Three dimensional stationary cyclic symmetric Einstein–Maxwell solutions; energy, mass, momentum, and algebraic tensors characteristics

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The main purpose of this contribution is to determine physical and geometrical characterizations of whole classes of stationary cyclic symmetric gravitational fields coupled to Maxwell electromagnetic fields within the (2 + 1)–dimensional gravity. The physical characterization is based on the determination of the local and global energy–momentum–mass quantities using the Brown–York approach. As far as to the algebraic–geometrical characterization is concerned, the eigenvalue problem for the electromagnetic field, energy–momentum and Cotton tensors is solved and their types are established.

The families of Einstein–Maxwell solutions to be considered are: all uniform electromagnetic solutions possessing electromagnetic fields with vanishing covariant derivatives (stationary uniform and spinning Clement classes), all fields having constant electromagnetic field and energy–momentum tensors’ invariants (Kamata–Koikawa solutions), the whole classes of hybrid electromagnetic Ayon–Cataldo–Garcia solutions, a new family of stationary electromagnetic solutions, the electrostatic and magnetostatic solutions with Peldan limit, the Clement spinning charged metric, the Martínez–Teitelboim–Zanelli black hole solution, and Dias–Lemos electromagnetic solution.

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I. INTRODUCTION

During the last two decades three-dimensional gravity has received some attention, in particular, in topics such as: black hole physics, search of exact solutions, quantization of fields coupled to gravity, cosmology, topological aspects, and others. This interest in part has been motivated by the discovery, in 1992, of the 2 + 1 stationary circularly symmetric black hole solution by Bañados, Teitelboim and Zanelli [1]–the BTZ black hole—see also [2–4], which possesses certain features inherent to 3 + 1 black holes. On the other hand, it is believed that 2 + 1 gravity may provide new insights towards a better understanding of the physics of 3 + 1 gravity. In the framework of 2 + 1 gravity the list of references on exact solutions is extremely vast; one finds works on point masses, perfect fluid solutions, dilaton and string fields, electromagnetic fields coupled to gravity, and cosmologies, among others.

The general form of electromagnetic fields for stationary cyclic symmetric 2+1 spacetimes is given by: $\ast F = a dt + b d\phi + c g_{rr}/\sqrt{-g} dr$, which splits into various sub–families: the electric $b \neq 0$ fields, the magnetic $a \neq 0$ fields, the uniform fields characterized by the vanishing of the covariant derivatives $F_{\alpha\beta\gamma} = 0$, the class of stationary fields with constant invariant $F_{\mu\nu} F^{\mu\nu}$, and consequently, due to the structure of the electromagnetic fields, with constant energy–momentum tensor invariants, the gravitational stationary cyclic solutions for the hybrid electromagnetic field $\ast F = c g_{rr}/\sqrt{-g} dr$: the explicit derivation of the solutions belonging to the quoted branches can be found in [5].

This report is organized as follows. Section II is devoted to the determination of the algebraic types of the Cotton tensor for a generic stationary cyclic symmetric metric. In section I a
detailed derivation of the energy, momentum and mass quantities for this stationary cyclic symmetric metric is accomplished; these characteristic expressions will be evaluated in the next sections for each of the spacetimes to be considered. In particular, since the static and the stationary BTZ solutions are considered as limits for vanishing electromagnetic fields, it is convenient to have at hand their energy–momentum characteristics, this is done in Section \[IV\]. Next sections, following all a similar pattern, are devoted to the determination of the energy and momentum densities as well as of the corresponding integral energy, momentum and mass. Emphasis is put on the asymptotic behavior of these quantities at spatial infinity.

The application of the Hayward black hole dynamics formulation and the Ashtekar isolated horizon approach to the reported here static and stationary black hole solutions is straightforward.

II. COTTON TENSOR ALGEBRAIC CLASSIFICATION

In \((n + 1)\)-dimensional space–times, for \(n > 3\), the invariant decomposition of the Riemannian curvature tensor gives rise to the conformal Weyl tensor, the traceless Ricci tensor, and the scalar curvature; for the classification of gravity one classifies the Weyl tensor, and the classification of matter is achieved through the classification of the traceless Ricci tensor. For details, in \((3 + 1)\)-dimensional space–times, see for instance, the book \[6\].

In \((2 + 1)\)-dimensional space–times there is no room for the conformal Weyl tensor, the Riemannian curvature tensor decomposes into the Ricci tensor, and the scalar curvature. The role of the conformal tensor in 2+1 gravity is played by the Cotton tensor, see \[6\], which is defined by means of the Ricci tensor and the scalar curvature through their covariant derivatives

\[ C^{\alpha\beta} = \epsilon^{\alpha\gamma\delta}(R^{\beta\gamma} - \frac{1}{4}R \delta^{\beta\gamma})_{;\delta}, \quad C^{\alpha}_{\alpha} = 0. \]  

(2.1)

For the standard stationary (static) cyclic symmetric metric

\[ ds^2 = -N^2 dt^2 + L^{-2} dr^2 + K^2 [d\phi + W dt]^2, \]

the traceless Cotton tensor, in the form \(C^{\alpha}_{\beta}\), occurs to be

\[
(C^{\alpha}_{\beta}) = \begin{bmatrix}
C^1_1 & 0 & C^1_3 \\
0 & C^2_2 & 0 \\
C^3_1 & 0 & C^3_3
\end{bmatrix}; \quad C^1_1 + C^2_2 + C^3_3 = 0. \]

(2.2)

Determining the eigenvalues and eigenvectors of the Cotton matrix \(C^{\alpha}_{\beta}\) one establishes the algebraic Cotton type of the space–time one is dealing with. Accordingly, the characteristic equation for the eigenvalue \(\lambda\) amounts to

\[ (C^2_2 - \lambda) \left((C^1_1 - \lambda)(C^3_3 - \lambda) - C^3_1 C^1_3\right) = 0, \]

(2.3)

or, in terms of its solutions, as

\[
(C^2_2 - \lambda) \left(\lambda + \frac{1}{2}C^2_2 + \frac{1}{2}\sqrt{(C^2_2 + 2C^1_1)^2 + 4C^3_1 C^1_3}\right) \times \\
\left(\lambda + \frac{1}{2}C^2_2 - \frac{1}{2}\sqrt{(C^2_2 + 2C^1_1)^2 + 4C^3_1 C^1_3}\right) = 0
\]

(2.4)
while the eigenvector equations are

\[
(C^1 - \lambda)V^1 + C^1_3 V^3 = 0, \\
(C^2 - \lambda)V^2 = 0, \\
C^3_1 V^1 + (C^3 - \lambda)V^3 = 0.
\]  

For each eigenvalue the corresponding solution is:

\[
\begin{align*}
\lambda_1 &= C^2, \ V1 = (0, V^2, 0), \\
\lambda_2 &= -\frac{1}{2}C^2 + \frac{1}{2}\sqrt{(C^2 + 2C^1)^2 + 4C^3_1C^1_3}, \ V2 = (V^1 = -\frac{C^1_3}{C^1_1 - \lambda_2} V^3, 0, V^3), \\
\lambda_3 &= -\frac{1}{2}C^2 - \frac{1}{2}\sqrt{(C^2 + 2C^1)^2 + 4C^3_1C^1_3}, \ V3 = (V^1 = -\frac{C^1_3}{C^1_1 - \lambda_3} V^3, 0, V^3).
\end{align*}
\]

Thus, the eigenvector \(V1\) – a real one – is oriented in the \(\rho\)–direction, the remaining two vectors \(V2\) and \(V3\) might be real vectors lying on the surface spanned by the \(t\) and \(\phi\) coordinate directions or complex eigenvectors depending, correspondingly, upon whether the value of the radical \((C^2^2 + 2C^1)^2 + 4C^3_1C^1_3\) is positive or negative.

The nomenclature to be used for eigenvectors and algebraic types of tensors is borrowed from Plebanski’s monograph \[7\], Chapter VI: time-like, space–like, null, and complex vectors are denoted respectively by \(T\), \(S\), \(N\), and \(Z\). For algebraic types are used the symbols: \(\{\lambda_1 T, \lambda_2 S, \lambda_3 S\} \equiv \{T, S, S\}\), meaning that the first real eigenvalue \(\lambda_1\) gives raise to a time–like eigenvector \(T\), the second real eigenvalue \(\lambda_2\) is associated with a space–like eigenvector \(S_2\), finally the third real eigenvalue \(\lambda_3\) is related to a space–like eigenvector \(S_3\); for the sake of simplicity I use the typing \(\{T, S, S\}\). It is clear that \(\{N, N, S\}\) stands for the algebraic type allowing for two different real eigenvalues giving rise to two null eigenvectors while the third real root is associated with a space–like eigenvector. When there are a single and a double real eigenvalues giving rise correspondingly to a time–like and space–like eigenvectors, the algebraic type is denoted by \(\{T, 2S\}\), consequently, for a triple real eigenvalue, if that were the case, the types could be \(\{3T\}\), \(\{3N\}\), or \(\{3S\}\). For a complex eigenvalue \(\lambda_Z\), in general, the related eigenvectors occur to be complex and are denoted by \(Z\) and \(\bar{Z}\) its complex conjugated, the possible types are \(\{T, Z, \bar{Z}\}\), \(\{N, Z, \bar{Z}\}\), or \(\{S, Z, \bar{Z}\}\).

In general, the spaces described by the stationary (static) cyclic symmetric metric above belong to the Cotton type \(I\); if the three eigenvectors are real the type is \(I_R\), otherwise the type is \(I_Z\) with eigenvectors \(S\), \(N\), \(T\), \(Z\), and \(\bar{Z}\). Following the notation above–proposed, the algebraic types for the Cotton tensor could be: \(\{S, S, S\}\), \(\{S, N, N\}\), \(\{S, Z, Z\}\), and so on.

An alternative treatment of the Cotton tensor and conformal symmetries for \((2 + 1)\)-dimensional spaces is given in \[8\], and also in \[9\], where also is developed the analysis on Cotton tensors in \(n+1\)–dimensions.

III. ENERGY, MASS, AND MOMENTUM FOR \(2 + 1\) STATIONARY CYCLIC SYMMETRIC METRIC

In this section is established the general form of the energy and the momentum functions for metrics with non-flat but anti–de Sitter asymptotic following the Brown–York approach.
The (2 + 1)–dimensional stationary cyclic symmetric metric to be used is given by
\[ ds^2 = -N^2 dt^2 + L^{-2} dr^2 + K^2 [d\phi + W dt]^2, \] (3.1)
where the structural functions \( N, \ L, \ K, \) and \( W \) depend on the variable \( r. \) The timelike vector \( u_\mu \) normal to the hypersurface \( \Sigma : t_\Sigma = \text{const}. \) and the spacelike vector \( n_\mu \) normal to the surface \( B : r_B = R = \text{const}. \) are given by
\[ u_\mu = -N \delta_\mu^t, \ u^\mu = \frac{1}{N} \delta_t^\mu - \frac{W}{N} \delta_\phi^\mu, \]
\[ n_\mu = \frac{1}{L} \delta_\mu^r, \ n^\mu = L \delta_r^\mu. \] (3.2)
Therefore the projection metrics are:
\[ ds^2 |_{\Sigma : t = \text{const}.} = L^{-2} dr^2 + K^2 d\phi^2 = h_{ij} dx^i dx^j, \]
\[ ds^2 |_{r \to R = \text{const}.} = -N^2 dt^2 + K^2 (d\phi + W dt)^2 = \gamma_{AB} dx^A dx^B, \]
\[ ds^2 |_{B : t = \text{const}. , r = \text{const}.} = K^2 d\phi^2 = \sigma_{ij} dx^i dx^j. \] (3.3)
The components of the projection tensor \( h \) are explicitly given by
\[ h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \]
\[ h_{\mu\nu} = K^2 W^2 \delta_\mu^t \delta_\nu^t + W K^2 (\delta_\mu^t \delta_\nu^\phi + \delta_\mu^\phi \delta_\nu^t) + L^{-2} \delta_\mu^r \delta_\nu^r + K^2 \delta_\mu^\phi \delta_\nu^\phi, \]
\[ h_{ij} = L^{-2} \delta^t_i \delta^t_j + K^2 \delta^\phi_i \delta^\phi_j, \ \det(h_{ij}) = K^2 / L^2, \]
\[ h_{\mu\nu} = L^2 \delta_\mu^t \delta_\nu^t + \delta_\mu^\phi \delta_\nu^\phi / K^2, \ h_{\mu\nu} = W \delta_\mu^t \delta_\nu^\phi + \delta_\mu^\phi \delta_\nu^t + \delta_\mu^\phi \delta_\nu^\phi, \]
\[ h_{\mu\nu} = W \delta_\mu^t \delta_\nu^t + \delta_\mu^r \delta_\nu^r + \delta_\mu^\phi \delta_\nu^\phi, \ h_{ij} = \delta_i^j. \] (3.4)
The components of the projection tensor \( \gamma \) amount to
\[ \gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu, \]
\[ \gamma_{\mu\nu} = -(N^2 - K^2 W^2) \delta_\mu^t \delta_\nu^t + W K^2 (\delta_\mu^t \delta_\nu^\phi + \delta_\mu^\phi \delta_\nu^t) + K^2 \delta_\mu^\phi \delta_\nu^\phi, \]
\[ \gamma_{\mu\nu} = -\frac{1}{N^2} \delta_\mu^t \delta_\nu^t - \frac{W}{N^2} (\delta_\mu^t \delta_\nu^\phi + \delta_\mu^\phi \delta_\nu^t) + \frac{N^2 - K^2 W^2}{K^2 N^2} \delta_\mu^\phi \delta_\nu^\phi, \]
\[ \gamma_{\mu\nu} = \delta_\mu^t \delta_\nu^t + \delta_\mu^\phi \delta_\nu^\phi, \ \det(\gamma_{AB}) = -K^2 N^2. \] (3.5)
Notice that the indices \( \mu, \nu \) associated to the three-dimensional spacetime can be replaced by indices \( A, B \) running 0 \( \sim t, \) 3 \( \sim \phi. \) The components of the projection tensor \( \sigma \) amount to
\[ \sigma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu - n_\mu n_\nu, \]
\[ \sigma_{\mu\nu} = K^2 W^2 \delta_\mu^t \delta_\nu^t + W K^2 (\delta_\mu^t \delta_\nu^\phi + \delta_\mu^\phi \delta_\nu^t) + K^2 \delta_\mu^\phi \delta_\nu^\phi, \]
\[ \sigma_{\mu\nu} = \delta_\mu^t \delta_\nu^t / K^2, \ \sigma_{ij} = \delta_\phi^i \delta_\phi^j / K^2, \ \det(\sigma_{ij}) = K^2, \] (3.6)
where \( i, j \) run only 3 \( \sim \phi. \)
To evaluate extrinsic curvatures one needs the expressions of the symmetric Christoffel symbols, which amount to

\[
\Gamma_{tr}^t = \frac{1}{2N^2} (2NN_r - K^2WW_r), \quad \Gamma_{t\phi}^t = -\frac{1}{2N^2} K^2W_r, \\
\Gamma_{tt}^r = L^2 (NN_r - KW^2K_r - K^2WW_r), \\
\Gamma_{t\phi}^r = -\frac{1}{2} L^2K (2WK_r + KW_r), \\
\Gamma_{rr}^\phi = \frac{1}{2} L^2W_r, \quad \Gamma_{\phi\phi}^r = -L^2KK_r, \\
\Gamma_{t\phi}^\phi = \frac{1}{2} L^2 (\frac{1}{2} W K^3W + W^2K_3W + 2N^2WK_r + N^3KW_r), \\
\Gamma_{r\phi}^\phi = \frac{1}{2} L^2 (W^3K + NW^2W_r), (3.7)
\]

while all other components vanish.

The extrinsic curvature to the hypersurface \( \Sigma : t = \text{const.} \) is given by the spatial tensor \( K_{\mu\nu} \), namely

\[
K_{\mu\nu} = -h_{\mu}^\alpha \nabla_\alpha u_\nu = -h_{\mu\beta} g^\beta\alpha u_\nu, \\
K_{\mu\nu} = \frac{1}{2N} K^2WW_r (\delta_\mu^\nu \delta_\sigma^\rho + \delta_\mu^\rho \delta_\sigma^\nu), \\
K^{\mu\nu} = \frac{1}{2N} L^2W_r (\delta_\mu^\nu \delta_\phi^\rho + \delta_\rho^\nu \delta_\phi^\mu), \\
K_{\mu^\nu} = \frac{1}{2N} L^2K^2WW_r \delta_\mu^\nu + \frac{1}{2N} L^2K^2W_r \delta_\mu^\rho \delta_\rho^\nu + \frac{1}{2N} W_r \delta_\mu^\nu \delta_\phi^\rho, (3.8)
\]

thus the trace of \( K_{\mu\nu} \) is zero, \( K_{\mu}^\mu = 0 \).

The momentum tensor \( P^{\mu\nu} = \frac{1}{2\kappa} \sqrt{\det(h_{ij})} [K^\alpha_{\mu} h^{\mu\nu} - K^{\mu\nu}] \) for the hypersurface \( \Sigma \) becomes

\[
P^{\mu\nu} = -\frac{1}{4\kappa N} LKW_r (\delta_\mu^\nu \delta_\phi^\rho + \delta_\rho^\nu \delta_\phi^\mu), (3.9)
\]

while the surface momentum density vector \( j_\mu = -2\sigma_{\mu\nu} P^{\nu\alpha}n_\alpha / \sqrt{\det(h_{ij})} \) amounts to

\[
j_\mu = \frac{1}{2\kappa N} L K^2WW_r \delta_\mu^\nu + \frac{1}{2\kappa N} L K^2W_r \delta_\mu^\phi. (3.10)
\]

Consequently the surface momentum density \( j_\phi \) reduces in the studied case to

\[
j_\phi = -2\frac{1}{K} \sigma_{\phi\phi} P^{\phi\nu} = \frac{1}{2\kappa N} L K^2W_r. (3.11)
\]

modulo the additive constant related to the reference spacetime.

The energy density is evaluated by using the tensor

\[
k_{\mu\nu} = -\sigma_{\mu}^\alpha h_{\alpha}^\beta n_\lambda \Theta_{\beta\nu}, \quad \Theta_{\mu\nu} = -\gamma_{\mu}^\beta n_{\nu\beta},
\]
which amounts to

$$k_{\mu\nu} = -LKWK_{,r}[W\delta_\mu^t \delta_\nu^t + 2\delta_\mu^\phi \delta_\nu^\phi] - LKK_{,r}\delta_\mu^\phi \delta_\nu^\phi,$$

(3.12)

while

$$k^\mu_\nu = \frac{1}{K}LWK_{,r}\delta_\mu^\phi \delta_\nu^t - \frac{1}{K}LKK_{,r}\delta_\mu^\phi \delta_\nu^\phi.$$  

(3.13)

Rising with $\sigma_{\phi\phi} = K^2 = \sigma^{\phi\phi}$ one of the indexes of the component $k_{\phi\phi} = -LKK_{,r}$ of the extrinsic curvature $k$ associated to the metric of $\mathring{\mathcal{B}}$, one arrives at

$$k := k_i^i = \sigma^{\phi\phi} k_{\phi\phi} = -\frac{1}{K}LKK_{,r}.$$  

(3.14)

Therefore the energy density $\epsilon$ becomes

$$\epsilon = \frac{1}{\kappa}k|_{cl} = -\frac{1}{\kappa K}LKK_{,r}|_R - \epsilon_0.$$  

(3.15)

As far as to the integral characteristics is concerned, the total quasilocal energy $E = \int_B d\mathbf{x}\sqrt{\sigma}\epsilon = 2\pi K \epsilon$ is given by

$$E = -2\pi \frac{LK}{K}K_{,r}|_R - 2\pi K(R)\epsilon_0,$$  

(3.16)

while the mass related to the timelike Killing vector $\xi^\mu = (\frac{\partial}{\partial t})^\mu = \delta_t^\mu$, $M(\frac{\partial}{\partial t}) = -\int_B d\mathbf{x}\sqrt{\sigma}(\epsilon u_\mu + j_\mu)\xi^\mu$ amounts to

$$M(\frac{\partial}{\partial t}) = -2\pi \frac{NKLK_{,r}|_R}{K} - \frac{\pi L}{K N}K^3WW_{,r}|_R - 2\pi NK|_R\epsilon_0.$$  

(3.17)

Finally the total momentum $J(\frac{\partial}{\partial \phi}) = \int_B d\mathbf{x}\sqrt{\sigma}j_\mu\zeta^\mu$ associated to the Killing vector $\zeta^\mu = (\frac{\partial}{\partial \phi})^\mu = \delta_\phi^\mu$, is given by

$$J(\frac{\partial}{\partial \phi}) = \int_0^{2\pi} d\phi J_\phi K = \frac{\pi L}{K N}K^3W_{,r}|_R, j_\phi = \frac{1}{2\pi}J(R).$$  

(3.18)

Incidentally, other representations of the mass and momentum density are:

$$M(\frac{\partial}{\partial t}) = N(R)E(R) - W(R)J(R).$$  

(3.19)

The extrinsic curvature $\Theta_{\mu\nu} = -\gamma_\mu^\alpha \nabla_\alpha n_\nu = -n_\nu;\alpha \gamma_\alpha^\mu$ of the surface boundary $\mathring{\mathcal{B}}$ reduces to

$$\Theta_{\mu\nu} = -L(KW^2 K_{,r} + K^2WW_{,r} - NN_{,r})\delta_\mu^t \delta_\nu^t - LKK_{,r}\delta_\mu^\phi \delta_\nu^\phi,$$

(3.20)

with trace $\Theta$ equals to

$$\Theta = -\frac{L}{NK}(KN_{,r} + NK_{,r}).$$  

(3.21)
which, used in the definition of the boundary momentum

\[ \pi^{\mu\nu} = -\frac{1}{2\kappa} \sqrt{-\det \gamma_{\alpha\beta}(\Theta^{\alpha\beta} - \Theta^{\mu\nu})}, \]

taking into account that \( \det \gamma_{AB} = -K^2 N^2 \), gives

\[ \pi^{\mu\nu} = -\frac{L}{2\kappa N} K, \delta_t^\mu \delta_t^\nu + \frac{L}{2\kappa N} (2WK, r + KW, r) \delta_t^\mu \delta_\phi^\nu \]
\[ + \frac{L}{2\kappa NK} (NN, r - W^2 KK, r - K^2 WW, r) \delta_\phi^\mu \delta_\phi^\nu. \]  
(3.22)

This tensor is used in the construction of the stress tensor

\[ s^{\alpha\beta} = \frac{2}{\sqrt{\sigma} N} \sigma^{\alpha}_\mu \pi^{\mu\nu} \sigma^{\beta}_\nu - s_0^{\alpha\beta}. \]  
(3.23)

IV. BAÑADOS–TEITELBOIM–ZANELLI BLACK HOLE SOLUTION

To get an insight of how efficiently the definitions of energy and momentum densities work for spacetimes with non-flat asymptotic at spatial infinity let us consider the asymptotically anti-de Sitter (2 + 1)–dimensional stationary black hole solution–the BTZ–metric-- which is given by

\[ ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + \rho^2 [d\phi + W(\rho)dt]^2, \]
\[ N^2(\rho) = L^2(\rho) = -M + \frac{\rho^2}{l^2} + \frac{J^2}{4\rho^2}, \]
\[ K(\rho) = \rho, \]
\[ W(\rho) = -\frac{J}{2\rho^2}. \]

(4.1)

For the choice

\[ u_\mu = -N \delta_t^\mu, u^\mu = \frac{1}{N} \delta_t^\mu - \frac{W}{N} \delta_\phi^\mu, n_\mu = \frac{1}{L} \delta_\mu^\rho, n^\mu = L \delta_\mu^\rho, \]

(4.2)

one has

\[ ds^2|_{\Sigma, t=\text{const.}} = L^{-2} d\rho^2 + \rho^2 d\phi^2 = h_{ij} dx^i dx^j, \]
\[ ds^2|_{B, \rho=R=\text{const.}} = -N^2 dt^2 + R^2 (d\phi + W dt)^2 = \gamma_{AB} dx^A dx^B, \]
\[ ds^2|_{B, t=\text{const.}, \rho=R=\text{const.}} = R^2 d\phi^2 = \sigma_{ij} dx^i dx^j. \]

(4.3)

In what follows \( \kappa \) is choosing as \( \kappa = \pi \). The surface tensors amount to

\[ P^{\mu\nu} = -\frac{1}{4\pi} \frac{J}{\rho^2} (\delta_\mu^\rho \delta_\phi^\nu + \delta_\rho^\mu \delta_\phi^\nu), \]
\[ j^\mu = -\frac{1}{4\pi} \frac{J^2}{\rho^3} \delta_\mu^\rho + \frac{1}{2\pi \rho} \delta_\phi^\rho, \]
\[ k = -\frac{1}{\rho} \sqrt{-M + \frac{\rho^2}{l^2} + \frac{J^2}{4\rho^2}}. \]

(4.4)

(4.5)

(4.6)
A. Energy, mass and momentum for the BTZ black hole

The corresponding surface energy and momentum densities are given by

\[ \epsilon(R, \epsilon_0) = -\frac{1}{\pi R} \sqrt{-M + \frac{R^2}{l^2} + \frac{J^2}{4R^2} - \epsilon_0}, \]
\[ j_\phi(R) = \frac{1}{2\pi R} J. \] (4.7)

Consequently the total momentum, energy, and mass are

\[ J(\partial/\partial \phi) = J, \]
\[ E(R, \epsilon_0) = -2 \sqrt{-M + \frac{R^2}{l^2} + \frac{J^2}{4R^2} - 2\pi \epsilon_0}, \]
\[ M(\partial/\partial t) = N(R) E(R, \epsilon_0) + \frac{J^2}{2R^2} = 2M - 2\frac{R^2}{l^2} - 2\pi \epsilon_0 \sqrt{-M + \frac{R^2}{l^2} + \frac{J^2}{4R^2}}. \] (4.8)

These expressions for surface densities and global quantities are in full agreement with the corresponding ones reported in Ref. [11], section IV.

Notice that the energy and mass independent of \( \epsilon_0 \) behave at infinity \( R \), which will be denoted from now on by the same coordinate Greek letter \( \rho \) accompanied by \( \rightarrow \infty \) and the approximation \( \approx \) sign, as

\[ \epsilon(\rho \rightarrow \infty, \epsilon_0 = 0) \approx -\frac{1}{\pi l} + \frac{l M}{2\pi \rho^2}, \]
\[ E(\rho \rightarrow \infty, \epsilon_0 = 0) \approx -\frac{2\rho}{l} + \frac{l M}{\rho}, \]
\[ M(\rho \rightarrow \infty, \epsilon_0 = 0) \approx 2M - 2\frac{\rho^2}{l^2}. \] (4.9)

Although the expression of \( M(\rho, \epsilon_0 = 0) \) holds in the whole spacetime and not only in the boundary at spatial infinity, the approximation \( \approx \) sign is used instead of the \( = \) equal to be consistent with the point under consideration.

The reference energy density to be used in this work is the one corresponding to the anti–de Sitter metric with parameter \( M_0 \), \( \epsilon_0(M_0) = -\frac{1}{\pi \rho} \sqrt{\frac{l^2}{\rho^2} - M_0}, \epsilon_0(\infty)(M_0) \approx -\frac{1}{\pi l} + \frac{M_0}{2\pi \rho^2} \), then the expansions of the physical characteristics at spatial infinity, \( \rho \rightarrow \infty \), are given as

\[ \epsilon(\rho \rightarrow \infty, \epsilon_0(\infty)(M_0)) \approx \frac{l}{2\pi \rho^2} (M - M_0), \]
\[ E(\rho \rightarrow \infty, \epsilon_0(\infty)(M_0)) \approx \frac{l (M - M_0)}{\rho}, \]
\[ M(\rho \rightarrow \infty, \epsilon_0(\infty)(M_0)) \approx M - M_0. \] (4.10)

Another reference energy density \( \epsilon_0 \) of common use is the one corresponding to the proper anti–de Sitter space with \( M_0 = -1 \), namely \( \epsilon_0 = -\frac{1}{\pi \rho} \sqrt{1 + \frac{l^2}{\rho^2}}, \epsilon_{AdS}(\infty) \approx -\frac{1}{\pi l} - \frac{l}{2\pi \rho^2}, \) then
the expansions of the functions (IV) at spatial infinity, $\rho \to \infty$, are given as
\[
\begin{align*}
\epsilon(\rho \to \infty, \epsilon_{AdS|\infty}) & \approx \frac{l}{2\pi \rho^2} (1 + M), \\
E(\rho \to \infty, \epsilon_{AdS|\infty}) & \approx l \left(1 + \frac{M}{\rho}\right), \\
M(\rho \to \infty, \epsilon_{AdS|\infty}) & \approx 1 + M. 
\end{align*}
\] (4.11)

These results, concerning approximations of energies and masses at spatial infinity for the BTZ solution, Eq. (4.9), Eq. (4.10), and Eq. (4.11), will be used for comparison with the corresponding expressions related to other solutions to be treated in what follows.

In the forthcoming sections, in some cases, the order in the approximations could appear high compared with the order needed but the adopted expansions will be done to establish to what extent the solutions are comparable or similar at spatial infinity.

**B. Mass, energy and momentum of the BTZ solution counterpart**

As it has been pointed by Peldan [13], the sentence after Eq.(84), one should be not able to integrate the magnetic branch of solutions if the Schwarzschild gauge, i.e. $g_{\phi\phi} = \rho^2$, were been adopted. For a $\rho$-gauge different of the Schwarzschild one, as the reference metric it is more adequate to consider the stationary or static BTZ solution counterpart and its mass, energy, and momentum characteristics at spatial infinity. In the representation
\[
g = -\rho^2 \left( dt + \frac{J}{2\rho^2} d\phi \right)^2 + \frac{d\rho^2}{f(\rho)} + f(\rho) d\phi^2
\]
\[
= -\rho^2 \frac{f(\rho)}{\rho^2/l^2 + M} dt^2 + \frac{d\rho^2}{f(\rho)} + (\rho^2/l^2 + M) [d\phi - J_0 \frac{J_{\phi}}{2(\rho^2/l^2 + M)} dt]^2,
\]
\[
f(\rho) = \frac{\rho^2}{l^2} + M + \frac{J^2}{4\rho^2},
\] (4.12)
of the BTZ metric the coordinate $\phi$ loses its interpretation of circular angular coordinate; the same fact takes place in the case of magnetic solutions to be studied below.

It becomes apparent that this metric form is just another real cut of the metric (4.1) when subjecting it to the complex transformations $t \to i \phi$ and $\phi \to i t$.

Moreover, by accomplishing in the above metric the transformation of the radial coordinate
\[
t \to t/l, \rho \to l \sqrt{\rho^2/l^2 - M}, \phi \to l \phi
\] (4.13)
one gets the standard BTZ metric (4.1).

As far as to the energy-momentum characteristics are concerned, for the second form of the metric (4.12), one has
\[
\begin{align*}
J_\phi(\rho) &= \frac{J}{2\pi l^2 \sqrt{\rho^2/l^2 + M}}, \quad J_\phi(\rho \to \infty) \approx \frac{J}{2\pi l^2}, \\
J(\rho \to \infty) &= \frac{J}{l^2}, \quad J_\phi(\rho \to \infty) \approx \frac{J}{l^2}
\end{align*}
\] (4.14a)
\[\epsilon(\rho, \epsilon_0) = -\frac{\rho}{l^2 \pi} \frac{\sqrt{f(\rho, M, J)}}{\sqrt{\rho^2/l^2 + M}} - \epsilon_0,\]
\[\epsilon(\rho \to \infty, 0) \approx -\frac{1}{\pi l} + \frac{l}{2\pi \rho^2},\]
\[\epsilon(\rho \to \infty, \epsilon_0|\infty(M_0)) \approx \frac{l}{2\pi} \frac{M - M_0}{\rho^2}.\]  
(4.14b)

\[E(\rho, \epsilon_0) = -\frac{2}{l^2} \rho \frac{\sqrt{f(\rho, M, J)}}{\sqrt{\rho^2/l^2 + M}} - 2\pi \epsilon_0 \sqrt{\rho^2/l^2 + M},\]
\[E(\rho \to \infty, 0) \approx -\frac{2}{l^2},\]
\[E(\rho \to \infty, \epsilon_0|\infty(M_0)) \approx \frac{1}{\rho} (M - M_0),\]  
(4.14c)

\[M(\rho, \epsilon_0) = -\frac{2}{l^2} \rho^2 - 2\pi \rho \epsilon_0 \sqrt{f(\rho, M, J)},\]
\[M(\rho, 0) = -\frac{2}{l^2} \rho^2,\]
\[M(\rho \to \infty, \epsilon_0|\infty(M_0)) \approx M - M_0,\]  
(4.14d)

\[f(\rho, M, J) = \frac{\rho^2}{l^2} + M + \frac{J^2}{4\rho^2},\]
\[\epsilon_0(\rho, M_0) = -\frac{\rho}{\pi l^2 \sqrt{\rho^2/l^2 + M_0}},\]
\[\epsilon_0|\infty(M_0) \approx -\frac{1}{\pi l} + \frac{l}{2\pi \rho^2}.\]  
(4.14e)

It is clear that the BTZ solution counterpart, for \(J = 0\), gives rise to the AdS metric in a slightly modified representation—which one may call “the AdS metric counterpart”. The evaluated energy and mass can be considered as the reference energy and mass at spatial infinity for magnetic solutions. The point is that for this class of magnet–static solutions or stationary solutions generated from them via \(SL(2, R)\) there is no room for a Schwarzschild radial \(\rho\) coordinate such that \(g_{\phi\phi} = \rho^2\).

C. Symmetries of the stationary and static cyclic symmetric BTZ solutions

Although it is known that the BTZ solution possesses two Killing vectors—the timelike symmetry along the time coordinate and the spacelike symmetry along the orbits of the periodic angular variable—in my opinion, some comments on this respect can be added to clarify how the number of six Killing vectors solutions for the BTZ metric structure reduces to the quoted two. In this framework, the six symmetries of the anti–de Sitter space with parameter \(M_0\), \(AdS(M_0)\), are derived; the \(AdS(M_0)\) allowing for time + polar coordinates possesses a timelike and one \(2\pi\)–periodic circular symmetries. For the anti–de Sitter space
with parameter $M_0 = -1$, denoted simply by $AdS$, there are six symmetries: time, circular, and four boots symmetries.

The study of the symmetries of the stationary and static cyclic symmetric BTZ families and AdS classes of solutions starts with the stationary metric for the standard BTZ solution

$$
g = -F(r)^2 \, dt^2 + \frac{dr^2}{F(r)^2} + r^2 \left[ d\phi - \frac{J}{2r^2} \, dt \right]^2, \quad F(r)^2 = \frac{r^2}{\ell^2} - M + \frac{J^2}{4r^2}. \quad (4.15)$$

The covariant Killing vectors are derived in Appendix B, the contravariant vectors’ components $k_i^\mu$, $\partial_{k_i} = k_i^\mu \frac{\partial}{\partial x^\mu} = C_i V_i^\mu$, $i = 1, \ldots 6$ are given below. The reason to include the integration constants in the definitions of the Killing vectors $k_i^\mu$ is related with the domain of definition of the spatial coordinates; in the case of the existence of a periodic coordinate some Killing vectors vanish, which can be easily achieved by setting certain structural constant equal to zero. Explicitly these Killing vectors are:

$$C_1, \partial_{k_1}; \ k_1^\mu = C_1 \exp \left( \frac{\sqrt{Ml - J}}{l^{3/2}} (l\phi + t) \right) \left[ \begin{array}{ccc} 1 & l J - 2r^2 & 1 \\ \frac{1}{4} \sqrt{Ml - J} \frac{r F(r)}{l^{3/2}} & 2 & l \\ -\frac{1}{4} \sqrt{Ml - J} \frac{r F(r)}{l^{5/2}} & 1 & 1 \end{array} \right],$$

$$C_2, \partial_{k_2}; \ k_2^\mu = C_2 \exp \left( \frac{\sqrt{J + Ml}}{l^{3/2}} (l\phi - t) \right) \left[ \begin{array}{ccc} 1 & l J + 2r^2 & 1 \\ \frac{1}{4} \sqrt{J + Ml} \frac{r F(r)}{l^{3/2}} & 2 & l \\ -\frac{1}{4} \sqrt{J + Ml} \frac{r F(r)}{l^{5/2}} & 1 & 1 \end{array} \right],$$

$$C_3, \partial_{k_3}; \ k_3^\mu = C_3 \exp \left( -\frac{\sqrt{J + Ml}}{l^{3/2}} (l\phi - t) \right) \left[ \begin{array}{ccc} -1 & l J + 2r^2 & 1 \\ \frac{1}{4} \sqrt{J + Ml} \frac{r F(r)}{l^{3/2}} & 2 & l \\ -\frac{1}{4} \sqrt{J + Ml} \frac{r F(r)}{l^{5/2}} & 1 & 1 \end{array} \right],$$

$$C_4, \partial_{k_4}; \ k_4^\mu = C_4 \exp \left( -\frac{\sqrt{Ml - J}}{l^{3/2}} (l\phi + t) \right) \left[ \begin{array}{ccc} 1 & J l - 2r^2 & 1 \\ \frac{1}{4} \sqrt{Ml - J} \frac{r F(r)}{l^{3/2}} & 2 & l \\ -\frac{1}{4} \sqrt{Ml - J} \frac{r F(r)}{l^{5/2}} & 1 & 1 \end{array} \right],$$

$$\partial_{k_5}; \ k_5^\mu = C_5 [0, 0, 1/2],$$

$$\partial_{k_6}; \ k_6^\mu = C_6 [-2, 0, 0].$$
For completeness, the list of the independent commutators is given

\[ \partial_{[k_6} \partial_{k_4]} = -2C_6 \frac{\sqrt{Ml - J}}{l^{3/2}} \partial_{k_1}, \partial_{[k_6} \partial_{k_3]} = -2C_6 \frac{\sqrt{Ml + J}}{l^{3/2}} \partial_{k_2}, \]

\[ \partial_{[k_5} \partial_{k_4]} = 2C_6 \frac{\sqrt{Ml - J}}{l^{3/2}} \partial_{k_1}, \partial_{[k_6} \partial_{k_2]} = 2C_6 \frac{\sqrt{Ml + J}}{l^{3/2}} \partial_{k_2}, \] (4.17a)

\[ \partial_{[k_3} \partial_{k_1]} = C_5 \frac{1}{2} \frac{\sqrt{Ml - J}}{l^{1/2}} \partial_{k_1}, \partial_{[k_5} \partial_{k_4]} = -C_5 \frac{1}{2} \frac{\sqrt{Ml + J}}{l^{1/2}} \partial_{k_1}, \]

\[ \partial_{[k_5} \partial_{k_4]} = -C_5 \frac{1}{2} \frac{\sqrt{Ml - J}}{l^{1/2}} \partial_{k_1}, \partial_{[k_6} \partial_{k_2]} = C_5 \frac{1}{2} \frac{\sqrt{Ml + J}}{l^{1/2}} \partial_{k_1}, \] (4.17b)

\[ \partial_{[k_1} \partial_{k_4]} = -\frac{1}{2} \frac{C_1 C_4}{C_6} \frac{1}{l^{3/2} \sqrt{Ml - J}} \partial_{k_6} + 2 \frac{C_1 C_4}{C_5} \frac{1}{l^{7/2} \sqrt{Ml - J}} \partial_{k_5}, \] (4.17c)

\[ \partial_{[k_3} \partial_{k_2]} = -\frac{1}{2} \frac{C_3 C_2}{C_6} \frac{1}{l^{5/2} \sqrt{Ml + J}} \partial_{k_6} + 2 \frac{C_3 C_2}{C_5} \frac{1}{l^{7/2} \sqrt{Ml + J}} \partial_{k_5}, \] (4.17d)

All anti–de Sitter metrics for coordinates \{t, ρ, φ\}–merely names–ranging \(-∞ \leq t \leq ∞, -∞ \leq ρ \leq ∞, -∞ \leq φ \leq ∞\) allows for six symmetries, i.e., six Killing vectors. All these spaces in these coordinates are maximally symmetric spaces.

Another is the situation if the spatial coordinates are constrained to range

\[ 0 \leq ρ \leq ∞, 0 \leq φ \leq 2π, \]

in such case ρ and φ are polar coordinates with φ being the angular coordinate with period 2π. Since the expressions of four of the Killing vector fields depending on φ the do not exhibit the angular symmetry in 2π, invariance under the change φ → φ + 2π, therefore there is no room for the corresponding symmetries and the integration constants associated with those vectors ought to be zero. Consequently the metric with positive M allowing for polar angular coordinate, and only that, possesses only two Killing vectors \(∂_t\) and \(∂_φ\) (two symmetries: the time translation and the 2π–periodic angular rotation). This spacetime is known as the stationary BTZ black hole. On this respect see also [12].

A similar situation takes place in the case of the static anti–de Sitter metric. By setting the rotation parameter equal to zero, \(J = 0\), the above–expressions (4.17) give the Killing vectors for the static anti–de Sitter space–time. Again, in the case of the coordinates restricted to ranges \(0 \leq ρ \leq ∞, 0 \leq φ \leq 2π\), the static anti–de Sitter metric allows only for two Killing vectors: \(∂_t, ∂_φ\), i.e., the time translation and the 2π–periodic angular rotation, otherwise, when there are six constants, the space is maximally symmetric.

D. Symmetries of the anti–de Sitter metric for negative \(M, M = -α^2\)

If \(M\) is negative, one equates it to \(-α^2\). Moreover, it results better to use trigonometric sine and cosine functions instead of complex exponential functions. Thus, one can give the
Killing vector components as

\[ \partial_{k_1}; k_1^\mu = C_1 \left[ \frac{r}{l \alpha \sqrt{\alpha^2 l^2 + r^2}} \sin (\alpha \phi) \cos \left( \frac{\alpha t}{l} \right), \frac{\sqrt{\alpha^2 l^2 + r^2}}{l^2 \alpha r} \sin (\alpha \phi) \sin \left( \frac{\alpha t}{l} \right) \right] \]

(4.18a)

\[ \partial_{k_2}; k_2^\mu = C_2 \left[ -\frac{r}{l \alpha \sqrt{\alpha^2 l^2 + r^2}} \sin (\alpha \phi) \sin \left( \frac{\alpha t}{l} \right), \frac{\sqrt{\alpha^2 l^2 + r^2}}{l^2 \alpha r} \sin (\alpha \phi) \cos \left( \frac{\alpha t}{l} \right) \right] \]

(4.18b)

\[ \partial_{k_3}; k_3^\mu = C_3 \left[ \frac{r}{l \alpha \sqrt{\alpha^2 l^2 + r^2}} \cos (\alpha \phi) \cos \left( \frac{\alpha t}{l} \right), \frac{\sqrt{\alpha^2 l^2 + r^2}}{l^2 \alpha r} \cos (\alpha \phi) \sin \left( \frac{\alpha t}{l} \right) \right] \]

(4.18c)

\[ \partial_{k_4}; k_4^\mu = C_4 \left[ -\frac{r}{l \alpha \sqrt{\alpha^2 l^2 + r^2}} \cos (\alpha \phi) \sin \left( \frac{\alpha t}{l} \right), \frac{\sqrt{\alpha^2 l^2 + r^2}}{l^2 \alpha r} \cos (\alpha \phi) \cos \left( \frac{\alpha t}{l} \right) \right] \]

(4.18d)

\[ \partial_{k_5}; k_5^\mu = C_5 (0, 0, 1) \]

(4.18e)

\[ \partial_{k_6}; k_6^\mu = C_6 (-l^2, 0, 0). \]

(4.18f)

For completeness, the commutators are given explicitly as

\[ \partial_{[k_0} \partial_{k_1]} = \alpha \frac{C_6}{C_3} \partial_{k_3}, \partial_{[k_4} \partial_{k_2]} = \alpha \frac{C_6}{C_4} \partial_{k_4}, \]

(4.19a)

\[ \partial_{[k_0} \partial_{k_3]} = -\alpha \frac{C_6}{C_1} \partial_{k_1}, \partial_{[k_6} \partial_{k_4]} = -\alpha \frac{C_6}{C_2} \partial_{k_2}, \]

(4.19b)

\[ \partial_{[k_5} \partial_{k_1]} = -l \alpha \frac{C_5}{C_2} \partial_{k_2}, \partial_{[k_5} \partial_{k_2]} = l \alpha \frac{C_5}{C_1} \partial_{k_1}, \]

\[ \partial_{[k_5} \partial_{k_3]} = -l \alpha \frac{C_5}{C_4} \partial_{k_4}, \partial_{[k_5} \partial_{k_4]} = l \alpha \frac{C_5}{C_3} \partial_{k_3}, \]

(4.19b)

\[ \partial_{[k_4] \partial_{k_3]} = -\frac{1}{\alpha l^5} \frac{C_4}{C_5} \partial_{k_5}, \partial_{[k_4] \partial_{k_2]} = \frac{1}{\alpha l^4} \frac{C_4}{C_6} \partial_{k_6}, \]

(4.19c)
\[ \partial_{[k_3} \partial_{k_1]} = \frac{1}{\alpha l^4} \frac{C_3 C_1}{C_6} \partial_{k_6}, \quad \partial_{[k_2} \partial_{k_1]} = \frac{1}{\alpha l^5} \frac{C_2 C_1}{C_5} \partial_{k_5}. \] (4.19d)

This anti-de Sitter metric, (cosmological constant negative–\( \lambda = -1/l^2 \)), for the coordinates \( \{t, \rho, \phi\} \)–merely names–ranging \(-\infty \leq t \leq \infty, \ -\infty \leq \rho \leq \infty; \ -\infty \leq \phi \leq \infty \) allows for six symmetries, i.e., six Killing vectors. For these coordinate ranges the space is maximally symmetric.

Moreover, if the spatial coordinates are restricted to range

\[ 0 \leq \rho \leq \infty, \ 0 \leq \phi \leq 2\pi, \]

and \( \alpha \) is equated to unity, \( \alpha = 1 = -M \), then in such case \( \rho \) and \( \phi \) become polar coordinates with \( \phi \) being the angular coordinate with period \( 2\pi \). This spacetime–the (proper) anti-de Sitter space (with \( M = -1 \))–allows for six symmetries, and as such it is maximally symmetric.

\section{Peldan Electrostatic Solution}

It seems that static Einstein–Maxwell solutions with cosmological constant were first derived in the Peldan’s work \cite{13}; in that publication Peldan mentioned his failure in finding any work done on explicit solutions to Einstein–Maxwell solutions with cosmological constant, although there was pointed the existence of static and rotation symmetric solutions with vanishing cosmological constant, namely the Deser–Mazar \cite{14}, and Melvin \cite{15} solutions.

The electrostatic Peldan solution, see also \cite{5} Eq. (4.15), is given by the metric

\[ ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + K(\rho)^2 [d\phi + W(\rho) dt]^2, \]

\[ L(\rho) = N(\rho) = \sqrt{\rho^2 - 2b^2 \ln \rho - M}, \quad K(\rho) = \rho, \quad W(\rho) = 0. \] (5.1)

\textbf{A. Mass, energy and momentum for the electrostatic Peldan solution}

The surface energy density \( \epsilon \) occurs to be

\[ \epsilon(\rho, \epsilon_0) = -\frac{1}{\pi \rho} N(\rho) - \epsilon_0. \] (5.2)

Consequently the global energy and mass are given by

\[ E(\rho, \epsilon_0) = -2N(\rho) - 2\pi \rho \epsilon_0, \]

\[ M(\rho, \epsilon_0) = -2\frac{\rho^2}{l^2} + 2m + 4b^2 \ln \rho - 2\pi \rho N(\rho) \epsilon_0. \] (5.3)
Thus, for the natural choice of a vanishing reference energy density \( \epsilon_0 = 0 \), one has at the spatial infinity \( \rho \to \infty \) that

\[
\epsilon(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{1}{\pi l} + \frac{l M}{2\pi \rho^2} + \frac{l b^2}{\pi \rho^2} \ln \rho,
\]

\[
E(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{2\rho}{l} + \frac{M l}{\rho} + \frac{2l b^2}{\rho} \ln \rho,
\]

\[
M(\rho \to \infty, \epsilon_0 = 0) \approx -2\frac{\rho^2}{l^2} + 2M + 4b^2 \ln \rho,
\]

(5.4)

while if the reference energy is the one corresponding to the anti–de Sitter spacetime \( AdS(M_0) \), \( \epsilon_0 = -\frac{1}{\pi \rho} \sqrt{\frac{L^2}{\rho^2} - M_0} \), \( \epsilon_0|_{\infty}(M_0) \approx -\frac{1}{\pi l} + \frac{l M_0}{2\pi \rho^2} \), then the energies and mass at spatial infinity are expressed as

\[
\epsilon(\rho \to \infty, \epsilon_0|_{\infty}(M_0)) \approx l \frac{M - M_0}{2\pi \rho^2} + \frac{l b^2}{\pi \rho^2} \ln \rho,
\]

\[
E(\rho \to \infty, \epsilon_0|_{\infty}(M_0)) \approx l \frac{M - M_0}{\rho} + \frac{2l b^2}{\rho} \ln \rho,
\]

\[
M(\rho \to \infty, \epsilon_0|_{\infty}(M_0)) \approx M - M_0 + 2b^2 \ln \rho.
\]

(5.5)

Comparing with the static BTZ one recognize \( M \) as the BTZ \( M \). Notice that the energy and mass include an amount of energy due to the electric field through the logarithmical term; because of this dependence, these quantities diverge at infinity logarithmically.

**B. Field, energy–momentum, and Cotton tensors for the electrostatic Peldan solution**

The electromagnetic tensor field associated with the Peldan solution is given by

\[
(F^\alpha_{\beta}) = \begin{bmatrix}
0 & \frac{b}{L^2 \rho} & 0 \\
\frac{b L^2}{\rho} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

(5.6)

Searching for its eigenvectors, one arrives at

\[
\lambda_1 = 0; V1 = [V^1 = 0, V^2 = 0, V^3 = V^3], V_\mu V^\mu = \rho^2 V^3; V1 = S1,
\]

\[
\lambda_2 = \frac{b}{\rho}; V2 = [V^1 = \frac{V^2}{L^2}, V^2 = V^2; V^3 = 0], V_\mu V^\mu = 0, V2 = N2,
\]

\[
\lambda_3 = -\frac{b}{\rho}; V3 = [V^1 = -\frac{V^2}{L^2}, V^2 = V^2; V^3 = 0], V_\mu V^\mu = 0, V3 = N3,
\]

Type :\{S, N, N\}. 

(5.7)

thus its type is

Type :\{S, N, N\}
As far as to the electromagnetic energy momentum tensor is concerned, its matrix amounts to

\[
(T^\alpha_\beta) = \begin{bmatrix} -\frac{1}{8} \frac{b^2}{\rho^2} & 0 & 0 \\ 0 & -\frac{1}{8} \frac{b^2}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{8} \frac{b^2}{\rho^2} \end{bmatrix},
\]

with the following eigenvalues and their corresponding eigenvectors

\[
\begin{align*}
\lambda_1 &= -\frac{1}{8\pi} \frac{b^2}{\rho^2}; \quad & \mathbf{V}1 &= [V^1, V^2, 0], \quad & V1_\mu &= V^1 g_{\mu \tau} + V^2 g_{\mu \rho}, \quad & V1^\mu V1_\mu &= (V^1)^2 g_{\mu \tau} + (V^2)^2 g_{\rho \rho}, \\
\mathbf{V}1 &= \{T1, S1, N1\}, \\
\lambda_2 &= -\frac{1}{8\pi} \frac{b^2}{\rho^2}; \quad & \mathbf{V}2 &= [\tilde{V}^1, \tilde{V}^2, 0], \quad & V2_\mu &= \tilde{V}^1 g_{\mu \tau} + \tilde{V}^2 g_{\mu \rho}, \quad & V2^\mu V2_\mu &= (\tilde{V}^1)^2 g_{\mu \tau} + (\tilde{V}^2)^2 g_{\rho \rho}, \\
\mathbf{V}2 &= \{T2, S2, N2\}, \\
\lambda_3 &= \frac{1}{8\pi} \frac{b^2}{\rho^2}; \quad & \mathbf{V}3 &= [0, 0, V^3], \quad & V_\mu &= V^3 g_{\mu \phi}, \quad & V^\mu V_\mu &= (V^3)^2 g_{\phi \phi}, \\
\mathbf{V}3 &= \mathbf{S}3.
\end{align*}
\]

For \(\mathbf{V}1\) and \(\mathbf{V}2\), the character of these vectors depends on the sign of their magnitudes; for instance, choosing

\[
V^1 = s \sqrt{g_{\rho \rho}} / \sqrt{|g_{\mu \tau}|} V^2, \quad s = \text{constant}, \quad V1^\mu V1_\mu = (1 - s^2) g_{\rho \rho} (V^2)^2;
\]

\[
s > 1 \rightarrow \mathbf{V}1 = \mathbf{T}, \quad s = \pm 1 \rightarrow \mathbf{V}1 = \mathbf{N}, \quad s < 1 \rightarrow \mathbf{V}1 = \mathbf{S}.
\]

The space–like vector \(\mathbf{V}3\) is aligned along the circular Killing direction \(\partial_\phi\). Thus one may have the space–time arraignment \(\mathbf{T1, S2, S3}\), or \(\mathbf{N1, N2, S3}\), and so on.

The Cotton tensor for electrostatic cyclic symmetric gravitational field is given by

\[
(C^\alpha_\beta) = \begin{bmatrix} 0 & 0 & \frac{b^2}{2\rho^4} \\ 0 & 0 & 0 \\ -\frac{1}{2} \frac{b^2}{2\rho^4} \left( \frac{t^2 M + \rho^2 - 2b^2 t^2 \ln(\rho)}{\rho^2} \right) & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{b^2}{2\rho^4} \\ 0 & 0 & 0 \\ -\frac{b^2}{2\rho^4} L^2 & 0 & 0 \end{bmatrix}.
\]

The search for its eigenvectors yields

\[
\begin{align*}
\lambda_1 &= 0; \quad & \mathbf{V}1 &= [0, V^2, 0], \quad & V1_\mu &= V^2 g_{\rho \rho} \delta^\rho_{\mu}, \quad & V1^\mu V1_\mu &= (V^2)^2 g_{\rho \rho}, \quad & \mathbf{V}1 &= \mathbf{S1}, \\
\lambda_2 &= \frac{i}{2} \frac{L b^2}{\rho^3}; \quad & \mathbf{V}2 &= [V^1, 0, V^3], \quad & V2^\mu &= \frac{i}{\rho} V1_\mu, \quad & \mathbf{V}2 &= \mathbf{Z}, \\
\lambda_3 &= -\frac{i}{2} \frac{L b^2}{\rho^3}; \quad & \mathbf{V}3 &= [V^1, 0, V^3], \quad & V3^\mu &= -\frac{i}{\rho} V1_\mu, \quad & \mathbf{V}3 &= \mathbf{Z}.
\end{align*}
\]

therefore the corresponding tensor type is

\[
\text{Type :} \{S, Z, \bar{Z}\}.
\]
The eigenvectors $\mathbf{V}_2$ and $\mathbf{V}_3$ are complex conjugated, or, if one wishes, one may consider the component $V^1$ different for each of the complex vectors. For the zero eigenvalue $\lambda_1$, the vector $\mathbf{V}_1$ is a spacelike vector, it points along the $\rho$-coordinate direction.

It is worthwhile to point out that the field and Cotton tensors of the solutions generated via coordinate transformations, in particular $SL(2, \mathbb{R})$ transformations, applied onto this electrostatic cyclic symmetric Peldan solution will shear the eigenvalues $\lambda_i$ of the corresponding field and Cotton tensors of the Peldan solution; recall that eigenvalues are invariant characteristics of tensors, although the components of the eigenvectors, in general, look different in different coordinate systems—this remark also applies to the (eigenvalues) eigenvectors of the seed and resulting solutions.

C. Field, energy–momentum, and Cotton tensors for a modified electrostatic Peldan solution

The solution to be studied is a slight modification of the Peldan electrostatic solution used in the previous paragraph, namely the one with metric

$$ds^2 = -\frac{L^2 \rho^2}{\rho^2 + Mg} dt^2 + \frac{1}{L^2} \rho^2 + (\rho^2 + Mg) d\phi^2,$$

$$L^2 := \frac{[K0 + \rho^2 + Mg - b^2 l^2 \ln (\rho^2 + Mg)] (\rho^2 + Mg)}{l^2 \rho^2},$$

and electromagnetic Maxwell field tensor

$$(F^\alpha_\beta) = \begin{pmatrix} 0 & b & 0 \\ \frac{b L^2}{\rho^2 + Mg} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

(5.12)

(5.13)

(5.14)

(5.15)

Searching for its eigenvectors, one arrives at

$$\lambda_1 = 0; \mathbf{V}_1 = [V^1 = 0, V^2 = 0, V^3 = V^3], V^\mu V^\nu = (\rho^2 + Mg) V^{32}, \mathbf{V}_1 = \mathbf{S}_1,$$

$$\lambda_2 = \frac{b}{\sqrt{\rho^2 + Mg}}; \mathbf{V}_2 = [V^1 = \frac{V^2 \sqrt{\rho^2 + Mg}}{L^2 \rho}, V^2 = V^2, V^3 = 0],$$

$$V^\mu V^\nu = 0, \mathbf{V}_2 = \mathbf{N}_2,$$

$$\lambda_3 = -\frac{b}{\sqrt{\rho^2 + Mg}}; \mathbf{V}_3 = [V^1 = -\frac{V^2 \sqrt{\rho^2 + Mg}}{L^2 \rho}, V^2 = V^2, V^3 = 0],$$

$$V^\mu V^\nu = 0, \mathbf{V}_3 = \mathbf{N}_3,$$

thus its type is

$$\{S, N, N\}.$$ 

For the energy–momentum tensor

$$(T^\alpha_\beta) = \begin{pmatrix} -\frac{1}{8 (\rho^2 + Mg)^\pi} & 0 & 0 \\ 0 & -\frac{1}{8 (\rho^2 + Mg)^\pi} & 0 \\ 0 & 0 & \frac{1}{8 (\rho^2 + Mg)^\pi} \end{pmatrix},$$

(5.15)
one has the following set of eigenvectors

\[
\lambda_{1,2} = -\frac{1}{8} \frac{b^2}{(\rho^2 + Mg)} \pi; \quad V_{1,2} = [V^1 = V^2 = V^3 = 0],
\]

\[
V_\mu V^\mu = -\frac{\rho^2 L^2}{\rho^2 + Mg} V^{12} + \frac{1}{L^2} V^{22}
\]

\[V_1 = T_1, N_1, S_1, \quad V_2 = T_2, N_2, S_2,\]

\[
\lambda_3 = \frac{1}{8} \frac{b^2}{(\rho^2 + Mg)} \pi; \quad V_3 = [V^1 = 0, V^2 = 0, V^3 = V^3],
\]

\[
V_\mu V^\mu = V^{32}(\rho^2 + Mg), \quad V_3 = S_3,
\]

(5.16)

hence its type present several possibilities

Type : \{2T, S\}, \{2N, S\}, \{2S, S\}.

The Cotton tensor

\[
(C^\alpha_{\beta}) = \begin{bmatrix}
0 & 0 & \frac{1}{2} \frac{b^2}{\rho^2 + Mg} \\
0 & 0 & 0 \\
-\frac{1}{2} \frac{b^2 L^2 \rho^2}{(\rho^2 + Mg)^2} & 0 & 0
\end{bmatrix},
\]

(5.17)

allows for the eigenvectors

\[
\lambda_1 = 0; \quad V_1 = [V^1 = 0, V^2 = V^2, V^3 = 0], \quad V_\mu V^\mu = V^{22}/L^2, \quad V_1 = S_1,
\]

\[
\lambda_2 = \frac{1}{2} \frac{\sqrt{-L^2} \rho}{(\rho^2 + Mg)^2}; \\
V_2 = [V^1 = V^1, V^2 = 0, V^3 = \frac{\sqrt{-L^2} \rho V^1}{\rho^2 + Mg}], \quad V_2 = Z.
\]

\[
\lambda_3 = -\frac{1}{2} \frac{\sqrt{-L^2} \rho}{(\rho^2 + Mg)^2}; \\
V_3 = [V^1 = V^1, V^2 = 0, V^3 = -\frac{\sqrt{-L^2} \rho V^1}{\rho^2 + Mg}], \quad V_3 = \bar{Z},
\]

(5.18)

hence the type of this Cotton tensor is

Type : \{S, Z, \bar{Z}\}.

D. Field, energy–momentum, and Cotton tensors for the generalized–via \(SL(2, R)\) transformations–Peldan solution

Many solutions in 2+1 gravity are generated via \(SL(2, R)\) transformations of the form

\[
t = \alpha T + \beta \Phi, \quad \alpha \delta - \beta \gamma = 1,
\]

\[
\phi = \gamma T + \delta \Phi,
\]

(5.19)
applied, in this particular case, onto the seed electrostatic solution (5.12) and (5.13), yielding to the stationary metric

\[
\begin{align*}
g &= \begin{bmatrix}
-\alpha^2 L^2 \rho^2 + \gamma^2 (\rho^2 + Mg)^2 / \rho^2 + Mg & 0 & -\alpha \beta L^2 \rho^2 + \gamma \delta (\rho^2 + Mg)^2 / \rho^2 + Mg \\
0 & 1 / L^2 & 0 \\
-\alpha \beta L^2 \rho^2 + \gamma \delta (\rho^2 + Mg)^2 / \rho^2 + Mg & 0 & -\beta^2 L^2 \rho^2 + \delta^2 (\rho^2 + Mg)^2 / \rho^2 + Mg
\end{bmatrix}, \\
L^2 &= \frac{[K0 + \rho^2 + Mg - b^2 L^2 \ln (\rho^2 + Mg)] (\rho^2 + Mg)}{l^2 \rho^2}, \tag{5.20}
\end{align*}
\]

accompanied by the electromagnetic field tensor

\[
\begin{align*}
(F^\alpha_\beta) &= \begin{bmatrix}
0 & \frac{\Delta b}{L^2 \rho} & 0 \\
\frac{\alpha L^2 b \rho}{\rho^2 + Mg} & 0 & \frac{\beta L^2 b \rho}{\rho^2 + Mg} \\
0 & -\frac{\nu \gamma}{L^2 \rho} & 0
\end{bmatrix}, \tag{5.21}
\end{align*}
\]

with eigenvalues

\[
\begin{align*}
\lambda_1 &= 0; V1 = [V^1 = -\frac{\beta}{\alpha} V^3, V^2 = 0, V^3 = V^3], V^3 \mu V^\mu = (\rho^2 + Mg) V^3^2 ; V1 = S1, \\
\lambda_2 &= \frac{b}{\sqrt{\rho^2 + Mg}}; \\
V2 = [V^1 = V^1, V^2 = \frac{L^2 \rho V^1}{\delta \sqrt{\rho^2 + Mg}}, V^3 = -\frac{\gamma}{\delta} V^1], V^\mu \nu V^\mu = 0, V2 = N2, \\
\lambda_3 &= -\frac{b}{\sqrt{\rho^2 + Mg}}; \\
V3 = [V^1 = V^1, V^2 = -\frac{L^2 \rho V^1}{\delta \sqrt{\rho^2 + Mg}}, V^3 = -\frac{\gamma}{\delta} V^1], V^\mu \nu V^\mu = 0, V3 = N3 \tag{5.22}
\end{align*}
\]

\[
\{S, N, N\},
\]

The evaluation of the electromagnetic energy–momentum tensor brings

\[
\begin{align*}
(T^\alpha_\beta) &= \begin{bmatrix}
-\frac{b^2 (\alpha \delta + \beta \gamma)}{8 \pi (\rho^2 + Mg)} & 0 & -\frac{b^2 \beta \delta}{4 \pi (\rho^2 + Mg)} \\
0 & -\frac{b^2}{8 \pi (\rho^2 + Mg)} & 0 \\
\frac{b^2 \alpha \gamma}{4 \pi (\rho^2 + Mg)} & 0 & \frac{b^2 (\alpha \delta + \beta \gamma)}{8 \pi (\rho^2 + Mg)}
\end{bmatrix}. \tag{5.23}
\end{align*}
\]
characterized by the following eigenvectors

$$\lambda_{1,2} = -\frac{1}{8\pi \rho^2 + Mg} b^2;$$

$$\mathbf{V}_1, \mathbf{V}_2 = [V^1 = V^1, V^2 = V^2, V^3 = -\frac{\gamma}{\delta} V^1], \mathbf{V}_\mu V^\mu = -\frac{V_1^2}{\delta^2} \frac{\rho^2 L^2}{(\rho^2 + Mg)} + \frac{V_2^2}{L^2},$$

$$\mathbf{V}_1 = T_1, N_1, S_1, \mathbf{V}_2 = T_2, N_2, S_2;$$

$$\lambda_3 = \frac{1}{8\pi \rho^2 + Mg} b^2;$$

$$\mathbf{V}_3 = [V^1 = -\frac{\beta}{\alpha} V^3, V^2 = 0, V^3 = V^3], \mathbf{V}_\mu V^\mu = \frac{V_3^2}{\alpha^2} (\rho^2 + Mg), \mathbf{V}_3 = S_3, \quad (5.24)$$

thus its type could be, among other variants:

$$\{2T, S\}, \{2N, S\}, \{2S, S\}.$$  

The transformed Cotton tensor is given by

$$\left( C^\alpha_{\beta} \right) = \begin{bmatrix}
\frac{1}{2} b^2 \alpha^2 \beta L^2 \rho^2 \frac{\rho^2}{(\rho^2 + Mg)^2} + \frac{1}{2} \frac{b^2 \gamma \delta}{(\rho^2 + Mg)^2} & 0 & \frac{1}{2} b^2 \beta^2 L^2 \rho^2 \alpha^2 (\rho^2 + Mg)^2 + \frac{1}{2} b^2 \gamma \delta \\
0 & 0 & 0 \\
-\frac{1}{2} \frac{b^2 \alpha \beta L^2 \rho^2}{(\rho^2 + Mg)^2} - \frac{1}{2} \frac{b^2 \gamma \delta}{(\rho^2 + Mg)^2} & 0 & -\frac{1}{2} \frac{b^2 \alpha \beta L^2 \rho^2}{(\rho^2 + Mg)^2} - \frac{1}{2} \frac{b^2 \gamma \delta}{(\rho^2 + Mg)^2}
\end{bmatrix}, \quad (5.25)$$

$$\lambda_1 = 0; \mathbf{V}_1 = [V^1 = 0, V^2 = V^2, V^3 = 0], \mathbf{V}_\mu V^\mu = V_2^2 / L^2, \mathbf{V}_1 = S_1,$$

$$\lambda_2 = \frac{i}{2} \frac{L b^2 \rho}{(\rho^2 + Mg)^2};$$

$$\mathbf{V}_2 = [V^1 = -\frac{V_3}{\gamma \delta (\rho^2 + Mg)^2 + \alpha \beta L^2 \rho^2 - i L \rho (\rho^2 + Mg)}, V^2 = 0, V^3 = V^3],$$

$$\mathbf{V}_2 = Z,$$

$$\lambda_3 = -\frac{i}{2} \frac{L b^2 \rho}{(\rho^2 + Mg)^2};$$

$$\mathbf{V}_3 = [V^1 = -\frac{V_3}{\gamma \delta (\rho^2 + Mg)^2 + \alpha \beta L^2 \rho^2 + i L \rho (\rho^2 + Mg)}, V^2 = 0, V^3 = V^3],$$

$$\mathbf{V}_3 = \bar{Z}. \quad (5.26)$$

The type of this generalized Cotton tensor is

$$\{S, Z, \bar{Z}\}.$$
VI. MARTINEZ–TEITELBOIM–ZANELLI SOLUTION

Martinez–Teitelboim–Zanelli solution \[16\] is defined by the metric

\[
ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + K(\rho)^2 [d\phi + W(\rho) dt]^2,
\]

where

\[
H(\rho) := \rho^2 + \frac{t^2 \omega^2}{t^2 - \omega^2} (M + \frac{Q^2}{4} \ln \rho^2),
\]

\[
L(\rho) = \sqrt{\frac{\rho^2}{t^2} - M - \frac{Q^2}{4} \ln \rho^2},
\]

\[
K(\rho) = \sqrt{H(\rho)}, \quad N(\rho) = \rho \frac{L(\rho)}{\sqrt{H(\rho)}},
\]

\[
W(\rho) = -\frac{\omega^2}{t^2 - \omega^2} \frac{1}{H(\rho)} (M + \frac{Q^2}{4} \ln \rho^2).
\]

This solution can be also derived from the rotating–under \(SL(2, R)\) transformations–Peldan solution \[5.20\] with

\[
\alpha = \frac{1}{\sqrt{1 - \frac{\omega^2}{t^2}}}, \quad \beta = -\frac{\omega}{\sqrt{1 - \frac{\omega^2}{t^2}}}, \quad \gamma = -\frac{\omega}{t^2 \sqrt{1 - \frac{\omega^2}{t^2}}}, \quad \delta = \frac{1}{\sqrt{1 - \frac{\omega^2}{t^2}}}.
\]

A. Mass, energy and momentum for the MTZ solution

The evaluation of the surface energy and momentum densities yield

\[
\epsilon(\rho, \epsilon_0) = -\frac{1}{2\pi} \frac{L}{K^2} \left( 2\rho + \frac{t^2 \omega^2}{t^2 - \omega^2} \frac{Q^2}{2\rho} \right) - \epsilon_0,
\]

\[
j(\rho) = \rho \frac{t^2 \omega}{\pi} \frac{L}{NK^2} \left( M + \frac{1}{2} Q^2 \ln \rho - \frac{Q^2}{4} \right),
\]

while the integral quantities can be evaluated from the generic expressions

\[
J(\rho) = 2\pi K(\rho) j(\rho),
\]

\[
E(\rho) = 2\pi K(\rho) \epsilon(\rho),
\]

\[
M(\rho) = N(\rho) E(\rho) - W(\rho) J(\rho).
\]

The evaluation of the corresponding functions with \(\epsilon_0 = 0\) behave at infinity according
to
\[ \epsilon(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{1}{\pi l} + \frac{[2M(l^2 + \omega^2) - \omega^2 Q^2]}{4\pi(l^2 - \omega^2)\rho^2} + \frac{\omega^2 l^2 + \omega^2}{4\pi\rho^2 l^2 - \omega^2} \ln \rho, \]
\[ j(\rho \to \infty) \approx \frac{\omega l^2 4M - Q^2}{4\pi\rho l^2 - \omega^2} + \frac{\omega Q^2 l^2}{2\pi\rho(l^2 - \omega^2)} \ln \rho, \]
\[ J(\rho \to \infty) \approx \frac{\omega l^2 4M - Q^2}{2 l^2 - \omega^2} + \frac{\omega Q^2 l^2}{l^2 - \omega^2} \ln \rho, \]
\[ = J_{\text{MTZ}[16, \text{Eq.83}]}(\omega \to \omega/l) + \frac{\omega Q^2 l^2}{l^2 - \omega^2} \ln \rho, \]
\[ E(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{2\rho}{l} + \frac{l}{2\rho} \frac{2Ml^2 - \omega^2 Q^2}{l^2 - \omega^2} + \frac{l^3 Q^2}{2\rho(l^2 - \omega^2)} \ln \rho, \]
\[ M(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{2\rho^2}{l^2} + \frac{4Ml^2 - \omega^2 Q^2}{2(l^2 - \omega^2)} + \frac{l^3 Q^2}{l^2 - \omega^2} \ln \rho. \]  

(6.5)

Using in the expressions (6.3) and (6.4) as reference energy density the quantity \( \epsilon_0 = -\frac{1}{\pi l} \sqrt{-M_0 + \frac{\rho^2}{l^2}}, \) which at the spatial infinity behaves as \( \epsilon_{0,\infty}(M_0) \approx -\frac{1}{l} + \frac{M_0}{2\pi \rho^2}, \) the series expansions of the corresponding quantities at \( \rho = \text{infinity} \) result in
\[ \epsilon(\rho \to \infty, \epsilon_{0,\infty}(M_0)) \approx \frac{1}{2\pi \rho^2} (M - M_0) + \frac{l\omega^2}{4\pi(l^2 - \omega^2)\rho^2} (4M - Q^2) + \frac{l Q^2 l^2 + \omega^2}{4\pi\rho^2 l^2 - \omega^2} \ln \rho, \]
\[ E(\rho \to \infty, \epsilon_{0,\infty}(M_0)) \approx \frac{l}{\rho} (M - M_0) + \frac{l\omega^2}{2\rho(l^2 - \omega^2)} (4M - Q^2) + \frac{l Q^2 l^2 + \omega^2}{2 l^2 - \omega^2} \ln \rho, \]
\[ M(\rho \to \infty, \epsilon_{0,\infty}(M_0)) \approx M - M_0 + \frac{\omega^2}{2(l^2 - \omega^2)} (4M - Q^2) + \frac{Q^2 l^2 + \omega^2}{2 l^2 - \omega^2} \ln \rho, \]
\[ = -M_0 + M_{\text{MTZ}[16, \text{Eq.81}]}(\omega \to \omega/l) + \frac{Q^2 l^2 + \omega^2}{2 l^2 - \omega^2} \ln \rho. \]  

(6.6)

Notice that the charges \( Q \) used above differs from \( Q_{\text{MTZ}[16, \text{Eq.83}]} \),
\[ Q_{\text{MTZ}[16, \text{Eq.83}]} = \frac{l}{\sqrt{l^2 - \omega^2}} Q \]  

(6.7)

Therefore, comparing with the energy characteristics of the BTZ solution, one concludes that the parameter \( M \) can be considered as the BTZ mass, and the energy and mass functions logarithmically diverges at spatial infinity.

B. Field, energy–momentum, and Cotton tensors for the MTZ solution

The Maxwell field tensor for this MTZ solution is given by
\[ (F^{\alpha\beta}) = \begin{bmatrix} 0 & \frac{1}{2} \frac{Q l}{\rho \sqrt{l^2 - \omega^2}} & 0 \\ \frac{1}{2} \frac{Q l \omega^2}{\rho \sqrt{l^2 - \omega^2}} & 0 & -\frac{1}{2} \frac{\omega l Q}{\rho \sqrt{l^2 - \omega^2}} \\ 0 & -\frac{1}{2} \frac{\omega l Q}{\rho \sqrt{l^2 - \omega^2}} & 0 \end{bmatrix}, \]
\[ L^2 = \frac{\rho^2}{l^2} - m - \frac{1}{4} Q^2 \ln \left( \frac{\rho^2}{l^2} \right), \]  

(6.8)
while its eigenvalues and the corresponding eigenvectors amount to

\[
\begin{align*}
\lambda_1 &= 0; \mathbf{V}1 = [V^1 = V^3 \omega, V^2 = 0, V^3 = V^3], \\
V_\mu V^\mu &= \frac{l^2 - \omega^2}{l^2} \rho^2 V^3, \mathbf{V}1 = \mathbf{S}1, \\
\lambda_2 &= -\frac{1}{2} \frac{Q}{\rho} \cdot \mathbf{V}2 = [V^1 = V^1, V^2 = \frac{V^1 \sqrt{l^2 - \omega^2}}{l}, V^3 = V^1 \frac{\omega}{l^2}], \\
V_\mu V^\mu &= 0, \mathbf{V}2 = \mathbf{N}2, \\
\lambda_3 &= \frac{1}{2} \frac{Q}{\rho} \cdot \mathbf{V}3 = [V^1 = V^1, V^2 = -\frac{V^1 \sqrt{l^2 - \omega^2}}{l}, V^3 = V^1 \frac{\omega}{l^2}], \\
V_\mu V^\mu &= 0, \mathbf{V}3 = \mathbf{N}3, \\
\text{Type} : \{S, N, N\} & \quad (6.9)
\end{align*}
\]

The energy–momentum tensor

\[
(T^\alpha_\beta) = \begin{bmatrix}
-\frac{1}{32\pi} \frac{Q^2 (l_2^2 + \omega^2)}{\rho^2 (l^2 - \omega^2)} & 0 & \frac{1}{16\pi} \frac{l_2 \omega^2 Q^2}{\rho^2 (l^2 - \omega^2)} \\
0 & -\frac{1}{32\pi} \frac{Q^2}{\rho^2} & 0 \\
-\frac{1}{16\pi} \frac{l_2 \omega^2 Q^2}{\rho^2 (l^2 - \omega^2)} & 0 & \frac{1}{32\pi} \frac{Q^2 (l_2^2 + \omega^2)}{\rho^2 (l^2 - \omega^2)}
\end{bmatrix}, \quad (6.10)
\]

possesses the following eigenvalues and eigenvectors

\[
\begin{align*}
\lambda_1 &= \frac{1}{32\pi} \frac{Q^2}{\rho^2} \cdot \mathbf{V}1 = [V^1 = V^3 \omega, V^2 = 0, V^3 = V^3], \\
V_\mu V^\mu &= \frac{V^3^2 \rho^2 (l_2^2 - \omega^2)}{l^2}, \mathbf{V}1 = \mathbf{S}1, \\
\lambda_{2,3} &= -\frac{1}{32\pi} \frac{Q^2}{\rho^2} \cdot \mathbf{V}2 = [V^1 = \frac{l_2 V^3}{\omega}, V^2 = V^2, V^3 = V^3], \\
V_\mu V^\mu &= -\frac{l_2 (l_2^2 - \omega^2) V^3^2}{\omega^2} L^g + \frac{V^2^2}{L^2}, \\
\mathbf{V}2 = \{\mathbf{T}2, \mathbf{N}2, \mathbf{S}2\}, \mathbf{V}3 = \{\mathbf{T}3, \mathbf{N}3, \mathbf{S}3\}, \\
\text{Type} : \{S, 2N\} \\
& \quad (6.11)
\end{align*}
\]

For the Cotton tensor

\[
(C^\alpha_\beta) = \begin{bmatrix}
-\frac{1}{8} \frac{(\rho^2 + l_2^2 L^2) \omega Q^2}{\rho^4 (l_2^2 - \omega^2)} & 0 & \frac{1}{8} \frac{(\rho^2 + \omega^2 L^2)^2 Q^2}{\rho^4 (l_2^2 - \omega^2)} \\
0 & 0 & 0 \\
-\frac{1}{8} \frac{(\omega^2 \rho^2 + l_4^2 L^2) Q^2}{\rho^4 (l_2^2 - \omega^2)} & 0 & \frac{1}{8} \frac{(\rho^2 + l_2^2 L^2) \omega Q^2}{\rho^4 (l_2^2 - \omega^2)}
\end{bmatrix}, \quad (6.12)
\]
the eigenvalues and the corresponding eigenvectors are

\[ \lambda_1 = 0; V_1 = [V^1 = 0, V^2 = V^2, V^3 = 0], V_\mu V^\mu = \frac{V^2}{L^2}, V_1 = S_1, \]

\[ \lambda_2 = \frac{1}{8} \frac{Q^2 \sqrt{-L^2}}{\rho^3}; V_2 = [V^1 = V^1, V^2 = 0, V^3 = \frac{V^1 (\omega + \sqrt{-L^2 l^2})}{l^2 (\rho + \sqrt{-L^2 \omega})}], V_2 = Z, \]

\[ \lambda_3 = -\frac{1}{8} \frac{Q^2 \sqrt{-L^2}}{\rho^3}; V_3 = [V^1 = V^1, V^2 = 0, V^3 = \frac{V^1 (\omega - \sqrt{-L^2 l^2})}{l^2 (\rho - \sqrt{-L^2 \omega})}], V_3 = \bar{Z}, \]

Type : \{S, Z, \bar{Z}\}. \quad (6.13)

\section*{VII. CLEMENT SPINNING SOLUTION}

Clement rotating charged solution \cite{17} is defined by the metric functions

\[ ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + K(\rho)^2 d\phi + W(\rho) dt^2, \]

\[ H(\rho) := \rho^2 + 4\pi G \omega^2 Q^2 \ln \left( \frac{L}{\rho_0} \right), \]

\[ F(\rho) := \frac{\rho^2}{l^2} - \frac{4\pi G (l^2 - \omega^2) Q^2}{l^2} \ln \left( \frac{\rho^2}{\rho_0^2} \right), \]

\[ L(\rho) = \sqrt{F(\rho)}, K(\rho) = \sqrt{H(\rho)}, N(\rho) = \rho \sqrt{\frac{F(\rho)}{H(\rho)}}, \]

\[ W(\rho) = -\omega \frac{4\pi G Q^2}{H(\rho)} \ln \left( \frac{\rho^2}{\rho_0^2} \right), \quad (7.1) \]

see also \cite{5} Eq. (11.24).

\subsection*{A. Mass, energy and momentum for the Clement spinning solution}

The evaluation of the surface energy and momentum densities yield

\[ \epsilon(\rho, \epsilon_0) = -\frac{1}{\pi \rho} \sqrt{\frac{F(\rho)}{H(\rho)}} (\rho^2 + 4\pi G \omega^2 Q^2) - \epsilon_0, \]

\[ j(\rho) = -4 \frac{4\pi G Q^2 \rho [1 - \ln \left( \frac{\rho^2}{\rho_0^2} \right)]}{\sqrt{H(\rho)}}, \quad (7.2) \]

while the integral quantities can be evaluated from the generic expressions

\[ J(\rho) = 2 \pi K(\rho) j(\rho), E(\rho) = 2 \pi K(\rho) \epsilon(\rho), M(\rho) = N(\rho) E(\rho) - W(\rho) J(\rho). \]
In this manner one arrives at

\[ J(\rho) = -8\pi G \omega Q^2 (1 - \ln \left( \frac{\rho^2}{\rho_0^2} \right)) , \]

\[ E(\rho, \epsilon_0) = -2 \frac{L}{\rho K} \left( \rho^2 + 4\pi G \omega^2 Q^2 \right) - 2\pi K(\rho) \epsilon_0 , \]

\[ M(\rho, \epsilon_0) = -2 \frac{\rho^2}{l^2} - 8\pi G \omega^2 Q^2 \frac{l^2}{l^2} + 8\pi G Q^2 \ln \left( \frac{\rho^2}{\rho_0^2} \right) - 2\pi K(\rho) N(\rho) \epsilon_0 . \quad (7.3) \]

The evaluation of the corresponding functions for the base energy \( \epsilon_0 = 0 \) yield at spatial infinity \( \rho \to \infty \)

\[ j(\rho \to \infty) \approx -4 \frac{G \omega Q^2}{\rho} + 8 \frac{G \omega Q^2}{\rho} \ln \left( \frac{\rho}{\rho_0} \right) , \]

\[ J(\rho \to \infty) \approx -8\pi G \omega Q^2 (1 - 2 \ln \left( \frac{\rho}{\rho_0} \right)) , \]

\[ \epsilon(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{1}{\pi l} - 4 \frac{G \omega^2 Q^2}{l \rho^2} + 4 \frac{G Q^2}{l \rho^2} (l^2 + \omega^2) \ln \left( \frac{\rho}{\rho_0} \right) , \]

\[ E(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{2 l}{\rho} - 8 \frac{\pi G \omega^2 Q^2}{l \rho} + 8 \frac{\pi G Q^2}{\rho} \ln \left( \frac{\rho}{\rho_0} \right) , \]

\[ M(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{2 \rho^2}{l^2} - 8 \frac{\pi G \omega^2 Q^2}{l^2} + 16\pi G Q^2 \ln \left( \frac{\rho}{\rho_0} \right) . \quad (7.4) \]

Using in the expressions (7.2) and (7.3) as the reference energy density the quantity \( \epsilon_0 = -\frac{1}{\pi \rho} \sqrt{-M_0 + \frac{\rho^2}{l^2}} \), which at the spatial infinity behaves as \( \epsilon(0,\infty)(M_0) \approx -\frac{1}{\pi l} + \frac{M_0}{2\pi \rho^2} \), the series expansions of the corresponding quantities at \( \rho \to \infty \) result in

\[ \epsilon(\rho \to \infty, \epsilon_0(\infty)(M_0)) \approx -\frac{l M_0}{2\pi \rho^2} - 4 \frac{G \omega^2 Q^2}{l \rho^2} + 4 \frac{G Q^2}{l \rho^2} (l^2 + \omega^2) \ln \left( \frac{\rho}{\rho_0} \right) , \]

\[ E(\rho \to \infty, \epsilon_0(\infty)(M_0)) \approx -\frac{l M_0}{\rho} - 8 \frac{\pi G \omega^2 Q^2}{l \rho} + 8 \frac{\pi G Q^2}{l \rho} (l^2 + \omega^2) \ln \left( \frac{\rho}{\rho_0} \right) , \]

\[ M(\rho \to \infty, \epsilon_0(\infty)(M_0)) \approx -M_0 - 8 \frac{\pi G \omega^2 Q^2}{l^2} + 8 \pi G Q^2 \frac{l^2 + \omega^2}{l^2} \ln \left( \frac{\rho}{\rho_0} \right) . \quad (7.5) \]

Comparing with the energy characteristics of the BTZ solution, one concludes that a mass parameter \( M \) similar to the BTZ mass is absent, instead a term in the mass function due to the product of the rotation \( \omega \) and the charge \( Q \) is present. Notice that \( E(\rho) \) and \( M(\rho) \) logarithmically diverges at spatial infinity. The momentum parameter is due to the product of \( \omega Q \), and hence is not a free parameter.

**B. Cotton tensor for the Clement spinning solution**

The Cotton characterization of this solution is given by

\[
Cotton = \begin{bmatrix}
\frac{1}{8} \frac{\omega (F(\rho)l^2 + \rho^2)}{\rho (l^2 - \omega^2)} \frac{d^3}{d\rho^3} F(\rho) & 0 & -\frac{1}{8} \frac{(F(\rho)\omega^2 + \rho^2)}{\rho (l^2 - \omega^2)} \frac{d^3}{d\rho^3} F(\rho) \\
0 & 0 & 0 \\
\frac{1}{8} \frac{(F(\rho)l^4 + \omega^2 \rho^2)}{l^2 \rho (l^2 - \omega^2)} \frac{d^3}{d\rho^3} F(\rho) & 0 & -\frac{1}{8} \frac{\omega (F(\rho)l^2 + \rho^2)}{\rho (l^2 - \omega^2)} \frac{d^3}{d\rho^3} F(\rho)
\end{bmatrix}, \quad (7.6)
\]
with electromagnetic vector potential

\[
F(\rho) = \frac{\rho^2}{l^2} + 4Q^2 \pi gr \ln(\rho_0^2) - 4Q^2 \pi gr \ln(\rho^2), \quad \frac{d^3}{d\rho^3} F(\rho) = -16 \frac{Q^2 \pi gr}{\rho^3}
\]  
(7.7)

\[
\lambda_1 = 0;
\]

\[
V_1 = [V^1 = 0, V^2 = V^3 = 0], \quad V_\mu V^\mu = V^{22}, \quad V_1 = S_1,
\]

\[
\lambda_2 = 2 \sqrt{-F(\rho)}Q^2 \pi gr;
\]

\[
V_2 = [V^1 = V^3 = 0, V^2 = \frac{V^1 \left( \rho \omega + \sqrt{-F(\rho)}l^2 \right)}{l^2 \left( \rho + \sqrt{-F(\rho)}\omega \right)}], \quad V_2 = Z,
\]

\[
\lambda_3 = -2 \sqrt{-F(\rho)}Q^2 \pi gr;
\]

\[
V_3 = [V^1 = V^3 = 0, V^2 = \frac{V^1 \left( \rho \omega - \sqrt{-F(\rho)}l^2 \right)}{l^2 \left( \rho - \sqrt{-F(\rho)}\omega \right)}], \quad V_3 = \bar{Z}.
\]  
(7.8)

### VIII. GARCIA SOLUTION

The Garcia solution [5] Eq. (10.9), is defined by the metric functions

\[
ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + K(\rho)^2 [d\phi + W(\rho) dt]^2,
\]

\[
H(\rho) := \frac{H_n}{H_d};
\]

\[
H_n = 4 \rho^2 (\rho^2 - M l^2)(M^2 l^2 - J^2) - J^2 Q^4 l^6 R_\pm \left( \ln|Z(\rho)| \right)^2 - 2Q^2 l^3 \sqrt{M^2 l^2 - J^2} \left[ M J^2 l^2 - 2 \rho^2 R_- \sqrt{M^2 l^2 - J^2} \ln|Z(\rho)| \right],
\]

\[
H_d = 4 (M^2 l^2 - J^2) (\rho^2 - M l^2) - 2 l^3 Q^2 R_\pm \sqrt{M^2 l^2 - J^2} \ln|Z(\rho)|,
\]

\[
L(\rho)^2 = \frac{\rho^2}{l^2} - M + \frac{J^2}{4 \rho^2} + \frac{1}{2 \rho^2} (2\rho^2 R_- - l J^2) \ln|Z(\rho)|,
\]

\[
K(\rho)^2 = H(\rho),
\]

\[
N(\rho)^2 = \rho^2 \frac{L(\rho)^2}{H(\rho)};
\]

\[
W(\rho) H_n = J Q^4 l^5 R_\pm \left( \ln|Z(\rho)| \right)^2 + Q^2 l^2 J \sqrt{M^2 l^2 - J^2} \left[ J^2 l + 2 l R_\pm - 2 \rho^2 R_- \right] \ln|Z(\rho)| - 2 J (M^2 l^2 - J^2) (\rho^2 - M l^2),
\]

\[
Z(\rho) := \rho^2 - \frac{l}{2} (M l - \sqrt{M^2 l^2 - J^2}) = \rho^2 - \frac{l R_-}{2},
\]

\[
R_\pm := M l \pm \sqrt{M^2 l^2 - J^2},
\]  
(8.1)

with electromagnetic vector potential

\[
A := -\sqrt[3]{\frac{Q}{2}} \frac{\ln|\rho|}{\sqrt{M^2 l^2 - J^2}} \left( R_- dt - \frac{J}{l} d\phi \right),
\]  
(8.2)
therefore the non–vanishing covariant components of the electromagnetic field tensor $F_{\mu\nu}$ are

$$F_{\tau\rho} = \frac{\sqrt{l}Q}{\sqrt{2\sqrt{l^2M^2 - J^2}}} \frac{R_-}{\sqrt{2\sqrt{l^2M^2 - J^2}}} \frac{1}{\rho}, \quad F_{\phi\rho} = \frac{Q}{\sqrt{2\sqrt{l^2M^2 - J^2}}} \frac{J}{\sqrt{2\sqrt{l^2M^2 - J^2}}} \frac{1}{\rho}. \tag{8.3}$$

Notice that the above gravitational–electromagnetic field, as it was pointed previously out, when the electromagnetic field is switched off, $Q = 0$, becomes the rotating BTZ solution, while for vanishing rotation, $J = 0$, the corresponding solution is represented by the static BTZ metric, i.e., the AdS metric with $M$–parameter.

### A. Mass, energy and momentum for the Garcia black hole

The evaluation of the surface energy density at spatial infinity $\rho \to \infty$ for $\epsilon_0 = 0$ yields

$$\epsilon(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{1}{\pi l} + \frac{l M}{2\pi \rho^2} - \frac{l^2 J^2 Q^2}{2\pi \rho^2 \sqrt{M^2 l^2 - J^2}} + \frac{l^3 Q^2 M}{\pi \rho^2 \sqrt{M^2 l^2 - J^2}} R_- \ln (\rho), \tag{8.4}$$

while the momentum quantities amount to

$$j(\rho \to \infty) \approx \frac{J}{2\pi \rho} - \frac{l^2 JQ^2}{2\pi \rho \sqrt{M^2 l^2 - J^2}} R_- + \frac{l^2 JQ^2}{\pi \rho \sqrt{M^2 l^2 - J^2}} R_- \ln (\rho),$$

$$J(\rho \to \infty) \approx J - \frac{l^2 JQ^2}{\sqrt{M^2 l^2 - J^2}} R_- + \frac{2l^2 JQ^2}{\sqrt{M^2 l^2 - J^2}} R_- \ln (\rho). \tag{8.5}$$

The integral energy and mass characteristic at spatial infinity can be evaluated from the generic expressions $E(\rho) = 2\pi K \epsilon(\rho), M(\rho) = N E(\rho) - W J(\rho)$.

The evaluation of the corresponding functions with $\epsilon_0 = 0$ behave at infinity as $\rho \to \infty$ according to

$$E(\rho \to \infty, \epsilon_0 = 0) \approx -2\rho \frac{M}{2\pi \rho^2} + \frac{l M}{\sqrt{M^2 l^2 - J^2}} - \frac{l^2 J^2 Q^2}{\rho \sqrt{M^2 l^2 - J^2}} + \frac{l^2 Q^2 M}{\rho \sqrt{M^2 l^2 - J^2}} R_- \ln (\rho),$$

$$M(\rho \to \infty, \epsilon_0 = 0) \approx -2\rho \frac{M}{2\pi \rho^2} + 2M - \frac{l J^2 Q^2}{\sqrt{M^2 l^2 - J^2}} + \frac{2l^2 Q^2}{\sqrt{M^2 l^2 - J^2}} R_- \ln (\rho). \tag{8.6}$$

The series expansions of the expressions of the energy and mass evaluated for the reference energy density $\epsilon_0 = -\frac{1}{\pi l} \sqrt{-M_0 + \frac{\rho^2}{l^2}},$ which at the spatial infinity behaves as $\epsilon_{0|\infty}(M_0) \approx -\frac{1}{\pi l} + \frac{l M_0}{2\pi \rho^2 l},$ at $\rho \to \infty$ occur to be

$$\epsilon(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx \frac{l}{2\pi \rho^2} (M - M_0) - \frac{l J^2 Q^2}{2\pi \rho^2 \sqrt{M^2 l^2 - J^2}} + \frac{l^3 Q^2 M R_-}{\pi \rho^2 \sqrt{M^2 l^2 - J^2}} \ln (\rho),$$

$$E(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx \frac{l (M - M_0)}{\rho} - \frac{l J^2 Q^2}{\rho \sqrt{M^2 l^2 - J^2}} + \frac{2l^3 Q^2 M}{\rho \sqrt{M^2 l^2 - J^2}} R_- \ln (\rho),$$

$$M(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx M - M_0 - \frac{l J^2 Q^2}{\sqrt{M^2 l^2 - J^2}} + \frac{2l^2 M Q^2}{\sqrt{M^2 l^2 - J^2}} R_- \ln (\rho). \tag{8.7}$$
For vanishing electromagnetic field charge $Q$, which gives rise to the rotating BTZ black hole, the mass, and the energy–momentum quantities become just the mass–energy–momentum characteristics of the BTZ solution Eqs. (4.7)–(4.10), hence one concludes that the parameters $M$ and $J$ are related with the mass and momentum respectively. Moreover in the electromagnetic case, the momentum, mass, energy functions logarithmically diverges at spatial infinity.

**B. Field, energy–momentum, and Cotton tensors for the Garcia solution**

To determine the eigenvector structure of the Garcia solution it is more convenient to use another of its representation in the coordinates $\{\tau, r, \sigma\}$, namely

$$g = \begin{bmatrix} -F/H + H^2 & 0 & HW \\ 0 & 1/F & 0 \\ HW & 0 & H \end{bmatrix}, \quad (8.8)$$

where the metric functions are

$$F(r) = 4 \frac{r^2}{l^2} + 2 \frac{r}{l} \left(l w_1 + \sqrt{l^2 w_1^2 - 4}\right) \left(w_0 + W0 \ln (r)\right),$$
$$H(r) = -w_0 - W0 \ln (r) + \frac{r}{l} \sqrt{l^2 w_1^2 - 4},$$
$$W(r) = \frac{(w_0 + W0 \ln (r) + w_1 r)}{(-w_0 - W0 \ln (r) + \frac{r}{l} \sqrt{l^2 w_1^2 - 4})},$$
$$W0 = -\frac{l^2 \alpha^2}{2} \left(l^2 w_1^2 - 2 - l w_1 \sqrt{l^2 w_1^2 - 4}\right) = -\frac{l^2 \alpha^2}{4} \left(l w_1 - \sqrt{l^2 w_1^2 - 4}\right)^2. \quad (8.9)$$

To achieve the metric structure studied in the previous paragraph, one subject the above metric to the coordinate transformation

$$\tau = \frac{1}{\sqrt{2} J l} \left(J t - \frac{l^2 M \phi}{\sqrt{l^2 M^2 - J^2}}\right), \quad r = -\beta^2 + \frac{l^2 M}{2} + \frac{1}{2} \sqrt{l^2 M^2 - J^2}, \quad \sigma = \frac{l}{\sqrt{2} \sqrt{J}} \sqrt{l^2 M^2 - J^2} \phi,$$

together with

$$w_0 = -\frac{\sqrt{l^2 M^2 - J^2}}{J} R_- \quad w_1 = \frac{2 M}{J} \quad W0 = -\frac{l^2 \alpha^2}{J^2} R_-, \quad R_- = \left(l M - \sqrt{l^2 M^2 - J^2}\right).$$

followed by the change of the charge $\alpha \to J^{1/2} Q$.

In these coordinates, the Maxwell electromagnetic field tensor is given by

$$\left(F^\alpha_{\beta}\right) = \begin{bmatrix} 0 & \frac{\alpha}{F} & 0 \\ \alpha \left[F - H^2 W (W - 1)\right]/H & 0 & -H \alpha (W - 1) \\ 0 & -\frac{\alpha}{F} & 0 \end{bmatrix}. \quad (8.10)$$
and it is characterized by the following set of eigenvectors

\[ \lambda_1 = 0; \mathbf{V}_1 = [V^1 = \frac{H^2(W - 1)}{F - H^2W(W - 1)}V^3, 0, V^3], \]
\[ V_\mu V^\mu = HF \frac{(F - H^2(W - 1)^2)}{[F - H^2W(W - 1)]^2} V^{32}, \mathbf{V}_1 = \mathbf{T}_1, \mathbf{S}_1, \]
\[ \lambda_2 = \frac{\sqrt{F - H^2(W - 1)^2}}{\sqrt{HF}} \alpha; \]
\[ \mathbf{V}_2 = [V^1 = \frac{\alpha V^2}{\lambda_2 F(r)}, V^2, V^3 = -\frac{\alpha V^2}{\lambda_2 F}] , V^\mu V_\mu = 0, \mathbf{V}_2 = \mathbf{N}_2, \mathbf{Z}, \]
\[ \lambda_3 = -\frac{\sqrt{F - H^2(W - 1)^2}}{\sqrt{HF}} \alpha; \]
\[ \mathbf{V}_3 = [V^1 = \frac{\alpha V^2}{\lambda_3 F}, V^2, V^3 = -\frac{\alpha V^2}{\lambda_3 F}], V^\mu V_\mu = 0, \mathbf{V}_3 = \mathbf{N}_3, \bar{\mathbf{Z}}. \quad (8.11) \]

For the Maxwell energy–momentum tensor

\[ (T^\alpha_\beta) = \begin{bmatrix} -\frac{1}{8\pi} \frac{\alpha^2[F-H^2(W-1)^2]}{FH} & 0 & \frac{1}{4\pi} \frac{H\alpha^2(W-1)}{F} \\ 0 & -\frac{1}{8\pi} \frac{\alpha^2[F-H^2(W-1)^2]}{FH} & 0 \\ \frac{1}{4\pi} \frac{\alpha^2[F-H^2W(W-1)]}{FH} & 0 & \frac{1}{8\pi} \frac{\alpha^2[F-H^2(W-1)(W+1)]}{FH} \end{bmatrix} , \quad (8.12) \]

one has the following eigenvalues and eigenvectors

\[ \lambda_{1,2} = -\frac{1}{8\pi} \frac{\alpha^2 [F - H^2 (W - 1)^2]}{FH}; \mathbf{V}_1, 2 = [V^1 = -V^3, V^2 = V^2, V^3 = V^3], \]
\[ V_\mu V^\mu = \frac{V^{22}}{F} - \frac{V^{32} [F - H^2 (W - 1)^2]}{H}, \mathbf{V}_1 = \mathbf{T}_1, \mathbf{S}_1, \mathbf{V}_2 = \mathbf{T}_2, \mathbf{S}_2, \]
\[ \lambda_3 = \frac{1}{8\pi} \frac{\alpha^2 [F - H^2 (W - 1)^2]}{FH}; \mathbf{V}_3 = [V^1 = \frac{H^2 V^3 (W - 1)}{F - H^2 W (W - 1)}, V^2 = 0, V^3 = V^3], \]
\[ V_\mu V^\mu = -\frac{V^{32} HF [F - H^2 (W - 1)^2]}{[F - H^2 W (W - 1)]^2}, \mathbf{V}_3 = \mathbf{T}_3, \mathbf{S}_3 \quad (8.13) \]

Finally, the Cotton tensor

\[ (C^\alpha_\beta) = \begin{bmatrix} C^1_1 & 0 & C^1_3 \\ 0 & C^2_2 & 0 \\ C^3_1 & 0 & -C^1_1 - C^2_2 \end{bmatrix} , \quad (8.14) \]

\[ C^1_1 = -\frac{1}{32 \pi} \frac{HW \alpha^2 (W - 1)^2 (HF_r - FH_r)}{F^2} + \frac{1}{32 \pi} \frac{\alpha^2 (W - 1) F_r}{F} \]
\[ -\frac{1}{32 \pi} \frac{\alpha^2 (3W - 2) H_r}{H} - \frac{1}{16 \pi} \alpha^2 W_r. \quad (8.15) \]
The set of eigenvector equations reduces to roots, namely

\[ C^1_3 = -\frac{1}{32\pi} \alpha^2 H (W - 1)^2 (H F_r - F H_r) \frac{F^2}{F^2} - \frac{1}{32\pi} \alpha^2 H_r \frac{F}{H}, \]  
\[ C^2_2 = -\frac{1}{16\pi} \alpha^2 (W - 1) (F_r H - 2 F H_r) \frac{F H}{F H} + \frac{1}{16\pi} \alpha^2 \left( H^2 (W - 1)^2 + F \right) W_r, \]
\[ C^3_1 = \frac{1}{32\pi} \alpha^2 (W - 1) \left( -F (1 + W) + W^2 H^2 (W - 1) \right) \frac{F_r}{F^2} \]
\[ - \frac{1}{32\pi} (F + H^2 W^2) \alpha^2 \left( -F + H^2 (W - 1)^2 \right) \frac{H_r}{H^3 F} \]
\[ - \frac{1}{16\pi} W \alpha^2 \left( -F + H^2 (W - 1)^2 \right) \frac{W_r}{F}, \]
\[ C^3_3 = \frac{1}{32\pi} \left( H^2 W \alpha^2 (W - 1)^2 F_r \frac{F^2}{F^2} - \frac{1}{32\pi} \alpha^2 \left( H^2 W (W - 1)^2 + F (W - 2) \right) \frac{H_r}{F H} \right. \]
\[ - \left. \frac{1}{16\pi} \alpha^2 H^2 (W - 1)^2 W_r \right), \]

possesses, in general, three different eigenvalues, with the possibility of complex conjugated roots, namely

\[ \lambda_1 = C^2_2, \]
\[ \lambda_2 = -1/2 C^2_2 + 1/2 \sqrt{(C^1_1 + C^2_2)^2 + 4 C^1_3 C^3_1}, \]
\[ \lambda_3 = -1/2 C^2_2 - 1/2 \sqrt{(C^1_1 + C^2_2)^2 + 4 C^1_3 C^3_1}. \]

The set of eigenvector equations reduces to

\[ V^1 (C^1_1 - \lambda) + C^1_3 V^3 = 0, \]
\[ (C^2_2 - \lambda) V^2 = 0, \]
\[ C^3_1 V^1 - V^3 (C^1_1 + C^2_2 + \lambda) = 0. \]

with solutions

\[ \lambda_1 = C^2_2; V1 = [V^1 = 0, V^2 = V^2, V^3 = 0], V^1 \mu V^\mu = V^2^2 / F, V1 = S1, \]
\[ \lambda_2 = -1/2 C^2_2 + 1/2 \sqrt{(C^1_1 + C^2_2)^2 + 4 C^1_3 C^3_1}; \]
\[ V2 = [V^1 = -\frac{C^1_3 V^3}{C^1_1 - \lambda_2}, V^2 = 0, V^3 = V^3], \]
\[ V^\mu \nu V^\mu = \frac{H^2 (C^1_1 - WC^1_3)^2 + H^2 \lambda_2 (\lambda_2 - 2 C^1_1 + 2 WC^1_3) - (C^1_3)^2 F}{(C^1_1 - \lambda_2)^2 H} V^3^2, \]
\[ V2 = S2, N2, Z, \]
\[ \lambda_3 = -1/2 C^2_2 - 1/2 \sqrt{(C^1_1 + C^2_2)^2 + 4 C^1_3 C^3_1}; \]
\[ V2 = [V^1 = -\frac{C^1_3 V^3}{C^1_1 - \lambda_3}, V^2 = 0, V^3 = V^3], \]
\[ V^\mu \nu V^\mu = \frac{H^2 (C^1_1 - WC^1_3)^2 + H^2 \lambda_3 (\lambda_3 - 2 C^1_1 + 2 WC^1_3) - (C^1_3)^2 F}{(C^1_1 - \lambda_3)^2 H} V^3^2, \]
\[ V3 = S3, N3, Z, \]
IX. SPINNING ELECTRO–MAGNETIC SOLUTION

By accomplishing in (9.13), the coordinate transformations
\[ t \rightarrow C_1/2 t + j_0 \phi, \quad \phi \rightarrow \phi, \quad r \rightarrow \rho^2 + M_p - C_0, \]
one arrives at a stationary electromagnetic solution

\[ ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + K(\rho)^2 [d\phi + W(\rho) dt]^2, \]

\[ h(\rho) := \frac{\rho^2}{l^2} + \frac{M_p - M}{l^2} - b^2 \ln (\rho^2 + M_p), \]

\[ H(\rho) := \rho^2 + M_p - j_0^2 h(\rho), \]

\[ L(\rho) = \frac{1}{\rho} \sqrt{(\rho^2 + M_p) h(\rho)}, \]

\[ K(\rho) = \sqrt{H(\rho)}, \]

\[ N(\rho) = \sqrt{\rho^2 + M_p} \sqrt{\frac{h(\rho)}{H(\rho)}}, \]

\[ W(\rho) = -j_0 \frac{h(\rho)}{H(\rho)}, \quad (9.1) \]

To establish the angular deficit of this metric at the neighborhood of the rotation point \( \rho = 0 \) one looks at the behavior of \( g_{\phi\phi} = H(\rho) \) as \( \rho \rightarrow 0 \), which gives rise to the series

\[ Mp - \left( \frac{M_p}{l^2} - M - b^2 \ln (M_p) \right) j_0^2 + \left( 1 - \frac{1}{l^2} j_0^2 + \frac{b^2}{M_p} j_0^2 \right) \rho^2 + ..., \]

since the order zero in \( \rho \) has to vanish, then one gets the condition

\[ Mp - \left( \frac{M_p}{l^2} - M - b^2 \ln (M_p) \right) j_0^2 = 0, \quad (9.2) \]

which solved for \( M_p \) yields

\[ M_p = \exp\left[ -\frac{M}{b^2} - LambertW\left( \frac{l^2 - j_0^2}{j_0^2 b^2 l^2} \exp\left( -\frac{M}{l^2} \right) \right) \right] \quad (9.3) \]

Using (9.2) one brings \( H(\rho) \) to the form

\[ K(\rho)^2 = H(\rho) = b^2 j_0^2 \ln \left( 1 + \frac{\rho^2}{M_p} \right) + \frac{\rho^2 (l^2 - j_0^2)}{l^2}, \]

which behaves at \( \rho \rightarrow 0 \) as it should be as: \( H(\rho \rightarrow 0) = \left( 1 - \frac{1}{l^2} j_0^2 + \frac{b^2}{M_p} j_0^2 \right) \rho^2 + O (\rho^4) \).

On the other hand \( L(\rho)^2 \) is given explicitly by

\[ L(\rho)^2 = (\rho^2 + M_p) \left[ \frac{\rho^2}{l^2} + \frac{M_p}{j_0^2} - b^2 \ln \left( \frac{\rho^2 + M_p}{M_p} \right) \right] \rho^{-2}. \]
and $\rho \to 0$ behaves as a polynomial in $\rho$. Thus the factor $1 - \frac{1}{\pi} j_0^2 + \frac{b^2}{M_0} j_0^2$ determines the deficit in the angle $\phi$, $0 \leq \phi \leq 2\pi/\sqrt{1 - \frac{1}{\pi} j_0^2 + \frac{b^2}{M_0} j_0^2}$, $\Delta\phi = 2\pi(1 - 1/\sqrt{1 - \frac{1}{\pi} j_0^2 + \frac{b^2}{M_0} j_0^2})$.

Moreover this solution allows for the existence of a black hole; the vanishing of the function $h(\rho)$ determines the horizons

$$\rho^2_+ = -M_0 - b^2 l^2 \text{LambertW} \left( -\frac{M_0}{b^2 l^2} \right). \quad (9.4)$$

This metric can be considered as a modified Clement rotating electro–magnetic solution [18] (Cl.24).

**A. Mass, energy and momentum for the spinning electro–magnetic black hole**

The evaluation of the surface energy and momentum densities yield

$$\epsilon(\rho) = -\frac{1}{\pi \rho^2} \sqrt{h(\rho)} \left[ (l^2 - j_0^2) (\rho^2 + M_0) + j_0^2 b^2 \right] / (\rho^2 + M_0) - \epsilon_0,$$

$$j(\rho) = -\frac{j_0}{\pi} \frac{1}{\sqrt{h(\rho)}} [M - b^2 + b^2 \ln (\rho^2 + M_0)], \quad (9.5)$$

while the integral quantities can be evaluated from the generic expressions

$$J(\rho) = 2\pi K(\rho) j(\rho) = -2j_0 [M - b^2 + b^2 \ln (\rho^2 + M_0)],$$

$$E(\rho) = 2\pi K(\rho) \epsilon(\rho),$$

$$M(\rho) = N(\rho) E(\rho) - W(\rho) J(\rho) = -\frac{2\rho^2}{l^2} + 2M - \frac{2}{l^2} M_0 + 2b^2 \ln (\rho^2 + M_0) - 2\pi NK\epsilon_0. \quad (9.6)$$

The momentum–energy functions, evaluated for $\epsilon_0 = 0$, behave at spatial infinity $\rho \to \infty$ according to

$$j(\rho \to \infty) \approx -\frac{j_0 l}{\pi \rho \sqrt{l^2 - j_0^2}} [M - b^2 + 2b^2 \ln \rho],$$

$$J(\rho \to \infty) \approx -2j_0 [M - b^2 + 2b^2 \ln \rho],$$

$$\epsilon(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{1}{\pi l} + \frac{b^2 l^2 + j_0^2}{\pi \rho^2 l^2 - j_0^2} \ln (\rho) + \frac{M l^2 + j_0^2}{2\pi \rho^2 l^2 - j_0^2} - \frac{j_0^2}{\pi \rho^2 l^2 - j_0^2};$$

$$E(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{2\rho^2}{l^2} \sqrt{l^2 - j_0^2} + \frac{Ml^2}{\rho \sqrt{l^2 - j_0^2}} + 2\frac{b^2 l^2}{\rho \sqrt{l^2 - j_0^2}} \ln (\rho)$$

$$- \frac{\sqrt{l^2 - j_0^2}}{l^2 \rho} M_0 - 2\frac{b^2 j_0^2}{\rho \sqrt{l^2 - j_0^2}};$$

$$M(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{2\rho^2}{l^2} + 2M - \frac{2}{l^2} M_0 + 4b^2 \ln (\rho). \quad (9.7)$$

Using in the expressions (9.5) and (9.6) as reference energy density the quantity $\epsilon_0 = -\frac{1}{\pi \rho} \sqrt{-M_0 + \frac{\rho^2}{l^2}}$, which at the spatial infinity behaves as $\epsilon_{0|\infty}(M_0) \approx -\frac{1}{\pi l} + \frac{1M_0}{2\pi \rho^2}$, their
series expansions at $\rho \to \infty$ result in
\[
\epsilon(\rho \to \infty, \epsilon_0(\infty(M_0)) \approx -\frac{1M_0}{2\pi \rho^2} + \frac{b^2 l_j^2}{\pi \rho^2 l^2 - j_0^2} \ln(\rho) + \frac{Ml^2 + j_0^2}{\pi \rho^2 l^2 - j_0^2} - \frac{j_0^2 b^2 l^2}{\pi \rho^2 l^2 - j_0^2},
\]
\[
E(\rho \to \infty, \epsilon_0(\infty(M_0)) \approx -\frac{M_0}{\rho} \sqrt{l^2 - j_0^2} + \frac{M}{\rho} \frac{l^2 + j_0^2}{\sqrt{l^2 - j_0^2}} - 2\frac{b^2 l_j^2}{\rho \sqrt{l^2 - M^2}} \ln(\rho),
\]
\[
M(\rho \to \infty, \epsilon_0(\infty(M_0)) \approx M - M_0 + 2b^2 \ln(\rho).
\]

Therefore, comparing with the energy characteristics of the BTZ solution, one concludes that the mass parameter is equal to $M$. The momentum parameter is proportional to $j_0$. The contribution of the charge $b$ in the mass–energy–momentum functions at infinity is through logarithmical terms and hence they diverges at spatial infinity. For vanishing rotation parameter $j_0$ one arrives at the expressions of the mass–energy–momentum functions of the Peldan electrostatic solution, Section V.4.

X. KAMATA-KOIKA SOLUTION

The Kamata–Koikawa solution \[19\], see also \[5\] Eq. (7.7), is defined by the metric and the structural functions
\[
ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + K(\rho)^2 d\phi + W(\rho) dt^2,
\]
\[
L(\rho) = \frac{\sqrt{\Lambda}}{\rho}(\rho^2 - \rho_0^2), \sqrt{\Lambda} = 1/l,
\]
\[
K(\rho) = \sqrt{\rho^2 + \frac{Q^2}{\Lambda} \ln(\rho^2/\rho_0^2 - 1)},
\]
\[
N(\rho) = \rho L/K = \sqrt{\Lambda}(\rho^2 - \rho_0^2)/\sqrt{\rho^2 + \frac{Q^2}{\Lambda} \ln(\rho^2/\rho_0^2 - 1)},
\]
\[
W(\rho) = \frac{(\rho^2 - \rho_0^2)\sqrt{\Lambda}}{[\rho^2 + \frac{Q^2}{\Lambda} \ln(\rho^2/\rho_0^2 - 1)]} - \sqrt{\Lambda}.
\]

A. Mass, energy and momentum for the KK solution

The surface energy and momentum densities are respectively given by
\[
\epsilon(\rho, \epsilon_0) = -\frac{1}{\pi \sqrt{\Lambda}} \frac{Q^2 - \Lambda \rho_0^2 + \Lambda \rho^2}{[\rho^2 + \frac{Q^2}{\Lambda} \ln(\rho^2/\rho_0^2 - 1)]} - \epsilon_0,
\]
\[
j(\rho) = \frac{1}{\pi \sqrt{\Lambda}} \frac{\Lambda \rho_0^2 - Q^2 + Q^2 \ln(\rho^2/\rho_0^2 - 1)}{\sqrt{\rho^2 + \frac{Q^2}{\Lambda} \ln(\rho^2/\rho_0^2 - 1)}},
\]
while the integral quantities amount to

\[ J(\rho) = \frac{2\Lambda\rho_0^2 - Q^2}{\sqrt{\Lambda}} + \frac{2Q^2}{\sqrt{\Lambda}} \ln \left( \frac{\rho^2}{\rho_0^2} \right) - 1 \],

\[ E(\rho, \epsilon_0) = -\frac{2}{\sqrt{\Lambda}} \frac{Q^2 - \Lambda\rho_0^2 + \Lambda\rho^2}{\sqrt{\rho^2 + \frac{Q^2}{\Lambda} \ln \left( \frac{\rho^2}{\rho_0^2} \right) - 1}} - 2\pi K \epsilon_0 \],

\[ M(\rho, \epsilon_0) = -2\Lambda \rho^2 + 2(2\Lambda\rho_0^2 - Q^2) + 2Q^2 \ln \left( \frac{\rho^2}{\rho_0^2} - 1 \right) - 2\pi \sqrt{\Lambda} (\rho^2 - \rho_0^2) \epsilon_0. \]  

(10.3)

The evaluation of the functions above for vanishing \( \epsilon_0 \), i.e. \( \epsilon_0 = 0 \), behave at \( \rho \to \infty \) according to

\[ \epsilon(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{\sqrt{\Lambda}}{\pi} \left( \frac{\rho}{\rho_0} \right) + \frac{\Lambda \rho_0^2 - Q^2}{\pi \sqrt{\Lambda} \rho^2} + \frac{2Q^2}{\pi \sqrt{\Lambda} \rho^2} \ln \left( \frac{\rho}{\rho_0} \right), \]

\[ j(\rho \to \infty) \approx \frac{\Lambda \rho_0^2 - Q^2}{\pi \sqrt{\Lambda} \rho} + \frac{2Q^2}{\pi \sqrt{\Lambda} \rho} \ln \left( \frac{\rho}{\rho_0} \right), \]

\[ J(\rho \to \infty) \approx 2\frac{\Lambda \rho_0^2 - Q^2}{\sqrt{\Lambda}} + 4\frac{Q^2}{\sqrt{\Lambda}} \ln \left( \frac{\rho}{\rho_0} \right), \]

\[ E(\rho \to \infty, \epsilon_0 = 0) \approx -2\sqrt{\Lambda} \rho + \frac{2\Lambda \rho_0^2 - Q^2}{\sqrt{\Lambda} \rho} + \frac{2Q^2}{\sqrt{\Lambda} \rho} \ln \left( \frac{\rho}{\rho_0} \right), \]

\[ M(\rho \to \infty, \epsilon_0 = 0) \approx -2\Lambda \rho^2 + 2(2\Lambda\rho_0^2 - Q^2) + 4Q^2 \ln \left( \frac{\rho}{\rho_0} \right). \]  

(10.4)

Using in the expressions (10.3) as reference energy density the quantity \( \epsilon_0 = \frac{-1}{\pi \rho} \sqrt{-M_0 + \rho^2} \), which at the spatial infinity behaves as \( \epsilon_{0|\infty}(M_0) \approx -\frac{1}{\pi} + \frac{M_0}{2\pi \rho^2} = -\frac{\sqrt{\Lambda}}{\pi} + \frac{M_0}{2\pi \sqrt{\Lambda} \rho^2} \), the series expansions of the corresponding quantities at \( \rho \to \infty \) result in

\[ \epsilon(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx -\frac{M_0}{2\pi \sqrt{\Lambda} \rho^2} + \frac{\Lambda \rho_0^2 - Q^2}{\pi \sqrt{\Lambda} \rho^2} + \frac{2Q^2}{\pi \sqrt{\Lambda} \rho^2} \ln \left( \frac{\rho}{\rho_0} \right), \]

\[ E(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx -\frac{M_0}{\sqrt{\Lambda} \rho} + \frac{2\Lambda \rho_0^2 - Q^2}{\sqrt{\Lambda} \rho} + 4\frac{Q^2}{\sqrt{\Lambda} \rho} \ln \left( \frac{\rho}{\rho_0} \right), \]

\[ M(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx -M_0 + 2(\Lambda\rho_0^2 - Q^2) + 4Q^2 \ln \left( \frac{\rho}{\rho_0} \right). \]  

(10.5)

Therefore, comparing with the energy characteristics of the BTZ solution, one arrives to the conclusion that there is no a mass parameter of the kind \( M \) present in the BTZ solution. All characteristic functions logarithmically diverges at spatial infinity.
B. Field, energy–momentum and Cotton tensors for the Kamata-Koikawa solution

The electromagnetic field tensor

\[
(F^\alpha_\beta) = \begin{bmatrix}
0 & \frac{-Q\rho q}{\Lambda(\rho^2 - \rho_0^2)^2} & 0 \\
\frac{qQ\sqrt{\Lambda}(\rho^2 - \rho_0^2)}{\rho} & 0 & \frac{-qQ\sqrt{\Lambda}(\rho^2 - \rho_0^2)}{\rho} \\
0 & \frac{-Q\rho q}{\sqrt{\Lambda}(\rho^2 - \rho_0^2)^2} & 0
\end{bmatrix}, \tag{10.6}
\]

allows for a triple zero eigenvalue and the following set of eigenvectors

\[
\lambda_{1,2,3} = 0; \ V = (V^1, V^2, V^3 = \sqrt{\Lambda} V^1), \ V^\mu V_\mu = 0, \ V = S,
\]

Type : \{3S\}. \tag{10.7}

The electromagnetic energy momentum tensor with vanishing invariants occurs to be

\[
(T^\alpha_\beta) = \frac{1}{4\pi} \frac{Q^2}{\sqrt{\Lambda} (\rho^2 - \rho_0^2)} \begin{bmatrix}
-\sqrt{\Lambda} & 0 & -\Lambda \\
0 & 0 & 0 \\
1 & 0 & \sqrt{\Lambda}
\end{bmatrix}, \tag{10.8}
\]

while the Cotton tensor for this electromagnetic–gravitational stationary cyclic symmetric field is given by

\[
(C^\alpha_\beta) = \frac{Q^2}{(\rho^2 - \rho_0^2)} \begin{bmatrix}
-\sqrt{\Lambda} & 0 & 1 \\
0 & 0 & 0 \\
-\Lambda & 0 & \sqrt{\Lambda}
\end{bmatrix}. \tag{10.9}
\]

It is clear that both the Cotton and Maxwell tensors possess the same eigenvalues, namely the triple zero eigenvalue \( \lambda = 0 \). Searching for the eigenvectors of these tensors, one arrives at

\[
\lambda_{1,2,3} = 0; \ V = (V^1, V^2, V^3 = \sqrt{\Lambda} V^1), \ V^\mu V_\mu = \frac{(V^2)^2 \rho^2}{\Lambda (\rho - \rho_0)^2}, \ V = S, \ V(V^2 = 0) = N,
\]

Type : \{3S\}, \{2S, N\}, \{S, 2N\}, \{3N\}. \tag{10.10}

The eigenvectors are spacelike or null vectors depending on the non-vanishing or vanishing value of the component \( V^2 \). One may consider them different one to another, having different \( V^1 \) and \( V^2 \) components. The most degenerate case are \{3S\} and \{S, 2N\}. 
C. Proper Kamata-Koikawa solution, $\rho_0 = \pm Q/\sqrt{\Lambda}$

The proper Kamata–Koikawa solution is defined by the metric and structural functions of (10.1) for $\Lambda \rho_0^2 - Q^2 = 0$, i.e., $\rho_0 = \pm Q/\sqrt{\Lambda}$, namely

$$ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + K(\rho)^2 [d\phi + W(\rho) dt]^2,$$

$$L(\rho) = \frac{\sqrt{\Lambda}}{\rho} (\rho^2 - \rho_0^2),$$

$$K(\rho) = \sqrt{\rho^2 + \frac{Q^2}{\Lambda} \ln \left( \frac{\rho^2}{\rho_0^2} - 1 \right)},$$

$$N(\rho) = \rho L(\rho)/K = \sqrt{\Lambda} (\rho^2 - \rho_0^2)/(\rho^2 + \frac{Q^2}{\Lambda} \ln \left( \frac{\rho^2}{\rho_0^2} - 1 \right)),$$

$$W(\rho) = \frac{(\rho^2 - \rho_0^2) \sqrt{\Lambda}}{[\rho^2 + \frac{Q^2}{\Lambda} \ln \left( \frac{\rho^2}{\rho_0^2} - 1 \right)]} - \sqrt{\Lambda}, \quad (10.11)$$

The surface energy and momentum densities are respectively given by

$$\epsilon(\rho, \epsilon_0) = -\frac{\sqrt{\Lambda}}{\pi} \frac{\rho^2}{[\rho^2 + \frac{Q^2}{\Lambda} \ln \left( \frac{\rho^2}{\rho_0^2} - 1 \right)]},$$

$$j(\rho) = \frac{1}{\pi \sqrt{\Lambda}} \frac{Q^2 \ln \left( \frac{\rho^2}{\rho_0^2} - 1 \right)}{\sqrt{\rho^2 + \frac{Q^2}{\Lambda} \ln \left( \frac{\rho^2}{\rho_0^2} - 1 \right)}}, \quad (10.12)$$

while the integral quantities amount to

$$J(\rho) = 2 \frac{Q^2}{\sqrt{\Lambda}} \ln \left( \frac{\rho^2}{\rho_0^2} - 1 \right),$$

$$E(\rho, \epsilon_0) = -2 \sqrt{\Lambda} \frac{\rho^2}{\sqrt{\rho^2 + \frac{Q^2}{\Lambda} \ln \left( \frac{\rho^2}{\rho_0^2} - 1 \right)}} - 2\pi K \epsilon_0,$$

$$M(\rho, \epsilon_0) = -2 \Lambda \rho^2 + 2\Lambda \rho_0^2 + 2Q^2 \ln \left( \frac{\rho^2}{\rho_0^2} - 1 \right) - 2\pi \sqrt{\Lambda} (\rho_0^2 - \rho_0^2) \epsilon_0. \quad (10.13)$$
The evaluation of the functions above for vanishing $\epsilon_0$, i.e. $\epsilon_0 = 0$, behave at spatial infinity according to

$$
\epsilon(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{\sqrt{\Lambda}}{\pi} + \frac{2Q^2}{\pi\sqrt{\Lambda}\rho^2} \ln\left(\frac{\rho}{\rho_0}\right),
$$

$$
j(\rho \to \infty) \approx \frac{2Q^2}{\pi\sqrt{\Lambda}} \ln\left(\frac{\rho}{\rho_0}\right),
$$

$$
J(\rho \to \infty) \approx \frac{4Q^2}{\sqrt{\Lambda}} \ln\left(\frac{\rho}{\rho_0}\right),
$$

$$
E(\rho \to \infty, \epsilon_0 = 0) \approx -2\sqrt{\Lambda}\rho + \frac{2Q^2}{\sqrt{\Lambda}} \ln\left(\frac{\rho}{\rho_0}\right),
$$

$$
M(\rho \to \infty, \epsilon_0 = 0) \approx -2\Lambda\rho^2 + 2\Lambda\rho_0^2 + 4Q^2 \ln\left(\frac{\rho}{\rho_0}\right). \quad (10.14)
$$

Using in the expressions (10.13) as reference energy density the quantity $\epsilon_0 = -\frac{1}{\pi\rho}\sqrt{-M_0 + \frac{\rho^2}{\pi}}$, which at the spatial infinity behaves as $\epsilon_0(\infty) \approx -\frac{\sqrt{\Lambda}}{\pi} + \frac{M_0}{2\pi\sqrt{\Lambda}\rho^2}$, the series expansions of the corresponding quantities at $\rho = \infty$ result in

$$
\epsilon(\rho \to \infty, \epsilon_0(\infty)(M_0)) \approx -\frac{M_0}{2\pi\sqrt{\Lambda}\rho^2} + 2\frac{Q^2}{\pi\sqrt{\Lambda}\rho^2} \ln\left(\frac{\rho}{\rho_0}\right),
$$

$$
E(\rho \to \infty, \epsilon_0(\infty)(M_0)) \approx -\frac{M_0}{\sqrt{\Lambda}\rho} + 4\frac{Q^2}{\sqrt{\Lambda}\rho} \ln\left(\frac{\rho}{\rho_0}\right),
$$

$$
M(\rho \to \infty, \epsilon_0(\infty)(M_0)) \approx -M_0 + 4Q^2 \ln\left(\frac{\rho}{\rho_0}\right). \quad (10.15)
$$

In the work by Chan [20] there are some comments addressed to the evaluation of the global momentum, energy and mass of the proper Kamata–Koikawa solution: the exact and the approximated expressions of the momentum $J$ coincide with the corresponding ones given in (20 Eq.9) and (20 Eq.7). Moreover, the mass $M$ at spatial infinity, (10.14), coincides with the $M$ (20 Eq.10.) for a zero background energy density with the correct extra term $-2\Lambda\rho^2$. From my point of view, it is recommendable to accomplish series expansions of the quantities under consideration to determine how fast they approach to zero or diverge at spatial infinity. From this perspective, the evaluation of the energy density $\epsilon$ and the global energy $E$ yield to quantities different from zero at spatial infinity, although they both approach faster to zero as $\rho \to \infty$ than the momentum and mass.

Comparing with the energy characteristics of the BTZ solution, one concludes that the mass, energy and momentum functions logarithmically diverge at spatial infinity.

XI. PELDAN MAGNETOSTATIC SOLUTION

The magnetostatic solution [13], see also [5] (4.30), with a negative cosmological constant is determined by the metric functions

$$
ds^2 = -N(\rho)^2dt^2 + \frac{1}{L(\rho)^2}d\rho^2 + K(\rho)^2[d\phi + W(\rho)dt]^2,
$$

$$
L(\rho) = K(\rho) = \sqrt{\frac{\rho^2}{l^2} + 2a^2 \ln \rho + m}, \quad N(\rho) = \rho, \quad W(\rho) = 0. \quad (11.1)
$$
Notice that the metric functions $L^2$ and $K^2$ are positive functions for values of $\rho > \rho_{\text{root}}$, where

$$\rho_{\text{root}} = \exp\left[ -m^2 \frac{l^2}{a^2} - \frac{1}{2} \text{LambertW} \left( \frac{-1}{l^2 a^2 e^{-m^2}} \right) \right], \text{LambertW}(x) \exp(\text{LambertW}(x)) = x,$$

where for short $\text{LambertW} := \text{LambertW}$, $\rho_{\text{root}}$ is solution of the equation $g^\rho(\rho_{\text{root}}) = L^2(\rho_{\text{root}}) = 0$, or explicitly,

$$\rho_{\text{root}}^2/l^2 + 2a^2 \ln \rho_{\text{root}} + m = 0.$$

Therefore the coordinate $\rho$ does not cover the expected range $0 \leq \rho \leq \infty$. For $\rho \leq \rho_{\text{root}}$ the metric suffers an unacceptable signature change. This fact also points out on the non-existence of a horizon $\rho = \text{const}$ for the Peldan solution. Consequently, one has to modify the choice of the $\rho$ coordinate in order to be able to reach the origin of coordinates; with this purpose in mind a new coordinate system is chosen in the forthcoming paragraph (XI.C), see also (XI.D).

### A. Mass, energy and momentum for the Peldan solution

The surface energy density $\epsilon$ is given by

$$\epsilon(\rho) = -\frac{1}{\pi} K \left( \frac{\rho}{l^2} + \frac{a^2}{\rho} \right) \epsilon_0. \quad (11.2)$$

Consequently the global energy and mass are given by

$$E(\rho, \epsilon_0) = -2\frac{\rho}{l^2} - 2\frac{a^2}{\rho} - 2\pi K \epsilon_0,$$

$$M(\rho, \epsilon_0) = -2\frac{\rho^2}{l^2} - 2a^2 - 2\pi \rho K \epsilon_0. \quad (11.3)$$

For the natural choice of a vanishing reference energy density $\epsilon_0 = 0$, one has at the spatial infinity $\rho \to \infty$ that

$$\epsilon(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{1}{\pi} l + l \frac{m - 2a^2}{2\pi \rho^2} + l \frac{a^2}{2\pi \rho^2} \ln \rho,$$

$$E(\rho \to \infty, \epsilon_0 = 0) = -2\frac{\rho}{l^2} - 2\frac{a^2}{\rho},$$

$$M(\rho \to \infty, \epsilon_0 = 0) = -2\frac{\rho^2}{l^2} - 2a^2, \quad (11.4)$$

while if the reference energy is the one corresponding to the anti–de Sitter spacetime with $M_0$ parameter, $\epsilon_0 = -\frac{\rho}{\pi l^2}/\sqrt{l^2 + M_0}$, $\epsilon_{0|\infty}(M_0) \approx -\frac{\sqrt{\Lambda}}{\pi} + \frac{M_0}{2\pi \sqrt{\Lambda} \rho^2}$, then the energies are expressed at spatial infinity as

$$\epsilon(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx l \frac{m - M_0 - 2a^2}{2\pi \rho^2} + \frac{a^2}{2\pi \rho^2} \ln \rho,$$

$$E(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx \frac{m - M_0 - 2a^2}{\rho} + \frac{2a^2}{\rho} \ln \rho,$$

$$M(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx m - M_0 - 2a^2 + 2a^2 \ln \rho. \quad (11.5)$$
Comparing these quantities with the corresponding ones of the static BTZ solution counterpart, Section [IV.B] one sees a complete correspondence for vanishing electromagnetic parameter \( a \), thus one recognize \( m \) as mass parameter. Notice that the energy and mass include an amount of energy due to the magnetic field, in a way similar to the electric one, through a logarithmical terms; because of this dependence, these quantities logarithmically diverge at infinity.

B. Field, energy-momentum and Cotton tensors for the magnetostatic Peldan solution

The electromagnetic field tensor for this solution is given by

\[
(F^\alpha_\beta) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \frac{L^2 a}{\rho} \\
0 & -\frac{a}{L^2 \rho} & 0 \\
\end{bmatrix},
\]

(11.6)

and is algebraically characterized by the following eigenvectors

\[
\lambda_1 = 0; \text{ } V1 = [V^1 = V^1, V^2 = 0, V^3 = 0], V^\mu V_\mu = -(V^1)^2(\rho^2), \text{ } V1 = T1,
\]

\[
\lambda_2 = i\frac{a}{\rho}; \text{ } V1 = [V^1 = 0, V^2 = V^2, V^3 = i\frac{1}{L^2}V^2], \text{ } V1 = Z,
\]

\[
\lambda_3 = -i\frac{a}{\rho}; \text{ } V1 = [V^1 = 0, V^2 = V^2, V^3 = -i\frac{1}{L^2}V^2], \text{ } V1 = \bar{Z},
\]

Type: \( \{T, Z, \bar{Z}\} \).

(11.7)

As far as to the electromagnetic energy momentum tensor is concerned, its matrix amounts to

\[
(T^\alpha_\beta) = \begin{bmatrix}
-\frac{1}{8\pi} \frac{a^2}{\rho^2} & 0 & 0 \\
0 & \frac{1}{8\pi} \frac{a^2}{\rho^2} & 0 \\
0 & 0 & \frac{1}{8\pi} \frac{a^2}{\rho^2} \\
\end{bmatrix}
\]

(11.8)

with the following eigenvalues and their corresponding eigenvectors

\[
\lambda_1 = -\frac{1}{8\pi} \frac{a^2}{\rho^2}; \text{ } V1 = (V^1, 0, 0), \text{ } V^\mu V_\mu = -(V^1)^2, \text{ } V1 = T1,
\]

\[
\lambda_2 = \frac{1}{8\pi} \frac{a^2}{\rho^2}; \text{ } V2 = (0, V^2, V^3), \text{ } V^\mu V_\mu = (V^2)^2/L^2 + (V^3)^2 L^2, \text{ } V2 = S2,
\]

\[
\lambda_3 = \frac{1}{8\pi} \frac{a^2}{\rho^2}; \text{ } V3 = (0, \tilde{V}^2, \tilde{V}^3), \text{ } V^\mu V_\mu = (\tilde{V}^2)^2/L^2 + (\tilde{V}^3)^2 L^2, \text{ } V3 = S3,
\]

Type: \( \{T, 2S\} \).

(11.9)

This tensor structure corresponds to that one describing a perfect fluid energy momentum tensor, but this time for the state equation: \textit{energy = pressure}. Again, the solutions generated from this metric by using coordinate transformations will possess this perfect fluid feature because the invariance of the eigenvalues.
The Cotton tensor for electrostatic cyclic symmetric gravitational field is given by
\[ (C^{\alpha \beta}) = \begin{bmatrix} 0 & 0 & \frac{1}{2} \frac{a^2 (\rho^2 + 2 a^2 l^2 \ln(\rho))}{\rho^4} \\ 0 & 0 & 0 \\ \frac{1}{2} \frac{a^2}{\rho^2} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{a^2 L^2}{2 \rho^4} \\ 0 & 0 & 0 \\ -\frac{a^2}{2 \rho^2} & 0 & 0 \end{bmatrix}. \quad (11.10) \]

Searching for its eigenvectors, one arrives at
\[ \lambda_1 = 0; V_1 = (0, V^2, 0), V^\mu V_\mu = (V^2)^2 g_{\rho \rho}, V_1 = S_1, \]
\[ \lambda_2 = \frac{i}{2} \frac{L a^2}{\rho^3}; V_2 = (V^1 = -\frac{i L}{\rho} V^3, 0, V^3), V_2 = Z, \]
\[ \lambda_3 = -\frac{i}{2} \frac{L a^2}{\rho^3}; V_3 = (V^1 = \frac{i L}{\rho} V^3, 0, V^3), V_3 = \bar{Z}, \]
Type : \{T, Z, \bar{Z}\}. \quad (11.11) \]

The eigenvectors \( V_2 \) and \( V_3 \) are complex conjugated while the vector \( V_1 \), associated to the zero eigenvalue, occurs to be spacelike—the only physically tractable \( \rho \)-direction vector in this case. It is worthwhile to point out that the solutions generated via coordinate transformations, in particular the \( SL(2, R) \) transformations, applied onto this magneto-static cyclic symmetric metric will shear the eigenvalues \( \lambda_i \) of the Cotton tensor quoted above; recall that eigenvalues are invariant characteristics of tensors, although their components in different coordinate systems are different—this last also applies to the eigenvectors of the seed and the resulting solutions.

C. Field, energy-momentum and Cotton tensors for a modified magnetostatic Peldan solution

One encounters in the literature a slightly modified Peldan solution, namely the one with metric
\[ (g_{\alpha \beta}) = \begin{bmatrix} -\rho^2 - Mg & 0 & 0 \\ 0 & 1/L^2 & 0 \\ 0 & 0 & \frac{L^2 \rho^2}{\rho^2 + Mg} \end{bmatrix}, \]
\[ L^2 := \left(-l^2 M + \rho^2 + Mg + a^2 l^2 \ln(\rho^2 + Mg)\right)\left(\rho^2 + Mg\right). \quad (11.12) \]

One easily establishes that in this coordinate system one may reach the origin \( \rho = 0 \). In fact,
\[ g_{\phi \phi} = \frac{L^2 \rho^2}{\rho^2 + Mg} = \frac{\rho^2}{l^2} + \frac{Mg}{l^2} - M + a^2 \ln(\rho^2 + Mg) = \frac{\rho^2}{l^2} + a^2 \ln\left((\rho^2 + Mg) \exp\left(\frac{Mg - M l^2}{a^2 l^2}\right)\right), \]
thus, adopting the choice
\[ Mg \exp\left(\frac{Mg - M l^2}{a^2 l^2}\right) = 1 \rightarrow Mg = l^2 LW\left(\frac{e^M}{l^2}\right), LW(x) \exp(LW(x)) = x, \]
where LW stand for LambertW–function, one arrives as \(\rho \to 0\) at

\[
g_{\phi\phi|\rho\to0} = \rho^2 a^2 \exp\left(\frac{Mg - Ml^2}{a^2 l^2}\right) = \rho^2 a^2 \frac{Mg}{l^2}, \quad g_{\rho\rho|\rho\to0} = \frac{1}{a^2},
\]

\[
\to ds^2|_{\rho\to0} = -(dt \sqrt{Mg})^2 + (d\rho/a)^2 + \left(\frac{\rho}{a}\right)^2 (d\sqrt{Mg}/\rho)^2,
\]

where it has been taken into account that \(1/g_{\rho\rho} = L^2 = g_{\phi\phi}|\rho\to0 = \rho^2 a^2 (\rho^2 + Mg)\). Therefore, the angular coordinate exhibit an angle deficit, assuming \(0 \leq \phi < 2\pi\sqrt{Mg/a^2}\), and \(\Delta\phi = 2\pi \left(1 - \frac{\sqrt{Mg}}{a^2}\right)\).

The electromagnetic field tensor is given by

\[
(F^\alpha_\beta) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & L^2 \rho a/(\rho^2 + Mg) \\
0 & -a/L^2 \rho & 0
\end{bmatrix},
\]

with eigenvectors

\[
\lambda_1 = 0; \quad V_1 = [V^1 = V^2 = 0, V^3 = 0], \quad V^\mu V_\mu = -(V^1)^2 (\rho^2 + Mg), \quad V_1 = T_1,
\]

\[
\lambda_2 = i \frac{a}{\sqrt{\rho^2 + Mg}}; \quad V_1 = [V^1 = 0, V^2 = V^3 = i \sqrt{\rho^2 + Mg}/L^2 \rho] V^2, \quad V_1 = Z,
\]

\[
\lambda_3 = -i \frac{a}{\sqrt{\rho^2 + Mg}}; \quad V_1 = [V^1 = 0, V^2 = V^3 = -i \sqrt{\rho^2 + Mg}/L^2 \rho] V^2, \quad V_1 = \bar{Z},
\]

Type : \(\{T, Z, \bar{Z}\}\). (11.14)

while the electromagnetic energy–momentum tensor amounts to

\[
(T^\alpha_\beta) = \begin{bmatrix}
-\frac{1}{8\pi} \frac{a^2}{(\rho^2 + Mg)} & 0 & 0 \\
0 & \frac{1}{8\pi} \frac{a^2}{(\rho^2 + Mg)} & 0 \\
0 & 0 & \frac{1}{8\pi} \frac{a^2}{(\rho^2 + Mg)}
\end{bmatrix}.
\]

(11.15)
with eigenvectors

\[
\lambda_1 = -\frac{1}{8 \pi \rho^2 + Mg} V_1 = [V^1 = V^1, V^2 = 0, V^3 = 0],
\]
\[
V^\mu V_\mu = -V^1 (\rho^2 + Mg), \quad V_1 = T_1,
\]
\[
\lambda_2 = \frac{1}{8 \pi \rho^2 + Mg} V_2 = [V^1 = 0, V^2 = V^2, V^3 = V^3],
\]
\[
V^\mu V_\mu = \frac{V^2}{\rho^2 + (\rho^2 + Mg)} V^1, \quad V_2 = S_2,
\]
\[
\lambda_3 = \frac{1}{8 \pi \rho^2 + Mg} V_3 = [V^1 = 0, V^2 = \tilde{V}^2, V^3 = \tilde{V}^3],
\]
\[
V^\mu V_\mu = \frac{\tilde{V}^2}{\rho^2 + (\rho^2 + Mg)} \tilde{V}^1, \quad V_3 = S_3,
\]
Type : \{T, 2S\}.  \hspace{1cm} (11.16)

The Cotton tensor

\[
(C^\alpha_{\beta}) = \begin{pmatrix}
0 & 0 & \frac{1}{2} \frac{a^2 \rho^2}{(\rho^2 + Mg)^2} \\
0 & 0 & 0 \\
\frac{-1}{2} \frac{a^2}{\rho^2 + Mg} & 0 & 0
\end{pmatrix},
\]
\hspace{1cm} (11.17)

possesses the following set of eigenvectors

\[
\lambda_1 = 0; \quad V_1 = [0, V^2, 0], V^\mu V_\mu = \frac{V^2}{\rho^2}, \quad V_1 = S_1,
\]
\[
\lambda_2 = i \frac{a^2}{2} \frac{\rho L}{(\rho^2 + Mg)^2}; \quad V_2 = [V^1 = -i \frac{L}{\rho^2 + Mg} V^3, V^2 = 0, V^3 = 0], \quad V_2 = Z,
\]
\[
\lambda_3 = -i \frac{a^2}{2} \frac{\rho L}{(\rho^2 + Mg)^2}; \quad V_2 = [V^1 = i \frac{L}{\rho^2 + Mg} V^3, V^2 = 0, V^3 = 0], \quad V_2 = \bar{Z},
\]
Type : \{T, Z, \bar{Z}\}.  \hspace{1cm} (11.18)

D. Hirschman–Welch solution; energy and mass

Accomplishing in the original Peldan solution, Eq. (11.1), the coordinate transformation

\[
t \rightarrow t, \quad \rho \rightarrow \sqrt{(\rho^2 + r_+^2 - m l^2)/l^2}, \quad \phi \rightarrow \phi l^2, \quad \chi^2 := a^2 l^2
\]
one obtains the Hirschman–Welch magnetostatic solution representation [21], see also [3] (4.33), which is given by the metric functions

\[
ds^2 = -N(\rho)^2dt^2 + \frac{1}{L(\rho)^2}d\rho^2 + K(\rho)^2[\phi + W(\rho)dt]^2,
\]

\[
H(\rho) = \frac{\rho^2 + r_+^2 - ml^2}{l^2}, \quad L(\rho) = \frac{\sqrt{H(\rho)}}{\rho} K(\rho),
\]

\[
K(\rho) = \sqrt{\rho^2 + r_+^2 + \chi^2 \ln H(\rho)},
\]

\[
N(\rho) = \frac{\sqrt{H(\rho)}}{\rho} W(\rho) = 0.
\]

In the original Hirschman–Welch work [21] there is a condition to be fulfilled by the parameter \( r_+ \), arising from the vanishing of \( K \) at \( \rho = 0 \), namely

\[
\frac{r_+^2}{l^2} + \chi^2 \ln \left( \frac{r_+^2}{l^2} - m \right) = 0 \rightarrow \left( \frac{r_+^2}{l^2} - m \right) e^{\left( \frac{r_+^2}{l^2} / \chi^2 \right)} = 1.
\]

This equation has been used in the quoted publication to determine the conical angle deficit: as \( \rho \to 0 \) the behavior of \( K^2/\rho^2 = 1 + \chi^2/\rho^2 \ln \left[ (\rho^2 + r_+^2 - ml^2)/(r_+^2 - ml^2) \right] \) is given by

\[
(K^2/\rho^2)|_{\rho \to 0} \to \frac{r_+^2 - ml^2 + \chi^2}{r_+^2 - ml^2} = 1 + \frac{\chi^2}{l^2} e^{\left( \frac{r_+^2}{l^2} / \chi^2 \right)},
\]

hence the spatial sector \( \left[ \frac{1}{l^2} d\rho^2 + K^2 d\phi^2 \right]_{\rho \to 0} \) of the studied metric behaves as

\[
\left[d(\rho \frac{e^{\left( \frac{r_+^2}{l^2} / \chi^2 \right)}}{\sqrt{1 + \frac{\chi^2}{l^2} e^{\left( \frac{r_+^2}{l^2} / \chi^2 \right)}}})\right]^2 + \frac{\rho^2}{1 + \frac{\chi^2}{l^2} e^{\left( \frac{r_+^2}{l^2} / \chi^2 \right)}} \left[d\phi e^{\left( \frac{-r_+^2}{l^2} / \chi^2 \right)} (1 + \frac{\chi^2}{l^2} e^{\left( \frac{r_+^2}{l^2} / \chi^2 \right)}) \right]^2 = \frac{d\rho^2 + \rho^2 d\phi^2}{1 + \frac{\chi^2}{l^2} e^{\left( \frac{r_+^2}{l^2} / \chi^2 \right)}},
\]

hence the angles ranges

\[
0 \leq \phi \leq 2\pi \rightarrow 0 \leq \phi \leq 2\pi \frac{e^{\left( \frac{r_+^2}{l^2} / \chi^2 \right)}}{1 + \frac{\chi^2}{l^2} e^{\left( \frac{r_+^2}{l^2} / \chi^2 \right)}} = 2\pi \frac{\rho^2}{r_+^2 - ml^2 + \chi^2},
\]

thus, the conical singularity at \( \rho = 0 \), as pointed out in the HW paper, arises in \( \phi \) with the period \( T_\phi = 2\pi \nu := 2\pi e^{\left( \frac{r_+^2}{l^2} / \chi^2 \right)} (1 + \frac{\chi^2}{l^2} e^{\left( \frac{r_+^2}{l^2} / \chi^2 \right)}) \), consequently the angle deficit is \( \delta T_\phi = 2\pi (1 - \nu) \) as reported also in [22].

### E. Mass, energy and momentum for the HW solution

For this electromagnetic field solution the surface energy density is given by

\[
\epsilon(\rho, \epsilon_0) = -\frac{1}{\pi l} \frac{\rho^2 + r_+^2 - ml^2 + \chi^2}{\sqrt{\rho^2 + r_+^2 + \chi^2 \ln H(\sqrt{\rho^2 + r_+^2} - l^2 m)}} - \epsilon_0,
\]

\[(11.21)\]
while the integral energy and mass amount to

\begin{align*}
E(\rho, \epsilon_0) &= -\frac{2}{l} \rho^2 + r_+^2 - ml^2 + \chi^2 - 2\pi K \epsilon_0, \\
M(\rho, \epsilon_0) &= -\frac{2}{l^2} (\rho^2 + r_+^2 - ml^2 + \chi^2) - 2\pi N K \epsilon_0
\end{align*}

(11.22)

The evaluation of the above functions independent of \(\epsilon_0\) behave at infinity according to

\begin{align*}
\epsilon(\rho \to \infty, \epsilon_0 = 0) &\approx -\frac{1}{\pi l} + \frac{ml^2 - 2\chi^2}{2\pi l \rho^2} + \frac{\chi^2}{\pi l \rho^2} \ln \left(\frac{\rho}{l}\right), \\
E(\rho \to \infty, \epsilon_0 = 0) &\approx -\frac{2}{l} \rho + \frac{ml^2 - r_+^2 - 2\chi^2}{l \rho}, \\
M(\rho \to \infty, \epsilon_0 = 0) &= -\frac{2}{l^2} (\rho^2 + r_+^2 - ml^2 + \chi^2).
\end{align*}

(11.23)

Using in the expressions (11.22) the energy density for the anti–de Sitter solution counterpart, namely \(\epsilon_0 = -\frac{1}{\pi l^2} \rho/\sqrt{M_0 + \frac{\rho^2}{l^2}}\), which at the spatial infinity behaves as \(\epsilon_0(\rho \to \infty) \approx -\frac{1}{\pi l} + \frac{M_0}{2\pi \rho^2}\), the series expansions of the corresponding quantities at \(\rho \to \infty\) result in

\begin{align*}
\epsilon(\rho \to \infty, \epsilon_0(\rho \to \infty)) &\approx -\frac{l M_0}{2\pi \rho^2} + \frac{ml^2 - 2\chi^2}{2\pi l \rho^2} + \frac{\chi^2}{\pi l \rho^2} \ln \left(\frac{\rho}{l}\right), \\
E(\rho \to \infty, \epsilon_0(\rho \to \infty)) &\approx -\frac{l M_0}{\rho} + \frac{ml^2 - 2\chi^2}{l \rho} + \frac{2\chi^2}{l \rho} \ln \left(\frac{\rho}{l}\right), \\
M(\rho \to \infty, \epsilon_0(\rho \to \infty)) &= m - M_0 - 2\frac{\chi^2}{l^2} + 2\frac{\chi^2}{l^2} \ln \left(\frac{\rho}{l}\right).
\end{align*}

(11.24)

Therefore, comparing with the energy characteristics of the BTZ solution, one concludes that the mass logarithmically diverges at spatial infinity, and that the role of mass is played by \(m\).

\section*{F. Field, energy-momentum and Cotton tensors for the generalized–via \(SL(2, R)\) transformations–Peldan solution}

Under \(SL(2, R)\) transformations of the form

\begin{align*}
t &= \alpha T + \beta \Phi, \quad \alpha \delta - \beta \gamma = 1, \\
\phi &= \gamma T + \delta \Phi,
\end{align*}

(11.25)

the metric transforms into

\[
\begin{pmatrix}
-\alpha^2 (\rho^2 + Mg) + \gamma^2 \frac{L^2 \rho^2}{(\rho^2 + Mg)} & 0 & -\alpha \beta (\rho^2 + Mg) + \gamma \delta \frac{L^2 \rho^2}{(\rho^2 + Mg)} \\
0 & \frac{1}{L^2} & 0 \\
-\alpha \beta (\rho^2 + Mg) + \gamma \delta \frac{L^2 \rho^2}{(\rho^2 + Mg)} & 0 & -\beta^2 (\rho^2 + Mg) + \delta^2 \frac{L^2 \rho^2}{(\rho^2 + Mg)}
\end{pmatrix},
\]

(11.26)
and the field tensor becomes

\[
(F^\alpha_\beta) = \begin{bmatrix}
0 & \frac{\beta a}{L^2 \rho} & 0 \\
\frac{\gamma L^2 \rho a}{(\rho^2 + Mg)} & 0 & \frac{\delta L^2 \rho a}{(\rho^2 + Mg)} \\
0 & -\frac{\alpha a}{L^2 \rho} & 0
\end{bmatrix},
\] (11.27)

\[
\lambda_1 = 0; \quad V_1 = [V^1 = V^1, V^2 = 0, V^3 = -\frac{\gamma}{\delta} V^1],
\]

\[
V^\mu V_\mu = -V^2 (\rho^2 + Mg)/\delta^2, \quad V_1 = T_1,
\]

\[
\lambda_2 = \frac{a}{\sqrt{\rho^2 + Mg}}; \quad V_2 = [V^1 = -\frac{\beta}{\alpha} V^3, V^2 = \frac{L^2 \rho V^3}{\sqrt{\rho^2 + Mg}}, V^3 = V^3], \quad V_2 = Z,
\]

\[
\lambda_3 = -\frac{a}{\sqrt{\rho^2 + Mg}}; \quad V_3 = [V^1 = -\frac{\beta}{\alpha} V^3, V^2 = -\frac{L^2 \rho V^3}{\sqrt{\rho^2 + Mg}}, V^3 = V^3], \quad V_3 = \bar{Z},
\]

and the energy tensor

\[
(T^\alpha_\beta) = \begin{bmatrix}
\frac{1}{8} \frac{a^2 (\beta + \alpha \delta)}{\rho^2 \pi} & 0 & -\frac{1}{4} \frac{\gamma a^2}{\rho^2 \pi} \\
0 & \frac{1}{8} \frac{a^2}{\rho^2 \pi} & 0 \\
\frac{1}{4} \frac{\alpha \delta a^2}{\rho^2 \pi} & 0 & -\frac{1}{8} \frac{a^2 (\beta + \alpha \delta)}{\rho^2 \pi}
\end{bmatrix},
\] (11.29)

allows for the following eigenvectors

\[
\lambda_1 = -\frac{1}{8 \pi} \frac{a^2}{\rho^2 + Mg}; \quad V_1 = [V^1 = V^1, V^2 = 0, V^3 = -\frac{\gamma}{\delta} V^1],
\]

\[
V^\mu V_\mu = -V^2 (\rho^2 + Mg)/\delta^2, \quad V_1 = T_1,
\]

\[
\lambda_2 = \frac{1}{8 \pi} \frac{a^2}{\rho^2 + Mg}; \quad V_2 = [V^1 = -\frac{\beta}{\alpha} V^3, V^2 = V^2, V^3 = V^3],
\]

\[
V^\mu V_\mu = \frac{V_2^2}{L^2} + \frac{L^2 \rho^2}{\rho^2 + Mg} \frac{V_2^2}{\alpha^2}, \quad V_2 = S_2,
\]

\[
\lambda_3 = \frac{1}{8 \pi} \frac{a^2}{\rho^2 + Mg}; \quad V_3 = [V^1 = -\frac{\beta}{\alpha} V^3, V^2 = \bar{V}^2, V^3 = \bar{V}^3],
\]

\[
V^\mu V_\mu = \frac{\bar{V}_2^2}{L^2} + \frac{L^2 \rho^2}{\rho^2 + Mg} \frac{\bar{V}_2^2}{\alpha^2}, \quad V_3 = S_3,
\]

Type : \{T, 2 S\}.

The transformed Cotton tensor is given by

\[
(C^\alpha_\beta) = \begin{bmatrix}
-\frac{1}{2} \frac{\alpha \beta a^2}{(\rho^2 + Mg)} & -\frac{1}{2} \frac{a^2 L^2 \rho^2 \gamma \delta}{(\rho^2 + Mg)^3} & 0 & -\frac{1}{2} \frac{a^2 \beta^2}{(\rho^2 + Mg)} & -\frac{1}{2} \frac{a^2 L^2 \rho^2 \gamma \delta}{(\rho^2 + Mg)^3} \\
0 & 0 & 0 & 0
\end{bmatrix},
\] (11.31)
and has the eigenvectors

\[ \lambda_1 = 0; \ V1 = [0, V2, 0], V^\mu V_\mu = \frac{V^2}{L^2}, \ V1 = S1, \]

\[ \lambda_2 = -\frac{1}{2} \frac{\rho a^2 L}{(\rho^2 + Mg)^2}; \]

\[ V2 = [V^1 = -\frac{V^3}{\beta \alpha (\rho^2 + Mg)^2 - iL\rho (\rho^2 + Mg) + L^2 \rho^2 \gamma \delta}], V^2 = 0, V^3], \ V2 = Z \]

\[ \lambda_3 = -\frac{i}{2} \frac{\rho a^2 L}{(\rho^2 + Mg)^2}; \]

\[ V3 = [V^1 = -\frac{V^3}{\beta \alpha (\rho^2 + Mg)^2 + iL\rho (\rho^2 + Mg) + L^2 \rho^2 \gamma \delta}], V^2 = 0, V^3], \ V3 = \bar{Z}, \]

Type \(\{S, Z, \bar{Z}\}\). (11.32)

**G. False Stationary Peldan magnetostatic solution**

The magnetostatic solution with a negative cosmological constant is determined by the metric

\[ ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + K(\rho)^2[d\phi + W(\rho)dt]^2, \]

\[ h(\rho) := \rho^2 + M_+, \]

\[ K(\rho) = \frac{1}{l} \sqrt{K_0 + \rho^2 + a^2 l^2 \ln h(\rho)}, \]

\[ L(\rho) = \frac{1}{\rho l} \sqrt{(K_0 + \rho^2 + a^2 l^2 \ln h(\rho))h(\rho)}, \]

\[ N(\rho) = \sqrt{h(\rho)}, \ W(\rho) = -2J_0. \] (11.33)

The surface energy and momentum densities are given by

\[ \epsilon(\rho) = -\frac{1}{\pi l^2 K(\rho) N(\rho)}(\rho^2 + M_+ + a^2 l^2) - \epsilon_0, \]

\[ j(\rho) = 0. \] (11.34)

Since the momentum \(j(\rho)\) is zero, the stationarity of this solution is fictitious, as a matter of fact one is dealing with a static metric.

**XII. DIAS–LEMOS SOLUTION**

Subjecting the static Hirschman–Welch metric to the \(SL(2, R)\) transformation

\[ t \rightarrow \sqrt{1 + \omega^2} t - \omega l \phi, \ \rho \rightarrow \rho, \ \phi \rightarrow -\frac{\omega}{l} t + \sqrt{1 + \omega^2} \phi, \]
one arrives at the Dias–Lemos solution [22], see also [3] (11.41), determined by the metric

\[
    ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + K(\rho)^2 [d\phi + W(\rho) dt]^2,
\]

\[
    H(\rho) = (\rho^2 + r_+^2 - m l^2) / l^2,
\]

\[
    L(\rho) = \sqrt{\frac{H(\rho)}{\rho}} \sqrt{\rho^2 + r_+^2 + \chi^2 \ln H(\rho)},
\]

\[
    K(\rho) = \sqrt{\rho^2 + r_+^2 + \omega^2 l^2 m + (1 + \omega^2) \chi^2 \ln H(\rho)},
\]

\[
    N(\rho) = \rho \frac{\sqrt{L(\rho)}}{\sqrt{K(\rho)}} W(\rho) = -\frac{\omega \sqrt{1 + \omega^2} [m l^2 + \chi^2 \ln H(\rho)]}{K(\rho)^2} (12.1)
\]

Notice that this metric, in the case of vanishing charge $\chi = 0$, yields to an alternative coordinate representation of the rotating BTZ solution, with parameter $\omega$, namely

\[
    ds^2 = -\left(\frac{\rho^2 + r_+^2}{l^2} - (1 + \omega^2) m l^2\right) dt^2 + \frac{\rho^2 l^2}{(\rho^2 + r_+^2) (\rho^2 + r_+^2 - m l^2)} d\rho^2
\]

\[
    -2\omega m l \sqrt{1 + \omega^2} d\phi dt + (\rho^2 + r_+^2 + l^2 \omega^2) d\phi^2,
\]

which differs from the standard BTZ solution representations.

Accomplishing in the above metric the transformations

\[
    t \to l^2 t, \quad \rho \to \sqrt{\frac{\rho^2}{l^2} - m l^2 \omega^2 - r_+^2 + M}
\]

and identifying the parameters according to

\[
    m = \frac{M}{1 + 2 \omega^2}, \quad m^2 \frac{(1 + \omega^2) \omega^2 l^2}{(2 \omega^2 + 1)^2} = J^2 / 4, \to
\]

\[
    \omega^2 = \frac{1}{2} \frac{M l \pm \sqrt{M^2 l^2 - J^2}}{\sqrt{M^2 l^2 - J^2}}, \quad m = \frac{\sqrt{M^2 l^2 - J^2}}{l^3},
\]

one arrives at the BTZ solution counterpart representation (4.12)

\[
    ds^2 = -\rho^2 dt^2 + \left(\frac{\rho^2}{l^2} + M + \frac{J^2}{4 \rho^2}\right)^{-1} d\rho^2 - J d\phi dt + \left(\frac{\rho^2}{l^2} + M\right) d\phi^2.
\]

On the other hand, by replacing $\rho \to \sqrt{\rho^2 - m l^2 \omega^2 - r_+^2}$ one arrives at the middle of the road metric

\[
    ds^2 = -\left(\frac{\rho^2}{l^2} - (1 + 2 \omega^2) m\right) dt^2 + \left(\frac{\rho^2}{l^2} - (1 + 2 \omega^2) m + \frac{l^2 m^2 \omega^2 (1 + \omega^2)}{\rho^2}\right)^{-1} d\rho^2
\]

\[
    -2\omega m l \sqrt{1 + \omega^2} d\phi dt + \rho^2 d\phi^2,
\]

which, identifying

\[
    m = \frac{M}{1 + 2 \omega^2}, \quad l^2 m^2 \omega^2 (1 + \omega^2) = J^2 / 4 \to
\]

\[
    \omega^2 = \frac{1}{2} \frac{M l \pm \sqrt{M^2 l^2 - J^2}}{\sqrt{M^2 l^2 - J^2}}, \quad m = \frac{\sqrt{M^2 l^2 - J^2}}{l}, \quad 2 \sqrt{1 + \omega^2} \omega m l \to J,
\]
gives rise to the standard description of the stationary BTZ black hole metric (1.1).

Therefore, as the vacuum limit of the DL metric (12.1) one may consider the rotating BTZ solution counterpart, and consequently one may think it of as the reference vacuum solution in the evaluation of the quasi local energy, momentum and mass.

A. Mass, energy and momentum for the DL solution

The surface energy and momentum densities are given by

$$\epsilon(\rho, \epsilon_0) = -\rho \frac{L}{\pi l^2 K^2 H} (\rho^2 + r_+^2 - ml^2 + (1 + \omega^2)\chi^2) - \epsilon_0$$

$$j(\rho, \epsilon_0) = \omega \sqrt{1 + \omega^2} \frac{L}{\pi l N K} [m l^2 - \chi^2 + \chi^2 \ln H],$$ (12.5)

while the integral quantities amount to

$$J(\rho, \epsilon_0) = 2 \omega \sqrt{1 + \omega^2} \frac{L}{\pi l N K} [m l^2 - \chi^2 + \chi^2 \ln H] = \frac{2 l}{\pi} \omega \sqrt{1 + \omega^2} [m l^2 - \chi^2 + \chi^2 \ln H],$$

$$E(\rho, \epsilon_0) = -2 \frac{\rho}{l^2 K H} P(\rho) - 2 \pi K \epsilon_0,$$

$$M(\rho, \epsilon_0) = -2 \frac{\rho}{l^2 K H} P(\rho) - W J - 2 \pi N K \epsilon_0$$

$$= -2 \frac{\omega}{l^2} (\rho^2 + \chi^2 + r_+^2 - ml^2) + 2 \omega^2 \frac{l^2}{l^2} [m l^2 - \chi^2 + \chi^2 \ln H] - 2 \pi N K \epsilon_0,$$

$$P(\rho) := \rho^2 + r_+^2 - ml^2 + (1 + \omega^2)\chi^2.$$ (12.6)

The evaluation of the main parts of above functions, i.e., the corresponding functions independent of $\epsilon_0$ behave at infinity according to

$$J(\rho \to \infty) \approx \frac{\omega}{l} \sqrt{1 + \omega^2} [m l^2 - \chi^2 + 2 \chi^2 \ln (\frac{\rho}{l})],$$

$$E(\rho \to \infty) \approx -2 \frac{\rho}{l^2} \sqrt{1 + \omega^2} [m l^2 - \chi^2 + 2 \chi^2 \ln (\frac{\rho}{l})],$$

$$\epsilon(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{1}{\pi l} + \frac{m l^2 - 2 \chi^2}{2 \pi l \rho^2} + \frac{\chi^2}{\pi l \rho^2} \ln (\frac{\rho}{l})$$

$$+ \omega^2 \frac{[m l^2 - \chi^2]}{l \rho^2} + 2 \omega^2 \frac{\chi^2}{l \rho^2} \ln (\frac{\rho}{l})],$$

$$M(\rho \to \infty, \epsilon_0 = 0) \approx 2 m - 2 \frac{1}{l^2} (\rho^2 + r_+^2 + \chi^2) + 2 \omega^2 \frac{l^2}{l^2} [m l^2 - \chi^2 + 2 \chi^2 \ln (\frac{\rho}{l})].$$ (12.7)

Using in the expressions (12.6) as reference energy density the quantity $\epsilon_0 = -\frac{1}{\pi l^2} \sqrt{\frac{\rho^2}{l^2} - M_0}$, which at the spatial infinity behaves as $\epsilon_0(\infty) \approx -\frac{1}{\pi l} + \frac{M_0}{2 \pi r^2}$, the series expansions of the
corresponding quantities at \( \rho = \infty \) result in

\[
\begin{align*}
\epsilon(\rho \to \infty, \epsilon_0) & \approx \frac{l}{2\pi \rho^2} (m - M_0) - \frac{1}{2\pi l \rho^2} (\chi^2 - \chi^2 \ln \left(\frac{\rho}{l}\right)) \\
& \quad + \frac{\omega^2}{\pi l \rho^2} [ml^2 - \chi^2 + 2\chi^2 \ln \left(\frac{\rho}{l}\right)], \\
E(\rho \to \infty, \epsilon_0) & \approx \frac{l}{\rho} (m - M_0) - 2\frac{\chi^2}{l \rho} + 2\frac{\chi^2}{l \rho} \ln \left(\frac{\rho}{l}\right) \\
& \quad + 2\frac{\omega^2}{l \rho} [ml^2 - \chi^2 + 2\chi^2 \ln \left(\frac{\rho}{l}\right)], \\
M(\rho \to \infty, \epsilon_0) & \approx m - M_0 - 2\frac{\chi^2}{l^2} + 2\frac{\chi^2}{l^2} \ln \left(\frac{\rho}{l}\right) + 2\frac{\omega^2}{l^2} [ml^2 - \chi^2 + 2\chi^2 \ln \left(\frac{\rho}{l}\right)].
\end{align*}
\]

(12.8)

Therefore, comparing with the energy characteristics of the BTZ solution, one concludes that the mass logarithmically diverges at spatial infinity. For vanishing rotation parameter \( \omega \) one recovers the static solution in the representation of Hirschman–Welch and certainly the corresponding energy quantities. The parameter \( m \) can be considered as the BTZ mass.

### B. Field, energy and Cotton tensors for the Dias–Lemos solution

To determine the algebraic types of the electromagnetic field, energy–momentum, and Cotton tensors it is more convenient to work with the DL metric in the form

\[
g = \begin{bmatrix}
- h - \omega^2 (h - L^2) & 0 & l \sqrt{1 + \omega^2 \omega} (h - L^2) \\
0 & \frac{\rho^2}{h L^2} & 0 \\
l \sqrt{1 + \omega^2 \omega} (h - L^2) & 0 & l^2 (-\omega^2 h + L^2 + L^2 \omega^2)
\end{bmatrix},
\]

(12.9)

where

\[
L^2 = \frac{\chi^2 \ln(h) + \rho^2}{l^2}, h = \frac{\rho^2 + Mg}{l^2}, Mg = r_+^2 - ml^2.
\]

(12.10)

In his representation, the electromagnetic field tensor becomes

\[
(F^\alpha)_\beta = \begin{bmatrix}
0 & -\frac{\rho \chi \omega}{l \rho} & \frac{\chi \sqrt{1 + \omega^2 L^2}}{\rho} \\
-\frac{\omega \chi L^2}{l \rho} & 0 & \frac{\chi \sqrt{1 + \omega^2 L^2}}{\rho} \\
0 & -\frac{\rho \chi \sqrt{1 + \omega^2 L^2}}{L^2 \rho} & 0
\end{bmatrix},
\]

(12.11)
with the following eigenvalues and their corresponding eigenvectors

\[ \lambda_1 = 0; \quad V_1 = [V^1 = V^1, V^2 = 0, V^3 = \frac{\omega}{l \sqrt{1 + \omega^2}} V^1], \quad V_\mu V^\mu = -\frac{h}{1 + \omega^2} V^{12} \]

\[ V_1 = T_1, \]

\[ \lambda_2 = -i \frac{\chi}{\sqrt{h} l^2}; \quad V_2 = [V^1 = i \frac{\omega \rho}{L^2 \sqrt{h}} V^2, V^2 = V^2, V^3 = i \frac{\sqrt{1 + \omega^2} \rho}{L^2 \sqrt{h}} V^2], \]

\[ V_2 = Z, \]

\[ \lambda_3 = i \frac{\chi}{\sqrt{h} l^2}; \quad V_3 = [V^1 = -i \frac{\omega \rho}{L^2 \sqrt{h}} V^2, V^2 = V^2, V^3 = -i \frac{\sqrt{1 + \omega^2} \rho}{L^2 \sqrt{h}} V^2], \]

\[ V_3 = \bar{Z}, \]

Type : \{T, Z, \bar{Z}\}.

As far as to the electromagnetic energy momentum tensor is concerned, its matrix is given by

\[ (T^\alpha_\beta) = \begin{bmatrix} -\frac{1}{8\pi} \frac{\chi^2(1 + 2\omega^2)}{l^2(\rho^2 + Mg)} & 0 & \frac{1}{1 + \omega^2} \frac{\omega^2 \chi^2(1 + 2\omega^2)}{l^2(\rho^2 + Mg)} \\ 0 & \frac{1}{8\pi} \frac{\chi^2}{l^2(\rho^2 + Mg)} & 0 \\ -\frac{1}{8\pi} \frac{\omega^2 \chi^2}{l^2(\rho^2 + Mg)} & 0 & \frac{1}{8\pi} \frac{\chi^2(1 + 2\omega^2)}{l^2(\rho^2 + Mg)} \end{bmatrix}, \]

(12.13)

with the following eigenvalues and their corresponding eigenvectors

\[ \lambda_1 = -\frac{1}{8\pi} \frac{\chi^2}{l^2 (\rho^2 + Mg)}; \quad V_1 = [V^1 = 0, \frac{\omega}{l \sqrt{1 + \omega^2}} V^1], \]

\[ V_\mu V^\mu = -\frac{h}{1 + \omega^2} V^{12}, \quad V_1 = T_1, \]

\[ \lambda_2 = \frac{1}{8\pi} \frac{\chi^2}{l^2 (\rho^2 + Mg)}; \quad V_2 = [V^1 = \frac{\omega l}{\sqrt{1 + \omega^2}} V^3, V^2 = V^2, V^3 = V^3], \]

\[ V_\mu V^\mu = \frac{l^2 L^2}{1 + \omega^2} V^{32} + \frac{\rho^2}{h L^2 l^2} V^{22}, \quad V_2 = S_2, \]

\[ \lambda_3 = \frac{1}{8\pi} \frac{\chi^2}{l^2 (\rho^2 + Mg)}; \quad V_3 = [\frac{\omega l}{\sqrt{1 + \omega^2}} \bar{V}^3, \bar{V}^2, \bar{V}^3], \]

\[ V_\mu V^\mu = \frac{l^2 L^2}{1 + \omega^2} (\bar{V}^3)^2 + \frac{\rho^2}{h L^2 l^2} (\bar{V}^2)^2, \quad V_3 = S_3, \]

Type : \{T, 2S\}.

(12.14)

This tensor structure corresponds to that one describing a perfect fluid energy momentum tensor, but this time for the state equation: energy = pressure. Again, the solutions generated from this metric by using coordinate transformations possesses this perfect fluid feature because of the invariance of the eigenvalues.

The Cotton tensor for stationary cyclic symmetric gravitational field is given by

\[ (C^\alpha_\beta) = \begin{bmatrix} -\frac{\chi^2}{2} \frac{\omega \sqrt{1 + \omega^2} (h + L^2)}{h^2 l^2} & 0 & \frac{\chi^2}{2} \frac{\omega^2 h + L^2 + L^2 \omega^2}{h^4 l^4} \\ 0 & 0 & 0 \\ -\frac{\chi^2}{2} \frac{(h + \omega^2 h + L^2 \omega^2)}{h^2 l^2} & 0 & \frac{\chi^2}{2} \frac{\omega \sqrt{1 + \omega^2} (h + L^2)}{h^2 l^2} \end{bmatrix}. \]

(12.15)
Searching for its eigenvectors, one arrives at

\[ \lambda_1 = 0; V_1 = [V^1 = 0, V^2 = V^2, V^3 = 0], \]
\[ V_\mu V^\mu = \frac{\rho^2}{h L^2 l^2} V^{12}, V_1 = S, \]
\[ \lambda_2 = \frac{i}{2} \frac{\chi^2 L}{h^{3/2} l^5}; \]
\[ V_2 = \left[ V^1 = \frac{V^3 \left(-i L \sqrt{h + \omega \sqrt{1 + \omega^2 h + \omega \sqrt{1 + \omega^2 L^2}}\right)}{h + \omega^2 h + L^2 \omega^2}, V^2 = 0, V^3 = V^3\right], V_2 = Z, \]
\[ \lambda_3 = \frac{-i}{2} \frac{\chi^2 L}{h^{3/2} l^5}; \]
\[ V_3 = \left[ V^1 = \frac{V^3 \left(i L \sqrt{h + \omega \sqrt{1 + \omega^2 h + \omega \sqrt{1 + \omega^2 L^2}}\right)}{h + \omega^2 h + L^2 \omega^2}, V^2 = 0, V^3 = V^3\right], V_3 = \bar{Z}, \]

Type : \{T, Z, \bar{Z}\}. \quad (12.16)

The eigenvectors \( V_2 \) and \( V_3 \) are complex conjugated while the vector \( V_1 \), associated to the zero eigenvalue, occurs to be the only physically meaningful spacelike direction in this case.

**XIII. MATYJASEK-ZASLAVSKI SOLUTION**

The uniform electrostatic solution \[23\], see also \[5\] (5.20), is given by the metric functions

\[ ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + K^2 [d\phi + W dt]^2, \]
\[ L(\rho) = N(\rho) = \sqrt{\frac{2}{l^2} \rho^2 + 4c_1 \rho + c_0}, K = 1, W = 0. \]

(13.1)

**A. Vanishing mass, energy and momentum of the MZ solution**

Since the surface energy density \( \epsilon \) occurs proportional to \( \epsilon_0 \), \( \epsilon = -\epsilon_0 \), consequently all the energy–mass quantities are given through it

\[ \epsilon = -\epsilon_0, M(\rho, \epsilon_0) = -2\pi N(\rho) \epsilon_0, E(\rho, \epsilon_0) = -2\pi \epsilon_0. \]

(13.2)

Thus, for the natural choice of a vanishing reference energy density \( \epsilon_0 = 0 \) all the energy quantities vanish: \( \epsilon = 0, M(\rho, 0) = 0 = E(\rho, 0) \). On the other hand, if the reference energy is the one corresponding to the anti–de Sitter spacetime, \( \epsilon_0 = -\frac{1}{\pi \rho} \sqrt{\frac{\rho^2}{l^2} - M_0} \), the energies \( M(\rho, \epsilon_0) \) and \( E(\rho, \epsilon_0) \) will be again expressed through \( \epsilon_0 \).

Metric

\[ g = \begin{bmatrix} -N^2 & 0 & 0 \\ 0 & N^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, N(\rho)^2 = 2 \frac{\rho^2}{l^2} + c_- l \rho + c_- \theta \]

(13.3)
B. Cotton, field, and energy–momentum tensors

As far as the eigenvalue–vector properties of this solution one establishes straightforwardly that the Cotton tensor ought to vanish because of uniform character of the electromagnetic field, hence the 2+1 Matyjasek-Zaslavski gravitational field is conformally flat, \( C^\alpha_{\beta} = 0 \). On the other hand the electromagnetic field tensor

\[
(F^\alpha_{\beta}) = \begin{bmatrix}
0 & \frac{1}{N^2} & 0 \\
\frac{N^2}{l} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

allows for the eigenvectors

\[
\lambda_1 = 0; \ V1 = [V^1 = 0, V^2 = 0, V^3], V_\mu V^\mu = V^3, \ V1 = S1,
\]
\[
\lambda_2 = \frac{1}{l}; \ V2 = [V^1 = V^1, V^2 = N^2V^1, V^3 = 0],
\]
\[
V_\mu V^\mu = 0, \ V2 = N2,
\]
\[
\lambda_3 = -\frac{1}{l}; \ V3 = [V^1 = V^1, V^2 = -N^2V^1, V^3 = 0],
\]
\[
V_\mu V^\mu = 0, \ V3 = N3,
\]

consequently its type is \( \{S, N, N\} \).

For the electromagnetic energy–momentum tensor we have

\[
(T^\alpha_{\beta}) = \begin{bmatrix}
-\frac{1}{8\pi l^2} & 0 & 0 \\
0 & -\frac{1}{8\pi l^2} & 0 \\
0 & 0 & \frac{1}{8\pi l^2}
\end{bmatrix},
\]

with eigenvectors

\[
\lambda_1 = \frac{1}{8\pi l^2}; \ V1 = [V^1 = 0, V^2 = 0, V^3 = V^3], V_\mu V^\mu = V^1, \ V1 = S1,
\]
\[
\lambda_{2,3} = -\frac{1}{8\pi l^2}; \ V2, 3 = [V^1 = V^1, V^2 = V^2, V^3 = 0],
\]
\[
V_\mu V^\mu = -\frac{(N^2V^1 - V^2)(N^2V^1 + V^2)}{N^2}
\]
\[
V2 = T2, S2, N2, \ V3 = T3, S3, N3,
\]

and therefore it allows for the types \( \{S, 2T\}, \{S, 2N\}, \{S, 2S\} \).
XIV. CATALDO SOLUTION

The structural functions of the Cataldo static solution \[24\], see \[5\] Eq. (4.46), are given by

\[
ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + K(\rho)^2 [d\phi + W(\rho) dt]^2,
\]

\[
N(\rho) = \rho^{(1/2-\sqrt{\alpha}/2)} (\rho^2/ l^2 - M)^{(1/4+\sqrt{\alpha}/4)},
\]

\[
L(\rho) = (\rho^2/l^2 - M)^{(1/2)},
\]

\[
K(\rho) = \rho^{(1/2+\sqrt{\alpha}/2)} (\rho^2/l^2 - M)^{(1/4-\sqrt{\alpha}/4)}.
\]

(14.1)

A. Mass, energy and momentum for the Cataldo solution

The corresponding surface densities occur to be

\[
\epsilon(\rho, \epsilon_0) = -\frac{1}{\pi l \rho} \frac{1}{\sqrt{\rho^2 - M l^2}} [\rho^2 - M l^2 \frac{(1 + \sqrt{\alpha})}{2}] - \epsilon_0,
\]

\[
j_\phi(\rho) = 0 = J(\rho),
\]

(14.2)

while the integral quantities amount to

\[
E(\rho, \epsilon_0) = -l^{\sqrt{\alpha}/2-3/2} (\rho^2 - M l^2)^{(-\sqrt{\alpha}/4-1/4)} \rho^{\sqrt{\alpha}/2-1/2} [2 \rho^2 - (1 + \sqrt{\alpha}) M l^2]
\]

\[- 2\pi \epsilon_0 l^{(\sqrt{\alpha}/2-1/2)} (\rho^2 - M l^2)^{(-\sqrt{\alpha}/4+1/4)} \rho^{\sqrt{\alpha}/2+1/2},
\]

\[
M(\rho, \epsilon_0) = -2\frac{\rho^2}{l^2} + (1 + \sqrt{\alpha}) M - \frac{2\pi}{l} \epsilon_0 \rho \sqrt{\rho^2 - M l^2}
\]

(14.3)

The evaluation of energy and mass functions independent of \(\epsilon_0\) behave at infinity as

\[
\epsilon(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{1}{\pi l} + \frac{l \sqrt{\alpha} M}{2\pi \rho^2},
\]

\[
E(\rho \to \infty, \epsilon_0 = 0) \approx l^{\frac{1}{2}(1+\sqrt{\alpha})} \left[ -2 \frac{\rho}{l^2} + \frac{(1 + \sqrt{\alpha}) M}{2 \rho} \right],
\]

\[
M(\rho \to \infty, \epsilon_0 = 0) \approx M (1 + \sqrt{\alpha}) - 2\frac{\rho^2}{l^2}.
\]

(14.4)

Using in the expressions \((14.2)\) and \((14.3)\) as the reference energy density the quantity

\[
\epsilon_0 = -\frac{1}{\pi \rho^2} \sqrt{-M_0 + \rho^2},
\]

which at the spatial infinity behaves as \(\epsilon_0|_\infty (M_0) \approx -\frac{1}{\pi l} + \frac{M_0}{2\pi \rho^2}\), the series expansions of the corresponding quantities at \(\rho \to \infty\) result in

\[
\epsilon(\rho \to \infty, \epsilon_0|_\infty (M_0)) \approx \frac{l}{2\pi \rho^2} (-M_0 + \sqrt{\alpha} M),
\]

\[
E(\rho \to \infty, \epsilon_0|_\infty (M_0)) \approx \frac{(-M_0 + \sqrt{\alpha} M)}{\rho} l^{\frac{1}{2}(1+\sqrt{\alpha})},
\]

\[
M(\rho \to \infty, \epsilon_0|_\infty (M_0)) \approx -M_0 + \sqrt{\alpha} M.
\]

(14.5)

Therefore, comparing with the energy characteristics of the BTZ solution, one concludes that the mass parameter at spatial infinity is determined by the product \(\sqrt{\alpha} M\), although
the mass function diverges at infinity as fast as $1/\rho^2$, a similar behavior is exhibited by the energy density in that spatial region.

### B. Field, energy–momentum, and Cotton tensors

The electromagnetic field of this solution is given by

$$(F^\alpha_\beta) = \frac{l M (1 - \alpha)^{1/2}}{\rho^2 - M l^2} \times \begin{bmatrix} 0 & 0 - \frac{1}{2} l \sqrt{\alpha} (\rho^2 - M l^2)^{1/2} \sqrt{\alpha} \rho^{-\sqrt{\alpha}} \\ 0 & 0 & 0 \\ -\frac{1}{2} l \sqrt{\alpha} (\rho^2 - M l^2)^{-1/2} \rho^{\sqrt{\alpha}} & 0 & 0 \end{bmatrix}, \quad (14.6)$$

and it is characterized by the following eigenvalues and eigenvectors

$$\lambda_1 = 0; \ V_1 = (0, V^2, 0), \ V^\mu V_\mu = (V^2)^2 g_{\rho \rho}, \ V_1 = S_1,$$

$$\lambda_2 = -1/2 \frac{M l \sqrt{1 - \alpha}}{\sqrt{\rho^2 - M l^2} \rho}; \ V_2 = [l \sqrt{\alpha} (\rho^2 - M l^2)^{-1/2} \rho^{\sqrt{\alpha}} V^3, 0, V^3],$$

$$V^\mu V_\mu = 0, \ V_2 = N_2,$$

$$\lambda_3 = 1/2 \frac{M l \sqrt{1 - \alpha}}{\sqrt{\rho^2 - M l^2} \rho}; \ V_3 = [-l \sqrt{\alpha} (\rho^2 - M l^2)^{-1/2} \rho^{\sqrt{\alpha}} V^3, 0, V^3],$$

$$V^\mu V_\mu = 0, \ V_3 = N_3,$$

Type : $\{S, N, N\}$. \quad (14.7)

On the other hand, the energy–momentum tensor, having the structure

$$(T^\alpha_\beta) = -\frac{1}{32} \frac{M^2 l^2 (1 - \alpha)}{\rho^2 (\rho^2 - M l^2) \pi} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (14.8)$$

allows for the eigenvalues $\lambda_1 = \frac{1}{32} \frac{M^2 l^2 (1 - \alpha)}{\rho^2 (\rho^2 - M l^2) \pi}$ and the other one, of multiplicity two, $\lambda_2 = \lambda_3 = -\frac{1}{32} \frac{M^2 l^2 (1 - \alpha)}{\rho^2 (\rho^2 - M l^2) \pi}$, with the corresponding eigenvectors

$$\lambda_1 = \frac{1}{32} \frac{M^2 l^2 (1 - \alpha)}{\rho^2 (\rho^2 - M l^2) \pi}; \ V_1 = (0, V^2, 0), \ V_\mu = V^2 g_{\rho \rho} \delta_\mu^\rho, \ V^\mu V_\mu = (V^2)^2 g_{\rho \rho}, \ V_1 = S_1,$$

$$\lambda_2 = -\frac{1}{32} \frac{M^2 l^2 (1 - \alpha)}{\rho^2 (\rho^2 - M l^2) \pi}; \ V_2 = (V^1, 0, V^3), \ V^\mu V_\mu = (V^1)^2 g_{tt} + (V^3)^2 g_{\phi \phi},$$

$$V_2 = T_2, S_2, N_2,$$

$$\lambda_3 = -\frac{1}{32} \frac{M^2 l^2 (1 - \alpha)}{\rho^2 (\rho^2 - M l^2) \pi}; \ V_3 = (\tilde{V}_1, 0, \tilde{V}_3), \ V^\mu V_\mu = (\tilde{V}^1)^2 g_{tt} + (\tilde{V}^3)^2 g_{\phi \phi},$$

$$V_3 = T_3, S_3, N_3. \quad (14.9)$$
For \( \mathbf{V}_2 \) and \( \mathbf{V}_3 \), the character of the vector depends on the sign of its norm; for instance, by choosing

\[
V^1 = s \sqrt{g_{\phi \phi}} / \sqrt{|g_{tt}|} \mathbf{V}^3, \quad s = \text{const.}, \quad \mathbf{V}^\mu_1 \mathbf{V}_1^\mu = (1 - s^2)g_{\phi \phi}(\mathbf{V}^3)^2;
\]

\( s > 1 \rightarrow \mathbf{V}_1 = \mathbf{T}, \quad s = \pm 1 \rightarrow \mathbf{V}_1 = \mathbf{N}, \quad s < 1 \rightarrow \mathbf{V}_1 = \mathbf{S}. \)

Recall that in 3+1 gravity the eigenvectors of the electromagnetic energy-momentum tensor (and at the same time of the electromagnetic field tensor) are null in pairs, i.e., they exhibit double coincidence. Hence in the 2+1 case under study one may think of the alignments \( \{\mathbf{S}, 2\mathbf{N}\} \) or \( \{\mathbf{N}, \mathbf{S}, \mathbf{N}\} \) as the corresponding reductions of electromagnetic field eigen-directions of 3+1 gravity.

To complete the characterization of this solution, it is reasonable to add some comments about the conformal Cotton tensor, which is given by

\[
(C^\alpha_\beta) = \frac{l^3 \sqrt{\alpha} (\alpha - 1) M^3}{8 (\rho^2 - Ml^2)^{3/2} \rho^3} \begin{bmatrix}
0 & 0 & l^{-\sqrt{\alpha} \rho \sqrt{\alpha}} (\rho^2 - Ml^2)^{-\sqrt{\alpha}/2} \\
0 & 0 & 0 \\
-ll^{-\sqrt{\alpha} \rho \sqrt{\alpha}} (\rho^2 - Ml^2)^{-\sqrt{\alpha}/2} & \rho^{-\sqrt{\alpha}} & 0
\end{bmatrix}, \quad (14.10)
\]

with eigenvectors

\[
\lambda_1 = 0; \quad \mathbf{V}_1 = (0, V^2, 0), \quad V^\mu \mathbf{V}_\mu = (V^2)^2 / L^2, \quad \mathbf{V}_1 = \mathbf{S},
\]

\[
\lambda_2 = -\frac{i \sqrt{\alpha} l^3 (1 - \alpha) M^3}{8 (\rho^2 - Ml^2)^{3/2} \rho^3}; \quad \mathbf{V}_2 = [V^1 = -i \rho \sqrt{\alpha} (\rho^2 - Ml^2)^{-\sqrt{\alpha}/2} l^{-\sqrt{\alpha} \mathbf{V}_3}, 0, V^3], \quad \mathbf{V}_2 = \mathbf{Z},
\]

\[
\lambda_3 = \frac{i \sqrt{\alpha} l^3 (1 - \alpha) M^3}{8 (\rho^2 - Ml^2)^{3/2} \rho^3}; \quad \mathbf{V}_3 = [V^1 = i \rho \sqrt{\alpha} (\rho^2 - Ml^2)^{-\sqrt{\alpha}/2} l^{-\sqrt{\alpha} \mathbf{V}_3}, 0, V^3], \quad \mathbf{V}_3 = \bar{\mathbf{Z}},
\]

Type : \( \{S, Z, \bar{Z}\} \). \quad (14.11)
XV. STATIONARY GENERALIZATION OF THE CATALDO STATIC SOLUTION

The structural functions corresponding to the stationary generalization of the Cataldo static solution, via $SL(2, R)$ transformations, are given by

$$ds^2 = -N(\rho)^2 dt^2 + \frac{1}{L(\rho)^2} d\rho^2 + K(\rho)^2 [d\phi + W(\rho) dt]^2,$$

$$K(\rho)^2 = \frac{\rho^2 - M l^2}{l^2} \left[ \delta_0^2 l^2 \sqrt{\alpha} \rho^2 \rho^{-2} - \delta_0^2 l^2 \sqrt{\alpha} (\rho^2 - M l^2)^{-1/2} \right],$$

$$N(\rho)^2 = \frac{\rho^2 l^2 - M l^2}{l^2},$$

$$W(\rho) = -\frac{1}{\Delta} \frac{\rho l^2}{K(\rho)^2} [\alpha_0 \rho^2 \rho^{-2} - \delta_0 l^2 \sqrt{\alpha} (\rho^2 - M l^2)^{-1/2} - \gamma_0 \rho^2 \rho^{-2} - \delta_0 l^2 \sqrt{\alpha} (\rho^2 - M l^2)^{-1/2}],$$

$$L(\rho)^2 = \frac{\rho^2}{l^2} - M.$$ (15.1)

A. Mass, energy and momentum for the generalized Cataldo solution

The corresponding surface densities occur to be

$$\epsilon(\rho, \epsilon_0) = \left[ \beta_0^2 \rho^2 \left( 1 - \frac{M l^2}{\rho^2} \right)^{1/2} \left( 2 - (1 + \sqrt{\alpha}) \frac{M l^2}{\rho^2} \right) - \delta_0^2 l^2 \sqrt{\alpha} (1 - \frac{M l^2}{\rho^2})^{-1/2} \right],$$

$$j(\rho) = \beta_0 \rho^2 \sqrt{\alpha} \frac{M}{\pi \Delta K(\rho)},$$ (15.2)

therefore the product $\beta_0 \delta_0$ is related with the rotation properties of the considered solution. The integral quantities amount to

$$J(\rho) = 2 \pi K(\rho) j(\rho) = 2 \beta_0 \rho^2 \sqrt{\alpha} M \frac{1}{\Delta},$$

$$E(\rho) = 2 \pi K(\rho) \epsilon(\rho),$$

$$M(\rho) = N(\rho) E(\rho) - W(\rho) J(\rho).$$ (15.3)
The evaluation of energy and mass functions independent of $\epsilon_0$ behave at infinity as

$$
e(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{1}{\pi l} \frac{\sqrt{\alpha} M \delta_0^2 l^2 \sqrt{\alpha} + \beta_0^2}{2\pi \rho^2 \delta_0^2 l^2 \sqrt{\alpha} - \beta_0^2},$$

$$E(\rho \to \infty, \epsilon_0 = 0) \approx -l^{1/2-\sqrt{\alpha}/2} \frac{M}{2\rho \sqrt{\Delta}} \left[ \frac{(\sqrt{\alpha} - 1) \beta_0^2 + 2\beta_0^2 \delta_0^4 l^2 \sqrt{\alpha} - (\sqrt{\alpha} + 1) \delta_0^4 l^4 \sqrt{\alpha}}{(\delta_0^2 l^2 \sqrt{\alpha} - \beta_0^2)^{3/2}} \right]$$

$$-2\rho l^{-3/2-\sqrt{\alpha}/2} \frac{\sqrt{\delta_0^2 l^2 \sqrt{\alpha} - \beta_0^2}}{\sqrt{\Delta}},$$

$$M(\rho \to \infty, \epsilon_0 = 0) = -2\rho^2 l^2 + M \left[ 1 + \frac{\sqrt{\alpha} \delta_0^2 l^2 \sqrt{\alpha} + \beta_0^2 \delta_0^2 l^2 \sqrt{\alpha} - \beta_0^2}{\delta_0^2 l^2 \sqrt{\alpha} - \beta_0^2} + 2\sqrt{\alpha} \beta_0 \delta_0 \alpha_0 \beta_0 - \gamma_0 \delta_0^4 l^2 \sqrt{\alpha} \right].$$

(15.4)

Using in the expressions (15.2)–(15.3) as reference energy density the quantity $\epsilon_0 = -\frac{\rho}{\pi l^2} / \sqrt{M_0 + \rho^2}$, which at the spatial infinity behaves as $\epsilon_0|_{\infty}(M_0) \approx -\frac{1}{\pi l} + \frac{M_0}{2\pi \rho^2}$, the series expansions of the corresponding quantities at $\rho \to \infty$ result in

$$
e(\rho \to \infty, \epsilon_0|_{\infty}(M_0)) \approx \frac{l}{2\pi \rho^2} (\sqrt{\alpha} M - M_0) - \frac{l}{\pi \rho^2} \frac{\beta_0^2}{\delta_0^2 l^2 \sqrt{\alpha} - \beta_0^2},$$

$$E(\rho \to \infty, \epsilon_0|_{\infty}(M_0)) \approx l^{1/2-\sqrt{\alpha}/2} \frac{\sqrt{\delta_0^2 l^2 \sqrt{\alpha} - \beta_0^2}}{\rho \sqrt{\Delta}} (\sqrt{\alpha} M - M_0) + 2l^{1/2-\sqrt{\alpha}/2} \frac{\beta_0^2 \sqrt{\alpha} M}{\rho \sqrt{\delta_0^2 l^2 \sqrt{\alpha} - \beta_0^2}},$$

$$M(\rho \to \infty, \epsilon_0|_{\infty}(M_0)) \approx \sqrt{\alpha} M - M_0 - \frac{\sqrt{\alpha} M \beta_0 \gamma_0 \delta_0^2 l^2 \sqrt{\alpha} - 2\alpha_0 \beta_0 \delta_0 + \beta_0^2 \gamma_0}{\Delta} \frac{\sqrt{\delta_0^2 l^2 \sqrt{\alpha} - \beta_0^2}}{\delta_0^2 l^2 \sqrt{\alpha} - \beta_0^2}. \quad (15.5)$$

Therefore, comparing with the energy characteristics of the BTZ solution, one concludes that role of the mass parameter is played by the product $\sqrt{\alpha} M$. At spatial infinity the mass function occurs to be finite, the energy density and global energy approach to infinity as fast as $1/\rho^2$ and $1/\rho$ correspondingly.
XVI. AYON–CATALDO–GARCIA HYBRID SOLUTION

The metric defining this kind of stationary electromagnetic solution [2, 5], see also [5], Eq.(8.15), can be given in the standard form as

\[ g = -N^2 \, dt^2 + \frac{1}{L^2} \, d\rho^2 + K^2 [d\phi + W \, dt]^2 = -\frac{\rho^2}{H} \, dt^2 + \frac{d\rho^2}{f} + H (d\phi + W \, dt)^2, \]

\[ f(\rho) = \frac{\rho^2}{l^2} - M + \frac{J^2}{4 \rho^2}, \]

\[ H(\rho) = \frac{1}{4K_1 \sqrt{\alpha S_M}} \sqrt{2 \rho^2 - lR_- \sqrt{2 \rho^2 - lR_+}} \left[ J^2 K_1^2 (2 \rho^2 - lR_-)^{-\sqrt{\alpha}/2} (2 \rho^2 - lR_+)^{\sqrt{\alpha}/2} \right. \]

\[ \left. - (2 \rho^2 - lR_-)^{\sqrt{\alpha}/2} (2 \rho^2 - lR_+)^{-\sqrt{\alpha}/2} \right], \]

\[ W(\rho) = -\frac{R_-}{jl} \left[ (2 \rho^2 - lR_+) \sqrt{\alpha} (2 \sqrt{\alpha} S_M + R_-) R_- - (2 \rho^2 - lR_-) \sqrt{\alpha} J^2 \right] \times \]

\[ \left[ (2 \rho^2 - lR_+) \sqrt{\alpha} R_+ - (2 \rho^2 - lR_-) \sqrt{\alpha} J^2 \right]^{-1}, \]

\[ R_\pm := M \, l \pm \sqrt{M^2 l^2 - J^2}, \quad K_1 := -\frac{R_-}{J^2}, \quad S_M := \sqrt{M^2 l^2 - J^2}. \tag{16.1} \]

The structural functions appearing in the definitions of the energy and momentum quantities are expressed as

\[ N(\rho) = \sqrt{\frac{\rho^2 f(\rho)}{H(\rho)}}, \quad L(\rho) = \sqrt{f(\rho)}, \quad K(\rho) = \sqrt{H(\rho)}, \quad W(\rho) = W(\rho). \tag{16.2} \]

The corresponding electromagnetic tensors are given as

\[ F_{\mu \nu} = -\frac{1 - \alpha}{l} \sqrt{M^2 l^2 - J^2} \delta_{[\mu} \delta_{\nu]}^\phi, \]

\[ 8 \pi T_{\mu \nu} = -\frac{1}{4 l^2} \frac{M^2 l^2 - J^2}{\rho^2 f(\rho)} \left[ \delta_{\mu}^T \delta_{\nu}^T - \delta_{\mu}^\rho \delta_{\nu}^\rho + \delta_{\mu}^\phi \delta_{\nu}^\phi \right]. \tag{16.3} \]

When the electromagnetic field is turned off, \( \alpha = 1 \), the above metric components reduce to

\[ g_{\Phi \Phi} = M - \frac{\rho^2}{l^2}, \quad g_{\Phi \Phi} = -\frac{J^2}{2}, \quad g_{\Phi \Phi} = \rho^2, \quad g_{\rho \rho} = \left( \frac{\rho^2}{l^2} - M + \frac{J^2}{4 \rho^2} \right)^{-1}, \]

which correspond to the BTZ ones.

A. Mass, energy and momentum for the ACG solution

In terms of the structural metric functions the momentum quantities they allow for very simple expressions

\[ j(\rho) = \frac{J}{2\pi} \frac{1}{\sqrt{H(\rho)}}, \quad J(\rho) = J. \tag{16.4} \]
while the energy and mass characteristics become

\[ \epsilon(\rho, \epsilon_0) = -\frac{1}{2\pi} \frac{\sqrt{f(\rho)}}{H(\rho)} \frac{d}{d\rho} H(\rho) - \epsilon_0, \]  

(16.5)

\[ E(\rho, \epsilon_0) = -\frac{\sqrt{f(\rho)}}{\sqrt{H(\rho)}} \frac{d}{d\rho} H(\rho) - 2\pi \epsilon_0 \sqrt{H(\rho)}, \]  

(16.6)

\[ M(\rho, \epsilon_0) = -\rho \frac{f(\rho)}{H(\rho)} \frac{d}{d\rho} H(\rho) - J W(\rho) - 2\rho \pi \epsilon_0 \sqrt{f(\rho)}, \]  

(16.7)

Because of the involved dependence of the metric functions upon the \( \rho \) coordinate, the evaluation of the energy quantities will be done in the approximation of the spatial infinity. The momentum density at infinity becomes

\[ j(\rho \to \infty) \approx \frac{J}{2\pi} \frac{\alpha^{1/4}}{\rho}, \]

(16.8)

while the global momentum remains constant in the whole space

\[ J(\rho) = J. \]

(16.9)

It becomes apparent then that the role of the momentum parameter is played and coincides with \( J \).

The approximated at \( \rho \to \infty \) surface energy density, global energy and mass, for zero base energy density \( \epsilon_0 \) are given by

\[ \epsilon(\rho \to \infty, \epsilon_0 = 0) \approx -\frac{1}{\pi l} + \frac{l M \sqrt{\alpha}}{2\pi \rho^2}, \]

\[ E(\rho \to \infty, \epsilon_0 = 0) \approx -2 \frac{\rho}{l \alpha^{1/4}} + \frac{l}{2 \rho \alpha^{1/4}} (1 + \sqrt{\alpha}) M, \]

\[ M(\rho \to \infty, \epsilon_0 = 0) \approx -2 \frac{\rho^2}{l^2} + 2M + \frac{\sqrt{\alpha} - 1}{l} \sqrt{M^2 l^2 - J^2}. \]  

(16.10)

Using in the expressions (16.5)–(16.7) as reference energy density the energy corresponding to the anti–de–Sitter metric, \( \epsilon_0 = -\frac{1}{\pi \rho} \sqrt{\frac{\rho^2}{l^2} - M_0}, \) \( \epsilon_0(\infty)(M_0) \approx -\frac{1}{\pi l} + \frac{l M_0}{2\pi \rho^2}, \) the series expansions of the global energy and mass quantities at \( \rho \to \infty \) result in

\[ \epsilon(\rho \to \infty, \epsilon_0(\infty)(M_0)) \approx \frac{l}{2\pi \rho^2} (\sqrt{\alpha} M - M_0), \]

\[ E(\rho \to \infty, \epsilon_0(\infty)(M_0)) \approx \frac{l (\sqrt{\alpha} M - M_0)}{\rho \alpha^{1/4}}, \]

\[ M(\rho \to \infty, \epsilon_0(\infty)(M_0)) \approx M - M_0 + \frac{\sqrt{\alpha} - 1}{l} \sqrt{M^2 l^2 - J^2}. \]  

(16.11)

Comparing with the energy characteristics of the BTZ solution, the mass parameter occurs to be an involved quantity depending on \( M \), the momentum \( J \), and the charge \( \alpha \), namely \( M + \frac{\sqrt{\alpha} - 1}{l} \sqrt{M^2 l^2 - J^2} \), although the mass function is finite at spatial infinity. On the other hand, if \( M l \gg J \) then \( \sqrt{\alpha} M \) becomes the mass parameter. The energy density and global energy are proportional at infinity to \( 1/\rho^2 \) and \( 1/\rho \) correspondingly.
B. Field, energy–momentum, and Cotton tensors

The coordinate system \( \{t, r, \phi\} \) occurs to be more adequate in the derivation of the eigenvalue–vector characteristics of the considered solution \([5]\), Eq.(8.11). The metric in \( \{t, r, \phi\}\)–coordinates is given by

\[
g = \begin{bmatrix}
-\frac{F}{H} + H W^2 & 0 & HW \\
0 & 1/F & 0 \\
HW & 0 & H
\end{bmatrix}, 
\]

(16.12)

with structural functions

\[
F(r) = 4 \frac{(r - r_1)(r - r_2)}{l^2},
\]

\[
H(r) = \frac{l}{2} \sqrt{(r - r_1)(r - r_2)} \left[ \left( \frac{r - r_1}{r - r_2} \right)^{1/\sqrt{\alpha}} - J^2 K_1^2 \left( \frac{r - r_1}{r - r_2} \right)^{-1/\sqrt{\alpha}} \right],
\]

\[
W(r) = W_0 - 2 K_1 J \left( \frac{r - r_1}{r - r_2} \right)^{-1/\sqrt{\alpha}} \sqrt{(r - r_1)(r - r_2)} \frac{dW}{dr} = \frac{-J}{H},
\]

(16.13)

The electromagnetic field tensor is given by

\[
\left( F^\alpha_{\beta} \right) = \begin{bmatrix}
-\frac{H W}{F} & 0 & -\frac{H}{F} \\
0 & 0 & 0 \\
-\frac{(F - H^2 W^2)}{HF} & 0 & \frac{H W}{F}
\end{bmatrix}, \quad c = \frac{(r_2 - r_1) \sqrt{1 - \alpha}}{l^2}.
\]

(16.14)

In the search of its eigenvectors, one arrives at

\[
\lambda_1 = 0; V_1 = [V^1 = 0, V^2 = V^3 = 0], V^\mu V_\mu = \frac{V^2}{F}, V_1 = S_1,
\]

\[
\lambda_2 = \frac{c}{\sqrt{F}};
\]

\[
V_2 = [V^1 = V^1, V^2 = 0, V^3 = -\frac{(H W + \sqrt{F}) V^1}{H}], V^\mu V_\mu = 0, V_2 = N_2,
\]

\[
\lambda_3 = -\frac{c}{\sqrt{F}};
\]

\[
V_3 = [V^1 = V^1, V^2 = 0, V^3 = \frac{-(H W + \sqrt{F}) V^1}{H}], V^\mu V_\mu = 0, V_3 = N_3,
\]

(16.15)

hence this tensor is of the type \( \{S, N_1, N_2\} \).

For the energy–momentum tensor

\[
\left( T^\alpha_{\beta} \right) = \begin{bmatrix}
-\frac{1}{8\pi} \frac{c^2}{F} & 0 & 0 \\
0 & \frac{1}{8\pi} \frac{c^2}{F} & 0 \\
0 & 0 & -\frac{1}{8\pi} \frac{c^2}{F}
\end{bmatrix},
\]

(16.16)
the eigenvectors are
\[ \lambda_1 = \frac{1}{8\pi F} c^2; \quad V_1 = [V^1 = 0, V^2 = V^2, V^3 = 0], \quad V_\mu V^\mu = \frac{V^{22}}{F}, \quad V_1 = S_1, \]
\[ \lambda_{2,3} = -\frac{1}{8\pi F} c^2; \quad V_{2,3} = [V^1 = V^1, V^2 = 0, V^3 = V^3], \]
\[ V_\mu V^\mu = -\frac{V^{12}}{H} + (V^1 W + V^3)^2 H = \frac{V^{12}}{H} (Z^2 - 1), \]
\[ V_2 = \{T_2, N_2, S_2\}, \quad V_3 = \{T_3, N_3, S_3\}, \]
(16.17)
hence it allows for the types:
\[ \{S, 2T\}, \{S, 2N\}, \{S, 2S\}. \]

The Cotton tensor
\[
(C^\alpha{}_{\beta}) = \begin{bmatrix}
C_1^{\alpha} & 0 & C_3^{\alpha} \\
0 & 0 & 0 \\
C_3^{\alpha} & 0 & -C_1^{\alpha}
\end{bmatrix},
\]
\[ C_1^\alpha = -C_3^\alpha = -\frac{c^2}{32\pi F^2} \left( WQ + 2 \frac{FJ}{H} \right), \]
\[ C_3^\alpha = -\frac{c^2}{32\pi F^2} Q, \]
\[ C_3^\alpha = -\frac{c^2}{32\pi F^2 H^2} \left[ -WH^2 \left( WQ + 4 \frac{FJ}{H} \right) - FQ \right], \]
\[ Q : = 2 F \frac{d}{dr} H - H \frac{d}{dr} F; \]
\[ Q = -\frac{2}{lK_1} \sqrt{(r - r_1)(r - r_2)} \left[ \left( \frac{r - r_1}{r - r_2} \right)^{1/2} + J^2 K_1^2 \left( \frac{r - r_1}{r - r_2} \right)^{-1/2} \right]. \]
(16.18)
possesses the following set of eigenvectors
\[ \lambda_1 = 0; \quad V_1 = [V^1 = 0, V^2 = V^2, V^3 = 0], \quad V_\mu V^\mu = \frac{V^{22}}{F}, \quad V_1 = S_1, \]
\[ \lambda_2 = \frac{1}{4i} \sqrt{\alpha c^2 (r_2 - r_1)} \frac{r_2 - r_1}{\pi F^{3/2} l^2}; \]
\[ V_2 = [V^1 = -c^2 V^3 Q \left( c^2 WQ + 2 \frac{c^2 F}{H} + 32 \lambda_2 F^2 \pi \right)^{-1}, V^2 = 0, V^3 = V^3], \quad V_2 = Z, \]
\[ \lambda_3 = -\frac{1}{4i} \sqrt{\alpha c^2 (r_2 - r_1)} \frac{r_2 - r_1}{\pi F^{3/2} l^2}; \]
\[ V_3 = [V^1 = -c^2 V^3 Q \left( c^2 WQ + 2 \frac{c^2 F}{H} + 32 \lambda_3 F^2 \pi \right)^{-1}, V^2 = 0, V^3 = V^3], \quad V_3 = \bar{Z}. \]
(16.19)
therefore its type is
\[ \{S, Z, \bar{Z}\}. \]
VII. CONCLUDING REMARKS

In the framework of the Einstein–Maxwell theory with negative cosmological constant different families of exact solutions for cyclic symmetric stationary (static) metrics have been studied, evaluated their energy–momentum densities and global energy–momentum–mass quantities using the Brown–York approach. As reference characteristics there have been used the ones corresponding to the stationary or static BTZ –AdS with parameter $M_0$–solutions. The electric and magnetic solutions, and their generalizations through $SL(2, R)$–transformations exhibit at the spatial infinity $\rho \to \infty$ the following generic behavior

\[
J(\rho \to \infty) \approx \alpha_J J + \beta_J \ln \rho,
\]
\[
\epsilon(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx \frac{l}{2\pi \rho^2}(\alpha_M M - \alpha_{M_0} M_0) + \alpha_Q \frac{Q^2}{2\pi \rho^2} + \frac{a}{2\pi \rho} J + \frac{A}{\pi \rho^2} \ln \rho,
\]
\[
E(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx \frac{l}{\rho}(\beta_M M - \beta_{M_0} M_0) + \beta_Q \frac{Q^2}{\rho} + \frac{b}{\rho} J + \frac{B}{\rho} \ln \rho,
\]
\[
M(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx \gamma_M M - \gamma_{M_0} M_0 + \gamma_Q Q^2 + c J + C \ln \rho,
\]

where $\alpha_J$, $\beta_M$, ..., $\gamma_Q$ are constant numerical factors related to the physical parameters: $J$ momentum, $M$ mass, ..., $Q$ electromagnetic charge.

The momentum, energy and mass of the hybrid solutions behaves at spatial infinity $\rho \to \infty$ as follows

\[
J(\rho \to \infty) \approx \alpha_J J + \beta_J,
\]
\[
\epsilon(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx \frac{l}{2\pi \rho^2}(\alpha_M M - \alpha_{M_0} M_0) + \alpha_Q \frac{Q^2}{2\pi \rho^2} + \frac{a}{2\pi \rho^2} J,
\]
\[
E(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx \frac{l}{\rho}(\beta_M M - \beta_{M_0} M_0) + \beta_Q \frac{Q^2}{\rho} + \frac{b}{\rho} J,
\]
\[
M(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx \gamma_M M - \gamma_{M_0} M_0 + \gamma_Q Q^2 + c J,
\]

where the charge $Q$ is related with the electromagnetic parameter $\alpha$.

Moreover, the eigenvectors for their electromagnetic field, energy–momentum and Cotton tensors have been explicitly determined; the static and stationary Peldan electric classes, the Martinez–Teitelboim–Zanelli and the Clement solutions exhibit the following algebraic types:

Field : \{S, N, N\}, Energy : \{S, 2T\}, \{S, 2N\}, \{S, 2S\}; Cotton : \{S, Z, Z\},

while the static and stationary Peldan magnetic families, the Hirschmann–Welch and the Dias–Lemos solutions exhibit the following algebraic types:

Field : \{T, Z, Z\}, Energy : \{T, 2S\}; Cotton : \{S, Z, Z\}.

The Garcia solution allows for the set of types:

Field : \{T, Z, Z\}, \{S, Z, Z\}, \{T, 2N\}, \{S, 2N\}, \{3N\},

Energy : \{T, 2T\}, \{T, 2S\}, \{T, 2N\}, \{T, 2N\}, \{S, 2T\}, \{S, 2S\}, \{S, 2N\}, \{3N\},

Cotton : \{S, Z, Z\}. 

\[
\epsilon(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx \frac{l}{2\pi \rho^2}(\alpha_M M - \alpha_{M_0} M_0) + \alpha_Q \frac{Q^2}{2\pi \rho^2} + \frac{a}{2\pi \rho} J + \frac{A}{\pi \rho^2} \ln \rho,
\]

\[
E(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx \frac{l}{\rho}(\beta_M M - \beta_{M_0} M_0) + \beta_Q \frac{Q^2}{\rho} + \frac{b}{\rho} J + \frac{B}{\rho} \ln \rho,
\]

\[
M(\rho \to \infty, \epsilon_{0|\infty}(M_0)) \approx \gamma_M M - \gamma_{M_0} M_0 + \gamma_Q Q^2 + c J + C \ln \rho,
\]
The hybrid Cataldo and Ayon–Cataldo–Garcia solutions fall into the types:

Field : \( \{ S, N, N \} \), Energy : \( \{ S, 2T \}, \{ S, 2S \}, \{ S, 2N \} \), Cotton : \( \{ S, Z, \bar{Z} \} \).

The Kamata–Koikawa belongs to types

Field : \( \{ 3S \} \), Energy : \( \{ 3S \}, \{ 3N \} \), Cotton : \( \{ 3S \}, \{ 3N \} \).

Finally, the Matyjasek–Zaslavski solution exhibits the types

Field : \( \{ S, N, N \} \), Energy : \( \{ S, 2T \}, \{ S, 2S \}, \{ S, 2N \} \), Cotton : \( \{ 0 \} \).

Recall that algebraic structures \( \{ T, 2S \} \) are thought of as perfect fluids. though

XVIII. ACKNOWLEDGMENTS

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Appendix A: Summary of the Brown–York approach to $\Sigma$–hypersurfaces and $B$–three–boundary

Following the Brown–York formulation [10], for a 4–dimensional spacetime with metric $g_{\mu \nu}$, a foliation $\Sigma$ determines and is determined by a congruence of timelike unit normal vectors $u_\mu$ proportional to the gradient of a scalar field $t$ labeling the hypersurfaces, i.e., $u_\mu = -N t_\mu$, where $N$ is the lapse function. The spatial metric tensor field $h_{\mu \nu}$ is defined on $\Sigma$ by $h_{\mu \nu} = g_{\mu \nu} + u_\mu u_\nu$. The covariant derivative $\nabla_\mu$ with respect to $g_{\mu \nu}$ induces on $\Sigma$ a covariant derivative $D_\mu$ by means of the projection $D_\mu = h_\mu^\alpha \nabla_\alpha$. For any spatial tensor field $T^\nu_\mu, T'^\nu_\mu u_\mu = 0$, the spatial covariant derivative is defined as $D_\mu T^\nu_\nu = h^\alpha_\mu h^\beta_\nu \nabla_\alpha T^\gamma_\beta$. The extrinsic curvature of $\Sigma$ occurs to be $K_{\mu \nu} = -h^\alpha_\mu \nabla_\alpha u_\nu = -D_\mu u_\nu$, which is a symmetric spatial tensor because the normal timelike vector $u_\mu$ is a gradient and consequently possesses vanishing rotation, $\omega_{\mu \nu} = (u_{\mu ; \nu} - u_{\nu ; \mu})/2 + (u_{\mu ; \alpha} u_\alpha u_\nu - u_{\nu ; \alpha} u_\alpha u_\mu)/2 = 0$, which implies $h^\alpha_\mu \nabla_\alpha u_\nu - h^\alpha_\nu \nabla_\alpha u_\mu = 0$.

Following Brown–York paper, for convenience, coordinates adapted to the foliation are introduced by choosing $t$ as the time coordinate while $x^i, i = 1, 2, 3$, lie in the surface $\Sigma$, consequently $\frac{\partial}{\partial x^i}$ are spacelike vectors. Accordingly, the spacetime metric can be written as

$$ds^2 = g_{\mu \nu} dx^\mu dx^\nu = (h_{\mu \nu} - u_\mu u_\nu) dx^\mu dx^\nu = -NdT^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt),$$

where $N$ and $V^i = h^i_0 = -N u^i$ are correspondingly the shift function and the shift vector. It follows that the spatial tensor $h^\mu_\nu = 2h^{ij} \delta^\mu_i \delta^\nu_j$, where $h^{ij}$ form a matrix inverse to the metric.
components $h_{ij}$ of the hypersurface, $h^{i*}h_{i} = \delta^i_j$. Spatial vector fields $T^\mu$ possess vanishing contravariant time components $T^0 = 0$, therefore their space components are lowered and raised by means of $h_{ij}$ and $h^{ij}$, $T_i = g_{\mu i}T^\mu = h_{ij}T^j$, and $T^i = g^{i\mu}T_\mu = h^{ij}T_j$. In particular, the spacetime tensors $D_\mu f$, $D_\mu T^\alpha$, for a spatial vector $T^\alpha$, and $K_{\mu\nu} = -\bar{D}_\mu u_\nu$ are spatial tensors, then $D_i f$, $D_i T^j$, and $K_{ij}$ are tensors on $\Sigma$ with indices raised and lowered by $h^{ij}$ and $h_{ij}$.

With respect to this coordinate decomposition, the space components of the extrinsic curvature $K_{ij}$ occurs to be

$$K_{ij} = -\frac{1}{2N} \left[ \frac{\partial}{\partial t} h_{ij} - 2D_i V_j \right],$$  

hence the momentum $P^{ij}$ for the hypersurfaces $\Sigma$ is defined as

$$P^{ij} = \frac{1}{2\kappa} \sqrt{-\det(h_{ij})} \left[ K h^{ij} - K_{ij} \right],$$

which is appropriate if the matter fields are minimally coupled to gravity, i.e., those fields do not contain derivatives of $g_{\mu\nu}$.

The extrinsic geometry of a three–boundary $\mathcal{B}$ is defined in a way similar to the one exhibited above for the hypersurfaces $\Sigma$. Nevertheless, as pointed in Ref. [10], the three–boundary is not thought of as a member of a foliation of the whole spacetime $\mathcal{M}$ since the extension of $\mathcal{B}$ throughout all $\mathcal{M}$ could be forbidden by the topology of $\mathcal{M}$. Denoting by $n^\mu$ the outward pointing spacelike normal to the three–boundary $\mathcal{B}$, $n^\mu n_\mu = 1$, the projection metric on $\mathcal{B}$ is given by $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. The extrinsic curvature is defined by $\Theta_{\mu\nu} = -\gamma^{\lambda} \nabla_{\lambda} n_\nu = -\bar{D}_\mu n_\nu$, where $\bar{D}_\mu = \gamma^{\lambda} \nabla_{\lambda}$ denote the induced covariant derivative for tensors that are tangent to the three–boundary $\mathcal{B}$.

Introducing intrinsic coordinates on $\mathcal{B}$ adapted to the choice of the spacelike normal $n_\mu = N^i \delta^3_\mu$ by means of $x^i, i = 0, 1, 2$, the spacetime metric becomes

$$ds^2 = (\gamma_{\mu\nu} + n_\mu n_\nu) dx^\mu dx^\nu = N^2(dx^3)^2 + \gamma_{ij}(dx^i + N^i dx^3)(dx^j + N^j dx^3).$$

Likewise, the components of tensors tangent to the three–boundary $\mathcal{B}$ are raised and lowered by the intrinsic metric tensors $\gamma^{ij}$ and $\gamma_{ij}$, in particular this hold for the components of the intrinsic curvature $\Theta_{ij}$.

The definition of the boundary momentum amounts to

$$\pi^{ij} = -\frac{1}{2\kappa} \sqrt{-\det(\gamma_{ij})} \left[ \Theta^{i\bar{j}} - \Theta^{i\bar{j}} \right].$$

Two dimensional subspace $V_2$ immersed in a spacetime $\mathcal{M}$

According to the theory of $V_n$ embedded in a $V_m, m \geq n$, in the specific case of $n = 2$ and $m = 4$ there exist two orthogonal vector fields, say $\xi_1$ and $\xi_2$, normal to two three–dimensional manifolds $\Sigma_1$ and $\Sigma_2$. The set of points where these ”hypersurfaces” intersect gives rise to a 2-subspace having as outward normals the vectors $\xi_1$ and $\xi_2$. This 2-subspace can be characterized by its metric and extrinsic curvature.
From the Brown–York paper it follows that the intersections of the families of hypersurfaces $\Sigma$ and the three–boundary $B$, such that for their normals $u$ and $n$ the orthogonality condition $(u \cdot n)|^3_B = 0$ holds, determine the two–boundaries $B$ on which the metric $\sigma_{\mu\nu}$, tangent to both $u$ and $n$ fields at the intersection, is defined by

$$\sigma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu - n_\mu n_\nu = h_{\mu\nu} - n_\mu n_\nu = \gamma_{\mu\nu} + u_\mu u_\nu,$$

(A4)

fulfilling $\sigma_{\mu\nu} u^\mu = 0 = \sigma_{\mu\nu} n^\mu$; notice that $\gamma_{\mu\nu} u^\mu = u_\nu$ and $h_{\mu\nu} n^\mu = n_\nu$ because of the tangent property of the vectors involved. The spacetime metric can be given as

$$ds^2 = (\sigma_{\mu\nu} + n_\mu n_\nu - u_\mu u_\nu) dx^\mu dx^\nu = -(u_\mu dx^\mu)^2 + (n_\nu dx^\nu)^2 + \sigma_{\mu\nu} dx^\mu dx^\nu.$$

The metric element on $B$, constructed via the space metric $\gamma_{ij}$, $i = 0, 1, 2$ through the metric $\sigma_{ab}$, $a, b = 1, 2$, of the two boundary $B$, and the components of $u_\mu = -N_\delta^0_\mu$ can be given as

$$\gamma_{ij} dx^i dx^j = -N^2 dt^2 + \sigma_{ab} (dx^a + V^a dt)(dx^b + V^b dt).$$

(A5)

If we were using the decomposition of the spacetime metric with respect to a normal spacelike vector $n_\mu = N^0_\mu$ one would arrive at

$$g_{\mu\nu} = \gamma_{\mu\nu} + n_\mu n_\nu, g_{ij} = \gamma_{ij}, g_{3j} = \gamma_{3j} = -N^i \gamma_{ij}, g_{33} = N^2 + N^i N^j \gamma_{ij},$$

where $i, j = 0, 1, 2$. Hence the four–dimensional metric would be written as

$$ds^2 = (\gamma_{\mu\nu} + n_\mu n_\nu) dx^\mu dx^\nu = N^2 (dx^3)^2 + \gamma_{ij} (dx^i - N^i dx^3) (dx^j - N^j dx^3).$$

Restricting oneself to lie on the surface $x^3 = \text{const.}$, the three–boundary $B$ metric reduces to

$$ds^2|_B = \gamma_{ij} dx^i dx^j.$$

This projection metric $\gamma_{ij}$, $i, j = 0, 1, 2$, can be considered as spanned by the two-dimensional metric $\sigma_{ab}$ of the two–boundary $B$ and the vector field $u_i = -N^0_\delta^0_\mu$ orthogonal to the hypersurface $\Sigma$. I consider illustrative to give certain details about the derivation of the metric of $B$ in terms of the tensor components $\sigma_{ab}$ and the components of $u^i$, $\gamma_{ij} = \sigma_{ij} - u_i u_j$, $i, j = 0, 1, 2$. Since $u_i = -N^0_\delta^0_\mu$, $u^j = \gamma^{ji} u_i$, $u_\mu = \gamma_{\mu\nu} u^\nu$, then $u^j = -N^j \gamma^{0j}$, and from the unit condition $u^\mu u^\mu = -1$ one gets $\gamma^{00} = -1/N^2$, consequently $u^0 = 1/N$ and $u^a = -N^a \gamma^{a0} =: N^a/N$, which yields $N^a = -N u^a$. On the other hand, from the condition of orthogonality $\sigma_{ij} u^j = 0$ one has $\sigma_{00} u^0 + \sigma_{ii} u^a = 0$, the substitution of $u^0$ and $u^a$ yields $\sigma_{00} = -\sigma_{0a} N^a$, or explicitly by components $\sigma_{00} = \sigma_{ab} N^b$, $\sigma_{00} = \sigma_{0b} N^b = \sigma_{ab} N^a N^b$. Gathering these results one has

$$\begin{align*}
\gamma_{00} &= \sigma_{00} - N^2 = \sigma_{ab} N^a N^b - N^2, \\
\gamma_{0b} &= \sigma_{0b} = \sigma_{ab} N^a, \\
\gamma_{ab} &= \sigma_{ab},
\end{align*}$$

(A6)

consequently

$$\begin{align*}
\gamma_{ij} dx^i dx^j &= \gamma_{00} dt^2 + 2 \gamma_{0a} dt dx^a + \gamma_{ab} dx^a dx^b \\
&= -N^2 dt^2 + \sigma_{ab} N^a N^b dt^2 + \sigma_{ab} N^a dx^b dt + \sigma_{ab} dx^a dx^b \\
&= -N^2 dt^2 + \sigma_{ab}(dx^a + N^a dt)(dx^b + N^b dt).
\end{align*}$$

(A7)
The three–boundary unit normal \( n^\mu \) is orthogonal to both the three–boundary \( \overline{B} \) embedded in the spacetime \( M \) and the two–boundary \( B \) as embedded in the hypersurface \( \Sigma \). The extrinsic curvature of the two–boundary \( B \) as embedded in the hypersurface \( \Sigma \) is defined by

\[
k_{\mu\nu} = -\sigma^\alpha_\mu D_\alpha n_\nu = -\sigma^\alpha_\mu h^{\beta}_\alpha h^\lambda_\nu \nabla_\beta n_\lambda = -\sigma_\mu h^\alpha_\beta n_\lambda h^\lambda_\nu , \tag{A8}
\]

where \( D_\alpha n_\nu = h^\beta_\alpha n_\lambda h^\lambda_\nu \) is the covariant derivative on \( \Sigma \) of \( n_\alpha \) obtained by projecting with the tensor \( h^\beta_\alpha \) the spacetime covariant derivative of the spacelike vector \( n_\alpha \). The tensors \( \sigma_\alpha^\beta \) and \( k_{\mu\nu} \) are defined only on \( \overline{B} \).

The relevance of this splitting of the spacetime \( M \) into hypersurfaces \( \Sigma \), the three–boundary \( \overline{B} \), and the two–boundary space \( B \) in the Brown–York approach resides in its use in the formulation of the quasilocal energy and conserved charges of gravitational and matter fields minimally coupled to gravity in a spatially bounded region given in detail in Ref. \[?\].

For the choice of the timelike field \( u_\alpha = -N \delta^0_\alpha = -\delta^0_\alpha / \sqrt{-g^{00}} \) normal to the hypersurface \( \Sigma \) one establishes that the contravariant components \( u^\alpha = -N g^{0\alpha} \) are given in terms of the metric components as \( u^\alpha = -g^{0\alpha} / \sqrt{-g^{00}} \), and the shift function is \( N = 1 / \sqrt{-g^{00}} \), while the shift vector components amount to \( N^i = -N u^i = -g^{i0} / g^{00} \). Hence the projection tensor can be given as:

\[
\begin{align*}
  h^{\alpha}_{\beta} &= \begin{bmatrix} h^{a0} = g_{ab} N^b = -g_{ab} g^{b0} \frac{1}{g^{00}} & h^{ab} = g_{ab} \\
g_{ab} N^a N^b = g_{ab} g^{a0} g^{b0} \frac{1}{(g^{00})^2} & h^{0b} = g_{bs} N^s = -g_{ab} g^{a0} \frac{1}{g^{00}} \end{bmatrix}, \\
  h^{\alpha}_{\beta} &= \begin{bmatrix} 0 & h^{ab} = g^{ab} - g^{a0} g^{b0} \frac{1}{(g^{00})^2} \\
 0 & 0 \end{bmatrix}, \\
  h^{\beta}_{\alpha} &= \begin{bmatrix} 0 & h^b_a = \delta^b_a \\
 0 & h^{0b} = N^b = -g^{b0} \frac{1}{g^{00}} \end{bmatrix}, \tag{A9}
\end{align*}
\]

where Latin letters run \( a, b, ..., i = 1, 2, 3 \).

If one chose the spacelike normal \( n_\alpha \) to the three–boundary \( \overline{B} \) as pointing along the coordinate \( x^3 \) then \( n_3 = N \delta^3_\alpha \). One would have that the contravariant components \( n^\alpha = N g^{3\alpha} \) are given in terms of the metric components as \( n^\alpha = g^{3\alpha} / \sqrt{g^{33}} \), and the function is \( N = 1 / \sqrt{g^{33}} \), while the vector components amount to \( N^A = N A^A = g^{A3} / g^{33} \). Moreover, the following decomposition will hold:

\[
g_{\mu\nu} = \gamma_{\mu\nu} + n_\mu n_\nu, g_{AB} = \gamma_{AB}, g_{3B} = \gamma_B = -N^A g_{AB}, g_{33} = N^2 + N^A N^B g_{AB},
\]

where capital Latin letters run \( A, B, ..., J = 0, 1, 2 \) have been introduced to avoid misunderstanding.

In the definition of the intrinsic curvature \( k_{\mu\nu} \), Eq. (A8), of the two–surface \( B \) enter the metric tensor \( \sigma_{\mu\nu} \), the covariant derivative on \( \Sigma \) and the normal \( n \) to \( B \). The metric on \( B \), i.e., the spacetime metric on \( x^3 = \text{const.} \) is given by

\[
ds^2 \big|_B = g_{AB} dx^A dx^B = \gamma_{AB} dx^A dx^B.
\]
One can relate \( \gamma_{AB} \) to \( \sigma_{\mu\nu} \) and the timelike normal vector \( u^\mu \) to the hypersurface \( \Sigma \) by defining \( \sigma_{AB} = \gamma_{AB} + u_A u_B \). Since \( u_A = -N \delta^0_A \), \( u^\mu = g^{\mu A} u_A = -N g^{00} \), and the unit condition \( u_B u^\mu = -1 \rightarrow N^2 g^{00} = -1 \rightarrow N = 1/\sqrt{-g^{00}} \). Moreover, due to \( (u \cdot n)_B \), therefore \( u^\mu n_\mu = 0 \rightarrow u^3 = 0 \), hence \( u^0 \rightarrow u^A \). Consequently \( u^B = g^{BA} u_A = -g^{00}/\sqrt{-g^{00}} \).

On the other hand \( \sigma_{\mu\nu} \) are defined as normal and tangential projections of \( \tau_{\mu\nu} \). Thus \( \sigma = u^0 \) and \( \tau_{\mu\nu} \) are normal and tangential projections of \( \tau_{\mu\nu} \). Consequently \( \sigma_{\mu\nu} = 0 \rightarrow \sigma_{\mu\nu} = 0 \rightarrow \sigma_{\mu\nu} = 0 \), and \( \gamma_{ij} \) thus \( \gamma_{ij} = 0 \) and \( \gamma_{ij} \) yields \( \gamma_{ij} = 0 \), hence \( \gamma_{ij} = 0 \). Consequently, \( \sigma_{ij} = 0 \) and \( \sigma_{ij} = 0 \). Moreover, \( \sigma_{ij} = 0 \) or \( \sigma_{ij} = 0 \) yields \( \sigma_{ij} = 0 \), which yields \( \sigma_{ij} = 0 \), hence \( \sigma_{ij} = 0 \). Consequently, \( \sigma_{ij} = 0 \) reduces to the components \( i, j = 1, 2 \), i.e., with \( i, j = 1, 2 \), namely

\[
\begin{align*}
\sigma_{ij} &= \gamma_{ij}, \\
\sigma_{0j} &= \gamma_{0j} = \gamma_{ij} N^i = \gamma_{ij} N^i, \\
\sigma_{00} &= \gamma_{00} + N^2 = \gamma_{ij} N^i N^j = \gamma_{ij} N^i N^j, \\
\end{align*}
\]

It remains still to give the expression of \( \sigma_{ij} \) in terms of the metric components \( g^{\mu\nu} \). Since \( u_B n^a = 0 \) on \( B^3 \), because of the choice of coordinates along \( u_\alpha \) and \( n_\alpha \), one has \( u^3 = 0 \) and \( n^0 = 0 \), hence \( g^{03} = 0 \). Consequently, \( \sigma_{ij} \) reduces to the components \( i, j = 1, 2 \), i.e., with \( i, j = 1, 2 \), namely

\[
\begin{align*}
\sigma_{ij} &= g^{ij} - n^i n^j + u^i u^j = g^{ij} - g^{ij} g^{00} / g^{00} = 0, \\
\end{align*}
\]

while \( \sigma_{ij} = \sigma_{ij} = 0 \), \( i, j = 1, 2, 3 \).

In this coordinate frame the intrinsic curvature of \( B \) becomes

\[
k_{ij} = -\sigma_{ij} \delta^3 h_{ij} = -\sigma_{ij} \delta^3 h_{ij} \left( \frac{\partial N}{\partial x^j} - N \Gamma^{3}_{ij} \right). \quad (A13)
\]

1. Energy and momentum surface densities, and spatial stress

The energy surface density \( \epsilon \), the momentum surface density \( j_a \), and the spatial stress \( s^{ab} \) are defined as normal and tangential projections of \( \tau^{ij} = 2/\sqrt{-\gamma} (\bar{\tau}^{ij} - \bar{\tau}^{ij}) \) on the two-surface \( B \)

\[
\begin{align*}
\epsilon &= u_i u_j \tau^{ij} = \frac{1}{\sqrt{\gamma}} \frac{\delta S_{cl}}{\delta N}, \\
\end{align*}
\]

\[
\begin{align*}
\tau^{ij} &= \frac{1}{\sqrt{\gamma}} \frac{\delta S_{cl}}{\delta V^a}, \\
\end{align*}
\]

\[
\begin{align*}
\tau^{ij} &= \frac{2}{\sqrt{\gamma}} \frac{\delta S_{cl}}{\delta \sigma_{ab}}. \quad (A14)
\end{align*}
\]

The subscript \( cl \) stands for classical. The surface stress–energy–momentum tensor \( \tau^{ij} \) includes contributions from both the gravitational and the matter fields. These quantities are tensors with respect to the metric \( \sigma_{ab} \) defined on the two–surface \( B \) and physically represent the corresponding quantities associated with matter and gravitational fields on the hypersurface \( \Sigma \) with boundary \( B \). In particular, the total energy for \( \Sigma \) is obtained by integrating \( \epsilon \) on the boundary \( B \), 
\[
E = \int_B d^2x \sqrt{\sigma} \epsilon.
\]

These tensors can be given as
\[
\begin{align*}
\epsilon &= \frac{1}{\kappa} k|_{cl} + \frac{1}{\sqrt{\sigma}} \frac{\delta S^0}{\delta N}, \\
\dot{j}_a &= -2 \left( \frac{1}{\sqrt{h}} \sigma^{ai} n_k P^{jk} \right) |_{cl} - \frac{1}{\sqrt{\sigma}} \frac{\delta S^0}{\delta V_a}, \\
s^{ab} &= \frac{1}{\kappa} \left[ k^{ab} + (n \cdot a - k) \sigma^{ab} \right] |_{cl} - \frac{2}{\sqrt{-\gamma}} \frac{\delta S^0}{\delta \sigma_{ab}}.
\end{align*}
\]

\( |_{cl} \) means evaluation for the classical solution, i.e., evaluation for a particular spacelike hypersurface \( \Sigma \) in the spacetime, indices that refer to coordinates on \( B^3 \) are denoted by \( i, j \), tensor indices that refer to coordinates on \( \Sigma \) are underlined \( i, k \), indices that refer to coordinates on \( B \) are denoted by \( a, b \).

There exist an ambiguity in the choice of the reference term \( S^0 \), which is eliminated in the Brown–York paper by demanding that \( \epsilon \) and \( \dot{j}_a \) of a particular \( \Sigma \) should depend only on the canonical variables \( h_{ij} \) and \( P^{ij} \) defined on \( \Sigma \). In particular, with this aim in mind, a possible choice is \( S^0 \) as a linear functional of the lapse function \( N \) and shift vector \( V^a \) on the two–boundary \( B \)
\[
S^0 = -\int_{B^3} d^3x \left( \sqrt{\sigma} N A + 2\sqrt{\sigma} V^a B_a \right)
\]
where \( A \) and \( B_a \) are arbitrary functions of the two–metric \( \sigma_{ab} \).

In Ref. \([10]\) a comment in extenso follows the choice of \( B_a = \left( \frac{1}{\sqrt{h}} \sigma^{ai} n_k P^{jk} \right) |_0 \) and \( A = k|_0 \) by choosing a reference space–a fixed spacelike slice of some fixed spacetime–and then consider a two–surface in the slice whose induced two–metric is \( \sigma_{ab} \). If the two–surface exists then one evaluates the specific functions \( A \) and \( B \) above yielding the sought functions of \( \sigma_{ab} \). For such choice of \( A \) and \( B \) one gets
\[
\begin{align*}
\epsilon &= \frac{1}{\kappa} k|_0, \\
\dot{j}_a &= -2 \left( \frac{1}{\sqrt{h}} \sigma^{ai} n_k P^{jk} \right) |_0.
\end{align*}
\]

In particular, such functions \( A \) and \( B \) are uniquely determined by the flat reference space, at least for all positive curvature two–metrics with two–sphere topology. Moreover, for a flat slice of flat spacetime \( \left( \frac{1}{\sqrt{h}} \sigma^{ai} n_k P^{jk} \right) |_0 = 0 \) because \( P^{jk} = 0 \) identically.

The total quasilocal energy is defined over the two–surface \( B \) as
\[
E = \int_B d^2x \sqrt{\sigma} \epsilon.
\]
For metric allowing the existence of symmetries associated with a Killing field $\xi$ on the boundary $\mathcal{B}$ a conserved charge can be defined as

$$Q_\xi = \int_B d^2x \sqrt{\sigma} (\epsilon \, u^i + j^i) \xi_i.$$  

(A19)

If the Killing field $\xi$ is timelike, then the negative of the corresponding charge defines a conserved mass $M := -Q_{\xi_{\text{timelike}}}$. If the Killing field $\xi$ is a rotational symmetry on $\mathcal{B}$, then the corresponding conserved charge defines the angular momentum $J := Q_{\xi_{\text{rotational}}}$; if the surface $B$ contains the orbits of $\xi_{\text{rotational}}$, then the angular momentum can be determined by

$$J = \int_B d^2x \sqrt{\sigma} j^i \xi_i.$$  

(A20)

As pointed out in Ref. [11], the distinction between mass $M$ and energy $E$ is relevant for spacetimes that are asymptotically anti–de Sitter due to the divergent character at spatial infinity of the magnitude of the timelike Killing vector in that case; the timelike Killing vector does not approach the unit normal to the stationary time slices at spatial infinity, consequently $E$ and $M$ do not coincide.

The reduction of the above theory to (2+1)–dimensional spacetime is straightforward.

**Appendix B: Symmetries of the stationary and static cyclic symmetric BTZ solutions**

The study of the symmetries of the stationary and static cyclic symmetric BTZ families and AdS classes of solutions starts with the stationary metric for the standard BTZ solution

$$g = -F(r)^2 dt^2 + \frac{dr^2}{F(r)^2} + r^2 [d\phi + V(r) dt]^2,$$

$$F(r)^2 = \frac{r^2}{l^2} - M + \frac{J^2}{4r^2}, \quad V(r) = -\frac{J}{2r^2}.$$  

(B1)

The covariant components of the Killing vectors $V_\alpha$ are denoted by $v_\alpha$, namely

$$V_1(t, r, \phi) = v_1, \quad V_2(t, r, \phi) = v_2, \quad V_3(t, r, \phi) = v_3.$$  

(B2)

The Killing equations $V_{\mu;\nu} + V_{\nu;\mu} = 0$ amount explicitly to

$$EQ11 = \frac{\partial}{\partial t} v_1 - \frac{rF(r)^2}{l^2} v_2,$$

(B3a)

$$2EQ12 = \frac{\partial}{\partial r} v_1 + \frac{\partial}{\partial t} v_2 - \frac{J}{l^2 r F(r)^2} v_3 - 2 \frac{r}{l^2 F(r)^2} v_1,$$

(B3b)

$$2EQ13 = \frac{\partial}{\partial \phi} v_1 + \frac{\partial}{\partial t} v_3.$$  

(B3c)
EQ22 = F(r) \frac{\partial}{\partial r} v^2 + v^2 \frac{d}{dr} F(r) \rightarrow v^2 = \frac{F_1 (t, \phi)}{2lF(r)}, \quad (B3d)

EQ23 = \frac{\partial}{\partial \phi} v^2 + \frac{\partial}{\partial r} v^3 + \frac{J}{rF(r)} v^2 + 2 \left( \frac{M^2 - r^2}{r^2 F(r)^2} \right) v^3, \quad (B3e)

EQ33 = \frac{\partial}{\partial \phi} v^3 + r F(r)^2 v^2, \quad (B3f)

where $F_i (t, \phi), i = 1, 2, 3,$ are integration functions.

Isolating $v1$ from (B3e) in terms of $v2$ and $v3$ and their derivatives one gets

$$r l^2 J v1 = 2 r (r^2 - Ml^2) v3 - r^2 l^2 F(r)^2 \left( \frac{\partial}{\partial \phi} v^2 + \frac{\partial}{\partial r} v^3 \right). \quad (B4)$$

Next, substituting $v1$ from above and the first integral of $v2$ from (B3d) into equation (B3b) one arrives at a linear second order equation for $v3$ with integrals

$$v3 = \frac{r^2}{2} F_2 (t, \phi) + F_3 (t, \phi) - \frac{1}{2} \frac{r l F(r)}{2 M^2 l^2 - J^2} \left( J \frac{\partial}{\partial t} F_1 (t, \phi) + M \frac{\partial}{\partial \phi} F_1 (t, \phi) \right) \quad (B5)$$

which substituted, together with $v2$ from (B3d), into Eq. (B4) for $v1$ gives

$$v1 = 2 \left( \frac{r^2 - Ml^2}{Ml^2} \right) F_3 (t, \phi) - \frac{1}{4} J F_2 (t, \phi) + \frac{1}{2} \frac{r l F(r)}{l^2 M^2 - J^2} \left( J \frac{\partial}{\partial l} F_1 (t, \phi) + M \frac{\partial}{\partial t} F_1 (t, \phi) \right). \quad (B6)$$

The dependence of the Killing vector components on the $r$ variable has been established; it remains still to determine their dependence on the $t$ and $\phi$ variables hiding in the $F_1 (t, \phi), F_2 (t, \phi)$ and $F_3 (t, \phi)$ functions. Substituting the expressions of $v1$ from (B6), $v3$ from (B5), and $v2$ from (B3d) one arrives at the independent equations

$$F_1 (t, \phi) J^2 + l^2 J \frac{\partial^2}{\partial t \partial \phi} F_1 (t, \phi) + l^2 \left( \frac{\partial^2}{\partial \phi^2} F_1 (t, \phi) \right) M - F_1 (t, \phi) M^2 l^2 = 0, \quad (B7a)$$

$$F_1 (t, \phi) J^2 + l^2 J \frac{\partial^2}{\partial t \partial \phi} F_1 (t, \phi) + M \left( \frac{\partial^2}{\partial l^2} F_1 (t, \phi) \right) l^4 - F_1 (t, \phi) M^2 l^2 = 0, \quad (B7b)$$

with integral

$$F_1 (t, \phi) = C_1 e^{\frac{\sqrt{4M^2 l (\phi+1)}}{Ml^{(l+1)}}} + C_2 e^{\frac{\sqrt{4M^2 l (\phi-1)}}{Ml^{(l+1)}}} + C_3 e^{-\frac{\sqrt{4M^2 l (\phi-1)}}{Ml^{(l+1)}}} + C_4 e^{-\frac{\sqrt{4M^2 l (\phi+1)}}{Ml^{(l+1)}}}. \quad (B8)$$

Furthermore, there have to be solved constraints on $F_2 (t, \phi)$ and $F_2 (t, \phi)$, namely

$$\frac{\partial}{\partial t} F_3 (t, \phi) = 0, \frac{\partial}{\partial \phi} F_3 (t, \phi) = 0, F_3 (t, \phi) = J C_6 = \text{const.},$$

$$\frac{\partial}{\partial t} F_2 (t, \phi) = 0, \frac{\partial}{\partial \phi} F_2 (t, \phi) = 0, F_2 (t, \phi) = C_5 = \text{const.} \quad (B9)$$
where the integration constants are denoted through $C_i, i = 1, ..., 6$. Finally, the covariant Killing vector components are

$$V_1 = \frac{1}{2} \frac{r F(r)}{\sqrt{M^2 t^2 - J^2}} \left( C_1 \sqrt{Ml + J} \exp\left(\frac{\sqrt{Ml + J}}{l^{3/2}}(l\phi + t)\right) - C_2 \sqrt{Ml - J} \exp\left(\frac{-\sqrt{Ml - J}}{l^{3/2}}(l\phi - t)\right) - C_4 \sqrt{Ml - J} \exp\left(-\frac{\sqrt{Ml - J}}{l^{3/2}}(l\phi + t)\right) \right) - \frac{J}{4} C_5 - 2 \frac{(Ml^2 - r^2)}{l^2} C_6,$$

$$V_2 = \frac{1}{2l F(r)} \left( C_1 \exp(\frac{\sqrt{Ml - J}}{l^{3/2}}(l\phi + t)) + C_2 \exp(\frac{-\sqrt{Ml - J}}{l^{3/2}}(l\phi - t)) + C_3 \exp(-\frac{\sqrt{Ml - J}}{l^{3/2}}(l\phi + t)) + C_4 \exp(\frac{-\sqrt{Ml - J}}{l^{3/2}}(l\phi - t)) \right),$$

$$V_3 = -\frac{1}{2l^{1/2}} \frac{r F(r)}{\sqrt{M^2 t^2 - J^2}} \left( C_1 \sqrt{Ml + J} \exp\left(\frac{\sqrt{Ml + J}}{l^{3/2}}(l\phi + t)\right) + C_2 \sqrt{Ml - J} \exp\left(\frac{-\sqrt{Ml + J}}{l^{3/2}}(l\phi - t)\right) - C_3 \sqrt{Ml - J} \exp\left(-\frac{\sqrt{Ml + J}}{l^{3/2}}(l\phi - t)\right) - C_4 \sqrt{Ml + J} \exp\left(-\frac{\sqrt{Ml - J}}{l^{3/2}}(l\phi + t)\right) \right) + \frac{r^2}{2} C_5 + J C_6.$$  

These expressions allow one to determine the Killing vector $k_i$ associated to its corresponding integration constant $C_i$ for each of the possible $i$, by means of $V_\mu = \sum_{i=1}^{6} k_i \nu$, where $k_i \nu = C_i V_i \mu$, for each fixed value of $i$. The contravariant vectors’ components $k_i \nu$, $\partial_{\nu} \equiv k_i \frac{\partial}{\partial x^\nu}$, are derived from the relationship $V^\mu = V_\nu g^{\nu\mu} = \sum_{i=1}^{6} k_i \nu g^{\nu\mu} = \sum_{i=1}^{6} k_i \mu = \sum_{i=1}^{6} C_i V_i \mu$. Explicitly these Killing vectors are reported in the main text, Section 2.

### 1. Symmetries of the anti–de Sitter space with positive $M, M > 0$

By a straightforward integration of the Killing equations for the anti–de Sitter metric

$$g = -F(r)^2 dt^2 + \frac{dr^2}{F(r)^2}, +r^2 d\phi^2, F(r)^2 = \frac{r^2}{l^2} - M,$$  

or as a limiting transition of the previously derived Killing vectors for the stationary BTZ by setting $J \to 0, C_a \to 2c_a, C_6 \to c_6/2$ one arrives at the covariant components of the Killing vectors: $V = V_\mu dx^\mu = C_a V_a dx^\mu$—one vector for each constant—$V = c_a V_a$:

$$V_1 = r \frac{\sqrt{r^2 - Ml^2}}{l^3 \sqrt{M}} \left( e^{\frac{\sqrt{Ml}}{l} (l\phi + t)} c_1 - e^{\frac{\sqrt{Ml}}{l} (l\phi - t)} c_2 + e^{\frac{\sqrt{Ml}}{l} (l\phi - t)} c_3 - e^{\frac{\sqrt{Ml}}{l} (l\phi + t)} c_4 \right) + c_6 \frac{r^2 - Ml^2}{l^2},$$

(B12)
\[ V_2 = \frac{1}{\sqrt{r^2 - Ml^2}} \left( e^{\sqrt{\frac{\pi}{l^2}} t} c_1 + e^{\sqrt{\frac{\pi}{l^2}} t - t} c_2 + e^{-\sqrt{\frac{\pi}{l^2}} t} c_3 + e^{-\sqrt{\frac{\pi}{l^2}} t + t} c_4 \right), \] (B13)

\[ V_3 = -\frac{r}{\sqrt{Ml^2}} \left( e^{\sqrt{\frac{\pi}{l^2}} t} c_1 + e^{\sqrt{\frac{\pi}{l^2}} t - t} c_2 - e^{-\sqrt{\frac{\pi}{l^2}} t} c_3 - e^{-\sqrt{\frac{\pi}{l^2}} t + t} c_4 \right) + r^2 c_5. \] (B14)

From the viewpoint of the group properties more important are the contravariant components of the Killing vectors \( \partial V = C_a \partial_{k_a} = C_a k_a \mu \partial_{\mu} \).

2. Symmetries of the anti–de Sitter metric for negative \( M, M = -\alpha^2 \)

If negative \( M \), one equates it to \(-\alpha^2\). Moreover, instead of complex exponential function, it will be better to use trigonometric sine and cosine functions. Thus, one can give the Killing vector components as

\[ V_1 = \frac{r \sqrt{\alpha^2 l^2 + r^2}}{\alpha l^3} \left( -C_1 \sin (\alpha \phi) \cos \left( \frac{\alpha t}{l} \right) + C_2 \sin (\alpha \phi) \sin \left( \frac{\alpha t}{l} \right) \right) - C_3 \cos (\alpha \phi) \cos \left( \frac{\alpha t}{l} \right) + C_4 \cos (\alpha \phi) \sin \left( \frac{\alpha t}{l} \right) \right) + C_5 (\alpha^2 l^2 + r^2), \] (B15)

\[ V_2 = \sqrt{\alpha^2 l^2 + r^2} \left( C_1 \sin (\alpha \phi) \sin \left( \frac{\alpha t}{l} \right) + C_2 \sin (\alpha \phi) \cos \left( \frac{\alpha t}{l} \right) \right) + C_3 \cos (\alpha \phi) \sin \left( \frac{\alpha t}{l} \right) + C_4 \cos (\alpha \phi) \cos \left( \frac{\alpha t}{l} \right) \), \] (B16)

\[ V_3 = \frac{r \sqrt{\alpha^2 l^2 + r^2}}{\alpha l^2} \left( C_1 \cos (\alpha \phi) \sin \left( \frac{\alpha t}{l} \right) + C_2 \cos (\alpha \phi) \cos \left( \frac{\alpha t}{l} \right) \right) - C_3 \sin (\alpha \phi) \sin \left( \frac{\alpha t}{l} \right) - C_4 \sin (\alpha \phi) \cos \left( \frac{\alpha t}{l} \right) \right) + C_6 r^2, \] (B17)

correspondingly, the contravariant components are given in the main text, Section [IVD].

This anti–de Sitter metric, (cosmological constant negative–\( \lambda = -1/l^2 \)), for the coordinates \( \{ t, \rho, \phi \} \)–merely names–ranging \(-\infty \leq t \leq \infty, -\infty \leq \rho \leq \infty; -\infty \leq \phi \leq \infty \) allows for six symmetries, i.e., six Killing vectors. For these ranges of determination of the coordinates, the space is maximally symmetric.

The same situation takes place if the spatial coordinates are restricted to range

\[ 0 \leq \rho \leq \infty, 0 \leq \phi \leq 2\pi, \]

and \( \alpha \) is set equal to unity, \( \alpha = 1 = -M \), then in such case \( \rho \) and \( \phi \) become polar coordinates with \( \phi \) being the angular coordinate with period \( 2\pi \). This spacetime—the (proper) anti–de Sitter space (with \( M = -1 \))–allows for six symmetries, and as such it is maximally symmetric.