The serpentine representation of the infinite symmetric group and the basic representation of the affine Lie algebra \( \widehat{\mathfrak{sl}_2} \)

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Abstract

We introduce and study the so-called serpentine representations of the infinite symmetric group \( \mathfrak{S}_N \), which turn out to be closely related to the basic representation of the affine Lie algebra \( \widehat{\mathfrak{sl}_2} \) and representations of the Virasoro algebra.

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1 Introduction. Infinite-dimensional Schur–Weyl duality and the serpentine representation of \( \mathfrak{S}_N \)

The serpentine representation is a remarkable representation of the infinite symmetric group \( \mathfrak{S}_N \), which has not yet been studied. Its importance is due to the fact that it is very closely related to the basic representation of the affine Lie algebra \( \widehat{\mathfrak{sl}_2} \) and representations of the Virasoro algebra. This representation belongs to the class of so-called Schur–Weyl representations. Recall that in [12] we suggested an infinite-dimensional generalization of the classical Schur–Weyl duality for the symmetric group \( \mathfrak{S}_N \) and the special

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linear group $SL(2, \mathbb{C})$ using a “dynamical” approach. Namely, we started from the classical Schur–Weyl duality (for definiteness, assume that $N = 2n$)

$$(C^2)^\otimes N = \sum_{k=0}^{n} M_{2k+1} \otimes H_{\pi_k},$$

(1)

where $H_{\pi_k}$ is the space of the irreducible representation $\pi_k$ of the symmetric group $\mathfrak{S}_N$ corresponding to the two-row Young diagram $(n + k, n - k)$ and $M_{2k+1}$ is the $(2k + 1)$-dimensional irreducible $SL(2, \mathbb{C})$-module, and considered isometric embeddings $(C^2)^\otimes N \hookrightarrow (C^2)^{(N+2)}$ that are equivariant with respect to both the actions of $SL(2, \mathbb{C})$ and $\mathfrak{S}_N$, which we called Schur–Weyl embeddings. Given an infinite chain

$$(C^2)^\otimes 0 \overset{\alpha_0}{\hookrightarrow} (C^2)^\otimes 2 \overset{\alpha_2}{\hookrightarrow} (C^2)^\otimes 4 \overset{\alpha_4}{\hookrightarrow} \ldots$$

(2)

of Schur–Weyl embeddings, we can consider the corresponding inductive limit. The class of all representations (called Schur–Weyl representations) that can be obtained in this way is described in \cite[Theorem 1]{12}: Let $\Pi^{(\alpha_N)}$ be the representation of the infinite symmetric group $\mathfrak{S}_N$ obtained as the inductive limit of the standard representations of $\mathfrak{S}_N$ in $(C^2)^\otimes N$ with respect to an infinite chain of Schur–Weyl embeddings \cite{2}. Then it decomposes into a countable direct sum of primary representations

$$\Pi^{(\alpha_N)} = \sum_{k=0}^{\infty} M_{2k+1} \otimes \Pi^{(\alpha_N)}_k,$$

(3)

where $\Pi^{(\alpha_N)}_k$ is the inductive limit of the irreducible representations of $\mathfrak{S}_{2k}$, $\mathfrak{S}_{2k+2}, \ldots$ corresponding to the Young diagrams $(2k), (2k+1, 1), (2k+2, 2), \ldots$.

As an important example of such a representation, in \cite{12} we considered the unique infinite Schur–Weyl scheme that satisfies a natural additional condition, namely, preserves the tensor structure of $(C^2)^\otimes N$. The main goal of this paper is to study another example of Schur–Weyl duality, namely, the unique Schur–Weyl scheme that satisfies the following additional condition: it preserves the so-called stable major index of a Young tableau. We show that this particular representation of the infinite symmetric group, which we call the serpentine representation, can be naturally equipped with the structure of the basic representation of the affine Lie algebra $\widehat{sl}_2$, with the irreducible $\mathfrak{S}_N$-modules corresponding to the irreducible Virasoro modules. This reveals
new interrelations between the representation theory of the infinite symmetric group and that of the affine Lie and Virasoro algebras. The precise form of the underlying natural grading-preserving isomorphism of $\hat{\mathfrak{sl}}_2$-modules is still unknown in the general case, and perhaps it is not a simple task to find it, but we present several properties of this isomorphism which are corollaries of the main theorem.

Our approach uses the result of [2] that the level 1 irreducible highest weight representations of $\hat{\mathfrak{sl}}_2$ can be realized as certain inductive limits of tensor powers $(\mathbb{C}^2)^\otimes N$ of the two-dimensional irreducible representation of $\mathfrak{sl}_2$. The construction of [2] is based on the notion of the fusion product of representations, whose main ingredient is, in turn, a special grading in the space $(\mathbb{C}^2)^\otimes N$. A key observation underlying the results of this paper, which relies on the computation presented in [7] of the $q$-characters of the multiplicity spaces of irreducible $\mathfrak{sl}_2$-modules with respect to this grading, is that the fusion product under consideration can be realized in an $S_N$-module so that this special grading essentially coincides with a well-known combinatorial characteristic of Young tableaux called the major index (see Proposition 1). Thus our results provide, in particular, a kind of combinatorial description of the fusion product and show that the combinatorial notion of the major index of a Young tableau has a new representation-theoretic meaning. For instance, Corollary 2 in Sec. 4 shows that the so-called stable major indices of infinite Young tableaux are the eigenvalues of the Virasoro $L_0$ operator, the Gelfand–Tsetlin basis of the Schur–Weyl module being its eigenbasis.

The paper is organized as follows. In Sec. 2 we introduce our main object, the so-called serpentine representation of the infinite symmetric group, as well as the notion of the stable major index of an infinite Young tableau, and formulate our main Theorem 1 which states that there is a grading-preserving isomorphism of $\mathfrak{sl}_2$-modules between the basic $\hat{\mathfrak{sl}}_2$-module $L_{0,1}$ and the space $H_{11}$ of the serpentine representation. The theorem is proved in Sec. 3. In Sec. 4 we study the above isomorphism in more detail, describing some of its properties and giving examples.

For definiteness, in what follows we consider only the even case $N = 2n$. The odd case can be treated in exactly the same way; instead of the basic representation $L_{0,1}$, it leads to the other level 1 highest weight representation $L_{1,1}$ of $\hat{\mathfrak{sl}}_2$.

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The main theorem

2.1 The serpentine representation, stable major index, and the statement of the theorem

Let $T_N$ be the set of all standard Young tableaux with $N$ cells and at most two rows. Consider the following natural embedding $i_N: T_N \to T_{N+2}$: given a standard Young tableau $\tau$ with $N$ cells, its image $i_N(\tau)$ is the standard Young tableau with $N+2$ cells obtained from $\tau$ by adding the element $N+1$ to the first row and the element $N+2$ to the second row. As shown in [12], it determines a Schur–Weyl embedding $(\mathbb{C}^2)^\otimes N \hookrightarrow (\mathbb{C}^2)^\otimes (N+2)$, which, by abuse of notation, we denote by the same symbol $i_N$.

**Definition 1.** The Schur–Weyl representation $\Pi := \Pi^{(i_N)}$ of the infinite symmetric group $S_N$ in the space $H_\Pi = \lim((\mathbb{C}^2)^\otimes N, i_N)$ will be called the serpentine representation.

According to the theorem on Schur–Weyl representations (see the introduction), we have

$$\Pi = \sum_{k=0}^{\infty} M_{2k+1} \otimes \Pi_k,$$

where the irreducible component $\Pi_k$, which will be called the $k$-serpentine representation, is the representation of $S_N$ associated with the infinite tableau

$$\tau_k = \begin{array}{cccccc}
1 & 2 & \cdots & 2k & 2k+1 & 2k+3 \\
2k+2 & 2k+4 & \cdots
\end{array}$$

(in particular, $\tau_0$ is the tableau with 1, 3, 5, \ldots in the first row and 2, 4, 6, \ldots in the second row), which can be realized in the space $H_{\Pi_k}$ spanned by the set $T_k$ of infinite two-row Young tableaux tail-equivalent to $\tau_k$. It has a discrete spectrum with respect to the Gelfand–Tsetlin algebra.

In what follows, the tableaux $\tau_k$, $k = 0, 1, \ldots$, will be called principal, and a tableau tail-equivalent to $\tau_k$ for some $k$ will be called a serpentine tableau; denote by $T = \cup T_k$ the set of all serpentine tableaux.
Now consider the well-known statistic on Young tableaux called the major index. It is defined as follows (see [11, Sec. 7.19]):

$$\text{maj}(\tau) = \sum_{i \in \text{des}(\tau)} i,$$

where, for $\tau \in T_N$,

$$\text{des}(\tau) = \{i \leq N - 1 : \text{the element } i + 1 \text{ in } \tau \text{ lies lower than } i\}$$

is the descent set of $\tau$.

Obviously,

$$\text{maj}(i_N(\tau)) = \text{maj}(\tau) + (N + 1). \tag{5}$$

This suggests the following important step. Given $N = 2n$ and $\tau \in T_N$, denote $r_N(\tau) = n^2 - \text{maj}(\tau)$. Then $r_{N+2}(i_N(\tau)) = r_N(\tau)$, so that we have a well-defined index on all serpentine tableaux $\tau \in \mathcal{T}$:

$$r(\tau) = \lim_{n \to \infty} r_{2n}([\tau]_{2n}) = \lim_{n \to \infty} (n^2 - \text{maj}([\tau]_{2n})), \tag{6}$$

where $[\tau]_l$ is the tableau with $l$ cells obtained from $\tau$ by removing all the cells with entries $k > l$. Obviously, for the principal tableaux we have $r(\tau_k) = k^2$.

**Definition 2.** We call $r(\tau)$ the stable major index of an infinite tableau $\tau \in \mathcal{T}$.

The stable major index determines a grading on all the spaces $H_{\Pi_k}$ and hence on the whole space $H_{\Pi}$: for $w = u \otimes v \in M_{2k+1} \otimes H_{\Pi_k}$ we just set $\text{deg}_r(w) = r(v)$.

Now consider the affine Lie algebra $\widehat{sl}_2 = sl_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, its basic module $L_{0,1}$ with the homogeneous grading $\text{deg}_H$, and the natural embedding $sl_2 \subset \widehat{sl}_2$ given by $sl_2 \ni x \mapsto x \otimes 1 \in \widehat{sl}_2$. Our main theorem is the following.

**Theorem 1.** There is a grading-preserving unitary isomorphism of $sl_2$-modules between $(L_{0,1}, \text{deg}_H)$ and $(H_{\Pi}, \text{deg}_r)$. The serpentine representation is the unique Schur–Weyl representation satisfying this condition.

**Remarks.** 1. As mentioned in the introduction, we consider only the even case just for simplicity of notation. Considering instead of [12] the chain
and reproducing exactly the same arguments, we will obtain a grading-preserving isomorphism of the corresponding Schur–Weyl module with the other level 1 highest weight module $L_{1,1}$ of $\mathfrak{sl}_2$.

2. The conditions from the statement of Theorem [4] do not uniquely determine the isomorphism, since there is a nontrivial group of transformations in $H_\Pi$ that commute with $\mathfrak{sl}_2$ and preserve the grading. For more details, see the remark after Corollary [2] in Sec. 4. To find an explicit form of this isomorphism is an intriguing problem.

3 Proof of the main theorem

1. Fusion product. Our proof relies on the result of B. Feigin and E. Feigin [2] on a finite-dimensional approximation of the basic representation of $\hat{\mathfrak{sl}}_2$, which, in turn, uses the notion of the fusion product of representations introduced in [4]. Since the corresponding construction is of importance for us, we describe it in some detail.

Given a representation $\rho$ of $\mathfrak{sl}_2$ and $z \in \mathbb{C}$, let $\rho(z)$ be the evaluation representation of the polynomial current algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$, defined as $(x \otimes t^i)v = z^i \cdot xv$. Now, given a collection $\rho_1, \ldots, \rho_N$ of irreducible representations of $\mathfrak{sl}_2$ with lowest weight vectors $v_1, \ldots, v_N$, and a collection $z_1, \ldots, z_N$ of pairwise distinct complex numbers, we consider the tensor product of the corresponding evaluation representations:

$$\rho_1(z_1) \otimes \ldots \otimes \rho_N(z_N).$$

The crucial step is introducing a special grading in the space $V_N$ of this representation. Set

$$V_N^{(m)} \subset V_N = U_m(e \otimes \mathbb{C}[t]) (v_1 \otimes \ldots \otimes v_N) \subset V_N,$$

where $e$ is the raising operator in $\mathfrak{sl}_2$ and $U_m$ is spanned by homogeneous elements of degree $m$ in $t$. In other words, $U_m$ is spanned by the monomials of the form $e_{i_1} \ldots e_{i_k}$ with $i_1 + \ldots + i_k = m$, where $e_j = e \otimes t^j$. Then we consider the corresponding filtration on $V_N$: $V_N^{(\leq m)} = \sum_{k \leq m} V_N^{(k)}$. The fusion product of $\rho_1, \ldots, \rho_N$ is the graded representation with respect to the above filtration, which is realized in the space

$$V_N^* = \text{gr } V_N = V_N^{(\leq 0)} \oplus V_N^{(1)} / V_N^{(\leq 0)} \oplus V_N^{(2)} / V_N^{(\leq 2)} \oplus \ldots.$$  \hspace{1cm} (7)

The space $V_N^*[k] = V_N^{(\leq k)} / V_N^{(\leq k-1)}$ is the subspace of elements of degree $k$, and elements of the form $x \otimes t^l \in \mathfrak{sl}_2 \otimes \mathbb{C}[t]$ send $V_N^*[k]$ to $V_N^*[k+l]$. The degree of an element with respect to this grading will be denoted by $\deg$. It is proved in [4] that $V_N^*$ is an $\mathfrak{sl}_2 \otimes (\mathbb{C}[t]/t^N)$-module that does not depend on $z_1, \ldots, z_N$. \hspace{1cm} (6)
provided that they are pairwise distinct. Moreover, $V_N^*$ is isomorphic to $\rho_1 \otimes \ldots \otimes \rho_N$ as an $\mathfrak{sl}_2$-module.

We apply this construction to the case where $\rho_1 = \ldots = \rho_N$ is the two-dimensional irreducible representation of $\mathfrak{sl}_2$ with the lowest weight vector $v_0$. In this case, $V_N^* \simeq (\mathbb{C}^2)^\otimes N$ as an $\mathfrak{sl}_2$-module. We equip $V_N^*$ with the inner product such that the corresponding representation of $\mathfrak{sl}_2$ is unitary. It is proved in [2] that an inductive limit of $V_N^*$ is isomorphic to the basic representation $L_{0,1}$ of $\widehat{\mathfrak{sl}}_2$, so that we first establish a grading-preserving isomorphism of the finite-dimensional $\mathfrak{sl}_2$-modules $V_N^*$ and $\sum_{k=0}^{n} M_{2k+1} \otimes H_{\pi_k}$ and then show that it can be extended to the inductive limits of the corresponding spaces.

2. The $q$-character, major index, and the finite-dimensional result.
Consider the decomposition of $V_N^*$ into irreducible $\mathfrak{sl}_2$-modules:

$$V_N^* = \bigoplus_{k=0}^{n} M_{2k+1} \otimes M_k.$$

By the classical Schur–Weyl duality (1), we know that the multiplicity space $M_k$ coincides with the space $H_{\pi_k}$ of the irreducible representation of $\mathfrak{S}_N$ with the Young diagram $(n+k,n-k)$. On the other hand, it inherits the grading from $V_N^*$:

$$M_k = \bigoplus_{i \geq 0} M_k[i],$$

where $M_k[i] = M_k \cap V^*[i]$. Consider the corresponding $q$-character

$$\text{ch}_q M_k = \sum_{i \geq 0} q^i \dim M_k[i].$$

It was proved by Kedem [7] that

$$\text{ch}_q M_k = q^{\frac{n(N-1)}{2}} \cdot K_{(n+k,n-k),1}^N(1/q),$$

where $K_{\lambda,\mu}$ is the Kostka–Foulkes polynomial (see [9] Sec. III.6]).

Now we use the well-known combinatorial description of the Kostka–Foulkes polynomial due to Lascoux and Schützenberger [8]. For a two-row partition $\lambda$, their formula reduces to

$$K_{\lambda,1}^N (q) = \sum_{\tau \in [\lambda]} q^{c(\tau)},$$

\[7\]
where $[\lambda]$ is the set of standard Young tableaux of shape $\lambda$ and $c(\tau)$ is the charge of a tableau $\tau \in T_N$, defined as the sum of $i \leq N - 1$ such that in $\tau$ the element $i + 1$ lies to the right of $i$ (see [9]). But, obviously, for $\tau \in T_N$ we have $\text{maj}(\tau) = \frac{N(N-1)}{2} - c(\tau)$. Then it follows from (9) and (10) that

$$\dim \mathcal{M}_k[i] = \# \{ \tau \in [(n+k, n-k)] : \text{maj}(\tau) = i \}. \quad (11)$$

The major index defines a grading in the space $H_{\pi_k}$ (spanned by the standard Young tableaux of shape $(n+k, n-k)$), and hence in the whole space $X_N = \sum_{k=0}^n M_{2k+1} \otimes H_{\pi_k}$, which we equip with the standard inner product. We obtain the following finite-dimensional analog of Theorem 1.

**Proposition 1.** There is a grading-preserving unitary isomorphism of $\mathfrak{sl}_2$-modules between $(V_N^*, \tilde{\deg})$ and $(X_N, \text{maj})$ such that the multiplicity space $M_k$ is spanned by the standard Young tableaux $\tau$ of shape $(n+k, n-k)$ (and hence $M_k[i]$ is spanned by $\tau$ with $\text{maj}(\tau) = i$).

**Proof.** Follows from the fact that the fusion product $V_N^*$ is isomorphic to $(C_2 \otimes \mathbb{C})^N$ as an $\mathfrak{sl}_2$-module and equation (11). \qed

**Remarks.**

1. Observe that the isomorphism from Proposition 1 is not unique.
2. The isomorphism from Proposition 1 determines an action of the symmetric group $\mathfrak{S}_N$ on the space $V_N^*$. It does not coincide with the original action of $\mathfrak{S}_N$ on $C^\otimes N$.
3. Embeddings and the limit. It is proved in [2] that there is an embedding $j_N : V_N^* \rightarrow V_{N+2}^*$ equivariant with respect to the action of $\mathfrak{sl}_2 \otimes (\mathbb{C}[t^{-1}]/t^{-n})$, and the corresponding inductive limit $V = \lim(V_N^*, j_N)$ is isomorphic to the basic representation $L_{0,1}$ of the affine Lie algebra $\hat{\mathfrak{sl}}_2$. This embedding satisfies

$$\tilde{\deg}(j_N x) = \tilde{\deg}(x) - (N + 1). \quad (12)$$

Since we are now considering $\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}]$ instead of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$, we should slightly modify the previous constructions to take the minus sign into account. Namely, instead of (8) we now have $\mathcal{M}_k = \oplus_{i \geq 0} \mathcal{M}_k[-i]$, and the isomorphism of Proposition 1 identifies $\mathcal{M}_k[-i]$ with the space spanned by the tableaux $\tau$ of shape $(n+k, n-k)$ such that $\text{maj}(\tau) = i$. Denote this isomorphism between $V_N^*$ and $X_N$ by $\rho_N$. Observe that the only conditions we
impose on $\rho_N$ are as follows: (a) $\rho_N$ is a unitary isomorphism of $\mathfrak{sl}_2$-modules and (b) $\rho_N \circ \deg = -\maj$.

Now, since $L_{0,1} \simeq \lim(V_{N}^*, j_N)$, $H_{\Pi} = \lim(\mathcal{X}_N, i_N)$, and Proposition 1 holds, in order to prove Theorem 1 it suffices to show that we can choose a sequence of isomorphisms $\rho_N$ such that the diagram

$$
\begin{array}{ccc}
V_N^* & \xrightarrow{\rho_N} & \mathcal{X}_N \\
\downarrow j_N & & \downarrow i_N \\
V_{N+2}^* & \xrightarrow{\rho_{N+2}} & \mathcal{X}_{N+2}
\end{array}
$$

is commutative for all $N$. We use induction on $N$. The base being obvious, assume that we have already constructed $\rho_N$, and let us construct $\rho_{N+2}$.

We have $V_{N+2}^* = j_N (V_N^*) \oplus (j_N (V_N^*))^\perp$. On the first subspace, we set $\rho_{N+2}(x) := i_N (\rho_N(j_N^{-1}(x)))$. On the second one, we define it in an arbitrary way to satisfy the desired conditions (a) and (b). The fact that this definition is correct and provides us with a desired isomorphism between $V_{N+2}^*$ and $\mathcal{X}_{N+2}$ follows from (12) and (5). The theorem is proved.

4 The key isomorphism in more detail

Our aim in this section is to study the isomorphism from Theorem 1 in more detail. For this, we first give necessary background on the Fock space realization of the basic $\hat{\mathfrak{sl}}_2$-module.

4.1 The Fock space realization of the basic $\hat{\mathfrak{sl}}_2$-module

Let $\mathcal{F}$ be the fermionic Fock space constructed as the infinite wedge space over the linear space with basis $\{u_k\}_{k \in \mathbb{Z}} \cup \{v_k\}_{k \in \mathbb{Z}}$. That is, $\mathcal{F}$ is spanned by the semi-infinite forms $u_{i_1} \wedge \ldots \wedge u_{i_k} \wedge v_{j_1} \wedge \ldots \wedge v_{j_l} \wedge u_N \wedge v_N \wedge u_{N-1} \wedge v_{N-1} \wedge \ldots$, $N \in \mathbb{Z}$, $i_1 > \ldots > i_k > N$, $j_1 > \ldots > j_l > N$, and is equipped with the inner product in which such monomials are orthonormal. Let $\phi_k$ be the exterior multiplication by $u_k$ and $\psi_k$ be the exterior multiplication by $v_k$, and denote by $\phi_k^*$, $\psi_k^*$ the corresponding adjoint operators. Then this family of operators satisfies the canonical anticommutation relations (CAR). We consider the generating functions $\phi(z) = \sum_{i \in \mathbb{Z}} \phi_i z^{-(i+1)}$, $\phi^*(z) = \sum_{i \in \mathbb{Z}} \phi_i^* z^i$, and the same for $\psi$ and $\psi^*$. 

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Let $a_n^\phi$ and $a_n^\psi$ be the systems of bosons constructed from the fermions $\{\phi_k\}$ and $\{\psi_k\}$, respectively: $a_n^\phi = \sum_{k\in\mathbb{Z}} \phi_k \phi_k^* + n$ for $n \neq 0$ and $a_0^\phi = \sum_{n=0}^\infty \phi_n \phi_n^* - \sum_{n=0}^\infty \phi_{-n} \phi_{-n}^*$, and similarly for $a_n^\psi$. They satisfy the canonical commutation relations (CCR), i.e., form a representation of the Heisenberg algebra $\mathfrak{A}$. Denote $a^\phi(z) = \sum_{n\in\mathbb{Z}} a_n^\phi z^{-(n+1)}$, and similarly for $a^\psi$.

Let $V$ be the operator in $\mathcal{F}$ that shifts the indices by 1:

$$V(w_i \wedge w_{i+1} \wedge \ldots) = V_0(w_i) \wedge V_0(w_{i+1}) \wedge \ldots, \quad V_0(u_i) = u_{i+1}, \quad V_0(v_i) = v_{i-1}.$$ 

The vacuum vector in $\mathcal{F}$ is $\Omega = u_{-1} \wedge v_{-1} \wedge u_{-2} \wedge v_{-2} \wedge \ldots$. We also consider the family of vectors $\Omega_0 = \Omega$, $\Omega_{2n} = V^{-n} \Omega_0$, $n \in \mathbb{Z}$.

In the space $\mathcal{F}$ we have a canonical representation of the affine Lie algebra $\widehat{\mathfrak{sl}_2} = \mathfrak{sl}_2 \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, which is given by the following formulas. Given $x \in \mathfrak{sl}_2$, denote $X(z) = \sum_{n \in \mathbb{Z}} x_n z^{-(n+1)}$. Then

$$E(z) = \psi(z) \phi^*(z), \quad F(z) = \phi(z) \psi^*(z),$$

$$h_n = a_n^\psi - a_n^\phi, \quad d = \frac{h_0^2}{2} + \sum_{n=1}^\infty h_{-n} h_n, \quad c = 1.$$ 

We have $\mathcal{F} = \mathcal{H}_0 \otimes \mathcal{K}_0 + \mathcal{H}_1 \otimes \mathcal{K}_1$, where $\mathcal{H}_0 \simeq L_{0,1}$ and $\mathcal{H}_1 \simeq L_{1,1}$ are the irreducible level 1 highest weight $\widehat{\mathfrak{sl}_2}$-modules and $\mathcal{K}_0$ and $\mathcal{K}_1$ are the multiplicity spaces. Observe also that $e_{-(N+1)} \Omega_{-N} = \Omega_{-(N+2)}$.

Note that the operators $a_n = \frac{1}{\sqrt{2}} h_n$ satisfy the CCR, i.e., form a system of free bosons, or generate the Heisenberg algebra $\mathfrak{A}_h$. The vectors $\{\Omega_{2n}\}_{n \in \mathbb{Z}}$ introduced above are exactly singular vectors for this Heisenberg algebra: $h_k \Omega_m = 0$ for $k > 0$, $h_0 \Omega_m = m \Omega_m$. The representation of $\mathfrak{A}_h$ in $\mathcal{H}_0$ breaks into a direct sum of irreducible representations:

$$\mathcal{H}_0 = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_0[2k],$$

where $\mathcal{H}_0[2k]$ is the charge $2k$ subspace, i.e., the eigenspace of $h_0$ with eigenvalue $2k$:

$$\mathcal{H}_0[2k] = \{ v \in \mathcal{H}_0 : h_0 v = 2kv \} = \mathbb{C}[h_0, h_1, \ldots] \Omega_{2k}.$$ 

Now, given a representation of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$, we can use the Sugawara construction to obtain the corresponding representation of the
Virasoro algebra Vir. It can also be described in the following way. As noted above, the operators \( a_n = \frac{1}{\sqrt{2}} h_n \) form a system of free bosons. Given such a system, a representation of Vir can be constructed as follows (\cite{13}; see also \cite{6 Ex. 9.17}):

\[
L_n = \frac{1}{2} \sum_{j \in \mathbb{Z}} a_{-j} a_{j+n}, \quad n \neq 0; \quad L_0 = \sum_{j=1}^{\infty} a_{-j} a_j.
\]  

(14)

Thus we obtain a representation of Vir in \( \mathcal{F} \) and, in particular, in \( \mathcal{H}_0 \). In this representation, the algebras generated by the operators of Vir and \( \mathfrak{sl}_2 \subset \hat{\mathfrak{sl}}_2 \) are mutual commutants, and we have the decomposition

\[
\mathcal{H}_0 = \bigoplus_{k=0}^{\infty} M_{2k+1} \otimes L(1, k^2),
\]

(15)

where \( M_{2k+1} \) is the \((2k + 1)\)-dimensional irreducible \( \mathfrak{sl}_2 \)-module and \( L(1, k^2) \) is the irreducible Virasoro module with central charge 1 and conformal dimension \( k^2 \).

The charge \( k \) subspace \( \mathcal{H}_0[k] \) contains a series of singular vectors \( \xi_{k,m} \) of Vir with energy \((k + m)^2\):

\[
L_n \xi_{k,m} = 0 \text{ for } n = 1, 2, \ldots, \quad L_0 \xi_{k,m} = (k + m)^2.
\]

Let us use the so-called homogeneous vertex operator construction of the basic representation of \( \hat{\mathfrak{sl}}_2 \) (\cite{5}, see also \cite{6 Sec. 14.8}). In this realization,

\[
E(z) = \Gamma_-(z) \Gamma_+(z) z^{-h_0} V^{-1}, \quad F(z) = \Gamma_+(z) \Gamma_-(z) z^{h_0} V,
\]

(16)

where

\[
\Gamma_{\pm}(z) = \exp \left( \mp \sum_{j=1}^{\infty} \frac{z^j}{j} h_{\pm j} \right)
\]

and the operators \( \Gamma_{\pm}(z) \) satisfy the commutation relation

\[
\Gamma_+(z) \Gamma_-(w) = \Gamma_-(w) \Gamma_+(z) \left( 1 - \frac{z}{w} \right)^2.
\]

(17)

Using the boson–fermion correspondence (see \cite{6 Ch. 14}), we can identify \( \mathcal{H}_0 \) with the space \( \Lambda \otimes \mathbb{C}[q, q^{-1}] \), where \( \Lambda \) is the algebra of symmetric functions (see \cite{9}). In particular, consider the charge 0 subspace \( \mathcal{H}[0] = \mathcal{H}_0[0] \), which is
identified with $\Lambda$. We can use the following representation of the Heisenberg algebra generated by $\{h_n\}_{n \in \mathbb{Z}}$:

$$h_n \leftrightarrow 2n \frac{\partial}{\partial p_n}, \quad h_{-n} = p_n, \quad n > 0,$$

(18)

where $p_j$ are Newton’s power sums. Note that the representation (18) of $A_h$, and hence the corresponding representation (14) of Vir, are not unitary with respect to the standard inner product in $\Lambda$. To make it unitary, we should consider the inner product in $\Lambda$ defined by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} \cdot z_\lambda \cdot 2^{l(\lambda)},$$

(19)

where $p_\lambda$ are the power sum symmetric functions, $z_\lambda = \prod_{i} i^{m_i} m_i!$ for a Young diagram $\lambda$ with $m_i$ parts of length $i$, and $l(\lambda)$ is the number of nonzero rows in $\lambda$.

Denote the singular vectors of Vir in $\mathcal{H}[0]$ by $\xi_m := \xi_{0,m}$. According to a result by Segal [10], in the symmetric function realization (18),

$$\xi_n \leftrightarrow c \cdot s_{(n^n)},$$

(20)

where $s_{(n^n)}$ is the Schur function indexed by the $n \times n$ square Young diagram and $c$ is a numerical coefficient.

### 4.2 Further analysis of the key isomorphism

Comparing the structure (11) of the serpentine representation with (15), we obtain the following result.

**Corollary 1.** The space $H_{\Pi_k}$ of the $k$-serpentine representation of the infinite symmetric group has a natural structure of the Virasoro module $L(1, k^2)$.

Our aim is to study this Virasoro representation in $\Pi_k$ (or, which is equivalent, the corresponding representation of the infinite symmetric group in the Fock space). In particular, from the known theory of the basic module $L_{0,1}$, we immediately obtain the following result.

**Corollary 2.** In the above realization of the Virasoro module $L(1, k^2)$, the Gelfand–Tsetlin basis of $H_{\Pi_k}$ (which consists of the infinite two-row Young tableaux tail-equivalent to $\tau_k$) is the eigenbasis of $L_0$, and the eigenvalues are given by the stable major index $r$:

$$L_0 \tau = r(\tau) \tau.$$
Now we see that the isomorphism in Theorem 1 is in fact defined up to the commutant of $L_0$ in each $\Pi_k$.

Let $\omega_{-2k}$ be the lowest vector in $M_{2k+1}$. Then a natural basis of $\mathcal{V}$ is $\{e_0^m \omega_{-2k} \otimes \tau : m = 0,1,\ldots,2k, \tau \in T_k\}$. Denoting $\mathcal{V}_k = M_{2k+1} \otimes H_{\Pi_k}$ and $\mathcal{V}_k[0] = \{v \in \mathcal{V}_k : h_0 v = 0\}$, we have $\mathcal{V}_k[0] = e_0^k \omega_{-2k} \otimes H_{\Pi_k}$, so that we may identify $\mathcal{V}_k[0]$ with $H_{\Pi_k}$ via the correspondence

$$c(t) \cdot e_0^k \omega_{-2k} \otimes t \leftrightarrow t,$$

where $c(t)$ is a normalizing constant. On the other hand, it is shown in [2] that $V^*_{2n} \simeq \mathbb{C}[e_0,\ldots,e_{-(2n-1)}] \Omega_{-2n} \subset \mathcal{F}$ as an $\mathfrak{sl}_2 \otimes (\mathbb{C}[t^{-1}]/t^{-2n})$-module, and the limit space $\mathcal{V}$ coincides with $H_0$. Under this correspondence, the charge 0 subspace $\mathcal{H}[0]$ is identified with $\mathcal{V}[0] = \{v \in \mathcal{V} : h_0 v = 0\}$. Thus we have

$$\mathcal{H}[0] \simeq H_{\Pi[0]} = \bigoplus_{k=0}^{\infty} H_{\Pi_k}, \quad (21)$$

where $H_{\Pi[0]}$ is the space spanned by all serpentine tableaux, which is the space of the countable sum of the $k$-serpentine representations $\Pi_k$ of $\mathfrak{S}_N$ without multiplicities, and the following corollary holds.

**Corollary 3.** The space $H_{\Pi[0]}$ has a structure of an irreducible representation of the Heisenberg algebra $\mathfrak{A}$.

Now, using results of [2], one can easily prove the following lemma.

**Lemma 1.** A basis in $F_{2n} = \mathbb{C}[e_0,\ldots,e_{-(2n-1)}] \Omega_{-2n}$ is

$$\{e_0^i_0 e_{-1}^i_1 \ldots e_{-(2n-1)}^i_{2n-1} : 0 \leq k \leq 2n - (i_0 + \ldots + i_{2n-1})\} \Omega_{-2n}.$$

In particular, a basis of $F_{2n}[0] = F_{2n} \cap \mathcal{H}[0]$ is

$$\{\prod e_0^i_0 e_{-1}^i_1 \ldots e_{-n}^i_n : i_0 + i_1 + \ldots + i_n = n\} \Omega_{-2n}. \quad (22)$$

On the other hand, as mentioned above, $\mathcal{H}[0]$ can be identified with the algebra of symmetric functions $\Lambda$ via (18). Denote by $\Phi$ the obtained isomorphism between $H_{\Pi[0]}$ and $\Lambda$, which thus associates with every serpentine tableau $\tau \in T$ a symmetric function $\Phi(\tau) \in \Lambda$ such that $r(\tau) = \deg \Phi(\tau)$.

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Proposition 2. Under the isomorphism $\Phi$, the principal tableaux correspond to the Schur functions with square Young diagrams:

$$\Phi(\tau_k) = \text{const} \cdot s_{(k \times k)}.$$  

Proof. Follows from Segal's [10] result (20), since it is not difficult to see that the singular vector of $\text{Vir}$ in $\mathcal{V}_k[0]$ is just $e_k^0 \omega_{-2k} \otimes \tau_k$. \hfill \qed

Denote by $T^{(N)}$ the (finite) set of infinite two-row tableaux that coincide with some $\tau_n$, $n = 0, 1, \ldots$, from the $N$th level.

Proposition 3. Let $H^{(N)}_\Pi$ be the subspace in $H_\Pi[0]$ spanned by all $\tau \in T^{(N)}$. Then

$$\Phi(H^{(2k)}_\Pi) = \Lambda_{k \times k},$$

where $\Lambda_{k \times k}$ is the subspace in $\Lambda$ spanned by the Schur functions indexed by Young diagrams lying in the $k \times k$ square.

Proof. From all the above identifications, $H^{(2k)}_\Pi \leftrightarrow F_{2k}[0]$. Now the claim follows from the result proved in [3] that in the symmetric functions realization $F_{2k}[0]$ corresponds to $\Lambda_{k \times k}$. \hfill \qed

In the next theorem we refine this result, giving an explicit formula for the Schur basis in $\Lambda_{k \times k}$ in terms of the basis (22) in $H^{(2k)}_\Pi \simeq F_{2k}[0]$. In fact, we would like to have an explicit formula for $\Phi$ or $\Phi^{-1}$, expressing, say, a Schur function in terms of serpentine tableaux. At the moment we cannot provide such a general formula, but the theorem below is a step toward solving this problem, reducing it to describing the action of the operators $e_{-m}$ in the space of serpentine tableaux. Besides, Propositions 2 and 3 can easily be derived from formula (23), the former by taking $\nu = (k^k)$ and the latter by counting the dimensions.

Theorem 2. In the symmetric functions realization, the correspondence between the Schur function basis in $\Lambda_{k \times k}$ and the basis (22) in $H^{(2k)}_\Pi \simeq F_{2k}[0]$ is given by

$$s_\nu = \sum_{\mu = (0^r \, 1^s \, 2^t \, \ldots) \subset (k^k)} K_{\nu \mu} \prod_{j=0}^k r_j! e_{-(k-\mu_1)} \cdots e_{-(k-\mu_k)} \Omega_{-2k}, \quad \nu \subset (k^k),$$

where $K_{\lambda \mu}$ are Kostka numbers.
Proof. We generalize Wasserman’s [14] proof of Segal’s result [20] (a similar computation is also given in an earlier paper [1]).

Let 0 ≤ i_1, ..., i_k ≤ k. Then, obviously,

$$e_{-i_1} \cdots e_{-i_k} \Omega_{-2k} = \left[ \prod_{j=1}^{k} z_j^{i_j-1} \right] E(z_k) \cdots E(z_1) \Omega_{-2k},$$

where by [monomial] \( F(z_1, \ldots, z_m) \) we denote the coefficient of this monomial in \( F(z_1, \ldots, z_m) \) (in particular, \([1] F(z_1, \ldots, z_m) \) is the constant term of \( F \)).

Now, using the representation (16), the commutation relation (17), and the obvious facts that \( V^{-k} \Omega_{-2k} = \Omega_0 \) and \( \Gamma_+(z) \Omega_0 = \Omega_0 \), we obtain

$$E(z_k) \cdots E(z_1) \Omega_{-2k} = \prod_{j=1}^{k} z_j^{2(k-j)} \prod_{1 \leq j < i \leq k} \left( 1 - \frac{z_i}{z_j} \right)^2 \Gamma_-(z_k) \cdots \Gamma_-(z_1) \Omega_0.$$  

Observe that, in view of (18) and the well-known fact from the theory of symmetric functions, \( \Gamma_-(z) \) is exactly the generating function of the complete symmetric functions. Hence, expanding the product \( \Gamma_-(z_k) \cdots \Gamma_-(z_1) \Omega_0 \) by the Cauchy identity ([9, I.4.3]) and making simple transformations, we obtain

$$E(z_k) \cdots E(z_1) \Omega_{-2k} = (-1)^{k(k-1)/2} \prod_{j=1}^{k} z_j^{k-1} a_\delta(z) a_\delta(z^{-1}) \sum_{\lambda : l(\lambda) \leq k} s_\lambda(z^{-1}) s_\lambda,$$

where

$$a_\delta(z) = \prod_{1 \leq i < j \leq k} (z_i - z_j) = \text{det} [z_i^{k-j}]_{1 \leq i, j \leq k}$$

is the Vandermonde determinant, \( a_\delta(z^{-1}) \) is the similar determinant for the variables \( z^{-1} = (z_1^{-1}, \ldots, z_k^{-1}) \), \( l(\lambda) \) is the length of the diagram \( \lambda \) (the number of nonzero rows), \( s_\lambda(z^{-1}) \) is the Schur function calculated at the variables \( z^{-1} \), and \( s_\lambda \) is the Schur function as an element of \( \Lambda \) identified with \( \mathcal{H}[0] \).

Thus we have

$$e_{-i_1} \cdots e_{-i_k} \Omega_{-2k} = (-1)^{k(k-1)/2} \cdot [1] \left( \prod_{j=1}^{k} z_j^{k-i_j} a_\delta(z) a_\delta(z^{-1}) \sum_{\lambda} s_\lambda(z^{-1}) s_\lambda \right).$$

For convenience, set \( \tilde{e}_p := e_{-(k-p)}, \) 0 ≤ p ≤ k. Given 0 ≤ \( \alpha_1, \ldots, \alpha_k \) ≤ k, we have

$$\tilde{e}_{\alpha_1} \cdots \tilde{e}_{\alpha_k} \Omega_{-2k} = [1] \left( \prod_{j=1}^{k} z_j^{\alpha_j} a_\delta(z) \sum_{l(\lambda) \leq k} a_{\lambda+\delta}(z^{-1}) s_\lambda \right), \quad (24)$$
where \( a_{\lambda+\delta}(x) = \det[x^j_i + k-j, 1 \leq i,j \leq k] = s_\lambda(x)a_\delta(x) \). Consider a Young diagram \( \mu = (\mu_1, \ldots, \mu_k) = (0^r 1^r 2^r \ldots) \). Let us sum (24) over all different permutations \( \alpha = (\alpha_1, \ldots, \alpha_k) \) of the sequence \( (\mu_1, \ldots, \mu_k) \). Note that the operators \( e_j \) commute with each other, so that the left-hand side does not depend on the order of the factors. In the right-hand side, \( \sum_\alpha \prod z_{\alpha j}^{\alpha_j} = m_\mu(z) \), a monomial symmetric function. Thus we have

\[
\frac{k!}{\prod_{j=0}^k r_j!} \tilde{e}_{\mu_1} \cdots \tilde{e}_{\mu_k} = [1] \left( m_\mu(z) a_\delta(z) \sum_{l(\lambda) \leq k} a_{\lambda+\delta}(z^{-1}) s_\lambda \right).
\] (25)

Let \( \nu \) be a Young diagram with at most \( k \) rows and at most \( k \) columns, i.e., \( \nu \subset (k^k) \). We have

\[
s_\nu(z) = \sum_\mu K_{\nu \mu} m_\mu(z),
\] (26)

where \( K_{\nu \mu} \) are Kostka numbers. It is well known that \( K_{\nu \mu} = 0 \) unless \( \mu \leq \nu \), where \( \leq \) is the standard ordering on partitions: \( \mu \leq \nu \iff \mu_1 + \ldots + \mu_i \leq \nu_1 + \ldots + \nu_i \) for every \( i \geq 1 \). In particular, \( \mu_1 \leq \nu_1 \leq k \). Besides, since we consider only \( k \) nonzero variables \( z_1, \ldots, z_k \), it also follows that \( m_\mu(z) = 0 \) unless \( l(\mu) \leq k \). Thus the sum in (26) can be taken only over diagrams \( \mu \subset (k^k) \), for which equation (25) holds. Multiplying this equation by \( K_{\nu \mu} \) and summing over \( \mu \) yields

\[
\sum_{\mu=(0^r 1^r 2^r \ldots) \subset (k^k)} K_{\nu \mu} \tilde{e}_{\mu_1} \cdots \tilde{e}_{\mu_k} = [1] \left( s_\nu(z) a_\delta(z) \sum_{l(\lambda) \leq k} a_{\lambda+\delta}(z^{-1}) s_\lambda \right).
\]

By the orthogonality relations, the right-hand side is equal to \( k! s_\nu \), and the desired formula (23) follows.

### 4.3 Examples

In this section, we present the results of computing \( \Phi(\tau) \) for the serpentine tableaux with \( r(\tau) \leq 4 \) (note that although the conditions of Theorem II do not determine the isomorphism uniquely, these relations hold for any isomorphism satisfying them) in terms of Newton’s power sums \( p_k \). We write down only the “nontrivial” part of a tableau, meaning that it should be continued up to an infinite tableau in the “serpentine” way. We also omit the normalizing coefficients of \( \Phi(\tau) \), which are their norms in the inner product (19).
| $r(\tau)$ | $\tau$ | $\Phi(\tau)$ up to a constant | $r(\tau)$ | $\tau$ | $\Phi(\tau)$ up to a constant |
|------------|--------|-------------------------------|------------|--------|-------------------------------|
| 0          | $\tau_0$ | 1 = $s_0$                     | 4          | $\tau_2 = 1234$ | $p_1^4 + 3p_2^2 - 4p_1p_3 = s_{(2)}$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(1)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(2)}$ | 4 | $\tau_2 = 134$ | $p_1^4 - 3p_2 + 2p_1p_3$ |
| 2          | $\tau_1 = 12$ | $p_1 = s_{(3)}$ | 4 | $\tau_2 = 134$ | $p_1^4 - 3p_2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(1)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(2)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(3)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(4)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(1)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(2)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(3)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(4)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(1)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(2)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(3)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(4)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(1)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(2)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(3)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |
| 1          | $\tau_1 = 12$ | $p_1 = s_{(4)}$ | 4 | $\tau_2 = 134$ | $p_1^4 + 3p_2^2 + 2p_1p_3$ |

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