambda_{t_1, t_3, k} \), we will apply Theorem 4.6. For notational simplicity, we let \( I_s := [n_q'] [n_q'] \). Since the matrix \( \Delta_{q, up} \) is positive semi-definite, both \( \Delta_{q, up} (I_{t_2}^s, I_{t_2}^s) \) and \( \Delta_{q, up} (I_{t_1}^s, I_{t_1}^s) \) are proper in \( \Delta_{q, up} \) (cf. Example 4.2). Moreover, \( \Delta_{q, up} (I_{t_2}^s, I_{t_2}^s) \) is proper in \( \Delta_{q, up} (I_{t_1}^s, I_{t_1}^s) \).

Then, by Lemma 4.4, \( \Delta_{q, up} (I_{t_2}^s, I_{t_2}^s) \) is the Schur complement of some proper principal submatrix in \( \Delta_{q, up} (I_{t_2}^s, I_{t_2}^s) \). By Lemma 4.5,

\[
\lambda_k \left( \Delta_{q, up} (I_{t_2}^s, I_{t_2}^s) \right) \leq \lambda_k \left( \Delta_{q, up} (I_{t_1}^s, I_{t_1}^s) \right), \quad k = 1, \ldots, n_q^s.
\]

Then, by Theorem 4.6, we have that \( \lambda_{t_2, t_3} \leq \lambda_{t_1, t_3} \) for all \( k = 1, \ldots, n_q^s \).

Note that when \( q = 0 \), \( \Delta_{s, t}^s = \Delta_{s, t}^s \) for \( s \leq t \). Then, we have the following corollary.

**Corollary 5.2.** Let \( K = \{K_t\}_{t \in T} \) be a simplicial filtration. Then for any \( t_1 \leq t_2 \leq t_3 \in T \), we have for each \( k = 1, \ldots, n_q^s \) that \( \lambda_{0, k}^s \leq \lambda_{0, k}^r \) and \( \lambda_{t_2, t_3} \leq \lambda_{t_1, t_3} \).

A simple adaptation of the proof of the formula \( \lambda_{q, up, k} \leq \lambda_{q, up, k} \) will give rise to the following monotonicity result for eigenvalues of persistent Laplacians.

**Corollary 5.3.** Let \( K = \{K_t\}_{t \in T} \) be a simplicial filtration. Given \( q \in \mathbb{N} \), then for any \( t_1 \leq t_2 \leq t_3 \in T \), we have for each \( k = 1, \ldots, n_q^s \) that \( \lambda_{q, k}^t \leq \lambda_{q, k}^r \).

**Stability of up-persistent eigenvalues with respect to the interleaving distance.**

**Lemma 5.4.** Let \( K_{t_1} \hookrightarrow K_{t_2} \hookrightarrow K_{t_3} \hookrightarrow K_{t_4} \) be a simplicial filtration over an index set \( \{t_1 \leq t_2 \leq t_3 \leq t_4\} \) with at most four points. Then, for any \( k = 1, \ldots, n_q^s \), we have \( \lambda_{q, up, k} \geq \lambda_{q, up, k} \).

**Proof.** By Theorem 5.1 we have that \( \lambda_{t_1, t_4} \geq \lambda_{t_1, t_3} \geq \lambda_{t_1, t_2} \). Then, \( \lambda_{q, up, k} \geq \lambda_{q, up, k} \).

**Definition 5.5** (Interleaving distance between simplicial filtrations over \( \mathbb{R} \)). Let \( K = \{K_t\}_{t \in \mathbb{R}} \) and \( L = \{L_t\}_{t \in \mathbb{R}} \) be two simplicial filtrations over \( \mathbb{R} \) with the same underlying vertex set \( V \) and the same index set \( \mathbb{R} \). We define the interleaving distance between \( K \) and \( L \) by

\[
d^K_L (K, L) := \inf \{ \varepsilon \geq 0 : \forall t, K_t \subseteq L_{t+\varepsilon} \text{ and } L_t \subseteq K_{t+\varepsilon} \}.
\]

**Definition 5.6** (Interleaving distance between simple closed intervals). Let \( \text{Int} \) denote the set of closed intervals in \( \mathbb{R} \). Let \( f : \text{Int} \to \mathbb{R}_{\geq 0} \) and \( g : \text{Int} \to \mathbb{R}_{\geq 0} \) be two non-negative functions. We then define the interleaving distance between \( f \) and \( g \) by:

\[
d_f (f, g) := \inf \{ \varepsilon \geq 0 : \forall I \in \text{Int}, f(I^\varepsilon) \geq g(I) \text{ and } g(I^\varepsilon) \geq f(I) \}.
\]

Above, for \( \text{Int} \ni I = [a, b] \) and \( \varepsilon > 0 \), we denoted \( I^\varepsilon := [a - \varepsilon, b + \varepsilon] \).

**Remark 5.7.** The stability theorem given below is structurally similar to claims about stability of the rank invariant, see [29, Theorem 22] and [17, Remarks 4.10 and 4.11].

With these definitions we now obtain the following stability theorem:
Theorem 5.8 (Stability theorem for up-persistent eigenvalues). Let \( K = \{K_t\}_{t \in \mathbb{R}} \) and \( L = \{L_t\}_{t \in \mathbb{R}} \) be two simplicial filtrations over the same underlying vertex set \( V \). Then,

\[
d_I(\lambda^K_{q,up,k}, \lambda^L_{q,up,k}) \leq d^V_I(K, L),
\]

where \( \lambda^K_{q,up,k} : \text{Int} \to \mathbb{R}_{\geq 0} \) is defined by \( \text{Int} \ni I = [a, b] \mapsto \lambda^{a,b}_{q,up,k}(K) \).

Proof. If \( d^V_I(K, L) = \infty \), then Equation (11) holds trivially. Otherwise we assume there exists \( \varepsilon \geq 0 \) such that \( K_t \subseteq L_{t+\varepsilon} \) and \( L_t \subseteq K_{t+\varepsilon} \) for all \( t \in \mathbb{R} \). For any \( I = [a, b] \in \text{Int} \), then \( L_{a-\varepsilon} \subseteq K_a \subseteq K_b \subseteq L_{b+\varepsilon} \) is a simplicial filtration related to the following interleaving diagram:

\[
\begin{array}{c}
K_a \\
\downarrow
\end{array}
\begin{array}{c}
K_b \\
\downarrow
\end{array}
\begin{array}{c}
L_{a-\varepsilon} \\
\leftarrow
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow
\end{array}
\begin{array}{c}
L_{b+\varepsilon}
\end{array}
\]

By Lemma 5.4, \( \lambda^{L_{a-\varepsilon}, L_{b+\varepsilon}}_{q,up,k} \geq \lambda^{K_a, K_b}_{q,up,k} \). This implies that \( \lambda^L_{q,up,k}(I^c) \geq \lambda^K_{q,up,k}(I) \) for all \( I \in \text{Int} \). Similarly, \( \lambda^K_{q,up,k}(I^c) \geq \lambda^L_{q,up,k}(I) \) for all \( I \in \text{Int} \). Therefore, \( d_I(\lambda^K_{q,up,k}, \lambda^L_{q,up,k}) \leq \varepsilon \) and thus \( d_I(\lambda^K_{q,up,k}, \lambda^L_{q,up,k}) \leq d^V_I(K, L) \). \( \square \)

6 Concluding remarks

Currently, the persistent Laplacian is formulated only for inclusion maps. It seems interesting to extend it to the setting of simplicial maps which can be useful for e.g. graph sparsification. Another natural question is that of developing a notion of persistent Laplacian for pairs of Riemannian manifolds. In [34], it is suggested that persistent eigenvalues could be useful for shape recognition purposes. This also motivates the question of finding interpretations of their meaning, especially beyond the zero-eigenvalue. Finally, it would be interesting to elucidate more general stability properties of invariants associated to the persistent Laplacian.

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A  Computation of matrix representations of up and down Laplacians

When $K$ is a graph, the matrix representation $\Delta^K_0$ of the graph Laplacian $\Delta^K_0$ can be computed as follows [7]:

$$\Delta^K_0 = D^K_0 - A^K_0,$$

where $D^K_0$ is the diagonal degree matrix and $A^K_0$ is the adjacency matrix. An analogous formula also holds for higher dimensional up and down Laplacians, which we review next.

Let $K$ be a simplicial complex. Note that given input simplicial complex $K$, we give each simplex an arbitrary but fixed orientation. We also assume that the dimension $q$ follows [7]:

$$K_1 = \bar{K}_q = \{\emptyset\}$$

where $K_1$ is the diagonal degree matrix and $\bar{K}_q$ is the zero matrix.

Let $K_{q+1}$ and $K_{q-1}$ be orderings of bases of $C^K_{q+1}$, $C^K_q$ and $C^K_{q-1}$, respectively.

Let $D^K_{q,up}$ and $D^K_{q,down}$ be two $n^K_q \times n^K_q$ diagonal degree matrices such that for each $i \in [n^K_q]$

$$D^K_{q,up}(i, i) := |\{\tau \in S^K_{q+1} : \sigma_i \text{ is a face of } \tau\}|,$$

$$D^K_{q,down}(i, i) := |\{\rho \in S^K_{q-1} : \rho_j \text{ is a face of } \sigma_i\}|.$$

We further define two $n^K_q \times n^K_q$ adjacency matrices $A^K_{q,up}$ and $A^K_{q,down}$ as follows: for each $i, j \in [n^K_q]$

$$A^K_{q,up}(i, j) := -[\sigma_i \cup \sigma_j : \sigma_i][\sigma_i \cup \sigma_j : \sigma_j],$$

$$A^K_{q,down}(i, j) := -[\sigma_i : \sigma_i \cap \sigma_j][\sigma_i : \sigma_i \cap \sigma_j].$$

(12)

Here $[\sigma_i \cup \sigma_j : \sigma_i]$ denotes the sign of $[\sigma_i]$ in $\partial^K_{q+1}(\sigma_i \cup \sigma_j)$ if $\sigma_i \cup \sigma_j \in S^K_{q+1}$ and is 0 otherwise. Similarly, $[\sigma_i : \sigma_i \cap \sigma_j]$ denotes the sign of $[\sigma_i \cap \sigma_j]$ in $\partial^K_q(\sigma_i)$ if $\sigma_i \cap \sigma_j \in S^K_{q-1}$ and is 0 otherwise.

Then,

$$\Delta^K_{q,up} = D^K_{q,up} - A^K_{q,up} \text{ and } \Delta^K_{q,down} = D^K_{q,down} - A^K_{q,down}.$$

See [12, Section 3.3] for more details.

Computation of $\Delta^K_{q,up}$ and $\Delta^K_{q,down}$ and complexity analysis. Now, given the boundary matrices $B^K_{q+1}$ and $B^K_q$, we describe how we construct the degree matrices $D^K_{q,up}$, $D^K_{q,down}$ and the adjacency matrices $A^K_{q,up}$, $A^K_{q,down}$, and give the time complexity of our constructions.

1. $D^K_{q,up}$. We start with a $n^K_q \times n^K_q$ zero matrix $D^K_{q,up}$. Next, we scan over each $\tau \in S^K_{q+1}$ and update $D^K_{q,up}$ by adding 1 to $D^K_{q,up}(i, i)$ if $\sigma_i$ is a face of $\tau$ (i.e., $B^K_{q+1}(i, j) \neq 0$). Since each $\tau$ has $q + 2$ faces, it takes $O((q + 2)n^K_{q+1}) = O(n^K_q)$ total time to construct $D^K_{q,up}$.

2. $D^K_{q,down}$. Since each $\sigma_i \in S^K_q$ has exactly $q + 1$ faces, $D^K_{q,down} = (q + 1) \cdot I_{n^K_q}$ where $I_{n^K_q}$ is the $n^K_q$-dim identity matrix. Therefore, it takes time $O(n^K_q)$ to construct $D^K_{q,down}$.

3. $A^K_{q,up}$. First, note that any two $q$-simplices $\sigma_i$ and $\sigma_j$ can only both be faces of at most one $(q + 1)$-simplex. Now for each $(q + 1)$-simplex $\tau_\ell \in S^K_{q+1}$, we need to enumerate any two co-dimension 1 faces $\sigma_i$ and $\sigma_j$ of $\tau_\ell$, and fill in the entry $A^K_{q,up}(i, j)$ via Equation (12) (more precisely,
This inner product $\langle \cdot, \cdot \rangle_{w^K}$. Then, for each pair $\sigma_i, \sigma_j \in S^K_q$, if they have no common face then $A^K_{q,\text{down}}(i,j) = 0$ and if they have a common face $\rho_\ell \in S^K_{q-1}$, then fill in the entry $A^K_{q,\text{down}}(i,j)$ by either ‘1’ or ‘-1’ based on the relative orientations of $\rho_\ell$ w.r.t. $\sigma_i$ and $\sigma_j$ via Equation (13) (more precisely, $A^K_{q,\text{down}}(i,j) = -B^K_q(\ell,i)B^K_q(\ell,j)$). Since we have $O\left((n^K_q)^2\right)$ many pairs of $\sigma_i, \sigma_j \in S^K_q$, the time complexity for obtaining $A^K_{q,\text{down}}$ is $O\left((n^K_q)^2\right)$.

This implies that computing $\Delta^K_{q,\text{up}}$ takes time $O\left(n^K_{q+1}\right)$ and computing $\Delta^K_{q,\text{down}}$ takes time $O\left((n^K_q)^2\right)$.

B Missing details from Section 2

B.1 Weighted (persistent) Laplacians

Given a simplicial complex $K$, a weight function on $K$ is any positive function $w^K : \bigcup_{q=0}^{\dim(K)} S^K_q \to \mathbb{R}_+$ assigning a “weight” to each simplex of $K$. Let $w^K_q := w|_{S^K_q}$. Given any weight function on $K$, define the inner product $\langle \cdot, \cdot \rangle_{w^K}$ on $C_q(K)$ as follows:

$$\langle [\sigma], [\sigma'] \rangle_{w^K} := \delta_{\sigma\sigma'} \cdot (w^K_q(\sigma))^{-1}, \forall \sigma, \sigma' \in S^K_q.$$  

(14)

**Remark B.1.** Consider the dual space of $C^K_q$: the cochain space $C^q(K) := \text{Hom}(C_q(K), \mathbb{R})$. Then, $\langle \cdot, \cdot \rangle_{w^K}$ on $C_q(K)$ induces an inner product $\langle \cdot, \cdot \rangle_{\hat{w}^K}$ on $C^q(K)$ such that for any $f, g \in C^q(K)$

$$\langle f, g \rangle_{\hat{w}^K} = \sum_{\sigma \in S^K_q} w^K_q(\sigma)f([\sigma])g([\sigma]).$$

This inner product $\langle \cdot, \cdot \rangle_{\hat{w}^K}$ on $C^q(K)$ coincides with the one defined in [16]. This explains why we take the reciprocal of the weight function in Equation (14) to define the inner product $\langle \cdot, \cdot \rangle_{w^K}$ on $C_q(K)$.

**Weighted Laplacians.** The $q$-th weighted Laplacian on $C^K_q$ is defined as follows:

$$\Delta^K_q := \partial^K_{q,\text{up}} \circ (\partial^K_{q+1})^* + (\partial^K_q)^* \circ \partial^K_q,$$  

(15)

By Lemma 3.2, we have the following matrix representation of $\Delta^K_q$:

$$\Delta^K_q = B^K_{q+1}W^K_{q+1} \left(B^K_{q+1}\right)^T \left(W^K_q\right)^{-1} + W^K_q \left(B^K_q\right)^T \left(W^K_{q-1}\right)^{-1} B^K_q,$$

where $W^K_q$ denotes the diagonal weight matrix representation of $w^K_q$ corresponding to a given choice of basis of $C^K_q$.
Remark B.2. If \( w^K_q \equiv 1 \), then \( \Delta^K_q = B^K_q + (B^K_q)^T (W^K_{q-1})^{-1} B^K_q \) is a symmetric positive semi-definite matrix.

The weighted Laplacian can also be formulated via the degree and the adjacency matrices as we did in Appendix A. Let \( S^K_q = \{[\rho_i]_{i=1}^{n^K_q} \} \) and \( \bar{S}^K_q = \{[\sigma_i]_{i=1}^{n^K_q} \} \) be orderings of bases of \( C^K_{q-1}, C^K_q \) and \( C^K_{q+1} \), respectively. Let \( D^K_{q,\text{up}} \) and \( D^K_{q,\text{down}} \) be two \( n^K_q \times n^K_q \) diagonal degree matrices such that for any \( i \in [n^K_q] \)

\[
D^K_{q,\text{up}}(i, i) := \sum_{\tau_j \in \bar{S}^K_{q+1} : \tau_j \text{ a face of } \sigma_i} \frac{w^K_{q+1}(\tau_j)}{w^K_q(\sigma_i)},
\]

\[
D^K_{q,\text{down}}(i, i) := \sum_{\rho_j \in S^K_{q-1} : \rho_j \text{ a face of } \sigma_i} \frac{w^K_q(\sigma_i)}{w^K_{q-1}(\rho_j)}.
\]

We further define two \( n^K_q \times n^K_q \) adjacency matrices \( A^K_{q,\text{up}} \) and \( A^K_{q,\text{down}} \) as follows: for any \( i, j \in [n^K_q] \)

\[
A^K_{q,\text{up}}(i, j) := -\frac{w^K_{q+1}(\sigma_i \cup \sigma_j)}{w^K_q(\sigma_j)} \cdot \frac{[\sigma_i \cup \sigma_j : \sigma_i] \cdot [\sigma_i \cup \sigma_j : \sigma_j]}{[\sigma_i \cap \sigma_j : \sigma_i \cap \sigma_j]},
\]

\[
A^K_{q,\text{down}}(i, j) := -\frac{w^K_q(\sigma_i)}{w^K_{q-1}(\sigma_i \cap \sigma_j)} \cdot \frac{[\sigma_i : \sigma_i \cap \sigma_j] \cdot [\sigma_j : \sigma_i \cap \sigma_j]}{[\sigma_i : \sigma_i \cap \sigma_j]}.
\]

Then, \( \Delta^K_{q,\text{up}} = D^K_{q,\text{up}} - A^K_{q,\text{up}} \) and \( \Delta^K_{q,\text{down}} = D^K_{q,\text{down}} - A^K_{q,\text{down}} \).

Remark B.3 (Connection with weighted graph Laplacians). When \( K = (V_K, E_K) \) is a weighted graph and \( w^K_0 \equiv 1 \), then

\[
\Delta^K_0 = \Delta^K_{0,\text{up}} = B^K_1 W^K_1 (B^K_1)^T = D^K_{0,\text{up}} - A^K_{0,\text{up}}.
\]

Here, for any \( i, j \in [n^K_0] \), we have the following: \( A^K_{0,\text{up}}(i, j) = \delta_{\{v_i, v_j\} \in E_K} \cdot w^K_1(\{v_i, v_j\}) \) and \( D^K_{0,\text{up}}(i, i) = \sum_{\{v_i, v_j\} \in E_K} w^K_1(\{v_i, v_j\}) \). Then, \( A^K_{0,\text{up}} \) is the usual adjacency matrix and \( D^K_{0,\text{up}} \) is the usual diagonal degree matrix of a weighted graph. So our definition of \( \Delta^K_0 \) agrees with the usual definition of weighted graph Laplacians [7].

Weighted persistent Laplacians. A simplicial pair \((K, w^K) \hookrightarrow (L, w^L)\) is called a weighted simplicial pair if \( w^K = w^L|_{\cup_q S^K_q} \). Then, we define the \( q \)-th weighted persistent Laplacian \( \Delta^K_{q,\text{up}} : C^K_q \rightarrow C^K_q \) over a weighted simplicial pair \((K, w^K) \hookrightarrow (L, w^L)\) by the following same formula as Equation (3):

\[
\Delta^K_{q,\text{up}} := \delta^K_{q+1} (\delta^K_{q+1})^* + (\delta^K_q)^* \circ \delta^K_q,
\]

where the adjoint operators correspond to the inner products \( \langle \cdot, \cdot \rangle_{w^K_q} \) for \( q \in \mathbb{N} \). By Lemma 3.2, we have the following matrix representation of \( \Delta^K_{q,\text{up}} \) given a basis of \( C^K_{q+1} \).
Theorem B.4. Assume that \( n_{q+1}^{L,K} = \dim (C_{q+1}^{L,K}) > 0 \). Choose any basis of \( C_{q+1}^{L,K} \subseteq C_{q+1}^{L} \) represented by a column matrix \( Z \in \mathbb{R}^{n_{q+1}^{L,K} \times n_{q+1}^{L}} \). Let \( B_q^K \) and \( B_{q+1}^{L,K} \) be matrix representations of boundary maps \( \partial_q^K \) and \( \partial_{q+1}^{L,K} \), respectively. Then, the matrix representation \( \Delta_q^{K,L} \) of \( \Delta_q^{K,L} \) is expressed as follows:

\[
\Delta_q^{K,L} = B_{q+1}^{L,K} \left( Z^T (W_{q+1}^L)^{-1} Z \right)^{-1} \left( B_{q+1}^{L,K} \right)^T (W_q^K)^{-1} + W_q^K (B_q^K)^T (W_q^{K-1})^{-1} B_q^K. \tag{17}
\]

Moreover, \( \Delta_q^{K,L} \) is invariant under the choice of basis for \( C_{q+1}^{L,K} \).

Analogously to Theorem 4.6, the weighted persistent Laplacian can also be computed via the Schur complement of some principal submatrix of the Laplacian matrix.

Theorem B.5 (Up-persistent Laplacian as Schur complement). Let \( (K,w^K) \hookrightarrow (L,w^L) \) be a weighted simplicial pair. Assume that \( n_q^K < n_q^L \) and let \( I^K_{L} : = [n_q^K] \setminus [n_q^L] \). Then,

\[
\Delta_{q,up}^{K,L} = \Delta_{q,up}^{L}/\Delta_{q,up}^{I^K_{L},I^K_{L}}.
\]

The proof is essentially the same as the one for Theorem 4.6 and we omit the details here.

B.2 Proofs from Section 2

Proof of Lemma 2.3. This follows directly from the following obvious observations

1. \( C_{q-1}^K = \bigoplus_{i=1}^{m} C_{q-1}^{K_i}, C_q^K = \bigoplus_{i=1}^{m} C_{q}^{K_i} \) and \( C_{q+1}^{L,K} = \bigoplus_{i=1}^{m} C_{q+1}^{L,K_i} \).

2. \( \partial_q^K = \bigoplus_{i=1}^{m} \partial_q^{K_i} \) and \( \partial_{q+1}^{L,K} = \bigoplus_{i=1}^{m} \partial_{q+1}^{L,K_i} \).

Proof of Theorem 2.4. For item 1, let \( c_0^K := \sum_{v \in S_0^K} [v] \in C_0^K \), we prove that \( \Delta_{0}^{K,L} c_0^K = 0 \) and thus \( \lambda_{0,1}^{K,L} = 0 \). Let \( c_0^L := \sum_{v \in S_0^L} [v] \). Then,

\[
c_0^L = \sum_{v \in S_0^L \setminus S_0^K} [v] + \sum_{v \in S_0^K} [v] = \sum_{v \in S_0^L \setminus S_0^K} [v] + c_0^K.
\]

For any \( c_1 \in C_1^{L,K} \), we have the following:

\[
\left\langle \left( \partial_1^{L,K} \right)^* c_0^K, c_1 \right\rangle_{C_1^{L,K}} = \left\langle c_0^K, \partial_1^{L,K} c_1 \right\rangle_{C_0^K} = \left\langle c_0^L, \partial_1^{L,K} c_1 \right\rangle_{C_0^L} - \left\langle \sum_{v \in S_0^L \setminus S_0^K} [v], \partial_1^{L,K} c_1 \right\rangle_{C_0^L}.
\]

Since \( \partial_1^{L,K} c_1 \in C_0^K \), we have that \( \sum_{v \in S_0^L \setminus S_0^K} [v], \partial_1^{L,K} c_1 \right\rangle_{C_0^L} = 0 \). Now, assume that \( c_1 = x_1[e_1] + \ldots + x_{\ell}[e_{\ell}] \) where each \( e_i \in S_1^L \) and \( x_i \in \mathbb{R} \). Since \( \partial_1^{L,K} [e_i] = \partial_1^L [e_i] = [v_i] - [w_i] \) for some \( v_i, w_i \in S_0^L \), we have that \( \left\langle c_0^L, \partial_1^{L,K} [e_i] \right\rangle_{C_0^L} = 0 \) for each \( i = 1, \ldots, \ell \) and thus \( \left\langle c_0^L, \partial_1^{L,K} c_1 \right\rangle_{C_0^L} = 0 \). Therefore,

\[
\left\langle \left( \partial_1^{L,K} \right)^* c_0^K, c_1 \right\rangle_{C_1^{L,K}} = 0, \forall c_1 \in C_1^{L,K}.
\]
and thus $\Delta_0^{K,L}c_0^K = \partial_1^{L,K}\left(\partial_1^{L,K}\right)^*c_1 = 0$.

Now, assume that $L$ is connected. Suppose there exists $0 \neq c_0 \in C_0^K$ such that $\Delta_0^{L,K}c_0 = 0$. Then, $\left(\partial_1^{L,K}\right)^*c_0 = 0$. For any $v, w \in S^K$, since $L$ is connected, there exists a 1-chain $c_1 \in C_1^L$ such that $\partial_1^{L,K}c_1 = [v] - [w]$ (for example, one can take a path in $L$ connecting $v$ and $w$ and let $c_1$ be the corresponding 1-chain). Then, $c_1 \in C_1^{L,K}$ and $\partial_1^{L,K}c_1 = [v] - [w]$. Note that,

$$\langle c_0, [v] - [w]\rangle_{C_0^K} = \left\langle c_0, \partial_1^{L,K}c_1 \right\rangle_{C_0^K} = \left\langle \left(\partial_1^{L,K}\right)^*c_0, c_1 \right\rangle_{C_1^{L,K}} = 0.$$

This implies that $\langle c_0, [v]\rangle_{C_0^K} = \langle c_0, [w]\rangle_{C_0^K}$ and thus there exists $\alpha \in \mathbb{R}$ such that $\langle c_0, [v]\rangle_{C_0^K} = \alpha$ for each $v \in S^K$. Then, $c_0 = \alpha \cdot c_0^K$ and thus the multiplicity of 0 eigenvalue is 1.

For item 2, suppose $K$ intersects with exactly $m$ connected components of $L$, denoted by $L_1, \ldots, L_m$. Then, by Lemma 2.3 we have that $\Delta_0^{K,L} = \bigoplus_{i=1}^m \Delta_0^{K_i,L_i}$. Then, the spectrum of $\Delta_0^{K,L}$ is the multiset union of the spectra of $\Delta_0^{K_i,L_i}$s. By item 1 and item 2 we have that the multiplicity of zero eigenvalue of $\Delta_0^{K,L}$ is then exactly $m$. \hfill \qed

**Proof of Theorem 2.5.** We first prove the following elementary linear algebra fact:

**Claim B.6.** Let $A \in \mathbb{R}^{m \times n}$ and let $B \in \mathbb{R}^{n \times p}$. Suppose $AB = 0$, then we have

$$\ker(A)/\text{im}(B) \cong \ker(A) \cap \ker(B^T) = \ker( BB^T + A^T A),$$

where $\cong$ denotes isomorphism between vector spaces.

**Proof of Claim B.6.** The isomorphism part follows from [22, Theorem 5.3] and the equality follows from [22, Theorem 5.2]. \hfill \qed

Note that the image of $H_q(K)$ under the inclusion map inside $H_q(L)$ is exactly $\ker(\partial_q^K) / \text{im}(\partial_q^{L,K})$. Let $B_q^K$ be the matrix representation of $\partial_q^K$. Choose an ONB of $C_{q+1}^{L,K}$ and let $B_{q+1}^{L,K}$ be the corresponding matrix representation of $\partial_{q+1}^{L,K}$ in this basis. Then, by Theorem 3.1

$$\Delta_q^{K,L} = B_{q+1}^{L,K} \left( B_{q+1}^{L,K} \right)^T + (B_q^K)^T B_q^K.$$

Note that $B_q^K B_{q+1}^{L,K} = 0$. Therefore, by Claim B.6 we have that

$$\beta_q^{K,L} = \dim\left(\ker(B_q^K)/\text{im}(B_{q+1}^{L,K})\right) = \dim(\ker(\Delta_q^{K,L})) = \text{nullity}(\Delta_q^{K,L}).$$

**Proof of Theorem 2.6.** By abuse of the notation, we represent each $c^L \in C_q^L$ by a vector $c^L \in \mathbb{R}^d$. Then, $c^K$ corresponds to the vector $c^K = c^L \left([n_q^K]\right) \in \mathbb{R}^d$. By Theorem 4.6, the matrix representation $\Delta_{q,\text{up}}^{K,L}$ of $\Delta_{q,\text{up}}^{K,L}$ can be computed as follows:

$$\Delta_{q,\text{up}}^{K,L} = \Delta_{q,\text{up}}^{L} \left([n_q^K]\right) - \Delta_{q,\text{up}}^{L} \left([n_q^K], I_K^L\right) \Delta_{q,\text{up}}^{L} \left(I_K^L, I_K^L\right)^\dagger \Delta_{q,\text{up}}^{L} \left(I_K^L, [n_q^K]\right),$$

where $I_K^L = [n_q^K]\setminus [n_q^K]$.

Suppose $\sigma_i \in S_q^K$ is an interior simplex, then the $i$-th row of $\Delta_{q,\text{up}}^{L} \left([n_q^K]\right)$ is 0 (cf. Appendix A). Then,
1. the $i$-th entry of $\Delta_{q,up}^L \left( [n_q^K], [n_q^K] \right)_{c^K}$ exactly coincides with the $i$-th entry of $\Delta_{q,up}^L$;

2. the $i$-th row of $\Delta_{q,up}^L \left( [n_q^K], I_L^K \right) \Delta_{q,up}^L \left( I_L^K, I_L^K \right)^\dagger \Delta_{q,up}^L \left( I_L^K, [n_q^K] \right)$ is 0.

Therefore, the $i$-th entry of $\Delta_{q,up}^{K,L} \left( [n_q^K], [n_q^K] \right)_{c^K}$ agrees with the $i$-th entry of $\Delta_{q,up}^L \left( [n_q^K], [n_q^K] \right)_{c^K}$ and thus

$$\langle \Delta_{q,up}^L c^K, [\sigma_i] \rangle_{C_L^K} = \langle \Delta_{q,up}^{K,L} c^K, [\sigma_i] \rangle_{C_L^K}.$$ 

\[ \square \]

## C Missing details from Section 3

### C.1 An example for Remark 3.4

The following example illustrates that simply considering a certain submatrix of the boundary matrix (in a way suggested in [34]) and then multiplying it by its transpose does not produce the correct up persistent Laplacian, in the sense that persistent Betti number cannot be recovered.

**Example C.1.** Consider the graph $L$ shown in Figure 1 with vertices labeled as in the figure. Let $K$ be the subgraph with vertex set $V_K = \{1, 2\}$. Choose orientations and an order of edges as follows: $S^L_1 = \{[1, 3], [3, 4], [4, 2]\}$. Then, $B_1^L = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$. It is suggested in [34] to use the following matrix $B = B_1^L (\{1, 2\}, :) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ as the matrix representation of $\partial_1^{L,K}$.

Then, $BB^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$. Note that nullity ($I_2$) = 0. However, it is obvious that $\beta_0^{K,L} = 1$. Then, $\beta_0^{K,L} \neq$ nullity ($BB^T$) and thus $BB^T$ cannot be the correct matrix representation of the persistent Laplacian $\Delta_0^{K,L}$.

### C.2 Proofs from Section 3

**Proof of Lemma 3.2.** For any $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ and $y = [y_1, \ldots, y_m]^T \in \mathbb{R}^m$, we have that

$$\langle Fx, y \rangle_{\mathbb{R}^m} = (Fx)^T W_m y = x^T F^T W_m y,$$

and

$$\langle x, F^* y \rangle_{\mathbb{R}^n} = x^T W_n F^* y.$$ 

Then, since $\langle Fx, y \rangle_{\mathbb{R}^m} = \langle x, F^* y \rangle_{\mathbb{R}^n}$ and $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ are arbitrary, we must have that $F^* = W_n^{-1} F^T W_m$.

\[ \square \]
Proof of Lemma 3.3. Consider \( \pi^\perp \circ \partial_{q+1}^L : C_{q+1}^L \rightarrow (C_q^K)^\perp \) where \( \pi^\perp : C_q^L \rightarrow (C_q^K)^\perp \) is the orthogonal projection. Then, \( D_{q+1}^L \) is the matrix representation of \( \pi^\perp \circ \partial_{q+1}^L \) and \( C_{q+1}^{L,K} = \ker (\pi^\perp \circ \partial_{q+1}^L) \). \( R_{q+1}^L = D_{q+1}^L Y \) is the matrix representation of \( \pi^\perp \circ \partial_{q+1}^L \) after a change of basis of \( C_{q+1}^L \).

1. If \( I = \emptyset \), then since \( R_{q+1}^L \) is column reduced, \( R_{q+1}^L \) has full column rank. This implies that \( \pi^\perp \circ \partial_{q+1}^L : C_{q+1}^L \rightarrow (C_q^K)^\perp \) is injective and thus \( C_{q+1}^{L,K} = \ker (\pi^\perp \circ \partial_{q+1}^L) = \{0\} \).

2. If \( I \neq \emptyset \), then the column space of \( Z = Y(:, I) \) coincides with \( \ker (\pi^\perp \circ \partial_{q+1}^L) = C_{q+1}^{L,K} \).

Since \( Z \) has full column rank, we have that the columns of \( Z \) constitute a basis of \( C_{q+1}^{L,K} \).

Proof of Lemma 4.5. If \( D \) is non-singular, then \( \ker(D) = 0 \) and thus \( D \) is obviously proper in \( M \). A proof of the case when \( D \) is non-singular can be found in [11, Theorem 3.1].

Now, we assume that \( D \) is singular. Let \( D = \sum_{i=1}^d \lambda_i \varphi_i \varphi_i^T \) be the eigen-decomposition of \( D \) where \( 0 \leq \lambda_1 \leq \ldots \leq \lambda_d \) are eigenvalues and \( \varphi_i \)s are their corresponding eigenvectors in \( \mathbb{R}^d \). Assume that \( 0 = \lambda_1 = \ldots = \lambda_\ell \) are all the 0 eigenvalues of \( D \). For each \( \varepsilon > 0 \), define \( D_\varepsilon := \varepsilon \cdot \sum_{i=1}^\ell \varphi_i \varphi_i^T + \sum_{i=\ell+1}^d \lambda_i \varphi_i \varphi_i^T \). Then, \( D_\varepsilon \) is positive definite and in particular, non-singular.

Define \( M_\varepsilon := \begin{pmatrix} A & B \\ C & D_\varepsilon \end{pmatrix} \), which is still positive semi-definite. Then, since \( D_\varepsilon \) is non-singular, we have that

\[
\lambda_k(M_\varepsilon) \leq \lambda_k(M_\varepsilon / D_\varepsilon) \leq \lambda_k(A), \quad \forall 1 \leq k \leq n - d.
\]

Note that \( D_\varepsilon^\dagger = D_\varepsilon^{-1} \sum_{i=1}^{d} \frac{1}{\lambda_i} \varphi_i \varphi_i^T \). Then,

\[
D_\varepsilon^{-1} = \frac{1}{\varepsilon} \sum_{i=1}^{\ell} \varphi_i \varphi_i^T + \sum_{i=\ell+1}^{d} \frac{1}{\lambda_i} \varphi_i \varphi_i^T = \frac{1}{\varepsilon} \sum_{i=1}^{\ell} \varphi_i \varphi_i^T + D_\varepsilon^\dagger.
\]

For each \( i = 1, \ldots, \ell \), \( D \varphi_i = 0 \). Since \( D \) is proper, then \( B \varphi_i = 0 \) for each \( i = 1, \ldots, \ell \). Then, \( BD_\varepsilon^{-1} = BD_\varepsilon^\dagger \) and thus \( BD_\varepsilon^{-1} C = BD_\varepsilon^\dagger C \). This implies that \( M/D = M_\varepsilon / D_\varepsilon \). Since \( M_\varepsilon \) converges to \( M \) when \( \varepsilon \to 0 \), by continuity of eigenvalues, we have that

\[
\lambda_k(M) \leq \lambda_k(M / D) \leq \lambda_k(A), \quad \forall 1 \leq k \leq n - d.
\]
D  Missing details from Section 4

D.1  A second algorithm for computing a matrix representation of $\Delta_{q}^{K,L}$

Algorithm 2  Persistent Laplacian: matrix representation via Schur complement

1: Data: $B_{q}^{K}$ and $B_{q+1}^{L}$
2: Result: $\Delta_{q}^{K,L}$
3: Compute $\Delta_{q}^{L}$ from $B_{q+1}^{L}$ and $\Delta_{q}^{K}$ from $B_{q}^{K}$
4: if $n_{q}^{K} = n_{q}^{L}$ then
5: return $\Delta_{q}^{L}$
6: $\Delta_{q}^{K,L} = \Delta_{q}^{L}/\Delta_{q}^{L}(I_{L}^{K}, I_{K}^{L})$
7: return $\Delta_{q}^{K,L}$

D.2  A conjecture about a stronger persistent Cheeger inequality.

For nonempty subsets $A, B \subseteq V_{K}$, we denote by $P_{K}^{s}(A, B)$ the subset of $P_{K}(A, B)$ consisting of all paths $p = (v_{0}, \ldots, v_{n}) \in P_{K}(A, B)$ that satisfy the following additional condition: $v_{i} \notin A \cup B$ for all $0 < i < n$. We have the following obvious properties of $P_{K}^{s}(A, B)$.

Lemma D.1. $P_{K}^{s}(A, B)$ satisfies:

1. $|P_{K}^{s}(A, B)| = |P_{K}(B, A)|$.
2. For any $\emptyset \neq A \subseteq V_{K}$, $P_{K}^{s}(A, V_{K}\setminus A) = E_{K}(A, V_{K}\setminus A)$.

We can then define a new persistent Cheeger constant for a graph pair $K \hookrightarrow L$ via

$$\lambda_{0,2}^{K,L} = 2 \sum_{i=1}^{m} \frac{1}{l_{i}} \leq 2m = h_{K,L}^{L}.$$  

Indeed, by abuse of notation, we denote by $p_{1}, \ldots, p_{m}$ the 1-chains in $C_{1}^{L,K}$ corresponding to the above mentioned paths. More precisely, if $p_{1} = (v_{0} = v, v_{1}, \ldots, v_{n} = w)$, then $p_{1} := [v_{0}, v_{1}] + \ldots + [v_{n-1}, v_{n}] \in C_{1}^{L,K}$. Then, it is easy to see that $p_{1}, \ldots, p_{m}$ constitute a basis $Z$ of $C_{1}^{L,K}$. Moreover, $\partial p_{i} = -[v] + [w]$ for each $i = 1, \ldots, m$. Then, we have that

$$Z^{T}Z = \begin{pmatrix} l_{1} & 0 & \cdots & 0 \\ 0 & l_{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l_{m} \end{pmatrix} \text{ and } B_{q}^{L,K} = \begin{pmatrix} -1 & -1 & \cdots & -1 \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$
Figure 2: **Illustration of Example D.2.** We illustrate in the figure a simple graph consisting of 3 non-intersecting paths connecting vertices $v$ and $w$.

Therefore,
\[
\Delta^{K,L}_0 = B^{L,K}_1 \left( Z^T Z \right)^{-1} B^{L,K}_1 = \left( \sum_{i=1}^m \frac{1}{l_i} - \sum_{i=1}^m \frac{1}{l_i} \right)
\]

Then, $\lambda^{K,L}_{0,2} = 2 \sum_{i=1}^m \frac{1}{l_i}$. As for the persistent Cheeger constant, it is easy to see that $P_L(v,w) = \{ p_i \}_{i=1}^m$ and thus
\[
\lambda^{K,L}_{0,2} = 2 \sum_{i=1}^m \frac{1}{l_i} \leq 2m = h^{K,L}_{K}. \]

The quantity $\sum_{i=1}^m \frac{1}{l_i}$ inspires us to define a new version of the persistent Cheeger constant as follows. Let $K = (V_K, E_K) \hookrightarrow L = (V_2, E_2)$ be a graph pair. We define the *strong persistent Cheeger constant* $h^{K,L}_s$ by
\[
h^{K,L}_s := \min_{\emptyset \neq A \subseteq V_K} \frac{|V_K||P^{+}_L(A, V_K \setminus A)|}{|A||V_K \setminus A|}, \tag{18}
\]
where $|P^{+}_L(A, B)| := \sum_{p \in P^{+}_L(A, B)} \frac{1}{\text{length}(p)}$. It is obvious that $h^{K,L}_s \leq h^{K,L}_{K}$ and we conjecture the following strengthening of Theorem 4.9:

**Conjecture D.3.** Let $K \hookrightarrow L$ be a graph pair, then $\lambda^{K,L}_{0,2} \leq h^{K,L}_s$.

### D.3 Persistent Laplacians, Kron reduction and effective resistances for simplicial networks

Let $L = (V_L, E_L)$ be a connected graph with a weight function $w^L$ such that $w^L_0 \equiv 1$, i.e., $w^L$ assigns constant weight 1 on the vertex set $S^L_0 = V_L$. Let $V_K \subseteq V_L$ be a proper subset such that $|V_K| \geq 2$ and let $K = (V_K, E_K)$ be a subgraph of $L$ with vertex set $V_K$. Consider an ordering $V_L = \{ v_1, \ldots, v_{n^L} \}$ such that $V_K = \{ v_1, \ldots, v_{n^K} \}$. Then, the Schur complement $\Delta^L_0 / \Delta^L_0 (I^L_K, I^L_K)$ (and thus the persistent Laplacian $\Delta^L_0$) is called the graph Kron-reduced matrix in [8], where $I^L_K = [n^L_K] \setminus [n^L_K]$.

**Proposition D.4** ([8, Lemma 2.1 (2)]). $\Delta^L_0 / \Delta^L_0 (I^L_K, I^L_K)$ is again a weighted Laplacian of some weighted graph with vertex set $V_K$. 

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We call this weighted graph the Kron reduction of graph $L$; see [8] for more details.

Physically speaking, if we regard $L$ as an electric network where each edge $e \in E_L$ is assigned a conductance $w_1^L(e)$, then the Kron reduction gives rise to a lower dimensional electrically-equivalent network to $L$, i.e., effective resistances between nodes in the Kron reduction of $L$ are the same as effective resistances between corresponding nodes in $L$ (cf. Theorem 4.12 or [8, Theorem 3.8]).

We want to generalize the notion of Kron reduction of graphs to the case of simplicial networks: for any positive $q_0 \in \mathbb{N}$, a $q_0$-dim simplicial network is a $q_0$-dim simplicial complex $K$ with a weight function $w^K_q$ such that $w^K_q \equiv 1$ for all $q \neq q_0$. As our first step, we generalize the notion of graph Kron-reduced matrices to the setting of simplicial networks via the up persistent Laplacians.

Let $(K, w^K) \leftrightarrow (L, w^L)$ be a weighted simplicial pair. Suppose that $L$ is a $q_0$-dim simplicial network and $n^K_{q_0} \geq q_0 + 1$. We then call the up persistent Laplacian $\Delta^{K,L}_{q_0-1,\text{up}}$ the simplicial Kron-reduced matrix. Note that when $q_0 = 1$, $\Delta^{K,L}_{q_0-1,\text{up}} = \Delta^{K,L}_{1,\text{up}}$ reduces to the graph Kron-reduced matrix. It is currently unknown to us whether there is a result analogous to Proposition D.4 for simplicial networks, namely, whether there exists a well-defined simplicial network with $\Delta^{K,L}_{q_0-1,\text{up}}$ being its $(q_0 - 1)$-th up-Laplacian. We leave this for future work.

**Physical interpretation of simplicial Kron-reduced matrices.** For each $q \in \mathbb{N}$ we assume an ordering $\tilde{S}_q^L = \{[\sigma_i]\}_{i=1}^{n^K_q}$ on $\bar{S}_q^L$ such that $\tilde{S}^K_q = \{[\sigma_i]\}_{i=1}^{n^K_{q_0}}$. We view $w^K_{q_0}(\sigma)$ as the electric conductance on a given $q_0$-simplex $\sigma \in S_{q_0}^L$. Then, we define the generalized current-balance equation for a $q_0$-dim simplicial network $L$ as follows:

$$J = \Delta^{L}_{q_0-1,\text{up}} U,$$

where $J, U \in \mathbb{R}^{n_{q_0}-1}$ are the vector representations of chains in $C_{q_0-1}^L$ reflecting current influxes and voltage potentials at $(q_0 - 1)$-simplices, respectively. Let $J_K := J ([n^K_{q_0-1}])$, $J_{L\setminus K} := J (I_L^K)$ and let $U_K := U ([n^K_{q_0-1}])$, $U_{L\setminus K} := U (I_L^K)$, where $I_L^K = [n^K_{q_0-1}] [n^K_{q_0-1}]$. Then,

$$J_K - \Delta^{L}_{q_0-1,\text{up}} ([n^K_{q_0-1}], I_L^K) (\Delta^{L}_{q_0-1,\text{up}} (I_L^K, I_L^K)) \dagger J_{L\setminus K} = \Delta^{K,L}_{q_0-1,\text{up}} U_K. \quad (20)$$

In particular, if we regard $S_{q_0-1}^L \setminus S_{q_0-1}^K$ (indexed by $I_L^K$) as “interior nodes” of the simplicial network $L$, we let $J_{L\setminus K} = 0$ and then the current influxes at “boundary nodes” in $S_{q_0-1}^K$ are completely determined by voltage potentials on $S_{q_0-1}^K$ and the simplicial Kron-reduced matrix:

$$J_K = \Delta^{K,L}_{q_0-1,\text{up}} U_K. \quad (21)$$

**Proof of Equation (20).** For notationally simplicity, we use abbreviations $\Delta := \Delta^{L}_{q_0-1,\text{up}}$, $\Delta_{KK} := \Delta ([n^K_{q_0-1}], [n^K_{q_0-1}])$, $\Delta_{KL} := \Delta ([n^K_{q_0-1}], I_L^K)$, $\Delta_{LK} := \Delta (I_L^K, [n^K_{q_0-1}])$ and $\Delta_{LL} := \Delta (I_L^K, I_L^K)$. Then,

$$
\begin{pmatrix}
J_K \\
J_{L\setminus K}
\end{pmatrix}
= 
\begin{pmatrix}
\Delta_{KK} & \Delta_{KL} \\
\Delta_{LK} & \Delta_{LL}
\end{pmatrix}
\begin{pmatrix}
U_K \\
U_{L\setminus K}
\end{pmatrix}.
$$

Therefore, we have that

$$J_K = \Delta_{KK} U_K + \Delta_{KL} U_{L\setminus K} \quad \text{and} \quad J_{L\setminus K} = \Delta_{LK} U_K + \Delta_{LL} U_{L\setminus K}. \quad (22)$$

(23)
Left multiply Equation (23) by \( \Delta_{KL} \Delta_{LL}^\dagger \) and obtain
\[
\Delta_{KL} \Delta_{LL}^\dagger J_{L\setminus K} = \Delta_{KL} \Delta_{LL}^\dagger \Delta_{LR} U_K + \Delta_{KL} \Delta_{LL}^\dagger \Delta_{LL} U_{L\setminus K}.
\]
By Example 4.2 we have that \( \ker (\Delta_{LL}) \subseteq \ker (\Delta_{KL}) \). This is equivalent to the condition \( \Delta_{KL} = \Delta_{KL} \Delta_{LL}^\dagger \Delta_{LL} \). Therefore,
\[
\Delta_{KL} \Delta_{LL}^\dagger J_{L\setminus K} = \Delta_{KL} \Delta_{LL}^\dagger \Delta_{LR} U_K + \Delta_{KL} U_{L\setminus K}.
\] (24)
Then, we obtain Equation (20) by Theorem 4.6 and by subtracting Equation (24) from Equation (22).

Next, we will show that the simplicial Kron-reduced matrix (or the persistent Laplacian) preserves effective resistances in a manner similarly to the graph case (Theorem 4.12). To this end, we first give a definition of effective resistances in the setting of simplicial networks.

**Effective resistances for simplicial networks.** We represent explicitly the vertex set \( S^K_0 \) of \( K \) by an ordered finite set \( \{ n^K_0 \} = \{ 1, \ldots, n^K_0 \} \). Given a positive \( q_0 \in \mathbb{N} \), a \( q_0 \)-dim current generator \( \sigma \) is a \( (q_0 + 1) \)-point subset of \( n^K_0 \) such that \( \partial^K q_0 \sigma \in \text{im}(\partial^K) \) [18]. Here \( \sigma \) may not be a simplex in \( K \), and \( \partial^K q_0 \sigma \) denotes the formal boundary of \( \sigma \) computed via Equation (1). Note that any \( q_0 \)-simplex \( \sigma \) in \( K \) is automatically a \( q_0 \)-dim current generator. Let \( \partial^\sigma := \partial^K q_0 \sigma \) and let \( D^K_\sigma \in \mathbb{R}^{n^K_{q_0 - 1}} \) denote the vector representation of \( \partial^\sigma \in C^K_{q_0 - 1} \). Then, we define the effective resistance \( R^K_\sigma \) on a current generator \( \sigma \) by
\[
R^K_\sigma := (D^K_\sigma)^T \left( \Delta^K_{q_0 - 1, \text{up}} \right)^\dagger D^K_\sigma.
\] (25)

**Remark D.5** (Connection with graph effective resistance). When \( q_0 = 1 \), if \( v, w \in S^K_0 \) belong to the same connected component of \( K \), then \([v, w]\) is a 1-dim current generator. It is clear that \( R^K_{v, w} \) defined via Equation (9) coincides with \( R^K_{q_0} \) as defined via Equation (25). Therefore, our definition of effective resistances on current generators is a generalization of effective resistances between vertices on graphs.

**Remark D.6.** In [28], a formula similar to Equation (25) has been used to define effective resistances on \( q_0 \)-simplices of a \( q_0 \)-dim simplicial network. Note that our setting is more general in that we define effective resistances on all \( q_0 \)-dim current generators, not just the set of \( q_0 \)-simplices.

**Lemma D.7.** Let \( \sigma \) be a \( q_0 \)-dim current generator and let \( j_\sigma \in \mathbb{R} \). Let \( U := j_\sigma \left( \Delta^K_{q_0 - 1, \text{up}} \right)^\dagger D^K_\sigma \in \mathbb{R}^{n^K_{q_0 - 1}} \) representing a chain in \( C^K_{q_0 - 1} \). Then, \( j_\sigma D^K_\sigma \) and \( U \) satisfy the current-balance equation (Equation (19)):
\[
 j_\sigma D^K_\sigma = \Delta^K_{q_0 - 1, \text{up}} U.
\] (26)
Moreover, if \( j_\sigma \neq 0 \), then we have
\[
 R^K_\sigma = \frac{(D^K_\sigma)^T U}{j_\sigma}.
\]

**Proof.** We need the following property of current generators.

**Claim D.8.** If \( \sigma \) is a current generator, then \( \partial^\sigma \perp \ker (\Delta^K_{q_0 - 1, \text{up}}) \).

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Proof of Claim D.8. Since \( \sigma \) is a current generator, there exists a chain \( c_\sigma \in C^K_{q_0} \) such that \( \partial_\sigma = \partial^K_{q_0} c_\sigma \). It is obvious that \( \ker (\Delta^K_{q_0-1, \text{up}}) = \ker (\partial^K_{q_0})^* \).

Then, for any \( c \in \ker (\Delta^K_{q_0-1, \text{up}}) \), we have that
\[
\langle c, \partial_\sigma \rangle_{\omega^K_{q_0-1}} = \langle c, \partial^K_{q_0} c_\sigma \rangle_{\omega^K_{q_0-1}} = \langle (\partial^K_{q_0})^* c, c_\sigma \rangle_{\omega^K_{q_0}} = 0.
\]

This implies that \( \partial_\sigma \perp \ker (\Delta^K_{q_0-1, \text{up}}) \).\( \square \)

Then, \( (\Delta^K_{q_0-1, \text{up}})^\dagger \Delta^K_{q_0-1, \text{up}} D^K_{\sigma} = D^K_{\sigma} \). Therefore,
\[
\Delta^K_{q_0-1, \text{up}} \mathbf{U} = j_\sigma \Delta^K_{q_0-1, \text{up}} (\Delta^K_{q_0-1, \text{up}})^\dagger D^K_{\sigma} = j_\sigma (\Delta^K_{q_0-1, \text{up}})^\dagger \Delta^K_{q_0-1, \text{up}} D^K_{\sigma} = j_\sigma D^K_{\sigma}.
\]

If \( j_\sigma \neq 0 \), then
\[
R^K_\sigma = (D^K_{\sigma})^T (\Delta^K_{q_0-1, \text{up}})^\dagger D^K_{\sigma} = \frac{(D^K_{\sigma})^T \mathbf{U}}{j_\sigma}.
\]

\( \square \)

Relation with the notion of effective resistance defined in [18]. Let \( q_0 \) be a positive integer. Given a \( q_0 \)-dim simplicial network \( K \), a version of effective resistance \( \tilde{R}^K_\sigma \) on a current generator \( \sigma \) is defined in [18] differently from Equation (25). In particular, \( \tilde{R}^K_\sigma \) is characterized in [18] via the following formula:

**Theorem D.9** ([18, Theorem 4.2]). Let \( K \) be a \( q_0 \)-dim simplicial network and let \( \sigma \) be a \( q_0 \)-dim current generator. If the \((q_0 - 1)\)-th reduced homology\(^3\) \( \tilde{H}_{q_0-1}(K) = 0 \), then \( \Delta^K_{q_0-1} \) is invertible and
\[
\tilde{R}^K_\sigma = (D^K_{\sigma})^T (\Delta^K_{q_0-1})^{-1} D^K_{\sigma}.
\]

It turns out that \( R^K_\sigma = \tilde{R}^K_\sigma \) when \( \tilde{H}_{q_0-1}(K) = 0 \):

**Theorem D.10.** Let \( K \) be a \( q_0 \)-dim simplicial network and let \( \sigma \) be a \( q_0 \)-dim current generator. Then,
\[
R^K_\sigma = (D^K_{\sigma})^T (\Delta^K_{q_0-1, \text{up}})^\dagger D^K_{\sigma} = (D^K_{\sigma})^T (\Delta^K_{q_0-1})^\dagger D^K_{\sigma}.
\]

In particular, when \( \tilde{H}_{q_0-1}(K) = 0 \), we have that \( R^K_\sigma = \tilde{R}^K_\sigma \).

The proof of this theorem is based on the following result about the relation between the generalized inverse of Laplacians and the generalized inverses of up and down Laplacians.

**Lemma D.11.** Let \( K \) be a weighted simplicial complex with a weight function \( w^K \). If \( w^K = 1 \) for a given positive \( q \in \mathbb{N} \), then
\[
(\Delta^K_q)^\dagger = (\Delta^K_{q, \text{up}})^\dagger + (\Delta^K_{q, \text{down}})^\dagger.
\]

\(^3\)The \( q \)-th reduced homology of a simplicial complex \( K \) is the \( q \)-th homology group of the extended chain complex \( \cdots \to C^K_q \xrightarrow{\partial^K_{q+1}} C^K_{q} \xrightarrow{\partial^K_q} C^K_{q-1} \to \cdots \), where \( \partial^K_q \) is a linear map sending each vertex \( v \in S^K_q \) to \( 1 \in \mathbb{R} \).
**Proof.** By Remark B.2 we have that $\Delta^K_{q,\text{up}}$, $\Delta^K_{q,\text{down}}$ and $\Delta^K_{q,\text{up}\downarrow}$ are symmetric positive semi-definite matrices. Then, consider the eigen-decompositions $\Delta^K_{q,\text{up}} = \sum_i \lambda_i \phi_i \phi_i^T$ and $\Delta^K_{q,\text{down}} = \sum_j \mu_j \psi_j \psi_j^T$ where $\lambda_i, \mu_j \neq 0$. Since $\text{im}(\Delta^K_{q,\text{up}}) \subseteq \ker(\Delta^K_{q,\text{down}})$ and $\text{im}(\Delta^K_{q,\text{down}}) \subseteq \ker(\Delta^K_{q,\text{up}})$ (see [16, Theorem 2.2]), we then have the following eigen-decomposition of $\Delta^K_{q}$:

$$\Delta^K_{q} = \sum_i \lambda_i \phi_i \phi_i^T + \sum_j \mu_j \psi_j \psi_j^T.$$

Therefore,

$$(\Delta^K_{q})^\dagger = \sum_i \lambda_i^{-1} \phi_i \phi_i^T + \sum_j \mu_j^{-1} \psi_j \psi_j^T = (\Delta^K_{q,\text{up}})^\dagger + (\Delta^K_{q,\text{down}})^\dagger.$$

**Proof of Theorem D.10.** When $q_0 = 1$, $\Delta^K_{q_0-1} = \Delta^K_0 = \Delta^K_{0,\text{up}} = \Delta^K_{q_0-1,\text{up}}$. Then, Equation (27) holds trivially.

Now, we assume that $q_0 > 1$. Since $w^{K}_{q_0-1} \equiv 1$, by Lemma D.11, we only need to show that

$$(D^K_{\sigma})^T (\Delta^K_{q_0-1,\text{down}})^\dagger D^K_{\sigma} = 0.$$

Since $\sigma$ is a current generator, there exists a chain $c_\sigma \in C_{q_0}(K)$ such that $\partial \sigma = \partial_{q_0} c_\sigma$. Consider the eigen-decomposition $\Delta^K_{q_0-1,\text{down}} = \sum_j \mu_j \psi_j \psi_j^T$ where $\mu_j \neq 0$. Each $\psi_j \in \mathbb{R}^{n_{q_0-1}}$ represents a chain in $C^{q_0-1}$, which we still denote by $\psi_j$. Then,

$$\psi_j^T D^K_{\sigma} = \langle \mu_j^{-1} \Delta^K_{q_0-1} \psi_j, \partial^K c_\sigma \rangle_{w^K_{q_0-1}} = \langle \mu_j^{-1} (\partial^K_{q_0-1})^* \partial^K_{q_0-1} \psi_j, \partial^K c_\sigma \rangle_{w^K_{q_0-1}} = \langle \mu_j^{-1} \partial^K_{q_0-1} \psi_j, \partial^K_{q_0-1} \partial^K_{q_0-1} c_\sigma \rangle_{w^K_{q_0-1}} = \langle \mu_j^{-1} \psi_j, 0 \rangle_{w^K_{q_0-2}} = 0.$$

Therefore,

$$(D^K_{\sigma})^T (\Delta^K_{q_0-1,\text{down}})^\dagger D^K_{\sigma} = 0.$$

**Relationship between the up persistent Laplacian and the effective resistance.** Let $K \hookrightarrow L$ be a weighted simplicial pair. For simplicity of presentation, we assume for each $q \in \mathbb{N}$ an ordering $S_q^L = \{\{\sigma_i\}\}^n_{i=1}$ on $S_q^L$ such that $S_q^K = \{\{\sigma_i\}\}^n_{i=1}$.

**Theorem D.12.** Let $K \hookrightarrow L$ be a weighted simplicial pair. Let $q_0$ be a positive integer and suppose that $L$ is a $q_0$-dim simplicial network. Let $\sigma$ be a $q_0$-dim current generator in $L$. If $\partial \sigma = \partial_{q_0} \sigma \in C^{q_0-1}$, then

$$R^K_{\sigma} = (D^K_{\sigma})^T (\Delta^K_{q_0-1,\text{up}})^\dagger D^K_{\sigma} = (D^K_{\sigma})^T (\Delta^K_{q_0-1,\text{up}})^\dagger D^K_{\sigma},$$

where $D^K_{\sigma} \in \mathbb{R}^{n_{q_0}}$ and $D^K_{\sigma} \in \mathbb{R}^{n_{q_0}}$ denote the vector representations of $\partial \sigma$ in $C^{q_0-1}$ and $C^{q_0}$, respectively.

**Proof.** Let $U := (\Delta^K_{q_0-1,\text{up}})^\dagger D^K_{\sigma} \in \mathbb{R}^{n_{q_0}}$. Then, by Equation (26) we have that

$$D^K_{\sigma} = \Delta^K_{q_0-1,\text{up}} U.$$
Note that \( D^L_\sigma ([n^K_{q_0-1}]) = D^K_\sigma \) and \( D^L_\sigma ([n^K_{q_0-1}][n^K_{q_0-1}]) = 0 \). Then, by Equation (20), we have that
\[
D^K_\sigma = \Delta^{K,L}_{q_0-1,up} U_K,
\]
where \( U_K = U ([n^K_{q_0-1}]) \).

Therefore, by Lemma D.7, we have that
\[
R^L_\sigma = (D^L_\sigma)^T U = (D^K_\sigma)^T U_K = (D^K_\sigma)^T ((I - \pi)U_K) = (D^K_\sigma)^T (\Delta^{K,L}_{q_0-1,up})^T D^K_\sigma,
\]
where in the third equality we used the fact \( \partial_\sigma \perp \ker (\Delta^{K,L}_{q_0-1,up}) \) whose proof is essentially the same as the one for Claim D.8.

**Remark D.13.** When \( K \leftrightarrow L \) is a (unweighted) graph pair and \( L \) is connected, if we let \( \sigma = [v, w] \) for distinct points \( v, w \in V_K \), then Theorem D.12 reduces to Theorem 4.12 or [8, Theorem 3.8].

### D.4 Proofs from Section 4

**Proof of Lemma 4.7.** We first assume that \( B_2 \) has full column rank. Then, \( \text{rank}(M) \leq \text{rank}(B) = \text{rank}(B_2) \). By Lemma 4.3, we have that
\[
\text{rank}(B_2) \geq \text{rank}(M) \geq \text{rank}(M/M_2) + \text{rank}(M/M_2)
\]
\[
= \text{rank}(B_2B_2^T) + \text{rank}(M/M_2) = \text{rank}(B_2) + \text{rank}(M/M_2).
\]

Therefore, \( \text{rank}(M/M_2) = 0 \) and thus \( M/M_2 = 0 \).

Now, we assume that \( \text{rank}(B_2) < m \). We let \( X := B_1Y_1(Y_1^TY_1)^{-1}(B_1Y_1)^T \). We first assume that \( Y \) is an orthonormal matrix. Then,
\[
X = B_1Y_1Y_1^TB_1^T.
\]

Now, we compute \( M \) in an alternative way:
\[
M = BB^T = BYY^TB^T = \begin{pmatrix} B_1Y_1Y_1^TB_1^T + B_1Y_2Y_2^TB_2^T & B_1Y_2Y_2^TB_2^T \\ B_2Y_2Y_2^TB_2^T & B_2Y_2Y_2^TB_2^T \end{pmatrix}
\]

where we have used that \( B_2Y_1 = 0 \).

Consider
\[
\begin{pmatrix} (B_1Y_2Y_2^TB_1^T + B_1Y_2Y_2^TB_2^T) \\ B_2Y_2Y_2^TB_2^T \end{pmatrix} = \begin{pmatrix} B_1Y_2 \\ B_2Y_2 \end{pmatrix} \begin{pmatrix} B_1Y_2 \\ B_2Y_2 \end{pmatrix}^T.
\]
Since \( B_1Y_2 \) is of full column rank, by Lemma 4.3 again we have that

\[
\text{rank}(B_1Y_2) \geq \text{rank} \begin{pmatrix} B_1Y_2Y_2^T & B_1Y_2Y_2^T B_2^T \\ B_2Y_2Y_2^T B_1^T & B_2Y_2Y_2^T B_2^T \end{pmatrix}
\]

\[
\geq \text{rank} \begin{pmatrix} B_1Y_2Y_2^T B_1^T \end{pmatrix} + \text{rank} \begin{pmatrix} B_1Y_2Y_2^T B_2^T (B_2Y_2Y_2^T B_2^T) \end{pmatrix}
\]

\[
= \text{rank}(B_1Y_2) + \text{rank} \begin{pmatrix} B_1Y_2Y_2^T B_1^T \end{pmatrix}.
\]

This implies that

\[
B_1Y_2Y_2^T B_1^T - B_1Y_2Y_2^T B_2^T (B_2Y_2Y_2^T B_2^T) B_2Y_2Y_2^T B_1^T = 0.
\]

Therefore,

\[
M/M_{22} = M_{11} - M_{12}M_{22}^\dagger M_{21}
\]

\[
= X + B_1Y_2Y_2^T B_1^T - B_1Y_2Y_2^T B_2^T (B_2Y_2Y_2^T B_2^T) B_2Y_2Y_2^T B_1^T = X.
\]

Now, suppose \( Y \) is not orthonormal, then consider the QR factorization of \( Y: Y = QR \) where \( Q \) is an \( m \times m \) orthonormal matrix and \( R \) is an \( m \times m \) non-singular upper-triangular matrix. Suppose \( Y_1 \) has size \( m \times \ell \). Write \( Q \) and \( R \) as block matrices as follows:

\[
Q = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},
\]

where \( Q_1 \) has size \( m \times \ell \) and \( R_{11} \) has size \( \ell \times \ell \). Then, both \( R_{11} \) and \( R_{22} \) are non-singular and \( R_{21} \) is a zero matrix. Then, by \( Y = QR \) we have that \( Y_1 = Q_1R_{11} \) and \( Y_2 = Q_1R_{12} + Q_2R_{22} \). This implies that

\[
B_2Q_1 = B_2Y_1R_{11}^{-1} = 0 \quad \text{and thus} \quad B_2Q_2 = B_2Y_2R_{22}^{-1} \quad \text{has full column rank.}
\]

Moreover,

\[
X = B_1Y_1 (Y_1^TY_1)^{-1} (B_1Y_1)^T = B_1Q_1R_{11} \left( (Q_1R_{11})^T Q_1R_{11} \right)^{-1} (B_1Q_1R_{11})^T
\]

\[
= B_1Q_1R_{11} \left( R_{11}^T Q_1^T Q_1 R_{11} \right)^{-1} R_{11}^T (B_1Q_1)^T = B_1Q_1R_{11} R_{11}^{-1} (Q_1^T Q_1)^{-1} (R_{11}^T)^{-1} R_{11}^T (B_1Q_1)^T
\]

\[
= B_1Q_1 \left( Q_1^T Q_1 \right)^{-1} (B_1Q_1)^T
\]

Then, to prove that \( X = M/M_{22} \), we can first reduce to the case when \( Y \) is orthonormal and thus conclude the proof. \( \square \)

**Proof of Lemma 4.10.** Since \( |V_K| = 2 \), we let \( V_K = \{v, w\} \subseteq V_L \). Then, \( h^{K,L} = 2|P_L(v, w)| \). If \( h^{K,L} = 0 \), then \( L \) is not connected and \( v \) and \( w \) belong to different connected components of \( L \). By Theorem 2.4, \( \lambda_{0,2}^{K,L} = 0 = h^{K,L} \).

Now, we assume that \( |P_L(v, w)| > 0 \). Then, there exists a path \( p = (v_0 = v, v_1, \ldots, v_k = w) \in P_L(v, w) \). By abuse of notation, we also denote by \( p \) the chain \([v_0, v_1] + \ldots + [v_{k-1}, v_k] \in C_1^L \). Then, \( \partial_1^L p = -[v] + [w] \) and thus \( p \in C_1^{L,K} \). Expand \( p \) to obtain a basis \( c_1, \ldots, c_{n_{1,K}} \), and \( p \) of \( C_1^{L,K} \).
Since the sum of coefficients of the boundary of each edge in \( E_L \) is 0, sum of coefficients of \( \partial_{L,K}^i c_i \) is also 0 for each \( i = 1, \ldots, n_{L,K} \). Then, the associate boundary matrix \( B_{L,K}^i \) (for \( \partial_{L,K}^i \)) has the following form

\[
B_{L,K}^i = \begin{pmatrix}
ap_1 & \cdots & a_{n_{L,K}-1} & -1 
\vdots & \ddots & \vdots & \vdots 
a_1 & \cdots & a_{n_{L,K}-1} & -1
\end{pmatrix},
\]

where \( a_i \in \mathbb{R} \). Without loss of generality, we assume that \( a_i = 0 \) for all \( i \) (otherwise we can add a multiple of \( p \) to each \( c_i \) to obtain a new basis such that \( a_i = 0 \).) Apply the Gram-Schmidt process to the basis \( c_1, \ldots, c_{n_{L,K}-1}, p \) to obtain an ONB \( \tilde{c}_1, \ldots, \tilde{c}_{n_{L,K}-1}, \tilde{p} \). Then, the corresponding boundary matrix becomes

\[
\tilde{B}_{L,K}^i = \begin{pmatrix}0 & \cdots & 0 & -b \\
0 & \cdots & 0 & b
\end{pmatrix}.
\]

Here \( b \in \mathbb{R} \) is such that \( \partial_{L,K}^i \tilde{p} = -[v] + b[w] \). Now, we will have a closer look at \( \tilde{p} \). Suppose \( \tilde{p} = x_1[e_1] + x_2[e_2] + \cdots + x_m[e_m] \), where \( e_i \in E_L \) for \( i = 1, \ldots, m \) are distinct edges and \( x_i \in \mathbb{R} \). Assume that the first \( \ell \) edges \( e_1, e_2, \ldots, e_{\ell} \) are all the edges containing \( v \) as one of the endpoints. Then, the coefficient \(-b \) of \([v]\) in \( \partial_{L,K}^i \tilde{p} \) is of the form \( \pm x_1 \pm \cdots \pm x_{\ell} \). Therefore,

\[
|b| = |\pm x_1 \pm \cdots \pm x_{\ell}| \leq \ell \sqrt{\frac{x_1^2 + \cdots + x_{\ell}^2}{\ell}} \leq \ell \sqrt{\frac{|\tilde{p}|^2}{\ell}} = \sqrt{\ell}.
\]

Claim D.14. \( \ell \leq |P_L(v, w)| \).

Proof of Claim D.14. Consider \( e_1 = \{v, v_1\} \) containing \( v \) as one endpoint. If \( v_1 = w \) then \( (v, v_1) \in P_L(v, w) \). If \( v_1 \neq w \), then there exist \( 1 \leq i_2 \leq m \) and \( v_2 \in V_L \) such that \( e_{i_2} = \{v_1, v_2\} \), otherwise the coefficient of \([v]\) in \( \partial_{L,K}^i \tilde{p} \) is not 0. If \( v_2 = w \), then the path \( (v, v_1, v_2) \in P_L(v, w) \); otherwise there exist \( 1 \leq i_3 \leq m \) and \( v_3 \in V_L \) such that \( e_{i_3} = \{v_2, v_3\} \) and we can continue this process. Since \( V_L \) is finite, eventually, there exists a path \( p_1 = (v_0 = v, v_1, \ldots, v_n = w) \in P_L(v, w) \) such that for each \( j = 1, \ldots, n \), there exists some \( 1 \leq i_j \leq m \) such that \( e_{i_j} = \{v_{j-1}, v_j\} \). Similarly, for each \( i = 1, \ldots, \ell \), there exists a path \( p_i = (v_0 = v, v_1, \ldots, v_n = w) \in P_L(v, w) \) such that \( e_i = \{v_0, v_1\} \). Obviously, these \( \ell \) paths are distinct since \( e_i \neq e_j \) for any \( i, j \in [m] \). Therefore, \( \ell \leq |P_L(v, w)| \).

Since we are using an ONB of \( C_{L,K}^i \), the persistent Laplacian has the following form:

\[
\Lambda_{0,L,K}^i = \tilde{B}_{L,K}^i \tilde{B}_{L,K}^i = \begin{pmatrix}b_1^2 & -b_2^2 \\
-b_2^2 & b_2^2
\end{pmatrix}.
\]

Then, we have that \( \lambda_{0,2}^{K,L} = 2b_2^2 \leq 2\ell \leq 2|P_L(v, w)| = h_{K,L} \).

Proof of Corollary 4.14. By Theorem 4.12 (or by [8, Lemma 3.10]), it is easy to show that

\[
\left(\Delta_{0,L,K}^i\right)^+ = \begin{pmatrix}\frac{R_{L,w}^i}{4} - \frac{R_{L,w}^i}{4} \\
-\frac{R_{L,w}^i}{4} & \frac{R_{L,w}^i}{4}
\end{pmatrix}.
\]

Therefore, \( \Delta_{0,L,K}^i = \begin{pmatrix}1 & -1 \\
-\frac{1}{R_{L,w}^i} & \frac{1}{R_{L,w}^i}
\end{pmatrix} \).
Proof of Corollary 4.15. Let \( K \) be the subgraph of \( L \) with vertex set \( \{v, w\} \). By Lemma 4.10 we have that \( \lambda_{0,2}^{K,L} \leq h^{K,L} \). Note that \( h^{K,L} = 2|P_L(v, w)| \). By Corollary 4.14, \( \lambda_{0,2}^{K,L} = \frac{2}{R_{v,w}} \). Therefore, \( R_{v,w} \geq \frac{1}{|P_L(v, w)|} \).

\[ \square \]

### E Missing details from Section 5: an algorithm for \( \Delta_{s,t}^q \)

Consider the simplicial filtration \( K_1 \hookrightarrow K_2 \hookrightarrow \ldots \hookrightarrow K_m \) where each \( K_{t+1} \) contains exactly one more simplex than \( K_t \) for \( t = 1, \ldots, m-1 \). In this section, we show that, for a fixed index \( t \in [m] \), we can compute the matrix representation \( \Delta_{s,t}^q \) of the persistent Laplacian \( \Delta_{q,t}^s \), for all \( 1 \leq s \leq t \), in time \( O \left( t \left( n_q^t \right)^2 + n_{q+1}^t \right) \), where \( n_q^t := n_q^{K_t} \) is the number of \( q \)-simplices in \( K_t \). Note that this is more efficient than applying the Schur complement formula for \( \Delta_{s,t}^q \) (Equation (7)) \( t \) times, which will lead to \( O \left( t \left( n_q^0 \right)^3 + t n_{q+1}^0 \right) \) total time. This result is again achieved via the relation between persistent Laplacian with Schur complement (cf. Theorem 4.6).

Recall from Equation (10) that for any \( 1 \leq s \leq t \), \( \Delta_{s,t}^q = \Delta_{s,t}^{q,up} + \Delta_{s,t}^{q,down} \). As mentioned in Appendix A, \( \Delta_{q,down}^s \) can be constructed in time \( O \left( t \left( n_q^t \right)^2 \right) \). Hence the set of \( \Delta_{q,down}^s \) for all \( 1 \leq s \leq t \) can be computed in \( O \left( t \left( n_q^t \right)^2 \right) \) time.

For simplicity, we assume that \( S_q^s = \{\sigma_1, \ldots, \sigma_s\} \) for each \( s = 1, \ldots, t \), that is, \( K_{s+1} \) contains exactly one more \( q \)-simplex than \( K_s \) for \( s = 1, \ldots, t-1 \). It then follows that \( \Delta_{q,up}^{s,t} = \Delta_{q,up}^t / \Delta_{q,up}^t (I_s^t, I_s^t) \), where \( I_s^t \) is the index set \( I_s^t = \{s + 1, s + 2, \ldots, t\} \). It turns out, following the Quotient Formula (Lemma 4.4), to compute \( \Delta_{q,up}^{s,t} \), one can perform an iterative reduction from \( \Delta_{q,up}^t \) to \( \Delta_{q,up}^{t-1,t} \), \ldots, \( \Delta_{q,up}^{s+1,t} \), and down to \( \Delta_{q,up}^{s,t} \). More precisely, for any \( \ell \leq t \),

\[
\Delta_{q,up}^{\ell-1,t}(i, j) = \begin{cases} 
\Delta_{q,up}^{\ell,t}(i, j) & \text{if } \Delta_{q,up}^{\ell,t}(\ell, \ell) \neq 0 \\
\Delta_{q,up}^{\ell,t}(i, j) & \text{if } \Delta_{q,up}^{\ell,t}(\ell, \ell) = 0 \end{cases}
\text{for any } i, j \in [\ell - 1].
\tag{28}
\]

Equation (28) reduces to the celebrated Kron reduction formula (see Equation (16) of [8]) when \( K_t \) is a connected graph and \( q = 0 \). In other words, \( \Delta_{q,up}^{\ell-1,t} \) can be computed from \( \Delta_{q,up}^{\ell,t} \) in time linear to the size of the matrix, which is bounded by \( O \left( t \left( n_q^t \right)^2 \right) \). Note that from Appendix A we know computing \( \Delta_{q,up}^t \) takes time \( O \left( t \left( n_q^t \right)^2 \right) \). It then follows that using Equation (28), we can compute \( \Delta_{q,up}^{s,t} \), for all \( 1 \leq s \leq t \) iteratively in \( O \left( t \left( n_q^t \right)^2 + n_{q+1}^t \right) \) total time. We summarize our discussion into the following theorem.

**Theorem E.1.** Let \( K_1 \hookrightarrow K_2 \hookrightarrow \ldots \hookrightarrow K_m \) be a simplicial filtration where each \( K_{t+1} \) contains exactly one more simplex than \( K_t \) for \( t = 1, \ldots, m-1 \). For any fixed \( 1 \leq t \leq m \), we can compute the whole set \( \{\Delta_{s,t}^q\}_{s=1}^t \) of persistent Laplace matrices in \( O \left( t \left( n_q^t \right)^2 + n_{q+1}^t \right) \) time. This also implies that we can compute all \( \Delta_{i,j}^q \), for any \( 1 \leq i \leq j \leq m \), in \( O \left( m^2 \left( n_q^m \right)^2 + m n_{q+1}^m \right) \) total time.