ON THE DYNAMICAL SYSTEM GENERATED BY
THE MÖBIUS TRANSFORMATION AT PRIME TIMES

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Abstract. We study the distribution of the sequence of elements of the discrete dynamical system generated by iterations of the Möbius map $x \mapsto (ax + b)/(cx + d)$ over a finite field of $p$ elements at the moments of time that correspond to prime numbers. In particular, we obtain nontrivial estimates of exponential sums with such sequences.

1. Introduction

1.1. Motivation and background. Let $p$ be a sufficiently large prime and let $\mathbb{F}_p$ be the field of $p$ elements which we identify with the least residue system modulo $p$, that is, with the set $\{0, \ldots, p-1\}$.

With any a nonsingular matrix

\begin{equation}
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{F}_p),
\end{equation}

we consider the Möbius transformation $x \mapsto \psi(x)$ associated with $A$ where

\begin{equation}
\psi(x) = \frac{ax + b}{cx + d}.
\end{equation}

Investigating the distributional properties of elements’ orbits of the discrete dynamical system $x \mapsto \psi(x)$ on $\mathbb{F}_p$ and of similar systems over residue rings has been a very active area of research, especially in the theory of pseudorandom number generators [11, 16, 18–20, 22], see also [23, 25] for a general background on this and other related pseudorandom number generators. In fact, in the theory of pseudorandom number generators, typically only the special case $\psi(x) = ax^{-1} + b$ is considered (which is computationally more efficient) but there is no doubt the above results can be extended to any transformations of the form (1.2).

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Here we interested in more arithmetic aspects of this problem and study the distribution of element in orbits of the Möbius transformation at the moments of time that correspond to prime numbers.

More precisely, let \( u_0, u_1, \ldots \) be an orbit of the dynamical system generated by \( \psi \) that originates at some \( u_0 \in \mathbb{F}_p \), that is,

\[
(1.3) \quad u_n = \psi(u_{n-1}), \quad n = 1, 2, \ldots ,
\]

where \( u_0 \) is the initial value.

We can also write

\[
\psi^0(u_0) = u_n \quad n = 1, 2, \ldots ,
\]

where \( \psi^0 \) is the identity map and \( \psi^n \) is the \( n \)th composition of \( \psi \).

Since for any \( A \in \text{GL}_2(\mathbb{F}_p) \) the Möbius transformation is reversible, it is obvious that the sequence (1.3) is purely periodic with some period \( t \leq p \), see [6,8] for several results about the possible values of \( t \). For example, it is known when such sequences achieve the largest possible period, which is obviously \( t = p \), see [8].

The series of works [11,18,19,22] is devoted to the special case of the transformation \( \psi(x) = ax^{-1} + b \) and several results about the distribution of elements of the sequence (1.3) are given. Quite naturally, these results are based on bounds of exponential sums such as

\[
(1.4) \quad S_h(N) = \sum_{n=1}^{N} e_p(hu_n),
\]

where for an integer \( q \) and a complex \( z \) we define

\[
e_p(z) = \exp(2\pi iz/q).
\]

We also remark that a version of [1, Lemma 5.3] improves and generalises the bounds of [18] on \( S_h(N) \), see Lemma 3.1. It can easily be extended to the multidimensional settings [11] and thus has direct applications to the theory pseudorandom number generators.

Here, motivated by recent results of Sarnak and Ubis [21] on much more complicated dynamical systems on \( \text{SL}_2(\mathbb{R}) \), we consider the distribution of the sequence (1.3) at prime moments of time \( n = \ell \). In turn, this is equivalent (see [7]) to studying exponential sums

\[
T_h(N) = \sum_{\ell \in N \text{ prime}} e_p(hu_{\ell}).
\]

We also note that the results of [2–4,9,15] have an interpretation as results on the behaviour at prime moments of time of the dynamical system generated by the linear transformation \( x \mapsto gx \) on \( \mathbb{F}_p \) (or other
residue rings), that is, of the sequence $u_{0 \ell}$, where $\ell$ runs through the primes up to $N$.

1.2. **Our result and approach.** Our main result is the following bound:

**Theorem 1.1.** Assume that the characteristic polynomial of the matrix $A$ given by (1.1) has two distinct roots in $\mathbb{F}_p$. For any $\varepsilon > 0$ there exist some $B$ such that if the period $t$ of the sequence (1.3) satisfies $t \geq p^{3/4+\varepsilon}$, then, for a sufficiently large $p$, any constant $C > B$ and $p^C \geq N \geq p^B$ we have

$$\max_{h \in \mathbb{F}_p^*} |T_h(N)| \leq Np^{-\eta},$$

where $\eta > 0$ depends only on $C$ and $\varepsilon$.

We note that the condition $t \geq p^{3/4+\varepsilon}$ in Theorem 1.1 is not very restrictive as it is easy to see that one actually expects $t = p^{1+o(1)}$ for randomly chosen matrix $A \in \text{GL}_2(\mathbb{F}_p)$, see also Section 5.

To establish Theorem 1.1 we take full advantage of the flexibility of the Heath-Brown identity [13, see also [14, Proposition 13.3]. In particular it allows us to form very skewed bilinear sums to which we can estimate nontrivially. This is crucial for our result since for sum over essentially square regions we do not have nontrivial bounds. Another new ingredient is using the Burgess bound on character sums, see [14, Theorem 12.6], to estimate multiple sums of high dimension. Again the agility of the Heath-Brown identity [13] allows us to arrange such sums.

1.3. **Notation.** Throughout the paper, the implied constants in the symbols ‘$O$’ and ‘$\ll$’ may occasionally, where obvious, depend on the real positive parameters $C$ and $\varepsilon$, and are absolute otherwise (we recall that $U \ll V$ is equivalent to $U = O(V)$).

2. **Tools from analytic number theory**

2.1. **Products in residue classes.** Let $\varphi(k)$ denote the Euler function and let $\tau(k)$ denote the number of positive integer divisors of an integer $k \geq 1$. We first recall the following well-known estimates

$$\tau(k) = k^{o(1)} \quad \text{and} \quad k \varphi(k) \gg \frac{k}{\log \log k}$$

as $k \to \infty$, see [12, Theorems 317 and 328].

First we need to recall the following bound on the distribution of products in residue classes.
Given \( \nu \geq 1 \) integers \( N_1, \ldots, N_\nu \geq 1 \) and an arbitrary integer \( n \), let \( R_t(N_1, \ldots, N_\nu; n) \) be the number of solutions to the congruence

\[
1 \leq n_i \leq N_i, \quad i = 1, \ldots, \nu.
\]

We show that for sufficiently large \( N_1, \ldots, N_\nu \), for \( \gcd(n, t) = 1 \) the value of \( N_t(K, M; n) \) is close to its expected value.

**Lemma 2.1.** For any fixed \( \kappa > 0 \) there are some \( i_0 \) and \( \eta > 0 \), which depend only on \( \kappa \), such that if \( \nu > i_0 \) then for any integers \( t \geq N_1, \ldots, N_\nu \geq t^{1/3+\kappa} \) and \( n \geq 1 \) with \( \gcd(n, t) = 1 \), we have

\[
R_t(N_1, \ldots, N_\nu; n) = \frac{1}{\varphi(t)} N_1^* \cdots N_\nu^* + O \left( N_1 \cdots N_\nu t^{-1-\eta} \right),
\]

where

\[
N_i^* = \# \{ 1 \leq n_i \leq N_i : \gcd(n_i, t) = 1 \}, \quad i = 1, \ldots, \nu.
\]

**Proof.** Let \( \chi_0 \) denote the set of multiplicative characters modulo \( t \) and let \( \chi_0 \) denote the principal character; we refer to [14, Chapter 3] for a background on multiplicative characters. We also denote by \( \chi_0^* = \chi_0 \setminus \{ \chi_0 \} \) the set of non principal characters.

Using the orthogonality of multiplicative characters, we can express \( R_t(N_1, \ldots, N_\nu; n) \) via the following character sums,

\[
R_t(N_1, \ldots, N_\nu; n) = \sum_{n_1=1}^{N_1} \cdots \sum_{n_\nu=1}^{N_\nu} \frac{1}{\varphi(t)} \sum_{\chi \in \chi_0^*} \chi \left( n_1 \cdots n_\nu n^{-1} \right)
\]

\[
= \frac{1}{\varphi(t)} \sum_{\chi \in \chi_0^*} \chi \left( n^{-1} \right) \prod_{i=1}^{\nu} \sum_{n_i=1}^{N_i} \chi(n_i).
\]

We now see that the contribution from the principal character gives the main term \( N_1^* \cdots N_\nu^*/\varphi(t) \).

For other characters, since \( N_i \geq p^{1/3+\kappa} \), by the Burgess bounds, see [14, Theorem 12.6], we see that there is some \( \eta > 0 \) which depends only on \( \kappa \) and such that for any \( \chi \in \chi_0^* \) we have

\[
\sum_{n_i=1}^{N_i} \chi(n_i) \ll N_i t^{-\eta}, \quad i = 1, \ldots, \nu.
\]

Hence for \( i_0 = \lceil \eta^{-1} \rceil \) and \( \nu > i_0 \) we have

\[
\prod_{i=1}^{\nu} \sum_{n_i=1}^{N_i} \chi(n_i) \ll N_1 \cdots N_\nu t^{-1-\eta}
\]

which concludes the proof. \( \square \)
2.2. The Heath-Brown identity. As usual, we use $\mu(n)$ to denote the Möbius function and $\Lambda(n)$ to denote the von Mangoldt function given by

$$\Lambda(n) = \begin{cases} \log \ell & \text{if } n \text{ is a power of the prime } \ell, \\ 0 & \text{if } n \text{ is not a prime power.} \end{cases}$$

We need the following decomposition of $\Lambda(n)$ which is due to Heath-Brown \cite{13}, see also \cite[Proposition 13.3]{14}.

Lemma 2.2. For any integer $J \geq 1$ and $n < 2X$, we have

$$\Lambda(n) = -\sum_{j=1}^{J} (-1)^j \binom{J}{j} \sum_{m_1, \ldots, m_j \leq Z} \mu(m_1) \cdots \mu(m_j) \sum_{m_1 \cdots m_j n_1 \cdots n_j = n} \log n_1,$$

where $Z = X^{1/J}$.

For a real $A > 0$ we use $a \sim A$ to denote $A \leq a < 2A$. We also write $A \asymp B$ as an equivalent of $A \ll B \ll A$.

Hence summing the identity of Lemma 2.2 over all $n \sim N$ and separating the other variables in dyadic ranges we obtain

Lemma 2.3. For any integer $J \geq 1$ and arithmetic function $f$, we have

$$\sum_{n \sim N} \Lambda(n) f(n) \ll \sum_{1 \leq j \leq J} |S_j(M_j, N_j)|$$

for some integer vectors

$$\begin{align*} (M_j, N_j) = (M_{j,1}, \ldots, M_{j,j}, N_{j,1}, \ldots, N_{j,j}) \end{align*}$$

satisfying

$$M_{j,1}, \ldots, M_{j,j} \leq N^{1/J} \quad \text{and} \quad M_{j,1} \cdots M_{j,j} N_{j,1} \cdots N_{j,j} = N,$$

where

$$S_j(M_j, N_j) = \sum_{m_1 \cdots m_j n_1 \cdots n_j \sim N \atop m_1 \sim M_{j,1}, n_1 \sim N_{j,1}} \mu(m_1) \cdots \mu(m_j) \log n_1 f(m_1 \cdots m_j n_1 \cdots n_j),$$

where $j = 1, \ldots, J$, and the implied constants may depend on $J$. 
3. Exponential sums

3.1. Single Sums. We start with recalling the following variant of [1, Lemma 5.4] which in particular improves the bound $S_h(N) \ll N^{1/2}p^{1/4}$ of [18, Theorem 1] on the single sums (1.4).

**Lemma 3.1.** Assume that the characteristic polynomial of the matrix $A$ given by (1.1) has two distinct roots in $\mathbb{F}_{p^2}$. Let $t$ be the period of the sequence (1.3). For any integer numbers $k, N, K \geq 1$, uniformly over $h \in \mathbb{F}_p^*$, we have

$$\sum_{n=K}^{N+K-1} e_p(hu_{kn}) \ll \gcd(k, t) \left(1 + \frac{N}{t}\right) p^{1/2} \log p.$$ 

**Proof.** For $N \leq t$ this is exactly [1, Lemma 5.4]. Splitting sums of length $N > t$ into $[N/t] \leq (1 + N/t)$ pieces of length at most $t$, we obtain the result. \hfill \Box

We now need a similar bound with a co-primality condition. To simplify for the notation we use $\Sigma^*$ to indicate that the summation is over values of the summations variable which are relatively prime to $t$, for example,

$$\sum_{1 \leq n \leq N}^* f(n) = \sum_{1 \leq n \leq N, \gcd(n, t) = 1} f(n)$$

for an arithmetic function $f(n)$.

**Lemma 3.2.** Assume that the characteristic polynomial of the matrix $A$ given by (1.1) has two distinct roots in $\mathbb{F}_{p^2}$. Let $t$ be the period of the sequence (1.3). For any integer numbers $k, N, K \geq 1$, uniformly over $h \in \mathbb{F}_p^*$, we have

$$\sum_{1 \leq n \leq N}^* e_p(hu_{kn}) \ll \gcd(k, t)^{1/2} (N^{1/2} + Nt^{-1/2}) p^{1/4 + o(1)}.$$ 

**Proof.** We recall, that $\mu(d)$ denotes the Möbius function. Then using inclusion-exclusion principle we write

$$\sum_{1 \leq n \leq N}^* e_p(hu_{kn}) = \sum_{d | t} \mu(d) \sum_{1 \leq n \leq N, \text{(mod } d)} e_p(hu_{kn}).$$
Writing \( n = dm \), we see that by Lemma 3.1 each inner sum is bounded
\[
\sum_{1 \leq n \leq N \pmod{d}} e_p(hu_{kn}) \ll \gcd(dk, t) \left(1 + \frac{N}{t}\right)p^{1/2} \log p
\]
(3.1)
\[
\leq \gcd(k, t)d \left(1 + \frac{N}{t}\right)p^{1/2} \log p.
\]

It is also trivially bounded by
\[
\sum_{1 \leq n \leq N \pmod{d}} e_p(hu_{kn}) \ll N/d.
\]
(3.2)

Multiplying the bounds (3.1) and (3.2), we see that for each \( d \mid t \) we have
\[
\sum_{n=0}^{N-1} e_p(hu_{kn}) \ll \sqrt{\gcd(k, t)N \left(1 + \frac{N}{t}\right)}p^{1/2} \log p.
\]

Using the bound (2.1) on the divisor function \( \tau(t) = t^{o(1)} \), we conclude the proof. \( \square \)

**Lemma 3.3.** Assume that the characteristic polynomial of the matrix \( A \) given by (1.1) has two distinct roots in \( \mathbb{F}_{p^2} \). Let \( t \) be the period of the sequence (1.3). For any integer numbers \( N, K \geq 1 \), \( s \geq 2 \) and \( M \geq m_s > \ldots > m_1 \geq 1 \), uniformly over \( a_1, \ldots, a_s \in \mathbb{F}_p \) not all zeros, we have
\[
\sum_{n=K}^{N+K-1} e_p(a_1u_{m_1n} + \ldots + a_su_{m_sn}) \ll sM \left(1 + \frac{N}{t}\right)p^{1/2} \log p.
\]

**Proof.** This bound is a slight generalisation of [1, Lemma 5.3], which corresponds to \( s = 2 \). The general case follows from the same arguments without any changes except that we need to establish that the rational function of the form
\[
F(X) = \sum_{j=1}^{s} \frac{a_j}{X^{m_j} + \gamma}
\]
with some \( \gamma \in \mathbb{F}_{p^2}^* \) is non-constant. Without loss of generality, we can assume that \( a_1 \neq 0 \). Then the desired property of \( F \) is obvious from examining the leading term \( a_1X^{m_1+\ldots+m_2} \) of the numerator. \( \square \)

**Remark 3.4.** We remark that for full sums, that is, for \( N = t \) the logarithmic term \( \log p \) is not needed so in both Lemma 3.1 and 3.3 the term \( (1 + N/t)p^{1/2} \log p \) can be replaced with \( p^{1/2} \log p + Nt^{-1}p^{1/2} \). This however does not affect the final result.
3.2. Multiple Sums. Next we need to estimate certain multiple sums.

**Lemma 3.5.** Assume that the characteristic polynomial of the matrix $A$ given by (1.1) has two distinct roots in $\mathbb{F}_p$. Let $t$ be the period of the sequence (1.3). For any fixed $\kappa > 0$ there are some $j_0$ and $\zeta > 0$, which depend only on $\kappa$, such that if

$$\nu > j_0 \quad \text{and} \quad t \geq p^{1/2+\kappa},$$

then for any integers $t \geq N_1, \ldots, N_\nu \geq t^{1/3+\kappa}$ and $h \geq 1$, uniformly over $h \in \mathbb{F}_p^*$, we have

$$\sum_{1 \leq n_1 \leq N_1}^{*} \cdots \sum_{1 \leq n_\nu \leq N_\nu}^{*} e_p(h k n_1 \ldots n_\nu) \ll \gcd(k, t) N_1 \ldots N_\nu t^{-1-\zeta}.$$

**Proof.** By Lemma 2.1 we have

$$\sum_{1 \leq n_1 \leq N_1}^{*} \cdots \sum_{1 \leq n_\nu \leq N_\nu}^{*} e_p(h k n_1 \ldots n_\nu)$$

$$= \sum_{\text{gcd}(n, t) = 1}^{t} R_t(N_1, \ldots, N_\nu; n) e_p(h k n)$$

$$= \sum_{\text{gcd}(n, t) = 1}^{t} \left( \frac{1}{\varphi(t)} N_1 \ldots N_\nu + O(N_1 \ldots N_\nu t^{-1-\eta}) \right) e_p(h k n)$$

$$= \frac{1}{\varphi(t)} N_1 \ldots N_\nu \sum_{\text{gcd}(n, t) = 1}^{t} e_p(h k n) + O(N_1 \ldots N_\nu t^{-\eta}),$$

provided that $\nu > j_0$, where $j_0$ and $\eta > 0$ depend only on $\kappa$.

Now using Lemma 3.2 with $N = t$ and then recalling that by (2.1) we have $\varphi(t) = t^{1+o(1)}$, we obtain

$$\sum_{1 \leq n_1 \leq N_1}^{*} \cdots \sum_{1 \leq n_\nu \leq N_\nu}^{*} e_p(h k n_1 \ldots n_\nu)$$

$$\ll \frac{1}{\varphi(t)} N_1 \ldots N_\nu \gcd(k, t)^{1/2} t^{1/2} p^{1/4+o(1)} + N_1 \ldots N_\nu t^{-\eta}$$

$$\ll N_1 \ldots N_\nu \gcd(h, t)^{1/2} t^{-1/2} p^{1/4+o(1)} + N_1 \ldots N_\nu t^{-\eta},$$

and the desired result follows. \hfill \Box

3.3. Bilinear Sums. Next we need to estimate certain bilinear sum.

**Lemma 3.6.** Assume that the characteristic polynomial of the matrix $A$ given by (1.1) has two distinct roots in $\mathbb{F}_p$. Let $t$ be the period of the sequence (1.3). For any positive integers $M, K \geq 1$ and any
two sequences $\alpha = (\alpha_k)_{k=1}^K$ and $\beta = (\beta_m)_{m=1}^M$ of complex numbers, uniformly over $h \in \mathbb{P}^*$, we have

$$\left| \sum_{k=1}^K \sum_{m=1}^M \alpha_k \beta_m e_p(hu_{km}) \right| \leq \|\alpha\|_{\infty} \|\beta\|_{\infty} KM \left( M^{-1/2} + K^{-1/2}M^{1/4}p^{1/4} + M^{1/2}p^{1/4}t^{-1/2} \right) p^{o(1)},$$

where

$$\|\alpha\|_{\infty} = \max_{k \leq K} |\alpha_k| \quad \text{and} \quad \|\beta\|_{\infty} = \max_{m \leq M} |\beta_m|.$$  

Proof. We have

$$\left(3.3\right) \quad \left| \sum_{k=1}^K \sum_{m=1}^M \alpha_k \beta_m e_p(hu_{km}) \right| \leq \|\alpha\|_{\infty} W,$$

where

$$W = \sum_{k=1}^K \left| \sum_{m=1}^M \beta_m e_p(hu_{km}) \right|.$$ 

Using the Cauchy inequality, we derive

$$W^2 \leq K \sum_{k=1}^K \left| \sum_{m=1}^M \beta_m e_p(hu_{km}) \right|^2 \leq \|\beta\|_{\infty}^2 K \sum_{m,n=1}^M \sum_{k=1}^K \left| e_p(a(u_{km} - u_{kn})) \right|.$$ 

We now use the trivial bound $K$ for $M$ choice $m = n$ and use Lemma 3.3 for the remaining values. Hence, we derive

$$W^2 \ll \|\beta\|_{\infty}^2 K \left( MK + M^3(1 + K/t)p^{1/2} \log p \right).$$

Substituting this bound in (3.3), after simple calculations we conclude the proof. \(\square\)

4. Proof of Theorem 1.1

4.1. Preliminaries. We recall the notation $a \sim A$ and $A \asymp B$ from Section 2.2.

Using partial summation we see that instead of $T_h(N)$ it is enough to estimate the sums

$$U_h(N) = \sum_{n \leq N} \Lambda(n)e_p(hu_n),$$
which again via partial summation, and discarding the contribution of order $N^o(1)$ from primes $\ell \mid t$, can be reduced to the sums

$$V_h(N) = \sum_{n \sim N}^{*} \Lambda(n)e_p(hu_n)$$

over dyadic intervals.

To estimate $V_h(N)$, we fix some $\kappa > 0$ and define an integer $J \geq 2$ by the inequalities

(4.1) \hspace{1cm} N^{1/J} \leq tp^{-1/2+\kappa/2} \leq N^{1/(J-1)}.

Since $N \geq p^{B}$ we see that if $B$ is large enough then so is $J$ (in particular $J \geq 2$). Furthermore, since $t \geq p^{3/4+\kappa} \geq p^{3/4}$,

(4.2) \hspace{1cm} N^{1/J} = N^{(J-1)/(J(J-1))} \geq (tp^{-1/2-\kappa/2})^{(J-1)/J} \geq tp^{-1/2-\kappa},

provided that $B$ and thus $J$ are large enough in terms of $\kappa$.

We also define $j_0$ be as in Lemma 3.5 in terms of $\kappa$.

Next we note that by Lemma 2.3 it is enough to estimate the sums of the form

$$S(M,N) = \sum_{m_1 \ldots m_j, n_1 \ldots n_j \sim N}^{*} \mu(m_1) \ldots \mu(m_j) \log n_1 e_p \left( au_{m_1 \ldots m_j, n_1 \ldots n_j} \right)$$

for some integer vectors $(M, N) = (M_1, \ldots, M_j, N_1, \ldots, N_j)$ with $j \leq J$ and satisfying

$$M_1, \ldots, M_j \leq N^{1/J} \quad \text{and} \quad Q = N,$$

where

$$Q = 2^{2J} \prod_{i=1}^{j} M_i \prod_{i=1}^{j} N_i.$$ 

In particular, it is enough to prove that under the above conditions we have

(4.3) \hspace{1cm} S(M,N) \ll Np^{-\rho}

for some $\rho > 0$ which depends only on $\varepsilon$.

4.2. Large values of the product $M_1 \ldots M_j$. Assume that

$$M_1 \ldots M_j \geq N^{\kappa}.$$ 

There for some $i$ we have $M_i \geq N^{\kappa/j} \geq N^{\kappa/J}$. Hence we apply Lemma 3.6 with $M = M_i$ and $K = Q/M_i$ and

$$\|\alpha\|_{\infty} \leq 1 \quad \text{and} \quad \|\beta\|_{\infty} \leq \log Q = N^o(1).$$
Therefore, we obtain
\[ S(M, N) \ll Q \left( M_i^{-1/2} + (Q/M_i)^{-1/2} M^{1/2} p^{1/4} + M_i^{1/2} p^{1/4} t^{-1/2} \right) N^{o(1)} \]
\[ \ll N \left( M_i^{-1/2} + M_i N^{-1/2} p^{1/4} + M_i^{1/2} p^{1/4} t^{-1/2} \right) N^{o(1)}. \]

Since by our choice of parameters (4.1) we have
\[ t \geq N^{1/J} p^{1/2+\kappa} \geq M_i p^{1/2+\kappa} \]
as well as
\[ M_i \geq N^\kappa/J \quad \text{and} \quad N \geq N^{2/J} p^{1/2+\kappa} \geq M_i^2 p^{1/2+\kappa} \]
(provided that $B$ and thus $J$ are large enough), we obtain a bound of the desired type (4.3).

4.3. **Large values among** $N_1, \ldots, N_j$. If $N_k \geq t$ for some $k = 1, \ldots, j$, then by Lemma 3.1, applied to each of the
\[ O(M_1 \ldots M_j N_1 \ldots N_j / N_k) = O(Q/ N_k) \]
sums over $n_k$ (and using partial summation to eliminate the effect of $\log n_1$ if $k = 1$) we obtain
\[ |S(M, N)| \leq p^{1/2+o(1)} \left( 1 + \frac{N_k}{t} \right) Q / N_k \]
\[ \leq p^{1/2+o(1)} t^{-1} Q \leq N p^{-1/4}. \]

4.4. **Remaining cases.** So we can now assume that
\[ (4.4) \quad M_1 \ldots M_j \leq N^\kappa \quad \text{and} \quad N_1, \ldots, N_j < t. \]

First we note that if there is a set $\mathcal{I} \subseteq \{1, \ldots, j\}$ of cardinality $\#\mathcal{I} > j_0$ (we recall that $j_0$ is chosen as in Lemma 3.5) and such that
\[ N_i \geq t^{1/3+\kappa}, \quad i \in \mathcal{I}, \]
then using Lemma 3.5 instead of Lemma 3.1, as in the above we obtain the desired bound (4.3).

Otherwise, that is, if no more than $j_0$ elements among $N_1, \ldots, N_j$ exceed or equal $t^{1/3+\kappa}$, using (4.4), we obtain
\[ N^{1-\kappa} \leq N_1 \ldots N_j \leq \left( t^{1/3+\kappa} \right)^{j-j_0} t^{j_0}. \]
From which we trivially derive
\[ N^{1-\kappa} \leq N_1 \ldots N_j \leq \left( t^{1/3+\kappa} \right)^J t^{2j_0/3} \]
and hence
\[ (4.5) \quad N^{1/J} \leq t^{(1/3+\kappa)/(1-\kappa)+2j_0/3J(1-\kappa)} < t^{1/3+4\kappa}, \]
provided that
\[ \kappa \leq 1/3 \quad \text{and} \quad J \geq \kappa^{-1}(1 - \kappa)^{-1}j_0, \]
which holds if \( B \) is large enough in terms of \( \kappa \) and \( j_0 \), and thus in terms of \( \kappa \) and \( \varepsilon \). Combining (4.2) and (4.5) we derive
\[ tp^{-1/2-\kappa} < t^{1/3+4\kappa}, \]
which contradicts the assumption \( t \geq p^{3/4+\varepsilon} \), provided
\[ \frac{1/2 + \kappa}{2/3 - 4\kappa} = \frac{3 + 6\kappa}{4 - 24\kappa} < 3/4 + \varepsilon \]
which holds for \( \kappa < \varepsilon/50 \). Hence, taking \( \kappa = \varepsilon/51 \) we conclude the proof.

5. Remarks

It is well-known that under the Generalised Riemann Hypothesis, uniformly over \( \chi \in \mathcal{X}_t^\ast \), we have the bound
\[ \left| \sum_{x=1}^{N} \chi(x) \right| \leq N^{1/2} t^{o(1)}, \]
see [17, Section 1]; it can also be derived from [10, Theorem 2]. This allows us to replace in Lemma 2.1 the condition \( N_1, \ldots, N_\nu \geq t^{1/3+\kappa} \) with \( N_1, \ldots, N_\nu \geq t^\kappa \). In turn, the inequality (4.5) becomes
\[ N^{1/J} \leq t^{J\kappa/(1-\kappa)+j_0/J(1-\kappa)} < t^{3\kappa}, \]
provided
\[ \kappa \leq 1/2 \quad \text{and} \quad J \geq \kappa^{-1}(1 - \kappa)^{-1}j_0. \]
Hence, we now easily see that under the Generalised Riemann Hypothesis the result of Theorem 1.1 holds already for \( t \geq p^{1/2+\varepsilon} \).

Our method also works for exponential sums with the sequence (1.3) twisted with the Möbius function:
\[ r_h(N) = \sum_{n \leq N} \mu(n) e_p(hu_n), \quad h \in \mathbb{Z}. \]
In fact, such sums have been estimated in [1] with some logarithmic saving for rather small values of \( N \). The method and results of this work, such as Lemmas 3.5 and 3.6, apply to longer sums, but yield a power saving.

One can also use our approach to estimate exponential sum with orbits of Möbius transformation along sequences with other arithmetic conditions. For example, using a combinatorial identity of Vaughan [24,
Lemma 10.1] one can relate the sums over smooth numbers (that numbers without large prime divisors) to double sums and then use our results such as Lemma 3.6.

Probably the most challenging open question here is to obtain non-trivial results in the case of the period $t < p^{1/2}$. We note that the striking results and method of Bourgain [4, 5] do not seem to apply even to the case of the sums (1.4) over consecutive elements.

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