Asymptotics of skew orthogonal polynomials

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Abstract

Exact integral expressions of the skew orthogonal polynomials involved in Orthogonal ($\beta = 1$) and Symplectic ($\beta = 4$) random matrix ensembles are obtained: the (even rank) skew orthogonal polynomials are average characteristic polynomials of random matrices. From there, asymptotics of the skew orthogonal polynomials are derived.
1 Introduction

Families of Orthogonal (or skew-orthogonal) Polynomials, have many applications to mathematics and physics [1,2].

Here, we will have in mind applications to Random Matrix Theory (RMT) [3,4], i.e. disordered solid state physics [4], QCD [7], or statistical physics on a random fluctuating lattice [3,8] (2D quantum gravity). In all these fields of physics, one is interested in the spectrum of a matrix (Hamiltonian, Transmission matrix, S-matrix, Dirac operator,...), which can be considered as random for various reasons (disorder, random impurities, quantum fluctuations, chaos or non-integrability,...). It was observed that the spectrum of a large random matrix shows universal properties [9,10] (2-point correlation function, in the short or long range regime; universal conductance fluctuations of mesoscopic conductors). One possible way to understand and prove that universality is through the “orthogonal polynomials” method, which we shall recall below. In order to extract some useful numerical results, it is important to have some asymptotics of the orthogonal polynomials in some special limit.

The type of orthogonal polynomials involved, depends on the symmetry of the matrix ensemble [3]. The case of a physical system with broken time-reversibility (for instance a mesoscopic conductor in the presence of a magnetic field), represented by a \(U(N)\) invariant matrix ensemble, was extensively studied, because it is the simplest [2].

Here, we shall focus on the \(O(N)\) and \(Sp(2N)\) invariant matrix ensembles, which appear for physical systems with time-reversibility and/or half-integer spin with broken rotational symmetry. These ensembles involve families of skew-orthogonal polynomials.

The aim of this article is to present a remarkable exact expression of the skew orthogonal polynomial as an integral, and deduce from it the required asymptotics.

Section 2 is a brief introduction to the orthogonal polynomial’s method in RMT, in section 3 we give and prove the remarkable exact expressions for the skew-orthogonal polynomials, and in section 4, we consider their asymptotics.

2 The Orthogonal Polynomials

Consider the partition function of a random matrix \(M\):

\[
Z_N^{(\beta)}[V] = \int_{M \in E_N^{(\beta)}} \, dM \, e^{-N \beta \text{tr} V(M)}
\]

(2.1)

where \(E_N^{(1)}\) is the set of all \(N \times N\) real symmetric matrices, \(E_N^{(2)}\) is the set of all \(N \times N\) hermitian matrices, \(E_N^{(4)}\) is the set of all \(2N \times 2N\) self-adjoint real quaternionic
matrices\textsuperscript{2}, and $dM$ is the Haar measure on $E_N^{(d)}$. $V(x)$ is a polynomial potential, bounded from below, and $N_\beta = N, N, N/2$ respectively for $\beta = 1, 2, 4$.

The angular degrees of freedom of $M$ can be integrated out, and \[2.1\] can be rewritten as an integral over the $N$ eigenvalues $(\lambda_1, \ldots, \lambda_N)$ of $M$ only \[11, 3\]:

$$Z_N^{(\beta)}[V] = U_N^{(\beta)} \int \prod_{i=1}^N d\lambda_i |\Delta(\lambda)|^\beta$$

(2.2)

where $U_N^{(\beta)}$ is the volume of the group $O(N)$, $U(N)$ or $Sp(2N)$ respectively for $\beta = 1, 2, 4$. $d\lambda = d\lambda e^{-NV(\lambda)}$ is the measure element, and

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$$

(2.3)

is the Vandermonde determinant, which can be rewritten as:

$$\Delta(\lambda) = \det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \ldots & \lambda_N^{N-1} \end{pmatrix} = \det(\lambda_j^i) = \det P_j(\lambda_i)$$

(2.4)

where $P_j(\lambda) = \lambda^j + \ldots$ is an arbitrary monic polynomial of degree $j$. The last equality is obtained by linearly mixing columns of the determinant, and the first equality is the well known Vandermonde determinant, which can be found in any math textbook \[3\].

The computation of integral \[2.2\] becomes easier with a special choice of the polynomials $P_j(\lambda)$, chosen orthogonal with respect to an appropriate scalar product \[11\]:

- In the unitary case $\beta = 2$, the scalar product under consideration is:

$$< f | g > = \int_{-\infty}^{\infty} \bar{f}(x) g(x)$$

(2.5)

and the polynomials $P_n(x)$ are chosen orthogonal:

$$< P_n | P_m > = h_n \delta_{nm}$$

(2.6)

- In the orthogonal case $\beta = 1$, the scalar product under consideration is skew-symmetric:

$$< f | g >= - < g | f > = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f}(x) g(y) \text{sgn}(x - y)\ d\bar{x}\ dy$$

(2.7)

and the polynomials $P_n(x)$ are chosen skew-orthogonal:

$$< P_{2n} | P_{2m} > = < P_{2n+1} | P_{2m+1} >= 0$$

$$< P_{2n+1} | P_{2m} > = h_n \delta_{nm}$$

(2.8)

(2.9)

\textsuperscript{2}$M \in E_N^{(d)}$ can be viewed either as a $2N \times 2N$ matrix with complex number entries or a $N \times N$ matrix with quaternion entries. It has $N$ eigenvalues, each twice degenerated. \[3\]
• In the symplectic case $\beta = 4$, the scalar product under consideration is skew-symmetric too:

$$< f | g > = - < g | f > = \int_{-\infty}^{\infty} dx \ (f(x)g'(x) - f'(x)g(x))$$ (2.10)

and the polynomials $P_n(x)$ are chosen skew-orthogonal:

$$< P_{2n} | P_{2m} >= < P_{2n+1} | P_{2m+1} >= 0$$ (2.11)

$$< P_{2n+1} | P_{2m} >= h_n \delta_{nm}$$ (2.12)

In all three cases, the partition function 2.2 reduces mainly to $Z = \prod_{n=1}^{n_F} h_{n-1}$, where $n_F$ is called the ”Fermi level” by analogy with a system of fermions:

$$n_F = \frac{\beta}{2} N \beta \quad \text{for} \quad \beta = (1, 2, 4)$$ (2.13)

When $N$ is large, most of the physical quantities and relevant observables are related to properties of $h_n$ in the vicinity of the Fermi level: $n \to \infty$, $N \to \infty$ and $n - n_F \sim O(1)$.

2.1 Determination of the orthogonal polynomials

For a generic potential $V(x)$, those orthogonal polynomials exist, and can be constructed by recurrence. Indeed, we start from $P_0(x) = 1$, then the coefficients of $P_1$ are determined by the orthogonality conditions, and by recurrence, we determine $P_n$ and $h_n$ for all $n$.

Note that for the skew-orthogonal polynomials, there is an ambiguity: $P_{2n+1}$ is defined only up to an arbitrary linear combination with $P_{2n}$. If one wants a unique definition, an extra condition should be added, for instance that the term of degree $2n$ in $P_{2n+1}$ vanishes. Anyway, the values of $h_n$ don’t depend on this ambiguity.

The determination of the orthogonal polynomials by recurrence is inefficient if one wants to compute $P_n$ for $n$ large. The aim of this article is to present a closed expression of $P_n$ for any $n$, and to derive from it some asymptotics in the large $n$ limit, and particularly near the Fermi level $n - n_F \sim O(1)$.

3 An exact expression of the skew-orthogonal polynomials

• In the unitary case $\beta = 2$, it is known that

\footnote{The actual result may depend on the parity of $N$. Details can be found in \[\].}
The \( n \)th orthogonal polynomial is the average of the characteristic polynomial of a \( n \times n \) hermitian matrix with respect to the weight \( e^{-N \text{tr} V(M)} \):

\[
P_n^{(2)}(x) = \langle \det (x - M) \rangle_{n \times n}
\]

This has been known for more than a century [12] (in the context of RMT, see e.g. [1, 2, 13]). We are now going to generalize this expression to \( \beta = 1 \) and 4.

- **Orthogonal case** \( \beta = 1 \). We will prove below that:

\[
P_{2n}^{(1)}(x) = \frac{1}{Z_{2n}^{(1)}} \int_{M \in E_{2n}^{(1)}} dM \ \det (x - M) \ e^{-N \text{tr} V(M)}
\]

\[
= \frac{U_{2n}^{(1)}}{Z_{2n}^{(1)}} \int d\lambda_1 \ldots d\lambda_{2n} \prod_{i < j} |\lambda_i - \lambda_j|^2 \prod_i (x - \lambda_i)
\]

and

\[
P_{2n+1}^{(1)}(x) = \frac{1}{Z_{2n}^{(1)}} \int_{M \in E_{2n}^{(1)}} dM \ (x + \text{tr} M + c_n) \det (x - M) \ e^{-N \text{tr} V(M)}
\]

\[
= \frac{U_{2n}^{(1)}}{Z_{2n}^{(1)}} \int d\lambda_1 \ldots d\lambda_{2n} \prod_{i < j} |\lambda_i - \lambda_j| \ (x + \sum_i \lambda_i + c_n) \prod_i (x - \lambda_i)
\]

the constants \( c_n \) can be chosen arbitrarily, the choice \( c_n = 0 \) is the one such that the term of degree 2 in \( P_{2n+1} \) vanishes.

- **Symplectic case** \( \beta = 4 \)

\[
P_{2n}^{(4)}(x) = \frac{1}{Z_{2n}^{(4)}} \int_{M \in E_n^{(4)}} dM \ \det (x - M) \ e^{-\frac{N}{4} \text{tr} V(M)}
\]

\[
= \frac{U_{n}^{(4)}}{Z_{n}^{(4)}} \int d\lambda_1 \ldots d\lambda_n \prod_{i < j} |\lambda_i - \lambda_j|^4 \prod_i (x - \lambda_i)^2
\]

\[
= \langle \det (x - M) \rangle_{n \times n}
\]
\[ P^{(4)}_{2n+1}(x) = \frac{1}{Z_n^{(4)}} \int_{M \in E_n^{(4)}} dM \left( x + \text{tr} \ M + c_n \right) \det (x - M) e^{-\frac{N}{2} \text{tr} \ V(M)} \] (3.19)
\[ = \frac{Z_n^{(4)}}{Z_n^{(4)}} \int d\lambda_1 \ldots d\lambda_n \prod_{i \neq j} |\lambda_i - \lambda_j|^4 (x + 2 \sum \lambda_i + c_n) \prod_i (x - \lambda_i)^2 \] 
\[ = \langle (x + \text{Tr} \ M + c_n) \det (x - M) \rangle_{n \times n} \]

### 3.1 Proof of 3.16

Note that it is sufficient to prove that
\[ < P_{2n} | x^m > = 0 \quad \text{and} \quad < P_{2n+1} | x^m > = 0 \quad \text{for all} \quad m \leq 2n - 1 \] (3.20)

Consider:
\[ < P_{2n} | x^m > \propto \int d\bar{x} \ d\bar{y} \ d\lambda_1 \ldots d\lambda_{2n} \prod_{i < j} (\lambda_i - \lambda_j) \prod_i (x - \lambda_i) \prod_{i < j} \text{sgn} (\lambda_i - \lambda_j) \text{sgn} (x - y) \ y^m \] (3.21)

then write \( x = \lambda_{2n+1} \):
\[ < P_{2n} | x^m > \propto \int d\bar{y} \ d\lambda_1 \ldots d\lambda_{2n+1} \ y^m \prod_{1 \leq i < j \leq 2n+1} (\lambda_i - \lambda_j) \prod_{1 \leq i < j \leq 2n+1} \text{sgn} (\lambda_i - \lambda_j) \prod_{i=1}^{2n} \text{sgn} (\lambda_i - \lambda_{2n+1}) \text{sgn} (\lambda_{2n+1} - y) \] (3.22)

symmetrize with respect to the first \( 2n + 1 \) variables:
\[ < P_{2n} | x^m > \propto \sum_{k=1}^{2n+1} \int d\bar{y} \ d\lambda_1 \ldots d\lambda_{2n+1} \ y^m \prod_{1 \leq i < j \leq 2n+1} (\lambda_i - \lambda_j) \prod_{1 \leq i < j \leq 2n+1} \text{sgn} (\lambda_i - \lambda_j) \prod_{1 \leq i \neq k \leq 2n+1} \text{sgn} (\lambda_i - \lambda_k) \text{sgn} (\lambda_k - y) \] (3.23)

Note the following identity:
\[ \prod_{i=1}^{2n+1} \text{sgn} (y - \lambda_i) = \sum_{k=1}^{2n+1} \text{sgn} (y - \lambda_k) \prod_{i=1, i \neq k}^{2n+1} \text{sgn} (\lambda_i - \lambda_k) \] (3.24)

which gives (and note \( y = \lambda_{2n+2} \)):
\[ < P_{2n} | x^m > \propto \int d\lambda_1 \ldots d\lambda_{2n+2} \left[ \lambda_{2n+2}^m \prod_{1 \leq i < j \leq 2n+1} (\lambda_i - \lambda_j) \right] \left[ \prod_{1 \leq i < j \leq 2n+2} \text{sgn} (\lambda_i - \lambda_j) \right] \] (3.25)
The second bracket is completely antisymmetric in the $2n + 2$ variables, so that we have to antisymmetrize the first bracket as well. The result is zero when $m \leq 2n$, because any non-zero antisymmetric polynomial of $2n + 2$ variables must have degree at least $2n + 1$, while the first bracket is a polynomial of degree at most $2n$ in any of its variables.

By the same argument, one would find that

$$< P_{2n+1} | x^m > \propto \frac{\Delta_{2n}(\lambda_i, -\lambda_j)}{\prod_{i=1}^{2n+1} (\lambda_i - \lambda_j)}$$

which, by antisymmetrization of the first bracket, vanishes when $m \leq 2n - 1$.

### 3.2 Proof of 3.18

Again, it is sufficient to prove that

$$< P_{2n} | x^m > = 0 \quad \text{and} \quad < P_{2n+1} | x^m > = 0 \quad \text{for all} \quad m \leq 2n - 1 \quad (3.27)$$

$$< P_{2n} | x^m > \propto \frac{\Delta_{2n}(x, \lambda_i, \mu_i)}{\prod_{i=1}^{2n+1} (x - \lambda_i)}$$

Introduce $n$ extra variables ($\mu_1, \ldots, \mu_n$), and consider the $2n \times 2n$ Vandermonde determinant of the $2n$ variables ($\lambda_i, \mu_i$), divide it by $\prod_i (\lambda_i - \mu_i)$ and take the limit $\mu_i \rightarrow \lambda_i$. You get:

$$\prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^4 = \lim_{\mu_i \rightarrow \lambda_i} \frac{\Delta_{2n}(\lambda_i, \mu_i)}{\prod_{i=1}^{2n+1} (\lambda_i - \mu_i)} = \det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{2n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{2n-1} \\ 0 & 1 & 2\lambda_1 & \ldots & (2n-1)\lambda_1^{2n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2\lambda_n & \ldots & (2n-1)\lambda_n^{2n-2} \end{pmatrix}$$

With the same trick, we have:

$$\prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^4 \prod_{i=1}^{2n} (x - \lambda_i)^2 = \lim_{\mu_i \rightarrow \lambda_i} \frac{\Delta_{2n}(x, \lambda_i, \mu_i)}{\prod_{i=1}^{2n+1} (\lambda_i - \mu_i)}$$

$$= \det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{2n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{2n-1} \\ 0 & 1 & 2\lambda_1 & \ldots & (2n-1)\lambda_1^{2n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2\lambda_n & \ldots & (2n-1)\lambda_n^{2n-2} \end{pmatrix}$$

(3.29)
which obviously vanishes when $m_{3.16}$ and $3.18$. 

\[ m_{n} \] of integral, but with
\[ \frac{\partial}{\partial x} \prod_{1 \leq i < j \leq n} (\lambda_{i} - \lambda_{j})^{4} \prod_{i=1}^{n} (x - \lambda_{i})^{2} = \det \left( \begin{array}{cccc}
1 & x & x^{2} & \ldots & x^{2n} \\
1 & \lambda_{1} & \lambda_{1}^{2} & \ldots & \lambda_{1}^{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_{n} & \lambda_{n}^{2} & \ldots & \lambda_{n}^{2n} \\
0 & 1 & 2\lambda_{1} & \ldots & 2n\lambda_{1}^{2n-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 2\lambda_{n} & \ldots & 2n\lambda_{n}^{2n-1}
\end{array} \right) \]

and

\[ \frac{\partial}{\partial x} \prod_{1 \leq i < j \leq n} (\lambda_{i} - \lambda_{j})^{4} \prod_{i=1}^{n} (x - \lambda_{i})^{2} = \det \left( \begin{array}{cccc}
1 & \lambda_{1} & \lambda_{1}^{2} & \ldots & \lambda_{1}^{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_{n} & \lambda_{n}^{2} & \ldots & \lambda_{n}^{2n} \\
0 & 1 & 2x & \ldots & (2n + 1)x^{2n-1} \\
0 & 1 & 2\lambda_{1} & \ldots & 2n\lambda_{1}^{2n-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 2\lambda_{n} & \ldots & 2n\lambda_{n}^{2n-1}
\end{array} \right) \] (3.31)

Therefore, the integrand in (3.28) is a $(2n + 2) \times (2n + 2)$ determinant:

\[ \det \left( \begin{array}{cccc}
1 & x & x^{2} & \ldots & x^{2n} & x^{m} \\
1 & \lambda_{1} & \lambda_{1}^{2} & \ldots & \lambda_{1}^{2n} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & \lambda_{n} & \lambda_{n}^{2} & \ldots & \lambda_{n}^{2n} & 0 \\
0 & 1 & 2x & \ldots & (2n + 1)x^{2n-1} & mx^{m-1} \\
0 & 1 & 2\lambda_{1} & \ldots & 2n\lambda_{1}^{2n-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 2\lambda_{n} & \ldots & 2n\lambda_{n}^{2n-1} & 0
\end{array} \right) \] (3.32)

we note $x = \lambda_{n+1}$, and by antisymmetrization, it becomes:

\[ \det \left( \begin{array}{cccc}
1 & \lambda_{1} & \lambda_{1}^{2} & \ldots & \lambda_{1}^{2n} & \lambda_{1}^{m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & \lambda_{n+1} & \lambda_{n+1}^{2} & \ldots & \lambda_{n+1}^{2n} & \lambda_{n+1}^{m} \\
0 & 1 & 2\lambda_{1} & \ldots & 2n\lambda_{1}^{2n-1} & m\lambda_{1}^{m-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 2\lambda_{n+1} & \ldots & 2n\lambda_{n+1}^{2n-1} & m\lambda_{n+1}^{m-1}
\end{array} \right) \] (3.33)

which obviously vanishes when $m \leq 2n$.

By the same argument, one would find that $< P_{2n+1} | x^{m} >$ reduces to the same kind of integral, but with $m$ replaced by $m + 1$, and vanishes when $m \leq 2n - 1$.

We have thus proven that the skew-orthogonal polynomials are indeed given by 3.16 and 3.18.
4 Large $N$ asymptotics

Most of the large $N$ universal statistical properties of a random $N \times N$ matrix $M$ belonging to one of the three ensembles $E_N^{(\beta)}$, can be expressed in terms of a few $h_n$, with $n$ close to the Fermi level

$$n_F = \frac{\beta}{2} N \beta$$  \hspace{1cm} (4.34)

More precisely, for $\beta = 2$, we need asymptotics of $P_n$ in the limit

$$N \to \infty \quad , \quad n \to \infty \quad , \quad n - N \sim O(1)$$  \hspace{1cm} (4.35)

for $\beta = 1$, we need asymptotics of $P_{2n}$ and $P_{2n+1}$ in the limit

$$N \to \infty \quad , \quad n \to \infty \quad , \quad 2n - N \sim O(1)$$  \hspace{1cm} (4.36)

and for $\beta = 4$, we need asymptotics of $P_{2n}$ and $P_{2n+1}$ in the limit

$$N \to \infty \quad , \quad n \to \infty \quad , \quad n - N = n - 2N_4 \sim O(1)$$  \hspace{1cm} (4.37)

4.1 The resolvant

We introduce the function $W(z)$ usually called the resolvant or Green function:

$$W(z) := W_m^{(\beta)}[\mathcal{V}](z) := \frac{1}{m} \left\langle \sum_{k=1}^{m} \frac{1}{z - \lambda_k} \right\rangle \propto \frac{1}{m} \left\langle \text{tr} \frac{1}{z - M} \right\rangle$$  \hspace{1cm} (4.38)

where $M \in E_m^{(\beta)}$ and the mean value is taken with respect to the weight:

$$e^{-m \beta \text{Tr} \mathcal{V}(M)}$$  \hspace{1cm} (4.39)

When there is no ambiguity, we will drop the $\beta$, $m$ or $\mathcal{V}$ indices, and write the resolvant as $W(z)$. Note that we have chosen a normalization such that:

$$W(z) \sim \frac{1}{z}$$  \hspace{1cm} (4.40)

The reason to introduce the resolvant is that the logarithmic derivative of $P_n(x)$ is proportional to the resolvant $W_m(z)$ (from $3.14, 3.16, 3.18$, at least when $n$ is even) for some appropriate value of $m$, and with a potential of the form:

$$\mathcal{V}(z) = \frac{1}{T} V(z) - r \ln(x - z).$$  \hspace{1cm} (4.41)

More precisely, we have:
• In the Unitary case \( \beta = 2 \):

\[
\frac{P_n^{(2)'}(x)}{P_n^{(2)}(x)} = n W_n(z) \big|_{z=x} \quad \text{with} \quad V(z) = \frac{N}{n} V(z) - \frac{1}{n} \ln (x - z) \quad (4.42)
\]

i.e. \( m = n, \ r = \frac{1}{n} \) and \( T = \frac{n}{N} \ (\rightarrow 1 \text{ when } n \rightarrow n_F) \).

• In the Orthogonal case \( \beta = 1 \):

\[
\frac{P_n^{(1)'}(x)}{P_n^{(1)}(x)} = 2n W_{2n}(z) \big|_{z=x} \quad \text{with} \quad V(x) = \frac{N}{m} V - \frac{1}{m} \ln(x - z) \quad (4.43)
\]

i.e. \( m = 2n, \ r = \frac{1}{2n} \) and \( T = \frac{2n}{N} \ (\rightarrow 1 \text{ when } n \rightarrow n_F) \).

• In the Symplectic case \( \beta = 4 \):

\[
\frac{P_n^{(4)'}(x)}{P_n^{(4)}(x)} = 2n W_n(z) \big|_{z=x} \quad \text{with} \quad V(x) = \frac{N}{n} V - \frac{2}{n} \ln(x - z) \quad (4.44)
\]

i.e. \( m = n, \ r = \frac{2}{n} \) and \( T = \frac{n}{N} \ (\rightarrow 1 \text{ when } n \rightarrow n_F) \).

In all three cases: \( T = \frac{n}{n_F} \) and \( r = \frac{\beta}{2n} \).

### 4.2 Asymptotics for the Resolvant

In a potential \( V \), the resolvant \( W(z) = W_m(z) \) satisfies the equations of motion (resulting from invariance of an integral like eq.2.1 under a change of variable \( M \rightarrow f(M) \)):

\[
W(z)^2 - \frac{\eta}{2n} W'(z) = \frac{2}{\beta} V'(z) W(z) - Q(z) + O(1/n^2) \quad (4.45)
\]

where \( \eta = (1, 0, -1) \) respectively for \( \beta = (1, 2, 4) \) and \( Q(z) \) is a polynomial of degree \( \deg V - 2 \), which is not determined by the equations of motions, it has to be determined by analytical considerations, for instance the one-cut assumption.

Here, we will consider a potential \( V \) of the form:

\[
V(z) = \frac{1}{T} V(z) - r \ln (x - z) \quad (4.46)
\]

and we will be interested in the limit where \( T \rightarrow 1 \) and \( r \) are small of order \( 1/n \).

The method is to find first the solution \( W(z) \) at \( T = 1 \) and \( r = 0 \). We write it:

\[
W(z) = W_0(z) + \frac{\eta}{2n} W_1(z) + O(1/n^2) \quad (4.47)
\]

\[\text{when } V' \text{ has poles, } Q \text{ may have poles too. } Q(z) \text{ is a rational function, whose poles must be chosen in order to cancel the poles of } W(z) \text{ in eq.4.45.} \]
and then, add the variations:

\[(T - 1) \frac{\partial}{\partial T} W_0 + r \frac{\partial}{\partial r} W_0 \quad (4.48)\]

(at order 1/n, we don’t need to consider the variations of \(W_1\) with respect to \(T\) and \(r\), the derivatives are taken at \(T = 1\) and \(r = 0\)).

### 4.3 Contribution of \(W_0\)

The function \(W_0(z)\), (as well as its derivatives with respect to \(T\) and \(r\)) has been extensively studied in RMT. Note that \(W_0\) is nearly the same for \(\beta = 1, 2\) or 4. Let us recall here some of the main features of \(W_0\) in order to fix the notations.

At \(n \to \infty\) (and \(T = 1\) and \(r = 0\)), \((4.45)\) reduces to a quadratic equation for \(W_0(z)\).

The one-cut-solution is:

\[W_0(z) = \frac{1}{\beta} \left(V'(z) - M(z) \sqrt{(z - a)(z - b)} \right) = \frac{1}{\beta} V'(z) - i\pi \rho(z) \quad (4.49)\]

Where \(M(z)\) is a polynomial of degree \(d - 1\) (\(d = \deg V'\)), which is completely determined by the large \(z\) limit condition \((4.40)\):

\[M(z) = \text{Pol} \frac{V'(z)}{\sqrt{(z - a)(z - b)}} \quad (4.50)\]

The end-points \(a\) and \(b\) too, are determined by \((4.40)\) which implies:

\[\oint \frac{V'(z)}{\sqrt{(z - a)(z - b)}} dz = 0 \quad , \quad \oint \frac{zV'(z)}{\sqrt{(z - a)(z - b)}} dz = 2i\pi \beta \quad (4.51)\]

where the contour encircles the cut \([a, b]\) in the trigonometric direction.

The imaginary part of the resolvant

\[\rho(z) = \frac{1}{\beta \pi} M(z) \sqrt{(z - a)(b - z)} \quad (4.52)\]

is the average density of eigenvalues of the random matrix (as an example, consider the gaussian case: \(V\) is quadratic, i.e. \(V'\) is of degree \(d = 1\), thus \(M(z)\) is a constant and \(\rho(z) = \sqrt{(z - a)(b - z)}\) is the famous semi-circle law).

**Some notations:**

It will be convenient to parametrize \(z\) as:

\[z = \frac{a + b}{2} + \frac{b - a}{2} \cos \phi \quad , \quad \alpha := \frac{b - a}{4} \quad (4.53)\]

We write:

\[\sigma(z) := (z - a)(z - b) \quad , \quad \sqrt{\sigma(z)} = 2i\alpha \sin \phi \quad (4.54)\]

Note that \(\phi(z)\) is a multi-valued function. We will see that both determinations \(\phi\) and \(-\phi\) will enter the asymptotic expression of the orthogonal polynomials when \(z \in [a, b]\).
4.4 Variations of $W_0$ with respect to $T$ and $r$

It can be proven (see [13] for instance) that

$$W_T(z) := \frac{d}{dT} T W_0(z) = \frac{1}{\sqrt{\sigma(z)}} \frac{d\phi(z)}{dz}$$  \hspace{1cm} (4.55)$$

and

$$W_r(z) := \beta \frac{d}{dr} W_0(z) = -\frac{1}{\sqrt{\sigma(z)}} \frac{\sqrt{\sigma(z)} - \sqrt{\sigma(x)}}{(z-x)} + \frac{1}{\sqrt{\sigma(z)}}$$  \hspace{1cm} (4.56)$$

In particular at $z = x$, we have:

$$W_r(x) = -\frac{\sigma'(x)}{2\sigma(x)} + \frac{1}{\sqrt{\sigma(x)}}$$  \hspace{1cm} (4.57)$$

4.5 Contribution of $W_1$

For $T = 1$ and $r = 0$, and at order $O(1/n)$, the equation of motion reduces to:

$$W^2(z) - \frac{\eta}{2n} W'(z) + O(1/n^2) = \frac{2}{\beta} V'(z) W(z) - Q(z)$$  \hspace{1cm} (4.58)$$

and we expand $W(z)$ at first order in $1/n$ as:

$$W(z) \sim W_0(z) + \frac{\eta}{2n} W_1(z) + O(1/n^2)$$  \hspace{1cm} (4.59)$$

At order $1/n$, eq. (4.58) gives (using the value of $W_0(z)$ from [14]):

$$\frac{2}{\beta} W_1(z) = \frac{Q_1(z) - W_0'(z)}{M(z) \sqrt{\sigma(z)}}$$  \hspace{1cm} (4.60)$$

where $Q_1(z)$ is a polynomial of degree $d - 2$.

Let us factorize $M(z)$ (recall that $d = \text{deg} V'$ and $g$ is the leading coef. of $V'$):

$$M(z) = g \prod_{k=1}^{d-1} (z - z_k)$$  \hspace{1cm} (4.61)$$

and decompose $W_1$ in single pole terms. The condition that $W_1(z)$ is regular when $z = z_k$ allows to determine the polynomial $Q_1(z)$, and eventually we get:

$$W_1(z) = \frac{\sigma'(z)}{4\sigma(z)} + \frac{1}{2} \sum_{k=1}^{d-1} \frac{\sqrt{\sigma(z)} - \sqrt{\sigma(z_k)}}{(z - z_k) \sqrt{\sigma(z)}} - \frac{d}{2} \frac{1}{\sqrt{\sigma(z)}}$$  \hspace{1cm} (4.62)$$

With the parametrization $z = \frac{a+b}{2} + 2\alpha \cos \phi$ and $z_k = \frac{a+b}{2} + 2\alpha \cos \phi_k$, we have

$$W_1(z) = \frac{d}{dz} \left[ \ln \frac{\phi_0}{2} + \sum_{k=1}^{d-1} \ln \left( \frac{\phi + \phi_k}{2} - \frac{d}{2} \phi \right) - \frac{d}{2} \phi \right]$$  \hspace{1cm} (4.63)$$

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4.6 Asymptotics of the skew-orthogonal polynomials

Eventually we have computed all the contributions to the asymptotics of the resolvant:

$$W(z) \sim \frac{1}{T} W_0(z) + \frac{T - 1}{T} W_T(z) + \frac{r}{\beta} W_r(z) + \frac{\eta}{2n} W_1(z) + O(1/n^2)$$  \hspace{1cm} (4.64)

i.e.:

$$2nW(z) \sim \beta N_\beta W_0(z) + (2n - \beta N_\beta) W_T(z) + W_r(z) + \eta W_1(z) + O(1/n)$$  \hspace{1cm} (4.65)

where $W_0, W_T, W_r, W_1$ are given by \[4.49, 4.55, 4.56 \text{ (or 4.57), 4.62 \text{ (or 4.63).}}

Combining everything together, we get:

- $\beta = 2$. From $P'_n/P_n = nW(x)$, we get the asymptotic orthogonal polynomial (already known \[14, 13, 2]):

$$P_n^{(2)}(x)e^{-\frac{N}{2}V(x)} \sim \frac{C_n^{(2)}}{\sqrt{2i\alpha \sin \phi}} e^{-\frac{Ni\pi}{2} \int_{\rho(y)dy} e^{i\left(n-N+\frac{1}{2}\right)\phi}} + \text{c.c.}$$  \hspace{1cm} (4.66)

The normalization constant $C_n^{(2)} = \alpha^{n+1/2}$ is such that $P_n(x) \sim x^n$ for large $x$.

- $\beta = 1$ and $\beta = 4$ cases contain an extra contribution from $W_1$.

- $\beta = 1$. From $P'_n/P_{2n} = 2nW(x)$ we get:

$$P_{2n}^{(1)}(x)e^{-NV(x)} \sim \frac{C_n^{(1)}}{\sqrt{2i\alpha \sin \phi}} e^{-\frac{Ni\pi}{2} \int_{\rho(y)dy} e^{i\left(2n+1-N-\frac{1}{2}\right)\phi} M_+ (\phi) + \text{c.c.}}$$  \hspace{1cm} (4.67)

where

$$M_+ (\phi) = M_- (-\phi) = \prod_{k=1}^{d-1} 2i \sin \left(\frac{\phi + \phi_k}{2}\right)$$  \hspace{1cm} (4.68)

note that $M(x) = g \alpha^{d-1} M_+ (\phi)M_- (\phi)$, where $g$ is the leading coefficient of $V'(x)$.

$$C_n^{(1)} = \alpha^{2n+1/2} \prod_{k=1}^{d-1} e^{-i\phi_k/2}$$  \hspace{1cm} (4.69)

is the normalization constant chosen so that $P_{2n}(x) \sim x^{2n}$ for large $x$.

The odd polynomial is found from $P_{2n+1}/P_{2n} = < x + \text{Tr } M + c_n >$ and $< \text{Tr } M > = 2n \lim_{z \to \infty} z^2(W(z) - 1)$ (note that we need \[4.56\] not \[4.57\]). The whole
The dependence of $P_{2n+1}/P_{2n}$ comes from $x + \lim_{z \to \infty} z^2(W_r(z) - 1) = \sqrt{\sigma(x)} - \frac{a+b}{2}$.

Therefore, (and up to an arbitrary linear combination with $P_{2n}$), we have:

$$P_{2n+1}^{(1)}(x)e^{-NV(x)} \sim C_n^{(1)} \sqrt{2i\alpha \sin \phi} \ e^{-N\alpha \int_a^x \rho(y)dy} \ e^{i(2n+1-N-\frac{d}{2})\phi} \ M_+(\phi) + \text{c.c.}$$

(4.70)

- $\beta = 4$. From $P'_{2n}/P_{2n} = 2nW(x)$ we get:

$$P_{2n}^{(4)}(x)e^{-\frac{1}{2}V(x)} \sim \frac{C_n^{(4)}}{\sqrt{2i\alpha \sin \phi}} \ e^{-2N\alpha \int_a^x \rho(y)dy} \ e^{i(2n+1-2N+\frac{d}{2})\phi} \ \frac{M_-(\phi)}{i\rho(x)} + \text{c.c.}$$

(4.71)

with normalization constant:

$$C_n^{(4)} = \frac{g}{4\pi} \alpha^{2n+d+\frac{d}{2}} \prod_{k=1}^{d-1} e^{\phi_k/2}$$

(4.72)

and

$$P_{2n+1}^{(4)}(x)e^{-\frac{1}{2}V(x)} \sim C_n^{(4)} \sqrt{2i\alpha \sin \phi} \ e^{-2N\alpha \int_a^x \rho(y)dy} \ e^{i(2n+1-2N+\frac{d}{2})\phi} \ \frac{M_-(\phi)}{i\rho(x)} + \text{c.c.}$$

(4.73)

Note that we have used: $i\rho(x) = \frac{g}{4\pi} \alpha^d M_+(\phi) M_-(\phi) \ 2i \sin \phi$.

**Some remarks:**

- The derivation presented here is actually valid only when $x \notin [a, b]$, giving only one exponential term, with the determination of $\phi(x)$ (from (4.53)) such that $P_n(x)e^{-NV(x)}$ decreases when $x \to \infty$. When $x \in [a, b]$, a careful analysis shows that both determinations of $\phi(x)$ must be taken into account. The only effect is to add the complex conjugate exponential (c.c.) to the asymptotics, so that $P_n(x)$ is indeed real when $x \in [a, b]$. Outside $[a, b]$, $P_n e^{-NV}$ decreases exponentially, and in $[a, b]$, it oscillates like a cosine function, and it indeed has $n$ zeroes.

- Our derivation was carried out only in the "one-cut" case. We have assumed that the support of $\rho(x)$ is connected and is made of one interval $[a, b]$.

- Those asymptotics are not valid when $x$ is close to $a$ or $b$.

- Note that the above expressions all have the correct large $x$ behaviour: $P_n(x) \sim x^n$. It can be seen easily if one remembers that $x \sim \alpha e^{i\phi}$ when $i\phi \to +\infty$. 

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4.7 Check of orthogonality

We have presented a derivation of the asymptotics (4.67-4.73), so that there should be no reason to doubt they fulfill the orthogonality condition. However, it is interesting to see how. We will just sketch the procedure:

In all cases, we have to compute integrals of $P_nP_me^{-NV}$, with $x$ running from $-\infty$ to $+\infty$. The contributions outside $[a, b]$ are exponentially small, the integrals can thus be computed inside $[a, b]$. Within $[a, b]$, terms which oscillate exponentially fast like $e^{Ni\pi}\rho$, average to zero at order $O(1/N)$, so that at leading order, it is sufficient to consider only the cross-terms in the product $P_nP_m$, with opposite signs for the two determinations of $\phi$.

In the $\beta=1$ case, the scalar product $<P_n|P_m>$ of (2.7) can be computed by integration by part. For that, you need a primitive of $P_ne^{-NV}$, which is achieved at leading order by dividing 4.67 or 4.70 by $\rho(x) = \text{cte} M_+(\phi)M_-(\phi)\sin\phi$.

In the $\beta=4$ case, you need a derivative of $P_ne^{\frac{2}{N}V}$, which is achieved at leading order by multiplying 4.71 or 4.73 by $\rho(x) = \text{cte} M_+(\phi)M_-(\phi)\sin\phi$.

Then you find that in both cases ($\beta=1$ and 4), and up to unimportant constant factors, you have at leading order in $1/n$:

\[
<P_{2n}|P_{2m}> \propto \int_0^\pi d\phi \frac{\sin 2(n-m)\phi}{\sin \phi} = 0 \quad (4.74)
\]

\[
<P_{2n+1}|P_{2m+1}> \propto \int_0^\pi d\phi \sin \phi \sin 2(n-m)\phi = 0 \quad (4.75)
\]

\[
<P_{2n+1}|P_{2m}> \propto \int_0^\pi d\phi \cos 2(n-m)\phi \propto \delta_{nm} \quad (4.76)
\]

which confirms that our asymptotics indeed fulfill the orthogonality properties.

Taking into account properly the constant factors, we can determine the $h_n$'s:

- $\beta = 2$:
  
  \[
  h_n^{(2)} \sim 2\pi \alpha^{2n+1} \quad (4.77)
  \]

- $\beta = 1$:
  
  \[
  h_n^{(1)} \sim \frac{16\pi}{Ng\alpha^{d+1}} \alpha^{4n+3} \quad (4.78)
  \]

- $\beta = 4$:
  
  \[
  h_n^{(4)} \sim 2N\pi g\alpha^{d+1} \alpha^{4n+1} \quad (4.79)
  \]
5 Conclusions

Therefore, we have obtained some exact integral expressions and asymptotics for the skew-orthogonal polynomials involved in the Orthogonal and Symplectic random matrix ensembles.

Our asymptotics were derived in the "one-cut" case only, though it seems obvious that the result could be extended easily to the multicut case, following the method of [15] or [16], it would involve hyper-elliptical theta-functions instead of exponentials.

Another possible extension of the method presented here is to "multi-matrix models", and a time dependant matrix, as in [13]. It seems that the same kind of asymptotics could be obtained.

The asymptotics of the skew-orthogonal polynomials are useful to evaluate the kernels:

\[
K(\lambda, \mu) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{h_n} (P_{2n}(\lambda)P_{2n+1}(\mu) - P_{2n+1}(\lambda)P_{2n}(\mu)) e^{-NV(\lambda)} e^{-NV(\mu)}
\]  

(5.80)

which give all the correlation functions. For instance with \(\beta = 4\), we have

\[
\rho(\lambda) = -\frac{\partial}{\partial \lambda} \left( K(\lambda, \mu) \right)_{\mu = \lambda} (5.81)
\]

\[
\rho_c(\lambda, \mu) = -\frac{\partial}{\partial \lambda} K(\lambda, \mu) \frac{\partial}{\partial \mu} K(\lambda, \mu) + K(\lambda, \mu) \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \mu} K(\lambda, \mu) (5.82)
\]

In order to use the asymptotics of the orthogonal polynomials in (5.80), one needs a generalization of the Darboux-Christoffel theorem, which allows to write \(K(\lambda, \mu)\) in terms of a few \(P_n\) only with \(n\) close to the Fermi level \(n_F\). With asymptotics of the type [4.66, 4.67, 4.70, 4.71] or [4.73], the Darboux-Christoffel Theorem merely amounts to a formal resummation of the geometrical series (it was proven in [13] for hermitian multi-matrix models, and we cannot see any reason why the same proof would not work here). For instance in the \(\beta = 4\) case, the generalization of the Darboux-Christoffel theorem reads:

\[
\sum_{n=0}^{N-1} e^{i(2n+3-2N)(\phi(\lambda) - \phi(\mu))} \sim \frac{1}{2i \sin (\phi(\lambda) - \phi(\mu))} (5.83)
\]

This trick allows to find asymptotics for the kernels \(K(\lambda, \mu)\), and then asymptotics for all the correlation functions. One can then easily check that in the short distance regime \(|\lambda - \mu| \sim O(1/N)\), the universal 2-point connected correlation function is well reproduced, and that in the long distance regime \(|\lambda - \mu| \sim O(1)\), the smoothed 2-point connected correlation function is correctly reproduced too. The leading behaviour of short and long distance correlation functions was already known from other methods [3].
so that our method does not provide any new result for the correlation functions. However, it seems that our asymptotics can be used to build a rigorous mathematical proof of the universality, following the method of \cite{14}, because they allow a good control of the approximations.

In addition, the fact that the skew-orthogonal polynomials are exactly the average characteristic polynomials of the random matrices is remarkable. It would be interesting to understand the generality of this result, and for instance try to generalize it to the other random matrix ensembles related to Cartan's classification of symmetric spaces \cite{17,18}.
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