The blocks and weights of finite special linear and unitary groups*

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Abstract

This paper has two main parts. Firstly, we give a classification of the $\ell$-blocks of finite special linear and unitary groups $\mathrm{SL}_n(eq)$ in the non-defining characteristic $\ell \geq 3$. Secondly, we describe how the $\ell$-weights of $\mathrm{SL}_n(eq)$ can be obtained from the $\ell$-weights of $\mathrm{GL}_n(eq)$ when $\ell \nmid \gcd(n, q - \epsilon)$, and verify the Alperin weight conjecture for $\mathrm{SL}_n(eq)$ under the condition $\ell \nmid \gcd(n, q - \epsilon)$. As a step to establish the Alperin weight conjecture for all finite groups, we prove the inductive blockwise Alperin weight condition for any unipotent $\ell$-block of $\mathrm{SL}_n(eq)$ if $\ell \nmid \gcd(n, q - \epsilon)$.

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1 Introduction

Let $q = p^\ell$ be a power of a prime $p$ and $\mathrm{SL}_n(eq)$ with $\epsilon = \pm 1$ be the finite special linear (when $\epsilon = 1$) and unitary (when $\epsilon = -1$) group ($\mathrm{SL}_n(-q)$ is understood as $\mathrm{SU}_n(q)$, for definitions, see Section 2.4). Let $\ell$ be a prime number different from $p$. We are interested in parametrizing $\ell$-blocks of $\mathrm{SL}_n(eq)$. In [19] and [8], the authors had classified the $\ell$-blocks of $\mathrm{GL}_n(eq)$ and $\mathrm{SL}_n(eq)$. In this paper, we first determine when the label in [19] and [8] is the label in [25] for an $\ell$-block of $\mathrm{GL}_n(eq)$ and then give the number of $\ell$-blocks of $\mathrm{SL}_n(eq)$ covered by $B$. Thus we obtain a classification of $\ell$-blocks of $\mathrm{SL}_n(eq)$ when $\ell$ is odd (see Remark 4.13).

The Alperin weight conjecture relates for a prime $\ell$ information about a finite group $G$ to properties of $\ell$-local subgroups of $G$, that is, normalizers of $\ell$-subgroups of $G$. For a finite group $G$ and a prime $\ell$, an $\ell$-weight means a pair $(R, \varphi)$, where $R$ is a $\ell$-subgroup of $G$ and $\varphi \in \text{Irr}(N_{G}(R))$ with $R \subseteq \ker \varphi$ is of $\ell$-defect zero viewed as a character of $N_{G}(R)/R$. When such a character $\varphi$ exists, $R$ is necessarily an $\ell$-radical subgroup of $G$. For an $\ell$-block $B$ of $G$, a weight $(R, \varphi)$ is called a $B$-weight if $\text{bl}_{R}(\varphi) = B$, where $\text{bl}_{R}(\varphi)$ is the $\ell$-block of $N_{G}(R)$ containing $\varphi$. We denote by $W_{R}(B)$ the set of all $G$-conjugacy classes of $B$-weights. In [11], Alperin gave the following conjecture.

Conjecture 1.1 (Alperin). Let $G$ be a finite group, $\ell$ a prime. If $B$ is an $\ell$-block of $G$, then $|W_{R}(B)| = |\text{IBr}_{R}(B)|$.

The (blockwise) Alperin weight Conjecture [11] (BAWC) was proved by I. M. Isaacs and G. Navarro [24] for $\ell$-solvable groups. It was also shown to hold for groups of Lie type in defining characteristic (see [11]). In [2], [3], [4] and [5], the authors gave a combinatorial description for the $\ell$-weights of general linear and unitary groups, and thus proved the (BAWC) for general linear and unitary groups for any prime. In this paper we give a description of the $\ell$-weights of special linear and unitary groups $\mathrm{SL}_n(eq)$ with the assumption $\ell \nmid \gcd(n, q - \epsilon)$ (see Remark 5.14). Here, the $\ell$-weights of $\mathrm{SL}_n(eq)$ are obtained from the $\ell$-weights of $\mathrm{GL}_n(eq)$. We will prove the following statement.

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Theorem 1.2. Let $X \in \{\text{SL}_n(q), \text{SU}_n(q)\}$ and $\ell \nmid |Z(X)|$. Then the Alperin weight Conjecture [7] holds for $X$.

Even though (BAWC) has been verified in many particular instances, it has not been possible so far to find a general proof for arbitrary finite groups. In the recent past, the conjecture has been reduced to certain (stronger) statements about finite (quasi-)simple groups (see [35] and [39]). More precisely, it was shown that in order for the (BAWC) to hold for all finite groups, it is sufficient that all non-abelian finite simple groups satisfy a system of conditions, which is called the inductive blockwise Alperin weight (iBAW) condition. In this paper we will use a version of the (iBAW) condition (see Definition 2.5) which is given in [28].

The (iBAW) condition has been verified for some cases, such as many of the sporadic groups, simple alternating groups and any prime, simple groups of Lie type and the defining characteristic. But for non-defining characteristic, only a few simple groups of Lie type have been proved to satisfy the (iBAW) condition (see [14], [33] and [39]).

It seems that there is no general method yet to verify the (iBAW) condition for finite simple groups of Lie type and the non-defining characteristic, even for simple groups of type $A$. Using the description of $\ell$-weights of $\text{GL}_n(eq)$ in [2], [4], [4] and [5], in [30], the authors proved that if the pair $(n,q)$ is such chosen that the outer automorphism groups of $\text{PSL}_n(eq)$ is cyclic, then $\text{PSL}_n(eq)$ satisfies the (iBAW) condition for any prime. In this paper, we will consider the (iBAW) condition for the unipotent blocks of $\text{SL}_n(eq)$ without any restriction for $n$ and $q$. Our results are the following:

Theorem 1.3. Let $X \in \{\text{SL}_n(q), \text{SU}_n(q)\}$ and $\ell \nmid |Z(X)|$. Suppose that $b$ is a unipotent $\ell$-block of $X$, then the inductive blockwise Alperin weight condition (see Definition 2.5) holds for $b$.

For the general $\ell$-blocks, we have:

Proposition 1.4. Let $q = p^f$ be a power of a prime $p$ and $\ell$ a prime different from $p$. Assume that $X \in \{\text{SL}_n(q), \text{SU}_n(q)\}$ such that $\gcd(f,2|Z(X)|) = 1$, $\ell \nmid |Z(X)|$ and $2 \nmid |Z(X)|$. Then there is a blockwise bijection between the $\ell$-Brauer characters of $X$ and the $\ell$-weights of $X$ which is $\text{Aut}(X)$-equivariant.

In particular, the conditions (i) and (ii) of Definition 2.5 hold for any $\ell$-block of $X$.

The paper is built up as follows: In Section 2 we introduce the general notation around characters, weights and general linear and unitary groups. In Section 3 we recall the results of [26] and [16] about irreducible Brauer characters of special linear and unitary groups. Then we compare the labeling of blocks of general linear and unitary groups in [19] and [8] with the labeling given by $\epsilon$-Jordan-cuspidal-pairs in [12] and [25], and then classify the blocks of special linear and unitary groups in non-defining characteristic in Section 4. In Section 5 we give a description of weights of special linear and unitary groups in non-defining characteristic and prove Theorem 1.2 and Proposition 1.4. Section 6 gives the extendibility of weight characters of unipotent blocks of special linear and unitary groups in non-defining characteristic, while Section 7 proves Theorem 1.3.

2 Notations and preliminaries

In this section we establish the notation around groups and characters that is used throughout this paper.

2.1 Clifford theory

Notation. The cardinality of a set, or the order of a finite group, $X$, is denoted by $|X|$. If a group $A$ acts on a finite set $X$, we denote by $A_x$ the stabilizer of $x \in X$ in $A$, analogously we denote by $A_{X'}$ the setwise stabilizer of $X' \subseteq X$. Let $\ell$ be a prime.

If $A$ acts on a finite group $G$ by automorphisms, then there is a natural action of $A$ on $\text{Irr}(G) \cup \text{IBr}_\ell(G)$ given by $a \cdot \chi(g) = \chi^a(g) = \chi(g^{a^\ell})$ for every $g \in G$, $a \in A$ and $\chi \in \text{Irr}(G) \cup \text{IBr}_\ell(G)$. For $P \leq G$ and $\chi \in \text{Irr}(G) \cup \text{IBr}_\ell(G)$, we denote by $A_{\chi|P}$ the stabilizer of $\chi$ in $A_P$. 

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We denote the restriction of $\chi \in \text{Irr}(G) \cup \text{IBr}(G)$ to some subgroup $H \leq G$ by $\text{Res}_H^G \chi$, while $\text{Ind}_H^G \chi$ denotes the character induced from $\psi \in \text{Irr}(H) \cup \text{IBr}(H)$ to $G$. For $N \leq G$ we similarly identify the characters of $G/N$ with the characters of $G$ whose kernel contains $N$.

For $N \leq G$, and $\chi \in \text{Irr}(G) \cup \text{IBr}(G)$, we denote by $\kappa_N^G(\chi)$ the number of irreducible constituents of $\text{Res}_N^G(\chi)$ forgetting multiplicities. Let $B$ be an $\ell$-block of $G$, we denote by $\kappa_B^G(B)$ the number of $\ell$-blocks of $N$ covered by $B$.

**Lemma 2.1.** Suppose that $G$ is a finite group and $N \leq G$ satisfies that $G/N$ is cyclic.

(i) Let $\chi \in \text{Irr}(G)$ and $\theta \in \text{Irr}(N \mid \chi)$, then every character in $\text{Irr}(G \mid \theta)$ has the form $\chi \eta$ for some $\eta \in \text{Irr}(G/N)$, and $\kappa_N^G(\chi)$ is equal to the cardinality of the set $\{\eta \in \text{Irr}(G/N) \mid \chi \eta = \chi\}$.

(ii) Let $\psi \in \text{IBr}(G)$ and $\varphi \in \text{IBr}(N \mid \chi)$, then every $\ell$-Brauer character in $\text{IBr}(G \mid \varphi)$ has the form $\psi \tau$ for some $\tau \in \text{IBr}(G/N)$ and the $\ell'$-part of $\kappa_N^G(\psi)$ is equal to the cardinality of the set $\{\tau \in \text{IBr}(G/N) \mid \psi \tau = \psi\}$.

**Proof.** This is a direct consequence of Clifford theory (see, for example, [23] §19 and [34] Chap. 8). For (ii), see also [26] Lem. 3.3 and 3.8. □

We will make use of the following result:

**Lemma 2.2** ([27] Lem. 2.3]. Let $K$ be a normal subgroup of finite group $A$ and $H$ a subgroup of $A$. Let $M = K \cap H$. Suppose that $b$ is an $\ell$-block of $M$ and $c$ is an $\ell$-block of $H$ such that $c$ covers $b$. If both $b^K$ and $c^A$ are defined, then $c^A$ covers $b^K$.

For a finite group $H$, we denote by $\text{Rad}_1(H)$ the set of all $\ell$-radical subgroups of $H$ and $\text{Rad}_c(H)/\sim_H$ a complete set of representatives of $H$-conjugacy classes of $\ell$-radical subgroups of $H$.

**Lemma 2.3.** Let $H$ be a finite group, $N \leq H$ and $\ell$ a prime.

(i) If $R$ is an $\ell$-radical subgroup of $H$, then $R \cap N$ is an $\ell$-radical subgroup of $N$.

(ii) The map $\text{Rad}_1(H) \to \text{Rad}_1(N)$, $R \mapsto R \cap N$ is surjective.

(iii) Let $S$ be an $\ell$-radical subgroup of $N$. Assume that there is only one $\ell$-radical subgroup $R$ of $H$ such that $R \cap N = S$. Then $R = \text{O}_\ell(\text{N}_H(S))$ and $\text{N}_H(S) = \text{N}_H(R)$.

**Proof.** (i) is [36] (2.1). For (ii), if $S$ is an $\ell$-radical subgroup of $N$, let $R = \text{O}_\ell(\text{N}_H(S))$, then we claim that $R$ is an $\ell$-radical subgroup of $H$ with $R \cap N = S$. Indeed, $R \cap N$ is a normal $\ell$-subgroup of $N$. Thus $S \leq R \cap N$. Then $\text{N}_H(R) \leq \text{N}_H(S)$. Now $R \leq \text{N}_H(S)$, so $\text{N}_H(R) = \text{N}_H(S)$, then $R$ is an $\ell$-radical subgroup of $H$ and the claim holds and (ii) holds and (iii) easily follows. □

**Lemma 2.4.** Let $H$ be a finite group, $N \leq H$ and $\ell$ a prime. Assume that $H/N$ is cyclic and the map $\text{Rad}_c(H) \to \text{Rad}_c(N)$, $R \mapsto R \cap N$ is bijective.

(i) If $(R, \varphi)$ is an $\ell$-weight of $H$, then $(S, \psi)$ is an $\ell$-weight of $N$ for $S = R \cap N$ and any $\psi \in \text{Irr}(N/S) \mid \varphi$.

(ii) Let $(S, \psi)$ be an $\ell$-weight of $N$, and $R \in \text{Rad}_c(H)$ such that $R \cap N = S$. Assume further that $\ell \nmid |\text{N}_H(S)|$. Then there exists an $\ell$-weight $(R, \varphi)$ of $H$ such that $\varphi \in \text{Irr}(\text{N}_H(R) \mid \psi)$.

**Proof.** Let $R$ be an $\ell$-radical subgroup of $H$, $S = R \cap N$. By Lemma 2.3(iii), $\text{N}_H(R) = \text{N}_H(S)$ and then $\text{N}_H(S) = \text{N}_H(R) \cap N$. By the assumptions, $\text{N}_H(R) \mid \text{N}_H(S)$ is cyclic. Now $\text{N}_H(S)/R \cong \text{N}_H(S)/S$, so there is a bijection $\Psi : \text{Irr}(\text{N}_H(S) \mid 1_S) \to \text{Irr}(\text{N}_H(S) \mid 1_R)$ such that if $\psi \in \text{Irr}(\text{N}_H(S) \mid 1_S)$ and $\psi' = \Psi(\psi)$, then $\psi'$ is an extension of $\psi$. Obviously, every character in $\text{Irr}(\text{N}_H(S) \mid 1_S)$ is $R$-invariant.

(i). Let $(R, \varphi)$ be an $\ell$-weight of $H$ and $\psi \in \text{Irr}(\text{N}_H(S) \mid \varphi)$, then $\text{Res}_{N/H}(\varphi)$ is multiplicity-free. So $\varphi(1) = t \psi(1)$ with $t = [\text{N}_H(R) : \text{N}_H(R) \psi]$. Hence $t \mid [\text{N}_H(R) : \text{N}_H(S) \psi]$. Notice that $\varphi(1)_t = [\text{N}_H(R) / R]_t$, $\varphi(1)_t$
so \( \psi(1)_\ell \geq |N_N(S)/S|_\ell \). Thus \( \psi(1)_\ell = |N_N(S)/S|_\ell \). Hence \( \psi \) is of \( \ell \)-defect zero as a character of \( N_N(S)/S \) and then \( (S, \psi) \) is an \( \ell \)-weight of \( N \).

(ii). Let \( (S, \psi) \) be an \( \ell \)-weight of \( N \), \( \psi' = \Psi(\psi) \) and \( \varphi \in \operatorname{Irr}(N_H(R) \mid \psi') \). Then the proof is similar to (i).

\( \square \)

2.2 Cuspidal pairs

We will make use of the classification of the blocks of finite groups of Lie type in non-defining characteristic in [12] and [25]. Algebraic groups are usually denoted by boldface letters. Let \( q \) be a power of some prime number \( p \) and \( \mathbb{F}_q \) the field of \( q \) elements. Suppose that \( G \) is a connected reductive linear algebraic group over the algebraic closure of \( \mathbb{F}_q \) and \( F : G \rightarrow G \) a Frobenius endomorphism endowing \( G \) with an \( \mathbb{F}_q \)-structure. The group of rational points \( G^F \) is finite. Let \( G^* \) be dual to \( G \) with corresponding Frobenius endomorphism also denoted \( F \).

Let \( d \) be a positive integer. We will make use of the terminology of Sylow \( d \)-theory (see for instance [9]). For an \( F \)-stable maximal torus \( T \) of \( G \), denotes \((T)_d \) its Sylow \( d \)-torus. An \( F \)-stable Levi subgroup \( L \) of \( G \) is called \( d \)-split if \( L = C_G(Z^d(L)) \), and \( \zeta \in \operatorname{Irr}(L^F) \) is called \( d \)-cuspidal if \( ^*R^L_{M,P}(\zeta) = 0 \) for all proper \( d \)-split Levi subgroups \( M < L \) and any parabolic subgroup \( P \) of \( L \) containing \( M \) as Levi complement.

Let \( s \in G^{*F} \) be semisimple, following [12] and [25], we say \( \chi \in \mathcal{E}(G^F, s) \) is \( d \)-Jordan-cuspidal if

- \( Z^d(C^*_G(s))_d = Z^d(G^*_d) \); and
- \( \chi \) corresponds under Jordan decomposition (see, for example, [31] Prop. 5.1) to the \( C_{G^*}(s)^F \)-orbit of a \( d \)-cuspidal unipotent character of \( C_{G^*}(s)^F \).

If \( L \) is a \( d \)-split Levi subgroup of \( G \) and \( \zeta \in \operatorname{Irr}(L^F) \) is \( d \)-Jordan-cuspidal, then \((L, \zeta) \) is called a \( d \)-Jordan-cuspidal pair of \( G \).

Let \( \ell \) be a prime number different from \( p \). Now we define an integer \( e_0 = e_0(q, \ell) \), which is denoted by “\( e \)” in [25] (in this paper, we will use “\( e \)” for another integer, see Section 2.4):

\[
e_0 = e_0(q, \ell) = \text{multiplicative order of } q \text{ modulo } \begin{cases} \ell, & \text{if } \ell > 2, \\ 4, & \text{if } \ell = 2. \end{cases} \tag{2.1}
\]

For a semisimple \( \ell' \)-element \( s \) of \( G^{*F} \), we denote by \( \mathcal{E}_\ell(G^F, s) \) the union of all Lusztig series \( \mathcal{E}(G^F, s^t) \), where \( t \in G^{*F} \) is a semisimple \( \ell' \)-element commuting with \( s \). By [10], the set \( \mathcal{E}_\ell(G^F, s) \) is a union of \( \ell \)-blocks of \( G^F \).

Also, we denote by \( \mathcal{E}(G^F, \ell') \) the set of irreducible characters of \( G^F \) lying in a Lusztig series \( \mathcal{E}(G^F, s) \), where \( s \in G^{*F} \) is a semisimple \( \ell' \)-element.

The paper [12] gave a label for arbitrary \( \ell \)-blocks of finite groups of Lie type for \( \ell \geq 7 \) and it was generalised in [25] to its largest possible generality. Under the condition of [25] Thm. A (e)], the set of \( G^F \)-conjugacy classes of \( e_0 \)-Jordan-cuspidal pairs \((L, \zeta) \) of \( G \) such that \( \zeta \in \mathcal{E}(L^F, \ell') \), is a labeling set of the \( \ell \)-blocks of \( G^F \).

2.3 The inductive blockwise Alperin weight conditions

**Notation.** For a finite group \( H \) and a prime \( \ell \), we denote by

- \( dz_\ell(H) \) the set of \( \ell \)-defect zero characters of \( H \);
- \( \chi^\circ \) the restriction of \( \chi \) to the set of all \( \ell' \)-elements of \( H \) for \( \chi \in \operatorname{Irr}(H) \); and
- \( b_{\ell}(\varphi) \) the \( \ell \)-block of \( H \) containing \( \varphi \), for \( \varphi \in \operatorname{Irr}(H) \cup \operatorname{IBr}(H) \).
If $Q$ is a radical $\ell$-subgroup of $H$ and $B$ an $\ell$-block of $H$, then we define the set

$$dz_\ell(N_H(Q)/Q, B) = \{ \chi \in dz_\ell(N_H(Q)/Q) \mid bl_\ell(\chi)^H = B \},$$

where we regard $\chi$ as an irreducible character of $N_G(Q)$ containing $Q$ in its kernel when considering the induced $\ell$-block $bl_\ell(\chi)^H$.

There are several versions of the (iBAW) condition. Apart from the original version given in [39, Def. 4.1], there is also a version treating only blocks with defect groups involved in certain sets of $\ell$-groups [39, Def. 5.17], or a version handling single blocks [28, Def. 3.2]. We shall consider the inductive condition for a single block here (in order to consider unipotent $\ell$-blocks of special linear or unitary groups).

**Definition 2.5** ([28, Def. 3.2]). Let $\ell$ be a prime, $S$ a finite non-abelian simple group and $X$ the universal $\ell'$-covering group of $S$. Let $b$ be an $\ell$-block of $X$. We say the **inductive blockwise Alperin weight condition** holds for $b$ if the following statements hold:

(i) There exist subsets $IBr_\ell(b \mid Q) \subseteq IBr_\ell(b)$ for $Q \in \text{Rad}_\ell(X)$ with the following properties:

(1) $IBr_\ell(b \mid Q)^a = IBr_\ell(b \mid Q^a)$ for every $Q \in \text{Rad}_\ell(X)$, $a \in \text{Aut}(X)_b$,

(2) $IBr_\ell(b) = \bigcup_{Q \in \text{Rad}_\ell(X)/X} IBr_\ell(b \mid Q)$.

(ii) For every $Q \in \text{Rad}_\ell(X)$ there exists a bijection

$$\Omega^X_Q : IBr_\ell(b \mid Q) \to dz_\ell(N_X(Q)/Q, b)$$

such that $\Omega^X_Q(\phi)^a = \Omega^X_Q(\phi^a)$ for every $\phi \in IBr_\ell(b \mid Q)$ and $a \in \text{Aut}(X)_b$.

(iii) For every $Q \in \text{Rad}_\ell(X)$ and every $\phi \in IBr_\ell(b \mid Q)$ there exist a finite group $A := A(\phi, Q)$ and $\tilde{\phi} \in IBr_\ell(A)$ and $\tilde{\phi}' \in IBr_\ell(N_A(Q))$, where we use the notation

$$\overline{Q} := QZ/Z$$

and $Z := Z(X) \cap \ker(\phi)$, with the following properties:

(1) for $\overline{X} := X/Z$ the group $A$ satisfies $\overline{X} \subseteq A$, $A/C_A(\overline{X}) \cong \text{Aut}(X)_\phi$, $C_A(\overline{X}) = Z(A)$ and $\ell \nmid |Z(A)|$

(2) $\tilde{\phi} \in IBr_\ell(A)$ is an extension of the $\ell$-Brauer character of $\overline{X}$ associated with $\phi$,

(3) $\tilde{\phi}' \in IBr_\ell(N_A(\overline{Q}))$ is an extension of the $\ell$-Brauer character of $N_X(\overline{Q})$ associated with the inflation of $\Omega^X_Q(\phi)^a \in IBr_\ell(N_X(Q)/Q)$ to $N_X(Q)$,

(4) $bl_\ell(\text{Res}_A^X(\tilde{\phi})) = bl_\ell(\text{Res}_{N_A(Q)}^X(\tilde{\phi}'))$ for every subgroup $J$ satisfying $\overline{X} \leq J \leq A$.

2.4 Some notations and conventions for $GL_n(eq)$

From now on to the end of this paper, we always assume that $p$ is a prime, $q = p^f$ with a positive integer $f$, and $\ell$ is a prime number different from $p$.

We follow mainly the notation from [19], [8], [3], [4] and [5]. We first give some notation and conventions used throughout this paper.

For a positive integer $d$, we denote by $I(d)$ the identity matrix of degree $d$ and by $I_d$ the identity matrix of degree $\ell^d$. Let $\epsilon = \pm 1$ and $G = GL_n(eq)$, where $\text{GL}_n(-q)$ denotes the general unitary group $GU_n(q) = \{ A \in GL_n(q^2) \mid F_q(A)^r A = I_{\omega(n)} \}$, where $F_q(A)$ is the matrix whose entries are the $q$-th powers of the corresponding entries of $A$, and $^r$ denotes the transpose operation of matrices.

Denote $X = SL_n(eq)$, where $\text{SL}_n(-q) = SU_n(q) = GU_n(q) \cap \text{SL}_n(q^2)$, where we also use the notation $GL(n, eq)$ (and $\text{SL}(n, eq)$, respectively) for $GL_n(eq)$ (and $\text{SL}_n(eq)$, respectively). Let $F_p$ be the automorphism of $G$ defined by $F_p((g_{i,j})) = (g_{p,j}^p)$ and $\gamma$ the automorphism of $G$ defined by $\gamma(A) = (A^{-1})^r$. Denote
$D = (F_p, \gamma)$. Then the group $G \rtimes D$ is well-defined. For the unitary groups, $D$ is cyclic of order $2f$. By [22] Thm. 2.5.1, the automorphisms of $X$ induced by $G \rtimes D$ generate $\text{Aut}(X)$. If $n = 2$, $\gamma$ is an inner automorphism. If $n \geq 3$, then $\text{Aut}(X) \cong G/Z(G) \rtimes D$. We denote by $\overline{F} = F_{eq} = F_q$ or $F_{q^2}$ the field of $q$ or $q^2$ elements when $\epsilon = 1$ or $\epsilon = -1$ respectively. Let $e$ be the multiplicative order of $eq$ modulo $\ell$.

For a positive integer $d$, we denote by $\mathcal{F}_d[x]$ (irreducible polynomials, respectively) over the field $\overline{F}_{q^2}$. For $(\Delta(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_0$ in $\mathcal{F}_{q^2}[x]$, we define $\Delta(x) = x^m a_0^{-qd} \Delta^{qd}(x^{-1})$, where $\Delta^{qd}(x)$ means the polynomial in $x$ whose coefficients are the $q$-th powers of the corresponding coefficients of $\Delta(x)$. Then $\alpha$ is a root of $\Delta$ if and only if $\alpha^{-qd}$ is a root of $\Delta$. Now, we denote by

$$
\mathcal{F}_0(d) = \left\{ \Delta \in \text{Irr}(\mathcal{F}_{q^2}[x]) \mid \Delta \neq x \right\},
$$

$$
\mathcal{F}_1(d) = \left\{ \Delta \in \text{Irr}(\mathcal{F}_{q^2}[x]) \mid \Delta \neq x, \Delta = \tilde{\Delta} \right\},
$$

$$
\mathcal{F}_2(d) = \left\{ \tilde{\Delta} \Delta \in \text{Irr}(\mathcal{F}_{q^2}[x]), \Delta \neq x, \Delta \neq \tilde{\Delta} \right\}.
$$

Let $\mathcal{F}(d) = \mathcal{F}_0(d)$ if $d^e = 1$; and $\mathcal{F}(d) = \mathcal{F}_1(d) \cup \mathcal{F}_2(d)$ if $d^e = -1$. In particular, we abbreviate $\mathcal{F} := \mathcal{F}(1)$ and $\mathcal{F}_i := \mathcal{F}_i(1)$ for $i = 0, 1, 2$. We denote by $d_\ell$ the degree of any polynomial $\Gamma$. For unitary groups, the polynomials in $\mathcal{F}_1 \cup \mathcal{F}_2$ serve as the “elementary divisors” as polynomials in $\mathcal{F}_0$ serve for linear groups (see, for example, [19] p.111-112). For $\Gamma \in \mathcal{F}$, if $\sigma$ is a root of $\Gamma$, then $\sigma^{qh}$ is also a root of $\Gamma$ for any positive integer $h$. So $d_\ell$ is the minimal integer $d$ such that $\sigma^{qh}d^e - 1 = 1$ and all the roots of $\Gamma$ are $\sigma, \sigma^{q}, \ldots, \sigma^{qh}$.

Note that the meaning of our notation here for unitary groups, such as $e$ and $GU_n(q)$, is the same as those in [4] and [5] which is slightly different from that in [19] (for details, see [5] p.6). In particular, with the notation adopted here, there is no need to introduce the reduced degrees $d_\ell$ for the unitary groups. (For the results in [19] for unitary groups where $\delta_\ell$ appears, it is easy to reformulate them with the notation adopted here and $d_\ell$ replacing $\delta_\ell$ as in [5]).

Let $\overline{F}$ be the algebraic closure of $\overline{F}_q$. As usual, we denote $G = \text{GL}_m(\overline{F})$ (a connected reductive algebraic group). Define $F_q := F_{q^2}$ and $F = \gamma^{\frac{1}{2}} \circ F_q$ which is a Frobenius endomorphism over $G$ defining an $\overline{F}_q$ structure on it. We write $G^F$ for the group of fixed points, then $G = G^F$.

Now, for $\Gamma \in \mathcal{F}$, let $G$ be the companion matrix of $\Gamma$. Let $s$ be a semisimple element of $G$ and $s = \prod_i s_i$ is the primary decomposition of $s$ (see, for example, [19] p.112)). If the multiplicity $m_{\ell}(s)$ of $s_i$ is not zero, we call $G$ an “elementary divisor" of $s$ although $G$ may not be irreducible in the unitary case. Then there exists $g_{\ell}(s)$ such that $g_{\ell}(s) = I_{m_{g}(s)} \otimes \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{\ell}^{(\ell)}}$ where $\sigma_{\ell}^{(\ell)}$ are all the roots of $\Gamma$, and $g_{\ell}(s) = g_{\ell}(s)^{-1} F(g_{\ell}(s))$ is a blockwise permutation matrix corresponding to a $d_\ell$-cycle. Now let $H = G_{\ell}(s)$, then $H = \prod \Gamma H_{\Gamma}$, where $H_{\Gamma} = C_{G}(s_i)$ with $G_{\ell} = \text{GL}(m_{\ell}(s) d_\ell, \overline{F}_q)$. Let $H_{\Gamma,0} := H_{\Gamma}^{g_{\ell}(s)}$, then $H_{\Gamma,0} = \text{GL}(m_{\ell}(s), \overline{F}) \times \cdots \times \text{GL}(m_{\ell}(s), \overline{F})$ with $d_\ell$ factors and $F$ acts on $H_{\ell}$ in the same way as $g_{\ell}(s) F$ acts on $H_{\Gamma,0}$. Let $H_{\ell} = H_{\ell,0}^{g_{\ell}(s)} \cong \text{GL}(m_{\ell}(s), (eq)\ell)$. Also, $C_{G}(s) = H_{\ell} = \prod \Gamma H_{\Gamma}$. Let $\mathcal{P}(s)$ be the set of the symbols $\mu = \prod \mu_{\Gamma}$, such that $\mu_{\Gamma}$ is a partition of $m_{\ell}(s)$. Then the unipotent characters of $C_{G}(s)$ are in bijection with $\text{Irr}(\prod \Gamma, \text{Z}(m_{\ell}(s)))$ and consequently with $\mathcal{P}(s)$ (see, for example, [8] §4.B2). For $\mu \in \mathcal{P}(s)$, we denote by $\chi_{\mu} = \prod \chi_{\mu_{\Gamma}}$ the unipotent character of $C_{G}(s)$ corresponding to $\mu$.

3 The characters and Brauer characters of $SL_n(eq)$

With the parametrization of pairs involving semisimple elements above, the irreducible characters of $G$ can be constructed by the Jordan decomposition. The irreducible characters of $G$ are in bijection with $G$-conjugacy classes of pairs $(s, \mu)$, where $s$ is a semisimple element of $G$ and $\mu \in \mathcal{P}(s)$. The bijection is given as

$$
\chi_{s,\mu} = \epsilon G \epsilon C_{G}(s) R_{C_{G}(s)}(\delta_{\chi_{\mu}}),
$$

where $
\delta_{\chi_{\mu}}$ is the Gelfand-Tsetlin character of $\chi_{\mu}$.
where $\chi_\mu$ is a unipotent character of $H = C_f(s)$ described as in the end of previous section, and $\delta$ is the image of $s$ under the isomorphism (see [19] (1.16))

$$Z(H) \cong \text{Hom}(H/[H,H], \mathbb{Q}_l).$$

(3.1)

Here, $\mathbb{Q}_l$ is an algebraic closure of the $\ell$-adic field $\mathbb{Q}_l$.

Denote $\mathfrak{z} := \{f \in \mathbb{F}^X \mid \epsilon_e^f = 1\}$. Then we may identify the elements of $\mathfrak{z}$ with the elements of $Z(G)$. For $\Gamma \in \mathcal{T}$, let $\xi$ be a root of $\Gamma$. For $z \in \mathfrak{z}$, define $z\Gamma$ to be the unique polynomial in $\mathcal{T}$ such that $z\xi$ is a root of $z\Gamma$. Note that $d_{\ell} = d_{\ell}r$. In fact, since all the roots of $\Gamma$ are $\xi, \xi^{\ell q}, \ldots, \xi^{\ell q^m-1}$, we know that all the roots of $z\Gamma$ are $z\xi, z\xi^{\ell q}, \ldots, z\xi^{\ell q^m-1}$. Now we define an action of $\mathfrak{z}$ on the set of pairs $(s, \mu)$ with $\mu \in \mathcal{P}(s)$. For $z \in \mathfrak{z}$, define $z\mu = \prod_i(z\mu_i)$ with $(z\mu)_i = \mu_i^z$. Then $z\mu \in \mathcal{P}(z\mu)$.

By Lemma 2.1 for $\chi \in \text{Irr}(G)$, in order to compute the number of irreducible constituents of $\text{Res}_G^x(\chi)$ (recall that $X = \text{SL}_n(q)$ is defined as in Section 2.4), we need to know when $\chi\xi = \chi$, for $\xi \in \text{Irr}(G/X)$. Note that the group $Z(G)$ (hence $\mathfrak{z}$) is isomorphic via $\delta$ to the group of linear characters of $G/X$. The following proposition follows from [16 Prop. 3.5].

**Proposition 3.1.** Let $s$ be a semisimple element $s \in G^F$ and $z \in \mathfrak{z}$, if we write $E(G^F, s) = \{ \chi_1, \ldots, \chi_k \}$, then $E(G^F, zs) = \{ z\chi_1, \ldots, z\chi_k \}$.

Thus, for a semisimple element $s \in G^F$ and $z \in \mathfrak{z}$, if we write $E(G^F, s) = \{ \chi_1, \ldots, \chi_k \}$, then $E(G^F, zs) = \{ z\chi_1, \ldots, z\chi_k \}$ for any $z \in O_C(3)$.

**Corollary 3.2.** Let $s$ be a semisimple element $s \in G^F$. Suppose that $\text{IBr}_e(E_i(G^F, s)) = \{ \phi_1, \ldots, \phi_k \}$, then $\text{IBr}_e(E_i(G^F, zs)) = \{ z\phi_1, \ldots, z\phi_k \}$ for any $z \in O_C(3)$.

**Proof.** By [13 Thm. 14.4], $E_i(G^F, s)$ is a basic set for $E_i(G^F, s)$. By [20], the decomposition matrix for $E_i(G^F, s)$ with respect to $E_i(G^F, s)$ is unitirangular. We may assume the corresponding ordering of $E_i(G^F, s)$ and $\text{IBr}_e(E_i(G^F, s))$ for the unitirangular decomposition matrix $x_1, \ldots, x_k$ and $\phi_1, \ldots, \phi_k$ and the corresponding decomposition matrix $D = (d_{ij})_{1 \leq i, j \leq k}$ so that $d_{ii} = 1$ for all $1 \leq i \leq k$ and $d_{ij} = 0$ for $1 \leq i < j \leq k$.

Let $E_i(G^F, zs) = \{ x_1', \ldots, x_k' \}$ with $x_i' = z\chi_i$ for all $1 \leq i \leq k$. Suppose that $\phi_1', \ldots, \phi_k'$ are the irreducible $\ell$-Brauer characters of $E_i(G^F, zs)$ and the corresponding decomposition matrix $D' = (d_{ij}')_{1 \leq i, j \leq k}$ is unitirangular.

Firstly, let $1 \leq i_1 \leq k$ be the integer such that $\chi_1' = z\phi_1$. Noting that $\chi_1'' = \phi_1, \chi_1'' = \phi_1'$, then $z\phi_1' = \phi_1'$. Hence $d_{ii} = 0$ for all $1 \leq i \leq k$. Similarly, $d_{ij} = 0$ for all $1 \leq i < j \leq k$.

Now suppose that for some $1 \leq m \leq k$, we have $z\chi_i = \chi_i' = \phi_i'$ for all $1 \leq i \leq m - 1$. Let $m \leq i_{m} \leq k$ be the integer such that $\chi_i' = z\chi_i$. Note that $\chi_m'' = \phi_m + \sum_{j=1}^{m-1} d_{mj}\phi_j$ and $\chi_m'' = \phi_m' + \sum_{j=1}^{m-1} d_{mj}'\phi_j'$, then

$$z\phi_m + \sum_{j=1}^{m-1} d_{mj}\phi_j = \phi_m' + \sum_{j=1}^{m-1} d_{mj}'\phi_j'.$$

Since $\phi_i' \neq \phi_i, \ldots, \phi_{m-1}$, by the linear independence of irreducible $\ell$-Brauer characters, $\phi_i' = z\phi_m - d_{mj} = 0$ for all $1 \leq j \leq m - 1$ and $d_{mj}' = d_{mj}$ for all $1 \leq j \leq m - 1$. Thus we may change the ordering and assume that $z\chi_i = \chi_i'$ and $z\phi_i = \phi_i'$ for all $1 \leq i \leq m$. The decomposition matrix of $E_i(G^F, zs)$ is still unitirangular with respect to the new ordering.

Thus we complete the proof by repeating the process above.
Remark 3.3. By the proof of Corollary 3.2, with a suitable ordering, the decomposition matrices associated with the basic sets $E(G^F, s)$ and $E(G^F, zs)$ of $E_\ell(G^F, s)$ and $E_\ell(G^F, zs)$, respectively, are the same.

We may use the parameterisation $(s, \mu)$ of irreducible characters in $E(G^F, s)$ for the corresponding irreducible $\ell$-Brauer character of $E_\ell(G^F, s)$. For a semisimple $\ell'$-element $s$ of $G^F$, let

$$\Theta : IBr_\ell(E_\ell(G^F, s)) \to E(G^F, s)$$

be the bijection such that the corresponding decomposition matrix is unitriangular (in the sense of [16 Def. 2.2]), and we denote by $\phi_{s, \mu} = \Theta^{-1}(1, s, \mu)$ for all $z \in O_\ell(3)$ by Proposition 3.1 and the proof of Corollary 3.2. (For $\epsilon = 1$, this is just [26 Lem. 4.1].)

The number of irreducible constituents of the restriction of irreducible $\ell$-Brauer characters of $G$ to $X$ was obtained by Kleshchev and Tiep for $\epsilon = 1$ (see [26 Thm. 1.1 and Cor. 1.2]), and generalized by Denoncin (for $\epsilon = \pm 1$) (see [16 Prop. 3.5, 4.2 and 4.9]). We will state it as the following remark.

Remark 3.4. We introduce the notations of the combinatorial description of irreducible $\ell$-Brauer characters of $G$ in [26]. For a partition $\mu = (\mu_1, \mu_2, \ldots)$, denote $|\mu| = \mu_1 + \mu_2 + \cdots$ and write $\mu'$ for the transposed partition. Set $\Delta(\mu) = \gcd(\mu_1, \mu_2, \ldots)$.

For $\sigma \in F^\ell$, we denote by $[\sigma]$ the set of all roots of the polynomial in $F$ which has $\sigma$ as its root. Denote by deg($\sigma$) the cardinality of $[\sigma]$. Then deg($\sigma$) is the minimal integer $d$ such that $\sigma^{(eq)^d-1} = 1$ and

$$[\sigma] = \{\sigma, \sigma^{eq}, \sigma^{eq^2}, \ldots, \sigma^{eq^{deg(\sigma)-1}}\}.$$  

An $(n, \ell)$-admissible tuple is a tuple

$$\\{(\{\sigma_1^i\}, \mu^{(i)}_1), \ldots, (\{\sigma_a^i\}, \mu^{(i)}_a)\\}$$

of pairs, where $\sigma_1, \ldots, \sigma_a \in F^\ell$ are $\ell'$-elements, and $\mu^{(1)}_1, \ldots, \mu^{(a)}_a$ are partitions such that

- $[\sigma_i] \neq [\sigma_j]$ for all $i \neq j$; and
- $\sum_{i=1}^{a} \deg(\sigma_i)\mu^{(i)} = n$.

An equivalence class of the $(n, \ell)$-admissible tuple (3.2) up to a permutation of pairs

$$\\{(\{\sigma_1^i\}, \mu^{(i)}_1), \ldots, (\{\sigma_a^i\}, \mu^{(i)}_a)\\}$$

is called an $(n, \ell)$-admissible symbol and is denoted as

$$s = [(\{\sigma_1^i\}, \mu^{(i)}_1), \ldots, (\{\sigma_a^i\}, \mu^{(i)}_a)].$$

The set of $(n, \ell)$-admissible symbols is the labeling set for irreducible $\ell$-Brauer characters of $G$. Denote by $\phi_s$ the irreducible $\ell$-Brauer character corresponding to the $(n, \ell)$-admissible symbol $s$.

The group $O_\ell(3)$ acts on the set of $(n, \ell)$-admissible symbols via

$$z \cdot [(\{\sigma_1^i\}, \mu^{(i)}_1), \ldots, (\{\sigma_a^i\}, \mu^{(i)}_a)] = [(z\sigma_1^i, \mu^{(i)}_1), \ldots, (z\sigma_a^i, \mu^{(i)}_a)]$$

for $z \in O_\ell(3)$. We denote by $\kappa_\ell(s)$ the order of the stabilizer group in $O_\ell(3)$ of an $(n, \ell)$-admissible symbol $s$. Next, for an $(n, \ell)$-admissible symbol $s$ as (3.3), let $\kappa_\ell(s)$ be the $\ell'$-part of

$$\gcd(n, q - 1, \Delta(\mu^{(1)}), \ldots, \Delta(\mu^{(a)})).$$

Let $\kappa(s) = \kappa_\ell(s)\kappa_\ell'(s)$. By [26] and [16], $\kappa_X^G(\phi_s) = \kappa(s)$ (i.e. $\text{Res}_X^G\phi_s$ is a sum of $\kappa(s)$ irreducible constituents). For two $(n, \ell)$-admissible symbols $s$ and $s'$, if they are in the same $O_\ell(3)$-orbit, then $\text{Res}_X^G\phi_s = \text{Res}_X^G\phi_{s'}$.

If moreover, we write the decomposition $\text{Res}_X^G\phi_s = \bigoplus_{j=1}^{\kappa(s)}(\phi_{s_j})$, then the set $\{(\phi_{s_j})\}$, where $s$ runs through the $O_\ell(3)$-orbit representatives of $(n, \ell)$-admissible symbols and $j$ runs through the integers between 1 and $\kappa(s)$, is a complete set of the irreducible $\ell$-Brauer characters of $X$.  

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Notice that Remark 3.4 also holds for complex irreducible characters if we set $\ell = 0$ by Proposition 3.1. (For $e = 1$, the complex irreducible characters of $\text{SL}_n(q)$ was obtained in [29].)

For an $\ell$-block $B$ of $G$ and an $(n, \ell)$-admissible symbol $s$, if $\phi_s \in \text{IBr}_{\ell}(B)$, then we say that $s$ belongs to $B$.

## 4 The blocks of $\text{SL}_n(eq)$

Let $X = \text{SL}_n(\mathbb{F})$, then $X = [G, G]$. The labeling of $\ell$-blocks of $G^F$ and $X^F$ (using $e_0$-Jordan-cuspidal pairs) described in [12] and [25] can be stated as following.

**Theorem 4.1.** Let $H \in \{G, X\}$ and $e_0 = e_0(q, \ell)$ is defined as in Equation (2.7).

(i) For any $e_0$-Jordan-cuspidal pair $(L, \zeta)$ of $H$ such that $\zeta \in E(L^F, \ell')$, there exists a unique $\ell$-block $b_{H^F}(L, \zeta)$ of $H^F$ such that all irreducible constituents of $\rho_{H^F}^L(\zeta)$ lie in $b_{H^F}(L, \zeta)$.

(ii) Moreover, the map $\Xi : (L, \zeta) \mapsto b_{H^F}(L, \zeta)$ is a surjection from the set of $H^F$-conjugacy classes of $e_0$-Jordan-cuspidal pairs $(L, \zeta)$ of $H$ such that $\zeta \in E(L^F, \ell')$ to the $\ell$-blocks of $H^F$.

(iii) If $\ell \geq 3$, then $\Xi$ is bijective.

**Remark 4.2.** By a result of Bonnafé [7], the Mackey formula holds for type $A$, hence the Lusztig induction in Theorem 4.1(i) is independent of the ambient parabolic subgroup (containing $L$). So throughout this paper we always omit the parabolic subgroups when considering the Lusztig inductions.

Note that we let $e$ be the multiplicative order of $eq$ modulo $\ell$ throughout this paper. Here, $e_0$ and $e$ may not equal. In fact,

(i) when $\ell \geq 3$,

- if $e = 1$, then $e = e_0$; and
- if $e = -1$, then $e = 2e_0$, $e_0/2$, if $e_0$ is respectively odd, congruent to 2 modulo 4, or divisible by 4; and

(ii) when $\ell = 2$, $e = 1$ while $e_0 = 1$ if $4 \mid q - 1$; and $e_0 = 2$ if $4 \mid q + 1$.

For a positive integer $d$, we let $\Phi_d(x) \in \mathbb{Z}[x]$ be the $d$-th cyclotomic polynomial over $\mathbb{Q}$, i.e., the monic irreducible polynomial whose roots are the primitive $d$-th roots of unity. So if $\ell$ is odd, then $\Phi_d(\epsilon x) = \pm \Phi_{d_0}(x)$.

We will use the following lemma.

**Lemma 4.3.** Assume that $\ell$ is odd. Let $\lambda$ be an $e$-core of a partition of $n$, and $w = e^{-1}(n - |\lambda|)$. Let $T^{(e)}$ be a Coxeter torus of $(\text{GL}(e, \mathbb{F}), F)$, $T = (T^{(e)})^w \times I_{|\lambda|}$, and $L = C_G(T) = (T^{(e)})^w \times \text{GL}(|\lambda|, \mathbb{F})$. Let $\phi_{\lambda}$ be the unipotent character of $\text{GL}(|\lambda|, eq)$ corresponding to $\lambda$ and $\phi = 1_{T^{(e)}} \times \phi_{\lambda} \in \text{Irr}(L^F)$. Then every irreducible constituent of $R_{L^F}^G(\phi)$ has the form $\chi_{\mu}$ such that $\lambda$ is the $e$-core of $\mu$.

**Proof.** Let $H = \text{GL}(e, \mathbb{F}) \times \text{GL}(|\lambda|, \mathbb{F})$, then $H$ is an $F$-stable Levi subgroup of $G$ (moreover, there exists a semisimple element $\rho \in G^F$ such that $H = C_G(\rho)$). Then every irreducible constituent of $R_{L^F}^H(\phi)$ has the form $\phi_{\nu} \times \phi_{\lambda}$, where $\phi_{\nu}$ is a unipotent character of $\text{GL}(e, eq)$ corresponding to some $\nu \in \mathcal{W}$. Since $R_{L^F}^G(\phi) = R_{H^F}^G(\rho_{H^F}^L(\phi))$, it suffices to prove that every irreducible constituent of $R_{H^F}^G(\phi_{\nu} \times \phi_{\lambda})$ has the form $\chi_{\mu}$ such that $\lambda$ is the $e$-core of $\mu$ and then the result follows by [19] (2.12) (a result from the Murnaghan-Nakayama formula) and the remark following it.

**Remark 4.4.** In fact, with the hypothesis and setup of Lemma 4.3, the pair $(L, \phi)$ is an $e_0$-cuspidal pair (note that $L = C_G(T_{e_0})$), and the set of the irreducible constituents of $R_{L^F}^G(\phi)$ is exactly the $e_0$-Harish-Chandra series above $(L, \phi)$. So Lemma 4.3 also follows from the proof of [9] Thm. 3.2 and 3.3.
Now we give the relationship between the $e_0$-cuspidal pairs of $G$ and the $e_0$-cuspidal pairs of $X$.

**Proposition 4.5.** (i) Let $(L, \zeta)$ be an $e_0$-cuspidal pair of $G$ and $b$ an $\ell$-block of $X$ covered by $B = b_{G^F}(L, \zeta)$, then $b = b_{X^F}(L_0, \zeta_0)$, where $L_0 = L \cap X$ and $\zeta_0$ is an irreducible constituent of $\text{Res}^L_{L_0} \zeta$.

(ii) Let $(L_0, \zeta_0)$ be an $e_0$-cuspidal pair of $X$ and $B$ an $\ell$-block of $G$ which covers $b = b_{X^F}(L_0, \zeta_0)$, then $B = b_{G^F}(L, \zeta)$ where the $e_0$-cuspidal pair $(L, \zeta)$ satisfies that $L_0 = L \cap X$ and $\zeta_0$ is an irreducible constituent of $\text{Res}^L_{L_0} \zeta$.

**Proof.** Note that if $L_0 = L \cap X$ and $\zeta_0$ is an irreducible constituent of $\text{Res}^L_{L_0} \zeta$, then by [25] Lem. 2.3], $(L, \zeta)$ is an $e_0$-cuspidal pair of $G$ if and only if $(L_0, \zeta_0)$ is an $e_0$-cuspidal pair of $X$. Thus (i) follows by [25] Lem. 3.7).

For (ii), set $L = L_0 Z(G)$, then $L_0 = L \cap X$. Also, $Z(L) = Z(L_0) Z(G)$ and $Z(L_0) \subseteq Z(L) \subseteq G = Z(G) X$. Hence $C_G(Z(L_0)) = Z(G) C_X(Z(L_0)) \subseteq Z(G) C_X(Z(L_0)) = Z(G) L_0 = L$, and then $L$ is an $e_0$-split Levi subgroup of $G$. Thus (ii) follow by [25] Lem. 3.8). \hfill $\Box$

**Remark 4.6.** Proposition 4.5 is not restricted to the case of type $A$. In fact, it holds for any connected reductive linear algebraic group $G$ and $X = [G, G]$.

**Lemma 4.7.** Let $L$ be an $F$-stable Levi subgroup of $G$, $\zeta \in \text{Irr}(L^F)$ and $L_0 = L \cap X$. Let $\Delta := \text{Irr}(L_0^F) \mid \zeta)$. then $N_{X^F}(L_0)_{\Delta}$ acts trivially on $\Delta$.

**Proof.** Let $L = L^F$ and $L_0 = L_0^F$. Note that there exist integers $n_i$, $a_i$, $b_i$ $(1 \leq i \leq s)$ and $r$ such that $n_i \neq n_j$ for $i \neq j$ and $L = L_0 \times L^1_{b_1} \times \cdots \times L^s_{b_s}$ where $L_0 \cong \text{GL}(r, eq)$ and $L_i \cong \text{GL}(n_i, (eq)^{a_i})$. Then $N_{G^F}(L) = L_0 \times \prod_{1 \leq i \leq s} N_i \zeta(b_i)$, where $N_i = (L_i, \sigma_i)$, and $\sigma_i$ act on $L_i \cong \text{GL}(n_i, (eq)^{a_i})$ as a field automorphism of order $a_i$. We denote by $\text{Out}_{N_{G^F}(L_0)^F}(L_0^F)$ the subgroup of $\text{Out}(L_0^F)$ induced by $N_{G^F}(L)$ (i.e. $\text{Out}_{N_{G^F}(L_0)^F}(L_0^F) \cong N_{G^F}(L_0^F) / L_0^F Z(L^F)$). By comparing orders, we have $\text{Out}_{N_{G^F}(L_0)^F}(L_0^F) = \text{Out}_{L_0^F} \times \text{Out}_{N_{X^F}(L_0)^F}(L_0^F)$ since $Z(L^F) = Z(F^F) \cap L_0^F$. Let $\Delta := \text{Irr}(L^F) \mid \zeta)$. Firstly, we consider the case $L = L_i = \text{GL}(n_i, (eq)^{a_i})$. Then $N_{G^F}(L_i) = N_i$ and $\text{Out}_{L_0^F} \times \text{Out}_{N_{X^F}(L_0)^F}(L_0^F)$ commute. Now by [15] Thm. 4.1, there exists $\zeta_0 \in \Delta$ such that $N_{G^F}(L_0) \zeta_0 = L_0^F N_{X^F}(L_0) \zeta_0$. So $\zeta_0$ is invariant under $N_{X^F}(L_0)_{\Delta}$ since

$$\text{Out}_{N_{G^F}(L_0)(L^F, L_0^F)} = \text{Out}_{L_0^F} \times \text{Out}_{N_{X^F}(L_0)(L^F, L_0^F)}.$$ 

Now $L^F$ acts transitively on $\Delta$, then $N_{X^F}(L_0)_{\Delta}$ acts trivially on $\Delta$ since the restriction of $\zeta_0$ to $[L, L]$ is multiplicity-free.

Now we consider the case $L = L_i \cong \text{GL}(n_i, (eq)^{a_i})$. Then $N_{G^F}(L_i) = N_i \zeta(b_i)$, where $\zeta_{b_i} \in \text{Irr}(L_i)$ for $1 \leq k \leq b_i$. Then $\Delta = \prod_{1 \leq k \leq b_i} \Delta_0, k$, where $\Delta_0, k = \text{Irr}([L_i, L_i])) \zeta_0$ for $1 \leq k \leq b_i$. Let $\zeta_0 \in \Delta_0$ and $\zeta_0 = \zeta_0, 1 \times \cdots \times \zeta_0, b_i$, where $\zeta_0, k \in \Delta_0, k$ for $1 \leq k \leq b_i$. Then $g \in N_{G^F}(L)$. If $\zeta_0, k \in \Delta_0$, then without loss of generality, we may assume that $g = (\sigma_1, \ldots, \sigma_{b_i}; \tau)$, where $\sigma_i \in N_i$, $\tau \in \zeta(b_i)$ and $\tau = (1, \ldots, b_i)$. Then $\zeta_0, k = \zeta_{0, 1} \times \zeta_{0, 1} \times \cdots \zeta_{0, b_i-1}$. Hence there exist $l_1, \ldots, l_{b_i-1}, l_i \in L_i$ such that $\zeta_{0, 1} = \zeta_{0, b_i}$ and $\zeta_{0, k} = \zeta_{0, k-1}$ for $2 \leq k \leq b_i - 1$. By the argument of above paragraph, it is easy to check that $\zeta_{0, b_i} = \zeta_{0, b_i}$ for $b_i = (\prod_{1 \leq k \leq b_i} l_i)^{-1}$. Now let $l = \text{diag}(l_1, \ldots, l_{b_i})$, then $l \in L_0$ and $\zeta_0 = \zeta_{0, l}$. Then there exists $\zeta_0 \in \Delta$, such that $\zeta_0, \zeta_0 \in \text{Irr}([L, L] \mid \zeta_0)$ since $\text{Res}^L_{[L, L]} \zeta$ is multiplicity-free. So $N_{X^F}(L_0)_{\Delta}$ acts trivially on $\Delta$.

The assertion in the general case now follows by reduction to the preceding cases. \hfill $\Box$

Let $J$ be a subgroup of some general linear or unitary group $\text{GL}_n(eq)$, we denote $D(J) := \{ \text{det}(M) \mid M \in J \}$. Then $D(J)$ is a subgroup of $\mathbb{Z}$ and $D(J) \cap \text{SL}_n(eq) \cong D(J)$. 

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Remark 4.8. Let \( L \) a Levi subgroup of \( G \), and \( L_0 = L \cap X \). Note that \( \mathcal{D}(L^F) = 3 \). Then \( G^F = X^F \cdot N_{GF}(L) \) and \( L^F/L_0^F \cong G^F/X^F \). So the \( G^F \)-conjugacy classes of \( e_0 \)-split Levi subgroups of \( G \) are just the \( X^F \)-conjugacy classes of \( e_0 \)-split Levi subgroups of \( G \).

We denote by \( \overline{L} \) a complete set of representatives of the \( G^F \)-conjugacy classes of \( e_0 \)-Jordan-cuspidal pairs of \( G \) such that \( \zeta \in \mathcal{E}(L^F, t') \). We may assume that for \((L, \zeta), (L', \zeta') \in \overline{L} \), if \( L \) and \( L' \) are \( G^F \)-conjugacy, then \( L = L' \).

Define an equivalence relation on \( \overline{L} : (L, \zeta) \sim (L', \zeta') \) if and only if \( L = L' \) and \( \text{Res}_{L_0}^{L_F} \zeta = \text{Res}_{L_0}^{L_F} \zeta' \) where \( L_0 = L \cap X \). Then by Lemma 2.4 \( 4.2 \) and Proposition 4.3 \( (L \cap X, \zeta_0) \) is a complete set of representatives of \( X^F \)-conjugacy classes of \( e_0 \)-Jordan-cuspidal pairs of \( X \) such that \( \zeta_0 \in \mathcal{E}((L \cap X)^F, t') \), where \((L, \zeta) \) runs through a complete set of representatives of the equivalence classes of \( \overline{L} / \sim \) and \( \zeta_0 \) runs through \( \text{Irr}((L \cap X)^F \mid \zeta) \).

The \( \ell \)-blocks of \( G^F \) were classified in [19] and [8]. For \( \Gamma \in \mathcal{T} \), we denote by \( e_\Gamma \) the multiplicative order of \( (eq)^{\ell} \) modulo \( \ell \). Obviously, \( e_\Gamma = \frac{\text{gcd}(\ell, d_\Gamma)}{\ell} \). Note that for \( \ell = 2 \), \( e = e_\Gamma = 1 \). Given a semisimple element \( s \) of \( G^F \), let \( C_\Gamma(s) \) be the \( e_\Gamma \)-core of partitions of \( m_\Gamma(s) \), and \( C(s) = \prod_{\Gamma} C_\Gamma(s) \). The following result is a combination of [19] (5D) and (7A) and [8] (3.2) and (3.9).

Theorem 4.9 (Fong-Srinivasan, Broué). There is a bijection from the \( \ell \)-blocks of \( G \) onto the set of \( G \)-conjugacy classes of pairs \((s, \lambda)\), where \( s \) is a semisimple \( \ell \)-element of \( G \) and \( \lambda \in \mathcal{C}(s) \).

Moreover, Suppose \( B \) is an \( \ell \)-block of \( G \) with label \((s, \lambda)\). Then an irreducible character \( \chi \) of \( G \) of the form \( \chi_t|_{\mu} \) belongs to \( B \) if and only if the \( \ell \)-part of \( t \) is \( G \)-conjugacy to \( s \) and for every \( \Gamma \in \mathcal{T} \), \( \mu_{\Gamma} \) has \( e_\Gamma \)-core \( \lambda_{\Gamma} \).

We denote by \( B(s, \lambda) \) the \( \ell \)-block of \( G \) with label \((s, \lambda)\). Note that, for \( \ell = 2 \), \((s, \lambda)\) is always of the form \((s, -)\).

Now we give an \( e_0 \)-Jordan-cuspidal pair for the \( \ell \)-block \( B(s, \lambda) \). Let \( s \in G^F \) be a semisimple element and \( \lambda \in \mathcal{C}(s) \). Take the primary decomposition \( s = \prod_{\Gamma} s_\Gamma \) with \( s_\Gamma = m_\Gamma(s)(s_\Gamma) \) and \( \mathcal{C}_\Gamma(s) \subseteq \bigcap_{\Gamma} C_\Gamma(s) \). Then \( \mathcal{C}_\Gamma(s) = \bigcap_{\Gamma} C_\Gamma(s) \). Let \( \mathcal{C}_\Gamma(s) \) be the Sylow \( e_\Gamma \)-core of \( s \) with label \((s_\Gamma, \lambda_{\Gamma}) \).

Let \( H_{\Gamma, 0} = M_{\Gamma, 1} \times \cdots \times M_{\Gamma, \ell} \) with \( m_\Gamma(s) ) \) factors. Then \( H_{\Gamma, 0} = H_{\Gamma, 0}^{e_\Gamma} \times (GL(1, (eq)^{\ell} \lambda_{\Gamma}) \times \text{GL}(1, \lambda_{\Gamma}((eq)^{\ell} \lambda_{\Gamma})^\ell) \times \text{GL}(1, \lambda_{\Gamma})) \). Let \( H = \Pi_{\Gamma} H_{\Gamma, 0} \). Obviously, \( s \in H^F \).

Now let \( T_{\Gamma, 0} = ((T_{\Gamma, 0}^{(e_\Gamma)} \lambda_{\Gamma})_{\ell} \times (T_{\Gamma, 0}^{(e_\Gamma)} \lambda_{\Gamma})_{\ell} \). Then \( \lambda_{\Gamma} \) is a torus of \( C_{\Gamma, \lambda}(s_{\Gamma}) \). Now let \( T_{\Gamma} \) be the Sylow \( e_0 \)-torus of \( \lambda_{\Gamma} \). Then \( T = \Pi_{\Gamma} T_{\Gamma} \) is an \( e_0 \)-torus of \( G \). Let \( L = C_G(T) \), then \( L \) is an \( e_0 \)-split Levi subgroups of \( G \). Also, \( s \in L \) and \( H = C_{\mathcal{C}(s)}(T) = C_L(s) \).

Let \( \phi_{\Gamma} \) be the unipotent character of \( GL(|\lambda_{\Gamma}|, (eq)^{\ell} \lambda_{\Gamma}) \) corresponding to \( \lambda_{\Gamma} \) and \( \phi_{\ell} = 1_{(GL(1, (eq)^{\ell} \lambda_{\Gamma}) \times \phi_{\ell}, \ell)} \). Then \( \phi_{\ell} = 1_{\Pi_{\Gamma} T_{\Gamma} | \phi_{\ell}} \cdot \phi_{\ell} \end{document}
**Corollary 4.12.** Let \( z \in \mathbb{F} \). Then \( \mathbb{F}^z = \mathbb{F} \) if and only if \( (s, \lambda) \) and \((zs, \lambda)\) are \( \mathbb{F}^e \)-conjugate and \( \mathbb{F}^z = \mathbb{F} \) for all \( \mathbb{F}^e \). 

**Proof.** Firstly we assume that \( \ell \) is odd. Abbreviate \( \mathbb{L}_{s, \lambda} = \mathbb{L} \). By Proposition 3.11, \( \mathbb{F}^z = \mathbb{F} \) if and only if \((s, \lambda)\) and \((zs, \lambda)\) are \( \mathbb{F}^e \)-conjugate. Note that \( \mathbb{L}^e = L_0 \times L_1 \), where \( L_0 \cong GL(\mathbb{F}^e, \mathbb{F}^e) \) and \( L_1 \cong \prod \mathbb{F}^e GL(\mathbb{F}^e, (q_{\mathbb{F}^e})^w(s)) \). Write \( s = s_0 \times s_1 \) the corresponding decomposition such that \( s_0 \in L_0 \) and \( s_1 \in L_1 \). Then \((s, \lambda)\) and \((zs, \lambda)\) are \( \mathbb{F}^e \)-conjugate if and only if \((s_0, \lambda)\) and \((zs_0, \lambda)\) are \( L_0 \)-conjugate and \( s_1 \) and \( zs_1 \) are \( L_1 \)-conjugate. The semisimple element of \( GL(\mathbb{F}^e, (q_{\mathbb{F}^e})^w(s)) \) corresponding to the part of
Remark 4.13. Analogously with the description for irreducible \( \ell \)-Brauer characters of \( X \) in Remark 3.4, now we give a description for \( \ell \)-blocks of \( X = \text{SL}_n(eq) \) by summarizing the argument above. We call a tuple
\[
((\sigma_1], m_1, \lambda^{(1)}], \ldots, ([\sigma_a], m_a, \lambda^{(a)}])
\] (4.1)
of triples an \((n, \ell)\)-admissible block tuple, if

- for every \( 1 \leq i \leq a \), \( \sigma_i \in \overline{E^F} \) is an \( \ell \)-element, and \( m_i \) is a positive integer such that \( \lambda^{(i)} \) is the \( e_i \)-core of some partition of \( m_i \), where \( e_i \) is the multiplicative order of \( (eq)^{\deg(\sigma_i)} \) modulo \( \ell \);
- \( [\sigma_i] \neq [\sigma_j] \) if \( i \neq j \); and
- \( \sum_{i=1}^a m_i \deg(\sigma_i) = n \).

An equivalence class of the \((n, \ell)\)-admissible block tuple (4.1) up to a permutation of triples
\[
([\sigma_1], m_1, \lambda^{(1)}], \ldots, ([\sigma_a], m_a, \lambda^{(a)}])
\]
is called an \((n, \ell)\)-admissible block symbol and is denoted as
\[
b = ([([\sigma_1], m_1, \lambda^{(1)}], \ldots, ([\sigma_a], m_a, \lambda^{(a)}])].
\] (4.2)

Thus by Theorem 4.9, the set of \((n, \ell)\)-admissible block symbols is a labeling set for \( \ell \)-blocks of \( G \). Denote by \( B_0 \) the \( \ell \)-block of \( G \) corresponding to the \((n, \ell)\)-admissible block symbol \( b \).

The group \( O_{\ell^2}(3) \) acts on the set of \((n, \ell)\)-admissible block symbols via
\[
z \cdot ([([\sigma_1], m_1, \lambda^{(1)}], \ldots, ([\sigma_a], m_a, \lambda^{(a)}]) = ([z\sigma_1], m_1, \lambda^{(1)}], \ldots, ([z\sigma_a], m_a, \lambda^{(a)}])
\]
for \( z \in O_{\ell^2}(3) \). Now we denote by \( C_1(b) \) the stabilizer group in \( O_{\ell^2}(3) \) of the \((n, \ell)\)-admissible block symbol \( b \).

For a positive integer \( d \) and \( \sigma \in \overline{E^F} \), if \([\sigma] = [\sigma, \sigma^{eq}, \ldots, \sigma^{(eq)^{\deg(\sigma)}}] \), then we let
\[
[\sigma]_d := [\sigma, \sigma^{eq^d}, \sigma^{eq^{2d}}, \ldots, \sigma^{eq^{(eq)^{\deg(\sigma)}-d}}] \}
\]
We also define
\[
z[\sigma]_d := [z\sigma, z\sigma^{eq^d}, z\sigma^{eq^{2d}}, \ldots, z\sigma^{eq^{(eq)^{\deg(\sigma)}-d}}] \}
\]
for \( z \in \mathcal{Z} \).

For an \((n, \ell)\)-admissible block symbol \( b \) as (4.2), we define the sets \([\sigma_i]_b \) for \( 1 \leq i \leq a \) as follows:

(i) When \( \ell \) is odd, if \( |\lambda^{(i)}| = m_i \), then \([\sigma_i]_b \) is empty; and if \( |\lambda^{(i)}| < m_i \), then \([\sigma_i]_b = [\sigma_i]_{(e)} \).

(ii) When \( \ell = 2 \),
- if \( 4 \mid q - e \) or \( \deg(\sigma_i) \) is even, then \([\sigma_i]_b = [\sigma_i]_e \).
Remark 4.14. Suppose that \( \mathcal{B} \) runs through the (\( n \)-admissible) symbols such that \( n \) be the empty set for \( b \)-constituents lying in \( \mathcal{O} \), where \( b \in \mathcal{B} \). Let \( b \) runs through the integers between 1 and \( \kappa \), if \( \kappa \) is odd. By Lemma 2.1, Proposition 4.5 and 4.11 and Corollary 4.12, \( \hat{\chi}(\gamma) \text{-blocks of } G \) of \( \hat{\chi} \) are the \( (s, \ell) \)-admissible symbols such that \( \gamma \) is a \( (s, \ell) \)-admissible symbol of \( \hat{\chi} \) with label \( (s, \ell) \). Moreover, if we let \( (B_b)_0, (B_b)_1, \ldots, (B_b)_a \) be the \( (s, \ell) \)-admissible block symbols \( b \) and \( b' \), if they are in the same \( \mathcal{O} \)-orbit, then the sets of the \( (s, \ell) \)-blocks of \( X \) covered by \( B_b \) and \( B_{b'} \) are the same.

If moreover, we let \( (B_b)_0, (B_b)_1, \ldots, (B_b)_a \) be the \( (s, \ell) \)-admissible block symbols \( b \) and \( b' \) such that \( \gamma \) is also an extension of \( \gamma \) and \( \gamma \) runs through the integers between 1 and \( \kappa \), then \( \hat{\chi}(\gamma) \text{-blocks of } G \) is the \( (s, \ell) \)-admissible symbols \( b \) such that \( \gamma \) is a \( (s, \ell) \)-admissible symbol of \( \hat{\chi} \) with label \( (s, \ell) \). Moreover, if we let \( (B_b)_0, (B_b)_1, \ldots, (B_b)_a \) be the \( (s, \ell) \)-admissible block symbols \( b \) and \( b' \), if they are in the same \( \mathcal{O} \)-orbit, then the sets of the \( (s, \ell) \)-blocks of \( X \) covered by \( B_b \) and \( B_{b'} \) are the same.

Remark 4.14. Suppose that \( b = [[\sigma_1], m_1, \lambda^{(1)}], \ldots, [[\sigma_n], m_n, \lambda^{(a)}] \) is an \( (s, \ell) \)-admissible block symbol. Then the set of \( (s, \ell) \)-blocks in unipotent \( \ell \)-Brauer characters is covered by \( \hat{\chi}(\gamma) \text{-blocks of } G \) for all \( \hat{\chi} \in \mathcal{O}(\gamma) \). Moreover, if we let \( (B_b)_0, (B_b)_1, \ldots, (B_b)_a \) be the \( (s, \ell) \)-admissible block symbols \( b \) and \( b' \), if they are in the same \( \mathcal{O} \)-orbit, then the sets of the \( (s, \ell) \)-blocks of \( X \) covered by \( B_b \) and \( B_{b'} \) are the same.

The following result follows by Remark 3.7, 4.10 and 4.14 immediately (also by [21 Thm. C]).

Lemma 4.16. Assume that \( \ell \equiv \text{gcd}(n, q - e) \).

(i) The restriction of \( \ell \)-Brauer characters gives a bijection from the set of irreducible \( \ell \)-Brauer characters in unipotent \( \ell \)-blocks of \( G \) to the set of \( \ell \)-Brauer characters in unipotent \( \ell \)-blocks of \( X \).

(ii) Let \( b \) be a unipotent \( \ell \)-block of \( X \), then there exists a unique unipotent \( \ell \)-block \( B \) of \( G \) which covers \( b \). Moreover, \( \text{Res}^G_X : \text{IBr}_\ell(B) \to \text{IBr}_\ell(b) \) is a bijection.

Now we consider the extendibility of the irreducible \( \ell \)-Brauer characters in unipotent \( \ell \)-blocks of \( X = \text{SL}_n(e) \).

Proposition 4.17. Let \( \chi \in \text{Irr}(G) \), then \( \chi \) extends to \( (G \rtimes D)_\chi \).

Proof. Firstly, \( (G \rtimes D)_\chi = G \rtimes D_\chi \). If \( D_\chi \) is cyclic, then \( \chi \) extends to \( (G \rtimes D)_\chi \). If \( D_\chi \) is not cyclic, then \( D_\chi = \langle \gamma, F^\ell \rangle \) for some \( \gamma \mid \ell \). By [6 Thm. 4.3.1 and Lem. 4.3.2], there exists an extension \( \hat{\chi} \) of \( \chi \) to \( G \rtimes (F^\ell)_p \) such that \( \hat{\chi}(F^\ell)_p \neq 0 \). Since \( \gamma \) fixes \( \chi \), \( \gamma \hat{\chi} \) is also an extension of \( \chi \). Also we have \( \hat{\chi}^\gamma(F^\ell)_p = \hat{\chi}(F^\ell)_p \neq 0 \) since \( \gamma \) and \( F^\ell \) commute. By a direct consequence of Gallagher’s theorem (see [38 Rmk. 9.3(a)]), we have \( \hat{\chi}^\gamma = \hat{\chi} \), hence \( \hat{\chi} \) is \( \gamma \)-invariant. So \( \hat{\chi} \) has an extension to \( G \rtimes D_\chi \), which is also an extension of \( \chi \). \( \square \)
The following lemma is \([37, \text{Lem. 1.27 and Prop. 1.29}].\)

**Lemma 4.18.** Let \(N \cong K\) be finite groups, \(B\) be an \(\ell\)-block of \(N\) and \(B \subseteq \text{Irr}(B)\) a basic set of \(B\). Suppose that the \(\ell\)-decomposition matrix associated with \(B\) and \(\text{IBr}(B)\) is unitriangular with respect to a suitable ordering. Assume that every character in \(B\) is invariant under \(K\). Then every irreducible \(\ell\)-Brauer character of \(B\) is invariant under \(K\). Moreover, if every character in \(B\) extends to \(K\), then every irreducible \(\ell\)-Brauer character of \(B\) extends to \(K\).

**Corollary 4.19.** Let \(\phi \in \text{IBr}(G)\) in a unipotent \(\ell\)-block of \(G\), then \(\phi\) extends to \(G \rtimes \Delta\).

**Proof.** It is well-known that every unipotent character of \(G\) is \(D\)-invariant (see, for example, \([32, \text{Thm. 2.5}].\)) Now by \([20]\), after a suitable arrangement, the decomposition matrix of \(G\) with respect to \(E(G^\ell, t')\) is unitriangular. Then the claim follows by Proposition \([4,17]\) and Lemma \([4,18].\)

Thus by Lemma \([4,16]\) and Corollary \([4,19]\) we have:

**Corollary 4.20.** Let \(\ell \nmid \gcd(n, q - \epsilon)\), and \(\theta \in \text{IBr}(X)\) in a unipotent \(\ell\)-block of \(X\), then \(\theta\) extends to \(G \rtimes \Delta\).

## 5 Weights of \(\text{SL}_n(\epsilon q)\)

### 5.1 Radical subgroups of \(\text{GL}_n(\epsilon q)\)

Firstly, we consider the case that \(\ell\) is an odd prime and let \(a = \nu_\ell((\epsilon q)^\ell - 1)\). We first recall the basic constructions in \([2]\) and \([5]\). Let \(\alpha, \gamma\) be non-negative integers, \(Z_\alpha\) be the cyclic group of order \(\ell^{\alpha+\theta}\) and \(E_\alpha\) be an extraspecial \(\ell\)-group of order \(\ell^{2\alpha+1}\). We may assume the exponent of \(E_\alpha\) is \(\ell\) by \([2, (4A)]\) and \([5, (2B)].\) Denote by \(Z_\alpha E_\gamma\) the central product of \(Z_\alpha\) and \(E_\gamma\) over \(\Omega_1(Z_\alpha) = Z(E_\gamma)\). Assume \(Z_\alpha E_\gamma = \langle z, x_j, y_j \mid j = 1, \ldots, \gamma \rangle\) with \(\langle z \rangle = Z_\alpha, E_\gamma = \langle x_j, y_j \mid j = 1, \ldots, \gamma \rangle\), \(o(z) = \ell^{\alpha+\theta}\), \(o(x_j) = \ell (1 \leq j \leq \gamma), \langle x_j, y_j \rangle = \langle x_j, y_j \rangle_1 = 1\) if \(i \neq j\), and \(\langle x_j, y_j \rangle = x_j y_j x_j^{-1} y_j^{-1} = z^{z^{\alpha+\theta}}\).

The group \(Z_\alpha E_\gamma\) can be embedded into \(\text{GL}(\ell^\gamma, (\epsilon q)^\ell^\alpha)\) uniquely up to conjugacy in the sense that \(Z_\alpha\) is identified with \(\text{Out}(Z(\text{GL}(\ell^\gamma, (\epsilon q)^\ell^\alpha)))\). We denote by \(\text{R}_{\alpha, \gamma}\) the image of \(Z_\alpha E_\gamma\) in \(\text{GL}(\ell^\gamma, (\epsilon q)^\ell^\alpha)\). Then by \([5, (1C)],\) \(\text{R}_{\alpha, \gamma}\) is unique up to conjugacy in \(\text{GL}(\ell^\gamma, (\epsilon q)^\ell^\alpha)\) in the sense that \(Z(\text{R}_{\alpha, \gamma})\) is primary.

Let \(\text{R}_{m, a, \gamma} = \text{R}_{m, a} \otimes I_{(m)}\). For each positive integer \(c\), let \(A_c\) denote the elementary abelian group of order \(\ell^c\). For a sequence of positive integers \(c = (c_1, \ldots, c_t)\) with \(t \geq 0\), we denote by \(A_c = A_{c_1} \times \cdots \times A_{c_t}\) and \(|c| = c_1 + \cdots + c_t\). Then \(A_c\) can be regarded as an \(\ell\)-subgroup of the symmetric group \(S(\ell^d)\). Groups of the form \(\text{R}_{m, a, \gamma} : = \text{R}_{m, a} \rtimes A_c\) are called the basic subgroups. \(\text{R}_{m, a, \gamma} : \text{R}_{m, a, \gamma}\) is just \(\text{R}_{m, a, \gamma}\) in \([19]\) which we will write as \(\text{R}_{m, a}\) here. By \([2, (4A)]\) and \([5, (2B)],\) any \(\ell\)-radical subgroup \(R\) of \(G\) is conjugate to \(R_0 \times R_1 \times \cdots \times R_n\), where \(R_0\) is a trivial group and \(R_i (i \geq 1)\) is a basic subgroup.

Let \(G_{m, a} = \text{GL}(m\ell^a, \epsilon q), G_{m, a, \gamma} = \text{GL}(m\ell^a\gamma, \epsilon q). C_{m, a} = C_{G_{m, a}}(R_{m, a, \gamma})\) and \(C_{m, a, \gamma} = C_{G_{m, a, \gamma}}(R_{m, a, \gamma})\), then \(C_{m, a, \gamma} = C_{m, a} \otimes I_{\gamma}\). Let \(G_{m, a, \gamma} : = \text{GL}(m\ell^a\gamma + \epsilon q, \epsilon q)\) and \(C_{m, a, \gamma} : = C_{G_{m, a, \gamma}}(R_{m, a, \gamma})\). Then \(C_{m, a, \gamma} : = C_{m, a} \otimes I_{\gamma}\). We will also use the notation that \(N_{m, a} : = N_{G_{m, a}}(R_{m, a, \gamma})\).

Now we consider the case that \(\ell = 2\). Assume that \(q\) is odd and let \(a\) be the positive integer such that \(2^{a+1} = (q^2 - 1) / 2\). We will use the following conventions:

- **Case 1** \(4 \mid q - \epsilon\) or \(4 \mid q + \epsilon\) and \(a \geq 1\);
- **Case 2** \(4 \mid q + \epsilon\) and \(a = 0\).

We first recall the basic constructions in \([3]\) and \([4]\).

Let \(\alpha, \gamma\) be non-negative integers. We denote by \(Z_\alpha\) the cyclic group of order \(2^{\alpha+\theta}\) in **Case 1** and of order \(2\) in **Case 2**. Let \(E_\alpha\) be an extraspecial group of order \(2^{2\alpha+1}\). Denote by \(Z_\alpha E_\gamma\) the central product of \(Z_\alpha\) and \(E_\gamma\) over \(\Omega_1(Z_\alpha) = Z(E_\gamma)\). Thus in **Case 2**, \(Z_\alpha E_\gamma = E_\gamma\). Assume \(Z_\alpha E_\gamma = \langle z, x_j, y_j \mid j = 1, \ldots, \gamma \rangle\) with \(\langle z \rangle = Z_\alpha, E_\gamma = \langle x_j, y_j \mid j = 1, \ldots, \gamma \rangle, [x_j, y_j] = x_j y_j x_j^{-1} y_j^{-1} = z^{z^{\alpha+\theta}}\) in **Case 1** and \(z \in C_{\text{R}_{m, a}}\) in **Case 2**. Assume furthermore that \(o(x_j) = o(y_j) = 2\) for \(j \geq 2\) and \(o(x_1) = o(y_1) = 2\) or \(2^2\) when \(E_\gamma\) is of plus type or minus type respectively, which means that \(\langle x_1, y_1 \rangle\) is isomorphic to \(D_8\) or \(Q_8\). We may assume \(E_\gamma\) is of plus type in **Case 1**.
The group $Z_nE_γ$ can be embedded into $GL(2^t,(eq)^{2^n})$ uniquely up to conjugacy in the sense that $Z_n$ is identified with $O_2(Z(GL(2^t,(eq)^{2^n})))$ by [3, p.509] and [4, p.266]. We denote by $R_{a,γ}$ the image of $Z_nE_γ$ in $GL(2^t,(eq)^{2^n})$. Then by [3, p.510] and [4, p.266], $R_{a,γ}$ is unique up to conjugacy in $GL(2^t,(eq)^{2^n})$ in the sense that $Z(R_{a,γ})$ is primary. Set $R_{m,a,γ} = R_{a,γ} ⊗ I_m$.

Now assume $4 | q + e$, then $GL(2, eq)$ has a Sylow 2-subgroup isomorphic to the semi-dihedral group $S_{a+2}$ of order $2^{a+2}$, thus $S_{a+2}$ is unique up to conjugacy in $GL(2, eq)$. Denote by $S_{a+2} E_γ$ the central product of $S_{a+2}$ and $E_γ$ over $Z(S_{a+2}) = Z(E_γ)$. We may assume $E_γ$ is of plus type by [3, (1F)] and [4, (11)]. Also, $S_{a+2} E_γ$ can be embedded into $GL(2^t,(eq)^{2^n})$ and we denote by $S_{1,γ}$ the image of $S_{a+2} E_γ$. By [3, (1F)] and [4, (11)], $S_{1,γ}$ is unique up to conjugacy in $GL(2^t+1, eq)$. Set $S_{m,1,γ} = S_{1,γ} ⊗ I_m$.

For each $α ≥ 0, γ ≥ 0, m ≥ 1$ and $1 ≤ i ≤ 2$, define

$$R_{m,a,γ}^i = \begin{cases} S_{m,1,γ-1}^i & \text{in Case 2 and } γ ≥ 1, i = 2; \\ R_{m,a,γ} & \text{otherwise.} \end{cases}$$

For each positive integer $c$, let $A_c$ denote the elementary abelian group of order $2^c$. For a sequence of positive integers $c = (c_1, \ldots, c_t)$ with $t ≥ 0$, we denote by $A_c = A_{c_1} \times \cdots \times A_{c_t}$ and $|c| = c_1 + \cdots + c_t$. Set $R^i_{m,a,γ,c} = R^i_{m,a,γ} \uparrow A_c$.

Groups of the form $R^i_{m,a,γ,c}$ are called the basic subgroups except in Case 2 and $γ = 0, c_1 = 1$. By [3, (2B)] and [4, (2B)], any 2-radical subgroup $R$ of $G$ is conjugate to $R_1 \times \cdots \times R_s \times R_{s+1} \times \cdots \times R_n$, where $R_i = \{±I_m\}$ for $1 ≤ i ≤ s$ and $R_i (i ≥ s+1)$ are basic subgroups. Moreover, if $4 | q - e$, then $s = 0$.

When considering further the weights instead of only radical subgroups, we can exclude some basic subgroups which do not afford any weight by the remark on [3, p.518] and [4, p.275]. Thus as in [3] and [4], we may assume every component of a 2-radical subgroup is of the form $D_{m,a,γ,c}$ defined as follows:

$$D_{m,a,γ,c} = \begin{cases} R_{m,a,γ,c} & \text{in "Case 1" or "Case 2 and } γ = 0, c_1 \neq 1"; \\ S_{m,1,γ-1,c} & \text{in Case 2 and } γ ≥ 1; \\ R_{m,0,1,c} & \text{in Case 2 and } γ = 0, c_1 = 1, \end{cases}$$

(5.1)

where $c' = (c_2, \ldots, c_t)$ for $c = (c_1, \ldots, c_t)$ and in Case 2 and $γ = 0, c_1 = 1, R_{m,0,1}$ is a quaternion group. We will use some obvious simplification of the notations, such as $D_{a,γ} = D_{0,a,γ,0}$. Note also that $D_{m,0,0,0}$ in Case 2 is just the group $\{±I_m\}$.

In order to deal with the two cases that $ℓ$ is odd and $ℓ = 2$ simultaneously, we use the notation $D_{m,a,γ,c}$ standing for the basic subgroups, so for an odd prime $ℓ$, $D_{m,a,γ,c} = R_{m,a,γ,c}$, and for $ℓ = 2$, $D_{m,a,γ,c}$ is as in (5.1).

Lemma 5.1. Assume that $ℓ ∤ gcd(n, q - e)$. Let $R$ be an $ℓ$-radical subgroup of $G$, then $D(R C_ℓ(R)) = D(NC_ℓ(R)) = 3$.

Proof. Note that $O_ℓ(Z(G)) ≤ R$, hence $D(R) = O_ℓ(3)$ since $ℓ ∤ gcd(n, q - e)$. So it suffices to show that $O_ℓ(R) ≤ D(C_ℓ(R))$. By the structure of $ℓ$-radical subgroups, it suffices to show that for a basic subgroup $D_{m,a,γ,c} ≤ G$, we have $O_ℓ(R) ≤ D(C_{m,a,γ,c})$.

By [2], [3], [4] and [5], $C_{m,a,γ,c} \cong C_{m,a} ⊗ I_{γ+|c|}$ where $C_{m,a} \cong GL(m,q^{e_m})$. The elements of $C_{m,a,γ,c}$ have the form $\text{diag}(g, \ldots, g)$ where $g \in C_{m,a}$. Also, $C_{m,a}$ is the image under the embedding $GL(m,(eq)^{e_m}) \hookrightarrow GL(mel^2, eq)$. Let $c$ be a generator of the group $\{x \in F_{(eq)^{e_m}} \mid x^{(eq)^{e_m+1}} = 1\}$ and $Δ$ the minimal polynomial of $c$ over $F$. Then $Δ$ has roots $c, cq, \ldots, c^{(eq)^{e_m+1}}$. Hence $\det((Δ)) = c^{(eq)^{e_m+1}}$. Hence $Δ((Δ))$ is a generator of the group $3$. Then $D(C_{m,a}) = 3$. So $O_ℓ(3) ≤ D(C_{m,a,γ,c})$ since $C_{m,a,γ,c} \cong C_{m,a} ⊗ I_{γ+|c|}$. This completes the proof. □

5.2 Radial subgroups of $SL_n(eq)$

Now we consider the $ℓ$-radical subgroups of $X$. Let $X = XZ(G)$. We will always assume $ℓ ∤ gcd(n, q - e)$ from now on to the end of this section.

By Lemma [2,3] the map $Rad_ℓ(G) \to Rad_ℓ(X)$ given by $R \mapsto R \cap X$ is surjective. In fact, we have:
Proposition 5.2. \( R \mapsto R \cap X \) gives a bijection from \( \text{Rad}_\ell(G) \) to \( \text{Rad}_\ell(X) \) with inverse given by \( S \mapsto \text{SO}_\ell(Z(G)) \).

**Proof.** Firstly, we have \( \text{Rad}_\ell(G) = \text{Rad}_\ell(\hat{X}) \), since \( \ell \nmid |G/\hat{X}| \).

Since \( \hat{X}/Z(X) \cong X/Z(X) \times Z(G)/Z(X) \) and \( Z(X) \) is a central \( \ell \)-subgroup of \( \hat{X} \), by the same argument as the proof of [18, Lem. 4.5] (use [18, Lem. 4.3 and 4.4]), there is a bijection \( \text{Rad}_\ell(\hat{X}) \to \text{Rad}_\ell(X) \) given by \( R \mapsto R \cap X \) with inverse given by \( S \mapsto \text{SO}_\ell(Z(G)) \). \( \Box \)

**Lemma 5.3.** Let \( R \) be an \( \ell \)-radical subgroup of \( G \) and \( S = R \cap X \). Then

(i) \( C_X(S) = C_G(R) \cap X, \quad SC_X(S) = RC_G(R) \cap X, \quad N_X(S) = N_G(R) \cap X \);

(ii) \( RC_G(R)/SC_X(S) \cong N_G(R)/N_X(S) \cong G/X \).

**Proof.** By Proposition 5.2, \( R = \text{SO}_\ell(Z(G)) \), so we have \( C_X(S) = C_G(R) \cap X, \quad N_X(S) = N_G(R) \cap X \). Also \( RC_G(R) \cap X = SC_G(R) \cap X = S(C_G(R) \cap X) = SC_X(S) \) and then we obtain (i). By Lemma 5.1 we have \( G = XRC_G(R) \) and then \( G = XN_G(R) \). Thus (ii) follows. \( \Box \)

Let \( R \) be an \( \ell \)-radical subgroup of \( G \), by Lemma 5.1 \( G = XN_G(R) \). So if two \( \ell \)-radical subgroups of \( G \) are \( G \)-conjugate, then they are \( X \)-conjugate. Thus by Proposition 5.2 and Lemma 5.3 we have:

**Corollary 5.4.** \( R \mapsto R \cap X \) gives a bijection from \( \text{Rad}_\ell(G)/\sim_G \) to \( \text{Rad}_\ell(X)/\sim_X \).

5.3 Weights of \( \text{SL}_n(\text{eq}) \)

Now we consider the \( \ell \)-weights of \( X \) with \( \ell \nmid \text{gcd}(n, q - \varepsilon) \).

**Proposition 5.5.** Assume that \( \ell \nmid \text{gcd}(n, q - \varepsilon) \). Let \( (R, \varphi) \) be an \( \ell \)-weight of \( G \) and \( S = R \cap X \), then \( (S, \psi) \) is an \( \ell \)-weight of \( X \) for every \( \psi \in \text{Irr}(N_X(S) \mid \varphi) \).

Conversely, let \( (S, \psi) \) be an \( \ell \)-weight of \( X \) and \( R = \text{SO}_\ell(Z(G)) \), then there exists \( \varphi \in \text{Irr}(N_G(R) \mid \psi) \) such that \( (R, \varphi) \) is an \( \ell \)-weight of \( G \).

**Proof.** By Lemma 2.4 and Proposition 5.2 and Lemma 5.3. \( \Box \)

**Remark 5.6.** Let \( \mathcal{W}_\ell(G) \) be a complete set of representatives of all \( G \)-conjugacy classes of \( \ell \)-weights of \( G \). We may assume that for \( (R_1, \varphi_1), (R_2, \varphi_2) \in \mathcal{W}_\ell(G), R_1 \) and \( R_2 \) are \( G \)-conjugate if and only if \( R_1 = R_2 \).

Now define an equivalence relation on \( \mathcal{W}_\ell(G) \) such that for \( (R_1, \varphi_1), (R_2, \varphi_2) \in \mathcal{W}_\ell(G), (R_1, \varphi_1) \sim (R_2, \varphi_2) \) if and only if \( R_1 = R_2 \) and \( \varphi_1 = \varphi_2 \eta \) for some \( \eta \in \text{Irr}(N_G(R_1)/N_X(R_1)) \). Then by Lemma 2.4 Corollary 5.4 and Proposition 5.5, the set \( \{(R \cap X, \psi)\} \), where \( (R, \varphi) \) runs through a complete set of representatives of the equivalence classes of \( \mathcal{W}_\ell(G)/\sim \) and \( \psi \) runs through \( \text{Irr}(N_X(R) \mid \varphi) \), is a complete set of representatives of all \( X \)-conjugacy classes of \( \ell \)-weights of \( X \).

**Remark 5.7.** Let \( (R, \varphi) \) be an \( \ell \)-weight of \( G \), \( (S, \psi) \) an \( \ell \)-weight of \( X \) such that \( S = R \cap X \) and \( \varphi \in \text{Irr}(N_G(R) \mid \psi) \). Let \( b = bl_\ell(\varphi), b_0 = bl_\ell(\psi) \) and \( B = b^G \) and \( B_0 = b_0^G \). By Lemma 2.3 if \( B \) covers \( B_0 \), then \( B \) covers \( B_0 \).

Let \( B_0 \) be an \( \ell \)-block of \( X \). Denote by \( B_0 \) the union of the \( \ell \)-blocks of \( X \) which are \( G \)-conjugate to \( B_0 \) and \( B \) the union of the \( \ell \)-blocks of \( G \) which cover \( B_0 \). Then

- if \( (R, \varphi) \) is an \( \ell \)-weight of \( G \) belonging to \( B \) and \( S = R \cap X \), then for every \( \psi \in \text{Irr}(N_X(S) \mid \varphi) \), \( (S, \psi) \) is an \( \ell \)-weight of \( X \) belonging to \( B_0 \); and

- if \( (S, \psi) \) is an \( \ell \)-weight of \( X \) belonging to \( B_0 \) and \( R = \text{SO}_\ell(Z(G)) \), then there exists \( \varphi \in \text{Irr}(N_G(R) \mid \psi) \) such that \( (R, \varphi) \) is an \( \ell \)-weight of \( G \) belonging to \( B \).
Let \((R, \varphi)\) be an \(\ell\)-weight of \(G\). For some \(\eta \in \text{Irr}(N_G(R)/N_X(R))\), \((R, \eta \varphi)\) is also an \(\ell\)-weight of \(G\), then \(O_{\ell}(Z(G)) \subseteq \ker \eta\) since \(O_{\ell}(Z(G)) \subseteq R\) by Proposition 5.2. Hence \(\eta \in O_{\ell}(\text{Irr}(N_G(R)/N_X(R)))\).

By Lemma 5.3, \(N_G(R)/N_X(R) \cong G/X\), now we identify \(\text{Irr}(N_G(R)/N_X(R))\) with \(\text{Irr}(G/X)\). So in order to compute \(\kappa_{N_G(R)}(\varphi)\), it suffices to consider when \(\text{Re}^G_{N_G(R)}(\hat{z}) \cdot \varphi = \hat{z}\) for \(z \in O_{\ell}(\hat{3})\). We often abbreviate \(\hat{z}\) for \(\text{Re}^G_{N_G(R)}(\hat{z})\).

Now we recall the description of \(\ell\)-weights of \(G\) in [2], [3], [4] and [5] and give some more notations and conventions.

We denote by \(\mathcal{F}'\) the subset of \(\mathcal{F}\) consisting of polynomials whose roots are of \(\ell\)'-orders. By [8 (3.2)], given any \(\Gamma \in \mathcal{F}'\), there is a unique \(\ell\)-block \(B_{\ell} \in G_{\ell}\) with \(\Gamma = \text{GL}(\mathbb{m}_\ell e^{\alpha \delta}, e_{\ell})\) with \(\ell \neq \mathbb{Z}\) and \(\ell = 1\). Also, note that there is no direct connection between \(m_{\ell}\) and \(m_{\ell}(s)\). These results have been proved for odd primes in [19, (5A)] and for \(\ell = 2\) on [3, p.520] and [4, p.276] using the results from [8]. Let \(C_{\Gamma} = C_{G_{\ell}}(D_{\ell})\) and \(N_{\ell} = N_{G_{\ell}}(D_{\ell})\). Then \(C_{\ell} \cong \text{GL}(m_{\ell}(s), (e_{\ell})^{d_{\ell}})\).

The polynomial \(\Gamma\) also determines a unique \(N_{\ell}\)-conjugacy classes of pairs (\(b_{\ell}\), \(\theta_{\ell}\)) where \(b_{\ell}\) is a root \(\ell\)-block of \(C_{\ell}D_{\ell} = C_{\ell}\) with defect group \(D_{\ell}\) and \(\theta_{\ell}\) is the canonical character of \(b_{\ell}\). The subgroup \((D_{\ell}, \phi_{\ell})\) has the label \((D_{\ell}, \phi_{\ell}, \phi_{\ell})\) as in [8 (3.2)]. Since \(d_{\ell} = d, \alpha_{\ell} = \alpha_{\ell}, \ell = m_{\ell}, \ell = m_{\ell}\), we may assume that \(\ell = m_{\ell}, \ell = m_{\ell}\).

Let \(\ell \in \mathcal{F}'\) and keep the notation of the previous sections. Let \(D_{\ell}, G_{\ell} = D_{m_{\ell}, \alpha_{\ell}, \ell} = \ell\) be a basic subgroup and let \(G_{\ell, \ell} = C_{\ell}G_{\ell, \ell}, N_{\ell, \ell} = C_{\ell}G_{\ell, \ell}\) be defined similarly. Then \(G_{\ell, \ell} = C_{\ell} \otimes I_{\ell}, I_{\ell}\). Let \(\theta_{\ell, \ell} = \theta_{\ell} \otimes I_{\ell}, I_{\ell}\). Hence \(\ell, \ell\) can be viewed as the canonical character of \(C_{\ell}G_{\ell, \ell}\) with \(\ell\) in the kernel and all canonical characters are of this form.

Note that the equations [3 (3.2)] and [4 (3.1)] can be written also uniformly in this form (see the remarks before [50 Prop. 4.2.4.3]). Let \(\mathcal{R}_{\ell, \ell}\) be the set of all the basic subgroups of the form \(D_{\ell, \ell} = \ell\) with \(\ell = \delta\) and denote \(D_{\ell} = D_{\ell, \ell}\). Denote the basic subgroups in \(\mathcal{R}_{\ell, \ell}\) as \(D_{\ell, \ell}, D_{\ell, \ell, \ell}, \ldots\). Denote the canonical character associated to \(D_{\ell, \ell}\) by \(\theta_{\ell, \ell}\). It is possible that there exists \(\ell \in \mathcal{F}'\) such that \(m_{\ell} = m_{\ell} = m\) and \(\alpha_{\ell} = \alpha_{\ell} = \alpha\). In this case, \(\mathcal{R}_{\ell, \ell} = \mathcal{R}_{\ell, \ell}\) and naturally we may consider the labeling of \(\mathcal{R}_{\ell, \ell}\) and \(\mathcal{R}_{\ell, \ell}\) such that \(D_{\ell, \ell} = D_{\ell, \ell}\) for \(i = 1, 2, \ldots\). We will denote \(D_{m_{\ell}, \alpha_{\ell}, \ell}\) as \(D_{\ell, \ell}\) or \(D_{\ell, \ell}\) depending on whether the related canonical character of \(C_{\ell}G_{\ell, \ell}\) \(C_{\ell}G_{\ell, \ell}\) is considered to be \(\theta_{\ell} = \theta_{\ell}\).

For \(\ell \in O_{\ell}(\hat{3})\), \(\hat{z}\) is a linear character of \(G_{\ell, \ell}\). By the proof of Lemma 5.1, \(O_{\ell}(D_{\ell, \ell}) = O_{\ell}(D_{\ell, \ell})\), so \(\hat{z}\) may be regarded as a character of \(G_{\ell, \ell}\) (by restriction). Here we need some precise information of \(\hat{z}\).

**Remark 5.8.** Now we recall the description of the map \(\hat{z}\) in [8]. As pointed in [8 note 2 (p.186)], the isomorphism in Equation (3.1) is not uniquely determined. Also, the author introduces a set \(S(G)\) to replace the set of semisimple elements of \(G\) in [8].

Firstly, denote by \(k\) a subfield of \(\mathbb{Q}\) of finite degree over \(\mathbb{Q}\). Also, assume that \(k\) is big enough for all finite groups considered. Suppose that we have chosen an algebraic closure \(\overline{\mathbb{F}}\) of \(\mathbb{F}\), an isomorphism \(\iota: \mu(\overline{\mathbb{Q}}) \rightarrow \overline{\mathbb{Q}}/\mathbb{Q}\) and an isomorphism \(\iota': \overline{\mathbb{F}}^\times \rightarrow (\mathbb{Q}/\mathbb{Z})_{p'}\).

Let \(s\) be a semisimple element of \(G\), then \(L = C_{\ell}(s) = \prod L_{\ell, s}\) with \(L_{\ell} \cong GL(m_{\ell}(s), (e_{\ell})^{d_{\ell}})\). If \(\ell_{\ell}\) denotes the field generated by \(Z(L_{\ell})\) in \(\text{End}_{\mathbb{C}}(\mathbb{F})\), the group \(Z(L_{\ell})\) is equal to the subgroup of order \(|(e_{\ell})^{d_{\ell}} - 1|\) of \(\mathbb{F}_{\ell}^\times\). Every family \(s\) of embeddings \(\iota_{\ell}: \ell \rightarrow \mathbb{F}_{\ell}^\times\) over \(\mathbb{F}\) is associated to a character \(\xi_{\ell}(s)\) of \(Z(L_{\ell})\) with values in \(k\) in the following way. Let \(\ell_{\ell}\) be the particular generator of \(Z(L_{\ell})\) defined by the corresponding embedding of \(\mathbb{F}_{\ell}^\times\) in \(\mathbb{Q}/\mathbb{Z}\). The character \(\xi_{\ell}(s)\) defined by the equation \(\iota(\xi_{\ell}(s)(s)) = \ell'(\sigma_{\ell}(s))\).

We denote by \(S(G)\) the set of pairs \((L, \xi)\) such that there exists semisimple element \(s\) of \(G\) and an embedding \(\mathbb{F} \subseteq \mathbb{F}_{\ell}\), \(\iota, \iota', \sigma, \alpha\) such that \(L = C_{\ell}(s)\) and \(\xi = \xi_{\ell}(s)\). Then by [8 (4.4)], the \(\mathcal{G}\)-conjugacy classes of \(S(G)\) are in bijection with the set of \(\mathcal{G}\)-conjugacy classes of semisimple elements of \(G\).

If \(s = (L, \xi) \in S(G)\), we denote by \(\hat{z}\) the linear character of \(L = C_{\ell}(s)\) with values in \(k\) obtained by composing \(\xi\) with the (surjective) morphism \(\text{det}_{L}: L \rightarrow Z(L)\) (defined in [8 p.171]. Indeed, if \(h \in L\), we write \(h = \prod h_{\ell}\) corresponding to the decomposition \(L = \prod L_{\ell}\). Also, we identify \(Z(L_{\ell})\) with
Remark 5.12. We choose the labeling of $GL(e,F)$ in $GL(_{finite}F)$ such that $\det_{\ell}(h) = [GL(e,F)] \cdot (eq)^{h}$ for all $h \in _{finite}GL(e,F)$.

Let $s = (C_{G}(s), \zeta) \in S(G)$, and $X$ an $\ell'$-subgroup of $C_{G}(s)$. We set $C = C_{G}(X)$, and we define an element $\gamma_{X} = (C(s), \zeta_{X})$ in the following way: we may suppose that $C_{G}(s) = \prod_{\ell} L_{\ell}$, where $L_{\ell} \cong GL(m_{\ell}, (eq)^{h})$. Then $Z(L_{\ell})$ is isomorphic to a product of $GL(1, F_{i})$ where $F_{i}$ is a certain extension of $F$. For any element $\ell$ of one such factor, we set $\zeta_{X}(\ell) = \zeta(N_{F_{i}/F_{i}}(eq^{h})(z))$. The surjectivity of the norm in the finite extensions of finite fields allows then to establish that $(C_{G}(s), \zeta_{X})$ belongs to $S(C)$. Noting that if $X$ is abelian, the linear character $\delta_{X}$ is simply the restriction to $C_{G}(s)$ of the linear character $\delta$ to $C_{G}(s)$. Also, the map form $S(G)$ into $S(C)$ which associates $s_{X}$ to $s$ is surjective. We often omit the index $X$ in $s_{X}$.

Remark 5.9. Abbreviate $\theta_{G} = d_{G, i}, \eta = e_{i, j}$ and hence it can be regarded as a generator of $GL_{\ell}(e,F)$.

Proof. By [8, Prop. 4.16], $\theta_{G, i} = \pm R_{G, i}(\zeta_{i})$, where $s$ is a semisimple $\ell'$-element of $C_{G, i}$, which has only one elementary divisor $\Gamma$ with multiplicity $\ell'_{G, i}$ (as in Remark 5.9). Note that $\theta_{G, i} = C_{G, i}(\zeta_{i}) = C_{G, i}(\zeta_{i})$. Then $\Res_{G, i}(\zeta_{i}) \cdot \theta_{G, i} = \pm \Res_{G, i}(\zeta_{i}) \cdot R_{G, i}(\zeta_{i}) \cdot \hat{\zeta}_{i}$. Notice that $\zeta_{i}$ is a semisimple $\ell'$-element of $C_{G, i}$, which has only one elementary divisor $\zeta_{i}$ with multiplicity $\ell'_{G, i}$. This completes the proof.

Let $\gamma_{G, i}$ be the set of characters of $N_{G, i}(\theta_{G, i})$ lying over $\theta_{G, i}$ and of defect zero as characters of $N_{G, i}(\theta_{G, i})/D_{\ell}$, and $\gamma_{G, i} = \bigcup_{i} \gamma_{G, i}$. By Clifford theory, this set is in bijection with the set of characters of $N_{G, i}$ lying over $\theta_{G, i}$ and of defect zero as characters of $N_{G, i}/D_{\ell}$ for all $i$. We assume $\gamma_{G, i} = \{\psi_{G, i, j}\}$ with $\psi_{G, i, j}$ a character of $N_{G, i}(\theta_{G, i})$. Note that for $\ell = 2$, $j$ has only one choice. Also, we may assume $D_{\ell} = D_{\ell, i, j}$, $N_{G, i} = N_{G, i, j}$, and $C_{G, i} = C_{G, i, j}$.

Convention 5.11. We choose the labeling of $\gamma_{G, i}$ and $\gamma_{G, i}$ such that $\Res_{G, i}(\zeta_{i}) \cdot \psi_{G, i, j} = \psi_{G, i, j}$.

Remark 5.12. We can make Convention 5.11 because if for some $z \in O_{G}(3)$, $\Res_{G, i}(\zeta_{i}) \cdot \theta_{G, i} = \theta_{G, i}$, then $\Res_{G, i}(\zeta_{i})$ fixes every element of $\gamma_{G, i}$. In fact, if $\ell = 2$, then $\gamma_{G, i}$ has only one element.
by [3] and [4]. If \( \ell \) is odd, and we assume that \( D_{\Gamma,\delta,i} = R_{m_{\nu_{\ell,i}},\gamma} \), then by [2] p.14 and [5] p.10, \( N_{T,\delta,i}/D_{\Gamma,\delta,i} \cong N_{m_{\nu_{\ell,i}},\gamma}/R_{m_{\nu_{\ell,i}},\gamma} \times Y_{\epsilon}/A_{\epsilon} \), for some subgroups \( Y_{\epsilon} \) and \( A_{\epsilon} \). Also, all elements of \( Y_{\epsilon} \) and \( A_{\epsilon} \) are permutation matrices and then have determinant 1. So we may assume that \( |\epsilon| = 0 \). By the construction of \( (N_{m_{\nu_{\ell,i}},\gamma})_{\delta,i} \) in [2] and [5], we may assume that \( \gamma = 0 \) and then \( D_{\Gamma,\delta,i} = R_{\Gamma} \) and \( N_{T,\delta,i} = N_{\Gamma} \). By [50] §4, up to conjugation, \( N_{\Gamma} = C_{\Gamma} \rtimes \langle P \rangle \), where \( P \) is a permutation matrix. Thus \( R_{\mu_{\ell,i}}(z) \) fixes every element of \( \mathcal{G}_{\Gamma,\delta,i} \).

We use the notation from [50] §5 now. Define \( \mathcal{W}_{\ell}(G) \) to be the \( G \)-conjugacy classes of the set

\[
\left\{ (s, \lambda, K) \mid s \text{ is a semisimple } \ell' \text{-element of } G, \lambda = \prod_{\Gamma} \lambda_{\Gamma}, \lambda_{\Gamma} \text{ is the } \ell_{\Gamma} \text{-core of a partition of } m_{\Gamma}(s), K = K_{\Gamma}, K_{\Gamma} : \bigcup_{\Theta} \mathcal{G}_{\Theta,\delta} \to \{ \ell \text{-cores } \} \text{ s.t.} \\
\sum_{\delta,i,j} \ell^{\delta}(\lambda_{\Gamma}(\Theta_{\delta,i,j})) = w_{\Gamma}, m_{\Gamma}(s) = |\lambda_{\Gamma}| + \ell_{\Gamma}w_{\Gamma} \right\}.
\]

Note that for \( \ell = 2 \), the triple becomes \( (s, -, K) \).

A bijection between \( \mathcal{W}_{\ell}(G) \) and \( \mathcal{W}_{\ell}(G) \) has been constructed implicitly in [2], [3], [4] and [5] and can be described as follows. Let \( (R, \varphi) \) be an \( \ell \)-weight of \( G \). Set \( C = C_{G}(R) \) and \( N = N_{C}(R) \). Then there exists an \( \ell \)-block of \( CR \) with a defect group such that \( \varphi = \text{Ind}_{\mathbb{G}(\theta)}^{\mathbb{Z}(\theta)} \psi \) where \( \theta \) is the canonical character of \( b \) and \( \psi \) is a character of \( N(\theta) \) lying over \( \theta \) and of \( \ell \)-defect zero as a character of \( N(\theta)/R \). Assume \( R = D_{0}D_{+} \) with \( D_{0} \) an identity group of degree \( n_{0} \) and \( D_{+} \) a product of basic subgroups. Note that for \( \ell = 2 \), \( R = D_{+} \). Then \( C, N, \varphi, \theta, \psi, N(\theta) \) can be decomposed accordingly.

First, we have \( C_{0} = N_{0} = \text{GL}(n_{0}, \epsilon_{0}) \) and \( \varphi_{0} = \psi_{0} = b_{0} \) a character of \( \text{GL}(n_{0}, \epsilon_{0}) \) of \( \ell \)-defect zero. So it is of the form \( \chi_{\delta,\lambda} \), where \( \delta_{0} \) is a semisimple \( \ell' \)-element of \( \text{GL}(n_{0}, \epsilon_{0}) \) and \( \lambda = \prod_{\Gamma} \lambda_{\Gamma} \) with \( \lambda_{\Gamma} \) a partition of \( m_{0_{\delta,\Gamma}} \) without \( \ell_{\Gamma} \)-hook which affords the second component of the triple \( (s, \lambda, K) \).

Secondly, assume we have the following decomposition \( \varphi_{+} = \prod_{\Gamma} \varphi_{+}(\Gamma,\delta,i) \) such that \( \lambda_{\Gamma} \) determines a \( \ell_{\Gamma} \)-element of canonical form \( \ell_{\Gamma}(\Gamma) \) in \( G_{\Gamma} \). Thus \( s = s_{0} \prod_{\Gamma} \varphi_{+}(\Gamma,\delta,i) \otimes I_{\Gamma}(\Gamma) \) is the first component of the triple \( (s, \lambda, K) \). We can view \( b \) as an \( \ell \)-block of \( C_{G}(R) \), then the Brauer pair \( (R, b) \) has a label \((R, s, \lambda) \) as in [5] (3.2). Thus \( (R, \varphi) \) belongs to an \( \ell \)-block \( B \) of \( G \) with label \((s, \lambda) \). In particular, \( \lambda_{\Gamma} \) is the \( \ell_{\Gamma} \)-core of a partition of \( m_{\Gamma}(s) \).

Finally, we have \( N_{s}(\zeta_{+}) = \prod_{\delta,i,j} N_{\delta,i,j}(\varphi_{+,\delta,i,j})(\zeta_{+,\delta,i,j}), \varphi_{+} = \prod_{\delta,i,j} \varphi_{+,\delta,i,j} \) with \( \varphi_{+,\delta,i,j} \) a character of \( N_{\delta,i,j}(\varphi_{+,\delta,i,j}) \) and of defect zero as a character of \( N_{\delta,i,j}(\varphi_{+,\delta,i,j}) \). Now \( \varphi_{+,\delta,i,j} \) is of the form

\[
\text{Ind}_{N_{\delta,i,j}(\varphi_{+,\delta,i,j})}^{N_{\delta,i,j}(\varphi_{+,\delta,i,j})} \zeta_{+,\delta,i,j} \prod_{j} \ell_{\delta,i,j}^{\delta}(\varphi_{+,\delta,i,j}) \prod_{j} \ell_{\delta,i,j}^{\delta}(\varphi_{+,\delta,i,j}) \cdot \prod_{j} \phi_{\delta,i,j}^{\delta}(\varphi_{+,\delta,i,j}),
\]

where \( t_{\delta,i,j} = \sum_{j} t_{\delta,i,j,j} \) is an extension of \( \prod_{j} \ell_{\delta,i,j,j}^{\delta}(\varphi_{+,\delta,i,j,j}) \) from \( N_{\delta,i,j}(\varphi_{+,\delta,i,j,j}) \) to \( N_{\delta,i,j}(\varphi_{+,\delta,i,j,j}) \) and \( \prod_{j} \ell_{\delta,i,j,j}^{\delta}(\varphi_{+,\delta,i,j,j}) \) without \( \ell \)-hook and \( \phi_{\delta,i,j}^{\delta}(\varphi_{+,\delta,i,j,j}) \) a character of \( \zeta_{+,\delta,i,j,j} \) corresponding to \( \lambda_{\delta,i,j} \).

Define \( K_{\Gamma} : \bigcup_{\Theta} \mathcal{G}_{\Theta,\delta} \to \{ \ell \text{-cores } \}, \varphi_{+,\delta,i,j} \mapsto \lambda_{\delta,i,j} \). Then we get the third component \( K = \prod_{\Gamma} K_{\Gamma} \) of the triple \((s, \lambda, K)\).

Now we define an action of \( O_{\ell}(3) \) on \( \mathcal{W}_{\ell}(G) \) by setting \( zK = \prod_{\Gamma}(z_{\Gamma}K_{\Gamma}) \) where \( (z_{\Gamma})_{\delta,i} = K_{\Gamma} \). For an \( \ell \)-weight \((R, \varphi) \) of \( G \) with label \((s, \lambda, K) \), we also write \( R = R_{s,\lambda,\delta} \) and \( \varphi = \varphi_{s,\lambda,\delta} \). Then by the conventions above, \( R_{s,\lambda,\delta} = R_{s,\lambda,\delta} \).

By Proposition 5.11, \( R_{s,\lambda,\delta} = R_{s,\lambda,\delta} \).

By Proposition 5.13, \( \zeta_{s,\lambda,\delta} = \zeta_{s,\lambda,\delta} \).

Proof. Let \((R, \varphi)\) be an \( \ell \)-weight of \( G \) corresponding to \((s, \lambda, K) \) and assume \( R \) can be decomposed as above. Let \( z \in O_{\ell}(3) \). We want to find which triple corresponds to \((R, \zeta_{s,\lambda,\delta})\). Assume it be \((s', \lambda', K') \).

Now, \( \zeta_{s,\lambda,\delta} \) is the \( \ell \)-block of \( G \) by construction. By Proposition 5.11 \( \zeta_{s,\lambda,\delta} = \chi_{s,\lambda,\delta} \).

Then we have \( s' = s, \lambda \).

Secondly, by Lemma 5.10 \( \zeta_{s,\lambda,\delta} = \theta_{s,\lambda,\delta} \) for \( z \in O_{\ell}(3) \). Note that \( \theta_{s,\lambda,\delta} \) corresponds to \( \ell_{\Gamma}^{\delta}(\Gamma) \) and \( \theta_{s,\lambda,\delta} \) corresponds to \( e_{s,\lambda}^{\delta}(\zeta, \Gamma) \). Up to conjugacy, we have \( s' = zs \).
Finally, by the conventions above, we may assume $D_{r,\delta,i} = D_{z,\Gamma,\delta,i}$, $N_{\Gamma,\delta,i} = N_{z,\Gamma,\delta,i}$, and $C_{\Gamma,\delta,i} = C_{z,\Gamma,\delta,i}$.

To determine $K'$, we note that $\tilde{\psi}_\lambda = \prod_{\delta,i} \tilde{\psi}_{\lambda,\delta,i}$. By (5.2), $\tilde{\psi}_\lambda$ is

\[ \tilde{\psi}_\lambda = \text{Ind}_{\gamma}^{N_{\Gamma,\delta,i}(\theta_{\theta,i})} \prod_j \psi_{\lambda,\delta,i,j} \cdot \prod_j \phi_{\theta,\delta,i,j} \]

\[ = \text{Ind}_{\gamma}^{N_{\Gamma,\delta,i}(\theta_{\theta,i})} \prod_j \psi_{\lambda,\delta,i,j} \cdot \prod_j \phi_{\theta,\delta,i,j}. \]

Since $\tilde{\psi}_\lambda = \tilde{\psi}_{\lambda,\delta,i}$, we have $N_{\Gamma,\delta,i}(\theta_{\theta,i}) = N_{z,\Gamma,\delta,i}(\theta_{\theta,i})$. Here, we note that the elements of $\Xi(\theta_{\theta,i})$ and $\Xi(\theta_{\theta,i,j})$ are permutation matrices and then have determinant 1 when $\ell$ is odd and have determinant $\pm 1$ when $\ell = 2$. In both cases, the elements of $\Xi(\theta_{\theta,i})$ and $\Xi(\theta_{\theta,i,j})$ are in the kernel of the linear character $\tilde{\psi}$ since $z \in O_{\ell}$. We can fix the way to extend $\prod_j \psi_{\lambda,\delta,i,j}$ as in [23, Lem. 25.5], then we have that $\tilde{\psi}_{\lambda,\delta,i,j} = \psi_{\lambda,\delta,i,j}$ by Convention 5.11 $\tilde{\psi}_\lambda$ would be

\[ \text{Ind}_{\gamma}^{N_{\Gamma,\delta,i}(\theta_{\theta,i})} \prod_j \psi_{\lambda,\delta,i,j} \cdot \prod_j \phi_{\theta,\delta,i,j}. \]

Then $K' = K$ which is just $K' = z.K$. Thus we complete the proof.

Now by Proposition 5.13 for an $\ell$-weight $(R, \varphi)$ of $G$, the number of irreducible constituents of $Res_{N_i(K)} \varphi$ can be obtained.

**Remark 5.14.** Analogous to the description of irreducible Brauer characters of $G$ and $X$ in Remark 3.4, now we give an analogous description of $\ell$-weights of $G$ and $X$ by summarizing the argument above.

For positive integers $h, w, d$, we define

\[ I_d(h) := \{ (d, k, j) \mid 1 \leq k \leq h, 1 \leq j \leq \ell^d \}, \]

\[ I(h) := \bigsqcup_{d \geq 0} I_d(h), \text{ and } \]

\[ \mathcal{A}(h, w) := \{ K : I(h) \rightarrow \{ \ell\text{-cores} \} | \sum_{d,k,j} \ell^d |K((d, k, j))| = w \}. \]

We call a tuple

\[ (([\sigma_1], m_1, \lambda^{(1)}, K^{(1)}), \ldots, ([\sigma_a], m_a, \lambda^{(a)}, K^{(a)})) \]

(5.3)
of tuples an $(n, \ell)$-admissible weight tuple, if

- for every $1 \leq i \leq a$, $\sigma_i \in \mathbb{F}^X$ is an $\ell$-element, and $m_i$ is positive integers such that $\lambda^{(i)}$ is an $e_i$-core of some partition of $m_i$ and $K^{(i)} \in \mathcal{A}(e_i, m_i)$ where $e_i$ is the multiplicatively order of $(eq)^{deg(\sigma_i)}$ modulo $\ell$ and $w_i = e_i^{-1}(m_i - \deg(\sigma_i));$
- $[\sigma_i] \neq [\sigma_j]$ if $i \neq j;$ and
- $\sum_{i=1}^a m_i \deg(\sigma_i) = n.$

An equivalence class of the $(n, \ell)$-admissible weight tuple (5.3) up to a permutation of tuples

\[ ([\sigma_1], m_1, \lambda^{(1)}, K^{(1)}), \ldots, ([\sigma_a], m_a, \lambda^{(a)}, K^{(a)}) \]
is called an $(n, \ell)$-admissible weight symbol and is denoted as

\[ w = [[[\sigma_1], m_1, \lambda^{(1)}, K^{(1)}], \ldots, ([\sigma_a], m_a, \lambda^{(a)}, K^{(a)})]. \]
Then by [2, 3, 4 and 5], the set of \((n, \ell)\)-admissible weight symbols is a labeling set for the \(G\)-conjugacy classes of \(\ell\)-weights of \(G\). We denote by \((R_\omega, \varphi_\omega)\) the \(\ell\)-weight of \(G\) corresponding to the \((n, \ell)\)-admissible weight symbol \(\omega\).

The group \(O_\ell(3)\) acts on the set of \((n, \ell)\)-admissible weight symbols via

\[
  z \cdot [(\{\sigma_1\}, m_1, \lambda^{(1)}), \ldots, (\{\sigma_a\}, m_a, \lambda^{(a)})] = [(z\sigma_1), m_1, \lambda^{(1)}], \ldots, (z\sigma_a), m_a, \lambda^{(a)}],
\]

for \(z \in O_\ell(3)\). We denote by \(\kappa(\omega)\) the order of the stabilizer group in \(O_\ell(3)\) of an \((n, \ell)\)-admissible weight symbol \(\omega\).

Assume that \(\ell \nmid \gcd(n, q - \epsilon)\). Then by Lemma 2.1 and Proposition 5.13, \(\kappa_{\text{res}}(R_\omega) = \kappa(\omega)\) (i.e., \(\text{Res}_{\text{res}}(R_\omega)\) is a sum of \(\kappa(\omega)\) irreducible constituents). For two \((n, \ell)\)-admissible weight symbols \(\omega\) and \(\omega'\), if they are in the same \(O_\ell(3)\)-orbit, then \(R_\omega = R_{\omega'}\) and the restrictions of \(\varphi_\omega\) and \(\varphi_{\omega'}\) to \(N_X(R_\omega \cap X)\) are the same.

If moreover, we write the decomposition \(\text{Res}_{\text{res}}(R_\omega)\) via \(\bigoplus_{j=1}^{\kappa(\omega)} (\varphi_{\omega_j})_j\), then by Remark 5.6, the set \((\varphi_{\omega_1}, \ldots, \varphi_{\omega_{\kappa(\omega)}})\), where \(\omega\) runs through the \(O_\ell(3)\)-orbit representatives of \((n, \ell)\)-admissible weight symbols and \(j\) runs through the integers between 1 and \(\kappa(\omega)\), is a complete set of representatives of \(X\)-conjugacy classes of the \(\ell\)-weights of \(X\).

**Remark 5.15.** Let \(b = [(\{\sigma_1\}, m_1, \lambda^{(1)}), \ldots, (\{\sigma_a\}, m_a, \lambda^{(a)})]\) be an \((n, \ell)\)-admissible block symbol. Then by [2, 3, 4 and 5], the set of \(\ell\)-weights \(\{(R_\omega, \varphi_\omega)\}\), where \(\omega\) runs through the \((n, \ell)\)-admissible symbols of the form

\[
  \omega = [(\{\sigma_1\}, m_1, \lambda^{(1)}), \ldots, (\{\sigma_a\}, m_a, \lambda^{(a)}),]
\]

is a complete set of representatives of \(G\)-conjugacy classes of \(\ell\)-weights of \(B_\omega\).

Assume that \(\ell \nmid \gcd(n, q - \epsilon)\). If we write \(W_\ell(B_\omega) = \{(R_1, \varphi_1), \ldots, (R_k, \varphi_k)\}\), then by Proposition 5.13, \(W_\ell(B_\omega) = \{(R_1, \tilde{\varphi}_1), \ldots, (R_k, \tilde{\varphi}_k)\}\) for all \(z \in O_\ell(3)\).

Assume that \(\ell \geq 3\). Let \(b\) be an \(\ell\)-block of \(X\) covered by \(B_\omega\), then the number of \(\ell\)-weights lying in \(b\) of the form \((R_\omega \cap X, \varphi')\) where \(\varphi' \in \text{Irr}(N_X(R_\omega))\) is \(\kappa(\omega)/\kappa(b)\).

For an \(\ell\)-block \(B\) and \((n, \ell)\)-admissible weight symbol \(\omega\), we say \(\omega\) belongs to \(B\), if \((R_\omega, \varphi_\omega)\) is a \(B\)-weight.

**Proof of Theorem 1.2.** If \(\ell = p\), then the assertion holds by II.1. Now we assume that \(\ell \neq p\). For an \(\ell\)-block \(b\) of \(X\), let \(B\) be an \(\ell\)-block associated to \(B\). By Remark 4.11 5.13 and 2 (1A), there is a natural bijection \(\mathcal{S}\) from the \((n, \ell)\)-admissible symbols belonging to \(B\) onto the \((n, \ell)\)-admissible weight symbols belonging to \(B\). For any two \((n, \ell)\)-admissible symbols \(s, s'\) which belong to \(B\), by Remark 5.13 and 5.14 and the construction of \(\mathcal{S}\) in 2 (1A), we have

- \(\kappa(s) = \kappa(\mathcal{S}(s))\).
- \(s\) and \(s'\) are in the same \(O_\ell(3)\)-orbit if and only if \(\mathcal{S}(s)\) and \(\mathcal{S}(s')\) are in the same \(O_\ell(3)\)-orbit.

Hence \(|\text{IBr}_\ell(b)| = |\mathcal{S}(\ell)|\) by Remark 5.7.

For a positive integer, we denote by \(C_d\) the cyclic group of order \(d\). We will make use of the following lemma to prove Proposition 1.4.

**Lemma 5.16.** Let \(B_1\) and \(E\) be cyclic groups of order \(n_1\) and \(n_2\) respectively. Suppose that \(H = B \times E\) satisfies that either

(i) \(B = B_1\); or

(ii) \(B = B_1 \times C_2\) is isomorphic to a dihedral group of order \(2n_1\) and \(n_1\) is odd.
Assume that \( \gcd(|B|, |E|) = 1 \). Let \( H_1 \) and \( H_2 \) be two subgroups of \( H \) such that \( |H_1| = |H_2| \), \( |H_1 \cap B| = |H_2 \cap B| \) and \( H_1 \cap B_1 = H_2 \cap B_1 \). Then \( H_1 \) and \( H_2 \) are conjugate in \( H \).

Proof. We first recall the result about the subgroups of direct products. A subgroup \( \pi \) and \( S \) of \( \Delta \) follows by Remark 5.14 and 5.15 immediately.

Assume that \( \gcd(|B|, |E|) = 1 \). Let \( \theta \in \Delta \). \( \theta \) is a unipotent \( \ell \)-block of \( X \) and \( B \) are conjugate in \( B \). So \( H_1 \) and \( H_2 \) are conjugate in \( H \).

Proof of Proposition 1.4. Thanks to [23] Thm. C, we can assume that \( \ell \neq p \). For any \( \theta \in IBr(X) \), let \( \phi \in IBr_\ell(G | \theta) \) and \( (R, \varphi) \) the \( \ell \)-weight of \( G \) corresponding to \( \phi \) under the bijection induced by \( \varphi \) (see the proof of Theorem 2.2). Let \( S = R \cap X \) and \( \psi \in \text{Irr}(N_X(S) | \psi) \). Now we consider the \( G \simeq \Delta \)-orbit of \( \theta \) and \( (S, \psi) \), respectively in \( IBr_\ell(X) \) and \( \mathcal{W}_\ell(X) \), respectively. Denote by \( \Delta_1 \) the \( \varphi \simeq \Delta \)-orbit of \( \theta \) in \( IBr_\ell(X) \) and \( \Delta_2 \) the \( \varphi \simeq \Delta \)-orbit of \( (S, \psi) \) in \( \mathcal{W}_\ell(X) \). By Remark 5.13 and 5.14 (and the construction of \( \mathcal{W} \) (see that it is \( D \)-equivariant by [30] Thm. 1.1)), \( \Delta_1 \) and \( \Delta_2 \) have the same cardinality. Obviously, \( \text{Out}(X) \) acts on \( \Delta \) (or \( \Delta_2 \), respectively) as \( \text{Out}(X) \) does. Also \( |\text{Out}(X)_\theta| = |\text{Out}(X)_\psi| \).

Now we denote by \( \text{Outdiag}(X) \) the outer automorphisms induced by \( G \) on \( X \) then \( \text{Outdiag}(X) \cong C_{\text{gcd}(n, q - \ell)} \) is cyclic. Thus by Remark 5.14 and 5.14 the stabilizers of \( \theta \) and \( \psi \) in \( \text{Outdiag}(X) \) are the same.

If \( n \geq 3 \), by a similar argument of the paragraph above (replace \( \text{Out}(X) \) by \( \text{Outdiag}(X, \gamma) \), where \( \gamma \) is defined as in Section 2.4), we have \( |\text{Outdiag}(X, \gamma)_\theta| = |\text{Outdiag}(X, \gamma)_\psi| \).

Thus by Lemma 5.16 \( \text{Out}(X)_\theta \) and \( \text{Out}(X)_\psi \) are conjugate in \( \text{Out}(X) \). Thus there exists an \( \text{Aut}(X) \)-equivariant bijection between \( \Delta_1 \) and \( \Delta_2 \), hence there exists an \( \text{Aut}(X) \)-equivariant bijection \( \mathcal{W} \) between \( IBr_\ell(X) \) and \( \mathcal{W}_\ell(X) \). Obviously, we can choose the bijection \( \mathcal{W} \) satisfies that if \( \theta \in IBr_\ell(X) \), \( \phi \in IBr(G | \theta) \), \( (R, \varphi) = \mathcal{W} \)(\( \phi \)), \( S = R \cap X \), then \( \mathcal{W}(\theta) = (S, \psi) \) for some \( \psi \in \text{Irr}(N_X(S) | \varphi) \). So \( \mathcal{W} \) preserves blocks. Moreover, the conditions (i) and (ii) of Definition 2.5 hold for any \( \ell \)-block of \( X \) (for details, see the proof of Corollary 5.19).

5.4 The unipotent blocks

Lemma 5.17. Assume that \( \ell \nmid \gcd(n, q - \ell) \). Let \( b \) be a unipotent \( \ell \)-block of \( X \) and \( B \) the unipotent \( \ell \)-block of \( G \) which covers \( b \). Then \( (R, \varphi) \mapsto (R \cap X, \text{Res}^{N_X(R)}_{N_X(R)} \varphi) \) gives a bijection from \( \mathcal{W}_\ell(B) \) to \( \mathcal{W}_\ell(b) \).

Proof. By Lemma 4.16, there is a unique unipotent \( \ell \)-block of \( G \) which covers \( b \). Then the claim follows by Remark 5.14 and 5.15 immediately.

Corollary 5.18. Assume that \( \ell \nmid \gcd(n, q - \ell) \). If \( b \) is a unipotent \( \ell \)-block of \( X \), then there exists an \( \text{Aut}(X) \)-equivariant bijection between \( IBr_\ell(B) \) and \( \mathcal{W}_\ell(b) \).

Proof. Let \( B \) be a unipotent \( \ell \)-block of \( G \) which covers \( b \). By [30] Thm. 1.1, there exists a \( D \)-equivariant bijection between \( IBr_\ell(B) \) and \( \mathcal{W}_\ell(B) \). Then the claim follows by Lemma 5.16 and 5.17 since the automorphisms of \( X \) induced by \( G \simeq D \) generate \( \text{Aut}(X) \).

Now note that the universal covering group of a simple group \( PSL_n(\epsilon q) \) is a group isomorphic to \( SL_n(\epsilon q) \), apart from a few exceptions, see [22] 6.1.8.

Corollary 5.19. Assume that \( \ell \nmid \gcd(n, q - \ell) \). Let \( b \) be a unipotent \( \ell \)-block of \( X \), then the conditions (i) and (ii) of Definition 2.5 hold for \( b \).
Proof. By Corollary 5.18 there is an Aut(X)-equivariant bijection $\Omega_b : \text{IBr}_\ell(b) \to \mathcal{W}_\ell(b)$. Now for every $Q \in \text{Rad}_\ell(X)$, we set
\[ \text{IBr}_\ell(b \mid Q) := \bigcup_{\psi \in \text{Inr}^b(N_X(Q), b)} \{ \Omega_b^{-1}((Q, \psi)) \} \]
and define a map
\[ \Omega_X^Q : \text{IBr}_\ell(b \mid Q) \to \text{dz}_\ell(N_X(Q), b), \]
such that $\phi \mapsto \tilde{\Omega}_b(\phi)$, where $\tilde{\Omega}_b(\phi)$ denotes the unique element in $\text{dz}_\ell(N_X(Q), b)$ whose inflation $\psi$ to $N_X(Q)$ satisfies that $\Omega_b(\phi) = (Q, \psi)$. Then by [37] Lem. 3.8, the subsets $\text{IBr}_\ell(b \mid Q)$ and maps $\Omega_X^Q$ defined here satisfy (i) and (ii) of Definition 2.5. 

Remark 5.20. In fact, we have a generalisation of Corollary 5.19. Assume that $\ell \not\equiv \text{gcd}(n, q - e)$. Suppose that $s$ is a semisimple $\ell'$-element of $G$ such that $zs$ and $s$ are not $G$-conjugate for any $z \in O_F(\zeta)$. Let $B$ be an $\ell$-block of $G$ with label $(s, \lambda)$ and $b$ the $\ell$-block of $X$ covered by $B$. Then by the same argument, there exists an Aut(X)-equivariant bijection between $\text{IBr}_\ell(b)$ and $\mathcal{W}_\ell(b)$, and then the conditions (i) and (ii) of Definition 2.5 hold for $b$. 

6 Extendibility of weight characters of unipotent blocks

In this section, we will prove the following result.

Proposition 6.1. Let $(R, \varphi)$ be an $\ell$-weight of $G$ which belongs to a unipotent $\ell$-block. Then $\varphi$ extends to $(G \rtimes D)_{R, \varphi}$.

We will use the following lemma.

Lemma 6.2. Suppose that $H$ is a finite group, $C \subseteq H$, $N \subseteq H$, $D_0 \leq D \leq H$, $\chi \in \text{Irr}(N)$ satisfies that
\begin{itemize}
  \item $H/N$ is abelian, $H = ND$, $N \cap D_0 \leq C_1$ and $H/N D_0$ is cyclic;
  \item there are normal subgroups $C_0, C_1, N_0$ and $N_1$ of $H$ such that $C = C_0 \times C_1$, $N = N_0 \times N_1$, $C_0 = N_0$ and $C_1 \leq N_1$;
  \item $D_0$ acts trivially on $N_1/C_1$;
  \item $N_0 D = N_0 \rtimes D$. Let $K$ be the kernel of the action of $D$ on $N_0$.
  \item $\chi \in \text{Irr}(N \mid \theta)$ where $\theta = \theta_0 \times \theta_1$ with $\theta_0 \in \text{Irr}(C_0)$ and $\theta_1 = 1_{C_1}$;
  \item $H_\chi = H$ and $\theta_0$ extends to $N_0 \rtimes (D/K)$;
\end{itemize}
Then $\chi$ extends to $H$.

Proof. Let $\chi = \chi_0 \times \chi_1$ where $\chi_0 = \theta_0$ and $\chi_1 \in \text{Irr}(N_1)$. Now $\theta_0$ extends to $N_0 \rtimes (D/K)$, so there exist an extension $\chi_0' \in \text{Irr}(N_0D)$ of $\chi_0$ and a representation $\rho_0'$ affording $\chi_0'$ such that if $n_0 \in N_0$, $d, d' \in D$ satisfy that $d$ and $d'$ induce the same automorphism on $N_0$, then $\rho_0'(n_0d) = \rho_0'(n_0d')$. Let $\tilde{\rho}_0 = \text{Res}_{N_0 D_0}^{N_0D} \rho_0'$.

Let $\rho_1 : N_1 \to \text{GL}_{\chi_1(1)}(\mathbb{C})$ a representation of $N_1$ affording $\chi_1$. Now let $\tilde{\rho} : ND_0 \to \text{GL}_{\chi(1)}(\mathbb{C})$ satisfy $\tilde{\rho}(n_0n_1d) = \tilde{\rho}_0(n_0d) \otimes \rho_1(n_1)$ for all $n_0 \in N_0$, $n_1 \in N_1$ and $d \in D_0$. Here, $\tilde{\rho}$ is well-defined. In fact, if $n_0, n_0' \in N_0$, $n_1, n_1' \in N_1$ and $d, d' \in D_0$ satisfy $n_0n_1d = n_0'n_1'd'$, then $n_0 = n_0'$ and there exists $c \in C_1$ such that $n_1 = n_1'c$ and $d = c^{-1}d'$. Hence $\rho_1(n_1) = \rho_1(n_1')$ since $C_1 \leq \ker \rho_1$. Also, by the paragraph above, $\tilde{\rho}_0(n_0d) = \tilde{\rho}_0(n_0'd')$. So $\tilde{\rho}(n_0n_1d) = \tilde{\rho}(n_0'n_1'd')$. We claim that $\tilde{\rho}$ is a representation of $ND_0$. In fact, let $n_0, n_0' \in N_0$, $n_1, n_1' \in N_1$ and $d, d' \in D_0$,
\[ \tilde{\rho}(n_0n_1dn_0'n_1'd') = \tilde{\rho}(n_0(d'n_0)n_1'(d'n_1')dd') \]
\[ = \tilde{\rho}_0(n_0(d'n_0)dd') \otimes \rho_1(n_1'(d'n_1')) \]
\[ = \tilde{\rho}_0(n_0dn_0'd') \otimes \rho_1(n_1'(d'n_1')). \]
On the other hand, \( \tilde{\rho}(n_0n_1d) = \tilde{\rho}(n_0d) \tilde{\rho}(n_1) \). Since \( D_0 \) acts trivially on \( N_1/C_1 \), then \( d_{n_1} = n_1'c \) for some \( c \in C_1 \). Hence \( \rho_1(n_1\gamma') = \rho_1(n_1)\rho_1(\gamma') = \rho_1(n_1)\rho_1(n_1') \) since \( C_1 \leq \ker(\rho_1) \). Thus the claim holds.

Let \( g \in D, n_0 \in N_0, n_1 \in N_1, d \in D_0 \), then

\[
\tilde{\rho}(\tilde{\rho}(n_0n_1d)) = \tilde{\rho}(\tilde{\rho}(n_0d)\tilde{\rho}(n_1d)) = \tilde{\rho}(n_0d) \tilde{\rho}(n_1d).
\]

Let \( \tilde{\chi} \) be the character afforded by \( \tilde{\rho} \), then \( \tilde{\chi}(n_0n_1d) = \tilde{\chi}(n_0d)\tilde{\chi}(n_1) \). Hence

\[
\tilde{\chi}(n_0n_1d) = \text{Trace}(\tilde{\rho}(n_0d)\tilde{\rho}(n_1d)) = \tilde{\chi}(n_0d)\tilde{\chi}(n_1d) = \tilde{\chi}(n_0n_1d)
\]

since \( \tilde{\chi}, \tilde{\chi} \) are \( D \)-invariant. Thus \( \tilde{\chi} \) is \( D \)-invariant. Then \( \tilde{\chi} \) extends to \( H \) since \( H/N_0 \) is cyclic. So \( \chi \) extends to \( H \).

Firstly, by the uniqueness of \( R_{m,a,\gamma} \) and \( R_{m,a,\gamma}^0 \) proved in [3, 4] and [5], \( D \) acts trivially on the set of \( G \)-conjugacy classes of \( \ell \)-radical subgroups of \( G \). Denote \( \alpha_1 = F_p \) and \( \alpha_2 = \gamma \). Then \( D = \langle \alpha_1, \alpha_2 \rangle \). So there exist \( g^{(k)} \in G \) such that \( g^{(k)}\alpha_k \in (G \rtimes D)_{\alpha_k} \) for \( k = 1, 2 \). Let \( D' = \langle g^{(1)}\alpha_1, g^{(2)}\alpha_2 \rangle \). Then we have the following result by direct calculation.

**Lemma 6.3.** With the notations above,

1. \( (G \rtimes D)_{\alpha} = N_G(R)D' \);
2. \( D' \cap G \equiv D \);
3. \( N_G(R)D'/N_G(R) \equiv D \).

If \( D \) is cyclic, then Proposition [6.1] holds immediately. So we will assume that \( D \) is not cyclic. Then \( e = 1 \), that is \( G = \text{GL}_n(q) \). Let \( q = p^f \) for some prime \( p \) and integer \( f \). Then \( f \) is even. In particular, if \( q \) is odd, then \( 4 \mid q - 1 \). Hence, by the description in Section [5] if \( \ell = 2 \), we always only have the *Case 1* when considering basic subgroups. Then \( D_{m,a,\gamma} = R_{m,a,\gamma}^0 \) whenever \( \ell \) is odd or \( \ell = 2 \).

One embedding of \( Z_{a,E_\gamma} \) can be constructed explicitly as follows (see, [3] and [5]). Let \( \xi \) be a fixed \( \ell^{a+r} \)-th primitive root of unity in \( \mathbb{F}_{(e\alpha)e^{\nu}} \) and \( \zeta = \xi^{e^{\nu}/\ell^{a+r-1}} \). We first let \( Z_0 = \xi I_\ell \) with \( I_\ell \) the identity matrix of degree \( \ell \)

\[
X_0 = \text{diag}(1, \xi, \ldots, \xi^{\ell-1}), \quad Y_0 = \begin{bmatrix} 0 & 1 \\ \\ 1 & 0 \end{bmatrix}.
\]

We then set \( X_{0,j} = I_\ell \otimes \cdots \otimes X_0 \otimes \cdots \otimes I_\ell \) and \( Y_{0,j} = I_\ell \otimes \cdots \otimes Y_0 \otimes \cdots \otimes I_\ell \) with \( X_0 \) and \( Y_0 \) appearing as the \( j \)-th components. Define

\[
\rho_{a,\gamma,0} : \quad Z_{a,E_\gamma} \quad \longrightarrow \quad \text{GL}(\ell^f, (e\alpha)e^{\nu})
\]

\[
\begin{array}{c}
\gamma \quad \longrightarrow \quad Z_0 \\
X_j \quad \longrightarrow \quad X_{0,j} \\
Y_j \quad \longrightarrow \quad Y_{0,j}
\end{array}
\]

Now, let \( \iota \) be an embedding of \( \text{GL}(\ell^f, (e\alpha)e^{\nu}) \) into \( \text{GL}(e^{\nu+r} \ell^f, e\alpha) \) with \( \iota(\xi) \) being the companion matrix \( (\Lambda_\alpha) \) of the polynomial \( \Lambda_\alpha \in \mathbb{F} \) having \( \xi \) as a root. Then we set \( R_{a,\gamma} \) the image of \( Z_{a,E_\gamma} \) under \( \rho_{a,\gamma} = \iota \circ \rho_{a,\gamma,0} \).

For later use, we replace \( R_{m,a,\gamma} \) and \( R_{m,a,\gamma,0} \) by one of their conjugates. Now define

\[
Z_{m,0} = I_{(m)} \otimes Z_0, \quad X_{m,0,j} = I_{(m)} \otimes X_{0,j}, \quad Y_{m,0,j} = I_{(m)} \otimes Y_{0,j}.
\]
Define
\[ \rho_{m,a,y} : Z_a E_y \rightarrow \mathrm{GL}(m \ell^y, (eq)^{eq}) \]
in the same way as \( \rho_{a,y} \) with \( Z_0, X_{0,j}, Y_{0,j} \) replaced by \( Z_{m,0}, X_{m,0,j}, Y_{m,0,j} \). Denote still by \( \iota \) the embedding of \( \mathrm{GL}(m \ell^y, (eq)^{eq}) \) into \( \mathrm{GL}(m \ell^y, eq) \) and \( \rho_{m,a,y} = \iota \circ \rho_{m,a,y} \). Then we set \( R_{m,a,y} \) the image of \( \rho_{m,a,y} \). Finally, we set \( R_{m,a,y} \in R_{m,a,y} \cap \mathbb{A}_C \).

Now we give some precise information for \( g^{(1)}, g^{(2)} \) above. Indeed, by [30] Prop. 4.2 and 4.3, if there is a decomposition \( R = R_0 \times R_1 \times \cdots \times R_k \) where \( R_0 \) is a trivial group and \( R_i \cong R_{m,a,y,e}(i \geq 1) \) is a basic subgroup, then \( g^{(k)} \) is blockwise diagonal corresponding to the decomposition \( g^{(k)} = \mathrm{diag}(g^{(1)}_1, g^{(2)}_1, \ldots, g^{(k)}_1) \) where \( g^{(k)}_i \) is identity matrix and \( g^{(k)}_i = g_{m,a,i} \otimes I_y_i \otimes I_e \) with \( g_{m,a,i} \in G_{m,a} \) such that \( g^{(k)}_i \sigma_k \) fixes \( R_i \) for all \( k = 1, 2 \) and \( 0 \leq i \leq u \). Obviously, the action of \( g^{(k)}_i \sigma_k \) on \( G_{m,a} \otimes I_y_i \otimes I_e \), \( C_{m,a} \otimes I_y_i \otimes I_e \), and \( N_{m,a} \otimes I_y_i \otimes I_e \) is just as the actions of \( g^{(k)}_i \sigma_k \) on \( G_{m,a} \), \( C_{m,a} \), and \( N_{m,a} \), respectively, for all \( k = 1, 2 \) and \( 0 \leq i \leq u \). We also regard the actions above as the actions of \( g^{(k)}_i \sigma_k \) (\( k = 1, 2 \)).

**Lemma 6.4.** With the notations above, there exists a subgroup \( D'_0 \) of \( D' \) independent of \( m, \alpha \) and \( \gamma \), such that \( D'_0 \) acts trivially on \( R_{m,a,y} \) and \( D'/(D' \cap G)D'_0 \) is cyclic. In particular, \( D'_0 \) acts trivially on \( N_{m,a,y}/C_{m,a,y} \).

**Proof.** Denote \( Z = \iota(Z_{m,0}), X_j = \iota(X_{m,0,j}), Y_j = \iota(Y_{m,0,j}) \) and \( B = \langle Z, X_j \mid j = 1, \ldots, \gamma \rangle, H = \langle Y_j \mid j = 1, \ldots, \gamma \rangle \). Then \( R_{m,a,y} = B \times H \). By the proof of [30] Lem. 4.1, for \( k = 1, 2 \),
\[ g^{k} \sigma_k(x) = \left\{ \begin{array}{ll} x h_1 & \text{if } x \in B \\ x & \text{if } x \in H \end{array} \right. \]
where \( h_1 = p \) and \( h_2 = -1 \).

Now let \( r \) be the multiplicative order of \( p \) modulo \( \ell \). We take \( D'_0 = \langle (g^{1} \sigma_1)^r \rangle \) when \( r \) is odd; and \( D'_0 = \langle (g^{1} \sigma_1)^{r/2} g^{2} \sigma_2 \rangle \) when \( r \) is even. Then \( D'_0 \) acts trivially on \( R_{m,a,y} \) and \( D'/(D' \cap G)D'_0 \) is cyclic. □

**Corollary 6.5.** With the notations above, \( D'_0 \) acts trivially on \( N_{m,a,y}/C_{m,a,y} \).

**Proof.** For \( c = (c_1, \ldots, c_l) \), we have \( N_{m,a,y} \in C_{m,a,y} \in R_{m,a,y} \otimes Y_e \) by [2] and [3]. Here \( Y_e \) is the normalizer of \( A_e \in \mathbb{A}(\ell^k) \) and then consists of permutation matrices. By Lemma 6.4, \( D'_0 \) acts trivially on \( N_{m,a,y}/R_{m,a,y} \). Hence \( D'_0 \) acts trivially on \( R_{m,a,y}/C_{m,a,y} \) since \( C_{m,a,y} \subseteq C_{m,a} \otimes I_e \). □

**Proof of Proposition 6.7.** By the argument after Lemma 6.3 we may assume that \( \epsilon = 1 \) and \( q = p^l \) with \( f \) even. Suppose that \( G = GL_{m}(q) = GL(V) \), where \( V \) is a vector space of dimension \( n \) over \( \mathbb{R} \). By [2] (4A)) and [3] (2B)), \( R = R_0 \times R_+ \) where \( R_0 \) is an identity group and \( R_+ \) is a direct product of basic subgroups. Let \( V = V_0 \times V_+ \) be the corresponding decomposition of \( V \), such that \( V_0 \) is the underlying space of \( R_0 \) and \( V_+ \) is the underlying space of \( R_+ \). Note that if \( \ell = 2 \), then \( \dim(V_0) = 0 \). Then \( C_0(R) = C_0 \times C_+ \) and \( N_0(R) = N_0 \times N_+ \), where \( C_0 = N_0 = GL(V_0), C_+ = GL(V_+)(R_+), N_+ = GL(V_+(R_+). \)

Let \( \theta \in \mathrm{Irr}(RC_0(R) \otimes \varphi) \), then \( \theta = \theta_0 \times \theta_+ \), where \( \theta_0 \in \mathrm{Irr}(R_0C_0) \) and \( \theta_+ \in \mathrm{Irr}(R_+C_+) \). We write \( \varphi = \phi_0 \otimes \varphi_+ \), with \( \phi_0 \in \mathrm{Irr}(N_0) \) and \( \varphi_+ \in \mathrm{Irr}(N_+) \). Obviously, \( \varphi_0 = \theta_0 \).

We write \( R_+ = R_1^{b_1} \times \cdots \times R_u^{b_u} \) as a direct product of basic subgroups, where \( R_i \) appears \( b_i \) times as a component of \( R_+ \). Let \( C_i = C_{GL(V_+)(R_+)} \), \( N_i = N_{GL(V_+)(R_+)} \), where \( V_+ \) is the underlying space of \( R_+ \). Then \( C_0 = C_1^{b_1} \times \cdots \times C_u^{b_u} \) and \( \theta_+ = \prod_{i=1}^{u} \prod_{j=1}^{b_i} \theta_{i,j}^{(i)} \), where \( \theta_1, \ldots, \theta_{b_1} \) are distinct irreducible characters of \( C_i \) trivial on \( R_i \) and \( \theta_{i,j} \) occurs \( b_j \) times as a factor in \( \theta_+ \).

Now \( (R, \varphi) \) belongs to a unipotent \( \ell \)-block, so for all \( 1 \leq i \leq u, 1 \leq j \leq v_i, \theta_{i,j} \) has the form \( \theta_{i,\delta,\ell,\delta} \) for some \( \delta \) and \( k \), where \( \ell = x - 1 \). By the construction of \( \theta_+ \), we have \( \theta_+ = 1_{C_1} \) when \( \Gamma = x - 1 \) (since \( \theta_+ = \pm \phi_2^{(1)}(1) = 1_{C_1} \) by [8] (4.16))). Hence \( \theta_+ = 1_{C_0R_0} \) for all \( 1 \leq i \leq u, 1 \leq j \leq v_i \). Also, \( \varphi_0 = \theta_0 \) is a unipotent character of \( N_0 \in C_0 \). By Proposition [4.17] \( \varphi_0 \) extends to \( N_0 \times D \).

Now \( N_+ = (\theta_+ = \prod_{i=1}^{u} \prod_{j=1}^{v_i} (N_0(\theta_{i,j})) \otimes (\theta_{i,j})) \). By the argument above, \( \theta_+ \) is the trivial character hence is invariant under \( N_+ \). By Corollary 6.3, \( D'_0 \) acts trivially on \( N_0/R_0 \). So \( D'_0 \) acts trivially on \( N_+/C_+ \) since \( D'_0 \) acts trivially on every \( \mathbb{A}(b_i) \). Also, \( N_0(R')D'/N_0(R')D'_0 \) is cyclic since \( D'/(D' \cap G)D'_0 \) is cyclic by Lemma 6.4. Hence \( \varphi \) extends to \((G \times D)_R \) by Lemma 6.2. This completes the proof. □
Corollary 6.6. Assume that $\ell \nmid \gcd(n, q - \epsilon)$. Let $(Q, \psi)$ be an $\ell$-weight of $X$ which belongs to a unipotent $\ell$-block. Then $\psi$ extends to $(G \rtimes D)_{Q}$.

Proof. By Lemma 5.17 there is an $\ell$-weight $(R, \varphi)$ of $G$ in a unipotent $\ell$-block of $G$, such that $Q = R \cap X$ and $\psi = \text{Res}_{N_{Q}(R)}^{N_{Q}(G)} \varphi$. So $(G \rtimes D)_{R, \varphi} \leq (G \rtimes D)_{Q, \psi}$. Note that $(G \rtimes D)_{R, \varphi} = (G \rtimes D)_{R}$ and $(G \rtimes D)_{R} = (G \rtimes D)_{Q}$. So $(G \rtimes D)_{Q, \psi} = (G \rtimes D)_{R}$. Now by Proposition 6.1, $\psi$ extends to $(G \rtimes D)_{R}$, then $\psi$ extends to $(G \rtimes D)_{Q}$. □

7 Proof of Theorem 1.3

Now we consider the condition (iii) of Definition 2.5

Proposition 7.1. Assume that $\ell \nmid \gcd(n, q - \epsilon)$ and $n \geq 3$. Let $b$ be a unipotent $\ell$-block of $X$, then the subsets $\text{IBr}_{b}(Q)$ and maps $\Omega^{X}_{Q}$ for every $Q \in \text{Rad}(X)$, defined as in the proof of Corollary 6.6.19 satisfy Definition 2.5.1(iii)(1)-(3) for $\phi \in \text{IBr}_{b}(Q)$, $A := A(\phi, Q) = (G \rtimes D)/O_{\ell}(Z(G))Z$ with $Z = Z(X) \cap \ker(\phi)$.

Proof. Now $X = X/Z$. It is easy to check (1) of Definition 2.5(iii). For (2), by Corollary 4.20 we have an extension $\phi' \in \text{IBr}_{b}(G \rtimes D)$ of $\phi$. Then $O_{\ell}(Z(G)) \leq O_{\ell}(G \rtimes D) \leq \ker(\phi')$. Also, $Z \leq \ker(\phi')$. Let $\hat{\phi}$ be the Brauer character of $A$ associated with $\phi'$, then $\hat{\phi}$ is an extension of the $\ell$-Brauer character of $A$ associated with $\phi$.

For (3), let $\psi \in \text{Irr}(N_{X}(Q))$ be the inflation of $\Omega^{X}_{Q}(\phi)$ to $N_{X}(Q)$ and $\overline{\psi}$ be the character of $N_{X}(Q) = N_{Q}(Q)$ associated with $\varphi$. Moreover, we have $(G \rtimes D)_{Q, \phi} = N_{G \rtimes D}^{-}(Q)$ by [37] Lem. 9.16 since $Z(G) = Z(G \rtimes D)$ and $\text{Aut}(X) = (G \rtimes D)/Z(G)$. Now by Corollary 6.6 $\psi \in \text{Irr}(N_{X}(Q))$ extends to a character $\tilde{\psi}$ of $(G \rtimes D)_{Q, \phi} = (G \rtimes D)_{Q}$, then there exists an extension of $\psi$ to $N_{A}(Q) = N_{G \rtimes D}^{-}(Q)/O_{\ell}(Z(G))Z$ since $O_{\ell}(Z(G)) \subseteq \ker(\tilde{\psi})$ by the proof of Corollary 6.6. Then $\overline{\tilde{\psi}}$ extends to $N_{A}(Q)$. This completes the proof. □

For condition (4) of Definition 2.5(iii), we have:

Lemma 7.2. Keep the hypotheses and setup of Proposition 7.1 let $(S, \psi)$ be an $\ell$-weight of $X$. Denote by $\psi'$ the inflation of $\Omega^{S}_{Q}(\psi)^{\ast}$ viewed as $\ell$-Brauer character in $\text{IBr}_{b}(N_{X}(S)/S)$ to $N_{X}(S)$. Let $\hat{\psi}'$ be an extension of $\psi'$ to $N_{A}(S)$. Then there exists an extension $\hat{\phi} \in \text{IBr}_{b}(A)$ of $\phi$ to $A$ satisfying

$$
\text{bl}(\text{Res}_{N_{X}(S)}^{N_{A}(S)} \hat{\psi}')^{} = \text{bl}(\text{Res}_{N_{S}(S)}^{N_{A}(S)} \hat{\phi})
$$

for any $\overline{X} \leq J \leq A$.

Proof. Since $A/\overline{X}$ is solvable, all Hall $\ell'$-subgroups of $A/\overline{X}$ are conjugate and every $\ell'$-element of $A$ is contained in some $J$ such that $J/\overline{X}$ is a Hall $\ell'$-subgroup of $A/\overline{X}$. Then by [27] Lem. 2.4 and 2.5(a)], to prove this proposition, it suffices to prove that $A = \overline{X}N_{A}(\overline{X})$ and that the proposition holds for certain (thus every) $\overline{X} \leq J \leq A$ such that $J/\overline{X}$ is a Hall $\ell'$-subgroup of $A/\overline{X}$ (for details, see the proof of [37] Prop. 9.21]).

Firstly, let $R = SO_{\overline{X}}(Z(G))$, then by Proposition 5.2 $R$ is an $\ell$-radical subgroup of $G$ such that $R \cap X = S$. As pointed in Section 6 (by the uniqueness of $R_{m,\alpha, Y}$ and $R'_{m,\alpha, Y}$ proved in [3], [4] and [5]), $G \rtimes D$ acts trivially on the $G$-conjugacy classes of $\ell$-radical subgroups of $G$. Hence $G \rtimes D = G_{G \rtimes D}(R)$ by Frattini’s argument. Then $A = \overline{X}N_{A}(\overline{X})$ since $N_{A}(\overline{X}) = N_{G \rtimes D}(S)/O_{\ell}(Z(G))Z$ and $N_{G \rtimes D}(S) = N_{G \rtimes D}(R)$.

Now $(G \rtimes D)_{\overline{X}} = G \rtimes D$. Let $\overline{G} := G/ZO_{\ell}(Z(G))$, then $A = (G \rtimes D)_{\overline{X}}/ZO_{\ell}(Z(G)) = \overline{G} \rtimes D$. Since $\ell \nmid \gcd(n, q - \epsilon)$, $\overline{G} = G/XO_{\ell}(Z(G))$ is an $\ell'$-group. Thus there is a unique Hall $\ell'$-subgroup $\hat{A}/\overline{X}$ of $A/\overline{X}$ such that $\overline{A} \leq A$. Let $\hat{\psi} = \text{Res}_{\overline{X}}^{N_{A}(\overline{X})} \hat{\phi}$, then $\text{bl}(\hat{\psi}')^{\overline{X}}$ covers $\text{bl}(\hat{\psi})^{\overline{X}} = \text{bl}(\phi)$. Note that $\phi$ extends to $\hat{\phi} \in \text{IBr}_{b}(\text{Res}_{\overline{X}}^{\overline{G}})_{\overline{X}}$ with $\hat{A}_{\overline{X}} = A_{\overline{X}} = A$ by Lemma 4.16 and Corollary 6.6. Replacing $\overline{X}$, $N_{A}(\overline{R})$, $\phi$, $\phi'$ by $\overline{G}$, $N_{G}(\overline{R})$, $\hat{\phi}$, $\hat{\phi}'$ respectively and noting that $A/\overline{G}$ is abelian, we can use the same arguments as in the first paragraph of the proof of [37] Prop. 9.21] to prove that the proposition holds for $A$. By the remarks at the beginning of the proof, the proposition holds for general $\overline{X} \leq J \leq A$. □
Proof of Theorem 1.3. If \( \ell = p \), then the assertion holds by [39, Thm. C]. Now we assume that \( \ell \neq p \). Now the case when \( n \geq 3 \) is completely solved by our results in Corollary 5.19, Proposition 7.1 and Lemma 7.2. Now we assume that \( n = 2 \). By Corollary 5.19, it suffices to check condition (iii) of Definition 2.5. Note that, if we define \( D := (F_p^\ell) \) for this case, then it is easy to see that Proposition 7.1 and Lemma 7.2 also hold by the same argument. This completes the proof. \( \square \)

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