Discrete eigenvalues of the spin-boson Hamiltonian with two photons: on a problem of Minlos and Spohn

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Abstract

Under minimal regularity conditions on the photon dispersion and the coupling function, we prove that the spin-boson model with two massless photons in \( \mathbb{R}^d \) cannot have more than two bound state energies whenever the coupling strength is sufficiently strong.

1 Introduction

In this paper we are concerned with the discrete spectrum analysis for the Hamiltonian of a quantum mechanical model which describes the interaction between a two-level atom and two massless photons. The energy operator is obtained from the spin-boson Hamiltonian by the compression onto the subspace of two bosons and acts on the Hilbert space which is given by the tensor product of \( \mathbb{C}^2 \) and the truncated Fock space

\[
\mathcal{F}_s^2 := \mathbb{C} \oplus L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^d \times \mathbb{R}^d).
\]

Here \( L^2(\mathbb{R}^d \times \mathbb{R}^d) \) stands for the subspace of the Hilbert space \( L^2(\mathbb{R}^2 \times \mathbb{R}^2) \) consisting of symmetric functions and equipped with the inner product

\[
(\phi, \psi) = \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(k_1, k_2) \overline{\psi(k_1, k_2)} \, dk_1 \, dk_2, \quad \phi, \psi \in L^2_{\mathrm{s}}(\mathbb{R}^d \times \mathbb{R}^d).
\]

For \( f = (f^{(\sigma)}_0, f^{(\sigma)}_1, f^{(\sigma)}_2) \in \mathbb{C}^2 \otimes \mathcal{F}_s^2 \), where \( \sigma = \pm \) is the discrete variable, the Hamiltonian of our system is given by the formal expression

\[
\begin{align*}
(H_\alpha f)_0^{(\sigma)}(k) &= \sigma \varepsilon f_0^{(\sigma)} + \alpha \int_{\mathbb{R}^d} \lambda(q) f_1^{(-\sigma)}(q) \, dq, \\
(H_\alpha f)_1^{(\sigma)}(k) &= \left(\sigma \varepsilon + \omega(k)\right) f_1^{(\sigma)}(k) + \alpha \lambda(k) f_0^{(-\sigma)} + \alpha \int_{\mathbb{R}^d} f_2^{(-\sigma)}(k, q) \lambda(q) \, dq, \\
(H_\alpha f)_2^{(\sigma)}(k_1, k_2) &= \left(\sigma \varepsilon + \omega(k_1) + \omega(k_2)\right) f_2^{(\sigma)}(k_1, k_2) + \alpha \lambda(k_1) f_1^{(-\sigma)}(k_2) + \alpha \lambda(k_2) f_1^{(-\sigma)}(k_1).
\end{align*}
\]

Here \( \varepsilon (\varepsilon > 0) \) and \( -\varepsilon \) are the excited and the ground state energies of the atom, respectively, \( \omega(k) = |k| \) is the photon dispersion relation, \( \alpha > 0 \) is the coupling constant and \( \lambda \) is the coupling function which is given by the product of \( \sqrt{\omega(k)} \) with a cut-off function for large \( k \).

In general, the dispersion relation \( \omega \geq 0 \) and the coupling function \( \lambda \) are fixed by the physics of the problem. Motivated by different applications of the spin-boson Hamiltonian one considers them as free parameter functions and imposes only some general conditions such as

\[
\lambda \in L^2(\mathbb{R}^d), \quad \frac{\lambda}{\sqrt{\omega}} \in L^2(\mathbb{R}^d),
\]

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provided that
\[ \inf_{k \in \mathbb{R}^d} \omega(k) = 0. \]  
(5)

Most of the work on the spectrum of the spin-boson model in the up-to-date literature assumes at least \([4]\), or its various strengthened versions, where, for example, the second condition in \([4]\) is replaced by the requirement that
\[ \frac{\lambda}{\omega} \in L^2(\mathbb{R}^d), \]  
(6)

which is known as the infrared regularity condition (cf. \([6]\)).

We notice that the natural domain of the unperturbed operator \(H_0\) is given by the tensor product of \(\mathbb{C}^2\) with \(\mathbb{C} \oplus H_1 \oplus H_2\), where \(H_1\) and \(H_2\) are the weighted \(L^2\)-Hilbert spaces
\[ H_1 := \left\{ f \in L^2(\mathbb{R}^d) : \int |\omega(k)|^2 |f(k)|^2 \, dk < \infty \right\} \]  
(7)

and
\[ H_2 := \left\{ g \in L^2_s(\mathbb{R}^2 \times \mathbb{R}^d) : \int |\omega(k_1) + \omega(k_2)|^2 |g(k_1, k_2)|^2 \, dk_1 \, dk_2 < \infty \right\}. \]  
(8)

The first condition in \([4]\) implies the boundedness of the perturbation \(H_\alpha - H_0\) and thus the expression for \(H_\alpha\) given in \([4]\) generates a self-adjoint operator in the Hilbert space \(\mathbb{C}^2 \otimes F^2_s\) on the natural domain of \(H_0\) (see \([11]\) Theorem V.4.3]). Throughout the paper we denote the corresponding self-adjoint operator again by \(H_\alpha\) for notational convenience. The spatial dimension, \(d \geq 1\), plays no particular role in our analysis and is left arbitrary.

Starting with the pioneering work of Hübner and Spohn \([9]\), spectral properties of the spin-boson model as well as of its finite photon approximations have been investigated extensively and the corresponding literature is enormous. In contrast to rigorous results of the weak coupling regime, it seems that spectral properties of the spin-boson Hamiltonian (even with particle number cut-off) for general coupling have not been fully understood yet. The interested reader is referred, for example, to \([8, 7, 2, 3, 4, 5]\), where the location of the essential spectrum is given for any value of the coupling constant, and results including the finiteness of the point spectrum or absence of the singular continuous spectrum are proven under various assumptions in addition to \([4]\). The ergodicity of the spin-boson Hamiltonian at arbitrary coupling strength was studied in \([12]\). In the recent work \([10]\) we have obtained an explicit description of the essential spectrum and proved the finiteness of the discrete spectrum for the spin-boson model with two photons for arbitrary coupling where the only requirement on the coupling function was its square integrability.

It is well known that, under appropriate conditions in addition to \([4]\)–\([6]\), there exists a sufficiently small coupling constant \(\alpha_0 > 0\) such that for all \(\alpha \in (0, \alpha_0)\) the self-adjoint operator generated by \([4]\) has a unique discrete eigenvalue (see \([14, 9]\)). Moreover, it is also well known that, if the coupling is “weak but not very weak”, then one further eigenvalue appears. This is the case for any finite photon approximation of the spin-boson Hamiltonian (see \([11, 7]\)). Unlike these observations, arbitrary coupling results of the up-to-date literature in this direction guarantee only finiteness of the number of discrete eigenvalues for a given coupling constant \(\alpha > 0\) (cf. \([10]\)). In fact, it has been an open problem for a long time whether the spin-boson Hamiltonian with two photons can have more than two bound state energies for some (i.e. strong) coupling (cf. \([13]\) p. 192 and \([7]\ p. 8\]). It is the goal of the present paper to answer this question for strong coupling by proving the following claim which holds under considerably relaxed conditions on the parameter functions.

**Theorem 1.** Let \(\lambda \in L^2(\mathbb{R}^d)\) and let \(\omega : \mathbb{R}^d \to [0, \infty)\) be an unbounded and almost everywhere continuous function satisfying \([4]\). Then the Hamiltonian \(H_\alpha\) cannot have more than two bound state energies for sufficiently strong coupling strength \(\alpha > 0\).

The detailed proof of this result is given in the next section. It relies on a simple and instructive method based on the splitting trick developed in author’s recent work \([10]\) and asymptotic analyses of zeros of Nevanlinna functions.

Throughout the paper we adopt the following notation. For a self-adjoint operator \(T\) acting in a Hilbert space and a constant \(\mu \in \mathbb{R}\) such that \(\mu \leq \min \sigma_{ess}(T)\), we denote by \(N(\mu; T)\) the
dimension of the spectral subspace of $T$ corresponding to the interval $(-\infty, \mu)$. The latter quantity coincides with the number of discrete eigenvalues (counted with multiplicities) of $T$ that are less than $\mu$. Integrals with no indication of limits imply integration over the whole space $\mathbb{R}^d$ or $\mathbb{R}^d \times \mathbb{R}^d$, and $\|\cdot\|$ denotes the usual $L^2$-norm.

2 Proof of Theorem [1]

Unless otherwise specified, we always assume that the discrete variable $\sigma = \pm$ is fixed. Moreover, without loss of generality we assume that the dispersion relation $\omega: \mathbb{R}^d \to [0, \infty)$ is a continuous function and the coupling function $\lambda: \mathbb{R}^d \to \mathbb{C}$ is not identically zero. All the arguments below plainly work for almost everywhere continuous $\omega$. If the coupling function is identically zero on $\mathbb{R}^d$, then the photons do not couple to the atom and the description of the spectrum becomes straightforward.

It is easy to see that the transformation $U: \mathbb{C}^2 \otimes F^2_\sigma \rightarrow F^2_\sigma \oplus F^2_\lambda$, defined by

$$U: \begin{pmatrix} (f_0^{(+)}), (f_1^{(+)}), (f_2^{(+)}) \\ (f_0^{(-)}), (f_1^{(-)}), (f_2^{(-)}) \end{pmatrix} \mapsto \begin{pmatrix} (f_0^{(\pm)}, f_1^{(\pm)}, f_2^{(\pm)}) \\ (f_0^{(\mp)}, f_1^{(\mp)}, f_2^{(\mp)}) \end{pmatrix},$$

is a unitary operator and block-diagonalizes the Hamiltonian $H_\alpha$ in (3), i.e.

$$U^* H_\alpha U = \text{diag}\{H_\alpha^{(+)}, H_\alpha^{(-)}\},$$

(10)

where the operator matrix

$$H_\alpha^{(\sigma)} := \begin{pmatrix} H_{00}^{(\sigma)} & \alpha H_{01}^{(\sigma)} & 0 \\ \alpha H_{10}^{(\sigma)} & H_{11}^{(\sigma)} & \alpha H_{12}^{(\sigma)} \\ 0 & \alpha H_{21}^{(\sigma)} & H_{22}^{(\sigma)} \end{pmatrix}, \quad \text{Dom}(H_\alpha^{(\sigma)}) := \mathbb{C} \oplus H_1 \oplus H_2,$$

(11)

acts in the truncated Fock space $F^2_\sigma$ with $H_1$ and $H_2$ defined in (1) and (3). The operator entries of $H_\alpha^{(\sigma)}$ are given by

$$(H_{00}^{(\sigma)} f_0)(k) = \sigma \varepsilon f_0, \quad (H_{01}^{(\sigma)} f_0)(k) = \int \lambda(q) f_1(q) \, dq, \quad (H_{10}^{(\sigma)} f_0)(k) = f_0 \overline{\lambda(k)},$$

(12)

$$(H_{11}^{(\sigma)} f_1)(k) = (-\sigma \varepsilon + \omega(k)) f_1(k), \quad (H_{12}^{(\sigma)} f_2)(k) = \int f_2(k, q) \lambda(q) \, dq,$$

$$(H_{21}^{(\sigma)} f_1)(k_1, k_2) = \lambda(k_1) f_2(k_2) + \lambda(k_2) f_1(k_1)$$

and

$$(H_{22}^{(\sigma)} f_2)(k_1, k_2) = (\sigma \varepsilon + \omega(k_1) + \omega(k_2)) f_2(k_1, k_2), \quad (f_0, f_1, f_2) \in F^2_\sigma.$$

It follows from the square-integrability of the coupling function that $H_{12}: L^2(\mathbb{R}^d \times \mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ and $H_{01}: L^2(\mathbb{R}^d) \rightarrow \mathbb{C}$ are bounded operators with $H_{10} = H_{01}^*$ and $H_{21} = H_{12}^*$. Hence, $H_\alpha^{(\sigma)}$ is a self-adjoint operator on the domain $\mathbb{C} \oplus H_1 \oplus H_2$ (see [11] Theorem V.4.3]).

In view of (10), in the sequel we focus on the study of the discrete spectrum of the operator matrices $H_\alpha^{(\pm)}$. We recall from (10) that

$$\min \sigma_{\text{ess}}(H_\alpha^{(+)}) = E_\varepsilon(\alpha) \quad \text{for all} \quad \alpha > 0,$$

$$\min \sigma_{\text{ess}}(H_\alpha^{(-)}) = \begin{cases} -\varepsilon, & \text{if} \quad \frac{\lambda}{\sqrt{\omega}} \in L^2(\mathbb{R}^d) \quad \text{and} \quad 0 < \alpha \leq \frac{\sqrt{\varepsilon}}{\| \lambda \|}, \\
E_{-\varepsilon}(\alpha), & \text{otherwise}, \end{cases}$$

(13)

where $E_{\sigma \varepsilon}(\alpha)$ is the unique zero of the continuous function

$$\Phi_\alpha^{(\sigma)}(z) = -\sigma \varepsilon - z - \alpha^2 \int \frac{[\lambda(q)]^2 \, dq}{\omega(q) + \sigma \varepsilon - z}, \quad z \in (-\infty, \sigma \varepsilon).$$

(14)
We notice that \( E_\varepsilon(\alpha) \) exists for all \( \alpha > 0 \), while the existence of \( E_{-\varepsilon}(\alpha) \) requires \( \alpha > 0 \) to be not too small whenever the condition \( \lambda / \sqrt{\omega} \in L^2(\mathbb{R}^d) \) holds. We make the convention \( E_{-\varepsilon}(\alpha) := -\varepsilon \) whenever \( \Phi(\cdot) \) does not have a zero in \( (-\infty, -\varepsilon) \). In view of (10), the bottom of the essential spectrum of the Hamiltonian \( H_\alpha \) is thus given by

\[
E(\alpha) := \min \sigma_{\text{ess}}(H^{(\alpha)}_\sigma) := \min \{ E_\varepsilon(\alpha), E_{-\varepsilon}(\alpha) \}.
\]

One can easily observe that \( E_\varepsilon(\alpha) < -\varepsilon \) for all \( \alpha > 0 \).

In the sequel we will be dealing with the non-linear pencil \( R^{(\sigma)}_\alpha : (-\infty, \sigma \varepsilon) \to \mathcal{L}(H_1, L^2(\mathbb{R}^d)) \), defined by

\[
R^{(\sigma)}_\alpha(z) := H^{(\sigma)}_{11} - z - \alpha^2 H_{12}(H^{(\sigma)}_{22} - z)^{-1} H_{12}^* - \alpha^2 H_{01}(H^{(\sigma)}_{00} - z)^{-1} H_{01}^*.
\]

It is easy to see that, for each \( z < \sigma \varepsilon \), the operator \( R^{(\sigma)}_\alpha(z) \) is well-defined, bounded from below and self-adjoint on the Hilbert space \( \mathcal{H}_1 \).

We split the proof of Theorem 1 into several steps as in the following lemmas.

**Lemma 2.1.** The number of the discrete eigenvalues of the operator matrix \( H^{(\sigma)}_\alpha \) is equal to the number of the negative eigenvalues of the operator \( R^{(\sigma)}_\alpha(E_{\sigma \varepsilon}(\alpha)) \), i.e.

\[
N\left( E_{\sigma \varepsilon}(\alpha); H^{(\sigma)}_\alpha \right) = N\left( 0; R^{(\sigma)}_\alpha\left( E_{\sigma \varepsilon}(\alpha) \right) \right).
\]

**Proof.** First, we justify the relation

\[
z \in \sigma_{\text{disc}}(H^{(\sigma)}_\alpha) \cap (-\infty, E_{\sigma \varepsilon}(\alpha)) \iff 0 \in \sigma_{\text{disc}}(R^{(\sigma)}_\alpha(z)).
\]

To this end, let us fix \( z < E_{\sigma \varepsilon}(\alpha) \) and note that \( z \) is a discrete eigenvalue for the operator matrix \( H^{(\sigma)}_\alpha \) with a non-zero eigenvector \( \Psi = (f_0, f_1, f_2)^t \in H^2_\alpha \) if and only if

\[
\begin{align*}
&\langle H^{(\sigma)}_{00}(z) f_0 + \alpha H_{01} f_1, 1 \rangle = 0, \\
&\langle \alpha H_{01} f_0 + (H^{(\sigma)}_{11}(z) - \alpha^2) f_1 + \alpha H_{12} f_2, 1 \rangle = 0, \\
&\langle \alpha H_{12} f_1 + (H^{(\sigma)}_{22}(z) - \alpha^2) f_2, 1 \rangle = 0.
\end{align*}
\]

Since the spectra of both \( H^{(\sigma)}_{00} \) and \( H^{(\sigma)}_{22} \) lie on the right of \( \sigma \varepsilon \), the first and the third equations in (19) can be equivalently written as \( f_0 = -\alpha (H^{(\sigma)}_{00}(z) - \alpha^2) f_1 \) and \( f_2 = -\alpha (H^{(\sigma)}_{22}(z) - \alpha^2) f_1 \), respectively. Inserting these into the second equation in (19), we obtain

\[
R^{(\sigma)}_\alpha(z)f_1 = 0.
\]

Note that \( f_1 \) must be a non-zero vector, for otherwise \( \Psi \) would be the zero element of \( H^2_\alpha \), contradicting our hypothesis. Since \( z < E_{\sigma \varepsilon}(\alpha) \), it follows that \( R^{(\sigma)}_\alpha(z) \) is a Fredholm operator as it is a rank-one perturbation of the Fredholm operator corresponding to the Schur complement of \( H^{(\sigma)}_{22} - z \) in the lower \( 2 \times 2 \) operator matrix in \( H^{(\sigma)}_\alpha \) (see [10]). That is why (20) is equivalent to the fact that 0 is a discrete eigenvalue for \( R^{(\sigma)}_\alpha(z) \) with an eigenvector \( f_1 \), thus justifying the claim in (18).

Now the claim in (17) follows from (18) if we show that \( R^{(\sigma)}_\alpha(z) \) is a strictly decreasing operator function in \( (-\infty, E_{\sigma \varepsilon}(\alpha)) \) and has the following property

\[
R^{(\sigma)}_\alpha(z) \uparrow +\infty \quad \text{as} \quad z \downarrow -\infty.
\]

To this end, let us fix \( \phi \in \mathcal{H}_1 \) and note that

\[
\langle R^{(\sigma)}_\alpha(z) \phi, \phi \rangle = \langle H^{(\sigma)}_{11}(z) \phi, \phi \rangle - z \| \phi \|^2 - \alpha^2 \langle (H^{(\sigma)}_{22} - z)^{-1} H_{12}^* \phi, H_{12} \phi \rangle - \alpha^2 \langle (H^{(\sigma)}_{00} - z)^{-1} H_{01}^* \phi, H_{01} \phi \rangle.
\]
Hence, it follows that, for $\phi \neq 0$,
\[
\frac{\partial}{\partial z} \langle R^{(\sigma)}(\alpha) z, \phi \rangle = -\|\phi\|^2 - \alpha^2 \| H^{(\sigma)}_{22} - z \|^{-1} H^{(\sigma)}_{12} \phi \|^2 - \alpha^2 \| H^{(\sigma)}_{00} - z \|^{-1} H^{(\sigma)}_{01} \phi \|^2 < 0. \tag{23}
\]
Moreover, we have the following standard estimates
\[
\| \langle R^{(\sigma)}(\alpha) z, \phi \rangle - \langle H^{(\sigma)}_{11} \phi, \phi \rangle \| \leq \| \langle H^{(\sigma)}_{22} - z \|^{-1} H^{(\sigma)}_{12} \phi \rangle + \| \langle H^{(\sigma)}_{00} - z \|^{-1} H^{(\sigma)}_{01} \phi \rangle \| \\
\leq \| H^{(\sigma)}_{22} - z \|^{-1} \| H^{(\sigma)}_{12} \phi \|^2 + \| H^{(\sigma)}_{00} - z \|^{-1} \| H^{(\sigma)}_{01} \phi \|^2 \\
\leq \frac{1}{\sigma \varepsilon - z} (\| H^{(\sigma)}_{12} \phi \|^2 + \| H^{(\sigma)}_{01} \phi \|^2). \tag{24}
\]
Now letting $z \downarrow -\infty$, we obtain the claim in (24).

We notice that the function $\Phi^{(\sigma)}_{\alpha}$ (see (23)) has a unique zero for all sufficiently large $\alpha > 0$. On the account of Lemma 2.1, from now on we always assume $\alpha > 0$ to be so large that the bottom of the essential spectrum of $H^{(\sigma)}_{\alpha}$ coincides with the unique zero in $(-\infty, \sigma \varepsilon)$ of $\Phi^{(\sigma)}_{\alpha}$.

**Lemma 2.2.** The bottom of the essential spectrum of $H^{(\sigma)}_{\alpha}$ has the following large coupling behavior
\[
E_{\sigma \varepsilon}(\alpha) = -\|\lambda\| \alpha + o(\alpha), \quad \alpha \uparrow +\infty. \tag{25}
\]

**Proof.** Consider the continuous function
\[
\psi(x, y) = y - \sigma \varepsilon x - \frac{1}{\|\lambda\|^2} \int \frac{|\lambda(q)|^2 dq}{(\omega(q) + \sigma \varepsilon) x + y} \tag{26}
\]
for $(x, y) \in [0, \frac{1}{2}) \times [\frac{1}{2}, \infty)$. We have $\psi(0, 1) = 0$ and the partial derivative $\frac{\partial \psi}{\partial y}$ exists and is right continuous at $(0, 1)$. Since $\frac{\partial \psi}{\partial y}(0, 1) = 2$, the implicit function theorem applies and yields the existence of a sufficiently small constant $\delta > 0$ and a unique continuous function $E_{\sigma \varepsilon} : [0, \delta) \to \mathbb{R}$ such that $E(0) = 1$ and $\psi(\beta, E_{\sigma \varepsilon}(\beta)) = 0$ for all $\beta \in [0, \delta)$. Therefore, we get
\[
\Phi^{(\sigma)}_{\alpha} \left( -\alpha \|\lambda\| E_{\sigma \varepsilon} \left( \frac{1}{\alpha \|\lambda\|} \right) \right) = \alpha \|\lambda\| \psi \left( 1 \frac{1}{\alpha \|\lambda\|}, E_{\sigma \varepsilon} \left( \frac{1}{\alpha \|\lambda\|} \right) \right) = 0 \tag{27}
\]
for all $\alpha \in \left( \frac{1}{\|\lambda\|}, \infty \right)$. Clearly, we have
\[
-\alpha \|\lambda\| E_{\sigma \varepsilon} \left( \frac{1}{\alpha \|\lambda\|} \right) < \sigma \varepsilon \tag{28}
\]
for all sufficiently large $\alpha > 0$. Since $E_{\sigma \varepsilon}(\alpha)$ is the unique zero of the function $\Phi^{(\sigma)}_{\alpha}$ in the interval $(-\infty, \sigma \varepsilon)$, we thus deduce from (27) that, for all sufficiently large $\alpha > 0$,
\[
E_{\sigma \varepsilon}(\alpha) = -\alpha \|\lambda\| E_{\sigma \varepsilon} \left( \frac{1}{\alpha \|\lambda\|} \right). \tag{29}
\]
This and the right-continuity at zero of the function $E_{\sigma \varepsilon}(\cdot)$ yield the desired asymptotic expansion.

It is easy to verify that
\[
R^{(\sigma)}_{00}(\alpha) = \Delta^{(\sigma)}_{\alpha}(\alpha) - \alpha^2 \hat{K}^{(\sigma)}_{00}(\alpha), \tag{30}
\]
where, $\Delta^{(\sigma)}_{\alpha} : (-\infty, \sigma \varepsilon) \to \mathcal{L}(H_1, L^2(\mathbb{R}^d))$ is the pencil of multiplication operators by the functions
\[
\Delta^{(\sigma)}_{\alpha}(k; z) := \Phi^{(\sigma)}_{\alpha}(z - \omega(k)) \tag{31}
\]
\[
= \omega(k) - \sigma \varepsilon - z - \alpha^2 \int \frac{|\lambda(q)|^2 dq}{\omega(k) + \omega(q) + \sigma \varepsilon - z}
\]

*In fact, here we are using the non-standard version (cf. (23)) of the implicit function theorem which requires only strict monotonicity in $y$ and no differentiability in a (right-) neighborhood of $(0, 1)$.*
and \( \tilde{K}^{(\sigma)}(z) : (-\infty, \sigma \varepsilon) \to \mathcal{L}(\mathcal{H}_1, L^2(\mathbb{R}^d)) \) is the pencil of integral operators with the kernels
\[
\tilde{\rho}^{(\sigma)}(k_1, k_2; z) := \frac{\lambda(k_1)\lambda(k_2)}{\omega(k_1) + \omega(k_2) + \sigma \varepsilon - z} + \frac{\lambda(k_1)\lambda(k_2)}{\sigma \varepsilon - z}. 
\] (32)

For fixed \( z \leq E_{\sigma \varepsilon}(\alpha) \), let us consider the decomposition
\[
\frac{1}{\omega(k_1) + \omega(k_2) + \sigma \varepsilon - z} = \Psi_1^{(\sigma)}(k_1, k_2; z) + \Psi_2^{(\sigma)}(k_1, k_2; z), 
\] (33)

where
\[
\Psi_1^{(\sigma)}(k_1, k_2; z) := \frac{1}{\omega(k_1) + \sigma \varepsilon - z} + \frac{1}{\omega(k_2) + \sigma \varepsilon - z} - \frac{1}{\sigma \varepsilon - z} 
\] (34)

and
\[
\Psi_2^{(\sigma)}(k_1, k_2; z) := \frac{1}{\omega(k_1) + \omega(k_2) + \sigma \varepsilon - z} - \Psi_1^{(\sigma)}(k_1, k_2; z), \quad k_1, k_2 \in \mathbb{R}^d. 
\] (35)

We denote by \( \tilde{K}_1^{(\sigma)}(z) \) and \( K_2^{(\sigma)}(z) \) the integral operators in \( L^2(\mathbb{R}^d) \) whose kernels are the functions
\[
(k_1, k_2) \mapsto \overline{\lambda(k_1)\lambda(k_2)} \left[ \Psi_1^{(\sigma)}(k_1, k_2; z) + \frac{1}{\sigma \varepsilon - z} \right] 
\] (36)

and
\[
(k_1, k_2) \mapsto \overline{\lambda(k_1)\lambda(k_2)} \Psi_2^{(\sigma)}(k_1, k_2; z), \quad k_1, k_2 \in \mathbb{R}^d. 
\] (37)

The next result confirms that the numerical range of the self-adjoint operator \( \Delta_{\alpha}^{(\sigma)}(E_{\sigma \varepsilon}(\alpha)) - \alpha^2 K_2^{(\sigma)}(E_{\sigma \varepsilon}(\alpha)) \) is contained in \((0, \infty)\) for all sufficiently large \( \alpha > 0 \).

**Lemma 2.3.** Let \( \psi \in \mathcal{H}_1 \) be arbitrary. For all sufficiently large \( \alpha > 0 \), we have
\[
\int \Delta_{\alpha}^{(\sigma)}(k; E_{\sigma \varepsilon}(\alpha))|\psi(k)|^2 \, dk \geq \alpha^2 \int (K_2^{(\sigma)}(E_{\sigma \varepsilon}(\alpha))\psi)(k)\overline{\psi}(k) \, dk. 
\] (38)

**Proof.** Since \( \Phi_{\alpha}^{(\sigma)}(E_{\sigma \varepsilon}(\alpha)) = 0 \), we have the following simple yet quite important pointwise estimate
\[
\Delta_{\alpha}^{(\sigma)}(k; E_{\sigma \varepsilon}(\alpha)) = \Delta_{\alpha}^{(\sigma)}(k; E_{\sigma \varepsilon}(\alpha)) - \Phi_{\alpha}^{(\sigma)}(E_{\sigma \varepsilon}(\alpha)) 
\]
\[
= \omega(k) \left( 1 + \alpha^2 \int \frac{|\lambda(q)|^2 \, dq}{(\omega(q) + \sigma \varepsilon - E_{\sigma \varepsilon}(\alpha))(\omega(q) + \omega(k) + \sigma \varepsilon - E_{\sigma \varepsilon}(\alpha))} \right) 
\]
\[
\geq \omega(k), \quad k \in \mathbb{R}^d. 
\]

This leads to the following estimate for the left-hand-side of (38)
\[
\int \Delta_{\alpha}^{(\sigma)}(k; E_{\sigma \varepsilon}(\alpha))|\psi(k)|^2 \, dk \geq \int \omega(k)|\psi(k)|^2 \, dk. 
\] (39)

Next, we recall from [10] the following elementary (yet quite important!) inequality
\[
0 \leq \frac{1}{a + b + c} - \frac{1}{a + c} - \frac{1}{b + c} + \frac{1}{c} \leq \frac{\sqrt{abc}}{2\varepsilon^2}, 
\] (40)

which holds for all \( a \geq 0, b \geq 0 \) and \( c > 0 \). Applying this inequality with \( a = \omega(k_1) \geq 0, b = \omega(k_2) \geq 0 \) and \( c = \sigma \varepsilon - E_{\sigma \varepsilon}(\alpha) > 0 \), we can estimate the function in (35) as follows
\[
0 \leq \Psi_2^{(\sigma)}(k_1, k_2; E_{\sigma \varepsilon}(\alpha)) \leq \frac{1}{2(\sigma \varepsilon - E_{\sigma \varepsilon}(\alpha))^2 \sqrt{\omega(k_1)}} \sqrt{\omega(k_2)}, \quad k_1, k_2 \in \mathbb{R}^d. 
\] (41)
Using Fubini’s theorem and the Cauchy-Schwarz inequality we thus obtain the following upper bound for the right-hand-side of (43)

\[ \left| \alpha^2 \int (K_2^{(\sigma)}(E_{\sigma}(\alpha)) \psi(k) \overline{\psi(k)}) \, dk \right| \leq \frac{\alpha^2}{2(\sigma \varepsilon - E_{\sigma}(\alpha))^2} \left( \int |\lambda(k)||\psi(k)| \sqrt{\omega(k)} \, dk \right)^2 \leq \frac{\alpha^2||\lambda||^2}{2(\sigma \varepsilon - E_{\sigma}(\alpha))^2} \left( \int \omega(k)|\psi(k)|^2 \, dk \right). \]  

(42)

On the other hand, the asymptotic expansion (25) guarantees that

\[ \frac{\alpha^2}{2(\sigma \varepsilon - E_{\sigma}(\alpha))^2} = \frac{1}{2||\lambda||^2} + o(1), \quad \alpha \uparrow +\infty. \]  

(43)

In view of this, (43) follows immediately from (42) combined with (39).

\[ \square \]

Lemma 2.4. For all sufficiently large \( \alpha > 0 \), the integral operator \( \hat{K}_1^{(\sigma)}(E_{\sigma}(\alpha)) \) has exactly one positive eigenvalue.

Proof. First, we recall that the kernel of the integral operator \( \hat{K}_1^{(\sigma)}(E_{\sigma}(\alpha)) \) is given by the function

\[ (k_1, k_2) \mapsto \overline{\lambda(k_1)} \lambda(k_2) \left[ \frac{1}{\omega(k_1) + \sigma \varepsilon - E_{\sigma}(\alpha)} + \frac{1}{\omega(k_2) + \sigma \varepsilon - E_{\sigma}(\alpha)} \right]. \]  

(44)

Observe that \( \hat{K}_1^{(\sigma)}(E_{\sigma}(\alpha)) \) is a well-defined rank-two operator in \( L^2(\mathbb{R}^d) \). In fact, its range coincides with the span of the linearly independent functions \( \overline{\lambda} \) and \( \overline{\frac{1}{\omega + \sigma \varepsilon - E_{\sigma}(\alpha)}} \). It is easy to check that the matrix \( M^{(\sigma)} \) of \( \hat{K}_1^{(\sigma)}(E_{\sigma}(\alpha)) \) with respect to the basis consisting of the latter two functions has the following entries

\[ m_{11} = \int_{\mathbb{R}^d} \frac{\lambda(q)^2}{\omega(q) + \sigma \varepsilon - E_{\sigma}(\alpha)} \, dq; \quad m_{12} = \int_{\mathbb{R}^d} \frac{\lambda(q)^2}{(\omega(q) + \sigma \varepsilon - E_{\sigma}(\alpha))^2} \, dq; \]  

\[ m_{21} = \int_{\mathbb{R}^d} \frac{\lambda(q)^2}{\omega(q) + \sigma \varepsilon - E_{\sigma}(\alpha)} \, dq; \quad m_{22} = m_{11}. \]  

(45)

By the Cauchy-Schwarz inequality, we have

\[ \det M^{(\sigma)} = \left( \int_{\mathbb{R}^d} \frac{\lambda(q)^2}{\omega(q) + \sigma \varepsilon - E_{\sigma}(\alpha)} \right)^2 - \int_{\mathbb{R}^d} \frac{\lambda(q)^2}{\omega(q) + \sigma \varepsilon - E_{\sigma}(\alpha)} \, dq \int_{\mathbb{R}^d} \frac{\lambda(q)^2}{(\omega(q) + \sigma \varepsilon - E_{\sigma}(\alpha))^2} \, dq \leq 0. \]  

(46)

However, the equality sign cannot hold as the functions \( \overline{\lambda} \) and \( \overline{\frac{1}{\omega + \sigma \varepsilon - E_{\sigma}(\alpha)}} \) are linearly independent. Hence, precisely one of the two eigenvalues of the matrix \( M^{(\sigma)} \in \mathbb{R}^{2 \times 2} \) (and thus of the integral operator \( \hat{K}_1^{(\sigma)}(E_{\sigma}(\alpha)) \)) is positive.

\[ \square \]

Proof of Theorem 11. In view of Lemma 2.3 we have

\[ R_1^{(\sigma)}(E_{\sigma}(\alpha)) = \Delta^{(\sigma)}(E_{\sigma}(\alpha)) - \alpha^2 K_1^{(\sigma)}(E_{\sigma}(\alpha)) - \alpha^2 K_{11}^{(\sigma)}(E_{\sigma}(\alpha)) \geq -\alpha^2 K_{11}^{(\sigma)}(E_{\sigma}(\alpha)) \]  

(47)

for all sufficiently large \( \alpha > 0 \). Hence, the variational principle (see e.g. [15]) and Lemma 2.4 imply that

\[ N(0; R_1^{(\sigma)}(E_{\sigma}(\alpha))) \leq N(0; -\alpha^2 \hat{K}_1^{(\sigma)}(E_{\sigma}(\alpha))) = 1. \]  

(48)

Combining (10), Lemma 2.1 and (48), we conclude that

\[ N(E(\alpha); H_\alpha) \leq \sum_{\sigma = \pm} N(E_{\sigma}(\alpha); H_\alpha^{(\sigma)}) \leq \sum_{\sigma = \pm} N(0; R_1^{(\sigma)}(E_{\sigma}(\alpha))) \leq 2. \]  

(49)

This completes the proof of Theorem 1.
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