Simpson- and Newton-Type Inequalities for Convex Functions via \((p,q)\)-Calculus

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Abstract: In this paper, we establish several new \((p,q)\)-integral identities involving \((p,q)\)-integrals by using the definition of a \((p,q)\)-derivative. These results are then used to derive \((p,q)\)-integral Simpson- and Newton-type inequalities involving convex functions. Moreover, some examples are given to illustrate the investigated results.

Keywords: Simpson inequality; Newton inequality; convex function; \((p,q)\)-derivative; \((p,q)\)-integral; \((p,q)\)-calculus

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1. Introduction

Mathematical inequalities, as critical tools, are employed in both areas of pure and applied mathematics [1–8]. These inequalities have been continuously improved because of their wide applications in those fields. A convex function has attracted interest because it can be applied in various techniques by many researchers (see [9–16] for more details and the references cited therein).

A convex function is defined as follows: A function \(f : [a, b] \rightarrow \mathbb{R}\) is convex if

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]

holds for all \(x, y \in [a, b]\) and \(t \in [0, 1]\).

In recent years, convex functions in mathematical inequalities have attracted considerable attention. The most famous inequalities used with the convex functions are Simpson- and Newton-type inequalities (see [17–25] for more details).

1. Simpson’s quadrature (Simpson’s 1/3 rule) is formulated as follows:

\[
\int_a^b f(x)dx \approx \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right],
\]

see [19] for more details.

2. Simpson’s second or Newton–Cotes quadrature (Simpson’s 3/8 rule) is formulated as follows:

\[
\int_a^b f(x)dx \approx \frac{1}{8} \left[ f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right],
\]
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...see [23] for more details.

The error estimations of Simpson- and Newton-type inequalities are as follows:

**Theorem 1** (Ref. [19]). If \( f : [a, b] \to \mathbb{R} \) is a four times continuously differentiable function on \((a, b)\) and

\[
\left\| f^{(4)} \right\|_{\infty} = \sup_{x \in (a, b)} \left| f^{(4)}(x) \right| < \infty,
\]

then

\[
\frac{1}{6} \left[ f(a) + 4f\left(\frac{a + b}{2}\right) + f(b) \right] - \frac{1}{b - a} \int_{a}^{b} f(x) dx \leq \frac{1}{2880} \left\| f^{(4)} \right\|_{\infty} (b - a)^5.
\]

**Theorem 2** (Ref. [23]). If \( f : [a, b] \to \mathbb{R} \) is a four times continuously differentiable function on \((a, b)\) and

\[
\left\| f^{(4)} \right\|_{\infty} = \sup_{x \in (a, b)} \left| f^{(4)}(x) \right| < \infty,
\]

then

\[
\frac{1}{8} \left[ f(a) + 3f\left(\frac{2a + b}{3}\right) + 3f\left(\frac{a + 2b}{3}\right) + f(b) \right] - \frac{1}{b - a} \int_{a}^{b} f(x) dx \leq \frac{1}{6480} \left\| f^{(4)} \right\|_{\infty} (b - a)^5.
\]

So far, Simpson- and Newton-type inequalities have been studied in the form of a convex function using quantum calculus by many researchers, and the results of quantum calculus can be found in [26–32], and the references cited therein.

Quantum calculus or \( q \)-calculus is the study of calculus with no limits, beginning by studying Newton’s infinite series, and was revealed by Euler (1707–1783). Then, many researchers, such as Gudermann (1798–1897), Weierstrass (1815–1897), and Heaviside (1850–1925) studied the properties of \( q \)-series. In 1910, the \( q \)-derivative and \( q \)-integral of a continuous function on the interval \((0, \infty)\), based on the \( q \)-calculus of an infinite series, were defined by Jackson [33]. In \( q \)-calculus, the main objective is to obtain the \( q \)-analogues of mathematical objects recaptured by taking \( q \to 1 \). In recent years, the \( q \)-calculus has attracted interest because of its various applications in mathematics and physics (see [34–40] for more details and the references cited therein).

Tariboon and Ntouyas [41] defined new \( q \)-derivatives and \( q \)-integrals of a continuous function on a finite interval. Recently, such definitions have been applied in various inequalities, such as Hermite–Hadamard inequalities [42–45], Hanh integral inequalities [46], Hermite–Hadamard-like inequalities [47], Ostrowski inequalities [48], Fejér-type inequalities [49], and Simpson- and Newton-type inequalities [50], and the references cited therein.

Another generalization of \( q \)-calculus on the interval \((0, \infty)\) is well-known as post-quantum calculus or \((p, q)\)-calculus. The \((p, q)\)-calculus includes two-parameter quantum calculus \((p \text{ and } q\text{-numbers})\) which are independent. In \((p, q)\)-calculus, we obtain the \( q \)-calculus formula for the case of \( p = 1 \), and then get the classical formula for the case of \( q \to 1 \). This generalization was first introduced by Chakrabarti and Jagannathan [51] in 1991. Then, Tunç et al. [52,53] presented new \((p, q)\)-derivatives and \( q \)-integrals of a continuous function on a finite interval in 2016. Based on the definitions of \((p, q)\)-calculus, many literatures have been published by many researchers (see [54–58] and the references cited therein).

In 2020, Budak et al. [50] presented Simpson- and Newton-type inequalities for convex functions via \( q \)-calculus. In this paper, we establish some new integral inequalities of Simpson- and Newton-type inequalities for convex functions via \((p, q)\)-calculus to generalize and extend the results given in the above-mentioned report. Furthermore, we give some examples to investigate the main results.
The rest of the paper is organized as follows: In Section 2, we recall some basic knowledge and notations used in the next part. In Section 3, we give Simpson-type inequalities for a convex function via the \((p, q)\)-calculus. In Section 4, we give Newton-type inequalities for convex function via \((p, q)\)-calculus. In Section 5, we show some examples to illustrate the investigated results. In the final part, we summarize the conclusions.

2. Preliminaries

In this section, we give basic knowledge used in our work. Throughout this paper, let \([a, b] \subseteq \mathbb{R}\) be an interval, and let \(a < b\) and \(0 < q < p \leq 1\) be constants.

**Definition 1** [Refs. \([52,53]\)]. Suppose that \(f : [a, b] \to \mathbb{R}\) is a continuous function. Then, the \((p, q)\)-derivative of function \(f\) at \(t \in [a, b]\) is defined by

\[
_aD_{p,q}f(t) = \begin{cases} 
(f(pt + (1 - p)a) - f(qt + (1 - q)a), & \text{if } t \neq a; \\
\lim_{t \to a^+} aD_{p,q}f(t), & \text{if } t = a.
\end{cases}
\]

The function \(f\) is said to be a \((p, q)\)-differentiable function on \([a, b]\) if \(_aD_{p,q}f(t)\) exists for all \(t \in [a, b]\).

In Definition 1, if \(p = 1\), then \(_aD_{1,q}f(t) = _aD_qf(t)\), and Equation (1) reduces to

\[
_aD_qf(t) = \begin{cases} 
\frac{f(t) - f(qt + (1 - q)a)}{(1 - q)(t - a)}, & \text{if } t \neq a; \\
\lim_{t \to a^+} aD_qf(t), & \text{if } t = a,
\end{cases}
\]

which is the \(q\)-derivative of function \(f\) defined in \([a, b]\) (see \([59–61]\) for more details). In addition, if \(a = 0\), then \(_0D_qf(t) = D_qf(t)\), and Equation (2) reduces to

\[
D_qf(t) = \begin{cases} 
\frac{f(t) - f(qt)}{(1 - q)t}, & \text{if } t \neq 0; \\
\lim_{t \to 0^+} D_qf(t), & \text{if } t = a,
\end{cases}
\]

which is the \(q\)-derivative of function \(f\) defined in \([0, b]\) (see \([62]\) for more details).

**Definition 2** [Ref. \([63]\)]. Suppose that \(f : [a, b] \to \mathbb{R}\) is a continuous function. Then, the \((p, q)\)-derivative of function \(f\) at \(t \in [a, b]\) is defined by

\[
_bD_{p,q}f(t) = \begin{cases} 
\frac{f(qt + (1 - q)b) - f(pt + (1 - p)b)}{(p - q)(b - t)}, & \text{if } t \neq b; \\
\lim_{t \to b^-} bD_{p,q}f(t), & \text{if } t = b.
\end{cases}
\]

The function \(f\) is said to be a \((p, q)\)-differentiable function on \([a, b]\) if \(_bD_{p,q}f(t)\) exists for all \(t \in [a, b]\).

In Definition 2, if \(p = 1\), then \(_bD_{1,q}f(t) = _bD_qf(t)\), and Equation (4) reduces to

\[
_bD_qf(t) = \begin{cases} 
\frac{f(qt + (1 - q)b) - f(t)}{(1 - q)(b - t)}, & \text{if } t \neq b; \\
\lim_{t \to b^-} bD_qf(t), & \text{if } t = b,
\end{cases}
\]

which is the \(q\)-derivative of function \(f\) defined in \([a, b]\) (see \([64,65]\) for more details).
The function $f$ is said to be a $(p, q)_a$-integrable function on $[a, b]$ if $f$ is a continuous function and a differentiable function on $[a, b]$. Then, the $(p, q)_a$-integral of function $f$ at $t \in [a, b]$ is defined by

$$\int_a^b f(t) \, d_{p,q} t = (p - q)(b - a) \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j b + \left(1 - \frac{q^j}{p^{j+1}}\right) a}{b}\right).$$

(6)

The function $f$ is said to be a $(p, q)_a$-differentiable function on $[a, b]$ if $f$ is a continuous function. Then, the $(p, q)_a$-Hermite–Hadamard inequalities are as follows:

$$f\left(\frac{pa + qb}{p + q}\right) \leq \frac{1}{p(b - a)} \int_a^b f(t) \, d_{p,q} t \leq \frac{qf(a) + pf(b)}{p + q}.$$

(10)

3. Simpson-Type inequalities for $(p, q)$-Calculus

**Theorem 5.** Let $f : [a, b] \rightarrow \mathbb{R}$ be a $(p, q)_b$-differentiable function on $(a, b)$. If $b_D_{p,q}f$ is a continuous function and a $(p, q)_b$-integrable function on $[a, b]$, then

$$b - a \int_0^1 \phi(t) \, b_D_{p,q} f(ta + (1 - t)b) \, d_{p,q} t = \frac{1}{p(b - a)} \int_a^b (p - q)(b - a) \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f(x) \, d_{p,q} x - \frac{1}{6} \left[ f(a) + 4f\left(\frac{a + b}{2}\right) + f(b) \right].$$

(13)
where
\[ \phi(t) = \begin{cases} 
q_t - \frac{1}{6}, & \text{for } 0 \leq t < \frac{1}{2}; \\
q_t - \frac{3}{6}, & \text{for } \frac{1}{2} \leq t \leq 1.
\end{cases} \]

**Proof.** From basic properties of the \((p, q)\)-integral and the definition of \(\phi(t)\), it follows that
\[
\int_0^1 \phi(t) \ b_{p,q} f(ta + (1-t)b) \, dp,q t = \frac{2}{3} \int_0^1 b_{p,q} f(ta + (1-t)b) \, dp,q t \\
+ \int_0^1 q t \ b_{p,q} f(ta + (1-t)b) \, dp,q t \\
- \frac{5}{6} \int_0^1 b_{p,q} f(ta + (1-t)b) \, dp,q t.
\]

By Definition 2, we have
\[
b_{p,q} f(ta + (1-t)b) = \frac{f(q(ta + (1-t)b) + (1-q)b) - f(p(ta + (1-t)b) + (1-p)b)}{(p-q)(b - (ta + (1-t)b))} \\
= \frac{f(qta + (1qt) + f(pita + (1pt)b)}{(p-q)(b-a)}.
\]

Then, we obtain
\[
\int_0^1 \frac{1}{b-a} \sum_{j=0}^{\infty} f \left( \frac{q^{j+1}}{2p^{j+1}} a + \left( 1 - \frac{q^{j+1}}{2p^{j+1}} \right) b \right) - \frac{1}{b-a} \sum_{j=0}^{\infty} f \left( \frac{q^j}{2p^j} a + \left( 1 - \frac{q^j}{2p^j} \right) b \right)
= \frac{1}{b-a} \left[ f(b) - f \left( \frac{a+b}{2} \right) \right].
\]

Similarly, we obtain
\[
\int_0^1 q t \ b_{p,q} f(ta + (1-t)b) \, dp,q t \\
= \int_0^1 q t \frac{f(qta + (1q) + f(pita + (1pt)b)}{(p-q)(b-a)} \, dp,q t \\
= \frac{1}{b-a} \left( \frac{p-q}{p} \right) \sum_{j=0}^{\infty} \frac{q^j}{p^j} f \left( \frac{q^j}{p^j} a + \left( 1 - \frac{q^j}{p^j} \right) b \right) - \frac{f(a)}{b-a} \\
= \frac{1}{p(b-a)^2} \int_{pa+(1-p)b}^{b} f(x) \ b_{p,q} x - \frac{f(a)}{b-a}.
\]

Finally, we observe that
\[
\int_0^1 b_{p,q} f(ta + (1-t)b) \, dp,q t = \frac{f(b) - f(a)}{b-a}.
\]
Substituting Equations (15)–(17) in Equation (14), we have
\[ \int_0^1 \phi(t) \, b \, D_{p,q} f(ta + (1-t)b) \, d_{p,q} t = \frac{2}{3} \left[ \frac{1}{b-a} \left( f(b) - f\left( \frac{a+b}{2} \right) \right) \right] + \frac{1}{p(b-a)^2} \int_{pa+(1-p)b}^{b} f(x) \, b \, d_{p,q} x - \frac{5}{6} \frac{f(b) - f(a)}{b-a} \]
\[ = \frac{1}{p(b-a)^2} \int_{pa+(1-p)b}^{b} f(x) \, b \, d_{p,q} x - \frac{1}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right]. \]

Multiplying the above equality with $b-a$, we obtain the required result. The proof is completed. \[ \square \]

**Remark 1.** If $p = 1$, then Equation (13) reduces to
\[ (b-a) \int_0^1 \phi(t) \, b \, D_{q} f(ta + (1-t)b) \, d_{q} t = \frac{1}{b-a} \int_a^b f(x) \, b \, d_{q} x - \frac{1}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right], \]
where
\[ \phi(t) = \begin{cases} qt - \frac{1}{b}, & \text{for } 0 \leq t < \frac{1}{2}; \\ qt - \frac{5}{b}, & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases} \]
which appeared in [50].

**Theorem 6.** Let $f : [a,b] \to \mathbb{R}$ be a $(p,q)^3$-differentiable function on $(a,b)$. If $|b \, D_{p,q} f|$ is a convex function and a $(p,q)^3$-differentiable function on $[a,b]$, then
\[ \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^{b} f(x) \, b \, d_{p,q} x - \frac{1}{6} \left[ f(a) + 4f\left( \frac{a+b}{2} \right) + f(b) \right] \right| \]
\[ \leq (b-a) \left[ |\Lambda_1(p,q) + \Lambda_3(p,q)| \, b \, D_{p,q} f(a) + |\Lambda_2(p,q) + \Lambda_4(p,q)| \, b \, D_{p,q} f(b) \right], \quad (18) \]
where $\Lambda_i(p,q)$, $i = 1,2,3,4$ are defined by
\[ \Lambda_1(p,q) = \int_0^{\frac{1}{2}} \left| qt - \frac{1}{b} \right| \, d_{p,q} t \]
\[ = \begin{cases} \frac{p^2 - 2pq - 2q^2}{24(p+q)(p^2 + pq + q^2)}, & \text{for } 0 < q < \frac{1}{2}; \\ \frac{18q^4 + 18pq^3 - 9pq^2q + 2q^2 + 2pq - 2q + 2p - 2p}{216q^2(p + q)(p^2 + pq + q^2)}, & \text{for } \frac{1}{2} \leq q < 1, \end{cases} \]
\[ \Lambda_2(p,q) = \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} (1-t) \right| \, d_{p,q} t \]
\[ = \begin{cases} \frac{2p^3 - p^2 + 2pq - 2p^2q + 2q^2 - 2pq^2 - 4q^3}{24(p+q)(p^2 + pq + q^2)}, & \text{for } 0 < q < \frac{1}{3}; \\ \frac{36q^5 + 18pq^4 - 6pq^4 + 18pq^3 + 6pq^3 - 12q^3 + 33p^3q^2}{216q^2(p + q)(p^2 + pq + q^2)}, & \text{for } \frac{1}{3} \leq q < 1, \end{cases} \]
Similarly, we have

\[ \begin{align*}
\Lambda_3(p, q) &= \int_0^1 \left| qt - \frac{5}{6} \right| \, t \, dp_q t \\
&= \begin{cases} 
15p^2 - 6pq - 6q^2, & \text{for } 0 < q < \frac{5}{6}; \\
\frac{24(p + q)(p^2 + pq + q^2)}{256q^2(p + q)(p^2 + pq + q^2)}, & \text{for } \frac{5}{6} \leq q < 1,
\end{cases} \\
\Lambda_4(p, q) &= \int_0^1 \left| qt - \frac{5}{6} \right|^3 (1 - t) \, dp_q t \\
&= \begin{cases} 
10p^3 - 15p^2 + 2p^2q + 6pq + 2pq^2 + 6q^2 - 8q^3, & \text{for } 0 < q < \frac{5}{6}; \\
\frac{[270pq^4 + 282q^4 - 270p^3q^4 + 582p^3q^3]}{216q^2(p + q)(p^2 + pq + q^2)}, & \text{for } \frac{5}{6} \leq q < 1.
\end{cases}
\end{align*} \]

**Proof.** Taking the absolute value of both sides of Theorem 5, we observe that

\[ \begin{align*}
&\left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b f(x) \, b \, dp_q x - \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \\
&= \mid (b-a) \int_0^1 \phi(t) \, b \, D_{p,q} f(ta + (1-t)b) \, dp_q t \mid \\
&= (b-a) \left| \int_0^1 \left( qt - \frac{5}{6} \right) \, b \, D_{p,q} f(ta + (1-t)b) \, dp_q t \right| \\
&\quad + \int_0^1 \left( qt - \frac{5}{6} \right) \, b \, D_{p,q} f(ta + (1-t)b) \, dp_q t \\
&\leq (b-a) \left( \int_0^1 \left| qt - \frac{1}{6} \right| \, b \, D_{p,q} f(ta + (1-t)b) \, dp_q t \right) \\
&\quad + \int_0^1 \left| qt - \frac{5}{6} \right| \, b \, D_{p,q} f(ta + (1-t)b) \, dp_q t \right). \quad (19)
\end{align*} \]

If the first \((p, q)\)-integral on the right side of the inequality is used to consider the convexity of \([b]D_{p,q} f\), from the case when \(a = 0\) of Lemma 1, then we obtain

\[ \begin{align*}
\int_0^1 \left| qt - \frac{1}{6} \right| \, b \, D_{p,q} f(ta + (1-t)b) \, dp_q t &\leq \left| b \, D_{p,q} f(a) \right| \int_0^1 \left| qt - \frac{1}{6} \right| \, dp_q t \\
&\quad + \left| b \, D_{p,q} f(b) \right| \int_0^1 \left| qt - \frac{1}{6} \right| \, (1-t) \, dp_q t \\
&= \Lambda_1(p, q) \left| b \, D_{p,q} f(a) \right| + \Lambda_2(p, q) \left| b \, D_{p,q} f(b) \right|. \quad (20)
\end{align*} \]

Similarly, we have

\[ \begin{align*}
\int_0^1 \left| qt - \frac{5}{6} \right| \, b \, D_{p,q} f(ta + (1-t)b) \, dp_q t &\leq \left| b \, D_{p,q} f(a) \right| \int_0^1 \left| qt - \frac{5}{6} \right| \, dp_q t \\
&\quad + \left| b \, D_{p,q} f(b) \right| \int_0^1 \left| qt - \frac{5}{6} \right| \, (1-t) \, dp_q t \\
&= \Lambda_3(p, q) \left| b \, D_{p,q} f(a) \right| + \Lambda_4(p, q) \left| b \, D_{p,q} f(b) \right|. \quad (21)
\end{align*} \]
Substituting Equations (20) and (21) into Equation (19), we obtain the required result. □

**Remark 2.** If \( p = 1 \), then Equation (18) reduces to

\[
\frac{1}{b-a} \int_a^b f(x) b_{a}^b x - \frac{1}{6} \left[ f(a) + 4 f \left( \frac{a+b}{2} \right) + f(b) \right] \leq (b-a) \left[ \left| \Lambda_1(q) + \Lambda_3(q) \right| |b D_q f(a)| + \left| \Lambda_2(q) + \Lambda_4(q) \right| |b D_q f(b)| \right],
\]

(22)

where \( \Lambda_i(q), i = 1, 2, 3, 4 \) are defined by

\[
\Lambda_1(q) = \int_{0}^{1} t \left| qt - \frac{1}{6} \right| t \, dq = \begin{cases} 
\frac{1-2q-2q^2}{24(1+q)(1+q+q^2)}, & \text{for } 0 < q < \frac{1}{3}; \\
\frac{1-4q^3}{216(1+q)(1+q+q^2)}, & \text{for } \frac{1}{3} \leq q < 1,
\end{cases}
\]

\[
\Lambda_2(q) = \int_{0}^{1} t \left| qt - \frac{1}{6} \right| (1-t) \, dq = \begin{cases} 
\frac{1-4q^3}{24(1+q)(1+q+q^2)}, & \text{for } 0 < q < \frac{5}{6}; \\
\frac{15-6q-6q^2}{18q^2+18q+25}, & \text{for } \frac{5}{6} \leq q < 1,
\end{cases}
\]

\[
\Lambda_3(q) = \int_{\frac{1}{2}}^{1} t \left| qt - \frac{5}{6} \right| t \, dq = \begin{cases} 
\frac{15-6q-6q^2}{24(1+q)(1+q+q^2)}, & \text{for } 0 < q < \frac{5}{6}; \\
\frac{18q^2+8q+5}{216(1+q)(1+q+q^2)}, & \text{for } \frac{5}{6} \leq q < 1,
\end{cases}
\]

\[
\Lambda_4(q) = \int_{\frac{1}{2}}^{1} t \left| qt - \frac{5}{6} \right| (1-t) \, dq = \begin{cases} 
\frac{-8q^3+8q^2+8q-5}{24(1+q)(1+q+q^2)}, & \text{for } 0 < q < \frac{5}{6}; \\
\frac{5+12q+12q^2}{216(1+q)(1+q+q^2)}, & \text{for } \frac{5}{6} \leq q < 1,
\end{cases}
\]

which appeared in [50]. Moreover, if \( q \to 1 \), then Equation (22) reduces to

\[
\frac{1}{6} \left[ f(a) + 4 f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{5(b-a)}{72} \left[ |f'(a)| + |f'(b)| \right],
\]

which appeared in [20].

**Theorem 7.** Let \( f : [a,b] \to \mathbb{R} \) be a \((p,q)\)-differentiable function on \((a,b)\). If \(|b D_{p,q} f|\) is a convex function and a \((p,q)\)-integrable function on \([a,b]\), and \(r > 1\) with \(1/r + 1/s = 1\), then

\[
\left| \frac{1}{p(b-a)} \int_{p_{a}+(1-p)b}^{b} f(x) b_{a}^b x - \frac{1}{6} \left[ f(a) + 4 f \left( \frac{a+b}{2} \right) + f(b) \right] \right| \\
\leq \frac{1}{6} (b-a) \left\{ 2^{1-1/s} \left( \frac{1}{4(p+q)} \right)^{r' \prime} + \frac{2p+2q-1}{4(p+q)} \right\}^{1/r} + (1-2^{1-1/s})^{1/s} \left( \frac{3}{4(p+q)} \right)^{r' \prime} \left[ b D_{p,q} f(a) \right]^{r' \prime}
\]

(23)
Proof. Taking the absolute value of both sides of Theorem 5, we have

\[
\left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^{b} f(x)x^{b} d_{p,q}x - \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right|
\]

\[
= \left| (b-a) \int_{0}^{1} \phi(t) b^{D_{p,q}}f(ta + (1-t)b) d_{p,q}t \right|
\]

\[
= (b-a) \int_{0}^{1/2} (qt - \frac{1}{6}) b^{D_{p,q}}f(ta + (1-t)b) d_{p,q}t
\]

\[
+ \int_{1/2}^{1} (qt - \frac{5}{6}) b^{D_{p,q}}f(ta + (1-t)b) d_{p,q}t
\]

\[
\leq (b-a) \left( \int_{0}^{1/2} \left| qt - \frac{1}{6} \right| b^{D_{p,q}}f(ta + (1-t)b) d_{p,q}t \right) 1/r
\]

\[
+ \int_{1/2}^{1} \left| qt - \frac{5}{6} \right| b^{D_{p,q}}f(ta + (1-t)b) d_{p,q}t
\]

\[
\leq (b-a) \left( \int_{0}^{1/2} \left| qt - \frac{1}{6} \right| b^{D_{p,q}}f(ta + (1-t)b) d_{p,q}t \right) 1/r
\]

\[
\int_{0}^{1/2} \left| qt - \frac{1}{6} \right| b^{D_{p,q}}f(ta + (1-t)b) d_{p,q}t
\]

\[
\leq \left( \int_{0}^{1/2} \left| qt - \frac{1}{6} \right| d_{p,q}t \right)^{1/s}
\]

\[
\times \left( \left| b^{D_{p,q}}f(a) \right| r \int_{0}^{1/2} t d_{p,q}t + \left| b^{D_{p,q}}f(b) \right| r \int_{1/2}^{1} (1-t) d_{p,q}t \right)^{1/r}
\]

Using Equation (7), we obtain

\[
\int_{0}^{1/2} \left| qt - \frac{1}{6} \right| d_{p,q}t = (p-q) \frac{1}{2} \sum_{j=0}^{\infty} \frac{q^j}{p^j+1} \frac{1}{2^j+1} \left| \frac{1}{6} \right|^{s}
\]

\[
\leq \frac{1}{2} (p-q) \sum_{j=0}^{\infty} \frac{q^j}{p^j+1} \frac{1}{2} \frac{1}{6} \left| \frac{1}{6} \right|^{s}
\]

\[
= \frac{1}{2 \cdot 3^{s}}
\]

Calculating the \((p,q)\)-integral in Equation (25) and substituting the inequality Equation (26) into Equation (25), we have

\[
\int_{0}^{1/2} \left| qt - \frac{1}{6} \right| b^{D_{p,q}}f(ta + (1-t)b) d_{p,q}t
\]

\[
\leq \left( \frac{1}{2 \cdot 3^{s}} \right)^{1/s} \left( \frac{1}{4(p+q)} \left| b^{D_{p,q}}f(a) \right| r + \frac{2p+2q-1}{4(p+q)} \left| b^{D_{p,q}}f(b) \right| r \right)^{1/r}.
\]

(27)
Similarly, we have

\[
\int_{\frac{1}{2}}^{1} \left| q t - \frac{5}{6} \right|^r b D_{p,q} f(ta + (1 - t)b) \, d_{p,q} t \\
\leq \left( \frac{1 - 2^{s-1}}{6^s} \right)^{1/s} \left( \frac{3}{4(p + q)} \right)^r b D_{p,q} f(a)^r + \frac{2p + 2q - 3}{4(p + q)} \left| b D_{p,q} f(b) \right|^r \right)^{1/r}. \tag{28}
\]

Substituting Equations (27) and (28) into Equation (24), we obtain the required result. \(\square\)

**Remark 3.** If \( p = 1 \), then Equation (23) reduces to

\[
\left| \frac{1}{b - a} \int_{a}^{b} f(x) \, b d_q x - \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] \right| \\
\leq \frac{1}{6} (b - a) \left\{ 2^{1 - 1/s} \left( \frac{1}{4(1 + q)} \right)^r b D_q f(a)^r + \frac{2q + 1}{4(1 + q)} \left| b D_q f(b) \right|^r \right\}^{1/r} \\
+ (1 - 2^{s-1})^{1/s} \left( \frac{3}{4(1 + q)} \right)^r b D_q f(a)^r + \frac{2q - 1}{4(1 + q)} \left| b D_q f(b) \right|^r \right\}^{1/r}.
\]

**Theorem 8.** Let \( f : [a, b] \to \mathbb{R} \) be a \((p, q)^b\)-differentiable function on \((a, b)\). If \( \left| b D_{p,q} f \right|^r, r \geq 1 \) is a convex function and a \((p, q)^b\)-integrable function on \([a, b]\), then

\[
\left| \frac{1}{p(b - a)} \int_{p a + (1 - p)b}^{b} f(x) \, b d_{p,q} x - \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] \right| \\
\leq (b - a) (\Lambda_5(p, q))^{1 - 1/r} \left( \Lambda_1(p, q) \right)^r b D_{p,q} f(a)^r + \left( \Lambda_6(p, q) \right)^r b D_{p,q} f(b)^r \right\}^{1/r} \\
+ (b - a) (\Lambda_6(p, q))^{1 - 1/r} \left( \Lambda_3(p, q) \right)^r b D_{p,q} f(a)^r + \left( \Lambda_6(p, q) \right)^r b D_{p,q} f(b)^r \right\}^{1/r}, \tag{29}
\]

where \( \Lambda_i(p, q), \ i = 1, 2, 3, 4 \) are given in Theorem 6 and \( \Lambda_j(p, q), \ j = 5, 6 \) are defined by

\[
\Lambda_5(p, q) = \int_{0}^{1} \left| q t - \frac{1}{6} \right|^r b d_{p,q} t = \begin{cases} \frac{p - 2q}{12(p + q)}, & \text{for } 0 < q < \frac{1}{3}; \\ 6q^2 - 3pq + 2q + 2p - 2, & \text{for } \frac{1}{3} \leq q < 1, \end{cases}
\]

and

\[
\Lambda_6(p, q) = \int_{\frac{1}{2}}^{1} \left| q t - \frac{5}{6} \right|^r b d_{p,q} t = \begin{cases} \frac{5p - 4q}{12(p + q)}, & \text{for } 0 < q < \frac{5}{6}; \\ -4pq + 50q + 50p - 50, & \text{for } \frac{5}{6} \leq q < 1. \end{cases}
\]
Proof. Using Theorem 5 and the Hölder inequality, we have
\[
\left| \frac{1}{p(b-a)} \int_a^b f(x)^b d_{pq}x - \frac{1}{6} \int f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right| \\
= \left| (b-a) \int_0^1 \phi(t)^b D_{pq} f(ta + (1-t)b) d_{pq}t \right| \\
= \left| (b-a) \left[ \frac{1}{2} \left( qt - \frac{1}{6} \right)^b D_{pq} f(ta + (1-t)b) d_{pq}t + \int \frac{1}{2} \left( qt - \frac{5}{6} \right)^b D_{pq} f(ta + (1-t)b) d_{pq}t \right] \right| \\
\leq (b-a) \left[ \left( \int_0^1 \left| qt - \frac{1}{6} \right|^2 d_{pq}t \right)^{1/2} \left( \int_0^1 \left| qt - \frac{5}{6} \right|^2 d_{pq}t \right)^{1/2} \right]^{1/2} \\
\leq (b-a) \left[ \left( \int_0^1 \left| qt - \frac{1}{6} \right|^2 d_{pq}t \right)^{1/2} \left( \int_0^1 \left| qt - \frac{5}{6} \right|^2 d_{pq}t \right)^{1/2} \right] \\
\times \left( \left| D_{pq} f(a) \right|^r \int_0^1 \left| qt - \frac{1}{6} \right|^t d_{pq}t + \left| D_{pq} f(b) \right|^r \int_0^1 \left| qt - \frac{1}{6} \right|^t d_{pq}t \right)^{1/2} \\
+ (b-a) \left( \int_0^1 \left( qt - \frac{5}{6} \right)^2 d_{pq}t \right)^{1/2} \\
\times \left( \left| D_{pq} f(a) \right|^r \int_0^1 \left| qt - \frac{5}{6} \right|^t d_{pq}t + \left| D_{pq} f(b) \right|^r \int_0^1 \left| qt - \frac{5}{6} \right|^t d_{pq}t \right)^{1/2} \right)^{1/2} \\
= (b-a) (\Lambda_5(p,q))^{1/2} (\Lambda_1(p,q) \left| D_{pq} f(a) \right|^r + \Lambda_2(p,q) \left| D_{pq} f(b) \right|^r) \right)^{1/2} \\
+ (b-a) (\Lambda_6(p,q))^{1/2} (\Lambda_3(p,q) \left| D_{pq} f(a) \right|^r + \Lambda_4(p,q) \left| D_{pq} f(b) \right|^r) \right)^{1/2}.
\]
Therefore, the proof is completed. □

Remark 4. If \( p = 1 \), then Equation (29) reduces to
\[
\left| \frac{1}{b-a} \int_a^b f(x)^b d_{pq}x - \frac{1}{6} \int f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right| \\
\leq (b-a) (\Lambda_5(q))^{1/2} (\Lambda_1(q) \left| D_{pq} f(a) \right|^r + \Lambda_2(q) \left| D_{pq} f(b) \right|^r) \right)^{1/2} \\
+ (b-a) (\Lambda_6(q))^{1/2} (\Lambda_3(q) \left| D_{pq} f(a) \right|^r + \Lambda_4(q) \left| D_{pq} f(b) \right|^r) \right)^{1/2},
\] (30)
where \( \Lambda_i(q), \ i = 1, 2, 3, 4 \) are given in Remark 2 and \( \Lambda_j(q), \ j = 5, 6 \) are defined by

\[
\Lambda_5(q) = \int_0^1 |qt - \frac{1}{6}| \, dq t = \begin{cases} 
\frac{1 - 2q}{12(1 + q)}, & \text{for } 0 < q < \frac{1}{3}; \\
\frac{6q - 1}{36(1 + q)}, & \text{for } \frac{1}{3} \leq q < 1,
\end{cases}
\]

and

\[
\Lambda_6(q) = \int_0^1 |qt - \frac{5}{6}| \, dq t = \begin{cases} 
\frac{5 - 4q}{12(1 + q)}, & \text{for } 0 < q < \frac{5}{6}; \\
\frac{5}{36(1 + q)}, & \text{for } \frac{5}{6} \leq q < 1,
\end{cases}
\]

which appeared in [50]. Moreover, if \( q \to 1 \), then Equation (30) reduces to

\[
\frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b - a} \int_a^b f(t) \, dt \leq \left\{ \frac{5^{1 - 1/r}}{72} (b - a) \left( \frac{29}{18} |bD_q f(a)|^r + \frac{61}{18} |bD_q f(b)|^r \right)^{1/r} \right. \\
\left. \times \left( \frac{61}{18} |bD_q f(a)|^r + \frac{29}{18} |bD_q f(b)|^r \right)^{1/r} \right\},
\]

which appeared in [20].

4. Newton-Type Inequalities for \((p, q)\)-Calculus

**Theorem 9.** Let \( f : [a, b] \to \mathbb{R} \) be a \((p, q)\)\(^p\)-differentiable function on \((a, b)\). If \( bD_{p,q} f \) is a continuous function and a \((p, q)\)\(^b\)-integrable function on \([a, b]\), then

\[
(b - a) \int_0^1 \psi(t) bD_{p,q} f(ta + (1 - t)b) \, d_{p,q} t
= \frac{1}{p(b - a)} \int_a^b \frac{b^{b + (1-p)a}}{f(x) d_{p,q} x} - \frac{1}{8} \left[ f(a) + 3f \left( \frac{2a + b}{3} \right) + 3f \left( \frac{a + 2b}{3} \right) + f(b) \right], \tag{31}
\]

where

\[
\psi(t) = \begin{cases} 
qt - \frac{1}{3}, & \text{for } 0 \leq t < \frac{1}{3}; \\
qt - \frac{2}{3}, & \text{for } \frac{1}{3} \leq t < \frac{2}{3}; \\
qt - \frac{7}{8}, & \text{for } \frac{2}{3} \leq t \leq 1.
\end{cases}
\]

**Proof.** From basic properties of the \((p, q)\)-integral and the definition of \( \psi(t) \), it follows that

\[
\int_0^1 \psi(t) bD_{p,q} f(ta + (1 - t)b) \, d_{p,q} t
= \frac{3}{8} \left[ \int_0^1 bD_{p,q} f(ta + (1 - t)b) \, d_{p,q} t + \int_0^2 bD_{p,q} f(ta + (1 - t)b) \, d_{p,q} t \right]
+ \int_0^1 \left( qt - \frac{7}{8} \right) bD_{p,q} f(ta + (1 - t)b) \, d_{p,q} t.
\]

The rest of this proof is similar to that of Theorem 5. \( \Box \)
Remark 5. If $p = 1$, then Equation (31) reduces to

\[
(b - a) \int_{0}^{1} \psi(t) b D_{a} f((1 - t)b) dt = \frac{1}{b - a} \int_{a}^{b} f(x) d_{p,q} x - \frac{1}{8} \left[ f(a) + 3 f \left( \frac{2a + b}{3} \right) + 3 f \left( \frac{a + 2b}{3} \right) + f(b) \right],
\]

where

\[
\psi(t) = \begin{cases} 
qt - \frac{1}{3}, & \text{for } 0 \leq t < \frac{1}{3}; \\
qt - \frac{1}{3}, & \text{for } \frac{1}{3} \leq t < \frac{2}{3}; \\
qt - \frac{2}{3}, & \text{for } \frac{2}{3} \leq t \leq 1,
\end{cases}
\]

which appeared in [50].

Theorem 10. Let $f : [a, b] \to \mathbb{R}$ be a $(p, q)$-differentiable function on $(a, b)$. If $|b D_{p,q} f|$ is a convex function and a $(p, q)$-integrable function on $[a, b]$, then

\[
\left| \frac{1}{p(b - a)} \int_{pa+(1-p)b}^{b} f(x) d_{p,q} x - \frac{1}{8} \left[ f(a) + 3 f \left( \frac{2a + b}{3} \right) + 3 f \left( \frac{a + 2b}{3} \right) + f(b) \right] \right| 
\]

\[
\leq (b - a) \left[ |\psi_{1}(p, q) + \psi_{3}(p, q) + \psi_{5}(p, q)| \right] |b D_{p,q} f(a)| 
\]

\[
+ |\psi_{2}(p, q) + \psi_{4}(p, q) + \psi_{6}(p, q)| \right] |b D_{p,q} f(b)|, 
\]

(32)

where $\psi_{i}(p, q), \ i = 1, 2, \ldots, 6$ are defined by

\[
\psi_{1}(p, q) = \int_{0}^{\frac{1}{3}} \left| qt - \frac{1}{8} \right| t d_{p,q} t
\]

\[
= \begin{cases} 
\frac{3p^{2} - 5pq - 5q^{2}}{216(p + q)(p^{2} + pq + q^{2})}, & \text{for } 0 < q < \frac{3}{8}; \\
\frac{160q^{6} + 160pq^{3} - 96p^{2}q^{2}}{6912q^{2}(p + q)(p^{2} + pq + q^{2})}, & \text{for } \frac{3}{8} \leq q < 1,
\end{cases}
\]

\[
\psi_{2}(p, q) = \int_{0}^{\frac{1}{3}} \left| qt - \frac{1}{8} \right| (1 - t) d_{p,q} t
\]

\[
= \begin{cases} 
\frac{9p^{3} - 3p^{2} - 6p^{2}q + 5pq - 6p^{2}q^{2} + 5q^{2} - 15q^{3}}{216(p + q)(p^{2} + pq + q^{2})}, & \text{for } 0 < q < \frac{3}{8}; \\
\frac{480q^{6} + 192pq^{4} + 56q^{4} + 192p^{2}q^{2} + 272pq^{3}}{6912q^{2}(p + q)(p^{2} + pq + q^{2})}, & \text{for } \frac{3}{8} \leq q < 1,
\end{cases}
\]

\[
\psi_{3}(p, q) = \int_{\frac{1}{3}}^{\frac{2}{3}} \left| qt - \frac{1}{2} \right| t d_{p,q} t
\]

\[
= \begin{cases} 
\frac{9p^{2} - 5pq - 5q^{2}}{54(p + q)(p^{2} + pq + q^{2})}, & \text{for } 0 < q < \frac{3}{4}; \\
\frac{6q^{4} + 6pq^{3} - 30p^{2}q^{2}}{108q^{2}(p + q)(p^{2} + pq + q^{2})}, & \text{for } \frac{3}{4} \leq q < 1,
\end{cases}
\]
\[ \psi_4(p, q) = \int_{\frac{1}{2}}^{\frac{3}{8}} \left( qt - \frac{1}{2} \right) \left( 1 - t \right) \, dt \]
\[ = \begin{cases} 
9p^3 - 9p^2 + 5pq + 5q^2 - 9q^3 \\
54(p + q)(p^2 + pq + q^2) \\
6q^5 - 48pq^4 + 48q^4 - 48p^2q^3 + 102pq^3 - 54q^3 \\
-54p^3 q^2 + 138p^2 q^2 - 54p^2 q^2 - 27q^2 + 54pq^2 \quad & \text{for } 0 < q < \frac{3}{4}; \\
-54pq^2 - 27pq + 27q - 27p + 27p \quad & \text{for } \frac{3}{4} \leq q < 1,
\end{cases} \]
\[
\psi_5(p, q) = \int_{\frac{1}{2}}^{\frac{7}{8}} \left( qt - \frac{7}{8} \right) \left( 1 - t \right) \, dt \\
= \begin{cases} 
105p^2 - 47pq - 47q^2 \\
216(p + q)(p^2 + pq + q^2) \\
224q^2 + 224pq^3 - 8736p^2 q^2 \\
+9261q^2 + 9261pq - 9261q + 9261p^2 - 9261p \quad & \text{for } 0 < q < \frac{7}{8}; \\
6912q^2(p + q)(p^2 + pq + q^2) \quad & \text{for } \frac{7}{8} \leq q < 1.
\end{cases} \]
\[
\psi_6(p, q) = \int_{\frac{1}{2}}^{\frac{7}{8}} \left( qt - \frac{7}{8} \right) \left( 1 - t \right) \, dt \\
= \begin{cases} 
63p^3 - 105p^2 + 6pq^2 + 47pq + 6pq^2 + 47q^2 - 57q^3 \\
216(p + q)(p^2 + pq + q^2) \\
-96q^5 - 10176pq^4 + 10360q^5 - 10176pq^3 \\
+20944pq^3 - 10584q^3 - 10080p^3 q^2 + 29904p^2 q^2 \\
-10584pq^2 - 9261q^2 + 10584pq^3 - 10584pq^2 \quad & \text{for } 0 < q < \frac{7}{8}; \\
-9261pq + 9261p^2 - 9261p + 9261q + 9261p^2 + 9261q \quad & \text{for } \frac{7}{8} \leq q < 1.
\end{cases} \]

Proof. The proof of Theorem 10 is similar to that of Theorem 6 and is omitted.

Remark 6. If \( p = 1 \), then Equation (32) reduces to
\[
\left| \frac{1}{b - a} \int_{a}^{b} f(x) \, dx - \frac{1}{8} \left[ f(a) + 3f \left( \frac{2a + b}{3} \right) + 3f \left( \frac{a + 2b}{3} \right) + f(b) \right] \right| \\
\leq \left| (b - a) \left[ (\psi_1(q) + \psi_3(q) + \psi_5(q)) |^{b} D_q f(a) | + [\psi_2(q) + \psi_4(q) + \psi_6(q)) |^{b} D_q f(b) | \right] \right|, \tag{33}
\]
where \( \psi_i(q) \), \( i = 1, 2, \ldots, 6 \) are defined by
\[
\psi_1(q) = \int_{0}^{\frac{1}{2}} \left( qt - \frac{1}{8} \right) \, dt = \begin{cases} 
3 - 5q - 5q^2 \\
216(1 + q)(1 + q + q^2) \\
160q^2 + 160q - 69 \quad & \text{for } 0 < q < \frac{3}{8}; \\
6912(1 + q)(1 + q + q^2) \quad & \text{for } \frac{3}{8} \leq q < 1,
\end{cases} \\
\psi_2(q) = \int_{0}^{\frac{1}{2}} \left( qt - \frac{1}{8} \right) \left( 1 - t \right) \, dt = \begin{cases} 
6 - q^2 - 15q^3 \\
216(1 + q)(1 + q + q^2) \\
480q^3 + 248q^2 + 248q - 3 \quad & \text{for } 0 < q < \frac{3}{8}; \\
6912(1 + q)(1 + q + q^2) \quad & \text{for } \frac{3}{8} \leq q < 1,
\end{cases}
\]
Let \( f \) be a differentiable function on \((a, b)\), and \( r, s > 1 \) with \( 1/r + 1/s = 1 \), then
\[
\int_a^b \left| f(x) \right|^{1/r} dx \leq (b - a) \left\{ \left( \frac{5^s}{3 \cdot 24^s} \right)^{1/s} \left( \frac{1}{9(p + q)} \right) \left| b \int_a^b f(x) dx \right| + \left( \frac{1}{3 \cdot 6^s} \right)^{1/s} \left( \frac{5^s}{9(p + q)} \right) \left| b \int_a^b f(x) dx \right| + \left( \frac{2^{s+1} - 2}{3 \cdot 24^s} \right)^{1/s} \left( \frac{5^s}{9(p + q)} \right) \left| b \int_a^b f(x) dx \right| \right\}. \tag{34}
\]

Proof. The proof of Theorem 11 is similar to that of Theorem 7. Hence it is omitted. \( \square \)
Remark 7. If $p = 1$, then Equation (34) reduces to
\[
\left| \frac{1}{b-a} \int_a^b f(x)^b d_q x - \frac{1}{8} \left[ f(a) + 3f \left( \frac{2a+b}{3} \right) + 3f \left( \frac{a+2b}{3} \right) + f(b) \right] \right| \\
\leq (b-a) \left\{ \left( \frac{5^s}{3^{1/24}} \right)^{1/s} \left( \frac{1}{9(1+q)} \beta D_q f(a) \right)^{1/r} + \frac{3q+2}{9(1+q)} \beta D_q f(b) \right\}^{1/r} \\
+ \left( \frac{1}{3-6^s} \right)^{1/s} \left( \frac{3}{9(1+q)} \beta D_q f(a) \right)^{1/r} + \frac{3q-2}{9(1+q)} \beta D_q f(b) \right\}^{1/r}\right\}.
\]
Moreover, if $q \to 1$, then Equation (35) reduces to
\[
\left| \frac{1}{8} \left[ f(a) + 3f \left( \frac{2a+b}{3} \right) + 3f \left( \frac{a+2b}{3} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b-a}{3} \left\{ \left( \frac{5^s}{3^{1/24}} \right)^{1/s} \left( \frac{1}{6} |f'(a)|^r + \frac{5|f'(b)|^r}{6} \right)^{1/r} \right\} + \left( \frac{1}{3-6^s} \right)^{1/s} \left( \frac{1}{6} |f'(a)|^r + \frac{|f'(b)|^r}{2} \right)^{1/r}
\]
\[
+ \left( \frac{3^{s+1} - 2 \cdot 5^s}{3^{1/24}} \right)^{1/s} \left( \frac{5}{6} |f'(a)|^r + |f'(b)|^r \right)^{1/r}\right\}.
\]

Theorem 12. Let $f : (a,b) \to \mathbb{R}$ be a $(p,q)^b$-differentiable function on $(a,b)$. If $\beta D_{p,q} f \in L^r$, $r \geq 1$ is a convex function and a $(p,q)^b$-integrable function on $[a,b]$, then
\[
\left| \frac{1}{p(b-a)} \int_a^b f(x)^{p(b-1)} d_{p,q} x - \frac{1}{8} \left[ f(a) + 3f \left( \frac{2a+b}{3} \right) + 3f \left( \frac{a+2b}{3} \right) + f(b) \right] \right| \\
\leq (b-a) \left\{ \left( \psi_2(p,q) \right)^{1-1/r} \left( \psi_3(p,q) \right)^{b D_{p,q} f(a)} \right\}^{1/r} + \left( \psi_2(p,q) \right)^{1/r} \left( \psi_3(p,q) \right)^{b D_{p,q} f(b)} \right\}^{1/r}
\]
\[
+ \left( \psi_3(p,q) \right)^{1-1/r} \left( \psi_4(p,q) \right)^{b D_{p,q} f(a)} \right\}^{1/r} + \left( \psi_3(p,q) \right)^{1/r} \left( \psi_4(p,q) \right)^{b D_{p,q} f(b)} \right\}^{1/r}\right\},
\]
where $\psi_i(p,q)$, $i = 1, 2, \ldots, 6$ are given in Theorem 10 and $\psi_j(p,q)$, $j = 7, 8, 9$ are defined by
\[
\psi_2(p,q) = \int_0^1 \left| q t - \frac{1}{8} \right| d_{p,q} t = \begin{cases} 
3p - 9q & \text{for } 0 < q < \frac{3}{8}; \\
72(p+q) & \text{for } \frac{3}{8} \leq q < 1,
\end{cases}
\]
\[
\psi_3(p,q) = \int_0^1 \left| q t - \frac{1}{8} \right| d_{p,q} t = \begin{cases} 
3p - 3q & \text{for } 0 < q < \frac{3}{4}; \\
18(p+q) & \text{for } \frac{3}{4} \leq q < 1,
\end{cases}
\]
\[
\psi_4(p,q) = \int_0^1 \left| q t - \frac{7}{8} \right| d_{p,q} t = \begin{cases} 
21p - 19q & \text{for } 0 < q < \frac{7}{5}; \\
72(p+q) & \text{for } \frac{7}{5} \leq q < 1.
\end{cases}
\]

Proof. The proof of Theorem 12 is similar to that of Theorem 8. \qed
Remark 8. If $p = 1$, then Equation (36) reduces to

\[
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{1}{8} \left[ f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right| \\
\leq (b-a) \left\{ \left( \psi_2(q) \right)^{1-1/r} \left( \psi_3(q) \right)^{bD_qf(a)} + \psi_4(q) \left| bD_qf(b) \right|^r \right\}^{1/r}
\]

Moreover, if $q \to 1$, then Equation (37) reduces to

\[
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{1}{8} \left[ f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right| \\
\leq \frac{b-a}{36} \left\{ \left( \frac{17}{16} \right)^{1-1/r} \left( \frac{251}{1152} |f'(a)|^r + \frac{937}{1152} |f'(b)|^r \right)^{1/r} + \left( \frac{1}{2} |f'(a)|^r + \frac{1}{2} |f'(b)|^r \right)^{1/r} \right\}
\]

which appeared in [50].

Moreover, if $q \to 1$, then Equation (37) reduces to

\[
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{1}{8} \left[ f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right| \\
\leq \frac{b-a}{36} \left\{ \left( \frac{17}{16} \right)^{1-1/r} \left( \frac{251}{1152} |f'(a)|^r + \frac{937}{1152} |f'(b)|^r \right)^{1/r} + \left( \frac{1}{2} |f'(a)|^r + \frac{1}{2} |f'(b)|^r \right)^{1/r} \right\}
\]

which appeared in [66].

5. Examples

In this section, we give some examples of our main theorems.

Example 1. Define function $f : [0,1] \to \mathbb{R}$ by $f(x) = 1 - x$. Then $|bD_{p,q}f(x)| = |bD_{p,q}(1-x)| = 1$ is a convex function and a $(p,q)$-integrable function on $[0,1]$. Applying Theorem 6 with $p = 1$ and $q = \frac{1}{2}$, the left side of Equation (18) becomes

\[
\left| \frac{1}{p(b-a)} \int_{p+(b-p)x}^{b} f(x) \, dx - \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \\
= \left| \frac{1}{1 \cdot (1-0)} \int_{1 \cdot (1-0)}^{1 \cdot (1-1)} (1-x) \, dx - \frac{1}{6} \left[ f(0) + 4f\left(\frac{0+1}{2}\right) + f(1) \right] \right|
\]
\[ = \left| \int_{0}^{1} (1 - x) \, 1d_{1/2} x - \frac{1}{6}[1 + 2 + 0] \right| = \left| \frac{2}{3} - \frac{1}{2} \right| = \frac{1}{6}, \]

and the right side of Equation (18) becomes

\[
(b - a) \left[ [\Lambda_1(p, q) + \Lambda_3(p, q)] \left| bD_{p,q}f(a) \right| + [\Lambda_2(p, q) + \Lambda_4(p, q)] \left| bD_{p,q}f(b) \right| \right] \\
= (1 - 0) \left[ [\Lambda_1(1, 1/2) + \Lambda_3(1, 1/2)] \left| 1D_{1/2} f(0) \right| + [\Lambda_2(1, 1/2) + \Lambda_4(1, 1/2)] \left| 1D_{1/2} f(1) \right| \right] \\
= \left[ \frac{13}{1134} + \frac{1}{6} \right] \cdot [1] + \left[ \frac{29}{1134} + 0 \right] \cdot [1] = \frac{11}{54}. \]

It is clear that

\[
\frac{1}{6} \leq \frac{11}{54}, \]

which demonstrates the result described in Theorem 6.

**Example 2.** Define function \( f : [0, 1] \to \mathbb{R} \) by \( f(x) = 1 - x \). Then \( \left| bD_{p,q}f(x) \right| = \left| bD_{p,q}(1 - x) \right| = 1 \) is a convex function and a \((p, q)\)-integrable function on \([0, 1]\). Applying Theorem 10 with \( p = 1 \) and \( q = \frac{1}{2} \), the left side of Equation (33) becomes

\[
\left| \frac{1}{p(b - a)} \int_{pa+(1-p)b}^{b} f(x) \, bD_{p,q}x - \frac{1}{8} \left[ f(a) + 3f\left( \frac{2a + b}{3} \right) + 3f\left( \frac{a + 2b}{3} \right) + f(b) \right] \right| \\
= \left| \frac{1}{1 \cdot (1 - 0)} \int_{1 + 0 \cdot (1 - 1)}^{1} (1 - x) \, 1d_{1/2} x - \frac{1}{8} \left[ f(0) + 3f\left( \frac{0 + 2 - 1}{3} \right) + 3f\left( \frac{0 + 2 \cdot 1}{3} \right) + f(1) \right] \right| \\
= \left| \int_{0}^{1} (1 - x) \, 1d_{1/2} x - \frac{1}{8}[1 + 2 + 1 + 0] \right| = \left| \frac{2}{3} - \frac{1}{2} \right| = \frac{1}{6}, \]

and the right side of Equation (33) becomes

\[
(b - a) \left[ [\psi_1(p, q) + \psi_3(p, q) + \psi_5(p, q)] \left| bD_{p,q}f(a) \right| + [\psi_2(p, q) + \psi_4(p, q) + \psi_6(p, q)] \left| bD_{p,q}f(b) \right| \right] \\
= (1 - 0) \left[ [\psi_1(1, 1/2) + \psi_3(1, 1/2) + \psi_5(1, 1/2)] \left| 1D_{1/2} f(0) \right| + [\psi_2(1, 1/2) + \psi_4(1, 1/2) + \psi_6(1, 1/2)] \left| 1D_{1/2} f(1) \right| \right] \\
= \left[ \frac{17}{6048} + \frac{1}{27} + \frac{31}{252} \right] \cdot [1] + \left[ \frac{3}{224} + \frac{1}{54} - \frac{25}{1512} \right] \cdot [1] = \frac{77}{432}. \]

It is clear that

\[
\frac{1}{6} \leq \frac{77}{432}, \]

which demonstrates the result described in Theorem 10.

**6. Conclusions**

In this work, we employed \((p, q)\)-calculus to establish new integral inequalities related to Simpson- and Newton-type inequalities for convex functions. The results in this study were the generalization and extension of some previously proved research in the literature of Simpson- and Newton-type inequalities. In addition, some examples were displayed to verify our main results.
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