COVARIANT ALGEBRAIC CALCULATION OF THE ONE-LOOP EFFECTIVE POTENTIAL IN NON-ABELIAN GAUGE THEORY AND A NEW APPROACH TO STABILITY PROBLEM

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Abstract

We use our recently proposed algebraic approach for calculating the heat kernel associated with the Laplace operator to calculate the one-loop effective action in the non-Abelian gauge theory. We consider the most general case of arbitrary space-time dimension, arbitrary compact simple gauge group and arbitrary matter and assume a covariantly constant gauge field strength of the most general form, having many independent color and space-time invariants (Savvidy type chromomagnetic vacuum) and covariantly constant scalar fields as a background. The explicit formulas for all the needed heat kernels and zeta-functions are obtained. We propose a new method to study the vacuum stability and show that the background field configurations with covariantly constant chromomagnetic fields can be stable only in the case when more than one independent field invariants are present and the values of these invariants differ not greatly from each other. The role of space-time dimension is analyzed in this connection and it is shown that this is possible only in space-times with dimensions not less than five \( d \geq 5 \).

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1 Introduction

It is well known at present that the seamy side of the asymptotic freedom in non-Abelian gauge theories is the fact that the effective interaction coupling becomes strong at long distances. This leads to the confinement of color, viz. the color objects (quarks and gluons) are ‘confined’ inside hadrons and can not be observed in real physical states \( [1, 2] \). Although the perturbation theory proved to be a very successful tool at high energies in asymptotically free theories, it is not valid any more at low energies (or long distances). Here an unknown nonperturbative mechanism takes place that ensures the confinement. However, the property of confinement is not understood well in field theory. It was proven rigorously only in some low-dimensional and lattice models \([2]\) and remains still a hypothesis confirmed by the experiments.

The main point of the confinement problem is the problem of investigation of the vacuum structure of the theory. The vacuum seems to be far more complicated than the perturbation theory admits. For the investigation of the vacuum problem various approaches were proposed. In the phenomenological approach the general parameters of the vacuum are estimated: non-zero vacuum expectation values of the local bilinear forms of the fields operators (gluon and quark condensates) and, consequently, the negative vacuum energy density, spontaneous breakdown of the classical symmetries, anomalies etc. \([3]\). The effects of instantons are investigated in \([4, 5]\).

Another possible approach is to put forward some explicit simple model, which allows the analytical investigation. This was initiated first by Savvidy \([6]\) who showed that in Yang-Mills \( SU(2) \) model the perturbative empty vacuum is unstable under the creation of a constant chromomagnetic field that leads to the negative vacuum energy. Such a calculation goes back to the Landau’s \([7]\) and Schwinger’s \([8]\) investigations of the electron in a constant magnetic field. Thus the space filled with a constant homogeneous chromomagnetic field can serve as a simple and visual model of the nonperturbative vacuum. The Savvidy’s result was significantly specified in further investigations \([9]\), where it was shown, in particular, that the Savvidy’s magneto-vacuum is still unstable. The real vacuum is likely to have a small domain structure with random constant chromomagnetic fields.

The natural tool for investigating the vacuum structure is the effective potential \([10]\), i.e. the low-energy limit of the effective action \([11]\). It is determined by general covariantly constant background fields. The calculation of the effective action in gauge theories is complicated by the gauge invariance. In defining the low-energy limit one has to factorize out the gauge degrees of freedom. That is why it is the covariantly constant background that should be used in defining the effective potential \([12, 13]\).

In his pioneering work \([6]\) Savvidy studied the simplest non-Abelian \( SU(2) \) gauge group and the only possible covariantly constant chromomagnetic field with only one nonvanishing color and space-time component. However, when investigating more complicated groups the covariantly constant background may have much more general structure with many independent color and space-time invariants. The main aim of present paper is to investigate the effective action for this general case of covariantly constant background and to analyze some new opportunities to ensure the stability of the vacuum that it provides.

We consider a Yang-Mills model with scalar and spinor matter fields (QCD, GUT models etc.) in flat \(d\)-dimensional space-time with Euclidean action of the form

\[
S = \int dx \left\{ -\frac{1}{2g^2} \text{tr}(F_{\mu\nu}^2) + i\bar{\psi}(\gamma^\mu \nabla_\mu + M(\varphi))\psi + \frac{1}{2} \varphi^T (-\Box) \varphi + V(\varphi) \right\} \tag{1}
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \) is the strength of the gauge fields taking the values in the Lie algebra of an arbitrary compact simple gauge group, \( g \) is the interaction coupling constant, \( \varphi = \{\varphi^A\} \) and \( \psi = \{\psi^i\} \) are the multiplets of real, for definiteness, scalar fields and the Dirac spinor ones, which belong to some, in general, different irreducible representations of the gauge group, \( M(\varphi) = \{M_k(\varphi)\} \) is a spinor mass matrix and \( V(\varphi) \) is a potential for scalar fields, \( \nabla_\mu = \partial_\mu + T(A_\mu) \) is the covariant derivative in the representation \( T \) and \( \Box = \nabla_\mu \nabla^\mu \).

As we will carry the calculations in a manifestly covariant way we will not need the explicit form of the covariant derivative. All the information we will use is contained in the commutators of the covariant derivatives

\[
[\nabla_\mu, \nabla_\nu] F_{\alpha\beta} = [F_{\mu\nu}, F_{\alpha\beta}], \quad [\nabla_\mu, \nabla_\nu] \psi = R_{\mu\nu} \psi, \quad [\nabla_\mu, \nabla_\nu] \varphi = \tilde{R}_{\mu\nu} \varphi \tag{2}
\]

where

\[
F_{\mu\nu} = F_{\mu\nu}^a C_a,
\]
\[ R_{\mu\nu} = T(F_{\mu\nu}) = F_{\mu\nu}^a T_a, \quad \tilde{R}_{\mu\nu} = \tilde{T}(F_{\mu\nu}) = F_{\mu\nu}^a \tilde{T}_a \]  
\( (3) \)

Here \( C_a = \{ C_{bc} \} \), with \( C_{bc}^a \) being the structure constants of the gauge group, are the generators of the gauge group in the adjoint representation and \( T_a \) and \( \tilde{T}_a \) are the generators in the representations realized by spinor and scalar fields respectively.

To avoid misunderstanding let us make some remarks about our notations. In this paper the symbol \( \text{tr} \) means the trace only over group indices, all other possible being left intact, the symbols \( \text{Sp} \) and \( \text{det} \) denote the trace and the determinant only over vector indices and the symbol \( \text{tr}_\gamma \) stands for the trace over spinor indices. The symbol \( \text{Tr} \) denotes the functional trace, viz. it means that not only the traces over all discrete indices should be taken but over the continuous (space-time coordinates) as well. The explicit meaning of this notation depends on the structure of the quality to which it is applied. In any case the functional trace is the trace over all present indices, including the continuous ones.

## 2 One-loop effective potential

From the technical point of view we are going to calculate the one-loop effective action in the situation when both the scalar background fields and the gauge background fields but not the spinor ones are present. The quantization of the model \( (1) \) in a general covariant gauge leads to the one-loop Euclidean effective action \( \Gamma_{(1)} \)

\[ \Gamma_{(1)} = \Gamma_{(1)YM} + \Gamma_{(1)mat} \]  
\( (4) \)

Here the contribution of the gauge fields (and ghosts) proper has the form

\[ \Gamma_{(1)YM} = \frac{1}{2} \text{Tr} \ln \Delta(\lambda)/\mu^2 - \text{Tr} \ln F(\lambda)/\mu^2 \]  
\( (5) \)

\[ \Delta(\lambda) = \Delta + \lambda H \]  
\( (6) \)

\[ \Delta^\mu_\nu = - \Box \delta^\mu_\nu - 2 F^\mu_\nu \]  
\( (7) \)

\[ H^\mu_\nu = \nabla^\mu \nabla_\nu \]  
\( (8) \)

where \( \Delta^\mu_\nu(\lambda) \) is the inverse propagator of gauge fields,

\[ F(\lambda) = \sqrt{(1 - \lambda)} F \]  
\( (9) \)

\[ F = - \Box \]  
\( (10) \)

is the inverse ghost propagator and \( \lambda \) is the gauge fixing parameter. Although the factor \( \sqrt{(1 - \lambda)} \) in Eq. \( (9) \) seems to be irrelevant, it ensures the gauge independence of the regularized effective action on the mass shell (see the proof below).

The contribution of the matter fields has the form

\[ \Gamma_{(1)mat} = - \text{Tr} \ln(\gamma^\mu \nabla_\mu + M(\phi))/\mu + \frac{1}{2} \text{Tr} \ln N/\mu^2 \]  
\( (11) \)

\[ N = - \Box + Q(\phi) \]  
\( (12) \)

where \( \phi \) is the background scalar field and the mass matrix of the scalar fields \( Q = \{ Q^A_B \} \) is of the form

\[ Q^A_B(\phi) = \frac{\partial^2}{\partial \phi^B \partial \phi^A} V(\phi) \]  
\( (13) \)

To calculate the effective potential in gauge theories, i.e. the low-energy limit of the effective action, one has to assume the background fields to be not simply constant but, more precisely, covariantly constant \( \nabla_\mu F_{\alpha\beta} = 0, \quad \nabla_\mu M = 0, \quad \nabla_\mu Q = 0 \)  
\( (14) \)

that means, in particular,

\[ [F_{\alpha\beta}, F_{\mu\nu}] = 0, \quad [R_{\alpha\beta}, M] = 0, \quad [\tilde{R}_{\alpha\beta}, Q] = 0 \]  
\( (15) \)
Further, it is assumed that the mass matrix of fermions $M$ does not contain the Dirac matrices or contains only even number of them, so that $[M, \gamma_\mu] = 0$. Then it is easy to show, that the contribution of fermions can be expressed in terms of the squared Dirac operator

$$\text{Tr} \ln(\gamma^\mu \nabla_\mu + M) = \frac{1}{2} \text{Tr} \ln K$$

where

$$K = (\gamma^\mu \nabla_\mu + M)(-\gamma^\nu \nabla_\nu + M) = -\Box - \frac{1}{2} \gamma^{\mu\nu} R_{\mu\nu} + M^2$$

and $\gamma_{\mu\nu} = \gamma_{[\mu} \gamma_{\nu]}$ are the generators of the orthogonal (in Euclidean case) group.

It is well known that the effective action does not depend on the gauge parameter $\lambda$ and on the parametrization of the quantum fields when the background fields lie on the mass shell, i.e. satisfy the classical equations of motion. It is obvious that the covariantly constant background satisfies the classical equations of motion, if the scalar fields satisfy additionally the condition of extremum

$$\frac{\partial V(\phi)}{\partial \phi} = 0$$

and, therefore, the covariantly constant background ensures the independence of the effective potential on the gauge.

It is not difficult to show this explicitly. Indeed, by differentiating $\Gamma_{(1)}$ with respect to $\lambda$ and using the Ward identities

$$\nabla_\mu \Delta^{-1} \mu_\nu(\lambda) = -\frac{1}{1-\lambda} \Box^{-1}(\nabla_\nu + J_\mu \Delta^{-1} \mu_\nu(\lambda))$$

$$\Delta^{-1} \mu_\nu(\lambda) \nabla_\nu = -\frac{1}{1-\lambda} \left(\nabla_\mu + \Delta^{-1} \mu_\nu(\lambda) J^\nu\right) \Box^{-1}$$

where

$$J_\mu = \nabla_\nu F_{\mu}$$

we get

$$\frac{\partial \Gamma_{(1)}}{\partial \lambda} = \frac{1}{2(1-\lambda)^2} \text{Tr} \left\{ \nabla_\mu J^\mu \Box^{-2} + J_\mu \Delta^{-1} \mu_\nu(\lambda) J^\nu \Box^{-2} \right\}$$

Hence it is obvious that on mass shell ($J = 0$) the effective potential really does not depend on the gauge

$$\left. \frac{\partial \Gamma_{(1)}}{\partial \lambda} \right|_{J=0} = 0$$

Generally speaking, this proof needs a substantiation since it is absolutely formal, as the expressions (5) and (22) for the effective action contain the ultraviolet divergences. More rigorously, one has to prove the gauge independence of the regularized effective action on mass shell. Using the $\zeta$-function regularization one can rewrite the effective action in the form

$$\Gamma_{(1)} = -\frac{1}{2} \zeta_{\text{tot}}(0)$$

$$\zeta'(0) = \left. \frac{d}{dp} \zeta(p) \right|_{p=0}$$

where

$$\zeta_{\text{tot}}(p) = \zeta_{YM}(p) + \zeta_{\text{mat}}(p)$$

is the total $\zeta$-function,

$$\zeta_{YM}(p) = \zeta_{\Delta(\lambda)}(p) - 2 \zeta_{F(\lambda)}(p)$$
is the Yang-Mills $\zeta$-function and

$$\zeta_{\text{mat}}(p) = -\zeta_K(p) + \zeta_N(p)$$  \hfill (28)

is the total matter fields one.

The $\zeta$-function of a differential operator $L$ is defined in terms of the heat kernel as usual \[15\]

$$\zeta_L(p) = \mu^{2p} \text{Tr} L^{-p} = \frac{\mu^{2p}}{\Gamma(p)} \int_0^\infty dt \ t^{p-1} \text{Tr} \exp(-tL)$$ \hfill (29)

The $\zeta$-functions \[29\] are analytic in the point $p = 0$ and, therefore, the expression \[24\] is finite and good defined. The matter fields $\zeta$-function does not depend on the gauge-fixing parameter $\lambda$ at all. Therefore, to prove the gauge independence of the effective action we have to check this for the Yang-Mills $\zeta$-function only.

Let us calculate the $\zeta$-functions of the operator $\Delta(\lambda)$ \[3\] on mass shell ($J_\mu = \nabla_\nu F^\nu_\mu = 0$). We have from Eq. \[6\]

$$\Delta(\lambda) = \Delta(1) - (1 - \lambda) H$$ \hfill (30)

Then it is easy to show that

$$\Delta_\lambda^\mu(1) H^\lambda_\nu = -J^\mu \nabla_\nu, \quad H^\mu_\lambda \Delta_\nu^\lambda(1) = -\nabla^\mu J^\nu$$ \hfill (31)

Therefore on mass shell (at $J_\mu = 0$) the operators $\Delta(1)$ and $H$ are orthogonal. Therefrom we obtain the heat kernel for the operator \[30\]

$$\exp(-t\Delta(\lambda)) = \exp(-t\Delta(1)) + \exp(t(1 - \lambda)H) - 1$$ \hfill (32)

Hence

$$\exp(-t\Delta(0)) = \exp(-t\Delta(1)) + \exp(tH) - 1$$ \hfill (33)

which allows to express the general heat kernel in terms of the heat kernel for 'minimal' ($\lambda = 0$) gauge operator $\Delta = \Delta(0)$ \[3\]

$$\exp(-t\Delta(\lambda)) = \exp(-t\Delta) + \exp(t(1 - \lambda)H) - \exp(tH)$$ \hfill (34)

At last making use of the relation

$$\text{Tr}(H)^n = \text{Tr}\Box^n$$ \hfill (35)

which follows obviously from the definition of the operator $H$ \[8\], we express the trace of the gauge heat kernel in terms of the minimal gauge heat kernel and the ghost one

$$\text{Tr} \exp(-t\Delta(\lambda)) = \text{Tr} \exp(-t\Delta) + \text{Tr} \{\exp(-t(1 - \lambda)F) - \exp(-tF)\}$$ \hfill (36)

Now by the definition of the $\zeta$-function \[23\] we get from Eqs. \[1\] and \[36\] the $\zeta$-functions for the gauge and ghost operators in general gauge

$$\zeta_{\Delta(\lambda)}(p) = \zeta_{\Delta}(p) + ((1 - \lambda)^{-p} - 1) \zeta_F(p)$$ \hfill (37)

$$\zeta_{F(\lambda)}(p) = (1 - \lambda)^{-p/2} \zeta_F(p)$$ \hfill (38)

Finally, by differentiating these $\zeta$-functions we obtain

$$\zeta'_{\Delta(\lambda)}(0) = \zeta'_\Delta(0) - \ln(1 - \lambda) \zeta_F(0)$$ \hfill (39)

$$\zeta'_{F(\lambda)}(0) = \zeta'_F(0) - \frac{1}{2} \ln(1 - \lambda) \zeta_F(0)$$ \hfill (40)

Therefore, the $\lambda$-depending terms in the Yang-Mills $\zeta$-function \[27\] and in the Yang-Mills effective action cancel exactly and we find

$$\zeta_{YM}(0) = \zeta'_\Delta(0) - 2 \zeta'_F(0)$$ \hfill (41)

$$\Gamma_{(1)YM} = -\frac{1}{2} \zeta_\Delta(0) + \zeta_F(0)$$ \hfill (42)

Thereby we have proven explicitly that the regularized effective action \[24\] does not depend on the gauge fixing parameter $\lambda$. Thus in what follows we choose the most convenient so called minimal (or diagonal) gauge by putting $\lambda = 0$. 5
3 Heat kernel on covariantly constant background

The operators $\Delta$ \ref{7} and $F$ \ref{10} are both of Laplace type

$$L = - \Box + P$$ \hfill (43)

where $\Box = \nabla_\mu^2$ and $P$ is a potential term (a matrix valued function). One can obtain the trace of the heat kernel for such operators in case of covariantly constant background using a very elegant algebraic theorem proven in \cite{13}.

Namely, it is proven in \cite{13} that for any nilpotent Lie algebra given by the commutation relations

$$[\nabla_\mu, \nabla_\nu] = R_{\mu\nu}, \quad [\nabla_\mu, R_{\alpha\beta}] = 0, \quad [R_{\mu\nu}, R_{\alpha\beta}] = 0$$ \hfill (44)

$$[\nabla_\mu, P] = 0, \quad [R_{\mu\nu}, P] = 0$$ \hfill (45)

the ‘operator heat kernel’ $\exp(-tL)$ for the ‘Laplace operator’ $L = - \Box + P$, with $\Box = \nabla_\mu^2$, can be presented in the form of an averaging over the Lie group with a Gaussian measure as follows \cite{13}

$$\exp(-tL) = (4\pi t)^{-d/2} \text{det} \left( \frac{t\hat{R}}{\sinh(t\hat{R})} \right)^{1/2} \exp(-tP) \times \int dk \exp \left\{ - \frac{1}{4t} k^\mu (t\hat{R} \coth(t\hat{R}))_{\mu\nu} k^\nu + k^\mu \nabla_\mu \right\}$$ \hfill (46)

Here and everywhere below in expressions of a similar nature the hat denotes a matrix with vector indices $\hat{R} = \{R_{\mu\nu}\}$. The analytical functions of a matrix like $\hat{R}$ are always assumed to be defined in terms of corresponding power series. In a general case when the matrix $\hat{R}$ also belongs to a representation $T$ of a Lie group, i.e. $R_{\mu\nu} = F^a_{\mu\nu} T_a$, the powers of this matrix should be understood as follows

$$\hat{R}^0 = 1, \quad \hat{R}^n = \{F^a_{\mu_1 \lambda_1} F^a_{\mu_2 \lambda_2} \cdots F^a_{\mu_n \lambda_n} \lambda_{n-1} T_{a_1} T_{a_2} \cdots T_{a_n}\}$$ \hfill (47)

The ‘potential term’ $P$ is, in general, also a matrix with representation indices (they can be not only the group ones but vector or spinor indices as well).

Acting now with the operator heat kernel \ref{46} on the $\delta$-function $\delta(x, x')$ and taking the coincidence limit $x = x'$ one obtains finally \cite{13}

$$\text{Tr} \exp(-tL) = (4\pi t)^{-d/2} \text{Tr} \left\{ \text{det} \left( \frac{t\hat{R}}{\sinh(t\hat{R})} \right)^{1/2} \exp(-tP) \right\}$$ \hfill (48)

This is the generalization of the famous Schwinger’s result for the Abelian $U(1)$ gauge field (QED) \cite{8}. As a matter of fact it is valid in much more general case of arbitrary semi-simple gauge group.

4 Contribution of gauge fields

Let us consider first the most essential contribution of gauge fields. Substituting the explicit forms of the minimal gauge operator \ref{7} and the ghost one \ref{10} we obtain from Eqs. \ref{48} and \ref{6} for the traces of the heat kernels appeared in previous section

$$\text{Tr} \exp(-t\Delta) = \int dx (4\pi t)^{-d/2} \text{tr} \left\{ \text{det} \left( \frac{t\hat{F}}{\sinh(t\hat{F})} \right)^{1/2} \text{Sp} \exp(2t\hat{F}) \right\}$$ \hfill (49)

$$\text{Tr} \exp(-tF) = \int dx (4\pi t)^{-d/2} \text{tr} \left( \text{det} \left( \frac{t\hat{F}}{\sinh(t\hat{F})} \right)^{1/2} \right)$$ \hfill (50)
where $\hat{F} = \{ F_{\mu\nu} \}$.

Let us now calculate the traces in formulae (ref(35)) and (50). The Eq. (ref(13)) means that the gauge fields take their values in the Cartan subalgebra, i.e. the nontrivial nonvanishing components of the gauge field exist only in the direction of the diagonal generators. The maximal number of independent fields is equal to the dimension of the Cartan subalgebra, i.e. the rank of the group $r$. Mention, first of all, that the generators of the Cartan subalgebra of the compact simple group in adjoint representation $C_a, (a = 1, \cdots , r)$ are the traceless antisymmetric (in real basis) commuting matrices (16)

$$[C_a, C_b] = 0, \quad a, b = 1, \cdots , r$$

Hence they can be diagonalized (in complex basis) simultaneously

$$C_a = \text{diag}(0, \cdots , 0, i\alpha_a^{(1)}, -i\alpha_a^{(1)}, \cdots , i\alpha_a^{(p)}, -i\alpha_a^{(p)})$$

where $\alpha^i$ are the positive roots of the algebra, $p = (n - r)/2$ is the number of positive roots and $n$ is the dimension of the group. The number of zeros on the diagonal of the generators of the Cartan subalgebra in adjoint representation equals the maximum number of commuting generators of the group, i.e. the rank of the group $r$.

Therefore, the heat kernels (49) and (50) can be rewritten in the form

$$\text{Tr} \exp(-t\Delta) = \int dx(4\pi t)^{-d/2} \left\{ rd + 2 \sum_{\alpha > 0} \det \left( \frac{t\hat{F}(\alpha)}{\sin(t\hat{F}(\alpha))} \right)^{1/2} \text{Sp cos}(2t\hat{F}(\alpha)) \right\}$$

$$\text{Tr} \exp(-t\hat{F}) = \int dx(4\pi t)^{-d/2} \left\{ r + 2 \sum_{\alpha > 0} \det \left( \frac{t\hat{F}(\alpha)}{\sin(t\hat{F}(\alpha))} \right)^{1/2} \right\}$$

where 2-forms

$$\hat{F}(\alpha) = \{ F_{\mu\nu}(\alpha) \}, \quad F_{\mu\nu}(\alpha) = F_{\mu\nu}^a \alpha_a$$

are introduced (not to be confused with the ghost operator $F(\lambda)$ (16) and the sums are to be taken over all positive roots.

To calculate the trace and the determinant over the vector indices we show first that for any analytic function $f(\hat{F})$ of a 2-form $\hat{F} = \{ F_{\mu\nu} \}$ with $f(0) \neq 0$ it takes place (14)

$$\text{Sp} f(\hat{F}) = (d - 2q) f(0) + \sum_{1 \leq j \leq q} (f(iH_j) + f(-iH_j))$$

$$\det f(\hat{F}) = (f(0))^{d-2q} \prod_{1 \leq j \leq q} f(iH_j) f(-iH_j)$$

where $H_i$ are the invariants of the 2-form and $q \leq [d/2]$ is the number of independent invariants.

The invariants $H_i$ are to be determined from the equations

$$\sum_{1 \leq i \leq q} H_i^{2k} = I_k, \quad (k = 1, 2, \cdots , [d/2])$$

or, equivalently, from an algebraic equation of power $[d/2]$

$$H^{2[d/2]} + c_1 H^{2([d/2]-1)} + \cdots + c_{[d/2]-1} H^2 + c_{[d/2]} = 0$$

where

$$I_k = \frac{1}{2} (-1)^k \text{Sp} \hat{F}^{2k}$$

are the basic invariants of the 2-form $\hat{F}$ and the coefficients $c_k$ have the form

$$c_k = \sum_{1 \leq j \leq k} (-1)^j \sum_{1 \leq k_1 \leq \cdots \leq k_j \leq k} \frac{1}{k_1 \cdots k_j} I_{k_1} \cdots I_{k_j}$$

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It is not difficult to show that the coefficients $c_k$ vanish identically for $k \geq \left\lfloor \frac{d}{2} \right\rfloor + 1$. Besides, in the case of Euclidean signature all invariants $H_i$ can be regarded to be positive.

By making use of these formulae, one can compute the trace and the determinant over vector indices in Eqs. (53) and (54) and obtain finally

$$\text{Tr} \exp(-t\Delta) = \int dx (4\pi t)^{-d/2} \times \left\{ r d + 2 \sum_{\alpha > 0, 1 \leq i \leq q} \prod_{1 \leq j \leq q} \left( \frac{tH_i(\alpha)}{\sinh(tH_i(\alpha))} \right) \left( d + 4 \sum_{1 \leq j \leq q} \sinh^2(tH_j(\alpha)) \right) \right\}$$

(62)

$$\text{Tr} \exp(-tF) = \int dx (4\pi t)^{-d/2} \left\{ r + 2 \sum_{\alpha > 0, 1 \leq i \leq q} \prod_{1 \leq j \leq q} \left( \frac{tH_i(\alpha)}{\sinh(tH_i(\alpha))} \right) \right\}$$

(63)

where $H_i(\alpha)$ are the invariants of the tensor $F_{\mu\nu}(\alpha)$ (55).

Hence the total $\zeta$-function for gauge fields (27) equals

$$\zeta_{YM}(p) = \int dx (4\pi)^{-d/2} \frac{\mu^{2p}}{\Gamma(p)} \int_0^\infty dt t^{-d/2-1} \times \left\{ r(d - 2) + 2 \sum_{\alpha > 0, 1 \leq i \leq q} \prod_{1 \leq j \leq q} \left( \frac{tH_i(\alpha)}{\sinh(tH_i(\alpha))} \right) \left( d - 2 + 4 \sum_{1 \leq j \leq q} \sinh^2(tH_j(\alpha)) \right) \right\}$$

(64)

Wherefrom it is immediately seen how the ghost fields effectively decrease the number of degrees of freedom of the gauge field $d \to (d - 2)$.

Further calculations are possible only for a particular gauge group, i.e. for a specific system of roots of the algebra.

5 Contribution of matter fields

Consider now the contribution of the matter fields to the one-loop effective potential (11). Using Eqs. (48) and (2) one can write down the heat kernels for the matter fields operators: the scalar (12) and the spinor (17) ones

$$\text{Tr} \exp(-tK) = \int dx (4\pi t)^{-d/2} \text{tr} \left\{ \exp(-tM^2) \det \left( \frac{t\hat{R}}{\sinh(t\hat{R})} \right) \right\}$$

(65)

$$\text{Tr} \exp(-tQ) = \int dx (4\pi t)^{-d/2} \text{tr} \left\{ \exp(-tQ) \det \left( \frac{t\hat{R}}{\sinh(t\hat{R})} \right) \right\}$$

(66)

As above the background fields lie in the Cartan subalgebra, and, therefore, the generators $T_a$ and $\hat{T}_a$ can be diagonalized simultaneously

$$T_a = \text{diag}(i\nu_a^{(1)}, \ldots, i\nu_a^{(D)}),$$

$$\hat{T}_a = \text{diag}(i\tilde{\nu}_a^{(1)}, \ldots, i\tilde{\nu}_a^{(D)}), \quad (a = 1, \ldots, r)$$

(67)

where $\nu$ and $\tilde{\nu}$ are the weights of the representations $T$ and $\hat{T}$, some of them being, in general, multiple or equal to zero [13].

By isolating in mass matrices the singlet contributions

$$M^2 = M^2_{(0)} + M^2_a T_a, \quad Q = Q_{(0)} + Q^a \hat{T}_a$$

(68)
and denoting

\[ M^2(\nu) = M_{\alpha}^2 \nu_\alpha, \quad Q(\tilde{\nu}) = Q^a \tilde{\nu}_a \]  \hspace{1cm} (69)

we get from this for the heat kernels

\[
\begin{align*}
\text{Tr} \exp(-tK) &= \int dx (4\pi t)^{-d/2} \exp(-tM^2_{(0)}) \\
&\times \left\{ R^2 \nu^2 + \sum_{\nu} d_\nu \exp(-itM^2(\nu)) \det \left( \frac{t\hat{F}(\nu)}{\sin(t\hat{F}(\nu))} \right)^{1/2} \text{tr}_\gamma \exp \left( \frac{i}{2} t\gamma^{\mu \nu} F_{\mu \nu}(\nu) \right) \right\} \\
\text{Tr} \exp(-tN) &= \int dx (4\pi t)^{-d/2} \exp(-tQ_{(0)}) \\
&\times \left\{ \tilde{R} + \sum_{\nu} d_\nu \exp(-itQ(\tilde{\nu})) \det \left( \frac{t\hat{F}(\tilde{\nu})}{\sin(t\hat{F}(\tilde{\nu}))} \right)^{1/2} \right\}
\end{align*}
\]  \hspace{1cm} (70)

\[
\text{Tr} \exp(-tK) = \int dx (4\pi t)^{-d/2} \exp(-tM^2_{(0)}) \\
&\times 2^{d/2} \left\{ R + \sum_{\nu} d_\nu \exp(-itM^2(\nu)) \prod_{1 \leq i \leq q} \left( tH_i(\nu) \coth(tH_i(\nu)) \right) \right\} \\
\text{Tr} \exp(-tN) &= \int dx (4\pi t)^{-d/2} \exp(-tQ_{(0)}) \\
&\times \left\{ \tilde{R} + \sum_{\nu} d_\nu \exp(-itQ(\tilde{\nu})) \prod_{1 \leq i \leq q} \left( \frac{tH_i(\tilde{\nu})}{\sinh(tH_i(\tilde{\nu}))} \right) \right\}
\]  \hspace{1cm} (71)

where it is denoted \( \hat{F}(\nu) = \{ F_{\mu \nu}^a \nu_\alpha \} \), \( \hat{F}(\tilde{\nu}) = \{ F_{\mu \nu}^a \tilde{\nu}_a \} \) and the summation is to be taken over all nonvanishing weights, \( d_\nu \) and \( d_\nu \) are the multiplicities of the weights and \( R \) and \( \tilde{R} \) are the numbers of zero weights, i.e. the multiplicities of the zero weights.

It is not difficult to get for the trace over spinor indices \[ \text{tr}_\gamma \exp \left\{ \frac{i}{2} t\gamma^{\mu \nu} F_{\mu \nu}(\nu) \right\} = 2^{d/2} \prod_{1 \leq i \leq q} \cosh(tH_i(\nu)) \]  \hspace{1cm} (72)

Taking into account Eqs. (53) and (72) the final result for the heat kernels of matter fields takes the form

\[
\begin{align*}
\text{Tr} \exp(-tK) &= \int dx (4\pi t)^{-d/2} \exp(-tM^2_{(0)}) \\
&\times 2^{d/2} \left\{ R + \sum_{\nu} d_\nu \exp(-itM^2(\nu)) \prod_{1 \leq i \leq q} \left( tH_i(\nu) \coth(tH_i(\nu)) \right) \right\} \\
\text{Tr} \exp(-tN) &= \int dx (4\pi t)^{-d/2} \exp(-tQ_{(0)}) \\
&\times \left\{ \tilde{R} + \sum_{\nu} d_\nu \exp(-itQ(\tilde{\nu})) \prod_{1 \leq i \leq q} \left( \frac{tH_i(\tilde{\nu})}{\sinh(tH_i(\tilde{\nu}))} \right) \right\}
\end{align*}
\]  \hspace{1cm} (73)

Thus it is found for the total \( \zeta \)-function for matter fields \[ \zeta_{\text{mat}}(p) = \int dx (4\pi)^{-d/2} \frac{\mu^{2p}}{\Gamma(p)} \int_0^\infty dt t^{p-d/2-1} \\
\times \left\{ -2^{d/2} \exp(-tM^2_{(0)}) \left( R + \sum_{\nu} d_\nu \exp(-itM^2(\nu)) \prod_{1 \leq i \leq q} \left( tH_i(\nu) \coth(tH_i(\nu)) \right) \right) \\
+ \exp(-tQ_{(0)}) \left( \tilde{R} + \sum_{\nu} d_\nu \exp(-itQ(\tilde{\nu})) \prod_{1 \leq i \leq q} \left( \frac{tH_i(\tilde{\nu})}{\sinh(tH_i(\tilde{\nu}))} \right) \right) \right\}
\]  \hspace{1cm} (75)

where \( H_i(\nu) \) and \( H_i(\tilde{\nu}) \) are the invariants of the tensors \( F_{\mu \nu}^a \nu_\alpha \) and \( F_{\mu \nu}^a \tilde{\nu}_a \) respectively, defined from the equations of the form [58] or [59].

After taking specific matter field representations one can obtain from here more explicit expressions for the \( \zeta \)-functions and the effective potential.
6 Asymptotic behavior of the heat kernel and the stability of the vacuum

Thus we have computed all the ingredients for the calculation of the effective potential. By making use of obtained heat kernels one can get simply the \( \zeta \)-functions as well as the effective potential. The exact values of the background fields can be determined then from the effective equations, viz. the condition of the extremum of the effective potential.

Let us first analyze shortly the stability of the considered model of the vacuum state, i.e. when the background gauge fields with covariantly constant field strength are present. As it is well known the vacuum of a model is stable only in the case when the corresponding operators determining the dynamics of the quantum fields on the given background do not have negative modes. The presence of the negative modes leads obviously to the imaginary part of the effective potential and, as a consequence, to the instability of the vacuum. Let us stress here that it is the extremum of the effective action that is important but not its sign. That is why the classical contribution, which is, of course, always positive, can not improve the situation when the negative modes pop up.

We will not derive here the explicit formulae but only analyze the problem of negative modes. This does not require the knowledge of the whole spectrum and can be done very naturally by investigating only the asymptotic behavior of the heat kernels at the infinity \( t \to \infty \). Indeed, one can show easily that the behavior of the heat kernel for any second-order operator of Laplace type \( L \) (32) on covariantly constant background at \( t \to \infty \) is determined by the minimal eigenvalue \( \lambda_{\min} \)

\[
\text{Tr} \exp(-tL) \bigg|_{t \to \infty} \sim t^{-(d-2p)/2} \exp(-t\lambda_{\min}), \tag{76}
\]

\( p \) being some integer. Thus one can calculate the minimal eigenvalues of the operators under consideration: gauge \( \Delta \), ghost \( F \), Dirac \( K \) and scalar \( N \) ones.

From Eqs. (62) and (63) we have

\[
\text{Tr} \exp(-t\Delta) \bigg|_{t \to \infty} \sim \int dx (4\pi)^{-d/2} \times \left\{ \sum_{\alpha>0} \prod_{1 \leq i \leq q} H_i(\alpha) \sum_{1 \leq k \leq q} \exp \left\{ -t \sum_{1 \leq j \leq q} H_j(\alpha) - 2H_k(\alpha) \right\} \right\} \tag{77}
\]

\[
\text{Tr} \exp(-tF) \bigg|_{t \to \infty} \sim \int dx (4\pi)^{-d/2} t^{-d/2} \tag{78}
\]

The heat kernel for the scalar ghost operator (78) behaves good, viz. decreases, at \( t \to \infty \). This means that the minimal eigenvalue of the operator \( F \) is equal to zero

\[
\lambda_{\min}(F) = 0 \tag{79}
\]

The heat kernel for vector gauge field operator behaves at \( t \to \infty \), in general, not good. In this case the second term in Eq. (77) can be exponentially large. It means that the operator \( \Delta \) can have, in general, the negative modes. This is caused by the self-interaction of the gauge fields, (viz. by the extremely large value of the anomalous moment of the gluon). From Eq. (77) one can conclude that the minimal eigenvalue of \( \Delta \) is either negative or equal to zero

\[
\lambda_{\min}(\Delta) = \begin{cases} -\max_{\alpha} \left\{ - \sum_{1 \leq i \leq q} H_i(\alpha) + H_{\max}(\alpha) \right\}, & \text{if this is negative} \\ 0, & \text{if the previous value is positive} \end{cases} \tag{80}
\]

(where \( H_{\max}(\alpha) = \max_{1 \leq i \leq q} H_i(\alpha) \) is the largest invariant of the field \( F_{\mu\nu}(\alpha) \) (55), the prime at the sum meaning that the sum does not include this maximal term).
Thus the vector operator $\Delta$ is positive definite and the heat kernel behaves good at $t \to \infty$ and, consequently, the vacuum is stable, only in the case when the background satisfies the condition of stability

$$\max_{1 \leq i \leq q} H_i(\alpha) < \sum_{i \leq q} H_i(\alpha), \text{ for any } \alpha$$

(81)

i.e. if for any root $\alpha$ the maximal invariant $H_{\text{max}}(\alpha)$ of the background field $F_{\mu\nu}(\alpha)$ is smaller than the sum of all other ones. This is possible only in the case when the number of independent invariants is equal or greater than two $q \geq 2$, i.e. when the dimension of the space is not less than four $d \geq 4$. This means that, for sure, the chromomagnetic vacuum model is unstable in dimensions less than four. There are, of course, in this case the unstable configurations too. Namely, those that do not satisfy the condition of stability (81).

In case $q = 1$, i.e. when there is only one independent invariant of the field $F_{\mu\nu}(\alpha)$ (this case for the $SU(2)$ group was considered first by Savvidy [1]), without fail the negative modes of the gauge field operator exist since the minimal eigenvalue is negative

$$\lambda_{\text{min}}(\Delta) = -\max_{\alpha} H(\alpha) < 0$$

(82)

(in this case $H(\alpha) = \sqrt{(F_{\mu\nu}^2 a_a)^2/2}$). It is this fact that leads to the well known instability of the Savvidy vacuum [1].

Only starting with the number of invariants equal to two ($q = 2$) the instability can (but not must!) disappear. In this case

$$\lambda_{\text{min}}(\Delta) = -\max_{\alpha} \{-H_{\text{min}}(\alpha) + H_{\text{max}}(\alpha)\}$$

(83)

where $H_{\text{min}}(\alpha)$ and $H_{\text{max}}(\alpha)$ are the minimal and the maximal invariants. Hence it is seen, that if $H_{\text{min}}(\alpha) \neq H_{\text{max}}(\alpha)$, then again $\lambda_{\text{min}}(\Delta) < 0$. The only possibility to achieve the absence of the negative modes in the case $q = 2$ is to choose these invariants equal to each other $H_1(\alpha) = H_2(\alpha) = H(\alpha)$.

Then the heat kernels (82) and (83) take the form

$$\text{Tr} \exp(-t\Delta) = \int dx(4\pi t)^{-d/2} \left\{ rd + 2 \sum_{\alpha > 0} \frac{t^2 H^2(\alpha)}{\sinh^2(tH(\alpha))} \left( d + 8 \sinh^2(tH(\alpha)) \right) \right\}$$

(84)

$$\text{Tr} \exp(-tF) = \int dx(4\pi t)^{-d/2} \left\{ r + 2 \sum_{\alpha > 0} \frac{t^2 H^2(\alpha)}{\sinh^2(tH(\alpha))} \right\}$$

(85)

and have both good decreasing power behavior at $t \to \infty$

$$\text{Tr} \exp(-t\Delta)_{t \to \infty} \sim \int dx(4\pi)^{-d/2} t^{2-d/2} (-2)t\text{tr}(F_{\mu\nu})$$

(86)

$$\text{Tr} \exp(-tF)_{t \to \infty} \sim \int dx(4\pi)^{-d/2} t^{-d/2}$$

(87)

Thus at $q = 2, H_1(\alpha) = H_2(\alpha)$ the operator $\Delta$ is positive definite (except for the zero modes), i.e. the instability that is characteristic to the Savvidy vacuum does not exist. It seems on the face of it, that this case can be realized also in four-dimensional space-time $d = 4$. However, as we show below, in case of four dimensions $d = 4$ it is not possible to make the analytic continuation of the equality of two invariants $H_1 = H_2$ to the pseudo-Euclidean space of Lorentzian signature, limiting thereby the possible physical applications of this result.

Consider the case of two invariants in four-dimensional space ($q = 2, d = 4$) at greater length. In this case the invariants $H_i(\alpha)$ given by the solutions of the Eq. (58) or (59) have the simple form

$$H_{1,2}(\alpha) = \sqrt{\frac{1}{2} I_1(\alpha) \pm \frac{1}{2} \sqrt{2 I_2(\alpha) - I_1^2(\alpha)}}$$

(88)

where $I_k(\alpha)$ are the invariants defined by Eq. (60), viz.

$$I_1(\alpha) = \frac{1}{2} F_{\mu\nu}(\alpha) F_{\mu\nu}(\alpha)$$

(89)

$$I_2(\alpha) = \frac{1}{2} F_{\mu\nu}(\alpha) F_{\nu\lambda}(\alpha) F_{\lambda\rho}(\alpha) F_{\rho\mu}(\alpha)$$

(90)
The situation of two equal invariants $H_1(\alpha) = H_2(\alpha)$ in four dimensional space considered above means, that a relation between the invariants $I_k$ takes place

\[ I_2(\alpha) = \frac{1}{2} I_1^2(\alpha) \quad (91) \]

But this is possible only in space of Euclidean signature when for any field $F_{\mu\nu}(\alpha)$

\[ I_2(\alpha) < I_1^2(\alpha) \quad (92) \]

When going to the pseudo-Euclidean (Lorentzian) signature the sign of inequality here changes to the opposite one

\[ I_2(\alpha) > I_1^2(\alpha) \quad (93) \]

that leads to the impossibility of the equality condition (91) to be satisfied in Minkowski space. This leads to the fact that in Euclidean case both invariants $H_1(\alpha)$ and $H_2(\alpha)$ are real, whereas in Minkowski space one of them is necessarily imaginary.

Thus only beginning with $d \geq 5$ there exists such a background that, on the one hand, ensures the operator $\Delta (5)$ to be positive definite and, on the other hand, assumes the analytic continuation on the pseudo-Euclidean space of Lorentzian signature. This is a consequence of the general fact that, when doing the analytic continuation of the results obtained in Euclidean signature to the Lorentzian signature, one should put, in general, one of the invariants of any 2-form to be imaginary, i.e.

\[ H_q(\alpha) = i E(\alpha) \quad (94) \]

One may call the real invariants the magnetic and the imaginary the electric ones. In other words, in Euclidean space all invariants are magnetic, while in the Minkowski space one of them has to be electric (it can also vanish).

The presence of the electric field leads to poles in heat kernels, indeterminacy in integrals over $t$ and, as a consequence, to imaginary part of the effective potential, i.e. to the creation of the particles and instability (although not so potent as in the presence of negative modes). It is not perfectly clear now how to do the analytical continuation of Euclidean effective potential to the space of Lorentzian signature for obtaining physical results in this general case. The methods of the proof of the possibility of such a continuation are based essentially on the perturbative expansion, i.e. are valid, strictly speaking, in weak background fields.

In contrast to the contribution of the gauge fields the heat kernels for matter fields (73) and (74) have good exponentially decreasing behavior at $t \to \infty$ (when some positive singlet contributions in mass matrices $M^2(0)$ and $Q(0)$ are present). When the singlet contributions are zero or even negative then the instability appear, that leads to a reconstruction of the vacuum and to other values of the background fields ensuring the stability of the vacuum state.

Thus we have shown that in the Yang-Mills model under consideration the Savvidy-like vacuum with constant chromomagnetic fields can be stable only in the case when more than one constant chromomagnetic fields are present and the values of these fields differ not greatly from each other. This is possible only in space-times with dimensions not less than five $d \leq 5$.

### 7 Concluding remarks

In this paper we continued the investigation of the low-energy effective action in quantum gravity and gauge theories initiated in [12, 13]. We considered the very wide and important class of non-Abelian gauge field theories and applied a very natural and elegant pure algebraic method for calculation the effective action on the covariantly constant background, developed in [12]. The generalization of such algebraic approach to the case of the curved manifolds is under investigation in [18].

Let us now underline shortly our results. First of all, we considered the most general case of \textit{arbitrary compact simple} gauge group and present the \textit{manifestly covariant} calculation of the one-loop effective potential for the \textit{most general covariantly constant background} with many chromomagnetic and space-time invariants. We showed first that the effective action does not depend on the gauge fixing parameter and obtained then \textit{explicit} formulas for all the heat kernels and zeta-functions of gauge, ghost, spinor and scalar fields.

Further, we proposed a \textit{new method to study the stability} of the vacuum via the asymptotic behavior of the heat kernels at the infinity. Using this method we found the minimal eigenvalues of all operators and analyzed therefrom
the conditions of absence of negative modes. Moreover, we formulated explicitly the condition of the stability of the vacuum (as an inequality to be satisfied by the background fields (63a)). It is shown that it is impossible to get a stable vacuum of chromomagnetic type in space-times of dimensions less than five. Nevertheless, in four-dimensional space, but only with Euclidean signature, it is found also a special stable background field configuration (with two equal field invariants). In higher dimensions, \( d > 4 \), there always exist stable field configurations, viz. those with many independent chromomagnetic invariants which do not differ greatly from each other. Of course, there are always the unstable configurations in this case too.

One should, perhaps, underline here that we calculated in present paper not the ultraviolet divergences of the effective action, i.e. the beta-functions, anomalies etc. (a much discussed subject in the literature), but the finite part of it. While the beta-functions are determined in terms of the zeta-functions \( \zeta(p) \) in the point \( p = 0 \), we calculated much more general object, viz. the zeta-functions at any point! That is why we did not discuss the asymptotic behavior of the effective coupling \( g(\mu) \) based on the renormgroup analysis. In higher dimensions \( (d > 4) \) the non-Abelian gauge theory has a dimensionful coupling constant and is above its critical dimension. Therefore, one could expect from the naive dimension consideration that the vacuum stabilizes within one-loop level. The point is, however, not so easy. The non-Abelian gauge theory is nonrenormalizable in higher dimensions \( (d > 4) \), therefore, one can not apply the renormgroup analysis in this case. There is not any effective coupling like \( g(\mu) \) in nonrenormalizable theories at all! That is why the behavior of considered model in the case \( d > 4 \) that is found in this paper is not trivial.

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