BBGKY DYNAMICS:
FROM LOCALIZATION TO PATTERN FORMATION

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1. Introduction

Kinetic theory is an important part of general statistical physics related to phenomena which cannot be understood on the thermodynamic or fluid models level. First of all we mean (local) fluctuations from equilibrium state and a lot of complex phenomena \(^1\). Also, it is well-known in plasma physics, e.g., that only a kinetic approach can describe Landau damping, intra-beam scattering, while Schottky noise and associated cooling techniques require understanding of the spectrum of local fluctuations of the beam charge density \(^2\). In this paper we consider the applications of a new numerical/analytical technique based on wavelet analysis approach for calculations related to the description of complex behaviour in the framework of the general BBGKY–hierarchy \(^3\). We restrict ourselves to rational/polynomial type of nonlinearities (with respect to the set of all dynamical variables (\(n\)-particle distribution functions in the case under consideration)) which allows us to use our results from Refs. \(^3\)–\(^9\), which are based on the application of wavelet analysis technique and variational formulation of initial nonlinear (pseudodifferential) problems.

Wavelet analysis is a set of mathematical methods which give us a pos-
sibility to work with well-localized bases in functional spaces and provide maximum sparse forms for the general type of operators (differential, integral, pseudodifferential) in such bases. It provides the best possible rates of convergence and minimal complexity of algorithms inside and, as a result, saves CPU time and HDD space. Our main goals are an attempt of classification and construction of possible nontrivial states in the system under consideration. First of all we are interested in the following states: localized, chaotic-like patterns, localized (stable) patterns. We start from the corresponding definitions (at this stage these definitions have only qualitative character).

1. By localized state (localized mode) we mean the corresponding (particular) solution of the system under consideration which is localized in maximally small region of the phase space.
2. By chaotic pattern we mean some solution (or asymptotics of solution) of the system under consideration which has equidistribution of energy spectrum in a full domain of definition.
3. By localized pattern (waveleton) we mean (asymptotically) stable solution localized in relatively small region of the whole phase space (or a domain of definition). In this case all energy is distributed during some time (sufficiently large) between few localized modes (from point 1) only. We believe, it is a good model for a plasma fusion (energy confinement) in the nearest future. It is also obvious that such states are very important in many areas of statistical physics.

In all cases above, by the system under consideration we mean the full BBGKY–hierarchy or some cut-off of it. Our construction of cut-off of the infinite system of equations is based on some criterion of convergence of the full solution. This criterion is based on a natural norm in a proper functional space, which takes into account (non-perturbatively) the underlying multiscale structure of complex statistical dynamics. According to our approach the choice of the underlying functional space is important for understanding the corresponding complex dynamics. It is obvious that we need to fix accurately the space in which we construct the solutions, evaluate convergence etc. and in which the very complicated infinite set of operators acts which appears in the BBGKY formulation. We underline that many concrete features of the complex dynamics are related not only to the concrete form/class of operators/equations but depend also on the proper choice of function spaces, where operators act. It should be noted that the class of smoothness of the functions under consideration plays a key role in the following (more details will be considered elsewhere).

In Sec. 2 the kinetic BBGKY–hierarchy is formulated and an important
particular case is described. In Sec. 3 we present the explicit analytical construction of solutions of the hierarchy, which is based on tensor algebra extensions of bases generated by the hidden multiresolution structure and proper variational formulation leading to an algebraic parametrization of the solutions. We give the explicit representation of the hierarchy of N-particle reduced distribution functions in the basis of highly-localized generalized coherent (regarding the underlying affine group) states given by polynomial tensor algebra of wavelets, which takes into account contributions from all underlying hidden multiscales, from the coarsest scale of resolution to the finest one, to provide full information about stochastic dynamical process. So, our approach resembles Bogolyubov’s and related approaches but we don’t use any perturbation technique (like virial expansion) or linearization procedures. Numerical modeling as in general case as in particular cases of the Vlasov-like equations shows the creation of various internal (coherent) structures from localized modes, which are related to stable (equilibrium) or unstable/chaotic type of behaviour and the corresponding pattern (waveletons) formation.

2. BBGKY-Hierarchy

Let \(M\) be the phase space of an ensemble of \(N\) particles (\(\dim M = 6N\)) with coordinates \(x_i = (q_i, p_i), \quad i = 1, \ldots, N, \quad q_i = (q_{i1}, q_{i2}, q_{i3}) \in \mathbb{R}^3, \quad p_i = (p_{i1}, p_{i2}, p_{i3}) \in \mathbb{R}^3, \quad q = (q_1, \ldots, q_N) \in \mathbb{R}^{3N}\). Individual and collective measures are: \(\mu_i = dx_i = dq_i dp_i, \quad \mu = \prod_{i=1}^N \mu_i\). The distribution function \(D_N(x_1, \ldots, x_N; t)\) satisfies Liouville’s equation of motion and the normalization constraint for an ensemble with the Hamiltonian \(H_N\):

\[
\frac{\partial D_N}{\partial t} = \{H_N, D_N\} \int D_N(x_1, \ldots, x_N; t) d\mu = 1. \quad (1)
\]

Our constructions can be applied to the following general Hamiltonians:

\[
H_N = \sum_{i=1}^N \left( \frac{p_i^2}{2m} + U_i(q_i) \right) + \sum_{1 \leq i \leq j \leq N} U_{ij}(q_i, q_j), \quad (2)
\]

where the potentials \(U_i(q) = U_i(q_1, \ldots, q_N)\) and \(U_{ij}(q_i, q_j)\) are restricted to rational functions of the coordinates.

Let \(L_s\) and \(L_{ij}\) be the Liouvillean operators (vector fields)

\[
L_s = \sum_{j=1}^s \left( \frac{p_j}{m} \frac{\partial}{\partial q_j} - \frac{\partial U_j}{\partial q} \frac{\partial}{\partial p_j} \right) - \sum_{1 \leq i \leq j \leq s} L_{ij}, \quad (3)
\]

\[
L_{ij} = \frac{\partial U_{ij}}{\partial q_i} \frac{\partial}{\partial p_j} + \frac{\partial U_{ij}}{\partial q_j} \frac{\partial}{\partial p_i}. \quad (4)
\]
Let \( F_N(x_1, \ldots, x_N; t) = \sum_{S_N} D_N(x_1, \ldots, x_N; t) \) be the \( N \)-particle distribution function (\( S_N \) is permutation group of \( N \) elements). Then we have the hierarchy of reduced distribution functions (\( V \) is the volume)

\[
F_s(x_1, \ldots, x_s; t) = V_s \int D_N(x_1, \ldots, x_N; t) \prod_{s+1 \leq i \leq N} \mu_i
\] (5)

After standard manipulations we arrive at the BBGKY–hierarchy 1:

\[
\frac{\partial F_s}{\partial t} + L_s F_s = \frac{1}{V} \int d\mu_{s+1} \sum_{i=1}^{s} L_{i,s} F_{s+1}
\] (6)

It should be noted that we may apply our approach even to more general formulation (nonlinear) than (6). For \( s=1,2 \) we have, from the general BBGKY–hierarchy:

\[
\frac{\partial F_1(x_1; t)}{\partial t} + \frac{p_1}{m} \frac{\partial}{\partial q_1} F_1(x_1; t) = \frac{1}{V} \int dx_2 L_{12} F_2(x_1, x_2; t),
\]

\[
\frac{\partial F_2(x_1, x_2; t)}{\partial t} + \left( \frac{p_1}{m} \frac{\partial}{\partial q_1} + \frac{p_2}{m} \frac{\partial}{\partial q_2} - L_{12} \right) F_2(x_1, x_2; t) = \frac{1}{V} \int dx_3 (L_{13} + L_{23}) F_3(x_1, x_2, x_3; t).
\] (7)

In most cases, one is interested in a representation of the form

\[
F_k(x_1, \ldots, x_k; t) = \prod_{i=1}^{k} F_1(x_i; t) + G_k(x_1, \ldots, x_k; t),
\]

where \( G_k \) are correlators. Additional reductions often lead to simplifications, the simplest one, \( G_k = 0 \), corresponding to the Vlasov approximation. Such physically motivated ansatzes for \( F_k \) formally replace the linear (in \( F_k \)) and pseudodifferential (in general case) infinite system (6), (7) by a finite-dimensional but nonlinear system with polynomial nonlinearities (more exactly, multilinearities) \(^{10} \). Our key point in the following consideration is the proper generalization of the perturbative multiscale approach of Bogolyubov.

3. Multiscale Analysis

The infinite hierarchy of distribution functions satisfying system (6) in the thermodynamical limit is:

\[
F = \{ F_0, F_1(x_1; t), F_2(x_1, x_2; t), \ldots, F_N(x_1, \ldots, x_N; t), \ldots \},
\] (8)

where \( F_p(x_1, \ldots, x_p; t) \in H^p, H^0 = R, H^p = L^2(\mathbb{R}^{6p}) \) (or any different proper functional space), \( F \in H^\infty = H^0 \oplus H^1 \oplus \ldots \oplus H^p \oplus \ldots \) with
the natural Fock space like norm (guaranteeing the positivity of the full measure):

\[
(F, F) = F_0^2 + \sum_i \int F_i^2(x_1, \ldots, x_i; t) \prod_{\ell=1}^i \mu_\ell.
\]  

First of all we consider \( F = F(t) \) as a function of time only, \( F \in L^2(R) \), via multiresolution decomposition which naturally and efficiently introduces the infinite sequence of the underlying hidden scales. Because the affine group of translations and dilations generates multiresolution approach, this method resembles the action of a microscope. We have the contribution to the final result from each scale of resolution from the whole infinite scale of spaces. We consider a multiresolution decomposition of \( L^2(R) \) (of course, we may consider any different and proper for some particular case functional space) which is a sequence of increasing closed subspaces \( V_j \in L^2(R) \) (subspaces for modes with fixed dilation value):

\[
\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots
\]

The closed subspace \( V_j(j \in Z) \) corresponds to the level \( j \) of resolution, or to the scale \( j \) and satisfies the following properties: let \( W_j \) be the orthonormal complement of \( V_j \) with respect to \( V_{j+1} \): \( V_{j+1} = V_j \oplus W_j \). Then we have the following decomposition:

\[
\{ F(t) \} = \bigoplus_{\infty > j > -\infty} W_j \quad \text{or} \quad \{ F(t) \} = V_0 \bigoplus_{j=0}^\infty W_j,
\]

in case when \( V_0 \) is the coarsest scale of resolution. The subgroup of translations generates a basis for the fixed scale number: \( \text{span}_{k \in Z} \{ 2^{j/2} \Psi(2^j t - k) \} = W_j \). The whole basis is generated by action of the full affine group:

\[
\text{span}_{k \in Z, j \in Z} \{ 2^{j/2} \Psi(2^j t - k) \} = \text{span}_{k, j \in Z} \{ \Psi_{j, k} \} = \{ F(t) \}.
\]

Let the sequence \( \{ V_j^{x_i} \}, V_j \subset L^2(R) \) correspond to multiresolution analysis on the time axis, \( \{ V_j^{x_i} \} \) correspond to multiresolution analysis for coordinate \( x_i \), then \( V_j^{n+1} = V_j^{x_1} \otimes \ldots \otimes V_j^{x_n} \otimes V_j^{t} \) corresponds to the multiresolution analysis for the \( n \)-particle distribution function \( F_n(x_1, \ldots, x_n; t) \). E.g., for \( n = 2 \): \( V_0^2 = \{ f : f(x_1, x_2) = \sum_{k_1, k_2} a_{k_1, k_2} \phi^2(x_1 - k_1, x_2 - k_2), \ a_{k_1, k_2} \in \ell^2(Z^2) \} \), where \( \phi^2(x_1, x_2) = \phi^1(x_1) \phi^2(x_2) = \phi^1 \otimes \phi^2(x_1, x_2) \), and \( \phi^j(x_i) \equiv \phi(x_i) \) form a multiresolution basis corresponding to \( \{ V_j^{x_i} \} \). If \( \{ \phi^1(x_i - t) \}, \ t \in Z \) form an orthonormal set, then \( \phi^2(x_i - k_1, x_2 - k_2) \) form an orthonormal basis for \( V_0^2 \). So, the action of the affine group generates multiresolution representation of \( L^2(R^2) \). After introducing the detail
spaces $W_j^2$, we have, e.g. $V_1^2 = V_0^2 \oplus W_0^2$. Then the 3-component basis for $W_2^2$ is generated by the translations of three functions

\[ \Psi_1^2 = \phi^2(x_1) \otimes \Psi^2(x_2), \quad \Psi_2^2 = \Psi^1(x_1) \otimes \phi^2(x_2), \quad \Psi_3^2 = \Psi^1(x_1) \otimes \Psi^2(x_2). \]

Also, we may use the rectangle lattice of scales and one-dimensional wavelet decomposition:

\[ f(x_1, x_2) = \sum_{i, j, k} \langle f, \Psi_{i,j} \otimes \Psi_{j,k} \rangle \Psi_{j,k}(x_1, x_2), \]

where the basis functions $\Psi_{i,j} \otimes \Psi_{j,k}$ depend on two scales $2^{-i}$ and $2^{-j}$.

After constructing the multidimensional basis we may apply one of the variational procedures from Refs. 3–9. We obtain our multiscale/multiresolution representations (formulae (17) below) via the variational wavelet approach for the following formal representation of the BBGKY system (6) (or its finite-dimensional nonlinear approximation for the $n$-particle distribution functions) with the corresponding obvious constraints on the distribution functions.

Let $L$ be an arbitrary (non)linear differential/integral operator with matrix dimension $d$ (finite or infinite), which acts on some set of functions from $L^2(\Omega^n)$:

\[ L \Psi \equiv L(Q, t, x_i)\Psi(t, x_i) = 0, \]

where

\[ Q \equiv Q_{i_0,i_1,i_2,...}(t, x_1, x_2, \ldots, \partial/\partial t, \partial/\partial x_1, \partial/\partial x_2, \ldots, \int \mu_k) = \sum_{i_0,i_1,i_2,...=1}^{d_{i_0,i_1,i_2,...}} \int q_{i_{0,i_1,i_2}}(t, x_1, x_2, \ldots) \left( \frac{\partial}{\partial t} \right)^{i_0} \left( \frac{\partial}{\partial x_1} \right)^{i_1} \left( \frac{\partial}{\partial x_2} \right)^{i_2} \ldots \int \mu_k. \]

Let us consider now the $N$ mode approximation for the solution as the following ansatz:

\[ \Psi^N(t, x_1, x_2, \ldots) = \sum_{i_{0,i_1,i_2,...}=1}^{N} a_{i_{0,i_1,i_2}} A_{i_0} \otimes B_{i_1} \otimes C_{i_2} \ldots (t, x_1, x_2, \ldots). \]

We shall determine the expansion coefficients from the following conditions (different related variational approaches are considered in 3–9):

\[ \hat{c}_{k_0,k_1,k_2,...}^N \equiv \int (L \Psi^N)A_{k_0} (t)B_{k_1}(x_1)C_{k_2}(x_2)dt dx_1 dx_2 \ldots = 0. \]
Thus, we have exactly $\mathcal{D}N^n$ algebraical equations for $\mathcal{D}N^n$ unknowns $a_{i_0,i_1,...}$. This variational approach reduces the initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second. We consider the multiresolution expansion as the second main part of our construction. So, the solution is parametrized by the solutions of two sets of reduced algebraical problems, one is linear or nonlinear (depending on the structure of the operator $L$) and the rest are linear problems related to the computation of the coefficients of the algebraic equations (16). These coefficients can be found by some wavelet methods by using the compactly supported wavelet basis functions for the expansions (15). As a result the solution of the equations (6) has the following multiscale or multiresolution decomposition via nonlinear high-localized eigenmodes

$$F(t, x_1, x_2, \ldots) = \sum_{(i,j) \in \mathbb{Z}^2} a_{ij} U^i \otimes V^j(t, x_1, x_2, \ldots),$$

$$V^j(t) = V_{N}^{j,\text{slow}}(t) + \sum_{l \geq N} V_{l}^{j}(\omega_l t), \quad \omega_l \sim 2^l,$$

$$U^i(x_s) = U_{M}^{i,\text{slow}}(x_s) + \sum_{m \geq M} U_{m}^{i}(k^s_m x_s), \quad k^s_m \sim 2^m,$$

which corresponds to the full multiresolution expansion in all underlying time/space scales. The formulae (17) give the expansion into a slow part $\Psi_{N,M}^{\text{slow}}$ and fast oscillating parts for arbitrary $N, M$. So, we may move from the coarse scales of resolution to the finest ones for obtaining more detailed information about the dynamical process. In this way one obtains contributions to the full solution from each scale of resolution or each time/space scale or from each nonlinear eigenmode. It should be noted that such representations give the best possible localization properties in the corresponding (phase)space/time coordinates. Formulae (17) do not use perturbation techniques or linearization procedures. Numerical calculations are based on compactly supported wavelets and related wavelet families and on evaluation of the accuracy on the level $N$ of the corresponding cut-off of the full system (6) regarding norm (9):

$$\|F^{N+1} - F^N\| \leq \varepsilon.$$  

(18)

To summarize, the key points are:

1. The ansatz-oriented choice of the (multidimensional) basis related to some polynomial tensor algebra.

2. The choice of a proper variational principle. A few projection (Galerkin-like) principles for constructing (weak) solutions are considered in $3^{-9}$.
The advantages of formulations related to biorthogonal (wavelet) decomposition should be noted.

3. The choice of basis functions in the scale spaces $W_j$. They correspond to highly-localized (nonlinear) oscillations/excitations, nontrivial local (stable) distributions/fluctuations, etc. Besides fast convergence properties we note the minimal complexity of all underlying calculations, especially by choosing wavelet packets which minimize Shannon’s entropy.

4. Operator representations providing maximum sparse representations for arbitrary (pseudo) differential/integral operators $d f / dx$, $d^n f / dx^n$, $\int T(x, y) f(y) dy$, etc. 10.

5. (Multi)linearization. Besides the variation approach we can consider also a different method to deal with (polynomial) nonlinearities: paraproducts-like decompositions 10,9.

4. Example: Vlasov equation

As a particular case we consider the Vlasov approximations $F_2 = F_1 F_1$, $F_3(x_1, x_2, x_3) = \sum S_3 F_1(x_1) F_2(x_j, x_k)$ in Eqs. (7), which are important in plasma physics.

This is a particular case of the general form (13) (for simplicity we consider only one variable)

$$Q(R, x) \Psi(x) = P(R, x) \Psi(x) \quad \text{or} \quad L \Psi \equiv L(R, x) \Psi(x) = 0,$$

where $R \equiv R(x, \partial / \partial x, \Psi)$ is not more than a rational (operator) function. We have the following representation for the $N$ mode approximation for the solution of the Vlasov equation via expansion in some high-localized wavelet-like basis (other independent variables are considered analogously):

$$\Psi^N(x) = \sum_{r=1}^{N} a_r \phi_r(x).$$

We shall determine the expansion coefficients from the following variational conditions:

$$L^N_k \equiv \int (L \Psi^N(x)) \phi_k(x) dx = 0.$$

We have exactly $dN$ algebraical equations for $dN$ unknowns $a_r$. So, the variational approach reduces the initial problem (7) to the problem of the solution of functional equations at the first stage and some algebraical problems at the second stage. As a result we have the following reduced algebraical system of equations (RSAE) on the set of unknown coefficients
\(a_i^N\) of the expansion (20):

\[
H(Q_{ij}, a_i^N, \alpha_I) = M(P_{ij}, a_i^N, \beta_J),
\]

where the operators \(H\) and \(M\) are algebraizations of the RHS and LHS of the initial problem (19). \(Q_{ij}\) \((P_{ij})\) are the coefficients of LHS (RHS) of the initial system of differential equations (19) and, as consequence, are the coefficients of the RSAE. \(I = (i_1, ..., i_{q+2})\), \(J = (j_1, ..., j_{p+1})\) are multiindexes labeling \(\alpha_I\) and \(\beta_J\), the other coefficients of (22) are

\[
\beta_J = \{\beta_{j_1}...\beta_{j_{p+1}}\} = \int \prod_{1 \leq j_k \leq p+1} \phi_{j_k},
\]

where \(p\) is the degree of the polynomial operator \(P\),

\[
\alpha_I = \{\alpha_{i_1}...\alpha_{i_{q+2}}\} = \sum \int \phi_{i_1}...\phi_{i_{q+2}},
\]
and $q$ is the degree of the polynomial operator $Q$, $i_\ell = (1, \ldots, q + 2)$, $\dot{\phi}_i = d\phi_i/dx$.

We may extend our approach to the case when there are additional constraints on the set of dynamical variables and additional averaged terms also. In these cases, by using the method of Lagrangian multipliers one again may apply the same approach, but for the extended set of variables. As a result, one obtains the expanded system of algebraical equations, analogous to the system (22). Then, one again can extract from its solution the coefficients of expansion (20). It should be noted that if one considers only the truncated expansion (22) with $N$ terms one has a system of $N \times d$ algebraical equations of degree $\ell = \max\{p, q\}$ which coincides with the degree of the initial system.

5. Demonstration of typical patterns

Formulae (15), (17), (20) provide, in principle, a fast convergent decomposition for the general solutions of the systems (6), (7) in terms of contributions from all underlying hidden internal scales. Of course, we cannot guarantee that each concrete system (6), (7) with fixed coefficients will have a priori a specific type of behaviour, either localized or chaotic. Instead, we can analyze if typical structures described by definitions 1-3 are present. To classify the qualitative behaviour we apply standard methods from general control theory or really use the control\(^9\). We will start from a priori unknown coefficients, the exact values of which will subsequently be recovered. Roughly speaking, we will fix only class of nonlinearity (polynomial in our case) which covers a broad variety of examples of systems (6), (7).
As a simple model we choose band-triangular non-sparse matrices \((a_{ij})\) from (17) or (20) in particular cases. These matrices provide tensor structure of bases in (extended) phase space and are generated by the roots of the reduced variational (Galerkin-like) systems (16) or (21), (22). As a second step we need to restore the coefficients of (6), (7) from these matrices by which we may classify the types of behaviour. We start with the localized mode, which is an “elementary” eigenfunction, Fig. 1, corresponding to def. 1, which was constructed as a tensor product of the two Daubechies functions. Fig. 2, corresponding to def. 2, presents the result of summation of series (17) up to value of the dilation/scale parameter equal to six on the bases of symmlets \(^{10}\) with the corresponding matrix elements equal to one. The size of matrix is 512x512 and as a result we provide modeling for one-particle distribution function corresponding to standard Vlasov-like system (7) with \(F_2 = F_1^2\). So, different possible distributions of the root values of the generical algebraical system (22) provide qualitatively different types of behaviour. The above choice provides us by chaotic-like equidistributional distribution. But, if we consider a band-like structure of matrix \((a_{ij})\) with the band along the main diagonal with finite size \((\ll 512)\) and values, e.g. five, while the other values are equal to one, we obtain localization in a fixed finite area of the full phase space, i.e. almost all energy of the system is concentrated in this small volume. This corresponds to definition 3 and is shown in Fig. 3, constructed by means of Daubechies-based wavelet packets. Depending on the type of solution, such localization may be present during the whole time evolution (asymptotically-stable) or up to the needed value from time scale (e.g. enough for plasma fusion/confinement).

Now we discuss how to solve the inverse/synthesis problem or how to restore the coefficients of the initial systems (6), (7). Let

\[
L^0(Q^0)\Psi^0 = 0
\]

be the system (13) with the fixed coefficients \(Q^0\). The corresponding solution \(\Psi^0\) is represented by formulae (15), (17) or (20), which are parametrized by roots of reduced algebraic system (22) and constructed by some choice of the tensor product bases from part 1. The proper counterpart of the system (25) with prescribed behaviour \(\Psi^u\), corresponding to a given choice of both tensor product structure and coefficients \(\{a_{ij}\}\) described above, corresponds to the class of systems like (13) but with undetermined coefficients \(Q^u\) and has the same form

\[
L^u(Q^u)\Psi^u = 0.
\]

Our goal is to restore coefficients \(Q^u\) from (25), (26) and explicit representations for solutions \(\Psi^0\) and \(\Psi^u\). This is a standard problem in the adaptive
control theory: one adds a controlling signal $u(x,t)$ which deforms the controlled signal $\Psi(x,t)$ from the fixed state $\Psi^0(x,t)$ to the prescribed one $\Psi^u(x,t)$. At the same time one can determine the parameters $Q^u$. Finally, we apply two variational constructions. The first one gives the systems of algebraic equations for unknown coefficients, generated by the following set of functionals

$$
\Phi_N = \int \left( (L^0 - L^u)\Psi_N^u, \Psi_N^0 \right) d\mu_N,
$$

where $N$ means the $N$-order approximation according to formulae (15). The unknown parameters $Q^*$ are given by $Q^* = \lim_{N \to \infty} Q_N^x$. The second is an important additional constraint on the region $\mu_0$ in the phase space where we are interested in localization of almost all energy $E = \int H(\Psi^u)d\mu$, where $E$ is the proper energy functional (Marsden-like).

We believe that the appearance of nontrivial localized patterns observed by these simple methods is a general effect which is also present in the full BBGKY–hierarchy, due to its complicated intrinsic multiscale dynamics and it depends on neither the cut-off level nor the phenomenological-like hypothesis on correlators. So, representations like (17) and the prediction of the existence of the (asymptotically) stable localized patterns/states (energy confinement states) in BBGKY-like systems are the main results of this paper.

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