Large singular solutions for conformal $Q$-curvature equations on $\mathbb{S}^n$

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September 25, 2020

Abstract

In this paper, we study the existence of positive functions $K \in C^1(\mathbb{S}^n)$ such that the conformal $Q$-curvature equation

$$P_m(v) = K v^{\frac{n+2m}{n-2m}} \quad \text{on } \mathbb{S}^n$$

has a singular positive solution $v$ whose singular set is a single point, where $m$ is an integer satisfying $1 \leq m < n/2$ and $P_m$ is the intertwining operator of order $2m$. More specifically, we show that when $n \geq 2m + 4$, every positive function in $C^1(\mathbb{S}^n)$ can be approximated in the $C^1(\mathbb{S}^n)$ norm by a positive function $K \in C^1(\mathbb{S}^n)$ such that (1) has a singular positive solution whose singular set is a single point. Moreover, such a solution can be constructed to be arbitrarily large near its singularity. This is in contrast to the well-known results of Lin [24] and Wei-Xu [36] which show that Eq. (1), with $K$ identically a positive constant on $\mathbb{S}^n$, $n > 2m$, does not exist a singular positive solution whose singular set is a single point.

Keywords: Conformal $Q$-curvature equations; Isolated singularity; Large singular solutions

Mathematics Subject Classification (2010): 35J30 ; 53C21

1 Introduction

Let $\mathbb{S}^n$ be the $n$-dimensional unit sphere endowed with the induced metric $g_{\mathbb{S}^n}$ from $\mathbb{R}^{n+1}$. The aim of this paper is to study the existence of positive functions $K \in C^1(\mathbb{S}^n)$ such that the conformal $Q$-curvature equation

$$P_m(v) = K v^{\frac{n+2m}{n-2m}} \quad \text{on } \mathbb{S}^n$$

has a singular positive solution $v$ whose singular set consists of a single point, where $m$ is an integer, $1 \leq m < n/2$, and $P_m$ is an intertwining operator (see, e.g., Branson [3]) of order $2m$. 

1
given by

$$P_m = \frac{\Gamma(B + \frac{1}{2} + m)}{\Gamma(B + \frac{1}{2} - m)}, \quad B = \sqrt{-\Delta_{g^{mn}} + \left(\frac{n-1}{2}\right)^2}$$ (3)

with $\Gamma$ being the Gamma function and $\Delta_{g^{mn}}$ being the Laplace-Beltrami operator on $(\mathbb{S}^n, g^{mn})$. The operator $P_m$ can be seen more concretely on $\mathbb{R}^n$ via the stereographic projection. Let $N$ be the north pole of $\mathbb{S}^n$ and $F$ be the inverse of the stereographic projection:

$$F : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}, \quad x \mapsto \left(\frac{2x}{1 + |x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1}\right).$$

Then it follows from the conformal invariance of $P_m$ that

$$(P_m(\phi)) \circ F = |J_F|^{-\frac{n+2m}{2n}} (-\Delta)^m |J_F|^{-\frac{n-2m}{2n}} (\phi \circ F) \quad \text{for } \phi \in C^\infty(\mathbb{S}^n),$$ (4)

where $\Delta$ is the Laplacian operator on $\mathbb{R}^n$ and $|J_F|$ is the determinant of the Jacobian of $F$, i.e.,

$$|J_F| = \left(\frac{2}{1 + |x|^2}\right)^n.$$

When $m = 1$, $P_1 = -\Delta_{g^{mn}} + \frac{n-2}{4(n-1)} R_0$ is the well-known conformal Laplacian associated with the metric $g^{mn}$, where $R_0 = n(n-1)$ is the scalar curvature of $(\mathbb{S}^n, g^{mn})$. In this case, the equation (2) reads as

$$-\Delta_{g^{mn}} v + \frac{n(n-2)}{4} v = K v^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{S}^n, \quad n \geq 3,$$ (5)

which is usually called the conformal scalar curvature equation. Equation (5) naturally arises in the study of the classical Nirenberg problem that asks: Which function $K$ on $(\mathbb{S}^n, g^{mn})$ is the scalar curvature of a metric $g$ that is conformal to $g^{mn}$? There have been many papers on this problem, see, for example, [1,5,6,9,10,22,23,31,37] and the references therein. Furthermore, the classical works of Schoen and Yau [29, 30] on conformally flat manifolds have indicated the importance of studying singular positive solutions of (5). When $K$ is identically a positive constant on $\mathbb{S}^n$, Caffarelli, Gidas and Spruck [4] proved that Eq. (5) does not have a singular positive solution whose singular set consists of a single point, while Schoen [29] constructed a singular positive solution of (5) whose singular set is any prescribed finite collection of at least two points. Mazzeo and Pacard in [26] provided another construction method for Schoen’s result. When $K$ is a non-constant positive function, Taliaferro [33] studied the existence of $K \in C^1(\mathbb{S}^n)$ such that (5) has a singular positive solution $v$ whose singular set is a single point in dimension $n \geq 6$, and the solution $v$ can be constructed to be arbitrarily large near its singularity. This shows that there does not exist an a priori estimate on the blow up rate of such a solution $v$ near its singularity. On the other hand, Chen and Lin in a series of papers [7, 8, 25] studied, among other things, that under what assumptions on $K$ a singular positive solution $v$ of (5) near its singularity $\xi_0$ satisfies the following a priori estimate

$$v(\xi) = O(|\xi - \xi_0|^{-\frac{n+2}{2}}).$$

We may also see [35, 39] for the similar a priori estimates of singular positive solutions of (5).
When $m = 2$, $P_2$ is the fourth order conformally invariant Paneitz operator associated with the metric $g_{\mathbb{S}^n}$. For a smooth compact $n$-dimensional Riemannian manifold $(M, g)$ with $n \geq 4$, the Branson’s $Q$-curvature (see Branson [2]) is given by

$$Q_g = -\frac{1}{2(n-1)} \Delta_g R_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2 - \frac{2}{(n-2)^2} |Ric_g|^2,$$

where $R_g$ and $Ric_g$ denote the scalar curvature and the Ricci tensor of $g$ respectively. The fourth order Paneitz operator (see [2, 27]) with respect to the metric $g$ is defined by

$$P_2^g = \Delta^2_g - \text{div}_g (a_n R_g g + b_n Ric_g) d + \frac{n-4}{2} Q_g$$

with $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$ and $b_n = -\frac{4}{n-2}$. When $n \geq 5$, the operator $P_2^g$ is conformally invariant: if $\tilde{g} = v^{n-4} g$ is a conformal metric to $g$, then for all $\phi \in C^\infty(M)$ we have

$$P_2^g(v\phi) = v^{\frac{n+4}{n-4}} P_2^\tilde{g}(\phi).$$

As a consequence one has the following conformal transformation law

$$P_2^g(v) = \frac{n-4}{2} Q_{\tilde{g}} v^{\frac{n+4}{n-4}} , \quad \tilde{g} = v^{\frac{4}{n-4}} g.$$  \tag{6}

Therefore, as in the Nirenberg problem, the problem of prescribing $Q$-curvature on $(\mathbb{S}^n, g_{\mathbb{S}^n})$ for $n \geq 5$ can be reduced to the study of existence of positive solutions to Eq. (2) with $m = 2$. The problem has been studied in [11–14, 28] and many others. More generally, higher order conformally invariant differential operators on Riemannian manifolds and their associated $Q$-curvatures have also been studied in [15, 16, 19] and so on. In particular, the operator $P_m$ (or more precisely, $P_m^{g_{\mathbb{S}^n}}$) on $\mathbb{S}^n$ involved in Eq. (2) is the most typical example, and prescribing its associated higher order $Q$-curvature on $\mathbb{S}^n$ is naturally reduced to the study of Eq. (2) with $m \geq 3$. One may see the work of Jin-Li-Xiong [17] for a unified approach for all $m \in (0, n/2)$.

When $K$ is identically a positive constant on $\mathbb{S}^n$, the fact that no solution of Eq. (2) exists which is singular at one point has been shown by Lin [24] for $m = 2$ and by Wei-Xu [36] for $m \geq 3$. In this paper, we study the existence of positive functions $K \in C^1(\mathbb{S}^n)$ such that Eq. (2) has a singular positive solution $v$ whose singular set consists of a single point when $m \geq 2$. Clearly this will be in contrast to the results of Lin [24] and Wei-Xu [36]. Moreover, we also investigate whether such a solution $v$ can always be constructed to be arbitrarily large near its singularity. This means that, for any given large continuous function $\varphi : (0, 1) \to (0, \infty)$ whether such a solution $v$ can be constructed to satisfy

$$v(\xi) \neq O(\varphi(|\xi - \xi_0|)) \quad \text{as} \quad \xi \to \xi_0,$$

where $\{\xi_0\}$ is the singular set of $v$ on $\mathbb{S}^n$.

The main result of this paper is the following theorem.

**Theorem 1.1.** Let $\varepsilon$ be a positive number and let $\varphi : (0, 1) \to (0, \infty)$ be a continuous function. Suppose that $k \in C^1(\mathbb{S}^n)$ is a positive function, $m \geq 1$ is an integer and $n \geq 2m + 4$. Then there exists $\xi_0 \in \mathbb{S}^n$, and a positive function $K \in C^1(\mathbb{S}^n)$ satisfying

$$\|K - k\|_{C^1(\mathbb{S}^n)} < \varepsilon \quad \text{and} \quad K(\xi) = k(\xi) \quad \text{for} \quad |\xi - \xi_0| \geq \varepsilon$$  \tag{8}

Therefore, as in the Nirenberg problem, the problem of prescribing $Q$-curvature on $(\mathbb{S}^n, g_{\mathbb{S}^n})$ for $n \geq 5$ can be reduced to the study of existence of positive solutions to Eq. (2) with $m = 2$. The problem has been studied in [11–14, 28] and many others. More generally, higher order conformally invariant differential operators on Riemannian manifolds and their associated $Q$-curvatures have also been studied in [15, 16, 19] and so on. In particular, the operator $P_m$ (or more precisely, $P_m^{g_{\mathbb{S}^n}}$) on $\mathbb{S}^n$ involved in Eq. (2) is the most typical example, and prescribing its associated higher order $Q$-curvature on $\mathbb{S}^n$ is naturally reduced to the study of Eq. (2) with $m \geq 3$. One may see the work of Jin-Li-Xiong [17] for a unified approach for all $m \in (0, n/2)$.
such that Eq. (2) has a positive solution $v \in C^{2m}(\mathbb{S}^n \setminus \{\xi_0\})$ satisfying
\[
v(\xi) \neq O(\varphi(|\xi - \xi_0|)) \quad \text{as } \xi \to \xi_0.
\] (9)

**Remark 1.2.** The solution $v$ constructed in Theorem 1.1 is a distributional solution of (2) on the whole sphere $\mathbb{S}^n$. That is, $v \in L^{n+2m}(\mathbb{S}^n)$ and $v$ satisfies
\[
\int_{\mathbb{S}^n} vP_m(\phi) = \int_{\mathbb{S}^n} Kv^n-m \phi \quad \text{for all } \phi \in C^\infty(\mathbb{S}^n).
\] (10)

At first glance, the above statement seems impossible since the solution $v$ in Theorem 1.1 satisfies (9) where no bound is imposed on the size of $\varphi$ near 0. However this is not the case. Indeed, if $v$ is a $C^{2m}$ positive solution of Eq. (2) in some punctured neighborhood $\Omega \setminus \{\xi_0\}$ of some point $\xi_0 \in \mathbb{S}^n$, then $v \in L^{n+2m}_{\text{loc}}(\Omega)$ and $v$ is a distributional solution of (2) in $\Omega$. See, e.g., [18, 38].

To prove Theorem 1.1, take $\xi_0 \in \mathbb{S}^n$ such that $\nabla k(\xi_0) = 0$ and let $\pi$ be the stereographic projection of $\mathbb{S}^n$ onto $\mathbb{R}^n \cup \{\infty\}$ with $\xi_0$ being the south pole. Then $v$ is a positive solution of (2) with singular set $\{\xi_0\}$ if and only if
\[
u(x) := \left(\frac{2}{1 + |x|^2}\right)^{n-2m} v(\pi^{-1}(x)), \quad x \in \mathbb{R}^n \setminus \{0\}
\]
is a positive solution of
\[
\begin{cases}
(-\Delta)^m u = K(x)u^{n+2m-2m} & \text{in } \mathbb{R}^n \setminus \{0\}, \\
u(x) \neq O(1) & \text{as } |x| \to 0^+,
\end{cases}
\]
\[
u(x) = O(|x|^{2m-n}) & \text{as } |x| \to \infty.
\]

Therefore, to show Theorem 1.1, it suffices to establish the following result on the equation
\[
(-\Delta)^m u = K(x)u^{n+2m-2m} \quad \text{in } \mathbb{R}^n \setminus \{0\}. \quad (11)
\]

**Theorem 1.3.** Suppose that $m \geq 1$ is an integer and $n \geq 2m + 4$. Let $k : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$ function which is bounded between two positive constants and satisfies $\nabla k(0) = 0$. Let $\varepsilon$ be a positive number and $\varphi : (0, 1) \to (0, \infty)$ be a continuous function. Then there exists a $C^1$ positive function $K : \mathbb{R}^n \to \mathbb{R}$ satisfying $\nabla K(0) = 0$, $K(0) = k(0)$, $K(x) = k(x)$ for $|x| \geq \varepsilon$ and
\[
\|K - k\|_{C^1(\mathbb{R}^n)} < \varepsilon \quad (12)
\]
such that Eq. (11) has a $C^{2m}$ positive solution $u$ satisfying
\[
u(x) \neq O(\varphi(|x|)) \quad \text{as } |x| \to 0^+, \quad (13)
\]
\[
u(x) = O(|x|^{2m-n}) \quad \text{as } |x| \to \infty, \quad (14)
\]
and
\[
(-\Delta)^s u > 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \quad s = 1, 2, \ldots, m - 1. \quad (15)
\]
Remark 1.4. When \( m \geq 2 \) and \( n = 2m + 1, 2m + 2 \) or \( 2m + 3 \), the existence of such \( K \) is still unknown. When \( m = 1 \), Theorem 1.3 is not true in dimension \( n = 3 \) (see Chen-Lin [7]) or \( n = 4 \) (see Taliaferro-Zhang [35]), but its validity in dimension \( n = 5 \) is unknown.

For when \( m = 1 \), Theorem 1.3 was proved by Taliaferro [33]. In the case of \( m = 1 \), we also mention that Taliaferro [32] proved the existence of positive functions \( K \in C(\mathbb{R}^n \setminus \{0\}) \) with \( 0 < a < K(x) < b \) in \( \mathbb{R}^n \setminus \{0\} \) such that Eq. (11) has a \( C^2 \) positive solution \( u \) satisfying (13) in dimension \( n \geq 3 \), where \( a \) and \( b \) are given positive constants satisfying \( b/a > \sqrt{2/(n-2)} \), and Taliaferro-Zhang [34] proved the existence of positive continuous functions \( K \in C(\mathbb{R}^n) \) such that Eq. (11) has a \( C^2 \) positive solution \( u \) not satisfying \( u(x) = O(|x|^{-(n-2)/2}) \) near the origin.

On the other direction, Jin and Xiong in [18] recently proved that every positive solution \( u \) of the equation

\[
\begin{cases}
(\Delta)^m u = u^{n+2m}/(n-2m), & n > 2m \\
(\Delta)^s u \geq 0, & s = 1, 2, \ldots, m - 1
\end{cases}
\]

in \( B_1 \setminus \{0\} \) (16) satisfies the a priori bound

\[
u(x) = O\left(|x|^{-n+2m}/2\right) \quad \text{as } |x| \to 0^+.
\]

Moreover, they also showed that every positive solution \( u \) of (16) is asymptotically radially symmetric near the origin. Clearly, the conclusion of Theorem 1.3 that the solution \( u \) can be constructed to satisfy (13) and (15) is also in contrast to the result of Jin and Xiong.

We will prove Theorem 1.3 in the next section. Our proof adapts that of Taliaferro [33] when \( m = 1 \). The basic idea is as follows. Without loss of generality we may assume \( k(0) = 1 \). Let \( w_\lambda(x) = c_{n,m} \left( \frac{\lambda}{\lambda^2 + |x|^2} \right)^{n-2m}/2 \)

be a smooth positive solution (which is usually called a bubble) of the equation \( (-\Delta)^m u = u^{n+2m}/(n-2m) \) on \( \mathbb{R}^n \) for a positive constant \( c_{n,m} > 0 \) and for each \( \lambda > 0 \). Notice that as \( \lambda \to 0^+ \), \( w_\lambda(x) \) and each of its partial derivatives with respect to \( x \) converge uniformly to zero on every closed subset of \( \mathbb{R}^n \setminus \{0\} \) and \( w_\lambda(0) \) tends to \( +\infty \). Define \( u_i(x) = w_\lambda(x - x_i) \), where \( \{x_i\}_{i=1}^\infty \) is a sequence of distinct points in \( B_\delta \setminus \{0\} \) for some small \( \delta \in (0, \varepsilon) \) which tends to the origin and \( \{\lambda_i\}_{i=1}^\infty \) is a sequence of positive numbers which tends sufficiently fast to \( 0 \). Then the function \( \tilde{u} := \sum_{i=1}^\infty u_i \in C(\mathbb{R}^n \setminus \{0\}) \) would satisfy \( \tilde{u}(x) \neq O(\varphi(|x|)) \) as \( |x| \to 0^+ \) and approximately satisfy

\[
(-\Delta)^m \tilde{u} = k(x)\tilde{u}^{n+2m}/(n-2m) \quad \text{in } B_\delta \setminus \{0\}.
\]

We then construct an appropriate positive bounded function \( u_0 \in C^{2m}(\mathbb{R}^n \setminus \{0\}) \) such that

\[
u := u_0 + \tilde{u} \quad \text{and} \quad K := \frac{(-\Delta)^m u}{u^{n+2m}/(n-2m)}
\]

satisfy the conclusion of Theorem 1.3. To this end, the sequences \( \{x_i\} \) and \( \{\lambda_i\} \) need to be selected very carefully. We will check at the end of the proof that \( K \) defined in (18) is \( C^1 \) on
the whole space \( \mathbb{R}^n \), where it becomes clear why we need \( n \geq 2m + 4 \). For the higher order equation (11), we need to establish several more delicate estimates on the gradient of \( K \). Indeed, if one completely follows the estimates for the gradient of \( K \) in Taliaferro [33], then the stronger condition \( n \geq 6m \) will be required.

**Acknowledgments.** Both authors would like to thank Prof. Tianling Jin for many helpful discussions and encouragement.

## 2 Proof of Theorem 1.3

We will use \( B_r(x) \) to denote the open ball of radius \( r \) in \( \mathbb{R}^n \) with center \( x \) and write \( B_r(0) \) as \( B_r \) for short. We write \( a_i \sim b_i \) if the sequence \( \{a_i/b_i\}_{i=1}^\infty \) is bounded between two positive constants depending only on \( n, m, \inf_{\mathbb{R}^n} k \) and \( \sup_{\mathbb{R}^n} k \). To prove Theorem 1.3, we also need the following simple lemma.

**Lemma 2.1.** ([33]) Suppose \( \lambda > 1 \), \( \{a_i\}_{i=1}^N \subset (0, \infty) \), and \( a_1 \geq a_i \) for \( 2 \leq i \leq N \). Then
\[
\frac{\sum_{i=1}^N a_i^\lambda}{\left(\sum_{i=1}^N a_i\right)^\lambda} \leq \frac{1 + \frac{a_2}{a_1}}{1 + \lambda \frac{a_2}{a_1}} < 1.
\]

**Proof of Theorem 1.3.** The proof consists of six steps.

**Step 0. Preliminaries.** Without loss of generality, we can assume that \( 0 < \varepsilon < 1 \) and \( k(0) = 1 \). Since \( \nabla k(0) = 0 \), there exists a \( C^1 \) positive function \( \tilde{k} : \mathbb{R}^n \to \mathbb{R} \) such that \( \tilde{k} \equiv 1 \) in a small neighborhood of the origin, \( \tilde{k}(x) = k(x) \) for \( |x| \geq \varepsilon \) and \( \|\tilde{k} - k\|_{C^1(\mathbb{R}^n)} < \varepsilon/2 \). Replacing \( k \) by \( \tilde{k} \), we can assume that \( k \equiv 1 \) in \( B_{\delta} \) for some \( 0 < \delta < \varepsilon \).

Let
\[
\psi(r, \lambda) = c_{n,m} \left(\frac{\lambda}{\lambda^2 + r^2}\right)^{\frac{n-2m}{2m}}
\]
with \( c_{n,m} = \left(\frac{(n+2m-2)!/(n-2m-2)!}{(n-2m)!}\right)^{\frac{n-2m}{4m}} \). It is easy to check that for every \( \lambda > 0 \), the function \( w_\lambda(x) := \psi(|x|, \lambda) \) satisfies
\[
\begin{align*}
(-\Delta)^m w_\lambda &= w_\lambda^{\frac{n-2m}{n+2m}} \\
(-\Delta)^s w_\lambda &> 0, \quad s = 1, 2, \ldots, m-1 \quad \text{in } \mathbb{R}^n.
\end{align*}
\]

After some calculations, one can find that there exist \( \delta_1 \) and \( \delta_2 \) satisfying
\[
0 < \delta_2 < \frac{\delta_1}{2} < \frac{\delta}{4}
\]
and for any \( |x| \leq \delta_2 \) or \( |x| \geq \delta \),
\[
\frac{1}{2} \left< \frac{w_\lambda(x-x_1)}{w_\lambda(x-x_2)} \right> < 2 \quad \text{when } |x_1| = |x_2| = \delta_1 \text{ and } 0 < \lambda \leq \delta_2.
\]
Recall that $k$ is bounded between two positive constants, we denote
\[
a = \frac{1}{2} \inf_{\mathbb{R}^n} k \quad \text{and} \quad b = \sup_{\mathbb{R}^n} k. \tag{21}
\]

Let $i_0$ be the smallest integer greater than 2 such that
\[
\frac{4m}{r_0^{n-4m}} > \frac{2^{3n+2m}}{(2\pi)^{\frac{n-2m}{4m}}}. \tag{22}
\]

Choose a sequence $\{x_i\}_{i=1}^{\infty}$ of distinct points in $\mathbb{R}^n$ and a sequence $\{r_i\}_{i=1}^{\infty}$ of positive numbers such that
\[
|x_1| = |x_2| = \cdots = |x_{i_0}| = \delta_1, \quad \lim_{i \to \infty} |x_i| = 0, \tag{23}
\]
\[
r_1 = r_2 = \cdots = r_{i_0} = \frac{\delta_2}{2}, \quad \lim_{i \to \infty} r_i = 0, \tag{24}
\]
\[
B_{4r_1}(x_i) \subset B_{\delta_2} \setminus \{0\} \quad \text{for } i > i_0 \tag{25}
\]
and
\[
\overline{B_{2r_1}(x_i)} \cap \overline{B_{2r_j}(x_j)} = \emptyset \quad \text{for } j > i > i_0. \tag{26}
\]

In addition, we require that the union of the line segments $x_1 x_2, x_2 x_3, \ldots, x_{i_0-1} x_{i_0}, x_{i_0} x_1$ be a regular $i_0$-gon. We will prescribe the side length of this polygon later. From (19), (23) and (24) we know that
\[
\overline{B_{2r_1}(x_i)} \subset B_{2\delta_1} \setminus \overline{B_{\delta_2}} \quad \text{for } 1 \leq i \leq i_0
\]
and hence by (25),
\[
\overline{B_{2r_1}(x_i)} \cap \overline{B_{2r_j}(x_j)} = \emptyset \quad \text{for } 1 \leq i \leq i_0 < j. \tag{27}
\]

Define three functions $f: [0, \infty) \times (0, \infty) \times (0, \infty) \to \mathbb{R}$ and $M, Z: (0, 1) \times (0, \infty) \to (0, \infty)$ by
\[
f(z_1, z_2, z_3) = z_2(z_1 + z_3) = \frac{n+2m}{n-2m} - \frac{n+2m}{n-2m},
\]
\[
M(z_2, z_3) = \frac{z_2 z_3}{1 - \frac{n-2m}{n-4m}} \quad \text{and} \quad Z(z_2, z_3) = \frac{z_3 z_2}{1 - \frac{n-2m}{n-4m}}. \tag{28}
\]

For each fixed $(z_2, z_3) \in (0, 1) \times (0, \infty)$, the function $f(\cdot, z_2, z_3): [0, \infty) \to \mathbb{R}$ is strictly increasing on $[0, Z(z_2, z_3)]$ and is strictly decreasing on $[Z(z_2, z_3), \infty)$, and attains its maximum value $M(z_2, z_3)$ at $z_1 = Z(z_2, z_3)$.

Define $F: [0, \infty) \times (0, \infty) \times (0, \infty) \to (0, \infty)$ by
\[
F(z_1, z_2, z_3) = \begin{cases} 
  f(z_1, z_2, z_3) & \text{if } 0 < z_2 < 1 \text{ and } 0 < z_1 \leq Z(z_2, z_3), \\
  M(z_2, z_3) & \text{if } 0 < z_2 < 1 \text{ and } z_1 > Z(z_2, z_3), \\
  f(z_1, z_2, z_3) & \text{if } z_2 \geq 1.
\end{cases}
\]

It is easy to see that $f$ and $F$ are $C^1$, $f \leq F$ and $F$ is non-decreasing in $z_1, z_2$ and $z_3$. 

7
Step 1. Selecting the sequences \( \{x_i\} \) and \( \{\lambda_i\} \). Let

\[
w(x) = (2b)^{-\frac{n}{2m}} \psi(\lambda |x|, 1) = \frac{c_{n,m}}{(2b)^{m}} \left(\frac{1}{1 + |x|^2}\right)^{\frac{n-2m}{2}} \quad \text{for } x \in \mathbb{R}^n.
\]

Then we have

\[
(-\Delta)^m w = (2b)^{\frac{2m}{n-2m}} w^{\frac{n+2m}{n-2m}} \quad \text{in } \mathbb{R}^n. \tag{29}
\]

Choose a sequence \( \{\varepsilon_i\}_{i=1}^{\infty} \) of positive numbers such that

\[
\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_i \quad \text{and} \quad \varepsilon_i \leq 2^{-i} \quad \text{for } i \geq 1. \tag{30}
\]

Now we introduce four sequences of real numbers as follows. For \( i \geq 1 \), let

\[
k_i \in \left(\frac{1}{2}, 1\right) \quad \text{with} \quad k_1 = k_2 = \cdots = k_i = \lambda_i,
\]

\[
M_i = \frac{M(k_i, 2w(0))}{(2w(0))^{\frac{n+2m}{n-2m}}} = \frac{k_i}{\left(1 - k_i^{-\frac{4m}{n-2m}}\right)^{\frac{4m}{n-2m}}},
\]

\[
\rho_i = \sup \left\{ \rho > 0 : I_{2m}(\chi_{B_{2\rho_i}(x_i)}) \leq \frac{w}{2^i + 1(2w(0))^{\frac{n+2m}{n-2m}} M_i} \right\},
\]

and

\[
\lambda_i = \sup \left\{ \lambda > 0 : w_\lambda(x - x_i) \leq \varepsilon_i a^{\frac{n-2m}{4m}} w(x) \quad \text{for } |x - x_i| \geq \rho_i \right\},
\]

where \( I_{2m} \) is the Riesz potential operator of order \( 2m \) and \( \chi_{B_{2\rho_i}(x_i)} \) is the characteristic function of the ball \( B_{2\rho_i}(x_i) \). Then we have

**Lemma 2.2.** For \( i \geq 1 \),

\[
M_i \sim \frac{1}{(1 - k_i^{-\frac{4m}{n-2m}})}, \quad \rho_i^{2m} \sim \frac{1}{2M_i} \quad \text{and} \quad \lambda_i \sim \varepsilon_i^{\frac{2m}{n-2m}} \rho_i^2.
\]

**Proof.** Since \( k_i \in (0, 1/2) \), the first estimate is obvious. For the second, we only need to notice that

\[
\frac{1}{C} \rho_i^{2m} \leq I_{2m}(\chi_{B_{2\rho_i}(x_i)}) \leq C \rho_i^{2m} \quad \text{for } |x - x_i| \leq 2\rho_i
\]

and

\[
\frac{1}{C} \rho_i^n |x - x_i|^{2m-n} \leq I_{2m}(\chi_{B_{2\rho_i}(x_i)}) \leq C \rho_i^n |x - x_i|^{2m-n} \quad \text{for } |x - x_i| \geq 2\rho_i
\]

for some positive constant \( C \) depending on \( n \) and \( m \).

For the last, the inequality \( w_\lambda(x - x_i) \leq \varepsilon_i a^{\frac{n-2m}{4m}} w(x) \) holds for \( |x - x_i| \geq \rho_i \) if and only if

\[
\lambda(1 + |x|^2) \leq \varepsilon_i^{\frac{2}{n-2m}} a^{\frac{1}{2m}} (2b)^{-\frac{n}{(m+n-2m)^2}} (\lambda^2 + |x - x_i|^2) \quad \text{for } |x - x_i| \geq \rho_i.
\]

8
After some rotations, the above inequality is equivalent to
\[ \varepsilon_i^{n-2m} a^{2n} 2b - m(n-2m) (\lambda^2 + |y'|^2 + |y_1 - |x_i||^2) \geq \lambda(1 + |y'|^2 + |y_1|^2), \]
where \( y_1 \in \mathbb{R}, y' \in \mathbb{R}^{n-1} \) and \( |y_1 - |x_i||^2 + |y'|^2 \geq \rho_i^2 \).
Comparing the coefficients of \(|y'|^2\), we get
\[ \varepsilon_i^{n-2m} a^{2n} 2b - m(n-2m) \geq \lambda. \]
We also need \( \lambda \) satisfying
\[ \varepsilon_i^{n-2m} a^{2n} 2b - m(n-2m) (\lambda^2 + |y_1 - |x_i||^2) \geq \lambda(1 + |y_1|^2) \geq 0 \quad \text{for} \quad |y_1 - |x_i|| \geq \rho_i. \]
This inequality holds if and only if
\[
\begin{cases}
\varepsilon_i^{n-2m} a^{2n} 2b - m(n-2m) \rho_i^2 \geq \lambda(1 + (|x_i| + \rho_i)^2 - \varepsilon_i^{n-2m} a^{2n} 2b - m(n-2m) \lambda), \\
\varepsilon_i^{n-2m} a^{2n} 2b - m(n-2m) \rho_i^2 \geq \lambda(1 + (|x_i| - \rho_i)^2 - \varepsilon_i^{n-2m} a^{2n} 2b - m(n-2m) \lambda).
\end{cases}
\]
Since both \( 1 + (|x_i| + \rho_i)^2 - \varepsilon_i^{n-2m} a^{2n} 2b - m(n-2m) \lambda \) and \( 1 + (|x_i| - \rho_i)^2 - \varepsilon_i^{n-2m} a^{2n} 2b - m(n-2m) \lambda \) are bounded between two positive constants depending only on \( n, m, a \) and \( b \), we get the desired estimate. \( \square \)

By Lemma 2.2, after increasing the values of \( k_i \) for certain values of \( i \) while holding \( \varepsilon_i \) fixed, we can assume for \( i \geq 1 \) that
\[ M_i > 3^4, \quad \rho_i \in (0, r_i), \quad \lambda_i \in (0, \delta_2), \]
and
\[ k_i^{a+2m} > \frac{1 + \left(\frac{2}{7}\right)^{n-2m}}{1 + \frac{n+2m}{m-2m} \left(\frac{2}{7}\right)^{n-2m}}, \quad M_i^a > \max \left\{ \frac{1}{\varepsilon_i^{n-2m}}, 2^i \right\}, \quad \lambda_i^2 < \frac{2^{m-2m}}{2^i}, \]
where \( \alpha = \alpha(m) \in (0, 1/6) \) is a constant to be specified later.

Notice that for \( 1 \leq i \leq i_0, \rho_i \) and \( \lambda_i \) do not change as \( x_i \) moves on the sphere \( |x| = \delta_1 \).
Therefore, we can require that the union of the line segments \( x_1 x_2, x_2 x_3, \ldots, x_{i_0-1} x_{i_0}, x_{i_0} x_1 \) be a regular \( i_0 \)-gon with side length \( 4\rho_1 \). Thus,
\[ \text{dist}(B_{\rho_i}(x_i), B_{\rho_j}(x_j)) \geq \rho_i + \rho_j \quad \text{for} \quad 1 \leq i < j \leq i_0. \]
(37)

By (26), (27) and (35), the inequality (37) also holds for \( 1 \leq i < j \).

For \( i \geq 1 \), define
\[ u_i(x) := w_{\lambda_i}(x - x_i). \]
Then one can check that
\[ \min_{x \in B_{2\rho_j}(x_j)} \frac{u_{j+1}(x)}{u_{j-1}(x)} > \left(\frac{1}{3}\right)^{n-2m} \quad \text{for} \quad 2 \leq j \leq i_0 - 1 \]
and a similar inequality holds when \( j = 1 \) or \( i_0 \).

Here we also give some inequalities which will be used later. By (36) and Lemma 2.2 we have for \( 1 \leq j \leq i_0 \) that

\[
\min_{x \in B_{2\rho_j}(x_j)} Z \left( k_j^{4m}, \sum_{i=1, i \neq j}^{i_0} u_i(x) \right) \\
= \min_{x \in B_{2\rho_1}(x_1)} Z \left( k_1^{4m}, \sum_{i=2}^{i_0} u_i(x) \right) \\
\geq \min_{x \in B_{2\rho_1}(x_1)} Z \left( k_1^{4m}, u_2(x) \right) \\
\geq Z \left( k_1^{\frac{4m}{4m}}, \psi(6\rho_2, \lambda_2) \right) \sim \frac{1}{1-k_1} \left( \frac{\lambda_1}{\lambda_1^2+(6\rho_1)^2} \right)^{\frac{n-2m}{2}} \sim \frac{1}{1-k_1} \left( \frac{\lambda_1}{\rho_1^2} \right)^{\frac{n-2m}{2}}.
\]

Thus, by increasing \( k_1 \) (recall that \( k_1 = k_2 = \cdots = k_{i_0} \)) we have

\[
\min_{x \in B_{2\rho_j}(x_j)} Z \left( k_j^{\frac{4m}{4m}}, \sum_{i=1, i \neq j}^{i_0} u_i(x) \right) > w(0) \quad \text{for} \quad 1 \leq j \leq i_0.
\]  

(40)

By Lemma 2.2,

\[
Z \left( k_j^{\frac{4m}{4m}}, \frac{1}{2M_j^{\frac{4m}{4m}}} \right) \sim \frac{1}{1-k_j} \frac{1}{M_j^{\frac{(1-\alpha)(n-2m)}{4m}}} \sim M_j^{\frac{(1-\alpha)(n-2m)}{4m}} \quad \text{for} \quad j \geq 1.
\]  

(41)

Therefore, by increasing each term of the sequence \( \{k_j\}_{j=1}^{\infty} \), we also have

\[
Z \left( k_j^{\frac{4m}{4m}}, \frac{1}{2M_j^{\frac{4m}{4m}}} \right) > w(0) \quad \text{for} \quad j \geq 1.
\]

Then, by (34), (21) and (30) we have for \( j \geq 1 \) and \( |x - x_j| \geq \rho_j \) that

\[
u_j(x) \leq \varepsilon_j a^{\frac{n-2m}{4m}} w(0) \\
< w(0) \leq Z \left( k_j^{\frac{4m}{4m}}, \frac{1}{2M_j^{\frac{4m}{4m}}} \right).
\]

(42)

It follows from Lemma 2.2 and (30) that

\[
\max_{s \geq \rho_j} \left| \frac{d}{ds} \psi(s, \lambda_j) \right| \sim \left( \frac{\lambda_j}{\rho_j^2} \right)^{\frac{n-2m}{2}} \frac{1}{\rho_j} \sim \varepsilon_j 2^{\frac{1}{2m}} M_j^{\frac{1}{2m}} < M_j^{\frac{1}{2m}} \quad \text{for} \quad j \geq 1.
\]  

(43)
Step 2. Estimating the sum of the bubbles $u_i$. By (34) and (30) we have

$$u_i \leq \varepsilon_i \alpha^{n-2m} w \quad \text{in } \mathbb{R}^n \setminus B_{\rho_i}(x_i)$$

and

$$\sum_{i=1}^{\infty} u_i \leq \alpha^{n-2m} w \quad \text{in } \mathbb{R}^n - \bigcup_{i=1}^{\infty} B_{\rho_i}(x_i).$$

By (30) and (44), we know that $\sum_{i=1}^{\infty} u_i \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfies

$$\left\{ \begin{array}{ll}
(\Delta)^m \left( \sum_{i=1}^{\infty} u_i \right) = \sum_{i=1}^{\infty} u_i^{n+2m} & \text{in } \mathbb{R}^n \setminus \{0\}, \\
(\Delta)^s \left( \sum_{i=1}^{\infty} u_i \right) = \sum_{i=1}^{\infty} (-\Delta)^s u_i > 0, & s = 1, 2, \ldots, m - 1
\end{array} \right.$$ \hspace{1cm} (46)

By increasing $k_i$ for each $i$, we can assume that

$$u_i(x_i) = c_{n,m} \lambda_i^{\frac{n-2m}{2}} > i\varphi(|x_i|) \quad \text{for } i \geq 1$$

and $u_i + |\nabla u_i| < 2^{-i}$ in $\mathbb{R}^n \setminus B_{2r_i}(x_i), i \geq 1$. Thus by (26) and (27),

$$u_i + |\nabla u_i| < 2^{-i} \quad \text{in } B_{2r_j}(x_j)$$

when $i \neq j$ and either $i \geq 1$ and $j > i_0$ or $i > i_0$ and $1 \leq j \leq i_0$. Again, by increasing $k_i$ for $i > i_0$, we can force $u_i$ and $M_i$ to satisfy

$$\sum_{i=i_0+1}^{\infty} u_i(x) < \frac{1}{2} \min_{1 \leq i \leq i_0} u_i(x) \quad \text{for } |x| \geq \delta_2,$$

$$\sum_{i=i_0+1, i \neq j}^{\infty} u_i(x) < \frac{u_j}{2} \quad \text{in } B_{2r_j}(x_j), j > i_0$$

and

$$\frac{1}{M_j^{\frac{n}{n-2m}}} < \min_{|x| \leq \delta} u_1(x) \quad \text{for } j > i_0.$$ \hspace{1cm} (51)

It follows from (48) and (44) that

$$\sum_{i=1, i \neq j}^{\infty} u_i + u_i^{\frac{n+2m}{n-2m}} \leq C \quad \text{in } B_{\rho_j}(x_j), j \geq 1.$$ \hspace{1cm} (52)

Similarly, by (48), (44), Lemma 2.2 and (22),

$$\sum_{i=1, i \neq j}^{\infty} |\nabla u_i| + u_i^{\frac{4m}{n-2m}} |\nabla u_i| \leq \sum_{i=1, i \neq j}^{i_0} |\nabla u_i| + u_i^{\frac{4m}{n-2m}} |\nabla u_i| + C$$

$$\leq \sum_{i=1, i \neq j}^{i_0} u_i \frac{1}{\rho_j} + u_i^{\frac{n+2m}{n-2m}} \frac{1}{\rho_j} + C$$

$$\leq C2^{\frac{4m}{2m}} M_j^{\frac{1}{m}} \leq C2^{\frac{i_0}{2m}} M_j^{\frac{1}{m}} \leq CM_j^{\frac{1}{m}} \quad \text{in } B_{\rho_j}(x_j), 1 \leq j \leq i_0.$$
and by (48) and (44),
\[ \sum_{i=1, i \neq j}^{\infty} |\nabla u_i| + u_i^{4m} |\nabla u_i| \leq C \quad \text{in } B_{\rho_j}(x_j), \ j > i_0. \]

Thus, we get
\[ \sum_{i=1, i \neq j}^{\infty} |\nabla u_i| + u_i^{4m} |\nabla u_i| \leq CM_j^{\frac{1}{2m}} \quad \text{in } B_{\rho_j}(x_j), \ j \geq 1. \quad (53) \]

**Step 3. Constructing the correction function \( u_0 \).** Since \( n \geq 2m + 4 \), by Lemma 2.2 and (35) we have
\[ 1 - k_i \rho_i \sim \frac{2^{\frac{4m}{2m}} M_i^{\frac{1}{2m}}}{M_i^{\frac{n-2m-2}{4m}}} \leq \frac{2^{\frac{4m}{2m}}}{M_i^{\frac{n-2m-2}{4m}}} \leq \frac{2^{\frac{4m}{2m}}}{3} \to 0 \quad \text{as } i \to \infty. \quad (54) \]

Let \( \eta : [0, \infty) \to [0, 1] \) be a \( C^\infty \) cut-off function satisfying \( \eta(t) = 1 \) for \( 0 \leq t \leq 1 \) and \( \eta(t) = 0 \) for \( t \geq 3/2 \). Define
\[ \kappa(x) = k(x) + \sum_{i=1}^{\infty} (k_i - k(x)) \eta_i(x) \quad \text{for } x \in \mathbb{R}^n, \quad (55) \]

where \( \eta_i(x) = \eta(\|x - x_i\|/\rho_i) \). Since \( \{\eta_i\}_{i=1}^{\infty} \) have disjoint supports contained in \( B_{2\delta_i} \setminus \{0\} \), \( \kappa \) is well-defined. Recall that \( k \equiv 1 \) in \( B_\delta \), we obtain \( \kappa(0) = k(0) = 1, \ k \leq k \) in \( \mathbb{R}^n \) and \( \kappa(x) = k(x) \) for \( |x| \geq 2\delta_1 \). By (21) and (31) we have
\[ \inf_{\mathbb{R}^n} \kappa \geq a. \quad (56) \]

Since \( k \equiv 1 \) in \( B_\delta \) and then
\[ \nabla \kappa(x) = \sum_{i=1}^{\infty} \frac{k_i - 1}{\rho_i} \eta_i' \left( \frac{|x - x_i|}{\rho_i} \right) \frac{x - x_i}{|x - x_i|} \quad \text{for } x \in B_\delta, \quad (57) \]

it follows from (54) that \( \kappa \in C^1(\mathbb{R}^n) \) and \( \nabla \kappa(0) = 0 \).

By (33),
\[ 0 < I_{2m} M < \frac{w}{2} \quad \text{in } \mathbb{R}^n, \quad (58) \]

where
\[ M(x) := \begin{cases} (2w(0))^{\frac{n+2m}{n-2m}} M_i & \text{in } B_{\rho_i}(x_i), \ i \geq 1, \\ 0 & \text{in } \mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} B_{2\rho_i}(x_i), \\ (2w(0))^{\frac{n+2m}{n-2m}} M_i \left( 2 - \frac{|x - x_i|}{\rho_i} \right) & \text{in } B_{2\rho_i}(x_i) \setminus B_{\rho_i}(x_i), \ i \geq 1. \end{cases} \]
Since $\overline{M}$ is locally Lipschitz continuous in $\mathbb{R}^n \setminus \{0\}$, we have $\overline{v} := u/(2b) + I_{2m} M \in C^{2m,\beta}_{loc}(\mathbb{R}^n \setminus \{0\})$ for any $0 < \beta < 1$. By (29),

\[
\left\{ \begin{array}{ll}
(\Delta)^s \overline{v} = (\Delta)^s w/(2b) + I_{2(m-s)} \overline{M} > 0, & s = 1, 2, \ldots, m - 1 \\
(\Delta)^{m-s} \overline{v} = (2b)^{n+2m/2} w^{n+2m/2} + \overline{M} & \text{in } \mathbb{R}^n \setminus \{0\},
\end{array} \right.
\]

where $I_{2(m-s)}$ is the Riesz potential operator of order $2(m-s)$. It follows from (58) that

\[
\frac{w}{2b} < \overline{v} < w \quad \text{in } \mathbb{R}^n.
\]

Define $H : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ by

\[
H(x, v) = \kappa(x) \left( v + \sum_{i=1}^{\infty} u_i(x) \right) - \sum_{i=1}^{\infty} u_i(x) \frac{v}{\kappa(x)}.
\]

Then we have

\[
H(x, v) = f(U(x), \kappa(x), P(x, v)),
\]

where

\[
U(x) := \left( \sum_{i=1}^{\infty} u_i(x) \right)^{\frac{n+2m}{n-2m}} \quad \text{and} \quad P(x, v) := v + \sum_{i=1}^{\infty} u_i(x) - U(x).
\]

Define $H : \mathbb{R}^n \times [0, \infty) \to (0, \infty)$ by

\[
H(x, v) = F(U(x), \kappa(x), P(x, v)).
\]

Then

\[
H(x, v) \leq M(\kappa(x), P(x, v)) \quad \text{when } \kappa(x) < 1.
\]

Moreover, by the definition of $F$ we have that $H(x, v) = H(x, v)$ if and only if either $\kappa(x) < 1$ and $U(x) \leq Z(\kappa(x), P(x, v))$ or $\kappa(x) \geq 1$.

For $x \in \mathbb{R}^n - \bigcup_{i=1}^{\infty} B_{\rho_i}(x_i)$ and $\kappa(x) < 1$, we have

\[
U(x) \leq \sum_{i=1}^{\infty} u_i(x) \leq d^{\frac{n+2m}{4m}} w(x) \quad \text{by (45)}
\]

\[
\leq \frac{w(x)\kappa(x)^{\frac{n-2m}{4m}}}{1 - \kappa(x)^{\frac{n-2m}{4m}}} \quad \text{by (56)}
\]

\[
\leq \frac{P(x, w(x))\kappa(x)^{\frac{n-2m}{4m}}}{1 - \kappa(x)^{\frac{n-2m}{4m}}}
\]

\[
= Z(\kappa(x), P(x, w(x))) \quad \text{by (28)}.
\]

Hence

\[
H(x, w(x)) = H(x, w(x)) \quad \text{for } x \in \mathbb{R}^n - \bigcup_{i=1}^{\infty} B_{\rho_i}(x_i).
\]
Thus, for $x \in (\mathbb{R}^n \setminus \{0\}) - \bigcup_{i=1}^{\infty} B_{\rho_i}(x_i)$ and $0 \leq v \leq w(x)$ we have

$$H(x, v) \leq H(x, w(x)) = H(x, w(x)) \leq \kappa(x) \left( w(x) + \sum_{i=1}^{\infty} u_i(x) \right)^{\frac{n+2m}{n-2m}}$$

by (45), (21) and (59).

For $x \in B_{\rho_i}(x_i)$ and $i \geq 1$ we have $\kappa(x) \equiv k_i < 1$. Hence, from (63) we obtain for $x \in B_{\rho_i}(x_i)$ and $0 \leq v \leq w(x)$ that

$$H(x, v) \leq M_i \left( v + \sum_{j=1}^{\infty} u_j(x) - U(x) \right)^{\frac{n+2m}{n-2m}} \quad \text{by (32)}$$

$$\leq M_i \left( v + \sum_{j=1, j \neq i}^{\infty} u_j(x) \right)^{\frac{n+2m}{n-2m}}$$

$$\leq M_i (2w(x))^{\frac{n+2m}{n-2m}} \quad \text{by (44)}$$

$$\leq M_i (2w(0))^{\frac{n+2m}{n-2m}} = \overline{M}(x) \leq (-\Delta)^m \overline{\psi}(x) \quad \text{by (59)}.$$  

This together with (64) implies that

$$H(x, v) \leq (-\Delta)^m \overline{\psi}(x) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\} \text{ and } 0 \leq v \leq w(x).$$

Hence, by the nonnegativity of $H$, (60) and (66) we can use $\underline{\psi} \equiv 0$ and $\overline{\psi}$ as sub- and supersolutions of the problem

$$(-\Delta)^m u = H(x, u) \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$  

Now, applying the method of sub- and super-solutions we can get the desired correction function $u_0$.

**Lemma 2.3.** There exists a $C^{2m}(\mathbb{R}^n \setminus \{0\})$ solution $u_0$ of

$$\begin{cases}
(-\Delta)^m u_0 = H(x, u_0) \\
0 \leq (-\Delta)^s u_0 \leq (-\Delta)^s \overline{\psi}, \ s = 1, 2, \ldots, m - 1 \quad \text{in } \mathbb{R}^n \setminus \{0\}.
\end{cases}$$

**Proof.** For each positive integer $l \geq 2$, we consider about the following problem

$$\begin{cases}
(-\Delta)^m v = H(x, v) \quad \text{in } B_l \setminus \bar{B}_{1/l}, \\
(-\Delta)^s v = 0 \quad \text{on } \partial(B_l \setminus \bar{B}_{1/l}), \ s = 0, 1, \ldots, m - 1.
\end{cases}$$

(68)
Notice that \( v, \overline{\nu} \in C^{2m, \beta}_{loc}(\mathbb{R}^n \setminus \{0\}) \) for any \( 0 < \beta < 1 \), \( H \in C^1(\mathbb{R}^n \setminus \{0\} \times [0, \infty)) \) and \( H \) is non-decreasing with respect to the last variable. By using the method of sub- and super-solutions (see, e.g., Theorem 2.1 in [20]) the problem \((68)\) has a \( C^{2m} \) solution \( v_l \) satisfying \( 0 \leq v_l \leq \overline{\nu} \) and \( 0 \leq (-\Delta)^s v_l \leq (-\Delta)^s \overline{\nu} \) for every \( s = 1, 2, \ldots, m - 1 \). It follows from standard elliptic theory that, after passing to a subsequence, \( \{v_l\} \) converges to a nonnegative function \( u_0 \in C^{2m}_{loc}(\mathbb{R}^n \setminus \{0\}) \) which satisfies \((67)\).

**Step 4. Defining the solution \( u \) and the function \( K \).** Define \( \overline{H} : \mathbb{R}^n \times [0, \infty) \to (0, \infty) \) by \( \overline{H}(x, v) = F(U(x), k(x), P(x, v)). \) Then \( \overline{H} \leq H \leq \overline{H} \) since \( \kappa \leq k \). In particular,

\[
\overline{H}(x, u_0(x)) \leq H(x, u_0(x)) \leq \overline{H}(x, u_0(x)) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}. \tag{69}
\]

For \( |x| > \delta \) we have

\[
U(x) = \sum_{i=1}^{\infty} u_i(x) \left( \frac{n+2m}{n-2m} \right)^i \\
\leq i_0 \left( \frac{n+2m}{n-2m} \right)^{m+1} u_1(x) \left( \frac{n+2m}{n-2m} \right)^{m+1} + u_1(x) \left( \frac{n+2m}{n-2m} \right)^{m+1} \quad \text{by (20) and (49)}
\]

\[
\leq i_0 \left( \frac{n+2m}{n-2m} \right)^{m+1} u_1(x) \left( \frac{n+2m}{n-2m} \right)^{m+1} = \frac{i_0}{4^{m+2}} 2^{m-2m} u_1(x) \left( \frac{n+2m}{n-2m} \right)^{m+1}
\]

\[
\leq \frac{(2a) i_0}{4^{m+2}} 2^{m-2m} u_1(x) \left( \frac{n+2m}{n-2m} \right)^{m+1} \quad \text{by (22)}
\]

\[
\leq k(x) \left( \frac{n+2m}{4m} \right)^{i_0} \left( \frac{n+2m}{n-2m} \right)^{m+1} \sum_{i=1}^{\infty} u_i(x) \quad \text{by (21)}
\]

\[
\leq k(x) \left( \frac{n+2m}{4m} \right)^{i_0} \left( \sum_{i=1}^{\infty} u_i(x) \right) \quad \text{by (20)}.
\]

Therefore, for \( k(x) < 1 \) (which implies that \( |x| > \delta \)) and \( v \geq 0 \),

\[
U(x) \leq k(x) \left( \frac{n+2m}{4m} \right)^{i_0} \sum_{i=1}^{\infty} u_i(x).
\]

From \((28)\), for \( k(x) < 1 \) and \( v \geq 0 \), we have

\[
U(x) \leq \left( \sum_{i=1}^{\infty} u_i(x) - U(x) \right) k(x) \left( \frac{n+2m}{4m} \right)^{i_0} \\
\leq Z(k(x), P(x, v)).
\]

Thus, by the definition of \( F \) we obtain for \( x \in \mathbb{R}^n \) and \( v \geq 0 \) that

\[
\overline{H}(x, v) = f(U(x), k(x), P(x, v))
\]

\[
= k(x) \left( v + \sum_{i=1}^{\infty} u_i(x) \right) \left( \frac{n+2m}{n-2m} \right)^{i_0} - \sum_{i=1}^{\infty} u_i(x) \left( \frac{n+2m}{n-2m} \right)^{i_0},
\]

\[
15
\]
which together with (46), (61), (67) and (69) implies that

$$u := u_0 + \sum_{i=1}^{\infty} u_i$$

is a $C^{2m}$ positive solution of

$$\begin{cases}
\kappa(x) u^{n+2m} \leq (-\Delta)^m u \leq k(x) u^{n+2m} \\
(-\Delta)^s u > 0, \quad s = 1, 2, \ldots, m - 1
\end{cases} \quad \text{in } \mathbb{R}^n \setminus \{0\}. \quad (70)$$

It follows from (47), (49) and (67) that $u$ satisfies (13) and (14).

Define $K : \mathbb{R}^n \to (0, \infty)$ by

$$K(x) = \frac{(-\Delta)^m u(x)}{u(x)^{n+2m}} \quad \text{for } x \in \mathbb{R}^n \setminus \{0\} \quad (71)$$

and $K(0) = 1$. Then

$$K(x) = \frac{H(x, u_0(x)) + \sum_{i=1}^{\infty} u_i(x)^{n+2m}}{(u_0(x) + \sum_{i=1}^{\infty} u_i(x))^{n+2m}} \quad \text{for } x \in \mathbb{R}^n \setminus \{0\} \quad (72)$$

and hence $K \in C^1(\mathbb{R}^n \setminus \{0\})$. It follows from (70) and (71) that

$$\kappa(x) \leq K(x) \leq k(x) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}. \quad (73)$$

Recall that $\kappa, k \in C^1(\mathbb{R}^n)$ and $\kappa(0) = K(0) = k(0) = 1$, we get $K \in C(\mathbb{R}^n)$,

$$\nabla \kappa(0) = \nabla K(0) = \nabla k(0) = 0 \quad (74)$$

and

$$\kappa(x) = K(x) = k(x) \quad \text{for } |x| \geq 2\delta_1. \quad (75)$$

**Step 5. Showing that $K \in C^1(\mathbb{R}^n)$**. We only need to show that

$$\lim_{|x| \to 0^+} \nabla K(x) = 0. \quad (76)$$

Let $S = \{x \in \mathbb{R}^n \setminus \{0\} : H(x, u_0(x)) < H(x, u_0(x))\}$. It follows from (61) and (72) that

$$S = \{x \in \mathbb{R}^n \setminus \{0\} : \kappa(x) < K(x)\}. \quad (77)$$

By (73), (74) and (77) we obtain

$$\nabla \kappa(x) = \nabla K(x) \quad \text{for } x \in \mathbb{R}^n \setminus S \quad (78)$$

and thus (76) holds for $x \in (\mathbb{R}^n \setminus \{0\}) - S$. Next we show that (76) holds for $x \in S$. It follows from (61) and (62) that

$$\begin{cases}
H(x, u_0(x)) = M(\kappa(x), P_0(x)) \\
U(x) > Z(\kappa(x), P_0(x))
\end{cases} \quad \text{for } x \in S, \quad (79)$$

and

$$\kappa(x) = K(x) = k(x) \quad \text{for } |x| \geq 2\delta_1. \quad (75)$$

Hence, by (73), (74) and (77) we obtain

$$\nabla \kappa(x) = \nabla K(x) \quad \text{for } x \in \mathbb{R}^n \setminus S \quad (78)$$

and thus (76) holds for $x \in (\mathbb{R}^n \setminus \{0\}) - S$. Next we show that (76) holds for $x \in S$. It follows from (61) and (62) that

$$\begin{cases}
H(x, u_0(x)) = M(\kappa(x), P_0(x)) \\
U(x) > Z(\kappa(x), P_0(x))
\end{cases} \quad \text{for } x \in S, \quad (79)$$

and
where $P_0(x) := P(x, u_0(x))$. Since $\kappa \geq k_j$ in $B_{2\rho_j}(x_j)$, by (28) we have
\[
U(x) > Z(k_j, P_0(x)) = M_j^{n-2m} P_0(x) \quad \text{for } x \in S \cap B_{2\rho_j}(x_j), \ j \geq 1. \tag{80}
\]
For $x' \in (\mathbb{R}^n \setminus \{0\}) - \bigcup_{i=1}^{\infty} B_{2\rho_i}(x_i)$, $\kappa(x') = k(x')$. Hence, by (73) and (77) we know that $x' \not\in S$. Consequently,
\[
S \subset \bigcup_{i=1}^{\infty} B_{2\rho_i}(x_i). \tag{81}
\]
For $j \geq 1$ and $x \in S \cap B_{2\rho_j}(x_j)$, by (80) we have
\[
U(x) > \frac{k_j^{\frac{n-2m}{4m}} \left( \sum_{i=1}^{\infty} u_i(x) - U(x) \right)}{1 - k_j^{\frac{n-2m}{4m}}},
\]
therefore
\[
U(x) > k_j^{\frac{n-2m}{4m}} \sum_{i=1}^{\infty} u_i(x).
\]
Hence
\[
\sum_{i=1, i \neq j}^{\infty} u_i(x)^{\frac{n+2m}{n-2m}} \geq f \left( u_j(x), k_j^{\frac{n+2m}{4m}}, \sum_{i=1, i \neq j}^{\infty} u_i(x) \right) \quad \text{for } x \in S \cap B_{2\rho_j}(x_j), \ j \geq 1. \tag{82}
\]
However, by Lemma 2.1, (38), (49) and (36), we have for $1 \leq j \leq i_0$ and $x \in S \cap B_{2\rho_j}(x_j)$ that
\[
\frac{\sum_{i=1, i \neq j}^{\infty} u_i(x)^{\frac{n+2m}{n-2m}}}{f \left( 0, k_j^{\frac{n+2m}{4m}}, \sum_{i=1, i \neq j}^{\infty} u_i(x) \right)} = \frac{\sum_{i=1, i \neq j}^{\infty} u_i(x)^{\frac{n+2m}{n-2m}}}{k_j^{\frac{n+2m}{4m}} \left( \sum_{i=1, i \neq j}^{\infty} u_i(x) \right)^{\frac{n+2m}{n-2m}}} \leq \frac{1 + (\frac{1}{3})^{n-2m}}{k_j^{\frac{n+2m}{4m}} \left( 1 + \frac{n+2m}{n-2m} (\frac{1}{3})^{n-2m} \right)} < 1.
\]
Thus, by the property of $f$, (82) and (40),
\[
u_j(x) > Z \left( k_j^{\frac{n+2m}{4m}}, \sum_{i=1, i \neq j}^{\infty} u_i(x) \right) > w(0) \quad \text{for } x \in S \cap B_{2\rho_j}(x_j), \ 1 \leq j \leq i_0, \tag{83}
\]
which together with (42) implies that
\[
S \cap B_{2\rho_j}(x_j) = S \cap B_{\rho_j}(x_j) \quad \text{for } 1 \leq j \leq i_0. \tag{84}
\]
It follows from (39) and (83) that
\[ u_j \geq CM_j \frac{(1-\alpha)(n-2m)}{4m} \quad \text{in } S \cap B_{2\rho_j}(x_j), 1 \leq j \leq i_0. \tag{85} \]

For \( j > i_0 \) and \( x \in B_{2\rho_j}(x_j) \), by Lemma 2.1, (20), (50) and (36) we get
\[
\frac{\sum_{i=1, i \neq j}^\infty u_i(x) \frac{n+2m}{n-2m}}{f \left( 0, k_j \frac{n+2m}{4m}, \sum_{i=1, i \neq j}^\infty u_i(x) \right)} = \frac{\sum_{i=1, i \neq j}^\infty u_i(x) \frac{n+2m}{n-2m}}{k_j \frac{n+2m}{4m} \left( \sum_{i=1, i \neq j}^\infty u_i(x) \right)^\frac{n+2m}{2M_j}} \leq \frac{1 + \frac{1}{2}}{k_j \frac{n+2m}{4m} \left( 1 + \frac{n+2m}{n-2m} \right)} < 1.
\]

Thus, by the property of \( f \), (82) and (51),
\[ u_j(x) > Z \left( k_j \frac{n+2m}{4m}, \sum_{i=1, i \neq j}^\infty u_i(x) \right) > Z \left( k_j \frac{n+2m}{4m}, \frac{1}{2M_j} \right) \quad \text{for } x \in S \cap B_{2\rho_j}(x_j), j > i_0. \]

Therefore it follows from (42) and (84) that
\[ S \cap B_{2\rho_j}(x_j) = S \cap B_{\rho_j}(x_j) \quad \text{for } j \geq 1 \tag{86} \]
and it follows from (41) and (85) that
\[ u_j \geq CM_j \frac{(1-\alpha)(n-2m)}{4m} \quad \text{in } S \cap B_{2\rho_j}(x_j), j \geq 1. \tag{87} \]

Recall (72) and (79), we have
\[
K(x) = \frac{M_j P_0(x) \frac{n+2m}{n-2m} + U(x) \frac{n+2m}{n-2m}}{(P_0(x) + U(x)) \frac{n+2m}{n-2m}} = \frac{M_j \left( \frac{P_0(x)}{U(x)} \right) \frac{n+2m}{n-2m}}{(P_0(x) + 1) \frac{n+2m}{n-2m}} \quad \text{for } x \in S \cap B_{\rho_j}(x_j), j \geq 1.
\]

Thus
\[
\nabla K = \frac{n + 2m}{n - 2m} \left( \frac{M_j \left( \frac{P_0}{U} \right) \frac{4m}{2m} - 1}{(P_0 + 1) \frac{4m}{2m}} \right) \left( \nabla \frac{P_0}{U} \right) \quad \text{in } S \cap B_{\rho_j}(x_j), j \geq 1
\]
and hence, by (80),
\[
|\nabla K| \leq \frac{n + 2m}{n - 2m} \left| \nabla \frac{P_0}{U} \right| \leq \frac{n + 2m}{n - 2m} \left( \left| \nabla \frac{u_0}{U} \right| + \left| \sum_{i=1, i \neq j}^\infty \frac{u_i}{U} \right| + \left| \frac{u_j}{U} \right| \right) \quad \text{in } S \cap B_{\rho_j}(x_j), j \geq 1. \tag{88}
\]
Now we estimate these three terms. By (53) and (87),
\[
\left| \nabla \frac{1}{U} \right| = \left| \nabla \left[ (U \frac{n+2m}{n-2m})^{\frac{2m-n}{2m+n}} \right] \right| = \left| \frac{n-2m}{n+2m} (U \frac{n+2m}{n-2m})^{\frac{n-2m}{2m+n}} \nabla U \frac{n+2m}{n-2m} \right|
\leq C \left( \frac{M_j^{\frac{1}{2m}}}{M_j^{\frac{(1-\alpha)(n-2m)}{2m}}} + \frac{\nabla u_j}{u_j^2} \right) \text{ in } S \cap B_{\rho_j}(x_j), \ j \geq 1.
\]
Hence, by (52), (53), (67) and (87),
\[
\left| \nabla \frac{u_0}{U} \right| = \left| \frac{n-2m}{n+2m} \frac{\nabla u_0}{U} + u_0 \frac{\nabla 1}{U} \right|
\leq C \left( \frac{\nabla u_0}{M_j^{\frac{(1-\alpha)(n-2m)}{2m}}} + \frac{M_j^{\frac{1}{2m}}}{M_j^{\frac{(1-\alpha)(n-2m)}{2m}}} + \frac{\nabla u_j}{u_j^2} \right) \text{ in } S \cap B_{\rho_j}(x_j), \ j \geq 1 \tag{89}
\]
and
\[
\left| \nabla \sum_{i=1, i \neq j}^\infty \frac{u_i}{U} \right| \leq \left| \nabla \frac{1}{U} \right| \sum_{i=1, i \neq j}^\infty u_i + \frac{1}{U} \left| \nabla \sum_{i=1, i \neq j}^\infty u_i \right|
\leq C \left( \frac{M_j^{\frac{1}{2m}}}{M_j^{\frac{(1-\alpha)(n-2m)}{2m}}} + \frac{\nabla u_j}{u_j^2} + \frac{M_j^{\frac{1}{2m}}}{M_j^{\frac{(1-\alpha)(n-2m)}{2m}}} \right) \tag{90}
\]
Since
\[
\nabla \frac{u_j}{U} = \nabla \left( \sum_{i=1}^\infty \frac{u_i^{\frac{n+2m}{n-2m}}}{u_j^{\frac{n+2m}{n-2m}}} \right) = \nabla \left( 1 + \sum_{i=1, i \neq j}^\infty \frac{u_i^{\frac{n+2m}{n-2m}}}{u_j^{\frac{n+2m}{n-2m}}} \right)
= -\frac{n-2m}{n+2m} \left( 1 + \sum_{i=1, i \neq j}^\infty \frac{u_i^{1+\frac{n+2m}{n-2m}}}{u_j^{\frac{n+2m}{n-2m}}} \right)
- \frac{n+2m}{n-2m} \left( \frac{\nabla u_j}{u_j^{\frac{n+2m}{n-2m}}} \sum_{i=1, i \neq j}^\infty \frac{u_i}{u_j^{\frac{n+2m}{n-2m}}} \right),
\]
and by (52), (53), (67) and (87), we get
\[
\left| \nabla \frac{u_j}{U} \right| \leq C \left( \frac{M_j^{\frac{1}{2m}}}{M_j^{\frac{(1-\alpha)(n+2m)}{4m}}} + \frac{\nabla u_j}{u_j^{\frac{n+2m}{n-2m}}} \right) \text{ in } S \cap B_{\rho_j}(x_j), \ j \geq 1. \tag{91}
\]
By (88)-(91), we obtain for \( j \geq 1 \) and \( x \in S \cap B_{\rho_j}(x_j) \) that
\[
|\nabla K| \leq C \left( \frac{|\nabla u_0|}{M_j^{\frac{1}{4m}}} + \frac{|\nabla u_j|}{u_j} + \frac{M_j^{\frac{1}{2m}}}{M_j^{\frac{1}{4m}}} + \frac{|\nabla u_j|}{u_j} \right). \tag{92}
\]

We now estimate \( \nabla u_0 \) in \( B_{\rho_j}(x_j) \). By (67) there exists a continuous function \( h : \overline{B_2} \to \mathbb{R} \) which satisfies \((-\Delta)^m h = 0\) in \( B_2 \) such that
\[
u_0(x) = \gamma_{n,m} \int_{B_2} \frac{H(y, u_0(y))}{|x - y|^{n-2m}} dy + h(x) \quad \text{for} \quad 0 < |x| \leq 2,
\]
where \( \gamma_{n,m} = \Gamma(n/2 - m)/(2^{2m} \pi^{n/2} \Gamma(m)) \). By (64), (65) and (67),
\[
H(x, u_0(x)) \leq \begin{cases} (2w(0))^\frac{n+2m}{n-2m} M_j & \text{in} \ B_{\rho_j}(x_j), \ j \geq 1, \\ (2w(0))^\frac{n+2m}{n-2m} b & \text{in} \ (\mathbb{R}^n \setminus \{0\}) - \bigcup_{i=1}^{\infty} B_{\rho_i}(x_i). \end{cases}
\]

It follows from (67) that \( |h(x)| < C \) for \( |x| \leq 2 \). Thus \( |\nabla h(x)| < C \) for \( |x| \leq 1 \). Hence, for \( x \in B_{\rho_j}(x_j) \),
\[
|\nabla u_0(x)| \leq C \int_{B_1} \frac{H(y, u_0(y))}{|x - y|^{n-2m+1}} dy + C \leq C |I_1(x) + I_2(x) + I_3(x)| + C
\]
where
\[
I_1(x) := \int_{B_{\rho_j}(x_j)} \frac{M_j}{|x - y|^{n-2m+1}} dy \leq CM_j \rho_j^{2m-1} \leq CM_j^{\frac{1}{2m}} \quad \text{for} \quad x \in B_{\rho_j}(x_j)
\]
by Lemma 2.2, and
\[
I_2(x) := \sum_{i=1, i \neq j}^{\infty} \int_{B_{\rho_i}(x_i)} \frac{M_i}{|x - y|^{n-2m+1}} dy \leq C \sum_{i=1, i \neq j}^{\infty} \frac{M_i \rho_i^n}{\text{dist}(B_{\rho_j}(x_j), B_{\rho_i}(x_i))^{n-2m+1}}
\]
\[
\leq C \sum_{i=1, i \neq j}^{\infty} \frac{\rho_i^{n-2m}}{2^i (\rho_i + \rho_j)^{n-2m+1}} \leq \frac{C}{\rho_j} \sim C 2^{\frac{j}{2m}} M_j^{\frac{1}{2m}} \leq CM_j^{\frac{1}{2m}} \quad \text{for} \quad x \in B_{\rho_j}(x_j)
\]
by Lemma 2.2, (37) and (36), and
\[
I_3(x) := \int_{B_4 \setminus \bigcup_{i=1}^{\infty} B_{\rho_i}(x_i)} \frac{1}{|x - y|^{n-2m+1}} dy \leq C \quad \text{for} \quad x \in B_{\rho_j}(x_j).
\]
Thus
\[
|\nabla u_0| \leq CM_j^{\frac{1}{2m}} \quad \text{in} \ B_{\rho_j}(x_j), \ j \geq 1. \tag{93}
\]
Since \( n \geq 2m + 4 \), it follows from (93) that
\[
\frac{|\nabla u_0|}{M_j^{\frac{1}{4m}}} \leq \frac{CM_j^{\frac{1}{2m}}}{M_j^{\frac{1}{2m}}} \leq \frac{C}{M_j^{\frac{1}{2m}}}. \tag{94}
\]
Finally, we estimate \(|\nabla u_j|/u_j^2\) and \(|\nabla u_j|/u_j^{2/n-2m}\) in \(S \cap B_{\rho_j}(x_j)\). Let
\[
s_j = \inf\{s > 0 : S \cap B_{\rho_j}(x_j) \subset B_s(x_j)\}
\]
and \(\bar{u}_j(s) = \psi(s, \lambda_j)\). Then \(s_j \leq \rho_j\) and \(\bar{u}_j(s) = u_j(x)\) when \(|x - x_j| = s\). By (87) we have
\[
\bar{u}_j(s) \geq CM_j^{(1-\alpha)(n-2m)}/4m \quad \text{for} \quad 0 \leq s \leq s_j, \ j \geq 1.
\]
It follows from Lemma 2.2 that
\[
\frac{\lambda_j}{(\lambda_j^2 + s_j^2)^{2m}} \geq CM_j^{1-\alpha} \geq C \left(\frac{2m}{\epsilon_j^{4m}}\right)^{1-\alpha} \lambda_j^{4m(1-\alpha)} \lambda_j^{\frac{3-\alpha}{4m}}
\]
and hence, by (36),
\[
s_j \leq C \left(\frac{2j}{2m} \right)^{\frac{1-\alpha}{4m}} \lambda_j^{\frac{3-\alpha}{4m}} \lambda_j^{\frac{3-\alpha}{4m}} \leq C(\lambda_j^{1-\alpha}) \lambda_j^{\frac{3-\alpha}{4m}} \lambda_j^{\frac{3-\alpha}{4m}} \lambda_j^{\frac{3m-\alpha(m+1)}{4m}} \quad \text{for} \quad j \geq 1.
\]
Since \(n \geq 2m + 4\), we have for \(0 \leq s \leq s_j\) and \(j \geq 1\) that
\[
\frac{-\bar{u}_j'(s)}{\bar{u}_j(s)^2} = n - 2m \left(\lambda_j^2 + s_j^2\right) \frac{n-2m-2}{2} s
\]
\[
\leq n - 2m \left(\lambda_j^2 + s_j^2\right) \frac{n-2m-2}{2} s
\]
\[
\leq \left(\frac{\lambda_j}{\lambda_j^2 + s_j^2}\right)^{\frac{n-2m-2}{2}} \lambda_j^{\frac{3m-\alpha(m+1)}{4m}} \lambda_j^{\frac{3m-\alpha(m+1)}{4m}}
\]
\[
\leq C \lambda_j^{\frac{n-2m-2}{4m} + \frac{m(n-2m-3) - \alpha(m+1)(n-2m-1)}{4m}}
\]
(95)

and
\[
\frac{-\bar{u}_j'(s)}{\bar{u}_j(s)^{\frac{2n-2m}{2}}} = \frac{n - 2m \left(\lambda_j^2 + s_j^2\right) \frac{n+2m-2}{2} s}{n \lambda_j^2}
\]
\[
\leq \left(\frac{\lambda_j}{\lambda_j^2 + s_j^2}\right)^{\frac{n+2m-2}{2}} \lambda_j^{\frac{3m-\alpha(m+1)}{4m}} \lambda_j^{\frac{3m-\alpha(m+1)}{4m}}
\]
\[
\leq C \lambda_j^{\frac{n+2m-2}{4m} + \frac{m(n+2m-3) - \alpha(m+1)(n+2m-1)}{4m}}
\]
(96)
We pick \( \alpha = m/(6m+6) \). Since \( n \geq 2m+4 \), by (92)-(96) we get

\[
|\nabla K| \leq C \left( \frac{1}{M_j^{\frac{m+2}{2m(m+1)}}} + \lambda_j^{\frac{1}{2}} \right) \quad \text{in } S \cap B_{\rho_j}(x_j), \ j \geq 1. \tag{97}
\]

Hence, it follows from Lemma 2.2, (36), (81) and (86) that (76) also holds for \( x \in S \). Thus we have \( K \in C^1(\mathbb{R}^n) \).

By sufficiently increasing \( k_i \) for each \( i \geq 1 \), we can force \( \kappa \) to satisfy

\[
\|k - \kappa\|_{C^1(\mathbb{R}^n)} < \frac{\varepsilon}{4} \tag{98}
\]

by (54), (55), (57), (72) and (75). We also can force \( K \) to satisfy

\[
|\nabla(K - \kappa)| = |\nabla(K - (\kappa - k))| \leq |\nabla K| + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2} \quad \text{in } S
\]

by (81), (86) and (97). Thus by (78), \( |\nabla(K - \kappa)| \leq \varepsilon/2 \) in \( \mathbb{R}^n \). It follows from (73) and (98) that \( K \) satisfies (12). The proof of Theorem 1.3 is completed. \( \square \)

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