Quantum conditional operator and a criterion for separability

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We analyze the properties of the conditional amplitude operator, the quantum analog of the conditional probability which has been introduced in quant-ph/9512022. The spectrum of the conditional operator characterizing a quantum bipartite system is invariant under local unitary transformations and reflects its inseparability. More specifically, it is shown that the conditional amplitude operator of a separable state cannot have an eigenvalue exceeding 1, which results in a necessary condition for separability. This leads us to consider a related separability criterion based on the positive map $\rho \rightarrow (\text{Tr}\rho) - \rho$, where $\rho$ is an Hermitian operator. Any separable state is mapped by the tensor product of this map and the identity into a non-negative operator, which provides a simple necessary condition for separability. In the special case where one subsystem is a quantum bit, $\Gamma$ reduces to time-reversal, so that the separability condition is equivalent to partial transposition. It is therefore also sufficient for $2 \times 2$ and $2 \times 3$ systems. Finally, a simple connection between this map and complex conjugation in the “magic” basis is displayed.

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I. INTRODUCTION

The state of a quantum bipartite system $AB$ is described as separable (or classically correlated) if it can be obtained by having both parties $A$ and $B$ preparing their subsystem according to some common instructions (see, e.g., [1]). Mathematically, this means that the density operator $\rho$ characterizing the state of the bipartite system can be written as a convex sum of product states, that is

$$\rho = \sum_i w_i \left( \rho_i^{(A)} \otimes \rho_i^{(B)} \right) \quad (1)$$

where the weights $w_i$ satisfy $\sum_i w_i = 1$ and $0 \leq w_i \leq 1$. The $w_i$’s can be viewed as the probability distribution of a classical random variable that is known to both parties $A$ and $B$ and used by them to prepare their subsystem. Namely, if the subsystem $A$ (and $B$) is prepared in state $\rho_i^{(A)}$ (and $\rho_i^{(B)}$) when the classical variable takes on value $i$, the state of the joint system $AB$ is given by Eq. (1). A separable state $\rho$ satisfies several interesting properties. The joint statistics of any pair of local observables $O_A$ and $O_B$ (measured separately on each subsystem) can be described classically, based on an underlying global “hidden” variable. For example, the quantum expectation value of the product $O_A O_B$ is given by

$$\text{Tr}[\rho( O_A \otimes O_B )] = \sum_i w_i \langle a_i \rangle_i \langle b_i \rangle_i \quad (2)$$

where $\langle a_i \rangle = \text{Tr}[\rho_i^{(A)} O_A]$ and $\langle b_i \rangle = \text{Tr}[\rho_i^{(B)} O_B]$. In other words, the joint statistics of $O_A$ and $O_B$ can be understood classically, by assuming that the local statistics of the outcomes can be described separately for each $\rho_i^{(A)}$ and $\rho_i^{(B)}$, and that the correlations originate from a hidden variable $i$ distributed according to $w_i$. Moreover, a separable system always satisfies Bell’s inequalities (the converse is not true), so that the latter represent a necessary condition for separability (see, e.g., [3]). Note that any joint probability distribution can be represented as a convex combination of product distributions, so that classical probabilities are always separable in the above sense.

The decomposition of a separable state $\rho$ into a convex mixture of product states is not unique in general, but the fact that $\rho$ is separable implies that there must exist at least one such decomposition. If no such decomposition can be written, then $\rho$ is termed inseparable, and it can be viewed as quantum correlated. Except for the special case where $\rho$ describes a pure state, the distinction between separable and inseparable states appears to be an extraordinary difficult problem. More precisely, some mixed states can be “weakly” inseparable, in the sense that it is very hard to establish with certainty their inseparability. This is basically due to the difficulty of enumerating explicitly all the possible convex combinations of product states in order to detect that a state is actually inseparable. Still, it is possible to find some conditions that all separable states must satisfy, therefore allowing the detection of inseparability.
when a state violates one such condition. The most common example of such a necessary condition for separability is the satisfaction of Bell’s inequalities. A state that violates Bell’s inequalities is inseparable, while a state satisfying them may be separable or weakly inseparable.\footnote{A map is defined as positive if it maps positive operators into positive operators.}

More recently, a surprisingly simple necessary condition for separability has been discovered by Peres\cite{2}, which has been shown by Horodecki et al.\cite{3} to be strong enough to guarantee separability for bipartite systems of dimension $2 \times 2$ and $2 \times 3$. If the state $\rho$ is separable, then the operator obtained by applying a partial transposition with respect to subsystem $A$ (or $B$) to $\rho$ must be positive, that is

$$\rho^{TA} = (\rho^{T_B})^* \geq 0$$

Thus, this criterion amounts to checking that all the eigenvalues of the partial transposition of $\rho$ are non-negative, which must be so for all separable states. In Hilbert spaces of dimensions $2 \times 2$ and $2 \times 3$, this condition is actually sufficient, that is, it suffices for ruling out all inseparable states. In larger dimensions, however, it is provably not sufficient, in the sense that it does not detect some weakly inseparable states. A general necessary and sufficient condition for separability in arbitrary dimensions has been found by Horodecki et al.\cite{3}, which states that $\rho$ is separable if and only if the tensor product of any positive map $\rho$ (acting on $A$) and the identity (acting on $B$) maps $\rho$ into a positive operator. Although very important in theory, this criterion is hardly more practical than the definition of separability itself since it involves the characterization of the set of all positive maps. It appears to be useful mainly for $2 \times 2$ and $2 \times 3$ bipartite systems, where such a general characterization has been found.

In this paper, we focus on the connection between quantum non-separability and the conditional amplitude operator which has been introduced recently in the context of quantum information theory. In Section I, we start by detailing the mathematical properties of such an operator (support, spectrum, connection with von Neumann entropies, etc.). We then derive a necessary condition for separability, based on the conditional von Neumann entropy and its underlying conditional amplitude operator. Namely, the eigenvalues of the latter operator cannot exceed 1 if the bipartite state is separable, as was conjectured in Refs.\cite{1–7}. Since the conditional von Neumann entropy can be negative only if the conditional amplitude operator admits an eigenvalue larger than 1, a related—and weaker—separability condition is that the conditional entropy is non-negative (see also Refs.\cite{1–7} and \cite{3}). This leads us to consider a positive map $\Gamma: \rho \rightarrow (\text{Tr}_B) - \rho$ which gives rise to a simple necessary condition for separability in arbitrary dimensions. More specifically, it is shown in Section III that any separable state is mapped by the tensor product of $\Gamma$ (acting on one subsystem, $A$) and the identity (acting on the other, $B$) into a non-negative operator. In the case where $\Gamma$ is applied to a two-state system (quantum bit or spin-1/2 particle), this corresponds to the time-reversal operation applied on one system with respect to the other one. Since Peres’ criterion has been shown to be unitarily equivalent to such a “local” time-reversal by Sanpera et al.\cite{9}, our separability criterion is simply equivalent to Peres’ for $2 \times n$ composite systems. Therefore, it also results in a sufficient condition for $2 \times 2$ and $2 \times 3$ systems, according to Ref.\cite{3}. It also has a very simple geometric representation in the Hilbert-Schmidt representation of the bipartite state. Finally, it appears that the map $\Gamma$ is connected to the complex conjugation operation in the “magic” basis introduced by Hill and Wootters.\cite{4}. In Appendix A, we illustrate the separability condition based on $\Gamma$ by applying it to several separable or inseparable states, and compare it to the separability condition based on partial transposition.

**II. Conditional “Amplitude” Operator**

Let us consider a bipartite system $AB$, characterized by a density operator $\rho_{AB}$ in the product Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Each subsystem, $A$ or $B$, is characterized by the reduced density operator $\rho_A = \text{Tr}_B[\rho_{AB}]$ or $\rho_B = \text{Tr}_A[\rho_{AB}]$, respectively.

**Definition 1:** Define the conditional amplitude operator of $A$ conditional on $B$ as

$$\rho_{A|B} \equiv \exp[\log \rho_{AB} - \log(1_A \otimes \rho_B)]$$

$$= \lim_{n \to \infty} \left(\rho_{AB}^{1/n}(1_A \otimes \rho_B)^{-1/n}\right)^n$$

(4)

which is a positive semi-definite Hermitian operator in the joint Hilbert space $\mathcal{H}_{AB}$ defined on the range of $\rho_{AB}$ (see Lemma 1).
The second expression in Eq. (4), based on the Trotter decomposition of \( \rho_{AB} \), explicitly emphasizes that the conditional amplitude operator is the natural quantum analog of the conditional probability, \( p(a|b) = p(a,b)/p(b) \). As \( \rho_{AB} \) and \( (1_A \otimes \rho_B)^{-1} \) do not necessarily commute, the Trotter symmetrization is used here to obtain an Hermitian operator.

**Lemma 1:** \( \text{Ran}(\rho_{AB}) \subseteq \text{Ran}(1_A \otimes \rho_B) \), where \( \text{Ran}(\rho) \) is the range of \( \rho \). Thus, the support of \( \rho_{AB} \) is included in the support of \( 1_A \otimes \rho_B \), and, consequently, the conditional amplitude operator \( \rho_{AB} \) is well-defined on the support of \( \rho_{AB} \).

We must prove that \( \text{Ker}(1_A \otimes \rho_B) \subseteq \text{Ker}(\rho_{AB}) \), where \( \text{Ker}(\rho) \) is the kernel of \( \rho \), i.e., that any eigenvector \( |\psi\rangle \) of \( (1_A \otimes \rho_B) \) with zero eigenvalue is such that \( \rho_{AB}|\psi\rangle = 0 \). First note that any such eigenvector \( |\psi\rangle \) can be written as a linear combination of states \( |\phi\rangle \) where

\[
|\phi\rangle = |a\rangle \otimes |b\rangle
\]

and \( |a\rangle \) is an arbitrary state vector in \( \mathcal{H}_A \) while \( |b\rangle \) is an eigenvector of \( \rho_B \) with zero eigenvalue, i.e., \( \rho_B|b\rangle = 0 \). Let us now consider the positive semi-definite operator \( \tilde{\rho} \equiv (1_A \otimes P_b)\rho_{AB}(1_A \otimes P_b) \), with \( P_b = |b\rangle \langle b| \). It is trivial to check that its partial trace over \( A \) vanishes, that is

\[
\text{Tr}_A[\tilde{\rho}] = P_b \rho_B P_b = 0
\]

This results from the general relation

\[
\text{Tr}_A[(1_A \otimes \lambda_B)\mu_{AB}] = \lambda_B \text{Tr}_A[\mu_{AB}]
\]

where \( \lambda_B \) and \( \mu_{AB} \) are arbitrary operators in \( \mathcal{H}_B \) and \( \mathcal{H}_{AB} \) respectively. Since \( \tilde{\rho} \) is positive semi-definite and traceless, we have \( \tilde{\rho} = 0 \). Thus, in particular, the expectation value of \( \tilde{\rho} \) in the state \( |\phi\rangle \) vanishes,

\[
\langle \phi | \tilde{\rho} | \phi \rangle = \langle \phi | \rho_{AB} | \phi \rangle = 0
\]

which in turn implies that \( \rho_{AB}|\phi\rangle = 0 \) since \( \rho_{AB} \) is positive semi-definite. As this is true for each term \( |\phi\rangle \) in the superposition, we conclude that \( \rho_{AB}|\psi\rangle = 0 \). \( \Box \)

**Remark:** Lemma 1 clearly implies that \( \text{Ker}(1_A \otimes \rho_B) \cap \text{Ran}(\rho_{AB}) = \emptyset \). Thus, the subspace spanned by the eigenvectors with zero eigenvalue of \( (1_A \otimes \rho_B) \) is disjoint from the support of \( \rho_{AB} \), so that the definition of \( \rho_{AB} \) contains no singularities in the support of \( \rho_{AB} \). Of course, there is a classical analog for probability distributions which ensures that \( p(a|b) = p(a,b)/p(b) \) is well defined if \( a,b \) are such that \( p(a,b) \neq 0 \). Indeed, if \( b \) is such that \( p(b) = 0 \), then \( p(a,b) = 0 \), \( \forall a \). This is obvious since \( p(b) = \sum_a p(a,b) \) and \( p(a,b) \geq 0 \).

**Definition 2:** The conditional von Neumann entropy is defined using the joint density operator \( \rho_{AB} \) and the conditional amplitude operator \( \rho_{AB} \) as \(^2\)

\[
S(A|B) = -\text{Tr}[\rho_{AB} \log_2 \rho_{AB}]
\]

in close analogy to the classical definition

\[
H(A|B) = -\sum_{a,b} p(a,b) \log_2 p(a|b)
\]

Thus, \( S(A|B) \) corresponds to the *quantum* entropy of \( A \) conditional on \( B \), and is mathematically well-defined as a consequence of Lemma 1. The trace in Eq. (8) can indeed be restricted to the support of \( \rho_{AB} \), using the fact that the common eigenvectors \( |\psi\rangle \) with zero eigenvalue of both \( \rho_{AB} \) and \( (1_A \otimes \rho_B)^{-1} \) yield a vanishing contribution to the entropy, as \( \lim_{x \to 0} (x \log x) = 0 \). (The same argument is used in classical information theory to discard zero probabilities when calculating Shannon entropies.)

\(^2\)For a linear and Hermitian operator \( \rho \), the range is the subspace of the domain of \( \rho \) that is spanned by the eigenvectors corresponding to non-zero eigenvalues, i.e., it is the support of \( \rho \).
**Theorem 1:** The definitions of $\rho_{A|B}$ and the conditional von Neumann entropy imply that $S(A|B) = S(AB) - S(B)$, as for Shannon entropies.

First, using Eqs. (1) and (3), we have

$$S(A|B) = -\text{Tr}[\rho_{AB} \log_2 \rho_{AB}] + \text{Tr}[\rho_{AB} \log_2 (1_A \otimes \rho_B)]$$

where the first term on the right-hand side is clearly equal to $S(AB)$. In order to calculate the second term on the right-hand side of Eq. (11), we write

$$\text{Tr}[\rho_{AB} \log_2 (1_A \otimes \rho_B)] = \text{Tr}[\rho_{AB} (1_A \otimes \log_2 \rho_B)]$$

$$= \text{Tr}[\rho_{AB}] \log_2 \rho_B$$

$$= \rho_B \log_2 \rho_B$$

where we have made use of Eq. (7). This implies that the second term on the right-hand side of Eq. (11) is

$$\text{Tr}[\rho_B \log_2 \rho_B] = -S(B)$$

resulting in $S(A|B) = S(AB) - S(B)$. □

**Lemma 2:** The spectrum of the conditional amplitude operator $\rho_{A|B}$ is invariant under unitary transformations of the product form $U_A \otimes U_B$ on $\rho_{AB}$.

Let us consider the isomorphism

$$\rho_{AB} \rightarrow \rho'_{AB} = (U_A \otimes U_B)\rho_{AB}(U_A^\dagger \otimes U_B^\dagger)$$

We first calculate the partial trace of the joint density operator over $A$ after this transformation, that is

$$\rho'_B = \text{Tr}_A[\rho'_{AB}]$$

$$= \text{Tr}_A[(U_A \otimes U_B)\rho_{AB}(U_A^\dagger \otimes U_B^\dagger)]$$

$$= \text{Tr}_A[(1_A \otimes U_B)(U_A \otimes 1_B)\rho_{AB}(U_A^\dagger \otimes 1_B)(1_A \otimes U_B^\dagger)]$$

$$= U_B \text{Tr}_A[(U_A \otimes 1_B)\rho_{AB}(U_A^\dagger \otimes 1_B)]U_B^\dagger$$

$$= U_B \rho_B U_B^\dagger$$

where we have used Eq. (3) and the basis invariance of the trace. This implies that the conditional amplitude operator transforms as

$$\rho_{A|B} \rightarrow \rho'_{A|B} = (U_A \otimes U_B)\rho_{A|B}(U_A^\dagger \otimes U_B^\dagger)$$

so that its spectrum is conserved under $U_A \otimes U_B$ on $\rho_{AB}$. Note that the classical analog of an $U_A \otimes U_B$ isomorphism corresponds to permuting the rows and columns of the joint probability distribution $p(a,b)$, so that the classical counterpart of Eq. (14) is straightforward. □

**Remark:** This Lemma suggests that the spectrum of $\rho_{A|B}$ could be related to the separability of the state $\rho_{AB}$, since separability (or inseparability) is conserved under a $U_A \otimes U_B$ isomorphism. This will be examined later on.

**Corollary:** The conditional von Neumann entropy $S(A|B)$ is invariant under a unitary transformation of the product form $U_A \otimes U_B$.

This property results from the definition of $S(A|B)$, Eq. (3), together with Eq. (14), or can be checked trivially from Theorem 1.

**Theorem 2:** The operator $\sigma_{AB} \equiv -\log \rho_{A|B} = \log(1_A \otimes \rho_B) - \log \rho_{AB}$ is positive semi-definite if the quantum bipartite system characterized by $\rho_{AB}$ is separable.

Let us consider a separable bipartite system $\rho_{AB}$, i.e., a convex combination of product states:

$$\rho_{AB} = \sum_i w_i \left( \rho^{(i)}_{A} \otimes \rho^{(i)}_{B} \right) \quad \text{with} \quad \sum_i w_i = 1 \quad \text{and} \quad 0 \leq w_i \leq 1$$

(17)
where $\rho_A^{(i)}$ and $\rho_B^{(i)}$ are states in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. We first define the operator

$$\lambda_{AB} \equiv (1_A \otimes \rho_B) - \rho_{AB} \tag{18}$$

It is easy to check that $\lambda_{AB}$ is positive semi-definite if $\rho_{AB}$ is separable. Indeed, in such a case we have

$$\lambda_{AB} \equiv \sum_i w_i \left( (1_A - \rho_A^{(i)}) \otimes \rho_B^{(i)} \right) \geq 0 \tag{19}$$

since a sum of positive operators is a positive operator. Now, we can use the fact that, if $X$ and $Y$ are two Hermitian operators such that $X \geq Y > 0$, then $\log X \geq \log Y$, as implied by Löwner’s theorem [11]. (Note that the converse is not true.) As a consequence, using $X = 1_A \otimes \rho_B$ and $Y = \rho_{AB}$, we conclude that $\lambda_{AB} \geq 0$ implies $\sigma_{AB} \geq 0$. □

**Corollary 1:** Any separable bipartite state is such that $\rho_{A|B} \leq 1$.

Since we have $\rho_{A|B} = \exp(-\sigma_{AB})$, Theorem 2 shows indeed that no eigenvalue of the conditional amplitude operator exceeds 1 for a separable state, as was conjectured in Ref. [5–7]. This yields a simple necessary (but not sufficient) condition for separability. The classical analog of this property is that $-\log p(a|b) \geq 0$, $\forall a, b$. The latter inequality simply results from the fact that $p(a|b) = p(a,b)/p(b) \leq 1$, $\forall a, b$, as $p(b) = \sum_a p(a,b)$ and $p(a,b) \geq 0$.

**Corollary 2:** The conditional von Neumann entropy $S(A|B)$ is non-negative for a separable bipartite state.

Since we have $S(A|B) = \text{Tr}[\rho_{AB} \sigma_{AB}]$, this simply follows from the fact that $\text{Tr}[XY] \geq 0$ if $X,Y \geq 0$. Thus, the non-negativity of the conditional entropy is another (weaker) necessary condition for separability [5–7]. This has been shown in general for “α-entropies” in Ref. [8]. It can also be related to the non-violation of entropic Bell inequalities [12].

Note that Corollary 2 can also be obtained by using the concavity of $S(A|B)$ in a convex combination of $\rho_{AB}$’s,

$$S(A|B) = S(\rho_{AB}) - S(\rho_B) \geq \sum_i w_i \left( S(\rho_{AB}^{(i)}) - S(\rho_B^{(i)}) \right) \quad \text{if} \quad \rho_{AB} = \sum_i w_i \rho_{AB}^{(i)} \tag{20}$$

a property related to the strong subadditivity of quantum entropies [12]. Using the fact that, for a separable state, each term $i$ gives $S(A|B) = S(A)$ because $\rho_{AB}^{(i)} = \rho_A^{(i)} \otimes \rho_B^{(i)}$ (i.e., $A$ and $B$ are independent), we obtain

$$S(A|B) \geq \sum_i w_i S(\rho_A^{(i)}) \geq 0 \tag{21}$$

Note that a negative conditional von Neumann entropy $S(A|B)$ necessarily implies that an eigenvalue of $\rho_{A|B}$ exceeds 1, but the converse is not true. Thus, weak inseparability (in the sense that $S(A|B) \geq 0$ despite the inseparability of $\rho_{AB}$) may be revealed by the spectrum of $\rho_{A|B}$.

**Theorem 3:** There exist inseparable bipartite states $\rho_{AB}$ such that the operator $\sigma_{AB}$ is positive semi-definite; consequently, $\sigma_{AB} \geq 0$ (or $\rho_{A|B} \leq 1$) is not a sufficient condition for separability.

Let us consider a bipartite system $AB$ characterized by $\rho_{AB}$, which we extend with another system $A'B'$ in the state $\rho_{A'B'}$. The joint system is then characterized by a density operator of the product form

$$\rho_{AA':BB'} = \rho_{AB} \otimes \rho_{A'B'} \tag{22}$$

We first calculate the conditional amplitude operator of the joint system ($AA'$ conditional on $BB'$)

$$\rho_{AA'|BB'} = \exp[\log \rho_{AA':BB'} - \log(1_{AA'} \otimes \rho_{BB'})] \tag{23}$$

where the reduced density operator describing $BB'$ is

$$\rho_{BB'} = \text{Tr}_{AA'}[\rho_{AA':BB'}] = \rho_B \otimes \rho_{B'} \tag{24}$$

\[^{3}\text{Here and below, the notation } X \geq Y \text{ means that } X - Y \text{ is a positive semi-definite operator.}\]
Using the identity $\log(X \otimes Y) = \log X \otimes 1 + 1 \otimes \log Y$ for operators $X, Y > 0$ as well as its exponential, i.e., $\exp X \otimes \exp Y = \exp(X \otimes 1 + 1 \otimes Y)$, we obtain

$$\rho_{A|B} = \exp[\log \rho_{AB} \otimes 1_{A'B'} + 1_A \otimes \log \rho_{A'} - 1_A \otimes \log \rho_{AB} \otimes 1_{A'B'} - 1_{AB} \otimes 1_{A'} \otimes \log \rho_{B'}]$$

$$= \exp[(\log \rho_{AB} - 1_A \otimes \log \rho_B) \otimes 1_{A'B'} + 1_A \otimes (\log \rho_{A'B'} - 1_{A'} \otimes \log \rho_{B'})]$$

$$= \exp[\log \rho_{AB} - 1_A \otimes \log \rho_B] \otimes \exp[\log \rho_{A'B'} - 1_{A'} \otimes \log \rho_{B'}]$$

(25)

Thus, we have

$$\rho_{A|B} = \rho_{A|B} \otimes \rho_{A'|B'}$$

(26)

which parallels the classical relation $p(a|b) = p(a|b')$ if $AB$ and $A'B'$ are independent bipartite systems, that is, if $p(a, a'; b, b') = p(a, b)p(a', b')$. In particular, we have

$$\sigma(\rho_{A|B}) = \sigma(\rho_{A|B}) \otimes \sigma(\rho_{A'|B'})$$

(27)

where $\sigma(\rho)$ stands for the spectrum of $\rho$.

Now, let us assume that $AB$ is an inseparable system with $\sigma_{AB} \not\geq 0$ or $\lambda_{AB} \not\leq 1$. In other words, the operator $\rho_{AB}$ admits an eigenvalue that exceeds 1. Assume also that $A'B'$ corresponds to two independent systems in a product state, that is, $\rho_{A'B'} = \rho_A \otimes \rho_{B'}$. The resulting conditional amplitude operator for $A'B'$ is then $\rho_{A'|B'} = \rho_{A'} \otimes \rho_{B'}$, just like its classical counterpart $p(a|b) = p(a)$ if $p(a, b) = p(a)p(b)$. Obviously, we then have $\rho_{A'|B'} \not\leq 1$, as expected since $A'B'$ is separable. According to Eq. (27), the eigenvalues of $\rho_{A|B}$ are the pairwise products of eigenvalues of $\rho_{A|B}$ with eigenvalues of $\rho_{A'|B'}$. Therefore, it is easy to find a system $A'B'$ with eigenvalues of $\rho_{A'|B'}$ small enough so that the product of any of them with an unclassical ($> 1$) eigenvalue of $\rho_{AB}$ results in eigenvalues of $\rho_{A|B}$ that are all $\leq 1$. The extended system is then characterized by $\sigma_{A|B} \geq 0$ or $\rho_{A'|B'} \not\leq 1$, while it obviously contains an inseparable component $AB$. Such a dilution of inseparability (or entanglement) is always achievable with a system $A'B'$ that is large enough and maximally disordered (i.e., $\rho_{A'B'} \sim \rho_A \otimes \rho_{B'}$). Consequently, the condition that $\sigma_{AB} \geq 0$ or $\rho_{AB} \not\leq 1$ cannot be a sufficient condition for separability. 

Remark 1: Eq. (27) implies that, if $AB$ and $A'B'$ are inseparable systems with $\rho_{AB} \not\leq 1$ and $\rho_{A'|B'} \not\leq 1$, then the inseparability of the joint system is revealed by $\rho_{A|B} \not\leq 1$.

Remark 2: While $\lambda_{AB} \geq 0$ is a sufficient separability condition for $2 \times 2$ and $2 \times 3$ systems (cf. Section III), it cannot be concluded that $\sigma_{AB} \geq 0$ or $\rho_{AB} \not\leq 1$ is also sufficient in these cases, as the converse of Löwner's theorem does not hold. Interestingly, numerical evidence reveals that only very few inseparable states of two qubits with $\rho_{AB} \not\leq 1$ can be found.

III. NECESSARY CONDITION FOR SEPARABILITY OF MIXED STATES

A. Bipartite system of arbitrary dimension

As we saw in the previous section, Theorem 2 results in a simple necessary condition for the separability of mixed states based on Eq. (19), which does not require the calculation of the conditional amplitude operator (although it is related to it).

Definition 3: Define a linear map $\Lambda$ which maps Hermitian operators on $\mathcal{H}_{AB}$ into Hermitian operators on $\mathcal{H}_{AB}$:

$$\Lambda : \rho_{AB} \rightarrow \lambda_{AB} = I_A \otimes \rho_B - \rho_{AB} \quad \text{with} \quad \rho_B = Tr_A[\rho_{AB}]$$

(28)

It commutes with a unitary transformation acting independently on $A$ and $B$. Indeed, according to Eq. (13), if $\rho_{AB}$ undergoes a unitary transformation of the product form $U_A \otimes U_B$, then

$$\lambda_{AB} \rightarrow \lambda_{AB}' = (U_A \otimes U_B)\lambda_{AB}(U_A \otimes U_B)^\dagger$$

(29)

e.g., $\lambda_{AB}$ transforms just like $\rho_{AB}$. Therefore, the spectrum of $\lambda_{AB}$ is invariant under a $U_A \otimes U_B$ isomorphism on $\rho_{AB}$, as expected.

Theorem 4: A necessary condition for the separability of the state $\rho_{AB}$ of a bipartite system $AB$ is that it is mapped by $\Lambda$ into a non-negative operator, i.e., $\Lambda \rho_{AB} \geq 0$. 

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The proof of this theorem is contained in the proof of Theorem 2 where we used the fact that the map $\Lambda$ reveals non-separability: if $\lambda_{AB} \not\geq 0$, then $\rho_{AB}$ is inseparable. Conversely, any separable state $\rho_{AB}$ is mapped into a positive semi-definite operator $\lambda_{AB}$. Moreover, it is easy to see that the map $\Lambda$ conserves separability since it is linear and maps product states into product states: if $\rho_{AB}$ is separable, then $\lambda_{AB} \geq 0$ is also separable. Let us now calculate the partial traces of the operator $\lambda_{AB}$:

$$\lambda_A = \text{Tr}_B[\lambda_{AB}] = 1_A - \rho_A$$

$$\lambda_B = \text{Tr}_A[\lambda_{AB}] = (d_A - 1)\rho_B$$

(30) (31)

where $d_A$ is the dimension of $\mathcal{H}_A$. This shows that the map $\Lambda$ does not preserve the trace in general. Indeed, the total trace is scaled by an integer factor under $\Lambda$, that is, $\text{Tr}[\lambda_{AB}] = (d_A - 1)\text{Tr}[\rho_{AB}]$. Thus, $\Lambda$ is trace-preserving only in the special case where $A$ is a 2-state system (i.e., $d_A = 2$). It is also interesting to note that the map $\Lambda$ is always reversible, the inverse map being given by

$$\Lambda^{-1} : \lambda_{AB} \rightarrow (d_A - 1)^{-1}(1_A \otimes \lambda_B) - \lambda_{AB} = \rho_{AB}$$

where $\lambda_B$ is defined as above. Notice that the map $\Lambda$ is equal to its inverse $\Lambda^{-1}$ only in the case where $d_A = 2$. This implies that, if $\lambda_{AB}$ is separable, then $\Lambda^{-1} : \lambda_{AB} \rightarrow \rho_{AB} \geq 0$. (The fact that the inverse map reveals inseparability is true in this case only.)

The separability condition based on Theorem 4 is illustrated in Appendix A, where we consider several separable and inseparable states. It appears that $\lambda_{AB} \geq 0$ results in the same condition as Peres’ in the case of two quantum bits, in which case it is sufficient (see Theorem 9). For larger dimensions, it is only necessary.

**Remark 1:** More generally, following the approach of Horodecki et al. [3], the map $\Lambda$ can be written as the tensor product of a positive linear map $\Gamma$ and the identity, that is

$$\Lambda = \Gamma \otimes I \quad \text{with} \quad \Gamma : \rho \rightarrow (\text{Tr}\rho) - \rho$$

(33)

where $\Gamma$ acts on Hermitian operators on $\mathcal{H}_A$ and the identity acts on operators on $\mathcal{H}_B$. Since $\Gamma$ is a positive map (i.e., it maps positive operators into positive operators), $\Lambda = \Gamma \otimes I$ maps separable states into positive operators [3]. It therefore results in a necessary condition for separability, according to Theorem 4. The map $\Gamma$ commutes with an arbitrary unitary transformation $U$, that is

$$\Gamma(U\rho U^\dagger) = U(\Gamma\rho)U^\dagger$$

(34)

which makes the separability condition based on $\Lambda = \Gamma \otimes I$ independent on the basis chosen for $A$ and $B$. In the same manner, the inverse map $\Lambda^{-1}$ can be written as

$$\Lambda^{-1} = \Gamma^{-1} \otimes I \quad \text{with} \quad \Gamma^{-1} : \rho \rightarrow \frac{\text{Tr}\rho}{d - 1} - \rho$$

(35)

where $d$ is the dimension of the Hilbert space of $\rho$. Note that $\Gamma^{-1}$ is not a positive map for $d > 2$, so that $\Lambda^{-1}$ is in general useless as far as detecting inseparability is concerned. This emphasizes that the separability criterion based on Theorem 4 is quite special in 2 dimensions (e.g., for a spin-1/2 particle or a quantum bit), as will be studied in detail later on. Let us only mention here that the map $\Gamma$ applied to a two-dimensional system amounts to a spin flip (this can be interpreted as time reversal). In order to see this, let us write an arbitrary state of a two-dimensional system as

$$\rho = \frac{1}{2}(1 + \vec{r} \cdot \vec{\sigma})$$

(36)

where $\vec{\sigma}$ represent the three Pauli matrices and $\vec{r} = \text{Tr}(\rho \vec{\sigma})$ is a real vector in the Bloch sphere (of radius 1). The vector $\vec{r}$ describes the statistics of measurements on the system, as, for example, the quantum expectation value of the spin component along an axis defined by the vector $\vec{v}$ is

$$\text{Tr}[\rho(\vec{v} \cdot \vec{\sigma})] = (\vec{v}, \vec{r})$$

(37)

Using Eq. (36), it is straightforward to check that
Thus, the map $\Gamma$ performs a spin-flip, or, equivalently, a parity transformation $\vec{r} \to -\vec{r}$ on the vector characterizing the system in the Bloch sphere.

**Remark 2:** It is interesting to consider the classical analog of the maps $\Gamma$ and $\Lambda = \Gamma \otimes I$ to gain some insight into their physical meaning. First, the map $\Gamma$ applied to a classical probability distribution $p_i$ (diagonal $\rho$) corresponds to the transformation:

$$p_i \to q_i = \sum_k p_k - p_i$$  \hspace{1cm} (39)

(Obviously, $q_i \geq 0$ is not normalized except for a binary distribution.) The classical analog of the map $\Lambda = \Gamma \otimes I$ is thus

$$p_{ij} \to q_{ij} = \left( \sum_k p_{k|j} - q_{k|j} \right) p_j = p_j - p_{ij}$$  \hspace{1cm} (40)

Since $p_{ij}$ is a probability distribution in $i$, we always have $1 - p_{ij} \geq 0$ so that $q_{ij} \geq 0$ and the separability criterion is fulfilled. This emphasizes that quantum inseparability ("$q_{ij} < 0"$) may be viewed as resulting from a conditional probability that exceeds 1 (more precisely, an eigenvalue of $\rho_{AB}$ which exceeds 1).

**Remark 3:** It is instructive to consider the action of the map $\Lambda = \Gamma \otimes I$ on a product state $|\psi\rangle = |a\rangle \otimes |b\rangle$. Using $\rho_{AB} = P_a \otimes P_b$ with $P_a = |a\rangle \langle a|$ and $P_b = |b\rangle \langle b|$, it is easy to see that

$$\lambda_{AB} = P_a^\perp \otimes P_b$$  \hspace{1cm} (41)

where $P_a^\perp = \Gamma(|a\rangle \langle a|) = 1_A - |a\rangle \langle a|$ is the projector on the subspace orthogonal to $|a\rangle$. In the special case where $A$ is a 2-state system ($d_A = 2$), $P_a^\perp$ is a rank-one projector as the total trace is preserved. We have then $P_a^\perp = |a^\perp\rangle \langle a^\perp|$, where $|a^\perp\rangle$ is a state vector orthogonal to $|a\rangle$ obtained by applying a complex conjugation on the components of $|a\rangle$ followed by a rotation by an angle $\pi$ about the $y$-axis. (Note that it can be shown that it is impossible to construct a state $|a^\perp\rangle$ that is orthogonal to an arbitrary state $|a\rangle$ by applying a unitary transformation alone.) Indeed, an arbitrary state

$$|a\rangle = \alpha |0\rangle + \beta |1\rangle$$  \hspace{1cm} (42)

with $|\alpha|^2 + |\beta|^2 = 1$, is transformed into

$$|a^\perp\rangle = -\beta^* |0\rangle + \alpha^* |1\rangle$$  \hspace{1cm} (43)

by applying to the state vector $\alpha^* |0\rangle + \beta^* |1\rangle$ the rotation

$$U_y = \exp(-i\pi \sigma_y / 2) = -i \sigma_y = \sigma_x \sigma_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$  \hspace{1cm} (44)

that is, a bit- and phase-flip. The transformed state $|a^\perp\rangle$ is such that $\langle a^\perp |a\rangle = 0$ and $|a^\perp\rangle \langle a^\perp| = 1_A - |a\rangle \langle a|$, as expected. Consequently, the $\Gamma$ map (i.e., the complex conjugation followed by $U_y$ rotation) is an antiunitary operation on state vectors in the 2-dimensional Hilbert space $\mathcal{H}_A$. Indeed, for any two state vectors $|a\rangle$ and $|\tilde{a}\rangle$, we have $\langle a^\perp |a\rangle^* = \langle \tilde{a}| \tilde{a} |a^\perp\rangle$. Since time-reversal $T$ is known to be an antiunitary operator given by complex conjugation $K$ followed by a rotation $R_y$ of angle $\pi$ about the $y$-axis, i.e., $T = R_y K$ (see, e.g., [14]), we conclude that the map $\Gamma$ is time-reversal when $d_A = 2$. [This is also clear from Eq. (43).] Consequently, when $d_A = 2$, $\Lambda = \Gamma \otimes I$ corresponds to physical time-reversal on the subsystem $A$ (while leaving the subsystem $B$ unchanged). Such a link between "local" time-reversal $T \otimes I$ and separability has recently been pointed out by Sanpera et al. [1]. The discussion of the case $d_A = d_B = 2$ will be continued in Section IIIb.

**Lemma 3:** If $\rho_{AB}$ is a separable state, then $\lambda_{AB}$ is a separable operator obtained by replacing the states $|a\rangle$ in $\mathcal{H}_A$ by projectors $P_a^\perp$ orthogonal to them.

Let us assume that the bipartite system $AB$ is separable, that is
\[
\rho_{AB} = \sum_i w_i (|a_i\rangle \langle a_i| \otimes |b_i\rangle \langle b_i|) \tag{45}
\]

where the \(|a_i\rangle \otimes |b_i\rangle\) are pure product states [using the spectral decomposition of \(\rho_A^{(i)}\) and \(\rho_B^{(i)}\), it is easy to rewrite Eq. (17) into this form]. As a result of Remark 3, we see that \(\rho_{AB}\) is mapped by \(\Lambda\) into a separable operator of the form

\[
\lambda_{AB} = \sum_i w_i \left( P_{a_i}^{\perp} \otimes |b_i\rangle \langle b_i| \right) \quad \square \tag{46}
\]

The operator \(\lambda_{AB}\) is a unit-trace operator in the case \(d_A = 2\) since each component pure state \(|a\rangle \otimes |b\rangle\) is mapped into a pure product state, \(|a^{\perp}\rangle \otimes |b\rangle\), in which case it simply reads

\[
\lambda_{AB} = \sum_i w_i \left( |a_i^{\perp}\rangle \langle a_i^{\perp}| \otimes |b_i\rangle \langle b_i| \right) \tag{47}
\]

This expression results in a simple necessary condition for separability (distinct from \(\lambda_{AB} \geq 0\)), inspired from the condition recently proposed by Horodecki \cite{4} as follows.

**Theorem 5:** A necessary separability condition for the bipartite state \(\rho_{AB}\) is that its support can be spanned by a set of product states which are such that the corresponding product states obtained by applying \(\Gamma\) to the state vector in the \(A\) space span the support of \(\lambda_{AB} = \Lambda \rho_{AB}\).

We only consider this condition in the case where \(d_A = 2\). The central point is to note that, if \(\rho_{AB}\) is separable as in Eq. (45), then the ensemble of product states \(|a_i\rangle \otimes |b_i\rangle\) span the entire support of \(\rho_{AB}\). Conversely, any state \(|a_i\rangle \otimes |b_i\rangle\) must belong to the support of \(\rho_{AB}\) and cannot have a non-vanishing component orthogonal to it.) Using Lemma 3, we see that the ensemble of states \(|a_i^{\perp}\rangle \otimes |b_i\rangle\) span the entire support of the corresponding separable state \(\lambda_{AB}\) obtained by applying \(\Lambda\) on \(\rho_{AB}\) [cf. Eq. (17)]. (Also, any state \(|a_i^{\perp}\rangle \otimes |b_i\rangle\) cannot be outside the support of \(\lambda_{AB}\).) This results in another necessary condition for separability which can be stated as follows. If a state \(\rho_{AB}\) is separable, then it must be possible to span its support by a set of product states \(|a\rangle |b\rangle\) which are such that their image (i.e., the product states obtained by rotating the complex conjugate of state vector \(|a\rangle\) in the \(A\) space by an angle \(\pi\) about the \(y\)-axis while leaving the state vector \(|b\rangle\) in the \(B\) space unchanged) span the support of the mapped state \(\lambda_{AB} = \Lambda \rho_{AB}\). \(\square\)

**Definition 4:** Two additional maps from operators on \(\mathcal{H}_{AB}\) to operators on \(\mathcal{H}_{AB}\) can be defined: the dual map

\[
\hat{\Lambda} : \rho_{AB} \rightarrow \hat{\lambda}_{AB} = \rho_A \otimes 1_B - \rho_{AB} \tag{48}
\]

and the symmetric map

\[
M : \rho_{AB} \rightarrow \mu_{AB} = 1_A \otimes 1_B - \rho_A \otimes 1_B - 1_A \otimes \rho_B + \rho_{AB} \tag{49}
\]

where \(\rho_A = \text{Tr}_B[\rho_{AB}]\) and \(\rho_B = \text{Tr}_A[\rho_{AB}]\).

The map \(\Lambda\) which we considered until now is related to the conditional amplitude operator of \(A\) conditionally on \(B\), that is \(\rho_{AB}\). Of course, a similar linear map can be defined using the amplitude operator \(\rho_{B|A}\), and exactly the same conclusions follow. This is the dual map \(\hat{\Lambda}\) defined in Eq. (18). It is trace-preserving and self-inverse in the special case where \(d_B = 2\). It can obviously be written as the tensor product \(\hat{\Lambda} = I \otimes \Gamma\), where the map \(\Gamma\) now acts on operators on \(\mathcal{H}_B\), and therefore commutes with a \(U_A \otimes U_B\) isomorphism. Since \(\Gamma\) is positive, \(\hat{\Lambda}\) maps separable states into positive (separable) operators, which results in another separability condition, i.e., \(\lambda_{AB} \geq 0\). As we will see in Section IIIb, the operators \(\lambda_{AB}\) and \(\hat{\lambda}_{AB}\) can be shown to have the same spectrum when \(d_A = d_B = 2\) (i.e., for two 2-state systems), in which case they result in the same separability condition. However, this property does not hold in larger dimensions, i.e., \(\lambda_{AB}\) and \(\hat{\lambda}_{AB}\) do not have the same spectrum in general (see Appendix A).

We can also construct another linear map by cascading \(\Lambda\) and \(\hat{\Lambda}\) (the order is irrelevant), which results in the symmetric map \(M = \Lambda \hat{\Lambda} = \Gamma \otimes \Gamma\) defined in Eq. (19). Any separable \(\rho_{AB}\) is mapped by \(M\) into a separable operator \(\mu_{AB} \geq 0\), as expected. The symmetric map also commutes with a \(U_A \otimes U_B\) isomorphism,

\[
M \left( (U_A \otimes U_B)\rho_{AB}(U_A^\dagger \otimes U_B^\dagger) \right) = (U_A \otimes U_B)(M\rho_{AB})(U_A^\dagger \otimes U_B^\dagger) \tag{50}
\]

so that the spectrum of \(\mu_{AB} = M\rho_{AB}\) is invariant under local transformations on \(\rho_{AB}\). It is also reversible, its inverse map \(M^{-1} = \Gamma^{-1} \otimes \Gamma^{-1}\) being given by

9
\[ M^{-1} : \mu_{AB} \rightarrow 1_A \otimes 1_B - (d_B - 1)^{-1}(\mu_A \otimes 1_B) - (d_A - 1)^{-1}(1_A \otimes \mu_B) + \mu_{AB} = \rho_{AB} \]  

(51)

where \( \mu_A = \text{Tr}_B[\mu_{AB}] = (d_B - 1)(1_A - \rho_A) \), \( \mu_B = \text{Tr}_A[\mu_{AB}] = (d_A - 1)(1_B - \rho_B) \), and \( d_A (d_B) \) is the dimension of \( \mathcal{H}_A (\mathcal{H}_B) \). As expected, this map is trace-preserving and self-inverse in the special case where \( d_A = d_B = 2 \). It corresponds then to a time-reversal operation applied to the joint system \( AB \). Note that, in this case, the symmetric map \( M \) by itself is not useful as far as revealing inseparability is concerned since it is positive, i.e., \( M \rho_{AB} \geq 0 \). Therefore, all inseparable states of two quantum bits are mapped into positive operators \( \mu_{AB} \) just as separable states. Still, \( M \) is important when analyzing the separability of two quantum bits as it is also equivalent to the conjugation operation in the “magic” basis introduced by Hill and Wootters [4] (see below, Theorem 10). The case of two quantum bits will be studied later on. Whether the positivity of \( M \) holds in arbitrary dimensions is not known.

**Theorem 6:** The criterion that \( \Lambda \) maps the state \( \rho_{AB} \) into a positive semi-definite operator \( \lambda_{AB} \) is not a sufficient condition for the separability of \( \rho_{AB} \).

The fact that \( \lambda_{AB} \geq 0 \) is necessary for separability (Theorem 4) was proven before. In order to prove that it is not sufficient, we will show that it is possible to have an inseparable system with \( \lambda_{AB} \geq 0 \), i.e., such that its inseparability is not revealed by the map \( \Lambda \). As before, we build an inseparable system \( \rho_{AB} \) by extending an inseparable component with a separable one. Let us consider an inseparable system \( A'B' \) with \( \lambda_{A'B'} \geq 0 \). We extend \( A'B' \) with a separable system \( A''B'' \), and apply the separability criterion to the joint system \( AB \) where \( A \equiv A'A'' \) and \( B \equiv B'B'' \). Since the joint system is characterized by the density operator \( \rho_{AB} = \rho_{A'B'} \otimes \rho_{A''B''} \), its associated operator under the map \( \Lambda \) is given by

\[
\lambda_{AB} = \Lambda \rho_{AB} = 1_A \otimes \rho_B - \rho_{AB} = (1_A \otimes \rho_B') \otimes (1_B \otimes \rho_{B''}) - \rho_{A'B''} \otimes \rho_{A''B''}
\]

(52)

Thus, using the operators \( \lambda_{A'B'} = \Lambda \rho_{A'B'} = 1_A \otimes \rho_B - \rho_{A'B'} \) and \( \lambda_{A''B''} = \Lambda \rho_{A''B''} = 1_{A''} \otimes \rho_{B''} - \rho_{A''B''} \), corresponding to \( \Lambda \) applied to both component systems, we obtain the simple expression

\[
\lambda_{AB} = \lambda_{A'B'} \otimes \lambda_{A''B''} + \lambda_{A'B''} \otimes \lambda_{A''B'} + \lambda_{A'B'} \otimes \lambda_{A'B''}
\]

(53)

with \( \lambda_{A'B'} \not\geq 0 \) and \( \lambda_{A''B''} \geq 0 \) (since \( A''B'' \) is separable). Thus, the sum of the first two terms on the right-hand side of Eq. (53) is not positive semi-definite, while only the third one is. Therefore, Eq. (53) does not guarantee that \( \lambda_{AB} \geq 0 \) even though the composite system \( AB \) contains an inseparable component as \( \lambda_{A'B'} \not\geq 0 \). □

Note that, if both components are inseparable systems that have \( \lambda_{A'B'} \not\geq 0 \) and \( \lambda_{A''B''} \not\geq 0 \), then \( \lambda_{AB} \not\geq 0 \) is not necessarily true, so that the inseparability of the joint system \( AB \) is not always revealed by the map \( \Lambda \). [This property contrasts with Eq. (29).] Conversely, Eq. (53) implies that, if both components have \( \lambda_{A'B'} \geq 0 \) and \( \lambda_{A''B''} \geq 0 \), then \( \lambda_{AB} \geq 0 \). The example of weakly inseparable states with a positive partial transpose (see Ref. [4]) is treated in Appendix A, to illustrate that \( \lambda_{AB} \geq 0 \) is not a sufficient condition in general.

**Remark:** The mechanism of dilution of inseparability can be understood by examining the action of the map \( \Gamma \) on product states. Indeed, when applying the map \( \Gamma = \Gamma \otimes I \) on the state \( \rho_{AB} = \rho_{A'B'} \otimes \rho_{A''B''} \), \( \Gamma \) acts on the state \( \rho_A \otimes \rho_{A'} \) (\( B \) and \( B' \) are left unchanged by \( I \)). Let us consider a density operator of the product form \( \rho = \rho' \otimes \rho'' \). Since we have \( \text{Tr}(\rho) = \text{Tr}(\rho')\text{Tr}(\rho'') \), we see that it is mapped to

\[
\Gamma(\rho' \otimes \rho'') = \text{Tr}(\rho')\text{Tr}(\rho'') - \rho' \otimes \rho''
\]

\[
= [\text{Tr}(\rho') - \rho'] \otimes [\text{Tr}(\rho'') - \rho''] + \text{Tr}(\rho') \otimes \rho'' + \rho' \otimes \text{Tr}(\rho'') - 2\rho' \otimes \rho''
\]

(54)

which implies the relation

\[
\Gamma = \Gamma' \otimes \Gamma'' + \Gamma' \otimes I'' + I' \otimes \Gamma''
\]

(55)

where \( \Gamma' \) (or \( \Gamma'' \)) stands for the same map but acting on the subspace of \( \rho' \) (or \( \rho'' \)) while \( \Gamma \) acts on the joint space. Using the same notation for the map \( \Lambda \) (i.e., \( \Lambda' \) acts on the subspace of \( A'B' \) while \( \Lambda'' \) acts on the subspace of \( A''B'' \)), the latter equation gives

\[
\Lambda = \Gamma \otimes I = \Lambda' \otimes \Lambda'' + \Lambda' \otimes I'' + I' \otimes \Lambda''
\]

(56)

which results in Eq. (53). The same reasoning can be applied to the dual map \( \hat{\Lambda} = I \otimes \Gamma \) and to the symmetric map \( M = \Gamma \otimes \Gamma \). Thus, even if the maps \( \Lambda' \) and \( \Lambda'' \) reveal inseparability by themselves, the combined map, Eq. (56), is not guaranteed to do so because the non-positivity of \( (\Lambda' \otimes \Lambda'') \rho \) can be masked by the two following terms.
B. Special case of two 2-dimensional systems

**Lemma 4:** The map $\Gamma$ acting on the state of a two-dimensional system corresponds to time-reversal, and is therefore equivalent to applying the complex conjugation operator $K$ followed by a rotation $R_y$ by an angle $\pi$ about the $y$-axis, that is, $\Gamma = R_y K$.

Let us start by summarizing the action of the map $\Gamma : \rho \rightarrow (\text{Tr}\rho) - \rho$ on the density operator $\rho$ characterizing a 2-dimensional system (e.g., a quantum bit). Since $\rho$ can be written as a linear combination of the unit matrix and the three Pauli matrices $\sigma$ with real coefficients, it is sufficient to consider the action of $\Gamma$ on these (Hermitian) basis matrices. It is straightforward to check that $\Gamma$ is an antiunitary operator that leaves the unit matrix unchanged and corresponds to a sign-flip of the Pauli matrices acting on $\rho$.

\[
1 \xrightarrow{\Gamma} 1, \quad \sigma_x \xrightarrow{\Gamma} -\sigma_x, \quad \sigma_y \xrightarrow{\Gamma} -\sigma_y, \quad \sigma_z \xrightarrow{\Gamma} -\sigma_z \tag{57}
\]

so that, in the Bloch-sphere picture, we obtain

\[
\rho = \frac{1}{2} (1 + \vec{r} \cdot \vec{\sigma}) \rightarrow \Gamma \rho = 1 - \rho = \frac{1}{2} (1 - \vec{r} \cdot \vec{\sigma}) \tag{58}
\]

where $\vec{r}$ is the Bloch vector. Thus, the map $\Gamma$ flips $\vec{\sigma}$ (or performs a parity transformation on the Bloch vector $\vec{r} \rightarrow -\vec{r}$), and corresponds to time-reversal. It can therefore be decomposed into complex conjugation $K$ followed by a rotation $R_y$ of an angle $\pi$ about the $y$-axis, that is $\Gamma = T = R_y K$ \[\square\].

**Remark:** The complex conjugation operator $K$ (or equivalently the transposition, as we deal with Hermitian operators) corresponds to an antiunitary operator which acts on the four basis matrices as:

\[
1 \xrightarrow{K} 1, \quad \sigma_x \xrightarrow{K} \sigma_x, \quad \sigma_y \xrightarrow{K} -\sigma_y, \quad \sigma_z \xrightarrow{K} \sigma_z \tag{59}
\]

(Remember that it is enough to consider the action of $K$ on the basis matrices as the coefficients are real.) Also, $R_y$ is a unitary operation characterized by the unitary matrix $U_y = \exp(-i\pi \sigma_y/2) = -i\sigma_y = \sigma_x\sigma_z$ which maps $\rho$ into $U_y \rho U_y^\dagger = \sigma_y \rho \sigma_y$, so that the basis matrices are transformed according to

\[
1 \xrightarrow{R_y} 1, \quad \sigma_x \xrightarrow{R_y} -\sigma_x, \quad \sigma_y \xrightarrow{R_y} \sigma_y, \quad \sigma_z \xrightarrow{R_y} -\sigma_z \tag{60}
\]

It is straightforward to check, using Eqs. (57), (59) and (60), that $\Gamma$ is the product of $K$ and $R_y$. (It is a general property of an antiunitary transformation that it can be written as the product of a unitary transformation and a fixed antiunitary operator such as time-reversal.)

In short, one can see that, if the system is in a (pure or mixed) state given by Eq. (60), then

\[
U_y \rho^\dagger U_y^\dagger = \sigma_y \rho^* \sigma_y = \frac{1}{2} (1 + \sigma_y (\vec{r} \cdot \vec{\sigma}^*) \sigma_y) = \frac{1}{2} (1 - \vec{r} \cdot \vec{\sigma}) = \Gamma \rho \tag{61}
\]

where we have used the fact that $\vec{r}$ is a real vector and that $\sigma_y \vec{\sigma} \sigma_y = -\vec{\sigma}^*$. This generalizes what was shown earlier for pure states, namely that if $|a\rangle = |\alpha\rangle + |\beta\rangle$ and $|a^\perp\rangle = U_y (\alpha^* |0\rangle + \beta^* |1\rangle) = -\beta^* |0\rangle + \alpha^* |1\rangle$, then we have

\[
|a^\perp\rangle (a^\perp | = \Gamma (|a\rangle \langle a|) \tag{62}
\]

**Corollary 1:** When writing the Hilbert-Schmidt decomposition of $\rho_{AB}$ for two 2-state systems, the map $\Lambda = \bigotimes (\rho < I)$ corresponds to a sign-flip of the Pauli matrices acting on $A$ while leaving the sign of those acting on $B$ unchanged.

Let us consider the Hilbert-Schmidt decomposition of an arbitrary state of $AB$ (e.g., two quantum bits or spin-1/2 particles) \[\mathcal{B}\]:

\[
\rho_{AB} = \frac{1}{4} \left( 1_A \otimes 1_B + \vec{r} \cdot \vec{\sigma}_A \otimes 1_B + 1_A \otimes \vec{s} \cdot \vec{\sigma}_B + \sum_{m,n=1}^{3} t_{n,m} \sigma_A^{(n)} \otimes \sigma_B^{(m)} \right) \tag{63}
\]

where $\sigma_A^{(n)}$ and $\sigma_B^{(m)}$ stand for the Pauli matrices (with $n = 1, 2, 3$) in the $A$ and $B$ space, respectively. Eq. (63) depends on 15 real parameters, the two 3-dimensional vectors $\vec{r}$ and $\vec{s}$, and the $3 \times 3$ real matrix $t_{n,m}$. The vectors $\vec{r}$ and $\vec{s}$ correspond to the state of $A$ and $B$ in the Bloch sphere since we have
\[ \rho_A = \text{Tr}_B [\rho_{AB}] = \frac{1}{2} (1_A + \vec{r} \cdot \vec{\sigma}_A) \]
\[ \rho_B = \text{Tr}_A [\rho_{AB}] = \frac{1}{2} (1_B + \vec{s} \cdot \vec{\sigma}_B) \]

They characterize the reduced systems \( A \) and \( B \), that is the local (marginal) statistics of any observable on \( A \) or \( B \).

The matrix \( t_{n,m} = \text{Tr} [\rho_{AB} (\sigma^{(n)}_A \otimes \sigma^{(m)}_B)] \) describes the joint statistics of \( A \) and \( B \), and yields for example a very simple expression for the correlation between the measured spin components along two axes (defined by the vectors \( \vec{a} \) and \( \vec{b} \)):

\[ \text{Tr} \left[ \rho (\vec{a} \cdot \vec{\sigma}_A \otimes \vec{b} \cdot \vec{\sigma}_B) \right] = (\vec{a}, \vec{b}) \]

It is checked by straightforward calculation that the map \( \Lambda = \Gamma \otimes I \) simply flips the sign of the terms in \( \vec{\sigma}_A \):

\[ \lambda_{AB} = 1_A \otimes \rho_B - \rho_{AB} = \frac{1}{4} \left( 1_A \otimes 1_B - \vec{r} \cdot \vec{\sigma}_A \otimes 1_B + 1_A \otimes \vec{s} \cdot \vec{\sigma}_B - \sum_{m,n=1}^{3} t_{n,m} \sigma^{(n)}_A \otimes \sigma^{(m)}_B \right) \]  \( \square \)

**Corollary 2:** The dual map \( \hat{\Lambda} = I \otimes \Gamma \) flips the sign of the Pauli matrices acting on \( B \) while leaving the sign of those acting on \( A \) unchanged. The action of the symmetric map \( M \) on the Hilbert-Schmidt decomposition of \( \rho_{AB} \) is to flip the sign of the Pauli matrices acting on both \( A \) and \( B \):

\[ \mu_{AB} = 1_A \otimes 1_B - \rho_A \otimes 1_B - 1_A \otimes \rho_B + \rho_{AB} = \frac{1}{4} \left( 1_A \otimes 1_B - \vec{r} \cdot \vec{\sigma}_A \otimes 1_B - 1_A \otimes \vec{s} \cdot \vec{\sigma}_B + \sum_{m,n=1}^{3} t_{n,m} \sigma^{(n)}_A \otimes \sigma^{(m)}_B \right) \]

This operation corresponds to time-reversal applied to \( A \) and \( B \) simultaneously, and can be shown to be equivalent to the conjugation operation in the “magic” basis introduced by Hill and Wootters \( \text{[10]} \) (see below).

It is worth noting that the set of states that remain invariant under the symmetric map \( M \) are mixtures of generalized Bell states, the latter being defined as the states obtained by applying any local transformation to the four Bell states. These states are also called “\( T \)-states” by Horodecki et al. \( \text{[8]} \), and are such that the entropy of \( A \) and \( B \) is maximal, that is \( S(\rho_A) = S(\rho_B) = 1 \). (The only pure states in this set are the fully entangled states of two qubits, i.e., the generalized Bell states.) Thus, in particular, the (generalized) Bell states are left unchanged by the action of \( M \). In contrast, a (separable) product state \( \rho_A \otimes \rho_B \) is mapped into the distinct state \( \mu_{AB} = (1_A - \rho_A) \otimes (1_B - \rho_B) \). Because of this property, \( \mu_{AB} \) by itself is uninteresting as far as revealing inseparability is concerned, as mentioned earlier.

**Theorem 7:** For two-dimensional systems \( A \) and \( B \), the map \( M \) conserves the spectrum, so that the separability criteria resulting from the map \( \Lambda \) and its dual \( \hat{\Lambda} \) are equivalent.

As we have seen before, the map \( \Gamma \) acting on subsystem \( A \) (or \( B \)) amounts to performing a complex conjugation operation \( K \) followed by a rotation \( R_y \) defined by \( U_y = \exp(-i \sigma_y / 2) = -i \sigma_y \) (i.e., a bit- and phase-flip). Thus, the symmetric map \( M = \Gamma \otimes \Gamma \) is equivalent to a complex conjugation \( K \) (or transposition) of the joint density operator in the Hilbert space \( \mathcal{H}_{AB} \), followed by a tensor product of rotations \( U_y \otimes U_y = -\sigma_y \otimes \sigma_y \). Note that, as we are dealing with Hermitian (density) operators, their spectrum is unchanged by \( K \). The same is true for the rotation \( U_y \otimes U_y \). As a consequence, the operator \( \mu_{AB} = M \rho_{AB} \) has the same spectrum as \( \rho_{AB} \) when \( d_A = d_B = 2 \). As \( \Gamma \) is self-inverse \( (\Gamma^2 = I) \) when \( d_A = d_B = 2 \), we have the relation \( I \otimes \Gamma = (\Gamma \otimes I)(\Gamma \otimes \Gamma) \) or in short \( \hat{\Lambda} = \Lambda M \). This implies that

\[ \hat{\lambda}_{AB} = \Lambda \left[ (U_y \otimes U_y) \rho_{AB}^* (U_y^\dagger \otimes U_y^\dagger) \right] \]

which in turn results in

\[ \lambda_{AB} = (U_y \otimes U_y) \lambda_{AB}^* (U_y^\dagger \otimes U_y^\dagger) \]

as \( \Lambda \) commutes with \( U_y \otimes U_y \) and complex conjugation. Since \( \lambda_{AB} \) is Hermitian (just as \( \rho_{AB} \)), the latter expression shows that the spectrum of \( \hat{\lambda}_{AB} \) and \( \lambda_{AB} \) are identical, so that the resulting criteria for separability are equivalent. \( \square \)
Theorem 8: For two-dimensional systems $A$ and $B$ which have maximal reduced entropy, i.e., $S(\rho_A) = S(\rho_B) = 1$, the positivity of the operator $\Lambda_{\rho_{AB}}$ results in a necessary and sufficient condition for the separability of $\rho_{AB}$.

The states with $\vec{r} = \vec{s} = 0$ are such that the reduced density operators are given by $\rho_A = \rho_B = 1/2$, so that the reduced entropies are $S(\rho_A) = S(\rho_B) = 1$. These “$T$-states” are thus completely characterized by the matrix $t_{n,m}$. It has been shown in Ref. [3] that any state belonging to this set of $T$-states can be transformed by a unitary transformation of the product form $U_A \otimes U_B$ into a state for which $t_{A_2}$ is diagonal. As far as separability is concerned, we can thus restrict ourselves to the class of all states with diagonal $t$, since these are representative of the entire set of $T$-states (up to an $U_A \otimes U_B$ isomorphism).

The class of states with diagonal $t$ is a convex subset of the set of $T$-states, and any state belonging to this subset can be characterized by the real vector $\vec{t} = (t_{11}, t_{22}, t_{33})$ made out of the diagonal elements of $t$. It has been shown in Ref. [3] that an operator $\rho_{AB}$ of the form given by Eq. (64) with $\vec{r} = \vec{s} = 0$ and diagonal $t$ corresponds to a state (i.e., a positive unit-trace operator) if and only if the vector $\vec{t}$ belongs to a tetrahedron with vertices $\vec{t}_1 = (-1,1,1)$, $\vec{t}_2 = (1,-1,1)$, $\vec{t}_3 = (1,1,-1)$, and $\vec{t}_4 = (-1,-1,-1)$. In other words, any state of this class can be represented by a point inside this tetrahedron. In this representation, the four Bell states $|\Phi^\pm\rangle = 2^{-1/2}(|00\rangle \pm |11\rangle)$ and $|\Psi^\pm\rangle = 2^{-1/2}(|01\rangle \pm |10\rangle)$ correspond to the vertices of the tetrahedron, that is

$$
\begin{align*}
\vec{t}_1 & : |\Phi^-\rangle\langle\Phi^-| = \frac{1}{4} \left( 1_A \otimes 1_B - \sigma_A^{(x)} \otimes \sigma_B^{(x)} + \sigma_A^{(y)} \otimes \sigma_B^{(y)} + \sigma_A^{(z)} \otimes \sigma_B^{(z)} \right) \\
\vec{t}_2 & : |\Phi^+\rangle\langle\Phi^+| = \frac{1}{4} \left( 1_A \otimes 1_B + \sigma_A^{(x)} \otimes \sigma_B^{(x)} - \sigma_A^{(y)} \otimes \sigma_B^{(y)} + \sigma_A^{(z)} \otimes \sigma_B^{(z)} \right) \\
\vec{t}_3 & : |\Psi^+\rangle\langle\Psi^+| = \frac{1}{4} \left( 1_A \otimes 1_B + \sigma_A^{(x)} \otimes \sigma_B^{(x)} + \sigma_A^{(y)} \otimes \sigma_B^{(y)} - \sigma_A^{(z)} \otimes \sigma_B^{(z)} \right) \\
\vec{t}_4 & : |\Psi^-\rangle\langle\Psi^-| = \frac{1}{4} \left( 1_A \otimes 1_B - \sigma_A^{(x)} \otimes \sigma_B^{(x)} - \sigma_A^{(y)} \otimes \sigma_B^{(y)} - \sigma_A^{(z)} \otimes \sigma_B^{(z)} \right)
\end{align*}
$$

In Ref. [3], it is also shown that a state $\rho_{AB}$ of this $T$-diagonal class is separable if and only if the vector $\vec{t}$ characterizing $\rho_{AB}$ belongs to an octahedron with vertices $\vec{o}_1^\pm = (\pm 1,0,0)$, $\vec{o}_2^\pm = (0,\pm 1,0)$, and $\vec{o}_3^\pm = (0,0,\pm 1)$. This results in a necessary and sufficient condition for separability within the class of $T$-states.

Let us consider the action of $\Lambda$ in this representation. As shown earlier, $\Lambda$ flips the “spin” $\vec{o}_A$. Within the set of $T$-states, this amounts to changing the sign of the $t_{n,m}$ matrix, that is, to flipping the sign of the vector $\vec{t}$ for $T$-diagonal states. Therefore, the criterion for separability $\lambda_{AB} = \Lambda_{\rho_{AB}} \geq 0$ translates, in this representation, to the condition that the “parity” operation on the vector $\vec{t}$ characterizing a separable state results in a positive operator (i.e., a legitimate state). Thus, $-\vec{t}$ must belong to the tetrahedron. It is easy to see that the set of points of the tetrahedron which are such that their image under parity still belongs to the tetrahedron corresponds exactly to the octahedron defined above. Therefore, no inseparable state exists that satisfies $\lambda_{AB} \geq 0$, so that $\Lambda$ provides a necessary and sufficient condition for separability within the class of $T$-states.

Theorem 9: A bipartite system of two-dimensional components $A$ and $B$ characterized by an arbitrary joint density operator $\rho_{AB}$ is separable if and only if the operator $\lambda_{AB} = \Lambda_{\rho_{AB}}$ is positive semi-definite.

It is enough to show that $\Lambda$ is equivalent to a partial transposition up to some completely positive map (in fact, a unitary transformation). As already mentioned, Peres’ separability criterion is based on the partial transposition operation, that is, on the map $T \otimes I$, where $T$ is the standard transposition of operators on the $\mathcal{H}_A$ subspace. Since we are dealing with Hermitian operators, $T \otimes I$ is equivalent to the “partial conjugation” operation $K \otimes I$, where $K$ is the complex conjugation operator acting on states on $\mathcal{H}_A$. Note that, although $K$ is well-defined, partial conjugation $K \otimes I$ is only defined for product state vectors in $\mathcal{H}_{AB}$. We can now use Lemma 4, i.e., $\Gamma = \mathcal{R}_y K$, together with the fact that any positive map $\Pi$ acting on density operators in a two-dimensional Hilbert space can be written as

$$
\Pi = \Pi_1^{\text{CP}} + \Pi_2^{\text{CP}} T
$$

where $\Pi_1^{\text{CP}}$ and $\Pi_2^{\text{CP}}$ are completely positive maps (which therefore do not reveal inseparability). With the identification $\Pi_1^{\text{CP}} = 0$ and $\Pi_2^{\text{CP}} = \mathcal{R}_y$, we see that the map $\Gamma$ can be used rather than the transposition operator $T$ (or $K$) in order to test the positivity of the operator resulting from applying any element of the set of maps $\Pi \otimes I$ on $\rho_{AB}$. This is simply due to the fact that the complex conjugation operator $K$ is unitarily equivalent to $\Gamma$. Thus, the reasoning used in Ref. [3] holds here, so that $\lambda_{AB} \geq 0$ results in a necessary and sufficient condition for the separability of $\rho_{AB}$.

Remark 1: Since the spectrum of an operator is conserved by a unitary transformation ($\mathcal{R}_y$), it is clear that the spectrum of the matrix obtained by partial transposition in subspace $A$, $\rho_{AB}^{T_A}$, is the same as the spectrum of...
\( \lambda_{AB} = \Lambda \rho_{AB} \). Therefore, testing Peres’ condition or the positivity of \( \lambda_{AB} \) is operationally equivalent, and these conditions can be used interchangeably in the case of two quantum bits, as illustrated in Appendix A. Since \( \Gamma \) corresponds to time-reversal (on the subsystem \( A \)), the map \( \Lambda = \Gamma \otimes I \) amounts to changing the arrow of time for subsystem \( A \) with respect to subsystem \( B \). Such a relation between time-reversal and Peres’ map has been pointed out previously by Sanpera et al. \[15\], where it was shown that the partial transposition operator is unitarily equivalent to “local” time-reversal.

**Remark 2:** As we know that \( \Gamma \) applied to a two-dimensional system is unitarily equivalent to the transposition of the unitary matrix \( T \) whenever subsystem \( A \) is two-dimensional. More precisely, \( \lambda_{AB} \) and \( \rho_{AB}^T \) have the same spectrum for \( 2 \times n \) systems, so that the conditions are equivalent if \( \Gamma \) is applied on the two-dimensional subsystem. As a consequence, the separability condition based on \( \Lambda \) is also necessary and sufficient for \( 2 \times 3 \) systems, while it is only necessary for \( 2 \times n \) systems with larger \( n \), just as Peres’ condition \[3\]. Numerical evidence suggests that, for systems with \( d_A, d_B > 2 \), the condition based on \( \Lambda \) (or \( \Lambda \)) is weaker than (or equivalent to) the one based on partial transposition.

**Theorem 10:** The symmetric map \( M = \Gamma \otimes \Gamma \) applied to a bipartite system of two-dimensional components (i.e., global time-reversal) is equivalent to complex conjugation in the “magic” basis introduced in Ref. \[14\]. Since \( \Gamma = R_y K \), where \( K \) denotes the conjugation operator and \( R_y \) is a rotation characterized by \( U_y = \exp(-i\pi \sigma_y/2) = -i\sigma_y \), the symmetric map \( M \) applied to the state \( \rho_{AB} \) of a bipartite systems results in

\[
M \rho_{AB} = (U_y \otimes U_y) \rho_{AB}^* (U_y^\dagger \otimes U_y^\dagger)
\]

where \( U_y \otimes U_y = -\sigma_y \otimes \sigma_y \). Since \( M \) is antiunitary and self-inverse \( (M^2 = I) \), it is a conjugation \[15\]. It can be written as the complex conjugation operator if expressed in a specific basis. Let us assume that \( V \) is the unitary operator (in the joint space) that transforms the product states into the states \( \{|e_i\} \) that form this specific basis, that is

\[
|e_1\rangle = V|00\rangle \quad |e_2\rangle = V|01\rangle \quad |e_3\rangle = V|10\rangle \quad |e_4\rangle = V|11\rangle
\]

We would like to show that \( M \) is equivalent to rotating the states \( |e_i\rangle \) into the product states, taking the complex conjugation of the density matrix (in the product basis), and then rotating the product states back to the \( |e_i\rangle \)'s:

\[
M \rho_{AB} = V(V^T \rho_{AB}^*) V^\dagger = (VV^T) \rho_{AB} (VV^T)^\dagger
\]

where \( V^T \) is the transpose of the unitary matrix \( V \). Identifying Eqs. (72) and (77), we obtain

\[
VV^T = U_y \otimes U_y = -\sigma_y \otimes \sigma_y = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

It is easy to prove that, if \( V \) is unitary, then \( V V^T \) is unitary and symmetric (but not necessarily Hermitian). In order to find a solution for \( V \) that satisfies Eq. (72), we first diagonalize the matrix \( \sigma_y \otimes \sigma_y \). Consider the unitary matrix

\[
W = \exp\left(-\frac{i\pi}{4}(1 - \sigma_x) \otimes (1 - \sigma_x)\right) = (1 \otimes 1 + 1 \otimes \sigma_x + \sigma_x \otimes 1 - \sigma_x \otimes \sigma_x)/2
\]

This matrix is self-inverse, that is, \( W^2 = 1 \), so that it is Hermitian \( W^\dagger = W \). It can easily be shown that it diagonalizes \( \sigma_y \otimes \sigma_y \), i.e.,

\[
W(\sigma_y \otimes \sigma_y)W = \sigma_z \otimes \sigma_z
\]

Replacing \( V \) in Eq. (76) by the product of a diagonal matrix \( D \) and \( W \), that is \( V = WD \), we obtain

\[
DD^T = -W(\sigma_y \otimes \sigma_y)W = -\sigma_z \otimes \sigma_z = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

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which implies that

$$D = \begin{pmatrix} \pm i & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm i \end{pmatrix}$$

(80)

This yields a (non-unique) solution for the unitary matrix $V = WD$ which defines the basis \{\ket{e_i}\}. It is worth noticing that the matrix

$$W = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

(81)

transforms the product states into the four maximally entangled states which are obtained by applying a local transformation $H \otimes 1$ on the four Bell states, i.e.,

$$W\ket{00} = (\ket{00} + \ket{01} + \ket{10} - \ket{11})/2 = (H \otimes 1)\Phi^+$$
$$W\ket{01} = (\ket{00} + \ket{01} - \ket{10} + \ket{11})/2 = (H \otimes 1)\Psi^+$$
$$W\ket{10} = (\ket{00} - \ket{01} + \ket{10} + \ket{11})/2 = (H \otimes 1)\Phi^-$$
$$W\ket{11} = (-\ket{00} + \ket{01} + \ket{10} + \ket{11})/2 = (H \otimes 1)\Psi^-$$

(82)

where $H$ is the Hadamard transform. (As a matter of fact, the unitary transformation $W$ corresponds simply to a controlled-NOT gate where the control is in the dual basis \{(\ket{0} + \ket{1}), (\ket{0} - \ket{1})\} rather than the standard basis.) Therefore, the unitary transformation $V = WD$ is such that the product states are rotated into the four generalized Bell states with the appropriate phases

$$\ket{e_1} = V\ket{00} = \pm i(H \otimes 1)\Phi^+$$
$$\ket{e_2} = V\ket{01} = \pm 1(H \otimes 1)\Psi^+$$
$$\ket{e_3} = V\ket{10} = \pm 1(H \otimes 1)\Phi^-$$
$$\ket{e_4} = V\ket{11} = \pm i(H \otimes 1)\Psi^-$$

(83)

These states $\ket{e_i}$ are therefore equivalent, up to a local change of basis $H \otimes 1$ and a phase $i$ that are irrelevant here, to the “magic” states introduced in Ref. [12]. (Any four states obtained from the $\ket{e_i}$’s up to an overall phase and a unitary transformation acting locally on each quantum bit are legitimate “magic” states.) This implies that, when expressed in this basis, the symmetric map $M = \Gamma \otimes \Gamma$ reduces the complex conjugation operation that was used in the context of the calculation of the entropy of formation of a pair of quantum bits (see Refs. [10][17]). □

IV. CONCLUSION

Given a bipartite system characterized by a density operator $\rho_{AB}$, one can define a conditional amplitude operator $\rho_{A|B}$ (a positive Hermitian operator defined on the range of $\rho_{AB}$) which plays the role of a conditional probability operator in quantum information theory. Specifically, it can be used to define a conditional von Neumann entropy, $S(A|B) = -\operatorname{Tr}[\rho_{AB} \log \rho_{A|B}]$, in perfect analogy with the conditional Shannon entropy. The quantum counterpart of many classical properties also holds: i) $\rho_{A|B}$ is defined on the support of $\rho_{AB}$, so that $S(A|B)$ is well-defined; ii) $S(A|B) = S(AB) - S(B)$; iii) $\rho_{A|B} = \rho_A \otimes \mathbf{1}_B$ if $A$ and $B$ are independent; iv) $\rho_{A|BB'} = \rho_{A|B} \otimes \rho_{A'|B'}$ if $\rho_{AA';BB'} = \rho_{AB} \otimes \rho_{A'B'}$; v) $\rho_{A|B}$ transforms as $(U_A \otimes U_B)\rho_{A|B}(U_A^\dagger \otimes U_B^\dagger)$ when performing a local unitary transformation $U_A \otimes U_B$ on $\rho_{AB}$, so that its spectrum and therefore $S(A|B)$ are invariant under such transformations on $\rho_{AB}$.

The main non-classical feature that appears when dealing with a quantum bipartite system rather than a classical one is that $\rho_{A|B}$ may have a “non-classical” spectrum, that is, eigenvalues of $\rho_{A|B}$ may exceed 1, which in turn implies that $S(A|B)$ can be negative. More specifically, we have shown that $\rho_{AB} \leq 1$ for any separable state, which also straightforwardly implies $S(A|B) \geq 0$. Therefore, a necessary condition for separability is that the conditional amplitude operator has a “classical” spectrum, or that the conditional entropy is non-negative (the latter is a weaker...
condition). These conditions are not sufficient, since extending an inseparable state with a separable one of large dimension may result in a *dilution* of inseparability, that is, it may give rise to a state with \( \rho_{AB} \leq 1 \). In other words, some inseparable states exist with \( \rho_{AB} \leq 1 \), and certainly some with \( S(A|B) \geq 0 \) (even if \( \rho_{AB} \not\leq 1 \)).

These considerations can be used to define a simpler necessary condition for separability, based on the positive linear map \( \Gamma: \rho \to (\text{Tr}\rho) - \rho \). Any separable state is mapped by the tensor product of \( \Gamma \) (acting on \( A \)) and the identity \( I \) (acting on \( B \)) into a positive operator. Therefore, a simple separability criterion is based on checking the positivity of the operator \( (\Gamma \otimes I) \rho_{AB} = 1_A \otimes \rho_B - \rho_{AB} \). This condition, along with the one based on the dual map \( I \otimes \Gamma \), can be shown to be non-sufficient for a system of arbitrary dimension. Since the map \( \Gamma \) commutes with any unitary transformation, the spectrum of the operator \( (\Gamma \otimes I) \rho_{AB} \) is invariant under a local unitary transformation \( U_A \otimes U_B \), making this condition independent of the basis in which \( A \) and \( B \) are expressed.

In the case of a 2-dimensional system, the map \( \Gamma \) can be shown to be the time-reversal operator, which flips the sign of the spin matrices (or, consequently, reverses the Bloch vector characterizing the state of the quantum bit). It is therefore unitarily equivalent to the transposition operator \( \sigma_y \), so that the condition based on \( \Gamma \otimes I \) is equivalent to the one based on Peres’ partial transposition for \( 2 \times n \) systems (when applying the map \( \Gamma \) on the 2-dimensional subsystem). As a consequence, it is necessary and sufficient for \( 2 \times 2 \) and \( 2 \times 3 \) systems while it is only necessary for larger systems, just as is Peres’ \( \sigma_y \).

Numerical evidence suggests that, for systems where \( d_A, d_B > 2 \), the condition based on \( \Gamma \otimes I \) or \( I \otimes \Gamma \) is weaker than (or equivalent to) the one based on partial transposition.

Finally, we consider the symmetric map, defined as \( (\Gamma \otimes \Gamma) \rho_{AB} = 1_A \otimes 1_B - \rho_A \otimes 1_B - 1_A \otimes \rho_B + \rho_{AB} \). The states which are left invariant under this map are mixes of generalized Bell states (or “T-states”), which include the maximally entangled pure states as well as the product of two independent (unentangled) random bits. It can be seen that the map \( \Gamma \otimes \Gamma \) is also related to quantum nonlocality of two quantum bits even though it does not directly reveal inseparability. Indeed, \( \Gamma \otimes \Gamma \) reduces to the complex conjugation in the “magic” basis that has been used in the context of the calculation of the entropy of formation of a pair of quantum bits (see Refs. \[1,17\]). Therefore might be interesting to look for a simple relation between the map \( \Gamma \) (related to inseparability) and the entropy of formation. This will be the subject of further work.

*Note:* Parts of Section IIIB of this paper are equivalent to results contained in Ref. \[17\], which was brought to our attention after completion of this work.

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**APPENDIX A: EXAMPLES**

Here we consider several examples illustrating the separability criterion \( \lambda_{AB} \geq 0 \), and compare it to Peres’ criterion \[3\]. Examples 1-4 deal with states of two quantum bits, and illustrate the fact that the \( \Lambda \)-criterion is necessary and sufficient (the spectrum of \( \lambda_{AB} \) is identical to the spectrum of \( \rho^{TA} \)). Examples 5-6 illustrate that the \( \Lambda \)-condition is not sufficient for systems in larger dimensions (\( 3 \times 3 \) and \( 2 \times 4 \)) whose partial transpose is positive (cf. Ref. \[3\]). In fact, the \( \Lambda \)-condition is equivalent to Peres’ condition for \( 2 \times n \) systems, so that it is also necessary and sufficient for \( 2 \times 3 \) systems \[3\] while it is only necessary for larger \( n \).

**Example 1:** Consider a Werner state \[1\] with parameter \( x \) (\( 0 \leq x \leq 1 \)), that is a mixture of a fraction \( x \) of the singlet state \( |\Psi^-\rangle \) and a random fraction \( (1 - x) \). We shall see that \( \lambda_{AB} \geq 0 \) is equivalent to Peres’ criterion, and is therefore sufficient in this case. Indeed, the joint density operator

\[
\rho_{AB} = x|\Psi^-\rangle\langle\Psi^-| + \frac{(1 - x)}{4}(1 \otimes 1) = \begin{pmatrix}
\frac{1}{4} & 0 & 0 & 0 \\
0 & \frac{1 + x}{4} & -\frac{x}{2} & 0 \\
0 & -\frac{x}{2} & \frac{1 + x}{4} & 0 \\
0 & 0 & 0 & \frac{1 - x}{4}
\end{pmatrix}
\]

is mapped by \( \Lambda \) into the operator

\[
\frac{1}{4} & 0 & 0 & 0 \\
0 & \frac{1 + x}{4} & -\frac{x}{2} & 0 \\
0 & -\frac{x}{2} & \frac{1 + x}{4} & 0 \\
0 & 0 & 0 & \frac{1 - x}{4}
\]
$$\lambda_{AB} = \begin{pmatrix} \frac{1+x}{2} & 0 & 0 & 0 \\ 0 & \frac{1-x}{2} & -x & 0 \\ 0 & -x & \frac{1+x}{2} & 0 \\ 0 & 0 & 0 & \frac{1-x}{2} \end{pmatrix}$$ \hspace{1cm} (A2)$$

which admits three eigenvalues equal to \((1+x)/4\) and a fourth equal to \((1-3x)/4\). The latter becomes negative if \(x > 1/3\), so that \(\lambda_{AB}\) is positive semi-definite only if \(x \leq 1/3\), which has been proven to be the \textit{exact} threshold for separability (any Werner state with \(x \leq 1/3\) is separable as it can be written as a mixture of product states \([8]\)). As expected, the spectrum of \(\lambda_{AB}\) is equal to the spectrum of the partial transpose of \(\rho_{AB}\), so that the A-condition is sufficient to ensure separability for Werner states.

Example 2: Consider a mixed state that is made out of a fraction \(x\) of the entangled state |ψ\> = \(a|01\rangle + b|10\rangle\), and fractions \((1-x)/2\) of the separable product states |00\> and |11\> (see \([8]\)). The joint density matrix is of the form

$$\rho_{AB} = x|\psi\rangle\langle\psi| + \frac{1-x}{2}|00\rangle\langle00| + \frac{1-x}{2}|11\rangle\langle11| = \begin{pmatrix} \frac{1+x}{2} & 0 & 0 & 0 \\ 0 & x|a|^2 & xab^* & 0 \\ 0 & xa^*b & x|b|^2 & 0 \\ 0 & 0 & 0 & \frac{1-x}{2} \end{pmatrix}$$ \hspace{1cm} (A3)

with \(a\) and \(b\) satisfying \(|a|^2 + |b|^2 = 1\). It is mapped by \(\Lambda\) into the matrix

$$\lambda_{AB} = \begin{pmatrix} x|b|^2 & 0 & 0 & 0 \\ 0 & \frac{1-x}{2} & -xab^* & 0 \\ 0 & -xa^*b & \frac{1+x}{2} & 0 \\ 0 & 0 & 0 & x|a|^2 \end{pmatrix}$$ \hspace{1cm} (A4)

The eigenvalues of \(\lambda_{AB}\) are \(x|a|^2\), \(x|b|^2\), and \((1-x\pm2x|ab|)/2\). This implies that \(\rho_{AB}\) is inseparable if \(x > (1+2|ab|)^{-1}\), exactly as predicted by Peres using the partial transpose of \(\rho_{AB}\). Since we are dealing with two qubits, this is the exact limit between separability and inseparability \([8]\).

Example 3: In the simpler case where \(\rho_{AB}\) is a mixture of a fraction \(x\) of the singlet state |Ψ\> = \(a|01\rangle + b|10\rangle\) and a fraction \((1-x)\) of the separable product state |00\>,

$$\rho_{AB} = x|\psi\rangle\langle\psi| + (1-x)|00\rangle\langle00| = \begin{pmatrix} 1-x & 0 & 0 & 0 \\ 0 & x/2 & -x/2 & 0 \\ 0 & -x/2 & x/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ \hspace{1cm} (A5)

we obtain

$$\lambda_{AB} = \begin{pmatrix} x/2 & 0 & 0 & 0 \\ 0 & 0 & x/2 & 0 \\ 0 & x/2 & 1-x & 0 \\ 0 & 0 & 0 & x/2 \end{pmatrix}$$ \hspace{1cm} (A6)

The latter matrix admits two eigenvalues equal to \(x/2\) and two eigenvalues equal to \((1-x \pm \sqrt{(1-x)^2 + x^2})/2\), so that its determinant is equal to \(-(x/2)^4\). Thus, this state is inseparable whenever \(x > 0\), as expected. (It is separable only if it is the pure product state |00\>.)

Example 4: Consider the class of 2-qubit inseparable states described by Horodecki et al. \([3]\), a mixture of two entangled states:

$$\rho_{AB} = p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2|$$ \hspace{1cm} (A7)

where |\psi_1\> = \(a|00\rangle + b|11\rangle\) and |\psi_2\> = \(a|01\rangle + b|10\rangle\), with \(a, b > 0\) and satisfying \(|a|^2 + |b|^2 = 1\). The joint density matrix

$$\rho_{AB} = \begin{pmatrix} pa^2 & 0 & 0 & pab \\ 0 & (1-p)a^2 & (1-p)ab & 0 \\ 0 & (1-p)ab & (1-p)b^2 & 0 \\ pab & 0 & 0 & pb^2 \end{pmatrix}$$ \hspace{1cm} (A8)
is mapped by $\Lambda$ to

$$\lambda_{AB} = \begin{pmatrix} (1 - p)b^2 & 0 & 0 & -pab \\ 0 & pb^2 & (p - 1)ab & 0 \\ 0 & (p - 1)ab & pa^2 & 0 \\ -pab & 0 & 0 & (1 - p)a^2 \end{pmatrix}$$  \hspace{1cm} (A9)$$

The latter matrix admits two eigenvalues equal to $\left( p \pm \sqrt{p^2 + 4a^2b^2(1 - 2p)} \right)/2$ and two eigenvalues equal to $\left( 1 - p \pm \sqrt{(1 - p)^2 + 4a^2b^2(2p - 1)} \right)/2$, so that its determinant is equal to $-a^4b^4(1 - 2p)^2$. This state is therefore inseparable whenever $ab \neq 0$ and $p \neq 1/2$, in perfect agreement with Ref. [3].

**Example 5:** Consider the $3 \times 3$ system in a weakly inseparable state introduced by Horodecki [4],

$$\rho_{AB} = \frac{1}{1 + 8a} \begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1 + a}{2}} & 0 & \sqrt{\frac{1 - a^2}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & a & 0 & \sqrt{\frac{1 - a^2}{2}} & 0 & \frac{1 + a}{2} \end{pmatrix}$$  \hspace{1cm} (A10)$$

where $a$ is a parameter ($a \neq 0, 1$). As shown in Ref. [3], the partial transpose of this state is positive, although $\rho_{AB}$ is inseparable, which makes the inseparability of $\rho_{AB}$ undetectable using Peres’ criterion. It is simple to check that the $\Lambda$-mapped operator

$$\lambda_{AB} = \frac{1}{1 + 8a} \begin{pmatrix} \frac{1 + 3a}{2} & 0 & \sqrt{\frac{1 - a^2}{2}} & 0 & -a & 0 & 0 & 0 & -a \\ 0 & 2a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{1 - a^2}{2}} & 0 & \frac{1 + 3a}{2} & 0 & \sqrt{\frac{1 - a^2}{2}} & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & \frac{3a}{2} & 2a & 0 & 0 & 0 & -a \\ 0 & 0 & 0 & \sqrt{\frac{1 - a^2}{2}} & 0 & \frac{1 + 3a}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2a & 0 & 0 \\ -a & 0 & 0 & 0 & -a & 0 & 0 & 0 & 2a \end{pmatrix}$$  \hspace{1cm} (A11)$$

is positive (with a trace equal to 2), so that $\Lambda$ cannot reveal the inseparability of $\rho_{AB}$ either. Accordingly, the determinant of $\lambda_{AB}$ is equal to $6a^7(1 - a)(5a + 3)/(1 + 8a)^9$ and thus positive. Note that the dual map also yields a positive operator $\check{\lambda}_{AB}$ (of trace 2), although the eigenvalues of $\check{\lambda}_{AB}$ are distinct from those of $\lambda_{AB}$, as is its determinant $\text{Det}(\check{\lambda}_{AB}) = 24a^7(1 - a^2)/(1 + 8a)^9$. This example emphasizes the fact that $\Lambda$ does not result in a sufficient separability condition for $3 \times 3$ systems, just as Peres’ condition [3].

**Example 6:** Following Horodecki [4], we consider a $2 \times 4$ system in an inseparable state

$$\rho_{AB} = \frac{1}{1 + 7b} \begin{pmatrix} b & 0 & 0 & 0 & b & 0 & 0 & b \\ 0 & b & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & b & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{1+b}{2}} & 0 & \sqrt{\frac{1-b}{2}} & 0 \\ b & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & b & 0 & \sqrt{\frac{1+b}{2}} & 0 & \sqrt{\frac{1-b}{2}} & 0 \end{pmatrix}$$  \hspace{1cm} (A12)$$

that has a positive partial transpose, where $b$ is a parameter ($b \neq 0, 1$). Applying $\Lambda$, we see that
\[ \lambda_{AB} = \frac{1}{1 + 7b} \begin{pmatrix} \frac{1 + b}{2} & 0 & 0 & \sqrt{1 - b^2} & 0 & -b & 0 & 0 \\ \frac{2}{b} & b & 0 & 0 & 0 & 0 & -b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & -b \\ 0 & 0 & 0 & \sqrt{1 - b^2} & 0 & 0 & 0 & 0 \\ -b & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & -b & 0 & 0 & 0 & 0 & b \end{pmatrix} \] 

(A13)

has eigenvalues 0, b, 2b, and \( \left(1 + 2b \pm \sqrt{(1 + 2b)^2 - 2b(3 + b)} \right)/2 \) so that it is always non-negative. Note that the spectrum of \( \lambda_{AB} \) is the same as the spectrum of the partial transpose \( \rho_{AB}^{T_A} \) (cf. [4]), as expected. This confirms that the condition based on \( \Lambda = \Gamma \otimes I \) and Peres’ separability condition are equivalent for 2 × n systems (when \( \Gamma \) is applied to the two-dimensional system and \( I \) to the n-dimensional one). In this example, applying the dual map \( \tilde{\Lambda} = I \otimes \Gamma \) yields a positive operator which traces to 3.

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