Universal scaling exponents in shell models of turbulence: Viscous effects are finite-size corrections to scaling

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In a series of recent works it was proposed that shell models of turbulence exhibit inertial range scaling exponents that depend on the nature of the dissipative mechanism. If true, and if one could imply a similar phenomenon to Navier-Stokes turbulence, this finding would cast strong doubts on the universality of scaling in turbulence. In this Letter we propose that these “nonuniversalities” are just corrections to scaling that disappear when the Reynolds number goes to infinity.

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The aim of this Letter is to question a growing consensus concerning the non-universality of the scaling exponents that characterize correlation functions that appear in the context of numerical studies of shell models of turbulence. Shell models, and in particular the so-called GOY model, became very popular due to their ease of simulation and the fact that they appear in the context of numerical studies of shell models of turbulence. Shell models, and in particular the so-called GOY model, became very popular due to their ease of simulation and the fact that they appear in the context of numerical studies of shell models of turbulence.

The GOY model is a simplified reduced wavenumber analog to the spectral Navier-Stokes equations. The wavenumbers are represented as shells, each of which is defined by a (real) wavenumber \( k_0 = k_0 \lambda^n \) where \( \lambda \) is the “shell spacing”. There are \( N \) degrees of freedom where \( N \) is the number of shells. The model specifies the dynamics of the “velocity” \( u_n \) which is considered a complex number, \( n = 1, \ldots, N \). The GOY model reads

\[
\frac{d}{dt} + \nu k_n^{2\alpha} u_n \equiv f_n + i \left[ k_{n+1} u_{n+1} u_{n+2} - \frac{k_n}{2} u_{n+1} u_{n-1} - \frac{k_n-1}{2} u_n u_{n-2} \right].
\]

In this equation * stands for complex conjugation, \( \nu \) is the “viscosity”, and to motivate the discussion we already specified a one parameter family of models in which the exponent \( \alpha \) determines the type of viscous dissipation. In the following we use \( \lambda = 2 \) and a forcing restricted to the first two shells.

The scaling exponents can be defined using the “correlation functions” \( \langle |u_n|^p \rangle \). It was shown however that the GOY model suffers from periodic oscillations superposed on the scaling laws obeyed by these functions [4]. It was proposed in [4] that it is advantageous therefore to focus on the scaling properties of correlation functions of the type

\[
F_q(k_n) \equiv \left| \text{Im} \left( u_n u_{n+1} u_{n+2} + \frac{1}{4} u_n u_{n-1} u_{n+1} \right) \right|^{q/3}.
\]

These functions scale with \( k_n \), \( F_q(k_n) \sim k_n^{-\zeta_q} \) as long as \( k_n \) is smaller than some dissipative cutoff scale \( k_d \) and sufficiently larger than \( k_1 \). The exponents \( \zeta_q \) are the scaling exponents whose universality is questioned. The definition \( F_q \) has the advantage that \( \zeta_2 \) is known exactly (if one assumes that inertial range scaling is universal) and it is \( \zeta_2 = 1 \) [8].

The issue to be discussed in this Letter is introduced in Fig. 1 which shows the results of numerical simulations of this family of models. The simulations were performed using the adaptive step-size backward differentiation algorithm (DDEBDF) from the SLATEC library [1,2]. We imposed a random forcing on the first two shells, \( f_1 = 5 \times 10^{-3} \) and \( f_2 = a f_1 \) where a typical value of \( a \) is 0.75. The time correlation of the forcing has been chosen to be of the magnitude of the natural forcing turn-over time-scale \( \tau = 1/\sqrt{k_1 f} = 40 \).

![Fig. 1. The correlation functions \( F_1 \) to \( F_5 \) averaged over about 1500 forcing turn-over scales for a model of 22 shells with a dissipative term proportional to \( k^2 \) (\( \nu = 5 \times 10^{-7} \)), \( k^4 \) (\( \nu = 10^{-14} \)) and \( k^6 \) (\( \nu = 2 \times 10^{-22} \)).](image-url)
Table 1. Simple scaling regression $k_n^{-\zeta}$ for regular and hyperviscosity, $\alpha = 1, 2, 3$.

| $k^d$ | $N$ | $\zeta_1$ | $\zeta_2$ | $\zeta_3$ | $\zeta_4$ | $\zeta_5$ |
|-------|-----|--------|--------|--------|--------|--------|
| $k^d$ | 22  | 0.38   | 0.71   | 1.00   | 1.27   | 1.52   |
| $k^d$ | 22  | 0.36   | 0.69   | 0.97   | 1.22   | 1.46   |
| $k^d$ | 22  | 0.34   | 0.62   | 0.85   | 1.06   | 1.25   |

is chosen (by fixing the parameter $\nu$) to keep $k_d \approx k_0 2^{17}$ for all values of $\alpha$. In Table 1 we show the results obtained by simple linear fits between the shells 4 and 13. Results of this type were interpreted in previous works as an indication that the inertial range scaling depends on $\alpha$, casting therefore severe doubts on the universality of the scaling exponents $\zeta_3$. It should be stressed that this is not a trivial issue; if indeed the scaling exponents depend on the dissipative mechanism this could either imply a major departure from the standard thinking about universality in turbulence, or an indication that shell models are fundamentally different from Navier-Stokes dynamics. The main point of this Letter is that none of this is necessary. We propose that these scaling plots should be interpreted as standard corrections to scaling that can be factored away by taking into account the effect of the viscous dissipation. These effects become less important when the number of shells increases and when the length of the inertial range increases, as we show below in detail.

To provide strong evidence that we are faced with finite size effects we integrated the equations of motion for different number of shells. We chose three sets of data with $N = 22, 28$ and 34 shells. In all cases the integration was performed over 1500 forcing turn-over time scales. The viscosities were chosen so that the width of the dissipative range is about 5-6 shells: for $\alpha = 1$ (normal dissipation), $\nu = 5 \times 10^{-7}, 8 \times 10^{-9}$ and $4 \times 10^{-11}$ for $N = 22, 28$ and 34 respectively; for $\alpha = 2$, $\nu = 10^{-14}, 2 \times 10^{-20}$ and $2 \times 10^{-26}$ and for $\alpha = 3$, $\nu = 2 \times 10^{-22}, 10^{-30}$ and $10^{-41}$. The results of this simulation are a direct illustration of the finite size effect: for any value of $\alpha$, we can replot our data for $N = 22, 28$ and 34, moving the abscissa to coalesce the near dissipative region. Consider for example $F_3(k_n)$ for which we have a theoretical expectation, i.e., $F_3(k_n) \sim k_n^{-1}$ in the inertial range, but for which the data in Table 1 indicate significant deviation from $\zeta_3 = 1$. In Fig. 2 we present double-logarithmic plots of $k_nF_3(k_n)$ vs. $k_n$ for $\alpha = 2$. We expect this function to be constant in the inertial range. Instead, we see a strong viscous effect, leading to deviation from constancy over 10 shells even before deep dissipation sets in. Increasing the number of shells moves the problematic region to higher shells. We have shifted the abscissa in Fig. 2 to collapse all our data together. The data indeed collapses, with an $N$ independent (fixed size) cross over region.

This is a sure indication that we are dealing with a finite size effect and not the breaking of universality. Any deviation of the apparent $\zeta_3$ from unity is going to disappear slowly when the size of the inertial range increases. Of course, the linear fit procedure is very sensitive to the existence of a bump of the type shown in Fig. 2 and the determination of the exponents in Table 1 is very unreliable.

As is commonly found in scaling systems that suffer strong finite-size effects, it is advantageous to fit the whole scaling function rather than its power law part alone. In other words, to extract reliable inertial range scaling exponents, we propose to fit the correlation functions over their full domain, including the deep viscous regime. To achieve such a fit, we need to consider first the viscous range, where we may guess a generalized exponential decay:

$$u_n \sim \exp \left[ - \frac{k_n}{k_d} \right], \quad (3)$$

where $k_d$ is an appropriate dissipative scale. In the stationary state we expect a balance between the non-linear transfer and the dissipiative damping for every shell. Due to the exponential decay, the most relevant non-linear term for the $n$th shell is $k_n^{-1}u_{n-1}u_{n-2}$, which has to balance with $k_n^{2\alpha}u_n$. Up to logarithmic corrections, the exponent $x$ is then determined from.
FIG. 3. Plots of $\log_2 |\log_2 S_3(k_n)|$ vs $\log_2(k_n)$ for indices of hyperviscosity $\alpha = 1, 2, 3$ and viscosity $\nu = 5 \times 10^{-5}, 10^{-9}$ and $8 \times 10^{-15}$. The dotted lines represent the asymptotic stretched exponential behavior. The slopes are respectively 0.69, 0.76 and 0.86 to compare to the theoretical expectation $x = 0.694$.

FIG. 4. Third order structure function $F_3(k_n)$ for $\alpha = 1, 2$ and 3. The symbols indicate the results of simulations with 34 shells; the lines are the fits performed between the 3rd and 32nd shells. The results have been shifted in the y-coordinate for clarity.

$$\exp\left[-\left(\frac{k_n-1}{k_d}\right)^x\right]\exp\left[-\left(\frac{k_n-2}{k_d}\right)^x\right]\exp\left[-\left(\frac{k_n}{k_d}\right)^x\right],$$

or equivalently,

$$1 + \lambda^x - \lambda^{2x} = 0,$$

so that we have finally

$$x = \log_\lambda \frac{1 + \sqrt{5}}{2}.$$

Our simulations provide excellent support for the deep dissipative stretched exponential dependence, but the predicted value of $x$ depends on the coefficient $\alpha$, see Fig. 3. In the inertial range we expect a power law, and to estimate the behavior in the cross over region we note that the ratio of the viscous to nonlinear term can be written in the form

$$(\lambda x^2 - \lambda^{3x})\exp\left[-\left(\frac{k_n}{k_d}\right)^x\right].$$

FIG. 5. Third order exponent $\zeta_3$ computed via the fitting procedure described in the text for $\alpha = 2$ and 3 obtained for $N = 22, 28, 34$. The results have been plotted against $1/(\Delta n)^2$ where $\Delta n$ is the estimated width of the inertial range. The straight lines (with respective slope -3.5 and -14) indicate that the calculated exponents converge linearly toward the predicted value $\zeta_3 = 1$.

FIG. 6. Same figure as above for the second order exponent $\zeta_2$. The illustrative lines have slopes -6 and -12. The final value $\zeta_2 = 0.711$ has been obtained in the normal dissipation case $\alpha = 1$.
\[ \frac{\nu k_{n}^{2\alpha}u_{n}}{k_{n}u_{n}^{2}} \sim \frac{\nu k_{d}^{2\alpha}u_{d} k_{d}u_{d}^{2}}{k_{d}u_{d}^{2}} \sim \left( \frac{k_{n}}{k_{d}} \right)^{2\alpha-2/3}, \tag{7} \]

where \(u_{d}\) is the velocity at the viscous cross-over shell with \(k_{n} = k_{d}\). For simplicity we assumed \(u_{n} \sim k_{n}^{-1/3}\) for this estimate. Taking (7) as a first order correction of the inertial range power law by viscous effects we can propose a functional form for \(F_{q}(k_{n})\) over its full domain:

\[ F_{q}(k_{n}) = A_{q}k_{n}^{-\zeta_{q}} \left( 1 + \beta_{q} \frac{k_{n}}{k_{d,q}} \right)^{2\alpha-2/3} \exp \left[ - \left( \frac{k_{n}}{k_{d,q}} \right)^{x_{n}} \right]. \tag{8} \]

By construction this form fits the scaling law and the deep dissipative form for \(k_{n} \ll k_{d}\) and \(k_{n} \gg k_{d}\) respectively. For crossover values of \(k_{n}\) the formula offers two additional free parameters, i.e. \(\beta_{q}, k_{d,q}\). This is the minimal number of fit parameters that is needed to interpolate between the inertial and the dissipative ranges.

As an example of the excellent fit that this formula provides we show in Fig.4 \(F_{3}(k_{n})\) for \(\alpha = 1, 2\) and \(3\). This simulation employed 34 shells, and the fit were performed between the 3rd and 32nd shells. Similarly good fits are found for all the data that we analyzed.

Accordingly, we recorded the apparent values of the scaling exponent \(\zeta_{q}\) for increasing values of the \(N\), and noticed that they converge to a given value. This convergence can be seen in Figs. 5 and 6 in which the scaling exponents \(\zeta_{2}\) and \(\zeta_{3}\) are plotted for \(\alpha = 2, 3\) as a function of \(1/(\Delta n)^{2}\) where \(\Delta n = \log_{2}(k_{d}/k_{2})\) is the estimated width of the inertial range, and \(k_{d}\) is obtained from the fit. It is obvious that the exponents converge towards a universal value that agrees with \(\alpha = 1\).

The conclusion of this Letter is that it is dangerous to measure scaling exponents in multiscaling situations without taking into account crossover and finite size effects. Even in the case of shell models which appear to have much longer inertial ranges than Navier-Stokes turbulence, simple log-log plots are misleading. We propose that in future determination of scaling exponents in numerical experiments similar care must be given to such issues.

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