Turbulent black holes

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We show that rapidly-spinning black holes can display turbulent gravitational behavior which is mediated by a new type of parametric instability. This instability transfers energy from higher temporal and azimuthal spatial frequencies to lower frequencies—a phenomenon reminiscent of the inverse energy cascade displayed by 2+1-dimensional turbulent fluids. Our finding reveals a path towards gravitational turbulence for rapidly-spinning black holes, and provides the first evidence for gravitational turbulence in an asymptotically flat spacetime. Interestingly, this finding predicts observable gravitational wave signatures from such phenomena in black hole binaries with high spins and gives a gravitational description of turbulence relevant to the fluid-gravity duality.

Black holes are fascinating objects. They play a fundamental role in a plethora of energetic phenomena in our universe, for example as the engines of active galactic nuclei, X-ray binaries, and possibly even as regulators of galactic structure. In addition, they have become central tools in the study of field theories through the framework of holography [1]. This includes attempts to understand superfluidity, superconductivity and quark-gluon plasmas obtained in energetic collisions (see e.g. [2,3]). One particularly exciting connection inspired by holography is the “fluid-gravity” duality, which indicates the dynamics of black holes in asymptotically anti-deSitter (AAdS) spacetimes in d + 1 dimensions can be mapped to the physics described by conformal fluids governed by viscous, relativistic hydrodynamics in d dimensions [5,6]. This opens the door to search for particular behavior known to exist on one side of the duality on the other. For instance, this duality has motivated studies showing that particular gravitational scenarios can become turbulent when their fluid counterparts have high Reynolds numbers [7–9]. Additionally, concepts in hydrodynamics, such as entropy, have geometric counterparts related to curvature quantities [7]. This duality can also shed light on poorly understood phenomena from a new perspective. Analyzing turbulence from an intrinsically gravitational point of view is thus an exciting prospect.

In this work we develop a method to do precisely this and consider realistic, asymptotically flat black holes. Our analysis describes how gravitational turbulence is mediated by a parametric instability in the gravitational field—which does not require the “confining properties” of asymptotically AdS spacetimes—and motivates the definition of a gravitational Reynolds number. We first review general properties of turbulent flows, salient features of the fluid-gravity duality, and parametric instability.

Hydrodynamic turbulence. Turbulence is a ubiquitous property of fluid flows with sufficiently high Reynolds number (\( \text{Re} \equiv \rho v \lambda / \eta > 1 \)) [10,11]. Here \( v \) and \( \lambda \) refer to the typical velocity and wavelength of characteristic modes of the solution, and \( \rho, \eta \) the fluid density and viscosity. At high \( \text{Re} \), nonlinear interactions prevail over dissipation due to viscosity, and chaotic behavior ensues. Turbulence displays several features: (i) an energy cascade (which can be towards higher frequencies in 3-spatial dimensions or lower ones in 2-spatial dimensions), (ii) an exponential growth—possibly transitory—of additional modes in the solution and (iii) a breaking of initial symmetries of the flow, which are only recovered in a statistical sense at later times. Further, in the absence of a driving force, global norms of the solution display a transient power-law decay, and viscous losses then decrease \( \text{Re} \) until turbulence ends. Beyond these broad aspects, a full understanding of turbulence is missing. A promising new road of study has been furnished through the fluid-gravity duality, provided a purely gravitational model for turbulence is available. Here we develop such a model and uncover possibly astrophysical consequences.

Fluid-gravity duality and black holes in AAdS vs AF. The fluid-gravity duality indicates long-wavelength perturbations of black holes in AAdS spacetimes can be described by relativistic hydrodynamic equations (with an equation of state given by \( p = \rho / d \)) [5]. In addition to connecting known hydrodynamic and gravitational effects, such as loss of energy through the black hole horizon to viscous dissipation, the duality can reveal new phenomena. The presence of turbulence in hydrodynamics indicates that a similar behavior appears in perturbed AAdS black holes, and this expectation has been confirmed by simulations of the gravitational side of the problem [5] which are direct counterparts of those in the hydrodynamical front [2,9]. Nevertheless, an analytical understanding of what mediates turbulence in gravity is an open question, as well as whether such striking behavior can take place in the realistic case of asymptotically flat (AF) spacetimes.

In considering these questions we recall the differences in how these two classes of spacetimes relate to hydrodynamics. Regardless of the class considered, a
parametric instability. However, only AAdS has a unique surface, lying at infinity, where the correspondence can be defined unambiguously. In both classes, perturbed black holes have a spectrum of free, damped oscillation modes known as quasinormal modes (QNMs, see e.g. [14, 15]). Black holes in AAdS only lose energy through the event horizon (as its boundary acts as a confining box), while energy in AF spacetimes can also be lost to infinity. Consequently, QNMs decay considerably more slowly in the AAdS case. From the hydrodynamic view, a slow decay of QNMs implies low viscosity and a correspondingly higher Reynolds number [9]. In what follows, we show that this slow decay is key for generating turbulent behavior, and how it might arise in the AF case. By doing so, we provide the first gravitational description of a turbulent mechanism acting in realistic black hole spacetimes.

**Damped parametric oscillator.** The parametric instability in black holes described below is analogous to the simple parametric oscillator. A parametrically driven oscillator can be described by the equation

$$\ddot{q} + \gamma \dot{q} + \omega^2 [1 + 2f(t)] q = 0,$$

where $\omega$ is the intrinsic harmonic frequency, $\gamma$ is a weak damping coefficient ($\gamma \ll \omega$) and $f(t)$ characterizes the parametric driving. The solution to this equation is bounded in time, except when $f(t)$ oscillates at approximately twice the intrinsic frequency: $f(t) = f_0 \cos \omega' t$, $\omega' \approx 2\omega$. In this case the time dependence of the solution is described by $e^{\Omega t}$, with the rate $2\Omega \approx \omega \sqrt{f_0^2 - \omega^{-4}[\omega^2 - (\omega')^2]^2} - \gamma$. When $\omega'$ is close to $2\omega$, a small parametric driving amplitude $f_0$ will be able to excite a growing solution, which is referred to as a parametric instability. For a given value of the damping coefficient $\gamma$, there is a critical relation that $f_0$ and $\omega$ satisfy at the separatix between growth or decay. This is related to the critical gravitational Reynolds number for the onset of turbulent behavior in perturbed black holes.

**Perturbed black holes in AF scenarios and turbulence.** In 4 dimensions, a stationary AF black hole is characterized by its mass $M$ and spin parameter $a$, which has a maximum value of $a/M = 1$. When $a/M \approx 1$ or $\epsilon \equiv 1 - a/M \ll 1$, there exists a family of quasinormal modes with a small damping rate proportional to $\sqrt{\epsilon}$ (referred as zero-damping-modes or ZDMs) [16, 17]. These modes have time dependence $e^{\omega_{l,m,n} t}$, with

$$\omega_{l,m,n} \equiv \omega_R - i\omega_I \approx \frac{m}{2} - \frac{\delta \sqrt{\epsilon}}{\sqrt{2}} - \frac{i}{2} \left( n + \frac{1}{2} \right) \sqrt{\frac{\epsilon}{2}},$$

(with $l, m, n$ denoting the angular, azimuthal and over-tone numbers respectively, and $\delta$ a function of $l, m$ and the spin-weight of the perturbation considered, see Supplemental Material). Consider as an example a black hole perturbed by a small mass falling towards the event horizon. This excites some of the ZDMs to a characteristic amplitude $h_0$. Once a particular ZDM is excited, at linear order its amplitude decays exponentially with a rate $\propto \sqrt{\epsilon}$ (in hydrodynamical terms this decay corresponds to laminar flow). However, nonlinear coupling between modes introduces a competing energy transfer between modes at a rate dependent on $h_0$. As we decrease $\epsilon$, the mode-mode coupling mechanism may overcome decay, even pumping up modes that are not initially excited, regardless of how weak the initial perturbation is. This is analogous to the onset of turbulence at high Re.

**Formalism.** As we go beyond linear perturbation theory, the spacetime metric $g$ can be expanded as $g = g_B + h^{(1)} + h^{(2)} + \ldots$, where $g_B$ is the background Kerr metric and $h^{(n)}$ is the $n$th order perturbation. We are interested in how an initial ZDM metric perturbation $h^{(1)}$ might trigger other modes through parametric resonance. One way to analyze the problem is to take $g_B = g_B + h^{(1)}$ as a dynamical background metric and study the evolution of $h^{(2)}$ on it. To avoid delicate gauge issues for the higher order metric perturbations, we adopt a simpler version of this approach, solving the evolution of a massless scalar field in the dynamical background $g_B$. This field obeys the wave equation

$$\Box g \Phi = 0,$$

and we bear in mind that $\Phi$ plays a role analogous to $h^{(2)}$. Since $\Box g \Phi$ is gauge invariant, our results concerning the parametric instability are gauge invariant.

The first-order perturbation $h^{(1)}$ corresponding to a quasinormal mode with index $(l, m, n)$ is

$$h_{lmn}^{(1)} = 2h_0 \Re [Z_{\mu\nu}(r, \theta) e^{-i\omega_{l,m} t + im \phi}],$$

where $h_0(t) = h_0 e^{-\omega I t}$. As we perturb the background metric $g_B$ to $g_B + h^{(1)}$, $\Phi$ obeys,

$$\Box g \Phi = \left[ \Box g + \frac{1}{\Sigma} \mathcal{H}(h^{(1)}) \right] \Phi.$$  

Here $\Sigma \equiv r^2 + a^2 \cos^2 \theta$ and $\mathcal{H}(.)$ is a time-dependent operator linear in its argument. The time dependence of $\mathcal{H}$ is crucial in triggering the parametric instability, which occurs when the temporal and azimuthal frequencies of the parent $h^{(1)}$ match the daughter mode $\Phi$. For rapidly-spinning Kerr black holes, this occurs when the daughter mode satisfies $m' = m/2$, as Eq. guarantees that $\omega_R \approx \omega_R/2$ as well. We make the ansatz

$$\Phi_{lm'n'}(x^\mu) = \left[ g_j(t) e^{(-1/2)\omega_{l,n} t + 2t - (-1)^j i m' \Phi} \right] e^{-\omega I t},$$

(summing over $j = 1, 2$), with $g_1, g_2$ characterizing the time dependence and $Y_{lm'n'}(r, \theta)$ the perturbed wave function. The equations of motion determining $g_1, g_2$ are closely related to the parametric instability previously
discussed. The solution to these equations are given by \( g_j = A_j e^{\alpha(t) dt'} \) with

\[
\alpha = \pm \sqrt{|H h_0(t)/Q m'|^2 - (\omega'_R - \omega_R/2)^2},
\]

where \( H \) has the physical meaning of mode-mode coupling strength and \( Q \) gives the susceptibility of the wave equation to a perturbation of the mode frequency. At leading order, \( Q \) is independent of \( m \). An exponential growth in \( \Phi \) will occur if \( \Omega \equiv \alpha(t) - \omega'_I > 0 \), i.e. when

\[
h_0(t)/(m' \omega'_f) - |Q/H| \sqrt{(\omega'_R - \omega_R/2)^2 / \omega'_f^2 + 1} > 0.
\]

We emphasize that given \( m' = m/2 \), both \( \omega'_R - \omega_R/2 \) and \( \omega'_R \) can be read off from Eq. (2), and both are \( \propto \sqrt{\gamma} \). We choose to normalize the radial wave function of the ZDMs such that \( |H/Q| \) is \( \epsilon \) independent — in other words, the effect of mode-mode coupling stays constant for varying black hole spins \( \eta \). These properties are useful in defining and interpreting the gravitational Reynolds number.

**Turbulent Black Holes.** Based on the above analysis, consider an initial ZDM mode with \( m = 2m' \) and amplitude \( h_0 \); as we increase \( h_0 \), all the secondary ZDMs with azimuthal quantum number \( m' \) satisfying Eq. (7) are parametrically excited. As these daughter modes grow, energy flows from the parent mode to the daughter modes, and the parent mode experiences back reaction due to the mode coupling. Ignoring this back reaction, these secondary modes grow as long as Eq. (7) holds, but in a realistic situation parametric growth terminates when the amplitudes of the parent mode and the secondary modes become comparable requiring a fully nonlinear treatment (or numerical study, e.g. [20]). The gravitational parametric instability displays an inverse cascade, as energy flows from modes with high azimuthal frequencies to modes with lower azimuthal frequencies, and from higher to lower temporal frequencies. An initial azimuthal mode \( m \) generates a series of of modes with azimuthal number \( m/2^p \) after \( p \) generations. This is similar to the inverse energy cascade in 2 + 1-dimensional turbulent fluids. Since modes with the same \( m' \) but high \( l \) can also be excited, there is also a direct transfer of energy towards higher overall angular frequencies.

From the criteria in Eq. (7) we define a gravitational Reynolds number \( \text{Re}_g \), taking \( m = 2m' \), and with \( \omega'_I \) chosen to be the lowest possible decay rate of all the ZDMs, \( \gamma_{\eta} = \sqrt{\epsilon/8} \). This gives

\[
\text{Re}_g \equiv h_0/(m \gamma_{\eta}).
\]

For a mode having \( \text{Re}_g \) below some critical value given in Eq. (7), no growth is expected, and the mode so the mode behaves in a “laminar” manner, decaying normally. For larger values of \( \text{Re}_g \), turbulent behavior ensues, driving growing modes and a richer angular structure. Once \( \text{Re}_g \) decreases below the critical value for a given mode, that mode again decays exponentially. Notice that the natural identifications \{ \eta/\rho \leftrightarrow \gamma_{\eta}, L \leftrightarrow 1/m, \nu \leftrightarrow h_0 \} gives \( \text{Re}_g \leftrightarrow \text{Re} \). Our definition arises from the criteria for the onset of instability, and it agrees with the one proposed in [21] motivated through the fluid-gravity duality.

Table 1 presents a list of numerical values of the critical \( \text{Re}_g \), beyond which the parametric instability for different driving and secondary modes will be turned on. We consider only the lowest overtone modes, \( n = n' = 0 \). We can see that for fixed \( \epsilon \) and \( m \), the critical \( \text{Re}_g \) asymptotes to a constant value at for high \( l \) modes. One may argue that this means modes with arbitrarily high \( l \) are all excited. However, as discussed in Yang et al. [18] there is a minimum, critical \( \epsilon \), beyond which the required phase-matching condition gradually fails to hold. A conservative estimate for this critical value is \( \epsilon_{\text{c}} \propto l^{-2} \). So for a given spin, there is a high angular frequency cut-off scale where the instability criteria is not satisfied and the energy-transfer stops.

![Table 1: Critical Re_g for different parent daughter modes with m = 2m'. These numbers are obtained in the ingoing radiation gauge using a value for |H/Q| evaluated at \( \epsilon = 10^{-5} \) (although they are expected to be \( \epsilon \)-independent, numerically we use a small \( \epsilon \) to reduce systematic error in the wave functions); extrapolation to lower spins and error in the matching of radial eigenfunctions are the dominant sources of error, which we estimate conservatively to be 10%. The parent mode of the 42 \( \rightarrow \) 11 driving has an imaginary value of \( \delta \), whereas the parents in the other two cases have real \( \delta \), which may explain the large critical Reynolds numbers in those cases. Note also that the 44 \( \rightarrow \) 22 driving is unique in the sense that both its parent and daughter mode have real \( \delta \).](image-url)

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Figure 1 illustrates the rich angular structure of the perturbed spacetime that arises due to the parametric instability, due to driving by the fundamental \( l = 2, m = 2, n = 0 \) QNM. We take for our fiducial example \( \epsilon = 2 \times 10^{-3} \) (\( \alpha/M = 0.998 \) [25] and \( h_0(t = 0) = (1/8)\sqrt{\gamma} \). This amplitude is motivated by the expected excitation following a large mass-ratio inspiral, such as can occur in supermassive binary black hole coalescence following galaxy mergers (see the Supplemental Material). Note that for such an \( h_0 \) the criteria for growth is independent of spin, so long as \( \epsilon \ll 1 \). In the fully gravitational case, we can expect a similar development of structure in both the far-field radiation and curvature quantities on the event horizon. Figure 2 shows the amplitudes of the driving gravitational mode and excited scalar modes for the same fiducial example as in Fig. 1. Though we focus on driving by the dominant \((2, 2)\) mode, Table 1
FIG. 1: Snapshots of parametrically driven modes on a sphere of constant radius. We plot the dominant \((2, 2)\) spin \(s = -2\) spheroidal harmonic from a collective driving mode, plus the spin-0 \((l, 1)\) spheroidal harmonics for all of the growing scalar modes. Initially \(h_0(t = 0) = (1/8)\sqrt{\epsilon}\), and \(\epsilon = 2 \times 10^{-3}\) \((a = 0.998)\). In this case, modes with \(2 \leq l \leq 6\) are resonantly excited, with the higher \(l\) modes growing faster; the \(l > 6\) modes are not ZDMs for this \(a\). At \(t/M = 0\), the scalar modes are seeded with equal amplitude 10\% of the gravitational mode, and random phases. (a) Reference spin \(s = -2, (2, 2)\) spheroidal harmonic. (b) At time \(t/M = 0\), the seed modes are visible only where the gravitational mode is weak. (c) At time \(t/M = 16\), more angular structure has developed. (d) The harmonics at \(t/M = 32\) when the amplitude of the \((6, 1)\) scalar mode is closest to the \((2, 2)\) mode.

indicates that modes can be driven by a \((4, 4)\) mode for even smaller values of \(h_0\).

During the inverse cascade, modes with frequencies \(2^{-p} (p \in \mathbb{Z})\) times the parent mode frequency are excited by parametric resonance. However, in a fully turbulent fluid, energy transfers throughout the entire spectrum. One possible mechanism for this is in the gravitational case is through resonant excitation of additional modes, as occurs in systems of coupled oscillators. For example, two oscillators with frequencies \(\omega_1\) and \(\omega_2\), and amplitudes \(A_1(t)\) and \(A_2(t)\) can drive modes with frequencies \(\omega' = \omega_1 \pm \omega_2\), resulting in amplitudes proportional to \(A_1 A_2\). These three-mode interactions are not as strong as the parametric resonance but they can redistribute energy to both higher and lower frequencies, and fill in the gaps in the spectrum.

**Observational consequences.** This parametric instability discussed here relies on the system having a rapidly spinning black hole. Theoretical models arguing for such scenarios have been developed \cite{21, 22} and, crucially, there is observational evidence for highly spinning black holes \cite{23, 24}. The turbulent instability has several possible signatures:

- **Gravitational wave structure.** In gravitational wave observations from large mass-ratio mergers involving a rapidly-spinning black hole. Such scenarios can arise for instance in the inspiral of supermassive binary black holes following galaxy mergers. After merger the final black hole rings down by emitting gravitational waves primarily through the \((2, 2)\) mode. The magnitude of the initial perturbation \(h_0\) is proportional to the mass ratio, and so for smaller \(\mu\) values Eq. (7) is not satisfied, and distant observers should see mainly the \((2, 2)\) mode during the entire ringdown. However, if the initial perturbation is strong enough, modes with \(m = 1\) will be parametrically excited. The growth of the modes can allow them to overtake the amplitude of the \((2, 2)\) mode, in which case a treatment of the back reaction is needed. However, it is possible that a distant observer could measure a growing amplitude of some modes during the ringdown, a clear evidence of the instability, perhaps followed by complicated and turbulent behavior in the mode structure of the observed signal. Gravitational wave signals from supermassive binary black hole mergers would be detectable by pulsar timing arrays (e.g. \cite{26}) while stellar mass systems are the target of LIGO/VIRGO/KAGRA \cite{26, 28}.

- **Jitter in the black hole geometry.** The phenomena discussed indicates that the geometry of the spacetime around a black hole can acquire a rich multipolar structure as as a result of an object falling into a rapidly spinning black hole. This structure will impact the surrounding region and, in particular, may cause angular time-dependent shifts in the location of the inner most stable circular orbit. This, in turn, can affect emission lines of accreting material.

- **Chaos in black holes?** We have seen that turbulent behavior occurs in nearly extremal black holes, where the mode-mode coupling concentrates near the horizon. This may be related to the fact that the black hole singularity moves “closer” to the horizon for higher black hole mass-ratios.
spins, and the chaotic region near the singularity may be reflected in the existence of turbulence near the horizon. Recently it has been suggested that chaotic behavior in the vicinity of the black hole singularity may be responsible for the information loss in the black hole information paradox [39]. Our work indicates that complicated behavior arises in a transitory way outside of the event horizon if the gravitational Reynolds number is high enough.

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SUPPLEMENTAL MATERIAL

Perturbative formalism and metric reconstruction. We are interested in the perturbations of a Kerr black hole beyond linear order, with the spacetime metric is expanded as \( g = g_B + h^{(1)} + h^{(2)} + \ldots \), where \( g_B \) is the background metric for a hole of mass \( M \) and spin parameter \( a \), given by

\[
\begin{align*}
\upsilon \Sigma = & -\left( 1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4Mr\upsin^2 \theta}{\Sigma} dtd\phi + \frac{\Sigma}{\Delta} dr^2 \\
& + \Sigma d\theta^2 + \left( r^2 + a^2 + \frac{2Mr^2 \upsin^2 \theta}{\Sigma} \right) \upsin^2 \theta d\phi^2,
\end{align*}
\]

(9a)

\[
\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2,
\]

(9b)

and \( h^{(n)} \) is the \( n \)th order perturbation field with amplitude \( h_B^{(n)} \). Given the initial excitation of a ZDM with metric perturbation \( h^{(1)} \), we wish to understand how other modes evolve when we take into account mode-mode coupling. As a model for the problem of nonlinear mode coupling of the gravitational perturbations of a Kerr background, we consider the scalar wave equation \( \square g \Phi = 0 \), in the dynamical background metric \( g_B = g_B + h^{(1)} \). In this model, \( \Phi \) is analogous to a higher order metric perturbation \( h^{(2)} \), and we expect it to have the same qualitative behavior as the problem of interest. Under small gauge transformations \( x^\mu \to x^\mu + \xi^\mu \), \( \Phi \) has the simple transformation \( \Phi(x) \to \Phi(x) + \xi^\mu \partial_\mu \Phi(x) \) to the order we are concerned with. We have adopted geometric units with \( G = c = 1 \), and from here, we measure length in units of the black hole mass \( M \), setting \( M = 1 \).

The first-order perturbed metric \( h^{(1)} \) corresponding to a quasinormal mode can be obtained from the Weyl scalar \( \Psi_4 \) or \( \Psi_0 \) using a specific gauge choice. For a mode \((l, m, n)\) with amplitude \( h_0 \), \( \Psi_4 \) is given by [31]

\[
\Psi_4 = h_0 e^{-i\omega_{lmn}t + i\omega_{lmn}r} - 2S_{lmn}(\theta) - 2R_{lmn}(r) .
\]

(10)

Here \( -2S_{lmn}(\theta) \) is spin-weighted spheroidal harmonic function (with spin weight \( s = -2 \)) and \( -2R_{lmn}(r) \) is the radial wave function of the quasinormal mode. The ZDM mode frequency is approximately [17,19]

\[
\omega_{lmn} \equiv \omega_R - i\omega_I \approx \frac{m}{2} - \frac{\delta \sqrt{q}}{2} - i \left( n + \frac{1}{2} \right) \frac{\sqrt{q}}{2} ,
\]

(11)

where \( \epsilon = 1 - a, \delta^2 \equiv 7m^2/4 - (s+1/2)^2 - A_{lm} \) and \( A_{lm} \) is the eigenvalue of the spin-weighted spheroidal harmonic function.

We construct the spin-weighted spheroidal harmonics at leading order in small \( \epsilon \) using the power series expansion discussed by Leaver in [32]. In this limit they are real. For the radial wave functions, we must use expressions for \( R_{lmn} \) that are appropriate for nearly extremal Kerr black holes. We discuss them, their normalization, and the expected size of \( h_0 \) below, where we detail our method of constructing an appropriate inner product on the radial functions.

With knowledge of \( \Psi_4 \), the corresponding metric perturbation \( h^{(1)} \) can be reconstructed. In the Kerr spacetime, metric reconstruction is performed in one of two gauges — the ingoing and outgoing radiation gauges, as first carried out by Chrzanowski [33] and developed by others (see e.g. [34,39]). The details of the metric reconstruction procedure are relatively lengthy and tedious, requiring the application of Newman-Penrose formalism [40], and so we will only summarize the major steps here. We compute \( h^{(1)} \) in ingoing radiation gauge, using the standard Kinnersley null tetrad vectors \( l^\mu, n^\mu, m^\mu, n^\mu \), as discussed in [39]. The metric \( h^{(1)} \) is built by applying a tensor differential operator to a scalar \( \Psi_H \) known as the Hertz potential,

\[
h_{\mu\nu} = \left( -l_\mu l_\nu \right) \left( \delta + \alpha^* + 3\beta - \tau \right) \left( \delta + 4\beta + 3\tau \right) - m_\mu m_\nu \left( D + \rho + 3\epsilon \right) \left( D + 3\rho + 3\epsilon \right) + l_\mu m_\nu \left[ \left( D + \rho^* - \rho + \epsilon^* + 3\epsilon \right) \left( \delta + 4\beta + 3\tau \right) + \left( \delta + 3\beta - \alpha^* - \pi^* - \tau \right) \left( D + 3\rho + 4\epsilon \right) \right] \Psi_H + \text{c.c.}
\]

(12)

Here \( \delta = m^\mu \partial_\mu, \) and \( D = l^\mu \partial_\mu \) are directional derivatives; \( \alpha, \beta, \tau, \rho, \epsilon, \pi \) are the scalar Newman-Penrose spin coefficients for the Kerr spacetime and Kinnersley tetrad and can be found in e.g. [31]; and c.c. indicates the complex conjugate of the preceding expression. The Hertz potential which generates the desired \( \Psi_4 \) in ingoing radiation gauge is given by

\[
\Psi_H^{\text{HKG}} = \sum_{lm\omega} e^{-i\omega t - i\omega r} - 2S_{lm\omega}(\theta) - 2X_{lm\omega}(r) ,
\]

(13)
with a the radial function given by
\[ -2X_{lmw} = 8\frac{(-1)^m D^*_m}{D^*_{lmw} - 12iM\omega^2} + 144M^2\omega^2 - 2R_{lmw}. \]  
(14)

Here,
\[ D^*_{lmw} = \frac{\lambda^2 C^2(C - 2)^2 - 8\lambda C(5\lambda_C + 6)(a^2\omega^2 - am\omega)}{\lambda C(a^2\omega^2 + 144(a^2\omega^2 - am\omega)^2),} \]  
(15)

and \( \lambda_C = A_{lm} + s + |s| - 2am\omega + a^2\omega^2 \) is the angular separation constant used by Chandrasekhar [41], which differs from that originally used by Teukolsky [31].

To accomplish in our case by choosing the convention that initially a quasinormal mode with \( \Psi_4 \) is injected into the black hole spacetime. Because \( \partial_t \) and \( \partial_\varphi \) are the two Killing-fields of the Kerr spacetime, after the metric reconstruction \( h^{(1)} \) shares the same periodic time, on any fixed time slice they tend to diverge as \( r \) approaches infinity or as \( r \) approaches horizon. This means that any inner product diverges if we follow a generalized inner product. This is a subtle problem here, because while the quasinormal mode solutions decay in time, on any fixed time slice they tend to diverge as \( r \) asymptotes to infinity or as \( r \) approaches horizon. This means that any inner product diverges if we follow a standard definition, keeping \( r \) a real coordinate variable. Moreover, after factoring out the \( t, \varphi \) dependence out of the wave equations, we must require that \( \Sigma\Delta \) (the Teukolsky equation for scalars [31]) is self-adjoint with respect to this inner product. In other words, we require for any \( \chi(r, \theta, \phi) \), \( \xi(r, \theta, \phi) \) that
\[ \langle \chi | \Sigma\Delta \chi | \xi \rangle = \langle \Sigma\Delta \chi | \xi \rangle. \]  
(20)

The first problem can be solved by moving the integration contour into the complex \( r \) plane. A similar integration technique has previously been used by Leaver to evaluate the amount of quasinormal mode excitation by initial data and matter sources [32]. The second requirement can be satisfied if we define the inner product on spin \( s \) wave functions to be
\[ \langle \psi | \chi \rangle = \int_0^\pi sin \theta d\theta \int_C dr \Delta^s \psi \chi, \]  
(21)

where \( C \) is the complex contour for integration over \( r \). In this case, the radial wave function has two branch points.
at \( r = r_{\pm} \), and we choose the branch cuts to point vertically upward starting from the branch points, running into the upper complex plane. Our contour \( C \) begins in the upper complex plane to the right of the branch cut, \( \Re[z] > r_{+} \) and a large \( \Im[z] \). The contour runs down into the lower half plane parallel to the branch cut, wraps around \( r_{+} \), and returns to large \( \Im[z] \) with \( \Re[z] < r_{+} \), running between the branch cut from \( r_{+} \) and \( r_{-} \) and remaining close the the former branch cut. The asymptotic behavior of the radial functions guarantee that they decay exponentially at large \( z \) in the upper half plane, which in turn guarantees that the inner product on \( C \) is finite.

The radial Teukolsky wave function is obtained analytically in two separate regions in the limit of \( \epsilon \ll 1 \), as discussed in e.g. [19, 43]. In the inner region, where \( |r - r_{+}| \ll M \), the approximate wave function in Boyer-Lindquist coordinates is

\[
s R_{in} \propto (z) - 2i \tau / \sigma - s (1 - z) 2i \tau / \sigma - 2i \omega - s 2F_1(\alpha, \beta, \gamma, z),
\]

(22)

where \( z \equiv -(r - r_{+}) / (r_{+} - r_{-}) \), \( \alpha \equiv (r_{+} - r_{-}) / r_{+} \), \( \tau \equiv \omega - ma / (2r_{+}) \), \( \omega \equiv \omega r_{+} \), and

\[
\alpha = -2i \omega - s + 1/2 + i \delta, \quad \beta = -2i \omega - s + 1/2 - i \delta, \\
\gamma = 1 - s - 4i \tau / \sigma.
\]

(23)

On the other hand, when \( |r - r_{+}| \gg \sqrt{\epsilon} \), the asymptotic form of the radial Teukolsky equation allows an outer form

\[
s R_{out} = A e^{-i \omega x} e^{-1/2 - s + i \delta} \\
\times \int F_1(1/2 - s + i \delta + 2i \omega, 1 + 2i \delta, 2i \omega) \\
+ B(\delta \rightarrow -\delta)
\]

(24)

where \( x \equiv (r - r_{+}) / r_{+} \). Here \( (\delta \rightarrow -\delta) \) means that we assign a minus sign to all the factors of \( \delta \) in the previous function. The outgoing-wave boundary condition \( (|r| \gg M) \) forces the ratio between \( A \) and \( B \) to be

\[
A / B = e^{\pi \delta + 2i \delta \ln(2 \omega)} G(-2i \delta) G(1/2 + s + i \delta - 2i \omega) / G(2i \delta) G(1/2 - s - i \delta - 2i \omega).
\]

(25)

and the overall scale of \( A, B \) can be determined by comparing \( R_{in} \) and \( R_{out} \) in the matching zone: \( \sqrt{\epsilon} \ll |r - r_{+}| \ll M \). In order to evaluate the contour integration, the above solutions are analytically continued to the complex \( r \) plane, with the subtlety that there are two separate outer-solutions on each side of the \( r_{+} \)-branch cut. These two outer solutions still obey Eq. (25), but the absolute magnitudes of their \( A, B \) are different from each other, according to the matching procedure. The contour integration is performed in these two outer regions and one inner region, but the result is dominated by the integration in the inner zone. Physically this means that mode-mode coupling between ZDMs mainly happens near the horizon.

We fix the overall normalization of the radial wave function in a way such that the effective mode-mode coupling strength \( [H / Q] \) (with \( H \) and \( Q \) defined explicitly below) stays constant with varying \( \epsilon \) for nearly extremal black holes. More specifically, we require that

\[
-2R_{ln}(r) = |\epsilon^{1/4 - i\delta/2}| r^{3} e^{i\omega_{lmn} r_{+}}, \quad r \rightarrow \infty,
\]

(26)

where the tortoise coordinate \( r_{+} \) is defined through \( dr_{+} / dr = (r^{2} + a^{2}) / \Delta \), and we fix the integration constant so that to leading order in \( \epsilon \), \( r_{+} \rightarrow r + 2ln \epsilon \) asymptotically.

Numerical simulations (e.g. [14]) indicate that following an inspiral the amplitude of a driving mode \( h_{0} \sim \mu \) at the onset of ringdown for non-extremal spins, where \( \mu \) is the mass ratio of the binary. We expect this to hold in the nearly extremal case, and in our example we take \( \mu = 1/8 \) (a larger \( \mu \) would require an accounting of backreaction). Additionally, our normalization of \( R_{ln} \) contributes a scaling \( \sim \epsilon^{1/4} \) to the expected \( h_{0} \) of a driving mode with \( \delta^{2} > 0 \). It is possible that the details of mode excitation introduce further dependence on \( \epsilon \), and we can infer that this dependence does exist in the following way. We consider the emission from an extreme-mass-ratio-inspiral (EMRI) into a nearly-extremal host black hole. The peak emission is associated with the plunge phase, which occurs in the near-zone with some amplitude \( h_{\text{max}} \), which also sets the initial amplitude of the ringdown. In the near-zone, the ZDM wavefunctions depend on the overtone \( n \), and so it is unlikely that they are collectively excited (note this goes against expectations of a power-law ringdown from [19, 29, 45] who studied initial data mostly supported away from the horizon). This means that the individual ZDMs receive characteristic amplitudes \( \sim h_{\text{max}} \). A recent calculation of the energy flux from a near-zone orbit at fixed \( z = z_{0} \) about an extremal Kerr indicates that the amplitude of emission is proportional to \( \sqrt{z_{0}} \propto \epsilon^{1/4} \), and is suppressed [46]. This implies a similar dependence of \( h_{\text{max}} \) and motivates \( h_{0}(t = 0) \sim \mu \sqrt{\epsilon} \) in our example, but more investigation is needed. We note that if \( h_{0} \) is suppressed by larger powers of \( \epsilon \), the instability may not occur.

Inserting our solution ansatz, Eq. (18), into Eq. (17) and using our definition of the inner product Eq. (21) defines equations for \( g_{1}, g_{2} \) at leading order,

\[
-im'Qg_{1} = g_{2}Hh_{0}(t) - m'Q \omega_{R} - \omega_{R}^{2} g_{1}, \quad (27a)
\]

\[
im'Q^{*}g_{2} = g_{1}H^{*}h_{0}(t) - m'Q^{*} \omega_{R} - \omega_{R}^{2} g_{2}, \quad (27b)
\]

where

\[
Q \equiv \langle Y|Q|Y \rangle, \quad H \equiv \langle Y|H(Z_{\mu
u})|Y^{*} \rangle,
\]

\[
Q = \frac{\omega_{R} / 2 - i \omega_{l} / m'}{\Delta} \left[ \frac{(r^{2} + a^{2}) / \Delta - a^{2} \sin^{2} \theta}{\Delta} - 4Mar \right].
\]

(28)
Note that $H$ has no explicit dependence on $h_0(t)$. Further, $Q$ has no explicit dependence on $n'$ to leading order in $\sqrt{\epsilon}$. With the ansatz $g_j = A_j e^{\int (t')^j dt'} (j = 1, 2)$ (with $A_j$ to be determined) and the requirement of obtaining a non-trivial solution to the above system, one obtains Eq. [6] for $\alpha$ and the condition [7] for mode growth.

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[47] This normalization means that the amplitude $h_0$ of a particular mode excited by a physical process will have an additional dependence on $\epsilon$, see the Supplemental Material.
[48] Note that our perturbative analysis is about an isolated black hole, and so a corresponds to the spin parameter of the final black hole in the case of a binary merger.