Limit theorem for a time-dependent coined quantum walk on the line

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Abstract. We study time-dependent discrete-time quantum walks on the one-dimensional lattice. We compute the limit distribution of a two-period quantum walk defined by two orthogonal matrices. For the symmetric case, the distribution is determined by one of two matrices. Moreover, limit theorems for two special cases are presented.

1 Introduction

The discrete-time quantum walk (QW) was first intensively studied by Ambainis \(\text{et al.} \) \cite{1}. The QW is considered as a quantum generalization of the classical random walk. The random walker in position \(x \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \} \) at time \(t (\in \{0, 1, 2, \ldots \}) \) moves to \(x-1\) at time \(t+1\) with probability \(p\), or \(x+1\) with probability \(q = 1-p\). In contrast, the evolution of the quantum walker is defined by replacing \(p\) and \(q\) with \(2 \times 2\) matrices \(P\) and \(Q\), respectively. Note that \(U = P + Q\) is a unitary matrix. A main difference between the classical walk and the QW is seen on the particle spreading. Let \(\sigma(t)\) be the standard deviation of the walk at time \(t\). That is, \(\sigma(t) = \sqrt{\mathbb{E}(X_{t}^{2}) - \mathbb{E}(X_{t})^{2}}\), where \(X_{t}\) is the position of the quantum walker at time \(t\) and \(\mathbb{E}(Y)\) denotes the expected value of \(Y\). Then the classical case is a diffusive behavior, \(\sigma(t) \sim \sqrt{t}\), while the quantum case is ballistic, \(\sigma(t) \sim t\) (see \cite{1}, for example).

In the context of quantum computation, the QW is applied to several quantum algorithms. By using the quantum algorithm, we solve a problem quadratically faster than the corresponding classical algorithm. As a well-known quantum search algorithm, Grover’s algorithm was presented. The algorithm solves the following problem: in a search space of \(N\) vertices, one can find a marked vertex. The corresponding classical search requires \(O(N)\) queries. However, the search needs only \(O(\sqrt{N})\) queries. As well as the Grover algorithm, the QW can also search a marked vertex with a quadratic speed up, see Shenvi \(\text{et al.} \) \cite{2}. It has been reported that quantum walks on regular graphs (e.g., lattice, hypercube, complete graph) give faster searching than classical walks. The Grover search algorithm can also be interpreted as a QW on complete graph. Decoherence is an important concept in quantum information processing. In fact, decoherence on QWs has been extensively investigated, see Kendon \cite{3}, for example. However, we should note that our results
are not related to the decoherence in QWs. Physically, Oka et al. [4] pointed out that the Landau-Zener transition dynamics can be mapped to a QW and showed the localization of the wave functions.

In the present paper, we consider the QW whose dynamics is determined by a sequence of time-dependent matrices, \{U_t : t = 0, 1, \ldots\}. Ribeiro et al. [5] numerically showed that periodic sequence is ballistic, random sequence is diffusive, and Fibonacci sequence is sub-ballistic. Mackay et al. [6] and Ribeiro et al. [5] investigated some random sequences and reported that the probability distribution of the QW converges to a binomial distribution by averaging over many trials by numerical simulations. Konno [7] proved their results by using a path counting method. By comparing with a position-dependent QW introduced by Wójcik et al. [8], Bañuls et al. [9] discussed a dynamical localization of the corresponding time-dependent QW.

In this paper, we present the weak limit theorem for the two-period time-dependent QW whose unitary matrix \(U_t\) is an orthogonal matrix. Our approach is based on the Fourier transform method introduced by Grimmett et al. [10]. We think that it would be difficult to calculate the limit distribution for the general n-period (n = 3, 4, \ldots) walk. However, we find out a class of time-dependent QWs whose limit probability distributions result in that of the usual (i.e., one-period) QW. As for the position-dependent QW, a similar result can be found in Konno [11].

The present paper is organized as follows. In Sect. 2, we define the time-dependent QW. Section 3 treats the two-period time-dependent QW. By using the Fourier transform, we obtain the limit distribution. Finally, in Sect. 4, we consider two special cases of time-dependent QWs. We show that the limit distribution of the walk is the same as that of the usual one.

\section{Time-dependent QW}

In this section we define the time-dependent QWs. Let \(|x\rangle (x \in \mathbb{Z})\) be infinite components vector which denotes the position of the walker. Here, \(x\)-th component of \(|x\rangle\) is 1 and the other is 0. Let \(|\psi_t(x)\rangle \in \mathbb{C}^2\) be the amplitude of the walker in position \(x\) at time \(t\), where \(\mathbb{C}\) is the set of complex numbers. The time-dependent QW at time \(t\) is expressed by

\[ |\psi_t\rangle = \sum_{x \in \mathbb{Z}} |x\rangle \otimes |\psi_t(x)\rangle. \tag{1} \]

To define the time evolution of the walker, we introduce a unitary matrix

\[ U_t = \begin{bmatrix} a_t & b_t \\ c_t & d_t \end{bmatrix}, \tag{2} \]

where \(a_t, b_t, c_t, d_t \in \mathbb{C}\) and \(a_tb_tcd_t \neq 0 (t = 0, 1, \ldots)\). Then \(U_t\) is divided into \(P_t\) and \(Q_t\) as follows:

\[ P_t = \begin{bmatrix} a_t & b_t \\ 0 & 0 \end{bmatrix}, \quad Q_t = \begin{bmatrix} 0 & 0 \\ c_t & d_t \end{bmatrix}. \tag{3} \]
The evolution is determined by

$$|\Psi_{t+1}\rangle = \sum_{x \in \mathbb{Z}} |x\rangle \otimes (P_t |\psi_t(x+1)\rangle + Q_t |\psi_t(x-1)\rangle).$$

(4)

Let $||y||^2 = \langle y|y \rangle$. The probability that the quantum walker $X_t$ is in position $x$ at time $t$, $P(X_t = x)$, is defined by

$$P(X_t = x) = |||\psi_t(x)\rangle||^2.$$

(5)

Moreover, the Fourier transform $|\hat{\Psi}_t(k)\rangle$ ($k \in [0, 2\pi]$) is given by

$$|\hat{\Psi}_t(k)\rangle = \sum_{x \in \mathbb{Z}} e^{-ikx} |\psi_t(x)\rangle,$$

(6)

with $i = \sqrt{-1}$. By the inverse Fourier transform, we have

$$|\psi_t(x)\rangle = \int_{0}^{2\pi} \frac{dk}{2\pi} e^{ikx} |\hat{\Psi}_t(k)\rangle.$$  
(7)

The time evolution of $|\hat{\Psi}_t(k)\rangle$ is

$$|\hat{\Psi}_{t+1}(k)\rangle = \hat{U}_t(k) |\hat{\Psi}_t(k)\rangle,$$

(8)

where $\hat{U}_t(k) = R(k)U_t$ and $R(k) = \begin{bmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{bmatrix}$. We should remark that $R(k)$ satisfies $R(k_1)R(k_2) = R(k_1 + k_2)$ and $R(k)^* = R(-k)$, where $*$ denotes the conjugate transposed operator. From (8), we see that

$$|\hat{\Psi}_t(k)\rangle = \hat{U}_{t-1}(k)\hat{U}_{t-2}(k)\cdots\hat{U}_0(k) |\hat{\Psi}_0(k)\rangle.$$  
(9)

Note that, when $U_t = U$ for any $t$, the walk becomes a usual one-period walk, and $|\hat{\Psi}_t(k)\rangle = \hat{U}(k)^t |\hat{\Psi}_0(k)\rangle$. Then the probability distribution of the usual walk is

$$P(X_t = x) = \left|\int_{0}^{2\pi} \frac{dk}{2\pi} e^{ikx} \hat{U}(k)^t |\hat{\Psi}_0(k)\rangle\right|^2.$$  
(10)

In Sect. 4, we will use this relation. In the present paper, we take the initial state as

$$|\psi_0(x)\rangle = \begin{cases} T[\alpha, \beta] & (x = 0) \\ T[0, 0] & (x \neq 0), \end{cases}$$

(11)

where $|\alpha|^2 + |\beta|^2 = 1$ and $T$ is the transposed operator. We should note that $|\hat{\Psi}_0(k)\rangle = |\psi_0(0)\rangle$. 


3 Two-period QW

In this section we consider the two-period QW and calculate the limit distribution. We assume that \( \{U_t : t = 0, 1, \ldots\} \) is a sequence of orthogonal matrices with \( U_{2s} = H_0 \) and \( U_{2s+1} = H_1 \) (\( s = 0, 1, \ldots \)), where

\[
H_0 = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}.
\]

Let

\[
f_K(x; a) = \frac{\sqrt{1 - |a|^2}}{\pi(1 - x^2)^{\frac{3}{2}}} I_{(-|a|, |a|)}(x),
\]

where \( I_A(x) = 1 \) if \( x \in A \), \( I_A(x) = 0 \) if \( x \notin A \). Then we obtain the following main result of this paper:

**Theorem 1.**

\[
\frac{X_t}{t} \Rightarrow Z,
\]

where \( \Rightarrow \) means the weak convergence (i.e., the convergence of the distribution) and \( Z \) has the density function \( f(x) \) as follows:

(i) If \( \det(H_1 H_0) > 0 \), then

\[
f(x) = f_K(x; a) \left[ 1 - \left\{ |\alpha|^2 - |\beta|^2 + \frac{\overline{\alpha \beta} + \overline{\alpha^2 \beta}}{a_0} \right\} x \right],
\]

where \( |a_\xi| = \min \{|a_0|, |a_1|\} \).

(ii) If \( \det(H_1 H_0) < 0 \), then

\[
f(x) = f_K(x; a_0 a_1) \left[ 1 - \left\{ |\alpha|^2 - |\beta|^2 + \frac{\overline{\alpha \beta} + \overline{\alpha^2 \beta}}{a_0} \right\} x \right].
\]

If the two-period walk with \( \det(H_1 H_0) > 0 \) has a symmetric distribution, then the density of \( Z \) becomes \( f_K(x; a_\xi) \). That is, \( Z \) is determined by either \( H_0 \) or \( H_1 \). Figure 1 (a) shows that the limit density of the two-period QW for \( a_0 = \cos(\pi/4) \) and \( a_1 = \cos(\pi/6) \) is the same as that for the usual (one-period) QW for \( a_0 \), since \( |a_0| < |a_1| \). Similarly, Fig. 1 (b) shows that the limit density of the two-period QW for \( a_0 = \cos(\pi/4) \) and \( a_1 = \cos(\pi/3) \) is equivalent to that for the usual (one-period) QW for \( a_1 \), since \( |a_0| > |a_1| \).
Proof. Our approach is due to Grimmett et al. [10]. The Fourier transform becomes

$$|\hat{\Psi}_{2t}(k)\rangle = \left(\hat{H}_1(k)\hat{H}_0(k)\right)^t|\hat{\Psi}_0(k)\rangle,$$

where $$\hat{H}_\gamma(k) = R(k)H_\gamma (\gamma = 0, 1)$$. We assume

$$H_\gamma = \begin{bmatrix} \cos \theta_\gamma & \sin \theta_\gamma \\ \sin \theta_\gamma & -\cos \theta_\gamma \end{bmatrix},$$

with $$\theta_\gamma \neq \frac{n\pi}{2} (n \in \mathbb{Z})$$ and $$\theta_0 \neq \theta_1$$. For the other case, the argument is nearly identical to this case, so we will omit it. The two eigenvalues $$\lambda_j(k)$$ ($$j = 0, 1$$) of $$\hat{H}_1(k)\hat{H}_0(k)$$ are given by

$$\lambda_j(k) = c_1c_2 \cos 2k + s_1s_2 + (-1)^j\sqrt{1 - (c_1c_2 \cos 2k + s_1s_2)^2},$$

where $$c_\gamma = \cos \theta_\gamma$$, $$s_\gamma = \sin \theta_\gamma$$. The eigenvector $$|v_j(k)\rangle$$ corresponding to $$\lambda_j(k)$$ is

$$|v_j(k)\rangle = \begin{cases} s_1c_2e^{2ik} - c_1s_2 \\ -c_1c_2 \sin 2k + (-1)^j\sqrt{1 - (c_1c_2 \cos 2k + s_1s_2)^2} \end{cases} i.$$  

The Fourier transform $$|\hat{\Psi}_0(k)\rangle$$ is expressed by normalized eigenvectors $$|v_j(k)\rangle$$ as follows:

$$|\hat{\Psi}_0(k)\rangle = \sum_{j=0}^{1} \langle v_j(k)|\hat{\Psi}_0(k)\rangle |v_j(k)\rangle.$$

Therefore we have

$$|\hat{\Psi}_{2t}(k)\rangle = \left(\hat{H}_1(k)\hat{H}_0(k)\right)^t|\hat{\Psi}_0(k)\rangle$$

$$= \sum_{j=0}^{1} \lambda_j(k)^t \langle v_j(k)|\hat{\Psi}_0(k)\rangle |v_j(k)\rangle.$$
The $r$-th moment of $X_{2t}$ is
\[
E((X_{2t})^r) = \sum_{x \in \mathbb{Z}} x^r P(X_{2t} = x) = \int_0^{2\pi} \frac{dk}{2\pi} \langle \hat{\Psi}_{2t}(k) | \left( D^r |\hat{\Psi}_{2t}(k)\right) \rangle = \int_0^{2\pi} \sum_{j=0}^{1} (t^r \lambda_j(k)^{-1} (D \lambda_j(k))^r |\langle v_j(k)|\hat{\Psi}_0(k)\rangle|^2 + O(t^{-1}),
\]
(23)
where $D = i(d/dk)$ and $(t^r) = t(t-1) \times \cdots \times (t-r+1)$. Let $h_j(k) = D\lambda_j(k)/2\lambda_j(k)$. Then we obtain
\[
E((X_{2t}/2t)^r) \to \int_{D_0} \frac{dk}{2\pi} \sum_{j=0}^{1} h_j^r(k) |\langle v_j(k)|\hat{\Psi}_0(k)\rangle|^2 \quad (t \to \infty). \tag{24}
\]
Substituting $h_j(k) = x$, we have
\[
\lim_{t \to \infty} E((X_{2t}/2t)^r) = \int_{-\epsilon x_1}^{\epsilon x_1} x^r f(x) \, dx, \tag{25}
\]
where
\[
f(x) = f_K(x; \epsilon x_1) \left[ 1 - \left\{ |\alpha|^2 - |\beta|^2 + \frac{(\alpha \bar{\beta} + \bar{\alpha} \beta)s_1}{c_1} \right\} x \right], \tag{26}
\]
and $|\epsilon x_1| = |\cos \theta|x_1 = \min \{ |\cos \theta_0|, |\cos \theta_1| \}$. Since $f(x)$ is the limit density function, the proof is complete. \qed

4 Special cases in time-dependent QWs

In the previous section, we have obtained the limit theorem for the two-period QW determined by two orthogonal matrices. For other two-period case and general $n$-period ($n \geq 3$) case, we think that it would be hard to get the limit theorem in a similar fashion. Here we consider two special cases in the time-dependent QWs and give the weak limit theorems.

4.1 Case 1

Let us consider the QW whose evolution is determined by the following unitary matrix:
\[
U_t = \begin{bmatrix} ae^{iw} & b \\ c & de^{-iw} \end{bmatrix}, \tag{27}
\]
with $a, b, c, d \in \mathbb{C}$. Here $w_t \in \mathbb{R}$ satisfies $w_{t+1} + w_t = \kappa_1$, where $\kappa_1 \in \mathbb{R}$ and $\mathbb{R}$ is the set of real numbers. Note that $\kappa_1$ does not depend on time. In this case, $w_t$ can be written as $w_t = (-1)^t(w_0 - \frac{\kappa_1}{2}) + \frac{\kappa_1}{2}$. Therefore the period of the QW becomes two. We should remark that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} (\equiv U)$ is a unitary matrix. Then we have
Theorem 2.

\[ \frac{X_t}{t} \Rightarrow Z_t, \quad (28) \]

where \( Z_t \) has the density function \( f_1(x) \) as follows:

\[ f_1(x) = f_K(x;a) \left\{ 1 - \left( |\alpha|^2 - |\beta|^2 + \frac{a\alpha\beta e^{i\omega_0} + \alpha^* \beta^* e^{-i\omega_0}}{|\alpha|^2} \right) x \right\}. \quad (29) \]

Proof. The essential point of this proof is that this case results in the usual walk. First we see that \( U_t \) can be rewritten as

\[ U_t = \begin{bmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{bmatrix} \]

\[ = R \left( \frac{\omega t}{2} \right) U R \left( \frac{\omega t}{2} \right). \quad (30) \]

From this, the Fourier transform \( \tilde{\psi}_t(k) \) can be computed in the following.

\[ |\tilde{\psi}_t(k)\rangle = \left\{ R(k) R \left( \frac{\omega_{t-1}}{2} \right) U R \left( \frac{\omega_{t-1}}{2} \right) \right\} \left\{ R(k) R \left( \frac{\omega_{t-2}}{2} \right) U R \left( \frac{\omega_{t-2}}{2} \right) \right\} \]

\[ \cdots \left\{ R(k) R \left( \frac{\omega_{0}}{2} \right) U R \left( \frac{\omega_{0}}{2} \right) \right\} |\tilde{\psi}_0(k)\rangle \]

\[ = R \left( \frac{\omega_{t}}{2} \right) \left\{ R(k + \kappa_1/2) U \right\}^{t} R \left( \frac{\omega_{0}}{2} \right) |\tilde{\psi}_0(k)\rangle. \quad (31) \]

Therefore we have

\[ |\psi_t(x)\rangle = \int_{\kappa_{1/2}}^{2\pi} \frac{dk}{2\pi} e^{ikx} |\tilde{\psi}_t(k)\rangle = \int_{\kappa_{1/2}}^{2\pi+\kappa_{1/2}} \frac{dk}{2\pi} e^{ikx} (k+\kappa_1/2) |\tilde{\psi}_t(k-\kappa_1/2)\rangle \]

\[ = e^{-i\kappa_1 x/2} R \left( -\frac{\omega_{t}}{2} \right) \int_{\kappa_{1/2}}^{2\pi+\kappa_{1/2}} \frac{dk}{2\pi} e^{ikx} (k) U \right\}^{t} |\tilde{\psi}_0^R(k)\rangle, \quad (32) \]

where \( |\tilde{\psi}_0^R(k)\rangle = R \left( \frac{\omega_{0}}{2} \right) |\tilde{\psi}_0(k-\kappa_1/2)\rangle \). Then the probability distribution is

\[ P(X_t = x) \]

\[ = \left\{ e^{i\kappa_1 x/2} \left( \int_{\kappa_{1/2}}^{2\pi+\kappa_{1/2}} \frac{dk}{2\pi} e^{ikx} (k) U \right) \right\}^{t} R \left( \frac{\omega_{t}}{2} \right) \]

\[ \times \left\{ e^{-i\kappa_1 x/2} R \left( -\frac{\omega_{t}}{2} \right) \left( \int_{\kappa_{1/2}}^{2\pi+\kappa_{1/2}} \frac{dk}{2\pi} e^{ikx} (k) U \right) \right\}^{t} |\tilde{\psi}_0^R(k)\rangle \]

\[ = \left| \int_{\kappa_{1/2}}^{2\pi+\kappa_{1/2}} \frac{dk}{2\pi} e^{ikx} U(k) \right|^2. \quad (33) \]
where $\hat{U}(k) = R(k)U$. This implies that Case 1 can be considered as the usual QW with the initial state $|\hat{\Psi}_0^R(k)\rangle = R\left(\frac{w_0}{2}\right)|\hat{\Psi}_0^R(k - \kappa_1/2)\rangle$ and the unitary matrix $U$. Then the initial state becomes

$$|\hat{\Psi}_0^R(k)\rangle = T[e^{iw_0/2}\alpha, e^{-iw_0/2}\beta],$$

that is,

$$|\psi_0(x)\rangle = \begin{cases} T[e^{iw_0/2}\alpha, e^{-iw_0/2}\beta] (x = 0) \\ T[0, 0] (x \neq 0) \end{cases}.$$  \hspace{1cm} (34)

Finally, by using the result in Konno [12, 13], we can obtain the desired limit distribution of this case. \hspace{1cm} $\Box$

### 4.2 Case 2

Next we consider the QW whose dynamics is defined by the following unitary matrix:

$$U_t = \begin{bmatrix} a & be^{iw_t} \\ ce^{-iw_t} & d \end{bmatrix}.$$  \hspace{1cm} (35)

Here $w_t \in \mathbb{R}$ satisfies $w_{t+1} = w_t + \kappa_2$, where $\kappa_2 \in \mathbb{R}$ does not depend on $t$. In this case, $w_t$ can be expressed as $w_t = \kappa_2 t + w_0$. Noting $U_t = R\left(\frac{w_t}{2}\right)UR\left(-\frac{w_t}{2}\right)$, we get a similar weak limit theorem as Case 1:

**Theorem 3.**

$$X_t \Rightarrow Z_2,$$  \hspace{1cm} (36)

where $Z_2$ has the density function $f_2(x)$ as follows:

$$f_2(x) = f_K(x; a) \left\{1 - \left(|\alpha|^2 - |\beta|^2 + \frac{\alpha\beta e^{-iw_0} + \alpha^* \beta^* e^{iw_0}}{|\alpha|^2}\right)x\right\}.$$  \hspace{1cm} (37)

If $w_t = 2\pi n/n (n = 1, 2, \ldots)$, $\{U_t\}$ becomes an $n$-period sequence. In particular, when $n = 2$ and $a, b, c, d \in \mathbb{R}$, $\{U_t\}$ is a sequence of two-period orthogonal matrices. Then Theorem 3 is equivalent to Theorem 1 (i).

### 5 Conclusion and Discussion

In the final section, we draw the conclusion and discuss our two-period walks. The main result of this paper (Theorem 1) implies that if $\det(H_1H_0) > 0$ and $\min\{|a_0|, |a_1|\} = |a_0|$, then the limit distribution of the two-period walk is determined by $H_0$. On the other hand, if $\det(H_1H_0) > 0$ and $\min\{|a_0|, |a_1|\} = |a_1|$, or $\det(H_1H_0) < 0$, then the limit distribution is determined by both $H_0$ and $H_1$.

Here we discuss a physical meaning of our model. We should remark that the time-dependent two-period QW is equivalent to a position-dependent
two-period QW if and only if the probability amplitude of the odd position in the initial state is zero. In quantum mechanics, the Kronig-Penney model, whose potential on a lattice is periodic, has been extensively investigated, see Kittel [14]. A derivation from the discrete-time QW to the continuous-time QW, which is related to the Schrödinger equation, can be obtained by Strauch [15]. Therefore, one of interesting future problems is to clarify a relation between our discrete-time two-period QW and the Kronig-Penney model.

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