Sufficient Lyapunov conditions for exponential mean square stability of discrete-time systems with markovian delays

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Abstract—This paper introduces sufficient Lyapunov conditions guaranteeing exponential mean square stability of discrete-time systems with markovian delays. We provide a transformation of the discrete-time system with markovian delays into a discrete-time Markov jump system. Then, we extend sufficient Lyapunov conditions existing for the global asymptotic stability of discrete-time systems with markovian delays. Finally, an example is provided to illustrate the efficiency and advantage of the proposed method.

I. INTRODUCTION

This paper aims to study the nonlinear discrete-time delay systems with delays constrained to vary on a Markov chain (see [1] for the linear case). The stability analysis for discrete-time delay systems is studied in [2]–[13]. Time-delays often lead to complex behaviors in the dynamics of a system and may lead to the failure of stability. Constraints on time-delays can be described by means of the delays digraphs notion (see [14], [15]). In recent years, the graph theory approach has been satisfactorily used in the development of stability theory for discrete-time switching systems with constrained switching signals (see [10], [16]–[18] and the references therein). The motivations of modeling the constraints through a digraph and the impact of this choice in establishing the stability results are presented in [14]. Constraints provided by bounded delay variations are studied in [1], [11], [19]. In [1], the regulation problem for discrete-time linear systems with bounded unknown random state delay is presented. In [19], the problem of disturbance rejection control for markovian jump linear systems is investigated. The modelling framework for systems subject to markovian switching is given by discrete-time markovian switching systems, also known as Markov jump systems. There is a wide literature investigating this kind of systems. Discrete-time markovian switching systems are particularly useful in the modeling of systems subject to abrupt changes, such as Wireless Control Networks (WCNs). Markovian switching systems are good approximations of the stochastic characterization of WCNs models in presence of packet losses and induced random delays. In [20]–[23] the use of markovian switching systems handles the challenges in analysis and co-design of wireless networked control systems, allows to verify instability of a system due to bursts of packet loss when Bernoulli-like channel models fail. Moreover, the Markov modelling of the Wireless channel (see [21], [22], [24]) allows performance improvement in stabilizing control synthesis, as it is shown in [23]. Motivated by the above discussions, we aim to study discrete-time systems with markovian delays, linking the methodologies available for Markov jump systems and discrete-time systems with constrained delays. The link between discrete-time systems with delays and switching delay-free systems is provided in [13], [14]. We provide an exponential mean square stability analysis for the class of considered systems, i.e., a stability analysis concerning the behaviour of the second moment of the state. The mean square stability of discrete-time markovian switching systems has been extensively analysed in the linear case (see [25], [26]), only few works presented in the literature investigate this stability notion in the nonlinear framework (see [27], [28]). On performing the analysis, we write the discrete-time delay system as a switching discrete-time system where the delays are constrained to adhere to a Markov chain. Then, we transform the switching systems with markovian delays to a Markov jump system. The Lyapunov conditions guaranteeing global asymptotic stability of discrete-time delay systems with delays switches digraphs already exist in literature (see [14]). Sufficient Lyapunov conditions guaranteeing exponential mean square stability of discrete-time Markov jump systems are introduced in [27], [29]. Our contribution consists in the extension of the Lyapunov conditions in [14] for discrete-time systems with delays digraph, to the study of exponential mean square stability of discrete-time systems with markovian delays. We provide a methodology which makes use of multiple Lyapunov functions (see [10], [29]) depending on the mode of the Markov chain, that governs the switching delay. The remainder of the paper is organized as follows. In Section II, discrete-time systems with markovian delay signals are introduced. In Section III, we provide the main result of the paper consisting of sufficient Lyapunov conditions guaranteeing exponential mean square stability. In Section IV, we illustrate a meaningful example showing the effectiveness of our result.

A. Notation and basic definitions

The symbols \( \mathbb{N} \), \( \mathbb{R} \), and \( \mathbb{R}^+ \) denote the set of non-negative integer numbers, the set of real numbers, and the set of non-negative real numbers, respectively. For a given finite
set $D$, $\text{card}(D)$ denotes its cardinality. The notation $\|x\|$ is used to denote the Euclidean norm of a vector $x \in \mathbb{R}^n$. For any positive real $\Delta$ and any positive integer $n$, the symbol $C$ denotes the space of functions mapping $\{-\Delta, -\Delta + 1, \ldots, 0\}$ into $\mathbb{R}^n$. For any non-negative integer $c$, (or for $c = +\infty$), for any function $x : \{-\Delta, -\Delta + 1, \ldots, c\} \rightarrow \mathbb{R}^n$, for any integer $k \in \mathbb{Z}$, $x_k$ is the function in $C$ defined, for $\tau \in \{-\Delta, -\Delta + 1, \ldots, 0\}$, as $x_k(\tau) = x(k + \tau)$. The function sat : $\mathbb{R} \rightarrow [-1, 1]$ is defined, for $x \in \mathbb{R}$, as $\text{sat}(x) = \min\{1, \max\{x, -1\}\}$. We consider the stochastic basis defined by the quadruple $(\Omega, \mathcal{G}, \{\mathcal{G}_k\}, P)$, where $\Omega$ is the sample space, $\mathcal{G}$ is the corresponding $\sigma$-algebra of events, $\{\mathcal{G}_k\}_{k \in \mathbb{N}}$ is the filtration, $P$ is the probability measure. Let $\mathbb{E}[\cdot]$ denote the expectation of a random variable with respect to $P$, and let $\mathbb{E}[\cdot|\mathcal{G}]$ denote the conditional expectation of a random variable on the filtration $\{\mathcal{G}_k\}_{k \in \mathbb{N}}$. The acronyms EMSS and GAS and stand for exponential mean square stability or exponentially mean square stable, and global asymptotic stability or globally asymptotically stable, respectively.

II. DISCRETE - TIME SYSTEMS WITH MARKOVIAN DELAYS

Let us consider the discrete-time delay system $\Sigma$ of the form (see [14])

$$x(k+1) = f(x(k), x(k-d_1(k)), \ldots, x(k-d_r(k))),$$
$$x(\theta) = \xi_0(\theta), \quad \theta \in \{-\Delta, -\Delta + 1, \ldots, 0\},$$

(1)

where: $k \in \mathbb{N}$; $\Delta$ is a known positive integer, the maximum involved time delay; $d_j(k) \in \{0, 1, \ldots, \Delta\}$ is a $\tau$-time varying delay, $r$ is a known positive integer; the function $f : \mathbb{R}^{r+1} \rightarrow \mathbb{R}^n$ satisfies the equality $f(0, 0, \ldots, 0) = 0$, $\xi_0 \in C$, $d(k) = [d_1(k), d_2(k), \ldots, d_r(k)]^T$, $k \in \mathbb{N}$, denote the vector collecting all time delays at time $k$. Let $D \subset \{0, 1, \ldots, \Delta\}^+$ be the set of allowed values for the time-delays vector $d(k)$. That is, for any $k \in \mathbb{N}$, $d(k) \in D$. The system (1) can be rewritten by using the following equation, (see [6] and the references therein):

$$x_{k+1} = F(x_k, d(k)), \quad k \in \mathbb{N},$$
$$x_0 = \xi_0, \quad \xi_0 \in C,$$

(2)

$$x_k \in C, \quad x_k(\theta) = x(k + \theta), \quad \theta \in \{-\Delta, -\Delta + 1, \ldots, 0\},$$

$k \in \mathbb{N}$. The map $F : C \times D \rightarrow C$ is defined, for $\phi \in C$, $d = [d_1, d_2, \ldots, d_r] \in D$,

$$F(\phi, d)(\theta) = \begin{cases} f(\phi(0), \phi(-d_1), \ldots, \phi(-d_r)), & \theta = 0, \\
\phi(\theta + 1), & \theta = -\Delta, -\Delta + 1, \ldots, 1. \end{cases}$$

(3)

Let us define the Markov chain (hereafter MC) as $\eta : \mathbb{N} \rightarrow S$, with $S \triangleq \{1, 2, \ldots, s\}$, $s = \text{card}(D)$. The transition probability matrix (hereafter TPM) of the MC is defined as $P \triangleq [p_{ij}]_{i,j \in S}$, $p_{ij} \triangleq \mathbb{P}(\eta(k+1) = j | \eta(k) = i)$, (4a)

for all $i, j \in S$, and

$$\sum_{j \in S} p_{ij} = 1, \quad \forall i \in S, \quad 0 \leq p_{ij} \leq 1, \quad \forall i, j \in S. \quad (4b)$$

Assume that the delay $d(k+1), k \in \mathbb{N}$, depends only on the delay at the previous step $d(k), k \in \mathbb{N}$, and assume that our prior knowledge on the transition from $d(k)$ to $d(k+1)$ is given by a transition probability. Let $H : D \rightarrow S$ be a bijective function defined for all $\delta_i \in D$, and for all $i \in S$, as

$$H(\delta_i) \triangleq i.$$  

(5)

The inverse function of $H$ is $H^{-1} : S \rightarrow D$, defined for all $i \in S$ and for all $\delta_i \in D$, as follows

$$H^{-1}(i) \triangleq \delta_i.$$  

(6)

Consider $p_{ij}$ defined in (4a). By applying the definition of $p_{ij}$ in (4a) and the definition of the functions $H$ and $H^{-1}$, the following equalities hold:

$$p_{ij} = \mathbb{P}(\eta(k+1) = j | \eta(k) = i)$$
$$= \mathbb{P}(H(d(k+1)) = H(\delta_i) | H(d(k)) = H(\delta_i))$$
$$= \mathbb{P}(d(k+1) = \delta_j | d(k) = \delta_i),$$

(7)

for all $\delta_i, \delta_j \in D$, for all $i, j \in S$. Consequently, the modes of the MC $\{\eta(k)\}_{k \in \mathbb{N}}$ with TPM defined by (4) are associated with the delays in the set $D$, through the function $H^{-1}$. Let $E(D)$ be the finite set of all pairs $(\delta_i, \delta_j) \in D \times D$, $i, j \in S$, such that, for any $k \in \mathbb{N}$, if $d(k) = \delta_i$, it is allowed $d(k+1) = \delta_j$. We define the set $E(D)$ as follows,

$$E(D) \triangleq \{(\delta_i, \delta_j) \in D \times D, \delta_i, \delta_j \in D, i, j \in S | p_{ij} > 0\}.$$  

(8)

In the following we define the Markov jump system that we consider throughout the paper (see [25], [26] and the references therein).

Definition 1: Let $\Sigma$ denote the Markov jump system defined on the stochastic basis $(\Omega, \mathcal{G}, \{\mathcal{G}_k\}, P)$, as

$$\Sigma \triangleq (D, P, H),$$  

(9)

where $D$ is the system described by (1) and rewritten in the form (2), $P$ is a known TPM defined by (4) modeling the stochastic switching of the delays, and $H$ is the bijective function defined by (5). From (2), the Markov jump system $\Sigma$ can be written as follows:

$$x_{k+1} = F(x_k, H^{-1}(\eta(k))), \quad k \in \mathbb{N},$$
$$x_0 = \xi_0, \quad \xi_0 \in C,$$

(10)

where $x_k \in C, x_k(\theta) = x(k + \theta), \theta \in \{-\Delta, -\Delta + 1, \ldots, 0\}, k \in \mathbb{N}$, $\eta(k) \in S, k \in \mathbb{N}$ is a MC with TPM $P$ defined by
Fig. 1: The Figure depicts the state diagram of the Markov chain $\eta(k)$ modeling the switching delay in the presented example: $p$ stands for the probability of having a delay $d(k+1) = 0$ provided that the previous delay is $d(k) = 0$, while $q$ stands for the probability of having a delay $d(k+1) = 2$, provided that the previous delay is $d(k) = 2$.

The TPM associated with the MC \( \{\eta_k\}_{k \in \mathbb{N}} \) is given by
\[
P = \begin{bmatrix}
p & 1-p \\
1-q & q
\end{bmatrix}, \quad p, q \in (0, 1).
\]

Hence, we obtain a Markov jump system $\Sigma$ where $D$ is (12), $P$ is defined in (16) and $H$ is defined in (14).

Remark 1: Notice that the variable $x(k, \xi_0) \in \mathbb{R}^n$, $\xi_0 \in \mathcal{C}$, $k \in \mathbb{N}$, is a random variable on the stochastic basis \((\Omega, \mathcal{G}, \mathcal{F}, \mathbb{P})\), since the delay evolves according to a discrete-time MC, with given transition probabilities. Thus, we are interested in the behaviour of the second moment of $x(k, \xi_0)$, $k \in \mathbb{N}$, $\xi_0 \in \mathcal{C}$.

Definition 2: The Markov jump system $\Sigma$ is $EMSS$ if there exist $M$, $\zeta \in \mathbb{R}^+$ with $M \geq 1$ and $0 < \zeta < 1$, such that for any $\xi_0 \in \mathcal{C}$, the following inequality holds for any $k \in \mathbb{N}$,
\[
\mathbb{E}[\|x(k, \xi_0)\|^2] \leq M\zeta^k (\|\xi_0\|_\infty)^2.
\]

III. MAIN RESULT

In this section, we provide the main result of the paper. We derive sufficient Lyapunov conditions guaranteeing the $EMSS$ of system $\Sigma$.

Let us consider a scalar function $V : \mathcal{C} \times D \to \mathbb{R}^+$. Let us associate to $V$ the operator $L^V : \mathcal{C} \times D \to \mathbb{R}$, defined for $\phi \in \mathcal{C}$, $i \in \mathcal{S}$, as
\[
L^V(\phi, H^{-1}(i)) = \sum_{j \in \mathcal{S}} p_{ij} V(F(\phi, H^{-1}(i)), H^{-1}(j)) - V(\phi, H^{-1}(i)),
\]
with $H^{-1}$ defined in (6).

Theorem 1: Assume there exist a function $V : \mathcal{C} \times D \to \mathbb{R}^+$ and real positive numbers $\alpha_1$, $i = 1, 2, 3$, such that, for all $\phi \in \mathcal{C}$, for all $i \in \mathcal{S}$, the following inequalities hold:
\[
\begin{align*}
i) & \quad \alpha_1\|\phi(0)\|^2 \leq V(\phi, H^{-1}(i)) \leq \alpha_2\|\phi\|_\infty^2, \\
ii) & \quad L^V(\phi, H^{-1}(i)) \leq -\alpha_3\|\phi(0)\|^2,
\end{align*}
\]
where $H^{-1}$ is defined in (6).

Then, the system $\Sigma$ is $EMSS$.

Proof: See the extended version [30].

Remark 2: Notice that the classical representation of system $\Sigma$ using a Markov jump system allows to use the conditions in [27], [29]. An important feature of the involved Lyapunov inequalities, shared with the cases of delay-dependent and delay-independent Lyapunov functions (see [6], [14]), is that lower bound of Lyapunov functions, as well as of the related difference operators, are given in a weaker form with respect to the Lyapunov conditions for Markov jump systems (see for instance [27], [29]). Indeed, the lower bound of condition (i) in Theorem 1 and the inequality in condition (ii) in Theorem 1 do not involve $\phi \in \mathcal{C}$, but only $\phi(0) \in \mathbb{R}^n$.  

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IV. EXAMPLE

In this section, we aim to study the EMSS property of the system \( \Sigma \) obtained in Example 1 starting from the discrete-time delay system (12).

We analyze the Markov jump system resulting by the application of our methodology. As mentioned before, we obtain a switching system with two modes: one is stable and the other one is unstable.

Notice that, as consequence of the structure of \( P \) in (16), the set \( \mathcal{E}(D) \) is given by \( \mathcal{E}(D) = \{(0, 0), (0, 2), (2, 0), (2, 2)\} \).

In the following, we want to verify whether conditions (i) and (ii) of Theorem 1 are satisfied.

We consider a candidate Lyapunov function \( V : \mathcal{C} \times D \to \mathbb{R}^+ \) defined, for \( \phi \in \mathcal{C}, i \in \mathcal{S} \), as

\[
V(\phi, H^{-1}(i)) = \lambda_i \sup_{j=0,1,2} 2^{j-1} \gamma^j c^{-j} \|\phi(-j)\|^2, \tag{19}
\]

with \( \lambda_i \in \mathbb{R}^+, i \in \mathcal{S}, \gamma \in [1, 1.2], c \geq e \).

Pick \( \alpha_1 = \min \{\lambda_i\}, \alpha_2 = 2\gamma^2 \max \{\lambda_i\} \).

Thus, condition (i) of Theorem 1 is satisfied. In order to verify condition (ii), we consider the expression of \( \mathcal{L}V(\phi, H^{-1}(i)) \), for all \( \phi \in \mathcal{C}, i \in \mathcal{S} \).

When we consider the expression of \( \mathcal{L}V(\phi, H^{-1}(1)) \), from (18) we obtain the following equality:

\[
\mathcal{L}V(\phi, H^{-1}(1)) = pV(F(\phi, H^{-1}(1)), H^{-1}(1)) + (1 - p)V(F(\phi, H^{-1}(1)), H^{-1}(2)) - V(\phi, H^{-1}(1)). \tag{20}
\]

From (19) and (20), we obtain the following equality:

\[
\mathcal{L}V(\phi, H^{-1}(1)) = (p\lambda_1 + (1 - p)\lambda_2) \sup_{j=0,1,2} 2^{j-1} \gamma^j c^{-j} \|F(\phi, H^{-1}(1))(-j)\|^2
- \lambda_1 \sup_{j=0,1,2} 2^{j-1} \gamma^j c^{-j} \|\phi(-j)\|^2. \tag{21}
\]

By (15), we have:

\[
\mathcal{L}V(\phi, H^{-1}(1)) \leq (p\lambda_1 + (1 - p)\lambda_2) 2^{-1} \| (1 - \gamma) \text{sat}(\phi(0))\|^2
+ (p\lambda_1 + (1 - p)\lambda_2) \sup_{j=1,2} 2^{j-1} \gamma^j c^{-j} \|\phi(-j + 1)\|^2
- \lambda_1 \sup_{j=0,1,2} 2^{j-1} \gamma^j c^{-j} \|\phi(-j)\|^2. \tag{22}
\]

By the properties of the supremum, the following inequality holds

\[
\sup_{j=1,2} 2^{j-1} \gamma^j c^{-j} \|\phi(-j + 1)\|^2 \leq \sup_{j=1,2,3} 2^{j-1} \gamma^j c^{-j} \|\phi(-j + 1)\|^2. \tag{23}
\]

By changing the index variable in the supremum, we can write

\[
\begin{align*}
\sup_{j=1,2,3} 2^{j-1} \gamma^j c^{-j} \|\phi(-j + 1)\|^2 &= 2 \gamma^{-1} \sup_{j=1,2,3} 2^{(j-1)-1} \gamma^{(j-1)} c^{-(j-1)} \|\phi(-j + 1)\|^2 \\
&= 2 \gamma^{-1} \sup_{\theta=0,1,2} 2^{-1} \gamma^\theta c^{-\theta} \|\phi(-\theta)\|^2. \tag{24}
\end{align*}
\]

Thus, from (22), (23), (24), we obtain the following inequalities

\[
\mathcal{L}V(\phi, H^{-1}(1)) \leq (p\lambda_1 + (1 - p)\lambda_2) (2^{-1} (1 - \gamma)^2 \|\phi(0)\|^2 + 2 \gamma^{-1} \sup_{\theta=0,1,2} 2^{-1} \gamma^\theta c^{-\theta} \|\phi(-\theta)\|^2) +
- \lambda_1 \sup_{\theta=0,1,2} 2^{-1} \gamma^\theta c^{-\theta} \|\phi(-\theta)\|^2 \leq (p\lambda_1 + (1 - p)\lambda_2) ((1 - \gamma)^2 + 2 \gamma c^{-1}) \\
&+ \sup_{\theta=0,1,2} 2^{-1} \gamma^\theta c^{-\theta} \|\phi(-\theta)\|^2
- \lambda_1 \sup_{\theta=0,1,2} 2^{-1} \gamma^\theta c^{-\theta} \|\phi(-\theta)\|^2. \tag{25}
\]

By defining \( \omega_1 \) as follows,

\[
\omega_1 = 1 - \left[ \frac{1}{\lambda_1} \left( (1 - \gamma)^2 + 2 \gamma c^{-1} \right) \right], \tag{26}
\]

we get

\[
\mathcal{L}V(\phi, H^{-1}(1)) \leq -\omega_1 \sup_{\theta=0,1,2} 2^{-1} \gamma^\theta c^{-\theta} \|\phi(-\theta)\|^2. \tag{27}
\]

When we consider the expression of \( \mathcal{L}V(\phi, H^{-1}(2)) \), we obtain:

\[
\begin{align*}
\mathcal{L}V(\phi, H^{-1}(2)) &= (1 - q) V(F(\phi, H^{-1}(2)), H^{-1}(1)) + q V(F(\phi, H^{-1}(2)), H^{-1}(2)) - V(\phi, H^{-1}(2)). \tag{28}
\end{align*}
\]

From (28), and (19), the following equality holds:

\[
\begin{align*}
\mathcal{L}V(\phi, H^{-1}(2)) &= (1 - q) \lambda_1 + q \lambda_2 \sup_{j=0,1,2} 2^{j-1} \gamma^j c^{-j} \|F(\phi, H^{-1}(2))(-j)\|^2
- \lambda_2 \sup_{j=0,1,2} 2^{j-1} \gamma^j c^{-j} \|\phi(-j)\|^2. \tag{29}
\end{align*}
\]

From (29), applying the properties of the supremum, it follows that

\[
\begin{align*}
\mathcal{L}V(\phi, H^{-1}(2)) &\leq (1 - q) \lambda_1 + q \lambda_2 \left( 2^{-1} \|F(\phi, H^{-1}(2))(0)\|^2 + \sup_{j=1,2} 2^{j-1} \gamma^j c^{-j} \|\phi(-j + 1)\|^2 \right)
- \lambda_2 \sup_{j=0,1,2} 2^{j-1} \gamma^j c^{-j} \|\phi(-j)\|^2. \tag{30}
\end{align*}
\]
From (30), we have
\[ \mathcal{L}V(\phi, H^{-1}(2)) \leq ((1 - q)\lambda_1 + q\lambda_2)(2\gamma c^-1 \sup_{j=1,2} 2^{j-1-1} \gamma^{j-1} e^{j+1} \|\phi(-j + 1)\|^2) \]
\[ - \lambda_2 \sup_{\theta \in [0,1,2]} 2^{2-1-1} \gamma^{1-1} c^{-\theta} \|\phi(-\theta)\|^2. \]
(31)

From (31), by applying the properties of the Euclidean norm, Young’s inequality and the properties of the function \( \text{sat} \), the following inequalities hold
\[ \mathcal{L}V(\phi, H^{-1}(2)) \leq ((1 - q)\lambda_1 + q\lambda_2)(2\gamma c^-1 \sup_{\theta \in [0,1,2]} 2^{2-1-1} \gamma^{1-1} c^{-\theta} \|\phi(-\theta)\|^2) \]
\[ - \lambda_2 \sup_{\theta \in [0,1,2]} 2^{2-1-1} \gamma^{1-1} c^{-\theta} \|\phi(-\theta)\|^2. \]
(32)

From (32), by defining \( \omega_2 \) as follows,
\[ \omega_2 \triangleq \lambda_2 \left( 1 - (q + (1-q)\lambda_1) \right)(2 + 2^{-1} c^2 + 2\gamma c^{-1}) \],
(33)
we obtain
\[ \mathcal{L}V(\phi, H^{-1}(2)) \leq -\omega_2 \sup_{\theta \in [0,1,2]} 2^{2-1-1} \gamma^{1-1} c^{-\theta} \|\phi(-\theta)\|^2. \]
(34)

Under the following constraints
\[ L_B < \frac{\lambda_2}{\lambda_1} < U_B, \]
with
\[ U_B = \frac{1 - ((1 - \gamma)^2 + 2\gamma c^{-1})p}{((1 - \gamma)^2 + 2\gamma c^{-1})(1 - p)}, \]
\[ L_B = \frac{4 + c^2 + 4\gamma c^{-1}}{2 - (4 + c^2 + 4\gamma c^{-1}) q}, \]
(35a, 35b, 35c)
where \((p, q) \in (0,1)\) and \(q < \frac{2}{4 + c^2 + 4\gamma c^{-1}}\);
we obtain that \( \omega_1, \omega_2 \in \mathbb{R}^+ \). Thus, from (27) and (34) the following inequality holds, for all \( \phi \in C \), for all \( i \in S \),
\[ \mathcal{L}V(\phi, H^{-1}(i)) \leq -\alpha_3 \|\phi(0)\|^2, \]
(36)

with \( \alpha_3 \in \mathbb{R}^+ \), defined as \( \alpha_3 \triangleq \frac{1}{2} \min\{\omega_1, \omega_2\} \). Thus, condition (ii) of Theorem 1 is satisfied and the system (12) with switches delays governed by the MC \( \{\eta(k)\}_{k \in \mathbb{N}} \), with TPM \( P \) defined in (16), and with the function \( H \) defined in (15) is \( \text{EMSS} \).

A. Statistical results
In Figure 2, we present Montecarlo simulations of the trajectories generated by the system (12), considering values of the pairs \((p, q)\) such that conditions (i) and (ii) of Theorem 1 are satisfied. The yellow trajectories correspond to the state trajectories associated with different switching paths (that are admissible according to \( P \)), the maximum and the minimum trajectory are plotted in blue and green, respectively. Finally, the red line corresponds to the average evolution of the state trajectories. From Figure 2, we observe that trajectories decrease exponentially and converge to zero. This result reflects the analysis presented in this section. In Figures 3, for different values of \( c \) in the candidate Lyapunov function (19), we show the regions of pairs \((p, q)\) such that conditions (i) – (ii) of Theorem 1 are satisfied (light blue region) and the evolution of the maximum \( q \) with respect to \((1 - p)\) such that the conditions (i) – (ii) of Theorem 1 are satisfied (dark blue line). From Figures 3, by comparing row-wise, we observe that when \( c \) spans from 5.2 to 6, the segment on \( 1 - p \) shrinks while segment on \( q \) expands. From Figures 3, by comparing column-wise, when \( q \) goes from 1.2 to 0.5, the segment on \( 1 - p \) shrinks, the aforementioned light blue region becomes smaller and smaller when the parameter \( c \) increases. Choosing \( c > e \), the conditions (35) lead to a widen set of values for \( 1 - p \), while, by condition (35d), the values of \( q \) are restricted.

V. CONCLUSIONS
In this paper, we provide sufficient Lyapunov conditions guaranteeing the \( \text{EMSS} \) property of discrete-time systems with markovian delays. Future work directions would be the extension of sufficient conditions for the exponential input-to-state stability in mean square sense, as well as necessary and sufficient conditions guaranteeing \( \text{EMSS} \) for this class of systems.

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Fig. 3: The figure shows the regions of pairs \((p, q) \in (0, 1) \times (0, 1)\) such that conditions of Theorem 1 are satisfied.

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