Relativistic Theory of Infinite Statistics Fields

Chao Cao, Yi-Xin Chen and Jian-Long Li
Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, China

Infinite statistics in which all representations of the symmetric group can occur is known as a special case of quon theory. However, the validity of relativistic quon theories is still in doubt. In this paper we prove that there exists a relativistic quantum field theory which allows interactions involving infinite statistics particles. We also give some consistency analysis of this theory such as conservation of statistics and Feynman rules.

PACS numbers:

I. INTRODUCTION

In conventional quantum theory the identical particles always obey Bose-Einstein statistics or Fermi-Dirac statistics, which are characterized by commutation or anti-commutation relations respectively. This restriction in fact requires a symmetrization postulate that all particles should be in a symmetric state or an anti-symmetric state. Without such postulate, new approaches to particle statistics with small violations of Bose or Fermi statistics are allowed. One famous approach is called quon theory in which the annihilation and creation operators obey the -deformed commutation relation, \( a_k a_l^\dagger - q a_l^\dagger a_k = \delta_{kl}, -1 \leq q \leq 1 \). There exist three special cases in such theory, Bose and Fermi statistics for \( q = \pm 1 \), infinite statistics for \( q = 0 \).

Infinite statistics with \( a_k a_l^\dagger = \delta_{kl} \) involves no commutation relation between two annihilation or creation operators. The quantum states are orthogonal under any permutation of the identical particles. So it allows all representations of the symmetric group to occur. Furthermore, the loss of local commutativity also implies violation of locality, which is an important character of quantum gravity. By virtue of these properties, infinite statistics has been applied to many subjects, such as black hole statistics, dark energy quanta, large \( N \) matrix theory and holography principle. Many of these applications involve discussions in relativistic case.

Unfortunately, the validity of relativistic theory obeying infinite statistics is still in doubt. Greenberg has showed that the infinite statistics theory is valid in non-relativistic case. This theory can also have relativistic kinematics. Cluster decomposition and the CPT theorem still hold for free fields. However, there are two difficulties for infinite statistics to have a consistent relativistic theory. For one thing, the physical observables do not commute at spacelike separation. It is bad news for a relativistic theory which requires Lorentz invariance for any physical scattering process (the time-ordering of the operator product in the \( S \)-matrix is not Lorentz invariant). For the other thing, by acquiring that the energies of systems that are widely spacelike separated should be additive, Greenberg shows that the conservation of statistics in a relativistic theory limits that \( q = \pm 1 \), which means it must be Bose or Fermi case. The \( q = 0 \) case for infinite statistics has been excluded.

In this paper we prove the existence of interacting relativistic field theory obeying infinite statistics by solving the two difficulties above. First, we directly analyze the Lorentz invariance of the \( S \)-matrix from the infinitesimal Lorentz transformations on \( S \). The loss of local commutativity does not destroy the invariance. Second, we can also show that this field theory obeys conservation of statistics rule by acquiring some special form of the interaction Hamiltonian. In addition, We also expect that the conventional Feynman rules still hold in this new theory.

This paper is organized as follows. In Sec. II, we introduce the elementary ingredients of infinite statistics in non-relativistic case. In Sec. III, we prove the Lorentz invariance of the \( S \)-matrix. In Sec. IV, we discuss the condition that the energies are additive for product states, and show that the conservation of statistics still holds. Sec. V discusses the Feynman rules and provides some simple examples. General conclusions are given in Sec. VI.

II. INFINITE STATISTICS

The basic algebra of infinite statistics is

\[ a_k a_l^\dagger = \delta_{kl}, \]

where the operator \( a_k \) annihilates the vacuum

\[ a_k |0\rangle = 0. \]

This relation determines a Fock-state representation in a linear vector space. \( n \)-particle state is constructed as

\[ |\phi_{m_1 m_2 ... m_j} \rangle = (a_{k_1})^{m_1} (a_{k_2})^{m_2} ... (a_{k_j})^{m_j} |0\rangle, \]

with \( m_1 + m_2 + ... m_j = m \). Such states have positive norms and the normalization factor equals one. Since
there is no commutation relation between two annihilation or creation operators, the states created by any permutations of creation operators are orthogonal. That’s why it is also called quantum Boltzmann statistics.

One can define a set of number operators \( n_i \) such that
\[
 n_i |\phi_m\rangle = m_i |\phi_m\rangle, \quad [n_i, a_j] = -\delta_{ij} a_j.
\]
Then the total number operator is \( N = \sum_i n_i \), and the energy operator is given by \( E = \sum_i \epsilon_i n_i \), where \( \epsilon_i \) is the single particle energy. The explicit form of \( n_i \) was given by Greenberg
\[
n_i = a_i^\dagger a_i + \sum_k a_k^\dagger a_i a_k a_k + \sum_{k_1,k_2} a_{k_1}^\dagger a_{k_2}^\dagger a_i a_k a_k a_k a_k + \cdots
\]
\[
\quad + \sum_{k_1,k_2,...,k_s} a_{k_1}^\dagger a_{k_2}^\dagger \cdots a_{k_s}^\dagger a_i a_k a_k a_k \cdots a_k a_k + \cdots,
\]
which is obviously a non-local operator. One can easily check that this definition obeys Eq. (4).

III. THE LORENTZ INVARIANCE OF THE S-MATRIX

It’s not difficult to construct infinite statistics fields that transform irreducibly under the Lorentz group. In momentum space, the annihilation field \( \psi^+ (x) \) and creation field \( \psi^- (x) \) are
\[
\psi^+_i (x) = \sum_{\sigma n} (2\pi)^{-3/2} \int d^3 p \, v_i^{(n)} (p, \sigma) e^{i p \cdot x} a_p^{(n)} (\sigma),
\]
\[
\psi^-_i (x) = \sum_{\sigma n} (2\pi)^{-3/2} \int d^3 p \, v_i^{(n)} (p, \sigma) e^{-i p \cdot x} a_p^{(n)} (\sigma),
\]
where \( p^\mu \) denotes four-momentum, \( \sigma \) labels spin z-components (or helicity for massless particles), and the superscript \( n \) labels particle species \( a_i^{(n)} a_j^{(m)} = \delta (nm) \delta (ij) \). With these fields we will be able to construct the interaction density as
\[
\mathcal{H} (x) = \sum_{N,M} \sum_{n_1} \cdots \sum_{n_M} \sum_{l_1} \cdots \sum_{l_M} g_{l_1}^{(n_1)} \cdots g_{l_M}^{(n_M)} \psi^-_{l_1} (x) \cdots \psi^-_{l_M} (x) \psi^+_{n_1} (x) \cdots \psi^+_{n_M} (x).
\]

In conventional field theory, we usually construct \( \mathcal{H} (x) \) out of a linear combination \( \psi (x) = \kappa \psi^+ (x) + \lambda \psi^- (x) \), where \( \epsilon \) is antiparticle. By the requirement of relativistic micro-causality \( \langle \psi (x), \psi^\dagger (y) \rangle = 0 \) for \( x - y \) spacelike, we always have \( \kappa = \lambda \). However, in a theory based on infinite statistics this local commutativity does not hold. So we can’t determine the relationship between \( \kappa \) and \( \lambda \).

Moreover, from Eq. 3 and other operator definitions such as charge operators one may guess that a general operator formulation is defined as
\[
A (\mathcal{O}) = \sum_{m=0}^{\infty} \sum_{n_1,\ldots,n_m} \sum_{k_1,\ldots,k_m} a_{k_1}^{(n_1)} \cdots a_{k_m}^{(n_m)} \mathcal{O} a_{k_m}^{(n_m)} \cdots a_{k_1}^{(n_1)}.
\]

We will see that this definition is important in the next section’s discussion.

With above operator definitions, we can see the Lorentz invariance of the S-matrix. One traditional condition comes from the Dyson series for the S-operator
\[
S = T \{ \exp (-i \int_{-\infty}^{\infty} dt V (t)) \}
\]
\[
= 1 + \sum_{N=1}^{\infty} \frac{(-i)^N}{N!} \int d^4 x_1 \cdots d^4 x_N T \{ \mathcal{H} (x_1) \cdots \mathcal{H} (x_N) \},
\]
where \( V (t) \) is the interaction term \( H = H_0 + V \) and \( T \) denotes the time-ordered product. Since the time-ordering of two spacetime points \( x_1, x_2 \) is invariant unless \( x_1 - x_2 \) is spacelike, so this condition that makes S Lorentz invariant is that \( \mathcal{H} (x) \) all commute at space-like separations
\[
[\mathcal{H} (x), \mathcal{H} (x')] = 0 \quad \text{for} \quad (x - x')^2 \geq 0.
\]

Now we compute \( [\mathcal{H} (x), \mathcal{H} (x')] \) under infinite statistics. First we write the interaction Hamiltonian density as a polynomial \( \mathcal{H} (x) = \sum_i g_{a_i} \mathcal{H}_a (x) \), each term \( \mathcal{H}_a \) is a product of definite numbers of annihilation fields and creation fields. Then we have
\[
[\mathcal{H} (x), \mathcal{H} (x')] = \sum_{\alpha} g^2_{a_\alpha} [\mathcal{H}_a (x), \mathcal{H}_a (x')] + \sum_{\alpha < \beta} g_{a_\alpha} g_{a_\beta} (\mathcal{H}_a (x), \mathcal{H}_b (x') - \mathcal{H}_b (x'), \mathcal{H}_a (x)) \]
\[
\quad \text{[10]}.
\]

By using Eqs. 3 - 9, we have
\[
[\mathcal{H}_a (x), \mathcal{H}_a (x')]
\]
\[
\sim \int d^3 p_1 \cdots d^3 p_j d^3 p'_1 \cdots d^3 p'_j \cdots \]
\[
a_{p_1}^\dagger \cdots a_{p_j}^\dagger \cdots a_{p'_1}^\dagger \cdots a_{p'_j}^\dagger \frac{1}{|x - p_1 - \ldots - p_j + x'|} \cdot \frac{1}{|x - p'_1 - \ldots - p'_j + x'|} \cdot \ldots \cdot \frac{1}{|x - p_{j-1} - \ldots - p_j + x'|} \cdot \frac{1}{|x - p'_{j-1} - \ldots - p'_j + x'|} \cdot \ldots \cdot \frac{1}{|x - x'|}.
\]
\[
\text{(13)}.
\]
\[
[H_\alpha(x), H_\beta(x')] - [H_\alpha(x'), H_\beta(x)] \\
\sim \int d^3 p_1 \cdots d^3 p_j d^3 p'_1 \cdots d^3 p'_l \cdots
\]

\(a^\dagger_{p_1} \cdots a^\dagger_{p_j} (a_{p_{j+1}} \cdots a_{p_l} a^\dagger_{p'_j} \cdots a^\dagger_{p'_l}) (e^{i[-(p_1 + \cdots + p_j) + (p_{j+1} + p_l)]x + \cdots} - (x \leftrightarrow x')) + \int d^3 p_1 \cdots d^3 p_j d^3 p'_1 \cdots d^3 p'_l \cdots
\]

\(e^{i[-(p_1 + \cdots + p_j) + (p_{j+1} + p_l)]x + \cdots} - (x \leftrightarrow x'))
\]

\[
= \langle \beta \rangle \int d^3 x_1 \cdots d^3 x_{i-1} d^4 x_{i+1} \cdots d^4 x_n
\]

\(T\{ [H(x_1) \cdots [H(x_i), H(x_{i+1})] \cdots H(x_n)] \}|\alpha\rangle
\]

(15)

for \((x_i - x_{i+1})^2 \geq 0\), where \(\langle \beta \rangle, |\alpha\rangle\) denote the final state and the initial state. Those annihilation and creation operators in Eqs. (13), (14) should contract with \(a^\dagger\)'s and \(a\)'s in the initial states, final states and other \(H\)’s (except \(H(x_i)\) and \(H(x_{i+1})\)) in Eq. (16), or they will directly annihilate the vacuum state and get zero. One should note that the S-matrix involves a four-momentum conservation relation \(S_{B\alpha} \sim \delta^4(p_{B} - p_{\alpha})\) (see chapter 3 in [18] for details). So after those annihilation and creation operators in Eqs. (13), (14) are totally contracted, we find the commutation relation \([H(x), H(x')]\) is constituted by terms of the form

\[
\sim \int \prod d^3 k f(k) [e^{i(\sum p + \sum k)(x-x')} - e^{-i(\sum p + \sum k)(x-x')}],
\]

(17)

where the terms including \(k\) come from self-contractions (contractions don’t involve the initial states or final states, such as contractions in \(a_{p_{j+1}} \cdots a_{p_j} a^\dagger_{p'_j} \cdots a^\dagger_{p'_l}\)), while \(\sum p\) is a sum of some particle momenta in the initial or final states, which is more explicitly, the sum of momentum in Eqs. (13), (14) \(-[(p_1 + \cdots + p_j) + (p_{j+1} + p_l)]\) or \(-[(p_1 + \cdots + p_k) + (p_{k+1} + p_j)]\) minus the self-contraction momenta. One can easily see that Eq. (17) is non-zero. As a result, the interaction density \(H(x)\) will not commute with \(H(x')\) at spacelike separations \(x - x'\), which means that this theory cannot be local.

However, the failure of the above commutation in \(T\{\}\) of Eq. (16) does not prohibit the existence of a relativistic field theory. There exists a less restrictive sufficient condition for Lorentz invariance of the \(S\)-matrix, which directly comes from the infinitesimal Lorentz transformations of \(S\)-operator (see Chapter 3, page 145 in [18]). This condition is

\[
0 = \int d^3 x \int d^3 y x [H(x, 0), H(y, 0)].
\]

(18)

We can also put this condition together with the initial states and the final states [21]. According to our analysis presented above, this condition becomes

\[
0 = \prod d^3 k f(k) \int d^3 x \int d^3 y x (e^{i(\sum p + \sum k)(x-y)} - e^{-i(\sum p + \sum k)(x-y)}),
\]

(19)

which is always satisfied because \(\int d^3 x x e^{ixp}\) is even in \(p\). So we conclude that the interacting field theory based on infinite statistics is Lorentz invariant.

A similar analysis can be applied to the commutation relation \([A(\rho_\alpha(x), A(\rho_\alpha(x'))] = A(\rho(x, x'))\) (see Appendix for details), in which the interaction density \(A(\rho)\) is defined in Eq. (9). This commutation can be decomposed into a sum of terms that are similar to Eqs. (13), (14). Noting that the momenta of \(a^\dagger\)’s and \(a\’s summed over in these terms (such as \(p_i\) in \(\sum a^\dagger_{p_i} \cdots \rho(x, x') \cdots a_{p_i}\) terms) have no contribution to the momentum conservation relation in \(\rho\), the final terms after total contraction are still as the form (17). Then by using condition (18), we conclude that the theory with interaction density of the form \(A(\rho(x))\) is also Lorentz invariant.

IV. CONSERVATION OF STATISTICS

Here we try to impose the condition that the energy should be additive for product states on the interaction \(H(x)\). For subsystems that are widely spacelike separated, the contribution to the energy should be additive if

\[
[H(x), \psi(x')] \rightarrow 0, \text{ as } x - x' \rightarrow \infty
\]

for all fields. One can check that this condition is equivalent to

\[
(e^{iH_A t} \psi_A) \otimes (e^{iH_B t} \psi_B) = e^{i(H_A + H_B)t} (\psi_A \otimes \psi_B),
\]

(21)

while subsystems \(A\) and \(B\) are widely spacelike separated. By using Eq. (21), Greenberg expected that the Hamiltonian operators should be effectively bosonic, which leads to “conservation of statistics” and acquires that \(q = \pm 1\) [2][17]. However, we think this restriction is too strong, and we provides a much less restrictive requirement on \(H(x)\), which also leads to conservation of statistics.
In order to satisfy the energy additive condition, we should replace the density $\mathcal{H}$ with $\mathcal{A}(\mathcal{H})$. Thus Eq. 20 becomes

$$[\mathcal{A}(\mathcal{H}(x)), \psi(x')] \rightarrow 0, \text{ as } x \rightarrow x' \rightarrow \infty \text{ spacelike.}$$

Noting that the basic fields are $\psi^{\pm}$ here, we can get

$$[\mathcal{A}(\mathcal{H}(x)), \psi^+(x')] = -\psi^+(x')\mathcal{H}(x)$$

$$[\mathcal{A}(\mathcal{H}(x)), \psi^-(x')] = \mathcal{H}(x)\psi^-(x').$$

(23)

We note that in infinite statistics

$$\psi^{+^{(n)}}(x)\psi^{-^{(m)}}(x) \sim [\cdots]\delta(nm)\Delta_+ (x - x'),$$

where the coefficient $[\cdots]$ may contain some derivative times such as $\gamma^\mu\partial_\mu$ for spin $\frac{1}{2}$ particles and $\partial^\mu\partial_\mu$ for spin one particles, while $\Delta_+ (x - x') \equiv \frac{1}{(2\pi)^2} \int \frac{d^4p}{2p^0} e^{ip(x-x')}$ is a standard function. Moreover, for $(x-x')^2 \geq 0$

$$\Delta_+ (x - x') = \frac{m}{4\pi^2}\sqrt{(x-x')^2},$$

(25)

in which $K_1()$ is the modified Hankel function of order 1. So $\Delta_+ (x - x')$ and its derivations are $\rightarrow 0$

as $x - x' \rightarrow \infty \text{ spacelike}$. Then by using Eqs. 20 and 22, we infer that the condition 22 is satisfied if the interaction density $\mathcal{H}(x)$ has the form Eq. 3 with $N, M \geq 1$. Moreover, in order to get condition 21 satisfied for our new definition $\mathcal{A}(H)$, we also need

$$[\mathcal{A}(\mathcal{H}(x)), \mathcal{A}(\mathcal{H}(x'))] \rightarrow 0, \text{ as } x \rightarrow x' \rightarrow \infty \text{ spacelike},$$

this proof is given in Appendix. Here we just exclude the terms in $\mathcal{H}$ which are products containing only annihilation fields $\psi^+(x)$ (or creation fields $\psi^-(x)$). Since the operation definition $\mathcal{A}(\mathcal{H})$ is quite normal in infinite statistics field theory, our requirement is much looser than the condition that Hamiltonian operator must be effectively bosonic.

Although the interaction density may not be bosonic, conservation of statistics still holds in our theory. To see this, let’s consider the case that infinite statistics fields couple to normal fields (we will denote infinite statistics fields by the subscript $I$ and normal statistics fields by the subscript $B$). According to conventional fields theory and our above discussion, all interactions must involve any number of bosons, an even number of fermions (including zero), at least one annihilation infinite statistics field and at least one creation infinite statistics field. These three kinds of particles commute with each other, so $\mathcal{A}(O_{1}O'_{B}) = \mathcal{A}(O_{I})O'_{B}$. Since we exclude the terms in $\mathcal{H}$ which are products containing only annihilation fields (or creation fields), then the term $T[\mathcal{A}(\mathcal{H}(x_1)) \cdots \mathcal{A}(\mathcal{H}(x_N))]$ in $S$-operator (11) must have both $\psi^+_I$ and $\psi^+_B$ fields and $\psi^-_I$ and $\psi^-_B$ fields with $\delta^{(n)}\delta^{(m)} = \delta(nm)\delta(ij)$ contractions. So there must be infinite statistics particles both in the initial and final states, which forbids any process that the in-state obeys infinite statistics (normal statistics) while the out-state obeys normal statistics (infinite statistics). Moreover, since the interaction vertices such as $\mathcal{A}(\psi^+_I\psi^+_B)\mathcal{A}(\psi^+_I\psi^+_B)$ do not exist, we also exclude those virtual processes such as annihilation of a pair of infinite (normal) statistics particles producing a normal (infinite) statistics particle, which also break conservation of statistics. So we conclude that our theory obeys the conservation of statistics. Some examples are presented in in the next section.

V. FEYNMAN RULES AND EXAMPLES

In order to derive Feynman rules, first we see “Wick’s theorem” for infinite statistics fields, by using the relation $\delta^{(n)}\delta^{(m)} = \delta(nm)\delta(ij)$, any product of a set of infinite statistics operators can be finally expressed as a normal product. This looks a bit different from conventional field theory, in which contractions can arise between any creator and annihilator pairs by permuting the operators, while in our theorem contractions can only arise between the neighboring operators. However, by inducing $\mathcal{A}(\mathcal{H})$, we can also realize some permutations. For example, in order to get contraction between a final particle $|0\cdots a$ and a creation field $\psi^-_I(.)$ in $|0\cdots a\mathcal{A}(\psi^+_I(.))^+|a\mathcal{A}(\psi^+_I(.))^+$, we can use $\sum a_i(\psi^+_I(.))a_i(\psi^+_I(.))^+ = \psi^-_I(.)\psi^+_I(.)^+$ to move $\psi^+_I(.)$ to the left. So by using the operator definition 3, we can get “Wick’s theorem” for infinite statistics fields.

Since the operators can’t be moved arbitrarily, there will be some limits on the Feynman rules. In fact, the step functions $\theta(x)$ do not just appear in propagators, but also affect the external lines. To see this, let’s take $\mathcal{H} = \psi^-\psi^-\psi^+ + \psi^-\psi^+\psi^+$ for example. Then $S$-operator contains a term $\theta(x-y)(\psi^-_I(.))^{+}(\psi^-_I(.))^{+}(\psi^-_I(.)^+)(\psi^-_I(.)^+)(\psi^+_I(.)^+)$, in which $\psi^-_I(.)^+$ and $\psi^+_I(.)^+$ are unchangeable. If the final state is $|p_1p_2|$ and the initial state is $|p_3p_4|$, such term will be a sum of two subgraphs, one with two external lines carrying momenta $p_1p_2$ at $x$ point, two external lines carrying momenta $p_3p_4$ at $y$ point and one internal line $\Delta(x - y)$, while the other with two external lines carrying momenta $p_3p_4$ at $x$ point, two external lines carrying momenta $p_1p_2$ at $y$ point and one internal line $\Delta(y - x)$. The propagator $\Delta(x - y)$ is defined as

$$\Delta_{lm}(x - y) \equiv i\theta(x-y)(\psi^+_I(x)\psi^-_I(y))$$

$$= (2\pi)^4 \int d^4q \frac{-P_{lm}^{(L)}(q)e^{i(q(x-y))}}{2\sqrt{q^2 + m^2(q^0 - \sqrt{q^2 + m^2} + i\epsilon)}},$$

in which $P_{lm}^{(L)}$ is defined in Chapter 6.2 in [18]. Noting that the position related term $e^{i(q(x-y))}$ is still the same as in conventional propagator, we can infer that the Feynman rules for external lines in momentum space (after integrating over the spacetime position $x, y$) are the same
as before. So the contribution of the external lines to the S-matrix are the same in the two subgraphs. If we denote \( \Delta_\alpha = (2\pi)^4 \int d^4 q (-P_{im}(q))/(2\sqrt{q^2 + m^2}(q^0 - \sqrt{q^2 + m^2} + ie)) \) as internal line contribution in momentum space, then the total S-matrix for this 1 \( \rightarrow \) 1' 2' progress is (external line terms) \( \cdot (\Delta_\alpha + \Delta_{-\alpha}) = \) (external line terms) \( \cdot \Delta_F \), in which \( \Delta_F \) is the conventional propagator in momentum space. So we expect that the Feynman rules for our new theory are still the same as before. This allows us to apply some traditional methods such as renormalization analysis.

Here we give two examples for infinite statistics field interactions. For simple, we consider a scattering process 1 \( \rightarrow \) 1' 2' and the interaction density is trilinear in a set of real scalar fields. First, for pure infinite statistics interaction, we take \( \mathcal{H} = \phi \phi^- \phi + \phi^- \phi^+ \phi^- \), the Feynman diagrams are presented in Fig. 1. Fig. 1.(a) describes the term \( \theta(x-y)(\phi^- (x)\phi^- (x)\phi^+ (y)) + x \leftrightarrow y \); Fig. 1.(b) describes the term \( \theta(x-y) \sum_i [ (\phi^- (x)\phi^- (x)\phi^+ (x))a_i (\phi^- (y)\phi^- (y)\phi^+ (y))] + x \leftrightarrow y \). Secondly, for the case that infinite statistics fields couple to a bosonic field, we take \( \mathcal{H} = \phi \phi \phi \phi B \), the Feynman diagrams are presented in Fig. 2. Fig. 2.(a) describes the term \( \theta(x-y)(\phi B (x)\phi B (x)\phi B (y)) + x \leftrightarrow y \); Fig. 2.(b) describes the term \( \theta(x-y) [ \phi B (x)\phi B (y)] \sum_i [ (\phi^+ (x)\phi^+ (x)\phi B (x))]a_i (\phi^- (y)\phi^- (y)\phi B (y)) + (\phi^- (x)\phi^+ (x))a_i (\phi^- (y)\phi^+ (y))] + x \leftrightarrow y \). We see that those processes breaking the conservation of statistics as presented in Fig. 3 are excluded.

VI. CONCLUSIONS

In this paper, we have showed that the scattering processes involving infinite statistics particles are Lorentz invariant. This proof is directly based on the infinitesimal Lorentz transformations on the S-matrix. By applying the condition that the energies are additive for product states, we have showed this theory can obey conservation of statistics with selected interaction Hamiltonian. For all the above reasons, we conclude that the relativistic quantum field theory can also involve infinite statistics particles. For infinite statistics part of this theory, the basic fields should be both annihilation fields \( \psi^+ (x) \) and creation fields \( \psi^- (x) \), while the interaction density has a nonlocal definition \( \mathcal{A}(\phi^+(x)) \) in which \( \mathcal{H}^I(x) \) should take the form Eq. [5] with \( N,M \geq 1 \). Since we have showed that the conventional Feynman rules are still valid, some traditional methods such as renormalization analysis can also be extended to our new theory.

Acknowledgments

This work is supported in part by the NNSF of China Grant No. 90503009, No. 10775116, and 973 Program Grant No. 2005CB724508.

APPENDIX

In this Appendix, we calculate the commutation relation \( [\mathcal{A}(\mathcal{H}(x)), \mathcal{A}(\mathcal{H}_J(x'))] \). Here we also consider one single species of particle for simple. As we have shown in Sec. [III] this commutation can be decomposed into a sum of \( g_{\alpha \beta} [\mathcal{A}(\mathcal{H}_\alpha (x)), \mathcal{A}(\mathcal{H}_J(x'))] \), in which \( \mathcal{H}_\alpha, \mathcal{H}_J \) are products of definite numbers of annihilation fields and creation fields. Here we denote the total numbers of fields in \( \mathcal{H}_\alpha, \mathcal{H}_J \) are \( j, l \), while the numbers of creation fields are \( i, k \). Then we only need to calculate the relation \( [\mathcal{A}(a_1^\dagger \cdots a_k^\dagger a_{p_1} \cdots a_{p_l}) , \mathcal{A}(a_{p_1}^\dagger \cdots a_{p_k}^\dagger a_{p_{i+1}} \cdots a_{p_l})] \).

Firstly, we write the product \( \mathcal{A}(a_{p_1}^\dagger \cdots a_{p_k}^\dagger a_{p_{i+1}} \cdots a_{p_l}) \).
\( \mathcal{A}(a^+_p \cdots a^+_p a_{p_{k+1}} \cdots a_{p_l}) \) as

\[
\mathcal{A}(a^+_p \cdots a^+_p a_{p_{k+1}} \cdots a_{p_l}) \cdot \mathcal{A}(a^+_p \cdots a^+_p a_{p_{k+1}} \cdots a_{p_l}) \\
= \mathcal{O}_{11} + \mathcal{O}_{12} + \mathcal{O}_{13} + \cdots + \mathcal{O}_{1n} + \cdots \\
\mathcal{O}_{21} + \mathcal{O}_{22} + \mathcal{O}_{23} + \cdots + \mathcal{O}_{2n} + \cdots \\
\mathcal{O}_{31} + \mathcal{O}_{32} + \mathcal{O}_{33} + \cdots + \mathcal{O}_{3n} + \cdots \\
\vdots \\
\mathcal{O}_{m1} + \mathcal{O}_{m2} + \mathcal{O}_{m3} + \cdots + \mathcal{O}_{mn} + \cdots \\
\ldots.
\]

(A.1)

in which \( \mathcal{O}_{m_n} \) is defined as the product of the \( m \)-th term in \( \mathcal{A}(\mathcal{H}_a) \) and the \( n \)-th term in \( \mathcal{A}(\mathcal{H}_b) \).

\[
\mathcal{O}_{m_n} = (a^+_{q_1} \cdots a^+_{q_{m-1}} (a^+_p \cdots a^+_p a_{p_{k+1}} \cdots a_{p_l}))_{a_{q_{m-1}} \cdots a_{q_l}} \\
= (a^+_{q_1} \cdots a^+_{q_{n-1}} (a^+_p \cdots a^+_p a_{p_{k+1}} \cdots a_{p_l}))_{a_{q_{n-1}} \cdots a_{q_l}}.
\]

(A.2)

It’s not difficult to find that \( \mathcal{O}_{m_n} \) have a recursion relation

\[
\mathcal{O}_{(m+1)(n+1)} = \sum_q a^+_q \mathcal{O}_{mn} a_q. 
\]

(A.3)

Thus we can denote \( \mathcal{O}_{m_n} + \mathcal{O}_{(m+1)(n+1)} + \mathcal{O}_{(m+2)(n+2)} + \cdots + \mathcal{O}_{(m+n)(n+n)} + \cdots \) by \( \mathcal{A}(\mathcal{O}_{m_n}) \). Then the product \( \mathcal{A}(\mathcal{O}_{m_n}) \) can be simplified as

\[
\mathcal{A}(a^+_p \cdots a^+_p a_{p_{k+1}} \cdots a_{p_l}) \cdot \mathcal{A}(a^+_p \cdots a^+_p a_{p_{k+1}} \cdots a_{p_l}) \\
= \mathcal{A}(\mathcal{O}_{11}) + \mathcal{A}(\mathcal{O}_{12}) + \mathcal{A}(\mathcal{O}_{13}) + \cdots + \mathcal{A}(\mathcal{O}_{1n}) + \cdots \\
\mathcal{A}(\mathcal{O}_{21}) + \mathcal{A}(\mathcal{O}_{22}) + \mathcal{A}(\mathcal{O}_{23}) + \cdots + \mathcal{A}(\mathcal{O}_{2n}) + \cdots \\
\mathcal{A}(\mathcal{O}_{31}) + \mathcal{A}(\mathcal{O}_{32}) + \mathcal{A}(\mathcal{O}_{33}) + \cdots + \mathcal{A}(\mathcal{O}_{3n}) + \cdots \\
\vdots \\
\mathcal{A}(\mathcal{O}_{m1}) + \mathcal{A}(\mathcal{O}_{m2}) + \mathcal{A}(\mathcal{O}_{m3}) + \cdots + \mathcal{A}(\mathcal{O}_{mn}) + \cdots \\
\ldots.
\]

(A.4)

We can also define \( \mathcal{O}'_{m_n} \) as the product of the \( m \)-th term in \( \mathcal{A}(\mathcal{H}_a) \) and the \( n \)-th term in \( \mathcal{A}(\mathcal{H}_b) \). Then the commutation becomes

\[
[A(a^+_p \cdots a^+_p a_{p_{k+1}} \cdots a_{p_l}), A(a^+_p \cdots a^+_p a_{p_{k+1}} \cdots a_{p_l})] \\
= A(\mathcal{O}_{11}) + A(\mathcal{O}_{12}) + A(\mathcal{O}_{13}) + \cdots + A(\mathcal{O}_{1n}) + \cdots \\
A(\mathcal{O}_{21}) + A(\mathcal{O}_{22}) + A(\mathcal{O}_{23}) + \cdots + A(\mathcal{O}_{2n}) + \cdots \\
A(\mathcal{O}_{31}) + A(\mathcal{O}_{32}) + A(\mathcal{O}_{33}) + \cdots + A(\mathcal{O}_{3n}) + \cdots \\
\vdots \\
A(\mathcal{O}_{m1}) + A(\mathcal{O}_{m2}) + A(\mathcal{O}_{m3}) + \cdots + A(\mathcal{O}_{mn}) + \cdots \\
\mathcal{A}(\mathcal{O}_{(j-i+m)}) - \mathcal{A}(\mathcal{O}_{(i+m)}) = 0 \\
\mathcal{A}(\mathcal{O}_{(k+n-1)}) - \mathcal{A}(\mathcal{O}_{(i+k+n)}) = 0, \quad m \geq 1.
\]

(A.5)

(A.6)

\( \mathcal{O}_{m_n} \) (or \( \mathcal{O}'_{m_n} \)) in the remaining \( j + l - 2 \) terms are some permutations of \( (a^+_p \cdots a^+_p a_{p_{k+1}} \cdots a_{p_l}) \cdot (a^+_p \cdots a^+_p a_{p_{k+1}} \cdots a_{p_l}) \). Moreover, if the interaction density \( \mathcal{H}(x) \) is defined in Eq. (8) with \( N, M \geq 1 \), then by using Eq. (A.4) each term in \( [\mathcal{A}(\mathcal{H}(x)), \mathcal{A}(\mathcal{H}(x'))] \) involves \( \psi^+(x)\psi^-(x') \) (or \( \psi^+(x')\psi^-(x) \)). So \( \mathcal{A}(\mathcal{H}(x)), \mathcal{A}(\mathcal{H}(x')) \rightarrow 0 \), as \( x \rightarrow x' \rightarrow \infty \) spacelike, which is a necessary condition for Eq. (21).

[1] O.W. Greenberg, AIP Conf.Proc.545:113-127 (2004), hep-th/0007054.
[2] O.W. Greenberg, Phys. Rev. D 43, 4111 (1991).
[3] O.W. Greeberg, Phys. Rev. Lett. 64, 705 (1990).
[4] A. Strominger, Phys. Rev. Lett. 71, 3397 (1993).
[5] I.V. Volovich, hep-th/9601837.
[6] D. Minic, hep-th/9712202.
[7] Y.J. Ng, Phys. Lett. B 657, 10 (2007).
[8] Y.J. Ng, arXiv:0801.2962 [hep-th].
[9] V. Jejjala, M. Kavic and D. Minic, Adv. High Energy Phys. 2007, 21586 (2007).
[10] M. Li, X.D. Li, C.S. Lin and Y. Wang, Commun. Theor. Phys. 51, 181 (2009).
[11] A.J.M. Medved, Gen. Rel. Grav. 41, 287 (2009).
[12] M. R. Douglas, Phys. Lett. B 344, 117 (1995); M. R. Douglas and M. Li, Phys. Lett. B 348, 360 (1995).
[13] R. Gopakumar and D. Gross, Nucl. Phys. B 451, 379 (1995).
[14] I.Arefeva and I.Volovich, Nucl. Phys. B 462, 600 (1996).
[15] V.Shevchenko, arXiv:0812.0185 [hep-th].
[16] Yi-Xin Chen and Yong Xiao, arXiv:0812.3466 [hep-th].
[17] Chi-Keung Chow and O.W. Greenberg, Phys. Lett. A 283, 20 (2001).
[18] S. Weinberg, The Quantum Theory of Fields, [Cambridge University Press, Cambridge, 1995], Vol.1.
[19] See Chapter 5.1 for details. The derivations of fields such as \( \mathcal{O}'_a \) can be seen as new fields with redefinitions of \( u_i, v_i \).
[20] Here we consider only one single species of particle which has no distinct antiparticle for simple. Since other cases can be treated as involving some intrinsic indices (i.e. adding some delta functions such as \( \delta(mn) \)), they won't affect our results.
[21] More explicitly, the correct formulation should be \( \langle \beta(\int d^3 x \int d^3 y \times [\mathcal{H}(x, 0), \mathcal{H}(y, 0)]) f(H, H_0) \rangle \); where \( f(H, H_0) \) is a polynomial of \( H, H_0 \). Since these Hamiltonians won’t change the momentum conservation relation in \( [\mathcal{H}(x, 0), \mathcal{H}(y, 0)] \) while acting on the initial or final state, we conclude that this won’t affect our results.
[22] One can see Chapter 5 in [18] for more details.
have proved the existence of relativistic infinite statistics field theory, those analyzes based on Lorentz group are still valid.