MASS DISTRIBUTION FOR TORAL EIGENFUNCTIONS VIA BOURGAIN’S DE-RANDOMISATION

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ABSTRACT. We study the problem of mass distribution of Laplacian eigenfunctions in shrinking balls for the standard flat torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$. By averaging over the centre of the ball we use Bourgain’s de-randomisation to compare the mass-distribution of toral eigenfunctions at Plank scale to the mass distribution of random waves in growing balls around the origin. We are then able to classify all possible limiting distributions and variances. Finally we give sufficient and necessary conditions so that the mass of “generic” eigenfunction equidistributes at Plank scale in almost all balls.

1. Introduction

1.1. Shnirelman’s Theorem in shrinking sets. Given a compact Riemannian manifold $(M, g)$ without boundary and normalised to have volume 1, let $\Delta_g$ be the Laplace-Beltrami operator on $M$, there exists and orthonormal basis for $L^2(M, dvol)$ consisting of eigenfunctions $\{f_{E_i}\}$

$$\Delta_g f_{E_i} = -E_i^2 f_{E_i}$$

with $0 = E_1 < E_2 \leq ...$ repeated accordingly to multiplicity and $E_i \rightarrow \infty$. The celebrated Quantum Ergodocity Theorem [8, 18, 22] asserts that, if the geodesic flow on $M$ is ergodic, there exists a density one subsequence of eigenfunctions $\{f_{E_j}\}$ of $\Delta_g$ such that

$$\int_A |f_{E_j}|^2 \rightarrow Vol(A) \quad j \rightarrow \infty$$

(1.1)

where $A$ is a sufficiently “nice” subset of $M$ and $\int_M |f_{E_j}|^2 = 1$. That is the $L^2$ mass of most eigenfunctions equidistributes on $M$. Berry [2, 3] conjectured a stronger form of this theorem: given a parameter $r = r(E)$ such that $r \cdot \sqrt{E} \rightarrow \infty$ for generic eigenfunctions we expect

$$\frac{1}{Vol(B(x,r))} \int_{B(x,r)} |f_{E_j}|^2 \rightarrow 1 \quad E, r \rightarrow \infty$$

(1.2)

uniformly for all $x \in M$ (here $B(x, r)$ denotes the geodesic ball centred at $x$ of radius $r$). That is the mass of generic eigenfunctions should equidistribute at scales as small as the Plank scale.

Luo and Sarnak [17] considered the case when $M$ is the modular surface and proved that there exists a density one sequence of eigenfunctions, which are also
eigenfunction of all Hecke operators, such that small scale distribution holds for \( r \gg E^{-\alpha} \) for some small \( \alpha > 0 \). Young [21] improved, under the Generalised Riemann Hypothesis, the scale of said result to \( r \gg E^{-1/6+o(1)} \) and to all eigenfunctions. Hezari, Rivièrè and independently Han [12] proved that when \( M \) has negative sectional curvature then small scale equidistribution holds along a density one sequence of eigenfunctions for \( r = \log(E)^{-\alpha} \) for some small \( \alpha > 0 \).

Here we are interested in the case when \( M \) is the flat torus \( T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \). Lester and Rudnick [17] proved that (1.2) holds for \( r > E^{-1/2+o(1)} \) for a density one subsequence of eigenfunctions. On the other hand they also proved that there exists eigenfunctions for which (1.2) fails at the point \( x = 0 \) at Plank-scale. This naturally raises the question of whether the failure of (1.2) is only limited to small set of points. To make this precise, Granville and Wigman [10] introduced the (pseudo-)random variable

\[
M_f(x, r) := \frac{1}{\text{Vol}(B(x, r))} \int_{B(x, r)} |f_E|^2
\]

(1.3)

where \( x \) is drawn uniformly at random from \( T^2 \) and showed, under some additional assumptions on \( f \) (see Remark (1.5) below), that the variance of \( M(x) \) is \( o(1) \). It follows that for these eigenfunctions (1.2) holds for almost all points \( x \in T^2 \). Further work has been carried out by Wigman and Yesha [20]; they proved that, under flatness assumptions and small variation of the coefficients (see Remark 1.5 below), the distribution of \( M_f(x, r) \) is Gaussian with mean zero and variance \( c \cdot (\sqrt{E}r)^{-1} \) where the constant \( c \) depends on the eigenfunction \( f \).

In this article, using a different method than [10, 20], we study \( M_f(x, r) \); this allows us to find its limit distribution and variance for a wider class of eigenfunctions. Moreover we also classify all the possible limit distribution of such eigenfunctions. Such approach of averaging over ball centres was also recently used by Humphries [13] for mass distribution of automorphic forms.

1.2. Toral eigenfunctions and random waves. An eigenfunction for \( \Delta \) on \( T^2 \) with eigenvalue \( E \) can be written explicitly as

\[
f(x) = \sum_{|\xi|^2 = E} a_{\xi} e(\langle x, \xi \rangle).
\]

(1.4)

where \(^1 e(\cdot) = e(2\pi i \cdot) \) and some complex numbers \( \{a_{\xi}\}_{\xi} \). The multiplicity of the eigenspace is the number of representations of \( E \) as a sum of two squares and we denote it by \( N = N(E) \). We also let \( \mathcal{E}(E) = \mathcal{E} = \{\xi \in \mathbb{Z}^2 : |\xi|^2 = E\} \) so we have that \( |\mathcal{E}| = N \). We will require that

\[
\bar{a}_{\xi} = a_{-\xi}
\]

(1.5)

so that \( f \) is real-valued. Finally, we normalise every eigenfunction so that

\(^1\) This normalisation implies that the eigenvalue is \( 4\pi^2 E \).
\(|f|_{L^2(\mathbb{T}^2)}^2 = \sum_{|\xi|^2 = E} |a_{\xi}|^2 = 1. \quad (1.6)\)

Thanks to (1.6) we can associate to \(f\) a probability measure on the unit circle \(S^1 = \mathbb{R}/\mathbb{Z}\) as

\[
\mu_f = \sum_{|\xi|^2 = E} |a_{\xi}|^2 \delta_{\xi/\sqrt{E}}. \quad (1.7)
\]

where \(\delta_{\xi/\sqrt{E}}\) is the Dirac delta function at the point \(\xi/\sqrt{E}\). These measures appear naturally in the study of \(f\); in fact, Bourgain [5] and subsequently Bucklely and Wigman [6] proved that “generic” \(f\) as in (1.4) approximate a centred stationary Gaussian field with spectral measure \(\mu_f\) when averaged over \(x \in \mathbb{T}^2\).

Importantly the sequence \(\mu_f\) does not have an unique weak limi: in fact the weak limits of \(\{\mu_f\}\), in the special case \(|a_{\xi}|^2 = 1/N\) for all \(\xi\), are called “attainable” and have been studied in [15].

We are now going to briefly recall some facts about Gaussian fields which will be useful later. A stationary centred Gaussian field \(F = F(y, \omega) = F_\omega(y)\), where \(y \in \mathbb{R}^2\) and \(\omega \in \Omega\) some arbitrary probability space, can be thought as a Gaussian random function (on \(\mathbb{R}^2\)) with mean zero and covariance

\[
\mathbb{E}[F(x) \cdot F(y)] = \mathbb{E}[F(x - y) \cdot F(0)].
\]

Notably, the covariance function is invariant under the action \((x, y) \rightarrow (x + \tau, y + \tau)\) for \(\tau \in \mathbb{R}\) (stationary). Moreover, it is also positive definite so, by Bochner’s theorem, it is the Fourier transform of some measure \(\mu\) on the plane satisfying \(\mu(I) = \mu(-I)\) (as the field is real-valued); that is

\[
\mathbb{E}[F(x)F(y)] = \int e(\langle x - y, \lambda \rangle) d\mu(\lambda).
\]

The measure \(\mu\) is called the spectral measure of \(F\). Since Gaussian fields are determined by their mean and covariance, \(\mu\) fully defines \(F\), therefore we write \(F = F_\mu\). We also make the technical assumption that all random fields are defined on a common probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with expectation \(\mathbb{E}\).

1.3. Statement of main results. We make the following two assumptions:

- A1. (Spectral correlations)

  Let \(0 < \gamma < 1/2\) and \(B = B(E)\) be an arbitrarily slow growing function of \(E\) taking integer values, then we say that \(E\) satisfies assumption A1 if for any \(2 \leq 2l \leq B\) the number of solutions of

  \[
  \xi_1 + \xi_2 + \ldots + \xi_{2l} = 0 \quad (1.8)
  \]
where $\xi_i \in \mathcal{E}(E)$, is equal to

$$\frac{(2l!)}{2^l \cdot l!} N^l + O(N^{\gamma l}).$$

- A2. (Flatness)

  Fix some function $g : \mathbb{R} \to \mathbb{R}$ such that for any $\epsilon > 0$ we have $g(N)/N^\epsilon \to 0$ as $N \to \infty$. A function $f$ of the form (1.4), normalised as in (1.6), satisfies assumption A2 if for any $\xi \in \mathcal{E}$

  $$|a_\xi|^2 \leq \frac{g(N)}{N}.$$ (1.9)

**Remark 1.1.** By [4, Theorem 17] (see also [5, Lemma 4]) assumption A1 is satisfied for a density one of energy levels. Moreover, if we regard the set $(a_\xi)_{\xi}$ as points on an $N$-dimensional sphere, by Levi-concentration of measure, assumption A2 (with $g(N) = \log^{O(1)} N$, say) is satisfied with probability asymptotic to 1. Thus “generic” eigenfunctions on $\mathbb{T}^2$ satisfy both conditions. Also see section 4.2 for a discussion on the assumptions.

We will then prove that:

**Theorem 1.2.** Let $\epsilon > 0$ and $t > 0$, $R > 1$ be fixed and $M_f(x, r)$ be as in (1.3), then there exists some $E_0 = E_0(\epsilon, t, R)$ such that for any $E > E_0$ satisfying A1 and $N \to \infty$ we have

$$\left| \text{Vol}(x \in \mathbb{T}^2 : M_f(x, r) \leq t) - \mathbb{P}\left( \frac{1}{\pi R^2} \int_{B(R)} |F_{\mu_f}|^2 \leq t \right) \right| \leq \epsilon$$

where $r = R/\sqrt{E}$, uniformly for all $f$ satisfying A2.

In section 2.2 we compute the distribution of the random variable $\frac{1}{\pi R^2} \int_{B(R)} |F_{\mu_f}|^2$, where $\mu$ is some probability measure on $\mathcal{S}^1$. Therefore, upon passing to a subsequence such that $\mu_f$ weak* converges to $\mu$, Theorem 1.2 allows us to find the distribution of $M_f(x, r)$. To state the next result we need to first introduce some notation: by Lebesgue decomposition theorem we can write every probability measure $\mu$ on $\mathcal{S}^1$ as

$$\mu = \alpha \mu_A + \beta \mu_B$$

where $\alpha, \beta \in \mathbb{R}$, $\mu_A$ is purely atomic and $\mu_B$ has no atoms. We then write $\mu_A = \sum_{\xi \in \text{spt}(\mu_A)} \sigma_\xi \delta_\xi$ for some positive numbers $\sigma_\xi$ with $\sum_\xi \sigma_\xi = 1$ and define

$$W(\mu) := \sum_{\xi \in \text{spt}(\mu)} |X_\xi|^2$$

where $X_\xi$ are independent $N(0, \sigma_\xi)$ random variables satisfying $\overline{X_\xi} = X_{-\xi}$.
Theorem 1.3. Under the assumptions of Theorem 1.2, suppose that $\mu_f \Rightarrow \mu$ as $E \to \infty$ and write $\mu = \alpha \mu_A + \beta \mu_B$ as in (1.11) then for any fixed $t > 0$ we have
\[
\lim_{R \to \infty} \lim_{E \to \infty} \text{Vol}(x \in \mathbb{T}^2 : M_f(x, r) \leq t) = \mathbb{P}(\alpha \cdot W(\mu_A) + \beta \leq t)
\]
where $r = R/\sqrt{E}$. Moreover
\[
\lim_{R \to \infty} \lim_{E \to \infty} \int_{\mathbb{T}^2} (M_f(x, r) - 1)^2 = \alpha^2 \text{Var}(W(\mu_A)).
\]

One direct consequence of the Theorem 1.3 is that asymptotic uniform distribution of the mass holds if and only if the limiting spectral measure is not atomic. More precisely, taking $\alpha = 0$ and thus $\beta = 1$ in Theorem 1.3 we have:

Corollary 1.4. Under the assumptions of Theorem 1.2, as $E \to \infty$ and $R \to \infty$, we have
\[
\int_{\mathbb{T}^2} (M_f(x, r) - 1)^2 \to 0
\]
if and only if $\mu$ has no atoms.

Remark 1.5. In [10, 20] the authors give sufficient conditions for the vanishing of the variance of $M(x, r)$. These are essentially flatness of $f$ and the lack of mass concentration for $\mu_f$ (see [10, equation (19)] and [20, Definition 2.4]). In hindsight Corollary 2.8 explain such conditions and show that they are sharp as they imply that $\mu$ is non-atomic.

It is well known (see [9] and [14]) that lattice points on generic circles equidistribute when projected onto $S^1$. Therefore, Corollary 1.4 addresses the case of generic measures $\mu_f$, provided that not too many coefficients are zero. At the other extreme, Cilleruelo [7] proved that there exists a sequence of $E$ such that $N \to \infty$ and $\mu = \frac{1}{4} \left( \delta_{(1,0)} + \delta_{(0,1)} + \delta_{(-1,0)} + \delta_{(0,-1)} \right)$ so we have the following example where the variance does not vanish.

Example 1.6. Suppose that $\mu$ is the Cilleruelo measure
\[
\mu = \frac{1}{4} \left( \delta_{(1,0)} + \delta_{(0,1)} + \delta_{(-1,0)} + \delta_{(0,-1)} \right)
\]
then
\[
M_f(x, r) \overset{d}{\to} \frac{\chi^2(2)}{2} \quad \int_{\mathbb{T}^2} (M_f(x, r) - 1)^2 \to 1
\]
as $E \to \infty$ and $R \to \infty$.

2. Proof of Theorem 1.3

In this section we prove Theorem 1.3, assuming Theorem 1.2. This will be done in two steps: firstly we take the limit as $E \to \infty$ and show that the limiting distribution of $M_f(x, r)$ for $R/\sqrt{E}$ tends to the random variable $\frac{1}{\sqrt{E}} \int_{B(R)} |F_\mu|^2$. Secondly we determine such random variable in the limit $R \to \infty$. 


2.1. Limit as $E \to \infty$. The aim of this section is to prove the following:

**Proposition 2.1.** Under the assumptions of Theorem 1.2, suppose that $\mu_f \Rightarrow \mu$ as $E \to \infty$ then

$$M_f(x, r) \xrightarrow{d} \frac{1}{\pi R^2} \int_{B(R)} |F_\mu|^2 \, dx.$$ 

To show this we need two simple results; the first one is a direct consequence of the Borell-TIS inequality (see [1, Theorem 2.1.1]).

**Lemma 2.2.** For any $\delta_1 > 0$ there exists some $M = M(R, \delta_1)$ such that

$$\mathbb{P} \left( \sup_{B(R)} |F_\mu| > M \right) \leq \delta_1.$$

**Proof.** Since $\mu$ is supported on the unit circle we have that $\int |\lambda|^3 d\mu(\lambda) \leq 1$, thus the covariance function is differentiable; then the field is almost surely continuous and thus almost surely bounded in $B(R)$. The Borell-TIS inequality then implies that $\mathbb{E}[\sup_{B(R)} |F_\mu|]$ is finite, therefore we can take $M = \delta_1^{-1} \mathbb{E}[\sup_{B(R)} |F_\mu|]$ and apply Markov’s inequality. \(\square\)

We also need the following lemma from [19] of which we give the proof for convenience.

**Lemma 2.3** (Lemma 4, [19]). Let $\mu_n$ be a sequence of probability measures on $S^1$ such that $\mu_n \Rightarrow \mu$, then for any $\alpha > 0$ and $\delta_2 > 0$ there exists some $n_0 = n_0(\alpha, \delta_2)$ such that for all $n \geq n_0$ we have

$$\sup_{B(R)} |F_{\mu_n} - F_\mu| \leq \alpha$$

outside an event of probability $\delta_2$.

**Proof.** We can associate to $\mu$ the field $V$ defined on $\mathbb{R}^2$ as follows: for any open and measurable (with respect to $\mu$) subset $A$ of $\mathbb{R}^2$ we have

$$V(A) = N(0, \mu(A))$$

and if $A \cap B = \emptyset$ then $V(A)$ and $V(B)$ are independent. We define $V_n$ with respect to $\mu_n$ similarly. Since $\mu$ is compactly supported we see that $V_n \xrightarrow{d} V$ and, since a normal random variable is square integrable we obtain that $V_n \to V$ in $L^2(\Omega)$ (we recall that $\Omega$ is the common probability space of our random objects). By [1, Theorem 5.4.2] we have the $L^2(\Omega)$ representations

$$F_{\mu_n}(x) = \int_{S^1} e(\langle x, \lambda \rangle) V_n(d\lambda) \
F_\mu(x) = \int_{S^1} e(\langle x, \lambda \rangle) V(d\lambda)$$

From this and the preceding discussion we deduce that that $\sup_{B(R)} |F_{\mu_n} - F_\mu| \to 0$ in $L^2(\Omega)$ form which the lemma follows. \(\square\)
**Proof of Proposition 2.1.** Fix some $t \in \mathbb{R}$ and $\epsilon > 0$ and let $X(\mu_f) = \int_{B_0(R)} |F_{\mu_f}|^2 / \pi R^2$ then by Lévy’s continuity Theorem and Theorem 1.2 we know that

$$\left| \int_{T^2} \exp(itM_f(x,r)) - \mathbb{E}[\exp(itX(\mu_f))] \right| \leq \epsilon/2$$

for all $E$ large enough. By the triangle inequality it is sufficient to prove that

$$|\mathbb{E}[\exp(itX(\mu_f))] - \mathbb{E}[\exp(itX(\mu))]| \leq \epsilon/2 \quad (2.1)$$

for all $E$ sufficiently large. To prove the above we are going to show that $X(\mu_f)$ and $X(\mu)$ are close outside and event of small probability. To see this we take $\delta_1 = \epsilon/8$ in Lemma 2.2 and let $V_1$ be the event that $\sup |F_{\mu_f}| > M$. Then we take $\delta_2 = \epsilon/8$, $\mu_n = \mu_f$, $\alpha = \sqrt{\epsilon/(8tM)}$ in Lemma 2.3 and denote by $V_2$ the event that $\sup |F_{\mu_f} - F_\mu| > \alpha$. Finally, let $V = V_1 \cup V_2$ then $\mathbb{P}(V) \leq \epsilon/4$ for all $E$ large enough. So outside $V$ we have that

$$|X(\mu_f) - X(\mu)| = \frac{1}{\pi R^2} \int_{B_0(R)} |F_{\mu} - (F_{\mu_f} - F_{\mu_f})|^2 - |F_{\mu}|^2$$

$$\leq 2\sup_{B(R)} |F_{\mu_f} - F_\mu| |F_{\mu_f}| + \sup_{B(R)} |F_{\mu_f} - F_\mu|^2 \leq 2M \alpha + \alpha^2 \leq \epsilon/4t \quad (2.2)$$

Moreover,

$$|\exp(itX(\mu_f)) - \exp(itX(\mu))| \leq |\exp(itX(\mu_f)) - X(\mu)| - 1 \leq t|X(\mu_f) - X(\mu)| \quad (2.3)$$

Now, combining (2.2) and (2.3) with the fact that $\mathbb{P}(V) \leq \epsilon/4$ we obtain (2.1) as required. □

**2.2. Mass distribution for random waves.** In this section we study the distribution of the random variable $1/\pi R^2 \int_{B(R)} |F_{\mu_f}|^2$ where $R \to \infty$ and $\mu$ is a probability measure on $\mathbb{S}^1$.

**Proposition 2.4.** Suppose that $\mu$ is a probability measure on $\mathbb{S}^1$ and write $\mu$ as in (1.11) then

$$\frac{1}{\pi R^2} \int_{B(R)} |F_{\mu_f}|^2 dx \xrightarrow{d} \alpha W(\mu_A) + \beta.$$

as $R \to \infty$ and the convergence is in distribution.

To prove the Proposition we need a few preliminary results.

**Lemma 2.5.** Suppose that $\mu$ is a probability measure on $\mathbb{S}^1$ and write $\mu$ as in (1.11) then

$$F_{\mu} = \sqrt{\alpha} F_{\mu_A} + \sqrt{\beta} F_{\mu_B}$$

where $F_{\mu_B}$ and $F_{\mu_A}$ are independent stationary Gaussian fields.
Proof. Since both the left and the right hand side above are Gaussian fields with mean zero they are fully determined by their covariance, the covariance of the right hand side is

\[
\mathbb{E}\left[ (\sqrt{\alpha} F_{\mu_A} + \sqrt{\beta} F_{\mu_B}) (x) (\sqrt{\alpha} F_{\mu_A} + \sqrt{\beta} F_{\mu_B}) (y) \right] \\
= \alpha \mathbb{E}[F_{\mu_A}(x)F_{\mu_B}(y)] + \beta \mathbb{E}[F_{\mu_A}(x)F_{\mu_B}(y)] \\
= \int_{S^1} e((x-y,\lambda)) d(\alpha \mu_A(\lambda) + \beta \mu_B(\lambda))
\]

where in the second line we have used the fact that the cross term vanishes by independence and both \( F_{\mu_B} \) and \( F_{\mu_A} \) have zero mean and in the last line we have used the definition of \( F_{\mu} \) and this proves the Lemma.

We can understand the distribution of the atomic part quite directly in fact we have the following:

**Lemma 2.6.** Suppose that \( \mu_A \) is a purely atomic measure supported on \( S^1 \), then

\[
\frac{1}{R^2} \int_{B(R)} |F_{\mu_A}(x)|^2 dx \xrightarrow{d} W(\mu_A)
\]

as \( R \to \infty \) where \( W(\mu_A) \) is as in (1.12).

**Proof.** We claim that we can write explicitly

\[
F_{\mu_A}(x) = \sum_{\xi \in \text{spt}(\mu_A)} X_\xi \epsilon(\langle \xi, x \rangle).
\]

where \( X_\xi \) are as in (1.12). Firstly, by Kolmogorov’s two-series theorem and the fact that \( \sum \sigma_\xi = 1 \), the sum (absolutely) converges almost surely. So we can compute the covariance function \( r(x-y) = \sum \sigma_\xi \epsilon(\langle x-y, \xi \rangle) \) which is the Fourier transform of \( \mu \). This proves the claim. Now we square \( F_{\mu_A}(x) \) and separate the diagonal terms to obtain

\[
\frac{1}{\pi R^2} \int_{B(R)} |F_{\mu_A}|^2 dx = \sum_\xi |X_\xi|^2 + \frac{1}{\pi R^2} \sum_{\xi \neq \xi'} X_\xi X_{\xi'} \int_{B(R)} e(\langle \xi - \xi', x \rangle) dx \\
= W(\mu_A) + O\left( \sum_{\xi \neq \xi'} X_\xi X_{\xi'} \int_{B(1)} e(\langle \xi - \xi', Rx \rangle) dx \right) \\
= W(\mu_A) + O\left( \sum_{\xi \neq \xi'} X_\xi X_{\xi'} \frac{J_1(R||\xi - \xi'||)}{R||\xi - \xi'||} \right) \quad (2.4)
\]

where \( J_1(\cdot) \) is the Bessel function of the first kind. Since \( J_1(T) \sim T^{-1/2} \) for \( T \) large enough and \( J_1(T) = O(1) \) for \( T \) small, the error term in (2.4) can be
bounded as
\[
\sum_{\xi \neq \xi'} X_{\xi} \overline{X_{\xi'}} J_1(R||\xi - \xi'||) \lesssim R^{-3/4} \sum_{|\xi - \xi'| > R^{-1/2}} |X_{\xi}||\overline{X_{\xi'}}| + \sum_{|\xi - \xi'| < R^{-1/2}} |X_{\xi}||\overline{X_{\xi'}}|
\]
(2.5)

Since \(E[\sum_{\xi \neq \xi'} |X_{\xi} \overline{X_{\xi'}}|] \leq (2/\pi)^{1/2} \sum_{\xi \neq \xi'} \sigma_{\xi} \sigma_{\xi'} \leq 1\) we can apply Markov inequality to obtain the bound
\[
P\left(\sum_{\xi \neq \xi'} |X_{\xi}||\overline{X_{\xi'}}| > R^{1/2}\right) \leq 1/R^{1/2}.
\]

Therefore the first term in (2.5) tends to zero outside an event of probability \(1/R^{1/2}\). To bound the second term we notice that
\[
E[\sum_{|\xi - \xi'| < R^{-1/2}} \sum_{\xi \neq \xi'} |X_{\xi}||\overline{X_{\xi'}}| \sigma_{\xi} \sigma_{\xi'} \to 0 \quad (2.6)
\]
as \(R \to \infty\) because the inner sum converges to 0 and the outer sum has value at most 1. It follows that also the second term in (2.5) converges in probability to 0 and hence the lemma. \(\square\)

Understanding the non-atomic part requires the following deep theorem due to Wiener and Grenander-Fomin-Maruyama (see [19, Theorem 3] and references therein):

**Theorem 2.7.** Suppose that the spectral measure \(\mu\) (supported on \(S^1\)) of a stationary Gaussian field \(F = F_\mu\) has no atoms, then for any random variable \(H(F)\) with \(E[|H(F)|] < \infty\) we have
\[
\lim_{R \to \infty} \frac{1}{\pi R^2} \int_{B(R)} H(F(x)) dx \to E[H(F)].
\]
almost surely and in \(L^1\).

Taking \(H(F) = F^2(0)\) we have that \(E[|F^2(0)|] = 1\) so we obtain the following lemma:

**Lemma 2.8.** Suppose that \(\mu_B\) is a probability measure supported on \(S^1\) with no atoms then
\[
\frac{1}{\pi R^2} \int_{B(R)} |F_{\mu_B}|^2 dx \overset{d}{\to} 1
\]
as \(R \to \infty\).

We are now ready to give the proof of Proposition 2.4:
Proof of Proposition 2.4. Lemma 2.5 implies that
\[
\frac{1}{\pi R^2} \int_{B(R)} |F_\mu|^2 dx = \alpha \frac{1}{\pi R^2} \int_{B(R)} |F_{\mu_A}|^2 dx + \beta \frac{1}{\pi R^2} \int_{B(R)} |F_{\mu_B}|^2 dx
+ 2\sqrt{\alpha\beta} \frac{1}{\pi R^2} \int_{B(R)} (F_{\mu_A} \cdot F_{\mu_B}) dx.
\] (2.7)
By Lemma 2.8, as \( R \to \infty \), the second term converges in probability to \( \beta \), by Lemma 2.6 the first term converges in distribution to \( W(\mu_A) \). We are only left to show that third term above converges in distribution to 0. By independence of \( F_{\mu_B} \) and \( F_{\mu_A} \) we have
\[
E \left[ \frac{1}{\pi R^2} \int_{B(R)} (F_{\mu_A} \cdot F_{\mu_B}) dx \right] = 0
\]
and
\[
E \left[ \left( \frac{1}{\pi R^2} \int_{B(R)} (F_{\mu_A} \cdot F_{\mu_B}) dx \right)^2 \right] =
\]
\[
= \frac{1}{\pi^2 R^4} \int_{B(R)} \int_{B(R)} E[F_{\mu_A}(x) \cdot F_{\mu_A}(y)]E[F_{\mu_B}(x)F_{\mu_B}(y)] dx dy =
\]
\[
= \frac{1}{\pi^2 R^4} \int_{B(R)} \int_{B(R)} r_{\mu_B}(x-y)r_{\mu_A}(x-y) dx dy
\] (2.8)
where \( r_{\mu_B} \) and \( r_{\mu_A} \) are the covariance function of \( F_{\mu_B} \) and \( F_{\mu_A} \) respectively. Writing
\[
r_{\mu}(x-y) = \int_{S^1} e((x-y,t)) d\mu(t)
\]
changing the order of integration and bearing in mind that \( \mu_A \) is invariant under \( \pi \) rotation, we see that (2.8) is equal to
\[
\frac{1}{\pi^2 R^4} \int_{B(R)} \int_{B(R)} \int_{S^1} \int_{S^1} e((x-y, t-w)) dx dy d\mu_B(t) d\mu_A(w)
\]
\[
= \int_{S^1} \int_{S^1} \left( \frac{J_1(R||t-w||)}{R||t-w||} \right)^2 d\mu_B(t) d\mu_A(w)
\] (2.9)
where the second line follows from a similar computation to (2.4). Now we can split the above integral as
\[
\left( \int_{||t-w|| > R^{-1/2}} + \int_{||t-w|| \leq R^{-1/2}} \right) \left( \frac{J_1(R||t-w||)}{R||t-w||} \right)^2 d\mu_B(t) d\mu_A(w)
\]
As \( J_1(T) \approx T^{-1/2} \) for \( T \) large enough, we can bound the first integral by \( R^{-3/2} \); for the second integral we observe that \( J_1(T)/T = O(1) \) for \( T \) small so we fix \( w \)
to see that
\[ \int_{|t-w| \leq R^{-1/2}} d\mu_B(t) = o(1) \]
as \( R \to \infty \) as \( \mu_B \) has no atoms. All in all we have shown that the third term in (2.7) has zero mean and variance tending to 0 as \( R \to \infty \) and therefore converges in probability to 0. Hence the right hand side of (2.4) converges in distribution to \( W(\mu_A) + \beta \) and this concludes the proof of the Proposition. \( \square \)

2.3. Concluding the proof of Theorem 1.3. To deduce the convergence of the first two moments of a random variable from the convergence in distributions we need to show that the third moment, say, is bounded. In our case we prove the following:

**Lemma 2.9.** Let \( \{E(E)\} \) be a sequence such that the number of solution to
\[ \xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 = 0 \]
where \( \xi_i \in E(E) \), is equal to
\[ 15N^3 + O(N^{5\gamma}) \]
for some \( 0 < \gamma < 1/2 \) and \( N \to \infty \). Moreover suppose that \( f \) satisfies assumption \( A_2 \) then for all fixed \( R > 1 \) we have the uniform bound
\[ \int_{T^2} dx \left( \frac{1}{\pi r^2} \int_{B(x,r)} |f(y)|^2 dy \right)^3 = O(1) \]
where \( r = R/\sqrt{E} \).

**Proof.** Changing variables and expanding the integral, we can write
\[ \left( \frac{1}{\pi r^2} \int_{B(x,r)} |f(y)|^2 dy \right)^3 = \left( \frac{1}{\pi} \int_{B(1)} |f(x + ry)|^2 dy \right)^3 \]
\[ = \left( \sum_{\xi,\xi'} a_{\xi} \overline{a}_{\xi'} e((\xi - \xi', x)) \frac{1}{\pi} \int_{B(1)} e((\xi - \xi', ry)) \right)^3 \]
\[ = \sum_{\xi_1,\xi_2,\xi_3} \sum_{\xi'_1,\xi'_2,\xi'_3} a_{\xi_1} \overline{a}_{\xi'_1} a_{\xi_2} \overline{a}_{\xi'_2} a_{\xi_3} \overline{a}_{\xi'_3} e((\xi_1 + \xi_2 + \xi_3 - \xi'_1 - \xi'_2 - \xi'_3, x)) \]
\[ \times \left( \frac{1}{\pi} \int_{B(1)} e((\xi - \xi', ry)) \right)^3 \]
Now we observe that \( \int_{T^2} dx e((\xi_1 + \xi_2 + \xi_3 - \xi'_1 - \xi'_2 - \xi'_3, x)) = 1 \) if \( \xi_1 + \xi_2 + \xi_3 - \xi'_1 - \xi'_2 - \xi'_3 = 0 \) and zero otherwise, so integrating the above with respect to \( x \) we pick up solutions to the above equation. We call a solution diagonal if it is
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given by pair-wise cancellation (like $\xi_i = \xi'_i$ for $i = 1, 2, 3$), we call all the other solution off-diagonal and split the sum accordingly

$$\int_{T^2} dx \left( \frac{1}{r^2} \int_{B(x,r)} |f(y)|^2 dy \right)^3 = \sum \limits_{\xi_1 + \xi_2 + \xi_3 - \xi'_1 - \xi'_2 - \xi'_3 = 0, \text{diagonal}} a_{\xi_1} \overline{a_{\xi_1}} a_{\xi_2} \overline{a_{\xi_2}} a_{\xi_3} \overline{a_{\xi_3}}$$

$$\times \left( \int_{B(1)} e(\langle \xi - \xi', ry \rangle) dy \right)^3$$

$$+ O \left( \sum \limits_{\xi_1 + \xi_2 + \xi_3 - \xi'_1 - \xi'_2 - \xi'_3 = 0, \text{off-diagonal}} |a_{\xi_1} \overline{a_{\xi_1}} a_{\xi_2} \overline{a_{\xi_2}} a_{\xi_3} \overline{a_{\xi_3}}| \right)$$

where we have bounded $|\int_{B(1)} dy e(\langle \xi - \xi', ry \rangle)| \leq \pi$. The number of diagonal solutions is the number of ways of dividing a set of 6 elements into pairs which are $6!/(3! \cdot 2^3) = 15$ so the main term is bounded by

$$15\pi^3 \sum \limits_{\xi_1, \xi_2, \xi_3} |a_{\xi_1}|^2 |a_{\xi_2}|^2 |a_{\xi_3}|^2 = 15 \left( \sum \limits_{\xi} |a_{\xi}|^2 \right)^3 = 15\pi^3$$

To bound the error term we use assumption $A_2$ to see that $|a_{\xi}| \leq \sqrt{g(N)/N}$ for all $\xi \in \mathcal{E}$ and the hypothesis that the number of off-diagonal solutions is bounded by $N^{76}$ to obtain

$$\sum \limits_{\xi_1 + \xi_2 + \xi_3 - \xi'_1 - \xi'_2 - \xi'_3 = 0, \text{off-diagonal}} |a_{\xi_1} \overline{a_{\xi_1}} a_{\xi_2} \overline{a_{\xi_2}} a_{\xi_3} \overline{a_{\xi_3}}| \leq (g(N)/N)^3 \cdot N^{76} = o(1)$$

as $N \to \infty$ and this proves the lemma.

We are finally ready to conclude the proof of Theorem 1.3:

**Proof of Theorem 1.3.** The first claim of the Theorem follows directly combining Proposition 2.1 and Proposition 2.4. By Lemma 2.9 we deduce that the first two moments are uniformly integrable and therefore

$$\int_{T^2} (M f(x,r) - 1)^2 \to Var(\alpha + \beta W(\mu_A)) = \beta^2 Var(W(\mu_A))$$

which concludes the proof.

3. **Proof of Theorem 1.2**

As mentioned in the introduction, in [5, 6] it was proved that $f(x)$, when considered in a small neighbourhood of $x \in T^2$ and averaged over $x$, approximate a Gaussian field. Formally we fix some large parameter $R > 0$ and write $f$ around the point $x$ as
\[ F_x(y) = f \left( x + \frac{R}{\sqrt{Ey}} \right) \]

Then we have the following Proposition:

**Proposition 3.1.** Let \( f_E \) be as in (1.4) and \( \Omega \) some abstract probability space, suppose that \( \mathcal{E}(E) \) satisfies assumption A1 and \( N \to \infty \), moreover let \( \epsilon_1, \epsilon_2 > 0 \) be given, then there exists some \( B_0 = B_0(\epsilon_1, \epsilon_2, R) \) and some \( E_0 = E_0(\epsilon_1, \epsilon_2, R, B_0) \) such that for all \( E > E_0 \) and for all \( B > B_0 \) the following holds:

1. There exists a subspace \( \Omega' \subset \Omega \) with \( \Pr[\Omega'] > 1 - \epsilon_2 \) and a measure-preserving function \( \tau : \Omega' \to \mathbb{T}^2 \) such that \( \text{vol}(\mathbb{T}^2 \setminus \tau(\Omega')) \leq \epsilon_2 \).

2. Moreover \( \tau \) satisfies the following: uniformly for all \( f_E \) satisfying assumption A2,

\[
\sup_{y \in [-1/2,1/2]^2} |F_{\tau(\omega)}(y) - F_{\mu_f}^R(y, w)| \leq \epsilon_1
\]

for all \( \omega \in \Omega' \), where the covariance of \( F_{\mu_f}^R(y) \) is given by

\[
\mathbb{E}[F_{\mu_f}^R(y)F_{\mu_f}^R(x)] = \int_{S^1} e(iR(x - y))d\mu_f.
\]

Assuming the Proposition, which we will prove after, we prove the main theorem

**Proof of Theorem 1.2.** Let \( \epsilon > 0 \) and \( t > 0 \) be fixed and let \( X(\mu_f) = \int_{B(R)} |F_{\mu_f}|^2/\pi R^2 \), by Lévy continuity theorem, we can equivalently (up to possibly changing \( \epsilon \)) prove that

\[
\left| \int_{\mathbb{T}^2} \exp(itM_f(x, r))dx - \mathbb{E}[\exp(itX(\mu_f))] \right| \leq \epsilon \quad \text{(3.1)}
\]

for all \( E \) large enough (depending on \( \epsilon, t \) and \( R \)). We take \( \epsilon_2 = \epsilon/8 \) and \( \epsilon_1 = \epsilon_1(\epsilon, t) \) to be chosen later in Proposition 3.1 and let \( \tau : \Omega' \to \mathbb{T}^2 \) and \( F_{\mu_f}^R \) be given as in the Proposition, taking \( E > E_0 \) and \( B > B_0 \). Then we observe that, by a change of variables,

\[
X(\mu_f) = \frac{1}{\pi} \int_{B(1)} |F_{\mu_f}^R|^2 dx \quad \text{(3.2)}
\]

and moreover by Lemma 2.2 with \( \delta_1 = \epsilon/8 \) we obtain some \( M = M(\epsilon) \) such that

\[
\Pr \left( |F_{\mu_f}^R|^2 > M \right) \leq \epsilon/8.
\]

Choosing \( \epsilon_1 = \sqrt{\epsilon}/(8Mt) \) we have proved that outside an event of size at most \( \epsilon/4 \) and a subset of \( \mathbb{T}^2 \) of size at most \( \epsilon/8 \) we have

\[
\sup_{y \in [-1/2,1/2]^2} |F_{\tau(\omega)}(y) - F_{\mu_f}^R(y)| \leq \epsilon_1 \quad |F_{\mu_f}^R|^2 \leq M. \quad \text{(3.3)}
\]
Arguing as in Lemma 2.1, i.e. under conditions (3.3) we have
\[
\left| \exp(it M_f(\tau(\omega), r)) - \exp\left( it \int_{B(1)} |F^{R}_{\tau}|^2 dx \right) \right| \leq \epsilon/2
\]  
so combining (3.2), (3.4) and the fact that (3.3) holds outside a set of size at most \( \epsilon/4 \) and volume \( \epsilon/4 \) we obtain (3.1) and this concludes the proof. □

4. Bourgain’s de-randomisation: proof of Proposition 3.1

The material of this section is contained, in various forms, in [5, 6]. We present it here for the convenience of the reader and since Proposition 3.1, as stated, does not appear in the literature.

4.1. Approximating \( f \) is small squares. The aim of this section is to approximate the function \( f \) (in small squares) by a more tractable function so we fix some large parameter \( R \) and recall the notation
\[
F_x(y) = f\left(x + \frac{R}{\sqrt{E}} y\right) = \sum_{|\xi|^2 = E} a_\xi e(\langle \xi, x \rangle) e\left(\frac{\langle \xi, R y \rangle}{\sqrt{E}}\right)
\]
The points \( \{\xi/\sqrt{E}\} \) lie on the unit circle so as \( N \to \infty \) they will accumulate somewhere, we first want to approximate these accumulation points. To this end we pick some large parameter \( K \) and divide the circle into arcs of length \( 1/2K \), namely
\[
I_k = \left(\frac{k - 1}{2K}, \frac{k}{2K}\right)
\]
for \(-K + 1 \leq k \leq K\) and let
\[
\mathcal{E}^{(k)} = \{\xi \in \mathcal{E} : \xi/\sqrt{E} \in I_k\}.
\]
Now we pick some small parameter \( 0 < \delta < 1 \) and further subdivide \( \{\mathcal{E}^{(k)}\} \) accordingly to the measure \( \mu_f \) as
\[
\mathcal{K} = \{-1 + K \leq k \leq K : \mu_E(I_k) \geq \delta\} \quad \mathcal{G} = \bigcup_{k \in \mathcal{K}} \mathcal{E}^{(k)}
\]
and finally we approximate the points in \( \mathcal{E}^{(k)} \) via the middle point \( \zeta^k \) of \( I_k \). We have thus obtained the functions
\[
\tilde{F}_x(y) = \sum_{k \in \mathcal{K}} \sum_{\xi \in \mathcal{E}^{(k)}} a_\xi e(\langle \xi, x \rangle) e\left(\frac{\langle \xi, R y \rangle}{\sqrt{E}} - \zeta^k\right) e(\zeta^k, R y)
\]
\[
\tilde{\psi}_x(y) = F_x(y) - \tilde{F}_x(y)
\]
Now we focus on the inner sum in (4.1), since \( |\xi/\sqrt{E} - \zeta^k| \) can be made arbitrarily small taking \( K \) large, we approximate the whole sum by the term with \( y = 0 \),
moreover suppose that for all $N$ and \[ Lemma 4.2. \]

Let $\mu_f(I_k)\approx \sum \mu_f(I_k)\delta_{\xi_k} / \sum \mu_f(I_k)$ (4.2)

The above approximations are justified by the following lemma [5, Lemma 1] and [6, Proposition 3.2]

**Lemma 4.1.** Let $R > 1$, $\epsilon_3, \epsilon_4 > 0$ be given then there exists $K_0 = K_0(R, \epsilon_1, \epsilon_2)$ and $\delta_0 = \delta_0(R, K, \epsilon_3, \epsilon_4)$ and a subset $V \subset \mathbb{T}^2$ with $\text{vol}(V) \leq \epsilon_4$ such that for all $x \in \mathbb{T}^2 \setminus V$ we have

$$\sup_{y \in [-1/2,1/2]^2} |F_x - \phi_x| \leq \epsilon_3$$

4.2. **Passage to random fields.** The advantage in passing to $\phi_x$ in the previous section is the following: under assumptions A1 and A2 one can prove that $b_k$ tends to a Gaussian random variable for every $k$ essentially by the central limit theorem. To apply the latter we need the coefficients $a_\xi$ to be roughly uniform in size (i.e. some sort of Lindeberg’s condition) and the functions $\epsilon(\langle \xi, x \rangle)$ to be “independent”. The first conditions can be made rigorous by imposing a bound on the growth of the $a_\xi$’s as $E \to \infty$, and thus we require $f$ to be essentially “flat”, formally assumption A1. The second condition, on the other hand, requires more care: to exhibit convergence to a Gaussian random variable one is lead, via the method of moments, to consider expressions of the form

$$\int_{\mathbb{T}^2} |f(x)|^2 dx = \sum_{\xi_1, \ldots, \xi_{2k}} a_{\xi_1} a_{\xi_2} \ldots a_{\xi_{2k}} \int_{\mathbb{T}^2} \epsilon(\xi_1 - \xi_2 + \ldots - \xi_{2k}) dx. \quad (4.3)$$

By orthogonality the inner integral is non-zero if only if $\xi_1 - \xi_2 + \ldots - \xi_{2k} = 0$. The trivial solution is given by pair cancellation, i.e. $\xi_1 = \xi_2$, $\xi_3 = \xi_4$, ..., $\xi_{2k-1} = \xi_{2k}$ which gives a contribution of $\frac{2^k}{2\pi} N^4$. Thus the “independence” assumption can be made rigorous if we demand that all other solutions do not contribute much to (4.3), formally assumption A2. This allows us to prove that $\phi_x$ is well approximated by a random field, more precisely we have the following lemma [5, Lemma 2] and [6, Proposition 3.3]:

**Lemma 4.2.** Let $R, K > 1$, $0 < \delta < 1$ be as above and $\epsilon_5, \epsilon_6 > 0$ be given, moreover suppose that $f_E$ satisfies assumption A1, $E(E)$ satisfies assumption A2 and $N \to \infty$, then there exists $B_0 = B_0(\epsilon_5, \epsilon_6, R)$ and $E_0 = E_0(\epsilon_5, \epsilon_6, R, B_0)$ such that for all $E > E_0$ and for all $B > B_0$ the following holds:
There exists a subspace \( \Omega' \subset \Omega \) with \( \mathbb{P}[\Omega'] > 1 - \epsilon_6 \) and a measure-preserving function \( \tau : \Omega' \rightarrow \mathbb{T}^2 \) such that \( \text{vol}(\mathbb{T}^2 \setminus \tau(\Omega')) \leq \epsilon_6 \).

Moreover \( \tau \) satisfies the following: uniformly for all \( f \) satisfying assumption A2

\[
\sup_{y \in [-1/2,1/2]^2} |\phi_{\tau(\omega)}(y) - F_{\mu_K}^R(y,\omega)| \leq \epsilon_5
\]

for all \( \omega \in \Omega' \), where the covariance of \( F_{\mu_K}^R(y) \) is given by

\[
\mathbb{E}[F_{\mu_K}^R(y)F_{\mu_K}^R(x)] = \int_{S^1} e(iR(x - y))d\mu_K.
\]

To conclude the proof of the Proposition we only have to show that \( F_{\mu_K}^R \) is close to \( F_{\mu_f}^R \) and this is the content of the following lemma:

**Lemma 4.3.** Let \( \mu_K \) be given as in (4.2) then

\( \mu_K \Rightarrow \mu_f \)

as \( K \rightarrow \infty \).

**Proof.** Let \( A \subset S^1 \) be an open subset then

\[
\mu_f(A) = \sum_{\xi \in A} |a_\xi|^2
\]

on the other hand

\[
\mu_K(A) = \sum_{\zeta^{(k)} \in A} \mu_f(I_k) / \sum_{k \in \mathcal{K}} \mu_f(I_k) = \sum_{\zeta^{(k)} \in A} \sum_{\xi \in I_k} |a_\xi|^2 / \sum_{k \in \mathcal{K}} \mu_f(I_k).
\]

By definition of \( \mathcal{K} \), we have

\[
\sum_{k \in \mathcal{K}} \mu_f(I_k) = 1 - \sum_{k \notin \mathcal{K}} \mu_f(I_k) = 1 + O(\delta K)
\]

and similarly

\[
\sum_{\zeta^{(k)} \in A} \sum_{\xi \in I_k} |a_\xi|^2 = \sum_{\xi \in A} |a_\xi|^2 + O(\delta K)
\]

hence, taking \( \delta < K^{-2} \), we obtain

\[
\mu_K(A) = \mu_f(A) + O\left(\frac{1}{K}\right)
\]

as required. \( \square \)

We are now ready to complete the proof of Proposition 3.1:

**Proof of Proposition 3.1.** Let \( \epsilon_1, \epsilon_2 \) be given firstly we apply Lemma 2.3 together with Lemma 4.3 with \( \alpha = \epsilon_1 / 4 \) and \( \delta_2 = \epsilon_2 / 4 \) to see that

\[
\sup_{y \in [-1/2,1/2]^2} |F_{\mu_f}^R(y) - F_{\mu_K}^R(y)| \leq \epsilon_1 / 4 \quad (4.4)
\]
for all $K > K_0$ outside an event $\Omega''''$ of probability at most $\epsilon^2/8$. Now pick $\epsilon_5 = \epsilon_1/4$ and $\epsilon_6 = \epsilon_2/4$ in Lemma 4.2 and let $\Omega''''$ and $\tau$ be given by the lemma taking $E > E_0$ and $B > B_0$. Then we define $\Omega'' = \Omega'''' \setminus \Omega'''''$, since $P[\Omega'''] > 1 - \epsilon_2/4$ we obtain that $P[\Omega'''] > 1 - \epsilon_1/2$. So by Lemma 4.1 and the triangular inequality we obtain that

$$\sup_{y \in [-1/2,1/2]^2} |\phi_\tau(\omega)(y) - F_{\mu_f}(y)| \leq \epsilon_1/2$$

(4.5)

for all $\omega \in \Omega''$. Now we let $\epsilon_3 = \epsilon_1/2$ and $\epsilon_4 = \epsilon_2/4$ in Lemma 4.1 to obtain a set $V$ of volume at most $\epsilon_2/4$ such that

$$|F_x - \phi_x| \leq \epsilon_1/2$$

(4.6)

for all $x \in \mathbb{T}^2 \setminus V$. Now since $\tau$ is measure-preserving we have that $P[\tau^{-1}(V)] \leq \epsilon_2/2$ so we can finally take $\Omega' = \Omega'' \setminus \tau^{-1}(V)$ and see that $P[\Omega'] > 1 - \epsilon_2$ and for all $\omega \in \Omega'$ in light of (4.5) and (4.6) we have

$$\sup_{y \in [-1,1/2]^2} |F_\tau(\omega)(y) - F_{\mu_f}(y)| \leq \epsilon_1$$

as required. \qed

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**References**

[1] R. J. Adler and J. E. Taylor. *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York, 2007.
[2] Berry, Michael, *Regular and irregular semiclassical wavefunctions*. J. Phys. A 10 (1977), no. 12.
[3] Berry, Michael, *Semiclassical mechanics of regular and irregular motion*. Chaotic behavior of deterministic systems (Les Houches, 1981), 171271, North-Holland, Amsterdam.
[4] Enrico Bombieri and Jean Bourgain, *A problem on sums of two squares*, IMRN 11 (2015), 33433407.
[5] Jean Bourgain, *On toral eigenfunctions and the random wave model*. Israel J. Math. 201 (2014), no. 2, 611-630, DOI 10.1007/s11856-014-1037-2. MR3265298.
[6] J. Buckley, and I. Wigman. *On The Number Of Nodal Domains Of Toral Eigenfunction*. Annales Henri Poincar 17.11 (2016): 3027-3062. Web.
[7] Cilleruelo, Javier. *The distribution of the lattice points on circles*. J. Number Theory 43 (1993), no. 2, 198202.
[8] Y. Colin de Verdière, *Ergodicité et fonctions propres du Laplacien*. Comm. Math. Phys. 102 (1985).
[9] Erdős, P., and R.R. Hall. *On The Angular Distribution Of Gaussian Integers With Fixed Norm*. Discrete Mathematics 200.1-3 (1999): 87-94. Web.
[10] Granville, A., Wigman, I., Planck-scale mass equidistribution of toral Laplace eigenfunctions, Communications in Mathematical Physics, 355(2), 2017,

[11] Hezari, Hamid; Rivi`ere, Gabriel. $L^p$ norms, nodal sets, and quantum ergodicity. Adv. Math. 290 (2016), 938966.

[12] Han, Xiaolong. Small scale quantum ergodicity in negatively curved manifolds. Nonlinearity 28 (2015), no. 9.

[13] Humphries, P., 2017. Equidistribution in Shrinking Sets and $L^4$-Norm Bounds for Automorphic Forms. arXiv preprint arXiv:1705.05488.

[14] KátaI, I. Környei, I. “On the distribution of lattice points on circles ”. Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 19 (1976), 8791 (1977). (Reviewer: Ekkehard Kratzel) 10J05 (10E05)

[15] Kurlberg P. and Wigman I. On Probability Measures Arising From Lattice Points On Circles. Mathematische Annalen (2016)

[16] Lester, Stephen; Rudnick, Ze`ev. Small scale equidistribution of eigenfunctions on the torus. Comm. Math. Phys. 350 (2017), no. 1, 279300.

[17] Luo, Wen Zhi; Sarnak, Peter. Quantum ergodicity of eigenfunctions on $P SL_2(\mathbb{Z}) \backslash \mathbb{H}$ 2. Inst. Hautes Etudes Sci. Publ. Math. No. 81 (1995), 207237.

[18] A. Snirelman, Ergodic properties of eigenfunctions, Uspekhi Mat. Nauk 180 (1974), 181-182.

[19] Sodin, M. Lectures on random nodal portraits. Probability and statistical physics in St. Petersburg, 91 (2016), 395422.

[20] I. Wigman, N. Yesha CLT for mass distribution of Toral Laplacian eigenfunctions https://arxiv.org/abs/1712.03318

[21] Young, M. The quantum unique ergodicity conjecture for thin sets. Adv. Math. 286 (2016), 958-1016

[22] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces, Duke Math. J. 55 (1987), 919-941