An Exact Convex Relaxation for the Freeway Network Control Problem in the Case of Controlled Merges

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Abstract

We consider the Freeway Network Control problem, that is, we aim to optimize the operation of traffic networks modeled by the Cell Transmission Model via ramp metering and partial mainline demand control. Optimal control problems for the Cell Transmission Model are usually nonconvex due to the nonlinear fundamental diagram, but a convex relaxation in which demand and supply constraints are relaxed is often used. Previous works have established conditions which ensure that solutions of the relaxation can be made feasible with respect to the original constraints, in particular by using mainline demand control. In addition, conditions on the network structure and the flow dynamics have been derived which ensure that ramp metering alone is sufficient to make solutions to the relaxation feasible if the objective is to minimize the Total Time Spent. In this work, we generalize these conditions and show that control of flows into the merge vertices in a traffic network is sufficient to make solutions to the relaxation feasible.

1 Introduction

We study the Freeway Network Control (FNC) problem, that is, the problem of optimal operation of freeway traffic for networks modeled using a variant of the Cell Transmission Model (CTM) [9, 10], with the standard objective of minimizing the Total Time Spent (TTS). The CTM is a first order traffic model obtained as a discretization of the kinematic wave model [17, 22]. It describes road traffic by a conservation law and the nonlinear fundamental diagram, which models the relationship between traffic flow and traffic densities.

Finite horizon optimal control problems for systems modeled by the CTM lead to nonconvex optimization problems in general, due to the nonlinear fundamental diagram. However, a convex optimization problem is obtained if the demand and supply constraints encoded in the fundamental diagram are relaxed. In particular, a Linear Program (LP) is obtained if the standard, triangular or trapezoidal fundamental diagram is used [20]. In general, the solution to the relaxation does not satisfy the dynamics of the CTM, but subsequent work has identified conditions for which the relaxation can be used in order to obtain solutions to the original problem. In particular, it turns out that for a freeway stretch with only onramp merge junctions and only offramp diverge junctions solutions to the relaxed problem are feasible with respect to the CTM dynamics [12]. This result relies on the assumption that onramps are metered and inflow from onramps is not obstructed by mainline congestion, while offramps are assumed to be uncongested and hence, they do not obstruct mainline flow via spill back effects. In [13] it has been shown that the corresponding CTM model is in fact a discrete-time, monotone system, a generalization of monotone maps [15] to systems with actuated inputs. Basic definitions and results on monotone systems are presented in [1] for the analogous continuous-time case.

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Consequently, it is natural to ask whether monotonicity properties can be leveraged to facilitate the analysis and control of systems based on the CTM. However, it turns out that First-In, First-Out (FIFO) diverge junction dynamics as used in the CTM are not monotone [7]. Alternative diverge models with monotone dynamics have been suggested [18, 19], but these models do not preserve the turning ratios. In [8] it is shown that the CTM dynamics satisfy a mixed-monotonicity property instead, but it is also suggested that the non-monotone dynamics of FIFO diverge junctions are exactly what dynamic traffic control should target in order to realize improvements over the uncontrolled case. Monotonicity has also been used to analyze robustness of optimal trajectories [4]. In addition, traffic routing problems have been considered. In such problems, the turning ratios are not fixed a-priori but they are (partially) actuated variables. With time-varying turning ratios, diverge junctions do not exhibit FIFO dynamics, which allows to circumvent the issues arising from the non-monotone effects. In particular, monotone routing policies show favorable resilience to non-anticipated capacity reductions in individual links [6, 5]. In subsequent work, a class of distributed, monotone routing policies was proposed, which make use of the implicit back-propagation of congestion to stabilize maximal-throughput equilibria [3].

It has also been suggested that solutions to the relaxed FNC problem (using relaxed demand and supply constraints) for arbitrary networks can be made feasible if traffic demand control is available in every cell of the CTM, for example via variable speed limits [20, 4]. However, it is questionable whether the assumption of demand control in every cell is realistic, in particular for freeway networks. Even if variable speed limits are installed, admissible operation is usually restrictive, with only few distinct speed limits to choose from and potentially additional constraints on how often these may change. Therefore, the question of whether demand control in every cell is necessary is highly important. A partial answer is known for the special case of a symmetric triangular fundamental diagram in which the congestion wave speed is equal to the free-flow velocity in every cell. In this case, [4, Proposition 2] states that the solution to the relaxed FNC problem can be made feasible by only using priority control for flows into merge junctions.

In this work, we generalize these results and consider CTM networks with FIFO-merge junctions and concave (but not necessarily symmetric or even PWA) fundamental diagrams. We show that control of the flows into merge junctions is sufficient to make solutions to the relaxed FNC problem feasible if the objective is to minimize the TTS. This result allows to use the convex, relaxed problem to efficiently compute solutions of the original nonconvex FNC problem, suitable for example for real-time optimization of traffic controls that can be applied in a receding horizon manner. The main result of this work relies on the analysis of a novel, alternative system representation of the CTM, obtained via a state transformation. It turns out that the system dynamics are concave and state-monotone in the new representation. This allows to employ results originally derived for convex, monotone systems [21] to show equivalence of the convex relaxation to the nonconvex optimal control problem.

We generalize existing results, in particular [4] which addresses a similar problem, in the following ways:

- Our main result applies to CTM networks with general concave, monotone fundamental diagrams. The existing result holds only for affine demand and supply functions with identical slope (of opposite sign). Real-world free-flow speeds, which are equal to the slope of the demand function are typically significantly larger than congestion wave speeds, which correspond to the slope of the supply function.

- We derive the main result by analyzing an alternative system representation obtained via a coordinate change, which transforms the system dynamics into a state-monotone, concave form. The reformulation of the system dynamics links the result to properties of the dynamical system itself and hence, it suggests a straightforward method to verify if the results continue to hold for extensions or modifications of the models studied in this work.

This paper is structured as follows: in Section 2, we introduce results on the optimal control of convex, state-monotone respectively concave, state-monotone systems that will be used subsequently. The freeway network model is introduced in Section 3 and we formulate the main problem. In Section 4, we perform a state transformation to derive an equivalent system representation and show that it is concave and state-monotone. This allows to prove the main result of this work, the derivation of an exact, convex relaxation of the FNC problem for networks with controlled merges. We contrast the properties of merge and diverge

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1 A different result in [4] allows for the more general fundamental diagrams, however, it requires demand control in every cell like in [20], as opposed to only in cells immediately upstream of merges.
juncions in the original system model to the ones in the alternative representation in Section 5 in order
to demonstrate the applicability of results in either formulation. In Section 6 we apply the main result to
compute optimal open-loop controls for two freeway network examples, a real world freeway stretch and an
artificial network designed to showcase the behavior of merges and diverges. Finally, we conclude in Section
7 and provide suggestions for future work.

2 Preliminaries

Consider the discrete-time, nonlinear dynamical system with state \( x(t) \in X \subseteq \mathbb{R}^n \), input \( u(t) \in U \subseteq \mathbb{R}^m \) and dynamics given as

\[
x(t + 1) = f(x(t), u(t))
\]

with \( f : X \times U \to X \). In this work, we are interested in the special case when the system dynamics are both
jointly convex (in \( x(t) \) and \( u(t) \)) and monotone in the states, as defined next:

**Definition 1.** A system is state-monotone if for all \( x_1 \in X \), \( x_2 \in X \) such that \( x_1 \geq x_2 \) it holds that
\[
f(x_1, u) \geq f(x_2, u) \quad \forall u \in U.
\]

A system is convex if the system equations \( f(x, u) \) are jointly convex in \( x \) and \( u \) and the sets \( X \) and \( U \) are closed and convex. We call systems that satisfy both properties convex,
state-monotone systems.

Our definition of a state-monotone system is closely related to the standard definition of a cooperative
system, a special case of the order preserving monotone system [1]. To be precise, cooperative systems are
systems that are monotone with respect to the order induced by the positive orthant \( \mathbb{R}^n_+ \). Note that the
previous reference considers continuous time systems while we exclusively consider the discrete time case: in
discrete time, finite horizon optimal control problems can be reformulated as finite dimensional optimization
problems which are suitable for numerical solution.

The idea of a convex-monotone system, that is, a system with system equations that are both monotone
and convex, has been introduced in [21] and much of the remainder of this section follows the reasoning and
results in this paper. However, different from the existing paper, we drop the assumption of monotonicity in
the inputs but impose additional constraints on the control objective and potential constraints. In particular,
we assume that the control objective is the minimization of a stage-wise cost \( c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) which is
convex and state-monotone, that is, \( x_1 \geq x_2 \) implies that \( c(x_1, u) \geq c(x_2, u) \) for all \( u \in U \). For ease
of notation, define a vector containing all states \( x := [x(0)^\top \ x(1)^\top \ \ldots \ x(T-1)^\top]^\top \) and a vector
\( u := [u(0)^\top \ u(1)^\top \ \ldots \ u(T-1)^\top]^\top \) containing all inputs over the entire horizon. We also allow convex,
state-monotone constraints on state and input \( g(x, u) \leq 0 \), that is, we assume that \( g : \mathbb{R}^{n} \times \mathbb{R}^{n} \to \mathbb{R}^c \) (with \( c \) the number of constraints) is convex and state-monotone. For convex, state-monotone systems equipped
with a stage-wise convex, state-monotone cost and convex, state-monotone constraints, we consider the
finite-horizon optimal control problem

\[
P^* := \min_{x(t), u(t)} \quad \sum_{t=0}^{T-1} c(x(t), u(t)) + c_T(x(T))
\]

subject to

\[
x(t + 1) = f(x(t), u(t)) \quad \forall t \in \{0, 1, \ldots, T-1\}
\]

\[
g(x, u) \leq 0
\]

\[
x(0) \text{ given}.
\]

This problem is nonconvex due to the nonlinear equality constraints encoding the system dynamics. However,
the convex relaxation

\[
R^* := \min_{z(t), v(t)} \quad \sum_{t=0}^{T-1} c(z(t), v(t)) + c_T(z(T))
\]

subject to

\[
z(t + 1) \geq f(z(t), v(t)) \quad \forall t \in \{0, 1, \ldots, T-1\}
\]

\[
g(z, v) \leq 0
\]

\[
z(0) = x(0) \text{ given}
\]

allows for efficient solution of the original problem.
Theorem 1. For a convex, state-monotone system \( x(t+1) = f(x(t), u(t)) \) with convex, state-monotone stage cost \( c(x(t), u(t)) \) respectively \( c_T(x(T)) \) and convex, state-monotone constraints \( g(x, u) \), problems \(^1\) and \(^2\) are equivalent in the sense that the optimal values coincide \( P^* = R^* \) and a feasible solution to \(^1\) can be constructed from a solution to \(^2\), by forward simulation using the known initial state and the sequence of inputs for any optimizer of the relaxed problem.

Proof. Assume \( z^*(t), v^*(t) \) is an optimizer of the relaxed problem. Consider now the candidate solution \( u^*(t) = v^*(t), x^*(0) = z^*(0) \) and \( x^*(t+1) = f(x^*(t), u^*(t)) \). Note that from \( x^*(0) = z^*(0) \) it follows inductively that

\[
x^*(t) = f(x^*(t-1), u^*(t-1)) \leq f(z^*(t-1), u^*(t-1)) \leq z^*(t) \quad \forall t \in \{1, T\}
\]

This in turn implies that

\[
g(x^*, u^*) \leq g(z^*, v^*) \leq 0
\]

that is, \( x^*(t), u^*(t) \) is a feasible solution to the original problem. Furthermore,

\[
P^* \leq \sum_{t=0}^{T-1} c(x^*(t), u^*(t)) + c_T(x^*(T)) \leq \sum_{t=0}^{T-1} c(z^*(t), v^*(t)) + c_T(z^*(T)) = R^*
\]

Since the second problem is a relaxation of the first, \( P^* \geq R^* \) and the claim follows when combining the two inequalities. \( \square \)

The existence of an exact, convex relaxation in the sense of Theorem \(^1\) is the main reason for our interest in convex, state-monotone systems, since it allows the efficient solution of finite horizon optimal control problems. An equivalent result can be found for concave, state-monotone systems when maximizing an objective function.

Corollary 1. For a concave, state-monotone system \( x(t+1) = f(x(t), u(t)) \) with concave, state-monotone cost \( c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \), the finite horizon optimal control problem

\[
P^* := \max_{x(t), u(t)} \sum_{t=0}^{T-1} c(x(t), u(t)) + c_T(x(T))
\]

subject to

\[
x(t+1) = f(x(t), u(t)) \quad \forall t \in \{0, 1, \ldots, T-1\}
\]

\[
g(x, u) \geq 0
\]

\[
x(0) \text{ given}
\]

is equivalent to its convex relaxation, in which the constraints encoding the state evolution are relaxed to \( x(t+1) \leq f(x(t), u(t)) \), in the sense that the optimal values coincide and a feasible solution to the original problem can be computed by forward simulation using the known initial state and the sequence of inputs for any optimizer of the relaxed problem.

Note that we now assume concavity of most functions instead of convexity, that the order of the inequalities changes and that we maximize an objective instead of minimizing a cost. The proof is perfectly analogous to the proof of Theorem \(^1\) and therefore omitted. In the next chapters, we will demonstrate how certain FNC problems can be cast as concave, state-monotone problems and hence, can be solved efficiently.

3 Problem statement

We consider the dynamic traffic assignment problem for freeway networks \(^4\) \(^20\). Our traffic model is very similar to the one used in the former reference, although we make additional assumptions, in particular on merge junctions, that will be detailed later. Similar models are also studied e.g. in \(^7\) \(^8\), where the focus is on stability of system equilibria. All these models are based on the Cell Transmission Model (CTM) \(^9\) \(^10\).

The freeway network is represented by a directed graph \( G = (V, E) \) in which the edges \( e \in E \subseteq V \times V \) represent cells, that is, parts of a road, and the vertices represent either junctions (merge or diverge) or interfaces between consecutive cells. We denote the origin of an edge \( e \) as \( \sigma_e \) and the tail as \( \tau_e \), i.e.,
Assumption 1. The directed network graph \( G \) does not contain self-loops. In addition, we assume that \( M \cap D = \emptyset \), that is, there do not exist vertices that are both merges and diverges. We also assume \( M \cap S = \emptyset \), that is, merges are not sinks.

The state of each cell is described by the traffic density \( \rho_c(t) \), i.e., the number of cars in a cell divided by the respective length of the cell \( l_c \). We adopt a discrete-time model in which the evolution of the system is described by flows of cars during discrete time intervals of duration \( \Delta t \). For each cell, we define the traffic flow \( \phi_t(t) \) as the traffic flow out of cell \( e \) during the time interval \( t \). To model how traffic flows are distributed on multiple downstream cells in case of a diverge, we introduce turn ratios \( 0 < \beta_{c,i} \leq 1 \) for any two adjacent edges \( e \) and \( i \), that is, if \( \tau_e = o_i \). The turn ratio \( \beta_{c,i} \) models the percentage of flow leaving cell \( e \) that travels to cell \( i \) and we assume that the turn ratios are invariant in time. To simplify notation, we also define \( \beta_{c,i} = 0 \) whenever \( \tau_e \neq o_i \). The conservation law of traffic requires that that \( \sum_{e \in \mathcal{E}} \beta_{c,i} \leq 1 \). We allow for \( \sum_{e \in \mathcal{E}} \beta_{c,i} < 1 \) and we assume that the percentage of flow that is not distributed onto the downstream cells has left the network, e.g. via an offramp.

In addition to the flows within the network, external flows \( w_e(t) \) may enter the network. The traffic densities evolve according to the conservation law

\[
\rho_c(t+1) = \rho_c(t) + \frac{\Delta t}{l_e} \left( \sum_{i \in \mathcal{E}} \beta_{i,c} \phi_i(t) - \phi_c(t) + w_e(t) \right) \quad \forall e \in \mathcal{E}.
\]

The CTM is a first-order model and the flows \( \phi(t) \) will be computed as functions of the states \( \rho(t) \). In general, the traffic flows depend on the traffic demand of cars that seek to travel downstream within a time interval and the supply of free space in downstream cells. To model this behavior, we introduce a demand function \( d_c(\rho_c(t)) : \mathbb{R} \to \mathbb{R} \) and a supply function \( s_c(\rho_c(t)) : \mathbb{R} \to \mathbb{R} \) for each cell \( e \in \mathcal{E} \). Demand- and supply functions are often combined in the fundamental diagram of a cell, as depicted in Figure 2. In the original work of \([9,10]\), a piecewise-affine (PWA) fundamental diagram was assumed, as it is obtained from the Godunov discretization of the Lighthill-Whitham-Richards (LWR) model \([17,22]\). In practice, one might want to consider more general shapes of the fundamental diagram and hence of the demand- and supply functions, in order to better approximate real world data, see e.g. \([3,8,18]\) for recent examples. In the remainder of this work, we will assume that:

\[ e = (\sigma_e, \tau_e) \]. Traffic moves along the directed edges and we define the set of downstream cells as \( \mathcal{E}^+(e) := \{ i : \sigma_i = \tau_e \} \subset \mathcal{E} \) and the set of upstream cells as \( \mathcal{E}^-(e) := \{ i : \tau_i = \sigma_e \} \subset \mathcal{E} \). Similarly, we define the downstream cells of a junction (vertex) \( v \) as \( \mathcal{E}^+(v) := \{ i : \sigma_i = v \} \subset \mathcal{E} \) and the upstream cells of a junction \( v \) as \( \mathcal{E}^-(v) := \{ i : \tau_i = v \} \subset \mathcal{E} \). The indegree, that is, the number of incoming edges of a vertex \( v \) is denoted \( \deg^- (v) \) and the outdegree, the number of edges leaving a vertex \( v \) is \( \deg^+ (v) \). Junctions with two or more outgoing edges are called diverge vertices and the set of all diverges is denoted \( D := \{ v : \deg^+(v) > 1 \} \subset \mathcal{V} \). Similarly, vertices with more than one incoming edge are called merge vertices (or simply merges) and we write \( M := \{ v : \deg^-(v) > 1 \} \subset \mathcal{V} \) for the set of all merges. We use \( \mathcal{N} := \{ e : \tau_e \in M \} \) to denote the set of all cells directly upstream of a merge junction. A vertex without outgoing edges is called a sink and \( S := \{ v : \deg^+(v) = 0 \} \).

\[ \sigma_e \in M \quad e_1 \in E \quad \tau_e \in D \quad e_4 \in E^+(e_1) \quad e_5 \in E^+(e_1) \]

(a) Example notation

\[ e = (w, w) \quad v \in M \cup D \]

(b) Violations of Assumption 1

Figure 1: On the left, the notation is exemplified on a subgraph. The subgraph on the right violates Assumption 1 in particular, edge \( e \) violates the assumption of no self loops and vertex \( v \) violates the assumption that vertices are not both merges and diverges.
Figure 2: Different shapes of the fundamental diagram may be desirable in order to approximate real-world data. Figure (a) depicts the traditional triangular version. Figure (b) shows a version with concave and monotone demand and supply functions, satisfying Assumption 2 for suitable $\Delta t$. Figure (c) shows a fundamental diagram with a capacity drop in congestion, modeled via a non-conave and non-monotone demand function, which does not satisfy Assumption 2.

**Assumption 2.** For every cell $e$, define a maximal density $\bar{\rho}_e$, called the traffic jam density. The demand $d_e(\rho_e(t))$, $d_e : [0, \bar{\rho}] \to \mathbb{R}_+$ is a concave, Lipschitz-continuous, nondecreasing function with $d_e(0) = 0$. Conversely, the supply $s_e(x)$, $s_e : [0, \bar{\rho}] \to \mathbb{R}_+$ is a concave, Lipschitz-continuous, nonincreasing function with $s_e(\bar{\rho}_e) = 0$. Furthermore, the sampling time $\Delta t$ is chosen such that it satisfies the bounds

$$
\Delta t \leq \frac{l_e}{\gamma_d} \quad \forall e \in \mathcal{E},
$$

$$
\Delta t \leq \frac{l_e}{\gamma_s} \quad \forall e \in \mathcal{E},
$$

with respect to the Lipschitz constant of the demand $\gamma_d$ and of the supply $\gamma_s$.

The classical, piecewise affine fundamental diagram satisfies this assumption: Here, we have that $d_e(\rho_e(t)) = \min \left\{ v_e \rho_e, \frac{w_e}{v_e + w_e} \rho_e \right\}$ and $s_e(\rho_e(t)) = \min \left\{ \frac{w_e}{v_e + w_e} \rho_e, (\bar{\rho}_e - \rho_e) w_e \right\}$ with $v_e$ the free-flow speed and $w_e$ the congestion wave speed. The demand is non-decreasing and the supply is non-increasing by definition, also both functions are concave since they are defined as the pointwise minimum of affine functions. The condition on $\Delta t$ can be recognized as the stability condition $v_e \cdot \Delta t \leq l_e$, $\forall e$ that arises if the CTM is derived as a discretization of the wave PDE using the Godunov scheme. Note that in practice, the congestion-wave speed $w_e$ is significantly lower than the free-flow speed, thus the upper bound in inequality (4b) in Assumption 2 is not restrictive. Different shapes of the fundamental diagram encountered in practice are depicted in Figure 2 for illustration. It is worth highlighting that there is empirical evidence for a so-called capacity drop, that is, a non-monotone demand function as depicted in Figure 2c, see e.g. [16] for an overview of modeling approaches in first-order models. Assumption 2 excludes such a behavior. This assumption helps to keep the problem tractable and it is in line with most of the previously cited research on the FNC problem.

Using demand and supply functions, we are now ready to state the equations for the flows. For non-merge flows $\phi_e(t)$, that is, for $e \notin \mathcal{N}$, the flows are given as the minimum of upstream traffic demand and downstream supply of free space. Cases in which not all demand can be served are modeled using the First-In, First-Out (FIFO) model: For every cell $e \notin \mathcal{N}$, the demand satisfaction is computed as

$$
\kappa_e(t) = \min \left\{ 1, \min_{i \in \mathcal{E}^+(e)} \left\{ \frac{s_i(\rho_i(t)) - w_i(t)}{\beta_{e,i} \cdot d_e(\rho_e(t))} \right\} \right\}.
$$

Note that for any sink $s$, the set of downstream cells is empty $\mathcal{E}^+(s) = \emptyset$ and $\kappa_s(t) = 1$. Using the demand satisfaction, the flows are then computed as

$$
\phi_e(t) = \kappa_e(t) \cdot d_e(\rho_e(t)) \quad \forall e \in \mathcal{E} \setminus \mathcal{N}
$$
Figure 3: Types of admissible merges. Controlled flows are highlighted in the symmetric controlled merge and the asymmetric controlled merge. In the latter, $\phi_2(t) = \min\{d_2(t), s_3(t) - \phi_1(t)\}$ is shorthand for $\phi_2(t) = \min\{d_2(\rho_2(t)), s_3(\rho_3(t)) - \phi_1(t)\}$, assuming that $w_3(t) = 0$ in this particular instance.

In diverges, the turn ratios $\beta_{e,i}$ are respected at all times in the FIFO model. This implies that a congestion in any one of the cells downstream of a merge will potentially also reduce the flow from the diverge to other downstream cells. We will have a closer look at this effect and its implications for optimal traffic control in Section 5.

It remains to define the merge flows, that is, $\phi_e(t)$ for $e \in \mathcal{N}$. In contrast to existing work, we do not introduce a general merge model, but we assume that any merge falls into one of three distinct categories:

(i) **Symmetric controlled merges**: Let $\mathcal{M}_S \subset \mathcal{M}$ denote the set of symmetric controlled merges and define the set of cells immediately upstream of a symmetric-controlled merge as $\mathcal{N}_S := \{e : \tau_e \in \mathcal{M}_S\} \subset \mathcal{N}$. In a symmetric controlled merge, we assume that all merge flows are controlled. That means the flows $\phi_e(t)$ for $e \in \mathcal{N}_S$ are control inputs, subject to the constraints

\[
0 \leq \phi_e(t) \leq d_e(\rho_e(t)) \quad \quad \forall e \in \mathcal{N}_S,
\]
\[
\sum_{e \in \mathcal{E}} \beta_{e,i} \cdot \phi_e(t) \leq s_i(\rho_i(t)) - w_i(t) \quad \quad \forall i : \sigma_i \in \mathcal{M}_S.
\]

Note that in the latter constraint, $\beta_{e,i} > 0$ implies that $e \in \mathcal{E}^-(i)$ and hence, the summation is in only over the cells $e \in \mathcal{N}_S$ immediately upstream of the merge.

To realize a symmetric controlled merges in the real world, all incoming roads might be equipped with controlled traffic lights to enable explicit control of the flows. Alternatively, the possibility of utilizing velocity control to control flows has also been discussed, e.g. in [20].

(ii) **Asymmetric controlled merges**: Intuitively speaking, all but one of the flows into an asymmetric merge are controlled. Prototypical asymmetric controlled merges are onramps equipped with ramp metering, where the inflow from the ramp is controlled, but the mainline flow is not. To formalize this idea, let $\mathcal{M}_A \subset \mathcal{M}$ denote the set of asymmetric controlled merges and we introduce the set of controlled cells immediately upstream of a controlled merge $\mathcal{N}_A$ and the set of of uncontrolled cells $\mathcal{N}_M$, with $\mathcal{N}_A \cap \mathcal{N}_M = \emptyset$ and $\mathcal{N}_A \cup \mathcal{N}_M = \{e : \tau_e \in \mathcal{M}_A\}$. We assume that for any particular asymmetric merge $v \in \mathcal{M}_A$ there is exactly one uncontrolled flow into this merge $\{e : \tau_e = v\}$ as stated before. The controlled flows $\phi_e(t)$ for $e \in \mathcal{N}_A$ are actuated inputs subject to the constraint

\[
0 \leq \phi_e(t) \leq d_e(\rho_e(t)) \quad \forall e \in \mathcal{N}_A.
\]

The flow from cells $e \in \mathcal{N}_M$ is assumed to be uncontrolled, that is, it is determined by the fundamental diagram

\[
\phi_e(t) = \min \left\{ d_e(\rho_e(t)) , \frac{s_i(\rho_i(t)) - w_i(t) - \sum_{j \in \mathcal{N}_A} \beta_{j,i} \cdot \phi_j(t)}{\beta_{e,i}} \right\} \quad \forall e \in \mathcal{N}_M, \ i \in \mathcal{E}^+(e).
\]

Note that since $\mathcal{D} \cap \mathcal{M} = \emptyset$, there is a unique cell $i \in \mathcal{E}^+(e)$ downstream of every asymmetric merge. In the former equation, priority is given to the controlled flows: from the supply of free space in cell
i, the controlled flows \( \phi_j(t) \) for \( j \in \mathcal{N}_A \) and the external demand \( w_i(t) \) are substracted first. Only the remainder of the free space is allocated for \( \phi_e(t) \).

It is important to highlight that the controlled flows are not constrained by the supply of free space downstream of the merge. To obtain a valid system model, this requires an additional assumption, that there is always sufficient free space to accommodate all of the demand originating from controlled cells

\[
s_i(\rho_e(t)) \geq w_i(t) + \sum_{j \in \mathcal{N}_A} \beta_{j,i} \cdot d(\rho_j(t)) \quad \forall i \in \mathcal{E}^+(e) : e \in \mathcal{M}_M. \tag{5}
\]

In reality, this assumption might be typically satisfied for certain merges, e.g. when a metered onramp is modeled as a controlled merge and the onramp demand is much smaller than the mainline demand. Then, it might be reasonable to assume that the onramp demand can always be released onto the mainline in typical operation conditions, if given priority. In fact, the main motivation for introducing the asymmetric controlled merge is to model metered ramps as shown in Section 6.1.

(iii) **Uncongested merges**: Let \( \mathcal{M}_U \subset \mathcal{M} \) denote the set of uncontrolled merges and define the set of cells immediately upstream of an uncontrolled merge as \( \mathcal{N}_U := \{ e : \tau_e \in \mathcal{M}_U \} \). In uncongested merges, all flows are equal to the upstream demands

\[
\phi_e(t) = d_e(\rho_e(t)) \quad \forall e \in \mathcal{N}_U
\]

at all times. The uncongested merge model might be suitable for intersections for which typical traffic volume is low, such that traffic densities remain far below the critical densities in all relevant operating conditions.

The different merge categories are depicted in Figure 3. We assume that the three types of merges partition the set of merges \( \mathcal{M} \) and call the system model the **CTM with controlled merges** to highlight that only these three types of merges are permissible.

In order to state the main problem, it remains to define the objective of active traffic control. As indicated before, we aim to compute open-loop optimal control inputs, for horizons such as a full day or a rush-hour period. A natural objective is to minimize the **Total Time Spent** (TTS) on the road

\[
TTS = \Delta t \cdot \sum_{t=1}^{T} \sum_{e \in \mathcal{E}} l_e \cdot \rho_e(t)
\]

over the whole horizon \( t \in \{1, 2, \ldots, T\} \), for all drivers. The FNC problem for the CTM with controlled merges is defined as

\[
\mathcal{P}_{\text{CTM}}^* = \begin{array}{l}
\text{minimize} \quad \Delta t \cdot \sum_{t=1}^{T} \sum_{e \in \mathcal{E}} l_e \cdot \rho_e(t) \\
\text{subject to} \\
\rho_e(t+1) = \rho_e(t) + \frac{\Delta t}{l_e} \cdot \left( \sum_{i \in \mathcal{E}(e)} \beta_{i,e} \phi_i(t) - \phi_e(t) + w_e(t) \right) \quad \forall e \in \mathcal{E} \\
\phi_e(t) = \min \left\{ d_e(\rho_e(t)), \min_{i \in \mathcal{E}(e)} \left\{ \frac{s_i(\rho_i(t) - w_i(t))}{\beta_{i,e}} \right\} \right\} \quad \forall e \in \mathcal{E} \setminus \mathcal{N} \\
\phi_e(t) = \min \left\{ d_e(\rho_e(t)), \frac{s_i(\rho_i(t) - w_i(t)) - \sum_{j \in \mathcal{N}_A} \beta_{j,i} \phi_j(t)}{\beta_{i,e}} \right\} \quad \forall e \in \mathcal{M}_M, i \in \mathcal{E}^+(e) \\
\phi_e(t) = d_e(\rho_e(t)) \quad \forall e \in \mathcal{N}_U \\
0 \leq \phi_e(t) \leq d_e(\rho_e(t)) \quad \forall e \in \mathcal{N}_S \cup \mathcal{N}_A \\
\sum_{i \in \mathcal{E}} \beta_{i,e} \cdot \phi_i(t) \leq s_e(\rho_e(t)) - w_e(t) \quad \forall e : \tau_e \in \mathcal{M}_S \\
\rho(0) \text{ given}
\end{array}
\tag{6}
\]

The optimization problem is non-convex, due to the nonlinear equality constraints that describe the fundamental diagram. We aim to find a convex relaxation of this problem that allows to construct an optimal solution for Problem (6). To do so, we will perform a state transformation in the next section to facilitate analysis.
Remark 1. The FNC problem assumes that predictions for the external traffic demands \( w_e(t) \) are available for the optimization horizon \( t \in \{0, \ldots, T-1\} \). Such predictions are highly uncertain in practice and hence, the inputs computed by solving \((6)\) should not be applied in open loop. This work focuses exclusively on the efficient solution of \((6)\), but it should be emphasized that control inputs obtained via solving the FNC problem should be applied e.g. in a receding horizon fashion to mitigate the effects of demand uncertainty and model uncertainty.

4 Conave, State-monotone Reformulation

In this section, we introduce an equivalent system representation of the traffic model presented in the previous section. To this end, we define the cumulative flow

\[
\Phi_e(t) := \Delta t \cdot \sum_{\tau=0}^{t-1} \phi_e(\tau)
\]

as the total flow that leaves a particular cell \( e \) up to time \( t \). Similarly, we define the cumulative external demand for a cell \( e \) as \( W_e(t) := \Delta t \cdot \sum_{\tau=0}^{t-1} w_e(\tau) \). With these quantities, we can compute the densities as

\[
\rho_e(t) = \rho_e(0) + \frac{1}{l_e} \cdot \left( \sum_{i \in E} \beta_{e,i} \Phi_i(t) - \Phi_e(t) + W_e(t) \right) \quad \forall e \in E,
\]

(7)

according to the conservation law. A similar reformulation of the CTM for the special case of a freeway stretch, that is, with only onramp merges and offramp diverges, was employed in \([23]\) to derive optimality conditions on non-predictive, distributed ramp metering. The cumulative quantities are also reminiscent of cumulative arrivals and cumulative departures used in network calculus, which are defined in a similar manner. Originally developed for communication networks, network calculus has also been applied to the control of traffic networks \([24]\). However, the aforementioned reference considers a more simple store-and-forward model which does not include congestion spill-back effects. In addition, the goal is only the stabilization of the network queues in the long term such that no queues grow unbounded.

Recall that the quantities \( \Phi_e(t) \) for \( e \in N_S \cup N_A \) are controlled flows, that is, actuated system inputs. The constraints on the controlled flows can be expressed equivalently as

\[
\Phi_e(t) \leq \Phi_e(t + 1) \leq \Phi_e(t) + \Delta t \cdot d_e(\rho_e(t)) \quad \forall e \in N_S \cup N_A
\]

for both symmetric and asymmetric controlled merges and in addition

\[
\sum_{i \in E} \beta_{e,i} \cdot \left( \Phi_i(t + 1) - \Phi_i(t) \right) \leq \Delta t \cdot \left( s_e(\rho_e(t)) - w_e(t) \right) \quad \forall e : \sigma_e \in M_S.
\]

(8)

for symmetric controlled merges. We can combine all equations to an alternative system representation of the CTM with FIFO diverges and merges corresponding to one of the three types as defined before.

**Definition 2.** We define the Cumulative Cell Transmission Model (CCTM) as the system with states \( \Phi_e(t) \), \( e \in E \setminus (N_S \cup N_A) \), inputs \( \Phi_e(t) \), \( e \in N_S \cup N_A \), defined on a graph \( G \) satisfying Assumption 7. Each edge is equipped with a fundamental diagram satisfying Assumption 2 and the state evolves according to

\[
\Phi_e(t + 1) = \Phi_e(t) + \Delta t \cdot \min \left\{ d_e(\rho_e(t)), \min_{i \in E \setminus \{e\}} \left\{ \frac{s_e(\rho_e(t)) - w_i(t) - \sum_{j \in N_A} \beta_{j,i} \cdot \Phi_j(t)}{\rho_e(t)} \right\} \right\}
\]

\[
\Delta t \cdot \left( s_e(\rho_e(t)) - w_e(t) \right) \quad \forall e : \sigma_e \in M_S.
\]

(8a)

\[
\Phi_e(t + 1) = \Phi_e(t) + \Delta t \cdot d_e(\rho_e(t)) \\
\Phi_e(t + 1) = \Phi_e(t) + \Delta t \cdot d_e(\rho_e(t)) \\
\Phi_e(t + 1) = \Phi_e(t) + \Delta t \cdot d_e(\rho_e(t))
\]

(8b)

with \( \Phi_e(0) := 0 \) and densities computed according to the conservation equation as

\[
\rho_e(t) = \rho_e(0) + \frac{1}{l_e} \cdot \left( \sum_{i \in E} \beta_{e,i} \Phi_i(t) - \Phi_e(t) + W_e(t) \right) \quad \forall e \in E.
\]

(9)
The controlled flows are subject to the constraints

\[ \Phi_e(t + 1) - \Phi_e(t) \geq 0 \quad \forall e \in \mathcal{N}_S \cup \mathcal{N}_A , \]  
\[ \Delta t \cdot d_e(\rho_e(t)) + \Phi_e(t) - \Phi_e(t + 1) \geq 0 \quad \forall e \in \mathcal{N}_S \cup \mathcal{N}_A , \]  
\[ \Delta t \cdot \left( s_e(\rho_e(t)) + w_e(t) \right) + \sum_{i \in \mathcal{E}} \beta_{i,e} \cdot \left( \Phi_i(t) - \Phi_i(t + 1) \right) \geq 0 \quad \forall e : \sigma_e \in \mathcal{M}_S . \]  

for symmetric and asymmetric controlled merges, respectively.

The CCTM is an alternative representation of the CTM with controlled merges, obtained via a state transformation. In the formulation of the CCTM, the densities \( \rho_e(t) \) are not states, but merely auxiliary variables used to express the state equations in a convenient manner. Note that the cumulative flow \( \Phi_e(t) \) may be either a state, if \( e \in \mathcal{E} \setminus (\mathcal{N}_S \cup \mathcal{N}_A) \) or an input, if \( e \in \mathcal{N}_S \cup \mathcal{N}_A \). Using the symbol \( \Phi_e(t) \) in either case allows to write all conservation equations \( \Phi \) in the same manner and thus avoids unnecessary case distinctions. Our main interest in the CCTM stems from the fact that it satisfies the conditions necessary to apply Corollary \( \mathbb{1} \) it turns out that the CCTM is a concave, state-monotone system, as we will show in the remainder of this section. Before analyzing the system dynamics of the CCTM, we introduce two intermediate results which will facilitate the analysis.

**Lemma 1.** For any \( e \in \mathcal{E} \), the auxiliary function

\[ D_e(\Phi(t)) := \Phi_e(t) + \Delta t \cdot d_e(\rho_e(t)) \]

called the cumulative demand is concave and monotone in \( \Phi(t) \).

The proof is provided in \( \mathbb{1} \). We can show a similar result for a second auxiliary function, constructed using the supply.

**Lemma 2.** For any pair of adjacent cells \( e,i \) (\( \tau_e = \sigma_i \)) with \( e \notin \mathcal{N}_A \cup \mathcal{N}_S \cup \mathcal{N}_U \), the auxiliary function

\[ S_i(\Phi(t)) := \Phi_e(t) + \frac{\Delta t}{\beta_{e,i}} \cdot \left( s_i(\rho_i(t)) - w_i(t) - \sum_{j \in \mathcal{N}_A} \beta_{j,i} \cdot \phi_j(t) \right) \]

called the cumulative supply is concave in \( \Phi(t) \) and state-monotone, that is, it is monotone in \( \Phi_e(t) \) for \( e \notin \mathcal{N}_S \cup \mathcal{N}_A \).

The proof is provided in \( \mathbb{2} \). Note that prioritized flows, that is, external inflows and controlled flows from asymmetric controlled merges, are subtracted from the available supply of free space. We can use the preceding Lemmas to analyze the system dynamics of the CCTM.

**Lemma 3.** The CCTM dynamics \( \Phi \) are concave and state-monotone.

**Proof.** The state evolution for cells \( e \in \mathcal{E} \setminus (\mathcal{N}_S \cup \mathcal{N}_A \cup \mathcal{N}_U) \) is given as the point-wise minimum

\[ \Phi_e(t + 1) = \Phi_e(t) + \Delta t \cdot \min_{e \in \mathcal{E} \setminus (\mathcal{E} \cup \mathcal{N}_U)} \left\{ d_e(\rho_e(t)), \min_{i \in \mathcal{E} \setminus (\mathcal{E} \cup \mathcal{N}_U)} \left\{ s_i(\rho_i(t)) - w_i(t) - \sum_{j \in \mathcal{N}_A} \beta_{j,i} \cdot \phi_j(t) \right\} \right\} \]

over the cumulative demand \( D_e(\Phi(t)) \) and the cumulative supply \( S_i(\Phi(t)) \). Concavity and state monotonicity of cumulative demand and supply have been shown in Lemma \( \mathbb{1} \) and in Lemma \( \mathbb{2} \) respectively. Taking the pointwise minimum over finitely many functions preserves both concavity \( \mathbb{2} \) and monotonicity of the arguments and hence, the system dynamics are concave and state-monotone.

The cumulative flow via uncongested merges

\[ \Phi_e(t + 1) = \Phi_e(t) + \Delta t \cdot d_e(\rho_e(t)) = D_e(\Phi(t)) \quad \forall e \in \mathcal{N}_u , \]

is equal to the cumulative demand and thus it is concave in \( \Phi \) and state-monotone.
We proceed to show that the constraints on the inputs of the CCTM are also concave and state-monotone.

**Lemma 4.** The CCTM constraints \(10\) are concave and state-monotone.

**Proof.** The constraints preventing negative controlled flows

\[\Phi_e(t + 1) - \Phi_e(t) \geq 0 \quad \forall e \in \mathcal{N}_S \cup \mathcal{N}_A\]

are linear and hence concave. State monotonicity is trivially satisfied, as they do not depend on any states. For analyzing the demand constraints

\[D_e(\Phi(t)) - \Phi_e(t + 1) \geq 0 \quad \forall e \in \mathcal{N}_S \cup \mathcal{N}_A,\]

recall that the cumulative demand \(D_e(\Phi(t))\) is concave in \(\Phi(t)\) and state-monotone. Concavity of the left-hand side (LHS) follows since it is the sum of a concave and a linear function in \(\Phi(t)\). Similarly, state monotonicity of the LHS follows since it is the sum of a state-monotone function and the term \(-\Phi_e(t + 1)\) with \(e \in \mathcal{N}_S \cup \mathcal{N}_A\), that does not depend on the state.

For analyzing the supply constraints

\[
\Delta t \cdot \left( s_e(\rho_e(t)) + w_e(t) \right) + \sum_{i \in E} \beta_{i,e} \cdot \left( \Phi_i(t) - \Phi_i(t + 1) \right) \geq 0 \quad \forall e : \sigma_e \in \mathcal{M}_S
\]

we first focus on the supply \(s_e(\rho_e(t))\), for the special case of \(\sigma_e \in \mathcal{M}_S\). The density \(\rho_e(t) = \rho_e(0) + \frac{1}{\ell_e} \cdot \left( \sum_{f \in \mathcal{E} \setminus \{e\}} \beta_{e,f} \Phi_f(t) - \Phi_e(t) + W_e(t) \right)\) is an affine function of \(\Phi(t)\). The demand \(d_e(\rho_e(t))\) is concave in \(\rho_e(t)\) by Assumption 2 and hence, it is concave in \(\Phi(t)\). Concavity of the left-hand side (LHS) of the supply constraint follows since it is sum of a concave function and affine terms.

Similarly, it is easy to see that \(s_e(\rho_e(t)) = s_e \left( \rho_e(0) + \frac{1}{\ell_e} \cdot \left( \sum_{i \in E} \beta_{i,e} \Phi_i(t) - \Phi_e(t) + W_e(t) \right) \right)\) is monotone in \(\Phi_e(t)\) since the supply function is non-increasing according to Assumption 3. In addition, \(\sigma_e \in \mathcal{M}_S\) implies that \(\beta_{i,e} = 0\) for all \(i \notin \mathcal{N}_S\) and hence, \(s_e(\rho_e(t))\) is state monotone. State monotonicity of the LHS follows since it is the sum of a state-monotone term \(\Delta t \cdot s_e(\rho_e(t))\) and additional terms that does not depend on the state.

It remains to transform the objective function. In the CCTM, we consider linear stage-wise objective functions of the form

\[c(\Phi(t)) = \sum_{e \in \mathcal{E}} c_e \cdot \Phi_e(t) \quad (11)\]

with \(c_e \geq 0\). This objective function is linear in states and inputs and hence concave. Also, \[\frac{\partial}{\partial \Phi_{e,t}} c(\Phi(t)) = c_e \geq 0\] implies that it is state-monotone. The choice \(\hat{c}_e := 1 - \sum_{i \in E} \beta_{e,i} \geq 0\) makes the stage-wise objective \(11\) equal to the total discharge flows, that is, all flows that leave the network within time step \(t\). One can

\[2\text{It is straightforward to extend the results to concave, state-monotone objective functions that are not necessarily defined stage-wise. However, relevant objectives for the FNC problem such as TTS can be expressed as stage-wise, linear functions and we restrict our attention to this case for simplicity of exposition.}\]
use the objective function \([11]\) to encode minimization of TTS. We can verify that

\[
\text{TTS} = \sum_{t=1}^{T} \sum_{e \in \mathcal{E}} l_e \cdot \rho_e(t)
\]

\[
= \sum_{t=1}^{T} \sum_{e \in \mathcal{E}} \left( l_e \cdot \rho_e(0) - \left( \Phi_e(t) - \sum_{j \in \mathcal{E}^{-}(e)} \beta_{je} \Phi_j(t) \right) + W_e(t) \right)
\]

\[
= \sum_{t=1}^{T} \sum_{e \in \mathcal{E}} \left( l_e \cdot \rho_e(0) - \left( \Phi_e(t) - \sum_{i \in \mathcal{E}^{+}(e)} \beta_{ei} \Phi_i(t) \right) + W_e(t) \right)
\]

\[
= \sum_{t=1}^{T} \sum_{e \in \mathcal{E}} \left( l_e \cdot \rho_e(0) + W_e(t) \right) - \sum_{t=1}^{T} \sum_{e \in \mathcal{E}} \hat{e}_e \Phi_e(t)
\]

\[
\begin{aligned}
\sum_{t=1}^{T} \sum_{e \in \mathcal{E}} (l_e \cdot \rho_e(0) + W_e(t) - \sum_{t=1}^{T} \sum_{e \in \mathcal{E}} \hat{e}_e \Phi_e(t))
\end{aligned}
\]

where \(C_{\text{TTS}}\) is a constant that does not affect the set of minimizers if added to (or subtracted from) the objective. Hence, maximization of the stage-wise discharge flows is equivalent to minimizing TTS over the whole horizon, in the sense that the set of optimizers is identical. Note that the third equality follows from rearranging flows within the network in the summation over \(e \in \mathcal{E}\). The relationship between total discharge flows and TTS is not new \([12, 20]\). We repeat the derivation here in order to be able to confirm that the objective of minimizing TTS is state-monotone and concave in the CCTM representation. Concavity and state-monotonicity of the TTS objective in the density state was already discussed in \([4]\).

We are now ready to state the main result of this work:

**Theorem 2.** Consider a traffic network modeled by the CTM defined on a graph satisfying Assumptions \([7]\) with fundamental diagrams satisfying Assumption \([2]\) and with only symmetric controlled, asymmetric controlled or uncongested merges. The FNC problem for the CTM with controlled merges \([6]\) is equivalent to the convex relaxation

\[
\mathcal{R}_{\text{CTM}}^{\ast} = \minimize_{\phi(t), \rho(t)} \Delta t \cdot \sum_{t=1}^{T} \sum_{e \in \mathcal{E}} l_e \cdot \rho_e(t)
\]

subject to

\[
\begin{aligned}
\rho_e(t+1) &= \rho_e(t) + \frac{\Delta t}{l_e} \cdot \left( \sum_{i \in \mathcal{E}} \beta_{ie} \phi_i(t) - \phi_e(t) + W_e(t) \right) \quad \forall e \in \mathcal{E} \\
\phi_e(t) &= d_e(\rho_e(t)) \quad \forall e \in \mathcal{E} \\
\sum_{i \in \mathcal{E} \setminus \mathcal{N}_e} \beta_{ie} \cdot \phi_i(t) &\leq s_e(\rho_e(t)) - w_e(t) \quad \forall e \in \mathcal{E} : \mathcal{E}^{+}(e) \neq \emptyset \\
\phi_e(t) &\geq 0 \quad \forall e \in \mathcal{E} \setminus \mathcal{N}_e \\
\rho(0) &\text{ given}
\end{aligned}
\]

called the relaxed FNC problem, in the sense that the objective values are equal and any optimizer of \([12]\) can be used to compute a solution to the original problem \([6]\) by forward simulation.

**Proof:** The problem of minimizing TTS in the CTM with controlled merges can be expressed equivalently using the CCTM as

\[
\mathcal{P}_{\text{CCTM}}^{\ast} = C_{\text{TTS}} - \maximize_{\Phi(t), \rho(t)} \sum_{t=1}^{T} \sum_{e \in \mathcal{E}} \hat{e}_e \Phi_e(t)
\]

subject to

\[
\begin{aligned}
\text{CCTM dynamics } \text{[8], [9]} \text{ and CCTM constraints } \text{[10]} \quad \rho(0) \text{ given}.
\end{aligned}
\]

According to Lemma \([3]\) the system dynamics of the CCTM concave and state-monotone, Lemma \([4]\) asserts that the constraints are concave and state-monotone and we have shown that the objective of minimizing TTS is concave and state-monotone. Hence, we can apply Corollary \([1]\) which implies that the finite horizon optimal control problem is equivalent to the relaxation in which the equality constraints encoding the system
dynamics are replaced with inequalities, that is,

\[
R_{\text{CCTM}}^* = C_{\text{TTS}} - \max_{\Phi(t), \rho(t)} \sum_{t=1}^{T} \sum_{e \in E} c_e \Phi_e(t)
\]

subject to \( \Phi_e(t + 1) \leq \Phi_e(t) + \Delta t \cdot \min \left\{ \frac{d_e(\rho_e(t)), \ldots}{\sum_{j \in \mathcal{N}_e} \beta_{j,e} \phi_j(t)} \right\} \) \forall e \in \mathcal{E} \setminus \mathcal{N} , \quad (14)

\( \Phi_e(t + 1) \leq \Phi_e(t) + \Delta t \cdot d_e(\rho_e(t)) \) \forall e \in \mathcal{N}_U ,

Conservation law \([9]\) and CCTM constraints \([10]\)

Hence, the optimal values of Problem \((14)\) and Problem \((13)\) (and in turn, Problem \((6)\)) are identical. It can be shown that Problem \((14)\) can be transformed into the equivalent Problem \((12)\) by reverting the state transformation, that is, by substituting \(\phi_e^*(t) = \frac{\Phi_e^*(t+1) - \Phi_e^*(t)}{\Delta t} \) with \(\Phi_e^*(0) := 0\). Details are given in \([3]\). It follows that the optimal values are equal \(P_{\text{CTM}}^* = P_{\text{CCTM}}^* = R_{\text{CCTM}}^* = R_{\text{CTM}}^*\).

It remains to be shown that one can use the optimal inputs obtained via solving the relaxed problem \((12)\) to compute a solution to the original problem \((6)\). Let \(\phi_e^*(t)\) for \(e \in \mathcal{N}_S \cup \mathcal{N}_A\) be an optimal input trajectory of Problem \((12)\). By forward simulating the CTM using inputs \(\phi_e^*(t)\), one obtains a valid trajectory \(\rho^*(t), \phi^*(t)\) for the CTM. Similarly, an optimal input trajectory to Problem \((14)\) can be computed as \(\Phi_e^*(t + 1) = \Phi_e^*(t) + \Delta t \cdot \phi_e^*(t), \Phi_e^*(0) = 0\) and by forward simulating the CCTM using inputs \(\Phi_e^*(t)\) \((e \in \mathcal{N}_S \cup \mathcal{N}_A)\), one obtains a valid trajectory \(\Phi(t)\) for the CCTM. Since the CCTM was defined as an alternative representation of the CTM, the trajectories defined by \((\rho^*(t), \phi^*(t))\) on one hand and by \(\Phi(t)\) on the other hand are equivalent in the sense that they jointly satisfy the state transformation equation \((7)\), and hence, they achieve the same objective value. But according to Corollary \([1]\), the trajectory \(\Phi^*(t)\) is an optimizer of Problem \((13)\) and since the optimal values are equal, \((\rho^*(t), \phi^*(t))\) is an optimizer of Problem \((6)\), which completes the proof.

Note that the relaxed FNC problem \((12)\) corresponds to the “natural” relaxation of the FNC problem, where the flows are not restricted to be equal to the fundamental diagram but are only constrained to the convex hypograph (the set of points below the graph) of the fundamental diagram.

**Remark 2.** It is worth highlighting that in the constraints of the relaxed FNC problem \((12)\), few distinctions between symmetric controlled merges, asymmetric controlled merges and uncongested merges are necessary, in particular in comparison to the variety of case distinctions used in the original FNC problem \((6)\). This is no coincidence. The purpose of control for merges is to prioritize either merge flow over the others, trading off the consequences of congestion spillback effects. The relaxed FNC problem implicitly assumes that prioritizing individual merge flows is possible (or that the merge is uncongested and hence, no tradeoff is necessary), but it is agnostic with respect to the means by which this is accomplished. The definitions of symmetric controlled merges and asymmetric controlled merges describe two different actuation schemes which make arbitrary prioritization of merge flows possible.

## 5 A closer look on Merge and Diverge Junctions

There already exists a body of work which uses monotonicity of the CTM. In particular, the CTM of a freeway stretch, that is, a freeway without merges and diverges expect for onramps and offramps, is monotone if a simplified onramp model (similar to the asymmetric merge) is used \([13]\). If the CTM is expressed in its standard form, that is, with the densities as states, the FIFO diverge dynamics are not monotone. In \([7, 8]\), non-monotonicity of FIFO diverge dynamics is discussed in detail, while the latter work also shows that the dynamics satisfy a mixed-monotonicity property. The authors suggest that the non-monotone effects are in fact important for achieving improvements via active freeway control. There also exist monotone diverge models which can be used to employ results from monotone system theory to analyze stability, for example \([18, 19]\). However, these models do not preserve the turn ratios.
In this work, we used the FIFO diverge model and found that it is in fact state monotone in the (uncontrolled) cumulative flows, even though it is not monotone in the density states. This means in particular that congestion spill back effects, the effects that \[7\] highlight as important for achieving benefits via active freeway control, are present in our model. We illustrate the effect in Example 1 using both density states and cumulative flow states. Note that in the following example and in subsequent sections, we will make use of a shorthand notation for indices: whenever cells are labeled using integers, that is, \(E = \{e_1, e_2, \ldots, e_n\}\), we will use the shorthand notation \(\rho_1(t), \rho_2(t) \ldots \rho_n(t)\) with only the integer as index instead of \(\rho_{e_1}(t), \rho_{e_2}(t) \ldots \rho_{e_n}(t)\) to denote any quantities that are indexed by cells.

Example 1. Consider a simple network as depicted in Figure 4. For simplicity, we chose a triangular fundamental diagram with identical parameters among all cells, in particular \(l_e = 1\) km, \(v_e = 100\) km/h, \(\rho_e^c = 50\) cars/km and \(\bar{\rho}_e = 250\) cars/km. The split ratios in the diverge are \(\beta_{1,2} = \beta_{1,3} = \frac{1}{2}\) and the external inflow \(w_1(t) = 4000\) cars/h is constant. The system starts in the free-flow equilibrium (where equilibrium means that the densities are constant, but not the cumulative flows) and the trajectories in this equilibrium are depicted as dashed lines. We compare these baseline trajectories against a scenario where cell \(e_2\) upstream of the diverge experiences congestion: at \(t = 6\) min, the controlled flow \(\phi_2(t)\) is reduced to zero (or equivalently, \(\Phi_2(t) = \Phi_2(6)\) for all \(t \geq 6\)), which leads to an increase of \(\rho_2(t)\) first and \(\rho_1(t)\) subsequently, depicted in Figure 4b. A monotone system model will preserve the ordering of the trajectories, but it can be seen that the density \(\rho_3(t)\) decreases, in comparison to the corresponding baseline trajectory, demonstrating that the model is not monotone with respect to the positive orthant in terms of the densities. By contrast, the reduction in \(\Phi_2(t)\) reduces the growth of both \(\Phi_1(t)\) and \(\Phi_3(t)\) in comparison to the baseline trajectory, which is consistent with a monotone model.

While FIFO diverge dynamics are concave and state-monotone in the cumulative flows, uncontrolled merge dynamics are not. This is the reason for restricting the merge dynamics to the three types introduced in Section 3. We illustrate the non-monotone effect of an uncontrolled merge in Example 2.

Example 2. Consider a simple network as depicted in Figure 4. We chose again a triangular fundamental diagram with identical parameters among all cells, in particular \(l_e = 1\) km, \(v_e = 100\) km/h, \(\rho_e^c = 50\) cars/km and \(\bar{\rho}_e = 250\) cars/km. The external inflow \(w_1(t) = 2500\) cars/h is constant, whereas \(w_2(t) = 2500\) cars/h for \(0 \leq t \leq 5\) and \(w_2(t) = 5000\) cars/h for \(5 < t\). The case with \(w_2(t) = 2500\) cars/h for all \(t\) serves as a reference, the corresponding trajectories are depicted as dashed lines. Again, a monotone model should preserve the ordering of the trajectories. We chose the proportional priority merge model used e.g. in \[3\], in which the supply of free space of a merge is allocated proportional to the demands of the upstream cells, whenever total demand exceeds supply. In particular, \(\phi_1(t) = d_1(\rho_1(t)) \cdot \min \left\{ 1, \frac{s_4(\rho_2(t))}{d_1(\rho_1(t)) + d_2(\rho_2(t))} \right\}\).
Figure 5: An increase in the external inflow \( w_2(t) \) leads to a monotone response in terms of the densities but to a non-monotone response in terms of the cumulative flows. The trajectories obtained if \( w_2(t) \) is not increased are shown in dashed.

\[
\phi_2(t) = d_2(\rho_2(t)) \cdot \min \left\{ 1, \frac{s_3(\rho_3(t))}{d_1(\rho_1(t)) + d_2(\rho_2(t))} \right\}.
\]

The merge dynamics are known to be monotone in the densities. It can be seen that the increased inflow leads to an increase in the density \( \rho_2(t) \). This in turn leads to an increase in \( \rho_1(t) \), in accordance with monotonicity of the dynamics. By contrast, one observes that in the same scenario \( \Phi_2(t) \) increases faster than its reference trajectory, whereas \( \Phi_1(t) \) falls below the reference trajectory. The merge is operating at its maximum capacity and an increase in the demand from cell 2 leads to a decrease in the flow from cell 1, which is a non-monotone effect.

The previous examples exemplify the well-known fact that cooperativity of a system, that is, monotonicity with respect to the positive orthant, might depend the choice of the state. Note that instead of transforming the state, one could also have analyzed monotonicity with respect to a different ordering. However, the states of the CCTM admit an intuitive interpretation as cumulative flows and hence, presentation of the results in terms of a state transformation seems preferable. It is worth highlighting that while monotonicity respectively mixed-monotonicity of certain CTM variants in the densities has been used to analyze stability properties \[18, 8\] and robustness of optimal trajectories \[4\], the system equations in the densities are neither concave nor convex. Therefore, even though the CTM in densities is monotone for certain networks, in particular for networks without diverges, Corollary \[1\] cannot be applied directly.

6 Application

In this section, we use the results of this work to compute optimal open-loop controls for two freeway networks. The first example is based on a real freeway in Grenoble, France and we focus on the asymmetric controlled merge as a potential onramp model. The second example is based on a fictional freeway network designed to incorporated both FIFO diverges and all types of merges considered in the definition of the CCTM.

6.1 Freeway segment with ramp metering

In this section, we consider the problem of ramp metering control of a freeway stretch with only onramp merges and offramp diverges. The freeway model in question is based on a congestion prone freeway in the vicinity of Grenoble \[11\], the Rocade Sud with 10 onramps and 7 offramps. The mainline flow dynamics are modeled using a piecewise-affine fundamental diagram with parameters as described in \[23\]. Ramp metering will be installed on this freeway in the near future.

We model the metered onramps using asymmetric controlled merges. The advantage of asymmetric controlled merges over symmetric controlled merges is that only control of the inflow from the onramp is
assumed in the former model. In particular, the onramp model using asymmetric controlled merges will be similar to the Asymmetric CTM (ACTM) \[13\]. In the ACTM, it is assumed that inflow from the onramps is not constrained by congestion on the mainline much in the same way as for the metered flow in an asymmetric controlled merge. Consider a mainline cell $e$ with an onramp. In the asymmetric CTM, the onramp is modeled via an integrator

$$\rho_e(t + 1) = \rho_e(t) + \frac{\Delta t}{l_e} \cdot (w_e(t) - r_e(t))$$

with $\rho_e(t)$ the onramp occupancy (here expressed as a density), $w_e(t)$ the external demand and $r_e(t)$ the metered inflow to the mainline. The dynamics are subject to an onramp capacity constraint $0 \leq \rho_e(t) \leq \bar{\rho}_e$ and metering constraints $0 \leq r_e(t) \leq \bar{r}_e$. Note that the former constraints imply $r_e(t) \leq w_e(t) + \frac{l_e}{\Delta t} \rho_e(t)$. We can replicate the behavior in our model by defining the onramp demand $d_e(\rho_e(t)) = \max\left\{\frac{l_e}{\Delta t} \rho_e(t), \bar{r}_e\right\}$ and including the ramp in the set of asymmetric controlled merges $e \in \mathcal{N}_A$. It should be noted that the shape of the demand function is purely to reproduce the integrator-like onramp dynamics of the ACTM and neither should the factor $\frac{l_e}{\Delta t}$ be interpreted as the free-flow velocity on the onramp nor should one assume that the density $\rho_e(t)$ is necessarily homogeneous on the onramp. The onramp capacity constraints can be expressed in the CCTM as

$$(\Phi_e(t) - W_e(t)) \cdot \Delta t + \bar{\rho}_e \geq 0,$$

$$(\Phi_e(t) - W_e(t)) \cdot \Delta t \leq 0,$$

for all cells $e$ which represent metered onramps. It is easy to see that both constraints are concave and state-monotone (recall that $\Phi_e(t)$ is a controlled variable for $e \in \mathcal{N}_A$). Therefore, the onramp capacity constraints can be added to the relaxed FNC problem without affecting exactness of the relaxation. Offramp diverges are modeled via constant turn ratios equal to the values stated in \[23\]. The densities on the offramps are not part of the model and hence, mainline flow will never be obstructed by the state of the offramps. However, outflow via the offramps is affected by spill back of congestion on the mainline, in accordance with the FIFO diverge rule.

The resulting system model satisfies all assumptions made on the CCTM and hence, Theorem \[2\] is applicable. It shall be noted that for the special case of a single freeway stretch without diverges and merges except for onramps and offramps as considered here, \[12\] proves equivalence of the relaxed FNC problem with the original problem\[^3\].

\[^3\]To be precise, the original paper introduces additional technical conditions on the sampling time and the existence of a low-demand “decongestion period” at the end of the horizon which ensure that the solution to the relaxed problem is feasible to the original problem. These technical assumptions are not critical, though, as even without them, one can create an optimal solution to the original problem by forward simulation, using the optimal inputs of the relaxation. Also, the paper only considers the triangular fundamental diagram but the extension to the concave fundamental diagram is straightforward.
(a) Distribution of metered onramp inflow divided by total cell inflow, while the mainline is congested.  
(b) Densities, uncontrolled case.  
(c) Densities, optimal LP solution.  
(d) Densities, forward simulation.  

Figure 7: Simulation results for the Rocade Sud, for the afternoon/evening rush-hour of March 5th, 2014. The optimal solution (c) does not satisfy the fundamental diagram everywhere, in particular during the duration of congestion in cells $e_7$ to $e_{10}$. No offramps are present in this part of the freeway and hence, there is no inherent incentive for the optimizer to maximize flows during times when downstream flow is obstructed by congestion. The forward simulation (d) creates a solution that does satisfy the fundamental diagram everywhere, with the same objective value (TTS).

We pose the finite horizon optimal control problem using the historical (external) traffic demands of the afternoon/evening rush-hour on March 5th, 2014, with the objective of minimizing TTS. Since we assumed a PWA fundamental diagram, the optimization problem (12) can be reformulated as a linear program (LP). For a sampling time of $15\text{sec}$, the resulting LP with a horizon of 5 hours has 67284 primal variables and it is solved by Gurobi in $66\text{sec}$. The results of the optimization are depicted in Figure 7c. It turns out that the optimizer is not unique and the particular solution found by the optimization routine does not coincide exactly with the fundamental diagram at all times. Therefore, it is necessary to perform a forward simulation using the inputs of the optimizer of the relaxed problem. The results of the forward simulation are depicted in Figure 7d. For comparison, we depict also the uncontrolled case without ramp metering in Figure 7b. Both the optimizer of the relaxed FNC problem and the solution obtained via forward simulation achieve the same cost, as predicted by Theorem 2. In comparison to the uncontrolled case, an improvement of $rac{TTS^\ast - TTS_{ol}}{TTS_{ol}} = 5.2\%$ is achieved. If one considers only the Total Waiting Time (TWT), defined as the

---

4By Assumption 1, $\Delta t \geq 20\text{sec}$ for this freeway. The critical cells have length $l_c = 0.5\text{km}$ and free-flow velocity $v_c = 90\text{km/h}$.  
5The device used was a 2013 MacBook Pro with 2.3GHz. Gurobi was interfaced via Matlab.
time surplus over a hypothetical free-flow time, that is, the time wasted in congestion and potentially on metered onramps, the relative savings are $\frac{TWT - TWT_t}{TWT_t} = 16.0\%$. As stated before, the onramp flow is not explicitly constrained by the supply of free space on the mainline in the ACTM. In [12] an a-posteriori check is performed, concluding that the supply constraints on the onramp inflows can be neglected as they never become active (or close to active) in the optimal solution. Here, we compare the inflows from the onramps $\phi_{on}(t)$ to the mainline flow $\phi_{ml}(t)$ at the same location during times of mainline congestion. The ratio $\frac{\phi_{on}(t)}{\phi_{on}(t) + \phi_{ml}(t)}$ is depicted in Figure 7a. It turns out that the onramp inflow typically accounts for less than one fourth of the total flow and it accounts for at most one third of the total flow in some rare instances, which means that condition (5) is satisfied by a large margin and the asymmetric merge model seems justified.

6.2 Networks with (partial) mainline demand control

Consider the FNC problem, that is, the problem of minimizing TTS for a freeway network, using ramp metering at selected onramps and in addition mainline demand control. We use the FIFO diverge model and the demand proportional merge model. For ramp metering, assume that the onramp model as described in the previous section is used. For mainline demand control, we follow the notation of [4] which introduces demand controls $\alpha_e(t) \in [0,1]$ for certain cells. The controlled demand of such a cell is defined as

$$d^e_c(\rho_e(t)) = \alpha_e(t) \cdot d_e(\rho_e(t))$$

and it replaces the uncontrolled demand $d_e(\rho_e(t))$ in the system equations. The usage of demand control in every cell allows to make any solution to the relaxed FNC problem (12) feasible, by selecting the control input as follows [4, Proposition 1]: Assume a desired flow $\phi^*_e(t)$ has been computed by solving the relaxed FNC problem, which implies that $\phi^*_e(t)$ satisfies the relaxed fundamental diagram (10b) and (10c). Then, set $\alpha_e(t) := \frac{\phi^*_e(t)}{\phi_e(\rho_e(t))}$. This result also holds true if Daganzo’s priority rule [10] is used to model merges. In [20], variable speed limits are suggested for demand control. In particular, the controlled demand of a cell with variable speed limit $v_e(t)$ is given as

$$d^e_c(\rho_e(t), v_e(t)) = \min \left( d_e(\rho_e(t)), v_e(t) \cdot \rho_e(t) \right),$$

assuming perfect compliance with the speed limit. By choosing $v^*_e(t) = \frac{\phi^*_e(t)}{\rho^*_e(t)}$, flow control can be realized.

In the following, we consider the freeway network depicted in Figure 8 with controlled merges. We assume that demand control as outlined above is provided for cells $e \in N_S$. The network contains two FIFO diverges (downstream of $e_2$ and $e_5$) within the network in addition to several offramp diverges, which can be interpreted as a special case of a FIFO diverge where the offramp is never congested. The corresponding turning rates are $\beta_{2,3} = \frac{2}{3}, \beta_{2,15} = \frac{1}{3}, \beta_{3,4} = \frac{4}{5}, \beta_{5,6} = \frac{1}{2}, \beta_{5,11} = \frac{1}{4}, \beta_{6,7} = \frac{4}{5}, \beta_{7,8} = \frac{4}{5}, \beta_{13,14} = \frac{4}{5}$. and
(a) Nonlinear fundamental diagram for a single lane. The nonlinear parts of demand and supply functions are modeled using 4th order polynomials.

(b) Solution of the relaxed problem $\rho_{15,16}(t)$ in comparison to the result of the forward simulation $\rho_{15,16}(t)$.

(c) Densities without control.

(d) Densities using optimal control.

Figure 9: The main bottleneck is the merge in cell 8. The optimal control alleviates the spill back of congestion from this bottleneck mainly by ramp metering using ramps 21 and 23. In addition, preference is given to flows from cell 7 (as opposed to cell 18), where spill back threatens to block several upstream FIFO diverges.

$\beta_{16,17} = \frac{4}{5}$. Four onramps ($e_{20}$, $e_{21}$, $e_{22}$ and $e_{23}$) are present and we assume that they are all used for ramp metering. We model the onramps as asymmetric controlled merges and use the onramp model presented in the previous section, which replicates a simple integrator behavior for the onramps, with limited capacity. For mainline cells, that is, all cells except the onramps, the flow-density relationship of a single lane is modeled using the concave fundamental diagram depicted in Figure 9(a). The number of lanes differs between cells. We assume cells $e_{1}$, $e_{2}$, $e_{3}$, $e_{4}$, $e_{5}$, $e_{9}$ and $e_{10}$ are composed of three lanes, cell $e_{14}$ of only one lane and the remainder of the mainline cells of two lanes. All cells are 0.5km long and the sampling time is chosen as 15sec, in accordance with Assumption 2. Two mainline merges are modeled as symmetric controlled merges. One merge is modeled as an uncongested merge and the parameters are chosen such that this holds true in the considered scenarios: Note that the number of lanes in cells $e_{9}$ and $e_{10}$ is equal to the sum of lanes in cells $e_{8}$ and $e_{19}$ combined, also, because of the network structure, there is no possibility of congestion spill back from a downstream bottleneck into cells $e_{9}$ and $e_{10}$. The network structure and the parameters are mainly chosen to exemplify the behavior of FIFO-diverges and symmetric controlled merges. Also, we chose a simple piecewise-constant demand pattern in order to facilitate visual analysis of the results as opposed to e.g. Figure 9, where demand fluctuations lead to large disturbances. To be precise, $w_{1}(t) = 2000$ cars/h, $w_{19}(t) = 1000$ cars/h and $w_{20}(t) = w_{21}(t) = w_{22}(t) = w_{23}(t) = 750$ cars/h for $1 \leq t \leq 60$ respectively, and
zero otherwise. All other external traffic demands are equal to zero.

Simulation results without control are depicted in Figure 9c. In this simulation, we use the proportional demand merge rule for the symmetric controlled merges and do not restrict the flow from onramps onto the mainline. The merge upstream of cell $e_8$ is the major bottleneck of this network, causing congestion that spills back via cells $e_7$ to $e_4$ and via cells $e_{14}$ to $e_{16}$. In addition, there is a minor bottleneck at cell $e_{14}$. We use TTS as the performance metric and obtain a total cost of $TTS_{ol} = 222.5h$.

Next, we solve the relaxed FNC problem \footnote{The device used was a 2013 MacBook Pro with 2.3GHz. IPOPT was interfaced via Matlab.}. For the given network and a horizon of $T = 100$, the resulting problem is a nonlinear, convex optimization problem with 5543 primal variables and 8280 constraints (in addition to lower and upper bounds on the primal variables). It is solved to optimality by IPOPT \footnote{The device used was a 2013 MacBook Pro with 2.3GHz. IPOPT was interfaced via Matlab.} using the L-BFGS algorithm within 34sec of CPU time. The optimizer is not unique and some cells of the optimizer found by IPOPT do not satisfy the non-relaxed fundamental diagram, that is, some flows are strictly lower than the corresponding minimum of traffic demand and supply. Therefore, it is necessary to perform a forward simulation using the optimal control actions of the relaxed problem. The results of the forward simulation are depicted in Figure 9d. The cost achieved in this simulation is $TTS^* = 201.1h$ and it is equal to the optimal cost of the relaxed FNC problem, as predicted by Theorem 2. In this case, differences between the particular solution of the relaxed problem and the results of the forward simulation occur for example in cells $e_{15}$ and $e_{16}$, as depicted in Figure 9d. No offramp is present in cell $e_{15}$ and therefore, there is no incentive in the optimization to maximize the flow from cell $e_{15}$ to $e_{16}$ as long as cell $e_{16}$ is congested.

The different objective values for the uncontrolled and the controlled case show an improvement of $\frac{TTS_{ol} - TTS^*}{TTS_{ol}} = 9.6\%$. The difference is a consequence of the elimination of congestion in cells $e_4$ to $e_7$ in the controlled case, as depicted in Figure 9d, which is mainly achieved by prioritizing the flow from cell $e_7$ over the flow from $e_{18}$ in the downstream merge. The congestion in the uncontrolled case restricts offramp flows in cells $e_5$ and $e_6$ and the flows to cell $e_{11}$ downstream of the diverge for the duration of the congestion since all the diverves act in a FIFO manner, which is avoided in the controlled case. In addition, ramps $e_{22}$ and $e_{23}$ (and to a limited extent, ramp $e_{21}$) engage in ramp metering to prevent or reduce local congestion such that the flows from the respective upstream offramps are not obstructed.

7 Conclusions

We have demonstrated that the FNC problem in the case of controlled merges allows for an equivalent system representation which is concave and state-monotone. Interestingly, the FIFO diverge model is monotone in this alternative representation even though its dynamics are not monotone with respect to the positive orthant if expressed in terms of the densities. Concavity and state monotonicity of the dynamics have been used to derive an exact convex relaxation that allows for the efficient solution of finite horizon optimal control problems, that is, the computation of optimal open-loop control inputs. This result generalizes existing results which have shown that the "natural" relaxation of certain FNC problems can be used to compute an optimal solution of the original problem, but under more restrictive assumptions on the model. It is worth highlighting that the main result in this work is based on a characterization of the system dynamics. This allows to generalize the results to modifications of the system dynamics as long as these modifications preserve concavity and state monotonicity in terms of the cumulative flows. For example, the onramp capacity constraints were added to the model in Section 6.1. To extend the proof of exactness of the convex relaxation of the FNC to this case, it was sufficient to verify that the additional constraints are concave and state monotone if expressed in the cumulative flows. Unfortunately, we have also seen that the arguably most relevant extension, the case of uncontrolled merges, does not satisfy these properties.

Future research might focus on uncontrolled merges nevertheless. In particular, it seems reasonable to ask whether it is possible to find "partial" relaxations of the FNC problem with only "few" nonconvex constraints relating to the uncontrolled merges that are tractable to solve numerically but can be used to compute a solution to the original problem. In addition, there seems to be few examples of convex-monotone systems reported in literature. The results in this work raise the question of whether there exist other relevant systems which can be transformed into a convex-monotone (or concave, state-monotone) form by a state transformation.
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Proof of Lemma 1

Proof. We first verify concavity by employing convex analysis, see e.g. [2]. The density $\rho_e(t) = \rho_e(0) + \frac{1}{l_e} \cdot \left( \sum_{i \in E} \beta_{i,e} \Phi_i(t) - \Phi_e(t) + W_e(t) \right)$ is an affine function of $\Phi(t)$. The demand $d_e(\rho_e(t))$ is concave in $\rho_e(t)$ by Assumption 2 and hence, it is concave in $\Phi(t)$. The cumulative demand $D_e(\Phi(t)) = \Phi_e(t) + \Delta t \cdot d_e(\rho_e(t))$ is the non-negative sum of two concave functions and therefore, it is concave in $\Phi(t)$.

Next, we verify state monotonicity, by resorting to the basic definition of monotonicity. For ease of notation, we will drop all time indices, i.e., write $\Phi_e$ instead of $\Phi_e(t)$ etc. in the following derivation. In the following, assume $\Delta \Phi \geq 0$. We find

$$D_e(\Phi + \Delta \Phi) - D_e(\Phi)$$

$$= \Delta \Phi_e + \Delta t \cdot d_e \left( \rho_e(0) + \frac{\sum_{i \in E} \beta_{i,e} (\Phi_i + \Delta \Phi_i)}{l_e} - \Phi_e - \Delta \Phi_e + W_e \right)$$

$$- \Delta t \cdot d_e \left( \rho_e(0) + \frac{\sum_{i \in E} \beta_{i,e} \Phi_i}{l_e} - \Phi_e + W_e \right)$$

$$\geq \Delta \Phi_e + \Delta t \cdot d_e \left( \rho_e(0) + \frac{\sum_{i \in E} \beta_{i,e} (\Phi_i + \Delta \Phi_i)}{l_e} - \Phi_e - \Delta \Phi_e + W_e \right)$$

$$- \Delta t \cdot d_e \left( \rho_e(0) + \frac{\sum_{i \in E} \beta_{i,e} \Phi_i}{l_e} - \Phi_e + W_e \right)$$

$$\geq \Delta \Phi_e - \Delta t \cdot \frac{\gamma_d}{l_e} \cdot \Delta \Phi_e = 0 ,$$

which proves monotonicity. 

\[\square\]
.2 Proof of Lemma 2

Proof. Again, we first verify concavity by employing convex analysis. The density \( \rho_i(t) = \rho_i(0) + \frac{1}{t_i} \cdot \left( \sum_{j \in S} \beta_{j,i}(t) - \Phi_i(t) + W_i(t) \right) \) is an affine function of \( \Phi(t) \). The supply \( s_i(\rho_i(t)) \) is concave in \( \rho_i(t) \) by Assumption 2 and hence, it is concave in \( \Phi(t) \). The cumulative supply

\[
S_i(\Phi(t)) := \Phi_e(t) + \frac{\Delta t}{\beta_{e,i}} \cdot \left( s_i(\rho_i(t)) - w_i(t) - \sum_{j \in N_A} \beta_{j,i} \cdot \phi_j(t) \right)
\]

for is the non-negative sum of two concave functions and two additional terms that do not depend on \( \Phi(t) \) and therefore, it is concave in \( \Phi(t) \).

To verify state monotonicity, we apply the basic definition of monotonicity in a similar manner as for the cumulative supply. In the following, assume \( \Delta \Phi \geq 0 \) and \( \Delta \Phi_j = 0 \) for any \( j \in N_S \cup N_A \). For any pair of adjacent cells \( e, i \) (\( \tau_e = \sigma_i \)) with \( e \notin N_A \cup N_S \cup N_U \) we find

\[
S_i(\Phi(t) + \Delta \Phi) - S_i(\Phi(t)) \\
= \Delta \Phi_e + \frac{\Delta t}{\beta_{e,i}} \cdot s_i \left( \rho_i(0) + \frac{1}{l_i} \cdot \left( \sum_{j \in E} \beta_{j,i}(t) + \Delta \Phi_j(t) \right) - \Phi_i(t) - \Phi_e(t) + W_i(t) \right) + \sum_{j \in N_A} \beta_{j,i} \cdot \Delta \Phi_j(t)
\]

which proves state monotonicity. In the first inequality, note that \( \beta_{j,i} \neq 0 \) implies that \( j = e \) or \( j \in N_A \Rightarrow \Delta \Phi_j = 0 \).

.3 Derivation of the constraints of the relaxed FNC problem

In this section, we show how to transform the relaxed FNC in cumulative states (14) into the relaxed FNC (12). The transformation of the objective is straightforward and here, we show how the CCTM constraints

\[
\rho_e(t) = \rho_e(0) + \frac{1}{l_e} \cdot \left( \sum_{i \in E} \beta_{i,e}(t) - \Phi_e(t) + W_e(t) \right) \quad \forall e \in E , \quad (15a)
\]

\[
\Phi_e(t+1) \leq \Phi_e(t) + \Delta t \cdot \min \left\{ d_e(\rho_e(t)) \right\} \quad \forall e \in E \setminus (N_S \cup N_A \cup N_U) , \quad (15b)
\]

\[
0 \leq \Phi_e(t+1) - \Phi_e(t) \quad \forall e \in N_S \cup N_A , \quad (15c)
\]

\[
0 \leq \Delta t \cdot d_e(\rho_e(t)) + \Phi_e(t) - \Phi_e(t+1) \quad \forall e \in N_S \cup N_A , \quad (15d)
\]

\[
0 \leq \Delta t \cdot \left( s_e(\rho_e(t)) - w_e(t) \right) + \sum_{i \in E} \beta_{i,e} \cdot (\Phi_i(t) - \Phi_i(t+1)) \quad \forall e : \sigma_e \in M_S , \quad (15f)
\]
are transformed. To do so, we first rearrange the constraints, by splitting the minimization in (15a) into distinct constraints and including the part describing the demand constraint in (16b), while the supply constraints are listed separately in (16e) (note the index change). Similarly, constraint (15e) has also been included in (16b), which is now imposed for all $e \in \mathcal{E}$.

\[
\rho_e(t) = \rho_e(0) + \frac{1}{l_e} \left( \sum_{i \in \mathcal{E}} \beta_{i,e} \Phi_i(t) - \Phi_e(t) + W_e(t) \right) \quad \forall e \in \mathcal{E},
\]

(16a)

\[
\Phi_e(t+1) - \Phi_e(t) \leq \Delta t \cdot d_e(\rho_e(t)) \quad \forall e \in \mathcal{E},
\]

(16b)

\[
\Phi_e(t+1) - \Phi_e(t) \geq 0 \quad \forall e \in \mathcal{N}_S \cup \mathcal{N}_A,
\]

(16c)

\[
\sum_{i \in \mathcal{E}} \beta_{i,e} \cdot (\Phi_i(t+1) - \Phi_i(t)) \leq \Delta t \cdot \left( s_e(\rho_e(t)) - w_e(t) \right)
\]

\forall e : \sigma_e \in \mathcal{M}_S,

(16d)

\[
\Phi_i(t+1) - \Phi_i(t) \leq \Delta t \cdot \frac{s_e(\rho_e(t)) - w_e(t) - \sum_{j \in \mathcal{N}_A} \beta_{j,e} \cdot \phi_j(t)}{\beta_{i,e}}
\]

\forall i \in \mathcal{E} \setminus (\mathcal{N}_S \cup \mathcal{N}_A \cup \mathcal{N}_U),

(16e)

with $(\star) = \forall i, e : i \in \mathcal{E} \setminus (\mathcal{N}_S \cup \mathcal{N}_A \cup \mathcal{N}_U), e \in \mathcal{E}^+(i)$. We now define $\phi(t) = \frac{\Phi(t+1) - \Phi(t)}{\Delta t}$ and substitute. The conservation equation of the CTM (17a) then follows from the conservation equation of the CCTM (16a) by computing

\[
\rho_e(t+1) - \rho_e(t) = \frac{1}{l_e} \left( \sum_{i \in \mathcal{E}} \beta_{e,i} \Phi_i(t+1) - \Phi_e(t+1) - \Phi_e(t) + W_e(t) - W_e(t-1) \right)
\]

for every $0 \leq t < T$. To show equivalence of (17a) and (16a), we notice that the reverse direction corresponds to the state transformation (7) used in the definition of the CCTM. The transformation of equations (16b) and (16c) into (17b) and (17c) follows directly from the substitution. We note that equation (16e) can be interpreted as a special case of equation (16d) if cell $e$ is not downstream of a merge and we combine the two equations into the final supply constraint (17d), to obtain

\[
\rho_e(t+1) = \rho_e(t) + \frac{\Delta t}{l_e} \left( \sum_{i \in \mathcal{E}} \beta_{i,e} \phi_i(t) - \phi_e(t) + w_e(t) \right)
\]

\forall e \in \mathcal{E},

(17a)

\[
\phi_e(t) \leq d_e(\rho_e(t))
\]

\forall e \in \mathcal{E},

(17b)

\[
\phi_e(t) \geq 0
\]

\forall e \in \mathcal{N}_S \cup \mathcal{N}_A,

(17c)

\[
\sum_{i \in \mathcal{E} \setminus \mathcal{N}_U} \beta_{i,e} \cdot \phi_i(t) \leq s_e(\rho_e(t)) - w_e(t)
\]

\forall e \in \mathcal{E} : \mathcal{E}^-(e) \neq \emptyset,

(17d)

which are exactly the constraints of the relaxed FNC problem (12).