Time-harmonic diffuse optical tomography: H"older stability of the derivatives of the optical properties of a medium at the boundary

JASON CURRAN\textsuperscript{*}, ROMINA GABURRO\textsuperscript{†}
CLIFFORD J. NOLAN\textsuperscript{‡} \text{and} ERKKI SOMERSALO\textsuperscript{§}

Abstract
We study the inverse problem in Optical Tomography of determining the optical properties of a medium \(\Omega \subset \mathbb{R}^n\), with \(n \geq 3\), under the so-called \textit{diffusion approximation}. We consider the time-harmonic case where \(\Omega\) is probed with an input field that is modulated with a fixed harmonic frequency \(\omega = \frac{k}{c}\), where \(c\) is the speed of light and \(k\) is the wave number. Under suitable conditions that include a range of variability for \(k\), we prove a result of H"older stability of the derivatives of the \textit{absorption coefficient} \(\mu_a\) of any order at the boundary \(\partial \Omega\) in terms of the measurements, in the case when the \textit{scattering coefficient} \(\mu_s\) is assumed to be known. The stability estimates rely on the construction of singular solutions of the underlying forward elliptic system, which extend results obtained in J. Differential Equations 84 (2): 252-272 for the single elliptic equation.

Keywords: Diffuse optical tomography, anisotropy, stability.

1 Introduction
In this paper we address the problem in diffuse Optical Tomography (OT) of determining the optical properties of a medium when light is radiated through the surface of the medium. Although Maxwell's equations provide a complete model for the light propagation in a scattering medium on a micro scale, on the scale suitable for medical diffuse OT an appropriate model is given by the \textit{radiative transfer equation} (or \textit{Boltzmann equation})\cite{13}. If \(\Omega\) is a domain in \(\mathbb{R}^n\), with \(n \geq 2\) with smooth boundary \(\partial \Omega\) and radiation is considered in the body \(\Omega\), then it is well known that if the input field is modulated with a fixed harmonic frequency \(\omega\), the so-called \textit{diffusion approximation} leads to the complex partial differential equation (see \cite{10}) for the energy current density \(u\),

\textsuperscript{*}Department of Mathematics and Statistics, CONFIRM-Science Foundation Ireland, University of Limerick, Ireland, Jason.Curran@ul.ie
\textsuperscript{†}Department of Mathematics and Statistics, CONFIRM-Science Foundation Ireland, Health Research Institute (HRI), University of Limerick, Ireland, Romina.Gaburro@ul.ie
\textsuperscript{‡}Department of Mathematics and Statistics, CONFIRM-Science Foundation Ireland, Health Research Institute (HRI), University of Limerick, Ireland, Clifford.Nolan@ul.ie
\textsuperscript{§}Department of Mathematics, Applied Mathematics and Statistics, Case Western Reserve University, Cleveland, OH 44106-7058, U.S., erikki.somersalo@case.edu
Here $k = \frac{\omega}{c}$ is the wave number, $c$ is the speed of light and, in the anisotropic case, the so-called diffusion tensor $K$, is the complex matrix-valued function

$$
K = \frac{1}{n} ((\mu_a - ik)I + (I - B)\mu_s)^{-1}, \quad \text{in } \Omega,
$$

where $B_{ij} = B_{ji}$ is a real matrix-valued function, $I$ is the $n \times n$ identity matrix and $I - B$ is positive definite ([10], [35], [37]) on $\Omega$. The diffusion equation (1.1) is obtained by projecting the Boltzmann equation to the subspace of first order spherical harmonics ($P_1$ approximation), and the matrix $B$ is the contribution of the scattering phase function. The isotropic case when $B$ and hence $K$ is a scalar multiple of the identity is obtained if the scattering phase function depends only on the angle between the incident and scattering direction ([36], [38]). The spatially dependent real-valued coefficients $\mu_a$ and $\mu_s$ are called the absorption and the scattering coefficients of the medium $\Omega$ respectively and represent the optical properties of $\Omega$. It is worth noticing that many tissues including parts of the brain, muscle and breast tissue have fibrous structure on a microscopic scale which results in anisotropic physical properties on a larger scale. Therefore, the model considered in this manuscript is appropriate for the case of medical applications of OT. Although it is common practise in OT to use the Robin-to-Robin map to describe the boundary measurements (see [10]), the Dirichlet-to-Neumann (D-N) map will be employed here instead. This is justified by the fact that in OT, prescribing its inverse, the Neumann-to-Dirichlet (N-D) map (on the appropriate spaces), is equivalent to prescribing the Robin-to-Robin boundary map. A rigorous definition of the D-N map for equation (1.1) will be given in section 2.

It is also well known that in the static case, where $k = 0$ in (1.1), (1.2), prescribing the N-D map is insufficient to recover both coefficients $\mu_a$ and $\mu_s$ uniquely ([12] unless a-priori smoothness assumptions are imposed [33]. See also [34] for a further discussion on the simultaneous unique determination, in the isotropic static case, of both optical coefficients in the case when such coefficients are piecewise analytic. The static anisotropic case, for which (1.1) is a single real elliptic equation, was studied in [29], where the author proved Lipschitz stability of $\mu_a$ and Hölder stability of the derivatives of $\mu_a$ at the boundary in terms of $\Lambda_{K,\mu_a}$, in the case when $\mu_s$ is assumed to be known. In the time-harmonic case, where the medium $s$ probed with an input field which is modulated with a fixed harmonic frequency $\omega = \frac{\omega}{c}$, with $k \neq 0$, the forward model (1.1) is a complex elliptic equation. A result of Lipschitz stability of the boundary values of $\mu_a$ in terms of the D-N map, when $\mu_s$ is again assumed known, was established in the time-harmonic anisotropic case by some of the authors in [25].

In this paper, we consider the anisotropic time-harmonic case and extend the result in [25], by stably determining the derivatives of the absorption coefficient $\mu_a$, $D^h \mu_a$, for any $h \geq 1$, at the boundary of an anisotropic medium $\Omega \subset \mathbb{R}^n$, $n \geq 3$, whose scattering coefficient $\mu_s$ is assumed to be known. More precisely, we show that, under suitable conditions, $D^h \mu_a$ at the boundary $\partial \Omega$, depends upon the D-N map of (1.1), $\Lambda_{K,\mu_a}$, with a modulus of continuity of Hölder type, if $k$ is chosen in certain intervals that depend on a-priori bounds on $\mu_a$, $\mu_s$ and on the ellipticity constant of $I - B$ (Theorem [23]). The intervals of variability for $k$ were determined in [25].

The case where $\mu_a$ is assumed to be known and the scattering coefficient $\mu_s$ is to be determined, can be treated in a similar manner. The choice in this paper, as well as in [25], of focusing on the determination of $\mu_a$ rather than the one of $\mu_s$ is driven by the medical application of OT we have in mind. While $\mu_s$ varies from tissue to tissue, it is the absorption coefficient $\mu_a$ that carries the
more interesting physiological information as it is related to the global concentrations of certain metabolites in their oxygenated and deoxygenated states \[21\].

Our main result (Theorem 2.5) is based on the construction of singular solutions to the complex elliptic equation \(1.1\), having an isolated singularity outside \(\Omega\). Such solutions were first constructed in \[1\] for the equation

\[
\text{div}(K \nabla u) = 0, \quad \text{in} \quad \Omega, \tag{1.3}
\]

when \(K\) is a real matrix-valued function belonging to \(W^{1,p}(\Omega)\), with \(p > n\). Such solutions have been employed to prove stability estimates at the boundary in \[1, 7, 8\] and \[30\] in the case of Calderón’s problem (see \[23\]) with global, local data and on manifolds (see also \[44\] and \[42\]). We also recall the seminal papers \[1, 40, 41, 43, 45\] of this extensively studied companion inverse problem, together with the review papers \[22\] and \[46\].

In this paper we extend the construction of the singular solutions of \[1\] to the case of elliptic equations of type \(1.1\) with complex coefficients. Such a construction is done by treating \(1.1\) as a strongly elliptic system with real coefficients, since \(\Re K \geq \tilde{\lambda}^{-1} I > 0\), where \(\tilde{\lambda}\) is a positive constant depending on the \textit{a-priori} information on \(\mu_s, B\) and \(\mu_a\). In \[25\], the authors extended the construction of singular solutions to the complex equation \(1.1\) having an isolated singularity of Green’s type only. This was enough to prove the Lipschitz continuity of the boundary values of \(\mu_a\) in terms of the D-N map. Here singular solutions with an isolated singularity of arbitrary high order for elliptic complex partial differential equations are constructed and employed to prove our main result of Hölder stability of the derivatives (of any order) of \(\mu_a\) at the boundary in terms of the D-N map, therefore further extending the original results of \[1\].

Our result also provides a first step towards a reconstruction procedure of \(\mu_a\) by boundary measurements based on a Landweber iterative method for non-linear problems studied in \[24\], where the authors provided an analysis of the convergence of such algorithm in terms of either a Hölder or Lipschitz global stability estimates (see also \[2, 18, 26, 27\]). We recall the important results of \[9\] and \[19\] of global stability estimates for Calderón’s inverse conductivity problem in the case of real and complex isotropic conductivities, respectively, and refer to the subsequent papers \[4, 5, 6, 14, 15, 16, 17, 20, 28, 31\] for an overview of the issue of stability estimates in related inverse problems. We also refer to \[11, 39\] for further reconstruction techniques of the optical properties of a medium and to \[32\] for a topical review on diffuse OT.

The paper is organized as follows. In Section 2 we rigorously formulate the problem, state the main result (Theorem 2.5) of Hölder stability of the derivatives of \(\mu_a\) on \(\partial \Omega\) and recall a previous result of Lipschitz stability of \(\mu_a\) on \(\partial \Omega\) (Theorem 2.4) for the sake of completeness. Section 3 is devoted to the construction of singular solutions for the complex partial differential equation \(1.1\) on a ball, having an isolated singularity (of any order) at the centre of the ball. In Section 4 we give the proof of Theorem 2.5.

2 Formulation of the problem

2.1 Main assumptions

We rigorously formulate the problem by introducing the following notation, definitions and assumptions. For \(n \geq 3\), a point \(x \in \mathbb{R}^n\) will be denoted by \(x = (x', x_n)\), where \(x' \in \mathbb{R}^{n-1}\) and \(x_n \in \mathbb{R}\). Moreover, given a point \(x \in \mathbb{R}^n\), we will denote with \(B_r(x), B'_r(x')\) the open balls in \(\mathbb{R}^n, \mathbb{R}^{n-1}\), centred at \(x\) and \(x'\) respectively with radius \(r\) and by \(Q_r(x)\) the cylinder
\[ Q_r(x) = B'_r(x') \times (x_n - r, x_n + r). \]

We will also denote \( B_r = B_r(0), \ B'_r = B'_r(0) \) and \( Q_r = Q_r(0) \).

**Definition 2.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), with \( n \geq 3 \). We shall say that the boundary of \( \Omega, \partial \Omega \), is of Lipschitz class with constants \( r_0, L > 0 \), if for any \( P \in \partial \Omega \) there exists a rigid transformation of coordinates under which we have \( P = 0 \) and

\[ \Omega \cap Q_{r_0} = \{(x', x_n) \in Q_{r_0} | x_n > \varphi(x')\}, \]

where \( \varphi \) is a Lipschitz function on \( B'_{r_0} \) satisfying

\[ \varphi(0) = 0 \]

and

\[ \|\varphi\|_{C^{0,1}(B'_{r_0})} \leq Lr_0. \]

We consider, for a fixed \( k > 0 \),

\[ L = -\text{div} \left( K \nabla \cdot \right) + q, \quad \text{in} \ \Omega, \quad (2.1) \]

where \( K \) is the complex matrix-valued function

\[ K(x) = \frac{1}{n} \left( (\mu_a(x) - ik)I + (I - B(x))\mu_s(x) \right)^{-1}, \quad \text{for any} \ x \in \Omega, \quad (2.2) \]

and \( q \) is the complex-valued function

\[ q = \mu_a - ik \quad \text{in} \ \Omega. \quad (2.3) \]

Here \( I \) denotes the \( n \times n \) identity matrix, where the matrix \( B \) is given by the OT physical experiment and it is such that \( B \in L^\infty(\Omega, \text{Sym}_n) \), where \( \text{Sym}_n \) denotes the class of \( n \times n \) real-valued symmetric matrices and such that \( I - B \) is a positive definite matrix (\([11], [35], [36], [37]\)). In this paper we assume that the scattering coefficient \( \mu_s \) is also known in \( \bar{\Omega} \) and it is the derivatives of the absorption coefficient \( \mu_a \) on \( \partial \Omega \) that we seek to estimate from boundary measurements.

We assume that there are positive constants \( \lambda, E \) and \( \mathcal{E} \) and \( p > n \) such that the known quantities \( B \in L^\infty(\Omega, \text{Sym}_n), \mu_s \in L^\infty(\Omega) \) and the unknown quantity \( \mu_a \in L^\infty(\Omega, \text{Sym}_n) \) satisfy the two assumptions below respectively.

**Assumption 2.1.** (Assumption on \( \mu_s \) and \( B \))

\[ \lambda^{-1} \leq \mu_s(x) \leq \lambda, \quad \text{for a.e.} \ x \in \Omega, \quad (2.4) \]

\[ \|\mu_s\|_{W^{1, p}(\Omega)}, \quad \|B\|_{W^{1, p}(\Omega)} \leq E, \quad (2.5) \]

\[ \mathcal{E}^{-1} |\xi|^2 \leq (I - B(x))\xi : \xi \leq \mathcal{E} |\xi|^2, \quad \text{for a.e.} \ x \in \Omega, \quad \text{for any} \ \xi \in \mathbb{R}^n, \quad (2.6) \]
Assumption 2.2. (Assumption on $\mu_a$)

$$\lambda^{-1} \leq \mu_a(x) \leq \lambda, \quad \text{for a.e. } x \in \Omega,$$

(2.7)

$$\| \mu_a \|_{W^{1,p}(\Omega)} \leq E.$$  

(2.8)

We state below some facts needed in the sequel of the paper. Most of them are straightforward consequences of our assumptions.

The inverse of $K$,

$$K^{-1} = n\left(\mu_a I + (I - B)\mu_s - ikI\right), \quad \text{on } \Omega$$  

has real and imaginary parts given by the symmetric, real matrix valued-functions on $\Omega$,

$$K_R^{-1} = n\left(\mu_a I + (I - B)\mu_s\right),$$  

(2.10)

$$K_I^{-1} = -nkI,$$  

(2.11)

respectively. As an immediate consequence of assumptions 2.1, 2.2 we have

$$n\lambda^{-1}(1 + E^{-1})|\xi|^2 \leq K_R^{-1}(x)\xi \cdot \xi \leq n\lambda(1 + E)|\xi|^2,$$

(2.12)

$$-K_I^{-1}(x)\xi \cdot \xi = nk|\xi|^2,$$  

(2.13)

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$. Moreover $K_R^{-1}$ and $K_I^{-1}$ commute, therefore the real and imaginary parts of $K$ are the symmetric, real matrix valued-functions on $\Omega$,

$$K_R = \frac{1}{n}\left(\left(\mu_a I + (I - B)\mu_s\right)^2 + k^2I\right)^{-1}\left(\mu_a I + (I - B)\mu_s\right),$$  

(2.14)

$$K_I = \frac{k}{n}\left(\left(\mu_a I + (I - B)\mu_s\right)^2 + k^2I\right)^{-1},$$  

(2.15)

respectively. Assumptions 2.1, 2.2 also imply that

$$K_R(x)\xi \cdot \xi \geq \frac{\lambda(1 + E)}{n}\left(\lambda^2(1 + E)^2 + k^2\right)^{-1}|\xi|^2,$$

(2.16)

$$K_I(x)\xi \cdot \xi \geq \frac{k}{n}\left(\lambda^2(1 + E)^2 + k^2\right)^{-1}|\xi|^2,$$  

(2.17)

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$ and the boundness condition

$$|K_R(x)|^2 + |K_I(x)|^2 \leq \left(\lambda^{-2}(1 + E^{-1})^2 + k^2\right)^{-2}\left(\frac{\lambda^2(1 + E)^2 + k^2}{n^2}\right),$$  

(2.18)

for a.e. $x \in \Omega$. 

5
Moreover $K = \{ K^{hk} \}_{h,k=1,...,n}$ and $q$ satisfy
\[
\| K^{hk} \|_{W^{1,p}(\Omega)} \leq C_1, \quad h, k = 1, \ldots, n \tag{2.19}
\]
and
\[
|q(x)| = |\mu_a(x) - ik| \leq \lambda + k, \quad \text{for a.e. } x \in \Omega, \tag{2.20}
\]
respectively, where $C_1$ is a positive constant depending on $\lambda$, $E$, $E$, $k$ and $n$.

By denoting $q = q_R + iq_I$, the complex equation
\[
- \text{div} (K \nabla u) + qu = 0, \quad \text{in } \Omega \tag{2.21}
\]
is equivalent to the system for the vector field $u = (u^1, u^2)$
\[
\begin{cases}
- \text{div}(K_R \nabla u^1) + \text{div}(K_I \nabla u^2) + (q_R u^1 - q_I u^2) = 0, & \text{in } \Omega,
- \text{div}(K_I \nabla u^1) - \text{div}(K_R \nabla u^2) + (q_I u^1 + q_R u^2) = 0, & \text{in } \Omega,
\end{cases} \tag{2.22}
\]
which can be written in a more compact form as
\[
- \text{div}(C \nabla u) + qu = 0, \quad \text{in } \Omega \tag{2.23}
\]
or, in components, as
\[
- \frac{\partial}{\partial x_h} \left( C^{hk}_{ij} \frac{\partial}{\partial x_k} u^j \right) + q_{lj} u^j = 0, \quad \text{for } l = 1, 2, \quad \text{in } \Omega, \tag{2.24}
\]
where $\left\{ C^{hk}_{ij} \right\}_{h,k=1,...,n}$ is defined by
\[
C^{hk}_{ij} = K_{R}^{hk} \delta_{ij} - K_{I}^{hk} (\delta_{i1} \delta_{j2} - \delta_{i2} \delta_{j1}), \tag{2.25}
\]
or, in block matrix notation,
\[
C = \begin{pmatrix} K_R & -K_I \\ K_I & K_R \end{pmatrix} \in \mathbb{R}^{2n \times 2n}, \tag{2.26}
\]
and $\left\{ q_{lj} \right\}_{l,j=1,2}$ is a $2 \times 2$ real matrix valued function on $\Omega$ defined by
\[
q_{lj} = q_R \delta_{lj} - q_I (\delta_{l1} \delta_{j2} - \delta_{l2} \delta_{j1}), \tag{2.27}
\]
or,
\[
q = \begin{pmatrix} q_R & -q_I \\ q_I & q_R \end{pmatrix} = \begin{pmatrix} \mu_a & k \\ -k & \mu_a \end{pmatrix} \in \mathbb{R}^{2 \times 2}. \tag{2.28}
\]
Observing that for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{2n}$, using the symmetry of $K_I$, we have
\[
C \xi \cdot \xi = K_R \xi_1 \cdot \xi_1 + K_R \xi_2 \cdot \xi_2, \tag{2.29}
\]
the estimate (2.16), together with (2.18) imply that system (2.22) is uniformly elliptic and bounded, therefore it satisfies the strong ellipticity condition
\[
C_{2}^{-1} |\xi|^2 \leq C \xi \cdot \xi \leq C_2 |\xi|^2, \quad \text{for a.e. } x \in \Omega, \quad \text{for all } \xi \in \mathbb{R}^{2n}, \tag{2.30}
\]
where $C_2 > 0$ is a constant depending on $\lambda$, $E$, $k$ and $n$. 

6
Remark 2.3. The matrix $q$ given by (2.28) is uniformly positive definite on $\Omega$ and it satisfies
\[ \lambda^{-1} |\xi|^2 \leq q \xi \cdot \xi = \mu_a |\xi|^2 \leq \Lambda |\xi|^2, \quad \text{for a.e. } x \in \Omega, \quad \text{for every } \xi \in \mathbb{R}^2. \] (2.31)

Definition 2.2. We will refer in the sequel to the set of positive numbers $r_0$, $L$, $\lambda$, $E$, $\mathcal{E}$ introduced above, along with the space dimension $n$, $p > n$, the wave number $k$ and the diameter of $\Omega$, $diam(\Omega)$, as to the a-priori data.

2.2 The Dirichlet-to-Neumann map

Let $K$ be the complex matrix-valued function on $\Omega$ introduced in (2.2) and $q = \mu_a - ik$, satisfying assumptions $2.1$, $2.2$ $B$ and $\mu_a$ are assumed to be known in $\Omega$ and satisfying assumption $2.1$ so that $K$ is completely determined by $\mu_a$, satisfying assumption $2.2$ on $\Omega$. Denoting by $\langle \cdot, \cdot \rangle$ the $L^2(\partial \Omega)$-pairing between $H^\frac{1}{2}(\partial \Omega)$ and its dual $H^{-\frac{1}{2}}(\partial \Omega)$, we will emphasise such dependence of $K$ on $\mu_a$ by denoting $K = K_{\mu_a}$.

For any $v, w \in \mathbb{C}^n$, with $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n)$, we will denote throughout this paper by $v \cdot w$, the expression
\[ v \cdot w = \sum_{i=1}^{n} v_i w_i. \]

Definition 2.3. The Dirichlet-to-Neumann (D-N) map corresponding to $\mu_a$ is the operator
\[ \Lambda_{\mu_a} : H^\frac{1}{2}(\partial \Omega) \longrightarrow H^{-\frac{1}{2}}(\partial \Omega) \] (2.32)
defined by
\[ \langle \Lambda_{\mu_a} f, g \rangle = \int_{\Omega} \left( K_{\mu_a}(x) \nabla u(x) \cdot \nabla \varphi(x) + (\mu_a(x) - ik)u(x)\varphi(x) \right) dx, \] (2.33)
for any $f, g \in H^\frac{1}{2}(\partial \Omega)$, where $u \in H^1(\Omega)$ is the weak solution of
\[ \begin{cases} -\text{div}(K_{\mu_a}(x) \nabla u(x)) + (\mu_a - ik)u(x) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial \Omega \end{cases} \]
and $\varphi \in H^1(\Omega)$ is any function such that $\varphi|_{\partial \Omega} = g$ in the trace sense.

Given $B$, $\mu_a$, $\mu_{a_i}$, and the corresponding diffusion tensors $K_{\mu_{a_i}}$, for $i = 1, 2$, satisfying assumptions $2.1$, $2.2$ the well known Alessandrini’s identity (see [1] (5.0.4), p.129))

\[ \langle (\Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}}) f, g \rangle = \int_{\Omega} (K_{\mu_{a_1}}(x) - K_{\mu_{a_2}}(x)) \nabla u(x) \cdot \nabla v(x) dx \]
\[ + \int_{\Omega} (\mu_{a_1}(x) - \mu_{a_2}(x)) u(x)v(x) dx, \] (2.34)
holds true for any $f, g \in H^\frac{1}{2}(\partial \Omega)$, where $u, v \in H^1(\Omega)$ are the unique weak solutions to the Dirichlet problems
\[ \begin{cases} -\text{div}(K_{\mu_{a_1}}(x) \nabla u(x)) + (\mu_{a_1} - ik)u(x) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial \Omega \end{cases} \]

\[ \begin{cases} -\text{div}(K_{\mu_{a_2}}(x) \nabla u(x)) + (\mu_{a_2} - ik)u(x) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial \Omega \end{cases} \]
and

\[
\begin{aligned}
-\text{div}(K\mu_2(x)\nabla v(x)) + (\mu_2 - ik)v(x) &= 0, \quad \text{in } \Omega, \\
v &= g, \quad \text{on } \partial\Omega,
\end{aligned}
\]

respectively.

We will denote in the sequel by \(\|\cdot\|_{L(H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega))}\) the norm on the Banach space of bounded linear operators between \(H^{\frac{1}{2}}(\partial\Omega)\) and \(H^{-\frac{1}{2}}(\partial\Omega)\).

### 2.3 Main result

Before stating the main result (Theorem 2.5), for the sake of completeness we recall the Lipschitz stability estimate at the boundary for \(\mu_a\) in [25].

**Theorem 2.4.** (Lipschitz stability of boundary values). Let \(n \geq 3\), and \(\Omega\) be a bounded domain in \(\mathbb{R}^n\) with Lipschitz boundary with constants \(L, r_0\) as in definition [2.1]. If \(p > n\), \(B, \mu_s\) and \(\mu_{a_i}\) for \(i = 1, 2\), satisfy assumptions [2.1, 2.2] and the wave number \(k\) satisfies either

\[
0 < k \leq k_0 := \frac{\sqrt{\lambda^2(1 + E)^2 + \lambda^{-2}(1 + E^{-1})^2 \tan^2 \left(\frac{\pi n}{2}\right)} - \lambda(1 + E)}{\tan \left(\frac{\pi n}{2}\right)},
\]

or

\[
k \geq k_0 := \frac{1 + \sqrt{1 + \tan^2 \left(\frac{\pi n}{2}\right)}}{\tan \left(\frac{\pi n}{2}\right)} \lambda(1 + E),
\]

where, \(\lambda\) and \(E\) are the positive numbers introduced in assumptions [2.1, 2.2]. Then

\[
\|\mu_{a_1} - \mu_{a_2}\|_{L^\infty(\partial\Omega)} \leq C \|\Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}}\|_{L(H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega))},
\]

where \(C > 0\) is a constant depending on \(n, p, L, r_0, \text{diam}(\Omega), \lambda, E,\) and \(k\).

Here we address the issue of stably determining the derivatives of any order \(h \in \mathbb{N}\) of \(\mu_a\) at the boundary in terms of the D-N map. We assume that \(\text{supp}(B)\) is compactly contained in \(\Omega\), i.e. that the material is isotropic while not necessarily homogenous near the boundary \(\partial\Omega\), which is a reasonable assumption in many imaging applications.

**Theorem 2.5.** (Hölder stability of boundary derivatives). Let \(n \geq 3\), and \(\Omega\) be a bounded domain in \(\mathbb{R}^n\) with Lipschitz boundary with constants \(L, r_0\) as in definition [2.1]. Let \(p > n\), \(B, \mu_s\) and \(\mu_{a_i}\) for \(i = 1, 2\), satisfy assumptions [2.1, 2.2] with \(\text{supp}(B) \Subset \Omega\). Assume that

\[
\|D^h(\mu_{a_1} - \mu_{a_2})\|_{L^\infty(\partial\Omega)} \leq C \|\Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}}\|_{L(H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega))},
\]

for some integer \(h \geq 1\), where \(\Omega_r = \{x \in \Omega : \text{dist}(x, \partial\Omega) < r\}\) and that the wave number \(k\) satisfies either [2.35] or [2.36], where \(\lambda\) and \(E\) are the positive numbers introduced in assumptions [2.1, 2.2]. Then

\[
\|D^h(\mu_{a_1} - \mu_{a_2})\|_{L^\infty(\partial\Omega)} \leq C \|\Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}}\|_{L(H^{\frac{1}{2}}(\partial\Omega), H^{-\frac{1}{2}}(\partial\Omega))},
\]
where $\delta_h = \Pi_{i=0}^h \frac{a_i}{a_i+k}$ and $C > 0$ is a constant depending on $n$, $p$, $L$, $r_0$, $\text{diam}(\Omega)$, $\lambda$, $E$, $\mathcal{E}$, $h$ and $k$.

A H"older stability estimate at the boundary of the derivatives of the diffusion tensor $K_{\mu_a}$ in terms of $\Lambda_{\mu_a}$ follows as a straightforward consequence of Theorem 2.5 under more stringent regularity conditions.

**Corollary 2.6.** Suppose the hypotheses of Theorem 2.5 are satisfied. Moreover, assume that

\begin{equation}
||\mu_i||_{C^{0,\alpha}(\Omega)} \leq E_h, \quad \text{for } i = 1, 2, \quad (2.40)
\end{equation}

\begin{equation}
||\mu_s||_{C^{0,\alpha}(\Omega)} \leq E_h, \quad (2.41)
\end{equation}

\begin{equation}
||B||_{C^{0,\alpha}(\Omega)} \leq E_h, \quad (2.42)
\end{equation}

then

\begin{equation}
||D^h(K_{\mu_1} - K_{\mu_2})||_{L^\infty(\partial\Omega)} \leq C ||\Lambda_{\mu_1} - \Lambda_{\mu_2}||_{L^2(H^\frac{1}{2}(\partial\Omega)), H^\frac{1}{2}(\partial\Omega)}, \quad (2.43)
\end{equation}

where $h$, $\delta_h$, $\alpha$, $\Omega_\delta$ are as in Theorem 2.5 and $C > 0$ is a constant depending on $n$, $p$, $L$, $r_0$, $\text{diam}(\Omega)$, $\lambda$, $E$, $\mathcal{E}$, $h$ and $k$.

### 3 Singular solutions

We consider

\begin{equation}
L = -\text{div}(K\nabla \cdot) + q, \quad \text{in } B_R(z) = \left\{ x \in \mathbb{R}^n \mid |x - z| < R \right\}, \quad (3.1)
\end{equation}

where $K = \{K_{kk}\}_{k=1,...,n}$ and $q$ are the complex matrix valued-function and the complex function respectively introduced in section 1 and satisfying assumptions 2.1, 2.2 on $B_R(z)$.

**Theorem 3.1.** (Singular solutions). Given $L$ on $B_R(z)$ as in (3.1) with $B(z) = 0$, for any $m = 0, 1, 2, \ldots$, there exists $u \in W^{2,p}_{\text{loc}}(B_R(z) \setminus \{z\})$ such that

\begin{equation}
Lu = 0, \quad \text{in } B_R(z) \setminus \{z\}, \quad (3.2)
\end{equation}

with

\begin{equation}
u(x) = (K^{-1}(z)(x - z) \cdot (x - z))^\frac{2-n-m}{2m} (K^{-1}_{nn}(z))^\frac{2}{m} C_m^{\frac{n-2}{m}} \left( \frac{K^{-1}_{(n)}(z)(x - z)}{(K^{-1}_{nn}(z)K^{-1}(z)(x - z) \cdot (x - z))^\frac{2}{m}} \right) + w(x),
\end{equation}

\begin{equation}
|w(x)| + |x - z| |Dw(x)| \leq C |x - z|^2 - n + \alpha, \quad \text{in } B_R(z) \setminus \{z\}, \quad (3.4)
\end{equation}

where $C_m^{\frac{n-2}{m}} : \mathbb{C} \to \mathbb{C}$ is the complex Gegenbauer polynomial of degree $m$ and order $\frac{n-2}{m}$ (see [2]).

Defined on $|z| \leq 1$ and $K_{(n)}^{-1}(z)$, $K_{nn}^{-1}(z)$ denotes the last row, last entry in the last row of matrix $K^{-1}(z)$, respectively. Moreover, $w$ satisfies
\[
\left( \int_{r < |x-z| < 2r} |D^2 u|^p \right)^{\frac{1}{p}} \leq C r^{\frac{n}{2} - n + \alpha}, \quad \text{for every } r, 0 < r < R/2.
\] (3.5)

Here \( \alpha \) is such that \( 0 < \alpha < 1 - \frac{2}{p} \), and \( C \) is a positive constant depending only on \( \alpha, n, p, R, \lambda, E, \mathcal{E} \) and \( k \).

Remark 3.2. Since \( K^{-1}(z) \) is a complex matrix, the expression

\[
\left( K^{-1}(z)(x-z) \cdot (x-z) \right)^{\frac{1}{2}}
\] (3.6)

appearing in (3.3) is defined as the principal branch of (3.6), where a branch cut along the negative real axis of the complex plane has been defined for \( z^{\frac{1}{2}} \), \( z \in \mathbb{C} \). Expressions like (3.6) will appear in the sequel of the paper and they will be understood in the same way.

Remark 3.3. Under assumption \( B(z) = 0 \) the singular solution in (3.3) simplifies to

\[
u(x) = m! \left( \mu_a(z) + \mu_s(z) - ik \right)^{\frac{2m}{p}} |x-z|^{2-n-m} C_0 \left( \frac{(x-z)_n}{|x-z|} \right) + w(x),
\] (3.7)

where \( w \) satisfies (3.4) and (3.5).

Next we consider three technical lemmas that are needed for the proof of Theorem 3.1. We set \( z = 0 \) to simply our notation.

Lemma 3.4. Let \( p > n \) and \( u \in W^{2,p}_{\text{loc}}(B_R \setminus \{0\}) \) be a complex-valued function such that, for some positive \( s \),

\[
|u(x)| \leq |x|^{2-s}, \quad \text{for any } x \in B_R \setminus \{0\},
\] (3.8)

\[
\left( \int_{r < |x| < 2r} |Lu|^p \right)^{\frac{1}{p}} \leq Ar^{\frac{n}{2} - s}, \quad \text{for any } r, 0 < r < R/2.
\] (3.9)

Then we have

\[
|Du(x)| \leq C|x|^{1-s}, \quad \text{for any } x \in B_R \setminus \{0\},
\] (3.10)

\[
\left( \int_{r < |x| < 2r} |D^2 u|^p \right)^{\frac{1}{p}} \leq C r^{\frac{n}{2} - s} \quad \text{for any } r, 0 < r < R/4,
\] (3.11)

where \( C \) is a positive constant depending only on \( A, n, p, \lambda, E, \mathcal{E} \) and \( k \).

Proof of Lemma 3.4. For a complete proof of this lemma we refer to [Lemma 3.3., 25], which is based on on the interior \( L^p \) - Schauder estimate for uniformly elliptic systems

\[
\left( \int_{r < |x| < 2r} |D^2 u|^p \right)^{\frac{1}{p}} \leq C \left\{ \left( \int_{\frac{r}{2} < |x| < 4r} |Lu|^p \right)^{\frac{1}{p}} + r^{-2} \left( \int_{\frac{r}{2} < |x| < 4r} |u|^p \right)^{\frac{1}{p}} \right\},
\] (3.12)

for every \( r, 0 < r < \frac{R}{4} \), where \( C > 0 \) depends on \( n, p, \lambda, E, \mathcal{E} \) and \( k \) only.
Lemma 3.5. Let \( f \in L^p_{loc}(B_R \setminus \{0\}) \) be a complex-valued function that satisfies
\[
\left( \int_{r < |x| < 2r} |f|^p \right)^{\frac{1}{p}} \leq Ar^{\frac{n}{p} - s}, \quad \text{for any } r, \quad 0 < r < \frac{R}{2},
\]
with \( 2 < s < n < p \). Then there exists \( u \in W^{2,p}_{loc}(B_R \setminus \{0\}) \) satisfying
\[
Lu = f, \quad \text{in } B_R \setminus \{0\}
\]
and
\[
|u(x)| \leq C|x|^{2-s}, \quad \text{for any } x \in B_R \setminus \{0\},
\]
where \( C \) is a positive constant depending only on \( A, s, n, p, R, \lambda, E, \mathcal{E} \) and \( k \).

Proof of Lemma 3.5. See [Lemma 3.4 [25]] for a proof of this lemma.

Let \( L_0 \) be the operator defined by
\[
L_0 := -\text{div}(K(0)\nabla \cdot), \quad \text{in } B_R.
\]
Note that if we assume that \( B(0) = 0 \), then
\[
K(0) = \frac{1}{n(\mu_a(0) + \mu_s(0) - ik)} f
\]
and \( L_0 \) simplifies to
\[
L_0 = \frac{1}{n(\mu_a(0) + \mu_s(0) - ik)} \Delta.
\]

Lemma 3.6. Let \( f \in L^p_{loc}(B_R \setminus \{0\}) \) be a complex-valued function that satisfies
\[
\left( \int_{r < |x| < 2r} |f|^p \right)^{\frac{1}{p}} \leq Ar^{\frac{n}{p} - s}, \quad \text{for any } r, \quad 0 < r < \frac{R}{2},
\]
where \( s > n \) is a non-integral real number and \( A \) a positive constant. Then there exists \( u \in W^{2,p}_{loc}(B_R \setminus \{0\}) \) satisfying
\[
\Delta u = f, \quad \text{in } B_R \setminus \{0\}
\]
and
\[
|u(x)| \leq C|x|^{2-s}, \quad \text{for any } x \in B_R \setminus \{0\},
\]
where \( C \) is a positive constant depending only on \( A, s, n, p, R, \lambda, E, \mathcal{E} \) and \( k \).
Proof. The proof of this lemma is a straightforward adaptation of the argument of [1, Proof of Lemma 2.3] to the case of $f$ complex-valued function. We provide the proof for the sake of completeness.

Letting $\Gamma(x - y) = -C_n |x - y|^{2-n}$ be the fundamental solution for the Laplace operator in $\mathbb{R}^n$, we have that for $|y| < |x|$,

$$
\Gamma(x - y) = -C_n \sum_{j=0}^{\infty} \frac{|y|^j}{|x|^{j+n-2}} C_j^{n-2} \left( \frac{x}{|x|}, \frac{y}{|y|} \right),
$$

(3.22)

where $C_j^{n-2}$ are the Gegenbauer polynomials and $|C_j^{n-2}(t)| \leq \text{Const} j^{-3}$, for $|t| \leq 1$. Defining for $\nu = 0, 1, 2, \ldots$

$$
\Gamma_\nu(x - y) = \Gamma(x - y) - C_n \sum_{j=0}^{\nu} \frac{|y|^j}{|x|^{j+n-2}} C_j^{n-2} \left( \frac{x}{|x|}, \frac{y}{|y|} \right),
$$

(3.23)

we have that

$$
\Delta_x \Gamma_\nu(x, y) = \delta(x - y), \quad \text{for } x \neq 0.
$$

(3.24)

Assuming without loss of generality (by means of a cut-off argument) that $f \in L^\infty(B_R)$ and setting

$$
u = [s] - n,$$

(3.25)

we have that

$$
\Delta u = f, \quad \text{in } B_R \setminus \{0\}.
$$

(3.26)

To prove [3.27], setting $P_j(x, y) := \frac{|y|^j}{|x|^{j+n-2}} C_j^{n-2} \left( \frac{x}{|x|}, \frac{y}{|y|} \right)$, we write

$$
\int_{B_R} \Gamma_\nu(x - y) f(y)dy = \int_{\{|y|<|x|<R\}} \Gamma(x - y) f(y)dy
$$

$$
- \sum_{j=0}^{\nu} \int_{\{|y|<|x|<R\}} P_j(x, y) f(y)dy
$$

$$
+ \sum_{j=\nu+1}^{\infty} \int_{\{|y|<|x|\}} P_j(x, y) f(y)dy.
$$

(3.27)

The first integral on the right hand side of [3.27] can be estimated as follows,

$$
\left| \int_{\{|y|<|x|<R\}} \Gamma(x - y) f(y)dy \right| \leq C \int_{\{|y|<|x|<R\}} |x - y|^{2-n} |f(y)|dy \leq C|x|^{2-s}.
$$

(3.28)

To estimate the second and third integrals on the right hand side of [3.27], we will make use of inequality

$$
\left| \int_{\{|y|<|x|\}} P_j(x, y) f(y)dy \right|
$$

$$
\leq C\int_{\{|y|<|x|\}} |x - y|^{2-n} |f(y)|dy \leq C|x|^{2-s}.
$$

(3.29)
\[
\int_{\{r < |y| < 2r\}} |y|^\mu |f(y)| dy \leq C(\mu, s, n, A) \int_{\{r < |y| < 2r\}} |y|^\mu - s dy, \tag{3.29}
\]
which holds true for any \( f \) complex-valued function satisfying (3.19), hence the second integral on the right hand side of (3.27) can be estimated by

\[
\left| \sum_{j=0}^{\nu} \int_{\left\{ \frac{|y|}{2} < |y| < R \right\}} P_j(x, y) f(y) dy \right| \leq C \sum_{j=0}^{\nu} j^{n-3} \int_{\left\{ \frac{|y|}{2} < |y| < R \right\}} \frac{|y|^j}{|x|^{j+n-2}} |f(y)| dy, \tag{3.30}
\]
where \( C \) is a positive constant depending only on \( n, \lambda, E, k \) and \( s \). Extending \( f \) outside \( B_R \) by setting \( f = 0 \) on \( \mathbb{R}^n \setminus B_R \) and noting that \( j - s + n < 0 \), for \( j = 0, \ldots, \nu \), we have

\[
\left| \sum_{j=0}^{\nu} \int_{\left\{ \frac{|y|}{2} < |y| < R \right\}} P_j(x, y) f(y) dy \right| \leq C \sum_{j=0}^{\nu} j^{n-3} |x|^{2-n-j} \int_{\left\{ |y| > \frac{|x|}{4} \right\}} |y|^{j-s} dy \leq C |x|^{2-s}, \tag{3.31}
\]
where \( C \) is a positive constant depending only on \( n, \lambda, E, k \) and \( s \). Similarly, for the third integral on the right hand side of (3.27), noticing that \( j - s + n > 0 \), for \( j > \nu \), we have

\[
\left| \sum_{j=\nu+1}^{\infty} \int_{\left\{ |y| < \frac{|x|}{4} \right\}} P_j(x, y) f(y) dy \right| \leq C \sum_{j=\nu+1}^{\infty} j^{n-3} |x|^{2-n-j} \int_{\left\{ |y| < \frac{|x|}{4} \right\}} |y|^{j-s} dy \leq C |x|^{2-s} \sum_{j=\nu+1}^{\infty} j^{n-4} 2^{-j} \leq C |x|^{2-s}, \tag{3.32}
\]
where \( C \) is a positive constant depending only on \( n, \lambda, E, k \) and \( s \).

Next we proceed with the proof of Theorem 3.1.

**Proof of Theorem 3.1.** We start by considering the fundamental solution to

\[
L_{K(z)} := \text{div} (K(z) \nabla \cdot ) \tag{3.33}
\]
in \( \mathbb{R}^n \) with pole at \( y \),

\[
u_0(x) := \Gamma_{K(z)}(x - y) = \left( K^{-1}(z)(x - y) \cdot (x - y) \right)^{\frac{2-n}{2}}, \quad \text{in} \quad B_R(z) \setminus \{z\}.
\]
By an induction argument, we have that for \( m = 0, 1, 2, \ldots \),

\[
\int_{\{r < |y| < 2r\}} |y|^\mu |f(y)| dy \leq C(\mu, s, n, A) \int_{\{r < |y| < 2r\}} |y|^\mu - s dy, \tag{3.29}
\]
\[ u_m := \left. \frac{\partial^m u_0}{\partial y^m} \right|_{y=z} = \left( K^{-1}(z)(x-z) \cdot (x-z) \right)^{\frac{2-n-m}{2}} \]

\[ \times \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \left( \prod_{k=0}^{m-j-1} \left( \frac{2-n-k}{2} \right) (2K^{-1}_{mn}(z)) \right) \]

\[ \times \frac{m!}{2^j (m-2j)! j!} \left( \frac{-2K^{-1}_{n}(z)(x-z)}{K^{-1}(z)(x-z)} \right)^{m-2j} \]  

(3.34)

which leads to

\[ u_m = \left( K^{-1}(z)(x-z) \cdot (x-z) \right)^{\frac{2-n-m}{2}} m! \left( K^{-1}_{mn}(z) \right)^{\frac{m}{2}} C_{\frac{m}{2}}^{n+2}(\bar{z}) \left[ \frac{K^{-1}_{n}(z)(x-z)}{K^{-1}_{mn}(z)K^{-1}(z)(x-z) \cdot (x-z)} \right]^{\frac{1}{2}} \]

(3.35)

where

\[ C_{\frac{m}{2}}^{n+2}(\bar{z}) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^j \Gamma \left( m - j + \frac{n-2}{2} \right) \Gamma \left( \frac{n-2}{2} \right) j! (m-2j)! (2\bar{z})^{m-2j} \]  

(3.36)

is the Gegenbauer polynomial of order \( m \) and degree \( \frac{n-2}{2} \) in the complex variable \( \bar{z} \).

Observing that for any \( m = 1, 2, \ldots \)

\[ L_{K(z)}u_m = 0, \quad \text{in } B_R(z) \setminus \{ z \}, \] 

(3.37)

we aim to find a solution \( w \) to

\[ Lw = -Lu_m, \quad \text{in } B_R(z) \setminus \{ z \}, \] 

(3.38)

satisfying (3.4), (3.5), where \( L \) is defined by (2.1). We have

\[ -Lu_m = \left( K_{ij}(x) - K_{ij}(z) \right) \frac{\partial^2 u_m}{\partial x_i \partial x_j} + \frac{\partial K_{ij}}{\partial x_i} \frac{\partial u_m}{\partial x_j} - qu_m \]  

(3.39)

and recalling that for any \( i, j = 1, \ldots, n \), \( K_{ij} \in W^{1,p}(B_R(z)) \), with \( p > n \), we have that \( K_{ij} \) is Hölder continuous on \( B_R(z) \) with Hölder coefficient \( \beta = 1 - \frac{n}{p} \) and hence
\[
\left( \int_{r<|x-z|<2r} |L_{u_m}|^p \right)^{\frac{1}{p}} \leq \left( \int_{r<|x-z|<2r} |x-z|^\beta p |x-z|^{-(n+m)p} \right)^{\frac{1}{p}} \\
+ \left( \int_{r<|x-z|<2r} \left| \frac{\partial K_{ij}}{\partial x_i} \right|^p |x-z|^{(1-n-m)p} \right)^{\frac{1}{p}} \\
+ \left( \lambda \int_{r<|x-z|<2r} |x-z|^{(2-n-m)p} \right)^{\frac{1}{p}} \\
\leq C r^{\frac{1}{p} - (n+m-\beta)}, \tag{3.40}
\]

where \( C \) is a positive constant depending on \( n, p, R, \lambda, E, \mathcal{E} \) and \( k \). Since \( n + m - \beta > n \), by Lemma \( 3.6 \) there is \( w_0 \in W^{2,p}_{\text{loc}}(B_R(z) \setminus \{z\}) \) solution to \( L_{K(z)} w_0 = -L_{u_m} \), satisfying

\[
|w_0(x)| \leq C |x - z|^{2-(n+m-\beta)}, \text{ for any } x \in B_R(z) \setminus \{z\}. \tag{3.41}
\]

Notice that

\[
L w_0 = (L - L_{K(z)}) w_0 - L u_m,
\]

and that by combining (3.41) together with Lemma 3.4 we obtain

\[
\left( \int_{r<|x-z|<2r} |(L_{K(z)} - L) w_0|^p \right)^{\frac{1}{p}} \leq \left( \int_{r<|x-z|<2r} |x-y|^\beta p |x-z|^{-(n+m)p} \right)^{\frac{1}{p}} \\
+ \left( \int_{r<|x-z|<2r} \left| \frac{\partial K_{ij}}{\partial x_i} \right|^p |x-y|^{(1-n-m)p} \right)^{\frac{1}{p}} \\
+ \left( \lambda \int_{r<|x-z|<2r} |x-y|^{(2-n-m)p} \right)^{\frac{1}{p}} \\
\leq C r^{\frac{1}{p} - (n+m-2\beta)}. \tag{3.42}
\]

Letting \( \alpha \) be an irrational number, with \( 0 < \alpha < \beta \) and setting \( J = \lfloor \frac{\alpha}{\beta} \rfloor \), note that \( n + m - j \alpha > n \), for \( j = 1, \ldots, J; n + m - (J+1) \alpha < n \) and \( n + m - (J+1) \alpha \) is a non-integral real number. Defining, for \( j = 1, \ldots, J - 1, w_j \) to be the solution to

\[
L_{K(z)} w_j = (L_{K(z)} - L) w_{j-1} \quad \text{in } B_R(z) \setminus \{z\},
\]

given by Lemma 3.6 we have

\[
|w_j(x)| \leq C |x - z|^{2-n-m+(j+1)\alpha}, \text{ for any } x, x \in B_R(z) \setminus \{z\}, \tag{3.43}
\]

\[
\left( \int_{r<|x-z|<2r} |(L_{K(z)} - L) w_j|^p \right)^{\frac{1}{p}} \leq C r^{\frac{1}{p} - n-m+(j+2)\alpha}, \text{ for any } r, 0 < r < \frac{R}{2}. \tag{3.44}
\]

For \( j = J \) define \( W_J \) to be the solution to
\[ LW_J = (L_K(z) - L)w_{J-1} \quad \text{in} \quad B_R(z) \setminus \{ z \} \]
given by Lemma 3.5, which satisfies
\[ |W_J(x)| \leq C|x - z|^{2-n-m+(J+1)\alpha}, \quad \text{for any} \ x \in B_R(z) \setminus \{ z \}. \quad (3.45) \]

Finally, forming
\[ w = \sum_{j=0}^{J-1} w_j + W_J, \quad (3.46) \]
we have that
\[ Lw = \sum_{j=0}^{J-1} (L - L_K(z))w_j + \sum_{j=0}^{J-1} L_K(z)w_j + (L_K(z) - L)w_{J-1} = -Lu_m \quad (3.47) \]
and
\[ |w(x)| \leq C \sum_{j=0}^{J} |x - y|^{2-n-m+(j+1)\alpha} \leq C|x - y|^{2-n-m+\alpha}, \quad \text{in} \quad B_R(z) \setminus \{ z \}, \quad (3.48) \]
where \( C > 0 \) is a constant depending on \( n, p, R, \lambda, E, E \) and \( k \). Properties (3.4), (3.5) follow from Lemma 3.4.

We shall also need the following lemma. We set again \( z = 0 \) to simply the notation.

\textbf{Lemma 3.7.} Let the hypotheses of Theorem 3.1 be satisfied. Then, for any \( m = 0, 1, 2, \ldots \), the singular solution \( u \) constructed in Theorem 3.1 on \( B_R \) and having an isolated singularity at \( z = 0 \) also satisfies
\[ |Du(x)| > C|x|^{1-(n+m)}, \quad \text{for every} \ x \in \Omega, \quad 0 < |x| < r_0, \quad (3.49) \]
where \( C > 0 \) and \( r_0 > 0 \) are constants depending on \( \lambda, E, E, m, \Omega, n, p \) and \( k \).

\textbf{Proof.} Recall that, for \( m = 0, 1, 2, \ldots \), the singular solution of Theorem 3.1 with singularity at \( z = 0 \) is given by
\[ u(x) = C \left( K^{-1}(0)x \cdot x \right)^{\frac{2-n-m}{2}} C_m^{\frac{n-2}{2}} \left( \frac{K^{-1}_{(0)}(0)x}{(K^{-1}_{(0)}(0)x \cdot x)^{\frac{1}{2}}} \right)^2 + w(x), \quad (3.50) \]
where \( C = m! \left[(K^{-1}(0)_{nn})^\frac{2}{m} \right]^2, K^{-1}_{(0)}(0) \) and \( K^{-1}_{nn}(0) \) denote the last row and the last entry in the last row of matrix
\[ K^{-1}(0) = n \left( \mu_a(0)I + (I - B(0))\mu_s(0) - ikI \right), \quad (3.51) \]
\( C_m^{\frac{n-2}{2}}(z) \) is the Gegenbauer polynomial on the complex plane and \( w \) satisfies (3.4) and (3.5). Recalling that under the assumption that \( B(0) = 0 \), (3.50) simplifies to (see lemma 3.3).
\[
u(x) = m! (\mu_\alpha(z) + \mu_s(z) - ik) \frac{x_n}{|x|^{2-n-m}} C_m^{-2} \left( \frac{x_n}{|x|} \right) + w(x), \quad (3.52)
\]
to prove \((3.49)\), it is enough to show that
\[
\left| D \left[ m! (\mu_\alpha(z) + \mu_s(z) - ik) \frac{x_n}{|x|^{2-n-m}} C_m^{-2} \left( \frac{x_n}{|x|} \right) \right] \right|^2 > C|x|^{-2(2m+n)}, \quad (3.53)
\]
where \(C > 0\) is a constant depending on \(\lambda, \mathcal{E}, E, k, \Omega, n\) and \(m\). Setting \(\frac{\tilde{w}}{|x|} := t\) and recalling \(\tilde{w} = \tau(\tilde{w}, \partial \Omega) \) from \[1\], \( (3.51) \), \( (3.54) \), we obtain
\[
\left| D \left[ m! (\mu_\alpha(z) + \mu_s(z) - ik) \frac{x_n}{|x|^{2-n-m}} C_m^{-2} \left( \frac{x_n}{|x|} \right) \right] \right|^2 \\
\geq C|x|^{-2(2m+n)} \left\{ (2-n-m)^2 \left( C_m^{-2} \right)^2 + \left( \frac{dC_m^{-2}}{dt} \right)^2 (1-t^2) \right\}. \quad (3.54)
\]
Recalling as in \[1\] Proof of Lemma 3.1 that \(C_m^{-2} (\pm 1) \neq 0\) and that \(C_m^{-2} (t)\) solves
\[
(t^2 - 1)w''' + 2t(n-1)w' - m(m+n-2)w = 0,
\]
it follows by the Cauchy uniqueness theorem that \(C_m^{-2} (t)\) and \(\frac{dC_m^{-2}}{dt} (t)\) cannot vanish simultaneously for any \(t, |t| < 1\), which concludes the proof.

\[\square\]

4 Proof of the main result

Since the boundary \(\partial \Omega\) is Lipschitz, the normal unit vector field might not be defined on \(\partial \Omega\). We shall therefore introduce a unitary vector field \(\tilde{\nu}\) locally defined near \(\partial \Omega\) such that: (i) \(\tilde{\nu}\) is \(C^\infty\) smooth, (ii) \(\tilde{\nu}\) is non-tangential to \(\partial \Omega\) and it points to the exterior of \(\Omega\) (see \[7\] Lemmas 3.1-3.3) for a precise construction of \(\tilde{\nu}\). Here we simply recall that any point \(z_\tau = x^0 + \tau \tilde{\nu}\), where \(x^0 \in \partial \Omega\), satisfies
\[
C \tau \leq d(z_\tau, \partial \Omega) \leq \tau, \quad \text{for any } \tau, \quad 0 \leq \tau \leq \tau_0, \quad (4.1)
\]
where \(\tau_0\) and \(C\) depend on \(L, r_0\) only.

Several constants depending on the a-priori data introduced in Definition \[22\] will appear in the proof of the main result below. In order to simplify our notation, we shall denote by \(C\) any of these constants, avoiding in most cases to point out their specific dependence on the a-priori data which may vary from case to case.

To simplify our notation in what follows, we will denote \(K_{\mu_\alpha}, K_{\mu_s}^{-1}\) and \(\Lambda_{\mu_\alpha}\) simply by \(K_i, K_i^{-1}\) and \(\Lambda_i\) respectively, for \(i = 1, 2\). We will also denote the operator norm \(||\Lambda_1 - \Lambda_2||_{L(H^{1/2}(\partial \Omega), H^{-1/2}(\partial \Omega))}\) simply by \(||\Lambda_1 - \Lambda_2||_{*,*}\).
Proof of Theorem 2.5. We start by recalling that by Alessandrini’s Identity, 
\begin{align*}
\langle (\Lambda_1 - \Lambda_2)u_1, \nu_2 \rangle &= \int_{\Omega} (K_1(x) - K_2(x)) \nabla u_1(x) \cdot \nabla u_2(x) \, dx \\
&= \int_{\Omega} (\mu_{a_1}(x) - \mu_{a_2}(x)) u_1(x) u_2(x) \, dx,
\end{align*}
for any $u_1, u_2 \in H^1(\Omega)$ that solve
\begin{align*}
\operatorname{div}(K_1 \nabla u_1) + (\mu_{a_1} - ik)u_1 &= 0, \quad \text{in } \Omega, \quad (4.2) \\
\operatorname{div}(K_2 \nabla u_2) + (\mu_{a_2} - ik)u_2 &= 0, \quad \text{in } \Omega. \quad (4.3)
\end{align*}
We set $x^0 \in \partial \Omega$ such that 
\begin{align*}
(-1)^{h} \frac{\partial^{h}}{\partial \nu^{h}} (\mu_{a_1} - \mu_{a_2})(x^0) &= \left\| \frac{\partial^{h}}{\partial \nu^{h}} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)}
\end{align*}
and $z_\tau = x^0 + \tau \nu$, with $0 < \tau \leq \tau_0$, where $\tau_0$ is the number fixed in (4.1). Let $m > 0$ an integer and $u_1, u_2 \in W^{2,p(\Omega)}$ be the singular solutions of Theorem 3.1 to (4.2), (4.3) respectively, having a singularity at $z_\tau$
\begin{align*}
u_{a_{i}}(x) &= \left\| (K_{i}^{-1}(z_\tau))_{nn} \right\|_{\Omega}, \quad \text{for } i = 1, 2 \quad \text{and}
C_{m,i}^{(n-2)}(x, z_\tau) := C_{m}^{(n-2)} \left( \frac{(K_{i}^{-1}(z_\tau)_{nn}(K_{i}^{-1}(z_\tau)_{nn} \nabla K_{i}^{-1}(z_\tau)(x - z_\tau))}{((K_{i}^{-1}(z_\tau))_{nn}K_{i}^{-1}(z_\tau)(x - z_\tau))^{\frac{1}{2}}} \right), \quad \text{for } i = 1, 2. \quad (4.5)
\end{align*}
To prove (2.39), we will prove
\begin{align*}
\left\| \frac{\partial^{j}}{\partial \nu^{j}} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} \leq C\|\Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}}\|_{\Lambda}^\delta, \quad \text{for any } j, j \leq h, \quad (4.6)
\end{align*}
by induction on $j$, where $\delta_j = \prod_{i=0}^{j} \frac{\alpha}{m+i}$. (2.39) will follow by (4.6) combined with an iterative use of the interpolation inequality
\begin{align*}
\|Df\|_{L^\infty(\partial \Omega)} \leq C(\Omega) \left\{ \left\| \frac{\partial f}{\partial \nu} \right\|_{L^\infty(\partial \Omega)} + \|f\|_{L^\infty(\partial \Omega)} + \|f\|_{L_{\infty}(\partial \Omega)} \right\}, \quad (4.7)
\end{align*}
which holds true for any $f \in C^{1+\gamma}(\Omega)$, $0 < \gamma \leq 1$ (see, for instance, (3.3) in [1 Lemma 3.2]).
For \( j = 0 \), (4.10) is given by (2.37). Assuming (1.6) holds true for any \( j, j \leq h - 1 \), we will prove it holds true for \( j = h \) too. By setting \( \rho = 2 \tau_0 \) we have \( \Omega \cap B_\rho(z_r) \neq \emptyset \) and from (4.2) we obtain

\[
\|A_1 - A_2\|_\infty \|u_1\|_{H^2(\partial \Omega)} \|u_2\|_{H^2(\partial \Omega)} \geq \left| \int_{\Omega \cap B_\rho(z_r)} (K_1(x) - K_2(x)) \nabla u_1(x) \cdot \nabla u_2(x) \, dx \right|
\]

\[
- \int_{\Omega \setminus B_\rho(z_r)} |K_1(x) - K_2(x)||\nabla u_1(x)||\nabla u_2(x)| \, dx
\]

\[
- \int_{\Omega \cap B_\rho(z_r)} |(\mu_{a_1} - \mu_{a_2})(x)||u_1(x)||u_2(x)| \, dx
\]

\[
- \int_{\Omega \setminus B_\rho(z_r)} |(\mu_{a_1} - \mu_{a_2})(x)||u_1(x)||u_2(x)| \, dx.
\] (4.8)

Recalling that \( K_i(x) := K(x, \mu_{a_i}) = \frac{1}{n} \left( (\mu_{a_i} - ik)I + (I - B)\mu_s \right)^{-1} \), where it is understood that \( B = B(x) \), \( \mu_s = \mu_s(x) \) and \( \mu_{a_i} = \mu_{a_i}(x) \) for \( i = 1, 2 \), then by Lagrange Theorem we have for any \( x \in \Omega \cap B_\rho(z_r) \)

\[
K_1(x) - K_2(x) = \left. \frac{\partial K(x,t)}{\partial t} \right|_{t=c(x)} (\mu_{a_1}(x) - \mu_{a_2}(x))
\] (4.9)

with \( \mu_{a_1}(x) < c(x) < \mu_{a_2}(x) \) and

\[
K(x,t) = \frac{1}{n} \left( (t-ik)I + (I - B)\mu_s \right)^{-1}.
\] (4.10)

Observing that \( \frac{\partial K(x,t)}{\partial t} = -nK^2(x,t) \), and that the integrals over \( \Omega \setminus B_\rho(z_r) \) on the right hand side of (4.8) are bounded by a constant \( C > 0 \) that depends on the \( a\text{-priori} \) data, we obtain from (4.8)

\[
\|A_1 - A_2\|_\infty \|u_1\|_{H^2(\partial \Omega)} \|u_2\|_{H^2(\partial \Omega)} \geq \left| \int_{\Omega \cap B_\rho(z_r)} (\mu_{a_1} - \mu_{a_2})(x) nK^2(x,c(x)) \nabla u_1(x) \cdot \nabla u_2(x) \, dx \right|
\]

\[- C - \int_{\Omega \setminus B_\rho(z_r)} |(\mu_{a_1} - \mu_{a_2})(x)||u_1(x)||u_2(x)| \, dx.
\] (4.11)

Next, to estimate from below the first integral on the right hand side of (4.11), we show that

\[
|K^2(x,c(x)) \nabla u_1(x) \cdot \nabla u_2(x)| \geq C|x - z_r|^{2-2(n+m)}, \quad \text{for any } x, \ x \in \Omega \cap B_\rho(z_r).
\] (4.12)

To prove (4.12) we observe that (4.2), (3.4) lead to

\[
|\nabla u_1 - \nabla u_2| \leq C \left\{ |x - z_r|^{1-n-m} |\mu_{a_1}(z_r) - \mu_{a_2}(z_r)| + |x - z_r|^{1-n-m+\alpha} \right\}
\]

\[
\leq C \left\{ |x - z_r|^{1-n-m} |\mu_{a_1}(x^0) - \mu_{a_2}(x^0)| + |x - z_r|^{1-n-m+\beta} + |x - z_r|^{1-n-m+\alpha} \right\}
\] (4.13)
and recalling that $|x - z_\tau| > C\tau$, for any $x \in \Omega \cap B_\rho(z_\tau)$ and $\alpha < \beta$,

$$\|\nabla u_1 - \nabla u_2\| \leq C\left\{|x - z_\tau|^{1-n-m}|\mu_{a_1}(x_0) - \mu_{a_2}(x_0)| + |x - z_\tau|^{1-n-m+\alpha}\right\}. \tag{4.14}$$

Hence, for almost every $x \in B_\rho(z_\tau) \cap \Omega$, we have

$$|K^2(x, c(x))\nabla u_1 \cdot \nabla u_2| \geq |K^2(x, c(x))\nabla u_1 \cdot \nabla u_1| - |K^2(x, c(x))| |\nabla u_1| |\nabla u_2 - \nabla u_1|$$

$$\geq C|x - z_\tau|^{2-2(n+m)} - C|x - z_\tau|^{1-n-m}\left\{|x - z_\tau|^{1-n-m}|\mu_{a_1}(x_0) - \mu_{a_2}(x_0)| + C|x - z_\tau|^{1-n-m+\alpha}\right\}$$

$$= C|x - z_\tau|^{2-2(n+m)} \left\{1 - |\mu_{a_1}(x_0) - \mu_{a_2}(x_0)| - |x - z_\tau|^{\alpha}\right\}. \tag{4.15}$$

Recalling that (4.10) for $j = 0$ leads to

$$|\mu_{a_1}(x_0) - \mu_{a_2}(x_0)| \leq C\|\Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}}\|_\ast \tag{4.16}$$

and that without loss of generality we can assume that

$$\|\Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}}\|_\ast \leq \frac{1}{2C^2}. \tag{4.17}$$

we obtain from (4.15) that

$$|K^2(x, c(x))\nabla u_1 \cdot \nabla u_2| \geq C|x - z_\tau|^{2-2(n+m)} \left\{1 - |x - z_\tau|^{\alpha}\right\}. \tag{4.18}$$

for almost every $x \in B_\rho(z_\tau) \cap \Omega$ and, by possibly reducing $\rho$ so that $|x - z_\tau| < \frac{1}{4}$, (4.12) follows. Hence, without loss of generality we can assume that

$$\Re\{K^2(x, c)\nabla u_1 \cdot \nabla u_2\} \geq C|x - z_\tau|^{2-2(n+m)}. \tag{4.19}$$

for almost every $x \in B_\rho(z_\tau) \cap \Omega$. Hence (4.19), combined together with (4.11), leads to

$$\|\Lambda_1 - \Lambda_2\|_\ast ||u_1||_{H^2(\partial \Omega)} ||u_2||_{H^2(\partial \Omega)} \geq \Re\left\{\int_{\Omega \cap B_\rho(z_\tau)} (\mu_{a_1} - \mu_{a_2})(x) i K^2(x, c(x))\nabla u_1 \cdot \nabla u_2 \, dx\right\}$$

$$- C - \int_{\Omega \cap B_\rho(z_\tau)} |(\mu_{a_1} - \mu_{a_2})(x)||u_1(x)||u_2(x)| \, dx, \tag{4.20}$$

therefore

$$\|\Lambda_1 - \Lambda_2\|_\ast ||u_1||_{H^2(\partial \Omega)} ||u_2||_{H^2(\partial \Omega)} \geq C \int_{\Omega \cap B_\rho(z_\tau)} (\mu_{a_1} - \mu_{a_2})(x)|x - z_\tau|^{2-2(n+m)} \, dx$$

$$- C - \int_{\Omega \cap B_\rho(z_\tau)} |(\mu_{a_1} - \mu_{a_2})(x)||u_1(x)||u_2(x)| \, dx \tag{4.21}$$
where, without loss of generality we assumed that \( (\mu_{a_1} - \mu_{a_2})(x) > 0 \), for almost every \( x \in B_p(z) \cap \Omega \). Noticing that any \( x \in \Omega \cap B_p(z) \) can uniquely be written as \( x = y - s\nu \), with \( y \in \partial \Omega \), then for any \( x \in B_p(z) \cap \Omega \) and by using Taylor’s theorem, we have

\[
\sum_{j=0}^{h-1} \frac{\partial^{j}}{\partial y^{j}}(\mu_{a_1} - \mu_{a_2})(y) \frac{(-s)^j}{j!} + \frac{\partial^{h}}{\partial y^{h}}(\mu_{a_1} - \mu_{a_2})(x) \frac{(-s)^h}{h!} - (\mu_{a_1} - \mu_{a_2})(x) \leq C|x - x_0|^\alpha s^h, \tag{4.22}
\]

for any \( x \in B_p(z) \cap \Omega \) and

\[
(\mu_{a_1} - \mu_{a_2})(x) \geq \left\| \frac{\partial^{h}}{\partial y^{h}}(\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} s^h - \sum_{j=0}^{h-1} \left\| \frac{\partial^{j}}{\partial y^{j}}(\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} s^j - C|x - x_0|^\alpha s^h, \tag{4.23}
\]

for any \( x \in B_p(z) \cap \Omega \). Observing that we also have

\[
(\mu_{a_1} - \mu_{a_2})(x) \leq \sum_{j=0}^{h-1} \left\| \frac{\partial^{j}}{\partial y^{j}}(\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} s^j + C s^h, \tag{4.24}
\]

for any \( x \in B_p(z) \cap \Omega \) and combing together (4.23), (4.24) and (4.21), leads to

\[
\left\| \Lambda_1 - \Lambda_2 \right\|_* \leq \left\| u_1 \right\|_{H^2(\partial \Omega)} \left\| u_2 \right\|_{H^4(\partial \Omega)} - C \left\{ \left\| \frac{\partial^{h}}{\partial y^{h}}(\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} \int_{\Omega \cap B_p(z)} |x - z|^2 |dx, \partial \Omega|^h dx \right.

\left. - \sum_{j=0}^{h-1} \left\| \frac{\partial^{j}}{\partial y^{j}}(\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} \int_{\Omega \cap B_p(z)} |x - z|^2 |dx, \partial \Omega|^j dx \right.

\left. - \int_{\Omega \cap B_p(z)} |x - x_0|^\alpha |x - z|^2 |dx, \partial \Omega|^h dx \right.

\left. - \sum_{j=0}^{h-1} \left\| \frac{\partial^{j}}{\partial y^{j}}(\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} \int_{\Omega \cap B_p(z)} |x - z|^4 |dx, \partial \Omega|^j dx \right.

\left. - \int_{\Omega \cap B_p(z)} |x - z|^4 |dx, \partial \Omega|^h dx \right\} - C, \tag{4.25}
\]

which leads to
Recalling that by the induction hypothesis we have

$$\left\| \frac{\partial^h}{\partial x^h} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} \int_{\Omega \cap B_{\rho}(z)} |x - z_t|^{2-2(n+m)} d(x, \partial \Omega)^h dx \leq$$

$$\leq C \left\{ \sum_{j=0}^{h-1} \left\| \frac{\partial^j}{\partial x^j} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} \int_{\Omega \cap B_{\rho}(z)} |x - z_t|^{2-2(n+m)} d(x, \partial \Omega)^j dx + \int_{\Omega \cap B_{\rho}(z)} |x - x_0|^\alpha |x - z_t|^{2-2(n+m)} d(x, \partial \Omega)^h dx + \int_{\Omega \cap B_{\rho}(z)} |x - z_t|^{4-2(n+m)} d(x, \partial \Omega)^h dx + \| \Lambda_1 - \Lambda_2 \|_{H^{\frac{1}{2}}(\partial \Omega)} \| u_1 \|_{H^{\frac{1}{2}}(\partial \Omega)} \| u_2 \|_{H^{\frac{1}{2}}(\partial \Omega)} \right\} + C. \quad (4.26)$$

The first integral on the right hand side of (4.26) can be estimated from above by observing that

$$\Omega \cap B_{\rho}(z) \subset \{ C \tau \leq |x - z_t| \leq 2\tau_0 \}$$

and that $d(x, \partial \Omega) \leq |x - z_t|$, hence

$$\int_{\Omega \cap B_{\rho}(z)} |x - z_t|^{2-2(n+m)} d(x, \partial \Omega)^j dx \leq \int_{C\tau \leq |x - z_t| \leq 2\tau_0} |x - z_t|^{2-2(n+m)} |x - z_t|^j dx = \int_{C\tau}^{2\tau_0} \sigma^{2-2n-2m+j+n-1} d\sigma \int_{|\xi|=1} d\Sigma_\xi \leq C\tau^{2-n-2m+j}, \quad (4.27)$$

where $d\Sigma_\xi$ denotes the surface measure on the unit sphere. Similarly, the other integrals on the right hand side of (4.26) can be estimated from above as

$$\int_{\Omega \cap B_{\rho}(z)} |x - x_0|^\alpha |x - z_t|^{2-2(n+m)} d(x, \partial \Omega)^h dx \leq C\tau^{2-n-2m+h}, \quad (4.28)$$

$$\int_{\Omega \cap B_{\rho}(z)} |x - z_t|^{4-2(n+m)} d(x, \partial \Omega)^h dx \leq C\tau^{2-n-2m+h+n-1}, \quad (4.29)$$

where the integral on the left hand side of (4.26) can be estimated from below (see [1] p.66)

$$\int_{\Omega \cap B_{\rho}(z)} |x - z_t|^{2-2(n+m)} d(x, \partial \Omega)^h dx \geq C\tau^{2-n-2m+h}. \quad (4.30)$$

Recalling that by the induction hypothesis we have

$$\left\| \frac{\partial^j}{\partial x^j} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} \leq C\| \Lambda_1 - \Lambda_2 \|_{H^{\frac{1}{2}}(\partial \Omega)} \| u_1 \|_{H^{\frac{1}{2}}(\partial \Omega)} \| u_2 \|_{H^{\frac{1}{2}}(\partial \Omega)} \quad \text{for any } j, j \leq h - 1. \quad (4.31)$$

By combining (4.27) - (4.31) and the $H^{\frac{1}{2}}(\partial \Omega)$-norms of $u_1, u_2$ (see [1], [7]) we get
\[ \left\| \frac{\partial^h}{\partial \nu^h} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} \leq \frac{C}{h} \sum_{j=0}^{h-1} \| \Lambda_1 - \Lambda_2 \|^j_{2-n-m+j} \]

\[ + C \tau^{2-n-m+\alpha+h} + C \tau^{4-n-2m+h} \]

\[ + \| \Lambda_1 - \Lambda_2 \|^\delta_{2-n-2m}. \]

\[ \leq C \| \Lambda_1 - \Lambda_2 \|^\delta_{2-n-2m} \]

\[ + C \tau^{2-n-m+\alpha+h} + C \tau^{4-n-2m+h}, \tag{4.32} \]

hence

\[ \left\| \frac{\partial^h}{\partial \nu^h} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} \leq C \left\| \| \Lambda_1 - \Lambda_2 \|^\delta_{2-n-2m} \right\|^\tau_{2-n-2m+h} \]

\[ \leq C \left\{ \| \Lambda_1 - \Lambda_2 \|^\delta_{2-n-2m} \tau^{2-n+m-2} + \tau^2 \right\}. \tag{4.33} \]

By choosing \( m \) sufficiently large, (4.33) becomes

\[ \left\| \frac{\partial^h}{\partial \nu^h} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} \leq C \left\{ \| \Lambda_1 - \Lambda_2 \|^\delta_{2-n-2m} \tau^{2-n+m-2} + \tau^2 \right\}. \tag{4.34} \]

Finally, optimising (4.34) with respect to \( \tau \) we obtain

\[ \left\| \frac{\partial^h}{\partial \nu^h} (\mu_{a_1} - \mu_{a_2}) \right\|_{L^\infty(\partial \Omega)} \leq C \| \| \Lambda_1 - \Lambda_2 \|^\delta_{2-n-2m} \|_{\tau^{2-n+m-2} + \tau^2}, \tag{4.35} \]

which concludes the proof of (4.35).

\[ \square \]

**Proof of Corollary 2.6.** By an induction argument, for every multi-index \( \beta, |\beta| \leq h \), we have

\[ \frac{\partial^{\beta}}{\partial x^{\beta}} K(x, \mu_a(x)) = \sum_{|\gamma| + |\delta| \leq |\beta|} P_{\gamma \delta}(\mu_a(x), \ldots, D^{\delta} \mu_a(x)) \frac{\partial^{\gamma}}{\partial x^{\gamma}} \frac{\partial^{\delta}}{\partial t} K(x, t) \bigg|_{t=\mu_a(x)}, \tag{4.36} \]

where \( P_{\gamma \delta} \) is a polynomial. By hypotheses (2.41, 2.42), we obtain that \( K(x, \mu_{a_i}(x)) \in C^{h,\alpha}(\Omega_r) \), for \( i = 1, 2 \), hence

\[ ||D^h (K(x, \mu_{a_1}) - K(x, \mu_{a_2})) ||_{L^\infty(\partial \Omega)} \leq C ||\mu_{a_1} - \mu_{a_2}||_{C^h(\Omega_r)}, \tag{4.37} \]

which, combined with (2.39), implies (2.43).

\[ \square \]
References

[1] G. Alessandrini, Singular solutions of elliptic equations and the determination of conductivity by boundary measurements, J. Differential Equations 84, (2) (1990), 252-272.

[2] G. Alessandrini, F. Faucher, M. V. de Hoop, R. Gaburro and E. Sincich, Inverse problem for the Helmholtz equation with Cauchy data: reconstruction with conditional well-posedness driven iterative regularization, ESAIM: Mathematical Modelling and Numerical Analysis 53 (3) (2019), 1005-1030.

[3] G. Alessandrini, M.V. de Hoop and R. Gaburro, Uniqueness for the electrostatic inverse boundary value problem with piecewise constant anisotropic conductivities, Inverse Problems 33 (12) (2017), 125013.

[4] G. Alessandrini, M. V. de Hoop, R. Gaburro and E. Sincich, Lipschitz stability for the electrostatic inverse boundary value problem with piecewise linear conductivities, Journal de Mathématiques Pures et Appliquées 107 (5) (2017), 638 - 664.

[5] G. Alessandrini, M. V. de Hoop, R. Gaburro and E. Sincich, Lipschitz stability for a piecewise linear Schrödinger potential from local Cauchy data, Asymptotic Analysis 108 (3) (2018), 115-149.

[6] G. Alessandrini, M. V. de Hoop, R. Gaburro and E. Sincich, EIT in a layered anisotropic medium, Inverse Problems and Imaging 12 (3) (2018), 667 - 676.

[7] G. Alessandrini and R. Gaburro, Determining conductivity with special anisotropy by boundary measurements, SIAM J. MATH. ANAL. 33 (2001), 153-171.

[8] G. Alessandrini and R. Gaburro, The local Calderón problem and the determination at the boundary of the conductivity, Comm. Partial Differential Equations 34 (2009), 918-936.

[9] G. Alessandrini and S. Vessella, Lipschitz stability for the inverse conductivity problem, Advances in Applied Mathematics 35 (2005), 207-241.

[10] S. R. Arridge, Optical tomography in medical imaging, Inverse Problems 15 (2) (1999), R41.

[11] S. R. Arridge and J. C. Hebdon, Optical imaging in medicine II: modelling and reconstruction, Physics in Medicine and Biology 42 (5) (1997), 841.

[12] S. R. Arridge, W.R.B Lionheart, Nonuniqueness in diffusion-based optical tomography, Optics Letters 23 (11) (1998), 882-884.

[13] S. R. Arridge and J.C. Schotland, Optical tomography: forward and inverse problems, Inverse Problems 25 (12) (2009), 123010.

[14] E. Beretta, M. De Hoop, F. Faucher and O. Scherzer, Inverse boundary value problem for the Helmholtz equation: quantitative conditional Lipschitz stability estimates, SIAM J. Math. Anal. 48 (2016), 3962-3983.

[15] E. Beretta, E. Francini and S. Vessella, Uniqueness and Lipschitz stability for the identification of Lamé parameters from boundary measurements, Inv. Probl. Imag. 8 (2014), 611-644.
[16] E. Beretta, M. V. de Hoop, E. Francini, S. Vessella and J. Zhai, Uniqueness and Lipschitz stability of an inverse boundary value problem for time-harmonic elastic waves, Inverse Problems 33 (3) (2017), 035013.

[17] E. Beretta, M. De Hoop and L. Qiu, Lipschitz stability of an inverse boundary value problem for a Schrödinger type equation, SIAM J. Math. Anal. 45 (2013), 679-699.

[18] E. Beretta E, M. V. de Hoop, L. Qiu L and O. Scherzer 2014 Inverse boundary value problem for the Helmholtz equation: multi-level approach and iterative reconstruction, arXiv:1406.2391 (2014).

[19] E. Beretta and E. Francini, Lipschitz stability for the electrical impedance tomography problem: the complex case, Communications in Partial Differential Equations 36 (2011), 1723-1749.

[20] E. Beretta, E. Francini, A. Morassi, E. Rosset and S. Vessella, Lipschitz continuous dependence of piecewise constant Lamé coefficients from boundary data: the case of non flat interfaces, Inverse Problems 30 (2014), 125005.

[21] D.A. Boas, D.H. Brooks, E.L. Miller, C.A. DiMarzio, M. Kilmer, R.J. Gaudette and Q. Zhang, Imaging the body with diffuse optical tomography. IEEE signal processing magazine, 18 (2001), 57-75.

[22] L. Borcea, Electrical impedance tomography, Inverse Problems 18 (2002), R99-R136.

[23] A. P. Calderón, On an inverse boundary value problem, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), 65–73, Soc. Brasil. Mat., Rio de Janeiro, 1980. Reprinted in: Comput. Appl. Math. 25 (2-3) (2006), 133-138.

[24] M. V. de Hoop, L. Qiu and O. Scherzer, Local analysis of inverse problems: Hölder stability and iterative regularization, Inverse Problems 23 (2012) 045001 (16pp).

[25] O. Doeva, R. Gaburro, W.R.B. Lionheart and C. J. Nolan, Lipschitz stability at the boundary for time-harmonic diffuse optical tomography, Applicable Analysis (2020) doi.org/10.1080/00036811.2020.1758314.

[26] F. Faucher, G. Alessandrini, H. Barucq, M.V. de Hoop, R. Gaburro and E. Sincich, Full reciprocity-gap waveform inversion enabling sparse-source acquisition, Geophysics 85 2020, R461-R476.

[27] F. Faucher, M.V. de Hoop and O. Scherzer, Reciprocity-gap misfit functional for distributed acoustic sensing, combining data from passive and active sources, Geophysics 86 (2021), doi.org/10.1190/geo2020-0305.1

[28] S. Foschiatti, R. Gaburro and E. Sincich, Stability for the Calderón’s problem for a class of anisotropic conductivities via an ad-hoc misfit functional, Inverse Problems (2021), in press.

[29] R. Gaburro, Stable determination at the boundary of the optical properties of a medium: the static case, Rend. Istit. Mat. Univ. Trieste 48 (2016), 407 - 431.

[30] R. Gaburro and W. R. B. Lionheart, Recovering Riemannian metrics in monotone families from boundary data, Inverse Problems 25 (4) (2009), 045004 (14pp).
[31] R. Gaburro and E. Sincich, Lipschitz stability for the inverse conductivity problem for a conformal class of anisotropic conductivities, Inverse Problems 31 (2015), 015008.

[32] A. P. Gibson, J. C. Hebden and S. R. Arridge, Recent advances in diffuse optical imaging, Phys. Med. Biol. 50 (2005), R1 - R43.

[33] B. Harrach, On uniqueness in diffuse optical tomography, Inverse Problems 25 (5) (2009), 055010.

[34] B. Harrach, Simultaneous determination of the diffusion and absorption coefficient from boundary data, Inverse Problems and Imaging 6 (4) (2012), 663 - 679.

[35] J. Heino and E. Somersalo, Estimation of optical absorption in anisotropic background, Inverse Problems 18 (3) (2002), 559.

[36] J. Heino, S. Arridge, J. Sikora, and E. Somersalo, Anisotropic effects in highly scattering media, Physical Review E 68 (3) (2003), 031908.

[37] N. Hyvönens, Characterizing inclusions in optical tomography, Inverse Problems 20 (3) (2004), 737.

[38] J. Kaipio and E. Somersalo, Statistical and computational inverse problems. Vol. 160 (2006). Springer Science & Business Media.

[39] V. Kolehmainen, M. Vauhkonen, J. P. Kaipio and S. R. Arridge, Recovery of piecewise constant coefficients in optical diffusion tomography, Optic Express 7 (13) (2000), 468 - 480.

[40] R. Kohn and M. Vogelius, Determining conductivity by boundary measurements, Comm. Pure Appl. Math. 37, (1984), 289-298.

[41] R. Kohn and M. Vogelius, Determining Conductivity by Boundary Measurements II. Interior Results, Comm. Pure. Appl. Math. 38 (1985), 643-667.

[42] V. Isakov, On the uniqueness in the inverse conductivity problem with local data, Inverse Problems and Imaging 1 (1) (2007), 95 - 105.

[43] A. Nachman, Global uniqueness for a two dimensional inverse boundary value problem, Ann. Math. 142 (1995), 71-96.

[44] M. Salo, Inverse problems for nonsmooth first order perturbations of the Laplacian, PhD Thesis, University of Helsinki, Helsinki (2004).

[45] J. Sylvester and G. Uhlmann, A Global Uniqueness Theorem for an Inverse Boundary Valued Problem, Ann. of Math., Vol.125 (1987), 153-169.

[46] G. Uhlmann, Electrical impedance tomography and Calderón’s problem, Inverse Problems 25 (12) (2009), 123011.