Modified gravity and Space-Time-Matter theory

F. Darabi*
Department of Physics, Azarbaijan University of Tarbiat Moallem, Tabriz 53741-161, Iran

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Abstract

The correspondence between $f(R)$ theories of gravity and model theories explaining induced dark energy in a 5D Ricci-flat universe, known as the Space-Time-Matter theory (STM), is studied. It is shown that such correspondence may be used to interpret the four dimensional expressions, induced from geometry in 5D STM theories, in terms of the extra terms appearing in $f(R)$ theories of gravity. The method is demonstrated by providing an explicit example in which a given $f(R)$ is used to predict the properties of the corresponding 5D Ricci-flat universe. The accelerated expansion and the induced dark energy in a 5D Ricci-flat universe characterized by a big bounce is studied and it is shown that an arbitrary function $\mu(t)$ in the 5D solutions can be rewritten, in terms of the redshift $z$, as a new arbitrary function $F(z)$ which corresponds to the 4D curvature quintessence models.

1 Introduction

The recent distance measurements from the light-curves of several hundred type Ia supernovae [1, 2] and independently from observations of the cosmic microwave background (CMB) by the WMAP satellite [3] and other CMB experiments [4, 5] suggest strongly that our universe is currently undergoing a period of acceleration. This accelerating expansion is generally believed to be driven by an energy source called dark energy. The question of dark energy and the accelerating universe has therefore the focus of a large amount of activities in recent years. Dark energy and the accelerating universe have been discussed extensively from various point of views over the past few years [6, 7, 8]. In principle, a natural candidate for dark energy could be a small positive cosmological constant. One approach in this direction is to employ what is known as modified gravity where an arbitrary function of the Ricci scalar is added to the Einstein-Hilbert action. It has been shown that such a modification may account for the late time acceleration and the initial inflationary period in the evolution of the universe [9, 10]. Alternative approaches have also been pursued, a few example of which can be found in [11, 12, 13]. These schemes aim to improve the quintessence approach overcoming the problem of scalar field potential, generating a dynamical source for dark energy as an intrinsic feature. The goal would be to obtain a comprehensive model capable of linking the picture of the early universe to the one observed today, that is, a model derived from some effective theory of quantum gravity which, through an inflationary period would result in today accelerated Friedmann expansion driven by some $\Omega_\Lambda$-term. However, the mechanism responsible for this acceleration is not well understood and many authors introduce a mysterious cosmic fluid, the so called dark energy, to explain this

*email: f.darabi@azaruniv.edu
As was mentioned above, it has been shown that such an accelerated expansion could be the result of a modification to the Einstein-Hilbert action [15]. A scenario where the issue of cosmic acceleration in the framework of higher order theories of gravity in 4D is addressed can be found in [28]. One of the first proposals in this regard was suggested in [9] where a term of the form $R^{-1}$ was added to the usual Einstein-Hilbert action. In $f(R)$ gravity, Einstein equations possess extra terms induced from geometry which, when moved to the right hand side, may be interpreted as a matter source represented by the energy-momentum tensor $T_{\text{Curv}}$, see equation (5).

In a similar fashion, the Space-Time-Matter (STM) theory, discussed below, results in Einstein equations in 4D with some extra geometrical terms which may be interpret as induced matter. It therefore seems plausible to make a correspondence between the geometrical terms in STM and $T_{\text{Curv}}$ resulting in $f(R)$ gravity. We shall explore this idea to show that different choices of the parameter $\mu(t)$ in STM may correspond to different choices of $f(R)$ in curvature quintessence models in modified gravity.

The correspondence discussed above is based on the idea of extra dimensions. The idea that our world may have more than four dimensions is due to Kaluza [17], who unified Einstein’s theory of General Relativity with Maxwell’s theory of Electromagnetism in a 5D manifold. Since then, higher dimensional or Kaluza-Klein theories of gravity have been studied extensively [18] from different angles. Notable amongst them is the STM theory mentioned above, proposed by Wesson and his collaborators, which is designed to explain the origin of matter in terms of the geometry of the bulk space in which our 4D world is embedded, for reviews see [20]. More precisely, in STM theory, our world is a hypersurface embedded in a five-dimensional Ricci-flat ($R_{AB} = 0$) manifold where all the matter in our world can be thought of as being manifestations of the geometrical properties of the higher dimensional space. The fact that such an embedding can be done is supported by Campbell’s theorem [21] which states that any analytical solution of the Einstein field equations in $N$ dimensions can be locally embedded in a Ricci-flat manifold in $(N + 1)$ dimensions. Since the matter is induced from the extra dimension, this theory is also called the induced matter theory. Applications of the idea of induced matter or induced geometry can also be found in other situations [22]. The STM theory allows for the metric components to be dependent on the extra dimension and does not require the extra dimension to be compact. The sort of cosmologies stemming from STM theory is studied in [23, 24, 26].

In this paper we consider the correspondence between $f(R)$ gravity and STM theory. In section 2 we present a short review of 4D dark energy models in the framework of $f(R)$ gravity. In section 3 the field equations are solved in STM theory by fixing a suitable metric and the resulting geometric terms are interpreted as dark energy. The cosmological evolution in STM are considered in section 4. Section 5 deals with an example for a special form of $f(R)$. Conclusions are drawn in the last section.

## 2 Modified $f(R)$ Gravity

General coordinate invariance in the gravitational action, without the assumption of linearity, allows infinitely many additive terms to the Einstein-Hilbert action [25]

$$S = \int d^4x \sqrt{-g} \left[ c_0 R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\lambda\delta} R^{\mu\nu\lambda\delta} + \cdots \right] + S_m, \quad (1)$$

where $R$, $R_{\mu\nu}$ and $R_{\mu\nu\lambda\delta}$ are Ricci scalar, Ricci tensor and Reimann tensor, respectively and $S_m$ is the action for the matter fields. The fourth order term $R_{\mu\nu\lambda\delta} R^{\mu\nu\lambda\delta}$ may be neglected as a consequence of the Gauss-Bonnet theorem. The action (1) is not canonical because the Lagrangian function contains derivatives of the canonical variables of order higher than one. This means that, not only do we expect higher order field equations, but also the validity of the Euler-Lagrange equations is compromised. This problem is particularly difficult in the general case, but can be solved for specific metrics. In homogeneous and isotropic spacetimes, the Lagrangian in (1) can be further simplified. Specifically,
the variation of the term $R_{\mu\nu}R^{\mu\nu}$ can always be rewritten in terms of the variation of $R^2$. Thus, the effective fourth order Lagrangian in cosmology contains only powers of $R$ and we can suppose, without loss of generality, that the general form for a non-linear Lagrangian is given by

$$S = \int d^4x \sqrt{-g} f(R) + S_m, \quad (2)$$

where $f(R)$ is a generic function of the Ricci scalar$^1$. Variation with respect to the metric $g_{\mu\nu}$ leads to the field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = f'(R)g^{\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}) + \tilde{T}_m^{\mu\nu}, \quad (3)$$

where

$$\tilde{T}_m^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \quad (4)$$

and the prime denotes a derivative with respect to $R$. It is easy to check that standard Einstein equations are immediately recovered if $f(R) = R$. When $f'(R) \neq 0$ the equation (3) can be recast in the more expressive form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{\text{Curv}}_{\mu\nu} + T^m_{\mu\nu}, \quad (5)$$

where an stress-energy tensor has been defined for the curvature contribution

$$T^{\text{Curv}}_{\mu\nu} = \frac{1}{f'(R)} \left\{ \frac{1}{2}g_{\mu\nu} \left[ f(R) - Rf'(R) \right] + f'(R)g^{\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}) \right\}, \quad (6)$$

and

$$T^m_{\mu\nu} = \frac{1}{f'(R)} \tilde{T}_m^{\mu\nu}, \quad (7)$$

is an effective stress-energy tensor for standard matter. This step is conceptually very important since a gravity model with a complicated structure converts to a model in which the gravitational field has the standard GR form with a source made up of two fluids: perfect fluid matter and an effective fluid (curvature fluid) that represents the non-Einsteinian part of the gravitational interaction.

We now consider the Robertson-Walker metric for the evolution of the universe

$$ds^2 = dt^2 - a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad (8)$$

where $k$ is the curvature of the space, namely, $k = 0, 1, -1$ for the flat, closed and open universes respectively. Substituting the above metric with $k = 0$ in equation (5) we obtain the 4D, spatially flat Friedmann equations as follows

$$H^2 = \frac{1}{3} (\rho_m + \rho_{\text{Curv}}), \quad (9)$$

and

$$\dot{H} = -\frac{1}{2} \left[ (\rho_m + p_m) + \rho_{\text{Curv}} + p_{\text{Curv}} \right], \quad (10)$$

where a dot represents derivation with respect to time. Such a universe is dominated by a barotropic perfect fluid with the equation of state (EOS) given by $p_m = w_m \rho_m$ ($w_m = 0$ for pressureless cold dark matter and $w_m = 1/3$ for radiation) and a spatially homogenous curvature quintessence.

The energy density and pressure of the curvature quintessence are

$$p_{\text{Curv}} = \frac{1}{f'(R)} \left\{ 2 \left( \frac{\dot{a}}{a} \right) \dot{R} f''(R) + \ddot{R} f'(R) + \dot{R}^2 f''(R) - \frac{1}{2} \left[ f(R) - Rf'(R) \right] \right\}, \quad (11)$$

$^1$We use units such that $8\pi G_N = c = \hbar = 1$. 

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\[ \rho_{\text{Curv}} = \frac{1}{f'(R)} \left\{ \frac{1}{2} \left[ f(R) - R f'(R) \right] - 3 \left( \frac{\dot{a}}{a} \right) \dot{R} f''(R) \right\}, \] (12)

respectively. The equation of state of the curvature quintessence is

\[ w_{\text{Curv}} = \frac{p_{\text{Curv}}}{\rho_{\text{Curv}}}. \] (13)

Recently, cosmological observations have indicated that our universe is undergoing an accelerated expanding phase. This could be due to the vacuum energy or dark energy which dominates our universe against other forms of matter such as dark matter and Baryonic matter. We thus concentrate on the vacuum sector i.e. \( \rho_m = p_m = 0 \), from which the evolution equation of curvature quintessence becomes

\[ \dot{\rho}_{\text{Curv}} + 3H (\rho_{\text{Curv}} + p_{\text{Curv}}) = 0, \] (14)

which yields

\[ \rho_{\text{Curv}}(z) = \rho_{0\text{Curv}} \exp \left[ 3 \int^z_0 (1 + w_{\text{Curv}}) d\ln(1 + z) \right] \]

\[ \equiv \rho_{0\text{Curv}} E(z), \] (15)

where, \( 1 + z = \frac{a_0}{a} \) is the redshift and the subscript 0 denotes the current value. In terms of the redshift, the first Friedmann equation can be written as

\[ H(z)^2 = H_0^2 \Omega_{0\text{Curv}} E(z), \] (16)

where \( \Omega_{0\text{Curv}} \) and \( H_0 \) are the current values of the dimensionless density parameter and Hubble parameter, respectively. Equation (16) is the Friedmann equation in terms of redshift, \( z \), which is suitable for cosmological observations. In fact, equations (16) and (28), obtained in section 4, are the cosmological connections between \( f(R) \) gravity and STM theory.

3 Space-Time-Matter theory

According to the old suggestion of Kaluza and Klein the 5D vacuum Kaluza-Klein equations can be reduced under certain conditions to the 4D vacuum Einstein equations plus the 4D Maxwell equations. Recently, the idea that our four-dimensional universe might have emerged from a higher dimensional spacetime is receiving much attention [19]. One current interest is to find out in a more general way how the 5D field equations relate to the 4D ones. In this regard, a proposal was made recently by Wesson [20] in that the 5D Einstein equations without sources \( R_{AB} = 0 \) (the Ricci flat assumption) may be reduced to the 4D ones with sources \( G_{ab} = 8\pi G T_{ab} \), provided an appropriate definition is made for the energy-momentum tensor of matter in terms of the extra part of the geometry. Physically, the picture behind this interpretation is that curvature in \( (4 + 1) \) space induces effective properties of matter in \( (3 + 1) \) spacetime. This idea is known as space time matter (STM) or modern KaluzaKlein theory.

In this popular non-compact approach to Kaluza-Klein gravity, the gravitational field is unified with its source through a new type of 5D manifold in which space and time are augmented by an extra non-compact dimension which induces 4D matter within four dimensional universe. Unlike the usual Kaluza-Klein theory in which a cyclic symmetry associated with the extra dimension is assumed, the new approach removes the cyclic condition and derivatives of the metric with respect to the extra coordinate are retained. This induces non-trivial matter on the hypersurface of \( l = \text{constant} \). This theory basically is guaranteed by an old theorem of differential geometry due to Campbell [21].

In the context of STM theory, a class of exact 5D cosmological solutions has been investigated and discussed in [27]. This solution was further pursued in [23] where it was shown to describe a
A cosmological model with a big bounce as opposed to the ubiquitous big bang. The 5D metric of this solution reads

$$dS^2 = B^2 dt^2 - A^2 \left( \frac{dr^2}{1 - k r^2} + r^2 d\Omega^2 \right) - dy^2,$$

(17)

where $d\Omega^2 \equiv (d\theta^2 + \sin^2 \theta d\phi^2)$ and

$$A^2 = (\mu^2 + k) y^2 + 2\nu y + \frac{\nu^2 + K}{\mu^2 + k},$$

$$B = \frac{1}{\mu} \frac{\partial A}{\partial t} = \frac{\dot{A}}{\mu}.$$

(18)

Here $\mu = \mu(t)$ and $\nu = \nu(t)$ are two arbitrary functions of $t$, $k$ is the 3D curvature index ($k = \pm 1, 0$), and $K$ is a constant. This solution satisfies the 5D vacuum equation $R_{AB} = 0$. The Kretschmann curvature scalar

$$I_3 = R_{ABCD} R^{ABCD} = \frac{72 K^2}{A^8},$$

(19)

shows that $K$ determines the curvature of the 5D manifold. Such a solution was considered in [27] with a different notation.

Using the 4D part of the 5D metric (17) to calculate the 4D Einstein tensor, we obtain

$$^{(4)}G_0^0 = \frac{3 (\mu^2 + k)}{A^2},$$

$$^{(4)}G_1^1 = ^{(4)}G_2^2 = ^{(4)}G_3^3 = \frac{2 \mu A}{A A} + \frac{\mu^2 + k}{A^2}.$$

(20)

As was mentioned earlier, since the recent observations show that the universe is currently going through an accelerated expanding phase, we assume that the induced matter contains only dark energy with $\rho_{DE}$, i.e. $\rho_m = 0$. We then have

$$\frac{3 (\mu^2 + k)}{A^2} = \rho_{DE},$$

(21)

$$\frac{2 \mu A}{A A} + \frac{\mu^2 + k}{A^2} = -p_{DE}.$$  

(22)

From equations (21) and (22), one obtains the EOS of dark energy

$$w_{DE} = \frac{p_{DE}}{\rho_{DE}} = -\frac{2 \mu A (\mu^2 + k)/(A^2)}{3 (\mu^2 + k)/(A^2)}.$$

(23)

The Hubble and deceleration parameters are given in [23, 26] and can be written as

$$H \equiv \frac{\dot{A}}{AB} = \frac{\mu}{A},$$

(24)

and

$$q(t, y) = -A \frac{d^2 A}{dt^2} \left( \frac{dA}{dt} \right)^2 = -\frac{\dot{A}^2}{\mu A},$$

(25)

from which we see that $\dot{\mu}/\mu > 0$ represents an accelerating universe while $\dot{\mu}/\mu < 0$ represents a decelerating one. The function $\mu(t)$ therefore plays a crucial role in defining the properties of the universe at late times.
4 Correspondence between modified $f(R)$ gravity and STM theory

In this section we will concentrate on the predictions of the cosmological evolution in the spatially flat case ($k = 0$). To avoid having to specify the form of the function $\nu(t)$, we change the parameter $t$ to $z$ and use $A_0 / A = 1 + z$ and define $\mu_0^2 / \mu^2 = F(z)$, noting that $F(0) \equiv 1$. We then find that equations (23)-(25) reduce to

$$w_{DE}(z) = -\frac{1 + (1 + z) d \ln F(z) / dz}{3},$$

(26)

and

$$q_{DE}(z) = \frac{1 + 3 \Omega_{DE} w_{DE}}{2} = - \frac{(1 + z) d \ln F(z)}{2 \frac{dz}{dz}}.$$  

(27)

There is an arbitrary function $\mu(t)$ in the present 5D model. Different choices of $\mu(t)$ may correspond to different choices of $f(R)$ in curvature quintessence models in modified gravity. Various choices of $\mu(t)$ correspond to the choices of $F(z)$. This enables us to look for the desired properties of the universe via equations (26) and (27). Using these definitions, the Friedmann equation becomes

$$H^2 = H_0^2 (1 + z)^2 F(z)^{-1}.$$  

(28)

This would allow us to use the supernovae observational data to constrain the parameters contained in the model or the function $F(z)$. By comparing equation (28) with equation (16), we find that there exists a correspondence between the functions $f(R)$ and $F(z)$. We thus take $F(z)$ as

$$F(z) = (1 + z)^2 [\Omega_{0,\text{curv}} E(z)]^{-1}.$$  

(29)

According to (15), it is easy to see that the function $E(z)$ is determined by the particular choice for $f(R)$ which, in turn, determines the function $F(z)$ through equation (29). The evolution of the density components and the EOS of dark energy may now be derived. To this end, we must determine the functional form of $f(R)$. Thus, for example, we choose $f(R)$ as a generic power law of the scalar curvature and assume for the scale factor a power law solution in 4D, investigated in [28]. Therefore

$$f(R) = f_0 R^n, \quad a(t) = a_0 \left( \frac{t}{t_0} \right)^{\alpha}.$$  

(30)

The interesting cases are for the values of $\alpha$ satisfying $\alpha > 1$ which would lead to an accelerated expansion of our universe. Let us now concentrate on the case $\rho_m = 0$. Inserting equation (30) into the dynamical system (9) and (10), for a spatially flat space-time we obtain an algebraic system for parameters $n$ and $\alpha$

$$\begin{cases} \alpha \left[ \alpha(n - 2) + 2n^2 - 3n + 1 \right] = 0, \\ \alpha \left[ n^2 - n + 1 + \alpha(n - 2) \right] = n(n - 1)(2n - 1), \end{cases}$$

(31)

from which the allowed solutions are

$$\alpha = 0 \rightarrow n = 0, \frac{1}{2}, 1,$$

$$\alpha = \frac{2n^2 - 3n + 1}{2 - n}, \forall n, \quad n \neq 2.$$  

(32)

The solutions with $\alpha = 0$ are not interesting since they provide static cosmologies with a non-evolving scale factor. Note that this result matches the standard General Relativity result $n = 1$ in the absence
of matter. On the other hand, the cases with generic $\alpha$ and $n$ furnish an entire family of significant cosmological models. Using equations (11) and (12) we can also deduce the equation of state for the family of solutions $\alpha = \frac{2n^2 - 3n + 1}{2 - n}$ as

$$w_{\text{Curv}}(n) = - \left( \frac{6n^2 - 7n - 1}{6n^2 - 9n + 3} \right),$$

where $w_{\text{Curv}} \to -1$ as $n \to \infty$. This shows that an infinite $n$ is compatible with recovering an infinite cosmological constant. Thus, using equation (33), $E(z)$ and $F(z)$ are given by

$$E(z) = (1 + z)^3 \left[ \frac{-2n^2 + 2n + 1}{6n^2 - 9n + 3} \right],$$

$$F(z) = (1 + z)^2 \left[ \Omega_{0\text{Curv}} (1 + z)^3 \left[ \frac{-2n^2 + 2n + 1}{6n^2 - 9n + 3} \right] \right]^{-1}.$$  

Now, using the above equations, equations (26) and (27) can be written as

$$w_{DE}(n) = - \left( \frac{6n^2 - 7n - 1}{6n^2 - 9n + 3} \right),$$

and

$$q_{DE}(n) = \frac{A\dot{\mu}}{\mu A} = - \frac{2n^2 + 2n + 1}{2n^2 - 3n + 1}.$$  

Therefore, within the context of the present investigation, the accelerating, dark energy dominated universe, can be obtained by using the correspondence between $F(z)$ and $f(R)$ in modified gravity theories. We observe that in STM theory, 5D dark energy cosmological models correspond to 4D curvature quintessence models. This result is consistent with the correspondence between exact solutions in Kaluza-Klein gravity and scalar tensor theory [29]. Note that, as is well known, with a suitable conformal transformation, $f(R)$ gravity reduces to the scalar tensor theory.

From equations (34) and (35), we can rewrite equation (28) as

$$h(z, n) = \Omega_{0\text{Curv}} (1 + z)^3 \left[ \frac{-2n^2 + 2n + 1}{6n^2 - 9n + 3} \right],$$

where $h(z, n) \equiv \frac{H(z)^2}{\Omega_0}$ and the contribution of ordinary matter has been neglected. Figure 1 shows the behavior of $h(n)$ as a function of $n$ for $z \sim 1.5$ and $\Omega_{0\text{Curv}} \simeq 0.70$. As can be seen, for $n \to \pm \infty$ and $z \to 0$ we have $h(z, n) \to \Omega_{0\text{Curv}}$, that is, the universe finally approaches the curvature dominant state, thus undergoing an accelerated expanding phase. Figure 2 shows the behavior of $h(z)$ as a function of $z$ for $n = 2, 10, -10$ and $\Omega_{0\text{Curv}} \simeq 0.70$. We see that for small $z$, $h(z) \to 0.70$. Thus, we have obtained late-time accelerating solutions only by using the correspondence between $f(R)$ gravity and STM theory. Here, we have interpreted the properties of 5D Ricci-flat cosmologies by dark energy models in modified gravity.

## 5 Conclusions

In this paper we have studied the correspondence between modified $f(R)$ gravity and Space-Time-Matter theory by investigation of the present accelerated expanding phase of the universe using a general class of 5D cosmological models, characterized by a big bounce as opposed to a big bang, which is the standard prediction in 4D cosmological models. Such an exact solution contains two arbitrary functions, $\mu(t)$ and $\nu(t)$, which are analogous to different forms of $f(R)$ in curvature quintessence
Figure 1: Behavior of $h(n)$ as a function of $n$ for $z \sim 1.5$ and $\Omega_{0\text{curv}} \simeq 0.70$. An accelerating universe occurs for $n \lesssim -2$ and $n \gtrsim 2$.

Figure 2: Behavior of $h(z)$ as a function of $z$ for $n = 2$ (solid line), $n = 10$ (dashed line), $n = -10$ (dot-dashed line) and $\Omega_{0\text{curv}} \simeq 0.70$. Note that for $n = 2, 10, -10$, $z \to 0$ and $h(z) \to 0.7$. 
models. Also, once the forms of the arbitrary functions are specified, the characteristic parameters determining the evolution of our universe are specified. We have noted that the correspondence between the functions $F(z)$ and $f(R)$ plays a crucial role and defines the form of the function $F(z)$. Finally, by taking a specific form for $f(R)$ we obtained solutions that describe the late-time acceleration of the universe. Explicitly, the induced dark energy and the resulting accelerated expansion in a 5D Ricci-flat universe is studied and it is shown that an arbitrary function $\mu(t)$ in the 5D solutions can be rewritten as a new arbitrary function $F(z)$ which corresponds to the 4D curvature quintessence models.

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