On the $L^2$ Cohomology of a Convex Cocompact Hyperbolic Manifold

Xiaodong Wang *

September 2002

Abstract

We prove a vanishing theorem for a convex cocompact hyperbolic manifold, which relates its $L^2$ cohomology and the Hausdorff dimension of its limit set. The borderline case is shown to characterize the manifold completely.

1 Introduction

The study of $L^2$ harmonic forms on a complete Riemannian manifold is a very interesting and important subject. In [1] the author has studied $L^2$ harmonic 1-forms on a conformally compact Einstein manifold and proved the following theorem.

**Theorem 1.1** Let $(M^{n+1}, g)$ be a conformally compact Einstein manifold.

1. If $\lambda_0(g) > n - 1$ then $\mathcal{H}^1(M) = 0$.

2. If $\lambda_0(g) = n - 1$ and $\mathcal{H}^1(M) \neq 0$, then $M$ is isometric to $\mathbb{R} \times \Sigma$, with warped product metric $dt^2 + \cosh^2(t)h$, where $\Sigma$ is compact and $h$ is a metric on $\Sigma$ with $\text{Ric}(h) = -(n-1)h$.

Here $\lambda_0(g)$ is the infimum of the $L^2$ spectrum of $-\Delta$. The proof hinges on the following inequality for a harmonic 1–form

$$|\nabla \theta|^2 \geq \frac{n+1}{n} |\nabla \theta||^2$$

and the characterization of the equality case.

*Department of Mathematics, MIT, Cambridge, MA 02139; Email:xwang@math.mit.edu
In this paper we use the same idea to study $L^2$ harmonic forms on a convex cocompact hyperbolic manifold. There has been much work on this topic. We simply mention the paper by Mazzeo and Phillips [7] and the recent work by Lott [5] and refer the reader to the reference therein for more background knowledge. Our main result is

**Theorem 1.2** Let $M = \mathbb{H}^{n+1}/\Gamma$ be a convex cocompact hyperbolic manifold and $\delta$ the Hausdorff dimension of the limit set of $\Gamma$. Suppose $\delta > n/2$. Let $\mathcal{H}^p(M)$ be the space of $L^2$ harmonic $p-$forms.

1. If $p < n - \delta$ then $\mathcal{H}^p(M) = 0$.

2. If $\delta$ is an integer and $\mathcal{H}^{n-\delta}(M) \neq 0$, then $M$ is a twisted warped product of $\mathbb{H}^{n-\delta}$ and a compact hyperbolic manifold of dimension $\delta + 1$ (described in detail in Section 3) and $\dim \mathcal{H}^{n-\delta}(M) = 1$.

Our proof is conceptually very simple. We use Bochner formula to prove the vanishing theorem, but to get the sharp result we need a technical lemma like (1.1). It turns out that this inequality is an example of a refined Kato inequality and there are many other examples in Riemannian geometry. Recently D. Calderbank, P. Gauduchon and M. Herzlich [2] have worked out a general principle which covers all known examples and gives interesting new ones (T. Branson [1] has a different approach). As a special case their theorem implies that

$$|\nabla \theta|^2 \geq \frac{n + 2 - p}{n + 1 - p} |\nabla |\theta||^2$$

(1.2)

for a harmonic $p-$form on an $(n + 1)$ dimensional Riemannian manifold. Moreover the equality case is fully characterized. This result plays a key role in the proof. The proof of the second part is a little bit involved and may have some independent interest. In this borderline case we have an $L^2$ harmonic form which satisfies an overdetermined system of first order PDEs. The existence of such a harmonic form gives rise to a splitting of $M$ and forces the metric to be a twisted warped product. It is surprising to have a situation where the Bochner formula gives sharp results on higher dimensional cohomology.

In closing the introduction we should mention the paper [8] by Nayatani who proved a similar result for compact Kleinian $n-$manifolds. His assumption requires $\delta < n/2 - 1$ while we assume that $\delta > n/2$. In some sense his result and ours are complementary.

**Acknowledgment:** I am indebted to Prof. Marc Herzlich for drawing my attention to his joint paper with Calderbank and Gauduchon [2] which plays an important
role in the proof of the main theorem. I wish to thank Professors Rick Schoen and Rafe Mazzeo for their constant encouragement and helpful discussions. Finally I want to thank the referee for many valuable comments and suggestions.

2 Preliminaries

A complete hyperbolic manifold \((M^{n+1}, g)\) is the quotient of the unit ball \(B^{n+1}\) by a torsion-free discrete group \(\Gamma\) of isometries of \(\mathbb{H}^{n+1}\). The limit set \(\Lambda(\Gamma)\) is defined to be the set of accumulation points in the sphere \(S^n = \partial B^{n+1}\) of an orbit \(\Gamma(x) = \{\gamma(x) | \gamma \in \Gamma\}\), where \(x\) is a point in \(B^{n+1}\). \(M\) is called geometrically finite if \(\Gamma\) has a fundamental domain bounded by finitely many geodesic hyperplanes. \(M\) is called convex cocompact if the action of \(\Gamma\) on the hyperbolic convex hull of \(\Lambda(\Gamma)\) in \(B^{n+1}\) has a compact fundamental region. Convex cocompact hyperbolic manifolds can be characterized as geometrically finite hyperbolic manifolds without cusps.

A convex cocompact hyperbolic manifold \(M\) is conformally compact in the sense that \(\overline{M} = M \sqcup (\Omega(\Gamma)/\Gamma)\) is a compact manifold with boundary and, if \(r\) is a defining function (i.e. a smooth function on \(M\) with first order zero on the boundary, positive on \(M\)), then \(\overline{g} = r^2g\) extends as a regular metric on \(\overline{M}\). Its conformal infinity is the compact Kleinian manifold \(\Sigma = \Omega(\Gamma)/\Gamma\).

The \(L^2\) cohomology of a geometrically finite hyperbolic manifold was studied by Mazzeo and Phillips [7]. We state their theorem for a convex cocompact hyperbolic manifold.

**Theorem 2.1** Let \(M = \mathbb{H}^{n+1}/\Gamma\) be an orientable convex cocompact hyperbolic manifold. There are natural isomorphisms

\[
\begin{align*}
\mathcal{H}^p &\simeq H^p(M, \partial M) & \text{if } p < (n+1)/2 \\
\mathcal{H}^p &\simeq H^p(M) & \text{if } p > (n+1)/2
\end{align*}
\]

If \(n+1\) is even then \(\mathcal{H}^{(n+1)/2}\) is infinite dimensional.

The asymptotics of such harmonic forms are also studied in detail in [7]. To formulate the result we consider a neighborhood \(U\) of \(\partial M\) and use standard upper-half-space coordinates \((x, y)\) so that \(U \cap \partial M = \{y = 0\}\). Express \(\omega = \alpha + dy \wedge \beta\), where \(\alpha\) and \(\beta\) are a \(p\) and \((p-1)\) form in \(x\), respectively, depending parametrically on \(y\).

**Theorem 2.2** Suppose \(\omega\) is an \(L^2\) harmonic \(p\)-form in a neighborhood \(U\) of \(\partial M\). Writing \(\omega = \alpha + dy \wedge \beta\), the terms \(\alpha, \beta\) have complete asymptotic expansions as
In all cases, the leading coefficients $\alpha_0, \alpha_1, \beta_0, \beta_1$ are $C^\infty$ and their estimates are uniform for $\omega$'s bounded in $L^2(\mathcal{U})$. Similar expansions and rates of decay hold when $p \geq n/2 + 1$.

The key to prove such asymptotic expansions is to construct a parametrix $Q$ for the Hodge Laplacian $\Delta = dd^* + d^*d$ near the conformal infinity such that

$$Q\Delta = I - R,$$

where $R$ is a smoothing operator. By construction the Schwartz kernel $R$ has an asymptotic expansion, hence $\omega$ has a similar expansion. In fact it is shown in [7] that

$$\begin{align*}
\alpha &\sim \sum_{j=0}^\infty \sum_{l=0} N_j \alpha_{jl} y^{n-2p-1} (\log y)^l, & N_0 = 0 \\
\beta &\sim \sum_{j=0}^\infty \sum_{l=0} M_j \beta_{jl} y^{n-2p+1} (\log y)^l,
\end{align*}$$

(2.1)

provided $p < n/2$. The forms $\alpha_{jl}, \beta_{jl}$ are $C^\infty$. For $p = n/2$ there are similar results. For details we refer to [6] and [7].

The above asymptotics given by Mazzeo and Phillips, while valid for solutions of $\Delta \omega = 0$ which are not necessarily closed and coclosed (e.g. might be needed when looking at solutions of $\Delta \omega = f$ where $f$ is compactly supported, hence zero in this boundary neighborhood), can be improved if $\omega$ is closed and coclosed. Write $\alpha = \sum_{|I|=p} \alpha_I dx^I$ and $\beta = \sum_{|J|=p-1} \beta_J dx^J$. From $d\omega = 0$ we get

$$\sum_{|I|=p} \frac{\partial \alpha_I}{\partial y} dy \wedge dx^I + \sum_{|I|=p,i} \frac{\partial \alpha_I}{\partial x^i} dx^i \wedge dx^I - \sum_{|J|=p-1,j} \frac{\partial \beta_J}{\partial x^j} dy \wedge dx^j \wedge dx^I = 0.$$

Therefore

$$\frac{\partial \alpha_I}{\partial y} = \sum_{I,J} \epsilon_{IJ} \frac{\partial \beta_J}{\partial x^j},$$

(2.2)
where $\epsilon^l_{j} = 0$ unless $I = J \cup \{j\}$, in which case it is the sign of the permutation $(j^I_j J)$. This equation combined with the asymptotic expansion (2.1) easily implies that the coefficients $\alpha_{jl} = 0$ in (2.1) for $j = 0, 1$. Similarly we can prove that $\beta_{0l} = 0$ for $l \neq 0$ by using the equation $d^* \omega = 0$ combined with the asymptotic expansion (2.1).

It is well-known that an $L^2$ harmonic form on a complete Riemannian manifold is both closed and coclosed. Therefore an $L^2$ harmonic form on a convex cocompact hyperbolic manifold satisfies the improved decay rate. Though it is an elementary observation, we state it as a theorem for later reference. In this improved version we do not need to formulate $p < n/2$ and $p = n/2$ separately.

**Theorem 2.3** Suppose $\omega$ is an $L^2$ harmonic $p$–form on a convex cocompact hyperbolic manifold $M$ with $p \leq n/2$. Then in a neighborhood $U$ of $\partial M$, writing $\omega = \alpha + dy \wedge \beta$, the terms $\alpha, \beta$ have complete asymptotic expansions as $y \to 0$, and in particular

$$\alpha = O(y^{n+2-2p} \log y),$$

$$\beta = O(y^{n+1-2p}).$$

Similar expansions and rates of decay hold when $p \geq n/2 + 1$.

For a geometrically finite hyperbolic manifold, the asymptotics of an $L^2$ harmonic form at a cusp are also analyzed in [7].

3 Special examples of convex cocompact hyperbolic manifolds

Let $(N, g_0)$ be a compact Riemannian manifold of dimension $k+1$ such that $\text{Ric}(g_0) = -kg_0$. Consider the following metric on $M = B^{n-k} \times N$

$$g = \frac{4}{1-|x|^2} \left( dx^2 + \frac{1 + |x|^2}{4} g_0 \right),$$

where $x$ is the coordinates on $B^{n-k}$. Then $g$ is a conformally compact Einstein metric. The conformal infinity is the $S^{n-k-1} \times N$ with the product metric. If we use polar coordinates on the hyperbolic space the metric can be written in the following form

$$g = dt^2 + \sinh^2(t) d\zeta^2 + \cosh^2(t) g_0, \quad (3.1)$$

where $d\zeta^2$ is the standard metric on $S^{n-k-1}$. It is obvious that $(M, g)$ is the warped product of $\mathbb{H}^{n-k}$ and $(N, g_0)$. 

5
If \((N,g_0)\) is hyperbolic, then \((M,g)\) is a convex cocompact hyperbolic manifold. To see this we first consider the hyperbolic space \(\mathbb{H}^{n+1}\) using the upper space model with coordinates \((r,x,y)\), where \(r > 0, x \in \mathbb{R}^k, y \in \mathbb{R}^{n-k}\). The hyperbolic metric is
\[
g = r^{-2}(dr^2 + dx^2 + dy^2).
\]
We introduce polar coordinates \(y = \rho \zeta\) on \(\mathbb{R}^{n-k}\), with \(\rho > 0, \zeta \in S^{n-k-1}\). Then
\[
g = r^{-2}(dr^2 + d\rho^2 + \rho^2 d\zeta^2 + dx^2),
\]
where \(d\zeta^2\) is the standard metric on \(S^{n-k-1}\). We change coordinates by setting
\[
r = s/\cosh(t), \quad \rho = s \tanh(t).
\]
Straightforward calculation shows that in the new coordinates
\[
g = dt^2 + \sinh^2(t)d\zeta^2 + \cosh^2(t)s^{-2}(ds^2 + dx^2).
\]
This demonstrates that \(\mathbb{H}^{n+1}\) is the warped product \(\mathbb{H}^{n-k} \times \mathbb{H}^{k+1}\), since
\[
dt^2 + \sinh^2(t)d\zeta^2
\]
is exactly the hyperbolic metric in geodesic polar coordinates, and since
\[
s^{-2}(ds^2 + dx^2)
\]
is the hyperbolic metric on \(\mathbb{H}^{k+1}\). This change of coordinates has a clear geometric meaning. \(\mathbb{H}^{k+1}\) sits in \(\mathbb{H}^{n+1}\) as the totally geodesic submanifold \(\{y = 0\}\). In our new coordinates we simply view \(\mathbb{H}^{n+1}\) by the exponential map on the normal bundle of \(\mathbb{H}^{k+1}\) in \(\mathbb{H}^{n+1}\). It is easy to verify that \(t\) as given by (3.3) is the distance from the point \((r,x,y)\) to \(\mathbb{H}^{k+1}\) with the closest point being \((s,x,0)\). Let \(\Gamma\) be the cocompact Kleinian group such that \(N = \mathbb{H}^{k+1}/\Gamma\). There is a natural way to extend the action of \(\Gamma\) to \(\mathbb{H}^{n+1}\) \((n > k)\) called the Poincaré extension. In terms of the above description of \(\mathbb{H}^{n+1}\) as the product \(\mathbb{H}^{n-k} \times \mathbb{H}^{k+1}\) with the warped product metric (3.4), the extension is that \(\Gamma\) only acts on the second component. Hence \(\mathbb{H}^{n+1}/\Gamma = \mathbb{H}^{n-k} \times N^{k+1}\) with the warped product metric given by (3.4).

There is a slightly more general construction. We give two equivalent descriptions here. Suppose that \(E \to N\) is a rank \(= n - k\) flat \(O(n-k)\) bundle. Such a bundle is determined by its holonomy \(\rho : \Gamma \to O(n-k)\). We can cover \(N\) by open sets \(\{U_\alpha\}\) with parallel trivialization \(E|_{U_\alpha} \to \mathbb{R}^{n-k} \times U_\alpha\). Then on \(E|_{U_\alpha}\) we can define a hyperbolic metric using formula (3.4). Since the transition functions \(U_\alpha \cap U_\beta \to O(n-k)\) are locally constant, we get a global hyperbolic metric which
is apparently conformally compact. We call such a hyperbolic manifold a twisted warped product. The second description we give is simpler. We define a generalized Poincare extension of $\Gamma$–action on $\mathbb{H}^{k+1}$ to $\mathbb{H}^{n+1} = \mathbb{H}^{n-k} \times \mathbb{H}^{k+1}$ by using the homomorphism $\rho : \Gamma \to O(n-k)$

$$\gamma \cdot (x, y) = (\rho(\gamma)x, \gamma \cdot y).$$

Then $M = \mathbb{H}^{n+1}/\Gamma$ is the twisted product. It is obvious that the limit set of $M$ is a totally geodesic $S^k \subset S^n$. The convex core is the compact $k + 1$ dimensional hyperbolic manifold $N$ which is totally geodesic and the full manifold $M$ is a vector bundle over $N$ with rank $n - k$. Conversely we have the following proposition whose proof is simple and hence omitted.

**Proposition 3.1** Let $M = \mathbb{H}^{n+1}/\Gamma$ be a convex cocompact hyperbolic manifold. Suppose the limit set is a totally geodesic $S^k \subset S^n$, then $M$ is a twisted warped product of $\mathbb{H}^{n-k}$ and a compact $k + 1$ dimensional hyperbolic manifold.

Suppose $k > n/2 - 1$ and both $N$ and $M$ are orientable, we can describe $L^2$ harmonic $(n - k)$–forms on $M$ explicitly. First note $H^*(M) = H^*(N)$ for $N$ is a deformation retract of $M$. By Lefschetz duality and Mazzeo-Phillips theorem

$$\mathcal{H}^{n-k}(M) \simeq H^{n-k}(M, \Sigma) \simeq H^{k+1}(M) \simeq H^{k+1}(N) \simeq \mathbb{R}.$$ 

If we introduce polar coordinates on the normal bundle of $N$ in $M$, by the previous discussion the metric $g = dt^2 + \sinh^2(t)d\zeta^2 + \cosh^2(t)h$, where $h$ is the metric on $N$ and $d\zeta^2$ is the standard metric on $S^{n-k-1}$. By calculation one can show that all $L^2$ harmonic $(n - k)$–forms on $M$ are given by the following formula

$$\omega = c\frac{\sinh^{n-k-1}(t)}{\cosh^{k+1}(t)}dt \wedge \Theta, \quad (3.5)$$

where $\Theta$ is the volume form on $S^{n-k-1}$ and $c$ is any constant. It is easy to see that $|\omega| = |c|\cosh^{-(k+1)}(t)$. Apparently the maximal level set is the convex core $N$. Note $t$ is the distance function to $N$.

### 4 Proof of the main theorem

We start to prove Theorem [1,2]. We prove part 1 by contradiction. Suppose we have a nonzero $L^2$ harmonic form $\xi$ of degree $p \leq n - \delta$. By Bochner formula

$$0 = (dd^* + d^*d)\xi = \nabla^*\nabla\xi + \mathcal{R}\xi, \quad (4.1)$$

where $\mathcal{R}\xi = \theta^k \wedge i_{\varepsilon_i}R(e_k, e_l)\xi$, if we choose orthonormal frame $\{e_i\}$ for the tangent bundle and $\{\theta^i\}$ the dual frame for the cotangent bundle. The following lemma is well known. For completeness we present the proof.
Lemma 4.1 Let \( \omega \) be a \( p \)-form on an \( (n + 1) \)-dimensional manifold of constant sectional curvature \( \kappa \). Then

\[
R \omega = p(n + 1 - p) \kappa \omega. \tag{4.2}
\]

Proof. As the metric has constant sectional curvature \( \kappa \) we have

\[
R(e_k, e_l)\theta^i = -\kappa (\delta_{li} \theta^k - \delta_{ki} \theta^l).
\]

Without loss of generality we assume \( \omega = \theta^1 \wedge \cdots \wedge \theta^p \). We compute

\[
\begin{align*}
\theta^k \wedge i_{e_l} R(e_k, e_l) \omega \\
&= -\kappa \theta^k \wedge i_{e_l} \left( \sum_{i=1}^p (-1)^{i-1} (\delta_{li} \theta^k - \delta_{ki} \theta^l) \wedge \theta^1 \wedge \cdots \wedge \hat{\theta}^i \wedge \cdots \wedge \theta^p \right) \\
&= \kappa \sum_{i=1}^p (-1)^{i} \left( \delta_{li} \delta^k - (n+1) \delta_{ki} \right) \theta^k \wedge \theta^1 \wedge \cdots \wedge \hat{\theta}^i \wedge \cdots \wedge \theta^p \\
&\quad + \kappa \sum_{i=1}^p (-1)^{i} \delta_{ki} \theta^k \wedge \theta^l \wedge i_{e_l} \left( \theta^1 \wedge \cdots \hat{\theta}^i \wedge \cdots \wedge \theta^p \right) \\
&= np \kappa \omega - p(p - 1) \kappa \omega \\
&= p(n + 1 - p) \kappa \omega.
\end{align*}
\]

By this lemma and (4.1) we get

\[
\nabla^* \nabla \xi = p(n + 1 - p) \xi.
\]

This easily implies

\[
\frac{1}{2} \triangle |\xi|^2 = |\nabla \xi|^2 - p(n + 1 - p) |\xi|^2.
\tag{4.3}
\]

To proceed we need the following lemma which is Theorem 6.3.(ii) in [2], but we state it in a way convenient for our purpose without introducing abstract notations.

Lemma 4.2 Let \( \xi \) be a harmonic \( p \)-form (i.e. \( d \xi = 0 \) and \( \delta \xi = 0 \)) on a Riemannian manifold of dimension \( n + 1 \), then

\[
|\nabla \xi|^2 \geq \frac{n + 2 - p}{n + 1 - p} |\nabla |\xi||^2.
\tag{4.4}
\]
Moreover the equality holds iff there exists a 1–form \( \alpha \) with \( \alpha \wedge \xi = 0 \) such that
\[
\nabla \xi = \alpha \otimes \xi - \frac{1}{n+2-p} \sum_{i=0}^{n} \theta^i \otimes (\theta^i \wedge i_{\alpha^\sharp} \xi),
\]
(4.5)
where \( \{\theta^0, \theta^1, \ldots, \theta^n\} \) is an orthonormal basis for the cotangent bundle and \( \alpha^\sharp \) is the vector dual to the 1–form \( \alpha \).

Let \( f = |\xi| \). By the above lemma, we get from (4.3)
\[
\frac{1}{2} \triangle f^2 \geq \frac{n+2-p}{n+1-p} |\nabla f|^2 - p(n+1-p)f^2,
\]
or, equivalently
\[
f \triangle f \geq \frac{1}{n+1-p} |\nabla f|^2 - p(n+1-p)f^2.
\]

Let \( \phi = f^\beta \). We compute
\[
\frac{1}{2} \triangle \phi^2 = \phi \triangle \phi + |\nabla \phi|^2
\]
\[
= f^\beta \left[ \beta f^{\beta-1} \triangle f + \beta (\beta - 1) f^{\beta-2} |\nabla f|^2 \right] + |\nabla \phi|^2
\]
\[
= \beta f^{2(\beta-1)} \left[ f \triangle f + (\beta - 1) |\nabla f|^2 \right] + |\nabla \phi|^2
\]
\[
\geq \beta f^{2(\beta-1)} \left[ \frac{1}{n+1-p} |\nabla f|^2 - p(n+1-p)f^2 + (\beta - 1) |\nabla f|^2 \right] + |\nabla \phi|^2
\]
\[
= \left( 2\beta - \frac{n-p}{n+1-p} \right) \frac{1}{\beta} |\nabla \phi|^2 - p(n+1-p)\beta \phi^2.
\]

Let \( \beta = \frac{n-p}{n+1-p} \), then we have
\[
\frac{1}{2} \triangle \phi^2 \geq |\nabla \phi|^2 - p(n-p)\phi^2.
\]
(4.6)

We now take a defining function \( r \) such that near the conformal infinity \( \Sigma \) the metric \( g = r^{-2}(dr^2 + h_r) \), where \( h_r \) is an \( r \)–dependent family of metrics on \( \Sigma \). Define \( M^\epsilon = \{x \in M \mid r(x) \geq \epsilon \} \). For \( \epsilon \) small enough this is a compact manifold with boundary. By (4.6)
\[
\int_{M^\epsilon} (|\nabla \phi|^2 - p(n-p)\phi^2) \, dV \leq \int_{\partial M^\epsilon} \phi \frac{\partial \phi}{\partial \nu} \, d\sigma,
\]
(4.7)
where \( \nu \) is the outer unit normal of \( \partial M^\epsilon \). By Theorem 2.3 we have
\[
\phi = O(r^{n-p}).
\]
Notice $\frac{\partial \phi}{\partial \nu}$ is of the same order as $\phi$. Therefore we get

$$\int_{\partial M^e} \phi \frac{\partial \phi}{\partial \nu} d\sigma = \int_{\Sigma} O(\epsilon^{n-2p}) dV_\epsilon,$$

where $dV_\epsilon$ is the volume form of $h_\epsilon$ on $\Sigma$. Under the condition $n - p \geq \delta > n/2$, the boundary term apparently goes to zero as $\epsilon \to 0$. Therefore

$$\int_M |\nabla \phi|^2 \leq p(n - p) \int_M \phi^2. \quad (4.8)$$

According to Sullivan [10] the infimum of the spectrum of $-\Delta$ is given by $\lambda_0 = \delta(n - \delta)$. If $n - p > \delta$, then $\lambda_0 > p(n - p)$ and the above inequality is impossible. Hence $H^p = 0$. This finishes the proof of the first part.

Next, we prove the second part of Theorem 1.2. Suppose that $\delta$ is an integer and $\xi$ is a nonzero $L^2$ harmonic form of degree $p = n - \delta$. Then from the previous discussion we must have

$$-\Delta \phi = \delta(n - \delta) \phi, \quad (4.9)$$

and, in view of Lemma 4.2

$$\nabla \xi = \alpha \otimes \xi - \frac{1}{\delta + 2} \sum_{i=0}^n \theta^j \otimes (\theta^j \wedge \iota_{\alpha^i} \xi) \quad (4.10)$$

for a 1–form $\alpha$ with $\alpha \wedge \xi = 0$.

Before we plunge into the details, we describe our strategy. The existence of a nontrivial solution of the overdetermined equations (4.10) will be used to show that the regular level sets of $\phi$ are compact hypersurfaces of constant mean curvature and they carry local splittings. Then by taking limit we prove that the maximum level set is a totally geodesic compact submanifold. The exponential map from its normal bundle is then a diffeomorphism onto $M$ and gives the twisted warped product structure.

**Step 1.** By Harnack inequality the function $\phi$ is everywhere positive on $M$. Let $c > 0$ be a regular value of $\phi$, then $\Sigma_c = \phi^{-1}(c)$ is a compact hypersurface in $M$. Near a point $x \in \Sigma_c$ we choose orthonormal basis of 1-forms $\{\theta^0, \ldots, \theta^n\}$ such that $\alpha = (\delta + 2)u \theta^0$ where $u$ is positive. As $\alpha \wedge \xi = 0$, we can write $\xi = \theta^0 \wedge \omega$ such that $\omega$ contains no components involving $\theta^0$. By (1.10) we obtain the following two equations

$$\nabla_{\epsilon_0} (\theta^0 \wedge \omega) = (\delta + 1)u \theta^0 \wedge \omega, \quad (4.11)$$

$$\nabla_{\epsilon_j} (\theta^0 \wedge \omega) = -u \theta^j \wedge \omega, \quad j = 1, \ldots, n. \quad (4.12)$$
Thus (recall $\phi = |\xi|^\delta/(\delta+1)$)

$$e_0 \phi = \frac{\delta}{\delta+1} |\xi|^{-(\delta+2)/(\delta+1)} \langle \nabla e_0 \xi, \xi \rangle = \delta u \phi,$$

$$e_j \phi = \frac{\delta}{\delta+1} |\xi|^{-(\delta+2)/(\delta+1)} \langle \nabla e_j \xi, \xi \rangle = 0, \quad j = 1, \ldots, n. \tag{4.14}$$

Therefore $e_0$ is the normal vector field of the hypersurface $\Sigma_c$ and $e_1, \ldots, e_n$ are tangent to $\Sigma_c$. Thus

$$\nabla \phi = \delta u \phi e_0. \tag{4.15}$$

We can write

$$\nabla e_i \theta^0 = \sum_{j=1}^n \Pi_{ij} \theta^j, \tag{4.16}$$

where $\Pi_{ij} = \langle \nabla e_i e_0, e_j \rangle$ is the second fundamental form of the hypersurface $\Sigma_c$.

Since $\nabla e_0 (\theta^0 \wedge \omega) = \nabla e_0 \theta^0 \wedge \omega + \theta^0 \wedge \nabla e_0 \omega$ and $\nabla e_0 \theta^0$ has no $\theta^0$-component, we obtain from (4.11)

$$\nabla e_0 \theta^0 \wedge \omega = 0, \tag{4.17}$$

$$\theta^0 \wedge (\nabla e_0 \omega - (\delta+1) u \omega) = 0. \tag{4.18}$$

Similarly from (4.12) we get

$$\theta^0 \wedge \nabla e_j \omega = 0, \tag{4.19}$$

$$\left(\nabla e_j \theta^0 + u \theta^j\right) \wedge \omega = 0. \tag{4.20}$$

The equation (4.19) implies that the tangential component of $\nabla e_j \omega$ is zero, i.e., $\omega$ restricted to $\Sigma_c$ is parallel.

We introduce a distribution $E$ on $\Sigma_c$ by defining

$$E_x = \{v \in T_x \Sigma_c | v^* \wedge \omega = 0\}, \forall x \in \Sigma_c. \tag{4.21}$$

Let $E^\perp$ be the orthogonal complement of $E$. Then we have a decomposition

$$T \Sigma_c = E \oplus E^\perp. \tag{4.22}$$

Both $E$ and $E^\perp$ are parallel for $\omega$ is parallel. Obviously $0 \leq \text{rank } E \leq p-1 = \deg \omega$.

The decomposition (4.22) gives a (local) splitting $\Sigma_c = \Sigma^1 \times \Sigma^2$ such that $g$ is the product of $g_1$ on $\Sigma^1$ and $g_2$ on $\Sigma_2$. Hence $R_{\Sigma_c}(X, Y, X, Y) = 0, \forall X \in E, Y \in E^\perp$.

By Gauss equation

$$-|X|^2|Y|^2 = -\Pi(X, X)\Pi(Y, Y) + \Pi(X, Y)^2. \tag{4.23}$$
We can choose orthonormal bases on $E$ and $E^\perp$ such that in the corresponding coordinates

$$
\Pi(X,X) = \sum_i \lambda_i x_i^2, \quad \Pi(Y,Y) = \sum_j \mu_j y_j^2.
$$

By (4.23) we have

$$
\Pi(X,Y)^2 = -\sum_i x_i^2 \sum_j y_j^2 + \sum_i \lambda_i x_i^2 \sum_j \mu_j y_j^2.
$$

Fixing $x$, view both sides as quadratic forms in $y$. The right hand side has no mixed terms $y_i y_j, i \neq j$. It follows that the linear form $\Pi(X,Y)$ in $y$ involves only one of $y_j$'s. The same argument works for $x$ while fixing $y$. Therefore by renumbering we can assume $\Pi(X,Y) = ax_1 y_1$. Then it is easy to see that $a = 0$ and

$$
\Pi(X,Y) = 0, \forall X \in E, Y \in E^\perp \quad (4.24)
$$

$$
\Pi(X,X) = |X|^2 / \lambda, \forall X \in E, \quad (4.25)
$$

$$
\Pi(Y,Y) = \lambda |Y|^2, \forall Y \in E^\perp, \quad (4.26)
$$

for some function $\lambda$. We choose our basis such that $\theta^1, \ldots, \theta^s \in E^\perp$ and the rest in $E$, where $\delta + 1 \leq s = \text{rank } E^\perp \leq n$. By (4.16) (4.20) and the above three equations

$$
\Pi_{ij} = \begin{cases} 
0, & i \neq j \\
-u, & i = j \leq s \\
-\frac{u}{n}, & i = j > s.
\end{cases} \quad (4.27)
$$

Hence the mean curvature of $\Sigma_c$ is given by

$$
H = -su - (n - s)/u. \quad (4.28)
$$

Again by Gauss equation and (4.27)

$$
R_{\Sigma^1}(X,Y,X,Y) = -1 + 1/u^2, \quad \text{for orthonormal } X,Y \in T\Sigma^1 \quad (4.29)
$$

$$
R_{\Sigma^2}(Z,W,Z,W) = -1 + u^2, \quad \text{for orthonormal } Z,W \in T\Sigma^2 \quad (4.30)
$$

It follows that $\text{Ric}_{\Sigma_c} = s(-1 + u^2) + (n - s)(-1 + 1/u^2)$. As $\dim \Sigma_c = n \geq 3$, it is a standard consequence of the second Bianchi identity that $u$ is constant on $\Sigma_c$. In particular both $(\Sigma^1, g_1)$ and $(\Sigma^2, g_2)$ have constant sectional curvatures.
Step 2. Thus for $j = 1, \ldots, n$

\[
\langle \nabla e_0 e_0, e_j \rangle = \langle \nabla e_0 (\nabla \phi / |\nabla \phi|), e_j \rangle \\
= \langle \nabla e_0 \nabla \phi, e_j / |\nabla \phi| \rangle \\
= \langle \nabla e_j \nabla \phi, e_0 / |\nabla \phi| \rangle \\
= \frac{1}{|\nabla \phi|} (e_j e_0 \phi - \nabla e_j e_0 \phi) \\
= \frac{1}{|\nabla \phi|} (\delta e_j (u \phi) - \Pi_{ij} e_i \phi) \\
= 0,
\]

where in the last step we use (4.13) and the fact that $\phi$ and $u$ are constant on $\Sigma_c$. Therefore

\[
\nabla e_0 e_0 = 0. \quad \text{(4.31)}
\]

We claim that the constant $u$ is not 1. Suppose that $u = 1$, then $\Sigma_c$ is a compact flat hypersurface in $M$. Its lifting in $\mathbb{H}^{n+1}$ is then a horosphere which can be taken to be the hyperplane $S = \{y = a\}$ in the upper space model for some $a > 0$. Let $x$ and $\gamma \cdot x$ be in $S$, which map to the same point in the quotient $M = \mathbb{H}^{n+1}/\Gamma$, where $\gamma \in \Gamma$. Then $\gamma S = S$, and it follows that $\gamma$ is a parabolic element. But this is impossible since $M$ has no cusps.

Moreover rank $E = \deg \omega$, i.e. $s = \delta + 1$. For otherwise we can write $\omega = \theta^{s+1} \ldots \theta^n \wedge \tau$ with $\tau$ a nontrivial parallel form on $\Sigma^2$. This would lead to a contradiction if we apply Lemma 4.1 on $\Sigma^2$ which has nonzero constant curvature $-1 + u^2$.

The equation (4.9) can be written as

\[
-\delta (n - \delta) \phi = D^2 \phi(e_0, e_0) + \triangle_{\Sigma^c} \phi + He_0 \phi. \quad \text{(4.32)}
\]

The function $\phi$ being constant on $\Sigma_c$, we get using (4.28) (4.41)

\[
D^2 \phi(e_0, e_0) = -\delta (n - \delta) \phi - He_0 \phi \\
= -\delta (n - \delta) \phi + ((\delta + 1) + (n - \delta - 1)/u) \delta u \phi \\
= \delta (1 + \delta) u^2 - \delta \phi. \quad \text{(4.35)}
\]

Combining these equations with (4.13) we obtain

\[
D^2 \phi(e_0, e_0) = \frac{1 + \delta}{\delta \phi} |\nabla \phi|^2 - \delta \phi. \quad \text{(4.36)}
\]
On the other hand \( D^2\phi(e_0, e_i) = \langle \nabla_{e_0}\nabla\phi, e_i \rangle = |\nabla\phi|\langle \nabla_{e_0}e_0, e_i \rangle = 0 \) for \( i = 1, \ldots, n \) while
\[
D^2\phi(e_i, e_j) = \langle \nabla_{e_i}\nabla\phi, e_j \rangle \\
= |\nabla\phi|\langle \nabla_{e_i}e_0, e_j \rangle \\
= \delta u \phi \Pi_{ij}.
\]

Therefore we have
\[
D^2\phi(e_i, e_j) = \begin{cases} 
0, & i \neq j \\
\frac{1 + \delta}{\delta \phi} |\nabla\phi|^2 - \delta \phi, & i = j = 0 \\
-\frac{|\nabla\phi|^2}{\delta \phi}, & 1 \leq i = j \leq \delta + 1 \\
-\delta \phi, & i = j > \delta + 1.
\end{cases}
\tag{4.37}
\]

This shows that at any critical point of \( \phi \) the Hessian \( D^2\phi \) has constant rank \( n + 1 - (\delta + 1) \). Therefore \( N = \{ x \mid \phi(x) = B \} \) is a nondegenerate critical manifold of dimension \( \delta + 1 \), where \( B = \max \phi \). Let \( h \) be the induced metric. We show that \( N \) is totally geodesic. Near \( N \) we decompose \( TM \) as the direct sum of two subbundles according to the eigenspaces of \( D^2\phi \). We choose orthonormal basis \( \{ e_1, \ldots, e_{n+1} \} \) such that the first \( \delta + 1 \) vector correspond to the eigenvalue \(-\frac{|\nabla\phi|^2}{\delta \phi}\). On each regular level surface \( \Sigma_c \), \( \{ e_1, \ldots, e_{\delta+1} \} \) span the distribution \( E^\perp \) introduced before. We know that \( E \) and \( E^\perp \) are parallel on \( \Sigma_c \), hence
\[
\langle \nabla_{e_i}e_k, e_j \rangle = 0
\]
for \( 1 \leq i, j \leq \delta + 1; k > \delta + 1 \) and \( \langle e_k, \nabla\phi \rangle = 0 \) while by (4.27)
\[
\langle \nabla_{e_i}\nabla\phi, e_j \rangle = -u = -\frac{|\nabla\phi|}{\delta \phi}.
\]
Therefore
\[
|\langle \nabla_{e_i}X, e_j \rangle| \leq \frac{|\nabla\phi|}{\delta \phi}
\]
for any unit vector \( X \) orthogonal to \( e_1, \ldots, e_{\delta+1} \). As \( \nabla\phi = 0 \) on \( N \), we conclude that \( N \) has zero second fundamental form i.e. totally geodesic.

**Step 3.** We use the Ricci equation to compute the curvature of the normal bundle \( \mathcal{N}(N) \)
\[
\langle R^\perp_{VW}X, Y \rangle = R(V, W, X, Y) + \sum_{i=1}^{\delta+1} (\Pi_X(V, e_i)\Pi_Y(W, e_i) - \Pi_X(W, e_i)\Pi_Y(V, e_i))
\]
\[
= 0.
\]

14
Hence the normal bundle is flat. Therefore we can choose our local orthonormal frame on an open subset \( U \subset N \) such that \( e_{d+2}, \ldots, e_{n+1} \) are parallel sections of \( N(N) \).

Finally we consider the exponential map \( N(N) \to M \) or locally

\[
\psi : \mathbb{R}^+ \times S^{n-\delta-1} \times U \to M,
\]

\[
\psi(t, \zeta, x) = \exp_x \left( t \sum_{i=d+2}^{n+1} \zeta^i e_i \right).
\]

Given \( V \in T_x N \) and \( X \in T_x S^{n-\delta-1} \) we get Jacobi fields \( V(t) = \psi_t(V) \) and \( X(t) = \psi_t(X) \) along the geodesic \( \gamma(t) = \psi(t, \zeta, x) \). Note \( V(0) = V \) and \( \dot{V}(0) = \sum_{i=d+2}^{n+1} \zeta^i \nabla_X e_i = 0 \) because \( e_{d+2}, \ldots, e_{n+1} \) are parallel sections of \( N(N) \) and \( N \) is totally geodesic. On the other hand \( X(0) = 0, \dot{X}(0) = X \). Since the metric is hyperbolic, the Jacobi equation is easy to solve to give

\[
V(t) = \cosh(t) P_t(V), \quad X(t) = \sinh(t) P_t(X),
\]

where \( P_t \) is the parallel translation from \( T_x M \) to \( T_{\gamma(t)} M \) along \( \gamma \). Therefore in the geodesic polar coordinates \( (t, \zeta, x) \) along \( N \) the metric takes the form

\[
g = dt^2 + \sinh^2(t)d\zeta^2 + \cosh^2(t)h.
\]

By (4.15) we have the following ODE on \( \gamma \)

\[
\frac{d\phi}{dt} = -\delta u \phi.
\]

We have negative sign here because \( \dot{\gamma} = -e_0 \) with our previous choice of \( e_0 \). We compute

\[
\frac{d^2\phi}{dt^2} = -\delta \phi \frac{du}{dt} + \delta u \frac{d\phi}{dt} = \delta \phi \frac{du}{dt} + \delta^2 u^2 \phi.
\]

On the other hand (4.33) gives us

\[
\frac{d^2\phi}{dt^2} = D^2\phi(e_0, e_0) = \delta(1 + \delta)u^2 - \delta \phi.
\]

Combining these two equations we obtain the ODE

\[
\begin{cases}
\frac{du}{dt} = 1 - u^2 \\
u(0) = 0.
\end{cases}
\]

(4.42)
This can be easily solved and we get

\[ u(t) = \frac{\sinh(t)}{\cosh(t)}, \]
\[ \phi(t) = B \cosh^{-\delta}(t). \]

This shows that outside \( \Sigma \) the function \( \phi \) is regular everywhere. Therefore \( \psi : \mathcal{N}(N) \to M \) is a diffeomorphism. This finishes the proof.

**Remark.** Part 1 of the theorem and its proof works for a geometrically finite hyperbolic manifold whose only cusps are of maximum rank. A cusp of maximum rank is isometric to \([1, \infty[ \times N\) with the metric \( t^{-2}(dt^2 + g_0) \), where \((N, g_0)\) is a compact flat manifold. If we write \( \omega = \alpha + dt \wedge \beta \), by [7] we have

\[
\alpha = \begin{cases} 
\alpha_0(x) + O(e^{-\lambda t}), & k < n/2 \\
O(e^{-\lambda t}), & n/2 \leq k \leq (n + 1)/2,
\end{cases}
\]

\[
\beta = \begin{cases} 
\beta_0(x)t + O(e^{-\lambda t}), & k < n/2 \\
O(e^{-\lambda t}), & n/2 \leq k \leq (n + 1)/2,
\end{cases}
\]
as \( t \to \infty \) for some \( \lambda > 0 \), where \( \alpha_0 \) and \( \beta_0 \) are harmonic forms on \( N \).

When we do integration by parts on a compact domain in (4.7), each cusp gives rise to a boundary term \( \int_{\{t\} \times N} \phi \frac{\partial \phi}{\partial \nu} d\sigma \) which tends to zero as \( t \to \infty \), by the asymptotics given above. Therefore the rest of the argument goes without any change.

The asymptotics of \( L^2 \) harmonic forms near cusps of intermediate ranks are also given in [7], but the results are much more intricate. We do not know whether the above proof can be generalized to cover the general case.

By Lefschetz duality and Theorem 2.1 we get the following corollary from Theorem 1.2.

**Corollary 4.1** Let \( M = \mathbb{H}^{n+1}/\Gamma \) be an orientable convex cocompact hyperbolic manifold and \( \delta \) the Hausdorff dimension of the limit set of \( \Gamma \). Suppose \( \delta > n/2 \).

1. If \( p > \delta + 1 \) then \( H^p(M, \mathbb{R}) = 0 \).

2. If \( \delta \) is an integer and \( H^{\delta+1}(M, \mathbb{R}) \neq 0 \), then \( M \) is a twisted warped product of \( \mathbb{H}^{n-\delta} \) and a compact hyperbolic manifold of dimension \( \delta + 1 \).

**Remark.** As shown by Izeki [3], part one of the above corollary can be proven by algebraic topology. Let \( \Sigma = \Omega(\Gamma)/\Gamma \) be the conformal infinity which is a compact
Kleinian $n$–manifold. First by an idea in Schoen-Yau $[9]$ one can prove the relative homotopy groups $\pi_i(M, \Sigma) = 0$ for $i < n - \delta$. Then by Hurewicz isomorphism theorem $H^i(M, \Sigma) = 0$ for $i < n - \delta$. By Lefschetz duality, $H^p(M, \mathbb{R}) = 0$ for $p > \delta - 1$.

By Theorem 2.1, this implies that the $L^2$ cohomology is actually trivial if $\delta < n/2$. This can also be easily seen from our approach, using (4.6) and the fact that $\lambda_0 = n^2/4$ when $\delta \leq n/2$. By Mazzeo-Phillips theorem $\mathcal{H}^p(M) = 0$ except for the middle dimension when $n+1$ is even. Therefore the $L^2$ cohomology contains no useful information. However the interesting work of Nayatani $[8]$ shows that one can then read off $\delta$ from the cohomology of $\Sigma$ when $\delta < n/2 - 1$.

References

[1] T. Branson, Kato Constants in Riemannian Geometry, Math. Res. Letters 7 (2000) 245–261.

[2] D. Calderbank, P. Gauduchon and M. Herzlich, Refined Kato Inequalities and Conformal Weights in Riemannian Geometry, J. Func. Analy. 173 (2000) 214–255.

[3] H. Izeki, Limit Sets of Kleinian groups and Conformally Flat Riemannian Manifolds, Invent. Math. 122 (1995) 603–625.

[4] J. Lee, The Spectrum of an Asymptotically Hyperbolic Einstein Manifold, Comm. Anal. Geom. 3 (1995) 253–271.

[5] J. Lott, Invariant Currents on Limit Sets, Comment. Math. Helv. 75 (2000) 319–350.

[6] R. Mazzeo, The Hodge Theory of a Conformally Compact Metrics, JDG 28 (1988) 171–185.

[7] R. Mazzeo and R. Phillips, Hodge Theory on Hyperbolic Manifolds, Duke Math. J. 60 (1990) 509–559.

[8] S. Nayatani, Patterson-Sullivan Measure and Conformally Flat Metrics, Math. Z. 225 (1997) 115–131.

[9] R. Schoen and S.-T. Yau, Conformally Flat Manifolds, Kleinian Groups, and Scalar Curvature, Invent. Math. 92 (1988) 47–71.
[10] D. Sullivan, *Related Aspects of Positivity in Riemannian Geometry*, J. Diff. Geom. 25 (1987) 327–351.

[11] X. Wang, *On Conformally Compact Einstein Manifolds*, Math. Res. Letters 8 (2001) 671-688.