Fermion doubling theorems in 2D non-Hermitian systems

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The fermion doubling theorem plays a pivotal role in Hermitian topological materials. It states, for example, that Weyl points must come in pairs in three-dimensional semimetals. Here, we present an extension of the doubling theorem to non-Hermitian lattice Hamiltonians. We focus on two-dimensional non-Hermitian systems without any symmetry constraints, which can host two different types of topological point nodes, namely, (i) Fermi points and (ii) exceptional points. We show that these two types of protected point nodes obey doubling theorems, which require that the point nodes come in pairs. To prove the doubling theorem for exceptional points, we introduce a generalized winding number invariant, which we call the discriminant number. Importantly, this invariant is applicable to any two-dimensional non-Hermitian Hamiltonian with exceptional points of arbitrarily high order. While the non-Hermitian doubling theorems must be satisfied for lattices with periodic boundary conditions, they can be violated in the presence of surfaces, which can host a single exceptional point with anomalous properties. We illustrate our results by discussing several examples that can be realized, e.g., photonic cavity arrays.

Fermion doubling theorems are an important concept in the topological band theory of condensed matter physics. They state that topological point nodes in the energy spectrum of lattice Hamiltonians must come in pairs. Hereby they prevent the occurrence of quantum anomalies in lattice systems. This is because for a single point node the low-energy physics is described by a field theory with a quantum anomaly, while for two point nodes the anomalies cancel. Well known examples of doubled point nodes include the two Dirac points of graphene and the two Weyl points of magnetic Weyl semimetals. While the doubling theorems must be fulfilled in the bulk of any lattice Hamiltonian, they may be violated on a lattice surface. For example, the three-dimensional topological insulator with time-reversal symmetry exhibits a single Dirac point with parity anomaly on its surface. These anomalous surface states lead to unusual physical responses and give a powerful diagnostic of the nontrivial bulk topology.

Recently, topological band theory has been extended to non-Hermitian Hamiltonians, which can be realized in, e.g., photonic cavity arrays, and provide effective descriptions of non-equilibrium systems, where energy is not conserved due to, e.g., dissipation or particle gain and loss. In contrast to the Hermitian case, two-dimensional (2D) non-Hermitian Hamiltonians can exhibit three different types of point nodes, namely (i) Fermi points (FPs), (ii) exceptional points (EPs), and (iii) non-defective degeneracy points (NDPs). Both at an EP and at an NDP two (or more) energy bands become degenerate as degeneracy points (DPs). However, at an EP also the corresponding eigenstates become identical, while at an NDP the eigenstates remain distinct. For this reason, non-Hermitian Hamiltonians at EPs are non-diagonalizable and can only be reduced to Jordan block forms. In the absence of symmetry, both FPs and EPs can be topologically stable in 2D, meaning that these point nodes cannot be removed by perturbations. NDPs on the other hand, are unstable, since they can be split or deformed into EPs by arbitrarily small deformations.

The physics of EPs has recently attracted a lot of attention, in particular in the context of photonic platforms, where they have many interesting applications, for example, as optical omnipolarizers, or as sensors with enhanced sensitivity. While the occurrence and stability of FPs and EPs has been studied in various settings, the existence of doubling theorems for these topological point nodes remains unknown. In this Letter we derive doubling theorems for FPs and EPs in 2D periodic lattice Hamiltonians without any symmetry constraints. For that purpose, we consider a non-Hermitian Hamiltonian \( \mathcal{H}(k) \) possessing well-separated complex energy bands, except for some possible EPs. The proof of the doubling theorems relies on the fact that both FPs and EPs carry nonzero topological charges, whose sum must vanish in the entire Brillouin zone (BZ) as charge neutralization, i.e.,

\[
\sum_{k_i \in \text{BZ}} C(k_i) = 0, \tag{1}
\]

where \( C(k_i) \) is the topological charge of an FP or EP located at \( k_i \) in the BZ. The charges \( C(k_i) \) are defined in terms of an integral of some topological charge density along a closed contour that encircles the FP or EP. Equation (1) then follows by continuously deforming the integration contours to the boundary of the BZ. We find that for FPs the appropriate charge density is the logarithmic derivative of \( \det[\mu - \mathcal{H}(k)] \).
while for EPs it is the logarithmic derivative of the discriminant of \( \mathcal{H}(\bm{k}) \). We illustrate our findings with several examples that can be realized in various metamaterials, e.g., in photonic cavity arrays or in electric circuit lattices. We also show that the doubling theorems can be violated at surfaces, which can host single FPs or EPs with anomalous properties. On the other hand, it was thought that EPs are branch point singularities \(^{14}\). However, we show that the statement is not generally true by providing several counter examples.

**Doubling theorem for FPs.**— We start with the doubling theorem for FPs of generic 2D non-Hermitian Hamiltonians \( \mathcal{H}(\bm{k}) \) with complex energy bands \( E_i(\bm{k}) \). The FPs of \( \mathcal{H}(\bm{k}) \) are defined as those points \( k^*_F \) in the BZ, where the complex chemical potential \( \mu \) intersects with one of the energy bands \( E_i(\bm{k}) \), i.e., \( \mu - E_i(k^*_F) = 0 \) for some \( i \). By choosing a proper basis, each entry in \( \mathcal{H}(\bm{k}) \) is single-valued in the entire BZ. Chemical potential \( \mu \), which can be a complex constant, determine the locations of FPs. To find the Fermi points, we use \( \mu \) to define the characteristic polynomial of \( \mathcal{H}(\bm{k}) \)

\[
    f(\mu, \bm{k}) \equiv \det(\mu - \mathcal{H}(\bm{k})) = \prod_i (\mu - E_i(\bm{k})).
\]

As \( f(\mu, k) = 0 \), the energy band \( E_i(\bm{k}) \) is the root of the polynomial. The Fermi points are located at \( k^*_F \) so that \( f(\mu, k^*_F) = 0 \). That is, the two constraints determine the locations of the Fermi points

\[
    f(\mu, k^*_F) = f_r(k^*_F) + f_i(k^*_F) i = 0
\]

Since each constraint leads to line loops in the 2D BZ, the crossings of the two different constraint loops form Fermi points as shown in Fig. 1(a). Pictorially, the minimal number of the crossings of the two loops is two, and the crossing points as Fermi points are stable under small perturbations as illustrated in Fig. 1(b); this panel shows stable Fermi points always come with at least a pair as the doubling theorem of Fermi points. However, the robustness of the Fermi points needs to be carefully quantified by characterizing Fermi points and providing a clear description of the doubling theorem. We define a global winding number to characterize Fermi points

\[
    W(k^*_F) = \frac{i}{2\pi} \oint_{\partial_{\text{BZ}}} d\bm{k} \cdot \nabla_k \ln|\det[\mu - \mathcal{H}(\bm{k})]|, \quad (4)
\]

where the integral path \( \Gamma(k^*_F) \) we choose is a loop encircling \( k^*_F \) counterclockwise as shown in Fig. 2(a). Since \( \det(\mu - \mathcal{H}(\bm{k})) \) is single-valued in the entire BZ, the winding number \( W \) is quantized. A non-zero winding number forbids the integral loop to contract and vanish through the Fermi point. In other words, the Fermi point is protected by the non-zero winding numbers and stable even in the presence of perturbations. We sum over the winding numbers for all of the Fermi points in the BZ

\[
    \sum_{k^*_F \in \text{BZ}} W(k^*_F) = \frac{i}{2\pi} \oint_{\partial_{\text{BZ}}} d\bm{k} \cdot \nabla_k \ln|\det[\mu - \mathcal{H}(\bm{k})]| = 0. \quad (5)
\]

Since the Fermi points can be the only singularity points in the integrand, the integral path can be continuously deformed to the boundary of the BZ (\( \partial \text{BZ} \)) as shown in Fig. 2. With the vanishing of the entire integral, the Fermi points protected by non-zero winding numbers obey the doubling theorem (1) in 2D non-Hermitian systems.

**Examples of doubled FPs.**— We use two examples to demonstrate the properties of the Fermi points. First, consider a single-band model described by the Bloch Hamiltonian \( \mathcal{H}(\bm{k}) = \cos k_x + \cos k_y - 3/2 + i \sin k_x \). Choosing \( \mu = 0 \), with vanishing \( f(0, \bm{k}) \) we have the Fermi points located at \( k^*_F = (0, \pm \pi/3) \). As shown in Fig. 3(a) these two Fermi points represent the zeros of \( |E(\bm{k})| \) and the winding numbers \( W(k^*_F) = \mp 1 \) obey the neutralization of the doubling theorem (5). Second, the next example is a two-band Hamiltonian

\[
    \mathcal{H}(\bm{k}) = h_0(\bm{k}) \sigma_0 + h(\bm{k}) \cdot \sigma, \quad (6)
\]

where \( h_0(\bm{k}) = \sin k_y/2, h_x(\bm{k}) = \sin k_x - i, h_y(\bm{k}) = \sin k_y \) and \( h_z(\bm{k}) = -2 + \cos k_x + \cos k_y \). With the chemical potential \( \mu = \sqrt{3}/4 \), the two Fermi points are located at \( k^*_F = (0, -0.479 \pi) \) and \( k^*_F = (0, \pi/3) \) and the winding numbers is given by \( W(k^*_F) = \mp 1 \) as shown in Fig. 3(b). On the other hand, regardless of the chemical potential, the two two-fold degeneracy points as two EPs appear at \( (0, \pm \pi/3) \) with different energies 0 and
where \( c \) represents the band dispersion of a band model and (b) two-band model. The surfaces and arrows represent degeneracy roots. Since the discriminant of the characteristic polynomial is a well-defined function searching for degeneracy points in a generic non-Hermitian Hamiltonian. The invariant has been well-studied in simple non-Hermitian Hamiltonians in the past \cite{59,60}. However, a generic and clear invariant is absent and needed to characterize degeneracy points in \( n \times n \) Hamiltonians \( \text{Disc}_E[H] \), and then we can use the invariant for the doubling theorem. To find this missing global invariant, we use the characteristic polynomial (2) \( f(E,k) \), where variable \( E \) replaces the chemical potential \( \mu \). A DP appears at \( k_d \) when \( E_i(k_d) = E_j(k_d) \) as \( i \neq j \). In this regard, the characteristic polynomial must have a degeneracy root. Mathematically, the discriminant of a characteristic polynomial is a well-defined function searching for degeneracy roots. Since the discriminant of \( f(E,k) \) is defined as

\[
\text{Disc}_E[H](k) = \prod_{i<j} |E_i(k) - E_j(k)|^2 ,
\]

momentum \( k_d \), which obeys \( \text{Disc}_E[H](k_d) = 0 \), is a degeneracy point. That is, using the vanishing discriminant as a tool can find any DPs at any energy levels in the entire BZ and can be also applied in Hermitian systems. This approach is distinct from Eq. 2, which identifies Fermi points at a fixed energy level. Furthermore, the explicit form of the discriminant can be easily obtained by building the Sylvester matrix, which is a \((2n-1) \times (2n-1)\) matrix formed by the coefficients of the polynomials \( f(E,k) \) and \( \partial_E f(E,k) \) (see the details in supplementary materials). That is, the discriminant equals the determinant of the Sylvester matrix. Since the coefficients of the characteristic polynomial \( f(E,k) \) are single-valued, \( \text{Disc}_E[H](k_d) \) is also single-valued. This property is a key to have a quantized invariant later. Furthermore, without directly computing all of the energy bands, using the vanishing discriminant to search for DPs is more efficient.

Similar to Eq. 3, the two constraints \( \text{Re}(\text{Disc}_E[H](k)) = 0 \) and \( \text{Im}(\text{Disc}_E[H](k)) = 0 \) form at least two loops in the 2D BZ. Since the DPs are generally located at the crossings of the two loops, the doubling theorem qualitatively holds for the DPs. It is possible that the number of degenerate bands at a DP can be more than two. However, an \( n \)-fold DP \((n > 2)\) without additional symmetry constraints is unstable and can be easily deformed to multiple 2-fold DPs. For example, the rising of a 3-fold DP requires momentum \( k \) to obey \( E_1(k) - E_2(k) = 0 \), \( E_3(k) - E_2(k) = 0 \) at the same time. The presence of a small perturbation can easily remove this requirement and then the 3-fold degeneracy point is split into two 2-fold DPs located at two different momenta separately (See the concrete model in the supplementary materials). Since in general in the absence of symmetry protection with small perturbations any \( n \)-fold DP \((n > 2)\) can be deformed to multiple two-fold DPs, our focus is only on two-fold DPs.

**Discriminant number.**—To flesh out the doubling theorem, we define a new topological invariant characterizing the DP at \( k_d \)

\[
\nu(k_d) = \frac{i}{2\pi} \oint_{\Gamma(k_d)} dk \cdot \nabla_k \ln \text{Disc}_E[H](k),
\]

which is a winding number invariant, which we call the discriminant number. The mathematical structure is almost identical to the winding number characterizing FPs. The only difference is that \( \text{det}[\mu - H(k)] \) in the integrand (4) is replaced by \( \text{Disc}_E[H](k) \). In the literature \cite{59}, DPs can also be characterized by vorticity invariant

\[
\nu_{ij}(k_d) = \frac{1}{2\pi} \oint_{\Gamma(k_d)} \nabla_k \arg [E_i(k) - E_j(k)] \cdot dk,
\]

where \( \arg(z) = -i \ln(|z|) \). After some algebra exercises (see supplementary materials), we find the discriminant number is equal to the summation of the vorticities \( \nu(k_d) = \sum_{i \neq j} \nu_{ij}(k_d) \). The reason to use the discriminant number in the form (8) for the doubling theorem is that this invariant \( \nu \) can identify any stable DPs in the entire BZ while \( \nu_{ij} \) can find only DPs at the \( i \)-th and \( j \)-th energy levels. Furthermore, using this form (9) to compute the invariant requires knowing all of the values of the energy bands \( E_i(k) \). Our new expression (8) circumvents this computational issue to obtain the invariant by directly using the discriminant of the characteristic polynomial (see the detailed calculations in supplementary materials). Since in the entire BZ \( \text{Disc}_E[H](k_d) \), which stems from the Sylvester matrix of \( f(E,k) \), is a single-valued function of \( k \), this discriminant number is always quantized. Similar to FP protection, non-zero invariant \( \nu(k_d) \) protects the DP, which cannot be annihilated alone in the presence of any small perturbation, although the DP can be deformed to multiple DPs. The summation of the discriminant numbers for all of the DPs in the BZ

\[
\sum_{k_d \in BZ} \nu(k_d) = \frac{i}{2\pi} \oint_{\partial BZ} dk \cdot \nabla_k \ln \text{Disc}_E[H](k) = 0
\]
vanishes, because the integral path can be deformed to the boundary of the BZ ($\partial$BZ) as shown in Fig. 2. Thus, the vanishing summation directly provides the doubling theorem of DPs. By using the definition of the discriminant number, in the example of Eq. 6 two EPs (0, $\pm \pi/3$) with $\nu = \pm 1$ obey the doubling theorem. However, since the DPs can be either EPs or NDPs, Eq. 10 is not sufficient to the doubling theorem of EPs. Let us study two examples to examine the properties of EPs and NDPs.

In principal, the value of the discriminant number cannot distinguish an EP and an NDP. An EP with $\nu = \pm 1$ is stable, while an NDP can possess discriminant number $\nu = \pm 1$. Consider a 2-band Hamiltonian near $k = 0$

$$\mathcal{H}(k) = \begin{pmatrix} 0 & (k_x + i k_y)^2 \\ k_x - i k_y & 0 \end{pmatrix}$$

(11)

The NDP is located at $k = 0$ and the discriminant is given by $\text{Disc}_E[\mathcal{H}(k)] = 4(k_x + i k_y)^2(k_x - i k_y)$; hence, this NDP possesses discriminant number $\nu = -1$. On the other hand, the energy spectrum $E_{\pm} = \pm \sqrt{(k_x + i k_y)^2(k_x - i k_y)}$ shows the branch point located at the NDP. This is the example showing a branch point is not an EP. However, the presence of this NDP is unstable since by adding small $\delta \sigma_z$, the point moves to $(-\frac{1}{2}, 0)$ and becomes an EP, which is the branch point evolving from the NDP.

In the 2D BZ, with non-zero discriminant number an EP pairing with a NDP can be shown by the following $2 \times 2$ Hamiltonian

$$\mathcal{H}(k) = \begin{pmatrix} A(k) & B(k) \\ 0 & -A(k) \end{pmatrix}$$

(12)

where $A(k) = 1 - \cos k_x - \cos k_y + i \sin k_x$ and $B(k) = 1 - \sin k_y$. The characteristic polynomial $f(E, k) = -A^2(k) + E^2$ leads to the discriminant $\text{Disc}_E[\mathcal{H}(k)] = 4A^2(k)$. Only one EP appears at $(0, -\pi/2)$ with invariant $\nu = -2$ and one NDP is located at $(0, \pi/2)$ with $\nu = 2$. The charge of the EP can be neutralized by the charge of the NDP. This EP is not a branch point since the energy dispersion $E_{\pm} = \pm \sqrt{A(k)}$ is single-valued without any branch cut. This example still fulfills the statement of the DP doubling theorem. On the other hand, the NDP and the EP are unstable in the presence of perturbations. By adding any small constant matrix $\eta \sigma_z$ in $\mathcal{H}(k)$, each of the two DPs (the EP and the NDP) can evolve to two stable EPs with $|\nu| = 1$. The reason is that the condition leading to the DPs with $|\nu| = 2$ is stricter than EPs with $\nu = \pm 1$. In fact, 6 constraint equations are needed to realize non-splitting DPs with $|\nu| = 2$. In general, the emergence of the stable EPs and NDPs with higher order $|\nu| > 1$ requires at least 6 constraint equations (see the details in the supplemental materials). In 2D BZ, without fine-tuning two momentum parameters $k_x, k_y$ can not obey 6 constraint equations at the same time. Therefore, the high-ordered EPs and DPs can be easily locally split into several EPs with $\nu = \pm 1$ in the presence of perturbations.

To show that a NDP can generally deformed to an EP, we consider a generic non-hermitian Hamiltonian at a NDP with momentum $k_d$

$$\mathcal{H}(k) = \mathcal{H}(k_d) + \delta \mathcal{H},$$

(13)

where $\delta \mathcal{H}$ is an arbitrary perturbation term. The left and right eigenstates of the Hamiltonian at $k_d$ obey

$$\langle \phi_i | \delta \mathcal{H}(k_d) | \phi_i \rangle = E_i - E_{\phi_i}(k)$$

(14)

With 2-fold degeneracy at the NDP, we choose $E_1 = E_2 = E_d$. The first order of the perturbation for the other energy level ($i > 2$) is given by

$$E'_i = E_i + \Delta_{ij}$$

(15)

where $\Delta_{ij} = \langle \phi_i | \delta \mathcal{H} | \phi_j \rangle$. The effective Hamiltonian for the degenerate states in the first order perturbation is written as

$$H_{eff} = E_d \sigma_0 + \delta_1 \sigma_x + \delta_2 \sigma_y + \delta_3 \sigma_z$$

(16)

leading to the first two energy levels $E'_1, E'_2$ in the first order. We rewrite the discriminant in Eq. 7 in terms of $E'_1, E'_2$

$$\text{Disc}_E[\mathcal{H}_d](k_d + \delta k) = (E'_1 - E'_2)^2 \prod_{2 < i < j} \{E_i(k) - E_j(k) + \Delta_{ij} - \Delta_{ji}\}^2$$

$$\times \prod_{2 < i} \{E'_1(k) - E_i(k) - \Delta_{ii} \}^2 \{E'_2(k) - E_i(k) - \Delta_{ii} \}^2$$

(17)

Since the perturbation does not dramatically change the spectrum so that $(E_i - E_j)^2 \gg 0$ for $i \neq 1, 2$, only $E'_1 - E'_2$ is near the zero point and the vanishing discriminant at $k_d$ is determined by

$$(E'_1 - E'_2)^2 = 4(\delta_1^2 + \delta_2^2 + \delta_3^2).$$

(18)

Any non-zero perturbation obeying $\delta_1^2 + \delta_2^2 + \delta_3^2 = 0$ transfers the DP to an EP. Hence, in the absence of any symmetry, NDP is unstable and easily evolves an EP. Thus, we have the weak doubling theorem for EPs — in the presence of arbitrary perturbations, stable EPs with $\nu = \pm 1$ always come in pairs in the 2D BZ. Although in principal an EP can pair an NDP, the emergence of the NDP requires fine-tuned parameters.

Violation of the doubling theorem on 3D surface.— Consider a 3D non-Hermitian Hamiltonian with the lattice constant $a = 1$

$$H_{3D} = (m(k) + \cos k_z) \Gamma_0 + \sqrt{2} \sin(k_x + \pi/4) \Gamma_1 + (i + \sin k_y) \Gamma_2 + \sin k_z \Gamma_3,$$

(19)

where $m(k) = -2.8 + \cos k_x + \cos k_y, \Gamma_0 = \rho_3 \sigma_0, \Gamma_1 = \rho_1 \sigma_1$. This bulk states are always 2-fold degenerate and have gap closing. With open boundary condition in the z direction, the Hamiltonian in the form of the second quantization is given by

$$H_{3D} = \sum_{0 \leq z \leq N} \left\{ c^\dagger \left[ \frac{\Gamma_0}{2} c_{z+1} + \frac{\Gamma_0 - i \Gamma_3}{2} c_{z-1} + c^\dagger \left[ m(k) \Gamma_0 + \sqrt{2} \sin(k_x + \pi/4) \Gamma_1 + (i + \sin k_y) \Gamma_2 \right] c_{z} \right] \right\},$$

(20)
By solving the recurrence equation as the Harper equation, we have two surface states near $z = 0$

$$|1\rangle = \sum_z \alpha_z (i 0 1 0)^T c_z^\dagger |0\rangle, \quad |2\rangle = \sum_z \alpha_z (0 1 0 i)^T c_z^\dagger |0\rangle,$$

where $\alpha_z$ normalizes the wavefunctions and obeys the recurrence relation $\alpha_{z+1} = -m(k)\alpha_z$. However, the surface states near $z = 0$ are not present in the entire surface BZ, since the localization requirement leads to the presence of the surface state in this region $\cos k_x + \cos k_y > 1.8$ ($m(k) < 1$). By projecting these two surface states to the bulk Hamiltonian (20), the surface Hamiltonian is written as

$$\mathcal{H}_{\text{surf}}(k) = \sqrt{2} \sin(k_x + \pi/4)\sigma_x + (i + \sin k_y)\sigma_y. \quad (22)$$

On the surface, there is the only one EP located at $(0, 0)$ with $\nu = -1$ as illustrated in Fig. 4. Hence, the 2D surface from the 3D bulk violates the doubling theorem.

Conclusion.— In conclusion, we have derived doubling theorems for FPs and EPs in generic 2D non-Hermitian lattice Hamiltonians. To derive the doubling theorem for exceptional points we have introduced a new topological invariant, which we call the discriminant number. This discriminant number endows EPs with a quantized topological charge, thereby guaranteeing their stability against perturbations. The doubling theorem ensures that a single EP of first order must be accompanied by another first-order EP with opposite topological charge as all of the energy bands are taken into account. We note that 2D non-Hermitian Hamiltonians with fine-tuned parameters can also exhibit higher-order EPs or non-defective degeneracy points. These are, however, unstable and can be removed by arbitrarily small perturbations, which gap them out or split them into EPs of first order. In this fine tuned situation it is possible that there exists a single higher-order EP in the BZ accompanied by a non-defective degeneracy point. In this case the topological charge of the EP is canceled by the topological charge of the fine-tuned non-defective degeneracy point. We have also shown by several examples that EPs do not need to be branch point singularities, and vice versa, that branch cuts do not need to be terminated by EPs. Last but not least, we have found that the doubling theorems can be violated on the surface, which can host single FPs or EPs. These surface EPs and FPs are expected to exhibit anomalous properties with interesting physical consequences. Extensions of the doubling theorems to higher dimensions and symmetry constrained Hamiltonians, as well as their applications to various physical systems, will be presented in future works.

Note added.— Upon submission of this manuscript we became aware of a related preprint\textsuperscript{61}, which makes use of the discriminant to classify non-Hermitian Hamiltonians.

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Supplementary Materials for
“Fermion doubling theorems in 2D non-Hermitian systems”

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This supplementary information is organized as follows. In Sec. I, we review the mathematical properties of discriminant in the literature, since the idea of discriminant is the key of our doubling theorem in non-Hermitian systems. In Sec. II, an example is provided to show that a 3-fold degeneracy EP can be deformed to two 2-fold degeneracy EPs. In Sec. III, we show the two different expressions of the discriminant number in the main text are equivalent. Since an EP with $\nu = \pm 1$ is stable in 2D BZ and other DPs are unstable, we study the requirement to realize the other types of DPs in Sec. IV.

I. MATHEMATICS OF DISCRIMINANT

The discriminant is the key to define the winding number characterizing DPs and EPs. Although the mathematical properties of the discriminant have been studied throughly in the literature$^{62-64}$, for the clarity of the manuscript, we review the mathematical details of the discriminant as a powerful and efficient tool to calculate the band degeneracies in a general $n$-band non-Hermitian Hamiltonian $\mathcal{H}(\mathbf{k})$. Another straightforward approach is to diagonalize the non-hermitian Hamiltonian and then to compare the set of the eigenenergies. However, it is not an efficient way for large size Hamiltonians. Hence, to circumvent the diagonalization problem, we adapt the discriminant approach to find energy degeneracies and to compute winding numbers characterizing the DPs. To be more precise, we start with the characteristic equation of the Bloch Hamiltonian $\mathcal{H}(\mathbf{k})$,

$$ f(E, \mathbf{k}) = \det [E - \mathcal{H}(\mathbf{k})] = \prod_{i=1}^{n} [E - E_i(\mathbf{k})], $$

(23)

where $E_i(\mathbf{k})$ is the $i$-th eigenvalue of the non-Hermitian Hamiltonian. A non-Hermitian degeneracy point at $\mathbf{k}_d$ should satisfy $E_i(\mathbf{k}_d) = E_j(\mathbf{k}_d)$ as $i \neq j$. In the following, we show that $\mathcal{H}(\mathbf{k})$ has non-Hermitian band degeneracy located at $\mathbf{k}_d$ if and only if

$$ \text{Disc}_E[\mathcal{H}](\mathbf{k}) = 0, $$

(24)

where $\text{Disc}_E[\mathcal{H}](\mathbf{k})$ is the discriminant of $f(E, \mathbf{k})$. Before proceeding to the discussion of discriminant, we reintroduce the mathematical concepts of resultants and Sylvester matrices. In the following, we focus on the relation between the discriminant and the characteristic equation $f(E, \mathbf{k})$; we relabel $\text{Disc}_E[\mathcal{H}](\mathbf{k})$ by $\Delta(f)$.

A. Resultant and Sylvester matrix

Here, we review the basic mathematics in the literature for the completeness of the manuscript.

**Definition A.1 (Polynomial).** A polynomial $f(x) \in F[x]$ is defined as

$$ f(x) = \prod_{i=1}^{n} (x - \xi_i) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \quad a_n \neq 0 $$

(25)

where each coefficient $a_i$ belongs to the field $F$ and each root $\xi_i$ belongs to the extension of $F$. For example, if $a_n, \ldots, a_0$ are real numbers, $\xi_1, \ldots, \xi_n$ are complex numbers.

For characteristic polynomials in non-Hermitian Hamiltonians, we choose the field $F$ to be the complex number system $\mathbb{C}$.

**Definition A.2 (Resultant).** Given two polynomials $f(x) = a_n x^n + \ldots + a_0, g(x) = b_m x^m + \ldots + b_0 \in F[x]$, their resultant relative to the variable $x$ is a polynomial over the field of coefficients of $f(x)$ and $g(x)$, and is defined as

$$ R(f, g) = a_n b_m \prod_{i,j} (\xi_i - \eta_j), $$

(26)
where $f(\xi_i) = 0$ for $1 \leq i \leq n$ and $g(\eta_j) = 0$ for $1 \leq j \leq m$.

**Theorem A.3.** Let $f(x) = a_n x^n + \ldots + a_0, g(x) = b_m x^m + \ldots + b_0 \in F[x]$.

1. Suppose that $f$ has $n$ roots $\xi_1, \ldots, \xi_n$ in some extension of $F$. Then
   \[ R(f, g) = a_n^m \prod_{i=1}^{n} g(\xi_i). \] (27)

2. Suppose that $g$ has $m$ roots $\eta_1, \ldots, \eta_m$ in some extension of $F$. Then
   \[ R(f, g) = (-1)^{mn} b_m^m \prod_{j=1}^{m} f(\eta_j). \] (28)

The proof can be found in Ref\textsuperscript{63,64}.

**Theorem A.4.** Let $f$ and $g$ be two non-zero polynomials with coefficients in a field $F$. Then $f$ and $g$ have a common root in some extension of $F$ if and only if their resultant $R(f, g)$ is equal to zero.

**Proof:** Suppose $\gamma$ is their common root, $R(f, g) \propto (\gamma - \gamma) = 0$. Conversely, if $R(f, g) = 0$, at least one of the factors of $R(f, g)$ must be zero, say $\xi_i - \eta_j = 0$, then, $\xi_i = \eta_j$ is their common root.

Hence, the resultant can be applied to determine whether two polynomials share a common root. However, by Definition A.2, obtaining the value of the resultant requires to know the roots of each polynomial. The following theorem enables us to calculate the resultant directly according to the coefficients of $f$ and $g$ by using the Sylvester matrix.

**Definition A.5.** The Sylvester matrix of two polynomials $f(x) = a_n x^n + \ldots + a_0, g(x) = b_m x^m + \ldots + b_0 \in F[x]$ is defined by

\[
\text{Syl}(f, g) = \begin{pmatrix}
  a_n & a_{n-1} & a_{n-2} & \cdots & 0 & 0 & 0 \\
  0 & a_n & a_{n-1} & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & a_1 & a_0 & 0 \\
  0 & 0 & 0 & \cdots & a_2 & a_1 & a_0 \\
  b_m & b_{m-1} & b_{m-2} & \cdots & 0 & 0 & 0 \\
  0 & b_m & b_{m-1} & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & b_1 & b_0 & 0 \\
  0 & 0 & 0 & \cdots & b_2 & b_1 & b_0
\end{pmatrix},
\] (29)

where $a_n, \ldots, a_0$ are the coefficients of $f$ and $b_m, \ldots, b_0$ are the coefficients of $g$.

**Theorem A.6.** The resultant of two polynomials $f, g$ equals the determinant of their Sylvester matrix, namely

\[ R(f, g) = \det[\text{Syl}(f, g)] \] (30)

For example, if $n = 3, m = 2$,

\[ R(f, g) = \det \begin{pmatrix}
  a_3 & a_2 & a_1 & a_0 & 0 \\
  0 & a_3 & a_2 & a_1 & a_0 \\
  b_2 & b_1 & b_0 & 0 & 0 \\
  0 & b_2 & b_1 & b_0 & 0 \\
  0 & 0 & b_2 & b_1 & b_0
\end{pmatrix}. \] (31)

The proof the this theorem can be found in Ref\textsuperscript{63,64}. 
B. Discriminant

Definition B.1 (Discriminant). Let \( f = a_n x^n + \ldots + a_0 \) be a polynomial with coefficients in an arbitrary field \( F \). Then the (standard) discriminant of \( f \) is defined as

\[
\Delta(f) := a_n^{2n-2} \Delta_0(f) = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\xi_i - \xi_j)^2 ,
\]

(32)

where \( \xi_1, \ldots, \xi_n \) are the roots of \( f \) in some extension of \( F \).

Using this definition of the discriminant, we can directly have the following theorem. That is, the discriminant can directly determine whether the polynomial \( f \) possesses degeneracy roots.

Theorem B.2. Let \( f \) be a polynomial of degree \( n \geq 1 \) with coefficients in a field \( F \). Then \( f \) has a double root in some extension of \( F \) if and only if \( \Delta(f) = 0 \).

Furthermore, since the resultant can be computed by using its Sylvester matrix, the theorem below provides the connection between the discriminant and the resultant.

Theorem B.3. The discriminant of \( f = a_n x^n + \ldots + a_0 \) can be expressed by the resultant of \( f \) and its derivative \( f' := \partial_x f \), namely

\[
\Delta(f) = (-1)^{n(n-1)/2} a_n^{-1} R(f, f').
\]

(33)

Proof: From \( f(x) = a_n \prod_{i=1}^n (x - \xi_i) \) and \( f'(\xi_i) = a_n \prod_{j \neq i} (\xi_i - \xi_j) \), one can obtain the following equation by using Theorem A.3,

\[
R(f, f') = a_n^{n-1} \prod_{i=1}^n \prod_{j \neq i} (\xi_i - \xi_j) = a_n^{2n-1} \prod_{1 \leq i < j \leq n} (\xi_i - \xi_j) (\xi_j - \xi_i)
\]

(34)

Without knowing any root values, the explicit form of the discriminant can be obtained directly from the coefficient of the polynomial. Back to non-Hermitian quantum systems, in the characteristic polynomial \( a_n = 1 \) so that we have

\[
\Delta(f) = (-1)^{n(n-1)/2} \det[Syl(f, g)].
\]

(35)

This is the main equation we use to compute the discriminant.

C. Some examples

We show some examples in the literature to compute discriminants.

1. \( n = 2 \) case

If \( f(x) = ax^2 + bx + c \), then

\[
\Delta(f) = -a^{-1} R(f, f') = -a^{-1} \det \begin{pmatrix} a & b & c \\ 2a & b & 0 \\ 0 & 2a & b \end{pmatrix} = b^2 - 4ac.
\]

(36)
2. \( n = 3 \) case

If \( f(x) = ax^3 + bx^2 + cx + d \), then

\[
\Delta(f) = -a^{-1} R(f, f') = -a^{-1} \det \begin{pmatrix}
a & b & c & d & 0 \\
0 & a & b & c & d \\
0 & a & b & c & d \\
0 & 3a & 2b & c & 0 \\
0 & 0 & 3a & 2b & c
\end{pmatrix}
\]

(37)

\[
= b^2 c^2 - 4ac^3 - 4b^3 d + 18abcd - 27a^2 d^2.
\]

3. \( n = 4 \) case

If \( f(x) = ax^4 + bx^3 + cx^2 + dx + e \), then

\[
\Delta(f) = a^{-1} R(f, f') = a^{-1} \det \begin{pmatrix}
a & b & c & d & e & 0 & 0 \\
0 & a & b & c & d & e & 0 \\
0 & 0 & a & b & c & d & e \\
0 & 0 & a & b & c & d & e \\
0 & 0 & 0 & 3a & 2b & 2c & d \\
0 & 0 & 0 & 4a & 3b & 2c & d \\
0 & 0 & 0 & 0 & 4a & 3b & 2c & d
\end{pmatrix}
\]

(38)

\[
= b^2 c^2 d^2 - 4b^2 c^3 e - 4b^3 d^3 + 18b^3 cde - 27b^4 e^2 - 4ac^3 d^2 + 16ac^3 e
\]

\[
+ 18abcd^3 - 80abc^2 de - 6a^2 b^2 d^2 e + 144a^2 b^2 e^2 - 27a^2 d^4
\]

\[
+ 144a^2 c^2 d^2 e - 128a^2 c^2 e^2 - 192a^2 bde^2 + 256a^3 e^3.
\]

D. Application to non-Hermitian systems

The energy spectrum is complex numbers in non-Hermitian Hamiltonians. We calculate the discriminant in two- and three-band models based on Eqs. 29,35 to find the detailed criterion of the non-Hermitian degenerate points.

1. Two-band example

Consider a generic two-band model

\[
\mathcal{H}(k) = h_0(k) + h_x(k)\sigma_x + h_y(k)\sigma_y + h_z(k)\sigma_z,
\]

(39)

where \( h_\mu(k) = h_\mu^r(k) + i h_\mu^i(k) \) are complex functions of \( k \). The characteristic equation of the two band model can be written as

\[
f(E, k) = E^2 + b(k)E + c(k),
\]

(40)

where \( b(k) = -2h_0(k) \) and \( c(k) = h_x^2(k) - h_y^2(k) - h_z^2(k) \). By using the discriminant of Eq. 40 as a function of \( E \), we have the equation to determine the DPs

\[
\Delta_f(k) = b^2(k) - 4c(k) = 4[h_x^2(k) + h_y^2(k) + h_z^2(k)] = 0.
\]

(41)
The band degeneracy condition also can be obtained by the energy spectrum \( E_\pm = h_0(k) \pm (h_x^2(k) + h_y^2(k) + h_z^2(k))^{1/2} \).

2. Three-band model

Since the identity matrix is the only one not affecting the band degeneracy, a generic three-band model without the identity matrix can be written as

\[
\mathcal{H}(k) = \sum_{\rho=1}^8 g_\rho(k)\lambda_\rho,
\]

(42)
where the eight Gell-Mann matrices are
\[
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (43)
\]
\[
\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (44)
\]

The characteristic equation can be written in economic form of
\[
f(E, k) = E^3 + c(k)E + d(k) = 0, \quad (45)
\]
where \(c = -\sum_{s=1}^{8} g_s^2\) and \(d = g_8 (-6 g_1^2 - 6 g_2^2 - 6 g_3^2 + 2 g_2^3 + 3 (g_1^2 + g_3^2 + g_6^2 + g_7^2)) / 3^{3/2} - 2g_1 (g_4 g_6 + g_5 g_7) + 2g_2 (g_4 g_7 - g_5 g_6) + g_3 (-g_1^2 - g_3^2 + g_6^2 + g_7^2).\) The discriminant of the characteristic equation is given by
\[
\Delta_f(k) = -4c^3(k) - 27d^2(k) = 0. \quad (46)
\]
This is a quite simple equation, which can be analytically dealt in 2D systems.

II. THE SPLITTING OF A 3-FOLD DEGENERACY EP

To address the splitting of a 3-fold degeneracy EP, we provide an low-energy Hamiltonian with a perturbation as an example
\[
\hat{H}(\delta k) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \delta k_p & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (47)
\]
where \(\delta k_p = \delta k_x + i \delta k_y\). The discriminant is given by \(\text{Disc}_E[\hat{H}](k) = 4\lambda^3 - 27(1 + \lambda)^2 \delta k_p^2\). As \(\Delta(k_d) = 0\), the two degeneracy points are located at \((\pm k_\lambda, 0)\) for \(\lambda \geq 0\) and \((0, \pm k_\lambda)\) for \(\lambda \leq 0\), where \(k_\lambda = \sqrt{4|\lambda|^3 (1 + \lambda)^{-2}/27}\). When the perturbation vanishes (\(\lambda = 0\)), the three-fold degeneracy EP, which is defective, is located at \((0, 0)\) as shown in Fig. S1(a,b). However, the three-fold degeneracy is unstable because a small perturbation (\(\lambda \neq 0\)) splits the point into two two-fold degeneracy EPs as shown in Fig. S1(c,d).

III. DERIVE THE EQUIVALENCE BETWEEN EQ. 9 AND EQ. 10

From Eq. 9 in the main text, one can define the discriminant number for the DP \(k_d^j\) as follows,
\[
\nu(k_d^j) = \frac{i}{2\pi} \oint_{\Gamma(k_d^j)} dk \cdot \nabla_k \ln \Delta_f(k), \quad (48)
\]
where \(\Delta_f(k)\) is the discriminant of the characteristic equation
\[
\Delta_f(k) = \prod_{1 \leq i < j \leq n} [E_i(k) - E_j(k)]^2. \quad (49)
\]
Then
\[
\nu(k_d^j) = \frac{i}{2\pi} \sum_{i \neq j} \oint_{\Gamma(k_d^j)} dk \cdot \nabla_k \ln \prod_{1 \leq i < j \leq n} [E_i(k) - E_j(k)]^2
\]
\[
= \frac{i}{2\pi} \sum_{i \neq j} \oint_{\Gamma(k_d^j)} dk \cdot \nabla_k \ln [E_i(k) - E_j(k)]
\]
\[
= \frac{-1}{2\pi} \sum_{i \neq j} \oint_{\Gamma(k_d^j)} dk \cdot \nabla_k \arg [E_i(k) - E_j(k)]
\]
\[
= \sum_{i \neq j} \nu_{ij}, \quad (50)
\]
where we have used \(\ln(z) = \ln(|z|) + i \arg(z)\) and \(\nu_{ij}(\Gamma) = -\frac{1}{2\pi} \oint_{\Gamma} \nabla_k \arg [E_i(k) - E_j(k)] \cdot dk\).
FIG. S1. The energy dispersions of the three-band model (47) (a,b) show that Re($E$) and Im($E$) near the three-fold degeneracy EP (green point) with $\lambda = 0$. The black arcs represent the branch cuts. (c,d) show the DP evolves to the two two-fold EPs in the presence of the perturbation ($\lambda = 1/2$).

IV. THE REQUIREMENT TO REALIZE EPS AND NDPS WITH DIFFERENT DISCRIMINANT NUMBERS

An EP and an NDP are two DPs and exhibit different stabilities. These two types of the DPs are characterized by winding numbers ($\nu$) as invariants, which are immune from any symmetry-preserving perturbations. In the 2D BZ, with $\nu = \pm 1$ an EP is always stable, whereas an NDP can be split into several stable EPs with $\nu = \pm 1$ in the presence of perturbations. To have a stable NDP without splitting, additional constraints are needed. Furthermore, to realize an EP with discriminant number different from $\pm 1$, extra constraints are also needed to be imposed. The numbers of the constraint equations for EPs and NDPs with different discriminant numbers is summarized in Table I.

In the main text of the paper, we have mentioned that the emerging of stable NDPS requires at least 6 constraint equations in a 2-band model. To show this, we start from the $d$-dimensional two-band non-Hermitian Hamiltonian $H(k)$ in Eq. 39. It has been shown that the vanishing discriminant of the characteristic equation determines DPs

$$\Delta_f(k) = h_x^2(k) + h_y^2(k) + h_z^2(k) = 0,$$

where $h_l(k) = h^*_l(k) + i h^*_l(k)$ with $l = x, y, z$ are complex functions of $k = (k_1, ..., k_d)$. Suppose there exists a DP located at $k_d$. As discussed in the main text, this DP $k_d$ can be either EP or NDP. The Hamiltonian $\mathcal{H}(k_d)$ can be unitary transformed to two possible forms

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$  

Hence we can classify EPs and NDPs as follows

$$\begin{align*}
\text{EP} & : h(k_d) \neq 0 \cap \Delta_f(k_d) = 0; \\
\text{NDP} & : h(k_d) = 0,
\end{align*}$$

where $h = (h_x, h_y, h_z)$. Since $h_{x/y/z}(k)$ are complex function, the 6 constraint equations determine NDPs and the vanishing discriminant provides 2 constraint equations for EPs. The 2 constraint equations lead to stable EPs in the 2D BZ. Similarly, in 6D space, 6 independent momenta can be adjusted to satisfy the 6 constraint equations so that NDPs can be stable.

As the discriminant characterizing a DP is not equal to $\pm 1$, in the presence of perturbation the DP can be destroyed ($\nu = 0$) or split to multiple DPs ($\nu \neq 0$). That is, these types of the DP cannot be unstable at one single point in the 2D BZ; it is important
to learn in which conditions these special DPs emerge. Therefore, we study the minimal number of the constraint equations leading to the presence of EPs and NDPs with $\nu \neq \pm 1$ separately in a generic 2-band model. Back to 2d momentum space, we expand the discriminant at $k_d$

$$\Delta_f(k_d + \delta k) = \Delta'_f(k_d + \delta k) + i\Delta''_f(k_d + \delta k) = \sum_{n+m=1}^{\infty} (\alpha_{nm}^r + i\alpha_{nm}^i)\delta k_x^n \delta k_y^m. \quad (54)$$

We consider some linear momentum terms do not vanish in the discriminant ($\alpha \neq 0$) and define

$$\alpha := \begin{pmatrix} \alpha_{10}^r & \alpha_{01}^r \\ \alpha_{10}^i & \alpha_{01}^i \end{pmatrix}. \quad (55)$$

The invariant is determined by the phase winding in the momentum path encircling $k_d$. As $\det[\alpha] = 0$, the complex phases are identical in the two different directions; hence, the invariant $\nu = 0$. Since each entry in $\alpha$ is real, $\det[\alpha] = 0$ and $\Delta_f(k_d) = 0$ are the three constraint equations leading to an EP with $\nu = 0$.

For an EP with $\nu = \pm N (N > 1)$, the order in $\Delta_f(k_d + \delta k)$ less than $\delta k^N$ must vanish. The reason is that $\delta k^j (j < N)$ never leads to the winding number $\nu = \pm N$. For all $n + m < N$, $\alpha_{nm}^r$ and $\alpha_{nm}^i$ should be zero at the EP. Hence, with $\Delta_f(k_d) = 0$, the number of the constraint equations is given by $2 + 2[2 + \ldots + N] = N(N + 1)$. We note that the $N(N + 1)$ constraint equations are necessary conditions for $\nu = \pm N$. The discriminant $\Delta_f(k_d + \delta k) \approx \sum_{n+m=N}^{\infty} (\alpha_{nm}^r + i\alpha_{nm}^i)\delta k_x^n \delta k_y^m$ can possess the limited discriminant number, of which the absolute value is less than or equal to $N (|\nu| \leq N$). The exact value of the winding number depends on the details of $\alpha_{nm}^r$, $\alpha_{nm}^i$.

For an NDP, all of the Pauli matrices must vanish at $k_d$; the coefficients of the Pauli matrices can be expanded in form of

$$h_i(k_d + \delta k) = \sum_{n+m=0}^{\infty} (\beta_{i,nm}^r + i\beta_{i,nm}^i)\delta k_x^n \delta k_y^m. \quad (56)$$

It is shown in the main text that the winding number of any NDP is always even. To have an NDP with $\nu = \pm 2N$, for all $n + m < 2N$, $\beta_{i,nm}^r$ and $\beta_{i,nm}^i$ should be zero at the NDP and the discriminant is written in the form of

$$\Delta_f(k_d + \delta k) = \sum_{l=x,y,z} \left[ \sum_{n+m=N}^{\infty} (\beta_{l,nm}^r + i\beta_{l,nm}^i)\delta k_x^n \delta k_y^m \right]^2. \quad (57)$$

Therefore, with $h_m(k_d) = 0$, the number of the constraint equations is calculated as $6 + 6[2 + \ldots + N] = 3N(N + 1)$. Thus, to realize an DP (EP or NDP) with winding number $\nu$ away from zero, the number of the constraint equations is approximately proportional to $\nu^2$. 

| EP/non-EP | Winding number $\nu$ | Evolution under perturbations | Number of constraint equations |
|-----------|---------------------|-------------------------------|-----------------------------|
| EP        | 0                   | Split/Gapped                  | 3                           |
| EP        | $\pm 1$             | Stable                        | 2                           |
| EP        | $\pm n(n \geq 2)$   | Split                         | $n(n + 1)$                  |
| NDP       | 0                   | Split/Gapped                  | 6                           |
| NDP       | $\pm 2n(n \geq 1)$ | Split                         | $3n(n + 1)$                 |