SECOND TIME SCALE OF THE METASTABILITY OF REVERSIBLE INCLUSION PROCESSES

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ABSTRACT. We investigate the second time scale of the metastable behavior of the reversible inclusion process in an extension of the study by [Bianchi, Dommers, and Giardinà, Electronic Journal of Probability, 22:1–34, 2017], which presented the first time scale of the same model and conjectured the scheme of multiple time scales. We show that \( N/d_N^2 \) is indeed the correct second time scale for the most general class of reversible inclusion processes, and thus prove the first conjecture of the foresaid study. Here, \( N \) denotes the number of particles, and \( d_N \) denotes the small scale of randomness of the system. The main obstacles of this research arise in calculating the sharp asymptotics for the capacities, and in the fact that the methods employed in the former study are not directly applicable due to the complex geometry of particle configurations. To overcome these problems, we first thoroughly examine the landscape of the transition rates to obtain a proper test function of the equilibrium potential, which provides the upper bound for the capacities. Then, we modify the induced test flow and precisely estimate the equilibrium potential near the metastable valleys to obtain the correct lower bound for the capacities.

1. INTRODUCTION

An interacting particle system was introduced in \([12,13]\) as a dual process of a certain class of energy diffusion models, known as Brownian momentum (energy) processes. In \([14]\), this process was first named as a (symmetric) inclusion process, which was treated as a bosonic counterpart of the well-known exclusion process. Since then, this particular random system has gathered the interest of numerous researchers. A general overview on the study of inclusion processes is provided in \([10,\text{Chapters 2 and 6}]\).

Condensation takes place in various particle systems that exhibit attractive interactions. It is defined by the situation in which a significant portion of particles in the system becomes concentrated at a single site (cf. \((2.6)\)), due to the bosonic interactions among them. This phenomenon has been a consistently popular research subject during the past few decades. Condensation of inclusion processes was first studied in \([15]\), where the authors presented the unique invariant measure of the system under some restrictions, together with the condensation result of particles in the dynamics. Since then, a variety of results were reported on condensation of inclusion processes under various conditions and geometries of the system \([2,10,18,23]\).

Key words and phrases. Metastability, multiple time scales, interacting particle systems, inclusion process.

\(^1\)Bosonic particle systems represent dynamics in which particles tend to attract each other. They are mostly used to represent dynamical systems in low temperatures.
Metastability represents the macroscopic phenomenon that occurs when certain observables in a system linger in one state for an extended period of time and at a random moment later evolve to another state within a relatively short time. In the context of particle systems, this is described as follows: After condensation occurs, the condensate of particles remains at its site for a relatively long time. However, on appropriately long time scales, it tends to move to another site within the system. This behavior, also referred to as tunneling, can be characterized by a suitable random walk of the condensate on the collection of sites. In the context of inclusion processes, this phenomenon was first characterized in [16], where the authors showed the asymptotic behavior of formation and evolution of the condensate. However, this striking result was obtained only for symmetric inclusion processes. Accordingly, the next objective was to find a similar result for a more general class of inclusion processes. This project has been steadily maturing over the past few years. [7] reported a result on the metastable behavior for reversible inclusion processes. Moreover, in [19], Seo and the author of the current paper worked on the setting of general non-reversible inclusion processes.

The metastable behavior of reversible inclusion processes is subjected to the scheme of multiple time scales, which was studied thoroughly in [4]. For completeness, we briefly recall the result from [7, Theorem 2.3], stating that metastable behavior exists among certain sites $S_\star$ (cf. Proposition 2.3) in the first time scale $1/d_N$, where $d_N$ denotes the control factor of randomness of the dynamics which vanishes as the number of particles $N$ tends to infinity. We must emphasize that the limiting metastable dynamics on $S_\star$ may not be irreducible, in contrast to the original underlying random walk. Because the original process is irreducible, it is expected that all metastable states are eventually achievable. Hence, the system is likely to exhibit completely novel metastable movements at longer time scales. The authors of [7] conjectured two additional time scales, $N/d_N^2$ and $N^2/d_N^3$, by proving the existence of such time scales in a simple one-dimensional setting. Moreover, they showed that $N^2/d_N^3$ represents the terminal level of metastability, in the sense that there are no time scales larger than $N^2/d_N^3$ in which metastable movements occur.

In this study, we extensively generalize the metastable result of reversible inclusion processes in [7], and we fully characterize the metastable behavior in the second time scale, $\theta_{N,2} = N/d_N^2$. Specifically, we prove that the conjectured second time scale $N/d_N^2$ is indeed the correct one for the most general class of reversible underlying random walks, and that there are no intermediate time scales between $1/d_N$ and $N/d_N^2$. This leads us to a complete analysis of the metastability of reversible inclusion processes up to the second level on the scheme of multiple time scales.

In this research, we encountered two mathematical obstacles. The first obstacle concerns with the investigation of the sharp asymptotics of the equilibrium potential, which is the main ingredient to apply the Dirichlet principle. To overcome this issue, we first analyze the simplest case (Condition 2.7) and carefully examine the Dirichlet form of the dynamics to find a proper test function (Subsection 5.2). This test function naturally induces a test flow (3.3) and Proposition 3.3). The second obstacle concerns with the control of the major
and minor parts of the divergence of this test flow. This is essential to apply Theorem 3.6, which originates from [24]. We deal with the minor part in Subsection 6.1 by replacing the test flow by its asymptotic limit. Subsequently, we address the major part in Subsections 6.2 and 6.3 using the fact that the equilibrium potential behaves well near metastable valleys (cf. Lemma 4.4). After settling these problems for the simple case, we address the general model by applying a similar method to obtain the main theorem. A more detailed explanation of the procedure is provided in Remark 2.9 and Section 3.

Moreover, we strongly agree with the other conjecture in [7], that $N^2/d_N^3$ is indeed the third time scale of this process, and that the given three time scales completely characterize the metastable behavior, indicating that there are no additional time scales in metastable movements. However, it remains the possibility that an intermediate step of metastable behavior emerges between them. Investigating the third time scale is out of the scope of the current machineries developed in this study (Remark 2.11(4)). Hence, this topic serves as the main objective of future research in this direction.

Notably, the degree of $d_N$ increases by 1 in the consecutive time scales; $1/d_N \to N/d_N^2 \to N^2/d_N^3$. This is attributed to the fundamental property of transition rates of the inclusion process. According to (2.1), the process has to wait a time of order $1/d_N$ to send a particle to an empty site. As long as a site is occupied with at least one particle, it requires roughly constant-scale time to send the rest of the particles there. Hence, the degrees of $d_N$ in the time scales represent the graph distance (see footnote 3) between the corresponding metastable states. Evidently, the scale grows as the distance increases. This serves as a milestone in constructing the exact test function representing the equilibrium potential (see Subsections 5.2 and 7.2 for further detail).

In contrast, the metastability of non-reversible inclusion processes occurs in an entirely different manner. It has been established in [19] that there are two types of first time scales in the system, namely $1/d_N$ if the limit dynamics is symmetric (and thus generalizes the reversible case), and $1/(d_NN)$ if it is asymmetric. Succeeding time scales are entirely unidentified and deserve extensive future research. Further information is provided in [19, Theorems 3.10 and 3.12].

Metastability of inclusion processes is often compared to that of the well-known (supercritical and critical) zero-range processes, as they both involve bosonic particle systems representing stickiness in low temperature. Moreover, both metastable behaviors can be proven by a similar series of techniques, known as the martingale approach [3,6]. The main difference between them is that unlike the inclusion process, the zero-range process exhibits single-step metastable behavior; hence, there is only one time scale. This is because particle movements in the zero-range process are affected only by the number of particles on the starting site, which is the reason behind the naming “zero-range.” Hence, on a suitable time scale, all metastable states are reachable simultaneously. Full details on the recent results on condensation and metastability of zero-range processes are provided in [1,5,21,24].
We assume throughout this article that the number of sites is fixed, and the number of particles diverges. Alternatively, a model, named the inclusion process in the thermodynamic limit assumes that the number of sites tends to infinity along with the number of particles, so that particle density converges to a certain target density $\rho > 0$. In this case, yet another type of condensation and metastability is detected. [9] provides formulation and computational data, while [17] reports various condensation results depending on the behavior of $d_N$, and [19, Theorems 3.21, 3.22, and 3.23] present the general result of metastability on the torus.

The main ingredients of the proof of our main result are the potential theory [8] and the martingale approach [3,6]. Compared to the classical pathwise approach to metastability, the potential-theoretic approach has the big advantage of being highly useful in calculating the sharp asymptotics of the mean hitting time between metastable states and the consecutive metastable movements among the sites in the limit. Based on this technology, Beltrán and Landim proposed an outstanding method of calculating the mean transition rates of the trace process by precisely estimating the corresponding capacities of the system. We explain these methodologies in more detail in Section 3.

2. Notation and Main Results

We first settle the basic notation in this article.

- The set of natural numbers, $\mathbb{N}$, includes 0, i.e., $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 0\}$.
- Writing $\alpha, \beta \in \mathbb{R}$ or $\{\alpha, \beta\} \subseteq \mathbb{R}$ implies that $\alpha$ and $\beta$ are different.
- For integers $a$ and $b$ with $a \leq b$, $[a, b]$ represents $[a, b] \cap \mathbb{Z}$, i.e., the set of integers from $a$ to $b$.
- For two sequences $\{\alpha_N\}_{N \geq 1}$ and $\{\beta_N\}_{N \geq 1}$ of real numbers, $\alpha_N$ and $\beta_N$ are asymptotically equal, or $\alpha_N \simeq \beta_N$ if $\lim_{N \to \infty} \alpha_N/\beta_N = 1$.
- In what follows, $C$ denotes a global positive constant which may vary among equations.
- For functions $f$ and $g$ of $N$, we write $f(N) = O(g(N))$ if there exists a constant $C$ with the property that $|f(N)| \leq Cg(N)$ for all $N \geq 1$. Moreover, we write $f(N) = o(g(N))$ if $\lim_{N \to \infty} f(N)/g(N) = 0$.

2.1. Reversible inclusion processes. We fix a finite state space $S$ which represents our collection of sites. Suppose that $r: S \times S \to [0, \infty)$ is a transition rate function which defines a continuous-time irreducible random walk on $S$. For convenience, we let $r(x, x) = 0$ for all $x \in S$. We further assume that the random walk is reversible with respect to a probability distribution $m$, namely,

$$m(x)r(x, y) = m(y)r(y, x)$$

for all $x, y \in S$.

The sites with maximal measure deserve particular attention, as they are precisely the sites where particles condensate (Proposition 2.3). We define

$$M_* = \max\{m(x) : x \in S\}, \quad S_* = \{x \in S : m(x) = M_*\}, \quad \text{and} \quad m_*(\cdot) = \frac{m(\cdot)}{M_*}.$$
Notably, \( m_*(x) \leq 1 \) for all \( x \in S \), and the equality holds if and only if \( x \in S_* \).

Based on the underlying random walk introduced above, we introduce the inclusion process on \( S \). First, the set of configurations corresponding to the distribution of \( N \) particles on \( S \) is denoted by

\[
\mathcal{H}_N = \left\{ \eta \in \mathbb{N}^S : \sum_{x \in S} \eta_x = N \right\}.
\]

Hence, \( \eta_x \) is regarded as the number of particles at \( x \in S \) of \( \eta \).

Now, we define the inclusion process to be a continuous-time Markov chain \( \{\eta_N(t)\}_{t \geq 0} \) on \( \mathcal{H}_N \) associated with generator \( \mathcal{L}_N \) acting on functions \( f : \mathcal{H}_N \to \mathbb{R} \) by

\[
(\mathcal{L}_N f)(\eta) = \sum_{x, y \in S} \eta_x (d_N + \eta_y) r(x, y) \{ f(\sigma^{x \to y} \eta) - f(\eta) \} \quad \text{for} \quad \eta \in \mathcal{H}_N.
\]

Here, \( \sigma^{x \to y} \eta \) is the configuration obtained from \( \eta \) by sending a particle, if possible, from \( x \) to \( y \). Hence, if \( \eta_x = 0 \), then \( \sigma^{x \to y} \eta = \eta \) and if \( \eta_x \geq 1 \), then \( (\sigma^{x \to y} \eta)_x = \eta_x - 1 \), \( (\sigma^{x \to y} \eta)_y = \eta_y + 1 \), and \( (\sigma^{x \to y} \eta)_z = \eta_z \) for \( z \neq x, y \). Moreover, \( \{d_N\}_{N \geq 1} \) is a sequence of positive real numbers converging to 0. We will further assume that \( d_N \) decays more quickly than the logarithmic scale;

\[
\lim_{N \to \infty} d_N \log N = 0.
\]

A typical choice for \( d_N \) in practice is the polynomial scale, \( d_N = 1/N^\alpha \), \( \alpha > 0 \). One can readily verify that \( \eta_N(\cdot) \) is irreducible. We denote the transition rate of this process by \( q_N : \mathcal{H}_N \times \mathcal{H}_N \to [0, \infty) \), and the law and expectation of the process starting at \( \eta \) by \( \mathbb{P}_\eta = \mathbb{P}_\eta^N \) and \( \mathbb{E}_\eta = \mathbb{E}_\eta^N \), respectively.

We conclude this subsection with a brief explanation of the dynamical characteristics of the inclusion process. Given a configuration \( \eta \in \mathcal{H}_N \), a particle moves from site \( x \) to site \( y \) at rate

\[
q_N(\eta, \sigma^{x \to y} \eta) = \eta_x (d_N + \eta_y) r(x, y) = d_N \eta_x r(x, y) + \eta_x \eta_y r(x, y).
\]

Here, \( d_N \eta_x r(x, y) \) denotes the diffusive part and \( \eta_x \eta_y r(x, y) \) denotes the inclusive part of the dynamics. More specifically, the diffusive part represents the random walk of each particle with respect to \( r(\cdot, \cdot) \), which is controlled by a parameter \( d_N \). In contrast, the inclusive part represents the attractive behavior of particles, because the rate from \( x \) to \( y \) increases as \( \eta_y \) increases, and particles tend to prefer more occupied sites. As \( d_N \) decays to 0, the inclusive behavior is expected to dominate the dynamics. Consequently, particles are very likely to assemble at a single site, forming a condensate (Proposition 2.3). However, the small diffusive interactions trigger a long-term evolution of this condensate among sites, which is referred to as tunneling or metastable behavior (Theorem 2.5). Precise interpretation of these concepts is provided in the following.

2.2. Condensation of reversible inclusion processes. Because the process \( \eta_N(\cdot) \) is irreducible, it exhibits a unique invariant distribution. We denote the unique distribution by \( \mu_N \).

The great advantage we gain by assuming reversibility of the underlying random walk is that \( \eta_N(\cdot) \) likewise becomes reversible with respect to \( \mu_N \), and that \( \mu_N \) admits an explicit formula.
This is stated in the following proposition, whose proof is straightforward. Hereafter, $\Gamma(\cdot)$ denotes the typical Gamma function.

**Proposition 2.1.** The inclusion process $\{\eta_N(t)\}_{t \geq 0}$ is reversible with respect to the invariant measure $\mu_N$, which satisfies

$$\mu_N(\eta) = \frac{1}{Z_N} \prod_{x \in S} w_N(\eta_x) m_*(x)^{\eta_x} \text{ for } \eta \in \mathcal{H}_N,$$

(2.3)

where

$$w_N(n) = \frac{\Gamma(d_N + n)}{n! \Gamma(d_N)}, \quad n \in \mathbb{N}, \quad \text{and } Z_N = \sum_{\eta \in \mathcal{H}_N} \prod_{x \in S} w_N(\eta_x) m_*(x)^{\eta_x}.$$

**Remark 2.2.** The following asymptotics hold for the functions introduced in Proposition 2.1:

$$1 \leq \frac{(d_N + k)w_N(k)}{d_N} = \frac{(k + 1)w_N(k + 1)}{d_N} \leq e^{d_N \log N}, \quad k \geq 0, \quad \text{and } \lim_{N \to \infty} \frac{NZ_N}{d_N} = |S_*|. \quad (2.4)$$

In particular, $(d_N + k)w_N(k) = (k + 1)w_N(k + 1) \simeq d_N$ by (2.2), which is uniform in $k \geq 0$. These convergence results are frequently applied in the following. The proofs are provided in [7, Lemma 3.1 and Proposition 3.2].

Next, we define the metastable valleys of the process. Let

$$\mathcal{E}^x_N = \{\xi^x\} = \{\eta \in \mathcal{H}_N : \eta_x = N\} \text{ for } x \in S.$$

Hence, $\xi^x$ represents the configuration where all particles are concentrated on the site $x$. Each $\mathcal{E}^x_N$ is referred to as a valley of the system. Moreover, we denote $\mathcal{E}_N(A) = \bigcup_{x \in A} \mathcal{E}^x_N$ for $A \subseteq S$. Valleys of further special interest are $\mathcal{E}^x_N$ for $x \in S_*$, as explained by the following proposition. The proof of this is provided in [7, Proposition 2.1].

**Proposition 2.3.** For each $x \in S_*$, it holds that

$$\lim_{N \to \infty} \mu_N(\mathcal{E}^x_N) = \frac{1}{|S_*|}. \quad (2.5)$$

Consequently, we have $\lim_{N \to \infty} \mu_N(\mathcal{H}_N \setminus \mathcal{E}_N(S_*)) = 0$.

**Remark 2.4.** In particular, $\mathcal{E}^x_N$, $x \in S_*$, are referred to as metastable valleys of the process.

For simplicity, we write $\mathcal{E}^x_N = \mathcal{E}_N(S_*)$. Proposition 2.3 implies that the (static) condensation occurs on $S_*$, i.e.,

$$\lim_{N \to \infty} \mu_N(\mathcal{E}^x_N) = 1. \quad (2.6)$$

This fact depends heavily on the explicit formula (2.3). If the underlying random walk is non-reversible, then the right-hand side of (2.3) is not necessarily the invariant distribution of the system. In fact, we do not have a closed formula of the invariant distribution in this case. Thus, even the basic condensation result on valleys is not a simple issue for non-reversible inclusion processes. Nevertheless, condensation on $\mathcal{E}_N(S)$ can be demonstrated for non-reversible systems by adding a few minor conditions on $d_N$ and $r(\cdot, \cdot)$. For a recent result on this topic, we refer to [19, Theorem 3.15].
2.3. **First time scale of the metastable behavior of reversible dynamics.** The first time scale is fully characterized in \[7\]. We recall the result in this subsection to motivate our main result of this study. To this end, we must first introduce the trace process.

A non-empty subset \( \mathcal{G} \) of \( \mathcal{H}_N \) is fixed, and a non-decreasing random variable \( T^\mathcal{G} \), the local time in \( \mathcal{G} \) of the process, is defined by

\[
T^\mathcal{G}(t) = \int_0^t \mathbb{1}\{\eta_N(s) \in \mathcal{G}\} ds, \quad t \geq 0.
\]

Let \( S^\mathcal{G} \) be its generalized inverse function:

\[
S^\mathcal{G}(t) = \sup\{s \geq 0 : T^\mathcal{G}(s) \leq t\}, \quad t \geq 0.
\]

Then, the **trace process** \( \{\eta^\mathcal{G}_N(t)\}_{t \geq 0} \) on \( \mathcal{G} \) is defined by

\[
\eta^\mathcal{G}_N(t) = \eta_N(S^\mathcal{G}(t)) \quad \text{for} \quad t \geq 0.
\]

The random time \( T^\mathcal{G}(t) \) measures the amount of time up to \( t \) that the process spends in \( \mathcal{G} \). Hence, the random function \( S^\mathcal{G} \) reconstructs the global time of the process, starting from the local time in \( \mathcal{G} \). In this sense, the trace process \( \eta^\mathcal{G}_N(\cdot) \) on \( \mathcal{G} \) is obtained from the original process \( \eta_N(\cdot) \) by turning off the clock whenever it is not in \( \mathcal{G} \). Therefore, \( \eta^\mathcal{G}_N(\cdot) \) becomes a continuous-time, irreducible Markov chain on \( \mathcal{G} \). Rigorous proof of this fact can be found in e.g., \[3\].

Here, we trace the original process \( \eta_N(\cdot) \) on \( \mathcal{E}^*_N \), where condensation occurs. For simplicity, it is denoted by

\[
\eta^*_N(\cdot) = \eta^\mathcal{E}^*_N(\cdot).
\]

As we are concerned only with the superscripts of the sets \( \{\mathcal{E}^*_N : x \in S_*\} \), we define a projection function \( \Psi_{1,N} : \mathcal{E}^*_N \to S_* \) as

\[
\Psi_{1,N}(\xi^x) = x \quad \text{for} \quad x \in S_*,
\]

The symbol 1 in the subscript of \( \Psi_{1,N} \) denotes the first time scale of metastability. Using this function, we define a process \( \{X_N(t)\}_{t \geq 0} \) on \( S_* \) by

\[
X_N(t) = \Psi_{1,N}(\eta^*_N(t)) \quad \text{for} \quad t \geq 0.
\]

In general, \( X_N(\cdot) \) is non-Markovian, as it is merely a process of labelling of the metastable valleys. However, in the case of inclusion processes, \( X_N(\cdot) \) is indeed a Markov process, as \( \Psi_{1,N} \) is a bijection between \( \mathcal{E}^*_N \) and \( S_* \).

Here, we can formulate the first metastable behavior in terms of the projected trace process \( X_N(\cdot) \). Proof of the following theorem is provided in \[7\] Section 4.

**Theorem 2.5 (First time scale of reversible inclusion processes).** Fix a site \( x_0 \in S_* \) and let \( \theta_{N,1} = 1/d_N \).

(1) The law of the rescaled process \( \{X_N(\theta_{N,1}t)\}_{t \geq 0} \) starting at \( x_0 \) converges (with respect to the Skorokhod topology) on the path space \( D([0, \infty); S_* ) \) to the law of the Markov process
\{X_{\text{first}}(t)\}_{t \geq 0} \text{ on } S_\star \text{ starting at } x_0, \text{ which is defined by the generator}

\mathcal{L}_1 f(x) = \sum_{y \in S_\star} r(x, y) \{ f(y) - f(x) \}, \quad x \in S_\star \quad \text{for } f : S_\star \to \mathbb{R}.

(2) The process spends negligible time outside the metastable valleys, i.e., for all \( t > 0 \),

\begin{align*}
\lim_{N \to \infty} \sup_{\eta \in E_N^\star} \mathbb{E}_\eta \left[ \int_0^t \mathbb{1}_{\{ \eta_N(\theta_{N,1}s) \notin E_N^\star \}} ds \right] &= 0.
\end{align*}

Remark 2.6. In Theorem 2.5, the limiting dynamics \( X_{\text{first}}(\cdot) \) is exactly the underlying random walk restricted to \( S_\star \). Here, we must note that even though the underlying system is irreducible, \( X_{\text{first}}(\cdot) \) can still not be irreducible. For example, let \( S = \{1, 2, 3\} \), \( r(1, 2) = r(3, 2) = 1 \), and \( r(2, 1) = r(2, 3) = 2 \), as in the left part of Figure 2.1. Then, we have \( S_\star = \{1, 3\} \); thus, \( X_{\text{first}}(\cdot) \) on \( S_\star \) represents the null Markov chain. This phenomenon suggests additional time scales of the metastable behavior of inclusion processes.

We further remark that non-reversible inclusion processes exhibit a completely different scheme of metastability in the first time scale. Namely, in non-reversible inclusion processes, the time scale is \( 1/d_N \) if the limiting Markov chain of the process (cf. \( X_{\text{first}}(\cdot) \) in Theorem 2.5) is symmetric, and it is \( 1/(d_NN) \) if the limiting Markov chain is not symmetric. This is a remarkable difference in the metastability of reversible and non-reversible inclusion processes, and the details are provided in [19, Theorems 3.10 and 3.12].

2.4. Second time scale of the metastable behavior of reversible dynamics: Simple case. In this subsection, we present a simple case of our general main result. Namely, we assume that the following condition holds throughout this subsection.

\textbf{Condition 2.7.} \( S = \{x_1, x_2, y_1, y_2\} \) with

\begin{align*}
r(y_p, x_i) &> r(x_i, y_p) > 0 \quad \text{for } 1 \leq i, p \leq 2, \quad (2.7) \\
r(x_1, x_2) &= r(x_2, x_1) = 0, \quad \text{and} \quad (2.8) \\
m(x_1) &= m(x_2). \quad (2.9)
\end{align*}

In this setting, because the process is reversible, we have \( m_\star(x_1) = m_\star(x_2) = 1 \), \( m_\star(y_1) < 1 \), and \( m_\star(y_2) < 1 \), so that \( S_\star = \{x_1, x_2\} \). See the right part of Figure 2.1 for a visualization of this simple model.

There are two reasons for providing a simple version of the theorem first, instead of directly addressing the general main result. The first reason is that this simple model already covers most of the mathematical essentials of the second level of metastable behavior. The second reason is that proposing the proof of the general main result in a straightforward manner would be confusing to the readers, while inspecting the proof of the simple case first is helpful.

We add the term \textit{spl}, which denotes simple, in the superscripts of some quantities defined in this subsection to avoid possible confusion with the general main result in the following subsection.
By (2.8), we do not observe any movements in the first time scale by Theorem 2.5. Thus, it is natural to seek the following time scale, in which metastable behavior is exhibited between $x_1$ and $x_2$. Similar to the first scale, we define a projection function $\Psi_{2,N}^{\text{spl}} : \mathcal{E}_N^* \to \{1, 2\}$ by $\Psi_{2,N}^{\text{spl}}(\xi_{x_i}) = i$ for $1 \leq i \leq 2$.

Then, we define a process $Y_{2,N}^{\text{spl}}(\cdot)$ on $\{1, 2\}$ by

$$Y_{2,N}^{\text{spl}}(t) = \Psi_{2,N}^{\text{spl}}(\eta_1^*N(t)) \quad \text{for } t \geq 0.$$ 

Following the notation of [7], we state that $d_N$ decays subexponentially, if

$$\lim_{N \to \infty} d_N e^{\epsilon N} = \infty \quad \text{for any } \epsilon > 0. \quad (2.10)$$

Hence, (2.10) indicates that $d_N$ decays more slowly relative to any exponential scales. Moreover, we define a positive constant $\mathcal{R}$ by

$$\mathcal{R} = \int_0^1 \sum_{p=1}^2 \frac{1}{(1-m(yp))^{-1} \left( r(x_1, y_p) + r(x_2, y_p) \right)} dt. \quad (2.11)$$

**Theorem 2.8 (Second time scale of reversible inclusion processes: Simple case).** Assume Condition 2.7. Suppose that $d_N$ decays subexponentially. Define the second time scale as $\theta_{N,2} = N/d_N^2$ and fix $i_0 \in \{1, 2\}$. Then, the law of the rescaled process $\{Y_{N}^{\text{spl}}(\theta_{N,2} t)\}_{t \geq 0}$ starting at $i_0$ converges (with respect to the Skorokhod topology) on the path space $D([0, \infty); \{1, 2\})$.
to the law of the Markov process on \( \{1, 2\} \), starting at \( i_0 \) and jumping back and forth at rate \( 1/\mathcal{R} \).

Remark 2.9. Note that Theorem 2.8 slightly generalizes [7, Theorem 2.5], and there is a sole additional non-metastable site in the system. However, the approach used in [7, Section 5] fails even in this simplest case, due to two important drawbacks. First, the test function given in [7, Subsection 5.2] does not provide a direct clue of the test function we need for this generalized model, as this step requires a high-level understanding of the whole landscape of the transition rates. This is provided in Subsection 5.2. Second, it is impossible to apply the Cauchy–Schwarz inequality consecutively as in [7, Subsection 5.1]. This is because the inequalities used there do not provide a consistent equality condition; hence, this merely yields a weaker lower bound for the capacities. To overcome this, we employ Theorem 3.6 to obtain the lower bound, which was proposed in [24]; see Section 6 for further detail.

2.5. Second time scale of the metastable behavior of reversible dynamics: General case. Finally, in this subsection, we present the main result of this article in the most general setting. To this end, we decompose \( S_* \) into irreducible components with respect to \( X_{\text{first}}(\cdot) \), which is the limiting dynamics in the first scale (see the left part of Figure 2.2):

\[
S_* = \bigcup_{i=1}^{\kappa_*} S_i^{(2)}, \quad \text{where } S_i^{(2)} = \{x_{i,1}, \ldots, x_{i,n(i)}\} \text{ for } 1 \leq i \leq \kappa_*.
\]  

(2.12)

Here, we use the second label (1 to \( n(i) \)) in the elements of \( S_i^{(2)} \) to emphasize that they belong to the same set \( S_i^{(2)} \). The common superscript (2) denotes the second time scale. More
specifically, the system with transition rates \( r(\cdot, \cdot) \) restricted to \( S^{(2)}_i \) is irreducible for each \( i \in [1, \kappa_*] \), and \( r(x_{i,n}, x_{j,m}) = 0 \) for all \( i \neq j \), \( 1 \leq n \leq n(i) \), and \( 1 \leq m \leq n(j) \). By definition, we have

\[
|S_*| = \sum_{i=1}^{\kappa_*} |S^{(2)}_i| = \sum_{i=1}^{\kappa_*} n(i).
\]

In this setting, our dynamics in the second scale \( \theta_{N,2} = N/d_N^2 \) takes place on the set of \( \kappa_* \) elements; \( \{\mathcal{E}_N(S^{(2)}_i) : 1 \leq i \leq \kappa_* \} \). All elements in \( S^{(2)}_i \) that are connected in the first scale \( \theta_{N,1} \) form a metastable group in the second scale (Theorem 2.10(1)). If \( \kappa_* = 1 \), we observe all possible metastable movements in the first time scale; thus, the metastable behavior is fully characterized by Theorem 2.5, and there is no need for an additional time scale. Hence, hereafter we assume that \( \kappa_* \geq 2 \). Moreover, we write \( S \setminus S_* = \{y_1, \ldots, y_{\kappa_0}\} \), such that we have

\[
S = \{x_{i,n} : 1 \leq i \leq \kappa_* \}, \quad 1 \leq n \leq n(i) \} \cup \{y_1, \ldots, y_{\kappa_0}\}.
\]

From \( \kappa_* \geq 2 \) and irreducibility of the underlying random walk, it is straightforward that \( \kappa_0 \geq 1 \). For \( A \subseteq [1, \kappa_*] \), we introduce the notation \( \mathcal{E}_N^{(2)}(A) = \bigcup_{i \in A} \mathcal{E}_N(S^{(2)}_i) \). If \( A = \{a\} \), we abbreviate \( \mathcal{E}_N^{(2)}(\{a\}) \) as \( \mathcal{E}_N^{(2)}(a) \).

As in the simple case, we define a projection function. Let \( \Psi_{2,N} : \mathcal{E}_N^{(2)} \to [1, \kappa_*] \) be defined by

\[
\Psi_{2,N}(\xi_{x,n}) = i \text{ for } 1 \leq i \leq \kappa_* \text{ and } 1 \leq n \leq n(i).
\]

Then, we define a process \( Y_N(\cdot) \) on \([1, \kappa_*]\) by

\[
Y_N(t) = \Psi_{2,N}(\eta_N^{*}(t)) \text{ for } t \geq 0.
\]

In contrast to \( X_N(\cdot) \) and \( Y_N^{\text{spl}}(\cdot) \), \( Y_N(\cdot) \) is not necessarily Markovian, since \( \Psi_{2,N} \) is generally not bijective.

We are ready to state our main theorem. We define constants \( \mathfrak{M}_{i,j} \) for \( i, j \in [1, \kappa_*] \):

\[
\mathfrak{M}_{i,j} = \int_0^1 \frac{1}{\sum_{n=1}^{n(i)} \sum_{m=1}^{n(j)} \sum_{p=1}^{\kappa_0} \frac{(1-m_p(y_p))^{-1}}{r(x_{i,n,y_p})^p r(x_{j,m,y_p})^p}} \, dt. \quad (2.13)
\]

In (2.13), we regard the fraction in the denominator as 0 if \( r(x_{i,n}, y_p)r(x_{j,m}, y_p) = 0 \). In this sense, we write \( \mathfrak{M}_{i,j} = \infty \) if \( r(x_{i,n}, y_p)r(x_{j,m}, y_p) = 0 \) for all \( 1 \leq n \leq n(i), 1 \leq m \leq n(j) \), and \( 1 \leq p \leq \kappa_0^2 \). It is clear that \( \mathfrak{M}_{i,j} = \mathfrak{M}_{j,i} \).

Further, for \( \mathcal{A} \subseteq \mathcal{H}_N \), let \( \tau_{\mathcal{A}} = \tau_{\mathcal{A}}^N \) be the hitting time of the set \( \mathcal{A} \).

Theorem 2.10 (Second time scale of reversible inclusion processes: General case).
Suppose that \( d_N \) decays subexponentially. Then, with \( \theta_{N,2} = N/d_N^2 \), the following statements hold.

\[\text{We take } 1/\infty \text{ to be } 0 \text{ in the following.}\]
(1) For each \(1 \leq i \leq \kappa_*, \mathcal{E}_N(S_i^{(2)})\) thermalizes before reaching another metastable set, i.e.,
\[
\lim_{N \to \infty} \inf_{\eta, \iota \in \mathcal{E}_N(S_i^{(2)})} \mathbb{P}_\eta[\tau(\zeta) < \tau_{\mathcal{E}_N(S_i \setminus S_i^{(2)})}] = 1. \tag{2.14}
\]

(2) Fix \(i_0 \in [1, \kappa_*].\) Then, the law of the rescaled process \(\{Y_N(\theta_{N,2t})\}_{t \geq 0}\) starting at \(i_0\) converges (with respect to the Skorokhod topology) on the path space \(D([0, \infty); [1, \kappa_*])\) to the law of the Markov process \(X_{\text{second}}(\cdot)\) on \([1, \kappa_*]\) starting at \(i_0\) and defined by the generator, acting on functions \(f : [1, \kappa_*] \to \mathbb{R},\) given by
\[
(\mathcal{L}_2 f)(i) = \sum_{j \in [1, \kappa_*] \setminus \{i\}} \frac{1}{|S_i^{(2)}|} |\mathcal{R}_{i,j}| \{f(j) - f(i)\} \text{ for } i \in [1, \kappa_*]. \tag{2.15}
\]

Consequently, \(S_*\) is decomposed into irreducible components with respect to \(X_{\text{second}}(\cdot).\)

We denote this partition by
\[
S_* = S_1^{(3)} \cup \cdots \cup S_{N}^{(3)}. \tag{2.16}
\]

(3) Fix \(1 \leq i \leq \kappa_*\) and \(1 \leq n \leq n(i).\) From (2.16), there is a unique \(i \in [1, \gamma_*]\) such that \(S_i^{(3)} \subseteq S_j^{(3)}\) (see the right part of Figure 2.2). Then, starting at \(\xi^{x_i, n},\) the process spends negligible time outside \(\mathcal{E}_N(S_i^{(3)})\), which is uniform in all choices of \((i, n),\) i.e., for all \(t > 0,
\[
\lim_{N \to \infty} \sup_{i \in [1, \kappa_*], n \in [1, n(i)]} \mathbb{E}_{\xi^{x_i, n}} \left[ \int_0^t 1\{\eta_N(\theta_{N,2s}) \not\in \mathcal{E}_N(S_i^{(3)})\} ds \right] = 0.
\]

Remark 2.11. We remark several issues regarding the main theorem.

(1) Note that Theorem 2.8 is indeed a special case of Theorem 2.10 where \(\kappa_* = 2, n(1) = n(2) = 1, x_{1,1} = x_1, x_{2,1} = x_2, \kappa_0 = 2,\) and \(r(x_{1,1}, y_p)r(x_{2,1}, y_p) > 0\) for \(p = 1, 2.\)

(2) Theorem 2.10 proves the conjecture in [7] that \(\theta_{N,2} = N/d_N^2\) is indeed the second time scale in the metastability of reversible inclusion processes, in the sense that there are no intermediate time scales between \(\theta_{N,1}\) and \(\theta_{N,2}\).

(3) By (2.15), for \(i, j \in [1, \kappa_*],\) the limit transition rate from \(S_i^{(2)}\) to \(S_j^{(2)}\) is \(1/(|S_i^{(2)}| |\mathcal{R}_{i,j}|).\) This vanishes if and only if \(\mathcal{R}_{i,j} = \infty,\) which is equivalent to state that the graph distance between \(S_i^{(2)}\) and \(S_j^{(2)}\) is bigger than \(2.\) In this sense, we cannot observe a metastable movement between \(S_i^{(3)}\) and \(S_j^{(3)},\) \(i, j \in [1, \gamma_*],\) in the second time scale \(\theta_{N,2} = N/d_N^2.\) Because the original underlying random walk is irreducible, it is natural to suggest the existence of a third time scale, where we can detect metastable movements among \(S_i^{(3)},\) \(1 \leq i \leq \gamma_*\). In [7], this scale is strongly expected to be \(\theta_{N,3} = N^2/d_N^3.\) Moreover, even though \(\theta_{N,3}\) is proven to be the longest scale possible in [7], there is a possibility that an intermediate time scale exists between \(\theta_{N,2}\) and \(\theta_{N,3}.\) This can be considered a fruitful future research topic.

\footnote{For two subsets \(A\) and \(B\) of \(S,\) the graph distance between \(A\) and \(B\) is defined as \(\min\{n \geq 0 : \exists x_0, \ldots, x_n \in S\text{ such that } x_0 \in A, x_n \in B, \text{ and } r(x_i, x_{i+1}) > 0\text{ for } 0 \leq i \leq n - 1\}.\)
According to the previous remark, we attempted to apply the methodology used in this study to address the third time scale of metastability of the inclusion process. The first obstacle is encountered in constructing an exquisite test function which approximates the equilibrium potential, as in Subsection 7.2. This becomes far more complicated when compared to what is done here, as the geometric property of the typical path is highly complex in the third time scale. The other obstacle is that the asymptotic value of the equilibrium potential, which is successfully determined in the second time scale in Subsections 6.3 and 8.3, is unknown in the third time scale. In order to apply a similar methodology in the third scale, a precise information on the equilibrium potential of the entire typical path between metastable valleys is needed. At this point, this is a technically difficult task.

Note that (2.14) is not included in the previous metastability results, i.e., Theorems 2.5 and 2.8. This is because in the setting of the previous theorems, each metastable valley is a singleton; hence, thermalization is obvious.

In this study, all convergence results are provided in terms of convergence of the trace process in the Skorokhod topology. In fact, there are alternatives to the stated results, represented by convergence of the original process in the soft topology [20] and convergence of finite-dimensional marginal distributions [22]. We remark that given our result, the other modes of convergence may be easily proven by verifying some additional technical conditions presented in the foresaid studies. In Section 10, we prove the convergence of finite-dimensional marginal distributions using [22, Proposition 2.1]. This result is needed to prove (3) of Theorem 2.10.

3. Outline of proof of Theorems 2.8 and 2.10

In this section, we review some potential-theoretic notions and explain how to apply these skills to prove the main theorems, namely Theorems 2.8 and 2.10. We remark that some claims in this section hold due to reversibility, and more general results are provided in e.g., [11] and [25].

3.1. Potential theory and discrete flows. In this subsection, we introduce some crucial notions from the potential theory, which are fundamental in stating and proving our results.

**Definition 3.1.** (1) The Dirichlet form $D_N(\cdot)$ associated to our inclusion process is defined by, for $f : H_N \rightarrow \mathbb{R}$,

$$D_N(f) = \frac{1}{2} \sum_{\eta, \zeta \in H_N} \mu_N(\eta)q_N(\eta, \zeta)\{f(\zeta) - f(\eta)\}^2$$

$$= \frac{1}{2} \sum_{\eta \in H_N} \sum_{x, y \in S} \mu_N(\eta)\eta_x(d_N + \eta_y)r(x, y)\{f(x, y - \eta) - f(\eta)\}^2. \quad (3.1)$$

In this study, all convergence results are provided in terms of convergence of the trace process in the Skorokhod topology. In fact, there are alternatives to the stated results, represented by convergence of the original process in the soft topology [20] and convergence of finite-dimensional marginal distributions [22]. We remark that given our result, the other modes of convergence may be easily proven by verifying some additional technical conditions presented in the foresaid studies. In Section 10, we prove the convergence of finite-dimensional marginal distributions using [22, Proposition 2.1]. This result is needed to prove (3) of Theorem 2.10.
(2) If \( A \) and \( B \) are disjoint and non-empty subsets of \( \mathcal{H}_N \), then the \emph{equilibrium potential} \( h_{A,B} \) between \( A \) and \( B \) is defined by

\[
h_{A,B}(\eta) = \mathbb{P}_\eta[\tau_A < \tau_B].
\]

Note that \( h_{A,B} = 1 \) on \( A \) and \( h_{A,B} = 0 \) on \( B \).

(3) The \emph{capacity} between \( A \) and \( B \) is defined by

\[
\text{Cap}_N(A, B) = D_N(h_{A,B}).
\]

Later in this section, we show that to study the metastable behavior of interacting particle systems, it suffices to obtain sharp asymptotics on the corresponding capacities between metastable valleys (cf. (3.8) and Proposition 3.8). In the following, we define the discrete flows corresponding to our system.

**Definition 3.2.** (1) A function \( \phi : \mathcal{H}_N \times \mathcal{H}_N \to \mathbb{R} \) is considered a \emph{(discrete) flow} on \( \mathcal{H}_N \), if

(a) \( \phi \) is \emph{anti-symmetric}; that is, \( \phi(\eta, \zeta) = -\phi(\zeta, \eta) \) for all \( \eta, \zeta \in \mathcal{H}_N \), and

(b) \( \phi \) is \emph{compatible} to \( q_N(\cdot, \cdot) \); that is, \( \phi(\eta, \zeta) > 0 \) only if \( q_N(\eta, \zeta) > 0 \).

(2) An inner product structure \( \langle \cdot, \cdot \rangle_N \) is defined for the flows on \( \mathcal{H}_N \): For flows \( \phi \) and \( \psi \),

\[
\langle \phi, \psi \rangle_N = \frac{1}{2} \sum_{\eta, \zeta \in \mathcal{H}_N : q_N(\eta, \zeta) > 0} \frac{\phi(\eta, \zeta)\psi(\eta, \zeta)}{\mu_N(\eta)q_N(\eta, \zeta)}.
\]

Consequently, a norm \( \| \cdot \|_N \) is established in the system; \( \| \phi \|_N = \sqrt{\langle \phi, \phi \rangle_N} \).

(3) Given a flow \( \phi \) on \( \mathcal{H}_N \), the \emph{divergence} of \( \phi \) at \( \eta \in \mathcal{H}_N \) is defined as

\[
(\text{div} \, \phi)(\eta) = \sum_{\zeta \in \mathcal{H}_N} \phi(\eta, \zeta) = \sum_{\zeta \in \mathcal{H}_N : q_N(\eta, \zeta) > 0} \phi(\eta, \zeta).
\]

Following this notation, the divergence of \( \phi \) on \( A \subseteq \mathcal{H}_N \) is

\[
(\text{div} \, \phi)(A) = \sum_{\eta \in A} (\text{div} \, \phi)(\eta).
\]

Here, we connect two notions defined above: For a function \( f : \mathcal{H}_N \to \mathbb{R} \), we define a flow \( \Phi_f \) on \( \mathcal{H}_N \), which is given by

\[
\Phi_f(\eta, \zeta) = \mu_N(\eta)q_N(\eta, \zeta)\{f(\eta) - f(\zeta)\}; \; \eta \in \mathcal{H}_N, \; \zeta \in \mathcal{H}_N.
\]

Then, the following proposition explains the relationship between potential-theoretic objects and discrete flows.

**Proposition 3.3.** For each \( f : \mathcal{H}_N \to \mathbb{R} \), we have \( \| \Phi_f \|_N^2 = D_N(f) \). Consequently, if \( A \) and \( B \) are disjoint and non-empty subsets of \( \mathcal{H}_N \),

\[
\| \Phi_{h_{A,B}} \|_N^2 = D_N(h_{A,B}) = \text{Cap}_N(A, B).
\]
Proof. The first statement follows by a simple calculation. Namely,
\[
\|\Phi_f\|_N^2 = \frac{1}{2} \sum_{\eta, \zeta \in \mathcal{H}_N: q_N(\eta, \zeta) > 0} \frac{\{\Phi_f(\eta, \zeta)\}^2}{\mu_N(\eta)q_N(\eta, \zeta)}
\]
\[
= \frac{1}{2} \sum_{\eta, \zeta \in \mathcal{H}_N: q_N(\eta, \zeta) > 0} \mu_N(\eta)q_N(\eta, \zeta)\{f(\eta) - f(\zeta)\}^2 = D_N(f).
\]
Then, (3.4) is straightforward by Definition 3.1(3). \(\Box\)

3.2. Dual variational principles: The Dirichlet–Thomson principle. According to the definitions in the last subsection, two important variational principles hold, namely the Dirichlet principle and the Thomson principle. These statements play a key role in calculating the explicit behavior of the capacities in the limit \(N \to \infty\).

**Theorem 3.4 (The Dirichlet–Thomson principle).** Suppose that \(A\) and \(B\) are two disjoint and non-empty subsets of \(\mathcal{H}_N\).

1. *(Dirichlet)* It holds that
   \[
   \text{Cap}_N(A, B) = \inf_F D_N(F),
   \]
   where the infimum runs over functions \(F : \mathcal{H}_N \to \mathbb{R}\) satisfying
   \[
   F|_A = 1 \text{ and } F|_B = 0.
   \] (3.5)
   Moreover, the unique optimizer of the infimum is \(h_{A,B}\).

2. *(Thomson)* It holds that
   \[
   \text{Cap}_N(A, B) = \sup_{\phi} \frac{1}{\|\phi\|_N^2},
   \]
   where the supremum runs over flows \(\phi\) on \(\mathcal{H}_N\) satisfying
   \[
   (\text{div } \phi)(A) = 1 \text{ and } (\text{div } \phi)(\eta) = 0 \text{ for all } \eta \in \mathcal{H}_N \setminus (A \cup B).\] (3.6)
   Moreover, the unique optimizer of the supremum is \(\Phi_{h_{A,B}}/\text{Cap}_N(A, B)\).

In fact, the two principles in Theorem 3.4 have generalizations to non-reversible dynamics. Namely, [11, Theorem 2.7] gives the non-reversible version of (1), and [25, Proposition 2.6] gives the non-reversible version of (2). We refer to the references [11, 25] for the proof of Theorem 3.4.

**Remark 3.5.** Like the Dirichlet principle, the Thomson principle can also be stated in terms of test functions; however, it is somewhat more complicated to state the result, which is provided in [25, Proposition 2.1(ii)].

The application of Theorem 3.4 runs generally as follows. First, we construct test functions and flows, say \(g\) and \(\psi\), which satisfy (3.5) and (3.6), respectively. Second, we apply the principles to obtain
\[
\frac{1}{\|\psi\|_N^2} \leq \text{Cap}_N(A, B) \leq D_N(g).
\]
Finally, we send $N$ to infinity to obtain the desired estimate. From (3.4), it is natural to take $g$ and $\psi$, which in some sense approximate $h_{A,B}$ and $\Phi_{h_{A,B}}/\text{Cap}_{N}(A,B)$, respectively.

According to the above methodology, the Dirichlet principle is relatively easy to apply, as the restriction (3.5) is feeble. In contrast, the Thomson principle has a strong restriction on the test flows, (3.6). In particular, it is practically impossible to find a test flow that has vanishing divergence in each configuration in $H_{N \setminus (A \cup B)}$.

To overcome this drawback, we need the following variant of the useful result [24 Theorem 5.3], which generalizes the Thomson principle.

**Theorem 3.6 (Generalized Thomson principle).** Suppose that $A$ and $B$ are two disjoint and non-empty subsets of $H_{N}$. Then, for any non-trivial flow $\psi$ on $H_{N}$, it holds that

$$\text{Cap}_{N}(A,B) \geq \frac{1}{\|\psi\|_{N}^{2}} \left[ \sum_{\eta \in H_{N}} h_{A,B}(\eta)(\text{div} \, \psi)(\eta) \right]^{2}. \quad (3.7)$$

Moreover, the equality holds if and only if $\psi = c\Phi_{h_{A,B}}$ for a non-zero constant $c$.

**Proof.** By [24 Proposition 5.1(3)] and the Cauchy–Schwarz inequality,

$$\left[ \sum_{\eta \in H_{N}} h_{A,B}(\eta)(\text{div} \, \psi)(\eta) \right]^{2} = \left( \Phi_{h_{A,B}}, \psi \right)_{N}^{2} \leq \|\Phi_{h_{A,B}}\|_{N}^{2} \times \|\psi\|_{N}^{2} = \text{Cap}_{N}(A,B) \times \|\psi\|_{N}^{2}. \quad (3.7)$$

Because $\psi$ is non-trivial, we divide both sides by $\|\psi\|_{N}^{2}$ to obtain the result. The equality condition is straightforward.

**Remark 3.7.** If $\psi$ satisfies the condition (3.6), then Theorem 3.6 is equivalent to Theorem (3.4)(2), as

$$\sum_{\eta \in H_{N}} h_{A,B}(\eta)(\text{div} \, \psi)(\eta) = \sum_{\eta \in A}(\text{div} \, \psi)(\eta) = (\text{div} \, \psi)(A) = 1. \quad (3.6)$$

Thus, using Theorem 3.6, we may choose a test flow $\psi$ that does not satisfy the strict constraint (3.6). This point is demonstrated in Sections 6 and 8.

### 3.3. Martingale approach and outline of proof.

A sharp estimate of the capacities can be used to calculate the transition rates of the process traced on $E_{N}^{*}$, employing the following formula from [3 Lemma 6.8]: We denote by $q_{N}^{*} : E_{N}^{*} \times E_{N}^{*} \rightarrow [0, \infty)$ the transition rate of the trace process $\eta_{N}^{*}(\cdot)$, and we define the mean transition rate $r_{N}^{*} : [1, \kappa_{*}] \times [1, \kappa_{*}] \rightarrow [0, \infty)$ by $r_{N}^{*}(i, j) = 0$ and

$$r_{N}^{*}(i, j) = \frac{1}{\mu_{N}(E_{N}^{*}(S_{i}^{(2)})))} \sum_{\eta \in E_{N}^{*}(S_{i}^{(2)})} \mu_{N}(\eta) \sum_{\zeta \in E_{N}^{*}(S_{j}^{(2)})} q_{N}^{*}(\eta, \zeta) \text{ for } i, j \in [1, \kappa_{*}]. \quad (3.8)$$

Then, for $i, j \in [1, \kappa_{*}]$,

$$\mu_{N}(E_{N}^{*}(i))r_{N}^{*}(i, j) = \frac{1}{2} \left[ \text{Cap}_{N}(E_{N}^{*}(i), E_{N}^{*} \setminus E_{N}^{*}(i)) \right. \left. + \text{Cap}_{N}(E_{N}^{*}(j), E_{N}^{*} \setminus E_{N}^{*}(j)) - \text{Cap}_{N}(E_{N}^{*}(\{i, j\}), E_{N}^{*} \setminus E_{N}^{*}(\{i, j\})) \right]. \quad (3.8)$$
The asymptotics on \( r_N^*(\cdot, \cdot) \) is the main ingredient to describe the metastable behavior. This is explained in the following proposition, which is a consequence of the martingale approach developed in [3]. We refer to [3] Theorem 2.7 for its proof.

**Proposition 3.8.** Suppose that there exists a sequence \( \{\theta_N\}_{N \geq 1} \) of positive real numbers such that

\[
\lim_{N \to \infty} \theta_N r_N^*(i, j) = a(i, j) \quad \text{for all} \quad i, j \in [1, \kappa_*],
\]

for some \( a : [1, \kappa_*] \times [1, \kappa_*] \to [0, \infty) \). Moreover, suppose that the following estimate holds for each \( 1 \leq i \leq \kappa_* \):

\[
\lim_{N \to \infty} \inf_{i, \zeta \in E_N^2(i)} \frac{\text{Cap}_N(E_N^2(i), E_N^2 \setminus E_N^2(i))}{\text{Cap}_N(\{\eta\}, \{\zeta\})} = 0.
\]

Then, for each \( 1 \leq i \leq \kappa_* \), the following statements hold.

1. \( E_N^2(i) \) thermalizes before reaching another metastable set, i.e.,

\[
\lim_{N \to \infty} \inf_{i, \zeta \in E_N^2(i)} \mathbb{P}_\eta[\tau(\zeta) < \tau_{E_N^2 \setminus E_N^2(i)}] = 1.
\]

2. For each \( 1 \leq i \leq \kappa_* \), the law of the rescaled process \( \{Y_N(\theta_N t)\}_{t \geq 0} \) starting at \( i \) converges (with respect to the Skorokhod topology) on the path space \( D([0, \infty); [1, \kappa_*]) \) to the law of the Markov process on \([1, \kappa_*]\) starting at \( i \) with transition rates \( a(\cdot, \cdot) \).

To prove statement (3) of Theorem 2.10, we also must know the mode of convergence of finite-dimensional distributions of the rescaled process \( \eta_N(\theta_N \cdot) \). [22] Proposition 2.1 provides a simple approach of proving this result.

**Proposition 3.9.** Suppose that statement (2) of Theorem 2.10 holds, and that the process spends negligible time outside the metastable valleys, i.e., for \( t > 0 \),

\[
\lim_{N \to \infty} \sup_{\eta \in E_N^*} \mathbb{E}_\eta \left[ \int_0^t 1_{\{\eta_N(\theta_N, 2s) \notin E_N^*\}} ds \right] = 0.
\]

In addition, suppose that the following holds:

\[
\lim \limsup_{\delta \to 0} \sup_{N \to \infty} \sup_{2\delta \leq s \leq 3\delta} \mathbb{P}_\eta[\eta_N(\theta_N, 2s) \notin E_N^*] = 0.
\]

Then, the rescaled original process \( \eta_N(\theta_N, 2 \cdot) \) converges to \( X_{\text{second}}(\cdot) \) in the sense of finite-dimensional marginal distributions, i.e., for all \( 0 \leq t_1 < \cdots < t_k, i \in [1, \kappa_*], n \in [1, \eta(i)] \), and \( A_1, \ldots, A_k \subseteq [1, \kappa_*] \), it holds that

\[
\lim_{N \to \infty} \mathbb{P}_{\xi_n \xi} \left[ \eta_N(\theta_N, t_1) \in E_N^2(A_1), \ldots, \eta_N(\theta_N, t_k) \in E_N^2(A_k) \right] = \mathbb{P}_i[X_{\text{second}}(t_1) \in A_1, \ldots, X_{\text{second}}(t_k) \in A_k],
\]

where \( \mathbb{P}_i \) denotes the law of \( X_{\text{second}}(\cdot) \) starting at \( i \).

The remainder of this study is organized as follows. In Section 4, we provide some preliminaries regarding hitting times on the tubes. These are used in Sections 6 and 8. Subsequently,
we calculate the upper and lower bounds for the capacities, respectively, in the simple case of Theorem 2.8. This procedure is performed by the variational principles given in Theorems 3.4 and 3.6. In Sections 7 and 8 we provide the estimate of the capacities in the general case of Theorem 2.10. Then, we prove the condition (3.10) in the general case in Section 9. Finally, in Section 10, we use the estimates given in Propositions 3.8 and 3.9 to prove our main result, stated in Theorem 2.10. This simultaneously proves Theorem 2.8 as well.

4. Hitting Times on Tubes

We recall crucial results from [19], which provide sharp estimates of hitting times on the tubes. These are used in Sections 6 and 8 to compute the asymptotic equilibrium potential (Lemmas 6.4 and 8.3).

To state the results, we first define some relevant subsets of $H_N$. The notation is mainly inherited from [19].

**Definition 4.1.** (1) For every subset $R$ of $S$, define the $R$-tube $A^R_N$ as

$$A^R_N = \{ \eta \in H_N : \eta_x = 0 \text{ for all } x \in S \setminus R \}. \quad (4.1)$$

For example, $A^S_N = H_N$ and $A^{\{x\}}_N = \mathcal{E}^x_N$. We may write the superscripts of $A^R_N$ by the explicit elements of $R$ without commas.

(2) Especially, if $R = \{x, y\}$, we write $A^{xy}_N = \{ \eta \in H_N : \eta_x + \eta_y = N \}$.

(3) For $x, y \in S$ and $0 \leq i \leq N$, define the configuration $\xi^{xy}_i$ by

$$\begin{cases} i & \text{if } z = x, \\ N-i & \text{if } z = y, \\ 0 & \text{otherwise}, \end{cases}$$

such that $A^{xy}_N = \{ \xi^{xy}_0, \xi^{xy}_1, \ldots, \xi^{xy}_N \}$. Note that $\xi^{xy}_N = \xi^x$ and $\xi^{xy}_0 = \xi^y$.

(4) Finally, for $x, y \in S$, define

$$\hat{A}^R_N = \{ \eta \in A^R_N : \eta_x \geq 1 \text{ for all } x \in R \}. \quad (4.2)$$

Clearly, $\hat{A}^{xy}_N = \{ \xi^{xy}_0, \ldots, \xi^{xy}_{N-1} \}$ and $A^{xy}_N = \hat{A}^{xy}_N \cup \{ \xi^x, \xi^y \}$.

(5) Generally, if $R = \{a_1, \ldots, a_r\}$, then we denote by $\xi^{a_1 \ldots a_r}_{n_1, \ldots, n_{r-1}} \in A^R_N$ the element which satisfies

$$\xi^{a_1 \ldots a_r}_{n_1, \ldots, n_{r-1}}(a) = \begin{cases} n_j & \text{if } a = a_j \text{ for } j \in [1, r-1], \\ N - (n_1 + \cdots + n_{r-1}) & \text{if } a = a_r, \\ 0 & \text{otherwise}. \end{cases}$$
The tube $\mathcal{A}_N^R$ with $R = \{x, y\}$ has the advantage that it is an one-dimensional bridge of typical paths between two valleys, $\xi^x$ and $\xi^y$. More precisely, the following estimate from \cite[Lemma 4.7]{19} holds.

**Lemma 4.2.** Suppose that $E$ is a subset of the path space which depends only on the hitting times of subsets of $\mathcal{H}_N \setminus \mathcal{A}_N^R$. Moreover, suppose that $x, y \in S$ satisfy $r(x, y) + r(y, x) > 0$. Then, there is a fixed constant $C > 0$ such that

$$\left| \mathbb{P}_{\xi_i^y}[E] - \frac{r(x, y)}{r(x, y) + r(y, x)} \mathbb{P}_{\xi_{i-1}^y}[E] - \frac{r(y, x)}{r(x, y) + r(y, x)} \mathbb{P}_{\xi_{i+1}^y}[E] \right| \leq C \frac{d_N N}{i(N-i)}$$

for all $1 \leq i \leq N - 1$.

**Remark 4.3.** In the above lemma, typical examples of subsets $E$ are the following.

$$\{ \tau_{\xi_i^y}^x < \tau_{\xi_i^y}^x \}, \{ \tau_{\xi_i^y}^x = \tau_{E_N}(A) \} \text{ for } A \subseteq S, \text{ and } \{ \tau_{\xi_i^y}^x = \tau_{\mathcal{H}_N \setminus \mathcal{A}_N^R} \}.$$

Lemma 4.2 can be iterated to formulate $\mathbb{P}_{\xi_i^y}[E], 1 \leq i \leq N - 1$ in terms of the boundary values $\mathbb{P}_{\xi_i^y}[E]$ and $\mathbb{P}_{\xi_i^y}[E]$. This imperatively relies on the fact that the system is approximated to be one-dimensional.

We conclude this section with the following lemma, which estimates the equilibrium potential on one-dimensional tubes. This lemma is the main ingredient to estimate the divergence of the test flow in Lemma 6.4.

**Lemma 4.4.** Suppose that $A$ and $B$ are two disjoint subsets of $S$. Further, assume that $a \in A, b \in B, \text{ and } c \in S$ satisfy $r(c, a) > r(a, c) > 0$ and $r(c, b) > r(b, c) > 0$.

Then, we have

$$\sup_{0 \leq i \leq \lfloor N/2 \rfloor} \left| h_{\xi_N(A), \xi_N(B)}(\xi_{N-i}^{ac}) - 1 \right| = o(1) \quad (4.3)$$

and

$$\sup_{0 \leq i \leq \lfloor N/2 \rfloor} \left| h_{\xi_N(A), \xi_N(B)}(\xi_{N-i}^{bc}) \right| = o(1). \quad (4.4)$$

**Proof.** It must be noticed that $\{ \tau_{\xi_i^y}^x < \tau_{E_N(B)} \}$ is a subset of the path space satisfying the assumption of Lemma 4.2, thus, we may apply Lemma 4.2 to the equilibrium potential $h_{\xi_N(A), \xi_N(B)}$.

It suffices to prove (4.3) and (4.4) for $1 \leq i \leq \lfloor N/2 \rfloor$, as they are trivial for $i = 0$. We abbreviate $h_{\xi_N(A), \xi_N(B)}$ as $h$. Because $a \in A$ and $b \in B$, we have $h(\xi^a) = 1$ and $h(\xi^b) = 0$. Next, write $q = r(a, c)/r(c, a) < 1$ and

$$\alpha_i = h(\xi_{N-i+1}^{ac}) - h(\xi_{N-i}^{ac}) \text{ for } 1 \leq i \leq N.$$ 

Then, Lemma 4.2 implies

$$\left| \alpha_{i+1} - \frac{1}{q} \alpha_{i} \right| \leq C \frac{d_N N}{i(N-i)} \text{ for } 1 \leq i \leq N - 1. \quad (4.5)$$
Now, fix $1 \leq i \leq \left\lfloor \frac{N}{2} \right\rfloor$. Because $h(\xi^a) - h(\xi^c) = \alpha_1 + \cdots + \alpha_N$, we may estimate,

$$\left| h(\xi^a) - h(\xi^c) - \frac{1}{q^{N-i}} \frac{1 - q^N}{1 - q^i} (\alpha_1 + \cdots + \alpha_i) \right|$$

$$= \frac{1 - q}{q^{N-i} - q^N} \sum_{j=1}^{i} \sum_{k=i+1}^{N} (q^{N-j} \alpha_k - q^{N-k} \alpha_j).$$

(4.6)

Applying (4.5), the last formula is bounded by

$$\left\| \frac{1 - q}{q^{N-i} - q^N} \sum_{j=1}^{i} \sum_{k=i+1}^{N} q^{N-j} \sum_{\ell=j}^{k-1} \frac{C d_N N}{q^\ell (N - \ell)} \right\|.$$

By simple double counting, this is bounded from above by

$$\frac{C d_N N}{q^{N-i} - q^N} \left( \sum_{\ell=1}^{i-1} q^{N-\ell} \sum_{j=1}^{N-i} \frac{iq^{N-\ell}}{N - \ell} + \sum_{\ell=i}^{N-i} \frac{N - i + 1}{N - \ell} + \sum_{\ell=N-i+1}^{N-1} \frac{q^{N-\ell}}{\ell} \right).$$

(4.7)

From $\alpha_1 + \cdots + \alpha_i = h(\xi^a) - h(\xi^c_{N-i})$, by (4.6) and (4.7), we have

$$\left| h(\xi^a_{N-i}) - \frac{1 - q^{N-i}}{1 - q^N} h(\xi^a) - \frac{q^{N-i} - q^N}{1 - q^i} h(\xi^c) \right|$$

$$\leq 2C d_N N \left( \frac{q^{N+i+1} (1 - q)^{-1}}{N - i + 1} + \frac{2q^i (1 - q)^{-1}}{N} + \frac{(1 - q)^{-1}}{N - i + 1} \right) \leq 16C (1 - q)^{-1} d_N.$$

Because $h(\xi^a) = 1$ and $0 \leq h(\xi^c) \leq 1$, (4.3) follows. Moreover, by a similar computation, we deduce that

$$\left| h(\xi^c_{N-i}) - \frac{1 - \tilde{q}^{N-i}}{1 - \tilde{q}^N} h(\xi^c) - \frac{\tilde{q}^{N-i} - \tilde{q}^N}{1 - \tilde{q}^i} h(\xi^c) \right| \leq 16C (1 - \tilde{q})^{-1} d_N,$$

where $\tilde{q} = r(b, c)/r(c, b) < 1$. Because $h(\xi^c) = 0$ and $0 \leq h(\xi^c) \leq 1$, we have (4.4). □

5. Upper Bound for Capacities: Simple Case

In this section, we assume Condition 2.7 and establish the upper bound for $\text{Cap}_N(\mathcal{E}^{x_1}_N, \mathcal{E}^{x_2}_N)$. As previously mentioned, this and the succeeding subsections have most of the mathematical essentials for proving the general main result. Notions from Subsection 2.4 are frequently employed.

**Proposition 5.1 (Upper bound for capacities: Simple case).** Under the conditions of Theorem 2.8, the following inequality holds.

$$\lim_{N \to \infty} \sup_{N} \frac{N}{d_N^2} \text{Cap}_N(\mathcal{E}^{x_1}_N, \mathcal{E}^{x_2}_N) \leq \frac{1}{29 \mathcal{R}}.$$

5.1. Preliminary notions. Let $m_* = \max_{y_p} m_*(y_p) < 1$ and recall the notation (4.1). For all $N$, we define the following discretized version of the constant $\mathcal{R}$ given in (2.11):
\[ \mathcal{R}^N = \sum_{t=1}^{\frac{N}{2}} \sum_{p=1}^{2} \frac{1}{\left( \frac{N-t}{r(x_1, y_p)} + \frac{t-1}{r(x_2, y_p)} \right)^{-1}}. \]

Clearly, we have \( N^{-2} \mathcal{R}^N \rightarrow \mathcal{R} \) as \( N \) tends to infinity.

The constant \( \mathcal{R}^N \) has the shape of an inverse effective conductance of an electrical network consisted of conductors. In this sense, \( \mathcal{R}^N \) can be regarded as the inverse conductance of \( N \) serially connected conductors \((1 \leq t \leq N)\). Moreover, each conductor can be decomposed into two parallelly connected conductors \((1 \leq p \leq 2)\), and each of them corresponds to the motion of a particle from site \( x_1 \) to site \( x_2 \), passing through \( y_p \), for \( p = 1, 2 \). In each individual conductance \((1 - m_*(y_p))^{-1}\left( \frac{N-t}{r(x_1, y_p)} + \frac{t-1}{r(x_2, y_p)} \right)^{-1}\), the former term corresponds to the sum of geometric series of ratio \( m_*(y_p) \), and the latter term corresponds to the serial connection of particle motions \( x_1 \leftrightarrow y_p \) and \( x_2 \leftrightarrow y_p \) with conductances \( \frac{r(x_1, y_p)}{N-t} \) and \( \frac{r(x_2, y_p)}{t-1} \), respectively. These heuristic explanations are rigorously formulated in the proof of Lemma 5.2.

Moreover, we define
\[ \mathcal{U}_N = \bigcup_{p=1}^{2} \mathcal{A}_N^{x_1 y_p x_2} \quad \text{and} \quad \mathcal{V}_N = \mathcal{H}_N \setminus \mathcal{U}_N. \quad (5.1) \]

5.2. Construction of test function \( f_{\text{test}} \). In this subsection, we define a test function \( f = f_{\text{test}} \) on \( \mathcal{H}_N \), which approximates the equilibrium potential \( h_{\mathcal{E}_N^{x_1}, \mathcal{E}_N^{x_2}} \). This procedure presents the first major achievement of this article. To this end, \( f \) is constructed in four steps. See Figure 5.1 for a graphical explanation of this process.

- First, we define \( f \) on \( \mathcal{E}_N(S) \):
  \[ f(\xi^{x_1}) = 1 \quad \text{and} \quad f(\xi^{x_2}) = f(\xi^{y_1}) = f(\xi^{y_2}) = 0, \quad (5.2) \]
  so that condition (3.5) is verified.

- Second, we define \( f \) on \( \mathcal{A}_N^{x_1 y_p} \) for \( 1 \leq i \leq 2 \) and \( 1 \leq p \leq 2 \) by
  \[ f(\xi_k^{x_1 y_p}) = f(\xi_k^{x_i}), \quad 1 \leq k \leq N - 1. \quad (5.3) \]

- Next, we define \( f \) on the remainder of \( \mathcal{U}_N \), i.e., on \( \mathcal{A}_N^{x_1 y_p x_2} \setminus (\mathcal{A}_N^{x_1 y_p} \cup \mathcal{A}_N^{x_2 y_p}) \) for \( 1 \leq p \leq 2 \). The main contribution to the Dirichlet form occurs in this part. For \( k \in [1, N-2] \) and \( \ell \in [0, N-k-1] \),
  \[ f(\xi_k^{x_1 y_p x_2}) = \frac{K_p^{k, \ell}}{\mathcal{R}^N}, \quad (5.4) \]
  where for \( k \geq 0 \) and \( \ell \geq 0 \),
  \[ K_p^{k, \ell} = \sum_{t=1}^{k} \frac{N-t}{r(x_1, y_p)} / \left( \frac{N-t}{r(x_1, y_p)} + \frac{t-1}{r(x_2, y_p)} \right) + \sum_{t=1}^{k+\ell} \frac{t-1}{r(x_2, y_p)} / \left( \frac{N-t}{r(x_1, y_p)} + \frac{t-1}{r(x_2, y_p)} \right) \]
  \[ + \sum_{t=1}^{k+\ell} \frac{t-1}{r(x_2, y_p)} / \left( \frac{N-t}{r(x_1, y_p)} + \frac{t-1}{r(x_2, y_p)} \right). \]

- Finally, we define \( f \) on \( \mathcal{V}_N \). Assume \( \eta \in \mathcal{V}_N \). There are three types, (V1), (V2), and (V3), denoted by \( \mathcal{V}_N^{1}, \mathcal{V}_N^{2}, \) and \( \mathcal{V}_N^{3} \), respectively, such that
  \[ \mathcal{V}_N = \mathcal{V}_N^{1} \cup \mathcal{V}_N^{2} \cup \mathcal{V}_N^{3}. \quad (5.5) \]
Figure 5.1. (Left) distribution of the test function in a model satisfying Condition 2.7 with $N = 4$. (Right) more detailed landscape of the test function on the tube $\mathcal{A}_N^{x_1y_1x_2}$.

(V1) If $\eta_{x_1} = 0$, then define
\[
f(\eta) = 0. \tag{5.6}\]

(V2) If $\eta_{x_1} \geq 1$ and $\eta_{x_2} = 0$, then define
\[
f(\eta) = 1. \tag{5.7}\]

(V3) If $\eta_{x_1} \geq 1$ and $\eta_{x_2} \geq 1$, we define
\[
f(\eta) = f(\xi_{x_1x_2}). \tag{5.8}\]

- By the above construction, $0 \leq f(\eta) \leq 1$ for all $\eta \in \mathcal{H}_N$.

Here, we divide the Dirichlet form into four parts:
\[
D_N(f) = \sum_{\{\eta, \zeta\} \subseteq \mathcal{H}_N} \mu_N(\eta) q_N(\eta, \zeta) \{f(\zeta) - f(\eta)\}^2
= \Sigma_1(f) + \Sigma_2(f) + \Sigma_3(f) + \Sigma_4(f).
\]
The four summations are defined as follows, according to where the movement $\eta \leftrightarrow \zeta$ occurs.

- The first part $\Sigma_1(f)$ consists of movements inside $\mathcal{A}_N^{x_1y_1x_2}$ for $1 \leq p \leq 2$.
- The second part $\Sigma_2(f)$ consists of movements between the set differences of $\mathcal{A}_N^{x_1y_1x_2}$ and $\mathcal{A}_N^{x_2y_2x_2}$.
- The third part $\Sigma_3(f)$ consists of movements between $\mathcal{U}_N$ and $\mathcal{V}_N$.
- The last part $\Sigma_4(f)$ consists of movements inside $\mathcal{V}_N$.

From (5.1), the above four members are disjoint, and they characterize $D_N(f)$ completely.
As shown below, $\Sigma_1(f)$ is the main contribution to $D_N(f)$, whereas the other summations vanish (compared to $\Sigma_1(f)$) as $N$ tends to infinity.
5.3. Main contribution of Dirichlet form. In this subsection, we calculate the main contribution to the Dirichlet form, which is provided by $\Sigma_1(f)$. This is executed in Lemma 5.2.

**Lemma 5.2.** Under the conditions of Theorem 2.8, it holds that 

$$\Sigma_1(f) \leq \frac{d_N}{2N} \left[ \frac{1}{N} + O\left( \frac{1}{N} \right) \right].$$

**Proof.** To calculate $\Sigma_1(f)$, we write down all movements inside $A_N^{x_1y_px_2}$ and sum it up for $1 \leq p \leq 2$. More precisely,

$$\Sigma_1(f) = \sum_{p=1}^{2} \sum_{\eta \in A_N^{x_1y_px_2}} \sum_{i=1}^{2} \mu_N(\eta) q_N(\eta, x_i) \{ f(x_i, y_\eta) - f(\eta) \}^2.$$

There are no overlaps because $r(x_1, x_2) = r(x_2, x_1) = 0$. By (2.3) and (2.4), the right-hand side is asymptotically equal to

$$\frac{d_N N}{2} \sum_{p=1}^{2} \sum_{\ell=0}^{N-1} m_\ast(y_p)^\ell \left[ \sum_{k=1}^{N-\ell} w_N(N - \ell - k)r(x_1, y_p) \{ f(x_1y_px_2) - f(x_1y_px_2) \}^2 \right. \left. + \sum_{k=0}^{N-\ell-1} w_N(k)r(x_2, y_p) \{ f(x_1y_px_2) - f(x_1y_px_2) \}^2 \right].$$

(5.9)

Here, we fix $1 \leq p \leq 2$ and divide the range $\{ 0 \leq \ell \leq N-1 \}$ into $\{ \ell > \lfloor N/2 \rfloor \}$ and $\{ \ell \leq \lfloor N/2 \rfloor \}$. First, using (2.4) and summing up the geometric series with respect to $m_\ast(y_p)$, summation in the first range $\{ \ell > \lfloor N/2 \rfloor \}$ is easily bounded from above by

$$Cd_N N \sum_{\ell > \lfloor N/2 \rfloor} m_\ast(y_p)^\ell = o\left( \frac{d_N^2}{N} \right).$$

(5.10)

Second, we calculate summation in the range $\{ \ell \leq \lfloor N/2 \rfloor \}$. By (5.3), we discard the movements inside $A_N^{x_1y_p}$ and $A_N^{x_2y_p}$. Hence, we rewrite this summation as

$$\sum_{\ell=0}^{\lfloor N/2 \rfloor} m_\ast(y_p)^\ell \left[ \sum_{k=2}^{N-\ell-1} w_N(N - \ell - k)r(x_1, y_p) \{ f(x_1y_px_2) - f(x_1y_px_2) \}^2 \right. \left. + \sum_{k=1}^{N-\ell-2} w_N(k)r(x_2, y_p) \{ f(x_1y_px_2) - f(x_1y_px_2) \}^2 \right].$$

(5.11)

By (2.4) and (5.4), the first line of (5.11) is asymptotically equivalent to

$$\frac{d_N}{(N N)^2} \sum_{\ell=0}^{\lfloor N/2 \rfloor} m_\ast(y_p)^\ell \sum_{k=2}^{N-\ell-1} \frac{r(x_1, y_p)}{N - \ell - k} \left\{ \sum_{q=1}^{2} \left( \frac{1 - m_\ast(y_p)^{q-1}}{r(x_1, y_p) + \frac{k-1}{r(x_2, y_p)}} \right) \right\}^2.$$
Dividing \( \frac{1}{N-\ell-k} = \frac{1}{N-k} + \frac{\ell}{(N-\ell-k)(N-k)} \) and using \( \frac{N-k}{N-\ell-k} \leq \ell + 1 \), the last line is bounded by

\[
\frac{d_N}{(R N)^2} \sum_{\ell=0}^{\lfloor N/2 \rfloor} m_\ast(y_p)^\ell \sum_{k=2}^{N-\ell-1} \left[ \frac{N-k}{r(x_1, y_p)} - \frac{k-1}{r(x_2, y_p)} \right]^2 + C \ell(\ell + 1) \].
\]

By the theory of Riemann integration, this is further bounded by

\[
\frac{d_N}{(R N)^2} \sum_{\ell=0}^{\lfloor N/2 \rfloor} m_\ast(y_p)^\ell \left[ N^2 \left( \int_0^1 \frac{1}{r(x_1, y_p)} \left( \frac{1}{r(x_1, y_p)} + \frac{t}{r(x_2, y_p)} \right)^2 \right) dt + O(N) + C N \ell(\ell + 1) \right].
\]

Calculating the geometric series in \( 0 \leq \ell \leq \lfloor N/2 \rfloor \), this asymptotically equals

\[
\frac{d_N}{R^2 N^2} \frac{1}{1 - m_\ast(y_p)} \left[ \left( \int_0^1 \frac{1}{r(x_1, y_p)} \left( \frac{1}{r(x_1, y_p)} + \frac{t}{r(x_2, y_p)} \right)^2 \right) dt + O(1/N) \right]. \tag{5.12}
\]

Similarly, the second line of (5.11) is asymptotically bounded from above by

\[
\frac{d_N}{R^2 N^2} \frac{1}{1 - m_\ast(y_p)} \left[ \left( \int_0^1 \frac{t}{r(x_2, y_p)} \left( \frac{1}{r(x_1, y_p)} + \frac{t}{r(x_2, y_p)} \right)^2 \right) dt + O(1/N) \right]. \tag{5.13}
\]

The remaining parts of (5.11) are asymptotically equal to

\[
\frac{d_N}{R^2 N^2} \sum_{\ell=0}^{\lfloor N/2 \rfloor} m_\ast(y_p)^\ell \left[ \left( \frac{r(x_1, y_p)}{N - \ell - 1} \{ f(\xi_{0, \ell+1}) - f(\xi_{1, \ell}) \} \{ f(\xi_{0, \ell+1}) - f(\xi_{1, \ell}) \} \right)^2 \right. \tag{5.14}
\]

\[+ \left. \frac{r(x_2, y_p)}{N - \ell - 1} \{ f(\xi_{N-\ell-1, \ell+1}) - f(\xi_{N-\ell-1, \ell}) \} \{ f(\xi_{N-\ell-1, \ell+1}) - f(\xi_{N-\ell-1, \ell}) \} \right)^2 \].

By (5.3) and (5.4),

\[
\left| f(\xi_{0, \ell+1}) - f(\xi_{1, \ell}) \right| = \frac{K^1, \ell}{R N} \leq \frac{1}{R N} \sum_{l=1}^{\ell+1} \sum_{q=1}^{2} \frac{(1-m_\ast(y_p))^{-1}}{r(x_1, y_p) + r(x_2, y_p)},
\]

which is of order \( (\ell + 1) \times O(1/N) \), and

\[
\left| f(\xi_{N-\ell-1, \ell+1}) - f(\xi_{N-\ell-1, \ell}) \right| = \frac{N - \ell - 1, \ell}{R N} \leq \frac{1}{R N} \sum_{l=N-\ell}^{N} \sum_{q=1}^{2} \frac{(1-m_\ast(y_p))^{-1}}{r(x_1, y_p) + r(x_2, y_p)},
\]

which is again of order \( (\ell + 1) \times O(1/N) \). Hence, (5.14) is bounded from above by

\[
Cd_N \frac{1}{N} \sum_{\ell=0}^{\lfloor N/2 \rfloor} m_\ast(\ell + 1)^2 = O(\frac{d_N}{N^3}). \tag{5.15}
\]
Therefore, by (5.12), (5.13), and (5.15), we have the following asymptotic upper bound for (5.11):

$$\frac{d_N}{29^2 N^2} \left[ \int_0^1 \frac{1}{2} \frac{(1-m_*(y_p))^{-1}(1-t/r(x_1, y_p)+t/r(x_2, y_p))^{-1}}{\{\sum_{q=1}^{N-1} (1-m_*(y_q))^{-1} \}} \frac{1}{r(x_1, y_q)+r(x_2, y_q)} dt + o\left(\frac{1}{N}\right) \right].$$  \hspace{1cm} (5.16)

Collecting (5.9), (5.10), and (5.16), and the fact that $d_N$ decays subexponentially, $\Sigma_1(f)$ has the following asymptotic upper bound:

$$\frac{d_N^2}{29^2 N^2} \left[ \int_0^1 \frac{1}{2} \frac{1}{\{\sum_{q=1}^{N-1} (1-m_*(y_q))^{-1} \}} \frac{1}{r(x_1, y_q)+r(x_2, y_q)} dt + O\left(\frac{1}{N}\right) \right] = \frac{d_N^2}{29^2 N^2} \left[ \int_0^1 \frac{1}{2} \frac{1}{\{\sum_{q=1}^{N-1} (1-m_*(y_q))^{-1} \}} dt + O\left(\frac{1}{N}\right) \right].$$

The integral in the last line is exactly $\mathcal{R}$. Hence, we have

$$\Sigma_1(f) \leq \frac{d_N^2}{2N} \left[ \frac{1}{\mathcal{R}} + O\left(\frac{1}{N}\right) \right].$$

The last formula yields our exact expectations. \hspace{1cm} \square

5.4. Remainder of Dirichlet form. Next, we deal with the remaining terms in the Dirichlet form, $\Sigma_2(f)$, $\Sigma_3(f)$, and $\Sigma_4(f)$. Lemma 5.3 deals with $\Sigma_2(f)$.

**Lemma 5.3.** Under the conditions of Theorem 2.8, it holds that

$$\Sigma_2(f) = O\left(\frac{d_N^2 \log N}{N^2}\right) = o\left(\frac{d_N^2}{N}\right).$$

**Proof.** Recall that $\Sigma_2(f)$ consists of dynamics between the set differences of $A_N^{x_1 y_1 x_2}$ and $A_N^{x_2 y_2 x_1}$. This happens when a sole particle moves between $y_1$ and $y_2$. Precisely,

$$\Sigma_2(f) = \sum_{k=0}^{N-1} \mu_N(x_{k+1}^{x_1 y_1 x_2}) d_N r(y_1, y_2) \{f(x_{k+1}^{x_1 y_1 x_2}) - f(x_{k+1}^{x_1 y_1 x_2})\}^2.$$  \hspace{1cm} (5.17)

If $k = 0$ or $N - 1$, then $f(x_{k+1}^{x_1 y_1 x_2}) = f(x_{k+1}^{x_1 y_1 x_2})$ by (5.3). If $1 \leq k \leq N - 2$, then by (5.4),

$$f(x_{k+1}^{x_1 y_1 x_2}) - f(x_{k+1}^{x_1 y_1 x_2}) = \frac{r(x_2, y_2)}{r(x_1, y_1)} \left( \frac{N-k-1}{r(x_1, y_1)} + \frac{k}{r(x_1, y_1)} \right) - \frac{r(x_2, y_2)}{r(x_1, y_1)} \left( \frac{N-k-1}{r(x_1, y_1)} + \frac{k}{r(x_1, y_1)} \right),$$

which is $O(1/N)$. Thus, (5.17) is bounded by

$$O \sum_{k=1}^{N-2} \frac{N k^2}{k(N-k-1)} \times \frac{1}{N^2} = O\left(\frac{d_N^2 \log N}{N^2}\right).$$

This concludes the proof. \hspace{1cm} \square

Next, we consider $\Sigma_3(f)$. 

Lemma 5.4. Under the conditions of Theorem 2.8, it holds that
\[ \Sigma_3(f) = O\left(d_N^3 m_*^N + \frac{d_N^{2q}}{N} \log N \right) = o\left(\frac{d_N^{2q}}{N} \right). \]

Proof. We can formulate
\[ \Sigma_3(f) = \sum_{\eta \in U_N} \sum_{\zeta \in V_N} \mu_N(\eta) q_N(\eta, \zeta) \{ f(\zeta) - f(\eta) \}^2. \]

We divide the summation into three cases, depending on which subset \( \eta \) belongs to.

(Case 1) \( \eta \in \mathcal{A}_N^{x_1 y_p} \): In this case, there are no particle movements with \( \zeta \in V_N \).

(Case 2) \( \eta \in \mathcal{A}_N^{x_1 y_p} \setminus \mathcal{E}_N^{x_i} \) for some \( 1 \leq i \leq 2 \) and \( 1 \leq p \leq 2 \): We divide again according to types of the particle movement.

- **(Case 2-1)** \( \zeta = \sigma^{y_p, y_q} \eta \), where \( q \in \{1, 2\} \setminus \{ p \} \): The corresponding summation becomes
  \[ \sum_{i=1}^{2} \sum_{p=1}^{2} \sum_{\ell=1}^{N} \frac{w_N(N-\ell)w_N(\ell)}{Z_N} m_*(y_p) \ell d_N r(y_p, y_q) \{ f(\xi_{N-\ell, \ell-1}) - f(\xi_{N-\ell}) \}^2. \]
  By (5.2), (5.3), (5.6) and (5.7), \( f(\xi_{N-\ell, \ell-1}) = f(\xi_{N-\ell}) \) for all \( p \), and \( \ell \). Therefore, the summation is 0 in this case.

- **(Case 2-2)** \( \zeta = \sigma^{x_1 y_p} \eta \), where \( q \in \{1, 2\} \setminus \{ p \} \): The corresponding summation becomes
  \[ \sum_{i=1}^{2} \sum_{p=1}^{2} \sum_{\ell=1}^{N-1} \frac{w_N(N-\ell)w_N(\ell)}{Z_N} m_*(y_p) (N-\ell) d_N r(x_i, y_q) \{ f(\xi_{N-\ell-1, \ell-1}) - f(\xi_{N-\ell-1}) \}^2. \]
  This vanishes unless \( \ell = N - 1 \), in which case it becomes \( O(d_N^2 m_*^N) \). Concluding, **(Case 2)** yields a contribution \( O(d_N^2 m_*^N) \).

(Case 3) \( \eta \in \tilde{\mathcal{A}}_N^{x_1 y_p x_2} \) for some \( 1 \leq p \leq 2 \):

In this case, we can write the summation as
\[
\sum_{p=1}^{2} \sum_{\ell=1}^{N-2} \sum_{k=1}^{N-\ell-1} \mu_N(\xi_{k, \ell-1}^{x_1 y_p x_2}) \cdot \ell d_N r(y_p, y_q) \{ f(\xi_{k, \ell-1, 1}^{x_1 y_p x_2}) - f(\xi_{k, \ell}^{x_1 y_p x_2}) \}^2
+ \sum_{p=1}^{2} \sum_{\ell=1}^{N-2} \sum_{k=1}^{N-\ell-1} \mu_N(\xi_{k, \ell-1}^{x_1 y_p x_2}) \cdot k d_N r(x_1, y_q) \{ f(\xi_{k-1, \ell, 1}^{x_1 y_p x_2}) - f(\xi_{k, \ell}^{x_1 y_p x_2}) \}^2
+ \sum_{p=1}^{2} \sum_{\ell=1}^{N-2} \sum_{k=1}^{N-\ell-1} \mu_N(\xi_{k, \ell-1}^{x_1 y_p x_2}) \cdot (N-\ell-k) d_N r(x_2, y_q) \{ f(\xi_{k, \ell-1, 1}^{x_1 y_p x_2}) - f(\xi_{k, \ell}^{x_1 y_p x_2}) \}^2,
\]
\[ (5.18) \]

where in the summation, \( q \in \{1, 2\} \setminus \{ p \} \). For the first line in (5.18), it is bounded by
\[ C \sum_{p=1}^{2} \sum_{\ell=1}^{N-2} \sum_{k=1}^{N-\ell-1} \frac{Nd_N^2 m_*}{k(N-\ell-k)} \{ f(\xi_{k, \ell-1, 1}^{x_1 y_p x_2}) - f(\xi_{k, \ell}^{x_1 y_p x_2}) \}^2. \]
By (5.8), \( f(x_{\ell_{1}}, \ell_{1}, y_{2}) = f(x_{\ell_{1}}, \ell_{2}) \), and thus by (5.4), this is bounded by
\[
C \sum_{\ell=2}^{N-2} \sum_{k=1}^{N-\ell-1} \frac{Nd_{N}^{3}m_{*}}{k(N - \ell - k)} \cdot \frac{\ell^{2}}{N^{2}} = O\left(\frac{d_{N}^{3}}{N^{2} \log N}\right).
\]
For the second line in (5.18), it is bounded by
\[
C \sum_{p=1}^{2} \sum_{\ell=1}^{N-\ell-1} \sum_{k=1}^{N-\ell} \frac{Nd_{N}^{3}m_{*}}{\ell(N - \ell - k)} \cdot \{f(x_{\ell_{1}}, \ell_{1}) - f(x_{\ell_{1}}, \ell_{2})\}^{2}.
\]
Similarly, this is bounded by
\[
C \sum_{p=1}^{2} \sum_{\ell=2}^{N-\ell-1} \sum_{k=1}^{N-\ell} \frac{Nd_{N}^{3}m_{*}}{\ell(N - \ell - k)} \cdot \frac{(\ell + 1)^{2}}{N^{2}} = O\left(\frac{d_{N}^{3}}{N \log N}\right).
\]
Similarly, the third line in (5.18) is also bounded by \( O\left(\frac{d_{N}^{3}}{N \log N}\right) \). Collecting all cases, we conclude that
\[
\Sigma_{3}(f) = O\left(\frac{d_{N}^{3}}{N} + \frac{d_{N}^{3}}{N \log N}\right).
\]

Finally, we deal with \( \Sigma_{4}(f) \).

**Lemma 5.5.** Under the conditions of Theorem 2.8, it holds that
\[
\Sigma_{4}(f) = O(d_{N}^{2}m_{*}^{N}) + O\left(\frac{d_{N}^{3}}{N \log N}\right) = o\left(\frac{d_{N}^{2}}{N}\right).
\]

**Proof.** By definition, we have
\[
\Sigma_{4}(f) = \frac{1}{2} \sum_{\eta, \zeta \in V_{N}} \mu_{N}(\eta)q_{N}(\eta, \zeta)\{f(\zeta) - f(\eta)\}^{2}.
\]
By (5.5), we divide the summation in \( \eta, \zeta \in V_{N} \) by where \( \eta \leftrightarrow \zeta \) happens.

**Case 1** \( V_{1}^{1} \leftrightarrow V_{1}^{1} \) or \( V_{N}^{2} \leftrightarrow V_{N}^{2} \): \( f \) remains unchanged by (5.6) and (5.7).

**Case 2** \( V_{1}^{1} \leftrightarrow V_{N}^{2} \): There are at least \( N - 1 \) particles in \( \{y_{1}, y_{2}\} \), so the summation behaves as \( O(d_{N}^{2}m_{*}^{N}) \).

**Case 3** \( V_{1}^{1} \leftrightarrow V_{2}^{1} \): This is impossible.

**Case 4** \( V_{N}^{2} \leftrightarrow V_{3}^{2} \): This case can be bounded by
\[
C \sum_{\ell=2}^{N-2} \sum_{\ell' = 1}^{N-\ell-1} \frac{Nd_{N}^{3}}{(N - \ell - \ell')^{\ell + \ell'}} m_{*}^{\ell + \ell'} \{1 - f(x_{\ell_{1}}, \ell_{2})\}^{2} \leq C \sum_{\ell=2}^{N-2} \sum_{\ell' = 1}^{N-\ell-1} \frac{Nd_{N}^{3}}{(N - \ell - \ell')^{\ell + \ell'}} m_{*}^{\ell + \ell'} \cdot \frac{(\ell + \ell')^{2}}{N^{2}} = O\left(\frac{d_{N}^{3}}{N^{2}}\right).
\]
**Proof of Proposition 5.1.** By Lemmas 5.2, 5.3, 5.4, and 5.5,
\[ D_N(f_{\text{test}}) \leq \frac{d_N^2}{2N^9} + O\left(\frac{d_N^2}{N^2}\right) + O(d_N^2m^N) + O\left(\frac{d_N^2}{N} \log N\right). \]

Sending \( N \to \infty \), as \( \lim_{N \to \infty} d_N \log N = 0 \) and \( d_N \) decays subexponentially, we have
\[ \limsup_{N \to \infty} \frac{N}{d_N^2} D_N(f_{\text{test}}) \leq \frac{1}{29}. \]

Therefore, by Theorem 3.4 we obtain the desired result. \( \square \)

**6. Lower Bound for Capacities: Simple Case**

In this section, we assume Condition 2.7 and establish the lower bound for \( \text{Cap}_N(\xi^{e_1}_N, \xi^{e_2}_N) \). Once more, we recall the notions from Subsection 2.4. The following proposition is our main objective.

**Proposition 6.1 (Lower bound for capacities: Simple case).** Under the conditions of Theorem 2.8, the following inequality holds.
\[ \liminf_{N \to \infty} \frac{N}{d_N^2} \text{Cap}_N(\xi^{e_1}_N, \xi^{e_2}_N) \geq \frac{1}{29}. \quad (6.1) \]

As mentioned after Remark 3.5, the procedure involves the use of a test flow, which is in some sense close to \( c\Phi_{f_{\text{test}}} \), as \( \psi \) in Theorem 3.6. The main difficulty is to find a suitable flow, such that
\[ \sum_{\eta \in \mathcal{H}_N} h_{\xi^{e_1}_N, \xi^{e_2}_N}(\eta)(\text{div} \psi)(\eta) \]
can be easily calculated. Here, the major obstacle is that the exact values of \( h_{\xi^{e_1}_N, \xi^{e_2}_N} \) are unknown except on the one-dimensional tubes, \( A_{ab}^N \) for \( a, b \in S \), as shown in Section 4. Thus, the objective is to find a proper approximating flow \( \psi_{\text{test}} \) whose divergence can be neglected outside those tubes.
6.1. Construction of test flow $\psi_{\text{test}}$. In this subsection, we build the test flow $\psi = \psi_{\text{test}}$ on $\mathcal{H}_N$. As mentioned above, the key here is as follows: We must construct $\psi$ such that

1. the flow norm of $\psi$ is asymptotically equal to $c\Phi_{f_{\text{test}}}$, $c \neq 0$, 
2. the divergence of $\psi$ can be summed up in the sense of the right-hand side of (6.7).

To overcome both issues, we modify $\Phi_{f_{\text{test}}}$ properly, so that the divergence vanishes on $\mathcal{A}^{x_1 y_{2p}} \backslash (\mathcal{A}_N^{x_1 y_{2p}} \cup \mathcal{A}_N^{x_2 y_{2p}})$:

First, we define, for $1 \leq p \leq 2$, $\ell \in [0, \frac{N}{2} - 1]$, and $k \in [1, N - \ell - 1]$,

$$
\psi_0(x_1 y_{2p}, \xi_{k, \ell}, \xi_{k-1, \ell+1}) = \frac{m_+ (y_p)^{\ell / \left( \frac{N - \ell - k - 1}{r(x_1, y_p)} + \frac{k + \ell}{r(x_2, y_p)} \right)}}{\Re \sum_{q=1}^{2} \frac{(1 - m_+ (y_q))^{-1}}{r(x_1, y_q)} + \frac{k - 1}{r(x_2, y_q)}},
$$

(6.2)

$$
\psi_0(x_1 y_{2p}, \xi_{k, \ell}, \xi_{k, \ell+1}) = -\frac{m_+ (y_p)^{\ell / \left( \frac{N - \ell - k - 1}{r(x_1, y_p)} + \frac{k + \ell}{r(x_2, y_p)} \right)}}{\Re \sum_{q=1}^{2} \frac{(1 - m_+ (y_q))^{-1}}{r(x_1, y_q)} + \frac{k + \ell}{r(x_2, y_q)}},
$$

(6.3)

and 0 otherwise.

Observe that by the above construction, $(\text{div} \, \psi_0)(\xi_{2i}) = 0$ for $1 \leq i \leq 2$ and $(\text{div} \, \psi_0)(\eta) = 0$ for all $\eta \in \mathcal{V}_N$.

However, it holds that $(\text{div} \, \psi_0)(\xi_{2i+1}^{x_1 y_{2p}}) \neq 0$ for $\ell \in \left[ 1, \frac{N}{2} \right]$ and $k \in \left[ 0, N - \ell \right]$. We overcome this issue by adding correction flows to $\psi_0$ and make the divergence to be zero.

Before the exact definition, we calculate the non-zero term $(\text{div} \, \psi_0)(\xi_{k, \ell}^{x_1 y_{2p}})$. We define, for $k \in [1, N - 1]$,

$$
\mathfrak{A}(N, k) := \frac{1}{\Re \left( \frac{N - k}{r(x_1, y_2)} + \frac{k - 1}{r(x_2, y_2)} \right) \left( \frac{N - k}{r(x_1, y_1)} + \frac{k - 1}{r(x_2, y_1)} \right) \left( \frac{N - k}{r(x_1, y_2)} + \frac{k - 1}{r(x_2, y_2)} \right) \left( \frac{N - k}{r(x_1, y_1)} + \frac{k + 1}{r(x_2, y_1)} \right)}.
$$

(6.4)

Then, by the estimate

$$
\frac{N - k}{r(x_1, y_2)} + \frac{k - 1}{r(x_2, y_2)} \geq \frac{N - k}{C} + \frac{k - 1}{C} = \frac{N - 1}{C},
$$

and three additional similar bounds, it is straightforward that

$$
\mathfrak{A}(N, k) \leq \frac{C}{N}.
$$

(6.5)

The next lemma represents $(\text{div} \, \psi_0)(\xi_{k, \ell}^{x_1 y_{2p}})$ in terms of $\mathfrak{A}(N, k + \ell)$.

**Lemma 6.2.** For $1 \leq p \leq 2$, $\ell \in \left[ 1, \frac{N}{2} \right]$, and $k \in [1, N - \ell - 1]$, we have

$$
(\text{div} \, \psi_0)(\xi_{k, \ell}^{x_1 y_{2p}}) = \frac{m_+ (y_p)^{\ell - 1}}{1 - m_+ (y_s)} \left[ \frac{1}{r(x_1, y_s)r(x_2, y_p)} - \frac{1}{r(x_2, y_s)r(x_1, y_p)} \right] \mathfrak{A}(N, k + \ell),
$$

(6.6)

where $\{p, s\} = \{1, 2\}$. 

Proof. By (6.2) and (6.3), \((\text{div } \psi_0)(\xi_{k, \ell}^{x_1 y_1 x_2})\) equals
\[
\psi_0(\xi_{k, \ell}^{x_1 y_1 x_2} ; \phi_{k, \ell}) + \psi_0(\xi_{k, \ell}^{x_1 y_1 x_2} ; \phi_{k, \ell+1}^{x_1 y_1 x_2})
\]
\[
= - \frac{m_\ast(y_p)\ell^{-1}/(N_{r(x_1, y_p)} + k + \ell \ r(x_2, y_p))}{1-m_\ast(y_p)} + \frac{m_\ast(y_p)\ell^{-1}/(N_{r(x_1, y_p)} + k + \ell \ r(x_2, y_p))}{1-m_\ast(y_p)},
\]
where the first line holds since the other two flow values cancel out with each other. Noting that \(\{p, s\} = \{1, 2\}\), we rearrange the right-hand side as \(m_\ast(y_p)\ell^{-1}/\Re\) times
\[
- \frac{N - \ell - k - 1}{r(x_1, y_s)} + \frac{k + \ell}{r(x_2, y_s)}\left(\frac{N - \ell - k}{r(x_1, y_s)} + \frac{k + \ell - 1}{r(x_2, y_s)}\right)
+ \left(\frac{N - \ell - k}{r(x_1, y_s)} + \frac{k + \ell - 1}{r(x_2, y_s)}\right)
\]
divided by
\[
\left(\frac{N - \ell - k}{r(x_1, y_s)} + \frac{k + \ell}{r(x_2, y_s)}\right)\left(\frac{N - \ell - k}{r(x_1, y_s)} + \frac{k + \ell - 1}{r(x_2, y_s)}\right)
\]
Thus, according to (6.4), it remains to prove that (6.7) equals
\[
\frac{N - 1}{1 - m_\ast(y_p)} \times \left[\frac{1}{r(x_1, y_s) r(x_2, y_p)} - \frac{1}{r(x_2, y_s) r(x_1, y_p)}\right].
\]
In (6.7), the two terms involving \(1 - m_\ast(y_p)\) cancel out with each other. Thus, (6.7) becomes \((1 - m_\ast(y_p))^{-1}\) times
\[
- \left(\frac{N - \ell - k - 1}{r(x_1, y_s)} + \frac{k + \ell}{r(x_2, y_s)}\right)\left(\frac{N - \ell - k}{r(x_1, y_s)} + \frac{k + \ell - 1}{r(x_2, y_s)}\right)
+ \left(\frac{N - \ell - k}{r(x_1, y_s)} + \frac{k + \ell - 1}{r(x_2, y_s)}\right),
\]
Again, the terms cancel out with each other so that (6.7) equals
\[
(1 - m_\ast(y_p))^{-1}\left[\frac{N - 1}{r(x_1, y_s) r(x_2, y_p)} - \frac{N - 1}{r(x_2, y_s) r(x_1, y_p)}\right],
\]
as wanted. \(\square\)

Now, for all \(1 \leq p \leq 2\) and \(k \in [1, N - 1]\), we define a correction flow \(\phi_{p, k}\) as follows.

- Suppose that \(N \over 2 < k \leq N - 1\). Then, for \(\ell \in [1, N \over 2]\),
\[
\phi_{p, k}(\xi_{k-\ell, \ell}^{x_1 y_1 x_2}, \xi_{k-\ell+1, \ell}^{x_1 y_1 x_2}) := - \sum_{t=\ell}^{N/2} (\text{div } \psi_0)(\xi_{k-t, t}^{x_1 y_1 x_2}),
\]
and \( \phi_{p,k} = 0 \) on all other edges.

- Suppose that \( 1 \leq k \leq \frac{N}{2} \). Then, for \( \ell \in [1, k-1] \),
  \[
  \phi_{p,k}(\xi_{k-\ell, \ell}, \xi_{k-\ell+1, \ell-1}) := - \sum_{t=\ell}^{k-1} (\text{div } \psi_0)(\xi_{k-t, t}^{x_1 y_p x_2}),
  \]  

(6.9) and \( \phi_{p,k} = 0 \) on all other edges.

Finally, we define a flow

\[
\psi = \psi_{\text{test}} := \psi_0 + \sum_{p=1}^{2} \sum_{k=1}^{N-1} \phi_{p,k}.
\]

Then, the flows \( \phi_{p,k} \) for \( 1 \leq p \leq 2 \) and \( k \in [1, N-1] \) cancel the divergence of \( \psi_0 \) at each \( \xi_{k-\ell, \ell}^{x_1 y_p x_2} \in A_N^{x_1 y_p x_2} \). Thus, we obtain that \( (\text{div } \psi)(\eta) = 0 \) for all \( \eta \) in

\[
A_N^{x_1 y_p x_2} \setminus (A_N^{x_1 y_p} \cup A_N^{x_2 y_p} \cup A_N^{x_1 x_2}) \text{ for } 1 \leq p \leq 2.
\]

### 6.2. Flow norm of \( \psi_{\text{test}} \)

In this subsection, we calculate the flow norm of the test flow \( \psi \).

#### Lemma 6.3

Under the conditions of Theorem 2.8, it holds that

\[
\|\psi\|_N^2 \leq (1 + o(1)) \frac{2N}{d^2N} \mathcal{R}.
\]

**Proof.** By [6.2], [6.3], and Definition 3.2, we have

\[
\|\psi\|_N^2 = \sum_{p=1}^{2} \sum_{\ell=0}^{\lfloor N/2 \rfloor - 1} \sum_{k=1}^{N-\ell-1} \left[ \frac{(\psi(\xi_{k-\ell, \ell}^{x_1 y_p x_2}, \xi_{k-\ell+1, \ell-1}^{x_1 y_p x_2}))^2}{\mu_N(\xi_{k-\ell}^{x_1 y_p x_2})k(d_N + \ell)r(x_1, y_p)} + \frac{(\psi(\xi_{k, \ell}^{x_1 y_p x_2}, \xi_{k, \ell-1}^{x_1 y_p x_2}))^2}{\mu_N(\xi_{k}^{x_1 y_p x_2})(N-\ell-k)(d_N + \ell)r(x_2, y_p)} \right].
\]

(6.10)

By [2.4] and [6.2], the part of (6.10) including the first fraction inside bracket is asymptotically equivalent to

\[
\frac{2}{d^2N} \sum_{p=1}^{2} \left\lfloor \frac{N/2}{2} \right\rfloor \sum_{\ell=0}^{N-\ell-1} \sum_{k=1}^{N-\ell-1} \frac{m_*(y_p)}{N^2} \sum_{q=1}^{N^2} \frac{1}{r^2(x_1, y_p)} \left\lfloor \frac{N-k-\ell-1}{r(x_1, y_p)} + \frac{k+\ell}{r(x_2, y_p)} \right\rfloor^2.
\]

Divide \( N-\ell-k = (N-k-\ell-1) + 1 \). Then, as in obtaining [6.12], the last formula can be bounded from above by

\[
\frac{2}{d^2N} \sum_{p=1}^{2} \frac{N^2 \mathcal{R} - 2}{1 - m_*(y_p)} \left\lfloor \int_0^1 \frac{1-t}{r(x_1, y_p)} \left\lfloor \frac{1-t}{r(x_1, y_p)} + \frac{t}{r(x_2, y_p)} \right\rfloor^2 dt + o(1) \right\rfloor.
\]

(6.11)
Similarly, the part of (6.10) including the second fraction inside bracket is asymptotically bounded from above by
\[ \frac{2}{d_N^2} \sum_{p=1}^{\lfloor N/2 \rfloor} 1 - m_\star(y_p) \left[ \int_0^1 \frac{t}{r(x_2, y_p)} \left( \frac{1-t}{r(x_1, y_p)} + \frac{t}{r(x_2, y_p)} \right)^2 dt + o(1) \right]. \quad (6.12) \]

Hence, by (6.10), (6.11), and (6.12), we have the following asymptotic upper bound for \( \|\psi\|_N^2 \):
\[ \frac{2}{d_N^2} \sum_{p=1}^{\lfloor N/2 \rfloor} \|\psi\|_N^2 \leq \frac{2N}{d_N^2} \int_0^1 \left( 1 - m_\star(y_p) \right)^{-1} \left( \frac{1-t}{r(x_1, y_p)} + \frac{t}{r(x_2, y_p)} \right)^{-1} \left( \sum_{q=1}^2 \frac{(1-m_\star(y_q))^{-1}}{r(x_1, y_q) + r(x_2, y_q)} \right)^2 dt = \frac{2N}{d_N^2} \times \frac{1}{\mathcal{R}}. \]

This concludes the proof. \( \square \)

### 6.3. Remaining terms

Here, we address the remaining terms on the right-hand side of (3.7) with respect to \( \psi \). To this end, Lemma 4.4 is used to calculate the equilibrium potential near the metastable valleys.

**Lemma 6.4.** Under the conditions of Theorem 2.8, it holds that
\[ \sum_{\eta \in \mathcal{H} \setminus \mathcal{H}_N(S_\star)} h_{\mathcal{E}_N^{x_1}, \mathcal{E}_N^{x_2}}(\eta)(\text{div} \psi)(\eta) = \frac{1 + o(1)}{\mathcal{R}}. \quad (6.13) \]

**Proof.** We will abbreviate \( h_{\mathcal{E}_N^{x_1}, \mathcal{E}_N^{x_2}} \) as \( h \). It follows from the last observation in Section 6.1 that we only need to sum up the configurations in \( A_N^{x_1 y, x_2} \setminus \mathcal{E}_N^{x_1}, A_N^{x_2 y, x_2} \setminus \mathcal{E}_N^{x_2} \), and \( A_N^{x_1 y} \setminus (\mathcal{E}_N^{x_1} \cup \mathcal{E}_N^{x_2}) \).

First, we claim that
\[ \sum_{p=1}^{\lfloor N/2 \rfloor} \left[ \sum_{\eta \in A_N^{x_1 y, x_2} \setminus \mathcal{E}_N^{x_1}} + \sum_{\eta \in A_N^{x_2 y, x_2} \setminus \mathcal{E}_N^{x_2}} \right] h(\eta)(\text{div} \psi_0)(\eta) = \frac{1 + o(1)}{\mathcal{R}}. \quad (6.14) \]

The left-hand side of (6.14) is
\[ \sum_{p=1}^{\lfloor N/2 \rfloor} \sum_{\ell=1}^{\lfloor N/2 \rfloor} h(\xi_{\ell}^{x_1 y})(\text{div} \psi_0)(\xi_{\ell}^{x_1 y}) + \sum_{p=1}^{\lfloor N/2 \rfloor} \sum_{\ell=1}^{\lfloor N/2 \rfloor} h(\xi_{\ell}^{x_2 y})(\text{div} \psi_0)(\xi_{\ell}^{x_2 y}). \quad (6.15) \]

By Lemma 4.4, we have
\[ \sup_{1 \leq \ell \leq \lfloor N/2 \rfloor} |h(\xi_{\ell}^{x_1 y}) - 1| = o(1), \quad (6.16) \]
and
\[ \sup_{1 \leq \ell \leq \lfloor N/2 \rfloor} h(\xi_{\ell}^{x_2 y}) = o(1). \quad (6.17) \]

Hence, (6.15) is equal to
\[ (1 + o(1)) \sum_{p=1}^{\lfloor N/2 \rfloor} \sum_{\ell=1}^{\lfloor N/2 \rfloor} (\text{div} \psi_0)(\xi_{\ell}^{x_1 y}) + o(1) \sum_{p=1}^{\lfloor N/2 \rfloor} \sum_{\ell=1}^{\lfloor N/2 \rfloor} (\text{div} \psi_0)(\xi_{\ell}^{x_2 y}). \quad (6.18) \]
By (6.2), the first term of (6.18) is asymptotically equivalent to
\[
\frac{1}{\mathfrak{R}} \sum_{p=1}^{2} \sum_{\ell=1}^{[N/2]} \frac{m_*(y_p)^{\ell-1}}{r(x_2, y_p)} \sum_{q=1}^{2} \frac{(1-m_*(y_q))^{\ell}}{r(x_2, y_q)} = \frac{1}{\mathfrak{R}} \sum_{p=1}^{2} \sum_{\ell=1}^{[N/2]} \frac{m_*(y_p)^{\ell-1} r(x_2, y_p)}{1-m_*(y_q)}.
\]
Summing for \(1 \leq \ell \leq \frac{N}{2}\), the last formula equals \(1/\mathfrak{R}\). Similarly, the second part of (6.18) equals \(o(1)/\mathfrak{R} = o(1)\). This concludes the proof of (6.14).

Next, from the definition, note that \((\operatorname{div} \phi_{p,k})(\eta)\) vanishes unless
\[
\eta \in \mathcal{A}_N^{x_1y_2x_2} \setminus (\mathcal{A}_N^{x_1y_2p} \cup \mathcal{A}_N^{x_2y_2p}).
\]
Moreover, it is verified right after the definition of \(\psi\) that \(\operatorname{div} \psi\) vanishes in
\[
\mathcal{A}_N^{x_1y_2x_2} \text{ for } 1 \leq p \leq 2.
\]
Finally, it is straightforward that \((\operatorname{div} \psi_0)(\eta) = 0\) for \(\eta \in \mathcal{A}_N^{x_1x_2}\). Combining these observations, it remains to prove that
\[
\sum_{k=1}^{N-1} \sum_{p=1}^{2} h(\eta)(\operatorname{div} \phi_{p,k})(\eta) = o(1).
\]
This can be rewritten as
\[
\sum_{k=1}^{N-1} \sum_{p=1}^{2} h(\xi_k^{x_1x_2})(\operatorname{div} \phi_{p,k})(\xi_k^{x_1x_2}) = o(1).
\]
Since \(0 \leq h \leq 1\), it suffices to prove that
\[
\sum_{k=1}^{N-1} \left| \sum_{p=1}^{2} (\operatorname{div} \phi_{p,k})(\xi_k^{x_1x_2}) \right| = o(1).
\]
For \(\frac{N}{2} < k \leq N - 1\), by (6.8) it holds that
\[
\sum_{p=1}^{2} (\operatorname{div} \phi_{p,k})(\xi_k^{x_1x_2}) = \sum_{p=1}^{2} \sum_{t=1}^{[N/2]} (\operatorname{div} \psi_0)(\xi_k^{x_1y_2x_2}).
\]
By (6.6), for fixed \(1 \leq p \leq 2\), the summation in \(1 \leq t \leq \lfloor N/2 \rfloor\) is calculated as
\[
\sum_{t=1}^{[N/2]} \frac{m_*(y_p)^{t-1}}{1-m_*(y_s)} \times \left[ \frac{1}{r(x_1, y_s)r(x_2, y_p)} - \frac{1}{r(x_2, y_s)r(x_1, y_p)} \right] \times \mathfrak{A}(N, k)
= \frac{\mathfrak{A}(N, k)}{1-m_*(y_s)} \times \left[ \frac{1}{r(x_1, y_s)r(x_2, y_p)} - \frac{1}{r(x_2, y_s)r(x_1, y_p)} \right] \times \left[ \frac{1}{1-m_*(y_p)} + O(m_*(y_p)^{\frac{N}{2}}) \right]
= \frac{\mathfrak{A}(N, k)}{(1-m_*(y_p))(1-m_*(y_s))} \left[ \frac{1}{r(x_1, y_s)r(x_2, y_p) - r(x_2, y_s)r(x_1, y_p)} \right] + O\left(\frac{m_*(y_p)^{\frac{N}{2}}}{N}\right),
\]
where \( \{p, s\} = \{1, 2\} \), where in the first equality we used
\[
\sum_{t = 1}^{\lfloor N/2 \rfloor} m_*(y_p)^{t-1} = \frac{1}{1 - m_*(y_p)} + \sum_{t > \lfloor N/2 \rfloor} m_*(y_p)^{t-1} = \frac{1}{1 - m_*(y_p)} + O(m_*(y_p)^{N/2}),
\]
and where in the second equality we used (6.5). Summing up for \( p \in \{1, 2\} \), the two terms involving the square bracket cancel out with each other. Hence, we conclude that
\[
\sum_{k = \lfloor N/2 \rfloor + 1}^{N-1} \left| \sum_{p=1}^2 (\text{div} \phi_{p,k})(\xi^{x_1 x_2}_k) \right| \leq \frac{N}{2} \times \sum_{p=1}^2 O\left(\frac{m_*(y_p)^k}{N}\right) = O\left(\frac{m_*(y_p)^k}{N}\right) = o(1).
\]
Therefore, it remains to prove that
\[
\sum_{k=1}^{\lfloor N/2 \rfloor} \left| \sum_{p=1}^2 (\text{div} \phi_{p,k})(\xi^{x_1 x_2}_k) \right| = o(1).
\]
By a similar calculation, we obtain that the left-hand side is bounded by
\[
\sum_{k=1}^{\lfloor N/2 \rfloor} \sum_{p=1}^2 O\left(\frac{m_*(y_p)^{k-1}}{N}\right) = O\left(\frac{1}{N}\right) = o(1).
\]
Thus, we conclude the proof. \( \square \)

6.4. Proof of Proposition 6.1 We are now ready to prove Proposition 6.1.

**Proof of Proposition 6.1.** By Lemmas 6.3 and 6.4, we have
\[
\frac{1}{\|\psi_{\text{test}}\|^2_N} \left[ \sum_{\eta \in \mathcal{H}_N \setminus \mathcal{E}_N} h_{E_{N}^{(2)}(\eta)}(\text{div} \psi_{\text{test}})(\eta) \right]^2 \geq (1 + o(1)) \frac{d_N^2}{2N^{1/3}}.
\]
Therefore, we deduce from Theorem 3.6 that
\[
\text{Cap}_N(\mathcal{E}_N^{x_1}, \mathcal{E}_N^{x_2}) \geq (1 + o(1)) \frac{d_N^2}{2N^{1/3}}.
\]
This concludes the proof of Proposition 6.1. \( \square \)

7. Upper Bound for Capacities: General Case

In this section, we omit Condition 2.7 and extend the results of Section 5 to the most general setting of Theorem 2.10 described in Subsection 2.5. Because proofs of the assertions here are fundamentally similar to those in Section 5, they will be written in a brief manner.

**Proposition 7.1 (Upper bound for capacities: General case).** Assume the conditions of Theorem 2.10. Then, for any non-trivial partition \( \{A, B\} \) of \([1, \kappa_*]\), the following inequality holds.
\[
\limsup_{N \to \infty} \frac{N}{d_N^2} \text{Cap}_N(\mathcal{E}_N^{(2)}(A), \mathcal{E}_N^{(2)}(B)) \leq \frac{1}{|S_*|} \sum_{i \in A} \sum_{j \in B} \frac{1}{\mathcal{R}_{i,j}}.
\]

**Remark 7.2.** In Proposition 7.1, it is crucial to have \( A \cup B = [1, \kappa_*] \); if \( A \cup B \subsetneq [1, \kappa_*] \), then the equilibrium potential is significantly more complicated. Moreover, we remark that if
\( \mathcal{R}_{i,j} = \infty \) for all \( i \in A \) and \( j \in B \), then Proposition 7.1 asserts that

\[
\text{Cap}_N(\mathcal{E}_N^{(2)}(A), \mathcal{E}_N^{(2)}(B)) = o\left( \frac{d_N^2}{N} \right).
\]

This equality indicates that we do not observe metastable transitions in the time scale \( N/d_N^2 \). Therefore, in this case, we expect that yet another time scale is required to observe the metastable transitions. This is conjectured to be \( N^2/d_N^2 \) in [7].

### 7.1. Preliminary notions.

Once more, we define \( m_* = \max_{p=1}^{\kappa_0} m_*(y_p) < 1 \). Because there are too many subscripts in the general case, we introduce a convenient notation that helps us calculate the objects.

We define the following discretized version of the constant \( \mathcal{R}_{i,j} \) for \( i, j \in [1, \kappa_*] \) given in (2.13):

\[
\mathcal{R}_{i,j}^N = \sum_{t=1}^{N} \sum_{n=1}^{n(i)} \sum_{m=1}^{n(j)} \sum_{p=1}^{\kappa_0} \frac{1}{(1 - m_*(y_p))^{-1} \left( \frac{N - t}{r(x_n, y_p)} + \frac{1}{r(x_n, y_p)} \right)}.
\]

As in the special case in Subsection 5.1 we write \( \mathcal{R}_{i,j}^N = \infty \) if \( r(x_i, n) r(x_j, m) = 0 \) for all \( 1 \leq n \leq n(i) \), \( 1 \leq m \leq n(j) \), and \( 1 \leq p \leq \kappa_0 \). Clearly, we have \( N^{-2} \mathcal{R}_{i,j}^N \to \mathcal{R}_{i,j} \) as \( N \) tends to infinity. Moreover, define

\[
I = \{(i, j) \in A \times B : \mathcal{R}_{i,j} < \infty\},
\]

and for \( (i, j) \in A \times B \),

\[
P_{i,n,j,m} = \{p : r(x_i, n) r(x_j, m) > 0\}
\]

and

\[
Q_{i,n,j,m} = \{p : r(x_i, n) + r(x_j, m) > 0\}.
\]

For example, \( (i, j) \in I \) if and only if \( r(x_i, n) r(x_j, m) > 0 \) for some \( n, m, \) and \( p \), which is also equivalent to

\[
\bigcup_{n=1}^{n(i)} \bigcup_{m=1}^{n(j)} P_{i,n,j,m} = \emptyset.
\]

Moreover, we have \( P_{i,n,j,m} \subseteq Q_{i,n,j,m} \). Finally, we define

\[
\mathcal{U}_N = \bigcup_{i \in A} \bigcup_{j \in B} \bigcup_{n=1}^{n(i)} \bigcup_{m=1}^{n(j)} \bigcup_{p=1}^{\kappa_0} \mathcal{A}_{N}^{x_i, n, y_p x_j, m} \text{ and } \mathcal{V}_N = \mathcal{H}_N \setminus \mathcal{U}_N.
\] (7.1)

### 7.2. Construction of test function \( f_{\text{test}}^A \).

In this subsection, we define a test function \( f = f_{\text{test}}^A \) on \( \mathcal{H}_N \), which approximates the equilibrium potential \( h_{\mathcal{E}_N^{(2)}(A), \mathcal{E}_N^{(2)}(B)} \). This procedure is a natural extension of the definition in Subsection 5.2.

- First, we define \( f \) on \( \mathcal{E}_N(S) \):

\[
f(\xi^{x_i, n}) = 1, \ i \in A \text{ and } f(\xi^z) = 0, \ z \in S \setminus \{x_i, n : i \in A, 1 \leq n \leq n(i)\},
\] (7.2)
such that we have \( f|_{\mathcal{E}^{(2)}_N(A)} = 1 \) and \( f|_{\mathcal{E}^{(2)}_N(B)} = 0. \)

- Second, we define \( f \) on \( \tilde{A}_N^{x_i, n y_p} \) for \( i \in [1, \kappa], n \in [1, n(i)], \) and \( p \in [1, \kappa_0] \) by
  \[
f(\xi^{x_i, n y_p}_{k, \ell}, \mathbf{y}) = f(\xi^{x_i, n}_{k, \ell}), \quad 1 \leq k \leq N - 1.
  \] (7.3)

- Next, we define \( f \) on the remainder of \( \mathcal{U}_N \), i.e., on \( \mathcal{A}_N^{x_i, n y_p x_j, m} \setminus (\mathcal{A}_N^{x_i, n y_p} \cup \mathcal{A}_N^{x_j, m y_p}) \) for \( (i, j) \in A \times B, n, m \geq 1, \) and \( p \in [1, \kappa_0] \). This part is the main technical obstacle in the definition of \( f_{\text{test}} \). There are four types, (U1) through (U4).

(U1) If \( (i, j) \in I \) and \( n, m \geq 1 \) with \( p \in Q_{i, n, j, m} \), then for \( \ell \in [1, N - 2] \) and \( k \in [1, N - \ell - 1] \),
  \[
f(\xi^{x_i, n y_p x_j, m}_{k, \ell}) = \frac{K^{k, \ell}_{i, n, p, j, m}}{R_{i, j}^N},
  \] (7.4)

where
  \[
  K^{k, \ell}_{i, n, p, j, m} = \sum_{t = 1}^{k} \frac{r(x_i, n, y_p)}{r(x_i, n, y_p) + r(x_j, m, y_p)} + \sum_{t = 1}^{k + \ell} \frac{r(x_j, m, y_p)}{r(x_i, n, y_p) + r(x_j, m, y_p)}.
  \]

By substituting \( \ell = 0 \), one can verify that (7.4) is well defined on \( \tilde{A}_N^{x_i, n x_j, m} \). The fractions inside summations are well defined, as \( (i, j) \in I \) implies that the common denominator is strictly positive. The numerators of the fractions must be understood naturally if \( r(x_i, n, y_p)r(x_j, m, y_p) = 0 \). Indeed, if e.g., \( r(x_i, n, y_p) > 0 \) and \( r(x_j, m, y_p) = 0 \), then the first one ("0/∞") is 0, and the second one ("∞/∞") is 1.

(U2) If \( (i, j) \in I \) and \( n, m \geq 1 \) with \( p \notin Q_{i, n, j, m} \), then for \( \ell \in [1, N - 2] \) and \( k \in [1, N - \ell - 1] \),
  \[
f(\xi^{x_i, n y_p x_j, m}_{k, \ell}) = \frac{1}{R_{i, j}^N} \sum_{t = 1}^{k} \frac{r(x_i, n, y_p)}{r(x_i, n, y_p) + r(x_j, m, y_p)} + \sum_{t = 1}^{k + \ell} \frac{1}{r(x_i, n, y_p) + r(x_j, m, y_p)}.
  \] (7.5)

Note that (7.5) is consistent with (7.4) on \( \tilde{A}_N^{x_i, n x_j, m} \).

(U3) If \( (i, j) \notin I \) and \( n, m \geq 1 \) with \( p \in Q_{i, n, j, m} \setminus P_{i, n, j, m} \), such that \( r(x_i, n, y_p) > 0 \) and \( r(x_j, m, y_p) = 0 \), then for \( \ell \in [0, N - 2] \) and \( k \in [1, N - \ell - 1] \),
  \[
f(\xi^{x_i, n y_p x_j, m}_{k, \ell}) = \frac{k + \ell}{N}.
  \] (7.6)

As done previously, one can substitute \( \ell = 0 \) to verify that (7.6) is well defined on \( \tilde{A}_N^{x_i, n x_j, m} \). Similarly, if \( p \in Q_{i, n, j, m} \setminus P_{i, n, j, m} \) with \( r(x_i, n, y_p) = 0 \) and \( r(x_j, m, y_p) > 0 \), then define
  \[
f(\xi^{x_i, n y_p x_j, m}_{k, \ell}) = \frac{k}{N}.
  \] (7.7)

(U4) If \( (i, j) \notin I \) and \( n, m \geq 1 \) with \( p \notin Q_{i, n, j, m} \), then for \( \ell \in [1, N - 2] \) and \( k \in [1, N - \ell - 1] \),
  \[
f(\xi^{x_i, n y_p x_j, m}_{k, \ell}) = \frac{k}{N}.
  \] (7.8)
is well defined on $\hat{A}_N^{x_i, n^x_j, m}$ and consistent with (7.6); substitute $\ell = 0$. 

- Finally, we define $f$ on $V_N$. Assume $\eta \in V_N$. There are three types, (V1), (V2), and (V3), denoted by $V_N^1$, $V_N^2$, and $V_N^3$, respectively, such that

\[
V_N = V_N^1 \cup V_N^2 \cup V_N^3. \tag{7.9}
\]

Define $\eta_A := \sum_{i \in A} \sum_{n=1}^{n(i)} \eta_{x_i, n}$ and $\eta_B := \sum_{j \in B} \sum_{m=1}^{n(j)} \eta_{x_j, m}$.

(V1) If $\eta_A = 0$, then define

\[
f(\eta) = 0. \tag{7.10}
\]

(V2) If $\eta_A \geq 1$ and $\eta_B = 0$, then define

\[
f(\eta) = 1. \tag{7.11}
\]

(V3) If $\eta_A, \eta_B \geq 1$, we define

\[
f(\eta) = \sum_i \sum_j \sum_{n=1}^{n(i)} \sum_{m=1}^{n(j)} \eta_{x_i, n} \eta_{x_j, m} \frac{f(\xi_{x_i, n^x_j, m})}{\eta_A \eta_B}. \tag{7.12}
\]

- By construction, we have $0 \leq f(\eta) \leq 1$ for all $\eta \in H_N$.

We divide the Dirichlet form into four parts:

\[
D_N(f) = \Sigma_1(f) + \Sigma_2(f) + \Sigma_3(f) + \Sigma_4(f).
\]

- The first part $\Sigma_1(f)$ consists of movements inside $A_N^{x_i, n^y_x, m}$ for all $i, j \in [1, \kappa_*], n \in [1, n(i)], m \in [1, n(j)]$, and $p \in [1, \kappa_0]$.
- The second part $\Sigma_2(f)$ consists of movements between the set differences of two distinct $A_N^{x_i, n^y_x, m}$-type sets.
- The third part $\Sigma_3(f)$ consists of movements between $U_N$ and $V_N$.
- The last part $\Sigma_4(f)$ consists of movements inside $V_N$.

7.3. **Main contribution of Dirichlet form.** In this subsection, we calculate $\Sigma_1(f)$, which is the main ingredient of $D_N(f)$.

**Lemma 7.3.** Under the conditions of Theorem 2.10, it holds that

\[
\Sigma_1(f) \leq \frac{d_N^2}{|S_*|N} \left[ \sum_{i \in A} \sum_{j \in B} \frac{1}{|R_{i,j}|} + o(1) \right].
\]

**Proof.** We write down all movements inside $A_N^{x_i, n^y_x, m}$ and sum it up for all $i, j, n, m, p$. Namely,

\[
\Sigma_1(f) \leq \sum_{i \in A} \sum_{j \in B} \sum_{n, m, p} \mu_N(\eta) \times
\]

\[
\left[ q_N(\eta, \sigma_{x_i, n^y_x}) \{ f(\sigma_{x_i, n^y_x}) - f(\eta) \}^2 + q_N(\eta, \sigma_{x_j, m^y_x}) \{ f(\sigma_{x_j, m^y_x}) - f(\eta) \}^2 \right]
\]

is the desired equation. The only overlapping terms on the right-hand side above are movements along $A_N^{x_i, n^y_x}$ for $i \in [1, \kappa_*], n \in [1, n(i)]$, and $p \in [1, \kappa_0]$. In fact, by (7.3), these
terms have an exponentially small effect on the entire summation. Thus, the inequality used above is actually sharp. By (2.3) and (2.4), the right-hand side is asymptotically equal to

\[
\frac{d_N N}{|S^*|} \sum_{i \in A} \sum_{j \in B} \sum_{n, m} \sum_{p \in Q_{i, n, j, m}} \sum_{\ell=0}^{N-1} m_*(y_p) \ell \times \\
\left[ \sum_{k=1}^{N-\ell} w_N(N - \ell - k) r(x_{i, n}, y_p) \{ f(\xi_{x_{i, n}, y_p x_{j, m}}^{\ell, k-1}) - f(\xi_{x_{i, n}, y_p x_{j, m}}^{\ell, k-1}) \}^2 \right. \\
+ \sum_{k=0}^{N-\ell-1} w_N(k) r(x_{j, m}, y_p) \{ f(\xi_{x_{i, n}, y_p x_{j, m}}^{\ell+1, k}) - f(\xi_{x_{i, n}, y_p x_{j, m}}^{\ell+1, k}) \}^2 .
\]

We only need to consider \( p \in Q_{i, n, j, m} \), as otherwise \( r(x_{i, n}, y_p) = r(x_{j, m}, y_p) = 0 \). Next, if \( p \in Q_{i, n, j, m} \setminus P_{i, n, j, m} \), then the terms inside the bracket vanish due to (7.4), (7.5), and (7.6). Gathering the preceding observations, the last formula is asymptotically equal to

\[
\frac{d_N N}{|S^*|} \sum_{(i, j) \in I \cap (A \times B)} \sum_{n, m} \sum_{p \in P_{i, n, j, m}} \sum_{\ell=0}^{N-1} m_*(y_p) \ell \times \\
\left[ \sum_{k=1}^{N-\ell} w_N(N - \ell - k) r(x_{i, n}, y_p) \{ f(\xi_{x_{i, n}, y_p x_{j, m}}^{\ell, k-1}) - f(\xi_{x_{i, n}, y_p x_{j, m}}^{\ell, k-1}) \}^2 \right. \\
+ \sum_{k=0}^{N-\ell-1} w_N(k) r(x_{j, m}, y_p) \{ f(\xi_{x_{i, n}, y_p x_{j, m}}^{\ell+1, k}) - f(\xi_{x_{i, n}, y_p x_{j, m}}^{\ell+1, k}) \}^2 .
\]

The rest of the proof is almost identical to that of Lemma 5.2, we obtain

\[
\Sigma_1(f) \leq \frac{d_N^2}{|S^*| N} \left[ \sum_{(i, j) \in I \cap (A \times B)} \frac{1}{R_{i, j}} + o(1) \right].
\]

Because \( R_{i, j} = \infty \) if \( (i, j) \notin I \), the last formula is exactly what we expect. \( \square \)

### 7.4. Remainder of Dirichlet form.

Here, we deal with the remaining terms in the Dirichlet form, \( \Sigma_2(f) \), \( \Sigma_3(f) \), and \( \Sigma_4(f) \). Lemma 7.4 deals with \( \Sigma_2(f) \).

**Lemma 7.4.** Under the conditions of Theorem 2.10, it holds that

\[
\Sigma_2(f) = O\left( \frac{d_N^3 \log N}{N^2} \right) = o\left( \frac{d_N^3 N}{N} \right).
\]

**Proof.** Recalling that \( \Sigma_2(f) \) consists of dynamics between the set differences of two distinct \( A_{x_{i, n}, y_p x_{j, m}}^N \)-type sets, there are two such types of movements.
(Case 1) The first case is represented when a sole particle moves between \(x_{i,n} \) and \(x_{i,\tilde{n}}\). More specifically, this is written as

\[
\sum_{i \in A} \sum_{j \in B} \sum_{n,m,p=0}^{N-1} \mu_N(\xi_{1,\ell}) \times \left[ \sum_{\tilde{n}} d_{N} r(x_{i,n}, x_{i,\tilde{n}}) \{f(\xi_{1,\ell}) - f(\xi_{1,\ell})\}^2 \right.
\]

\[
+ \sum_{\tilde{m}} d_{N} r(x_{j,m}, x_{j,\tilde{m}}) \{f(\xi_{1,\ell}) - f(\xi_{1,\ell})\}^2 \right].
\]

If \(\ell = 0\), then this vanishes by (7.4) and (7.6). If \(\ell = N - 1\), then this vanishes by (7.2) and (7.3). If \(\ell \in [1, N - 2]\), then

\[
f(\xi_{1,\ell}) - f(\xi_{1,\ell}) = O\left(\frac{\ell}{N}\right),
\]

by (7.4), (7.5), (7.6), (7.7), and (7.8). Therefore, (7.13) is bounded from above by

\[
C \sum_{\ell=1}^{N-2} \frac{N d_{N}^{3} m_{*} \ell^2}{\ell(N - \ell - 1)} N^2 = O\left(\frac{d_{N}^{3}}{N^2}\right).
\]

(Case 2) The second case is represented when a sole particle moves between \(y_{p}\) and \(y_{q}\). This case is identical to Lemma 5.3, which is bounded by

\[
C \sum_{k=1}^{N-2} \frac{Nd_{N}^{3} m_{*}}{k(N - k - 1)} \frac{1}{N^2} = O\left(\frac{d_{N}^{3} \log N}{N^2}\right).
\]

Gathering the cases, we have by (7.14) and (7.15) that \(\Sigma_{2}(f) = O(d_{N}^{3} N^{-2} \log N)\). This concludes the proof.

Next, we consider \(\Sigma_{3}(f)\).

Lemma 7.5. Under the conditions of Theorem 2.10, it holds that

\[
\Sigma_{3}(f) = O\left(d_{N}^{3} m_{*} N + d_{N}^{3} N \log N\right) = o\left(\frac{d_{N}^{2}}{N}\right).
\]

Proof. We formulate

\[
\Sigma_{3}(f) = \sum_{\eta \in \mathcal{U}_{N}} \sum_{\zeta \in \mathcal{V}_{N}} \mu_N(\eta) q_{N}(\eta, \zeta) \{f(\zeta) - f(\eta)\}^2.
\]

We divide this into several cases depending on which subset \(\eta\) belongs to.

(Case 1) \(\eta \in \mathcal{A}_{N}^{x_{i,n} y_{j,m}}\) for some \((i,j) \in A \times B\) and \(n, m \geq 1\): In this case, the movement must occur between sites in \(S_{i}^{(2)}\) or between sites in \(S_{j}^{(2)}\). Otherwise, \(\zeta \notin \mathcal{V}_{N}\). Hence, \(f\) remains unchanged by (7.4), (7.6), and (7.12).

(Case 2) \(\eta \in \mathcal{A}_{N}^{x_{i,n} y_{p}} \setminus \mathcal{E}_{N}^{x_{i,n}}\) for some \(i \in [1, \kappa_{*}]\) and \(n, p \geq 1\): We divide again by types of the particle movement.
• (Case 2-1) Movement from $y_p$ into $S \setminus \{x_{i,n}\}$: We have $\eta_{y_p} \leq N - 1$, since otherwise $\zeta \notin \mathcal{V}_N$. Hence, $f$ remains unchanged by (7.3), (7.10), and (7.11).

• (Case 2-2) Movement from $y_p$ into $S \setminus (S_* \cup \{y_p\})$: This is identical to (Case 2-1) of Lemma 5.4, so that the summation is 0.

• (Case 2-3) Movement from $x_{i,n}$ into $S \setminus \{x_{i,n}\}$: $f$ remains unchanged by (7.3), (7.10), and (7.11).

• (Case 2-4) Movement from $x_{i,n}$ into $S \setminus (S_* \cup \{y_p\})$: This is same with (Case 2-2) of Lemma 5.4. We obtain the upper bound $O(d_N^2m_*)$. In conclusion, (Case 2) yields $O(d_N^2m_*)$.

(Case 3) $\eta \in \hat{\mathcal{A}}^{x_{i,n},y_p} x_{j,m}$ for some $(i,j) \in A \times B$ and $n,m,p \geq 1$: This case is almost identical to (Case 3) of Lemma 5.4. The only different thing is that the particle movement might occur as $x_{i,n} \leftrightarrow x_{i,n}$ or $x_{j,m} \leftrightarrow x_{j,m}$. In this other case, we may bound the summation as

$$O \left( \sum_{k=1}^{N-2} \sum_{\ell=1}^{N-1-k} \frac{Nd_N^3m_\star^{\ell}}{(N-k-\ell)} \cdot \frac{(\ell+1)^2}{N} \right) = O \left( \frac{d_N^3N^2 \log N}{N} \right).$$

Thus, we can bound (Case 3) by $O(\frac{d_N^3N^2 \log N}{N})$. Summarizing all cases, we conclude that $\Sigma_3(f) = O(d_N^2m_\star^N + \frac{d_N^3}{N} \log N)$.

Our final aim of this subsection is $\Sigma_4(f)$.

Lemma 7.6. Under the conditions of Theorem 2.10, it holds that

$$\Sigma_4(f) = O(d_N^2m_\star^N) + O(\frac{d_N^3}{N} \log N) = o(\frac{d_N^2}{N} N).$$

Proof. By definition, we have

$$\Sigma_4(f) = \frac{1}{2} \sum_{\eta,\zeta \in \mathcal{V}_N} \mu_N(\eta) q_N(\eta, \zeta) \{f(\zeta) - f(\eta)\}^2.$$

Recalling (7.9), we divide the summation in $\eta, \zeta \in \mathcal{V}_N$ by where $\eta$ and $\zeta$ belong.

(Case 1) $\mathcal{V}_N^1 \leftrightarrow \mathcal{V}_N^1$ or $\mathcal{V}_N^2 \leftrightarrow \mathcal{V}_N^2$: $f$ remains unchanged by (7.10) and (7.11).

(Case 2) $\mathcal{V}_N^1 \leftrightarrow \mathcal{V}_N^2$: Similarly to (Case 2) of Lemma 5.5, the summation is exponentially small scaling as $O(d_N^2m_\omega^{*N})$.

(Case 3) $\mathcal{V}_N^1 \leftrightarrow \mathcal{V}_N^3$: This is impossible.

(Case 4) $\mathcal{V}_N^2 \leftrightarrow \mathcal{V}_N^3$: We can bound this case by $O(d_N^3)$ in a similar way as in (Case 4) of Lemma 5.5.
(Case 5) $Y_N^2 \leftrightarrow Y_N^2$: In the same reasoning with (Case 5) of Lemma 5.5, with some additional care, we may bound this case by

$$C \frac{d^3 N}{N} \sum_{\ell=1}^{\infty} (d_N \log N)^\ell = C \frac{d^3 N}{N} \log N \cdot \frac{1}{1 - d_N \log N} = O\left(\frac{d^3 N}{N} \log N\right).$$

\[\square\]

7.5. Proof of Proposition 7.1

Now, we are in position to prove Proposition 7.1.

Proof of Proposition 7.1. By Lemmas 7.3, 7.4, 7.5, and 7.6,

$$D_N(f^A_{\text{test}}) \leq \frac{d^2 N}{|S_*|^2} \sum_{i \in A, j \in B} \frac{1}{R_{i,j}} + O\left(\frac{d^2 N}{N^2}\right) + O(d^2 m_N) + O\left(\frac{d^3 N}{N} \log N\right).$$

Sending $N \to \infty$, as $\lim_{N \to \infty} d_N \log N = 0$ and $d_N$ decays subexponentially, we have

$$\limsup_{N \to \infty} \frac{N}{d^2 N} D_N(f^A_{\text{test}}) \leq \frac{1}{|S_*|} \sum_{i \in A} \sum_{j \in B} \frac{1}{R_{i,j}}.$$

Therefore, by Theorem 3.4, we obtain the desired result. \[\square\]

8. Lower Bound for Capacities: General Case

In this section, we establish the lower bound for the capacities in the most general setting given in Subsection 2.5. The following proposition explains the result. The proofs in this section will be stated concisely.

Proposition 8.1 (Lower bound for capacities: General case). Assume the conditions of Theorem 2.10. Suppose that \{A, B\} is a non-trivial partition \([1, \kappa_*]\). Then, the following inequality holds.

$$\liminf_{N \to \infty} \frac{N}{d^2 N} \text{Cap}_N(\mathcal{E}_N^{(2)}(A), \mathcal{E}_N^{(2)}(B)) \geq \frac{1}{|S_*|} \sum_{i \in A} \sum_{j \in B} \frac{1}{R_{i,j}}.$$

(8.1)

We construct a test flow, whose divergence can be handled outside the one-dimensional tubes.

8.1. Construction of test flow $\psi^A_{\text{test}}$. In this subsection, we build the test flow $\psi = \psi^A_{\text{test}}$ on $\mathcal{H}_N$.

- We define, for $(i, j) \in I \cap (A \times B)$, $n, m \geq 1$, $p \in P_{i,n,j,m}$, $k \geq 1$, $N - \ell - k \geq 1$, and $\ell \in [0, \frac{N}{2} - 1]$,

$$\psi^A_{0}(s_{k, \ell}^{x_i, n, y_p, x_j, m}, s_{k-1, \ell+1}^{x_i, n, y_p, x_j, m}) = \frac{m_*(y_p)^{\ell/2}}{\mathcal{R}_{i,j} \sum_{n=1}^{n(i)} \sum_{m=1}^{m(j)} \sum_{q=1}^{q_0} \frac{(1-m_*(y_q))^{-1}}{r^{x_i, n, y_p} + r^{x_j, m, y_q}}},$$

(8.2)

$$\psi^A_{0}(s_{k, \ell}^{x_i, n, y_p, x_j, m}, s_{k-1, \ell+1}^{x_i, n, y_p, x_j, m}) = - \frac{m_*(y_p)^{\ell/2}}{\mathcal{R}_{i,j} \sum_{n=1}^{n(i)} \sum_{m=1}^{m(j)} \sum_{q=1}^{q_0} \frac{(1-m_*(y_q))^{-1}}{r^{x_i, n, y_p} + r^{x_j, m, y_q}}},$$

(8.3)
and 0 otherwise.

- Then, for all \((i, j) \in I \cap (A \times B), n, m \geq 1, p \in P_{i, n, j, m}, \) and \(k \in [1, N - 1],\) we define a correction flow \(\phi_{i, j, p, k}^A\) as follows.
  - Suppose that \(\frac{N}{2} < k \leq N - 1.\) Then, for \(\ell \in [1, \frac{N}{2}],\)
    \[
    \phi_{i, j, p, k}^A(x_{k - \ell, \ell}, x_{k, k+1, \ell-1}) := -\sum_{t=\ell}^{\lfloor N/2 \rfloor} (\text{div } \psi_0)(\zeta_{k-t, t}^{x_{i, j, n, y_j, m}}),
    \]
    and \(\phi_{i, j, p, k}^A = 0\) on all other edges.
  - Suppose that \(1 \leq k \leq \frac{N}{2}.\) Then, for \(\ell \in [1, k-1],\)
    \[
    \phi_{i, j, p, k}^A(x_{k-\ell, \ell}, x_{k, k+1, \ell-1}) := -\sum_{t=\ell}^{k-1} (\text{div } \psi_0)(\zeta_{k-t, t}^{x_{i, j, n, y_j, m}}),
    \]
    and \(\phi_{i, j, p, k}^A = 0\) on all other edges.

- Finally, we define a flow
  \[
  \psi = \psi_\text{test}^A := \psi_0^A + \sum_{(i, j) \in I \cap (A \times B)} \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} \sum_{p \in P_{i, n, j, m}} \phi_{i, j, p, k}^A.
  \]

Then, observe that \((\text{div } \psi)(\zeta_{i, j}^{x_{i, j, n, y_j, m}}) = 0\) for all \(i \in [1, \kappa_i]\) and \(n \in [1, n(i)]\). Moreover, it holds that \((\text{div } \psi)(\eta) = 0\) for all \(\eta\) in

\[
A_{i, j}^{x_{i, j, n, y_j, m}} \setminus (A_{i, j}^{x_{i, j, n, y_j, m}} \cup A_{i, j}^{x_{i, j, n, y_j, m}}) \text{ for } i \neq j \text{ and } n, m, p \geq 1.
\]

### 8.2. Flow norm of \(\psi_\text{test}^A.\)

In this subsection, we calculate the flow norm of the test flow \(\psi.\)

**Lemma 8.2.** Suppose that \(I \cap (A \times B) \neq \emptyset.\) Then, under the conditions of Theorem 2.10

\[
\|\psi\|^2_N \leq (1 + o(1)) \left( \frac{|S_N|}{d_N^2} \sum_{i \in A} \sum_{j \in B} \frac{1}{R_{i, j}} \right).
\]

**Proof.** The proof is almost identical to that of Lemma 6.3; therefore, we omit it. \(\square\)

### 8.3. Remaining terms.

We estimate the remaining terms on the right-hand side of (3.7) with respect to \(\psi.\) Lemma 4.4 is employed once more.

**Lemma 8.3.** Suppose that \(I \cap (A \times B) \neq \emptyset.\) Then, under the conditions of Theorem 2.10

\[
\sum_{\eta \in H_N \setminus E_+} h_{\mathcal{E}_N^1(A), \mathcal{E}_N^1(B)}(\eta)(\text{div } \psi)(\eta) = (1 + o(1)) \left( \sum_{i \in A} \sum_{j \in B} \frac{1}{R_{i, j}} \right).
\]

**Proof.** We omit the proof due to its similarity to Lemma 6.4 \(\square\)

### 8.4. Proof of Proposition 8.1

We are now ready to prove Proposition 8.1.

**Proof of Proposition 8.1.** There remains nothing to prove if \(I \cap (A \times B) = \emptyset,\) as then the right-hand side of (8.1) equals 0. Thus, we may assume \(I \cap (A \times B) \neq \emptyset.\) Then, by Lemmas
By Proposition 2.1 and (2.4), the last line equals 4.1, and by reversibility, we calculate
\[ D \]
for all \( 0 \leq S \)

By the definition of \( \eta, \zeta \)

Using the Cauchy–Schwarz inequality once more on \( 0 \leq \eta \leq \zeta = \zeta^y \) with \( x, y \) in \( S_i^{(2)} \).

By the Cauchy–Schwarz inequality on \( 1 \)

Therefore, we deduce from Theorem 3.6 that
\[ \text{Cap}_N(E_N^{(2)}(A), E_N^{(2)}(B)) \geq (1 + o(1)) \frac{d_N^2}{|S_*|N} \sum_{i \in A} \sum_{j \in B} \frac{1}{\mathfrak{g}_{i,j}}. \]

This concludes the proof of Proposition 8.1.

9. Proof of Condition (3.10)

In this section, we prove the condition (3.10) formulated in Proposition 3.8.

**Proposition 9.1.** The condition (3.10) holds for every \( i \in [1, \kappa_*] \).

**Proof.** The numerator in (3.10) can be dealt with using Proposition 5.1
\[ \text{Cap}_N(E_N^{(2)}(i), E_N^{(2)}(i)) = \frac{d_N^2}{N} \cdot O(1). \]

For the denominator in (3.10), fix \( \eta, \zeta \in E_N^{(2)}(i) \) and write \( \eta = \xi^u \) and \( \zeta = \xi^y \) with \( x, y \in S_i^{(2)} \).

By the definition of \( S_i^{(2)} \), there exist \( x = x_0, x_1, \ldots, x_t = y \) in \( S_i^{(2)} \) so that \( t \leq |S_i^{(2)}| \) and
\[ r(x_n, x_{n+1}) = r(x_{n+1}, x_n) > 0 \]

for all \( 0 \leq n \leq t - 1 \). Take any \( F : \mathcal{H}_N \to \mathbb{R} \) with \( F(\eta) = 1 \) and \( F(\zeta) = 0 \). Recalling Definition 4.1 and by reversibility, we calculate \( D_N(F) \) by
\[ \frac{1}{2} \sum_{\eta \in \mathcal{H}_N} \sum_{a, b \in S} \mu_N(\eta) \eta_a(d_N + \eta_b) r(a, b) \{ F(\sigma^{a,b}_\eta) - F(\eta) \}^2 \]
\[ \geq \sum_{n=0}^{t-1} \sum_{j=1}^{N} \mu_N(\xi_j^{x_nx_{n+1}}) j(d_N + N - j) r(x_n, x_{n+1}) \{ F(\xi_j^{x_nx_{n+1}}) - F(\xi_j^{x_{n+1}}) \}^2. \]

By Proposition 2.1 and (2.4), the last line equals
\[ (1 + o(1)) \frac{1}{N} \sum_{n=0}^{t-1} \frac{d_N}{|S_*|} r(x_n, x_{n+1}) \sum_{j=1}^{N} \{ F(\xi_j^{x_nx_{n+1}}) - F(\xi_j^{x_{n+1}}) \}^2. \]

By the Cauchy–Schwarz inequality on \( 1 \leq j \leq N \), the above is bounded from below by
\[ (1 + o(1)) \sum_{n=0}^{t-1} \frac{d_N}{|S_*|} r(x_n, x_{n+1}) \{ F(\xi_n) - F(\xi^{x_{n+1}}) \}^2. \]

Using the Cauchy–Schwarz inequality once more on \( 0 \leq n \leq t - 1 \), we obtain the following lower bound for \( D_N(F) \):
\[ (1 + o(1)) \frac{d_N}{|S_*|} \sum_{n=0}^{t-1} \frac{1}{r(x_n, x_{n+1})} \geq (1 + o(1)) \frac{d_N}{|S_*|} \sum_{u, v \in S_i^{(2)}} \frac{1}{|S_i^{(2)}|} \min\{r(u, v) > 0 : u, v \in S_i^{(2)}\}. \]
As $F$ was arbitrary, by the Dirichlet principle given in Theorem 3.4 (1), we have

$$\text{Cap}_N(\{\eta\}, \{\zeta\}) \geq (1 + o(1)) \frac{d_N}{|S^*_i|} \min \{r(u, v) > 0 : u, v \in S_N^{(2)}\},$$

(9.2)

Therefore, by (9.1) and (9.2), we obtain

$$\limsup_{N \to \infty} \text{Cap}_N(\mathcal{C}_N^{(2)}(i), \mathcal{E}_N \setminus \mathcal{C}_N^{(2)}(i)) \leq C \limsup_{N \to \infty} \frac{d^2_N/N}{d_N} = C \limsup_{N \to \infty} \frac{d_N}{N} = 0.\]$$

The last formula concludes the proof of Proposition 9.1. □

10. Proof of the Main Theorem

Now, we are in position to prove the main theorem given in Theorem 2.10. First, we provide sharp asymptotics for the transition rate of the trace process $\eta^*_N(\cdot)$.

Proposition 10.1 (Transition rates of the trace process). Suppose that $d_N$ decays subexponentially. Then, for $i, j \in [1, \kappa_*]$,

$$\lim_{N \to \infty} \frac{N}{d^2_N} r^*_N(i, j) = \frac{1}{|S^*_i| |\mathcal{R}_{i,j}|}.$$

Proof. By Proposition 2.3, $\lim_{N \to \infty} \mu_N(\mathcal{C}_N^{(2)}(i)) = |S^*_i|/|S^*_i|$ for each $i \in [1, \kappa_*]$. Hence, by Propositions 5.1, 6.1, and (3.8), we have

$$\frac{|S^*_i|}{|S^*_i|} \frac{1}{|\mathcal{R}_{i,j}|} \left| \frac{\sum_{k_i, k_j} \frac{1}{\mathcal{R}_{i,k}} + \sum_{k_i, k_j, k_k} \frac{1}{\mathcal{R}_{j,k}} - \sum_{k_i, k_j, k_k} \left( \frac{1}{\mathcal{R}_{i,k}} + \frac{1}{\mathcal{R}_{j,k}} \right) + o(1)}{\sum_{k_i, k_j, k_k} \frac{1}{\mathcal{R}_{i,k}} + o(1)} \right| = \frac{d^2_N}{|S^*_i| |\mathcal{R}_{i,j}|} = o(1).$$

Multiplying $|S^*_i|/|S^*_i|d^2_N$ on both sides, we obtain the desired result. □

Finally, we provide the proof of Theorem 2.10

Proof of Theorem 2.10. By Propositions 9.1 and 10.1, the conditions (3.9) and (3.10) are verified for

$$a(i, j) = \frac{1}{|S^*_i| |\mathcal{R}_{i,j}|} \text{ for } i, j \in [1, \kappa_*] \text{ and } \theta_N = \theta_{N, 2} = \frac{d^2_N}{N}.$$ 

Therefore, Proposition 8.8 establishes the thermalization result stated in (1) and the convergence result stated in (2).

For the last statement in (3), we first show that

$$\lim_{N \to \infty} \sup_{i \in [1, \kappa_*], n \in [1, n(i)]} \mathbb{E}_{\xi^i_n} \left[ \int_0^t 1 \{\eta_N(\theta_{N, 2}s) \notin \mathcal{E}_N^*\} ds \right] = 0.$$

(10.1)

To this end, fix $i$ and $n$. Note that

$$\mathbb{P}_{\xi^i_n} [\eta_N(\theta_{N, 2}s) \notin \mathcal{E}_N^*] \leq \frac{1}{\mu_N(\mathcal{E}_N^*)} \mu_N [\eta_N(\theta_{N, 2}s) \notin \mathcal{E}_N^*] = \frac{\mu_N(\mathcal{H}_N \setminus \mathcal{E}_N^*)}{\mu_N(\mathcal{E}_N^*)}.$$

(10.2)
Here, $\mathbb{P}_{\mu_N}$ is the law of the process whose initial distribution is $\mu_N$. The identity holds, as $\mu_N$ is the invariant distribution. Therefore,

$$
\mathbb{E}_{\xi^{i,n}} \left[ \int_0^t 1 \{ \eta_N(\theta_N, 2s) \notin \mathcal{E}_N^x \} ds \right] = \int_0^t \mathbb{P}_{\xi^{i,n}} \left[ \eta_N(\theta_N, 2s) \notin \mathcal{E}_N^x \right] ds 
\leq \int_0^t \mu_N(\mathcal{H}_N \setminus \mathcal{E}_N^x) \mu_N(\mathcal{E}_N^{x,i,n}) \mu_N(\mathcal{E}_N^{x,i,n}) ds = t \cdot \mu_N(\mathcal{H}_N \setminus \mathcal{E}_N^x),
$$

which vanishes uniformly in the limit $N \to \infty$ by Proposition 2.3. This proves (10.1).

It remains to show that

$$
\lim_{N \to \infty} \sup_{i \in [1, \kappa], n \in [1, n(i)]} \mathbb{E}_{\xi^{i,n}} \left[ \int_0^t 1 \{ \eta_N(\theta_N, 2s) \in \mathcal{E}_N^x \setminus \mathcal{E}_N(S^{(3)}_i) \} ds \right] = 0. \quad (10.3)
$$

We apply Proposition 3.9. Because the first two conditions are already proven, it suffices to prove (3.11). This is clear from (10.2). Hence, we have the convergence of finite-dimensional marginal distributions. Therefore, for each pair $(i, n)$ and $s \in [0, t]$,

$$
\lim_{N \to \infty} \mathbb{P}_{\xi^{i,n}} \left[ \eta_N(\theta_N, 2s) \in \mathcal{E}_N^x \setminus \mathcal{E}_N(S^{(3)}_i) \right] = \mathbb{P}_i \left[ X_{\text{second}}(s) \in S_s \setminus S^{(3)}_i \right] = 0.
$$

The last equality holds, as starting at $i$, $X_{\text{second}}(\cdot)$ never visits $S_s \setminus S^{(3)}_i$ by (2.16). Because $S_s$ is finite, we have (10.3). Finally, (10.1) and (10.3) conclude the proof of Theorem 2.10.

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