Low Complexity Sparse Bayesian Learning Using Combined BP and MF with a Stretched Factor Graph

Chuanzong Zhang, Zhengdao Yuan, Zhongyong Wang and Qinghua Guo

Abstract—This paper concerns message passing based approaches to sparse Bayesian learning (SBL) with a linear model corrupted by additive white Gaussian noise with unknown variance. With the conventional factor graph, mean field (MF) message passing based algorithms have been proposed in the literature. In this work, instead of using the conventional factor graph, we modify the factor graph by adding some extra hard constraints (the graph looks like being ‘stretched’), which enables the use of combined belief propagation (BP) and MF message passing. We then propose a low complexity BP-MF SBL algorithm based on which an approximate BP-MF SBL algorithm is also developed to further reduce the complexity. Thanks to the use of BP, the BP-MF SBL algorithms show their merits compared with state-of-the-art MF SBL algorithms: they deliver even better performance with much lower complexity compared with the vector-form MF SBL algorithm and they significantly outperform the scalar-form MF SBL algorithm with similar complexity.

Index Terms—sparse Bayesian learning, message passing, BP-MF.

I. INTRODUCTION

RECENTLY, compressed sensing [1], [2] has received tremendous attention and it has found wide applications in a large variety of engineering areas, e.g. biomagnetic imaging, sparse channel estimation, bandlimited extrapolation and spectral estimation, echo cancellation and image restoration [3]. In compressed sensing, a vector $\alpha \in \mathbb{C}^L \times 1$ which exhibits sparsity is estimated based on the measurement vector $y \in \mathbb{C}^N \times 1$ with the following model

$$y = \Phi \alpha + \omega \quad (1)$$

where $\Phi \in \mathbb{C}^{N \times L}$ is called dictionary matrix and $\omega$ represents an additive white Gaussian noise (AWGN) vector with zero mean and covariance matrix $\lambda^{-1} I$. In this work, we are particularly interested in the case that the variance of the AWGN (or the precision parameter $\lambda$) is unknown.

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C. Zhang is with the School of Information Engineering, Zhengzhou University, Zhengzhou 450001, China, and the Department of Electronic Systems, Aalborg University, Aalborg 9220, Denmark (e-mail: ieczhang@gmail.com).

Z. Yuan is with the Zhengzhou Institute of Information Science and Technology, Zhengzhou 450001, China (e-mail: yuan.zhengdao@foxmail.com).

Z. Wang is with the School of Information Engineering, Zhengzhou University, Zhengzhou 450001, China (e-mail: yuan.zhengdao@foxmail.com).

Q. Guo is with the School of Electrical, Computer and Telecommunications Engineering, University of Wollongong, Wollongong, NSW 2522, Australia, and also with the School of Electrical, Electronic and Computer Engineering, University of Western Australia, Crawley, WA 6009, Australia (e-mail: gguo@uow.edu.au).

Besides convex [4] and greedy [5] methods, sparse Bayesian learning (SBL) [6]–[8] is an alternative method of sparse signal estimation, which aims at finding a sparse maximum a posteriori (MAP) estimate $\hat{\alpha} = \arg\max p(\alpha|y)$ of the vector $\alpha$ by specifying a priori probability density function (pdf) $p(\alpha)$. Instead of working directly with a prior $p(\alpha)$, SBL typically employs a two-layer (2-L) hierarchical structure [9] that assumes a conditional prior pdf $p(\alpha|\gamma)$ and a hyper-prior pdf $p(\gamma)$, so that $p(\alpha) = \int p(\alpha|\gamma)p(\gamma)d\gamma$ has a sparsity-inducing nature. Most recently, SBL has been efficiently implemented using belief propagation (BP) [10], [11] and approximate message passing [12], [13]. However, these methods assume that $\lambda$ is known, which may not be true in many applications. This work deals with message passing based approaches to SBL with unknown $\lambda$.

Mean field (MF) based message passing [14]–[16], which is also often referred to as variational message passing (VMP), has been widely used for approximate Bayesian inference, especially for exponential distributions. With 2-L or 3-L hierarchical priori structures, Pedersen et al. proposed an MF SBL algorithm (with unknown $\lambda$) [17], which was applied to sparse channel estimation in OFDM. As the MF SBL algorithm deals with the sparse signal $\alpha$ in a vector-form, matrix inversion is involved in each iteration and its computational complexity is as high as $O(L^3)$. To address the issue of complexity, a low complexity MF SBL algorithm [13] is then proposed, where the inverse of a large matrix is decomposed into a number of matrix inverses with smaller size. Flexible trade-off between complexity and performance can be achieved by adjusting the size of smaller matrices, which means that the reduction of complexity comes at the cost of performance loss. Apparently, the size of the smaller matrices can be set to be 1, so that the matrix inverses are avoided and we call it scalar-form MF SBL algorithm. Recently, the scalar-form MF SBL algorithm was used for channel gain and delay estimation in [19]. We note that an efficient hyperprior $p(\alpha)$ with 2-L structure was proposed in [6], which performs better than the 2-L and 3-L structures in [17].

Different from MF which supposes all the beliefs of variable nodes are independent, BP considers the joint belief of variable nodes neighbouring a factor node and makes the most of their correlation. BP, which may achieve exact Bayesian inference, is efficient to deal with discrete probability models and linear Gaussian models. However, BP may have a high complexity, when especially dealing with models involving both discrete
and continuous random variables. Recently, a unified message passing framework was proposed in [20], where BP and MF are merged to keep the merits of both while avoiding their drawbacks.

In this work, a low complexity BP-MF SBL algorithm with a 2-L hierarchical prior is proposed. Instead of using the conventional factor graph shown in Fig. 1(a) we modify the factor graph by adding a number of extra hard constraint factors as shown in Fig. 1(b), i.e., the factor graph looks like being 'stretched'. The hard constraint factors seem redundant, which however facilitates the use of BP in the graph, leading to considerable performance improvement. As we assume that the noise variance $\lambda^{-1}$ is unknown, MF can be used to tackle the exponential factors, while BP is used to handle the hard constraint factors. As we factorize the signal $\alpha$ in a scalar form, the developed BP-MF SBL algorithm avoids matrix inversion and has a low complexity. Inspired by the derivation of the generalized approximate message passing (GAMP) [21], we further simplify the BP message passing by ignoring some minimal terms and develop an approximate BP-MF SBL algorithm. Numerical examples show that the proposed BP-MF SBL algorithms provide even better mean-square-error (MSE) performance with much lower complexity compared with the vector-form MF SBL algorithm [17], and achieve noticeable MSE performance gain with similar complexity compared with the scalar-form MF SBL algorithm [18, 19].

Notation- Boldface lowercase and uppercase letters denote vectors and matrices, respectively. The expectation operator with respect to a pdf $g(x)$ is expressed by $\langle f(x) \rangle_{g(x)} = \int f(x)g(x)dx/\int g(x)dx'$, while $\text{var}[x]_{g(x)} = \langle x^2 \rangle_{g(x)} - \langle x \rangle_{g(x)}^2$ stands for the variance. The pdf of a complex Gaussian distribution with mean $\mu$ and variance $\nu$ is represented by $\mathcal{CN}(x; \mu, \nu)$. The relation $f(x) = cg(x)$ for some positive constant $c$ is written as $f(x) \propto g(x)$.

II. FACTOR GRAPH MODEL

The joint a posteriori pdf of $\alpha, \gamma$ and $\lambda$ in (1) with a 2-L hierarchical prior [9] can be factorized as

$$p(\alpha, \gamma, \lambda|x) \propto f_{\lambda}(\lambda) \prod_n f_{y_n|\alpha}(\alpha, \gamma) \prod_l f_{\alpha_l|\gamma_l}(\gamma_l),$$

(2)

where $f_{y_n|\alpha}(\alpha, \gamma) \triangleq p(y_n|\alpha, \gamma) = \mathcal{CN}(y_n; \Phi_n\alpha, \lambda^{-1})$, with $\Phi_n$ being the $n$-th row of matrix $\Phi$, and $f_{\lambda}(\lambda)$ denotes the prior of noise precision parameter $\lambda$. The factor $f_{\alpha_l|\gamma_l}(\gamma_l)$ denotes the conditional pdf $p(\alpha_l|\gamma_l) = \mathcal{CN}(\alpha_l; 0, \gamma_l^{-1})$, which is chosen as a Gaussian prior of $\alpha_l$ and $f_{y_l|\gamma_l}(\gamma_l)$ represents a hyperprior $p(\gamma_l) = \text{Ga}(\gamma_l; c, d)$ of the hyperparameter $\gamma_l$. The factorization in (2) can be visually depicted on the factor graph [22] shown in Fig. 1(a) which is similar to those in [18] and [19]. We assume that $\lambda$ is unknown, and MF can be used to deal with factor nodes $\{f_{y_n}, \forall n \in [1 : N]\}$, which leads to the scalar-form MF SBL algorithm [18]. In [17], the vector-form MF SBL algorithm is derived based on a conventional factor graph, where the vector $\alpha$ is treated as a single variable node.

To facilitate the use of both BP and MF, we modify the factor graph in Fig. 1(a) by adding hard constraint factors $\{f_{\delta_n}(h_n, \alpha) = \delta(h_n - \Phi_n\alpha), \forall n \in [1 : N]\}$ with a new variable vector $h = \Phi\alpha$. Therefore, factor $f_{y_n}$ denotes the likelihood function $p(y_n|h_n, \lambda) = \mathcal{CN}(y_n; h_n, \lambda^{-1})$. The new factor graph, shown in Fig. 1(b) looks like a stretched version of the graph in Fig. 1(a). In the new graph, MF rules with fixed points equations can be used to compute the messages for the exponential factors, while BP rules, often yielding better performance, can be used to deal with the hard constraint factors. The message computations and scheduling are detailed in the following section.

III. BP-MF BASED SBL

In this section, with the combined BP-MF message update rule [20], we detail the message computations and scheduling on the factor graph shown in Fig. 1(b) to perform sparse signal estimation. All the factors in Fig. 1(b) are represented by set $\mathcal{A}$, and it is divided into two disjoint subsets, a BP subset and an MF subset, which are denoted by $\mathcal{A}_{\text{BP}} = \{f_{\delta_n}, \forall n\}$ and $\mathcal{A}_{\text{MF}} = \mathcal{A} \setminus \mathcal{A}_{\text{BP}}$, respectively.

A. Message Computations

The computations for messages passing from left to right (forward) and from right to left (backward) are elaborated. The computations of some forward messages may need relevant backward messages, which we assume are produced from the previous iteration.

1) Forward message computations: Assuming that the belief $b(\lambda)$, later defined in [23], of noise precision $\lambda$ is known, the message $m_{f_{y_n} \rightarrow h_n}(h_n)$ from observation factor $f_{y_n} \in \mathcal{A}_{\text{MF}}$ to $h_n$ is calculated by the MF rule, as follows

$$m_{f_{y_n} \rightarrow h_n}(h_n) = \text{exp}\left\{\langle \log f_{y_n}(h_n, \lambda) \rangle_{b(\lambda)} \right\} \propto \mathcal{CN}\left(h_n; y_n, \lambda^{-1}\right),$$

(3)

where $\hat{\lambda} = \langle \lambda \rangle_{b(\lambda)}$.

The message $m_{f_{\delta_n} \rightarrow \alpha_l}(\alpha_l)$ from the hard factor $f_{\delta_n} \in \mathcal{A}_{\text{BP}}$ to variable node $\alpha_l$ is computed by the BP rule with the messages

$$m_{n_{\alpha_l} \rightarrow f_{\delta_n}}(h_n) = m_{f_{y_n} \rightarrow h_n}(h_n) \prod_{n \neq n'} n_{\alpha_l} \rightarrow f_{\delta_n}(\alpha_l),$$

(4)

where

$$\hat{\alpha}_n \rightarrow l \triangleq \frac{y_n - \hat{\mu}_n + \Phi_{nl}\hat{\alpha}_{l \rightarrow n}}{\Phi_{nl}} \quad \hat{\mu}_{\alpha_l \rightarrow n} \triangleq \hat{\lambda}^{-1} + \nu_{\alpha_l \rightarrow n} = \frac{|\Phi_{nl}|^2}{|\Phi_{nl}|^2 + \nu_{\alpha_l \rightarrow n}^2} \quad \hat{\nu}_{\alpha_l \rightarrow n} \triangleq \sum_l |\Phi_{nl}|^2 \nu_{\alpha_l \rightarrow n},$$

(5)

(6)

(7)

(8)
For convenience of description, the product of all the Gaussian messages \( \{m_{f_{\delta n} \rightarrow \alpha_l}(\alpha_l), \forall n \in [1 : N] \} \) is denoted by

\[
q_l(\alpha_l) = \prod_n m_{f_{\delta n} \rightarrow \alpha_l}(\alpha_l)
\]
\[
\propto \mathcal{CN}(\alpha_l; \hat{\alpha}_l, \nu_{\alpha_l}),
\]

where

\[
\nu_{\alpha_l} \triangleq \left( \sum_n \frac{1}{\nu_{\alpha_{n \rightarrow l}}} \right)^{-1},
\]
\[
\hat{\alpha}_l \triangleq \nu_{\alpha_l} \left( \sum_n \frac{\hat{\alpha}_{n \rightarrow l}}{\nu_{\alpha_{n \rightarrow l}}} \right).
\]

Given the message \( m_{f_{\alpha_l} \rightarrow \alpha_l}(\alpha_l) \propto \mathcal{CN}(\alpha_l; 0, \gamma_l^{-1}) \), later defined in (16), the belief \( b(\alpha_l) \) of variable \( \alpha_l \) is obtained as

\[
b(\alpha_l) \propto q_l(\alpha_l)m_{f_{\alpha_l} \rightarrow \alpha_l}(\alpha_l)
\]
\[
\propto \mathcal{CN}(\alpha_l; \hat{\alpha}_l, \nu_{\alpha_l}),
\]

where

\[
\hat{\alpha}_l \triangleq \frac{\hat{q}_l}{1 + \nu_{\alpha_l} \gamma_l^{-1}},
\]
\[
\nu_{\alpha_l} \triangleq \left( 1/\nu_{\alpha_l} + \hat{\gamma}_l \right)^{-1}.
\]

Since the factor \( f_{\alpha_l} \) is classified into the MF subset, the message \( m_{f_{\alpha_l} \rightarrow \gamma_l}(\gamma_l) \) is calculated by using the MF rule,

\[
m_{f_{\alpha_l} \rightarrow \gamma_l}(\gamma_l) = \exp \left\{ \log f_{\alpha_l}(\alpha_l, \gamma_l)b(\alpha_l) \right\}
\]
\[
\propto \gamma_l \exp \left\{-\gamma_l(|\hat{\alpha}_l|^2 + \nu_{\alpha_l}) \right\},
\]

so that the belief \( b(\gamma_l) \) of hyperparameter \( \gamma_l \) reads

\[
b(\gamma_l) \propto m_{f_{\alpha_l} \rightarrow \gamma_l}(\gamma_l)f_{\gamma_l}(\gamma_l)
\]
\[
\propto \gamma_l^{t+1} \exp \left\{-\gamma_l(\nu + |\hat{\alpha}_l|^2 + \nu_{\alpha_l}) \right\}.
\]

2) Backward Message: We firstly compute the message \( m_{f_{\gamma_l} \rightarrow \alpha_l}(\alpha_l) \) from \( f_{\gamma_l} \) to \( \alpha_l \) by the MF rule, as follows

\[
m_{f_{\gamma_l} \rightarrow \alpha_l}(\alpha_l) = \exp \left\{ \log f_{\gamma_l}(\alpha_l, \gamma_l)b(\gamma_l) \right\}
\]
\[
\propto \mathcal{CN}(\alpha_l; 0, \hat{\gamma}_l^{-1}),
\]

where

\[
\hat{\gamma}_l \triangleq \langle \gamma_l \rangle_{b(\gamma_l)} = \frac{\epsilon + 1}{\eta + |\hat{\alpha}_l|^2 + \nu_{\alpha_l}}.
\]

Since factor node \( f_{\delta n} \in \mathcal{A}_{BP} \), the message \( n_{\alpha_l \rightarrow f_{\delta n}}(\alpha_l) \) from variable node \( \alpha_l \) to \( f_{\delta n} \) is updated by the BP rule,

\[
n_{\alpha_l \rightarrow f_{\delta n}}(\alpha_l) \propto \frac{b(\alpha_l)}{m_{f_{\delta n} \rightarrow \alpha_l}(\alpha_l)} \propto \mathcal{CN}(\alpha_l; \hat{\alpha}_{l \rightarrow n}, \nu_{\alpha_{l \rightarrow n}}),
\]

where

\[
\nu_{\alpha_{l \rightarrow n}} \triangleq \left( 1/\nu_{\alpha_l} - 1/\nu_{\alpha_{n \rightarrow l}} \right)^{-1},
\]
\[
\hat{\alpha}_{l \rightarrow n} \triangleq \nu_{\alpha_{l \rightarrow n}} \left( \frac{\hat{\alpha}_l}{\nu_{\alpha_l}} - \frac{\hat{\alpha}_{n \rightarrow l}}{\nu_{\alpha_{n \rightarrow l}}} \right).
\]

Then the message \( m_{f_{\delta n} \rightarrow h_n}(h_n) \) can be computed with the BP rule for \( f_{\delta n} \in \mathcal{A}_{BP} \), yielding

\[
m_{f_{\delta n} \rightarrow h_n}(h_n) = \langle f_{\delta n}(h_n, \alpha) \rangle_{\prod_{\alpha_l \rightarrow f_{\delta n}}(\alpha_l)}
\]
\[
\propto \mathcal{CN}(h_n; \hat{h}_n, \nu_{h_n}),
\]

where

\[
\nu_{h_n} \triangleq \left( \lambda + 1/\nu_{\gamma_n} \right)^{-1},
\]
\[
\hat{h}_n \triangleq \nu_{h_n} \left( \gamma_n \lambda + \hat{\gamma}_n / \nu_{\gamma_n} \right).
\]

The message \( m_{f_{\delta n} \rightarrow \lambda}(\lambda) \propto \lambda \exp \{-\langle |y_n - h_n|^2 \rangle_{b(h_n)} \} \) is calculated with the MF rule. With the conjugate prior pdf \( f_{\lambda}(\lambda) \propto 1/\lambda \), the belief \( b(\lambda) \) is updated by

\[
b(\lambda) \propto m_{f_{\gamma_l} \rightarrow \lambda}(\lambda)f_{\lambda}(\lambda)
\]
\[
\propto \lambda^{N-1} \exp \left\{-\lambda \sum_n \langle |y_n - h_n|^2 \rangle_{b(h_n)} \right\},
\]

and the parameter \( \hat{\lambda} \) in (5) is computed as

\[
\hat{\lambda} = \langle \lambda \rangle_{b(\lambda)} = \frac{N}{\sum_n \langle |y_n - h_n|^2 \rangle_{b(h_n)}}.
\]
B. Message Scheduling for BP-MF SBL Algorithm

The factors in Fig. 1(b) are very densely connected and thus there are a multitude of different options for message scheduling. In this paper, we simply choose a schedule, where the messages are sequentially updated in both forward and backward directions, while the messages in vertical direction are simultaneously computed for all $n \in [1 : N]$ and $l \in [1 : L]$. The BP-MF SBL algorithm with such scheduling is summarized in Algorithm 1.

Algorithm 1 BP-MF SBL Algorithm

1: Initialize $\hat{p}_n, \nu_p, \hat{\alpha}_{l \to n}, \nu_{\alpha_{l \to n}}, \hat{\gamma}_l, \forall n, \forall l$ and $\lambda$.
2: for $t = 1 \rightarrow \# \text{ of iterations}$ do
3: \hspace{1em} $\forall n, l$: update $\hat{\alpha}_{n \to l}$ and $\nu_{\alpha_{n \to l}}$ by (5) and (6).
4: \hspace{1em} $\forall l$: update $\nu_q$ and $\hat{q}_l$ by (10) and (11).
5: \hspace{1em} $\forall n$: update $\hat{\alpha}_l$ and $\nu_{\alpha_l}$ by (13) and (14).
6: \hspace{1em} $\forall l$: update $\hat{\gamma}_l$ by (17).
7: \hspace{1em} $\forall l$: update $\hat{\alpha}_l$ and $\nu_{\alpha_l}$ again, by (13) and (14).
8: $\forall n, l$: update $\nu_{\alpha_{l \to n}}$ and $\hat{\alpha}_{l \to n}$ by (19) and (20).
9: $\forall n$: update $\hat{p}_n$ and $\nu_p$ by (7) and (8).
10: $\forall n$: update $\hat{h}_n$ and $\hat{h}_n$ by (22) and (23).
11: update $\lambda$ by (25), with $b(h_n) = \mathcal{N}(\hat{h}_n; \hat{h}_n, \nu_{h_n})$.
12: end for $t$

IV. APPROXIMATE BP-MF SBL

It is observed that there are $NL$ edges between variable nodes $\{\alpha_l, \forall l\}$ and factor nodes $\{\delta_n, \forall n\}$, so we have to compute $2NL$ messages (see Lines 3 and 8 in Algorithm 1) for both forward and backward directions in each iteration. To simplify the BP-MF SBL, we approximate the means and variances of Gaussian messages in the BP part by eliminating some small terms, leading to the approximate BP-MF SBL algorithm.

A. Approximation of Messages

By substituting (14) into (19),

$$\nu_{\alpha_{l \to n}} = \left(1/\nu_q + \hat{\gamma}_l - 1/\nu_{\alpha_{n \to l}}\right)^{-1} \approx \nu_{\alpha_l}, \quad (26)$$

can be obtained as $1/\nu_q \gg 1/\nu_{\alpha_{n \to l}}$ from (10) when the number $N$ is large enough. Similarly, substituting (5) and (6) into (20), yields

$$\hat{\alpha}_{l \to n} = \nu_{\alpha_{l \to n}} \cdot \frac{\hat{\alpha}_l - \Phi_{nl}(y_n - \hat{p}_n + \Phi_{nl}\hat{\alpha}_{l \to n}^{-1})}{\lambda^{-1} + \nu_p + |\Phi_{nl}|^2\nu_{\alpha_{l \to n}}^{-1}} \approx \hat{\alpha}_l - \nu_{\alpha_l} \frac{y_n - \hat{p}_n + \Phi_{nl}}{\lambda^{-1} + \nu_p + |\Phi_{nl}|^2}, \quad (27)$$

where

$$s_n \triangleq \frac{y_n - \hat{p}_n}{\lambda^{-1} + \nu_p}. \quad (28)$$

The above approximation is made by assuming that the length $L$ of variable vector $\alpha$ is very large, so that $\hat{p}_n \gg \Phi_{nl}\hat{\alpha}_{l \to n}$ and $\nu_p \gg |\Phi_{nl}|^2\nu_{\alpha_{l \to n}}$ from (7) and (8).

Substituting (26) and (27) into (8) and (7) respectively, we obtain the approximate variance and mean

$$\nu_{p_n} \approx \sum_l |\Phi_{nl}|^2 \nu_{\alpha_l} \quad (29)$$

$$\hat{p}_n \approx \sum_l \Phi_{nl}(\hat{\alpha}_l - \nu_{\alpha_l}s_n\Phi_{nl}) \quad (30)$$

We further substitute (9) and (5) into (10) and (11), and approximate them for a large $L$, as follows,

$$\nu_q = \left(\frac{|\Phi_{nl}|^2}{\lambda^{-1} + \nu_p - |\Phi_{nl}|^2\nu_{\alpha_{l \to n}}^{-1}}\right)^{-1} \approx \sum_l |\Phi_{nl}|^2 \lambda^{-1} + \nu_p \quad (31)$$

$$\hat{q}_l = \nu_q \left(\sum_n \Phi_{nl}s_n + |\Phi_{nl}|^2\nu_{\alpha_{l \to n}}^{-1}\hat{\alpha}_{l \to n}\right) \approx \nu_q \left(\sum_n \Phi_{nl}s_n + |\Phi_{nl}|^2\nu_{\alpha_{l \to n}}^{-1}\hat{\alpha}_{l \to n}\right) \approx \hat{\alpha}_l + \nu_q \sum_n \Phi_{nl}s_n. \quad (32)$$

The approximation in (32) is according to $|\Phi_{nl}|^2 \lambda^{-1} + \nu_p \ll \nu_{\alpha_l}^{-1}$, since $\nu_{\alpha_l}^{-1} = \sum_n |\Phi_{nl}|^2/\lambda^{-1} + \nu_p$, $\hat{\gamma}_l$ is obtained by inserting (31) into (17).

B. Message Scheduling for Approximate BP-MF SBL Algorithm

We choose the similar message scheduling to BP-MF SBL shown in Algorithm 1 where the corresponding parameters are replaced by the above approximate computations. The parameters $\nu_q$ and $\hat{q}_l$ are updated by (31) and (32) instead of (10) and (11). The parameters $\nu_{p_n}$ and $\hat{p}_n$ are calculated by (29) and (30) rather than (8) and (7). In addition, the computations of parameters $\nu_{\alpha_{l \to n}}, \nu_{\alpha_{n \to l}}, \nu_{\alpha_{l \to n}}$ and $\alpha_{l \to n}$ in Lines 3 and 8 of Algorithm 1 are avoided, while a set of intermediate parameters $s_n, \forall n$, have to be inserted. We summarize the approximate BP-MF SBL in Algorithm 2. It is interesting that the message computations for the densely connected BP subgraph as shown in Fig. 1(b) coincide with the GAMP [21] algorithm.

V. NUMERICAL SIMULATION RESULTS

In this section, we assess the proposed SBL algorithms by means of Monte Carlo simulations. Consider the sparse signal model (1) with a random $M \times N (M = 100, N = 200)$ dictionary matrix $\Phi$, whose entries are independent and identically
distributed (i.i.d.) zero-mean complex Gaussian random variables with unit variance. We assume that the length-$N$ vector $\alpha$ has $K$ nonzero elements which are randomly dispersed in vector $\alpha$. In addition, the nonzero elements are i.i.d. and also drawn from a zero-mean complex Gaussian distribution with unit variance. All curves are produced based on 200 Monte-Carlo runs, and for each run with a new realization of the dictionary matrix $\Phi$, the vector $\alpha$ and the AWGN vector $\omega$ are generated.

We compare the MSE performance of our proposed algorithms and the state-of-the-art algorithms. “BP-MF” and “A-BP-MF” denotes our proposed BP-MF and approximate BP-MF SBL algorithms, i.e., Algorithms 1 and 2, respectively. “MF-vector” and “MF-scalar” stand for MF SBL algorithms in vector-form [17] and in scalar-form (sequentially estimating each element of the sparse signal $\alpha$) [19], respectively. For a fair companion, all the above algorithms use 2-L hierarchical structure with the hyperprior proposed in [6]. In addition, we also provide the performance of the vector-form MF algorithm using 3-L hierarchical prior in [17], denoted by “MF-vector-3L”.

In Fig. 2 the MSE performance of the algorithms is shown over a wide range of signal-to-noise ratios (SNRs), where all algorithms run 20 iterations and the number of nonzero elements $K = 26$. We can observe that the proposed BP-MF and A-BP-MF algorithms deliver slightly better MSE performance than MF-vector, and significantly outperform MF-scalar and MF-vector-3L. Fig. 3 depicts MSE performance with an SNR of 14dB versus the number of non-zero elements $K$. It shows that all the algorithms have similar performance when $K$ is small. However, with the increase of $K$, the MF SBL algorithms exhibit considerable performance loss compared to the proposed BP-MF and A-BP-MF SBL algorithms. It is also seen that BP-MF performs slightly better than A-BP-MF.

Fig. 4 illustrates the convergence of the algorithms, where SNR = 14dB and $K = 26$. We can see that MF-scalar has the fastest convergence rate due to its sequential message updating schedule. Our proposed BP-MF algorithms converge slower but achieve better MSE performance compared to MF-scalar and MF-vector-3L. It can also be seen that our proposed algorithms have similar convergence rate and performance compared to MF-vector.

In addition, our simulation results in Figs. 2, 3 and 4 also show that the 2-L hierarchical priori structure proposed in [6] outperforms 3-L hierarchical priori structure [17].

A. Computational Complexity

As the message computations for updating $\lambda$ and $\gamma$ are the same for all the algorithms, we only analyze the complexity of message computations related to $h$ and $\alpha$. Due to the
matrix inversion involved, MF-vector has a complexity of $O(L^3)$ per iteration, while MF-scalar $O(NL)$. Since the proposed BP-MF and A-BP-MF algorithms using scalar-form factor graph shown in Fig. 1(b) they have similar complexity to MF-scalar. In details, BP-MF needs to compute $O(NL)$ messages and $O(NL)$ memory cells to store the parameters (means and variances) of messages (see Lines 3 and 8 in Algorithm 1), while MF-scalar and A-BP-MF only need to update and store $O(N + L)$ messages. However, in updating the belief $b_\alpha(l), \forall l \in [1 : L]$ MF-scalar with sequential message schedule may take longer running time than BP-MF algorithms.

VI. CONCLUSION

In this paper, we have investigated message passing based approaches to SBL. Two low complexity BP-MF SBL algorithms have been proposed based on a stretched factor graph which is obtained by adding extra hard constraint factors to the conventional factor graph. It has been shown that the BP-MF SBL algorithms outperform the state-of-the-art MF SBL algorithms in terms of computational complexity or performance.

REFERENCES

[1] D. L. Donoho, “Compressed sensing,” IEEE Trans. Inform. Theory, vol. 52, no. 4, pp. 1289–1306, April 2006.
[2] E. Candès and M. Wakin, “An introduction to compressive sampling,” IEEE Signal Processing Magazine, vol. 25, no. 2, pp. 21 – 30, March 2008.
[3] D. P. Wipf and B. D. Rao, “Sparse Bayesian learning for basis selection,” IEEE Trans. Signal Processing, vol. 52, no. 8, pp. 2153 – 2164, Aug. 2004.
[4] S. Chen, D. L. Donoho, and M. A. Saunders, “Atomic decomposition by basic pursuit,” SIAM Journal on Scientific Computing, no. 1, pp. 33–61, 1998.
[5] J. A. Tropp, “Greed is good: algorithmic results for sparse approximation,” IEEE Trans. Inform. Theory, pp. 2231–2242, Oct. 2004.
[6] M. E. Tipping, “Sparse Bayesian learning and the relevance vector machine,” Journal of Machine Learning Research, pp. 211–244, 2001.
[7] M. E. Tipping and A. C. Faul, “Fast marginal likelihood maximisation for sparse Bayesian models,” Proc. 9th international Workshop on Artificial Intelligence and Statistics, 2003.
[8] D. Shutin, T. Buchgraber, S. R. Kulkarni, and H. V. Poor, “Fast variational sparse Bayesian learning with automatic relevance determination for superimposed signals,” IEEE Trans. Signal Processing, vol. 59, no. 12, pp. 6257–6261, Dec. 2011.
[9] N. L. Pedersen, C. N. Manchón, M.-A. Badiu, D. Shutin, and B. H. Fleury, Sparse estimation using Bayesian hierarchical prior modeling for real and complex linear models, Signal Processing, vol. 115, no. 0, pp. 1566 – 1570.
[10] X. Tan and J. Li, “Computationally efficient sparse Bayesian learning via belief propagation,” in 2009 Conference Record of the Forty-Third Asilomar Conference on Signals, Systems and Computers, Nov. 2009, pp. 1566 – 1570.
[11] D. Baron, S. Sarvotham, and R. Baraniuk, “Bayesian compressive sensing via belief propagation,” IEEE Trans. Signal Processing, vol. 58, no. 1, pp. 269 – 280, Jan. 2010.
[12] S. Som and P. Schniter, “Compressive imaging using approximate message passing and a Markov-tree prior,” IEEE Trans. Signal Processing, vol. 60, no. 7, pp. 3439 – 3448, July 2012.
[13] M. Al-Shoukairi and B. Rao, “Sparse Bayesian learning using approximate message passing,” in 2014 48th Asilomar Conference on Signals, Systems and Computers, Nov. 2014, pp. 1957 – 1961.
[14] E. P. Xing, M. I. Jordan, and S. Russell, “A generalized mean field algorithm for variational inference in exponential families,” in Proceedings of the Nineteenth Conference on Uncertainty in Artificial Intelligence, ser. UAI03, San Francisco, CA, USA, 2003, pp. 583–591.
[15] C. M. Bishop and J. Winn, “Structured variational distributions in VIBES,” Proceedings Artificial Intelligence and Statistics, pp. 3–6, 2003.
[16] J. Dauwels, “On variational message passing on factor graphs,” in Proc. IEEE International Symposium on Information Theory (ISIT 2007), Jun. 2007, pp. 2546–2550.
[17] N. L. Pedersen, C. N. Manchón, D. Shutin, and B. H. Fleury, “Application of Bayesian hierarchical prior modeling to sparse channel estimation,” IEEE International Conference on Communications (ICC 2012), pp. 3487–3492, June 2012.
[18] N. L. Pedersen, C. N. Manchón, and B. H. Fleury, “Low complexity sparse Bayesian learning for channel estimation using generalized mean field,” 20th European Wireless Conference, pp. 838–843, June 2014.
[19] T. L. Hansen, P. B. Jørgensen, M. Badiu, and B. H. Fleury, “Joint sparse channel estimation and decoding: Continuous and discrete domain sparsity,” CoRR, vol. abs/1507.02954, 2015. [Online]. Available: http://arxiv.org/abs/1507.02954
[20] E. Riegler, G. E. Karkelund, C. N. Manchón, M.-A. Badiu and B. H. Fleury, “Merging belief propagation and the mean field approximation: A free energy approach,” IEEE Trans. Inform. Theory, vol. 59, no. 1 pp. 588–602, Jan. 2013.
[21] S. Rangan, “Generalized approximate message passing for estimation with random linear mixing,” in Proc. IEEE Int. Symp. on Inform. Theory (ISIT 2011), Aug. 2011, pp. 2168 C 2172.
[22] F. Kschischang, B. Frey, and H.-A. Loeliger, “Factor graphs and the sum-product algorithm,” IEEE Trans. Inform. Theory, vol. 47, no. 2, pp. 498–519, Feb. 2001.