WEAK COMPACTNESS AND STRONGLY SUMMING MULTILINEAR OPERATORS

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Abstract. Every absolutely summing linear operator is weakly compact. However, for strongly summing multilinear operators and polynomials – one of the most natural extensions of the linear case to the non linear framework – weak compactness does not hold in general. We show that a subclass of the class of strongly summing multilinear operators/polynomials, sharing its main properties such as Grothendieck’s Theorem, Pietsch Domination Theorem and Dvoretzky–Rogers Theorem, has even better properties like weak compactness and a natural factorization theorem.

1. Introduction

The theory of absolutely summing linear operators has its roots in the 1950s with A. Grothendieck’s pioneer ideas; in its modern presentation, it appeared in 1966-67 in the works of A. Pietsch [40] and B. Mitiagin and A. Pełczyński [30]. A cornerstone in the theory is the remarkable paper of J. Lindenstrauss and A. Pełczyński [25], which clarified Grothendieck’s ideas, without the use of tensor products. Lindenstrauss and Pełczyński were also responsible for the reformulation of Grothendieck’s inequality, which is still a fundamental result of Banach Space Theory and Mathematical Analysis in general (see [12]). Nowadays absolutely summing operators is a current subject in books of Banach Space Theory (see, for instance, [1, 20]). For a detailed approach to the linear theory of absolutely summing operators we refer to the excellent book of J. Diestel, H. Jarchow and A. Tonge [17].

It is then comprehensive that a big effort has been made, since Pietsch’s proposal [41], to try and generalize the linear theory to non linear operators. Many families of non linear operators have been considered such as multilinear operators, homogeneous polynomials, holomorphic mappings, $\alpha$-homogeneous mappings, Lipschitz mappings among others. However, extending summability properties to non linear operators has been proved difficult and intriguing. For instance, there are several extensions of absolutely $p$-summing linear operators to the multilinear setting that have been considered in the literature. Besides its intrinsic interest, the multilinear theory of absolutely summing operators has shown important connections, including applications to Quantum Information Theory (see [31]). This proliferation of classes of summing multilinear maps has lead to the appearance of works that compare different approaches (see [14, 39]). The first challenging task when dealing with multilinear operators is probably to identify the class of multilinear operators that best inherits the spirit of the absolutely summing linear operators. According to [30, 38], one of the most natural extensions of the notion of absolutely $p$-summing linear operators to

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the multilinear setting is the notion of strongly $p$-summing multilinear operators, due to V. Dimant (18). This class lifts to the multilinear framework most of the main properties of absolutely $p$-summing linear operators: Grothendieck’s Theorem, Pietsch Domination Theorem, Inclusion Theorem. However, as we will see, a natural version of the Pietsch Factorization Theorem does not hold for this class.

The good behavior of multilinear extensions has found no echo when considering extensions of absolutely summing operators to polynomials. In this non linear setting, several attempts have been made but all of them have found rough edges to succeed in. This is the case of $p$-dominated homogeneous polynomials, for which a Pietsch type factorization theorem has been pursuit (see 25, 29, 8, 13) and succeeded just when the domain is separable. Recently, the second and third authors 43 have isolate the class of $p$-dominated polynomials that satisfy a Pietsch type factorization theorem: the factorable $p$-dominated polynomials. However, even if this makes a big difference with $p$-dominated polynomials, they still lack good properties as evidenced by the fact that factorable $p$-dominated polynomial do not define a composition ideal or, equivalently, the linearization of a factorable $p$-dominated polynomial may not be absolutely $p$-summing.

Our aim in this paper is to introduce factorable strongly $p$-summing multilinear operators and homogeneous polynomials to the full extent of absolutely $p$-summing linear operators. These new classes of summing polynomials/multilinear operators stand apart from previous generalizations as they keep a big amount of the fundamental properties that are satisfied in the linear theory and are not satisfied by former non linear classes. Factorable strongly $p$-summing multilinear operators is a subclass of strongly $p$-summing multilinear operators that has in addition a quite natural Pietsch Factorization type theorem and weak compactness. Factorable strongly $p$-summing homogeneous polynomials also fulfills a factorization theorem in the spirit of Pietsch, are weakly compact and a polynomial belongs to the class if and only if its second adjoint (in the sense of Aron and Schottenloher) is in the class. Actually, an homogeneous polynomial is factorable strongly $p$-summing if and only if its associated multilinear map is factorable $p$-summing or, equivalently, its linearization is absolutely $p$-summing. This brings deep strengths that are not shared by former classes of summing polynomials as dominated or strongly summing polynomials.

This paper is organized as follows. The next section contains the basics (definitions and main results) on linear and non linear summability that are in order for our purposes. In Section 3 we show that a slight modification of the notion of strongly $p$-summing operator (inspired in a recent paper of the second and third author) generates a subclass that keeps its main properties and also has a factorization theorem in the lines of the approach above. These are the factorable strongly $p$-summing multilinear operators. As a consequence we have weak compactness, as in the linear case. In Section 4 we deal with homogeneous polynomials, proving that a polynomial is factorable strongly $p$-summing if and only if its linearization is absolutely $p$-summing. The connection between $m$-homogeneous polynomials and $m$-linear operators is established: an $m$-homogeneous polynomial is factorable strongly $p$-summing if and only if its associated symmetric $m$-linear map is factorable strongly $p$-summing. These results yield to obtain in Section 5 proper generalizations of fundamental properties related to summability for linear operators to multilinear maps and homogeneous polynomials. Among other results, we show that a Dvoretzky-Rogers type theorem, a Lindenstrauss–Pełczyński type theorem or a Grothendieck type theorem work for factorable strongly summing polynomials. Finally, in Section 6 we show that the sequence formed by the ideals of factorable strongly summing homogeneous polynomials and factorable strongly summing multilinear operators is coherent and compatible with the ideal of absolutely summing linear operators.
2. Background: linear and multilinear summability

If 1 ≤ p < ∞ and X, Y are Banach spaces, a continuous linear operator \( u : X \to Y \) is absolutely p-summing (\( u \in \Pi_p(X; Y) \)) if there is a constant \( C \geq 0 \) such that

\[
\left( \sum_{j=1}^{m} \| u(x_j) \|^p \right)^{1/p} \leq C \left( \sup_{\varphi \in B_{X^*}} \sum_{j=1}^{m} |\varphi(x_j)|^p \right)^{1/p}
\]

for all \( x_1, \ldots, x_m \in X \) and all positive integers \( m \). The infimum of all \( C \) that satisfy the above inequality defines a norm, denoted by \( \pi_p(u) \), and \( (\Pi_p(X, Y), \pi_p) \) is a Banach space. The cornerstones of the theory of absolutely summing linear operators are the following theorems:

- (Dvoretzky-Rogers theorem) If \( p \geq 1 \), then \( \Pi_p(X; X) = \mathcal{L}(X; X) \) if and only if \( \dim X < \infty \).
- (Grothendieck’s theorem) Every continuous linear operator from \( \ell_1 \) to \( \ell_2 \) is absolutely 1-summing.
- (Lindenstrauss–Pełczyński theorem) If \( X \) and \( Y \) are infinite-dimensional Banach spaces, \( X \) has an unconditional Schauder basis and \( \Pi_1(X; Y) = \mathcal{L}(X; Y) \) then \( X = \ell_1 \) and \( Y \) is a Hilbert space.
- (Pietsch Domination theorem) If \( X \) and \( Y \) are Banach spaces, a continuous linear operator \( u : X \to Y \) is absolutely p-summing if and only if there exist a constant \( C \geq 0 \) and a Borel probability measure \( \mu \) on the closed unit ball of the dual of \( X \), \( (B_X^*, \sigma(X^*, X)) \), such that

\[
\| u(x) \| \leq C \left( \int_{B_X^*} |\varphi(x)|^p \, d\mu \right)^{1/p}
\]

for all \( x \in X \).

- (Inclusion theorem) If \( 1 \leq p \leq q < \infty \), then every absolutely p-summing operator is absolutely \( q \)-summing.
- (Pietsch Factorization theorem) A continuous linear operator \( u : X \to Y \) is absolutely p-summing if, and only if, there exist a regular Borel probability measure \( \mu \) on \( B_{X^*} \), a closed subspace \( X_p \) of \( L_p(\mu) \) and a continuous linear operator \( \tilde{u} : X_p \to Y \) such that

\[
j_p \circ i_X(X) \subset X_p \quad \text{and} \quad \tilde{u} \circ j_p \circ i_X = u,
\]

where \( i_X : X \to C(B_{X^*}) \) and \( j_p : C(B_{X^*}) \to L_p(\mu) \) are the canonical inclusions. Moreover, every absolutely p-summing linear operator is weakly compact.

From now on \( p \in [1, \infty) \) and \( X, X_1, \ldots, X_n, Y \) are Banach spaces over the same scalar field \( K = \mathbb{R} \) or \( \mathbb{C} \). A continuous \( n \)-linear operator \( T : X_1 \times \cdots \times X_n \to Y \) is \( p \)-dominated if there is a constant \( C \geq 0 \) such that

\[
\left( \sum_{j=1}^{m} \| T(x^1_j, \ldots, x^n_j) \|_p \right)^{n/p} \leq C \left( \sup_{\varphi \in B_{X^*_1}} \sum_{j=1}^{m} |\varphi(x^1_j)|^p \right)^{1/p} \cdots \left( \sup_{\varphi \in B_{X^*_n}} \sum_{j=1}^{m} |\varphi(x^n_j)|^p \right)^{1/p}
\]

for all \( x^k_j \in X_k \), all \( m \in \mathbb{N} \) and \( (j, k) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \). This concept is essentially due to Pietsch (see [24, 26]) and lifts several important properties of the original linear ideal of absolutely summing operators to the multilinear framework. The terminology “\( p \)-dominated”, coined by M.C. Matos, is motivated by the following Pietsch-Domination type theorem:

**Theorem 2.1** (Pietsch, Geiss, 1985). A continuous \( n \)-linear operator \( T : X_1 \times \cdots \times X_n \to Y \) is \( p \)-dominated if and only if there exist \( C \geq 0 \) and regular probability measures \( \mu_j \)
on the Borel $\sigma$-algebras of $B_{X^*_j}$ endowed with the weak star topologies such that
\[
\|T(x_1,\ldots,x_n)\| \leq C \prod_{j=1}^n \left( \int_{B_{X^*_j}} |\varphi_j(x_j)|^p \, d\mu_j(\varphi) \right)^{1/p}
\]
for every $x_j \in X_j$ and $j = 1,\ldots,n$.

Corollary 2.2. If $1 \leq p \leq q < \infty$, then every $p$-dominated multilinear operator is $q$-dominated.

The notion of $p$-semi-integral operator is another possible multilinear generalization of the class of absolutely summing linear operators. If $p \geq 1$, a continuous $n$-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is $p$-semi-integral if there exists a $C \geq 0$ such that
\[
\left( \sum_{j=1}^m \|T(x_1^j,\ldots,x_n^j)\|^p \right)^{1/p} \leq C \left( \sup_{(\varphi_1,\ldots,\varphi_n) \in B_{X^*_1} \times \cdots \times B_{X^*_n}} \sum_{j=1}^m |\varphi_1(x_1^j)\cdots\varphi_n(x_n^j)|^p \right)^{1/p}
\]
for every $m \in \mathbb{N}$, $x_j^k \in X_k$ with $k = 1,\ldots,n$ and $j = 1,\ldots,m$.

This class dates back to the research report $[2]$ of R. Alencar and M.C. Matos. As in the case of $p$-dominated multilinear operators, a Pietsch Domination theorem is valid in this context:

Theorem 2.3. A continuous $n$-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is $p$-semi-integral if and only if there exist $C \geq 0$ and a regular probability measure $\mu$ on the Borel $\sigma$-algebra $\mathcal{B}(B_{X^*_1} \times \cdots \times B_{X^*_n})$ of $B_{X^*_1} \times \cdots \times B_{X^*_n}$ endowed with the product of the weak star topologies $\sigma(X^*_1;X_l), l = 1,\ldots,n$, such that
\[
\|T(x_1,\ldots,x_n)\| \leq C \left( \int_{B_{X^*_1} \times \cdots \times B_{X^*_n}} |\varphi_1(x_1)\cdots\varphi_n(x_n)|^p \, d\mu(\varphi_1,\ldots,\varphi_n) \right)^{1/p}
\]
for all $x_j \in X_j$, $j = 1,\ldots,n$.

Corollary 2.4. If $1 \leq p \leq q < \infty$, every $p$-semi-integral multilinear operator is $q$-semi-integral.

This class is strongly connected to the class of $p$-dominated multilinear operators. For example, in $[14]$ it is shown that every $p$-semi integral $n$-linear operator is $np$-dominated.

The following result shows that we cannot expect to lift coincidence results of the linear case to dominated multilinear operators:

Theorem 2.5 (Jarchow, Palazuelos, Pérez-García and Villanueva, 2007). $([24])$ For every $n \geq 3$ and every $p \geq 1$ and every infinite dimensional Banach space $X$ there exists a continuous $n$-linear operator $T : X \times \cdots \times X \to \mathbb{K}$ that fails to be $p$-dominated.

Since $p$-semi-integral $n$-linear operators are $np$-dominated, we have:

Corollary 2.6. For every $n \geq 3$, every $p \geq 1$ and every infinite dimensional Banach space $X$ there exists a continuous $n$-linear operator $T : X \times \cdots \times X \to \mathbb{K}$ that fails to be $p$-semi-integral.

So, in view of the previous result, it is obvious that we cannot expect a Grothendieck type theorem for dominated or semi-integral operators. In this direction, the classes of multiple summing multilinear operators $([8],[27])$, strongly multiple summing multilinear operators $([2])$ and strongly summing multilinear operators $([15])$ are other possible generalizations, with a quite better behavior if we are interested in lifting coincidence theorems,
like Grothendieck’s theorem. But, as a matter of fact, none of these classes lifts all the main properties of absolutely summing linear operators to the multilinear setting.

In [43], a variant of the notion of p-dominated polynomials which satisfy (in a very natural form) a Pietsch factorization type theorem, is introduced. A continuous n-homogeneous polynomial $P : X \to Y$ is factorable p-dominated if there is a $C \geq 0$ such that for every $x_j^i \in X$, and scalars $\lambda_j^i$, $1 \leq j \leq m_1$, $1 \leq i \leq m_2$ and all positive integers $m_1, m_2$, we have

$$\left( \sum_{j=1}^{m_1} \| \sum_{i=1}^{m_2} \lambda_j^i P(x_j^i) \|^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_{X^*}} \left( \sum_{j=1}^{m_1} \| \sum_{i=1}^{m_2} \lambda_j^i \varphi(x_j^i) \|^n \right)^{\frac{1}{n}}.$$

The natural multilinear version of the notion of “factorable p-dominated polynomials” seems to be:

**Definition 2.7.** A continuous n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is factorable p-dominated if there is a constant $C \geq 0$ such that for every $x_{k,j}^i \in X_k$, and scalars $\lambda_j^i$, $1 \leq j \leq m_1$, $1 \leq i \leq m_2$ and all positive integers $m_1, m_2$, we have

$$\left( \sum_{j=1}^{m_1} \| \sum_{i=1}^{m_2} \lambda_j^i T(x_{1,j}^i, \ldots, x_{n,j}^i) \|^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi_k \in B_{X_k^*}} \left( \sum_{j=1}^{m_1} \| \sum_{i=1}^{m_2} \lambda_j^i \varphi_1(x_{1,j}^i) \cdots \varphi_n(x_{n,j}^i) \|^p \right)^{\frac{1}{p}}.$$

These notions have some connection with the idea of weighted summability, sketched in [47]. It is likely that this class has a nice factorization theorem (like its polynomial version) but a simple calculation shows that any factorable p-dominated multilinear operator is p-semi-integral and thus we have:

**Proposition 2.8.** For every $n \geq 3$ and every $p \geq 1$ and every infinite dimensional Banach space $X$ there exists a continuous n-linear operator $T : X \times \cdots \times X \to \mathbb{K}$ that fails to be factorable p-dominated. A fortiori, regardless of the Banach space $Y$, there exists a continuous n-linear operator $T : X \times \cdots \times X \to Y$ that fails to be factorable p-dominated.

So, since we are looking for classes that also lift coincidence results to the multilinear setting, the class of factorable p-dominated multilinear operators is not what we are searching.

A continuous n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is strongly p-summing if there exists a constant $C \geq 0$ such that

$$(2) \quad \left( \sum_{j=1}^{m} \| T(x_j^1, \ldots, x_j^n) \|^p \right)^{\frac{1}{p}} \leq C \left( \sup_{\varphi \in B_{\ell_\infty}} \sum_{j=1}^{m} | \varphi(x_j^1, \ldots, x_j^n) |^p \right)^{\frac{1}{p}}.$$

for every $m \in \mathbb{N}$, $x_j^i \in X_k$ with $k = 1, \ldots, n$ and $j = 1, \ldots, m$.

The class of strongly p-summing multilinear operators is due to V. Dimant [18] and according to [36, 38] it is perhaps the class that best translates to the multilinear setting the properties of the original linear concept. For example, a Grothendieck type theorem and a Pietsch-Domination type theorem are valid:

**Theorem 2.9** (Grothendieck-type theorem). ([18]) Every continuous n-linear operator $T : \ell_1 \times \cdots \times \ell_1 \to \ell_2$ is strongly 1-summing.

**Theorem 2.10** (Pietsch Domination type theorem). ([18]) A continuous n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is strongly p-summing if, and only if, there are a probability measure
\( \mu \) on \( B(X_1 \tilde{\otimes} \cdots \tilde{\otimes} X_n)^*, \) with the weak-star topology, and a constant \( C \geq 0 \) so that

\[
\| T(x_1, \ldots, x_n) \| \leq C \left( \int_{B(X_1 \tilde{\otimes} \cdots \tilde{\otimes} X_n)^*} |\varphi(x_1 \otimes \cdots \otimes x_n)|^p \, d\mu(\varphi) \right)^{\frac{1}{p}}
\]

for all \( (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n. \)

Corollary 2.11. If \( p \leq q \) then every strongly \( p \)-summing multilinear operator is strongly \( q \)-summing.

It is not hard to prove that a Dvoretzky-Rogers Theorem is also valid for this class:

Theorem 2.12 (Dvoretzky-Rogers type theorem). Every continuous \( n \)-linear operator \( T : X \times \cdots \times X \to X \) is strongly \( p \)-summing if, and only if, \( \dim X < \infty. \)

A property fulfilled by the class of absolutely summing operators which is not lifted to the multilinear framework by the notion of strong summability is the weak compactness. In fact, it is well known that every absolutely \( p \)-summing linear operator is weakly compact, but Carando and Dimant have shown that there exist strongly \( p \)-summing multilinear operators that fail to be weakly compact [13]. This result implies that a natural version of the Pietsch Factorization Theorem is not valid for strongly summing multilinear operators, as we will see below.

Suppose that the following factorization theorem holds: \( T : X_1 \times \cdots \times X_n \to Y \) is strongly \( p \)-summing if and only if there is a regular Borel probability measure \( \mu \) on \( B(X_1 \tilde{\otimes} \cdots \tilde{\otimes} X_n)^* \), with the weak-star topology, a closed subspace \( Z_p \) of \( L_p \left( B(X_1 \tilde{\otimes} \cdots \tilde{\otimes} X_n)^*, \mu \right) \) and a continuous linear operator \( \hat{T} : Z_p \to Y \) such that

\[
j_p \circ i_{X_1 \times \cdots \times X_n} : X_1 \times \cdots \times X_n \to Z_p \quad \text{and} \quad \hat{T} \circ j_p \circ i_{X_1 \times \cdots \times X_n} = T,
\]

where

\[
i_{X_1 \times \cdots \times X_n} : X_1 \times \cdots \times X_n \to C \left( B(X_1 \tilde{\otimes} \cdots \tilde{\otimes} X_n)^* \right)
\]

is the canonical \( n \)-linear map \( i_{X_1 \times \cdots \times X_n}(x_1, \ldots, x_n)(\varphi) = \varphi(x_1 \otimes \cdots \otimes x_n) \) and

\[
j_p : C \left( B(X_1 \tilde{\otimes} \cdots \tilde{\otimes} X_n)^* \right) \to L_p \left( B(X_1 \tilde{\otimes} \cdots \tilde{\otimes} X_n)^*, \mu \right)
\]

is the canonical linear inclusion.

Since \( j_p \) is absolutely \( p \)-summing (and thus weakly compact), then we conclude that the set \( j_p \circ i_{X_1 \times \cdots \times X_n}(B(X_1 \times \cdots \times X_n)) \) is relatively weakly compact in \( Z_p \). Since \( \hat{T} \) is continuous and linear, then \( T(B(X_1, \ldots, X_n)) = \hat{T} \circ j_p \circ i_{X_1 \times \cdots \times X_n}(B(X_1 \times \cdots \times X_n)) \) is relatively weakly compact in \( Y \) and thus \( T \) is weakly compact, but this is not true in general [13].

In this paper we combine the idea of factorable summability from [13] with the notion of strongly \( p \)-summing multilinear operators and we show that the new class we introduce recovers all these lacks suffered by the former multilinear extensions.

3. FACTORABLY STRONGLY \( p \)-SUMMING MULTILINEAR OPERATORS

The following definition is inspired in ideas from [13], adapted to the notion of strongly summing multilinear operators:

Definition 3.1. A continuous \( n \)-linear operator \( T : X_1 \times \cdots \times X_n \to Y \) is factorably strongly \( p \)-summing if there is a constant \( C \geq 0 \) such that for every \( x_{k,j}^i \in X_k \), and scalars \( \lambda_{k,j}^i \), \( 1 \leq j \leq m_1, \ 1 \leq i \leq m_2 \) and all positive integers \( m_1, m_2 \), we have

\[
\left( \sum_{j=1}^{m_1} \left| \sum_{i=1}^{m_2} \lambda_{k,j}^i T(x_{1,j}^i, \ldots, x_{n,j}^i) \right|^p \right)^{\frac{1}{p}} \leq C \sup_{\| \varphi \| \leq 1} \left( \sum_{j=1}^{m_1} \left| \sum_{i=1}^{m_2} \lambda_{k,j}^i \varphi(x_{1,j}^i, \ldots, x_{n,j}^i) \right|^p \right)^{\frac{1}{p}}.
\]
where the supremum is taken over all the continuous \( n \)-linear functionals \( \varphi : X_1 \times \cdots \times X_n \to K \) of norm less or equal than 1. The class of all factorable strongly \( p \)-summing \( n \)-linear operators \( T : X_1 \times \cdots \times X_n \to Y \) is denoted by \( \Pi_{FSt,p}(X_1, \ldots, X_n; Y) \) and endowed with the norm \( \| T \|_{FSt,p} \), where \( \| T \|_{FSt,p} \) is given by the infimum of all constant \( C \) fulfilling the above inequality.

Note that if \( T \) is factorable strongly \( p \)-summing then making \( m_2 = 1 \) and \( \lambda_j^1 = 1 \) for all \( j = 1, \ldots, m_1 \), we have

\[
\left\| \left( T \left( x_{1,j}, \ldots, x_{n,j} \right) \right) \right\|_{p}^{m_1} \leq C \sup_{\| \varphi \| \leq 1} \left( \sum_{j=1}^{m_1} |\varphi \left( x_{1,j}, \ldots, x_{n,j} \right)|^p \right)^{\frac{1}{p}}.
\]

i.e., \( T \) is strongly \( p \)-summing. In particular, whenever \( n = 1 \), \( \Pi_{FSt,p}(X_1; Y) = \Pi_p(X_1; Y) \) is the class of all absolutely \( p \)-summing operators from \( X_1 \) to \( Y \).

The ideal property is straightforward. It is also trivial that every scalar-valued \( n \)-linear operator is factorable strongly \( p \)-summing. Straightforward calculations show that this class forms a Banach multi-ideal.

As we will see in Section 3, this class preserves the nice properties of the class of strongly summing multilinear operators and has extra desirable properties: weak compactness and a factorization theorem.

**Theorem 3.2** (Pietsch-Domination type theorem). A continuous \( n \)-linear operator \( T : X_1 \times \cdots \times X_n \to Y \) is factorable strongly \( p \)-summing if and only if there is a regular probability measure \( \mu \) on \( B_{(X_1 \hat{\otimes} \cdots \hat{\otimes} X_n)^*} \), endowed with the weak-star topology, and a constant \( C \geq 0 \), such that

\[
\left\| \sum_{i=1}^{m} \lambda^i x^i \right\|_{p} \leq C \left( \int_{B_{(X_1 \hat{\otimes} \cdots \hat{\otimes} X_n)^*}} \left( \sum_{i=1}^{m} \lambda^i |\varphi \left( x_{1,i}^i, \ldots, x_{n,i}^i \right)|^p \right)^{\frac{1}{p}} d\mu (\varphi) \right).
\]

Proof. The notion of factorable strongly \( p \)-summing multilinear operator is precisely the concept of RS-abstract \( p \)-summing (see [10, 35, 38]) for

\[
R : B_{(X_1 \hat{\otimes} \cdots \hat{\otimes} X_n)^*} \times (K \times X_1 \times \cdots \times X_n)^n \times \{ 0 \} \to [0, \infty)
\]
given by

\[
R \left( \varphi, (\lambda^1, x_1^1, \ldots, x_n^1), \ldots, (\lambda^m, x_1^m, \ldots, x_n^m), 0 \right) = \left| \sum_{i=1}^{m} \lambda^i \varphi \left( x_{1,i}^i \otimes \cdots \otimes x_{n,i}^i \right) \right|
\]

and

\[
S : L(X_1, \ldots, X_n; Y) \times (K \times X_1 \times \cdots \times X_n)^n \times \{ 0 \} \to [0, \infty)
\]
given by

\[
S \left( T, (\lambda^1, x_1^1, \ldots, x_n^1), \ldots, (\lambda^m, x_1^m, \ldots, x_n^m), 0 \right) = \left| \sum_{i=1}^{m} \lambda^i T \left( x_{1,i}, \ldots, x_{n,i} \right) \right|
\]

Since \( R \) and \( S \) satisfy the hypotheses of the general Pietsch Domination Theorem, the result follows straightforwardly.

**Theorem 3.3** (Pietsch-Factorization type theorem). A continuous \( n \)-linear operator \( T : X_1 \times \cdots \times X_n \to Y \) is factorable strongly \( p \)-summing if and only if there exist a regular probability measure \( \mu \) on \( B_{(X_1 \hat{\otimes} \cdots \hat{\otimes} X_n)^*} \), endowed with the weak-star topology, a constant
\( C \geq 0 \), a closed subspace \( Z_p \) of \( L_p \left( B_{(X_1 \otimes \ldots \otimes X_n)^*}, \mu \right) \) and a continuous linear operator \( \hat{T} : Z_p \to Y \) such that

\[
j_p \circ i_{X_1 \times \ldots \times X_n}(X_1 \times \ldots \times X_n) \subset Z_p \quad \text{and} \quad \hat{T} \circ j_p \circ i_{X_1 \times \ldots \times X_n} = T,
\]

where

\[
i_{X_1 \times \ldots \times X_n} : X_1 \times \ldots \times X_m \to C \left( B_{(X_1 \otimes \ldots \otimes X_n)^*} \right)
\]

is the canonical \( n \)-linear map \( i_{X_1 \times \ldots \times X_n}(x_1, \ldots, x_n) (\varphi) = \varphi(x_1 \otimes \ldots \otimes x_n) \) and

\[
\hat{j}_p : C \left( B_{(X_1 \otimes \ldots \otimes X_n)^*} \right) \to L_p \left( B_{(X_1 \otimes \ldots \otimes X_n)^*}, \mu \right)
\]

is the canonical linear inclusion.

**Proof.** Suppose that \( T \) is factorable strongly \( p \)-summing. Let \( \mu \) be the measure given by the Pietsch Domination Theorem (Theorem \( 3.2 \)) applied to \( T \). Let \( W_p \) be the subspace of \( L_p \left( B_{(X_1 \otimes \ldots \otimes X_n)^*}, \mu \right) \) given by the linear span of \( j_p \circ i_{X_1 \times \ldots \times X_m}(X_1 \times \ldots \times X_m) \). Define the linear operator \( \hat{T} : W_p \to Y \) by

\[
\hat{T}(z) = \sum_{i=1}^{n} \lambda_i T(x_1^i, \ldots, x_n^i)
\]

for

\[
z = \sum_{i=1}^{n} \lambda_i (\cdot, (x_1^i \otimes \ldots \otimes x_n^i)) \in W_p.
\]

Note that \( \hat{T} \) is well-defined. In fact, if

\[
z_1 = \sum_{i=1}^{m_1} \lambda_i (\cdot, (x_1^i \otimes \ldots \otimes x_n^i)) \quad \text{and} \quad z_2 = \sum_{i=1}^{m_2} \alpha_i (\cdot, (y_1^i \otimes \ldots \otimes y_n^i))
\]

coincide in \( W_p \), then considering

\[
w := \sum_{i=1}^{m_1} \lambda_i (\cdot, (x_1^i \otimes \ldots \otimes x_n^i)) - \sum_{i=1}^{m_2} \alpha_i (\cdot, (y_1^i \otimes \ldots \otimes y_n^i)),
\]

we have \( w = 0 \) almost everywhere in \( W_p \), i.e.,

\[
\int_{B_{(X_1 \otimes \ldots \otimes X_n)^*}} \left| \sum_{i=1}^{m_1} \lambda_i \varphi(x_1^i \otimes \ldots \otimes x_n^i) - \sum_{i=1}^{m_2} \alpha_i \varphi(y_1^i \otimes \ldots \otimes y_n^i) \right|^p \, d\mu(\varphi) = 0.
\]

Thus, from the domination theorem,

\[
\left\| \sum_{i=1}^{m_1} \lambda_i T(x_1^i, \ldots, x_n^i) - \sum_{i=1}^{m_2} \alpha_i T(y_1^i, \ldots, y_n^i) \right\|^{1/p} \leq C \left( \sum_{i=1}^{m_1} \lambda_i \varphi(x_1^i \otimes \ldots \otimes x_n^i) - \sum_{i=1}^{m_2} \alpha_i \varphi(y_1^i \otimes \ldots \otimes y_n^i) \right)^{1/p} d\mu(\varphi) = 0
\]

and we conclude that

\[
\hat{T}(z_1) - \hat{T}(z_2) = \sum_{i=1}^{m_1} \lambda_i T(x_1^i, \ldots, x_n^i) - \sum_{i=1}^{m_2} \alpha_i T(y_1^i, \ldots, y_n^i) = 0.
\]
Note also that for \( z = \sum_{i=1}^{m} \lambda_i \langle \cdot, (x_1^i \otimes \cdots \otimes x_m^i) \rangle \in W_p \) we have
\[
\left\| \hat{T}(z) \right\| = \left\| \sum_{i=1}^{m} \lambda_i T(x_1^i, \ldots, x_m^i) \right\|
\leq C \left( \int_{B_{X_1^1 \otimes \cdots \otimes X_n^m}} \left\| \sum_{i=1}^{m} \lambda_i \varphi(x_1^i \otimes \cdots \otimes x_m^i) \right\|^p \, d\mu(\varphi) \right)^{1/p}
= C \|z\|_{L_p(\mu)}
\]
and \( \hat{T} \) is continuous. It is obvious that from the very definition of \( \hat{T} \) we have \( \hat{T} \circ j_p \circ i_{X_1 \times \cdots \times X_n} = T \). Now we extend \( \hat{T} \) to \( Z_p = \overline{W_p} \). The converse is immediate. \( \square \)

4. Factorable strongly \( p \)-summing polynomials

The \( m \)-fold symmetric tensor product of \( X \) is the linear span of all tensors of the form \( x \otimes \cdots \otimes x, x \in X \), and is denoted by \( \otimes_{m,s}^X \). This space is endowed with the \( s \)-projective tensor norm, defined as
\[
\pi_s(z) = \inf \left\{ \sum_{j=1}^{k} \|\lambda_j\| \|x_j\|^m : k \in \mathbb{N}, z = \sum_{j=1}^{k} \lambda_j x_j \otimes \cdots \otimes x_j \right\},
\]
for \( z \in \otimes_{m,s}^X \). Let \( \otimes_{m,s}^\ast X \) denote the completion of \( \otimes_{m,s}^X \).

Given \( P \in \mathcal{P}(^m X; Y) \), the linearization of \( P \) is the unique linear operator \( P_{L,s} : \otimes_{m,s}^X \to Y \) such that \( P_{L,s}(\alpha \otimes \cdots \otimes \alpha) = P(\alpha) \) for all \( \alpha \in X \). Ryan [44] proved that the correspondence \( P \leftrightarrow P_{L,s} \) establishes a isometric isomorphism between the space \( \mathcal{P}(^m X) \), endowed with the usual sup norm, and the strong dual of \( \otimes_{m,s}^X \). Another map associated to \( P \in \mathcal{P}(^m X; Y) \) is the unique continuous symmetric \( m \)-linear mapping \( \hat{P} \) that satisfies \( \hat{P}(x, \ldots, x) = P(x) \), for all \( x \in X \). It is well known that \( \|\hat{P}\| \leq c(m, X)\|P\| \) for all \( P \in \mathcal{P}(^m X) \), where \( c(m, X) \) is the \( m \)-th polarization constant of \( X \). For the general theory of homogeneous polynomials we refer to [19] and [32].

Concomitantly to multilinear mappings, factorable strongly \( p \)-summing homogeneous polynomials can be introduced. Our aim is to prove that both classes coincide in the sense that a polynomial is factorable strongly \( p \)-summing if and only if its associated symmetric multilinear mapping is factorable strongly \( p \)-summing. Moreover, we will see the deep relationship between factorable strong summability and absolute summability by proving that, for an homogeneous polynomial, it is equivalent that the polynomial is factorable strongly summing to that its linearization is an absolutely summing operator. To attain this purpose, we will show that both, factorable strongly \( p \)-summing polynomials and factorable strongly \( p \)-summing multilinear operators, form composition ideals.

**Definition 4.1.** A continuous \( n \)-homogeneous polynomial \( P : X \to Y \) is factorable strongly \( p \)-summing if there is a \( C \geq 0 \) such that for every \( x_j \in X \), and scalars \( \lambda_j^i, 1 \leq j \leq m_1, 1 \leq i \leq m_2 \) and all positive integers \( m_1, m_2 \), we have that
\[
\left\| \left( \sum_{i=1}^{m_2} \lambda_j^i P(x_j^i) \right) \right\|_p \leq C \sup_{\|\varphi\| \leq 1, q \in \mathcal{P}(^m X)} \left( \sum_{j=1}^{m_1} \left\| \sum_{i=1}^{m_2} \lambda_j^i q(x_j^i)^p \right\| \right)^{1/p}.
\]
The class of all factorable strongly \( p \)-summing \( m \)-homogeneous polynomials from \( X \) to \( Y \) is denoted by \( \mathcal{P}_{FS,t,p}^{(m)X;Y} \) and endowed with the norm \( \left\| \cdot \right\|_{FS,t,p} \) given by the infimum of all constants \( C \) fulfilling the above inequality.
It is clear that factorable $p$-dominated polynomials are factorable strongly $p$-summing. An easy calculation shows the following ideal property:

**Proposition 4.2.** If $P \in \mathcal{P}_{FSt,p}(mX;Y)$ and $u : G \to X$, $v : Y \to Z$ are continuous linear operators then $v \circ P \circ u \in \mathcal{P}_{FSt,p}(mG;Z)$ and $\|v \circ P \circ u\|_{FSt,p} \leq \|v\| \cdot \|P\|_{FSt,p}\|u\|^m$.

It is not difficult to complete Proposition 4.2 and show that factorable strongly $n$-homogeneous polynomials form an ideal of polynomials (for the definition of ideal of polynomials we refer to [4]).

Dimant [13] introduced the class of strongly $p$-summing $m$-homogeneous polynomials from $X$ to $Y$ as those $m$-homogeneous polynomials $P : X \to Y$ that satisfy that there exists $K > 0$ such that for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in X$,

$$\left(\sum_{j=1}^{n} \|P(x_j)\|^p\right)^{1/p} \leq K \sup_{\|q\| \leq 1, q \in \mathcal{P}(mX)} \left(\sum_{j=1}^{n} |q(x_j)|^p\right)^{1/p}.$$  

In [13] Proposition 3.2 it is proved that if the linearization $P_{L,s}$ of $P \in \mathcal{P}(mX;Y)$ is absolutely $p$-summing then $p$ is strongly $p$-summing. However, the converse is not true (see [15] Example 3.3]). The reason, as for $p$-dominated polynomials, is that not every strongly $p$-summing polynomial is weakly compact. So, once again, the lack of connection with weak compactness turns out to be a deep inconvenience in the way that strongly $p$-summing polynomials generalize absolutely $p$-summing linear operators. Even if a domination holds also for strongly $p$-summing polynomials [13] Proposition 3.2], no factorization theorem is expected. Let us prove a factorization theorem for factorable strongly $p$-summing polynomials. We first need a domination theorem, that is obtained as a particular case of [10, Theorem 2.2]. We denote by $\delta : X \to C(B_{\mathcal{P}(mX)})$ the $m$-homogeneous polynomial given by $\delta(x) := \delta_x : B_{\mathcal{P}(mX)} \to \mathbb{K}$, where $\delta_x(P) := P(x)$. Considering that the space of continuous $m$-homogeneous polynomials is a dual space (see [44]), its closed unit ball $B_{\mathcal{P}(mX)}$ is a weak-star compact set.

**Theorem 4.3** (Pietsch-Domination type theorem). Let $P \in \mathcal{P}(mX;Y)$. Then $P$ is factorable strongly $p$-summing if and only if there exists a regular Borel probability measure $\mu$ on $B_{\mathcal{P}(mX)}$, endowed with the weak-star topology, such that

$$\|\sum_{i=1}^{k} \lambda^i P(x^i)\| \leq C(\int_{B_{\mathcal{P}(mX)}} \left|\sum_{i=1}^{k} \lambda^i q(x^i)\right|^p d\mu)^{1/p}$$

for all $x^1, \ldots, x^k \in X$ and $\lambda^1, \ldots, \lambda^k \in \mathbb{K}$.

**Proof.** It is a particular case of [10, Theorem 2.2] analogous to the proof of Theorem 3.2. \qed

We shall need the following result to prove the sufficiency of the Factorization Theorem. Besides, Proposition 4.3 will be the key for our purposes to obtain that factorable strongly $p$-summing homogeneous polynomials form a composition ideal.

**Proposition 4.4.** If $Q \in \mathcal{P}(mG;X)$ and $u : X \to Y$ is an absolutely $p$-summing linear operator, then $u \circ Q \in \mathcal{P}(mG;Y)$ and $\|u \circ Q\|_{FSt,p} \leq \pi_p(u)\|Q\|$.

In [18, Proposition 3.2] it is proved that if the linearization $P_{L,s}$ of $P \in \mathcal{P}(mX;Y)$ is absolutely $p$-summing then $p$ is strongly $p$-summing. However, the converse is not true (see [15] Example 3.3]). The reason, as for $p$-dominated polynomials, is that not every strongly $p$-summing polynomial is weakly compact. So, once again, the lack of connection with weak compactness turns out to be a deep inconvenience in the way that strongly $p$-summing polynomials generalize absolutely $p$-summing linear operators. Even if a domination holds also for strongly $p$-summing polynomials [13] Proposition 3.2], no factorization theorem is expected. Let us prove a factorization theorem for factorable strongly $p$-summing polynomials. We first need a domination theorem, that is obtained as a particular case of [10, Theorem 2.2]. We denote by $\delta : X \to C(B_{\mathcal{P}(mX)})$ the $m$-homogeneous polynomial given by $\delta(x) := \delta_x : B_{\mathcal{P}(mX)} \to \mathbb{K}$, where $\delta_x(P) := P(x)$. Considering that the space of continuous $m$-homogeneous polynomials is a dual space (see [44]), its closed unit ball $B_{\mathcal{P}(mX)}$ is a weak-star compact set.

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$$\|\sum_{i=1}^{k} \lambda^i P(x^i)\| \leq C(\int_{B_{\mathcal{P}(mX)}} \left|\sum_{i=1}^{k} \lambda^i q(x^i)\right|^p d\mu)^{1/p}$$

for all $x^1, \ldots, x^k \in X$ and $\lambda^1, \ldots, \lambda^k \in \mathbb{K}$.

**Proof.** It is a particular case of [10, Theorem 2.2] analogous to the proof of Theorem 3.2. \qed

We shall need the following result to prove the sufficiency of the Factorization Theorem. Besides, Proposition 4.3 will be the key for our purposes to obtain that factorable strongly $p$-summing homogeneous polynomials form a composition ideal.

**Proposition 4.4.** If $Q \in \mathcal{P}(mG;X)$ and $u : X \to Y$ is an absolutely $p$-summing linear operator, then $u \circ Q \in \mathcal{P}(mG;Y)$ and $\|u \circ Q\|_{FSt,p} \leq \pi_p(u)\|Q\|$.
Proof. Let \( m_1, m_2 \) be positive integers, \( x_j \in X \), and scalars \( \lambda_{ij} \), \( 1 \leq j \leq m_1, 1 \leq i \leq m_2 \). Then,
\[
\left( \sum_{j=1}^{m_1} \left( \sum_{i=1}^{m_2} \lambda_{ij} u \circ Q(x_j) \right)^p \right)^{1/p} = \left( \sum_{j=1}^{m_1} \left( \sum_{i=1}^{m_2} \lambda_{ij} Q(x_j) \right)^p \right)^{1/p}
\]
\[
\leq \pi_p(u) \sup_{\|x^*\| \leq 1, x^* \in X^*} \left( \sum_{j=1}^{m_1} \left( \sum_{i=1}^{m_2} \lambda_{ij} Q(x_j) \right)^p \right)^{1/p}
\]
\[
\leq \pi_p(u) ||Q|| \sup_{\|x^*\| \leq 1, x^* \in X^*} \left( \sum_{j=1}^{m_1} \left( \sum_{i=1}^{m_2} \lambda_{ij} (x^*, Q(x_j)) \right)^p \right)^{1/p}
\]
\[
\leq \pi_p(u) ||Q|| \sup_{\|q\| \leq 1, q \in P^{(m)}(G)} \left( \sum_{j=1}^{m_1} \left( \sum_{i=1}^{m_2} \lambda_{ij} Q(x_j) \right)^p \right)^{1/p}.
\]
\[\square\]

Theorem 4.5 (Pietsch-Factorization type theorem). Let \( P \in P^{(m)}(X; Y) \). Then \( P \) is factorable strongly \( p \)-summing if and only if there exists a regular Borel probability measure \( \mu \) on \( B_{\hat{P}^{(m)}} \), a closed subspace \( G_\mu \) of \( L_p(\mu) \) and a continuous linear operator \( v_0 : G_\mu \to Y \) such that \( j_P \circ \delta(\mu) \subseteq G_\mu \) and \( v_0 \circ j_P \circ \delta = P \), where \( j_P : C(B_{\hat{P}^{(m)}}) \to L_p(B_{\hat{P}^{(m)}}), \mu \) is the canonical inclusion.

Proof. Assume first that \( P \) is factorable strongly \( p \)-summing. Let \( \mu \) be given by Theorem 4.3. Take \( G_\mu \) the completion of the image by \( j_P \) of the linear span of \( \delta(\mu) \). Define \( v_0(j_P(\sum_{i=1}^k \lambda_i x_i)) := \sum_{i=1}^k \lambda_i P(x_i) \). To see that \( v_0 \) is well defined, consider that \( j_P(\sum_{i=1}^k \lambda_i x_i) = j_P(\sum_{i=1}^l \eta_i y_i) \). Then \( w := \sum_{i=1}^k \lambda_i \delta x_i - \sum_{i=1}^l \eta_i \delta y_i = 0 \) a.e. on \( B_{\hat{P}^{(m)}} \). Hence,
\[
\| \sum_{i=1}^k \lambda_i P(x_i) - \sum_{i=1}^l \eta_i P(y_i) \| \leq \| P \|_{\text{FSt,p}}(\int_{B_{\hat{P}^{(m)}}} | \sum_{i=1}^k \lambda_i q(x_i) - \sum_{i=1}^l \eta_i q(y_i) |^p d\mu)^{1/p} = 0.
\]
Thus, \( v_0(w) = 0 \). That proves that \( v_0 \) is well defined. The continuity of \( v_0 \) follows from the calculations:
\[
\| v_0(z) \| = \left\| \sum_{i=1}^k \lambda_i P(x_i) \right\| \leq \| P \|_{\text{FSt,p}}(\int_{B_{\hat{P}^{(m)}}} | \sum_{i=1}^k \lambda_i q(x_i) |^p d\mu)^{1/p} \]
\[
= \| P \|_{\text{FSt,p}} \| \sum_{i=1}^k \lambda_i \delta x_i \|_{L_p(\mu)} = \| P \|_{\text{FSt,p}} \| v \|_{L_p(\mu)}
\]
for any \( z = j_P(\sum_{i=1}^k \lambda_i x_i) \). The desired linear operator is just the continuous extension of \( v_0 \) to \( G_\mu \). The converse follows from Proposition 4.3. \[\square\]

Corollary 4.6. Let \( P \in P^{(m)}(X; Y) \). Then \( P \in P_{\text{FSt,p}}^{(m)}(X; Y) \) if and only if \( P = u \circ Q \) for some continuous \( m \)-homogeneous polynomial \( Q \) and some absolutely \( p \)-summing linear operator \( u \). In that case \( \| P \|_{\text{FSt,p}} = \inf \{ \pi_p(u)||Q|| : P = u \circ Q \} \).

Proof. It follows from Theorem 4.5 and Proposition 4.3. \[\square\]

Corollary 4.6 says that the ideal of all factorable strongly \( p \)-summing \( m \)-homogeneous polynomials is the composition ideal with all absolutely \( p \)-summing linear operators, that is, \( P_{\text{FSt,p}} = \Pi_p \circ P \) (see [9]). An analogous argument for multilinear operators instead of polynomials yields to prove that the ideal of all factorable strongly \( p \)-summing \( m \)-linear
operators is the composition ideal with all absolutely $p$-summing linear operators, that is, $\Pi_{FSt,p} = \Pi_p \circ \mathcal{L}$.

**Remark 4.7.** In [29] it is shown an example of a continuous $m$-homogeneous polynomial $P : X \to Y$ and $\phi \in \Pi_{as,r}(Y;Z)$ such that $\phi \circ P : X \to Z$ is not $r$-dominated. By Proposition 4.4 $\phi \circ P$ is factorable strongly $r$-summing. Therefore, the class of dominated polynomials differs from the class of factorable strongly $r$-summing polynomials.

**Theorem 4.8.** Let $P \in \mathcal{P}^{(m)X;Y}$. The following are equivalent:

1. $P \in \mathcal{P}_{FSt,p}^{(m)X;Y}$.
2. $P_{L,s}$ is absolutely $p$-summing.
3. $P \in \Pi_{FSt,p}^{(m)X;Y}$.

In that case, $\|P\|_{FSt,p} = \pi_p(P_{L,s})$.

**Proof.** (1)⇒(2) Assume first that $P$ is factorable strongly $p$-summing. Then

$$\left(\sum_{j=1}^{m_1} \left\| P_{L,s} \left( \sum_{i=1}^{m_2} \lambda_i x_i^j \otimes \cdots \otimes x_i^j \right) \right\|^p \right)^{1/p} = \left(\sum_{j=1}^{m_1} \left\{ \sum_{i=1}^{m_2} \lambda_i^j P_{L,s}(x_i^j \otimes \cdots \otimes x_i^j) \right\}^p \right)^{1/p}
$$

$$= \left(\sum_{j=1}^{m_1} \left\{ \sum_{i=1}^{m_2} \lambda_i^j P(x_i^j) \right\}^p \right)^{1/p}
$$

$$\leq \|P\|_{FSt,p} \sup_{\|g\| \leq 1, q \in \mathcal{P}^{(m)X'}} \left(\sum_{j=1}^{m_1} \left\{ \sum_{i=1}^{m_2} \lambda_i^j g(x_i^j) \right\}^p \right)^{1/p}
$$

$$= \|P\|_{FSt,p} \sup_{\|g\| \leq 1, q \in \mathcal{P}^{(m)X'}} \left(\sum_{j=1}^{m_1} \left\{ \sum_{i=1}^{m_2} \lambda_i^j g(\otimes_m x_i^j) \right\}^p \right)^{1/p}
$$

(2)⇒(1) follows from Proposition 4.4.

(1)⇒(3) As $\mathcal{P}_{FSt,p}^{(m)X;Y} = \Pi_p \circ \mathcal{P}^{(m)X;Y}$, it follows from [9, Proposition 3.2] that $P \in \mathcal{P}_{FSt,p}^{(m)X;Y}$ if and only if $P \in \Pi_p \circ \mathcal{L}^{(m)X;Y}$ and, similarly to Proposition 4.3 it can be proved that $\Pi_p \circ \mathcal{L}^{(m)X;Y} = \Pi_{FSt,p}^{(m)X;Y}$.

We finish this section with a Grothendieck type theorem:

**Theorem 4.9** (Grothendieck type theorem). If $m \geq 1$ is a positive integer, then $\mathcal{P}^{(m)\ell_1;\ell_2} = \mathcal{P}_{FSt,1}^{(m)\ell_1;\ell_2}$.

**Proof.** Let $P \in \mathcal{P}^{(m)X;Y}$. Then $P_{L,s} \in \mathcal{L}(\otimes_m \ell_1;\ell_2) = \mathcal{L}(\ell_1;\ell_2) = \Pi_{as,1}(\ell_1;\ell_2)$. Theorem 4.8 yields the result. 

5. The wealth of factorable strong $p$-summability

In this section it is shown that factorable strong $p$-summability is an excellent non linear frame where linear results for absolute summability are properly generalized to multilinear operators and polynomials. This evidences the interest of this new class as it really reflects the good behavior of absolute summability in the non linear context. Some of these results are established for multilinear operators and some for homogeneous polynomials. However, as a consequence of Theorem 4.8 it is clear that one can pass easily from one to each other.

The following results are consequences of Theorem 4.8 and their linear analogs (see [17, Theorems 3.15 and 3.17]):

**Proposition 5.1** (Composition Theorem). If $u \in \Pi_p(X;Y)$ and $P \in \mathcal{P}_{FSt,q}^{(m)G;X}$ then $u \circ P \in \mathcal{P}_{FSt,r}^{(m)G;Y}$ for $1/r := \min\{1, 1/p + 1/q\}$. 

Proof. By Theorem 4.8 $P_{L,s}$ is absolutely $q$-summing. Then $u \circ P_{L,s}$ is $r$-summing for $1/r := \min\{1, 1/p + 1/q\}$ (see Theorem 2.22). Since $u \circ P_{L,s} = (u \circ P)_{L,s}$, a second application of Theorem 4.8 yields the result. \qed

**Theorem 5.2** (Extrapolation type theorem). Let $1 < r < p < \infty$, and let $X$ be a Banach space. If $P_{FST,r}(mX; \ell_p) = P_{FST,r}(mX; \ell_p)$, then $P_{FST,q}(mX; Y) = P_{FST,q}(mX; Y)$ for every Banach space $Y$.

Recall that given $1 \leq p \leq \infty$ and $\lambda > 1$, a Banach space $X$ is said to be an $\ell_{p,\lambda}$-space if every finite dimensional subspace $E$ of $X$ is contained in a finite dimensional subspace $F$ of $X$ for which there is an isomorphism $v : F \to \ell_p^{\dim F}$ with $\|v\| : \|v^{-1}\| < \lambda$.

**Theorem 5.3** (Lindenstrauss–Pełczyński type theorem). Let $1 \leq p \leq 2$ and $2 < q < \infty$. If $X$ is a Banach space and $Y$ is a subspace of an $\ell_{p,\lambda}$-space, then $P_{FST,q}(mX; Y) = P_{FST,q}(mX; Y)$.

We have already proved that Domination/Factorization Theorems are fulfilled in the multilinear and polynomial classes of factorable strongly $p$-summing maps. As a straightforward consequence of the Factorization Theorem 4.8 we get

**Theorem 5.4.** Any factorable strongly $p$-summing polynomial is weakly compact.

An alternative way to prove it is the following: by Theorem 4.8 the linearization of a factorable strongly $p$-summing polynomial $P$ is absolutely $p$-summing and hence weakly compact. By 4.14 this is equivalent to the weak compactness of $P$. The same holds for the case of multilinear operators.

The Domination Theorem 5.2 also yields to the following inclusion theorem.

**Proposition 5.5** (Inclusion Theorem). If $1 \leq p \leq q < \infty$ then every factorable strongly $p$-summing polynomial is factorable strongly $q$-summing.

The forthcoming lemmas 5.7, 5.8, 5.9 and its consequences show that, besides its good properties, the classes of factorable strongly $p$-summing multilinear operators and polynomials have a coherent size.

**Lemma 5.6.** If every continuous $n$-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is factorable strongly $p$-summing, then every continuous linear operator $u_j : X_j \to Y$ is absolutely $p$-summing for every $j = 1, \ldots, n$.

Proof. For the sake of simplicity, let us suppose $j = 1$. Let $u : X_1 \to Y$ be a continuous linear operator and $\varphi_j \in X_j^*$, $j = 2, \ldots, n$ be non-null linear functionals. Then $T(x_1, \ldots, x_n) := u(x_1) \varphi_2(x_2) \cdots \varphi_n(x_n)$ is factorably strongly $p$-summing. Thus, in particular, there is a $C > 0$ such that

$$\left( \sum_{j=1}^{m_1} \| T(x_{1,j}, \ldots, x_{n,j}) \|^p \right)^{1/p} \leq C \sup_{\|\varphi\| \leq 1} \left( \sum_{j=1}^{m_1} \| \varphi(x_{1,j}, \ldots, x_{n,j}) \|^p \right)^{1/p}. $$

Choose $a_j \in X_j$ such that $\varphi_j(a_j) = 1$ for all $j = 2, \ldots, n$. Thus

$$\left( \sum_{j=1}^{m_1} \| T(x_{1,j}, a_2, \ldots, a_n) \|^p \right)^{1/p} \leq C \sup_{\|\varphi\| \leq 1} \left( \sum_{j=1}^{m_1} \| \varphi(x_{1,j}, a_2, \ldots, a_n) \|^p \right)^{1/p} $$

and it follows that $u$ is absolutely $p$-summing. \qed

The following two theorems are immediate consequences of the previous lemma and of the respective linear results:
Theorem 5.7 (Dvoretzky-Rogers type theorem). Let $Y$ be a Banach space. Every continuous $n$-linear operator $T : Y \times \cdots \times Y \to Y$ is factorable strongly $p$-summing if, and only if, $\dim Y < \infty$.

Theorem 5.8 (Lindenstrauss–Pełczyński type theorem). Let $m$ be a positive integer. If $X$ and $Y$ are infinite-dimensional Banach spaces, $X$ has an unconditional Schauder basis and $\Pi_{\text{FSt},1}(m; X; Y) = \mathcal{L}(m; X; Y)$ then $X = \ell_1$ and $Y$ is a Hilbert space.

For polynomials we have a natural version of Lemma 5.6

Lemma 5.9. If every continuous $n$-homogeneous polynomial $P : X \to Y$ is factorable strongly $p$-summing, then every continuous linear operator $u : X \to Y$ is absolutely $p$-summing.

Proof. Let $u : X \to Y$ be a continuous linear operator and $\varphi \in X^*$ be a non-null linear functional and $a \in X$ be so that $\varphi(a) = 1$. Then $P(x) := u(x) \varphi (x)^{n-1}$ is factorable strongly $p$-summing. Thus $P$ is factorable strongly $p$-summing. From the proof of Lemma 5.6 we conclude that the linear operator $v : X \to Y$ defined by $v(x) = P(a, \ldots, a, x)$ is absolutely $p$-summing. But $v$ is a linear combination of $u(a) \varphi$ and $u$; since $u(a) \varphi$ is absolutely $p$-summing it follows that $u$ is absolutely $p$-summing. □

An immediate consequence of the previous lemma is that the analogs of theorems 5.7 and 5.8 work for polynomials. For instance:

Theorem 5.10 (Dvoretzky-Rogers type theorem for polynomials). Let $Y$ be a Banach space. Every continuous $n$-homogeneous polynomial $P : Y \to Y$ is factorable strongly $p$-summing if, and only if, $\dim Y < \infty$.

Given $P \in \mathcal{P}(m; X; Y)$ let us consider its transpose $P^t : Y^* \to \mathcal{P}(m; X)$ given by $P^t(y^*) := y^* \circ P$. Note that $P^t$ is a continuous linear operator. Let $P^{tt} : \mathcal{P}(m; X)^* \to Y^{**}$ be the transpose of $P^t$. It is well known (see [17, Theorem 2.21]) that, if $Y = H$ is a Hilbert space then a continuous linear operator is absolutely 1-summing whenever its transpose is absolutely $p$-summing for some $1 \leq p < \infty$. Let us see that the analogous result is true for polynomials.

Proposition 5.11. Let $H$ be a Hilbert space and $P \in \mathcal{P}(m; X; H)$. If $P^t \in \Pi_p(\EFs m,s X; H)$ for some $1 \leq p < \infty$ then $P \in \mathcal{P}_{\text{FSt},1}(m; X; H)$.

Proof. From the equality $P^{tt}_L = \delta \circ P^t$, where $\delta : \mathcal{P}(m; X) \to (\EFs m,s X)^*$ is the canonical isomorphism, it follows that $P^{tt}_L$ is absolutely $p$-summing and then $P_L$ is absolutely 1-summing. By Theorem 5.8 we conclude that $P$ is factorable strongly 1-summing. □

Proposition 5.12. Let $P \in \mathcal{P}(m; X; Y)$. Then $P \in \mathcal{P}_{\text{FSt},p}(m; X; Y)$ if and only if $P^{tt} \in \Pi_p(\mathcal{P}(m; X)^*; Y^{**})$.

Proof. It is a consequence of Theorem 4.8 the fact that $P^{tt}_L = P^{tt} \circ \delta^t$ and the analogous well known property for linear operators (see [17, Proposition 2.19]). □

6. Coherence and compatibility

Let us denote the ideal of factorable strongly $p$-summing $n$-homogeneous polynomials by $\mathcal{P}_{\text{FSt},p}$ whereas $\Pi_{\text{FSt},p}$ denotes the ideal of factorable strongly $p$-summing $n$-linear operators. The notions of coherent and compatible ideals of polynomials were introduced by Carando, Dimant and Muro [19] in order to evaluate what polynomial approaches preserve the spirit of a given operator ideal. Standard calculations show that $(\mathcal{P}_{\text{FSt},p})_{q=1}^\infty$ is coherent and compatible with $\Pi_p$. Very recently, in [34], the notions of coherence and compatibility
were extended to pairs of ideals of polynomials and multi-ideals. It is also possible to show that \((P_{\text{FS},p}^n \Pi_{\text{FS},p}^n)_{n=1}^\infty\) is coherent and compatible with \(\Pi_p\).

We have shown that \(P_{\text{FS},p}^n\Phi\) coincides with the composition ideal with the absolutely \(p\)-summing operators. However, we cannot apply [34, Theorem 5.7] to get the coherence and compatibility as the topology involved in that result comes from the multilinear operators space norm and it does not coincide with \(\| \cdot \|_{\text{FS},p}\) (see [9]). Despite of this, standard calculations also allow to get the following:

**Theorem 6.1.** The sequence \((P_{\text{FS},p}^n \Pi_{\text{FS},p}^n)_{n=1}^\infty\) is coherent and compatible with \(\Pi_p\).

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