Stokes Phenomena and Non-perturbative Completion in the Multi-cut Two-matrix Models

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http://web.phys.ntu.edu.tw/string/index.htm

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References

Stokes phenomena and strong-coupling side
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Spectral curves and weak-coupling side
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The worldsheet description
5. [Irie ‘09] “Fractional supersymmetric Liouville theory and the multi-cut matrix models,” Nucl.Phys. B819 (2009) 351-374
The horizon of perturbation theory

Current understanding of perturbative expansion

Non-critical string (pure-gravity)

\[ Z = \sum_{\text{possible WS}} g^{2n-2} \]

(genus: \( n=0,1,\ldots \))

2D Pure-Gravity on the worldsheets

\[ S_{WS} = \int d^2\sigma \sqrt{g} R + t \int d^2\sigma \sqrt{g} \]

Cosmological constant
The horizon of perturbation theory

**Current understanding of perturbative expansion**

**Non-critical string (pure-gravity)**

\[ Z = \sum_{\text{possible WS}} g^{2n-2} \]  
(genus: \( n=0,1,\ldots \))

**Large N expansion of matrix models**

\[ Z_{\text{MM}} = \int dM e^{-N \text{Tr}V(M)} = \sum_{\text{Feynman Graph: } G} N^{2-2n} \]  
(genus: \( n=0,1,\ldots \))

The matrix models know higher-order behavior and non-perturbative strings!!
The horizon of perturbation theory

**The answer from the matrix models**

\[ \mathcal{F} = \ln Z \approx \sum_{n=0}^{\infty} g^{2n-2} \mathcal{F}_n(t) + \sum_{I} \theta_I \exp \left[ \sum_{n=0}^{\infty} g^{2n+b_I-2} \mathcal{F}_{n}^{(I)}(t) \right] \]

1. Perturbative amplitudes of WS\(_n\):
   \[ \mathcal{F}_n(t) \leftrightarrow \ldots \]

2. Non-perturbative amplitudes are D-instantons! [Shenker ’90, Polchinski ‘94]
   \[ \mathcal{F}_{n}^{(I)}(t) \leftrightarrow \ldots \]
   **calculable within perturbative string theory!!**

3. Only the overall weight \( \theta \)'s (=Chemical Potentials) are undetermined within the perturbation theory
   \[ \theta_I \]
   \[ \mathcal{F}_{n}^{(I)}(t) \leftrightarrow \ldots \]
   **WS with Boundaries = open string theory**
\[ \theta \text{ is an integration constant} \]

Then, \( u(t) \) satisfies non-linear differential equation (String equation, Painlevé I):

\[ \frac{g^2}{2} u'' + 3u^2 - t = 0 \quad (g : \text{string coupling}) \]

Perturbation theory of string coupling \( g \):

\[ u(t) \simeq u_p(t) + O(e^{-\ast/g}) \]

\[ u_p = \sqrt{\frac{t}{3}} \sum_{n=0}^{\infty} u_n g^{2n} t^{-\frac{5}{2}n} \quad \text{with} \quad \frac{g^2}{2} u_p'' + 3u_p^2 - t = 0 \]

(this recursively fixes the perturbative amplitudes \( u_p \))

Then, the next 1-instanton sector:

\[ u(t) \simeq u_p(t) + u_{NP}^{(1)}(t) + O((u_{NP}^{(1)})^2) \]

\[ \frac{g^2}{2} \frac{d^2 u_{NP}^{(1)}(t)}{dt^2} + 6u_p(t) u_{NP}^{(1)}(t) = 0 \]

Therefore, there remains an integration constant in \( t \):

\[ u_{NP}^{(1)}(t) = C \sqrt{\frac{tg}{t^{\frac{5}{4}}} f \left( \frac{g^2}{t^{\frac{5}{2}}} \right) \exp \left[ \sum_{n=0}^{\infty} g^{2n-1} t^{-\frac{5}{2}n + \frac{5}{4}} u_n^{(1)} \right]} \]

Similarly, higher order is given with an arbitrary parameter, say \( \theta \):

\[ u(t) \simeq u_p(t) + \sum_{n=1}^{\infty} \theta^n u_{NP}^{(n)}(t) \]
Universality of the chemical potentials, $\phi$

$$\mathcal{F} = \ln Z \simeq \sum_{n=0}^{\infty} g^{2n-2} \mathcal{F}_n(t) + \sum I \theta_I \exp \left[ \sum_{n=0}^{\infty} g^{2n+b_I-2} \mathcal{F}_n^{(I)}(t) \right]$$

1. It seems that the chemical potentials are arbitrary

2. But it is known as a **universal observable** in the matrix models
   (independent from the choice of MM potentials)
   [Hanada-Hayakawa-Ishibashi-Kawai-Kuroki-Matsuo-Tada ’03]

3. At least, these calculations seem to require a **finite N analysis**
   (calculate in the finite N, then take the continuum limit)
   $\rightarrow$ Very hard !!

4. Then, how do we obtain **within the continuum formulation**?
   And/Or What is **the physical requirements**?

   *In this talk, we will give the answer to this question!!*
Contents

1. Introduction
   (the horizon of perturbation theory)

2. Spectral curves and Stokes phenomena

3. Concrete solutions in the higher-cut system

4. Conclusion and prospects
2. Spectral curves and Stokes phenomena
The resolvent operator

\[ W(x) = \left\langle \frac{1}{N} \text{tr} \frac{1}{x - M} \right\rangle \]

Diagonalization: \[ U^\dagger M U = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) \]

\[ Z = \int dM e^{-N\text{tr}V(M)} \]

In Large N limit (= semi-classical)

\[ Z = \int d^N \lambda \prod_{i>j} (\lambda_i - \lambda_j)^2 e^{-N \sum_i V(\lambda_i)} \]

N-body problem in the potential V

The resolvent operator allows us to read this information

\[ W(x) = \left\langle \frac{1}{N} \text{tr} \frac{1}{x - M} \right\rangle = \int_{\text{cuts}} d\lambda \frac{\rho(\lambda)}{x - \lambda} \]

Eigenvalue density

\[ W(x \pm i\epsilon) = \frac{V'(x)}{2} \mp \pi i \rho(x) \]

Cuts = Position of Eigenvalues
The perturbative amplitudes from the spectral curve

Topological Recursions [Eynard’04, Eynard-Orantin ‘07]

All the perturbative amplitudes are calculable in terms of algebraic observables on the spectral curve

\[
W_{n+1}^{(g)} = \sum_i \text{Res} \frac{K(z_0, z)}{z - a_i} \left[ W_{n+2}^{(g-1)}(z, \bar{z}, J) + \sum_{h=0}^{g} \sum_{I \subset J} W_1^{(h)}(z, I) W_{1+n-|I|}^{(g-h)}(\bar{z}, J \setminus I) \right]
\]

\[
W_n^{(g)}(J) = \left( \prod_{j=1}^{n} \frac{1}{N} \text{tr} \left( \frac{1}{z_j - M} \right)^{(g)} \right)^{(g)}
\]

\[K(z_0, z) \sim W_2^{(0)}(z_0, z)\]

Here we only use analytic structure around branch points \(a_i\)

Where is the information of the Position of Cuts?

Of course, it doesn’t matter!!
Position of cuts and Stokes phenomena

Maybe, there is a natural path by the effective potential? Say,

\[ V'_\text{eff}(x) = 0 \quad V_{\text{eff}}(x) = \int^x dx' \left[ W(x' + i\varepsilon) + W(x' - i\varepsilon) - V'(x') \right] \]

Or which defines a real eigenvalue density:

\[ W(x \pm i\varepsilon) = \frac{V'(x)}{2} \mp \pi i \rho(x) \]

However, we can also add infinitely long cuts on the spectral curve:

Cf) [CIY1 ‘10]

This degree of freedom does not appear in the perturbation theory!
Simple analogy?

1. **Perturbative amplitudes** do not know the position of cuts
   \[\Leftrightarrow\] **perturbative analysis** cannot fix \(\theta\)

2. **The position of cuts** has the physical meaning
   \[(The \ position \ of \ eigenvalues)\]
   \[\Leftrightarrow\] \(\theta\) is universal and should have a physical meaning

This correspondence is true!!

1. We give the mathematical definition of the physical cuts
   \[(cf \ [Maldacena-Moore-Seiberg-Shih \ '05])\]
2. formulate the above constraint resulting the physical \(\theta\)!!

The Key is **Stokes phenomena** on the spectral curve
Orthonormal polynomials and spectral curves

The spectral curve can be read from determinant operator: \( \Psi_n(x) \)

\[
W(x) = \frac{1}{N}\langle \text{tr} \frac{1}{x-M} \rangle \rightarrow \Psi_n(x) = \langle \text{det}(x-M) \rangle_{n \times n}
\]

which is identified as orthonormal polynomial of matrix models [Gross-Migdal ‘90]:

\[
\int dx \ e^{-NV(x)} \Psi_n(x)\Psi_m(x) = h_n \delta_{n,m}
\]

Then they satisfy recursive equations:

\[
x \Psi_n(x) = \hat{A}(n; e^{d_n}) \Psi_n(x), \quad N^{-1} \frac{d}{dx} \Psi_n(x) = \hat{B}(n; e^{d_n}) \Psi_n(x)
\]

After the continuum (scaling) limit,

\[
\partial_n \rightarrow a^{1/2} \partial \equiv a^{1/2} g_s \partial_t, \quad N^{-1} = a^{(p+q)/2} g_s, \quad n/N = e^{-a^{(p+q-1)/2} t}
\]

They become differential equations:

\[
x \Psi(t; x) = \mathcal{A}(t; \partial) \Psi(t; x), \quad g \frac{d}{dx} \Psi(t; x) = \mathcal{B}(t; \partial) \Psi(t; x)
\]

with \( \mathcal{A} = A_0 \partial^p + \cdots + A_p \), \( \mathcal{B} = B_0 \partial^q + \cdots + B_p \)
Orthonormal polynomials and spectral curves

The spectral curve can be read from determinant operator: \( \Psi_n(x) \)

\[
W(x) = \frac{1}{N} \langle \text{tr} \frac{1}{x - M} \rangle \rightarrow \Psi_n(x) = \langle \det(x - M) \rangle_{n \times n}
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\[
\int dx \, e^{-NV(x)} \psi_n(x) \psi_m(x) = h_n \delta_{n,m}
\]

Then they satisfy recursive equations:

After the continuum (scaling) limit, they become differential equations:

\[
\begin{align*}
\mathcal{W} & \leftrightarrow \mathcal{B} \\
\partial_n & \rightarrow a \\
J &= e^{-a^{(p+q-1)/2} t}
\end{align*}
\]

They become differential equations:

\[
x \psi(t; x) = \mathcal{A}(t; \partial) \psi(t; x), \quad g \frac{d}{dx} \psi(t; x) = \mathcal{B}(t; \partial) \psi(t; x)
\]

with \( \mathcal{A} = A_0 \partial^p + \cdots + A_p, \quad \mathcal{B} = B_0 \partial^q + \cdots + B_p \)
Orthonormal polynomials and spectral curves

For simplicity, we put $p=1$:

\[ x \Psi(t; x) = \mathcal{A}(t; \partial) \Psi(t; x), \quad g \frac{d}{dx} \Psi(t; x) = \mathcal{B}(t; \partial) \Psi(t; x) \]

with $\mathcal{A} = A_0 \partial^p + \cdots + A_p$, $\mathcal{B} = B_0 \partial^q + \cdots + B_p$
Orthonormal polynomials and spectral curves

For simplicity, we put $p=1$:

\[ x\Psi(t; x) = \left[ A_0 \partial + A_1 \right] \Psi(t; x) , \quad g \frac{d}{dx} \Psi(t; x) = B(t; \partial) \Psi(t; x) \]

with \[ B = B_0 \partial^q + \cdots + B_p \]

\[ \partial\Psi(t; x) = A_0^{-1} \left[ x - A_1(t) \right] \Psi(t; x) \]

\[ g \frac{d}{dx} \Psi(t; x) = Q(t; x) \Psi(t; x) \]

\[ Q(t; x) = Q_0 x^q + \cdots + Q_q(t) \]

\[ = B_0 A_0^{-q} x^q + \cdots + Q_q(t) \]
Orthonormal polynomials and spectral curves

For simplicity, we put $p=1$:

$$x \Psi(t; x) = \left[ A_0 \partial + A_1 \right] \Psi(t; x), \quad g \frac{d}{dx} \Psi(t; x) = \mathcal{B}(t; \partial) \Psi(t; x)$$

with

$$\mathcal{B} = B_0 \partial^q + \cdots + B_p$$

$$\partial \Psi(t; x) = Q(t; x) \Psi(t; x)$$

$$g \frac{d}{dx} \Psi(t; x) = Q(t; x) \Psi(t; x)$$

$$Q(t; x) = Q_0 x^q + \cdots + Q_q(t)$$

$$= B_0 A_0^{-q} x^q + \cdots + Q_q(t)$$

ODE!
NP definition of CUTs

For simplicity, we put $p=1$:

$$\partial \Psi(t; x) = A_0^{-1} [x - A_1(t)] \Psi(t; x)$$
$$\frac{d}{dx} \Psi(t; x) = Q(t; x) \Psi(t; x)$$

Then we consider weak coupling limit: $g \to 0$

**NOTE:** $g \to 0 \iff g = 1, t \to \infty, x \to \infty$

The asymptotic solution is

$$\Psi_{\text{orth}}(t; x) \approx \exp \left[ \int_{x}^{\infty} Q^{(j)}(t; x') dx' \right] \quad x \to \infty \in D$$

$$g = \lambda^{p+q} \bar{g}, \quad x = \lambda^p \bar{x}, \quad t = \lambda^{p+q-1} \bar{t}$$

$$g \to 0, \quad (x, t : \text{fixed}) \iff \bar{x}, \bar{t} \to \infty, \quad (\bar{g} = 1)$$
NP definition of CUTs

For simplicity, we put $p=1$:

$$
\partial \Psi(t; x) = A_0^{-1}[x - A_1(t)] \Psi(t; x)
$$

$$
g \frac{d}{dx} \Psi(t; x) = Q(t; x) \Psi(t; x)
$$

Then we consider weak coupling limit: $g \to 0$

**NOTE:** $g \to 0 \iff g = 1, t \to \infty, x \to \infty$

The asymptotic solution is

$$
\Psi_{\text{orth}}(t; x) \simeq \exp \left[ \int x Q^{(j)}(t; x') dx' \right] \quad x \to \infty \in D
$$

**NOTE:**

Not absolutely convergent (convergent radius $= 0$) \(\rightarrow\) convergence is restricted in some angular domain
NP definition of CUTs

For simplicity, we put $p=1$:

$$
\partial\Psi(t; x) = A_0^{-1}[x - A_1(t)]\Psi(t; x)
$$

$$
g \frac{d}{dx} \Psi(t; x) = Q(t; x) \Psi(t; x)
$$

Then we consider weak coupling limit:

$$
g \rightarrow 0
$$

**NOTE:** $g \rightarrow 0 \iff g = 1, t \rightarrow \infty, x \rightarrow \infty$

The asymptotic solution is

$$
\Psi_{\text{orth}}(t; x) \approx \exp \left[ \int_{x}^{x} Q^{(j)}(t; x') dx' \right]
$$

$x \rightarrow \infty \in D$

This behavior would change if $x$ crosses the **Stokes lines:**

$$
\text{Re} \int_{x}^{x} dx' Q^{(j)}(x') = \text{Re} \int_{x}^{x} dx' Q^{(i)}(x')
$$

Therefore, the following combination

$$
\Psi_{\text{orth}}(t; x) \approx \exp \left[ \int_{x}^{x} Q^{(j)}(t; x') dx' \right] + s \exp \left[ \int_{x}^{x} Q^{(i)}(t; x') dx' \right]
$$

has a **CUT along the Stokes line**

This is the non-perturbative definition of cuts

(a real eigenvalue density: $W(x \pm i\epsilon) = \frac{V'(x)}{2} \mp \pi i \rho(x)$)
NP definition of CUTs

For simplicity, we put $p=1$:

$$\partial \Psi(t; x) = A_0^{-1}[x - A_1(t)] \Psi(t; x)$$

$$g \frac{d}{dx} \Psi(t; x) = Q(t; x) \Psi(t; x)$$

Then we consider weak coupling limit: $g \to 0$

**NOTE:** $g \to 0 \iff g = 1, t \to \infty, x \to \infty$

The asymptotic solution is

$$\Psi_{\text{orth}}(t; x) \simeq \exp \left[ \int^x Q^{(j)}(t; x') dx' \right] \quad x \to \infty \in D$$

This behavior would change if $x$ crosses the **Stokes lines**

$$Q(t; x) \leftrightarrow W(x)$$

Therefore, the following combination

$$\Psi_{\text{orth}}(t; x) \simeq \exp \left[ \int^x Q^{(j)}(t; x') dx' \right] + s \exp \left[ \int^x Q^{(i)}(t; x') dx' \right]$$

has a **CUT** along the **Stokes line**

This is the non-perturbative definition of cuts

(a real eigenvalue density: $W(x \pm i\epsilon) = \frac{V'(x)}{2} \pm \pi i \rho(x)$)
Stokes Phenomena

There is a maximal domain $D$ for expansion

The complete set of solutions:

$$\Psi^{(j)}(t; x) \simeq \chi_j \times \exp\left[ \int^x Q^{(j)}(t; x') dx' \right]$$

$$\chi_j = t^{(0, \cdots, 0, 1, 0, \cdots, 0)}$$

$$\Psi_{\text{asym}}(t; x) \equiv (\Psi^{(1)}(t; x), \cdots, \Psi^{(k)}(t; x))(\text{upto normalization})$$

$$= \left[ I_k + \frac{Y_1(t)}{x} + \cdots \right] \exp\left[ \frac{\varphi - r}{r} x^r + \cdots \right]$$

Stokes Phenomena

$$\hat{\Psi}_2(t; x) = \hat{\Psi}_1(t; x) S, \quad x \in D_1 \cap D_2$$

with

$$\begin{cases} 
\hat{\Psi}_1(t; x) \simeq \Psi_{\text{asym}}(t; x), \quad x \in D_1 \\
\hat{\Psi}_2(t; x) \simeq \Psi_{\text{asym}}(t; x), \quad x \in D_2
\end{cases}$$

$$\Psi^{(j)}_2 = \Psi^{(j)}_1 + \sum_i \Psi^{(i)}_1 s_{i,j}, \quad x \in D_1 \cap D_2$$

when

$$\text{Re} \int^x dx' Q^{(j)}(x') > \text{Re} \int^x dx' Q^{(i)}(x'), \quad x \in D_1 \cap D_2$$
**Isomonodromy property**

Stokes matrices satisfy

\[
\frac{dS}{dt} = \frac{dS}{dx} = 0
\]

Therefore, they are **integration constants** of t-flow:

**String equation**

\[
\frac{g^2}{2} u'' + 3u^2 - t = 0
\]

On the other hand, the chemical potentials \( \theta \)

are also **integration constants**

Therefore,

\[
\theta \leftrightarrow S
\]

Direct relation is realized by the Riemann-Hilbert approach

For mathematical references of Stokes phenomena, Isomonodromy deformation and the RH problem, see

[Fokas-Its-Kapaev-Novokshenov ’06]

“Painlevé transcendents: the Riemann-Hilbert approach”
The multi-cut boundary condition [CIY2 '10]

Requirement

Existence of Orthonormal polynomial which has necessary and sufficient physical cuts

The result of the Two-cut case (pure-supergravity)

1. Generally 6 physical cuts around \( x = \infty \) on the spectral curve
2. Mathematically, there are 2 real free Stokes multipliers
3. The above BC (+\( \alpha \)) completely fixes the D-instanton chemical potential (results in the Hastings-McLeod solution (’80)) [CIY2 ‘10]
4. Cf) [Bleher-Its ‘02]: a finite-N approach in the two-cut critical point
5. Our procedure is after the continuum limit, and very easy!!
3. Concrete solutions in the higher-cut system
E.g.) $q=1, k=5$ case

Let's define the Fine Stokes Matrices:

$$S_l = \Psi_l^{-1} \Psi_{l+1} = (S_{l,i,j})_{i,j}$$

Thm [CIY2 '10]

$$\exists (i | j)_l \iff S_{l,j,i} : \text{non-trivial}$$
The system we solve:

\[ \{ S_n \}_{n=0}^{2(q+1)k-1} \]

1. \( Z_k \)-symmetry condition

\[ S_{n+2(q+1)} = \Gamma^{-1} S_n \Gamma \]

2. Hermiticity condition

\[ S_n^* = \Delta \Gamma S_{(2q+1)k-n} \Delta \Gamma \]

3. Monodromy Free condition

\[ S_0 S_1 S_2 \cdots S_{2(q+1)k-1} = I_k \]

4. The multi-cut boundary condition

\[ \Psi_{\text{orth}}(t; x) = \Psi_n(t; x) X^{(n)} \]

\[ X^{(n)} = S_n X^{(n+1)} \]

Information of BC
Solutions for multi-cut cases ($q=1$):

Discrete solutions

\[
S_{l,i,j} = (-1)^{l-1} \sigma_{L_{l,i,j}} \left( \left\{ -\omega^{-(l-1) n_m} \right\}_{m=1}^{k/2} \right)
\]

\[
\tilde{S}_{l,i,j} = (-1)^{l-1} \sigma_{L_{l,i,j}} \left( \left\{ -\omega^{-(l-1) \tilde{n_m}} \right\}_{m=1}^{k/2} \right)
\]

\[
0 \leq L_{l,i,j} < k, \quad L_{l,i,j} \equiv (-1)^{l-1} (i - j)
\]

Characterized by \((n_1, n_2, \ldots, n_{k/2}; \tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_{k/2})\)

Which is also written with Young diagrams (**avalanches**):

![Diagram](image-url)
Solutions for multi-cut cases (q=1):

Continuum solutions

\[
\begin{align*}
S_{l,i,j} &= (-1)^{l-1} \sigma_{L_{l,i,j}} \left( \{-\omega^{-(1)^{l-1}n_m}\}_{m=1}^{[k/2]} \right) \\
\tilde{S}_{l,i,j} &= \theta_{L_{l,i,j}}
\end{align*}
\]

with \( \theta_n = S_n(\{\sigma_m\}_m) + \theta_{\left[\frac{k}{2}\right]-n+1}^* \sigma_{[k/2]}^* (\{\sigma_m\}_m) \)

The polynomials \( S_n \) are related to Schur polynomials \( P_n \):

\[
S_n(\{x_m\}_m) = P_n(\{y_m\}_m) \quad x_n = P_n(\{-y_m\}_m)
\]
Conclusion

1. D-instanton chemical potentials $\theta$ are beyond the horizon of perturbation theory, which is the last information for non-perturbative completion.

2. We identified the physical and geometrical meaning of $\theta$ and formulated the physical constraints resulting from the matrix models.

3. Our constraint specifies some physical section of $\theta$ which generalize the Hastings-McLeod solution in the two-cut cases.

4. We obtained several solutions valid for general-cut cases, which labelled by charges of Young diagrams.
Other prospects

1. We obtained several constraints on the D-instanton chemical potentials. We should note that some are fixed and some are not.

2. It was proposed [CIY1 ‘10] that strong-coupling dual theory is non-critical M theory. Then, these (non-perturbative) information would be the moduli space of the M theory.

3. Is there integrable system which governs these solutions?

4. We here consider the multi-cut critical points, but how about 1-cut critical points? How does the solution looks like?

5. The function G(z) in the RH problem is understood as off-shell background of the non-critical string theory -> Off-shell formulation? (see CIY2 ‘10)