Belinskii-Zakharov Formulation for
Bianchi Models and Painlevé III Equation

Nenad Manojlović* and Aleksandar Mikovic†‡
Área Departamental de Matemática, UCEH, Universidade
do Algarve, Campus de Gambelas, 8000 Faro, Portugal

ABSTRACT

We show that $\alpha < 0, \beta > 0, \gamma = \delta = 0$ Painlevé III equation arises as a zero-
curvature condition in the Belinskii-Zakharov inverse scattering formulation for
Bianchi cosmological models. For special values of the parameters this Painlevé
III equation becomes the dynamical equation for Bianchi I, II, VI₀ and VII₀ models.

* E-mail:nmanoj@ualg.pt
† On leave of absence from Institute of Physics, Belgrade, Yugoslavia
‡ E-mail:amikovic@ualg.pt
1. Introduction

Belinskii and Francaviglia showed in [1] that the Einstein equations for Bianchi I, II, VI₀ and VII₀ spacetimes admit a zero-curvature representation, i.e. they found a linear system whose integrability condition is the dynamical equation for the Bianchi model. This was done by using a more general framework of Belinskii-Zakharov (BZ) inverse scattering method for the spacetimes admitting two commuting spacelike Killing vectors [2]. The results of [1] demonstrated that the Bianchi models which admit two commuting spacelike Killing vectors are solvable dynamical systems. However, not much work has been done on the issue of what kind of integrable nonlinear dynamical equations can be obtained from this approach.

In [3] it has been shown that in the case of the Bianchi VII₀ model one obtains a special Painlevé III (PIII) equation, which is \( \alpha = -2, \beta = 2, \gamma = \delta = 0 \) case of the standard PIII form [4]

\[
\frac{d^2 u}{dt^2} = \frac{1}{u} \left( \frac{du}{dt} \right)^2 - \frac{1}{t} \left( \frac{du}{dt} \right) + \frac{1}{t} \left( \alpha u^2 + \beta \right) + \gamma u^3 + \delta u^{-1}. \tag{1.1}
\]

In this paper we consider the Belinskii-Zakharov formulation for all Bianchi models which admit two commuting Killing vectors, i.e. types I, II, VI₀ and VII₀. We only consider the equations of motion for local degrees of freedom, and we do not discuss the problems related with non-trivial topology of the spatial manifold [3]. By considering a larger class of Bianchi models, we obtain a more general PIII equation as the dynamical equation, namely \( \alpha < 0, \beta > 0, \gamma = \delta = 0 \). Although the relevant Bianchi models correspond to special values of the parameters, we show that the zero curvature representation is valid for all other values of the
parameters. Consequently we obtain a zero-curvature representation of $\alpha < 0$, $\beta > 0$, $\gamma = \delta = 0$ Painlevé III equation in the Belinskii-Zakharov inverse scattering formulation.

2. Zero-curvature formulation for Bianchi models

Belinskii-Francaviglia approach to solving the dynamics of Bianchi models [1] is derived from the Belinskii-Zakharov method for solving the Einstein equations for spacetimes with two commuting spacelike Killing vectors [2]. Such spacetimes have the following form of the metric

$$ds^2 = f(t, z)(-dt^2 + dz^2) + g_{ab}(t, z)\, dx^a dx^b, \quad (2.1)$$

where $a, b = 1, 2$, $\{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$, $f$ is a positive function and $g_{ab}$ is a symmetric two-by-two matrix. It is convenient to introduce the null co-ordinates $(\xi, \eta) = (1/2(z + t), 1/2(z - t))$, since the form of the metric (2.1) is preserved by the conformal co-ordinate transformations $(\xi, \eta) \rightarrow (\tilde{\xi}(\xi), \tilde{\eta}(\eta))$. The positivity of the function $f$ is preserved if $\partial_\xi \tilde{\xi} \partial_\eta \tilde{\eta} > 0$.

The complete set of vacuum Einstein equations for the metric (2.1) decomposes into two groups of equations [2]. The first group determines the matrix $g_{ab}$ and can be written as a single matrix equation, called the Ernst equation

$$\partial_\eta (\sigma \partial_\xi g^{-1}) + \partial_\xi (\sigma \partial_\eta g^{-1}) = 0, \quad (2.2)$$

where $\sigma^2 = \text{det} g$. The second group of equations determines the function $f(\xi, \eta)$ in terms of a given solution of the Ernst equation

$$\partial_\xi (\ln f) = \frac{\partial^2_\xi (\ln \sigma)}{\partial_\xi (\ln \sigma)} + \frac{1}{4\sigma \sigma_\xi} \text{tr} A^2, \quad (2.3)$$
\[
\partial_\eta (\ln f) = \frac{\partial^2_\eta (\ln \sigma)}{\partial_\eta (\ln \sigma)} + \frac{1}{4\sigma \sigma_\eta} \text{tr} B^2 ,
\]

(2.4)

where \(\sigma_\xi = \partial_\xi \sigma , \sigma_\eta = \partial_\eta \sigma\) and the matrices \(A\) and \(B\) are defined by

\[
A = -\sigma \partial_\xi g \ g^{-1} , \quad B = \sigma \partial_\eta g \ g^{-1} .
\]

(2.5)

Thus the dynamics of the system is determined by the Ernst equation (2.2). An important consequence of the Ernst equation is that \(\sigma_\xi \eta = 0\) so that \(\sigma = c(\xi) + d(\eta)\). By using the conformal transformations, one can bring the functions \(c(\xi)\) and \(d(\eta)\) to a prescribed form.

The crucial step in the inverse scattering method is to define the linearized system whose integrability conditions are the equations of interest, in our case the equation (2.2). Following the ref. [2], we define two differential operators

\[
D_1 = \partial_\xi - \frac{2\sigma_\xi \lambda}{\lambda - \sigma} \partial_\lambda , \quad D_2 = \partial_\eta + \frac{2\sigma_\eta \lambda}{\lambda + \sigma} \partial_\lambda ,
\]

(2.6)

where \(\lambda\) is a complex parameter independent of the co-ordinates \(\{\xi, \eta\}\). The differential operators \(D_1\) and \(D_2\) commute since \(\sigma\) satisfies the wave equation, and hence one can consider the following linear system

\[
D_1 \psi = \frac{A}{\lambda - \sigma} \psi , \quad D_2 \psi = \frac{B}{\lambda + \sigma} \psi ,
\]

(2.7)

where \(\psi(\lambda, \xi, \eta)\) is a complex matrix function. The integrability condition for the system (2.7) is given by the Ernst equation (2.2). Furthermore, a solution \(\psi(\lambda, \xi, \eta)\) yields a matrix \(g(\xi, \eta)\) that satisfies the Ernst equation (2.2). Namely, the matrix \(g(\xi, \eta)\) is given by

\[
g(\xi, \eta) = \psi(\lambda, \xi, \eta) \big|_{\lambda=0} .
\]

(2.8)
In order to take into account that $g(\xi, \eta)$ is real and symmetric we have to impose two additional conditions, see [2]. Also, it is easy to see that the equations (2.7) for $\lambda = 0$, imply equations (2.5).

Bianchi spacetimes (see [5] for a review and references) have finitely many degrees of freedom, and only the Bianchi types I, II, VI$_0$ and VII$_0$ admit two commuting spacelike Killing vectors. The metric for these Bianchi spacetimes has the form (2.1), and this can be shown by considering the general Bianchi spacetime metric

$$ds^2 = -dT^2 + g_{ij}(T, x^k) \, dx^i dx^j , \quad (2.9)$$

where $g_{ij} = g_{IJ}(T) \chi^I_i (x^k) \chi^J_j (x^k)$ and $\chi^I (x^k)$ are the one-forms associated with the spatial manifold. These one-forms satisfy the Maurer-Cartan equations

$$d\chi^I + \frac{1}{2} C^I_{JK} \chi^J \wedge \chi^K = 0 \quad ,$$

where the structure constants $C^I_{JK}$ correspond to the Lie algebra of the symmetry group of the Bianchi model. For the relevant models the structure constants satisfy

$$C^I_{JK} = \epsilon_{JKL} S^{LI} \quad , \quad (2.10)$$

where $\epsilon_{JKL}$ is a totally antisymmetric tensor density and $S$ is a symmetric matrix.

In this case the one forms $\chi^I$ take the following form

$$\chi^1 = l^1_1 (z) \, dx + l^1_2 (z) \, dy , \quad \chi^2 = l^2_1 (z) \, dx + l^2_2 (z) \, dy , \quad \chi^3 = dz . \quad (2.11)$$

An important consequence of the Maurer-Cartan equations for the one forms $\chi^I$ is that the matrix $l = ||l^a b||$ satisfies the following linear differential equation

$$\frac{dl}{dz} = C^T c l \quad , \quad (2.12)$$
where the matrix $C$ is the upper two-by-two block on the principal diagonal of the matrix $S^{IJ}$ and $\epsilon$ is the antisymmetric matrix with $\epsilon_{12} = 1$.

After a time redefinition $t = t(T)$, the metric (2.9) can be written in the form (2.1)

$$ds^2 = f(t) \left(-dt^2 + dz^2\right) + g_{ab}(t, z) \, dx^a dx^b , \quad (2.13)$$

where

$$g(t, z) = l^T(z) \hat{\gamma}(t) l(z) , \quad (2.14)$$

and $\hat{\gamma}$ is a two-by-two symmetric matrix. Notice that now $\sigma^2 = (\det l)^2 \det \hat{\gamma}$, and since $\det l = 1$ we get

$$\sigma^2(t) = \det \hat{\gamma}(t) . \quad (2.15)$$

In addition, $\sigma$ has to satisfy the wave equation, so that $\ddot{\sigma}(t) = 0$, and hence $\sigma$ can only be a linear function of time.

The linearized system (2.7) can be simplified for the models described by the metric (2.13). The first step is to define a two-by-two matrix function $\varphi$ by

$$\psi(t, z, \lambda) = l^T(z) \varphi(t, z, \lambda) l(z) , \quad (2.16)$$

and a constant two-by-two matrix $R = \epsilon C$. The second step is to substitute (2.14) into (2.5) and use the definition of the coordinates $\xi$ and $\eta$. Then the results of these calculations, together with the definition (2.16), can be used to simplify the equations (2.7). The crucial step in which a simplification occurs is to perform
a conformal coordinate transformation \( \{t, z, \lambda\} \rightarrow \{t, w, \lambda\} \), where \( w \) is given by \( w = \frac{1}{2}(\frac{\sigma^2}{\lambda} + 2\beta + \lambda) \). The linear system after this coordinate transformation involves only derivatives in \( t \) and \( \lambda \) since all the terms involving derivatives in \( w \) are canceled. Finally, it is useful to make some simple linear combinations of the two equations and to use the fact that \( \sigma \) is a linear function of time. In this way one obtains a new linear system

\[
\partial_t \varphi = \frac{t}{\lambda} \left( \hat{\gamma} R^T \hat{\gamma}^{-1} \varphi - \varphi R^T \right),
\]

\[
\partial_\lambda \varphi = \frac{1}{2} \left( -R \varphi - \varphi R^T + \frac{t^2}{\lambda^2} \hat{\gamma} \hat{\gamma}^{-1} \varphi + \frac{t^2}{\lambda^2} \varphi R^T - \frac{t^2}{\lambda^2} \hat{\gamma} R^T \hat{\gamma}^{-1} \varphi \right),
\]  

(2.17)

where we have set \( \sigma = t \).

Although the matrix function \( \varphi(t, \lambda, w) \) depends on all three variables, the right-hand side of the system (2.17) does not have any \( w \) dependence. The integrability condition for the system (2.17) is

\[
\frac{1}{t} \frac{d}{dt} \left( t \hat{\gamma} \hat{\gamma}^{-1} \right) = R \hat{\gamma} R^T \hat{\gamma}^{-1} - \hat{\gamma} R^T \hat{\gamma}^{-1} R.
\]  

(2.18)

Equivalently, one can derive the equation (2.18) by a direct substitution of the formula (2.14) into equation (2.2). Thus the dynamics of these Bianchi models is determined by the equation (2.18).
3. Zero-curvature representation for Painlevé III

The linear system (2.17) and the corresponding nonlinear equation (2.18) were derived for special matrices $R$, which correspond to $C$ matrices of the relevant Bianchi models. Note that the $C$ matrix is symmetric, and the only relevant information about the Bianchi model is contained in its signature, so that there are four distinct possibilities:

1. $C = diag(1, 1)$ for Bianchi VII$_0$,
2. $C = diag(1, -1)$ for Bianchi VI$_0$,
3. $C = diag(1, 0)$ for Bianchi II,
4. $C = diag(0, 0)$ for Bianchi I.

However, if we consider the linear system (2.17) independently of Bianchi models, then we can take $C$ to be an arbitrary symmetric two-by-two matrix. In this case we have

$$C = \begin{pmatrix} c & d \\ d & k \end{pmatrix},$$

with $R = \epsilon C$, $\hat{\gamma} = diag(a, b)$ and $ab = t^2$. Then the consistency condition (2.18) gives

$$t^{-1} \frac{d}{dt} (ta^{-1} \dot{a}) = k^2 a^{-1} b - c^2 ab^{-1}$$

$$0 = d(k + cab^{-1}).$$

The second equation implies either $d = 0$ or $a^2 = -(k/c)t^2$. The second possibility gives a linear in time solution, and the first possibility is more interesting. $d = 0$
By making a change of variables $u = a^2/t^2$, $\tau = t^2/4$, the equation (3.3) takes the standard Painlevé III form (1.1)

$$
\frac{d^2u}{d\tau^2} = \frac{1}{u} \left( \frac{du}{d\tau} \right)^2 - \frac{1}{\tau} \left( \frac{du}{d\tau} \right) + \frac{2}{\tau} \left( -c^2 u^2 + k^2 \right),
$$

(3.4)
so that $\alpha = -2c^2$, $\beta = 2k^2$ and $\gamma = \delta = 0$.

Let us now consider the Bianchi models. In the case of Bianchi VII$_0$ model the spatial hyper-surface is a three torus $T^3$. The matrix $C$ is $diag(1,1)$ and the matrix $l$ is given by

$$
l(z) = \begin{pmatrix}
\cos z & \sin z \\
-\sin z & \cos z
\end{pmatrix},
$$

(3.5)
The matrix $R$ is given by

$$
R = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
$$

(3.6)
so that the local dynamics is given by (3.3) for $c = k = 1$. By making a change of variables $u = e^q$, this PIII equation takes a more symmetric form

$$
\frac{d}{d\tau} \left( \tau \frac{dq}{d\tau} \right) = -4 \sinh q.
$$

(3.7)
Bianchi VI$_0$ model corresponds to $C = diag(1,-1)$, so that the equation (2.12) gives

$$
l(z) = \begin{pmatrix}
\cosh z & \sinh z \\
\sinh z & \cosh z
\end{pmatrix}.
$$

(3.8)
This model has a non-compact spatial manifold, which is locally compact, and the local dynamics is the same as in the Bianchi VII$_0$ case, because $c^2 = k^2 = 1$. 

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In the Bianchi II case $C = \text{diag}(1, 0)$, and the dynamics is given by \( (3.3) \) with $c = 1$ and $k = 0$. The $l$ matrix is given by

$$l(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad (3.9)$$

and this model allows compact spatial sections [6]. In the Bianchi I case $C = 0$, so that $c = k = 0$ and $l = \text{diag}(1, 1)$.

### 4. Conclusions

We have shown that a class of PIII equations ($\alpha > 0, \beta < 0, \gamma = \delta = 0$) arises as a zero-curvature condition in the Belinskii-Zakharov inverse scattering method applied to Bianchi spacetime metrics. For the particular values of the parameters these PIII equations become the dynamical equations for Bianchi I, II, VIo and VIIo models. Note that the linear system (2.17) can be transformed into the standard form

$$\frac{\partial \Psi}{\partial \lambda} = \hat{A}\Psi, \quad \frac{\partial \Psi}{\partial t} = \hat{B}\Psi \quad (4.1)$$

where $\hat{A}$ and $\hat{B}$ are four-by-four matrices given by

$$\hat{A} = \frac{t}{\lambda} \left( \gamma R^T \gamma^{-1} \otimes I_2 + I_2 \otimes R^T \right)$$

$$\hat{B} = \frac{1}{2} \left( -R + \frac{t}{\lambda} \gamma^{-1} - \frac{t^2}{\lambda^2} \gamma R^T \gamma^{-1} \right) \otimes I_2 + \frac{1}{2} \left( \frac{t^2}{\lambda^2} - 1 \right) I_2 \otimes R \quad (4.2)$$

where $I_2$ is the identity matrix and $\Psi$ is a column formed from the columns of the matrix $\varphi$. 

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This linear system is different from the linear system which is used for the study of Painlevé III equation within the isomonodromic deformation (IMD) method [7,8,9]. Although it is not obvious what are the advantages of the new linear system in comparison with the IMD linear system, there are some interesting features of the new system which can be investigated.

In the Belinskii-Zakharov inverse scattering approach, it is natural to consider path-ordered exponentials of matrices (holonomies) associated to the linear system (2.7) in order to find the integrals of motion [10],[11]. In the special case of Bianchi metrics, the holonomy construction would simplify, and one could try to see what kind of expressions one would obtain for the PIII equation.

Note that the new linear system can be interpreted as a Lax pair for a dynamical system with a time-dependent Hamiltonian

\[ H = \frac{1}{2\tau} p^2 + 2(k^2 e^{-q} + c^2 e^q). \]  

(4.3)

This is analogous and complementary to the results of Harnad and Routhier [12], where a Lax pair for PIII equation with \( \alpha\beta\gamma\delta \neq 0 \) was constructed. It is interesting that in that case the Lax pair contains \( \int d\tau u \), which does not happen in our case of PIII equation.

In the context of Bianchi models, it is more natural to work with a dynamically equivalent Hamiltonian to (4.3)

\[ \tilde{H} = \frac{1}{2} \tilde{p}^2 + 2e^\tilde{t}(k^2 e^{-q} + c^2 e^q), \]  

(4.4)

where \( \tau = e^\tilde{t} \). One can now examine the physical properties of the solutions, like small and large time asymptotic, as well as the singularities, since these properties of the Painlevé III solutions have been thoroughly studied [9].
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