DIRECT IMAGE OF LOGARITHMIC COMPLEXES
AND INFINITESIMAL INVARIANTS OF CYCLES

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ABSTRACT. We show that the direct image of the filtered logarithmic de Rham complex is a direct sum of filtered logarithmic complexes with coefficients in variations of Hodge structures, using a generalization of the decomposition theorem of Beilinson, Bernstein and Deligne to the case of filtered $D$-modules. The advantage of using the logarithmic complexes is that we have the strictness of the Hodge filtration by Deligne after taking the cohomology group in the projective case. As a corollary, we get the total infinitesimal invariant of a (higher) cycle in a direct sum of the cohomology of filtered logarithmic complexes with coefficients, and this is essentially equivalent to the cohomology class of the cycle.

Introduction

Let $X$, $S$ be complex manifolds or smooth algebraic varieties over a field of characteristic zero. Let $f : X \to S$ be a projective morphism, and $D$ be a divisor on $S$ such that $f$ is smooth over $S \setminus D$. We have a filtered locally free $\mathcal{O}$-module $(V^i, F)$ on $S \setminus D$ underlying a variation of Hodge structure whose fiber $V^i_s$ at $s \in S \setminus D$ is the cohomology of the fiber $H^i(X_s, \mathbb{C})$. If $D$ is a divisor with normal crossings on $S$, let $\tilde{V}^i$ denote the Deligne extension [7] of $V^i$ such that the eigenvalues of the residue of the connection are contained in $[0, 1)$. The Hodge filtration $F$ is naturally extended to $\tilde{V}^i$ by [25]. We have the logarithmic de Rham complex

$$\text{DR}_{\log}(\tilde{V}^i) = \Omega^\bullet_X(\log D) \otimes_{\mathcal{O}} \tilde{V}^i,$$

which has the Hodge filtration $F^p$ defined by $\Omega^p_X(\log D) \otimes_{\mathcal{O}} F^{p-j}\tilde{V}^i$. In general, $V^i$ can be extended to a regular holonomic $D_S$-module $M^i$ on which a local defining equation of $D$ acts bijectively. By [23], $M^i$ and hence the de Rham complex $\text{DR}(M^i)$ have the Hodge filtration $F$. If $Y := f^*D$ is a divisor with normal crossings on $X$, then $\Omega^\bullet_X(\log Y)$ has the Hodge filtration $F$ defined by the truncation $\sigma$ (see [8]) as usual, i.e. $F^p\Omega^\bullet_X(\log Y) = \Omega^\bullet_X^{\geq p}(\log Y)$.

Theorem 1. Assume $Y = f^*D$ is a divisor with normal crossings. There is an increasing split filtration $L$ on the filtered complex $Rf_*(\Omega^\bullet_X(\log Y), F)$ such that we have noncanonical and canonical isomorphisms in the filtered derived category:

$$Rf_*(\Omega^\bullet_X(\log Y), F) \simeq \bigoplus_{i \in \mathbb{Z}} (\text{DR}(M^i), F)[-i],$$

$$\text{Gr}^L_i Rf_*(\Omega^\bullet_X(\log Y), F) = (\text{DR}(M^i), F)[-i].$$
If $D$ is a divisor with normal crossings, we have also
\[
\mathbf{R}f_*(\Omega^*_X(\log Y), F) \simeq \bigoplus_{i \in \mathbb{Z}} (\text{DR}_{\log}(\tilde{V}^i), F)[-i],
\]
\[
\text{Gr}^i_{\text{h}} \mathbf{R}f_*(\Omega^*_X(\log Y), F) = (\text{DR}_{\log}(\tilde{V}^i), F)[-i].
\]

This follows from the decomposition theorem (see [2]) extended to the case of the direct image of $(\mathcal{O}_X, F)$ as a filtered $D$-module, see [22]. Note that Hodge modules do not appear in the last statement if $D$ is a divisor with normal crossings. The assertion becomes more complicated in the non logarithmic case, see Remark (i) in (2.5). A splitting of the filtration $L$ is given by choosing the first noncanonical isomorphism in the filtered decomposition theorem, see (1.4.2). A canonical choice of the splitting is given by choosing an relatively ample class, see [9].

Let $\text{CH}^p(X \setminus Y, n)$ be Bloch's higher Chow group, see [3]. In the analytic case, we assume for simplicity that $f : (X, Y) \to (S, D)$ is the base change of a projective morphism of smooth complex algebraic varieties $f' : (X', Y') \to (S', D')$ by an open embedding of complex manifolds $S \to S'_{\text{an}}$, and an element of $\text{CH}^p(X \setminus Y, n)$ is the restriction of an element of $\text{CH}^p(X' \setminus Y', n)$ to $X \setminus Y$. If $n = 0$, we may assume that it is the restriction of an analytic cycle of codimension $p$ on $X$. From Theorem 1, we can deduce

**Corollary 1.** With the above notation and assumption, let $\xi \in \text{CH}^p(X \setminus Y, n)$. Then, choosing a splitting of the filtration $L$ in Theorem 1 (or more precisely, choosing the first noncanonical isomorphism in the filtered decomposition theorem (1.4.2)), we have the total infinitesimal invariant
\[
\delta_{S,D}(\xi) = (\delta^i_{S,D}(\xi)) \in \bigoplus_{i \geq 0} H^i(S, F^p \text{DR}(M^{2p-n-i})),
\]
\[
(\text{resp. } \overline{\delta}_{S,D}(\xi) = (\overline{\delta}^i_{S,D}(\xi)) \in \bigoplus_{i \geq 0} H^i(S, \text{Gr}^i_{\text{h}} F^p \text{DR}(M^{2p-n-i})), )
\]

where $\delta^i_{S,D}(\xi)$ (resp. $\overline{\delta}^i_{S,D}(\xi)$) is independent of the choice of a splitting if the $\delta^j_{S,D}(\xi)$ (resp. $\overline{\delta}^j_{S,D}(\xi)$) vanish for $j < i$. In the case $D$ is a divisor with normal crossings, the assertion holds with $\text{DR}(M^{2p-n-i})$ replaced by $\text{DR}_{\log}(\tilde{V}^{2p-n-i})$.

This shows that the infinitesimal invariants in [14], [13], [27], [5], [1], [24] can be defined naturally in the cohomology of filtered logarithmic complexes with coefficients in variations of Hodge structures if $D$ is a divisor with normal crossings, see (2.4) for the compatibility with [1]. Note that if $S$ is Stein or affine, then $H^i(S, F^p \text{DR}_{\log}(\tilde{V}^q))$ is the $i$-th cohomology group of the complex whose $j$-th component is $\Gamma(S, \Omega^j_S(\log D) \otimes \mathcal{O} F^{p-j} \tilde{V}^q)$. If $D$ is empty, then an inductive definition of $\delta^i_{S,D}(\xi)$, $\overline{\delta}^i_{S,D}(\xi)$ was given by Shuji Saito [24] using the filtered Leray spectral sequence together with the $E_2$-degeneration argument in [6]. He also showed that the infinitesimal invariants depend only on the cohomology class of the cycle. If $S$ is projective, then it follows from [8] that the total infinitesimal invariant $(\delta^i_{S,D}(\xi))$ is equivalent to the cycle class of $\xi$ in $H^2_{\text{DR}}(X \setminus Y)$ by the strictness of the Hodge filtration, and the filtration $L$ comes from the Leray filtration on the cohomology of $X \setminus Y$, see Remark (iii) in (2.5).
Corollary 1 is useful to study the behavior of the infinitesimal invariants near the boundary of the variety. If \( D = \emptyset \), let \( \delta_i^S(\xi) \) denote \( \delta_i^{S,D}(\xi) \). We can define \( \delta_i^{DR,S}(\xi) \) as in [19] by omitting \( F^p \) before \( DR \) in Corollary 1 where \( D = \emptyset \).

**Corollary 2.** Assume \( S \) is projective. Let \( U = S \setminus D \). Then for each \( i \geq 0 \), \( \delta_i^{S,D}(\xi) \), \( \delta_i^U(\xi) \), and \( \delta_i^{DR,U}(\xi) \) are equivalent to each other, i.e. one of them vanishes if and only if the others do.

Indeed, \( (\delta_i^{DR,U}(\xi)) \) is determined by \( (\delta_i^U(\xi)) \), and \( (\delta_i^U(\xi)) \) by \( (\delta_i^{S,D}(\xi)) \). Moreover, \( (\delta_i^{S,D}(\xi)) \) is equivalent to \( (\delta_i^{DR,U}(\xi)) \) by the strictness of the Hodge filtration [8] applied to \((X, Y)\) together with Theorem 1, see (2.3). For the relation with \( \delta_i^{S,D}(\xi) \), see (2.1). Note that the equivalence between \( \delta_i^U(\xi) \) and \( \delta_i^{DR,U}(\xi) \) in the case of algebraic cycles (i.e. \( n = 0 \)) was first found by J.D. Lewis and Shuji Saito in [19] (assuming a conjecture of Brylinski and Zucker and the Hodge conjecture and using an \( L^2 \)-argument). The above arguments seem to be closely related with their question, see also Remark (i) in (2.5) below.

As another corollary of Theorem 1 we have

**Corollary 3.** Assume \( f \) induces an isomorphism over \( S \setminus D \), and \( Y = f^*D \) is a divisor with normal crossings on \( X \). Then

\[
R^i f_* \Omega^p_X (\log Y) = 0 \quad \text{if } i + p > \dim X.
\]

This follows immediately from Theorem 1 since \( M^i = 0 \) for \( i \neq 0 \). Corollary 3 is an analogue of the vanishing theorem of Kodaira-Nakano. However, this does not hold for a non logarithmic complex (e.g. if \( f \) is a blow-up with a point center). This corollary was inspired by a question of A. Dimca.

I would like to thank Dimca, Lewis and Shuji Saito for good questions and useful suggestions.

In Section 1, we prove Theorem 1 after reviewing some basic facts on filtered differential complexes. In Section 2 we explain the application of Theorem 1 to the infinitesimal invariants of (higher) cycles. In Section 3 we give some examples using Lefschetz pencils.

### 1. Direct image of logarithmic complexes

#### 1.1. Filtered differential complexes

Let \( X \) be a complex manifold or a smooth algebraic variety over a field of characteristic zero. Let \( D^b F(D_X) \) (resp. \( D^b F(D_X)^r \)) be the bounded derived category of filtered left (resp. right) \( D_X \)-modules. Let \( D^b F(O_X, \text{Diff}) \) be the bounded derived category of filtered differential complexes \((L, F)\) where \( F \) is exhaustive and locally bounded below (i.e. \( F_p = 0 \) for \( p \ll 0 \) locally on \( X \)), see [22], 2.2. We have an equivalence of categories

\[
(1.1.1) \quad \text{DR}^{-1} : D^b F(O_X, \text{Diff}) \to D^b F(D_X)^r,
\]
whose quasi-inverse is given by the de Rham functor $\text{DR}^r$ for right $\mathcal{D}$-modules, see (1.2) below. Recall that, for a filtered $\mathcal{O}_X$-module $(L, F)$, the associated filtered right $\mathcal{D}$-module $\text{DR}^{-1}(L, F)$ is defined by

\begin{equation}
\text{DR}^{-1}(L, F) = (L, F) \otimes_{\mathcal{O}} (\mathcal{D}, F),
\end{equation}

and the morphisms $(L, F) \to (L', F)$ in $\mathcal{M}F(\mathcal{O}_X, \text{Diff})$ correspond bijectively to the morphisms of filtered $\mathcal{D}$-modules $\text{DR}^{-1}(L, F) \to \text{DR}^{-1}(L', F)$. More precisely, the condition on $(L, F) \to (L', F)$ is that the composition

\[ F_p L \to L \to L' \to L'/F_q L' \]

is a differential operator of order $\leq p - q - 1$. The proof of (1.1.1) can be reduced to the canonical filtered quasi-isomorphism for a filtered right $\mathcal{D}$-module $(M, F)$

\[ \text{DR}^{-1} \circ \text{DR}^r(M, F) \to (M, F), \]

which follows from a calculation of a Koszul complex.

Note that the direct image $f_*$ of filtered differential complexes is defined by the sheaf-theoretic direct image $Rf_*$, and this direct image is compatible with the direct image $f_*$ of filtered $\mathcal{D}$-modules via (1.1.1), see [22], 2.3. So we get

\begin{equation}
Rf_* = \text{DR}^r \circ f_* \circ \text{DR}^{-1} : D^b F(\mathcal{O}_X, \text{Diff}) \to D^b F(\mathcal{O}_S, \text{Diff}),
\end{equation}

where we use $\text{DR}^r$ for right $\mathcal{D}$-modules (otherwise there is a shift of complex).

### 1.2. De Rham complex.

The de Rham complex $\text{DR}^r(M, F)$ of a filtered right $\mathcal{D}$-module $(M, F)$ is defined by

\begin{equation}
\text{DR}^r(M, F)^i = \bigwedge^{-i} \Omega^i_X \otimes_{\mathcal{O}} (M, F[-i]) \quad \text{for } i \leq 0.
\end{equation}

Here $(F[-i])_p = F_{p+i}$ in a compatible way with $(F[-i])^p = F^{p-i}$ and $F_p = F^{-p}$. Recall that the filtered right $\mathcal{D}$-module associated with a filtered left $\mathcal{D}$-module $(M, F)$ is defined by

\begin{equation}
(M, F)^r := (\Omega_{\text{lim}}^X, F) \otimes_{\mathcal{O}} (M, F),
\end{equation}

where $\text{Gr}_p F_{\text{lim}}^X = 0$ for $p \neq -\dim X$. This induces an equivalence of categories between the left and right $\mathcal{D}$-modules. The usual de Rham complex $\text{DR}(M, F)$ for a left $\mathcal{D}$-module is defined by

\begin{equation}
\text{DR}(M, F)^i = \Omega^i_X \otimes_{\mathcal{O}} (M, F[-i]) \quad \text{for } i \geq 0,
\end{equation}

and this is compatible with (1.2.1) via (1.2.2) up to a shift of complex, i.e.

\begin{equation}
\text{DR}(M, F) = \text{DR}^r(M, F)^r[-\dim X].
\end{equation}

### 1.3. Logarithmic complex.

Let $X$ be as in (1.1), and $Y$ be a divisor with normal crossings on $X$. Let $(V, F)$ be a filtered locally free $\mathcal{O}$-module underlying a polarizable variation of Hodge structure on $X \setminus Y$. Let $(\tilde{V}, F)$ be the Deligne extension of $(V, F)$ to $X$ such that the eigenvalues of the residue of the connection are contained in $[0, 1)$. Then we have the filtered logarithmic de Rham complex $\text{DR}_{\log}(\tilde{V}, F)$ such that $F^p$ of its $i$-th component is

\[ \Omega^i_X(\log Y) \otimes F^{p-i}\tilde{V}. \]
If \((M, F) = (\mathcal{O}_X, F)\) with \(\text{Gr}_p^F \mathcal{O}_X = 0\) for \(p \neq 0\), then
\[
\text{DR}_{\log}^* (\mathcal{O}_X, F) = (\Omega^*_X (\log X), F).
\]

Let \(\tilde{V}(*Y)\) be the localization of \(\tilde{V}\) by a local defining equation of \(Y\). This is a regular holonomic left \(\mathcal{D}_X\)-module underlying a mixed Hodge module, and has the Hodge filtration \(F\) which is generated by the Hodge filtration \(F\) on \(\tilde{V}\), i.e.
\[
F_p \tilde{V}(*Y) = \sum \partial^{\nu} F^{-p+|\nu|} \tilde{V},
\]
where \(F_p = F^{-p}\) and \(\partial^{\nu} = \prod_i \partial_i^{\nu_i}\) with \(\partial_i = \partial / \partial x_i\). Here \(x_1, \ldots, x_n\) is a local coordinate system such that \(Y\) is contained in \(\{x_1 \cdots x_n = 0\}\). By [23], 3.11, we have a filtered quasi-isomorphism
\[
(\Omega^*_X (\log Y), F) \sim \text{DR}(\mathcal{O}_X (*Y), F).
\]

This generalizes the filtered quasi-isomorphism in [7]
\[
(\Omega^*_X (\log Y), F) \sim \text{DR}(\mathcal{O}_X (*Y), F).
\]

Note that the direct image of the filtered \(\mathcal{D}_X\)-module \((\tilde{V}(*Y), F)\) by \(X \to pt\) in the case \(X\) projective (or proper algebraic) is given by the cohomology group of the de Rham complex \(\text{DR}(\tilde{V}(*Y), F)\) (up to a shift of complex) by definition, and the Hodge filtration \(F\) on the direct image is strict by the theory of Hodge modules. So we get
\[
F^p \mathcal{H}^i (X \setminus Y, \text{DR}(V)) := F^p \mathcal{H}^i (X, \text{DR}(\tilde{V}(*Y)))
\]
\[
= \mathcal{H}^i (X, F^p \text{DR}(\tilde{V}(*Y))) = \mathcal{H}^i (X, F^p \text{DR}_{\log}^* (\tilde{V})).
\]

1.4. Decomposition theorem. Let \(f : X \to S\) be a projective morphism of complex manifolds or smooth algebraic varieties over a field of characteristic zero. Then the decomposition theorem of Beilinson, Bernstein and Deligne [2] is extended to the case of Hodge modules ([22], [23]), and we have noncanonical and canonical isomorphisms
\[
f_* (\mathcal{O}_X, F) \simeq \bigoplus_{j \in \mathbb{Z}} \mathcal{H}^j f_* (\mathcal{O}_X, F)[-j] \quad \text{in } \mathcal{D}^b F(\mathcal{D}_S),
\]
\[
\mathcal{H}^j f_* (\mathcal{O}_X, F) \simeq \bigoplus_{Z \subset S} (M^j_Z, F) \quad \text{in } MF(\mathcal{D}_S),
\]
where \(Z\) are irreducible closed analytic or algebraic subsets of \(S\), and \((M^j_Z, F)\) are filtered \(\mathcal{D}_S\)-modules underlying a pure Hodge module of weight \(j + \dim X\) and with strict support \(Z\), i.e. \(M^j_Z\) has no nontrivial sub nor quotient module whose support is strictly smaller than \(Z\). (Here \(MF(\mathcal{D}_S)\) denotes the category of filtered left \(\mathcal{D}_S\)-modules.) Indeed, the second canonical isomorphism follows from the strict support decomposition which is part of the definition of pure Hodge modules, see [22], 5.1.6. The first noncanonical isomorphism follows from the strictness of the Hodge filtration and the relative hard Lefschetz theorem for the direct image (see [22], 5.3.1) using the \(E_2\)-degeneration argument in [6] together with the equivalence of categories \(\mathcal{D}^b F(\mathcal{D}_S) \simeq \mathcal{D}^b G(\mathcal{B}_S)\). Here \(\mathcal{B}_S = \bigoplus_{i \in \mathbb{N}} F_i \mathcal{D}_S\) and \(\mathcal{D}^b G(\mathcal{B}_S)\) is the derived category of bounded complexes of graded left \(\mathcal{B}_S\)-modules \(M^*_i\) such that \(M^*_i = 0\) for \(i \ll 0\) or \(|j| \gg 0\), see [22], 2.1.12. We need a derived category associated
to some abelian category in order to apply the argument in [6] (see also [9]). In
the algebraic case, we can also apply [6] to the derived category of mixed Hodge
modules on $S$ and it is also possible to use [23], 4.5.4 to show the first noncanonical
isomorphism.

If $f$ is smooth over the complement of a divisor $D \subset S$ and $Y := f^*D$ is a divisor
with normal crossings, then the filtered direct image $f_*(\mathcal{O}_X(*Y), F)$ is strict (see
[23], 2.15), and we have noncanonical and canonical isomorphisms

$$
\begin{align*}
& f_*(\mathcal{O}_X(*Y), F) \simeq \bigoplus_{j \in \mathbb{Z}} \bigwedge^j \mathcal{H}^* f_*(\mathcal{O}_X(*Y), F)[-j] \quad \text{in } D^b F(D_S), \\
& \mathcal{H}^j f_*(\mathcal{O}_X(*Y), F) = (M_S^j(*D), F) \quad \text{in } MF(D_S).
\end{align*}
$$

(1.4.2)

Here $(M_S^j(*D), F)$ is the ‘localization’ of $(M_S^j, F)$ along $D$ which is the direct
image of $(M_S^j, F)|_U$ by the open embedding $U := S \setminus D \to S$ in the category of
filtered $\mathcal{D}$-modules underlying mixed Hodge modules. (By the Riemann-Hilbert
correspondence, this gives the direct image in the category of complexes with con-
structible cohomology because $D$ is a divisor.) The Hodge filtration $F$ on the direct
image is determined by using the $V$-filtration of Kashiwara and Malgrange, and
$(M_S^j(*D), F)$ is the unique extension of $(M_S^j, F)|_U$ which underlies a mixed Hodge
module on $S$ and whose underlying $D_S$-module is the direct image in the category of
regular holonomic $D_S$-modules, see [23], 2.11. So the second canonical isomorphism
follows because the left-hand side satisfies these conditions. (Note that $(M_S^j, F)$
for $Z \neq S$ vanishes by the localization, because $Z \subset D$ if $(M_S^j, F) \neq 0$ .) The first
noncanonical isomorphism follows from the strictness of the Hodge filtration and
the relative hard Lefschetz theorem by the same argument as above.

1.5. Proof of Theorem 1. Let $r = \dim X - \dim S$. By (1.1.3), (1.3.2) and (1.4.2),
we have isomorphisms

$$
\begin{align*}
\mathcal{R}f_*(\Omega_X^\bullet(\text{log } Y), F) &= DR^r f_* DR^{-1}(\Omega_X^\bullet(\text{log } Y), F) \\
&= DR^r f_*(\mathcal{O}_X(*Y), F)[\dim X] \\
&\simeq \bigoplus_{i \in \mathbb{Z}} DR(M_S^i(*D), F)[-r-i],
\end{align*}
$$

(1.5.1)

where the shift of complex by $r$ follows from the difference of the de Rham complex
for left and right $\mathcal{D}$-modules. Furthermore, letting $L$ be the filtration induced by
$\tau$ on the complex of filtered $D_S$-modules $f_*(\mathcal{O}_X(*Y), F)[-r]$, we have a canonical
isomorphism

$$
\text{Gr}_i^L f_*(\mathcal{O}_X(*Y), F)[-r] = (M_S^{i-r}(*D), F)[-i],
$$

(1.5.2)

and the first assertion follows by setting $M^i = M_S^{i-r}(*D)$. The second assertion
follows from the first by (1.3.1). This competes the proof of Theorem 1.

2. Infinitesimal invariants of cycles

2.1. Cycle classes. Let $X$ be a complex manifold, and $\mathcal{C}^{\bullet, \bullet}$ denote the double
complex of vector spaces of currents on $X$. The associated single complex is denoted
by $\mathcal{C}^\bullet$. Let $F$ be the Hodge filtration by the first index of $\mathcal{C}^{\bullet, \bullet}$ (using the truncation
σ in [8]). Let ξ be an analytic cycle of codimension p on X. Then it is well known that ξ defines a closed current in $F^pΩ^{2p}$ by integrating the restrictions of $C^\infty$ forms with compact supports on X to the smooth part of the support of ξ (and using a triangulization or a resolution of singularities of the cycle). So we have a cycle class of ξ in $H^{2p}(X, F^pΩ^X_\bullet)$.

Assume X is a smooth algebraic variety over a field k of characteristic zero. Then the last assertion still holds (where $Ω^\bullet_X$ means $Ω^\bullet_{X/k}$), see [11]. Moreover, for the higher Chow groups, we have the cycle map (see [4], [10], [12], [15], [16])

$$cl : CH^p(X, n) \to F^pH^{2p-n}_{DR}(X),$$

where the Hodge filtration $F$ is defined by using a smooth compactification of X whose complement is a divisor with normal crossings, see [8]. This cycle map is essentially equivalent to the cycle map to $Gr_F^pH^{2p-n}_{DR}(X)$ because we can reduce to the case $k = \mathbb{C}$ where we have the cycle map

$$cl : CH^p(X, n) \to \text{Hom}_{\text{MHS}}(\mathbb{Q}, H^{2p-n}(X, \mathbb{Q})(p)),$$

and morphisms of mixed Hodge structures are strictly compatible with the Hodge filtration $F$.

2.2. Proof of Corollary 1. By (2.1) ξ has the cycle class in $H^{2p-n}(X, F^pΩ^\bullet_X(\log Y))$.

By theorem 1, this gives the total infinitesimal invariant

$$δ_{S,D}(ξ) = (δ_{S,D}^{2p-n-i}(ξ)) \in \bigoplus_{i \geq 0} H^i(S, F^pDR(M^i)),
$$

and similarly for $δ_{S,D}(ξ)$. So the assertion follows.

2.3. Proof of Corollary 2. Choosing the first noncanonical isomorphism in the filtered decomposition theorem (1.4.2), we get canonical morphisms compatible with the direct sum decompositions

$$\bigoplus_{i \geq 0} H^i(S, F^pDR(M^q-i)) \to \bigoplus_{i \geq 0} H^i(S \setminus D, F^pDR(M^q-i))$$

$$\to \bigoplus_{i \geq 0} H^i(S \setminus D, DR(M^q-i)),$$

and these are identified with the canonical morphisms

$$H^p(X, F^pΩ^\bullet_X(\log Y)) \to H^q(X \setminus Y, F^pΩ^\bullet_{X\setminus Y})$$

$$\to H^q(X \setminus Y, Ω^\bullet_{X\setminus Y}).$$

By Deligne [8], the composition of the last two morphisms is injective because of the strictness of the Hodge filtration, see also (1.3). So we get the equivalence of $δ_{S,D}^i(ξ), δ_U^i(ξ), δ_{DR,U}^i(ξ)$. The equivalence with $δ_{S,D}^i(ξ)$ follows from (2.1).

2.4. Compatibility with the definition in [1]. When D is empty, the infinitesimal invariants are defined in [1] by using the extension groups of filtered $D$-modules together with the forgetful functor from the category of mixed Hodge modules to that of filtered $D$-modules. Its compatibility with the definition in this paper follows from the equivalence of categories (1.1.1) and the compatibility of the direct image functors (1.1.3).
Note that for \((L, F) \in D^bF(O_X, \text{Diff})\) in the notation of (1.1), we have a canonical isomorphism
\[
\text{Ext}^i((\Omega^*_X, F), (L, F)) = H^i(X, F_0L),
\]
where the extension group is taken in \(D^bF(O_X, \text{Diff})\). Indeed, the left-hand side is canonically isomorphic to
\[
\begin{align*}
\text{Ext}^i(DR^{-1}(\Omega^*_X, F), DR^{-1}(L, F)) \\
= H^i(X, F_0\mathcal{H}om_D(DR(D_X, F), DR^{-1}(L, F))), \\
= H^i(X, F_0DR'\text{DR}^{-1}(L)),
\end{align*}
\]
and the last group is isomorphic to the right-hand side of (2.4.1) which is independent of a representative of \((L, F)\). If \(X\) is projective, then this assertion follows also from the adjoint relation for filtered \(D\)-modules.

If \(X\) is smooth projective and \(Y\) is a divisor with normal crossings, then the cycle class can be defined in
\[
\text{Ext}^{2p}((\Omega^*_X, F), \Omega^*_X(\log Y), F[p])) = H^{2p}(X, F^p\Omega^*_X(\log Y)) = F^pH^{2p}(X, \Omega^*_X(\log Y)).
\]

2.5. Remarks. (i) If we use (1.4.1) instead of (1.4.2) we get an analogue of Theorem 1 for non logarithmic complexes. However, the assertion becomes more complicated, and we get noncanonical and canonical isomorphisms
\[
\begin{align*}
Rf_*(\Omega^*_X, F) &\simeq \bigoplus_{i \in \mathbb{Z}, Z \subset S}(DR(M^{i-r}_S), F)[-i]. \\
\text{Gr}^H_i Rf_*(\Omega^*_X, F) &\simeq \bigoplus_{Z \subset S}(DR(M^{i-r}_Z), F)[-i].
\end{align*}
\]
This implies an analogue of Corollary 1. If \(D\) is a divisor with normal crossings, we have a filtered quasi-isomorphism for \(Z = S\)
\[
(DR_{\log}(\tilde{M}^{i-r}_S), F) \xrightarrow{\sim} (DR(M^{i-r}_S), F),
\]
where \(DR_{\log}(\tilde{M}^{i-r}_S)\) is the intersection of \(DR(M^{i-r}_S)\) with \(DR_{\log}(\tilde{V}^i_S)\). This seems to be related with a question of Lewis and Shuji Saito, see also [19].

(ii) If \(\dim S = 1\), we can inductively define the infinitesimal invariants in Corollary 1 by an argument similar to [24] using [26].

(iii) Assume \(S\) is projective and \(D\) is a divisor with normal crossings. Then the Leray filtration for \(X \to S \to \text{pt}\) is given by the truncation \(\tau\) on the complex of filtered \(\mathcal{D}_S\)-modules \(f_*(\mathcal{O}_X(*Y), F)\), and gives the Leray filtration on the cohomology of \(X \setminus Y\) (induced by the truncation \(\tau\) as in [8]). Indeed, the graded pieces \(H^i f_*(\mathcal{O}_X(*Y), F)\) of the filtration \(\tau\) on \(S\) coincide with \((\tilde{V}^{j+r}(*D), F)\), and give the open direct images by \(U \to S\) of the graded pieces \((\tilde{V}^{j+r}, F)\) of the filtration \(\tau\) on \(U\) as filtered \(\mathcal{D}\)-modules underlying mixed Hodge modules. Note that the morphism \(U \to S\) is open affine so that the direct image preserves regular holonomic \(\mathcal{D}\)-modules.
3. Examples

3.1. Lefschetz pencils. Let $Y$ be a smooth irreducible projective variety of dimension $n$ embedded in a projective space $P$ over $\mathbb{C}$. We assume that $Y \neq P$ and $Y$ is not contained in a hyperplane of $P$ so that the hyperplane sections of $Y$ are parametrized by the dual projective spaces $P^\vee$. Let $D \subset P^\vee$ denote the discriminant. This is the image of a projective bundle over $Y$ (consisting of hyperplanes tangent to $Y$), and hence $D$ is irreducible. At a smooth point of $D$, the corresponding hyperplane section of $Y$ has only one ordinary double point. We assume that the associated vanishing cycle is not zero in the cohomology of $H^{n-1}(Y) \to H^{n-1}(X)$.

A Lefschetz pencil of $Y$ is a line $\mathbb{P}^1$ in $P$ intersecting the discriminant $D$ at smooth points of $D$ (corresponding to hyperplane sections having only one ordinary double point). We have a projective morphism $\pi: \tilde{Y} \to \mathbb{P}^1$ such that $\tilde{Y}_t := \pi^{-1}(t)$ is the hyperplane section corresponding to $t \in \mathbb{P}^1 \subset P$ and $\tilde{Y}$ is the blow-up of $Y$ along a smooth closed subvariety $Z$ of codimension 2 which is the intersection of $\tilde{Y}_t$ for any (or two of) $t \in \mathbb{P}^1$.

A Lefschetz pencil of hypersurface sections of degree $d$ is defined by replacing the embedding of $Y$ using $\mathcal{O}_Y(d)$ so that a hyperplane section corresponds to a hypersurface section of degree $d$. Here $\mathcal{O}_Y(d)$ for an integer $d$ denote the invertible sheaf induced by that on $P$ as usual.

3.2. Hypersurfaces containing a subvariety. Let $Y, P$ be as in (3.2). Let $E$ be a closed subvariety (which is not necessarily irreducible nor reduced). Let

$$ E_{(i)} = \{ x \in E : \dim T_x E = i \}. $$

Let $\mathcal{I}_E$ be the ideal sheaf of $E$ in $Y$. Let $\delta$ be a positive integer such that $\mathcal{I}_E(\delta)$ is generated by global sections. By [18], [20] (or [21]) we have the following

(3.2.1) If $\dim Y > \max\{\dim E_{(i)} + i\}$ and $d \geq \delta$, then there is a smooth hypersurface section of degree $d$ containing $E$.

We have furthermore

(3.2.2) If $\dim Y > \max\{\dim E_{(i)} + i\} + 1$ and $d \geq \delta + 1$, then there is a Lefschetz pencil of hypersurface sections of degree $d$ containing $E$.

Indeed, we have a pencil such that $\tilde{Y}_t$ has at most isolated singularities, because $\tilde{Y}_t$ is smooth near the center $Z$ which is the intersection of generic two hypersurface sections containing $E$, and hence is smooth, see [18], [20] (or [21]). Note that a local equation of $\tilde{Y}_t$ near $Z$ is given by $f - tg$ if $t$ is identified with an appropriate affine coordinate of $\mathbb{P}^1$ where $f, g$ are global sections of $\mathcal{I}_E(d)$ corresponding to smooth hypersurface sections.

To get only ordinary double points, note first that the parameter space of the hypersurfaces containing $E$ is a linear subspace of $P^\vee$. So it is enough to show that this linear subspace contains a point of the discriminant $D$ corresponding to an ordinary double point. Thus we have to show that an isolated singularity
can be deformed to ordinary double points by replacing the corresponding section
$h \in \Gamma(Y, \mathcal{L}_E(d))$ with $h + \sum_i t_i g_i$ where $g_i \in \Gamma(Y, \mathcal{L}_E(d))$ and $t_i \in \mathbb{C}$ are general with sufficiently small absolute values. Since $d \geq \delta + 1$, we see that $\Gamma(Y, \mathcal{L}_E(d))$ generates the 1-jets at each point of the complement of $E$. So the assertion follows from the fact that for a function with an isolated singularity $f$, the singularities of \{ $f + \sum_i t_i x_i = 0$ \} are ordinary double points if $t_1, \ldots, t_n$ are general, where $x_1, \ldots, x_n$ are local coordinates. (Note that $f$ has an ordinary double point if and only if the morphism defined by $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is locally biholomorphic at this point.)

3.3. Construction. For $Y, \mathcal{P}$ be as in (3.1), let $i_{Y, \mathcal{P}} : Y \to \mathcal{P}$ denote the inclusion. Assume

(3.3.1) $i_{Y, \mathcal{P}}^*: H^j(\mathcal{P}) \to H^j(Y)$ is surjective for any $j \neq \dim Y$,

where cohomology has coefficients in any field of characteristic zero. This condition is satisfied if $Y$ is a complete intersection.

Let $E_1, E_2$ be $m$-dimensional irreducible closed subvarieties of $Y$ such that

$E_1 \cap E_2 = \emptyset, \quad \deg E_1 = \deg E_2.$

Here $\dim Y = n = 2m + s + 1$ with $m \geq 0, s \geq 1$. Let $E = E_1 \cup E_2$. With the notation of (3.2), assume

(3.3.2) $d > \delta, \quad \dim Y > \max\{ \dim E_{i} + i \} + s,$

(3.3.3) $i_{X(j), Y}^*: H^{n-j}(Y) \to H^{n-j}(X(j))$ is not surjective for $j \leq s$,

where $X(j)$ is a general complete intersection of multi degree $(d, \ldots, d)$ and of codimension $j$ in $Y$. (This is equivalent to the condition that the vanishing cycles for a hypersurface $X^{(j)}$ of $X^{(j-1)}$ are nonzero.)

Let $X$ be a general hypersurface of degree $d$ in $Y$ containing $E$, see (3.2.1). Let $L$ denote the intersection of $X$ with a general linear subspace of codimension $m+s$ in the projective space. Then $[E_a]$ $(a = 1, 2)$ and $c[L \cap X]$ have the same cohomology class in $H^{2m+2s}(X)$ for some $c \in \mathbb{Q}$, because $\dim H^{2m+2s}(X) = 1$ by the weak and hard Lefschetz theorems together with (3.3.1). Let

$\xi_a = [E_a] - c[L \cap X] \in \text{CH}^{m+s}(X)_\mathbb{Q} \quad (a = 1, 2).$

These are homologous to zero. It may be expected that one of them is non torsion, generalizing an assertion in [24]. More precisely, let $S$ be a smooth affine rational variety defined over a finitely generated subfield $k$ of $\mathbb{C}$ and parametrizing the smooth hypersurfaces of degree $d$ containing $E$ as above so that there is the universal family $\mathcal{X} \to S$ defined over $k$ (see [2], [28]). Assume $X$ corresponds to a geometric generic point of $S$ with respect to $k$, i.e. $X$ is the geometric generic fiber for some embedding $k(S) \to \mathbb{C}$. Let

$\xi_{a, X} = [E_a \times_k S] - c[L] \in \text{CH}^{m+s}(\mathcal{X})_\mathbb{Q},$

where $[L] \in \mathcal{X}$ is the pull-back of $[L]$ by $\mathcal{X} \to Y$. Since the local system $\{ H^{2m+2s-j}(X_a) \}$ on $S$ is constant for $j < s$ and $S$ is smooth affine rational, we see that $\delta^j_S(\xi_{a, X}) = 0$ for $j < s$. Then it may be expected that $\delta^j_S(\xi_{a, X}) \neq 0$ for one of $a$, where $S$ can be
replaced by any non empty open subvariety. We can show this for \( s = 1 \) as follows. (For \( s > 1 \), it may be necessary to assume further conditions on \( d \), etc.)

3.4. Case \( s = 1 \). Consider a Lefschetz pencil \( \pi : \tilde{Y} \to \mathbb{P}^1 \) such that \( \tilde{Y}_t := \pi^{-1}(t) \) for \( t \in \mathbb{P}^1 \) is a hypersurface of degree \( d \) in \( Y \) containing \( E \). Here \( \tilde{Y} \) is the blow-up of \( Y \) along a smooth closed subvariety \( Z \), and \( Z \) is the intersection of \( \tilde{Y}_t \) for any \( t \in \mathbb{P}^1 \). Note that \( \tilde{Y}_t \) has an ordinary double point for \( t \in \Lambda \subset \mathbb{P}^1 \), where \( \Lambda \) denotes the discriminant, see (3.2.2).

Since \( Z \) has codimension 2 in \( Y \), we have the isomorphism

\[
H^n(\tilde{Y}) = H^n(Y) \oplus H^{n-2}(Z),
\]

so that the cycle class of \( [E_a \times \mathbb{P}^1] - c[L]_{\tilde{Y}} \in \text{CH}^{m+1}(\tilde{Y})_\mathbb{Q} \) in \( H^n(\tilde{Y}) \) is identified with the difference of the cycle class \( cl_Z(E_a) \in H^{n-2}(Z) \) and the cycle class of \( L \) in \( H^n(Y) \). Indeed, the injection \( H^{n-2}(Z) \to H^n(\tilde{Y}) \) in the above direct sum decomposition is defined by using the projection \( Z \times \mathbb{P}^1 \to Z \) and the closed embedding \( Z \times \mathbb{P}^1 \to \tilde{Y} \), and the injection \( H^n(Y) \to H^n(\tilde{Y}) \) is the pull-back by \( \tilde{Y} \to Y \), see [17].

By assumption, one of the \( cl_Z(E_a) \) is not contained in the non primitive part, i.e. not a multiple of the cohomology class of the intersection of general hyperplane sections. Indeed, if both are contained in the non primitive part, then \( cl_Z(E_1) = cl_Z(E_2) \) and this implies the vanishing of the self intersection number \( E_a \cdot E_a \) in \( Z \).

We will show that the cycle class of \( [E_a \times \mathbb{P}^1] - c[L]_{\tilde{Y}} \) does not vanish in the cohomology of \( \pi^{-1}(U) \) for any non empty open subvariety of \( \mathbb{P}^1 \), in other words, it does not belong to the image of \( \bigoplus_{t \in \Lambda'} H^n(\tilde{Y})_\mathbb{Q} \) where \( \Lambda' \) is any finite subset of \( \mathbb{P}^1 \) containing \( \Lambda \). (Note that the condition for the Lefschetz pencil is generic, and for any proper closed subvariety of the parameter space, there is a Lefschetz pencil whose corresponding line is not contained in this subvariety.)

Thus the assertion is reduced to that \( \dim H^n(\tilde{Y}) \) is independent of \( t \in \mathbb{P}^1 \) because this implies that the image of \( H^n(\tilde{Y})_t \to H^n(\tilde{Y}) \) is independent of \( t \). (Note that the Gysin morphism \( H^{n-2}(\tilde{Y}_t) \to H^n(\tilde{Y}) \) for a general \( t \) can be identified with the direct sum of the Gysin morphism \( H^{n-2}(\tilde{Y}_t) \to H^n(Y) \) and the restriction morphism \( H^{n-2}(\tilde{Y}_t) \to H^{n-2}(Z) \) up to a sign, and the image of the last morphism is the non primitive part by the weak Lefschetz theorem.) By duality, this is equivalent to that \( R^m\pi_*\mathbb{Q}_\tilde{Y} \) is a local system on \( \mathbb{P}^1 \). Then it follows from the assumption that the vanishing cycles are nonzero, see (3.3.3).

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