Lower Bounds Implementing Mediators in Asynchronous Systems

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Abstract

Abraham, Dolev, Geffner, and Halpern [2019] proved that, in an asynchronous systems, a \((k, t)\)-robust equilibrium for \(n\) players and a trusted mediator can be implemented without the mediator as long as \(n > 4(k + t)\), where an equilibrium is \((k, t)\)-robust if, roughly speaking, no coalition of \(t\) players can decrease the payoff of any of the other players, and no coalition of \(k\) players can increase their payoff by deviating. We prove that this bound is tight, in the sense that if \(n \leq 4(k + t)\) there exist \((k, t)\)-robust equilibria with a mediator that cannot be implemented by the players alone. Even though implementing \((k, t)\)-robust mediators seems closely related to implementing asynchronous multiparty \((k + t)\)-secure computation [Ben-Or, Canetti, and Goldreich 1993], to the best of our knowledge there is no known straightforward reduction from one problem to another. Nevertheless, we show that there is a non-trivial reduction from a slightly weaker notion of \((k + t)\)-secure computation, which we call \((k + t)\)-strict secure computation, to implementing \((k, t)\)-robust mediators. We prove the desired lower bound by showing that there are functions on \(n\) variables that cannot be \((k + t)\)-strictly securely computed if \(n \leq 4(k + t)\). This also provides a simple alternative proof for the well-known lower bound of \(4t + 1\) on asynchronous secure computation in the presence of up to \(t\) malicious agents [Abraham, Dolev, and Stern 2020; Ben-Or, Kelmer, and Rabin 1994; Canetti 1996].

1 Introduction

Ben-Or, Goldwasser, and Wigderson [1988] (BGW from now on) showed that given a finite domain \(D\), a function \(f : D^n \to D\) can be \(t\)-securely computed by \(n\) agents in a synchronous network with private authenticated channels as long as \(n > 3t\), where \(t\) is a bound on the number of malicious players. Roughly speaking, “\(t\)-securely computed” means that all honest agents correctly compute the output of \(f\), while a group of up to \(t\) malicious agents can learn nothing about the players’ inputs beyond what can be learned the output of \(f\). Ben-Or, Canetti, and Goldreich [1993] (BCG from now on) later provided analogous results for the asynchronous case: a function \(f : D^n \to D\) can be \(t\)-securely computed by \(n\) agents if \(n > 4t\).

Abraham, Dolev, Gonen, and Halpern [2006] consider a problem related to secure function computation that has deep roots in the game-theory literature. The agents in this case have an input and play a game \(\Gamma\). They make a move in the game and get a payoff that depends on the action profile (i.e., the move made by each agent). Of course, if the moves consist of outputting a value in \(D\), then we can view function computation as a game, where the agents (or players\(^1\)) get a payoff depending on the value that they output.

Secure function computation is often viewed as a game with a trusted third party, or mediator. Roughly speaking, we want the outcome to be the same as if the agents had sent their input values \(\vec{x}\) to the mediator, who then sends back \(f(\vec{x})\). Motivated by this viewpoint, Abraham et al. considered

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\(^2\)We typically use the term agent when there is no underlying game and player when there is.
two extensions of a Bayesian game $\Gamma$ (a game where the agents each have an input, or type, make a single move, and then get a payoff that depends on the type profile—the tuple consisting of each agent’s type—and move profile). The first is a game $\Gamma_d$ with a trusted mediator, where, after a communication phase in which the players can communicate with the mediator, they make a move in the underlying game $\Gamma$ and get the same payoffs as they would in $\Gamma$. The second is a communication game, denoted $\Gamma_{ACT}$, where there is no mediator, but players can communicate with each other before making a move in the underlying game.

Combining ideas from game theory and distributed computing, Abraham et al. were interested in what they called $(k, t)$-robust equilibria. These are strategy profiles (i.e., a strategy for each agent) where, roughly speaking, no coalition of $t$ players can decrease the payoff of any of the other players, and no coalition of $k$ players can increase their payoff by deviating. They showed, among other things, that if $n > 3(k + t)$ and there exists a $(k, t)$-robust equilibrium $\sigma + \sigma_d$ in the mediator game $\Gamma_d$ (where $\sigma$ represents the players’ strategies and $\sigma_d$ represents the mediator’s strategy), then there exists a $(k, t)$-robust equilibrium $\sigma_{ACT}$ in $\Gamma_{ACT}$ in a synchronous setting such that, for all inputs, $\sigma_{ACT}$ and $\sigma + \sigma_d$ produce the same distribution over outputs if no player deviates. They also proved a matching lower bound [Abraham, Dolev, and Halpern 2008].

Abraham, Dolev, Geffner, and Halpern [2019] (ADGH from now on) extended this result to the asynchronous setting. They showed that if $n > 4(k + t)$ and there exists a $(k, t)$-robust equilibrium $\sigma + \sigma_d$ in the mediator game $\Gamma_d$, then there exists a $(k, t)$-robust equilibrium $\sigma_{ACT}$ in $\Gamma_{ACT}$ in an asynchronous setting such that, for all inputs, $\sigma_{ACT}$ and $\sigma + \sigma_d$ produce the same set of possible distributions over outputs (note that agents have no control over how long the messages take to be delivered, and this can affect the output).

Our goal in this paper is to prove a lower bound that matches the upper bounds of ADGH. To do so, we would like to reduce implementing $(k + t)$-secure computation to implementing $(k, t)$-robust mediators. If such a reduction were possible, the $n > 4(k + t)$ lower bound for implementing $(k, t)$-robust mediators would follow immediately from the same lower bound for secure computation [Abraham, Dolev, and Stern 2020; Ben-Or, Kelmer, and Rabin 1994; Canetti 1996]. Unfortunately, there does not seem to be such a reduction. However, we show that there exists a nontrivial reduction from a slightly weaker notion of $(k + t)$-secure computation, which we call $(k + t)$-strict secure computation, to implementing $(k, t)$-robust mediators. We thus start by providing a careful proof of the lower bound for $(k + t)$-strict secure computation in the asynchronous setting. In the process, we also give a simple alternative proof for the lower bound on asynchronous secure computation.\footnote{As Ran Canetti [private communication] agreed, there is a nontrivial problem with the proof given in his thesis [Canetti 1996], a different technique is needed. We thank him for his comments.}

Intuitively, a protocol $t$-strictly securely computes a function $f$ if it satisfies the properties of secure computation but only for adversaries consisting of exactly $t$ malicious agents. It might seem that $t$-secure computation should be equivalent to $t$-strict secure computation. After all, if a function can be securely computed with adversaries of maximal size, surely it can be securely computed with smaller adversaries! As we show by example in Section 2.2.3 this is not the case. Intuitively, the problem is that an adversary consisting of fewer than $t$ agents may not be permitted to learn as much as an adversary consisting of $t$ agents. While investigating these issues, we noted an ambiguity in the definition of $t$-secure computation in BCG, which led us to consider yet another notion that we call $t$-weak secure computation. As the name suggests, it is weaker than $t$-secure computation; we show that it is actually equivalent to $t$-strict secure computation. By considering these variants of secure computation, we gain a deeper understanding of its subtleties.

2 Basic Definitions

2.1 The Asynchronous Model

The model used throughout this paper is the one used by ADGH [Abraham, Dolev, Geffner, and Halpern 2019], which consists of an asynchronous network in which there is a reliable, authenticated and asyn-
chronous channel between all pairs of players. This means that all messages sent by player \(i\) to player \(j\) are guaranteed to be delivered eventually, and that \(j\) can identify that these messages were sent by \(i\). However, that these messages may be delayed arbitrarily. The order in which these messages are received is decided by an adversarial entity called the scheduler. The scheduler also decides in which order the players are scheduled.

We define the local history \(h_i\) of player \(i\) to be the ordered sequence of local computations (including random coin tosses), messages sent and received (including senders and recipients), in addition to all the times in between in which \(i\) has been scheduled. Similarly, we define the local history \(h_T\) of a subset \(T\) of players as the collection of local histories \(h_i\) with \(i \in T\). Note that in the distributed computing literature, it is generally assumed that the players are scheduled automatically right after receiving a message. However, in this model we allow the scheduler to decide separately when the messages are delivered and when the players are scheduled. This means that whenever it is the turn of a player to act, that player may have received no messages since its last turn, or it may have received more than one (as opposed to exactly one). It is straightforward to check that all of our results also hold if we use the more standard model.

### 2.2 Secure Computation

For the main definitions in this section, we need the following notation, largely taken from BCG. Given a finite domain \(D\), let \(\vec{x}\) be a vector in \(D^n\). Given a set \(C \subseteq [n]\), denote by \(\vec{x}_C\) the vector obtained by projecting \(\vec{x}\) onto the indices of \(C\). Also, given a vector \(\vec{z} \in D^{|C|}\), let \(\vec{x}/(C,\vec{z})\) be the vector obtained by replacing the entries of \(\vec{x}\) indexed by \(C\) by the corresponding entries of \(\vec{z}\). To simplify notation, given a function \(f : D^n \rightarrow D\), we write \(f_C(\vec{x})\) rather than \(f(\vec{x}/(\sigma,\vec{z}))\) to denote the output of evaluating \(f\) on \(\vec{x}\) with the entries in \(\vec{x}\) not indexed by an element of \(C\) replaced by some default value \(x_0 \in D\).

Suppose that a group of \(n\) agents wants to compute the output of a function \(f : D^n \rightarrow D\), but the \(i\)th input \(x_i\) is known only by agent \(i\). A protocol securely computes \(f\) if (a) all agents correctly compute \(f\), regardless of the deviations of malicious players, and (b) malicious agents do not learn anything about the input of honest agents beyond what can be deduced from the output of \(f\). Before going on, we need to make precise what it means to correctly compute \(f\), since a malicious agent can lie about its input or not participate in the computation at all. Roughly speaking, the idea is to accept as correct any output of \(f\) that can be obtained from an input profile that differs from the actual input profile in at most \(t\) coordinates (intuitively, these coordinates are ones corresponding to inputs of malicious agents who did not submit a value or lied about their actual input.) More precisely, we have the following definition:

**Definition 1.** A protocol \(\pi\) t-securely computes \(f\) in synchronous systems if for every coalition \(T\) of at most \(t\) malicious agents and every strategy \(\vec{r}\) for players in \(T\), there exist functions \(h : D^{|T|} \rightarrow D^{|T|}\) and \(O : D^{|T|} \times D \times T \rightarrow \{0,1\}^*\) such that, for each input \(\vec{x}\),

(a) each agent \(i \notin T\) outputs \(f(\vec{x}/(T,h(\vec{x}_T)))\);

(b) each agent \(i \in T\) outputs \(O(\vec{x}_T,f(\vec{x}/(T,h(\vec{x}_T))),i)\).

Note that \(h\) and \(O\) encode how malicious agents might lie about their inputs (if a malicious agent does not participate in the computation, its input is assumed to be the default value \(x_0 \in D\)) and what they output, respectively. We thus consider an output to be correct if only the inputs of agents in \(T\) used in the computation of \(f\) differ from their actual inputs, and if the output of malicious agents output is just a function of the output of \(f\) and their own inputs. Note that this last requirement captures the fact that malicious agents do not learn anything besides the (honest agents’ ) output of the secure computation protocol, since otherwise they could use this extra information to generate outputs that cannot be written as such a function \(O\). Since malicious agents can randomize, we assume that both \(h\) and \(O\) have an extra input \(r\), a bitstring chosen uniformly at random from \(\{0,1\}^\omega\) (the set of all finite bitstrings), and that agent \(i\)'s output is distributed identically to \(f(\vec{x}/(T,h(\vec{x}_T)))\) or \(O(\vec{x}_T,f(\vec{x}/(T,h(\vec{x}_T))),i)\), depending on whether \(i\) is honest.
over outputs. BCG proved the following result.

\[ f \] and there exists a trusted-party adversary such that for all inputs, gives the same distribution for all inputs.

In the synchronous case, except that here we must take into account the subset of malicious agents, their strategy \( \tilde{\tau} \), and the scheduler’s strategy \( \sigma_e \). The existence of such adversaries implies that there are deviations that are possible in asynchronous systems that are not possible in synchronous systems; specifically, the scheduler can delay a subset of agents until the other agents terminate the protocol. If the number of agents delayed is less than the number of malicious agents that the protocol tolerates, delayed honest agents are indistinguishable from malicious agents that never engage in the communication, and thus the remaining agents must be able to terminate regardless of the delay. Since the inputs of delayed honest agents are not taken into consideration, the adversary can choose a set \( C \subseteq [n] \) of size at least \( n - t \) and force the computation to ignore the inputs of agents not in \( C \).

To define asynchronous secure computation, BCG introduced another type of adversary that they called a trusted-party adversary. A \( t \)-trusted-party adversary is defined as a quadruple \( A = (T, h, c, O) \) where:

- \( T \) is the set of malicious agents;
- \( h : D^{|T|} \times \{0, 1\}^\omega \rightarrow D^{|T|} \) is the input substitution function;
- \( c : D^{|T|} \times \{0, 1\}^\omega \rightarrow \{C \subseteq [n] | |C| \geq n - t\} \) is a subset of agents (intuitively, the ones whose inputs are taken into consideration);
- \( O : D^{|T|} \times \{0, 1\}^\omega \times D \times T \rightarrow \{0, 1\}^* \) is the output function for the malicious agents.

In the sequel, we use “trusted-party adversary" to refer to such a tuple \((T, h, c, O)\), and reserve the term adversary for a tuple of the form \((A, \tilde{T}, \sigma_e)\), as defined earlier.

Given a function \( f : D^n \rightarrow D \), a trusted-party adversary \( A = (T, h, c, O) \), and an input vector \( \bar{x} \), let \( C = c(\tilde{T}, r) \) and \( \bar{y} = \bar{x}/(T, h(\tilde{T}, r)) \). Intuitively, \( C \) is the set of agents whose inputs are considered and \( \bar{y} \) is the input profile obtained by replacing the actual inputs of agents in \( T \) with the output of \( h \). The output of \( f \) with trusted-party adversary \( A \) and input \( \bar{x} \) is an \( n \)-vector of random variables \( \rho(A, \bar{x}; f) \) such that:

\[
\rho_i(A, \bar{x}; f) = \begin{cases} (C, f_C(\bar{y})) & \text{if } i \notin T \\ O(\bar{x}_B, r, f_C(\bar{y}), i) & \text{if } i \in T. \end{cases}
\]

Note that the outputs of trusted-party adversaries are analogous to the outputs of secure computation in the synchronous case, except that here we must take into account the subset \( C \) of agents that provide their inputs. In asynchronous systems, secure computation is defined as follows:

**Definition 2 (Secure computation).** Let \( f : D^n \rightarrow D \) be a function of \( n \) variables over some finite domain \( D \). The protocol \( \bar{\sigma} \) \( t \)-securely computes \( f \) in an asynchronous setting if the following hold for all (standard) adversaries \( A = (T, \tilde{T}, \sigma_e) \) with \( |T| \leq t \):

- on all inputs, agents not in \( T \) terminate the protocol with probability 1;
- there exists a \( t \)-trusted-party adversary \( A^{tr} = (T, h, c, O) \) such that, for all inputs \( \bar{x} \in D^n \), we have \( \bar{\sigma}(\bar{x}, A) \sim \rho(A^{tr}, \bar{x}; f) \) (i.e., \( \bar{\sigma}(\bar{x}, A) \) and \( \rho(A^{tr}, \bar{x}) \) are identically distributed).

In other words, a protocol \( \bar{\sigma} \) \( t \)-securely computes some function \( f \) if it terminates with probability 1 and there exists a trusted-party adversary such that, for all inputs, gives the same distribution over outputs. BCG proved the following result.
Theorem 2. [Ben-Or, Canetti, and Goldreich 1993] If $D$ is a finite domain, $n > 4t$, and $f : D^n \to D$, then there exists a protocol $\pi$ that $t$-securely computes $f$ in asynchronous systems.

2.3 Weaker Notions of Secure Computation

Note that the $T$ in the second condition of Definition[2] that is, the $T$ in the trusted-party adversary $(T, h, c, O)$, is the same as the $T$ in the adversary. This is also true in the BGW definition of $t$-secure computation. While we believe that this was also the intention of BCG, their definition simply says that there exists a $t$-trusted-party adversary, without specifying $T$ (the set of malicious agents) that satisfies the second bullet of Definition[2]. Taking this definition seriously leads to a slightly weaker notion of secure computation that we call $t$-weak secure computation, which is defined just as $t$-secure computation except that the $t$-trusted-party adversary $A^\prime$ may involve any subset $T'$ of malicious agents such that $|T'| = t$ and $T' \supseteq T$, as opposed to consisting of the same set $T$ of malicious agents as $A$.

We show next that $t$-weak secure computation is strictly weaker than the standard notion of secure computation. To do so, first we introduce an intermediate notion of secure computation called $t$-strict secure computation; it is defined just as $t$-secure computation, except that we require only that the properties are satisfied for adversaries of size exactly $t$ (i.e., for $|T| = t$). As we mentioned in the introduction, somewhat surprisingly, $t$-strict secure computation is strictly weaker than $t$-secure computation, but, as we show next, it is actually equivalent to $t$-weak secure computation.

Theorem 3.

(a) If a protocol $\pi$ $t$-securely computes a function $f$, it also $t$-strictly securely computes $f$.

(b) A protocol $\pi$ $t$-strictly securely computes a function $f$ if and only if it $t$-weakly securely computes $f$.

(c) If $t > 1$ and $n > 4t$, there exists a function $f$ on $n$ variables and a protocol $\pi$ such that $\pi$ $t$-strictly securely computes $f$ but does not $t$-weakly securely computes $f$.

Proof. Part (a) follows immediately from the definition of secure computation and strict secure computation. For part (b), first suppose that a protocol $\pi$ $t$-strictly securely computes $f : D^n \to D^m$. Given an adversary $A = (T, \pi_T, \sigma_e)$ with $|T| \leq t$, consider an adversary of the form $A' = (T \cap T', \pi_T + \pi_{T'}, \sigma_e)$ (Since the agents in $T'$ play $\pi$, they in fact do not deviate.) Because $\pi$ such that $T \cap T' = \emptyset$ and $|T \cup T'| = t$. Since $\pi$ $t$-strictly securely computes $f$, there exists a trusted-party adversary $A^{tr} = (T \cup T', h, c, O)$ such that $\pi(x, A') = \overline{\rho}(A^{tr}, x; f)$. By construction, $\pi(x, A') \sim \pi(x, A)$, since the additional malicious agents in $A'$ do not deviate from the protocol. Therefore, $\pi(x, A) \sim \overline{\rho}(A^{tr}, x; f)$, so $\pi$ $t$-weakly securely computes $f$.

The converse is almost immediate from the definitions. Suppose that protocol $\pi$ $t$-weakly securely computes $f : D^n \to D^n$ for some $t$. Given an adversary $A = (T, \pi_T, \sigma_e)$ with $|T| = t$, then, by assumption, there is a $t$-trusted-party adversary $A' = (T, h, c, O)$ such that $\pi(x, A') = \pi(x, A)$.

For part (c), consider the following setup. Let $\mathbb{F}_2$ be the field with domain $\{0, 1\}$. Given $n$ and $t$ such that $t > 1$ and $n > 4t$, consider a function $f : (\mathbb{F}_2)^n \to (\mathbb{F}_2)^n$ that does the following. Given the input $(x^i, c^i, y^i, z^i) \in (\mathbb{F}_2)^n$ of each agent $i$, where $x^i \in \mathbb{F}_2^n$, $c^i \in (\mathbb{F}_2)^n$, $y^i \in (\mathbb{F}_2)^{t-1}$, and $z^i$ consists of the remaining $n^2 - n - t$ coordinates (which do not affect the function $f$; they are needed because in Definition[2] the input space of each agent must be the same as the output space of the function), let $p_i \in \mathbb{F}_2[X]$ be the unique polynomial of degree $t-1$ such that $p_i(0) = x^i$ and $p_i(j) = y^i_j$ for all $j = 1, 2, \ldots, t-1$. The output of $f$ is then $\{p_i(j) + c^i_j\}_{1 \leq i \leq n}$. In other words, $f$ encodes the first coordinate of each agent’s input using Shamir’s agent secret sharing scheme [Shamir 1979]. The polynomial $p_i$ that each agent $i$ uses to do the encoding and the one-time pads $c^i_j$ added by $i$ to each of the shares are part of $i$’s input, and not known by the other agents. However, a coalition $T$ of $t$ malicious agents can reconstruct the values $p_i(j)$ for all $i \in [n]$ and $j \in T$, and thus is able to reconstruct each $x^i$, as well, since the agents in $T$ know $t$ points on each polynomial $p_i$, although no coalition of size strictly smaller than $t$ knows those values.
Consider a protocol $\bar{\pi}$ that consists of the following: each agent $i$ performs its part of BCG’s $t$-secure computation protocol to compute $f$ and then, if $i$ is included in the core set of the output, $i$ broadcasts the first bit of its input. By the earlier argument, if the adversary is of size exactly $t$, it can reconstruct the first coordinate of the inputs of the agents in the core-set from the output of $f$ and its own inputs, which means that the values broadcast after BCG’s secure computation protocol do not give any extra information about the inputs of honest agents to the adversary. However, this is not true for smaller adversaries. Thus, $\bar{\pi}$ $t$-strictly securely computes $f$, but does not $t$-securely compute $f$. \hfill \Box

2.4 Implementing mediators

We now formalize the notion of $(k, t)$-robust equilibrium. Recall that in this setting, there are three games, an underlying game $\Gamma$ for $n$ players, which is technically a Bayesian game, a mediator game $\Gamma_d$, and a communication game $\Gamma_{ACT}$. In a Bayesian game, players have inputs, and their payoff depends on the profile of moves made and the input profile. The set of players is the same in all three games, except that in the mediator game, there is also a mediator, who can be viewed as a special non-strategic player (i.e., there is no utility function for the mediator) and uses a commonly-known strategy, denoted $\sigma_d$. In the mediator game, the players just communicate with the mediator (although deviating or malicious players are allowed to communicate with each other). In the communication game, they communicate among themselves using a point-to-point network. After communicating in the mediator game and the communication game, the players make a move in the underlying game $\Gamma$, and get payoffs as in $\Gamma$. As is standard, we use $\bar{\sigma} := (\sigma_1, \ldots, \sigma_n)$ to denote a strategy profile for $n$ players in which each player $i$ plays $\sigma_i$; we use $\bar{\sigma} + \sigma_d$ to denote the strategy profile for $n$ players and a mediator in which each player $i$ plays $\sigma_i$ and the mediator plays $\sigma_d$; finally, we use $(\sigma_{-T}, \tau_T)$ to denote the strategy where each player $i \notin T$ uses the strategy $\sigma_i$ while $j \in T$ uses the strategy $\tau_j$.

In this game-theoretic setting, we are interested in protocols that are $(k, t)$-robust. To define $(k, t)$-robustness, we need two preliminary definitions.

Definition 3. Given a game $\Gamma$, a strategy profile $\bar{\sigma}$ is $t$-immune if for all subsets $T$ of size at most $t$ and all strategies $\bar{\tau}_T$ for players in $T$ \[ u_i(\bar{\sigma}_{-T}, \bar{\tau}_T) \geq u_i(\bar{\sigma}) \text{ for all } i \notin T, \] where $u_i(\bar{\sigma})$ is the payoff of player $i$ when players play $\bar{\sigma}$.

Intuitively, a strategy profile is a $t$-immune equilibrium if no subset of at most $t$ players can decrease the payoff of other players by deviating.

Definition 4. A strategy profile $\bar{\sigma}$ is a $(k, t)$-resilient (resp., strongly $(k, t)$-resilient) equilibrium of a game $\Gamma$ if, for all disjoint subsets $K$ and $T$ of sizes at most $k$ and $t$, respectively, and all strategy profiles $\bar{\tau}_{K \cup T}$ for players in $K \cup T$, \[ u_i(\bar{\sigma}_{-(K \cup T)}, \bar{\tau}_{K \cup T}) \leq u_i(\bar{\sigma}_{-T}, \bar{\tau}_T) \text{ for some (resp., for all) } i \in K. \]

Intuitively, a strategy protocol is a $(k, t)$-resilient if no subset of at most $k$ players can all increase their payoffs, even if they can collude with up to $t$ malicious players. It is a strong $(k, t)$-resilient equilibrium if not even one player in the set can increase its payoff.

Definition 5. A strategy profile is a $(k, t)$-robust (resp., strongly $(k, t)$-robust) equilibrium in a game $\Gamma$ if it is $t$-immune and a $(k, t)$-resilient (resp., strongly $(k, t)$-resilient) equilibrium.

The notion of $(k, t)$-robustness was introduced by Abraham, Dolev, Gonen and Halpern [2006], who also proved the following:

Theorem 4. [Abraham, Dolev, Gonen, and Halpern 2006] If $\bar{\sigma} + \sigma_d$ is a $(k, t)$-robust equilibrium for a synchronous game $\Gamma_d$ that extends some game $\Gamma$ and $n > 3(k + t)$, then there exists a $(k, t)$-robust equilibrium $\bar{\sigma}_{ACT}$ for $\Gamma_{ACT}$ such that, for all input profiles, the distribution over outcomes induced by $\bar{\sigma} + \sigma_d$ is identical to that induced by $\bar{\sigma}_{ACT}$.

ADGH proved an analogous result for asynchronous systems. Making the statement precise required a little care since, even for a fixed input, the output distribution induced by a protocol depends on the scheduler. This observation motivates the following definition.
Definition 6. Protocol $\sigma$ implements protocol $\tau$ in an asynchronous network if, for all input profiles $\vec{x}$ and all schedulers $\sigma_e$, there exists a scheduler $\sigma'_e$ such that the distribution over output profiles induced by $\sigma$ with input profile $\vec{x}$ and scheduler $\sigma_e$ is identical to the distribution over output profiles induced by $\tau$ with input profile $\vec{x}$ and scheduler $\sigma'_e$.

Essentially, this definition says that $\sigma$ implements $\tau$ if, for all input profiles $\vec{x}$, the set of possible output distributions of $\sigma$ is the same as that of $\tau$.

Theorem 5. [Abraham, Dolev, Geffner, and Halpern 2019] If $\sigma + \sigma_d$ is a $(k, t)$-robust strategy for an game $\Gamma_d$ that extends some game $\Gamma$ and $n > 4(\kappa + t)$, then there exists a $(k, t)$-robust protocol $\sigma_{ACT}$ for $\Gamma_{ACT}$ that implements $\sigma + \sigma_d$.

It is easy to $t$-securely compute a function $f$ with the help of a mediator: Each player sends its input to the mediator, the mediator waits until it receives an input from at least $n - t$ agents (in synchronous systems it just waits one round), then it computes the output of $f$ given the input of the players, and sends it to all players. However, despite the fact that we think of $t$-secure computation in terms of mediators, it is not obvious that Theorem 5 follows from Theorem 2 due to the differences between the definitions of $(k, t)$-robustness and secure computation. At the end of Section 6, we sketch how to reduce $t$-secure computation to implementing $(t, 0)$-robust strategies in mediator games.

3 Main Results

In this paper we show that the bound in Theorem 5 is tight:

Theorem 6. If $k + t + 1 < n \leq 4k + 4t$ there exists a $(k, t)$-robust (resp., strongly $(k, t)$-robust) strategy profile $\sigma + \sigma_d$ for $n$ players and a mediator such that there is no $(k, t)$-robust (resp., strongly $(k, t)$-robust) strategy profile $\sigma_{ACT}$ that implements $\sigma + \sigma_d$.

The proof of Theorem 6 is divided in two parts.

3.1 Case 1: $3k + 3t \leq n \leq 4k + 4t$

Here, we show that that $(k + t)$-strictly securely computing a function $f$ reduces to implementing a $(k, t)$-robust strategy $\sigma + \sigma_d$ for some game $\Gamma^{f,k,t}$. To make this precise, we need the following definition:

Definition 7. If $g : A \rightarrow B$, and $\sigma$ is a strategy that plays actions in $A$, then $g(\sigma)$ is the strategy that is identical to $\sigma$ except that each action $a \in A$ is replaced by $g(a) \in B$. If $\sigma$ is a strategy profile where each player $i$ plays actions in $A$, then $g(\sigma) = (g(\sigma_1), \ldots, g(\sigma_n))$.

Theorem 7. If $f : D^n \rightarrow D$, $D$ is a finite domain, and $2(k + t) < n$, then there exists a game $\Gamma_d^{f,k,t}$ in which all players have the same set $A$ of possible actions, a function $g : A \rightarrow D$, and a $(k, t)$-robust strategy (resp., strongly $(k, t)$-robust strategy) $\sigma + \sigma_d$ for $n$ players and the mediator in $\Gamma_d^{f,k,t}$ such that if $\sigma_{ACT}$ is $(k, t)$-robust implementation (resp., strongly $(k, t)$-robust implementation) of $\sigma + \sigma_d$, then $g(\sigma_{ACT})$ $(k, t)$-strictly securely computes $f$.

Theorem 7 shows that being able to implement all $(k, t)$-robust mediators with $n$ players implies that all functions on $n$ variables can be $(k + t)$-strictly securely computed. The proof of Theorem 7 for $3(k + t) \leq n \leq 4(k + t)$ follows from the fact that if $3(k + t) \leq n \leq 4(k + t)$ there exist functions that cannot be $(k + t)$-weakly securely computed:

Theorem 8.

(a) If $n > 4t$ or $n \leq 2t$, every function $f : D^n \rightarrow D$ can be $t$-weakly securely computed in asynchronous systems.
(b) If \( 3t \leq n \leq 4t \), there exists a domain \( D \) and a function \( f: D^n \to D \) that cannot be \( t \)-weakly securely computed in asynchronous systems.

The proof of Theorem 9 is given in Section [4]. A slight variation of the proof provides a simple proof for the well-known lower bound for secure computation on asynchronous systems:

**Theorem 9.**

(a) If \( n > 4t \) or \( n \leq t \), for all domains \( D \), every function \( f: D^n \to D \) can be \( t \)-securely computed in asynchronous systems.

(b) If \( t < n \leq 4t \) there exists a domain \( D \) and a function \( f: D^n \to D \) that cannot be \( t \)-securely computed in asynchronous systems.

### 3.2 Case 2: \( k + t + 1 < n \leq 3k + 3t \)

If \( k + t + 1 < n \leq 3k + 3t \) we show that implementing \((k,t)\)-resilient weak consensus with \( n \) players can be reduced to implementing \((k,t)\)-robust mediators:

**Theorem 10.** If \( n > k + t + 1 \), then there exists a game \( 1^{k,t}_d \) in which all players have the same set \( A \) of possible actions, a function \( g: A \to \{0,1\} \), and a \((k,t)\)-robust (resp., strongly \((k,t)\)-robust) strategy \( \vec{\sigma} + \sigma_d \) for \( n \) players and the mediator in \( 1^{k,t}_d \) such that if \( \vec{\sigma}_{ACT} \) is a \((k,t)\)-robust (resp., strongly \((k,t)\)-robust) implementation of \( \vec{\sigma} + \sigma_d \), then \( g(\vec{\sigma}_{ACT}) \) is a \((k,t)\)-resilient implementation of weak consensus.

The proof of Theorem 10 for \( k + t + 1 \leq n \leq 3(k + t) \) follows from Lamport’s lower bound for weak consensus [2].

### 4 Proof of Theorems 12 and 11

In this section we prove the \( n > 4t \) lower bound for \( t \)-secure computation (also proved or claimed in [Abraham, Dolev, and Stern 2020] [Ben-Or, Kelmer, and Rabin 1994] [Canetti 1996]), which we strengthen slightly by showing that it applies to \( t \)-weak secure computation.

Our proof is similar to that of Canetti [1996] at a high level: We construct a function \( f \) with four inputs, the scheduler schedules the agents so that the fourth agent never gets to participate in the computation, and one of the three remaining agents is malicious and manages to trick the other two participating agents into outputting something inappropriate. Canetti then claims that conversations between agents (where a conversation is just the collection of messages sent by two given agents) must be independent of the inputs of the agents, agent 3 can send messages to agents 1 and 2 in such a way that agents 1 and 2 believe they should output different values. However, there are two significant problems with this approach:

(a) First, the conversations between the agents might not be totally independent of their inputs, since they can depend on the output of the computation, and this ultimately does depend on the inputs. For example, agents can run Bracha’s [1984] consensus protocol (which tolerates \( t \) malicious agents if \( n > 3t \)) after terminating the secure computation protocol to decide the output. This would guarantee that all honest agents output the same value at the end of the computation, so their conversations are certainly not independent.

(b) Second, there is a more subtle issue when trying to simultaneously trick agents 1 and 2 into outputting some given values \( a \) and \( b \), respectively. Even though Canetti proves that for the function \( f \) that he uses and a particular input \( \vec{x} \), for each conversation \( h_{1,2} \) between 1 and 2, there is a protocol for player 3 that results in a conversation \( h_{1,3} \) between 1 and 3 such that 1 outputs \( a \), and that for each conversation \( h_{1,2} \) between 1 and 2 there exists a protocol for player 3 that results in a conversation \( h_{2,3} \) between 2 and 3 such that 2 outputs \( b \), there might not exist a protocol for agent 3 that results in 1 and 2 having conversation \( h_{1,3} \) and agents 2 and 3 having conversation \( h_{2,3} \) simultaneously. In fact, if \( a \) and \( b \) are different and agents run a consensus protocol as in (a), there is not.
Roughly speaking, we deal with these issues as follows. We prove that for our function $f$, a malicious agent can make all honest agents output the same incorrect value, and we show that in our case there does exist a conversation $h_{1,2}$ between 1 and 2 such that agent 3 can trick both of them simultaneously, as desired (see Lemma 1). Some of these techniques can also be applied to prove lower bounds for weak secure computation.

## END

In this section we prove the $n > 4t$ lower bound for $t$-secure computation (also proved or claimed in [Abraham, Dolev, and Stern 2020; Ben-Or, Kelmer, and Rabin 1994; Canetti 1996]), which we strengthen slightly by showing that it applies to $t$-weak secure computation.

Our proof is similar to that of Canetti [1996] at a high level: We construct a function $f$ with four inputs, the scheduler schedules the agents so that the fourth agent never gets to participate in the computation, and one of the three remaining agents is malicious and manages to trick the other two participating agents into outputting something inappropriate. Canetti then claims that conversations between agents (where a conversation is just the collection of messages sent by two given agents) must be independent of the inputs of the agents, agent 3 can send messages to agents 1 and 2 in such a way that agents 1 and 2 believe they should output different values. However, there are two significant problems with this approach:

(a) First, the conversations between the agents might not be totally independent of their inputs, since they can depend on the output of the computation, and this ultimately does depend on the inputs. For example, agents can run Bracha’s [1984] consensus protocol (which tolerates $t$ malicious agents if $n > 3t$) after terminating the secure computation protocol to decide the output. This would guarantee that all honest agents output the same value at the end of the computation, so their conversations are certainly not independent.

(b) Second, there is a more subtle issue when trying to simultaneously trick agents 1 and 2 into outputting some given values $a$ and $b$, respectively. Even though Canetti proves that for the function $f$ that he uses and a particular input $x$, for each conversation $h_{1,2}$ between 1 and 2, there is a protocol for player 3 that results in a conversation $h_{1,3}$ between 1 and 3 such that 1 outputs $a$, and that for each conversation $h_{1,2}$ between 1 and 2 there exists a protocol for player 3 that results in a conversation $h_{2,3}$ between 2 and 3 such that 2 outputs $b$, there might not exist a protocol for agent 3 that results in 1 and 2 having conversation $h_{1,3}$ and agents 2 and 3 having conversation $h_{2,3}$ simultaneously. In fact, if $a$ and $b$ are different and agents run a consensus protocol as in (a), there is not.

Roughly speaking, we deal with these issues as follows. We prove that for our function $f$, a malicious agent can make all honest agents output the same incorrect value, and we show that in our case there does exist a conversation $h_{1,2}$ between 1 and 2 such that agent 3 can trick both of them simultaneously, as desired (see Lemma 1). Some of these techniques can also be applied to prove lower bounds for weak secure computation.

**Theorem 11.**

(a) If $n > 4t$ or $n \leq t$, for all domains $D$, every function $f : D^n \to D$ can be $t$-securely computed in asynchronous systems.

(b) If $t < n \leq 4t$ there exists a domain $D$ and a function $f : D^n \to D$ that cannot be $t$-securely computed in asynchronous systems.

**Theorem 12.**

(a) If $n > 4t$ or $n \leq 2t$, every function $f : D^n \to D$ can be $t$-weakly securely computed in asynchronous systems.

(b) If $3t \leq n \leq 4t$, there exists a domain $D$ and a function $f : D^n \to D$ that cannot be $t$-weakly securely computed in asynchronous systems.
To prove part (b) of Theorems 11 and 12 for $3t \leq n \leq 4t$ we show that the majority function $f^+ : \{0, 1, \bot\}^4 \rightarrow \{0, 1, \bot\}$ that outputs 1 if there are at least as many inputs equal to 1 as inputs equal to 0, and outputs 0 otherwise, cannot be 1-weakly securely computed. In fact, we show that player 3 can get players 1 and 2 to output 1 even when all agents have input 0. The full proof can be found in Appendix A.

6 Reducing $(k+t)$-Strict Secure Computation to Implementing a $(k,t)$-Robust Equilibrium

In this section we show that $(k+t)$-strictly securely computing a function $f$ reduces to implementing a $(k,t)$-robust strategy $\vec{\sigma} + \sigma_d$ for some game $\Gamma_{d}^{f,k,t}$. To make this precise, we need the following definition:

**Definition 8.** If $g : A \rightarrow B$, and $\sigma$ is a strategy that plays actions in $A$, then $g(\sigma)$ is the strategy that is identical to $\sigma$ except that each action $a \in A$ is replaced by $g(a) \in B$. If $\vec{\sigma}$ is a strategy profile where each player $i$ plays actions in $A$, then $g(\vec{\sigma}) = (g(\sigma_1), \ldots, g(\sigma_n))$.

**Theorem 13.** If $f : D^n \rightarrow D$, $D$ is a finite domain, and $2(k+t) < n$, then there exists a game $\Gamma_{d}^{f,k,t}$ in which all players have the same set $A$ of possible actions, a function $g : A \rightarrow D$, and a $(k,t)$-robust strategy (resp., strongly $(k,t)$-robust strategy) $\vec{\sigma} + \sigma_d$ for $n$ players and the mediator in $\Gamma_{d}^{f,k,t}$ such that if $\vec{\sigma}_{ACT}$ is a $(k,t)$-robust implementation (resp., strongly $(k,t)$-robust implementation) of $\vec{\sigma} + \sigma_d$, then $g(\vec{\sigma}_{ACT})$ $(k+t)$-strictly securely computes $f$.

The proof of Theorem 13 is surprisingly nontrivial. Given a function $f$, it is easy to check that there is a $(k,t)$-robust strategy $\vec{\sigma} + \sigma_d$ with a mediator that $(k+t)$-securely computes $f$, where we assume that actions in the mediator game have the form $(a,tp)$, where $a$ is a possible output of $f$ and $tp$ is the player’s self-declared type (i.e., whether the player is honest, rational, or malicious), honest players get a payoff of 1 if they all agree on a valid output of $f$ and 0 otherwise, and self-declared rational players get a payoff of 1 if they disrupt the output of honest players: players send their inputs to the mediator, the mediator waits to receive $n-k-t$ inputs, sends to every player the output of the computation (taking the remaining $k+t$ inputs to be $\bot$), and then the players play the output received. We would expect that any $(k,t)$-robust implementation of $\vec{\sigma} + \sigma_d$ also $(k+t)$-securely computes $f$ (without the mediator), but this is not the case. For example, if players perform a secure computation of $f$ and, right after that, they broadcast their inputs, the resulting protocol would still be a $(k,t)$-robust implementation of $\vec{\sigma} + \sigma_d$. However, using this strategy, all rational and malicious players would learn the honest players outputs. This example shows that the game must somehow reward players that declare themselves to be rational if they manage to learn something that they shouldn’t. A more detailed discussion of this issue, our solution, and a full proof of Theorem 13 can be found in Appendix B. As an immediate corollary of Theorems 12 and 13 we get the desired lower bound for implementing mediators.

**Corollary 1.** If $3k + 3t \leq n \leq 4k + 4t$ there exists a $(k,t)$-robust (resp., strongly robust) protocol $\vec{\sigma} + \sigma_d$ for $n$ players and a mediator such that there is no $(k,t)$-robust (resp., strongly robust) protocol $\vec{\sigma}_{ACT}$ that implements $\vec{\sigma} + \sigma_d$.

7 The Lower Bound on Implementing Mediators

In the previous section, we showed that $(t+k)$-strict secure computation can be reduced to implementing certain $(k,t)$-robust (or strongly robust) strategies, and thus that if $3(t+k) \leq n \leq 4(t+k)$, then there exist $(k,t)$-robust (resp., strongly robust) strategies with a mediator that cannot be implemented without a mediator. In this section, we use a different construction to extend this impossibility result to $t+k+1 \leq n \leq 3k+3t$. That is, we have the following strengthening of Corollary 1.
**Theorem 14.** If \( k + t + 1 < n \leq 4k + 4t \) there exists a \((k, t)\)-robust (resp., strongly \((k, t)\)-robust) strategy profile \( \bar{\sigma} + \sigma_d \) for \( n \) players and a mediator such that there is no \((k, t)\)-robust (resp., strongly \((k, t)\)-robust) strategy profile \( \sigma_{ACT} \) that implements \( \bar{\sigma} + \sigma_d \).

Corollary 1 shows that Theorem 14 holds if \( 3k + 3t \leq n \leq 4k + 4t \). We prove the remaining cases by reducing weak consensus to implementing mediators, much like as we did in the previous section for secure computation.

**Theorem 15.** If \( n > k + t + 1 \), then there exists a game \( \Gamma^{k,t}_d \) in which all players have the same set \( A \) of possible actions, a function \( g : A \to \{0, 1\} \), and a strongly \((k, t)\)-robust strategy \( \bar{\sigma} + \sigma_d \) for \( n \) players and the mediator in \( \Gamma^{k,t}_d \) such that if \( \bar{\sigma}_{ACT} \) is a strongly \((k, t)\)-robust implementation of \( \bar{\sigma} + \sigma_d \), then \( g(\bar{\sigma}_{ACT}) \) is a \((k + t)\)-resilient implementation of weak consensus.

The proof of Theorem 15 can be found in Appendix C.

Since, as proved by Lamport [1983], there are no \((t + k)\)-resilient implementations of weak consensus if \( n \leq 3(t + k) \), it follows from Theorem 15 that Theorem 14 holds for \( t + k + 1 < n \leq 3k + 3t \) as well, completing its proof.

## 8 Conclusion

We have shown that both \((k+t)\)-secure computation and the problem of implementing a \((k, t)\)-robust equilibrium with a mediator have a lower bound of \( n > 4k + 4t \). Moreover, we have shown that this is also a lower bound for weaker notions of secure computation such as \((k + t)\)-strict secure computation and \((k + t)\)-weak secure computation. Finally, by considering a number of variants of the definition of secure computation, we also highlighted some of the subtleties in the definition.

ADGH showed that protocols can tolerate more malicious behavior if honest players can punish rational players if they are caught deviating. Honest players can perform this punishment by playing an action profile that results in all players getting an expected payoff that is worse than their payoff in equilibrium. Not all games have such a punishment profile, but ADGH showed that for games that do, every \((k, t)\)-robust strategy with a mediator can be implemented if \( n > 3k + 4t \). Finding a matching lower bound for this case remains an open problem.
A Proof of Theorems 11 and 12

For Theorem 11(a), note that if \( n > 4t \), Theorem 2 shows that every function \( f : D^n \rightarrow D \) can be \( t \)-securely computed, and thus \( t \)-weakly securely computed as well. If \( n \leq t \), let \( \perp \) be the input assigned to the agents that did not submit an input. It can be easily shown that the protocol where each agent sends no messages and outputs \( (\perp, f(\perp^n)) \) \( t \)-securely computes \( f \). Similarly, for Theorem 12(a), it can be easily checked that if \( n \leq 2t \), the protocol where each agent sends nothing and outputs \( (n - t), f(\perp^n) \) \( t \)-weakly securely computes \( f \).

For the lower bounds (Theorems 11(b) and 12(b)), we proceed as follows. Consider the function \( f^n : \{0, 1, \perp\}^n \rightarrow \{0, 1, \perp\} \) that essentially takes majority between 0 and 1: it outputs 1 if the number of agents with input 1 is greater or equal to the number of agents with input 0, otherwise it outputs 0. Players who do not submit an input are assumed to have input \( \perp \). We start by showing that \( f^n \) cannot be 1-weakly securely computed by four agents.

Suppose that \( f^4 \) can be 1-weakly securely computed using a protocol \( \overline{\sigma} \). Let \( \sigma^4_n \) be the scheduler that schedules agents 1, 2, and 3 cyclically, and right before scheduling an agent, it delivers the messages that were sent by the other agents the last time they were scheduled. After scheduling each of the first three agents \( N \) times, it schedules agent 4 as well, adding it to the cyclic order.

Given a history \( H \), let \( \overline{x}_H \) denote the input profile of agents in \( H \), let \( H_i \) denote agent \( i \)'s local history in \( H \), let \( H_e \) denote the scheduler’s local history in \( H \), and let \( H_{(i,j)} \) denote the conversation between agents \( i \) and \( j \) (i.e., the messages sent and received between \( i \) and \( j \), in addition to the relative times at which \( i \) and \( j \) were scheduled). We can now prove essentially what BCG claimed to prove (albeit, as we said, this claim does not hold for the BCG construction).

**Lemma 1.** There exist \( N \) and two (finite) histories \( H \) and \( H' \) of \( \overline{\sigma} \) where the scheduler uses \( \sigma^4_N \), \( \overline{x}_H = (1, 0, 1, 1) \), \( \overline{x}_{H'} = (0, 1, 1, 1) \), agents 1, 2, and 3 all output 1 in \( H \), agent 4 is never scheduled in either \( H \) or \( H' \), \( H_{1,2} = H_{1,2}' \), and \( H_e = H'_e \).

To prove Lemma 1 we first need to prove what seems to be obvious: if all agents are honest, at most \( t \) agents have input 0, and \( n \geq 3t \), then the output of a weakly secure computation of \( f^n \) will be 1. While this seems obvious (and true), it is not quite so trivial. For example, it is not true if \( n = 3t - 1 \). In this case, if we consider a trusted-party adversary \( A = (T, c, h, O) \), in which \( |T| = t \), \( h \) replaces all inputs of malicious players with 0, and \( c \) chooses all \( t \) additional honest players and \( t - 1 \) additional honest players, it is easy to check that the output of honest players is 0.

**Lemma 2.** Let \( n \geq 3t \) and let \( \overline{\sigma} \) be a protocol that \( t \)-weakly securely computes \( f^n \). Then for all schedulers, in all histories of \( \overline{\sigma} \) in which all agents are honest and at most \( t \) agents have input 0, all agents output 1.

**Proof.** Let \( S \) be the subset of agents that have input 0. Given a scheduler \( \sigma^e \), consider an adversary \( A = (T, \overline{\tau}, \sigma^e) \) such that \( T \supseteq S \), \( |T| = t \), and \( \overline{\tau}_T = \overline{\tau}_T \) (so all the malicious agents follow protocol \( \overline{\sigma} \)). By definition of \( t \)-weak secure computation, the output of honest agents with adversary \( A \) should be one that is possible with a trusted-party adversary of the form \( A' = (T, c, h, O) \). However, no matter what the output \( C \) of \( c \) is, since \( |C| \geq n - t \), there will be at least \( n - 2t \) honest agents in \( C \), all of them with input 1. Since \( n \geq 3t \), this suffices to guarantee that all players not in \( T' \) output 1.

Since malicious agents play \( \overline{\sigma} \), they are indistinguishable from honest agents. Thus, if all agents are honest, all agents not in \( T \) output 1. To see that agents in \( T \) also output 1 if all players are honest, consider an adversary \( A = (T, \overline{\tau}, \sigma^e) \) such that \( T \cap T' = \emptyset \), \( |T'| = t \) (such a set \( T' \) always exists since \( n \geq 3t \)), and \( \overline{\tau}_T = \overline{\tau}_T \). Since honest agents not in \( T \cup T' \) (note that \( [n] \setminus (T \cup T') \neq \emptyset \)) have the same histories with \( A \) and \( A' \), they must output the same value with both adversaries, and so must output 1 with adversary \( A' \). By definition of \( t \)-weak secure computation, since \( |T'| = t \), all agents not in \( T' \) must output the same value. Thus, since \( T \cap T' = \emptyset \), all agents in \( T \) also output 1 with adversary \( A' \). Again, since agents in \( T' \) are indistinguishable from honest agents, this implies that agents in \( T \) also output 1 if all agents are honest.

**Proof of Lemma 1.** By Lemma 2 there exists an integer \( N \) such that if agents 1, 2, and 3 are honest, with nonzero probability, they will output 1 with scheduler \( \sigma^N \) at or before the \( N \)th time they are
scheduled. Let $H$ be a history where the agents use $\vec{\sigma}$, the scheduler uses $\sigma^N_e$, the input is $(1, 0, 1, 1)$, agents 1, 2, and 3 are honest and have been scheduled at most $N$ times and all three have outputted 1. By the properties of secure computation, in particular, the secrecy of the inputs, there must exist a history $H''$ such that $\vec{x}_{H''} = (1, 1, 1, 1)$, $H''_1 = H_1$, and $H''_2 = H_e$. (Note that this means that we can assume, without loss of generality, that the scheduler uses strategy $\sigma^N_e$.) If this were not the case and agent 1 were malicious in $H''$, then it would know that the input profile can’t be $(1, 1, 0, 1)$ given histories $H_1$ and $H_e$. (Recall that we can assume without loss of generality that the malicious agents can communicate with the scheduler.) Similarly, there exists a history $H'$ with $\vec{x}_{H'} = (0, 1, 1, 1)$ such that $H'_2 = H''_2$ and $H'_e = H''_e$. The fact that the scheduler has the same local history in $H$, $H'$, and $H''$ and that $H_1 = H'_1$ and $H''_2 = H'_2$ implies that $H_{1,2} = H'_{1,2} (= H''_{1,2})$, as desired. In more detail, since $H'_2 = H''_2$, agent 2 sends the same messages to and receives the same messages from agent 1 in $H'$ and $H''$, so 1 receives the same messages from and sends the same messages to 2 in both $H'$ and $H''$. Thus, $H'_{1,2} = H''_{1,2}$. A similar argument shows that $H_{1,2} = H''_{1,2}$.

Now suppose that agents have input profile $\vec{x} = (0, 0, 0, 0)$. We show that there exists a strategy $\tau_3$ for agent 3 such that if all other agents play $\vec{\sigma}$ and the scheduler plays $\vec{\sigma}^N_e$, then with non-zero probability, agents 1 and 2 output 1. This suffices to show that $f^4$ cannot be 1-weakly securely computed, since honest agents should output 0 when playing with any trusted-party adversary with at most one malicious agent.

**Lemma 3.** If the agents have input profile $(0, 0, 0, 0)$, then there exists a strategy $\tau_3$ for agent 3 such that if all other agents play $\vec{\sigma}$ and the scheduler plays $\vec{\sigma}^N_e$, then with non-zero probability, agents 1 and 2 output 1.

**Proof.** Let $H$ and $H'$ be the two histories guaranteed to exist by Lemma 1. The strategy $\tau_3$ for agent 3 consists of sending agent 1 the messages that agent 3 sends to agent 1 in $H'$ while sending agent 2 the messages that agent 3 sends to agent 2 in $H$. Suppose that agent 1 has the same random bits as in $H'$, while agent 2 has the same random bits as in $H$. An easy induction now shows that, in the resulting history, agent 1 will have history $H'_1$ and agent 2 will have history $H'_2$ after each having been scheduled at most $N$ times, using the fact that, as shown in Lemma 1, $H_{1,2} = H'_{1,2}$. Thus, by Lemma 1, agents 1 and 2 output 1 in this case. This contradicts the fact that $\vec{\sigma}$ 1-weakly securely computes $f^4$, since Lemma 2 shows that, with input profile $(0, 0, 0, 0)$, all honest players output 0.

It is straightforward to extend this argument to all $n$ and $t$ such that $3t \leq n \leq 4t$. Given $n$ and $t$ such that $3t \leq n \leq 4t$, we divide the agents into four disjoint sets $S_1$, $S_2$, $S_3$, and $S_4$ such that $0 < |S_i| \leq t$ for all $i \in \{1, 2, 3\}$ and $0 \leq |S_4| \leq t$. Consider a scheduler $\sigma^N_e$ that schedules agents in $S_1$, $S_2$, and $S_3$ cyclically and, right before scheduling an agent, it delivers the messages that were sent by the other agents the last time they were scheduled. After scheduling each of the agents in $S_1 \cup S_2 \cup S_3$ $N$ times, it schedules the agents in $S_4$ as well. Suppose that $\vec{\sigma}$ is a strategy for $n$ agents that $t$-securely computes $f^n$.

**Lemma 4.** There exist $N$ and two (finite) histories $H$ and $H'$ of $\vec{\sigma}$ where the scheduler uses $\sigma^N_e$, $\vec{x}_H = (1_{S_1}, 0_{S_2}, 1_{S_3}, 1_{S_4})$, $\vec{x}_{H'} = (0_{S_1}, 1_{S_2}, 1_{S_3}, 1_{S_4})$, agents in $S_1 \cup S_2 \cup S_3$ output 1 in $H$, agents in $S_4$ are never scheduled in either $H$ or $H'$, $H_{S_1, S_2} = H'_{S_1, S_2}$ (which is the conversation between the agents in $S_1$ and the agents in $S_2$) and $H_e = H''_e$.

**Proof.** The proof is analogous to the proof of Lemma 1 the subsets $S_1$, $S_2$, $S_3$, and $S_4$ play the roles of agents 1, 2, 3, and 4, respectively.

We now have the tools we need to prove Theorem 12(b). Given $H$ and $H'$ from Lemma 4, consider a strategy $\vec{\tau}_{S_2}$ for agents in $S_2$ that consists of sending agents in $S_1$ and $S_2$ exactly the same messages they would send in $H'$ and $H$ respectively. Again, if agents have input 0, with non-zero probability, agents in $S_2$ will eventually have history $H_{S_2}$, and thus will output 1, contradicting the assumption that $\vec{\sigma}$ $t$-weakly securely computes $f^n$. This completes the proof of Theorem 12(b).

The proof of Theorem 11(b) follows similar lines. We start with an analogue of Lemma 2 which holds for a larger range of values of $n$:
Lemma 5. Let \( n \geq t + 2 \) and \( \tilde{\sigma} \) be a protocol that t-securely computes \( f^n \). Then, in all histories of \( \tilde{\sigma} \) in which all agents are honest and at most \( (n - t)/2 \) agents have input 0, all agents output 1.

Proof. Given any scheduler \( \sigma_e \), if all agents are honest, their output should be one that is possible with a trusted-party adversary of the form \( A = (\emptyset, c, h, O) \). No matter what the output \( C \) of \( c \) is, at most \( (n - t)/2 \) agents in \( C \) have input 0. Since \( |C| \geq n - t \), at least half of the agents in \( C \) have input 1 and thus all honest agents output 1.

We also need the following technical result:

Lemma 6. If \( t + 2 \leq n \leq 4t \) then

(a) \( n \geq 3 \left\lceil \frac{n - t}{3} \right\rceil \);

(b) \( \left\lceil \frac{n - t}{3} \right\rceil \leq \frac{n - t}{2} \).

Proof. If \( t + 2 \leq 4t \) then \( t > 0 \). To prove part (a), note that if \( t = 1 \), then \( n \) can be only 3 or 4. In both cases, the inequality is satisfied. If \( t \geq 2 \) then \( \left\lceil \frac{n - t}{3} \right\rceil \leq \left\lceil \frac{n - t}{3} \right\rceil \leq \frac{n - t}{2} \), from which the desired result immediately follows. To prove part (b), let \( a \) and \( b \) be the two positive integers such that \( n - t = 3a + b \) with \( 1 \leq b \leq 3 \). Then \( \left\lceil \frac{n - t}{a} \right\rceil = a + 1 \) and \( \frac{n - t}{2} = \frac{3a + b}{2} = a + \frac{b}{2} \). Since \( n - t \geq 2 \), then either \( a > 0 \) or \( b > 1 \). Since \( b \geq 1 \), in both cases, \( a + 1 \leq a + \frac{b}{2} \).

Given \( n \) and \( t \) such that \( t + 2 \leq n \leq 4t \), we divide the agents into four disjoint sets \( S_1, S_2, S_3, S_4 \) such that \( |S_i| = \left\lceil \frac{n - t}{3} \right\rceil \) for \( i \leq 3 \) and \( |S_4| \leq t \) (which is always possible, by Lemma 5(a)). If \( n \geq t + 2 \), then by Lemma 6(b), \( \left\lceil \frac{n - t}{3} \right\rceil \leq \frac{n - t}{2} \), and thus by Lemma 5 in all histories in which all agents are honest and have inputs \((0, 1, 1, 1, 1, 1, 1), (1, 0, 0, 1, 1, 1, 1), (1, 1, 0, 0, 1, 1, 1), (1, 1, 1, 0, 0, 1, 1)\) or \((1, 1, 1, 1, 0, 0, 1), (1, 1, 1, 1, 1, 0, 1)\), all the agents output 1. Reasoning analogous to that used in the proof of Theorem 12(b) then shows that \( f^n \) cannot be t-securely computed for \( t + 2 \leq n \leq 4t \).

It remains to deal with the case where \( n = t + 1 \). To show that there exist functions that cannot be t-securely computed if \( n = t + 1 \), we reduce t-resilient weak consensus to t-secure computation.

Definition 9. A protocol \( \tilde{\sigma} \) for \( n \) agents is a t-resilient implementation of weak consensus if the following holds for all adversaries \( A = (T, \tilde{\tau}, \sigma_e) \) with \( |T| \leq t \) and all histories:

(a) All agents not in \( T \) output the same value.

(b) If all agents are honest and have the same input \( x \), all agents output \( x \).

Lampert [1983] showed that if \( n \leq 3t \) there is no t-resilient implementation of weak consensus. Thus, if there exists a reduction from t-resilient weak consensus to t-secure computation for \( n > t \), then there are functions \( f : D^n \rightarrow D \) with \( n = t + 1 \) that cannot be t-securely computed.

The reduction proceeds as follows: Consider a function \( g^n : \{0, 1\}^n \rightarrow \{0, 1\} \) such that \( g^n(\perp, \ldots, \perp) = \perp \), and \( g^n(x_1, \ldots, x_n) = x_i \) if \( x_i \neq \perp \) and \( x_j = \perp \) for all \( j < i \); that is, \( g^n \) outputs the first non-\( \perp \) value if there is one, and otherwise outputs \( \perp \). Suppose, by way of contradiction, that \( \tilde{\sigma} \) t-securely computes \( g^n \). Let \( \tilde{\tau} \) be identical to \( \tilde{\sigma} \), except hat, whenever agent \( i \) would have output \((C, v)\) with \( \sigma_e \), it outputs \( v \) instead if \( v \neq \perp \), and otherwise it outputs 0. By the properties of t-secure computation, all honest agents output the same value when using \( \tilde{\tau} \). Moreover, if all honest agents have input 0 or all of them have input 1, if \( n > t \), then the output of the secure computation has the form \((C, 0)\) or \((C, 1)\), respectively. Thus, if there exists a protocol that t-securely computes \( g^n \) for \( n = t + 1 \), then there exists also a t-resilient implementation of weak consensus for \( t + 1 \) agents, contradicting Lampert’s result. This proves Theorem 11.

### B Proof of Theorem 13

We prove Theorem 13 only for the case of \( (k, t) \)-robustness; the proof in the case of \( (k, t) \)-strong robustness is identical.

A naive construction of \( \Gamma_d \) and \( \tilde{\sigma} + \sigma_d \) proceeds as follows. The set of actions of each player consists of all possible outputs of a secure computation of \( f \) in addition to their type (more precisely,
actions are of the form \( (C, z, Q) \) with \( C \subseteq [n] \) and \( z \in D \) and \( Q \in \{ H, R, M \} \), where \( H \) stands for honest, \( R \) stands for rational, and \( M \) stands for malicious). If there is no subset \( S \) of at least \( n - k - t \) honest players such that players in \( S \) securely compute \( f \), that is, every subset \( S \) of \( n - k - t \) players either do not output the same value or they all output a value that is not a possible output of a secure computation of \( f \), then rational players get a higher payoff and/or the honest players get a lower payoff. In \( \vec{\sigma} + \sigma_d \), each player sends its input to the mediator when it is scheduled for the first time. The mediator waits until it receives the input \( x_i \) from a set \( C \) of players with \( |C| \geq n - k - t \), then computes \( z := f_C(\vec{x}) \) and sends \( (C, z, H) \) to all players. Players play \( (C, z, H) \) when they receive the message.

It would seem that any \((k, t)\)-robust implementation of \( \vec{\sigma} + \sigma_d \) also \((k + t)\)-strictly securely computes \( f \). In fact, any \((k + t)\)-secure computation of \( f \) is also a \((k, t)\)-robust implementation of \( \vec{\sigma} + \sigma_d \), but not all \((k, t)\)-robust implementations \((k + t)\)-securely compute \( f \). As we suggested in the main text, consider a protocol \( \sigma_{\text{ACT}} \) in which players perform BCG’s \((k + t)\)-secure computation protocol and broadcast their inputs immediately afterwards. It is easy to check that \( \sigma_{\text{ACT}} \) is a \((k, t)\)-robust implementation of \( \vec{\sigma} + \sigma_d \) whenever \( n > 4(k + t) \), but that \( \sigma_{\text{ACT}} \) does not \((k + t)\)-securely compute \( f \), since it leaks the honest players’ inputs to all other players.

This shows that it is necessary to somehow encode all information that malicious players can learn into the set of actions of \( \Gamma_d \) in such a way that they can increase their payoff if they manage to learn anything about the other players’ inputs besides what can be learned from the output of the computation. The idea for doing this is that, besides the output, the action of each player should include a guess as to what the input profile \( \vec{x} \) is (they can also guess \( \bot \) if they have no guess). If a player \( i \) guesses correctly it receives an additional positive payoff \( q_i \), while if it guesses wrong, its payoff decreases by 1. The value of \( q_i \) should be chosen in such a way that (a) it is never worthwhile deviating if \( i \) is not able to learn anything besides the output, and (b) it is always worthwhile deviating if \( i \) is able to learn something (otherwise, \( \sigma_{\text{ACT}} \) may not \((k + t)\)-strictly securely compute \( f \) even if it is \((k, t)\)-robust, as in the example above). Given the set \( C \) of players whose inputs are included in the computation, the output \( z \) of \( f \), and the input profile \( \vec{x} \), let \( p_i \) be the probability that a player \( i \) guesses \( \vec{x} \) conditional on its own input \( x_i \), \( \vec{C} \), and \( z \). Conditions (a) and (b) imply that \( p_i q_i + (1 - p_i)(-1) \leq 0 \) and \( p_i q_i + (1 - p_i)(-1) \geq 0 \) respectively, which means that \( p_i q_i + p_i - 1 = 0 \) and thus that \( q_i = p_i^{-1} - 1 \).

This approach cannot be generalized easily to a situation where a coalition of players may deviate. In this case, a player in the coalition will know the values of all players in the coalition, not just its own input. Moreover, if a player in the coalition plays just like an honest player, except that it tells the other coalition members its input, then this is completely indistinguishable (by the honest players) from the scenario in which that player is honest and the other members of the coalition were just lucky guessing its input. In addition, a player can lie about its input if it is easier to guess the input profile with a different input than its own. Since the payoffs of a Bayesian game depend only on their actions and their real input profile, it is always worthwhile for a player to lie about its input if this is the case. For instance, suppose that \( D = \mathbb{F}_2 \) and that \( f(\vec{x}) = \prod_{i=1}^n (1 - x_i) \). If \( i \) has input 1 and plays honestly, then it learns absolutely nothing about the other players’ inputs, since the output will be 0 no matter what. However, if \( i \) pretends to have input 0, \( i \) will learn more information: if the output is 1, then all players have input 0; otherwise, at least one player has input 1. In this case, it would always be worthwhile for player \( i \) to act as if it has input 0, regardless of its actual input. This shows that to compute the probability that the adversary guesses the inputs correctly, it is critical to know who is malicious and what inputs the malicious players are pretending to use in the computation.

To deal with the fact that we may not be able to tell which players are deviating, we require that exactly \( k + t \) players must try to guess a non-\( \bot \) value in order to get an additional (positive or negative) payoff. Moreover, their guesses must be identical. If honest players always guess \( \bot \), this suffices to identify the coalition of \( k + t \) deviating players given their action profile. Note that this is why we require strict secure computation in Theorem 13. If we required only (standard) secure computation, smaller adversaries wouldn’t be able to get a better payoff, even if they managed to guess the inputs of everyone else (thus it wouldn’t satisfy condition (b)). To deal with players lying about their inputs, we require that the action profile of the players encode the inputs used
by the players for the computation (even though these inputs may differ from their actual inputs). The probability of guessing the input profile is based on the inputs used, not players’ actual inputs. Note that these values must be encoded into the action profile without any coalition of \( k + t \) players learning anything about them. This can be done as follows: each player \( i \), in addition to the set \( C_i \), the output \( z_i \) and their guess \( b_i \), also outputs \( n \) values \( s_{i1}, \ldots, s_{in} \) such that the values \( s_{ij} \) for a fixed \( j \) encode the value used by \( j \) for the computation (using Shamir’s scheme).

There is one final issue. The definition of \((0, t)\)-robustness is equivalent to that of \( t \)-immunity, which means that no coalition of \( t \) players can decrease the payoff of other players by deviating. In this case, the effect of a coalition of \( t \) players being able to learn something about the inputs should be to decrease the payoffs of the remaining players, rather than increasing their own payoff. To deal with this, we require players to declare whether they are \( G \) (good), \( R \) (rational), or \( M \) (malicious). If a coalition of \((k + t)\) players tries to guess the inputs of everyone else, if they all declare \( R \), then they get an additional payoff as described above. Otherwise, everyone gets the negative of that value.

We next formalize these ideas. Given \( f \) and integers \( k \) and \( t \) such that \( n > 2k + 2t \), consider the game \( \Gamma^{f,k,t}_n \), defined as follows. The input profile of the players is chosen uniformly at random from \( D^n \). The set of actions of each player in \( \Gamma^{f,k,t}_n \) is \( \{G, R, M\} \times \mathbb{R} \), so an action of player \( i \) has the form \( a_i = (Q_i, C_i, z_i, s_i, b_i) \), where \( Q_i \in \{G, R, M\}, C_i \subseteq [n], z_i \in D, x_i \in D, \) and \( b_i \in D^n \). Intuitively, \( Q_i \) denotes if \( i \) is good (\( G \)), rational (\( R \)), or malicious (\( M \)), \((C_i, z_i)\) is \( i \)-\'s output in the secure computation of \( f \); \( s_i \) is \( i \)-\’s share of \( j \)-\’s input (this will be made clear below), and \( b_i \) is \( i \)-\’s guess of the (supposedly secret) input, where \( b_i = \perp \) if \( i \) has no guess.

We next define the utility function. We take \( u_i = u_i^1 + u_i^2 \), where, intuitively, \( u_i^1 \) is the utility that \( i \) gets if honest players either output different values or some honest player outputs a value that is not a possible output of a secure computation of \( f \) and \( u_i^2 \) is the utility that \( i \) gets from guessing the correct input of the other players. To define \( u_i^1 \), we first define what it means for an action profile \( \vec{a} \) to be safe for an input profile \( \vec{x} \). This is the case if there exist subsets \( C, T \subseteq [n] \) with \( |C| \geq n - t - k \) and \( |T| = k + t \), a vector \( \vec{\sigma} \in D^{k+t} \), and \( n \) polynomials \( p_1, \ldots, p_n \) of degree \( k + t \) (where, intuitively, \( p_j \) encodes j\’s input, so \( p_j(i) \) is \( i \)-\’s share of \( j \)-\’s input) such that, for each player \( j \in T \), the action \( a_j = (Q_j, C_j, z_j, s_j, b_j) \) of player \( j \) satisfies (1) \( Q_j = G \), (2) \( C_j = C \), (3) \( b_j = \perp \), (4) \( z_j = f(\vec{\sigma}) \), \( \vec{\sigma} = (\vec{x} / (T, \vec{\sigma})) / (\vec{\sigma}, \vec{a}) \) \( j \)-\’s utility for the computation—which may differ from the the actual input profile due to deviating players lying about their inputs—were shared correctly between the players. If \( \vec{a} \) is secure for \( \vec{x} \), then \( u_i^1(\vec{a}, \vec{x}) = 0 \) for all \( i \in [n] \); if \( \vec{a} \) is not secure for \( \vec{x} \) and at least one player \( i \) played \( R \) (i.e., played an action with \( Q_i = R \)), then \( u_i^1(\vec{a}, \vec{x}) = 1 \) for all players \( j \); otherwise, \( u_i^1(\vec{a}, \vec{x}) = -1 \) for all players \( j \).

If \( \vec{a} \) is not secure for \( \vec{x} \), then \( u_i^2(\vec{a}, \vec{x}) = 0 \). If \( \vec{a} \) is secure for \( \vec{x} \), let \( K \) be the subset of players that did not play \( G \). Note that if \( \vec{a} \) is secure for \( \vec{x} \), then \( |K| \leq k + t \). If \( |K| < k + t \) or not all players in \( K \) guess the same value \( b \) (i.e., not all players in \( K \) have the same value \( b \) as the last component of their actions), then \( u_i^2(\vec{a}, \vec{x}) = 0 \) for all \( i \). Otherwise, let \( b \) be the common guess of players in \( K \) and let \( p \) be the probability that a vector \( \vec{a} \) sampled uniformly from \( D^n \) is equal to \( \vec{x} \), conditional on \( \vec{a} \), and fixed input of the other players. Then, \( u_i^2(\vec{a}, \vec{x}) = 0 \) for all \( i \). If at least one player \( i \) played \( R \) in its action, then, if \( b = \vec{x} \), \( u_i^2(\vec{a}, \vec{x}) = p^{t-1} - 1 \) for all \( i \in K \); otherwise, \( u_i^2(\vec{a}, \vec{x}) = -1 \) for all \( i \in K \). On the other hand, if no player \( i \in K \) played \( R \) in its action, then, if \( b = \vec{x} \), \( u_i^2(\vec{a}, \vec{x}) = 1 - p^{t-1} \) for all \( i \in K \); otherwise, \( u_i^2(\vec{a}, \vec{x}) = 1 \) for all \( i \). Note that since \( p^{t-1} \) is the probability that the adversary can, in expectation, either increase its payoff (if there are any rational players) or decrease the payoff of everyone else (if there are no rational players) if it can guess the inputs of honest players with higher probability than \( p \), it is the probability of guessing the honest players’ inputs if the adversary knew nothing but the output of the function and its own strategy and inputs).

Consider the following strategy \( \vec{\sigma} + \sigma_d \) for \( \Gamma^{f,k,t}_n \). According to \( \sigma_d \), player \( i \) sends its input to the mediator at the beginning of the game. If \( i \) receives a message \( msg \) from the mediator, it
get a payoff of 1 rather than 0. It follows that \( \vec{\sigma} \) plays \( \vec{a} \), which all players use the same randomization as in \( \vec{a} \), first component, and none of the players in \( T \) plays an action \( | A \).

Proof. Suppose that \( u_i(\vec{\sigma}, A) \) be the expected payoff of player \( i \) when playing \( \vec{\sigma} \) with adversary \( A \). It follows by construction that \( u_i^1(\vec{\sigma} + \sigma_d, A) = 0 \) for all adversaries \( A = (T, \vec{\tau}) \) of size at most \( k + t \) since, no matter what \( T \) is, the output profile \( \vec{a} \) is \( (T, \vec{x}) \)-secure for all input profiles \( \vec{x} \).

Thus, \( \vec{\sigma} + \sigma_d \) is not \( (k, t) \)-robust if and only if there exists an adversary \( A = (T, \vec{\tau}) \) with \( |T| \leq k + t \) and an input profile \( \vec{x}_T \) such that, in expectation, (a) \( u_i^1(\vec{\sigma} + \sigma_d, A, \vec{x}_T) > 0 \) for some \( i \in T \), or (b) \( u_i^2(\vec{\sigma} + \sigma_d, A, \vec{x}_T) < 0 \) for all \( i \in K \). The definition of \( u_i^2 \) guarantees that, in both cases, the adversary must consist of exactly \( k + t \) players and these players must all play a non-\( G \) action. Moreover, these players must guess the input of honest players with a probability higher than they could guess it by just knowing the output of the function, their strategy, and their inputs. However, the construction of \( \vec{\sigma} + \sigma_d \) guarantees that don’t have any extra information (note that \( C \) depends only on the adversary, and that the adversary does not get any information about the input of the honest players besides the value of \( z \), since the polynomials \( p_i \) are of degree \( k + t \)).

We next show that if there exists a \( (k, t) \)-robust strategy that implements \( \vec{\sigma} + \sigma_d \), then that strategy also \( (k + t) \)-securely computes \( f \). We first need the following lemma.

**Proposition 1.** \( \vec{\sigma} + \sigma_d \) is \( (k, t) \)-robust and the equilibrium payoff is 0.

Proof. Suppose that \( k > 0 \). If there exists an input \( \vec{x} \) and an adversary \( A = (T, \vec{\tau}, \sigma_e) \) with \( |T| = k + t \) such that, for some history \( H \), the action profile \( \vec{a} \) played in \( H \) is \( (T, \vec{x}) \)-secure, consider the adversary \( A' = (T, \vec{\tau}', \sigma_e) \) where \( \vec{\tau}' \) is identical to \( \vec{\tau} \), except that if a player \( i \in T \) plays an action \( a \) with \( \tau_i \), then instead plays \( (R, \emptyset, 0, 0, \bot) \) with \( \tau_i' \). Thus, if an action profile \( \vec{\sigma} \) played in some history \( H' \) when \( \vec{\sigma} \) is \( (T', \vec{x}) \)-secure, then \( T' \subseteq T \) (note that for an action profile \( \vec{a} \) to be \( (T', \vec{x}) \)-secure, we require that all players not in \( T' \) play \( G \) in the first component, and none of the players in \( T \) plays \( G \)) and, since \( |T'| = k + t \), \( T' \subseteq T \). Since the histories generated by playing with adversaries \( A \) and \( A' \) are indistinguishable by honest players, if there exists a history \( H \) with adversary \( A \) and input \( \vec{x} \) such that the action profile \( \vec{a} \) played in \( H \) is not \( (T, \vec{x}) \)-secure, the resulting action profile \( \vec{\sigma} \) of playing \( \vec{\sigma} \) with adversary \( A' \) and input \( \vec{x} \) in which all players use the same randomization as in \( H \) is not \( (T, \vec{x}) \)-secure, and all players in \( T \) would get a payoff of 1 rather than 0. It follows that \( \vec{\sigma} \) is not \( (k, t) \)-robust. If \( k = 0 \), the argument is analogous, except that players in \( T \) play \( (M, \emptyset, 0, 0, \bot) \) rather than \( (R, \emptyset, 0, 0, \bot) \).

To complete the proof of Theorem 13, we must show that if \( \vec{\sigma} \) is a \( (k, t) \)-robust implementation of \( \vec{\sigma} + \sigma_d \), the output of an adversary \( A = (T, \vec{\tau}, \sigma_e) \) with \( |T| = k + t \) is a (randomized) function of its input \( \vec{x} \) and the output \( v \) of the function. To do this, we need the following lemma:

**Lemma 7.** If \( \vec{\sigma} \) is a \( (k, t) \)-robust strategy that implements \( \vec{\sigma} + \sigma_d \), then for all adversaries 

Proof. Suppose that \( k > 0 \). If there exists an input \( \vec{x} \) and an adversary \( A = (T, \vec{\tau}, \sigma_e) \) with \( |T| = k + t \) such that, for some history \( H \), the action profile \( \vec{a} \) played in \( H \) is \( (T, \vec{x}) \)-secure, consider the adversary \( A' = (T, \vec{\tau}', \sigma_e) \) where \( \vec{\tau}' \) is identical to \( \vec{\tau} \), except that if a player \( i \in T \) plays an action \( a \) with \( \tau_i \), then instead plays \( (R, \emptyset, 0, 0, \bot) \) with \( \tau_i' \). Thus, if an action profile \( \vec{\sigma} \) played in some history \( H' \) when \( \vec{\sigma} \) is \( (T', \vec{x}) \)-secure, then \( T' \subseteq T \) (note that for an action profile \( \vec{a} \) to be \( (T', \vec{x}) \)-secure, we require that all players not in \( T' \) play \( G \) in the first component, and none of the players in \( T \) plays \( G \)) and, since \( |T'| = k + t \), \( T' \subseteq T \). Since the histories generated by playing with adversaries \( A \) and \( A' \) are indistinguishable by honest players, if there exists a history \( H \) with adversary \( A \) and input \( \vec{x} \) such that the action profile \( \vec{a} \) played in \( H \) is not \( (T, \vec{x}) \)-secure, the resulting action profile \( \vec{\sigma} \) of playing \( \vec{\sigma} \) with adversary \( A' \) and input \( \vec{x} \) in which all players use the same randomization as in \( H \) is not \( (T', \vec{x}) \)-secure, and all players in \( T \) would get a payoff of 1 rather than 0. It follows that \( \vec{\sigma} \) is not \( (k, t) \)-robust. If \( k = 0 \), the argument is analogous, except that players in \( T \) play \( (M, \emptyset, 0, 0, \bot) \) rather than \( (R, \emptyset, 0, 0, \bot) \).

To complete the proof of Theorem 13, we must show that if \( \vec{\sigma} \) is a \( (k, t) \)-robust implementation of \( \vec{\sigma} + \sigma_d \), the output of an adversary \( A = (T, \vec{\tau}, \sigma_e) \) with \( |T| = k + t \) is a (randomized) function of its input \( \vec{x} \) and the output \( v \) of the function. To do this, we need the following lemma:

**Lemma 8.** Consider two random variables \( X \) and \( Y \) that take values on countable spaces \( S_1 \) and \( S_2 \) respectively. Then, \( \Pr[X = x \mid Y = y] = \Pr[X = x \mid Y = y'] \) for all \( x \in S_1 \) and \( y, y' \in S_2 \) if and only if \( X \) and \( Y \) are independent.

Proof. If \( \Pr[X = x \mid Y = y] \) does not depend on \( y \), there exists a constant \( \lambda_x \) such that \( \Pr[X = x \mid Y = y] = \lambda_x \) for all \( y \in S_2 \). Then, since \( \Pr[X = x \mid Y = y] = \Pr[X = x \mid Y = y'] \), it follows that \( \Pr[X = x, Y = y] = \lambda_x \Pr[Y = y] \). Therefore, \( \sum_{y \in S_2} \Pr[X = x, Y = y] = \Pr[Y = y] \), which gives that \( \lambda_x = \Pr[X = x] \), as desired. The converse is straightforward.
We can now complete the proof of Theorem 13. Suppose that \( k > 0 \). Recall that if all players \( i \in T \) set \( b_i \) to some input \( \bar{x} \), they get a payoff of \( p_{\bar{x}} - 1 \), where \( p_{\bar{x}} \in [0, 1] \) if the input profile is indeed \( \bar{x} \), and otherwise they get \(-1\). Given a history \( H_T \) in which the adversary has input \( \bar{x}_T \) and honest players output \( v \), let \( p^{\text{HT}}(\bar{x}) \) be the probability that the input profile is \( \bar{x} \) conditional on \( v \) and \( H_T \). If \( p^{\text{HT}}(\bar{x}) > p_{\bar{x}} \), then \( p^{\text{HT}}(\bar{x})(p_{\bar{x}} - 1) + (-1)(1 - p^{\text{HT}}(\bar{x})) > 0 \), which means that taking \( b_i = \bar{x} \) is strictly better than taking \( b_i = \perp \) for each of the players in \( T \), contradicting the assumption that \( \sigma_{\text{ACT}} \) is \((k, t)\)-robust. Thus, \( p^{\text{HT}}(\bar{x}) \leq p_{\bar{x}} \) for all \( \bar{x} \). Since both \( \sum_{\bar{x}} p^{\text{HT}}(\bar{x}) \) and \( \sum_{\bar{x}} p_{\bar{x}} \) are 1, it must be the case that \( p^{\text{HT}}(\bar{x}) = p_{\bar{x}} \) for all \( \bar{x} \). This shows that for every history \( \bar{H}_T \) of the adversary, the distribution of possible inputs of honest players conditional on \( \bar{H}_T \) depends only on their inputs and what honest players output. By Lemma 8 this implies that the input of honest players and the history of the adversary are independent (given the input \( \bar{x}_T \) of the adversary and the output \( v \) of honest players), and thus, again by Lemma 8 it follows that the distribution of possible histories of the adversary depends only on \( \bar{x}_T \) and \( v \). This shows that every possible output function of the adversary can be simplified to a function that has as inputs only \( \bar{x}_T \) and \( v \), as desired. The argument for \( k = 0 \) is analogous, except that in this case, if the distribution of possible histories of the adversary is not independent of the inputs of the honest players, the adversary decreases the payoffs of the honest players, rather than increasing the payoffs of the deviating players. This completes the proof of Theorem 13.

Note that a \((k, t)\)-robust implementation of \( \bar{\sigma} + \sigma_{\text{ACT}} \) may not necessarily (non-strictly) \((k + t)\)-securely compute \( f \), since if the adversary consists of fewer than \( k + t \) malicious players, the malicious players might be able to deduce information about the honest players’ inputs without being able to take advantage of it (recall that a subset \( K \) consisting of \( k + t \) players must all guess the same value for \( u_i^2(\bar{a}, \bar{x}) \) to be non-zero). However, a small variation in the construction of \( u_i^2 \) in \( G^{(k,t)} \) allows us to construct a game \( G^{(k,t)} \) such that any strongly \((k, 0)\)-robust implementation of the strategy used in Proposition 10 also \( k \)-securely computes \( f \), so secure computation can be reduced to implementing strategies for certain mediator games. The idea is that instead of requiring the subset \( K \) of players who do not play \( G \) to have size exactly \( k \), we only require it to have size at most \( k \). This modification of \( u_i^2 \) leads to some of the problems discussed at the beginning of this section, namely, that if some rational players act like honest players, except that they share their inputs with other rational players, the latter players might be able to guess the input profile and get a strictly positive expected payoff. This scenario is indistinguishable from one in which the players who shared their input are actually honest and rational players are just lucky. To deal with this issue, we further modify the payoffs in \( G^{(k,t)} \) so that if the players in \( K \) guess the inputs correctly, then everyone else gets a huge negative payoff (rather than 0, as in the original construction). We can show that if this payoff is sufficiently small (e.g., \( -n \) times the winnings), then if there exists a strategy in which rational players get a positive payoff from \( u^2 \), then there exists a strategy in which rational players get a positive payoff from \( u^2 \) and they all guess the same value in every possible history (if the negative payoffs are small enough, rational players not guessing any value gives a negative total payoff for rational players, regardless if some of them guess the correct value).

Note that this modification works only for strong \((k, t)\)-robustness, since if we require only \((k, t)\)-robustness, a rational player may decrease its own payoff if that helps other rational players, even if the total gain from doing so is negative. This is enough to show that the strategy used in Proposition 10 is strongly \((k, 0)\)-robust with these payoffs.

**C Proof of Theorem 15**

Consider the game \( G^{(k,t)} \) in which the set of actions of each player is \( \{G, R\} \times \{0, 1\} \). Given an action profile \( \bar{a} \), in which each player \( i \) plays \( a_i = (Q_i, y_i) \) with \( Q_i \in \{G, R\} \) and \( y_i \in \{0, 1\} \), let \( T \) be the subset of players \( i \) such that \( Q_i = R \). If \( |T| > k + t \), if \( k = 0 \) all players get a payoff of \(-1\), otherwise all players get a payoff of \( 1 \). If \( |T| = t + k \) and there exist two players \( i, j \notin T \) such that \( y_i \neq y_j \), if \( k = 0 \) all players get a payoff of \(-1\), otherwise all players get a payoff of \( 1 \). In all remaining cases, all players get a payoff of \( 0 \). Let \( g \) be the function such that \( g(Q, y) = y \).

Consider the following protocol \( \bar{\sigma} + \sigma_i \) for \( n \) players and a mediator. With \( \sigma_i \), each player \( i \) sends
the mediator its input $x_i$ the first time it is scheduled. The mediator waits until receiving a message containing either 0 or 1, and sends that value $y$ to all players. The players play $(G, y)$ whenever they receive $y$ from the mediator. Clearly, this strategy is $(k, t)$-robust (resp., strongly $(k, t)$-robust), since the only way that players get a payoff other than 0 with an adversary of size at most $k + t$ is if two honest players output different values, but they all receive the same value from the mediator.

Suppose a strategy $\tilde{\sigma}_{ACT}$ is a $(k, t)$-robust (resp., strongly $(k, t)$-robust) implementation of $\tilde{\sigma} + \sigma_d$. We show next that (a) for all adversaries $A = (T, \tilde{\tau}_T, \sigma_e)$ with $|T| \leq k + t$, all honest players play the same value $y_i$, and (b) if all players are honest and have the same input $x$, then they output $x$.

Property (b) follows trivially from the fact that $\tilde{\sigma}_{ACT}$ implements $\tilde{\sigma} + \sigma_d$: if all players are honest and have the same input $x$, the value received by the mediator in $\tilde{\sigma} + \sigma_d$ is guaranteed to be $x$, and thus, in $\tilde{\sigma} + \sigma_d$, all honest players play $(G, x)$.

To prove (a), suppose that there exists an adversary $A = (T, \tilde{\tau}, \sigma_e)$ with $|T| \leq k + t$ such that, in some history $H$ of $\tilde{\sigma}_{ACT}$ with $A$, there exist two players $i, j \notin T$ that play $(Q_i, y_i)$ and $(Q_j, y_j)$, respectively, with $y_i \neq y_j$, $Q_i = R$, or $Q_j = R$. Consider an adversary $A' = (T', \tilde{\tau}_{T'}, \sigma_e)$ such that $|T'| = k + t$, $T \subseteq T'$, and $i, j \notin T'$ (we know that such a subset $T'$ exists, since $n > t + k + 1$), and such that players in $T$ act as in $\tilde{\tau}_T$ and players in $T' - T$ act like honest players, except that all of them play $(R, 0)$. Since histories generated by playing with $A$ and $A'$ are indistinguishable by honest players, there exists a history $H'$ in $\tilde{\sigma}_{ACT}$ with adversary $A'$ in which all honest players send and receive the same messages, and perform the same actions. If $Q_i = R$ in $H$, then there are $k + t + 1$ players that play $R$ in $H'$: the $k + t$ players in $T'$ and $i$. Thus, all players get a payoff of 1 if $k > 0$, contradicting the assumption that $\tilde{\sigma}_{ACT}$ is $(k, t)$-resilient, or all players get a payoff of $-1$ if $k = 0$, contradicting the assumption that $\tilde{\sigma}_{ACT}$ is $t$-immune. The same argument shows that $Q_i = G$ in $H$ and $H'$ and, indeed, that all honest players must play $G$ in $H$ and $H'$. Now if $q_i \neq q_j$ in $H$, then $q_i \neq q_j$ in $H'$, so (since all honest players play $G$, so exactly $k + t$ players in $H'$ play $R$), again, all players in $H'$ get a payoff of 1 if $k > 0$ and a payoff of $-1$ if $k = 0$, so we again get the same contradiction as before.

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