Universal multiport interferometers that can be programmed to perform any unitary or linear transformation turn into an important building block for both classical and quantum photonics. These interferometers typically utilize the mathematical framework of U(2) unitary matrix decomposition techniques and comprise a mesh of $2 \times 2$ beam splitters and phase shifters. All of them are, however, inherently fidelity limited as their U(2)-based deployment approach leads to imbalanced path losses without supporting any fidelity restoration mechanism. Herein, a novel Universal Generalized Mach–Zehnder Interferometer (UGMZI) architecture is presented that migrates from U(2) decomposition techniques and adopts a recursive U(N) decomposition, exploiting cascaded size-augmenting $N \times N$ beam splitters and phase shifters. The design can natively support fidelity restoration and safeguard absolute fidelity, while outperforming the state-of-the-art designs also with respect to phase-error-induced fidelity performance properties. Finally, it is demonstrated that its fidelity restoration properties turn the design into the optimal architecture for constructing any real/complex-valued matrix via the singular value decomposition (SVD) scheme.

1. Introduction

Unitary and linear transformations enabled in the field of optics by universal multiport interferometers are critical functions in a broad field of applications including optical networking,\cite{1,2} photonic neural networks,\cite{3,4} microwave photonics,\cite{5,6} and quantum photonics.\cite{7,9} These devices comprise typically a multiport interferometric mesh that is capable of transferring any real- or complex-valued matrix into the optical domain\cite{7,9}—following the singular value decomposition (SVD) technique,\cite{10,11,12} which in turn rests upon the factorization of unitary operators. Consequently, decomposition of unitary matrices and their practical transfer into the field of integrated photonics is a fundamental step in a wide range of linear algebra problems required to be implemented by optical circuitry. Although several unitary matrix decomposition methods have been formulated mathematically over the years,\cite{13,14,15,16} photonic unitary matrix deployments have relied so far exclusively on U(2) factorization techniques that are translated into circuit layouts through a mesh of phase shifters (PSs) and $2 \times 2$ Mach–Zehnder Interferometric (MZI) nodes serving as beam splitters.\cite{17,18,19} Based on the parametrization of unitary and rotation groups proposed by Murnaghan,\cite{13} Reck et al.\cite{17} pioneered a seminal triangular mesh of $(N-1)/2$ lossless tunable MZI-based beam-splitters followed by N lossless PSs that can turn any given $N \times N$ unitary operator into a photonic circuit requiring N-1 steps of programming. More recently, a unitary matrix decomposition method supporting a simpler programming procedure but retaining the same optical depth and relying again on lossless nodes has been proposed by De Guise et al.\cite{20}

In practice, however, neither the $2 \times 2$ beam splitters nor the PSs elements are lossless, as assumed by Reck et al. and De Guise et al.,\cite{17,18} meaning that the fidelity of the circuit architecture degrades when real lossy elements are employed. Fidelity degradation suggests that the photonically realized matrix deviates from the targeted unitary matrix, with all U(2)-based optical layouts demonstrated so far inherently failing in restoring loss-induced fidelity and matrix representation accuracy. The optimal multiport interferometric layout reported so far by Clements et al.\cite{19} uses the same number of beam splitter and PS nodes but relies on a more compact decomposition topology, yielding half of the optical depth compared to the Reck’s and De Guise designs and supporting a more loss-resistive performance. However, even this design can’t yield an absolute match between the targeted unitary matrix and its experimental implementation, being the result of the inherently absent mechanism for fidelity restoration in the U(2)-based layouts. The situation becomes even worse in the case of SVD-based experimental schemes that can implement any, even non-unitary, linear transformation, where two cascaded unitary optical designs have to be incorporated leading to accumulation of the fidelity degradation. The fidelity performance of all these unitary optical designs can be only improved either by inserting dummy components that obviously intervenes a priori at the core hardware design and modifies the respective architecture, as shown in refs. \cite{20,21} or by sacrificing the universality of the configuration.\cite{12}

In this work, we demonstrate a universal unitary matrix decomposition scheme that demarcates from U(2)-based
decomposition schemes and relies on recursive U(N) matrix factorization, outperforming all state-of-the-art unitary optical multiport interferometers in three key aspects: first, it allows for a fidelity restoration mechanism so that loss-induced fidelity can always turn into the optimal 100%. Second, its fidelity restoration properties can be naturally transferred into fidelity restorable SVD-based linear optical transformations and, finally, it provides improved robustness also to phase-related imperfections. Our findings rely on a recursive U(N) mathematical decomposition scheme that has been first formulated by Dita\cite{16} and the introduction of a Special Generalized Mach–Zehnder Interferometer (SGMZI) as the architectural candidate for realizing the U(N) matrix elements required in Dita’s recursive factorization. SGMZI serve as \(N \times N\) beam splitters followed by \(N\) PSs, with every \(N \times N\) beam splitter comprising two \(N \times N\) optical couplers and \(N\) PSs within a simple Generalized MZI design. We provide the mathematical framework for our new Universal Generalized Mach–Zehnder Interferometer (UGMZI) architecture that comprises \(N\) size-augmenting SGMZIs and we compare its performance with the Clements and Reck layouts in terms of their loss- and phase-error resistivity. We show that the UGMZI scheme can always restore loss-induced fidelity to the maximum of 100%, showing also increased robustness to phase errors compared to its counterparts. Finally, we prove that the fidelity restoration properties of the UGMZI, when transferred to SVD-based optical designs, provide absolute fidelity without requiring any architectural changes, creating an optimal multiport interferometric arrangement for the realization of any real/complex-valued matrix implementation.

2. Background

The prevailing method for decomposition of any unitary matrix \(U(N)\) relies on its segmentation to elementary \(U(2)\) matrices, that can be implemented in photonic domain by MZI structures supplemented by two PSs.\cite{17,19} Such decomposition techniques are based on a recursive procedure, where the targeted matrix \(U(N)\) is sequentially transformed by modified Givens rotation matrices\cite{22} \(R_{r,c}\), of the form equivalent to

\[
R_{r,c} = \begin{bmatrix}
I_{c-1} & 0 & \cdots & 0 \\
0 & e^{i\theta} \cos \theta & \cdots & -e^{i\theta} \sin \theta \\
\vdots & \vdots & \ddots & \vdots \\
0 & e^{i\phi} \sin \theta & \cdots & e^{i\phi} \cos \theta \\
0 & 0 & \cdots & I_{N-r}
\end{bmatrix}
\]

(1)

where \(c \leq r \leq N\) and which differ from the identity matrix of size \(N\), denoted \(I_{N}\), only in elements indexed by permutations of \(r\) and \(c\) as follows: \(r,c = e^{i\theta} \cos \theta, r,c = e^{i\theta} \sin \theta, r,c = -e^{i\phi} \sin \theta, r,c = \cos \theta\). Matrices \(R_{r,c}\) can be experimentally realized by MZI nodes that couple the signal of the \(c\)-th and \(r\)-th lanes of an \(N\) input \(-N\) output configuration, as shown in Figure 1. These constituent nodes comprise two 3 dB directional couplers placed in an MZI configuration with a 20-PS with

\[
\theta \in [0, \pi/2] \text{ at one of its arms. A } \varphi\text{-PS with } \varphi \in [0, 2\pi] \text{ is employed at one of its inputs.}
\]

The targeted matrix \(U(N)\) is decomposed into a product of a diagonal phase matrix \(D\), with elements having unitary magnitude, implemented by a PS at each lane and \(N(N-1)/2\) Givens rotation matrices \(R_{r,c}\), equaling

\[
U_{Cl/R} = D \prod_{(r,c) \in W_{Cl/R}} R_{r,c}
\]

(2)

where \(W_{Cl/R}\) represents the particular sorted of ordered pairs \((r,c)\) in which the transformations are applied. The main differentiator between Clements’ (Cl) and Reck’s (R) implementations is that \(W_{Cl} \neq W_{R}\), resulting in different topologies.

3. Universal Generalized MZI Unitary Decomposition Method

Here, we introduce a novel unitary decomposition algorithm and the corresponding photonic architecture that is based on concatenated size-augmenting SGMZIs\cite{23-25} scheme, exploiting the unitary decomposition formula proposed by Dita.\cite{16} Decomposition procedure rests upon two facts: 1) each unitary matrix of size \(n\), \(P_{n}\), can be written as a product of a special unitary matrix of size \(n\), \(Q_{n}\), uniquely defined by its first column vector with \(2n - 1\) free real parameters, and a unitary block matrix of size \(n\), consisting of an arbitrary unitary matrix of size \(n - 1\), \(P_{n-1}\), and an element 1 on its diagonal supplementing up to the size of \(n\); and 2) each unitary matrix of size \(n\), \(Q_{n}\), can be written in a form of a product of a diagonal matrix with phase elements, \(D_{n}\), (magnitude of all diagonal elements is one) and a unitary matrix of size \(n\), \(Q_{n}\), generated by its first column vector, which is populated by non-negative numbers.

The first fact allows for sequential problem dimensionality reduction (i.e., size-augmenting), opening the possibility for implementing a recursive algorithm, whereas the second allows for the problem to be treated independently with respect to magnitudes and phases, a property particularly of interest when translating the algorithm into a photonic platform layout. Following a recursive procedure and relying on the previously stated, the work of Dita\cite{16} concludes that any unitary matrix \(U(N)\) can be decomposed into a product of \(N\) matrices of the form

\[
U(N) = \prod_{n=N}^{1} Q_{n}^{N-n}, \quad Q_{n}^{N-n} = \begin{bmatrix}
I_{N-n} & 0 \\
0 & Q_{n}
\end{bmatrix}
\]

(3)

where \(Q_{N-n}\) is an \(N \times N\) unitary block diagonal matrix, constructed of an identity matrix of the dimension \(N - n\) in its upper-left block, \(I_{N-n}\), and an \(n \times n\) unitary matrix \(Q_{n}\) in
its lower-right block, uniquely defined by its first column vector $q_{n,1} \in \mathbb{S}_{2n-1} \in \mathbb{C}^n$ and parametrized by $2n-1$ real variables, where the $\mathbb{S}_{2n-1}$ stands for the unit sphere of the Hilbert space $\mathbb{C}^n$, whose real dimension is $2n-1$. Matrices $Q_n$ are determined sequentially, starting from $Q_1$, following identical procedure, which single step assumes solving a system of $n$ complex linear equations with respect to $n$ complex variables, out of which, only $2n-1$ parameters are free to preserve unitarity property of the matrices.

Following the second statement from,\cite{23} matrix $Q_n$ can be written in a form of $Q_n = D_n \tilde{Q}_n$, by factoring out the phases of all elements in the first column of $Q_n$ to a diagonal matrix, leaving the first column of $\tilde{Q}_n$ populated by non-negative numbers. In this manner, $D_n$ will have $n$ free real parameters related to phases $\arg(d_{r,k}) = \theta_{r,k}$, whereas the remaining $n-1$ free parameters, related to the magnitudes, will remain in the first column of $\tilde{Q}_n$.

In translating $\tilde{Q}_n$ matrix to a photonic platform, we employ a modified version of the $n \times n$ Generalized Mach-Zehnder Interferometer (GMZI) that has been proven to serve as variable ratio power splitter, able to cast the light from any or all input to any or all output port(s) according to the assigned ratio,\cite{23-25} implying that it is capable of implementing any unit vector of the real sphere of dimension $n$ via its electrical field magnitudes. Typical $n \times n$ GMZI devices comprise two $n \times n$ Multimode Interferometers (MMI) interposed by $n$ PSs, as illustrated in the yellow frame of Figure 2. Their transfer matrix can be expressed as

$$G_n = T_n \Theta_n T_n$$

with $T_n$ and $\Theta_n$ being the transfer matrices of the $n \times n$ MMI couplers and the interleaved PSs, respectively

$$T_n = \frac{1}{\sqrt{n}} \left[ e^{i\theta_{0,1}}, \ldots, e^{i\theta_{0,n}} \right] \ldots \left[ e^{i\theta_{n,1}}, \ldots, e^{i\theta_{n,n}} \right]$$

$$\Theta_n = \text{diag}(e^{i\theta_0}, e^{i\theta_1}, \ldots, e^{i\theta_n})$$

both of which are unitary, making also $G_n$, defined by Equation (4), a unitary matrix. Phases $\varphi_{k,x,y}$ in Equation (5), associated with imaging an input $x$ to an output $y$ in an $n \times n$ MMI coupler, can be determined from ref. [26], and calculated as follows

$$\varphi_{k,x,y} = \begin{cases} \varphi_0 + \frac{\pi}{4n} (n + 1 - x - y)(n - 1 + x + y), & n + x - y \text{odd} \\ \varphi_0 + \frac{\pi}{4n} (n + x - y)(n - x + y), & n + x - y \text{even} \end{cases}$$

where $\varphi_0$ denotes a constant phase associated with the design parameters of the device (MMI coupler length, number of inputs/outputs, propagation constant). From the experimental perspective, thus far, demonstrations as high as $16 \times 16$ have already been developed\cite{27} in the form of MMI couplers, while several theoretical design methods have been proposed, presenting general mathematical formulas that analytically describe the transfer function and calculate the phase and amplitude relations between the inputs and outputs of any dimension MMI coupler.\cite{28,29} To this end, as the silicon photonic technology progressively matures, higher-dimension MMI couplers are expected to be fabricated and exploited into photonic architectures. In contrast, directional couplers could, also, be employed as an alternative solution for the construction of an $N \times N$ coupler, but in turn, the dimension $N$ is required to be power of 2. Other alternatives, such as the method described in ref. [30], any dimension coupler (without any restriction, e.g., $n$ equals power of 2) is able to be implemented using only $3 \text{ dB}$ couplers, and as such it can be adopted in our architecture. In either of the two cases, GMZI transfer function reveals $n$ free real parameters $\theta_k$, which serve for intensity redistribution, or, in other words, adjust the magnitudes of matrix elements if used in our matrix decomposition scheme. This implies that the magnitudes of the $G_n$ first column elements need to be equal to the magnitudes of the corresponding first column elements of the targeted matrix $Q_n$, or

$$|q_{r,1}| = \frac{1}{n} \left| \sum_{k=1}^{n} e^{i\theta_k} e^{i(\varphi_{r,k} + \varphi_{r,1})} \right|$$

where $r \in [0,n]$. In the system of equations given by Equation (9), not all n parameters $\theta_k$ are free. In addition, it employs $n-1$ free $\theta_1$ parameters with the $n$th parameter being used to set the reference point. Once Equation (9) is solved, as described in refs. [23–25], all elements of the matrix $G_n$ can be determined. Even though the magnitudes of the first columns’ elements of $G_n$ and $\tilde{Q}_n$ will be the same, elements of $G_n$ will have nonzero phases. To meet the final target, that is $Q_n$, phase of each element in the first column of $G_n$ needs to be adjusted to cancel the accumulated phase shift within GMZI and introduce a new phase shift equal to $\arg(q_{r,1})$. We achieve this by placing $n$ PSs at the output of GMZI, resulting in a configuration that we call special-, or SGMZI, and is illustrated in Figure 2. The newly introduced PSs are described by a diagonal matrix

$$\Omega_n = \text{diag}(e^{i\omega_1}, e^{i\omega_2}, \ldots, e^{i\omega_n})$$

where $\omega_k = \arg(q_{r,1}) - \arg(\tilde{g}_{r,1})$ with $\tilde{g}_{r,1}$ denoting the $r$th element of the first column vector of $\tilde{Q}_n$. Finally, the transfer matrix of the whole SGMZI unit cell can be written as
Concatenating $N$ size-augmenting SGMZIs of appropriate dimensions, from $n = N$ to $n = 1$, as shown in Figure 3, we reach a universal-, or UGMZI layout, able to represent an arbitrary unitary matrix of size $N$. In accordance with Equation (3), (4), (9), and (10), the transfer matrix of UGMZI reads

$$U_{\text{UGMZI}}(N) = \prod_{n=N}^1 S_{n}^{N-n}, \quad S_{n}^{N-n} = \begin{bmatrix} I_{N-n} & 0 \\ 0 & S_{n} \end{bmatrix}$$

and has $(2N-1) + (2N-3) + \cdots + 1 = N^2$ free real parameters, equal to $\dim_{U}(U(N))$, which is in agreement with the analysis of ref. [16] and Equation (3), confirming that the UGMZI can be used for representing any unitary matrix. We note here that an arbitrary unitary matrix, and consequently UGMZI, can also be constructed based on the elementary building blocks with transfer matrices of the form

$$U_{\text{UGMZI}}(N) = \prod_{n=N}^1 \tilde{S}_{n}^{N-n}, \quad \tilde{S}_{n}^{N-n} = \begin{bmatrix} S_{n} & 0 \\ 0 & I_{N-n} \end{bmatrix}$$

where the identity block matrix is placed in the lower-right instead of upper-left part of the elementary SGMZI matrix, which, in practical implementation, implies that every $n \times n$ SGMZI block gets connected to the ports 1 to $n$ with the ports $n + 1$ until $N$ comprising just free waveguides, that is, the free waveguides will be below instead of above the SGMZI. This means that the UGMZI layout formed in this way will be identical to the UGMZI configuration shown in Figure 3 but vertically (i.e., along the horizontal axis) flipped, otherwise unchanged, with the SGMZI elements concatenated again in the increasing order of dimensionality up to $N$. This design still complies with the decomposition procedure analyzed in ref. [16], provided that the decomposition procedure is modified so that Equation (8) is now solved with respect to the last column instead of the first column elements of $Q_{n}$, specifically $q_{n,n}'$, instead of $q_{r,1}$. The rest of the steps remain unchanged, suggesting that this flipped UGMZI design is also capable of representing any unitary matrix.

4. Performance Evaluation and Comparison

4.1. Loss and Phase-Induced Fidelity of UGMZI, Triangular and Rectangular Meshes toward Implementing Any Unitary Matrix

Implementation of an arbitrary unitary matrix with realistic, nonideal optical components will result in a discrepancy between the achieved and targeted matrix element values. To quantify this discrepancy, we rely on the fidelity parameter, typically used in tolerance analysis of a photonic architecture versus its idealized counterpart,[19] and we benchmark our UGMZI-based unitary matrix decomposition approach against the state-of-the-art architectures pioneered by Reck et al.[17] and Clements et al.[19] The standard fidelity measure, based on the Frobenius inner product of two matrices and normalized to its balanced losses, that is, the equal losses enforced along all paths, is given as follows

$$F(U, U_{\text{exp}}) = \frac{\text{tr}(U^\dagger U_{\text{exp}})}{\sqrt{\text{tr}(U^\dagger U) \text{tr}(U_{\text{exp}}^\dagger U_{\text{exp}})}}^2$$

where $U$ is the targeted unitary matrix of size $N$ and $U_{\text{exp}}$ is its experimental counterpart, with $U^\dagger$ and $U_{\text{exp}}^\dagger$ denoting their conjugate transposes, respectively. Knowing that the targeted matrix $U$ is indeed unitary, we have $\text{tr}(U^\dagger U) = N$, which does not generally hold true for its experimental implementation, $U_{\text{exp}}$, due to the deviations originating from lossy optical elements and phase errors in the PS structures.

Accounting for loss in optical elements and relying on Equation (10), we define the experimental implementation of the targeted unitary matrix as

$$U_{\text{exp,UGMZI}} = \prod_{n=N}^1 S_{\text{exp,n}}^{N-n}$$

where the transmittivity factor of the lossy nodes, $k_{n} \leq 1$, associated with the $n$th node losses as $I_{n,\text{node,db}} = -10\log_{10}(k_{n}^2)$, modifies the ideal matrices $S_{n}^{N-n}$ to

$$S_{\text{exp,n}}^{N-n} = k_{n} \begin{bmatrix} I_{N-n} & 0 \\ 0 & S_{n} \end{bmatrix}$$

After an iterative process of $N$ mid-to-edge matrix multiplications (see extended analysis in Section A of the Supporting Information), we derive...
\[ U_{\text{exp,UGMZI}}^{U} U_{\text{exp,UGMZI}} = \prod_{n=1}^{N} k_n \cdot B_U^{-1} \]  

\[ U_{\text{exp,UGMZI}}^{U} U_{\text{exp,UGMZI}} = \prod_{n=1}^{N} k_n \cdot (B_U)_{n}^{-2} \]  

where \( B_U^{-1} \) is a diagonal matrix that embraces the unbalanced loss factors of the system and reads

\[ B_U^{-1} = \text{diag}\left[ \prod_{n=1}^{N-1} k_n^{-1}, \prod_{n=2}^{N-2} k_n^{-1}, \ldots, k_1^{-1}, 1 \right] \]  

By substituting Equation (16,17) into Equation (13), we find

\[ F_{\text{UGMZI}} = \left| \frac{\sum_{n=1}^{N} \prod_{n=1}^{N} k_n}{\sqrt{N \sum_{n=1}^{N} \prod_{n=1}^{N} k_n^2}} \right|^2 \]  

Equations (16,17) and (18) indicate that \( U_{\text{exp,UGMZI}}^{U} U_{\text{exp,UGMZI}} \) and \( U_{\text{exp,UGMZI}}^{U} U_{\text{exp,UGMZI}} \) depend solely on the loss-associated factors \( k_n \) and the dimension of the problem \( N \). This implies that by applying compensating components to the inputs of our baseline UGMZI architecture in the form of either variable optical attenuators (VOAs), as shown in the inset of Figure 3, or amplifiers, we can fully balance the losses, achieving balanced, or B-UGMZI. The addition of the balancing components algebraically translates to the multiplication of \( U_{\text{exp,UGMZI}} \) with a diagonal matrix \( B_U \) (i.e., the inverse matrix of \( B_U^{-1} \)) from the right in case of attenuators, or by \( \prod_{n=1}^{N} k_n^{-1} \cdot B_U \) in the case of amplifiers. For simplicity, and to mitigate employing the active components, in what follows we choose VOAs for balancing components, yielding

\[ U_{\text{exp,B-UGMZI}} = \prod_{n=1}^{N} S_{\text{exp,n}} \cdot B_U \]  

Exploiting the balanced experimental matrix and relying on the fact that \( B_U B_U^{-1} = I \), and since the matrix \( B_U \) is diagonal and populated by real elements, following the previously outlined fidelity calculation procedure, we acquire

\[ U_{\text{exp,B-UGMZI}}^{U} U_{\text{exp,B-UGMZI}} = \prod_{n=1}^{N} k_n \cdot I_N \]  

\[ U_{\text{exp,B-UGMZI}}^{U} U_{\text{exp,B-UGMZI}} = \prod_{n=1}^{N} k_n \cdot I_N \]  

which, according to definition (13), yields absolute fidelity, \( F_{\text{B-UGMZI}} = 1 \). This key attribute of the proposed architecture can be always applied and is totally independent of the differential loss of its constituent nodes \( k_n \).

Having identified that fidelity mainly depends on the losses per node and the dimension of the matrix, excluding for the time being any deviations originating from the PSSs, we employ Monte Carlo method for comparing the UGMZI approach with the state-of-the-art MZI-mesh architectures, arranged per Clements’ and Reck’s decomposition schemes. For every given set of inputs \( N \), we generate 500 random unitary matrices \( U(N) \) and, by considering ideal components, we decompose the targeted matrices via the three decomposition formulas, using Equation (2) to calculate \( D \) and \( R_{\text{exp}} \) following rectangular (Clements) or triangular (Reck) MZI arrangement via the matrix multiplication order \( W_{\text{Cl/R}} \) and Equation (12) to determine \( \Omega_{\text{r,c}} \). For simplicity, in determining the experimental matrices, we consider a node loss of \( L_{\text{node,db}} \) for every node of the compared architectures, which in the case of the UGMZI corresponds to an average loss of all the constituent nodes (SMZIs), while in the case of the Reck’s and Clements’ schemes, the loss corresponds to the average of the loss of its elementary nodes (MZIs). This is, in turn, applied to the ideal \( S_n \) and \( R_{\text{exp}} \) matrices to form the corresponding \( S_{\text{exp,n}} \) and \( R_{\text{exp,r,c}} \). The experimentally obtained unitary matrices read

\[ U_{\text{exp,B-UGMZI}} = \prod_{n=1}^{N} S_{\text{exp,n}} \]  

\[ U_{\text{exp,B-UGMZI}} = \prod_{n=1}^{N} R_{\text{exp,r,c}} \]  

**Figure 4**: Fidelity comparison of UGMZI (red squares [restored] and dark red circles [unbalanced]), Clements’ (black up-pointing triangles) and Reck’s (blue down-pointing triangles) unitary matrix decomposition schemes versus the number of inputs \( N \in [4, 20] \) for a constant loss of a) \( L_{\text{node,db}} = 0.5 \text{ dB} \) and b) \( L_{\text{node,db}} = 1.5 \text{ dB per node and c) versus the losses per node } L_{\text{node,db}} \in [0, 2] \text{ dB for } N = 20 \). Insets: close-ups of the fidelity curves of the restored UGMZI and the Clements’ architecture.
to the circuit size and the losses per node, of UGMZI, Clements’ and Reck’s architectures, that are represented with red, black, and blue colors, respectively. More specifically, Figure 4a,b depicts the respective curves when the losses per node $I_{\text{node,db}}$ retain a constant value of 0.5 and 1.5 dB, respectively, while the number of inputs varies from 4 to 20 in both cases. Figure 4c shows the fidelity performance in the decomposition of a $20 \times 20$ matrix, when the losses per node range from 0 to 2 dB. It can be observed that the unbalanced UGMZI (dark red circles) scheme yields a degraded fidelity behavior in comparison with both the Clements’ and Reck’s schemes, yielding up to $\approx70\%$ and $\approx30\%$ accuracy in a $20 \times 20$ implementation for $II_{\text{node,db}} = 0.5$ dB and $II_{\text{node,db}} = 1.5$ dB, respectively, while the respective Clements’ layouts yield $\approx99.5\%$ and $\approx95\%$ accuracy. Moreover, in the case of $II_{\text{node,db}} = 2$ dB and $N = 20$, Figure 4c, the unbalanced UGMZI degrades even more, to $\approx22\%$, while Clements’ implementation remains $>90\%$. Nevertheless, balancing the losses in UGMZI architecture allows for full fidelity restoration, as confirmed by red squares of Figure 4 outperforming its counterparts for an arbitrary node loss or matrix dimension. The fidelity gap for $N = 20$ unitary matrix implementation with 2 dB losses per node reaches $\approx8\%$ between UGMZI and Clements’ and $>80\%$ in comparison to Reck’s implementation. It should be noted that each $n \times n$ SGMZI structure with $n > 2$ employed within an $N \times N$ UGMZI will probably have higher losses than the simpler $2 \times 2$ MZIs required within the Reck and Clements architectures, implying that the fidelity curve of the UGMZI would be expected in reality to degrade faster than depicted in Figure 4. Yet, fidelity restoration is completely independent of the node loss values in the UGMZI layout (see Section A, Supporting Information), so that the restored fidelity curves shown in Figure 4 would have exactly the same value irrespective of the choice on the SGMZI node losses. Apart from the loss-induced fidelity, phase mismatches can play a weighty factor to an experimental implementation of a unitary matrix. Within this scope, we followed the Monte Carlo method to calculate and compare the phase-error tolerance of our architecture with respect to the state-of-the-art architectures (Reck, Clements). More specifically, for each targeted matrix $U(N)$ generated for the loss-induced fidelity calculation, we applied 100 sets of random phase errors that follow the Gaussian distribution with a mean value of zero and a standard deviation $\sigma$, to the phase shifts of the ideal matrices $\Omega^{N-n}_{n}$, $\Theta^{N-n}_{n}$ (of the UGMZI scheme), and $D$, $R_{r,c}$ (of Reck’s and Clements’ architectures), producing the corresponding $\Omega^{N-n\exp}_{n\exp}$, $\Theta^{N-n\exp}_{n\exp}$, and $D_{\exp}$, $R_{\exp,r,c}$ to calculate the respective experimental matrices

$$U_{\text{UGMZI,exp}} = \prod_{n=N}^{1} \Omega^{N-n\exp}_{n\exp} T^{N-n\exp}_{n\exp} \Theta^{N-n}_{n\exp} T^{N-n}_{n\exp}$$ (25)

$$U_{C_{IL}/R_{\exp}} = D_{\exp} \prod_{(r,c) \in W_{C_{IL}/R_{\exp}}} R_{\exp,r,c}$$ (26)

We calculate the phase-error tolerance of the comparing layouts via the fidelity definition given by Equation (13), when lossless nodes are considered, for different circuit sizes and phase-error standard deviations. Figure 5a,b depicts the fidelity averaged over the 100 sample matrices for the $4 \times 4$ and $8 \times 8$ implementations, respectively, revealing that UGMZI provides higher tolerance to phase mismatches than either Reck’s or Clements’ layouts. It can be observed that, as the standard deviation $\sigma$ increases, the fidelity gap between the UGMZI and its counterparts increase. For the maximum analyzed $\sigma$ of 0.2 rad, UGMZI yields $\approx98.5\%$ ($4 \times 4$) and $\approx95.6\%$ ($8 \times 8$) accuracy, while the accuracy of the two comparing architectures remains lower than 97% and 93%, respectively. Figure 5c shows the fidelity performance of the three architectures with a constant $\sigma = 0.1$ rad versus the number of inputs. As the size of the matrices scales up, the fidelity of Reck’ and Clements’ schemes degrades faster than comparing to UGMZI architecture, resulting in a gap greater than 3.5% in the $20 \times 20$ implementation. The 95% confidence intervals of the deployed Monte Carlo simulations were, also, included in the fidelity curves, showing the confidence level of the respective fidelity values. Consequently, it can be deduced that our architecture outperforms the state-of-the-art counterparts both in the loss-induced fidelity, where it can yield the absolute accuracy, and in the phase-inaccuracies behavior, where even for the implementations of large matrices or for large phase deviations, it preserves its high accuracy.

Finally, a summary of the key attributes of the compared architectures is stated in Table 1.

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**Figure 5.** Fidelity comparison of UGMZI’s (red squares), Clements’ (black up-pointing triangles) and Reck’s (blue down-pointing triangles) schemes with respect to the zero-mean Gaussian distributed phase error with the standard deviation of $\sigma \in [0, 0.2]$ rad for a number of inputs a) $N = 4$ and b) $N = 8$ and c) the number of inputs $N \in [4, 20]$ for a phase error standard deviation of $\sigma = 0.1$ rad.
and is typically realized by the SVD decomposition procedure.\[10\] Unitary but any real or complex matrix can be factorized to produce the matrix representation, described by Equation (11), and a baseline UGMZI design for realizing any real/complex matrix.

The SVD assumes factorization of any matrix \( D \) in the form of \( D = U \Sigma V^\dagger \), where \( U \) and \( V \) are unitary matrices and \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_N) \) is a diagonal matrix with non-negative real elements represented by VOAs in photonic platform. To validate the performance of an SVD-based matrix implementation using the UGMZI design for realizing \( U \) and \( V \) unitary matrices, we compare its loss tolerance with respect to an SVD-based deployment where the Clements approach (proven to be the optimal so far) is utilized for the \( U \) and \( V \) matrix layouts. Moreover, as loss imbalance has been observed in UGMZI architecture, instead of concatenating two baseline UGMZIs, separated by a column of VOAs, we use UGMZI flipped around its horizontal axis for \( V \) representation, described by Equation (12), and a baseline UGMZI for \( U \) representation, described by Equation (11), resulting in

\[
\begin{align*}
U &= \prod_{n=1}^{1} S_{U, n}^n = \begin{bmatrix} I_{N-n} & 0 \\ 0 & S_{U,n} \end{bmatrix} \\
V &= \prod_{n=1}^{1} S_{V, n}^n = \begin{bmatrix} S_{V,n} & 0 \\ 0 & I_{N-n} \end{bmatrix}
\end{align*}
\]

(27)

(28)

As photonic platform should represent \( V \) rather than \( V \), the UGMZI that has been flipped along the horizontal axis for representing the matrix \( V \) has also to be flipped along its vertical axis to produce the \( V^\dagger \), complying with \( V^\dagger = \prod_{n=1}^{N} (S_{V,n}^n)^\dagger \), where the constituent GMZIs acting on an optical input vector will be concatenated from \( n = N \) to \( n = 1 \) following the direction of light propagation, as shown in Figure 6. The experimental realization of the matrices is again expected to deviate from targeted matrices \( U \), \( \Sigma \), and \( V^\dagger \). In a similar manner to Equation (14) and (15), we define the experimental matrices as

\[
\begin{align*}
U_{\text{exp}} &= \prod_{n=1}^{1} S_{U, n, \text{exp}}^n = k_n \begin{bmatrix} I_{N-n} & 0 \\ 0 & S_{U, n} \end{bmatrix} \\
V_{\text{exp}} &= \prod_{n=1}^{1} S_{V, n, \text{exp}}^n = k_n \begin{bmatrix} S_{V,n} & 0 \\ 0 & I_{N-n} \end{bmatrix}
\end{align*}
\]

(29)

(30)

\[
\Sigma_{\text{exp}} = k_n \Sigma
\]

(31)

assuming that the SGMZI blocks of the identical size in both \( U \) and \( V \) introduce the same losses \( (k_{U,n} = k_{V,n} = k_n) \), which generally differs for the SGMZIs of different sizes \((k_n \neq k_m \text{ if } n \neq m)\). We also assume that all VOAs within the diagonal \( \Sigma \) matrix have identical insertion losses, captured by the transmission factor \( k_n \leq 1 \). According to the general fidelity expression given by Equation (13), to quantify the performance of SVD-UGMZI approach, we determine the products \( D^\dagger D \), \( D^\dagger D_{\text{exp}} \), and \( D_{\text{exp}} D_{\text{exp}} \), following a similar approach as in determining the fidelity of the UGMZI alone, with the detailed derivation given in Section B of Supporting Information.

\[
\begin{align*}
D^\dagger D &= V \Sigma^2 V^\dagger \\
D^\dagger D_{\text{exp}} &= k_n \prod_{n=1}^{N} k_n \cdot V \Sigma^2 (B_{n}^\dagger B_{n}) V^\dagger \\
D_{\text{exp}} D_{\text{exp}} &= k_n^2 \prod_{n=1}^{N} k_n^2 \cdot V \Sigma^2 (B_{n}^\dagger B_{n})^2 V^\dagger
\end{align*}
\]

(32)

(33)

(34)

Figure 6. UGMZI scheme in a singular value decomposition (SVD) configuration for the photonic realization of any real/complex matrix.

Table 1. UGMZI, Reck, and Clements architectures performance summary.

| Architecture | No of PSs | No of couplers \(^{a, b}\) | Loss-induced fidelity | Phase-error-induced fidelity | Path loss difference dependence | Scalability |
|--------------|----------|-------------------------|-----------------------|-----------------------------|---------------------------------|-------------|
| UGMZI        | \( N^2 + N \) | 2N | Fully restorable | Low degradation | Independent | Fabrication technology & node loss limited |
| Reck         | \( N^2 \) | \( N^2 \) | Significant degradation | Moderate degradation | High | Node loss limited |
| Clements     | \( N^2 \) | \( N^2 \) | Low degradation | Moderate degradation | Low | Node loss limited |

\(^{a}\)UGMZI: \( N \times N \) MMI couplers; \(^{b}\)Reck, Clements: \( 2 \times 2 \) couplers.
where

\[ B_V = \text{diag} \left[ 1, k_1^{-1}, \ldots, \prod_{n=1}^{N-2} k_n^{-1}, \prod_{n=1}^{N-1} k_n^{-1} \right] \quad (35) \]

The product \( B_V^{-1} B_U \) results in a diagonal matrix, symmetrical around its antidiagonal, capturing the unbalanced losses of the SVD–UGMZI implementation, similar as \( B_V^{-1} \), given by Equation (18), was doing in the case of UGMZI alone. This implies that, without a restoration mechanism and assuming different losses per SGZMs of different sizes, fidelity remains below 1 and reads

\[ F_{\text{SVD–UGMZI}}(D, D_{\exp}) = \frac{\text{tr}(\Sigma^2 B_U^{-1} B_V)}{\sqrt{\text{tr}(\Sigma^2) \text{tr}(\Sigma^2 B_U^{-1} B_V)^2}} \quad (36) \]

Nonetheless, unlike MZI-mesh-based implementations (e.g., Reck, Clements), the SVD–UGMZI can have its fidelity fully restored, following the same principle previously applied in the case of UGMZI alone. By including a column of VOAs per UGMZI, such that diagonal matrix \( B_U \) is implemented to multiply \( U_{\exp} \) from the right side (appearing physically before \( U \)-UGMZI) and diagonal matrix \( B_V^{-1} \) is implemented to multiply \( V_{\exp} \) from the left side (appearing physically after \( V^\dagger \)-UGMZI), the disbalance in Equation (37) vanishes and the fidelity of the balanced layout reaches 100%. Based on the requirement that \( B_V^{-1}, \Sigma \) and \( B_U \) are arranged such that they are physically next to each other and between UGMZIs, the two additional columns of attenuators required for loss balancing can be joined with the already present VOAs, used for \( \Sigma \) implementation, requiring no physical change of the SVD–UGMZI layout. Rather, three logically different sources of losses are joined within the experimentally implemented \( \Sigma \) matrix such that it becomes

\[ \Sigma_{\exp, B_{\text{UGMZI}}} = \Sigma_{\exp} B_U^{-1} B_U = k_0 \Sigma B_U^{-1} B_U \quad (37) \]

Finally, in the special case of identical losses per GZMs node, irrespective of their size \( k_0 = k, \forall n \), having \( k^{-N-1} B_V = B_U \) results in absolute fidelity without the need to activate any restoration mechanism.

We proceed with the numerical study of the proposed photonic SVD implementations, following the Monte Carlo method. For any given number of inputs \( N \), we generate 500 random matrices, where we apply the SVD, producing the ideal \( U, \Sigma \), and \( V^\dagger \) matrices. Thereafter, we implement both the Clements and UGMZI-based decomposition method in each \( U, V^\dagger \) pair of matrices and accounting for losses find their experimental counterparts, to finally calculate the corresponding fidelities.

Figure 7 illustrates the fidelity curves averaged over the 500 sample matrices per combination of \( N \) and losses per node (for simplicity assumed to be the same for GZMsIs of different sizes, VOAs, and MZI-nodes) for both SVD–UGMZI (red squares) and SVD–Clements (black triangles), parametrized to the circuit size and the losses per individual node. Specifically, Figure 7a depicts the fidelity values for \( 4 \times 4 \) and \( 8 \times 8 \) arbitrary matrices when the UGMZI and the Clements unitary decomposition methods were applied. The fidelity gap among the comparing architectures increases rapidly when the losses per node scale up, reaching to \( >10% \) and \( \approx 35% \) for \( I_{\text{node,db}} = 2 \text{dB} \) in the decomposition of the \( 4 \times 4 \) and \( 8 \times 8 \) matrices, respectively. Figure 7b,c illustrates the fidelity performance of the SVD implementations when the UGMZI and the Clements methods are applied where circuit size of the targeted matrices ranges from 4 to 20 and the losses per node equal 0.5 and 1.5 dB, respectively. The resulting plots reveal, again, that SVD–UGMZI layout outperforms the SVD–Clements design, since the former is constantly 100%, resulting to an increasing fidelity degradation gap when the number of inputs \( N \) scales up, that becomes even \( >50% \) in the case of \( I_{\text{node,db}} = 1.5 \text{dB} \) and \( N = 20 \).

5. Conclusion

Demarcating from the conventional unitary transformations that are based on U(2) parametrization, we have developed a recursive unitary decomposition method via a special GMZI-based architecture formulating its mathematical framework and theoretically validating its functionality. The proposed UGMZI architecture relies on cascaded size-augmenting GZMsIs layouts and is capable of decomposing an arbitrary unitary matrix, while supporting a fidelity restoration mechanism that can secure absolute fidelity performance in a realistic environment with lossy optical elements. We compare UGMZI’s fidelity behavior with respect to the formerly state-of-the-art Clements architecture,
revealing that the fidelity performance gap of the two increases significantly with the size of the targeted matrix and the loss value per individual node. We also compare their phase-error tolerance, concluding that the UGMZI scheme offers improved robustness to phase errors comparing to the Clements implementation. Finally, we utilize the UGMZI configuration to construct universal linear operators relying on the SVD technique, highlighting that the UGMZI loss–balance restoration properties can lead to inherently loss-balanced linear optics where any real/complex-valued matrix can be accurately represented.

Supporting Information
Supporting Information is available from the Wiley Online Library or from the author.

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Conflict of Interest
The authors declare no conflict of interest.

Data Availability Statement
The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

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