Superfluidity of spin-1 bosons in optical lattices

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In this paper we show that the superfluidity of cold spin–1 Bose atoms of weak interactions in an optical lattice can be realized according to the excitation energy spectrum which is derived by means of Bogliubov transformation. The characteristic of the superfluid-phase spectrum is explained explicitly in terms of the nonvanishing critical velocity, i.e., the Landau criterion. It is observed that critical velocities of superfluid are different for three spin components and, moreover, can be controlled by adjusting the lattice parameters in practical experiments to detect the superfluid phase.

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I. INTRODUCTION

Originally discovered in the system of liquid helium and later in the context of superconductors, superfluidity is a hallmark property of interacting quantum fluids and encompasses a whole class of fundamental phenomena such as the absence of viscosity, persistent currents and quantized vortices. With the achievement of Bose-Einstein condensation (BEC) in alkali-metal atoms, the weakly interacting Bose gases have served as an idealized model of the superfluid [1] and a test ground of macroscopic quantum effects at low temperatures. Recently, rapid advances of experimental techniques in optical traps [2–4] open up a prospect for the study of the superfluidity of BEC trapped in periodic potentials, which has attracted fast growing interests both experimentally and theoretically [5,6]. The main reason is that the optical lattices possess controllable potential depths and lattice constants by adjusting the intensity of the laser beams in realistic experiments and moreover the lattice is basically defect free. In addition, the great advantage of optical traps is that it liberates the spin degree of freedom and provides us an opportunity to test spin-dependent quantum phe-
nomena that are absent in the scale-condensate cases. Theoretical studies have predicted a variety of novel phenomena of spinor condensates such as fragmented condensation [7], skyrmion excitations [8] and propagation of spin waves [8,9]. The quantum phase transition from a superfluid to a Mott-insulator (SF-MI) phase in spinor BEC has been observed in experiments [4]. In the Mott-insulator phase (MIP), atoms are localized; the particle-number fluctuations at each lattice site are suppressed so that there is no phase coherence across the lattice. When the tunnel coupling through the interwell barriers becomes large compared to the atom–atom interactions, the system undergoes a phase transition into the superfluid phase (SFP) in which the atom number per site is random and hence wave function exhibits long-range phase coherence. That means one can go from the regime in which the interaction energy dominates (high barrier of periodic potential) to the regime where the kinetic energy is the leading part (low barrier of periodic potential) by varying the intensity of laser beams and vice versa. The SF-MI transition has been also investigated in Refs. [10–14], where the Bose–Hubbard model is introduced as the starting point, and the analytic phase-transition condition and phase diagram have been obtained by using a perturbation expansion over the superfluid order parameter with on-site zero-order energy spectrum which, although gives rise to a reasonable description of the MIP, the SFP is not described explicitly. In the present paper we study the spinor BEC in a parameter region such that its ground state stands deeply in the SFP [14]. The Bogliubov approach is used to obtain the explicit excitation energy spectra of weakly interacting spin–1 atoms in an optical lattice and the superfluidity is explained explicitly in terms of the energy spectra. The critical velocities known as the Landau criterion for the superfluid phase are evaluated for the $^{23}\text{Na}$ atoms and are seen to be realizable in practical experiments. It is also demonstrated that the critical velocities are spin component dependent and controllable by adjusting of the lattice parameters. Our result may throw light on the experimental observation of the persistent atom-current density for the spinor-atom matter waves.
II. BÖGLIUBOV METHOD AND ENERGY SPECTRUM

Alkali-metal atoms with nuclear spin \( I = \frac{3}{2} \), such as \( ^{23}\text{Na},^{39}\text{K}, \) and \( ^{87}\text{Rb} \), behave at low temperatures like simple bosons with a hyperfine spin \( f = 1 \). The most general model Hamiltonian for the dilute gas of bosonic atoms with hyperfine spin \( f = 1 \) trapped in the optical potential can be written, in the second-quantization notation, as

\[
\hat{H} = \sum_{\alpha} \int d^3X \hat{\psi}^\dagger_{\alpha}(X) \left( -\frac{\nabla^2}{2M} + V_0(X) + V_T(X) \right) \hat{\psi}_{\alpha}(X) \\
+ \frac{C_0}{2} \sum_{\alpha,\beta} \int d^3X \hat{\psi}^\dagger_{\alpha}(X) \hat{\psi}^\dagger_{\beta}(X) \hat{\psi}_{\beta}(X) \hat{\psi}_{\alpha}(X) \\
+ \frac{C_2}{2} \sum_{\alpha,\beta,\alpha',\beta'} \int d^3X \hat{\psi}^\dagger_{\alpha}(X) \hat{\psi}^\dagger_{\beta}(X) F_{\alpha\alpha'} F_{\beta\beta'} \hat{\psi}_{\beta'}(X) \hat{\psi}_{\alpha'}(X),
\]

where \( M \) is the mass of a single atom; \( \hat{\psi}_{\alpha}(X) \) is the atomic field annihilation operator associated with atoms in the hyperfine spin state \( |f = 1, m_f = \alpha\rangle \) and the indices \( \alpha, \beta, \alpha', \beta' \) label the three spin components \( (\alpha, \beta, \alpha', \beta' = -1, 0, 1) \). \( V_0(X) = V_0(\sin^2 kX_1 + \sin^2 kX_2 + \sin^2 kX_3) \) is the optical lattice potential formed by laser beams, which is assumed to be the same for all three spin components, where \( k = 2\pi/\lambda \) is the wave vector of the laser light with \( \lambda \) being the wavelength of the laser light and \( V_0 \) is the tunable depth of the potential well and, hence, the lattice constant is \( d = \lambda/2 \). \( V_T(X) \) denotes an additional (slowly varying) external trapping potential, e.g., a magnetic trap. The \( 3 \times 3 \) spin matrices \( F \) denote the conventional three-dimensional representation (corresponding to the spin \( f = 1 \)) of the angular momentum operator with \( F^x_{\alpha\beta} = (\delta_{\alpha,-1} + \delta_{\alpha,1})/\sqrt{2}, F^y_{\alpha\beta} = i(\delta_{\alpha,-1} - \delta_{\alpha,1})/\sqrt{2}, F^z_{\alpha\beta} = \alpha \delta_{\alpha\beta} \). The coefficients \( C_0 \) and \( C_2 \) are related to scattering lengths \( a_0 \) and \( a_2 \) of two colliding bosons with total angular momenta 0 and 2, respectively, by \( C_0 = 4\pi\hbar^2 (2a_2 + a_0)/3M \) and \( C_2 = 4\pi\hbar^2 (a_2 - a_0)/3M \). For atoms \( ^{23}\text{Na}, \) we have \( a_2 > a_0 \), that is \( C_2 > 0 \) and the interaction is antiferromagnetic. While, for \( ^{87}\text{Rb} \) atoms the situation is just opposite that \( a_2 < a_0 \) (this leads to \( C_2 < 0 \)) and the interaction is ferromagnetic [8]. For the periodic potential, the energy eigenstates are Bloch states. We can expand the field operator \( \hat{\psi}_{\alpha}(X) \) in the Wannier basis, which is a superposition of Bloch...
states such that
\[ \hat{\psi}_\alpha(X) = \sum_i \hat{a}_{\alpha i} w(X - X_i), \]  
(2)
where \( w(X - X_i) \) are the Wannier functions localized in the lattice site \( i \) and \( \hat{a}_{\alpha i} \) corresponds to the bosonic annihilation operator on the \( i \)th lattice site. Also Eq. (2) suggests that atoms in different spin states are approximately described by the same coordinate wave function, which is seen to be the case when the spin-symmetric interaction is strong compared with the asymmetric part, i.e., \( |C_0| \gg |C_2| \) [11]. This is relevant to experimental conditions for \(^{23}\text{Na}\) and \(^{87}\text{Rb}\) atoms. Using Eq. (2), the general Hamiltonian (1) reduces to the Bose–Hubbard Hamiltonian
\[
\hat{H} = -J \sum_{\langle i,j \rangle} \alpha \sum_{\alpha} \hat{a}_{\alpha i} \hat{a}_{\alpha j} + \sum_{i} \sum_{\alpha} \varepsilon_i \hat{a}_{\alpha i} \hat{a}_{\alpha i} + \frac{1}{2} U_0 \sum_{i} \sum_{\alpha,\beta} \hat{a}_{\alpha i} \hat{a}_{\beta i} F_{\alpha \beta} F_{\beta \alpha} \hat{a}_{\beta i} \hat{a}_{\alpha i} + \frac{1}{2} U_2 \sum_{i} \sum_{\alpha,\beta,\alpha',\beta'} \hat{a}_{\alpha i} \hat{a}_{\beta i} F_{\alpha \alpha'} F_{\beta \beta'} \hat{a}_{\beta' i} \hat{a}_{\alpha' i}.
\]  
(3)
Here the first term in Eq. (3) describes the strength of the spin-symmetric tunneling, which is characterized by the hopping matrix element between adjacent sites \( i \) and \( j \). The tunneling constant \( J = -\int d^3X w^* (X - X_i) (-\nabla^2/2M + V_0(X)) w(X - X_j) \) depends exponentially on the depth of potential well \( V_0 \) and can be varied experimentally by several orders of magnitude. The second term denotes energy offset of the \( i \)th lattice site due to the external confinement of the atoms, where the parameter \( \varepsilon_i = \int d^3X w^* (X - X_i) V_T(X) w(X - X_i) \) is assumed to be of the same value \( \varepsilon \) for all lattice sites in the present paper. The third and fourth terms characterize the repulsion interaction between two atoms on a single lattice site, which is quantified by the on-site interaction matrix element \( U_{0(2)} = C_{0(2)} \int |w(X - X_i)|^4 d^3X \). In our case the interaction energy is very well determined by the single parameters \( U_0 \) and \( U_2 \), due to the short range of the interactions, which is smaller than the lattice spacing.

Now we use Bogliubov approach to diagonalize the Hamiltonian and obtain the excitation energy spectrum of spin–1 bosons with weak interaction [14] in an optical lattice and hence
to study the SFP property explicitly. To this end we firstly express the site space operator \( \hat{a}_{\alpha i} \) in terms of the wave–vector space operator \( \hat{a}_{k,\alpha} \) as

\[
\begin{align*}
\hat{a}_{\alpha i} &= \frac{1}{\sqrt{N_s}} \sum_k \hat{a}_{k,\alpha} e^{i k \cdot X_i}, \\
\hat{a}_{\alpha i}^\dagger &= \frac{1}{\sqrt{N_s}} \sum_k \hat{a}_{k,\alpha}^\dagger e^{-i k \cdot X_i},
\end{align*}
\]

(4)

where \( N_s \) is the total number of the lattice sites and \( X_i \) is the coordinate of site \( i \). The wave vector \( k \) runs only over the first Brillouin zone. In the tight–binding approximation (TBA) if we limit our description to simple cubic lattice and substitute Eq. (4) into the Hamiltonian (3), we can obtain

\[
\hat{H} = \sum_{\alpha} \sum_k \varepsilon (k) \hat{a}_{k,\alpha}^\dagger \hat{a}_{k,\alpha} + \hat{H}_{\text{int}},
\]

\[
\hat{H}_{\text{int}} = \frac{U_0}{2N_s} \sum_{\alpha,\beta, k, p, k', p'} \delta_{k+p, k'+p'} \hat{a}_{k,\alpha}^\dagger \hat{a}_{p,\beta}^\dagger \hat{a}_{p',\beta} \hat{a}_{k',\alpha},
\]

\[
+ \frac{U_2}{2N_s} \sum_{\alpha, \beta, \alpha', \beta', k, p, k', p'} \delta_{k+p, k'+p'} \hat{a}_{k,\alpha}^\dagger \hat{a}_{p,\beta}^\dagger F_{\alpha\alpha'} F_{\beta\beta'} \hat{a}_{p',\beta'} \hat{a}_{k',\alpha'},
\]

(5)

where \( \varepsilon (k) = \varepsilon - J z \cos (kd) \) with \( z \) the number of nearest neighbors of each site. Since the number of atoms condensed in the zero-momentum state is much larger than one, we have

\[
\sum_{\alpha} \hat{a}_{\alpha 0} \hat{a}_{\alpha 0}^\dagger = \sum_{\alpha} \hat{a}_{\alpha 0}^\dagger \hat{a}_{\alpha 0} + 1 \approx \sum_{\alpha} N_{\alpha 0} = N_0 \gg 1.
\]

(6)

\( N_{\alpha 0} \) is the number of condensed atoms of spin-\( \alpha \) component in the zero-momentum state and \( N_0 \) is the total number of condensed atoms. So, we can replace the operator \( \hat{a}_{\alpha 0} \) and \( \hat{a}_{\alpha 0}^\dagger \) with a “ \( c \)” number \( \sqrt{N_{\alpha 0}} \). Thus we have

\[
N_{\alpha 0} = N_\alpha - \sum_{k \neq 0} \hat{a}_{k,\alpha}^\dagger \hat{a}_{k,\alpha},
\]

(7)

where \( N_\alpha \) is the total number of atoms of spin-\( \alpha \) component and the term of \( k \neq 0 \) is exclusive in the wave-vector sum. Moreover, in the interacting part of the Hamiltonian when \( k \neq 0 \) we regard \( \hat{a}_{k,\alpha}^\dagger, \hat{a}_{k,\alpha} \) as the small deviation from the operators of vanishing momentum, thus all products of four boson operators are approximated as quadratic form, for example,

\[
\hat{a}_{0,\alpha}^\dagger \hat{a}_{0,\alpha}^\dagger \hat{a}_{0,\alpha} \hat{a}_{0,\alpha} = N_{\alpha 0}^2 \approx N_\alpha^2 - 2N_\alpha \sum_{k \neq 0} \hat{a}_{k,\alpha}^\dagger \hat{a}_{k,\alpha},
\]

(8)
\[
\sum_{\alpha,\beta,k \neq 0} \hat{a}^\dagger_{0,\alpha} \hat{a}^\dagger_{k,\beta} \hat{a}_{k,\beta} \hat{a}_{0,\alpha} = \sum_{\alpha,\beta,k \neq 0} \hat{a}^\dagger_{k,\alpha} \hat{a}_{k,\beta} \hat{a}_{0,\alpha} = \left[ \sum_{\alpha} \left( N_{\alpha} - \sum_{k \neq 0} \hat{a}^\dagger_{k,\alpha} \hat{a}_{k,\alpha} \right) \right] \sum_{\beta,k \neq 0} \hat{a}^\dagger_{k,\beta} \hat{a}_{k,\beta} \approx N \sum_{\alpha,k \neq 0} \hat{a}^\dagger_{k,\alpha} \hat{a}_{k,\alpha}
\]

(9)

where \( N = \sum_{\alpha} N_{\alpha} \) is the total number of atoms. With the approximation Eqs. (8) and (9) the total Hamiltonian (5) can be written as

\[
\hat{H} = \frac{U_0}{2N_s} N^2 + N (\varepsilon - z J) + \frac{U_2}{2N_s} [(N_1 - N_{-1})^2 + 2N_0(\sqrt{N_1} + \sqrt{N_{-1}})^2] + \sum_{\alpha} \left[ \sum_{\gamma \neq \pm 1} \varepsilon_{\gamma} \right] \hat{a}^\dagger_{k,\alpha} \hat{a}_{k,\alpha}
\]

\[
+ \frac{U_0}{2N_s} \sum_{\alpha,\beta} \sqrt{N_{\alpha 0}} \sqrt{N_{0 \beta}} (\hat{a}^\dagger_{k,\alpha} \hat{a}^\dagger_{-k,\beta} \hat{a}_{k,\beta} \hat{a}_{-k,\alpha} + 2\hat{a}^\dagger_{k,\alpha} \hat{a}_{k,\beta}) + \frac{U_2}{2N_s} \left( 2N_0 (\hat{a}^\dagger_{k,1} \hat{a}^\dagger_{-k,-1} + \hat{a}_{k,1} \hat{a}_{-k,-1}) + \sum_{\gamma = \pm 1} \hat{a}_{k,\gamma} \hat{a}_{-k,\gamma} \hat{a}_{k,\gamma} + 2\hat{a}_{k,0} \hat{a}_{k,\gamma} \right)
\]

\[
+ 2\sqrt{N_0} \sqrt{N_{-10}} (\hat{a}^\dagger_{k,0} \hat{a}^\dagger_{-k,0} - \hat{a}^\dagger_{k,1} \hat{a}^\dagger_{-k,-1} + \hat{a}_{k,0} \hat{a}_{-k,0} - \hat{a}_{k,1} \hat{a}_{-k,1} - \hat{a}^\dagger_{k,1} \hat{a}_{k,-1} - \hat{a}^\dagger_{k,-1} \hat{a}_{k,1} - 2\hat{a}^\dagger_{k,0} \hat{a}_{k,0}) \right] .
\]

Note that the ratio \( U_2/U_0 \) is proportional to the ratio of \( C_2/C_0 \) for all lattice geometries and hence \( U_2/U_0 \) is small enough in general. Therefore the spin-asymmetric part of the interaction is much smaller than the spin-symmetric one.

The Hamiltonian (10) is quadratic in the operators \( \hat{a}_{k,\alpha}, \hat{a}^\dagger_{-k,\alpha} \) and can be diagonalized by the linear transformation

\[
\hat{a}_{k,\alpha} = u_{k,\alpha} \hat{b}_{k,\alpha} - v_{k,\alpha} \hat{b}_{-k,\alpha},
\]

\[
\hat{a}^\dagger_{k,\alpha} = u_{k,\alpha} \hat{b}^\dagger_{k,\alpha} - v_{k,\alpha} \hat{b}^\dagger_{-k,\alpha},
\]

(11)

known as the Bogoliubov transformation. This transformation introduces a new set of operators \( \hat{b}_{k,\alpha} \) and \( \hat{b}^\dagger_{k,\alpha} \) to which we impose the same Bose-operator commutation relations

\[
[\hat{b}_{k,\alpha}, \hat{b}^\dagger_{k',\beta}] = \delta_{k,k'} \delta_{\alpha \beta} .
\]

It is easy to check that the commutation relations are fulfilled if the parameters \( u_{k,\alpha} \) and \( v_{k,\alpha} \) satisfy the relation

\[
u_{k,\alpha}^2 - v_{k,\alpha}^2 = 1,
\]

(12)

where the auxiliary parameters \( u_{k,\alpha} \) and \( v_{k,\alpha} \) are to be chosen in order to have the vanishing coefficients of the nondiagonal terms \( \hat{b}^\dagger_{k,\alpha} \hat{b}^\dagger_{-k,\alpha} \) and \( \hat{b}_{k,\alpha} \hat{b}_{-k,\alpha} \) in the Hamiltonian (10) (see the
Appendix for detail). In virtue of the Bogliubov transformation (11), we finally obtain the diagonalized Hamiltonian as

$$\hat{H} = E_c + \sum_{k \neq 0} E_{k,\alpha,\alpha} \hat{b}_{k,\alpha}^\dagger \hat{b}_{k,\alpha}$$

with

$$E_c = \frac{1}{2} U_0 N \varepsilon^2 + N (\varepsilon - zJ) + \frac{U_2}{2N_s}[(N_1 - N_{-1})^2 + 2N_0(\sqrt{N_1} + \sqrt{N_{-1}})^2],$$

where the energy spectra $E_{k,\alpha,\alpha}$ ($\alpha = 0, \pm 1$) of the quasiparticle are given by

$$E_{k,\gamma,\gamma} = \sqrt{\varepsilon_k (\varepsilon_k + 2U_0 n_\gamma + 2U_2 n_\gamma)},$$

$$E_{k,0,0} = \sqrt{\varepsilon_k (\varepsilon_k + 2U_0 n_0 + 4U_2 \sqrt{n_{-1} n_1})},$$

where $\gamma = \pm 1$ and

$$\varepsilon_{k(k\neq 0)} = zJ [1 - \cos (kd)].$$

The symbol $n_\alpha = N_\alpha / N_s$ represents the average atom number of spin-$\alpha$ component per lattice site. In the experiments of Ref. [4] the number of atoms per lattice site is shown to be around $1 - 3$.

### III. CRITICAL VELOCITY OF SUPERFLUID

The energy spectra Eqs. (14) and (15) are typical for the superfluid. To this end we look at the dispersion relations of energy spectra $E_{k,\alpha,\alpha}$ for the limit case $k \to 0$

$$E_{k,\gamma,\gamma} \sim [zJd^2 (U_0 + U_2) n_\gamma]^{1/2} k,$$

$$E_{k,0,0} \sim [zJd^2 (U_0 n_0 + 2U_2 \sqrt{n_{-1} n_1})]^{1/2} k.$$
\[ v_{s,\gamma} = \left( \partial E_{k,\gamma,\gamma} / \partial k \right)_{k \to 0} = \frac{1}{\hbar} [zJd^2 (U_0 + U_2) n_\gamma]^{1/2}, \]  
(18)

\[ v_{s,0} = \left( \partial E_{k,0,0} / \partial k \right)_{k \to 0} = \frac{1}{\hbar} [zJd^2 (U_0 n_0 + 2U_2 \sqrt{n_{-1} \sqrt{n_1}})]^{1/2}, \]  
(19)

which reduce to the critical velocity of superfluid given in Ref. [13] for the spin-zero case when \( U_2 = 0 \), where \( 1/\hbar \) is dimension correction. The nonvanishing velocity is nothing but the Landau criterion for the superfluid phase. As seen from the above formulas (18) and (19), whether there exist critical velocities of superfluid or not depend on appropriate values of \( J \) and \( U_{0(2)} \) which are related to the Wannier functions determined essentially by the potential of optical lattice. Thus, \( J \) and \( U_{0(2)} \) can be controlled dependently by adjusting the laser parameters. Since the spin-asymmetric interaction \( U_2 \) is typically one to two orders of magnitude less than the spin-symmetric interaction \( U_0 \), it can ensure that the nonvanishing \( v_{s,\alpha} \) (\( \alpha = 0, \pm 1 \)) exist, whether for the antiferromagnetic interaction \( (U_2 > 0) \) or for the case of ferromagnetic interaction \( (U_2 < 0) \).

Certainly, it is important to see whether or not the superfluid phase can be realized practically with the recent progress of experiments on the confinement of atoms in the light-induced trap. To see this we evaluate the values of critical velocities of superfluid adopting the typical experimental data in Ref. [15] for a spin-1 condensate of \(^{23}\text{Na} \) atoms in the optical lattice created by three perpendicular standing laser beams with \( \lambda = 985 \text{ nm} \). The scattering lengths for \(^{23}\text{Na} \) atoms are \( a_0 = (46 \pm 5)a_B \) and \( a_2 = (52 \pm 5)a_B \), where \( a_B \) is the Bohr radius (corresponding to a ratio value \( U_2/U_0 = 0.04 \)). The valid condition of the Bogliubov approach that \( U << J \) (for example, \( U/J << 0.1 \)) can be fulfilled in a region of barrier-height values of the optical lattice potential \( V_0 \) from 0 to two or three times \( E_R \), where \( E_R \) is the recoil energy. It turns out that the magnitude of the critical velocities of the superfluid is the order of mm/s which is seen to be in the range of experimental values [15].

The critical velocities of superfluid \( v_{s,\alpha} \) different for three spin components are functions of densities. Therefore, the critical velocities of superfluid can be detected experimentally by
counting the atom–number populations. Moreover, the component-dependent velocities may imply component separation of spinor BEC in an optical lattice similar to the experimentally observed component separation in a binary mixture of BECs [16] since one can control the subsequent time evolution of the mixture of a three-condensate system and detect the relative motions of the three components which tend to preserve the density profiles, respectively. In particular, we may consider a condensate of spin–polarized atoms which are all in the state of spin component $\alpha = 0$ at initial time $t = 0$, i.e., $|\psi(0)\rangle = |0, N_0, 0\rangle$. In this case a pair of atoms in the $\alpha = 0$ state can be excited into the $\alpha = \pm 1$ states respectively. Thus after a time $t_c$ the number of atoms of the $\alpha = 0$ component becomes steady such that $N_0(t_c) = N_0/2$ [11] and the number of atoms of $\alpha = \pm 1$ components is about half of $N_0(t_c)$, i.e., $N_{-1}(t_c) = N_1(t_c) = N_0(t_c)/2 \approx N_0/4$. Consequently, the critical velocities of superfluid for $\alpha = -1$ ($v_{s,-1}$) and $\alpha = 1$ ($v_{s,1}$) are equal and different from that for the $\alpha = 0$ component ($v_{s,0}$). There exists a simple relation that $v_{s,-1} = v_{s,1} = v_{s,0}/\sqrt{2}$ for the case considered and therefore the atoms of the $\alpha = 0$ component can be separated from the mixture of BECs.

IV. CONCLUSION

The energy–band structure of excitation spectra derived in terms of Bogliubov transformation for spin–1 cold bosons in an optical lattice is shown to be typical for the superfluid phase from the viewpoint of the Landau criterion. Our observation is that the critical velocities of the superfluid flow are spin-component dependent and can be controlled by adjusting the laser lights that form the optical lattice. The theoretical values of critical velocities obtained are in agreement with experimental observations and possible experiments to detect the superfluid phase are also discussed.
V. ACKNOWLEDGMENT

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VI. APPENDIX

The inverse transformation of Eq. (11) is

\[
\begin{align*}
\hat{b}_{k,a} &= u_{k,a} \hat{a}_{k,a} + v_{k,a} \hat{a}_{-k,a}^\dagger, \\
\hat{b}_{k,a}^\dagger &= u_{k,a} \hat{a}_{k,a}^\dagger + v_{k,a} \hat{a}_{-k,a}.
\end{align*}
\]

\[\text{(A1)}\]

The Hamiltonian Eq. (10) can be written in terms of the quasiboson operators \(\hat{b}_{k,a}\) and \(\hat{b}_{k,a}^\dagger\) as

\[
\hat{H} = E_c + \hat{H}_1 + \hat{H}_2,
\]

\[\text{(A2)}\]

where

\[
E_c = \frac{U_0}{2N_s} N^2 + N (\varepsilon - zJ) + \frac{U_2}{2N_s} [(N_1 - N_{-1})^2 + 2N_0(\sqrt{N_1} + \sqrt{N_{-1}})^2]
\]

\[\text{(A3)}\]

is a constant. \(\hat{H}_1\) and \(\hat{H}_2\) denote the diagonal and off-diagonal parts, respectively, with

\[
\begin{align*}
\hat{H}_1 &= \sum_{k \neq 0} \left( \left[ \left( \varepsilon_k + \frac{U_0}{N_s} N_0 - 2 \frac{U_2}{N_s} \sqrt{N_{-10}} \sqrt{N_{10}} \right) \left( u_{k,0}^2 + v_{k,0}^2 \right) - \left( \frac{2U_0}{N_s} N_0 + \frac{4U_2}{N_s} \sqrt{N_{-10}} \sqrt{N_{10}} \right) u_{k,0} v_{k,0} \right] \hat{b}_{k,0}^\dagger \hat{b}_{k,0} \\
&+ \sum_{\gamma} \left[ \left( \varepsilon_k + \frac{U_0}{N_s} N_\gamma + \frac{U_2}{N_s} N_{\gamma_0} \right) \left( u_{k,\gamma}^2 + v_{k,\gamma}^2 \right) - \left( \frac{2U_0}{N_s} + \frac{2U_2}{N_s} \right) N_{\gamma} u_{k,\gamma} v_{k,\gamma} \right] \hat{b}_{k,\gamma}^\dagger \hat{b}_{k,\gamma} + \sum_{\gamma} \left[ \left( \frac{U_0}{N_s} \right) \sqrt{N_{\gamma_0} \sqrt{N_{-\gamma_0}}} - \frac{U_2}{N_s} N_{\gamma_0} \right] \left( u_{k,\gamma} v_{k,-\gamma} + u_{k,-\gamma} v_{k,\gamma} \right) \right) \hat{b}_{k,\gamma}^\dagger \hat{b}_{k,\gamma} \\
&+ \sum_{\gamma} \left[ \left( \frac{U_0}{N_s} + \frac{U_2}{N_s} \right) \sqrt{N_{\gamma_0} \sqrt{N_{-\gamma_0}}} + \frac{2U_2}{N_s} \sqrt{N_{-\gamma_0} \sqrt{N_{00}}} \right] \left( u_{k,\gamma} u_{k,0} + u_{k,0} v_{k,\gamma} \right) - \left( \frac{U_0}{N_s} + \frac{U_2}{N_s} \right) \sqrt{N_{\gamma_0} \sqrt{N_{-\gamma_0}}} \left( u_{k,\gamma} u_{k,-\gamma} + u_{k,-\gamma} v_{k,\gamma} \right) \right) \hat{b}_{k,\gamma}^\dagger \hat{b}_{k,\gamma} \right) + \text{const},
\end{align*}
\]

\[
\hat{H}_2 = \sum_{k \neq 0} \left( \left[ \frac{\varepsilon_k}{2N_s} N_0 + \frac{U_2}{N_s} \sqrt{N_{-10}} \sqrt{N_{10}} \right] \left( u_{k,0}^2 + v_{k,0}^2 \right) - \left( \varepsilon_k + \frac{U_0}{N_s} N_0 + 2 \frac{U_2}{N_s} \sqrt{N_{-10}} \sqrt{N_{10}} \right) u_{k,0} v_{k,0} \right] \hat{b}_{k,0}^\dagger \hat{b}_{k,0} \right)
\]

\[\text{(A3)}\]
In order to eliminate the off-diagonal part $\tilde{H}_2$ we require that the coefficients of all terms $\hat{b}_{k,0}^\dagger \hat{b}_{-k,0}$ and $\hat{b}_{k,\alpha} \hat{b}_{-k,\alpha}$ vanish. In view of condition (12), it is easy to introduce a set of parameters $\phi_{k,\alpha}$ such that

\[
\begin{align*}
    u_{k,\alpha} &= \cosh \phi_{k,\alpha}, \\
    v_{k,\alpha} &= \sinh \phi_{k,\alpha}
\end{align*}
\]  

(A6)

for the convenience of calculation. Conditions (12) and (A6) lead to the useful relations

\[
\begin{align*}
    \tanh 2\phi_{k,\alpha} &= \frac{2u_{k,\alpha}v_{k,\alpha}}{u_{k,\alpha}^2 + v_{k,\alpha}^2}, \\
    \cosh (2\phi_{k,\alpha}) &= \frac{u_{k,\alpha}^2 + v_{k,\alpha}^2}{\sqrt{1 - \tanh^2 2\phi_{k,\alpha}}}.
\end{align*}
\]

with which the Hamiltonian (10) can be finally reduced to the diagonal form as given in Eq. (13).

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