Fractional Gamma process and fractional Gamma-subordinated processes

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Abstract

We define and study fractional versions of the well-known Gamma subordinator \( \Gamma := \{ \Gamma(t), t \geq 0 \} \), which are obtained by time-changing \( \Gamma \) by means of an independent stable subordinator or its inverse. Their densities are proved to satisfy differential equations expressed in terms of fractional versions of the shift operator (with fractional parameter greater or less than one, in the two cases). As a consequence, the fractional generalization of some Gamma subordinated processes (i.e. the Variance Gamma, the Geometric Stable and the Negative Binomial) are introduced and the corresponding fractional differential equations are obtained.

Keywords: Gamma subordinator; Variance Gamma process; Geometric Stable subordinator; Negative Binomial process; Fractional shift operator.

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1 Introduction and preliminaries

The Gamma subordinator \( \Gamma(t), t > 0 \) is a very well-known process, applied to many different fields such as, for example, engineering reliability, maintenance theory, risk theory, option pricing and so on. It can be considered as a particular case of the tempered stable subordinators and thus used in financial modelling (see e.g. [8]).

We define here two fractional versions of \( \Gamma \), obtained by a random time-change by means of an independent stable subordinator \( A_{1/\nu} \), or, alternatively, the inverse stable subordinator \( L_{\nu} \) (see section 2.2 for their exact definitions). Thus we define, for any \( t \geq 0 \),

\[
\begin{align*}
\Gamma_{\nu}(t) &:= \Gamma(L_{\nu}(t)), & 0 < \nu < 1 \\
\overline{\Gamma}_{\nu}(t) &:= \Gamma(A_{1/\nu}(t)), & \nu > 1
\end{align*}
\]

(1)

(the case \( \nu = 1 \) corresponds to the standard Gamma process \( \Gamma(t) \)). Only for \( \nu > 1 \), \( \overline{\Gamma}_{\nu} \) represents itself a Lévy process, since it is obtained by subordinating a Lévy process to a stable subordinator.

The processes defined in (1) can be considered as fractional versions of \( \Gamma \), since we prove that, in both cases, their distributions satisfy differential equations expressed in terms of a new operator, that we call ”fractional shift operator”. We recall the definition of the (integer order) shift operator: let \( D^n_x := d^n / dx^n \), for any \( n \in \mathbb{N} \), then

\[
e^{cD_x} f(x) := \sum_{n=0}^{\infty} \frac{c^n D^n_x f(x)}{n!} = f(x + c),
\]

(2)

for any analytic function \( f : \mathbb{R} \to \mathbb{R} \) and \( c \in \mathbb{R} \).

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The fractional counterpart of (2) is obtained by replacing the \( n \)-th order derivative by the \( n \)-fold iterated fractional derivative. We will adopt the Caputo definition of fractional derivative, for \( 0 < \nu \leq 1 \), i.e.

\[
D^\nu x u(x) := \frac{1}{\Gamma(1-\nu)} \int_0^x \frac{1}{(x-s)^\nu} ds, 
\]

(3)

while, for \( \nu \geq 1 \), we use the right sided fractional Riemann-Liouville derivative on \( \mathbb{R}^+ \), i.e.

\[
D^\nu_{-x} u(t) := \frac{1}{\Gamma(m-\nu)} \left( -\frac{d}{dx} \right)^m \int_x^{+\infty} \frac{1}{(s-x)^{1+\nu-m}} ds u(s), \quad m-1 < \nu < m,
\]

(4)

for \( x > 0 \) (see (2.2.4) of [13], p.80).

Correspondingly we define the two operators \( O^\nu_{c,x} \), for \( \nu \leq 1 \), and \( \overline{O}^\nu_{c,x} \), for \( \nu \geq 1 \), as fractional variants of (2) (see Definitions 1 and 2 below).

A generalized exponential operator of fractional order has been presented in [9] (see also [2]): when applied to power functions, i.e. \( f(x) = x^k \), \( k \in \mathbb{N} \), it is proved to produce the so-called Hermite-Kampé de Feriét polynomials. More recently, another fractional version of (2) has been proposed in [17] (where the exponential is replaced by the Mittag-Leffler function).

By means of the operators \( O^\nu_{c,x} \) and \( \overline{O}^\nu_{c,x} \), we obtain, in the next section, the fractional differential equations satisfied by the one-dimensional distributions of \( \Gamma^\nu \) and \( \overline{\Gamma}^\nu \), in the two ranges of the fractional parameter \( \nu \). The expedience of using (3) and (4) in the definitions of \( O^\nu_{c,x} \) and \( \overline{O}^\nu_{c,x} \) for \( \nu < 1 \) and \( \nu > 1 \), respectively, has been suggested by the results on stable subordinators and their inverse: indeed it is known that the law of \( A_{1/\nu} \) (that we will denote as \( h_{1/\nu}(x,t) \)) satisfies the following fractional equation of order \( \nu > 1 \)

\[
D^\nu_{-x} h_{1/\nu} = \frac{\partial}{\partial x} h_{1/\nu}, \quad x, t \geq 0, \quad h_{1/\nu}(x,0) = \delta(x)
\]

(with other appropriate initial conditions, see [10] and [4] for details). On the other hand the density of the process \( L^\nu(t) := \inf \{ z : A^\nu(z) > t \} \) (denoted hereafter as \( l^\nu(x,t) \)) satisfies the fractional equation of order \( \nu < 1 \)

\[
D^\nu_{x} l^\nu = -\frac{\partial}{\partial x} l^\nu, \quad x, t \geq 0, \quad l^\nu(x,0) = \delta(x),
\]

(see [11]).

Moreover, we prove that the one-dimensional distributions of the processes in (1) satisfy, alternatively, a differential equation expressed in terms of the fractional version of the operator \( P_{c,x} \), defined as

\[
P_{c,x} f(x) := \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j c^j} D^j_{x} f(x), \quad c, x \in \mathbb{R},
\]

(5)

for any infinitely differentiable function \( f \).

As a consequence of all the previous results, we derive in Section 3 the differential equations satisfied by the fractional versions of the following Gamma subordinated processes: the Variance Gamma process, the Geometric Stable subordinator and the Negative Binomial process.

The Variance Gamma (hereafter VG) process (alternatively defined as Laplace motion) is obtained by subordinating a Brownian motion to an independent Gamma subordinator:

\[
X(t) := B(\Gamma(t)), \quad t \geq 0,
\]

where \( B \) is a standard Brownian motion. The VG process is a particular case of a symmetric geometric \( \nu \)-stable process, for \( \nu = 2 \), and it is widely used in the financial theory, in order to
model the logarithm of stock prices (see, e.g., [16], [14]). It has been already proposed a fractional version of the VG process in [19], defined by subordinating a fractional Brownian motion $B_H$ to an independent Gamma subordinator, i.e., as $X_H(t) := B_H(\Gamma(t))$, where $H \in (0, 1)$ is the Hurst exponent. This process is useful to model hydraulic conductivity fields in geophysics, as well as financial time series. We propose here different fractional versions of $X$, defined as $X_\nu(t) := B(\Gamma_\nu(t))$, $t \geq 0$, for $\nu < 1$, and $\overline{X}_\nu(t) := B(\overline{\Gamma}_\nu(t))$, $t \geq 0$, for $\nu > 1$. Again, in the last case, we get a Lévy process and the corresponding Lévy symbol is obtained. Moreover, by definition, it is clear that the marginal distributions of the fractional VG processes are scale mixtures of normal laws: indeed it is

$$X_\nu(t) \overset{d}{=} \Gamma_\nu(t)Z, \text{ for } \nu < 1 \text{ and } \overline{X}_\nu(t) \overset{d}{=} \overline{\Gamma}_\nu(t)Z, \text{ for } \nu > 1,$$

where $Z$ is a standard Gaussian variable and $\overset{d}{=} \text{ denotes the equality of one-dimensional distributions. By comparing (6) with formula (1.5) of [19], we can note that here $\Gamma_\nu$ and $\overline{\Gamma}_\nu$ play the same role of the generalized Gamma (or Amoroso) random process $G_t^{2H}$ process (whose law is reported there in (2.1)). Thus they can represent the stochastic variance or volatility, in financial terms.

Also the Geometric Stable (hereafter GS) subordinator is widely studied and applied, especially in financial contexts (see [18]); it is one of the special subordinators for which the potential measure has a decreasing density, thus a wide potential theory has been established for it (see [6], [23]).

The GS process is defined as a stable subordinator time-changed by means of a Gamma process (see (47) below). The differential equation satisfied by its density has been obtained in [3] and we generalize it to the fractional case.

The Negative Binomial (hereafter NB) process is a discrete valued process, which can be defined, alternatively, as a compound Poisson process with logarithmic jumps or as a mixed Poisson process (i.e. a Poisson process subordinated to an independent Gamma subordinator, see, e.g., [20]).

Through the first definition, a fractional version of the NB process has been introduced in [5] and the corresponding densities are proved to solve fractional recursive differential equations, which generalize the Kolmogorov ones. By exploiting the mixing representation, we obtain here alternative differential equations, involving the fractional shift operators.

## 2 Fractional Gamma processes

We first recall the following preliminary result, proved in [3]: the one-dimensional distribution of the Gamma subordinator $\Gamma(t), t \geq 0$, of parameter $b > 0$, i.e.

$$f_\Gamma(x, t) := \Pr \{\Gamma(t) \in dx\} = \begin{cases} \frac{b^t}{1(t)} x^{t-1} e^{-bx}, & x \geq 0 \\ 0, & x < 0 \end{cases},$$

satisfies the following Cauchy problem, for $x, t \geq 0$,

$$\begin{cases} \frac{\partial}{\partial x} f_\Gamma = -b(1 - e^{-\partial x}) f_\Gamma \\ f_\Gamma(x, 0) = \delta(x) \\ \lim_{|x| \to +\infty} f_\Gamma(x, t) = 0 \end{cases},$$

where $e^{-\partial x}$ is the partial derivative version of the shift operator defined in (2) and $\delta(x)$ is the Dirac delta function.

Then we need to introduce the definition of the fractional shift operators, for the two cases $\nu \leq 1$ and $\nu \geq 1$. 

Definition 1 Let $f: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with fractional derivative $D_\nu^x$ defined in (3), for $\nu \in (0, 1]$, then
\[
\mathcal{O}_\nu^{c,x} f(x) := \sum_{n=0}^{\infty} \frac{c^n}{n!} D_\nu^x \ldots D_\nu^x f(x),
\] (9)
provided that the series converges.

Definition 2 Let $f: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with fractional derivative $D_\nu^{\nu-} x$ defined in (4), for $\nu \geq 1$, then
\[
\mathcal{O}_\nu^{c,x} f(x) := \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} D_\nu^{\nu-} \ldots D_\nu^{\nu-} f(x),
\] (10)
provided that the series converges.

The semigroup property does not hold for the fractional derivatives $D_\nu^x$ and $D_\nu^{\nu-} x$ and thus for the operators (9) and (10) cannot be used the formalism $e^{cD_x}$ adopted in [9].

It is easy to check that, for $\nu = 1$, the fractional shift operator defined in (9) coincides with the standard shift operator in (2): indeed we get
\[
\mathcal{O}_1^{c,x} f(x) = \sum_{n=0}^{\infty} \frac{c^n}{n!} D_x^n f(x) = e^{cD_x} f(x).
\]

On the other hand, for $\nu = 1$, formula (10) reduces to
\[
\mathcal{O}_1^{c,x} f(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} D_x^n f(x) = e^{cD_x} f(x),
\]
since $D_\nu^{\nu-} x = (-1)^n D_x^n$ (see (2.2.5) of [13]).

We note that (9) and (10) do not coincide with the fractional analogue of the Taylor’s series expansion introduced in [21] and its generalizations presented in [24], [12].

2.1 The case $\nu < 1$: the fractional Gamma process

We consider now the first fractional Gamma process, defined as
\[
\Gamma_\nu(t) := \Gamma(\mathcal{L}_\nu(t)), \quad \nu \in (0, 1], \ t \geq 0,
\] (11)
where $\Gamma$ is a Gamma process independent of $\mathcal{L}_\nu$ and by $\mathcal{L}_\nu$ we denote the inverse of a stable subordinator $\mathcal{A}_\nu$ of index $\nu$ (with parameters $\mu = 0$, $\beta = 1$, $\sigma = (t \cos \pi \nu/2)^{1/\nu}$, in the notation of [22]). Thus, by definition, $\mathcal{L}_\nu(t) := \inf \{ z \geq 0 : \mathcal{A}_\nu(z) > t \}$ and we recall that
\[
\mathbb{E} e^{-k\mathcal{L}_\nu(t)} = E_{\nu,1}(-kt^\nu), \quad k > 0,
\] (12)
where
\[
E_{\nu,\beta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\nu j + \beta)}, \quad \Re(\nu) > 0, \ \beta, x \in \mathbb{C}
\]
is the Mittag-Leffler function. We start by deriving the fractional equation satisfied by the one-dimensional distribution of the process $\Gamma_\nu$ defined in (11).
Theorem 3 Let \( f_{\Gamma_{\nu}}(x,t) := \Pr\{\Gamma_{\nu}(t) \in dx\} \), for \( x,t \geq 0 \), and \( l_{\nu}(x,t) := \Pr\{L_{\nu}(t) \in dx\} \), then the density 
\[
f_{\Gamma_{\nu}}(x,t) = \int_0^\infty f_{\Gamma}(x,z)l_{\nu}(z,t)dz \tag{13}
\]
satisfies, for \( \nu \in (0,1) \) and \( t > 1 \), the following equation 
\[
\frac{\partial}{\partial x} f_{\Gamma_{\nu}} = -b(1 - O_{-1,t}^{\nu}) f_{\Gamma_{\nu}}, \quad x \geq 0,
\]
with initial condition 
\[
f_{\Gamma_{\nu}}(0,t) = 0. \tag{15}
\]

Proof. The initial condition is immediately satisfied by (13), since 
\[
f_{\Gamma_{\nu}}(0,t) = 0, \quad \text{for} \quad t > 1.
\]
In order to verify equation (14) we evaluate the Laplace transform of (13), with respect to \( x \), by denoting 
\[
\tilde{f}_{\Gamma_{\nu}}(\theta,t) := \int_0^\infty e^{-\theta x} f_{\Gamma_{\nu}}(x,t)dx,
\]
\[
\tilde{f}_{\Gamma_{\nu}}(\theta,t) = \int_0^\infty \tilde{f}_\Gamma(z,t)l_{\nu}(z,t)dz \tag{16}
\]
\[
= \int_0^\infty \exp \left\{ -z \log \left( 1 + \frac{\theta}{b} \right) \right\} l_{\nu}(z,t)dz
\]
\[
= \left[ \text{by (12) with } k = \log[1 + \theta/b] > 0 \right]
\]
\[
= E_{\nu,1} \left( -\log \left( 1 + \frac{\theta}{b} \right) t^\nu \right).
\]

By taking the Laplace transform of the r.h.s. of (14) we get 
\[
-b(1 - O_{-1,t}^{\nu}) \tilde{f}_{\Gamma_{\nu}}(\theta,t) \tag{17}
\]
\[
= -bE_{\nu,1} \left( -\log \left( 1 + \frac{\theta}{b} \right) t^\nu \right) + b \sum_{n=0}^\infty \frac{(-1)^n}{n!} D_{-1}^{\nu} \underbrace{D_{-1}^{\nu} \cdots D_{-1}^{\nu}}_{\text{n-times}} E_{\nu,1} \left( -\log \left( 1 + \frac{\theta}{b} \right) t^\nu \right)
\]
\[
= \left[ \text{by } n \text{ applications of formula (2.4.58) of [13]} \right]
\]
\[
= -bE_{\nu,1} \left( -\log \left( 1 + \frac{\theta}{b} \right) t^\nu \right) + bE_{\nu,1} \left( -\log \left( 1 + \frac{\theta}{b} \right) t^\nu \right) \sum_{n=0}^\infty \frac{(\log (1 + \theta/b))^n}{n!}
\]
\[
= -b \left[ 1 - 1 - \frac{\theta}{b} \right] E_{\nu,1} \left( -\log \left( 1 + \frac{\theta}{b} \right) t^\nu \right) = \theta E_{\nu,1} \left( -\log \left( 1 + \frac{\theta}{b} \right) t^\nu \right),
\]

which coincides with the Laplace transform of the l.h.s. of (14), i.e. 
\[
\theta \tilde{f}_{\Gamma_{\nu}}(\theta,t) = \theta E_{\nu,1} \left( -\log \left( 1 + \frac{\theta}{b} \right) t^\nu \right) - f_{\Gamma_{\nu}}(0,t) \tag{18}
\]
\[
= \left[ \text{by [13]} \right]
\]
\[
= \theta E_{\nu,1} \left( -\log \left( 1 + \frac{\theta}{b} \right) t^\nu \right).
\]

Remark 4 We can alternatively prove that the density (13) satisfies the following time-fractional equation, for any \( \nu \in (0,1) \) and \( t > 1 \): 
\[
D_{t}^{\nu} f_{\Gamma_{\nu}} = -P_{b,x} f_{\Gamma_{\nu}}, \quad x \geq 0, \tag{19}
\]
with initial conditions
\[ \frac{\partial^j}{\partial x^j} f_{\Gamma_{\nu}}(x, t) \bigg|_{x=0} = 0, \quad j = 0, 1, \ldots \] (20)
where \( D_{\nu}^x \) denotes, as usual, the Caputo fractional derivative. Indeed by taking the Laplace transform of (19) we get
\[
\mathcal{L} \{ D_{\nu}^x f_{\Gamma_{\nu}}(\cdot, t); \theta \} = D_{\nu}^x E_{\nu,1} \left( - \log \left( 1 + \frac{\theta}{b} \right) t^\nu \right)
\]
(21)
\[
= - \log \left( 1 + \frac{\theta}{b} \right) E_{\nu,1} \left( - \log \left( 1 + \frac{\theta}{b} \right) t^\nu \right)
\]
\[
= - \mathcal{L} \{ P_{b,x} f_{\Gamma_{\nu}}(\cdot, t); \theta \},
\]
where \( \mathcal{L} \{ f(\cdot); \theta \} := \int_0^\infty e^{-\theta x} f_{\Gamma}(x, t)dx \). The last equality in (21) is obtained by considering the well-known formula
\[
\mathcal{L} \{ D_{\nu}^x f(\cdot); \theta \} = \theta^l \bar{f}(\theta) - \sum_{j=0}^{l-1} \theta^j D_{\nu}^{l-1-j} x f(x) \bigg|_{x=0}
\]
(22)
Indeed, by the definition of \( P_{c,x} \) given in (5), for \( c = b \), we get
\[
\mathcal{L} \{ P_{b,x} f_{\Gamma_{\nu}}(\cdot, t); \theta \} = \sum_{l=1}^{\infty} \frac{(-1)^{l+1} \Gamma(l)}{b^l} \int_0^\infty e^{-\theta x} D_{\nu}^x f_{\Gamma_{\nu}}(x, t)dx
\]
\[
= \frac{1}{b} \int_0^\infty z \int_0^\infty e^{-\theta x} D_{\nu}^x f_{\Gamma_{\nu}}(z, t)dz dt
\]
\[
= \log \left( 1 + \frac{\theta}{b} \right) E_{\nu,1} \left( - \log \left( 1 + \frac{\theta}{b} \right) t^\nu \right)
\]
\[
= \log \left( 1 + \frac{\theta}{b} \right) E_{\nu,1} \left( - \log \left( 1 + \frac{\theta}{b} \right) t^\nu \right)
\]
We note that the process \( \Gamma_{\nu} \) is no longer a subordinator, since it is clear from (16) that its density is not infinitely divisible. To avoid this problem we present in the next section a fractional version of the Gamma process which is still infinitely divisible, and, being increasing, is also a subordinator.

We analyze here some properties of the process \( \Gamma_{\nu} \), such as its moments. The expected value is finite and can be obtained by taking its Laplace transform and considering the well-known result
\[
\int_0^\infty e^{-st} l_{\nu}(x,t)dt = s^{\nu-1}e^{-sx}.
\]
Indeed we get
\[
\mathcal{L} \{ \mathbb{E} \Gamma_{\nu}(t); s \} = \int_0^\infty e^{-st} \int_0^\infty \mathbb{E} \Gamma(z) l_{\nu}(z,t)dz dt
\]
\[
= \frac{1}{b} \int_0^\infty z \int_0^\infty e^{-syt} l_{\nu}(z,t)dt dz
\]
\[
= \frac{s^{\nu-1}}{b} \int_0^\infty ze^{-sx} dz = \frac{1}{b s^{\nu+1}}
\]
and thus
\[
\mathbb{E} \Gamma_{\nu}(t) = \frac{t^\nu}{b \Gamma(\nu + 1)},
\]
which, for \( \nu = 1 \), reduces to the well-known expected value of the Gamma process.
Lemma 5 The $r$-th absolute moments of $\Gamma_\nu$ are given by

$$\mathbb{E}\Gamma_\nu(t)^r = \frac{1}{b^r} \sum_{k=0}^{r} \left[ \frac{r}{k} \right] \frac{k! \nu^k}{\Gamma(\nu k + 1)}, \quad r \in \mathbb{Z},$$

(23)

where $\left[ \frac{r}{k} \right]$ denotes the (unsigned) Stirling numbers of the first kind.

Proof. Recall that, for the Gamma subordinator,

$$\mathbb{E}\Gamma_\nu(t)^r = \frac{1}{b^r} \frac{\Gamma(r + t)}{\Gamma(t)},$$

(24)

so that we get

$$\mathcal{L}\left\{ \mathbb{E}\Gamma_\nu(\cdot)^r; s \right\} = \int_{0}^{\infty} e^{-st} \int_{0}^{\infty} \mathbb{E}\Gamma(z)^r \nu(z,t) dzdt$$

(25)

$$= s^{\nu - 1} \int_{0}^{\infty} \frac{\Gamma(r + z)}{\Gamma(z)} e^{-s^\nu z} dt$$

$$= \frac{s^{\nu - 1}}{b^r} \int_{0}^{\infty} z^{(r)} e^{-s^\nu z} dt,$$

where $z^{(r)}$ denotes the rising factorial defined as $z^{(r)} := \Gamma(r + z)/\Gamma(r)$. We recall the following expansion for the rising factorials:

$$z^{(r)} = \sum_{k=0}^{r} \left[ \frac{r}{k} \right] z^k,$$

(26)

therefore (25) can be rewritten as

$$\mathcal{L}\left\{ \mathbb{E}\Gamma_\nu(\cdot)^r; s \right\} = \frac{s^{\nu - 1}}{b^r} \sum_{k=0}^{r} \left[ \frac{r}{k} \right] \int_{0}^{\infty} z^k e^{-s^\nu z} dt$$

$$= \frac{s^{\nu - 1}}{b^r} \sum_{k=0}^{r} \left[ \frac{r}{k} \right] \frac{k!}{s^{\nu k + 1}} = \frac{1}{b^r} \sum_{k=0}^{r} \left[ \frac{r}{k} \right] \frac{k!}{s^{\nu k + 1}}.$$

By inverting the Laplace transform we get (23).

In order to obtain the variance of $\Gamma_\nu$ we choose $r = 2$ in (23), so that we get

$$var(\Gamma_\nu(t)) = \frac{2t^{2\nu}}{b^2 \Gamma(2\nu + 1)} + \frac{t^\nu}{b^2 \Gamma(\nu + 1)} - \frac{t^{2\nu}}{b^2 \Gamma^2(\nu + 1)}.$$

Again, for $\nu = 1$, we obtain the variance of the Gamma process.

2.2 The case $\nu > 1$: the fractional Gamma subordinator

The second fractional Gamma process we present here is defined as

$$\Gamma_\nu(t) := \Gamma(\mathcal{A}_{1/\nu}(t)), \quad \nu > 1, \ t \geq 0,$$

(27)

where $\Gamma$ is a Gamma process and $\mathcal{A}_{1/\nu}$ is the (independent) stable subordinator of index $1/\nu$ (with parameters $\mu = 0, \theta = 1, \sigma = (t \cos \pi/2\nu)^\nu$). It is well-known that

$$\mathbb{E}e^{-k \mathcal{A}_{1/\nu}(t)} = e^{-k^{1/\nu} t}, \quad k > 0.$$
Theorem 6 Let \( f_{\Gamma_{\nu}}(x,t) := \text{Pr}\{\Gamma_{\nu}(t) \in dx\}, \ x, t \geq 0, \) and \( h_{1/\nu}(x,t) := \text{Pr}\{A_{1/\nu}(t) \in dx\}, \) then the density

\[
f_{\Gamma_{\nu}}(x,t) = \int_0^\infty f_{\Gamma}(x,z) h_{1/\nu}(z,t)\,dz
\]  

(29)
satisfies, for \( \nu > 1 \) and \( t > 1 \), the following equation

\[
\frac{\partial}{\partial x} f_{\Gamma_{\nu}} = -b(1 - \overline{\rho}_{-1,t}) f_{\Gamma_{\nu}}, \quad x \geq 0,
\]

(30)
with initial condition

\[
f_{\Gamma_{\nu}}(0,t) = 0.
\]

(31)

Proof. The condition (31) is immediately satisfied by (29). As far as equation (30) is concerned, as in the proofs of Theorem 3, we take the Laplace transform of the r.h.s. of (30) which reads

\[
-\frac{\partial}{\partial x} f_{\Gamma_{\nu}}(x,t) = -b(1 - \overline{\rho}_{-1,t}) \int_0^\infty \tilde{f}_{\Gamma}(\theta,z) h_{1/\nu}(z,t)\,dz
\]

(28)

\[
= -b(1 - \overline{\rho}_{-1,t}) \exp \left\{ -t \left( \log \left( 1 + \frac{\theta}{b} \right) \right)^{1/\nu} \right\}
\]

(29)

\[
= -b \exp \left\{ -t \left( \log \left( 1 + \frac{\theta}{b} \right) \right)^{1/\nu} \right\} + b \sum_{n=0}^\infty \frac{1}{n!} \frac{D^{\nu}_{-t} \cdot \cdots \cdot D^{\nu}_{-t}}{n\text{ times}} \exp \left\{ -t \left( \log \left( 1 + \frac{\theta}{b} \right) \right)^{1/\nu} \right\}
\]

(30)

\[
= -b \exp \left\{ -t \left( \log \left( 1 + \frac{\theta}{b} \right) \right)^{1/\nu} \right\} + b \exp \left\{ -t \left( \log \left( 1 + \frac{\theta}{b} \right) \right)^{1/\nu} \right\} \sum_{n=0}^\infty \frac{1}{n!} \left( \log \left( 1 + \frac{\theta}{b} \right) \right)^n
\]

(31)

\[
= -b \exp \left\{ -t \left( \log \left( 1 + \frac{\theta}{b} \right) \right)^{1/\nu} \right\} + b \exp \left\{ -t \left( \log \left( 1 + \frac{\theta}{b} \right) \right)^{1/\nu} \right\} \frac{b + \theta}{b},
\]

which coincides with the Laplace transform of the l.h.s. of (30), i.e.

\[
\mathcal{L} \left\{ \frac{\partial}{\partial x} f_{\Gamma_{\nu}}(\cdot,t) ; \theta \right\} = \theta \tilde{f}_{\Gamma_{\nu}}(\theta,t) - f_{\Gamma_{\nu}}(0,t)
\]

(32)

\[
= \theta \exp \left\{ -t \left( \log \left( 1 + \frac{\theta}{b} \right) \right)^{1/\nu} \right\}.
\]

Remark 7 We can alternatively prove that the density (29) satisfies the following time-fractional equation, for any \( \nu \in (1, +\infty) \) and \( t > 1 \):

\[
D^{\nu}_{-t} f_{\Gamma_{\nu}} = -\mathcal{P}_{b,x} f_{\Gamma_{\nu}}, \quad x \geq 0,
\]

(33)

with initial conditions

\[
\left. \frac{\partial^j}{\partial x^j} f_{\Gamma_{\nu}}(x,t) \right|_{x=0} = 0, \quad j = 0, 1, ...
\]
where \( \mathcal{D}^\nu_{-t} \) denotes the Riemann-Liouville fractional derivative defined in (4). Indeed by taking the Laplace transform of (32) we get

\[
\mathcal{L}\{\mathcal{D}^\nu_{-t} f_{\Gamma^{\nu}}(\cdot, t) ; \theta\} = \mathcal{D}^\nu_{-t} \exp \left\{-t \left( \log \left( 1 + \frac{\theta}{b} \right) \right)^{1/\nu} \right\}
\]

\[
= \log \left( 1 + \frac{\theta}{b} \right) \exp \left\{-t \left( \log \left( 1 + \frac{\theta}{b} \right) \right)^{1/\nu} \right\}
\]

\[
= \left[ \text{by (22) and (33)} \right] - \log \left( 1 + \theta b \right) \exp \left\{-t \left( \log \left( 1 + \theta b \right) \right)^{1/\nu} \right\}.
\]

We remark that the process \( \Gamma^{\nu} \) defined in (27) is obtained by subordinating a Lévy process to the independent stable subordinator \( A_{1/\nu} \). Thus it is itself a Lévy process (see, e.g. [1], Theorem 1.3.25) and, being also real valued and increasing, it is a subordinator. Its Laplace exponent can be evaluated directly:

\[
\psi^{\Gamma^{\nu}}(\theta) := -\frac{1}{t} \log \mathbb{E} e^{-\theta \Gamma^{\nu}(t)} = -\frac{1}{t} \log f^{\Gamma^{\nu}}(\theta, t) = \left( \log \left( 1 + \frac{\theta}{b} \right) \right)^{1/\nu},
\]

which reduce, for \( \nu = 1 \), to the Laplace exponent of \( \Gamma \). It can be checked that (34) is a Bernstein function, by verifying that \((-1)^{n} d^{(n)} \psi^{\Gamma^{\nu}}(\theta)/d\theta^{(n)} \geq 0 \). Moreover we have \( \lim_{\theta \to 0} \psi^{\Gamma^{\nu}}(\theta) = 0 \). Thus, by Theorem 1.3.4, p.45 in [1], for \( \psi^{\Gamma^{\nu}}(\theta) \) there exists the following representation

\[
\psi^{\Gamma^{\nu}}(\theta) = a + b\theta + \int_{0}^{\infty} (1 - e^{-y\theta}) \lambda(dy),
\]

for a measure \( \lambda \) s.t. \( \int_{0}^{\infty} (y \wedge 1) \lambda(dy) < \infty \), for all \( \theta > 0, a, b \geq 0 \).

Loosely speaking formula (34) implies that \( \Gamma^{\nu} \) grows more slowly than the standard Gamma subordinator, as \( x \to +\infty \). Indeed \( \Gamma \) increases at a logarithmic rate (see, for example, [15]).

The Lévy symbol \( \eta^{\Gamma^{\nu}}(u) := \log \mathbb{E} e^{iu\Gamma^{\nu}(t)/t} \) of \( \Gamma^{\nu} \) can be obtained by applying Proposition 1.3.27 in [1], for any \( u \in \mathbb{R} \),

\[
\eta^{\Gamma^{\nu}}(u) = -\psi_{A_{1/\nu}}(-\eta^{\Gamma}(u)),
\]

where \( \psi_{A_{1/\nu}}(\theta) = \theta^{1/\nu} \) is the Laplace exponent of the stable subordinator and \( \eta^{\Gamma}(u) = -\log(1 - iu/b) \) the Lévy symbol of \( \Gamma \). Thus we get

\[
\eta^{\Gamma^{\nu}}(u) = -\left[ \log(1 - iu/b) \right]^{1/\nu},
\]

which, for \( \nu = 1 \) reduces to \( \eta^{\Gamma} \).

Finally we note that the process \( \Gamma^{\nu} \) does not possess any finite moment, since the same is true for the subordinator \( A_{1/\nu} \).

### 3 Applications to Gamma-subordinated processes

We will define and analyze the fractional versions of well-known processes, such as the VG, the GS and the NB processes, which are all expressed through a random time change by the Gamma subordinator. By applying the previous results, we will be able to derive the fractional equations satisfied by their distributions, expressed by means of the fractional shift operators \( \mathcal{O}^{\nu}_{-1,t} \) and \( \overline{\mathcal{O}}^{\nu}_{-1,t} \).
3.1 Fractional Variance Gamma processes

We consider now the process obtained by the composition of a Brownian motion with one of the two fractional versions of the Gamma process studied so far. Thus we define

\[
\begin{align*}
X_\nu(t) := B(\Gamma_\nu(t)), & \quad t \geq 0, \ \nu \in (0, 1) \\
\bar{X}_\nu(t) := B(\Gamma_\nu(t)), & \quad t \geq 0, \ \nu \in (1, \infty)
\end{align*}
\]

(35)

where \( B \) is a standard Brownian motion and \( \Gamma_\nu \) and \( \Gamma_\nu \) are independent of \( B \). For \( \nu = 1 \), the processes in \eqref{35} coincide with the Variance Gamma process. Let, for simplicity, \( b = 1 \) from now onwards. Let us denote the one-dimensional distributions of \( X_\nu \) and \( \bar{X}_\nu \) as

\[
f_{X_\nu}(x, t) = \int_0^\infty f_B(x, z) f_{\Gamma_\nu}(z, t) dz, \quad x, t \geq 0, \ \nu \in (0, 1),
\]

(36)

and

\[
f_{\bar{X}_\nu}(x, t) = \int_0^\infty f_B(x, z) f_{\bar{\Gamma}_\nu}(z, t) dz, \quad x, t \geq 0, \ \nu > 1,
\]

(37)

where \( f_B \) is the transition density of the standard Brownian motion \( B \).

**Theorem 8** The density \( f_{X_\nu} \) of \( X_\nu \) satisfies the following fractional differential equation

\[
\frac{1}{2} \frac{\partial^2}{\partial x^2} f_{X_\nu} = (1 - O_{\nu-1,t}) f_{X_\nu}, \quad x \geq 0, \ t > 1, \ \nu \in (0, 1),
\]

(38)

while the density \( f_{\bar{X}_\nu} \) of \( \bar{X}_\nu \) satisfies the equation

\[
\frac{1}{2} \frac{\partial^2}{\partial x^2} f_{\bar{X}_\nu} = (1 - O_{\nu-1,t}) f_{\bar{X}_\nu}, \quad x \geq 0, \ t > 1, \ \nu > 1.
\]

(39)

The boundary conditions for both equations are

\[
\begin{align*}
\lim_{|x| \to \infty} f(x, t) &= 0 \\
\lim_{|x| \to \infty} \frac{\partial}{\partial x} f(x, t) &= 0
\end{align*}
\]

(40)

where \( f := f_{X_\nu} \), for \( \nu \in (0, 1) \) and \( f := f_{\bar{X}_\nu} \), for \( \nu \in (1, \infty) \).

**Proof.** The conditions \eqref{40} can be easily checked by considering that their analogues are satisfied by \( f_B \). For \( \nu \in (0, 1) \), in order to prove \eqref{35}, we take the Fourier transform with respect to \( x \) of its r.h.s.: let

\[
\tilde{f}(u) := \mathcal{F} \{ f(\cdot); u \} = \int_{-\infty}^{+\infty} e^{iux} f(x) dx,
\]

then

\[
\mathcal{F} \{(1 - \mathcal{O}_{-1,t}) f_{X_\nu}(\cdot, t); u \} = (1 - \mathcal{O}_{-1,t}) \tilde{f}_{X_\nu}(u, t)
\]

(41)

\[
= (1 - \mathcal{O}_{-1,t}) E_{\nu,1} \left( -t^\nu \log \left( 1 + \frac{u^2}{2} \right) \right)
\]

\[
= \frac{u^2}{2} E_{\nu,1} \left( -t^\nu \log \left( 1 + \frac{u^2}{2} \right) \right),
\]

where for the last step we have performed some calculations similar to \eqref{17}. For the l.h.s. of \eqref{35}, by considering the conditions \eqref{40}, we get instead

\[
\mathcal{F} \left\{ \frac{1}{2} \frac{\partial^2}{\partial x^2} f_{X_\nu}(\cdot, t); u \right\} = -\frac{u^2}{2} \tilde{f}_{X_\nu}(u, t),
\]

(42)
which coincides with (41). Analogously, for \( \nu > 1 \), we get
\[
\mathcal{F} \left\{ \frac{1}{1 - \mathcal{O}'_{-1,t}} \mathcal{F}_{X_{\nu}}(\cdot,t); u \right\} = (1 - \mathcal{O}'_{-1,t}) \mathcal{F}_{X_{\nu}}(u,t) = (1 - \mathcal{O}'_{-1,t}) \exp \left\{ -t \left( \log \left( 1 + \frac{u^2}{2} \right) \right)^{1/\nu} \right\}
\]
which is equal to the analogue of (42), with \( f_{X_{\nu}} \) replaced by \( \hat{f}_{X_{\nu}} \).

Alternatively we can prove the previous result directly, without resorting to the Fourier transform, by considering the heat equation together with (14), for \( \nu \in (0,1) \), and (30), for \( \nu > 1 \): indeed, in the first case, we get
\[
\frac{1}{2} \frac{\partial^2}{\partial x^2} f_{X_{\nu}}(x,t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \int_0^\infty f_B(x,z) f_{\Gamma_{\nu}}(z,t) dz
\]
\[
= \int_0^\infty \frac{\partial}{\partial z} f_B(x,z) f_{\Gamma_{\nu}}(z,t) dz
\]
\[
= [f_B(x,z) f_{\Gamma_{\nu}}(z,t)]_{z=0}^{z=\infty} - \int_0^\infty f_B(x,z) \frac{\partial}{\partial z} f_{\Gamma_{\nu}}(z,t) dz
\]
\[
= [\text{by (15)}] - \int_0^\infty f_B(x,z) \frac{\partial}{\partial z} f_{\Gamma_{\nu}}(z,t) dz,
\]
which coincides with the right-hand side of (38). We can obtain (39) analogously, for \( \nu > 1 \).

We analyze now the properties of the two versions of fractional VG process, starting with their absolute moments.

**Corollary 9** The absolute \( q \)-moments of \( X_{\nu} \) are given by
\[
E|X_{\nu}(t)|^q = \begin{cases} \frac{2^q}{\sqrt\pi} \Gamma \left( r + \frac{1}{2} \right) \sum_{k=0}^r \binom{r}{k} \frac{k! t^{r-k}}{\Gamma(k+1)}, & \text{for } q = 2r \\ \frac{2^q \Gamma (r+1)}{\sqrt\pi} \int_0^\infty \frac{\Gamma(r+1/2)}{\Gamma(z)} l_{\nu}(z,t) dz, & \text{for } q = 2r+1 \end{cases}
\]
for \( \nu < 1 \), while \( E|X_{\nu}(t)|^q = \infty \), for any \( q \geq 1 \), for \( \nu > 1 \).

**Proof.** For \( \nu < 1 \), by considering (36) and taking into account the well-known form of the absolute moments of the Brownian motion \( B \), we can write
\[
E|X_{\nu}(t)|^q = \int_0^{+\infty} |x|^q \int_0^\infty f_B(x,z) f_{\Gamma_{\nu}}(z,t) dz dx
\]
\[
= \int_0^\infty E|B(z)|^q f_{\Gamma_{\nu}}(z,t) dz
\]
\[
= \frac{\sqrt{2^q}}{\sqrt\pi} \Gamma \left( \frac{q+1}{2} \right) \int_0^\infty z^{q/2} f_{\Gamma_{\nu}}(z,t) dz.
\]
For \( q = 2r \), by considering (23), with \( b = 1 \), we immediately get the first line in (44). For \( q = 2r+1 \) we rewrite (45) as
\[
E|X_{\nu}(t)|^q = \frac{\sqrt{2^q}}{\sqrt\pi} \Gamma \left( \frac{q+1}{2} \right) \int_0^\infty E|\Gamma(z)|^{q/2} l_{\nu}(z,t) dz,
\]
which, by (24), coincides with the second line in (44).
Remark 10 It is easy to check that, for \( \nu = 1 \), formula (44) reduces to
\[
\mathbb{E}|X(t)|^q = \frac{2^{q/2}}{\sqrt{\pi}} \Gamma \left( \frac{q}{2} + \frac{1}{2} \right) \frac{\Gamma \left( \frac{q}{2} + t \right)}{\Gamma(t)}
\]
which is the well-known formula for the absolute moments of the VG process (see [19], formula (2.2) for \( H = 1/2 \)). Indeed, for \( q = 2r \), we get from (44)
\[
\mathbb{E}|X_\nu(t)|^{2r} = \frac{2^r}{\sqrt{\pi}} \Gamma \left( r + \frac{1}{2} \right) \sum_{k=0}^{r} \binom{r}{k} t^k
\]
while, for \( q = 2r + 1 \), it is
\[
\mathbb{E}|X_\nu(t)|^{2r+1} = \frac{2^{r+\frac{1}{2}}}{\sqrt{\pi}} \Gamma \left( r + 1 \right) \frac{\Gamma \left( r + t + \frac{1}{2} \right)}{\Gamma(t)}
\]

since \( l_\nu(z,t) = \delta(z - t) \), for \( \nu = 1 \). It is evident from (44) that the even-order absolute moments are never linear in \( t \): in particular the variance is given by
\[
\text{var}(X_\nu(t)) = \frac{t^\nu}{\Gamma(\nu + 1)}
\]

The most important feature of the fractional VG process is that, for \( \nu > 1 \), it is a Lévy process and thus infinitely divisible for any \( t \), since it is obtained by subordinating the Brownian motion (which is a Lévy process) to the subordinator \( \Gamma_\nu \).

The Lévy symbol \( \eta_{X_\nu}(u) \) of \( \overline{\overline{X}}_\nu \) can be obtained by applying again Proposition 1.3.27 in [1]: for any \( u \in \mathbb{R} \),
\[
\eta_{X_\nu}(u) = -\psi_{\Gamma_\nu}(-\eta_B(u)),
\]
where \( \psi_{\Gamma_\nu}(\theta) = (\log(1 + \theta))^{1/\nu} \) is the Laplace exponent of \( \overline{\overline{\Gamma}}_\nu \) (see (34), for \( b = 1 \)) and \( \eta_B(u) = -u^2/2 \) the Lévy symbol of \( B \). Thus we get
\[
\eta_{\overline{\overline{X}}_\nu}(u) = -\left( \log \left( 1 + \frac{u^2}{2} \right) \right)^{1/\nu}
\]
which, for \( \nu = 1 \), reduces to \( \eta_X \) (see [1], p.57).

From (46) it is evident that \( \overline{\overline{X}}_\nu \) cannot be represented as difference of two independent Gamma subordinators, as happens for \( X \). Let \( \overline{\overline{\Gamma}}' \) and \( \overline{\overline{\Gamma}}'' \) be two independent Gamma random processes with parameters \( a = 1, b = \sqrt{2} \). Then it is well-known that \( X(t) \overset{d}{=} \overline{\overline{\Gamma}}'(t) - \overline{\overline{\Gamma}}''(t), t \geq 0 \). But, in the fractional case, for \( \nu > 1 \), the characteristic functions are respectively
\[
\mathbb{E} e^{iu\overline{\overline{X}}_\nu(t)} = \exp \left\{ -t \left( \log \left( 1 + \frac{u^2}{2} \right) \right)^{1/\nu} \right\},
\]
and
\[
\mathbb{E} e^{iu\overline{\overline{\Gamma}}'(t)-iu\overline{\overline{\Gamma}}''(t)} = \exp \left\{ -t \left[ \log(1 - iu/\sqrt{2}) \right]^{1/\nu} - t \left[ \log(1 + iu/\sqrt{2}) \right]^{1/\nu} \right\},
\]
for \( \overline{\overline{\Gamma}}'_{\nu} \) and \( \overline{\overline{\Gamma}}''_{\nu} \) independent and defined in (27) with \( b = \sqrt{2} \). The previous expressions coincide only in the special case \( \nu = 1 \).
3.2 Fractional Geometric Stable subordinator

The Geometric Stable (GS) subordinator of index $\alpha$ is defined by the following subordinating relationship

$$G_\alpha(t) := A_\alpha(\Gamma(t)), \quad t \geq 0, \; 0 < \alpha \leq 1,$$

where $A_\alpha$ denotes a stable subordinator of index $\alpha$ and $\Gamma$ an independent Gamma subordinator with $b = 1$, for simplicity (see [23] and [18]). Its Laplace exponent is

$$\psi_{G_\alpha}(\theta) = \log (1 + \theta^\alpha)$$

so that its Lévy measure is equal to

$$\lambda(dx) = \alpha x^{\alpha-1} E_{\alpha,1}(x) dx.$$

For $\alpha = 1$, the process $G_\alpha$ reduces to the Gamma subordinator. The fractional equation satisfied by the density of $G_\alpha$ has been already obtained in [3], in the general case of the GS process (i.e. not necessarily totally skewed to the right). It is expressed in terms of the fractional Riesz-Feller derivative. We derive here an analogous equation, in terms of the Caputo fractional derivative $D^\alpha_x$ of order $\alpha \in (0,1]$ and then we extend it to the fractional version of the GS subordinator. We define the latter as

$$G_\nu^\alpha(t) := A_\alpha(\Gamma_\nu(t)), \quad t \geq 0, \; 0 < \nu < 1,$$

and

$$G_\nu^\alpha(t) := A_\alpha(\Gamma_\nu(t)), \quad t \geq 0, \; \nu > 1,$$

where $\alpha \in (0,1]$, $\Gamma_\nu$ and $\Gamma_\nu'$ are independent of $B$. For $\nu = 1$, the processes in (48) reduce to the GS subordinator. Let us denote the one-dimensional distributions of $G_\nu^\alpha$ and $G_\nu^\alpha$ as

$$f_{G_\nu^\alpha}(x,t) = \int_0^\infty h_\alpha(x,z)f_{\Gamma_\nu}(z,t)dz, \quad x,t \geq 0, \; \nu \in (0,1),$$

and

$$f_{G_\nu^\alpha}(x,t) = \int_0^\infty h_\alpha(x,z)f_{\Gamma_\nu'}(z,t)dz, \quad x,t \geq 0, \; \nu > 1.$$

**Theorem 11** The density (49) of $G_\nu^\alpha$ satisfies the following doubly fractional differential equation

$$D^\alpha_x f_{G_\nu^\alpha} = (1 - \partial^\nu_{-1,t}) f_{G_\nu^\alpha}, \quad x \geq 0, \; t > 1, \; \alpha \in (0,1], \; \nu \in (0,1)$$

while the density (50) of $G_\nu^\alpha$ satisfies the equation

$$D^\alpha_x f_{G_\nu^\alpha} = (1 - \partial^\nu_{-1,t}) f_{G_\nu^\alpha}, \quad x \geq 0, \; t > 1, \; \alpha \in (0,1], \; \nu > 1,$$

The initial condition for both equations is $f(0,t) = 0$ (where $f := f_{G_\nu^\alpha}$, for $\nu \in (0,1)$ and $f := f_{G_\nu^\alpha}$, for $\nu > 1$).

**Proof.** We can apply the result given in [5] to obtain the differential equation satisfied by the density of $G_\alpha$. By (47), the density of the GS subordinator can be written as

$$f_{G_\alpha}(x,t) = \int_0^\infty h_\alpha(x,z)f_{\Gamma}(z,t)dz,$$

where $h_\alpha(x,t)$ is the density of $A_\alpha(t), \; t \geq 0$. Then $f_{G_\alpha}$ satisfies the following space-fractional differential equation

$$D^\alpha_x f_{G_\alpha} = -(1 - e^{-\partial t}) f_{G_{\alpha/2}}, \quad x,t \geq 0,$$

For $\alpha = 1$, the process $G_\alpha$ reduces to the Gamma subordinator. The fractional equation satisfied by the density of $G_\alpha$ has been already obtained in [3], in the general case of the GS process (i.e. not necessarily totally skewed to the right). It is expressed in terms of the fractional Riesz-Feller derivative. We derive here an analogous equation, in terms of the Caputo fractional derivative $D^\alpha_x$ of order $\alpha \in (0,1]$ and then we extend it to the fractional version of the GS subordinator. We define the latter as

$$G_\nu^\alpha(t) := A_\alpha(\Gamma_\nu(t)), \quad t \geq 0, \; 0 < \nu < 1,$$

and

$$G_\nu^\alpha(t) := A_\alpha(\Gamma_\nu(t)), \quad t \geq 0, \; \nu > 1,$$

where $\alpha \in (0,1]$, $\Gamma_\nu$ and $\Gamma_\nu'$ are independent of $B$. For $\nu = 1$, the processes in (48) reduce to the GS subordinator. Let us denote the one-dimensional distributions of $G_\nu^\alpha$ and $G_\nu^\alpha$ as

$$f_{G_\nu^\alpha}(x,t) = \int_0^\infty h_\alpha(x,z)f_{\Gamma_\nu}(z,t)dz, \quad x,t \geq 0, \; \nu \in (0,1),$$

and

$$f_{G_\nu^\alpha}(x,t) = \int_0^\infty h_\alpha(x,z)f_{\Gamma_\nu'}(z,t)dz, \quad x,t \geq 0, \; \nu > 1.$$
with initial condition
\[ f_{G_\alpha}(0, t) = 0. \] (55)

In order to get (54) we apply a well-known result on stable subordinators: the density \( h_\alpha \) satisfies, for \( \alpha \in (0, 1] \), the following equation
\[ D_x^\alpha h_\alpha = -\frac{\partial}{\partial t} h_\alpha, \quad h_\alpha(x, 0) = \delta(x), \quad x, t \geq 0. \] (56)

Thus
\[
D_x^\alpha f_{G_\alpha}(x, t) = \int_0^\infty D_x^\alpha h_\alpha(x, z) f_{\Gamma}(z, t) dz = -\int_0^\infty \frac{\partial}{\partial z} h_\alpha(x, z) f_{\Gamma}(z, t) dz = \left[ h_\alpha(x, z) f_{\Gamma}(z, t) \right]_{z=0}^{z=\infty} + \int_0^\infty h_\alpha(x, z) \frac{\partial}{\partial z} f_{\Gamma}(z, t) dz,
\]
which, by considering (5), with \( b = 1 \), gives (54). The initial condition is trivially satisfied since \( h_\alpha(0, z) = 0 \). Equations (51) and (52) can be easily obtained by considering theorems 3 and 6.

Also in this case, only for \( \nu > 1 \), the fractional GS process is still a subordinator and its Lévy symbol can be obtained as follows:
\[
\eta_{G_\nu}(u) = -\psi_{\Gamma_v}(-\eta_{A_\alpha}(u)) = -\left\{ \log[1 + (-iu)^\alpha] \right\}^{1/\nu},
\]
since \( \eta_{A_\alpha}(u) = -(-iu)^\alpha \).

### 3.3 Fractional Negative Binomial process

The NB process \( M(t), t > 0 \), is a jump Lévy process with the following distribution
\[ q_k(t) := \Pr \{ M(t) = k \} = \binom{t + k - 1}{k} p^k (1 - p)^{t-k}, \quad t > 0, \ k \in \mathbb{N}, \ p \in (0, 1), \] (57)
where \( \binom{x}{k} \) is the generalized binomial coefficient defined, for any \( x \in \mathbb{R} \), as
\[
\binom{x}{k} := \frac{(x)_k}{k!} = \frac{x(x-1)...(x-k+1)}{k!}
\]
and \( (x)_k \) is the falling factorial.

The NB process has two alternative representations (see [20]). The first one is in terms of compound Poisson process: let \( X_j \) be i.i.d. random variables with discrete logarithmic distribution
\[ \Pr \{ X_j = k \} = -\frac{(1 - p)^k}{k \ln p}, \quad k = 1, 2, \ldots \] (58)
for any \( j = 1, 2, \ldots \), then the process defined as
\[ M(t) := \sum_{j=1}^{N(t)} X_j, \quad t \geq 0, \] (59)
where \( N_\lambda \) denotes an homogeneous Poisson process of parameter \( \lambda = \ln(1/p) \) (independent from 
\( X_j \), for any \( j \)), is proved to be equivalent to the process defined by (57). Thus \( M \) can be considered 
as a particular case of the so called continuous-time random walks.

Another possible representation of the NB process is in terms of Cox process with Gamma 
distributed directing measure, i.e. \( N_{\Lambda(0,t]}, t \geq 0 \), where \( \Lambda(0,t] \) is a Gamma r.v. with shape 
parameter \( t \) and scale parameter \((1 - p)/p\). Thus we immediately get the following equality in 
distribution

\[
M(t) \overset{d}{=} N_1(\Gamma(t)), \quad t \geq 0,
\]

where \( \Gamma \) has density (7) with \( a = 1, b = p/(1 - p) \).

**Lemma 12** The distribution of the NB process of parameter \( p \), given in (57), satisfies the following differential equation:

\[
\begin{aligned}
\frac{d}{dt} q_0(t) &= \ln p q_0(t) \\
\frac{d}{dt} q_k(t) &= \left[ \ln p + \sum_{j=0}^{k-1} \frac{1}{1+j} \right] q_k(t), \quad k \geq 1,
\end{aligned}
\]

for \( t \geq 0 \), with initial conditions

\[
q_k(0) = \begin{cases} 
1, & k = 0 \\
0, & k > 0 
\end{cases}.
\]

**Proof** We take the derivative of (57), by considering that

\[
\binom{t + k - 1}{k} = \frac{(t)^{(k)}}{k!},
\]

where \( (x)^{(k)} \) is the rising factorial defined as \( (x)^{(k)} := \Gamma(x + k)/\Gamma(x) \):

\[
\frac{d}{dt} q_k(t) = \frac{(1 - p)^k}{k!} \frac{d}{dt} \left[ p^t(t)^{(k)} \right] \\
= \frac{(1 - p)^k}{k!} \left[ p^t \ln p (t)^{(k)} + p^t \frac{d}{dt} (t)^{(k)} \right] \\
= \ln p q_k(t) + \frac{(1 - p)^k}{k!} p^t(t)^{(k)} \left[ \Psi^{(0)}(t + k) - \Psi^{(0)}(t) \right],
\]

where, in the last step, we have used the well-known result on the derivative of the rising factorial and \( \Psi^{(0)} \) denotes the Digamma function defined as \( \Psi^{(0)}(x) := d\ln \Gamma(x)/dx \) (see, for example, [7]).

By the properties of the Digamma function it is easy to see that \( \Psi^{(0)}(t + k) - \Psi^{(0)}(t) = \sum_{j=0}^{k-1} \frac{1}{t+j} \),

so that we get (61). The initial conditions are trivially satisfied, by considering that \((0)^{(k)} = 0\) for 
any \( k > 0 \) and \((0)^{(k)} = 1\) for \( k = 0 \).

We now define a fractional version of the NB process by substituting in (60) the standard 
Gamma subordinator with its fractional counterpart defined in (1). Thus we set

\[
\begin{aligned}
\{ M_\nu(t) := N_1(\Gamma_\nu(t)), \quad \nu \in (0, 1) \\
\overline{M}_\nu(t) := N_1(\overline{\Gamma}_\nu(t)), \quad \nu \in (1, \infty)
\end{aligned}
\]

for any \( t \geq 0 \). For \( \nu = 1 \), \( M_\nu(t) := M(t) \).

From (11) and (27), it is clear that

\[
\begin{aligned}
\{ M_\nu(t) &\overset{d}{=} M(\mathcal{L}_\nu(t)), \quad \nu \in (0, 1) \\
\overline{M}_\nu(t) &\overset{d}{=} M(\mathcal{A}_{1/\nu}(t)), \quad \nu \in (1, \infty)
\end{aligned}
\]

(64)
Unfortunately, due to the presence of non-constant coefficients in [61], we are not able to derive its fractional analogue. By resorting to the compound Poisson representation, it is proved in [5] that, for \( \nu < 1 \), the distribution \( q^\nu_k(t) := \Pr\{M_\nu(t) = k\} \) satisfies the following birth-type (or Kolmogorov forward) fractional equations

\[
\begin{align*}
\{ D^\nu_t q^\nu_0 &= \ln p q^\nu_0 \\
D^\nu_t q^\nu_k &= \ln p q^\nu_k - \ln p \sum_{i=1}^{k} \frac{(1-p)^i}{i!} q^\nu_{k-i}, \quad k > 0
\end{align*}
\] (65)

with initial conditions

\[ q^\nu_0(0) = 1, \quad q^\nu_k(0) = 0, \text{ for all integer } k > 0, \]

Analogously, for \( \nu > 1 \), the distribution \( \overline{q}^\nu_k(t) := \Pr\{\overline{M}_\nu(t) = k\} \) satisfies

\[
\begin{align*}
\{ D^-\nu_t \overline{q}^\nu_0 &= -\ln p \overline{q}^\nu_0 \\
D^-\nu_t \overline{q}^\nu_k &= -\ln p \overline{q}^\nu_k + \ln p \sum_{i=1}^{k} \frac{(1-p)^i}{i!} \overline{q}^\nu_{k-i}, \quad k > 0
\end{align*}
\] (66)

with initial conditions

\[ \overline{q}^\nu_0(0) = 1, \quad \overline{q}^\nu_k(0) = 0, \text{ for all integer } k > 0. \]

By applying the results of section 2, we can obtain alternative equations satisfied by the distribution of the NB process and its fractional versions defined in (63), in terms of the shift and fractional shift operators.

**Theorem 13** The distribution of the NB process \( M \) solves the following fractional differential equations, for \( t > 1 \),

\[
e^{-D_t q_k} = \frac{1}{p} q_k - \frac{1 - p}{p} q_{k-1}, \quad k \geq 0,
\] (67)

with initial condition

\[ q_0(0) = 1, \quad q_k(0) = 0, \text{ for any integer } k > 0
\]

and \( q_{-1}(t) = 0 \), for any \( t \).

**Proof.** From (60) we can write that

\[ q_k(t) = \int_0^{+\infty} p_k(z) f_\Gamma(z, t) dz, \] (68)

where \( p_k(t) := \Pr\{N_1(t) = k\} \) is the distribution of a Poisson process with intensity 1. By applying to (68) the shift operator (2) with \( c = -1 \), we get, for any \( k \geq 0 \),

\[
e^{-D_t q_k(t)} = \int_0^{+\infty} p_k(z) e^{-\partial_t f_\Gamma(z, t)} dz = \int_0^{+\infty} \frac{1 - p}{p} \left[ \frac{\partial}{\partial z} f_\Gamma(z, t) \right] dz + \int_0^{+\infty} p_k(z) f_\Gamma(z, t) dz
\] (69)

\[ = \left[ \text{integrating by parts} \right]
\]

\[ = \frac{1 - p}{p} \int_0^{+\infty} \left[ p_k(z) f_\Gamma(z, t) \right]_{z=0}^{z=+\infty} - \frac{1 - p}{p} \int_0^{+\infty} \frac{\partial}{\partial z} p_k(z) f_\Gamma(z, t) dz + q_k(t)
\]

\[ = \left[ \text{by (7)-(8)} \right]
\]

\[ = -\frac{1 - p}{p} \int_0^{+\infty} D_z p_k(z) f_\Gamma(z, t) dz + q_k(t),
\]
which coincides with (67), by considering the well-known equation satisfied by the Poisson distribution.

By applying theorems 3 and 6, we obtain the fractional equations satisfied by the distribution of the fractional NB processes defined in (64).

**Theorem 14** The distribution $q^\nu_k(t)$ satisfies the following fractional differential equations:

$$
O_{-1,t}^{\nu} q^\nu_k = \frac{1}{p} q^\nu_k - \frac{1 - p}{p} q^{\nu}_{k-1}, \quad k \geq 0, \ t > 1, \ \nu \in (0,1)
$$

(70)

(where $O_{-1,t}^{\nu}$ is defined in (69)). The distribution $\overline{q}^\nu_k(t)$ satisfies

$$
O_{-1,t}^{\nu} \overline{q}^\nu_k = \frac{1}{p} \overline{q}^\nu_k - \frac{1 - p}{p} \overline{q}^{\nu}_{k-1}, \quad k \geq 0, \ t > 1, \ \nu > 1
$$

(71)

(where $O_{-1,t}^{\nu}$ is defined in (70)). The initial conditions for both equations are

$q_0(0) = 1, \ q_k(0) = 0$, for any integer $k > 0$

and $q_{-1}(t) = 0$, for any $t$, where we set $q := q^\nu$, for $\nu < 1$ and $q := \overline{q}^\nu$, for $\nu > 1$.

**Proof.** By (63) we get, for $\nu < 1$,

$$
q^\nu_k(t) = \int_{0}^{+\infty} p_k(z) f_{\nu}(z, t) dz,
$$

thus we can write that

$$
O_{-1,t}^{\nu} q^\nu_k = \int_{0}^{+\infty} p_k(z) O_{-1,t}^{\nu} f_{\nu}(z, t) dz
$$

$$
= \left[ \text{by (14), with } b = p/(1 - p) \right]
$$

$$
= \frac{1 - p}{p} \int_{0}^{+\infty} p_k(z) \frac{\partial}{\partial z} f_{\nu}(z, t) dz + \int_{0}^{+\infty} p_k(z) f_{\nu}(z, t) dz.
$$

We obtain equation (70), by some steps similar to (69). The case $\nu > 1$ can be treated analogously.

As happens with the fractional VG process, by definition (63), it is clear that, in the case $\nu > 1$, also $\overline{M}_\nu$ is a Lévy process, since the same is true for $M$ and $A_{1/\nu}$ is a subordinator. The Lévy symbol can be obtain by evaluating the characteristic function of the process:

$$
E e^{iu\overline{M}_\nu(t)} = \sum_{k=0}^{\infty} e^{iku} q^\nu_k(t)
$$

$$
= \sum_{k=0}^{\infty} e^{iku} \int_{0}^{+\infty} q_k(z) h_{1/\nu}(z, t) dz
$$

$$
= \int_{0}^{+\infty} E e^{iuM(z)} h_{1/\nu}(z, t) dz
$$

$$
= \int_{0}^{+\infty} \left( \frac{p}{1 - (1 - p)e^{iu}} \right)^{z} h_{1/\nu}(z, t) dz
$$

$$
= \int_{0}^{+\infty} \exp \left\{ -z \log \frac{1 - (1 - p)e^{iu}}{p} \right\} h_{1/\nu}(z, t) dz
$$

$$
= \exp \left\{ -t \left( \log \frac{1 - (1 - p)e^{iu}}{p} \right)^{1/\nu} \right\}.
$$
Thus we get

$$\eta \nu(u) = -\log \left(\frac{1 - (1 - p)e^{iu}}{p}\right)^{1/\nu},$$

which coincides with

$$-\psi_{1/\nu}(-\eta_M(u)) = -(-\eta_M(u))^{1/\nu},$$

where $\eta_M(u) = \log \frac{p}{1 - (1 - p)e^{iu}}$ (see formula (1.1) in [20]).

For the reader’s convenience we sum up the results on the Lévy symbols obtained so far in the following table: recall that in all the cases below it is $\nu > 1$. For $\nu = 1$ we obtain the well-known Lévy symbols of the corresponding non-fractional processes.

| Process          | Lévy symbol                                           |
|------------------|-------------------------------------------------------|
| Fractional Gamma $\Gamma_\nu$ | $\eta_{\Gamma_\nu}(u) = -\log(1 - iu/b)^{1/\nu}$   |
| Fractional VG $X_\nu$             | $\eta_{X_\nu}(u) = -\log \left(1 + \frac{u^2}{2}\right)^{1/\nu}$ |
| Fractional GS $G_\alpha$          | $\eta_{G_\alpha}(u) = -\log (1 + (-iu)^\alpha)^{1/\nu}$ |
| Fractional NB $M_\nu$             | $\eta_{M_\nu}(u) = -\log \frac{1 - (1 - p)e^{iu}}{p}^{1/\nu}$ |

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