PRESENTATION OF IMMERSED SURFACE-LINKS BY
MARKED GRAPH DIAGRAMS

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Abstract. It is well known that surface-links in 4-space can be presented
by diagrams on the plane of 4-valent spatial graphs with makers on the ver-
tices, called marked graph diagrams. In this paper we extend the method
of presenting surface-links by marked graph diagrams to presenting immersed
surface-links. We also give some moves on marked graph diagrams that pre-
serve the ambient isotopy classes of their presenting immersed surface-links.

1. Introduction

A surface-link, or an embedded surface-link, is a closed surface embedded in
Euclidean 4-space \( \mathbb{R}^4 \). An immersed surface-link is a closed surface immersed in \( \mathbb{R}^4 \)
such that the multiple points are transverse double points. It is well known that
surface-links can be presented by diagrams on the plane of 4-valent spatial graphs
with makers on the vertices, called marked graph diagrams (cf. [1, 3, 6, 7, 8, 9, 10]).

In this paper we extend the method of presenting surface-links by marked graph
diagrams to presenting immersed surface-links. We also give some moves on marked
graph diagrams that preserve the ambient isotopy classes of their presenting im-
mersed surface-links, which are extension of moves given by Yoshikawa [10] for
presentation of embedded surface-links.

2. Marked graph diagrams of immersed surface-links

In this section, we introduce a marked graph presentation of immersed surface-
links. First, we recall quickly the notion of marked graph diagrams and links with
bands from [9][10].

Let \( A \) be the square \( \{(x, y) | -1 \leq x, y \leq 1\} \), \( X \) be the diagonals in \( A \) presented
by \( x^2 = y^2 \), and \( M_h \) (or \( M_v \)) be a thick interval in \( A \) given by \( \{(x, y) | -1/2 \leq x \leq
1/2, -\delta \leq y \leq \delta \} \) (or \( \{(x, y) | -1/2 \leq y \leq 1/2, -\delta \leq x \leq \delta \} \)), where \( \delta \) is a small
positive number.

A marked graph (in \( \mathbb{R}^3 \)) is a spatial graph \( G \) in \( \mathbb{R}^3 \) which satisfies the following:

1. \( G \) is a finite regular graph with 4-valent vertices.
2. Each vertex \( v \) is rigid; that is, there is a neighborhood \( N(v) \) of \( v \) which is
   identified with thickened \( A \) such that \( v \) corresponds to the origin and the
   edges restricted to \( N(v) \) correspond to \( X \).
(3) Each $v$ has a marker, which is a thick interval in $N(v)$ which corresponds to $M_h$ or $M_v$ under the identification in (2).

An orientation of a marked graph $G$ is a choice of an orientation for each edge of $G$ such that around every vertex $v$, two edges incident to $v$ in a diagonal position are oriented toward $v$ and the other two incident edges are oriented outward. For example, see Fig. 1

\[
\begin{array}{c}
\text{Figure 1. An orientation around a marked vertex}
\end{array}
\]

Not every marked graph admits an orientation. A marked graph is called orientable (or non-orientable) if it admits (or does not admit) an orientation. A marked graph depicted in Fig. 2 is non-orientable. An oriented marked graph is a marked graph equipped with an orientation. Two (oriented) marked graphs are said to be equivalent if they are ambient isotopic in $\mathbb{R}^3$ with respect to markers as subsets of $\mathbb{R}^3$ (and the orientations).

\[
\begin{array}{c}
\text{Figure 2. A non-orientable marked graph}
\end{array}
\]

A banded link $\mathcal{BL}$ (or a link with bands) is a pair $(L, B)$ of a link $L$ in $\mathbb{R}^3$ and a set of mutually disjoint bands in $\mathbb{R}^3$ attached to $L$. It is called oriented if $L$ is oriented and all bands are oriented coherently with respect to the orientation of $L$. In this case, the link obtained from $L$ by surgery along the bands inherits an orientation, see Fig. 3. Two (oriented) banded links are equivalent if there is an ambient isotopy of $\mathbb{R}^3$ carrying the (oriented) link and (oriented) bands of one to those of the other.

\[
\begin{array}{c}
\text{Figure 3. Surgery}
\end{array}
\]

For a marked graph $G$, we obtain a banded link $(L, B)$ by replacing a neighborhood of each 4-valent vertex with a band such that the core of the band corresponds to the marker as in Fig. 4 (b). The banded link is called the banded link associated with $G$ and is denoted by $\mathcal{BL}(G)$. Conversely, a marked graph $G$ is recovered from a banded link $\mathcal{BL}$ by shortening and replacing each band to a 4-valent vertex as in Fig. 4 (a). If $G$ is oriented, then $\mathcal{BL}(G)$ is oriented, and vice versa.
For a banded link $BL = (L, B)$, the lower resolution $L_-(BL)$ is $L$ and the upper resolution $L_+(BL)$ is the surgery result. For a marked graph $G$, the lower resolution $L_-(G)$ and the upper resolution $L_+(G)$ are defined to be those of the banded link $BL(G) = (L, B)$ associated with $G$.

We present a marked graph by a diagram on the plane, which we call a marked graph diagram, in a usual way in knot theory.

Let $D$ be a marked graph diagram. We denote by $BL(D)$, $L_-(D)$, and $L_+(D)$ the banded link, the lower resolution and the upper resolution of the marked graph presented by $D$. See Fig. 5.

A link is called H-trivial if it is a split union of trivial knots and Hopf links \[4\]. A trivial link is regarded as an H-trivial link without Hopf links.

**Definition 2.1.** A marked graph diagram $D$ (or a marked graph $G$) is H-admissible if both resolutions $L_-(D)$ and $L_+(D)$ (or $L_-(G)$ and $L_+(G)$) are H-trivial links.

Now we discuss a marked graph presentation of an immersed surface-link.

A marked graph diagram $D$ (or a marked graph $G$) is called admissible if both resolutions $L_-(D)$ and $L_+(D)$ (or $L_-(G)$ and $L_+(G)$) are trivial links. By definition, an admissible marked graph (diagram) is H-admissible.

For a subset $A \subset \mathbb{R}^3$ and an interval $I \subset \mathbb{R}$, let

$$AI = \{(x, t) \in \mathbb{R}^4 | x \in A, t \in I\}.$$
Let $D$ be an H-admissible marked graph diagram, and $\mathcal{B}(D) = (L, \mathcal{B})$ the banded link associated with $D$. Consider a surface $S_{t}^{1}$ in $\mathbb{R}^{3}[-1,1]$ satisfying

$$S_{t}^{1} \cap \mathbb{R}^{3}[t] = \begin{cases} L_{+}(D)[t] & \text{for } 0 < t \leq 1, \\ (L_{-}(D) \cup |\mathcal{B}|)[t] & \text{for } t = 0, \\ L_{-}(D)[t] & \text{for } -1 \leq t < 0, \end{cases}$$

where $|\mathcal{B}|$ denotes the union of the bands belonging to $\mathcal{B}$.

When $D$ is oriented, we assume that the surface $S_{t}^{1}$ is oriented so that the orientation of $L_{+}(D)[1]$ as the boundary of $S_{t}^{1}$ coincides with the orientation of $L_{+}(D)$ induced from $D$.

Let $L$ be an H-trivial link with trivial knot components $O_{i} (i = 1, \ldots, m)$ and Hopf link components $H_{j} (j = 1, \ldots, n)$, where $m \geq 0$ and $n \geq 0$. For an interval $[a, b]$, let $L_{\gamma}[a, b]$ be the union of disks $\Delta_{i} (i = 1, \ldots, m)$ and $n$ pairs of disks $C_{j} (j = 1, \ldots, n)$ in $\mathbb{R}^{3}[a, b]$ such that (1) $\partial \Delta_{i} = O_{i}[a]$ and $\partial C_{j} = H_{j}[a]$, (2) $\Delta_{i}$ has a unique maximal point, (3) each disk of $C_{j}$ has a unique maximal point, and (4) the two disks of $C_{j}$ intersect in a point transversely. We call $\Delta_{i} (i = 1, \ldots, m)$ a cone system with base $O_{i} (i = 1, \ldots, m)$ and $C_{j} (j = 1, \ldots, n)$ a cone system with base $H_{j} (j = 1, \ldots, n)$.

We often assume an additional condition: (5) for each cone $C_{j}$ over $H_{j}$, the intersection point of the two disks of $C_{j}$ in condition (4) is the unique maximal point of each of the disks in condition (3). Similarly, for an H-trivial link $L'$ with trivial knot components $O'_{i} (i = 1, \ldots, m')$ and Hopf link components $H'_{j} (j = 1, \ldots, n')$, where $m' \geq 0$ and $n' \geq 0$, let $L'_{\gamma}[a, b]$ be the disjoint union of a cone system $\Delta'_{i}$ in $\mathbb{R}^{3}[a, b]$ with base $O'_{i}$ in $\mathbb{R}^{3}[a]$ ($i = 1, \ldots, m'$) and a cone system $C'_{j}$ in $\mathbb{R}^{3}[a, b]$ with base $H'_{j}$ in $\mathbb{R}^{3}[a]$ ($i = 1, \ldots, n'$), where each disk in the cone system has a unique minimal point.

Let $D$ be an H-admissible marked graph diagram. Consider the union

$$S(D) = L'_{\gamma}[-2, -1] \cup S_{t}^{1} \cup L_{\gamma}[1, 2],$$

which is an immersed surface-link in $\mathbb{R}^{4}$, where $L$ and $L'$ be upper and lower resolutions of $D$, respectively.

By an argument in [H] [5] it is seen that the ambient isotopy class of the immersed surface-link $S(D)$ is uniquely determined from $D$. We call the immersed surface-link $S(D)$ the immersed surface-link constructed from $D$.

**Theorem 2.2.** Let $\mathcal{L}$ be an immersed surface-link. There is an H-admissible marked graph diagram $D$ such that $\mathcal{L}$ is ambient isotopic to $S(D)$.

In the situation of this theorem, we say that $\mathcal{L}$ is presented by $D$.

**Proof.** The following argument is based on an argument in [5] where embedded and oriented surface-links are discussed (cf. [3]). Let $\mathcal{L}$ be an immersed surface-link. Let $d_{1}, \ldots, d_{n}$ be the double points of $\mathcal{L}$, and let $N(d_{1}), \ldots, N(d_{n})$ be regular neighborhoods of them. Moving $\mathcal{L}$ by an ambient isotopy, we may assume the following conditions:

1. All critical points of $\mathcal{L}$, except the double points, with respect to the projection $\mathbb{R}^{4} = \mathbb{R}^{3} \times \mathbb{R} \to \mathbb{R}$ are elementary critical points, that are maximal points, saddle points and minimal points.
2. $\mathcal{L}$ is in $\mathbb{R}^{3}(-2, 2)$.
3. All double points are in $\mathbb{R}^{3}[1]$.
(4) For each \( i = 1, \ldots, n \) such that \( N(d_i) = N^3(d_i)[1 - \epsilon, 1 + \epsilon] \) for a 3-disk \( N^3(d_i) \), and \( N(d_i) \cap \mathcal{L} \) is the cone of a Hopf link \( H_i = \int N^3(d_i)[1 - \epsilon] \) with the cone point \( d_i \in \mathbb{R}^3[1] \). Here \( \epsilon \) is a sufficiently small positive number. The 3-disks \( N^3(d_1), \ldots, N^3(d_n) \) are mutually disjoint.

Move double points into \( \mathbb{R}^3[3] \) such that the condition (4) is preserved although the 3-disk \( N^3(d_i) \) may change and the time level of \( d_i \) changes from 1 to 3, i.e.,

(1) All critical points of \( \mathcal{L} \), except the double points, with respect to the projection \( \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \) are elementary critical points, that are maximal points, saddle points and minimal points.

(2) \( \mathcal{L} \) is in \( \mathbb{R}^3[(-2, 4)] \).

(3) All double points are in \( \mathbb{R}^3[3] \). All maximal, saddle and minimal points are in \( \mathbb{R}^3[(-2, 2)] \).

(4) For each \( i = 1, \ldots, n \) such that \( N(d_i) = N^3(d_i)[3 - \epsilon, 3 + \epsilon] \) for a 3-disk \( N^3(d_i) \), and \( N(d_i) \cap \mathcal{L} \) is the cone of a Hopf link \( H_i = \int N^3(d_i)[3 - \epsilon] \) with the cone point \( d_i \). The 3-disks \( N^3(d_1), \ldots, N^3(d_n) \) are mutually disjoint.

Let \( p_1, \ldots, p_m \) be the maximal points of \( \mathcal{L} \), \( q_1, \ldots, q_m' \) be the minimal points of \( \mathcal{L} \), and let \( N(p_1), \ldots, N(p_m), N(q_1), \ldots, N(q_m') \) be regular neighborhoods of them. Moving \( \mathcal{L} \) by an ambient isotopy, we may assume the following conditions:

(1) All critical points of \( \mathcal{L} \), except the double points, with respect to the projection \( \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \) are elementary critical points, that are maximal points, saddle points and minimal points.

(2) \( \mathcal{L} \) is in \( \mathbb{R}^3[(-4, 4)] \).

(3) All double points and all maximal points are in \( \mathbb{R}^3[3] \). All minimal points are in \( \mathbb{R}^3[-3] \). All saddle points are in \( \mathbb{R}^3[(-2, 2)] \).

(4) For each \( i = 1, \ldots, n \) such that \( N(d_i) = N^3(d_i)[3 - \epsilon, 3 + \epsilon] \) for a 3-disk \( N^3(d_i) \), and \( N(d_i) \cap \mathcal{L} \) is the cone of a Hopf link \( H_i = \int N^3(d_i)[3 - \epsilon] \) with the cone point \( d_i \). The 3-disks \( N^3(d_1), \ldots, N^3(d_n) \) are mutually disjoint.

(5) For each \( i = 1, \ldots, m \) such that \( N(p_i) = N^3(p_i)[3 - \epsilon, 3 + \epsilon] \) for a 3-disk \( N^3(p_i) \), and \( N(p_i) \cap \mathcal{L} \) is the cone of a trivial knot \( O_i \subset \int N^3(p_i)[3 - \epsilon] \) with the cone point \( p_i \). The 3-disks \( N^3(p_1), \ldots, N^3(p_m) \) are mutually disjoint, and also disjoint from \( N^3(d_1), \ldots, N^3(d_n) \).

(6) For each \( i = 1, \ldots, m' \) such that \( N(q_i) = N^3(q_i) [-3 - \epsilon, -3 + \epsilon] \) for a 3-disk \( N^3(q_i) \), and \( N(q_i) \cap \mathcal{L} \) is the cone of a trivial knot \( O_i' \subset \int N^3(q_i)[-3 - \epsilon] \) with the cone point \( q_i \). The 3-disks \( N^3(q_1), \ldots, N^3(q_{m'}) \) are mutually disjoint.

Finally, applying the argument in [5], we can move all saddle points into the same hyperplane \( \mathbb{R}^3[0] \). Then we see the result.

\[ \square \]

**Remark 2.3.** Theorem 1.4 of [4] states that any immersed and oriented surface-link is ambient isotopic to an immersed surface-link satisfying a certain condition. Applying the argument in [5], we can obtain an immersed surface-link required in Theorem 2.22.

3. Moves on marked graph diagrams

We discuss moves on marked graph diagrams which preserve the ambient isotopy classes of the immersed surface-links presented by the diagrams.

The moves depicted in Figs. 6 and 7 were introduced by Yoshikawa [11] as moves on marked graph diagrams which do not change the ambient isotopy classes of their
presenting surface-links. The moves and their mirror images are called Yoshikawa moves. Furthermore, we call the moves in Fig. 6 (Fig. 7) and their mirror images moves of type I (moves of type II). Moves of type I do not change the ambient isotopy classes of marked graphs in \( \mathbb{R}^3 \), and moves of type II do. Note that Yoshikawa moves preserve \( H \)-admissibility and admissibility.

It is known that two admissible marked graph diagrams present ambient isotopic surface-links if and only if they are related by Yoshikawa moves (cf. [7, 9, 10]).

Let \( D \) be a link diagram of an \( H \)-trivial link \( L \). A crossing point \( p \) of \( D \) is an unlinking crossing point if it is a crossing between two components of the same Hopf link of \( L \) and if the crossing change at \( p \) makes the Hopf link into a trivial link.
Definition 3.1. Let $D$ be an $H$-admissible marked graph diagram and let $D_-$ and $D_+$ be the diagrams of the lower resolution $L_-(D)$ and the upper resolution $L_+(D)$, respectively. A crossing point $p$ of $D$ is a lower singular point (or an upper singular point, resp.) if $p$ is an unlinking crossing point of $D_-$ (or $D_+$, resp.).

We introduce new moves for $H$-admissible marked graph diagrams. They are the moves $\Omega_9$, $\Omega'_9$ and $\Omega_{10}$ in Fig. 8 and their mirror images, which we call moves of type III. Here we assume that the moves of type III are defined only if two diagrams appearing before and after the move are $H$-admissible. For example, for the move $\Omega_9$ (or $\Omega'_9$, resp.) in Fig. 8, we require that the component $l$ in the resolution $L_+(D)$ (or $L_-(D)$, resp.) is trivial and that $p$ is an upper (or lower, resp.) singular point.

![Figure 8. Moves of Type III: $\Omega_9$, $\Omega'_9$ and $\Omega_{10}$](image)

Definition 3.2. The generalized Yoshikawa moves are Yoshikawa moves (moves of type I and II) and moves of type III introduced above. Two marked graph diagrams are stably equivalent if they are related by a finite sequence of generalized Yoshikawa moves.

Theorem 3.3. Let $L$ and $L'$ be immersed surface-links presented by marked graph diagrams $D$ and $D'$, respectively. If $D$ and $D'$ are stably equivalent, then $L$ and $L'$ are ambient isotopic.

Proof. It suffices to show that $L$ and $L'$ are ambient isotopic when $D'$ is obtained from $D$ by a move of $\Omega_9$, $\Omega'_9$ or $\Omega_{10}$. The moves $\Omega_9$ and $\Omega'_9$ correspond to a creation or removal of a saddle point, and the move $\Omega_{10}$ corresponds to a change the level of double point singularity. See Fig. 9 which shows partial pictures of broken surface diagrams in 3-space in the sense of [2]. (In [2], embedded surfaces are discussed. However, broken surface diagrams are considered for immersed surface-links and it is true that if two broken surface diagrams are ambient isotopic in 3-space then the immersed surface-links are ambient isotopic in 4-space.) Since the moves $\Omega_9$, $\Omega'_9$ and $\Omega_{10}$ do not change the ambient isotopy classes of broken surface diagrams in 3-space, we see that $L$ and $L'$ are ambient isotopic.

Let $\Omega_{9}^*$ and $\Omega'_{9}^*$ be the moves depicted in Fig. 10 or their mirror images. They are equivalent to $\Omega_9$ and $\Omega'_9$ modulo Yoshikawa moves (of type I) as shown in Fig. 11.

We conclude the paper by proposing a question.

Question 3.4. Suppose that $D$ and $D'$ are marked graph diagrams presenting ambient isotopic immersed surface-links. Is $D$ stably equivalent to $D'$?
Figure 9. Immersed surface-links presented by $\Omega_9$, $\Omega'_9$, and $\Omega_{10}$

Figure 10. Moves $\Omega_9^*$ and $\Omega'_9^*$

Figure 11. Moves $\Omega_9^*$, $\Omega'_9^*$ are equivalent to $\Omega_9$, $\Omega'_9$.

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