On homeomorphisms with finite distortion in the plane

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Abstract

It is shown that every homeomorphism of finite distortion in the plane is the so-called lower $Q$-homeomorphism with $Q(z) = K_f(z)$, and, on this base, it is developed the theory of the boundary behavior of such homeomorphisms.

1 Introduction

The concept of the generalized derivative was introduced by Sobolev in [31]. Given a domain $D$ in the complex plane $\mathbb{C}$, the Sobolev class $W^{1,1}(D)$ consists of all functions $f : D \rightarrow \mathbb{C}$ in $L^1(D)$ with first partial generalized derivatives which are integrable in $D$. A function $f : D \rightarrow \mathbb{C}$ belongs to $W^{1,1}_{\text{loc}}(D)$ if $f \in W^{1,1}(D_*)$ for every open set $D_*$ with its compact closure $\overline{D_*} \subset D$.

Recall that a homeomorphism $f$ between domains $D$ and $D'$ in $\mathbb{C}$ is called of finite distortion if $f \in W^{1,1}_{\text{loc}}$ and

$$||f'(z)||^2 \leq K(z) \cdot J_f(z)$$

(1.1)

with a.e. finite function $K$ where $||f'(z)||$ denotes the matrix norm of the Jacobian matrix $f'$ of $f$ at $z \in D$ and $J_f(z) = \det f'(z)$, see [10]. Later on, we use the notion $K_f(z)$ for the minimal function $K(z) \geq 1$ in (1.1). Note that $||f'(z)|| = |f_z| + |f_{\bar{z}}|$ and $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2$ at the points of total differentiability of $f$. Thus, $K_f(z) = ||f'(z)||^2 / J_f(z) = (|f_z| + |f_{\bar{z}}|) / (|f_z| - |f_{\bar{z}}|)$ if $J_f(z) \neq 0$, $K_f(z) = 1$ if $f'(z) = 0$, i.e. $|f_z| = |f_{\bar{z}}| = 0$, and $K_f(z) = \infty$ at the rest points.

A continuous mapping $\gamma$ of an open subset $\Delta$ of the real axis $\mathbb{R}$ or a circle into $D$ is called a dashed line, see e.g. Section 6.3 in [23]. Recall that every open set $\Delta$ in $\mathbb{R}$ consists of a countable collection of mutually disjoint intervals. This is the motivation for the term.
Given a family $\Gamma$ of dashed lines $\gamma$ in complex plane $\mathbb{C}$, a Borel function $\varrho : \mathbb{C} \to [0, \infty]$ is called \textbf{admissible} for $\Gamma$, write $\varrho \in \text{adm } \Gamma$, if
\[ \int \gamma \varrho \, ds \geq 1 \quad (1.2) \]
for every $\gamma \in \Gamma$. The \textbf{(conformal) modulus} of $\Gamma$ is the quantity
\[ M(\Gamma) = \inf_{\varrho \in \text{adm } \Gamma} \int_{\mathbb{C}} \varrho^2(z) \, dm(z) \quad (1.3) \]
where $dm(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$. We say that a property $P$ holds for \textbf{a.e.} (almost every) $\gamma \in \Gamma$ if a subfamily of all lines in $\Gamma$ for which $P$ fails has the modulus zero, cf. [4]. Later on, we also say that a Lebesgue measurable function $\varrho : \mathbb{C} \to [0, \infty]$ is \textbf{extensively admissible} for $\Gamma$, write $\varrho \in \text{ext adm } \Gamma$, if (1.2) holds for a.e. $\gamma \in \Gamma$, see e.g. 9.2 in [23].

The following concept was motivated by Gehring’s ring definition of quasiconformality in [5]. Given domains $D$ and $D'$ in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $z_0 \in \overline{D} \setminus \{\infty\}$, and a measurable function $Q : D \to (0, \infty)$, we say that a homeomorphism $f : D \to D'$ is a \textbf{lower $Q$-homeomorphism at the point} $z_0$ if
\[ M(f \Sigma_\varepsilon) \geq \inf_{\varrho \in \text{ext adm } \Sigma_\varepsilon} \int_{D \cap R_\varepsilon} \frac{\varrho^2(x)}{Q(x)} \, dm(x) \quad (1.4) \]
for every ring
\[ R_\varepsilon = \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}, \quad \varepsilon \in (0, \varepsilon_0), \quad \varepsilon_0 \in (0, d_0), \]
where
\[ d_0 = \sup_{z \in D} |z - z_0|, \]
and $\Sigma_\varepsilon$ denotes the family of all intersections of the circles
\[ S(r) = S(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}, \quad r \in (\varepsilon, \varepsilon_0), \]
with the domain $D$.

The notion can be extended to the case $z_0 = \infty \in \overline{D}$ in the standard way by applying the inversion $T$ with respect to the unit circle in $\overline{\mathbb{C}}$, $T(x) = z/|z|^2$, $T(\infty) = 0$, $T(0) = \infty$. Namely, a homeomorphism $f : D \to D'$ is a \textbf{lower $Q$-homeomorphism at} $\infty \in \overline{D}$ if $F = f \circ T$ is a lower $Q_*$-homeomorphism with $Q_* = Q \circ T$ at 0. We also say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a \textbf{lower $Q$-homeomorphism in} $\partial D$ if $f$ is a lower $Q$-homeomorphism at every point $z_0 \in \partial D$.

Further we show that every homeomorphism of finite distortion in the plane is a lower $Q$-homeomorphism with $Q(z) = K_f(z)$ and, thus, the whole theory of the boundary behavior in [12], see also Chapter 9 in [23], can be applied.
2 Preliminaries

Recall first of all the following topological notion. A domain $D \subset \mathbb{C}$ is said to be **locally connected at a point** $z_0 \in \partial D$ if, for every neighborhood $U$ of the point $z_0$, there is a neighborhood $V \subseteq U$ of $z_0$ such that $V \cap D$ is connected. Note that every Jordan domain $D$ in $\mathbb{C}$ is locally connected at each point of $\partial D$, see e.g. [35], p. 66.

We say that $\partial D$ is **weakly flat at a point** $z_0 \in \partial D$ if, for every neighborhood $U$ of the point $z_0$ and every number $P > 0$, there is a neighborhood $V \subset U$ of $z_0$ such that

\[ M(\Delta(E, F; D)) \geq P \]  

for all continua $E$ and $F$ in $D$ intersecting $\partial U$ and $\partial V$. Here and later on, $\Delta(E, F; D)$ denotes the family of all paths $\gamma : [a, b] \rightarrow \overline{\mathbb{C}}$ connecting $E$ and $F$ in $D$, i.e. $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for all $t \in (a, b)$. We say that the boundary $\partial D$ is **weakly flat** if it is weakly flat at every point in $\partial D$.

We also say that a point $z_0 \in \partial D$ is **strongly accessible** if, for every neighborhood $U$ of the point $z_0$, there exist a compactum $E$ in $D$, a neighborhood $V \subset U$ of $z_0$ and a number $\delta > 0$ such that

\[ M(\Delta(E, F; D)) \geq \delta \]  

for all continua $F$ in $D$ intersecting $\partial U$ and $\partial V$. We say that the boundary $\partial D$ is **strongly accessible** if every point $z_0 \in \partial D$ is strongly accessible.
Here, in the definitions of strongly accessible and weakly flat boundaries, one can take as neighborhoods $U$ and $V$ of a point $z_0$ only balls (closed or open) centered at $z_0$ or only neighborhoods of $z_0$ in another fundamental system of neighborhoods of $z_0$. These conceptions can also be extended in a natural way to the case of $\overline{\mathbb{C}}$ and $z_0 = \infty$. Then we must use the corresponding neighborhoods of $\infty$.

It is easy to see that if a domain $D$ in $\mathbb{C}$ is weakly flat at a point $z_0 \in \partial D$, then the point $z_0$ is strongly accessible from $D$. Moreover, it was proved by us that if a domain $D$ in $\mathbb{C}$ is weakly flat at a point $z_0 \in \partial D$, then $D$ is locally connected at $z_0$, see e.g. Lemma 5.1 in [12] or Lemma 3.15 in [23].

The notions of strong accessibility and weak flatness at boundary points of a domain in $\mathbb{C}$ defined in [11] are localizations and generalizations of the corresponding notions introduced in [21]–[22], cf. with the properties $P_1$ and $P_2$ by Väisälä in [33] and also with the quasiconformal accessibility and the quasiconformal flatness by Nääkkä in [26]. Many theorems on a homeomorphic extension to the boundary of quasiconformal mappings and their generalizations are valid under the condition of weak flatness of boundaries. The condition of strong accessibility plays a similar role for a continuous extension of the mappings to the boundary. In particular, recently we have proved the following significant statements, see either Theorem 10.1 (Lemma 6.1) in [12] or Theorem 9.8 (Lemma 9.4) in [23].

**Proposition 2.1.** Let $D$ and $D'$ be bounded domains in $\mathbb{C}$, $Q : D \to (0, \infty)$ a measurable function and $f : D \to D'$ a lower $Q$-homeomorphism in $\partial D$. Suppose that the domain $D$ is locally connected on $\partial D$ and that the domain $D'$ has a (strongly accessible) weakly flat boundary. If

$$\int_0^{\delta(z_0)} \frac{dr}{||Q||_1(z_0, r)} = \infty \quad \forall \ z_0 \in \partial D \quad (2.3)$$

for some $\delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$ and

$$||Q||_1(z_0, r) = \int_{D \cap \mathcal{S}(z_0, r)} Q(z) \, ds ,$$

then $f$ has a (continuous) homeomorphic extension $\overline{f}$ to $\overline{D}$ that maps $\overline{D}$ (into) onto $\overline{D'}$.

Here as usual $\mathcal{S}(z_0, r)$ denotes the circle $|z - z_0| = r$.

A domain $D \subset \mathbb{C}$ is called a **quasiextremal distance domain**, abbr. **QED-domain**, see [7], if

$$M(\Delta(E, F; \overline{\mathbb{C}}) \leq K \cdot M(\Delta(E, F; D)) \quad (2.4)$$

where $\Delta(E, F; \overline{\mathbb{C}})$ is the extremal length of $E$ with respect to $F$. This condition is equivalent to the extremal distance of $E$ with respect to $F$ being bounded by $\frac{1}{K}$ times the extremal distance of $E$ with respect to $D$, see [7].
for some $K \geq 1$ and all pairs of nonintersecting continua $E$ and $F$ in $D$.

It is well known, see e.g. Theorem 10.12 in [33], that

$$M(\Delta(E, F; \mathbb{C})) \geq \frac{2}{\pi} \log \frac{R}{r}$$

(2.5)

for any sets $E$ and $F$ in $\mathbb{C}$ intersecting all the circles $S(z_0, \rho)$, $\rho \in (r, R)$. Hence a QED-domain has a weakly flat boundary. One example in [23], Section 3.8, shows that the inverse conclusion is not true even among simply connected plane domains.

A domain $D \subset \mathbb{C}$ is called a uniform domain if each pair of points $z_1$ and $z_2 \in D$ can be joined with a rectifiable curve $\gamma$ in $D$ such that

$$s(\gamma) \leq a \cdot |z_1 - z_2|$$

(2.6)

and

$$\min_{i=1,2} s(\gamma(z_i, z)) \leq b \cdot d(z, \partial D)$$

(2.7)

for all $z \in \gamma$ where $\gamma(z_i, z)$ is the portion of $\gamma$ bounded by $z_i$ and $z$, see [24]. It is known that every uniform domain is a QED-domain but there exist QED-domains that are not uniform, see [7]. Bounded convex domains and bounded domains with smooth boundaries are simple examples of uniform domains and, consequently, QED-domains as well as domains with weakly flat boundaries.

A closed set $X \subset \mathbb{C}$ is called a null-set for extremal distances, abbr. NED-set, if

$$M(\Delta(E, F; \mathbb{C})) = M(\Delta(E, F; \mathbb{C}\setminus X))$$

(2.8)

for any two nonintersecting continua $E$ and $F \subset \mathbb{C}\setminus X$.

Remark 2.1. It is known that if $X \subset \mathbb{C}$ is a NED-set, then

$$|X| = 0$$

(2.9)

and $X$ does not locally separate $\mathbb{C}$, see [34], i.e.,

$$\dim X \leq 0,$$

(2.10)

and hence they are totally disconnected, see e.g. p. 22 and 104 in [9]. Conversely, if a set $X \subset \mathbb{C}$ is closed and is of length zero,

$$H^1(X) = 0,$$

(2.11)

then $X$ is a NED-set, see [34]. Note also that the complement of a NED-set in $\mathbb{C}$ is a very particular case of a QED-domain.
Here $H^1(X)$ denotes the 1-dimensional Hausdorff measure (length) of a set $X$ in $\mathbb{C}$. Also we denote by $C(X, f)$ the cluster set of the mapping $f : D \to \overline{\mathbb{C}}$ for a set $X \subset \overline{D}$,

$$C(X, f) : = \left\{ w \in \overline{\mathbb{C}} : w = \lim_{k \to \infty} f(z_k), \ z_k \to z_0 \in X, \ z_k \in D \right\}. \quad (2.12)$$

Note that the inclusion $C(\partial D, f) \subseteq \partial D'$ holds for every homeomorphism $f : D \to D'$, see e.g. Proposition 13.5 in [23].

3 The main lemma

**Theorem 3.1.** Let $f : D \to \mathbb{C}$ be a homeomorphism with finite distortion. Then $f$ is a lower $Q$-homeomorphism at each point $z_0 \in \overline{D}$ with $Q(z) = K_f(z)$.

**Proof.** Let $B$ be a (Borel) set of all points $z$ in $D$ where $f$ has a total differential with $J_f(z) \neq 0$ a.e. It is known that $B$ is the union of a countable collection of Borel sets $B_l$, $l = 1, 2, \ldots$, such that $f_l = f|_{B_l}$ is a bi-Lipschitz homeomorphism, see e.g. Lemma 3.2.2 in [3]. With no loss of generality, we may assume that the $B_l$ are mutually disjoint. Denote also by $B_*$ the set of all points $z \in D$ where $f$ has a total differential with $f'(z) = 0$.

Note that the set $B_0 = D \setminus (B \cup B_*)$ has the Lebesgue measure zero in $\mathbb{C}$ by Gehring–Lehto–Menchoff theorem, see [6] and [19]. Hence by Theorem 2.11 in [13], see also Lemma 9.1 in [23], length$(\gamma \cap B_0) = 0$ for a.e. paths $\gamma$ in $D$. Let us show that length$(f(\gamma) \cap f(B_0)) = 0$ for a.e. circle $\gamma$ centered at $z_0$.

The latter follows from absolute continuity of $f$ on closed subarcs of $\gamma \cap D$ for a.e. such circle $\gamma$. Indeed, the class $W_{1,1}^{\text{loc}}$ is invariant with respect to local quasi-isometries, see e.g. Theorem 1.1.7 in [25], and the functions in $W_{1,1}^{\text{loc}}$ is absolutely continuous on lines, see e.g. Theorem 1.1.3 in [23]. Applying say the transformation of coordinates $\log(z - z_0)$, we come to the absolute continuity on a.e. such circle $\gamma$.

Thus, length$(\gamma_* \cap f(B_0)) = 0$ where $\gamma_* = f(\gamma)$ for a.e. circle $\gamma$ centered at $z_0$. Now, let $\varrho_* \in \text{adm} f(\Gamma)$ where $\Gamma$ is the collection of all dashed lines $\gamma \cap D$ for such circles $\gamma$ and $\varrho_* \equiv 0$ outside $f(D)$. Set $\varrho \equiv 0$ outside $D$ and

$$\varrho(z) : = \varrho_*(f(z)) (|f_\bar{z}| + |f_\bar{z}|) \quad \text{for a.e. } z \in D$$

Arguing piecewise on $B_l$, we have by Theorem 3.2.5 under $m = 1$ in [3] that

$$\int_\gamma \varrho \, ds \geq \int_{\gamma_*} \varrho_* \, ds_* \geq 1 \quad \text{for a.e. } \gamma \in \Gamma$$
because \( \text{length}(f(\gamma) \cap f(B_0)) = 0 \) and \( \text{length}(f(\gamma) \cap f(B_*)) = 0 \) for a.e. \( \gamma \in \Gamma \), consequently, \( \varrho \in \text{ext adm } \Gamma \).

On the other hand, again arguing piecewise on \( B_l \), we have the inequality
\[
\int_{D} \frac{\varrho^2(z)}{K_f(z)} \, dm(z) \leq \int_{f(D)} \tilde{\varrho}_*^2(w) \, dm(w)
\]
because \( J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \) and \( K_f(z) = (|f_z| + |f_{\bar{z}}|)/(|f_z| - |f_{\bar{z}}|) \) on \( B \) and \( K_f(z) = 1 \) and \( \varrho(z) = 0 \) on \( B_* \). Consequently, we obtain that
\[
M(f\Gamma) \geq \inf_{\varrho \in \text{ext adm } \Gamma} \int_{D} \frac{\varrho^2(z)}{K_f(z)} \, dm(z),
\]
i.e. \( f \) is really a lower \( Q \)-homeomorphism with \( Q(z) = K_f(z) \).

4 On the removability of isolated singularities

In view of Theorem 3.1 we obtain by Theorem 4.1 in [12] or Theorem 9.3 in [23] the following statement.

**Theorem 4.1.** Let \( D \) be a domain in \( \mathbb{C} \), \( z_0 \in D \), and \( f \) be a homeomorphism with finite distortion of \( D \setminus \{z_0\} \) into \( \overline{\mathbb{C}} \). Suppose that
\[
\int_{0}^{\varepsilon_0} \frac{dr}{r \cdot k_f(r)} = \infty \quad (4.1)
\]
where \( \varepsilon_0 < \text{dist}(z_0, \partial D) \) and
\[
k_f(r) = \int_{|z - z_0| = r} K_f(z) \, |dz| \quad (4.2)
\]
Then \( f \) has a continuous extension to \( D \) in \( \overline{\mathbb{C}} \).

From here we have, in particular, the following consequences.

**Corollary 4.1.** Let \( D \) be a domain in \( \mathbb{C} \) and let \( f \) be a homeomorphism with finite distortion of \( D \setminus \{z_0\} \) into \( \overline{\mathbb{C}} \). If
\[
\int_{|z - z_0| = r} K_f(z) \, |dz| = O\left(\log \frac{1}{r}\right) \quad \text{as } r \to 0, \quad (4.3)
\]
then $f$ has a continuous extension to $D$ in $\overline{C}$.

**Corollary 4.2.** Let $D$ be a domain in $\mathbb{C}$, $x_0 \in D$, and $f$ be a homeomorphism with finite distortion of $D \setminus \{z_0\}$ into $\overline{C}$. If

$$\int_{|z-z_0|=r} K_f(z) \, |dz| = O \left( \log \frac{1}{r} \cdot \log \log \frac{1}{r} \cdot \ldots \cdot \log \ldots \log \frac{1}{r} \right) \quad \text{as } r \to 0 \ ,$$

then $f$ has a continuous extension to $D$ in $\overline{C}$.

### 5 On a continuous extension to boundary points

In view of Theorem 3.1 we have by Theorem 6.1 in [12] or Lemma 9.4 in [23] the next statement.

**Lemma 5.1.** Let $D$ and $D'$ be domains in $\mathbb{C}$, $z_0 \in \partial D$, and $f : D \to D'$ be a homeomorphism with finite distortion. Suppose that the domain $D$ is locally connected at $z_0 \in \partial D$ and $\partial D'$ is strongly accessible at least at one point of the cluster set $C(z_0, f)$. If

$$\int_0^{\varepsilon_0} \frac{dr}{\| K_f \|_1(r)} = \infty \quad (5.1)$$

where $0 < \varepsilon_0 < d_0 = \sup_{z \in D} |z - z_0|$, and

$$\| K_f \|_1(r) = \int_{D \cap S(z_0, r)} K_f \, ds \ , \quad (5.2)$$

then $f$ extends by continuity to $z_0$ in $\overline{C}$.

In particular, we have the following consequence of Lemma 5.1.

**Corollary 5.1.** Let $D$ and $D'$ be QED domains in $\mathbb{C}$, $z_0 \in \partial D$, and $f : D \to D'$ be a homeomorphism of finite distortion. If (5.1) holds, then $f$ extends by continuity to $z_0$ in $\overline{C}$.

Note that the complements of NED sets in $\mathbb{C}$ give very particular cases of QED domains. Thus, arguing locally, by Theorem 5.1, we obtain the following statement.
Theorem 5.1. Let $D$ be a domain in $\mathbb{C}$, $X \subset D$, and $f$ be a homeomorphism with finite distortion of $D \setminus X$ into $\overline{\mathbb{C}}$. Suppose that $X$ and $C(X, f)$ are NED sets. If

$$\int_0^{\varepsilon_0} \frac{dr}{\|K_f\|_1(r)} = \infty$$

(5.3)

where

$$0 < \varepsilon_0 < d_0 = \text{dist} (z_0, \partial D)$$

(5.4)

and

$$\|K_f\|_1(r) = \int_{|z-z_0|=r} K_f(z) \, |dz|,$$  

(5.5)

then $f$ can be extended by continuity in $\overline{\mathbb{C}}$ to $z_0$.

6 The extension of the inverse mappings to the boundary

The base of the proof for extending the inverse mappings for homeomorphisms of finite distortion is the following lemma on the cluster sets.

Lemma 6.1. Let $D$ and $D'$ be domains in $\mathbb{C}$, $z_1$ and $z_2$ be distinct points in $\partial D$, $z_1 \neq \infty$, and let $f$ be a homeomorphism with finite distortion of $D$ onto $D'$. Suppose that the function $K_f$ is integrable on the dashed lines

$$D(r) = \{ z \in D : |z - z_1| = r \} = D \cap S(z_1, r)$$

(6.1)

for some set $E$ of numbers $r < |z_1 - z_2|$ of a positive linear measure. If $D$ is locally connected at $z_1$ and $z_2$ and $\partial D'$ is weakly flat, then

$$C(z_1, f) \cap C(z_2, f) = \emptyset.$$  

(6.2)

The of Lemma 6.1 follows by Theorem 3.1 from Lemma 9.1 in [12] or Lemma 9.5 in [23].

As an immediate consequence of Lemma 6.1, we have the following statement.

Theorem 6.1. Let $D$ and $D'$ be domains in $\mathbb{C}$, $D$ locally connected on $\partial D$ and $\partial D'$ weakly flat. If $f$ is a homeomorphism with finite distortion of $D$ onto $D'$ with $K_f \in L^1(D)$, then $f^{-1}$ has an extension by continuity in $\overline{\mathbb{C}}$ to $\overline{D'}$.

Proof. By the Fubini theorem, the set

$$E = \{ r \in (0, d) : K_f|_{D(r)} \in L^1(D(r)) \}$$

(6.3)
has a positive linear measure because $K_f \in L^1(D)$.

**Remark 6.1.** It is clear from the proof that it is even sufficient to assume in Theorem 6.1 that $K_f$ is integrable only in a neighborhood of $\partial D$.

Moreover, in view of Theorem 3.1 we obtain by Theorem 9.2 in [12] or Theorem 9.7 in [23] the following conclusion.

**Theorem 6.2.** Let $D$ and $D'$ be domains in $\mathbb{C}$, $D$ locally connected on $\partial D$ and $\partial D'$ weakly flat, and let $f : D \to D'$ be a homeomorphism with finite distortion such that the condition

$$
\delta(z_0) \int_0^r \frac{dr}{\|K_f\|_1(z_0, r)} = \infty
$$

holds for all $z_0 \in \partial D$ with some $\delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$ and

$$
\|K_f\|_1(z_0, r) = \int_{D(z_0, r)} K_f \, ds
$$

is the $L_1$-norm of $K_f$ over $D(z_0, r) = \{z \in D : |z - z_0| = r\} = D \cap S(z_0, r)$. Then there is an extension of $f^{-1}$ by continuity in $\overline{C}$ to $\overline{D'}$.

7 On homeomorphic extension to the boundary

Combining Lemma 5.1 and Theorem 6.2, we obtain the following statements.

**Theorem 7.1.** Let $D$ and $D'$ be bounded domains in $\mathbb{C}$ and let $f : D \to D'$ be a homeomorphism with finite distortion in $D$. Suppose that the domain $D$ is locally connected on $\partial D$ and that the domain $D'$ has a weakly flat boundary. If

$$
\delta(z_0) \int_0^r \frac{dr}{\|K_f\|_1(z_0, r)} = \infty \quad \forall \ z_0 \in \partial D
$$

for some $\delta(z_0) \in (0, d(z_0))$ where $d(z_0) = \sup_{z \in D} |z - z_0|$ and

$$
\|K_f\|_1(z_0, r) = \int_{D \cap S(z_0, r)} K_f \, ds
$$

then $f$ has a homeomorphic extension to $\overline{D}$.
In particular, as a consequence of Theorem 7.1 we obtain the following generalization of the well-known Gehring-Martio theorem on a homeomorphic extension to the boundary of quasiconformal mappings between QED domains, see [7].

**Corollary 7.1.** Let $D$ and $D'$ be bounded domains with weakly flat boundaries in $\mathbb{C}$ and let $f : D \to D'$ be a homeomorphism with finite distortion in $D$. If the condition (7.1) holds at every point $z_0 \in \partial D$, then $f$ has a homeomorphic extension to $\overline{D}$.

By Theorem 3.1 we have also the following, see Theorem 10.3 in [12] or Theorem 9.10 in [23].

**Theorem 7.2.** Let $D$ be a bounded domain in $\mathbb{C}$, $X \subset D$, and $f : D \setminus \{X\} \to \mathbb{C}$ a homeomorphism with finite distortion. Suppose that $X$ and $C(X, f)$ are NED sets. If the condition (7.1) holds at every point $z_0 \in X$ for $\delta(z_0) < \text{dist}(z_0, \partial D)$ where

$$||K_f||_1(z_0, r) = \int_{|z-z_0|=r} K_f(z) \, |dz|, \quad (7.3)$$

then $f$ has a homeomorphic extension to $D$.

**Remark 7.1.** In particular, the conclusion of Theorem 7.2 is valid if $X$ is a closed set with

$$H^1(X) = 0 = H^1(C(X, f)). \quad (7.4)$$

### 8 On some integral conditions

Recall theorems on interconnections between some integral conditions from [29] and [30].

For every non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$, the **inverse function** $\Phi^{-1} : [0, \infty] \to [0, \infty]$ can be well defined by setting

$$\Phi^{-1}(\tau) = \inf_{\Phi(t) \geq \tau} t. \quad (8.1)$$

Here inf equal to $\infty$ if the set of $t \in [0, \infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function $\Phi^{-1}$ is non-decreasing, too.

Further, the integral in (8.4) is understood as the Lebesgue–Stieltjes integral and the integrals in (8.3) and (8.5)–(8.8) as the ordinary Lebesgue integrals. In (8.3) and (8.4) we complete the definition of integrals by $\infty$ if $\Phi(t) = \infty$, correspondingly,
$H(t) = \infty$, for all $t \geq T \in [0, \infty)$. 

**Theorem 8.1.** Let $\Phi : [0, \infty] \to [0, \infty]$ be a non-decreasing function and set 

$$H(t) = \log \Phi(t).$$  \hspace{1cm} (8.2) 

Then the equality 

$$\int_{\Delta}^{\infty} \frac{H'(t)}{t} \, dt = \infty$$  \hspace{1cm} (8.3) 

implies the equality 

$$\int_{\Delta}^{\infty} \frac{dH(t)}{t} = \infty$$  \hspace{1cm} (8.4) 

and (8.4) is equivalent to 

$$\int_{\Delta}^{\infty} \frac{H(t)}{t^2} \, dt = \infty$$  \hspace{1cm} (8.5) 

for some $\Delta > 0$, and (8.4) is equivalent to every of the equalities: 

$$\int_{0}^{\delta} H \left( \frac{1}{t} \right) \, dt = \infty$$  \hspace{1cm} (8.6) 

for some $\delta > 0$, 

$$\int_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty$$  \hspace{1cm} (8.7) 

for some $\Delta_* > H(+0)$, 

$$\int_{\delta_*}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty$$  \hspace{1cm} (8.8) 

for some $\delta_* > \Phi(+0)$.

Moreover, (8.3) is equivalent to (8.4) and hence (8.3)–(8.8) are equivalent each to other if $\Phi$ is in addition absolutely continuous. In particular, all the conditions (8.3)–(8.8) are equivalent if $\Phi$ is convex and non-decreasing.

**Remark 8.1.** It is necessary to give one more explanation. From the right hand sides in the conditions (8.3)–(8.8) we have in mind $+\infty$. If $\Phi(t) = 0$ for $t \in [0, t_*]$, then 

$H(t) = -\infty$ for $t \in [0, t_*]$ and we complete the definition $H'(t) = 0$ for $t \in [0, t_*]$. Note, the conditions (8.4) and (8.5) exclude that $t_*$ belongs to the interval of integrability.
because in the contrary case the left hand sides in (8.4) and (8.5) are either equal to $-\infty$ or indeterminate. Hence we may assume in (8.3)–(8.6) that $\Delta > t_0$ where $t_0 := \sup t$, $t_0 = 0$ if $\Phi(0) > 0$, and $\delta < 1/t_0$, correspondingly.

**Theorem 8.2.** Let $Q : \mathbb{D} \to [0, \infty]$ be a measurable function such that

$$\int_{\mathbb{D}} \Phi(Q(z)) \, dx \, dy < \infty$$  \hspace{1cm} (8.9)

where $\Phi : [0, \infty] \to [0, \infty]$ is a non-decreasing convex function such that

$$\int_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty$$  \hspace{1cm} (8.10)

for some $\delta_0 > \Phi(0)$. Then

$$\int_{0}^{1} \frac{dr}{rq(r)} = \infty$$  \hspace{1cm} (8.11)

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$.

Here $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$. Combining Theorems 8.1 and 8.2 we obtain also the following.

**Corollary 8.1.** If $\Phi : [0, \infty] \to [0, \infty]$ is a non-decreasing convex function and $Q : \mathbb{D} \to [0, \infty]$ satisfies (8.9), then every of the conditions (8.3)–(8.8) implies (8.11).

**9 On the mappings quasiconformal in the mean**

Integral conditions of the type

$$\int_{\mathbb{D}} \Phi(K(x)) \, dm(x) < \infty$$  \hspace{1cm} (9.1)

are often applied in the mapping theory, see e.g. [1], [2], [8], [15], [18], [27a], [28] and [32].

Combining Theorem 8.2 with Lemma 5.1 and Theorem 7.1, we come to the following statement.
Theorem 9.1. Let $D$ and $D'$ be bounded domains in $\mathbb{C}$ such that $D$ is locally connected at $\partial D$ and $D'$ has a weakly flat (strongly accessible) boundary. Suppose that $f : D \to D'$ is a homeomorphism with finite distortion and
\[
\int_D \Phi(K_f(z)) \, dm(z) < \infty \tag{9.2}
\]
for a convex non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$. If
\[
\int_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{9.3}
\]
for some $\delta_0 > \Phi(0)$, then $f$ has a homeomorphic (continuous) extension $\overline{f}$ to $\overline{D}$ that maps $\overline{D}$ onto (into) $\overline{D'}$.

Remark 9.1. In particular, the conclusion on homeomorphic extension is valid for domains $D$ and $D'$ with smooth boundaries and for convex domains. Note also that by Theorem 8.1 the condition (9.3) can be replaced by each of the conditions (8.3)–(8.7). The example in [14] shows that each of the given conditions are not only sufficient but also necessary for continuous extension of $f$ to the boundary.

References

[1] Ahlfors L.: On quasiconformal mappings. J. Analyse Math. 3, 1–58 (1953/54).
[2] Biluta P.A.: Extremal problems for mappings which are quasiconformal in the mean. Sib. Mat. Zh. 6, 717–726 (1965).
[3] Federer H.: Geometric Measure Theory. Springer-Verlag, Berlin (1969).
[4] Fuglede B.: Extremal length and functional completion. Acta Math. 98, 171–219 (1957).
[5] Gehring F.W.: Rings and quasiconformal mappings in space. Trans. Amer. Math. Soc. 103, 353–393 (1962).
[6] Gehring F.W., Lehto O.: On the total differentiability of functions of a complex variable. Ann. Acad. Sci. Fenn. A1. Math. 272, 1–9 (1959).
[7] Gehring F.W., Martio O.: Quasiextremal distance domains and extension of quasiconformal mappings. J. Anal. Math. 45, 181–206 (1985).
[8] Golberg A.: Homeomorphisms with finite mean dilatations. Contemporary Math. 382, 177–186 (2005).

[9] Hurewicz W., Wallman H.: Dimension theory. Princeton Univ. Press, Princeton, NJ (1948).

[10] Iwaniec T., Martin G.: Geometrical Function Theory and Non-linear Analysis. Clarendon Press, Oxford (2001).

[11] Kovtonyuk D., Ryazanov V.: On boundaries of space domains. Proc. Inst. Appl. Math. & Mech. NAS of Ukraine 13, 110–120 (2006) [in Russian].

[12] Kovtonyuk D., Ryazanov V.: On the theory of lower $Q$-homeomorphisms. Ukrainian Math. Bull. 5 (2), 157–181 (2008).

[13] Kovtonyuk D., Ryazanov V.: On the theory of mappings with finite area distortion. J. Anal. Math., 104, 291–306 (2008).

[14] Kovtonyuk D., Ryazanov V.: On the boundary behavior of generalized quasi-isometries. ArXiv: 1005.0247, 20 p. (2010)

[15] Kruglikov V.I.: Capacities of condensors and quasiconformal in the mean mappings in space. Mat. Sb. 130 (2) (1986), 185–206.

[16] Krushkal’ S.L.: On mappings that are quasiconformal in the mean. Dokl. Akad. Nauk SSSR 157 (3), 517–519 (1964).

[17] Krushkal’ S.L., Kühnau R.: Quasiconformal mappings: new methods and applications, Novosibirsk, Nauka (1984) (in Russian).

[18] Kühnau R.: Über Extremalprobleme bei im Mittel quasiconformen Abbildungen. Lecture Notes in Math. 1013, 113–124 (1983) (in German).

[19] Menchoff D.: Sur les differentielles totales des fonctions univalentes. Math. Ann. 105, 75–85 (1931).

[20] Martio O., Ryazanov V., Srebro U., Yakubov E.: Mappings with finite length distortion. J. d’Anal. Math. 93, 215–236 (2004).

[21] Martio O., Ryazanov V., Srebro U., Yakubov E.: $Q$-homeomorphisms. Contemporary Math. 364, 193–203 (2004).

[22] Martio O., Ryazanov V., Srebro U., Yakubov E.: On $Q$-homeomorphisms. Ann. Acad. Sci. Fenn. 30, 49–69 (2005).

[23] Martio O., Ryazanov V., Srebro U., Yakubov E.: Moduli in Modern Mapping Theory. Springer, New York (2009).
[24] Martio O., Sarvas J.: Injectivity theorems in plane and space. Ann. Acad. Sci. Fenn. Ser. A1 Math. **4**, 384–401 (1978/1979).

[25] Maz’ya V.: Sobolev Classes. Springer-Verlag, Berlin (1985).

[26] Nakki R.: Boundary behavior of quasiconformal mappings in $n$–space. Ann. Acad. Sci. Fenn. Ser. A1. Math. **484**, 1–50 (1970).

[27] Pesin I.N.: Mappings quasiconformal in the mean. Dokl. Akad. Nauk SSSR **187** (4), 740–742 (1969).

[28] Ryazanov V.I.: On mappings that are quasiconformal in the mean. Sibirsk. Mat. Zh. **37** (2), 378–388 (1996).

[29] Ryazanov V., Srebro U., Yakubov E.: Integral conditions in the theory of the Beltrami equations. ArXiv 1001.2821v11, 26 p. (2010)

[30] Ryazanov V., Srebro U., Yakubov E.: Integral conditions in the mapping theory. Ukrainian Math. Bull. **7**, 73–87 (2010).

[31] Sobolev S.L.: Applications of functional analysis in mathematical physics. Izdat. Gos. Univ., Leningrad (1950); English transl, Amer. Math. Soc., Providence, R.I. (1963).

[32] Ukhlov A., Vodopyanov S.K.: Mappings associated with weighted Sobolev spaces. Complex Anal. Dynam. Syst. III, Contemp. Math. **455**, 369–382 (2008).

[33] Väisälä J.: Lectures on $n$-Dimensional Quasiconformal Mappings. Lecture Notes in Math. **229**, Springer–Verlag, Berlin etc. (1971).

[34] Väisälä J.: On the null-sets for extremal distances. Ann. Acad. Sci. Fenn. Ser. A1. Math. **322**, 1–12 (1962).

[35] Wilder R.L.: Topology of Manifolds. AMS, New York (1949).

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