ON THE NONLINEAR SCHRÖDINGER EQUATION WITH A TOROIDAL TRAP IN THE STRONG CONFINEMENT REGIME

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ABSTRACT. We consider the 3D cubic nonlinear Schrödinger equation (NLS) with a strong toroidal trap. In the first part, we show that as the confinement is strengthened, a large class of global solutions to the time-dependent model can be described by 1D flows solving the 1D periodic NLS (Theorem 1.4). In the second part, we construct a steady state as a constrained energy minimizer, and prove its dimension reduction to the well-known 1D periodic ground state (Theorem 1.6 and 1.7). Then, employing the dimension reduction limit, we establish the local uniqueness and the orbital stability of the 3D ring soliton (Theorem 1.8). These results justify the emergence of stable quasi-1D periodic dynamics for Bose-Einstein condensates on a ring in physics experiments.

1. INTRODUCTION

1.1. Bose-Einstein condensate in a toroidal trap. A Bose-Einstein condensate (BEC) is a state of matter where almost all particles have the same quantum state. It is produced by cooling a dilute Bose gas down to near absolute 0K. This exotic material form was first proposed in 1924-25 by Bose and Einstein. Many decades later, in 1995, Cornell and Wieman in JILA group [2] and Ketterle’s group [20] independently succeeded in creating BECs in laboratory using alkali vapor. For weakly interacting BECs, via the mean-field approximation, the particle dynamics is described by a single wave function governed by the cubic nonlinear Schrödinger equation (NLS) also known as the Gross-Pitaevskii equation [21, 24, 35].

In most experiments, BECs are confined in a trapping potential, and different trapping geometries determine different shapes of condensates. In particular, controlling confining frequencies in an anisotropic harmonic trap, quasi-lower dimensional cigar-shaped and disc-shaped condensates can be produced. In this way, BECs for attractive particles are possibly stabilized in a quasi-lower dimensional regime [4, 12, 31, 38]. For various dimension reduction processes, we refer to the survey articles [3, 13] and to [8–10, 17–19, 28, 30] for rigorous proofs in various settings.

In this paper, we are particularly concerned with a BEC confined in a toroidal trap. In experiments, a toroidal trap can be realized by optical and/or magnetic potentials [38, 40].
For instance, the NIST atomic physics group constructed a potential of the form\footnote{In [38], a fully anisotropic potential \( \frac{m}{2} (\omega_1^2 y_1^2 + \omega_2^2 y_2^2 + \omega_z^2 z^2) + V_0 e^{-2(y_1^2+y_2^2)/w_0^2} \) is employed with \( \frac{\omega_1}{\omega_2} = 36, \frac{\omega_1}{\omega_z} = 51 \) and \( \frac{\omega_2}{\omega_z} = 25 \) (Hz), where \( \omega_{y_1}, \omega_{y_2} \) and \( \omega_z \) are harmonic trapping frequencies, \( V_0 \) is the maximum optical potential and \( w_0 \) is the Gaussian beam waist. However, in this article, we set \( \omega_{y_1} = \omega_{y_2} \) for mathematical simplicity.}:

\[
\frac{m}{2} (\omega_1^2 y^2 + \omega_z^2 z^2) + V_0 \exp \left( -\frac{2|y|^2}{w_0^2} \right), \quad (y, z) \in \mathbb{R}^2 \times \mathbb{R}
\]  

(1.1)

plugging a repulsive Gaussian laser beam in the \( z \)-direction in the middle of a magnetic harmonic trap \cite{38}. An important property of this toroidal trap potential is that it has the minimum value on a ring provided that the Gaussian beam is strong enough. Precisely, if \( V_0 > V_{\text{crit}} := \frac{m \omega_1^4 w_0^4}{4} \), then the potential attains its minimum on the ring \( \{(y, 0) : |y| = r_\ast \} \) with \( r_\ast = \frac{w_0}{\sqrt{2}} (\ln(V_0/V_{\text{crit}}))^{1/2} \), and it is formally expanded as

\[
V_{\text{crit}} \left[ 1 + \ln \left( \frac{V_0}{V_{\text{crit}}} \right) \right] + m \omega_\ast^2 \ln \left( \frac{V_0}{V_{\text{crit}}} \right) (|y| - r_\ast)^2 + \frac{m}{2} \omega_\ast^2 z^2 + \cdots
\]

near the ring.\footnote{If the Gaussian beam is weak \( (V_0 \leq V_{\text{crit}}) \), then the potential can be approximated by a harmonic + quartic potential \( V_0 + \frac{2(V_{\text{crit}}-V_0)}{w_0^2} |y|^2 + \frac{4V_0}{w_0^2} |y|^4 + \frac{m}{2} \omega_\ast^2 z^2 \) near the origin \cite{11, 25, 26}.} Thus, a trapped BEC can be placed on a ring-shaped region if the potential is strong enough or if a Bose gas is sufficiently diluted. As a consequence, a quasi-1D periodic dynamics may arise from the unbounded 3D system, and a BEC becomes easier to analyze as well as it exhibits rich dynamics. In addition, in a toroidal trap, a persistent flow of a BEC with 10-second lifetime is observed \cite{38, 11}; however, quantum vortices will not be discussed in this article.

1.2. Model description. Let \( \omega \geq 1 \) be sufficiently large. Throughout the article, we assume that a smooth function

\[
U_\omega = U_\omega(s) : [-\sqrt{\omega}, \infty) \to [0, \infty)
\]

is close to \( s^2 \) near the origin, and that it has global quadratic lower and upper bounds;

(H1) \(|U_\omega(s) - s^2| \lesssim \frac{1}{\sqrt{\omega}} s^2 \) on \([-\sqrt{\omega}, \sqrt{\omega}]\)

(H2) \(|U_\omega(s)| \sim s^2 \) for all \( s \geq -\sqrt{\omega} \)

where the implicit constants are independent of large \( \omega \geq 1 \) (see Section 1.6 for notations). Then, we consider a general strong toroidal trap potential of the form

\[
\omega U_\omega(\sqrt{\omega}|y| - 1) + \omega^2 z^2, \quad (y, z) \in \mathbb{R}^2 \times \mathbb{R}.
\]

Note that by the assumptions,

\[
\omega U_\omega(\sqrt{\omega}|y| - 1) + \omega^2 z^2 \sim \omega^2 ((|y| - 1)^2 + z^2)
\]

on the disk \( \{(y, 0) : |y| \leq 2 \} \) and it has a lower bound \( \sim \omega^2 ((|y| - 1)^2 + z^2) \) in \( \mathbb{R}^3 \).
Example 1.1. (i) If \( U_\omega(s) = s^2 \), then

\[
\omega U_\omega(\sqrt{\omega}|y| - 1)) + \omega^2 z^2 = \omega^2(|y| - 1)^2 + z^2).
\]

(ii) \( U_\omega(s) = \frac{m\omega}{2}\{(1 + \frac{s}{\sqrt{\omega}})^2 + m e^{\frac{1}{\omega^4}(1-(1+\frac{s}{\sqrt{\omega}})^2)} - (m + 1)\} \) satisfies (H1) and (H2). In this case, the toroidal potential is given by

\[
\omega U_\omega(\sqrt{\omega}|y| - 1)) + \omega^2 z^2 = \frac{m}{2} \left( \omega^2 |y|^2 + \frac{2\omega}{m} \right) + \frac{m^2 \omega^2}{2} e^{\frac{1}{\omega^4}(1-|y|^2)} - \frac{m(m + 1)\omega^2}{2},
\]

which is, up to constant addition, the trapping potential \([13]\) in the experiment \([38]\) with \( \omega^2 = \omega^2, \omega_z^2 = \frac{2m\omega}{m}, m = \frac{\omega^2}{2}, V_{\text{crit}} = \frac{m^2 \omega^2}{2} \) and \( V_0 = V_{\text{crit}}^2 \). Here, the choice of physical parameters seems rather restricted, but that is because the trap potential is chosen to be asymptotically \( \omega^2((|y| - 1)^2 + z^2) \) just for simplicity. Indeed, the potential is allowed to be close to \( \omega^2(c_r^2(|y|-r_s)^2 + c_z^2 z^2) \) without any essential change in analysis.

Then, we have three additional degree of freedom for physical coefficients.

Suppose that a BEC is confined in the above toroidal trap. Then, its mean-field dynamics is described by the nonlinear Schrödinger equation (NLS)

\[
i \partial_t \psi_\omega = -\Delta \psi_\omega + \omega U_\omega(\sqrt{\omega}|y| - 1)) \psi_\omega + \omega^2 z^2 \psi_\omega + \frac{\kappa}{\omega} |\psi_\omega|^2 \psi_\omega,
\]

where \( \psi_\omega = \psi_\omega(t, x) : I(\subset \mathbb{R}) \times \mathbb{R}^3 \to \mathbb{C} \) and \( \omega \gg 1 \) represents the strength of the confinement and the weakness of particle interactions. The normalized coefficient \( \kappa = \pm 1 \) determines the repulsive/attractive pair particle interaction in the condensate; the equation (1.2) is called defocusing (resp., focusing) when \( \kappa = 1 \) (resp., \( \kappa = -1 \)).

By the assumptions, low energy states would be concentrated on the ring \( \{(y, 0) \in \mathbb{R}^2 \times \mathbb{R} : |y| = 1\} \). In order to observe the quasi-1D dynamics in the strong confinement regime, we consider the wave function of the form

\[
\psi_\omega(t, x) = e^{-it\Lambda_\omega} \omega^{-\frac{d}{2}} u_\omega(t, \sqrt{\omega}x),
\]

where \( \Lambda_\omega \) is the lowest eigenvalue for the Schrödinger operator

\[
H_\omega = -\Delta + U_\omega(|y| - \sqrt{\omega}) + z^2,
\]

and deduce the rescaled NLS

\[
i \partial_t u_\omega = \omega(H_\omega - \Lambda_\omega) u_\omega + \kappa \sqrt{\omega} |u_\omega|^2 u_\omega.
\]

The Cauchy problem for the equation (1.5) is locally well-posed in the energy space \( \Sigma = \{u \in H^1(\mathbb{R}^3) : (1 + |x|^2)^{1/2} u \in L^2(\mathbb{R}^3)\} \) equipped with the weighted Sobolev norm

\[
\|u\|_\Sigma := \left\{ \|u\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 + \|x u\|_{L^2(\mathbb{R}^3)}^2 \right\}^{1/2}.
\]

Moreover, solutions preserve the mass

\[
M[u] = \int_{\mathbb{R}^3} |u|^2 dx
\]
and the energy
\[ E_\omega[u] = \frac{\omega}{2} \langle (H_\omega - \Lambda_\omega)u, u \rangle_{L^2(\mathbb{R}^2)} + \frac{\kappa \sqrt{\omega}}{4} \int_{\mathbb{R}^3} |u|^4 \, dx. \] (1.7)

Remark 1.2. By the phase shift (see (1.3)), the lowest eigenvalue \( \Lambda_\omega \) is subtracted in the definition of the energy (1.7). Otherwise, even the energy of the lowest eigenfunction for the Schrödinger operator \( H_\omega \) blows up as \( \omega \to \infty \).

The structure of a toroidal trap naturally leads us to decompose the Schrödinger operator into the radial-axial part and the angular part in cylindrical coordinates,
\[ H_\omega = H^\perp_\omega - \frac{1}{r^2} \partial_\theta^2, \]
where
\[ H^\perp_\omega := -\partial_r^2 - \frac{1}{r} \partial_r - \partial_z^2 + U_\omega(r - \sqrt{\omega}) + z^2 \] (1.8)
acts on the partially radial class \( L^2_{rad}(\mathbb{R}^2) \times L^2(\mathbb{R}) \). We note that by the assumptions, the lowest eigenfunction for \( H^\perp_\omega \) is localized on a large ring \( r = |y| = \sqrt{\omega} \) so that \( -\frac{1}{r} \partial_r \) vanish effectively as \( \omega \to \infty \). As a consequence, the lowest eigenvalue \( \Lambda_\omega \) and the corresponding eigenfunction are approximated by 2 and \( \omega^{-\frac{1}{4}} \Phi_0(|y| - \sqrt{\omega}, z) \) respectively, where \( \Phi_0 = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2+y^2}{2}} \) is the normalized lowest eigenfunction for the 2D hermite operator (see Section 3). Therefore, we may expect that as the potential is strengthened, small energy solutions to the 3D NLS (1.5) get concentrated on \( \omega^{-\frac{1}{4}} \Phi_0(|y| - \sqrt{\omega}, z) \). As a consequence, two degrees of freedom (for \( |y| \) and \( z \)) are frozen and only one degree of freedom (for \( \theta \)) remains.

The purpose of this paper is to justify the emergence of the 1D periodic motion in the aforementioned physical experiment by answering the two problems.

1. Find a large class of global-in-time solutions to the NLS (1.5), including all solutions near the ring soliton in (2). Then, we show that they are concentrated on the ring, and their angular dynamics can be approximately described by the time-dependent 1D periodic NLS.

2. Construct an orbitally stable ring soliton concentrated on the ring whose angular profile is a ground state for the 1D periodic NLS.

Both (1) and (2) concern the dimension reduction to the 1D periodic NLS. The dimension reduction by anisotropic harmonic potentials is verified theoretically in various settings, for example, for cigar-shaped and disc-shaped condensates [8–10, 17–19, 28, 30]. To authors’ knowledge, the dimension reduction for the toroidal trap model (both time-dependent and independent cases) and construction of a stable ring soliton have not been studied in the literature.

1.3. Dimension reduction for the time-dependent equation. The first part of this paper is devoted to the dimension reduction for a class of global-in-time solutions to the time-dependent rescaled NLS (1.5). Indeed, in the defocusing case \( \kappa = 1 \), the conservation laws yield global existence of all solutions in the energy space (see Proposition 6.1). On the
other hand, in the focusing case $\kappa = -1$, the equation has blow-up solutions (see [14] for instance). However, when the trap is strong enough ($\omega \gg 1$), we have a simple criteria for global existence, that is,

\[(H3) \quad \left\langle (H^\perp_\omega - \Lambda_\omega) u_{\omega,0}, u_{\omega,0} \right\rangle_{L^2(\mathbb{R}^3)} \leq \delta \omega\]

for some sufficiently small $\delta > 0$ (see Proposition [5.2]).

**Remark 1.3.** The assumption (H3) says that the portion of high energy states for $H^\perp_\omega$ in the initial data $u_{\omega,0}$ is not too large. This restriction vanishes as $\omega \to \infty$.

Our first main result establishes the emergence of the effective 1D periodic dynamics for such global solutions. It is stated in two parts. First, we show that a 3D global NLS flow $u_\omega(t,x)$ can be approximated by a 1D periodic global flow $v_\omega|| (t,\theta)$. Secondly, we prove that the 1D flow $v_\omega|| (t,\theta)$ weakly converges to a solution to the periodic 1D cubic NLS

\[i \partial_t w = -\partial_x^2 w + \frac{\kappa}{2\pi} |w|^2 w, \quad (1.9)\]

where $w = w(t,\theta) : I(\subset \mathbb{R}) \times S^1 \to \mathbb{C}$.

**Theorem 1.4** (Dimension reduction for the time-dependent 3D NLS (1.5)). Let $\kappa = \pm 1$, and fix $m, E > 0$. Suppose that (H1) and (H2) holds, and that initial data $u_{\omega,0} \in \Sigma$ satisfies

\[M[u_{\omega,0}] = m \quad \text{and} \quad E_\omega[u_{\omega,0}] \leq E. \quad (1.10)\]

In the focusing case $\kappa = -1$, we further assume that (H3) holds for all sufficiently large $\omega \gg 1$ and for some small constant $\delta = \delta(m,E) > 0$ independent of large $\omega \gg 1$ (see Corollary [5.4]). Let $u_\omega(t) \in C([0,\infty); \Sigma)$ be the global solution to the 3D NLS (1.5) with initial data $u_{\omega,0}$. Then, the following hold.

(i) (Dimension reduction to a 1D periodic flow) There exists a global 1D flow $v_\omega|| (t) \in C([0,\infty); H^1(S^1))$ such that

\[\sup_{t \geq 0} \left\| u_\omega(t,x) - v_\omega|| (t,\theta) \left( \omega^{-\frac{1}{4}} \Phi_0(|y| - \sqrt{\omega}, z) \right) \right\|_{L^2(\mathbb{R}^3)} \lesssim \omega^{-\frac{1}{2}}, \quad (1.11)\]

where $\theta$ denotes the angle from the $y_1$-axis on the $y = (y_1, y_2)$-plane and $\Phi_0(s,z) = \frac{1}{\sqrt{\pi}} e^{-s^2 + z^2}$.

(ii) (Derivation of the 1D periodic NLS) By the assumptions on the 3D initial data $u_{\omega,0}$, there exist $w_{\infty,0} \in H^1(S^1)$ and a sequence $\{\omega_j\}_{j=1}^\infty$, with $\omega_j \to \infty$, such that

\[u_{\omega_j,0}(x) - w_{\infty,0}(\theta) \left( \omega_j^{-\frac{1}{4}} \Phi_0(|y| - \sqrt{\omega_j}, z) \right) \rightharpoonup 0 \quad \text{in} \quad H^1(\mathbb{R}^3) \quad (1.12)\]

(see Section 6.3.4 for the construction of $w_{\infty,0}$). Then, for any fixed $T > 0$, we have the weak-* convergence

\[w^* - \lim_{j \to \infty} v_{\omega_j||} (t) = w_\infty(t) \quad \text{in} \quad L^\infty([0,T]; H^1(S^1)), \]
where \( w_\infty(t) \in C([0, \infty); H^1(S^1)) \) is the global solution to the 1D NLS (1.9) with initial data \( w_{\infty,0} \).

**Remark 1.5.**

(i) The dimension reduction (Theorem 1.4 (i)) is proved in strong sense. However, we are currently able to derive the 1D NLS only in weak sense due to a technical difficulty mentioned in Remark 6.4.

(ii) In Theorem 1.4 (ii), the 1D NLS is derived sub-sequentially. That is just because the 1D initial data is prepared as a sub-sequential limit (see (1.12)) under the general assumptions on the 3D initial data ((H3) and (1.10)). Indeed, if we assume more that \( u_{\omega,0}(x) - w_{\infty,0}(\theta) \left( \omega^{-\frac{4}{7}} \Phi_0(|y| - \sqrt{\omega}, z) \right) \xrightarrow{\omega \to \infty} 0 \) in \( H^1(\mathbb{R}^3) \), for instance, if \( u_{\omega,0} \) is factorized as \( w_{\infty,0}(\theta)(\omega^{-\frac{4}{7}} \Phi_0(|y| - \sqrt{\omega}, z)) \) for some fixed profile \( w_{\infty} \in H^1(S^1) \), then we have

\[ v_{\omega,\theta}(t) \xrightarrow{\omega \to \infty} w_{\infty}(t) \text{ in } L^\infty([0,T]; H^1(S^1)). \]

1.4. **Construction of constrained energy minimizers and their dimension reduction.** In the second part of the paper, we aim to construct stable low energy steady states and to prove their dimension reduction. Before presenting our second main result, we note that in the defocusing case \( \kappa = 1 \), a ground state under a mass constraint can be constructed by standard calculus of variation techniques, and its key properties, such as uniqueness, orbital stability and dimension reduction, can be proved relatively easily. Thus, for readers’ convenience, we put aside the discussion on the defocusing case in Appendix B. From now on, we only consider the focusing case \( \kappa = -1 \).

On the other hand, in the focusing case, a mass constraint energy minimization problem is not properly formulated, because the 3D cubic NLS is mass-supercritical and the energy is not bounded from below. Therefore, motivated by earlier works [6, 29, 30] on dimension reduction in different settings, we instead consider the following *constrained* energy minimization problem

\[
J^{(3D)}(m) := \min \left\{ E_\omega[u] : u \in \Sigma, \ M[u] = m, \ (H_\omega^\perp - \Lambda_\omega)u, u \right\}_{L^2(\mathbb{R}^3)} \leq \delta \omega \right\} \tag{1.13}
\]

where the mass and the energy are given by (1.6) and (1.7) with \( \kappa = -1 \) respectively, and \( \delta > 0 \) is a sufficiently small number. Note that we here impose the constraint for the time-dependent problem (see (H3)). Similarly as before, since we consider sufficiently large \( \omega \geq 1 \) but \( \delta > 0 \) will be chosen independently of large \( \omega \geq 1 \), this additional constraint restricts the portion of high energy states with respect to \( H_\omega^\perp \) but the restriction gets weaker as \( \omega \to \infty \).

For this modified energy minimization problem, we construct a minimizer.

**Theorem 1.6** (Existence of a constrained energy minimizer; focusing case). Let \( \kappa = -1 \) and let \( \delta > 0 \) be a small number chosen in Corollary 5.4. We assume that \( \omega \gg 1 \) is
sufficiently large. Then, for any minimizing sequence \( \{u_n\}_{n=1}^{\infty} \) to the variational problem (1.13), there exist \( \{O_n\}_{n=1}^{\infty} \subset \text{SO}(2) \) and \( \{\gamma_n\}_{n=1}^{\infty} \subset \mathbb{R} \) such that passing to a subsequence,
\[
\lim_{n \to \infty} \|e^{i\gamma_n} u_n(O_n, y, z) - Q_\omega\|_{\Sigma} = 0
\]
for some positive minimizer \( Q_\omega \). Moreover, \( Q_\omega \) solves the nonlinear elliptic equation
\[
\omega(H_\omega - \Lambda_\omega)Q_\omega - \sqrt{\omega}Q_\omega^3 = -\mu_\omega Q_\omega
\]
with a Lagrange multiplier \( \mu_\omega \in \mathbb{R} \).

For the constrained minimizer in Theorem 1.6, we are concerned with its dimension reduction and orbital stability. Indeed, analogously to the time-dependent model, the 3D problem (1.13) is closely related to the 1D energy minimization problem
\[
J^{(1D)}(\infty)(m) = \inf \left\{ E_\infty(w) : w \in H^1(S^1) \text{ and } \|w\|_{L^2(S^1)}^2 = m \right\},
\]
where
\[
E_\infty[w] = \frac{1}{2} \|\partial_\theta w\|_{L^2(S^1)}^2 - \frac{1}{8\pi} \|w\|_{L^4(S^1)}^4.
\]
We recall from [27] that the variational problem \( J^{(1D)}(\infty)(m) \) occupies a positive minimizer \( Q_\infty \) with \( Q_\infty(0) = \max_{\theta \in S^1} Q_\infty(\theta) \), solving the Euler-Lagrange equation
\[
-\partial_\theta^2 Q_\infty - \frac{1}{2\pi} Q_\infty^3 = -\mu_\infty Q_\infty.
\]
Moreover, \( Q_\infty \) is a unique minimizer up to phase shift and spatial translation. Indeed, \( Q_\infty \) is given by a dnoiadal function when \( m > \frac{2\pi^2}{2} \) while it is constant when \( 0 < m \leq \frac{2\pi^2}{2} \) (see Proposition A.1). More detailed properties of the ground state is provided in Appendix A.

The next theorem establishes the dimension reduction from the 3D to the 1D minimizer.

**Theorem 1.7** (Dimension reduction for a constrained minimizer). Under the assumption in Theorem 1.6, let \( Q_\omega \) be the positive minimizer for the constrained problem \( J^{(3D)}_\omega(m) \). Then, there exists \( \{O_\omega\}_{\omega \gg 1} \subset \text{SO}(2) \) such that
\[
\left\| Q_\omega(O_\omega y, z) - Q_\infty(\theta) \left( \chi(|y|) e^{-\frac{s^2}{2}} \Phi_0(|y| - \sqrt{\omega}, z) \right) \right\|_{\Sigma} + |\mu_\omega - \mu_\infty| \to 0 \quad \text{as } \omega \to \infty,
\]
where \( \Phi_0(s, z) = \frac{1}{\sqrt{\pi}} e^{-\frac{s^2}{2} + |z|^2} \) and \( \chi : [0, \infty) \to [0, 1] \) is a smooth function such that \( \chi \equiv 0 \) on \([0, 1]\) and \( \chi \equiv 1 \) on \([2, \infty)\).

By the variational characterization together with global existence for the time-dependent NLS (1.5) (Proposition 6.2), it immediately follows that the set of 3D minimizers in Theorem 1.6 is orbitally stable (see [15]). However, for the orbital stability of a minimizer by itself, the possibility of transforming one minimizer to another should be eliminated.

Our last main result asserts that if the confinement is strong enough, this can be done by proving uniqueness up to symmetries.
**Theorem 1.8** (Uniqueness and orbital stability of a constrained minimizer). Let $\omega \gg 1$ be sufficiently large, and suppose that $m \neq 2\pi^2$. Under the assumption in Theorem 1.7, let $Q_\omega$ be the minimizer for the constrained problem $J_\omega^{(3D)}(m)$.

(i) (Uniqueness) The minimizer $Q_\omega$ is unique up to a rotation on the plane and phase shift. In other words, any minimizer for the problem $J_\omega^{(3D)}(m)$ can be expressed as $e^{i\gamma} Q_\omega(Oy, z)$ for some $O \in SO(2)$ and $\gamma \in S^1$.

(ii) (Orbital stability) For any $\epsilon > 0$, there exists $\delta > 0$ such that if $\|u_0 - Q_\omega\|_\Sigma < \delta$, then the global solution $u_\omega(t) \in C_t(\mathbb{R}, \Sigma)$ to 3D NLS (1.5) with the initial data $u_0$ satisfies

$$\inf_{O \in SO(2), \gamma \in S^1} \|u_\omega(t, x) - e^{i\gamma} Q_\omega(Oy, z)\|_\Sigma < \epsilon \quad \text{for all} \ t \geq 0.$$ 

**Remark 1.9.** The assumption $m \neq 2\pi^2$ is from that we do not know the desired coercivity estimate for the linearized operator at the 1D periodic ground state $Q_\infty$ (see Remark A.2).

1.5. **Ideas of the proofs, and the outline of the paper.** Since the dimension reduction by a toroidal trap model has been less explored, some considerable efforts are taken into developing analysis tools from the basic ones (Section 2-5). First of all, due to the geometry of the trapping potential, it is natural to employ cylindrical coordinates and translate in the axial direction so that localization is achieved at the origin in the new coordinates. In Section 2, we introduce the reformulated problem and function spaces accordingly.

Next, we note that in our setting, the lowest eigenstate for the operator $H_{\omega}^{\perp}$ has a different $\omega \to \infty$ behavior from other higher eigenstates (see the energy functional (1.7) and its reformulation (2.11)). Thus, for the proof, we need detailed properties for the lowest eigenstate and the spectral projection on it (as well as its orthogonal complement). An important technical remark is that when we take the projection to the lowest eigenstate, we need to truncate out near the $z$-axis (or $s = -\sqrt{\omega}$ in the new coordinates), because the projected state may have infinite energy (see Remark 4.1). Fortunately, it turns out that this truncation is not very harmful, because the lowest eigenstate enjoys Gaussian decay. In Section 3, we provide the properties of the eigenstates including its decay and convergence to the lowest eigenstate to the hermite operator. In Section 4, we introduce the truncated projection operator, and present its useful various mapping properties.

In Section 5, based on the previous two sections, we introduce our key analysis tool, that is, the refined Gagliardo-Nirenberg inequality (Proposition 5.1 and Corollary 5.3); by (2.1), it is equivalent to

\[
\|u\|_{L^4(\mathbb{R}^3)} \lesssim \|u\|_{L^2(\mathbb{R}^3)} \left(\sqrt{\|H_{\omega}^{\perp} - \Lambda_{\omega} u\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_\theta u\|_{L^2(\mathbb{R}^3)}^2} + \frac{1}{\sqrt{\omega}} \|u\|_{L^2(\mathbb{R}^3)}\right)^{\frac{1}{2}}
\]

(1.15)

The inequality (1.15) is improved in that compared to the standard inequality $\|u\|_{L^4(\mathbb{R}^3)} \lesssim \|u\|_{L^2(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)}^2$, the degree of the angular derivative term $\|\partial_\theta u\|_{L^2(\mathbb{R}^3)}$ is reduced in the
upper bound. Therefore, under the assumptions (1.10) and (H3), a sub-quadratic bound
\[ \|u\|_{L^4(R^3)}^4 \lesssim \sqrt{\delta m \omega} \left( \sqrt{H_\omega - \Lambda_\omega} u \right)_{L^2(R^3)}^2 + m^2 \|\partial_\theta u\|_{L^2(R^3)} + \frac{m^2}{\sqrt{\omega}} \]
becomes available. By the inequality, the sub-critical nature can be captured under the constraint (H3) for the super-critical problem. Note also that the inequality immediately implies concentration to the lowest eigenstate for $H_\omega^\perp$ localized on a ring (see Corollary 5.4 for the forbidden region in the function space due to (H3)). Indeed, similar refined inequalities and their consequences have been employed in different settings [29, 30].

After preparing tools, in Section 6 we give a proof of the first main result (Theorem 1.4). As mentioned above, as an almost direct consequence of the refined Gagliardo-Nirenberg inequality, we prove global existence for solutions considered in the theorem and the dimension reduction (Theorem 1.4 (i)), and obtain a uniform-in-$\omega$ bounds for global solutions. Then, using the uniform bounds, we derive the time-dependent 1D periodic NLS (Theorem 1.4 (ii)).

The last two sections (Section 7 and 8) are devoted to the constrained minimization problem (1.13). In Section 7 we construct a minimizer (Theorem 1.6) and prove its dimension reduction (Theorem 1.7). As mentioned above, the energy minimization problem is super-critical without the constraint $\langle (H_\omega^\perp - \Lambda_\omega) u, u \rangle_{L^2(R^3)} \leq \delta \omega$. However, with the additional constraint, it has sub-critical nature can be captured by the refined Gagliardo-Nirenberg inequality (1.15) so that a minimizer is constructed by concentration-compactness principle. Moreover, we can show that higher eigenstates of an energy minimizer vanish. Consequently, together with the energy convergence, the dimension reduction limit to the 1D periodic ground state (Theorem 1.7) follows. Finally, in Section 8 we establish the local uniqueness of a minimizer (Theorem 1.8). In the proof, an important step is to obtain a coercivity property of a linearized operator at a 3D minimizer (Proposition 8.1). That can be done by transferring the coercivity of the linearized operator at the 1D periodic ground state via the dimension reduction limit.

In Appendix A we review the properties of the 1D periodic ground state, and give a proof of the coercivity of its linearized operator, which is a key ingredient for the coercivity of the linearized operator at a 3D minimizer. In Appendix B we present the analogous results in the defocusing case.

1.6. Notations. We denote $A \lesssim B$ (resp., $A \gtrsim B$) ignoring the implicit constant $C > 0$ in the inequality $A \leq CB$ (resp., $A \geq CB$) unless there is a confusion. In the same context, $\frac{1}{C} B \leq A \leq CB$ is expressed as $A \sim B$. In this article, we employ various Lebesgue spaces of the form $L^p(S, g(x) dx)$ with $S \subset \mathbb{R}^d$ and weighted measure $g(x) dx$ for some non-negative function $g$. If the domain and the variables for the given function space are clearly given in the context, $L^p(S, g(x) dx)$ is abbreviated to $L^p(g dx)$. 
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2. **Reformulation of the problem**

For a clearer picture for the dimension reduction, we derive a reformulated problem making suitable changes of variables via the cylindrical coordinates.

2.1. **Modified mass and energy.** Since we are concerned with wave functions localized on a ring of radius $\sqrt{\omega}$, we convert everything into the cylindrical coordinates and then translate in the axial distance axis, introducing the $(s, z, \theta)$-coordinates given by

$$(x, y, z) = \left((s + \sqrt{\omega}) \cos \theta, (s + \sqrt{\omega}) \sin \theta, z\right) : [-\sqrt{\omega}, \infty) \times \mathbb{R} \times S^1 \to \mathbb{R}^3.$$ 

Note that the $(s, z, \theta)$-variable domain formally converges to $\mathbb{R}^2 \times S^1$ as $\omega \to \infty$. In the new coordinates, the 3D toroidal potential $U_\omega(|y| - \sqrt{\omega}) + z^2$ becomes effectively a 2D harmonic potential $s^2 + z^2$.

Accordingly, we change the variables as

$$v(s, z, \theta) = \omega^{\frac{3}{2}} u(s + \sqrt{\omega}, z, \theta), \quad |y| = s + \sqrt{\omega},$$

and we define the modified mass

$$M_\omega[v] := \int_{S^1} \int_{-\infty}^{\infty} \int_{-\sqrt{\omega}}^{\infty} |v(s, z, \theta)|^2 \sigma_\omega(s) ds dz d\theta$$

and the modified energy

$$E_\omega[v] := \frac{\omega}{2} \int_{S^1} \int_{-\infty}^{\infty} \int_{-\sqrt{\omega}}^{\infty} \left(|\nabla (s, z)v|^2 + (U_\omega(s) + z^2)|v|^2 - \Lambda_\omega|v|^2\right) \sigma_\omega(s) ds dz d\theta$$

$$+ \frac{1}{2} \int_{S^1} \int_{-\infty}^{\infty} \int_{-\sqrt{\omega}}^{\infty} \frac{1}{\sigma_\omega(s)} |\partial_\theta v|^2 \sigma_\omega(s) ds dz d\theta + \frac{\kappa}{4} \int_{S^1} \int_{-\infty}^{\infty} \int_{-\sqrt{\omega}}^{\infty} |v|^4 \sigma_\omega(s) ds dz d\theta$$

with the weight function $\sigma_\omega : [-\sqrt{\omega}, \infty) \to [0, \infty)$ is given by

$$\sigma_\omega = \sigma_\omega(s) = 1 + \frac{s}{\sqrt{\omega}}.$$

**Remark 2.1.** The mass and the energy, given (1.6) and (1.7), remain same by the above modification and the transformation (2.1); $M[u] = M_\omega[v]$ and $E_\omega[u] = E_\omega[v]$.

**Remark 2.2.** Our reformulation requires to deal with functions on the rather uncommon domain $[-\sqrt{\omega}, \infty) \times \mathbb{R} \times S^1$ with the weight $\sigma_\omega(s) = 1 + \frac{s}{\sqrt{\omega}}$, but we may analyze the problem invoking that they formally converge to $\mathbb{R}^2 \times S^1$ and 1, respectively.
2.2. Function spaces. From now on, we minimize the modified energy under the modified constraints. Thus, it is convenient to employ the following function spaces which fit better into our reformulated problem.

For notational convenience, given an $s$-variable weight function $g = g(s) : [-\sqrt{\omega}, \infty) \rightarrow [0, \infty)$ and $1 \leq p \leq \infty$, we abbreviate the weighted space $L^p([-\sqrt{\omega}, \infty) \times \mathbb{R} \times S^1, g(s) ds dz d\theta)$ to $L^p(g)$ equipped with the norm

$$\|v\|_{L^p(g)} = \begin{cases} \left\{ \int_{S^1} \int_{-\infty}^{\infty} \int_{-\sqrt{\omega}}^{\infty} |v(s, z, \theta)|^p g(s) ds dz d\theta \right\}^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \text{ess sup} \left( (s, z, \theta) \in [-\sqrt{\omega}, \infty) \times \mathbb{R} \times S^1 \right) |v(s, z, \theta)| & \text{if } p = \infty. \end{cases}$$

(2.5)

In particular, $L^2(g)$ is the Hilbert space with the inner product

$$\langle v_1, v_2 \rangle_{L^2(g)} = \int_{S^1} \int_{-\infty}^{\infty} \int_{-\sqrt{\omega}}^{\infty} v_1(s, z, \theta) \overline{v_2(s, z, \theta)} g(s) ds dz d\theta.$$

To be specific, we will mostly employ the two weight functions $\sigma_\omega(s)$ and $\frac{1}{\sigma_\omega(s)}$. Note that both $L^p(\sigma_\omega)$ and $L^p(\frac{1}{\sigma_\omega})$ formally converge to $L^p(\mathbb{R}^2 \times S^1)$, since $\sigma_\omega(s) = 1 + \frac{s}{\sqrt{\omega}} \rightarrow 1$.

We define the differential operator

$$H^{(2D)}_{\omega} := -\partial_s^2 - \frac{1}{\sqrt{\omega} \sigma_\omega} \partial_s - \partial_z^2 + U_\omega(s) + z^2$$

(2.6)

as a quadratic form acting on the weighted space $L^2(\sigma_\omega ds dz)$. The spectral properties of the operator will be presented in Section 3. Indeed, it will be shown that the lowest eigenvalue $\Lambda_\omega$ is simple and it converges to 2 as $\omega \rightarrow \infty$ (see Corollary 3.3).

Referring to the first term in the modified energy (2.3) as well as the additional constraint, we introduce the semi-norm

$$\|v\|_{\Sigma_{\omega(s, z)}} := \langle (H^{(2D)}_{\omega} - \Lambda_\omega)v, v \rangle_{L^2(\sigma_\omega)}^{\frac{1}{2}},$$

(2.7)

and define

$$\|v\|_{\Sigma_\omega} := \left\{ \|v\|_{\Sigma_{\omega(s, z)}}^2 + \frac{1}{\omega} \|\partial_\theta v\|_{L^2(\frac{1}{\sigma_\omega})}^2 \right\}^{\frac{1}{2}}.$$

(2.8)

3With abuse of notation, we denote $L^\infty(g) \equiv L^\infty(1)$.
4By integration by parts,

$$\|v\|_{\Sigma_{\omega(s, z)}}^2 = \int_{S^1} \int_{-\infty}^{\infty} \int_{-\sqrt{\omega}}^{\infty} \left( |\nabla_{(s, z)} v|^2 + (U_\omega(s) + z^2) |v|^2 - \Lambda_\omega |v|^2 \right) \sigma_\omega(s) ds dz d\theta,$$

provided that the both sides are finite.
Finally, collecting all quadratic terms in the modified mass and the energy, we denote by $\Sigma_\omega$ the Hilbert space with the norm
\[
\|v\|_{\Sigma_\omega} := \left\{(1 + \Lambda_\omega)\|v\|_{L^2(\sigma_\omega)}^2 + \|v\|_{\Sigma_\omega}^2\right\}^{\frac{1}{2}}.
\] (2.9)

2.3. Reformulated variational problem. By the definitions of the function spaces and the norms introduced in the previous subsection, the modified mass and the modified energy can be expressed in a compact form as
\[
\mathcal{M}_\omega[v] = \|v\|_{L^2(\sigma_\omega)}^2
\] (2.10) and
\[
\mathcal{E}_\omega[v] = \frac{\omega}{2}\|v\|_{\Sigma_\omega}^2 + \frac{\kappa}{4}\|v\|_{L^4(\sigma_\omega)}^4 = \frac{\omega}{2}\|v\|_{\Sigma_\omega(s,z)}^2 + \frac{1}{2}\|\partial_\theta v\|_{L^4(\sigma_\omega)}^4 + \frac{\kappa}{4}\|v\|_{L^4(\sigma_\omega)}^4.
\] (2.11)

Consequently, by (2.11), the energy minimization problem (1.13) is rephrased as
\[
\mathcal{J}_0^{(3D)}(m) := \min \left\{ \mathcal{E}_\omega[v] : v \in \Sigma_\omega, \mathcal{M}_\omega[v] = m \text{ and } \|v\|_{\Sigma_\omega(s,z)} \leq \delta \sqrt{\omega} \right\}.
\] (2.12)

Remark 2.3. The minimization problem (2.12) is in essence nothing but a reformulation of the original problem (1.13), but it has many advantages to observe the strong confinement effect and to catch a sub-critical nature of the super-critical problem. Indeed, the energy $\mathcal{E}_\omega(v)$ is formally approximated by
\[
\frac{\omega}{2} \langle (-\Delta_{(s,z)} + s^2 + z^2 - 2)v, v \rangle_{L^2(\mathbb{R}^2 \times \mathbb{S}^1)} + \frac{1}{2}\|\partial_\theta v\|_{L^4(\mathbb{R}^2 \times \mathbb{S}^1)} + \frac{\kappa}{4}\|v\|_{L^4(\mathbb{R}^2 \times \mathbb{S}^1)}^4.
\]

Thus, an energy minimizer is expected to be concentrated at a factorized state $\Phi_\infty(s,z)w(\theta)$, where $\Phi_\infty(s,z) = \frac{1}{\sqrt{\pi}}e^{-\frac{s^2+z^2}{2}}$ is the lowest eigenstate for the 2d hermite operator $-\Delta_{(s,z)} + s^2 + z^2$ (with eigenvalue 2). Subsequently, the minimum energy is further reduced to
\[
\mathcal{E}_\omega[w(\theta)\Phi_\infty(s,z)] \approx \frac{1}{2\sqrt{\pi}} \int_{\mathbb{S}^1} |\partial_\theta w|^2 + \frac{\kappa}{8\pi} \int_{\mathbb{S}^1} |w|^4 d\theta = E_\infty[w].
\]

On the other hand, the additional constraint $\|v\|_{\Sigma_\omega(s,z)} \leq \delta \sqrt{\omega}$ gets weaker. Therefore, a ground state on the unit circle is derived in the strong confinement regime.

Accordingly, the main results in the second part of this paper are reworded as follows.

Theorem 2.4 (Existence of a constrained energy minimizer; reformulated). Let $\kappa = -1$ and let $\delta > 0$ be a small number chosen in Corollary 5.4. We assume that $\omega \gg 1$ is sufficiently large. For any minimizing sequence $\{v_n\}_{n=1}^{\infty}$ of the variational problem (2.12), there exists $\{\gamma_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $\{\theta_n\}_{n=1}^{\infty} \subset \mathbb{S}^1$ such that passing to a subsequence,
\[
e^{i\gamma_n}v_n(s,z,\theta-\theta_n) \to Q_\omega > 0 \text{ in } \Sigma_\omega
\]

5By (2.1),
\[
\|v\|_{\Sigma_\omega}^2 = \int_\mathbb{R}^3 |\nabla u|^2 + (U_\omega(|y| - \sqrt{\omega}) + z^2 + 1)|u|^2 dx.
\]
and $Q_\omega$ solves
\[ \omega(\mathcal{H}_\omega^{(2D)} - \Lambda_\omega) Q_\omega - \frac{1}{\sigma_\omega} \partial_s^2 Q_\omega - Q_\omega = -\mu_\omega Q_\omega. \]

**Theorem 2.5** (Dimension reduction; reformulated). A minimizer $Q_\omega$ constructed in Theorem 2.4 satisfies
\[ \lim_{\omega \to \infty} \|Q_\omega(s, z, \theta) - Q_{\infty}(\theta)\chi_\omega(s)\Phi_0(s, z)\|_{\Sigma_\omega} = 0, \]
where $\Phi_0(s, z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2 + z^2}{2}}$ and $\chi_\omega = \chi(\cdot + \sqrt{\omega})$ for a smooth cut-off $\chi : [0, \infty) \to [0, 1]$ such that $\chi \equiv 0$ on $[0, 1]$ and $\chi \equiv 1$ on $[2, \infty)$.

**Theorem 2.6** (Uniqueness; reformulated). Let $\omega \gg 1$ be sufficiently large, and suppose that $m \neq 2\pi$. For the problem (2.12), every minimizer is of the form $e^{i\gamma} Q_\omega(s, z, \theta - \theta_0)$ for some $\gamma \in \mathbb{R}$ and $\theta_0 \in S^1$, where $Q_\omega$ is a positive minimizer obtained in Theorem 2.4.

3. **Spectral properties of the Schrödinger operator $\mathcal{H}_\omega^{(2D)}$**

Before entering into our main nonlinear problem, we study spectral properties of the differential operator
\[ \mathcal{H}_\omega^{(1D)} = -\partial_s^2 - \frac{1}{\sqrt{\omega} \sigma_\omega} \partial_s + U_\omega(s) \quad \text{(resp., } \mathcal{H}_\omega^{(2D)} = \mathcal{H}_\omega^{(1D)} - \partial_z^2 + z^2) \] (3.1)
acting on the Hilbert space
\[ L^2(\sigma_\omega ds) = L^2([-\sqrt{\omega}, \infty), \sigma_\omega ds) \quad \text{(resp., } L^2(\sigma_\omega ds dz) = L^2([-\sqrt{\omega}, \infty) \times \mathbb{R}, \sigma_\omega ds dz)). \]

**Remark 3.1.** (i) The properties of the 2D operator immediately follows from those of the 1D operator, because the 2D operator is separated as $\mathcal{H}_\omega^{(2D)} = \mathcal{H}_\omega^{(1D)} \otimes I_{L^2(\mathbb{R}, dz)} + I_{L^2(\sigma_\omega ds)} \otimes \mathcal{H}_\omega^{(1D)}$ and the 1D hermite operator $\mathcal{H}_\omega^{(1D)} = -\partial_z^2 + z^2$ is well-known.

(ii) By the relation (2.1), the operator $\mathcal{H}_\omega^{(2D)}$ is equivalent to the Schrödinger operator $-\Delta_x + U_\omega(|y| - \sqrt{\omega}) + z^2$ acting on the partially radial class $L^2_{\text{rad}}(\mathbb{R}^2_y) \times L^2(\mathbb{R}^2_z).

(iii) By (i), $\mathcal{H}_\omega^{(2D)}$ has a positive normalized ground state $\Phi_\omega = \phi_\omega(s) \phi_\infty(z)$, where $\phi_\omega(s)$ (resp., $\phi_\infty(z)$) is the $L^2$-normalized ground state for $\mathcal{H}_\omega^{(1D)}$ (resp., $\mathcal{H}_\omega^{(1D)}$) corresponding to the lowest eigenvalue $\lambda_\omega$ (resp., $\lambda_\infty$). By (ii), referring to the equivalent radially symmetric $\mathbb{R}^2$ problem (see [37], Section XIII.11 and XIII.12 for instance), we have that the normalized ground state $\phi_\omega$ is unique up to phase shift, and $\phi_\omega$ and $\phi_\omega'$ decrease exponentially. Moreover, $\phi_\omega(-\sqrt{\omega}) > 0$, which corresponds to the fact that the ground state for $-\Delta_y + U_\omega(|y| - \sqrt{\omega})$ in radial class is not zero at the origin. Our goal here is to obtain properties of eigenfunctions which hold uniformly in $\omega \gg 1$.

The main result of this section provides the asymptotics of the lowest eigenvalue $\lambda_\omega$ and the ground state $\phi_\omega$. 
Proposition 3.2 (1D ground state). Suppose that $U_\omega$ satisfies (H1) and (H2). Let $\lambda_\omega < \lambda'_\omega$ be the two lowest eigenvalues for the Schrödinger operator $H^{(1D)}_\omega$, and let $\phi_\omega$ be its unique positive $L^2(\sigma_\omega ds)$ normalized ground state. Then there exists $\omega_0 > 0$ such that, for any $c \in (0, \omega_0)$, there exists $\omega_c \geq 1$ such that if $\omega \geq \omega_c$, the following hold.

(i) (Lowest eigenvalue asymptotic)
\[
\lambda_\omega = \min \left\{ \mathcal{E}^{(1D)}_\omega(v) := \frac{1}{2} \langle H^{(1D)}_\omega v, v \rangle_{L^2(\sigma_\omega)} : \|v\|_{L^2(\sigma_\omega ds)} = 1 \right\} = 1 + O(\omega^{-\frac{1}{2}}).
\]

(ii) (Exponential decay)
\[
\phi_\omega(s) + \left| \phi'_\omega(s) \right| \lesssim e^{-c s^2} \text{ for all } s \geq -\sqrt{\omega}.
\]

Moreover, we have
\[
\| (1 - \chi_\omega)\phi_\omega \|_{L^2(\sigma_\omega ds)} \lesssim e^{-c_1 \omega}, \quad \| \chi_\omega \phi_\omega \|_{L^2(\frac{1}{\omega_\omega} ds)} = 1 + O(\omega^{-\frac{1}{2}}),
\]
where $\chi_\omega$ is a cut-off such that $\chi_\omega(s) = 0$ on $[-\sqrt{\omega}, -\sqrt{\omega} + 1]$ and $\chi_\omega(s) = 1$ for $s \geq -\sqrt{\omega} + 2$, precisely given in Theorem 2.7.

(iii) (Convergence of the 1D ground state)
\[
\int_{-\infty}^{\infty} \left\{ (\phi_\omega - \phi_\infty)^2 + (\phi'_\omega - \phi'_\infty)^2 + s^2 (\phi_\omega - \phi_\infty)^2 \right\} ds = O(\omega^{-1})
\]
and
\[
\int_{-\infty}^{\infty} (\phi_\omega - \phi_\infty)^4 ds = O(\omega^{-2}),
\]
where $\phi_\infty(z) = \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{\omega}}$.

(iv) (Spectral gap asymptotic)
\[
\lambda'_\omega - \lambda_\omega = 2 + O(\omega^{-\frac{1}{2}}).
\]

By Remark 3.1 (i), the following properties of the 2D operator are immediately obtained.

Corollary 3.3 (2D ground state). Suppose that $U_\omega$ satisfies (H1) and (H2). Let $\Lambda_\omega < \Lambda'_\omega$ be the first two lowest eigenvalues for the Schrödinger operator $H^{(2D)}_\omega$, and let $\Phi_\omega$ be its unique positive $L^2(\sigma_\omega ds)$ normalized ground state. Then, we have
\[
\Phi_\omega(s, z) = \phi_\omega(s) \phi_\infty(z), \quad \Lambda_\omega = \lambda_\omega + 1 = 2 + O(\omega^{-\frac{1}{2}}) \quad \text{and} \quad \Lambda'_\omega = 4 + O(\omega^{-\frac{1}{2}}).
\]

Proof of Corollary 3.3 assuming Proposition 3.2. By the non-degeneracy of ground state solution for $H^{(2D)}_\omega$ (see [37] Theorem XIII.47), it is obvious that $\Lambda_\omega = \lambda_\omega + 1$ and $\Phi_\omega(s, z) =$

---

\footnote{By integration by parts,}
\[
\langle H^{(1D)}_\omega v, v \rangle_{L^2(\sigma_\omega ds)} = \int_{-\infty}^{\infty} \left\{ |\partial_s v|^2 + U_\omega(s) |v|^2 \right\} \sigma_\omega(s) ds dz d\theta.
\]
and \( \omega \). Moreover, by [31, Theorem XIII.47] and separation of the variables (see Remark 3.1), we have \( \lambda'_\omega = \min\{\lambda'_\omega + 1, \lambda_\omega + 3\} \to 4 \), because 3 is the second lowest eigenvalue for \(-\partial_z^2 + z^2\). \( \square \)

The proof of the proposition 3.2 will be broken into several steps. First, we obtain an upper bound on the lowest energy level.

**Lemma 3.4** (Upper bound on the ground state energy \( \lambda_\omega \)). *In Proposition 3.2 we have*

\[
\lambda_\omega \leq 1 + O(\omega^{-\frac{1}{2}}).
\]

**Proof.** Let \( \chi_\omega \) be a cut-off given in Theorem 2.5. Then, direct calculation with \( \sigma_\omega = 1 + \frac{1}{\sqrt{\omega}} \) yields \( \|\chi_\omega \phi_\infty\|_{L^2(\sigma_\omega ds)} = 1 + O(\omega^{-\frac{1}{2}}) \). Similarly but by the assumptions on \( U_\omega \), one can show that \( \langle H_\omega^{(1D)}(\chi_\omega \phi_\infty), \chi_\omega \phi_\infty \rangle_{L^2(\sigma_\omega ds)} = \langle (-\partial_s^2 + s^2)\phi_\infty, \phi_\infty \rangle_{L^2(\mathbb{R})} + O(\omega^{-\frac{1}{2}}) = 1 + O(\omega^{-\frac{1}{2}}) \). Thus, we conclude that \( \lambda_\omega \leq \mathcal{E}_\omega^{(1D)}(\chi_\omega \phi_\infty) = 1 + O(\omega^{-\frac{1}{2}}) \).

Next, following the argument of Agmon [1], we prove the decay of the ground state.

**Proof of Proposition 3.2 (ii).** We claim that there exists \( \alpha_0 > 0 \) such that for any \( c \in (0, \alpha_0) \), if \( \omega \geq 1 \) is large enough, then

\[
\phi_\omega(s) \lesssim \sigma_\omega(s)^{-\frac{1}{2}} e^{-cs^2} \quad \text{for all } s > -\sqrt{\omega}. \quad (3.7)
\]

For the proof, we take \( c \in (0, \alpha_0) \), and let \( g \) be a bounded smooth function on \([-\sqrt{\omega}, \infty)\) with \( g(s) = 1 \) for large \( s > 1 \), where \( \alpha_0 \) and \( g \) will be chosen later. Then, a direct calculation with \( H_\omega^{(1D)} \phi_\omega = \lambda_\omega \phi_\omega \) yields

\[
\langle (H_\omega^{(1D)} - \lambda_\omega)(\phi_\omega g), \phi_\omega g \rangle_{L^2(\sigma_\omega ds)} = \int_{-\sqrt{\omega}}^{\infty} (-2\phi_\omega \phi'_\omega gg' - \phi^2_\omega gg''')\sigma_\omega - \frac{1}{\sqrt{\omega}} \phi^2_\omega gg'ds
\]

\[
= \int_{-\sqrt{\omega}}^{\infty} (\phi_\omega g')^2 \sigma_\omega ds. \quad (3.8)
\]

where in the second identity, we did integration by parts for \(-\phi^2_\omega gg'' \sigma_\omega \). Hence, we have

\[
0 \geq -\int_{-\sqrt{\omega}}^{\infty} (\phi_\omega g')^2 \sigma_\omega ds \geq \int_{-\sqrt{\omega}}^{\infty} (U_\omega - \lambda_\omega)g^2 - (g')^2 \phi^2_\omega \sigma_\omega ds,
\]

where we used integration by parts (with Remark 3.1 (iii)). For large \( L \geq \sqrt{\omega} \), we take \( g = e^{f_L}, \) where \( f_L = cs^2\eta(\frac{s}{\sqrt{\omega}}) \) and \( \eta \) is a smooth cutoff such that \( \eta \) is supported on \([-\frac{1}{2}, \frac{1}{2}]\) and \( \eta \equiv 1 \) on \([-\frac{1}{4}, \frac{1}{4}]\). Then, it follows that

\[
0 \geq -\| (e^{f_L} \phi_\omega) |angle_{L^2(\sigma_\omega ds)}^2 \geq \int_{-\sqrt{\omega}}^{c} (U_\omega - \lambda_\omega - (f'_L)^2 \phi^2_\omega) \sigma_\omega ds.
\]

Note that by the assumption (H2), \( U_\omega \) has a quadratic lower bound. Thus, if \( \alpha_0 > 0 \) small and \( \omega \geq 1 \) is large enough, there exists \( R \in (1, \sqrt{\omega}) \), independent of \( \omega \geq 1 \) and \( L \geq \sqrt{\omega} \), such that \( U_\omega - \lambda_\omega - (f'_L)^2 \geq 1 \) for all \( |s| \geq R \), while there exists \( C_R > 0 \), independent of
ω ≥ 1 and $L ≥ \sqrt{\omega}$, such that $|U_ω - λ_ω - (f'_L)^2| e^{2f_L} ≤ C_R^2 < ∞$ for all $|s| ≤ R$. Thus, it follows that

$$∥1_{|s| ≥ R} e^{f_L} φ_ω∥_{L^2(σ_ω ds)}^2 ≤ \int_{−\sqrt{ω}}^∞ 1_{|s| ≥ R} \{U_ω - λ_ω - (f'_L)^2\} (e^{f_L} φ_ω)^2 σ_ω ds$$
$$≤ - \int_{−∞}^∞ 1_{|s| ≤ R} \{U_ω - λ_ω - (f'_L)^2\} (e^{f_L} φ_ω)^2 σ_ω ds ≤ C_R^2.$$ 

Subsequently, taking $L → ∞$, we obtain $∥1_{|s| ≥ R} e^{cs^2} φ_ω∥_{L^2(σ_ω ds)} ≤ C_R$. On the other hand, we have $∥1_{|s| ≤ R} e^{cs^2} φ_ω∥_{L^2(σ_ω ds)} ≤ e^{cr^2} ∥φ_ω∥_{L^2(σ_ω ds)} = e^{cr^2}$. Since $R$ does not depend on large $ω ≥ 1$, we prove that

$$∥e^{cs^2} φ_ω∥_{L^2(σ_ω ds)} ≤ 1.$$  

(3.9)

On the other hand, inserting $g = e^{cs^2} φ(\frac{s}{c})$ in (3.8), one can show that $∥e^{cs^2} φ_ω∥_{L^2(σ_ω ds)} ≤ 1$, because by (3.9) with $c_0 ∈ (c, α_0)$, $∥φ_ω g∥_{L^2(σ_ω ds)}$ is bounded uniformly in $ω ≥ 1$ and $L ≥ \sqrt{ω}$. Therefore, the claim (3.7) follows from (5.6).

For (3.2), we first estimate the derivative $φ'_ω(s)$. For $-\sqrt{ω} ≤ s ≤ 0$, by the fundamental theorem of calculus and the equation $H^{(1D)}_ω φ_ω = λ_ω φ_ω$, we have

$$(σ_ω φ'_ω)(s) = (σ_ω φ'_ω)(s) - (σ_ω φ'_ω)(−\sqrt{ω}) = \int_{−\sqrt{ω}}^s (σ_ω φ'_ω)' ds_1$$
$$= \int_{−\sqrt{ω}}^s (σ_ω φ''_ω + \frac{1}{ω} φ'_ω) ds_1 = \int_{−\sqrt{ω}}^s σ_ω (U_ω - λ_ω) φ_ω ds_1.$$ 

Thus, applying (3.7) with $c_0 ∈ (c, α_0)$, by the assumption on $U_ω$, we prove that

$$|(σ_ω φ'_ω)(s)| ≤ σ_ω(s) ∫_{−\sqrt{ω}}^s |U_ω(s_1) - λ_ω| \frac{1}{σ_ω(s_1)} e^{-c_0 s_1^2} ds_1 ≤ σ_ω(s) e^{-cs^2},$$

and consequently, $|φ'_ω(s)| ≤ e^{-cs^2}$. For $s ≥ 0$, we write

$$(σ_ω φ'_ω)(s) = (σ_ω φ'_ω)(s) - (σ_ω φ'_ω)(∞) = - ∫_{−∞}^∞ (σ_ω φ'_ω)' ds_1.$$ 

Then, by the same way, we can prove the bound $|φ'_ω(s)| ≤ e^{-cs^2}$. It remains to estimate $φ_ω(s)$. Indeed, for $s ≥ -\sqrt{ω}$, it is obvious that (3.7) with $c_0 ∈ (c, α_0)$ implies $φ_ω(s) ≤ σ_ω(-\sqrt{ω})^{-\frac{1}{2}} e^{-cs^2}$. On the other hand, when $-\sqrt{ω} + 1 ≤ s ≤ -\sqrt{ω}$, we use (3.7) with $c_0 ∈ (c, α_0)$ to obtain $φ_ω(s) ≤ σ_ω(-\sqrt{ω} + 1)^{-\frac{1}{2}} e^{-c_0 s^2} = ω^{\frac{1}{2}} e^{-c_0 s^2} ≤ e^{-cs^2}$. Finally, if $-\sqrt{ω} ≤ s ≤ -\sqrt{ω} + 1$, by the fundamental theorem of calculus and the derivative estimate, we prove that

$$φ_ω(s) ≤ φ_ω(-\sqrt{ω} + 1) + ∫_{−\sqrt{ω}}^{-\sqrt{ω} + 1} |φ'_ω(τ)| dτ ≤ e^{-cs} ∼ e^{-cs^2}.$$ 

By fast decay (3.2), the former inequality in (3.3) immediately follows. For the other one, we write $∥∥χ_ω φ_ω∥∥_{L^2(\frac{1}{σ_ω ds})} - 1 = ∥∥\frac{χ_ω}{σ_ω} φ_ω∥∥_{L^2(σ_ω ds)} - ∥φ_ω∥_{L^2(σ_ω ds)}∥ ≤ (\frac{c}{σ_ω} - 1) φ_ω∥_{L^2(σ_ω ds)}$. 

We note that
\[
\begin{cases}
|\frac{\chi_\omega}{\sigma_\omega} - 1| = 1 & \text{if } -\sqrt{\omega} \leq s \leq -\sqrt{\omega} + 1, \\
|\frac{\chi_\omega}{\sigma_\omega} - 1| \leq \frac{1}{\sigma_\omega} + 1 \leq \frac{1}{\sigma_\omega(-\sqrt{\omega} + 1)} + 1 \sim \sqrt{\omega} & \text{if } -\sqrt{\omega} + 1 \leq s \leq -\frac{\sqrt{\omega}}{2}, \\
|\frac{\chi_\omega}{\sigma_\omega} - 1| = |\frac{s}{\sqrt{\omega} + s} - 1| = \frac{|s|}{\sqrt{\omega} + s} \lesssim \frac{|s|}{\sqrt{\omega}} & \text{if } s \geq -\frac{\sqrt{\omega}}{2}.
\end{cases}
\]

Thus, it follows from (3.3) that \(\left\|\left(\frac{\chi_\omega}{\sigma_\omega} - 1\right)\phi_\omega\right\|_{L^2(\sigma_\omega ds)} \lesssim \omega^{-\frac{1}{2}}\). \(\square\)

**Proof of Proposition 3.2 (i) and (iii).** For (i), by Lemma 3.4, it suffices to show a lower bound. Let us consider \(\chi_\omega \phi_\omega\) as its trivial extension to the real line. Then, repeating the argument to prove Lemma 3.4 but with Proposition 3.2 (ii), one can show that \(\left\|\chi_\omega \phi_\omega\right\|_{L^2(\mathbb{R})} = 1 + O(\omega^{-\frac{1}{2}})\) and \(\left\langle (-\partial_s^2 + s^2)(\chi_\omega \phi_\omega), (\chi_\omega \phi_\omega) L^2(\mathbb{R}, ds) = \lambda_\omega + O(\omega^{-\frac{1}{2}})\). Hence, it follows that \(1 \leq \left\langle (-\partial_s^2 + s^2)(\frac{\chi_\omega}{\sigma_\omega} \phi_\omega), (\frac{\chi_\omega}{\sigma_\omega} \phi_\omega) L^2(\mathbb{R}, ds) = \lambda_\omega + O(\omega^{-\frac{1}{2}})\).

For (iii), assume that (3.4) holds. We observe that for \(s > \sqrt{\omega}\),
\[
|v(s)|^2 = -2 \int_\sigma^1 vv' ds \leq 2 \left\|v\right\|_{L^2(1, \sqrt{\omega}]} \left\|v'\right\|_{L^2(1, \sqrt{\omega}]} ds,
\]
then by (3.4), we have
\[
\left\|\phi_\omega - \phi_\infty\right\|_{L^2(1, \sqrt{\omega}]} ds \leq \left\{\sup_{s > \sqrt{\omega}} \left\|\phi_\omega - \phi_\infty\right\|^2\right\} \left\|\phi_\omega - \phi_\infty\right\|^2_{L^2(1, \sqrt{\omega}]} ds \lesssim \omega^{-2},
\]
which proves (3.5).

Next, we prove (3.4). We note that by Proposition 3.2 (ii), \(\left\|\left(1 - \chi_\omega\right)\phi_\omega\right\|_{\Sigma(1, \sqrt{\omega}]} ds \lesssim \omega^{-\frac{1}{2}},\) where \(J \subset \mathbb{R}, \left\|v\right\|_{\Sigma(1, ds)} = \left\langle v, v\right\rangle_{\Sigma(1, ds)} \) and
\[
\left\langle v, w\right\rangle_{\Sigma(1, ds)} = \int_\sigma^1 (vw + v'w' + s^2 vw) \textbf{1}_J(s) ds.
\]
Thus, it suffices to show \(\left\|\chi_\omega \phi_\omega - \phi_\infty\right\|_{\Sigma(ds)} \lesssim \omega^{-\frac{1}{2}},\) where \(\left\|v\right\|_{\Sigma(ds)} := \left\|v\right\|_{\Sigma(1, ds)}\). For the proof, we decompose \(\chi_\omega \phi_\omega - \phi_\infty = c_\omega \phi_\infty + r_\omega\) with
\[
c_\omega = \frac{\left\langle \chi_\omega \phi_\omega - \phi_\infty, \phi_\infty\right\rangle_{\Sigma(ds)}}{\left\|\phi_\infty\right\|^2_{\Sigma(ds)}} = \left\langle \chi_\omega \phi_\omega, \phi_\infty\right\rangle_{L^2(\mathbb{R}, ds)} - 1,
\]
because \((1 - \partial_s^2 + s^2)\phi_\infty = 2\phi_\infty\). On the other hand, by Proposition 3.2 (ii), \(c_\omega = \frac{1}{2} \left(\left\|\chi_\omega \phi_\omega\right\|_{L^2(\mathbb{R}, ds)}^2 - \left\|\chi_\omega \phi_\omega - \phi_\infty\right\|_{L^2(\mathbb{R}, ds)}^2\right) + \|\phi_\infty\|_{L^2(\mathbb{R}, ds)}^2 - 1 = -\frac{1}{2} \left\|\chi_\omega \phi_\omega - \phi_\infty\right\|_{L^2(\mathbb{R}, ds)}^2 + O(\omega^{-\frac{1}{2}})\). Thus, we have
\[
\left\|\chi_\omega \phi_\omega - \phi_\infty\right\|_{\Sigma(ds)} \leq |c_\omega| \left\|\phi_\infty\right\|_{\Sigma(ds)} + \|r_\omega\|_{\Sigma(ds)} \leq \frac{1}{\sqrt{2}} \left\|\chi_\omega \phi_\omega - \phi_\infty\right\|_{L^2(\mathbb{R}, ds)} + \|r_\omega\|_{\Sigma(ds)} + O\left(\omega^{-\frac{1}{2}}\right).
\]
Now, we recall from the proof of Proposition 3.2 (i), \(\left\|\chi_\omega \phi_\omega\right\|_{L^2(\mathbb{R}, ds)} \to 1\). Hence, it follows that \(\chi_\omega \phi_\omega \to \phi_\infty\) in \(L^2(\mathbb{R}, ds)\), and consequently \(\left\|\chi_\omega \phi_\omega - \phi_\infty\right\|_{\Sigma(ds)} \lesssim \|r_\omega\|_{\Sigma(ds)} + O(\omega^{-\frac{1}{2}})\).

It remains to show \(\|r_\omega\|_{\Sigma(ds)} \lesssim \omega^{-\frac{1}{2}}\). Indeed, \(\|r_\omega\|_{\Sigma(ds)} = 2 \int_\mathbb{R} r_\omega \phi_\infty ds = 0\) and \(-\partial_s^2 + s^2\) has a simple lowest eigenvalue 1 (with the eigenfunction \(\phi_\infty\)) and the second
eigenvalue is 3. Thus, it follows that \( \| r_\omega \|_{\Sigma(ds)} = \| \sqrt{1 - \partial_s^2 + s^2} (-\partial_s^2 + s^2) r_\omega \|_{L^2(\mathbb{R}, ds)} \leq \frac{2}{3} \| (-\partial_s^2 + s^2) r_\omega \|_{L^2(\mathbb{R}, ds)} \), where we used that \( \sqrt{1 + \lambda} \leq \frac{2}{3} \) for \( \lambda \geq 3 \). Moreover, we have

\[
(-\partial_s^2 + s^2) r_\omega = (-\partial_s^2 + s^2) (\chi_\omega \phi_\omega - (c_\omega + 1) \phi_\infty) = (-\phi''_\omega + s^2 \phi_\omega) \chi_\omega - 2 \phi'_\omega \chi'_\omega - \chi''_\omega \phi_\omega - (c_\omega + 1) \phi_\infty,
\]

which is equal to, by the assumptions on \( U_\omega \), the equation \( \mathcal{H}^{(1D)}(\phi_\omega) = \lambda_\omega \phi_\omega \) and Proposition 3.2 (i) and (ii),

\[
\left( \frac{1}{\sqrt{\omega_\sigma}} \phi'_\omega + (s^2 - U_\omega) \phi_\omega + \lambda_\omega \phi_\omega \right) \chi_\omega - (c_\omega + 1) \phi_\infty + O_{L^2(ds)}(\omega^{-\frac{1}{2}})
= \lambda_\omega \chi_\omega \phi_\omega - (c_\omega + 1) \phi_\infty + O_{L^2(ds)}(\omega^{-\frac{1}{2}}) = r_\omega + O_{L^2(ds)}(\omega^{-\frac{1}{2}}),
\]

where \( v = O_{L^2(ds)}(\omega^{-\frac{1}{2}}) \) is defined by \( \| v \|_{L^2(ds)} = O(\omega^{-\frac{1}{2}}) \). Thus, it follows that \( \| r_\omega \|_{\Sigma(ds)} \leq \frac{2}{3} \| r_\omega \|_{\Sigma(ds)} + O(\omega^{-\frac{1}{2}}) \), which completes the proof. \( \square \)

**Sketch of Proof of Proposition 3.2 (iv).** Proposition 3.2 (iv) follows almost identically as the ground state case, thus we omit the proof. Indeed, for the second eigenvalue for \( \mathcal{H}^{(1D)} \), we may consider the variational problem

\[
\lambda'_\omega = \min \left\{ \mathcal{E}^{(1D)}(v) = \frac{1}{2} \langle \mathcal{H}^{(1D)}(v, v) \rangle_{L^2(\sigma, ds)} : \| v \|_{L^2(\sigma, ds)} = 1 \text{ and } \langle v, \phi_\omega \rangle_{L^2(\sigma, ds)} = 0 \right\}.
\]

(3.10)

Since the second eigenvalue of \(-\partial_s^2 + s^2\) is 3 and the corresponding eigenfunction is square exponentially decreasing, repeating the proof of Proposition 3.2 (i)-(iii), one can obtain the second lowest eigenvalue asymptotic. \( \square \)

4. **Truncated projection onto the 2D lowest eigenspace**

For both time-dependent/independent cases, we are concerned with 3D states mostly concentrated on the 2D lowest energy state \( \Phi_\omega(s, z) = \phi_\omega(s) \phi_\infty(z) \).

**Remark 4.1.** At first glance, one might try to approximate such a 3D state by the projected state

\[
(P_{\Phi_\omega}v)(s, \theta, z) := \langle v(\cdot, \cdot, \theta), \Phi_\omega \rangle_{L^2(\sigma, dsdz)} \Phi_\omega(s, z).
\]

(4.1)

However, it turns out to be impossible, because the projected state \( P_{\Phi_\omega}v \) has infinite modified energy. Indeed, we have

\[
\| \partial_\theta (P_{\Phi_\omega}v) \|_{L^2(\mathbb{S}^1)} = \| \partial_\theta (v(\cdot, \cdot, \theta), \Phi_\omega) \|_{L^2(\sigma, dsdz)} \|_{L^2(\mathbb{S}^1)} \int_0^{\infty} \frac{\phi_\omega(s)^2}{1 + \frac{s}{\sqrt{\omega}}} ds = \infty,
\]

(4.2)

because \( \phi_\omega(-\sqrt{\omega}) > 0 \) (see Remark 3.1 (iii)).
In this section, we introduce a modified projection to bypass non-integrability in $\mathbb{R}^2$, and collect some useful properties. Let $\chi : [0, \infty) \to [0, 1]$ be a smooth cut-off such that $\chi \equiv 0$ on $[0, 1]$ and $\chi \equiv 1$ on $[2, \infty)$ given in Theorem 1.3. For the lowest energy state $\Phi_\omega$, let

$$\bar{\Phi}_\omega(s, z) := \frac{\chi_\omega(s)\Phi_\omega(s, z)}{||\chi_\omega\Phi_\omega||_{L^2(\sigma_{s, ds})}}$$

be a cut-offed normalized function, where $\chi_\omega(s) = \chi(s + \sqrt{\omega})$. Note that $\chi_\omega \equiv 0$ on $[-\sqrt{\omega}, -\sqrt{\omega} + 1]$ and $\chi_\omega \equiv 1$ on $[-\sqrt{\omega} + 2, \infty)$. We define the 2D $\bar{\Phi}_\omega$-directional projection $\mathcal{P}_{\bar{\Phi}_\omega}$ by

$$(\mathcal{P}_{\bar{\Phi}_\omega})^{(\theta)}v(s, \theta, z) := v_\parallel(\theta)\bar{\Phi}_\omega(s, z),$$

(4.3)

where

$$v_\parallel(\theta) := \langle v, \cdot, \theta \rangle_{\bar{\Phi}_\omega} \equiv \int_{-\infty}^{\infty} \int_{-\sqrt{\omega}}^{\infty} v(s, z, \theta)\bar{\Phi}_\omega(s, z)\sigma_\omega(s)dsdz$$

is the $\bar{\Phi}_\omega(s, z)$-directional component, and define its orthogonal complement by

$$\mathcal{P}_{\bar{\Phi}_\omega}^\perp := 1 - \mathcal{P}_{\bar{\Phi}_\omega}.$$  

Obviously, by orthogonality, we have

$$||v||_{L^2(\sigma_\omega)}^2 = ||\mathcal{P}_{\bar{\Phi}_\omega}v||_{L^2(\sigma_\omega)}^2 + ||\mathcal{P}_{\bar{\Phi}_\omega}^\perp v||_{L^2(\sigma_\omega)}^2.$$  

(4.4)

By definition, it is obvious that $||v||_{L^2(\mathbb{S}^1)} \leq ||v||_{L^2(\sigma_\omega)}$. Indeed, the directional component satisfies more bounds.

**Lemma 4.2** (Bounds for the $\bar{\Phi}_\omega(s, z)$-directional component). For any $1 \leq p < \infty$, we have

$$||v||_{L^p(\mathbb{S}^1)} \lesssim ||v||_{L^p(\sigma_\omega)}, \quad ||v||_{L^2(\mathbb{S}^1)} \lesssim ||v||_{L^2(\frac{1}{\sigma_\omega})}.$$  

**Proof.** By the Gaussian bound for $\phi_\omega$ (Proposition 3.2 (ii)) and the Hölder inequality, we show that $v = \langle v, \cdot, \theta \rangle_{\bar{\Phi}_\omega} \equiv \int_{-\infty}^{\infty} \int_{-\sqrt{\omega}}^{\infty} v(s, z, \theta)\bar{\Phi}_\omega(s, z)\sigma_\omega(s)dsdz$ obeys $||v||_{L^p(\mathbb{S}^1)} \leq ||\bar{\Phi}_\omega||_{L^p(\sigma_{s, ds})}||v||_{L^p(\sigma_\omega)} \sim ||v||_{L^p(\sigma_\omega)}$ and $||v||_{L^2(\mathbb{S}^1)} \leq \frac{1}{\sigma_\omega} ||v||_{L^2(\sigma_\omega)} ||\sigma_\omega\bar{\Phi}_\omega||_{L^p(\sigma_{s, ds})} \lesssim ||v||_{L^2(\frac{1}{\sigma_\omega})}$. \hfill \Box

The following lemma asserts that as $\omega$ increases, the modified projection gets closer to the lowest eigenstate projection $\mathcal{P}_{\Phi_\omega}$ (see (4.1)) except the angular derivative semi-norm $||\partial_\theta v||_{L^2(\frac{1}{\sigma_\omega})}$.

**Lemma 4.3** (Truncation error bounds). Let $1 \leq p < \infty$ and $\alpha_0 > 0$ be the constant given in Proposition 3.2. For any $c \in (0, \alpha_0)$, there exists $\omega_c \geq 1$ such that if $\omega \geq \omega_c$, then

$$\left|\left|\left(\mathcal{P}_{\Phi_\omega} - \mathcal{P}_{\bar{\Phi}_\omega}\right)v\right|\right|_{L^2(\sigma_\omega)} \lesssim \left|\left|\left(\mathcal{P}_{\Phi_\omega}^\perp - \mathcal{P}_{\bar{\Phi}_\omega}^\perp\right)v\right|\right|_{L^2(\sigma_\omega)} \lesssim e^{-\omega c}||v||_{L^2(\sigma_\omega)};$$

$$\left|\left|\left(\mathcal{P}_{\Phi_\omega} - \mathcal{P}_{\bar{\Phi}_\omega}\right)v\right|\right|_{L^p(\sigma_\omega)} = \left|\left|\left(\mathcal{P}_{\Phi_\omega}^\perp - \mathcal{P}_{\bar{\Phi}_\omega}^\perp\right)v\right|\right|_{L^p(\sigma_\omega)} \lesssim e^{-\omega c}||v||_{L^p(\sigma_\omega)},$$

where $\mathcal{P}_{\bar{\Phi}_\omega}^\perp := 1 - \mathcal{P}_{\bar{\Phi}_\omega}$. 

Proof. Since $\mathcal{P}_{\bar{\Phi}_\omega}^\perp - \mathcal{P}_{\bar{\Phi}_\omega}^\perp = \mathcal{P}_{\bar{\Phi}_\omega} - \mathcal{P}_{\Phi_\omega}$, it suffices to show the lemma for

$$(\mathcal{P}_{\Phi_\omega} - \mathcal{P}_{\bar{\Phi}_\omega})v = (v(\cdot, s, \theta), \Phi_\omega - \bar{\Phi}_\omega)_{L^2(\sigma_\omega \, ds \, dz)} \Phi_\omega + v\| (\theta)(\Phi_\omega - \bar{\Phi}_\omega) = \mathcal{P}_{\Phi_\omega} - \mathcal{P}_{\bar{\Phi}_\omega}.$$ 

Indeed, by the fast decay of $\bar{\Phi}_\omega$ (Proposition 3.2 (ii)), we have $\| \Phi_\omega - \bar{\Phi}_\omega \|_{L^2(\sigma_\omega \, ds \, dz)} \leq e^{-c_\omega r}$ for any $r > 1$. Hence, the Hölder inequality and Lemma 4.2 imply that

$$\| (\mathcal{P}_{\Phi_\omega} - \mathcal{P}_{\bar{\Phi}_\omega})v \|_{L^p(\sigma_\omega)} \leq e^{-c_\omega} (\| v \|_{L^p(\sigma_\omega)} + \| v \|_{L^p(\sigma_\omega)}).$$

By the same way, one can show that $\| (\mathcal{P}_{\Phi_\omega} - \mathcal{P}_{\bar{\Phi}_\omega})v \|_{L^2(\sigma_\omega)} \leq e^{-c_\omega} \| v \|_{L^2(\sigma_\omega)}$. \qed

Moreover, the orthogonal complement projection satisfies the following bounds.

Lemma 4.4 (Bounds for the orthogonal complement). Let $\alpha_0 > 0$ be the constant given in Proposition 3.2. For any $c \in (0, \alpha_0)$, there exists $\omega_c \geq 1$ such that if $\omega \geq \omega_c$, then

$$\| \mathcal{P}_{\Phi_\omega} \|_{\mathcal{L}(\sigma_\omega \, ds \, dz)} \leq \frac{1}{\omega_c - \alpha_0} \| v \|_{\sigma_\omega \, ds \, dz}^2.$$ 

Proof. By Lemma 4.3, it suffices to prove that

$$\| \mathcal{P}_{\Phi_\omega} \|_{\mathcal{L}(\sigma_\omega \, ds \, dz)} \leq \frac{1}{\omega_c - \alpha_0} \| v \|_{\sigma_\omega \, ds \, dz}^2.$$ 

A key advantage of using the modified projection is that if $\| \partial_\theta v \|_{L^2(\frac{1}{\sigma_\omega})}$ is bounded, then so is $\| \partial_\theta (\mathcal{P}_{\Phi_\omega} v) \|_{L^2(\frac{1}{\sigma_\omega})}$, since the possible singularity at $s = -\sqrt{\omega}$ is eliminated (see (1.2)). The following almost orthogonality proves that besides boundedness, the projection asymptotically reduces the semi-norm, i.e., $\| \partial_\theta (\mathcal{P}_{\Phi_\omega} v) \|_{L^2(\frac{1}{\sigma_\omega})} \leq \| \partial_\theta v \|_{L^2(\frac{1}{\sigma_\omega})} + o_\omega(1)$.

Lemma 4.5 (Asymptotic Pythagorean identities). Let $\alpha_0 > 0$ be a constant given in Proposition 3.2. For any $c \in (0, \alpha_0)$, there exists $\omega_c \geq 1$ such that if $\omega \geq \omega_c$, then

$$\| v \|_{L^2(\frac{1}{\sigma_\omega})} \leq \| \mathcal{P}_{\Phi_\omega} v \|_{L^2(\frac{1}{\sigma_\omega})}^2 + \| \mathcal{P}_{\Phi_\omega} v \|_{L^2(\frac{1}{\sigma_\omega})}^2 \left| 1 - \sigma_\omega \right| \| \mathcal{P}_{\Phi_\omega} v \|_{L^2(\frac{1}{\sigma_\omega})} \right| \leq -\omega^{-\frac{3}{2}} \left( 1 - \sigma_\omega \right) \| \mathcal{P}_{\Phi_\omega} v \|_{L^2(\frac{1}{\sigma_\omega})}.$$ 

Proof. It suffices to show that $\| \partial_\theta (\mathcal{P}_{\Phi_\omega} v) \|_{L^2(\frac{1}{\sigma_\omega})} \leq \omega^{-\frac{3}{2}} \| v \|_{L^2(\frac{1}{\sigma_\omega})}^2$. Indeed, we have

$$\langle \mathcal{P}_{\Phi_\omega} v, \mathcal{P}_{\Phi_\omega} v \rangle_{L^2(\frac{1}{\sigma_\omega})} = \langle (1 - \sigma_\omega) \mathcal{P}_{\Phi_\omega} v, \mathcal{P}_{\Phi_\omega} v \rangle_{L^2(\frac{1}{\sigma_\omega})} = -\omega^{-\frac{3}{2}} \langle (1 + \sigma_\omega) \mathcal{P}_{\Phi_\omega} v, \mathcal{P}_{\Phi_\omega} v \rangle_{L^2(\frac{1}{\sigma_\omega})}.$$


because $\langle \mathcal{P}_{\phi} v, \mathcal{P}_{\phi}^\perp v \rangle_{L^2(\sigma)} = 0$ by orthogonality and $\sigma - 1 = \frac{q}{\sqrt{2}}$. Hence, it follows that

$$|\langle \mathcal{P}_{\phi} v, \mathcal{P}_{\phi}^\perp v \rangle_{L^2(\frac{1}{\sqrt{2}})}| \leq \omega^{-\frac{1}{2}} \|s(1 + \sigma)\mathcal{P}_{\phi} v\|_{L^2(\frac{1}{\sqrt{2}})} \{\|v\|_{L^2(\frac{1}{\sqrt{2}})} + \|\mathcal{P}_{\phi} v\|_{L^2(\frac{1}{\sqrt{2}})}\}.$$ 

Note that by the fast decay of $\phi$ (see Proposition 3.2) and the $v_\|_\perp$-bound (Lemma 4.2),

$$\|s(1 + \sigma)\mathcal{P}_{\phi} v\|_{L^2(\frac{1}{\sqrt{2}})} = \frac{|s(1 + \sigma)\chi_\omega \Phi_\omega \|L^2(\sigma dsdz)\|v\|_{L^2(\Omega)}}{\chi_\omega \Phi_\omega \|L^2(\sigma dsdz)\|v\|_{L^2(\Omega)}} \lesssim \|v\|_{L^2(\Omega)} \lesssim \|v\|_{L^2(\frac{1}{\sqrt{2}})};$$

$$\|\mathcal{P}_{\phi} v\|_{L^2(\frac{1}{\sqrt{2}})} = \frac{|\chi_\omega \Phi_\omega \|L^2(\sigma dsdz)\|v\|_{L^2(\Omega)}}{\chi_\omega \Phi_\omega \|L^2(\sigma dsdz)\|v\|_{L^2(\Omega)}} \lesssim \|v\|_{L^2(\Omega)} \lesssim \|v\|_{L^2(\frac{1}{\sqrt{2}})}.$$ 

Therefore, (4.5) follows. □

5. Refined Gagliardo-Nirenberg inequality

For the modified problem in Section 2, we translate basic inequalities in the new $(s, z, \theta)$-coordinates via the relation

$$v(s, z, \theta) = \omega^\frac{1}{2} u(s + \sqrt{\omega}, z, \theta)$$

with $\|u\|_{L^p(\mathbb{R}^3)} = \omega^{-\frac{1}{4} + \frac{1}{p}} \|v\|_{L^p(\sigma)}$ and $\|\nabla u\|^2_{L^2(\mathbb{R}^3)} = \|\nabla_{(s,z)} v\|^2_{L^2(\sigma)} + \omega \|\partial_\theta v\|^2_{L^2(\frac{1}{\sqrt{2}})}$. For instance, the Sobolev inequality $\|u\|_{L^p(\mathbb{R}^3)} \lesssim \|\nabla u\|_{L^2(\mathbb{R}^3)}$ and the Gagliardo-Nirenberg inequality $\|u\|^4_{L^4(\mathbb{R}^3)} \lesssim \|\nabla u\|^2_{L^2(\mathbb{R}^3)}$ are restated respectively as

$$\|v\|^4_{L^4(\sigma)} \lesssim \omega^\frac{1}{4} \left\{ \|\nabla_{(s,z)} v\|^2_{L^2(\sigma)} + \frac{1}{\omega} \|\partial_\theta v\|^2_{L^2(\frac{1}{\sqrt{2}})} \right\}^{\frac{1}{2}}$$

and

$$\|v\|^4_{L^4(\sigma)} \lesssim \sqrt{\omega} \|v\|_{L^2(\sigma)} \left\{ \|\nabla_{(s,z)} v\|^2_{L^2(\sigma)} + \frac{1}{\omega} \|\partial_\theta v\|^2_{L^2(\frac{1}{\sqrt{2}})} \right\}^{\frac{1}{2}}.$$  

However, these inequalities are not good enough to capture the subcritical nature of the problem under the additional constraint $\|v\|^2_{\Sigma_{(s,z)}(\sigma)} \leq \delta_\omega$.

Our key analysis tool is the following refined version of the inequality (5.2).

**Proposition 5.1** (Refined Gagliardo-Nirenberg inequality).

$$\|v\|^4_{L^4(\sigma)} \lesssim \sqrt{\omega} \|v\|_{L^2(\sigma)} \|\nabla_{(s,z)} v\|^2_{L^2(\sigma)} \left\{ \|\nabla_{(s,z)} v\|^2_{L^2(\sigma)} + \frac{1}{\omega} \|\partial_\theta v\|^2_{L^2(\frac{1}{\sqrt{2}})} \right\}^{\frac{1}{2}}.$$  

For the proof, we recall the following standard inequalities.

**Lemma 5.2.**  (i) (Gagliardo-Nirenberg inequality on $\mathbb{R}^d$) If $\frac{1}{q} = \frac{1}{2} - \frac{\beta}{d}$, $q > 2$ and $0 < \beta < 1$, then

$$\|u\|_{L^q(\mathbb{R}^d)} \lesssim \|u\|_{L^2(\mathbb{R}^d)}^{1-\beta} \|\nabla u\|_{L^2(\mathbb{R}^d)}^\beta.$$  

(ii) (Gagliardo-Nirenberg inequality on the unit circle $S^1$) If $q > 2$, then

$$\|u\|_{L^q(S^1)} \lesssim \|u\|_{L^2(S^1)} \|u\|_{H^1(S^1)}^{q-2}.$$
(iii) (Radial Gagliardo-Nirenberg inequality\footnote{It simply follows from the well-known inequality \( \sup_{y \in \mathbb{R}^d \backslash \{0\}} |y||u(y)|^2 \lesssim \|u\|_{L^p(\mathbb{R}^d)}^2 \|\nabla_y u\|_{L^q(\mathbb{R}^d)} \) for radial functions, via \( u(|y|) = \omega^{-\frac{d}{2}} v(|y| - \sqrt{\omega}) \) (see \cite{16} for instance).})

\[
\sup_{s > -\sqrt{\omega}} \sigma_\omega(s)|v(s)|^2 \lesssim \|v\|_{L^2(\sigma_\omega ds)} \|\partial_s v\|_{L^2(\sigma_\omega ds)} \tag{5.6}
\]

Proof of Proposition \[5.1\]. First, we apply the inequality \[5.5\] on \( S^1 \),

\[
\|v\|_{L^4(\sigma_\omega)}^4 = \|\|v\|_{L^4(\sigma_\omega)}^2\|^2_{L^1(\sigma_\omega ds)}
\lesssim \|\|v\|_{L^2(\sigma_\omega)}^3\|\partial_\sigma v\|_{L^2(\sigma_\omega)} + \|\|v\|_{L^4(\sigma_\omega)}^2\|\partial_\sigma v\|_{L^2(\frac{1}{\sqrt{\omega}})} =: I + II.
\]

For \( I \), by the Hölder and the Minkowski inequalities, we have

\[
I \leq \||\sigma_\omega^{\frac{3}{2}} v\|_{L^6(\sigma_\omega ds)}^3 \|\partial_\sigma v\|_{L^2(\sigma_\omega ds)} + \||\sigma_\omega^{\frac{2}{3}} v\|_{L^6(\sigma_\omega ds)}^3 \|\partial_\sigma v\|_{L^2(\sigma_\omega ds)}
\]

Then, it follows from the radial inequality \[5.6\] and the inequality \[5.5\] with \( d = 1 \) for the \( z \)-variable that

\[
\||\sigma_\omega^{\frac{1}{2}} v\|_{L^6(\sigma_\omega ds)}^6 = \int_{-\infty}^{\infty} \int_{-\sqrt{\omega}}^{\infty} (\sigma_\omega|v|^2|v|^2) \sigma_\omega ds dz \lesssim \int_{-\infty}^{\infty} \||v\|_{L^2(\sigma_\omega ds)}^4 \|\partial_s v\|_{L^2(\sigma_\omega ds)}^2 dz
\]

Consequently, by the Hölder inequality in \( \|\cdot\|_{L^2(\mathbb{S}^1)} \), we obtain

\[
\||v\|_{L^6(\sigma_\omega ds)} \|\sigma_\omega^{\frac{3}{2}} v\|_{L^2(\mathbb{S}^1)} \|\nabla(s, v)\|_{L^2(\sigma_\omega ds)} \|\nabla(s, v)\|_{L^2(\sigma_\omega ds)} \|\partial_\sigma v\|_{L^2(\sigma_\omega ds)} \leq \||v\|_{L^2(\sigma_\omega)}^3 \|\nabla(s, v)\|_{L^2(\sigma_\omega)}^3 \|\nabla(s, v)\|_{L^2(\sigma_\omega)}^3.
\]

Hence, we conclude that \( I \lesssim \||v\|_{L^2(\sigma_\omega)}^2 \|\nabla(s, v)\|^2_{L^2(\sigma_\omega)} \|\partial_\sigma v\|_{L^2(\sigma_\omega)} \). On the other hand, for \( II \), applying a Gagliardo-Nirenberg type inequality\footnote{For \( u = u(|y|, z) \), the inequality \( \||v\|_{L^4(\mathbb{R}^3)}^4 \lesssim \||v\|_{L^2(\mathbb{R}^3)}^3 \|\nabla u\|_{L^6(\mathbb{R}^3)}^3 \) can be written as

\[
\int_{-\infty}^{\infty} \int_{0}^{\infty} |u(r, z)|^4 r dr dz \lesssim \left\{ \int_{-\infty}^{\infty} \int_{0}^{\infty} |u(r, z)|^2 r dr dz \right\}^{\frac{3}{4}} \left\{ \int_{-\infty}^{\infty} \int_{0}^{\infty} |\nabla(r, z) u(r, z)|^2 r dr dz \right\}^{\frac{3}{4}}.
\]

Then, inserting \( u(r, z) = \omega^{-\frac{d}{2}} v(r - \sqrt{\omega}, z) \), we obtain the desired inequality.}

\[
\||v\|_{L^4(\sigma_\omega ds)}^4 \lesssim \sqrt{\omega} \||v\|_{L^2(\sigma_\omega ds)} \|\nabla(s, v)\|^2_{L^2(\sigma_\omega ds)} \|\nabla(s, v)\|^2_{L^2(\sigma_\omega ds)}.
\]
we prove that
\[ II = \||v||_{L^2(S^1)}^4 \leq \|v\|_{L^4(\sigma_dsdz)}^4 \leq \|v\|_{L^4(\sigma_dsdz)}^4 \leq \sqrt{\omega} \||v||_{L^2(\sigma_dsdz)}^4 \|\nabla v\|_{L^2(\sigma_dsdz)}^4 \leq \sqrt{\omega} \||v||_{L^2(\sigma_dsdz)}^4 \|\nabla v\|_{L^2(\sigma_dsdz)}^4. \]

Therefore, combining the bounds for \( I \) and \( II \), we complete the proof. \( \square \)

In our applications, we modify the inequality expressing in terms of the quantities in the modified mass and energy (see (2.10) and (2.11)) but we also emphasize uniformity (in \( \omega \)) for the implicit constant in the statement.

**Corollary 5.3.** There exists \( C_{GN} > 0 \), independent of large \( \omega \geq 1 \), such that
\[ \|v\|_{L^4(\sigma_d)}^4 \leq C_{GN} \|v\|_{L^2(\sigma_d)}^4 \left\{ \sqrt{\omega} \|v\|_{S_{\omega}}^2 \|v\|_{S_{\omega}} + \|v\|_{L^2(\sigma_d)}^2 \|\partial \theta v\|_{L^2(\sigma_d)} + \|v\|_{L^2(\sigma_d)}^3 \right\}, \]
where \( \|v\|_{S_{\omega}}^2 = \|v\|_{S_{\omega,s}}^2 + \frac{1}{\omega} \|\partial \theta v\|_{L^2(\sigma_d)}^2 \).

**Proof.** We decompose \( v = P_{\tilde{\Phi}_\omega} v + P_{\perp} v \), where \( P_{\tilde{\Phi}_\omega} \) and \( P_{\perp} \) are the modified projections in Section 4. For the orthogonal complement \( \tilde{v}_\omega \) = \( P_{\perp} v \), we apply the refined Gagliardo-Nirenberg inequality (Proposition 5.1), the orthogonality (4.4) and the asymptotic Pythagorean identities (4.5) to obtain
\[ \|\tilde{v}_\omega\|_{L^4(\sigma_d)}^4 \lesssim \sqrt{\omega} \|\tilde{v}_\omega\|_{L^2(\sigma_d)} \|\nabla v\|_{L^2(\sigma_d)} \left\{ \|\nabla v\|_{L^2(\sigma_d)}^2 + \frac{1}{\omega} \|\partial \theta \tilde{v}_\omega\|_{L^2(\sigma_d)}^2 \right\}^{1/2}, \]
\[ \lesssim \sqrt{\omega} \|v\|_{L^2(\sigma_d)} \|\nabla v\|_{L^2(\sigma_d)} \left\{ \|\nabla v\|_{L^2(\sigma_d)}^2 + \frac{1}{\omega} \|\partial \theta v\|_{L^2(\sigma_d)}^2 \right\}^{1/2}. \]

Then, it follows from the bound for \( P_{\perp} \) (Lemma 4.2) that
\[ \|\tilde{v}_\omega\|_{L^4(\sigma_d)}^4 \lesssim \|v\|_{L^2(\sigma_d)} \left\{ \sqrt{\omega} \|v\|_{S_{\omega,s}}^2 \|v\|_{S_{\omega}} + \|v\|_{L^2(\sigma_d)}^2 \|\partial \theta v\|_{L^2(\sigma_d)} + \|v\|_{L^2(\sigma_d)}^3 \right\}. \]

For the \( \tilde{\Phi}_\omega \)-directional component \( P_{\tilde{\Phi}_\omega} v \), we apply the Gagliardo-Nirenberg inequality on \( S^1 \) (5.1). Then, it follows that
\[ \|P_{\tilde{\Phi}_\omega} v\|_{L^4(\sigma_d)} \sim \|v\|_{L^4(S^1)}^4 \lesssim \|v\|_{L^2(S^1)}^4 \|\partial \theta v\|_{L^2(S^1)} + \|v\|_{L^2(S^1)}^2 \|v\|_{L^2(S^1)}, \]
where the \( v \)-bounds (Lemma 4.2) is used in the last inequality. Therefore, combining the bounds for \( P_{\tilde{\Phi}_\omega} v \) and \( \tilde{v}_\omega \), we prove the desired inequality. \( \square \)

As a direct consequence, we show that in the focusing case, a wave function having uniformly bounded energy is forbidden to exist in the region \( \frac{2}{\omega} < \|v\|_{S_{\omega}}^2 \leq \delta \omega \).
Corollary 5.4 (Forbidden region). Let \( \kappa = -1 \) and \( \omega \geq 1 \) be sufficiently large. Then, for fixed \( m, E > 0 \), there exist small \( \delta = \delta(m, E) > 0 \) and large \( C = C(m), K = K(m, E) > 0 \), independent of large \( \omega \geq 1 \), such that if \( v \in \Sigma_\omega, M_\omega[v] = m \),

\[
\mathcal{E}_\omega[v] \leq E \quad \text{and} \quad \|v\|_{\Sigma_\omega(s,z)}^2 \leq \delta \omega,
\]

then

\[
\mathcal{E}_\omega[v] \geq -C \quad \text{and} \quad \|v\|_{\Sigma_\omega(s,z)}^2 = \|v\|_{\Sigma_\omega(s,z)}^2 + \frac{1}{\omega} \|\partial_\theta v\|^2_{L^2(\Sigma_\omega)} \leq \frac{K}{\omega}.
\]

Proof. By the assumptions on \( v \), the refined Gagliardo-Nirenberg inequality (Corollary 5.3) yields a lower bound on the energy,

\[
\mathcal{E}_\omega[v] \geq \frac{\omega}{2} \|v\|_{\Sigma_\omega}^2 - \frac{C_{GN}}{4} \sqrt{m} \left\{ \sqrt{\delta \omega} \|v\|_{\Sigma_\omega(s,z)} \|v\|_{\Sigma_\omega} + m \|\partial_\theta v\|_{L^2(\Sigma_\omega)}^2 + m^2 \right\}.
\]

Consequently, it follows from the Cauchy-Schwarz inequality and smallness of \( \delta > 0 \) that

\[
E \geq \mathcal{E}_\omega[v] \geq \frac{\omega}{4} \|v\|_{\Sigma_\omega(s,z)}^2 + \frac{1}{4} \|\partial_\theta v\|_{L^2(\Sigma_\omega)}^2 - C(m) = \frac{\omega}{4} \|v\|_{\Sigma_\omega}^2 - C(m)
\]

for some \( C(m) \). Thus, we obtain the desired bound on \( \|v\|_{\Sigma_\omega}^2 \). \( \square \)

Remark 5.5. By the relation (2.1), Corollary 5.4 is translated into that if \( u \in \Sigma, M[u] = m, E_\omega[u] \leq E \) and \( \langle (H_{\omega}^1 - \Lambda_u)u, u \rangle_{L^2(\mathbb{R}^3)} \leq \delta \omega \), then \( \langle (H_{\omega} - \Lambda_u)u, u \rangle_{L^2(\mathbb{R}^3)} \leq \frac{K}{\omega} \).

6. Global-in-time dimension reduction for the time-dependent NLS: Proof of Theorem 1.4

This section is devoted to the first part of the paper where the full 3D time-dependent NLS is discussed. First of all, using the refined Gagliardo-Nirenberg inequality, we provide a simple criteria for global existence of solutions (Proposition 6.1 and 6.2). An important remark is that the family of global solutions we consider includes solutions which evolve from a neighborhood of a constrained minimizer (Theorem 1.6). It is necessary to obtain its orbital stability (Theorem 1.8 (ii)), because orbital stability a priori-ly requires solutions nearby to exist for all time. Secondly, we establish the dimension reduction for those global solutions in the strong confinement limit, and derive the time-dependent periodic 1D NLS (Theorem 1.4).

6.1. Well-posedness of the time-dependent 3D NLS, and a criteria for global existence. We consider the initial-value problem for the 3D NLS (1.3), which is in a strong form represented as

\[
u(t) = e^{-it\omega(H_{\omega} - \Lambda_u)}u_{\omega,0} - i\kappa \int_0^t e^{-i(t-t_1)(H_{\omega} - \Lambda_u)}(|u_{\omega}|^2u_{\omega})(t_1)dt_1, \quad (6.1)
\]

where \( H_{\omega} = -\Delta + U_\omega(|y| - \sqrt{\omega}) + \omega \). This equation is locally well-posed in the weighted Sobolev space \( \Sigma \) with the norm

\[
\|u\|_{\Sigma} := \left\{ \|u\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 + \|xu\|_{L^2(\mathbb{R}^3)}^2 \right\}^{\frac{1}{2}}.
\]
Proposition 6.1 (Local well-posedness for the time-dependent 3D NLS \((1.5)\)). Let \(\kappa = \pm 1\).

(i) (Local well-posedness in \(\Sigma\)) For initial data \(u_{\omega,0} \in \Sigma\), there exist \(T > 0\) and a unique solution \(u_{\omega}(t) \in C([0,T];\Sigma) \cap L^2_t([0,T];W^{1,0})\) to the integral equation \((6.1)\).

(ii) (Conservation laws) \(u_{\omega}(t)\) preserves the mass \(M[u_{\omega}] = \|u_{\omega}\|^2_{L^2(\mathbb{R}^3)}\) and the energy

\[
E_\omega[u_{\omega}] = \frac{\omega}{2} \langle (H_\omega - \Lambda_\omega)u_{\omega}, u_{\omega} \rangle_{L^2(\mathbb{R}^3)} + \frac{\kappa \sqrt{\omega}}{4} \int_{\mathbb{R}^3} |u_{\omega}|^4 \, dx.
\]

(iii) (Blow-up criteria) Let \(T_{\text{max}} \in (0,\infty)\) be the maximal existence time, i.e., the supremum of \(T > 0\) in (i). If \(T_{\text{max}} < \infty\), then

\[
\lim_{t \to T_{\text{max}}} \langle (H_\omega - \Lambda_\omega)u_{\omega}(t), u_{\omega}(t) \rangle_{L^2(\mathbb{R}^3)} = \infty.
\]

(iv) (Global existence; defocusing case) If \(\kappa = 1\), then \(T_{\text{max}} = \infty\) and

\[
\sup_{t \geq 0} \langle (H_\omega - \Lambda_\omega)u_{\omega}(t), u_{\omega}(t) \rangle_{L^2(\mathbb{R}^3)} \leq \frac{1}{\omega} E_\omega[u_{\omega,0}].
\]

Proof. By a standard contraction mapping argument, (i) and (ii) follow. Moreover, if \(T_{\text{max}} < \infty\), then \(\|u_{\omega}(t)\|_\Sigma \to \infty\) as \(t \to T_{\text{max}}\) (see [14] for instance). Hence, together with the mass conservation and the assumption on the potential \(U_\omega\), we prove (iii). For (iv), the energy conservation yields that if \(\kappa = 1\), then \(\langle (H_\omega - \Lambda_\omega)u_{\omega}(t), u_{\omega}(t) \rangle_{L^2(\mathbb{R}^3)} \leq \frac{2}{\omega} E_\omega[u_{\omega}(t)] = \frac{2}{\omega} E_\omega[u_{\omega,0}]\), and thus (iii) implies global existence. \(\square\)

In the focusing case \((\kappa = -1)\), the 3D NLS \((1.5)\) is not globally well-posed; it admits finite blow-up solutions. Nevertheless, the following proposition asserts that solutions exist for all time under some seemingly weak additional constraint \((6.2)\).

Proposition 6.2 (A criteria for global existence; focusing case). Let \(\kappa = -1\) and \(\omega \geq 1\) be sufficiently large. Then, for fixed \(m, E > 0\), there exist small \(\delta = \delta(m, E) > 0\) and large \(K = K(m, E) > 0\), independent of large \(\omega \geq 1\), such that if \(u_{\omega,0} \in \Sigma, M[u_{\omega,0}] = m\),

\[
E_\omega[u_{\omega,0}] \leq E \quad \text{and} \quad \langle (H_{\omega}^{-\frac{1}{2}} - \Lambda_\omega)u_{\omega,0}, u_{\omega,0} \rangle_{L^2(\mathbb{R}^3)} \leq \delta_\omega,
\]

where \(H_{\omega}^{-\frac{1}{2}} := -\partial_t^2 - \frac{1}{r} \partial_r - \partial_\theta^2 + U(|y| - \sqrt{\omega}) + z^2\) with \(r = |y|\), then the solution \(u_{\omega}(t)\) to the 3D NLS \((1.5)\) with \(u_{\omega}(0) = u_{\omega,0}\) exists globally in time, and

\[
\sup_{t \geq 0} \langle (H_\omega - \Lambda_\omega)u_{\omega}(t), u_{\omega}(t) \rangle_{L^2(\mathbb{R}^3)} \leq K_{\omega}^{-1}.
\]

Proof. For the proof, a key observation is that there is an area in the function space \(\Sigma_\omega\) prohibited by the refined Gagliardo-Nirenberg inequality (Corollary 5.3). Indeed, by the assumption and Corollary 5.4 (see Remark 5.5), the initial data \(u_{\omega,0}\) satisfies the stronger bound \(\langle (H_\omega - \Lambda_\omega)u_{\omega,0}, u_{\omega,0} \rangle_{L^2(\mathbb{R}^3)} \leq K_{\omega}^{-1}\). Hence, by continuity (in \(\Sigma\)) of the nonlinear solution \(u_{\omega}(t)\), we have \(\langle (H_{\omega}^{-\frac{1}{2}} - \Lambda_\omega)u_{\omega}(t), u_{\omega}(t) \rangle_{L^2(\mathbb{R}^3)} \leq \delta_\omega\) at least on a short time interval. Then, by the conservation laws and Corollary 5.4, this bound is immediately improved to \(\langle (H_\omega - \Lambda_\omega)u_{\omega}(t), u_{\omega}(t) \rangle_{L^2(\mathbb{R}^3)} \leq K_{\omega}^{-1}\). Therefore, by iteration, we conclude that the same bound holds for all \(t \geq 0\), and \(u_{\omega}(t)\) never blows up. \(\square\)
6.2. Global well-posedness for the time-dependent periodic 1D NLS. Next, following [14, Section 3], we summarize a well-posedness result for the periodic 1D NLS,

\[ \begin{align*}
  i \partial_t w + \partial_x^2 w - \frac{\kappa}{2\pi} |w|^2 w &= 0, \\
  w(0) &= w_0 \in H^1(S^1),
\end{align*} \tag{6.3} \]

where \( w = w(t, \theta) : I(\subset \mathbb{R}) \times S^1 \rightarrow \mathbb{C} \). Note that due to some technical difficulty (see Remark 6.4 below), the dimension reduction in Theorem 1.4 is restricted to a weak-* sense. For this reason, we need uniqueness of weak solutions, and thus we provide a well-posedness result including weak solutions.

Given \( T > 0 \), we say that

\[ w = w(t, \theta) \in L^\infty([0, T]; H^1(S^1)) \cap W^{1,\infty}([0, T]; H^{-1}(S^1)) \]

is a weak (resp., strong) \( H^1(S^1) \)-solution on the interval \([0, T]\) if \( i \partial_t w + \partial_x^2 w - \frac{\kappa}{2\pi} |w|^2 w = 0 \) holds in \( H^{-1}(S^1) \) for almost every \( t \in [0, T] \) (resp., for all \( t \in [0, T] \)) and \( w(0) = w_0 \), equivalently

\[ w(t) = e^{it\partial_x^2} w_0 - \frac{i\kappa}{2\pi} \int_0^t e^{i(t-t_1)\partial_x^2} (|w|^2 w)(t_1) dt_1 \quad \text{in} \quad H^1(S^1) \]

for almost every \( t \in [0, T] \) (resp., for all \( t \in [0, T] \)) (see [14, Proposition 3.1.3]).

**Proposition 6.3** (Global well-posedness for the time-dependent 1D periodic NLS).

(i) (Global existence for strong solutions [14, Theorem 3.5.1 and Corollary 3.5.2]) For \( w_0 \in H^1(S^1) \), the initial-value problem [6.3] has a strong \( H^1(S^1) \)-solution \( w(t) \) on the interval \([0, \infty)\). Moreover, the solution \( w(t) \) preserves the mass \( M[w] = ||w||_{L^2(S^1)}^2 \) and the energy \( E[w] = \frac{1}{2} ||\partial_x w||_{L^2(S^1)}^2 + \frac{\kappa}{4\pi} ||w||_{L^4(S^1)}^4 \).

(ii) (Uniqueness of weak solutions [14, Proposition 4.2.1]) If \( w_1(t) \) and \( w_2(t) \) are weak \( H^1(S^1) \)-solutions to (6.3) with initial data \( w_0 \in H^1(S^1) \) on \([0, T]\), then \( w_1(t) = w_2(t) \) for almost every \( t \in [0, T] \).

6.3. Derivation of the time-dependent 1D periodic NLS; Proof of Theorem 1.4

We prove our first main result. Let \( u_\omega(t) \) be a solution to the time-dependent 3D NLS (1.5) constructed in Proposition 6.1. Then, by the assumptions in the theorem and Proposition 6.1 in the defocusing case/Proposition 6.2 in the focusing case, \( u_\omega(t) \) exists globally in time. In both cases, we have

\[ \sup_{t \geq 0} ((H_\omega - \Lambda_\omega) u_\omega(t), u_\omega(t))_{L^2(\mathbb{R}^3)} \lesssim \omega^{-1}. \tag{6.4} \]

6.3.1. Decomposition. For dimension reduction, it is convenient to us the transformation

\[ v_\omega(t, s, z, \theta) = \omega^{\frac{1}{4}} u_\omega(t, s + \sqrt{\omega}, z, \theta), \tag{6.5} \]
where \( u_\omega(t, r, z, \theta) \) denotes, with abuse of notation, \( u_\omega(t, x) \) in cylindrical coordinates. By construction, \( v_\omega(t) \) is a global solution to the equation

\[
i \partial_t v_\omega = \omega (\mathcal{H}_\omega^{(2D)} - \Lambda_\omega) v_\omega - \frac{1}{\sigma_\omega^2(s)} \partial_\theta^2 v_\omega + \kappa |v_\omega|^2 v_\omega
\]

(6.6) satisfying

\[
\sup_{t \geq 0} \|v_\omega(t)\|_{\Sigma_\omega} \lesssim \omega^{-\frac{1}{2}}.
\]

Let

\[
v_{\omega,||}(t, \theta) = \langle v_\omega(t, s, z, \theta), \tilde{\Phi}_\omega(s, z) \rangle_{L^2(\sigma_\omega dsdz)}
\]

(6.7) be the \( \tilde{\Phi}_\omega \)-directional component of \( v_\omega \), where \( \tilde{\Phi}_\omega = \frac{\chi_\omega(s) \Phi_\omega(s, z)}{\|\chi_\omega(s) \Phi_\omega(s, z)\|_{L^2(\sigma_\omega dsdz)}} \) is the truncated 2D lowest eigenfunction and \( \chi_\omega \) is the smooth cut-off such that \( \chi_\omega \equiv 1 \) on \( [-\sqrt{\omega} + 2, \infty) \) and \( \text{supp} \chi_\omega \subset [-\sqrt{\omega} + 1, \infty) \) (see Section 3). We decompose

\[
v_\omega(t, s, z, \theta) = v_{\omega,||}(t, \theta) \chi_\omega(s) \Phi_0(s, z) + r_\omega(t, s, z, \theta),
\]

(6.9) where the remainder is given by

\[
r_\omega(t, s, z, \theta) = v_{\omega,||}(t, \theta) \left( \tilde{\Phi}_\omega(s, z) - \chi_\omega(s) \Phi_0(s, z) \right) + (P_{\tilde{\Phi}_\omega} v_\omega)(t, s, z, \theta).
\]

(6.10) For the core part, we observe from the 3D equation (6.6) that \( v_{\omega,||} \) solves

\[
i \partial_t v_{\omega,||} = \omega ((\mathcal{H}_\omega^{(2D)} - \Lambda_\omega) v_\omega, \tilde{\Phi}_\omega)_{L^2(\sigma_\omega dsdz)} - \langle \partial_\theta^2 v_\omega, \tilde{\Phi}_\omega \rangle_{L^2(\frac{1}{\sigma_\omega} dsdz)} + \kappa \langle |v_\omega|^2 v_\omega, \tilde{\Phi}_\omega \rangle_{L^2(\sigma_\omega dsdz)}.
\]

(6.11)

Our goal is then to show that \( v_{\omega,||} \) converges to a solution to the 1D periodic NLS (6.3), and that the remainder vanishes as \( \omega \to \infty \).

Remark 6.4. For the derivation of the 1D periodic NLS, we need to extract the linear term \( \partial_\theta^2 v_{\omega,||} \) from \( \langle \partial_\theta^2 v_\omega, \tilde{\Phi}_\omega \rangle_{L^2(\frac{1}{\sigma_\omega} dsdz)} \) by showing that as \( \omega \to \infty \),

\[
\langle \partial_\theta^2 v_\omega, \tilde{\Phi}_\omega \rangle_{L^2(\frac{1}{\sigma_\omega} dsdz)} - \partial_\theta^2 v_{\omega,||} = \langle \partial_\theta^2 v_\omega, (1 - \sigma_\omega^2) \tilde{\Phi}_\omega \rangle_{L^2(\frac{1}{\sigma_\omega} dsdz)} \to 0.
\]

Indeed, we have \( (1 - \sigma_\omega^2) \tilde{\Phi}_\omega = \frac{\sigma_\omega}{\sqrt{\omega}} (2 + \frac{1}{\sqrt{\omega}}) \tilde{\Phi}_\omega \to 0 \) in a suitable norm. On the other hand, if one wishes to obtain a strong convergence, one needs to control \( \partial_\theta^2 v_\omega \). However, the 3D problem (1.5) has no conservation law controlling higher-order derivative norms, and thus it does not seem possible to obtain a global-in-time strong convergence. To avoid this difficulty, we consider weak convergence.

6.3.2. Uniform bounds on the flow \( v_{\omega,||}(t) \). We aim to show that \( v_{\omega,||}(t) \) converges weak-*ly to a solution to the periodic NLS. However, before proving this, we need to confirm that \( \|v_{\omega,||}(t)\|_{C(\mathbb{R}^3)} \) and \( \|v_{\omega,||}(t)\|_{C^1(\mathbb{R}^3)} \) are uniformly bounded in \( \omega \geq \omega_\ast \gg 1 \), where \( \|w\|_{H^{-1}(\mathbb{S}^1)} := \sup_{g \in H^1(\mathbb{S}^1)} \|g(w)\|_{L^2(\mathbb{S}^1)} \); then the Banach-Alaoglu theorem can be applied.

For \( \|v_{\omega,||}(t)\|_{H^1(\mathbb{S}^1)} \), we observe that by the change of variables (6.5) and the mass conservation law, \( \|v_\omega(t)\|_{L^2(\sigma_\omega)} = \|u_\omega(t)\|_{L^2(\mathbb{R}^3)} = \|u_0\|_{L^2(\mathbb{R}^3)} = \sqrt{m} \) for all \( t \geq 0 \). On the other
Indeed, it follows from (6.11) that the left hand side of the equation (6.11) are bounded in \( \omega \). Moreover, we have
\[
\| \frac{\partial_{\theta} v_\omega}{\omega} \|_{L^2(\frac{1}{\omega^2})} \lesssim 1
\]
for all \( t \geq 0 \). Hence, \( v_\omega, (t, \theta) = \langle v_\omega(t, \cdot, \cdot, \theta), \tilde{\Phi}_\omega \rangle_{L^2(\sigma_d s dz)} \) satisfies
\[
\| v_\omega, (t) \|_{L^2(S^1)} \leq \| v_\omega(t) \|_{L^2(\sigma_\omega)} = \sqrt{m}
\]
and
\[
\| \partial_{\theta} v_\omega, (t) \|_{L^2(S^1)} \leq \left\| \| \partial_{\theta} v_\omega(t) \|_{L^2(\frac{1}{\omega^2} s dz)} \| \sigma_\omega \tilde{\Phi}_\omega \|_{L^2(\sigma_\omega)} \right\|_{L^2(S^1)} \lesssim \| \partial_{\theta} v_\omega(t) \|_{L^2(\frac{1}{\omega^2})} \lesssim 1,
\]
because \( \Phi_\omega \) is a rapidly decreasing function. Thus, we conclude that for all \( t \geq 0 \),
\[
\sup_{\omega \geq \omega_*} \| v_\omega, (t) \|_{H^1(S^1)} \lesssim 1.
\]

For a uniform bound on \( \| \partial_{\theta} v_\omega, (t) \|_{H^{-1}(S^1)} \), it suffices to show that all terms on the right hand side of the equation (6.11) are bounded in \( H^{-1}(S^1) \). First, we claim that for all \( t \geq 0 \),
\[
\lim_{\omega \to \infty} \| \omega \langle (H_w^{(2D)} - \Lambda_\omega) v_\omega(t), \tilde{\Phi}_\omega \rangle_{L^2(\sigma_\omega s dz)} \|_{L^2(S^1)} = 0. \tag{6.14}
\]
Indeed, it follows from \( (H_w^{(2D)} - \Lambda_\omega) \Phi_\omega = 0 \), with \( \tilde{\Phi}_\omega (s, z) = \chi_\omega(s) \Phi_\omega(s, z) \), that
\[
\| \langle \omega (H_w^{(2D)} - \Lambda_\omega) v_\omega(t), \tilde{\Phi}_\omega \rangle_{L^2(\sigma_\omega s dz)} \|_{L^2(S^1)}
\]
\[
= \omega \langle v_\omega, (-\chi_\omega'' \Phi_\omega - 2\chi_\omega' \partial s \Phi_\omega - \frac{1}{\sqrt{\omega} \sigma_\omega} \chi_\omega' \Phi_\omega) \rangle_{L^2(\sigma_\omega s dz)} \|_{L^2(S^1)}
\]
\[
\leq \| v_\omega \|_{L^2(\sigma_\omega) \omega} \| \chi_\omega'' \Phi_\omega + 2\chi_\omega' \partial s \Phi_\omega + \frac{1}{\sqrt{\omega} \sigma_\omega} \chi_\omega' \Phi_\omega \|_{L^2(\sigma_\omega s dz)}.
\]
However, since \( \Phi_\omega \) is rapidly decreasing and \( \chi_\omega'(s) \) and \( \chi_\omega''(s) \) are supported only on \([-\sqrt{\omega} + 1, -\sqrt{\omega}] \), we have
\[
\| \langle \omega (H_w^{(2D)} - \Lambda_\omega) v_\omega(t), \tilde{\Phi}_\omega \rangle_{L^2(\sigma_\omega s dz)} \|_{L^2(S^1)} \leq \sqrt{m} \omega \omega (1) \to 0.
\]
Moreover, we have
\[
\| \langle \partial_{\theta}^2 v_\omega(t), \tilde{\Phi}_\omega \rangle_{L^2(\frac{1}{\omega^2} s dz)} \|_{H^{-1}(S^1)} \lesssim \| \partial_{\theta} v_\omega(t), \tilde{\Phi}_\omega \rangle_{L^2(\frac{1}{\omega^2} s dz)} \|_{L^2(S^1)}
\]
\[
\leq \| \partial_{\theta} v_\omega(t) \|_{L^2(\frac{1}{\omega^2})} \| \tilde{\Phi}_\omega \|_{L^2(\frac{1}{\omega^2} s dz)} \lesssim 1 \quad (\text{by (6.12)}).
\]
On the other hand, the energy conservation law \( E_\omega[v_\omega(t)] = E_\omega[v_\omega(0)] \) (equivalent to \( E_\omega[u_\omega(t)] = E_\omega[u_\omega(0)] \)) and (6.7) yield
\[
\sup_{\omega \geq \omega_*} \| v_\omega(t) \|_{L^2(\sigma_\omega)} \lesssim 1
\]
for all \( t \geq 0 \). Consequently, we obtain
\[
\| \langle v_\omega^2 v_\omega(t), \tilde{\Phi}_\omega \rangle_{L^2(\sigma_\omega s dz)} \|_{H^{-1}(S^1)} \leq \| \| v_\omega(t) \|_{L^2(\sigma_\omega s dz)} \|_2 \| \tilde{\Phi}_\omega \|_{L^4(\sigma_\omega s dz)} \|_{L^4(S^1)}
\]
\[
\lesssim \| v_\omega(t) \|_{L^4(\sigma_\omega)} \lesssim 1.
\]
Therefore, collecting all, by the equation (6.11), we conclude that for all \( t \geq 0 \),
\[
\sup_{\omega \geq \omega_*} \| \partial_t u_{\omega_*} (t) \|_{H^{-1}(\mathbb{S}^1)} \lesssim 1.
\]

6.3.3. Dimension reduction: Proof of Theorem 1.4 (i). For the dimension reduction, it suffices to show that for all \( t \geq 0 \),
\[
\| u_{\omega_*} (t, \theta) (\Phi_\omega(s, z) - \chi_\omega (s) \Phi_0(s, z)) \|_{L^2(\sigma_\omega)} + \| (P^+_{\Phi_\omega} u_{\omega_*} (t)) \|_{L^2(\sigma_\omega)} \lesssim \omega^{-\frac{1}{2}}, \tag{6.15}
\]
because
\[
\left\| (1 - \chi(|y|)) u_{\omega_*} (t, \theta) \omega^{-1/4} \Phi_0(|y| - \sqrt{\omega}, z) \right\|_{L^2(\mathbb{R}^3)} \lesssim \omega^{-\frac{1}{2}}
\]
and by the decomposition (6.10) and the change of variables by (6.5), (6.15) implies
\[
\left\| u_{\omega_*} (t, x) - u_{\omega_*} (t, \theta) \left( \chi(|y|) \omega^{-1/4} \Phi_0(|y| - \sqrt{\omega}, z) \right) \right\|_{L^2(\mathbb{R}^3)} = \| r_{\omega_*} (t) \|_{L^2(\sigma_\omega)}.
\]

Indeed, since both \( \phi_0 \) and \( \phi_\omega \) are rapidly decreasing and \( \| \phi_\omega - \phi_0 \|_{L^2(\sigma_\omega)} \lesssim \omega^{-\frac{1}{2}} \) (see Proposition 3.2 and Corollary 3.3), by the uniform bound on \( \| u_{\omega_*} (t) \|_{H^1(\mathbb{S}^1)} \), we prove that
\[
\| u_{\omega_*} (t, \theta) (\Phi_\omega(s, z) - \chi_\omega (s) \Phi_0(s, z)) \|_{L^2(\sigma_\omega)} = \| u_{\omega_*} (t) \|_{L^2(\mathbb{S}^1)} \| \Phi_\omega(s, z) - \chi_\omega (s) \Phi_0(s, z) \|_{L^2(\sigma_\omega)} \lesssim \omega^{-\frac{1}{2}}.
\]

On the other hand, Lemma 4.3 implies that \( \| (P^+_{\Phi_\omega} u_{\omega_*} (t)) - (P^+_{\Phi_\omega} u_{\omega_*} (t)) \|_{L^2(\sigma_\omega)} \lesssim \omega^{-\frac{1}{2}} \). Moreover, by the spectral gap (Corollary 3.3 and (6.7), we have
\[
\| (P^+_{\Phi_\omega} u_{\omega_*} (t)) \|_{L^2(\sigma_\omega)} \lesssim \| (P^+_{\Phi_\omega} u_{\omega_*} (t)) \|_{\Sigma_{\omega, (s, z)}} \leq \| u_{\omega_*} (t) \|_{\Sigma_\omega} \lesssim \omega^{-\frac{1}{2}}.
\]

6.3.4. Initial data. Now, we present how the initial data for the 1D NLS is prepared in Theorem 1.4 (ii). We consider the decomposition (6.9) at \( t = 0 \). Indeed, by (6.4) and the dimension reduction (Theorem 1.4 (i)), we have \( \| u_{\omega_0, 0} \|_{H^1(\mathbb{R}^3)} \lesssim 1 \) and
\[
\lim_{\omega \to \infty} \| u_{\omega_0, 0} (x) - u_{\omega_0} (0, \theta) \left( \omega^{-\frac{1}{4}} \Phi_0(|y| - \sqrt{\omega}, z) \right) \|_{L^2(\mathbb{R}^3)} = 0.
\]

On the other hand, by the \( H^1(\mathbb{S}^1) \)-uniform bound (6.13), it follows from the Banach-Alaoglu theorem that there exists \( \{ \omega_j \}_{j=1}^\infty \), with \( \omega_j \to \infty \), such that \( v_{\omega_j} (0, \theta) \to u_{\infty, 0} \) in \( H^1(\mathbb{S}^1) \).
Therefore, we conclude that as \( j \to \infty \),
\[
u_{\omega_j, 0} (x) - w_{\infty, 0} (\theta) \left( \omega_j^{-\frac{1}{4}} \Phi_0(|y| - \sqrt{\omega_j}, z) \right) \to 0 \quad \text{in} \quad H^1(\mathbb{R}^3).
\]
6.3.5. Derivation of the 1D NLS. Fix arbitrary $T > 0$, and let $\{v_{\omega}(t)\}_{\omega > 1}$ be a one-parameter family of solutions to (6.11) with $v_{\omega}(0) = (v_{\omega}(0), \tilde{\Phi}_\omega)_{L^2(\sigma_\omega dsdz)}$. We have $v_{\omega_j}(0) \to w_{\infty,0}$ in $H^1(S^1)$ for some $\{\omega_j\}_{j=1}^\infty$ with $\omega_j \to \infty$. On the other hand, by the uniform bounds obtained previously and the Banach-Alaoglu theorem, we have subsequential convergences $v_{\omega_{j_k}} \rightharpoonup w_{\infty}$ in $L^\infty([0,T]; H^1(S^1))$ and $\partial_t v_{\omega_{j_k}} \rightharpoonup \partial_t w_{\infty}$ in $L^\infty([0,T]; H^{-1}(S^1))$ as $j_k \to \infty$. Therefore, by uniqueness of weak solutions (Proposition 6.3), it suffices to show that the limit $w_{\infty}(t)$ is a weak $H^1(S^1)$-solution to the 1D NLS (6.11).

To prove it, we fix an arbitrary $g = g(t, \theta) \in L^1([0,T]; H^1(S^1))$, and test $g$ against the equation (6.11). Then, by (6.11) and Fubini’s theorem, one can write

$$
\langle g, i\partial_t v_{\omega} \rangle = \langle g, \omega \langle (H_\omega^{(2D)} - \Lambda_\omega) v_{\omega}, \tilde{\Phi}_\omega \rangle_{L^2(\sigma_\omega dsdz)} \rangle
+ \langle (\partial_\theta g, \partial_\theta v_{\omega}) \rangle + \langle (\partial_{\theta g})(1 - \sigma_\omega^2) \tilde{\Phi}_\omega, \partial_\theta v_{\omega}, \|L^2([0,T]; L^2(\sigma_\omega)) \rangle
+ \kappa \langle g \tilde{\Phi}_\omega, |v_{\omega}||\chi_\omega \Phi_0|^2 |(v_{\omega}||\chi_\omega \Phi_0)|_{L^2([0,T]; L^2(\sigma_\omega))} \rangle
+ \kappa \langle g \tilde{\Phi}_\omega, |v_{\omega}|^2 v_{\omega} - |v_{\omega}||\chi_\omega \Phi_0|^2 |(v_{\omega}||\chi_\omega \Phi_0)|_{L^2([0,T]; L^2(\sigma_\omega))} \rangle
= : I_\omega + II_\omega + III_\omega + IV_\omega + \kappa V_\omega,
$$

where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2([0,T]; L^2(S^1))}$. By construction, it is obvious that $\langle g, \partial_t v_{\omega_{j_k}} \rangle \to \langle g, \partial_t w_{\infty} \rangle$ and $II_{\omega_{j_k}} \to \langle g, -\partial^2_\theta w_{\infty} \rangle$. By (6.14), $I_\omega \to 0$. We will show that $III_\omega, V_\omega \to 0$ and

$$IV_{\omega_{j_k}} \to \langle g, \frac{1}{2\pi} |w_{\infty}|^2 w_{\infty} \rangle.
$$

Then, we conclude that $w_{\infty}(t)$ is a weak solution, so the proof is completed.

For $III_\omega$, by the Hölder inequality, we obtain that

$$|III_\omega| = \left| \langle (\partial_\theta g)(1 - \sigma_\omega^2) \tilde{\Phi}_\omega, \partial_\theta v_{\omega}, \|L^2([0,T]; L^2(\frac{1}{\sigma_\omega})) \rangle \right|
\leq \|\partial_\theta g\|_{L^1([0,T]; L^2(\sigma_1))} \|1 - \sigma_\omega^2\|_{L^1(\sigma_\omega dsdz)} \|\partial_\theta v_{\omega}\|_{L^\infty([0,T]; L^2(\sigma_\omega))}.
$$

Because $1 - \sigma_\omega(s) = \frac{1}{\sqrt{\omega}}$ and $\tilde{\Phi}_\omega$ is rapidly decreasing, $\|1 - \sigma_\omega^2\|_{L^2(\frac{1}{\sigma_\omega})} \leq 1$. Thus, it follows that $III_\omega \to 0$. For $IV_\omega$, by the convergence $\Phi_\omega \to \Phi_0$ in $L^2(\sigma_\omega dsdz)$ with their Gaussian decay and $\|\Phi_0\|_{L^4(\mathbb{R}^2)}^4 = \frac{1}{2\pi}$, we write

$$IV_\omega = \langle \tilde{\Phi}_\omega, \chi_\omega \Phi_0 |^2 \chi_\omega \Phi_0 \rangle_{L^2(\sigma_\omega dsdz)} \langle g, |v_{\omega}|^2 v_{\omega} \rangle = \langle g, \frac{1}{2\pi} |v_{\omega}|^2 v_{\omega} \rangle + o_1(1).
$$

Then, the Rellich-Kondrachov theorem $H^1(S^1) \to L^2(S^1)$ implies $IV_{\omega_{j_k}} \to \langle g, \frac{1}{2\pi} |w_{\infty}|^2 w_{\infty} \rangle$. For the convergence $V_\omega \to 0$, by the Hölder inequality and the decomposition (6.9), it suffices to show that $\|r_{\omega}\|_{L^4(\sigma_\omega)} \to 0$, where $r_{\omega} = v_{\omega} - (\tilde{\Phi}_\omega - \chi_\omega \Phi_0) + P \frac{1}{P_\omega} v_{\omega}$ (see (6.10)). Indeed, $\|v_{\omega}\|_{L^4(\sigma_\omega)} \leq \|v_{\omega}\|_{L^4(S^1)} \|\tilde{\Phi}_\omega - \chi_\omega \Phi_0\|_{L^4(\sigma_\omega dsdz)} \leq \|v_{\omega}\|_{H^1(S^1)} \|\tilde{\Phi}_\omega - \chi_\omega \Phi_0\|_{L^4(\sigma_\omega dsdz)} \to 0$. On the other hand, by the refined Gagliardo-Nirenberg inequality

$9w_{\infty}(0) = w_{\infty,0}$, because $v_{\omega_{j_k}}$ is a subsequence of $v_{\omega_j}$ and $v_{\omega_j}(0) \to w_{\infty,0}$ in $H^1(S^1)$. 


(Corollary 5.3), we can show that \( \|P_{\perp} \tilde{\Phi} \omega v \|_{L^4(\sigma)} \to 0 \), because
\[
\| \partial_\theta P_{\perp} \tilde{\Phi} \omega v \|_{L^2(\frac{1}{\sigma})} \lesssim \| \partial_\theta v \|_{L^2(\frac{1}{\sigma})} \leq \sqrt{\omega} \| v \|_{\Sigma_\omega} \lesssim 1 \quad \text{(by (4.5) and (6.7))},
\]
\[
\|P_{\perp} \tilde{\Phi} \omega v \|_{L^4(\sigma)} \lesssim \| \partial_\theta v \|_{L^2(\frac{1}{\sigma})} \lesssim \sqrt{\omega} \| v \|_{\dot{\Sigma}_\omega} \lesssim \omega^{-\frac{1}{2}} \quad \text{(by Lemma 4.4 and (6.7))}.
\]
Therefore, it follows that \( \| r_{\omega} \|_{L^4(\sigma)} \to 0 \).

7. Existence of a minimizer and its dimension reduction limit: Proof of Theorem 1.6 and 1.7

We consider the modified nonlinear minimization problem
\[
J_\omega^{(3D)}(m) := \min \left\{ E_\omega(v) : v \in \Sigma_\omega, \ M_\omega(v) = m \text{ and } \| v \|_{\dot{\Sigma}_\omega; (s,z)} \leq \delta \sqrt{\omega} \right\}.
\]
(7.1)

In this section, we prove 1) existence of a minimizer (Proposition 7.1); 2) its uniform decay/bound properties (Proposition 7.5 and Proposition 7.7); 3) the dimension reduction to the 1D periodic ground state (Proposition 7.11).

7.1. Construction of a minimizer. By a concentration-compactness argument, we establish existence of a minimizer for the variational problem \( J_\omega^{(3D)}(m) \), but we also present some basic properties directly coming from its construction.

**Proposition 7.1** (Existence of a minimizer and its preliminary properties). For sufficiently large \( \omega \geq 1 \), the following hold.

(i) The minimization problem \( J_\omega^{(3D)}(m) \) occupies a positive minimizer, denoted by \( Q_\omega \).

(ii) A minimizer \( Q_\omega \) solves the Euler-Lagrange equation
\[
\omega(\mathcal{H}_\omega^{(2D)} - \Lambda_\omega)Q_\omega - \frac{1}{\sigma_\omega^2} \partial_\theta^2 Q_\omega - |Q_\omega|^2 Q_\omega = -\mu_\omega Q_\omega,
\]
(7.2)
where \( \mu_\omega \in \mathbb{R} \) is a Lagrange multiplier. Thus, it satisfies
\[
\omega \| Q_\omega \|_{\Sigma_\omega; (s,z)}^2 + \| \partial_\theta Q_\omega \|_{L^2(\frac{1}{\sigma_\omega})}^2 - \| Q_\omega \|_{L^1(\sigma_\omega)}^4 + \mu_\omega \| Q_\omega \|_{L^2(\sigma_\omega)}^2 = 0.
\]
(7.3)

(iii) \( \sqrt{\omega} \| Q_\omega \|_{\Sigma_\omega; (s,z)} \), \( \| \partial_\theta Q_\omega \|_{L^2(\frac{1}{\sigma_\omega})} \), \( \| Q_\omega \|_{L^6(\sigma_\omega)} \) and \( |\mu_\omega| \) are bounded uniformly in \( \omega \).

**Remark 7.2.** As mentioned in the introduction, the energy minimization only under a mass constraint is super-critical in that the energy is not bounded from below, but it is also considered partially sub-critical in a certain regime (\( \omega \gg 1 \)). To capture the sub-critical nature of the problem, the additional constraint \( \| v \|_{\dot{\Sigma}_\omega; (s,z)} \leq \delta \sqrt{\omega} \) is imposed, and we analyze the energy minimization problem using the refined Gagliardo-Nirenberg inequality (Corollary 5.3). We refer to [3, 29, 30, 36] for similar settings.

For the proof, we first show a uniform negative upper bound on the lowest energy.
Lemma 7.3 (Negative minimum energy). For sufficiently large $\omega \geq 1$, we have

$$J_0^{(3D)}(m) \leq J_1^{(1D)}(m) + O(\omega^{-\frac{1}{2}}) < 0.$$ 

Proof. We employ $\chi_{\omega} Q_\omega Q_\infty = \chi_{\omega}(s) \Phi_\omega(s, z) Q_\infty(\theta)$, where $Q_\infty$ is the ground state for 1D circle problem $J_0^{(1D)}(m)$, $\Phi(s, z) = \phi_\omega(s) \phi_\infty(z)$, $\chi_{\omega} = \chi(\cdot + \sqrt{\omega})$ and $\chi : [0, \infty) \to [0, 1]$ is given by $\chi \equiv 0$ on $[0, 1]$ and $\chi \equiv 1$ on $[2, \infty)$. Then, it follows from the fast decay of $\phi_\omega$ (Proposition 7.2 (ii)) and $(\mathcal{H}_\omega^{(2D)} - \Lambda_\omega) \Phi_\omega = 0$ that $\|\chi_{\omega} \Phi_\omega Q_\infty\|_{L^2(\sigma_\omega)}^2 = m + O(\omega^{-\frac{1}{2}})$, $\|\chi_{\omega} \Phi_\omega Q_\infty\|_{\Sigma \omega(s, z)} = O(\omega^{-\frac{1}{2}})$ and $E(\chi_{\omega} \Phi_\omega Q_\infty) = E(\Phi_\omega Q_\infty) + O(\omega^{-\frac{1}{2}})$. Therefore, we conclude that $J_0^{(3D)}(m) \leq \epsilon E(\chi_{\omega} \Phi_\omega Q_\infty) = E(\chi_{\omega} \Phi_\omega Q_\infty) + O(\omega^{-\frac{1}{2}}) = J_1^{(1D)}(m) + O(\omega^{-\frac{1}{2}})$. In addition, using the constant function $\sqrt{\frac{m}{2\pi}}$, we get $J_0^{(1D)}(m) < 0$ (refer to [27] Proposition 3.2). \[\Box\]

Remark 7.4. Lemma 7.3 assures that Corollary 5.4 can be applied to a minimizing sequence for $J_0^{(3D)}(m)$. Consequently, the minimum energy level is finite, and the constraint $\|v\|_{\Sigma \omega(s, z)} \leq \delta \sqrt{\omega}$ is immediately strengthened to $\|v\|_{\Sigma \omega} \leq \sqrt{\frac{m}{2\pi}} \omega^{-\frac{1}{2}}$. Therefore, a minimizer (if it exists) does not meet the boundary $\|v\|_{\Sigma \omega(s, z)} = \delta \sqrt{\omega}$ of the constraint so that the Euler-Lagrange equation can be expressed with one multiplier rather than two.

Proof of Proposition 7.1. Suppose that $\omega \geq 1$ is sufficiently large. Let $\{v_n\}_{n=1}^\infty$ be a minimizing sequence for $J_0^{(3D)}(m)$, that is, $E_\omega(v_n) \to J_0^{(3D)}(m)$ as $n \to \infty$. Then, Lemmas 7.3 and 5.4 imply that $\|v_n\|_{\Sigma \omega}^2$ is bounded uniformly in $n$ (see (2.9) for the definition of the norm $\|\cdot\|_{\Sigma \omega}$), and thus we may assume that $v_n \to v$ in $\Sigma \omega$ as $n \to \infty$.

We claim that

$$\lim_{n \to \infty} \|v_n - v\|_{L^2(\sigma_\omega)} = 0. \tag{7.4}$$

Indeed, by the assumption on $U_\omega$ (H2), for any $0 > \epsilon > 0$, there exists $R_\epsilon > 0$, independent of $n$, such that

$$\int_{B^1} \int_{U_\omega(s, z)} |v_n - v|^2 \sigma_\omega(s) dsdzd\theta = \epsilon \int_{B^1} \int_{-\infty}^{\infty} \int_{-\sqrt{\omega} \cdot \epsilon}^{\infty} (U_\omega(s) + z^2) |v_n - v|^2 \sigma_\omega(s) dsdzd\theta \leq \epsilon \|v_n - v\|_{\Sigma \omega} \lesssim \epsilon \sup_{n \in \mathbb{N}} \|v_n\|_{\Sigma \omega},$$

where the implicit constant is independent of $n$. On the other hand, it follows from the Rellich-Kondrachov compactness theorem\textsuperscript{10} that

$$\|1_{(s, z) \in [-R_\epsilon, R_\epsilon]^2 \cap ((-\sqrt{\omega} \cdot \epsilon) \times \mathbb{R})} (v_n - v)\|_{L^2(\sigma_\omega)} \to 0.$$ 

Thus, the claim (7.4) follows.

\textsuperscript{10}By $u_n(s, z, \theta) = \omega^{-\frac{1}{2}} u_n(s + \sqrt{\omega}, z, \theta)$ (see (2.1)), $\|u_n\|_{H^1(\mathbb{R}^3)} \leq \|v_n\|_{\Sigma \omega}$. Then we have $u_n \to u$ in $L^2_{\text{loc}}(\mathbb{R}^3)$, and this implies that $\|1_{(s, z) \in [-R_\epsilon, R_\epsilon]^2 \cap ((-\sqrt{\omega} \cdot \epsilon) \times \mathbb{R})} (v_n - v)\|_{L^2(\sigma_\omega)} \to 0$. 


By the claim (7.4), we have \( \|v\|_{L^2(\sigma_\omega)}^2 = m \). Moreover, the Gagliardo-Nirenberg inequality (Corollary 5.3) implies \( v_n \to v \) in \( L^4(\sigma_\omega) \). Hence, it follows from the weak convergence \( v_n \to v \) in \( \Sigma_\omega \) that

\[
J_\omega^{(3D)}(m) + o_n(1) = \mathcal{E}_\omega(v_n) - \frac{\omega}{2}\|v_n\|_{\Sigma_\omega}^2 - \frac{1}{4}\|v_n\|_{L^4(\sigma_\omega)}^4
= \mathcal{E}_\omega(v) + \mathcal{E}_\omega(v_n - v) + o_n(1)
\geq J_\omega^{(3D)}(m) + \frac{\omega}{2}\|v_n - v\|_{\Sigma_\omega}^2 + o_n(1).
\]

Therefore, we conclude that \( \|v_n - v\|_{\Sigma_\omega} \to 0 \) and the limit \( v \) is a minimizer.

We denote a minimizer \( v \) by \( Q_\omega \). Then, by Remark 7.4 and integration by parts, one can derive the Euler-Lagrange equation (7.2) and the identity (7.3). On the other hand, we recall that by the relation (2.1), \( R_\omega(x) = R_\omega(r, z, \theta) = \omega^{-\frac{1}{4}}Q_\omega(r - \sqrt{\omega}, z, \theta) \) solves the equivalent nonlinear elliptic equation

\[
(-\Delta_x + U(\|y\| - \sqrt{\omega}) + z^2 + \Lambda_\omega)R_\omega - \omega^{-\frac{1}{4}}|R_\omega|^2R_\omega = -\frac{\mu_\omega}{\omega}R_\omega \quad \text{in} \quad \mathbb{R}^3.
\]

Then we may assume that \( Q_\omega \) is non-negative, because \( \mathcal{E}_\omega(Q_\omega) \geq \mathcal{E}_\omega(\|Q_\omega\|) \). Thus, the strong maximum principle can be applied to \( R_\omega \) to prove that \( Q_\omega \) is positive.

For (iii), we observe from Corollary 5.3 Lemma 7.3 and the refined Gagliardo-Nirenberg inequality (Corollary 5.3) that \( \sqrt{\omega}\|Q_\omega\|_{\Sigma_\omega} \) and \( \|Q_\omega\|_{L^4(\sigma_\omega)} \) are uniformly bounded, and so is \( |\mu_\omega| \) by the identity (7.3). It remains to estimate \( \|Q_\omega\|_{L^6(\sigma_\omega)} \). We will estimate \( P_{\Phi_\omega} Q_\omega \) and \( P_{\Phi_\omega}^\perp Q_\omega \) separately. Indeed, it is obvious that

\[
\|P_{\Phi_\omega} Q_\omega\|_{L^6(\sigma_\omega)} \leq \|Q_\omega\|_{L^6(\Sigma^1)} \|\Phi_\omega \chi_\omega\|_{L^6(\sigma_\omega, dsdz)} \|\Phi_\omega \chi_\omega\|_{L^6(\sigma_\omega, dsdz)} \lesssim \|Q_\omega\|_{L^6(\Sigma^1)}.
\]

Hence, it follows from the Gagliardo-Nirenberg inequality on \( S^1 \) (5.5), the \( v_n \)-bounds (Lemma 4.2) and uniform boundedness of \( \|\partial_\theta Q_\omega\|_{L^2(\frac{1}{\omega})} \) that

\[
\|P_{\Phi_\omega} Q_\omega\|_{L^6(\sigma_\omega)} \lesssim \|Q_\omega\|_{L^2(\Sigma^1)} \|\partial_\theta Q_\omega\|_{L^2(\Sigma^1)} + \|Q_\omega\|_{L^2(\Sigma^1)} \lesssim \|Q_\omega\|_{L^2(\sigma_\omega)} \|\partial_\theta Q_\omega\|_{L^2(\frac{1}{\omega})} + \|Q_\omega\|_{L^2(\sigma_\omega)} \lesssim 1.
\]

For \( P_{\Phi_\omega}^\perp Q_\omega \), by the Sobolev inequality (5.1), we obtain

\[
\|P_{\Phi_\omega}^\perp Q_\omega\|_{L^6(\sigma_\omega)} \lesssim \omega^{-\frac{1}{2}} \left\{ \omega \|\nabla_{(s,\varepsilon)}P_{\Phi_\omega}^\perp Q_\omega\|_{L^2(\sigma_\omega)}^2 + \|\partial_\theta P_{\Phi_\omega}^\perp Q_\omega\|_{L^2(\frac{1}{\omega})}^2 \right\}^{\frac{1}{2}}.
\]

Then, applying Lemma 4.4 to the first term and the asymptotic Pythagorean identity (4.5) to the second term on the right hand side of the above inequality, we prove that

\[
\|P_{\Phi_\omega}^\perp Q_\omega\|_{L^6(\sigma_\omega)} \lesssim \omega^{-\frac{1}{4}} \left\{ \omega \|Q_\omega\|_{\Sigma_\omega}^2 + \omega^{-\frac{1}{2}} \right\}^{\frac{1}{2}}.
\]

Therefore, \( \|P_{\Phi_\omega}^\perp Q_\omega\|_{L^6(\sigma_\omega)} \) is also uniformly bounded. \( \square \)
7.2. Uniform Gaussian decay of an energy minimizer. The purpose of this subsection is to provide more detailed information about an energy minimizer constructed in Proposition 7.1.

**Proposition 7.5** (Uniform Gaussian weighted $L^2$ bounds). For any $c \in (0, \alpha_0)$, if $\omega \geq 1$ is large enough, then we have
\[
\|e^{c(s^2+z^2)}Q_\omega\|_{L^2(\omega)}^2 + \|e^{c(s^2+z^2)}Q_\omega\|_{\Sigma_{\omega}(s,z)}^2 + \frac{1}{\omega}\|\partial_\theta Q_\omega\|_{L^2(\frac{1}{\omega})}^2 \lesssim 1, \tag{7.6}
\]
\[
\|\sigma_\omega e^{c(s^2+z^2)}(H_\omega^{(2D)} - \lambda_\omega)Q_\omega\|_{L^2(\omega)}^2 + \frac{1}{\omega\omega^2}\|e^{c(s^2+z^2)}\partial_\theta^2 Q_\omega\|_{L^2(\frac{1}{\omega})}^2 \lesssim 1, \tag{7.7}
\]
where $\alpha_0 > 0$ is a constant given in Proposition 7.1.

**Remark 7.6.**
(i) Proposition 7.5 proves very strong localization near the ring $s = 0$ and $z = 0$ (or $|y| = \sqrt{\omega}$ and $z = 0$).
(ii) By the definition of the semi-norm (see (2.7)), $\|e^{c(s^2+z^2)}\nabla(s,z)Q_\omega\|_{L^2(\omega)} \lesssim 1$.
(iii) For the angle derivatives, we only have inequalities $\|e^{c(s^2+z^2)}\partial_\theta Q_\omega\|_{L^2(\frac{1}{\omega})} \lesssim \sqrt{\omega}$ and $\|e^{c(s^2+z^2)}\partial_\theta^2 Q_\omega\|_{L^2(\frac{1}{\omega})} \lesssim \omega$ with increasing upper bounds. However, we still have good localization; there exists $K > 0$ such that
\[
\|1_{(s,z)}|K\sqrt{\omega}\partial_\theta Q_\omega\|_{L^2(\frac{1}{\omega})} \lesssim \omega^{-cK^2+\frac{1}{2}}, \|1_{(s,z)}|K\sqrt{\omega}\partial_\theta^2 Q_\omega\|_{L^2(\frac{1}{\omega})} \lesssim \omega^{-cK^2+1}.
\]

As mentioned in Remark 7.6, the angle derivative bounds in Proposition 7.5 are rather weak near the ring $s = 0$ and $z = 0$. For the first-order derivative, we, on the other hand, have $\|\partial_\theta Q_\omega\|_{L^2(\frac{1}{\omega})} \lesssim 1$ (Proposition 7.1 (iii)). The following result refines some crudeness for the second-order derivative.

**Proposition 7.7** (Uniform $L^2(\omega)$ bounds for angle derivatives).
\[
\|\partial_\theta^2 Q_\omega\|_{L^2(\frac{1}{\omega})} \lesssim 1.
\]

**Remark 7.8.** By Proposition 7.1 (iii) and Proposition 7.7, $\|\partial_\theta Q_\omega\|_{L^2(\frac{1}{\omega})}$ and $\|\partial_\theta^2 Q_\omega\|_{L^2(\frac{1}{\omega})}$ are uniformly bounded, and so is $\|Q_\omega\|_{L^2(S^1)}$ due to Lemma 7.2. Hence, by the Sobolev embedding $H^2(S^1) \hookrightarrow C^1(S^1)$, $Q_\omega$ and $Q_\omega$ are uniformly bounded point-wisely.

As a first step, we prove a primitive $L^\infty$-bound.

**Lemma 7.9** ($L^\infty$-bound).
\[
\text{ess sup}_{(s,z,\theta) \in [-\sqrt{\omega}, \infty) \times \mathbb{R} \times S^1} |Q_\omega(s,z,\theta)| = \|Q_\omega\|_{L^\infty} \lesssim \omega^\frac{1}{4}.
\]

For the proof, we employ a Sobolev-type inequality from the semi-group theory, since it is a convenient tool to deal with non-negative external potentials.

**Lemma 7.10** (Semigroup Sobolev inequality). It holds that
\[
\text{ess sup}_{(s,z,\theta) \in [-\sqrt{\omega}, \infty) \times \mathbb{R} \times S^1} |v(s,z,\theta)| = \|v\|_{L^\infty} \lesssim \omega^\frac{3}{4} \left\| \left( H_\omega^{(2D)} - \frac{1}{\omega \sigma_\omega} \partial_\theta^2 + 1 \right) v \right\|_{L^2(\omega)}.
\]
Proof. By $v(s, z, \theta) = \omega^\frac{4}{3} u(s + \sqrt{\omega}, z, \theta)$ with $|y| = s + \sqrt{\omega}$, the left and the right hand sides of the inequality respectively can be written as $\|v\|_{L^\infty} = \omega^\frac{4}{3} u\|u\|_{L^\infty(\mathbb{R}^3)}$ and

\[
\left\| \left( \mathcal{H}_\omega^{(2D)} - \frac{1}{\omega \sigma^2_\omega} \nabla^2 \right) v \right\|_{L^3(\sigma_\omega)} = \|(-\Delta + U(|y| - \sqrt{\omega}) + z^2 + 1)u\|_{L^2(\mathbb{R}^3)}.
\]

Thus, the inequality is equivalent to $\|u\|_{L^\infty(\mathbb{R}^3)} \lesssim \|(-\Delta_x + U_s(|y| - \sqrt{\omega}) + z^2 + 1)u\|_{L^2(\mathbb{R}^3)}$, which indeed holds true by the theory of Schrödinger semigroups [39, Theorem B.2.1]. \(\square\)

Proof of Lemma 7.9. By the Euler-Lagrange equation (7.2), it follows from Proposition 7.11 (iii) that $\|H_{\omega}^{(2D)} - \frac{1}{\omega \sigma^2_\omega} \nabla^2 + 1\| L^2(\sigma_\omega) \leq \|\Lambda + 1 - \frac{\mu_\omega}{\omega}\| L^2(\sigma_\omega) + \frac{1}{\omega}\|Q_\omega\|_{L^3(\sigma_\omega)}^3$ is uniformly bounded. Hence, Lemma 7.10 proves the desired bound. \(\square\)

Next, we show Gaussian weighted $L^2(\sigma_\omega)$-bounds following the argument of Agmon [1] as we did in the proof of Proposition 3.2 (ii).

Proof of Proposition 7.5. In the proof, if there is no confusion, we omit the integral domain

\[
\int \int \int dsdzd\theta = \int_{\mathbb{R}^1} \int_{-\infty}^{\infty} \int_{-\sqrt{\omega}}^{\sqrt{\omega}} dsdzd\theta.
\]

Let $g(s, z) : [-\sqrt{\omega}, \infty) \times \mathbb{R} \to \mathbb{R}$ be a bounded smooth function with $g(s, z) = 1$ for large $|(s, z)| > 1$ to be chosen later. Then, it is obvious from the Euler-Lagrange equation (7.2) that

\[
\left( \mathcal{H}_\omega^{(2D)} - \Lambda - \frac{1}{\omega \sigma^2_\omega} \nabla^2 \right)(Q_\omega g) = \frac{Q_\omega^2 - \mu_\omega}{\omega} Q_\omega g - \frac{1}{\sqrt{\omega} \sigma_\omega} Q_\omega(\partial_s g) \\
- 2(\nabla_{(s,z)} Q_\omega) \cdot (\nabla g) - Q_\omega(\Delta g).
\]

Recalling the definition of the semi-norm $\|\cdot\|_{\Sigma_\omega}$ (see (2.3)), we write

\[
\|Q_\omega g\|_{\Sigma_\omega}^2 = \left\| \frac{Q_\omega^2 - \mu_\omega}{\omega} Q_\omega g - \frac{1}{\sqrt{\omega} \sigma_\omega} Q_\omega(\partial_s g) - 2(\nabla_{(s,z)} Q_\omega) \cdot (\nabla g) - Q_\omega(\Delta g), Q_\omega g \right\|_{L^2(\sigma_\omega)}.
\]

Note that for the inner product including the term (*), by integration by parts, we have

\[
- \langle 2(\nabla_{(s,z)} Q_\omega) \cdot \nabla g, Q_\omega g \rangle_{L^2(\sigma_\omega)} = - \int \int \int ((\nabla_{(s,z)} Q_\omega^2) \cdot \nabla g) g \sigma_\omega dsdzd\theta \\
= \int \int \int Q_\omega^2 \left\{ (\Delta g) g \sigma_\omega + |\nabla g|^2 \sigma_\omega + \frac{1}{\sqrt{\omega}} g(\partial_s g) \right\} dsdzd\theta.
\]

Thus, it follows that

\[
\|Q_\omega g\|_{\Sigma_\omega}^2 = \int \int \int \left\{ \frac{Q_\omega^2 - \mu_\omega}{\omega} g^2 + |\nabla g|^2 \right\} Q_\omega^2 \sigma_\omega dsdzd\theta,
\]

Proof of Proposition 7.9. By the Euler-Lagrange equation (7.2), it follows from Proposition 7.11 (iii) that $\|H_{\omega}^{(2D)} - \frac{1}{\omega \sigma^2_\omega} \nabla^2 + 1\| L^2(\sigma_\omega) \leq \|\Lambda + 1 - \frac{\mu_\omega}{\omega}\| L^2(\sigma_\omega) + \frac{1}{\omega}\|Q_\omega\|_{L^3(\sigma_\omega)}^3$ is uniformly bounded. Hence, Lemma 7.10 proves the desired bound. \(\square\)
and consequently by definition of the semi-norm $\| \cdot \|_{\Sigma_\omega}$ again,
\[
0 \leq \| \nabla_{(s,z)} (Q_{\omega}g) \|_{L^2(\sigma_{\omega})}^2 + \frac{1}{\omega} \| \partial_\theta(Q_{\omega}g) \|_{L^2(\frac{1}{\omega} s, \frac{1}{\omega} z)}^2 \\
\leq - \iiint \left\{ U_{\omega}(s) + z^2 - \Lambda_{\omega} - \frac{Q_{\omega}^2 - \mu_{\omega}}{\omega} - \frac{\| g \|}{g^2} \right\} (Q_{\omega}g)^2 \sigma_{\omega} ds dz d\theta.
\]
(7.11)

Now, we insert $g(s,z) = e^{f_{L}(s,z)}$, where $L \geq \sqrt{\omega}$ is a large number and $f_{L}(s,z) = c(s^2 + z^2)\eta(\frac{|(s,z)|}{L})$ for some $c \in (0,\alpha_0)$ and a smooth cut-off $\eta$ such that $\eta \equiv 1$ on $[-\frac{1}{4}, \frac{1}{4}]$ and $\eta$ is supported on $[-\frac{1}{2}, \frac{1}{2}]$. Then, it follows that
\[
0 \geq \iiint \left\{ U_{\omega}(s) + z^2 - \Lambda_{\omega} - \frac{Q_{\omega}^2 - \mu_{\omega}}{\omega} - \| f_{L} \| \right\} (e^{f_{L}Q_{\omega}})^2 \sigma_{\omega} ds dz d\theta.
\]

We observe that by the assumptions on $U_{\omega}$ (H2) and the $L^\infty$-bound (Lemma 7.9), if $\omega \geq 1$ is large, there exists $R \in (1, \sqrt{\omega})$, independent of $\omega \geq 1$ and $L \geq \sqrt{\omega}$, such that $U_{\omega}(s) + z^2 - \Lambda_{\omega} - \frac{Q_{\omega}^2 - \mu_{\omega}}{\omega} - \| f_{L} \|^2 \geq 1$ for all $|(s,z)| \geq R$, while there exists $C_R > 0$, independent of $\omega \geq 1$ and $L \geq \sqrt{\omega}$, such that $|U_{\omega}(s) + z^2 - \Lambda_{\omega} - \frac{Q_{\omega}^2 - \mu_{\omega}}{\omega} - \| f_{L} \|^2|e^{f_{L}L} \leq C_R^{2}$ for all $|(s,z)| \leq R$. Therefore, for a smooth cut-off $\eta$ such that $\eta \equiv 1$ on $[-\frac{1}{4}, \frac{1}{4}]$ and $\eta(s) = 0$ for $|s| \geq \frac{1}{2}$, it follows that
\[
\| 1_{|(s,z)| \geq R}(e^{f_{L}Q_{\omega}}) \|^2_{L^2(\sigma_{\omega})} \\
\leq \iiint 1_{|(s,z)| \geq R} \left\{ U_{\omega}(s) + z^2 - \Lambda_{\omega} - \frac{Q_{\omega}^2 - \mu_{\omega}}{\omega} - \| f_{L} \|^2 \right\} (e^{f_{L}Q_{\omega}})^2 \sigma_{\omega} ds dz d\theta \\
\leq - \iiint 1_{|(s,z)| \leq R} \left\{ U_{\omega}(s) + z^2 - \Lambda_{\omega} - \frac{Q_{\omega}^2 - \mu_{\omega}}{\omega} - \| f_{L} \|^2 \right\} (e^{f_{L}Q_{\omega}})^2 \sigma_{\omega} ds dz d\theta \leq C_R^{2},
\]
and taking $L \to \infty$, we obtain $\| 1_{|(s,z)| \geq R}(e^{(s^2+z^2)Q_{\omega}}) \|^2_{L^2(\sigma_{\omega})} \leq C_R$. On the other hand, we have $\| 1_{|(s,z)| \leq R}(e^{(s^2+z^2)Q_{\omega}}) \|^2_{L^2(\sigma_{\omega})} \leq e^{CR^2} \| Q_{\omega} \|^2_{L^2(\sigma_{\omega})} = e^{CR^2} m$. Since $R$ does not depend on $\omega \geq 1$, we prove
\[
\| e^{(s^2+z^2)Q_{\omega}} \|^2_{L^2(\sigma_{\omega})} \lesssim 1.
\]
(7.12)

To estimate the second and the third terms in (7.10), we take $g = e^{(s^2+z^2)\eta(\frac{|(s,z)|}{L})}$ in (7.10). Then, (7.12) with $c_0 \in (c, \alpha_0)$ implies that the right hand side of (7.10) is bounded uniformly in large $L \geq 1$. Thus, sending $L \to \infty$, we prove $\| e^{(s^2+z^2)Q_{\omega}} \|^2_{\Sigma_\omega} \lesssim 1$.

For (7.7), we insert $g = e^{(s^2+z^2)}$ in (7.5) and multiply both sides by $\sigma_{\omega}$. Then, estimating the right side by (7.6) with $c_0 \in (c, \alpha_0)$, together with the $L^\infty$-bound (Lemma 7.9), yields
\[
\left\| \sigma_{\omega}(H^{(2D)}_{\omega} - \Lambda_{\omega})(Q_{\omega}e^{(s^2+z^2)}) - \frac{1}{\omega \sigma_{\omega}} \partial_{\theta}^2 (Q_{\omega}e^{(s^2+z^2)}) \right\|^2_{L^2(\sigma_{\omega})} \lesssim 1.
\]
Finally, expanding the left hand side, we can deduce (7.7), because the cross inner product $(H^{(2D)}_{\omega} - \Lambda_{\omega})u, -\partial_{\theta}^2 v)_{L^2(\sigma_{\omega})} = \| \partial_{\theta} v \|^2_{\Sigma_{\omega}(s,z)}$ is non-negative. □

Next, we prove uniform $L^2(\sigma_{\omega})$ bounds for the second-order angle derivatives.
Proof of Proposition 7.7. By Proposition 7.5 (see Remark 7.6), it is enough to show that \( \| \eta(\sqrt{\omega}) \partial_\theta Q_\omega \|_{L^2(\frac{1}{\omega^2})} \lesssim 1 \), where \( \eta \) is a smooth cut-off such that \( \eta \equiv 1 \) on \([-\frac{1}{2}, \frac{1}{2}]\) and \( \eta \) is supported on \([-\frac{1}{2}, \frac{1}{2}]\). For the proof, we take \( g = \eta(\sqrt{\omega}) \) in (7.8), we write

\[
\left( H_\omega^{(2D)} - \Lambda_\omega - \frac{1}{\omega \sigma_\omega^2} \partial_\theta^2 \right) (Q_\omega \eta(\sqrt{\omega})) = \frac{Q_\omega^2}{\omega} - \frac{\mu_\omega}{\omega} Q_\omega \eta(\sqrt{\omega}) - \frac{1}{\sqrt{\omega \sigma_\omega}} Q_\omega (\eta(\sqrt{\omega}))' - 2(\partial_s Q_\omega) (\eta(\sqrt{\omega}))' - Q_\omega (\eta(\sqrt{\omega}))''.
\]

(7.13)

We multiply both sides of (7.13) by \( \sigma_\omega \). Then, we can show that the \( L^2(\sigma_\omega) \)-norm of the right hand side is bounded by \( \omega^{-1} \), because \( \| Q_\omega \|_{L^6(\sigma_\omega)} \lesssim 1 \) (see Proposition 7.1 (iii)), the derivatives of \( \eta(\sqrt{\omega}) \) is supported on \( \sqrt{\omega} \leq |s| \leq \frac{\sqrt{\omega}}{2} \), and \( Q_\omega \) and \( \partial_s Q_\omega \) can take Gaussian weights (Proposition 7.5). Therefore, it follows that

\[
\frac{1}{\omega^2} \geq \left\| \sigma_\omega (H_\omega^{(2D)} - \Lambda_\omega) (Q_\omega \eta(\sqrt{\omega})) - \frac{1}{\omega \sigma_\omega} \partial_\theta^2 (Q_\omega \eta(\sqrt{\omega})) \right\|^2_{L^2(\sigma_\omega)}
\]

\[
\geq \left\| \sigma_\omega (H_\omega^{(2D)} - \Lambda_\omega) (Q_\omega \eta(\sqrt{\omega})) \right\|^2_{L^2(\sigma_\omega)} + \frac{1}{\omega^2} \left\| \eta(\sqrt{\omega}) \partial_\theta Q_\omega \right\|^2_{L^2(\frac{1}{\omega^2})},
\]

because the cross-inner product is non-negative. \( \square \)

7.3. Dimension reduction. We close this section by proving the dimension reduction in the strong confinement limit \( \omega \to \infty \).

Proposition 7.11 (Dimension reduction). For sufficiently large \( \omega \geq 1 \), the following hold.

(i) (Improved vanishing higher eigenstates)

\[
\omega \| Q_\omega \|_{\Sigma_{\omega, (s,z)}}^2 + \| \partial_\theta (P_{\Phi_\omega} Q_\omega) \|_{L^2(\frac{1}{\omega^2})} = O(\omega^{-\frac{1}{2}}).
\]

(ii) (Minimum energy convergence)

\[
J_\infty^{(1D)}(m) = J_\infty^{(3D)}(m) + O(\omega^{-\frac{1}{2}}).
\]

(iii) (Strong convergences to the circle ground state) Translating along the \( \theta \)-axis if necessary, \( Q_\omega \) satisfies

\[
\lim_{\omega \to \infty} \| Q_\omega \|_{H^1(\mathbb{S}^1)} = 0 \quad \text{and} \quad \lim_{\omega \to \infty} \| \partial_\theta Q_\omega - \partial_\theta Q_\infty(\theta) \Phi_\omega(s,z) \|_{L^2(\sigma_\omega)} = 0.
\]

(iv) (Lagrange multiplier convergence)

\[
\mu_\omega = \mu_\infty + o_\omega(1).
\]

Proof. We prove (i) and (ii) by extracting the mass and the energy of \( Q_\omega \). Indeed, by orthogonality, \( m = \| Q_\omega \|_{L^2(\sigma_\omega)} = \| Q_\omega \|_{L^2(\mathbb{S}^1)} + \| P_{\Phi_\omega} Q_\omega \|_{L^2(\sigma_\omega)} \). However, by the bound for \( P_{\Phi_\omega} \) (Lemma 4.1) and a uniform bound in Proposition 7.5 (iii), we have \( \| P_{\Phi_\omega} Q_\omega \|_{L^2(\sigma_\omega)} \lesssim \| Q_\omega \|_{\Sigma_{\omega, (s,z)}}^2 + e^{-\alpha_\omega} \| Q_\omega \|_{L^2(\mathbb{S}^1)} \lesssim \omega^{-1} \). Thus, we obtain

\[
M_\infty(\| Q_\omega \|) = \| Q_\omega \|_{L^2(\mathbb{S}^1)}^2 = m + O(\omega^{-1}).
\]
On the other hand, we observe that the potential energy is well approximated by the projection, because by the H"older inequality and Proposition 7.1 (iii),
\[
\| Q_{\omega} \|_{L^4(\sigma)}^4 - \| P_{\Phi_{\omega}} Q_{\omega} \|_{L^4(\sigma)}^4 \lesssim \left\{ \| Q_{\omega} \|_{L^6(\sigma)}^3 + \| P_{\Phi_{\omega}} Q_{\omega} \|_{L^6(\sigma)}^3 \right\} \| P_{\Phi_{\omega}}^\perp Q_{\omega} \|_{L^2(\sigma)} \lesssim \omega^{-1/2}
\]
and by Proposition 8.2 (iii),
\[
\| P_{\Phi_{\omega}} Q_{\omega} \|_{L^4(\sigma)}^4 = \frac{\| \phi_\infty \|_{L^4(\Sigma^1)}^4 \| \chi_{\omega} \phi_\omega \|_{L^4(\sigma)}^4 \| Q_{\omega} \|_{L^4(\Sigma^1)}^4}{\| \chi_{\omega} \phi_\omega \|_{L^2(\sigma)}^2} = \left( \frac{1}{2\pi} + O(\omega^{-1/2}) \right) \| Q_{\omega} \|_{L^4(\Sigma^1)}^4.
\]
Hence, by \(3.3\) and the asymptotic Pythagorean identity \(4.5\), the energy \( E_\omega(Q_{\omega}) \) can be written as
\[
E_\omega(Q_{\omega}) = E_\infty(Q_{\omega}) + \frac{\omega}{2} \| Q_{\omega} \|_{L^2(\Sigma^1)}^2 + \frac{1}{2} \| \partial_\theta(P_{\Phi_{\omega}} Q_{\omega}) \|_{L^2(\omega)}^2 + O(\omega^{-1/2}).
\]
Thus, it follows that
\[
J_{\infty}(1D)(m) \leq E_\infty \left( \frac{\sqrt{m} Q_{\omega}}{\| Q_{\omega} \|_{L^2(\Sigma^1)}} \right) = E_\infty(Q_{\omega}) + O(\omega^{-1/2})
\]
\[
= E_\omega(Q_{\omega}) - \frac{\omega}{2} \| Q_{\omega} \|_{L^2(\Sigma^1)}^2 - \frac{1}{2} \| \partial_\theta(P_{\Phi_{\omega}} Q_{\omega}) \|_{L^2(\omega)}^2 + O(\omega^{-1/2})
\]
\[
= J_{\omega}(3D)(m) - \frac{\omega}{2} \| Q_{\omega} \|_{L^2(\Sigma^1)}^2 - \frac{1}{2} \| \partial_\theta(P_{\Phi_{\omega}} Q_{\omega}) \|_{L^2(\omega)}^2 + O(\omega^{-1/2}).
\]
Combining the upper bound on \( J_{\omega}(3D)(m) \) (Lemma 7.3), we conclude that \( J_{\infty}(1D)(m) = J_{\omega}(3D)(m) + O(\omega^{-1/2}) \) and \( \omega \| Q_{\omega} \|_{L^2(\Sigma^1)}^2 + \| \partial_\theta(P_{\Phi_{\omega}} Q_{\omega}) \|_{L^2(\omega)}^2 = O(\omega^{-1/2}). \)

For \((iii)\), we note from the proof of Proposition 7.11 (i) and (ii) that \( \{ \frac{\sqrt{m} Q_{\omega}}{\| Q_{\omega} \|_{L^2(\Sigma^1)}} \} \) is shown to be a minimizing sequence for \( J_{\infty}(1D)(m) \). Hence, by the concentration-compactness property for \( J_{\infty}(1D)(m) \) and the uniqueness of the minimizer \( Q_\infty \) (see \(27,33\)), we conclude that up to a translation, \( Q_{\omega} \rightarrow Q_\infty \) in \( H^1(\Sigma^1) \) as \( \omega \rightarrow \infty \). We assume that translating along the \( \theta \)-axis if necessary, \( Q_{\omega} \rightarrow Q_\infty \) in \( H^1(\Sigma^1) \) as \( \omega \rightarrow \infty \). On the other hand, this convergence immediately implies that \( \| \partial_\theta(P_{\Phi_{\omega}} Q_{\omega}) - \partial_\theta Q_\infty(\theta) \Phi_{\omega}(s,z) \|_{L^2(\sigma)} = \| \partial_\theta(Q_{\omega} - Q_\infty) \|_{L^2(\Sigma^1)} + o_\omega(1) \rightarrow 0 \) as \( \omega \rightarrow \infty \). Thus, it remains to show that \( \| \partial_\theta(P_{\Phi_{\omega}}^\perp Q_{\omega}) \|_{L^2(\sigma)} \rightarrow 0 \). Indeed, since \( P_{\Phi_{\omega}}^\perp \) and \( \partial_\theta \) commute, using Proposition 7.11 (i), we obtain
\[
\| \partial_\theta(P_{\Phi_{\omega}}^\perp Q_{\omega}) \|_{L^2(\sigma)}^2 = \int_{\Sigma^1} \int_{-\infty}^\infty \int_{-\sqrt{\omega}}^{\sqrt{\omega}} \frac{\sigma_\omega^2 - 1}{\sigma_\omega} | \partial_\theta(P_{\Phi_{\omega}} Q_{\omega}) |^2 dsdz d\theta + o_\omega(1)
\]
\[
\lesssim \| \sqrt{\sigma_\omega^2 - 1} \| \partial_\theta Q_{\omega} \|_{L^2(\omega)}^2 + \| \sqrt{\sigma_\omega^2 - 1} \| \partial_\theta(P_{\Phi_{\omega}} Q_{\omega}) \|_{L^2(\omega)}^2 + o_\omega(1).
\]
For the first term, we have
\[
\| \sqrt{\sigma_\omega^2 - 1} \| \partial_\theta Q_{\omega} \|_{L^2(\omega)}^2 \leq \omega^{-1/2} \| \delta_{|s| \leq \omega^{1/2}} \sqrt{s(s + 2\sqrt{\omega})} \|_{L^2(\omega)} + \omega^{-1/2} \| \delta_{|s| \geq \omega^{1/2}} \sqrt{s(s + 2\sqrt{\omega})} \|_{L^2(\omega)}
\]
\[
= o_\omega(1) \quad \text{(by Proposition 7.11 (iii) and Proposition 7.5)}.
\]
For the second term on the bound, we have
\[ \| \sqrt{\frac{\sigma^2}{\omega}-1} \partial_\theta (P_{\Phi, \omega} \Phi) \|_{L^2(\frac{1}{\omega})} = \| \chi_\omega \Phi_{\omega} \|_{L^2(\frac{1}{\omega})} \| \sqrt{\frac{\sigma^2}{\omega}-1} \chi_\omega \Phi_{\omega} \|_{L^2(\frac{1}{\omega})} \| \partial_\theta \Phi_{\omega} \|_{L^2(\frac{1}{\omega})} \leq o_\omega(1) \| \partial_\theta \Phi_{\omega} \|_{L^2(\frac{1}{\omega})} \] (by Lemma 4.2).

Therefore, we conclude that \( \| \partial_\theta (P_{\Phi, \omega} \Phi) \|_{L^2(\frac{1}{\omega})} = o_\omega(1). \)

For (iv), we note that by (3.3), (4.5), (7.3) and Proposition 7.11 (i),
\[ E(\omega, \Phi) = \frac{\omega}{4} \| \Phi \|_{L^2(S^1)}^2 + \frac{1}{4} \| \partial_\theta \Phi \|_{L^2(\frac{1}{\omega})}^2 - \frac{\mu_\omega}{4} \| \Phi \|_{L^2(\frac{1}{\omega})}^2 \]
\[ = \frac{1}{4} \| \partial_\theta \Phi_{\omega} \|_{L^2(S^1)}^2 - \frac{\mu_\omega}{4} + o_\omega(1), \]
while
\[ E_m(\omega) = \frac{1}{2} \| \partial_\theta \Phi_{\omega} \|_{L^2(S^1)}^2 - \frac{1}{8\pi} \| \Phi \|_{L^2(S^1)}^4 = \frac{1}{4} \| \partial_\theta \Phi_{\omega} \|_{L^2(S^1)}^2 - \frac{\mu_m}{4}. \]

Thus, combining with the two convergences \( J^{(3D)}_\omega(m) \to J^{(1D)}_\omega(m) \) and \( \Phi_{\omega} \to \Phi \) in \( H^1(S^1) \), we conclude \( \mu_\omega \to \mu_\infty. \)

**Proof of Theorem 1.7.** We write
\[ \Phi_{\omega}(s, z, \theta) - \Phi_\infty(\theta) \chi_\omega(s) \Phi_0(s, z) = \left( \Phi_{\omega}(\theta) - \Phi_\infty(\theta) \right) \frac{\chi_\omega(s) \Phi_{\omega}(s, z)}{\| \chi_\omega \Phi_{\omega} \|_{L^2(\sigma, dz)}^2} \right) + P_{\Phi, \omega} \Phi_{\omega} \]
\[ + \left( \| \chi_\omega \Phi_{\omega} \|_{L^2(\sigma, dz)}^{-1} \chi_\omega(s) \Phi_0(s, z) - \Phi_0(s, z) \right) \Phi_\infty(\theta) \chi_\omega(s). \]

Then we see that by the bound for \( P_{\Phi, \omega} \) (Lemma 4.4), the fast decay and convergence of \( \chi_\omega(\Phi_0) \) (Proposition 3.2), strong convergences of \( \Phi_{\omega} \) and vanishing higher eigenstates (Proposition 7.11),
\[ \| \Phi_{\omega} - \Phi_\infty \chi_\omega \Phi_0 \|_{L^2(\sigma, dz)} \leq \| \Phi_{\omega} \|_{L^2(S^1)} - \Phi_\infty \|_{L^2(S^1)} + \| P_{\Phi, \omega} \Phi_{\omega} \|_{L^2(\sigma, dz)} \chi_\omega(s, z) + o_\omega(1) = o_\omega(1) \]
and
\[ \| \partial_\theta (\Phi_{\omega} - \Phi_\infty \chi_\omega \Phi_0) \|_{L^2(\frac{1}{\omega})} \leq \| \partial_\theta (\Phi_{\omega} - \Phi_\infty) \|_{L^2(S^1)} + \| \partial_\theta (P_{\Phi, \omega} \Phi_{\omega}) \|_{L^2(\frac{1}{\omega})} + o_\omega(1) = o_\omega(1), \]
and therefore we get
\[ \lim_{\omega \to \infty} \| \Phi_{\omega}(s, z, \theta) - \Phi_\infty(\theta) \chi_\omega(s) \Phi_0(s, z) \|_{\Sigma_\omega} = 0. \)

8. **Uniqueness of a ground state: Proof of Theorem 1.8**

In this section, we derive a coercivity estimate of the linearized operator at the constrained minimizer \( \Phi_{\omega} \) invoking that for the 1D periodic NLS (see [27] and Appendix A) and the dimension reduction limit. Then, finally, we complete the proof uniqueness of a minimizer \( \Phi_{\omega} \).
8.1. Linearized operator at a 3D ground state. For our 3D variational problem \( J^{(3D)}(m) \), the operator

\[
\mathcal{L}_\omega := \omega (\mathcal{H}^{(3D)}_{\omega} - \Lambda_{\omega}) - \frac{1}{\sigma_{\omega}} \partial_\theta^2 + \mu_{\omega} - 3Q_{\omega}^2,
\]

acting on \( L^2(\sigma_{\omega}) \), arises when linearizing about a ground state \( Q_{\omega} \). By rotation invariance with respect to the z-axis in the formulation in \( \mathbb{R}^3 \) (see (7.5)), its kernel contains \( \partial_\theta Q_{\omega} \). Moreover, it has at least one negative eigenvalue, because \( \langle \mathcal{L}_\omega Q_{\omega}, Q_{\omega} \rangle_{L^2(\sigma_{\omega})} < 0 \).

The following proposition asserts that in the strong confinement regime, the linearized operator \( \mathcal{L}_\omega \) is strictly positive on the subspace \( \text{span}\{Q_{\omega}, \partial_\theta Q_{\omega}\} \subset L^2(\sigma_{\omega}) \).

**Proposition 8.1** (Coercivity of the linearized operator \( \mathcal{L}_\omega \)). Suppose that \( m \neq 2\pi^2 \), \( \omega \geq 1 \) is large enough, and \( Q_{\omega} \) satisfies

\[
\lim_{\omega \to \infty} \|Q_{\omega}\| - Q_{\infty}\|H^1(S^1)\| = \lim_{\omega \to \infty} \|\partial_\theta Q_{\omega} - \partial_\theta Q_{\infty}(\theta)\Phi_{\omega}(s,z)\|_{L^2(\sigma_{\omega})} = 0. \tag{8.2}
\]

Then, we have

\[
\langle \mathcal{L}_\omega \varphi, \varphi \rangle_{L^2(\sigma_{\omega})} \geq \frac{L_{\infty}}{8} \|\varphi_{\omega}\|_{H^1(S^1)}^2 + \frac{\omega}{4} \|\varphi\|_{L^2(\sigma_{\omega}, s,z)}^2
\]

for all \( \varphi \in \Sigma_{\omega} \) such that \( \langle \varphi, Q_{\omega} \rangle_{L^2(\sigma_{\omega})} = \langle \varphi, \partial_\theta Q_{\omega} \rangle_{L^2(\sigma_{\omega})} = 0 \), where \( L_{\infty} > 0 \) is the constant given in Proposition 8.4.

**Remark 8.2.** Proposition 8.1 implies the non-degeneracy, i.e., \( \text{Ker}(\mathcal{L}_\omega) = \text{span}\{\partial_\theta Q_{\omega}\} \), and it also shows that the linearized operator has only one negative eigenvalue.

**Proof of Proposition 8.1.** Suppose that \( \varphi \) is orthogonal to \( Q_{\omega} \) and \( \partial_\theta Q_{\omega} \) in \( L^2(\sigma_{\omega}) \). For the proof, we introduce an auxiliary 3D linear operator

\[
\tilde{\mathcal{L}}_{\omega} := -\frac{1}{\sigma_{\omega}} \partial_\theta^2 + \mu_{\omega} - 3(\Phi_{\omega}(s,z))^2,
\]

where \( \Phi(s,z) = \frac{\chi_{\omega}(s)\Phi_{\omega}(s,z)}{\|\chi_{\omega}\|_{L^2(\sigma_{\omega}, s,z)}} \) (see (8.3)). Then, by the definitions, \( \langle \mathcal{L}_\omega \varphi, \varphi \rangle_{L^2(\sigma_{\omega})} \) can be written as

\[
\langle \mathcal{L}_\omega \varphi, \varphi \rangle_{L^2(\sigma_{\omega})} = \omega \|\varphi\|_{L^2(\sigma_{\omega})}^2 + \langle \tilde{\mathcal{L}}_{\omega} \varphi, \varphi \rangle_{L^2(\sigma_{\omega})} + (\mu_{\omega} - \mu_{\infty}) \|\varphi\|_{L^2(\sigma_{\omega})}^2 - 3\langle (Q_{\omega}, \varphi)_{L^2(\sigma_{\omega})} - Q_{\infty}\|\Phi_{\omega}\|_{L^2(\sigma_{\omega})}^2 \rangle_{L^2(\sigma_{\omega})} - 3\langle (Q_{\omega}, \varphi)_{L^2(\sigma_{\omega})} - Q_{\infty}\|\Phi_{\omega}\|_{L^2(\sigma_{\omega})}^2 \rangle_{L^2(\sigma_{\omega})},
\]

where \( -3Q_{\omega}^2 \) in \( \mathcal{L}_\omega \) is expanded by the decomposition \( Q_{\omega} = Q_{\omega,\|\Phi_{\omega}\|_{L^2(\sigma_{\omega})}} + \mathcal{P}_{\Phi_{\omega}} Q_{\omega} \). We recall from Proposition 7.11 (iv) that \( \mu_{\omega} - \mu_{\infty} = \alpha_{\omega}(1) \). On the other hand, by the Hölder inequality and the convergence \( Q_{\omega,\|\Phi_{\omega}\|_{L^2(\sigma_{\omega})}} = Q_{\infty} \) in \( H^1(S^1) \), we have

\[
\|\langle Q_{\omega,\|\Phi_{\omega}\|_{L^2(\sigma_{\omega})}}^2 - Q_{\omega}^2 \|\varphi\|_{L^2(\sigma_{\omega})}^2 \rangle_{L^2(\sigma_{\omega})} \| \lesssim \|Q_{\omega,\|\Phi_{\omega}\|_{L^2(\sigma_{\omega})}} - Q_{\omega}\|_{L^4(S^1)} \|Q_{\omega,\|\Phi_{\omega}\|_{L^2(\sigma_{\omega})}} - Q_{\omega}\|_{L^4(S^1)} \|\varphi\|_{L^2(\sigma_{\omega})}^2 = \alpha_{\omega}(1) \|\varphi\|_{L^4(\sigma_{\omega})}^2,
\]

where \( \alpha_{\omega}(1) \) is the constant given in Proposition 7.11 (iv).
and by the fact that $\mathcal{P}_{\Phi_{\omega}}^\perp Q_{\omega} \to 0$ in $L^4(\sigma_{\omega})$ (by Lemma 4.4, Proposition 5.1, Proposition 7.1) (i),

$$\langle (Q_{\omega}, \Phi_{\omega} + Q_{\omega}) \mathcal{P}_{\Phi_{\omega}}^\perp Q_{\omega}, \wp \rangle \vert_{L^2(\sigma_{\omega})} \leq (\|Q_{\omega}, \Phi_{\omega} \|_{L^4(\sigma_{\omega})} + \|Q_{\omega} \|_{L^1(\sigma_{\omega})}) \|\mathcal{P}_{\Phi_{\omega}}^\perp Q_{\omega} \|_{L^4(\sigma_{\omega})} \|\wp \|_{L^1(\sigma_{\omega})} = o_{\omega}(1) \|\wp \|_{L^1(\sigma_{\omega})}^2.$$  

Thus, it follows that

$$\langle \mathcal{L}_{\omega} \wp, \wp \rangle \vert_{L^2(\sigma_{\omega})} \geq \omega \|\wp \|_{L^2(\sigma_{\omega})}^2 + \langle \tilde{\mathcal{L}}_{\omega} \wp, \wp \rangle \vert_{L^2(\sigma_{\omega})} - o_{\omega}(1) \|\wp \|_{L^2(\sigma_{\omega})}^2 - o_{\omega}(1) \|\wp \|_{L^1(\sigma_{\omega})}^2.$$  

We note that by Lemma 4.4,

$$\|\wp \|_{L^2(\sigma_{\omega})}^2 = \|\mathcal{P}_{\Phi_{\omega}} \wp \|_{L^2(\sigma_{\omega})}^2 + \|\mathcal{P}_{\Phi_{\omega}}^\perp \wp \|_{L^2(\sigma_{\omega})}^2 \lesssim \|\wp \|_{L^2(\Sigma^1)}^2 + \|\wp \|_{L^1(\Sigma_{w(s,z)})}^2,$$  

and consequently by the refined Gagliardo-Nirenberg inequality (Corollary 5.3) and Young’s inequality, one can easily show that

$$\|\wp \|_{L^1(\sigma_{\omega})}^2 \lesssim \|\wp \|_{L^2(\sigma_{\omega})}^2 + \|\partial_\theta \wp \|_{L^2(\frac{1}{\sigma_{\omega}})}^2 + \omega \|\wp \|_{L^2(\Sigma_{w(s,z)})}^2 \lesssim \|\wp \|_{L^1(\Sigma^1)}^2 + \|\partial_\theta \wp \|_{L^2(\frac{1}{\sigma_{\omega}})}^2 + (1 + \omega) \|\wp \|_{L^1(\Sigma_{w(s,z)})}^2.$$  

Hence, applying (8.3), we obtain

$$\langle \mathcal{L}_{\omega} \wp, \wp \rangle \vert_{L^2(\sigma_{\omega})} \geq (1 - o_{\omega}(1)) \omega \|\wp \|_{L^1(\Sigma_{w(s,z)})}^2 + (\tilde{\mathcal{L}}_{\omega} \wp, \wp \rangle \vert_{L^2(\sigma_{\omega})} - o_{\omega}(1) \|\partial_\theta \wp \|_{L^2(\frac{1}{\sigma_{\omega}})}^2 - o_{\omega}(1) \|\wp \|_{L^1(\Sigma^1)}^2.$$  

Note that in the above inequality, the lower bound includes an unfavorable negative term $- o_{\omega}(1) \|\partial_\theta \wp \|_{L^2(\frac{1}{\sigma_{\omega}})}^2$. In order to absorb it, we take a small number $c_{\omega} > 0$ such that $c_{\omega} \to 0$ to be specified later, and employ a simple lower bound

$$\langle \mathcal{L}_{\omega} \wp, \wp \rangle \vert_{L^2(\sigma_{\omega})} \geq (1 - o_{\omega}(1)) \omega \|\wp \|_{L^1(\Sigma_{w(s,z)})}^2 - \omega \omega_{\omega}(1) \|\partial_\theta \wp \|_{L^2(\frac{1}{\sigma_{\omega}})}^2 - o_{\omega}(1) \|\wp \|_{L^2(\Sigma^1)}^2$$

$$\geq (1 - c_{\omega}) \langle \tilde{\mathcal{L}}_{\omega} \wp, \wp \rangle \vert_{L^2(\sigma_{\omega})} + c_{\omega} \langle \mathcal{L}_{\omega} \wp, \wp \rangle \vert_{L^2(\sigma_{\omega})} \geq (1 - o_{\omega}(1)) \omega \|\wp \|_{L^1(\Sigma_{w(s,z)})}^2 + (1 - c_{\omega}) \langle \tilde{\mathcal{L}}_{\omega} \wp, \wp \rangle \vert_{L^2(\sigma_{\omega})}$$

$$+ (c_{\omega} - o_{\omega}(1)) \|\partial_\theta \wp \|_{L^2(\frac{1}{\sigma_{\omega}})}^2 - (Kc_{\omega} + o_{\omega}(1)) \|\wp \|_{L^2(\Sigma^1)}^2.$$  

We assume that $0 < c_{\omega} \leq \frac{1}{2}$, $(c_{\omega} - o_{\omega}(1)) \geq \frac{c}{2}$ and $(Kc_{\omega} + o_{\omega}(1)) \leq \frac{L_{\infty}}{2}$. Then, it is refined as

$$\langle \mathcal{L}_{\omega} \wp, \wp \rangle \vert_{L^2(\sigma_{\omega})} \geq \frac{\omega}{2} \|\wp \|_{L^1(\Sigma_{w(s,z)})}^2 + \frac{1}{2} \langle \tilde{\mathcal{L}}_{\omega} \wp, \wp \rangle \vert_{L^2(\sigma_{\omega})} + \frac{\omega}{2} \|\partial_\theta \wp \|_{L^2(\frac{1}{\sigma_{\omega}})}^2 - \frac{L_{\infty}}{8} \|\wp \|_{L^2(\Sigma^1)}^2.$$
As a consequence, the proof of the proposition is reduced to that of a lower bound on the auxiliary operator

\[
\langle \hat{L}_\omega \varphi, \varphi \rangle_{L^2(\sigma_\omega)} \geq \frac{L_\infty}{2} \| \varphi_{\omega,\|} \|^2_{H^1(S^1)} - \frac{\omega}{2} \| \varphi \|^2_{\Sigma_{\omega,(s,z)}} - \frac{c_\omega}{2} \| \partial_\theta \varphi \|^2_{L^2(\frac{1}{\omega})}. \tag{8.5}
\]

To show (8.5), we insert the decomposition\textsuperscript{11} \( \varphi = P_{\hat{\Phi}_\omega} \varphi + P_{\hat{\Phi}_\omega}^\perp \varphi \) in its left hand side, where \( P_{\hat{\Phi}_\omega} \) is the modified projection defined in Section 4 and expand as

\[
\langle \hat{L}_\omega \varphi, \varphi \rangle_{L^2(\sigma_\omega)} = \langle \hat{L}_\omega P_{\hat{\Phi}_\omega} \varphi, P_{\hat{\Phi}_\omega} \varphi \rangle_{L^2(\sigma_\omega)} + 2 \langle \hat{L}_\omega P_{\hat{\Phi}_\omega}^\perp \varphi, P_{\hat{\Phi}_\omega} \varphi \rangle_{L^2(\sigma_\omega)}
\]

Then, it follows from Proposition 3.2 that

\[
\langle \hat{L}_\omega P_{\hat{\Phi}_\omega} \varphi, P_{\hat{\Phi}_\omega} \varphi \rangle_{L^2(\sigma_\omega)} = \langle \hat{L}_\omega (\varphi_{\omega,\|} \hat{\Phi}_\omega), \varphi_{\omega,\|} \hat{\Phi}_\omega \rangle_{L^2(\sigma_\omega)} = \| \hat{\Phi}_\omega \|_{L^2(\frac{1}{\omega})}^2 \langle \partial_\omega \varphi_{\omega,\|}, \varphi_{\omega,\|} \rangle_{L^2(S^1)} + \mu_\infty \| \hat{\Phi}_\omega \|^2_{L^2(S^1)} \| \varphi_{\omega,\|} \|^2_{L^2(S^1)}
\]

\[
- 3 \| \hat{\Phi}_\omega \|_{L^2(\frac{1}{\omega})}^2 \langle Q_{\omega,\|} \varphi_{\omega,\|}, \varphi_{\omega,\|} \rangle_{L^2(S^1)}
\]

\[
= \langle \hat{L}_\omega \varphi_{\omega,\|}, \varphi_{\omega,\|} \rangle_{L^2(S^1)} - \sigma_\omega(1) \| \varphi_{\omega,\|} \|^2_{H^1(S^1)}.
\]

On the other hand, by Lemma 4.4 and the asymptotic orthogonality (4.5)\textsuperscript{12}, we have

\[
\langle \hat{L}_\omega P_{\hat{\Phi}_\omega}^\perp \varphi, P_{\hat{\Phi}_\omega} \varphi \rangle_{L^2(\sigma_\omega)} \geq \langle P_{\hat{\Phi}_\omega}^\perp (\partial_\theta \varphi), P_{\hat{\Phi}_\omega} (\partial_\theta \varphi) \rangle_{L^2(\frac{1}{\omega})} + \mu_\infty \langle P_{\hat{\Phi}_\omega}^\perp \varphi, P_{\hat{\Phi}_\omega} \varphi \rangle_{L^2(\sigma_\omega)}
\]

\[
- 3 \| Q_{\omega,\|} \varphi_{\omega,\|} \|_{L^2(\frac{1}{\omega})}^2 \| P_{\hat{\Phi}_\omega} \varphi \|_{L^2(\sigma_\omega)} \| P_{\hat{\Phi}_\omega}^\perp \varphi \|_{L^2(\sigma_\omega)}
\]

\[
\geq - \sigma_\omega(1) \| \partial_\theta \varphi \|^2_{L^2(\frac{1}{\omega})} - (\| \varphi \|^2_{\Sigma_{\omega,(s,z)}} + e^{\omega \varphi} \| \varphi_{\omega,\|} \|^2_{L^2(S^1)}) \| \varphi_{\omega,\|} \|_{L^2(S^1)}
\]

\[
\geq - \sigma_\omega(1) \| \varphi_{\omega,\|} \|_{L^2(S^1)}^2 + \omega \| \varphi \|^2_{\Sigma_{\omega,(s,z)}} + \| \varphi_{\omega,\|} \|_{L^2(S^1)}^2,
\]

while Lemma 4.4 implies that

\[
\langle \hat{L}_\omega P_{\hat{\Phi}_\omega}^\perp \varphi, P_{\hat{\Phi}_\omega}^\perp \varphi \rangle_{L^2(\sigma_\omega)} \geq - 3 \| Q_{\omega,\|} \varphi_{\omega,\|} \|_{L^2(\frac{1}{\omega})}^2 \| P_{\hat{\Phi}_\omega} \varphi \|_{L^2(\sigma_\omega)} - \mu_\infty \| P_{\hat{\Phi}_\omega} \varphi \|^2_{L^2(\sigma_\omega)}
\]

\[
\geq - \| \varphi \|^2_{\Sigma_{\omega,(s,z)}} - \sigma_\omega(1) \| \varphi_{\omega,\|} \|_{L^2(S^1)}.
\]

Collecting all, we conclude that

\[
\langle \hat{L}_\omega \varphi, \varphi \rangle_{L^2(\sigma_\omega)} \geq \langle \hat{L}_\omega \varphi_{\omega,\|}, \varphi_{\omega,\|} \rangle_{L^2(S^1)} - \frac{\omega}{4} \| \varphi \|^2_{\Sigma_{\omega,(s,z)}}
\]

\[
- \sigma_\omega(1) \| \partial_\theta \varphi \|^2_{L^2(\frac{1}{\omega})} - \sigma_\omega(1) \| \varphi_{\omega,\|} \|_{H^1(S^1)}^2
\]

Hence, assuming \( \sigma_\omega(1) \leq \frac{1}{\omega} \), the proof of (8.5) is further reduced to that of the 1D lower bound,

\[
\langle \hat{L}_\omega \varphi_{\omega,\|}, \varphi_{\omega,\|} \rangle_{L^2(S^1)} \geq \frac{2L_\infty}{3} \| \varphi_{\omega,\|} \|_{H^1(S^1)}^2 - \frac{\omega}{4} \| \varphi \|^2_{\Sigma_{\omega,(s,z)}}. \tag{8.6}
\]

\textsuperscript{11}Our goal is to extract the 1D core part from the 3D operator \( \hat{L}_\omega \).

\textsuperscript{12}\( \langle P_{\Phi_\omega}^\perp \varphi, P_{\Phi_\omega} \varphi \rangle_{L^2(\frac{1}{\omega})} = O(\omega^{-1}) \| \varphi \|_{L^2(\frac{1}{\omega})} \).
For the lower bound \((8.6)\), we employ the coercivity of the linearized operator \(L_{\infty}\) on \(S^1\) (Proposition \(A.4\)). To do so, we extract the core part \(\tilde{\varphi}_{\omega,||}\) from \(\varphi_{\omega,||}\) in the form
\[
\varphi_{\omega,||} = \tilde{\varphi}_{\omega,||} + \frac{\langle \varphi, Q_{\infty} \rangle_{L^2(S^1)}}{\|Q_{\infty}\|_{L^2(S^1)}} Q_{\infty} + \frac{\langle \varphi, \partial_\theta Q_{\infty} \rangle_{L^2(S^1)}}{\|\partial_\theta Q_{\infty}\|_{L^2(S^1)}} \partial_\theta Q_{\infty},
\]
such that \(\tilde{\varphi}_{\omega,||}\) is orthogonal to \(Q_{\infty}\) and \(\partial_\theta Q_{\infty}\) in \(L^2(S^1)\). Then, using that \(L_{\infty} Q_{\infty} = -\frac{1}{\pi} Q_{\infty}^3\) and \(L_{\infty}(\partial_\theta Q_{\infty}) = 0\), we write
\[
\langle L_{\infty} \tilde{\varphi}_{\omega,||}, \tilde{\varphi}_{\omega,||} \rangle_{L^2(S^1)} = \langle L_{\infty} \tilde{\varphi}_{\omega,||}, \tilde{\varphi}_{\omega,||} \rangle_{L^2(S^1)} = \frac{2 \langle \tilde{\varphi}_{\omega,||}, Q_{\infty} \rangle_{L^2(S^1)} (Q_{\infty}, \varphi_{\omega,||})_{L^2(S^1)}}{\pi \|Q_{\infty}\|_{L^2(S^1)}} - \frac{\langle \varphi_{\omega,||}, Q_{\infty} \rangle_{L^2(S^1)}^2}{\pi \|Q_{\infty}\|_{L^2(S^1)}} \|Q_{\infty}\|_{L^2(S^1)},
\]
Note that by the assumptions \(\varphi \perp Q_{\omega}, \partial_\theta Q_{\omega}\) in \(L^2(\sigma_\omega)\) and the dimension reduction \((8.2)\), one may expect that \(\varphi_{\omega,||}\) is almost orthogonal to \(Q_{\infty}\) and \(\partial_\theta Q_{\infty}\). Indeed, we have
\[
\langle \varphi_{\omega,||}, Q_{\infty} \rangle_{L^2(S^1)} = \langle \varphi, Q_{\infty} \tilde{\Phi}_{\omega} \rangle_{L^2(\sigma_\omega)} = \langle \varphi, Q_{\infty} \tilde{\Phi}_{\omega} - Q_{\omega} \rangle_{L^2(\sigma_\omega)} = \langle \varphi, (Q_{\infty} - Q_{\omega}) \tilde{\Phi}_{\omega} \rangle_{L^2(\sigma_\omega)} = o_{\omega}(1) \|\varphi\|_{L^2(\sigma_\omega)}
\]
and
\[
\langle \varphi_{\omega,||}, \partial_\theta Q_{\infty} \rangle_{L^2(S^1)} = \langle \varphi, (\partial_\theta Q_{\infty}) \tilde{\Phi}_{\omega} \rangle_{L^2(\sigma_\omega)} = \langle \varphi, (\partial_\theta Q_{\infty}) \tilde{\Phi}_{\omega} - Q_{\omega} \rangle_{L^2(\sigma_\omega)} = o_{\omega}(1) \|\varphi\|_{L^2(\sigma_\omega)}.
\]
Therefore, it follows that
\[
\langle L_{\infty} \tilde{\varphi}_{\omega,||}, \tilde{\varphi}_{\omega,||} \rangle_{L^2(S^1)} \geq \langle L_{\infty} \tilde{\varphi}_{\omega,||}, \tilde{\varphi}_{\omega,||} \rangle_{L^2(S^1)} - o_{\omega}(1) \|\varphi\|_{L^2(\sigma_\omega)} \|\varphi_{\omega,||}\|_{L^2(S^1)} - o_{\omega}(1) \|\varphi\|_{L^2(\sigma_\omega)}^2
\]
and
\[
\|\tilde{\varphi}_{\omega,||}\|_{H^1(S^1)}^2 \geq \|\varphi_{\omega,||}\|_{H^1(S^1)}^2 - o_{\omega}(1) \|\varphi\|_{L^2(\sigma_\omega)}^2.
\]
Consequently, the coercivity estimate for \(L_{\infty}\) (see \((A.10)\)) and the Cauchy-Schwarz inequality yield
\[
\langle L_{\infty} \varphi_{\omega,||}, \varphi_{\omega,||} \rangle_{L^2(S^1)} \geq (L_{\infty} - o_{\omega}(1)) \|\varphi_{\omega,||}\|_{H^1(S^1)}^2 - o_{\omega}(1) \|\varphi\|_{L^2(\sigma_\omega)}^2 \geq (L_{\infty} - o_{\omega}(1)) \|\varphi_{\omega,||}\|_{H^1(S^1)}^2 - o_{\omega}(1) \|\varphi\|_{L^2(\sigma_\omega)}^2.
\]
Finally, applying \((8.3)\), we complete the proof of \((8.6)\).\[\square\]
8.2. **Uniqueness of a ground state.** To prove uniqueness, we introduce the functional

\[ I_\omega(v) = E_\omega(v) + \frac{\mu_\omega}{2} M_\omega(v). \]

For contradiction, we assume that two positive minimizers \( Q_\omega \) and \( \tilde{Q}_\omega \) exist for \( J_\omega^{(3D)}(m) \) such that \( \langle Q_\omega, (\cdot, \cdot, - \theta_s) \rangle \neq \langle Q_\omega, (\cdot, \cdot, - \theta) \rangle \) for all \( 0 \leq \theta_s \leq 2\pi \). By the dimension reduction limit (Proposition 7.11 (iii)), translating in \( \theta \) if necessary, we may assume that the \( \Phi, \) directional components \( Q_\omega \) and \( \tilde{Q}_\omega \) both converge to \( Q_\infty \) in \( H^1(S^1) \) as \( \omega \to \infty \), and \( \lim_{\omega \to \infty} \| \partial_\theta Q_\omega - \partial_\theta Q_\infty(\theta) \Phi_\omega(s, z) \|_{L^2(\sigma_\omega)} = 0 \). On the other hand, we take the angle translation parameter \( \theta_\omega \in [0, 2\pi] \) such that

\[ \Vert \tilde{Q}_\omega - Q_\omega(\cdot, \cdot, + \theta_\omega) \Vert_{L^2(\sigma_\omega)}, = \min_{\theta_\in [0,2\pi]} \Vert \tilde{Q}_\omega - Q_\omega(\cdot, \cdot, + \theta) \Vert_{L^2(\sigma_\omega)}. \]

Then, we have \( \langle \tilde{Q}_\omega, (\partial_\theta Q_\omega)(\cdot, \cdot, + \theta_\omega) \rangle = \langle Q_\omega, (\cdot, \cdot, + \theta_\omega) \rangle \|_{L^2(\sigma_\omega)} = 0 \). Indeed, we have \( \theta_\omega \to 0 \), because \( Q_\omega \to Q_\infty \). Hence, \( Q_\omega(\cdot, \cdot, + \theta_\omega) \) is also a minimizer whose \( \Phi \) directional component \( Q_\omega(\cdot, \cdot, + \theta_\omega) \) converges to \( Q_\infty(\theta) \Phi_\omega(s, z) \) in \( L^1(S^1) \) as \( \omega \to \infty \), and \( \lim_{\omega \to \infty} \| \partial_\theta Q_\omega(s, z, \theta + \theta_\omega) - \partial_\theta Q_\infty(\theta) \Phi_\omega(s, z) \|_{L^2(\sigma_\omega)} = 0 \). Thus, translating the angle variable for \( Q_\omega \) by \( -\theta_\omega \), we may assume that

\[ \langle \tilde{Q}_\omega, \partial_\theta Q_\omega \rangle = 0 \]

and

\[ \lim_{\omega \to \infty} \| Q_\omega \|_{L^1(S^1)} = \lim_{\omega \to \infty} \| \partial_\theta Q_\omega - \partial_\theta Q_\infty(\theta) \Phi_\omega(s, z) \|_{L^2(\sigma_\omega)} = 0. \]

Next, we decompose

\[ \tilde{Q}_\omega = \sqrt{1 - \delta_\omega^2} Q_\omega + R_\omega, \]

where \( \delta_\omega > 0 \) is chosen so that the remainder \( R_\omega \) is orthogonal to \( Q_\omega \), i.e.,

\[ \langle R_\omega, Q_\omega \rangle = 0. \]

Indeed, by construction, \( R_\omega \) is perpendicular to \( \partial_\theta Q_\omega \) too, because

\[ \langle R_\omega, \partial_\theta Q_\omega \rangle = \langle \tilde{Q}_\omega, \partial_\theta Q_\omega \rangle - \sqrt{1 - \delta_\omega^2} \langle Q_\omega, \partial_\theta Q_\omega \rangle = 0. \]

Note that the parameter \( \delta_\omega \) and the remainder \( R_\omega \) are small in the sense that

\[ \delta_\omega = \frac{1}{\sqrt{m}} \| R_\omega \|_{L^2(\sigma_\omega)} = o(1), \tag{8.7} \]

because \( m = \| \tilde{Q}_\omega \|_{L^2(\sigma_\omega)} = (1 - \delta_\omega^2) m + \| R_\omega \|_{L^2(\sigma_\omega)} \) and \( 0 \leftarrow \| Q_\omega - \tilde{Q}_\omega \|_{L^2(\sigma_\omega)} = (1 - \sqrt{1 - \delta_\omega^2}) m + \| R_\omega \|_{L^2(\sigma_\omega)} \).

Now, we are ready to deduce a contradiction extracting \( I_\omega(Q_\omega) \) from

\[ I_\omega(\tilde{Q}_\omega) = \frac{\omega}{2} \| \sqrt{1 - \delta_\omega^2} Q_\omega + R_\omega \|_{L^2(\sigma_\omega)}^2 + \frac{1}{2} \| \partial_\theta (\sqrt{1 - \delta_\omega^2} Q_\omega + R_\omega) \|_{L^2(\sigma_\omega)}^2 - \frac{1}{4} \| \sqrt{1 - \delta_\omega^2} Q_\omega + R_\omega \|_{L^2(\sigma_\omega)}^4 + \frac{\mu_\omega}{2} \| \sqrt{1 - \delta_\omega^2} Q_\omega + R_\omega \|_{L^2(\sigma_\omega)}^2. \]
Indeed, expanding and then reorganizing terms with respect to degree of \( R_\omega \), it can be expressed as

\[
\mathcal{I}_\omega(Q_\omega) = \frac{1 - \delta^2}{2} \left\{ \| Q_\omega \|_{2,\Sigma_{(s,z)}}^2 + \| \partial_\theta Q_\omega \|_{L^2(\sigma_0/2)}^2 - \frac{1 - \delta^2}{2} \| Q_\omega \|_{L^4(\sigma_0)}^4 \right\} \\
+ \sqrt{1 - \delta^2} \left\langle \left( \omega(H_\omega^{(2D}} - \Lambda_\omega) - \frac{1}{\sigma_\omega} \partial_\theta^2 + \mu_\omega \right) Q_\omega - (1 - \delta^2) Q_\omega^3, R_\omega \right\rangle_{L^2(\sigma_0)} \\
+ \frac{1}{2} \left\langle \left( \omega(H_\omega^{(2D}} - \Lambda_\omega) - \frac{1}{\sigma_\omega} \partial_\theta^2 + \mu_\omega \right) R_\omega - 3(1 - \delta^2) Q_\omega^2 R_\omega, R_\omega \right\rangle_{L^2(\sigma_0)} \\
- \sqrt{1 - \delta^2} (Q_\omega, R_\omega^3)_{L^2(\sigma_0)} - \frac{1}{4} \| R_\omega \|_{L^4(\sigma_0)}^4,
\]

For the zeroth order terms in the above expansion, extracting \( \mathcal{I}_\omega(Q_\omega) \), we write

\[
\frac{1 - \delta^2}{2} \left\{ \| Q_\omega \|_{2,\Sigma_{(s,z)}}^2 + \| \partial_\theta Q_\omega \|_{L^2(\sigma_0/2)}^2 + \mu_\omega \| Q_\omega \|_{L^2(\sigma_0)}^2 - \frac{1 - \delta^2}{2} \| Q_\omega \|_{L^4(\sigma_0)}^4 \right\} \\
= \mathcal{I}_\omega(Q_\omega) - \frac{\delta^2}{2} \left\{ \| Q_\omega \|_{2,\Sigma_{(s,z)}}^2 + \| \partial_\theta Q_\omega \|_{L^2(\sigma_0/2)}^2 + \mu_\omega \| Q_\omega \|_{L^2(\sigma_0)}^2 - \frac{\delta^2}{2} \| Q_\omega \|_{L^4(\sigma_0)}^4 \right\} \\
= \mathcal{I}_\omega(Q_\omega) - \frac{\delta^4}{4} \| Q_\omega \|_{L^4(\sigma_0)}^4 = \mathcal{I}_\omega(Q_\omega) + o_\omega(1) \| R_\omega \|_{L^2(\sigma_0)}^2,
\]

where the elliptic equation (or (7.3)) is used in the second identity and (8.7) is employed in the last step. For the first order term, cancelling by the equation (7.2), we obtain

\[
\sqrt{1 - \delta^2} \left\langle \left( \omega(H_\omega^{(2D}} - \Lambda_\omega) - \frac{1}{\sigma_\omega} \partial_\theta^2 + \mu_\omega \right) Q_\omega - (1 - \delta^2) Q_\omega^3, R_\omega \right\rangle_{L^2(\sigma_0)} \\
= \sqrt{1 - \delta^2} \delta^2 \| Q_\omega \|_{L^2(\sigma_0)}^2 \leq \delta^2 \| Q_\omega \|_{L^4(\sigma_0)}^3 \| R_\omega \|_{L^2(\sigma_0)} \\
= o_\omega(1) \| R_\omega \|_{L^2(\sigma_0)}^2
\]

For the higher-order terms, by Young’s inequality, Corollary 5.3 Proposition 7.1 (iii) and Proposition 7.11 (i) (\( \| R_\omega \|_{L^4(\sigma_0)} = O(\omega^{-3/2}) \)), we get

\[
\| R_\omega \|_{L^4(\sigma_0)}^2 \leq \| R_\omega \|_{L^2(\sigma_0)}^2 \| R_\omega \|_{2,\Sigma_{(s,z)}}^2 \| R_\omega \|_{L^2(\sigma_0)}^2 \| R_\omega \|_{L^4(\sigma_0)}^3 + o_\omega(1) \| R_\omega \|_{L^2(\sigma_0)}^2 \\
= o_\omega(1) (\| R_\omega \|_{L^2(\sigma_0)}^2 + \| R_\omega \|_{2,\Sigma_{(s,z)}}^2)
\]

and

\[
\| \langle Q_\omega, R_\omega^3 \rangle \|_{L^2(\sigma_0)} \leq \| Q_\omega \|_{L^4(\sigma_0)} \| R_\omega \|_{L^4(\sigma_0)}^3 \|
\leq \| R_\omega \|_{L^4(\sigma_0)}^3 \| R_\omega \|_{2,\Sigma_{(s,z)}}^3 \| R_\omega \|_{L^2(\sigma_0)}^2 \| R_\omega \|_{L^4(\sigma_0)}^3 + o_\omega(1) \| R_\omega \|_{L^2(\sigma_0)}^2 \\
= o_\omega(1) (\| R_\omega \|_{L^2(\sigma_0)}^2 + \| R_\omega \|_{2,\Sigma_{(s,z)}}^2).
\]
For the second order terms, we extract the linearized operator

\[
\left\langle \left( \omega (H^{(2D)}_\omega) - \Lambda_\omega \right) - \frac{1}{\sigma^2} \partial^2_\theta + \mu_\omega, R_\omega \right\rangle_{L^2(\sigma_\omega)} \rightarrow \left\langle L_\omega R_\omega, R_\omega \right\rangle_{L^2(\sigma_\omega)} + 3\delta_\omega^2 \left\langle Q_\omega^2 R_\omega, R_\omega \right\rangle_{L^2(\sigma_\omega)},
\]

and then apply the $L^4$ estimate (8.8) and the coercivity estimate (Proposition 8.1) to obtain

\[
\left\langle \left( \omega (H^{(2D)}_\omega) - \Lambda_\omega \right) - \frac{1}{\sigma^2} \partial^2_\theta + \mu_\omega, R_\omega \right\rangle_{L^2(\sigma_\omega)} \geq \frac{L_\omega}{8} \| R_\omega \|_{H^1(S^1)}^2 + \frac{\omega}{4} \| R_\omega \|_{S_{\omega,(s,z)}}^2 - o_\omega(1) \| R_\omega \|_{L^2(\sigma_\omega)}^2 \geq \frac{L_\omega}{16} \| R_\omega \|_{L^2(\sigma_\omega)}^2 + \frac{\omega}{8} \| R_\omega \|_{S_{\omega,(s,z)}}^2 \quad \text{(by (8.3))},
\]

provided that $\omega \geq 1$ is large enough.

Putting all together, we obtain

\[
\mathcal{I}_\omega(Q_\omega) \geq \mathcal{I}_\omega(Q_\omega) + \frac{L_\omega}{32} \| R_\omega \|_{L^2(\sigma_\omega)}^2,
\]

which contradicts to our assumptions that $\mathcal{I}_\omega(Q_\omega) = \mathcal{I}_\omega(Q_\omega)$ and $R_\omega \neq 0$.

**APPENDIX A. ENERGY MINIMIZATION FOR THE 1D PERIODIC NLS**

Following [27], we review characterization of a ground state for the 1D periodic energy minimization

\[
J^{(1D)}(m) = \inf \left\{ E_\infty(w) : w \in H^1(S^1) \text{ and } \| w \|_{L^2(S^1)}^2 = m \right\},
\]

where

\[
E_\infty[w] = \frac{1}{2} \| w' \|_{L^2(S^1)}^2 + \frac{\kappa}{8\pi} \| w \|_{L^4(S^1)}^4 \quad (\kappa = \pm 1).
\]

Then, we prove a coercivity estimate for the linearized operator at a ground state; it is a key ingredient for the proof of uniqueness of a 3D constrained energy minimizer.

**A.1. Ground states on the unit circle; a review.** To begin with, we summarize basic facts about the Jacobi elliptic functions, because a ground state can be represented by them; we refer to [32] for more details of elliptic functions. Given $k \in (0, 1)$, we define the incomplete elliptic integral of the first kind (resp., the second kind) by

\[
F(\phi; k) := \int_0^\phi \frac{ds}{\sqrt{1 - k^2 \sin^2(s)}}, \quad \text{resp., } E(\phi; k) := \int_0^\phi \sqrt{1 - k^2 \sin^2(s)} ds.
\]

In particular, when $\phi = \frac{\pi}{2}$, it is called the complete elliptic integral of the first kind (resp., the second kind), i.e.,

\[
K(k) := F\left(\frac{\pi}{2}; k\right), \quad \text{resp., } E(k) := E\left(\frac{\pi}{2}; k\right),
\]

satisfying that $K(k) \rightarrow \frac{\pi}{2}$ as $k \rightarrow 0^+$ and $K(k) \rightarrow \infty$ as $k \rightarrow 1^-$. 

Next, using the inverse of 
\[ z = F(\phi; k), \]
we define the \textit{snoidal} (resp., \textit{cnoidal}, \textit{dnoidal}) function by

\[ \text{sn}(z; k) := \sin \phi(z; k) \quad \text{resp.,} \quad \text{cn}(z; k) := \cos \phi(z; k), \quad \text{dn}(z; k) := \sqrt{1 - k^2 \text{sn}^2(z; k)}. \]

These three functions are called the \textit{Jacobi elliptic functions}. Note that \( \text{sn}(z; k) \) and \( \text{cn}(z; k) \) are \( 4K(k) \)-periodic, but \( \text{dn}(z; k) \) is \( 2K(k) \)-periodic. Moreover, we have

\[ 1 = \text{sn}^2 + \text{cn}^2 = k^2 \text{sn}^2 + \text{dn}^2, \]

\[ \text{sn}' = (\text{cn})(\text{dn}), \quad \text{cn}' = -(\text{sn})(\text{dn}), \quad \text{dn}' = -k^2(\text{cn})(\text{sn}) \quad (A.3) \]

and

\[ E(k) = \int_0^{K(k)} \text{dn}^2(z; k)dz. \quad (A.4) \]

Thus, it follows that \( \text{dn}(z; k) \) solves the nonlinear elliptic equation on \( (0, 2K(k)) \),

\[ -\text{dn}'' - 2\text{dn}^3 = -(2 - k^2)\text{dn}. \quad (A.5) \]

Coming back to the minimization problem \( (A.1) \), we may assume that

\[ Q_\infty(0) = \max_{\theta \in S^1} |Q_\infty(\theta)| > 0 \quad (A.6) \]

without loss of generality, since the variational problem is invariant under translation and phase shift. The following proposition asserts that a minimizer is uniquely determined, and that in the focusing case, a ground state is given by the dnoidal function provided that the mass is large enough; otherwise, it is constant.

**Proposition A.1** (Ground state on the unit circle; \cite{27} Proposition 3.2). Suppose that \( Q_\infty \) is a minimizer for the problem \( J_\infty^{(1D)}(m) \) with \( (A.6) \).

(i) If \( \kappa = 1 \) or if \( \kappa = -1 \) and \( 0 < m \leq 2\pi^2 \), then \( Q_\infty \equiv \sqrt{\frac{m}{2\pi}} \).

(ii) If \( \kappa = -1 \) and \( m > 2\pi^2 \), then \( Q_\infty = \frac{4}{3} \text{dn}(\frac{\theta}{\beta}; k) \), where \( \alpha, \beta \) and \( k \) are uniquely determined by

\[ 4\pi \alpha^2 = \beta^2, \quad K(k)\beta = \pi, \quad m = 8E(k)K(k). \quad (A.7) \]

**Remark A.2.** The minimizer \( Q_\infty \) solves the nonlinear elliptic equation

\[ -Q_\infty'' + \frac{\kappa}{2\pi}(Q_\infty)^3 = -\mu_\infty Q_\infty \quad (A.8) \]

with the Lagrange multiplier

\[ \mu_\infty = \begin{cases} 
-\frac{m}{(2\pi)^2} & \text{if } \kappa = 1, \\
\frac{m}{(2\pi)^2} & \text{if } \kappa = -1 \text{ and } 0 < m \leq 2\pi^2, \\
\frac{(2 - k^2)K(k)^2}{\pi^2} & \text{if } \kappa = -1 \text{ and } m > 2\pi^2.
\end{cases} \quad (A.9) \]

\footnote{When \( \kappa = -1 \) and \( m > 2\pi^2 \), it follows from \((A.3)\) and \((A.7)\).}
A.2. Linearized operator at the ground state \( Q_\infty \). Our next goal is to establish a coercivity estimate for the linearized operator

\[
\mathcal{L}_\infty = -\partial_x^2 + \mu + \frac{3\kappa}{2\pi}(Q_\infty)^2
\]

at the ground state \( Q_\infty \), that is, if \( \kappa = 1 \) or \( \kappa = -1 \) and \( m \in (0, 2\pi^2) \cup (2\pi^2, \infty) \), then there exists \( L_\infty > 0 \) such that

\[
\langle \mathcal{L}_\infty \varphi, \varphi \rangle_{L^2(S^1)} \geq L_\infty \| \varphi \|_{H^1(S^1)}^2
\]

for all \( \varphi \in H^1(S^1) \) such that \( \langle \varphi, Q_\infty \rangle_{L^2(S^1)} = \langle \varphi, Q'_\infty \rangle_{L^2(S^1)} = 0 \).

Remark A.3. If \( Q_\infty \) is constant (see (A.9)), then \( \mathcal{L}_\infty \) has the following simple lower bound. If \( \kappa = 1 \), then \( \mathcal{L}_\infty = -\partial_x^2 + \frac{m}{2\pi} \) is obviously strictly positive. On the other hand, if \( \kappa = -1 \) and \( 0 < m \leq 2\pi^2 \), then \( \mathcal{L}_\infty = -\partial_x^2 - \frac{m}{2\pi} \) satisfies \( \langle \mathcal{L}_\infty \varphi, \varphi \rangle_{L^2(S^1)} \geq \frac{1}{2}(1 - \frac{m}{2\pi})\| \varphi \|_{H^1(S^1)}^2 \) for all \( \varphi \in H^1(S^1) \) such that \( \langle \varphi, Q_\infty \rangle_{L^2(S^1)} = 0 \).

By Remark A.3 it suffices to consider the case \( \kappa = -1 \) and \( m > 2\pi^2 \). We prove the following coercive estimate.

Proposition A.4. If \( \kappa = -1 \) and \( m > 2\pi^2 \), then there exists \( L_\infty > 0 \) such that

\[
\langle \mathcal{L}_\infty \varphi, \varphi \rangle_{L^2(S^1)} \geq L_\infty \| \varphi \|_{H^1(S^1)}^2
\]

for all \( \varphi \in H^1(S^1) \) such that \( \langle \varphi, Q_\infty \rangle_{L^2(S^1)} = \langle \varphi, Q'_\infty \rangle_{L^2(S^1)} = 0 \).

We recall \( Q_\infty = \frac{1}{\alpha}dn(\frac{\theta}{2}, k) \) if \( \kappa = -1 \) and \( m > 2\pi^2 \) (see Proposition A.1 (ii)). Thus, for the proof, we need the spectral property for the operator \( L^dn_+ := -\partial_x^2 + (2 - k^2) - 6dn^2 \).

Lemma A.5 (Spectrum of \( L^dn_+ [27, Section 4.1] \)). For \( k \in (0, 1) \), the first three lowest eigenvalues of \( L^dn_+ \) are given by

\[
\lambda_0 = (k^2 - 2) - 2\sqrt{k^4 - k^2 + 1}, \quad \lambda_1 = 0, \quad \lambda_2 = (k^2 - 2) + 2\sqrt{k^4 - k^2 + 1}
\]

with the corresponding eigenstates (in order)

\[
\chi_- = 1 - (k^2 + 1 - \sqrt{k^4 - k^2 + 1})sn^2, \quad dn', \quad \chi_+ = 1 - (k^2 + 1 + \sqrt{k^4 - k^2 + 1})sn^2.
\]

Proof of Proposition A.4. In order to prove the result, it suffices to show that

\[
\langle \mathcal{L}_\infty \varphi, \varphi \rangle_{L^2(S^1)} \geq \| \varphi \|_{L^2(S^1)}^2
\]

for all \( \varphi \in L^2(S^1) \) such that \( \langle \varphi, Q_\infty \rangle_{L^2(S^1)} = \langle \varphi, Q'_\infty \rangle_{L^2(S^1)} = 0 \). Indeed, if (A.11) holds but Proposition A.4 is false, there exists a sequence \( \{ \varphi_n \}_{n=1}^\infty \subset H^1(S^1) \) such that \( \langle \varphi_n, Q_\infty \rangle_{L^2(S^1)} = \langle \varphi_n, Q'_\infty \rangle_{L^2(S^1)} = 0 \) and \( \| \varphi_n \|_{H^1(S^1)} = 1 \) and \( \langle \mathcal{L}_\infty \varphi_n, \varphi_n \rangle_{L^2(S^1)} \to 0 \) as \( n \to \infty \). Then, (A.11) implies that \( \| \varphi_n \|_{L^2(S^1)} \to 0 \) as \( n \to \infty \), and so is \( \| \varphi_n \|_{L^4(S^1)} \to 0 \) by the Gagliardo-Nirenberg inequality (5.5). Thus, it follows that

\[
o_n(1) = \langle \mathcal{L}_\infty \varphi_n, \varphi_n \rangle_{L^2(S^1)} \geq \min\{1, \mu_\infty\} \| \varphi_n \|_{H^1(S^1)} + o_n(1) \to \min\{1, \mu_\infty\},
\]

which deduces a contradiction.
To prove (A.11), for sufficiently small \( \epsilon \in \mathbb{R} \) and any \( \psi \in H^1(\mathbb{S}^1) \) such that \( \|\psi\|_{L^2(\mathbb{S}^1)} = 1 \) and \( \langle Q_\infty, \psi \rangle_{L^2(\mathbb{S}^1)} = 0 \), we define
\[
 g(\epsilon) := E_\infty \left( \sqrt{m} Q_\infty + \epsilon \psi \right) / \|Q_\infty + \epsilon \psi\|_{L^2(\mathbb{S}^1)}.
\]
By direct computations with the identity \( \|\partial_\theta Q_\infty\|_{L^2(\mathbb{S}^1)}^2 - \frac{1}{2\pi} \|Q_\infty\|_{L^4(\mathbb{S}^1)}^4 = -\mu_\infty m \), it can be written as
\[
 g(\epsilon) = g(0) + \frac{\epsilon^2}{2} \langle L_\infty \psi, \psi \rangle_{L^2(\mathbb{S}^1)} + O(\epsilon^3).
\]
However, since \( Q_\infty \) is a positive minimizer to the 1D energy minimization problem (1.14), we must have
\[
 \langle L_\infty \psi, \psi \rangle_{L^2(\mathbb{S}^1)} \geq 0
\]
for all \( \psi \in H^1(\mathbb{S}^1) \) with \( \|\psi\|_{L^2(\mathbb{S}^1)} = 1 \) and \( \langle Q_\infty, \psi \rangle_{L^2(\mathbb{S}^1)} = 0 \). On the other hand, it is known that \( \ker L_\infty = \text{span}\{Q'_\infty\} \) and \( L_\infty \) has discrete eigenvalues (see Proposition A.1 (ii) and Lemma A.5). Therefore, we prove the claim (A.11). □

**Appendix B. Defocusing case**

In this appendix, we briefly state the results of the existence/uniqueness of an energy minimizer and its dimension reduction for the defocusing case. In the defocusing case, we consider the standard mass constraint energy minimization problem
\[
 J_\omega^{(3D)}(m) := \min \left\{ E_\omega[u] : u \in \Sigma, M[u] = m \right\}
\]
where the mass and the energy are given by (1.6) and (1.7) with \( \kappa = 1 \) respectively. The following existence and uniqueness result was proved in [34, Theorem 2.1].

**Theorem B.1** (Existence and uniqueness of an energy minimizer; defocusing case). Let \( \kappa = 1 \) and \( \omega \geq 1 \). Then the problem (1.1) has a positive unique minimizer \( Q_\omega \), up to phase shift, and it solves the nonlinear elliptic equation
\[
 \omega(H_\omega - \Lambda_\omega)Q_\omega + \sqrt{\omega}Q_\omega^3 = -\mu_\omega Q_\omega,
\]
where \( \mu_\omega \in \mathbb{R} \) is a Lagrange multiplier.

Next, we state the dimension reduction from 3D to the 1D minimizer, and we briefly sketch the ideas of the proof since the arguments are almost same with the focusing case(\( \kappa = -1 \)).

**Theorem B.2** (Dimension reduction; defocusing case \( \kappa = 1 \)). Let \( Q_\omega \) and \( \mu_\omega \) be the minimizer and a Lagrange multiplier in Theorem B.1. Then there exists \( \{O_\omega\}_{\omega \gg 1} \subset \text{SO}(2) \) such that
\[
 \left\| Q_\omega(O_\omega y, z) - \sqrt{\frac{m}{2\pi}} \left( \chi(|y|)\omega^{-\frac{1}{4}} \Phi_0(|y| - \sqrt{\omega}, z) \right) \right\|_\Sigma + \left| \mu_\omega + \frac{m}{(2\pi)^2} \right| \to 0 \text{ as } \omega \to \infty,
\]

where \( \chi(r) = \sum_{n \in \mathbb{Z}} e^{\frac{i}{\sqrt{\omega}} \left| r - \frac{n}{\sqrt{\omega}} \right|} \), \( \Phi_0(x) = e^{-|x|^2} \), and \( \Sigma \) is the two-sphere.
where \( \Phi_0(s, z) = \frac{1}{\sqrt{\pi}} e^{-\frac{s^2+z^2}{2}} \) and \( \chi : [0, \infty) \to [0, 1] \) is a smooth function such that \( \chi \equiv 0 \) on \([0, 1]\) and \( \chi \equiv 1 \) on \([2, \infty)\).

**Proof.** As in Section 2, we reformulate the problem with suitable changes of variables and recall that the problem \( J_\infty^{(1D)}(m) \) has a minimizer \( \sqrt{\frac{m}{2\pi}} \) with the Lagrange multiplier \( \mu_\infty = -\frac{m}{(2\pi)^2} \) for defocusing case \( \kappa = 1 \) (see Proposition A.1).

**Step 1.** Energy upper bound \((J_\omega^{(3D)}(m) \leq J_\infty^{(1D)}(m) + O(\omega^{-\frac{1}{2}}))\). Using the test function \( \sqrt{\frac{m}{2\pi}} \chi_\omega(s) \Phi_\omega(s, z) \), where \( \sqrt{\frac{m}{2\pi}} \) is the ground state for 1D circle problem \( J_\infty^{(1D)}(m) \) in the defocusing case, we can prove the energy upper bound (see Lemma 7.3).

**Step 2.** Dimension reduction. Applying the energy bound (Step 1) and arguments in Proposition 7.1 (iii), we can prove the following uniform bound:

\[
\sqrt{\omega} \| Q_\omega \|_{L^2(\Omega_\omega(s, z))}, \| \partial_\theta Q_\omega \|_{L^2(\omega)} \), \| Q_\omega \|_{L^6(\sigma_\omega)} \text{ and } |\mu_\omega|
\]

are bounded uniformly in \( \omega \). Then following the arguments in subsections 7.2, 7.3 we can obtain uniform Gaussian decay of an energy minimizer (Proposition 7.5) and dimension reduction (Proposition 7.11), and hence we can prove the result. □

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