Black Hole Collisions, Analytic Continuation, and Cosmic Censorship

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Abstract: Exact solutions of the Einstein-Maxwell equations that describe moving black holes in a cosmological setting are discussed with the aim of discovering the global structure and testing cosmic censorship. Continuation beyond the horizons present in these solutions is necessary in order to identify the global structure. Therefore the possibilities and methods of analytic extension of geometries are briefly reviewed. The global structure of the Reissner-Nordström-de Sitter geometry is found by these methods. When several black holes are present, the exact solution is no longer everywhere analytic, but less smooth extensions satisfying the Einstein equations everywhere are possible. Some of these provide counterexamples to cosmic censorship.

1 Introduction

The recently discovered exact dynamical solution of Einstein’s equations provides a common thread that ties together three items of the title. This solution describes a cosmology with several charged black holes in motion and capable of collision. To discover the details of the collision one needs to continue the solution beyond the region in which it was originally defined. The spacetime so continued can then contain a naked singularity and provide a counterexample to some versions of the cosmic censorship hypothesis.

Because interesting applications of Einstein’s equations frequently occur in manifolds of complicated topology, which cannot be covered by a single coordinate patch, the region in which a typical exact solution is first known is often incomplete. The simplest way to complete it, if possible, is by analytic extension. It is remarkable how little is known in a systematic way about this frequently encountered problem. Here we will not materially improve on this situation, but merely recall some of what is known about analytic continuation, and discuss the most common class of geometries for which a method exists.

Although it is tempting to expect that such a method can also analytically continue the colliding black holes of interest here, we will find that, surprisingly,
these holes are in general not everywhere analytic. There is of course nothing
unphysical about such behavior; it is what one expects in the presence of a grav-
itational waves pulse. We exhibit continuations that are as smooth as possible
across the associated Cauchy horizon. The naked singularities of interest for
cosmic censorship are then found on the other side of such horizons. (Other
horizons are cosmological and do not hide singularities, but an “antipodal” part
of the universe.)

2 Analytic Continuation of Spacetimes

The purpose of spacetime, as originally conceived, is to describe the history of
all inertial observers. A (timelike) geodesically incomplete spacetime fails to
do this, so it behooves us to extend it as far as possible. (If even the maximal
extension is incomplete we can begin to ask questions about cosmic censorship.)
An analytic extension, if possible, is preferred because of its uniqueness and
“permanence” (i.e., the continuation of Einstein’s equations is automatic).

Actually, if not only the metric but also the manifold needs to be continued,
then the continuation is not necessarily unique. A simple example is a finite
part of (flat) Minkowski space, which can be continued either to the complete
Minkowski space, or to one of the several locally Minkowskian spaces, such as
the torus. More generally, any simply connected part of spacetime that can be
continued to a multiply connected one can also be continued to a covering of
the latter. We will encounter such ambiguities in the cosmological black hole
case below. A somewhat more subtle example is the Taub-Nut space, which
has two distinct and inequivalent analytic extensions [2]. In these ambiguous
cases one needs to decide on such properties as the topology of the extended
manifold along with the extension of the metric. Once the whole (smooth)
manifold is known, its analytic structure is essentially unique. Because our
manifolds are real, the relevant notion is real analyticity — existence of local
power series expansions. Analyticity of the manifold means that the coordinate
transformations between neighborhoods are real analytic functions.

In this context the analytic continuation of functions, such as the metric co-
cefficients, is then unique, and the continuation satisfies the continued differential
equation if the latter is itself analytic, as in the case of Einstein’s equations [1].

To know this is, however, of little help in practical problems where the metric
is given in some coordinates, and we desire to extend across a boundary where
both the metric coefficients and the coordinates are non-analytic. There appears
to be no systematic criterion for deciding whether analytic continuation is pos-
sible, and one usually has to rely on ingenuity to find suitable new coordinates
in which the metric is analytic in the relevant region.

It can happen that neither the metric nor the coordinates are analytic func-
tions on the manifold, but the metric coefficients are analytic functions of the

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1 Analyticity is not assured, nor is it necessarily to be expected on physical grounds,
if there are source terms present. Well-known examples are stellar models that are
non-analytic on the stellar surface.
coordinates; or they may be extendable to another (real) range of the coordinates by an excursion in the complex plane. Examples of this are found in the Schwarzschild metric at \( r = 2m \) and \( r = 0 \) respectively. In either case we obtain solutions of the Einstein equations in the new coordinate range, but it is a separate question whether and how the geometry so described fits together with the original geometry, and if so, whether the fit is analytic. Finding a proper overlap seems to be the only way to assure the latter. For a class of metrics to be discussed below, of which the Schwarzschild metric is a member, one knows how to fit together the pieces across horizons like \( r = 2m \).

One can, of course, give criteria that establish non-analyticity, for example the divergence of invariants formed from the Riemann tensor and/or its derivatives. This happens, for example, at \( r = 0 \) in the Schwarzschild metric, so no real analytic extension is possible there. It is worth noting that, in the case of indefinite metrics, not all divergences of the Riemann tensor can be found in this way; this happens when there is a “null” infinity, as in a Riemann tensor of the type \( R_{\mu \nu \alpha \beta} = l_{[\mu} m_{\nu]} l_{[\alpha} m_{\beta]} \), with \( l^\mu l_\mu = 0, l^\mu m_\mu = 0 \) and divergence in \( l,m \). In that case the components in an orthonormal frame diverge. (To avoid spurious infinities due to the frame becoming null the orthonormal frame should be parallelly propagated).

If one admits an excursion to complex coordinate values one may find other real metrics analytically related to the original one. Such extensions are however not unique, and may have nothing directly to do with the original geometry. In fact, the “extension” may have a different signature than the original metric. An example is the Euclidean Schwarzschild geometry that is used to describe instanton or thermal effects. An example of a Lorentzian, complex analytic but hardly physical relation of the Schwarzschild metric is

\[
ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} - r^2 d\theta^2 + r^2 \cosh^2 \theta d\phi^2.
\]

Another example is the continuation of the de Sitter space metric,

\[
ds^2 = -\frac{dt^2}{t^2} + e^{2Ht}(dx^2 + dy^2 + dz^2)
\]

across \( t = 0 \), which effectively changes the cosmological expansion parameter \( H \) into its negative.

As remarked above, a null surface is a natural analyticity boundary, because “new” information can propagate along such surfaces. On the other hand, analyticity of a region would be expected to extend to the domain of dependence of that region. The features of simplicity that allow exact solutions may have

\( ^2 \)It is remarkable that some solutions known in closed form (which is sometimes — loosely — called “analytic”) can be extended with a high degree of differentiability (e.g. \( C^{122} \) as in [3]) but not analytically.

\( ^3 \)To restore uniqueness it has been suggested [4] that the slightly complex path should be a geodesic. It remains to be seen whether one does not still lose physical significance in this unorthodox continuation.
a similar extent, so that the coordinates in which a metric is originally found tend to be analytic only in such domains, even when the geometry itself can be extended. Therefore an approach worth trying is to introduce null coordinates in which the boundary is one of the coordinate surfaces. The following class of metrics provides an example.

3 Walker’s Spacetimes and their Maximal Extension

Walker [5] considers spherically symmetric “static” metrics of the form

\[ ds^2 = -F dt^2 + \frac{dr^2}{F} + r^2 d\Omega^2 \]

where \( F = F(r) \) is the norm of the Killing vector \( \partial/\partial t \). Here this Killing vector is not necessarily timelike (hence the quotes around “static”) because we allow \( F \) to be positive or negative. \( F \) may be an analytic function of \( r \), satisfying the Einstein equation, and range over positive and negative values, but the metric is clearly non-analytic at the zeros of \( F \); the problem is to find the continuation across these zeros. (Infinities of \( F \) imply infinities of the Riemann tensor, so no analytic continuation is possible there.) Because the angular part is regular for \( r > 0 \), it suffices to confine attention to the two-dimensional \( r, t \) part of the metric.

By “factoring” this two-dimensional part into two integrable null differential forms,

\[ du = dt + \frac{dr}{F}, \quad dv = dt - \frac{dr}{F} \]

we can give the metric the double null form, \( ds^2 = -F(u-v) du dv \), but as a metric this is still singular at \( F = 0 \). If instead, following Finkelstein’s trick, we use \( r \) and only one null coordinate, say \( u \), the metric assumes the nonsingular form

\[ ds^2 = -F(r) du^2 + 2 du dr, \]

which is analytic wherever \( F \) is analytic as a function. This metric, then, provides the overlap necessary to connect two regions with opposite signs of \( F \). This analytic connection between two neighboring regions is illustrated in the conformal diagram shown in Fig. 1. Here we have assumed that the region \( r \to \infty \) has the usual asymptotically flat structure, and that there is another zero of \( F \) at finite \( r \) below \( r = a \) (corresponding to the “roof” of the figure). If these structures are different, the blocks may not have a diamond shape, but the region around the zero, \( r = a \), will look the same.

\[ 4 \]

In this sentence the word “analytic” could be validly replaced everywhere by “smooth” or “\( C^n \)”. In those cases the extension would, of course, not be unique.
Fig. 1. Conformal diagram of region of Walker metric surrounding \( r = a \), the largest zero of \( F \). The \( r, t \) coordinates are degenerate on the boundaries of the diamond-shaped regions. For example, the lower left boundary can also be labeled \( r = a \).

The spacetime as shown is still not complete. For example, the middle left boundary \( r = a \) is at a finite distance, and so is the “roof.” By using Finkelstein coordinates \( r \) and \( v \) we obtain a system that overlaps the middle left boundary, and by repeating this procedure around the next smaller zero below \( r = a \) we can extend beyond the “roof.” Thus in passing through each zero of \( F \) we add several new diamond-shaped regions to the conformal diagram, according as we cross the boundary along \( u = \text{const} \) or \( v = \text{const} \). All the regions so generated at the zero \( r = a \) are shown in Fig. 2.

The overlapping coordinates constructed so far reach across all of the diagonal lines, but not across the intersection points P, Q, R, . . . . These come in two types, those like P and R being characterized by the vanishing of the \( \partial/\partial t \) Killing vector, and those like Q by different values of \( r \) trying to come together. It is therefore not surprising that no analytic continuation is possible or necessary across points of type Q; they are at an infinite distance \((t \to \infty)\) along Killing orbits and not part of the manifold. (They can also provide a “safe haven” for observers who might otherwise experience a naked singularity.) At points of type P the blocks may fit together smoothly or analytically, depending on the form of \( F \). The proof is not immediate; Walker \[5\] introduces lightlike coordinates \( U, V \) related to the \( u, v \) of (1) via an adjustable constant \( c \),

\[
dU/U = cd\ u \quad dV/V = -cdv.
\]

The metric then takes the form

\[
ds^2 = G\ dU\ dV \quad \text{with} \quad G = \frac{F}{c^2} \exp\left( -2c \int \frac{dr}{F} \right).
\]
For functions $F$ of the type

$$F(r) = \frac{\prod_i (r - a_i)}{K(r)},$$

(2)

with $K(r)$ a polynomial with zeros differing from the $a_i$, he shows that $c$ can be chosen so that $G \neq 0$ at any one $a_i$.

Fig. 2. Blocks that fit together at their $r = a$ boundary. The next lower zero of $F$ occurs at $r = b$.

If two roots coincide, the picture looks different. There are no points of type P because the double-root horizon is at an infinite spatial distance; a spacelike section does not have the “wormhole” shape, but is an infinite funnel or “cornucopion.” Figure 3 shows how the blocks fit together in that case. All lines and curves that are shown correspond to $r = \text{const}$. This diagram looks like what one would obtain by continuing Fig. 1 towards the upper left by the usual rules to another block with $F > 0$, and then eliminating the $F < 0$ block and moving the two $F > 0$ blocks together. Thus the coincidence limit of two roots of $F$ does not appear continuous in the conformal picture. This happens, for example, for the Reissner-Nordström geometry, where the coincidence limit corresponds to an “extremally” charged black hole, $Q^2 \rightarrow M^2$. As long as the roots are distinct the two $F > 0$ blocks have a finite size $F < 0$ block between them. Instead of eliminating the $F < 0$ block, one can keep its physical size constant by rescaling the metric. The resulting spacetime is the
Bertotti-Robinson universe. (More detail on the relation between the extremal Reissner-Nordström and the Bertotti-Robinson geometries is found in [7].)

Beyond the coincidence limit it can happen that a pair of (real) roots disappear. This is also not a continuous change in the conformal diagram — the two regions of Fig. 3 merge into one.

Metrics of the Walker type occur in the cosmological black hole context when there is a “single” black hole of mass $M$ and charge $Q$ in a universe with cosmological constant $\Lambda$. In that case, $F$ takes the form

$$F(r) = \left(\frac{-\frac{1}{3}\Lambda r^4 + r^2 - 2Mr + Q^2}{r^2}\right), \quad (3)$$

which is of the type (2). (As there, we will denote the zeros of the numerator, in decreasing order, by $a_1 \ldots a_4$.) Thus we know that the maximal analytic extension is given by the Walker construction, and once we know how the blocks fit together we can do all calculations in the original $r, t$ coordinates.

4 Global Structure of de Sitter and Reissner-Nordström-de Sitter Cosmos

From the above block-gluing rules it is clear that the conformal diagrams for the Reissner-Nordström-de Sitter metrics depend only on the number of zeros of the function $F$ of (3). Only three of the roots are positive (for positive $M$ and $\Lambda$). The simplest example is de Sitter space itself, $M = 0, Q = 0$, so $a_1 = \sqrt{3}/\Lambda, a_2 = a_3 = 0$. The blocks look different in this case, because $r = 0$ is a regular origin. Also, $r = \infty$ is infinite distance in time (since $F < 0$ for $r > a_1$), so we can identify it with timelike and null infinity, $\mathbb{I}$. Figure 4a
shows an embedding of an $r, t$ subspace of this geometry in flat 3-dimensional Minkowski space, and Fig. 4b is the corresponding conformal diagram. Note that the conformal diagram corresponds to only half of the embedded surface, because the latter shows both “sides” of the origin ($\phi = 0$ and $\phi = \pi$, for example).

De Sitter space is often described in coordinates different from the $r, t$ used above, in which the space sections are conformally flat. These new coordinates $r', t'$ are called isotropic or cosmological coordinates [8].

$$r = r' e^{Ht'} \quad t = t' - \frac{1}{2H} \ln(1 - H^2 r'^2).$$

Here $H = \sqrt{\Lambda/3} = 1/a_1$ is the “Hubble constant” or cosmological expansion parameter. The metric then becomes

$$ds^2 = -dt'^2 + e^{2Ht'} \left( dr'^2 + r'^2 d\Omega^2 \right). \quad (4)$$

These coordinates cover more of de Sitter space than one patch of the $r, t$ coordinates, but there still is a horizon at $t' = -\infty$. Another block of $r', t'$ coordinates but with opposite sign of $H$ can be analytically connected to the original one. We call the coordinates “expanding” in the block with $H > 0$, and “contracting” in the other one. Figure 5 shows some of the spacelike surfaces $t' = \text{const}$.

That the de Sitter universe can appear as either static (3), or expanding or contracting (4), is an accident due to the high degree of symmetry of this model. In fact, each “observer” (timelike geodesic) has a static frame centered around him/her. When there is a “single” black hole present, a static frame still exists (but only the one that is centered about the black hole). The analytic
extension of this case can therefore be easily treated by Walker’s method. In
discussing the global properties it is appropriate also to show the expanding
and contracting, cosmological frames, because only their analog exists in the
spacetimes \[1\] with several black holes.

The static spherically symmetric solution of the Einstein-Maxwell equations
with a cosmological constant is the Reissner-Nordström-de Sitter metric and
EM potential (abbreviated RNdS),

\[
ds^2 = -F dT^2 + \frac{dR^2}{F} + R^2 d\Omega^2
\]

\[
F = 1 - \frac{2m}{R} + \frac{Q^2}{R^2} - \frac{1}{3} \Lambda R^2 \quad A_T = -\frac{Q}{R}.
\]

Here we have used capital letters to denote the static frame in order to dis-
tinguish it from the cosmological coordinates, which will be in lower case. By
means of a somewhat involved coordinate transformation \[8\] one finds the form
of the metric in cosmological coordinates,

\[
ds^2 = V^{-2} dt^2 + U^2 e^{2Ht} (dr^2 + r^2 d\Omega^2),
\]

where

\[
U = 1 + \frac{M}{\rho} + \frac{M^2 - Q^2}{4\rho^2} \quad V = \frac{U}{1 - \frac{M^2 - Q^2}{4\rho^2}} \quad \rho = e^{Ht} r.
\]

By means of the simple coordinate change

\[
\tau \equiv H^{-1} e^{Ht}
\]
we can extend the region covered by these coordinates, by allowing \( \tau \) to be negative as well as positive. The metric and EM potential then become

\[
 ds^2 = -\frac{d\tau^2}{U^2} + U^2(d\rho^2 + r^2d\Omega^2) \quad U = H\tau + \frac{M}{r} \quad A_\tau = \frac{1}{U}
\]  

Figure 6 shows the analytic extension constructed according to the prescription of Sect. 3 for the generic case, when there are three roots of \( F \).
spacelike surface $T = 0$ that cuts the figure horizontally in its center then has the geometry akin to a string of beads: as we move along this surface, the radius of the sphere in the other two ($\theta, \phi$) directions alternately reaches maxima and minima. The maxima correspond to the large regions of the universe (the “background de Sitter space”), and the minima are throats of wormholes that connect one de Sitter region with the next. Thus each de Sitter region contains two wormhole mouths, placed in antipodal regions of each large universe. (This is the reason for the quotes above when calling this a “single” black hole in a de Sitter universe.) Alternatively and more compactly we can imagine the left and right halves of the figure identified, so that the horizontal spacelike surface of the figure is a closed circle, and the 3-D spacelike topology is $S^1 \times S^2$.

In this case the electric flux of the charge $Q$ also describes closed circles: it emerges from one wormhole mouth, spreads out to the maximum universe size, reconverges on the other mouth, and flows through the wormhole back to the first mouth. Seen from the large universe, the first mouth appears positively charged, and the antipodal one, negatively charged — an example of Wheeler’s “charge without charge”!

In this universe (as in the $\Lambda = 0$, asymptotically flat Reissner-Nordström (RN) geometry) a geodesic observer that wants to experience the singularity can do so, for example by moving along the vertical symmetry axis of the figure. In the RN case this is not considered a serious challenge to cosmic censorship, because the interior of the black hole, through which the observer in search of a singular experience must travel, is not stable under small perturbations of the exterior: radiation falling into the hole from the exterior would have a large blueshift at this observer — it would not only burn her up, but also change the nature of the singularity. It is remarkable that this does not necessarily happen in RNdS universes, for certain values of the parameters $\Lambda$. Thus these solutions are a counterexample to a strong interpretation of cosmic censorship. But there are, of course, many other geodesics that can lead observers who do not take the plunge to their safe haven at $\mathbb{I}^+$.

The picture of Fig. 6 changes, for example as in Fig. 3, when roots of $F$ coincide or cease to be real (see [8] and [10]). Another special case is of interest here because it can be generalized to the multi-black-hole case, namely $Q^2 = M^2$. In contrast to the RN case, when $\Lambda \neq 0$ this choice does not force a double root, so the global structure and conformal diagram of Fig. 6 still applies. What changes is the way the cosmological coordinates cover the diagram: the lens-shaped region degenerates into the line $\tau = 0$, so that a continuous region between the singularity and $3^+$ is covered. Therefore in this case the entire analytic extension of the metric (6) can be obtained by gluing together alternate copies of expanding and contracting cosmological coordinate patches. Figure 7 shows a pair of such regions, with the contracting one outlined by thick lines. (For easy comparison with Fig. 6 it may help to turn the latter upside down.) We see that the expanding patch contains $3^+$, and the contracting one contains $3^-$. It is therefore not immediately clear, if we calculate in contracting coordinates only, how to identify the black hole event horizon as the boundary of the past
of $\mathbb{S}^+$. We can follow outgoing light rays to $r = \infty$, but at that point their geometrical distance $R$, measured by the Schwarzschild coordinate $R$ (or by the area of the sphere $r = \text{const}$) is still finite, $R = a_1$. However, there is this difference between such light rays and those that fall into the black hole, that the latter reach the geometrical singularity, which is contained in the contracting cosmological coordinates. Furthermore, timelike geodesics heading toward the point $S$ in the figure have an infinite proper time. Thus $S$ is a safe haven for observers who desire to avoid the black hole, and it can be regarded as a small piece of $\mathbb{S}^+$ that can be asymptotically reached in contracting cosmological coordinates, just enough to be able to identify the event horizon.

The $Q^2 = M^2$ RNdS geometries still depend on two parameters, $M$ and $\Lambda$ resp. $H$, or on one dimensionless parameter $p = 4M |H|$ up to scale change. If $p < 1$ we have the “undermassive” case discussed so far. If $p > 1$ there is only one real root of $F(R) = 0$. The outer black hole horizon and the de Sitter horizon have disappeared, only what used to be the inner black hole horizon remains. One could also interpret the remaining horizon as a cosmological one, separating two naked singularities at antipodal regions of a background de Sitter.

Fig. 7. Two patches of cosmological coordinates for the RNdS geometry for the case $Q^2 = M^2$ and $p < 1$ (“undermassive”).
space. The conformal diagram for this case, constructed according to the block
 gluing rules, is shown in Fig. 8.

![Conformal diagram for RNdS geometry](image)

**Fig. 8.** Two patches of cosmological coordinates for the RNdS geometry for the case
 \(Q^2 = M^2\) and \(p > 1\) ("overmassive"). The thick outline shows the contracting patch.

### 4.1 Collapsing Dust

The solutions discussed so far correspond to "eternal" black holes (or a combi-
nation of white holes and black holes). The future part of the usual black hole
geometry can be generated from matter initial data, say by the collapse of a
sphere of dust. A similar process is possible for \(Q^2 = M^2\) RNdS.

To generate such a geometry from dust, the dust must itself have \(Q^2 = M^2\),
so that electrical forces largely balance gravitational forces. The dust can still
collapse if it has the right initial velocity. A simple situation is dust that is
at rest in cosmological coordinates: in any metric and potential of form (6),
for an arbitrary \(U\), such dust remains at rest (\(r = \text{const}\)), if considered as test
particles. A typical wordline is shown in Fig. 7. To understand why it be-
haves this way we look more closely at the electric field associated with this
geometry. We know that the parameter \(Q\) of the black hole on the left is opposite to
that of the one on the right. Suppose the left hole is negative and the right
one positive. On the spacelike surface corresponding to a horizontal line drawn
through the center of Fig. 7, the electric field will then point from right to left.
By flux conservation, the electric field will therefore also point to the left in the
region near the upper singularity shown in the Figure. Thus we can associate a
negative charge with this singularity. (The singularity to the right of the \(r = 0\)
inner horizon, which is not shown in the Figure, correspondingly has positive
charge). The dust particle on the \(r = \text{const}\) trajectory has the same charge as
the black hole that is included in that coordinate patch, namely positive. It
therefore ends up on the negative singularity.

Any point in Fig. 7 can be considered to be in either of two possible cosmo-
logical coordinate patches. For example, a point near \(\mathbb{S}^-\) is in the contracting
patch shown, which contracts about the right black hole. By reflecting this
patch about a vertical line through the center of the Figure we obtain a patch that contracts about the left black hole. It also contains the region near \( \mathfrak{I}^- \). Its radial coordinate will be denoted by \( r' \). The trajectory \( r' = \text{const} \), obtained by reflecting the \( r = \text{const} \) trajectory shown in the Figure, describes a negative test charge that falls into the positive singularity of the left (negative) black hole. Similarly the region between the de Sitter and the event horizon in the contracting patch has trajectories of positive charges that fall into the positive black hole, and of negative charges that go to \( \mathfrak{I}^+ \). The region inside the event horizon has trajectories that go to one or the other singularity, depending on their charges. Of course all these trajectories that are simply described by \( r = \text{const} \) or \( r' = \text{const} \) satisfy special initial conditions — their initial position is arbitrary, but their initial velocity is then determined.

To take into account the effect of the dust on the metric and potential, one needs to match the vacuum region to an interior solution. For spherical symmetry the matching conditions are equivalent to the demand that the dust at the boundary of the interior region move on a test particle path of the vacuum region. Thus a possible boundary for a collapsing (expanding) ball of dust is \( r = \text{const} \) in collapsing (expanding) cosmological coordinates. The surface area \( 4\pi r^2 U^2 \) of the dust ball collapses to zero when \( U = 0 \), which is also the location of the geometrical singularity; at that point the center of the dust ball must coincide with the surface. The corresponding conformal diagram therefore looks as shown by the thick curves in Fig. 9. The region filled with dots denotes the location of the dust.

**Fig. 9.** Conformal diagram for a RNdS black hole generated by the collapse of a ball of charged dust in a de Sitter background (thick lines). The dust region is shown dotted. The dotted line is the black hole event horizon. The thin lines show a dust ball that is symmetrically placed at the antipodal region of the RNdS background, and has opposite charge. If both dust balls are present only the region between the curves \( r = 0 \) resp. \( r' = 0 \) applies.

Because the dust includes the origin \( r = 0 \), there is now no continuation to the right of the heavily outlined region necessary or possible. There is still a cosmological horizon, \( r = \infty \), and the continuation on the other side could
be analytic (a semi-infinite “string of beads”) or reflection-symmetric about a vertical axis through ℑ (a collapsing dust ball at the antipodal region). Of the two geometrical singularities associated with one RNdS black hole, the dust covers up only the one that would have been on the right in the figure, with the same total charge as that of the dust. The other singularity could be covered up by another dust ball, with the opposite charge. This is shown by the thin curves in Figure 9. If both dust spheres are present, then the physical part of the geometry lies between the \( r' = 0 \) curve on the left and the \( r = 0 \) curve on the right. If conditions are as shown, they uncover and then cover up again a (small) region of vacuum RNdS geometry between them.

5 The Multi-Black-Hole Solutions

A solution representing any number \( n \) of arbitrarily placed charged black holes in a de Sitter background is given by a metric of type (6), with a different potential \( U \):

\[
ds^2 = -\frac{dt^2}{U^2} + U^2(dr^2 + r^2dΩ^2) \quad U = Hτ + \sum_{i=1}^{n} \frac{M_i}{|r - r_i|} \quad A_τ = \frac{1}{U}. \quad (7)
\]

We will call this the KT solution. Each mass of the KT solution has a charge proportional to the mass, \( Q_i = M_i \), and only the location \( r_i \) (not the initial velocity) is arbitrary. Here \( |r - r_i| \) denotes the Euclidean distance between the field point \( r \) and the fixed location \( r_i \) in a Euclidean space of cosmological coordinates. For \( n = 1 \) this reduces to the \( Q^2 = M^2 \) case of the RNdS solution (6). Also, in the limit of large \( r \), and for \( r_i \) in a compact region of Euclidean coordinate space, (7) approaches the RNdS solution with \( M = \sum M_i \). One therefore expects the horizon structure at \( r \rightarrow \infty \) to be similar to that of RNdS, which suggests that a \( H > 0 \) and a \( H < 0 \) version of (7) can be glued together as extensions of each other, similar to Fig. 7. The surprise is that, although this can be done with some degree of smoothness, it cannot be done analytically [11]. This means that there is no unique extension. An observer at rest in contracting \((H < 0)\) cosmological coordinates, whose entire past can be described in these coordinates, has no way of telling what is in the other “half” of the de Sitter background (he can only guess that the total charge over there must be \(-\sum M_i\), to balance the charge that he sees). If he moves and crosses the cosmological horizon \( r = \infty \), the other, expanding half suddenly comes into view, seen at a very early time when all the masses are very close together. It is therefore reasonable that there is a pulse of gravitational and EM radiation associated with the horizon: this is the physical description of the lack of analyticity.

It is instructive to note how the various solutions differ in respect to possible coordinate choices. Pure de Sitter space has a static and two cosmological (expanding and contracting) coordinate systems centered about any timelike geodesic. In RNdS these coordinates are centered about the black hole only, but one can still choose between expanding and contracting frames near either
of the holes. In KT there is one set of black holes, all with the same sign of charge, that is uniquely expanding (the distances between holes increasing as $e^{H\tau}$), and another, oppositely charged set that is uniquely contracting (the distances decreasing as $e^{-|H|\tau}$). The expanding set is described by cosmological coordinates that include $\mathcal{I}^+$, whereas for the contracting set, $\mathcal{I}^-$ is included.

5.1 Merging Black Holes

Since black (as opposed to white) holes are determined from $\mathcal{I}^+$, it is easiest to identify the black hole horizons for the expanding set. Consider the expanding coordinates in RNdS as in the left half of Fig. 7. The boundary of the past of $\mathcal{I}^+$ that lies in those coordinates is the left-hand line labeled $r = 0$ — it is the event horizon of the black hole in the “left” part of RNdS space. Similarly the event horizon of expanding KT space is given by $r - r_i = 0$. The Euclidean coordinate space with the $n$ points $r_i$ removed is a representation of $\mathcal{I}^+$ of expanding KT space. The $n$ missing points represent $n$ disjoint boundary components, and there is a finite distance between them at all times. Thus we have $n$ black holes that remain separate for all times.

It is not as easy to identify the black holes in the contracting KT space, because that does not contain much of $\mathcal{I}^+$. But because the cosmological horizon at $r = \infty$ is so similar to that of the RNdS space it does contain a “point” like S of Fig. 7 that is just enough to define the event horizon. So, to find the event horizon while staying within one coordinate patch we must find the surface that divides light rays reaching $r = \infty$ from those that reach the singularity. But where in the KT world is the singularity? One can show that metrics of type (7) are singular where $U = 0$. In the contracting case, $H < 0$ (and of course $M_i > 0$), this happens only for positive $\tau$. Thus we need to find the “last” light rays that just make it out to $r = \infty$ at $\tau = 0$. More precisely (since $r = \infty$ is not a very precise place) we need $Hr\tau$ finite in this limit.\footnote{This is so because part of the “somewhat involved” coordinate transformation leading to (5) is actually rather simple, $R = Hr\tau + M$. For RNdS this is the static $R$, which is finite on the black hole horizon. Because near $r = \infty$ the KT geometry is so close to RNdS, $R$ should also be finite for KT.}

If we know these last light rays (the 3D horizon surface) we can then intersect them with a spacelike surface to find the shape of the 2D horizon at different times. An interesting question is whether this 2D horizon changes topology with time. This would, for example, describe the merger of two black holes into one. If black holes merge we can try to make an overmassive (hence nakedly singular) one from two undermassive ones, to test cosmic censorship. This is of course possible only in a contracting part of the KT solution, because we have already seen that the black holes in the expanding part remain separate for all times.

All contracting black holes do eventually merge into one. To show this for the case of a pair of holes, we show that the horizon must consist of two parts at early times, and be a single surface at late times. One can see rather directly
that light starting at \( r \) sufficiently close to any one of the \( r_i \) at any time and in any direction will reach the singularity, because it will spend its entire history in a geometry sufficiently close to the single black hole, RNdS geometry. Thus points sufficiently close to \( r_i \) will always lie within the event horizon. So, for the case of two black holes, a key question is what happens to light that starts on the midplane between the holes. In fact, if the light starts early enough it will always be able to escape to \( r = \infty \). Thus at early times the midplane does not meet the horizon — the two black holes are disjoint. To show this we center our Euclidean coordinates at the midpoint between the holes, which have a Euclidean separation \( d \). From \( ds^2 = 0 \) in (7) we then find, for radial outgoing null geodesics in the midplane,

\[
\frac{d\tau}{dr} = U^2 = \left( H\tau + \frac{M}{\sqrt{r^2 + d^2}} \right)^2 < \left( H\tau + \frac{\sqrt{2}M}{r + d} \right)^2.
\]

Now let

\[
R_* = H\tau(r + d) \quad y_* = \ln(r + d)
\]

to find

\[(r + d)\frac{dR_*}{dr} > R_* + H(R_* + \sqrt{2}M)^2. \tag{8}\]

Standard analysis of this equation shows that if \( R_* \) is larger than the lower root of the RHS of (8), it will stay positive for all larger \( r \). In terms of \( r \) and \( \tau \) this means that if \( \tau \) is sufficiently negative for any \( r \) (remember \( H < 0 \)) then \( \tau \) will remain negative as \( r \) increases — the lightray avoids the singularity. By a similar estimate one can show [11] that for each sufficiently late (but negative) \( \tau \) there is a sphere surrounding both \( r_i \) such that all outgoing null geodesics will reach the \( U = 0 \) singularity. This means that the horizon surrounds both \( r_i \), and the black holes have merged.

5.2 Continuing Beyond the Horizons

We could now discuss the merging of two undermassive into one overmassive black hole. Since the geometry near \( r = \infty \) will be determined by the overmassive sum of the two individual masses, that neighborhood will look like the corresponding part of a single overmassive RNdS, that is, like the left hand side of Fig. 8. That contains a naked singularity (the left multiply-crossed line), which has nothing to do with the black hole merger, because it is located at the opposite side of the universe. But this is the only place in the coordinate patch where a \( U = 0 \) singularity occurs. Our patch does not describe enough of the history of the interesting region, where \( |r - r_i| \) is small, just as the thick outline does not extend far to the right side of Fig. 8. To find out whether black hole merger generates its own naked singularity we must continue the KT metric beyond the inner black hole horizons at \( r = r_i \). As we are continuing the KT geometry it is of course also interesting to look beyond the cosmological horizon.
at $r = \infty$, which exists only if the total mass is undermassive ($4|H| \sum M_i < 1$). So we look at null geodesics that approach these horizons.

Let us choose the origin $r = 0$ of our Euclidean coordinates at the location of the $i^{th}$ mass (so that $r_i = 0$). The equation $ds^2 = 0$ satisfied by an ingoing null geodesic then takes the form, for small $r$,

$$
\frac{dr}{d\tau} = -U^2 = - \left( H\tau + \frac{M_i}{r} + \sum_{j \neq i} \frac{M_j}{r_j} \right)^2.
$$

We can eliminate the last (constant) term on the right by defining a new time coordinate $\tau' = \tau + H^{-1} \sum (M_j/r_j)$. The equation then becomes an equality version of (8), and by analyzing it in the same way as above one finds [11] the limiting forms

$$
2H^2 r\tau' \rightarrow 1 - 2M_i H - \sqrt{1 - 4M_i H}.
$$

To assess any incompleteness we need to know how a null geodesic $(r(s), \tau(s))$ depends on the affine parameter $s$, and we can get that from the variational principle,

$$
\delta \int \left( -\frac{1}{U^2} \left( \frac{dr}{ds} \right)^2 + U^2 \left( \frac{dr}{ds} \right)^2 \right) ds = 0.
$$

The Euler-Lagrange equation for $\tau(s)$ together with the first equality of (9) yields

$$
\frac{d^2 r}{ds^2} - 2HU \left( \frac{dr}{ds} \right)^2 = 0.
$$

Substituting (10) we find, in the limit $r \rightarrow 0$,

$$
\frac{d^2 r}{ds^2} - \frac{1}{r} \frac{\sqrt{1 - 4M_i H}}{r} \left( \frac{dr}{ds} \right)^2 = 0
$$

with the solution

$$
r \sim (s - s_{\text{hor}})^{\frac{1}{\sqrt{1 - 4M_i H}}}, \quad \text{hence} \quad \tau \sim (s - s_{\text{hor}})^{-\frac{1}{\sqrt{1 - 4M_i H}}}.
$$

So the inner horizon is reached at a finite parameter value $s_{\text{hor}}$. Similarly one finds that the cosmological horizon is reached in a finite parameter interval,

$$
r \sim (s - s_{\text{Hor}})^{\frac{1}{\sqrt{1 + 4(\sum M_i) H}}}, \quad \tau \sim (s - s_{\text{Hor}})^{\frac{1}{\sqrt{1 + 4(\sum M_i) H}}}.
$$

This behavior of the coordinates $r$ and $\tau$ gives us important information about the differentiability and analyticity of the geometry near the horizon. We can first eliminate whichever of the two is infinite on a given horizon in favor of $\bar{R} = H\tau\tau$, which is always finite on the horizon. The metric is then an analytic function of remaining, finite coordinates, so the Riemann tensor will also be analytic in these coordinates. But in order that the geometry be differentiable, the Riemann tensor should be differentiable in the affine parameter $s$ along null
geodesics. Thus the differentiability of the geometry is measured by that of \( r \) resp. \( \tau \) as a function of \( s \), as given by (11) and (12).

Consider first the neighborhood of the inner horizon, where \( r \) is finite. Since \( H < 0 \) we have \( 1/\sqrt{1 - 4M_i H} < 1 \), \( r \) is not a differentiable function of \( s \) at \( r = 0 \), and the Riemann tensor will be singular there. A more careful analysis [1], using a transformation to coordinates that are not singular on the horizon, shows that the metric is \( C^1 \) but in general not \( C^2 \). There is therefore no unique, analytic extension across the inner horizon. One can match differentially essentially any KT solution with the same mass \( M_i \). One can increase the differentiability by arranging the other masses carefully around the \( i^{th} \) one, so as to make the potential \( U \) approximately spherically symmetric (by eliminating multipoles to some order). The neighborhood of \( M_i \) then becomes approximately RNdS and hence “more nearly analytic” — i.e., of increased differentiability.

The situation near the cosmological horizon offers more variety. To have this horizon at all the total mass must be undermassive, \( 4|H|\Sigma(M_i) < 1 \). Here \( \tau \) is the finite one, and the corresponding power of \( s \) is \( 1/\sqrt{1 - 4(\Sigma M_i)|H|} > 1 \). Thus \( \tau \) is always at least \( C^1 \). The transformation to coordinates that are good on the horizon shows that the metric is always at least \( C^2 \). In the special cases when the power is an integer \( n \), i.e., for masses such that

\[
4H \sum M_i = 1 - \frac{1}{n^2},
\]

the metric is \( C^\infty \). For these values the smooth continuation matches the KT spacetime at the cosmological horizon to one with the same position and magnitudes of all the masses (so that all multipole moments agree), but with the opposite sign of \( H \). We do not understand the physical significance of these special masses.

To show that the geometry at these horizons is not more differentiable than claimed one can compute the Riemann tensor. This infinity is of the null type mentioned in Sect. 2, and does not show up in invariants formed from the Riemann tensor. To see this for the horizon at \( r = 0 \) (or \( r = \infty \)) we write the metric (7) in terms of the coordinate \( \hat{R} = Hr\tau \) and \( y = \ln r \), and the quantity \( W = rU \). Then the horizon occurs at \( r \to \pm \infty \), where \( \hat{R} \) and \( W \) are finite,

\[
ds^2 = -\frac{(d\hat{R} - \hat{R}dy)^2}{H^2W^2} + W^2(dy^2 + d\Omega^2).
\]

Now an invariant formed from the curvature tensor involves terms in derivatives of the metric and its inverse, multiplied by powers of the metric and its inverse. All these reduce to derivatives of \( W \) and \( \hat{R} \) divided by powers of \( W \). But all derivatives of \( W \) remain bounded as \( y \to \pm \infty \), and \( W \) is finite on the horizon. Thus the invariants cannot blow up.

Singularities do show up in the components of the Riemann tensor in a parallelly propagated frame, for example along the null geodesic \((r(s), \tau(s))\) discussed above. Let \( l = \partial/\partial s \) be the parallelly propagated tangent. Because of the asymptotic symmetry near the horizon, \( \eta = \partial/\partial \theta \) is also asymptotically
parallly propagated. Now the frame component \( R_{\mu\nu\rho\sigma} l^{\mu} l^{\nu} l^{\rho} l^{\sigma} \) contains the term \( g_{\theta\theta,ss} \). If \( g_{\theta\theta} = W^2 \) depends on \( r \) (and not just on the regular \( R \)), it will not be a smooth function of \( s \). In the RNdS (“single mass”) case, \( g_{\theta\theta} \) depends only on \( R \). In the KT case the corrections to that behavior near an inner \( (r = 0) \) horizon start with the power \( r^2 = (s - s_{\text{hor}}) \sqrt{1 - 4M_i H} \) unless there is special symmetry; thus one finds the differentiability of the metric as claimed above.

6 Naked Singularities?

Naked singularities visible to observers safely outside the strong curvature regions do not form in realistic gravitational collapse — this is the essential notion behind cosmic censorship. It can be made more precise in various ways [12]. At present none of these have been proved to be true in generic cases. When a proof seems difficult, it may be easier to obtain a convincing counterexample. Even if the conjecture is correct under certain assumptions, counterexamples are useful to test the necessity of these assumptions. For example, it may be that a version of cosmic censorship holds in pure general relativity, but fails when the theory is modified, say by a cosmological constant or by applying it to higher dimensions. There are in fact indications that cosmic censorship fails in the higher-dimensional theory inspired by string theory: certain 5D black strings (objects that appear four-dimensionally like black holes) are unstable, it is entropically favorable for them to decay into a set of 5D black holes, and during this decay naked singularities would form [13]. In the present contribution we want to test whether cosmic censorship fails in the special circumstances that are afforded when a cosmological constant is present.

The idea of using KT spacetimes to test cosmic censorship is to start with two or more small (and hence not naked) black holes and let them collapse to form a single large (and maybe naked) one. We have seen that the KT solutions indeed can describe coalescing black holes, in the sense that the event horizons coalesce. But if it is possible to define the event horizon as we did, by observers who live for an infinite proper time in one KT coordinate patch, then these observers will see no signal from any singularity — all singularities that form from the initial data, including those in any of the (non-analytic) extensions, lie inside this event horizon. To find a situation where there is no “safe haven,” so that the generic observers does see a singularity, we must suppose that \( \sum M_i \) is overmassive, so the initial black holes cannot be defined by their event horizon. An alternative to starting with black holes that have event horizons is to start with regular initial data on a compact surface. The KT solution cannot provide this either, because each \( M_i \) has an infinite throat. But such throats are the next best thing: each throat is undermassive, it is surrounded by a trapped surface, so one would not expect that the asymptotic regions down the throat could

\footnote{A similar test for Einstein-Maxwell-dilaton theory with a cosmological constant inspired by string theory has been discussed in [14].}
influence the solution in the interior. Can we, then, construct a KT solution of undermassive throats that has regular initial data and a naked singularity in its time development?

To decide this we must explore the global structure of the KT geometry. Unfortunately this cannot be completely represented by 2D conformal diagrams, because there is insufficient symmetry to suppress the additional dimensions. If we confine attention to the case of two equal masses, they lie on a line in the Euclidean coordinate space (which is an axis of symmetry of the spacetime). We can represent the essential features of the spacetime by drawing the conformal diagram for the spacetime spanned by the part of the axis going from one of the masses to infinity. This is shown in Fig. 10a. The part of the diagram representing the region near \( r = \infty \) is to be read like a normal conformal diagram (i.e., each point represents a 2-sphere), whereas the region near \( r = 0 \) is to be thought of as doubled (each point represents two 2-spheres).

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**Fig. 10a.** Conformal diagram of history of axis from one black hole to “infinity” for a two-black-hole contracting KT geometry. The black holes correspond to the region on the right, and the surrounding “de Sitter background” is the region on the left. The total mass is assumed to be overmassive, so this “background” is really an overmassive RN\(\text{dS}\) geometry with the singularity shown on the left. Therefore the axis extends to \( r = \infty \) only for \( \tau < 0 \), after that it hits this singularity at \( U(r, \tau) = 0 \). The \( C^1 \) extension across the upper \( r = 0 \) horizon was chosen to be the time-reverse of the heavily outlined region.

**Fig. 10b.** In this diagram the cosmological singularity of Fig. 10a is covered up by a sphere of dust, as discussed in Sect. 4.1. The dust region is shown dotted. The thick curve is a spacelike surface with nonsingular initial data, containing two infinite charged black hole throats and, on the antipodal point of the universe, a collapsing sphere of dust. All observers have to cross the future \( r = 0 \) Cauchy horizon and can thereafter see the naked singularity shown on the right, the result of the merger of the two black holes. The spacetime ends after a finite proper time in a “big crunch.”
The region near \( r = \infty \) of a KT solution (7) always behaves like an RNdS geometry with mass equal to the total mass \( \sum M_i \). If this is overmassive, there will be a curvature singularity in this region, whether the individual (undermassive) holes have merged or not. This singularity at the antipodal point of the universe has nothing directly to do with the black hole merger. To obtain nonsingular initial data we can eliminate this singularity by replacing it with a collapsing sphere of dust, as in Sect. 4.1. The resulting diagram is shown in Fig. 10b. The initial data induced on the spacelike surface shown by the thick curve are now nonsingular. In the time development shown, beyond the future Cauchy horizons \( r = 0 \), a curvature singularity appears. It comes in from infinity through the infinite throats of the merged black holes and spreads to \( r' = 0 \), i.e. to the antipodal point of the universe. The fact that the infinite throats are hidden behind trapped surfaces does not seem to be sufficient to prevent the singularity from coming “out of” the throat. Perhaps a less coordinate-oriented way of saying this is that all of space collapses down the throats, carrying all observers with it.

It is clear from the diagram that all observers originating on the initial surface will reach the Cauchy horizon, and if they extend beyond, they will see the singularity. We have seen that the Cauchy horizon surrounding a typical KT throat is not smooth, so that delicate observers may not survive the crossing. But by distributing several KT masses symmetrically about a given one, we can make that one as differentiable as necessary to ensure an observer’s survival. So it is reasonable to conclude that cosmic censorship is violated in these examples.

The initial data in these examples already contain the black holes’ infinite throats, and are not compact. Can we first form the black holes from collapsing dust, and then let them go through the above scenario? We have seen in Fig. 9 that we can have simultaneous collapse of a dust ball to form a black hole, and simultaneously remove the overmassive singularity at the antipodal point by another dust ball. The problem is now that in the KT solution, as in the RNdS case shown in Fig. 9, the two balls collide before any singularities have formed, at least if the balls move on \( r = 0 \) resp. \( r' = 0 \) trajectories in KT (cosmological) coordinates. Even if we allow more general trajectories we know for charged test particles that the trajectories tend to avoid singularities of the same charge; and even if naked singularities were formed later in the evolution, one would not know whether they were a fundamental property of the theory, or due to the dust approximation (“shell crossing singularities,” which occur also in the absence of gravity and hence have nothing to do with cosmic censorship).

Even if we accept the KT solution’s infinite throats in place of compact initial data, we still do not yet have a serious violation of cosmic censorship, because the general KT solution is still quite special. The initial position and masses can be specified arbitrarily, but not the initial velocities. The constraints on the initial values can be solved in more general (but still not quite generic) contexts, for example one can drop the \( Q_i^2 = M_i^2 \) condition [11]. These initial data can be analytic, but we do not know what happens beyond the Cauchy horizon. In the general KT solution we have seen that one has to cross the Cauchy horizon...
to see the naked singularity. It is not clear whether more generic solutions have a Cauchy horizon with a stronger singularity than the KT solution. If so, then cosmic censorship would be preserved.

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