The kinetic theory of quasi-stationary collisionless accretion disc plasmas

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Astrophysical plasmas in accretion discs are usually treated in the framework of fluid or MHD approaches but there are some situations where these treatments become inadequate and one needs to revert to the more fundamental underlying kinetic theory. This occurs when the plasma becomes effectively collisionless or weakly-collisional such as, for example, in radiatively inefficient accretion flows onto black holes. In this paper, we lay down the basics of kinetic theory in these contexts. In particular, we formulate the kinetic theory for quasi-stationary collisionless accretion disc plasmas in the framework of a Vlasov-Maxwell description, taking the plasma to be non-relativistic, axisymmetric, gravitationally-bound and subject to electromagnetic fields. Quasi-stationary solutions for the kinetic distribution functions are constructed which are shown to admit temperature anisotropies. The physical implications of the theory are then investigated and the equations of state and angular momentum conservation law are discussed. Analysis of the Ampere equation reveals the existence of a quasi-stationary kinetic dynamo which gives rise to self-generation of poloidal and azimuthal magnetic fields and operates even in the absence of turbulence and/or instability phenomena.

I. INTRODUCTION

Fluids are usually treated as continuous media with their time evolution being determined by a suitable set of fluid equations, even if the underlying fundamental description is in terms of a discrete system of particles, whose dynamics is deterministic but characterized by random initial conditions. A continuum description for such systems applies when the kinetic description has a continuous phase-space probability density, given by a single-particle kinetic distribution function (KDF), which satisfies a suitable kinetic equation. Once the KDF is prescribed, all of the continuum fluid moments can be represented in terms of well-defined constitutive equations, determined via appropriate velocity moments of the KDF. When binary Coulomb collisions are negligible (collisionless or weakly-collisional plasmas), one should return to the kinetic-theory description. In these cases, “stand-alone” fluid or magneto-hydrodynamics (MHD) approaches formulated independently of an underlying kinetic theory can usually provide, at best, a partial description of the plasma phenomenology. This is because of two possible inconsistencies which may arise. Firstly, the set of fluid equations is generally not closed, and so requires the independent prescription of equations of state which may not give rise to a self-consistent system. Secondly, in these approaches no account is usually given of microscopic phase-space particle dynamics (including single-particle conservation laws) or of phase-space plasma collective phenomena (kinetic effects). These issues are naturally addressed within a kinetic treatment, where the fluid fields are determined from the underlying self-consistent KDF. This means that both the equations of state and the constitutive equations for the fluid fields (see Sections 7-9) then follow uniquely from the microscopic dynamics.

This paper is concerned with the description of stationary or slowly-time-varying (quasi-stationary) phenomena occurring in astrophysical plasmas, focusing particularly on accretion discs around compact objects. For matter in the inner parts of such discs, the plasma may be either collisional (as in the majority of cases) or collisionless, depending on the circumstances. A notable example of the latter case is provided by radiatively inefficient flows (RIAFs) arising in low-density geometrically-thick discs around black holes ([1]). Matter in these systems is thought to consist of a two-temperature plasma, with the ion temperature being much higher than the electron one, and the Coulomb collision timescale being much longer than the inflow time. Other interesting applications for collisionless or weakly-collisional plasmas concern accretion discs around neutron stars and white dwarfs. In the inner regions of such discs, the magnetic field of the central object may become dominant and ions and electrons can be collisionally decoupled so as to sustain different temperatures. This happens, in particular, if the radiative cooling time-scale of the electrons is much shorter than the time-scale for electron-ion collisions so that the electrons and ions are thermally decoupled and can have different temperatures ([2, 3]).
Treatments of accretion discs have often been made in terms of a purely fluid-dynamical approach to which was added an “anomalous” form of viscosity (i.e. one not due to binary particle collisions), following simple intuitive of the effective viscosity lies in magnetic phenomena (such as the magneto-rotational instability, MRI) and that the medium needs to be treated as a magnetized plasma, when making detailed investigations, rather than as a simple unmagnetized neutral fluid. Almost always, these calculations are then performed within the stand-alone continuum MHD treatments. However, as indicated above, the underlying plasma kinetic theory needs to be directly considered when treating collisionless or weakly-collisional plasmas. An approach of this type has been presented in two recent MHD treatments. However, as indicated above, the underlying plasma kinetic theory needs to be directly considered when treating collisionless or weakly-collisional plasmas. An approach of this type has been presented in two recent papers (4, 5), hereafter referred to as Paper I and Paper II respectively).

The aim of the present paper is to address in detail the astrophysical aspects of the theory previously developed. We focus on accretion discs composed of collisionless plasma which can be subject to strong electro-magnetic (EM) fields and which are gravitationally bound, in the sense of being confined by the combined gravitational potential of the central compact object and the disc. We deal with plasma which is sufficiently far from the central object so that it can be treated non-relativistically, with both $k_B T/m_s c^2$ and $(R \Omega/c)^2$ being < 0.1, where $k_B$ is the Boltzmann constant, $T$ is the temperature, $m_s$ is the rest-mass of the species of plasma particle labelled with the index $s$ ($s = e, i$ for an electron-ion plasma), $R$ is the cylindrical radial distance from the central object ($z$ will be the vertical cylindrical coordinate), and $\Omega$ is the local angular velocity in the disc. We do not include any interactions with radiation since we are not considering any background radiation field and radiation-reaction effects (associated with the self-emission of radiation) are negligible for our non-relativistic plasma. We envisage magnetic fields, on the other hand, which can in principle be either external ($B^{ext}$), or self-generated within the disc itself ($B^{self}$). Estimated values coming from observations of accretion discs around compact objects include magnetic-field strengths in the range $B \sim 10^4 - 10^8 G$ (4, 5). For these systems, estimates for species temperatures usually lie in the ranges $T_i \sim 10^4 - 10^{12} K$ and $T_e \sim 10^4 - 10^9 K$ for ions and electrons respectively. Particle densities in different types of accretion disc around compact objects span a very wide range of values, but here we focus on the bottom end of it. In particular, for discussing the collisionless regime of non-relativistic AD plasmas, the number density is taken to be in the interval $n_s \sim 10^6 - 10^{15} cm^{-3}$. In the case of hydrogen-ion systems, the Spitzer collision time $\tau_{Ci}$, the Larmor rotation time $\tau_{Li}$, and the Langmuir time $\tau_{pi}$ are then respectively in the ranges $\tau_{Ci} \simeq 10^{-2} - 10^{19} s$, $\tau_{Li} \simeq 10^{-4} - 10^{-11} s$ and $\tau_{pi} \simeq 1 - 10^{-4} s$, while the corresponding estimates for the Debye length $\lambda_D$, the mean free path $\lambda_{mfp,i}$ and the Larmor radius $r_{Li}$ are respectively $\lambda_D \simeq 10^{-2} - 10^{6} cm$, $\lambda_{mfp,i} \simeq 10^8 - 10^{31} cm$ (an extremely large number!) and $r_{Li} \simeq 10^{-4} - 10^7 cm$ (the lower values corresponding to lower temperature and higher density in the first and second cases and to lower temperature and higher magnetic field in the third one). For a plasma consisting of ion and electron species ($s = i, e$), one can determine the range of values of these characteristic parameters for each species. For any phenomena occurring on timescales $\Delta t$ and length scales $\Delta L$, satisfying

\[
\tau_{ps}, \tau_{Ls} \ll \Delta t \ll \tau_{Cs},
\]

\[
\lambda_D \ll \Delta L \ll \lambda_{mfp,s},
\]

these plasmas can be considered as:

- (#1) Collisionless: due to the inequalities between $\Delta t$ and $\tau_{Cs}$ and between $\Delta L$ and $\lambda_{mfp,s}$. Any plasma is effectively collisionless for processes whose timescales and lengthscales are short enough and, under those circumstances, one needs to use kinetic theory. For a plasma which is sufficiently diffuse, this can include almost all relevant processes.

- (#2) Characterized by a mean-field EM interaction: due to the inequalities between $\Delta t$ and $\tau_{ps}$ and between $\Delta L$ and $\lambda_D$, charged particles of the accretion disc interact with the others only via a continuum mean EM field.

- (#3) Quasi-neutral: due to the inequality between $\Delta L$ and $\lambda_D$, the plasma can be taken as being quasi-neutral on the lengthscale $\Delta L$.

When conditions #1 and #2 are satisfied, the medium is referred to as a Vlasov-Maxwell plasma and kinetic theory needs to be used. It is this which we will be considering in the following. The plasma is treated as an ensemble of particle species, each being described by a KDF $f_s(y, t)$ where $y$ is the particle state vector and $t$ is the time, with $y \equiv (r, v)$ where $r$ is the position and $v$ is the velocity. The species KDFs satisfy the Vlasov kinetic equation $\frac{\partial}{\partial t} f_s(y, t) = 0$, with the velocity moments of the KDFs determining the source of the EM self-field $\{E^{self}, B^{self}\}$.
identified with the plasma charge and current density \( \{ \rho(r, t), J(r, t) \} \):

\[
\rho(r, t) = \sum_s Z_e e \int d^3 v f_s(y, t),
\]

\[
J(r, t) = \sum_s Z_e e \int d^3 v v f_s(y, t).
\]

We work here within the framework of perturbative kinetic theory, extending the approach developed in Papers I and II. This approach allows construction of analytical solutions for the KDF for slowly-time-varying axisymmetric gravitationally-bound systems (referred to as quasi-stationary KDFs). The treatment is developed for both strongly and weakly-magnetized plasmas, distinguishing between magnetic field configurations with closed and open magnetic surfaces (\([8-11]\)). In the present paper, we are focusing on equilibrium or quasi-stationary configurations as a preliminary to subsequently studying perturbations around these solutions. Note that what is meant here by the term “equilibrium” is in general a quasi-stationary KDF, expressed in terms of the relevant first integrals of motion and adiabatic invariants (see Papers I and II and \([10-13]\)), which can also include a non-vanishing stationary radial accretion flow (\([13]\)). One of our main results is the demonstration that for strongly-magnetized collisionless plasmas, kinetic theory gives the possibility of having quasi-stationary accretion flows even in the absence of any additional forms of effective viscosity, such as those arising from turbulence phenomena (which lie beyond the equilibrium solutions). Within the framework of kinetic treatment, quasi-stationary accretion flows can occur in collisionless AD plasmas as part of the equilibrium configuration, with the role of viscous stresses being played by the anisotropic pressure tensor generated by phase-space anisotropies. We note that a somewhat similar magnetically-driven non-turbulent mechanism for driving accretion has been pointed out before (\([15]\)), based on a purely fluid treatment. While that has some similarity with what is being described here, in the sense that magnetic phenomena in stationary configurations are giving rise to the redistribution of angular momentum required for having the accretion flow, it is very different in other respects.

We are concerned here particularly with investigating the role of specifically kinetic effects which do not appear in a fluid treatment. These can arise due to individual-particle dynamics (e.g. finite Larmor-radius effects and ones arising from microscopic conservation laws), as well as due to statistical properties of the equilibrium KDF (e.g. temperature anisotropy and non-uniform fluid fields). Their specific influence is found to depend critically on the topology of the magnetic surfaces, the statistical properties of the plasma and the strength of the EM field. Instabilities and turbulent phenomena (see for example \([16, 17]\)) are not considered here.

A number of issues arise for accretion-disc plasmas of this type, including ones related to:

1) Existence of kinetic equilibria for both strongly and weakly-magnetized plasmas;
2) Dynamo effects occurring in quiescent accretion-disc plasmas, which can explain the self-generation of both azimuthal and poloidal magnetic fields;
3) Collisionless non-turbulent quasi-stationary accretion processes;
4) Closure conditions yielding a finite set of fluid equations for the relevant fluid fields;
5) Equations of state for the pressure tensor components.

Investigation of these issues is relevant for correctly understanding the equilibrium and dynamical properties of collisionless accretion-disc plasmas.

The paper is organized as follows. In Section 2 we provide a classification of accretion-disc plasmas together with the basic assumptions and definitions being used. In Section 3, the first integrals and adiabatic invariants of the system are derived, and their physical meaning is discussed. Section 4 deals with the construction of the quasi-stationary KDF for strongly-magnetized plasmas with open magnetic surfaces. Similar calculations are then presented in Section 5 for strongly-magnetized plasmas with closed nested magnetic surfaces, and in Section 6 for weakly-magnetized plasmas. In Section 7 the number density and flow velocity of each species are computed for all of the configurations considered and their physical properties are analyzed, stressing their connection with accretion-disc dynamics. In Section 8 the pressure tensor for each species is explicitly calculated. A separate discussion is provided concerning the determination of equations of state for the pressure tensor components. Section 9 deals with the calculation of the fluid angular momentum conservation law, while Section 10 deals with the kinetic accretion process which does not depend on the presence of perturbative processes or an explicit viscosity. Section 11 contains a demonstration of the existence of a kinetic dynamo mechanism and presents equations giving the poloidal and toroidal components of the magnetic field. Finally, Section 12 contains a summary of the main results and closing remarks. In the technical parts of this paper, we will often abbreviate “accretion-disc plasma” to “AD plasma” for conciseness.
II. BASIC PHYSICAL ASSUMPTIONS

In the following we will distinguish between strongly-magnetized, intermediate and weakly-magnetized accretion-disc plasmas, the distinction being made on the basis of asymptotic conditions expressed in terms of suitable species-dependent dimensionless physical parameters. These parameters will also be used for characterizing the EM fields and for constructing perturbative kinetic solutions for the quasi-stationary KDFs describing AD plasmas. Three parameters are required:

1) The first parameter is \( \varepsilon_{M,s} \equiv \frac{r_{L,s}}{\epsilon_m}, \) where \( s = i, e \) again denotes the species index. Here \( r_{L,s} = v_{\perp,\text{ths}}/\Omega_{e,s} \) is the species average Larmor radius, with \( v_{\perp,\text{ths}} = \{k_B T_{\perp,s}/M_s\}^{1/2} \) denoting the species thermal velocity perpendicular to the magnetic field and \( \Omega_{e,s} = Z_s e B/M_s c \) being the species Larmor frequency. Note, that we are here always measuring temperatures in degrees Kelvin, with \( k_B \) denoting the Boltzmann constant. \( L \) is the characteristic length-scale of the spatial inhomogeneities of the EM field, defined as \( L \sim L_B \sim L_E \), where \( L_B \) and \( L_E \) are the characteristic magnitudes of the gradients of the absolute values of the magnetic field \( \mathbf{B}(x,t) \) and the electric field \( \mathbf{E}(x,t) \), defined as \( \frac{1}{L_B^2} \equiv \max \left\{ \left| \frac{\partial}{\partial r} \ln B \right|, i = 1, 3 \right\} \) and \( \frac{1}{L_E^2} \equiv \max \left\{ \left| \frac{\partial}{\partial x} \ln E \right|, i = 1, 3 \right\} \), where the vector \( \mathbf{x} \) denotes \( \mathbf{x} = (R, z) \). From \( \varepsilon_{M,s} \) it is possible to define a unique parameter \( \varepsilon_M \equiv \max \{ \varepsilon_{M,s}, s = i, e \} \).

2) The second parameter is defined as \( \varepsilon_s \equiv \left| \frac{L_{\varphi,s}}{p_{\varphi,s}} \right| = \left| \frac{M_s R v_{\varphi,s}}{k_B T_{\perp,s}} \right| \), i.e. it is the ratio between the toroidal angular momentum of the particle \( L_{\varphi,s} = M_s R v_{\varphi,s} \) and the magnetic contribution to its toroidal canonical momentum \( p_{\varphi,s} \), with \( p_{\varphi,s} = L_{\varphi,s} + \mathbf{e}_\varphi \cdot \psi \). Here \( v_{\varphi,s} \equiv \mathbf{v} \cdot \mathbf{e}_\varphi \), with \( \mathbf{e}_\varphi \) being the unit vector along the azimuthal direction \( \varphi \), while \( \psi \) denotes the flux function of the poloidal magnetic field (see definition below).

3) The third parameter is \( \sigma_s \equiv \frac{M_s v_{\perp,s}^2}{Z_se^2} \simeq \frac{k_B T_{\perp,s}}{Z_se^2} \) which represents the ratio between the kinetic energy of the particle \( M_s v_{\perp,s}^2 \) and its potential energy. The latter is represented in terms of the effective potential \( \Phi^{eff} \), which, in turn, is related to the electrostatic and intermediate magnetized plasmas is similar but, on the other hand, they are importantly different in that the magnetic flux \( \psi \) completely dominates the particle canonical momentum for strongly-magnetized plasmas, so that the contribution due to the particle angular momentum \( L_{\varphi,s} \) is negligible in the corresponding conservation law, whereas the contributions are comparable for intermediate magnetized plasmas.

To give an indication of the circumstances in which conditions 1) and 2) above apply, we consider two situations representative of stellar mass and galactic-centre mass black holes. Taking mass \( M = 10^8 \odot \) (giving a Schwarzschild radius \( R_{\text{Sch}} \sim 30 km \)) as representative of a stellar-mass black hole, we focus on AD plasma located at \( R \sim 10 R_{\text{Sch}} \) from the central object, with ion temperature \( T_i \sim 10^8 K \), and characteristic length-scale \( L \sim 0.1R = R_{\text{Sch}} \) for the EM field. Then taking \( \varepsilon_{M,i} \lesssim 10^{-3} \) as representative of \( \varepsilon_{M,i} \lesssim 1 \), we get \( B \gtrsim 10^7 G \) which is at the lower end of the range of magnetic field strengths given above, and so the condition \( \varepsilon_{M,i} \lesssim 1 \) can easily be realized. For a galactic-centre black hole with mass \( M = 10^8 \odot \), the equivalent estimate, for the same radial distance in terms of Schwarzschild radii, gives \( B \gtrsim 10^{-6} G \). Considering now the condition \( \varepsilon_s \lesssim 1 \), we make similar choices as above, plus taking the representative azimuthal particle velocity \( v_{\varphi,s} \) as being between the ion thermal velocity and the Keplerian velocity, and the magnetic field strength as being related to the magnetic flux \( \psi \) by \( B \sim \frac{\psi}{L^2} \). Then requiring, for example, \( \varepsilon_i \lesssim 10^{-2} \) gives the (very approximate) bounds of \( B \gtrsim 10^8 G \) for the stellar-mass case and \( B \gtrsim 10^7 G \) for the galactic-centre one. The bound for the stellar-mass case is at the upper end of the range for magnetic field strengths given above, while that for the galactic-centre one is far more modest (and we note that this is the more interesting context in practice for the appearance of RIAF-type accretion flows). However, concerning these estimates, one should bear in mind that, in comparison with the case for \( \varepsilon_{M,s} \), the bound on \( B \) coming from \( \varepsilon_s \) is much more sensitive to the species being considered (electrons or ions) as well as to the value of the particle angular momentum and to the approximate order-of-magnitude relationship used for estimating \( B \) from \( \psi \). Depending on the circumstances being considered, the condition \( \varepsilon_s \lesssim 1 \) may or may not be satisfied, and so one should investigate both possibilities.

From the above considerations, it follows that both strongly and intermediate-magnetized plasmas may be found in the inner regions of accretion discs around compact objects. Weakly-magnetized plasmas would be located further out, in the outer regions of discs, where lower temperatures and weaker magnetic fields are expected. We will focus here
on studying the configurations for strongly and weakly-magnetized collisionless AD plasmas, leaving the corresponding investigation of intermediate-magnetized plasmas to a separate study.

Ignoring possible weakly-dissipative effects (Coulomb collisions and turbulence), we will assume that the KDF and the EM fields associated with the plasma obey the system of Vlasov-Maxwell equations, with Maxwell’s equations being considered in the quasi-static approximation. For definiteness, we will consider here a plasma consisting of two species of charged particles: one species of ions \( i \) and one of electrons \( e \).

Following the treatment presented in Papers I and II, we are taking the AD plasma to be: a) non-relativistic, in the sense that it has non-relativistic species flow velocities and \( k_B T / m_c c^2 \), that the gravitational field can be treated within the classical Newtonian theory, and that the non-relativistic Vlasov kinetic equation can be used as the dynamical equation for the KDF; b) collisionless, so that the mean free path of the plasma particles is much longer than the largest relevant characteristic scale length of the plasma; c) axisymmetric, so that the relevant dynamical variables characterizing the plasma (e.g., the fluid fields) are independent of the azimuthal angle \( \varphi \), when referred to a set of cylindrical coordinates \( (R, \varphi, z) \); d) acted on by both gravitational and EM fields.

We focus here on solutions for the equilibrium magnetic field \( B \) which admit a family of locally-nested axisymmetric toroidal magnetic surfaces \( [8, 9] \). We recall that a magnetic surface is defined as a surface on which the poloidal magnetic flux \( \psi \) is constant, and that the condition \( \nabla \psi \cdot B = 0 \) is then identically satisfied on each magnetic surface. A schematic view of nested magnetic surfaces is shown in Figure 1. For the equilibrium configurations which we are considering here (prior to any subsequent perturbation) it is reasonable to think that the magnetic field would be rather ordered, at least on a local scale, so that having nested magnetic surfaces is likely on that local scale. We distinguish between the cases of locally closed magnetic surfaces, discussed in Paper I, and locally open magnetic surfaces, discussed in Paper II. Note that here the meaning of open and closed surfaces has to be interpreted with reference to the local domain occupied by the AD plasma. See Figure 1 for a schematic comparison of the two topologies. For both of the configurations, a set of magnetic coordinates \( (\psi, \varphi, \theta) \) can be defined locally, where \( \theta \) is a curvilinear angle-like coordinate on the magnetic surfaces \( \psi(x) = \text{const} \). Each relevant physical quantity \( G(x, t) \) can then be conveniently expressed either in terms of the cylindrical coordinates or as a function of the magnetic coordinates, i.e. \( G(x, t) = \mathcal{G}(\psi, \theta, t) \), where the \( \varphi \) dependence has been suppressed due to the axisymmetry.

For the configuration of closed nested magnetic surfaces, we will assume “small inverse aspect ratio ordering” to hold, as expressed by the requirement that \( 0 < \delta \ll 1 \). Here the dimensionless quantity \( \delta \equiv r_{\text{max}} / R_0 \) is referred to as the inverse aspect ratio parameter, where \( R_0 \) is the radial distance from the vertical axis to the centre of the nested magnetic surfaces and \( r_{\text{max}} \) is the average cross-sectional poloidal radius of the largest closed toroidal magnetic surface. This ordering is consistent with the results presented by \( [8, 11] \) and with the assumption of nested and closed magnetic surfaces which are assumed to be localized in space (see also further discussion in Paper I).

In the following we will consider AD plasmas which are characterized by slowly time-varying phenomena. In particular, the EM field is taken to be given by an analytic function of the form

\[
[E(x, \lambda^k t), B(x, \lambda^k t)] \tag{5}
\]

with \( k \) being an integer \( \geq 1 \) and \( \lambda \equiv \min(\varepsilon_M, \sigma) \). This time dependence is connected with either external sources or
boundary conditions for the KDF. In particular, we will assume that the magnetic field is of the form

\[ \mathbf{B} = \nabla \times \mathbf{A} = \mathbf{B}^{self}(\mathbf{x}, \lambda^k t) + \mathbf{B}^{ext}(\mathbf{x}, \lambda^k t), \]  

(6)

where \( \mathbf{B}^{self} \) and \( \mathbf{B}^{ext} \) denote the self-generated magnetic field produced by the AD plasma and a finite external magnetic field produced by the central object (in the case of neutron stars or white dwarfs). For greater generality, we will not prescribe any relative orderings between the various components of the total magnetic field, which are taken to be of the form

\[ \mathbf{B}^{self} = I(\mathbf{x}, \lambda^k t) \nabla \varphi + \nabla \psi_p(\mathbf{x}, \lambda^k t) \times \nabla \varphi, \]

(7)

\[ \mathbf{B}^{ext} = \nabla \psi_D(\mathbf{x}, \lambda^k t) \times \nabla \varphi. \]

(8)

In particular, here \( \mathbf{B}_T = I(\mathbf{x}, \lambda^k t) \nabla \varphi \) and \( \mathbf{B}_P = \nabla \psi_p(\mathbf{x}, \lambda^k t) \times \nabla \varphi \) are the toroidal and poloidal components of the self-field, while the external magnetic field \( \mathbf{B}^{ext} \) is defined in terms of the vacuum potential \( \psi_D(\mathbf{x}, \lambda^k t) \). As a consequence, the magnetic field can also be written in the equivalent form

\[ \mathbf{B} = I(\mathbf{x}, \lambda^k t) \nabla \varphi + \nabla \psi(\mathbf{x}, \lambda^k t) \times \nabla \varphi, \]

(9)

where the function \( \psi(\mathbf{x}, \lambda^k t) \) is defined as \( \psi(\mathbf{x}, \lambda^k t) \equiv \psi_p(\mathbf{x}, \lambda^k t) + \psi_D(\mathbf{x}, \lambda^k t) \), with \( k \geq 1 \) and \( (\psi, \varphi, \theta) \) defining a set of local magnetic coordinates (as implied by the equation \( \mathbf{B} \cdot \nabla \psi = 0 \) which is identically satisfied). Finally, it is also assumed that the charged particles of the plasma are subject to the action of effective EM potentials \( \{ \Phi^{eff}_s(\mathbf{x}, \lambda^k t), A(x, \lambda^k t) \} \), where \( \Phi^{eff}_s(\mathbf{x}, \lambda^k t) \) is given by

\[ \Phi^{eff}_s(\mathbf{x}, \lambda^k t) = \Phi(\mathbf{x}, \lambda^k t) + \frac{M_s}{Z_s e} \Phi_G(\mathbf{x}, \lambda^k t), \]

(10)

with \( \Phi^{eff}_s(\mathbf{x}, \lambda^k t), \Phi(\mathbf{x}, \lambda^k t) \) and \( \Phi_G(\mathbf{x}, \lambda^k t) \) denoting the effective electrostatic potential and the electrostatic and gravitational potentials. In particular, the gravitational potential \( \Phi_G \) is expressed as \( \Phi_G = \Phi_G^{(ext)} + \Phi_G^{(self)} \), where \( \Phi_G^{(ext)} \) and \( \Phi_G^{(self)} \) denote the gravitational potentials due to the external sources (i.e., primarily the central object) and the same accretion disc respectively. The latter can be computed from the Poisson equation

\[ \nabla^2 \Phi^{(self)}_G(x, \lambda^k t) = 4\pi G_N \rho_m (x, \lambda^k t), \]

(11)

where \( G_N \) is the Newton gravitational constant and the source term \( \rho_m (x, \lambda^k t) \) describes the distribution of matter of the AD plasma. In the following it is assumed that the gravitational potential \( \Phi_G \) is a known quantity.

III. FIRST INTEGRALS AND ADIABATIC INVARIANTS FOR ACCRETION-DISC PLASMAS

In this section we first define relevant kinetic conserved quantities, namely dynamical variables depending on the states of the individual charged particles of the plasma. We refer here in particular to collisionless AD plasmas which satisfy the assumptions introduced in the previous section. Then, on the basis of this, we will investigate the basic physical implications of the conservation laws for the qualitative properties of single-particle dynamics. The determination of dynamical invariants (i.e., first integrals of motion and adiabatic invariants) is a basic requirement for the construction of equilibrium solutions for the KDF (kinetic equilibria). We recall here that an adiabatic invariant \( P \) of order \( n \) with respect to \( \lambda \) is a quantity which is conserved only asymptotically, i.e., in the sense that \( \frac{\partial}{\partial \lambda} \ln P = 0 + O(\lambda^{n+1}) \), where \( n \geq 0 \) is a suitable integer. In other words, adiabatic invariants are dynamical variables which change slowly on the time-scale of the Larmor rotation time. These invariants can be derived from the Lagrangian formulation of the single particle dynamics and the corresponding gyrokinetic theory (see subsection below), which can be obtained by means of a suitable asymptotic expansion in terms of the dimensionless parameter \( \varepsilon_{M,s} \) as long as this is \( \ll 1 \). Because of this requirement, and in view of the previous classification, gyrokinetic theory can only be formulated for strongly-magnetized and intermediate-magnetized AD plasmas.

We first consider the treatment of the first integrals and adiabatic invariants which are conserved for both strongly-magnetized and weakly-magnetized AD plasmas. By assumption, the only first integral of motion is the toroidal canonical momentum \( p_{\varphi s} \) conjugate to the ignorable azimuthal angle \( \varphi \):

\[ p_{\varphi s} = M_s R v \cdot e_\varphi + \frac{Z_s e}{c} \psi \equiv \frac{Z_s e}{c} \psi_{s s}. \]

(12)
The total particle energy

\[ E_s = \frac{M_s}{2} v^2 + Z_s e \Phi_s^{\text{eff}}(x, \lambda^n t), \]  

is instead considered, by construction, to be an adiabatic invariant of order \( n \), with \( n \geq 1 \).

### A. Guiding-centre adiabatic invariants for strongly-magnetized plasmas

Additional adiabatic invariants can be determined for strongly-magnetized plasmas based on gyrokinetic (GK) theory (see Paper II, \[18, 21\]). We recall that gyrokinetic theory provides a convenient formulation for single charged-particle dynamics in the presence of EM fields which are strong enough so that the motion of the particle can be well-represented as spiraling around a single field line, following an imaginary guiding centre which moves along the field line. Gyrokinetic theory clearly displays the characteristic symmetry of the spiral motion, allowing further conservation laws to be derived in addition to those mentioned previously. We refer to Paper II for the mathematical details concerning the formulation of non-relativistic gyrokinetic theory in the presence of both EM and gravitational fields. Here, for the sake of completeness, we recall the basic notation for this and the main conclusions which can be inferred from it. In the following, we will use a prime “’” to denote a dynamical variable defined at the guiding-centre position \( \mathbf{r}' \) (or \( \mathbf{x}' \equiv (R', z') \) in axisymmetry). The single particle velocity is decomposed as

\[ \mathbf{v} = u' \mathbf{b}' + \mathbf{w}' + \mathbf{V}_{\text{eff}}', \]

where \( u' \mathbf{b}' \) and \( \mathbf{w}' \) denote respectively the parallel and perpendicular components of the guiding-centre velocity with respect to the magnetic field direction, while the effective velocity \( \mathbf{V}_{\text{eff}}' \) is defined as

\[ \mathbf{V}_{\text{eff}}'(\mathbf{x}, \varepsilon_M t) \equiv \frac{c}{B'} \mathbf{E}_{s \text{eff}}' \times \mathbf{b}', \]

with \( \mathbf{b}' = \mathbf{b}(\mathbf{x}', \varepsilon_M t) \), \( \mathbf{b}(\mathbf{x}, \varepsilon_M t) \equiv \mathbf{B}(\mathbf{x}, \varepsilon_M t) / B(\mathbf{x}, \varepsilon_M t) \) being the unit vector of the local magnetic field and where we have identified \( \lambda = \varepsilon_M \). It then follows immediately that an adiabatic invariant is provided by the guiding-centre canonical momentum \( p_{\phi s}' \). Correct to \( O(\varepsilon_M^2) \), with \( k \geq 1 \), this is given by

\[ p_{\phi s}' \equiv \frac{M_s}{B'} \left( u' I' - \frac{c \nabla' \psi'}{B'} \cdot \nabla' \Phi_s^{\text{eff}}' \right) + \frac{Z_s e}{c} \psi'. \]

A further adiabatic invariant is the magnetic moment \( m_s' \), which is proportional to the canonical momentum \( p_{\phi s}' \), conjugate to the gyrophase angle \( \phi' \), which can be determined in principle to arbitrary order in \( \varepsilon_M \) (21). In particular, the leading-order approximation yields \( m_s' \equiv \mu_s' \equiv \frac{M_s w'^2}{2} \). Finally, by construction, the guiding-centre Hamiltonian \( \mathcal{H}_s' \) is also an adiabatic invariant. Accurate to order \( n \) this is given by

\[ \mathcal{H}_s'^{(1)} \equiv m_s'B' + \frac{M_s}{2} (u' \mathbf{b}' + \mathbf{V}_{\text{eff}}')^2 + Z_s e \Phi_s^{\text{eff}}'. \]

When the above assumptions hold, the invariants determined in this way necessarily exist for arbitrary initial conditions.

### B. Physical implications of the conservation laws

We now discuss the physical meaning of the conservation laws introduced here and their implications for particle dynamics in magnetized accretion discs.

Consider first the conservation of the toroidal canonical momentum [12]. For a charged particle it follows that this is the sum of two terms: the particle angular momentum \( L_{\phi s} \equiv M_s R \omega z_s \) and a magnetic contribution \( \frac{2eZ_s}{M_s} \psi \). It follows that the angular momentum by itself is generally not conserved. As a consequence, the canonical momentum conservation law allows for the existence of radial particle motion inside a disc. In fact, since in AD plasmas the magnetic flux function \( \psi \) is necessarily spatially non-homogeneous, a moving particle must change its angular momentum \( L_{\phi s} \) while fulfilling the constraint \( \psi_{s \text{ss}} = \text{const.} \), namely staying on a \( \psi_{s \text{ss}} \)-surface. Depending on the geometry of the magnetic surfaces, such particle motion may correspond to either a vertical or radial velocity towards regions of higher or lower magnetic flux. Since the magnetic contribution to \( \psi_{s \text{ss}} \) depends on the sign of the charges, single ions
and electrons exhibit motions in different directions while keeping $\psi_s$ constant. This feature is different from the situation for neutral particles, for which the angular momentum itself is conserved. Because of the presence of plasma boundaries, this can lead to the self-generation of quasi-stationary electric fields in the accretion disc as a result of charge separation.

We next focus on the conservation of the guiding-centre Hamiltonian \( H \) and the magnetic moment $\mu'_s$ (to leading-order approximation). These can be combined to represent the parallel velocity $u'$ as

$$u' = \pm \sqrt{\frac{2}{M_s} \left[ H'_s - \mu'_s B' - Z_s e \Phi^\text{eff}_s - \frac{M_s}{2} V^2_{\text{eff}} \right]}.$$

Therefore $u'$ is a local function of the guiding-centre position vector $x'$ and, due to axisymmetry, of the corresponding flux coordinates $(\psi', \vartheta')$. The above relationship is the basis of particle trapping phenomena, corresponding to the existence of allowed and forbidden regions of configuration space for the motion of charged particles. In fact, since $u'$ is only defined in the subset of the configuration space spanned by $(\psi', \vartheta')$ where the argument of the square root is non-negative, it follows from Eq. (18) that particles must undergo spatial reflections when $u' = 0$. The points of the configuration space where this occurs are the so-called mirror points and the occurrence of such points may generate various kinetic phenomena. In particular, particles can in principle experience zero, one or two reflections corresponding respectively to passing particles (PPs), bouncing particles (BPs) and trapped particles (TPs). In the present case, since the right hand side of Eq. (18) depends on the magnitude of the magnetic field $(B')$, the effective potential energy ($Z_s e \Phi^\text{eff}_s$) and the centrifugal potential ($\frac{M_s}{2} V^2_{\text{eff}}$), we will refer to the TP case as gravitational EM trapping.

Finally, an important qualitative property of collisionless magnetized plasmas follows from the conservation of the magnetic moment $\mu'_s$. The expression for this relates the magnitude of the perpendicular velocity $u'$ to that of the local magnetic field $B'$. Conservation of the adiabatic invariant $\mu'_s$ implies that when a charge is subject to a non-uniform or a non-stationary magnetic field, its kinetic energy of perpendicular motion, $M_s \frac{u'^2}{2}$, must change accordingly so as to keep $\mu'_s$ constant. On the other hand, particles moving on $\psi_s$—surfaces generally necessarily experience a non-uniform magnetic field $B(x, \zeta, t)$. It can be shown that this property implies also the phenomenon of having a non-isotropic kinetic temperature (i.e. there being different effective temperatures parallel and perpendicular to the local direction of the magnetic field). From the statistical point of view of kinetic theory, this temperature anisotropy corresponds to an anisotropy in the kinetic energy of random motion of particles subject to the magnetic field. Such a feature is a characteristic kinetic phenomenon arising in magnetized collisionless plasmas. This physical mechanism operating at the level of single particle dynamics has important consequences also for the macroscopic properties of such plasmas. As we will see, conservation of $\mu'_s$ allows the effects of temperature anisotropy to be included consistently in the quasi-stationary solution for the KDF, and for its physical implications for the dynamics of the corresponding fluid system to be inferred. Another candidate source of temperature anisotropy is radiation emission (cyclotron radiation) due to Larmor rotation in the presence of a strong magnetic field. The signature of this is the simultaneous occurrence of radiation emission corresponding to the Larmor frequencies of the different plasma species.

### IV. THE QUASI-STATIONARY KDF FOR STRONGLY-MAGNETIZED PLASMAS: THE CASE OF OPEN SURFACES

In this section and the following one, we point out the most relevant physical aspects of the kinetic treatment of quasi-stationary strongly-magnetized collisionless AD plasmas in the presence of magnetic field configurations with nested magnetic surfaces. In this section, we consider the case of magnetic surfaces which are open in the domain of the plasma. The case of closed magnetic surfaces is treated in the following section. Following the treatment presented in Paper II, here we want to emphasize the physical aspects of the theory and the role of the kinetic approach adopted here. For greater generality, it is assumed that each species in the plasma is associated with a set of sub-species (the PPs, BPs and TPs mentioned above), each one having a different KDF. The existence of temperature anisotropy is allowed for all of the species, involving the introduction of different temperatures parallel and perpendicular to the local direction of the magnetic field. It is also assumed that a non-vanishing species-dependent poloidal flow velocity can exist, related to possible inward or outward matter flows in the disc (see also the discussion below). Under these assumptions, as pointed out in Papers I and II, a solution for the quasi-stationary KDF can be obtained, which in the following is denoted as $\tilde{f}_{ss}$. This is expressed in terms of the integrals of motion and the adiabatic invariants identified in the previous section. It has the form

$$\tilde{f}_{ss} = \tilde{f}_{ss} \left( E_s, \psi_{ss}, p^s_{\psi_s}, m'_s, (E_s, \psi_{ss}), \lambda^k t \right),$$

(19)
where \( k \geq 1 \) and \( \lambda \) is to be identified with \( \varepsilon M \). Here, by construction, \( \hat{f}_s \) is only defined in the subset of the phase-space where the adiabatic invariants are defined, while the variables in the round brackets \( \langle E_s, \psi_s \rangle \) will be involved in the perturbative expansions to be defined below. It follows that \( \hat{f}_s \) is suitable for describing passing, bouncing and trapped particles. Hence, by construction, the KDF is itself an adiabatic invariant, and is therefore an asymptotic solution of the Vlasov equation. Due to the arbitrariness of the definition of the KDF, for each plasma sub-species it is always possible to identify it with a superposition of Gaussian distributions or, more generally, suitably generalized Gaussian distributions. However, for a collisionless plasma, each of these functions must actually itself be a quasi-stationary solution and so, due to the requirement (19), \( \hat{f}_s \) can always be prescribed to be asymptotically “close” to a local bi-Maxwellian KDF. Regarding this, in Paper II it was proved that \( \hat{f}_s \) can be identified with a properly-defined Generalized bi-Maxwellian KDF with parallel velocity perturbations.

In this paper we are mainly interested in the astrophysical applications of the kinetic analysis and so we omit here all of the mathematical details of the derivation, referring to Paper II for an exhaustive discussion. It is sufficient to mention here that a characteristic feature of the quasi-stationary KDF is that it contains implicit dependences in terms of the single particle velocities. As shown in Paper II, these dependences can be made explicit for strongly magnetized and gravitationally bound plasmas so as to allow an asymptotic analytical treatment of the velocity moments.

An interesting feature of the theory is the double Taylor-expansion which is performed on \( \hat{f}_s \) to reach this goal, achieving a systematic solution method for the Vlasov equation. More precisely, this is done in terms of the two dimensionless parameters \( \varepsilon_s \) and \( \sigma_s \), requiring that the inequalities \( \varepsilon_s \ll 1 \) and \( \sigma_s \ll 1 \) are satisfied when the particle velocity is taken to be of the order \( |v| \lesssim \nu_{\perp,\text{th}} \text{ or } |v| \lesssim \nu_{\parallel,\text{th}} \), where \( \nu_{\perp,\text{th}} = \{k_B T_{\perp s}/M_s\}^{1/2} \) and \( \nu_{\parallel,\text{th}} = \{k_B T_{\parallel s}/M_s\}^{1/2} \) denote respectively the parallel and perpendicular thermal velocities. For greater generality, \( \varepsilon_s \) and \( \sigma_s \) are here treated in the perturbative expansions as being infinitesimals of the same order. No expansion is performed in the ratio \( \frac{\nu_{\parallel,\text{th}}}{\nu_{\perp,\text{th}}} \). The Taylor expansion of \( \hat{f}_s \) with respect to the two parameters can be formally carried out at any order in \( \varepsilon_s \) and \( \sigma_s \). In particular, to leading-order this is done by setting \( \psi_s \equiv \psi + O(\varepsilon_s) \) and \( E_s \equiv Z_e e \Phi_{s}^{\text{eff}} + O(\sigma_s) \). It is then straightforward to prove that the following relation holds to leading-order for the quasi-stationary KDF:

\[
\hat{f}_s \equiv \tilde{f}_s \left[ 1 + h_{D_s}^1 + h_{D_s}^2 \right],
\]

where the leading-order distribution \( \tilde{f}_s \) is of the form \( \tilde{f}_s(\langle E_s, \psi_s, \psi_s, p_{\phi s}, m_s, (\psi, \vartheta), \lambda^s t \rangle) \). Here \( h_{D_s}^1 \) and \( h_{D_s}^2 \) represent the so-called FLR-diamagnetic and energy-correction parts of \( f_{s\ast} \) which are polynomial functions of the particle velocity, while \( \tilde{f}_s \) can be written as

\[
\tilde{f}_s = \frac{n_s}{(2\pi k_B/M_s)^{3/2} (T_{||s})^{1/2} T_{\perp s}} \exp \left\{ -\frac{M_s (v - V_s - U'_{||s}b')^2}{2k_B T_{||s}} - m_s^s \frac{B'}{k_B T_{||s}} \right\},
\]

which we refer to as the bi-Maxwellian KDF with parallel velocity perturbations. Again we stress that \( \tilde{f}_s \) is only defined in the subset of phase-space where the parallel velocity \( |u'| \) is a real function. Here we note that the form of \( \tilde{f}_s \) has been prescribed in order to allow the existence of:

1) A finite azimuthal flow velocity \( V_s = e_s R \Omega_s \), with \( \Omega_s \) being a suitable rotational frequency (a toroidal angular velocity).

2) Finite parallel flows (with respect to the local magnetic field direction) associated with \( U'_{||s} = \frac{L'}{\Omega_s} \xi_s \). These can include inward or outward radial flows of matter. The guiding-centre parallel flow velocity \( U'_{||s} \) is uniquely prescribed in terms of \( \xi_s \) with \( \xi_s \) being a suitable frequency (see Paper II).

3) A finite toroidal magnetic field, which is related to \( U'_{||s} \). In fact \( U'_{||s} \) is non-vanishing only for magnetic configurations in which the toroidal magnetic field is non-zero.

4) A finite temperature anisotropy identified with \( \frac{1}{k_B T_{\perp s}} = \frac{1}{k_B T_{\perp s}} - \frac{1}{k_B T_{||s}} \).

5) A non-uniform effective number density defined as

\[
n_s = \eta_s \exp \left\{ \frac{X_s}{k_B T_{||s}} \right\},
\]

(22)
with \( \eta_s \) denoting the *pseudo-density*. Here the function \( X_s \) is prescribed in such a way as to take into account the effects of the electrostatic and gravitational energy of the particle, the centripetal potential and azimuthal and parallel flows (the precise definition of this is given in Paper II).

6) Separate treatments of species and sub-species contributions, for which the previous asymptotic orderings are assumed to hold. In fact, for the different populations the analytical expansion can lead to different contributions for the terms appearing in the diamagnetic and energy-correction parts, depending on the relative magnitudes of the parameters \( \varepsilon_s \) and \( \sigma_s \). On the other hand, because of the double expansion and the energy dependence, the asymptotic solution for the two species can hold also in different spatial domains.

Finally we note that the expansion given in Eq. (20) shows that, in general, a bi-Maxwellian KDF cannot be an exact stationary solution of the Vlasov equation. Instead, the actual (asymptotic) equilibrium is necessarily described by the quasi-stationary KDF \( f\hat{\alpha}s \). By construction this is asymptotically close to a bi-Maxwellian when the expansion (20) holds.

There are two important physical implications which follow from the quasi-stationary solution for the KDF (20). The first one concerns the existence of kinetic constraints, namely functional dependences which need to be imposed on the quasi-stationary KDF in order to guarantee that this is an adiabatic invariant of the prescribed order. To outline this point, consider the set of functions

\[
\Lambda_s = \left( \beta_s, \alpha_s, T_s \parallel, \Omega_s, \xi_s \right),
\]

which we will refer to as *structure functions* (see also Paper II). In particular, here

\[
\beta_s \equiv \frac{\eta_s}{k_B T_s \parallel},
\]

\[
\alpha_s \equiv \frac{B'}{k_B \Delta T_s},
\]

which depend on the pseudo-density, the magnitude of the guiding-centre magnetic field and the parallel and perpendicular temperatures. It is important to point out that the kinetic constraints actually prescribe well-defined functional dependences for the structure functions, imposing for them the form

\[
\Lambda_s = \Lambda_s (\psi, Z_s e^{\Phi^s_{\text{eff}}}) + O (\varepsilon_s) + O (\sigma_s).
\]

The effective potential \( \Phi^s_{\text{eff}} \) is generally a function of the form \( \Phi^s_{\text{eff}} = \Phi^s_{\text{eff}}(x, \varepsilon^k_M t) \), with \( x = (R, z) \), while neither the gravitational potential nor the electrostatic potential are expected to be functions only of \( \psi \). Therefore, in magnetic coordinates, the structure functions are of the general form \( \Lambda_s \equiv \overline{\Lambda}_s (\psi, \partial, \varepsilon^k_M t) \). Hence, the functional forms of the leading-order effective number density, the parallel and azimuthal flow velocities and the temperatures carried by the bi-Maxwellian KDF, are uniquely determined in terms of \( \psi \) and \( \partial \). As a consequence, the azimuthal angular velocity of the general form \( \Omega_s = \overline{\Omega}_s (\psi, \partial, \varepsilon^k_M t) \). We stress that in the customary treatment of collisionless AD plasmas based on ideal-MHD these constraints are missing. Instead, they follow in a natural way from kinetic theory. By adopting a kinetic treatment it is possible to prescribe the correct form for the fluid fields, as required by the presence of kinetic constraints.

A further important consequence of the kinetic constraints is the relationship between the magnitude of the temperature anisotropy and the guiding-centre magnetic field at two different spatial locations. In fact, the quantity \( \frac{\overline{B'}^2}{\Delta T_s} \) in the KDF is necessarily an adiabatic invariant. To leading-order in the GK expansion, this implies that the asymptotic equation

\[
\frac{\Delta T_s}{\Delta T} \cdot [\Omega] \approx \frac{\overline{B'}^2}{\Delta T_s} [\Omega]\]

must hold identically for any two arbitrary positions “1” and “2”, with \( ([\Delta T_s], [\Omega]) \) and \( ([\Delta T], [\Omega]) \) denoting the temperature anisotropy and the magnitude of the magnetic field at these positions respectively.

The second striking aspect of the kinetic treatment concerns the diamagnetic and energy-correction contributions \( h^1_{\Delta s} \) and \( h^2_{\Delta s} \) of \( f\hat{\alpha}s \). These carry the contributions from the expansions of the particle toroidal canonical momentum and particle total energy respectively. The perturbative-correction to the KDF is a polynomial function of the particle velocity which depends linearly on the so-called effective *thermodynamic forces*. The latter are here denoted as \( A_{is} \) and \( C_{is} \), with \( i = 1, 5 \). In analogy with classical thermodynamics, it is natural to identify them with the gradients of the structure functions \( \Lambda_s \). Hence, in the present case they are associated with partial derivatives taken with respect to the magnetic flux \( \psi \) and the effective potential \( \Phi_{\text{eff}}^s \) (\( \frac{\partial}{\partial \psi} \)). There are the following definitions: \( A_{is} \equiv \frac{\partial \Omega_{is}}{\partial \psi} \),
with \( k \) case the perturbative contributions with respect to the expansion in our set of assumptions holds, the Taylor expansion of which is referred to as the bi-Maxwellian KDF with open magnetic surfaces, as shown in Fig. 1. In this context, the quasi-stationary KDF electrostatic energy, the centripetal potential and the azimuthal velocity (see the definition in Paper I).

\[ A_{2s} = \frac{\partial \ln T_s}{\partial \hat{\xi}}, \quad A_{3s} = \hat{\alpha} \frac{\partial \Omega_s}{\partial \hat{\xi}}, \quad A_{4s} = \partial \hat{\xi}, \quad A_{5s} = \frac{\partial \ln \xi}{\partial \hat{\xi}} \text{ and } C_{1s} = \frac{\partial \ln T_s}{\partial \hat{\xi}}, \quad C_{2s} = \frac{\partial \ln T_s}{\partial \hat{\psi}}, \quad C_{3s} = \frac{\partial \ln \Omega_s}{\partial \hat{\psi}}, \quad C_{4s} = \frac{\partial \hat{\xi}}{\partial \hat{\psi}}, \quad C_{5s} = \frac{\partial \ln \xi}{\partial \hat{\psi}}. \]

The diamagnetic and energy-correction effects carried by \( h_{Ds}^1 \) and \( h_{Ds}^2 \) cannot be ignored: the construction of kinetic equilibria cannot be achieved without them, as pointed out in Papers I and II. From a physical point of view, the perturbative contribution to the KDF determines first-order corrections to the fluid moments of \( \hat{f}_{ss} \) produced by finite Larmor-radius (FLR) effects. These carry the contributions of all of the thermodynamic forces which can arise in collisionless plasmas characterized by non-uniform differential rotation, density and temperature gradients and temperature anisotropy.

\[ \text{V. THE QUASI-STATIONARY KDF FOR STRONGLY-MAGNETIZED PLASMAS: THE CASE OF CLOSED SURFACES} \]

In this section we present the kinetic solution for strongly-magnetized plasmas in the case in which the equilibrium magnetic field admits locally a family of closed and nested magnetic surfaces. This is the configuration considered in Paper I; the geometry is illustrated schematically in Fig. 1.

The quasi-stationary KDF for collisionless plasmas with closed magnetic surfaces can be found as a particular asymptotic limit of the general solution \( \hat{f}_{ss} \) holding for open surfaces. Besides considering closed surfaces, this is obtained by imposing the requirement of small inverse aspect ratio ordering. As a side assumption, we must now also impose vanishing of the velocity perturbation \( U'_s \), which requires setting \( \xi_s \) to zero. In fact, the case of closed surfaces corresponds to plasma magnetic self-confinement in which no local net radial flow can take place, the latter being associated with \( U'_s \) (see Section 10). Note however that a general situation can include both locally-closed and open magnetic surfaces, as shown in Fig. 1. In this context, the quasi-stationary KDF \( \hat{f}_{ss} \) is reduced to a Generalized bi-Maxwellian KDF of the form

\[ \hat{f}_{ss} = \hat{f}_s \left( E_s, \psi_s, m'_s, (\psi_s), \lambda^k T_s \right), \]

with \( k \geq 1 \) and \( \lambda \) being identified with \( \varepsilon_M \), while the perturbative expansion is applied only to the variable \( (\psi_s) \). When our set of assumptions holds, the Taylor expansion of \( \hat{f}_{ss} \) can be performed only with respect to the dimensionless parameter \( \varepsilon_s \), while again no expansion is performed in the ratio \( \Omega_s / T_s \). Also, it is possible to prove that in the present case the perturbative contributions with respect to the expansion in \( \sigma_s \) all become negligible. More precisely, correct to first-order in \( \varepsilon_s \), the asymptotic expansion of \( \hat{f}_{ss} \) gives:

\[ \hat{f}_{ss} \approx \hat{f}_s \left[ 1 + h_{Ds}^1 \right], \]

with \( \hat{f}_s \) being of the form \( \hat{f}_s = \hat{f}_s \left( E_s, \psi_s, m'_s, (\psi), \lambda^k T_s \right) \). The notation here is similar to that adopted in the previous section, with \( h_{Ds}^1 \) representing the diamagnetic part of \( \hat{f}_{ss} \), while the leading-order distribution \( \hat{f}_s \) is now given by

\[ \hat{f}_s = \frac{n_s}{(2\pi k_B/M_s)^{3/2} \left( T_{ss}^{1/2} \right) T_s \Delta s \exp \left\{ -\frac{M_s (V - V_s)^2}{2k_B T_{ss}} - m'_s B' - \frac{k_B \Delta T_s}{2} \right\}}, \]

which is referred to as the bi-Maxwellian KDF. The form of \( \hat{f}_s \) has been prescribed in order to allow the existence of:

1) A finite non-uniform azimuthal flow velocity \( V_s = e_p R \Omega_s \), where \( \Omega_s \) is the rotational frequency (the toroidal angular velocity).

2) A finite temperature anisotropy characterized by \( \frac{1}{k_B \Delta T_s} = \frac{1}{k_B T_{ss}} - \frac{1}{k_B T_{ss}} \).

3) A finite toroidal magnetic field, related just to FLR and diamagnetic effects driven by temperature anisotropy.

4) A non-uniform number density defined as

\[ n_s = n_s \exp \left[ \frac{X_s}{k_B T_{ss}} \right], \]

with \( n_s \) again denoting the pseudo-density and the function \( X_s \) being prescribed in terms of the particle effective electrostatic energy, the centripetal potential and the azimuthal velocity (see the definition in Paper I).
5) Separate treatments of the species and sub-species contributions, for which the previous asymptotic ordering is assumed to hold.

Note that, unlike in the open-surface case, the energy-correction contribution $h^2_{Ds}$ coming from the $\sigma_s$-expansion does not appear in the asymptotic solution [29], as this becomes of higher order than $h^1_{Ds}$.

In this case the structure functions $\Lambda_s$ are

$$\Lambda_s \equiv (\beta_s, \alpha_s, T_{\parallel s}, \Omega_s) ,$$

whose physical meaning has been pointed out in the previous section. As before, to leading order in the asymptotic expansion [29] and in the GK expansion, the structure functions prescribe the fluid fields carried by the bi-Maxwellian KDF [30]. In particular, for closed nested magnetic surfaces and for strongly-magnetized plasmas, the kinetic constraints give:

$$\Lambda_s = \Lambda_s (\psi) + O (\varepsilon_s) .$$

Comparison with Eq. (26) shows that the energy dependence no longer appears, so that the fluid fields are only $\psi$-flux functions. This is because in the present case, to leading order, the effective potential is itself reduced to a flux-function, i.e. $\Phi^{\text{eff}}_s = \Phi^{\text{eff}}(\psi, \varepsilon^s_{\parallel}, t)$. This conclusion is in agreement with the small inverse aspect ratio ordering. Hence, the functional forms of the leading-order number density, azimuthal flow velocities and temperatures carried by the bi-Maxwellian KDF, are uniquely determined in terms of $\psi$.

Concerning the diamagnetic part $h^1_{Ds}$, it can easily be shown that it carries the contributions arising from Taylor-expanding the particle toroidal canonical momentum. As a consequence, the diamagnetic KDF is a polynomial function of the particle velocity, depending linearly on the thermodynamic forces $A_i$, with $i = 1, 4$. From Eq. (33) and following the treatments of Papers I and II, the latter are related to the gradients of the structure functions with respect to the magnetic flux $\psi$, and are defined as follows: $A_{1s} \equiv \partial \ln \beta_s / \partial \psi$, $A_{2s} \equiv \partial \ln T_{\parallel s} / \partial \psi$, $A_{3s} \equiv \partial \ln \Omega_s / \partial \psi$ and $A_{4s} \equiv \partial \varepsilon_s / \partial \psi$. The $A_i$, with $i = 1, 4$ carry the contributions due to density, temperature, angular velocity and temperature anisotropy gradients respectively. This general form of $h^1_{Ds}$ follows from the assumed kinetic equilibrium defined by Eq. (29).

The diamagnetic effects carried by $h^1_{Ds}$ determine the first-order FLR corrections to the fluid moments of $f_{\text{ws}}$. They are important for characterizing collisionless plasmas in the presence of non-uniform differential rotation, density and temperature gradients and non-uniform temperature anisotropy.

### VI. THE QUASI-STATIONARY KDF FOR WEAKLY-MAGNETIZED PLASMAS

In this section we derive a solution for the quasi-stationary KDF describing weakly-magnetized collisionless AD plasmas. The basic difference from the strongly-magnetized regime is that gyrokinetic theory does not hold for weakly-magnetized plasmas because single particles are effectively not magnetically confined (the Larmor radius is of the same order as the characteristic equilibrium scale-length of the plasma, or larger). Hence, contrary to the case for strongly-magnetized plasmas, guiding-centre adiabatic invariants can no longer be obtained. Here we retain, however, a number of physical features relevant for modelling weakly-magnetized plasmas:

1) Isotropic temperature: for all of the species it is assumed that the temperature is isotropic.
2) Nested magnetic flux surfaces: the magnetic field is assumed to allow quasi-stationary solutions with magnetic flux lines belonging either to closed or open nested magnetic surfaces.
3) Azimuthal flow velocity: the plasma is characterized by having a primarily differential azimuthal flow velocity, whose leading-order expression is $V_{\varphi s} \simeq \Omega_s R^2 \nabla \varphi$.
4) Fluid fields: the collisionless plasma is characterized by non-uniform fluid fields, defined in terms of velocity moments of the quasi-stationary KDF.
5) Kinetic constraints: suitable functional dependences must be imposed so as to ensure that the KDF is an asymptotic solution of the collisionless Vlasov equation. For weakly-magnetized plasmas, the kinetic constraints are found to differ from those considered before for strongly-magnetized plasmas. They include, in particular, constraints on the species angular frequency $\Omega_s$.
6) Analytic form: the KDF is required to be a smooth analytic function.

Given the requirements 1)-6), the solution for the KDF cannot, in general, be a Maxwellian. However, it is possible to show that they can all be satisfied by a suitably-generalized Maxwellian, the new solution being expressed only in terms of the first integral [12] and the adiabatic invariant [13]. For clarity of notation, in the following we will label with $f_{\text{ws}}$ the quasi-stationary KDF for weakly-magnetized plasmas. This has the general form

$$f_{\text{ws}} = f_{\text{ws}} (E_s, p_{\varphi s}, (E_s), \lambda^k t) ,$$

(34)
with \( k \geq 1 \) and \( \lambda \) being identified with \( \sigma \). In this case, the perturbative expansion is carried out only with respect to the variable \( (E_s) \). In agreement with the above requirements, a particular solution for \( f_{ws} \) is given by:

\[
f_{ws} = \frac{n_{ws}}{(2\pi k_B/M_s)^{3/2} T_{ws}^{3/2}} \exp\left\{ -\frac{|E_s - \Omega_{ws} p_{\varphi s}|}{k_BT_{ws}} \right\},
\]

(35)

which will be referred to as the Generalized Maxwellian KDF for weakly-magnetized plasmas, with \( E_s \) and \( p_{\varphi s} \) being defined respectively by Eqs. (13) and (12). In analogy with the previous treatment, we now introduce the following structure functions:

\[
\Lambda_{ws} \equiv (\eta_{ws}, T_{ws}, \Omega_{ws}).
\]

(36)

In order that Eq. (35) defines an adiabatic invariant, the following functional dependences must be imposed on the structure functions:

\[
\Lambda_{ws} = \Lambda_{ws}(E_s),
\]

(37)

which represent the kinetic constraints for weakly-magnetized plasmas. Note that, for this configuration, only a dependence in terms of the particle total energy is retained. Due to the assumption of having a gravitationally-bound plasma, this is the only physically admissible choice for the structure functions.

Using Eqs. (12)-(13), an equivalent representation for \( f_{ws} \) is provided by the expression

\[
f_{ws} = \frac{n_{ws}}{(2\pi k_B/M_s)^{3/2} T_{ws}^{3/2}} \exp\left\{ -\frac{M_s (v - V_{ws})^2}{2k_BT_{ws}} \right\}.
\]

(38)

Here \( V_{ws} \equiv \Omega_{ws} R^2 \nabla \varphi \), while the function \( n_{ws} \) is defined as

\[
n_{ws} \equiv \eta_{ws} \exp \left\{ \frac{X_{ws}}{k_BT_{ws}} \right\},
\]

(39)

with

\[
X_{ws} \equiv \frac{M_s |V_{ws}|^2}{2} + \frac{Z_se}{c} \psi \Omega_{ws} - Z_se \Phi_{eff}.
\]

(40)

We conclude that the quasi-stationary KDF is an adiabatic invariant because of the kinetic constraints (37).

### A. Analytical expansion of \( f_{ws} \)

Imposing the kinetic constraints introduces an implicit velocity dependence in the structure functions. This can be explicitly dealt with by performing an asymptotic analytic expansion of \( f_{ws} \), similar to that made for strongly-magnetized plasmas. In view of the form of Eq. (37), a convenient Taylor expansion for \( f_{ws} \) can be obtained here in terms of the parameter \( \sigma_s \): there is no expansion in the ratio \( \frac{\Omega_{ws} R}{v_{th,s}} \).

For gravitationally bound plasmas, we can set \( E_s \approx Z_se \Phi_{eff} + O(\sigma_s) \) to leading order. Correspondingly, the linear approximation for the structure functions, obtained neglecting corrections of \( O(\sigma_{k,s}) \) with \( k \geq 2 \), is

\[
\Lambda_{ws} \approx \Lambda_s + (E_s - Z_se \Phi_{eff}) \left[ \frac{\partial \Lambda_{ws}}{\partial E_s} \right]_{E_s = Z_se \Phi_{eff}}.
\]

(41)

where

\[
\Lambda_s \equiv [\Lambda_{ws}]_{E_s = Z_se \Phi_{eff}}.
\]

(42)

Then, the following relation for \( f_{ws} \) holds, correct to first-order in \( \sigma_s \):

\[
f_{ws} \approx f_{ws}^0 [1 + h_{ws}],
\]

(43)
where \( f^0_{ws} \) (\( E_s, p_{es}, (Z_s e \Phi^{eff}_s), \lambda^k t \)) is the leading-order solution and \( h_{ws} \) represents the first-order perturbative contribution coming from the Taylor expansion. The leading-order solution \( f^0_{ws} \) can be expressed as

\[
f^0_{ws} = \frac{n_s}{(2\pi k_B/M_s)^{3/2} T_s^{3/2}} \exp \left\{ -\frac{M_s (v - V_s)^2}{2k_B T_s} \right\},
\]

which we will refer to as the \textit{drifted Maxwellian KDF for weakly-magnetized plasmas} (i.e., a Maxwellian in the species co-moving frame having velocity \( V_s \)), and which depends on the number density \( n_s \), temperature \( T_s \) and azimuthal flow velocity \( V_s \equiv \Omega_s R^2 \nabla \phi \), with \( \Omega_s \) representing the leading-order azimuthal rotational frequency. The leading-order number density is then defined as

\[
n_s \equiv \eta_s \exp \left[ \frac{X_s}{k_B T_s} \right],
\]

with

\[
X_s \equiv M_s \frac{|V_s|^2}{2} + \frac{Z_s e}{c} \psi \Omega_s - Z_s e \Phi^{eff}_s.
\]

From Eq. (42), to leading-order the structure functions \( \Lambda_s \equiv (\eta_s, T_s, \Omega_s) \) must satisfy the following kinetic constraints:

\[
\Lambda_s = \Lambda_s (\Phi^{eff}_s) + O(\sigma_s).
\]

Finally, the first-order correction \( h_{ws} \) is again a polynomial function of the particle velocity and has the form:

\[
h_{ws} \equiv h^1_{ws} v^2 + h^2_{ws} \langle v \cdot e \phi \rangle,
\]

where we have explicitly singled out the dependences on the particle velocity. The contributions appearing in Eq. (48) are defined as follows:

\[
h^1_{ws} = \frac{M_s}{2Z_s e} \left[ \frac{D_{1s} + \frac{Z_s e \psi E_t}{c k_B T_s} D_{3s}^+}{D_{3s} - D_{2s}} \right],
\]

\[
h^2_{ws} = \frac{M_s^2 R^2 \Omega_s}{2Z_s e k_B T_s} [D_{3s} - D_{2s}],
\]

where we have introduced the following definitions for the \textit{energy gradients} of the structure functions: \( D_{1s} \equiv \frac{\partial \ln \eta_s}{\partial \Phi^{eff}_s}, \)

\( D_{2s} \equiv \frac{\partial \ln T_s}{\partial \Phi^{eff}_s}, \)

\( D_{3s} \equiv \frac{\partial \ln \Omega_s}{\partial \Phi^{eff}_s}. \) These quantities can again be interpreted as generalized thermodynamic forces.

\[\text{B. Properties and discussion}\]

A number of comments should be made about the properties of the kinetic solution obtained here and its physical meaning:

1. The existence of Eq. (48) demonstrates that quasi-stationary drifted Maxwellian kinetic solutions exist also for weakly-magnetized collisionless AD plasmas.

2. Similarly to the case of strongly-magnetized plasmas, the Maxwellian KDF obtained here is generally not an exact solution in a strict sense. However, within the validity of the asymptotic expansion (43), this becomes an asymptotic equilibrium solution.

3. The leading-order expressions for the number density, temperature and azimuthal flow velocity appearing in the Maxwellian KDF (41) are found to be functions of \( \Phi^{eff}_s \).

4. The first-order perturbation \( h_{ws} \) allows one to include kinetic effects arising in weakly-magnetized plasmas. These contributions are due to the thermodynamic forces \( D_{1s}, i = 1, 3 \), which carry information about the energy gradients of the structure functions.

5. In this case the equilibrium is compatible only with an azimuthal flow velocity, so that accretion flows can only occur as a result of turbulence phenomena and cannot be described as equilibrium solutions (see Sections 9 and 10).

6. The choice (47) adopted here for the kinetic constraints is an intrinsic feature of gravitationally bound weakly-magnetized plasmas. For this plasma regime, the particle dynamics is mainly determined by the gravitational potential and, in principle, also by the electrostatic potential, as shown by Eq. (47). In contrast, for strongly-magnetized plasmas with closed surfaces, the dynamics is determined mainly by the magnetic field and the structure functions are functions of the poloidal magnetic flux \( \psi \).
VII. NUMBER DENSITY AND FLOW VELOCITY

In this section we address the calculation of the relevant fluid fields associated with the quasi-stationary KDF for both strongly and weakly-magnetized plasmas. By definition, given a distribution function \( f_s \), a generic fluid field is expressed as an integral of the distribution over the velocity space, having the form

\[
\int_{\Gamma_u} d^3v Z(x) f_s,
\]

where \( Z(x) \) is an arbitrary velocity-weight function and \( \Gamma_u \) denotes the appropriate velocity space for the integration. In this section we focus attention on computing the species number density and flow velocity for plasmas in the high and low magnetic field regimes. Using \( Z(x) = 1 \) and \( Z(x) = v \), one obtains:

a) species number density

\[
n_{s}^{\text{tot}} \equiv \int_{\Gamma_u} d^3v f_s;
\]

b) species flow velocity

\[
V_{s}^{\text{tot}} \equiv \frac{1}{n_{s}^{\text{tot}}} \int_{\Gamma_u} d^3v v f_s.
\]

Knowledge of these two fluid moments is required in order to write the Poisson and Ampere equations for studying the self-generated EM fields. A basic feature of the present calculation is that the fluid fields are computed analytically in closed form by adopting the asymptotic analytic expansions of the quasi-stationary KDFs for \( \hat{f}_s \) and \( f_{ws} \). Note that these velocity moments are uniquely determined once the quasi-stationary KDFs \( \hat{f}_s \) and \( f_{ws} \) are prescribed in terms of the structure functions \( \Lambda_s \) and \( \Lambda_{ws} \). On the other hand, the equilibrium fluid moments which follow from this calculation are identically solutions of the corresponding fluid moment equations. These can be obtained as velocity integrals of the Vlasov equation, of the form

\[
\int_{\Gamma_u} d^3v Z(x) \frac{d}{dt} f_s = 0.
\]

Finally, concerning the notation adopted, here and in the rest of the paper the suffix “tot” is used to label fluid fields expressed in terms of \( \hat{f}_s \) or \( f_{ws} \), and to distinguish them from their leading-order solutions computed by means of the asymptotic expansions of the same KDFs.

A. Strongly-magnetized plasmas

For strongly-magnetized plasmas, the fluid fields must be computed by first performing a transformation of all of the guiding-centre quantities appearing in the quasi-stationary KDF to the actual particle position (guiding-centre back-transformation). The order of accuracy of this transformation is measured in terms of the parameter \( \varepsilon_{M,s} \) and depends on the corresponding order of accuracy of the adiabatic invariants used in the solution for the KDF. Contributions coming from this transformation carry FLR corrections to the fluid fields, which operate together with the FLR-diamagnetic and energy-correction contributions carried by the first-order perturbations of the KDFs. Here a distinction must be made between the cases of open and closed magnetic surfaces. In fact, for open-field configurations it is possible to show that \( \varepsilon_{M,s} \lesssim \varepsilon_s \), while for closed-field configurations, within the validity of the inverse aspect ratio ordering, it follows that \( \varepsilon_{M,s} \sim O(\delta) \ll 1 \). Hence, FLR effects from the guiding-centre back-transformation are negligible for strongly-magnetized plasmas with closed magnetic surfaces. Also, as indicated above, for closed-field configurations the terms contributing to the KDF coming from the \( \sigma_s \)-expansion are also negligible with respect to those proportional to \( \varepsilon_s \) and compared with the open-field case. From these considerations it follows that, when only first-order contributions are retained in the asymptotic expansion, then the corresponding first-order corrections coming from the guiding-centre back-transformation need to be retained only for the leading-order KDF. Finally, because of the existence of multiple-species plasmas, which for strong magnetic fields may include also velocity-space sub-species, the velocity sub-space \( \Gamma_u \) of integration must be properly prescribed. In fact, charged particles in both open and closed configurations can have mirror points (TPs and BPs) or be PPs, which are free to stream through the boundaries of the domain. These populations give different contributions to the relevant fluid fields and therefore require separate statistical treatments.
We consider first the species number density. For the general case of open magnetic surfaces the analytical expansion for the quasi-stationary KDF is given by Eq. (20). Then, to first-order in all of the expansion parameters ($\varepsilon_s$, $\sigma_s$ and $\varepsilon_{M,s}$), the number density is given by

$$n_{s}^{\text{tot}}(V) = \int_{T_{\text{n}}} d^3v \left\{ \left[ \hat{f}_s \right]_GK \left[ 1 + h_1^{\parallel s} + h_2^{\parallel s} \right] \right\},$$

(55)

where $\left[ \hat{f}_s \right]_GK$ denotes the leading-order KDF to which the guiding-centre back-transformation must be applied up to first-order in $\varepsilon_{M,s}$. The corresponding expression holding for closed surfaces in the case of small inverse aspect ratio ordering reduces to

$$n_{s}^{\text{tot}}(V) = \int_{T_{\text{n}}} d^3v \left\{ \hat{f}_s \left[ 1 + h_1^{\parallel s} \right] \right\},$$

(56)

where we have made use of Eq. (29). In both cases the constitutive equation for the total number density can be written as

$$n_{s}^{\text{tot}} \approx n_s \left[ 1 + \Delta n_s \right],$$

(57)

where the leading-order contribution $n_s$ is given by Eq. (31) for closed surfaces and by Eq. (22) for open surfaces when the ordering $\varepsilon_{M,s} \ll \varepsilon_s$ holds, while $\Delta n_s$ carries the first-order corrections associated with FLR, diamagnetic and energy-correction effects.

Next, let us consider the species flow velocity. Velocity-space integrals analogous to (55) and (56) can be written also in this case. The species flow velocity can be generally represented in terms of the constitutive equation

$$n_{s}^{\text{tot}} V_{s}^{\text{tot}} \equiv n_s \left[ U_s + \Delta U_s \right].$$

(58)

For open magnetic surfaces and when $\varepsilon_{M,s} \ll \varepsilon_s$, $U_s \equiv V_s + U_{\parallel s} b$ is the leading-order flow velocity carried by the bi-Maxwellian KDF with a parallel velocity perturbation, in which $V_s = \Omega_s R^2 \nabla \varphi$ and $U_{\parallel s} = \frac{1}{\tau} \xi_s$. Moreover, the two frequencies $\Omega_s$ and $\xi_s$ are subject to the kinetic constraints given by Eq. (26). The second term $\Delta U_s$ represents the first-order correction which can be decomposed as follows:

$$\Delta U_s = (\Delta U_{s,\varphi} \nabla \varphi \times \nabla \psi, \Delta U_{s,\varphi} \nabla \varphi, \Delta U_{s,\vartheta} \nabla \psi \times \nabla \varphi).$$

(59)

Note that for open magnetic surfaces:

1) All components of $\Delta U_s$ are linear functions of the thermodynamic forces appearing in the first-order perturbation of the KDF.

2) The component $\Delta U_{s,\varphi}$ provides a correction to the leading-order azimuthal velocity. This is non-vanishing even in the absence of any guiding-centre contribution in the KDF and also in the case of isotropic temperature.

3) The component $\Delta U_{s,\vartheta}$ is related to the temperature anisotropy as well as to both FLR-diamagnetic and energy-correction effects.

4) The component $\Delta U_{s,\psi}$ is associated with FLR effects coming from the guiding-centre back-transformation.

5) Both $U_{\parallel s} b$ and $\Delta U_s$ can give rise to inward or outward flows of matter, both in the radial and vertical directions. The existence of a non-vanishing parallel velocity $U_{\parallel s}$ in the kinetic solution is allowed by the gyrokinetic conservation laws. This means that it can always be suitably prescribed in agreement with the kinetic constraints. On the other hand, the correction $\Delta U_s$ acquires a precise physical meaning within the present formulation. In fact $\Delta U_s$ is generated by the existence of a non-uniform and non-isotropic plasma. Its precise form is determined automatically once the kinetic constraints for the leading-order structure functions are prescribed.

6) The expression for $\Delta U_s$ depends on the particle sub-species (i.e., the distinction between PPs, TPs, and BPs). In fact, each of these populations gives different contributions to the components of the velocity $\Delta U_s$. As indicated below, they give rise to interesting physical phenomena for the disc dynamics, in relationship with the Ampere equation and the self-generation of both toroidal and poloidal equilibrium magnetic fields.

We consider now the case of closed magnetic surfaces. The calculation is here made simpler because, by assumption, $U_{\parallel s}$ vanishes and only the $\varepsilon_s$-expansion is relevant in the asymptotic expansion. Thus, the flow velocity can still be written as in Eq. (58), but now only an azimuthal leading-order flow velocity $U_s \equiv \Omega_s R^2 \nabla \varphi$ can arise, with $\Omega_s = \Omega_s(\psi)$ as indicated above. On the other hand, the first-order correction $\Delta U_s$ now reduces to

$$\Delta U_s = \left( 0, \Delta U_{s,\varphi} \nabla \varphi, \Delta U_{s,\vartheta} \nabla \psi \times \nabla \varphi \right).$$

(60)
In particular, the $\nabla \vartheta \times \nabla \varphi$ component vanishes in this approximation since it is related to terms coming from the guiding-centre back transformation which are of higher-order for closed surfaces. Again, the poloidal component $\Delta U_{s,\vartheta}$ is due to diamagnetic FLR velocity corrections produced by temperature anisotropy (see also Paper I). However, under the hypothesis of closed nested magnetic surfaces, it cannot give rise to a net accretion velocity.

We therefore conclude that in both cases plasma temperature anisotropy affects the existence of non-vanishing species poloidal flow velocities. This feature only occurs for strongly-magnetized plasmas. The importance of this result for AD plasmas lies in the fact that equilibrium poloidal flow velocities may also give rise to a net poloidal current density. The latter in turn will generate a finite equilibrium toroidal magnetic field. Therefore, species temperature anisotropies in collisionless AD plasmas actually provide an effective physical mechanism for the self-generation of toroidal magnetic field in these systems.

\section*{B. Weakly-magnetized plasmas}

For weakly-magnetized plasmas, the explicit calculation of the fluid fields is made simpler by the fact that the kinetic equilibrium does not contain any guiding-centre adiabatic invariant. In this case, only contributions arising from the $\sigma_s$-expansion need to be taken into account. To first-order in $\sigma_s$ the species number density can then be easily calculated as follows:

$$n_{s,\text{tot}} \approx \int_{\Gamma_n} d^3v \left\{ f^0_{ws} [1 + h_{ws}] \right\}, \quad (61)$$

which recovers again the constitutive equation

$$n_{s,\text{tot}} \approx n_s [1 + \Delta n_s]. \quad (62)$$

Here the leading-order term $n_s$ is defined by Eq.(45) and is the contribution carried by the Maxwellian KDF. On the other hand, the first-order correction $\Delta n_s$ is found to be given by

$$\Delta n_s \equiv h_{ws}^1 \left[ V_s^2 + \frac{3 k_B T_s}{M_s} \right] + h_{ws}^2 \left[ \frac{9 k_B T_s}{M_s} + \frac{\Omega_s^2 R^2}{M_s} \right] \Omega_s R, \quad (63)$$

where $V_s^2 \equiv \Omega_s^2 R^2$ and the quantities $h_{ws}^1$ and $h_{ws}^2$ are defined by Eqs.(49)-(50). Similarly, in the same approximation, the flow velocity is given by the velocity integral

$$V_{s,\text{tot}} \approx \frac{1}{n_{s,\text{tot}}} \int_{\Gamma_n} d^3v \left\{ f^0_{ws} [1 + h_{ws}] \right\}. \quad (64)$$

After explicit calculation, this gives

$$n_{s,\text{tot}} V_{s,\text{tot}} \approx n_s [V_s + \Delta V_s], \quad (65)$$

where $V_s \equiv \Omega_s R^2 \nabla \varphi$ is the leading-order flow velocity carried by the Maxwellian KDF. The first-order contribution $\Delta V_s$ is similarly found to be given by

$$\Delta V_s \equiv V_s \Delta n_s + V_s h_{ws}^1 \frac{6 k_B T_s}{M_s} +$$

$$+ h_{ws}^2 \left( \frac{15 k_B T_s}{M_s} + V_s^2 + 2 \Omega_s^2 R^2 \right) \frac{k_B T_s}{M_s} e_x. \quad (66)$$

The following features should be noted:

1) The flow velocity is purely azimuthal. No further components of the flow velocity are allowed for weakly-magnetized plasmas, in contrast with the case of strongly-magnetized plasmas.

2) When the gravitational potential energy is dominant over the electrostatic energy, the functional dependence of the structure functions as implied by the kinetic constraints is determined primarily by the gravitational potential. For example, the leading-order azimuthal rotational frequency $\Omega_s$ is simply of the form $\Omega_s \approx \Omega_s (\Phi_G)$. This means that its functional dependence is compatible with the Keplerian rotational frequency.
VIII. PRESSURE TENSOR AND EQUATIONS OF STATE

In this section we focus on the calculation of the pressure tensor corresponding to the kinetic equilibria obtained for both strongly and weakly-magnetized plasmas. The species pressure tensor (or partial pressure tensor) is defined with respect to the species flow velocity as the velocity moment

$$\Pi_{s}^{\text{tot}} = \int_{\mathbb{V}_s} d^3v M_s (\mathbf{v} - \mathbf{v}_{s}^{\text{tot}}) (\mathbf{v} - \mathbf{v}_{s}^{\text{tot}}) f_s.$$  \hspace{1cm} (67)

In the context of the present kinetic treatment $\Pi_{s}^{\text{tot}}$ is then uniquely prescribed in terms of the quasi-stationary KDF. As indicated below, this enables us to determine to the requisite accuracy also the corresponding equations of state relating the components of the tensor to the structure functions, giving a self-consistent treatment of the physical properties of quasi-stationary collisionless AD plasmas. In the following we provide explicit expressions for $\Pi_{s}^{\text{tot}}$; the overall pressure tensor of the system can then be obtained by summing over the contributions from the separate species: $\Pi^{\text{tot}} = \sum_{s=i,e} \Pi_{s}^{\text{tot}}$.

We consider first the case of strongly-magnetized plasmas, adopting respectively for configurations with open and closed magnetic surfaces the asymptotic expansions given by Eqs. (20) and (29). As a result of the perturbative calculation, the species pressure tensor is represented as

$$\Pi_{s}^{\text{tot}} \simeq \Pi_{s} + \Delta \Pi_{s},$$ \hspace{1cm} (68)

where $\Pi_{s}$ is the leading-order term (with respect to all of the expansion parameters), while $\Delta \Pi_{s}$ represents the first-order correction. For both closed and open magnetic surfaces the tensor $\Pi_{s}$ is obtained using a bi-Maxwellian KDF with temperature anisotropy. To represent $\Pi_{s}$, we introduce for convenience the set of right-handed orthogonal unit vectors $(\mathbf{b}, \mathbf{e}_1, \mathbf{e}_2)$, where $\mathbf{b} = \frac{\mathbf{B}}{B}$, while $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthogonal vectors in the plane perpendicular to the magnetic field. In terms of this basis, the unit tensor can be represented as: $\mathbb{I} \equiv \mathbf{b} \mathbf{b} + \mathbf{e}_1 \mathbf{e}_1 + \mathbf{e}_2 \mathbf{e}_2$. Then, it follows that with respect to the unit tensor $\mathbb{I}$, the pressure tensor $\Pi_{s}$ is symmetric, diagonal and non-isotropic, with a representation of the form:

$$\Pi_{s} = n_s k_B T_{s} \mathbb{I} + n_s k_B \left( T_{s\parallel} - T_{s\perp} \right) \mathbf{b} \mathbf{b}.$$ \hspace{1cm} (69)

Here $n_s$ is the leading-order species number density and $T_{s\parallel}, T_{s\perp}$ are the leading-order species parallel and perpendicular temperatures. Instead, the precise form of $\Delta \Pi_{s}$ is geometry-dependent and contains FLR corrections. It is always possible to represent it in terms of the general decomposition:

$$\Delta \Pi_{s} \equiv \Delta \Pi_{s}^1 \mathbb{I} + \Delta \Pi_{s}^2 \mathbf{b} \mathbf{b} + \Delta \Pi_{s}^3,$$ \hspace{1cm} (70)

in which $\Delta \Pi_{s}^1$ and $\Delta \Pi_{s}^2$ are diagonal first-order anisotropic corrections to the pressure tensor, while $\Delta \Pi_{s}^3$ in this basis is generally non-diagonal. For strongly-magnetized plasmas and closed magnetic surfaces, the precise form of the tensor pressure has been given in [22]. The physical properties of the solutions (69) and (70) can be summarized as follows:

1) The total tensor pressure $\Pi_{s}^{\text{tot}}$ is symmetric.

2) The leading-order pressure tensor $\Pi_{s}$ calculated in this approximation is diagonal but non-isotropic. We note that the source of anisotropy in Eq. (69) is provided by the temperature anisotropy.

3) The first-order correction $\Delta \Pi_{s}$ instead is generally non-diagonal and non-isotropic in the $(\mathbf{b}, \mathbf{e}_1, \mathbf{e}_2)$ basis. Two different physical mechanisms contribute to generating this effect. The first one is again the temperature anisotropy, while the second is produced by first-order perturbative corrections to the KDF, and so depends linearly on the thermodynamic forces.

We next consider the case of weakly-magnetized plasmas. Performing a similar calculation, it is possible to prove that, to first-order in $\sigma_s$, the pressure tensor is symmetric and isotropic and can be written as

$$\Pi_{s}^{\text{tot}} \simeq [ \Pi_{s} + \Delta \Pi_{s} ] \mathbb{I}.$$ \hspace{1cm} (71)

In particular, the leading-order term is defined as

$$\Pi_{s} \equiv n_s k_B T_{s},$$ \hspace{1cm} (72)

with the number density being given by Eq. (15). Both the number density $n_s$ and the temperature $T_{s}$ are subject to the kinetic constraints expressed by Eq. (17). The explicit representation of the first-order isotropic term $\Delta \Pi_{s}$ is as
follows:

\[
\Delta \Pi_s \equiv h_{ws}^1 n_s \left[ \frac{15 k_B T_s^2}{M_s^2} + V_s^2 \frac{3 k_B T_s}{M_s} \right] + \nonumber \\
+ h_{ws}^2 n_s \Omega_s \left[ \frac{45 k_B T_s^2}{M_s^2} + V_s^2 \frac{3 k_B T_s}{M_s} \right]. 
\]  

(73)

A striking feature of Eq. (73) is the explicit dependence in terms of the thermodynamic forces, which take into account the gradients of the structure functions. Again, it should be noted that the pressure tensor (71) is isotropic because the associated KDF has an isotropic temperature and does not contain guiding-centre or FLR effects.

\[A. \text{ Equilibrium equations of state}\]

An important result of the present theory is the explicit construction of \textit{equations of state} for the various components of the species pressure tensor. For definiteness, let us consider here only the leading-order contributions to \(\Pi_{s}^{\text{tot}}\) with respect to the relevant expansion parameters. This provides finite-term equations (leading-order equations of state) of the form \(\Pi_{s}^{\text{tot}} = \Pi_{s}^{\text{tot}} (\Lambda_s)\), with \(\Lambda_s\) being the appropriate structure functions. Note that in principle the solution for \(\Pi_{s}^{\text{tot}}\) allows one to obtain equations of state which include also first-order corrections which are linearly proportional to the thermodynamic forces.

In particular, in the case of strongly-magnetized plasmas this recovers for the leading-order \textit{perpendicular} and \textit{parallel pressures} the expressions

\[
p_{\perp s} \equiv n_s k_B T_{\perp s}, \quad \text{(74)}
\]

\[
p_{\| s} \equiv n_s k_B T_{\| s}. \quad \text{(75)}
\]

The number density and the parallel and perpendicular temperatures are subject to their respective kinetic constraints (see the discussion above). Eqs. (22) and (31) provide a clear representation of the physical effects contributing to the equations of state for strongly-magnetized plasmas in the case of open and closed magnetic surfaces respectively. In particular, the functions \(X_s\) allow corrections due to particle electrostatic and gravitational energy, centripetal potential and azimuthal and parallel flows to be explicitly taken into account.

An analogous equation of state can be obtained in the case of weakly-magnetized plasmas for the isotropic pressure tensor:

\[
p_{s} \equiv n_s k_B T_{s}, \quad \text{(76)}
\]

with \(p_{s}\) denoting the species scalar pressure and \(n_s\) being given by Eq. (45) to leading-order. This equation of state allows one to clearly display the contributions due to the gravitational and electrostatic potentials as well as the azimuthal flow velocity.

Note that Eqs. (74) and (75), as well as Eq. (76), provide only the leading-order solution for the corresponding equations of state. In fact, a more accurate solution should necessarily include also higher-order terms coming from the diamagnetic and energy-correction contributions, both of which result from the Taylor expansions performed on the KDF. These corrections are responsible for the appearance of the terms \(\Delta \Pi_{\perp}\) and \(\Delta \Pi_{s}\) in Eqs. (68) and (71) respectively. This implies that, for both strongly and weakly magnetized collisionless plasmas, the equations of state for the partial pressures cannot be of the frequently-used polytropic type \(p = \kappa \rho^\gamma\) with \(\kappa\) and \(\gamma\) both being constants (and \(\rho\) again here being the mass density). This is not surprising because the quasi-stationary kinetic solutions considered here are not thermodynamic equilibria (as required for deriving this polytropic equation of state from microscopic considerations). This is clear from the appearance of non-vanishing thermodynamic forces.

The present kinetic approach leads naturally to the use of temperature and density as thermodynamic variables, since these fluid fields are directly related to the (leading-order) structure functions contained in the equilibrium KDF. On the other hand, for a proton-electron plasma, which does not possess internal binding energy, the temperature also represents a statistical measure of the specific internal energy of the system, which by definition does not include potential or EM energy. We note that equivalent representations can be given in terms of temperature or of specific internal energy although the more general case with a non-isotropic pressure tensor requires the introduction of the concept of “directional” specific internal energy (related to the anisotropic temperatures).
IX. ANGULAR MOMENTUM

In this section we discuss the implications of the kinetic treatment for the law of conservation of fluid angular momentum. For doing this, we first define the species fluid canonical toroidal momentum as

\[ L_{\text{tot}}^{s} \equiv \frac{1}{n_{\text{tot}}^{s}} \int_{\Gamma_{n}} d^{3}v \frac{Z_{s}e}{c} \psi_{ss} f_{s}, \]  

for \( f_{s} = f_{ss} \) or \( f_{s} = f_{ws} \) respectively in the cases of strongly and weakly-magnetized plasmas. Consider then the corresponding conservation law for the species total canonical momentum. This can be recovered by identifying the weight function \( Z = \psi_{ss} \) in Eq.\((54)\), i.e. by setting

\[ \int_{\Gamma_{n}} d^{3}v \frac{d}{dt} [\psi_{ss} f_{s}] = 0. \]  

In the equilibrium case this implies the species fluid angular momentum conservation law

\[ \nabla \cdot [R^{2} \Pi_{\text{tot}}^{s} \cdot \nabla \varphi + n_{s}^{\text{tot}} V_{\text{tot}}^{s} L_{\text{tot}}^{s}] + \frac{Z_{s}e}{c} \nabla \psi \cdot n_{s}^{\text{tot}} V_{\text{tot}}^{s} = 0 \]  

for the species angular momentum

\[ L_{s}^{\text{tot}} \equiv M_{s} R^{2} V_{\text{tot}}^{s} \cdot \nabla \varphi, \]  

where expressions for the number density, flow velocity and pressure tensor have been derived in the previous sections.

In Eq.\((79)\) a key role is played by the divergence of the species pressure tensor. For strongly-magnetized plasmas, using the leading-order expression \((69)\), this is given by:

\[ \nabla \cdot \Pi_{\text{tot}}^{s} \sim = \nabla p_{\perp} + \text{bB} \cdot \nabla \left( \frac{p_{||} - p_{\perp}}{B} \right) - \Delta p_{s} Q, \]  

where \( Q \equiv [\text{bB} \cdot \nabla \ln B + \frac{2}{cB} \text{b} \times \text{J} - \nabla \ln B] \) and \( \Delta p_{s} \equiv (p_{||} - p_{\perp}) \). It is clear that in this case \( \nabla \cdot \Pi_{\text{tot}}^{s} \) has non-vanishing components in arbitrary spatial directions, including the azimuthal direction along \( \nabla \varphi \). On the other hand, for weakly-magnetized plasmas, Eq.\((71)\) gives

\[ \nabla \cdot \Pi_{\text{tot}}^{s} \approx \nabla [\Pi_{s} + \Delta \Pi_{s}] \cdot \text{I}. \]  

Since the pressure tensor is isotropic and we are assuming axisymmetry, it follows that the component of \( \nabla \cdot \Pi_{\text{tot}}^{s} \) along \( \nabla \varphi \) must vanish identically.

For a single species, the total canonical momentum \( L_{\text{tot}}^{s} \) and the total angular momentum \( L_{s}^{\text{tot}} \) in general differ because of the contribution of the magnetic part proportional to the flux function \( \psi \). However, a different conclusion can be drawn for the corresponding canonical momentum density \( n_{s}^{\text{tot}} L_{\text{tot}}^{s} \) and angular momentum density \( n_{s}^{\text{tot}} L_{s}^{\text{tot}} \). If one considers summation over species for both these quantities and imposes the quasi-neutrality condition

\[ \sum_{s} Z_{s}e n_{s}^{\text{tot}} = 0, \]  

then one obtains the identity

\[ \sum_{s} n_{s}^{\text{tot}} L_{s}^{\text{tot}} = \sum_{s} n_{s}^{\text{tot}} L_{\text{tot}}^{s}. \]  

We next investigate the consequences of Eq.\((79)\) for the dynamical properties of collisionless plasmas. Note the following aspects:

1) In the usual interpretation and also for weakly-magnetized collisionless plasmas within our treatment (see Eq.\((85)\) below), the directional derivative of \( L_{s}^{\text{tot}} \) along the flow velocity \( V_{\text{tot}}^{s} \) vanishes. However, for strongly-magnetized plasmas Eq.\((79)\) shows that equilibrium configurations are possible in which this is generally non-zero. This arises because of the non-isotropic pressure tensor and the poloidal components of the flow velocity which, in turn, are consequences of temperature anisotropy, the first-order energy-correction and FLR-diamagnetic effects which are not included in standard MHD treatments.
2) According to Eq. (79), spatial variation in the species angular momentum implies the possibility of having quasistationary radial matter flows in the disc without departing from the unperturbed equilibrium solution. These can correspond either to local outflows or inflows; both can be described consistently within the present kinetic solution for open magnetic field lines in strongly-magnetized plasmas. Local inflows and outflows can occur independently and are described consistently by their respective quasi-stationary KDFs. Radial flows arise due both to the parallel velocities $U_{\parallel s}$ and to the kinetic effects driven by the first-order energy-correction and FLR-diamagnetic effects. Therefore, species radial flows appear necessarily together with a non-isotropic pressure tensor and a non-vanishing toroidal magnetic field (see also Papers I and II and the discussion below).

3) For weakly-magnetized plasmas, the tensor pressure is isotropic and Eq. (79) reduces to

$$n_s^{\text{tot}} V_s^{\text{tot}} \cdot \nabla L_s^{\text{tot}} = 0,$$

which, thanks to axisymmetry and recalling Eq. (65), is identically satisfied. Hence, in this case equilibrium radial flows are excluded.

In conclusion, for the case of weakly-magnetized plasmas, it follows from the present treatment that quasi-stationary equilibrium configurations with isotropic pressure tensor would not have any net radial flow. This is as expected. In order to have an accretion flow in this context one needs to have some form of effective viscosity, either appearing explicitly or coming from perturbations around the equilibrium state such as those leading to MRI in the conventional picture. A particular goal of our future work will be to investigate perturbations around the equilibrium states presented here, to see whether a process analogous to MRI then appears also for collisionless plasmas.

On the other hand, in the case of strongly-magnetized plasmas the situation is different and net radial inflow can occur even in the absence of effective viscosity being explicitly added or coming from perturbative mechanisms. For strongly-magnetized plasmas, Eq. (79) implies that particles of one species can move radially in a quasi-stationary configuration independently of those of other species, with the species angular momentum not being conserved, since the conservation law involves the canonical momentum (including a magnetic-field contribution) and not just the standard angular momentum. The flow velocity $V_s^{\text{tot}}$ is different for each species and, also, the pressure tensor is non-isotropic. Hence, even with quasi-neutrality, the total angular momentum of the matter can change due to balance with the torque produced by the non-isotropic species pressure tensor and/or a net current flow across magnetic surfaces (see also the discussion below). As indicated above, the non-isotropic nature of $\Pi^{\text{tot}}$ is caused by temperature anisotropy as well as by first-order and FLR effects, which are determined self-consistently in the kinetic approach.

X. THE KINETIC ACCRETION LAW

In this section we discuss how quasi-stationary accretion flows could occur in collisionless AD plasmas as a result just of the equilibrium configuration without requiring additional effective viscosity of the sort mentioned above. Here the role of viscous stresses is played by the anisotropic pressure tensor, which is part of the equilibrium solution.

For the physical conditions considered in the Introduction to which the theory applies, the characteristic time for the inward accretion flow in accretion discs is typically longer than the characteristic Larmor time $\tau_{\nu L_s}$ as well as the Langmuir time $\tau_{ps}$ and smaller than the Spitzer ion collision time so that an accretion flow can be consistently described by the present collisionless kinetic treatment for quasi-stationary equilibria. In this section we will demonstrate that equilibrium accretion flows cannot arise in the case of strongly-magnetized plasmas with closed magnetic surfaces and weakly-magnetized plasmas. Instead, equilibrium accretion flows are permitted for general open-field configurations. Note, however, that in general there are both open and closed surfaces co-existing, as illustrated in Fig.1.

Let us consider first domains which are locally characterized by open flux surfaces. In such domains, parallel flows can be included in the quasi-stationary KDF only if the guiding-centre canonical momentum $p_{\nu L_s}^{\nu L_s}$ is conserved, i.e. the plasma is strongly-magnetized (see Section 4). In fact, in such cases, the quasi-stationary KDF can sustain both poloidal and radial species flow velocities, which are defined respectively as

$$V_{ps} \equiv V_s^{\text{tot}} \cdot e_\theta = \frac{1}{n_s^{\text{tot}}} \int_{\Gamma_u} d^3v \left| \mathbf{v} \cdot e_\theta \right| \hat{f}_s^{s,\nu},$$

$$V_{Rs} \equiv V_s^{\text{tot}} \cdot e_R = \frac{1}{n_s^{\text{tot}}} J_{Rs}^{\nu L_s},$$

where $e_\theta \equiv \frac{\nabla^\perp}{|\nabla^\perp|}$, $e_R \equiv \frac{\nabla_R}{|\nabla_R|}$ and the species mass radial current density $J_{Rs}^{\nu L_s}$ is defined as

$$J_{Rs}^{\nu L_s} \equiv \int_{\Gamma_u} d^3v \left| \mathbf{v} \cdot e_R \right| \hat{f}_s^{s,\nu}.$$
In the velocity-space integrals indicated above, the contributions from PPs, BPs and TPs need to be distinguished. The physically-relevant situations are those in which there is a non-vanishing net radial species accretion flow, i.e. where the average species radial mass current \( \langle \langle J_{Rs} \rangle \rangle \equiv \frac{1}{2z_2 - z_1} \int_{z_1}^{z_2} J_{Rs} dz \) is negative (inward flow), with \( z_1 \) and \( z_2 \) being suitably prescribed. Although local contributions to \( \langle \langle J_{Rs} \rangle \rangle \) can arise from TPs, BPs and PPs, the overall accretion flow is mainly associated with PPs. Notice also that, to leading-order, the presence of poloidal accretion is on the same flux surface, the kinetic accretion law being suitably prescribed. Although local contributions to \( \psi \) species number densities must vary on a given \( \vartheta \)-surface according to the following relation:

\[
\frac{\langle \langle J_{Rs} \rangle \rangle_1}{\langle \langle J_{Rs} \rangle \rangle_2} \equiv \frac{[T_{\parallel s}]_1}{[T_{\parallel s}]_2}
\]

for any two arbitrary positions “1” and “2” prescribed in terms of the magnetic coordinates \((\psi, \vartheta)\). Then consider the case \( \xi_M, s \ll \xi_s \), which allows one to approximate the guiding-centre quantities with the expression for them evaluated at the particle position. Assuming that \( I = I(\psi) \) (see Paper II) and considering the two positions \((\psi, \vartheta_1)\) and \((\psi, \vartheta_2)\) on the same flux surface, the kinetic accretion law follows

\[
\frac{U_{\parallel s}(\psi, \vartheta_1)}{U_{\parallel s}(\psi, \vartheta_2)} \equiv \frac{B(\psi, \vartheta_1) T_{\parallel s}(\psi, \vartheta_1)}{B(\psi, \vartheta_1) T_{\parallel s}(\psi, \vartheta_2)}. \tag{90}
\]

In this case, under the same assumptions, it follows from the continuity equation that the ratio of the corresponding species number densities must vary on a given \( \psi \)-surface according to the following relation:

\[
\frac{n_s(\psi, \vartheta_1)}{n_s(\psi, \vartheta_2)} \equiv \frac{B^2(\psi, \vartheta_1) T_{\parallel s}(\psi, \vartheta_2)}{B^2(\psi, \vartheta_2) T_{\parallel s}(\psi, \vartheta_1)}. \tag{91}
\]

Therefore, on a given \( \psi \)-surface:

1) the species parallel flow velocity increases with the parallel temperature while decreasing with respect to the magnitude of the magnetic field;

2) the species number density instead increases with the magnetic pressure and decreases with the parallel temperature.

The physical interpretation for both of these is clear: higher magnetic pressure slows down the matter accretion rate while increasing the number density, whereas higher parallel temperature corresponds to higher radial fluid mobility, thus decreasing the local species number density.

Next, we consider domains of strongly-magnetized plasmas with closed field lines, again using the inverse aspect ratio expansion. We want to prove that the poloidal-averaged radial flow velocity \( \langle V_{Rs} \rangle_\vartheta \) vanishes identically, when \( V_{Rs} \) is computed in terms of the quasi-stationary KDF \((89)\). Given a generic function of the form \( C(\psi, \vartheta) \), the operator \( \langle C(\psi, \vartheta) \rangle_\vartheta \) is defined as

\[
\langle C(\psi, \vartheta) \rangle_\vartheta \equiv \frac{1}{\kappa} \int_0^{2\pi} \frac{d\vartheta}{|\mathbf{B}_p \cdot \nabla \vartheta|} C(\psi, \vartheta), \tag{92}
\]

with \( \kappa \equiv \int_0^{2\pi} \frac{d\vartheta}{|\mathbf{B}_p \cdot \nabla \vartheta|} \) and so, from Eq.(90) it follows that, to leading-order in the inverse aspect ratio

\[
\langle V_{Rs} \rangle_\vartheta \equiv \frac{1}{\kappa} \int_0^{2\pi} \frac{d\vartheta}{|\mathbf{B}_p \cdot \nabla \vartheta|} \mathbf{B}_p \cdot \nabla R \Delta U_{\psi, \vartheta}(\psi) \neq 0 \tag{93}
\]

which vanishes because, to leading-order in \( \delta, \mathbf{B}_p \cdot \nabla R \cong \mathbf{B}_p \cdot \nabla \vartheta \cdot \nabla R \) is antisymmetric with respect to the transformation \( \vartheta \rightarrow \vartheta + \pi \). Higher-order corrections in the inverse aspect ratio can be included, but they require also performing the asymptotic expansion of the KDF to higher-order in both \( \sigma_s \) and \( \varepsilon_s \). Therefore, in sub-domains of the plasma where magnetic surfaces are closed and nested no net equilibrium accretion flow can arise.

Finally, the corresponding treatment for weakly-magnetized plasmas can be recovered from Eq.(13). Since the species flow velocity \( \mathbf{V}_{s}^{\text{tot}} \) has only an azimuthal component, it follows that \( V_{Rs} \) is identically zero. Hence, under the present assumptions, no net quasi-stationary radial flow can arise in the case of weakly-magnetized plasmas.

To summarize: the present theory provides a possible new collisionless physical mechanism giving an equilibrium accretion process in AD plasmas. In particular, we note that:

1) Only strongly-magnetized plasmas with open magnetic surfaces can sustain these equilibrium accretion flows.

2) The primary source of this equilibrium accretion flow mechanism is the appearance of equilibrium radial flows driven by temperature anisotropies and phase-space anisotropies. These are directly connected with the existence of
non-isotropic species pressure tensors, which in turn play the role of an effective viscosity in driving quasi-stationary accretion flows.

3) Quasi-stationary accretion flows are consistent with the basic conservation laws (for mass density and canonical momentum) and with the existence of a non-isotropic species pressure tensor (see also the discussion in Section 11).

4) The accretion law could, in principle, be tested experimentally if one had suitable observations, since it relates the magnitude of the species parallel flow velocity $U_{s||}$ to the local values of the magnetic field magnitude and the parallel temperature.

5) First-order (as well as higher-order) perturbative corrections, can in principle be included consistently in the present theory.

XI. THE KINETIC DYNAMO

Here we address the problem of the self-generation of magnetic field in quasi-stationary collisionless AD plasmas. We refer here to this phenomenon of self-generation of both poloidal and toroidal magnetic fields as a quasi-stationary kinetic dynamo effect. This is described by the Ampere equation with current density $\mathbf{J}$ defined as

$$\mathbf{J} = \sum_s \mathbf{J}_s = \sum_s Z_s e \int_{\Gamma_u} d^3 v f_s$$

for $f_s = \hat{f}_s$ or $f_s = f_{ws}$ respectively in the cases of strongly and weakly-magnetized plasmas. In all cases considered here, the current density can be generally represented in terms of the magnetic coordinates $(\psi, \varphi, \theta)$ as

$$\mathbf{J} = (J_\psi \nabla \vartheta \times \nabla \varphi, J_\varphi \nabla \varphi, J_\theta \nabla \psi \times \nabla \varphi).$$

In particular, based on the calculation of the flow velocity (see Section 10), it can be shown that:

a) For strongly-magnetized plasmas with open magnetic surfaces all of the three components $(J_\psi, J_\varphi, J_\theta)$ are generally non-vanishing. These include the contributions carried by TPs, BPs and PPs.

b) For strongly-magnetized plasmas with closed magnetic surfaces, $J_\psi$ vanishes identically. In this case the current includes contributions from both TPs and PPs.

c) For weakly-magnetized plasmas both $J_\psi$ and $J_\varphi$ vanish identically.

The toroidal component of the Ampere equation gives the generalized Grad-Shafranov equation for the poloidal flux function $\psi_p$:

$$\Delta^* \psi_p = -\frac{4\pi}{c} J_\varphi,$$

where $\Delta^* \equiv R^2 \nabla \cdot (R^{-2} \nabla)$ is an elliptic differential operator. The components of Ampere’s equation along the directions $\nabla \vartheta \times \nabla \varphi$ and $\nabla \psi \times \nabla \varphi$ provide instead two PDEs for the toroidal component of the magnetic field $|B_T| = I/R$ (see the definition in Eq.(7)):

$$\frac{\partial I}{\partial \psi} = \frac{4\pi}{c} J_\theta,$$

$$\frac{\partial I}{\partial \theta} = \frac{4\pi}{c} J_\psi,$$

which give the general solubility constraint

$$\frac{\partial J_\psi}{\partial \psi} = \frac{\partial J_\theta}{\partial \theta},$$

which is equivalent to imposing the charge continuity equation $\nabla \cdot \mathbf{J} = 0$. In the case of closed surfaces, the additional solubility condition

$$\oint J_\psi d\theta = 0$$

must also be imposed. Both constraint equations must be taken as being solubility conditions to be satisfied due to the arbitrariness of the structure functions.
For open surfaces, the function $I$ is of the general form $I(\psi, \vartheta)$ and so Eq. (99) can always be satisfied. Instead, for closed surfaces $I = I(\psi)$, which can always be asymptotically satisfied if inverse aspect ratio ordering applies. Constraints (99) and (100) are then both satisfied.

Various scenarios can be envisaged in which quasi-stationary kinetic dynamos can be present. We will list the basic features of this mechanism:

1) In all of the cases discussed above, $J_\varphi$ is generally non-vanishing, implying the existence of a self-generated poloidal magnetic field. Instead, a toroidal component of the magnetic field only arises if a poloidal current is present.

2) For strongly-magnetized plasmas, first-order FLR-diamagnetic and energy-correction effects, driven by temperature anisotropy, are responsible for the generation of poloidal currents, and hence of toroidal magnetic field. Gyrophase-dependent contributions can arise in this case, driven by the same thermodynamic forces. These originate from the guiding-centre back-transformation, which is characteristic of open-field configurations, and are responsible for the generation of $J_\psi$. For strongly-magnetized plasmas with open magnetic surfaces, the parallel velocity $U_\parallel$ also contributes to the generation of poloidal currents.

3) For weakly-magnetized plasmas, only the poloidal component of the magnetic field can be self-generated. However, when the azimuthal angular velocity coincides to leading-order with the Keplerian frequency, the azimuthal current density vanishes due to quasi-neutrality (to leading-order). Therefore, in this case the current is necessarily produced by first-order corrections. This conclusion is consistent with the assumption of weak magnetic field.

We stress that, in contrast to customary MHD treatments, the quasi-stationary kinetic dynamo effect described here can occur even in the absence of possible instabilities or turbulence phenomena. In particular, configurations with closed magnetic surfaces or contributions from TPs in the case of open surfaces could be responsible for the self-generation of toroidal field even without needing any net accretion flow in the domain of interest. This toroidal field is associated with the existence of kinetic torques which cause redistribution of angular momentum, as discussed above.

XII. CONCLUSIONS

In this paper a kinetic description for collisionless quasi-stationary accretion disc plasmas has been formulated within the framework of Vlasov-Maxwell theory. A perturbative approach has been developed, which enables the systematic analytical construction of equilibrium kinetic distribution functions for non-relativistic axisymmetric collisionless plasmas subject to both gravitational and electromagnetic fields. The cases of both weakly and strongly-magnetized plasmas have been investigated, for configurations with both open and closed magnetic surfaces. The main aim was to establish kinetic equilibrium configurations to use as a starting point for subsequent perturbation analysis, looking for kinetic mechanisms which could give rise to an effective viscosity able to drive accretion flows within the collisionless regime.

Equilibrium solutions for the Vlasov equation with non-uniform number density, azimuthal rotation, possible accretion flows and non-uniform temperature anisotropy have been constructed. For doing this, the distribution function has been represented in terms of first integrals and adiabatic invariants, as follows from conservation laws of the single-particle dynamics, and taking into account suitable kinetic constraints. As a consequence, based on the perturbative kinetic approach, explicit constitutive equations for the fluid fields have been determined, which are accurate to first-order in the relevant expansion parameters. This permits the construction of asymptotic solutions of the corresponding MHD fluid equations systematically retaining all first-order corrections, including effects due to FLR-diamagnetic and energy-correction contributions.

Several physical issues have been analyzed. These concern: a) deriving equilibrium equations of state for the species pressure tensor components; b) establishing a fluid angular momentum conservation law and comparing it with the predictions of standard fluid treatments; c) investigating possible kinetic accretion processes within equilibrium AD configurations (without perturbing the equilibrium) and deriving a related kinetic accretion law; d) demonstrating the existence of a quasi-stationary kinetic dynamo mechanism for the self-generation of poloidal and toroidal magnetic field.

Potential applications of the theory include, in particular, the case of radiatively inefficient accretion flows. As mentioned above, future work will address making stability analysis of the kinetic equilibria presented here, so as to investigate further mechanisms which could be responsible for driving the accretion flow in the collisionless regime.

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