RESIDUES FOR MAPS GENERICALLY TRANSVERSE TO DISTRIBUTIONS

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Abstract. We show a residues formula for maps generically transversal to regular holomorphic distributions.

1. Introduction

Let \( f : X \to Y \) be a singular holomorphic map between complex manifolds \( X \) and \( Y \), with \( \dim(X) := n \geq m =: \dim(Y) \), having generic fiber \( F \). Consider the singular set of \( f \) defined by
\[
S := \text{Sing}(f) = \{ p \in X : \text{rank}(d f(p)) < m \}.
\]
If \( Y = C \) is a curve, Iversen in [11] proved the following multiplicity formula
\[
\chi(X) - \chi(F) \cdot \chi(C) = (-1)^n \sum_{p \in \text{Sing}(f)} \mu_p(f),
\]
where \( \mu_p(f) \) is the Milnor number of \( f \) at \( p \). Izawa and Suwa [14] generalized Iversen’s result for the case where \( X \) is possibly a singular variety.

A generalization of the multiplicity formula for maps was given by Diop in [7]. In his work he generalized some formulas involving the Chern classes given previously by Iversen [11], Brasselet [3, 4], and Schwartz [17]. More precisely, Diop showed that if \( S \) is smooth and \( \dim(S) = m - 1 \) then
\[
\chi(X) - \chi(F) \cdot \chi(Y) = (-1)^{n-m+1} \sum_j \mu_j \int_{S_j} c_{q-1}(f^*(TY)|_{S_j} - L_j),
\]
where \( S = \cup S_j \) is the decomposition of \( S \) into irreducible components, \( \mu_j = \mu(f|_{\Sigma_j}) \) is the Milnor number of the restriction of \( f \) to a transversal section \( \Sigma_j \) to \( S_j \) at a regular point \( p_j \in S_j \), and \( L_j \) is the line bundle over \( S_j \) given by the decomposition \( f^*d f(TX|_{S_j}) \oplus L_j = f^*(TY)|_{S_j} \).

On the other hand, Brunella in [5] introduced the notion of tangency index of a germ of curve with respect to a germ of holomorphic foliation: given a reduced curve \( C \) and a foliation \( \mathcal{F} \) (possibly singular) on a complex compact surface. Suppose that \( C \) is not invariant by \( \mathcal{F} \) and that \( C \) and \( \mathcal{F} \) are given locally by \( \{ f = 0 \} \) and a vector field \( v \), respectively. The tangency index \( I_p(\mathcal{F}, C) \) of \( C \) with respect to \( \mathcal{F} \) at \( p \) is given by the intersection number
\[
I_p(\mathcal{F}, C) = \dim \mathcal{O}_2/(f, v(f)).
\]
Using this index, Brunella proved the following formula
\[
c_1(\mathcal{O}(C))^2 - c_1(T_{\mathcal{F}}) \cap c_1(\mathcal{O}(C)) = \sum_{p \in \text{Tang}(\mathcal{F}, C)} I_p(\mathcal{F}, C),
\]
where \( T_{\mathcal{F}} \) is the tangent bundle of \( \mathcal{F} \) and \( \text{Tang}(\mathcal{F}, C) \) denotes the non-transversality loci of \( C \) with respect to \( \mathcal{F} \). In [9] and [10], T. Honda also studied Brunella’s tangency formula. Distributions...
and foliations transverse to certain domains in \( \mathbb{C}^n \) has been studied by Bracci and Scárdua in [2] and Ito and Scárdua in [12].

Recently, Izawa [13] generalized certain results due to Diop [7] in the foliated context. More precisely, let \( f : X \to (Y, \mathcal{F}) \) be a holomorphic map such that \( \mathcal{F} \) is a regular holomorphic foliation of codimension one in \( Y \). Let \( S(f, \mathcal{F}) \) be the set of points where \( f \) fails to be transverse to \( \mathcal{F} \). Suppose \( S(f, \mathcal{F}) \) is given by isolated points and let \( \overline{\mathcal{F}} := f^* \mathcal{F} \). Since \( \mathcal{F} \) is regular, we may find local coordinates in a neighborhood of \( p \in \text{Sing}(f) \) and \( f(p) \) in such a way that \( f = (f_1, \cdots, f_m) \) and \( \overline{\mathcal{F}} \) is given by \( \ker(df_m) \) nearby \( p \). If we pick \( g_i := \frac{\partial g_i}{\partial x_i} \) (i.e., \( df_m = g_1 dx_1 + \cdots + g_n dx_n \)), then

\[
\chi(X) - \sum_{i=1}^{r} f_*(c_{n-i}(TX) \cap [X]) \cap c_1(\mathcal{N}_\mathcal{F})^i = (-1)^n \sum_{p \in S(f, \mathcal{F})} \text{Res}_p \left[ \frac{dg_1 \wedge \cdots \wedge dg_m}{g_1, \cdots, g_m} \right],
\]

where \( \mathcal{N}_\mathcal{F} \) denotes the normal sheaf of \( \mathcal{F} \).

In this paper we generalize the above results for a regular distribution \( \mathcal{F} \) in \( Y \) of any codimension with the following residual formula for the non-transversality point \( s \) of \( \mathcal{F} \) with respect to \( \mathcal{F} \).

In order to state our main result, let us introduce some notions. Let \( f : X \to (Y, \mathcal{F}) \) be a holomorphic map and suppose that \( X \) and \( Y \) are projective manifolds. We say that the set of points in \( X \) where \( f \) fails to be transversal to \( \mathcal{F} \) is the ramification locus of \( f \) with respect to \( \mathcal{F} \), and denote it by \( S(f, \mathcal{F}) \). The set \( R(f, \mathcal{F}) := f(S(f, \mathcal{F})) \) is called the branch locus or the set of branch points of \( f \) with respect to \( \mathcal{F} \). Let \( S(f, \mathcal{F}) = \cup S_j \) be the decomposition of \( S \) into irreducible components, then we denote by \( \mu(f, \mathcal{F}, S_j) \) the multiplicity of \( S_j \) and call it the ramification multiplicity of \( f \) along \( S_j \) with respect to \( \mathcal{F} \). As usual, we denote by \( [W] \) the class in the Chow group of \( X \) of the subvariety \( W \subset X \). The class \( f_*[S_j] = [R_j] \) is called a branch class of \( f \). Observe that \( R(f, \mathcal{F}) \) is the set of tangency points between \( f(X) \) and \( \mathcal{F} \) if \( \dim(X) \leq \dim(Y) \).

**Theorem 1.1.** Let \( f : X \to (Y, \mathcal{F}) \) be a holomorphic map of generic rank \( r \) and \( \mathcal{F} \) a non-singular distribution of codimension \( k \) on \( Y \). Suppose the ramification locus of \( f \) with respect to \( \mathcal{F} \) has codimension \( n - k + 1 \), then

\[
f_*(c_{n-k+1}(TX) \cap [X]) + \sum_{i=1}^{r} (-1)^i f_*(c_{n-k+1-i}(TX) \cap [X]) \cap s_i(\mathcal{N}_\mathcal{F})^i = (-1)^n \sum_{R_j \subset R} \mu(f, \mathcal{F}, S_j)[R_j],
\]

where \( s_i(\mathcal{N}_\mathcal{F}) \) is the \( i \)-th Segre class of \( \mathcal{N}_\mathcal{F} \).

Some consequences of this result are the following.

**Corollary 1.2** (Izawa). If \( k = 1 \), then

\[
\chi(X) - \sum_{i=1}^{r} f_*(c_{n-i}(TX) \cap [X]) \cap c_1(\mathcal{N}_\mathcal{F})^i = (-1)^n \sum_{p \in S(f, \mathcal{F})} \text{Res}_p \left[ \frac{dg_1 \wedge \cdots \wedge dg_m}{g_1, \cdots, g_m} \right].
\]

In fact, if \( k = 1 \) we have \( c_n(T_X) \cap [X] = \chi(X) \) by the Chern-Gauss-Bonnet Theorem. Since \( \mathcal{N}_\mathcal{F} \) is a line bundle, then \( s_i(\mathcal{N}_\mathcal{F})^i = (-1)^i c_1(\mathcal{N}_\mathcal{F})^i \) for all \( i \). The above Izawa’s formula [13] Theorem 4.1] implies the multiplicity formula

\[
\chi(X) - \chi(F) \cdot \chi(C) = (-1)^n \sum_{p \in \text{Sing}(f)} \mu_p(f).
\]

**Corollary 1.3** (Tangency formulae). Let \( X \subset Y \) be a \( k \)-dimensional submanifold generically transverse to a non-singular distribution \( \mathcal{F} \) on \( Y \) of codimension \( k \). Then

\[
[c_1(\mathcal{N}_X|Y) - c_1(T_X)] \cap [X] = \sum_{R_j \subset R} \mu(f, \mathcal{F}, S_j)[R_j].
\]
In particular, if $\det(T_F)|_X - \det(N_{X|Y})$ is ample, then $X$ is tangent to $F$.

If $X = C$ is a curve on a surface $Y$, we have $[C] = c_1(\mathcal{O}(C)) = c_1(N_{X|Y})$. This yields Brunella’s formula

$$c_1(\mathcal{O}(C))^2 - c_1(T_F) \cap c_1(\mathcal{O}(C)) = \sum_{p \in \text{Tang}(F, C)} I_p(F, C).$$

Moreover, this formula coincides with the Honda’s formula \cite{10} in case $F$ is a one-dimensional foliation and $X$ is a curve.

In Section 3, we prove Theorem \cite{11} and Corollary \cite{12}.

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2. Holomorphic distributions

Let $X$ be a complex manifold of dimension $n$.

Definition 2.1. A codimension $k$ distribution $F$ on $X$ is given by an exact sequence

$$F : 0 \rightarrow N^*_F \rightarrow \Omega^k_X \rightarrow \Omega^k_F \rightarrow 0,$$

where $N^*_F$ is a coherent sheaf of rank $k \leq \dim(X) - 1$ and $\Omega^k_F$ is a torsion free sheaf. We say that $F$ is a foliation if at the level of local sections we have $d(N^*_F) \subset N^*_F \wedge \Omega^k_X$. The singular set of the distribution $F$ is defined by $\text{Sing}(F) := \text{Sing}(\Omega^k_F)$. We say that $F$ is regular if $\text{Sing}(F) = \emptyset$.

Taking determinants of the map $N^*_F \rightarrow \Omega^k_X$, we obtain a map:

$$\det(N^*_F) \rightarrow \Omega^k_X,$$

which induces a twisted holomorphic $k$-form $\omega \in H^0(X, \Omega^k_X \otimes \det(N^*_F)^*)$. Therefore, a distribution can be induced by a twisted holomorphic $k$-form $H^0(X, \Omega^k_X \otimes \det(N^*_F)^*)$ which is locally decomposable outside the singular set of $F$. That is, for each point $p \in X \setminus \text{Sing}(F)$ there exists a neighborhood $U$ and holomorphic 1-forms $\omega_1, \ldots, \omega_k \in H^0(U, \Omega^1_U)$ such that

$$\omega|_U = \omega_1 \wedge \cdots \wedge \omega_k.$$

Moreover, if $F$ is a foliation then by Definition 2.1 we have

$$d\omega_1 \wedge \cdots \wedge \omega_k = 0$$

for all $i = 1, \ldots, k$. The tangent sheaf of $F$ is the coherent sheaf of rank $(n - k)$ given by

$$T_F = \{ v \in T_X; i_v \omega = 0 \}.$$

The normal sheaf of $F$ is defined by $N_F = T_X/T_F$. It is worth noting that $N_F \neq (N^*_F)^*$ whenever $\text{Sing}(F) \neq \emptyset$. Dualizing the sequence (1) one obtains the exact sequence

$$0 \rightarrow T_F \rightarrow T_X \rightarrow (N^*_F)^* \rightarrow \text{Ext}^1(\Omega^k_F, \mathcal{O}_X) \rightarrow 0,$$

so that there is an exact sequence

$$0 \rightarrow N_F \rightarrow (N^*_F)^* \rightarrow \text{Ext}^1(\Omega^k_F, \mathcal{O}_X) \rightarrow 0.$$

Definition 2.2. Let $V \subset X$ an analytic subset. We say that $V$ is tangent to $F$ if $T_p V \subset (T_F)_p$, for all $p \in V \setminus \text{Sing}(V)$. 
3. Proof of the main results

We begin by proving the main theorem.

Proof of Theorem 1.1 Consider a map \( f : X \rightarrow Y \) and let \((U, x)\) and \((V, y)\) be local systems of coordinates for \(X\) and \(Y\) such that \(f(U) \subset V\). Since \(F\) is a regular distribution, we may suppose that it is induced on \(U\) by the \(k\)-form \(\omega_1 \wedge \cdots \wedge \omega_k\). Therefore, the ramification locus of \(f\) with respect to \(F\) on \(U\) is given by

\[
S(f, F)|_U = \{ f^*(\omega_1 \wedge \cdots \wedge \omega_k) = f^*(\omega_1) \wedge \cdots \wedge f^*(\omega_k) = 0 \}.
\]

In other words, the ramification locus \(S(f, F)\) coincides with \(\text{Sing}(f^*(F))\).

Let us denote \(\tilde{F} := f^*(F)\). Let \(\{U_\alpha\}\) be a covering of \(Y\) such that the distribution \(F\) is induced on \(U_\alpha\) by the holomorphic 1-forms \(\omega_1^\alpha, \ldots, \omega_k^\alpha\). Hence, on \(U_\alpha \cap U_\beta \neq \emptyset\) we have \((\omega_1^\alpha \wedge \cdots \wedge \omega_k^\alpha) = g_{\alpha\beta}(\omega_1^\beta \wedge \cdots \wedge \omega_k^\beta)\), where \(\{g_{\alpha\beta}\}\) is a cocycle generating the line bundle \(\text{det}(N^*_{\tilde{F}})^*\). Then the distribution \(\tilde{F}\) is induced locally by \(f^*(\omega_1^\alpha), \ldots, f^*(\omega_k^\alpha)\). This shows that \(N^*_{\tilde{F}}\) is locally free. Therefore the singular set of \(\tilde{F}\) is the loci of degeneracy of the induced map

\[
N^*_{\tilde{F}} \rightarrow \Omega^1_X.
\]

By hypothesis, the ramification locus of \(f\) with respect to \(F\), which is given by \(\text{Sing}(\tilde{F})\), has codimension \(n - k + 1\), then it follows from Thom-Porteous formula \([8]\) that

\[
c_{n-k+1}(\Omega^1_X - N^*_{\tilde{F}}) \cap [X] = \sum_j \mu_j[S_j],
\]

where \(\mu_j\) is the multiplicity of the irreducible component \(S_j\). It follows from \(c(\Omega^1_X - N^*_{\tilde{F}}) = c(\Omega^1_X) \cdot s(N^*_{\tilde{F}})\) that

\[
c_{n-k+1}(\Omega^1_X - N^*_{\tilde{F}}) = \sum_{i=0}^{n-k+1} c_{n-k+1-i}(\Omega^1_X) \cap s_i(N^*_{\tilde{F}}),
\]

where \(s_i(N^*_{\tilde{F}})\) is the \(i\)-th Segre classe of \(N^*_{\tilde{F}}\). Since \(X_0 := X - \text{Sing}(\tilde{F})\) is a dense and open subset of \(X\), then by taking the cap product we have

\[
c_{n-k+1}(\Omega^1_X - N^*_{\tilde{F}}) \cap [X] = c_{n-k+1}(\Omega^1_X - N^*_{\tilde{F}}) \cap [X_0] = \sum_{i=0}^{n-k+1} (c_{n-k+1-i}(\Omega^1_X)) \cap [X_0] \cap s_i(f^*N^*_{\tilde{F}}).
\]

It follows from the projection formula that

\[
f_*(c_{n-k+1}(\Omega^1_X - N^*_{\tilde{F}})) \cap [X] = \sum_{i=0}^{n-k+1} f_*(c_{n-k+1-i}(\Omega^1_X) \cap [X]) \cap s_i(N^*_{\tilde{F}}) = \sum_j \mu_j f_*[S_j].
\]

Now, we prove our tangency formulæ as a consequence of the main theorem.

Proof of Corollary 1.3 Let \(i : X \hookrightarrow Y\) be the inclusion map. It follows from Theorem 1.1 that

\[
i_*(c_1(T_X) \cap [X]) - i_*(|[X]) \cap s_1(N^*_{\tilde{F}}) = - \sum_{R_j \subset R} \mu(f_*F, S_j)[R_j].
\]
On the one hand, we have $c_1(T_Y|_X) = c_1(N_{X|Y}) + c_1(T_X)$, and on the other hand, we have $c_1(T_Y|_X) = c_1(T_X|_X) + c_1(N_X|_X)$. Since $s_1(N_X^*) = -c_1(N_X^*) = c_1(N_X|_X)$, we obtain

$$[c_1(N_{X|Y}) - c_1(T_X)] | [X] = \sum_{R_j \subset R} \mu(f, F, S_j)[R_j].$$

Now notice that, by construction, the cycle

$$Z = \sum_{R_j \subset R} \mu(f, F, S_j)[R_j]$$

is an effective divisor on $X$, since $\mu(f, F, S_j) \geq 0$. If the line bundle $\det(T_Y)|_X - \det(N_X|_Y) = -[\det(N_X|_Y) - \det(T_Y)|_X]$ is ample, we obtain

$$0 < -[\det(N_X|_Y) - \det(T_Y)|_X] \cdot C = -Z \cdot C,$$

for all irreducible curve $C \subset X$. If $X$ is not invariant by $F$ and $\det(T_Y)|_X - \det(N_X|_Y)$ is ample, we obtain an absurd. In fact, in this case $Z \cdot C < 0$, contradicting the fact that $Z$ is effective.

4. Examples

4.1. Integrable example. This example is inspired by an example due to Izawa [13]. Consider $Y = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the subvariety $X = F^{-1}(0) \cap g^{-1}(0)$ given by the homogenous equations

$$F(x, y, z) = \sum_{i=0}^{3} x_i^\ell, \quad G(x, y, z) = \sum_{i=0}^{1} x_i y_i,$$

where $([x], [y], [z]) = ((x_0 : x_1 : x_2 : x_3), (y_0 : y_1), (z_0 : z_1)) \in Y$ are homogeneous coordinates. By a straightforward calculation one may verify that $X$ is smooth. In $Y$ we consider the foliation $F$ given by the fibers of the map $\pi : \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ and let $f : X \to Y$ be the inclusion map. We will analyze the branch points of the $f$ with respect to $F$.

A simple but exhaustive calculation shows that there is no branch point in the hypersurface $x_0 = 0$, thus we concentrate in the Zariski open set $x_0 \neq 0$.

The affine charts for $y_0 \neq 0$. In the affine charts for $x_0 \neq 0$ and $y_0 \neq 0$ the equations defining $X$ assume the form

$$1 + x^\ell + y^\ell + z^\ell = 0,$$

$$1 + u x = 0,$$

where $(1 : x : y : z) = (1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} : \frac{x_3}{x_0})$ and $(1 : v) = (1 : \frac{y_1}{y_0})$. This yields the parametrization of $X$ given by

$$x = (-1)^\frac{\ell}{2}(y^\ell + z^\ell + 1)^\frac{1}{2},$$

$$v = (-1)^\frac{\ell}{2}(y^\ell + z^\ell + 1)^{-\frac{1}{2}}.$$

Now, recall that the leaves of $F$ are given by $\{const\} \times C$, hence the tangency points between $X$ and $F$ are the solutions to the equation $du = u_a dy + u_z dz = 0$. Thus the set of tangency points coincides with the solutions of the following system of equations

$$0 = \frac{\partial v}{\partial y} = (-1)^{\frac{\ell}{2}}(y^\ell + z^\ell + 1)^{-\frac{\ell}{2}},$$

$$0 = \frac{\partial v}{\partial z} = (-1)^{\frac{\ell}{2}}(y^\ell + z^\ell + 1)^{-\frac{\ell}{2}}.$$
or in other words
\[
\begin{cases}
  x = (-1)^{\frac{1}{\ell}} \\
y^\ell - 1 = 0 \\
z^\ell - 1 = 0 \\
v = -(-1)^{-\frac{1}{\ell}}
\end{cases}
\]

The solutions to this system of equations are given in terms of homogeneous coordinates by

\[S^{0,0}_k = \{(1 : \alpha_k : 0 : 0)\} \times \{(1 : -1/\alpha_k)\} \times \mathbb{P}^1,\]

where \(\alpha_k = \exp((2k+1)\pi i/\ell), k = 0, \ldots, \ell - 1\). Note that \(S^{0,0}_k\) is a solution with multiplicity \((\ell - 1)^2\) and that these solutions are contained in the codimension 2 variety given by \(x_2 = x_3 = 0\).

The affine chart for \(y_1 \neq 0\). On the other hand in the affine charts for \(x_0 \neq 0\) and \(y_1 \neq 0\) the equations defining \(X\) assume the form

\[1 + x^\ell + y^\ell + z^\ell = 0,\]
\[u + x = 0,
\]
where \((1 : x : y : z) = (1 : \frac{x_1}{x_0} : \frac{y_1}{x_0} : \frac{z_1}{x_0})\) and \((u : 1) = (\frac{u_1}{y_1} : 1)\). This leads to the parametrization of \(X\) given by

\[x = (-1)^{\frac{1}{\ell}}(y^\ell + z^\ell + 1)^{\frac{1}{\ell}} + 1\]
\[u = (-1)^{\frac{1}{\ell}}(y^\ell + z^\ell + 1)^{\frac{1}{\ell}}.\]

Since the leaves of \(\mathcal{F}\) are given by \(\{\text{const}\} \times \mathbb{C}\), then the tangency points between \(X\) and \(\mathcal{F}\) are the solutions to the equation \(du = u_y dy + u_z dz = 0\). Therefore the set of tangency points coincides with the solution to the system of equations

\[0 = \frac{\partial u}{\partial y} = (1)\frac{\ell+1}{\ell} y^{\ell-1}(y^\ell + z^\ell + 1)^{\frac{1}{\ell}}\]
\[0 = \frac{\partial u}{\partial z} = (1)\frac{\ell+1}{\ell} z^{\ell-1}(y^\ell + z^\ell + 1)^{\frac{1}{\ell}}\]

or in other words with the solutions to the system of equations

\[
\begin{cases}
  x = (-1)^{\frac{1}{\ell}} \\
y^\ell - 1 = 0 \\
z^\ell - 1 = 0 \\
u = -(-1)^{\frac{1}{\ell}}
\end{cases}
\]

In homogeneous coordinates the solutions to this system of equations are given by

\[S^{0,1}_k = \{(1 : \alpha_k : 0 : 0)\} \times \{(\alpha_k : 1)\} \times \mathbb{P}^1,\]

where \(\alpha_k = \exp((2k+1)\pi i/\ell), k = 0, \ldots, \ell - 1\). Note that \(S^{0,1}_k\) is a solution with multiplicity \((\ell - 1)^2\) and that this solution is contained in the codimension 2 variety \(x_2 = x_3 = 0\). Notice also that \(S^{0,1}_k = S^{0,0}_k\) for all \(k = 0, \ldots, \ell - 1\).

The residual formula. Consider the projections \(\pi_1 : Y = \mathbb{P}^3 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3, \pi_2 : Y = \mathbb{P}^3 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1, \pi_3 : Y = \mathbb{P}^3 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1\) and \(\rho : Y = \mathbb{P}^3 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1\). As usual, we denote a line bundle on \(Y\) by \(O(a, b, c) := \pi_1^* O_{\mathbb{P}^3}(a) \otimes \pi_2^* O_{\mathbb{P}^1}(b) \otimes \pi_3^* O_{\mathbb{P}^1}(c)\), with \(a, b, c \in \mathbb{Z}\). Now denote \(h_3 = c_1(O(1, 0, 0)), h_{1,1} = c_1(O(0, 1, 0)),\) and \(h_{1,2} = c_1(O(0, 0, 1))\).

Summing up, the set of non-transversal points is given by the following cycle

\[S = \sum_{k=0}^{\ell-1}(\ell - 1)^2 S^{0,0}_k.\]
Since \([S^0_k] = h_3^3 \cdot h_{1,1}\), we concluded that
\[
[S] = (\ell - 1)^2 \sum_{k=0}^{\ell-1} [S^0_k] = \ell(\ell - 1)^2 h_3^3 \cdot h_{1,1}.
\]

Recall that \(n = 3, k = 2, r = 2\), thus the left side of the formula stated in Theorem 1.1 assumes the form
\[
f_* (c_{n-k+1}(T_X) \cap [X]) = \sum_{i=1}^{r} (-1)^i f_* (c_{n-k+1-i}(T_X) \cap [X]) \cap s_i(N_X^r) = c_2(T_X) \cap [X] - c_1(T_X) \cap [X] \cap s_1(N_X^r) + c_0(T_X) \cap [X] \cap s_2(N_X^r) = \{ c_2(T_X) - c_1(T_X) \cap s_1(N_X^r) + s_2(N_X^r) \} \cap [X].
\]

Since the associated line bundles of \(V(x_0^3 + x_1^4 + x_2^4 + x_3^4)\) and \(V(x_0 y_0 + x_1 y_1)\) are \(\mathcal{O}(\ell, 0, 0)\) and \(\mathcal{O}(1, 1, 0)\), respectively, we have the short exact sequence
\[
0 \rightarrow T_X \rightarrow T_Y|_X \rightarrow \mathcal{O}(\ell, 0, 0) \oplus \mathcal{O}(1, 1, 0)|_X \rightarrow 0.
\]

Now let \(h_3 = c_1(\mathcal{O}(0, 0, 0)), h_{1,1} = c_1(\mathcal{O}(0, 1, 0)), h_{1,2} = c_1(\mathcal{O}(0, 0, 1))\), then by the Euler sequence for multiprojective spaces \([9]\), we conclude that
\[
c(T_Y) = (1 + h_3)^4 (1 + h_{1,1})^2 (1 + h_{1,2})^2.
\]

with relations \((h_3)^4 = (h_{1,1})^2 = (h_{1,2})^2 = 0\). Since \(c(\mathcal{O}(\ell, 0, 0) \oplus \mathcal{O}(1, 1, 0)) = (1 + \ell h_3)(1 + h_3 + h_{1,1})\) and
\[
c(T_Y)|_X = c(T_X) \cdot c(\mathcal{O}(\ell, 0, 0) \oplus \mathcal{O}(1, 1, 0)|_X)
\]
it follows that
\[
c_1(T_X) = (3 - \ell)h_3 + h_{1,1} + 2h_{1,2}, \quad c_2(T_X) = (4 - \ell)h_3 h_{1,1} + (6 - 2\ell)h_3 h_{1,2} + (3 - 3\ell + \ell^2)h_3^2 + 2h_{1,1} h_{1,2}.
\]

We calculate the Segre classes \(s_i(N_X^r)\) for \(i = 1, \ldots, r\). Since in our example \(r = 2\), then it is enough to calculate \(s_i(N_X^r), i = 1, 2\). The foliation \(\mathcal{F}\) is the restriction of \(\rho : Y = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1\) to \(X\), then the normal bundle of \(\mathcal{F}\) is
\[
N_{\mathcal{F}} = \rho^*(T_{\mathbb{P}^3} \oplus T_{\mathbb{P}^1})|_X = (\mathcal{O}(0, 2, 0) \oplus \mathcal{O}(0, 0, 2))|_X.
\]
Thus \(N_X^r = (\mathcal{O}(0, -2, 0) \oplus \mathcal{O}(0, 0, -2))|_X\). Since \((h_{1,1})^2 = (h_{1,2})^2 = 0\) we get
\[
s_1(N_X^r) = 2(h_{1,1} + h_{1,2}), \quad s_2(N_X^r) = 4h_{1,1} h_{1,2}.
\]

Observe that
\[
c_1(T_X) \cap s_1(N_X^r) = ((3 - \ell)h_3 + h_{1,1} + 2h_{1,2}) \cdot (2(h_{1,1} + h_{1,2})) = (6 - 2\ell)h_3 h_{1,1} + (6 - 2\ell)h_3 h_{1,2} + 6h_{1,1} h_{1,2}.
\]

Thus
\[
c_2(T_X) - c_1(T_X) \cap s_1(N_X^r) + s_2(N_X^r) = (4 - \ell)h_3 h_{1,1} + (6 - 2\ell)h_3 h_{1,2} + (3 - 3\ell + \ell^2)h_3^2 + 2h_{1,1} h_{1,2} - ((6 - 2\ell)h_3 h_{1,1} + (6 - 2\ell)h_3 h_{1,2} + 6h_{1,1} h_{1,2}) + 4h_{1,1} h_{1,2} = (\ell - 2)h_3 h_{1,1} + (3 - 3\ell + \ell^2)h_3^2.
\]

Moreover, we have
\[
[X] = [V(x_0^4 + x_1^4 + x_2^4 + x_3^4)] \cap [V(x_0 y_0 + x_1 y_1)] = \ell h_3 + h_{1,1} = \ell h_3^2 + \ell h_{3} h_{1,1}.
\]
Thus
\[\{c_2(T_X) - c_1(T_X) \cap s_1(N^*_2) + s_2(N^*_2)\} \cap [X] = [(\ell - 2)h_3h_{1,1} + (3 - 3\ell + \ell^2)h^3] \cdot [h^2_3 + \ell h_3h_{1,1}].\]

Finally, we obtain
\[\{c_2(T_X) - c_1(T_X) \cap s_1(N^*_2) + s_2(N^*_2)\} \cap [X] = \ell(\ell - 2 + 3 - 3\ell + \ell^2)[h^3_3h_{1,1}]
\[= \ell(\ell - 1)^2h^3_3h_{1,1} = [S].\]

4.2. Non-integrable example. Let \(X\) be a complex-projective manifold of dimension \(\dim(X) = 2n + 1\). A contact structure on \(X\) is a regular distribution \(\mathcal{F}\) induced by a twisted 1-form
\[\omega \in H^0(X, \Omega^1_X \otimes L),\]
such that \(\omega \wedge (d\omega)^n \neq 0\) and \(L\) is a holomorphic line bundle. Suppose that the second Betti number of \(X\) is \(h_2(X) = 1\) and that \(X\) is not isomorphic to the projective space \(\mathbb{P}^{2n+1}\). Then it follows from [15] that there exists a compact irreducible component \(H \subset \text{RatCurves}^n(X)\) of the space of rational curves on \(X\) such that the intersection of \(L\) with the curves associated with \(H\) is 1. Moreover, if \(C \subset X\) is a generic element of \(H\), then \(C\) is smooth, tangent to \(\mathcal{F}\), and
\[TX|_C = \mathcal{O}_C(2) \oplus \mathcal{O}_C(1)^{n-1} \oplus \mathcal{O}_C^{n+1},\]
\[TF|_C = \mathcal{O}_C(2) \oplus \mathcal{O}_C(1)^{n-1} \oplus \mathcal{O}_C^{n-1} \oplus \mathcal{O}_C(-1).\]

See [16] Fact 2.2 and Fact 2.3. In particular, we obtain that \(N_{C|X} = \mathcal{O}_C(1)^{n-1} \oplus \mathcal{O}_C^{n+1}\), since \(T_C = \mathcal{O}_C(2)\). Then
\[\det(T_F)|_C - \det(N_{C|X}) = \mathcal{O}_C(1)\]
is ample. Examples of such manifolds are given by homogeneous Fano contact manifolds, cf. [1]. This example satisfies the conditions of Corollary [13].

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