

L² CONCENTRATION OF BLOW-UP SOLUTIONS FOR THE MASS-CRITICAL NLS WITH INVERSE-SQUARE POTENTIAL

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ABSTRACT. In this paper, we prove a refined version of a compactness lemma and we use it to establish mass-concentration for the focusing nonlinear Schrödinger equation with an inverse-square potential.

1. Introduction

We consider the following L²-critical nonlinear Schrödinger equation (NLS) with an attractive inverse-square potential:

\[
\begin{aligned}
&i\partial_t u + \Delta u + \frac{c}{|x|^2}u + |u|^4 u = 0, \quad x \in \mathbb{R}^d, \ t > 0, \\
u(0, x) = u_0(x),
\end{aligned}
\]

with \( d \geq 3 \) and \( c \in (0, c_*) \), where \( c_* = \frac{(d-2)^2}{4} \) is the best constant in Hardy’s inequality:

\[
c_* \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^d} |\nabla u|^2 \, dx, \quad u \in H^1(\mathbb{R}^d).
\]

The Schrödinger equation (1) appears in a variety of physical settings, such as quantum field equations or black hole solutions of the Einstein’s equations [4].

As in the classical case, i.e., with \( c = 0 \), (1) is invariant under the scaling

\[u \to u_\lambda : (t, x) \mapsto \lambda^{d/2} u(\lambda^2 t, \lambda x), \quad \lambda > 0,\]

that is why the equation is called L²-critical.

We have also invariance under time-translation and phase shift. However, the strict positivity of the parameter \( c \) breaks the space-translation symmetry as well as the Galilean transformation.

A recent result of Okazawa, Suzuki and Yokota [10] shows that the Cauchy problem (1) is locally well-posed in \( H^1 \): there exists \( T^* \in (0, +\infty) \) and a
maximal solution $u \in C([0, T^*), H^1)$. Moreover, we have the following blow-up alternative: either $T^* = \infty$ (the solution is global) or $T^* < +\infty$ (the solution blows up in finite time) and

$$ \lim_{t \uparrow T^*} \|u(t, \cdot)\|_{H^1} = +\infty. $$

The unique solution has the following conserved quantities:

$$ M(t) := \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx = M(0), $$

$$ E(t) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \frac{c}{2} \int_{\mathbb{R}^d} |u|^2 \, dx - \frac{d}{4 + 2d} \int_{\mathbb{R}^d} |u|^\frac{4}{d+2} \, dx $$

$$ = E(0). $$

From the definition of the energy, we see that it is convenient to introduce the following Hardy functional

$$ H(u) := \int |\nabla u|^2 \, dx - c \int \frac{|u|^2}{|x|^2} \, dx. $$

The hypothesis on the parameter $c$ implies that $H$ defines a semi-norm on $H^1$ equivalent to $\|\nabla u\|_2$. In particular, $u$ blows up at $T^* > 0$ if and only if $\lim_{t \to T^*} H(u(t)) = \infty$.

The blow-up theory for (1) is mainly connected to the notion of ground state, which is a non-zero, non-negative and radially symmetric $H^1$-solution of the elliptic problem

$$ \Delta Q + \frac{c}{|x|^2} Q - |Q|^\frac{4}{d+2} Q = 0. $$

The existence of ground state solutions to (3) was recently obtained in [5, 6, 2] via Weinstein’s variational approach, but unlike the standard problem (i.e., $c = 0$) where the ground state is unique (up to the symmetries), we do not know if it is the case when $c \in (0, c_*)$.

In addition, the authors in [2] exhibited the following precised Gagliardo-Nirenberg inequality: for all $\psi \in H^1$

$$ \|\psi\|^{\frac{4}{d+2}}_{L^d} \leq C_d H(\psi) \|\psi\|^\frac{4}{d} \|_{L^2}, $$

where $C_d := \frac{d+2}{d} \|Q\|^{-\frac{4}{d}}_{L^2}$.

With this estimate in hand, one can prove that the $L^2$-norm of the ground state is the mass threshold for the formation of singularities. Besides, all solutions to (1) with a mass equal to that of a ground state are all equal to a ground state up to the symmetries.

We note that most of the previously mentioned phenomena (singularity formation, universality of the blow-up profile, etc.) were settled first for the standard problem and there is an abundant literature on that. We refer
the interested reader to [1].

Our aim here is to establish a concentration result for solutions to (NLS) with an inverse-square potential. That is, blowing-up solutions to (1) concentrates a minimal amount of mass, or more precisely

**Theorem 1.** Denote by $Q$ a ground state solution to (3). Let $u$ be a solution of (1) which blows up at finite time $T^* > 0$, and $a(t) > 0$ any function, such that $a(t)\|\nabla u(t)\|_{L^2} \to +\infty$ as $t \to T^*$. Then, there exists $x(t) \in \mathbb{R}^d$, such that

$$\liminf_{t \to T^*} \int_{\{|x-x(t)| \leq a(t)\}} |u(t, x)|^2 dx \geq \int_{\mathbb{R}^d} Q^2.$$

**Remark 2.** Results of this type where firstly obtained for equation (1) with $c = 0$ in [13, 7, 11].

**Remark 3.** Adapting the arguments in [8], one could establish the following lower bound on the blow-up rate for blowing-up solutions

$$\|\nabla u(t, \cdot)\|_{L^2} \geq \frac{C}{\sqrt{T^*-t}}.$$

Thus, any function $a(t) > 0$, such that $\frac{\sqrt{T^*-t}}{a(t)} \to 0$ as $t \to T^*$, fulfills the conditions of the above theorem.

The paper is organized as follows. In section 2 we prove a compactness lemma adapted to equation (1). In section 3, we apply the aforementioned lemma to prove our main result, Theorem 1. We conclude the paper with an appendix.

## 2. Compactness tools

This section is devoted to the proof of our key result which is crucial in establishing the $L^2$ concentration phenomenon for solutions to (1). It is equivalent to the concentration-compactness lemma used in [2], but expressed in terms of $H^1$-profiles.

**Theorem 4.** Let $v = \{v_n\}_{n=1}^{\infty}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Then, there exist a subsequence of $\{v_n\}_{n=1}^{\infty}$ (still denoted $\{v_n\}_{n=1}^{\infty}$), a family $\{x^j\}_{j=1}^{\infty}$ of sequences in $\mathbb{R}^d$ and a sequence $\{V^j\}_{j=1}^{\infty}$ of $H^1$-functions, such that

i) for every $k \neq j$, $|x^k_n - x^j_n| \to +\infty$;

ii) for every $\ell \geq 1$ and every $x \in \mathbb{R}^d$, we have

$$v_n(x) = \sum_{j=1}^{\ell} V^j(x - x^j_n) + v^\ell_n(x),$$

with

$$\limsup_{n \to \infty} \|v^\ell_n\|_{L^p(\mathbb{R}^d)} \to 0$$
for every $p \in ]2, 2^*[$.

Moreover, we have, as $n \to +\infty$,

\begin{equation}
\|v_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|V_j\|_{L^2}^2 + \|v_{\ell n}\|_{L^2}^2 + o_n(1) \tag{6}
\end{equation}

and

\begin{equation}
H(v_n) = \sum_{j=1}^{\ell} H(V_j(\cdot - x_n^j)) + H(v_{\ell n}) + o_n(1). \tag{7}
\end{equation}

**Proof.** Let $\mathcal{V}(v)$ be the set of functions obtained as weak limits in $H^1$ of subsequences of the translated $v_n(\cdot + x_n)$ with $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^d$. Set

$$
\eta(v) = \sup\{\|V\|_{H^1}, \ V \in \mathcal{V}(v)\}.
$$

Clearly

$$
\eta(v) \leq \limsup_{n \to \infty} \|v_n\|_{H^1}.
$$

We claim the existence of a sequence $\{V_j\}_{j=1}^\infty$ of $\mathcal{V}(v)$ and a family $\{x^j\}_{j=1}^\infty$ of sequences of $\mathbb{R}^d$, such that

$$
k \neq j \Rightarrow |x_n^k - x_n^j| \to \infty,
$$

and, up to extracting a subsequence, the sequence $\{v_n\}_{n=1}^\infty$ can be written as

$$
v_n(x) = \sum_{j=1}^{\ell} V_j(x - x_n^j) + v_{\ell n}(x), \quad \eta(v^\ell) \to 0,
$$

such that the identities (6)-(7) hold. Indeed, if $\eta(v) = 0$, one can take $V^1 \equiv 0$ for all $j$, otherwise one chooses $V^1 \in \mathcal{V}(v)$, such that

$$
\|V^1\|_{H^1} \geq \frac{1}{2} \eta(v) > 0.
$$

By definition, there exists some sequence $x^1 = \{x_n^1\}_{n=1}^\infty$ of $\mathbb{R}^d$, such that, up to extracting a subsequence, we have

$$
v_n(\cdot + x_n^1) \to V^1 \quad \text{weakly in } H^1.
$$

Define

$$
v_n^1(\cdot + x_n^1) := v_n - V^1(\cdot - x_n^1).
$$

Since $v_n^1(\cdot + x_n^1) \to 0$, we get

$$
\|v_n\|_{L^2}^2 = \|V^1\|_{L^2}^2 + \|v_{\ell n}\|_{L^2}^2 + o(1),
$$

$$
\|\nabla v_n\|_{L^2}^2 = \|\nabla V^1\|_{L^2}^2 + \|\nabla v_{\ell n}\|_{L^2}^2 + o(1), \quad \text{as } n \to \infty.
$$

It remains to show the following identity

$$
\int \frac{|v_n(x)|^2}{|x|^2} \ dx = \int \frac{|V^1(x - x_n^1)|^2}{|x|^2} \ dx + \int \frac{|v_{\ell n}(x)|^2}{|x|^2} \ dx + o(1), \quad \text{as } n \to \infty.
$$

We have
\[ |v_n(x)|^2 = |V^1(x - x_n^1)|^2 + |v_n^1(x)|^2 + 2\mathcal{R}[V^1(x - x_n^1)v_n^1(x)], \]

where \( \mathcal{R}(z) \) denotes the real part of the complex number \( z \). Thus, it suffices to prove that
\[
\int_{\mathbb{R}^d} \frac{V^1(x - x_n^1)v_n^1(x)}{|x|^2} \, dx \to 0, \quad \text{as} \quad n \to \infty.
\]

Without loss of generality, we suppose that \( V^1 \) is continuous and compactly supported in \( B(0, R), R > 0 \). We distinguish two cases:

- **Case 1:** \( |x_n^1| \to \infty. \)

  We have
  \[
  \int_{\mathbb{R}^d} \frac{V^1(x - x_n^1)v_n^1(x)}{|x|^2} \, dx = \int_{B(0,R)} \frac{V^1(x)v_n^1(x + x_n^1)}{|x + x_n^1|^2} \, dx.
  \]
  Since \( |x_n^1| \to \infty \), there exists \( n(R) \in \mathbb{N}^* \) such that for all \( n \geq n(R) \)
  \[ |x_n^1| \geq 2R. \]
  Therefore, for all \( n \geq n(R) \) and all \( x \in B(0, R) \)
  \[ |x_n^1 + x| \geq R, \]
  and then
  \[
  \int_{B(0,R)} \frac{|V^1(x)||v_n^1(x + x_n^1)|}{|x + x_n^1|^2} \, dx \leq \frac{1}{R^2} \int |V^1(x)||v_n^1(x + x_n^1)| \, dx.
  \]
  The right-hand side term tends to zero as \( n \) tends to infinity, since \( |v_n^1(x + x_n^1)| \to 0 \) in \( H^1 \) (see appendix for a proof).

- **Case 2:** Up to extracting a subsequence, we assume that \( x_n^1 \to x^1 \)
  for some \( x^1 \in \mathbb{R}^d \). It suffices to study the case when \( x^1 = 0 \).

Let \( \epsilon > 0 \). By the dominated convergence theorem, there exists \( \delta(\epsilon) > 0 \), such that
\[
\int_{B(0,2\delta(\epsilon))} \frac{|V^1(x)|^2}{|x|^2} \leq \frac{\epsilon^2}{2}.
\]

Now, write
\[
\left| \int_{\mathbb{R}^d} \frac{V^1(x)v_n^1(x + x_n^1)}{|x + x_n^1|^2} \, dx \right| \leq \int_{B(0,\delta(\epsilon))} \frac{|V^1(x)||v_n^1(x + x_n^1)|}{|x + x_n^1|^2} \, dx \\
+ \int_{B^c(0,\delta(\epsilon))} \frac{|V^1(x)||v_n^1(x + x_n^1)|}{|x + x_n^1|^2} \, dx.
\]
Since \( x_n^1 \to 0 \), there exists \( n_1(\epsilon) \) such that, for all \( n \geq n_1(\epsilon) \)
\[ |x_n^1| < \frac{\delta(\epsilon)}{2}. \]
This implies for all $n \geq n_1(\epsilon)$

\[ \int_{B^c(0,\delta(\epsilon))} \frac{|V^1(x)||v^1_n(x + x^1_n)|}{|x + x^1_n|^2} \, dx \leq \frac{4}{\delta(\epsilon)^2} \int |V^1(x)||v^1_n(x + x^1_n)| \, dx. \]  

(10)

Since $\int |V^1(x)||v^1_n(x + x^1_n)| \, dx \xrightarrow{n \to \infty} 0$, there exists $n_3(\epsilon)$ such that for all $n \geq n_3(\epsilon)$

\[ \int |V^1(x)||v^1_n(x + x^1_n)| \, dx \leq \frac{\epsilon \delta(\epsilon)^2}{4}. \]

Combining the latter estimate with (10) one gets, for all $n \geq \max(n_1(\epsilon), n_3(\epsilon))$

\[ \int_{B^c(0,\delta(\epsilon))} \frac{|V^1(x)||v^1_n(x + x^1_n)|}{|x + x^1_n|^2} \, dx \leq \epsilon. \]

(11)

Now, apply successively Cauchy-Schwarz and Hardy’s inequalities to get

\[ \int_{B(0,\delta(\epsilon))} \frac{|V^1(x)||v^1_n(x + x^1_n)|}{|x + x^1_n|^2} \, dx \lesssim \left( \int_{B(0,\delta(\epsilon))} \frac{|V^1(x)|^2}{|x + x^1_n|^2} \, dx \right)^{\frac{1}{2}} \|\nabla v^1_n\|_{L^2}. \]

The sequence \{v^1_n\} is bounded in $H^1$, we infer that

\[ \int_{B(0,\delta(\epsilon))} \frac{|V^1(x)||v^1_n(x + x^1_n)|}{|x + x^1_n|^2} \, dx \lesssim \left( \int_{B(0,\delta(\epsilon))} \frac{|V^1(x)|^2}{|x + x^1_n|^2} \, dx \right)^{\frac{1}{2}}. \]

(12)

We claim that there exists $n(\epsilon)$ such that for all $n \geq n(\epsilon)$

\[ \int_{B(0,\delta(\epsilon))} \frac{|V^1(x)|^2}{|x + x^1_n|^2} \, dx \leq \epsilon^2. \]

(13)

Set $K(\epsilon, d) := \frac{\sigma_d}{d(2\delta(\epsilon))^{d-2}}$, where $\sigma_d$ is the measure of $S^{d-1}$. The function $|V^1(\cdot)|^2$ is continuous on the compact $\bar{B}(0, 3\delta(\epsilon))$, hence uniformly continuous. That is, there exists $\alpha(\epsilon) \in (0, \delta(\epsilon))$, such that, for all $x, y \in \bar{B}(0, 3\delta(\epsilon))$

\[ |x - y| < \alpha(\epsilon) \Rightarrow ||V^1(x)|^2 - |V^1(y)|^2| < \frac{\epsilon^2}{2K(\epsilon, d)}. \]

Since $x^1_n \to 0$, there exists $n_2(\epsilon)$ such that, for all $n \geq n_2(\epsilon)$

\[ |x^1_n| < \alpha(\epsilon) < \delta(\epsilon). \]

So that, for all $x \in B(0, 2\delta(\epsilon))$ and all $n \geq n_2(\epsilon)$

\[ ||V^1(x - x^1_n)|^2 - |V^1(x)|^2| < \frac{\epsilon^2}{2K(\epsilon, d)}. \]

(14)

The fact that, for all $n \geq n_2(\epsilon)$, $B(x^1_n, \delta(\epsilon)) \subseteq B(0, 2\delta(\epsilon))$, yields along with (13)

\[ \int_{B(x^1_n, \delta(\epsilon))} \frac{|V^1(x - x^1_n)|^2}{|x|^2} \, dx \leq \int_{B(0, 2\delta(\epsilon))} \frac{|V^1(x)|^2}{|x|^2} \, dx + \frac{\epsilon^2}{2}, \quad \text{for all } n \geq n_2(\epsilon). \]

\[ \int_{B(x^1_n, \delta(\epsilon))} \frac{|V^1(x - x^1_n)|^2}{|x|^2} \, dx \leq \int_{B(0, 2\delta(\epsilon))} \frac{|V^1(x)|^2}{|x|^2} \, dx + \frac{\epsilon^2}{2}, \quad \text{for all } n \geq n_2(\epsilon). \]
One obtains (13) by applying estimate (9). At final, for all \( n \geq n_2(\epsilon) \)
\[
\int_{B(0,\delta(\epsilon))} \frac{|V^1(x)||v^1_n(x + x_n^1)|}{|x + x_n^1|^2} \, dx \leq \epsilon.
\] (15)

From (11) and (15), we have, for all \( n \geq \max(n_1(\epsilon), n_2(\epsilon), n_3(\epsilon)) \)
\[
\left| \int_{\mathbb{R}^d} \frac{V^1(x)}{|x + x_n^1|^2} \, dx \right| \leq \epsilon.
\]

This achieves the proof of (8).

Now, replace \( v \) by \( v^1 \) and repeat the same process. If \( \eta(v^1) > 0 \), one gets \( V^2, x^2 \) and \( v^2 \). Moreover, we have
\[
|x_n^1 - x_n^2| \rightarrow \infty, \quad \text{as} \quad n \rightarrow \infty.
\]

Otherwise, up to extracting a subsequence, one gets
\[
x_n^1 - x_n^2 \rightarrow x_0
\]
for some \( x_0 \in \mathbb{R}^d \). Since
\[
v^1_n(\cdot + x_n^2) = v^1_n(\cdot + (x_n^2 - x_n^1) + x_n^1)
\]
and \( v^1_n(\cdot + x_n^1) \) converge weakly to 0, then \( V^2 = 0 \). Thus, \( \eta(v^1) = 0 \), which is a contradiction. An argument of iteration and orthogonal extraction allows us to construct the family \( \{x^j\}_{j=1}^\infty \) and \( \{V^j\}_{j=1}^\infty \) satisfying the claims above. The rest of the proof remains the same as in [3], we omit the details. \( \square \)

As a consequence of Theorem 4, we get the following compactness lemma

**Lemma 5.** Let \( \{v_n\}_{n=1}^\infty \) be a bounded family of \( H^1 \)-functions, such that
\[
\limsup_{n \to \infty} H(v_n) \leq M \quad \text{and} \quad \limsup_{n \to \infty} \|v_n\|_{L^{\frac{4d}{d+2}}} \geq m.
\] (16)

Then, there exists \( \{x_n\}_{n=1}^\infty \subset \mathbb{R}^d \) such that, up to a subsequence,
\[
v_n(\cdot + x_n) \rightharpoonup V \quad \text{in} \quad H^1,
\]
with \( \|V\|_{L^2} \geq \left( \frac{d}{d+2} \right)^{d/4} \frac{m^{\frac{d+1}{2}}}{M^{1/4}} \|Q\|_{L^2} \).

**Proof.** According to Proposition 4, the sequence \( \{v_n\}_{n=1}^\infty \) can be written, up to a subsequence, as
\[
v_n(x) = \sum_{j=1}^\ell V^j(x - x_n^j) + e_n^\ell(x)
\]
such that (5), (6) and (7) hold. This implies, in particular,
\[
m^{\frac{d+2}{2}} \leq \limsup_{n \to \infty} \|v_n\|_{L^{\frac{4d}{d+2}}}^{\frac{4d+2}{2}} = \limsup_{n \to \infty} \left\| \sum_{j=1}^\infty V^j(\cdot - x_n^j) \right\|_{L^{\frac{4d}{d+2}}}^{\frac{4d+2}{2}}.
\]
The elementary inequality
\[
\left| \sum_{j=1}^{l} a_j |a_j|^{4/d+2} - \sum_{j=1}^{l} |a_j|^{4/d+2} \right| \leq C \sum_{j \neq k} |a_j||a_k|^{4/d+1}.
\]
along with the pairwise orthogonality of the family \(\{x_j\}_{j=1}^{\infty}\) leads the mixed terms in the sum above to vanish and we get
\[
m^{4/d+2} \leq \sum_{j=1}^{\infty} \|V_j\|^{4/d+2}_{L_4^{d+2}}.
\]

We claim that
\[
(17) \quad \sum_{j=1}^{\infty} \|V_j\|^{4/d+2}_{L_4^{d+2}} \leq C_d \sup\{\|V_j\|^{4/d}_{L_2}, j \geq 1\} M.
\]

Indeed, let \(\epsilon > 0\). On the one hand, we have from (16)
\[
\exists N_\epsilon \quad \forall n \geq N_\epsilon \quad H(v_n) < M + \frac{\epsilon}{2}.
\]

Let \(l \geq 1\) be fixed. From (7), there exists \(n(l, \epsilon)\) such that for all \(n \geq n(l, \epsilon)\)
\[
\left| H(v_n) - \sum_{j=1}^{l} H(V^j_n) - H(v^l_n) \right| < \frac{\epsilon}{2},
\]
where \(V^j_n(\cdot) := V^j(\cdot - x^j_n)\). Thus, using the fact that the functional \(H\) is positive, we obtain
\[
\sum_{j=1}^{l} H(V^j_{N_\epsilon + n(l, \epsilon)}) \leq \sum_{j=1}^{l} H(V^j_{N_\epsilon + n(l, \epsilon)}) + H(v^j_{N_\epsilon + n(l, \epsilon)}) \leq H(v_{N_\epsilon + n(l, \epsilon)}) + \frac{\epsilon}{2} \leq M + \epsilon.
\]

From the Gagliardo-Nirenberg inequality and the translation-invariance of the \(L^p\)-norms, one has, for all \(l \geq 1\) and all \(\epsilon > 0\)
\[
\sum_{j=1}^{l} \|V^j\|^{4/d}_{L_4^{d+2}} \leq C_d \sup\{\|V^j\|^{4/d}_{L_2}, j \geq 1\} \sum_{j=1}^{l} H(V^j_{N_\epsilon + n(l, \epsilon)}) \leq C_d \sup\{\|V^j\|^{4/d}_{L_2}, j \geq 1\} (M + \epsilon),
\]
that is
\[
\sum_{j=1}^{l} \|V^j\|^{4/d}_{L_4^{d+2}} \leq C_d \sup\{\|V^j\|^{4/d}_{L_2}, j \geq 1\} (M + \epsilon),
\]
which proves (17). Therefore,
\[
\sup_{j \geq 1} \|V_j\|^{4/d}_{L_2} \geq \frac{m^{4/d}}{MC_d}.
\]
Since the series \(\sum \|V_j\|_{L^2}^2\) converges, the supremum above is attained. Therefore, there exists \(j_0\), such that
\[
\|V_{j_0}\|_{L^2} \geq \frac{m^{\frac{d}{2}+1}}{(C_d M)^{d/4}} = \left(\frac{d}{d+2}\right)^{d/4} \frac{m^{\frac{d}{2}+1}}{M^{d/4}} \|Q\|_{L^2}.
\]
On the other hand, a change of variables gives for all \(l \geq j_0\)
\[
v_n(x + x_{j_0}^n) = V_{j_0}(x) + \sum_{1 \leq j \leq \ell} V_j(x + x_{j_0}^n - x_j^n) + \tilde{v}_n^\ell(x),
\]
where \(\tilde{v}_n^\ell(x) = v_n^\ell(x + x_{j_0}^n)\). The pairwise orthogonality of the family \(\{x_j^j\}_{j=1}^\infty\) implies
\[
V_j(\cdot + x_{j_0}^n - x_j^n) \rightharpoonup 0 \quad \text{weakly in } H^1
\]
for every \(j \neq j_0\). Thus
\[
v_n(\cdot + x_{j_0}^n) \rightharpoonup V_{j_0} + \tilde{v}_n^\ell,
\]
derives
\[
\|\tilde{v}_n^\ell\|_{L^\frac{4}{d+2}} \leq \limsup_{n \to \infty} \|\tilde{v}_n^\ell\|_{L^\frac{4}{d+2}} = \limsup_{n \to \infty} \|v_n^\ell\|_{L^\frac{4}{d+2}} \to 0.
\]
The uniqueness of the weak limit yields
\[
\tilde{v}_n^\ell = 0
\]
for every \(\ell \geq j_0\) and then
\[
v_n(\cdot + x_{j_0}^n) \rightharpoonup V_{j_0}.
\]
This closes the proof of the lemma. \(\square\)

3. \(L^2\) CONCENTRATION PHENOMENON

Now with Lemma 4 in hand, one can prove Theorem 1.

**Proof.** Define
\[
\rho(t) := \left(\frac{H(Q)}{H(u(t, \cdot))}\right)^\frac{1}{2} \quad \text{and} \quad v(t, x) := \rho(t)^{d/2} u(t, \rho(t)x).
\]
Let \(\{t_n\}_{n=1}^\infty\) be an arbitrary sequence such that \(t_n \uparrow T^*\). We set \(\rho_n = \rho(t_n)\) and \(v_n = v(t_n, \cdot)\). Since \(u\) conserves its mass, the sequence \(\{v_n\}_{n=1}^\infty\) satisfies
\[
\|v_n\|_{L^2} = \|u_0\|_{L^2} \quad \text{and} \quad H(v_n) = H(Q).
\]
The conservation of energy and the blow-up criteria imply
\[
\mathcal{E}(v_n) = \rho_n^2 \mathcal{E}(0) \to 0, \quad \text{as } n \to \infty.
\]
In particular,
\[
\|v_n\|_{L^\frac{4}{d+2}} \to \frac{d+2}{d} H(Q), \quad \text{as } n \to \infty.
\]
The family \( \{v_n\}_{n=1}^{\infty} \) satisfies the assumptions of Lemma 5 with 
\[
m^{\frac{4}{d} + 2} = \frac{d + 2}{d} H(Q) \quad \text{and} \quad M = H(Q).
\]

It follows that 
\[
\liminf_{n \to +\infty} \int_{|x| \leq \alpha} \rho_n^d |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x| \leq \alpha} |V|^2 dx,
\]
for every \( \alpha > 0 \). Thus, 
\[
\liminf_{n \to +\infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq \alpha(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x| \leq \alpha} |V|^2 dx.
\]

The fact that \( \frac{\rho_n}{N(t_n)} \to 0 \) implies 
\[
\liminf_{n \to +\infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq a(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x| \leq \alpha} |V|^2 dx \geq \int Q^2.
\]

Since the sequence \( \{t_n\} \) is arbitrary we get finally 
\[
\liminf_{t \to T^*} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq a(t)} |u(t, x)|^2 dx \geq \int Q^2.
\]

Since the function \( y \to \int_{|x-y| \leq a(t)} |u(t, x)|^2 dx \) is continuous and goes to 0 at infinity, there exists \( x(t) \) such that 
\[
\sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq a(t)} |u(t, x)|^2 dx = \int_{|x-x(t)| \leq a(t)} |u(t, x)|^2 dx,
\]
which concludes the proof of Theorem 1. \( \square \)

4. Appendix

Lemma 6. Let \( d \geq 1 \) be an integer. Let \( \{u_n\}_{n \geq 0} \) be a sequence of \( H^1(\mathbb{R}^d) \)-
functions such that 
\[
u_n \to 0 \quad \text{in} \quad H^1(\mathbb{R}^d),
\]

Then we have 
\[
|u_n| \to 0 \quad \text{in} \quad H^1(\mathbb{R}^d),
\]

where \( |u_n| \) denotes the modulus of \( u_n \).

Proof. Since \( \{u_n\}_{n \geq 0} \) converges weakly to 0 in \( H^1(\mathbb{R}^d) \) and \( H^1(\mathbb{R}^d) \hookrightarrow L^1_{\text{loc}}(\mathbb{R}^d) \) with compact embedding, the sequence \( \{u_n\}_{n \geq 0} \) converges strongly to 0 in \( L^1_{\text{loc}}(\mathbb{R}^d) \), so that \( \{u_n\}_{n \geq 0} \) converges strongly to 0 in \( L^1_{\text{loc}}(\mathbb{R}^d) \). On the one hand, we deduce from the preceding, using the Riesz representation.
theorem, that \( \{ |u_n| \}_{n \geq 0} \) converges weakly to 0 in \( L^2(\mathbb{R}^d) \). On the other hand, the Diamagnetic inequality \[ \int_{\mathbb{R}^d} |\nabla u|^2 dx \geq \int_{\mathbb{R}^d} |\nabla|u||^2 dx \] which holds true for all \( u \in H^1(\mathbb{R}^d) \), implies the existence of a function \( v \in H^1(\mathbb{R}^d) \) such that \( \{ |u_n| \}_{n \geq 0} \) converges weakly in \( H^1(\mathbb{R}^d) \) to \( v \), and hence weakly in \( L^2(\mathbb{R}^d) \) to \( v \). The uniqueness of the weak limit implies that \( v \equiv 0 \). This achieves the proof of the lemma. \[ \square \]

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References

[1] Cazenave, T. “Semilinear Schrödinger equations.” Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[2] Csobo, E. and F. Genoud, “Minimal mass blow-up solutions for the \( L^2 \)-critical NLS with inverse-square potential.” (2017): preprint [arXiv:1707.01421]

[3] Hmidi, T. and S. Keraani. “Blow-up theory for the critical nonlinear Schrödinger equations revisited.” International Mathematics Research Notices 46 (2005): 2815-2828.

[4] Kalf, H., U.-W. Schmincke, J. Walter and R. Wust. “On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials.” in: Spectral Theory and Differential Equations (Proceedings Symposium Dundee, 1974), Lecture Notes in Mathematics, 448, Springer, (1975): 182-226.

[5] Killip, R., C. Miao, M. Visan, J. Zhang and J. Zheng. “Sobolev spaces adapted to the Schrödinger operator with inverse-square potential.” (2015): preprint [arXiv:1503.02716]

[6] Killip, R., C. Miao, M. Visan, J. Zhang, J. Zheng. “The energy-critical NLS with inverse-square potential.” Discrete and Continuous Dynamical Systems 37 (2017), 3831-3866.

[7] Merle, F. and Y. Tsutsumi. “\( L^2 \) concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power nonlinearity.” Journal of Differential Equations 84, no. 2 (1990): 205–214.

[8] Merle, F. “Lower bounds for the blow-up rate of solutions of the Zakharov equation in dimension two.” Communications in Pure and Applied Mathematics 49 (1996): 765–794.

[9] Montefusco, E. “Lower Semi-continuity of Functionals via the Concentration-Compactness Principle.” J. of Mathematical Analysis and Applications 263 (2001): 264-276.

[10] Okazawa, N., T. Suzuki and T. Yokota. “Energy methods for abstract nonlinear Schrödinger equations.” Evolution Equations and Control Theory 1 (2012): 337-354.

[11] Tsutsumi, Y. “Rate of \( L^2 \) concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power.” Nonlinear Analysis 15, no. 8 (1990): 719–724.

[12] Tao, T. “Nonlinear Dispersive Equations: Local and Global Analysis.” CBMS Regional Conference Series in Mathematics 106, American Mathematical Society, 2006.
[13] Weinstein, M. I. “On the structure and formation of singularities in solutions to the nonlinear dispersive evolution equations.” Communications in Partial Differential Equations 11 (1986): 545-565.

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