A NEW QUILLEN MODEL FOR THE MORITA HOMOTOPY THEORY OF DG CATEGORIES

GONÇALO TABUADA

Abstract. We construct a new Quillen model, based on the notions of Drinfeld’s DG quotient, [1], and localization pair, for the Morita homotopy theory of DG categories. This new Quillen model carries a natural closed symmetric monoidal structure and allow us to re-interpret Toën’s construction of the internal Hom-functor for the homotopy category of DG categories as a total right derived internal Hom-functor.

Contents

1. Introduction 1
2. Acknowledgments 2
3. Preliminaries 2
4. Homotopy of DG functors 2
5. Q-model structure 4
6. Closed symmetric monoidal structure 13
7. Derived internal Hom-functor 15
8. Relation with dgcat 19
References 20

1. Introduction

In this article we propose a solution to the following problem stated by Toën in [13], where we suppose that the commutative ground ring $k$ is a field:

The model category $\text{dgcat}$ together with the symmetric monoidal structure $- \otimes -$ is not a symmetric monoidal model category, as the tensor product of two cofibrant objects in $\text{dgcat}$ is not cofibrant in general. A direct consequence of this fact is that the internal Hom object between cofibrant-fibrant objects in $\text{dgcat}$ can not be invariant by quasi-equivalences, and thus does not provide internal Hom’s for the homotopy categories $\text{Ho}(\text{dgcat})$.

In [13], Toën has constructed the internal Hom-functor $\text{rep}_{dg}(-, -)$ for $\text{Ho}(\text{dgcat})$, using a certain dg category of right quasi-representable bimodules.

Key words and phrases. Quillen model structure, DG quotient, localizing pair, closed symmetric monoidal structure, derived internal Hom-functor, DG category.

Supported by FCT-Portugal, scholarship SFRH/BD/14035/2003.
In [1], Drinfeld has given an explicit construction of the dg quotient of a dg category \( \mathcal{A} \) modulo a full dg subcategory \( \mathcal{B} \), under certain flatness assumptions that are satisfied if one works over a field.

In this article, we construct a Quillen model structure on the category \( \mathcal{L}_p \) of localization pairs using Drinfeld’s explicit dg quotient construction. We show that this new model is Quillen equivalent to the one constructed in [10][11], and carries a natural closed symmetric monoidal structure. The tensor product and internal \( \text{Hom} \)-functor in \( \mathcal{L}_p \) are shown to be derivable functors, which correspond, under the equivalence between \( \text{Ho}(\mathcal{L}_p) \) and \( \text{Ho}(\text{dgcat}) \), to the derived tensor product \( - \otimes - \) and \( \text{rep}_{dg}(-, -) \) constructed in [13]. In particular we re-interpret the functor \( \text{rep}_{dg}(-, -) \) as a total right derived internal \( \text{Hom} \)-functor \( \mathcal{R}\text{Hom}(-, -) \) in our new Quillen model.

2. Acknowledgments

This article is part of my Ph. D. thesis under the supervision of Prof. B. Keller. I deeply thank him for several useful discussions and generous patience. I am very grateful to B. Toën for pointing out an error in a previous version of this article.

3. Preliminaries

In what follows, \( k \) will denote a field. The tensor product \( \otimes \) will denote the tensor product over \( k \). Let \( \text{Ch}(k) \) denote the category of complexes over \( k \). By a \textit{dg category}, we mean a differential graded \( k \)-category, see [1] [7] [8]. For a dg category \( \mathcal{A} \), we denote by \( \mathcal{C}_{dg}(\mathcal{A}) \) the dg category of right \( \mathcal{A} \) dg modules and by \( ^*: \mathcal{A} \to \mathcal{C}_{dg}(\mathcal{A}) \) the Yoneda dg functor. We write \( \text{dgcat} \) for the category of small dg categories. It is proven in [10] [11] [12], that the category \( \text{dgcat} \) admits a structure of cofibrantly generated model category whose weak equivalences are the Morita equivalences defined in [10][11]. Recall that we dispose of an explicit set \( I = \{Q, S(n)\} \) of generating cofibrations and an explicit set \( J = \{R(n), F(n), I_n(k_0, \ldots, k_n), L_n(k_0, \ldots, k_n), C\} \) of generating trivial cofibrations.

4. Homotopy of DG functors

Let \( \mathcal{B} \) be a dg category.

\textbf{Definition 4.1.} Let \( P(\mathcal{B}) \) be the dg category, see [1], whose objects are the closed morphisms of degree zero in \( \mathcal{B} \)

\[ X \xrightarrow{f} Y, \]

that become invertible in \( H^0(\mathcal{B}) \). We define the complex of morphisms

\[ \text{Hom}_{P(\mathcal{B})}(X \xrightarrow{f} Y, X' \xrightarrow{f'} Y') \]

as the homotopy pull-back in \( \text{Ch}(k) \) of the diagram

\[ \begin{array}{ccc}
\text{Hom}_\mathcal{B}(Y, Y') & \xrightarrow{f^*} & \text{Hom}_\mathcal{B}(X, Y') \\
\text{Hom}_\mathcal{B}(X, X') & \xrightarrow{f'} & \text{Hom}_\mathcal{B}(X, Y') \\
\end{array} \]
Remark 4.1. We dispose of the natural commutative diagram in \( \text{dgcat} \)

\[
\begin{array}{c}
\mathcal{B} \\
\downarrow \Delta \\
\text{Pretr}(\mathcal{B})
\end{array}
\begin{array}{c}
\mathcal{B} \times \mathcal{B} \\
\downarrow \text{id} \\
\text{Pretr}(\mathcal{B}) \times \text{Pretr}(\mathcal{B})
\end{array}
\]

where \( i \) is the dg functor that associates to an object \( B \) of \( \mathcal{B} \) the morphism \( B \xrightarrow{\text{id}} B \) and \( p_0, \text{ resp. } p_1, \) is the dg functor that sends a closed morphism \( X \xrightarrow{\ell} Y \) to \( X, \text{ resp. } Y. \)

Lemma 4.1. The dg category \( P(\mathcal{B}) \) is a path object for \( \mathcal{B}, \) see \([3]\), in the Quillen model structure described in \([12]\).

Proof. We prove that the dg functor \( i \) is a quasi-equivalence. Clearly the dg functor \( i \) induces a quasi-isomorphism in \( \text{Ch}(k) \)

\[
\text{Hom}_B(X, Y) \xrightarrow{\sim} \text{Hom}_{P(\mathcal{B})}(i(X), i(Y)),
\]

for every object \( X, Y \in \mathcal{B}. \) Remark that the functor \( \text{H}^0(i) \) is also essentially surjective. In fact, let \( X \xrightarrow{\ell} Y \) be an object of \( P(\mathcal{B}). \) Consider the following morphism in \( P(\mathcal{B}) \) from \( i(X) \) to \( X \xrightarrow{\ell} Y, \)

\[
\begin{array}{c}
X \\
\downarrow h=0 \\
X \xrightarrow{f} Y,
\end{array}
\]

where \( h \) denotes de zero homotopy. Remark that it becomes an isomorphism in \( \text{H}^0(P(\mathcal{B})) \) simply because \( f \) becomes an isomorphism in \( \text{H}^0(\mathcal{B}). \) This proves that the dg functor \( i \) is a quasi-equivalence. We will now show that the dg functor \( p_0 \times p_1 \) is a fibration in the Quillen model structure described in \([12]\). Remark first, that by definition of \( P(\mathcal{B}) \) the dg functor \( p_0 \times p_1 \) induces a surjective morphism in \( \text{Ch}(k) \)

\[
\text{Hom}_{P(\mathcal{B})}(X \xrightarrow{\ell} Y, X' \xrightarrow{\ell'} Y') \xrightarrow{p_0 \times p_1} \text{Hom}_B(X, X') \times \text{Hom}_B(Y, Y')
\]

for every object \( X \xrightarrow{\ell} Y \) and \( X' \xrightarrow{\ell'} Y' \) in \( P(\mathcal{B}). \) We will now show that contractions lift along the dg functor \( P(\mathcal{B}) \xrightarrow{p_0 \times p_1} \mathcal{B} \times \mathcal{B} \). Let \( X \xrightarrow{\ell} Y \) be an object of \( P(\mathcal{B}). \) Remark that a contraction of \( X \xrightarrow{\ell} Y \) in \( P(\mathcal{B}) \) corresponds exactly to the following morphisms in \( \mathcal{B}, \) \( c_X \in \text{Hom}_B^{-1}(X, X), \) \( c_Y \in \text{Hom}_B^{-1}(Y, Y) \) and \( h \in \text{Hom}_B^2(X, Y) \) satisfying the relations \( d(c_X) = 1_X, d(c_Y) = 1_Y \) and \( d(h) = c_Y \circ f + f \circ c_X. \) Suppose now, that we dispose of a contraction \( (c_1, c_2) \) of \( (X, Y) \) in \( \mathcal{B} \times \mathcal{B}. \) We can lift this contraction by considering \( c_X = c_1, c_Y = c_2 \) and \( h = c_2 \circ f \circ c_1. \) This shows that contractions lift along the dg functor \( P(\mathcal{B}) \xrightarrow{p_0 \times p_1} \mathcal{B} \times \mathcal{B}. \) We dispose of the following equivalence of dg categories

\[
\text{pretr}(P(\mathcal{B})) \xrightarrow{\sim} P(\text{pretr}(\mathcal{B}))
\]

where \( \text{pretr} \) denotes the pre-triangulated hull of a dg category, see \([1]\). This implies that the dg functor \( p_0 \times p_1 \) is a fibration in the Quillen model structure described in \([12]\). This proves the proposition. √
Let $\mathcal{A}$ be a cofibrant dg category and $F, G : \mathcal{A} \to \mathcal{B}$ dg functors. The dg functors $F$ and $G$ are homotopic in the Quillen model structure described in [12] if and only if there exists a dg functor $H : \mathcal{A} \to P(\mathcal{B})$ that makes the following diagram commute

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{H} & P(\mathcal{B}) \\
F \downarrow & & \downarrow P_b \\
\mathcal{B} & & \\
G \downarrow & & \downarrow P_a \\
\mathcal{B} & & \\
\end{array}
\]

see [3].

Remark 4.2. Remark that a dg functor $H$ as above corresponds exactly, see [5], to:

- a morphism $\eta \mathcal{A} : F(\mathcal{A}) \to G(\mathcal{A})$ of $Z^0(\mathcal{B})$ which becomes invertible in $H^0(\mathcal{B})$ for all $A \in \mathcal{A}$ (but which will not be functorial in $A$, in general) and
- a morphism of graded $k$-modules homogeneous of degree $-1$

$$h = h(A, B) : \text{Hom}_\mathcal{A}(A, B) \to \text{Hom}_\mathcal{B}(F(A), G(B)),$$

for all $A, B \in \mathcal{A}$ such that we have

$$(\eta B)(F(f)) - (G(f))(\eta A) = d(h(f)) + h(d(f))$$

and

$$h(fg) = h(f)(F(g)) + (-1)^n(G(f))h(g)$$

for all composable morphisms $f, g$ of $\mathcal{A}$, where $f$ is of degree $n$.

It is shownd in [5] that if we dispose of a dg functor $H$ as above and the dg category $\mathcal{B}$ is stable under cones, we can construct a sequence of dg functors

$$F \to I \to G[1],$$

where $I(A)$ is a contractible object of $\mathcal{B}$, for all $A \in \mathcal{B}$.

5. $Q$-MODEL STRUCTURE

Definition 5.1. A localization pair $\mathcal{A}$ is given by a small dg category $\mathcal{A}_1$ and a full dg subcategory $\mathcal{A}_0 \subset \mathcal{A}_1$. A morphism $F : \mathcal{A} \to \mathcal{B}$ of localization pairs is given by a commutative square

\[
\begin{array}{ccc}
\mathcal{A}_0 & \xrightarrow{F_0} & \mathcal{A}_1 \\
F_1 \downarrow & & \downarrow F_1 \\
\mathcal{B}_0 & \xrightarrow{F_1} & \mathcal{B}_1 \\
\end{array}
\]

of dg functors.

We denote by $Lp$ the category of localization pairs.

Let $\mathcal{A}$ be a localization pair.

Definition 5.2. The dg quotient of $\mathcal{A}$, see [1], is the dg category $\mathcal{A}_1/\mathcal{A}_0$ obtained from $\mathcal{A}_1$ by introducing a new morphism $h_X$ of degree $-1$ for every object $X$ of $\mathcal{A}_0$ and by imposing the relation $d(h_X) = 1_X$. 
5.1. **Morita model structure.** Let \( L \) be the category with two objects 0 and 1 and with a unique non identity morphism 0 \( \to \) 1.

**Remark 5.1.** An immediate application of Theorem 11.6.1 from [3] implies that the category \( \text{dgcat}^L \), i.e. the category of morphisms in \( \text{dgcat} \), admits a structure of cofibrantly generated model category whose weak equivalences \( W \) are the componentwise Morita equivalences and with generating cofibrations \( F^L_I \) and generating trivial cofibrations \( F^L_J \), where we use the notation of [3]:

The functor \( F^i_i \), \( i = 0, 1 \), from \( \text{dgcat} \) to \( \text{dgcat}^L \) is left adjoint to the evaluation functor \( Ev_i, i = 0, 1, \) from \( \text{dgcat}^L \) to \( \text{dgcat} \). By definition, we have \( F^L_I = F^0_I \cup F^1_I \) and \( F^L_J = F^0_J \cup F^1_J \).

The inclusion functor \( U : Lp \to \text{dgcat}^L \) admits a left adjoint \( S \) which sends an object \( G : B_0 \to B_1 \) to the localization pair formed by \( B_1 \) and its full dg subcategory \( \text{Im} \ G \).

**Proposition 5.1.** The category \( Lp \) admits a structure of cofibrantly generated model category whose weak equivalences \( W \) are the componentwise Morita equivalences and with generating cofibrations \( F^L_I \) and generating trivial cofibrations \( F^L_J \).

**Proof.** We first prove that \( Lp \) is complete and cocomplete. Let \( \{ X_i \}_{i \in I} \) be a diagram in \( Lp \). We remark that

\[
\text{colim}_{i \in I} X_i \sim \to S(\text{colim}_{i \in I} U(X_i)),
\]

which implies that \( Lp \) is cocomplete. The category \( Lp \) is also complete, since it is stable under products and equalizers in \( \text{dgcat}^L \). We now prove that conditions (1) and (2) of Theorem 11.3.2 from [3] are satisfied:

- Since \( S(F^L_I) = F^L_I \) and \( S(F^L_J) = F^L_J \) condition (1) is verified.
- Since the functor \( U \) clearly commutes with filtered colimits, it is enough to prove the following: let \( Y \xrightarrow{G} Z \) be an element of the set \( F^L_J \), \( X \) an object in \( Lp \) and \( Y \to X \) a morphism in \( Lp \). Consider the following push-out in \( Lp \):

\[
\begin{array}{ccc}
Y & \xrightarrow{G} & X \\
\downarrow & & \downarrow \\
Z & \xrightarrow{Y} & \text{im} \ X
\end{array}
\]

We prove that \( U(G_*) \) is a weak equivalence in \( \text{dgcat}^L \). We consider two situations:

- if \( G \) belongs to the set \( F^0_J \subset F^L_J \), then \( U(G_*) \) is a weak-equivalence simply because \( J - \text{cell} \subset W \) in \( \text{dgcat} \), see [10][11][12].
- if \( G \) belongs to the set \( F^1_J \subset F^L_J \), then \( Ev_1(U(G_*)) \) is a Morita equivalence. In particular it induces a quasi-isomorphism in the Hom spaces and since the 0-component of \( G_* \) is the identity on objects, the functor \( Ev_0(U(G_*)) \) is also a Morita equivalence. This implies that \( U(G_*) \) is a weak equivalence and so condition (2) is proven.

This proves the proposition.
We will now slightly modify the previous Quillen model structure on \( L_p \).

Let \( \sigma \) be the morphism of localization pairs:

\[
\begin{array}{ccc}
(\text{End}_K(1) \subset K) & \xrightarrow{inc} & (K \subset K),
\end{array}
\]

where \( \text{End}_K(1) \) is the dg algebra of endomorphisms of the object 1 in \( K \), see [12], and \( inc \) is the natural inclusion dg-functor. Clearly \( \sigma \) is a componentwise Morita equivalence. We write \( \tilde{F}^L_I \) resp. \( \tilde{F}^L_J \) for the union of \( \{\sigma\} \) with \( F^L_I \) resp. \( F^L_J \).

**Proposition 5.2.** The category \( L_p \) admits a structure of cofibrantly generated model category whose weak equivalences \( W \) are the componentwise Morita equivalences and with generating cofibrations \( \tilde{F}^L_I \) and generating trivial cofibrations \( \tilde{F}^L_J \).

**Proof.** The proof will consist in verifying that conditions (1) – (6) of Theorem 2.1.19 from [2] are satisfied. Condition (1) is clear. Since the localization pair \( (\text{End}_K(1) \subset K) \) is small in \( L_p \), conditions (2) and (3) are also satisfied. We have

\[
F^L_I - \text{inj} = F^L_J - \text{inj} \cap W
\]

and so by construction

\[
\tilde{F}^L_I - \text{inj} = \tilde{F}^L_J - \text{inj} \cap W.
\]

This shows conditions (5) and (6). We now prove that \( \tilde{F}^L_I - \text{cell} \subset W \). Since \( F^L_j - \text{cell} \subset W \) it is enough to prove that pushouts with respect to \( \sigma \) belong to \( W \).

Let \( \mathcal{A} \) be a localization pair and

\[
T : (\text{End}_K(1) \subset K) \to (A_0 \subset A_1)
\]

a morphism in \( L_p \). Consider the following push-out in \( L_p \):

\[
\begin{array}{ccc}
(\text{End}_K(1) \subset K) & \xrightarrow{T} & (A_0 \subset A_1) \\
\sigma \downarrow & & \downarrow \sigma \\
(K = K) & \xrightarrow{R} & (U_0 \subset U_1).
\end{array}
\]

We remark that the morphism \( T \) corresponds to specifying an object \( X \) in \( A_0 \) and a homotopy equivalence from \( X \) to an object \( Y \) in \( A_1 \). Clearly \( U_1 = A_1 \) and \( U_0 \) identifies with the full dg-subcategory of \( U_1 \) whose objects are \( Y \) and those of \( A_0 \).

Since \( X \) and \( Y \) are homotopy equivalent, the natural dg-functor \( R_0 : A_0 \leftrightarrow U_0 \) is a quasi-equivalence. This proves condition (4). The proposition is now proven. \( \sqrt{\ } \)

**Remark 5.2.** Remark that in this new Quillen model structure on \( L_p \) we dispose of more cofibrations and less fibrations than the Quillen model structure of proposition 5.1 since the weak equivalences are the same.

From now on, by Quillen model structure on \( L_p \) we mean that of proposition 5.2.

**Lemma 5.1.** A localization pair \( (A_0 \subset A_1) \) is fibrant in \( L_p \) if and only if \( A_0 \) and \( A_1 \) are Morita fibrant dg categories and \( A_0 \) is stable under homotopy equivalences in \( A_1 \).
Proof. A localization pair \((\mathcal{A}_0 \subset \mathcal{A}_1)\) is fibrant in \(\mathbb{L}_p\) if and only if for every morphism \(F\) in \(\mathcal{F}^L\), the following extension problem in \(\mathbb{L}_p\) is solvable:

\[
\begin{array}{c}
X \\
\downarrow F \\
Y
\end{array}
\xrightarrow{\sim} \begin{array}{c}(\mathcal{A}_0 \subset \mathcal{A}_1) \\
\end{array}
\]

If \(F\) belongs to \(\mathcal{F}^L\) this means that \(\mathcal{A}_0\) and \(\mathcal{A}_1\) are fibrant and if \(F = \sigma\), remark that it corresponds exactly to the statement that \(\mathcal{A}_0\) is stable under homotopy equivalences in \(\mathcal{A}_1\).

**Lemma 5.2.** If the localization pair \(\mathcal{A}\) is cofibrant in \(\mathbb{L}_p\) then \(\mathcal{A}_1\) is cofibrant in \(\text{dgcat}\).

**Proof.** We need to construct a lift to the following problem:

\[
\begin{array}{c}
\mathcal{C} \\
\downarrow p \\
\mathcal{A}_1 \longrightarrow \mathcal{B},
\end{array}
\]

where \(P\) is a trivial fibration in \(\text{dgcat}\), see proposition 5.2 and \(\mathcal{A}_1 \rightarrow \mathcal{B}\) is a dg-functor. Consider the following diagram in \(\mathbb{L}_p\):

\[
\begin{array}{c}
\mathcal{F}_C^0 \\
\uparrow \sim \downarrow \mathcal{F}_C^p \end{array}
\xrightarrow{\sim} \begin{array}{c}
\mathcal{F}_B^0 \\
\end{array}
\]

where \(\mathcal{A} \rightarrow \mathcal{F}_B^0\) is the natural morphism of localization pairs. Remark that \(\mathcal{F}_C^0\) belongs to \(\sigma - \text{inj} \cap \mathcal{F}_I^L - \text{inj}\) and so is a trivial fibration in \(\mathbb{L}_p\). Since \(\mathcal{A}\) is cofibrant in \(\mathbb{L}_p\) we dispose of a lifting \(\mathcal{A} \rightarrow \mathcal{F}_C^0\) that when restricted to the 1-component gives us the searched lift \(\mathcal{A}_1 \rightarrow \mathcal{C}\). This proves the lemma.

5.2. *Q*-model structure.

**Definition 5.3.** Let \(Q : \mathbb{L}_p \rightarrow \mathbb{L}_p\) be the functor that sends a localization pair \(\mathcal{A}\) to the localization pair \(\overline{\mathcal{A}_0} \hookrightarrow \mathcal{A}_1/\mathcal{A}_0\), where \(\overline{\mathcal{A}_0}\) is the full dg-subcategory of \(\mathcal{A}_1/\mathcal{A}_0\) whose objects are those of \(\mathcal{A}_0\).

**Remark 5.3.** Remark that we dispose of natural morphisms

\[
\eta_{\mathcal{A}} : (\mathcal{A}_0 \subset \mathcal{A}_1) \rightarrow (\overline{\mathcal{A}_0} \subset \mathcal{A}_1/\mathcal{A}_0)
\]

in \(\mathbb{L}_p\).

**Definition 5.4.** A morphism of localization pairs \(F : \mathcal{A} \rightarrow \mathcal{B}\) is a *Q*-weak equivalence if the induced morphism \(Q(F)\) is a weak equivalence in the Quillen model structure of proposition 5.2.

**Remark 5.4.** Remark that since the objects of \(\overline{\mathcal{A}_0}\) and \(\overline{\mathcal{B}_0}\) are all contractible, the dg-functor \(\overline{\mathcal{A}_0} \rightarrow \overline{\mathcal{B}_0}\) is clearly a Morita equivalence and so the morphism \(F\) is a *Q*-weak equivalence if and only if the induced dg-functor \(\mathcal{A}_1/\mathcal{A}_0 \rightarrow \mathcal{B}_1/\mathcal{B}_0\) is a Morita equivalence.
Definition 5.5. A morphism in $L_p$ is a cofibration if it is one for the Quillen model structure of Proposition 5.2 and it is a $Q$-fibration if it has the right lifting property with respect to all cofibrations of $L_p$ which are $Q$-weak equivalences.

Theorem 5.1. The category $L_p$ admits a structure of Quillen model category whose weak equivalences are the $Q$-weak equivalences, whose cofibrations are the cofibrations of $L_p$ and whose fibrations are the $Q$-fibrations.

The proof will consist in adapting the general arguments from chapter X from [4] to our situation. We start with some remarks:

A1 Since $k$ is a field, the conditions of theorem 3.4 from [1] are satisfied and so the functor $Q$ preserves weak equivalences.

A2 The morphisms of localization pairs:

\[ Q(A) \xrightarrow{\eta_A} QQ(A) \]

are weak equivalences in $L_p$. This follows from the fact that in both cases we are introducing contractions to objects that are already contractible and that the functor $Q$ is the identity functor on objects.

Lemma 5.3. A morphism $F : A \to B$ is a fibration and a weak equivalence of $L_p$ if and only if it is a $Q$-weak equivalence and a $Q$-fibration.

Proof. Since condition A1 is verified we can use the proof of lemma 4.3 in chapter X from [4].

√

Counterexample 1. Remark that the Quillen model structure of proposition 5.2 is not right proper, see [3].

Let $B$ be your favorite Morita fibrant dg category, whose derived category $D(B)$ is not trivial. In particular the dg functor $B \to 0$, where 0 denotes the terminal object in $dgcat$ is a fibration. Let $A$ be the dg category with one object 1 and whose dg algebra of endomorphisms of 1 is $k$. Consider the following diagram :

\[
\begin{array}{ccc}
B & \xrightarrow{i_0 \circ P} & A \\
\downarrow & & \downarrow i_A \\
A & \xrightarrow{0} & \bigoplus A.
\end{array}
\]

Clearly $i_A$ is a Morita equivalence and remark that the dg functor $i_0 \circ P$ is a fibration, since the object 1 in $A$ is not contractible. This implies that in the fiber product

\[
\begin{array}{ccc}
0 & \xrightarrow{\tau} & B \\
\downarrow & & \downarrow i_0 \circ P \\
A & \xrightarrow{i_A} & \bigoplus A,
\end{array}
\]

the dg functor $\emptyset \to B$ is not a Morita equivalence and so this Quillen model structure is not right proper. This implies that the Quillen model structure of proposition 5.2 is also not right proper. Apply the functor $F^0_? \colon dgcat \to L_p$ to the previous fiber
A NEW QUILLEN MODEL

\[ \emptyset = F_0 \xrightarrow{r} F_0 \]

We dispose of a fiber product since the functor \( F_0 \) preserves limits. Clearly \( F_0 \) is a weak equivalence in \( \text{Lp} \) and remark that the morphism \( F_0 \circ \sigma \) belongs to \( \sigma - \text{inj} \cap F^l - \text{inj} \), which implies that it is a fibration in \( \text{Lp} \).

Nevertheless we have the following lemma.

**Lemma 5.4.** Let \( \mathcal{A} \) be a localization pair such that the natural morphism

\[ \eta_{\mathcal{A}} : \mathcal{A} \rightarrow Q(\mathcal{A}) \]

is a weak equivalence in \( \text{Lp} \). Let \( F : \mathcal{W} \rightarrow Q(\mathcal{A}) \) be a fibration in \( \text{Lp} \). Then the morphism

\[ \eta^*_\mathcal{A} : \mathcal{W} \times_{Q(\mathcal{A})} \mathcal{A} \rightarrow \mathcal{W} \]

is a weak equivalence in \( \text{Lp} \).

**Proof.** We remark that each component of the morphism \( \eta_{\mathcal{A}} \) is the identity functor on the objects of the dg categories involved. Since fiber products in \( \text{Lp} \) are calculated componentwise, we conclude that each component of the morphism \( \eta^*_{\mathcal{A}} \) is the identity functor on the objects. Let \( X \) and \( Y \) be arbitrary objects of \( \mathcal{W}_1 \). We remark that we dispose of the following fiber product in \( \text{Ch}(k) \):

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{W}_1} \times_{\mathcal{A}_1} (X, Y) & \xrightarrow{r} & \text{Hom}_{\mathcal{A}_1}(F_1 X, F_1 Y) \\
\eta^*(F_1 X, F_1 Y) & \sim & \eta(F_1 X, F_1 Y)
\end{array}
\]

Since \( F \) is a fibration in \( \text{Lp} \), \( F_1(X, Y) \) is a fibration in the projective model structure on \( \text{Ch}(k) \) and since this Quillen model structure on \( \text{Ch}(k) \) is right proper, \( \eta^*(F_1 X, F_1 Y) \) is a quasi-isomorphism. We could do the same argument for \( X \) and \( Y \) objects in \( \mathcal{W}_0 \) instead of \( \mathcal{W}_1 \). This proves the lemma.

**Lemma 5.5.** Suppose that \( F : \mathcal{A}_0 \rightarrow \mathcal{B} \) is a fibration in \( \text{Lp} \) and that \( \eta_{\mathcal{A}} \) and \( \eta_{\mathcal{B}} \) are weak equivalences of \( \text{Lp} \). Then \( F \) is a \( Q \)-fibration.

**Proof.** Consider exactly the same proof as for lemma 4.4 in chapter X from [4], but use lemma 5.4 instead of the right properness assumption on \( \text{Lp} \).

**Lemma 5.6.** Any morphism \( F : Q(\mathcal{A}) \rightarrow Q(\mathcal{B}) \) has a factorization \( F = P \circ I \) where \( P : \mathcal{Z} \rightarrow Q(\mathcal{B}) \) is a \( Q \)-fibration and \( I : Q(\mathcal{A}) \rightarrow \mathcal{Z} \) is a cofibration and a \( Q \)-weak equivalence.

**Proof.** Since lemma 5.5 and conditions A1 and A2 are satisfied, we consider the proof of lemma 4.5 in chapter X from [4].
Let $\mathcal{A}$ be a localization pair. By condition $A_2$ we know that the natural morphism:

$$\eta_A : (A_0 \subset A_1) \to (\overline{A_0} \subset A_1/A_0)$$

is a $Q$-weak equivalence in $L_p$.

**Lemma 5.7.** Let $F : Z \to Q(\mathcal{A})$ be a fibration in $L_p$. Then the induced morphism

$$\eta_A^* : Z \times_{Q(\mathcal{A})} A \to Z$$

is a $Q$-weak equivalence in $L_p$.

**Proof.** We need to prove that $Q(\eta_A^*)$ is a weak equivalence in $L_p$.

(1) We prove that the induced morphism:

$$Q(\eta_A)^* : Q(Z) \times_{QQ(\mathcal{A})} Q(\mathcal{A}) \to Q(Z)$$

is a weak equivalence in $L_p$. Remark first that since $F$ is a fibration in $L_p$, the dg functors $F_0$ and $F_1$ are Morita fibrations, see [10], and so they are surjective at the level of $\text{Hom}$-spaces. We now show that the dg functor $F_0 : Z \to \underline{A_0}$ is surjective on objects. If $A_0$ is the empty dg category then so is $Z_0$ and the claim is showed. If $\underline{A_0}$ is not empty, every object $X$ in $\underline{A_0}$ is contractible and since the dg functor $F_0$ belongs to $C - \text{inj}$ there exists an object $Y$ in $Z_0$ such that $F_0(Y) = X$. This implies that each component of the morphism

$$Q(F) : Q(Z) \to QQ(\mathcal{A})$$

is a dg functor that is surjective at the level of $\text{Hom}$-spaces. Since by condition $A_2$ the morphism

$$(Q\eta_A) : Q(\mathcal{A}) \to QQ(\mathcal{A})$$

is a weak equivalence an argument analogue to the proof of lemma 5.4 (we have just proved that $F_1(X, Y)$ is a fibration in the projective model structure on $\text{Ch}(k)$), proves the condition (1).

(2) We prove that the induced morphism:

$$Q(Z) \times_{Q(\mathcal{A})} A \to Q(Z) \times_{QQ(\mathcal{A})} Q(A)$$

is an isomorphism in $L_p$. Since by construction the functor $Q$ is the identity functor on objects, both components of the above morphism are also the identity on objects. Let us consider de 1-component of the above morphism. Let $X$ and $Y$ be objects of $Z_1/Z_0$. We dispose of the following fiber product in $\text{Ch}(k)$:

\[
\begin{array}{ccc}
\text{Hom}_{Z_1/Z_0} & \times & \text{Hom}_{A_1/A_0}(F_1(X), F_1(Y)) \\
\text{Hom}_{Z_1/Z_0}(X, Y) & \overset{QF_1}{\to} & \text{Hom}_{(A_1/A_0)/\underline{A_0}}(F_1(X), F_1(Y)) \\
\end{array}
\]

Remark that the functor $Q\eta_A$, resp. $QF_1$, sends the contractions in $A_1/A_0$, resp. $Z_1/Z_0$, associated with the objects of $A_0$, resp. $Z_0$, to the new
contractions in \((\mathcal{A}_1/\mathcal{A}_0)/\mathcal{A}_0\) associated with the objects of \(\mathcal{A}_0\). Recall that we dispose of the following fiber product in \(\text{Ch}(k)\):

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{Z}_1} (X, Y) & \xrightarrow{r} & \text{Hom}_{\mathcal{A}_1}(F_1 X, F_1 Y) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathcal{Z}_0} (X, Y) & \xrightarrow{F_1} & \text{Hom}_{\mathcal{A}_1/\mathcal{A}_0}(F_1 X, F_1 Y).
\end{array}
\]

A analysis of the above fiber products shows that the induced morphism

\[
\text{Hom}_{\mathcal{Z}_1} (X, Y) \xrightarrow{F_1} \text{Hom}_{\mathcal{A}_1/\mathcal{A}_0}(F_1 X, F_1 Y).
\]

is an isomorphism in \(\text{Ch}(k)\). The same argument applies to the 0-component of the above morphism. This proves condition (2).

Now, conditions 1) and 2) imply that the morphism

\[
Q(Z \times_{\mathcal{A}} A) \xrightarrow{(Q\eta)^*_{\mathcal{A}}} Q(Z)
\]

is a weak equivalence in \(\mathcal{L}_p\), which is exactly the statement of the lemma. The lemma is then proved.

√

Lemma 5.8. Any morphism \(F : \mathcal{A} \to \mathcal{B}\) of \(\mathcal{L}_p\) has a factorization \(F = Q \circ J\) where \(Q : \mathcal{Z} \to \mathcal{B}\) is a \(Q\)-fibration and \(J : \mathcal{A} \to \mathcal{Z}\) is a cofibration and a \(Q\)-weak equivalence.

Proof. Consider exactly the same proof as for lemma 4.6 in chapter X from [4], but use lemma 5.7 instead of condition \(A_3\).

√

We now prove theorem 5.1.

Proof. We will prove that conditions \(M_1 - M_5\) of definition 7.1.3 from [3] are satisfied. By the proof of proposition 5.1, the category \(\mathcal{L}_p\) is complete and cocomplete and so condition \(M_1\) is verified. By definition the \(Q\)-weak equivalences in \(\mathcal{L}_p\) satisfy condition \(M_2\). Clearly the \(Q\)-weak equivalences and \(Q\)-fibrations in \(\mathcal{L}_p\) are stable under retractions. Since the cofibrations are those of proposition 5.2 condition \(M_3\) is verified. Finally lemma 5.3 implies condition \(M_4\) and lemmas 5.3 and 5.8 imply condition \(M_5\).

√

We denote by \(\text{Ho}(\mathcal{L}_p)\) the homotopy category of \(\mathcal{L}_p\) given by theorem 5.1.

Let \(\mathcal{A}\) be a localization pair.

Lemma 5.9. If \(\mathcal{A}\) is fibrant, in the Quillen model structure of proposition 5.2 and the morphism \(\eta_{\mathcal{A}} : \mathcal{A} \to Q(\mathcal{A})\) is a weak equivalence in \(\mathcal{L}_p\) then \(\mathcal{A}\) is \(Q\)-fibrant.

Proof. We need to show that the morphism \(\mathcal{A} \xrightarrow{\eta} 0\) is a \(Q\)-fibration, where 0 denotes the terminal object in \(\mathcal{L}_p\). Consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\eta_{\mathcal{A}}} & Q(\mathcal{A}) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\eta} & Q(0).
\end{array}
\]

√
Factorize the morphism \( Q(P) \) as

\[
\begin{array}{ccc}
Q(A) & \xrightarrow{i} & Z \\
\downarrow & & \downarrow q \\
Q(P) & \xrightarrow{q} & Q(0),
\end{array}
\]

where \( i \) is a trivial cofibration and \( q \) a fibration in \( \mathbb{L}_p \). By the proof of lemma \( \ref{lemma:Q-fibration} \), \( q \) is a \( Q \)-fibration. Since the morphism \( 0 \to Q(0) \) is a weak equivalence, lemma \( \ref{lemma:weak-equivalence} \) implies that the induced morphism \( 0 \times A \to Z \) is a weak equivalence. Since \( \eta_A \) is a weak equivalence the induced morphism

\[
\theta : A \to 0 \times Z
\]

is also a weak equivalence. Factorize the morphism \( \theta \) as

\[
\begin{array}{ccc}
A & \xrightarrow{j} & W \\
\downarrow \theta & & \downarrow \pi \\
0 \times Z, & \xrightarrow{q_\ast \circ \pi} & 0.
\end{array}
\]

where \( \pi \) is a trivial fibration of \( \mathbb{L}_p \) and \( j \) is a trivial cofibration. Then \( q_\ast \circ \pi \) is a \( Q \)-fibration and the lifting exists in the diagram :

Thus \( P \) is a retract of a \( Q \)-fibration, and is therefore a \( Q \)-fibration itself. This proves the lemma. \( \checkmark \)

**Lemma 5.10.** If \( A \) is \( Q \)-fibrant, then \( A \) is fibrant in \( \mathbb{L}_p \) and the natural morphism

\[
\eta_A : A \to Q(A)
\]

is a weak equivalence.

**Proof.** Since the \( Q \)-model structure on \( \mathbb{L}_p \) has less fibrations than the Quillen model structure of proposition \( \ref{prop:Q-fibration} \) the localization pair \( A \) is fibrant in \( \mathbb{L}_p \). Consider the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & Q(A) \\
\downarrow P & & \downarrow Q(P) \\
0 & \xrightarrow{\eta} & Q(0).
\end{array}
\]
Factorize $Q(P) = q \circ i$ as in the previous lemma. We dispose of the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\theta} & 0 \times Z \\
\downarrow & & \downarrow \\
0 & \xleftarrow{q_*} & Q(0)
\end{array}
\]

Since $p$ and $q_*$ are $Q$-fibrations, $A$ and $Z$ are $Q$-fibrant objects in $Lp$ and $\theta$ is a $Q$-weak equivalence in $Lp$. By application of lemma 7.7.1 b) from [3] to $\theta$ and using lemma [5,3] we conclude that $\theta$ is a weak equivalence. Since so is $i$, we conclude that $\eta_A$ is also a weak equivalence. This proves the lemma.

**Remark 5.5.** By lemmas [5,3] and 5.10 a localization pair $A$ is $Q$-fibrant if and only if it is fibrant in $Lp$ and the natural morphism $\eta_A : A \rightarrow Q(A)$ is a weak equivalence.

We now describe explicitly the $Q$-fibrant objects in $Lp$.

**Proposition 5.3.** A localization pair $A$ is $Q$-fibrant, i.e. fibrant in the model structure of Theorem 5.1, if and only if it is isomorphic in $Lp$ to a localization pair of the form:

\[(B_{\text{contr}} \subset B),\]

where $B$ is a fibrant dg category and $B_{\text{contr}}$ is the full dg subcategory of contractible objects in $B$.

**Proof.** Suppose first that $A$ is $Q$-fibrant. Since it is also fibrant in $Lp$ the dg category $A_1$ is fibrant in $\text{dgcat}$. Since the morphism $\eta_A : (A_0 \subset A_1) \rightarrow (A_0 \subset A_1/A_0)$ is a weak equivalence all the objects of $A_0$ are contractible. Since $A$ is fibrant in $Lp$ by lemma [5,1] $A_0$ is stable under homotopy equivalences in $A_1$. This implies that $A_0$ is in fact the full dg subcategory of contractible objects of $A_1$. Consider now a localization pair $(B_{\text{contr}} \subset B)$ as in the statement of the proposition. We remark that since $B$ is fibrant in $\text{dgcat}$, then $B_{\text{contr}}$ it is also fibrant. Clearly $(B_{\text{contr}} \subset B)$ satisfies the extension condition in what regards $\sigma$ and the morphism $\eta : (B_{\text{contr}} \subset B) \rightarrow (B_{\text{contr}} \subset B/B_{\text{contr}})$ is a weak equivalence in $Lp$. This proves the proposition.

6. Closed symmetric monoidal structure

Let $A$ and $B$ be small dg categories. We denote by $A \otimes B$ the tensor product of $A$ and $B$, see [6] [13], and by $\text{Fun}_{dg}(A, B)$ the dg category of dg functors from $A$ to $B$, see [6] [8].

**Definition 6.1.** The internal Hom functor in $Lp$

\[\text{Hom}(-, -) : Lp^{op} \times Lp \rightarrow Lp,\]

associates to the localization pairs $(A_0 \subset A_1)$, $(B_0 \subset B_1)$ the localization pair:

\[(\text{Fun}_{dg}(A_1, B_0) \subset \text{Fun}_{dg}(A_0, B_0) \times \text{Fun}_{dg}(A_1, B_1)).\]
Definition 6.2. The tensor product functor in $\mathcal{L}_p$

$- \otimes : \mathcal{L}_p \times \mathcal{L}_p \rightarrow \mathcal{L}_p$

associates to the localization pairs $(\mathcal{A}_0 \subset \mathcal{A}_1)$, $(\mathcal{B}_0 \subset \mathcal{B}_1)$ the localization pair :

$(\mathcal{A}_0 \otimes \mathcal{B}_1 \cup \mathcal{A}_1 \otimes \mathcal{B}_0 \subset \mathcal{A}_1 \otimes \mathcal{B}_1)$,

where $\mathcal{A}_0 \otimes \mathcal{B}_1 \cup \mathcal{A}_1 \otimes \mathcal{B}_0$ is the full dg subcategory of $\mathcal{A}_1 \otimes \mathcal{B}_1$ consisting of those objects $a \otimes b$ of $\mathcal{A}_1 \otimes \mathcal{B}_1$ such that $a$ belongs to $\mathcal{A}_0$ or $b$ belongs to $\mathcal{B}_0$.

Let $\mathcal{A} = (\mathcal{A}_0 \subset \mathcal{A}_1)$, $\mathcal{B} = (\mathcal{B}_0 \subset \mathcal{B}_1)$ and $\mathcal{C} = (\mathcal{C}_0 \subset \mathcal{C}_1)$ be localization pairs.

Proposition 6.1. The category $\mathcal{L}_p$ endowed with the functors $\text{Hom}(\cdot, \cdot)$ and $- \otimes -$ is a closed symmetric monoidal category. In particular we dispose of a natural isomorphism in $\mathcal{L}_p$:

$\text{Hom}_{\mathcal{L}_p}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \text{Hom}_{\mathcal{L}_p}(\mathcal{A}, \text{Hom}(\mathcal{B}, \mathcal{C}))$.

Proof. Consider the following commutative square in $\text{dgcat}$:

$$
\begin{array}{ccc}
A_0 \otimes B_1 \cup A_1 \otimes B_0 & \rightarrow & A_1 \otimes B_1 \\
A_0 \otimes B_0 & \rightarrow & \text{Fun}_{\text{dg}}(B_1, C_0) \\
A_1 \otimes B_0 & \rightarrow & \text{Fun}_{\text{dg}}(B_0, C_0) \times \text{Fun}_{\text{dg}}(B_0, C_1) \\
A_1 & \rightarrow & \text{Fun}_{\text{dg}}(B_0, C_0) \\
A_1 \otimes B_1 & \rightarrow & C_1.
\end{array}
$$

which corresponds exactly to an element of $\text{Hom}_{\mathcal{L}_p}(\mathcal{A}, \text{Hom}(\mathcal{B}, \mathcal{C}))$. Recall from [6] that $\text{dgcat}$ endowed with the functors $- \otimes -$ and $\text{Hom}(\cdot, \cdot)$ is a closed symmetric monoidal category. This implies by adjunction that the commutative square above corresponds to the following commutative square in $\text{dgcat}$:

$$
\begin{array}{ccc}
A_0 \otimes B_1 & \times & A_1 \otimes B_0 \\
A_0 \otimes B_0 & \rightarrow & A_1 \otimes B_0 \\
A_1 \otimes B_1 & \rightarrow & C_1.
\end{array}
$$

This commutative square can be seen simply as a morphism in $\text{dgcat}^L$ from

$A_0 \otimes B_1 \times A_1 \otimes B_0 \rightarrow A_1 \otimes B_1$

to the localization pair $(C_0 \subset C_1)$. Remark that the morphism

$A_0 \otimes B_1 \times A_1 \otimes B_0 \rightarrow A_1 \otimes B_1$

defined by $A_0 \otimes B_0 \rightarrow A_0 \otimes B_0$ is injective on objects and that its image consists of those objects $a \otimes b$ of $A_1 \otimes B_1$ such that $a$ belongs to $\mathcal{A}_0$ or $b$ belongs to $\mathcal{B}_0$. This implies that

$\text{Im}(A_0 \otimes B_1 \times A_1 \otimes B_0 \rightarrow A_1 \otimes B_1) = A \otimes B,$

and by the adjunction $(S, U)$ from subsection 5.1 this last commutative square in $\text{dgcat}$ corresponds exactly to an element of $\text{Hom}_{\mathcal{L}_p}(\mathcal{A}, \mathcal{B}, \mathcal{C})$. This proves the proposition.

Remark 6.1. Remark that the unit object is the localization pair $(\emptyset \subset \mathcal{A})$, where $\mathcal{A}$ is the dg category with one object and whose dg algebra of endomorphisms is $k$. √
7. Derived internal Hom-functor

Let $\mathcal{A}$ be a cofibrant dg category and $\lambda$ an infinite cardinal whose size is greater or equal to the cardinality of the set of isomorphism classes of objects in the category $H^0(\mathcal{A})$. Let $\mathcal{B}$ be a Morita fibrant dg category.

**Definition 7.1.** Let $\mathcal{B}_{\lambda}$ be the full dg subcategory of $C_{dg}(\mathcal{B})$, whose objects are:
- the right $\mathcal{B}$ dg modules $M$ such that $M \oplus D$ is representable for a contractible right $\mathcal{B}$ dg module $D$ and
- the right $\mathcal{B}$ dg modules of the form $\tilde{B} \oplus C$, where $B$ is an object of $\mathcal{B}$ and the right $\mathcal{B}$ dg module $C$ is a direct factor of $\bigoplus_{i \in I} \text{cone}(1_{B_i})$, with $B_i$ an object of $\mathcal{B}$ and $I$ a set of cardinality bounded by $\lambda$.

Let $\text{rep}_{dg}(\mathcal{A}, \mathcal{B})$ be the dg category as in [6] [13].

**Remark 7.1.** Remark that we dispose of a quasi-equivalence $B \xrightarrow{h} B_{\lambda}$ and that the objects of $B_{\lambda}$ are cofibrant and quasi-representable as right $\mathcal{B}$ modules, see [13]. This implies that we dispose of a natural dg functor:

$$\overline{\text{Fun}}_{dg}(\mathcal{A}, B_{\lambda}) := \text{Fun}_{dg}(\mathcal{A}, B_{\lambda})/\text{Fun}_{dg}(\mathcal{A}, (B_{\lambda})_{\text{contr}}) \xrightarrow{\Phi} \text{rep}_{dg}(\mathcal{A}, \mathcal{B}).$$

**Theorem 7.1.** The natural induced dg functor:

$$\text{Fun}_{dg}(\mathcal{A}, B_{\lambda})/\text{Fun}_{dg}(\mathcal{A}, (B_{\lambda})_{\text{contr}}) \xrightarrow{\Phi} \text{rep}_{dg}(\mathcal{A}, \mathcal{B}),$$

is a quasi-equivalence.

**Proof.** We prove first that $H^0(\Phi)$ is essentially surjective. We dispose of the following composition of dg functors

$$\text{Fun}_{dg}(\mathcal{A}, \mathcal{B}) \xrightarrow{I} \overline{\text{Fun}}_{dg}(\mathcal{A}, B_{\lambda}) \xrightarrow{\Phi} \text{rep}_{dg}(\mathcal{A}, \mathcal{B}).$$

Since $\mathcal{A}$ is a cofibrant dg category, lemma 4.3 and sub-lemma 4.4 from [13] imply that $H^0(\Phi \circ I)$ is essentially surjective and so we conclude that so is $H^0(\Phi)$.

We now prove also that the functor $H^0(I)$ is essentially surjective. Let $F : \mathcal{A} \rightarrow \mathcal{B}_{\lambda}$ be a dg functor. Since $\mathcal{A}$ is a cofibrant dg category and $h$ is a quasi-equivalence, there exists a dg functor $F' : \mathcal{A} \rightarrow \mathcal{B}$ such that $F$ and $h \circ F'$ are homotopic in the Quillen model structure constructed in [12]. Remark that since $\mathcal{B}$ is a Morita fibrant dg category so is $\mathcal{B}_{\lambda}$. In particular $\mathcal{B}_{\lambda}$ is stable under cones up to homotopy, see [10] [11]. Since a cone can be obtained from a cone up to homotopy, by adding or factoring out contractible modules, we conclude that by definition, $\mathcal{B}_{\lambda}$ is also stable under cones. By remark 122 we dispose of a sequence of dg functors

$$F \rightarrow I \rightarrow h \circ F'[1],$$

such that $I$ belongs to $\text{Fun}_{dg}(\mathcal{A}, (B_{\lambda})_{\text{contr}})$. This implies that $F$ and $h \circ F'$ become isomorphic in $H^0(\text{Fun}_{dg}(\mathcal{A}, B_{\lambda}))$. This proves that the functor $H^0(\Phi)$ is essentially surjective. Let us now prove that the functor $H^0(\Phi)$ is fully faithful. Let $F$ belong to $\text{Fun}_{dg}(\mathcal{A}, B_{\lambda})$. Since $H^0(I)$ is essentially surjective, we can consider $F$ as belonging to $\text{Fun}_{dg}(\mathcal{A}, \mathcal{B})$. We will construct a morphism of dg functors

$$F' \xrightarrow{\mu} F,$$

where $\mu$ becomes invertible in $H^0(\text{Fun}_{dg}(\mathcal{A}, B_{\lambda}))$ and $F'$ belongs to the left-orthogonal of the category $H^0(\text{Fun}_{dg}(\mathcal{A}, (B_{\lambda})_{\text{contr}}))$. Consider the $\mathcal{A}$-$\mathcal{B}$-bimodule $X_F$ naturally
associated to $F$. Consider $X_F$ as a left $A$-module and let $P X_F$ denote the bar resolution of $X_F$. Remark that $P X_F$ is naturally a right $B$-module and that it is cofibrant in the projective model structure on the category of $A$-$B$-bimodules. Let $A$ be an object of $A$. Since the dg category $A$ is cofibrant in $\text{dgcat}$, $(P X_F)(?, A)$ is cofibrant as a $B$-module. We dispose of the following homotopy equivalence

$$(P X_F)(?, A) \xrightarrow{H_A} X_F(?, A),$$

since both $B$-modules are cofibrant. This implies that the $B$-module $(P X_F)(?, A)$ is isomorphic to a direct sum $X_F(?, A) \oplus C$, where $C$ is a contractible and cofibrant $B$-module. The $B$-module $C$ is in fact isomorphic to a direct factor of a $B$-module

$$\bigoplus_{i \in I}(\text{cone}(B))[n_i],$$

where $I$ is a set whose cardinality is bounded by $\lambda$, $B_i, i \in I$ is an object of $B$ and $n_i, i \in I$ is an integer, see [7].

This implies, by definition of $B_\lambda$, that the $B$-module

$$X_F(?, A) \oplus C$$

belongs to $B_\lambda$ and so the $A$-$B$-bimodule $P X_F$ is in fact isomorphic to $X_F$, for a dg functor $F' : A \to B_\lambda$. Remark that the previous construction is functorial in $A$ and so we dispose of a morphism of dg functors

$$F' \xrightarrow{\mu} F.$$

Since for each $A$ in $A$, the morphism $\mu_A : F'A \to FA$ is a retraction with contractible kernel, the morphism $\mu$ becomes invertible in

$$H^0(\text{Fun}_{dg}(A, B_\lambda)).$$

Let now $G$ belong to $\text{Fun}_{dg}(A, (B_\lambda)_{\text{contr}})$. We remark that

$$\text{Hom}_{H(A_{\text{op}} \otimes B)}(F', G) \xrightarrow{\sim} \text{Hom}_{H(A_{\text{op}} \otimes B)}(P X_F, X_G),$$

where $H(A_{\text{op}} \otimes B)$ denotes the homotopy category of $A$-$B$ bimodules. Since $P X_F$ is a cofibrant $A$-$B$-bimodule and $X_G(?, A)$ is a contractible $B$-module, for every object $A$ in $A$, we conclude that the right hand side vanishes and $F'$ belongs to the left-orthogonal of $H^0(\text{Fun}_{dg}(A, (B_\lambda)_{\text{contr}}))$. This implies that the induced functor

$$H^0(\text{Fun}_{dg}(A, B_\lambda)/\text{Fun}_{dg}(A, (B_\lambda)_{\text{contr}}) \to H^0(\text{rep}_{dg}(A, B))$$

is fully faithful. This proves the theorem. \hfill \checkmark

**Proposition 7.1.** The internal Hom functor

$$\text{Hom}(\cdot, \cdot) : L^p_{\text{op}} \times L^p \to L^p,$$

admits a total right derived functor

$$\mathcal{R}\text{Hom}(\cdot, \cdot) : \text{Ho}(L^p)^{\text{op}} \times \text{Ho}(L^p) \to \text{Ho}(L^p)$$

as in definition 8.4.7 from [3].

**Proof.** Let $A$ and $B$ be localization pairs. We are now going to define $\mathcal{R}\text{Hom}(A, B)$ and the morphism $\epsilon$ as in definition 8.4.7 from [3]. We denote by $A_\epsilon \xrightarrow{\epsilon} A$ a functorial cofibrant resolution of $A$ in $L^p$ and by $B \xrightarrow{\epsilon_f} B_f$ a functorial $Q$-fibrant resolution of $B$ in $L^p$. Remember, that by proposition 5.3 $B_f$ is of the form

$$B_f = ((B_f)_{\text{contr}} \subset B_f),$$
where $\mathcal{B}_f$ is a Morita fibrant dg category. Let $\lambda$ be an infinite cardinal whose size is greater or equal to the cardinality of the set of isomorphism classes in the category $H^0((\mathcal{A}_c)_1)$. Consider now the following localization pair

$$(B_f)_\lambda := ((B_f)_\lambda)_{\text{contr}} \subset (B_f)_\lambda,$$

where $(B_f)_\lambda$ is as in definition \[\text{[1]}\]
Remark that we dispose of a canonical weak equivalence in $L_p$

$$B_f \longrightarrow (B_f)_\lambda.$$

We now define $\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}(\mathcal{A}, \mathcal{B})$ as $\mathcal{H}\mathcal{o}\mathcal{m}(\mathcal{A}_c, (B_f)_\lambda)$ and we consider for morphism $\epsilon$ the image in $H^0(L_p)$ of the following $Q$-equivalence in $L_p$

$$\eta : (A, B) \xrightarrow{(P, F)} (A_c, B_f) \xrightarrow{(Id, F)} (A_c, (B_f)_\lambda)$$

under the functor $\mathcal{H}\mathcal{o}\mathcal{m}(-, -)$. We will now show that the dg category associated with the localization pair $\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}(\mathcal{A}, \mathcal{B})$ is canonically Morita equivalent to

$$\text{rep}_{dg}((\mathcal{A}_c)_1/(\mathcal{A}_c)_0, B_f).$$

Remark that since $\mathcal{A}_c$ is a cofibrant object in $L_p$, by lemma \[\text{[2]}\] $(\mathcal{A}_c)_1$ is cofibrant in $\text{dgcat}$ and so we dispose of an exact sequence in $H^0$, see \[\text{[6]}\]

$$(A_c)_0 \hookrightarrow (A_c)_1 \twoheadrightarrow (A_c)_1/(A_c)_0.$$

Since the dg category $(B_f)$ is Morita fibrant, the application of the functor $\text{rep}_{dg}(-, B_f)$ to the previous exact sequence induces a new exact sequence in $H^0$

$$\text{rep}_{dg}((\mathcal{A}_c)_0, B_f) \hookrightarrow \text{rep}_{dg}((\mathcal{A}_c)_1, B_f) \twoheadrightarrow \text{rep}_{dg}((\mathcal{A}_c)_1/(\mathcal{A}_c)_0, B_f).$$

Remember that:

$$\text{Hom}(A_c, (B_f)_\lambda)_1 = \text{Fun}_{dg}((A_c)_0, ((B_f)_\lambda)_{\text{contr}}) \times \text{Fun}_{dg}((A_c)_1, (B_f)_\lambda).$$

Now, since the dg categories $(A_c)_1$ and $(B_f)_\lambda$ satisfy the conditions of theorem \[\text{[7.1]}\] we dispose of a natural inclusion of dg categories

$$\text{Hom}(A_c, (B_f)_\lambda)_1 \hookrightarrow \text{Fun}_{dg}((A_c)_1, ((B_f)_\lambda)_{\text{contr}}) \twoheadrightarrow \text{rep}_{dg}((A_c)_1, B_f).$$

Now remark that this inclusion induces the following Morita equivalence

$$\text{Hom}(A_c, (B_f)_\lambda)_1 \hookrightarrow \text{Fun}_{dg}((A_c)_1, ((B_f)_\lambda)_{\text{contr}}) \twoheadrightarrow \text{rep}_{dg}((A_c)_1/(A_c)_0, B_f).$$

We now show that the functor $\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}(-, -)$ preserves $Q$-weak equivalences in $L_p^{op} \times L_p$. Consider a $Q$-weak equivalence

$$(A, B) \rightarrow (\tilde{A}, \tilde{B}),$$

in $L_p^{op} \times L_p$. By construction it will induce a Morita equivalence

$$(\tilde{A}_c)_1/(\tilde{A}_c)_0 \twoheadrightarrow (A_c)_1/(A_c)_0$$

and also a Morita equivalence

$$B_f \twoheadrightarrow \tilde{B}_f.$$

This implies that the induced dg functor

$$\text{rep}_{dg}((A_c)_1/(A_c)_0, B_f) \twoheadrightarrow \text{rep}_{dg}((\tilde{A}_c)_1/(\tilde{A}_c)_0, \tilde{B}_f)$$
is a Morita equivalence. Now, remark that we dispose of the natural zig-zag of $Q$-weak equivalences in $L p$:

\[
\begin{align*}
\text{Fun}_{dg}(\mathcal{A}_c, ((B_f)_\text{contr})) & \subset \text{Hom}(\mathcal{A}_c, (B_f)_\lambda) \\
\text{Fun}_{dg}(\mathcal{A}_c, ((B_f)_\text{contr})) & \subset \text{Hom}(\mathcal{A}_c, (B_f)_\lambda) \cup \text{Fun}_{dg}(\mathcal{A}_c, ((B_f)_\text{contr})) \\
\emptyset & \subset \text{Hom}(\mathcal{A}_c, (B_f)_\lambda) \cup \text{Fun}_{dg}(\mathcal{A}_c, ((B_f)_\text{contr}))
\end{align*}
\]

This allows us to conclude that the the functor $\mathcal{R}\text{Hom}(-, -)$ preserves $Q$-weak equivalences in $L p^{op} \times L p$. This proves the proposition.

**Lemma 7.1.** Let $\mathcal{A}$ be a cofibrant object in $L p$. The induced internal tensor product functor

\[\mathcal{A} \otimes - : L p \rightarrow L p,\]

preserves $Q$-weak equivalences.

**Proof.** Let $F : \mathcal{B} \rightarrow \mathcal{C}$ be a $Q$-weak equivalence in $L p$ between cofibrant objects. We prove that the induced morphism in $L p$

\[\mathcal{A} \otimes \mathcal{B} \xrightarrow{F} \mathcal{A} \otimes \mathcal{C},\]

is a $Q$-weak equivalence. By lemma [52], $\mathcal{A}_1$, $\mathcal{B}_1$ and $\mathcal{C}_1$ are cofibrant dg categories in $\text{dgcat}$ and so we dispose of a morphism of exact sequences in $\text{Hmo}$:

\[
\begin{array}{ccc}
B_0 & \longrightarrow & B_1 \\
\downarrow & & \downarrow \sim \\
C_0 & \longrightarrow & C_1
\end{array}
\]

\[
\begin{array}{ccc}
B_1/B_0 & \sim & C_1/C_0
\end{array}
\]

where the last column is a Morita equivalence. Since $\mathcal{A}_1$ is cofibrant in $\text{dgcat}$, by applying the functor $\mathcal{A} \otimes -$, we obtain the following morphism of exact sequences in $\text{Hmo}$:

\[
\begin{array}{ccc}
\mathcal{A}_1 \otimes B_0 & \longrightarrow & \mathcal{A}_1 \otimes B_1 \\
\downarrow & & \downarrow \sim \\
\mathcal{A}_1 \otimes C_0 & \longrightarrow & \mathcal{A}_1 \otimes C_1
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{A}_1 \otimes B_1/B_0 & \sim & \mathcal{A}_1 \otimes C_1/C_0
\end{array}
\]

This implies that we dispose of the following Morita equivalence:

\[(\mathcal{A}_1 \otimes B_1)/(\mathcal{A}_1 \otimes B_0) \longrightarrow (\mathcal{A}_1 \otimes C_1)/(\mathcal{A}_1 \otimes C_0).\]

Let $\mathcal{H}$ be the full dg subcategory of $(\mathcal{A}_1 \otimes B_1)/(\mathcal{A}_1 \otimes B_0)$, whose objects are $a \otimes b$, where $a$ belongs to $\mathcal{A}_0$ and $\mathcal{P}$ the full dg subcategory of $(\mathcal{A}_1 \otimes C_1)/(\mathcal{A}_1 \otimes C_0)$ whose objects are $a \otimes c$, where $a$ belongs to $\mathcal{A}_0$. We dispose of the following diagram:

\[
\begin{array}{ccc}
\mathcal{H}^\wedge & \longrightarrow & (\mathcal{A}_1 \otimes B_1)/(\mathcal{A}_1 \otimes B_0) \\
\downarrow & & \downarrow \sim \\
\mathcal{P}^\wedge & \longrightarrow & (\mathcal{A}_1 \otimes C_1)/(\mathcal{A}_1 \otimes C_0)
\end{array}
\]
Remark that the dg categories $A \otimes B$ and $A \otimes C$ are Morita equivalent dg subcategories of $((A_1 \otimes B_1)/(A_1 \otimes B_0))/\mathcal{H}$, resp. $((A_1 \otimes C_1)/(A_1 \otimes C_0))/\mathcal{P}$ and so we have the commutative square:

$$
\begin{array}{ccc}
((A_1 \otimes B_1)/(A_1 \otimes B_0))/\mathcal{H} & \xrightarrow{\sim} & A \otimes B \\
\downarrow{\sim} & & \downarrow{\sim} \\
((A_1 \otimes C_1)/(A_1 \otimes C_0))/\mathcal{P} & \xrightarrow{\sim} & A \otimes C.
\end{array}
$$

This implies the lemma.

**Remark 7.2.** Since the internal tensor product $- \otimes -$ is symmetric, lemma [7.1] implies that the total left derived functor $- \otimes -$ exists, see definition 8.4.7 of [3].

### 8. Relation with dgcat

We dispose of the following adjunction:

$$
\begin{array}{ccc}
\text{Lp} & \xrightarrow{F} & \text{dgcat} \\
\downarrow{E_{v1}} & & \downarrow{\text{Ev}_1} \\
\text{dgcat} & \xleftarrow{\text{F}} & \text{Lp}
\end{array}
$$

where $E_{v1}$ is the evaluation functor on the 1-component and $F$ associates to a dg category $A$ the localization pair $(\emptyset \subset A)$.

**Lemma 8.1.** If we consider on dgcat the Quillen model structure of [10] [11] and on Lp the Quillen model structure of theorem [7.1] the previous adjunction is a Quillen equivalence, see [3].

**Proof.** The functor $F$ clearly sends Morita equivalences to weak equivalences. By lemma [6.4] the evaluation functor $E_{v1}$ preserves trivial fibrations. This shows that $F$ is a left Quillen functor. Let $A$ be a cofibrant object in dgcat and $(B_{contr} \subset B)$ a $Q$-fibrant object in Lp. Let $A \xrightarrow{F} B$ be a dg-functor in dgcat. We need to show that $F$ is a Morita equivalence if and only if the induced morphism of localization pairs $(\emptyset \subset A) \rightarrow (B_{contr} \subset B)$ is a $Q$-weak equivalence. But since the dg functor $B \rightarrow B/B_{contr}$ is a Morita equivalence this automatically follows.

**Lemma 8.2.** The total derived functors, $- \otimes -$ and $\mathcal{R}{\text{Hom}}(-,-)$ in the category \text{Ho}(\text{Lp}) correspond, under the equivalence:

$$
\begin{array}{ccc}
\text{Ho}(\text{Lp}) & \xrightarrow{F} & \text{Ho}(\text{dgcat}) \\
\downarrow{\mathcal{R}{\text{Ev}}_1} & & \downarrow{\mathcal{R}{\text{Ev}}_1} \\
\text{Ho}(\text{dgcat}) & \xleftarrow{\mathcal{R}{\text{Ev}}_1} & \text{Ho}(\text{Lp})
\end{array}
$$

to the functors, $- \otimes -$ and $\mathcal{R}{\text{rep}}_{\text{dg}}(-,-)$, see [6] [13], in the category \text{Ho}(\text{dgcat}).
Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be dg categories. Then $\frac{1}{\mathcal{A} \otimes \mathcal{B}}$ identifies with $\mathcal{A}_c \otimes \mathcal{B}$, where $\mathcal{A}_c$ is a cofibrant resolution of $\mathcal{A}$ in $\mathbf{dgcat}$. Since $F(\mathcal{A}_c)$ is cofibrant in $\mathbf{Lp}$ by lemma 8.2 we have the following zig-zag:

$$F(\mathcal{A}) \overset{1}{\otimes} F(\mathcal{B}) \cong F(\mathcal{A}_c) \otimes F(\mathcal{B}) = F(\mathcal{A}_c \overset{1}{\otimes} \mathcal{B}) \cong F(\mathcal{A} \otimes \mathcal{B}),$$

of weak equivalences in $\mathbf{Lp}$. This proves that the total left derived tensor products in $\text{Ho}(\mathbf{Lp})$ and $\text{Ho}(\mathbf{dgcat})$ are identified. Now, $\operatorname{rep}_{dg}(\mathcal{A}, \mathcal{B})$ identifies with $\operatorname{rep}_{dg}(\mathcal{A}_c, \mathcal{B}_f)$, where $\mathcal{B}_f$ is a fibrant resolution of $\mathcal{B}$ in $\mathbf{dgcat}$. By definition

$$\mathcal{R}\text{Hom}(F(\mathcal{A}), F(\mathcal{B})) = \text{Hom}((F(\mathcal{A})_c, (F(\mathcal{B}_f)_\lambda)),$$

where $\lambda$ denotes an infinite cardinal whose size is greater or equal to the cardinality of the set of isomorphism classes of objects in the category $\mathcal{H}^0(\mathcal{A}_c)$. We dispose of the following $Q$-weak equivalent objects in $\mathbf{Lp}$:

$$\mathcal{R}\text{Hom}(F(\mathcal{A}), F(\mathcal{B}))$$

$$\text{Hom}((F(\mathcal{A})_c, (F(\mathcal{B}_f)_\lambda))$$

$$\text{Hom}((\emptyset \subset \mathcal{A}_c), ((\mathcal{B}_f)_\lambda)_{\text{contr}} \subset (\mathcal{B}_f)_\lambda))$$

$$\text{Fun}_{dg}(\mathcal{A}_c, ((\mathcal{B}_f)_\lambda)_{\text{contr}}) \subseteq \text{Fun}_{dg}(\mathcal{A}_c, (\mathcal{B}_f)_\lambda))$$

$$\text{Fun}_{dg}(\mathcal{A}_c, (\mathcal{B}_f)_\lambda) / \text{Fun}_{dg}(\mathcal{A}_c, ((\mathcal{B}_f)_\lambda)_{\text{contr}})$$

$$\emptyset \subset \text{rep}_{dg}(\mathcal{A}_c, \mathcal{B}_f))$$

This proves that the total right derived functor $\mathcal{R}\text{Hom}(\cdot, \cdot)$ in $\text{Ho}(\mathbf{Lp})$ corresponds to the functor $\text{rep}_{dg}(\cdot, \cdot)$, as in [6] [13]. \hfill \checkmark

References

[1] V. Drinfeld, DG quotients of DG categories, J. Algebra 272 (2004), 643–691.
[2] M. Hovey, Model categories, Mathematical Surveys and Monographs, Vol. 63.
[3] P. Hirschhorn, Model categories and their Localizations, Mathematical Surveys and Monographs, Vol. 99.
[4] P. Goerss, J. Jardine, Simplicial homotopy theory, progress in Mathematics, Birkhäuser.
[5] B. Keller, On the cyclic homology of exact categories, J. Pure Appl. Algebra 136 (1999), no. 1, 1–56.
[6] B. Keller, On differential graded categories, preprint math. KT/0601185.
[7] B. Keller, Deriving DG categories, Ann. Scient. Ec. Norm. Sup. 27 (1994), 63–102.
[8] B. Keller, On differential graded categories, ICM talk 2006.
[9] M. Kontsevich, Triangulated categories and geometry, Course at the École Normale Supérieure, Paris, Notes taken by J. Bellaïche, J.-F. Dat, I. Marin, G. Racinet and H. Randriambololona, 1998.
[10] G. Tabuada, Invariants additifs de dg-catégories, Int. Math. Res. Notices 53 (2005), 3309–3339.
[11] G. Tabuada, Addendum à Invariants additifs de dg-catégories, to appear in Int. Math. Res. Notices.
[12] G. Tabuada, Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories, C. R. Math. Acad. Sci. Paris 340 (2005), no. 1, 15–19.
[13] B. Toën, The homotopy theory of dg-categories and derived Morita theory, preprint, math.AG/0408337, to appear in Inventiones Mathematicae.

Université Paris 7 - Denis Diderot, UMR 7586 du CNRS, case 7012, 2 Place Jussieu, 75251 Paris cedex 05, France.
E-mail address: tabuada@math.jussieu.fr