Surface triads with optical properties

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Abstract. A geometric model of formation of surfaces comprising an interconnected triple of emitter, reflector and receiver is presented in the paper. The model is based on cyclographic mapping of a spatial curve to the plane. In such map any given point \((x, y, z)\) of the curve corresponds to a cycle with center \((x, y)\) and radius equal to \(z\) applicate. The entire curve corresponds to a directed envelope of cycles consisting, in the general case, of two branches. It is shown that the triad of curves consisting of two branches of the envelope and the orthogonal projection of the original curve within the plane \((xy)\) corresponds to a triad of developable surfaces. The triad of curves in the plane \((xy)\) and the original curve together form a triad of ruled surfaces. Both triads have an optical property. Any ray of light emerging from the point of the emitter surface along the normal to it and falling on the surface of the reflector afterwards is directed along the normal vector to the surface of the receiver.

The direct and inverse problems of formation of the triad of surfaces are solved. In the first case, a one-parameter set of triads of surfaces is defined from a given spatial curve. In the second case, a single triad of surfaces is defined from a pair of curves "emitter-receiver" defined on the plane \((xy)\). Numerical examples of solutions of the direct and inverse problems are considered and the corresponding visualizations are given.

The results of the work can be used in the design of reflector antennas in radar systems and systems for converting solar energy into electric and thermal energy. Keywords — geometric model, cyclographic mapping, triad of surfaces, optical property.

1. Introduction

Reflection and focusing of electromagnetic waves are studied and applied in geometric and computer optics [1, 2, 3]. The use of reflective surfaces in modern antennas in radar-location allows solving problems of telemetry, navigation, cartography, detection of minerals, determining the coordinates and shape of the target and its support. Traditionally, parabolic, cylindrical, spherical surfaces and their combinations are used as reflective surfaces of reflector antennas [4]. It is known to use wide-angle helicoidal type antennas with parabolic generators, which have a number of advantages in comparison with antennas with a simple reflector profile [5, 6].

An important and promising direction of reflective surfaces application is development of solar radiation concentrators with such surfaces [7, 8, 9] for solving a number of energy-consuming space problems, such as destruction of space debris, energy generation in space with its subsequent transmission to the Earth, lighting the Earth from the orbit [7]. The concentrator receives solar radiation, reflects it, multiplies the density of reflected solar flux by many times, and directs it to the heat receiver in which solar energy conversion system functions. The efficiency of a space solar power plant largely depends on the geometry of concentrator reflective surface and heat receiver surface [7]. Thus, there is a need to solve the problem of research and development of reflective surfaces with new geometric shapes.
2. Formulation of the problem
The task is to develop a geometric model for forming reflective surfaces of the reflectors with the simplest, after plane, form and the simplest geometry, namely, linear developable surfaces and at the same time to obtain the same simplest form of the heat receiver surface. At the same time, the heat receiver surface should be correlated with the reflector surface, while variation of shape and position of the heat receiver, allowed by the model, should allow achieving effective operating characteristics of heat receiver. Thus, for example, reflector antenna emitter is often moved beyond the opening of the reflector and placed on the focal arc of the reflective surface to avoid shading effect of emitter and reflector [5], and heat receiver of space solar power plant is often placed in the concentrator focal plane to achieve necessary heat transfer parameters on heat receiver surface [7].

3. Theory

3.1 Mathematical model of a triad of surfaces including the surface of reflection and radiation.
A generalized cyclographic projection of a spatial curve
\[ \mathbf{P}(t) = \{x(t), y(t), z(t)\}, \mathbf{P}'(t) \neq 0, t \in \mathbb{R}, T_0 \leq t \leq T, (1) \]
can be formed on a plane \( \Pi_b(x, y) \). Mathematical model of such cyclographic projection is a system of parametric equations:
\[ x_\beta(t) = x(t) + z(t) \cdot e(t) - x'(t) \cdot \mu(t) \frac{y'(t) \sqrt{\lambda(t) - \mu^2(t)}}{\lambda(t)}, \]
\[ y_\beta(t) = y(t) + z(t) \cdot e(t) - y'(t) \cdot \mu(t) \frac{x'(t) \sqrt{\lambda(t) - \mu^2(t)}}{\lambda(t)}, \]
where \( \mu(t) = e(t) \cdot z'(t) + z(t) \cdot e'(t) \),
\( \lambda(t) = x'(t)^2 + y'(t)^2 \),
\( e(t) = \tan \beta(t) \),
\[ \mathbf{P}' = \frac{d\mathbf{P}}{dt}, x'(t) = \frac{dx}{dt}, y'(t) = \frac{dy}{dt}, \]
\[ z'(t) = \frac{dz}{dt}, e'(t) = \frac{de}{dt}. \]

In case \( e(t) = \text{const} \), the equations of the envelope of directed circles can be obtained from the equations (2) for the case of \( \beta = \text{const} \), and in case \( e(t) = 1 \) – the known envelope equations for cyclographic projection for the case of \( \beta = 45^\circ \) [2,10].

A geometric scheme of spatial curve \( \mathbf{P}(t) \) cyclographic projection formation is shown in figure 1.

A cyclographic projection of any point of the spatial curve \( \mathbf{P}(t) \) is a cycle in the plane \( \Pi_1(x,y) \). The center of the cycle has coordinates \( (x, y) \) and the radius of the cycle \( R = \pm z \), where \( R = |z| \) corresponds to location of the point above the plane \( \Pi_1 \) and \( R = -|z| \) to location below the plane \( \Pi_1 \). In the first case the cycle has a positive direction and in the second case - a negative one.

In general case the envelope (2) of cycles in the plane \( \Pi_1 \) consists of two branches \( \mathbf{P}_{\beta(1)} \) and \( \mathbf{P}_{\beta(2)} \).

These lines together with the line \( \mathbf{P}_1 \) of cycle centers form a triad of cylindrical surfaces \( \Phi_1, \Phi_{(1)}, \) and \( \Phi_{(2)} \), which have projecting location relative to the plane \( \Pi_1 \) (figure 2).

Such triad of surfaces has the following optical property: a ray of light emerging from one of the surfaces, \( \Phi_{(1)} \) or \( \Phi_{(2)} \), along the normal to it, after reflection from the medial surface \( \Phi_1 \) is directed along the normal vector to the other. This optical property of the surface triad under consideration follows from known optical property of lines \( \mathbf{P}_1, \mathbf{P}_{\beta(1)} \) and \( \mathbf{P}_{\beta(2)} \) [2], which are directing lines of surfaces \( \Phi_1, \Phi_{(1)} \) and \( \Phi_{(2)} \), respectively.
Figure 1. A geometric scheme of spatial curve $\vec{P}(t)$ cyclographic projection formation.

Figure 2. The triad of cylindrical surfaces.

Consider pairs of lines $\vec{P}, \vec{P}_{\beta(1)}$ and $\vec{P}, \vec{P}_{\beta(2)}$ as directing lines of expanding surfaces $\Psi_{(1)}$ and $\Psi_{(2)}$ respectively. The mathematical model of these surfaces in the form of a system of parametric equations is obtained as follows:
The ruled surfaces $\Psi(1)$ and $\Psi(2)$ represent two parts of the envelope surface of a one-parameter set of cyclographic mapping cones, vertices of which belong to the spatial line $P(t)$, and bases of which belong to the plane of the projections $\Pi_1$ and represent cycles. The half-angles $\beta$ at the vertexes of cones can be constant $\beta = \text{const}$ for all points of the spatial line $\overline{P}(t)$, or can be expressed by a certain function $\beta = f(t): f(t) \subset C^1, t \in R: T_0 \leq t \leq T$.

The triad of ruled surfaces $\Phi_1, \Psi(1)$ and $\Psi(2)$ also has the optical property: a ray of light emerging from one of them along the normal to it after a reflection from the medial surface $\Phi_1$ is directed along the normal vector to the other.

For half-angle $\beta \in R: B_0 \leq \beta \leq B$ a one-parameter set of triads of ruled surfaces with constant reflection surface $\Phi_1$ is obtained (figure 3). A similar situation occurs for triads composed of triples of cylindrical surfaces with constant reflection surface $\Phi_1$ (see figure 2).

\begin{align*}
X(t,l) &= x(t) + l \cdot \left[ x_\beta(t) - x(t) \right], \\
Y(t,l) &= y(t) + l \cdot \left[ y_\beta(t) - y(t) \right], \\
Z(t,l) &= z(t) \cdot (1 - l), \\
t, l \in R: T_0 \leq t \leq T, L_0 \leq l \leq L.
\end{align*}

Figure 3. A system of triads of ruled surfaces of a spatial line.

The presence of a set of surface triads with constant medial reflection surface is convenient for constructing the surfaces $\Phi(1)$ and $\Phi(2)$ or $\Psi(1)$ and $\Psi(2)$ as emitter and heat receiver surfaces. This requires additional conditions that determine the geometric shape and position of these surfaces relative to the reflective surface $\Phi_1$.

3.2 Algorithm for solving the inverse surface triads formation problem.

The problem is formulated as follows: given the guiding curves $\overline{P}_\lambda(\lambda)$ and $\overline{P}_\mu(\mu)$ of emitter and receiver surfaces within the plane, where $\lambda, \mu \in R: \lambda_0 \leq \lambda \leq \lambda, M_0 \leq \mu \leq M$, it is required to determine the reflective surface of the triad.
In order to simplify the solution of the problem, it is assumed that the half-angles $\beta$ at the vertices of the mapping cones have constant values. Then the algorithm for solving the inverse problem comes to the following:

1. Evolutes $\overline{Q}_{(i)}(\lambda)$ and $\overline{G}_{(i)}(\mu)$ of the curves $\overline{P}_{(i)}(\lambda)$ and $\overline{P}_{(i)}(\mu)$ within the plane $\Pi_t(xy)$ are plotted respectively (figure 4).

$$\overline{Q}_{(i)}(\lambda) \rightarrow \overline{Q}(\lambda), \overline{G}_{(i)}(\mu) \rightarrow \overline{G}(\mu).$$

2. At each point of both evolutes the perpendiculars to the plane $\Pi_1$ are plotted with length equal to curvature radius of corresponding initial curve. All perpendiculars plotted from one evolute have to be aligned in the same direction along the $z$ axis.

On figure 5 all perpendiculars to the plane $\Pi_1$ are plotted from both evolutes in the positive direction of the $z$ axis. Thus for each evolute a corresponding curve in space is obtained: $\overline{Q}_{(i)}(\lambda) \rightarrow \overline{Q}(\lambda)$ and $\overline{G}_{(i)}(\mu) \rightarrow \overline{G}(\mu)$.

3. Consider pairs of curves $\overline{P}_{(i)}(\lambda)$, $\overline{Q}(\lambda)$ and $\overline{P}_{(i)}(\mu)$, $\overline{G}(\mu)$. As follows from the formation scheme of the curves $\overline{Q}(\lambda)$ and $\overline{G}(\mu)$, their geometric form is determined by the form parameter $\lambda$ and $\mu$ of the corresponding curve $\overline{P}_{(i)}(\lambda)$ or $\overline{P}_{(i)}(\mu)$. The pairs of curves $\overline{P}_{(i)}(\lambda) \rightarrow \overline{Q}(\lambda)$ and $\overline{P}_{(i)}(\mu) \rightarrow \overline{G}(\mu)$ form the ruled surfaces $\Psi_{(i)}$ and $\Psi_{(2)}$ of emitter (or receiver).

4. The curve of intersection of the surfaces $\Psi_{(1)} \cap \Psi_{(2)} = \overline{P}$ and its horizontal projection $\overline{P}$ are defined. The curves $\overline{P}$ and $\overline{P}_1$ are the guiding curves of the desired reflective surface $\Phi_{(i)}$ with projecting generatrix lines with respect to $\Pi_t(xy)$.

4. Experimental results
To perform a numerical experiment in order to construct a set of triads of ruled surfaces along a given spatial curve $\overline{P}(t)$, the arc of the helical curve was used:

$$\overline{P}(t) = \left\{ 10 \cos(t), 10 \sin(t), 2t \right\}, t \in R: \frac{\pi}{2} \leq t \leq \pi.$$

The calculations were performed by means of computer algebra on the basis of equations (2) and (3) and their visualization was obtained (figure 3).

To solve the inverse problem the curves (arcs of ellipses) were taken as initial data:

$$\overline{P}_{(i)}(t) = \left\{ 5 + 3 \cos(t), 3 + 2 \sin(t) \right\}, t \in R: -\pi \leq t \leq -\frac{\pi}{2},$$

$$\overline{P}_{(i)}(t) = \left\{ 6 + 4 \cos(t), 2 + 3 \sin(t) \right\}, t \in R: \frac{\pi}{2} \leq t \leq \pi.$$
On the basis of the algorithm for solving the inverse problem the results of a numerical experiment were obtained and their visualization was performed (figure 5).

![Formation of the reflective surface $\Phi_I$](image)

**Figure 5.** Formation of the reflective surface $\Phi_I$.

5. **Discussion of the results**

The analysis of the results of numerical experiments and computational algorithms capabilities that obtain them shows that the definition of triads of surfaces along a given curve $P(\lambda)$ is a simple task from computational point of view. This facilitates making the optimal choice of a triad from a one-parameter set of triads designed for a certain technical application.

The solution of the inverse problem, i.e. the definition of a triad by a set of two curves $P(\lambda)$ and $P(\mu)$ in a plane, is much more complicated from the computational point of view. As a rule, numerical methods have to be applied in the mathematical model of solving this problem.

6. **Conclusions**

The paper proposes a geometric model and presents its mathematical description for the formation of triad surfaces possessing an optical property. The surfaces of the triad are geometrically interconnected, mathematically formalized and can be used, for example, in technical systems of radar as reflector, receiver and emitter surfaces.

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