ON GR-FUNCTORS BETWEEN GR-CATEGORIES:
OBSTRUCTION THEORY FOR GR-FUNCTORS OF
THE TYPE \((\varphi, f)\)

Nguyen Tien Quang

April 20, 2009

Abstract

Each Gr-functor of the type \((\varphi, f)\) of a Gr-category of the type \((\Pi, C)\) has the ob-
struction be an element \(k \in H^3(\Pi, C)\). When this obstruction vanishes, there exists a
bijection between congruence classes of Gr-functors of the type \((\varphi, f)\) and the cohomol-
y group \(H^2(\Pi, C)\). Then the relation of Gr-category theory and the group exten-
sion problem can be established and used to prove that each Gr-category is Gr-equivalent
to a strict one.

1 Introduction and Preliminaries

1.1 Introduction

A monoidal category can be “refined” to become a category with group structure when the
definition of an invertible object is added (see [5], [8]). Then, if the background category is a
groupoid, we have the definition of a monoidal category group-like (see [2]), or a Gr-category
(see [9]).

The structure of a Gr-category relates closely to the group extension problem and the
group cohomology theory. Each Gr-category is Gr-equivalent to its reduced Gr-category.
That is a Gr-category of the type \((\Pi, A, \xi)\), where \(\Pi\) is a group, \(A\) is a \(\Pi\)-module and \(\xi\) is
a 3-cocycle of \(\Pi\), with coefficients in \(A\). Therefore, we can classify Gr-categories of the type
\((\Pi, A)\) with the cohomology group \(H^3(\Pi, A)\). This result has been applied and extended in
some studies on graded extension, where Gr-categories are called categorical groups (see [1]).

In this paper, let us consider the Gr-functor classification problem. This problem can be
reduced to consider Gr-functors of Gr-categories of the type \((\Pi, A)\). Each such Gr-functor is
a functor of the type \((\varphi, f)\). Conversely, each functor of the type \((\varphi, f)\) induces an element
in the cohomology group \(H^3(\Pi, A)\), called the obstruction of the functor of the type \((\varphi, f)\).
When this obstruction vanishes, Gr-functors exist and can be classified with the group
\(H^2(\Pi, A)\).

The definition of the obstruction of Gr-functors relates to the definition of the group
extension problem. Applying these 2 theories, we can prove that an arbitrary Gr-category
is equivalent to a strict one. (The analogous problem for a monoidal category can be solved
in many different way (see [4])).

1.2 Elementary concepts

Let us start with some elementary concepts of a monoidal category.

A monoidal category \((C, \otimes, I, a, l, r)\) is a category \(C\) together with a tensor product
⊗ : C × C → C; an object I, called the unitivity object of the category; and the natural isomorphisms

\[ a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C \]

\[ l_A : I \otimes A \to A \]

\[ r_A : A \otimes I \to A \]

respectively, called the associativity constraint, the left unitivity constraint and the right unitivity constraint. These constraints must satisfy the Pentagon Axiom

\[ (a_{A,B,C} \otimes \text{id}_D) a_{A,B \otimes C,D} (\text{id}_A \otimes a_{B,C,D}) = a_{A \otimes B,C,D} a_{A,B,C \otimes D}, \quad (1.1) \]

and the Triangle Axiom

\[ \text{id}_A \otimes l_B = (r_A \otimes \text{id}_B)a_{A,I,B}. \quad (1.2) \]

A monoidal category is strict if the associativity \( a \) and unitivity constraints \( l, r \) are all identities.

Let \( C = (C, \otimes, I, \alpha, \lambda, \rho) \) and \( C' = (C', \otimes', I', \alpha', \lambda', \rho') \) be monoidal categories. A monoidal functor from \( C \) to \( C' \) is a triple \( (F, \tilde{F}, \hat{F}) \) where \( F : C \to C' \) is a functor, \( \tilde{F} \) is an isomorphism from \( I' \) to \( FI \), and \( \hat{F} \) is a natural isomorphism

\[ \tilde{F}_{A,B} : FA \otimes FB \to F(A \otimes B) \quad (1.3) \]

satisfying following commutative diagrams

\[ \begin{array}{c}
FA \otimes (FB \otimes FC) \xrightarrow{id \otimes \tilde{F}} FA \otimes F(B \otimes C) \xrightarrow{\tilde{F}} F(A \otimes (B \otimes C)) \\
\downarrow \alpha' \quad \quad \quad \quad \quad \quad \downarrow F(\alpha) \\
(FA \otimes FB) \otimes FC \xrightarrow{\tilde{F} \otimes \text{id}} F(A \otimes B) \otimes FC \xrightarrow{\tilde{F}} F((A \otimes B) \otimes C)
\end{array} \quad (1.4) \]

\[ \begin{array}{c}
FA \otimes FI \xrightarrow{\tilde{F}} F(I \otimes A) \\
\downarrow \text{id} \otimes \tilde{F} \quad \quad \quad \quad \quad \downarrow F(r) \\
FI \otimes FA \xrightarrow{\tilde{F}} F(A \otimes I) \\
\downarrow \tilde{F} \otimes \text{id} \quad \quad \quad \quad \quad \downarrow F(l) \\
I' \otimes FA \xrightarrow{l'} FA
\end{array} \quad (1.5a) \]

\[ \begin{array}{c}
FI \otimes FA \xrightarrow{\tilde{F}} F(A \otimes I) \\
\downarrow \text{id} \otimes \tilde{F} \quad \quad \quad \quad \quad \downarrow F(l) \\
I' \otimes FA \xrightarrow{l'} FA
\end{array} \quad (1.5b) \]

A natural monoidal transformation \( \alpha : (F, \tilde{F}, \hat{F}) \to (G, \tilde{G}, \hat{G}) \) between monoidal functors from \( C \) to \( C' \) is a natural transformation \( \alpha : F \to G \), such that the following diagrams

\[ \begin{array}{c}
FX \otimes FY \xrightarrow{\alpha_X \otimes \alpha_Y} GX \otimes GY
\end{array} \quad (1.6) \]

\[ \begin{array}{c}
F(X \otimes Y) \xrightarrow{\alpha_{X,Y}} G(X \otimes Y)
\end{array} \quad (1.7) \]

commute, for all pairs \((X, Y)\) of objects in \( C \).

An monoidal isomorphism is a monoidal transformation as well as a natural isomorphism.
A monoidal equivalence between monoidal categories is a monoidal functor $F : \mathcal{C} \to \mathcal{C}'$, such that there exists a monoidal functor $G : \mathcal{C}' \to \mathcal{C}$ and natural monoidal isomorphism $\alpha : GF \to \text{id}_\mathcal{C}$ and $\beta : FG \to \text{id}_{\mathcal{C}'}$.

$\mathcal{C}$ and $\mathcal{C}'$ are monoidal equivalent if there exists a monoidal equivalence between them.

### 1.3 A reduced Gr-category and canonical equivalences

Let $\mathcal{C}$ be a Gr-category. Then, the set of congruence classes of invertible objects $\Pi_0(\mathcal{C})$ of $\mathcal{C}$ is a group with the operations induced by the tensor product in $\mathcal{C}$, and $\Pi_1(\mathcal{C})$ of automorphisms of the unitivity object $I$ is an abelian group with the operation, denoted by $+$, induced by the composition of the arrows. Moreover, $\Pi_1(\mathcal{C})$ is a $\Pi_0(\mathcal{C})$-module with the action

$$su = \gamma^{-1}_A\delta_A(s), \ A \in s, \ s \in \Pi_0(\mathcal{C}), \ u \in \Pi_1(\mathcal{C})$$

where $\delta_A, \gamma_A$ are defined by the following commutative diagrams

$$\begin{align*}
X & \xrightarrow{\gamma_X(u)} X \\
\downarrow i_X & \downarrow i_X \\
I \otimes X & \xrightarrow{u \otimes \text{id}} I \otimes X
\end{align*}$$

$$\begin{align*}
X & \xrightarrow{\delta_X(u)} X \\
\downarrow r_X & \downarrow r_X \\
X \otimes I & \xrightarrow{\text{id} \otimes u} X \otimes I
\end{align*}$$

The reduced Gr-category $\mathcal{S}$ of a Gr-category $\mathcal{C}$ is the category whose objects are the elements of $\Pi_0(\mathcal{C})$ and whose arrows are automorphisms of the form $(s, u)$, where $s \in \Pi_0(\mathcal{C}), \ u \in \Pi_1(\mathcal{C})$. The composition of two arrows is induced by the addition in $\Pi_1(\mathcal{C})$ as follows

$$(s, u).(s, v) = (s, u + v).$$

The category $\mathcal{S}$, equivalent to $\mathcal{C}$ thanks to the canonical equivalence, is built as follows. For each $s = X \in \Pi_0(\mathcal{C})$, we choose a representative $X_s \in \mathcal{C}$ and for each $X \in s$, we choose an isomorphism $i_X : X_s \to X$ such that $i_{X_s} = \text{id}$. The family $(X_s, i_X)$ is called a stick of the Gr-category $\mathcal{C}$. For any $(X_s, i_X)$, we obtain two functors

$$\begin{align*}
G : \mathcal{C} & \to \mathcal{S} \\
G(X) = X & = s \\
G(X \xrightarrow{f} Y) & = (s, \gamma^{-1}_X((i^{-1}_Y f)i_X))
\end{align*}$$

$$\begin{align*}
H : \mathcal{S} & \to \mathcal{C} \\
H(s) = X_s & \\
H(s, u) & = \gamma_X(u)
\end{align*}$$

The functors $G$ and $H$ are equivalences of categories together with the natural transformations

$$\alpha = (i_X) : HG \cong \text{id}_\mathcal{C} ; \beta = \text{id} : GH \cong \text{id}_\mathcal{S}$$

They are called canonical equivalences.

With the structure conversion (see [8]) by the quadruple $(G, H, \alpha, \beta)$, $\mathcal{S}$ can be equipped with the following operation to become a Gr-category

$$s \otimes t = s.t, \ s, t \in \Pi_0(\mathcal{C})$$

$$(s, u) \otimes (t, v) = (st, u + sv), \ u, v \in \Pi_1(\mathcal{C}).$$

The Gr-category $\mathcal{S}$ has the unitivity constraint be strict and the associativity constraint be a normalized 3-cocycle $\xi$ in $\Pi_0(\mathcal{C})$ with coefficients in $\Pi_0(\mathcal{C})$-module $\Pi_1(\mathcal{C})$. Moreover, the equivalence $G, H$ become Gr-equivalences together with natural transformations

$$\begin{align*}
\tilde{G}_{A,B} & = G(i_A \otimes i_B), \ \tilde{H}_{s,t} = i^{-1}_{A_s \otimes A_t}
\end{align*}$$

The Gr-category $\mathcal{S}$ is called a reduction of the Gr-category $\mathcal{C}$. We also say that $\mathcal{S}$ is of the type $(\Pi, A, \xi)$ or simply the type $(\Pi, A)$ when $\Pi_0(\mathcal{C}), \Pi_1(\mathcal{C})$ are, respectively, replaced with the group $\Pi$ and the $\Pi$-module $A$. 

3
2 The obstruction and classification of Gr-functors of the type \((\varphi, f)\)

In this section, we will show that each Gr-functor \((F, \tilde{F}) : C \to C'\) induces a Gr-functor \(\tilde{F}\) on their reduced Gr-categories, and this correspondence is 1-1. This allows us to study the Gr-functor existence problem and classify them on Gr-categories of the type \((\Pi, A)\). First, we have

**Proposition 1.** [9] Let \((F, \tilde{F}) : C \to C'\) be a Gr-functor. Then, \((F, \tilde{F})\) induces the pair of group homomorphisms

\[
F_0 : \Pi_0(C) \to \Pi_0(C') \ ; \ \overline{A} \mapsto \tilde{F}A \\
F_1 : \Pi_1(C) \to \Pi_1(C') \ ; \ u \mapsto \gamma_{F1}(Fu)
\]

satisfying \(F_1(su) = F_0(s)F_1(u)\).

The pair \((F_0, F_1)\) is called the pair of induced homomorphisms of the Gr-functor \((F, \tilde{F})\).

Let \(S, S'\) be, respectively, the reduced Gr-categories of \(C, C'\). Then, the functor \(F : S \to S'\) given by

\[
F(s) = F_0(s), \quad F(s, u) = (F_0s, F_1u)
\]

is called the induced functor of \((F, \tilde{F})\) on reduced Gr-categories.

**Proposition 2.** Let \(\overline{F}\) be the induced functor of the Gr-functor \((F, \tilde{F}) : C \to C'\). Then the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{F} & C' \\
H \downarrow & & \downarrow G' \\
S & \xrightarrow{\overline{F}} & S'
\end{array}
\]

commutes, where \(H, G'\) are canonical equivalences, and therefore \(\overline{F}\) induces a Gr-functor.

In order to prove this proposition, we consider the two following lemmas

**Lemma 3.** Let \(C, C'\) be \(\otimes\)-categories with, respectively, constraints \((I, l, r)\) and \((I', l', r')\). Let \((F, \tilde{F}) : C \to C'\) be a \(\otimes\)-functor which is compatible with the unitality constraints. Then, the following diagram

\[
\begin{array}{ccc}
FI & \xrightarrow{\gamma_{F1}(u)} & FI \\
\tilde{F} \downarrow & & \downarrow \tilde{F} \\
I' & \xrightarrow{u} & I'
\end{array}
\]

commutes, where \(\tilde{F}\) is the isomorphism induced by \((F, \tilde{F})\).

**Proof.** It is clear that \(\gamma_{I'}(u) = u\). Moreover, the family \((\gamma_{A'}(u)), A' \in C'\), is a morphism of the identity functor \(id_{C'}\). So the above diagram commutes. \(\square\)

**Lemma 4.** If the assumption of Lemma 3 holds, we have

\[F\gamma_A(u) = \gamma_{F\lambda}(\gamma_{F1}^{-1}Fu).\]

**Proof.** Consider the following diagram
In this diagram, the regions (4) and (5) commute thanks to the compatibility of the functor \((F, \tilde{F})\) with the associativity constraints. The region (3) commutes thanks to the definition of \(\gamma_A\) (image through \(F\)), the region (1) commutes by Lemma 3. The region (2) commutes thanks to the naturality of the isomorphism \(\tilde{F}\). Therefore, the outside commutes, i.e.,

\[ F \gamma_A(u) = \gamma_{FA}(\gamma_{FI}^{-1}Fu). \]

\[ \square \]

**The proof of Proposition 2.** Set \(K = G'FH\). It is easy to verify that \(K(s) = \overline{F}(s)\), for \(s \in \Pi_0(C)\). We now prove that \(K(s, u) = \overline{F}(s, u)\) for each arrow \(u : I \to I\):

\[ K(u) = G'FH(u) = G'F_{\gamma_A}(u). \]

Since \(H'G' \simeq id_{C'}\) thanks to natural equivalence \(\beta = (i'_{A'})\), the following diagram

\[ A'_s \xrightarrow{i'} FA_s \]

\[ H'G'F_{\gamma_A}(u) \downarrow \quad \downarrow F_{\gamma_A}(u) \]

\[ A'_s \xrightarrow{i'} FA_s \]

commutes. (Note that \(A'_s = H'G'FA_s\)).

According to Lemma 4, we have

\[ F_{\gamma_A}(u) = \gamma_{FA}(\gamma_{FI}^{-1}Fu). \]

Besides, since the family \((\gamma_{A'})\) is a natural equivalence of the identity functor \(id_{C'}\), the following diagram

\[ A'_s \xrightarrow{i'} FA_s \]

\[ \gamma_{A'_s}(\gamma_{FI}^{-1}Fu) \downarrow \quad \downarrow \gamma_{FA}(\gamma_{FI}^{-1}Fu) \]

\[ A'_s \xrightarrow{i'} FA_s \]

commutes.

Hence \(H'G'F_{\gamma_A}(u) = \gamma_{A'_s}(\gamma_{FI}^{-1}Fu)\). From the definition of \(H'\), we have

\[ G'F_{\gamma_A}(u) = \gamma_{FI}^{-1}Fu = F_s(u). \]

This means \(G'FH = \overline{F}\).

Now we describe the Gr-functors on Gr-categories of the type \((\Pi, A)\)
Definition 2.1. Let $S = (\Pi, A, \xi)$, $S' = (\Pi', A', \xi')$ be Gr-categories. A functor $F : S \to S'$ is called a functor of the type $(\varphi, f)$ if

$$F(x) = \varphi(x), \quad F(x, u) = (\varphi(x), f(u))$$

and $\varphi : \Pi \to \Pi'$, $f : A \to A'$ is a pair of group homomorphisms satisfying $f(xa) = \varphi(x)f(a)$ for $x \in \Pi, a \in A$.

Theorem 5. Let $S = (\Pi, A, \xi)$, $S' = (\Pi', A', \xi')$ be Gr-categories and $(F, \tilde{F})$ be a Gr-functor from $S$ to $S'$. Then, $(F, \tilde{F})$ is a functor of the type $(\varphi, f)$.

Proof. For $x, y \in \Pi$, $\tilde{F}_{x,y} : Fx \otimes Fy \to F(x \otimes y)$ is an arrow in $S'$. It follows that $Fx, Fy = F(xy)$, so the function $\varphi : \Pi \to \Pi'$, defined by $\varphi(x) = F(x)$ is a group homomorphism on objects.

Assume that $F(x, a) = (Fx, f_x(a))$. Since $F$ is a functor, we have

$$F((x, a). (x, b)) = F(x, a). F(x, b).$$

It follows that

$$f_x(a + b) = f_x(a) + f_x(b). \quad (2.1)$$

So $f_x : A \to A'$ is a group homomorphism for each $x \in \Pi$. Besides, since $(F, \tilde{F})$ is a $\otimes$-functor, the diagram

$$\begin{array}{ccc}
Fx.Fy & \xrightarrow{\tilde{F}} & F(xy) \\
F(u \otimes v) \downarrow & & \downarrow F(u \otimes v) \\
Fx.Fy & \xrightarrow{\tilde{F}} & F(xy)
\end{array}$$

commutes for all $u = (x, a), v = (y, b)$. Hence, we have

$$F(u \otimes v) = F(u) \otimes F(v)$$

$$\iff f_{xy}(a + xb) = f_x(a) + Fx.f_y(b)$$

$$\iff f_{xy}(a) + f_{xy}(xb) = f_x(a) + Fx.f_y(b) \quad (2.2)$$

Applying the relation (2.1) for $x = 1$, we obtain $f_y(a) = f_1(a)$. Thus, $f_y = f_1$ for all $y \in \Pi$. Set $f_y = f$ and use (2.1'), we obtain

$$f(xb) = F(x)f(b) = \varphi(x)f(b) \quad (2.3)$$

Note that if we regard $\Pi'$-module $A'$ as a $\Pi$-module by the action $x \alpha' = F(x). \alpha'$, then from (2.1), (2.3), it results $f : A \to A'$ is a homomorphism of $\Pi$-modules. $\square$

To find the sufficient condition to make a functor of the type $(\varphi, f)$ become a Gr-functor, let us present the definition of obstruction as in the case of Ann-functors (see [7]).

Definition 2.2. If $F : (\Pi, A, \xi) \to (\Pi', A', \xi')$ is a functor of the type $(\varphi, f)$, then $F$ induces 3-cocycles $\xi = f_\ast \xi, \xi'^{\ast} = \varphi^\ast \xi'$, in which

$$(f_\ast \xi)(x, y, z) = f(\xi(x, y, z))$$

$$(\varphi^\ast \xi')(x, y, z) = \xi'(\varphi x, \varphi y, \varphi z)$$

The function $k = \varphi^\ast \xi' - f_\ast \xi$ is called an obstruction of the functor of the type $(\varphi, f)$.

Theorem 6. The functor $F : (\Pi, A, \xi) \to (\Pi', A', \xi')$ of the type $(\varphi, f)$ is a Gr-functor iff the obstruction $k = 0$ in $H^3(\Pi, A')$.  

6
Theorem 7. 1) There exists a bijection between the set of congruence classes of Gr-functors of the type \((\varphi, f)\) and the cohomology group \(H^2(\Pi, A')\) of the group \(\Pi\) with coefficients in the \(\Pi\)-module \(A'\), where \(A'\) is a \(\Pi\)-module with the action \(\pi a' = \varphi(x)a'\).

2) If \(F : (\Pi, A, \xi) \to (\Pi', A', \xi')\) is a Gr-functor, there exists a bijection

\[\text{Aut}(F) \to Z^1(\Pi, A').\]

Proof. 1) Suppose that \((F, \tilde{F}) : (\Pi, A, \xi) \to (\Pi', A', \xi')\) is a Gr-functor, then from Theorem 5, \(F = (\varphi, f)\) and

\[\xi'^{\ast} - \xi_\ast = \delta k\]

where \(k\) is the associated function with \(\tilde{F}\). Let \(k\) be fixed. Now if

\[(G, \tilde{G}) : (\Pi, A, \xi) \to (\Pi', A', \xi')\]

is a Gr-functor of the type \((\varphi, f)\), then \(\tilde{G} \equiv g\) and

\[\xi'^{\ast} - \xi_\ast = \delta g.\]

So \(k - g\) is a 2-cocycle. Consider the correspondence:

\[\Phi : \text{class}(G, \tilde{G}) \to \text{class}(k - g)\]

between the set of congruence classes of Gr-functors of the type \((\varphi, f)\) from \((\Pi, A, \xi)\) to \((\Pi', A', \xi')\) and the group \(H^2(\Pi, A')\).

First, we will show that the above correspondence is a function. Indeed, suppose that

\[(G', \tilde{G}') : (\Pi, A, \xi) \to (\Pi', A', \xi')\]

is a Gr-functor of the type \((\varphi, f)\), and \(\alpha : G \to G'\) is a Gr-natural transformation, then for all \(x, y \in \Pi\), the diagram

\[\begin{array}{ccc}
Gx.Gy & \xrightarrow{\tilde{G}} & G(xy) \\
(\bullet, \alpha_x) \otimes (\bullet, \alpha_y) & \downarrow & (\bullet, \alpha_{xy}) \\
G'x.G'y & \xrightarrow{\tilde{G}'} & G'(xy)
\end{array}\]
commutes. From the definition of the tensor product in the category \((\Pi', A')\), we have
\[
\alpha_x \otimes \alpha_y = \alpha_x + G_x.\alpha_y.
\]
It follows that
\[
x\alpha_y - \alpha_{xy} + \alpha_x = g - g',
\]
where \(g, g'\) are, respectively, associated functions with \(\tilde{G}, \tilde{G}'\).
(Since \(\tilde{G} = g\), \(\tilde{G}' = g'\) are 2-cocyles and \(\alpha\) is an 1-cochain, it implies)
\[
g - g' = \delta \alpha.
\]
(2.4)
So \(k - g = k - g' \in H^2(\Pi, A')\). We now prove that \(\Phi\) is an injection. Suppose that
\[(G, \tilde{G}), (G', \tilde{G}') : (\Pi, A, \xi) \to (\Pi', A', \xi')\]
are Gr-functors of the type \((\varphi, f)\) and satisfying
\[
\overline{k - g} = \overline{k - g'} \in H^2(\Pi, A').
\]
Then, there exists an 1-cochain and such that
\[
k - g = k - g' + \delta \alpha
\]
i.e. \(g' = g + \delta \alpha\). So the above-mentioned diagram commutes, i.e., \(\alpha : G \to G'\) is an \(\otimes\)-morphism. So
\[
\text{class}(G, \tilde{G}) = \text{class}(G', \tilde{G}').
\]
Finally, we will show that the correspondence \(\Phi\) is a surjection. Indeed, assume that \(g\) is an arbitrary 2-cocycle. We have
\[
\delta (k - g) = \delta k - \delta g = \delta k = \xi'^* - \xi*.
\]
Then, from Theorem 6, there exist the Gr-functor
\[(G, \tilde{G}) : (\Pi, A, \xi) \to (\Pi', A', \xi')\]
of the type \((\varphi, f)\) and the functorial isomorphism \(\tilde{G} = (\bullet, k - g)\). Clearly, \(\Phi(G) = \overline{\gamma}\). So \(\Phi\) is a surjection.

2) Let \(F = (F, \tilde{F}) : (\Pi, A, \xi) \to (\Pi', A', \xi')\) be a Gr-functor and \(\alpha \in Aut(F)\). Then, the equality (2.4) implies that \(\delta \alpha = 0\), i.e., \(\alpha \in Z^1(\Pi, A')\).

3 The relation between a Gr-category and the group extension problem

In this section, we built a Gr-category of an abstract kernel and apply it to create the constraints of a Gr-category strict.

3.1 The Gr-category \(\mathcal{A}_G\) of the group \(G\)

For the given group \(G\), we construct a Gr-category \(\mathcal{A}_G\) whose objects are elements of the group of automorphisms \(Aut(G)\). For the objects \(\alpha, \beta\) of \(\mathcal{A}_G\), we denote
\[
\text{Hom}(\alpha, \beta) = \{c \in G | \alpha = \mu_c \circ \beta\}.
\]
where $\mu_c$ is the automorphism induced by $c$. For the arrows $c : \alpha \to \beta$, $d : \beta \to \gamma$ of $A_G$, their composition is defined by $d \circ c = d + c$ (the addition in $G$). Then, the category $A_G$ is a strict Gr-category with the tensor product defined by
\[
\alpha \otimes \beta = \alpha \circ \beta \\
c \otimes d = c + \alpha'(d)
\]
where $c : \alpha \to \alpha'$, $d : \beta \to \beta'$.

The following proposition describes the reduced Gr-category of the Gr-category of an abstract kernel.

**Theorem 8.** Let $(\Pi, G, \psi)$ be abstract kernel and $\overline{\xi} \in H^3(\Pi, Z(G))$ be its obstruction. Let $S' = (\Pi', C, \xi')$ be the reduced Gr-category of the strict one $A_G$. Then

\[
II' = AutG/IntG, C = Z(G),
\]
and $\psi^* \xi'$ belongs to the cohomology class of $\xi$.

**Proof.** From the definitions of the category $A_G$, and reduced Gr-categories, we have

\[
II' = AutG/IntG, C = Z(G)
\]

We just have to prove that $\overline{\psi^* \xi'} = \overline{\zeta}$.

Indeed, let $(H, \tilde{H})$ be a canonical Gr-equivalence from $S'$ to $A_G$. Then, the diagram

\[
\begin{array}{ccc}
H_r(H_s H_t) & \overset{id \otimes \tilde{H}}{\longrightarrow} & H_r H_{st} \\
\downarrow & & \downarrow H(\cdot, \xi', \cdot) \\
(H, H_s)H_t & \overset{\tilde{H} \circ id}{\longrightarrow} & H_r \circ H_t \\
\end{array}
\]  

(3.1)

commutes for all $r, s, t \in II'$. Since $A_{UG}$ is strict, we have

$\gamma_r(u) = u, \forall r \in A_{UG}, \forall u \in Z(G) = C$.

Together with the definition of $H$, we obtain $H(\cdot, c) = c, \forall c \in C$. The fact that the diagram (3.1) commutes gives us

\[
H_r(h_{s,t}) + h_{r,st} = -\xi'_{r,s,t} + h_{r,s} + h_{r,s,t}
\]

(3.2)

where $h_{s,t} = \tilde{H}_{s,t}$ is a function from $\Pi' \times \Pi'$ to $G$.

For the abstract kernel $(\Pi, G, \psi)$, we choose the function $\varphi = H\psi : \Pi \to Aut(G)$. Clearly, $\varphi(1) = id_G$. Moreover, since

\[
\tilde{H}(\psi(x), \psi(y)) : H(\psi(x)H(\psi(y)) \to H(\psi(xy))
\]

is a morphism in $A_{UG}$, for all $x, y \in II$, we have

\[
\varphi(x)\varphi(y) = H(\psi(x)H(\psi(y)) = \mu_{f(x,y)}H(\psi(x)) = \mu_{f(x,y)}\varphi(xy)
\]

where $f(x, y) = \tilde{H}(\psi(x), \psi(y))$. So the pair $(\varphi, f)$ is a factor set of the abstract kernel $(\Pi, G, \psi)$.

So, there exists an obstruction $k(x, y, z) \in C = Z(G)$ satisfying

\[
\varphi(x)[f(y, z)] + f(x, yz) = k(x, y, z) + f(x, y) + f(xy, z).
\]

From the supposition, $\overline{k} = \overline{\xi}$.

Now for $r = \psi(x)$, $s = \psi(y)$, $t = \psi(z)$, the equality (3.2) becomes

\[
\varphi(x)[f(y, z)] + f(x, yz) = -(\psi^* \xi')(x, y, z) + f(x, y) + f(xy, z)
\]

So

\[
\overline{\psi^* \xi'} = \overline{k} = \overline{\xi}.
\]

$\square$
3.2 The equivalence between a Gr-category and a strict one

Using Theorem 8 and Theorem of the realization of the obstruction in the group extension problem, we prove the following theorem

**Theorem 9.** Each Gr-category is Gr-equivalent to a strict one.

First, we prove the following lemma

**Lemma 10.** Let $C'$ be a strict Gr-category whose the reduced Gr-category is $S' = (\Pi', C', \xi')$. Then, for each group homomorphism $\psi : \Pi \to \Pi'$, there exists a strict Gr-category $C$, Gr-equivalent to the Gr-category $S = (\Pi, C, \xi)$, where $C$ is regarded as a $\Pi$-module with the operation $xc = \psi(x)c$, and $\xi$ belongs to the same cohomological class of $\psi^*\xi'$.

**Proof.** We construct the strict Gr-category $C$ as follows

$$\text{Ob}(C) = \{(x, X) | x \in \Pi, X \in \psi(x)\}$$
$$\text{Hom}((x, X), (x, Y)) = \{x\} \times \text{Hom}_{C'}(X, Y).$$

The tensor product on objects and arrows of $C$ are defined by

$$(x, X) \otimes (y, Y) = (xy, X \otimes Y)$$
$$(x, u) \otimes (y, v) = (xy, u \otimes v).$$

The unitivity object of $C$ is $(1, I)$ where $I$ is the unitivity object of $C'$. Readers may easily verify that the category $C$ is a strict Gr-category. Moreover, we have the isomorphisms

$$\lambda : \Pi_0(C) \to \Pi \quad \rho : \Pi_1(C) \to \Pi_1(C') = C$$
$$(x, X) \mapsto x \quad (1, c) \mapsto c$$

and a Gr-functor $(F, \widetilde{F}) : C \to C'$ is given by

$$F(x, X) = X, \quad F(x, u) = u, \quad \widetilde{F} = \text{id}.$$ Let $\mathcal{F} = (F_0, F_1)$ be the functor induced by $(F, \widetilde{F})$ on the reduced categories $S, S'$, we have

$$F_0(x, X) = \widetilde{F}(x, X) = \overline{X} = \psi(x)$$
$$F_1(1, u) = \gamma_{F_0(1, u)} F_1(1, u) = \gamma_{\psi(u)} = u.$$ This means $F_0 = \psi \lambda$ and $F_1 = f$, or $\widetilde{F}$ is the functor of the type $(\psi \lambda, f)$.

Now assume that $u$ is the associativity constraint of $S$. Let $(\phi, \tilde{\phi})$ denote the Gr-functor from $S$ to $S'$ determined by the composition

$$(\phi, \tilde{\phi}) = (G', \tilde{G'}) \circ (F, \widetilde{F}) \circ (H, \tilde{H})$$

for canonical equivalences $(H, \tilde{H}), (G', \tilde{G}')$. From the Proposition 2, we have $\phi = \mathcal{F}$.

Besides, from Theorem 6, the obstruction of the pair $(\psi \lambda, f)$ must vanish in $H^3(\Pi_0, C')$, i.e.,

$$(\psi \lambda)^* \xi' = f_* u + \delta \tilde{\phi}$$

Now if we denote $\xi = f_* u$, the pair $J = (\lambda, f), \tilde{J} = \text{id}$ is a Gr-functor from $S$ to $(\Pi, C, \xi)$. Then the composition $(J, \tilde{J}) \circ (G, \tilde{G})$ is a Gr-equivalence from $C$ to $(\Pi, C, \xi)$. Finally, we will prove that $\xi$ belongs to the same cohomological class as $\psi^* \xi'$. Let $K = (\lambda, J^{-1}) : (\Pi, C, \xi) \to S$. Then $K$ together with $\tilde{K} = \text{id}$ is a Gr-category, and the composition

$$(\phi, \tilde{\phi}) \circ (K, \tilde{K}) : (\Pi, C, \xi) \to S'$$
is a Gr-functor.

\[
\begin{array}{c}
S \\
\Phi \\
\downarrow K \\
\Phi \circ K
\end{array}
\]

\[(\Pi, C, \zeta)\]

Clearly, \(\phi \circ K\) is a functor of the type \((\psi, id)\) and therefore its obstruction vanishes, i.e., \(\zeta\) belongs to the same cohomological class as \(\psi^* \zeta'\).

**The proof of Theorem 9**

Let \(\mathcal{A}\) be a Gr-category whose the reduced Gr-category is \(\mathcal{I} = (\Pi, C, \zeta)\), the theorem on the realization of the obstruction (Theorem 9.2, Chapter IV [6]), there exists the group \(G\) whose center is \(Z(G) = C\) and such that the abstract kernel \(\Pi, G, \psi\) has the obstruction \(\mathcal{S}' = (\Pi', C, \zeta')\), where \(\psi^* \zeta' = \zeta\).

From Lemma 10, the homomorphism \(\psi : \Pi \rightarrow AutG/IntG\), defines a strict Gr-category \(\mathcal{C}\), Gr-equivalent to \(\mathcal{I} = (\Pi, C, \zeta)\). Therefore, \(\mathcal{C}\) and \(\mathcal{A}\) are Gr-equivalent. The theorem is completely proved.

Readers may find a different proof of Theorem 9 in [10].

**References**

[1] P. Carrasco and A. R. Garzón, *Obstruction theory for extensions of categorical groups*, Applied Categorical Structure, 12 (2004), 35-61.

[2] A. Fröhlich and C. T. C. Wall, *Graded monoidal categories*, Compositio Mathematica, tom 28, No 3 (1974), 229-285.

[3] A. Joyal and R. Street, *Braided tensor categories*, Adv. Math. Vol 2, No 1 (1993) 20-78.

[4] C. Kassel, *Quantum Groups*, Graduate Texts in Math, No 155, Springer (1995).

[5] M. L. Laplaza, *Coherence for categories with group structure: an antenative approach*, J. Algebra, 84 (1983), 305-323.

[6] S. Mac Lane, *Homology*, Springer, 1975.

[7] N.T. Quang, *Ann-categories and the Mac Lane-Shukla cohomology of rings*. Abelian groups and modules, No 11,12 (Russian), 166-183, Tomsk. Gos. Univ., Tomsk,1994.

[8] N. Saavedra Rivano, *Categories Tannakiennes*, Lecture Notes in Math. Vol 265, Springer-Verlag, Berlin and New York (1972).

[9] H. X. Sinh, *Gr-categories*, Universite Paris VII, Thèse de doctorat (1975).

[10] H. X. Sinh, *Gr-categoriesstrictes*, Acta mathematica Vietnamica Tom 3, No 2 (1978), pp 47-59.

Address: Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy Street, Cau Giay district, Hanoi, Vietnam.
Email: nguyenquang272002@gmail.com