A ONE DIMENSIONAL FAMILY OF $K3$ SURFACES WITH A $\mathbb{Z}_4$ ACTION

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ABSTRACT. The minimal resolution of the degree four cyclic cover of the plane branched along a GIT stable quartic is a $K3$ surface with a non symplectic action of $\mathbb{Z}_4$. In this paper we study the geometry of the one dimensional family of $K3$ surfaces associated to the locus of plane quartics with five nodes.

INTRODUCTION

Let $V \subset |O_{P^2}(4)|$ be the space of plane quartics with five nodes and $\mathcal{V}$ be the one dimensional family given by the quotient of $V$ for the action of $PGL(3,\mathbb{C})$. The minimal resolution $X_C$ of the degree four cyclic cover of the plane branched along a quartic $C \in V$ is a $K3$ surface equipped with a non-symplectic automorphism group $G_C \cong \mathbb{Z}_4$. Since the isomorphism class of $X_C$ only depends on the projective equivalence class of $C$, this construction gives a one dimensional family $\mathcal{X}$ of $K3$ surfaces. Moreover, as proved in [1], it defines an injective period map $P : \mathcal{V} \rightarrow \mathcal{M}$ where $\mathcal{M}$ is a moduli space for couples $(X_C, G_C)$ (as defined in [9]).

This paper describes the geometry of the family $\mathcal{X}$ by studying the structure of the moduli space $\mathcal{M}$, the behavior of the period map on the closure of $\mathcal{V}$ and the occurrence of singular $K3$ surfaces.

In the first section we introduce the $K3$ surface associated to a GIT stable plane quartic according to the construction given by S. Kondō in [9].

The period domain of these polarized $K3$ surfaces is isomorphic to the complex one dimensional ball, the second section shows that their moduli space $\mathcal{M}$ is the Fricke modular curve of level two.

Any $K3$ surface $X_C$ carries an elliptic fibration induced by the pencil of lines through one node of $C \in V$. In section 3 we prove that the fibration is isotrivial and the generic fiber is isomorphic to the elliptic curve $E = \mathbb{C}/\mathbb{Z}[i]$. In fact, after a base change and a normalization, the fibration is the product $B_C \times E$ where $B_C$ is a genus two curve with splitting Jacobian $J(B_C) = E_C \times E_C$.

In section 4 we describe the behavior of the period map on the closure $\overline{\mathcal{V}}$ of $\mathcal{V}$. We prove that the period map can be extended to $\overline{\mathcal{V}}$, giving an isomorphism with the projective line.

The last section shows that there is a correspondence between $X_C$ and the Kummer surface $Km(E \times E_C)$. In particular, the occurrence of singular $K3$ surfaces in

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the family is due to the existence of isogenies between the elliptic curves \( E \) and \( E_C \). This is also connected to the existence of certain rational “splitting curves” for \( C \) (see \([2]\)). We finally give a partial characterization of transcendental lattices of singular K3 surfaces in the family. In particular we prove that the Fermat quartic, the Klein quartic and Vinberg’s K3 surface (see \([23]\)) belong to the family.

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1. Plane quartics and K3 surfaces

Let \( C \) be a GIT stable plane quartic i.e. having at most ordinary nodes and cusps (see \([10]\)). The degree four cyclic cover of the plane branched along \( C \) has at most rational double points, hence its minimal resolution \( X_C \) is a K3 surface with an order four automorphism group \( G_C \) (see \([9]\) and \([1]\)). In this paper we consider the locus \( V \) of stable plane quartics with five nodes. By taking the quotient of \( V \) for the natural action of \( \text{PGL}(3, \mathbb{C}) \) we get a one dimensional family \( V \). The isomorphism class of the cover only depends on the class of the quartic in \( V \), hence this construction defines a period map

\[
P : V \rightarrow \mathcal{M}, \quad [C] \mapsto [(X_C, G_C)]
\]

where \( \mathcal{M} \) is a one dimensional moduli space parametrizing couples \( (X_C, G_C) \) (the precise definition is given in \([2,3]\)).

We now choose a parametrization for \( V \). Consider the plane quartic \( C_\alpha, \alpha \in \mathbb{P}^1 \) which is the union of the following conic and two lines:

\[
Q : y^2 - xz = 0, \quad L : y = 0, \quad M_\alpha : \alpha x + 2y + z = 0.
\]

This gives a one parameter non-constant family of plane quartics with five nodes, hence the general point in \( V \) is represented by a curve in this family.

We denote with \( \pi_\alpha \) the four cyclic cover of the plane branched along \( C_\alpha \)

\[
\pi_\alpha : Y_\alpha \rightarrow \mathbb{P}^2
\]

and with \( \nu_\alpha \) its minimal resolution

\[
\nu_\alpha : X_\alpha \rightarrow Y_\alpha.
\]

By the previous remark, the general \( X_\alpha \) is a K3 surface. Let \( G_\alpha \cong \mathbb{Z}_4 \) be the order four automorphism group of covering transformations of \( \pi_\alpha \).

2. The period domain

In this section we describe the moduli space parametrizing couples \( (X_\alpha, G_\alpha) \) where \( X_\alpha \) is a K3 surface associated to a plane quartic with five nodes and \( G_\alpha \) is the corresponding order four covering transformation group.

Let \( \sigma_\alpha \) be a generator of \( G_\alpha \) and \( \rho_\alpha \) be the induced isometry on the cohomology lattice \( H^2(X_\alpha, \mathbb{Z}) \). In \([1]\) it is proved that \( \sigma_\alpha \) acts as a primitive 4-th root of unity on \( H^{2,0}(X_\alpha) \). In fact we can assume

\[
\rho_\alpha(\omega_\alpha) = i\omega_\alpha
\]
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where $\omega_\alpha \neq 0$ is a holomorphic two-form on $X_\alpha$. In particular, the invariant lattice of the involution $\tau_\alpha = \sigma_\alpha^2$ is contained in the Picard lattice of $X_\alpha$. In fact we will show that the invariant lattice is the Picard lattice of the generic $K3$ surface $X_\alpha$.

2.1. The generic point. Let $T_\alpha$ and $N_\alpha$ be the transcendental lattice and the Picard lattice of the generic $K3$ surface $X_\alpha$ respectively.

**Lemma 2.1.** The isomorphism classes of $T_\alpha$ and $N_\alpha$ are given by

$$T_\alpha = A_1^{\oplus 2} \oplus A_1(-1)^{\oplus 2}, \quad N_\alpha = U \oplus E_7^{\oplus 2} \oplus A_1^{\oplus 2}.$$

Moreover, in the natural basis of $T_\alpha$, the action of the isometry $\rho_\alpha$ is given by the matrix:

$$J = A \oplus A$$

where:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

**Proof.** Let $\iota_\alpha = \rho_\alpha^2$ and $L_\pm(\alpha) \subset H^2(X_\alpha, \mathbb{Z})$ be its eigenspaces. It can be easily seen that the fixed locus of the involution $\tau_\alpha$ is the disjoint union of eight smooth rational curves. By Theorem 4.2.2, this implies:

$$r(L_+(\alpha)) = 18, \quad \ell(L_+(\alpha)) = 4,$$

where $r(\cdot)$ denotes the rank of a lattice and $\ell(\cdot)$ the minimal number of generators of its discriminant group. Since the family $\{C_\alpha\}_{\alpha \in \mathbb{P}^1}$ is one dimensional, the Picard lattice equals $L_+(\alpha)$ for general $\alpha$, in particular $r(T_\alpha) = 4$. By Theorem 3.1, there is an isomorphism of $\mathbb{Z}[i]$-modules:

$$T_\alpha \cong \mathbb{Z}[i] \oplus \mathbb{Z}[i].$$

Notice that an even symmetric lattice $\Lambda$ which is a free $\mathbb{Z}[i]$-module of rank one with $i \in O(\Lambda)$, $i^2 = -id$ is of the form:

$$\Lambda \cong A_1(n) \oplus A_1(n), \quad n \in \mathbb{Z},$$

where the action of the isometry $i$ is given by the matrix $A$. Hence, in a suitable integral basis, the transcendental lattice $T_\alpha$ has intersection matrix of the form:

$$B = \begin{pmatrix} A_1(n)^{\oplus 2} & C \\ C & A_1(m)^{\oplus 2} \end{pmatrix},$$

where:

$$C = \begin{pmatrix} b & c \\ -c & b \end{pmatrix}.$$
2.2. The moduli space. Let $L$ be the abstract $K3$ lattice and $\rho = \rho_\alpha$, $\tau = \tau_\alpha$. We denote with $N$ and $T$ the positive and negative eigenlattices of $\tau$ on $L$ respectively. By the remarks in the previous section it follows that the period domain for $K3$ surfaces in the family is given by:

$$D = \{ z \in \mathbb{P}(T \otimes \mathbb{C}) : \rho(z) = iz, (z, \bar{z}) > 0 \}.$$ 

Since $T$ has rank 4 it can be easily seen that $D$ is a one dimensional complex ball. By taking the quotient for the arithmetic group:

$$\Gamma = \{ \gamma \in O(T) : \gamma \circ \rho = \rho \circ \gamma \}$$

we get the moduli space

$$M = D/\Gamma.$$ 

Let $T_{-2} = \{ \delta \in T : \delta^2 = -2 \}$, $H_{\delta} = \delta^\perp \cap D$ and

$$\Delta = \bigcup_{\delta \in T_{-2}} H_{\delta}.$$ 

**Proposition 2.2.** The quotient $(D \setminus \Delta)/\Gamma$ parametrizes isomorphism classes of couples $(X_\alpha, G_\alpha)$. Moreover, the period map $P : \mathcal{V} \to (D \setminus \Delta)/\Gamma$ is an isomorphism.

**Proof.** See [1]. □

Lemma 2.1 allows us to describe in detail the structure of the moduli space $M$. Consider the following subgroups of $SL(2, \mathbb{Z})$:

$$G_0 = SU(1,1) \cap M(2, \mathbb{Z}[i]),$$

$$H_0 = \{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}) : a + d \equiv b + c \equiv 0 \ (\text{mod } 2) \}.$$ 

**Proposition 2.3.** We have the isomorphisms:

$$M \cong B/G \cong S/H$$

where $B = \{ z \in \mathbb{C} : |z| < 1 \}$ is the complex 1-ball, $S = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ is the Siegel upper half space and

$$G = G_0 \cup LH_0,$$

$$H = H_0 \cup MH_0$$

where

$$L = \left( \begin{array}{cc} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{array} \right),$$

$$M = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right).$$

**Proof.** The period domain $D$ is given by points $z = (z_1, \ldots, z_4) \in \mathbb{P}(T \otimes \mathbb{C})$ such that:

1) $^t zT\bar{z} > 0$

2) $Jz = iz$.

Hence $z$ is of the form:

$$z = (iz_2, z_2, iz_4, z_4), \quad |z_2|^2 - |z_4|^2 > 0.$$ 

Thus we get the isomorphism:

$$\Psi_1 : D \to B = \{ w \in \mathbb{C} : |w| < 1 \}, \quad z \mapsto z_4/z_2.$$
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We are interested in the following subgroup of the isometries of $T$:

$$\Gamma = \{ M \in O(T) : MJ = JM \}.$$ 

Under the identification:

$$\mathbb{Z}[J] \cong \mathbb{Z}[i]$$ 

we have the isomorphism $T \cong \mathbb{Z}[i]^2$ as $\mathbb{Z}[i]$-modules. It can be easily seen that in the natural basis for $\mathbb{Z}[i]^2$ the intersection form on $T$ is given by:

$$Q(z, w) = 2(z \bar{z} - w \bar{w}).$$ 

Then we get:

$$\Gamma = U(Q) \cap M_2(\mathbb{Z}[i]) \cong U(1, 1) \cap M_2(\mathbb{Z}[i]).$$

Let $M \in \Gamma$:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{Z}[i]$. The action of $M$ on $D$ induces an action of $M$ on $B$ which is given by the Möbius transformation:

$$z \mapsto \psi_M(z) = \frac{c + dz}{a + bz}.$$ 

Since two matrices in $M_2(\mathbb{C})$ give the same Möbius transformation if and only if they are the same up to multiplication for a nonnegative scalar, the group of Möbius transformations of $\mathbb{C}$ is isomorphic to the quotient:

$$T \cong SL(2, \mathbb{C})/\pm I.$$ 

Consider the homomorphism:

$$T : GL(2, \mathbb{C}) \to T \quad M \mapsto \frac{1}{\sqrt{\det(M)}} M.$$ 

Notice that the kernel of $T$ is isomorphic to $\mathbb{C}^*$. Let $T_{\Gamma}$ be the restriction of $T$ to $\Gamma$, then $ker(T_{\Gamma}) \cong F_4$ where $F_4$ is isomorphic to the group of 4-th roots of unity. Notice that $G = Im(T_{\Gamma}) \subset SU(1, 1)/\pm I$ is given by:

$$G = \{ M \in SU(1, 1)/\pm I \mid \exists \epsilon \in \mathbb{C}^* : \epsilon M \in M(2, \mathbb{Z}[i]) \}.$$ 

Let:

$$\Gamma_0 = SU(1, 1) \cap M(2, \mathbb{Z}[i]) \subset G$$

and $G_0$ its image in $T$. Let:

$$L' = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \in \Gamma$$

and

$$L = T(L') = [\begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}].$$

Notice that $L^{-1}G_0L = G_0$ and $L^2 \in G_0$. In fact, if $M \in G$ then $LM \in G_0$. Hence:

$$G = G_0 \cup LG_0.$$ 

A biholomorphic map between $B$ and $S = \{ z \in \mathbb{C} : Im(z) > 0 \}$ is given by the Möbius transformation $\Psi_2 = \psi_K$:

$$\psi_K : B \to S \quad z \mapsto \frac{i + z}{1 + iz}.$$
associated to the matrix:

\[ K = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \]

The map \( \psi_K \) induces the isomorphism between the groups of automorphisms:

\[ \Upsilon : \text{Aut}(B) \to \text{Aut}(S) \quad \phi \mapsto \phi' = \psi_K \phi \psi_K^{-1}. \]

Let \( \psi_M \) be the Möbius transformation corresponding to a matrix \( M \in SU(1,1) \):

\[ M = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \]

where \( a, b \in \mathbb{C} \). Then the map \( \phi' \) is the Möbius transformation associated to the matrix \( KMK^{-1} \in SL_2(\mathbb{R}) \):

\[ KMK^{-1} = \begin{pmatrix} \text{Re}(a) + \text{Im}(b) & \text{Re}(b) + \text{Im}(a) \\ \text{Re}(b) - \text{Im}(a) & \text{Re}(a) - \text{Im}(b) \end{pmatrix}. \]

Conversely, let \( N \in SL_2(\mathbb{R}) \):

\[ N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \]

Then \( N = KMK^{-1} \) where \( M \in SU(1,1) \) is given by:

\[ a = \frac{1}{2}[\alpha + \delta + i(\beta - \gamma)], \quad b = \frac{1}{2}[\beta + \gamma + i(\alpha - \delta)]. \]

This gives an isomorphism between the groups of Möbius transformations associated to \( SU(1,1) \) and that associated to \( SL(2,\mathbb{R}) \). The image of \( G_0 \) is the following subgroup of \( H \):

\[ H_0 = \{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) : \alpha + \delta \equiv \beta + \gamma \equiv 0 \ (mod \ 2) \}. \]

The image of \( L \) in \( SL(2,\mathbb{R}) \) is given by:

\[ \Upsilon(L) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \]

Then we have the following description:

\[ H = H_0 \cup \Upsilon(L)H_0. \]

Consider the level 2 congruence subgroup:

\[ H[2] = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \mid c \equiv 0 \ (mod \ 2) \}. \]

The order two element

\[ F = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1/\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \in PSL(2,\mathbb{R}) \]

lies in the normalizer of \( H[2] \) in \( PSL(2,\mathbb{R}) \) and it is called Fricke involution. The group:

\[ H[2]^+ = H[2] \cup FH[2] \subset PSL(2,\mathbb{R}) \]

is called Fricke modular group of level 2 and the quotient:

\[ C(2)^+ = S/H[2]^+ \]

is the Fricke modular curve of level 2.
Corollary 2.4. We have the isomorphisms:
\[ \mathcal{M} \cong C(2)^+ \cong \mathbb{A}^1. \]

Proof. The group \( H_0 \) is conjugated to \( H[2] \) in \( SL(2, \mathbb{Z}) \):
\[ TH_0 T^{-1} = H_0, \quad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \]

Besides, it can be easily proved that:
\[ T(\Upsilon(L)H_0)T^{-1} = FH_0. \]

Hence the group \( H \) is isomorphic to the Fricke modular group of level two and \( \mathcal{M} \) is isomorphic to \( C(2)^+ \). The last isomorphism follows by Proposition 7.3 and Corollary 7.4, [5]. □

Remark 2.5. In [5] it is proved that the Fricke modular curve of level 2 is also the moduli space for the mirror family of degree 4 polarized \( K3 \) surfaces. It would be interesting to understand if there is any geometric correspondence between the two families.

3. An elliptic pencil

In this section we show that \( X_\alpha \) carries a natural elliptic fibration, induced by the pencil of lines through one of the nodes of \( C_\alpha \).

3.1. Definition. Note that the conic \( Q \) intersects the line \( L \) in \( p_1 = (0 : 0 : 1) \) and \( p_2 = (1 : 0 : 0) \).

Proposition 3.1. The pencil of lines through the point \( p_1 \) induces an isotrivial elliptic fibration \( E_\alpha : X_\alpha \to \mathbb{P}^1 \). After a base change \( B_\alpha \to \mathbb{P}^1 \) and a normalization, the fibration is the trivial fibration:
\[ E \times B_\alpha \to B_\alpha, \]
where \( E = \mathbb{C}/\mathbb{Z}[i] \) is the elliptic curve with \( j = 1728 \) and \( B_\alpha \) is a genus two curve.

Proof. The pencil of lines through the point \( p_1 \in Q \cap L \) is given by the equation:
\[ y = \lambda x, \]
where \( \lambda \in \mathbb{P}^1 \). We substitute \( y = \lambda x \) in the equation of \( C_\alpha \) and we restrict to the affine subset where \( x = 1 \):
\[ \lambda(z - \lambda^2)(z + (2\lambda + \alpha)) = 0. \]
In general we have:
\[ (z - a)(z - b) = z^2 - (a + b)z + ab = (z - \frac{1}{2}(a + b))^2 - (\frac{1}{2}(a - b))^2. \]
Introducing a new variable \( z_1 \) by:
\[ z = \frac{1}{2}(a - b)z_1 + \frac{1}{2}(a + b), \]
we get:
\[ (z - a)(z - b) = \frac{1}{4}(a - b)^2 (z_1^2 - 1). \]
In our case \( a = \lambda^2, b = -(2\lambda + \alpha) \), so:
\[ \lambda(z - \lambda^2)(z + (2\lambda + \alpha)) = \frac{1}{4}\lambda(\lambda^2 + 2\lambda + \alpha)^2(z_1^2 - 1). \]
Thus we are considering the fibration on $Y_{\alpha}$:

$$Y_{\alpha} \longrightarrow \mathbb{P}^{1}_{\lambda}, \quad w^4 = \frac{1}{4} \lambda(\lambda^2 + 2\lambda + \alpha)^2(z_1^2 - 1).$$

This induces an elliptic fibration on $X_{\alpha}$ with fibers isomorphic to the elliptic curve:

$$E : w^4 = z_1^2 - 1.$$ 

Notice that $E \cong \mathbb{C}/(\mathbb{Z}[i])$ since $E$ has an automorphism of order 4 which fixes a point:

$$(z_1, w) \mapsto (z_1, iw),$$

(the point $(z_1, w) = (1, 0)$ is fixed).

To get the trivial fibration, we first make a base change

$$\mathbb{P}^{1}_{\rho} \longrightarrow \mathbb{P}^{1}_{\lambda}, \quad \rho \mapsto \lambda = \rho^2,$$

which gives the equation:

$$w^4 = \left(\frac{1}{2} \rho(\rho^4 + 2\rho^2 + \alpha)\right)^2(z_1^2 - 1).$$

Next we consider the genus two curve:

$$B_{\alpha} : \tau^2 = \rho(\rho^4 + 2\rho^2 + \alpha)).$$

We make the base change:

$$B_{\alpha} \longrightarrow \mathbb{P}^{1}_{\rho}, \quad (\rho, \tau) \mapsto \rho$$

and we define $w = \tau w_1/\sqrt{z}$, so we get

$$w_1^4 = z_1^2 - 1.$$ 

Hence the normalization of the pull-back of the family is the product $B_{\alpha} \times E$. □

3.2. The Weierstrass model. We now determine the Weierstrass model for the isotrivial elliptic fibration defined in Proposition 3.1.

Lemma 3.2. The Weierstrass form for the elliptic fibration $E_{\alpha}$ is given by:

$$E_{\alpha} : Y_{\alpha} \longrightarrow \mathbb{P}^{1}_{\lambda}, \quad v^2 = u^3 - \lambda^3(\lambda^2 + 2\lambda + \alpha)^2u.$$

Proof. Recall that $E_{\alpha}$ is given by

$$E_{\alpha} : Y_{\alpha} \longrightarrow \mathbb{P}^{1}_{\lambda}, \quad w^4 = \frac{1}{4} \lambda(\lambda^2 + 2\lambda + \alpha)^2(z_1^2 - 1).$$

Let

$$\beta = \left(\frac{1}{4} \lambda(\lambda^2 + 2\lambda + \alpha)\right)^{-2},$$

then the fibration can be rewritten as:

$$z_1^2 = \beta w_1^2 + 1.$$ 

Introducing the new coordinates $w = 1/s$, $z_1 = t/s^2$ we get:

$$t^2 = s^4 + \beta = (t - s^2)(t + s^2) = \beta.$$

Next we put (see [3]):

$$x = t + s^2 \implies t - s^2 = \beta/x, \quad 2s^2 = x - \beta/x.$$ 

Multiply the last equation by $x^2$ and put $y = sx$:

$$2s^2x^2 = x^3 - \beta x \implies 2y^2 = x^3 - \beta x.$$ 

We finally put $x = u/2$, $y = v/4$ and multiply the equation by 8:

$$v^2 = u^3 - 4\beta u.$$
where we have:

\[ u = 2z_1 + 1 \quad \text{and} \quad v = 4z_1 + 1 \quad \text{with} \quad w. \]

Hence the family is:

\[ v^2 = u^3 - \frac{16}{\lambda(\lambda^2 + 2\lambda + \alpha)^2} u. \]

The transformation to the form \( w^4 = \frac{1}{4}\lambda(\lambda^2 + 2\lambda + \alpha)^2(z_1^2 - 1) \) is given by:

\[ u = \left( \frac{\lambda(\lambda^2 + 2\lambda + \alpha)}{w} \right)^2 z_1 + 1, \quad v = \left( \frac{\lambda(\lambda^2 + 2\lambda + \alpha)}{w} \right)^3 z_1 + 1. \]

The original variables \( x, y, z, w \) can be obtained from:

\[ z_1 = 2z - \frac{\lambda^2 - 2\lambda - \alpha}{\lambda^2 + 2\lambda + \alpha}. \]

\[ \square \]

3.3. **Singular fibers.**

The singular fibers in a Weierstrass fibration with equation:

\[ v^2 = u^3 - f(\lambda)u, \quad f \in \mathbb{C}[\lambda], \]

correspond to the values \( \lambda \) where \( f(\lambda) = 0 \) and to \( \lambda = \infty \) if \( \operatorname{deg}(f) \) is not divisible by 4. Let \( f = (\lambda - a)^kg \) with \( g(a) \neq 0 \), then we may always assume that \( 0 \leq k \leq 3 \) and we have bad reduction in \( a \) only if \( k \neq 0 \). Thus we get three types of bad fibers for \( k = 1, 2, 3 \), they are:

- \( k = 1 \), type \( III \) (two tangent rational curves), \( \chi = 3 \),
- \( k = 2 \), type \( I^*_0 \) (a double component and 4 reduced comp.), \( \chi = 6 \),
- \( k = 3 \), type \( III^* \) (eight components), \( \chi = 9 \).

It is now easy to find the bad fibers in our case:

\[ v^2 = u^3 - \lambda^3(\lambda^2 + 2\lambda + \alpha)^2u. \]

**Lemma 3.3.** The elliptic fibration \( E_\alpha \) has the following configuration of singular fibers:

- **type \( III^* \) over \( \lambda = 0 \),**
- **type \( I^*_0 \) over the solutions \( \lambda_1, \lambda_2 \) of \( \lambda^2 + 2\lambda + \alpha \),**
- **type \( III \) over \( \lambda = \infty \).**

Notice that, for \( \lambda = 0 \) we get the line \( L \), for \( \lambda = \infty \) we get the line \( x = 0 \), which is tangent to the conic \( Q \) in \( p_1 \). Finally, the values \( \lambda_i, i = 1, 2 \) correspond to lines through the intersection points of \( Q \) and \( M_\alpha \).

**Remark 3.4.**

i) The elliptic fibration \( E_\alpha \) has two sections, given by the line \( M_\alpha \) and the conic \( Q \), which cut on each fiber the two fixed points of the order 4 automorphism (defined by \( (z_1, w) = (\pm 1, 0) \)).

ii) It follows easily from Lemma 2.4 that the generic \( K3 \) surface \( X_\alpha \) has also an elliptic fibration with two fibers of type \( III^* \) and two of type \( III \). In fact, this is the elliptic fibration induced by the pencil of lines through the intersection point of \( L \) and \( M_\alpha \).
4. Compactification

If the quartic $C_\alpha$ is not stable, then $X_\alpha$ is not a $K3$ surface. However, we show that in some cases proper modifications of the family still give $K3$ surfaces in the limit. In other words, we study the behavior of the period map $P$ on the closure $\overline{V}$ of $V$.

Note that $M_\alpha$ (see section 3.1, [11]), in particular its rank is a multiple of 4.

**Lemma 4.1.** There exists a modification $X'_\alpha$ of the family $X_\alpha$ such that the fiber $X'_\alpha$ is a $K3$ surface with

i) an elliptic fibration with the same configuration of singular fibers of Lemma 3.3;

ii) an automorphism of order eight with transcendental value $\beta$;

iii) Picard number 18.

**Proof.** We consider the elliptic fibration in Weierstrass form from section 3.2

$$E_\alpha : Y_\alpha \longrightarrow \mathbb{P}_\lambda^1, \quad v^2 = u^3 - \lambda^3(\lambda^2 + 2\lambda + \alpha)^2u.$$  

We put

$$\alpha := \beta^{-8}, \quad u := \beta^{-14}u, \quad v = \beta^{-21}v, \quad \lambda = \beta^{-4}\lambda.$$  

Then, after multiplying the equation by $\beta^{-42}$, we get:

$$Y'_\beta : \quad v^2 = u^3 - \lambda^3(\lambda^2 + 2\beta^4\lambda + 1)^2u.$$  

This modified family has a good reduction for $\beta \to 0$. The fibration $Y'_\infty \longrightarrow \mathbb{P}_\lambda^1$ has 4 bad fibers with the same configuration of the general case. Moreover, the surface $Y'_\infty$ has an extra automorphism $\varphi$ given by:

$$u := \zeta^2u, \quad v := \zeta^3v, \quad \lambda := -\lambda \quad (\zeta^4 = -1).$$

Note that the holomorphic two form on $X'_\infty$ is locally given by $\omega = (d\lambda \wedge du)/v$ and $\varphi^*\omega = (-1\zeta^2/\zeta^3)\omega = -\zeta^{-1}\omega$. Hence the transcendental value is equal to eight and the transcendental lattice of $X'_\infty$ allows the action of the ring $\mathbb{Z}[\zeta]$ (see Theorem 3.1, [11]), in particular its rank is a multiple of 4.

**Lemma 4.2.** There exists a modification $X''_\alpha$ of the family $X_\alpha$ such that the fiber $X''_\alpha$ is the “Vinberg’s $K3$ surface” and carries an elliptic fibration with two fibers of type $III^*$ and one of type $I_0^*$.  

**Proof.** We consider again the elliptic fibration in Weierstrass form from section 3.2

$$E_\alpha \longrightarrow \mathbb{P}_\lambda^1, \quad v^2 = u^3 - \lambda^3(\lambda^2 + 2\lambda + \alpha)^2u.$$  

When $\alpha \to 0$ we get $\lambda^5(\lambda + 2)$, and changing coordinates allows to reduce to the case $\lambda(\lambda + 2)$, which gives no longer a $K3$ surface. We consider the fibration near $\lambda = \infty$, so we put:

$$\mu = \lambda^{-1}, \quad u := u/\mu^4, \quad v := v/\mu^6$$  

and multiply throughout by $\mu^{12}$:

$$v^2 = u^3 - \mu(1 + 2\mu + \alpha\mu^2)^2u.$$
We make a base change and a coordinate change:
\[ \alpha = \beta^4, \quad \mu := \mu/\beta^4, \quad u := u/\beta^6, \quad y := y/\beta^9, \]
and multiply throughout by \( \beta^{18} \):
\[ v^2 = u^3 - \mu (\beta^4 + 2 \mu + \mu^2) u. \]
It is now obvious that for \( \beta \to 0 \) we get an elliptic fibration on a K3 surface \( X''_0 \) associated to:
\[ v^2 = u^3 - \mu^3 (2 + \mu)^2 u. \]
Notice that there are 2 fibers of type III* over \( \mu = 0, \infty \) and one of type I* over \( \mu = -1/2. \) It follows from the Shioda-Tate formula (Corollary 1.5, [19]) that the rank of the Picard lattice of \( X''_0 \) is at least 20. The table of Shimada-Zhang (see [18], case 279) shows that a fibration with this fiber type is unique and has transcendental lattice isomorphic to \( A_1(-1)^{\oplus 2}. \) Hence, the surface \( X''_0 \) is the “Vinberg’s K3 surface” (see [23]).

**Proposition 4.3.** The period map \( P \) can be extended to an isomorphism
\[ P : \overline{\mathcal{V}} \to \overline{\mathcal{M}} \]
where \( \overline{\mathcal{M}} = \mathcal{M} \cup \{X\} \cong \mathbb{P}^1 \) is the Baily Borel compactification of \( \mathcal{M}. \) The curve \( C_0 \) is mapped to \( \Delta/\Gamma, \) \( C_1 \) to \( X \) and \( C_\infty \) to \( (D/\Delta)/\Gamma. \)

**Proof.** It follows from Theorem 3.5, [1] that \( P \) gives an isomorphism between the closure of \( \mathcal{V} \) in the GIT quotient of the space of plane quartics to the Baily Borel compactification of \( \mathcal{M}. \) Moreover, strictly semistable points in the closure are mapped to the boundary. Lemma 4.2 and Lemma 4.1 show that the family \( C_\alpha \) has a stable reduction in \( \alpha = 0, \infty. \) In particular it follows from Lemma 4.2 and [28] that the stable reduction of \( C_0 \) is the plane quartic with six nodes i.e. the union of four lines. This implies that \( X_0 \) has period point in \( \Delta \) since the extra node gives a \( (-2) \) curve in \( T. \) By Lemma 4.1 \( X_\infty \) is a K3 surface with Picard number 18, hence its period point is not in \( \Delta \) (i.e. the stable reduction of \( C_\infty \) has only 5 nodes). Finally, note that \( C_1 \) is a strictly semistable quartic, hence it is mapped to \( \{X\}. \)

5. **Singular K3 surfaces and isogenies**

In this section we study the occurrence of singular K3 surfaces in the family and we prove that this is connected to the existence of isogenies between certain elliptic curves.

5.1. **The curve \( B_\alpha. \)** We consider the genus two curve in Proposition 4.1
\[ B_\alpha : \quad \tau^2 = \rho (\rho^4 + 2 \rho^2 + \alpha). \]
For further remarks it is convenient to take
\[ \alpha = \beta^{-8}. \]
Now we define \( \rho := \beta^{-2} \rho, \tau := \beta^{-5} \tau \) and the equation for \( B_\beta \) becomes:
\[ B_\beta : \quad \tau^2 = \rho (\rho^4 + 2 \beta^4 \rho^2 + 1). \]
It is now easy to see that \( B_\beta \) carries the involution
\[ \iota : B_\beta \to B_\beta, \quad (\rho, \tau) \mapsto (\rho^{-1}, \tau \rho^{-3}). \]
The quotient by \( \iota \) is the elliptic curve:

\[
E_\beta : \quad v^2 = u(u^2 + 4u + 2(1 + \beta^4))
\]

with quotient map:

\[
f : B_\beta \longrightarrow E_\beta, \quad (\rho, \tau) \mapsto (u, v) = \left( \frac{2(1 + \beta^4)\rho}{(\rho - 1)^2}, \frac{2(1 + \beta^4)\tau}{(\rho - 1)^3} \right).
\]

This formula shows that the hyperelliptic involution \( (\rho, \tau) \mapsto (\rho, -\tau) \) on \( B_\beta \) induces the involution \( (u, v) \mapsto (u, -v) \) on \( E_\beta \).

**Lemma 5.1.** The Jacobian of \( B_\beta \) is isogenous to the product \( E_\beta \times E_\beta \).

**Proof.** The reducibility of the Jacobian of \( B_\beta \) follows from Theorem 14.1.1, Ch.14, since it is clear that \( B_\beta \) is equivalent to a curve of the form:

\[
y^2 = x(x - 1)(x + 1)(x - b)(x + b).
\]

In fact, the curve \( B_\beta \) has the automorphism \( \iota' \):

\[
\iota'(\rho, \tau) = (-\rho, i\tau).
\]

This gives another map \( f \circ \iota' : B_\beta \rightarrow E_\beta \). Notice that:

\[
H^{1,0}(B_\beta) = \langle dp/\tau, \rho dp/\tau \rangle.
\]

We have:

\[
H^{1,0}(B_\beta) = \langle dp/\tau + \rho dp/\tau \rangle \oplus \langle dp/\tau - \rho dp/\tau \rangle = f^*H^{1,0}(E_\beta) \oplus (f \circ \iota')^*H^{1,0}(E_\beta).
\]

Hence the Jacobian is isogenous to the product \( E_\beta \times E_\beta \). \( \square \)

5.2. The elliptic curve \( E \). Notice that we have the isomorphism of genus one curves:

\[
E' = (y^2 = x^3 - x) \cong E = (w_1^4 = z_1^2 - 1), \quad (x, y) \mapsto (w_1, z_1) = (y/(\sqrt{2}x), (x + x^{-1})/2)).
\]

Moreover, the automorphism of order four on \( E \):

\[
(z_1, w_1) \mapsto (z_1, iw_1)
\]

is induced by the automorphism on \( E' \):

\[
(x, y) \mapsto (x^{-1}, iyx^{-2}).
\]

We call standard involution the automorphism \( (x, y) \mapsto (x, -y) \) on \( E' \equiv E \) (sometimes we simply write \( p \mapsto -p \) for this map).

5.3. Isogenies. The construction in the previous section gives a rational map of degree four from \( B_\alpha \times E \) to the quartic surface \( Y_\alpha \subset \mathbb{P}^3 \). In coordinates, it is given by:

\[
\Upsilon : B_\alpha \times E \longrightarrow Y_\alpha \subset \mathbb{P}^3,
\]

\[
((\rho, \tau), (z_1, w_1)) \mapsto \begin{cases} 
  x &= 1, \\
  y &= \rho^2, \\
  z &= \frac{1}{2}(\rho^4 + 2\rho^2 + \alpha)z_1 + \frac{1}{4}(\rho^4 - 2\rho^2 - \alpha), \\
  w &= \tau w_1 / \sqrt{2}.
\end{cases}
\]

It can be proved that the image of \( B_\beta \times E \) in \( \mathbb{P}^3 \) is the quotient by the order four automorphism

\[
\phi : B_\beta \times E \longrightarrow B_\beta \times E, \quad ((\rho, \tau), (z_1, w_1)) \mapsto ((-\rho, i\tau), (z_1, -iw_1)).
\]
Note that the square of the automorphism is the product of the hyperelliptic involution on $B_\beta$ and the standard involution on $E$. 

**Remark 5.2.** The rational map $\Upsilon$ has 9 base points, one of multiplicity 4 and 8 of multiplicity 2.

We now consider the Kummer surface associated to the abelian surface $E_\beta \times E$:

$$K_\beta = \text{Km}(E_\beta \times E).$$

From the previous remarks it follows that we have a diagram:

$$
\begin{array}{c}
\ \ \ \ \ \ \ \ \ \ \ \ (B_\beta \times E)/\phi^2 \\
\downarrow \phi \downarrow \downarrow \phi \\
K_\beta \searrow \nearrow \swarrow X_\beta
\end{array}
$$

where the left arrow is the quotient by the involution $\iota$ and the right arrow is the quotient by $\phi$ (composed with birational maps). We now prove the following

**Theorem 5.3.** The $K3$ surface $X_\beta$ is singular if and only if $K_\beta$ is singular (i.e. $E_\beta$ is isogenous to $E$).

**Proof.** Let $\omega = dp/\tau \in H^{1,0}(B_\beta)$, $\omega_\iota = dw_1/z_1 \in H^{1,0}(E)$. Notice that:

$$(H^1(B_\beta) \otimes H^1(E))^{\phi} = (\omega \otimes \omega_\iota, \rho \omega \otimes \omega_\iota, \overline{\rho} \omega \otimes \overline{\omega}_\iota).$$

Let $\tilde{B_\beta \times E}$ be the blow up of $B_\beta \times E$ along the indeterminacy locus of $\Upsilon$ and $\tilde{\Upsilon}$ be the map $\tilde{B_\beta \times E} \to X_\beta$ induced by $\Upsilon$. We have

$$\tilde{\Upsilon}^*(T_\beta) \subset (H^1(B_\beta) \otimes H^1(E))^{\phi} \subset H^2(\tilde{B_\beta \times E})$$

and the first inclusion is an equality for general $\beta \in \mathbb{P}$. The transcendental lattice $T_\beta$ has rank two if the space $(H^1(B_\beta, \mathbb{Q}) \otimes H^1(E, \mathbb{Q}))^{\phi}$ contains a cycle of type $(1, 1)$. It can be proved by easy computations that $H^1(B_\beta, \mathbb{Q}) \otimes H^1(E, \mathbb{Q})$ is the direct sum of the eigenspaces (with respect to the eigenvalues $\pm 1$) of the automorphism $\phi$. Moreover, the involution $\iota$ interchanges the eigenspaces of $\phi$. This implies that if $H^1(B_\beta, \mathbb{Q}) \otimes H^1(E, \mathbb{Q})$ contains a $(1, 1)$ cycle, then the same is true for the positive eigenspace of $\phi$. Notice that:

$$H^1(B_\beta) \otimes H^1(E) \cong H^1(B_\beta)^* \otimes H^1(E) \cong \text{Hom}(H^1(B_\beta), H^1(E)).$$

Hence we can associate to each element $\omega \in H^1(B_\beta) \otimes H^1(E)$ a homomorphism $\psi_\omega : H^1(B_\beta) \to H^1(E)$. Moreover $\omega$ is of type $(1, 1)$ iff $\psi_\omega$ preserves the Hodge decomposition i.e. $\psi_\omega(H^{1,0}(B_\beta)) \subset H^{1,0}(E)$ (see [22]). By Lemma 5.1, $J(B_\beta) \cong E_\beta \times E_\beta$, hence this existence is equivalent to the existence of a homomorphism:

$$\psi'_\omega : H^1(E_\beta, \mathbb{Q}) \to H^1(E, \mathbb{Q})$$

preserving the Hodge structure i.e. of an isogeny between $E_\beta$ and $E$. It is known that the Kummer surface associated to the product of two elliptic curves is singular if and only if the two curves are isogenous with complex multiplication an thus the result follows. $\Box$

Assume now that $\beta$ is such that there is an isogeny of elliptic curves:

$$g : E_\beta \to E.$$
Composing with the quotient map \( f \) (see \ref{5.1}) we have:

\[ h : B_{\beta} \to E_{\beta} \to E. \]

Let \( \Gamma_h \) be the graph of \( h \). By the proof of Theorem \ref{5.3} \( \Gamma_h \) is the \((1, 1)\) cycle in \( H^1(B_{\beta}) \otimes H^1(E) \) corresponding to \( g \).

**Lemma 5.4.** The image \( \Upsilon(\Gamma_h) \) is a rational curve in \( \mathbb{P}^3 \).

**Proof.** As observed in \ref{5.1} the hyperelliptic involution \( i \) on \( B_{\beta} \) induces the standard involution on \( E_{\beta} \). Since \( g \) is an isogeny (so a homomorphism of groups) the hyperelliptic involution on \( E_{\beta} \) composed with \( g \) is the standard involution on \( E \).

Thus if \((p, h(p)) \in \Gamma_h \cong B_{\beta} \), then also \((i(p), h(i(p))) = (i(p), -h(p)) \) lies in \( \Gamma_h \). This means that the graph \( \Gamma_h \) is invariant under \( i^2 \), therefore the composition \( B_{\beta} \cong \Gamma_h \to B_{\beta} \times E \to Y_{\beta} \subset \mathbb{P}^3 \) factors over \( B_{\beta}/i \cong \mathbb{P}^1 \). In particular, the image of the graph is a rational curve. \( \square \)

### 5.4. A special case

Theorem 4.5 in \cite{2} predicts that the curve \( \Upsilon(\Gamma_h) \) is a “splitting curve” for \( C_{\beta} \) i.e. its inverse image by the cover \( \pi_{\beta} \) is the union of four distinct curves. We prove this in a special example where the isogeny \( g : E_{\beta} \to E \) is an isomorphism.

**Example:** We consider the curve \( B_{\beta} \) from section \ref{5.1} with \( \bar{\beta}^4 = 7/9 \). Then we have:

\[ B_{\beta} : \quad \tau^2 = \rho(\rho^4 + \frac{4}{9}\rho^2 + 1) \]
\[ E_{\beta} : \quad v^2 = u(u + \frac{4}{9})(u + \frac{8}{9}). \]

Notice that, by putting \( u = \frac{2}{3}x - \frac{1}{9} \) we get an isomorphism with the curve \( E' : y^2 = x(x^2 - 1) \).

We fix the isomorphisms \( E_{\beta} \cong E' \cong E \) (the last one as in section \ref{5.2}). We denote by \( D_{\beta} \) the projection to \( \mathbb{P}^2 \) of the image of \( \Gamma_h \) in \( \mathbb{P}^3 \). Then we have:

**Lemma 5.5.** The image of \( \Gamma_h \subset B_{\beta} \times E \) in \( \mathbb{P}^3 \) is a rational curve of degree six. Moreover, the inverse image of the curve \( D_{\beta} \) splits in four components on the quartic surface \( Y_{\beta} \subset \mathbb{P}^3 \).

**Proof.** An explicit computation gives that the curve \( D_{\beta} \subset \mathbb{P}^2 \) is the image of the following map:

\[ \psi : \mathbb{P}^1_r \to \mathbb{P}^2, \]
\[ r \mapsto \left\{ \begin{array}{lll} x &=& 49(r - 1)^2, \\ y &=& 63r^2(r - 1)^2, \\ z &=& 3r^2(48 - 32r + 75r^2 - 54r^3 + 27r^4). \end{array} \right. \]

Recall that \( Y_{\beta} \) totally ramifies over the plane quartic:

\[ Q : \quad (y^2 - xz) \cdot y \cdot (\bar{\alpha}x + 2y + z) = 0, \quad \bar{\alpha} = 81/49. \]

Substituting for \( x, y, z \), we get:

\[ (-2352(r - 1)^2r^2(3 - 2r + 3r^2)) \cdot (63r^2(r - 1)^2) \cdot (3(3 - 2r + 3r^2)^3). \]

Thus this product is a fourth power in \( \mathbb{C}[r] \), hence the 4:1 cover of the curve splits into 4 components. \( \square \)
We remark that the curve $D_\bar{\beta}$ defined in the previous section defines a 2-section for the elliptic fibration $\mathcal{E}_\beta$, i.e. it meets every fiber in two points. Consider the 2:1 base change:

$$D_\bar{\beta} \rightarrow \mathbb{P}_1$$

given by the projection of the 2-section to the base. The pull-back $\mathcal{E}_r$ of the Weierstrass fibration $\mathcal{E}_\beta \rightarrow \mathbb{P}_1$ along this base change has two 'new' sections which are the irreducible components of the pull-back of the 2-section $D_\bar{\beta}$. The sum of these sections of $\mathcal{E}_r \rightarrow \mathbb{P}_1$ actually defines a section of $\mathcal{E}_\beta \rightarrow \mathbb{P}_1$.

**Lemma 5.6.** The elliptic fibration $\mathcal{E}_\beta$ with $\beta = \bar{\beta}$ has a new section. The inverse image by $\pi_\beta$ of the image of this curve in $\mathbb{P}^2$ splits in four components.

**Proof.** The parameter $\lambda$ was defined as $y/x$, (see section 3) hence the base change $D_\beta \rightarrow \mathbb{P}_1$ is defined by:

$$\lambda = y/x = (9/7)r^2.$$ 

On the other hand, the coordinates of the 2-section are polynomials in $r$, so we need to make a base change with $\sqrt{\lambda}$ or, equivalently, with $r$. Then the pull-back surface $\mathcal{E}_r$ has the sections $r \mapsto x_i(r)$ and $r \mapsto x_i(-r)$ (here $x_i = x, y, z, w$ or $u, v$ in the Weierstrass model).

Let $u_1 = u(r), u_2 = u(-r), v_1 = v(r), v_2 = v(-r)$. The coordinates $(u_3, v_3)$ of the sum of the two sections in the Weierstrass model can be found by using the formula

$$u_3 = (v_2 - v_1)^2/(u_2 - u_1)^2 - u_1 - u_2$$

from [20] (the $v_3$-coordinate is easy to find from the Weierstrass equation of $\mathcal{E}_\beta$).

The coordinate $u_3$ is a function of $r^2$ (since $u_1, v_1$ and $u_2, v_2$ are permuted under $r \mapsto -r$), hence we get a section of $\mathcal{E}_\beta$. Explicitly, the section of the fibration (note $\bar{\alpha} = 81/49$):

$$v^2 = u^3 - 7^{-4}\lambda^3(81 + 98\lambda + 49\lambda^2)^2u,$$

is given by:

$$u = \frac{(27 + 7\lambda)^2(81 + 98\lambda + 49\lambda^2)}{2^{17/2}},$$

$$v = \frac{(81 - 7\lambda)(27 + 7\lambda)(81 + 98\lambda + 49\lambda^2)^2}{2^{67/2}}.$$ 

The equation of the corresponding curve in $Y_\alpha$ can be easily found by using the inverse transformations. The projection of this curve to $\mathbb{P}^2$ is given by (after having replaced $r^2$ by $r$ throughout):

$$\zeta : \mathbb{P}_1^r \rightarrow \mathbb{P}^2, \quad r \mapsto \begin{cases} x = 49(-9 + r)^2, \\ y = 63r(-9 + r)^2, \\ z = 9r^2(729 + 94r + 9r^2). \end{cases}$$

Recall that $Y_\alpha$ totally ramifies over

$$(y^2 - xz) \cdot y \cdot (\alpha x + 2y + z), \quad \bar{\alpha} = 81/49.$$ 

Substituting for $x, y, z$, we get:

$$(-9144576r^3(-9 + r)^2) \cdot (63(-9 + r)^2) \cdot (81(r + 3)^3).$$

Thus this product is a fourth power in $\mathbb{C}[r]$, hence the 4:1 cover of the curve splits into four components in $Y_\alpha$. 
Remark 5.7. The elliptic fibration $\mathcal{E}_\alpha$ on the $K3$ surface $X_\alpha$ has a fiber of type $III$, one of type $III^*$, two of type $I_0^*$ and a section. The Shioda-Tate formula (Corollary 1.5, [19]) implies that the Picard number of $X_\alpha$ is at least 19. Since the transcendental lattice is a $\mathbb{Z}[i]$ module, the Picard number is an even integer, hence the $K3$ surface is indeed singular.

5.5. The transcendental lattice. In this section we prove the following:

**Proposition 5.8.** The intersection matrix of the transcendental lattice of a singular $K3$ surface $X_\alpha$ is of the form:

$$T_n = \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}, \quad n \in \mathbb{Z}, \ n > 0.$$

Conversely, if $n$ is a positive integer with $n \not\equiv 2 \ (mod \ 4)$, then the rank two lattice $T_n$ is the transcendental lattice of a $K3$ surface $X_\alpha$.

**Proof.** Let $T_\alpha$ be the transcendental lattice of a singular $K3$ surface $X_\alpha$ in the family. Notice that $X_\alpha$ carries an order four automorphism $\rho'$ such that the induced isometry $\rho'$ on $H^2(X_\alpha, \mathbb{Z})$ satisfies $\rho'^2 = -id$ on $T_\alpha$. Moreover, the transcendental lattice is isomorphic to $\mathbb{Z}[i]$ if we identify $i$ with $\rho'$. It follows (as in the proof of Lemma 2.1) that $T_\alpha \cong A_1(-n)^{\oplus 2}$ for some $n$ positive integer.

Conversely, if $n$ is a positive integer, we prove that there exists $a = (a_1, \ldots, a_4) \in \mathbb{Z}^4$ such that:

$$n = a_1^2 + a_2^2 - a_3^2 - a_4^2,$$

with $a_1^2 + a_2^2 > a_3^2 + a_4^2$ and such that the rank two lattice $\Lambda(a) = \langle a, \rho(a) \rangle$ is primitive in $\mathbb{Z}^4$. This is equivalent to the request that the rank two minors of the matrix:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & -a_1 & -a_4 & a_3 \end{pmatrix}$$

have no common factors. Let $n = 2k + 1$ be an odd integer, then we can choose: $a_1 = (k + 1)^2$, $a_3 = k^2$ and $a_2 = a_4 = 0$. For $n = 2(k + 1)$ with odd $k$ we can choose $a_2 = 1$, $a_4 = 0$ and $a_1, a_3$ as before. Assume that $k = 2\ell$ is even, then $n = 2(2\ell + 1) = 4\ell + 2$. Notice that:

$$(a_1^2 + a_2^2) - (a_3^2 + a_4^2) \equiv 2 \ (mod \ 4).$$

Since $x^2 \in \{0, 1\}$ for $x \in \mathbb{Z}_4$ we have only two possibilities:

1) $(a_1^2, a_2^2, a_3^2, a_4^2) \equiv (1, 1, 0, 0) \ (mod \ 4),$

2) $(a_1^2, a_2^2, a_3^2, a_4^2) \equiv (0, 1, 1, 0) \ (mod \ 4).$

In case 1) we have that $a_1, a_2$ are odd and $a_3, a_4$ is even. Hence we immediately get that $\Lambda(a)$ is not primitive (all minors are even integers). The second case is analogous.

We now assume that $n \not\equiv 2 \ (mod \ 4)$ and we choose $a_1, \ldots, a_4$ as before. Define $z(a) \in T \otimes \mathbb{C}$ by (with respect to the usual basis):

$$z(a) = (a_1 + ha_2, a_2 - ha_1, a_3 + ha_4, a_4 - ha_3).$$

We consider the sublattices of $L$:

$$N(a) = z(a)^\perp \cap L_{K3}, \quad T(a) = N(a)^\perp.$$
A one dimensional family of $K3$ surfaces with a $\mathbb{Z}_4$ action

Notice that $T(a) = \langle a, \rho(a) \rangle$ with intersection matrix given by:

$$T(a) \cong \begin{pmatrix}
2(a_1^2 + a_2^2 - a_3^2 - a_4^2) & 0 \\
0 & 2(a_1^2 + a_2^2 - a_3^2 - a_4^2)
\end{pmatrix}.$$

By the surjectivity of the period map, for every $a \in \mathbb{Z}_4$ as above there exists a marked $K3$ surface $X(a)$ with period point $z(a)$. In particular, the transcendental lattice of $X(a)$ is $T(a)$. □

We now give some examples of $K3$ surfaces with transcendental lattice isomorphic to $T_n$:

a) $n = 1$ for Vinberg’s $K3$ surface (see [23]),
b) $n = 3$ for the $K3$ surface described in [7],
c) $n = 4$ for the Fermat quartic (see [15]),
d) $n = 7$ for the Klein quartic (see [17]).

Lemma 5.9. Let $E \cong \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ and $E' \cong \mathbb{C}/\mathbb{Z} + mi\mathbb{Z}$, then the Kummer surface $Km(E \times E')$ has transcendental lattice of the form $T_{2m}$, $m \in \mathbb{Z}$, $m > 0$.

Proof. It follows easily from the proof of Theorem 4, [21]. □

Lemma 5.10. The family $\{X_\alpha\}$ contains the $K3$ surfaces a), b), c), d) and all Kummer surfaces in Lemma 5.9 with even $m$. The surface $Km(E \times E)$ is not in the family.

Proof. The first assertion is a corollary of Proposition 5.8. The transcendental lattice of $X = Km(E \times E)$ is isomorphic to $T_2$ (see also [8]), so Proposition 5.8 cannot be applied. Assume that $X = X_\alpha$, $\alpha \in \mathbb{P}^1$. Notice that $X$ can not correspond to the fibers $\alpha = 0, \infty$, since it is singular and it is not isomorphic to Vinberg’s $K3$ surface. Hence the elliptic fibration $E_\alpha$ on $X$ has the same configuration of singular fibers of the general case i.e. of Lemma 5.8. Since the rank of the Picard lattice is 20, it follows that the Mordell-Weil group modulo torsion is isomorphic to $\mathbb{Z}^{\oplus 2}$. However, the table in [13], shows that there exists no elliptic fibration on $X$ with these properties. □

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