HASSE PRINCIPLE FOR ROST MOTIVES

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Abstract. We prove a Hasse principle for binary direct summands of the Chow motive of a smooth projective quadric $Q$ over a number field $F$. Besides, we show that such summands are twists of Rost motives. In the case when $F$ has at most one real embedding we describe a complete motivic decomposition of $Q$.

1. Introduction

The classical Hasse–Minkowski theorem says that a non-degenerate quadratic form $q$ over a number field $F$ is isotropic if and only if it is isotropic over all completions of $F$. This assertion can be reformulated in the language of algebraic cycles and Chow motives. Namely, for the projective quadric $Q$ given by the equation $q = 0$, the Tate motive $\mathbb{Z}(0)$ is a direct summand of the motive of $Q$ if and only if it is a direct summand of the motive of $Q_{F_v}$ for each completion $F_v$ of $F$.

Moreover, the Hasse–Minkowski theorem readily implies that for every non-negative integer $m$, the Tate motive $\mathbb{Z}(m)$ is a direct summand of the motive of $Q$ over $F$ if and only if it is a direct summand of the motive of $Q$ over all completions of $F$. Indeed, this is equivalent to the condition that the Witt index of $q$ is greater than $m$.

In the present article we prove a generalization of the above assertion replacing the Tate motive $\mathbb{Z}(m)$ by a binary motive.

We say that a motive $N$ over an arbitrary field $F$ is a split motive (resp. a binary split motive) if it is a direct sum of a finite number of Tate motives over $F$ (resp. if it is a direct sum of two Tate motives over $F$). We say that $N$ is a binary motive over $F$ if becomes binary split over an algebraic closure $\overline{F}$ of $F$.

We work in the category of Chow motives with $\mathbb{F}_2$-coefficients (see [Ma68], [EKM]). We consider only non-degenerate quadratic forms (see [EKM] 7.A). For a non-degenerate quadratic form $q$ over a field $F$ we denote by $M(q)$ the Chow motive of the corresponding projective quadric given by the equation $q = 0$. For a number field $F$ and a place $v$ of $F$, we denote by $q_v$ the corresponding quadratic form over the completion $F_v$ of $F$ at $v$.

Our main result is the following theorem (see Theorem 5.1):

Main Theorem 1. Let $q$ be a quadratic form over a number field $F$. Let $N$ be a binary split motive over $F$. Assume that for every place $v$ of $F$ there exists a direct summand $M(q_v)$ of $M(q_v)$ that is isomorphic to $N$ over an algebraic closure $\overline{F_v}$ of $F_v$. Then there exists a direct summand $M$ of $M(q)$ that is isomorphic to $N$ over $\overline{F}$.

Key words and phrases. Pfister and excellent quadratic forms, number fields, Hasse principle, Chow groups and motives, Rost motives.

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It follows from our proof of Main Theorem \[1\] that every indecomposable binary direct summand of quadric over a number field is a twist of a Rost motive; see Corollary 5.14. Recall that the motive of a Pfister form \(\pi\) over an arbitrary field is isomorphic to a direct sum of twists of one binary motive, which is called the Rost motive of \(\pi\) (see \[Ro98\]). Rost motives over an arbitrary field appear in the proof of the Milnor conjecture by Voevodsky \[Vo03a\], \[Vo03b\].

We remark that over every completion of a number field, all quadratic forms are excellent, and therefore, the structure of their Chow motives is known (see Section 2). Moreover, due to Vishik \[Vi00\] and Haution \[Ha13\], every motivic decomposition of a quadric with \(\mathbb{F}_2\)-coefficients can be uniquely lifted to a motivic decomposition with integer coefficients. Therefore, our main theorem holds for motives with integer coefficients as well.

Finally, as an application of the Hasse principle for binary motives we obtain a complete motivic decomposition of the motive of a quadric over any number field with at most one real embedding, for example, over the field of rational numbers \(\mathbb{Q}\) (see Corollary 5.16):

**Corollary 2.** In the case when the field \(F\) has at most one real embedding, Main Theorem \[1\] holds for every split motive \(N\) (not necessarily binary).

## 2. Motivic decomposition of excellent forms

In this section \(F\) is an arbitrary field of characteristic not 2.

Let \(q\) be an anisotropic excellent quadratic form over \(F\) of dimension \(2d+1\) given by a strictly decreasing sequence of embedded Pfister forms \(\pi_0 \supset \pi_1 \supset \ldots \supset \pi_r\), where \(r\) is a positive integer and \(\dim(\pi_{r-1}) > 2 \dim(\pi_r)\) (see \[Kn77\] \S7, \[EKM\] \S28). Recall that an \(n\)-fold Pfister form is a form of the type \((1,a_1) \otimes \ldots \otimes (1,a_n)\) for some \(a_i \in F^*\). Let \(\dim(\pi_i) = 2^n_i\) for every \(i \in [0,r]\), \(n_{r-1} > n_r + 1\). We have

\[
\dim q = 2^{n_0} - 2^{n_1} + \ldots + (-1)^r 2^{n_r}.
\]

For an anisotropic Pfister form \(\pi\) we denote by \(R(\pi)\) the Rost motive of \(\pi\) (see \[Ro98\] Section 4]). By \[KM02\] Corollary 7.2 we have

\[
(2.1) \quad M(q) \simeq \bigoplus_{i=0}^{m_0} R(\pi_0)(i) \bigoplus \bigoplus_{i=m_0}^{m_0+m_1-1} R(\pi_1)(i) \bigoplus \ldots \bigoplus_{i=m_0+\ldots+m_{\tilde{r}-1}}^{m_0+\ldots+m_{\tilde{r}-1}} R(\pi_{\tilde{r}})(i),
\]

where \(\tilde{r} = r\) if \(\dim q\) is even and \(\tilde{r} = r - 1\) if \(\dim q\) is odd, and

\[
m_j = 2^{n_j-1} - 2^{n_{j+1}} + \ldots + (-1)^{j+r} 2^{n_r}
\]

for every \(j \in [0,\tilde{r}]\). If we want to specify the excellent form \(q\), we write \(m_j(q), n_j(q), r(q)\) etc.

More generally, for every \(k \in [0,\tilde{r}]\) we have

\[
(2.2) \quad m_0 + \ldots + m_k = \begin{cases} 
\frac{(\dim [q]_k)}{2} & \text{if } k \text{ is odd}, \\
\frac{(\dim [q]_k)}{2} - 2^{n_{k+1}} + \ldots + (-1)^{r} 2^{n_r} & \text{if } k \text{ is even},
\end{cases}
\]

where \([q]_k\) is the excellent quadratic form given by a strictly decreasing sequence of embedded Pfister forms \(\pi_0 \supset \pi_1 \supset \ldots \supset \pi_k\). It follows that

\[
(2.3) \quad m_0 + \ldots + m_k = (\dim q - \dim [q]_k)/2.
\]
Note that $m_0 + \ldots + m_r = \left\lfloor \frac{\text{dim} q}{2} \right\rfloor$, and this is the number of indecomposable direct summands in the complete motivic decomposition of $q$.

Finally, for an arbitrary $n$-dimensional quadratic form $q$ over $F$ with the Witt index $m$ we have by [Ro98, Proposition 2] (see also [EKM, Example 66.7]) the following motivic decomposition:

$$
(2.4) \quad M(q) \simeq \mathbb{F}_2 \oplus \ldots \oplus \mathbb{F}_2(m-1) \oplus M(q_{\text{an}})(m) \oplus \mathbb{F}_2(n-m-1) \oplus \ldots \oplus \mathbb{F}_2(n-2),
$$

where $q_{\text{an}}$ is the anisotropic part of $q$.

### 3. Construction of some Pfister forms

In this section $F$ be a number field. We denote by $\Omega(F)$ the set of all places of $F$. For $v \in \Omega(F)$, we denote by $F_v$ the completion of $F$ at $v$. For two non-zero elements $a, b \in F_v$, we denote by $(a, b)_v \in \{\pm 1\}$ their Hilbert symbol (see [O'Meara, §63.B]). Recall that $(a, b)_v = 1$ if $z^2 - ax^2 - by^2 = 0$ has a non-zero solution $(x, y, z)$ in $F_v^3$ and $(a, b)_v = -1$ otherwise. If $a, b \in F^*$, we write $(a, b)_v$ for $(a_v, b_v)_v$, where $a_v, b_v \in F_v$ are the images of $a$ and $b$, respectively.

For a diagonal quadratic form $p = \langle a_1, \ldots, a_n \rangle$, $a_i \in F_v^*$, over $F_v$ we define the Hasse invariant $\varepsilon_v(p)$ as in [Lam] and [Serre]:

$$
(3.1) \quad \varepsilon_v(p) = \prod_{i<j} (a_i, a_j)_v \in \{\pm 1\}.
$$

Let $q$ be a quadratic form over $F$. For $v \in \Omega(F)$, the embedding $F \hookrightarrow F_v$ induced by $v$ allows one to regard $q$ as a quadratic form over $F_v$, which we denote by $q_v$.

In this section we construct Pfister forms with prescribed local behavior. We shall use these forms in the proof of the main theorem.

We start by recalling several well-known facts about quadratic forms over local fields $F_v$, where $v \in \Omega(F)$ is a finite place (see [Lam], [O'Meara]). First of all, a quadratic form $p$ over $F_v$ is determined by its invariants: dimension, determinant $d(p) \in F_v^*/F_v^{*2}$ and the Hasse invariant $\varepsilon_v(p) \in \{\pm 1\}$. Every quadratic form over $F_v$ of dimension greater than 4 is isotropic (see [O'Meara 63:19]), and there exists a unique (up to isomorphism) anisotropic form of dimension 4. This form is a Pfister form and therefore has trivial determinant (see [O'Meara, 63:17]). Its Hasse invariant can be also computed:

**Lemma 3.2.** The 4-dimensional quadratic form $f$ with trivial determinant over $F_v$ is anisotropic if and only if its Hasse invariant is equal to $-(1, -1)_v$.

**Proof.** The split 4-dimensional form $(1, -1, 1, -1)$ has trivial determinant and the Hasse invariant $(-1, -1)_v$. Since $f$ has trivial determinant too and $f \not\cong \langle 1, -1, 1, -1 \rangle$, it must have a different Hasse invariant. We conclude that $\varepsilon_v(f) = -(1, -1)_v$. $\square$

The following proposition provides necessary and sufficient conditions for the existence of a quadratic form over $F$ with a prescribed local behavior.

**Proposition 3.3.** Let $n$ be a positive integer and for every $v \in \Omega(F)$ let $p^v$ be an $n$-dimensional quadratic form over $F_v$. For the existence of an $n$-dimensional quadratic form $p$ over $F$ such that $p_v \simeq p^v$ for all $v \in \Omega(F)$, it is necessary and sufficient that the following conditions hold:

1. There exists $c \in F^*$ such that $d(p^v) = c_v \cdot F_v^{*2}$ for all $v \in \Omega(F)$. 


(2) \( \varepsilon_v(p^v) = 1 \) for almost all \( v \in \Omega(F) \).
(3) \( \prod_{v \in \Omega(F)} \varepsilon_v(p^v) = 1 \).

Proof. The statement is proved in [O’Meara, Theorem 72:1], but with \( \varepsilon'_v(p^v) \) instead of \( \varepsilon_v(p^v) \), where for a diagonal quadratic form \( f = \langle b_1, \ldots, b_n \rangle, b_i \in F^*_v \), O’Meara considers the invariant
\[
\varepsilon'_v(f) = \prod_{i \leq j} (b_i, b_j)_v = \varepsilon_v(f) \cdot (d(f), d(f))_v \in \{\pm 1\}.
\]

Note that by [O’Meara, Theorem 71:18] we have \( (c, c)_v = 1 \) for almost all \( v \in \Omega(F) \) and
\[
\prod_{v \in \Omega(F)} (c, c)_v = 1.
\]

Therefore, conditions (1–3) of Proposition 3.3 are satisfied for an element \( c \in F^* \) and the numbers \( \varepsilon_v(p^v) \in \{\pm 1\} \) if and only if they are satisfied for the same \( c \) and the numbers \( \varepsilon'_v(p^v) = (c, c)_v \cdot \varepsilon_v(p^v) \in \{\pm 1\} \). This completes the proof of the proposition. \( \square \)

Recall that a quadratic form of dimension \( n \) is called split, if its Witt index has the maximal value \( [n/2] \).

Proposition 3.5. Let \( F \) be a number field. Let \( q \) be an anisotropic quadratic form over \( F \) of odd dimension \( 2d + 1 \). Assume that for every place \( v \in \Omega(F) \) there is a direct summand \( ^*M \) of \( M(q_v) \) such that \( ^*M_\Omega \simeq \mathbb{F}_2(d - 1) \oplus \mathbb{F}_2(d) \). Then there exists a 2-fold Pfister form \( \pi \) over \( F \) that splits exactly at those places \( v \) of \( F \) at which \( q \) splits.

Proof. Without loss of generality we may assume that the form \( q \) has trivial determinant. Indeed, we can replace \( q \) by the form \( d(q) \cdot q \), which has trivial determinant and the same motive as \( q \) and splits exactly at those places \( v \) of \( F \) at which \( q \) splits.

We apply Proposition 3.3 to the family \( \mathcal{F} = \{ \pi^v \mid v \in \Omega(F) \} \) of quadratic forms, where for every place \( v \in \Omega(F) \) the form \( \pi^v \) is defined as follows:

\[
\pi^v = \begin{cases} 
\text{a split form of dimension 4, if } q_v \text{ is split;} \\
\text{(1, 1, 1, 1), if } q_v \text{ is not split and } v \text{ is finite;} \\
\text{a (unique) anisotropic form of dimension 4, if } q_v \text{ is not split and } v \text{ is real.}
\end{cases}
\]

We take \( c = 1 \). Since \( d(\pi^v) = 1 \) for all \( v \in \Omega(F) \), condition (1) in Proposition 3.3 is satisfied.

Assume that conditions (2) and (3) in Proposition 3.3 are also satisfied for the family \( \mathcal{F} \). It follows then that there exists a quadratic form \( \pi \) such that for every \( v \in \Omega(F) \) we have \( \pi_v \simeq \pi^v \). Note that \( \pi \) represents 1 and \( d(\pi) = 1 \), since this holds locally for every \( v \in \Omega(F) \). Since \( \pi \) represents 1, it is equivalent to a diagonal form \( \langle 1, a, b, c \rangle \) for some \( a, b, c \in F^* \). Since \( d(\pi) = 1 \), we may take \( c = ab \). We see that \( \pi \) is a 2-fold Pfister form \( \langle 1, a \rangle \otimes \langle 1, b \rangle \). Clearly, by the construction of \( \mathcal{F} \), the form \( \pi \) satisfies the conclusion of the proposition.

Finally, the following lemma shows that conditions (2) and (3) in Proposition 3.3 are indeed satisfied for the family \( \mathcal{F} \). \( \square \)

Lemma 3.6. Under the hypotheses of Proposition 3.5 for every \( v \in \Omega(F) \) we have \( \varepsilon_v(q) = \varepsilon_v(\pi^v) \).

Proof. Since \( q \) is anisotropic over \( F \), there exists a place \( w \in \Omega(F) \) such that \( q_w \) is anisotropic over \( F_w \). Note that every anisotropic form over \( F_w \) is an excellent form. Therefore, the motivic decomposition of \( q_w \) is given by formula (2.1). The existence of the motivic direct summand of
Lemma 3.9. \□

Conclusion of the proposition.

(3.7)

Observe that for every finite place \(v\) of \(\mathbb{Q}\), the signature \((q_v)\) is such that \(q_v\) is split and has trivial determinant. Thus, \(\varepsilon_v(q) = (-1, -1)_v\). Since \(q_v\) is split and has trivial determinant, we have \(q \simeq d\mathbb{H} \perp (-1)^d\), where \(\mathbb{H}\) denotes the hyperbolic plane. Therefore, \(\varepsilon_v(q) = (-1, -1)_v\). Since \(d = 2d + 1\) is 3 or 5 modulo 8, we obtain in both cases \(\varepsilon_v(q) = (-1, -1)_v\).

2nd case: \(v\) is finite and \(\dim(q_v)_\text{an} = 3\). By Lemma 3.2, we have \(\varepsilon_v(q) = (-1, -1)_v\). Since a split quadratic form of dimension 2d + 1 and with trivial determinant has the Hasse invariant \((-1, -1)_v\), we obtain \(\varepsilon_v(q) = (-1, -1)_v\).

3rd case: \(v\) is real and \(\dim(q_v)_\text{an} > 1\). It is clear that \(\varepsilon_v(q) = 1\). Every quadratic form \(f\) over \(\mathbb{R}\) is equivalent to \(x_1^2 + \ldots + x_n^2 - y_1^2 - \ldots - y_m^2\) and the Hasse invariant of \(f\) is given by the following formula:

\[
\varepsilon_v(f) = (-1, -1)_v^{\frac{m(m-1)}{2}} = (-1)^{\frac{m(m-1)}{2}}.
\]

Note that by the same argument as in the beginning of the proof we have \(\dim(q_v)_\text{an}\) is either 3 or 5 modulo 8. Let \(m\) be the number of \(-1\)'s in the signature of \(q_v\).

If \(\dim(q_v)_\text{an} \equiv \dim(q_v)\) modulo 8, then \(d((q_v)_\text{an}) = d(q_v) = 1\). Since \((q_v)_\text{an}\) is of odd dimension, then \((q_v)_\text{an}\) is positive definite. It follows that \(m = (\dim(q_v) - \dim(q_v)_\text{an})/2\) and that \(m\) is divisible by 4. By formula (3.7) \(\varepsilon_v(q) = 1\).

If \(\dim(q_v)_\text{an} \not\equiv \dim(q_v)\) modulo 8, then \(d((q_v)_\text{an}) = -d(q_v) = -1\) and \((q_v)_\text{an}\) is negative definite. It follows that \(m = (\dim(q_v) + \dim(q_v)_\text{an})/2\) and that \(m\) is divisible by 4. By formula (3.7) \(\varepsilon_v(q) = 1\).

Proposition 3.8. Let \(F\) be a number field. Let \(q\) be an anisotropic quadratic form over \(F\) of even dimension \(2d + 2\). Assume that for every place \(v\) of \(\Omega(F)\) there is a direct summand \(\mathbb{M}\) of \(M(q_v)\) such that \(\mathbb{M}_{F_v} \simeq \mathbb{F}_2(d - 1) \oplus \mathbb{F}_2(d)\). Then there exists a 2-fold Pfister form \(\pi\) over \(\mathbb{F}_2\) that splits exactly at those places \(v\) of \(F\) at which \(q\) splits.

Proof. The proof is similar to that of Proposition 3.5.

Observe that for every finite place \(v\) of \(\Omega(F)\) the condition on \(M(q_v)\) implies that \(q_v\) is either split or \(\dim(q_v)_\text{an} = 4\). We define the family \(\mathcal{F} = \{\pi_v \mid v \in \Omega(F)\}\) as follows:

\[
\pi_v = \begin{cases} 
\text{a split form of dimension 4, if } q_v \text{ is split}; \\
\text{a (unique) anisotropic form of dimension 4, if } q_v \text{ is not split and } v \text{ is finite}; \\
(1, 1, 1, 1), \text{ if } q_v \text{ is not split and } v \text{ is real.}
\end{cases}
\]

The lemma below shows that the hypotheses of Proposition 3.8 are satisfied for the family \(\mathcal{F}\). It follows that there exists a quadratic form \(\pi\) such that for every \(v \in \Omega(F)\) we have \(\pi_v \simeq \pi_v\). Note that \(\pi\) represents 1 and \(d(\pi) = 1\). Therefore, \(\pi\) is a 2-fold Pfister form and satisfies the conclusion of the proposition.

Lemma 3.9. Under the hypotheses of Proposition 3.8 for every \(v \in \Omega(F)\) we have \(\varepsilon_v(q) = \varepsilon_v(\pi_v)\).
Proof. Since $q$ is anisotropic over $F$, there exists a place $w \in \Omega(F)$ such that $q_w$ is anisotropic over $F_w$. Note that every anisotropic form over $F_w$ is an excellent form. Therefore, the motivic decomposition of $q_w$ is given by formula (2.1). The existence of the motivic direct summand of $M(q_w)$ as in the statement of Proposition 3.3 implies that $\dim q$ is 4 modulo 8. In particular, $\text{disc}(q) = d(q)$, where $\text{disc}(q)$ denotes the discriminant of $q$ (recall that, by definition, $\text{disc}(q) = (-1)^{n(n-1)/2}d(q)$, where $n = \dim q$).

Note also that for every $v \in \Omega(F)$ by formula (2.1) the motive $M(q_v)$ does not have an indecomposable direct summand over $F_v$ which becomes $\mathbb{F}_2(d) \oplus \mathbb{F}_2(d)$ over $\overline{F_v}$. Therefore, for every $v \in \Omega(F)$ we have $\text{dim}(q_v) = 1$. Thus, $\text{disc}(q_v) = 1$ and $d(q_v) = 1$.

Let us now check case by case that for every $v \in \Omega(F)$ we have $\varepsilon_v(q) = \varepsilon_v(\pi^v)$.

1st case: $v$ is such that $q_v$ is split. In this case $\pi^v$ is split. Since $2d + 2$ is 4 modulo 8, we have $$\varepsilon_v(q) = (-1, -1)_v^{d(d+1)/2} = (-1, -1)_v = \varepsilon_v(\pi^v).$$

2nd case: $v$ is finite and $\dim(q_v)_{an} = 4$. By Lemma 3.2 we have $\varepsilon_v(\pi^v) = -(-1, -1)_v$. Since a split quadratic form of dimension $2d + 2$ has trivial determinant and the Hasse invariant $(-1, -1)_v$, the form $q_v$, which is of the same dimension and also has the trivial determinant, has the Hasse invariant $-(-1, -1)_v$.

3rd case: $v$ is real and $\dim(q_v)_{an} > 0$. It is clear that $\varepsilon_v(\pi^v) = 1$. Note that by the same argument as in the beginning of the proof we have $\dim(q_v)_{an}$ is 4 modulo 8. Let $m$ be the number of $-1$’s in the signature of $q_v$.

Depending on whether $(q_v)_{an}$ is positive definite or not, we have $m = (\dim q_v \pm \dim(q_v)_{an})/2$. Thus, in any case $m$ is divisible by 4. Then by formula (3.7) we have $\varepsilon_v(q) = 1$. □

4. Vishik’s results

In this section $F$ is an arbitrary field of characteristic not 2. We reformulate two Vishik’s results for our convenience in order to use them in the proof of the main theorem.

Proposition 4.1 ([V100], Theorem 5.1). Let $p = f \otimes \pi$ be a quadratic form over $F$, where $\pi$ is an $n$-fold Pfister form and $f$ is an odd-dimensional form. Then $M(\pi)(s)$ is a direct summand of $M(p)$, where $s = (\dim p - \dim \pi)/2$.

Let $R$ be a geometrically split motive over $F$ and $r$ an integer. We say that $R$ “starts at $r$”, if $r$ is the minimal integer $i$ such that the Tate motive $\mathbb{F}_2(i)$ is a direct summand of $R_{\overline{F}}$.

Proposition 4.2 ([V100], Theorem 4.17). Let $p$ and $q$ be two anisotropic quadratic forms over a field $F$. Let $m_p$, $m_q$ be fixed non-negative integers. Assume that $M(p)$ has an indecomposable direct summand of the form $R(m_p)$, where $R$ is a motive, which “starts at 0”. If for all field extensions $E/F$ we have $$i_W(q_E) > m_q \iff i_W(p_E) > m_p,$$

where $i_W(q_E)$ and $i_W(p_E)$ are the corresponding Witt indices over $E$, then $R(m_p)$ is an indecomposable direct summand of $M(p)$. 
5. Proof of the main theorem

We say that a motive $N$ over a field $F$ is binary if it becomes isomorphic to a direct sum of two Tate motives over $F$. We say that $N$ is a binary split motive if it is a direct sum of two Tate motives over $F$.

**Theorem 5.1.** Let $q$ be a quadratic form over a number field $F$. Let $N$ be a binary split motive over $F$. Assume that for every place $v \in \Omega(F)$ there exists a direct motivic summand ${}^vM$ of $M(q_v)$ such that ${}^vM_{F_v} \simeq N_{F_v}$. Then there exists a direct motivic summand $M$ of $M(q)$ such that $M_F \simeq N_F$.

**Remark 5.2.** By the Krull–Schmidt principle for the Chow motives of quadrics (see [Vi00], [ChM06]) the motive $M$ is isomorphic to ${}^vM$ over $F_v$ for all $v$.

**Proof.** If the quadratic form $q$ is isotropic, then using formula (2.3) we can reduce to the anisotropic part of $q$. Therefore, without loss of generality we may assume that the form $q$ is anisotropic. We denote by $\Omega_2(F)$ (resp. $\Omega_{1+1}(F)$) the subset of $\Omega(F)$ consisting of the places $v$ such that ${}^vM$ is indecomposable (resp. ${}^vM$ is split).

Note that for every $v \in \Omega(F)$ the form $q_v$ as well as its anisotropic part $(q_v)_{an}$ is an excellent form. Throughout the proof of the theorem we denote simply by $n_j(v)$, $m_j(v)$ and $r(v)$ respectively the numbers $n_j((q_v)_{an})$, $m_j((q_v)_{an})$ and $r((q_v)_{an})$ from Section 2 corresponding to the motivic decomposition of the anisotropic excellent form $(q_v)_{an}$.

Since $q$ is anisotropic, the set $\Omega_2(F)$ is not empty. Let $v \in \Omega_2(F)$. It follows from decomposition (2.1) and the Krull–Schmidt principle for the motives of quadrics (see [Vi00], [ChM06]) that ${}^vM \simeq {}^vR(t)$, where $t \geq 0$ and ${}^vR$ is the Rost motive of some anisotropic $n$-fold Pfister form over $F_v$, $n \geq 1$. Note that $t$ and $n$ do not depend on $v \in \Omega_2(F)$. Indeed, this follows from the hypothesis of the theorem, namely we have ${}^vM_{F_v} \simeq {}^wM_{F_w}$ for every $v, w \in \Omega_2(F)$.

It follows that for every $v \in \Omega_2(F)$ the summand $\pm 2^n$ is presented in the decomposition of $\dim (q_v)_{an}$ into an alternating sum of powers of 2, $\dim (q_v)_{an} = \sum_{i=0}^{r(v)}(-1)^i2^{n_i(v)}$. We denote by $k(v)$ the index satisfying the equality $n_{k(v)} = n$ in this decomposition. We also define

$$Q(v) := \sum_{i=0}^{k(v)}(-1)^i2^{n_i(v)}, \quad A(v) := \sum_{i=k(v)+1}^{r(v)}(-1)^{i-k(v)-1}2^{n_i(v)}$$

i.e. $Q(v) = 2^{n_0} - 2^{n_1} + \ldots \pm 2^n$ is a part (up to $2^n$) of the decomposition of $\dim (q_v)_{an}$ and $A(v) = |\dim (q_v)_{an} - Q(v)|$. Note that in terms of the notation of Section 2 we also have $Q(v) = \dim [(q_v)_{an}]k(v)$.

From now we assume that $n > 1$, and we shall consider the case $n = 1$ at the very end of the proof.

In order to prove the theorem we shall apply Proposition 4.1 to the quadratic form $q$ and a suitable form $p$. The form $p$ will be constructed below as a product $f \otimes \pi$, where $\pi$ is an $n$-fold Pfister form, and will satisfy the following properties:

1. For every place $v \in \Omega(F)$:

   $$(5.3) \quad \pi_v \text{ is split} \iff {}^vM \text{ is split}.$$ 

2. For every place $v \in \Omega_2(F)$ the following two equalities hold
We define \( f \) construct. Note that \( Q \) is divisible by 2.
\[
\dim(p_v)_{an} = Q(v)
\]
and
\[
\dim(q_v - p_v)_{an} = \left| \dim(q_v)_{an} - \dim(p_v)_{an} \right| = A(v).
\]

**Lemma 5.6. (Construction of \( \pi \))** There exists an \( n \)-fold Pfister form over \( F \), which satisfies Property (5.3) for every \( v \in \Omega(F) \).

**Proof.** First, assume that \( n = 2 \). Then for every \( v \in \Omega(F) \) the motive \( ^vM \) is split if and only if the form \( q_v \) is split. If \( \dim q = 2d + 1 \), then \( \dim \mathbb{F}_2(d - 1) \oplus \mathbb{F}_2(d) \) and the result follows from Proposition 3.5.

Now consider the case \( \dim q = 2d + 2 \). Denote by \( S \) the motive \( \mathbb{F}_2(d - 1) \oplus \mathbb{F}_2(d) \). Then either \( N \simeq S \) or \( N \simeq S(1) \). In the first case the conclusion of the lemma follows from Proposition 3.8 and the second case can be reduced to the first one. Indeed, if \( N \simeq S(1) \), then, by duality, for every \( v \in \Omega(F) \) the motive \( ^vM(-1) \) is a direct motivic summand of \( q_v \).

Assume that \( n > 2 \). By the Weak Approximation Theorem there exists \( a \in F \) such that the following property holds for every \( v \in \Omega_\mathbb{R}(F) \), where \( \Omega_\mathbb{R}(F) \) is the set of all real places of \( F \):

\[
a_v < 0 \iff ^vM \text{ is split}.
\]

Consider the \( n \)-fold Pfister form \( \pi = \langle 1, a \rangle \otimes \langle 1, 1 \rangle \otimes \ldots \otimes \langle 1, 1 \rangle \). We claim that \( \pi \) satisfies Property (5.3) for every \( v \in \Omega(F) \). Indeed, if \( v \in \Omega(F) \) is finite or complex, then the form \( \pi_v \) and the motive \( ^vM \) are both split. Thus, in this case Property (5.3) holds. Moreover, by the construction of \( a \) the form \( \pi \) satisfies Property (5.3) for every real place \( v \in \Omega(F) \).

**Lemma 5.7. (Construction of \( f \))** There exists a quadratic form \( f \) over \( F \) such that for every place \( v \in \Omega_2(F) \) the form \( p := f \otimes \pi \) satisfies Properties (5.4) and (5.5).

**Proof.** Let \( \Omega_2,\mathbb{R}(F) = \{v_1, \ldots, v_t\} \) be the set of all real places \( v \in \Omega(F) \) such that \( ^vM \) is indecomposable, i.e. \( \Omega_2,\mathbb{R}(F) = \Omega_2(F) \cap \Omega_\mathbb{R}(F) \). For every \( i = 1, \ldots, t \) we construct a quadratic form \( f_i \) over \( F \) as follows.

We fix an integer \( i \in [1, t] \) and denote \( v_i \) simply by \( v \). Since \( v \in \Omega_2(F) \), the power \( \pm 2^n \) appears in the alternating sum

\[
\dim(q_v)_{an} = \sum_{j=0}^{r(v)} (-1)^j 2^{n_j(v)}.
\]

Note that \( Q(v) = \sum_{j=0}^{k(v)} (-1)^j 2^{n_j(v)} \), where \( n_{k(v)}(v) = n \), is divisible by \( 2^n = \dim \pi \). We construct \( f_i \) in the form \( \langle 1, a_i \rangle \otimes \langle 1, \ldots, 1 \rangle \), \( a_i \in F \) such that the following equality holds

\[
\dim(f_i) + (-1)^{k(v)}(\dim \pi) = Q(v).
\]

By the Weak Approximation Theorem we can choose \( a_i \in F \) such that for every \( j = 1, \ldots, t \) the following property holds

\[
(a_i)_{v_j} > 0 \iff i = j.
\]

We define \( f := \alpha(f_1 \perp \ldots \perp f_t \perp (b))_{an} \), where \( \alpha, b \in F \) satisfy respectively the following properties for every \( v \in \Omega_2,\mathbb{R}(F) \)

\[
\text{sgn} b_v = \text{sgn}(-1)^{k(v)}.
\]

and

\[
a_v > 0 \iff (q_v)_{an} \text{ is positive definite}.
\]
Note that the existence of $\alpha$ and $b$ is once again guaranteed by the Weak Approximation Theorem.

We claim that $p = f \otimes \pi$ satisfies Properties (5.4) and (5.5) for every place $v \in \Omega_2(F)$.

Indeed, let $v = v_1 \in \Omega_2_q(F)$. Note that all forms $f_j$, $j \in \{1, \ldots, t\}$, $j \neq i$, become hyperbolic over $v$. Thus, we have in the Witt ring $W(F_v)$:

$$p_v = \alpha_v((f_i)_v + (b_{v}))\pi_v = \pm((\dim f_i)(1) + ((-1)^{k(v)})\pi_v).$$

It follows from formula (5.8) that $\dim(p_v)_{an} = Q(v)$. By the construction of $\alpha$, equality (5.5) also holds.

Assume now that there is a finite place $v \in \Omega_2(F)$. It is possible only if $n = 2$. Since the dimension of $f$ is odd and $\pi_v$ is anisotropic, we have $(p_v)_{an} = (f_v \otimes \pi_v)_{an} \cong \pi_v$. We also have $\dim(q_v)_{an} = 4$, $Q(v) = 4$, $A(v) = 0$ if $\dim q$ is even, and $\dim(q_v)_{an} = 3$, $Q(v) = 4$, $A(v) = 1$ if $\dim q$ is odd. In both cases the equality $\dim(p_v)_{an} = Q(v)$ holds.

Recall that over $F_v$ there is a unique anisotropic quadratic form of dimension 4. Thus, this form is $\pi_v$. Therefore, we obtain $(q_v)_{an} \cong \pi_v$ if $q$ is even-dimensional and $(q_v)_{an}$ is a subform of $\pi_v$ if $q$ is odd-dimensional. In both cases we obtain $\dim(p_v - q_v)_{an} = A(v)$. It follows that the form $p = f \otimes \pi$ satisfies Properties (5.4) and (5.5) for every finite place $v \in \Omega_2(F)$. $\square$

By Proposition 4.1 $M(\pi)(s)$ is a direct summand of $M(p)$, where $s = (\dim p - \dim \pi)/2$. Let $R$ be the Rost motive of $\pi$. Then $R(s)$ is also a direct summand of $M(p)$.

Recall that $^tR(t)_\pi \cong N_{^tR}$ for every place $v \in \Omega_2(F)$. Since $R$ and $^tR$ are both Rost motives of some anisotropic $n$-fold Pfister forms, we also have $R(t)_{\pi} \cong N_{^tR}$. In order to complete the proof (in the case $n > 1$), we shall show that $R(t)$ is a direct summand of $M(q)$. By Proposition 4.2 this follows from the lemma below.

**Lemma 5.9.** For every field extension $E/F$ the following equivalence holds

$$i_W(q_E) > t \iff i_W(p_E) > s,$$

where $i_W(q_E)$ and $i_W(p_E)$ are the corresponding Witt indices over $E$.

**Proof.** Throughout the proof we use the following observation which follows from formula (2.4). Namely, a twist of a Rost motive $R(i)$ appears in the complete decomposition of $M((q_v)_{an})$ if and only if the motive $R(i + i_W(q_v))$ appears in the complete decomposition of $M(q_v)$.

$\iff$. Assume that $i_W(p_E) > s$ for some field extension $E$ of $F$ (note that $E$ is not necessarily a number field). Recall that $R(s)$ is a binary motive, which is a direct summand of $M(p)$. It follows that $R_E$ is split. Then the motive of the Pfister form $\pi_E$ is also split, since it is a sum of shifts of the motive $R_E$. It follows that $\pi_E$ and, thus, $p_E = \pi_E \otimes \pi_E$ are split quadratic forms. Therefore, we have the following equality in the Witt ring $W(E)$

$$q_E = (q - p)_E + p_E = (q - p)_E.$$

It follows that $(q_E)_{an} \cong (q_E - p_E)_{an}$. Therefore, we have

$$i_W(q_E) = \frac{\dim q_E - \dim(q_E - p_E)_{an}}{2} \geq \frac{\dim q - \dim(q - p)_{an}}{2}.$$

We claim that the following inequality holds for every $v \in \Omega(F)$

$$\frac{\dim q_v - \dim(q_v - p_v)_{an}}{2} > t.$$
Note that this will imply the necessary inequality $i_W(q_v) > t$, since by the Hasse principle we have $\dim(q - p)_{\text{an}} = \max_{v \in \Omega(F)} \dim(q_v - p_v)_{\text{an}}$. In order to prove formula (5.10) we consider two cases: $v \in \Omega_2(F)$ and $v \in \Omega_{1+1}(F)$.

Assume $v \in \Omega_2(F)$. By the construction of $p$ we have

$$\dim(q_v - p_v)_{\text{an}} = |\dim(q_v)_{\text{an}} - \dim(p_v)_{\text{an}}| = |\dim(q_v)_{\text{an}} - \dim([q_v]_{k(v)})|$$

Therefore,

$$\frac{\dim q_v - \dim(q_v - p_v)_{\text{an}}}{2} = \frac{2i_W(q_v) + \dim(q_v)_{\text{an}} - |\dim(q_v)_{\text{an}} - \dim([q_v]_{k(v)})|}{2} = \frac{i_W(q_v) + m_0((q_v)_{\text{an}}) + \ldots + m_{k(v)}((q_v)_{\text{an}})}{t}.$$  

Note that the last equality follows from formula (2.3) and the last inequality is a consequence of the decomposition of $\mathcal{M}((q_v)_{\text{an}})$, see (2.1).

Assume now $v \in \Omega_{1+1}(F)$. Then $^*\mathcal{M}$ decomposes into a sum of two Tate motives, and one of them is $\mathbb{F}_2(t)$. Therefore, $\mathbb{F}_2(t)$ is a direct summand of $\mathcal{M}(q_v)$. Hence $i_W(q_v) > t$.

Since by the construction of $p$, $p_v$ is split, we obtain

$$\frac{\dim q_v - \dim(q_v - p_v)_{\text{an}}}{2} = \frac{\dim q_v - \dim(q_v)_{\text{an}}}{2} = i_W(q_v) > t.$$  

“⇒”. Assume that $i_W(q_E) > t$ for some field extension $E$ of $F$. In the Witt ring $W(E)$ we have

$$p_E = (p_E - q_E) + q_E.$$  

Therefore,

$$\dim(p_E)_{\text{an}} \leq \dim(p_E - q_E)_{\text{an}} + \dim(q_E)_{\text{an}} \leq \dim(p - q)_{\text{an}} + \dim q - 2t - 2$$

Note that in order to show $i_W(p_E) > s$, it is sufficient to prove the following inequality for every $v \in \Omega(F)$:

$$\dim(p_v - q_v)_{\text{an}} + \dim q_v - 2t - 2 < 2^n.$$  

Indeed, it follows from the Hasse principle that $\dim(p - q)_{\text{an}} + \dim q - 2t - 2 < 2^n$. Thus, from formula (5.11) we obtain $\dim(p_E)_{\text{an}} < 2^n$. Hence $i_W(p_E) > (\dim p_E - 2^n)/2 = s$.

In order to prove formula (5.10) we consider two cases: $v \in \Omega_2(F)$ and $v \in \Omega_{1+1}(F)$. Assume first $v \in \Omega_2(F)$. Since $^*\mathcal{R}(t - i_W(q_v))$ is present in the motivic decomposition of the excellent form $(q_v)_{\text{an}}$ (see (2.1)), we have $t \geq m_0(v) + \ldots + m_{k(v)-1}(v) + i_W(q_v)$. Therefore,

$$2t \geq 2(m_0(v) + \ldots + m_{k(v)}(v)) - 2m_{k(v)}(v) + 2i_W(q_v) = \dim(q_v) - |\dim(q_v)_{\text{an}} - \dim(p_v)_{\text{an}}| - 2m_{k(v)}(v),$$

where the equality follows from (2.3) and property (5.4) of $p$.

Using the above inequality for the left-hand side of (5.12) and property (5.5) of $p$ we obtain

$$\dim(p_v - q_v)_{\text{an}} + \dim q_v - 2t - 2 \leq 2A(v) + 2m_{k(v)}(v) - 2 = 2^n - 2 < 2^n,$$

where the last equality follows from the explicit expression of $A(v)$ and $m_{k(v)}(v)$, see Section 2.

Assume now that $v \in \Omega_{1+1}(F)$. Then, by construction, the form $p_v$ is split and we have $\dim(p_v - q_v)_{\text{an}} = \dim(q_v)_{\text{an}} = \dim q_v - 2i_W(q_v)$. 

Since \( \mathbf{^aM} \) is split, we have \( \mathbf{^aM} \simeq \mathbb{F}_2(t) \oplus \mathbb{F}_2(t + 2^{n-1} - 1) \). Therefore, the Tate motive \( \mathbb{F}_2(t + 2^{n-1} - 1) \) is present in the motivic decomposition of \( p_v \), and, by duality, the same is true for the trivial Tate motive \( \mathbb{F}_2 \) twisted by

\[
\dim q_v - 2 - (t + 2^{n-1} - 1) = \dim q_v - t - 2^{n-1} - 1.
\]

It follows that \( i_W(q_v) > \dim q_v - t - 2^{n-1} - 1 \). Using this we obtain the necessary inequality \([5.12]\).

Indeed,

\[
dim(p_v - q_v)_{an} + \dim q_v - 2t - 2 = 2 \dim q_v - 2i_W(q_v) - 2t - 2
\]

\[
< 2 \dim q_v - 2(\dim q_v - t - 2^{n-1} - 1) - 2t - 2 = 2^n.
\]

\( \square \)

In order to complete the proof of the theorem it remains to consider the case \( n = 1 \). Note that in this case the quadratic form \( q \) is even-dimensional: \( \dim q = 2d + 2 \). For every \( v \in \Omega_{1+1}(F) \) the motive \( \mathbf{^aM} \) is split, \( \mathbf{^aM} \simeq \mathbb{F}_2(d) \oplus \mathbb{F}_2(d) \), and therefore the form \( q_v \) is split as well. Let \( v \in \Omega_{2}(F) \). Then the motive \( \mathbf{^aM} \) is indecomposable. It follows that the quadratic form \( q_v \) has a non-trivial discriminant. Moreover, \( \mathbf{^aM} \simeq M(\text{Spec}(\sqrt{\text{disc } q_v}))(d) \). In both cases the following equivalence holds: for every \( v \in \Omega(F) \) and for every field extension \( E \) of \( F \) we have

\[
(q_v)_E \text{ is split} \quad \iff \quad (q_v)_E \text{ has trivial discriminant}.
\]

Note that \( \Omega_{2}(F) \) is non-empty, since \( q \) is anisotropic. Therefore, the discriminant of \( q \) is non-trivial and it becomes trivial over the quadratic field extension \( K = F(\sqrt{\text{disc } q}) \) of \( F \).

We claim that the quadratic form \( q \) is split over \( K \). Since \( K \) is a number field, by the Hasse principle, it is sufficient to show that \( (q_K)_w \) is split for every \( w \in \Omega(K) \). Note that \( K_w \) is an extension of \( F_v \) for some \( v \in \Omega(F) \). Since \( \text{disc } q_{K_w} \) is trivial, the claim follows from \([5.13]\).

It follows that \( q \) becomes hyperbolic over the function field of the 1-fold Pfister form \( \pi = (1, -\text{disc } q) \). Then, by \([\text{Lam}]\) Theorem 4.11, Chapter X] the form \( q \) is divisible by \( \pi \). Finally, by Proposition 4.11 \( M(\pi)(d) \) is a direct motivic summand of \( q \).

\( \square \)

As an immediate corollary of the proof we obtain:

**Corollary 5.14.** Let \( F \) be a number field and let \( q \) be a quadratic form over \( F \). Then every indecomposable binary direct summand of \( M(q) \) is a twist of a Rost motive.

**Remark 5.15.** Corollary 5.14 also follows from \([\text{L2hV00}]\) Theorem 6.9], where motivic cohomology of simplicial varieties and the Milnor operations are used. Conjecturally Corollary 5.14 holds over any field. Voevodsky formulated in \([\text{V011}]\) Remark 5.4 an even more general conjecture.

**Corollary 5.16.** In the case when the field \( F \) has at most one real embedding, Theorem 5.7 holds for every split motive \( N \).

**Proof.** Without loss of generality we may assume that the form \( q \) is anisotropic, \( \dim q \geq 4 \), and \( F \) has one real place.

First we apply Theorem 5.7 to all split binary motives \( N \) and split off all binary direct summands in \( M(q) \). Let us denote the remaining direct summand as \( S \) and assume that it is non-zero.

If \( \dim q = 2d + 1 \geq 5 \), then \( S \) over \( \overline{F} \) is isomorphic to

\[
\mathbb{F}_2(s) \oplus \mathbb{F}_2(d - 1) \oplus \mathbb{F}_2(d) \oplus \mathbb{F}_2(2d - s - 1)
\]
with some $0 \leq s \leq d - 2$, $d = s + 2^r - 1$ for some $r \geq 2$. By the assumptions, over $\mathbb{R}$ the motive of $q$ contains two indecomposable direct summands that over a splitting field of $q$ become isomorphic to $\mathbb{F}_2(s) \oplus \mathbb{F}_2(d)$ and $\mathbb{F}_2(d - 1) \oplus \mathbb{F}_2(2d - s - 1)$, resp. Over a finite place the motive of $q$ contains an indecomposable direct summand that over a splitting field becomes isomorphic to $\mathbb{F}_2(d - 1) \oplus \mathbb{F}_2(d)$. Therefore, $S$ is indecomposable over $F$.

Similarly, if $\dim q = 2d + 2$, then $S$ over $\overline{F}$ is isomorphic either to 

$$\mathbb{F}_2(s) \oplus \mathbb{F}_2(d)^{\oplus 2} \oplus \mathbb{F}_2(2d - s)$$

(for some $0 \leq s \leq d - 1$, $d = s + 2^r - 1$ for some $r \geq 1$) or to 

$$\mathbb{F}_2(s) \oplus \mathbb{F}_2(d - 1) \oplus \mathbb{F}_2(d)^{\oplus 2} \oplus \mathbb{F}_2(2d - s)$$

(for some $0 \leq s \leq d - 2$, $d = s + 2^r - 2$ for some $r \geq 2$) or to 

$$\tilde{N} := \mathbb{F}_2(s) \oplus \mathbb{F}_2(s + 1) \oplus \mathbb{F}_2(d - 1) \oplus \mathbb{F}_2(2d)^{\oplus 2} \oplus \mathbb{F}_2(d - 1) \oplus \mathbb{F}_2(2d - s - 1) \oplus \mathbb{F}_2(2d - s)$$

(for some $0 \leq s \leq d - 2$, $d = s + 2^r - 1$ for some $r \geq 2$). In the first two cases the motive $S$ is indecomposable over $F$ by the same reasoning as in the odd-dimensional case. In the third case the motive $S$ is indecomposable over $F$, if the discriminant of $q$ is non-trivial, since in this case the motive of $q$ over some finite place contains an indecomposable direct summand that over a splitting field becomes isomorphic to $\mathbb{F}_2(d)^{\oplus 2}$.

Therefore, it remains to consider the case when the discriminant of $q$ is trivial and $S$ over $\overline{F}$ is isomorphic to $\tilde{N}$. Note that in this case the hypotheses of Corollary 5.10 are satisfied for the motive $\tilde{N}$, but also for the (unique) motive $N'$ such that $\tilde{N} = N' \oplus N'(1)$.

From now we assume without loss of generality that $q_\mathbb{R}$ is positive definite. Since $S$ over $\overline{F}$ is isomorphic to $\tilde{N}$, we have $\dim q = 2^n m$, where $n = r + 1 \geq 3$ and $m$ is odd.

First we shall reduce the problem to the case $\dim q = 2^n$. There exists a quadratic form $\tilde{q}$ of dimension $2^n$ over $F$ such that $\tilde{q}_\mathbb{R} \simeq (1, 1)^{\otimes n}$ and $(\tilde{q}_v)_{\text{an}} \simeq (q_v)_{\text{an}}$ for every finite place $v \in \Omega(F)$. Indeed, the existence of $\tilde{q}$ follows from Proposition 3.3, since the equalities $\varepsilon_v(\tilde{q}) = \varepsilon_v(q)$ and $d(\tilde{q}_v) = 1$ hold for every place $v \in \Omega(F)$. Let $U$ be the upper indecomposable motive of $M(\tilde{q})$ (see [Ka13, §2b]). Lemma 5.17 below shows that $U(l)$ is a direct summand of $M(q)$, where $l = (\dim q - \dim \tilde{q})/2$. It follows that we can replace $q$ by $\tilde{q}$, that is we can assume $\dim q = 2^n$.

Before we prove the lemma, let us note that the following isomorphisms hold for the forms $q$ and $\tilde{q}$: 

$$(1, 1) \otimes \tilde{q} \simeq \pi \quad \text{and} \quad q \simeq \tilde{q} \otimes f \simeq \tilde{q} \perp (\pi \otimes g),$$

where $\pi = (1, 1)^{\otimes (n+1)}$, $f = (1)^{\otimes m}$ and $g = (1)^{\otimes (m-1)/2}$. One can easily check each of these isomorphisms locally for every $v \in \Omega(F)$.

**Lemma 5.17.** The motive $U(l)$ is a direct summand of $M(q)$.

**Proof.** By Proposition 4.2 it is enough to show that for every field extension $E/F$ the following equivalence holds 

$$i_W(q_E) > l \iff i_W(\tilde{q}_E) > 0.$$ 

"$\iff$". Assume that $i_W(\tilde{q}_E) > 0$ for some field extension $E$ of $F$. Then the $(n + 1)$-fold Pfister form $\pi_E \simeq (1, 1) \otimes \tilde{q}_E$ is isotropic and, therefore, is hyperbolic over $E$. Since $q \simeq \tilde{q} \perp (\pi \otimes g)$, we get $q_E = \tilde{q}_E$ in the Witt ring $W(E)$. It follows that $i_W(q_E) > l$.

"$\Rightarrow$". Assume that $i_W(q_E) > l$ for some field extension $E$ of $F$. Then the form $q_E$ can be represented in $W(E)$ by a form of dimension $< 2^n$. Therefore the form $(1, 1) \otimes q_E$ can be
represented in $W(E)$ by a form of dimension $< 2^n+1$. Note that
\[(1,1) \otimes q_E \simeq \pi_E \otimes f_E \in I^{n+1}(E),\]
where $I$ denotes the fundamental ideal in the Witt ring. It follows from [Lam, Hauptsatz X.5.1] that $\pi_E \otimes f_E = 0$ in $W(E)$, that is the form $\pi_E \otimes f_E$ is hyperbolic. By [Kahn, Cor. 3.3.4] the $(n + 1)$-fold Pfister form $\pi_E$ is also hyperbolic over $E$. Finally, since $q \simeq \tilde{q} \perp (\pi \otimes g)$, we get $q_E = \tilde{q}_E$ in $W(E)$. Now it follows from $i_W(q_E) > l$ that $i_W(\tilde{q}_E) > 0$. \qed

Next we prove the following lemma.

**Lemma 5.18.** The first Witt index of $q$ is greater than $1$.

**Proof.** By [Vio00, Lemma 5.2] it is enough to prove that $q$ is divisible by a binary quadratic form over $F$. Consider the quadratic form $\varphi \perp (-q)$, where $\varphi = (1,1)^{\otimes n}$, and denote by $\rho$ the anisotropic part of this form. It is easy to see that $\rho$ has dimension 4, trivial discriminant and represents 1. In particular, $\rho$ is a 2-fold Pfister form. We claim that $\rho$ and $(1,1)^{\otimes n}$ are divisible by a common binary form $(1,a)$ for some $a \in F^*$. Let $\rho'$ and $\varphi'$ be respectively the pure subforms of $\rho$ and $\varphi$ (see Lam §X.1]). By Lam Theorem X.1.5] it is enough to show that $\rho'$ and $\varphi'$ represent the same $a \in F^*$. We can take any $a \in F^*$ such that $\rho'$ represents $a$ and $a_R > 0$. Such an $a$ exists, since $\rho_R \simeq (1, -1, 1, -1)$ and therefore $\rho'_R \simeq (-1, 1, -1)$ is not negative definite. Since $a_R > 0$, the form $\varphi'$ also represents $a$ (since this holds locally for every place $v \in \Omega(F)$). We conclude that the both forms $\rho$ and $\varphi$ are divisible by $(1,a)$.

It follows that we have $\varphi - \rho = (1,a) \otimes \tau$ in the Witt ring $W(F)$ for some form $\tau$, hence, by Lam Theorem X.4.11] the form $q$ is divisible by $(1,a)$. \qed

It follows now from Lemma 5.18 and [Vio00, Theorem 4.13] that the motive $S$ from the beginning of the proof of the present Corollary is isomorphic to $S' \oplus S''(1)$ for some indecomposable motive $S'$ over $F$. Therefore the statement of the Corollary holds also in the remaining case for the motive $N'$.

**Example 5.19.** Let $q$ be an 11-dimensional quadratic form over a number field $F$. Assume that the only values of the Witt index of $q$ over the real places are 0 and 2, that is
\[\{i_W(q_v) \mid v \in \Omega_R(F)\} = \{0, 2\} .\]
Then we can find a complete decomposition of $M(q)$. Indeed, if $i_W(q_v) = 0$, we have
\[(5.20) \quad M(q_v) = R_3 \oplus R_4(1) \oplus R_4(2) \oplus R_3(3) \oplus R_2(4)\]
and if $i_W(q_v) = 2$, then
\[(5.21) \quad M(q_v) = F_2 \oplus F_2(1) \oplus R_3(2) \oplus R_3(3) \oplus R_3(4) \oplus F_2(8) \oplus F_2(9) ,\]
where $R_i$ is the Rost motive of a unique anisotropic $i$-fold Pfister form over $F_v = R$, $(R_i)_{F_v} \simeq F_2 \oplus F_2(2^{i-1} - 1)$.

Note that the hypotheses of Theorem 5.1 hold for $N = F_2(1) \oplus F_2(8)$ and for $N = F_2(3) \oplus F_2(6)$. It follows that $M(q)$ has two direct summands $\tilde{R}_4(1)$ and $\tilde{R}_3(3)$, where $\tilde{R}_i$ is the Rost motive of some anisotropic $i$-fold Pfister form over $F$. Therefore, for some motive $U$ we have
\[(5.22) \quad M(q) = U \oplus \tilde{R}_4(1) \oplus \tilde{R}_3(3) .\]
It follows from decompositions (5.20) and (5.21) that the motive $U$ is indecomposable. Thus, decomposition (5.22) of the motive $M(q)$ is complete. Note that the motive $U$ in the decomposition is the upper motive of $M(q)$ in terms of [Ka13 §2b].

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