ON K3 SURFACES WHICH DOMINATE KUMMER SURFACES

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Abstract. We study isogeny relations between K3 surfaces and Kummer surfaces. Specifically, we prove a Torelli-type theorem for the existence of rational maps from K3 surfaces to Kummer surfaces, and a Kummer sandwich theorem for K3 surfaces with Shioda-Inose structure.

1. Introduction

In the present paper we study rational maps between K3 surfaces in terms of their periods. Let X be a complex algebraic K3 surface and $T_X$ be the transcendental lattice of X, which is endowed with a natural Hodge structure. For a natural number $n > 0$ let $T_X(n)$ be the lattice obtained by multiplying the quadratic form on $T_X$ by $n$. In [7] Shafarevich posed the following question.

Problem 1.1 ([7] Question 1.1). Let X and Y be complex algebraic K3 surfaces. Is it true that there exists a dominant rational map $X \rightarrow Y$ if and only if there exists a Hodge isometry $T_X \otimes \mathbb{Q} \simeq T_Y(n) \otimes \mathbb{Q}$ for some natural number $n$?

Shafarevich’s question is a variation of a Torelli-type problem. It proposes to consider the $\mathbb{Q}$-Hodge structures $T_X \otimes \mathbb{Q}$ (up to scaling) for the existence of rational maps. A recent paper of Chen [1] implies that the answer is in general negative. On the other hand, the problem has been solved affirmatively in certain cases: for K3 surfaces X with Picard number $\rho(X) = 20$ (“singular K3 surfaces”) by Inose and Shioda [9, 2] already before [7]; for K3 surfaces X with $\rho(X) = 19$ by Nikulin-Shafarevich [7]; Nikulin [7] studied rational maps obtained as compositions of double coverings. When both X and Y are Kummer surfaces, Problem 1.1 is obviously true by the corresponding property of Abelian surfaces. The first purpose of this paper is to answer Problem 1.1 affirmatively when the target Y is a Kummer surface.

Theorem 1.2. Let X and Y be complex algebraic K3 surfaces. Assume that Y is dominated by some Kummer surface; e.g., Y admits a Shioda-Inose structure or Y itself is a Kummer surface. Then there exists a dominant rational map $X \rightarrow Y$ if and only if there exists a Hodge isometry $T_X \otimes \mathbb{Q} \simeq T_Y(n) \otimes \mathbb{Q}$ for some natural number $n$.

In order to produce a desired map $X \rightarrow Y$, we will compose the following three types of rational maps: (1) double coverings, (2) rational maps between Kummer surfaces induced by isogenies of Abelian surfaces, and (3) multiplication maps from...
elliptic $K3$ surfaces to the associated Jacobian fibrations. The first two have been also used in [9], [2], and [7]. A new ingredient of this paper is a systematic use of the third type of rational maps. In the course of the proof, we shall characterize those $K3$ surfaces which dominate Kummer surfaces by their Hodge structures.

The approach of Inose and Shioda for Problem 1.1 was to use a Kummer sandwich theorem, which roughly says that a singular $K3$ surface is two-isogenous to a Kummer surface. Recently the Kummer sandwich theorem has been extended to a larger class of $K3$ surfaces by Shioda [8] and has found some arithmetic applications. The $K3$ surfaces studied in [8] are characterized by the existence of Shioda-Inose correspondences with products of elliptic curves. The second purpose of this paper is to prove a Kummer sandwich theorem for all complex algebraic $K3$ surfaces with Shioda-Inose structure (Theorem 2.5). It is independent of Theorem 1.2 and shows a more precise isogeny relation between Kummer surfaces and $K3$ surfaces with Shioda-Inose structure.

Throughout this paper, the varieties are assumed to be complex algebraic. The transcendental lattice of an algebraic surface $X$ will be denoted by $T_X$. By $U$ we denote the rank 2 even indefinite unimodular lattice. By $E_8$ we denote the rank 8 even negative-definite unimodular lattice. For a lattice $L = (L, (\cdot, \cdot)_L)$ and a natural number $n$, we denote by $L(n)$ the scaled lattice $(L, n(\cdot, \cdot)_L)$.

2. Kummer sandwich theorem

Let $X$ be an algebraic $K3$ surface. Recall that a Nikulin involution of $X$ is an involution $\iota : X \to X$ which acts trivially on $H^{2,0}(X)$. A Nikulin involution of $X$ canonically corresponds to a double covering $X \dashrightarrow Y$ to another $K3$ surface $Y$. Indeed, if we have a double covering $\pi : X \dashrightarrow Y$, then the covering transformation of $\pi$ is a Nikulin involution of $X$. Conversely, for a Nikulin involution $\iota$ of $X$, the minimal resolution $Y = X/\langle \iota \rangle$ of the quotient surface is a $K3$ surface ([6]), and we have the rational quotient map $\pi : X \dashrightarrow Y$ of degree 2. The transcendental lattices $T_X$ and $T_Y$ are related by the chain of inclusions

\begin{equation}
2T_Y \subseteq \pi_* T_X = T_X(2) \subseteq T_Y,
\end{equation}

which preserves the quadratic forms and the Hodge structures.

Nikulin [6], [7] and Morrison [5] developed the lattice-theoretic aspect of Nikulin involution. Let us denote

\begin{align*}
\Lambda_0 &:= E_8(2) \oplus U^3, \\
\Lambda_1 &:= \frac{1}{2} E_8(2) \oplus U^3.
\end{align*}

We regard $\Lambda_0$ as a submodule of $\Lambda_1$ in a natural way. Then $\Lambda_1$ is the dual lattice of $\Lambda_0$. The following proposition reduces the construction of a Nikulin involution to a purely arithmetic problem.

Proposition 2.1 ([7] Section 2.1 and Lemma 2.2.4). Let $X$ be an algebraic $K3$ surface. Suppose that one is given a primitive embedding $T_X \subset \Lambda_0$ of lattices. Then there exists a Nikulin involution $\iota : X \to X$ such that if we denote $Y = X/\langle \iota \rangle$, then $T_Y$ is Hodge isometric to the lattice

\begin{equation}
T := (T_X \otimes \mathbb{Q} \cap \Lambda_1)(2),
\end{equation}

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where the Hodge structure of $T$ is induced from $T_X$. Conversely, if one has a rational map $X \dasharrow Y$ of degree 2 to a K3 surface $Y$, then there exists a primitive embedding $T_X \subset \Lambda_0$ such that $T_Y$ is Hodge isometric to the lattice $T$ defined by (2.4).

The Shioda-Inose structure is a special kind of Nikulin involution.

**Definition 2.2** ([5]). An algebraic K3 surface $X$ admits a Shioda-Inose structure if there exists a Kummer surface $Y = \text{Km}A$ and a rational map $\pi : X \dasharrow Y$ of degree 2 such that $\pi_*$ induces a Hodge isometry $T_X(2) \cong T_Y$.

There is a lattice-theoretic characterization of K3 surfaces admitting Shioda-Inose structures due to Morrison.

**Theorem 2.3** ([5], Theorem 6.3). An algebraic K3 surface $X$ admits a Shioda-Inose structure if and only if there exists a primitive embedding $T_X \hookrightarrow U^3$ of lattices.

Shioda [8], extending the work of Inose [2], proved a Kummer sandwich theorem for elliptic K3 surfaces with section and with two II*-fibers over an arbitrary algebraically closed field of characteristic $\neq 2, 3$. When the ground field is $\mathbb{C}$, one can characterize the K3 surfaces studied in [8] by the existence of Shioda-Inose structures such that the corresponding Abelian surfaces are products of elliptic curves. Here we shall derive in a transcendental way a Kummer sandwich theorem for all complex algebraic K3 surfaces with Shioda-Inose structure. We denote the Dynkin diagram of $E_8$ by

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  v1  v2  v3  v4  v5  v6  v7
    \downarrow
  v8
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We identify the $\mathbb{Z}$-modules underlying $E_8$ and $E_8(2)$ in a natural way and regard the above set $\{v_i\}_{i=1}^8$ as a basis of $E_8(2)$. Then we have $(v_i, v_j) = -4$ for $i = 1, \ldots, 8$, $(v_i, v_j) = 2$ if $v_i$ and $v_j$ are connected by an edge, and $(v_i, v_j) = 0$ otherwise. Let $\{e_i, f_j\}_{i=1}^3$ be the standard basis of $U^3$. We have $(e_i, e_j) = (f_i, f_j) = 0$ and $(e_i, f_j) = \delta_{ij}$. For $i = 1, 2, 3$, we define the vectors $l_i, m_i \in \Lambda_0 = E_8(2) \oplus U^3$ by

- $l_1 = -v_5 + v_7 + 2(e_1 + f_1)$,
- $m_1 = -v_4$,
- $l_2 = v_1 + v_8 + 2(e_2 + f_2)$,
- $m_2 = v_2$,
- $l_3 = v_7 + v_8 + 2(e_1 + e_2 + e_3 + f_3)$,
- $m_3 = v_6$.

and put $L := \langle l_1, m_1, l_2, m_2, l_3, m_3 \rangle$.

**Lemma 2.4.** The sublattice $L \subset \Lambda_0$ has the following properties:

1. $L \cong U(2)^3$.
2. $L \subset 2\Lambda_1$.
3. $L$ is a primitive sublattice of $\Lambda_0$.

**Proof.** We can extend the set $\{l_i, m_i\}_{i=1}^3$ to a $\mathbb{Z}$-basis of $\Lambda_0$ by adding the set of vectors $\{v_3, v_5, e_1, f_1, e_2, f_2, e_3, f_3\}$. Thus $L$ is primitive in $\Lambda_0$. The assertion (2) is obvious, and the assertion (1) is proved by direct calculations. $\square$
Theorem 2.5. Let $X$ be an algebraic K3 surface admitting a Shioda-Inose structure $X \rightarrow Y = \text{Km}$. Then there exists a Nikulin involution $\iota$ on $Y$ such that the minimal resolution of the quotient surface $Y/\langle \iota \rangle$ is isomorphic to $X$. In particular, one has the following sequence of rational maps of degree 2:

\[(2.5) \quad \text{Km} \rightarrow X \rightarrow \text{Km}.\]

Proof. By the definitions, we have the Hodge isometries

$T_Y \simeq T_A(2), \quad T_X \simeq T_A.$

Since $T_A$ is embedded into $H^2(A, \mathbb{Z}) \simeq U^3$ primitively, there exists a primitive embedding $\varphi : T_Y \hookrightarrow U(2)^3$. By composing $\varphi$ with an isometry $U(2)^3 \simeq L$, we obtain a primitive embedding $\psi : T_Y \hookrightarrow \Lambda_0$ such that $\psi(T_Y) \subset 2\Lambda_1$. We have

$\psi(T_Y) \otimes \mathbb{Q} \cap \Lambda_1 = \frac{1}{2} \psi(T_Y).$

By Proposition 2.1, there exists a Nikulin involution $\iota : Y \rightarrow Y$ such that for the minimal resolution $Z$ of $Y/\langle \iota \rangle$ the transcendental lattice $T_Z$ is Hodge isometric to

$\frac{1}{2} T_Y(2) \simeq \frac{1}{2} T_A(4) \simeq T_A \simeq T_X.$

Since a Hodge isometry $T_Z \simeq T_X$ can be extended to a Hodge isometry $H^2(Z, \mathbb{Z}) \simeq H^2(X, \mathbb{Z})$ (cf. [5], Corollary 2.10), we have $Z \simeq X$ by the Torelli theorem. □

The rational quotient map $\pi : \text{Km} \rightarrow X$ constructed in Theorem 2.5 induces a Hodge isometry $\pi^* : T_X(2) \rightarrow T_{\text{Km}}$. Thus a K3 surface $X$ with Shioda-Inose structure can be defined not only as a double cover of a Kummer surface Km but also as a double quotient of Km, which exhibits an isogeny relation between $X$ and Km. Unfortunately, as we rely on the Torelli theorem, our Kummer sandwich theorem is not explicit as in [2], [8], and our argument works only over $\mathbb{C}$.

K3 surfaces with Shioda-Inose structure are particular double covers of Kummer surfaces. Now, is it true in general that a double cover $X$ of a Kummer surface Km admits a double covering Km $\rightarrow X$ of the opposite direction, as do isogenies of elliptic curves? Here is a negative example.

Example 2.6. Let $A$ be an Abelian surface with $T_A \simeq U \oplus \langle 2 \rangle \oplus \langle -2 \rangle$ and $X$ be the K3 surface with $T_X$ Hodge isometric to $2T_A$. Then there exists a rational map $X \rightarrow \text{Km}$ of degree 2, but there does not exist a rational map Km $\rightarrow X$ of degree 2.

Proof. The existence of a double covering $X \rightarrow \text{Km}$ follows from Mehran’s criterion for double covers of Kummer surfaces ([4], Theorem 3.1). Suppose that we have a rational map Km $\rightarrow X$ of degree 2. By Proposition 2.1 there exists a primitive embedding $T_A(2) \hookrightarrow \Lambda_0$ such that

\[(2.6) \quad T_A(2) \otimes \mathbb{Q} \cap \Lambda_1 = T_A(2).\]

Via this embedding, we regard $T_A(2)$ as a primitive sublattice of $\Lambda_0$. Let $\pi : T_A(2) \rightarrow U^3$ be the orthogonal projection, which is injective by the condition (2.6). Let $M$ be the lattice $\pi(T_A(2))$ and $N$ be the primitive closure of $M$ in $U^3$. By the condition (2.6) again, the Abelian group $N/M$ has no 2-component. For an
even lattice \( L \), let \( L' \) be the dual lattice of \( L \), \( D_L = L'/L \) be the discriminant group of \( L \), and \( (D_L)_2 \) be the 2-component of \( D_L \). We see from the inclusions \( M \subset N \subset N' \subset M' \) that
\[
(D_M)_2 \simeq (D_N)_2 \simeq (D_{N'\cap U^3})_2.
\]
The second isomorphism follows from the fact that \( N \) is a primitive sublattice of the unimodular lattice \( U^3 \). In particular, the length of \( (D_M)_2 \) is less than or equal to 2. On the other hand, we have \( (v, w) \in 2\mathbb{Z} \) for every \( v, w \in M \). Thus we have \( \frac{1}{2} M \subset M' \), which is absurd. \( \square \)

Remark 2.7. It follows from \([7] \), Theorem 1.3, that for \( X \) and \( A \) as in Example 2.6, there nevertheless exists a rational map \( \text{Km}A \to X \) of degree \( 2^\mu \) for some \( \mu > 1 \).

### 3. Rational maps to Kummer surfaces

In this section we study rational maps from K3 surfaces to Kummer surfaces in general. We shall use the following.

**Proposition 3.1** ([3], Section 4). Let \( X \) and \( Y \) be algebraic K3 surfaces with \( \text{rk}(T_X) = \text{rk}(T_Y) \leq 9 \) such that there exists an embedding \( T_X \to T_Y \) of lattices preserving the periods. Then there exists a sequence \( X_1 = X, X_2, \ldots, X_n = Y \) of K3 surfaces such that \( X_{i+1} \) is isomorphic to the surface underlying the Jacobian fibration of an elliptic fibration \( \pi_i : X_i \to \mathbb{P}^1 \). In particular, for a line bundle \( L \in \text{Pic}(X) \) we have a rational map \( X_i \to X_{i+1} \) defined by \( x \mapsto \mathcal{O}_F(dx) \otimes L^{-1} \), where \( F \) is the \( \pi_i \)-fiber containing \( x \in X_i \) and \( d = (L.F) \).

We shall characterize K3 surfaces \( X \) dominating Kummer surfaces by the lattices \( T_X \).

**Proposition 3.2.** For an algebraic K3 surface \( X \) the following conditions are equivalent.

(i) There exists a dominant rational map \( X \to \text{Km}A \) to some Kummer surface \( \text{Km}A \).

(ii) There exists an embedding \( T_X \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q} \) of quadratic spaces.

(iii) There exists an embedding \( T_X \hookrightarrow U^3 \) of lattices.

**Proof.** (i) \( \Rightarrow \) (ii): A rational map \( f : X \to \text{Km}A \) of finite degree \( d \) induces a Hodge isometry
\[
f_* : T_X(d) \otimes \mathbb{Q} \isom T_{\text{Km}A} \otimes \mathbb{Q} \simeq T_A(2) \otimes \mathbb{Q}.
\]
Then the quadratic space \( T_X \otimes \mathbb{Q} \) is isometric to \( T_A(2d) \otimes \mathbb{Q} \) and thus is embedded into \( H^2(A, \mathbb{Q})(2d) \simeq U^3(2d) \otimes \mathbb{Q} \). By the property \( U^3(2d) \otimes \mathbb{Q} \simeq U^3 \otimes \mathbb{Q} \) of the lattice \( U \), we obtain an embedding \( T_X \otimes \mathbb{Q} \hookrightarrow U^3 \otimes \mathbb{Q} \) of quadratic spaces.

(ii) \( \Rightarrow \) (iii): Recall that an even lattice of rank \( r \) can be embedded (primitively) into \( U^r \). In particular, we may assume that \( \text{rk}(T_X) = 4 \) or 5. When \( \text{rk}(T_X) = 4 \), the condition (ii) is equivalent to the existence of an embedding \( U \otimes \mathbb{Q} \to T_X \otimes \mathbb{Q} \) by Witt’s theorem for \( (T_X \otimes \mathbb{Q})^\perp \cap U^3 \otimes \mathbb{Q} \). Thus we have an isotropic vector in \( T_X \). Let \( T \) be a maximal even overlattice of \( T_X \). A primitive isotropic vector \( v \in T \) induces an embedding \( U \to T \) because \( (v, T) = 0 \). Hence \( T \simeq U \oplus L \) for some rank 2 lattice \( L \) so that \( T \) can be embedded into \( U^3 \). When \( \text{rk}(T_X) = 5 \), as in the case...
of \( \text{rk}(T_X) = 4 \), the condition (ii) is equivalent to the existence of a rank 2 totally isotropic sublattice of \( T_X \). Then every maximal even overlattice of \( T_X \) is of the form \( T = U^2 \oplus L \), \( \text{rk}(L) = 1 \), and thus can be embedded into \( U^3 \).

(iii) \( \Rightarrow \) (i): We fix an embedding \( T_X \subset U^3 \). Let \( T \) be the primitive closure of \( T_X \) in \( U^3 \) and endow \( T \) with the Hodge structure induced from \( T_X \). We regard \( T \) as a primitive sublattice of \( U^3 \oplus E^2_8 \). By the surjectivity of the period map, there exists a \( K3 \) surface \( Y \) with \( T_Y \) Hodge isometric to \( T \). We have an embedding \( T_X \hookrightarrow T_Y \) of finite index which preserves the periods. It follows from Proposition 3.1 that there exists a dominant rational map \( X \dashrightarrow Y \). Since the lattice \( T_Y \) can be embedded primitively into \( U^3 \), the \( K3 \) surface \( Y \) admits a Shioda-Inose structure \( Y \dashrightarrow \text{Km}A \) by Theorem 2.3. □

Proposition 3.2 is analogous to Theorem 2.3, replacing Shioda-Inose structures by general rational maps corresponds to replacing primitive embeddings of lattices by embeddings of rational quadratic spaces.

An Abelian surface \( A \) is a product of two elliptic curves if and only if \( T_A \) can be embedded primitively into \( U^2 \). Hence by a similar argument as in the above proof we have the following variant of Proposition 3.2.

**Proposition 3.3.** For an algebraic \( K3 \) surface \( X \) the following conditions are equivalent.

1. There exists a dominant rational map \( X \dashrightarrow \text{Km}A \) to some Kummer surface \( \text{Km}A \), where \( A \) is a product of two elliptic curves.
2. There exists an embedding \( T_X \otimes \mathbb{Q} \hookrightarrow U^2 \otimes \mathbb{Q} \) of quadratic spaces.
3. There exists an embedding \( T_X \hookrightarrow U^2 \) of lattices.

By using Proposition 3.2 we deduce the next theorem, from which Theorem 1.2 follows immediately.

**Theorem 3.4.** Let \( X \) be an algebraic \( K3 \) surface and \( \text{Km}B \) be an algebraic Kummer surface. Then there exists a dominant rational map \( X \dashrightarrow \text{Km}B \) if and only if there exists a Hodge isometry \( T_X \otimes \mathbb{Q} \simeq T_A(n) \otimes \mathbb{Q} \) for some natural number \( n \).

**Proof.** It suffices to prove the “if” part. Assume the existence of a Hodge isometry \( T_X \otimes \mathbb{Q} \simeq T_A(n) \otimes \mathbb{Q} \). As \( T_A \otimes \mathbb{Q} \) is embedded into \( U^3 \otimes \mathbb{Q} \), by Proposition 3.2 we can find a Kummer surface \( \text{Km}B \) and a finite rational map \( X \dashrightarrow \text{Km}B \). Since we have a Hodge isometry \( T_B(m) \otimes \mathbb{Q} \simeq T_A \otimes \mathbb{Q} \) for some natural number \( m \), the Abelian surface \( B \) is isogenous to the Abelian surface \( A \). Thus there exists a dominant rational map \( \text{Km}B \dashrightarrow \text{Km}A \). □

**Corollary 3.5.** Let \( X \) be an algebraic \( K3 \) surface and \( A \) be an Abelian surface. If we have a dominant rational map \( A \dashrightarrow X \), then there exists a dominant rational map \( X \dashrightarrow \text{Km}A \).

**Corollary 3.6.** Let \( X \) and \( Y \) be algebraic \( K3 \) surfaces dominated by some Kummer surfaces. Then there exists a dominant rational map \( X \dashrightarrow Y \) if and only if there exists a dominant rational map \( Y \dashrightarrow X \).

Thus, as in Inose’s paper [2], we are able to define a notion of isogeny for those \( K3 \) surfaces dominated by Kummer surfaces by the existence of a rational map.
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