BILATERAL SMALL LEBESGUE SPACES.

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Abstract.

In this article we investigate the so-called Bilateral Small Lebesgue Spaces: prove that they are associated to the Grand Lebesgue spaces, calculate its fundamental functions and Boyd’s indices, find its dual spaces etc.

Key words and phrases: Grand and Small Lebesgue spaces, rearrangement invariant (r.i.) spaces, fundamental functions, Boyd’s index, associate and dual spaces.

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1. Definitions. Preliminaries. Grand Lebesgue Spaces.

Let \((X, \Sigma, \mu)\) be a measurable space with non-trivial measure \(\mu: \exists A \in \Sigma, \mu(A) \in (0, \mu(X))\). We will consider in this article the following conditions on the measure \(\mu\) or more exactly on the measurable space \((X, \Sigma, \mu)\):

1. **Finiteness, or the probabilistic case:** \(\mu(X) = 1\).
It is easy to understand that the case \(\mu(X) \in (0, \infty)\) can be considered by means of renorming of measure.

2. **Sigma-finiteness:** There exists the sequence \(\{A(n), A(n) \in \Sigma, \}\) such that
\[
\mu(A(n)) \in (0, \infty), \ \cup_{n=1}^{\infty} A(n) = X.
\]

3. **Diffuseness:**
\[
\forall A \in \Sigma, 0 < \mu(A) < \infty \ \exists B \subset A, \mu(B) = \mu(A)/2.
\]

4. **Separability:** The metric space \((\Sigma, \rho)\), where the distance \(\rho\) is defined as usually as
\[
\rho(A, B) = \mu(A \setminus B) + \mu(B \setminus A), \ A, B \in \Sigma
\]
is separable.

5. **Resonant property.** This means that the measure \(\mu\) is nonatomic or the set \(X\) consists only on the countable (finite case is trivial) set of points with equal non-zero finite measure.
More information about these definitions see in the classical monograph M.M.Rao [23].

We suppose in this report the measure µ is sigma-finite.

Define as usually for arbitrary measurable function \( f : X \to \mathbb{R} \)

\[
|f|_p = \left( \int_X |f(x)|^p \mu(dx) \right)^{1/p}, \quad p \geq 1;
\]

\( L_p = L(p) = L(p; X, \mu) = \{ f, |f|_p < \infty \} \). Let \( a = \text{const} \geq 1, b = \text{const} \in (a, \infty) \), and let \( \psi = \psi(p) \) be some positive continuous on the open interval \((a, b)\) function, such that there exists a measurable function \( f : X \to \mathbb{R} \) for which

\[
f(\cdot) \in \cap_{p \in (a,b)} L_p, \quad \psi(p) = |f|_p, \quad p \in (a,b).
\]

We will say that the equality (1.0) and the function \( f(\cdot) \) from (1.0) is the representation of the function \( \psi \).

The set of all those functions we will denote \( \Psi : \Psi = \Psi(a, b) = \{ \psi(\cdot) \} \). The complete description of this functions see, for example, in [17], p. 21-27, [18].

We extend the set \( \Psi \) as follows: for arbitrary \( \psi(\cdot) \in \Psi \) define

\[
EX \Psi \overset{\text{def}}{=} EX \Psi(a, b) = \{ \nu = \nu(p) \} = \{ \nu : \exists \psi(\cdot) \in \Psi : 0 < \inf_{p \in (a, b)} \psi(p)/\nu(p) \leq \sup_{p \in (a, b)} \psi(p)/\nu(p) < \infty \},
\]

\[ U \Psi \overset{\text{def}}{=} U \Psi(a, b) = \{ \psi = \psi(p), \inf_{p \in (a, b)} \psi(p) > 0 \} \]

and the function \( p \to \psi(p), \quad p \in (a, b) \) is continuous.

We define in the case \( b = \infty \) \( \psi(b - 0) = \lim_{p \to \infty} \psi(p) \).

**Definition 1.** Let \( \psi(\cdot) \in U \Psi(a, b) \). The space \( BSGL(\psi) = G(\psi) = G(X, \psi) = G(X, \psi, \mu) = G(X, \psi, \mu, a, b) \) (Bilateral Grand Lebesgue Space) consist on all the measurable functions \( f : X \to \mathbb{R} \) with finite norm

\[
||f|| G(\psi) \overset{\text{def}}{=} \sup_{p \in (a,b)} [||f|_p/\psi(p)||].
\]

We denote \( B(\psi) = \{ p : |\psi(p)| < \infty \} \) the bounded part of a function \( f \).

Note that if \( \max(\psi(a + 0), \psi(b - 0)) < \infty, b < \infty \), then the space \( G(\psi) \) coincides with the direct sum \( L_a + L_b \), and if \( \min(\psi(a + 0), \psi(b - 0)) = 0 \), then \( G(\psi) = \{0\} \).

**Hence, we can and will assume further that**

\[
\max(\psi(a + 0), \psi(b - 0)) = \infty, \quad \inf_{p \in [a, b]} \psi(p) > 0.
\]

In the considered case the spaces \( G(\psi), \quad \psi \in U \Psi \) are non-trivial: arbitrary bounded: vraisup \( |f(x)| < \infty \) measurable function \( f : X \to \mathbb{R} \) with finite support:

\[
\mu(\text{supp}(f)) < \infty, \quad \text{supp}(f) \overset{\text{def}}{=} \{ x, |f(x)| > 0 \}
\]

belongs to arbitrary space \( G(\psi) \forall \psi \in U \Psi \).
The Bilateral Small Lebesgue space \( \text{BSL}(\psi) = \text{SL}(\psi) = S(\psi) \) may be defined as an associate space to the r.i. Grand Lebesgue space

\[
\text{BSL}(\psi) = S(\psi) = \text{SL}(\psi) \overset{\text{def}}{=} G'(\psi).
\]  

Recall that the associate space to the Banach functional space \( B \) over introduced measurable space is defined as a (closed) linear subspace of dual (or conjugate) space \( B^* \), consisting on all the functionals of a view:

\[
l(f) = l_g(f) = \int_X f(x) \, g(x) \, d\mu, \; f \in B, \; g \in B'.
\]  

We may say that the function \( g \) generates the functional \( l_g \) and will identify as ordinary the functional \( l_g \) and the function \( g \).

We investigate in this paper some properties of the Bilateral Small Lebesgue \( \text{BSL} \) spaces: find its explicit view, give some examples of norm calculations and estimations for the functions belonging to this spaces, prove that they obeys the Fatou, Lebesgue and Absolute continuous norm property, state its separability and non-reflexivity, find its fundamental functions and Boyd’s indices, conditions for convergence etc.

The Small Lebesgue spaces were intensively studied in the last ten years, see e.g.,[2], [5], [6], [7], [8], [9], [10] etc.

We intend to continue this investigations: consider some examples, calculate Boyd’s indices etc., and generalize to the case of bilateral spaces.

This means that we can consider the case when \( p \in (a, b) \), where \( a \) may differ from 1 and \( b \) may differ from \( \infty \), i.e. we may consider not only the cases \( p \in [1, p_0) \) or \( p \in (p_0, \infty) \).

For example, the space \( L^b, b \in [1, \infty) \) (in the notations of the articles [2], [5], [6], [7]) coincides with our space \( G(1, b, 0, 1/b) \).

All the properties of the BSL spaces which have a proof alike one in the case one-side spaces, for instance, provided in the articles [2], [5], [6], [7], [8], [9], [10], will be described here very briefly.

Note that the \( G(\psi) \) spaces are the particular case of interpolation spaces (so-called \( \Sigma- \) spaces) [3], [4], [11], [12], [25]. But we hope that the our direct representation of these spaces (definition 3) is more convenient for investigation and application.

In the probabilistic case \( \mu(X) = 1 \) the spaces \( G(\psi) \) spaces appeared in the article [14], where are applied to the theory of random fields.

Now we consider a very important for further considerations the examples of \( G(\psi) \) spaces. Let \( a = \text{const} \geq 1, b = \text{const} \in (a, \infty]; \alpha, \beta = \text{const} \). Assume also that at \( b < \infty \) \( \min(\alpha, \beta) \geq 0 \) and denote by \( h \) the (unique) root of equation

\[
(h - a)^\alpha = (b - h)^\beta, \; a < h < b; \; \zeta(p) = \zeta(a, b; \alpha, \beta; p) =
\]

\[
(p - a)^\alpha, \; p \in (a, h); \; \zeta(a, b; \alpha, \beta; p) = (b - p)^\beta, \; p \in [h, b);
\]

and in the case \( b = \infty \) assume that \( \alpha \geq 0, \beta < 0 \); denote by \( h \) the (unique) root of equation \( (h - a)^\alpha = h^\beta, h > a \); define in this case
\[ \zeta(p) = \zeta(a, b; \alpha, \beta; p) = (p-a)^\alpha, \quad p \in (a, h); \quad p \geq h \Rightarrow \zeta(p) = p^\beta. \]

Note that at \( b = \infty \) \( \Rightarrow \zeta(p) \propto (p-a)^\alpha p^{-\alpha+\beta} \propto \min\{(p-a)^\alpha, p^\beta\}, \quad p \in (a, \infty) \) and that at \( b < \infty \) \( \Rightarrow \zeta(p) \propto (p-a)^\alpha (b-p)^\beta \propto \min\{(p-a)^\alpha, (b-p)^\beta\}, \quad p \in (a, b). \)

In the case \( \alpha = 0, b < \infty \) we define \( \zeta(p) = (b-p)^\beta, \quad p \in (a, b); \) analogously, if \( \beta = 0, b < \infty \) \( \zeta(p) = (p-a)^\alpha, \quad p \in (a, b). \)

We will denote also by the symbols \( C_j, j \geq 1 \) some "constructive" finite non-essentially positive constants which does not depend on the \( p, n, x \) etc. By definition, the indicator function of a measurable set \( A, A \in \Sigma \) may be defined as usually:

\[ I(A) = I(A, x) = I(x \in A) = 1, \quad x \in A; \quad I(A) = 0, \quad x \notin A. \]

**Definition 2.**

The space \( G = G_X = G_X(a; b; \alpha, \beta) = G(a, b) = G(a, b; \alpha, \beta) \) consists on all measurable functions \( f : X \to \mathbb{R}^1 \) with finite norm

\[ ||f||G(a, b; \alpha, \beta) = \sup_{\mu \in (a, b)} ||f||_p \cdot \zeta(a, b; \alpha, \beta; p). \quad (1.3) \]

On the other word, the space \( G_X(a; b; \alpha, \beta) \) is the particular case of \( G(\psi) \) spaces with the \( \psi \) function of a view: \( \psi(p) = 1/\zeta(p) \), and \( p \in (a, b). \)

Notice that if \( \psi \in \Psi, \quad p \in (a, b), \quad b < \infty, \) and

\[ \psi(p) \sim (p - a)^{-\alpha}, \quad p \to a + 0; \quad \psi(p) \sim (b - p)^{-\beta}, \]

\( p \to b \) \( 0, \) \( \alpha, \beta \geq 0, \) then the space \( G(\psi, a, b) \) is equivalent to the space \( G(a, b; \alpha, \beta). \)

**Corollary 2.** As long as the cases \( \alpha \leq 0, b < \infty, \beta \leq 0, \) and \( b = \infty, \beta \geq 0, \) are trivial, we will assume further that either \( 1 \leq a < b < \infty, \min(\alpha, \beta) > 0, \) or \( 1 \leq a, b = \infty, \alpha \geq 0, \beta < 0. \)

The introduced \( G(\psi) \) spaces are some generalization of the so-called Grand Lebesque spaces, see [16], [18].

The complete description of the spaces conjugated to the (linear topological) spaces \( \cap_p L_p, \) see in [4], [25]. The spaces which are conjugate to Orlicz’s spaces are described, e.g., in [21], pp. 128-135, [22], chapter 3.

We denote by \( G^0 = G^0_X(\psi), \quad \psi \in U\Psi \) the closed subspace of \( G(\psi) \), consisting on all the functions \( f, \) satisfying the following condition:

\[ \lim_{\psi(p) \to \infty} ||f||_p/\psi(p) = 0; \]

and denote by \( GB = GB(\psi) \) the closed span in the norm \( G(\psi) \) the set of all bounded: \( \text{vraisup}_x |f(x)| < \infty \) measurable functions with finite support: \( \mu(\text{supp} \ |f|) < \infty. \)

Another definition: for a two functions \( \nu_1(\cdot), \nu_2(\cdot) \in U\Psi \) we will write \( \nu_1 << \nu_2, \)

\[ \lim_{\nu_2(p) \to \infty} \nu_1(p)/\nu_2(p) = 0. \]
If for some \( \nu_1(\cdot), \nu_2(\cdot) \in U\Psi, \ \nu_1 << \nu_2 \) and \( ||f||G(\nu_1) < \infty \), then \( f \in G^0(\nu_2) \).

**Examples.**

We consider now some important examples, which are some generalizations of considered one in the works [16], [18].

Let \( X = \mathbb{R}, \ \mu(dx) = dx, 1 \leq a < b < \infty, \gamma = \text{const} > -1/a, \ \nu = \text{const} > -1/b, \ p \in (a, b), \)

\[
f_{a, \gamma} = f_{a, \gamma}(x) = I(|x| \geq 1) \cdot |x|^{-1/a}(|\log |x||)\gamma, \\
g_{b, \nu} = g_{b, \nu}(x) = I(|x| < 1) \cdot |x|^{-1/b} \log |x||\nu, \\
h_m(x) = (\log |x|)^{1/m}I(|x| < 1), \ m = \text{const} > 0,
\]

\[
f_{a, b, \gamma, \nu}(x) = f_{a, \gamma}(x) + g_{b, \nu}(x), \ g_{a, \gamma, m}(x) = h_m(x) + f_{a, \gamma}(x), \]

\[
\psi^p_{a, b, \gamma, \nu}(p) = 2(1 - p/b)^{-p\nu - 1} \Gamma(p\gamma + 1) + 2(p/a - 1)^{-p\gamma - 1} \Gamma(p\nu + 1), \\
\psi^p_{a, \gamma, m}(x) = 2(p/a - 1)^{-p\gamma - 1} \Gamma(p\gamma + 1) + 2\Gamma((p/m) + 1),
\]

\( \Gamma(\cdot) \) is usually Gamma-function.

We find by the direct calculation:

\[
|f_{a, b, \gamma, \nu}|_p = \psi^p_{a, b, \gamma, \nu}(p); \ |g_{a, \gamma, m}|_p = \psi^p_{a, \gamma, m}(p).
\]

Therefore,

\[
\psi_{a, b, \gamma, \nu}(\cdot) \in \Psi(a, b); \ \psi_{a, \gamma, m}(\cdot) \in \Psi(a, \infty).
\]

Further,

\[
f_{a, b, \gamma, \nu}(\cdot) \in G(a, b; \gamma + 1/a, \nu + 1/b) \setminus G^a(a, b; \gamma + 1/a, \nu + 1/b), \\
g_{a, \gamma, m}(\cdot) \in G \setminus G^0(a, \infty; \gamma + 1/a, -1/m),
\]

and \( \forall \Delta \in (0, 1) \Rightarrow f_{a, b, \alpha, \beta} \notin \)

\[
G(a, b; (1 - \Delta)(\gamma + 1/a), \nu + 1/b)) \cup G(a, b; 1/a, (1 - \Delta)(\nu + 1/b), \\
g_{a, \gamma, m}(\cdot) \in G \setminus G^0(a, \infty; \gamma + 1/a, -1/m).
\]

More generally, let us consider the following examples. Let \( L = L(z), \ z \in (0, \infty) \) be slowly varying as \( z \to \infty \) continuous positive function.

The reader can receive more information about the slowly varying function in the monograph [24].

Denote for the function \( \psi = \psi(a, b; \alpha, \beta; p); \ p \in (a, b) \)

\[
\psi_{La}(a, b; \alpha, \beta; p) = \psi(a, b; \alpha, \beta) L(a/(p - a)); \\
\psi_{Lb}(a, b; \alpha, \beta; p) = \psi(a, b; \alpha, \beta) L(b/(b - p));
\]

\[
\psi_{La, Lb}(a, b; \alpha, \beta; p) = \max(\psi_{La}(a, b; \alpha, \beta; p), \psi_{Lb}(a, b; \alpha, \beta; p)).
\]
We consider here the case $X = \mathbb{R}^n$ with usually norm for the $n$-dimensional vector $\vec{x} = x = (x_1, x_2, \ldots, x_n) \in X: |x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ equipped with the (weight) measure

$$\mu_\sigma(A) = \int_A |x|^\sigma \, dx_1 dx_2 \ldots dx_n, \; \sigma = \text{const}. \quad (1.4)$$

Define the function

$$f_L(x) = I(|x| < 1) \, |x|^{-1/b} |\log |x||^\gamma \, L(\log |x|);$$

$$b = \text{const}, \; (n + \sigma)b > 1, \; p \in [1, (n + \sigma)b].$$

We get after some calculations using multidimensional polar coordinates and well-known properties of slowly varying functions [28, p. 30-44]:

$$f_L(\cdot) \in G \backslash G_0(\psi_Lb)(a, b(n + \sigma); 0, \gamma + 1/b; p) \, L(b/(b(n + \sigma)p)), \quad 1 \leq a < b(n + \sigma).$$

We define analogously the function

$$g_L(x) = I(|x| > 1) \, |x|^{-1/a} |\log |x||^\gamma \, L(\log |x|);$$

$$a = \text{const}, \; (n + \sigma)a \geq 1, \; p \in (a(n + \sigma), b), \; b \in (a(n + \sigma), \infty).$$

We receive:

$$g_L(\cdot) \in G \backslash G_0(\psi_{La})(a(n + \sigma), b; \gamma + 1/a, 0; p) \, L(a/(p - a(n + \sigma))).$$

Correspondingly, if $1 \leq a(n + \sigma) < b(n + \sigma) < \infty$, then

$$f_L + g_L \in G \backslash G_0(\psi_{La,Lb}).$$

Let now

$$\omega(n) = \pi^{n/2}/\Gamma(n/2 + 1), \; \Omega(n) = n\omega(n) = 2\pi^{n/2}/\Gamma(n/2),$$

$$R = R(\sigma, n) = [(\sigma + n)/\Omega(n)]^{1/(\sigma + n)}, \; \sigma + n > 0,$$

such that

$$\mu_\sigma \{x: |x| < R\} = 1,$$

and let $h = h(|x|)$ be some nonnegative measurable function, $h(x) = 0$ if $|x| \geq R(\sigma, n); \; u \geq \exp(2) \Rightarrow$

$$\mu_\sigma \{x: h(|x|) > u\} = \min(1, \exp (-W(\log u))),$$

where $W = W(z)$ is twice differentiable strong convex in the domain $z \in [2, \infty)$ strong increasing function. Denote by

$$W^*(p) = \sup_{z>2} (pz - W(z))$$

the Young Fenchel transform of the function $W(\cdot)$, and define the function

$$\psi(p) = \exp (W^*(p)/p).$$

It follows from the theory of Orliczs spaces [22, p. 12 - 18] that at $p \in [1, \infty)$
Another examples. Put for \( X = R \), \( \sigma = 0 \),

\[
f(a,b; \alpha, \beta)(x) = |x|^{-1/b} \exp \left( C_1 |\log |x| |^{1-\alpha} \right) I(|x| < 1) +
\]

\[
I(|x| \geq 1) |x|^{1/a} \exp \left( C_2 (\log |x|)^{1-\beta} \right);
\]

\( 1 \leq a < b < \infty; \alpha, \beta = \text{const} \in (0, 1) \). We obtain by direct computation using the saddle - point method:

\[
\log \left| f(a,b; \alpha, \beta)(\cdot) \right|_p \asymp (p-a)^{1-1/\alpha} + (b-p)^{1-1/\beta}, \ p \in (a,b).
\]

It is known that the spaces \( G(\psi) \) with respect to the ordinary operations and introduced norm \( || \cdot ||_{G(\psi)} \) are Banach functional Spaces in the terminology of a book [1], they obey the Fatou property etc.

**Properties. Fundamental function.**

See also [5], [6], [2], [16], [18] etc.

Moreover, the spaces \( G(\cdot) \) are rearrangement invariant (r.i.) spaces with the fundamental function \( \phi(G(\psi), \delta) = \phi(\delta) = \)

\[
\sup \{ ||I(A)||G, \ A \in \Sigma, \ \mu(A) \leq \delta \}, \ \delta \in (0, \infty).
\]

If the measure \( \mu \) is nonatomic, \( \phi(G(\psi), \delta) = ||I(A)||G(\psi), \ \mu(A) = \delta, \) we have for the spaces \( G(\psi), \ \psi(\cdot) \in U\Psi, \ B(\psi) = (a,b), \ b \leq \infty \)

\[
\phi(G(\psi), \delta) = \sup_{p \in (a,b)} \left[ \delta^{1/p}/\psi(p) \right]. \quad (1.5)
\]

As a slight consequence (for nonatomic measures):

\[
\phi(G(\psi), 0+) = 0; \ \lim_{\delta \rightarrow 0^+} \delta/\phi(G(\psi), \delta) = 0.
\]

Note that in the case \( b < \infty \)

\[
\delta \leq 1 \ \Rightarrow \ C_1 \delta^{1/a} \leq \phi(G, \delta) \leq C_2 \delta^{1/b},
\]

\[
\delta > 1 \ \Rightarrow \ C_3 \delta^{1/b} \leq \phi(G, \delta) \leq C_4 \delta^{1/a}.
\]

For instance, define in the case \( b < \infty \ \delta_1 = \exp(\alpha h^2/(h - a)), \ \delta \geq \delta_1 \Rightarrow \)

\[
p_1 = p_1(\delta) = \log \delta/(2\alpha) - \left[ 0.25 \alpha^{-2} \log^2 \delta - a \alpha^{-1} \log \delta \right]^{1/2},
\]

\[
\phi_1(\delta) = \delta^{1/p_1}(p_1 - a)^{\alpha};
\]

\[
\delta \in (0, \delta_1) \ \Rightarrow \ \phi_1(\delta) = \delta^{1/h}(h - a)^{\alpha};
\]

\[
\delta_2 = \exp(-h^2/\beta/(b - h)), \ \delta \in (0, \delta_2) \ \Rightarrow
\]
\[ p_2 = p_2(\delta) = -|\log \delta|/2\beta + \left[ \log^2(\delta/(4\beta^2)) + b|\log \delta|/\beta \right]^{1/2}, \]
\[ \phi_2(\delta) = \delta^{1/p_2(\delta)}(b - p_2(\delta))^{\beta}; \]
\[ \delta \geq \delta_2 \Rightarrow \phi_2(\delta) = \delta^{1/b}(b - h)^{\beta}. \]

We obtain after some calculations:

\[ b < \infty \Rightarrow \phi(G(a, b; \alpha, \beta), \delta) = \max \left[ \phi_1(\delta), \phi_2(\delta) \right]. \tag{1.6} \]

Note that as \( \delta \to 0+ \)

\[ \phi(G(a, b, \alpha, \beta), \delta) \sim (\beta b^2/e)^{\beta} \delta^{1/b} |\log \delta|^{-\beta}, \]

and as \( \delta \to \infty \)

\[ \phi(G(a, b, \alpha, \beta), \delta) \sim (a^2 \alpha/e)^{\alpha} \delta^{1/a} (\log \delta)^{-\alpha}. \tag{1.7} \]

In the case \( b = \infty, \beta < 0 \) we have denoting

\[ \phi_3(\delta) = (\beta/e)^{\beta} |\log \delta|^{-|\beta|}, \delta \in (0, \exp(-h|\beta|)), \]

\[ \phi_3(\delta) = h^{-|\beta|} \delta^{1/h}, \delta \geq \exp(-h|\beta|) : \]

\[ \phi(G(a, \infty; \alpha, -\beta), \delta) = \max(\phi_1(\delta), \phi_3(\delta)), \tag{1.8} \]

and we receive as \( \delta \to 0+ \) and as \( \delta \to \infty \) correspondingly:

\[ \phi(G(a, \infty; \alpha, -\beta), \delta) \sim (\beta)^{|\beta|} |\log \delta|^{-|\beta|}, \]
\[ \phi(G(a, \infty; \alpha, -\beta), \delta) \sim (a^2 \alpha/e)^{\alpha} \delta^{1/a} (\log \delta)^{-a}. \]

**Boyds indices.**

At the end of this section we give using this results the expression for the so-called Boyds (and other) indices of \( G(\psi, a, b) \) spaces in the case \( X = [0, \infty) \) with usually Lebesgue measure. This indices play a very important role in the theory of operators interpolation, theory of Fourier series in the r.i. spaces etc.; see [1], [15].

Recall the definitions. Introduce the (linear) operators

\[ \sigma_s f(x) = f(x/s), \ s > 0, \]

then for arbitrary r.i. space \( G \) on the set \( X = R^1_+ \)

\[ \gamma_1(G) = \lim_{s \to 0+} \log \||\sigma_s||/\log s; \]
\[ \gamma_2(G) = \lim_{s \to \infty} \log \||\sigma_s||/\log s; \]

We obtained (see [20])

\[ \gamma_1(G(\psi, a, b)) = 1/b, \ \gamma_2(G(\psi, a, b)) = 1/a. \tag{1.10} \]
Note that also

\[ \gamma_1(G^o(\psi, a, b)) = 1/b, \quad \gamma_2(G^o(\psi, a, b)) = 1/a. \]  

1.11

2. Associate and dual spaces.

The complete description of the spaces conjugated (or, on the other words, dual) to the (linear topological) spaces \( \cap_p L_p \) or conversely \( \cup_p L_p \) see in [4], [25].

The spaces which are conjugate to the Orlicz’s spaces are described, e.g., in [22], chapter 3; [21], pp. 123 - 142.

It is easy to verify using the well-known theorem of Radon - Nikodim that the structure of linear continuous functionals \( l = l(f) \) over the space \( G^0(\psi) = GA = GB \) is follow:

\[ l(f) = l_g(f) = \int_X f(x)g(x) \mu(dx), \]  

(2.0)

where \( g \) is some local integrable function:

\[ \forall A \in \Sigma, \mu(A) \in (0, \infty) \Rightarrow \int_A |g| d\mu < \infty. \]

We will call as usually the space of all continuous in \( G(\psi, a, b) \) space functionals of a view (2.0) as associated space and will denote as \( G^/(\psi) = G^/(\psi, a, b) \). It is evident that \( G^/(\psi) \) is closed subspace of \( G^*(\psi) \).

Definition 3.

Analogously to the works [2], [5], [6], [7] we can describe the associated spaces to the \( G(\psi, a, b) \) spaces.

Let us introduce for the set \( A = (a, b) \subset [1, \infty) \) its adjoin set:

\[ A' \overset{\text{def}}{=} \left( b', a' \right), \quad a' = a/(a - 1), \quad b' = b/(b - 1). \]

We denote by the symbol \( DSL = DSL(\psi) = DSL(\Psi, a, b) = DS \) a Banach space of all the measurable functions \( g : X \rightarrow R^1 \) with finite norm

\[ ||g||_{DSL(\psi)} \overset{\text{def}}{=} \inf \left\{ \sum_{k=1}^{\infty} \psi(q(k)) \, |g_k|_{q(k)'}/ \right\}, \]  

(2.1)

where ” \( \inf \)” is calculated over all the sequences of measurable functions \( g_k = g_k(x) \) such that

\[ \sum_k g_k(x) = g(x) \]  

(2.2)

a.e., \( g_k \in L_{q(k)}/ \), and all sequences of numbers \{q(k)\} belonging to the set \((a, b) : q(k) \in (a, b)\).

Remark 2.1 It may be proved analogously [2], [5], [7] etc. that the \( DSL(\psi) \) spaces relative the introduced norm are really Banach function spaces and, moreover, are r.i. spaces.
Remark 2.2 It is easy to verify that in the decomposition (2.2) in the case if \( g(x) \geq 0 \) (almost everywhere) all the functions \( g_k(\cdot) \) may be choose to be non-negative.

Remark 2.3. The sequence of the numbers \( \{q(k)\} \) in the definition (2.1) may be choose such that

\[
\lim_{k \to \infty} q(k) = a, \quad q(k) > a,
\]

and

\[
\lim_{k \to \infty} q(k) = b, \quad q(k) < b.
\]

Theorem 2.1. Let \( \psi \in U\Psi \). Let also the triplet \((X, \Sigma, \mu)\) be resonant. Then the spaces \( DSL(\psi) \) and \( SL(\psi) \) coincides (up to norm equality):

\[
DSL(\psi) = SL(\psi).
\] (2.3)

On the other words, the spaces \( DSL(\psi) \) are associate to the Bilateral Grand Lebesgue spaces \( G(\psi) \).

Proof. This proposition may be proved analogously to the case one-side Grand Lebesgue spaces considered in [2], [5]-[7].

Step 1. We prove at first the implication \( g \in SL(\psi) \implies l_g(\cdot) \in GL'(\psi) \), with norm equality.

Namely, if \( g = \sum_k g_k \), \( f \in G(\psi) \), \( ||f||G(\psi) = 1 \) and for some sequence \( \{q(k)\} \in (a,b) \) and for any \( \epsilon \in (0,1) \)

\[
\sum_k \psi(q(k))|g_k|_{(q(k))}^q \leq ||g||GS(\psi) + \epsilon = C < \infty,
\]
then \( |f|_{q(k)} \leq \psi(q(k)), \quad q(k) \in (a,b) \). We have using Hölder inequality:

\[
|l_g(f)| \leq \sum_k |f|_{q(k)} \cdot |g_k|_{(q(k))}^q \leq \sum_k \psi((q(k))) \cdot |g_k|_{(q(k))}^q = C < \infty,
\]
or, equally,

\[
|l_g(f)| = \left| \int_X f(x) \ g(x) \ \mu(dx) \right| \leq ||f||G(\psi) \ ||g||SL(\psi). \quad (2.4)
\]

Note that the inequality (2.4) is called as usually as generalized Hölder inequality.

Step 2. Now we prove the inverse implication.

It is sufficient to consider only the case of the space \( G^a(\psi; a, b) \).

Let \( f(\cdot) \) be arbitrary element of the space \( G^a(\psi) \) with unit norm:

\[
|f|_p \leq \psi(p), \quad p \in (a,b);
\]
and let \( l_g \) be a linear functional on the space \( G^a(\psi) \) of the standard view:

\[
l_g(f) = \int_X f(x) \ g(x) \ d\mu.
\]

We deduce using the classical Hölder inequality:
\[
\forall p \in (a, b) \ |l_g(f)| \leq |f|_p |g|_{p'} \leq \psi(p) |g|_{p'}.
\]

Therefore,

\[
||g||_{SL(\psi)} \leq \inf_{q \in A'} \psi(q') |g|_q.
\]

(2.5)

It remain to prove the finiteness of the right side of inequality (2.5), or equally the following implication:

\[
\exists q \in A', \ |g|_q < \infty.
\]

(2.6)

We will prove this proposition by the method reduction ad absurdum. Let us consider for definiteness the case of purely atomic measure \(\mu : X = (1, 2, \ldots) \mu(k) = 1\).

Then the linear functional \(l_g\) has a view:

\[
l_g(f) = \sum_{n=1}^{\infty} f_n g_n
\]

and we suppose

\[
\forall q \in A' \ \sum_{n=1}^{\infty} (|g|_n)^q = \infty.
\]

(2.7)

We will suppose without loss of generality that \(f_n \geq 0, \ g_n \geq 0, \ n = 1, 2, \ldots\)

Define the truncated sequence

\[
f^{(N)}(n) = f_n, \ n \leq N, \ f^{(N)}(n) = 0, \ n > N;
\]

and the correspondent truncated sum

\[
S_N(\epsilon) = \sum_{n=1}^{N} |q_n|^{a'-\epsilon},
\]

(2.8)

\(\epsilon = \text{const} \in (0, \epsilon_0)\).

It follows from the uniform boundedness principle that there exists the positive finite constant \(K, \ K \in (0, \infty)\), such that

\[
L := \sum_{n=1}^{\infty} f_n g_n \leq K \cdot \sup_{p \in (a,b)} \frac{|f|_p}{\psi(p)}.
\]

(2.9)

We choose in the inequality (2.9) the sequence \(f\) as follows:

\[
f_n = (g_n)^{a-(\epsilon+1)}, \ n \leq N
\]

and \(f_n = 0\) in other case.

Substituting into (2.9), we get:

\[
L \leq K \cdot \sup_{p \in (a,b)} \frac{(S_N(\epsilon))^{1/p}}{\psi(p)} = K\phi(G(\psi), S_N(\epsilon)),
\]

(2.11)

where, recall, \(\phi(\cdot, \cdot)\) denotes the fundamental function.

We obtain from (2.11), denoting \(X = S_N(\epsilon)\) :
\[ X \leq K \cdot \phi(G(\psi), X). \]  

(2.12)

The solving of inequality (2.12) has a view: \( X \leq C \), where the constant \( C \) does not depend on the \( \epsilon \) and \( N \). Therefore,

\[
\sup_{\epsilon} \sup_{N} S_N(\epsilon) \leq C,
\]

or equally

\[
\sum_{n=1}^{\infty} |g_n|^q < \infty
\]

and hence for all the values \( q \in (a^{'}, b^{'}) \)

\[
\sum_{n=1}^{\infty} |g_n|^q < \infty,
\]

in contradiction.

This completes the proof of theorem 2.1.

**Step 3.**

Note analogously to the article [5] that the inequality (2.5) is exact.

We intend to prove that for all \( f \in G^o(\psi) \), \( f \geq 0 \) there exists a function \( g \in \bigcup_{q \in (b^{'}, a^{'})} L_q \) for which

\[
|l_g(f)| = ||f||G(\psi) \cdot ||g||SL(\psi).
\]

Indeed, the function \( h(p) = |f|p/\psi(p), p \in [a, b] \) is continuous and \( h(a) = h(b) = 0 \); therefore there exists a value \( \sigma \in (a, b) \) for which \( |f|_\sigma = ||f||\cdot\psi(\sigma) \). We have choosing the function \( g \) such that

\[
l_g(f) = |f|_\sigma \cdot |g|_{\sigma^{'}}:
\]

\[
||f||G(\psi) \cdot ||g||GS(\psi) \geq l_g(f) = \psi(\sigma)||f|| \cdot |g|_{\sigma^{'}} = ||f|| \left[ |g|_{\sigma^{'}}(\psi(\sigma)/\sigma^{'}) \right] \geq ||f||G(\psi) \cdot ||g||GS(\psi).
\]

**Remark 2.4.** As a corollary: if (in addition) our measure \( \mu \) is separable, then we infer:

\[
G(\psi)/ = GB(\psi)^* = GA(\psi)^* = G_0(\psi)^*,
\]

see [1], p. 20 - 22.

3. Norm's absolutely continuity. Compactness.

We will say as usually, see [1], pp. 14-16 that the function \( f \) belongs to some r.i. space \( Y \) over source triplet \((X, \Sigma, \mu)\) with the norm \( || \cdot ||_Y \) has absolutely continuous norm in this space and write \( f \in YA, \) if
\[
\lim_{\delta \to 0} \sup_{A: \mu(A) \leq \delta} \| f I_A \| Y = 0. \tag{3.1}
\]

We will write in the case \( Y = G(\psi) \): \( G(\psi)A = GA(\psi) \), \( G(\psi)B = GB(\psi) \) and analogously \( SL(\psi)B = SLB(\psi) \), \( SL(\psi)A = SLA(\psi) \).

**Theorem 3.1.** The space \( SL(\psi; a, b) \) satisfies the absolutely continuity norm property.

**Proof.** The proof is at the same as in [5]. Namely, let \( g \in SL(\psi) \), \( ||g||_{SL(\psi)} = 1 \). We write the following consequence from the decomposition for this function:

\[
\sum_{k=1}^{\infty} \left[ \psi(q(k)) |g_k|_{q(k)/} \right] \leq 2.
\]

Let also \( \{E(n)\}, n = 1, 2, \ldots \) be monotonically decreasing sequence of measurable sets such that as \( n \to \infty \)

\[
\mu(E(n)) \downarrow 0.
\]

We can suppose without loss of generality \( \mu(E(1)) \leq 1 \). Let us denote as in [5]

\[
a(k, n) = \psi(q(k)) |g_k|_{I(E(n))}_{q(k)/}.
\]

We have for all values \( n \):

\[
||g I(E(n))||_{SL(\psi)} \leq \sum_{k=1}^{\infty} a(k, n) < \infty.
\]

Since

\[
\sum_{k=1}^{\infty} a(k, n) < \infty
\]

and \( \forall k \Rightarrow a(k, n) \downarrow 0 \) as \( n \uparrow \infty \), we deduce:

\[
\lim_{n \to \infty} ||g I(E(n))||_{SL(\psi)} = 0, \tag{3.2}
\]

QED.

**Consequences.**

The next assertions follows from the theorem 3.1 and from the well-known facts about the general theory of Banach functional and rearrangement invariant spaces ([1], chapters 1 - 2).

In the Bilateral Small Lebesgue spaces are true the Levi’s theorem of a monotone convergence, Fatou property and Lebesgue majoring convergence theorem.

Both the spaces \( G(\psi) \) and \( SL(\psi) \) are the interpolation spaces between the spaces \( L_1(X, \mu) \) and \( L_\infty(X, \mu) \) relative the real method of interpolation.

If we assume in addition to the conditions of theorem 3.1 that the measure \( \mu \) is nonatomic and separable, then the space \( BSL(\psi) \) is also separable, coincides with \( BSLA(\psi), BSLB(\psi) \) and moreover
\[
[BSL(\psi)]^* = [BSL(\psi)]' = BGL(\psi).
\]  

(3.3)

but they are not reflexive.

Note for comparison: the subspaces \(BGA(\psi), BGB(\psi), BG^0(\psi)\) are closed strictly subspaces of the space \(BG(\psi)\). The following important property of the Grand Lebesgue spaces is proved in [16], [17].

If \(\psi \in U\Psi\), then

\[G^0(\psi) = GB(\psi) = GA(\psi).\]

4. Fundamental functions and Boyd’s indices.

We suppose in this section that the measure \(\mu\) again is sigma-finite and nonatomic.

The next fact follows from the well-known result about the connection between the fundamental functions of r.i. spaces and its associate (see, for example, [1], chapter 2, theorem 5.2.)

**Theorem 4.1.**

\[
\phi(SL(\psi), \delta) = \frac{\delta}{\phi(G(\psi), \delta)}, \ \delta \in (0, \mu(X)).
\]

(4.1)

We will use further for brevity the notation \(\chi(\delta) = \phi(SL(\psi, \delta))\).

The assertion of the theorem 4.1 allows us to calculate the exact value of \(SL(\psi)\) norm of the indicator function.

Notice that

\[\chi(0+) = 0; \ \lim_{\delta \to \infty} \chi(\delta) = \infty.\]

**Lemma 4.1.** Let \(A\) be a measurable set: \(A \in \Sigma\) such that \(0 < \mu(A) < \infty\). The following equality holds:

\[\| I(A) \|_{SL(\psi)} = \frac{\mu(A)}{\phi(G(\psi), \mu(A))} = \chi(\mu(A)).\]

(4.2)

Recall that the exact values for the fundamental functions for some \(G(\psi)\) spaces there are in the formulas (1.6), (1.8).

Let us give an other example for \(SL(\psi)\) norm exact value calculation.

**Example 4.1.** Assume that on the source triplet \((X, \Sigma, \mu)\) and assume that for some interval \((a, b)\), \(1 \leq a < b \leq \infty\) there exists a value \(q_0, a < q_0 < b\) and a measurable non-negative function \(g : X \to \mathbb{R}\) for which

\[|g|_q < \infty \Leftrightarrow q = q_0.\]

This condition is satisfied, for instance, when \(X = \mathbb{R}^d\) and \(\mu\) is \(d -\) dimensional Lebesgue measure.

We deduce using the explicit expression for the \(SL(\psi)\) norm that for this function \(g\) the inequality (2.5) transforms to the equality:
\[ \|g\|_{SL(\psi)} = \psi \left( q_0^g \right) |g|_{q_0}. \]  

(4.3)

The assertions of the theorem 4.1 and lemma 4.1 give us the possibility to compute the important for applications Boyd’s \( \gamma_1, \gamma_2 \) and Shimogaki’s \( \beta_1, \beta_2 \) indices for Bilateral Small Lebesgue spaces on the basis of ones results for Grand Spaces, which are calculated in [18], [20]. Indeed:

**Lemma 4.2.** Let \( \psi \in \Psi(a, b), \ 1 \leq a < b \leq \infty \). Then:

\[ A. \gamma_1(\text{SL}(\psi)) = 1 - 1/a; \ \gamma_2(\text{SL}(\psi)) = 1 - 1/b; \]  

(4.4)

\[ B. \beta_1(\text{SL}(\psi)) = 1 - 1/a; \ \beta_2(\text{SL}(\psi)) = 1 - 1/b; \]  

(4.5)

C. The Hardy - Littlewood maximal operator \( M \) in the case \( X = \mathbb{R}^d \) is in bounded in the Small Lebesgue space \( \text{SL}(\psi) \).

D. The Hilbert’s transform \( H \) in the case \( X = \mathbb{R}^d \) is in bounded in the Small Lebesgue space \( \text{SL}(\psi) \) iff \( a > 1 \).

E. Let here \( X = [0, 2\pi) \). The Fourier series for arbitrary function \( f \in \text{SL}(\psi) \) convergence to \( f \) in the \( \text{SL}(\psi) \) norm iff \( a > 0, \ b < \infty \).

5. **Improving of the norm estimation.**

Recall that

\[ \|I(A)\|_{\text{SL}(\psi)} = \chi(\mu(A)). \]  

(5.1)

The equality (5.1) gives us the possibility to *estimate* the \( \text{SL}(\psi) \) norm of arbitrary function belonging to the space \( \text{SL}(\psi) \).

Consider a *simple* function of a view:

\[ f(x) = \sum_{k=1}^{n} c(k) \ I(H_k, x), \ n < \infty, \]  

(5.2)

\( H_k \in \Sigma, \ k \neq l \Rightarrow H_k \cap H_l =, \ c(k) \in R; \) i.e. \( f \in \text{SLB}(\psi) \) and conversely the set of all the functions of a view (5.2) is dense in the \( \text{SLB}(\psi) \) space and following in all space \( \text{SL}(\psi) \).

We will call the decomposition (5.2) as a representation of a function \( f \) and will denote the set of all simple functions as \( \text{Sim}(\psi) \).

We have for these functions:

\[ \|f\|_{\text{SL}(\psi)} \leq \sum_{k=1}^{n} \left| c(k) \right| \chi(\mu(H_k)) \overset{\text{def}}{=} \int_X f(x) \ d\chi. \]  

(5.3)

Let now \( f(\cdot) \) be arbitrary non - negative function from the space \( \text{SL}(\psi) \). Denote by \( R(f) \) the set of all the simple functions \( \{g\} \) greatest than \( f : g \geq f \). We define:

\[ \int_X f(x) \ d\chi \overset{\text{def}}{=} \inf_{g \in R(f)} \int_X g(x) \ d\chi. \]  

(5.4)

For the function \( f \in \text{SL}(\psi) \) which may not to be non-negative we define
\[
\int_X f(x) \, d\chi = \int_X f_1(x) - \int_X f_2(x) \, d\chi, \tag{5.5}
\]
where the non-negative functions \(f_1\) and \(f_2\) in (5.5) from the set \(\text{Sim}(SL)\) are such that the function \(f\) is the difference of the functions \(f_1\) and \(f_2: f(x) = f_1(x) - f_2(x)\) and

\[
\int_X |f_1(x)| + |f_2(x)| \, d\chi = \inf \left\{ \int_X |g_1(x)| + |g_2(x)| \, d\chi \right\},
\]
where \(g_1, g_2 \in \text{Sim}(SL)\), \(g_1, g_2 \geq 0\) and \(g_1(x) - g_2(x) = f(x)\).

For the unbounded functions \(f = f(x)\) we may define

\[
\int_X f(x) \, d\chi = \lim_{N \to \infty} \int_X [f(x) \, I(|f(x)| \leq N)] \, d\chi,
\]
if there exists.

Let us denote

\[
\|f\| = \|f\|_{SL(\psi)} = \int_X |f(x)| \, d\chi. \tag{5.6}
\]

Note that the functional \(f \to \|f\|_{SL(\psi)}, f \in SL(\psi)\) obeys the following norm properties:

\[
\|f\| \geq 0; \quad \|f\| = 0 \iff f = 0 \pmod{\mu};
\]

\[
\|\lambda f\| = |\lambda| \cdot \|f\|, \quad \lambda \in \mathbb{R};
\]

\[
\|f + g\| \leq \|f\| + \|g\|.
\]

Since the set of the simple functions is dense in the space \(SL(\psi)\), we obtain the following convenient for applications estimation.

**Theorem 5.1.**

\[
\|f\|_{SL(\psi)} \leq \|f\|_{SL(\psi)}. \tag{5.7}
\]

**Remark 5.1.** Note that still in the case \(\mu(X) = 1\) the functional \(f \to \|f\|_{SL(\chi)}\) is discontinued in the *uniform* norm \(\|f\|_\infty = \text{vraisup}_{x \in X} |f(x)|\) in all the points aside from the origin. Let us consider the following example.

**Example 5.1.** Let \(X = [0, 1]\) and \(n = 2, 3, 4, \ldots\). Define the functions \(f(x) = 1\) and the sequence of a functions

\[
f_n(x) = I(x \in [0, 1/2]) + (1 + 1/n)I(x \in (1/2, 1)). \tag{5.8}
\]

Let also \(\chi(\delta) = \sqrt{\delta}, \ \delta \in (0, 1)\). We observe:

\[
\lim_{n \to \infty} |f_n - f|_\infty = 0; \quad |f|_\infty = 1;
\]

but

\[
\|f_n\|_{SL(\chi)} = \sqrt{0.5} + (1 + 1/n)\sqrt{0.5} \to \sqrt{2}, \ n \to \infty.
\]
6. Convergence and compactness.

A. Note that in the case if $X$ is the convex bounded subset of $R^n$ with usually bounded Lebesgue measure, for the spaces $G^o(\psi)$ and $SL(\psi)$ are true the classical conditions of Riesz’s and Kolmogorov’s for compactness of some subset $F = \{f_\alpha\} \subset G^o(\psi)$, as long as these spaces are separable.

B. If some subset $F = \{f_\alpha\} \subset G^o(\psi)$ is closed, bounded, is compact set in the sense of $L_p, p \in (a, b)$ convergence and has uniform absolutely continuous norm: $F \in UCN$, then $F$ is compact set in the space $G(\psi)$.

C. The direct calculation of the norm $||f||SL(\psi)$ is very hard. But we can replace the distance $||f - g||SL(\psi)$ by the distance

\[ d_\psi(f, g) \overset{\text{def}}{=} |||f - g|||SL(\psi) \] (6.1)

in order to formulate the sufficient conditions for convergence and compactness in the Bilateral Small Lebesgue spaces. For instance, if $f_n, f \in SL(\psi)$ and $|||f_n - f||| \to 0, n \to \infty$, then $|||f_n - f|||SL(\psi) \to 0, n \to \infty$.

If some closed subset $U$ of the space $SL(\psi)$ is compact set in the distance $d_\psi$, then $U$ is compact set relative the source norm of the space $SL(\psi)$ etc.

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