Coherent State for a Relativistic Spinless Particle

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Abstract

The Klein-Gordon equation with scalar potential is considered. In the Feshbach-Villars representation the annihilation operator for a linear potential is defined and its eigenstates are obtained. Although the energy levels in this case are not equally-spaced, depending on the eigenvalues of the annihilation operator, the states are nearly coherent and squeezed. The relativistic Poschl-Teller potential is introduced. It is shown that its energy levels are equally-spaced. The coherence of time evolution of the eigenstates of the annihilation operator for this potential is evaluated.

1 Introduction

Nowadays, the coherent state has found widespread application in many branches of physics such as non linear optics, laser, nuclear and particle physics. In the non-relativistic limit, the coherent states are well known and one can produce them using three different ways \cite{1}:

I. They can be constructed as eigenstates of the annihilation operator.

II. They can be defined as quantum states with minimum uncertainty relationship.

III. They can be obtained by operating the Glauber’s displacement operator on the vacuum state.

The above definitions are known to be equivalent for the Schrödinger equation with the harmonic-oscillator potential. In references \cite{2-12} coherent states for systems other than the harmonic-oscillator are also investigated.

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To extend the coherence to relativistic region one may use manifestly covariant equations \[13\]. Of course in this way, it is difficult to describe the wave functions in terms of four vectors. Thus, in many areas of physics, one may use the Schrödinger equation with relativistic corrections or the Klein-Gordon (KG) or Dirac equation with specific potentials to describe a relativistic particle or even bound state in a non-covariance manner. To describe the relativistic coherent states using the KG or Dirac equation one encounters particle pair creation that makes one particle interpretation nonsense. To avoid this problem one should consider the theory under conditions that the pair creation is impossible. Furthermore in relativistic quantum mechanics the consistency of the one particle description is limited and in this case the only valid operators are even operators which do not mix different charge states. In this direction authors of reference \[14\] have studied a spin 0 charged particle in two cases: free particle and a particle in a constant homogeneous magnetic field.

In this paper we examine the possibility of coherence for KG equation with a pure scalar potential and adopt the definition $I$ to extend the coherence to the relativistic region. Such a potential describes a position dependent valence and conduction-band edge of semiconductors near the $\Gamma$ or $L$ point in the Brillouin zone. In fact the position dependent mass characterizes a position dependent band gap. Examples for such materials are given in ref. \[15\].

In section 2 we obtain $x$ and $p$ operators in the Feshbach-Villars representation for a general scalar potential. Subsequently we derive eigensolutions of a linear scalar potential in the Schrödinger representation to show that pair creation does not occur. Consequently we construct the eigenstates of the annihilation operator for this potential. The time evolution of the states is numerically discussed in section 3. In section 4 we introduce a potential with equally spaced energy levels and examine their coherence with time. The resolution of unity for the obtained states is also discussed in this section.

2 KG equation for a Scalar potential

In order to explore the coherence of a relativistic particle in the simplest case we consider the KG equation in one dimension with a vector and a scalar potential ($\hbar = c = 1$)\[15,16\]

$$\frac{\partial^2}{\partial x^2} + \left(i \frac{\partial}{\partial t} - V(x)\right)^2 - \left(m + S(x)\right)^2 \psi(x,t) = 0, \quad (1)$$

where $m$ is the mass of the particle, $S(x)$ is the scalar potential and $V(x)$ is the time component of a 4-vector potential. In non-relativistic limit, Eq.(1) leads to a Schrödinger equation with an effective potential as $S + V$. In fact this equation gives all relativistic corrections to the Schrödinger equation with such an effective potential. For a pure scalar potential we have

$$H_s \psi(x,t) = -\frac{1}{2m} \frac{\partial^2}{\partial t^2} \psi(x,t), \quad (2)$$

in which

$$H_s = -\frac{1}{2m} \frac{d^2}{dx^2} + \frac{(m + S(x))^2}{2m}. \quad (3)$$
Now by introducing two functions $\phi$ and $\chi$ and the ansatz
\[ \psi = \phi + \chi ; \quad i \frac{\partial \psi}{\partial t} = m(\phi - \chi), \] (4)
we transform Eq.(2) to the Schrödinger form as follows
\[ i \frac{\partial \Psi}{\partial t} = \left[ (\sigma_3 + i\sigma_2)(H_s - \frac{m}{2}) + \sigma_3 m \right] \Psi, \] (5)
where $\sigma_i^s$, $i = 1, 2, 3$, are the Pauli matrices and $\Psi$ is a two components column vector as
\[ \Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \] (6)
It is easy to show that each component of $\Psi$ individually satisfies the KG equation i.e. Eq.(2). If we define $u_n$ as eigenstate of the Hamiltonian $H_s$ with corresponding eigenvalue $\epsilon_n$ with the ansatz
\[ \Psi = \begin{pmatrix} \phi_0 \\ \chi_0 \end{pmatrix} u_n e^{-i\epsilon_n t}, \] (7)
one obtains the positive and negative energy solutions as follows
\[ \Psi^\pm = N_{\pm} \begin{pmatrix} m \pm E_n \\ m \mp E_n \end{pmatrix} u_n e^{\mp iE_n t}, \] (8)
where $N_{\pm}$ are the appropriate normalization constants and
\[ E_n = \sqrt{2m\epsilon_n}. \] (9)
Now an operator
\[ U = \frac{(m + H) - \sigma_1(m - H)}{\sqrt{4mH}}, \] (10)
in which
\[ H = \sqrt{2mH_s}, \] (11)
can transform the Schrödinger representation to the Feshbach-Villars representation therefore it can be shown that the Hamiltonian in this representation is
\[ H_{\text{FV}} = \sigma_3 H = \sigma_3 \sqrt{2mH_s}, \] (12)
which is even and the even parts of operators $x$ and $p$ are also the same as the corresponding operators in the non-relativistic region:
\[ x_{\text{FV}} = iU \frac{\partial}{\partial p} U^{-1} = i \frac{\partial}{\partial p} - i \frac{p\sigma_1}{2E_n^2}, \] (13)
for $x$-operator in momentum space and
\[ p_{\text{FV}} = -iU \frac{\partial}{\partial x} U^{-1} = -i \frac{\partial}{\partial x} + i \frac{(\frac{\partial S}{\partial x})(m + S)\sigma_1}{2E_n^2}, \] (14)
for p-operator in configuration space. Now we are ready to study the coherent states for potentials without pair creation. First, we choose a linear potential as follows

\[ S(x) = k|x| - m, \]  

where \( k \) is a coupling constant. Introducing parameters \( \epsilon_n = E^2/(2m) \) and \( \omega = k/m \), by using the ansatz of Eq.(7), Eq.(2) can be written in a familiar form - the Schrödinger equation with a harmonic-oscillator potential. Therefore one can easily find

\[ \Psi_n^\pm \propto \left( \begin{array}{c} m \pm E_n \\ m \mp E_n \end{array} \right) e^{-\frac{m \omega x^2}{2}} H_n(\sqrt{mwx})e^{\pm iE_n t}, \]  

in which \( H_n(x) \) is the Hermite polynomial of order \( n \) and

\[ E = \pm \sqrt{(2n + 1)k} = \pm E_n, \]  

where \( n = 0, 1, 2, \ldots \). One can easily see from Eq.(17) that pair creation in this case never occurs. For a pure scalar potential the KG equation is independent of the sign of \( E \). In fact the scalar interaction is independent of the charge of the particle and has the same effect on particles and anti-particles. Thus, naively speaking, this kind of potential can describe the confining part of the potential of a quarkonium.

Since the even part of an operator is a measurable quantity, therefore Eq.(13) and Eq.(14) allow us to consider the usual annihilation operator for a non-relativistic harmonic oscillator potential to construct the coherent state for our linear potential. Therefore the coherent state \( |\alpha\rangle \) at \( t=0 \) in the Feshbach-Villars representation in terms of the energy eigenstate for a non-relativistic harmonic oscillator \( |n\rangle \), can be obtained as

\[ |\alpha, +\rangle = e^{-|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) |n\rangle, \]  

for positive energy state and

\[ |\alpha, -\rangle = e^{-|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) |n\rangle, \]  

for negative energy state. Thus the state given in Eq.(18), when we restrict ourselves to the positive energy states, evolves as

\[ |\alpha(t), +\rangle = e^{-|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) e^{-iE_n t}|n\rangle, \]  

where \( E_n = \sqrt{(2n + 1)k} \). Although \( |\alpha\rangle \) is coherent, its time evolution is not the eigenstate of the annihilation operator. Indeed, this is due to the non-equally-spaced of the energy levels given in Eq.(17).

One of the significant properties of the coherent states is an over completeness relation or resolution of unity as follows

\[ \int |\alpha\rangle d\mu(\alpha)\langle\alpha| = 1, \]  

(21)
where the measure $d\mu(\alpha)$ for the obtained states in this section (i.e. Eq.(18)) can be easily determined as

$$d\mu(\alpha) = \frac{d[Re\alpha]d[Im\alpha]}{\pi}. \quad (22)$$

3 The Coherence of the Solutions for Linear Potential

To explore the behavior of the solutions given in Eq.(20) we first, examine the expectation values of the appropriate quantities as follows

$$\langle x \rangle = \sqrt{\frac{2}{k}} e^{-|\alpha|^2} \sum_n \left[ a \cos(E_n - E_{n+1})t - b \sin(E_n - E_{n+1})t \right] \frac{|\alpha|^{2n}}{n!},$$

$$\langle p \rangle = \sqrt{2k} e^{-|\alpha|^2} \sum_n \left[ a \sin(E_n - E_{n+1})t + b \cos(E_n - E_{n+1})t \right] \frac{|\alpha|^{2n}}{n!},$$

$$\langle x^2 \rangle = \frac{1}{k}(|\alpha|^2 + \frac{1}{2}) + e^{-|\alpha|^2} \sum_n [(a^2 - b^2) \cos(E_n - E_{n+2})t - 2ab \sin(E_n - E_{n+2})t] \frac{|\alpha|^{2n}}{n!},$$

$$\langle p^2 \rangle = k(|\alpha|^2 + \frac{1}{2}) - ke^{-|\alpha|^2} \sum_n [(a^2 - b^2) \cos(E_n - E_{n+2})t - 2ab \sin(E_n - E_{n+2})t] \frac{|\alpha|^{2n}}{n!}, \quad (23)$$

where

$$\alpha = a + ib. \quad (24)$$

Time variations of $\Delta x$, $\Delta p$ and $\Delta x \cdot \Delta p$, for various values of $\alpha$, are illustrated in Figs.(1)-(11).

For $\alpha = \alpha_1 = 0.1 + 0.2i$ the value of $\Delta x \cdot \Delta p$ oscillates between 0.50 and 0.51 and the state is nearly coherent, Fig.(1). In contrast $\Delta x \cdot \Delta p$ for $\alpha = \alpha_2 = 1 + 2i$ increases with time though its value for long time is bounded and oscillates about a certain value, Figs.(2-3).

The numerical values of $\Delta x$ and $\Delta p$ in both cases are not constant as shown in Figs.(4-7). When $\alpha = \alpha_1$ they oscillate almost periodically about their minimum values. Furthermore, their oscillations are exactly out of phase and the states are squeezed, Figs.(4-5). When $\alpha$ increases to $\alpha_2$, the values

Figure 1: $\Delta x \cdot \Delta p$ for $\alpha = 0.1 + 0.2i$ as a function of time.
Figure 2: $\Delta x \cdot \Delta p$ for $\alpha = 1 + 2i$ as a function of time for $0 < t < 50$.

Figure 3: $\Delta x \cdot \Delta p$ for $\alpha = 1 + 2i$ as a function of time for $50 < t < 100$.

Figure 4: $\Delta x$ for $\alpha = 0.1 + 0.2i$ as a function of time.

Figure 5: $\Delta p$ for $\alpha = 0.1 + 0.2i$ as a function of time.
Figure 6: $\Delta x$ for $\alpha = 1 + 2i$ as a function of time.

Figure 7: $\Delta p$ for $\alpha = 1 + 2i$ as a function of time.

of $\Delta x$ and $\Delta p$ increase with time and finally oscillate about a finite value, Figs.(6-7).

The expectation values of $x$ and $p$ for $\alpha = \alpha_1$ and $\alpha = \alpha_2$ are plotted in Figs.(8-9) and Figs.(10-11), respectively. In the case $\alpha = \alpha_1$ the values $\langle x \rangle$ and $\langle p \rangle$ oscillate with an amplitude and a period nearly constant which is very similar to a classical harmonic oscillator. Again for $\alpha = \alpha_2$ the situation is exactly different and the values of $\langle x \rangle$ and $\langle p \rangle$ oscillate with a variable period and amplitude.

In the evaluation of the various quantities we have summed terms up to $n = 50$ and assumed $k = 1$, though the general behaviors do not depend on $k$.

Figure 8: The expectation value of $x$ for $\alpha = 0.1 + 0.2i$ as a function of time.
Figure 9: The expectation value of $p$ for $\alpha = 0.1 + 0.2i$ as a function of time.

Figure 10: The expectation value of $x$ for $\alpha = 1 + 2i$ as a function of time.

Figure 11: The expectation value of $p$ for $\alpha = 1 + 2i$ as a function of time.
4 Potentials with Equally-Spaced Levels

In the preceding section we saw that a relativistic scalar particle with a linear scalar potential, in general, is not coherent. The main reason for deviation from coherence is the fact that the eigenenergies are not equally spaced. Now, we would like to construct potentials with equally-spaced energy levels. For this purpose we rewrite $H_s$ as follows

$$H_s = -\frac{1}{2m} \frac{d^2}{dx^2} + \frac{U^2 + m^2}{2m},$$

(25)

where $U^2(x) = S^2(x) + 2mS(x)$. We now introduce a potential as follows

$$U(x) = m \tan(\omega x),$$

(26)

in the region $-\frac{\pi}{2}\omega \leq x \leq \frac{\pi}{2}\omega$ and is infinite elsewhere, or in the same region

$$S(x) = -m \pm \frac{m}{\cos(\omega x)},$$

(27)

where $\omega$ is a constant. Substituting this potential in Eq.(25) and using ansatz (7) equation (2) leads to the non-relativistic Poschl-Teller (PT) equation [18] and consequently $u_n$ in Eq.(8) can be obtained as

$$u_n(x) = \left[ \frac{\omega(\lambda + n)\Gamma(2\lambda + n)}{\Gamma(n + 1)} \right]^{\frac{1}{2}} (\cos \omega x)^{\frac{1}{2}} P_{n+\lambda-\frac{1}{2}}^{\frac{1}{2}-\lambda} (\sin \omega x)$$

(28)

and for the energy eigenvalues one has

$$E = \pm E_n = \pm \omega(n + \lambda) ; \quad n = 0, 1, 2, ...$$

(29)

where

$$\lambda = \frac{1}{2} + \frac{1}{2} \sqrt{4m^2 \omega^2 + 1}. \quad (30)$$

It should be noted that although in the relativistic PT potential, Eq.(27), the energy levels are equally-spaced but the PT potential in the Schrödinger equation has non-equally-spaced levels [3,4]. Here again pair creation does not occur and one-particle sector of the theory is applicable. Now we need the annihilation operators to construct the coherent states. To this end, one can use the Schrödinger method to determine the ladder operators as follows [3]:

$$A_{\pm} = (\sin \omega x) \left[ \frac{2m(H + \frac{m}{2})}{\omega} \right]^{\frac{1}{2}} \mp \frac{1}{\omega} (\cos \omega x) \frac{d}{dx}$$

(31)

where $H$ is the Hamiltonian of the Schrödinger equation for the PT potential. Therefore in the Feshbach-Villars representation we have

$$A_{\pm}\psi_n^{FV}(t = 0) = (n + \lambda)D(n - \frac{1}{2} \pm \frac{1}{2}, \lambda)\psi_n^{FV}(t = 0)$$

(32)

where $\psi^{FV}$ is the wave function in the Feshbach-Villars representation

$$\psi_n^{FV}(t) = N_{\pm} \begin{pmatrix} 1 \pm 1 \\ 1 \mp 1 \end{pmatrix} u_n e^{\mp i E_n t},$$

(33)
in which ± stands for positive and negative energy solutions, respectively, and

\[
D(n, \lambda) = \left[ \frac{(n+1)(2\lambda+n)}{(n+\lambda)(n+1+\lambda)} \right]^{\frac{1}{2}}.
\]

Let us now determine \( \psi_\alpha \) as the eigenstate of the annihilation operator \( A_- \), with eigenvalue \( \alpha \), by using an expansion on the states \( \psi^{FV}_n \):

\[
\psi_\alpha = \sum_n c_n \psi^{FV}_n(t = 0),
\]

we then have

\[
A_- \psi_\alpha = \sum_n c_n(n + \lambda)D(n - 1, \lambda)\psi^{FV}_{n-1}(t = 0) = \alpha \psi_\alpha,
\]

Therefore

\[
c_{n+1} = \alpha \left[ \frac{(n+\lambda)}{(n+1)(2\lambda+n)(n+1+\lambda)} \right]^{\frac{1}{2}} c_n,
\]

and, finally \( \psi_\alpha \) can be obtained as

\[
\psi_\alpha = N_\alpha \sum_n \alpha^n \left[ \frac{\lambda \Gamma(2\lambda)}{n!(n+\lambda)\Gamma(2\lambda+n)} \right]^{\frac{1}{2}} \psi^{FV}_n(t = 0),
\]

where

\[
N_\alpha = [\lambda \Gamma(2\lambda)S(\alpha)]^{-\frac{1}{2}},
\]

and

\[
S(\alpha) = \sum_n \frac{|\alpha|^{2n}}{n!(n+\lambda)\Gamma(2\lambda+n)}.
\]

Now we examine the evolution of \( \psi_\alpha \) at every instant. For this purpose we restrict ourselves to the positive energy states which implies

\[
\psi_\alpha(t) = N_\alpha \sum_n \alpha^n \left[ \frac{\lambda \Gamma(2\lambda)}{n!(n+\lambda)\Gamma(2\lambda+n)} \right]^{\frac{1}{2}} e^{-iE_n t} \psi^{FV}_n(t = 0),
\]

where \( E_n \) is given in Eq.(29). In this case since the eigenenergies are equally spaced one has

\[
\psi_\alpha(t) = e^{-i\omega t} \psi_\alpha(t),
\]

where

\[
\alpha(t) = e^{-i\omega t} \alpha.
\]

Therefore, the coherent state wave packet remains an eigenvector of \( A_- \) with an eigenvalue \( \alpha e^{-i\omega t} \).

Now we investigate the resolution of unity for the obtained states. As (38) shows the measure in this case is different from the standard form of (22) but one can follow the method of references [14] and [19] to find

\[
d\mu(\alpha) = S(|\alpha|^2)W(|\alpha|^2) \frac{d[Re\alpha]d[Im\alpha]}{\pi},
\]

where \( W(|\alpha|^2) \) can be determined by solving the following equation

\[
\int_0^\infty |\alpha|^{2n} W(|\alpha|^2) d|\alpha|^2 = n!(n+\lambda)\Gamma(2\lambda+n).
\]
summary

To summarize, in the Feshbach-Villars representation we obtained $x$ and $p$ operators for a general scalar potential. The even parts of these operators coincide with their counterparts in the Schrödinger representation, see Eqs.(13,14). Consequently, we constructed the annihilation operator coherent state for a purely linear scalar potential for a relativistic spinless particle. The properties of these states in $(1 + 1)$ dimension are:

i. The eigenfunctions for the small eigenvalues Eq.(24) are quasi-coherent and squeezed. Furthermore $\langle x \rangle$ and $\langle p \rangle$ oscillate like a classical harmonic oscillator. In the other words for the small values of $\alpha$, which is not the non-relativistic limit of Eq.(2), the states regain their coherence though the relativistic corrections destroy the coherence of harmonic potential in the non-relativistic Schrödinger equation.

ii. For both small and large eigenvalues the minimum uncertainties do not increase unbounded with time. In fact this means that wave packet does not spread unbounded.

iii. The expectation values of $x$ and $p$ for large $\alpha$ oscillate with decreasing amplitude and approach zero which is equal to the values of $\langle x \rangle$ and $\langle p \rangle$ in the energy basis.

We also introduced the relativistic Poschl-Teller potential and showed that its energy levels are equally-spaced, Eqs.(27,29). We obtained the eigenfunctions of the annihilation operator for this potential, Eq.(38). It is consequently shown that the time evolution of the obtained functions are still eigenstates of the annihilation operator, Eq.(41).

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