M-Branes on $k$-center Instantons

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Abstract

We present analytic solutions for membrane metric function based on transverse $k$-center instanton geometries. The membrane metric functions depend on more than two transverse coordinates and the solutions provide realizations of fully localized type IIA D2/D6 and NS5/D6 brane intersections. All solutions have partial preserved supersymmetries.

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1 Introduction

Fundamental M-theory in the low-energy limit is generally believed to be effectively described by $D = 11$ supergravity \[1, 2, 3\]. This suggests that brane solutions in the latter theory furnish classical soliton states of M-theory, motivating considerable interest in this subject. There is particular interest in finding $D = 11$ M-brane solutions that reduce to supersymmetric $p$-brane solutions (that saturate the Bogomol’nyi-Prasad-Sommerfield (BPS) bound) upon reduction to 10 dimensions. Some supersymmetric BPS solutions of two or three orthogonally intersecting 2-branes and 5-branes in $D = 11$ supergravity were obtained some years ago [4], and more such solutions have since been found [5].

Recently interesting new supergravity solutions for localized D2/D6, D2/D4, NS5/D6 and NS5/D5 intersecting brane systems were obtained [6, 7, 8, 9]. By lifting a D6 (D5 or D4)-brane to four-dimensional self-dual geometries embedded in M-theory, these solutions were constructed by placing M2- and M5-branes in different self-dual geometries. A special feature of this construction is that the solution is not restricted to be in the near core region of the D6 (or D5) brane, a feature quite distinct from the previously known solutions [10]. For all of the different BPS solutions, 1/4 of the supersymmetry is preserved as a result of the self-duality of the transverse metric. Moreover, in [11], partially localized D-brane systems involving D3, D4 and D5 branes were constructed. By assuming a simple ansatz for the eleven dimensional metric, the problem reduces to a partial differential equation that is separable and admits proper boundary conditions.

Motivated by this work, the aim of this paper is to construct the fully localized supergravity solutions of D2 (and NS5) intersecting D6 branes without restricting to the near core region of the D6 by reduction of ALE geometries lifted to M-theory.

In ref [12], the authors obtained several different supersymmetric BPS solutions of interest, based on transverse embedded 2-center Gibbons-Hawking space. All the solutions preserve eight supersymmetries and the metric functions depend on more than two transverse coordinates. The main motivation in this paper is extension of the results in [12], to embed multi-center (and in particular three-center) Gibbons-Hawking space in M-theory.

The outline of paper is as follows. In section 2 we discuss briefly the ALE geometries and present the eleven dimensional supergravity equations for M2-brane with an embedded transverse $k$-center instanton.

In section 3 we present the solutions to membrane equations of motion for a transverse embedded $k$-center Gibbons-Hawking space where $r$ is greater than a multiple of $a$.

In sections 4 we present membrane solutions for an embedded 3-center instanton and we find solutions in region $r > a$.

In section 5 we then discuss embedding products of Gibbons-Hawking instantons in M2-brane solutions as well as $M5$ brane solutions with one embedded Gibbons-Hawking instanton. We show all of the solutions presented in chapters 3, 4 and 5 preserve some of the supersymmetry.

In section 6 we consider the decoupling limit of our solutions and find evidence that in the limit of vanishing string coupling, the theory on the world-volume of the NS5-branes is a
new little string theory. Moreover, we apply T-duality transformations on type IIA solutions and find type IIB NS5/D5 intersecting brane solutions and discuss the decoupling limit of the solutions. We wrap up then by some concluding remarks and future possible research directions.

2 M-brane Solutions On \( k \)-center Instantons

We consider an M2-brane, given by the metric
\[
\text{ds}_{11}^2 = H(y, r, \theta)^{-2/3} (-dt^2 + dx_1^2 + dx_2^2) + H(y, r, \theta)^{1/3} (ds_4^2(y) + ds_4^2(r, \theta))
\]
and four-form field strength
\[
F_{tx_1x_2y} = -\frac{1}{2H^2} \frac{\partial H}{\partial y} \tag{2.2}
\]
\[
F_{tx_1x_2r} = -\frac{1}{2H^2} \frac{\partial H}{\partial r} \tag{2.3}
\]
\[
F_{tx_1x_2\theta} = -\frac{1}{2H^2} \frac{\partial H}{\partial \theta} \tag{2.4}
\]

For an M5-brane, the metric reads as
\[
\text{ds}^2 = H(y, r, \theta)^{-1/3} (-dt^2 + dx_1^2 + \ldots + dx_5^2) + H(y, r, \theta)^{2/3} (dy^2 + ds_4^2(r, \theta))
\]
and four-form field strength is
\[
F_{m_1\ldots m_4} = \alpha \epsilon_{m_1\ldots m_5} \partial^{m_5} H, \tag{2.6}
\]
where \( ds_4^2(y) \) and \( ds_4^2(r, \theta) \) are two four-dimensional (Euclideanized) metrics, depending on the non-compact coordinates \( y \) and \( r \), respectively and the quantity \( \alpha = \pm 1 \), which corresponds to an M5-brane and an anti-M5-brane respectively. The general solution, where the transverse coordinates are given by a flat metric, admits a solution with 16 Killing spinors \[13\]. As it is well known, the metric of \( k \)-center \( A \) series instantons could be written in closed form, given by:
\[
ds^2 = V^{-1}(dt + \vec{A} \cdot d\vec{x})^2 + V \gamma_{ij} dx^i \cdot dx^j \tag{2.7}
\]
where \( V, A_i \) and \( \gamma_{ij} \) are independent of \( t \) and \( \nabla V = \pm \nabla \times \vec{A} \); hence \( \nabla^2 V = 0 \). The most general solution for \( V \) is then \( V = \sum_{i=1}^{k} \frac{m}{||x-x_i||} \). The metric (2.7) describes the Gibbons-Hawking multi-center instantons. The \( k = 0 \) corresponds to flat space and \( k = 1 \) corresponds to Eguchi-Hanson metric. The different M2 and M5 brane solutions with one (or two) transverse \( k = 2 \) Gibbons-Hawking space have been constructed and studied extensively in \[12\]. In particular, the authors explicitly found exact supergravity solutions for fully localized D2/D6 and NS5/D6 brane intersections without restricting to the near core region of the D6 branes. The metric functions of all the solutions depend on three (or four) transverse
coordinates. The common feature of all of these solutions is that the brane function is a convolution of a decaying function with a damped oscillating one. The metric functions vanish far from the M2 and M5 branes and diverge near the brane cores.

In this paper we consider the extension of metrics (2.7) by considering

\[ V = \epsilon + \sum_{i=1}^{k} \frac{m_i}{|\vec{x} - \vec{x}_i|}. \] (2.8)

especially with \( k = 3 \). The hyper-Kahler metrics (2.7) with \( V \) pose a translational self-dual (or anti-self-dual) Killing vector \( K_\mu \), that means

\[ \nabla_\mu K_\nu = \pm \frac{1}{2} \sqrt{\det g} \rho^A_\mu \nabla_\rho K_\lambda. \] (2.9)

This (anti-) self-duality condition (2.9) implies the three-dimensional Laplace equation for \( V \) with solutions (2.8). For \( \epsilon \neq 0 \) in (2.8), the metrics (2.7) describe the asymptotically locally flat (ALF) multi Taub-NUT spaces. The removal of nut singularities implies \( m_i = m \) and \( t \) a periodic coordinate of period \( \frac{8\pi m}{k} \).

We consider the Gibbons-Hawking space with \( k = 3 \) and metric function \( V \) with \( \epsilon \neq 0 \), as a part of transverse space to M2 and M5-branes. The four-dimensional Gibbons-Hawking metric with \( k = N_1 + N_2 + 1 \) is

\[ ds^2_{GH} = V(r, \theta) \{dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\} + \frac{(d\psi + \omega(r, \theta)d\phi)^2}{V(r, \theta)} \] (2.10)

where

\[ \omega(r, \theta) = \sum_{m=-N_2}^{m=N_1} \frac{n(a + r \cos \theta)}{\sqrt{r^2 + (ma)^2 + 2mar \cos \theta}} \] (2.11)

\[ V(r, \theta) = \epsilon + \frac{n}{r} + \sum_{k=1}^{N_1} \frac{n}{\sqrt{r^2 + (ka)^2 + 2kar \cos \theta}} + \sum_{k=1}^{N_2} \frac{n}{\sqrt{r^2 + (ka)^2 - 2kar \cos \theta}}. \] (2.12)

For later convenience, we define \( N = \max(N_1, N_2) \). The eleven dimensional M2-brane with an embedded transverse Gibbons-Hawking space is given by the following metric

\[ ds^2_{11} = H(y, r, \theta)^{-2/3} (-dt^2 + dx_1^2 + dx_2^2) + H(y, r, \theta)^{1/3} (dy^2 + y^2d\Omega_3^2 + ds^2_{GH}) \] (2.13)

and non-vanishing four-form field components are given by eqs. (2.2), (2.3) and (2.4).

The metric (2.13) is a solution to the eleven dimensional supergravity equations provided \( H(y, r, \theta) \) is a solution to the differential equation

\[ \begin{align*}
2ry \sin \theta \frac{\partial H}{\partial r} + y \cos \theta \frac{\partial H}{\partial \theta} + r^2y \sin \theta \frac{\partial^2 H}{\partial r^2} + y \sin \theta \frac{\partial^2 H}{\partial \theta^2} & + \\
+ (r^2y \sin \theta \frac{\partial^2 H}{\partial y^2} + 3r^2 \sin \theta \frac{\partial H}{\partial \theta})V(r, \theta) & = 0.
\end{align*} \] (2.14)
We notice that solutions to the harmonic equation (2.14) determine the M2-brane metric function everywhere except at the location of the brane source. To maximize the symmetry of the problem, hence simplify the analysis, we consider the M2-brane source is placed at the point $y = r = 0$. Separating the coordinates by taking

$$H(y, r, \theta) = 1 + Q_{M2} Y(y) R(r, \theta)$$

(2.15)

where $Q_{M2}$ is the charge on the M2-brane, the equation (2.14) reduces to two separated differential equations for $Y(y)$ and $R(r, \theta)$. The solution of the differential equation for $Y(y)$ is

$$Y(y) \sim \frac{J_1(cy)}{y}$$

(2.16)

which has a damped oscillating behavior at infinity. The differential equation for $R(r, \theta)$ is

$$2 r \frac{\partial R(r, \theta)}{\partial r} + r^2 \frac{\partial^2 R(r, \theta)}{\partial r^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial R(r, \theta)}{\partial \theta} + \frac{\partial^2 R(r, \theta)}{\partial \theta^2} = c^2 r^2 V(r, \theta) R(r, \theta)$$

(2.17)

where $c$ is the separation constant.

3 Supergravity Solutions for $M_2$-Brane with Embedded $k$-center Instantons where $r > N a$

We try to find solutions to (2.17) in the presence of $k = N_1 + N_2 + 1$ charges (Figure 3.1) where the functional form of $V(r, \theta)$ is given by (2.12).

In general it is unlikely to find exact analytic solutions to (2.17), hence we need to make some approximations. In this section and appendix A, we find the solutions of (2.17) in region $r > N a$ and region $r < a$, respectively.

In region $r > N a$, the metric function (2.12) reduces to

$$V(r, \theta) \approx \epsilon + \frac{n(1 + N_1 + N_2)}{r} + \left[ \frac{N_2(N_2 + 1) - N_1(N_1 + 1)}{2} \right] \frac{an \cos \theta}{r^2}$$

(3.1)

where we keep the terms up to the second-order in $1/r$.

The separated differential equations after applying (3.1) are

$$r^2 \frac{d^2 f(r)}{dr^2} + 2r \frac{df(r)}{dr} - c^2(\epsilon r^2 + n(N_1 + N_2 + 1)r + M^2)f(r) = 0$$

(3.2)

$$\frac{d^2 g(\theta)}{d\theta^2} + \frac{\cos \theta}{\sin \theta} \frac{dg(\theta)}{d\theta} + c^2(M^2 + \tilde{m} \cos \theta)g(\theta) = 0$$

(3.3)

where

$$\tilde{m} = \frac{(N_1(N_1 + 1) - N_2(N_2 + 1))}{2} na$$

(3.4)
and the constants $c$ and $M$ are considered as real positive numbers.

The solution to equation (3.2) is given by

$$f(r) \sim \frac{1}{r} W_W\left(-\frac{cn(N_1 + N_2 + 1)}{2\sqrt{\epsilon}}, \frac{\sqrt{1 + 4M^2c^2}}{2}, 2c\sqrt{r}\right)$$  \hspace{1cm} (3.5)$$

where $W_W$ is a Whittaker function and the solution to equation (3.3) is given by

$$g(\xi) = C_{c,M} H_C(0, 0, 0, 2\tilde{m}c^2, -(M^2 + \tilde{m})c^2, \frac{\xi}{2}) +$$

$$C'_{c,M} H_C(0, 0, 0, 2\tilde{m}c^2, -(M^2 + \tilde{m})c^2, \frac{\xi}{2}) \int \frac{d\xi}{\xi(\xi - 2)H_C(0, 0, 0, 2\tilde{m}c^2, -(M^2 + \tilde{m})c^2, \frac{\xi}{2})^2}$$  \hspace{1cm} (3.6)$$

where $H_C$ is the Heun-C function (see appendix B), $\xi = 1 - \cos \theta$ and $C_{c,M}, C'_{c,M}$ are constants.

Figure (3.2) shows the behaviour of the first and second lines of (3.6) where the constants are set to $a = 1, \ n = 1, \ \tilde{m} = 12 \ (N_1 = 5 \text{ and } N_2 = 2), \ M = 1, \text{ and } c = 1$. As it’s shown in appendix C, the second line of (3.6) has a logarithmic divergence at $\xi = 1$. Knowing the general solution to (2.17), given by $R(r, \theta) = f(r)g(\xi)$; we can write the membrane metric function as

$$H(y, r, \theta) = 1 + Q_{M2} \int_0^\infty dc \int_0^\infty dM Y(y)f(r)g(\xi)$$  \hspace{1cm} (3.7)$$
in region \( r > N a \). As we notice, the solution (3.7) depends on two measure functions \( C_{c,M} \) and \( C'_{c,M} \). Each of these functions has dimension of inverse length to four. So, the measure functions should be considered as series expansions of the form \( e^{\alpha+4} M^\alpha \) where \( \alpha \in \mathbb{Z}_+ \).

In appendix A, the solutions to equation (2.17) are presented in other region of interest where \( r < a \). We are not able to find the analytic solutions in region \( a < r < Na \) for embedded \( k \)-center instantons where \( k > 3 \). For \( k = 2 \), the analytic solutions are already presented in [12] where \( r \) takes any value \( r \geq 0 \). In next section, we consider the case of embedded \( k = 3 \) center embedded Gibbons-Hawking space and we find the solutions on region \( r > a \).

4 Supergravity Solutions for \( M_2 \)-Brane with Embedded 3-center Instantons where \( r > a \)

To find the solutions to (2.17) over region \( r > a \), we define a pair of new independent coordinates \( \mu, \lambda \) given by

\[
\mu = \frac{R_2 + R_1}{2} = \frac{\sqrt{r^2 + a^2 + 2ar \cos \theta} + \sqrt{r^2 + a^2 - 2ar \cos \theta}}{2}, \quad (4.1)
\]

\[
\lambda = \frac{R_2 - R_1}{2} = \frac{\sqrt{r^2 + a^2 + 2ar \cos \theta} - \sqrt{r^2 + a^2 - 2ar \cos \theta}}{2}. \quad (4.2)
\]

A geometrical interpretation of \( \mu \) and \( \lambda \) can be obtained using Figure (4.1). According to Figure (4.1) we can easily show that \( |R_2 - R_1| < 2r < (R_1 + R_2) \) and \( |R_2 - R_1| < 2a < (R_1 + R_2) \) or in other words \( \lambda < r < \mu \) and \( \lambda < a < \mu \).

In region \( r > a \), we have \( R_1 \approx r - a \cos \theta \) and \( R_2 \approx r + a \cos \theta \). So, in terms of new coordinates...
\[ \mu \text{ and } \lambda, \text{ the equation } (2.17) \text{ turns into} \]
\[ (\mu^2 - a^2) \frac{\partial^2 R(\mu, \lambda)}{\partial \mu^2} + 2\mu \frac{\partial R(\mu, \lambda)}{\partial \mu} + (a^2 - \lambda^2) \frac{\partial^2 R(\mu, \lambda)}{\partial \lambda^2} - 2\lambda \frac{\partial R(\mu, \lambda)}{\partial \lambda} = \]
\[ c^2 \left[ \epsilon (\mu^2 - \lambda^2) + 3\mu n \right] R(\mu, \lambda). \quad (4.3) \]

This differential equation (4.3) separates into two ordinary second-order differential equations, given by
\[ (\mu^2 - a^2) \frac{d^2 G(\mu)}{d\mu^2} + 2\mu \frac{dG(\mu)}{d\mu} - c^2 (\epsilon \mu^2 + 3\mu n + M^2) G(\mu) = 0 \quad (4.4) \]
\[ (a^2 - \lambda^2) \frac{d^2 F(\lambda)}{d\lambda^2} - 2\lambda \frac{dF(\lambda)}{d\lambda} + c^2 (\epsilon \lambda^2 + M^2) F(\lambda) = 0. \quad (4.5) \]

For \( \mu \geq 2a \), introducing the new coordinate \( 0 \leq q \leq \tanh^{-1}(\frac{1}{2}) \) related to \( \mu \) by \( \mu = \frac{a}{\tanh(q)} \), the equation (4.4) changes to
\[ \frac{d^2 G(q)}{dq^2} - \left( \frac{M^2 c^2}{\sinh^2(q)} + \frac{\beta^2 \cosh(q)}{\sinh^3(q)} + \frac{\alpha^2 \cosh^2(q)}{\sinh^4(q)} \right) G(q) = 0 \quad (4.6) \]
where \( \beta^2 = 3nc^2a, \alpha^2 = \epsilon c^2a^2. \)

The solutions to (4.6) can be obtained as
\[ G_1(q) = g_1 q \mathcal{W}_n \left( -1/2 \frac{\beta^2}{\alpha}, 1/2 \sqrt{1 + 4 \gamma^2}, 2 \frac{\alpha}{q} \right) \quad (4.7) \]
where \( \gamma^2 = M^2 c^2 + 1/3 \alpha^2 \) and \( g_1 \) is a constant. For \( a < \mu \leq 2a \), the solutions to (4.4)
become

\[ G_2(z) = e^{-ca\sqrt{z}} \mathcal{H}_C \left( 4ca\sqrt{\epsilon}, 0, 0, 6c^2an, -c^2(3na + M^2 + \epsilon a^2), -\frac{z}{2} \right) \times \]

\[ (1 + g_2 \int \frac{e^{2ca\sqrt{z}}}{z(z+2)} \mathcal{H}_C \left( 4ca\sqrt{\epsilon}, 0, 0, 6c^2an, -c^2(3na + M^2 + \epsilon a^2), -\frac{z}{2} \right)^2 dz) \]  (4.8)

where \( z = \frac{\mu}{a} - 1 \) and \( g_2 \) is a constant. We should note by choosing proper values for \( g_1 \) and \( g_2 \), two solutions (4.7) and (4.8) are \( C^\infty \) continuous at \( \mu = 2a \).

For the second differential equation (4.5), the solutions are given by

\[ F(\lambda) = f_{cM} \mathcal{H}_C(0, -\frac{1}{2}, 0, -\frac{a^2c^2\epsilon}{4}, \frac{1}{4} - \frac{M^2c^2}{4}, \frac{\lambda^2}{a^2}) + f'_{cM} \mathcal{H}_C(0, \frac{1}{2}, 0, -\frac{a^2c^2\epsilon}{4}, \frac{1}{4} - \frac{M^2c^2}{4}, \frac{\lambda^2}{a^2}) \lambda \]  (4.9)

where \( f_{cM} \) and \( f'_{cM} \) are constants.

For completeness, we also numerically solve the equation (4.5) and the results are illustrated in Figure (4.2).

![Figure 4.2: Numerical solutions to equation (4.5).](image)

As the final result, the most general solution for the M2-brane metric function in region \( r > a \), is given by:

\[ H(y, r, \theta) = 1 + Q_{M2} \int_0^\infty dC \int_0^\infty dM \frac{J_1(cy)}{y} G_{c}(\mu) F(\lambda) \]  (4.10)
where \( G_t(\mu) = G_1(\tanh^{-1} (\frac{\mu}{p}))\theta(\frac{\mu}{p} - 2) + G_2(\frac{\mu}{p} - 1)\theta(2 - \frac{\mu}{p}) \).

Dimensional reduction of M2-brane metric (2.13) with the metric functions (2.15) along the coordinate \( \psi \) of the metric (2.10) gives type IIA supergravity metric

\[
d s_{10}^2 = H^{-1/2}(y, r, \theta) V^{-1/2}(r, \theta) \left( -dt^2 + dx_1^2 + dx_2^2 \right) + H^{1/2}(y, r, \theta) V^{1/2}(r, \theta) \left( dy^2 + y^2 d\Omega^2_3 \right) + H^{1/2}(y, r, \theta) V^{1/2}(r, \theta) (dr^2 + r^2 d\Omega^2_3) \tag{4.11}
\]

which describes a localized D2-brane at \( y = r = 0 \) along the world-volume of D6-brane. The only non-vanishing NSNS field in ten dimensions is given by

\[
\Phi = \frac{3}{4} \ln \left\{ \frac{H^{1/3}(y, r, \theta)}{V(r, \theta)} \right\} \tag{4.12}
\]

while the Ramond-Ramond (RR) fields are

\[
C_{\phi} = \omega(r, \theta) \tag{4.13}
\]

\[
A_{tx_1x_2} = \frac{1}{H(y, r, \theta)}. \tag{4.14}
\]

The intersecting configuration is BPS since it has been obtained by compactification along a transverse direction from the BPS membrane solution with harmonic metric function (2.15)[14].

5 M5-Brane Solutions, M2-Brane Solutions With Two Transverse Gibbons-Hawking Spaces and the Number of Preserved Supersymmetries

To embed the Gibbons-Hawking space into the eleven dimensional M5-brane metric, we consider

\[
d s_{11}^2 = H(y, r, \theta)^{-1/3} \left( -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 \right) + H(y, r, \theta)^{2/3} \left( dy^2 + ds^2_{GH} \right) \tag{5.1}
\]

with field strength components

\[
F_{\psi\phi y} = \frac{\alpha}{2} \sin(\theta) \frac{\partial H}{\partial \theta},
\]

\[
F_{\psi\phi y} = -\frac{\alpha}{2} r^2 \sin(\theta) \frac{\partial H}{\partial r},
\]

\[
F_{\psi\phi y} = \frac{\alpha}{2} r^2 \sin(\theta) V(r, \theta) \frac{\partial H}{\partial y}. \tag{5.2}
\]
The M5-brane corresponds to $\alpha = +1$; while the $\alpha = -1$ case corresponds to an anti-M5 brane.

The metric (5.1) along with (5.2) are solutions to the supergravity equations provided $H(y, r, \theta)$ satisfies the differential equation

$$2r \sin \theta \frac{\partial H}{V(r, \theta) \partial r} + \frac{\cos \theta}{V(r, \theta)} \frac{\partial H}{\partial \theta} + r^2 \sin \theta \frac{\partial^2 H}{\partial y^2} + \frac{\sin \theta}{V(r, \theta)} \left\{ \frac{\partial^2 H}{\partial \theta^2} + r^2 \frac{\partial^2 H}{\partial r^2} \right\} = 0.$$  

(5.3)

Upon substituting $H(y, r, \theta) = 1 + Q_{M5} Y(y) R(r, \theta)$, where $Q_{M5}$ is the charge on the M5-brane, the equation (5.3) straightforwardly separates. The solution to the differential equation for $Y(y)$ is a sine-harmonic function and the differential equation for $R(r, \theta)$ is the same equation as (2.17). Hence the most general M5-brane function, corresponding to embedded Gibbons-Hawking space with $k = 3$ is given by

$$H(y, r, \theta) = 1 + Q_{M5} \int_0^\infty dc \int_0^\infty dM \cos(cy + c') \times R(r, \theta)$$  

(5.4)

where $c'$ is a constant, $R(r, \theta)$ is given by (4.11) for region $r < a$ and (4.10) for region $r > a$, respectively. Reducing (5.1) to ten dimensions gives the following NSNS dilaton

$$\Phi = \frac{3}{4} \ln \left\{ \frac{H^{2/3}(y, r, \theta)}{V(r, \theta)} \right\}.$$  

(5.5)

The NSNS field strength of the two-form associated with the NS5-brane, is given by

$$\mathcal{H}(3) = F_{\phi y \psi} d\phi \wedge dy \wedge dr + F_{\phi y \theta \psi} d\phi \wedge dy \wedge d\theta + F_{\phi r \theta \psi} d\phi \wedge dr \wedge d\theta$$  

(5.6)

where the different components of 4-form $F$, are given by (5.2). The RR fields are

$$C_{(1)} = \omega(r, \theta)$$  

(5.7)

$$A_{\alpha \beta \gamma} = 0$$  

(5.8)

where $C_{\alpha}$ is the field associated with the D6-brane, and the metric in ten dimensions is given by:

$$ds^2_{10} = V^{-1/2}(r, \theta) \left( -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 \right) + H(y, r, \theta) V^{-1/2}(r, \theta) dy^2 + H(y, r, \theta) V^{-1/2}(r, \theta) (dr^2 + r^2 d\Omega_5^2).$$  

(5.9)

From (5.6), (5.7), (5.8) and the metric (5.9), we can see the above ten dimensional metric is an NS5⊥D6(5) brane solution. We have explicitly checked the BPS 10-dimensional metric (5.9), with the other fields (the dilaton (5.5), the 1-form field (5.7), and the NSNS field strength (5.6)) make a solution to the 10-dimensional supergravity equations of motion. In addition to the solutions presented in sections 3 and 4, we can also embed two four dimensional Gibbons-Hawking spaces into the eleven dimensional membrane metric. Here
we consider the embedding of two metrics of the form (2.10) with \( k = 3 \). The M-brane metric is

\[
ds_{11}^2 = H(y, \alpha, r, \theta)^{-2/3} \left(-dt^2 + dx_1^2 + dx_2^2\right) + H(y, \alpha, r, \theta)^{1/3} \left(ds_{GH(1)}^2 + ds_{GH(2)}^2\right) \tag{5.10}
\]

where \( ds_{GH(i)}, i = 1, 2 \) are two copies of the metric (2.10) with coordinates \((r, \theta, \phi, \psi)\) and \((y, \alpha, \beta, \gamma)\). The non-vanishing components of four-form field are

\[
F_{tx_1x_2} = -\frac{1}{2H^2} \frac{\partial H(y, \alpha, r, \theta)}{\partial x} \tag{5.11}
\]

where \( x = r, \theta, y, \alpha \). The metric (5.10) and four-form field (5.11) satisfy the eleven dimensional equations of motion if

\[
2ry \sin(\alpha) \sin(\theta) \{V(r, \theta) y \frac{\partial H}{\partial r} + V(y, \alpha) r \frac{\partial H}{\partial y}\} + \\
+ \sin(\alpha) y^2 \cos(\theta) V(r, \theta) \frac{\partial H}{\partial \theta} + r^2 \sin(\theta) \cos(\alpha) V(y, \alpha) \frac{\partial H}{\partial \alpha} + \\
+ r^2 \sin(\alpha) y^2 \sin(\theta) \{V(r, \theta) \frac{\partial^2 H}{\partial r^2} + V(y, \alpha) \frac{\partial^2 H}{\partial y^2}\} + \\
+ \sin(\theta) \sin(\alpha) \{r^2 V(y, \alpha) \frac{\partial^2 H}{\partial \alpha^2} + y^2 V(r, \theta) \frac{\partial^2 H}{\partial \theta^2}\} = 0 \tag{5.12}
\]

where \( V(y, \alpha) = \epsilon + \hat{n} \left(\frac{1}{y} + \frac{1}{\sqrt{y^2 + b^2}} + \frac{1}{\sqrt{y^2 + b^2 - 2by \cos(\alpha)}}\right)\). The equation (5.12) is separable if we set \( H(y, \alpha, r, \theta) = 1 + Q_{M2} R_1(y, \alpha) R_2(r, \theta) \). This gives two equations

\[
2x_i \frac{\partial R_i}{\partial x_i} + x_i \frac{\partial^2 R_i}{\partial x_i^2} + \cos y_i \frac{\partial R_i}{\partial y_i} + \frac{\partial^2 R_i}{\partial y_i^2} = u_i \epsilon x_i^2 V(x_i, y_i) R_i \tag{5.13}
\]

where \((x_1, y_1) = (y, \alpha)\) and \((x_2, y_2) = (r, \theta)\). There is no summation on index \( i \) and \( u_1 = +1, u_2 = -1 \), in equation (5.13). We already know the solutions to the two differential equations (5.13) as given by (A.11) for region \( r < a \) and (4.10) region \( r > a \). So the most general solution to (5.12) is

\[
H(y, \alpha, r, \theta) = 1 + Q_{M2} \int_0^\infty dc \int_0^\infty dM \int_0^\infty d\tilde{M} R(y, \alpha) \tilde{R}(r, \theta). \tag{5.14}
\]

We can choose to compactify down to ten dimensions by compactifying on either \( \psi \) or \( \gamma \) coordinates. In the first case, we find the type IIA string theory with the only non-vanishing NSNS field as

\[
\Phi = \frac{3}{4} \ln \left(\frac{H^{1/3}}{V(r, \theta)}\right) \tag{5.15}
\]

and RR fields

\[
C_\phi = \omega(r, \theta) \tag{5.16}
\]

\[
A_{tx_1x_2} = H(y, \alpha, r, \theta)^{-1}. \tag{5.17}
\]
The metric is given by

\[
ds_{10}^2 = H(y, \alpha, r, \theta)^{-1/2} V(r, \theta)^{-1/2} \left( -dt^2 + dx_1^2 + dx_2^2 \right) + \\
+ H(y, \alpha, r, \theta)^{1/2} V(r, \theta)^{-1/2} \left( ds_{GH(1)}^2 \right) + \\
+ H(y, \alpha, r, \theta)^{1/2} V(r, \theta)^{1/2} \left( dr^2 + r^2 \left( d\theta^2 + \sin^2(\theta) d\phi^2 \right) \right). \tag{5.18}
\]

In the latter case, the type IIA fields and metric are in the same form as (5.15), (5.16), (5.17) and (5.18), just by replacements \((r, \theta, \phi, \psi) \leftrightarrow (y, \alpha, \beta, \gamma)\). In either cases, we get a fully localized D2/D6 brane system. We can further reduce the metric (5.18) along the \(\gamma\) direction of the first Gibbons-Hawking space. However the result of this compactification is not the same as the reduction of the M-theory solution (5.10) over a torus, which is compactified type IIB theory. The reason is that to get the compactified type IIB theory, we should compactify the T-dual of the IIA metric (5.18) over a circle, and not directly compactify the 10D IIA metric (5.18) along the \(\gamma\) direction. We note also an interesting result in reducing the 11D metric (5.10) along the \(\psi\) (or \(\gamma\)) direction of the \(GH(1)\) (or \(GH(2)\)) in large radial coordinates. As \(y\) (or \(r\)) \(\to \infty\) the transverse geometry in (5.10) locally approaches \(\mathbb{R}^3 \otimes S^1 \otimes GH(2)\) (or \(GH(1) \otimes \mathbb{R}^3 \otimes S^1\)). Hence the reduced theory, obtained by compactification over the circle of the Gibbons-Hawking, is IIA. Then by T-dualization of this theory (on the remaining \(S^1\) of the transverse geometry), we find a type IIB theory which describes the D5 defects. The solutions (5.10) (with \(\epsilon = 0\) or \(\epsilon \neq 0\)) are BPS and also preserve 1/4 of the supersymmetry similar to all other solutions in this paper. Generically a configuration of \(n\) intersecting branes preserves \(\frac{1}{2^n}\) of the supersymmetry. In general, the Killing spinors are projected out by product of Gamma matrices with indices tangent to each brane. If all the projections are independent, then \(\frac{1}{2^n}\)-rule can give the right number of preserved supersymmetries. On the other hand, if the projections are not independent then \(\frac{1}{2^n}\)-rule can’t be trusted. There are some important brane configurations when the number of preserved supersymmetries is more than that by \(\frac{1}{2^n}\)-rule \cite{15,16}. The number of non-trivial solutions to the Killing spinor equation

\[
\partial_M \epsilon + \frac{1}{4} \omega_{abM} \Gamma^{ab} \epsilon + \frac{1}{144} \Gamma^M_{npqr} F_{npqr} \epsilon - \frac{1}{18} \Gamma^{npqr} F_{npqr} \epsilon = 0 \tag{5.19}
\]

determine the amount of supersymmetry of the solution where the indices \(M, N, P, \ldots\) are eleven dimensional world indices and \(a, b, \ldots\) are eleven dimensional non-coordinate tangent space indices. In \cite{12}, the authors presented the calculations explicitly to find how many supersymmetries are preserved for M2 and M5 brane solutions where the transverse space contains at least one Gibbons-Hawking of \(k = 2\) geometry. The explicit calculation enjoys the independence on explicit form of metric function \(V(r, \theta)\) and \(\omega(r, \theta)\). Hence we conclude all our solutions presented in previous sections preserve eight supersymmetries. In fact, half of the supersymmetry is removed by the projection operator that is due to the presence of the brane, and another half is removed due to the self-dual nature of the Gibbons-Hawking metric with \(k = 3\) or in general for any value of \(k\).
6 Decoupling Limits of Solutions

In this section we consider the decoupling limits of the solutions in different regions which are presented in sections 3, 4, 5 and appendix A. Since the specifics of calculating the decoupling limit are shown in detail elsewhere (see for example [17]), so we will only provide a brief outline here. The process is the same for all cases, so we will also only provide specific examples of a few of the solutions in different regions that presented in sections 3, 4, 5 and appendix A.

At low energies, the dynamics of the D2 brane decouple from the bulk, with the region close to the D6 brane corresponding to a range of energy scales governed by the IR fixed point [18]. For D2 branes localized on D6 branes, this corresponds in the field theory to a range of energy scales governed by the IR fixed point. We note in (6.2) we use \( \ell_s^{-1} \) = fixed. In this limit the gauge couplings in the bulk go to zero, so the dynamics decouple there. In each of our cases above, we scale the coordinates \( y \) and \( r \) given by \( y = Y \ell_s^2 \) and \( r = U \ell_s^2 \) respectively, such that \( Y \) and \( U \) are fixed. We note that this will change the harmonic function of the D6 brane in the Gibbons-Hawking case \( (k = 3) \) to the following

\[
V(U, \theta) = \epsilon + g_{YM2}^2 N_6 \left\{ \frac{1}{U} + \frac{1}{\sqrt{U^2 + A^2 + 2AU \cos \theta}} + \frac{1}{\sqrt{U^2 + A^2 - 2AU \cos \theta}} \right\} \quad (6.1)
\]

where we rescale \( a \) to \( a = A \ell_s^2 \) and generalize to the case of \( N_6 \) D6 branes. We also recall that to avoid any conical singularity, we should have \( n_1 = n_2 = n_3 = n \), hence the asymptotic radius of the 11th dimension is \( R_\infty = n = g_s \ell_s \). We show that the metric function \( H(y, r, \theta) \) always scales as \( H(Y, U, \theta) = \ell_s^{-4} h(Y, U, \theta) \) if the coefficients of solutions in different regions, obey some specific scaling. The scaling behavior of \( H(Y, U, \theta) \) causes then the D2-brane to warp the ALE region and the asymptotically flat region of the D6-brane geometry. As the first example, we consider the solutions given by (A.6) and (A.7) and calculate \( h(Y, U, \theta) \).

After scaling, we get

\[
h(Y, U, \theta) = 32 \pi^2 N_2 g_{YM}^4 \int_0^\infty dC \int_0^\infty M dM J_1(CY) \left[ \frac{CN}{2 \sqrt{\epsilon + A}} \right] \left[ \sqrt{1 + 4M^2C^2} \right] \frac{d\zeta}{2C \sqrt{\epsilon + AU}}
\]

\[
\times e^{-\tilde{\beta} \zeta} F(\Xi, 1 - \Xi, 1, 1, 1, \frac{1}{2}(1 - \zeta))(G_1 + G_2) \int \frac{d\zeta}{(C^2 - 1)F(\Xi, 1 - \Xi, 1, 1, \frac{1}{2}(1 - \zeta))^2}
\]

(6.2)

where we scale the coefficients to \( \tilde{\beta} = BNC^2A, C = c \ell_s^2 \) and \( M = M \ell_s^2 \). Moreover \( N = \frac{4}{\ell_s^2}, a = \ell_s^2A \) and \( \Xi = \frac{1}{2} + \frac{\sqrt{1 + 4M^2C^2}}{2} \) or \( \Xi = -\frac{1}{2} - \frac{\sqrt{1 + 4M^2C^2 - 4\tilde{\beta}^2}}{2} \) and \( \tilde{A} = (\sum_{k=1}^{N_1} \frac{1}{k} + \sum_{k=1}^{N_2} \frac{1}{k}) \times \frac{N}{A} \). We should note in (6.2) we use \( \ell_p = g_s^{1/3} \ell_s \) to rewrite \( Q_{M2} = 32 \pi^2 N_2 \ell_p^6 \) in terms of \( \ell_s \) given by
\[ Q_{M2} = 32\pi^2 N_2 g_{Y, M2}^4 \ell_s^2. \] For the second example, we consider solutions given by (1.10). The rescaled metric function \( h(Y, U, \theta) \) read as

\[ h(Y, U, \theta) = 32\pi^2 N_2 g_{Y, M2}^4 \int_0^\infty dC \int_0^{\infty} M dM \frac{J_1(CY)}{Y} G_t(\Psi) F(\Lambda). \tag{6.3} \]

In (6.3), \( G_t(\Psi) = G_1(\tanh^{-1}(\frac{A}{r}) \Theta(\frac{A}{r} - 2) + G_2(Z) \Theta(2 - \frac{A}{r}) \) in terms of scaled coordinate \( \Psi = \frac{A}{r}^\frac{5}{2}, \) where

\[ G_2(Z) = e^{-CA(\sqrt{\epsilon}, 0, 0, 6C^2 \epsilon, -A^2 \epsilon, -\frac{Z}{2}} \times \]

\[ (1 + G_2 \int \frac{e^{2CA(\sqrt{\epsilon}, 0, 0, 6C^2 \epsilon, -A^2 \epsilon, -\frac{Z}{2}}^2 dZ)}{Z^2} \tag{6.4} \]

The scaled quantities in (6.4) are \( a = A\ell_s^2, \) \( n = N\ell_s^2. \) \( G_2 = g_2 \) and \( Z \) is given by \( Z = \frac{A}{r} - 1. \) The other part of integrand in (6.3) is

\[ F(\Lambda) = F_{CM} H_C(0, -\frac{1}{2}, 0, -\frac{A^2 C^2 \epsilon}{4}, -\frac{M^2 C^2}{4}, \frac{\Lambda^2}{A^2}) + F_{CM}' H_C(0, 1, 0, -\frac{A^2 C^2 \epsilon}{4}, -\frac{M^2 C^2}{4}, \frac{\Lambda^2}{A^2}) \tag{6.5} \]

where \( \lambda = A\ell_s^2, F_{CM} = f_{eM} \ell_6^6 \) and \( F_{CM}' = f_{eM}' \ell_8. \)

In all other cases we can show we have the same scaling behavior as \( h(Y, U, \theta) = \ell_s^4 H(Y, U, \theta). \) In any case, the respective ten-dimensional supersymmetric metric (1.11) scales as

\[ \frac{ds_{10}^2}{\ell_s^2} = h^{-1/2}(Y, U, \theta)V^{-1/2}(U, \theta) (-dt^2 + dx_1^2 + dx_2^2) + \]

\[ + h^{1/2}(Y, U, \theta)V^{-1/2}(U, \theta) \{(dy^2 + Y^2 d\Omega_3^2) + V(U, \theta)(dU^2 + U^2 d\Omega_3^2) \} \tag{6.6} \]

that shows only one overall normalization factor of \( \ell_s^2 \) in the metric (6.6). This is the expected result for a solution that is a supergravity dual of a QFT. We now consider an analysis of the decoupling limits of M5-brane solution given by metric function (5.4).

At low energies, the dynamics of IIA NS5-branes will decouple from the bulk [10]. Near the NS5-brane horizon \( (H >> 1) \), we are interested in the behavior of the NS5-branes in the limit where string coupling vanishes \( g_s \to 0 \) while \( \ell_s = \text{fixed}. \) In these limits, we rescale the radial coordinates by \( Y = \frac{y}{g_s \ell_s} \) and \( U = \frac{u}{g_s \ell_s} \) such that they can be kept fixed. This causes the Gibbons-Hawking harmonic function of the D6-brane solution (3.9), change to

\[ V(U, \theta) = \epsilon + \frac{N_6}{\ell_s} \left\{ \frac{1}{U} + \frac{1}{\sqrt{U^2 + A^2 + 2AU \cos \theta}} + \frac{1}{\sqrt{U^2 + A^2 - 2AU \cos \theta}} \right\} \tag{6.7} \]

where we generalize to \( N_6 \) D6-branes and rescale \( a = A\ell_s^2 g_s. \)

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Similar to what we did for M2-branes, we easily can show the harmonic functions for M5-branes (5.4), rescale according to $H(Y, U, \theta) = g_s^{-2} h(Y, U, \theta)$ such that $h(Y, U, \theta)$ doesn't have any $g_s$ dependence [12].

As a result, in decoupling limit, the ten-dimensional metric (5.9) becomes,

$$ds_{10}^2 = V^{-1/2}(U, \theta) \left( -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 \right) + \ell_s^4 \left\{ h(Y, U, \theta)V^{-1/2}(U, \theta) dY^2 + h(Y, U, \theta) V^{1/2}(U, \theta) \left( dU^2 + U^2 d\Omega_2^2 \right) \right\}.$$  \hspace{1cm} (6.8)

In the limit of vanishing $g_s$ with fixed $\ell_s$, the decoupled free theory on NS5-branes should be a little string theory [20] (i.e. a 6-dimensional non-gravitational theory in which modes on the 5-brane interact amongst themselves, decoupled from the bulk). We note that our NS5/D6 system is obtained from M5-branes by compactification on a circle of self-dual transverse geometry. Hence the IIA solution has T-duality with respect to this circle. The little string theory inherits the same T-duality from IIA string theory, since taking the limit of vanishing string coupling commutes with T-duality. Moreover T-duality exists even for toroidally compactified little string theory. In this case, the duality is given by an $O(d, d, Z)$ symmetry where $d$ is the dimension of the compactified toroid. These are indications that the little string theory is non-local at the energy scale $\ell_s^{-1}$ and in particular in the compactified theory, the energy-momentum tensor can’t be defined uniquely [21].

As the last case, we consider the analysis of the decoupling limits of the IIB solution that can be obtained by T-dualizing the compactified M5-brane solution (5.1). The type IIA NS5⊥D6(5) configuration is given by the metric (5.9) and fields (5.5), (5.6), (5.7) and (5.8).

We apply the T-duality [22] in the $x_1$-direction of the metric (5.9), that yields gives the IIB dilaton field

$$\Phi = \frac{1}{2} \ln \frac{H}{f} \hspace{1cm} (6.9)$$

the 10D type IIB metric, as

$$\tilde{ds}_{10}^2 = V^{-1/2}(r, \theta) \left( -dt^2 + V(r, \theta) dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 \right) +\hspace{1cm}$$

$$H(y, r, \theta)V^{-1/2}(r, \theta) dy^2 + H(y, r, \theta) V^{1/2}(r, \theta) \left( dr^2 + r^2 d\Omega_2^2 \right) \hspace{1cm} (6.10)$$

The metric (6.10) describes a IIB NS5⊥D5(4) brane configuration (along with the dualized dilaton, NSNS and RR fields).

At low energies, the dynamics of IIB NS5-branes will decouple from the bulk. Near the NS5-brane horizon ($H > > 1$), the field theory limit is given by

$$g_{YM} = \ell_s = \text{fixed} \hspace{1cm} (6.11)$$

The harmonic function of the D5-brane is

$$V(r, \theta) = \epsilon + \frac{N_5}{g_{YM}} \left\{ \frac{1}{U} + \frac{1}{\sqrt{U^2 + A^2 + 2AU \cos \theta}} + \frac{1}{\sqrt{U^2 + A^2 - 2AU \cos \theta}} \right\} \hspace{1cm} (6.12)$$

where $N_5$ is the number of D5-branes.
The harmonic function of the NS5⊥D5 system (6.10), rescales according to $H(Y,U,\theta) = g_s^{-2}h(Y,U,\theta)$, and the ten-dimensional metric (6.10), in the decoupling limit, becomes

$$
\tilde{ds}_{10}^2 = V^{-1/2}(U,\theta) \left(-dt^2 + V(U,\theta)dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2\right) + 
+ g_{YM}^2h(Y,U,\theta)\{V^{-1/2}(U,\theta)dY^2 + V^{1/2}(U,\theta)\left(dU^2 + U^2d\Omega_2^2\right)\}.
\quad (6.13)
$$

The decoupling limit illustrates that the decoupled theory in the low energy limit is super Yang-Mills theory with $g_{YM} = \ell_s$. In the limit of vanishing $g_s$ with fixed $\ell_s$, the decoupled free theory on IIB NS5-branes (which is equivalent to the limit $g_s \to \infty$ of decoupled S-dual of the IIB D5-branes) reduces to a IIB (1,1) little string theory with eight supersymmetries.

7 Concluding Remarks

The central thrust of this paper is the construction of supergravity solutions for fully localized D2/D6 and NS5/D6 brane intersections without restricting to the near core region of the D6 branes. The metric functions of these solutions is the dependence of the metric function depend to three (and four) transverse coordinates. These solutions are new M2 and M5 brane metrics that are presented in equations (3.7), (4.10), (5.4) and (5.14), which are the main results of this paper. The common feature of all of these solutions is that the brane function is a convolution of an decaying function with a damped oscillating one. The metric functions vanish far from the M2 and M5 branes and diverge near the brane cores.

Dimensional reduction of the M2 solutions to ten dimensions gives us intersecting IIA D2/D6 configurations that preserve 1/4 of the supersymmetry. For the M5 solutions, dimensional reduction yields IIA NS5/D6 brane systems overlapping in five directions. The latter solutions also preserve 1/4 of the supersymmetry and in both cases the reduction yields metrics with acceptable asymptotic behaviors.

We considered the decoupling limit of our solutions and found that D2 and NS5 branes can decouple from the bulk, upon imposing proper scaling on some of the coefficients in the integrands.

In the case of M2 brane solutions; when the D2 brane decouples from the bulk, the theory on the brane is 3 dimensional $\mathcal{N} = 4$ $SU(N_2)$ super Yang-Mills (with eight supersymmetries) coupled to $N_6$ massless hypermultiplets [23]. This point is obtained from dual field theory and since our solutions preserve the same amount of supersymmetry, a similar dual field description should be attainable.

In the case of M5 brane solutions; the resulting theory on the NS5-brane in the limit of vanishing string coupling with fixed string length is a little string theory. In the standard case, the system of $N_5$ NS5-branes located at $N_6$ D6-branes can be obtained by dimensional reduction of $N_5N_6$ coinciding images of M5-branes in the flat transverse geometry. In this case, the world-volume theory (the little string theory) of the IIA NS5-branes, in the absence of D6-branes, is a non-local non-gravitational six dimensional theory [24]. This theory has (2,0) supersymmetry (four supercharges in the 4 representation of Lorentz symmetry $Spin(5,1)$) and an R-symmetry $Spin(4)$ remnant of the original ten dimensional Lorentz
symmetry. The presence of the D6-branes breaks the supersymmetry down to (1,0), with eight supersymmetries. Since we found that some of our solutions preserve 1/4 of supersymmetry, we expect that the theory on NS5-branes is a new little string theory. By T-dualization of the 10D IIA theory along a direction parallel to the world-volume of the IIA NS5, we find a IIB NS5⊥D5(4) system, overlapping in four directions. The world-volume theory of the IIB NS5-branes, in the absence of the D5-branes, is a little string theory with (1,1) supersymmetry. The presence of the D5-brane, which has one transverse direction relative to NS5 world-volume, breaks the supersymmetry down to eight supersymmetries. This is in good agreement with the number of supersymmetries in 10D IIB theory: T-duality preserves the number of original IIA supersymmetries, which is eight. Moreover we conclude that the new IIA and IIB little string theories are T-dual: the actual six dimensional T-duality is the remnant of the original 10D T-duality after toroidal compactification.

A useful application of the exact M-brane solutions in our paper is to employ them as supergravity duals of the NS5 world-volume theories with matter coming from the extra branes. More specifically, these solutions can be used to compute some correlation functions and spectrum of fields of our new little string theories.

In the standard case of $A_{k-1}$ (2,0) little string theory, there is an eleven dimensional holographic dual space obtained by taking appropriate small $g_s$ limit of an M-theory background corresponding to M5-branes with a transverse circle and $k$ units of 4-form flux on $S^3 \otimes S^1$. In this case, the supergravity approximation is valid for the (2,0) little string theories at large $k$ and at energies well below the string scale. The two point function of the energy-momentum tensor of the little string theory can be computed from classical action of the supergravity evaluated on the classical field solutions [20].

Near the boundary of the above mentioned M-theory background, the string coupling goes to zero and the curvatures are small. Hence it is possible to compute the spectrum of fields exactly. In [21], the full spectrum of chiral fields in the little string theories was computed and the results are exactly the same as the spectrum of the chiral fields in the low energy limit of the little string theories. Moreover, the holographic dual theories can be used for computation of some of the states in our little string theories.

We conclude with a few comments about possible directions for future work. Investigation of the different regions of the metric (5.1) or alternatively the 10D string frame metric (6.8) with a dilaton for small and large Higgs expectation value $U$ would be interesting, as it could provide a means for finding a holographical dual relation to the new little string theory we obtained. Moreover, the Penrose limit of the near-horizon geometry may be useful for extracting information about the high energy spectrum of the dual little string theory [24]. The other open issue is the possibility of the construction of a pp-wave spacetime which interpolates between the different regions of the our new IIA NS5-branes.

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A Solutions around origin

In this appendix, we present the solutions for M-brane metric functions in near region where $r < a$. In this region, we notice

$$V(r, \theta) \approx \epsilon + \frac{n}{r} + \sum_{k=1}^{N_1} \frac{n}{ka} + \sum_{k=1}^{N_2} \frac{nr \cos \theta}{a^2} \left[ \sum_{k=1}^{N_2} \frac{1}{k^2} - \sum_{k=1}^{N_1} \frac{1}{k^2} \right]$$

(A.1)

and the equation of motion (2.17) becomes

$$2r \frac{\partial R(r, \theta)}{\partial r} + r^2 \frac{\partial^2 R(r, \theta)}{\partial r^2} + \frac{\cos \theta \partial R(r, \theta)}{\sin \theta} + \frac{\partial^2 R(r, \theta)}{\partial \theta^2} = c^2 r^2 \left( \epsilon + A + \frac{n}{r} + \frac{nr \cos \theta}{a^2} \right) R(r, \theta)$$

(A.2)

where we assume $B \neq 0$ ($N_1 \neq N_2$). If $B = 0$, we should consider higher order terms in (A.1) which we will consider the case of $N_1 = N_2 = N_0$ later in this appendix. We redefine $R(r, \theta)$ as follows

$$R(r, \theta) = e^{\beta \cos \theta} \Psi(r, \theta)$$

where $\beta = \frac{naBc^2}{2}$. As we already know ($\frac{r}{a} < 1$), so the partial differential equation in terms of $\Psi(r, \theta)$ approximates to be

$$2r \frac{\partial \Psi(r, \theta)}{\partial r} + r^2 \frac{\partial^2 \Psi(r, \theta)}{\partial r^2} + \left( \frac{\cos \theta}{\sin \theta} - 2\beta \sin \theta \right) \frac{\partial \Psi(r, \theta)}{\partial \theta} + \frac{\partial^2 \Psi(r, \theta)}{\partial \theta^2} + (\beta \sin \theta)^2 \Psi(r, \theta) - 2\beta \cos \theta \Psi(r, \theta) - c^2 \left[ (\epsilon + A)r^2 + nr \right] \Psi(r, \theta) = 0.$$  

(A.3)

The partial differential equation (A.3) separates into

$$r^2 \frac{d^2 f(r)}{dr^2} + 2r \frac{df(r)}{dr} - c^2 \left[ (\epsilon + A)r^2 + nr + M^2 \right] f(r) = 0$$

(A.4)

$$\frac{d^2 g(\theta)}{d\theta^2} + \left( \frac{\cos \theta}{\sin \theta} - 2\beta \sin \theta \right) \frac{dg(\theta)}{d\theta} + (M^2 c^2 - 2\beta \cos \theta + (\beta \sin \theta)^2) g(\theta) = 0.$$  

(A.5)

Solution to (A.4) is a Whittaker M function

$$f(r) = \frac{f_0}{r} \mathcal{W}_M \left( -\frac{cn}{2\sqrt{\epsilon + A}}, \sqrt{1 + 4M^2c^2}, 2c\sqrt{\epsilon + Ar} \right).$$

(A.6)
The solutions to \((A.3)\), in terms of coordinate \(\zeta = \cos \theta\), are given by
\[
g(\zeta) = e^{-\beta \xi} F(\nu, 1 - \nu, 1, \frac{1}{2}(1 - \zeta)) \left[ g_1 + g_2 \int \frac{d\zeta}{(\zeta^2 - 1) F(\nu, 1 - \nu, 1, \frac{1}{2}(1 - \zeta))^2} \right] \quad (A.7)
\]
where \(F\) is the hypergeometric function and \(\nu = \frac{1}{2} + \frac{\sqrt{1 + 4M^2c^2}}{2}\). The solution can be expressed in the series forms as
\[
g(\xi) = C_1 \left( 1 + \frac{2\beta - M^2c^2}{2} \xi + \cdots \right) + C_2 \left( \ln(\xi) \left( 1 + \frac{2\beta - M^2c^2}{2} \xi + \cdots \right) + \left( \frac{1}{2} + M^2c^2 \right) \xi + \cdots \right) \quad (A.8)
\]
where \(\xi = 1 - \zeta\).

As we mentioned before, if \(N_1 = N_2 = N_0\), we should keep higher order terms in \((A.1)\). Starting from \((2.17)\) and changing the coordinates to
\[
x = \cos(\theta), \quad z = \frac{r}{a}
\]
we get
\[
z^2 \frac{\partial^2 R(z, x)}{\partial z^2} + 2z \frac{\partial R(z, x)}{\partial z} + (1 - x^2) \frac{\partial^2 R(z, x)}{\partial x^2} - 2x \frac{\partial R(z, x)}{\partial x} - \left[ c^2(a^2 \epsilon + 2n a A_0) z^2 + n a c^2 z + n a B_0 c^2 z^4 (3x^2 - 1) \right] R(z, x) = 0 \quad (A.10)
\]
where \(A_0 = \sum_{k=1}^{N_0} \frac{1}{k}\) and \(B_0 = \sum_{k=1}^{N_0} \frac{1}{k^2}\). To solve \((A.10)\), we introduce the function \(\Omega(x, z)\) as follows
\[
R(z, x) = e^{\beta z} \Omega(z, x) \quad (A.11)
\]
where \(\beta = \sqrt{3n a B_0 c}\). Hence the differential equation \((A.10)\) in terms of \(\Omega(z, x)\) becomes
\[
(2\beta - 2x - 2x^2 \beta) \frac{\partial}{\partial x} \Omega(z, x) + 2z \frac{\partial}{\partial z} \Omega(z, x) + (1 - x^2) \frac{\partial^2}{\partial x^2} \Omega(z, x) + z^2 \frac{\partial^2}{\partial z^2} \Omega(z, x) + (\beta^2 - 2\beta x - x^2 \beta^2) \Omega(z, x) + (n a c^2 B_0 z^4 - n a c^2 z + (-c^2 a^2 \epsilon - 2c^2 n a A) z^2) \Omega(z, x) = 0. \quad (A.12)
\]
Separating the variables in \(\Omega(z, x)\) by \(\Omega(z, x) = \Upsilon(z) \Theta(x)\) and substituting into \((A.12)\), we find two separated second order differential equations for \(\Theta(x)\) and \(\Upsilon(z)\), as follows
\[
(1 - x^2) \frac{d^2}{dx^2} \Theta(x) + 2 \left( (1 - x^2) \beta - x \right) \frac{d}{dx} \Theta(x) - (2x\beta + \beta^2 x^2 - M^2c^2 - \beta^2) \Theta(x) = 0 \quad (A.13)
\]
\[
z^2 \frac{d^2}{dz^2} \Upsilon(z) + 2z \frac{d}{dz} \Upsilon(z) + (-M^2c^2 + n a c^2 B_0 z^4 - n a c^2 z + (-c^2 a^2 \epsilon - 2c^2 n a A) z^2) \Upsilon(z) = 0. \quad (A.14)
\]
The solutions to (A.13) are given by (A.7) as \( \Theta(x) = g(\zeta)|_{\zeta=x} \) while the solutions to (A.14) can be written as

\[
\Upsilon(z) = z^{-\frac{\sqrt{4M^2c^2+1}}{2}}\Upsilon_1(z) + z^{\frac{\sqrt{4M^2c^2+1}}{2}}\Upsilon_2(z)
\]  

(A.15)

where \( \Upsilon_i(z) \), \( i = 1, 2 \) are two independent polynomials of \( z \).

B The Heun-C functions

The Heun-C function \( H\_C(\alpha, \beta, \gamma, \delta, \lambda, z) \) is the solution to the confluent Heun’s differential equation \[26\]

\[
H''_C + \left( \alpha + \frac{\beta + 1}{z} + \frac{\gamma + 1}{z - 1} \right) H'_C + \left( \frac{\mu}{z} + \frac{\nu}{z - 1} \right) H_C = 0 \tag{B.1}
\]

where \( \mu = \frac{\alpha - \beta - \gamma + \alpha \beta - \beta \gamma}{2} - \lambda \) and \( \nu = \frac{\alpha + \beta + \gamma + \alpha \beta + \beta \gamma}{2} + \delta + \lambda \). The equation (B.1) has two regular singular points at \( z = 0 \) and \( z = 1 \) and one irregular singularity at \( z = \infty \). The \( H_C \) function is regular around the regular singular point \( z = 0 \) and is given by \( H_C = \sum_{n=0}^{\infty} h_n(\alpha, \beta, \gamma, \delta, \lambda) z^n \), where \( h_0 = 1 \). The series is convergent on the unit disk \( |z| < 1 \) and the coefficients \( h_n \) are determined by the recurrence relation

\[
h_n = \Theta_n h_{n-1} + \Phi_n h_{n-2} \tag{B.2}
\]

where we set \( h_{-1} = 0 \) and

\[
\Theta_n = \frac{2n(n - 1) + (1 - 2n)(\alpha - \beta - \gamma) + 2\lambda - \alpha \beta + \beta \gamma}{2n(n + \beta)} \tag{B.3}
\]

\[
\Phi_n = \frac{\alpha(\beta + \gamma + 2(n - 1)) + 2\delta}{2n(n + \beta)} \tag{B.4}
\]

C Series expansion of some solutions

The angular function (3.6) has a series expansion around \( \xi = 0 \), given by

\[
g(\xi) = C_{c,M} \left[ 1 - \frac{1}{2} c^2 (M^2 + \bar{m}) \xi + \cdots \right] +
\]

\[
C'_{c,M} \left[ (1 - \frac{1}{2} c^2 (M^2 + \bar{m}) \xi + \cdots) \ln(\xi) + \left( \frac{1}{2} + c^2 (M^2 + \bar{m}) \right) \xi + \cdots \right] \tag{C.1}
\]

We notice an explicit logarithmically divergent behavior at \( \xi = \theta = 0 \) as well as on figure 3.2. The other divergent behavior of \( g_2(\xi) \) at \( \xi = 2 \) (in figure 3.2) could be obtained easily by expansion of (3.6) around \( \xi = 2 \).
The series solution of (4.9) is given by

\[ F(\lambda) = F_I \left[ 1 - \frac{c^2 M^2}{2a^2} \lambda^2 + \frac{c^2(c^2 M^4 - 6M^2 - 2a^2 \epsilon) \lambda^4 + \cdots}{24a^4} \right] + \]

\[ F_{II} \left[ \lambda + \frac{2 - M^2 c^2}{6a^2} \lambda^3 + \frac{24 + c^4 M^4 - 14M^2 c^2 - 6a^2 \epsilon c^2}{120a^4} \lambda^5 \cdots \right] \]

(C.2)

where \( F_I \) and \( F_{II} \) are constants. We verify for different values of constants, the series (C.2) has an appropriate radius of convergence. As an example, for \( \epsilon = 1, a = 2, n = 1, M = 1 \) and \( c = 1 \), the series is convergent for \( |\lambda| < 2 \) (Figure C.1). The recursion relation that we have used to derive (C.2), is

\[ 4k(k-1)Q_k - (k^2 - 3k + 1)Q_{k-2} + Q_{k-4} = 0 \]  

(C.3)

where \( Q_k \) is the coefficient of \( \lambda^k \).

![Figure C.1: \( F_1(\lambda) \) and \( F_2(\lambda) \) as given in (C.2).](image)

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