DOES THE QUENCHED KARDAR-PARISI-ZHANG EQUATION DESCRIBE THE DIRECTED PERCOLATION DEPINNING MODELS?

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(March 24, 2022)

Abstract

The roughening of interfaces moving in inhomogeneous media is investigated by numerical integration of the phenomenological stochastic differential equation proposed by Kardar, Parisi, and Zhang [Phys. Rev. Lett. 56, 889, (1986)] with quenched noise (QKPZ). We express the evolution equations for the mean height and the roughness into two contributions: the local and the lateral one. We compare this two contributions with the ones obtained for two directed percolation deppining models (DPD): the Tang and Leschhorn model [Phys. Rev. A 45, R8309 (1992)] and the Buldyrev et al. model [Phys. Rev. A 45, R8313 (1992)] by Braunstein et al. [J. Phys. A 32, 1801 (1999); Phys. Rev. E 59, 4243 (1999)]. Even these models have being classified in the same universality class that the QKPZ the contributions to the growing mechanisms are quite different. The lateral contribution in the DPD models, leads to an increasing of the roughness near the criticality while in the QKPZ
equation this contribution always flattens the roughness. These results suggest that the QKPZ equation does not describe properly the DPD models even when the exponents derived from this equation are similar to the one obtained from simulations of these models.

PACS numbers: 47.55.Mh, 68.35.Ja, 05.10.-a
I. INTRODUCTION

The description of the noise-driven growth that leads to self-affine interface far from equilibrium is a challenging problem. The interface has been characterized through scaling of the interfacial width $w$ with time $t$ and lateral size $L$. The result is the determination of two exponents $\beta$ and $\alpha$ called dynamical and roughness exponents, respectively. It is well known that interfacial width $w \sim L^\alpha$ for $t \gg t^*$ and $w \sim t^\beta$ for $t \ll t^*$, where $t^* \simeq L^{\alpha/\beta}$ is the saturation time. These properties occur in many experimental situations and many models of surface growth. The values of the exponents leads to their classification in different universality classes. Several models, belonging to the same directed percolation depinning (DPD) universality class, have been introduced to explain experiments on fluid imbibition in porous media, roughening in slow combustion of paper, growth of bacterial colonies, etc. It is currently accepted that the quenched disorder plays an essential role in those experiments. The DPD models take into account the most important features of the experiments [1,2]. On the other hand the phenomenological stochastic differential equation proposed by Kardar, Parisi, and Zhang [3] with quenched noise (QKPZ) is used to describe the interface growth in disordered media and drives to the same universality class than the DPD models, in the sense that they have the same exponents. Moreover, processes with the same exponent may not belong to the same universality class. For example, 1 + 1-dimensional lattice gas simulations of roughening of immiscible fluid-fluid interface [4] lead to the same exponents as the 1 + 1-dimensional KPZ [3] ($\beta = 1/3$ and $\alpha = 1/2$) for surface growth, but this model is completely linear, so there is no obvious mathematical relationship between these two processes.

The two first DPD models were simultaneously introduced by Buldyrev et al. [1] and Tang and Leschhorn [5] to explain the fluid imbibition in paper sheet. Several authors have been focused their attention on scaling properties and relationships between the dynamical and the static exponent for these models. These two models have been recently reviewed by Braunstein et al. [6,7] from a different point of view than the traditional one. The
principal contribution was the restatement of the Microscopic Equation for each model. These equations allow the separation into two contributions: the local and the lateral (or contact) one. They found that the lateral contribution to the temporal derivative of the interface width (DSIW) may be either negative or positive and that the behavior of this contribution depends on the pressure $p$, where $p$ is the microscopic driving force. The negative contribution tends to smooth out the surface, this case dominate for $p \gg p_c$ (where $p_c$ is the critical pressure). The positive contribution enhances the roughness. At the critical pressure the local contribution to the DSIW is practically constant, but the lateral contribution is very strong. This last contribution, has important duties on the power law behavior in the DPD models.

In this paper we focus the attention in the QKPZ equation in order to compare the different contributions with the ones obtained for the DPD models. In this context we show that the results obtained from this equation are quite different from the ones obtained in the DPD models. We separate the QKPZ equation into two contributions: the local contribution and the lateral one. In this context we study the mean height speed (MHS) and the DSIW as a function of time. The paper is organized as follows. In Section II we separate the QKPZ equation into two contributions for the mean height and the roughness. We study the MHS, analyzing the local and the lateral contributions. Also, the two contributions to the DSIW are analyzed. In Section III we compare the DPD models with the QKPZ model. Finally, we present the main conclusions in Section IV.

II. MACROSCOPIC CONTRIBUTIONS FROM THE QKPZ EQUATION

A. Equations

The QKPZ equation for the surface height $h = h(x,t)$, in $1 + 1$-dimension, is given by

$$\partial_t h = F + \nu \partial_x^2 h + \lambda (\partial_x h)^2 + \eta(x,h)$$  \hspace{1cm} (1)
where \( \nu \) and \( \lambda \) are constants and the noise depends on position \( x \) and height \( h \) with the properties that \( \langle \eta(x, h) \rangle = 0 \) and \( \langle \eta(x, h) \eta(x', h') \rangle = 2D \delta(x - x') \delta(h - h') \) where \( 2D \) is the noise intensity.

We can distinguish two contributions to this equation, the local growth \( S = F + \eta(x, h) \) and the lateral one \( L = \nu \partial_x^2 h + \lambda (\partial_x h)^2 \). So we can write the evolution equation for the mean height as

\[
\partial_t h = S + L .
\]

Taking the derivative of the square interface width, \( w^2 = \langle (h - \langle h \rangle)^2 \rangle \), its evolution equation is given by

\[
\partial_t w^2 = 2\langle (h - \langle h \rangle)\partial_t h \rangle = 2\langle (h - \langle h \rangle)S \rangle + 2\langle (h - \langle h \rangle)L \rangle
\]

where \( \langle \ldots \rangle \) means average over the lattice. The first term can be identified as the local growth contribution, and the second term as the lateral growth contribution. The separation into these two analytical terms allows us to compare the mechanisms of growth in the QKPZ equation with ones of the DPD models. In the present paper we focus only on the dynamical behavior, i.e. \( t \ll t^* \approx L \) (in these models \( \alpha \sim \beta \)) for the mean height and roughness.

We have performed the direct numerical integration of Eq. (1) in one dimension in the discretized version

\[
h(x, t + \Delta t) = h(x, t) + \Delta t \left\{ h(x - 1, t) + h(x + 1, t) - 2h(x, t) \right\} + \frac{\lambda}{8} \left\{ h(x + 1, t) - h(x - 1, t) \right\}^2 + F + \eta(x, [h(x, t)]) \right. ,
\]

where \([\ldots]\) denotes the integer part and \( \eta \) is uniformly distributed in \([-a/2, a/2]\], where \( a = 10^{2/3} \) is selected. We use \( L = 8192 \) and \( \Delta t = 0.01 \). The initial condition is \( h(x, 0) = 0 \) and periodic boundary conditions are used. The averages was taken over 100 samples. We study this equation for different values of \( \lambda \) with \( \nu = 1 \).
B. Mean height

In Fig. 1 we show the MHS as a function of time in the three regimes (moving, critical and pinning phases) for $\lambda = 1$. The initial condition for the MHS is $F$ in all regimes. At the criticality we found for the mean height a power law behavior with approximately the same dynamical exponent that the roughness one, $\beta = 0.67 \pm 0.01$ for $F_c = 0.464$, where $F_c$ is the critical driving force. In the moving and pinning phases we can see that this power law does not hold. Below the criticality, in the pinning phase, the MHS goes to zero. In the moving phase ($F > F_c$), the MHS goes to certain constant value.

In Fig. 2 we show the contributions to the MHS: the local one $\langle S \rangle$ and the lateral one $\langle L \rangle$. The local contribution, which is equal to $F$ at $t = 0$, is stronger in the early time regime. This is because in this regime the difference of heights between nearest neighbors is very low and the contribution of the lateral term is negligible. We see that the local contribution takes negative values from $t \gtrsim 7$, so in this regime the local contribution brakes the growing of $\langle h \rangle$. The behavior of both contributions, lateral and local one, is not very different in every phases although their sum only drive to a power law at the criticality. In every phases both contributions are equal at $t \simeq 2.5$. This means that at this time both mechanisms are equal independently of $F$. Increasing $\lambda$ the lateral contribution is enhanced at shorter time, for $\lambda = 2$ the crossover is at $t \simeq 1$. In the asymptotic dynamic regime both contributions are important and neither dominates over the other one.

C. Roughness

Fig. 3 shows the temporal DSIW as a function of time for various values of $F$ and $\lambda = 1$. Here we found that the DSIW increases continuously from zero to a maximum value. The power law holds only at the criticality. The DSIW goes asymptotically to zero at the pinning and moving phases. The dynamical exponent obtained for the roughness was $\beta = 0.66 \pm 0.02$.

In Fig. 4, we show the two contributions to the DSIW for different values of $F$. The
local contribution \(2\langle(h - \langle h\rangle)S\rangle\) to the DSIW is always positive. As \(\mathcal{F}\) decreases, this contribution also decreases slowly, but always roughen the interface. On the other hand, the lateral contribution \(2\langle(h - \langle h\rangle)\mathcal{L}\rangle\) takes negative values in every phases, smoothing out the surface.

In Fig. 5, we plot the DSIW for three values of \(\lambda\) at the criticality. The slope \(\beta\) is independent of \(\lambda\) although we found some differences in these plots. The scaling dynamical regime is reached before at greater values of \(\lambda\). This is due to the fact that the lateral contribution, which is the main responsible of the generation of correlations, becomes more important earlier for larger values of \(\lambda\).

III. COMPARISONS WITH THE DPD MODELS

In this section we present the similarities and differences between the DPD models and the QKPZ equation. In previous work Braunstein et al. [6,7] wrote the microscopic equation for the TL and Buldyrev models. They identified two separate contributions for the MHS and the DSIW: the local and the lateral (contact) one. In the present paper, we have also separate the QKPZ equation into these two contributions.

We found that, in the asymptotic dynamical regime, the behavior of the MHS are similar in the QKPZ equation and in the DPD models in every phases [6,7]. At short time we found a maximum value which depend on \(\lambda\). This maximum value is not found in the DPD models. Differences at short time are expected because the QKPZ only describes the scaling limit or hydrodynamic limit. The lateral and local contributions to the MHS in the DPD models are qualitative and quantitative different from the QKPZ ones. At long time the local contribution brakes the growing of \(\langle h \rangle\) in the QKPZ equation (in the DPD models this contribution is always positive). Also, the contributions of the QKPZ equation are not so much different in each phase, while in the DPD models both contributions are very different behavior in each phase.

In the asymptotic regime, the behavior of the DSIW is similar in the QKPZ equation and
in the DPD models. In the DPD models $p$ is the initial condition in all regimes, this is due to the fact that in the early regime the dynamics is like random deposition with probability $p$ \cite{10, 11}. In the QKPZ equation the DSIW increases continuously from zero to a maximum value, the macroscopic equation presented by Braunstein et al. \cite{10, 12} for the DPD models holds in the scaling limit or hydrodynamic limit, but breaks down at short times as was expected. The lateral and the local contributions to the DSIW is quite different in each model. In the QKPZ the lateral contribution is always negative. This is the main difference with the DPD models where for $p \lesssim p_c$, the lateral contribution is always positive roughening the interface \cite{6, 7}. In the DPD models the lateral contribution is enhanced by local growth, the lateral growth may also increase the probability of local growth. This crossing interaction mechanism makes the lateral growth dominant near the criticality and this is due to the fact that the quenched noise is coupled to the dynamic of the interface \cite{13}. In the QKPZ model this cross mechanism between contributions is not taken into account because the noise is additive. Moreover, the model are said to belong to the same universality class that the QKPZ equation because the exponents are quantitatively similar. Even thought the behaviors of the contributions to the growth are qualitatively and quantitatively different.

In the experiments the advancement of the interface is determined by the coupled effect of the random distribution of the capillary sizes, the surface tension and the local properties of the flow, so it is not surprising that all these effect give rise to a multiplicative noise. This multiplicative noise must be taken into account at the time to pose a model with the essential features of the experiment of surface growth in disordered media. In the TL and the Buldyrev models the growing rules for the evolution of the local height are strongly coupled to the quenched noise in a multiplicative way. In both models the microscopic rules that allows the growth from an unblocked cell \cite{6, 7} depends in some way on the local slope. In that sense this coupled effect is not taken into account in the QKPZ equation. The effect of a multiplicative noise as being proposed by Csahók et al. \cite{8} by means of a phenomenological equation. They found a crossover between two temporal regimes with $\beta = 0.65$ to $\beta = 0.26$ but the value of $\alpha \simeq 0.47$ was obtained over a short range spatial scale. Braunstein et al.
derived the continuous equation for the TL model [13]. This equation as the same term that
the QKPZ equation but its coefficients depends on the competition between the driving
force and the quenched noise. In that sense the noise is multiplicative. This results joint to
the results obtained in this work supports that the QKPZ does not describe fully the DPD
models even if the exponents are quantitatively similar.

IV. CONCLUSIONS

We express the evolution equations of the QKPZ model for the mean height and the
roughness into two contributions: the local and the lateral one. We found that the contrib-
tutions to the growing mechanisms are quite different from the DPD models. In the scaling
regime, the local contribution to the MHS brakes the growing of ⟨h⟩ in the QKPZ model.
The lateral contribution to the DSIW is negative in all phases for the QKPZ model smoothen-
ing out the interface. In the DPD models the lateral contribution is always positive for
\( p \lesssim p_c \) roughening the interface. Nevertheless, the DSIW and MHS gives the same scaling
exponents in these models, moreover it is not clear why. Our results suggest that the QKPZ
equation does not describe properly the dynamics of the DPD models even if the exponents
are similar.
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FIGURES

FIG. 1. Plots of $\mathcal{F}^{-1} \, dh/dt$ vs $t$ for $\lambda = 1$. The parameter $\mathcal{F}$ is 0.51 (○), 0.464 (□), and 0.43 (△).

FIG. 2. Semi-ln plots of the different contributions to the MHS as a function of time for different values of $\mathcal{F}$ and $\lambda = 1$. The circles (○) represent the local contribution, the triangles (△) represent the lateral contribution, and the squares (□) represent the total MHS. The (a) plot shows the critical phase $\mathcal{F} = 0.464$. The (b) plot shows the pinning phase $\mathcal{F} = 0.43$. The (c) plot shows the moving phase $\mathcal{F} = 0.51$.

FIG. 3. DSIW as a function of time in the critical, pinning, and moving phases for $\lambda = 1$. The parameter $\mathcal{F}$ is 0.464 (solid line), 0.43 (dashed line), and 0.54 (doted line).

FIG. 4. Semi-ln plots of the different contributions to the DSIW as a function of time for different values of $\mathcal{F}$ and $\lambda = 1$. The circles (○) represent the local contribution, the triangles (△) represent the lateral contribution, and the squares (□) represent the total DSIW. The (a) plot shows the critical phase $\mathcal{F} = 0.464$. The (b) plot shows the pinning phase $\mathcal{F} = 0.43$. The (c) plot shows the moving phase $\mathcal{F} = 0.54$.

FIG. 5. DSIW as a function of time in the critical regime for different values of $\lambda$: 0.5 (solid line), 1 (doted line), and 2 (dashed line)
Fig. 1
Fig. 2.a

(a)
Fig. 3

\[ \ln \left( \frac{dw^2}{dt} \right) \]

\[ \ln t \]
Fig. 4.a
Fig. 4.c
Fig. 5