COMPATIBLE CONTACT STRUCTURES OF FIBERED
POSITIVELY-TWISTED GRAPH MULTILINKS IN THE 3-SPHERE

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Abstract. We study compatible contact structures of fibered, positively-twisted graph multilinks in $S^3$ and prove that the contact structure of such a multilink is tight if and only if the orientations of its link components are all consistent with or all opposite to the orientation of the fibers of the Seifert fibrations of that graph multilink. As a corollary, we show that the compatible contact structures of the Milnor fibrations of real analytic germs of the form $(f\bar{g}, O)$ are always overtwisted.

1. Introduction

A contact structure on a closed, oriented, smooth 3-manifold $M$ is the kernel of a 1-form $\alpha$ on $M$ satisfying $\alpha \wedge d\alpha \neq 0$ everywhere. In this paper, we only consider positive contact forms i.e., contact forms $\alpha$ satisfying $\alpha \wedge d\alpha > 0$. The idea of contact structures compatible with fibered links in $M$ was first introduced by W.P. Thurston and H. Winkelnkemper in [11] and developed by E. Giroux in [4]. In the previous work [7], the compatible contact structures of fibered Seifert multilinks in Seifert fibered homology 3-spheres $\Sigma(a_1, a_2, \ldots, a_k)$ were studied. Here $a_i$’s are the denominators of the Seifert invariants. Especially, we determined their tightness in case $a_1 a_2 \cdots a_k > 0$. The 3-sphere with positive Hopf fibration is a typical example of Seifert fibered homology 3-spheres satisfying this inequality.

A graph multilink is obtained from Seifert multilinks by iterating a certain gluing operation, called a splicing. We focus on fibered graph multilinks obtained as a splice of Seifert multilinks in homology 3-spheres with $a_1 a_2 \cdots a_k > 0$, which we call positively-twisted graph multilinks in homology 3-spheres. For convenience, we may assume that the denominators of the Seifert invariants of the Seifert fibered homology 3-spheres constituting the graph multilink are all positive. This is always possible as mentioned in [2, Proposition 7.3]. In this setting, we say that the orientation of a graph multilink is canonical if the multiplicities of its link components are either all positive or all negative.

In this paper, we determine the tightness of positively-twisted graph multilinks in $S^3$.

Theorem 1.1. The compatible contact structure of a fibered, positively-twisted graph multilink in $S^3$ is tight if and only if its orientation is canonical.

A typical example of positively-twisted graph multilinks in $S^3$ is an oriented link obtained from a trivial knot in $S^3$ by iterating “positive” cablings. Here a “positive” cabling

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means that the cabling coefficients are positive with respect to the framing of the Seifert surface of the link component for the cabling.

The situation in Theorem 1.1 occurs when we consider the Milnor fibration of a real analytic germ of the form \((f\bar{g}, O)\), where \(f, g : (\mathbb{C}^2, O) \to (\mathbb{C}, 0)\) are holomorphic germs at the origin \(O \in \mathbb{C}^2\) and \(\bar{g}\) represents the complex conjugation of \(g\). In [9, 10], it is proved that 
\[
\frac{f\bar{g}}{|f\bar{g}|} : S_\varepsilon \setminus \{fg = 0\} \to S^1
\]
is a locally trivial fibration in most cases, called the Milnor fibration of \((f\bar{g}, O)\), where \(S_\varepsilon\) is the 3-sphere centered at \(O \in \mathbb{C}^2\) with sufficiently small radius \(\varepsilon > 0\). The next result answers a question of A. Pichon asked in her talk in Luminy, May, 2006 (cf. [6]).

**Corollary 1.2.** Suppose that 
\[
\frac{f\bar{g}}{|f\bar{g}|} : S_\varepsilon \setminus \{fg = 0\} \to S^1
\]
is a locally trivial fibration. Then its compatible contact structure is overtwisted.

This paper is organized as follows. In Section 2, we fix the notations of Seifert fibered homology 3-spheres, Seifert multilinks, and graph multilinks, following the book [2]. The notion of their compatible contact structures is also introduced in this section. The next two sections are devoted to preparations for the proof of Theorem 1.1. In Section 3, we give a compatible contact structure of a fibered, positively-twisted graph multilink in a special case. We then give another compatible contact structure in Section 4 based on the Thurston-Winkelnkemper’s construction in [11]. These compatible contact structures will be used to prove Theorem 1.1 in Section 5. Corollary 1.2 will also be proved in this section.

2. Preliminaries

In the following, \(\text{int}X\) and \(\partial X\) represent the interior and the boundary of a topological space \(X\) respectively.

2.1. **Notation of Seifert fibered homology 3-spheres.** We follow the notation used in [7], which originally appears in [2].

Let \(S = S^2 \setminus \text{int}(D_1^2 \cup \cdots \cup D_k^2)\) be a 2-sphere with \(k\) holes and make an oriented, closed, smooth 3-manifold \(\Sigma\) from \(S \times S^1\) by gluing solid tori \((D^2 \times S^1)_i, \cdots, (D^2 \times S^1)_k\) along the boundary \(\partial(S \times S^1)\) in such a way that \(a_iQ_i + b_iH\) is null-homologous in \((D^2 \times S^1)_i\), where

\[
Q_i = (-\partial S^{sec}) \cap (D^2 \times S^1)_i
\]
\[
H = \text{typical oriented fiber of } \pi\text{ in } \partial(D^2 \times S^1)_i,
\]
with \(S^{sec}\) a section of \(\pi : S \times S^1 \to S\) and \((a_i, b_i) \in \mathbb{Z}^2 \setminus \{(0, 0)\}\) are chosen such that \(\sum_{i=1}^k b_ia_1 \cdots a_{i-1}a_{i+1} \cdots a_k = 1\). The obtained 3-manifold does not depend on the ambiguity of the choice of \(b_i\)'s, so we may denote it simply as \(\Sigma = \Sigma(a_1, \cdots, a_k)\). The core curve \(S_i\) of each solid torus \((D^2 \times S^1)_i\) is a fiber of the Seifert fibration after the gluings. We assign to \(S_i\) an orientation in such a way that the linking number of \(S_i\) and \(a_iQ_i + b_iH\) equals 1. This orientation is called the working orientation.
Let \((m_i, l_i)\) be the preferred meridian-longitude pair of the link complement \(\Sigma \setminus S_i\) chosen such that the orientation of the longitude \(l_i\) agrees with the working orientation of \(S_i\). In this setting, \((m_i, l_i)\) and \((Q_i, H)\) are related by the following equations, see [2, Lemma 7.5]:

\[
(2.1) \quad \begin{pmatrix} m_i \\ l_i \end{pmatrix} = \begin{pmatrix} a_i & b_i \\ -\sigma_i & \delta_i \end{pmatrix} \begin{pmatrix} Q_i \\ H \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q_i \\ H \end{pmatrix} = \begin{pmatrix} \delta_i & -b_i \\ \sigma_i & a_i \end{pmatrix} \begin{pmatrix} m_i \\ l_i \end{pmatrix},
\]

where \(\sigma_i = a_1 \cdots \hat{a}_i \cdots a_k\) and \(\delta_i = \sum_{i \neq j} b_j a_1 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_k\). Note that they satisfy \(a_i \delta_i + b_i \sigma_i = 1\).

Set \(A = a_1 \cdots a_k\). Under the assumption \(A \neq 0\), the orientations of the fibers of the Seifert fibration in \(S \times S^1 \to S\) are canonically extended to the fibers in \((D^2 \times S^1)_i\) for each \(i = 1, \ldots, k\), which we call the orientation of the Seifert fibration.

### 2.2. Seifert multilinks in \(\Sigma(a_1, \ldots, a_k)\).

A Seifert link \(L\) is a link in \(\Sigma(a_1, \ldots, a_k)\) consisting of a finite number of fibers of the Seifert fibration. We may choose the core curves \(S_1, \ldots, S_k\) of the solid tori \((D^2 \times S^1)_i\) such that \(S_1 \cup S_2 \cup \cdots \cup S_n\) is the Seifert link \(L\) for some \(n \leq k\). A Seifert multilink is a Seifert link each of whose link components is equipped with a non-zero integer, called the multiplicity. We denote the set of multiplicities as \(m = (m_1, \ldots, m_n)\) and the Seifert multilink as

\[
(\Sigma, L(m)) = (\Sigma(a_1, \ldots, a_k), m_1 S_1 \cup \cdots \cup m_n S_n).
\]

We may denote it simply as \(L(m)\).

Each link component of \(L(m)\) is canonically oriented according to its multiplicity, i.e., the orientation is defined to be consistent with the working orientation if \(m_i > 0\) and opposite to it if \(m_i < 0\). In this paper, we allow \(m_i\) to be 0 for convenience, which means that \(S_i\) is not a component of \(L(m)\). We may call such an \(S_i\) also a link component of \(L(m)\) even though it is an empty component.

**Definition 2.1.** A Seifert multilink \((\Sigma, L(m))\) is called positively-twisted (or PT for short) if \(A = a_1 \cdots a_k > 0\).

**Remark 2.2.** The notion of positivity is usually used for oriented links, as positive braids and positive links. We here say that “a Seifert multilink is positively-twisted” because this is a notion for (multi-)links without specific orientation.

**Definition 2.3.** Suppose \(A \neq 0\) and fix an orientation of the Seifert fibration. We say that a link component \(m_i S_i\) of a Seifert multilink \(L(m)\) with \(m_i \neq 0\) is positive (resp. negative) if its orientation is consistent with (resp. opposite to) the orientation of the Seifert fibration.

### 2.3. Splicing and graph multilinks.

Let \((\Sigma_1, L_1)\) and \((\Sigma_2, L_2)\) be links in homology 3-spheres \(\Sigma_1\) and \(\Sigma_2\) respectively. Choose a link component \(S_1\) of \(L_1\) and also \(S_2\) of \(L_2\). For each \(i = 1, 2\), let \(N(S_i)\) be a compact tubular neighborhood of \(S_i\) and \((m_i, l_i)\) be a preferred meridian-longitude pair of \(\Sigma_i \setminus \text{int} N(S_i)\). Remark that the longitude \(l_i\) is chosen such that it is null-homologous in the exterior \(\Sigma_i \setminus \text{int} N(S_i)\). We then glue the two exteriors \(\Sigma_1 \setminus \text{int} N(S_1)\) and \(\Sigma_2 \setminus \text{int} N(S_2)\) in such a way that \((m_1, l_1)\) are identified with \((m_2, l_2)\) along the boundaries. The link \((L_1 \setminus S_1) \cup (L_2 \setminus S_2)\) in the glued manifold is called the splice of \(L_1\) and \(L_2\) along \(S_1\) and \(S_2\). Note that the glued manifold again becomes a homology 3-sphere.

Let \(\Gamma\) be a connected, simply-connected finite graph with the following decorations:
• Each terminal vertex is either a boundary vertex or an arrowhead vertex. Here the former is a usual vertex and the latter means the arrowhead of an edge $\rightarrow$ of arrow shape.
• Each non-terminal vertex has sign + or −. We call such a vertex an inner vertex and an edge connecting two inner vertices an inner edge.
• For each inner vertex, integers $a_1, \ldots, a_k$ representing a Seifert fibered homology 3-sphere $\Sigma(a_1, \ldots, a_k)$ are assigned to the roots of the edges connected to that vertex.
• A non-zero integer $m_i$ is assigned to each arrowhead vertex.

Such a diagram is called a splice diagram. In this paper, we only consider splice diagrams whose inner vertices have only + signs. So, we do not need to mind these signs.

We define a multilink from a given splice diagram $\Gamma$ as follows. First we prepare a Seifert link $(\Sigma(a_1, \ldots, a_k), S_1 \cup \cdots \cup S_n)$ for each inner vertex with integers $a_1, \ldots, a_k$ at the roots of the adjacent edges. If two inner vertices are connected by an edge, then we apply a splicing to the corresponding Seifert links. Applying splicings for all inner edges successively, we obtain a new link $L$ in a homology 3-sphere $\Sigma$. We then define the multiplicity of each link component of $L$ to be the integer assigned to the corresponding arrowhead vertex. Note that the working orientations of the link components of $L$ are defined to be those of the Seifert fibered homology 3-spheres. The multilink obtained from $\Gamma$ is called a graph multilink and denoted as $L(\Gamma)$.

Now we consider the inverse operation of splicing. Namely, decompose $\Sigma$ along a torus $T$ and then fill the two boundary components by solid tori $N(S_1)$ and $N(S_2)$ with new link components $S_1$ and $S_2$ being the core curves of $N(S_1)$ and $N(S_2)$ respectively, so that we obtain two new graph links $(\Sigma_1, L_1)$ and $(\Sigma_2, L_2)$. The restriction of the Seifert surface of $L(\Gamma)$ to $\Sigma_i \setminus \text{int} N(S_i)$ is canonically extended into $N(S_i)$. If the Seifert surface in $N(S_i)$ is a disjoint union of meridional disks, then we define the multiplicity of $S_i$ to be 0. Otherwise, it is defined to be the number of local leaves of the Seifert surface along $S_i$, with sign − if the orientation of $S_i$ as the boundary of the Seifert surface of $L_i$ is opposite to its working orientation. The multiplicities of the other link components of $L_1$ and $L_2$ are defined to be those of $L(\Gamma)$. We denote the obtained multilinks as $(\Sigma_i, L(\Gamma_i))$, where $\Gamma_1$ and $\Gamma_2$ are the corresponding splice diagrams, and their splice as

$$(\Sigma, L(\Gamma)) = \left[ (\Sigma_1, L(\Gamma_1))_{S_1} \right. \left. (\Sigma_2, L(\Gamma_2))_{S_2} \right].$$

Applying such a decomposition successively, we can represent $(\Sigma, L(\Gamma))$ as a splice of several Seifert multilinks in homology 3-spheres each of which corresponds to an inner vertex of $\Gamma$. In other words, there exists a set of disjoint tori $T_1, \ldots, T_r$ corresponding to the inner edges of $\Gamma$ such that each connected component of $\Sigma \setminus \bigcup_{j=1}^{r} T_j$ is a part of a Seifert fibered homology 3-sphere. We call these connected components the Seifert pieces of $(\Sigma, L(\Gamma))$.

In this paper, we only consider the following special class of graph multilinks.

**Definition 2.4.** A graph multilink is called positively-twisted (or PT for short) if it is obtained as a splice of only PT Seifert multilinks in homology 3-spheres.

To simplify the argument, we hereafter assume that the denominators of the Seifert invariants of the Seifert fibered homology 3-spheres before the splicing are all positive. We can always assume this by [2, Proposition 7.3]. Note that, under this assumption,
the orientation of the Seifert fibration coincides with the working orientations of the link components for each Seifert multilink before the splicing.

In [2, Theorem 8.1], six operations to produce equivalent splice diagrams are introduced and a splice diagram is called minimal if there is no equivalent splice diagram with fewer edges. In this paper, we will only deal with splice diagrams of fibered PT graph multilinks whose inner vertices have sign + and each of whose Seifert fibered homology 3-spheres before the splicing has the Seifert invariants with positive denominators. In this setting, two splice diagrams are equivalent if they are connected by the following two operations and their inverses:

3) Let \( v \) be an inner vertex. If an edge connected to \( v \) is assigned the integer 1 at the root and has a boundary vertex at the other endpoint then remove the edge and the boundary vertex. Furthermore, if the number of remaining edges connected to \( v \) is 2 then remove the inner vertex \( v \) and connect the two edges so that they become a single edge.

6) Let \( v \) and \( v' \) be inner vertices connected by an inner edge. Let \( a_0, a_1, \ldots, a_r \) and \( a'_0, a'_1, \ldots, a'_s \) be the denominators of the Seifert invariants assigned to \( v \) and \( v' \) respectively such that \( a_0 \) and \( a'_0 \) are assigned to the inner edge connecting them. If \( a_0a'_0 = a_1 \cdots a_r a'_1 \cdots a'_s \) is satisfied then replace the vertices \( v_1 \) and \( v_2 \) and the edge connecting them by a single inner vertex.

The numbers 3) and 6) correspond to those in [2, Theorem 8.1]. We say that a minimal splice diagram is of type \( \leftrightarrow \) if it consists of one edge with arrowhead vertices at both endpoints. This will be an exceptional case as in [2, Theorem 11.2].

2.4. Fibered graph multilinks and contact structures. We first briefly recall the terminologies in 3-dimensional contact topology. See [3, 8] for general references.

A contact structure on \( M \) is the 2-plane field given by the kernel of a 1-form \( \alpha \) satisfying \( \alpha \wedge d\alpha \neq 0 \) everywhere on \( M \). In this paper, we always assume that a contact structure is positive, i.e., it is given as the kernel of a 1-form \( \alpha \) satisfying \( \alpha \wedge d\alpha > 0 \), called a positive contact form on \( M \). A vector field \( R_\alpha \) on \( M \) determined by the conditions \( d\alpha(R_\alpha, \cdot) \equiv 0 \) and \( \alpha(R_\alpha) \equiv 1 \) is called the Reeb vector field of \( \alpha \). The 3-manifold \( M \) equipped with a contact structure \( \xi \) is called a contact manifold and denoted by \( (M, \xi) \). Two contact manifolds \( (M_1, \xi_1) \) and \( (M_2, \xi_2) \) are said to be contactomorphic if there exists a diffeomorphism \( \varphi : M_1 \to M_2 \) such that \( d\varphi : T\Sigma_1 \to T\Sigma_2 \) satisfies \( d\varphi(\xi_1) = \xi_2 \). A disk \( D \) in \( (M, \xi) \) is called overtwisted if \( D \) is tangent to \( \xi \) at each point on \( \partial D \). If \( (M, \xi) \) has an overtwisted disk then we say that \( \xi \) is overtwisted and otherwise that \( \xi \) is tight.

A graph multilink \( L(\Gamma) \) is called fibered if there is a fibration \( \Sigma \setminus L(\Gamma) \to S^1 \) such that

- the intersection of the fiber surface and a small tubular neighborhood \( N(S_i) \) of each link component \( S_i \) of \( L(m) \) consists of \( |m_i| > 0 \) leaves meeting along \( S_i \) if \( m_i \neq 0 \), and
- the working orientation of \( S_i \) is consistent with (resp. opposite to) the orientation as the boundary of the fiber surface if \( m_i > 0 \) (resp. \( m_i < 0 \)).

A fibered graph multilink \( L(\Gamma) \) in \( \Sigma \) is said to be compatible with a contact structure \( \xi \) on \( \Sigma \) if there exists a contact form \( \alpha \) on \( \Sigma \) whose kernel is contactomorphic to \( \xi \) and which satisfies that \( L(\Gamma) \) is positively transverse to \( \ker \alpha \) and \( d\alpha \) is a volume form on the interiors of the fiber surfaces of \( L(\Gamma) \); in other words, the Reeb vector field \( R_\alpha \) of \( \alpha \) is
tangent to \( L(m) \) in the same direction and positively transverse to the interiors of the fiber surfaces of \( L(m) \), see [7, Lemma 3.2].

We remark that multilinks, fibered multilinks, and their compatible contact structures are defined for any closed, oriented, smooth 3-manifolds, though we need to assign some working orientations to the link components at the beginning, see [7]. The same notion appears in [1], in which the fibration of a fibered multilink is called a rational open book decomposition of that 3-manifold. It is known that any fibered multilink in a closed, oriented, smooth 3-manifold admits a compatible contact structure and two contact structures compatible with the same fibered multilink are contactomorphic, see [7, Proposition 3.3 and 3.4] or [1, Theorem 1.7].

We close this section with introducing a way of describing a contact structure on \( D^2 \times S^1 \), which we used in [7]. Let \( \gamma \) be a curve on an \( xy \)-plane with parameter \( r \in [0, 1] \) which moves around \((0, 0)\) in clockwise orientation. For each point \( \gamma(r) = (x(r), y(r)) \) we set \((-h_1(r), h_2(r)) = (x(r), y(r))\) and define a 1-form on \( D^2 \times S^1 \) as
\[
\alpha = h_2(r) d\mu + h_1(r) d\lambda,
\]
where \((r, \mu, \lambda)\) are coordinates of \( D^2 \times S^1 \) with polar coordinates \((r, \mu)\) of \( D^2 \), and \( h_1 \) and \( h_2 \) are real-valued smooth functions with parameter \( r \). Since the curve \( \gamma \) rotates in clockwise orientation it satisfies the inequality \( h_1 h_2' - h_2 h_1' > 0 \), and this implies the inequality \( \alpha \wedge d\alpha > 0 \) except for the points at \( r = 0 \). Near \( r = 0 \), we may assume that either \((-h_1, h_2) = (-c, r^2)\) or \((-h_1, h_2) = (c, -r^2)\) for some positive real number \( c \), so that \( \alpha \) becomes a positive contact form on the whole \( D^2 \times S^1 \). See the right figure in Figure 1.

![Figure 1. How to read \( \ker \alpha \) and \( R_\alpha \) from the curve \( \gamma(r) = (-h_1(r), h_2(r)) \).](image)

Let \( T(r) \) be a torus parallel to \( \partial D^2 \times S^1 \) of radius \( r > 0 \). Since the positive normal vector to the contact structure \( \ker \alpha \) is \((h_2(r), h_1(r))\) on \( T(r) \), the line on the \( xy \)-plane connecting \((0, 0)\) and \((-h_1(r), h_2(r))\) represents the slope of \( \ker \alpha \) at \((r, \mu, \lambda)\). Moreover, since the Reeb vector field of \( \alpha \) is given as
\[
R_\alpha = \frac{1}{h_1 h_2' - h_2 h_1'} \left( -h_1' \frac{\partial}{\partial \mu} + h_2' \frac{\partial}{\partial \lambda} \right),
\]
the speed vector $\gamma'(r) = (-h'_1(r), h'_2(r))$ is parallel to $R_\alpha$ on $T(r)$ in the same direction. If the curve $\gamma$ reaches the positive $x$-axis on the $xy$-plane, say at $r = r_3$, then the contact structure $\ker \alpha$ has an overtwisted disk \{(r, \mu, \lambda) \in D^2 \times S^1 \mid \lambda = \text{constant}, r \leq r_3\}. This is a typical example of overtwisted contact structures, called a half Lutz twist.

3. Compatible contact structures in a special case

Let $\Gamma$ be a minimal splice diagram not of type $\leftrightarrow$. If a graph multilink $(\Sigma, L(\Gamma))$ is fibered then the interiors of the fiber surfaces of $L(\Gamma)$ intersect the fibers of the Seifert fibration in each Seifert piece transversely, see [2, Theorem 11.2] and the proof therein. Note that the orientations of the fibers of the Seifert fibrations determine by formula (2.1).

**Definition 3.1.** For a minimal splice diagram $\Gamma$ not of type $\leftrightarrow$, we define $\hat{\Gamma}$ to be the diagram obtained from $\Gamma$ by applying the following modifications:

1. Replace each inner vertex of $\Gamma$ by $\oplus$ (resp. $\ominus$) if the fibers of the Seifert fibration in the corresponding Seifert piece is positively (resp. negatively) transverse to the fiber surface of $L(\Gamma)$.
2. For each inner vertex $v$, assign $+$ (resp. $-$) to the root of each edge connected to $v$ if the multiplicity of the corresponding link component of the Seifert multilink is positive or zero (resp. negative). In particular, we assign $+$ to each edge with boundary vertex at the other endpoint since it is regarded as an empty link component having multiplicity 0.

Recall that we assumed that the denominators of the Seifert invariants of the Seifert fibered homology 3-spheres before the splicing are all positive.

**Proposition 3.2.** Let $(\Sigma, L(\Gamma))$ be a fibered PT graph multilink of a minimal splice diagram $\Gamma$ not of type $\leftrightarrow$. Suppose that $\hat{\Gamma}$ has only $\oplus$ vertices and that all inner edges have only $+$ signs. If $L(\Gamma)$ has a component with negative multiplicity then its compatible contact structure is overtwisted.

We prove this assertion by showing the overtwisted contact structure explicitly according to the construction of a graph multilink in Section 2. Let $C_i$ denote the boundary component $(-\partial S) \cap D_i^2$ of $S$. The next lemma is a refinement of [7, Lemma 4.4], where we added the conditions (4) and (5) for our purpose.

**Lemma 3.3.** Suppose $A > 0$ and $n \geq 2$. For $i = 1, \ldots, n$, let $U_i$ be a collar neighborhood of $C_i$ in $S$ with coordinates $(r_i, \theta_i) \in [1, 2) \times S^1$ satisfying \{(r_i, \theta_i) \mid r_i = 1\} = C_i$. Then, for a sufficiently small $\varepsilon' > 0$, there exists a 1-form $\beta$ on $S$ which satisfies the following properties:

1. $d\beta > 0$ on $S$.
2. If $i \geq 2$ and $\frac{h_i}{a_i} \leq 0$ then $\beta = R_i r_i d\theta_i$ with $-\frac{h_i}{a_i} < R_i$ near $C_i$ on $U_i$.
3. If $i \geq 2$ and $\frac{h_i}{a_i} > 0$ then $\beta = \frac{h_i}{r_i} d\theta_i$ with $-\frac{h_i}{a_i} < R_i < 0$ near $C_i$ on $U_i$.
4. If $\frac{h_i}{a_i} \leq 0$ then $\beta = R_1 r_1 d\theta_1$ with $R_1 = -\frac{h_1}{a_1} + \frac{1}{A} - \varepsilon' > 0$ near $C_1$ on $U_1$.
5. If $\frac{h_i}{a_i} > 0$ then $\beta = \frac{h_i}{r_i} d\theta_1$ with $R_1 = -\frac{h_1}{a_1} + \frac{1}{A} - \varepsilon' < 0$ near $C_1$ on $U_1$. 

Proof. Since \((- \frac{b_i}{a_i} + \frac{1}{A}) + \sum_{i=2}^{k} \left(- \frac{b_i}{a_i}\right) = 0\), we can choose \(R_1, \ldots, R_k\) such that they satisfy the above conditions. The rest of the proof is same as that of [7, Lemma 4.4]. □

Lemma 3.4. Let \((\Sigma, L(m)) = (\Sigma(a_1, \ldots, a_k), m_1S_1 \cup \cdots \cup m_nS_n)\) be a fibered PT Seifert multilink in a homology 3-sphere \(\Sigma\). Suppose that the fibers of the Seifert fibration intersect the interiors of the fiber surfaces of \(L(m)\) positively transversely. For a sufficiently small positive real number \(\varepsilon\) given, there exists a positive contact form \(\alpha\) on \(\Sigma\) with the following properties:

1. \(L(m)\) is compatible with the contact structure \(\text{ker } \alpha\).
2. The Reeb vector field \(R_\alpha\) of \(\alpha\) is tangent to the fibers of the Seifert fibration on \(S \times S^1\).
3. On a neighborhood of \(\partial(D^2 \times S^1)_i\), \(\alpha\) is given as \(\alpha = h_{1,2}(r_i) d\mu_1 + h_{1,1}(r_i) d\lambda_1\) with \(h_{1,1}(1)/h_{1,2}(1) = \varepsilon\) and \(h_{1,2}(1) > 0\).
4. On a neighborhood of \(\partial(D^2 \times S^1)_i\), for \(i = 2, \ldots, n\), \(\alpha = h_{i,2}(r_i) d\mu_i + h_{i,1}(r_i) d\lambda_i\) with \(h_{i,2}(1) > 0\).

Here \((r_i, \mu_i, \lambda_i)\) are coordinates of \((D^2 \times S^1)_i\) chosen such that \((r_i, \mu_i)\) are the polar coordinates of \(D^2\) of radius 1 and the orientation of \(\lambda_i\) agrees with the working orientation of \(S_i\), and \(h_{1,1}\) and \(h_{i,2}\) are real-valued smooth functions with parameter \(r_i\).

Proof. Let \(B_i = [1, 2) \times S^1 \times S^1\) be a neighborhood of \(\partial(D^2 \times S^1)_i\) in \(S \times S^3\) with coordinates \((r_i, \theta_i, t)\). The solid torus \((D^2 \times S^1)_i\) is glued to \(B_i\) as

\[
\mu_i, \lambda_i = (a_i \mu_i - \sigma_i \lambda_i) Q_i + (b_i \mu_i + \delta_i \lambda_i) H_i
\]

where \((m_i, l_i)\) is the standard meridian-longitude pair on \(\partial(D^2 \times S^1)_i\), \(Q_i\) is the oriented curve given by \(\{1\} \times S^1 \times \{a\ \text{point}\} \subset \partial B_i\), \(H\) is a typical fiber of the projection \([1, 2) \times S^1 \times S^1 \to [2, 1) \times S^1\) which omits the third entry, and \(a_i, b_i, \sigma_i, \delta_i \in \mathbb{Z}\) are given according to relations (2.1). As in the proof of [7, Proposition 4.1], the contact form on \(S \times S^1\) is set as \(\alpha_0 = \beta + dt\), where \(\beta\) is a 1-form chosen in Lemma 3.3. The Reeb vector field \(R_{\alpha_0} = \frac{\partial}{\partial t}\) satisfies the condition (1) in the assertion on \(S \times S^1\).

For the gluing map \(\phi_i\) of \((D^2 \times S^1)_i\) to \(B_i\), \(\phi_i^* \alpha_0\) is given as

\[
\phi_i^* \alpha_0 = R_i r_i d(a_i \mu_i - \sigma_i \lambda_i) + d(b_i \mu_i + \delta_i \lambda_i) = (b_i + a_i R_i r_i) d\mu_i + (\delta_i - \sigma_i R_i r_i) d\lambda_i
\]

\[
= a_i \left( \frac{b_i}{a_i} + R_i r_i \right) d\mu_i + \frac{1}{a_i} \left( 1 - a_i \sigma_i \left( \frac{b_i}{a_i} + R_i r_i \right) \right) d\lambda_i.
\]

Hence the inequality \(h_{i,2}(1) > 0\) holds for \(i = 1, \ldots, k\). Moreover, since \(R_1\) is chosen as \(R_1 = - \frac{b_1}{a_1} + \frac{1}{A} - \varepsilon'\) in Lemma 3.3, we have

\[
h_{1,1}(1) = \frac{1}{a_1} \left( 1 - a_1 \sigma_1 \left( \frac{b_1}{a_1} + R_1 \right) \right) = \frac{1}{a_1} \left( 1 - A \left( \frac{1}{A} - \varepsilon' \right) \right) = \frac{A \varepsilon'}{a_1},
\]

\[
h_{1,2}(1) = a_1 \left( \frac{b_1}{a_1} + R_1 \right) = a_1 \left( \frac{1}{A} - \varepsilon' \right)
\]

and hence

\[
\frac{h_{1,1}(1)}{h_{1,2}(1)} = \frac{A \varepsilon'}{a_1^2 \left( \frac{1}{A} - \varepsilon' \right)} > 0.
\]

Since \(\lim_{\varepsilon' \to 0} h_{1,1}(1)/h_{1,2}(1) = 0\), we can choose \(\varepsilon' > 0\) such that \(h_{1,1}(1)/h_{1,2}(1) = \varepsilon\).
We finally extend the contact form $\alpha_0$ on $S \times S^1$ into each $(D^2 \times S^1)_i$. If $m_i > 0$ then we describe a curve $\gamma_i(r_i) = (-h_{i,1}(r_i), h_{i,2}(r_i))$ on the $xy$-plane representing a positive contact form on $(D^2 \times S^1)_i$ in such a way that

1. $(-h_{i,1}, h_{i,2}) = (-c_i, r_i^2)$ near $r_i = 0$ with some constant $c_i > 0$,
2. $h_{i,2}d\mu_i + h_{i,1}d\lambda_i = \varphi_i^*\alpha_0$ near $r_i = 1$, and
3. $\gamma_i'(r_i)$ rotates monotonously.

The Reeb vector field of this contact form is positively transverse to the interiors of the fiber surfaces of $L(m)$ on $(D^2 \times S^1)_i$ as shown in Figure 2. The same observation works even in the case where $m_i = 0$.

If $m_i < 0$ then we describe a curve on the $xy$-plane representing a positive contact form on $(D^2 \times S^1)_i$ in such a way that

1. $(-h_{i,1}, h_{i,2}) = (c_i, -r_i^2)$ near $r_i = 0$ with some constant $c_i > 0$,
2. $h_{i,2}d\mu_i + h_{i,1}d\lambda_i = \varphi_i^*\alpha_0$ near $r_i = 1$, and
3. $\gamma_i'(r_i)$ rotates monotonously.

In this case, we also have the positive transversality as shown in Figure 3. This completes the proof.

**Lemma 3.5.** Let $(\Sigma, L(\Gamma))$ be a fibered PT graph multilink of a minimal splice diagram $\Gamma$ not of type $\leftrightarrow$. Suppose that $\hat{\Gamma}$ has only $\oplus$ vertices and that all inner edges have only $+$ signs. Then there exists a positive contact form $\alpha$ on $\Sigma$ with the following properties:

1. $L(\Gamma)$ is compatible with the contact structure $\ker \alpha$.
2. Each component $m_iS_i$ of $L(\Gamma)$ with negative multiplicity has a tubular neighborhood which contains a half Lutz twist. In particular, it contains an overtwisted disk.

**Proof.** Let $L_1(m_1), \ldots, L_\ell(m_\ell)$ denote the fibered Seifert multilinks before the splicing of $L(\Gamma)$. Suppose that $L(\Gamma)$ is obtained from $L_1(m_1)$ by splicing $L_2(m), \ldots, L_\ell(m_\ell)$ successively. The assertion for $L_1(m_1)$ had been proved in [7, Proposition 4.1]. Assume that the assertion holds for the fibered PT graph multilink $(\Sigma_{\ell-1}, L(\Gamma_{\ell-1}))$ obtained from
\[ L_1(m_1) \] by splicing \( L_2(m_2), \ldots, L_{\ell-1}(m_{\ell-1}) \) successively, where \( 2 \leq \ell \leq \ell' \), and then consider the next splice

\[ (\Sigma_{\ell}, L(\Gamma_\ell)) = \left( (\Sigma_{\ell-1}, L(\Gamma_{\ell-1})) \bigg|_{S}^{S_{1,1}} (\Sigma', L(\Gamma_{\ell}(m_\ell))) \right), \]

where \( \Sigma' \) is the Seifert fibered homology 3-sphere containing \( L_{\ell}(m_\ell) \).

Let \( \hat{\alpha}_\ell \) be a contact form on \( \Sigma' \) obtained according to Lemma 3.4, where \( (D^2 \times S^1)_1 \) in the lemma corresponds to the neighborhood \( N(S_{\ell,1}) \) of \( S_{\ell,1} \) for the splicing. From the lemma, \( \hat{\alpha}_\ell \) has the form \( \hat{\alpha}_\ell = h_{\ell,2}(r_\ell)d\mu_\ell + h_{\ell,1}(r_\ell)d\lambda_\ell \) near \( \partial N(S_{\ell,1}) \) with \( h_{\ell,1}(1)/h_{\ell,2}(1) > 0 \) being sufficiently small and \( h_{\ell,2}(1) > 0 \), where \( (r_\ell, \mu_\ell, \lambda_\ell) \) are coordinates of \( N(S_{\ell,1}) = D^2 \times S^1 \) chosen such that \( (r_\ell, \mu_\ell) \) are the polar coordinates of \( D^2 \) of radius 1 and the orientation of \( \lambda_\ell \) agrees with the working orientation of \( S_{\ell,1} \).

On the other hand, there is a contact form \( \alpha_{\ell-1} \) on \( \Sigma_{\ell-1} \) satisfying the required properties by the assumption of the induction. Set the neighborhood \( N(S) \) of \( S \) for the splicing to be the neighborhood specified in Lemma 3.4 in the inductive construction of \( L(\Gamma_{\ell-1}) \). Then, on a small neighborhood of \( \partial N(S) \), \( \alpha_{\ell-1} \) has the form

\[ \alpha_{\ell-1} = h_2(r)d\mu + h_1(r)d\lambda \]

with \( h_2(1) > 0 \), where \( (r, \mu, \lambda) \) are coordinates of \( N(S) = D^2 \times S^1 \) chosen such that \( (r, \mu) \) are the polar coordinates of \( D^2 \) of radius 1 and the orientation of \( \lambda \) agrees with the working orientation of \( S \). Set the gluing map of the splice as \( (r_\ell, \mu_\ell, \lambda_\ell) = (2 - r, \lambda, \mu) \), then on a small neighborhood \( N(T) \) of the torus \( T = \partial N(S_{\ell,1}) = \partial N(S) \) for the splicing we have

\[ \hat{\alpha}_\ell = h_{\ell,1}(2 - r)d\mu + h_{\ell,2}(2 - r)d\lambda. \]

Now we plot the two points corresponding to the contact forms (3.1) and (3.2) on the \( xy \)-plane and describe a curve connecting them to obtain a contact form on \( N(T) \) gluing \( \alpha_{\ell-1} \) and \( \hat{\alpha}_\ell \) smoothly. The Reeb vector fields of \( \alpha_{\ell-1} \) and \( \hat{\alpha}_\ell \) were chosen such that they are positively transverse to the fiber surfaces of \( L(\Gamma_\ell) \) in \( N(T) \). Recall that \( h_2(1) > 0, h_{\ell,2}(1) > 0 \) and that \( h_{\ell,1}(1)/h_{\ell,2}(1) > 0 \) can be sufficiently small. Since \( \hat{\alpha}_{\ell-1} \) at \( r = 1 \) corresponds to the point \((-h_{\ell,2}(1), h_{\ell,1}(1)) \) on the \( xy \)-plane representing a contact
form on \( N(S) \), by choosing \( h_{\ell,1}(1)/h_{\ell,2}(1) > 0 \) sufficiently small and multiplying a positive constant to \( \hat{\alpha}_\ell \) if necessary, we can describe a curve \( \gamma(r) \) on that \( xy \)-plane which defines a positive contact form \( \alpha_{N(T)} \) on \( N(T) \) connecting \( \alpha_{\ell-1} \) and \( \hat{\alpha}_\ell \) smoothly and whose speed vector \( \gamma'(r) \) rotates monotonously; see Figure 4. Set the slope of the fiber surface of \( L(\Gamma_\ell) \) to be constant in \( N(T) \), then the monotonous rotation of \( \gamma'(r) \) ensures that \( \ker \alpha_{N(T)} \) is compatible with \( L(\Gamma_\ell) \) on \( N(T) \). We then glue \( \alpha_{\ell-1} \) and \( \hat{\alpha}_\ell \) by \( \alpha_{N(T)} \) and obtain a contact form \( \alpha_\ell \) on \( \Sigma_\ell \). Since the Reeb vector field of \( \alpha_\ell \) satisfies the compatibility condition outside \( N(T) \), \( \ker \alpha_\ell \) is compatible with \( L(\Gamma_\ell) \) on the whole \( \Sigma_\ell \).

![Figure 4](image)

**Figure 4.** Connect the contact forms \( \hat{\alpha}_{\ell-1} \) and \( \alpha_\ell \) smoothly.

By induction, we obtain a contact form \( \alpha \) whose kernel is compatible with \( L(\Gamma) \) and which satisfies the inequality \( h_2(1) > 0 \) for each link component of \( L(\Gamma) \). If the link component is negative then the starting point of the curve on the \( xy \)-plane becomes the point \((c,0)\) with a positive constant \( c \). Therefore, the inequality \( h_2(1) > 0 \) ensures that \( \ker \alpha \) has a half Lutz twist in the neighborhood of that negative component.

**Proof of Proposition 3.2.** The assertion follows from Lemma 3.5.

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4. **Compatible contact structures via the Thurston-Winkelnkemper’s construction**

In the proof of Theorem 1.1, we will compare two compatible contact structures; one is constructed in the previous section and the other is obtained by applying the Thurston-Winkelnkemper’s construction to the fibration of the fibered multilink such that the contact form is “standard” on the tori for the splicings, which we will show in this section.

We prepare one lemma before showing the construction.

**Lemma 4.1.** Let \( L(m) \) be a fibered PT Seifert multilink in a homology 3-sphere \( \Sigma \) with link components \( m_1S_1, \ldots, m_nS_n \) with \( n \geq 2 \) and \( m_i \neq 0 \), and let \( c_1, \ldots, c_n \) be positive real numbers satisfying the inequality \(-c_1 + \sum_{i=2}^{n} c_i > 0\). Then there exists a contact form \( \alpha \) on \( \Sigma \) satisfying the following properties:
Then, as in [11], we can find a 1-form $\beta$ agrees with that of $m\partial$ given as $\theta$

The manifold $M$ where $R > (\cdots)$ such that $\lambda$

We denote by $\phi$ a diffeomorphism $F_0 \rightarrow F_1$ of the fibration of $L(m)$ in such a way that

$$\phi_t(r, \mu, \lambda) = \left( r, \mu + \frac{t}{|m_i|}, \lambda_i \right)$$

on each $(D^2 \times S^1)_i$. Let $\theta_i$ be the coordinate function on the curve $(-F_0) \cap \partial(D^2 \times S^1)_i$ given as $\theta_i = -\lambda_i$. Set $\tilde{F}_0 = F_0 \cap (S \times S^1)$ and let $\Omega$ be a volume form on $\tilde{F}_0$ which satisfies

1. $\int_{\tilde{F}_0} \Omega = -c_1 + \sum_{i=2}^{n} c_i > 0$,
2. $\Omega = c_1 \theta_1 \wedge d\theta_1$ near $\tilde{F}_0 \cap \partial(D^2 \times S^1)_1$, and
3. $\Omega = \frac{c_i}{r_i} \theta_i \wedge d\theta_i$ near $\tilde{F}_0 \cap \partial(D^2 \times S^1)_i$ for $i = 2, \cdots, n$.

Then, as in [11], we can find a 1-form $\beta$ on $\tilde{F}_0$ such that

1. $d\beta$ is a volume form on $\tilde{F}_0$,
2. $\beta = c_1 r_1 d\theta_1 = -c_1 r_1 d\lambda_1$ near $\tilde{F}_0 \cap \partial(D^2 \times S^1)_1$, and
3. $\beta = -\frac{c_i}{r_i} d\theta_i = \frac{c_i}{r_i} d\lambda_i$ near $\tilde{F}_0 \cap \partial(D^2 \times S^1)_i$ for $i = 2, \cdots, n$.

The manifold $M$ is constructed from $\tilde{F}_0 \times [0, 1]$ by identifying $(x, 1) \sim (\phi_1(x), 0)$ for each $x \in \tilde{F}_0$ and then filling the boundary components by the solid tori $(D^2 \times S^1)_i$’s. Using this construction, we define a 1-form $\alpha_0$ on $S \times S^1$ as

$$\alpha_0 = (1 - t)\beta + t\phi_1^*(\beta) + Rd\theta,$$

where $R > 0$. Near the boundary component $\partial(D^2 \times S^1)_i$, this 1-form is given as

$$\alpha_0 = \beta + Rd\theta = \begin{cases} -c_1 r_1 d\lambda_1 + R(U_1 d\mu_1 + V_1 d\lambda_1) & \text{for } i = 1, \cdots, n, \\ \frac{c_i}{r_i} d\lambda_i + R(U_i d\mu_i + V_i d\lambda_i) & \text{for } i = 2, \cdots, n. \end{cases}$$

We choose $R$ sufficiently large such that $\alpha_0$ becomes a positive contact form on $S \times S^1$. Since $\lambda_i$ is oriented in the same direction as the oriented link component $m_i S_i$, we always have $U_i > 0$ for $i = 1, \cdots, n$, see Figure 5. Hence the conditions (2) and (3) are satisfied.
successively and, for $i$ of $L$

Lemma 4.1. Suppose that the assertion holds for a fibered PT graph multilink $(\Sigma)$ inductive splicings.

Let

Proof.

D plane used to represent the contact form on $(D^2 \times S^1)_i$. Thus ker $\alpha$ set as shown in Figure 2. Thus ker $\alpha$ is also satisfied on each $(D^2 \times S^1)_i$. Since $U_i > 0$, $c_i > 0$ and $R$ is sufficiently large, we can describe a curve on the $xy$-plane as we did in the end of the proof of Lemma 3.4, which gives a contact form $\alpha$ on $\Sigma$. On $S \times S^1$, since the Reeb vector field $R_\alpha$ of $\alpha$ is given as $\frac{1}{3} \frac{\partial}{\partial \alpha}$, it is positively transverse to the fiber surfaces of $L(m)$. This property is also satisfied on each $(D^2 \times S^1)_i$ since $R_\alpha$ and the fiber surface of $L(m)$ are set as shown in Figure 2. Thus ker $\alpha$ is compatible with $L(m)$ on the whole $\Sigma$. 

Proposition 4.2. Let $(\Sigma, L(\Gamma))$ be a fibered PT graph multilink in a homology 3-sphere $\Sigma$. There exists a contact form $\alpha$ with the following properties:

1. $L(\Gamma)$ is compatible with the contact structure ker $\alpha$.
2. Let

$$(\Sigma, L(\Gamma)) = \left[ (\Sigma_1, L(\Gamma_1)) \right]_{S_1 - S_2} \left[ (\Sigma_2, L(\Gamma_2)) \right]$$

be a splice of $(\Sigma, L(\Gamma))$. On the neighborhood $N(S_1)$ of $S_1$ for the splicing, $\alpha$ has the form $\alpha = h_2(r)d\mu + h_1(r)d\lambda$, where $(r, \mu, \lambda)$ are coordinates of $N(S_1) = D^2 \times S^1$ chosen such that $(r, \mu)$ are the polar coordinates of $D^2$ of radius 1 and the orientation of $\lambda$ agrees with that of $m_i S_1$ if $m_1 \neq 0$ and is positively transverse to the meridional disk if $m_1 = 0$, and $h_1$ and $h_2$ are real-valued smooth functions with parameter $r \in [0, 1]$ such that the argument of $-h_1(r) + \sqrt{-1}h_2(r)$ varies in $(0, \pi]$.

Proof. Let $L_1(m_1), \ldots, L_{\ell'}(m_{\ell'})$ denote the fibered Seifert multilinks before the splicing of $L(\Gamma)$. Suppose that $L(\Gamma)$ is obtained from $L_1(m_1)$ by splicing $L_2(m_2), \ldots, L_{\ell'}(m_{\ell'})$ successively and, for $i = 2, \ldots, \ell'$, let $S_{i,1}$ denote the link component of $L_i(m)$ for this inductive splicings.

We prove the assertion by induction. The assertion holds for $L_1(m_1)$ as proved in Lemma 4.1. Suppose that the assertion holds for a fibered PT graph multilink $(\Sigma_{\ell-1}, L(\Gamma_{\ell-1}))$ obtained from $L_1(m_1)$ by splicing $L_2(m_2), \ldots, L_{\ell-1}(m_{\ell-1})$ successively, where $2 \leq \ell \leq \ell'$,
and consider the next splice

\[(\Sigma, L(\Gamma)) = \left[ (\Sigma_{\ell-1}, L(\Gamma_{\ell-1})) \overline{S_{\ell}} (\Sigma', L(\mu)) \right].\]

Let \(\alpha_{\ell-1}\) denote the contact form on \(\Sigma_{\ell-1}\) satisfying the required properties in the assertion, and \(m\) and \(m_{\ell,1}\) denote the multiplicities of \(S\) and \(S_{\ell,1}\) respectively.

We first deal with the case where \(m \neq 0\) and \(m_{\ell,1} \neq 0\). Let \(\hat{\alpha}_\ell\) be the contact form on \(\Sigma'\) whose kernel is compatible with \(L(\mu_{\ell})\), obtained by applying Lemma 4.1 such that the specified component \(S_1\) in the lemma corresponds to the link component \(S_{\ell,1}\) of \(L(\mu_{\ell})\). Set the neighborhood \(N(S_{\ell,1})\) of \(S_{\ell,1}\) for the splicing to be \((D^2 \times S^1)_1\) in the lemma and fix coordinates \((r_{\ell,1}, \mu_{\ell,1}, \lambda_{\ell,1})\) of \(N(S_{\ell,1}) = D^2 \times S^1\) such that \((r_{\ell,1}, \mu_{\ell,1})\) are the polar coordinates of \(D^2\) of radius 1 and the orientation of \(\lambda_{\ell,1}\) agrees with that of \(m_{\ell,1}S_{\ell,1}\). On \(N(S_{\ell,1})\), the contact form \(\hat{\alpha}_\ell\) is given as

\[\hat{\alpha}_\ell = R\ell U_{\ell,1} d\mu_{\ell,1} + (-c_{\ell,1} r_{\ell,1} + R\ell V_{\ell,1}) d\lambda_{\ell,1}\]

with \(U_{\ell,1} > 0\), where \((U_{\ell,1}, V_{\ell,1})\) is a vector positively normal to the fiber surface of \(L(\mu_{\ell})\) on \(\partial N(S_{\ell,1})\) with coordinates \((\mu_{\ell,1}, \lambda_{\ell,1})\) and \(c_{\ell,1}\) is some positive constant.

Similarly, set the neighborhood \(N(S)\) of \(S\) for the splicing to be the one specified in Lemma 4.1 in the inductive construction of \(L(\Gamma_{\ell-1})\) and fix coordinates \((r, \mu, \lambda)\) of \(N(S) = D^2 \times S^1\) such that \((r, \mu)\) are the polar coordinates of \(D^2\) of radius 1 and the orientation of \(\lambda\) agrees with that of \(mS\). On a neighborhood of \(\partial N(S)\), the contact form \(\alpha_{\ell-1}\) is given as

\[\alpha_{\ell-1} = RU d\mu + \left( \frac{c}{r} + RV \right) d\lambda\]

with \(U > 0\), where \((U, V)\) is a vector positively normal to the fiber surface of \(L(\Gamma_{\ell})\) on \(\partial N(S)\) with coordinates \((\mu, \lambda)\) and \(c\) is some positive constant. The gluing map of the splice can be written as \((r, \mu, \lambda) = (2 - r_{\ell,1}, \pm \lambda_{\ell,1}, \pm \mu_{\ell,1})\), where the sign \(\pm\) depends on if \(\lambda_{\ell,1}\) and \(\lambda\) are consistent with the working orientations or not, which will be clarified later. Let \(T\) denote the torus \(\partial N(S_{\ell,1}) = \partial N(S)\) for the splicing and \(N(T)\) denote its small neighborhood.

Consider the case \(V_{\ell,1} > 0\). We assume that the orientation of \(\lambda\) agrees with the working orientation of \(S_{\ell,1}\). The argument below works in the case where they are opposite, so we omit the proof in that case. In the case under consideration, since these orientations coincide, we have the left figure in Figure 6. By observing the identification of the splicing, the orientation of the fiber surface is fixed as shown on the right, which implies that the orientation of \(\lambda\) agrees with the working orientation of \(S\). Hence the positive normal vectors \((U_{\ell,1}, V_{\ell,1})\) and \((U, V)\) of the fiber surface are identified on \(T\) as \((U, V) = K(V_{\ell,1}, U_{\ell,1})\) with some constant \(K > 0\), see Figure 6. This means that the sign \(\pm\) above is \(+\) and the gluing map is given as \((r, \mu, \lambda) = (2 - r_{\ell,1}, \lambda_{\ell,1}, \mu_{\ell,1})\). Thus the contact form \(\alpha_{\ell-1}\) is written as

\[\alpha_{\ell-1} = \left( \frac{c}{2 - r_{\ell,1}} + RKU_{\ell,1} \right) d\mu_{\ell,1} + RKV_{\ell,1} d\lambda_{\ell,1}.\]

We now choose \(R\) sufficiently large relative to \(R\ell\) such that the positive normal vector of \(\ker \alpha_{\ell-1}\) lies between that of \(\ker \hat{\alpha}_\ell\) and that of the fiber surface, see Figure 7. Note that the figure is described with the coordinates \((r_{\ell,1}, \mu_{\ell,1}, \lambda_{\ell,1})\). By multiplying a positive constant to \(\hat{\alpha}_\ell\) if necessary, we can describe a curve \(\gamma(r)\) on the \(xy\)-plane which represents
a positive contact form $\alpha_{N(T)}$ on $N(T)$ connecting $\alpha_{\ell-1}$ and $\hat{\alpha}_\ell$ smoothly. Moreover we can choose $\gamma(r)$ such that $\gamma'(r)$ rotates monotonously. Set the slope of the fiber surface of $L(\Gamma_\ell)$ to be constant in $N(T)$, then the monotonous rotation of $\gamma'(r)$ ensures that $\alpha_{N(T)}$ is compatible with $L(\Gamma_\ell)$. Figure 7 shows that the argument of $-h_1(r) + \sqrt{-1}h_2(r)$ varies in $(0, \pi]$.

\[\text{Figure 6. The positive normal vector of the fiber surface on } T \text{ in case } V_{\ell,1} > 0. \text{ Here } (m_{\ell,1},l_{\ell,1}) \text{ and } (m,l) \text{ are the preferred meridian-longitude pairs of } \Sigma' \setminus \text{int } N(S_{\ell,1}) \text{ and } \Sigma_{\ell-1} \setminus \text{int } N(S) \text{ for the splicing respectively.}\]

\[\text{Figure 7. Glue } \alpha_{\ell-1} \text{ and } \hat{\alpha}_\ell \text{ on } N(T) \text{ in case } V_{\ell,1} > 0.\]

We next consider the case $V_{\ell,1} < 0$. Assume again that the orientation of $\lambda_{\ell,1}$ agrees with the working orientation of $S_{\ell,1}$, and we omit the proof in the other case. In this case, the mutual positions of the fiber surface, $\lambda_{\ell,1}$ and $\lambda$ become as shown in Figure 8. In particular, the orientation of $\lambda$ is opposite to the working orientation of $S$. Let $\vec{n}$ be the vector positive normal to $\ker \alpha_{\ell-1}$, which is given as $(U,V)$ with the coordinates $(\mu,\lambda)$. This means that $\vec{n}$ is given as $(-U,-V)$ with the coordinates corresponding to the preferred meridian-longitude pair $(m,l)$ of $\Sigma_{\ell-1} \setminus \text{int } N(S)$ for the splicing. Hence, the positive normal vectors $(U_{\ell,1},V_{\ell,1})$ and $(U,V)$ of the fiber surface are identified on $T$ as
\[ (U, V) = -K(V_{\ell,1}, U_{\ell,1}) \text{ with some constant } K > 0. \]

Since the gluing map of the splicing is given as \((r, \mu, \lambda) = (2 - r_{\ell,1}, -\lambda_{\ell,1}, -\mu_{\ell,1})\), we have

\[ \alpha_{\ell-1} = \left( -\frac{c}{2 - r_{\ell,1}} + RKU_{\ell,1} \right) d\mu_{\ell,1} + RKV_{\ell,1} d\lambda_{\ell,1}. \]

Plot \(\alpha_{\ell,1}\) above and \(\hat{\alpha}_\ell\) on the \(xy\)-plane and obtain a contact form connecting \(\alpha_{\ell-1}\) and \(\hat{\alpha}_\ell\) on \(N(T)\) smoothly such that it satisfies the required conditions, see Figure 9. Note that the figure is described with the coordinates \((r_{\ell,1}, \mu_{\ell,1}, \lambda_{\ell,1})\). The figure shows that the argument of \(-h_1(r) + \sqrt{-1}h_2(r)\) varies in \((0, \pi]\). This completes the proof in case \(V_{\ell,1} < 0\).

To finish the proof of the proposition, we need to prove the assertion in case \(m_{\ell,1} = 0\) and case \(m = 0\). We prove the former case here and omit the latter since the proof is similar. We further assume that the orientation of \(\lambda_{\ell,1}\) agrees with the working orientation of \(S_{\ell,1}\) and omit the proof of the other case. In the case under consideration, we can set \((U_{\ell,1}, V_{\ell,1}) = (0, 1)\) and \((U, V) = (1, 0)\). In particular, we have \(m > 0\). Let \(\hat{\alpha}_\ell\) be a contact form on \(\Sigma'\) obtained by applying Lemma 4.1. The empty link component \(S_{\ell,1}\) has...
a tubular neighborhood $N(S_{\ell,1})$ with a contact form
\[ \hat{\alpha}_{\ell} = c_{\ell,1}(r_{\ell,1}^2 d\mu_{\ell,1} + d\lambda_{\ell,1}), \]
where $c_{\ell,1}$ is a positive constant and the radius of $N(S_{\ell,1})$, say $\varepsilon > 0$, is sufficiently small.

We choose the neighborhood $N(S)$ of $S$ for the splicing as before. Now we perform the splice for the above $N(S_{\ell,1})$ and $N(S)$. The gluing map is given as $(r, \mu, \lambda) = (1 + \varepsilon - r_{\ell,1}, \lambda_{\ell,1}, \mu_{\ell,1})$ and hence the contact form $\alpha_{\ell-1}$ is

\[ \alpha_{\ell-1} = \frac{c}{1 + \varepsilon - r_{\ell,1}} d\mu_{\ell,1} + Rd\lambda_{\ell,1}. \]

We choose $R$ sufficiently large such that the positive normal vector of ker $\alpha_{\ell-1}$ lies between that of ker $\hat{\alpha}_{\ell}$ and that of the fiber surface, see Figure 10. Then the two contact forms $\alpha_{\ell-1}$ and $\hat{\alpha}_{\ell}$ are connected smoothly by describing a curve on the $xy$-plane as before. The figure shows that the argument of $-h_1(r) + \sqrt{-1}h_2(r)$ varies in $(0, \pi]$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10.png}
\caption{Glue $\alpha_{\ell-1}$ and $\hat{\alpha}_{\ell}$ on $N(T)$ in case $m_{\ell,1} = 0$.}
\end{figure}

5. Proofs of Theorem 1.1 and Corollary 1.2

Theorem 1.1 is included in the next assertion.

**Theorem 5.1.** Let $L(\Gamma)$ be a fibered PT graph multilink in $S^3$. Then the following three statements are equivalent:

1. The compatible contact structure of $L(\Gamma)$ is tight.
2. The compatible contact structure of each Seifert multilink before the splicing of $L(\Gamma)$ is tight.
3. Suppose that the denominators of the Seifert invariants of each Seifert fibered homology 3-sphere before the splicing of $L(\Gamma)$ are all positive. Note that we can always choose such Seifert invariants by [2, Proposition 7.3]. In this setting, the multiplicities assigned to the link components of $L(\Gamma)$ are either all positive or all negative.

**Proof.** The implication (3) $\Rightarrow$ (2) is proved as follows. If all the multiplicities of $L(\Gamma)$ are negative then we change the orientation of $L(\Gamma)$ by the involution in [2, Proposition 8.1] so that they are positive. The remaining multiplicities on each Seifert multilink before
the splicing are also positive by the formula in [2, Corollary 10.6]. Hence the compatible contact structure of each Seifert multilink is tight by [7, Theorem 1.1]. Note that this proof works not only for PT graph multilinks in $S^3$ but also for those multilinks in any homology 3-spheres.

We will use the condition that the manifold is $S^3$ essentially in the rest of the proof. It is known in [2, Theorem 9.2] that a graph link in $S^3$ is obtained from a trivial knot in $S^3$ by using cabling and summing operations and the proof still works for graph multilinks in $S^3$. Since we are considering only PT graph multilinks, we can think that a PT graph multilink $L(\Gamma)$ is obtained from a Seifert multilink in $S^3$ along a link component ambient isotopic to the trivial knot in $S^3$.

Let $L_1(m_1)$ denote the initial Seifert multilink in $S^3$ and suppose that $L(\Gamma)$ is obtained from $L_1(m_1)$ by splicing Seifert multilinks $L_2(m_2), \cdots, L_\ell(m_\ell)$ successively. Let $(\Sigma_{\ell-1}, L(\Gamma_{\ell-1}))$ denote the fibered PT graph multilink obtained from $L_1(m_1)$ by splicing $L_2(m_2), \cdots, L_{\ell-1}(m_{\ell-1})$ successively, where $2 \leq \ell \leq \ell'$.

We prove the implication $(2) \Rightarrow (1)$ by induction. Without loss of generality, we assume by [2, Proposition 7.3] that the denominators of the Seifert invariants of each Seifert fibered homology 3-sphere before the splicing are all positive. The assertion is obvious for $(\Sigma_1, L(\Gamma_1))$. Assume that the assertion holds for $i = 1, \cdots, \ell - 1$ and consider the next splice

$$
(\Sigma_{\ell}, L(\Gamma_{\ell})) = \left[ (\Sigma_{\ell-1}, L(\Gamma_{\ell-1}))_{S_{\ell,1}} (\Sigma', L_\ell(m_\ell)) \right].
$$

We prepare a contact form $\hat{\alpha}_\ell$ whose kernel is compatible with $L_\ell(m_\ell)$ as follows. The ambient space $S^3$ of $L_\ell(m_\ell)$ is decomposed into two solid tori $S^3 \setminus \text{int} N(S_{\ell,1})$ and $N(S_{\ell,1})$, where $N(S_{\ell,1})$ is the neighborhood of the link component $S_{\ell,1}$ of $L_\ell(m_\ell)$ for the splicing. Let $(m_1, l_1)$ and $(m_2, l_2)$ denote the preferred meridian-longitude pairs of these solid tori respectively. Note that they are glued in such a way that $(m_1, l_1) = (l_2, m_2)$. The curves in Figure 11 represent a contact form $\hat{\alpha}_\ell$ on $S^3$ whose Reeb vector field is tangent to the fibers of the Seifert fibration of $L_\ell(m_\ell)$ except for small tubular neighborhoods of its two singular fibers. Note that the curve on the left figure represents the contact form on $S^3 \setminus \text{int} N(S_{\ell,1})$ and the one on the right represents that on $N(S_{\ell,1})$.

![Figure 11. A contact form on $S^3$ whose Reeb vector field is tangent to the fibers of the Seifert fibration of $L(m)$ except small neighborhoods of the two singular fibers.](image-url)
On the other hand, by the assumption of the induction, there exists a contact form \( \alpha_{\ell-1} \) whose kernel is compatible with \( L(\Gamma_{\ell-1}) \). We choose a neighborhood \( N(S) \) of the link component \( S \) of \( L(\Gamma_{\ell-1}) \) for the splicing such that \( \alpha_{\ell-1} \) is given on \( N(S) \) as \( \alpha_{\ell-1} = c(r^2d\mu + d\lambda) \), where \( c \) is a positive constant, \( (r, \mu, \lambda) \) are coordinates of \( N(S) = D^2 \times S^1 \) chosen such that \( (r, \mu) \) are the polar coordinates of \( D^2 \) with sufficiently small radius and the orientation of \( \lambda \) is consistent with that of \( mS \). Note that this agrees with the working orientation of \( S \) because \( m > 0 \) by [7, Theorem 1.1] and [2, Corollary 10.6].

The splice (5.1) is equivalent to the replacement of \( N(S) \) in \( \Sigma_{\ell-1} \) by \( S^3 \setminus \text{int} N(S_{\ell,1}) \). We now choose \( S \) on \( S^3 \setminus \text{int} N(S_{\ell,1}) \) sufficiently small and deform \( \alpha_{\ell-1} \) such that \( \hat{\alpha}_\ell = \alpha_{\ell-1} \) on \( S^3 \setminus \text{int} N(S_{\ell,1}) = N(S) \), which is done by applying the Gray’s theorem [5]. We set the contact form \( \alpha_\ell \) on \( S^3 \) to be this deformed \( \alpha_{\ell-1} \). Since the Reeb vector fields of \( \alpha_{\ell-1} \) and \( \hat{\alpha}_\ell \) are both positively transverse to the fiber surfaces of \( L(\Gamma_\ell) \) on \( \Sigma_{\ell-1} \setminus \text{int} N(S) \) and \( S^3 \setminus \text{int} N(S_{\ell,1}) \) respectively, the Reeb vector field of \( \alpha_\ell \) also satisfies the same property. Hence ker \( \alpha_\ell \) is compatible with \( L(\Gamma_\ell) \).

By induction, we obtain a contact form \( \alpha_\ell \) on \( S^3 \) whose kernel is compatible with \( L(\Gamma) \) and contactomorphic to ker \( \alpha_1 \). Since ker \( \alpha_1 \) is tight, ker \( \alpha_\ell \) is also. This completes the proof of the implication (2) \( \Rightarrow \) (1).

Finally we prove the implication (1) \( \Rightarrow \) (3). We can assume that the splice diagram is minimal since the operations and their inverses to make equivalent splice diagrams do not change the multiplicities at the arrowhead vertices of \( \Gamma \). If the minimal splice diagram is of type \( \leftrightarrow \) then the assertion follows from [7, Theorem 1.1]. So, we further assume that the minimal splice diagram is not of type \( \leftrightarrow \). In particular, in this setting, the fibers of the Seifert fibration intersect the interiors of the fiber surfaces of \( L(\Gamma) \) transversely in each Seifert piece of \( (S^3, L(\Gamma)) \), so the diagram \( \hat{\Gamma} \) is defined.

Let \( \nu_\ell \) denote the inner vertex corresponding to \( L_\ell(\mu_i) \) and \( \hat{\nu}_\ell \) the corresponding vertex in \( \hat{\Gamma} \). Since the graph multilink is invertible by [2, Theorem 8.1], we change the signs of all the multiplicities at the arrowhead vertices of \( L(\Gamma) \) if necessary such that the fibers of the Seifert fibration in \( \Sigma_1 \) intersect the interiors of the fiber surfaces of \( L_1(\mu_i) \) positively transversely. By this assumption, we have \( \hat{\nu}_1 = \oplus \). Let \( \hat{\Gamma}_\ell \) denote the subgraph of \( \hat{\Gamma} \) consisting of the inner vertices \( \hat{\nu}_1, \ldots, \hat{\nu}_\ell \) and the inner edges connecting them.

Claim 5.2. Suppose that all vertices of \( \hat{\Gamma}_\ell \) are \( \oplus \) and all edges have sign \( + \). Suppose further that there exists an edge \( e \) of \( \hat{\Gamma} \) connected to, but not included in, \( \hat{\Gamma}_\ell \) with sign \( - \) at the root connected to \( \hat{\Gamma}_\ell \). Then the compatible contact structure of \( L(\Gamma) \) is overtwisted.

Proof. Let \( \alpha \) be a contact form on \( S^3 \) compatible with \( L(\Gamma) \) obtained in Proposition 4.2, and let \( \alpha_\ell \) be a contact form on \( \Sigma_\ell \) compatible with \( L(\Gamma_\ell) \) which was obtained in the proof of Proposition 4.2. We denote by \( m_\ell S_1, \ldots, m_\ell S_r \) the link components of \( L(\Gamma_\ell) \) corresponding to the inner edges of \( \hat{\Gamma} \) connected to, but not included in, \( \hat{\Gamma}_\ell \). From the construction, we have \( \alpha = \alpha_\ell \) on \( \Sigma_\ell \setminus \text{int} \cup_{i=1}^r N(S_i) \). Note that, by Proposition 4.2, \( \alpha_\ell \) has the form \( \alpha_\ell = h_2(r)d\mu + h_1(r)d\lambda \) on each \( N(S_i) \) such that the argument of \( -h_1(r) + \sqrt{-1}h_2(r) \) varies in \( (0, \pi] \). In particular, ker \( \alpha_\ell \) does not have a half Lutz twist in \( N(S_i) \).

On the other hand, let \( \alpha' \) be a contact form on \( \Sigma_\ell \), whose kernel is compatible with \( L(\Gamma_\ell) \), obtained according to Proposition 3.2. Remark that the edge \( e \) specified in the assertion is either an inner edge or an edge with arrowhead vertex. Since the root of \( e \) connected to \( \hat{\Gamma}_\ell \) has sign \( - \), \( (\Sigma_\ell, \ker \alpha') \) has an overtwisted disk \( D \) in a tube of a half
Lutz twist along the link component of $L(\Gamma_0)$ corresponding to $e$. By deforming $\alpha'$ and applying the Gray’s theorem [5], we can find a contact form $\alpha''$ on $\Sigma_0$ such that $\ker \alpha''$ is contactomorphic to $\ker \alpha'$, the equality $\alpha'' = \alpha'$ is satisfied on $\cup_{i=1}^r N(S_i)$, and $\partial D$ does not intersect $\cup_{i=1}^r N(S_i)$ in $(\Sigma_0, \ker \alpha'')$. Now we just follows the proof of [8, Proposition 9.2.7]. Prepare a 1-form $\eta$ such that $d\eta$ is a volume form on the fiber surfaces and vanishes near $\cup_{i=1}^r N(S_i)$, and then show that $ker \alpha''$ is contactomorphic to $ker \alpha'$ with preserving the contact form $\alpha'' = \alpha'$ on $\cup_{i=1}^r N(S_i)$, by connecting the contact forms $\alpha'' + s\eta$ and $\alpha' + s\eta$ with $s \geq 0$ by a one parameter family of contact forms. Thus we conclude that the boundary $\partial D$ of the overtwisted disk $D$ in the contact manifold $(\Sigma_0, \ker \alpha)$ is included in $\Sigma_0 \setminus \cup_{i=1}^r N(S_i)$. This $D$ is still an overtwisted disk in $(S^3, \ker \alpha)$ since $\alpha = \alpha'$ on $\Sigma_0 \setminus \text{int } \cup_{i=1}^r N(S_i)$. 

Claim 5.3. Suppose that all vertices of $\hat{\Gamma}_0$ are $\oplus$ and all edges have sign $+$. Suppose further that all edges of $\hat{\Gamma}$ connected to $\hat{\Gamma}_0$ have sign $+$ at the endpoints connected to $\hat{\Gamma}_0$. Let $e$ be an inner edge of $\hat{\Gamma}$ connected to, but not included in, $\hat{\Gamma}_0$. Then the sign of the other endpoint of $e$ is $+$ and the inner vertex at the other endpoint is $\oplus$.

Proof. The first assertion follows from [2, Corollary 10.6]. We prove the second assertion. Let $$(\Sigma, L(\Gamma)) = \left[ (\Sigma_\ell, L(\Gamma_\ell))_{\overline{S_1}}^{S_t} (\Sigma', L(\Gamma')) \right]$$ be the splice at the inner edge $e$. By the assumption and the first assertion, the multiplicities $m_1$ of $S_1$ and $m_2$ of $S_2$ are both non-negative. For $i=1,2$, let $N(S_i) = D^2 \times S^1$ be the tubular neighborhood of $S_i$ for the splicing and set coordinates $(r_i, \mu_i, \lambda_i)$ on $N(S_i)$ such that $(r_i, \mu_i)$ are the polar coordinates of $D^2$ and the orientation of $\lambda_i$ is consistent with the working orientation of $S_i$. Let $(U_i, V_i)$ be a vector positively normal to the fiber surface of $L(\Gamma)$ on $\partial N(S_i)$. We assume, for a contradiction, that the inner vertex at the other endpoint is $\ominus$.

Suppose that $m_1 > 0$ and $m_2 > 0$. Since the vertex at the root of $e$ connected to $\hat{\Gamma}_0$ is $\oplus$, we have $U_1 > 0$, see Figure 12. On the other hand, since $m_2 > 0$ and the vertex at the other endpoint is $\ominus$, we have $U_2 < 0$. We now observe the identification of the positive normal vectors $(U_1, V_1)$ and $(U_2, V_2)$ after the gluing of the splice. If $V_1 > 0$ then $(V_2, U_2) = K(U_1, V_1)$ with some constant $K > 0$. In particular, we have $U_2 = KV_1 > 0$, which contradicts $U_2 < 0$. If $V_1 < 0$ then $(V_2, U_2) = -K(U_1, V_1)$ with some constant $K > 0$. Hence $U_2 = -KV_1 > 0$, which again contradicts $U_2 < 0$.

If $m_1 = 0$ then $U_1 = 0$, $V_1 > 0$ and hence $U_2 > 0$, $V_2 = 0$ and $m_2 > 0$. However $m_2 > 0$ implies $U_2 < 0$ as before, which is a contradiction. If $m_2 = 0$ then $U_2 = 0$ and $V_2 < 0$ and hence $U_1 < 0$ and $V_1 = 0$, which contradicts $m_1 \geq 0$. This completes the proof.

We continue the proof of Theorem 5.1, i.e., prove the implication $(1) \Rightarrow (3)$. By Claim 5.2, all edges connected to $\hat{\nu}_1$ have sign $+$ at the roots connected to $\hat{\nu}_1$, otherwise ker $\alpha$ is overtwisted. Then, by Claim 5.3, the inner edge connecting $\hat{\nu}_1$ and $\hat{\nu}_2$ has sign $+$ at the endpoint connected to $\hat{\nu}_2$ and moreover we have $\hat{\nu}_2 = \oplus$. We then use Claim 5.2 again for $\hat{\Gamma}_2$ and conclude that all edges connected to $\hat{\Gamma}_2$ have sign $+$ at the roots connected to $\hat{\Gamma}_2$. We continue this argument successively for $\hat{\nu}_3, \hat{\nu}_4, \cdots, \hat{\nu}_r$ and finally obtain that all inner vertices of $\hat{\Gamma}$ are $\oplus$ and all edges have sign $+$. In particular, every non-empty link component of $L(\Gamma)$ has a positive multiplicity.
Figure 12. The positive normal vector to the fiber surface on $\partial N(S_1)$. Here $(m_1, l_1)$ is the preferred meridian-longitude pair on $\partial N(S_1)$ for the splicing.

Proof of Corollary 1.2. The intersection $L_{fg} = \{fg = 0\} \cap S_\epsilon$ is an oriented link in the 3-sphere $S_\epsilon$ and the orientation is given as the boundary of the fiber surface of the Milnor fibration $\frac{f}{g} : S_\epsilon \setminus \{fg = 0\} \to S^1$. By [2, Appendix to Chapter I], we know that $L_{fg}$ is realized by a splice diagram with only + signs, positive denominators and positive multiplicities. On the other hand, the oriented link $L_{f\bar{g}} = \{\bar{f} \bar{g} = 0\} \cap S_\epsilon$ of the fibration $\frac{f}{g} : S_\epsilon \setminus \{fg = 0\} \to S^1$ is obtained from $L_{fg}$ by reversing the orientations of the link components corresponding to $\{g = 0\}$, as mentioned in [9, Proposition 3.1]. Hence the assertion follows from Theorem 1.1.

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