PAIRWISE INTERSECTING HOMOTHETS OF A CONVEX BODY

A. POLYANSKII

Abstract. We show that the maximum number of pairwise intersecting positive homothets of a $d$-dimensional centrally symmetric convex body, none of which contains the center of another in its interior, is at most $3^d + 1$. Also, we improve upper bounds for cardinalities of $k$-distance sets in Minkowski spaces.

1. Introduction

A convex body $K$ in the $d$-dimensional Euclidean space $\mathbb{R}^d$ is a compact convex set with non-empty interior, and it is o-symmetric if $K = -K$. A homothet of $K$ is a set of the form $v + \lambda K := \{v + \lambda k : k \in K\}$, where $\lambda \in \mathbb{R}$ is the homothety ratio, and $v \in \mathbb{R}^d$ is a translation vector. A homothet of $K$ is called positive if its homothety ratio is positive. We will consider only positive homothets of o-symmetric bodies here, and thus we will omit the word "positive" most of the time. Also, we write $\left\lfloor n \right\rfloor$ for the set $\{1,2,\ldots,n\}$, $\text{dist}(h_1,h_2)$ for the Euclidean distance between two parallel hyperplanes $h_1$ and $h_2$, $\dim(h)$ for the dimension of a flat $h$. $\text{conv}(A)$, $\text{aff}(A)$, $\text{vol}(A)$ and $\partial A$ stand for the convex hull, the affine hull, the volume and the boundary of a set $A \subset \mathbb{R}^d$ respectively.

A Minkowski arrangement of an o-symmetric convex body $K$ is called a family $\{v_i + \lambda_i K\}$ of positive homothets of $K$ such that none of the homothets contains the center of any other homothet in its interior (see [7]). We write $\kappa(K)$ for the largest number of homothets that a pairwise intersecting Minkowski arrangement of $K$ can have. Z. Füredi and P.A. Loeb [2] proved that $\kappa(K) \leq 5^d$. Recently, M. Naszódi, J. Pach and K. Swanepoel [4] improved this result to $\kappa(K) \leq O(3^d d \log d)$. The authors of [4] noted that it is obvious that for the $d$-dimensional cube $C^d$ we have $\kappa(C^d) = 3^d$. We prove the following upper bound for $\kappa(K)$, which is sharp up to the constant factor.

Theorem 1. For any $d$-dimensional o-symmetric convex body $K$, 

$$\kappa(K) \leq 3^d + 1.$$ 

Also, some generalization of a Minkowski arrangement for non-symmetric bodies (the role of the center is played by an arbitrary interior point) was studied in [4]. Unfortunately, it is impossible to generalize our approach for non-symmetric bodies.

We call a subset $S$ of a metric space a $k$-distance set if the set of non-zero distances occurring between points of $S$ is of size at most $k$. A 1-distance set is called an equilateral set. For $d$-dimensional Minkowski spaces it is well known that the maximal cardinality of an equilateral (that is, a 1-distance) set is $2^d$ with equality iff the unit ball of the space is a parallelootope, see [6]. K. Swanepoel [8] proved that if the unit ball of a $d$-dimensional Minkowski space is a parallelootope then a $k$-distance set has cardinality at most $(k + 1)^d$, where the bound is tight. Therefore, he [8] conjectured that the maximal cardinality of $k$-distance sets in Minkowski spaces is $(k + 1)^d$. Also, it was proved in [8] that the cardinality of a $k$-distance set in a $d$-dimensional Minkowski space is at most $\min\{2^{kd}, (k+1)^{(11d-9d)/2}\}$. Moreover, the last bound was recently replaced by $(k+1)^{5d+o(d)}$, see [9]. Our second result is the following improvement.

Date: March 9, 2022.
Theorem 2. The cardinality of a k-distance set \((k > 1)\) in a \(d\)-dimensional Minkowski space is at most \(k^{O(d^4)}\), where the constant in \(O(\cdot)\) does not depend on \(d\) and \(k\).

Our proof is based on Theorem 3 which seems to be of independent interest.

Theorem 3. Assume that \(v_1, v_2, \ldots, v_n\) are points in a \(d\)-dimensional Minkowski space with an \(o\)-symmetric convex body \(K\) as the unit ball, such that \(\|v_i - v_j\|_K = \lambda_i\) for any \(1 \leq i < j \leq n\), where \(\lambda_i, i \in [n-1]\), are some positive numbers. Then

\[
n \leq d \left(1 + \frac{2}{2 - 2^{1/(d-1)}}\right)^{d+1} = O(3^d d).
\]

It is important to note that M. Naszódi, J. Pach and K. Swanepoel [4] proved that if the conditions of Theorem 3 hold then \(n = O(6^d(d \log d)^2)\).

For more links dealing with \(k\)-distance sets we refer the interested readers to [8, 9].

One of the main ingredients of the proofs of Theorems 1 and 3 is the following simple lemma which is a generalization of the well-known Danzer-Grünbaum Theorem about the maximal cardinality antipodal sets, i.e. such sets that satisfy conditions of Lemma 1 when \(\lambda = 1\) (see [1] and also Lemma 7 in [4]).

Lemma 1. Suppose that \(\lambda \geq 1\) is a real number and \(X = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d\) is a set of points such that for any \(i \neq j \in [n]\) there are two distinct parallel hyperplanes \(k_{i,j}\) and \(k_{j,i}\) with \(X \subset \text{conv}(k_{i,j}, k_{j,i})\) and

\[
\frac{\text{dist}(k_{i,j}, k_{j,i})}{\text{dist}(g_{i,j}, g_{j,i})} \leq \lambda,
\]

where \(g_{i,j}\) and \(g_{j,i}\) are hyperplanes passing through \(x_i\) and \(x_j\), respectively and parallel to \(k_{i,j}\) (and \(k_{j,i}\)). Then \(n \leq (1 + \lambda)^d\).

Another key tool in our proofs is the lifting method developed in [5] (see also [3]), where M. Naszódi showed that the maximal number of pairwise touching positive homothets of a convex body \(K\) that is not necessary \(o\)-symmetric is at most \(2^{d+1}\). We develop this method further by new ideas.

The article is organized in the following way. In Section 2.1 we prove Lemma 1. In Section 2.2 we discuss some properties of a set of pairwise intersecting homothets, which we will use in Sections 3.1 and 3.2 where we present the proofs of Theorem 1 and Theorem 3 respectively. In Section 3.3 we prove Theorem 2 using Theorem 3.

2. Auxiliary Lemmas

2.1. Proof of Lemma 1. We may clearly assume that \(P := \text{conv}(X)\) is a \(d\)-dimensional polytope in \(\mathbb{R}^d\), otherwise \(d_1 := \dim(\text{aff}(P)) < d\), i.e. by induction hypothesis, we have \(n \leq (1 + \lambda)^{d_1} < (1 + \lambda)^d\). It is easy to see that \(P_i = x_i + \frac{1}{1+\lambda}(P - x_i) \subset P\). Without loss of generality we assume that \(x_i\) is closer to \(k_{i,j}\) than \(x_j\). We claim that \(P_i\) and \(P_j\) do not share a common interior point. Indeed, \(P_i \subset \text{conv}(k_{i,j} \cup l_{i,j})\), \(P_j \subset \text{conv}(k_{j,i} \cup l_{j,i})\), where \(l_{i,j} = x_i + \frac{1}{1+\lambda}(k_{j,i} - x_i)\), \(l_{j,i} = x_j + \frac{1}{1+\lambda}(k_{i,j} - x_j)\). Note that \(\text{conv}(k_{i,j} \cup l_{i,j})\) and \(\text{conv}(k_{j,i} \cup l_{j,i})\) do not have a common interior point because

\[
\text{dist}(k_{i,j}, l_{i,j}) + \text{dist}(k_{j,i}, l_{j,i}) =
\]

\[
= \text{dist}(k_{i,j}, g_{i,j}) + \frac{1}{1+\lambda} \text{dist}(g_{i,j}, k_{j,i}) + \text{dist}(k_{j,i}, g_{j,i}) + \frac{1}{1+\lambda} \text{dist}(g_{j,i}, k_{i,j}) =
\]

\[
= \text{dist}(k_{i,j}, k_{j,i}) - \text{dist}(g_{i,j}, g_{j,i}) + \frac{1}{1+\lambda} \text{dist}(k_{j,i}, k_{j,i}) + \frac{1}{1+\lambda} \text{dist}(g_{j,i}, g_{j,i}) \leq \text{dist}(k_{i,j}, k_{j,i}).
\]

The last inequality holds because of (11). Therefore, \(\sum_{i=1}^n \text{vol}(P_i) \leq \text{vol}(P)\), i.e. \(\frac{n}{(1+\lambda)^d} \text{vol}(P) \leq \text{vol}(P), n \leq (1 + \lambda)^d\). Lemma 1 is proved.
2.2. Properties of pairwise intersecting homothets. Throughout Section 2.2, \( \ell(x, y) \) denotes the line passing through points \( x \) and \( y \), \( \ell(x, l) \) and \( h(x, h) \) stand for the line and the \( k \)-dimensional flat passing through a point \( x \) and parallel to a line \( l \) and to a \( k \)-dimensional flat \( h \) respectively, we write \([x, y]\) for the segment with endpoints \( x \) and \( y \), \( \triangle(x, y, z) \) denotes the triangle with vertices \( x, y, z \). We write \( \Delta(x_1, y_1, z_1) \sim \Delta(x_2, y_2, z_2) \) if the triangles \( \Delta(x_1, y_1, z_1) \) and \( \Delta(x_2, y_2, z_2) \) are similar. \((x_1, x_2; x_3, x_4)\) stands for the cross-ratio of points \( x_1, x_2, x_3, x_4 \) on the real line, i.e.

\[
(x_1, x_2; x_3, x_4) = \frac{x_1 - x_3}{x_2 - x_3} : \frac{x_1 - x_4}{x_2 - x_4},
\]

where \( x_1, x_2, x_3, x_4 \) are coordinates of the points \( x_1, x_2, x_3, x_4 \) respectively. If one of the points is the point at infinity then the two distances involving that point are dropped from the formula. Also, we will use the fact that if \( p : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a projective transformation and distinct points \( x_1, x_2, x_3, x_4 \in \mathbb{R}^d \) are collinear then \( p(x_1), p(x_2), p(x_3), p(x_4) \) are also collinear and

\[
(p(x_1), p(x_2); p(x_3), p(x_4)) = (x_1, x_2; x_3, x_4).
\]

Let us identify \( \mathbb{R}^d \) with the \( d \)-dimensional flat

\[
h := \{(x_1, \ldots, x_{d+2}) \in \mathbb{R}^{d+2} : x_{d+1} = 0, x_{d+2} = 1\} \quad \text{in} \quad \mathbb{R}^{d+2}
\]

and consider the following hyperplanes

\[
h_0 := \{(x_1, \ldots, x_{d+2}) \in \mathbb{R}^{d+2} : x_{d+2} = 1\} \quad \text{in} \quad \mathbb{R}^{d+2},
\]

\[
h_1 := \{(x_1, \ldots, x_{d+2}) \in \mathbb{R}^{d+2} : x_{d+1} = 1\} \quad \text{in} \quad \mathbb{R}^{d+2}.
\]

Note that \( h \subset h_0 \). Let \( \{e_i : i \in [d+2]\} \) be the standard basis of \( \mathbb{R}^{d+2} \).

Let \( K \) be an \( o \)-symmetric \( d \)-dimensional convex body such that

\[
K \subset h' := \{(x_1, \ldots, x_{d+2}) \in \mathbb{R}^{d+2} : x_{d+1} = 0, x_{d+2} = 0\}.
\]

Note that the \( d \)-dimensional flat \( h' \) is parallel to the \( d \)-dimensional flat \( h \). Suppose that \( \{v_i : i \in [n]\} \subset h \) is a set of \( n \) distinct vectors, \( \{\lambda_i : i \in [n]\} \subset \mathbb{R}^+ \) is a set of \( n \) positive scalars and \( \{v_i + \lambda_i K : i \in [n]\} \subset h \) is a finite family of pairwise intersecting positive homothets of \( K \).

Section 2.2 is organized in the following way. First, we define the set \( X_0 := \{x_i := v_i + \lambda_i e_{d+1} : i \in [n]\} \subset h_0 \) of \( n \) points and prove some properties of \( X_0 \). Second, we apply on \( X_0 \) the central projection \( \text{pr} : h_0 \rightarrow h_1 \) from the origin of \( \mathbb{R}^{d+2} \) onto the hyperplane \( h_1 \). Finally, we check that the image \( X_1 := \{y_i := \text{pr}(x_i) : i \in [n]\} \subset h_1 \) of \( X_0 \) satisfies some properties.

![Figure 1. Plane π](image)
Choose \( i \neq j \in [n] \). Write \( r := r_{i,j} := \ell(v_i, v_j) \subset h \) and let \( r := r_{i,j} \) and \(-r\) be the points of intersection of \( \partial K \) and \( \ell(o, r) \), here we assume that the vectors \( r \) and \( v_j - v_i \) have the same direction (see Figure 1). Denote by \( f := f_{i,j} \) a supporting hyperplane of \( v_i + \lambda_i K \) in \( h \) passing through \( v_i + \lambda_i r \), i.e. \( f \) is a \((d - 1)\)-dimensional flat.

Let \( v'_k := v'_{k,i,j}, x'_k := x'_{k,i,j} \) and \( t_k := t_{k,i,j} := \{v'_k - \lambda_k r, v'_k + \lambda_k r\} \subset r \) be the projections of \( v_k, x_k \) and \( v_k + \lambda_k K \) in the direction of \( f \) onto the two-dimensional plane \( \pi \), where \( \pi := \pi_{i,j} \) passes through \( v_i, v_j, x_i \) and \( x_j \) (see Figure 1). It follows immediately that \( v_i = v'_i, v_j = v'_j, x_i = x'_i, x_j = x'_j, x'_k = v'_k + \lambda_k e_{d+1}, t_i = [v_i - \lambda_i r, v_i + \lambda_i r] \) and \( t_j = [v_j - \lambda_j r, v_j + \lambda_j r] \).

We claim that the segments \( t_k \) share a common point, which we will denote as \( x := x_{i,j} \). Indeed, any two segments \( t_p \) and \( t_q \) share a common point otherwise \( \{v_p + \lambda_p K\} \) and \( \{v_q + \lambda_q K\} \) do not intersect each other. Therefore, by Helly’s theorem for \( \mathbb{R} \), we get that \( t_k \) have a common point \( x \).

Let \( u_i := u_{i,j} \) and \( u_j := u_{j,i} \) be the real numbers such that

\[
x - v_i = u_i r \quad \text{and} \quad v_j - x = u_j r.
\]

Set (see Figure 1)

\[
a_i := a_{i,j} := \ell(v_i + \lambda_i r, x_i), \quad a_j := a_{j,i} := \ell(v_j - \lambda_j r, x_j),
\]

\[
b_i := b_{i,j} := \ell(x, a_i), \quad b_j := b_{j,i} := \ell(x, a_j),
\]

\[
f_0 := f_{0,i,j} := h(x, f), \quad B_i := B_{i,j} := \text{aff}(b_i \cup f_0), \quad B_j := B_{j,i} := \text{aff}(b_j \cup f_0).
\]

Note that the set \( X_0 \) lies in the wedge formed by \( B_i \) and \( B_j \) in \( h_0 \) that lies in the halfspace \( \{(x_1, \ldots, x_{d+2}) \in \mathbb{R}^{d+2} : x_{d+1} \geq 0\} \). Indeed, points \( x'_k \) lie in the angle formed by \( b_i \) and \( b_j \) that lies in the halfspace \( \{(x_1, \ldots, x_{d+2}) \in \mathbb{R}^{d+2} : x_{d+1} \geq 0\} \) (see Figure 1). Since \( x'_k \) are the projections of \( x_k \) in the direction of \( f \) onto the plane \( \pi \), the points \( x_k \) lie in the corresponding wedge formed by \( B_i \) and \( B_j \).

Next, we apply the central projection \( \text{pr} : h_0 \to h_1 \) from the origin of \( \mathbb{R}^{d+2} \) onto the hyperplane \( h_1 \). The image of \( h \) is the ”hyperplane at infinity” in \( h_1 \). Therefore, we proved the following lemma.

**Lemma 2.** \( k_{i,j} := \text{pr}(B_i) \) and \( k_{j,i} := \text{pr}(B_j) \) are parallel hyperplanes in \( h_1 \) and \( X_1 = \text{pr}(X_0) \) lies in the slab \( \text{conv}(k_{i,j} \cup k_{j,i}) \).
Denote by $z_i := z_{i,j}$ and $z_j := z_{j,i}$, the points of intersection of $r_0 := r_{0,i,j} = \ell(x_i, x_j)$ with $B_i$ (or $B_j$) and $B_i$ (or $B_j$) respectively (see Figure 2). Recall that $y_k = pr(x_k)$. Let $s_i := s_{i,j} = pr(z_i)$, $s_j := s_{j,i} = pr(z_j)$. Of course, $s_i$ and $s_j$ are the points of intersection of $\ell(y_i, y_j)$ with $k_{i,j}$ and $k_{j,i}$ respectively because central projections preserve lines. Denote by $g_{i,j}$ and $g_{j,i}$ the hyperplanes in $h_1$ that are parallel to $k_{i,j}$ and $k_{j,i}$ and pass through $y_i$ and $y_j$ respectively.

Lemma 3. We have

$$\frac{\text{dist}(k_{i,j}, k_{j,i})}{\text{dist}(g_{i,j}, g_{j,i})} = \frac{\|s_i - s_j\|}{\|y_i - y_j\|} = \frac{2\lambda_i \lambda_j}{\lambda_i u_j + \lambda_j u_i}.$$  

Proof. Denote by $c$ the point of intersection $r_0$ with $r$, where, if $r_0$ and $r$ are parallel, then we consider $c$ as the corresponding point at infinity. Let $c' := pr(c)$. Since $c \in h$, the point $c'$ is a point at infinity. Without loss of generality we assume that points on the line $r_0$ lie in the following order: $z_i, x_i, x_j, z_j$. Denote by $w_i$ and $w_j$ the orthogonal projections of $z_i$ and $z_j$ onto the line $r$ respectively. Note that points on the line $r$ lie in the following order: $w_i, v_i, v_j, w_j$. Moreover, $x$ must lie between $v_j - \lambda_j r$ and $v_i + \lambda_i r$.

Using the fact that $c'$ is a point at infinity, $\{s_i, y_i, y_j, s_j\} = pr(\{z_i, x_i, x_j, z_j, c\})$ and $pr_0(\{z_i, x_i, x_j, z_j, c\}) = \{w_i, v_i, v_j, w_j, c\}$, where $pr_0 : r_0 \rightarrow r$ is the orthogonal projection onto the line $r$, we easily get

$$\frac{\|s_i - s_j\|}{\|y_i - y_j\|} = \frac{(s_i, y_i; s_j, c') \cdot (s_j, y_j; y_i; c')}{(z_i, x_i; z_j, c) \cdot (z_j, x_j; x_i, c)} = (w_i, v_i; w_j, c) \cdot (w_j, v_j; v_i, c) = \frac{\|w_i - w_j\|}{\|v_i - v_j\|} \cdot \frac{\|v_i - c\|}{\|w_j - c\|} \cdot \frac{\|v_j - c\|}{\|w_i - c\|}. \quad (3)$$

If $c$ is not a point at infinity then using $\Delta(c, v_i, x_i) \sim \Delta(c, w_i, z_i)$ and $\Delta(c, v_j, x_j) \sim \Delta(c, w_j, z_j)$, we have

$$\frac{\|v_i - c\|}{\|w_i - c\|} = \frac{\|v_i - x_i\|}{\|w_i - z_i\|} = \frac{\lambda_i}{\|w_i - z_i\|} \quad \text{and} \quad \frac{\|v_j - c\|}{\|w_j - c\|} = \frac{\|v_j - x_j\|}{\|w_j - z_j\|} = \frac{\lambda_j}{\|w_j - z_j\|}.$$  

Note that if $c$ is a point at infinity then these equalities are obvious. Substituting the last equality into (3), we get

$$\frac{\|s_i - s_j\|}{\|y_i - y_j\|} = \frac{\|w_i - w_j\|}{\|v_i - v_j\|} \cdot \frac{\lambda_i}{\|w_i - z_i\|} \cdot \frac{\lambda_j}{\|w_j - z_j\|}. \quad (4)$$

Since $\Delta(w_i, z_i, x) \sim \Delta(v_i, x_i, v_i + \lambda_i r)$ and $\|v_i - x_i\| = \lambda_i$, we get

$$\frac{\|w_i - x\|}{\|w_i - z_i\|} = \frac{\|v_i - v_i - \lambda_i r\|}{\|v_i - x_i\|} \leftrightarrow \frac{\|w_i - x\|}{\|r\|} = \|w_i - z_i\|. \quad (5)$$

By a similar argument, we obtain

$$\frac{\|x - w_j\|}{\|r\|} = \|w_j - z_j\|. \quad (6)$$

From (5), (6) and (2) we conclude that

$$\frac{\|w_i - w_j\|}{\|v_i - v_j\|} = \frac{\|w_i - z_i\| + \|w_j - z_j\|}{u_i + u_j}.$$  

Substituting the last equality into (4), we have

$$\frac{\|s_i - s_j\|}{\|y_i - y_j\|} = \frac{\|w_i - z_i\| + \|w_j - z_j\|}{u_i + u_j} \cdot \frac{\lambda_i}{\|w_i - z_i\|} \cdot \frac{\lambda_j}{\|w_j - z_j\|}. \quad (7)$$

Now we are ready to apply twice the following simple fact.
Lemma 4. Suppose that $a_i$ and $b_i$ for $1 \leq i \leq 3$ are points in $\mathbb{R}^d$ such that $\theta_1(a_1 - a_2) = \theta_2(a_2 - a_3)$ and $\theta_1(b_1 - b_2) = \theta_2(b_2 - b_3)$, where $\theta_1$ and $\theta_2$ are real numbers. Then

$$b_2 - a_2 = \frac{\theta_1}{\theta_1 + \theta_2}(b_1 - a_1) + \frac{\theta_2}{\theta_1 + \theta_2}(b_3 - a_3).$$

Proof. A simple exercise. □

Denote by $x'$ the point of intersection of $\ell(x_0, \ell(v_i, x_i))$ with $r_0$ (see Figure 2). Using Lemma 4 for $w_i, z_i, x, x', w_j$ and $z_j$, we obtain

$$\|x - x'\| = \frac{2\|w_i - z_i\|\|w_j - z_j\|}{\|w_i - z_i\| + \|w_j - z_j\|}. \quad (8)$$

Using Lemma 4 for $v_i, x_i, x', v_j$ and $x_j$, we have

$$\|x - x'\| = \frac{u_j}{u_i + u_j} \lambda_i + \frac{u_i}{u_i + u_j} \lambda_j = \frac{\lambda_i u_j + \lambda_j u_i}{u_i + u_j}. \quad (9)$$

The comparison of (8) and (9) shows that

$$\frac{\|w_i - z_i\| + \|w_j - z_j\|}{\|w_i - z_i\|\|w_j - z_j\|} = \frac{2}{\|x - x'\|} = 2 \frac{u_i + u_j}{\lambda_i u_j + \lambda_j u_i}.$$

Substituting the last equality into (7), we get

$$\frac{\|s_i - s_j\|}{\|y_i - y_j\|} = \frac{2\lambda_i \lambda_j}{\lambda_i u_j + \lambda_j u_i}.$$

Lemma 3 is proved. □

Lemma 5. If $t_i \cap t_j \subset [v_i, v_j]$ then

$$\frac{2\lambda_i \lambda_j}{\lambda_i u_j + \lambda_j u_i} \leq 2.$$

Proof. Without loss of generality we assume that $\lambda_i \geq \lambda_j$. Note that by definition $u_i, u_j$ are such numbers that $x - v_i = u_i r$ and $v_j - x = u_j r$. Thus if $x \in t_i \cap t_j \subset [v_i, v_j]$ then $u_i, u_j \geq 0$ and $u_i + u_j \geq \lambda_i \geq \lambda_j$, i.e. $\lambda_i \lambda_j \leq (u_i + u_j) \lambda_j \leq (\lambda_j u_i + \lambda_i u_j)$. The last inequality proves the statement of Lemma 5. □

3. Proofs of theorems

3.1. Proof of Theorem 1. Using the notations of Section 2.2 we consider $X_1 \subset h_1$, where $h_1$ is a $(d + 1)$-dimensional plane. Moreover, by Lemmas 2 and 3 for any $i \neq j \in [n]$ there exist two parallel $d$-dimensional planes $k_{i,j}$ and $k_{j,i}$ such that $y_k \in \text{conv}(k_{i,j} \cup k_{j,i})$ for any $k \in [n]$ and

$$\frac{\text{dist}(k_{i,j}, k_{j,i})}{\text{dist}(g_{i,j}, g_{j,i})} = \frac{2\lambda_i \lambda_j}{\lambda_i u_j + \lambda_j u_i}. \quad (10)$$

Since these homothets form a Minkowski arrangement, we have $t_i \cap t_j \subset [v_i, v_j]$, i.e. by Lemma 5 we have that (10) is less than or equal to 2. Therefore, $X_1$ satisfies conditions of Lemma 1 with $\lambda = 2$. Thus $n \leq 3^{d+1}$. 

6
3.2. **Proof of Theorem 3**. Consider the following family of pairwise intersecting homothets \( \{v_i + \lambda K : i \in [n]\} \), where \( \lambda_i := \lambda_{i-1} \). Without loss of generality assume that \( \max_{i \in [n]} \lambda_i = 1 \). Let us divide the set \([n]\) into \( d \) subsets. For any \( l \in [d] \) we consider

\[
J_l = \{i \in [n] : \lambda_i \in \mu^{l-1} I\},
\]

where \( \mu = 2^{-1/(d-1)} < 1 \), i.e. \( \mu^d = \mu/2 \), and

\[
I = I_1 \cup I_2 \cup I_3 \cup \cdots := (\mu, 1] \cup (\mu^{d+1}, \mu^d] \cup (\mu^{2d+1}, \mu^{2d}] \cup \cdots.
\]

Obviously, the \( J_l \)s are not pairwise intersecting sets and their union is \([n]\). We claim that

\[
|J_l| \leq \left( 1 + \frac{2}{2 - \mu^{-1}} \right)^{d+1}
\]

(11)

Clearly, (11) implies the statement of Theorem 3

\[
n \leq d \left( 1 + \frac{2}{2 - \mu^{-1}} \right)^{d+1}.
\]

It is enough to prove (11) for \( l = 1 \). Consider the set of homothets \( \{v_k + \lambda_k K : k \in J_1\} \). Using the notations of Section 2.2, we have that for any \( i \neq j \) there exist two parallel \( d \)-dimensional planes \( k_{i,j} \) and \( k_{j,i} \) in the \( (d+1) \)-dimensional plane \( h_{1} \) such that \( y_k \in \text{conv}(k_{i,j} \cup k_{j,i}) \) for any \( k \in J_1 \) and

\[
\frac{\text{dist}(k_{i,j}, k_{j,i})}{\text{dist}(g_{i,j}, g_{j,i})} = \frac{2\lambda_i \lambda_j}{\lambda_i u_j + \lambda_j u_i},
\]

(12)

By Lemma 1 it is enough to prove that the right hand side of (12) is at most \( \frac{2}{2 - \mu^{-1}} > 2 \).

Consider two cases:

1) \( i, j \in I_k \) for some \( k \). Assume that \( i < j \) thus \( v_j - v_i = \lambda_i r \). If \( \lambda_i \geq \lambda_j \) then we have \( t_i \cap t_j \subset [v_i, v_j] \), i.e. by Lemma 5 we have that (12) is at most 2. Assume that \( \lambda_j > \lambda_i \). Since \( x \in [v_j - \lambda_j r, v_i + \lambda_i r] = [v_j - \lambda_i r, v_j] \), we have \( u_i + u_j = \lambda_i, 0 \leq u_j \leq \lambda_j \).

Therefore, using \( \lambda_j/\lambda_i < \mu^{-1} \) (because \( i, j \in I_k \)) we have

\[
\frac{\lambda_i u_j + \lambda_j u_i}{\lambda_i \lambda_j} = u_j \left( \frac{1}{\lambda_j} - \frac{1}{\lambda_i} \right) + \frac{u_i + u_j}{\lambda_i} \geq \lambda_j \left( \frac{1}{\lambda_j} - \frac{1}{\lambda_i} \right) + 1 > 2 - \mu^{-1},
\]

i.e. the right hand side of (12) is at most \( \frac{2}{2 - \mu^{-1}} \).

2) \( i \in I_k, j \in I_l \) for some \( k < l \). Note that \( \lambda_i > 2\lambda_j \) (see the definition of \( I_m \)) thus it is impossible that \( v_j - v_i = \lambda_j r \). Indeed, in such case \( v_i + \lambda_i \partial K \) and \( v_j + \lambda_j \partial K \) do not intersect each other because of the triangle inequality, a contradiction. Therefore, \( v_j - v_i = \lambda_i r \), i.e. \( t_i \cap t_j \subset [v_i, v_j] \), thus (12) is at most 2.

Theorem 3 is proved.

3.3. **Proof of Theorem 2**. Assume that there exists a \( k \)-distance set \( \{x_i : i \in [n]\} \) in the \( d \)-dimensional Minkowski space with an \( o \)-symmetric convex body \( K \) as the unit ball, where

\[
n = k^{f(d)}, \quad f(d) = \left\lfloor d \left( 1 + \frac{2}{2 - 2^{1/(d-1)}} \right)^{d+1} \right\rfloor = O(3^d d).
\]

We will construct a set \( Y = \{y_i : i \in [f(d) + 1]\} \) in the same \( d \)-dimensional Minkowski space such that \( ||y_i - y_j||_K = \lambda_i \) for any \( 1 \leq i < j \leq f(d) + 1 \), where \( \lambda_i \) are some positive real numbers, using the following algorithm.

0. Set \( A := [n], Y := \{y_1 := x_1\}, l := 1. \)
1. Let $\lambda_l$ be a positive real number such that the cardinality of the set 
\[ A' := \{ j : \|y_l - x_j\|_K = \lambda_l, j \in A \} \]
is at least $k^{f(d)-l}$ (such $i_l$ exists because $|A| \geq k^{f(d)-l+1}$ and there are $k$ distances occurring between points of $\{x_i : i \in A \subseteq [n] \}$). Put $A := A'$.

2. Choose any $j \in A$ and put $y_{l+1} := x_j$. Add $y_{l+1}$ to the set $Y$.

3. If $l < f(d)$ then $l := l + 1$ and return to Step 1, else, output $Y$, and finish.

Obviously, the existence of the set $Y$ contradicts Theorem 3, therefore, we get a contradiction with our assumption that there exists a $k$-distance set consisting of $k^{f(d)}$ points in $\mathbb{R}^d$.

**Acknowledgment.**

We are grateful to Márton Naszódi and Konrad Swanepoel for stimulating and fruitful discussions, to anonymous referees for valuable comments that helped to significantly improve the presentation of the paper. We wish to thank one of the referees for bringing to our attention the idea to use cross-ratios in the proof of Lemma 3. The author was partially supported by ISF grant no. 409/16, and by the Russian Foundation for Basic Research, grants № 15-31-20403 (mol_a ved), № 15-01-99563 A, № 15-01-03530 A.

**References**

[1] L. Danzer, B. Grünbaum, Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee, Math. Z. 79 (1962), 95–99.

[2] Z. Füredi, P. A. Loeb, On the best constant for the Besicovitch covering theorem, Proc. Amer. Math. Soc. 121(4) (1994), 1063–1073.

[3] Zs. Lángi, M. Naszódi, On the Bezdek-Pach conjecture for centrally symmetric convex bodies, Canad. Math. Bull. 52(3) (2009), 407–415.

[4] M. Naszódi, J. Pach, K. Swanepoel, Arrangements of homothets of a convex body, [arXiv:1608.04639](arXiv:1608.04639) submitted.

[5] M. Naszódi, On a conjecture of Károly Bezdek and János Pach, Period. Math. Hungar. 53(1-2) (2006), 227–230.

[6] M. Petty, Equilateral sets in Minkowski spaces, Proc. Amer. Math. Soc. 29 (1971), 369–374.

[7] L. F. Tóth, Research problem, Period. Math. Hungar. 31(2) (1995), 165–166.

[8] K. Swanepoel, Cardinalities of $k$-distance sets in Minkowski spaces, Discrete Mathematics 197/198 (1999), 759–767.

[9] K. Swanepoel, Combinatorial distance geometry in normed spaces, [arXiv:1702.00066](arXiv:1702.00066) submitted.

Moscow Institute of Physics and Technology, Technion, Institute for Information Transmission Problems RAS.

E-mail address: alexander.polyanskii@yandex.ru

8