A Fenchel-Moreau-Rockafellar type theorem on the Kantorovich-Wasserstein space with Applications in Partially Observable Markov Decision Processes

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Abstract

By using the fact that the space of all probability measures with finite support can be somehow completed in two different fashions, one generating the Arens-Eells space and another generating the Kantorovich-Wasserstein (Wasserstein-1) space, and by exploiting the duality relationship between the Arens-Eells space with the space of Lipschitz functions, we provide a dual representation of Fenchel-Moreau-Rockafellar type for proper convex functionals on Wasserstein-1. We retrieve dual transportation inequalities as a Corollary and we provide examples where the theorem can be used to easily prove dual expressions like the celebrated Donsker-Varadhan variational formula. Finally our result allows to write convex functions as the supremum over all linear functions that are generated by roots of its conjugate dual, something that we apply to the field of Partially observable Markov decision processes (POMDPs) to approximate the value function of a given POMDP by iterating level sets. This extends the method used in Smallwood and Sondik (1973) for finite state spaces to the case where the state space is a Polish metric space.

Key words: Wasserstein metric; conjugate duality; Fenchel-Moreau-Rockafellar theorem; Donsker-Varadhan variational formula; weighted norm; optimal control; partially observable Markov decision processes

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1 Introduction

Dual representation of the Fenchel-Moreau-Rockafellar type (see e.g., Bot (2010); Ioan-Bot et al. (2009); Zălinescu (2002)), plays an important role in convex analysis and has wide applications in various fields (Borwein and Lewis, 2006; Boyd and Vandenberghe, 2004). In this paper, we are interested in a conjugate dual representation of functions on probability measures, for which the “choice of dual space” will allow for real-world applications in the field of Partially observable Markov decision processes (POMDPs) along the lines of Smallwood and Sondik (1973). In that direction, we are going to use the special connection of the Wasserstein-1 space with the space of Lipschitz functions.

More specifically, for \((X,d)\) being a Polish space, \((\mathcal{P}_1(X), W_1)\) the Wasserstein-1 space over \(X\), and \(\mathcal{L}(X)\) the space of all real-valued Lipschitz functions on \((X,d)\), the conjugate function \(\rho : \mathcal{L}(X) \rightarrow \mathbb{R}\) of a function \(\phi : \mathcal{P}_1(X) \rightarrow \mathbb{R}\) is defined as

\[
\rho(f) := \sup_{\mu \in \mathcal{P}_1(X)} \left( \int f d\mu - \phi(\mu) \right)
\]
and the second conjugate function $\phi^\ast : \mathcal{P}_1(X) \to \mathbb{R}$ is defined as
\begin{equation}
\phi^\ast (\mu) := \sup_{f \in \mathcal{L}(X)} \left( \int f d\mu - \rho(f) \right),
\end{equation}
A justification that $\rho$ is well defined and proper can be found in the proof of Ioan-Bot et al. (2004, Theorem 2.3.5). The first main result of this paper is the following conjugate duality.

**Theorem 1.1.** Let $\phi : \mathcal{P}_1(X) \to \mathbb{R}$ be a proper convex function on $(\mathcal{P}_1(X), W_1)$, i.e. a convex and lower semicontinuous function, satisfying $\phi(\mu) > -\infty$ for all $\mu \in \mathcal{P}_1(X)$ and $\phi(\mu_0) \in \mathbb{R}$ for some $\mu_0 \in \mathcal{P}_1(X)$. Then $\phi(\mu) = \phi^\ast (\mu)$, $\forall \mu \in \mathcal{P}_1(X)$.

In Villani (2003, Theorem 5.26), one has to assume that $\phi(\mu) = \phi^\ast (\mu)$, were the conjugates there are defined by taking the supremum over $C_b(X)$, in order to establish a pair of dual inequalities connecting an optimal transport distance to $\phi(\mu)$. Theorem 1.1 implies that, in the case of the Wasserstein-1 distance, this assumption is always satisfied, provided that $\mathcal{L}(X)$ is used in place of $C_b(X)$ for defining the conjugate dual. We will continue by providing a Wasserstein-1 specific version of Villani (2003, Theorem 5.26), and an example of a pair of well known dual functions where is it very simple to calculate $\phi^\ast = \phi$.

**Corollary 1.2.** Let $\phi : \mathcal{P}_1(X) \to \mathbb{R}$ be a proper convex function on $(\mathcal{P}_1(X), W_1)$. Let $\rho$ be its conjugate as in (11). Let finally $\Phi$ a real increasing and convex function with $\Phi(0) = 0$. We have
\begin{equation}
\Phi(W_1(\mu, \nu)) \leq \phi(\mu), \forall \mu \in \mathcal{P}_1(X) \Leftrightarrow \rho \left( \int_X tf d\nu - t \Phi^\ast (t) \right) \leq 0, \forall f \in [\mathcal{L}(X)]_1, t \in \mathbb{R}
\end{equation}
where $\Phi^\ast$ is the Legendre dual, i.e. is given by the formula $\Phi^\ast (s) = \sup \{ st - \Phi(t) \}$, and $[\mathcal{L}(X)]_1$ is the set of all Lipschitz functions with constant 1 (see next section for definition).

The proof of the Corollary is straightforward and can be found in the Appendix. We would like to remark that $[\mathcal{L}(X)]_1$ can be substituted with any subset $A$ of $[\mathcal{L}(X)]_1$ that satisfies $A + \mathbb{R} = [\mathcal{L}(X)]_1$. We proceed now with an example.

**Example 1.3.** We will show that a direct application of Theorem 1.1 can provide the celebrated Donsker-Varadhan variational formula, which has many fundamental applications in the theory of large deviations (see for example Dupuis and Ellis (1997), where a whole class of large deviation principles are proved by applying the formula) and in statistical physics in general.

It is known that the following pair of dual equation hold:
\begin{equation}
\log \int_X e^\mu d\nu = \sup_{\mu \in \mathcal{P}(X)} \left\{ \int_X g d\mu - \mathcal{R}(\mu|\nu) \right\}, \quad \forall g \in C_b(X)
\end{equation}
and
\begin{equation}
\mathcal{R}(\mu|\nu) = \sup_{g \in C_b(X)} \left\{ \int_X g d\mu - \log \int_X e^g d\nu \right\}, \quad \forall \mu \in \mathcal{P}(X)
\end{equation}
where $\mathcal{R}(\mu|\nu)$ is the relative entropy functional given by
\begin{equation}
\mathcal{R}(\mu|\nu) = \int_X \frac{d\mu}{d\nu} \log \left( \frac{d\mu}{d\nu} \right) d\nu \quad \text{if} \quad \mu << \nu,
\end{equation}
otherwise.

As it is shown in Lemma 5.4 in the Appendix, the first formula is straightforward to prove, even when $C_b(X)$ is replaced by $\mathcal{L}(X)$ (the original proof is even simpler and can be found in page 34 of Dupuis and Ellis (1997)). On the other hand, the second requires a more technical proof as one can see in Dupuis and Ellis (1997, Lemma 1.4.3.). By applying Theorem 1.1 we get the following alternative variational form for the relative entropy, i.e.:
\begin{equation}
\mathcal{R}(\mu|\nu) = \sup_{g \in \mathcal{L}(X)} \left\{ \int_X g d\mu - \log \int_X e^g d\nu \right\}.
\end{equation}
If one wishes to further retrieve (4) from (5), then it is a matter of a simple approximation argument as it is illustrated in Lemma 5.2. Finally, applying Corollary 1.2 with $\Phi(t) = \frac{1}{2}t^2$
on our example, we can retrieve Bobkov and Götze theorem (Bobkov and Götze, 1999).

\[
\int_X e^{t f} d\nu \leq e^{ct^2/2}, \quad \forall f \in \{ f \in \mathcal{L}(X) \mid \int_X f d\nu = 0 \}, \quad t \in \mathbb{R} \quad \Leftrightarrow \quad W_1(\mu, \nu) \leq \frac{1}{c} \sqrt{R(\mu, \nu)}, \quad \forall \mu \in \mathcal{P}_1(X).
\]

One could probably derive more refined versions of the inequality by playing around with the choice of \( \Phi \).

Another fundamental application of Theorem 1.1 is to derive a conjugate dual form for the optimality equation for POMDPs. In order to do that we first are going to show that each function that satisfies the condition in Theorem 1.1 has a representation as a supremum over a suitable class of linear functionals. More specifically we have the following.

Let \( \phi \) be a function on \( \mathcal{P}_1(X) \) and \( \rho \) be its conjugate as in (1). Consider the following sets

\[
\mathcal{N}_\phi := \{ f \in \mathcal{L}(X) \mid \rho(f) = 0 \}, \quad \text{and} \quad \bar{\mathcal{N}}_\phi := \{ f \in \mathcal{L}(X) \mid f \neq \phi \}, \forall \mu \in \mathcal{P}_1(X) \}.
\]

We call \( \mathcal{N}_\phi \) the null level-set of \( \phi \), whereas the latter set \( \bar{\mathcal{N}}_\phi \) is called the acceptance set of \( \phi \) (cf. Föllmer and Schied, 2004, Section 4.1)). Note that since \( \rho \) is convex and lower semicontinuous, \( \mathcal{N}_\phi \) is convex and closed (see e.g. Ioan-Bot et al., 2009, Theorem 2.2)).

We have the following dual representation.

**Corollary 1.4.** Let \( \phi : \mathcal{P}_1(X) \to \mathbb{R} \) be a function satisfying the condition in Theorem 1.1. Then

\[
\phi(\mu) = \sup_{f \in \mathcal{N}_\phi} \int_X f d\mu = \sup_{f \in \bar{\mathcal{N}}_\phi} \int_X f d\mu.
\]

The proof is given in the end of the next section. Before we proceed we will provide some background on POMDPS and we are going to explain how Corollary 1.4 is meant to be used in that setting.

A POMDP is a tuple of controlled stochastic processes where it is assumed that the system dynamics are determined by a Markov process, but the agent cannot directly observe the underlying state. POMDPs have important applications in various fields, such as operations research (Lovejoy, 1991), robotics (Pineau et al., 2006) and artificial intelligence (Kaelbling et al., 1998). It is known that a POMDP can be reduced to a standard Markov decision process (MDP) by using appropriate probability distributions over the hidden states. We refer to Sondik (1978) for finite spaces, to Sawaragi and Yoshikawa (1970) for countable spaces, and to Hernández-Lerma (1989, Chapter 4) and Feinberg et al. (2016) for Borel spaces.

In the setting of finite state spaces with a discounted infinite-horizon objective, the value function encoding the maximum reward can be found by solving the following equation

\[
\phi(\mu) = \max_{a} \left\{ \bar{r}(\mu, a) + \alpha \sum_{y} \phi(\mu' | \mu, a, y) \left( \sum_{x'} P(x' | \mu, a) Q(y | x', a) \right) \right\}, \forall \mu \in \mathcal{P}(X).
\]  

Here, \( x' \), \( a \) and \( y \) denote the hidden (or latent) state, the action and the observation, respectively, and \( \mu \in \mathcal{P}(X) \) is the distribution over states, while \( \mu' \) is the posterior distribution of the successive state given by

\[
\mu'(\cdot | \mu, a, y) = \frac{P(\cdot | \mu, a) Q(y | \cdot, a)}{\sum_{x'} P(x' | \mu, a) Q(y | x', a)}.
\]

\( P \) controls the transition probability between states, \( Q \) models the observation probability of \( y \) given states and actions, \( r \) denotes the reward function, and \( \alpha \in (0, 1) \) serves as a discount factor. A formal introduction of POMDPs on Borel spaces can be found in Section 3.2.
In spite of knowing the existence of such a theoretical solution, POMDPs were notoriously difficult to solve in practice [Shani et al., 2013]. In the case where the underlying space is finite, a fast algorithm called point-based value iteration [Pineau et al., 2006] was designed to overcome this numerical difficulty. This algorithm is mainly based on the property first observed by Smallwood and Sondik (1973), that the optimal solution to equation (6) can be arbitrarily well approximated by a function of the following dual representation 

$$\phi(\mu) = \max_{h \in \mathcal{N}} \sum_{x} h(x)\mu(x),$$  

where $\mathcal{N}$ can be chosen to be a finite collection of real functions on the hidden state space.

To our best knowledge, it is still an open question whether a similar approximation is also possible for POMDPs on continuous state spaces. The second major contribution of this paper is to provide an affirmative answer to this open question (see Theorem 4.19), by applying the conjugate duality on Wasserstein-1 spaces that we obtained in Theorem 1.1. We would like to remark, that contrary to the finite state space case, the set appearing in the counterpart of (7) is uncountable, so a computable algorithm is still elusive, and it remains an open problem to find one. However, new advances in the field of neural networks involving the Wasserstein-1 distance, made substitution of collections of Lipschitz functions by appropriate finite sets a necessity (Arjovsky, Chintala and Bottou, 2017), and we are planning to perform further research in the immediate future.

2 Preliminaries

A Polish space $\mathbf{(X,d)}$ is a complete separable metric space and a Borel space is a Borel subset of a Polish space. We denote by $C_b(X)$ the set of all real bounded continuous functions on $X$. We further denote the set of all probability measures on $X$ by $\mathcal{P}(X)$. Furthermore, we define the set

$$\mathcal{P}_1(X) := \left\{ \mu \in \mathcal{P}(X) \mid \int d(x_0,x)\mu(dx) < \infty \right\},$$  

where $x_0 \in X$ is arbitrary. If $Y$ is a Borel space, its Borel $\sigma$-algebra is denoted by $B(Y)$.

2.1 The Wasserstein-1 space

Let $(X,d)$ be a Polish space with metric $d$. The Wasserstein-1 space over $(X,d)$ is the set $\mathcal{P}_1(X)$, defined in (8), equipped with the Wasserstein-1 metric given by

$$W_1(\mu,\nu) := \inf \left\{ \mathbb{E}[d(X,Y)] \mid \text{law}(X) = \mu, \text{law}(Y) = \nu \right\}. $$  

For any real-valued function $f : X \to \mathbb{R}$, its Lipschitz seminorm is defined as

$$\|f\|_{\text{Lip}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$  

Denote by $\mathcal{L}(X) := \{ f : X \to \mathbb{R} \mid \|f\|_{\text{Lip}} < \infty \}$ the space of all real-valued Lipschitz functions on $X$ and by $[\mathcal{L}(X)]_1 := \{ f \in \mathcal{L} \mid \|f\|_{\text{Lip}} \leq 1 \}$ the unit ball of $\mathcal{L}(X)$. Then, it can be shown (Villani, 2009, Chapter 5) that $W_1$ has the following representation:

$$W_1(\mu,\nu) = \sup_{f \in [\mathcal{L}(X)]_1} \left( \int f d\mu - \int f d\nu \right).$$  

We have the following property (Villani, 2009, Theorem 6.18): If $(X,d)$ is Polish, $(\mathcal{P}_1(X),W_1)$ is also Polish. In the sequel, we are going to make use of the following set $\mathcal{P}(X)$ of all probability measures with finite support. More specifically, we define

$$\mathcal{P}(X) := \left\{ \sum_{i=1}^{n} a_i \delta_{x_i} \mid n \in \mathbb{N}, a_i \in \mathbb{R}_+, x_i \in X, i = 1, 2, \ldots, n, \sum_{i=1}^{n} a_i = 1 \right\}.$$  

By Theorem 6.18 in Villani (2009), $\mathcal{P}(X)$ is dense in $(\mathcal{P}_1(X),W_1)$. 

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2.2 The Arens-Eells space

We recall some results of the Arens-Eells space based on Weaver (1999, Section 2.2 and 2.3).

Definition 2.1. Let $(X, d)$ be a metric space. A molecule of $X$ is a function $m : X \to \mathbb{R}$ which is supported on a finite set and which satisfies $\sum_{x \in \mathbb{X}} m(x) = 0$.

For any $x, y \in X$ define the molecule $m_{xy} = 1_x - 1_y$, where $1_x$ denotes the indicator function on the singleton set \{x\}. Define the following seminorm for every molecule $m$:

$$\|m\|_{\mathcal{E}(X)} := \inf \left\{ \sum_{i=1}^{n} |a_i| d(x_i, y_i) \left| m = \sum_{i=1}^{n} a_i m_{x_i y_i} \right. \right\}, \quad (12)$$

and let $\mathcal{E}(X)$ be the completion of the space of molecules, which is also called Arens-Eells space.

The following theorem states that $\mathcal{E}(X)$ is a predual of the space of Lipschitz functions $\mathcal{L}(X)$. This predual is unique in many important cases (Weaver, 2016).

Theorem 2.2. Weaver (1999, Theorem 2.2.2) Let $(X, d)$ be a metric space with at least one point $x_0$. Then $(\mathcal{E}(X), \| \cdot \|_{\mathcal{E}(X)}) \cong (\mathcal{L}(X), \| \cdot \|_{\mathcal{L}(X)})$.

For any $f \in \mathcal{L}(X)$ and $m \in \mathcal{E}(X)$, we define

$$\langle f, m \rangle := \sum_{x \in \mathbb{X}} f(x)m(x).$$

Corollary 2.3. (i) $\|m\|_{\mathcal{E}(X)} = \max_{f \in \mathcal{L}(X)} \langle f, m \rangle$. (ii) $\| \cdot \|_{\mathcal{E}(X)}$ is a norm on $\mathcal{E}$.

For the proof, see Weaver (1999, Corollary 2.2.3).

Replacing the Dirac measure in $\mathcal{D}(X)$ by the indicator function, we define the following set of real-valued functions on $X$

$$\mathcal{D}(X) := \left\{ \sum_{i=1}^{n} a_i 1_{x_i} \left| n \in \mathbb{N}, a_i \in \mathbb{R}_+, i = 1, 2, x \in \mathbb{X}, \sum_{i=1}^{n} a_i = 1 \right. \right\}. \quad (13)$$

Let $\psi : \mathcal{D}(X) \to \mathcal{D}(X)$ be defined by

$$\psi(\nu)(x) := \nu(\{x\}) \quad (14)$$

Obviously, $\psi$ is a bijection between $\mathcal{D}(X)$ and $\mathcal{D}(X)$. Let now $\Psi : \mathcal{D}(X) \times \mathcal{D}(X) \to \mathcal{E}(X)$ be defined by

$$\Psi(\nu, \nu_0) := \psi(\nu) - \psi(\nu_0). \quad (15)$$

To see that $\Psi(\nu, \nu_0)$ is actually an element of $\mathcal{E}(X)$, notice that

$$\sum_{x \in \mathbb{X}} \Psi(\nu, \nu_0)(x) = \sum_{x \in \mathbb{X}} (\psi(\nu)(x) - \psi(\nu_0)(x)) = 0, \forall \nu, \nu_0 \in \mathcal{D}(X).$$

We remark that for every $\nu_0 \in \mathcal{D}(X)$, $\Psi$ is an injection from $\mathcal{D}(X) \times \{\nu_0\}$ into $\mathcal{E}(X)$. However, this obviously is not true for the whole product $\mathcal{D}(X) \times \mathcal{D}(X)$. Furthermore, $\Psi$ is a surjection into the set of molecules with “total mass” $\sum_{x \in \mathbb{X}} |m(x)|$ equal or less than 2.

2.3 Connecting Arens-Eells and Wasserstein-1 spaces.

Recall that $\mathcal{D}(X)$ is the subset of $\mathcal{P}(X)$, that contains all probability measures with finite support, and $\mathcal{D}(X)$ defined in (13) is its corresponding space of functions on $X$, with $\psi : \mathcal{D}(X) \to \mathcal{D}(X)$ being the bijective map.

Proposition 2.4. $W_1(\nu, \nu_0) = \|\Psi(\nu, \nu_0)\|_{\mathcal{E}(X)} = \|\psi(\nu) - \psi(\nu_0)\|_{\mathcal{E}(X)}$, for every $\nu, \nu_0 \in \mathcal{D}(X)$. 

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Proof. By the dual representation (10) of $W_1$, we have

$$W_1(\nu, \nu_0) = \sup_{f \in \mathcal{D}(X)_1} \left( \int f \nu - \int f \nu_0 \right) = \sup_{f \in \mathcal{D}(X)_1} \sum_{x \in K} (\psi(\nu)(x) - \psi(\nu_0)(x)) f(x)$$
\[= \|\psi(\nu) - \psi(\nu_0)\|_{\mathcal{E}(X)} = \|\Psi(\nu, \nu_0)\|_{\mathcal{E}(X)},\]

where the second to the last equality is due to Corollary 2.3(i). \(\square\)

Before we proceed, we would like to highlight the connection of the Wasserstein-1 space with the Arens-Eells space and its dual, namely the space of Lipschitz functions. We saw in the previous subsection and in Proposition 2.4 that we can embed the set $\mathcal{D}(X) \times \mathcal{D}(X)$ of pairs $(\nu, \nu_0) \in \mathcal{D}(X) \times \mathcal{D}(X)$ in the vector space $(\mathcal{E}(X), \|\cdot\|_{\mathcal{E}(X)})$, in a way that the Wasserstein-1 distance $W_1(\nu, \nu_0)$ is equal to the norm of the vector $\|\Psi(\nu, \nu_0)\|_{\mathcal{E}(X)}$. Moreover, for every element $m$ of the set of molecules (which is dense in $\mathcal{E}(X)$), one can find a pair $(\nu^{(m)}, \nu_0^{(m)}) \in \mathcal{D}(X) \times \mathcal{D}(X)$, and a positive number $a^{(m)}$, such that $m = a^{(m)} \Psi(\nu^{(m)}, \nu_0^{(m)})$ and $\|m\|_{\mathcal{E}(X)} = a^{(m)} W_1(\nu^{(m)}, \nu_0^{(m)})$. One can picture $\mathcal{D}(X) \times \mathcal{D}(X)$ as an absorbing set (Schaefer, 1971) of a dense subspace of $\mathcal{E}(X)$, which hints that $\mathcal{D}(X) \times \mathcal{D}(X)$ may be enough for characterizing its dual space $\mathcal{L}(X)$.

In the case where $X$ is compact, one can get a better intuition by reading the exposition on the relation between the so-called Kantorovich-Rubenstein space, the Arens-Eells space and the space of Lipschitz functions in Weaver (1999), Chapter 2, Section 3) or Kantorovich and Akilov (1982, Section VIII.4).

In addition, due to (10), the Wasserstein-1 space can be considered as a subspace of the dual of $\mathcal{D}(X)$. In what follows, we are going to exploit these relationships to prove a separation theorem on the Wasserstein space, and then our first main result, Theorem 2.6.

\textbf{Open Problem.} It is tempting to generalize our approach to establish duality w.r.t. the Wasserstein-$p$ distance with $p > 1$. To end this, one could try to generalize the Arens-Eells space to some "$p$-version", for example by using $d(x, y)^p$ instead of $d(x, y)$ in (12). However, in this case, the seminorm constructed in an analogous way to (12), is equal to zero everywhere. The approach we are following in order to prove the duality result, heavily depends on the fact that adding the same finite measure on two measures $\nu, \nu_0$ does not change its distance; here, one can understand the Wasserstein distance not between probability measures anymore, but between measures with the same total mass. Therefore, it remains an open problem, how one can generalize the result for Wasserstein-$p$ spaces with $p > 1$.

2.4 A separation theorem on $\mathcal{D}(X)$

We are now ready to state a separation theorem on $\mathcal{D}(X)$. We first restate the Nirenberg-Luenberger theorem (see Luenberger (1962), Section 5.13), also known as the minimum norm duality theorem).

\textbf{Definition 2.5.} Let $K$ be a convex set in a real normed vector space $X$. Let $X^*$ be its dual space. The function $h_K(x^*) = \sup_{x \in K} \langle x, x^* \rangle$ on $X^*$ is called the support functional of $K$.

\textbf{Theorem 2.6.} Let $x_1$ be a point in a real normed space $X$ and let $d > 0$ denote its distance from the convex set $K$ having support functional $h_K$. Then

$$d = \inf_{x \in K} \|x - x_1\| = \max_{\|x^*\| \leq 1} \left( \langle x_1, x^* \rangle - h_K(x^*) \right)$$

where the maximum on the right is attained by some $x_0^* \in X^*$.

\textbf{Theorem 2.7.} Let $K$ be a convex set of $\mathcal{D}(X)$ and $\nu_0 \in \mathcal{D}(X)$ be a point not contained in $K$ satisfying $W_1(\nu, \nu_0) \geq \epsilon > 0, \forall \nu \in K$. Then, there exists a function $f \in [\mathcal{L}(X)]_1$ such that $\int f \nu \geq \int f \nu_0 + \epsilon, \forall \nu \in K$.

\textbf{Proof.} Since $K$ is convex, $K^* := \{\psi(\nu) \mid \psi \in K\}$ is also convex in $\mathcal{E}(X)$. To apply Theorem 2.6, we set $X = \mathcal{E}(X)$ and its dual is $X^* = \mathcal{E}^*(X) = \mathcal{L}(X)$. By the definition of $h_K$, we have

$$h_K(f) = \sup_{m \in K^*} (f, m) = \sup_{\nu \in K} \left\{ \int f \nu - \int f \nu_0 \right\}, \forall f \in \mathcal{L}(X).$$
Then, by Theorem 2.8 and Proposition 2.4, we obtain
\[
\epsilon \leq \inf_{\nu \in K} W_1(\nu, \nu_0) = \inf_{m \in K} \|m\|_{\|\cdot\|} = \max_{f \in [\mathcal{L}(X)]_1} \left[-h_K(f)\right],
\]
which yields $\max_{f \in [\mathcal{L}(X)]_1} \inf_{\nu \in K} \int (-f) \, d\nu = \int (-f) \, d\nu_0 \geq \epsilon$. Suppose the maximum is attained at $f_0$. Then $\inf_{f \in K} \int (-f_0) \, d\nu - \int (-f_0) \, d\nu_0 \geq \epsilon$, which implies $\int (-f_0) \, d\nu \geq \int (-f_0) \, d\nu_0 + \epsilon, \forall \nu \in K$. □

In the following sections, we consider the dual space of the Cartesian product $\mathcal{E}(X) \times \mathbb{R}$, where we can obtain a similar result as above. The operator $\vee$ is defined as $a \vee b := \max(a, b)$.

**Theorem 2.8.** Let $\tilde{K}$ be a convex subset of $\mathcal{P}(X) \times \mathbb{R}$ and $(\nu_0, r_0) \in \mathcal{P}(X) \times \mathbb{R}$ be a point not contained in $\tilde{K}$. Suppose $W_1(\nu, \nu_0) \vee |r - r_0| \geq \epsilon, \forall (\nu, r) \in \tilde{K}$. Then, there exists a tuple $(f, \alpha) \in [\mathcal{L}(X)]_1 \times [-1, 1]$ satisfying
\[
\|f\|_{\text{Lip}} + |\alpha| \leq 1 \quad \text{and} \quad \int fd\nu + \alpha r \geq \int f \, d\nu_0 + \alpha r_0 + \epsilon, \forall (\nu, r) \in \tilde{K}.
\]

**Proof.** Let $\mathcal{E}(X) \times \mathbb{R}$ be equipped with the canonical norm $\|\cdot\| := \|\cdot\|_{\mathcal{E}(X)} \vee |\cdot|$. An extension of Theorem 2.8 shows that its dual space is isometric to $[\mathcal{L}(X)]_1 \times [-1, 1]$ to the proof of Theorem 2.7. □

### 2.5 A separation theorem on Wasserstein-1 space

In this subsection, we extend Theorem 2.7 to the whole Wasserstein-1 space.

**Theorem 2.9.** Let $A$ be a convex and closed subset of $(\mathcal{P}(X), W_1)$. Let $\mu_0 \in \mathcal{P}(X)$ be a point not contained in $A$. Then, there exists a function $f \in [\mathcal{L}(X)]_1$ such that
\[
\int fd\mu > \int fd\mu_0, \quad \forall \mu \in A.
\]

**Proof.**

Step 1. Since $A$ is closed and $\mu_0 \notin A$, there exists a positive constant $\epsilon_0 > 0$ satisfying $W_1(\mu, \mu_0) \geq \epsilon_0, \forall \mu \in A$.

Step 2. By Theorem 6.18 in [Villani (2009)], we can approximate $\mu$ by a point $\nu \in \mathcal{P}(X)$ with any accuracy. Finally, Theorem 2.8 can be applied to find the required separation function in $[\mathcal{L}(X)]_1$. More specifically, take $\epsilon_1 = \epsilon_0/5$ and define subsets of $\mathcal{P}(X)$ as follows
\[
B_{\epsilon_1}(\mu) := \{\nu \in \mathcal{P}(X) \mid W_1(\nu, \mu) < \epsilon_1\}, \quad A_{\epsilon_1} := \bigcup_{\mu \in A} B_{\epsilon_1}(\mu).
\]

Since $\mathcal{P}(X)$ is dense in $(\mathcal{P}(X), W_1)$, $B_{\epsilon_1}(\mu) \neq \emptyset, \forall \mu \in \mathcal{P}(X)$. Hence, for any $\nu_i \in A_{\epsilon_1}$ there exists a $\mu_i \in A$ such that $W_1(\mu_i, \nu_i) < \epsilon_1$, $i = 1, 2$. Let $\nu_0 := (1 - \alpha)\nu_2$ and $\mu_0 := \alpha\mu_1 + (1 - \alpha)\mu_2, \alpha \in (0, 1)$. We have, by the dual representation (10) of $W_1$
\[
W_1(\nu_0, \mu_0) \leq \alpha W_1(\nu_1, \mu_1) + (1 - \alpha)W_1(\nu_2, \mu_2) < \epsilon_1,
\]
which implies that $A_{\epsilon_1}$ is convex. Similarly, we can find a $\nu_0 \in \mathcal{P}(X)$ such that $W_1(\mu_0, \nu_0) < \epsilon_1$. Note that $W_1(\nu, \mu) \geq W_1(\mu_0, \nu) - W_1(\mu_0, \mu) \geq \epsilon_0 - \epsilon_1 - W_1(\nu_0, \nu), \forall \nu \in A_{\epsilon_1}, \mu \in A$, yields $\inf_{\nu \in A \setminus A_{\epsilon_1}} W_1(\nu, \mu) \geq \epsilon_0 - \epsilon_1 - W_1(\nu_0, \nu), \forall \nu \in A_{\epsilon_1}$. Because $\epsilon_0 \geq \inf_{\nu \in A \setminus A_{\epsilon_1}} W_1(\nu, \mu)$, we have $W_1(\nu_0, \nu) \geq \epsilon_0 - 2\epsilon_1 > 0, \forall \nu \in A_{\epsilon_1}$. Hence, by Theorem 2.7 there exists a function $f_0 \in [\mathcal{L}(X)]_1$ with $\|f_0\|_{\text{Lip}} \leq 1$, such that
\[
\int fd\nu \geq \int fd\nu_0 + \epsilon_0 - 2\epsilon_1, \forall \nu \in A_{\epsilon_1}, \tag{16}
\]

Step 3. Finally, we show that $f_0$ is the required Lipschitz function. For each $\mu \in A$, there exists an $\nu_0 \in A_{\epsilon_1}$ such that $W_1(\mu, \nu_0) < \epsilon_1$ and hence $\int fd\mu \geq \int fd\nu_0 - \epsilon_1, \forall f \in [\mathcal{L}(X)]_1$. In particular,
\[
\int fd\mu \geq \int fd\nu_0 - \epsilon_1. \tag{17}
\]
Similarly, \( W_1(\mu_0, \nu_0) < \epsilon_1 \) implies
\[
\int f_0 d\nu_0 \geq \int f_0 d\mu_0 - \epsilon_1.
\] (18)

Combining (16) – (18), we obtain
\[
\int f_0 d\mu \geq \int f_0 d\mu_0 + \epsilon_0 - 4\epsilon_1 = \int f_0 d\mu_0 + \frac{\epsilon_0}{\mu}, \forall \mu \in A,
\]
which yields the required separability. \( \square \)

Analogously, we can obtain the same separation result in the space \( \mathcal{P}_1(X) \times \mathbb{R} \) which will be used in the following subsection.

**THEOREM 2.10.** Let \( \tilde{A} \) be a convex and closed subset of \( \mathcal{P}_1(X) \times \mathbb{R} \) equipped with the metric
\[
\tilde{d}(\mu, \nu) := W_1(\mu, \nu) + |r_1 - r_2|.
\]

Let \( (\mu_0, r_0) \in \mathcal{P}_1(X) \times \mathbb{R} \) be a point not contained in \( A \). Then, there exists a tuple \( (f, a) \in \mathcal{P}(X) \times \mathbb{R} \), and \( \epsilon_0 > 0 \), such that
\[
\int f d\mu + \alpha r \geq \int f d\mu_0 + \alpha r_0 + \epsilon_0, \forall (\mu, r) \in \tilde{A}.
\]

The proof is similar to the proof of Theorem 2.9.

### 2.6 Proof of the duality theorem

We shall prove Theorem 1.1 in this subsection. We want to stress that the proofs of Lemma 2.11 and the Theorem 1.1 mostly follow the line of proof found in Ioann-Bot et al. (2009, Theorem 2.2.15).

**DEFINITION 2.11.** For a function \( \phi : \mathcal{P}_1(X) \to \tilde{\mathbb{R}} \), its domain is defined by \( \text{dom}(\phi) := \{ \mu \in \mathcal{P}_1(X) : \phi(\mu) < +\infty \} \). \( \phi \) is said to be proper if \( \phi(\mu) > -\infty \) for all \( \mu \in \mathcal{P}_1(X) \) and \( \text{dom}(\phi) \neq \emptyset \). The epigraph of \( \phi \) is defined as \( \text{epi}(\phi) := \{ (\mu, r) \in \mathcal{P}_1(X) \times \mathbb{R} | \phi(\mu) \leq r \} \).

Now we provide the following proposition whose proof is standard and will be omitted.

**PROPOSITION 2.12.** If \( \phi : \mathcal{P}_1(X) \to \tilde{\mathbb{R}} \) is convex and lower semicontinuous (equipped with the metric \( W_1 \)), then \( \text{epi}(\phi) \) is convex and closed w.r.t.
\[
\tilde{d}(\mu, a), (\nu, b)) := W_1(\mu, \nu) + |a - b|.
\]

For any \( \phi : \mathcal{P}_1(X) \to \tilde{\mathbb{R}} \), we define the following set
\[
M_{\phi} := \left\{ (f, \eta) \in \mathcal{P}(X) \times \mathbb{R} \left| \int f d\mu + \eta \leq \phi(\mu), \forall \mu \in \mathcal{P}_1(X) \right. \right\}.
\] (19)

**LEMMA 2.13.** Let \( \phi : \mathcal{P}_1(X) \to \tilde{\mathbb{R}} \) be proper, convex and lower semicontinuous with respect to \( W_1 \). Then, \( M_{\phi} \) is not empty.

**PROOF.** Since \( \phi \) is proper, there exists an element \( \nu \in \mathcal{P}_1(X) \) such that \( \phi(\nu) \in \mathbb{R} \). Then \( \text{epi}(\phi) \neq \emptyset \) and \( (\nu, \phi(\nu) - 1) \notin \text{epi}(\phi) \). By Theorem 2.10 there exists \( (f_0, \eta_0) \in \mathcal{P}(X) \times \mathbb{R} \) such that
\[
\int f_0 d\nu + \eta_0(\phi(\nu) - 1) < \int f_0 d\mu + \eta_0 r, \forall (\mu, r) \in \text{epi}(\phi).
\]

Since \( (\nu, \phi(\nu)) \in \text{epi}(\phi) \), we have \( \eta_0 > 0 \) and \( (1/\eta_0) \left( \int f_0 d\nu - \int f_0 d\mu \right) + \phi(\nu) - 1 < r, \forall (\mu, r) \in \text{epi}(\phi) \). For any \( \mu \in \text{dom}(\phi) \), we have \( (\mu, \phi(\mu)) \in \text{epi}(\phi) \), and hence
\[
(1/\eta_0) \left( \int f_0 d\nu - \int f_0 d\mu \right) + \phi(\nu) - 1 < \phi(\mu).
\]

If \( \mu \notin \text{dom}(\phi) \), this inequality holds trivially. Thus,
\[
\left( -f_0/\eta_0, \phi(\nu) - 1 + \int f_0 d\nu/\eta_0 \right) \in M_{\phi}.
\]

\( \square \)

It is easy to verify that under the same conditions as in the above lemma, we have the following equation for the second conjugate dual \( \phi^*_c \) of \( \phi \):
\[
\phi^*_c(\mu_0) = \sup_{(f, \eta) \in M_{\phi}} \left\{ \int f d\mu_0 + \eta \right\}.
\]

We now proceed with the proof of Theorem 1.1.
Proof of Theorem 1.1. By definition, we have $\rho(f) \geq \int fd\mu - \phi(\mu), \forall (\mu, f) \in \mathcal{P}_1(\mathcal{X}) \times \mathcal{L}(\mathcal{X})$, which implies $\int fd\mu - \rho(f) \leq \phi(\mu), \forall (\mu, f) \in \mathcal{P}_1(\mathcal{X}) \times \mathcal{L}(\mathcal{X})$, and hence $\phi'(\mu) \leq \phi(\mu), \forall \mu \in \mathcal{P}_1(\mathcal{X})$.

Next we show that $\phi(\mu) \leq \phi'(\mu), \forall \mu \in \mathcal{P}_1(\mathcal{X})$. Assume towards a contradiction that there exists a $\mu_0 \in \mathcal{P}_1(\mathcal{X})$ and $r_0 \in \mathbb{R}$ such that

$$\phi(\mu_0) > r_0 > \phi'(\mu_0) = \sup_{(f, \eta) \in M_\phi} \left\{ \int fd\mu_0 + \eta \right\}.$$  \hspace{1cm} (20)

It is clear that $(\mu_0, r_0) \notin \text{epi}(\phi)$. Note that by Proposition 2.12, $\text{epi}(\phi)$ is closed. Furthermore, it is convex and non-empty. Hence, by Theorem 2.18 there exists a $(f_0, \eta_0) \in \mathcal{L}(\mathcal{X}) \times \mathbb{R}$ such that

$$\int fd\mu_0 + \eta_0 r_0 > \int fd\mu_0 + \eta_0 r_0 + \epsilon, \forall (\mu, r) \in \text{epi}(\phi),$$  \hspace{1cm} (21)

Note that if $(\mu, r) \in \text{epi}(\phi)$, then $(\mu, r + s) \in \text{epi}(\phi)$ for any $s \geq 0$. Thus $\eta_0 \geq 0$.

Suppose $\phi(\mu_0) \in \mathbb{R}$. Using the fact that $(\mu_0, \phi(\mu_0)) \in \text{epi}(\phi)$, we obtain $\eta_0(\phi(\mu_0) - r_0) > 0$. Hence $\eta_0 > 0$. For $\mu \in \text{dom}(\phi)$, we obtain $\phi(\mu) > \frac{1}{\eta_0} \int fd\mu_0 - \frac{1}{\eta_0} \int fd\mu + r_0$. Now setting $\eta = r_0 + \frac{1}{\eta_0}$ and $f = -\frac{1}{\eta_0}$ in (20), we have $\int fd\mu_0 + \eta = r_0$, which contradicts (20).

Suppose now that $\phi(\mu_0) = +\infty$. If $\eta_0 > 0$, the contradiction remains. Thus $\eta_0 = 0$ and (21) becomes $\int fd\mu > \int fd\mu_0 + \epsilon, \forall (\mu, r) \in \text{epi}(\phi)$. By Lemma 2.13 the set $\mathcal{M}_\phi$ defined in (10) is not empty. Hence, there exists $(f_1, a_1) \in \mathcal{M}_\phi$ such that $\int fd\mu_0 + a_1 \leq \phi(\mu), \forall \mu \in \mathcal{P}_1(\mathcal{X})$. By the assumption made in (20), $b := (r_0 - \int fd\mu_0 - a_1)/\epsilon > 0$. For all $\mu \in \text{dom}(\phi)$,

$$\int (f_1 - bf_0) d\mu_0 + \int bf_0 d\mu_0 + a_1 + b\epsilon = \int f_1 d\mu + a_1 + b(-\int f_0 d\mu + \int f_0 d\mu + \epsilon) \leq \int f_1 d\mu + a_1 \leq \phi(\mu).$$  \hspace{1cm} (22)

This can be extended to all $\mu \in \mathcal{P}_1(\mathcal{X})$. Hence, $(f_1 - bf_0, \int bf_0 d\mu_0 + a_1 + b\epsilon) \in \mathcal{M}_\phi$. Taking $\mu = \mu_0$ in (22), we have $\int (f_1 - bf_0) d\mu_0 + \int bf_0 d\mu_0 + a_1 + b\epsilon = r_0$, which again contradicts (20).

We will end this section with the proof Corollary 1.4.

Proof of Corollary 1.4. (a) We show first $\phi(\mu) = \sup_{f \in \mathcal{N}_\phi} \int fd\mu$. Indeed, Theorem 1.1 yields

$$\phi(\mu) = \sup_{f \in \mathcal{L}(\mathcal{X})} \left( \int fd\mu - \rho(f) \right) \geq \sup_{f \in \mathcal{N}_\phi} \left( \int fd\mu - \rho(f) \right) = \sup_{f \in \mathcal{N}_\phi} \int fd\mu.$$  

Since there exists at least one $\mu$ such that $\phi(\mu) \in \mathbb{R}$, we have $\rho(f) > -\infty, \forall f \in \mathcal{L}(\mathcal{X})$.

Thus,

$$\sup_{f \in \mathcal{L}(\mathcal{X})} \left( \int fd\mu - \rho(f) \right) = \sup_{f \in \mathcal{L}(\mathcal{X}): \rho(f) < \infty} \left( \int fd\mu - \rho(f) \right) = \sup_{f \in \mathcal{L}(\mathcal{X}): \rho(f) \in \mathbb{R}} \left( \int fd\mu - \rho(f) \right).$$

Due to the translation invariance, $f' := f - \rho(f)$ satisfies that $\rho(f') = 0$ if $\rho(f) \in \mathbb{R}$.

Hence,

$$\phi(\mu) = \sup_{f \in \mathcal{L}(\mathcal{X}): \rho(f) \in \mathbb{R}} \left( \int fd\mu - \rho(f) \right) = \sup_{f' \in \mathcal{N}_\phi} \int f' d\mu.$$  \hspace{1cm} (23)

Combining the above two inequalities yields $\phi(\mu) = \sup_{f \in \mathcal{N}_\phi} \int fd\mu$.

(b) We show now $\phi(\mu) = \sup_{f \in \mathcal{N}_\phi} \int fd\mu$. Theorem 1.1 yields

$$\phi(\mu) = \sup_{f \in \mathcal{L}(\mathcal{X})} \left( \int fd\mu - \rho(f) \right) \geq \sup_{f \in \mathcal{N}_\phi} \left( \int fd\mu - \rho(f) \right) \geq \sup_{f \in \mathcal{N}_\phi} \int fd\mu.$$  

On the other hand, (23) yields $\phi(\mu) \leq \sup_{f' \in \mathcal{N}_\phi} \int f' d\mu \leq \sup_{f' \in \mathcal{N}_\phi} \int fd\mu$. \hfill \square
3 POMDPs

3.1 Outline and related literature.

Most early literature on POMDPs (see e.g., Sondik (1978); Hernández-Lerma (1989)) considers finite state spaces or bounded reward function. In their recent work, Feinberg et al. (2016) extend the ideas from Sondik (1978); Föllmer and Schied (2004), and which we expect to inspire a computable set iteration algorithm for approximations of the value function.

Although our approach covers cases with two-sided unbounded reward functions, these cases can be reduced to MDPs with bounded reward functions by applying an algebraic transformation (Van Der Wal, 1981). The contribution of our paper on POMDPs, is the conjugate approach for the computation of the value function described in Section 5 that extends the ideas from Sondik (1978); Pöllner and Schied (2004), and which we expect to inspire a computable set iteration algorithm for approximations of the value function.

3.2 Setup

A partially observable Markov decision process (POMDP, see e.g., Hernández-Lerma (1989), Chapter 4) is described by a tuple \((X, Y, A, P, Q, \mu, r)\), where:

(a) \(X\) is the (hidden or latent) state space, a Polish space with metric \(d\).

(b) \(Y\) is the space of observations, a Borel space.

(c) \(A\) is the action space, a Borel space.

(d) \(P(dx'|x, a)\) is the state transition law, a stochastic kernel on \(X\) given \(K := X \times A\). \(K\) is also a Borel space.

(e) \(Q(dy|a, x)\) is the observation kernel a stochastic kernel on \(Y\) given \(K\). \(Q_0\) is the initial observation kernel, a stochastic kernel on \(Y\) given \(X\).

(f) \(\mu \in \mathcal{P}(X)\) is the initial distribution.

(g) \(r : K \to \mathbb{R}\) is the one-step reward function, which is \(\mathcal{B}(K)\)-measurable.

The POMDP evolves as follows: At time \(t = 0\), the initial (hidden or latent) state \(x_0\) follows a given prior distribution \(\mu\), while the initial observation \(y_0\) is generated according to the initial observation kernel \(Q_0(\cdot|x_0)\). If, at time \(t\), the state of the system is \(x_t\) and the control \(a_t \in A\) is applied, then the agent receives a reward \(r(x_t, a_t)\) and the system transits to state \(x_{t+1}\) according to the transition law \(P(dx_{t+1}|x_t, a_t)\). The observation \(y_{t+1}\) is generated by the observation kernel \(Q(dy_{t+1}|a_t, x_{t+1})\). The observed history is defined as

\[
h_0 := \{\mu, y_0\} \in H_0 \quad \text{and} \quad h_t := \{\mu, y_0, a_0, \ldots, y_{t-1}, a_{t-1}, y_t\} \in H_t, t = 1, 2, \ldots, \tag{24}
\]

where \(H_0 := \mathcal{P}(X) \times Y\) and \(H_{t+1} = H_t \times Y \times A, t = 1, 2, \ldots\). Notably, comparing with the canonical Markov decision processes (MDPs, see e.g., Hernández-Lerma (1989)), the states \(\{x_t\}\) are not observable and hence, a policy depends only on the observed history.

A deterministic policy \(\pi := [\pi_0, \pi_1, \ldots]\) is composed of a sequence of one-step policies \(\pi_t : H_t \to A\), given the observed history up to time \(t\). Let \(\Pi\) be the set of all deterministic policies. Note that even with extension to nondeterministic policies, it is known (Hernández-Lerma (1989), Chapter 4) that an optimal policy to a POMDP is always deterministic. Hence, we consider in this paper only deterministic policies. The Ionescu-Tulcea theorem (Bertsekas and Shreve 1978, pp. 140–141) implies that for each \(\pi \in \Pi\) and an initial \(\mu \in \mathcal{P}(X)\), along with \(P, Q\) and \(Q_0\), a probability measure \(\mathbb{P}_\mu^\pi\) and a stochastic process \(\{X_t, Y_t, A_t\}\) can be defined in a canonical way. We denote by \(E^\pi_\mu\) the expectation with respect to this probability measure \(\mathbb{P}_\mu^\pi\).
We consider the following discounted cumulative rewards
\[ J_T(\pi, \mu) := \mathbb{E}_\mu^\pi \left[ \sum_{t=0}^{T} \alpha^t r(X_t, A_t) \right], \]  
where \( \alpha \in (0, 1) \) stands for a discount factor, and \( T \in \mathbb{N} \cup \{\infty\} \). The objective is now to maximize the expected reward over the set of deterministic policies \( \Pi \),
\[ \phi^*_T(\mu) := \sup_{\pi \in \Pi} J_T(\pi, \mu), \quad \mu \in \mathcal{P}(X). \]

We will finally use the notation
\[ \phi^* := \phi^*_\infty. \]  

3.3 Reduction to Markov decision process

We show briefly in this subsection that the POMDP can be reduced to a Markov decision process (MDP, see e.g. Hernández-Lerma and Lasserre 1999). We follow mostly the derivation by Hernández-Lerma and Lasserre (1999, Chapter 4). We first introduce the following notation:
\[ \tilde{r}(\mu, a) := \int r(x, a) \rho(dx), \quad \text{and} \quad \tilde{P}(B|\mu, a) := \int \rho(dx) P(B|x, a), B \in \mathcal{B}(X), \] where \( \mu \in \mathcal{P}(X), \) and \( a \in A \). For any \( C \in \mathcal{B}(Y) \) and \( B \in \mathcal{B}(X) \), we define
\[ R(B, C|\mu, a) := \int_B Q(C|a, x') \tilde{P}(dx'|\mu, a), \quad \text{and} \quad \tilde{R}(C|\mu, a) := R(X, C|\mu, a). \]  

**Proposition 3.1.** There exists a stochastic kernel \( M \) from \( \mathcal{P}(X) \times A \times Y \) to \( X \) such that for each \( \mu \in \mathcal{P}(X), \) \( a \in A, B \in \mathcal{B}(X) \) and \( C \in \mathcal{B}(Y) \),
\[ R(B, C|\mu, a) = \int_Y M(B|\mu, a, y) \tilde{R}(dy|\mu, a). \]  

**Proof.** Direct application of Bertsekas and Shreve (1978, Corollary 7.27.1). \( \square \)

\( M \) can also be viewed as a mapping \( \mathcal{P}(X) \times A \times Y \to \mathcal{P}(X) \). With a slight abuse of notation, let \( M(\mu, a, y) := M(\cdot|\mu, a, y) \in \mathcal{P}(X) \). Then we define the following stochastic kernel
\[ \tilde{Q}(D|\mu, a) := \int 1_D(M(\mu, a, y)) \tilde{R}(dy|\mu, a), \quad D \in \mathcal{B}(\mathcal{P}(X)). \]  

Let \( \mu_t \in \mathcal{P}(X) \) be the distribution at time \( t \). Then, given an action \( a_t \in A \) and an observation \( y_{t+1} \in Y \), the **successive** distribution \( \mu_{t+1} \in \mathcal{P}(X) \) is given by
\[ \mu_{t+1} = M(\mu_t, a_t, y_{t+1}). \]  

Note that \( \mu_{t+1} \) is a random measure since \( y_{t+1} \) is a random variable with the distribution \( \tilde{R}(\cdot|\mu_t, a_t) \). Hence, the POMDP can be reduced to a Markov decision process (MDP) with the belief state space \( \mathcal{P}(X) \), the action space \( A \), the reward function \( \tilde{r} \) on \( \mathcal{P}(X) \times A \) and the transition kernel on belief states \( \tilde{Q} \) defined above.

Let \( h_t \) be a t-stage history for the MDP described above:
\[ h_t := \{\tilde{\mu}_0, a_0, \ldots, \tilde{\mu}_{t-1}, a_{t-1}, \tilde{\mu}_t\} \in \hat{H}_t := \mathcal{P}(X) \times (A \times \mathcal{P})^t, t = 0, 1, \ldots, \] where \( \tilde{\mu}_0 \) is the initial distribution and the \( \tilde{\mu}_t \in \mathcal{P}(X) \) are recursively defined by (31).

Given the original t-stage history \( h_t \) defined in (24), let \( m_t : H_t \to \hat{H}_t \) be the mapping such that \( m_t(h_t) = h_t, \forall h_t \in H_t, t = 0, 1, \ldots, \) For a history-dependent MDP-policy \( \delta := [\delta_0, \delta_1, \ldots] \), where \( \delta_t : H_t \to A \), we define its counterpart POMDP-policy as \( \pi^\delta = [\pi^\delta_0, \pi^\delta_1, \ldots], \) with \( \pi^\delta_t(h_t) = \delta_t(m_t(h_t)) \). If a policy \( \delta \) is optimal for the MDP, then its counterpart policy \( \pi^\delta \) is also optimal for the POMDP. For more details, we refer to Hernández-Lerma (1989, Chapter 4) and references therein.
After the reduction to a MDP, it is well known that under proper assumptions (see, e.g., Hernández-Lerma [1989], Chapter 4) and Peinberg et al. (2016), the optimal \( \phi^* \) for the infinite-stage case satisfy the following optimality equation:

\[
\phi(\mu) = T(\phi)(\mu) := \sup_{a \in A} \left( r(\mu, a) + \alpha \int \phi(M(\mu, a, y)) d\mu(a) \right), \quad \forall \mu \in \mathcal{P}(X),
\]

where \( T \) is an operator on the space of Borel measurable functions on \( \mathcal{P}(X) \). We will show in Section 4 that under general assumptions, the existence of a solution of the above equation is guaranteed, as well as the existence of an optimal deterministic policy. The value iteration algorithm gives a sequence that converges to the value function.

4 Optimal solution on weighted space

4.1 Weighted norm

The weighted norm has proven to be very useful when dealing with MDPs (see e.g. Hernández-Lerma and Lasserre [1999]). The weighted norm induced by the Wasserstein-1 metric is defined as follows: First, we specify a weight function \( w \) on \( X \). Then, for any bounded continuous function \( f \) on \( X \), it follows that \( \int f(x) \mu_n(dx) \to \int f(x) \mu(dx) \) as \( n \to \infty \); and (ii) \( \int w(x) \mu_n(dx) \to \int w(x) \mu(dx) \) as \( n \to \infty \). Let \( \mathcal{P}_w(X) \subset \mathcal{P}(X) \) be the set of probability measures \( \mu \) on \( X \) satisfying \( \int w \, d\mu < \infty \). We equip \( \mathcal{P}_w(X) \) with the following concept of weak convergence:

**Definition 4.1.** (a) \( \mu_n \) is said to converge weakly to \( \mu \) in \( \mathcal{P}_w(X) \), if (i) for any bounded continuous function \( f \) on \( X \), it follows that \( \int f(x) \mu_n(dx) \to \int f(x) \mu(dx) \) as \( n \to \infty \); and (ii) \( \int w(x) \mu_n(dx) \to \int w(x) \mu(dx) \) as \( n \to \infty \). (b) A function \( \phi : \mathcal{P}_w(X) \to \mathbb{R} \) is said to be lower (resp. upper) semicontinuous if

\[
\liminf_{n \to \infty} \phi(\mu_n) \geq \phi(\mu) \ (\text{resp.} \ \limsup_{n \to \infty} \phi(\mu_n) \leq \phi(\mu)),
\]

whenever \( \mu_n \) converges weakly to \( \mu \). \( \phi \) is said to be continuous, if \( \phi \) is both lower and upper continuous.

**Remark 4.2.** It is worth to be mentioned that the weak convergence defined above is stronger than the usual weak convergence, which requires only (i). Hence, to emphasize this difference, we call the latter canonical weak convergence throughout the rest of this paper.

**Proposition 4.3.** The following two statements are equivalent: (i) \( \mu_n \) converges weakly to \( \mu \) in \( \mathcal{P}_w(X) \) and (ii) \( \int f \, d\mu_n \to \int f \, d\mu, \forall f \in \mathcal{L}_w(X) \).

For a proof, see Villani (2009, Definition 6.8).

**Proposition 4.4.** Let \( w \) be defined as in (33) with some fixed \( k > 0 \) and \( x_0 \in X \). Then (i) \( \mathcal{P}_1(X) = \mathcal{P}_w(X) \) and (ii) the weak convergence defined in Definition 4.1 is equivalent to the convergence in \( (\mathcal{P}_1(X), W_1) \), in other words, \( W_1 \) metrizes the weak convergence.

**Proof.** (i) This is obvious by definition. (ii) This is a direct result of Villani (2009, Theorem 6.9). \( \square \)

In the rest of this paper, we always assume that the weight function satisfies (33) and hence \( \mathcal{P}_w(X) \) is used interchangeably with \( \mathcal{P}_1(X) \). Then, the belief state space is Polish, and therefore, a Borel space.

We next specify the weight function and its weighted norm on \( \mathcal{P}_w(X) \). Define \( \tilde{w} : \mathcal{P}_w(X) \to [1, \infty) \) as

\[
\tilde{w}(\mu) := \int w \, d\mu.
\]
It is easy to check that \( \tilde{w} \) is a continuous function and hence measurable on \((P_\infty(X), W_1)\).

Define the following space of functions on \(P_\infty(X)\) with bounded \(\tilde{w}\)-norm:

\[
\mathcal{B}_w(X) := \left\{ \phi : P_\infty(X) \to \mathbb{R} \mid \phi \text{ is } B(P_\infty(X))-\text{measurable}, \|\phi\|_w := \sup_{\mu \in P_\infty(X)} \left| \frac{\phi(\mu)}{\tilde{w}(\mu)} \right| < \infty \right\}.
\]

In the next subsection, we shall specify some assumptions on the original POMDP in order to ensure the assumptions needed for MDPs as in Theorem 8.3.6 in Hernández-Lerma and Lasserre (1999).

4.2 Assumptions

We introduce the following assumption for the reward function.

**Assumption 4.5.** (i) There exists a positive constant \( \bar{r} > 0 \) such that \( r(x, a) \leq \bar{r} \tilde{w}(x) \), for each \((x, a) \in K\). (ii) For each \(x \in X\), \( a \mapsto r(x, a) \) is upper semicontinuous.

**Proposition 4.6.** Under Assumption 4.5, (i) \( |\tilde{r}(\mu, a)| \leq \bar{r} \tilde{w}(\mu), \forall (\mu, a) \in P_\infty(X) \times A\); (ii) for each \(\mu \in P_\infty(X), a \mapsto \tilde{r}(\mu, a) \) is upper semicontinuous; (iii) for each \(a \in A\), \( \mu \mapsto \tilde{r}(\mu, a) \) is continuous in \(P_\infty(X)\).

**Proof.** (i) For each \((\mu, a) \in P_\infty(X) \times A\), we have \( |\tilde{r}(\mu, a)| \leq \int |r(x, a)| d\mu(x) \leq \bar{r} \int d\mu = \bar{r} \tilde{w}(\mu)\). (ii) Let \(\{a_n, n = 1, 2, \ldots\} \) be a sequence of actions converging to \(a_0\) and set \(r_n(x) := r(x, a_n), n \in \mathbb{N}\). By Assumption 4.3(i), \(r_n \leq \bar{r}\tilde{w}\). Applying the reversed Fatou’s lemma, we obtain

\[
\limsup_{n \to \infty} \int r_n d\mu \leq \int \limsup_{n \to \infty} r_n d\mu \leq \int r d\mu,
\]

where the last inequality is due to Assumption 4.3(ii). Finally, (iii) is a direct result of Proposition 4.3(ii). \(\square\)

Similar to the assumptions made in the literature of MDPs (Hernández-Lerma and Lasserre, 1999, Assumptions 8.3.2 and 8.3.3), we introduce the following assumption on the transition kernel \(P\):

**Assumption 4.7.** (i) There exists a constant \(\beta \in (0, \alpha^{-1})\) such that

\[
\int w(x')P(dx'|x, a) \leq \beta w(x), \forall (x, a) \in X \times A.
\]

(ii) For each \(x \in X\), \( a \mapsto \int w(x')P(dx'|x, a) \) is continuous.

Under the above assumption, we show that the new probability measure \(M(\mu, a, y)\) belongs to \(P_\infty(X)\) almost surely, and \(a \mapsto \tilde{w}(\mu')\tilde{Q}(dy'|\mu, a)\) is continuous.

**Proposition 4.8.** Suppose Assumption 4.7 holds. Then for each \(\mu \in P_\infty(X)\) and \(a \in A\), (i) \(M(\mu, a, y) \in P_\infty(X), \tilde{R}(\cdot|\mu, a)\)-almost surely; (ii) \(\tilde{w}(\mu')\tilde{Q}(dy'|\mu, a) \leq \beta \tilde{w}(\mu)\); and (iii) for each \(\mu \in P_\infty(X), \text{ the map } a \mapsto \tilde{w}(\mu')\tilde{Q}(dy'|\mu, a)\) is continuous.

**Proof.** Fix an arbitrary \((\mu, a) \in P_\infty(X) \times A\). (i) Let \(C \in B(Y)\) be a subset such that \(\tilde{R}(C|\mu, a) > 0\). Then, we have

\[
\int_C \int_X w(x')M(dx'|\mu, a, y)\tilde{R}(dy'|\mu, a) = \int_C \int_X Q(C|\mu', a')w(x')\tilde{P}(dx'|a, a) \leq \\
\int_X Q(\emptyset|\mu', a')\tilde{w}(\mu')\tilde{Q}(dy'|\mu, a) \leq \int w(x')\tilde{P}(dx'|\mu, a) \leq \beta \int w d\mu < \infty.
\]

This implies that \(\int w(x')M(dx'|\mu, a, y) < \infty, \tilde{R}(\cdot|\mu, a)\)-almost surely, and hence (i) holds.

(ii) By definition, we have

\[
\int \tilde{w}(\mu')\tilde{Q}(dy'|\mu, a) = \int \tilde{w}(M(\mu, a, y))\tilde{R}(dy'|\mu, a)
\]

(by 24) \(= \int \int w(x')M(dx'|\mu, a, y)\tilde{R}(dy'|\mu, a)
\]

(by 29 and Fubini’s theorem) \(= \int \int Q(\emptyset|\mu', a')w(x')\tilde{P}(dx'|\mu, a) \leq \beta \int w d\mu = \beta \tilde{w}(\mu).
\]

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Finally, by a suitable application of Theorem 8.3.6 in Hernández-Lerma and Lasserre (1999), we obtain the following convergence. 

\[ f_n(x) := \int x \cdot P(dx|\cdot,x,a), n \in \mathbb{N}. \]

Hence, the required continuity is equivalent to showing that \( \lim_{n \to \infty} f_n d\mu = \int f_0 d\mu. \)

Indeed, by Assumption 4.7(i), we have \( f_n \leq \beta w, \forall n \in \mathbb{N}. \) The reversed Fatou’s lemma implies \( \limsup_{n \to \infty} f_n d\mu \leq \limsup_{n \to \infty} f_n d\mu = \int f_0 d\mu. \) On the other hand, we have \( f_n \geq -\beta w, \forall n \in \mathbb{N}. \) Then, the extended Fatou’s lemma implies \( \liminf_{n \to \infty} f_n d\mu \geq \liminf_{n \to \infty} f_n d\mu = \int f_0 d\mu. \) Combining the above two inequalities yields the convergence. \( \square \)

**Assumption 4.9.** For each \( \mu \in \mathcal{P}_w(X), \) there exist stochastic kernels \( M \) on \( \mathcal{P}(X) \) given \( \mathcal{P}_w(X) \times A \times Y \) and \( \tilde{R} \) on \( Y \) given \( \mathcal{P}_w(X) \times A \) satisfying (29) such that, if \( \{a_n \in A, n = 1, 2, \ldots\} \) converges to \( a_0 \in A \) as \( n \to \infty, \)

(i) there exists a subsequence \( \{a_{n_k} \} \subset \{a_n \} \) and a measurable set \( C \in B(Y) \) such that \( \tilde{R}(C|\mu,a_0) = 1 \) and for all \( y \in C, M(\mu,a_{n_k},y) \) converges canonically weakly to \( M(\mu,a_0),y) ; \)

(ii) for each \( C \in B(Y), \tilde{R}(C|\mu,a_n) \to \tilde{R}(C|\mu,a_0) \) as \( n \to \infty. \)

This assumption is inspired by Condition (c) in Feinberg et al. (2016, Theorem 3.2). A sufficient condition for it will be discussed in the next section (see Remark 4.11).

**Proposition 4.10.** Under Assumptions 4.6 and 4.7 for each \( \mu \in \mathcal{P}_w(X), a \to \tilde{Q}(|\mu,a) \) is canonically weakly continuous.

**Proof.** This canonical weak continuity is guaranteed by Feinberg et al. (2016, Theorem 3.4). \( \square \)

Finally, to guarantee the existence of one “selector”, we assume

**Assumption 4.11.** \( A \) is compact.

Note that the operator \( T : \mathcal{R}_\phi(X) \to \mathcal{R}_\phi(X) \) is defined as follows

\[ T_\alpha(\phi)(\mu) := \tilde{r}(\mu,a) + \alpha \int \phi(M(\mu,a,y)) \tilde{R}(dy|\mu,a) \quad (35) \]

and

\[ T(\phi)(\mu) := \sup_{a \in A} T_\alpha(\phi)(\mu). \quad (36) \]

Under Assumptions 4.6 - 4.9 it is guaranteed that for each \( \mu \in \mathcal{P}_w(X), a \to T_\alpha(\phi)(\mu) \) is upper-semicontinuous (for a proof, see Hernández-Lerma and Lasserre (1999, Lemma 8.3.7(a))). Hence, under the additional Assumption 4.11 the optimal \( a \) in the above optimization problem is always attainable in \( A \) (see, e.g., Hernández-Lerma and Lasserre (1999, Lemma 8.3.8(a))). Hence, from now on, we replace “sup” with “max”.

### 4.3 Value iteration

The following value iteration is a widely used method to compute the optimal solution for POMDPs, and MDPs as well. Starting from arbitrary value function in \( \mathcal{R}_\phi(X), \phi_0, \) at time \( t, \) we update value function as follows

\[ \phi_{t+1} = T(\phi_t)(\mu) = \max_{a \in A} \left( \tilde{r}(\mu,a) + \alpha \int \phi(M(\mu,a,y)) \tilde{R}(dy|\mu,a) \right) \]

Finally, by a suitable application of Theorem 8.3.6 in Hernández-Lerma and Lasserre (1999), we obtain the following convergence.

**Theorem 4.12.** Suppose that Assumptions 4.6 - 4.11 hold. Let \( \beta \) be the constant in Assumption 4.7(i) and \( \tilde{r} \) be the constant in 4.7(ii) and define \( \gamma := \alpha \beta \in (0,1). \) Then

(a) the optimal value function \( \phi^* \) is the unique fixed point of the operator \( T \) satisfying \( \phi^* = T(\phi^*) \) in \( \mathcal{R}_\phi(X) \) and \( ||\phi_t - \phi^*||_\phi \leq \tilde{r} \gamma^t/(1-\gamma), t = 1, 2, \ldots. \)
there exists a selector \( f^* : \mathcal{P}_w(X) \to A \) such that
\[
\phi^*(\mu) = \tilde{r}(\mu, f^*(\mu)) + \alpha \int \phi(M(\mu, f^*(\mu), y)) \tilde{R}(dy|\mu, f^*(\mu)), \forall \mu \in \mathcal{P}_w(X).
\]
and \( \pi^* = (f^*)^\infty \) is one optimal policy satisfying \( \phi^*(\mu) = J(\mu, \pi^*), \forall \mu \in \mathcal{P}_w(X). \)

**Proof.** The original POMDP specified in Subsection 3.2 can be reduced to an MDP with \( (\mathcal{P}_w(X), A, \tilde{r}, \tilde{Q}) \). Under Assumptions [3.3, 3.11] Propositions [1.6, 1.10] hold, and therefore, the conditions required by Theorem 8.3.6 in Hernández-Lerma and Lasserre [1992] are satisfied. The assertion is then a direct application of that Theorem. □

### 4.4 Application to POMDPs

Now we apply the conjugate duality obtained in Corollary 4.4 to POMDPs. Recall that the operators \( T_\alpha \) and \( T \) are defined in equations (35) and (36).

**Lemma 4.13.** If \( \phi : \mathcal{P}_w(X) \to \mathbb{R} \) is convex, then \( T_\alpha(\phi) \) is convex, \( \forall \alpha \in A \), and therefore, \( T(\phi) \) is convex as well.

**Proof.** It is sufficient to show that \( \mu \mapsto \tilde{r}(\mu, a) + \alpha \int \phi(M(\mu, a, y)) \tilde{R}(dy|\mu, a) \) is convex for each \( \mu \in A \). Indeed, take any action \( a \in A \) and let \( \mu_1 \) and \( \mu_2 \) be two arbitrary elements in \( \mathcal{P}_w(X) \). Take any \( \kappa \in (0, 1) \) and define \( \mu_\kappa := \kappa \mu_1 + (1 - \kappa) \mu_2 \). By the definition, for any \( B \in \mathcal{B}(X) \) and \( C \in \mathcal{B}(Y) \), we have
\[
\tilde{R}(B, C|\mu_\kappa, a) = \kappa \tilde{R}(B, C|\mu_1, a) + (1 - \kappa) \tilde{R}(B, C|\mu_2, a) \quad (37)
\]
and
\[
\tilde{R}(C|\mu_\kappa, a) = \kappa \tilde{R}(C|\mu_1, a) + (1 - \kappa) \tilde{R}(C|\mu_2, a), \forall C \in \mathcal{B}(Y). \quad (39)
\]
Hence, \( \tilde{R}(C|\mu_\kappa, a) = 0 \) implies \( \tilde{R}(C|\mu_1, a) = 0 \) and \( \tilde{R}(C|\mu_2, a) = 0, \forall C \in \mathcal{B}(Y) \). By Radon-Nikodym theorem, there exist functions \( f_i : Y \times A \to [0, \infty), i = 1, 2 \), which are both \( \mathcal{B}(Y) \)-measurable for the fixed \( a \), such that
\[
\tilde{R}(C|\mu_\kappa, a) = \int_C f_i(y, a) \tilde{R}(dy|\mu_\kappa, a), i = 1, 2. \quad (40)
\]
Applying these two equations in (39) accordingly, we obtain
\[
\tilde{R}(C|\mu_\kappa, a) = \int_C (\kappa f_1(y, a) + (1 - \kappa)f_2(y, a)) \tilde{R}(dy|\mu_\kappa, a), \forall C \in \mathcal{B}(Y),
\]
which implies that \( \kappa f_1(y, a) + (1 - \kappa)f_2(y, a) = 1, \tilde{R}(\cdot|\mu_\kappa, a) \)-almost surely. In other words, there exists a Borel set \( C \in \mathcal{B}(Y) \) such that \( \tilde{R}(C|\mu_\kappa, a) = 1 \) and \( \kappa f_1(y, a) + (1 - \kappa)f_2(y, a) = 1, \forall y \in C \).

Applying (10) to (39), we obtain
\[
\tilde{R}(B, C|\mu_\kappa, a) = \int_C [\kappa M(B|\mu_1, a, y)f_1(y, a) + (1 - \kappa)M(B|\mu_2, a, y)f_2(y, a)] \tilde{R}(dy|\mu_\kappa, a),
\]
and for each \( \kappa \in (0, 1) \), \( M(|a, y, \kappa) := \kappa M(|\mu_1, a, y)f_1(y, a) + (1 - \kappa)M(|\mu_2, a, y)f_2(y, a) \) is a valid stochastic kernel satisfying \( \tilde{R}(B, C|\mu_\kappa, a) = \int_C M(B|a, y, \kappa) \tilde{R}(dy|\mu_\kappa, a), \forall B \in \mathcal{B}(X), C \in \mathcal{B}(Y) \). Finally, the convexity of \( \phi \) implies
\[
\int \phi(M(|a, y, \kappa))) \tilde{R}(dy|\mu_\kappa, a) = \int \phi(M(|a, y, \kappa))) \tilde{R}(dy|\mu_\kappa, a)
\]
\[
\leq \int C (\kappa f_1(y, a)\phi(M(\mu_1, a, y)) + (1 - \kappa)f_2(y, a)\phi(M(\mu_2, a, y))) \tilde{R}(dy|\mu_\kappa, a)
\]
\[
\leq \kappa \int \phi(M(\mu_1, a, y))) \tilde{R}(dy|\mu_1, a) + (1 - \kappa) \int \phi(M(\mu_2, a, y))) \tilde{R}(dy|\mu_2, a),
\]
which yields the required convexity. □

We introduce the following assumption accompanying Assumption 4.4.
These steps will be repeated until some stopping criterion is satisfied.

4.5 Set iteration

Continuity. This confirms the necessity of Assumption 4.14(ii).

that the latter continuity in total variation cannot be weakened to the canonical weak total variation. In addition, it is demonstrated in (Feinberg et al., 2016, Example 4.1) using the following two steps:

(i) Suppose both Assumptions 4.9 and 4.14 is that (i) The stochastic kernel

\[ \phi > 0 \text{ such that } |\phi(\mu)| \leq \phi \int w \, d\mu. \]

Let \( \phi(\mu) := \phi(\mu) + \phi \int w \, d\mu \), which is nonnegative. Hence, it is a limit of a nondecreasing sequence of measurable bounded function \( \{\phi_n\} \) such that \( \phi_n \uparrow \phi \). Let \( \{\mu_n \in \mathcal{P}_w(X)\} \) be a converging sequence under \( W_1 \) to a limit \( \mu_0 \in \mathcal{P}_w(X) \).

We have then

\[ \liminf_{n \to \infty} \int \phi(\mu') \tilde{Q}(d\mu'|\mu_n,a) \geq \liminf_{n \to \infty} \int \phi_n(\mu') \tilde{Q}(d\mu'|\mu_n,a) = \int \phi_n(\mu') \tilde{Q}(d\mu'|\mu_0,a). \]

Hence, letting \( m \to \infty \), monotone convergence yields that

\[ \liminf_{n \to \infty} \int \phi(\mu') \tilde{Q}(d\mu'|\mu_n,a) \geq \int \phi(\mu') \tilde{Q}(d\mu'|\mu_0,a). \]  

4.5 Set iteration

Recall that Corollary 4.3 imply that a convex and lower semicontinuous function \( \phi \) admits a representation of \( \phi(\mu) = \sup_{f \in N} \int f \, d\mu \) with some set \( N \subset \mathcal{L}_w(X) \). Hence, instead of iterating the value function, we can iterate the acceptance set, which is described as follows.

Algorithm 4.18. Start with any set \( \tilde{N}_0 \subset \mathcal{L}(X) \). At time \( t \), update the acceptance set using the following two steps:

\[ \tilde{N}_{t+1} = \left\{ f \in \mathcal{L}(X) \mid \phi_{t+1}(\mu) \geq \int f \, d\mu, \forall \mu \in \mathcal{P}_w(X) \right\}. \]
An iteration of null level-sets can be analogously designed as above and is therefore omitted. Note that in course of iteration, Lemma 4.13 and 4.14 guarantee that $\phi_t$ is convex and lower semicontinuous for each $t = 1, 2, \ldots$. Hence, Corollary 1.4 ensures that $\phi_t(\mu) = \sup_{f \in \mathcal{F}_n} \int f d\mu, \forall \mu \in \mathcal{P}_w(X)$, and for each $t = 1, 2, \ldots$ By Theorem 4.12(a), we immediately obtain the following result.

**THEOREM 4.19.** Suppose Assumptions 4.3, 4.4, 4.6, 4.10 and 4.11 hold. Let $\phi^*$ be the optimal value function for the POMDP, $r > 0$ and $\gamma \in (0, 1)$ be the constants as in Theorem 4.12(a). Then,

$$\|\phi^* - \phi_t\|_{\mathcal{W}} \leq \tilde{r} \gamma^t / (1 - \gamma), \quad \text{where} \quad \phi_t(\mu) = \sup_{f \in \mathcal{F}_n} \int f d\mu, \forall \mu \in \mathcal{P}_w(X).$$

This implies that the optimal value function $\phi^*$ can be arbitrarily well approximated by a convex and lower semicontinuous function $\phi$ of the dual form.

**COROLLARY 4.20.** Suppose Assumptions 4.3, 4.4, 4.6, 4.10 and 4.11 hold. For any $\epsilon > 0$, there exists a set $N^\epsilon \subset \mathcal{L}(X)$ satisfying $\|\phi - \phi^*\|_{\mathcal{W}} \leq \epsilon$, where $\phi(\mu) := \sup_{f \in N^\epsilon} \int f d\mu, \forall \mu \in \mathcal{P}_w(X)$.

**A special case: $Q$ is supported by a reference measure** Let us assume that there exists a reference (probability) measure $\nu$ on $X$ such that $Q(=\nu)$ for all $(x', a) \in X \times A$. Note that POMDPs in many applications satisfy this assumption. For example, the assumption holds automatically if the observation space is finite. Hence, the density of $Q$ w.r.t. $\nu$ exists and is denoted by $q(y|x', a)$. In this case, the iteration can be further simplified. Indeed, it is easy to verify that

$$M(dx'|\mu, a, y) = \frac{\tilde{P}(dx'|\mu, a)q(y|x', a)}{\int_x \tilde{P}(dx'|\mu, a)q(y|x', a)} \quad \text{and} \quad \tilde{R}(dy|\mu, a) = \int_x \tilde{P}(dx'|\mu, a)q(y|x', a)\nu(dy)$$

satisfy (43). Under this setup, the calculation of iteration becomes much simpler. Suppose $\phi \in \mathcal{P}_w$ is convex and lower semicontinuous, then we have by Corollary 4.14

$$T_n(\phi)(\mu) = \tilde{r}(\mu, a) + \alpha \int \left( \sup_{f \in \mathcal{F}_n} \int f(x') \tilde{P}(dx'|\mu, a)q(y|x', a) \right) \nu(dy). \quad (42)$$

In particular, the continuity of $Q$ in total variation mentioned in Remark 4.17 is

$$\int |q(y|x_n, a_n) - q(y|x_{n-1}, a_{n-1})| \nu(dy) \to 0, \quad \text{as} \quad (x_n, a_n) \to (x_0, a_0).$$

## 5 Appendix

**LEMMA 5.1.** For $\mu \in \mathcal{P}_1(X)$, it holds that

$$\log \int_X e^\delta d\mu = \sup_{\nu \in \mathcal{P}_1(X)} \left\{ \int_X g d\nu - R(\nu|\mu) \right\}, \quad \forall g \in \mathcal{L}(X) \quad (43)$$

**Proof.** Let $\mu_n$ with $\frac{d\mu_n}{d\mu} = e^{\min(g, n)} \cdot \frac{1}{\int_x \min(g, n) d\mu}$. For an arbitrary $\nu \in \mathcal{P}_1(X)$ with $\mathcal{R}(\nu|\mu) < \infty$, we have that $\nu$ is absolutely continuous with respect to $\mu_n$ and therefore

$$\int_X g d\nu - R(\nu|\mu) = \int_X g d\nu - \int_X \log \left( \frac{d\nu}{d\mu} \right) d\nu \quad (44)$$

Now by using the positivity of $\mathcal{R}$ and applying the monotone convergence theorem we get

$$\int_X g d\nu - R(\nu|\mu) \leq \log \int_X e^\delta d\mu.$$
If $\mathcal{R}(\nu|\mu) = \infty$, the above inequality holds trivially.

Now, on the other hand, by setting $r = \mu_n$ in (44), observing that $\int_X g d\mu_n - \int_X \min(g, n) d\mu_n \geq 0$, and applying the monotone convergence theorem one more time, we get

$$\lim_{n \to \infty} \left( \int_X g d\mu_n - \mathcal{R}(\mu_n|\mu) \right) \geq \log \int_X e^g d\mu,$$

which yields our result. $\Box$

**Lemma 5.2.** For $\mu, \nu \in \mathcal{P}_1(X)$, it holds that

$$\sup_{g \in C_b(X)} \left\{ \int_X g d\nu - \log \int_X e^g d\mu \right\} = \sup_{g \in C_b(X)} \left\{ \int_X g d\nu - \log \int_X e^g d\mu \right\}.$$  (45)

**Proof.** For simplicity we will set $F(g) = \int_X g d\nu - \log \int_X e^g d\mu$. By properties of the supremum, we have

$$\sup_{g \in C_b(X)} F(g) \leq \sup_{g \in C_b(X)} \sup_{f \in C_b(X)} F(f), \quad \sup_{f \in C_b(X)} F(f) \leq \sup_{g \in C_b(X)} \sup_{f \in C_b(X)} F(g).$$

So it will be enough to prove that

$$\sup_{g \in C_b(X)} F(g) \leq \sup_{g \in C_b(X)} F(g).$$

Let $g \in C_b(X) \cup \mathcal{L}(X)$, with $F(g) \neq -\infty$. It is now enough to prove that for every $\epsilon > 0$, it exists $\tilde{g} \in C_b(X) \cap \mathcal{L}(X)$ such that $F(g) - F(\tilde{g}) \leq \epsilon$. Now if we further set $g_m = \max(g_m, -n)$, and we apply the dominated convergence theorem, we can find $\tilde{g} = g_m$ such that

$$|F(g) - F(\tilde{g})| \leq \frac{\epsilon}{2}. \quad (46)$$

To get the Lipschitz property, we will first apply Prohorov's theorem, and we will find a compact set $K$ such that $\mu(X \setminus K), \nu(X \setminus K) \leq \epsilon'$. Now, we can approximate any function $\tilde{g}$ in $C_b(K)$, $\epsilon'$-uniformly by a function in $\mathcal{L}(K)$ through the formula

$$\tilde{g}_n(x) = \inf_{y \in K} (\tilde{g}(y) + nd(x, y)). \quad (47)$$

for sufficiently large $n$. By taking $n$ big enough we can also have that $e^{\tilde{g}_n}, e^{\tilde{g}}$, are at most $\epsilon'$-uniformly apart in $K$. This formula actually defines $\tilde{g}_n$ to be Lipschitz on the whole space $X$. We can further bound by using $\tilde{g}(x) = \max (\min (\tilde{g}_n(x), \|\tilde{g}\|_\infty), -\|\tilde{g}\|_\infty)$. Now we have

$$\left| \int_K \tilde{g}_n d\nu - \int_K \tilde{g} d\nu \right| \leq \epsilon', \quad \left| \int_{X \setminus K} \tilde{g}_n d\nu - \int_{X \setminus K} \tilde{g} d\nu \right| \leq 2\epsilon' \|\tilde{g}\|_\infty \quad (48)$$

Now by using the modulus of uniform continuity $\omega$ for the logarithm on $[e^{-\|\tilde{g}\|_\infty}, e^{\|\tilde{g}\|_\infty}]$ (or a simple mean value theorem), we get the following estimate $|F(\tilde{g}) - F(\tilde{g})| \leq 2\epsilon' + 2\epsilon' \|\tilde{g}\|_\infty + \omega(2\epsilon' e^{\|\tilde{g}\|_\infty})$. Now if $\epsilon'$ becomes sufficiently small we have $|F(\tilde{g}) - F(\tilde{g})| \leq \frac{\epsilon}{2}$, and by combining with (46) we get our claim. $\Box$

We conclude by providing a proof for Corollary 1.2.
Proof of Corollary 1.2. First assume that $\Phi(W_1(\mu, \nu)) \leq \phi(\mu)$. For $f \in [L(X)]_1$, we have

$$
\rho \left( t \int_X f d\nu - tf - \Phi^*(t) \right) = \sup_{\mu \in P_1(X)} \left[ t \int_X f d\nu - \Phi^*(t) \right] d\mu - \phi(\mu)
$$

$$
= \sup_{\mu \in P_1(X)} \left[ t \left( \int_X f d\nu - \int_X f d\mu \right) - \Phi^*(t) - \phi(\mu) \right]
$$

$$
= \sup_{\mu \in P_1(X)} \left[ \Phi(W_1(\mu, \nu)) - \phi(\mu) \right] \leq 0.
$$

Conversely if $\rho \left( t \int_X f d\nu - tf - \Phi^*(t) \right) \leq 0$, we have

$$
t \left( \int_X f d\nu - \int_X f d\mu \right) - \Phi^* (t) = \int_X \left( t \int_X f d\nu - tf - \Phi^*(t) \right) d\mu
$$

$$
\leq \rho \left( t \int_X f d\nu - tf - \Phi^*(t) \right) + \phi(\mu) \leq \phi(\mu).
$$

Taking the supremum over $f \in [L(X)]_1$, we get

$$
t W_1(\mu, \nu) - \Phi^* (t) \leq \phi(\mu)
$$

(50)

By taking the supremum over $t \geq 0$, we have $\Phi(W_1(\mu, \nu)) \leq \phi(\mu)$.

\[\square\]

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