On an approach for computing the generating functions of the characters of simple Lie algebras

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Abstract
We describe a general approach to obtain the generating functions of the characters of simple Lie algebras which is based on the theory of the quantum trigonometric Calogero–Sutherland model. We show how the method works in practice by means of a few examples involving some low rank classical algebras.

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1. Introduction
The characters of the irreducible representations of the simple Lie algebras are systems of orthogonal polynomials which enjoy many interesting properties and have a distinguished role in pure mathematics and mathematical physics [1]. Many features of orthogonal systems of this kind can be studied by means of their generating functions, defined as formal power series in some auxiliary variables whose coefficients give the polynomials entering in the system. In the case of the simple Lie algebras, after pioneering works such as [2] which deals with weight multiplicities or [3] which refers to the characters themselves, several approaches for the computation of the generating functions have been proposed, see for instance [4] and references therein. In general, these approaches are based on the Weyl character formula [5, 6] and require, in consequence, to perform in one way or other two quite cumbersome tasks: first, one has to sum over the elements of the Weyl group, which can be a rather involved combinatorial issue and, second, to finish with a Weyl-symmetric generating function, one has
to divide two functions which are alternating with respect to Weyl reflections. While strategies of this type are powerful and can be used to establish formulas of great generality, see for instance [4, 7], we feel that it would be desirable to put forward an alternative approach which, by avoiding these difficulties, be able to yield the generating functions of the characters of each particular algebra along a few simple steps.

The main idea is to rely on the trigonometric Calogero–Sutherland model [8–12], rather than on the Weyl formula, as the tool for obtaining the characters. This point of view makes it possible to define a concrete and quite versatile procedure, which is suitable to be applied separately to each particular algebra, and uses Weyl symmetric instead of alternating formulas. The viability of such a procedure comes from two facts. First, in the last few years it has been shown explicitly how the quantum theory of integrable systems, in particular that of the Calogero–Sutherland model, can be used to compute the characters of the simple Lie algebras by identifying them with the eigenfunctions of the Hamiltonian for some particular values of the coupling constants, and several lists of characters obtained from this method are now available [13–15]. And, second, although the procedure that we are going to develop involves some long calculations, they are quite straightforward and can be performed rather quickly by means of symbolic calculus languages like Mathematica or Maple, nowadays of common use. In this respect, we remark that, although we shall here illustrate the approach by applying it to some low rank algebras, the use of these programs turns the method also useful for the higher rank ones.

The paper is organized as follows. In the next section, after presenting a quick review of the theory of the quantum trigonometric Calogero–Sutherland systems, we deduce the differential equation which must satisfy the generating function of the characters and develop a method to solve it. In section 3 we apply the procedure to some explicit examples, namely the classical Lie algebras up to rank two. Section 4 offers some concluding comments. Finally, the results concerning the algebra C\textsubscript{2} needed in section 3 are collected in the appendix. All along the paper we make an extensive use of our previous results, especially those contained in the last four articles in [15], but both the approach described here and the results reported are completely new and were not the subject of the researches carried out in these references.

2. Description of the method

2.1. Lie algebra characters and Calogero–Sutherland wave functions

Let \( A \) be a simple Lie algebra of rank \( r \) with simple roots \( \alpha_1, \alpha_2, \ldots, \alpha_r \) and fundamental weights \( \lambda_1, \lambda_2, \ldots, \lambda_r \). Let us denote \( R_\lambda \) the irreducible representation of \( A \) with highest weight \( \lambda = m_1 \lambda_1 + m_2 \lambda_2 + \cdots + m_r \lambda_r \). The character of this representation is defined as

\[
\chi_{m_1, m_2, \ldots, m_r} = \sum_w n_w e(w)
\]

where the sum extends to all weights \( w \) entering in the representation, \( n_w \) is the multiplicity of the weight \( w \) and, if \( w = n_1 \lambda_1 + n_2 \lambda_2 + \cdots + n_r \lambda_r \), then

\[
e(w) = \exp \left( i \sum_{j=1}^{r} n_j \varphi_j \right) = x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r},
\]

where \( \varphi_1, \varphi_2, \ldots, \varphi_r \) are coordinates on the maximal torus and \( x_i \) are complex phases, \( x_i = e^{i \varphi_i} \).

We can use the root system of \( A \) to define a very remarkable dynamical system, the trigonometric Calogero–Sutherland model. Here, we limit ourselves to mention the most salient features of this model which are useful for our present purposes and refer the reader to
The Hamiltonian has the form
\[ H = \frac{1}{2} p^2 + U(q) \]
where the coordinates \( q = (q_1, q_2, \ldots, q_r) \) and momenta \( p = (p_1, p_2, \ldots, p_r) \) are elements of a \( r \)-dimensional space \( V \). The roots and weights of the algebra \( A \) can also be seen as elements of this same space, and the potential term is
\[ U(q) = \sum_{\alpha \in \mathbb{R}^+} \kappa_\alpha (\kappa_\alpha - 1) \sin^{-2}(\alpha, q), \]
where \( \mathbb{R}^+ \) is the set of positive roots of \( A \) and \( \langle \cdot, \cdot \rangle \) is the Euclidean scalar product on \( V \). The constants \( \kappa_\alpha \) must be chosen in such a way that the couplings \( g^2 = \kappa_\alpha (\kappa_\alpha - 1) \) are equal for roots of equal length. The model, which has many physical and mathematical applications \([16, 17]\), is integrable both at the classical and quantum levels \([11, 12]\). In the latter case, it turns out that the energy eigenfunctions depend on quantum numbers \( m = (m_1, m_2, \ldots, m_r) \) and are of the form
\[ \Psi^\kappa_0 m = \psi_0 \cdot \Phi^\kappa m \]
where \( \psi_0 \) is the wave function of the ground state and the \( \Phi^\kappa m \) are solutions of the related Schrödinger equation
\[ \Delta^\kappa \Phi^\kappa m = \varepsilon(m, \kappa) \Phi^\kappa m \]
where \( \Delta^\kappa \) is the linear differential operator
\[ \Delta^\kappa = -\frac{1}{2} \sum_{j=1}^r \frac{\partial^2}{\partial q_j^2} - \sum_{\alpha \in \mathbb{R}^+} (\alpha, \alpha) \kappa_\alpha \cot(\alpha, q) (\alpha, \partial_i) \]
and the eigenvalues are
\[ \varepsilon(m, \kappa) = 2(\lambda + 2\rho(\kappa), \lambda) \]
for \( 2\rho(\kappa) = \sum_{\alpha \in \mathbb{R}^+} \kappa_\alpha \alpha \) and \( \lambda \) the highest weight \( \lambda = m_1 \lambda_1 + m_2 \lambda_2 + \ldots + m_r \lambda_r \) defined by \( m \). The most relevant fact for us is that if we tune all coupling constants \( \kappa_\alpha \) to one, the eigenfunctions of this Schrödinger operator are precisely the characters of the irreducible representations of the algebra \([12]\)
\[ \Phi^1 m = \chi m, \]
where the \( \varphi \)-angles are given in terms of the \( \alpha \)-coordinates as \( \varphi_j = 2(\lambda_j, q) \). This comes about as follows. Although the potential vanishes for \( \kappa_\alpha \to 1 \), there is a remnant of the interaction in that, to take the limit consistently, we have to choose fermionic boundary conditions ensuring that the wave functions are zero when \( \sin(\alpha, q) = 0 \) for any positive root. As a consequence, the wave function \( \psi^1 m \) is given by a Weyl-alternating sum of free-particle exponentials which turns out to coincide exactly with the numerator of the Weyl character formula. The ground state wave function, on the other hand, can be rewritten as the denominator of the Weyl formula, and the \( \Phi^1 m \) are the characters of the Lie algebra thereby. (The particles are free also for \( \kappa_\alpha = 0 \), but in that case bosonic boundary conditions are appropriate and the \( \Phi^0 m \) are the monomial symmetric functions associated to the root system; the \( \Phi^\kappa m \) for other values of the couplings are systems of orthogonal polynomials which interpolate between the monomial symmetric functions and the characters.)

Thus, we can obtain the characters by solving a second order differential equation. Furthermore, if we change variables and describe the dynamical system by means of the
characters \( z_k = \chi_{\lambda_k} \) of the fundamental representations \( R_{\lambda_k}, k = 1, 2, \ldots, r \), the differential operator \( \Delta^1 \) takes the form

\[
\Delta^1_z = \sum_{j,k=1}^r a_{jk}(z) \partial_z \partial_z + \sum_{j=1}^r b_j(z) \partial_z,
\]

with \( a_{jk}(z) \) and \( b_j(z) \) polynomials in the \( z_k \) with integer coefficients, and the Schrödinger equation can be solved by iterative methods \([13–15]\). This operator can be given an explicit form taking into account that:

- \( b_j(z) = \Delta^1_z z_j = \varepsilon(0, \ldots, 1^j, \ldots, 0; 1)z_j \), and
- \( \Delta^1_z(z_j z_k) = 2a_{jk}(z) + b_j(z)z_k + b_k(z)z_j \),

while \( z_j z_k \) is the character of the tensor product \( R_j \otimes R_k \). Hence, knowing all the quadratic Clebsch–Gordan series of the algebra we will be able to determine the \( a_{jk}(z) \) coefficients. For more explicit details, see \([15]\) and the appendix of this paper.

Once we know the definite expression for the operator \( \Delta^1_z \), it is possible to compute the characters \( \chi_m \) as polynomials in the \( z \)-variables by solving the Schrödinger equation (3) (with \( \kappa = 1 \)) in a recursive way as follows. Given a weight \( m_1 \lambda_1 + m_2 \lambda_2 + \cdots \), let us denote \( z^m = \Pi_{\lambda \in \Lambda} z_\lambda^{m_\lambda} \); thus the differential operator acting on \( z^m \) gives

\[
\Delta^1_z z^m = \varepsilon(m; 1) z^m + \sum_{\beta \in \Lambda} R_{m, \beta} z^{m - \beta},
\]

where \( \Lambda \) includes only integral linear combinations of simple roots with positive coefficients. The polynomials \( \chi_m \) can be written as \( \chi_m(z) = z^m + \sum_{\mu \in \Psi} S_{m, \mu} z^{m - \mu} \), where again only positive powers of the \( z \)'s appear. By substituting in the Schrödinger equation (3) we find the recursive formula for the coefficients of the polynomials,

\[
S_{m, \mu} = \frac{1}{\varepsilon(m; 1) - \varepsilon(m - \mu; 1)} \sum_{\beta \in \Lambda} R_{m - (\lambda - \beta), \beta} S_{m - \beta, \mu - \beta}.
\]

This recursive formula is suitable for implementation in programs like Mathematica, Maple or others. For further details or an alternative method, see \([15]\).

### 2.2. A differential equation for the generating function

Our goal is to compute the generating function for the characters of \( A \), which is the formal series

\[
G(t; z_k) = \prod_{m_1=0}^{\infty} \prod_{m_2=0}^{\infty} \cdots \prod_{m_r=0}^{\infty} \sum_{m_1^1, m_2^1, \ldots, m_r^1} \chi_{m_1, m_2^1, \ldots, m_r^1}(z_1, z_2, \ldots, z_r) \]

in the auxiliary variables \( t_1, t_2, \ldots, t_r \) and where we will treat the characters as polynomials in the \( z \)-variables. Now, from this definition and (3) and (4), we see that the generating function satisfies the differential equation

\[
(\Delta_t - \Delta^1_t^*) G(t; z_k) = 0,
\]

where \( \Delta_t \) is the differential operator

\[
\Delta_t = \varepsilon(t_1 \partial_t_1, t_2 \partial_t_2, \ldots, t_r \partial_t_r; 1).
\]

So, in principle, one could formulate the problem of finding the generating function as the problem of solving (7) with some suitable boundary conditions. Of course, posed in this abstract form, the problem seems to be difficult, but as we want the solution to be of the form (6), we can proceed in the reverse way, i.e. we will take (6) as the basis to build up a sensible ansatz and will afterward check that the ansatz satisfies (7).
2.3. Solving the differential equation

We expect that the solution of (7) which corresponds to the generating function is a rational function [3–6]

\[ G(t_j; z_k) = \frac{N(t_j; z_k)}{D(t_j; z_k)} \]

where both the numerator \( N(t_j; z_k) \) and the denominator \( D(t_j; z_k) \) are polynomials in the \( t \)-variables with coefficients depending on the \( z \)-variables. We will find a solution of this form through four successive steps. In the first three steps we will try to build up \( D(t_j; z_k) \) and \( N(t_j; z_k) \) in a reasonable fashion. Then the fourth step is to check that the tentative form of \( G(t_j; z_k) \) arising in this way does indeed satisfy the differential equation (7).

(i) By Weyl invariance, the term \( t_1^{m_1}t_2^{m_2} \cdots t_r^{m_r}X_{m_1,m_2,\ldots,m_r} \) of the generating function includes a summand of the form \( \prod_{h=1}^\infty e(hu_p)^{m_h} \) for each possible combination formed by picking up a weight \( w_j \) of each Weyl orbit \( W\lambda_j \) of the fundamental weights. We can obtain all these products multiplied by \( t_1^{m_1}t_2^{m_2} \cdots t_r^{m_r} \) if we multiply the geometric series \( \sum_{p=0}^\infty (t_j)^p e(hw_j)^p \) for all weights \( w \) entering in the Weyl orbit of each \( \lambda_j \). Therefore, the denominator of the generating function should be of the form

\[ D = D_1 \times D_2 \times \cdots \times D_r, \]  

where the factor \( D_j \) is given by the formula

\[ D_j = \prod_w (1 - t_j e(w)) \]  

with the product extended to all the weights of \( W\lambda_j \). Thus, \( D_j \) is a polynomial in \( t_j \) of degree equal to the cardinal \( |W\lambda_j| \) of the orbit. The coefficients depend on the phases \( x_l \) but, as the polynomial is by construction invariant under Weyl reflections, can be rewritten in terms of the Weyl invariant variables \( z_k \). So, in this first step we compute each \( D_j \) by means of (9) and use (1) and (2) to put it as a function of the \( z \)-variables. Finally, (8) yields \( D(t_j; z_k) \).

(ii) In the second step we compute the generating function

\[ F(t_1, t_2, \ldots, t_r) = \sum_{m_1=0}^\infty \sum_{m_2=0}^\infty \cdots \sum_{m_r=0}^\infty t_1^{m_1}t_2^{m_2} \cdots t_r^{m_r} \dim R_{m_1\lambda_1 + m_2\lambda_2 + \cdots + m_r\lambda_r} \]  

for the dimensions of the irreducible representations of the algebra \( \mathcal{A} \). This generating function is related to \( G(t_j; z_k) \) through the replacement \( x_l \to 1 \) or, equivalently, \( z_k \to \dim R_{\lambda_k} \), i.e. \( F(t_j) = G(t_j; \dim R_{\lambda_k}) \). The Weyl formula for dimensions implies that the coefficients of the series (10) are polynomials in the exponents \( m_j \). Thus, \( F(t_j) \) is a rational function which goes to zero for \( t_j \to \infty \) at least as \( t_j^{-1} \). So, we can write

\[ F(t_j) = \frac{P(t_j)}{Q(t_j)} \]  

where the denominator is known,

\[ Q(t_j) = D(t_j; \dim R_{\lambda_k}) = (1 - t_1)^{|W\lambda_k|} (1 - t_2)^{|W\lambda_k|} \cdots (1 - t_r)^{|W\lambda_k|}, \]

and the numerator should be of the form

\[ P(t_1, t_2, \ldots, t_r) = \sum_{I_1=0}^{[W\lambda_k]-1} \sum_{I_2=0}^{[W\lambda_k]-1} \cdots \sum_{I_r=0}^{[W\lambda_k]-1} t_1^{I_1}t_2^{I_2} \cdots t_r^{I_r} p_{I_1,I_2,\ldots,I_r}. \]  

In fact, one can see that the use of the Weyl character formula leads to this form of \( P \) as the common denominator in (6) [4].
Here, $p_{0,0,...,0} = 1$ and the remaining numerical coefficients $p_{l_1,l_2,...,l_t}$ are to be determined by comparing the Taylor expansion of (11) with the right member of (10). It will turn out that most of these coefficients are zero and the final form of $F(t_1)$ can be fixed after a few steps involving some very simple equations.

(iii) In the third step we come back to the numerator $N(t_1; z_k)$ through the ansatz

$$N(t_1; z_k) = \sum_{l_1,l_2,...,l_t} C_{l_1,l_2,...,l_t}(z_k) t_1^{l_1} t_2^{l_2} \cdots t_t^{l_t},$$

where, to simplify matters, we make the reasonable assumption that the only terms appearing in $N(t_1; z_k)$ are those corresponding to the nonvanishing $p$-coefficients. Often, we can guess some of the $C_{l_1,l_2,...,l_t}(z_k)$ directly from the numerical value of the coefficient $p_{l_1,l_2,...,l_t}$. For the others, we will have to write explicitly some low order terms of the right member of (6) and compare them with the Taylor expansion of $G(t_1; z_k)$. The characters needed to accomplish this task are polynomials in the $z$-variables which are easy to obtain by solving (3) with $\Delta^1_z$ written as in (5). Once this is done, we have a tentative expression for $G(t_1; z_k)$.

(iv) The remaining step is to check that the simplifying assumption made in step (iii) is valid and our tentative generating function is indeed correct. So, we substitute it in (7) and check that the differential equation is satisfied. Notice that, as we have manufactured our tentative generating function using only some low-order characters, this fourth step is necessary to prove that it gives (6) to all orders in the $t$-variables.

To sum up, the main ingredients needed to carry on the proposed procedure are: (i) the weights entering in the Weyl orbits and representations of the fundamental weights of the algebra, which can be obtained from sources like [18–21]; (ii) the Weyl formula for dimensions [18, 19]; and (iii) the Calogero–Sutherland Hamiltonian in $z$-variables and an iterative method for computing characters from it, see [15] for several examples. With this information and the help of some program for symbolic calculus, steps (i)–(iv) can be carried out for any simple Lie algebra.

3. Some examples

Let us now apply the method described in the previous section to some Lie algebras in order to appreciate how it works. We will work out the cases of the simple classical Lie algebras up to rank two, i.e. $A_1 = B_1 = C_1 = D_1, A_2$ and $C_2 \simeq B_2$, since $D_2 \simeq A_1 \oplus A_1$ is semisimple.

3.1. The generating function for the algebra $A_1$

This is the simplest case. The algebra has only one fundamental weight $\lambda_1$ and the Weyl orbit $W\lambda_1$ has weights $\{\lambda_1, -\lambda_1\}$. The fundamental representation is $R_{\lambda_1} = W\lambda_1$, so the character $z_1$ is

$$z_1 = e^{i\varphi_1} + e^{-i\varphi_1} = x_1 + \frac{1}{x_1}. \quad (13)$$

The generating functions for characters and dimensions are

$$G(t_1; z_1) = \sum_{m_1=0}^{\infty} t_1^{m_1} \chi_{m_1}(z_1), \quad F(t_1) = \sum_{m_1=0}^{\infty} t_1^{m_1} \dim R_{m_1}.$$

Let us now apply our four-step method. As there is only one weight, the denominator $D(t_1; z_1)$ is simply $D(t_1; z_1) = D_1$ with

$$D_1 = (1 - t_1 x_1) \left( 1 - t_1 \frac{1}{x_1} \right) = 1 - t_1 z_1 + t_1^2.$$
Now, as the dimension of $R_{\lambda_1}$ is two, the denominator of the generating function for dimensions is $Q(t_1) = D(t_1, 2) = (1 - t_1)^2$. For the numerator, we try the ansatz $P(t_1) = 1 + p_1 t_1$. The Weyl formula for dimensions gives $\dim R_{m_1} = m_1 + 1$ and comparing this ansatz with the first few terms of the series $F(t_1)$, one readily finds that $p_1 = 0$. Therefore

$$F(t_1) = \frac{1}{(1 - t_1)^2}. \tag{14}$$

Given the form of $F(t_1)$, our ansatz for the numerator for the generating function for characters is

$$N(t_1; z_1) = C_0(z_1) \tag{15}$$

but, given that $d_0 = 1$, the only sensible guess is $C_0(z_1) = 1$ and the tentative expression for the generating function is

$$G(t_1; z_1) = \frac{1}{1 - t_1 z_1 + t_1^2}. \tag{16}$$

It remains to confirm that this is correct. For that, we resort to the Calogero–Sutherland theory related to the algebra $A_1$ (see for example [22]), which leads to the differential operator

$$\Delta_1^t = (z_1^2 - 4) \partial^2_{z_1} + 3 z_1 \partial_{z_1},$$

with eigenvalues given by

$$\varepsilon(m_1; 1) = m_1^2 + 2 m_1. \tag{17}$$

Thus, the operator $\Delta_t$ is $\Delta_t = t_1^2 \partial^2_{z_1} + 3 t_1 \partial_{z_1}$ and one can check that

$$\left(\Delta_t - \Delta_1^t\right) \frac{1}{1 - t_1 z_1 + t_1^2} = 0,$$

as it should be. Therefore (14) gives correctly the generating function for the characters of $A_1$.

### 3.2. The generating function for the algebra $A_2$

In this case there are two fundamental weights $\lambda_1, \lambda_2$, and the corresponding Weyl orbits $W_{\lambda_1}$ and $W_{\lambda_2}$ have, respectively, weights $\{\lambda_1, \lambda_2 - \lambda_1, -\lambda_1\}$ and $\{\lambda_2, \lambda_1 - \lambda_2, -\lambda_1\}$. The fundamental representations $R_{\lambda_1}$ and $R_{\lambda_2}$ have only one Weyl orbit. Therefore, the characters $z_1$ and $z_2$ are

$$z_1 = e^{i \phi_1} + e^{-i \phi_2} + e^{i (\phi_2 - \phi_1)} = x_1 + \frac{1}{x_2} + \frac{x_2}{x_1} \tag{18}$$

$$z_2 = e^{i \phi_2} + e^{-i \phi_1} + e^{i (\phi_1 - \phi_2)} = x_2 + \frac{1}{x_1} + \frac{x_1}{x_2}. \tag{19}$$

The generating functions to be computed are

$$G(t_1, t_2; z_1, z_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \dim R_{m_1, m_2} x_{m_1, m_2} (z_1, z_2) \tag{20}$$

$$F(t_1, t_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \dim R_{m_1, m_2} \tag{21}$$
and we follow the same steps than before. The denominator of the generating function for characters is $D(t_1, t_2; z_1, z_2) = D_1 \times D_2$ where

$$D_1 = (1 - t_1 x_1) \left(1 - t_1 \frac{x_1}{x_2}\right) = 1 - t_1 z_1 + t_1^2 z_2 - t_1^3$$

$$D_2 = (1 - t_2 x_2) \left(1 - t_2 \frac{x_1}{x_2}\right) = 1 - t_2 z_2 + t_2^2 z_1 - t_2^3.$$

Since both $R_m$ and $R_n$ have dimension three, the denominator of the generating function for dimensions is $Q(t_1, t_2) = D(t_1, t_2; 3, 3) = (1 - t_1)^3(1 - t_2)^3$. For the numerator, we try the ansatz

$$P(t_1, t_2) = \sum_{l_1=0}^{2} \sum_{l_2=0}^{2} p_{l_1,l_2} t_1^{l_1} t_2^{l_2}$$

and, using that $\dim R_m \cdot R_n = \frac{1}{2} (m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)$ to compare with the right member of (18), we find that the only nonvanishing coefficients are $p_{0,0} = -p_{1,1} = 1$. Thus, the generating function for dimensions is

$$F(t_1, t_2) = \frac{1 - t_1 t_2}{(1 - t_1)^3(1 - t_2)^3}.$$

In view of that, the ansatz for the numerator for the generating function for characters is

$$N(t_1, t_2; z_1, z_2) = C_{0,0}(z_1, z_2) + C_{1,1}(z_1, z_2) t_1 t_2$$

and the reasonable guess is $C_{0,0}(z_1, z_2) = C_{1,1}(z_1, z_2) = 1$. One can confirm that this guess is correct by comparing the ansatz with the right member of (17): to do so, the only characters needed are

$$\chi_{1,0}(z_1, z_2) = z_1, \quad \chi_{0,1}(z_1, z_2) = z_2, \quad \chi_{1,1}(z_1, z_2) = z_1 z_2 - 1.$$

Therefore, the tentative form of the generating function is

$$G(t_1, t_2; z_1, z_2) = \frac{1 - t_1 t_2}{(1 - t_1 z_1 + t_1^2 z_2 - t_1^3)(1 - t_2 z_2 + t_2^2 z_1 - t_2^3)}.$$

Now, the second order differential operator associated to the Calogero–Sutherland model for the algebra $A_2$ whose eigenfunctions are the characters is $[13, 14]$:

$$\Delta = (z_1^2 - 3 z_2) \partial_{z_1}^2 + (z_2^2 - 3 z_1) \partial_{z_2}^2 + (z_1 z_2 - 9) \partial_z \partial_{z_1} + 4 z_1 \partial_{z_2} + 4 z_2 \partial_z$$

and their eigenvalues are

$$\varepsilon(m_1, m_2; 1) = m_1^2 + m_2^2 + m_1 m_2 + 3 m_1 + 3 m_2.$$

Thus, the operator $\Delta_r$ is $\Delta_r = t_1^2 \partial_{z_1}^2 + t_2^2 \partial_{z_2}^2 + t_1 t_2 \partial_z \partial_{z_1} + 4 t_1 \partial_{z_2} + 4 t_2 \partial_z$. One can check that, in fact,

$$\left(\Delta_r - \Delta_r \right) \frac{1 - t_1 t_2}{(1 - t_1 z_1 + t_1^2 z_2 - t_1^3)(1 - t_2 z_2 + t_2^2 z_1 - t_2^3)} = 0$$

so that (19) is the correct generating function for the characters of $A_2$. 
3.3. The generating function for the algebra $C_2$

Again, there are two weights, and the weights in the Weyl orbits $W\lambda_1$ and $W\lambda_2$ are, respectively, 
\{\lambda_1, -\lambda_1, \lambda_2, -\lambda_2\} and \{\lambda_2, -\lambda_2, 2\lambda_1 - \lambda_2, -2\lambda_1 + \lambda_2\}. The fundamental representation $R_1$ has only one Weyl orbit, but $R_2$ includes also the null weight. Thus, the characters $z_1$ and $z_2$ are

$$z_1 = e^{i\phi_1} + e^{-i\phi_1} + e^{i(\phi_1 - \phi_2)} + e^{i(\phi_2 - \phi_1)} = x_1 + \frac{1}{x_1} + \frac{x_1}{x_2} + \frac{x_2}{x_1}$$

(20)

$$z_2 = 1 + e^{i\phi_2} + e^{-i\phi_2} + e^{i(2\phi_1 - \phi_2)} + e^{i(\phi_1 - 2\phi_2)} = 1 + x_2 + \frac{1}{x_2} + \frac{x_1^2}{x_2} + \frac{x_2}{x_1^2}.$$  

(21)

The denominator of the generating function for characters is $D(t_1, t_2; z_1, z_2) = D_1 \times D_2$ with

$$D_1 = (1 - t_1 x_1) \left(1 - t_1 \frac{1}{x_1}\right) \left(1 - t_2 \frac{x_1}{x_2}\right) = 1 - t_1 z_1 + t_1^2 (z_2 + 1) - t_1^0 z_1 + t_1^4$$

$$D_2 = (1 - t_2 x_2) \left(1 - t_2 \frac{1}{x_2}\right) \left(1 - t_2 \frac{x_1^2}{x_2^2}\right) = 1 - t_2 (z_2 - 1) + t_2^2 (z_2^2 - 2z_2 - 1) + t_2^4,$$

while the denominator of the generating function for dimensions is

$$Q(t_1, t_2) = D(t_1, t_2; 4, 5) = (1 - t_1)^4(1 - t_2)^4.$$

For the numerator, we try the ansatz

$$P(t_1, t_2) = \sum_{l_1=0}^{3} \sum_{l_2=0}^{3} p_{l_1,l_2} t_1^{l_1} t_2^{l_2}$$

and, by means of $\dim R_{m_1,m_2} = \frac{1}{6} (m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)(m_1 + 2m_2 + 3),$ we find that the only nonvanishing coefficients are $p_{0,0} = p_{0,1} = p_{2,1} = p_{2,2} = 1$ and $p_{1,1} = -4.$ Thus

$$F(t_1, t_2) = \frac{(1 + t_2) (1 + t_1 t_2) - 4t_1 t_2}{(1 - t_1)^4(1 - t_2)^4}.$$  

Given this result, the ansatz for the numerator of the generating function for characters is

$$N(t_1, t_2; z_1, z_2) = C_{0,0} (z_4) + C_{0,1} (z_4) t_1 + C_{1,1} (z_4) t_1 t_2 + C_{1,2} (z_4) t_1^2 + C_{2,2} (z_4) t_1^2 t_2^2$$

and the reasonable guess is $C_{0,0} = C_{0,1} = C_{2,1} = C_{2,2} = 1$ and $C_{1,1} = -z_1.$ Again, one can check that this guess is accurate by comparing with the explicit series of characters: the only characters needed are $X_{1,0}, X_{0,11}, X_{2,1},$ and $X_{2,2},$ which are given in the appendix. Therefore the tentative form of the generating function is

$$G(t_1, t_2; z_1, z_2)$$

$$= \frac{1 + t_2 - z_1 t_1 t_2 + t_1^2 t_2 + t_1^3 t_2^2}{(1 - (t_1 + t_1^2) z_1 + t_1^2 (z_2 + 1) + t_1^4) (1 - (t_2 + t_2^2) (z_2 - 1) + t_2^3 (z_2^2 - 2z_2) + t_2^4)}.$$  

(22)

Now we need the Calogero–Sutherland differential operator for $C_2,$ which is shown in the appendix to be

$$\Delta_i = (\xi^2 - 2z_2 - 6) \partial_{\xi_{i+1}}^2 + (2z_2^2 - 4z_2^2 + 4z_2 - 6) \partial_{\xi_i}^2 + (2z_1 z_2 - 10z_1) \partial_{\xi_i} \partial_{\xi_{i+1}} + 5z_1 \partial_{\xi_i} + 4z_2 \partial_{\xi_i}.$$  

Since their eigenvalues are

$$(\varepsilon (m_1, m_2; 1) = m_1^2 + 2m_2^2 + 2m_1 m_2 + 4m_1 + 6m_2,$$

we see that the operator $\Delta_i$ is $\Delta_i = t_1^2 \partial_{\xi_i}^2 + 2t_1^2 \partial_{\xi_i}^2 + 2t_2 t_1 \partial_{\xi_i} \partial_{\xi_{i+1}} + 5t_1 \partial_{\xi_i} + 8t_2 \partial_{\xi_i}.$ We now can check that

$$\Delta_i (\Delta_i - \Delta_{i+1}^\perp) \frac{1 + t_2 - z_1 t_1 t_2 + t_1^2 t_2 + t_1^3 t_2^2}{(1 - (t_1 + t_1^2) z_1 + t_1^2 (z_2 + 1) + t_1^4) (1 - (t_2 + t_2^2) (z_2 - 1) + t_2^3 (z_2^2 - 2z_2) + t_2^4)} = 0$$

and this proves that (22) is the correct generating function for the characters of $C_2.$
3.4. Generating functions for some subsets of characters

If in the previous examples for $A_2$ and $C_2$ we take $t_1 = 0$ or $t_2 = 0$, we obtain the generating function for the characters of the form $X_{m,0}(z_1, z_2)$ or $X_{0,m}(z_1, z_2)$ of these algebras. It is also possible to use the method that we have been describing to compute the generating function of some other subsets of characters. Let us, for instance, consider the case of the diagonal characters of $A_2$ and seek for their generating function

$$G_{\text{diag}}(t; z_1, z_2) = \sum_{m=0}^{\infty} t^m X_{m,m}(z_1, z_2).$$

Now, according to the reasoning of section 2, the denominator should be a product including all factors of the form $(1 - t e(w_1) e(w_2))$ for all couples of nonzero weights $w_1$ and $w_2$ entering in the Weyl orbits $W_{\lambda_1}$ and $W_{\lambda_2}$ of $A_2$. This gives

$$D_{\text{diag}}(t; z_1, z_2) = 1 + t^6 + (3 - z_1 z_2) (t + t^2) + (6 + z_1^3 - 5 z_1 z_2 + z_2^3) (t^2 + t^4) + (7 + 2 z_1^3 - 6 z_1 z_2 - z_1^2 z_2 + 2 z_2^3) t^6.$$

From this, we can find easily the generating function for the dimensions of the diagonal characters, which is

$$F_{\text{diag}}(t) = \frac{1 + 2t - 6t^2 + 2t^3 + t^4}{(1 - t)^6}.$$

Thus, we try the ansatz

$$N_{\text{diag}}(t; z_1, z_2) = \sum_{l=0}^{4} C_l(z_1, z_2) t^l$$

for the numerator, and using the explicit form of the characters $X_{m,m}(z_1, z_2)$ for $m = 0$ to 4, we arrive to a tentative form of $G_{\text{diag}}(t; z_1, z_2)$ as

$$G_{\text{diag}}(t; z_1, z_2) = \frac{1 + 2t - (z_1 z_2 - 3) t^2 + 2 t^3 + t^4}{D_{\text{diag}}(t; z_1, z_2)}.$$  \hspace{1cm} (23)

For this subset of characters, the operator $\Delta_t$ takes the form $\Delta_t = 3 t^2 \partial_t^2 + 9 t \partial_t$, and one can check that $(\Delta_t - \Delta^0_t) G_{\text{diag}}(t; z_1, z_2) = 0$. Thus, (23) gives in fact the correct generating function.

The same procedure applied to $C_2$ gives the generating function of diagonal characters of that algebra. The result is

$$G_{\text{diag}}(t; z_1, z_2) = \frac{(1 - t^2)(1 + t^4 + 2 t z_1 + 2 t^3 z_1 + t^4(2 z_1^2 - z_1 z_2 + z_2 + z_2^2))}{1 + t^8 - (t + t^3) d_1 + (t^2 + t^6) d_2 + (t^3 + t^5) d_3 + t^4 d_4}$$

where

- $d_1 = z_1 (-3 + z_2)$
- $d_2 = -1 + z_1^4 + z_1^2 (3 - 6 z_2) + z_2 + 3 z_2^2 + z_2^3$
- $d_3 = z_1 (-3 + 2 z_1^4 - 2 z_2 + 8 z_2^2 + 3 z_2^3 - z_1^2 (-2 + 9 z_2 + z_2^2))$
- $d_4 = z_1^4 + z_1^2 (4 - 6 z_2) + z_1 (-5 - 6 z_2 + 5 z_2^2) + z_2 (-2 + 3 z_2 + 4 z_2^2 + z_2^3)$.
4. Concluding remarks

We have presented an approach for the computation of the generating function of the characters of a simple Lie algebra which is based on the theory of an integrable mechanical system, namely the quantum trigonometric Calogero–Sutherland model. The key point is that the Schrödinger equation of that model leads to a differential equation for the generating function. This equation can be solved by means of a convenient ansatz which has been described in detail. The procedure involves the computation of some low order characters and this can be done by solving the Calogero–Sutherland Schrödinger equation. The approach is by design Weyl invariant. We avoid the use of alternating functions by formulating all computations in terms of a set of dynamical variables \( z \) which correspond to the characters of the fundamental representations of the algebra. We have illustrated our approach by applying it to some low rank classical algebras, but we expect it to be equally useful for the classical or exceptional higher rank ones. Of course, in these cases the calculations are longer, but the type of mathematical objects that they involve are polynomials in \( z_k \) variables with integer coefficients. This make the computations especially well suited for the use of programs like Mathematica or Maple. In fact, we find it likely that this approach is more efficient than others based on the Weyl character formula. At any event, when one is faced to some complicated problem as it is the computation of the generating functions for characters in closed form, it is always desirable to have the possibility of choosing among different approaches.

There are other interesting mathematical objects, such as the Weyl invariant monomial functions of some zonal spherical functions in symmetric spaces, which are also identical to the eigenfunctions of the Calogero–Sutherland model for adequate values of the coupling constants. It could be that an approach similar to that advocated for in this paper be useful for obtaining generating functions for them.

Finally, let us take the example of \( A_2 \) to comment on an alternative way to obtain the denominator \( D(t; z) \) which could be useful for higher rank algebras. If we take \( t_2 = 0 \) in (19) we have:

\[
D_1 G(t_1, 0; z_1, z_2) = (1 - t_1 z_1 + t_1^2 z_2 - t_1^3) \sum_{m_1=0}^{\infty} t_1^{m_1} \chi_{m_1,0}(z_1, z_2) = 1
\]

and this implies the recurrence relation

\[
\chi_{m_1,0} - z_1 \chi_{m_1-1,0} + z_2 \chi_{m_1-2,0} - \chi_{m_1-3,0} = 0 \quad \text{for} \quad m_1 > 3. \tag{24}
\]

This recurrence relation can also be easily obtained by combining two well-known Clebsch–Gordan series of \( A_2 \), see for instance [13]

\[
\begin{align*}
z_1 \chi_{m_1,0} &= \chi_{m_1+1,0} + \chi_{m_1-1,1} \\
z_2 \chi_{m_1,0} &= \chi_{m_1,1} + \chi_{m_1-1,0}.
\end{align*}
\]

Thus, by working backwards, we could have obtained \( D_1 \) from the Clebsch–Gordan series. This has the advantage that the result appears directly written in the \( z \) variables. For an arbitrary simple algebra, \( D_j \) is of degree \( |W \lambda_j| \) and the characters \( \chi_{m_0,0,0,0,0,0,0} \) for \( m_j > |W \lambda_j| \) will obey a recurrence relation of \( |W \lambda_j| + 1 \) terms similar to (24). Now, if we know the Clebsch–Gordan series of type \( z_k \chi_m \) for the algebra, we can combine them to obtain this recurrence relation and, in this way, we can compute the factors \( D_j \) appearing in the denominator of the generating function. Some useful sources for Clebsch–Gordan series are [18–21] and for examples of how to compute the Clebsch–Gordan series using the Calogero–Sutherland Schrödinger equation see [13] and the second and third papers in [15].
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Appendix

In this appendix, we present the Calogero–Sutherland differential operator whose eigenfunctions are the characters of the algebra $C_2$ and give a short collection of characters of $C_2$ which includes those needed to fix the numerator of the generating function $G(t_1, t_2; z_1, z_2)$. For a more detailed account of the use of the quantum Calogero–Sutherland models to deal with the characters or other generalized orthogonal polynomials related to Lie algebras, see [15] and references therein.

The general form of the differential operator we are looking for is

$$\Delta_1 = a_{1,0}(z_1, z_2) \partial_{z_1}^2 + a_{0,1}(z_1, z_2) \partial_{z_2}^2 + a_{1,1}(z_1, z_2) \partial_{z_1} \partial_{z_2} + b_1(z_1, z_2) \partial_{z_1} + b_2(z_1, z_2) \partial_{z_2}$$

and we will fix the coefficients using that the character $\chi_{m_1, m_2}$ of the irreducible representation of $C_2$ with highest weight $\lambda = m_1 \lambda_1 + m_2 \lambda_2$ is an eigenfunction of the Schrödinger equation

$$\Delta_1 \chi_{m_1, m_2} = \epsilon(m_1, m_2; 1) \chi_{m_1, m_2}$$

with eigenvalue

$$\epsilon(m_1, m_2; 1) = \langle \lambda, \lambda \rangle + 2 \langle \lambda, \rho \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in the space spanned by the orthonormal basis $\{e_1, e_2\}$, the two fundamental weights of $C_2$ are represented in this basis by

$$\lambda_1 = e_1, \quad \lambda_2 = e_1 + e_2,$$

the four positive roots are

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = 2e_2, \quad \alpha_3 = e_1 + e_2, \quad \alpha_4 = 2e_1,$$

and $\rho = \lambda_1 + \lambda_2$ is the Weyl vector of the algebra [18, 19], so that

$$\epsilon(m_1, m_2; 1) = m_1^2 + 2m_2^2 + 2m_1m_2 + 4m_1 + 6m_2.$$

On the other hand, given that

$$\chi_{1,0}(z_1, z_2) = z_1 \quad \chi_{0,1}(z_1, z_2) = z_2,$$

the direct products of the representations of $C_2$ [18, 19]

$$R_{1,0} \otimes R_{1,0} = R_{2,0} + R_{0,1} + R_{0,0}$$
$$R_{1,0} \otimes R_{0,1} = R_{1,1} + R_{1,0}$$
$$R_{0,1} \otimes R_{0,1} = R_{0,2} + R_{2,0} + R_{0,0}$$

allow us to solve for

$$\chi_{2,0}(z_1, z_2) = z_1^2 - z_2 - 1$$
$$\chi_{1,1}(z_1, z_2) = z_1z_2 - z_1$$
$$\chi_{0,2}(z_1, z_2) = z_2^2 - z_1^2 + z_2.$$
Using all these characters in the Schrödinger equation one finds

\[ a_{1,0}(z_1, z_2) = z_1^3 - 2z_2 - 6 \]
\[ a_{0,1}(z_1, z_2) = 2z_1^2 - 4z_1^2 + 4z_2 - 6 \]
\[ a_{1,1}(z_1, z_2) = 2z_1z_2 - 10z_1 \]
\[ b_1(z_1, z_2) = 5z_1 \]
\[ b_1(z_1, z_2) = 4z_2. \]

Once the operator \( \Delta_1 \) is known, other characters of \( C_2 \) can be computed by solving the Schrödinger equation. We give here a few of them:

\[ \begin{align*}
\chi_{0,3}(z_1, z_2) &= -1 + z_1^3 - 2z_1^2z_2 + 2z_2^3 + z_2^3 \\
\chi_{1,2}(z_1, z_2) &= z_1 - z_1^3 + z_1z_2^2 \\
\chi_{2,1}(z_1, z_2) &= 1 - z_1^3 - z_2 + z_1^2z_2 - 2z_1z_2^2 \\
\chi_{3,0}(z_1, z_2) &= -z_1 + z_1^3 - 2z_1z_2 \\
\chi_{0,4}(z_1, z_2) &= -z_1^3 + z_1^3 - 2z_1^2z_2 + 2z_2^3 + 3z_2^3 + z_2^3 \\
\chi_{1,3}(z_1, z_2) &= -2z_1 + 2z_1^3 - 2z_1^2z_2 + z_2^3z_2 + z_1z_2^2 \\
\chi_{2,2}(z_1, z_2) &= 2z_1^2 - z_1^3 + z_1^2z_2 - 2z_2^3 + z_1^2z_2^2 - z_1^2 \\
\chi_{3,1}(z_1, z_2) &= 2z_1 - z_1^3 + z_1^2z_2 - 2z_1^2z_2 \\
\chi_{4,0}(z_1, z_2) &= -z_1 + z_1^3 + 2z_2 - 3z_1^2z_2 + z_2^3 \\
\chi_{0,5}(z_1, z_2) &= 3z_1^3 - 2z_1^3 - 2z_1^2z_2 + 3z_1z_2^3 - 3z_2^3 - 3z_1^3 + 3z_2 - 4z_1^3z_2 + 4z_2^3 + z_2^3 \\
\chi_{1,4}(z_1, z_2) &= 2z_1 - 3z_1^3 + z_1^3 - 2z_1^2z_2 + 2z_2^3 + 3z_1^3 + 2z_1z_2^3 + z_2^3 \\
\chi_{2,3}(z_1, z_2) &= 1 - 4z_1^3 + 2z_1^3 + 2z_2 + z_1^3z_2 - 2z_1^2z_2 - z_1^3z_2 + 3z_1^3 + z_2z_1^2 - z_1^2 \\
\chi_{3,2}(z_1, z_2) &= -z_1 + 2z_1^3 - z_1^3 - 2z_1^2z_2 + z_2^3z_2 + z_1z_2^2 - 2z_1^2z_2 \\
\chi_{4,1}(z_1, z_2) &= -1 + 2z_1^3 - z_1^3 - z_2 + z_1^2z_2 + z_1^2z_2 + z_2^3 + z_1^2z_2^3 + z_2^3 \\
\chi_{5,0}(z_1, z_2) &= -z_1 - z_1^3 + z_1^3 + 4z_1z_2 - 4z_1^3z_2 + 3z_1z_2^3 \\
\chi_{0,6}(z_1, z_2) &= 1 - 3z_1^3 + 3z_1^3 - z_1^3 - 6z_1^3z_2 - 3z_1z_2 - 3z_1^3 + 3z_1^3 + 6z_1^3z_2 - 8z_1^3z_2 + 6z_2^3 \\
&- 5z_1^3z_2^3 + 5z_1^3 + z_2^3 \\
\chi_{1,5}(z_1, z_2) &= -2z_1 + 6z_1^3 - 3z_1^3 - 2z_1^3z_2 + 3z_1z_2^3 - 3z_1z_2 - 4z_1^3z_2 + 3z_1z_2 + z_1z_2^3 \\
\chi_{2,4}(z_1, z_2) &= -1 + 4z_1^3 - z_1^3 - 2z_2 + 3z_1^3z_2 + z_1^2z_2 + 3z_1^3 + 3z_1^3z_2 + 3z_1^3z_2^3 \quad + 3z_1^3z_2^3 + 3z_1^3z_2^3 + z_2^3 \\
&+ 5z_1^3z_2^3 + z_1^3z_2^3 - z_1^3 \\
\chi_{3,3}(z_1, z_2) &= 2z_1 - 5z_1^3 + 2z_1^3 + 4z_1z_2 - z_1^3z_2 - 2z_1^3z_2 + 5z_1^3z_2^3 - 4z_1^3z_2 - z_1^3z_2 + z_1^3z_2^3 \\
\chi_{4,2}(z_1, z_2) &= -z_1^3 + 2z_1^3 - z_1^3 - z_2 + 3z_1^3z_2 + 3z_1^3z_2 + 3z_1^3z_2 + z_1^3z_2 + 3z_1^3 + 3z_1^3z_2 + z_1^3z_2 \\
\chi_{5,1}(z_1, z_2) &= -z_1 + 2z_1^3 - z_1^3 - 5z_1z_2 + 2z_1^3z_2 + z_1^3z_2 + 4z_1^3z_2 + 3z_1^3z_2 + 3z_1^3z_2 \\
\chi_{6,0}(z_1, z_2) &= -1 + 2z_1^3 - z_1^3 + z_1^3 - z_2 + 6z_1^3z_2 - 5z_1^3z_2 - 3z_1^3z_2 + 6z_1^3z_2 - z_1^3z_2.
\end{align*} \]

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