THE NON-CUTOFF VLASOV-POISSON-BOLTZMANN AND VLASOV-POISSON-LANDAU SYSTEMS IN UNION OF CUBES

DINGQUN DENG

Abstract. This work concerns the Vlasov-Poisson-Boltzmann system without angular cutoff and Vlasov-Poisson-Landau system including Coulomb interaction in bounded domain, namely union of cubes. We establish the global stability, exponential large-time decay with specular-reflection boundary condition when an initial datum is near Maxwellian equilibrium. We provide the compatible specular boundary condition for high-order derivatives and a velocity weighted energy estimate.

Contents

1. Introduction 1
  1.1. Equation 1
  1.2. Spatial Domain 3
  1.3. Notations 5
  1.4. Main Results 7
  2. Preliminary 8
  3. Macroscopic Estimates 10
  4. Global Existence 28
  5. Local Existence 37

References 40

1. INTRODUCTION

1.1. Equation. We consider the Vlasov-Poisson-Boltzmann (VPB) and Vlasov-Poisson-Landau (VPL) systems describing the motion of plasma particles of two species in domain $\Omega$:

$$\begin{align*}
\partial_t F_+ + v \cdot \nabla_x F_+ - \nabla_x \phi \cdot \nabla_v F_+ &= Q(F_+, F_+) + Q(F_+, F_+), \\
\partial_t F_- + v \cdot \nabla_x F_- + \nabla_x \phi \cdot \nabla_v F_- &= Q(F_-, F_-) + Q(F_-, F_-), \\
- \Delta_x \phi &= \int_{\mathbb{R}^3} (F_+ - F_-) \, dv, \\
F_{\pm}(0, x, v) &= F_{0,\pm}(x, v), \quad E(0, x) = E_0(x).
\end{align*}$$

(1.1)

Here, the unknown $F = [F_+, F_-]$ is the velocity distribution functions for the particles of ions ($+$) and electrons ($-$), respectively, at position $x \in \Omega$ and velocity $v \in \mathbb{R}^3$ and time $t \geq 0$. The self-consistent electrostatic field takes the form $E(t, x) = -\nabla_x \phi(t, x)$. The boundary condition for $(f, E)$ will be given in (1.7) and (1.8). Next we introduce the collision operator $Q$ first.
For Vlasov-Poisson-Landau system, the collision operator $Q$ is given by
\[ Q(G, F) = \nabla_v \cdot \int_{\mathbb{R}^3} \phi(v - v') \left[ G(v') \nabla_v F(v) - F(v) \nabla_v G(v') \right] dv'. \]
The non-negative definite matrix-valued function $\phi = [\phi_{ij}(v)]_{1 \leq i, j \leq 3}$ takes the form of
\[ \phi_{ij}(v) = \left\{ \delta_{ij} - \frac{v_i v_j}{|v|^2} \right\} |v|^\gamma + 2, \quad (1.2) \]
with $\gamma \geq -3$. It is convenient to call it hard potential when $\gamma \geq -2$ and soft potential when $-3 \leq \gamma < -2$. The case $\gamma = -3$ corresponds to the physically realistic Coulomb interactions; cf. [19].

For Vlasov-Poisson-Landau system, the collision operator $Q$ is defined by
\[ Q(G, F) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_\sigma) \left[ G(v'_\sigma) F(v) - G(v) F(v'_\sigma) \right] d\sigma dv_\sigma. \]
In this expression $v, v_\sigma$ and $v', v'_\sigma$ are velocity pairs given in terms of the $\sigma$-representation by
\[ v' = \frac{v + v_\sigma}{2} + \frac{|v - v_\sigma|}{2} \sigma, \quad v'_\sigma = \frac{v + v_\sigma}{2} - \frac{|v - v_\sigma|}{2} \sigma, \quad \sigma \in \mathbb{S}^2, \]
that satisfy conservation laws of momentum and energy:
\[ v + v_\sigma = v' + v'_\sigma, \quad |v|^2 + |v_\sigma|^2 = |v'|^2 + |v'|^2. \]
The Boltzmann collision kernel $B(v - v_\sigma, \sigma)$ depends only on $|v - v_\sigma|$ and the deviation angle $\theta$ through $\cos \theta = \frac{v \cdot \sigma - v_\sigma \cdot \sigma}{|v - v_\sigma|}$. Without loss of generality we can assume $B(v - v_\sigma, \sigma)$ is supported on $0 \leq \theta \leq \pi/2$, since one can reduce the situation with symmetrization: $B(v - v_\sigma, \sigma) = B(v - v_\sigma, \sigma) + B(v - v_\sigma, -\sigma)$. Moreover, we assume
\[ B(v - v_\sigma, \sigma) = |v - v_\sigma|^s b(\cos \theta), \]
and there exist $C_b > 0$ and $0 < s < 1$ such that
\[ \frac{1}{C_b \theta^{1+2s}} \leq \sin \theta b(\cos \theta) \leq \frac{C_b}{\theta^{1+2s}}, \quad \forall \theta \in (0, \frac{\pi}{2}). \]
It is convenient call it hard potential when $\gamma + 2s \geq 0$ and soft potential when $-3 < \gamma + 2s < 0$. Throughout the paper, we will assume
\[ -3 \leq \gamma \leq 1 \text{ for Landau case,} \]
\[ \max\{-3, -2s - \frac{3}{2}\} < \gamma < 1 - 2s, \quad \forall 0 < s < 1 \text{ for Boltzmann case.} \]
Note that we consider the full range of $0 < s < 1$ for Boltzmann case.

We reformulate problem (1.1) near a global Maxwellian as the following. Let $\mu$ be the global Maxwellian equilibrium state:
\[ \mu = \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}. \]
We construct a solution to (1.1) of the form
\[ F(t, x, v) = \mu + \mu^{1/2} f(t, x, v). \]
Then $f = [f_+, f_-]$ satisfies
\[ \begin{align*}
\partial_t f \pm v \cdot \nabla_x f \pm \frac{1}{2} \nabla_x \phi \cdot v f \pm \nabla_x \phi \cdot \nabla_v f \pm \nabla_x \phi \cdot v \mu^{1/2} - L f &= \Gamma_{\pm}(f, f), \\
- \Delta_x \phi &= \int_{\mathbb{R}^3} (f_+ - f_-) \mu^{1/2} dv, \\
f(0, x, v) &= f_0(x, v), \quad E(0, x) = E_0(x),
\end{align*} \quad (1.3) \]
where the linearized collision operator $L = [L_+, L_-]$ and nonlinear collision operator $\Gamma = [\Gamma_+, \Gamma_-]$ are given respectively by

$$L_{\pm}f = \mu^{-1/2}\left\{2Q(\mu, \mu^{1/2}f_{\pm}) + Q(\mu^{1/2}(f_{\pm} + f_{\mp}), \mu)\right\},$$

and

$$\Gamma_{\pm}(f, g) = \mu^{-1/2}\left\{Q(\mu^{1/2}f_{\pm}, \mu^{1/2}g_{\pm}) + Q(\mu^{1/2}f_{\mp}, \mu^{1/2}g_{\mp})\right\}.$$  

The kernel of $L$ on $L^2_{\Omega} \times L^2_{\Omega}$ is the span of $\{(1, 0)\mu^{1/2}, (0, 1)\mu^{1/2}, (1, 1)v\mu^{1/2}, (1, 1)|v|^2\mu^{1/2}\}$ and we define the projection $P = [P_+, P_-]$ from $L^2_{\Omega} \times L^2_{\Omega}$ onto ker $L$ to be

$$Pf = (a_+(t, x)[1, 0] + a_-(t, x)[0, 1] + v \cdot b(t, x)[1, 1] + (|v|^2 - 3)c(t, x)[1, 1])\mu^{1/2},$$

where functions $a_\pm, b, c$ are given by

$$a_\pm = (\mu^{1/2}, f_{\pm})_{L^2_{\Omega}},$$

$$b_j = \frac{1}{2}(v_j \mu^{1/2}, f_+ + f_-)_{L^2_{\Omega}},$$

$$c = \frac{1}{12}(|v|^2 - 3)\mu^{1/2}, f_+ + f_-)_{L^2_{\Omega}}.$$  

Then for given $f$, one can decompose $f$ uniquely as the macroscopic part microscopic part:

$$f = Pf + (I - P)f.$$  

It’s well-known that the solution to (1.3) satisfies the conservation laws on mass and energy. That is, the solution $f$ to (1.3) satisfies the following identities whenever it’s satisfied initially at $t = 0$:

$$\begin{cases}
\int_{\Omega \times \mathbb{R}^3} f_+(t)\mu^{1/2}\, dv\, dx = \int_{\Omega \times \mathbb{R}^3} f_-(t)\mu^{1/2}\, dv\, dx = 0, \\
\int_{\Omega \times \mathbb{R}^3} (f_+(t) + f_- (t))|v|^2\mu^{1/2}\, dv\, dx + \int_{\Omega} |E(t)|^2\, dx = 0.
\end{cases}$$

1.2. Spatial Domain. In this paper, we consider a domain $\Omega$ that is the union of finitely many cubes:

$$\Omega = \bigcup_{i=1}^{N} \Omega_i,$$

where $\Omega_i = (a_{i,1}, b_{i,1}) \times (a_{i,2}, b_{i,2}) \times (a_{i,3}, b_{i,3})$ with $a_{i,j}, b_{i,j} \in \mathbb{R}$ such that $a_{i,j} < b_{i,j}$. Then $\partial \Omega = \bigcup_{i=1}^{3} \Gamma_i$ is the union of three kinds of boundary $\Gamma_i$ ($i = 1, 2, 3$), where $\Gamma_i$ is orthogonal to axis $x_i$ and is the union of finitely many connected sets. We further assume that $\Gamma_i$ is of non-zero spherical measure. Since the boundary of $\Gamma_i$’s are of zero spherical measure, we don’t distinguish $\Gamma_i$ and the interior of $\Gamma_i$. Note that $\Omega$ could be non-convex and be closed to general bounded domains arbitrarily.

The unit normal outer vector $n(x)$ exists on $\partial \Omega$ almost everywhere with respect to spherical measure. On the interior of $\Gamma_i (i = 1, 2, 3)$, we have $n(x) = e_i$ or $-e_i$, where $e_i$ is the unit vector with $i$th-component being 1. We will denote vectors $\tau_1(x), \tau_2(x)$ on boundary $\partial \Omega$ such that $(n(x), \tau_1(x), \tau_2(x))$ forms an unit orthonormal basis for $\mathbb{R}^3$ such that for $j = 1, 2, \tau_j = e_k$ or $-e_k$ for some $k$. This implies that $\partial_{\tau_j}$ is the tangent derivative on $\partial \Omega$ for $j = 1, 2$.

The boundary of the phase space is

$$\gamma := \{(x, v) \in \partial \Omega \times \mathbb{R}^3\}.$$  

Denoting $n = n(x)$ to be the outward normal direction at $x \in \partial \Omega$, we decompose $\gamma$ as

$$\gamma_\pm = \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}, \quad \text{(the incoming set)},$$
\[
\gamma_+ = \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v > 0\}, \quad \text{(the outgoing set)},
\]
\[
\gamma_0 = \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v = 0\}, \quad \text{(the grazing set)}.
\]

Correspondingly, we assume that \( F(t, x, v) \) satisfies the specula-reflection boundary condition:

\[
F(t, x, R_xv) = F(t, x, v), \quad \text{on } \gamma_-,
\]

where for \((x, v) \in \gamma\),

\[
R_xv = v - 2n(x)(n(x) \cdot v).
\]

This is equivalent to the specular reflection boundary condition for perturbation \( f \):

\[
f(t, x, R_xv) = f(t, x, v), \quad \text{on } \gamma_-.
\]

(1.7)

For the boundary condition of electric potential \( \phi \), we further assume that

\[
\partial_n \phi = 0, \quad \text{on } x \in \partial \Omega.
\]

(1.8)

In particular, the Poisson equation for potential \( \phi \) is a pure Neumann boundary problem and we require zero-mean condition

\[
\int_{\Omega} \int_{\mathbb{R}^3} (f_+ - f_-) \mu^{1/2} \, dv \, dx = 0, \quad \text{for } t \geq 0,
\]

to ensure its existence, which follows from (1.5). Also, the zero-mean condition

\[
\int_{\Omega} \phi(t, x) \, dx = 0, \quad \text{for } t \geq 0
\]

ensures the uniqueness of solutions.

For the general theory of Vlasov-Poisson-Boltzmann and Vlasov-Poisson-Landau systems, we refer to \([12, 20, 22, 30]\) and reference therein. Mischler \([30]\) generalized the existence theory of Diperna-Lions renormalized solutions (cf. \([10]\)) to Vlasov-Poisson-Boltzmann system for the initial boundary value problem. Guo \([20]\) gives the global solution of VPB system near a global Maxwellian for the cutoff case. Guo \([22]\) establishes the global existence of VPL system with Coulomb potential by introducing a weight \( e^{\pm \phi} \). Using this method, Duan-Liu \([12]\) proves the global existence for VPB system without angular cutoff.

For the boundary theory of collisional kinetic problem such as Landau and Boltzmann equations, we refer to \([4, 5, 11, 15, 21, 23–26, 29, 30, 33]\). In the framework of perturbation near a global Maxwellian, initiating by Guo \([21]\), which established the \( L^2 - L^\infty \) method, many results are developed for Boltzmann equation and Landau equation. For instance, Guo, Kim, Tonon and Trescases \([24]\) give regularity of cutoff Boltzmann equation with several physical boundary conditions in short time. Esposito, Guo, Kim and Marra \([15]\) construct a non-equilibrium stationary solution. Kim and Lee \([26]\) study cutoff Boltzmann equation with specular boundary condition with external potential in \( C^3 \) bounded domain. Liu and Yang \([28]\) extend the result in \([21]\) to cutoff soft potential case. Cao, Kim and Lee \([4]\) prove the global existence for Vlasov-Poisson-Boltzmann with diffuse boundary condition. Guo, Hwang, Jang and Ouyang \([23]\) give the global stability of Landau equation with specular reflection boundary. Duan, Liu, Sakamoto and Strain \([13]\) prove the low regularity solution for Landau and non-cutoff Boltzmann equation in finite channel. Dong, Guo and Ouyang \([11]\) find the global existence for VPL system in general bounded domain with specular boundary condition.

Unfortunately, the boundary theory for non-cutoff VPB system remains open since many tools for cutoff Boltzmann theory are not applicable to non-cutoff case. Our main target is to consider the global stability of VPB system and VPL system in union of cubes. Compare to \([7]\), the boundary value for electric potential \( \phi \) creates new difficulties and we introduce Neumann boundary condition for \( \phi \) to overcome it in an elegant way. This work gives the
1.3. Notations. Now we give some notations throughout the paper. Let \( \langle v \rangle = \sqrt{1 + |v|^2} \) and \( 1_S \) be the indicator function on a set \( S \). Let \( \partial^\alpha_{\beta} = \partial^\alpha_{\beta_1} \partial^\alpha_{\beta_2} \partial^\alpha_{\beta_3}, \) where \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta = (\beta_1, \beta_2, \beta_3) \) are multi-indices. If each component of \( \beta' \) is not greater than that of \( \beta \)'s, we denote by \( \beta' \leq \beta \). The notation \( a \approx b \) (resp. \( a \gg b \), \( a \lesssim b \)) for positive real function \( a, b \) means there exists \( C > 0 \) not depending on possible free parameters such that \( C^{-1}a \leq b \leq Ca \) (resp. \( a \geq C^{-1}b \), \( a \leq Cb \)) on their domain. We will write \( C > 0 \) (large) to be a generic constant, which may change from line to line. Denote spaces \( L^2 \), \( L^2 S \) and \( L^2 S_T L^2_x \) for \( 1 \leq r, s \leq \infty \), respectively, as

\[
|f|^2_{L^2_r} = \int_{\mathbb{R}^3} |f|^2 \, dv, \quad \|f\|_{L^2_r L^2_s} = \left( \int_{\Omega} \|f\|^2_{r,s} \, dx \right)^{\frac{1}{2}}, \quad \|f\|_{L^r L^s L^2} = \|\|f(t)\|_{L^r_x L^s_x}\|_{L^2([0,T])}.
\]

Also, for velocity weighted space, we write

\[
|f|^2_{L^2_r} = \int_{\mathbb{R}^3} \langle v \rangle^{2k} |f|^2 \, dv.
\]

We will use some tools from pseudo-differential calculus. One may refer to [27, Chapter 2] for more details. Set \( \Gamma = |dv|^2 + |d\eta|^2 \) and let \( M \) be an \( \Gamma \)-admissible weight function. That is, \( M : \mathbb{R}^{2d} \to (0, +\infty) \) satisfies the following conditions: (a) (slowly varying) there exists \( \delta > 0 \) such that, for any \( X, Y \in \mathbb{R}^{2d}, |X - Y| \leq \delta \) implies

\[
M(X) \approx M(Y);
\]

(b) (temperance) there exists \( C > 0, N \in \mathbb{R} \), such that for \( X, Y \in \mathbb{R}^{2d}, \)

\[
\frac{M(X)}{M(Y)} \leq C \langle X - Y \rangle^N.
\]

We say that a symbol \( a \in S(M) = S(M, \Gamma) \), if for \( \alpha, \beta \in \mathbb{N}^d, v, \eta \in \mathbb{R}^3, \)

\[
|\partial_v^\alpha \partial_\eta^\beta a(v, \eta)| \leq C_{\alpha, \beta} M,
\]

with \( C_{\alpha, \beta} \) being a constant depending only on \( \alpha \) and \( \beta \). We formally define the Weyl quantization by

\[
a^w u(v) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{2\pi i (v - u) \cdot \eta} \frac{a(v + u, \eta)}{2} u(u) \, dv \, d\eta,
\]

for \( f \in \mathcal{S} \). A Weyl quantization \( a^w \) is said to be in \( Op(M) \) if \( a \in S(M) \).

To study the global well-posedness of problem (1.3) in union of cubes, we will consider the following function spaces and energy functionals. Firstly We let \( \nu \geq 0 \) and consider weight function

\[
w_{t, \nu}(\alpha, \beta) = \begin{cases} 
(v)^{t-r|\alpha|-q|\beta|}, & \text{for hard potential,} \\
(v)^{t-r|\alpha|-q|\beta|} \exp(\nu\langle v \rangle), & \text{for soft potential,}
\end{cases}
\]

where

\[
r = 1, \quad q = 1, \quad \text{for hard potential,}
\]

\[
r = -\gamma - \frac{2\gamma(1-s)}{s} + 1, \quad q = -\frac{2\gamma}{s} + 1, \quad \text{for soft potential,}
\]

and we let \( s = 1 \) for Landau case.

For Landau case, we denote

\[
\sigma^{ij}(v) = \phi^{ij} * \mu = \int_{\mathbb{R}^3} \phi^{ij}(v - v') \mu(v') \, dv',
\]
\[
\sigma^i(v) = \sigma^{ij} \frac{v_j}{2} = \phi^{ij} \{ \frac{v_j}{2} \mu \}.
\]

Here and after repeated indices are implicitly summed over. Define
\[
|f|^2_{L^2_{D,w}} = \sum_{i,j=1}^3 \int_{\mathbb{R}^3} w^2 (\sigma^{ij} \partial_{v_i} f \partial_{v_j} f + \sigma^{ij} \frac{v_i v_j}{2} |f|^2) dv, \quad \|f\|^2_{L^2_{D,w}} = \int_{\Omega} |f|^2_{L^2_{D,w}} dx,
\]
and \[
|f|^2_{L^2} = |f|^2_{L^2_{D,1}}. \]
Then by [31, Corollary 1] and [19, Lemma 5], we have
\[
|f|^2_{L^2_{D,w}} = |w\langle v\rangle^2 P_v \partial_{v_j} f|^2_{L^2} + |w\langle v\rangle^{2+s} (I - P_v) \partial_{v_j} f|^2_{L^2} + |w\langle v\rangle^{2+2s} f|^2_{L^2},
\]
where \( P_v \xi = \frac{\xi}{\|\xi\|_{\mathbb{R}^3}} \).

For Boltzmann case, as in [17], we denote
\[
|f|^2_{L^2_D} := |\langle v\rangle^{2+2s} f|^2_{L^2} + \int_{\mathbb{R}^3} dv \langle v\rangle^{\gamma+2s+1} \int_{\mathbb{R}^3} dv' \frac{(f' - f)^2}{d(v', v)^{\gamma+2s+1}} I_{(v,v') \leq 1},
\]
and
\[
|f|^2_{L^2_{D,w}} = |wf|^2_{L^2_D}, \quad \|f\|^2_{L^2_{D,w}} = \int_{\Omega} |f|^2_{L^2_{D,w}} dx.
\]

The fractional differentiation effects are measured using the anisotropic metric on the lifted paraboloid \( d(v,v') := |v - v'|^2 + \frac{1}{4}(|v|^2 - |v'|^2)^2 \). Then by [17, eq. (2.15)], we have
\[
|\langle v\rangle^2 (D_v)^s f|^2_{L^2} + |\langle v\rangle^{2+2s} f|^2_{L^2} \lesssim |f|^2_{L^2_D} \lesssim |\langle v\rangle^{2+2s} (D_v)^s f|^2_{L^2}.
\]

We will consider function space \( H^2_{x,v} \) for our analysis. Correspondingly, we define the “instant energy functional” \( \mathcal{E}_\nu(t) \) and “dissipation rate functional” \( \mathcal{D}_\nu(t) \) respectively by
\[
\mathcal{E}_\nu(t) = \sum_{|\alpha| + |eta| \leq 3} \left( \| w_{1,\nu}(\alpha, \beta) \partial_\beta^\gamma f \|^2_{L^2_{D,w}} + \| \partial^\alpha E \|^2_{L^2_D} \right), \quad (1.11)
\]
and
\[
\mathcal{D}_\nu(t) = \sum_{|\alpha| + |eta| \leq 3} \left( \| w_{1,\nu}(\alpha, \beta) \partial_\beta^\gamma f \|^2_{L^2_{D,w}} + \| \partial^\alpha E \|^2_{L^2_D} \right). \quad (1.12)
\]

We will let \( \mathcal{E}(t) = \mathcal{E}_0(t) \) and \( \mathcal{D}(t) = \mathcal{D}_0(t) \) to be the energy functional without exponential weight.

To obtain the rate of convergence, associated with weight function \( w_{1,\nu}(\alpha, \beta) \) given by (1.9), we let \( p \in (0, 1] \) be defined by
\[
p = \begin{cases} 
1, & \text{for hard potential in both Boltzmann and Landau case,} \\
-\gamma - 2s + 1, & \text{for soft potential in Boltzmann case}, \\
1, & \text{for soft potential in Landau case.} 
\end{cases} \quad (1.13)
\]

We then are able to show that the obtained solutions decay in time as
\[
\mathcal{E}(t) \lesssim e^{-\delta t p} \mathcal{E}(0),
\]
with some \( \delta > 0 \).
1.4. Main Results. In this section, we state our main results on global well-posedness of Vlasov-Poisson-Landau systems and Vlasov-Poisson-Boltzmann systems.

Theorem 1.1. Let $\Omega$ be defined by (1.6) and $w_{t,\nu}(\alpha, \beta)$ be given by (1.9). Let $\gamma \geq -3$ for Landau case and $(\gamma, s) \in \{-\frac{3}{2} < \gamma + 2s \leq 1, \ 0 < s < 1\}$ for Boltzmann case. Let $l \geq 3q$ with $q$ is given in (1.10). There exists $\varepsilon_0, \nu > 0$ such that if $F_0(x, v) = \mu + \mu^{1/2}f_0(x, v) \geq 0$ satisfying (1.5) and

$$
\sum_{|\alpha| + |\beta| \leq 3} \left( \|w_{t,\nu}(\alpha, \beta)\partial t^2 f_0\|^2_{L^2_x,v} + \|\partial \nu E_0\|^2_{L^2_x} \right) \leq \varepsilon_0,
$$

then there exists a unique solution $f(t, x, v)$ to the specular reflection boundary problem (1.3), (1.7) and (1.8), satisfying that $F(t, x, v) = \mu + \mu^{1/2}f(t, x, v) \geq 0$ and for any $T > 0$,

$$
\sup_{0 \leq t \leq T} e^{\delta \nu} E(t) + \sup_{0 \leq t \leq T} E_\nu(t) \lesssim \varepsilon_0,
$$

where $E_\nu(t)$ and $E(t)$ is defined by (1.11).

As in [7], we consider the bounded domain $\Omega$ as union of cubes. In this case, normal derivatives $\partial_n$ on $\partial \Omega$ are also derivatives along axis. By using

$$
v \cdot \nabla f = v \cdot n(x)\partial_n f + v \cdot \tau_1(x)\partial r_1 f + v \cdot \tau_2(x)\partial r_2 f,
$$

and equation (1.3), one can obtain the compatible high-order specular boundary condition in Lemma 3.1, 3.2 and 3.3. On the other hand, $\partial \nu$ can be rewritten into normal derivative $\partial_n$ and tangent derivative $\partial r_1$ and $\partial r_2$ on the boundary. Hence, the boundary term generated from ($\partial_3 (v \cdot \nabla f), \partial_3 f)_{L^2_x,v}$ vanishes by using high-order specular-reflection boundary condition.

In this work, we use space $H^3_{x,v}$ up to third derivatives. In order to obtain the specular boundary condition, we need to assume the Neumann boundary condition for potential $\phi$. With the Poisson equation, we can also derive the third order boundary values for $\phi$; see (3.19):

$$
\partial_{x,x,x} \phi = 0, \quad \text{on } \Gamma_1.
$$

Correspondingly, we can obtain the boundary values for macroscopic parts $\partial \nu[a_\pm, b, c]$ up to third derivatives:

$$
\partial_x c(x) = \partial_x a_\pm(x) = \partial_x b_1(x) = b_1(x)
$$

$$
= \partial_{x,x,x} c(x) = \partial_{x,x,x} a_\pm(x) = \partial_{x,x,x} b_1(x) = \partial_{x,x} b_1(x) = 0.
$$

These boundary values enable us to estimate the boundary terms and take integration by parts suitably with respect to $x$.

For the dissipation rate of $(a_\pm, b, c)$ in Section 3, we use the solutions to Poisson equation $-\Delta_x \phi_h = h$ with mixed Dirichlet-Neumann boundary condition or pure Neumann boundary condition. One should be careful when dealing with pure Neumann boundary condition. In this case, we need to assume the function $h$ on the right hand side has zero mean:

$$
\int_\Omega h \, dx = 0.
$$

Correspondingly, we need to assume the zero mean condition for $\phi_h$ to ensure the uniqueness for pure Neumann boundary problem. By using Poincaré’s inequality, one can obtain the elliptic estimate for Poisson equation with Neumann boundary. The case of mixed Dirichlet-Neumann boundary problem is much easier since there’s zero condition on the boundary and one can apply Sobolev embedding. We will illustrate these calculations in Theorem 3.4 in details.
With the nice property of macroscopic parts, we use $e^{\pm \phi}$ as in [22] to derive the energy estimates. The exponential weight in (1.9) is designed to generate velocity decay after taking derivative. The $2|\beta|$ in (1.9) for soft potential is designed to obtain velocity decay when estimating $w_{1,\mu}(\alpha, \beta)\partial_{\beta} v \cdot \nabla_x \partial_{\beta} \beta, f$. That is, when $|\beta_1| = 1$, one can generate velocity decay by using $w_{1,\mu}(\alpha, \beta) \lesssim (v)^{\gamma + 2}\mu_{1,\mu}(\alpha + \epsilon, \beta - \epsilon)$. Then we can control it by using dissipation rate.

Using the boundary values carefully, we will use the integration by parts with respect to spatial variable $x$ again and again in our analysis. Moreover, we will derive that

$$\sum_{i,j=1}^{3} \| \partial_{x_i} f \|^2_{L^2_x L^2_t} = \| \Delta_x f \|^2_{L^2_x L^2_t}.$$  

Similarly,

$$\sum_{i,j=1}^{3} \| \partial_{x_i} E \|^2_{L^2_x L^2_t} = \| \Delta_x E \|^2_{L^2_x L^2_t}.$$  

Then we only need to estimate $\Delta_x f$ in our proof, which is one of the key points.

The paper is organized as follows. In Section 2, we give some basic estimates on collision operators. In Section 3, we give the dissipation macroscopic estimates for VPL and VPB systems. In Section 4, we prove the global existence with large-time behavior via estimates on instant energy (1.11) and dissipation rate (1.12). In Section 5, we give the proof of local-in-time existence to close the a priori estimate.

2. Preliminary

In this section, we give some basic estimate on collision operator $L$ and $\Gamma(\cdot, \cdot)$. We begin with splitting $L_\pm$. For the Landau case, let $\varepsilon > 0$ small and choose a smooth cutoff function $\chi(|v|) \in [0, 1]$ such that $\chi(|v|) = 1$ if $|v| < \varepsilon$; $\chi(|v|) = 0$ if $|v| > 2\varepsilon$. Then we split $L_\pm f = -A_\pm f + K_\pm f$ as in [32, Section 4.2], where

$$-A_\pm f = 2\partial_{v_i} \{ \sigma^{ij} \partial_{v_j} f_\pm \} - 2\sigma^{ij} \frac{v_i v_j}{2} f_\pm + 2\partial_{v_i} \sigma^i \mathbb{1}_{|v| > R} f_\pm + A_1 f$$

$$+ (K_1 - \mathbb{1}_{|v| \leq R} K_1 \mathbb{1}_{|v| \leq R}) f,$$

$$K_\pm f = 2\partial_{v_i} \sigma^i \mathbb{1}_{|v| \leq R} f_\pm + \mathbb{1}_{|v| \leq R} K_1 \mathbb{1}_{|v| \leq R} f,$$

and $R > 0$ is to be chosen large, $\varepsilon > 0$ is to be chosen small, and $A_1$ and $K_1$ are given respectively by

$$A_1 f = -\sum_{\pm} \mu^{-1/2} \partial_{v_i} \left\{ \mu \left[ \left( \phi^{ij} \chi \right) \ast \left( \mu \partial_{v_j} \left[ \mu^{-1/2} f_\pm \right] \right) \right] \right\},$$

$$K_1 f = -\sum_{\pm} \mu^{-1/2} \partial_{v_i} \left\{ \mu \left[ \left( \phi^{ij} (1 - \chi) \right) \ast \left( \mu \partial_{v_j} \left[ \mu^{-1/2} f_\pm \right] \right) \right] \right\},$$

with the convolution taken with respect to the velocity variable $v$. Then [32, eq. (4.33), (4.32)] shows that

$$\sum_{\pm} (A_\pm f, f_\pm)_{L^2_x} \geq c_0 |f|^2_{L^2_t},$$

for some $c_0 > 0$, and

$$|(K_1 g, h)_{L^2_x}| \lesssim |\mu^{-1/10} g|_{L^2_x} |\mu^{1/10} h|_{L^2_t}.$$  

From [19, Lemma 3], we know that

$$|\partial_{\beta} \sigma^{ij}(v)| + |\partial_{\beta} \sigma^i(v)| \leq C_{\beta} (1 + |v|)^{\gamma + 2 - |\beta|}. $$  

(2.2)
Thus, (2.1) and (2.2) implies that $K$ is a bounded operator on $L^2_v$ with estimate
\[
|Kf|_{L^2_v} \lesssim |\mu|^{1/10} f|_{L^2_v}.
\] (2.3)
For Boltzmann case, we split $L_{\pm} f = -A_{\pm} f + K f$ with
\[
-A_{\pm} f = 2\mu^{-1/2} Q(\mu, \mu^{1/2} f_{\pm}),
\]
\[
K f = \mu^{-1/2} Q(\mu^{1/2}(f_{+} + f_{-}), \mu).
\]
Then by [2, Lemma 2.15], we have
\[
|Kf|_{L^2_v} \lesssim |\mu|^{1/10^3} f|_{L^2_v}.
\] (2.4)

**Lemma 2.1.** Let $w = \omega_{l,\nu}(\alpha, \beta)$ be given by (1.9). Let $\gamma > \max\{-3, -2s - \frac{3}{2}\}$ for Boltzmann case and $\gamma \geq -3$ for Landau case. Then
\[
\sum_{\pm}(L_{\pm} f, f_{\pm})_{L^2_v} \gtrsim \|(\mathbf{I} - \mathbf{P}) f\|_{L^2_v}^2,
\] (2.5)
\[
\sum_{\pm}(w^2 L_{\pm} g, g_{\pm})_{L^2_v} \geq c_0 |g|_{L^2_{D,w}}^2 - C|g|_{L^2(B_C)}^2.
\] (2.6)
and for $|\beta| \geq 1,
\[
\sum_{\pm}(w^2 A_{\pm} g, g_{\pm})_{L^2_v} \geq c_0 |g|_{L^2_{D,w}}^2 - C|g|_{L^2(B_C)}^2,
\] (2.8)
and for $|\beta| \geq 1,
\[
\sum_{\pm}(w^2 A_{\pm} g, g_{\pm})_{L^2_v} \geq c_0 |g|_{L^2_{D,w}}^2 - C|g|_{L^2(B_C)}^2.
\] (2.9)

Moreover, for any $|\alpha| + |\beta| \leq 3$, we have
\[
(w^2 \partial^\alpha_1 \Gamma_{\pm}(g_1, g_2), \partial^\beta_3 g_3)_{L^2_v} \lesssim \sum_{\alpha_1 + \alpha_2 = \alpha, \beta_1 + \beta_2 = \beta} \|\partial^\alpha_1 g_1\|_{L^2_v}^2 \sum_{\alpha_1 + \alpha_2 = \alpha, \beta_1 + \beta_2 = \beta} \|\partial^\beta_3 g_3\|_{L^2_v}^2.
\] (2.10)
where the summation is taken over $\alpha_1 + \alpha_2 = \alpha$, $\beta_1 + \beta_2 = \beta$. Consequently, taking integration over $x$,
\[
\sum_{|\alpha_1| + |\beta_1| \leq 3} \|\partial^\alpha_1 g_1\|_{L^2_v}^2 \sum_{|\alpha_1| + |\beta_1| \leq 3} \|\partial^\beta_3 g_3\|_{L^2_v}^2.
\] (2.11)

**Proof.** The proof of (2.5), (2.6) and (2.7) can be found in [31, Lemma 5, Lemma 9] for Landau case and [17, eq. (2.13)] as well as [14, Lemma 2.7] for Boltzmann case. The proof of (2.8) can be found in [31, Lemma 7 and Lemma 8] for Landau case and [14, Lemma 2.6] for Boltzmann case. Using (2.6) and boundedness of $K$ from (2.3), (2.4), we can obtain (2.9). The proof of (2.10) is given by [31, Lemma 10] for Landau case and [14, 16, Lemma
2.4 and Lemma 2.4] for Boltzmann case. Finally, we will give the proof of (2.11). For $|\alpha| + |\beta| \leq 3$, from (2.10), we know that

$$(w^2 \partial_\beta^0 \Gamma \pm (g_1, g_2), \partial_\beta^0 g_3)_{L^r_x, w} \lesssim \sum_{\alpha_1 + \alpha_2 = \alpha, \beta_1 + \beta_2 \leq \beta} \left\{ \int_{\Omega} |w \partial_{\beta_1}^\alpha g_1|_{L^2_x}^2 |\partial_{\beta_2}^\beta g_2|_{L^2_{v, w}}^2 \, dx \right\}$$

$$+ \int_{\Omega} |\partial_{\beta_1}^\alpha g_1|_{L^2_x}^2 |w \partial_{\beta_2}^\beta g_2|_{L^2_v}^2 \, dx \right\}^{\frac{1}{2}} \| \partial_\beta^0 g_3 \|_{L^2_x L^2_{v, w}}.$$

We apply $L^\infty - L^2$ and $L^3 - L^6$ Hölder’s inequality to the first term inside the brace:

$$\left( \int_{\Omega} |w \partial_{\beta_1}^\alpha g_1|_{L^2_x}^2 |\partial_{\beta_2}^\beta g_2|_{L^2_{v, w}}^2 \, dx \right)^{\frac{1}{2}} \leq \sum_{|\alpha_1| + |\beta_1| = 0} \| w \partial_{\beta_1}^\alpha g_1 \|_{L^\infty_L L^2_x} \| \partial_{\beta_2}^\beta g_2 \|_{L^2_x L^2_{v, w}}$$

$$+ \sum_{|\alpha_1| + |\beta_1| = 1} \| w \partial_{\beta_1}^\alpha g_1 \|_{L^2_x L^2_v} \| \partial_{\beta_2}^\beta g_2 \|_{L^2_x L^2_{v, w}}$$

$$+ \sum_{2 \leq |\alpha_1| + |\beta_1| \leq 3} \| w \partial_{\beta_1}^\alpha g_1 \|_{L^2_x L^2_v} \| \partial_{\beta_2}^\beta g_2 \|_{L^2_x L^2_{v, w}}$$

where we used embedding $\| f \|_{L^2_v(\Omega)} \lesssim \| f \|_{H^1_v(\Omega)}$, $\| f \|_{L^2_v(\Omega)} \lesssim \| f \|_{H^1_v(\Omega)}$, and $\| f \|_{L^\infty_v(\Omega)} \lesssim \| f \|_{H^1_v(\Omega)}$. Similarly,

$$\left( \int_{\Omega} |\partial_{\beta_1}^\alpha g_1|_{L^2_x L^2_v}^2 |w \partial_{\beta_2}^\beta g_2|_{L^2_v}^2 \, dx \right)^{\frac{1}{2}} \lesssim \sum_{|\alpha_1| + |\beta_1| \leq 3} \| \partial_{\beta_1}^\alpha g_1 \|_{L^2_x L^2_{v, w}} \sum_{|\alpha_1| + |\beta_1| \leq 3} \| w \partial_{\beta_1}^\beta g_2 \|_{L^2_x L^2_v}.$$

Combining the above estimate, we obtain (2.11). This completes the proof of Lemma 2.1. □

3. Macroscopic Estimates

In this section, we consider the macroscopic estimates for union of cubes. Let $\Omega$ be given by (1.6) and consider the following problem

$$\partial_t f_\pm + v \cdot \nabla_x f_\pm \pm \nabla_x \phi \cdot v \mu^{1/2} - L_\pm f = g_\pm, \quad (3.1)$$

with initial data $(f_0, E_0)$ and boundary condition

$$f(t, x, R_x v) = f(t, x, v), \quad \text{on} \quad x \in \partial \Omega, \quad (3.2)$$

where $g_\pm$ is chosen to be zero or given by

$$g_\pm = \pm \nabla_x \phi \cdot \nabla_v f_\pm + \frac{1}{2} \nabla_x \phi \cdot v f_\pm + \Gamma_\pm(f, f), \quad (3.3)$$

and the potential is determined by Poisson equation

$$-\Delta_x \phi = a_+ - a_-, \quad (3.4)$$

with the zero Neumann boundary condition

$$\partial_n \phi = 0, \quad \text{on} \quad \partial \Omega. \quad (3.5)$$

We denote $\zeta(v)$ to be a smooth function satisfying $\zeta(v) \lesssim e^{-\lambda |v|^2}$, for some $\lambda > 0$, which may change from line to line.

In order to discover the macroscopic dissipation, we take the following velocity moments

$$\mu^\frac{1}{2}, v_j \mu^\frac{1}{2}, \frac{1}{6}(|v|^2 - 3)\mu^\frac{1}{2}, (v_j v_m - 1)\mu^\frac{1}{2}, \frac{1}{10}(|v|^2 - 5) v_j \mu^\frac{1}{2}.$$
with $1 \leq j, m \leq 3$ for the equation (3.1). By taking the average and difference on $\pm$ of the resultant equations, one sees that the coefficient functions $[a_{\pm}, b, c] = [a_{\pm}, b, c](t, x)$ satisfy the fluid-type system; see [9, Section 3]:

\[
\begin{align*}
\partial_t \left( \frac{a_+ + a_-}{2} \right) + \nabla_x \cdot b &= 0, \\
\partial_t b_j + \partial_{x_1} \left( \frac{a_+ + a_-}{2} + 2c \right) + \frac{1}{2} \sum_{m=1}^{3} \partial_{x_m} \Theta_{jm}(\{I - P\}f \cdot [1, 1]) &= \frac{1}{2} \sum_\pm (g_\pm, v_{j1} \mu^{1/2})_{L^2}, \\
\partial_c + \frac{1}{3} \nabla_x \cdot b + \frac{5}{6} \sum_{j=1}^{3} \partial_{x_j} \Lambda_j(\{I - P\}f \cdot [1, 1]) &= \frac{1}{12} \sum_\pm (g_\pm, (|v|^2 - 3)\mu^{1/2})_{L^2}, \\
\partial_t \left( \frac{1}{2} \Theta_{jm}(\{I - P\}f \cdot [1, 1]) + 2c \delta_{jm} \right) + \partial_{x_j} b_m + \partial_{x_m} b_j &= \frac{1}{2} \sum_\pm \Theta_{jm}(g_\pm + h_\pm), \\
\frac{1}{2} \partial_t \Lambda_j(\{I - P\}f \cdot [1, 1]) + \partial_{x_j} c &= \frac{1}{2} \Lambda_j(g_+ + g_- + h_+ + h_-),
\end{align*}
\]

for $1 \leq j, m \leq 3$, where

\[
h_\pm = -v \cdot \nabla_x (I_\pm - P_\pm) f + L_\pm f,
\]

\[
\Theta_{jm}(f_\pm) = (f_\pm, (v_{j_1} v_{m_1} - 1)\mu^{1/2})_{L^2}, \quad \Lambda_j(f_\pm) = \frac{1}{10} (f_\pm, (|v|^2 - 5)v_{j1} \mu^{1/2})_{L^2},
\]

and

\[
\begin{align*}
\partial_t (a_+ - a_-) + \nabla_x \cdot G &= 0, \\
\partial_t G + \nabla_x (a_+ - a_-) - 2E + \nabla_x \cdot \Theta(\{I - P\}f \cdot [1, -1]) &= ((g + Lf) \cdot [1, -1], v\mu^{1/2})_{L^2},
\end{align*}
\]

where

\[
G = (\{I - P\}f \cdot [1, -1], v\mu^{1/2})_{L^2}. \tag{3.8}
\]

Here we first write the Lemma for high-order specular reflection boundary conditions. These conditions can be regarded as compatible condition.

Lemma 3.1. Let $(f, E)$ be the solution to (3.1), (3.2), (3.4) and (3.5). Fix $i \in \{1, 2, 3\}$. Then we have the following identities on boundary $\{(x, v) : v \cdot n(x) \neq 0 \text{ and } x \text{ belongs to the interior of } \Gamma_i\}$:

\[
f(x, v) = f(x, R_x v), \tag{3.9}
\]

and

\[
\begin{align*}
\partial_{\tau_j} f(x, R_v x) &= \partial_{\tau_j} f(x, v), \\
\partial_{\tau_j \tau_k} f(x, R_v x) &= \partial_{\tau_j \tau_k} f(x, v), \\
\partial_{\tau_j \tau_k \tau_m} f(x, R_v x) &= \partial_{\tau_j \tau_k \tau_m} f(x, v),
\end{align*} \tag{3.10}
\]

for $j, k, m = 1, 2$, where $(n, \tau_1, \tau_2)$ forms an unit normal basis in $\mathbb{R}^3$. For the normal derivatives, we have that on $\{(x, v) : v \cdot n(x) \neq 0 \text{ and } x \text{ belongs to the interior of } \Gamma_i\}$,

\[
\begin{align*}
\partial_n f(x, R_v x) &= -\partial_n f(x, v), \\
\partial_{\tau_j} \partial_n f(x, R_v x) &= -\partial_{\tau_j} \partial_n f(x, v), \\
\partial_{\tau_j \tau_k} \partial_n f(x, R_v x) &= -\partial_{\tau_j \tau_k} \partial_n f(x, v),
\end{align*} \tag{3.11}
\]

for $j, k = 1, 2$, and

\[
\begin{align*}
\partial_n^2 f(x, R_v x) &= \partial_n^2 f(x, v), \\
\partial_{\tau_j} \partial_n^2 f(x, R_v x) &= \partial_{\tau_j} \partial_n^2 f(x, v), \\
\partial_{\tau_j \tau_k} \partial_n^2 f(x, R_v x) &= \partial_{\tau_j \tau_k} \partial_n^2 f(x, v),
\end{align*} \tag{3.12}
\]

for $j = 1, 2$. 

Proof. Note that $R_x v$ maps $\gamma_-$ onto $\gamma_+$. Then it’s direct to obtain (3.9) from (3.2). On $\Gamma_i$, $\partial_{\tau_j}(x)$ ($j = 1, 2$) is the derivative with direction lies in $\Gamma_i$, where $\tau_1(x)$, $\tau_2(x)$ are tangent vector such that $(n, \tau_1, \tau_2)$ forms a unit normal basis in $\mathbb{R}^3$. Then we can obtain (3.10) by taking tangent derivatives on (3.9). For normal derivatives, we will apply the equation (3.1). We claim that

$$L f(x, v) = L f(x, R_x v)$$

and $g(x, v) = g(x, R_x v)$, on $n(x) \cdot v \neq 0$, (3.13)

for any $x$ belongs to the interior of $\Gamma_i$. Indeed, it suffices to show that

$$\nabla_x \phi \cdot \nabla_v f_{\pm}(R_x v) = \nabla_x \phi \cdot \nabla_v f_{\pm}(v),$$
and

$$\mu^{-1/2}Q(\mu^{1/2} f, \mu^{1/2} g)(R_x v) = \mu^{-1/2}Q(\mu^{1/2} f(R_x v), \mu^{1/2} g(R_x v)), (3.14)$$
on $n(x) \cdot v \neq 0$. By (3.5), we have $\partial_x \phi = 0$ on $\Gamma_i$. Notice that $R_x v$ sends $v_i$ to $-v_i$ and preserve the other components on $\Gamma_i$. Then for $j = 1, 2, 3$ such that $j \neq i$, we have $\partial_{v_j} f_{\pm}(R_x v) = \partial_{v_j} f_{\pm}(v)$ on $\Gamma_i$. Thus, on $\Gamma_i$, we have

$$\nabla_x \phi \cdot \nabla_v f_{\pm}(R_x v) = \sum_{j \neq i} \partial_{x_j} \phi \partial_{v_j} f_{\pm}(R_x v) = \nabla_x \phi \cdot \nabla_v f_{\pm}(v),$$
and

$$\nabla_x \phi \cdot R_x v f_{\pm}(R_x v) = \sum_{j \neq i} \partial_{x_j} \phi v_j f_{\pm}(v) = \nabla_x \phi \cdot v f_{\pm}(v).$$

Next we prove (3.14). For the Boltzmann case, we apply the Carleman representation as in [3, Appendix] to find that

$$\mu^{-1/2}Q(\mu^{1/2} f, \mu^{1/2} g)(R_x v)$$

$$= \int_{\mathbb{R}^3} \int_{E_{0,h}} \bar{b}(\alpha, h) 1_{|\alpha| \geq |h|} \frac{[\alpha + h]^{1+2s}}{|h|^{3+2s}} \mu^{1/2}(R_x v + \alpha - h)$$

$$\times \left( f(R_x v + \alpha) g(R_x v - h) - f(R_x v + \alpha - h)g(R_x v) \right) d\alpha dh$$

$$= \mu^{-1/2}Q(\mu^{1/2} f(R_x v), \mu^{1/2} g(R_x v)),$$

where we use change of variable $(\alpha, h) \mapsto (R_x \alpha, R_x h)$.

For Landau case, we will use representation from [19, Lemma 1]:

$$\mu^{-1/2}Q(\mu^{1/2} f, \mu^{1/2} g) = \partial_{v_j} \left[ \left\{ \phi^{jk} \ast [\mu^{1/2} f] \right\} \partial_{v_i} g \right] - \left\{ \phi^{jk} \ast \left[ \frac{\mu^{1/2}}{2} \partial_{v_i} f \right] \right\} \partial_{v_i} g$$

$$- \partial_{\tau_j} \left[ \left\{ \phi^{jk} \ast [\mu^{1/2} f] \right\} g \right] - \left\{ \phi^{jk} \ast \left[ \frac{\mu^{1/2}}{2} \partial_{\tau_i} f \right] \right\} g. (3.15)$$

Notice that $\partial_{v_j} f(R_x v) = -\partial_{v_i} f(R_x v)$ and $\partial_{v_j} f(R_x v) = \partial_{v_i} f(R_x v)$ on $\Gamma_i$, for $j \neq i$. Then on $\Gamma_i$, 

$$\sum_{j,k=1}^{3} \partial_{v_j} \left[ \left\{ \phi^{jk} \ast [\mu^{1/2} f] \right\} \partial_{v_k} g \right](R_x v)$$

$$= \sum_{k=1}^{3} \partial_{v_k} \left[ \left\{ \phi^{jk} \ast [\mu^{1/2} f] \right\} (R_x v) \partial_{v_k} g(R_x v) \right] + \sum_{j \neq i} \sum_{k=1}^{3} \partial_{v_j} \left[ \left\{ \phi^{jk} \ast [\mu^{1/2} f] \right\} (R_x v) \partial_{v_k} g(R_x v) \right]$$

$$= \partial_{v_i} \left[ \left\{ \phi^{ij} \ast [\mu^{1/2} f(R_x v)] \right\} \partial_{v_i} g(R_x v) \right] + \sum_{k \neq i} \partial_{v_i} \left[ \left\{ \phi^{ij} \ast [\mu^{1/2} f(R_x v)] \right\} \partial_{v_k} g(R_x v) \right]$$

$$+ \sum_{j \neq i} \partial_{v_j} \left[ \left\{ \phi^{ij} \ast [\mu^{1/2} f(R_x v)] \right\} \partial_{v_i} g(R_x v) \right] + \sum_{j \neq i, k \neq i} \partial_{v_j} \left[ \left\{ \phi^{ij} \ast [\mu^{1/2} f(R_x v)] \right\} \partial_{v_k} g(R_x v) \right]$$
where we apply (1.2) to deduce that \( \phi^{ik}(R_x v) = -\phi^{jk}(v), \phi^{ji}(R_x v) = -\phi^{ij}(v) \) when \( k \neq i, j \neq i \). Similar calculation can be applied to the second to forth terms of (3.15) and we obtain (3.14) for Landau case. This completes the claim (3.13).

Noticing that

\[
v \cdot \nabla_x f = v \cdot n(x)\partial_n f + v \cdot \tau_1(x)\partial_{\tau_1} f + v \cdot \tau_2(x)\partial_{\tau_2} f,
\]

we can rewrite (3.1) as

\[
v \cdot n(x)\partial_n f \pm = -v \cdot \tau_1(x)\partial_{\tau_1} f \pm - v \cdot \tau_2(x)\partial_{\tau_2} f \pm - \partial_t f \pm + \nabla_x \phi \cdot v \mu^{1/2} + L \pm f + g \pm.
\]

Applying (3.9) and (3.13) to the right hand side, we can obtain that on \( \partial \Omega \),

\[
R_x v \cdot n(x)\partial_n f \pm (R_x v) = v \cdot n(x)\partial_n f \pm (v).
\]

Since \( R_x v \cdot n(x) = -v \cdot n(x) \), this implies (3.11) by taking tangent derivative. Apply \( \partial_n \) to (3.1) twice and rewrite it to be

\[
v \cdot n\partial_n \partial_n f \pm = -v \cdot \tau_1(x)\partial_{\tau_1} \partial_n f \pm - v \cdot \tau_2(x)\partial_{\tau_2} \partial_n f \pm - \partial_t \partial_n f \pm + L \pm \partial_n f + \partial_n \nabla_x \phi \cdot v \mu^{1/2} + \partial_n g \pm. \tag{3.16}
\]

Here, on \( \Gamma_i \), by taking tangent derivatives on (3.5), we have \( \partial_n \partial_{x_j} \phi = 0 \) for \( j \neq i \) and hence

\[
\partial_n \nabla_x \phi \cdot R_x v \mu^{1/2}(R_x v) = \partial_n \partial_{x_i} \phi(R_x v)\mu^{1/2}(v) = -\partial_n \partial_{x_i} \phi v_i \mu^{1/2}.
\]

When \( g \pm \) is given by (3.3), we have on \( \Gamma_i \) that

\[
\partial_n g \pm = \pm \partial_n \nabla_x \phi \cdot \nabla_v f \pm + \nabla_x \phi \cdot \partial_n \nabla_v f \pm + \frac{1}{2} \partial_n \nabla_x \phi \cdot v f \pm + \frac{1}{2} \nabla_x \phi \cdot v \partial_n f \pm + \Gamma \pm(\partial_n f, f) + \Gamma \pm(f, \partial_n f) = \pm \partial_n \partial_{x_i} \phi (v_i f) + \frac{1}{2} \partial_n \partial_{v_j} \phi v_i f \pm + \frac{1}{2} \partial_n \partial_{x_i} \phi v_i f \pm + \Gamma \pm(\partial_n f, f) + \Gamma \pm(f, \partial_n f).
\]

Together with (3.11) and (3.14), we know that on \( \Gamma_i \),

\[
\partial_n g \pm (R_x v) = -\partial_n g \pm (v).
\]

Combining the above identities and (3.16), we have

\[
R_x v \cdot n(x)\partial_n^2 f (x, R_x v) = -v \cdot n(x)\partial_n^2 f (x, v).
\]

This gives (3.12)_1 and (3.12)_2 follows by taking tangent derivatives. This completes the proof of Lemma 3.1.

As a corollary, by definition (1.4), we have the following boundary values for \([a \pm, b, c]\).

**Lemma 3.2.** Let \((f, E)\) be the solution to (3.1), (3.2), (3.4) and (3.5). Define \([a \pm, b, c]\) by (1.4). For \(i = 1, 2, 3\) and any \(x \in \Gamma_i\), we have

\[
\partial_{x_i} c(x) = \partial_{x_i} a \pm(x) = \partial_{x_i} b_j(x) = \partial_{x_i x} b_i(x) = b_i(x) = 0, \tag{3.17}
\]
for \(j \neq i\). As a consequence,
\[
\sum_{i,j=1}^{3} \| \partial_{x,i,j} a_{\pm} \|_{L^2}^2 = \| \Delta_x a_{\pm} \|_{L^2}^2,
\]
\[
\sum_{i,j=1}^{3} \| \partial_{x,i,j} b_{\pm} \|_{L^2}^2 = \| \Delta_x b_{\pm} \|_{L^2}^2,
\]
\[
\sum_{i,j=1}^{3} \| \partial_{x,i,j} c \|_{L^2}^2 = \| \Delta_x c \|_{L^2}^2.
\]  

(3.18)

Moreover, on \(\Gamma_i\), we have
\[
\partial_{x,i} \phi = 0,
\]

(3.19)

Proof. Fix \(x \in \Gamma_i\). Notice that on the boundary of union of cubes, we have \(\partial_n f = \partial_{x,i} f\) or \(-\partial_{x,i} f\). Then by (3.11) and change of variable \(v \mapsto R_x v\), we have on \(\Gamma_i\) that
\[
\partial_{x,i} c = \frac{1}{12} \int_{\mathbb{R}^3} \partial_{x,i} \left( f_+(x, R_x v) + f_-(x, R_x v) \right) |R_x v|^2 \mu^{1/2}(R_x v) \, dv
\]
\[
= -\frac{1}{12} \int_{\mathbb{R}^3} \partial_{x,i} \left( f_+(x, v) + f_-(x, v) \right) |v|^2 \mu^{1/2}(v) \, dv = 0.
\]

Similarly, on interior of \(\Gamma_i\), we have
\[
\partial_{x,i} a_{\pm} = \int_{\mathbb{R}^3} \partial_{x,i} f_{\pm}(x, R_x v) \mu^{1/2}(R_x v) \, dv = -\int_{\mathbb{R}^3} \partial_{x,i} f_{\pm}(x, v) \mu^{1/2}(v) \, dv = 0,
\]
and for \(j \neq i\),
\[
\partial_{x,i} b_j = \frac{1}{2} \int_{\mathbb{R}^3} \partial_{x,i} \left( f_+(x, R_x v) + f_-(x, R_x v) \right) (R_x v)_j \mu^{1/2}(R_x v) \, dv
\]
\[
= -\frac{1}{2} \int_{\mathbb{R}^3} \partial_{x,i} \left( f_+(x, v) + f_-(x, v) \right) v_j \mu^{1/2}(v) \, dv = 0.
\]

On \(\Gamma_i\), we have \((R_x v)_i = -v_i\) and hence by (3.9) and (3.12), we have
\[
b_i(x) = \frac{1}{2} \int_{\mathbb{R}^3} \left( f_+(x, R_x v) + f_-(x, R_x v) \right) (R_x v)_i \mu^{1/2}(R_x v) \, dv
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^3} \left( f_+(x, v) + f_-(x, v) \right) v_i \mu^{1/2}(v) \, dv = 0,
\]
and
\[
\partial_{x,i} b_i = \frac{1}{2} \int_{\mathbb{R}^3} \left( \partial_{x,i} f_+(x, R_x v) + \partial_{x,i} f_-(x, R_x v) \right) (R_x v)_i \mu^{1/2}(R_x v) \, dv
\]
\[
= -\frac{1}{2} \int_{\mathbb{R}^3} \left( \partial_{x,i} f_+(x, v) + \partial_{x,i} f_-(x, v) \right) v_i \mu^{1/2}(v) \, dv = 0.
\]

For any \(\varphi = \varphi(x)\) satisfying that \(\partial_{x_i} \varphi = 0\) or \(\partial_{x_k} \varphi = 0\) on \(\Gamma_k\) for any \(k = 1, 2, 3\). We have
\[
\int_{\Omega} |\partial_{x,l} \varphi|^2 \, dx = \int_{\Gamma_j} \partial_{x,l} \varphi \partial_{x,l} \varphi \, dS(x) - \int_{\Gamma_j} \partial_{x,l} \varphi \partial_{x,l} \varphi \, dx + \int_{\Omega} \partial_{x,l} \varphi \partial_{x,l} \varphi \, dx
\]
\[
= \int_{\Omega} \partial_{x,l} \varphi \partial_{x,l} \varphi \, dx,
\]
where \(dS\) is the spherical measure. Then we have \(\sum_{i,j} \| \partial_{x,i,j} \varphi \|_{L^2}^2 = \| \Delta_x \varphi \|_{L^2}^2\). Replacing \(\varphi\) to be \(a_{\pm}, b_j\) and \(c\), we obtain (3.18).
For the proof of (3.19), by taking tangent derivatives on (3.5), we have $\partial_{x^j}x_j \phi = 0$ on $\Gamma_i$ for $j \neq i$. Then by (3.4), we have on $\Gamma_i$ that

$$
\partial_{x^i,x^j} \phi = - \sum_{j \neq i} \partial_{x^j,x^j} \phi - \partial_{x^i}(a_+ - a_-) = 0,
$$

where we used (3.17) for $\partial_{x^i}a_\pm = 0$. This completes Lemma 3.2.

With the help of (3.19), we are able to obtain the third derivative version for Lemma 3.1 and 3.2.

**Lemma 3.3.** Assuming the same conditions in Lemma 3.1. Then we have for $x \in \partial \Omega$ that

$$
\partial^3_n f_\pm(R_x v) = \partial^3_n f_\pm(v), \text{ on } v \cdot n(x) \neq 0. \tag{3.20}
$$

Consequently, we have on $\Gamma_i(i = 1, 2, 3)$ that

$$
\partial_{x^i,x_i} \phi(x) = \partial_{x^i,x_i} a_\pm(x) = \partial_{x^i,x_i} b_j(x) = 0, \tag{3.21}
$$

for $j \neq i$.

**Proof.** By taking normal derivative of (3.16), we have

$$
v \cdot n \partial_n \partial_n \partial_n f_\pm = -v \cdot \tau_1(x) \partial_n \partial_n f_\pm - v \cdot \tau_2(x) \partial_n \partial_n f_\pm - \partial_n \partial_n f_\pm + L_\pm \partial_n \partial_n f_\pm + \partial_n \partial_n g_\pm. \tag{3.22}
$$

Notice that on $\Gamma_i$, by (3.19), we have

$$
\partial_n \partial_n \nabla_x \phi \cdot v \mu^{1/2} = \sum_{j \neq i} \partial_{x^j,x^j} \phi \ v_j \mu^{1/2},
$$

and when $g$ is given by (3.3), one has

$$
\partial_n \partial_n g_\pm = \pm \partial_n \partial_n \nabla_x \phi \cdot \nabla_v f_\pm \pm \partial_n \nabla_x \phi \cdot \partial_n \partial_n \nabla_v f_\pm \pm \frac{1}{2} \partial_n \partial_n \nabla_x \phi \cdot \nabla_v f_\pm \pm \frac{1}{2} \partial_n \nabla_x \phi \cdot v \partial_n \partial_n f_\pm

+ \frac{1}{2} \partial_n \partial_n \nabla_x \phi \cdot v \partial_n \partial_n f_\pm + \frac{1}{2} \partial_n \nabla_x \phi \cdot v \partial_n \partial_n f_\pm + \partial_n \nabla_x \phi \cdot \partial_n \partial_n f_\pm + \frac{1}{2} \partial_n \partial_n \nabla_v \phi \cdot v \partial_n \partial_n f_\pm + \frac{1}{2} \partial_n \partial_n \nabla_v \phi \cdot v \partial_n \partial_n f_\pm

+ \Gamma_\pm(\partial_n \partial_n f, f) + 2 \Gamma_\pm(\partial_n f, \partial_n f) + \Gamma_\pm(f, \partial_n \partial_n f).
$$

Applying Lemma 3.1 and identity (3.14), we have on $\Gamma_i$ that

$$
\sum_{j \neq i} \partial_{x^j,x^j} \phi(R_x v_j) \mu^{1/2}(R_x v_j) = \sum_{j \neq i} \partial_{x^j,x^j} \phi v_j \mu^{1/2}(v),
$$

and

$$
\partial_n \partial_n g_\pm(R_x v) = \partial_n \partial_n g_\pm(v).
$$

Combining the above identities and Lemma 3.1, we have from (3.22) that on $\Gamma_i$

$$
R_x v \cdot n \partial_n \partial_n f_\pm(R_x v) = v \cdot n \partial_n \partial_n f_\pm(v).
$$

Note that $R_x v \cdot n(x) = -v \cdot n(x)$ on $\Gamma_i$, we obtain (3.20). The proof of (3.21) is similar to (3.17) by using specular reflection boundary condition (3.12) and (3.20) for high-order
Next we give the estimates on macroscopic parts \([a_\pm, b, c]\). The idea is similar to \([7]\). However, with the electric potential \(\phi\), we need more careful calculations.

**Theorem 3.4.** Let \(K = 2, 3\) be the total order of derivative. Let \(\gamma \geq -3\) for Landau case, \(\gamma > \max\{-3, -2s - 3/2\}\) for Boltzmann case and \(T > 0\). Let \((f, E)\) be the solution of (3.1), (3.2), (3.4) and (3.5) in bounded domain \(\Omega\) with initial data \((f_0, E_0)\). Then there exists an instant energy functional \(\mathcal{E}_{int}(t)\) satisfying

\[
\mathcal{E}_{int}(t) \lesssim \sum_{|\alpha| \leq K} \|\partial^\alpha f\|_{L^2}^2,
\]

such that

\[
\partial_t \mathcal{E}_{int}(t) + \lambda \sum_{|\alpha| \leq K} \|\partial^\alpha [a_+, a_-, b, c]\|_{L^2}^2 + \lambda \sum_{|\alpha| \leq K} \|\partial^\alpha E\|_{L^2}^2 
\lesssim \sum_{|\alpha| \leq K} \|\{I - P\} \partial^\alpha f\|_{L^2}^2 + \sum_{|\alpha| \leq K} \|\partial^\alpha g, \zeta(v)\|_{L^2}^2 + \|E\|_{L^2}^2,
\]

for some constant \(\lambda > 0\), where \(g = [g_+, g_-]\) is zero or given by (3.3).

**Proof.** Let \(|\alpha| \leq K\) and we restrict

\[
\partial^\alpha = \partial_{x_i}, \quad \text{for some } i = 1, 2, 3 \text{ when } |\alpha| = 2.
\]

(3.23)

Using Lemma 3.2, we only need to consider \(\|\Delta_x [a_+, a_-, b, c]\|_{L^2}^2\) when estimating the second order derivatives of \([a_+, a_-, b, c]\). Applying \(\partial^\alpha\) to (3.1), we have

\[
\partial_t \partial^\alpha f_\pm + v \cdot \nabla_x \partial^\alpha f_\pm \pm \partial^\alpha \nabla_x \phi \cdot v \mu^{1/2} - L_\pm \partial^\alpha f = \partial^\alpha g_\pm.
\]

(3.24)

To state the proof in a unified way, we let \(\Phi(t, x, v) \in C^1((0, +\infty) \times \Omega \times \mathbb{R}^3)\) be a test function. Taking the inner product of (3.24) with \(\Phi(t, x, v)\) with respect to \((x, v)\), we obtain

\[
\partial_t (\partial^\alpha f_\pm, \Phi)_{L_{2,v}} - (\partial^\alpha f_\pm, \partial_t \Phi)_{L_{2,v}} - (\partial^\alpha f_\pm, v \cdot \nabla_x \Phi)_{L_{2,v}} + \int_{\partial \Omega} (v \cdot n(x)) \partial^\alpha f_\pm, \Phi)_{L_v^2} dS(x)
\]

\[
\pm (\partial^\alpha \nabla_x \phi \cdot v \mu^{1/2}, \Phi)_{L_{2,v}} - (L_\pm \partial^\alpha f, \Phi)_{L_{2,v}} = (\partial^\alpha g_\pm, \Phi)_{L_{2,v}}.
\]

Using the decomposition \(f_\pm = P f_\pm + \{I - P\} f_\pm\), we rewrite the above equation to be

\[
\partial_t (\partial^\alpha f_\pm, \Phi)_{L_{2,v}} - (\partial^\alpha P f_\pm, v \cdot \nabla_x \Phi)_{L_{2,v}} = \sum_{j=1}^5 S_j,
\]

(3.25)

where \(S_j\)’s are defined by

\[
S_1 = (\partial^\alpha f_\pm, \partial_t \Phi)_{L_{2,v}},
\]

\[
S_2 = (\partial^\alpha (I_\pm - P_\pm) f_\pm + \partial^\alpha f_\pm, v \cdot \nabla_x \Phi)_{L_{2,v}},
\]

\[
S_3 = (L_\pm \partial^\alpha f_\pm + (\partial^\alpha g_\pm, \Phi)_{L_{2,v}},
\]

\[
S_4 = \mp(\partial^\alpha \nabla_x \phi \cdot v \mu^{1/2}, \Phi)_{L_{2,v}},
\]

\[
S_5 = -\int_{\partial \Omega} (v \cdot n(x)) \partial^\alpha f_\pm, \Phi)_{L_v^2} dS(x).
\]

**Step 1. Estimate on \(c(t, x)\):** In this step, we will let \(|\alpha| \geq 1\). Choose test function

\[
\Phi = \Phi_c = (|v|^2 - 5)(v \cdot \nabla_c(t, x)) \mu^{1/2},
\]

\[
|\partial^\alpha c|_{L_v^2} \lesssim \sum_{j=1}^5 S_j
\]

such that

\[
\partial_t (\partial^\alpha c, \Phi)_{L_{2,v}} - (\partial^\alpha P c, v \cdot \nabla_x \Phi)_{L_{2,v}} = \sum_{j=1}^5 S_j
\]

(3.26)
where $\phi_c$ solves
\[
\begin{aligned}
-\Delta x \phi_c &= \partial^\alpha c \text{ in } \Omega, \\
\phi_c(x) &= 0 \text{ on } x \in \Gamma_i, \text{ if } \alpha_i = 1 \text{ or } 3, \\
\partial_n \phi_c(x) &= 0 \text{ on } x \in \Gamma_i, \text{ if } \alpha_i = 0 \text{ or } 2.
\end{aligned}
\] (3.26)

The existence of solution to (3.26) is given by [18, Lemma 4.4.3.1]. In particular, when $|\alpha| = 2$, we deduce from (3.23) that $\alpha_i = 2$ for some $i$ and $\alpha_k = 0$ for $k \neq i$. Thus, (3.26) is pure Neumann problem and we need $\int_{\Omega} \partial_x x_i \partial_t c \, dx = \int_{\Gamma_1} \partial_x x_i \partial_t c \, dS(x) = 0$ from Lemma 3.2 to ensure the existence of (3.26). Similar to the proof for (3.18), by using boundary value of $\phi_c$, we have
\[
\sum_{i,j=1}^{3} \|\partial_{x_i} x_j \phi_c \|^2_{L^2_2} = \|\Delta x \phi_c \|^2_{L^2_2} \lesssim \|\partial^\alpha c\|^2_{L^2_2}.
\] (3.27)

We discuss the value of $\alpha$ in the following cases. If $|\alpha| = 1$, then $\alpha_i = 1$ for some $1 \leq i \leq 3$. Hence, $\phi_c(x) = 0$ on $\Gamma_i$ and $\partial_n \phi_c(x) = 0$ on $\Gamma_j$ for $j \neq i$. It follows that
\[
\|\nabla x \phi_c\|^2_{L^2_2} = \sum_{j=1}^{3} \int_{\Gamma_j} \partial_{x_j} x_i \phi_c \, dx - \int_{\Omega} \Delta x \phi_c \, dx \\
= \int_{\Omega} \partial^\alpha c \, dx \lesssim \|\partial^\alpha c\|_{L^2_2} \|\phi_c\|_{L^2_2}.
\] (3.28)

Since $\phi_c = 0$ on $\Gamma_1$, by Sobolev embedding [6, Theorem 6.7.5], we have $\|\phi_c\|_{L^2_2} \lesssim \|\nabla x \phi_c\|_{L^2_2}$. Then from (3.28), we have
\[
\|\nabla x \phi_c\|_{L^2_2} \lesssim \|\partial^\alpha c\|_{L^2_2} \lesssim \sum_{|\alpha|=1} \|\partial^\alpha c\|_{L^2_2}.
\] (3.29)

Similarly, since derivative $\partial_t$ doesn’t affect the boundary value for $\phi_c$, we have
\[
\|\partial_t \nabla x \phi_c\|_{L^2_2} \lesssim \sum_{|\alpha|=1} \|\partial_t \partial^\alpha c\|_{L^2_2}.
\] (3.30)

If $|\alpha| = 2$, at stated before, we only consider $\alpha_i = 2$ for some $1 \leq i \leq 3$. Then for this $i$, similar to (3.28), by using boundary values $\partial_{x_i} c = 0$ on $\Gamma_i$ from (3.2), we have
\[
\|\nabla x \phi_c\|^2_{L^2_2} = \int_{\Omega} \partial_{x_i} \partial_{x_i} x_i \phi_c \, dx = \int_{\Omega} \partial_{x_i} c \partial_{x_i} \phi_c \, dx \lesssim \|\partial_{x_i} c\|^2_{L^2_2} \|\partial_{x_i} \phi_c\|_{L^2_2}.
\]

This implies that
\[
\|\nabla x \phi_c\|_{L^2_2} \lesssim \|\partial_{x_i} c\|_{L^2_2}.
\] (3.31)

Similarly, since $\partial_t$ doesn’t affect the boundary value for $\phi_c$, we have
\[
\|\partial_t \nabla x \phi_c\|_{L^2_2} \lesssim \|\partial_t \partial_{x_i} c\|_{L^2_2}.
\] (3.32)

If $|\alpha| = 3$, then there exists $1 \leq i \leq 3$ such that $\alpha_i = 1$ or $3$. Then the boundary value for $\phi_c$ gives that $\phi_c = 0$ on $\Gamma_i$. Denote $\partial^\alpha = \partial_{x_i} x_k$, for some $1 \leq j, k \leq 3$. Then taking inner product of (3.26) with $\phi_c$, we have
\[
\|\nabla x \phi_c\|^2_{L^2_2} = \int_{\Omega} \partial^\alpha c \, dx = \int_{\Gamma_1} \partial_{x_i} c \phi_c \, dx - \int_{\Omega} \partial_{x_j x_k} c \partial_{x_i} \phi_c \, dx \\
\lesssim \|\nabla^2_{x} c\|_{L^2_2} \|\nabla x \phi_c\|_{L^2_2}.
\]
Thus,
\[
\|\nabla x \phi_c\|_{L^2_2} \lesssim \|\nabla^2_{x} c\|_{L^2_2}.
\] (3.33)
Similarly,
\[ \| \partial_t \nabla_x \phi_c \|_{L^2} \lesssim \| \partial_t \nabla^2 c \|_{L^2}. \] (3.34)
Now we can compute (3.25). For the second term on left hand side of (3.25), we have
\begin{align*}
&- (\partial^\alpha P \pm f, v \cdot \nabla_x \Phi_c)_{L^2,v} \\
&= - \sum_{j,m=1}^3 ((\partial^\alpha a_{\pm} + \partial^\alpha b \cdot v + (|v|^2 - 3)\partial^\alpha c)\mu^{1/2}, v_j v_m (|v|^2 - 5)\mu^{1/2}\partial_x_j \partial_x_m \phi_c)_{L^2,v} \\
&= 10 \sum_{j=1}^3 (\partial^\alpha c, -\partial^\alpha \phi)_{L^2,v} = 10\| \partial^\alpha c \|_{L^2}^2.
\end{align*}
Note that \( \int_{\mathbb{R}^3} |v|^2v_j^2 \mu \, dv = 35, \int_{\mathbb{R}^3} |v|^2v_j^2 \mu \, dv = 5 \) and \( \int_{\mathbb{R}^3} v_j^2 \mu \, dv = 1 \). For \( S_1 \), we obtain from (3.30), (3.32) and (3.34) that for any \( \eta > 0 \),
\begin{align*}
|S_1| &\lesssim \eta \| \partial_t \nabla_x \phi_c \|_{L^2}^2 + C_\eta \| (\mathbf{I} - P) \partial^\alpha f \|_{L^2_{1/2}}^2 \\
&\lesssim \eta \sum_{1 \leq |\alpha| \leq K} \| \partial^\alpha b \|_{L^2}^2 + \eta \sum_{1 \leq |\alpha| \leq K} \| \partial^\alpha g, \zeta \|_{L^2}^2 + C_\eta \sum_{|\alpha| \leq K} \| (\mathbf{I} - P) \partial^\alpha f \|_{L^2_{1/2}}^2.
\end{align*}
where we used (3.6)_3 in the last inequality. By (3.27), \( S_2 \) can be estimated as
\[ |S_2| \lesssim \eta \sum_{j=1}^3 \| \partial_x x_j \phi_c \|_{L^2}^2 + C_\eta \| \partial^\alpha \{ \mathbf{I} - P \} f \|_{L^2_{1/2}}^2 \lesssim \eta \| \partial^\alpha c \|_{L^2}^2 + C_\eta \| \partial^\alpha \{ \mathbf{I} - P \} f \|_{L^2_{1/2}}^2, \]
for any \( \eta > 0 \). For \( S_3 \), applying (3.29), (3.31) and (3.33), we have
\[ |S_3| \leq \eta \sum_{|\alpha| \leq K} \| \partial^\alpha c \|_{L^2}^2 + C_\eta \| \partial^\alpha \{ \mathbf{I} - P \} f \|_{L^2_{1/2}}^2 + C_\eta \| (\partial^\alpha g, \zeta) \|_{L^2}^2. \]
For \( S_4 \), we obtain from (3.29), (3.31) and (3.33) that
\[ |S_4| \lesssim C_\eta \| \partial^\alpha \nabla_x \phi_c \|_{L^2}^2 + \eta \sum_{|\alpha| \leq K} \| \partial^\alpha c \|_{L^2}^2. \]
For \( S_5 \), we need to use the boundary condition from Lemma 3.1 and (3.20): \[ S_5 = - \int_{\partial \Omega} (v \cdot n(x)) \partial^\alpha f (x), \Phi_c(x))_{L^2} \, dS(x). \]
Divide the integral on \( \partial \Omega \) into three parts, \( \Gamma_i \) (\( i = 1, 2, 3 \)), and consider each component \( \Gamma_i \) separately. Fix \( i = 1, 2, 3 \). Then on \( \Gamma_i \), we have \( \partial_n = \partial_{x_i} \) or \( -\partial_{x_i} \). Then
\begin{align*}
\int_{\Gamma_i} (v \cdot n(x)) \partial^\alpha f (x), \Phi_c(x))_{L^2} \, dS(x) \\
&= \int_{\Gamma_i} \int_{\mathbb{R}^3} v \cdot n(x) \partial^\alpha f (t, x, v)(|v|^2 - 5)(v \cdot \nabla_x \phi_c(t, x)) \mu^{1/2} \, dv \, dS(x). \tag{3.35}
\end{align*}
If \( \alpha_i = 0 \) or \( 2 \), then we deduce from (3.9), (3.10) and (3.12) that \( \partial^\alpha f (R_x v) = \partial^\alpha f (v) \) and from (3.26) that \( \partial_x \phi_c = 0 \). Applying change of variable \( v \mapsto R_x v \), (3.35) becomes
\begin{align*}
\int_{\Gamma_i} \int_{\mathbb{R}^3} v \cdot n(x) \partial^\alpha f (t, x, v)(|v|^2 - 5) \sum_{j \neq i} (v_j \partial_x_j \phi_c(t, x)) \mu^{1/2} \, dv \, dS(x) \\
= \int_{\Gamma_i} \int_{\mathbb{R}^3} R_x v \cdot n(x) \partial^\alpha f (t, x, R_x v)(|R_x v|^2 - 5) \sum_{j \neq i} (R_x v_j \partial_x_j \phi_c(t, x)) \mu^{1/2} \, dv \, dS(x)
\end{align*}
If \( \alpha_i = 1 \) or \( 3 \), then from boundary conditions, we have \( \partial^\alpha f(R_x v) = -\partial^\alpha f(v) \) and \( \partial_x \phi_c = 0 \) on \( \Gamma_i \) for any \( j \neq i \). Applying change of variable \( v \mapsto R_x v \) to (3.35) and using (3.11), we obtain

\[
\int_{\Gamma_i} \int_{\mathbb{R}^3} v \cdot n(x) \partial^\alpha f(t, x, v)(|v|^2 - 5)v_i \partial_x \phi_c(t, x) \mu_{1/2}^1 \, dv \, dS(x) = 0.
\]

Since the above estimates are valid for \( i = 1, 2, 3 \), we obtain

\[
S_5 = 0. \tag{3.36}
\]

Combining the above estimates for \( S_j \) (\( 1 \leq j \leq 5 \)), taking summation over \( 1 \leq |\alpha| \leq K \) and letting \( \eta \) suitably small, we obtain

\[
\partial_t \sum_{1 \leq |\alpha| \leq K} (\partial^\alpha f, \Phi_c)_{L^2_{x,v}} + \lambda \sum_{1 \leq |\alpha| \leq K} \|\partial^\alpha c\|^2_{L^2_{x,v}} \lesssim \eta \sum_{1 \leq |\alpha| \leq K} \|\partial^\alpha b\|^2_{L^2_{x,v}} + \sum_{|\alpha| \leq K} \|\partial^\alpha \nabla_x \phi\|^2_{L^2_{x,v}} + C_\eta \sum_{|\alpha| \leq K} \|(I - P) \partial^\alpha f\|^2_{L^2_{x,v}} + C_\eta \sum_{|\alpha| \leq K} \|(\partial^\alpha g, \zeta)\|^2_{L^2_{x,v}}, \tag{3.37}
\]

for some \( \lambda > 0 \) and any \( \eta > 0 \). Note that we have applied (3.18). The estimate (3.37) gives derivatives estimate on \( c \). For the zeroth derivative of \( c \), we apply the Poincaré’s inequality and (1.5) to obtain that

\[
\|c\|_{L^2_x} \lesssim \|\nabla_x c\|_{L^2_x} + \int_{\Omega} c \, dx \lesssim \|\nabla_x c\|_{L^2_x} + \|E\|_{L^2_x}^2.
\]

Plugging this estimate into (3.37), we have

\[
\partial_t \sum_{1 \leq |\alpha| \leq K} (\partial^\alpha f, \Phi_c)_{L^2_{x,v}} + \lambda \sum_{|\alpha| \leq K} \|\partial^\alpha c\|^2_{L^2_{x,v}} \lesssim \eta \sum_{1 \leq |\alpha| \leq K} \|\partial^\alpha b\|^2_{L^2_{x,v}} + \sum_{|\alpha| \leq K} \|\partial^\alpha \nabla_x \phi\|^2_{L^2_{x,v}} + C_\eta \sum_{|\alpha| \leq K} \|(I - P) \partial^\alpha f\|^2_{L^2_{x,v}} + C_\eta \sum_{|\alpha| \leq K} \|(\partial^\alpha g, \zeta)\|^2_{L^2_{x,v}} + \|E\|_{L^2_x}^4, \tag{3.38}
\]

**Step 2. Estimate of** \( b(t, x) \). Next we consider the estimate of \( b \). For this purpose we choose

\[
\Phi = \Phi_b = \sum_{m=1}^3 \Phi_b^{j,m}, \quad j = 1, 2, 3,
\]

where

\[
\Phi_b^{j,m} = \begin{cases} (|v|^2 v_m v_j \phi_m - \frac{7}{2} (v_m^2 - 1) \phi_j)^{1/2}, & m \neq j, \\ \frac{7}{2} (v_j^2 - 1) \phi_j^{1/2}, & m = j, \end{cases}
\]

and \( \phi_j(1 \leq j \leq 3) \) solves

\[
\begin{cases} -\Delta_x \phi_j = \partial^\alpha b_j \text{ in } \Omega, \\
\phi_k(x) = \partial_n \phi_m(x) = 0 \text{ on } x \in \Gamma_m, \text{ for } k \neq m, \text{ if } \alpha_m = 1 \text{ or } 3, \\
\phi_m(x) = \partial_n \phi_k(x) = 0 \text{ on } x \in \Gamma_m, \text{ for } k \neq m, \text{ if } \alpha_m = 0 \text{ or } 2, \forall m = 1, 2, 3. \tag{3.39} \end{cases}
\]
The existence of solutions to (3.39) is given by [18, Lamma 4.4.3.1]. We will explain the sufficient and necessary conditions for existence to pure Neumann type problem later. By using the boundary value of \( \phi_j \), similar to (3.27), we have

\[
3 \sum_{i,k=1}^{3} \| \partial_{x_i x_k} \phi_j \|_{L_x^2}^2 = \| \Delta_x \phi_j \|_{L_x^2}^2 \lesssim \| \partial^\alpha b_j \|_{L_x^2}^2.
\] (3.40)

Then \( S_2 \) can be estimated as

\[
|S_2| \lesssim \| \partial^\alpha (I - P) f \|_{L_x^2 L_{t/2}^2} \sum_{i,j,m=1}^{3} \| \partial_{x_i x_m} \phi_j \|_{L_x^2} \lesssim C_\eta \| \partial^\alpha (I - P) f \|_{L_x^2 L_{t/2}^2} + \eta \| \partial^\alpha b \|_{L_x^2}^2.
\] (3.41)

Next we fix \( 1 \leq j \leq 3 \) and discuss the value of \( |\alpha| \) in the following cases. If \( |\alpha| = 0 \), then (3.39) is mixed Neumann-Dirichlet boundary problem. Then by standard elliptic estimates, we have

\[
\| \nabla_x \phi_j \|_{L_x^2} \lesssim \| b_j \|_{L_x^2}, \quad \| \partial_t \nabla_x \phi_j \|_{L_x^2} \lesssim \| \partial_t b_j \|_{L_x^2},
\] (3.42)

If \( |\alpha| = 1 \). Then \( \alpha_i = 1 \) for some \( 1 \leq i \leq 3 \) and \( \alpha_k = 0 \) for \( k \neq i \). In particular, if \( j = i \), then \( \partial_{x_i} \phi_i = 0 \) on \( \Gamma_i \) and \( \partial_{x_k} \phi_i = 0 \) on \( \Gamma_k \) for \( k \neq i \). In this case, (3.39) is a pure Neumann boundary problem and we need \( \int_{\Omega} \partial_{x_i} b_i \, dx = \int_{\Gamma_i} b_i \, dS(x) = 0 \) to ensure the existence of (3.39), which follows from (3.17). In this case, \( \partial_{x_m} \phi_i = 0 \) on a subset of boundary \( \partial \Omega \) with non-zero spherical measure for any \( m = 1, 2, 3 \). By Sobolev embedding [6, Theorem 6.7-5], we have from (3.40) that

\[
\| \partial_{x_m} \phi_i \|_{L_x^2} \lesssim \| \nabla_x \partial_{x_m} \phi_i \|_{L_x^2} \lesssim \| \partial^\alpha b_i \|_{L_x^2},
\] (3.43)

and

\[
\| \partial_t \partial_{x_m} \phi_i \|_{L_x^2} \lesssim \| \partial_t \nabla_x \partial_{x_m} \phi_i \|_{L_x^2} \lesssim \| \partial_t \partial^\alpha b_i \|_{L_x^2},
\] (3.44)

for any \( m = 1, 2, 3 \).

If \( j \neq i \), then \( \phi_j = 0 \) on \( \Gamma_i \) and \( \Gamma_j \) while \( \partial_{x_j} \phi_j = 0 \) on \( \Gamma_k \) for \( k \neq j, i \). (3.39) is a mixed Dirichlet-Neumann boundary problem. By Sobolev embedding [6, Theorem 6.7-5], we have

\[
\| \partial_t \phi_j \|_{L_x^2} \lesssim \| \partial_t \nabla_x \phi_j \|_{L_x^2} \quad \text{and} \quad \| \phi_j \|_{L_x^2} \lesssim \| \nabla_x \phi_j \|_{L_x^2}
\]

Thus, by standard elliptic estimates for (3.39), we have

\[
\| \nabla_x \phi_j \|_{L_x^2} \lesssim \| \partial^\alpha b_j \|_{L_x^2}, \quad \| \partial_t \nabla_x \phi_j \|_{L_x^2} \lesssim \| \partial_t \partial^\alpha b_j \|_{L_x^2}.
\] (3.45)

Next we assume \( |\alpha| = 2 \) and \( \partial^\alpha = \partial_{x_j x_i} \) for some \( 1 \leq i \leq 3 \). Then for \( j = 1, 2, 3 \), \( \phi_j = 0 \) on \( \Gamma_j \) and \( \partial_{x_j} \phi_j = 0 \) on \( \Gamma_k \) for \( k \neq j \). Thus (3.39) is a mixed Dirichlet-Neumann boundary problem and by Sobolev embedding [6, Theorem 6.7-5], we know that \( \| \phi_j \|_{L_x^2} \lesssim \| \nabla_x \phi_j \|_{L_x^2} \). Then by standard elliptic estimates for (3.39), we have

\[
\| \nabla_x \phi_j \|_{L_x^2} = \int_{\Omega} \partial_{x_i} b_j \, dx + \int_{\Omega} \partial_{x_j} \phi_j \, dx,
\]

where we used \( \partial_{x_i} b_j = 0 \) on \( \Gamma_i \) from (3.17) for \( j \neq i \) and \( \phi_j = 0 \) on \( \Gamma_i \) if \( j = i \). Then we have

\[
\| \nabla_x \phi_j \|_{L_x^2} \leq \| \partial_{x_i} b_j \|_{L_x^2}, \quad \| \partial_t \nabla_x \phi_j \|_{L_x^2} \lesssim \| \partial_t \partial_{x_i} b_j \|_{L_x^2}.
\] (3.46)

If \( |\alpha| = 3 \), then \( \alpha_i = 1 \) or \( 3 \) for some \( 1 \leq i \leq 3 \). If further \( \alpha_k = 2 \) or \( 0 \) and \( \alpha_m = 0 \) for some \( k \neq i \) and \( m \neq k, i \), then (3.39) is pure Neumann problem when \( i = j \). Here we need \( \int_{\Omega} \partial_{x_i x_j x_i} b_i \, dx = \int_{\Gamma_i} \partial_{x_i x_i} \phi_i \, dS(x) = 0 \) or \( \int_{\Omega} \partial_{x_j x_i x_i} b_i \, dx = \int_{\Gamma_i} \partial_{x_i x_i} b_i \, dS(x) = 0 \) to ensure the existence of (3.39), which follows from (3.17) and (3.21). In any cases, we write
\( \partial^\alpha = \partial_{x_1 x_2 x_3} \) for some \( 1 \leq k, m \leq 3 \). Here either \( k = m = i \) or \( k, m \neq i \). Then taking inner product of (3.39) with \( \phi_j \), we have

\[
\| \nabla_x \phi_j \|_{L^2_v}^2 = \int \partial_{x_1 x_2 x_3} b_j \phi_j \, dx = \int \partial_{x_1 x_2 x_3} b_j \phi_j \, dx - \int \partial_{x_1 x_2 x_3} b_j \partial_{x_1} \phi_j \, dx \leq \| \partial_{x_1 x_2 x_3} b_j \|_{L^2_v} \| \partial_{x_1} \phi_j \|_{L^2_v},
\]

where we used the fact that \( \partial_{x_1 x_2 x_3} b_j = 0 \) on \( \Gamma_i \) when \( i = j \) and \( \phi_j = 0 \) on \( \Gamma_i \) when \( i \neq j \), which is from (3.17) and boundary condition (3.39). This implies that

\[
\| \nabla_x \phi_j \|_{L^2_v} \leq \sum_{|\alpha| = 2} \| \partial^\alpha b_j \|_{L^2_v}, \quad \| \partial_t \nabla_x \phi_j \|_{L^2_v} \leq \sum_{|\alpha| = 2} \| \partial_t \partial^\alpha b_j \|_{L^2_v}. \tag{3.47}
\]

As a summary, for \( |\alpha| \leq K \), we have from (3.42), (3.43), (3.44), (3.45), (3.46) and (3.47) that

\[
\| \nabla_x \phi_j \|_{L^2_v} \leq \sum_{|\alpha| \leq K-1} \| \partial^\alpha b_j \|_{L^2_v}, \quad \| \partial_t \nabla_x \phi_j \|_{L^2_v} \leq \sum_{|\alpha| \leq K-1} \| \partial_t \partial^\alpha b_j \|_{L^2_v}. \tag{3.48}
\]

Now we let \( |\alpha| \leq K \). For \( S_1 \), we have from (3.48) that

\[
\| S_1 \| \leq (P \partial^\alpha f, \partial_t \Phi_b)_{L^2_v} + (\{ I - P \} \partial^\alpha f, \partial_t \Phi_b)_{L^2_v, v} \\
\leq C_\eta \| \partial^\alpha c \|_{L^2_v}^2 + C_\eta \| \partial^\alpha f \|_{L^2_v}^2 + \eta \sum_{|\alpha| \leq K-1} \| \partial_t \partial^\alpha b \|_{L^2_v}^2 \\
\leq \sum_{|\alpha| \leq K} \| \partial^\alpha c \|_{L^2_v}^2 + C_\eta \sum_{|\alpha| \leq K} \| \partial^\alpha (I - P) f \|_{L^2_v}^2 \\
+ \eta \sum_{1 \leq |\alpha| \leq K} \| \partial^\alpha (a_+ + a_-) \|_{L^2_v}^2 + \eta \sum_{|\alpha| \leq K} \| (\partial^\alpha g, \zeta) \|_{L^2_v}^2,
\]

where we used (3.6). For \( S_3 \), by (3.48), we have

\[
\| S_3 \| \leq C_\eta \| \partial^\alpha (I - P) f \|_{L^2_v}^2 + C_\eta \| (\partial^\alpha g, \zeta) \|_{L^2_v}^2 + \eta \| \nabla_x \phi_j \|_{L^2_v} \\
\leq C_\eta \| \partial^\alpha (I - P) f \|_{L^2_v}^2 + C_\eta \| (\partial^\alpha g, \zeta(v)) \|_{L^2_v}^2 + \eta \sum_{|\alpha| \leq K-1} \| \partial^\alpha b \|_{L^2_v}, \tag{3.50}
\]

for any \( \eta > 0 \). For \( S_4 \), we apply (3.48) to obtain

\[
\| S_4 \| \leq C_\eta \| \partial^\alpha \nabla_x \phi_j \|_{L^2_v}^2 + \eta \sum_{|\alpha| \leq K-1} \| \partial^\alpha b \|_{L^2_v}, \tag{3.51}
\]

for any \( \eta > 0 \). For the second term on left hand side of (3.25), we have

\[
- \sum_{m=1}^3 (P_{\pm \alpha} f, v \cdot \nabla_x \Phi_{b,m}^v)_{L^2_v, v} = \sum_{m=1, m \neq j}^3 \left( v_m v_j \mu^{1/2} \partial^\alpha \Phi_j, |v|^2 v_m v_j \mu^{1/2} \partial^\alpha_x \phi_j \right)_{L^2_v, v} \\
- \sum_{m=1, m \neq j}^3 \left( v_m v_j \mu^{1/2} \partial^\alpha b_m, |v|^2 v_m v_j \mu^{1/2} \partial^\alpha_x \phi_j \right)_{L^2_v, v} \\
+ 7 \sum_{m=1, m \neq j}^3 \left( \partial^\alpha b_m, \partial_x m \partial_x \phi_j \right)_{L^2_v} - 7(\partial^\alpha b_m, \partial_x \phi_j)_{L^2_v}
\]
\[
= -7 \sum_{m=1}^{3} (\partial^a b_j, \partial_{x_m}^2 \phi_j)_{L^2} = 7 ||\partial^a b_j||_{L^2}^2. \tag{3.52}
\]

Note that \( \int_{\mathbb{R}^3} v_m^2 (v_m^2 - 1) \mu \, dv = 2, \int_{\mathbb{R}^3} v_m^2 (v_j^2 - 1) \mu \, dv = 0, \int_{\mathbb{R}^3} v_j^2 v_m^2 \mu \, dv = 7 \) and \( \int_{\mathbb{R}^3} (v_j^2 - 1) \mu \, dv = 0, \) when \( m \neq j. \) Now we consider the boundary term \( S_5. \) As in the 

estimate on \( c(t,x), \) we consider \( \Gamma_i \) for fixed \( i = 1, 2, 3: \)

\[
\int_{\Gamma_i} (v \cdot n(x) \partial^a f(x), \Phi_b(x))_{L^2} dS(x) = \sum_{m=1}^{3} \int_{\Gamma_i} \int_{\mathbb{R}^3} v \cdot n(x) \partial^a f(t,x,v) \Phi_b^{j,m}(x,v) dv dS(x). \tag{3.53}
\]

If \( \alpha_i = 0 \) or \( 2, \) then applying boundary condition (3.39), we have that for \( x \in \Gamma_i, \)

\[
\partial_{x_i} \phi_j(x) = \partial_{x_j} \phi_i(x) = 0, \quad \text{for } j \neq i.
\]

This shows that \( \Phi_b^{j,m}(x,v) \) is even with respect to \( v_i \) when \( x \in \Gamma_i. \) Noticing \( R_x v = v - 2v \cdot e_j e_j \) maps \( v_i \) to \( -v_i \) on \( \Gamma_i, \) we know that

\[
\Phi_b^{j,m}(x,R_x v) = \Phi_b^{j,m}(x,v), \quad \text{for } m = 1, 2, 3.
\]

Applying change of variable \( v \mapsto R_x v \) and using identities (3.9), (3.10) and (3.12), (3.53) becomes

\[
\sum_{m=1}^{3} \int_{\Gamma_i} \int_{\mathbb{R}^3} R_x v \cdot n(x) \partial^a f(t,x,v) \Phi_b^{j,m}(x,v) dv dS(x)
\]

\[
= \sum_{m=1}^{3} \int_{\Gamma_i} \int_{\mathbb{R}^3} R_x v \cdot n(x) \partial^a R_x f(t,x,v) \Phi_b^{j,m}(x,R_x v) dv dS(x)
\]

\[
= \sum_{m=1}^{3} \int_{\Gamma_i} \int_{\mathbb{R}^3} -v \cdot n(x) \partial^a f(t,x,v) \Phi_b^{j,m}(x,v) dv dS(x) = 0.
\]

If \( \alpha_i = 1 \) or \( 3, \) then boundary condition (3.39) shows that on \( x \in \Gamma_i, \)

\[
\partial_{x_j} \phi_j(x) = 0, \quad \text{for } j = 1, 2, 3,
\]

\[
\partial_{x_m} \phi_j(x) = 0, \quad \text{for } j, m \neq i.
\]

Note that \( \partial_{x_m} \) is tangent derivative on \( \Gamma_i \) when \( m \neq i. \) Then we know that \( \Phi_b^{j,m}(x,v) \) is odd with respect to \( v_i \) when \( x \in \Gamma_i \) and hence,

\[
\Phi_b^{j,m}(x,R_x v) = -\Phi_b^{j,m}(x,v).
\]

Now applying change of variable \( v \mapsto R_x v \) and using identities (3.11), (3.53) becomes

\[
\sum_{m=1}^{3} \int_{\Gamma_i} \int_{\mathbb{R}^3} v \cdot n(x) \partial^a f(t,x,v) \Phi_b^{j,m}(x,v) dv dS(x)
\]

\[
= \sum_{m=1}^{3} \int_{\Gamma_i} \int_{\mathbb{R}^3} R_x v \cdot n(x) \partial^a f(t,x,v) \Phi_b^{j,m}(x,R_x v) dv dS(x)
\]

\[
= \sum_{m=1}^{3} \int_{\Gamma_i} \int_{\mathbb{R}^3} v \cdot n(x) \partial^a f(t,x,v)(-\Phi_b^{j,m})(x,v) dv dS(x) = 0.
\]

Therefore,

\[
S_5 = 0. \tag{3.54}
\]
Combining estimates (3.41), (3.49), (3.50), (3.51), (3.52) and (3.54), taking summation over \(|\alpha| \leq K\) of (3.25) and letting \(\eta\) sufficiently small, we have
\[
\begin{align*}
\partial_t \sum_{|\alpha| \leq K} (\partial^\alpha f, \Phi_b)_{L^2_{t,v}} + \lambda \delta^\alpha \sum_{|\alpha| \leq K} \|\partial^\alpha b\|^2_{L^2_{t,v}} & \lesssim \eta \sum_{1 \leq |\alpha| \leq K} \|\partial^\alpha (a_+ + a_-)\|^2_{L^2_{t,v}} + C_\eta \sum_{|\alpha| \leq K} \|\partial^\alpha c\|^2_{L^2_{t,v}} \\
+ \sum_{|\alpha| \leq K} \|\partial^\alpha \nabla_x \phi\|^2_{L^2_{t,v}} & \lesssim C_\eta \sum_{|\alpha| \leq K} \|\partial^\alpha (I - P) f\|^2_{L^2_{t,v}} + C_\eta \sum_{|\alpha| \leq K} \|\partial^\alpha g, \zeta\|_{L^2_{t,v}},
\end{align*}
\]  
for some \(\lambda > 0\) and any \(\eta > 0\). Note that we have applied (3.18).

**Step 3. Estimate on \(a_+(t,x) + a_-(t,x)\):** We choose the following two test functions
\[
\Phi = \Phi_{a\pm} = (|v|^2 - 10)(v \cdot \nabla_x \Phi_{a\pm}(t,x))^{1/2},
\]
where \(\phi_a = (\phi_{a+}(x), \phi_{a-}(x))\) solves
\[
\left\{ \begin{array}{l}
- \Delta_x \phi_{a+} = - \Delta_x \phi_{a-} = \partial^\alpha (a_+ + a_-) \quad \text{in } \Omega, \\
\phi_a(x) = 0 \quad \text{on } x \in \Gamma_1, \quad \text{if } \alpha_i = 1 \text{ or } 3, \\
\frac{\partial \phi_a}{\partial n}(x) = 0 \quad \text{on } x \in \Gamma_1, \quad \text{if } \alpha_i = 0 \text{ or } 2.
\end{array} \right.
\]  
(3.56)
The existence and uniqueness of solution to (3.56) is guaranteed by [18, Lamma 4.4.3.1]. When \(|\alpha| = 0\), (3.56) is pure Neumann problem and we need \(\int_\Omega (a_+ + a_-) \, dx = 0\) from conservation laws (1.5) to ensure the existence of (3.56). When \(|\alpha| = 2\) and \(\alpha_i = 2\) for some \(i\), (3.56) is pure Neumann problem and we need \(\int_\Omega \partial_x (a_+ + a_-) \, dx = \int_{\Gamma_1} \partial_n (a_+ + a_-) \, dS(x) = 0\) from Lemma 3.2 to ensure the existence of (3.56). Now we compute (3.25). For the second term on left hand side of (3.25), taking summation on \(\pm\), we have
\[
- \sum_{\pm} (\partial^\alpha P_{\pm} f, v \cdot \nabla_x \Phi_{a\pm})_{L^2_{t,v}} = - \sum_{\pm} \sum_{j,m=1}^3 (\partial^\alpha a_{\pm} + \partial^\alpha b \cdot v + (|v|^2 - 3) \partial^\alpha c, v_j v_m (|v|^2 - 10) \mu (\partial_x, \partial_x, \Phi_{a\pm}))_{L^2_{t,v}} = \sum_{j=1}^3 (\partial^\alpha a_{\pm}, - \partial^\alpha \Phi_{a\pm})_{L^2} = \|\partial^\alpha a_+ + \partial^\alpha a_-\|^2_{L^2}.
\]
The estimates for \(S_j\) (\(1 \leq j \leq 5\)) are similar to the case of \(c(t,x)\) from (3.27) to (3.38), since \(\Phi_a\) and \(\Phi_c\) has similar structure. Then following the calculation from (3.27) to (3.34), we have that for \(|\alpha| \leq K\),
\[
\sum_{i,j=1}^3 \|\partial_x, \partial_a\|_{L^2} \lesssim \sum_{i,j=1}^3 \|\partial_x, \partial_{a+}\|_{L^2} \lesssim \sum_{|\alpha| \leq K-1} \|\partial^\alpha (a_+ + a_-)\|_{L^2},
\]  
(3.57)
and
\[
\|\partial_t \nabla_x \Phi_a\|_{L^2} \lesssim \sum_{|\alpha| \leq K-1} \|\partial_t \partial^\alpha (a_+ + a_-)\|_{L^2} \lesssim \sum_{1 \leq |\alpha| \leq K} \|\partial^\alpha b\|_{L^2},
\]  
(3.58)
where the last inequality follows from (3.6)1. Then for \(S_1\), we apply (3.59) to obtain
\[
|S_1| \lesssim \|((I - P) \partial^\alpha f, \partial_t \Phi_a)_{L^2_{t,v}}| + \|P \partial^\alpha f, \partial_t \Phi_a)_{L^2_{t,v}}| \lesssim \|((I - P) \partial^\alpha f\|_{L^2_{t,v}}^2 + \|\partial^\alpha b\|_{L^2}^2 + \|\partial_t \nabla_x \Phi_a\|_{L^2}^2.
\]
\[ \| \{ I - P \} \partial^\alpha f \|_{L^2_{2/2}}^2 + \sum_{|\alpha| \leq K} \| \partial^\alpha b \|_{L^2_2}^2. \]

For \( S_2 \), by (3.57), we have
\[ |S_2| \lesssim C_\eta \| \{ I - P \} \partial^\alpha f \|_{L^2_{2/2}}^2 + \eta \| \partial^\alpha (a_+ + a_-) \|_{L^2_2}^2. \]

For \( S_3 \), by (3.58), we have
\[ |S_3| \lesssim C_\eta \| \{ I - P \} \partial^\alpha f \|_{L^2_{2/2}}^2 + C_\eta \| (\partial^\alpha g, \zeta) \|_{L^2_2}^2 + \eta \sum_{|\alpha| \leq K-1} \| \partial^\alpha (a_+ + a_-) \|_{L^2_2}^2. \]

For \( S_4 \), we apply (3.58) to obtain
\[ |S_4| \leq C_\eta \| \partial^\alpha \nabla_x \phi \|_{L^2_2}^2 + \eta \sum_{|\alpha| \leq K-1} \| \partial^\alpha (a_+ + a_-) \|_{L^2_2}. \]

For \( S_5 \), since \( \Phi_a \) and \( \Phi_c \) has the same structure, following the arguments deriving (3.36), we have \( S_5 = 0 \). Combining the above estimates, taking summation \( |\alpha| \leq K \) and \( \pm \) of (3.25) and letting \( \eta > 0 \) small enough, we have
\[ \partial_t \sum_{|\alpha| \leq K} (\partial^\alpha f, \Phi_a)_{L^2_{2/2}} + \lambda \sum_{|\alpha| \leq K} \| \partial^\alpha (a_+ + a_-) \|_{L^2_2}^2 \lesssim \sum_{|\alpha| \leq K} \| \partial^\alpha \{ I - P \} f \|_{L^2_{2/2}}^2 \]
\[ + \sum_{|\alpha| \leq K} \| \partial^\alpha \nabla_x \phi \|_{L^2_2}^2 + \sum_{|\alpha| \leq K} \| \partial^\alpha b \|_{L^2_2}^2 + \sum_{|\alpha| \leq K} \| (\partial^\alpha g, \zeta) \|_{L^2_2}^2. \]

Note that we have applied (3.18).

**Step 4. Estimate on** \( a_+(t, x) - a_-(t, x) \) and \( E(t, x) \): We choose the following two test functions
\[ \Phi = \Phi_{a_\pm} = ((|v|^2 - 10)(v \cdot \nabla_x \phi_{a_\pm}(t, x)))^{1/2}, \]
where \( \phi_a = (\phi_{a_+}(x), \phi_{a_-}(x)) \) solves
\[ \begin{align*}
-\Delta_x \phi_{a_+} &= \partial^\alpha (a_+ - a_-) \quad \text{in } \Omega, \\
-\Delta_x \phi_{a_-} &= \partial^\alpha (a_- - a_+) \quad \text{in } \Omega, \\
\phi_a(x) &= 0 \quad \text{on } x \in \Gamma, \quad \text{if } \alpha_i = 1 \text{ or } 3, \\
\partial_n \phi_a(x) &= 0 \quad \text{on } x \in \Gamma, \quad \text{if } \alpha_i = 0 \text{ or } 2.
\end{align*} \]

The existence and uniqueness of solution to (3.56) is guaranteed by [18, Lemma 4.4.3.1]. When \( |\alpha| = 0 \), (3.61) is a pure Neumann problem and we need \( \int_{\Omega} (a_+ - a_-) dx = 0 \) from conservation laws (1.5) to ensure the existence of (3.61). When \( |\alpha| = 2 \) and \( \alpha_i = 2 \) for some \( i \), (3.61) is also pure Neumann problem and we need \( \int_{\Omega} \partial_{x_i} (a_+ - a_-) dx = \int_{\Gamma_i} \partial_{x_i} (a_+ - a_-) dS(x) = 0 \) from Lemma 3.2 to ensure the existence of (3.61). For the second term on left hand side of (3.25), taking summation on \( \pm \), we have
\[ -\sum_{\pm} (\partial^\alpha P_{\pm} f, v \cdot \nabla_x \Phi_{a\pm})_{L^2_{2/2}}, \]
\[ = -\sum_{\pm \pm} \sum_{j, m=1}^3 (\partial^\alpha a_{\pm} + \partial^\alpha b \cdot v + (|v|^2 - 3) \partial^\alpha c, v_j v_m (|v|^2 - 10) \mu (\partial_{x_j} \partial_{x_m} \phi_{a_{\pm}}))^)_{L^2_{2/2}}, \]
\[ = \sum_{\pm} \sum_{j=1}^3 (\partial^\alpha a_{\pm}, -\partial^2 \phi_{a\pm})_{L^2_{2/2}} = \| \partial^\alpha a_+ - \partial^\alpha a_- \|_{L^2_{2/2}}^2. \]
Then we estimate the $S_j$ ($1 \leq j \leq 5$). The same as the case of $a_+ + a_-$, the case of $a_+ - a_-$ is also similar to estimate of $c(t,x)$. Following the calculation from (3.27) to (3.34), we have that for $|\alpha| \leq K$,

$$\sum_{i,j=1}^{3} \|\partial_{x_i x_j} \phi_\alpha\|_{L^2_v}^2 = \|\Delta_\alpha \phi_\alpha\|_{L^2_v}^2 \lesssim \|\partial^\alpha (a_+ - a_-)\|_{L^2_v}^2,$$  

(3.62)

and

$$\|\nabla_\alpha \phi_\alpha\|_{L^2_v} \lesssim \sum_{|\alpha| \leq K-1} \|\partial^\alpha (a_+ - a_-)\|_{L^2_v},$$  

(3.63)

where the last inequality follows from (3.7). Then for $S_1$, we apply (3.64) to obtain

$$|S_1| \lesssim \left| \left( (\mathbf{I} - \mathbf{P}) \partial^\alpha f, \partial_t \phi_\alpha \right)_{L^2_v} \right| + \left| \left( \mathbf{P} \partial^\alpha f, \partial_t \phi_\alpha \right)_{L^2_v} \right|$$

$$\lesssim \|\mathbf{I} - \mathbf{P}\| \|\partial^\alpha f\|_{L^2_v}^2 + \eta \|\partial^\alpha b\|_{L^2_v}^2 + C_\eta \|\partial_t \nabla_\alpha \phi_\alpha\|_{L^2_v}^2$$

$$\lesssim C_\eta \sum_{|\alpha| \leq K} \|\mathbf{I} - \mathbf{P}\| \|\partial^\alpha f\|_{L^2_v}^2 + \eta \|\partial^\alpha b\|_{L^2_v}^2.$$  

(3.64)

For $S_2$, by (3.62), we have

$$|S_2| \lesssim C_\eta \|\mathbf{I} - \mathbf{P}\| \|\partial^\alpha f\|_{L^2_v}^2 + \eta \|\partial^\alpha (a_+ - a_-)\|_{L^2_v}^2.$$  

For $S_3$, by (3.63), we have

$$|S_3| \lesssim C_\eta \|\mathbf{I} - \mathbf{P}\| \|\partial^\alpha f\|_{L^2_v}^2 + C_\eta \|\partial^\alpha g, \zeta\|_{L^2_v}^2 + \eta \sum_{|\alpha| \leq K-1} \|\partial^\alpha (a_+ - a_-)\|_{L^2_v}^2.$$  

For $S_4$, from (3.61) we know that $\phi_{a+} = -\phi_{a-}$. Thus

$$\sum_{\pm} S_4 = \sum_{\pm} \mp (\partial^\alpha \nabla_\alpha \phi \cdot v \mu^{1/2} \phi_{a \pm})_{L^2_v}$$

$$= 5 \sum_{\pm} \mp (\partial^\alpha \partial_{x_j} \phi_\alpha, \partial_{x_j} \phi_{a \pm}(t,x))_{L^2_v}$$

$$= -10 \sum_{j=1}^{3} (\partial^\alpha \partial_{x_j} \phi_\alpha, \partial_{x_j} \phi_{a \pm}(t,x))_{L^2_v}.$$  

(3.65)

By using the boundary value from (3.61) and (3.5), we know that if $\alpha_j = 0$ or 2, then $\partial_{x_j} \phi_\alpha = 0$ on $\Gamma_j$. If $\alpha_j = 1$ or 3, then $\partial^\alpha \phi = 0$ on $\Gamma_j$, which is from (1.8) and (3.19). Thus,

$$\sum_{j=1}^{3} (\partial^\alpha \partial_{x_j} \phi_\alpha, \partial_{x_j} \phi_{a \pm})_{L^2_v} = \sum_{j=1}^{3} \int_{\Gamma_j} \partial^\alpha \phi_\alpha \partial_{x_j} \phi_{a \pm} \, dS(x) - (\partial^\alpha \phi_\alpha, \Delta_{x} \phi_{a \pm})_{L^2_v}$$

$$= (\partial^\alpha \phi_\alpha, \partial^\alpha (a_+ - a_-))_{L^2_v}.$$  

By (3.4), we know that $-\Delta_{x} \phi = a_+ - a_-$. If $|\alpha| = 0$, then by boundary condition (1.8), we have

$$\|\phi, (a_+ - a_-)\|_{L^2_v} = (\phi_\alpha, -\Delta_{x} \phi)_{L^2_v} = \|\nabla_\alpha \phi\|_{L^2_v}^2.$$  

If $\partial^\alpha = \partial_{x_i}$, then by the fact that $\partial_{x_i} \phi_\alpha = 0$ on $\Gamma_i$ and $\partial_{x_j} \phi_\alpha = 0$ on $\Gamma_j$ for $j \neq i$, we have

$$\|\partial^\alpha \phi_\alpha, (a_+ - a_-)\|_{L^2_v} = (\partial^\alpha \phi, -\partial^\alpha \Delta_{x} \phi)_{L^2_v} = \|\partial^\alpha \nabla_\alpha \phi\|_{L^2_v}^2.$$
If $\partial^\alpha = \partial_{x,x_1}$, then
\[
(\partial^\alpha \phi, \partial^\alpha (a_+ - a_-))_{L^2_\alpha} = \int_{\Gamma_i} \partial_{x,x_1} \phi \partial_{x_1} (a_+ - a_-) \, dS(x) - (\partial_{x,x_1} \phi, \partial_{x_1} (a_+ - a_-))_{L^2_\alpha}.
\]
The first term on the right hand side is zero because $\partial_{x_i} a_\pm = 0$ on $\Gamma_i$ from (3.17). Noticing $\partial_{x_i,x_j} \phi = 0$ on $\Gamma_j$ and $\Gamma_1$ for $j \neq i$, we have
\[
(\partial^\alpha \phi, \partial^\alpha (a_+ - a_-))_{L^2_\alpha} = (\partial_{x,x_1} \phi, \partial_{x_1} \Delta x \phi)_{L^2_\alpha}
= (\partial_{x,x_1} \phi, \partial_{x,x_1} \phi)_{L^2_\alpha} - \sum_{j \neq i} (\partial_{x,x_1,x_j} \phi, \partial_{x_j} \phi)_{L^2_\alpha}
= (\partial_{x,x_1} \phi, \partial_{x,x_1} \phi)_{L^2_\alpha} + \sum_{j \neq i} (\partial_{x,x_1,x_j} \phi, \partial_{x_j} \phi)_{L^2_\alpha} = \|\partial^\alpha \nabla x \phi\|_{L^2_{\alpha}}^2.
\]
If $\partial^\alpha = \partial_{x,x_1,x_k}$, then we need more discussion. Note that we will apply (1.8) and (3.19) frequently. If $i, j, k$ are pairwise different, then from (3.5), we have $\partial_{x,x_1,x_k} \phi = 0$ on $\Gamma_i$, $\Gamma_j$ and $\Gamma_k$ and hence
\[
(\partial^\alpha \phi, \partial^\alpha (a_+ - a_-))_{L^2_\alpha} = (\partial^\alpha \phi, -\partial^\alpha \Delta x \phi)_{L^2_\alpha} = \|\partial^\alpha \nabla x \phi\|_{L^2_{\alpha}}^2.
\]
If $i = j \neq k$, then by $\partial_{x,x_1}(a_+ - a_-) = 0$ on $\Gamma_i$ and $\partial_{x,x_1,x_k} \phi = 0$ on $\Gamma_k$, we have
\[
(\partial^\alpha \phi, \partial^\alpha (a_+ - a_-))_{L^2_\alpha} = (\partial_{x,x_1,x_k} \phi, \partial_{x_1}(a_+ - a_-))_{L^2_\alpha}
= (\partial_{x,x_1,x_k} \phi, -\partial_{x,x_1,x_k} \phi)_{L^2_\alpha}
+ \sum_{m=k} \partial_{x,x_1,x_k,x_k} \phi, \partial_{x,x_1,x_k} \phi)_{L^2_\alpha}
+ \sum_{m \neq k} \partial_{x,x_1,x_k} \phi, \partial_{x,x_1,x_k} \phi)_{L^2_\alpha}
= \|\partial^\alpha \nabla x \phi\|_{L^2_{\alpha}}^2.
\]
The last identity follows from suitable integration by parts. Note that if $m = i$, we have $\partial_{x,x_1,x_k} \phi = 0$ on $\Gamma_k$. If $m = k$, we have $\partial_{x,x_1,x_k} \phi = 0$ on $\Gamma_i$. If $m \neq k, i$, we have $\partial_{x,x_1,x_k} \phi = 0$ on $\Gamma_k$, $\partial_{x,x_1,x_k} \phi = 0$ on $\Gamma_i$, and $\partial_{x,x_1,x_k} \phi = 0$ on $\Gamma_m$. If $i = j = k$, then by $\partial_{x,x_1,x_k} \phi = 0$ on $\Gamma_i$ and $\partial_{x,x_1,x_k} \phi = 0$ on $\Gamma_m$ for $m \neq i$, we have
\[
(\partial^\alpha \phi, \partial^\alpha (a_+ - a_-))_{L^2_\alpha} = (\partial_{x,x_1,x_k} \phi, -\partial_{x,x_1}(a_+ - a_-))_{L^2_\alpha}
= (\partial_{x,x_1,x_k} \phi, \partial_{x,x_1,x_k} \phi)_{L^2_\alpha}
+ \sum_{m \neq i} (\partial_{x,x_1,x_k} \phi, \partial_{x,x_1,x_k} \phi)_{L^2_\alpha}
= \|\partial^\alpha \nabla x \phi\|_{L^2_{\alpha}}^2.
\]
Plugging the above estimates into (3.65), we have
\[
\sum_{\pm} S_1 = -10 \|\partial^\alpha \nabla x \phi\|_{L^2_{\alpha}}^2.
\]
Similar to the calculation we used to derive (3.36), we know that $S_5 = 0$. Combining the above estimates, taking summation $|\alpha| \leq K$ and $\pm$ and letting $\eta > 0$ small enough, we have
\[
\partial_t \sum_{|\alpha| \leq K} (\partial^\alpha f, \Phi_\alpha)_{L^2_{\alpha}} + \lambda \sum_{|\alpha| \leq K} \left( \|\partial^\alpha (a_+ - a_-)\|_{L^2_{\alpha}}^2 + \|\partial^\alpha \nabla x \phi\|_{L^2_{\alpha}}^2 \right)
\lesssim \sum_{|\alpha| \leq K} \left( C_\eta \|\partial^\alpha (I - P) f\|_{L^2_{\alpha}}^2 + \|\partial^\alpha g, \zeta\|_{L^2_{\alpha}}^2 + \eta \|\partial^\alpha b\|_{L^2_{\alpha}}^2 \right). \quad (3.66)
\]
Now we take the linear combination (3.66) + $\kappa \times (3.38) + \kappa^2 \times (3.55) + \kappa^3 \times (3.60)$ and let $\kappa, \eta$ sufficiently small, then

$$\partial_t \mathcal{E}_{int}(t) + \lambda \sum_{|\alpha| \leq K} \|\partial^\alpha [a_+, a_-, b, c]\|^2_{L^2_x} \lesssim \sum_{|\alpha| \leq K} \left( \|\partial^\alpha \{1-P\} f\|^2_{L^2_{x/2}} + \|\partial^\alpha g, \zeta\|_{L^2_x}^2 \right) + \|E\|_{L^2_x}^4,$$

where

$$
\mathcal{E}_{int}(t) = \sum_{|\alpha| \leq K} \left( \langle \partial^\alpha f, \Phi_\alpha \rangle_{L^2_{x,v}} + \sum_{\pm} \left( K \langle \partial^\alpha f_\pm, \Phi_c \rangle_{L^2_{x,v}} + \kappa^2 \langle \partial^\alpha f_\pm, \Phi_b \rangle_{L^2_{x,v}} \right) \right)
+ \kappa^3 \langle \partial^\alpha f, \Phi_\alpha \rangle_{L^2_{x,v}} \right).
$$

Note that $|a_+ - a_-|^2 + |a_+ + a_-|^2 = 2|a_+|^2 + 2|a_-|^2$. Using (3.29), (3.31), (3.33), (3.43), (3.45), (3.46), (3.47), (3.58) and (3.63), we know that

$$\mathcal{E}_{int}(t) \lesssim \sum_{|\alpha| \leq K} \|\partial^\alpha f\|^2_{L^2_{x} L^2_{t}}.$$

This completes the Theorem 3.4. \qquad \square

Now we estimate $\|\langle \partial^\alpha g, \zeta \rangle_{L^2_x} \|^2_{L^2_t}$ when $g$ is given by (3.3). For $|\alpha| \leq K$, by (2.10), we apply $L^3 - L^6$ and $L^\infty - L^2$ Hölder’s inequality to obtain

$$
\int_\Omega \|\langle \partial^\alpha \Gamma(f, f), \zeta(v) \rangle_{L^2_x} \|^2_{L^2_t} \lesssim \int_\Omega \sum_{|\alpha| \leq \alpha} \|\partial^\alpha f\|^2_{L^6_x L^2_t} \|\partial^\alpha f\|^2_{L^2_{x} L^2_{t}} \times \|
$$

where we used embedding $\|f\|_{L^6_x(\Omega)} \lesssim \|f\|_{H^1_x(\Omega)}$, $\|f\|_{L^6_x(\Omega)} \lesssim \|\nabla f\|_{L^2_x(\Omega)}$ and $\|f\|_{L^\infty_x(\Omega)} \lesssim \|\nabla f\|_{H^1_x(\Omega)}$ from [1, Section V and (V.21)]. Similarly, we have

$$
\int_\Omega \|\langle \partial^\alpha (\nabla_x \phi \cdot \nabla f_\pm), \zeta(v) \rangle_{L^2_x} \|^2_{L^2_t} \lesssim \int_\Omega \sum_{|\alpha| \leq \alpha} \|\partial^\alpha \zeta (\nabla_x \phi)\|^2_{L^2_x} \|\partial^\alpha f_\pm\|^2_{L^2_x} \times \|
$$

and

$$
\int_\Omega \|\langle \partial^\alpha (\nabla_x \phi \cdot vf_\pm), \zeta(v) \rangle_{L^2_x} \|^2_{L^2_t} \lesssim \int_\Omega \sum_{|\alpha| \alpha} \|\partial^\alpha \zeta (\nabla_x \phi)\|^2_{L^2_x} \|\partial^\alpha f_\pm\|^2_{L^2_x} \times \|
$$
Lemma 4.1. Let $T > 0$ and $l \in \mathbb{R}$. Let $\gamma \geq -3$ for Landau case, $(\gamma, s) \in \left\{-\frac{3}{2} < \gamma + 2s < 0, \frac{1}{2} \leq s < 1\right\} \cup \left\{\gamma + 2s \geq 0, 0 < s < 1\right\}$ for Boltzmann case. If $1 \leq |\alpha| \leq 3$, then

\[
\sum_{\alpha_1 < \alpha} \left| \partial^{\alpha - \alpha_1} \nabla_x \phi \cdot v \partial^{\alpha_1} f_\pm, e^{\pm \phi} w_{2l,2v}(2\alpha, 0) \partial^\alpha f_\pm \right|_{L^2_{x,v}} \lesssim \sqrt{\mathcal{E}_\nu(t)} \mathcal{D}_\nu(t),
\]

where $\mathcal{E}_\nu(t)$ and $\mathcal{D}_\nu(t)$ are given by (1.11) and (1.12) respectively. If $1 \leq |\alpha| + |\beta| \leq 3$, then

\[
\sum_{\alpha_1 + \beta_1 < \alpha + \beta} \left| \partial^{\alpha - \alpha_1} \nabla_x \phi \cdot v \partial^{\beta_1} f_\pm, e^{\pm \phi} w_{2l,2v}(2\alpha, 2\beta) \partial^\alpha f_\pm \right|_{L^2_{x,v}} \lesssim \sqrt{\mathcal{E}_\nu(t)} \mathcal{D}_\nu(t).
\]

Moreover, if $|\alpha| \leq 3$, we have

\[
(\partial^\alpha (\nabla_x \phi \cdot v f_\pm), e^{\pm \phi} w_{2l,2v}(2\alpha, 0) \partial^\alpha f_\pm)_{L^2_{x,v}} \lesssim \sqrt{\mathcal{E}_\nu(t)} \mathcal{D}_\nu(t) + \|\nabla_x \phi\|_{H^2} \mathcal{E}_\nu(t).
\]

If $|\alpha| + |\beta| \leq 3$, we have

\[
(\partial^\beta_{\beta_1} (\nabla_x \phi \cdot v f_\pm), e^{\pm \phi} w_{2l,2v}(2\alpha, 2\beta) \partial^\beta f_\pm)_{L^2_{x,v}} \lesssim \sqrt{\mathcal{E}_\nu(t)} \mathcal{D}_\nu(t) + \|\nabla_x \phi\|_{H^2} \mathcal{E}_\nu(t).
\]

**Proof.** Let $s = 1$ in Landau case. Note that $\langle v \rangle w_{l,v}(\alpha, 0) \lesssim \langle v \rangle^{\gamma + 2s} w_{l,v}(\alpha - e_i, 0)$ for $i = 1, 2, 3$. Then (4.3) can be estimated by

\[
\sum_{\alpha_1 < \alpha} \left\| \partial^{\alpha - \alpha_1} \nabla_x \phi w_{l,v}(\alpha - e_i, 0) \langle v \rangle^{\frac{2s}{2s + 1}} \partial^{\alpha_1} f_\pm \right\|_{L^2_{x,v}} \|w_{l,v}(\alpha, 0)\langle v \rangle^{\frac{2s}{2s + 1}} \partial^\alpha f_\pm\|_{L^2_{x,v}}^2 \lesssim \left( \sum_{|\alpha_1| = 0} \left\| \partial^{\alpha - \alpha_1} \nabla_x \phi \|_{L^2_{x,v}} w_{l,v}(\alpha - e_i, 0) \partial^{\alpha_1} f_\pm \right\|_{L^2_{x,v}} \right)^2
\]

\[
+ \sum_{|\alpha_1| = 1} \left\| \partial^{\alpha - \alpha_1} \nabla_x \phi \right\|_{L^2_{x,v}} w_{l,v}(\alpha - e_i, 0) \partial^{\alpha_1} f_\pm \|_{L^2_{x,v}}^2
\]

\[
+ \sum_{|\alpha_1| = 2} \left\| \partial^{\alpha - \alpha_1} \nabla_x \phi \right\|_{L^2_{x,v}} w_{l,v}(\alpha - e_i, 0) \partial^{\alpha_1} f_\pm \|_{L^2_{x,v}}^2.
\]
\[ \lesssim \| E \|_{L^3} \sum_{|\alpha| \leq 2} \| w_{l,\nu}(\alpha_1,0) \partial^{\alpha_1} f_\pm \|_{L^2_x L^2_D} \| w_{l,\nu}(\alpha,0) \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \]
\[ \lesssim \sqrt{\mathcal{E}_\nu(t)} \mathcal{D}_\nu(t). \] (4.7)

For (4.4), notice that \( \partial_{\beta_2} v w_{l,\nu}(\alpha,\beta) \lesssim \min \{ \langle v \rangle^{\gamma+2s} w_{l,\nu}(\alpha-\epsilon_i,\beta), \langle v \rangle^{\gamma+2s} w_{l,\nu}(\alpha,\beta-\epsilon_i) \} \).

Then similar to (4.7), (4.4) is bounded above by

\[ \sum_{\alpha_1+\beta_1<\alpha+\beta} \| \partial^{\alpha_1} \nabla \phi w_{l,\nu}(\alpha-\epsilon_i,\beta) \langle v \rangle^{\frac{\gamma+2s}{2}} \partial^{\alpha_1} f_\pm \|_{L^2_x L^2_D} \| w_{l,\nu}(\alpha,\beta) \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \]
\[ \lesssim \| E \|_{L^3} \sum_{|\alpha_1|+|\beta_1| \leq 2} \| w_{l,\nu}(\alpha_1,\beta_1) \partial^{\alpha_1} f_\pm \|_{L^2_x L^2_D} \| w_{l,\nu}(\alpha,\beta) \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \]
\[ \lesssim \sqrt{\mathcal{E}_\nu(t)} \mathcal{D}_\nu(t). \]

For (4.5), we have

\[ \left| \left( \partial^{\alpha} (\nabla \phi \cdot \nabla f_\pm), e^{\pm \phi} w_{2l,2\nu}(2\alpha,0) \partial^{\alpha} f_\pm \right)_{L^2_x} \right| \]
\[ \lesssim \sum_{\alpha_1 \leq \alpha} \left| \left( \partial^{\alpha_1} \nabla \phi \cdot \partial^{\alpha} \nabla f_\pm, e^{\pm \phi} w_{2l,2\nu}(2\alpha,0) \partial^{\alpha} f_\pm \right)_{L^2_x} \right|. \]

When \( \alpha_1 = \alpha \), taking integration by parts with respect to \( \nabla \phi \), we have

\[ \left| \left( \nabla \phi \cdot \partial^{\alpha} \nabla f_\pm, e^{\pm \phi} w_{2l,2\nu}(2\alpha,0) \partial^{\alpha} f_\pm \right)_{L^2_x} \right| \]
\[ = \| \nabla \phi \|_{L^\infty} \int_{\Omega} \int_{\mathbb{R}^3} |\partial^{\alpha} f_\pm|^2 |\nabla (w_{l,\nu}(\alpha,0)) w_{l,\nu}(\alpha,0) | \, dv \, dx \]
\[ \lesssim \| \nabla \phi \|_{H^2} \| w_{l,\nu}(\alpha,0) \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \]
\[ \lesssim \| \nabla \phi \|_{H^2} \mathcal{E}_\nu(t). \]

Note that \( |\nabla (w_{l,\nu}(\alpha,0))| \leq w_{l,\nu}(\alpha,0) \).

When \( \alpha_1 < \alpha \) in Landau case, we have \( w_{l,\nu}(\alpha,0) = \langle v \rangle^{\gamma+1} w_{l,\nu}(\alpha-\epsilon_i,0) \). Regarding \( \langle v \rangle^{\gamma/2} \nabla f \) as the dissipation term, we have

\[ \sum_{\alpha_1 < \alpha} \left| \left( \partial^{\alpha_1} \nabla \phi \cdot \partial^{\alpha} \nabla f_\pm, e^{\pm \phi} w_{2l,2\nu}(2\alpha,0) \partial^{\alpha} f_\pm \right)_{L^2_x} \right| \]
\[ \lesssim \sum_{\alpha_1 < \alpha} \| \partial^{\alpha_1} \nabla \phi \|_{L^2_x} \| \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \| w_{l,\nu}(\alpha,0) \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \]
\[ \lesssim \left( \sum_{|\alpha-\alpha_1|=3} \| \partial^{\alpha_1} \nabla \phi \|_{L^2_x} \| \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \right) \| w_{l,\nu}(\alpha,0) \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \]
\[ + \sum_{|\alpha-\alpha_1|=2} \| \partial^{\alpha_1} \nabla \phi \|_{L^2_x} \| \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \| w_{l,\nu}(\alpha,0) \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \]
\[ + \sum_{|\alpha-\alpha_1|=1} \| \partial^{\alpha_1} \nabla \phi \|_{L^\infty} \| \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \| w_{l,\nu}(\alpha,0) \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \]
\[ \lesssim \| E \|_{L^3} \sum_{|\alpha| \leq 2} \| \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \| w_{l,\nu}(\alpha,0) \partial^{\alpha} f_\pm \|_{L^2_x L^2_D} \]
\[ \lesssim \sqrt{\mathcal{E}_\nu(t)} \mathcal{D}_\nu(t). \]
When \( \alpha_1 < \alpha \) in Boltzmann case for hard potential, we regard \( \nabla_v \) as derivative and add it into \( \partial^2 \) with \(|\alpha| + |\beta| \leq 3 \). Note that in this case, \(| \cdot |_{L^2_v} \lesssim | \cdot |_{L^2_v}^\alpha \). Then
\[
\sum_{\alpha_1 < \alpha} \left| \left( \partial^{\alpha_1} \nabla_x \phi \cdot \partial^{\alpha_1} \nabla_v f_\pm , e^{\pm \phi} w_{2l,2\nu}(2\alpha,0) \partial^{\alpha} f_\pm \right)_{L^2_v} \right| \\
\lesssim \sum_{\alpha_1 < \alpha} \left| \left( \partial^{\alpha_1} \nabla_x \phi \cdot \partial^{\alpha_1} \nabla_v f_\pm \right)_{L^2_v} \right| \left| w_{l,\nu}(\alpha,0) \partial^{\alpha} f_\pm \right|_{L^2_v}
\lesssim \left( \sum_{|\alpha_1| = 3} \left| \partial^{\alpha_1} \nabla_x \phi \right|_{L^2_v} \left| w_{l,\nu}(\alpha,0) \partial^{\alpha_1} \nabla_v f_\pm \right|_{L^2_v} \right) \left| w_{l,\nu}(\alpha,0) \partial^{\alpha} f_\pm \right|_{L^2_v}
\lesssim \left| \nabla w_{l,\nu}(\alpha,0) \right|_{L^2_v} \left| \partial^{\alpha_1} f_\pm \right|_{L^2_v} \left| \partial^{\alpha} f_\pm \right|_{L^2_v} \\
\lesssim \sqrt{E_{\nu}(t)D_{\nu}(t)}.
\]

When \( \alpha_1 < \alpha \) in Boltzmann case for soft potential, we consider
\[
\sum_{\alpha_1 < \alpha} \left| \left( \partial^{\alpha_1} \nabla_x \phi \cdot \partial^{\alpha_1} \nabla_v f_\pm , e^{\pm \phi} w_{2l,2\nu}(2\alpha,0) \partial^{\alpha} f_\pm \right)_{L^2_v} \right|.
\]

By Young’s inequality, \( \langle \eta \rangle \lesssim \langle \eta \rangle^s \langle v \rangle^k + \langle \eta \rangle^{1+s} \langle v \rangle^{\frac{ks}{s+1}} \) for any \( k \in \mathbb{R} \) and hence \( \langle \eta \rangle \) is a symbol in \( S(\langle \eta \rangle^s \langle v \rangle^k + \langle \eta \rangle^{1+s} \langle v \rangle^{\frac{ks}{s+1}}) \) (cf. [8]), where \( \eta \) is the Fourier variable of \( v \). Then by [8, Lemma 2.3 and Corollary 2.5], we have
\[
|f|_{H^s_x} \lesssim |f(v)|_{H^s} + |f(v)|^{ks/(1-s)}_{H^{1+s}},
\]
for \( k \in \mathbb{R} \). From our choice (1.10), we have
\[
0 \leq rs + (r - q)(1 - s) + \gamma,
\]
and hence,
\[
w_{l,\nu}(\alpha, \beta) \lesssim \langle v \rangle^{\gamma} w_{l,\nu}(\alpha - e_i, \beta)^s w_{l,\nu}(\alpha - e_i, \beta + e_i)^{1-s}.
\]

Applying (4.9) with \( \langle v \rangle^k = w_{l,\nu}(\alpha - e_i, \beta)^{1-s} w_{l,\nu}(\alpha - e_i, \beta + e_i)^{(1-s)} \), we have
\[
\sum_{|\alpha_1| < |\alpha|} \left| \langle v \rangle^{\gamma} w_{l,\nu}(\alpha - e_i, \beta) \partial^{\alpha_1} \nabla_v f_\pm \right|_{L^2_v} \lesssim \sum_{\alpha_1 < \alpha} \left| \langle v \rangle^{\gamma} w_{l,\nu}(\alpha - e_i, \beta) \partial^{\alpha_1} f_\pm \right|_{L^2_v} \lesssim \sqrt{D_{\nu}(t)}.
\]

Then the first right-hand term of (4.8) can be estimated by
\[
\sum_{\alpha_1 < \alpha} \left| \left( \partial^{\alpha_1} \nabla_x \phi \cdot \partial^{\alpha_1} \nabla_v f_\pm , e^{\pm \phi} w_{2l,2\nu}(2\alpha,0) \partial^{\alpha} f_\pm \right)_{L^2_v} \right| \\
\lesssim \left( \sum_{|\alpha_1 - \alpha_1| = 3} \left| \partial^{\alpha_1} \nabla_x \phi \right|_{L^2_v} \left| \langle v \rangle^{\gamma} w_{l,\nu}(\alpha,0) \partial^{\alpha_1} \nabla_v f_\pm \right|_{L^2_v} \right) \left| \partial^{\alpha} f_\pm \right|_{L^2_v} \\
+ \sum_{|\alpha_1 - \alpha_1| = 2} \left| \partial^{\alpha_1} \nabla_x \phi \right|_{L^2_v} \left| \langle v \rangle^{\gamma} w_{l,\nu}(\alpha,0) \partial^{\alpha_1} \nabla_v f_\pm \right|_{L^2_v} \left| \partial^{\alpha} f_\pm \right|_{L^2_v}.
\]
\[+ \sum_{|\alpha - \alpha_1| = 1} \| \partial^{\alpha - \alpha_1} \nabla_x \phi \|_{L^\infty_x} \| (v)^{-\frac{2}{3}} w_{l,\nu}(\alpha, 0) \partial^{\alpha_1} \nabla_v f_\pm \|_{L^2_x L^2_v} \times \| (v)^{-\frac{2}{3}} w_{l,\nu}(\alpha, 0) \partial^\alpha f_\pm \|_{L^2_x v} \]\[\lesssim \| E \|_{H^2_x} \sum_{|\alpha| < |\alpha|} \| (v)^{-\frac{2}{3}} w_{l,\nu}(\alpha, 0, \beta) \partial^{\alpha_1} \nabla_v f_\pm \|_{L^2_x L^2_v} \| w_{l,\nu}(\alpha, 0) \partial^\alpha f_\pm \|_{L^2_x L^2_v} \]\[\lesssim \sqrt{\mathcal{E}_\nu(t)} \mathcal{D}_\nu(t).\]

The above estimates give (4.5). The proof of (4.6) is similar to (4.5) and we omit the details for brevity.

Now we are ready to prove our Main Theorem 1.1.

Proof of Theorem 1.1. Let $|\alpha| + |\beta| \leq 3$. If $|\alpha| = 2$, we restrict $\alpha_i = 2$ for some $i = 1, 2, 3$. Then applying $\partial^\beta_3$ to (1.3), we have

\[\partial_t (\partial^\beta_3 f_\pm) + \partial_\beta (v \cdot \nabla_x \partial^\alpha f_\pm) \pm \frac{1}{2} \partial^\beta_3 (\nabla_x \phi \cdot v f_\pm) \mp \partial^\beta_3 (\nabla_x \phi \cdot \nabla_v f_\pm) \pm \partial^\beta_3 (\nabla_x \phi \cdot v \mu^{1/2}) - \partial_\beta L_{\pm} \partial^\alpha f = \partial^\beta_3 \Gamma_{\pm}(f, f). \quad (4.10)\]

Step 1. Estimate with Spatial Derivatives. We begin with the following estimate. Taking integration by parts with respect to $\nabla_x$, we have

\[\langle v \cdot \nabla_x \partial^\alpha f_\pm, e^{\pm \phi} w_{2l,2\nu}(2\alpha, 0) \partial^\alpha f_\pm \rangle_{L^2_x v} \pm \frac{1}{2} \langle \partial^\alpha (\nabla_x \phi \cdot v f_\pm), e^{\pm \phi} w_{2l,2\nu}(2\alpha, 0) \partial^\alpha f_\pm \rangle_{L^2_x v}\]

where we apply Lemma 3.1 and $R_x v \cdot n(x) = -v \cdot n(x)$. This is what $e^{\pm \phi}$ designed for; cf. [22]. Then letting $|\beta| = 0$ in (4.10) and taking inner product with $e^{\pm \phi} w_{2l,2\nu}(2\alpha, 0) \partial^\alpha f_\pm$ of (4.10) over $\Omega \times \mathbb{R}^3$, we have

\[\frac{1}{2} \partial_t \| w_{l,\nu}(\alpha, 0) \partial^\alpha f_\pm \|_{L^2_x v}^2 + \frac{1}{2} \langle \partial^\beta \phi w_{l,\nu}(\alpha, 0) \partial^\alpha f_\pm, e^{\pm \phi} w_{2l,2\nu}(2\alpha, 0) \partial^\alpha f_\pm \rangle_{L^2_x v}\]

We denote the second to eighth term in (4.11) to be $I_1$ to $I_7$ and estimate them term by term. For $I_1$, we have

\[|I_1| \lesssim \| \partial_\beta \phi \|_{L^\infty_x} \mathcal{E}_\nu(t). \quad (4.12)\]

By Lemma 4.1, we know that

\[|I_2| + |I_3| \lesssim \sqrt{\mathcal{E}_\nu(t)} \mathcal{D}_\nu(t) + \| \nabla_x \phi \|_{H^2_x} \mathcal{E}_\nu(t). \quad (4.13)\]
For $I_4$, when $l = p|\alpha|$ and $\nu = 0$, we take summation on $\pm$ to obtain
\[ \sum_{\pm} I_4 = (\partial^a \nabla_x \phi \cdot v \mu^{1/2}, \partial^a (I - P)(f_+ - f_-))_{L^2_{x, v}} = (\partial^a \nabla_x \phi, \partial^a G)_{L^2_x}, \]
where $G$ is defined by (3.8). Similar to the proof of (3.17) and (3.21), we know that $\partial_{x_i} G_t(x) = G_t(x) = 0$ on $x \in \Gamma_i$ for $i = 1, 2, 3$. Also, by (1.8) and (3.19), we know that $\partial_{x_i} \phi = \partial_{x_i} \phi = 0$ on $\Gamma_i$ for $i = 1, 2, 3$. Then by (3.7) and integration by parts, we have
\[ (\partial^a \nabla_x \phi, \partial^a G)_{L^2_x} = -(\partial^a \phi, \partial^a \nabla_x G)_{L^2_x} = (\partial^a \phi, \partial^a \partial_t (a_+ - a_-))_{L^2_x}. \]
If $|\alpha| \leq 1$, from (1.8) we know that $\partial^a \phi = 0$ or $\partial^a \partial_{x_j} \phi = 0$ on $\Gamma_j$ for $j = 1, 2, 3$. Then
\[ (\partial^a \phi, \partial^a \partial_t (a_+ - a_-))_{L^2_x} = (\partial^a \phi, -\partial^a \partial_t \Delta_x \phi)_{L^2_x} = (\partial^a \nabla_x \phi, \partial^a \partial_t \nabla_x \phi)_{L^2_x}. \]
If $|\alpha| = 2$, then $\partial^a = \partial_{x_i} \partial_{x_j}$ for some $i, j = 1, 2, 3$. Since $\partial_{x_i} (a_+ - a_-) = 0$ and $\partial_{x_i} \phi = 0$ on $\Gamma_i$, by integration by parts, we know that
\[ (\partial^a \phi, \partial^a \partial_t (a_+ - a_-))_{L^2_x} = (\partial_{x_i} \partial_{x_j} \phi, \partial_{x_i} \partial_t (a_+ - a_-))_{L^2_x} \]
\[ = (\partial_{x_i} \partial_{x_j} \phi, \partial_{x_i} \partial_{x_j} \partial_t \phi)_{L^2_x} + \sum_{k \neq i} (\partial_{x_i} \partial_{x_j} \phi, \partial_{x_{i+k}} \partial_t \phi)_{L^2_x} \]
\[ = \frac{1}{2} \partial_t ||\partial^a \nabla_x \phi||^2_{L^2_x}, \]
where we used the following facts from (1.8) and (3.19), $\partial_{x_{i+k}} \phi = 0$ or $\partial_{x_{i+k}} \phi = 0$ on $\Gamma_k$ for $k \neq i$.

If $|\alpha| = 3$, then $\partial^a = \partial_{x_i} \partial_{x_j} \partial_{x_k}$ and we need more discussion. Note that we will apply (1.8) and (3.19) frequently. If $i, j, k$ are pairwise different, then $\partial_{x_i} \partial_{x_j} \phi = 0$ on $\Gamma_i$, $\Gamma_j$ and $\Gamma_k$. Thus,
\[ (\partial^a \phi, \partial_t \partial^a (a_+ - a_-))_{L^2_x} = (\partial^a \phi, -\partial_t \partial^a \Delta_x \phi)_{L^2_x} = \frac{1}{2} \partial_t ||\partial^a \nabla_x \phi||^2_{L^2_x}. \]

If $i = j \neq k$, then by $\partial_{x_i} \phi = \partial_{x_j} \phi = 0$ on $\Gamma_i$, and $\partial_{x_{i+k}} \phi = 0$ on $\Gamma_k$, we have
\[ (\partial^a \phi, \partial_t \partial^a (a_+ - a_-))_{L^2_x} = (\partial_{x_{i+k}} \phi, \partial_t \partial_{x_{i+k}} (a_+ - a_-))_{L^2_x} \]
\[ = (\partial_{x_{i+k}} \phi, \partial_t \partial_{x_{i+k}} \phi)_{L^2_x} + \sum_{m \neq k} (\partial_{x_{i+k}} \phi, \partial_t \partial_{x_{m+k}} \phi)_{L^2_x} \]
\[ = \frac{1}{2} \partial_t ||\partial^a \nabla_x \phi||^2_{L^2_x}. \]

The last identity follows from suitable integration by parts. Note that if $m = i$, we have $\partial_{x_{i+k}} \phi = 0$ on $\Gamma_k$. If $m = k$, we have $\partial_{x_{i+k}} \phi = 0$ on $\Gamma_i$. If $m \neq k, i$, we have $\partial_{x_{i+k}} \phi = 0$ on $\Gamma_k$, $\partial_{x_{i+k+m}} \phi = 0$ on $\Gamma_i$ and $\partial_{x_{i+k+m}} \phi = 0$ on $\Gamma_m$. If $i = j = k$, then by $\partial_{x_{i+k}} \phi = 0$ on $\Gamma_i$ and $\partial_{x_{i+k+m}} \phi = 0$ on $\Gamma_m$ for $m \neq i$, we have
\[ (\partial^a \phi, \partial_t \partial^a (a_+ - a_-))_{L^2_x} = (\partial_{x_{i+k}} \phi, \partial_t \partial_{x_{i+k}} (a_+ - a_-))_{L^2_x} \]
\[ = (\partial_{x_{i+k}} \phi, \partial_t \partial_{x_{i+k}} \phi)_{L^2_x} + \sum_{m \neq i} (\partial_{x_{i+k}} \phi, \partial_t \partial_{x_{i+k+m}} \phi)_{L^2_x} \]
\[ \lambda > \text{estimate with weight:} \]

\[ \sum_{\pm} I_4 = \frac{1}{2} \partial_l \| \partial^\alpha \nabla_x \phi \|_{L^2_x}^2. \quad (4.14) \]

When \( l \neq p|\alpha| \) or \( \nu \neq 0 \), we write an upper bound for \( I_4 \):

\[ |I_4| \lesssim C \eta \| \partial^\alpha E \|_{L^2_x}^2 + \eta \| w_{l,\nu}(\alpha,0) \partial^\alpha f_{\pm} \|_{L^2_x L_d^2}. \quad (4.15) \]

For \( I_5 \), when \( l = p|\alpha| \) and \( \nu = 0 \), using Lemma 2.1, we have

\[ \sum_{\pm} I_5 \gtrsim \| (I - P) \partial^\alpha f \|_{L^2_x L_d^2}^2. \quad (4.16) \]

When \( l \neq p|\alpha| \) or \( \nu \neq 0 \), by (2.6), we have

\[ \sum_{\pm} I_5 \geq c_0 \| \partial^\alpha f \|_{L^2_{d,w_{l,\nu}(\alpha,0)}}^2 - C \| \partial^\alpha f \|_{L^2(B_c)}. \quad (4.17) \]

For \( I_6 \), note that \( |e^{\pm \phi} - 1| \lesssim \| \phi \|_{L^\infty} \lesssim \| \nabla_x \phi \|_{H^1_x} \). Then by (2.10),

\[ |I_6| \lesssim \| \nabla_x \phi \|_{H^1_x} \left( \| \partial^\alpha f_{\pm} \|_{L^2_x L^2_d} \| \partial^\alpha f_{\pm} \|_{L^2_x L^2_d} + \| \partial^\alpha f_{\pm} \|_{L^2_x L^2_d}^2 \right) \lesssim \sqrt{\mathcal{E}_c(t)} D_c(t). \quad (4.18) \]

The estimate of \( I_7 \) can be obtained from (2.8) and it follows that

\[ |I_7| \lesssim \sqrt{\mathcal{E}_c(t)} D_c(t). \quad (4.19) \]

Therefore, if \( l = p|\alpha|, \nu = 0 \), plugging estimate (4.12), (4.13), (4.14), (4.16), (4.18) and (4.19) into (4.11), we have the energy estimate without weight:

\[ \frac{1}{2} \partial_t \left( \| \partial^\alpha f \|_{L^2_{x,v}}^2 + \| \partial^\alpha E \|_{L^2_x}^2 \right) + \lambda \| (I - P) \partial^\alpha f \|_{L^2_x L_d^2}^2 \lesssim \left( \| \partial_t \phi \|_{L^\infty} + \| \nabla_x \phi \|_{H^2_x} \right) \mathcal{E}_c(t) + \sqrt{\mathcal{E}_c(t)} D_c(t), \quad (4.20) \]

for some constant \( \lambda > 0 \). If \( l \neq p|\alpha| \) or \( \nu \neq 0 \), plugging estimate (4.12), (4.13), (4.15), (4.17), (4.18) and (4.19) into (4.11) and letting \( \eta > 0 \) small enough, we have the energy estimate with weight:

\[ \frac{1}{2} \partial_t \| w_{l,\nu}(\alpha,0) \partial^\alpha f_{\pm} \|_{L^2_{x,v}}^2 + \lambda \| \partial^\alpha f \|_{L^2_x L^2_d, w_{l,\nu}(\alpha,0)}^2 \lesssim \left( \| \partial_t \phi \|_{L^\infty} + \| \nabla_x \phi \|_{H^2_x} \right) \mathcal{E}_c(t) + \sqrt{\mathcal{E}_c(t)} D_c(t) + \| \partial^\alpha f \|_{L^2_x L^2_d}^2 + \| \partial^\alpha E \|_{L^2_x}^2, \quad (4.21) \]

for some constant \( \lambda > 0 \).

**Step 2. Estimate with Mixed Derivatives.** Let \( 1 \leq |\beta| \leq 3 \), then \(|\alpha| \leq 2 \). From Lemma 3.1, we have \( \partial^\alpha f(R_x v) \) equal to \( \partial^\alpha f(v) \) or \( -\partial^\alpha f(v) \), which implies that \( \partial^\alpha f(R_x v) \) equal to \( \partial^\alpha f(v) \) or \( -\partial^\alpha f(v) \), since \( R_x v \) maps \( v_i \) to \( -v_i \) for some \( i = 1, 2, 3 \) and derivatives on velocity variable would produce only sign \( \pm \). Then we have

\[ (v \cdot \nabla_x \partial^\alpha f_{\pm}, e^{+\phi} w_{2l,2\nu}(2\alpha, 2\beta) \partial^\alpha f_{\pm}) \mid_{L^3_x} \pm \frac{1}{2} \left( \nabla_x \phi \cdot v \partial^\alpha f_{\pm}, e^{+\phi} w_{2l,2\nu}(2\alpha, 2\beta) \partial^\alpha f_{\pm} \right) \mid_{L^3_x} = \frac{1}{2} \int_{\partial \Omega} \int_{R^3} v \cdot n(x) w_{l,\nu}(\alpha, \beta) \partial^\alpha f_{\pm}(v)^2 \ dv \ ds(x) \]

\[ = \frac{1}{2} \int_{\partial \Omega} \int_{R^3} R_x v \cdot n(x) w_{l,\nu}(\alpha, \beta) \partial^\alpha f_{\pm}(R_x v)^2 \ dv \ ds(x) \]

\[ = - \frac{1}{2} \int_{\partial \Omega} \int_{R^3} v \cdot n(x) w_{l,\nu}(\alpha, \beta) \partial^\alpha f_{\pm}(v)^2 \ dv \ ds(x) = 0. \]
Taking inner product with \( w_{2l,2\nu}(2\alpha,2\beta)\partial^\alpha f_\pm \) of (4.10) over \( \Omega \times \mathbb{R}^3 \), we have
\[
\frac{1}{2}\partial_t \| w_{l,\nu}(\alpha,\beta)\partial^\beta f_\pm \|_{L^2_{x,v}}^2 + (\partial_t \phi \partial^\beta f_\pm, e^{\pm\phi} w_{2l,2\nu}(2\alpha,2\beta)\partial^\beta f_\pm)_{L^2_{x,v}} + \sum_{|\beta_1|=1} (\partial_{\beta_1} v \cdot \nabla_x \partial^\beta_{\alpha-\beta_1} f_\pm, e^{\pm\phi} w_{2l,2\nu}(2\alpha,2\beta)\partial^\beta f_\pm)_{L^2_{x,v}} + \frac{1}{2} \sum_{\alpha_1 + \alpha = \alpha_1 + \beta - 1} \left( \partial^\beta (\nabla_x \phi \cdot v f_\pm), e^{\pm\phi} w_{2l,2\nu}(2\alpha,2\beta)\partial^\beta f_\pm \right)_{L^2_{x,v}} + (\partial^\beta (\nabla_x \phi \cdot v f_\pm), e^{\pm\phi} w_{2l,2\nu}(2\alpha,2\beta)\partial^\beta f_\pm)_{L^2_{x,v}} + (\partial^\beta (\nabla_x \phi \cdot v f_\pm), e^{\pm\phi} w_{2l,2\nu}(2\alpha,2\beta)\partial^\beta f_\pm)_{L^2_{x,v}} - (\partial^\beta \partial^\alpha f_\pm, w_{2l,2\nu}(2\alpha,2\beta)\partial^\beta f_\pm)_{L^2_{x,v}} - (\partial^\beta \partial^\alpha f_\pm, (e^{\pm\phi} - 1) w_{2l,2\nu}(2\alpha,2\beta)\partial^\beta f_\pm)_{L^2_{x,v}} = (\partial^\beta \Gamma_\pm (f,f), e^{\pm\phi} w_{2l,2\nu}(2\alpha,2\beta)\partial^\beta f_\pm)_{L^2_{x,v}}. 
\]
(4.22)

Denote the second to ninth terms of (4.22) to be \( J_1 \) to \( J_8 \). Then for \( J_1 \), we have
\[
|J_1| \lesssim \| \partial_t \phi \|_{L^\infty_x \mathcal{E}_\nu(t)}.
\]
For \( J_2 \), note that for hard potential case, we have \( | \cdot |_{L^2_x} \lesssim | \cdot |_{L^2_{x,v}} \) and \( w_{l,\nu}(\alpha,\beta) = w_{l,\nu}(\alpha + e_1, \beta - \beta_1) \). For soft potential case, we have \( w_{l,\nu}(\alpha,\beta) \leq \langle v \rangle^\gamma w_{l,\nu}(\alpha + e_1, \beta - \beta_1) \) for any \( |\beta_1| = 1 \), where we let \( s = 1 \) in Landau case. Then
\[
|J_2| = \sum_{|\beta_1|=1} \left| (\partial_{\beta_1} v \cdot \nabla_x \partial^\beta_{\alpha-\beta_1} f_\pm, e^{\pm\phi} w_{2l,2\nu}(2\alpha,2\beta)\partial^\beta f_\pm)_{L^2_{x,v}} \right| \lesssim \sum_{|\beta_1|=1} \int_{\Omega \times \mathbb{R}^3} \langle v \rangle^{\gamma+2s} w_{l,\nu}(\alpha + e_1, \beta - \beta_1) |\nabla_x \partial^\beta_{\alpha-\beta_1} f_\pm| w_{l,\nu}(\alpha,\beta) |\partial^\beta f_\pm| \, dx \, dv \lesssim C \eta \sum_{i=1}^3 \sum_{|\beta_1|=1} \| \partial^\beta_{\alpha+e_i-\beta_1} f_\pm \|_{L^2_x L^2_{x,v}(\alpha+e_i,\beta-\beta_1)} \cdot \| \partial^\beta f_\pm \|_{L^2_x L^2_{x,v}(\alpha,\beta)}.
\]
By Lemma 4.1, we have
\[
|J_4| + |J_4| \lesssim \sqrt{\mathcal{E}_\nu(t) D_\nu(t)} + \| \nabla_x \phi \|_{L^2_x \mathcal{E}_\nu(t)}.
\]
For \( J_5 \), we write an upper bound:
\[
|J_5| \lesssim C \| \partial^\alpha E \|_{L^2_x} \cdot \eta \| w_{l,\nu}(\alpha,\beta) \|_{L^2_x L^2_B}.
\]
For \( J_6 \), by (2.7), we have
\[
\sum_{\pm} J_6 \geq \sum_{\pm} c_0 \| \partial^\beta f_\pm \|_{L^2_x L^2_{x,v}(\alpha,\beta)} - C \sum_{|\beta_1| < |\beta|} \| \partial^\beta_{\alpha-\beta_1} f_\pm \|_{L^2_x L^2_{x,v}(\alpha,\beta)} - C \| \partial^\alpha f_\pm \|_{L^2_x L^2(B_C)}.
\]
The estimate for \( J_7 \) is similar to (4.18) and we have
\[
|J_7| \lesssim \sqrt{\mathcal{E}_\nu(t) D_\nu(t)}.
\]
For \( J_8 \), by (2.11), we have
\[
|J_8| \lesssim \sqrt{\mathcal{E}_\nu(t) D_\nu(t)}.
\]
Combining the above estimates on \( J_k \ (1 \leq k \leq 8) \) and letting \( \eta > 0 \) small enough, we have from (4.22) that
\[
\frac{1}{2}\partial_t \|w_{l,\nu}(\alpha, \beta)\partial_\beta^\alpha f_\pm\|_{L^2_{x,v}}^2 + \lambda \|\partial_\beta^\alpha h\|_{L^2_{x,v}}^2 \leq \|\partial_t \phi\|_{L^\infty} \mathcal{E}_\nu(t) + \sqrt{\mathcal{E}_\nu(t)} \mathcal{D}_\nu(t)
\]
\[
+ C \sum_{|\beta_1| < |\beta|} \|\partial_\beta^\alpha h\|_{L^2_{x,v}}^2 + \sum_{i=1}^3 \sum_{|\beta_1| = 1} \|\partial_\beta^\alpha + \epsilon_i f_\pm\|_{L^2_{x,v}}^2 \leq \|\partial^\alpha E\|_{L^2_x}^2.
\]
(4.23)
for some \(\lambda > 0\).

**Step 3. Energy Estimate.** From Theorem 3.4 with \(K = 3\), there exists \(\mathcal{E}_{int}(t)\) satisfying
\[
\mathcal{E}_{int}(t) \lesssim \sum_{|\alpha| \leq 3} \|\partial^\alpha f\|_{L^2_x},
\]
(4.24)
such that
\[
\partial_t \mathcal{E}_{int}(t) + \lambda \sum_{|\alpha| \leq 3} \|\partial^\alpha[a_+, a_-, b, c]\|_{L^2_x}^2 + \lambda \sum_{|\alpha| \leq 3} \|\partial^\alpha E\|_{L^2_x}^2
\]
\[
\lesssim \sum_{|\alpha| \leq 3} \| (I - P) \partial^\alpha f\|_{L^2_x}^2 + \sum_{|\alpha| \leq 3} \| (\partial^\alpha g, \zeta(v))\|_{L^2_x}^2 + \|E\|_{L^2_x}^4,
\]
(4.25)
for some constant \(\lambda > 0\). Also, by (3.67), we know that
\[
\| (\partial^\alpha g, \zeta L^2_x + \|E\|_{L^2_x}^4 \lesssim \mathcal{E}_\nu(t) \mathcal{D}_\nu(t).
\]
(4.26)
Taking linear combination \(\sum_{|\alpha| \leq 3} (4.20) + \kappa \times (4.25)\) with \(\kappa > 0\) small enough and applying (4.26), we have
\[
\frac{1}{2} \partial_t \sum_{|\alpha| \leq 3} \left( \|\partial^\alpha f_\pm\|_{L^2_{x,v}}^2 + \|\partial^\alpha E\|_{L^2_x}^2 \right) + \kappa \partial_t \mathcal{E}_{int}(t) + \lambda \sum_{|\alpha| \leq 3} \|\partial^\alpha f_\pm\|_{L^2_{x,v}}^2 + \lambda \sum_{|\alpha| \leq 3} \|\partial^\alpha E\|_{L^2_x}^2
\]
\[
\lesssim \left( \|\partial_t \phi\|_{L^\infty} + \|\nabla_x \phi\|_{H^2_x} \right) \mathcal{E}(t) + \left( \sqrt{\mathcal{E}_\nu(t)} + \mathcal{E}_\nu(t) \right) \mathcal{D}_\nu(t).
\]
(4.27)
Then taking linear combination (4.27) + \(\kappa \sum_{|\alpha| \leq 3} (4.21) + \kappa^2 \sum_{1 \leq |\beta| \leq 3} \sum_{|\alpha| \leq 3 - |\beta|} \kappa^2 \times (4.23)\) with \(0 < \kappa_3 \ll \kappa_2 \ll \kappa_1 \ll \kappa\) and \(\delta\) small enough, we have
\[
\partial_t \mathcal{E}_\nu(t) + \lambda \mathcal{D}_\nu(t) \lesssim \left( \|\partial_t \phi\|_{L^\infty} + \|\nabla_x \phi\|_{H^2_x} \right) \mathcal{E}_\nu(t) + \left( \sqrt{\mathcal{E}_\nu(t)} + \mathcal{E}_\nu(t) \right) \mathcal{D}_\nu(t),
\]
(4.28)
for any \(\nu \geq 0\), where \(\mathcal{E}_\nu(t)\) is given by
\[
\mathcal{E}_\nu(t) = \frac{1}{2} \sum_{|\alpha| \leq 3} \left( \|\partial^\alpha f_\pm\|_{L^2_{x,v}}^2 + \|\partial^\alpha E\|_{L^2_x}^2 \right) + \kappa \mathcal{E}_{int}(t)
\]
\[
+ \frac{\kappa}{2} \sum_{|\alpha| \leq 3} \left( \|w_{l,\nu}(\alpha, 0)\partial^\alpha f_\pm\|_{L^2_{x,v}}^2 + \|\partial^\alpha E\|_{L^2_x}^2 \right)
\]
\[
+ \frac{\kappa^2}{2} \sum_{1 \leq |\beta| \leq 3, |\alpha| \leq 3 - |\beta|} \kappa^2 \|w_{l,\nu}(\alpha, \beta)\partial^\beta f_\pm\|_{L^2_{x,v}}^2,
\]
and \(\mathcal{D}_\nu(t)\) is given by (1.12). It’s direct to check that \(\mathcal{E}_\nu(t)\) satisfies (1.11) by using (4.24). Thus, using the *a priori* assumption (4.2), (4.28) becomes
\[
\partial_t \mathcal{E}_\nu(t) + \lambda \mathcal{D}_\nu(t) \lesssim \left( \|\partial_t \phi\|_{L^\infty} + \|\nabla_x \phi\|_{H^2_x} \right) \mathcal{E}_\nu(t).
\]
(4.29)
For the hard potential case, we have \(\|w_{l,\nu}(\alpha, \beta)\partial^\beta f_\pm\|_{L^2_{x,v}}^2 \lesssim \|w_{l,\nu}(\alpha, \beta)\partial^\beta f_\pm\|_{L^2_{x,v}}^2\) and hence \(\mathcal{E}(t) \lesssim \mathcal{D}(t)\). Notice from (1.8) that \(\partial_t \phi = 0\) on \(\Gamma_t\). Then by Sobolev embedding [6, Theorem 6.7-5], we have \(\|\partial_t \nabla_x \phi\|_{L^2_x} \lesssim \|\partial_t \nabla_x \phi\|_{L^2_x} \lesssim \|\partial_t \Delta_x \phi\|_{L^2_x}\) and hence,
\[
\|\partial_t \phi\|_{L^\infty} \lesssim \|\partial_t \nabla_x \phi\|_{H^2_x} \lesssim \|\partial_t \nabla_x \phi\|_{L^2_x} = \|\partial_t \Delta_x \phi\|_{L^2_x}
\]
\[ \| \partial_t (a_+ - a_-) \|_{L^2_x} \lesssim \| \nabla_x G \|_{L^2_x} \lesssim \| \nabla_x (I - P) f \|_{L^2_x L^2_t} \lesssim \sqrt{E(t)}. \]  

(4.30)

Here, the first inequality follows from Sobolev inequality; cf. [1]. The first identity follows from boundary value \( \partial_{x_i} \phi = 0 \) on \( \Gamma_i \) and \( \Gamma_j \) for \( j \neq i \). The second identity is from (1.3)$_2$. The third inequality comes from (3.7)$_1$. Thus, when \( \nu = 0 \), (4.29) becomes

\[ \partial_t E(t) + \lambda E(t) \leq \sqrt{E(t)} E(t) \]

Under the smallness (1.14), we have

\[ \partial_t E(t) + \delta E(t) \leq 0, \]

(4.31)

and hence

\[ E(t) \leq e^{\delta t} E(0), \]

(4.32)

for some constant \( \delta > 0 \). This close the a priori assumption (4.2) by choosing \( \varepsilon_0 \) in (1.14) small enough for hard potential case.

For soft potential, we need more calculations. Recall definition (4.1) for \( X(t) \) and assume (4.2). Then from (4.30) and (4.2), we have \( e^{\delta t p/2} (\| \partial_t \phi \|_{L^\infty_x} + \| \nabla_x \phi \|_{H^2_x}) \lesssim \sqrt{X(t)} \lesssim \sqrt{\varepsilon_0}, \) for some \( \delta > 0 \). Solving (4.29), we have

\[ E_\nu(t) \lesssim E_\nu(0) e^{- \int_0^t \left( \| \partial_t \phi \|_{L^\infty_x} + \| \nabla_x \phi \|_{H^2_x} \right) d \tau} \lesssim E_\nu(0) \lesssim \varepsilon_0. \]

(4.33)

Next we claim that for \( T > 0 \),

\[ \sup_{0 \leq t \leq T} e^{\delta t p} E(T) \lesssim \varepsilon_0 + X^{3/2}(t). \]

(4.34)

Indeed, as in [12, 31], for \( p' > 0 \) to be chosen depending on \( p \), we define

\[ E = \{ \langle \nu \rangle \leq t^{p'} \}, \quad E^c = \{ \langle \nu \rangle > t^{p'} \}. \]

Corresponding to this splitting, we define \( E^{low}(t) \) to be the restriction of \( E(t) \) to \( E \) and similarly \( E^{high}(t) \) to be the restriction of \( E(t) \) to \( E^c \). We define \( p' = \frac{p}{\gamma + 2} \) for Boltzmann case and \( p' = \frac{p - 1}{\gamma + 2} \) for Landau case. Then on \( E \), we have \( t^{p - 1} \leq \langle \nu \rangle \frac{1}{t^{p'}} \), and hence \( t^{p - 1} E^{low}(t) \lesssim D(t) \). It follows from (4.29) with \( \nu = 0 \) that

\[ \partial_t E(t) + \lambda t^{p - 1} E(t) \lesssim \left( \| \partial_t \phi \|_{L^\infty_x} + \| \nabla_x \phi \|_{H^2_x} \right) E(t) + \lambda t^{p - 1} E^{high}(t). \]

By solving this ODE, we have

\[ E(t) \lesssim t^{\lambda p} E(0) + \int_0^t e^{- \lambda (t - \tau)} \left( \left( \| \partial_t \phi(\tau) \|_{L^\infty_x} + \| \nabla_x \phi(\tau) \|_{H^2_x} \right) E(\tau) + \lambda t^{p - 1} E^{high}(\tau) \right) d \tau. \]

(4.35)

In what follows we estimate the terms in the time integral on the right-hand side of (4.35). Firstly, by (4.30) and (4.1), we have

\[ \left( \| \partial_t \phi \|_{L^\infty_x} + \| \nabla_x \phi \|_{H^2_x} \right) E(t) \lesssim E^{\frac{3}{2}}(t) \lesssim e^{- \frac{3}{2} \lambda t} X^{\frac{3}{2}}(t). \]

On the other hand, choose \( p = p' \theta \), i.e. choose \( p \) satisfying (1.13). Then on \( E^c \), we have

\[ e^{- \nu \langle \nu \rangle} \lesssim e^{- \nu t^{p'}} = e^{- \nu p}. \]

Recalling the exponential weight in (1.9), by (4.33), we have

\[ E^{high}(t) \lesssim e^{- \nu t p} E_\nu(t) \lesssim e^{- \nu t p} \varepsilon_0. \]

Plugging the above estimates into (4.35), we have

\[ E(t) \lesssim t^{\lambda p} E(0) + \int_0^t e^{- \lambda (t - \tau)} \left( e^{- \frac{3}{2} \lambda t} X^{\frac{3}{2}}(\tau) + \varepsilon_0 e^{- \nu t p} \right) d \tau \lesssim e^{\delta t p} (\varepsilon_0 + X^{\frac{3}{2}}(t)), \]
by choosing $\lambda < \nu$. This completes the claim (4.34). Recalling definition (4.1) for $X(t)$ and using (4.33) and (4.34), we have

$$X(t) \lesssim \varepsilon_0 + X^\frac{2}{3}(t).$$

Then choosing $\delta_0$ in (4.2) sufficiently small, we have the \textit{a priori} estimate:

$$X(t) \lesssim \varepsilon_0.$$  \hspace{1cm} (4.36)

With (4.31), (4.32) and (4.36) in hand, under the smallness of (1.14), it’s now standard to apply the continuity argument with local existence from Section 5 to obtain the global existence, uniqueness and large time decay for initial boundary problem (1.3), (1.7) and (1.8) in bounded domain $\Omega$. The positivity of the solutions can be obtained from [22, Lemma 12, page 800] for VPL systems and [20, page 1121] for VPB systems. This complete the proof of Theorem 1.1.

\qed

5. Local Existence

In this section, we are concerned with the local-in-time existence of solutions to problem (1.3) in union of cubes. For brevity, we only consider the proof of Vlasov-Poisson-Landau systems when $\gamma \geq -3$, since the Vlasov-Poisson-Boltzmann case is similar.

**Theorem 5.1.** Let $\gamma \geq -3$, $\Omega$ be given by (1.6) and $w_{l,\nu}(\alpha, \beta)$ be given by (1.9). Then there exists $\varepsilon_0 > 0$, $T_0 > 0$ such that if $F_0(x, v) = \mu + \mu^{1/2} f_0(x, v) \geq 0$ and

$$\sum_{|\alpha| + |\beta| \leq 3} \left( \|w_{l,\nu}(\alpha, \beta)\partial_\beta^\alpha f_0\|^2_{L^2_{x,v}} + \|\partial^\alpha E_0\|^2_{L^2_x} \right) \leq \varepsilon_0,$$

then the specular reflection boundary problem for VPL systems (1.3), (1.7) and (1.8) admits a unique solution $f(t, x, v)$ on $t \in [0, T_0]$, $x \in \Omega$, $v \in \mathbb{R}^3$, satisfying the uniform estimate

$$\sup_{0 \leq t \leq T_0} \mathcal{E}_\nu(t) + \int_0^{T_0} \mathcal{D}_\nu(t) dt \lesssim \sum_{|\alpha| + |\beta| \leq 3} \left( \|w_{l,\nu}(\alpha, \beta)\partial_\beta^\alpha f_0\|^2_{L^2_{x,v}} + \|\partial^\alpha E_0\|^2_{L^2_x} \right), \hspace{1cm} (5.1)$$

where $\mathcal{E}_\nu(t)$, $\mathcal{D}_\nu(t)$ are defined by (1.11) and (1.12) respectively.

We begin with the following linear inhomogeneous problem on the union of cubes:

$$\begin{aligned}
\partial_t f_\pm + v \cdot \nabla_x f_\pm \pm \frac{1}{2} \nabla_x \psi \cdot v f_\pm + \nabla_x \psi \cdot \nabla_x f_\pm \pm \nabla_x \phi \cdot v \mu^{1/2} - A_\pm f
&= \Gamma_\pm(g, h) + K h,
- \Delta_x \phi = \int_{\mathbb{R}^3} (f_+ - f_-) \mu^{1/2} dv, \\
f(0, x, v) = f_0(x, v), \quad E(0, x) = E_0(x), \\
f(t, x, R_x v) = f(t, x, v), \quad \text{on } \gamma_-,
\partial_n \phi = 0, \quad \text{on } x \in \partial \Omega,
\end{aligned} \hspace{1cm} (5.2)$$

for a given $h = h(t, x, v)$ and $\psi = \psi(t, x)$.

**Lemma 5.2.** Let the same assumption in (5.1) be satisfied. There exists $\varepsilon_0 > 0$, $T_0 > 0$ such that if

$$\sum_{|\alpha| + |\beta| \leq 3} \left\{ \|w_{l,\nu}(\alpha, \beta)\partial_\beta^\alpha f_0\|_{L^2_{x,v}} + \|\partial_\beta^\alpha h\|_{L^2_{x,v}} \right\} + \|\partial^\alpha \nabla_x \psi\|_{L^\infty_{x,v}} \lesssim \varepsilon_0. \hspace{1cm} (5.3)$$
then the initial boundary value problem (5.2) admits a unique solution \( f = f(t, x, v) \) on \( \Omega \times \mathbb{R}^3 \) satisfying

\[
\sup_{0 \leq t \leq T_0} \mathcal{E}_\nu(t) + \int_0^{T_0} \mathcal{D}_\nu(t) \, dt + \| \partial_t \psi \|_{L^2_0 \mathbb{R}^3} \lesssim \sum_{|\alpha|+|\beta| \leq 3} \left( \| w_{1, \nu}(\alpha, \beta) \partial_\beta^2 f_0 \|_{L^2_0 \mathbb{R}^3}^2 + \| \partial_\alpha E_0 \|_{L^2_0 \mathbb{R}^3}^2 \right) + T_0^{1/2} \sum_{|\alpha|+|\beta| \leq 3} \| w_{1, \nu}(\alpha, \beta) \partial_\beta^2 h \|_{L^2_0 \mathbb{R}^3}^2, \quad (5.4)
\]

where \( \mathcal{E}_\nu(t) \) and \( \mathcal{D}_\nu(t) \) are defined by (1.11) and (1.12) respectively.

**Proof.** We consider equation (5.2) with initial data \((f_0, E_0)\). Similar to (4.11), applying \( \partial^\alpha \) to (5.2), and taking inner product of the resultant equation with \( w_{2, \nu}(2\alpha, 0)e^{\pm \psi} \partial^\alpha f_\pm \), we have

\[
\frac{1}{2} \partial_t \| w_{1, \nu}(\alpha, 0) \partial^\alpha f_\pm \|_{L^2_0 \mathbb{R}^3}^2 + \frac{1}{2} (\partial_t \psi \, w_{1, \nu}(\alpha, 0) \partial^\alpha f_\pm, e^{\pm \psi} w_{1, \nu}(\alpha, 0) \partial^\alpha f_\pm)_{L^2_0 \mathbb{R}^3} \\
\pm \frac{1}{2} \sum_{|\alpha|<|\beta|} \left( \partial^\alpha \psi \, v \partial^\beta f_\pm, e^{\pm \psi} w_{2, \nu}(2\alpha, 0) \partial^\alpha f_\pm \right)_{L^2_0 \mathbb{R}^3} \\
\mp (\partial^\alpha (\nabla \psi \cdot \nabla v, f_\pm), e^{\pm \psi} w_{2, \nu}(2\alpha, 0) \partial^\alpha f_\pm)_{L^2_0 \mathbb{R}^3} + \left( \partial^\alpha \psi \, v \mu^{1/2}, e^{\pm \psi} w_{2, \nu}(2\alpha, 0) \partial^\alpha f_\pm \right)_{L^2_0 \mathbb{R}^3} \\
- (L_\pm \partial^\alpha f, w_{2, \nu}(2\alpha, 0) \partial^\alpha f_\pm)_{L^2_0 \mathbb{R}^3} - (L_\pm \partial^\alpha f, (e^{\pm \psi} - 1) w_{2, \nu}(2\alpha, 0) \partial^\alpha f_\pm)_{L^2_0 \mathbb{R}^3} \\
= (\partial^\alpha \Gamma_\pm(f, f), e^{\pm \psi} w_{2, \nu}(2\alpha, 0) \partial^\alpha f_\pm)_{L^2_0 \mathbb{R}^3} + \left( \partial^\alpha h, e^{\pm \psi} w_{2, \nu}(2\alpha, 0) \partial^\alpha f_\pm \right)_{L^2_0 \mathbb{R}^3}. 
\]

Similar to (4.22), applying \( \partial^\alpha_\beta \) to (5.2) and taking inner product of the resultant equation with \( w_{2, \nu}(2\alpha, 2\beta)e^{\pm \psi} \partial^\alpha_\beta f_\pm \), we have

\[
\frac{1}{2} \partial_t \| w_{1, \nu}(\alpha, \beta) \partial^\alpha_\beta f_\pm \|_{L^2_0 \mathbb{R}^3}^2 + (\partial_t \psi \, \partial^\alpha_\beta f_\pm, e^{\pm \psi} w_{2, \nu}(2\alpha, 2\beta) \partial^\alpha_\beta f_\pm)_{L^2_0 \mathbb{R}^3} \\
+ \sum_{|\beta_1| = 1} (\partial_{\beta_1} v \cdot \nabla_\beta \partial^\alpha_\beta f_\pm, e^{\pm \psi} w_{2, \nu}(2\alpha, 2\beta) \partial^\alpha_\beta f_\pm)_{L^2_0 \mathbb{R}^3} \\
\pm \frac{1}{2} \sum_{|\alpha|+|\beta|<|\alpha|+|\beta|} \left( \partial^\alpha_\beta (\nabla \psi \cdot \nabla v, f_\pm), e^{\pm \psi} w_{2, \nu}(2\alpha, 2\beta) \partial^\alpha_\beta f_\pm \right)_{L^2_0 \mathbb{R}^3} \\
\mp (\partial^\alpha_\beta (\nabla \psi \cdot \nabla v, f_\pm), e^{\pm \psi} w_{2, \nu}(2\alpha, 2\beta) \partial^\alpha_\beta f_\pm)_{L^2_0 \mathbb{R}^3} + \left( \partial^\alpha_\beta \psi \, v \mu^{1/2}, e^{\pm \psi} w_{2, \nu}(2\alpha, 2\beta) \partial^\alpha_\beta f_\pm \right)_{L^2_0 \mathbb{R}^3} \\
- (L_\pm \partial^\alpha_\beta f, w_{2, \nu}(2\alpha, 2\beta) \partial^\alpha_\beta f_\pm)_{L^2_0 \mathbb{R}^3} - (L_\pm \partial^\alpha_\beta f, (e^{\pm \psi} - 1) w_{2, \nu}(2\alpha, 2\beta) \partial^\alpha_\beta f_\pm)_{L^2_0 \mathbb{R}^3} \\
= (\partial^\alpha_\beta \Gamma_\pm(f, f), e^{\pm \psi} w_{2, \nu}(2\alpha, 2\beta) \partial^\alpha_\beta f_\pm)_{L^2_0 \mathbb{R}^3} + \left( \partial^\alpha_\beta h, e^{\pm \psi} w_{2, \nu}(2\alpha, 2\beta) \partial^\alpha_\beta f_\pm \right)_{L^2_0 \mathbb{R}^3}. 
\]

Following the similar argument from (4.10) to (4.28), applying smallness (5.3), using the estimate (2.8) for \( L_\pm \) to replace the estimates on \( L_\pm \) and macroscopic estimates, we have

\[
\partial_t \mathcal{E}_\nu(t) + \lambda \mathcal{D}_\nu(t) \lesssim \| \partial_t \psi \|_{L^2 \mathbb{R}^3} \mathcal{E}_\nu(t) + \sum_{|\alpha|+|\beta| \leq 3} \| w_{1, \nu}(\alpha, \beta) \partial^\alpha_\beta h \|_{L^2_0 \mathbb{R}^3}^2 + \mathcal{E}_\nu(t) \\
+ \sum_{|\alpha|+|\beta| \leq 3} \| \partial^\alpha_\beta h \|_{L^2_0 \mathbb{R}^3}^2 \sqrt{\mathcal{E}_\nu(t) \mathcal{D}_\nu(t)}. \quad (5.5)
\]
By using (5.3), we have $\|\partial_{y} \psi\|_{L_{\infty}^{\infty}} \lesssim \varepsilon_{0}$. Then solving (5.5), we obtain
\begin{equation}
\sup_{0 \leq t \leq T} \left( e^{C_{t}E_{\nu}(t)} \right) + \lambda \int_{0}^{T} D_{\nu}(t) dt \lesssim T \sum_{|\alpha|+|\beta| \leq 3} \sup_{0 \leq t \leq T} \|w_{\nu,\mu}(\alpha,\beta)\partial_{\beta}^{3} h\|_{L_{x,v}^{2}}^{2},
\end{equation}
for some large constant $C > 0$. Since (5.2) is a linear equation, with (5.6) in hand, it’s standard to apply the theory for linear evolution equation to find the local-in-time existence for (5.2). In fact, we can obtain the local-in-time solution $(f, \phi)$ to (5.2) with estimate: for some $T_{0} > 0$,
\begin{equation}
\sup_{0 \leq t \leq T_{0}} \left( e^{C_{t}E_{\nu}(t)} \right) + \lambda \int_{0}^{T_{0}} D_{\nu}(t) dt \lesssim T_{0} \sum_{|\alpha|+|\beta| \leq 3} \sup_{0 \leq t \leq T_{0}} \|w_{\nu,\mu}(\alpha,\beta)\partial_{\beta}^{3} h\|_{L_{x,v}^{2}}^{2}.
\end{equation}

Similar to (4.30), we can obtain
\begin{equation}
\sup_{0 \leq t \leq T_{0}} \|\partial h\|_{L_{\infty}^{\infty}} \lesssim \sup_{0 \leq t \leq T_{0}} \sqrt{E(t)} \lesssim T_{0}^{1/2} \sup_{0 \leq t \leq T_{0}} \|w_{\nu,\mu}(\alpha,\beta)\partial_{\beta}^{3} h\|_{L_{x,v}^{2}}^{2}.
\end{equation}

The above two estimates implies (5.4). This completes Lemma 5.1.

\textbf{Proof of Theorem 5.1.} Write $(f_{0}^{\varepsilon}, E_{0}^{\varepsilon})$ to be the mollification of $(f_{0}, E_{0})$ as the following. Let $\eta_{v}$ and $\eta_{x}$ be the standard mollifier in $\mathbb{R}^{3}$ and $\Omega$: $\eta_{v}, \eta_{x} \in C_{c}^{\infty}$, $0 \leq \eta_{v}, \eta_{x} \leq 1$, $\int \zeta_{v} dv = \int \zeta_{x} dx = 1$. For $\varepsilon > 0$, let $\eta_{v}^{\varepsilon}(v) = \varepsilon^{-3} \zeta_{v}(\varepsilon^{-1} v)$ and $\eta_{x}^{\varepsilon}(x) = \varepsilon^{-3} \zeta_{x}(\varepsilon^{-1} x)$. Then we mollify the initial data as $f_{0}^{\varepsilon} = f_{0} * \eta_{v}^{\varepsilon} * \eta_{x}^{\varepsilon}$, $E_{0}^{\varepsilon} = E_{0} * \eta_{x}^{\varepsilon}$. Then
\begin{equation}
\|\partial_{\beta}^{a} f_{0}^{\varepsilon}\|_{L_{x,v}^{2}} \lesssim \|\partial_{\beta}^{a} f_{0} * \eta_{v}^{\varepsilon} * \eta_{x}^{\varepsilon}\|_{L_{x,v}^{2}} \lesssim \|\eta_{v}\|_{L_{x}^{2}} \|\eta_{x}\|_{L_{x}^{2}} \|\partial_{\beta}^{a} f_{0}\|_{L_{x,v}^{2}} \lesssim \|\partial_{\beta}^{a} f_{0}\|_{L_{x,v}^{2}},
\end{equation}
and similarly,
\begin{equation}
\|\partial^{a} E_{0}^{\varepsilon}\|_{L_{x}^{2}} \lesssim \|\partial^{a} E_{0}\|_{L_{x}^{2}}.
\end{equation}

Also, $f_{0}^{\varepsilon} \rightarrow f_{0}^{0}$ and $E_{0}^{\varepsilon} \rightarrow E_{0}$ in $L_{x,v}^{2}$ and $L_{x}^{2}$ respectively as $\varepsilon \rightarrow 0$. We now construct the approximation solution sequence as
\begin{equation}
\{(f^{n}(t, x, v), \phi^{n}(t, x))\}_{n=0}^{\infty}
\end{equation}
by using the following iterative scheme:
\begin{equation}
\begin{cases}
\partial_{t} f_{\pm}^{n+1} + v \cdot \nabla x f_{\pm}^{n+1} + \frac{1}{2} \nabla x \phi^{n} \cdot v f_{\pm}^{n+1} \mp \nabla x \phi^{n} \cdot \nabla v f_{\pm}^{n+1} \\
\pm \Delta x \phi^{n+1} = \int_{\mathbb{R}^{3}} (f_{+}^{n+1} - f_{-}^{n+1}) \mu_{1/2} dv, \\
f_{n+1}(0, x, v) = f_{0}^{\frac{1}{n+1}}(x, v), \quad E^{n+1}(0, x) = E_{0}^{\frac{1}{n+1}}(x), \\
f_{n+1}(t, x, R_{v} x) = f_{n+1}(t, x, v), \quad \text{on } \gamma_{-}, \\
\partial_{\nu} \phi^{n+1} = 0, \quad \text{on } x \in \partial \Omega,
\end{cases}
\end{equation}
for $n = 0, 1, 2, \ldots$, where we set $f_{0}(t, x, v) = f_{0}(x, v)$ and $\phi^{0}$ given by $-\Delta x \phi^{0} = \int_{\mathbb{R}^{3}} (f_{+}^{0} - f_{-}^{0}) \mu_{1/2} dv$ and $\partial_{\nu} \phi^{0} = 0$ on $\partial \Omega$. With Lemma 5.2, it is a standard procedure to apply the induction argument to show that there exists $\varepsilon_{0} > 0$ and $T_{0} > 0$ such that if
\begin{equation}
\sum_{|\alpha| \leq 3} (\|w^{\alpha} f_{0}\|_{L_{x,v}^{2}}^{2} + \|\partial^{a} E_{0}\|_{L_{x}^{2}}^{2}) \leq \varepsilon_{0},
\end{equation}

for some constant $C > 0$. Since (5.2) is a linear equation, with (5.6) in hand, it’s standard to apply the theory for linear evolution equation to find the local-in-time existence for (5.2). In fact, we can obtain the local-in-time solution $(f, \phi)$ to (5.2) with estimate: for some $T_{0} > 0$,
then the approximate solution sequence \( \{f^n\} \) is well-defined with estimate
\[
\sup_{0 \leq t \leq T_0} \mathcal{E}_\nu(f^n, t) + \int_0^{T_0} \mathcal{D}_\nu(f^n, t) \, dt + \|\partial_t \phi^n\|_{L_t^\infty L_x^\infty} \\
\lesssim \sum_{|\alpha| + |\beta| \leq 3} \left( \|w_{I, \nu}(\alpha, \beta) \partial_\beta \phi f_0\|_{L_x^2}^2 + \|\partial^\alpha E_0\|_{L_x^2}^2 \right) + T_0^{1/2} \sum_{|\alpha| + |\beta| \leq 3} \|w_{I, \nu}(\alpha, \beta) \partial_\beta f^{n-1}\|_{L_t^\infty L_x^2}^2 \\
\lesssim \sum_{k=0}^n T_0^{k/2} \sum_{|\alpha| + |\beta| \leq 3} \left( \|w_{I, \nu}(\alpha, \beta) \partial_\beta f_0\|_{L_x^2}^2 + \|\partial^\alpha E_0\|_{L_x^2}^2 \right) \lesssim \varepsilon_0^2, \tag{5.7}
\]
by choosing \( T_0 > 0 \) small enough, where we write \( \mathcal{E}_\nu(f^n, t) \) and \( \mathcal{D}_\nu(f^n, t) \) to show the dependence on \( f^n \). Notice that \( f^{n+1} - f^n \) solves
\[
\partial_t (f^{n+1}_\pm - f^n_\pm) + v \cdot \nabla_x (f^{n+1}_\pm - f^n_\pm) \pm \frac{1}{2} \nabla_x \phi^n \cdot v (f^{n+1}_\pm - f^n_\pm) \pm \frac{1}{2} \left( \nabla_x \phi^n - \nabla_x \phi^{n-1} \right) \cdot \nu f^n_\pm \\
\mp \nabla_x \phi^n \cdot \nabla_v (f^{n+1}_\pm - f^n_\pm) \mp \nabla_v f^n_\pm \pm \nabla_x \phi^n (f^{n+1}_\pm - f^n_\pm) \cdot \nu \mu^{1/2} \\
- A_\pm (f^{n+1}_\pm - f^n_\pm) = \Gamma_\pm (f^n_\pm, f^{n+1}_\pm - f^n_\pm) + \Gamma_\pm (f^n_\pm - f^{n-1}_\pm, f^n_\pm) + K(f^n_\pm - f^{n-1}_\pm),
\]
for \( n = 1, 2, 3, \cdots \). Using the method for deriving (5.4) and (5.7); see also [22], we know that \( f^{n+1} - f^n \) is Cauchy sequence with estimate
\[
\sup_{0 \leq t \leq T_0} \mathcal{E}_\nu(f^{n+1}_\pm - f^n_\pm, t) + \int_0^{T_0} \mathcal{D}_\nu(f^{n+1}_\pm - f^n_\pm, t) \, dt \\
\lesssim \sum_{|\alpha| + |\beta| \leq 3} \left( \|w_{I, \nu}(\alpha, \beta) \partial_\beta \phi (f^{n+1}_\pm - f^n_\pm)\|_{L_x^2}^2 + \|\partial^\alpha (E^{n+1}_0 - E^n_0)\|_{L_x^2}^2 \right) \to 0, \text{ as } n \to \infty.
\]
Then the limit function \( f(t, x, v) \) is indeed a unique local-in-time solution to (1.3), (1.7) and (1.8) satisfying estimate (5.1). For the positivity, we refer to the argument from [22, Lemma 12, page 800]; the details are omitted for brevity. The proof of Theorem 5.1 is complete. \( \Box \)

**Acknowledgements.** D.-Q Deng was supported by Direct Grant from BIMSA.

**References.**

[1] Robert Adams and John Fournier, *Sobolev Spaces*, Elsevier LTD, Oxford, 2003.
[2] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang, *The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential*, J. Funct. Anal. 262 (2012), no. 3, 915–1010.
[3] Radjesvarane Alexandre, Frédéric Hérau, and Wei-Xi Li, *Global hypelliptic and symbolic estimates for the linearized Boltzmann operator without angular cutoff*, J. Math. Pures Appl. 126 (2019), 1–71.
[4] Yunbai Cao, Chanwoo Kim, and Douchyun Lee, *Global Strong Solutions of the Vlasov–Poisson–Boltzmann System in Bounded Domains*, Arch. Ration. Mech. Anal. 233 (2019), no. 3, 1027–1130.
[5] Carlo Cercignani, *On the initial-boundary value problem for the Boltzmann equation*, Arch. Ration. Mech. Anal. 116 (1992), no. 4, 307–315.
[6] Philippe G. Ciarlet, *Linear and Nonlinear Functional Analysis with Applications: With 401 Problems and 52 Figures*, CAMBRIDGE, 2013.
[7] Dingqun Deng, *The Landau and Non-cutoff Boltzmann Equation in Union of Cubes*, Preprint.
[8] , *Dissipation and Semigroup on \( H^1_x \): Non-cutoff Linearized Boltzmann Operator with Soft Potential*, SIAM J. Math. Anal. 52 (2020), no. 3, 3093–3113.
[9] , *Regularity of the Vlasov–Poisson–Boltzmann System Without Angular Cutoff*, Comm. Math. Phys. 387 (2021), no. 3, 1603–1654.
[10] R. J. Diperna and P. L. Lions, *On the Cauchy Problem for Boltzmann Equations: Global Existence and Weak Stability*, Ann. of Math. 130 (1989), no. 2, 321.
[11] Houjie Dong, Yan Guo, and Zhimeng Ouyang, *The Vlasov-Poisson-Landau System with the Specular-Reflection Boundary Condition*.  


[12] Renjun Duan and Shuangqian Liu, *The Vlasov-Poisson-Boltzmann System without Angular Cutoff*, Comm. Math. Phys. 324 (2013), no. 1, 1–45.

[13] Renjun Duan, Shuangqian Liu, Shota Sakamoto, and Robert M. Strain, *Global Mild Solutions of the Landau and Non-Cutoff Boltzmann Equations*, Comm. Pure Appl. Math. 74 (2020), no. 5, 932–1020.

[14] Renjun Duan, Shuangqian Liu, Tong Yang, and Huijiang Zhao, *Stability of the nonrelativistic Vlasov-Maxwell-Boltzmann system for angular non-cutoff potentials*, Kinetic & Related Models 6 (2013), no. 1, 159–204.

[15] R. Esposito, Y. Guo, C. Kim, and R. Marra, *Non-Isothermal Boundary in the Boltzmann Theory and Fourier Law*, Comm. Math. Phys. 323 (2013), no. 1, 177–239.

[16] Yingzhe Fan, Yuanjie Lei, Shuangqian Liu, and Huijiang Zhao, *The non-cutoff Vlasov-Maxwell-Boltzmann system with weak angular singularity*, Science China Mathematics 61 (2017), no. 1, 111–136.

[17] Philip T. Gressman and Robert M. Strain, *Global classical solutions of the Boltzmann equation without angular cut-off*, J. Amer. Math. Soc. 24 (2011), no. 3, 771–771.

[18] P Grisvard, *Elliptic problems in nonsmooth domains*, Pitman Advanced Pub. Program, Boston, 1985.

[19] Yan Guo, *The Landau Equation in a Periodic Box*, Comm. Math. Phys. 231 (2002), no. 3, 391–434.

[20] ______, *The Vlasov-Poisson-Boltzmann system near Maxwellians*, Comm. Pure Appl. Math. 55 (2002), no. 9, 1104–1135.

[21] ______, *Decay and Continuity of the Boltzmann Equation in Bounded Domains*, Arch. Ration. Mech. Anal. 197 (2009), no. 3, 713–809.

[22] ______, *The Vlasov-Poisson-Landau system in a periodic box*, J. Amer. Math. Soc. 25 (2012), no. 3, 759–812.

[23] Yan Guo, Hyung Ju Hwang, Jin Woo Jang, and Zhimeng Ouyang, *The Landau Equation with the Specular Reflection Boundary Condition*, Arch. Ration. Mech. Anal. 236 (2020), no. 3, 1389–1454.

[24] Yan Guo, Chanwoo Kim, Daniela Tonon, and Ariane Trescases, *Regularity of the Boltzmann equation in convex domains*, Invent. Math. 207 (2016), no. 1, 115–290.

[25] K. Hamdache, *Initial-Boundary value problems for the Boltzmann equation: Global existence of weak solutions*, Arch. Ration. Mech. Anal. 119 (1992), no. 4, 309–353.

[26] Chanwoo Kim and Donghyun Lee, *The Boltzmann Equation with Specular Boundary Condition in Convex Domains*, Comm. Pure Appl. Math. 71 (2017), no. 3, 411–504.

[27] Nicolas Lerner, *Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators*, Birkhäuser Basel, Switzerland, 2010.

[28] Shuangqian Liu and Xiongfeng Yang, *The Initial Boundary Value Problem for the Boltzmann Equation with Soft Potential*, Arch. Ration. Mech. Anal. 223 (2016), no. 1, 463–541.

[29] Tai-Ping Liu and Shih-Hsien Yu, *Initial-boundary value problem for one-dimensional wave solutions of the Boltzmann equation*, Comm. Pure Appl. Math. 60 (2006), no. 3, 295–356.

[30] Stéphane Mischler, *On the Initial Boundary Value Problem for the Vlasov-Poisson-Boltzmann System*, Comm. Math. Phys. 210 (2000), no. 2, 447–466.

[31] Robert M. Strain and Yan Guo, *Exponential Decay for Soft Potentials near Maxwellian*, Arch. Ration. Mech. Anal. 187 (2007), no. 2, 287–339.

[32] Tong Yang and Hongjun Yu, *Spectrum Analysis of Some Kinetic Equations*, Arch. Ration. Mech. Anal. 222 (2016), no. 2, 731–768.

[33] Tong Yang and Hui-Jiang Zhao, *A Half-space Problem for the Boltzmann Equation with Specular Reflection Boundary Condition*, Comm. Math. Phys. 255 (2005), no. 3, 683–726.