**AN ℝ-MOTIVIC \(v_1\)-SELF-MAP OF PERIODICITY**

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**Abstract.** We consider a nontrivial action of \(C_2\) on the type 1 spectrum \(\mathcal{Y} := M_2(1) \wedge C(\eta)\), which is well-known for admitting a 1-periodic \(v_1\)-self-map. The resultant finite \(C_2\)-equivariant spectrum \(\mathcal{Y}^{C_2}\) can also be viewed as the complex points of a finite ℝ-motivic spectrum \(\mathcal{Y}^R\). In this paper, we show that one of the 1-periodic \(v_1\)-self-maps of \(\mathcal{Y}\) can be lifted to a self-map of \(\mathcal{Y}^{C_2}\) as well as \(\mathcal{Y}^R\). Further, the cofiber of the self-map of \(\mathcal{Y}^R\) is a realization of the subalgebra \(\mathbb{A}^2(1)\) of the ℝ-motivic Steenrod algebra. We also show that the \(C_2\)-equivariant self-map is nilpotent on the geometric fixed-points of \(\mathcal{Y}^{C_2}\).

1. **Introduction**

In classical stable homotopy theory, the interest in periodic \(v_n\)-self-maps of finite spectra lies in the fact that one can associate to each \(v_n\)-self-map an infinite family in the chromatic layer \(n\) stable homotopy groups of spheres. Therefore, interest lies in constructing type \(n\) spectra and finding \(v_n\)-self-maps of lowest possible periodicity on a given type \(n\) spectrum. This, in general, is a difficult problem, though progress has been made sporadically throughout the history of the subject [T, DM, BP, BHHM, N, BEM, BE]. With the modern development of motivic stable homotopy theory, one may ask if there are similar periodic self-maps of finite motivic spectra.

Classically any non-contractible finite \(p\)-local spectrum admits a periodic \(v_n\)-self-map for some \(n \geq 0\). This is a consequence of the thick-subcategory theorem [HS, Theorem 7], aided by a vanishing line argument [HS, §4.2]. In the classical case all the thick tensor ideals of \(\mathbf{Sp}_{p,\text{fin}}\) (the homotopy category of finite \(p\)-local spectra) are also prime (in the sense of [B]). The thick tensor-ideals of the homotopy category of cellular motivic spectra over \(\mathbb{C}\) or \(\mathbb{R}\) are not completely known (but see [HO]). However, one can gather some knowledge about the prime thick tensor-ideals in \(\text{Ho}(\mathbf{Sp}_{2,\text{fin}}^\mathbb{R})\) (the homotopy category of 2-local cellular ℝ-motivic spectra) through the Betti realization functor

\[
\beta : \text{Ho}(\mathbf{Sp}_{2,\text{fin}}^\mathbb{R}) \longrightarrow \text{Ho}(\mathbf{Sp}_{2,\text{fin}}^{C_2})
\]

using the complete knowledge of prime thick-subcategories of \(\text{Ho}(\mathbf{Sp}_{2,\text{fin}}^{C_2})\) [BS].

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The prime thick tensor-ideals of \( \text{Ho}(\text{Sp}_{2,\text{fin}}^{C_2}) \) are essentially the pull-back of the classical thick subcategories along the two functors, the geometric fix point functor

\[
\Phi^{C_2} : \text{Ho}(\text{Sp}_{2,\text{fin}}^{C_2}) \longrightarrow \text{Ho}(\text{Sp}_{2,\text{fin}})
\]

and the forgetful functor

\[
\Phi^e : \text{Ho}(\text{Sp}_{2,\text{fin}}^{C_2}) \longrightarrow \text{Ho}(\text{Sp}_{2,\text{fin}}).
\]

Let \( C_n \) denote the thick subcategory of \( \text{Ho}(\text{Sp}_{2,\text{fin}}) \) consisting of spectra of type at least \( n \). The prime thick-subcategories,

\[
C(e,n) = (\Phi^e)^{-1}(C_n) \quad \text{and} \quad C(C_2,n) = (\Phi^{C_2})^{-1}(C_n),
\]

are the only prime thick subcategories of \( \text{Ho}(\text{Sp}_{2,\text{fin}}^{C_2}) \).

**Definition 1.1.** We say a spectrum \( X \in \text{Ho}(\text{Sp}_{2,\text{fin}}^{C_2}) \) is of type \((n,m)\) iff \( \Phi^e(X) \) is of type \( n \) and \( \Phi^{C_2}(X) \) is of type \( m \).

For a type \((n,m)\) spectrum \( X \), a self-map \( f : X \to X \) is periodic if and only if at least one of \( \{\Phi^e(f), \Phi^{C_2}(f)\} \) are periodic (see [BGH, Proposition 3.17]).

**Definition 1.2.** Let \( X \in \text{Ho}(\text{Sp}_{2,\text{fin}}^{C_2}) \) be of type \((n,m)\). We say a self-map \( f : X \to X \) is

(i) a \((v,(n,m))\)-self-map of mixed periodicity \((i,j)\) if \( \Phi^e(f) \) is a \( v_n \)-self-map of periodicity \( i \) and \( \Phi^{C_2}(f) \) is a \( v_m \)-self-map of periodicity \( j \),

(ii) a \((v,(n,\text{nil}))\)-self-map of periodicity \( i \) if \( \Phi^e(f) \) is a \( v_n \)-self-map of periodicity \( i \) and \( \Phi^{C_2}(f) \) is nilpotent, and,

(iii) a \((v,(\text{nil},m))\)-self-map of periodicity \( j \) if \( \Phi^e(f) \) is a nilpotent self-map and \( \Phi^{C_2}(f) \) is a \( v_m \)-self-map of periodicity \( j \).

**Example 1.3.** The sphere spectrum \( S_{C_2} \) is of type \((0,0)\). The degree 2 map is a \((v,(0,0))\)-self-map. In general, if we consider the \( v_n \)-self-map of a type \( n \) spectrum with trivial action of \( C_2 \), then the resultant equivariant self-map is a \((v,(n,n))\)-self-map.

**Example 1.4.** Let \( S_{C_2}^{1,1} \) denote the \( C_2 \)-equivariant sphere which is the one-point compactification of the real sign representation. The unstable twist-map

\[
\epsilon_u : S_{C_2}^{1,1} \wedge S_{C_2}^{1,1} \longrightarrow S_{C_2}^{1,1} \wedge S_{C_2}^{1,1}
\]

stabilizes to a nonzero element \( \epsilon \in \pi_{0,0}(S_{C_2}) \). Let \( h = 1 - \epsilon \in \pi_{0,0}(S_{C_2}) \) be the stabilization of the map

\[
h_u = 1 - \epsilon_u : S_{C_2}^{3,2} \longrightarrow S_{C_2}^{3,2}.
\]

Note that on the underlying space \( \epsilon \) is of degree \(-1\), while on the fixed points it is the identity. Therefore \( \Phi^e(h) \) is multiplication by 2, whereas \( \Phi^{C_2}(h) \) is trivial. Thus \( h \) is a \((v,(0,\text{nil}))\)-self-map. Thus \( C^{C_2}(h) \) is of type \((1,0)\).
Example 1.5. The equivariant Hopf-map \( \eta_{1,1} \in \pi_{1,1}(S_{C_2}) \) is the Betti realization of the \( \mathbb{R} \)-motivic Hopf-map \( \eta \) [M2, D13]. Up to a unit, it is the stabilization of the projection map

\[
\pi : S^3_{C_2} \simeq \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{CP}^1 \cong S^2_{C_2},
\]

where the domain and the codomain are given the \( C_2 \)-structure using complex conjugation. On fixed-points, the map \( \pi \) is the projection map

\[
\pi : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{RP}^1,
\]

which is a degree 2 map. From this we learn that while \( \Phi^e(\eta_{1,1}) \) is nilpotent, \( \Phi^{C_2}(\eta_{1,1}) \) is the periodic \( v_0 \)-self-map. Hence, \( \eta_{1,1} \) is a \( v_{(\text{nil},0)} \)-self-map and the cofiber \( C(\eta_{1,1}) \) is of type \( (0, 1) \).

Remark 1.6. In the \( C_2 \)-equivariant stable homotopy groups, the usual Hopf-map (sometimes referred to as the ‘topological Hopf-map’) is different from \( \eta_{1,1} \) of Example 1.5. The ‘topological Hopf-map’ \( \eta_{1,0} \in \pi_{1,0}(S_{C_2}) \) should be thought of as the stabilization of the unstable Hopf-map

\[
(\eta_{1,0})_u : S^{3,0}_{C_2} \longrightarrow S^{2,0}_{C_2}
\]

where both domain and codomain are given the trivial \( C_2 \)-action.

Definition 1.7. We say a spectrum \( X \in \text{Ho}(\mathbb{SP}_{2,\text{fin}}^R) \) is of type \( (n, m) \) if \( \beta(X) \) is of type \( (n, m) \). We call an \( \mathbb{R} \)-motivic self-map

\[
f : X \rightarrow X
\]

a \( v_{(n,m)} \)-self-map, where \( m \) and \( n \) are in \( \mathbb{N} \cup \{\text{nil}\} \) (but not both nil), if \( \beta(f) \) is a \( C_2 \)-equivariant \( v_{(n,m)} \)-self-map.

Remark 1.8. The maps ‘multiplication by 2’ (of Example 1.3), \( h \) (of Example 1.4), and \( \eta_{1,1} \) (of Example 1.5) admit \( \mathbb{R} \)-motivic lifts along \( \beta \) and provide us with examples of a \( v_{(0,0)} \)-self-map, \( v_{(\text{nil},0)} \)-self-map and \( v_{(\text{nil},0)} \)-self-map of the \( \mathbb{R} \)-motivic sphere spectrum \( S_{\mathbb{R}} \), respectively.

A theorem of Balmer and Sanders [BS] asserts that \( C(e, n) \subset C(C_2, m) \) if and only if \( n \geq m + 1 \). In particular, \( C(e, n) \) is contained in \( C(C_2, n - 1) \). Consequently, there are no type \( (n, m) \) \( (C_2 \)-equivariant or \( \mathbb{R} \)-motivic) spectra if \( n \geq m + 2 \). Their result also implies the following:

Proposition 1.9. Let \( X \in \text{Ho}(\mathbb{SP}_{2,\text{fin}}^{C_2}) \) be of type \( (n + 1, n) \) for some \( n \). Then \( X \) cannot support a \( v_{(n+1,\text{nil})} \)-self-map.

The proposition holds since the cofiber of such a self-map would be of type \( (n + 2, n) \), contradicting the results of Balmer-Sanders. In particular, neither \( C^{C_2}(h) \) nor \( C^R(h) \) supports a \( v_{(\text{nil},0)} \)-self-map. However, it is possible that \( C^{C_2}(h) \) as well as \( C^R(h) \) can admit a \( v_{(1,0)} \)-self-map or a \( v_{(\text{nil},0)} \)-self-map. In fact, \( \eta_{1,1} \in \pi_{1,1}(S_{\mathbb{R}}) \) and \( \eta_{1,1} \in \pi_{1,1}(S_{C_2}) \) induce \( v_{(\text{nil},0)} \)-self-maps of \( C^R(h) \) and \( C^{C_2}(h) \) respectively. In Section 5, we show that:

Theorem 1.10. The spectrum \( C^R(h) \) does not admit a \( v_{(1,0)} \)-self-map.
However, it is possible that \( C^{C_2}(h) \) admits a \( v_{(1,0)} \)-self-map (for details see Remark 5.6). In contrast to the classical case, there is no guarantee that a finite \( C_2 \)-equivariant or \( \mathbb{R} \)-motivic spectrum will admit any periodic self-map, or at least nothing concrete is known yet. This question must be studied!

The goal of this paper is rather modest. We consider the classical spectrum

\[
\mathcal{Y} := M_2(1) \wedge C(\eta)
\]

that admits, up to homotopy, 8 different \( v_1 \)-self-maps of periodicity 1 [DM, Section 2] (see also [BEM]). We ask ourselves if the \( v_1 \)-self-maps can preserve symmetries upon providing \( \mathcal{Y} \) with interesting \( C_2 \)-equivariant structures. We will consider four \( C_2 \)-equivariant lifts of the spectrum \( \mathcal{Y} \),

(i) \( \mathcal{Y}^{C_2}_{\text{triv}} \), where the action of \( C_2 \) is trivial,

(ii) \( \mathcal{Y}^{C_2}_{(2,1)} := C^{C_2}(2) \wedge C^{C_2}(\eta_{1,1}) \), with \( \Phi^{C_2}(\mathcal{Y}^{C_2}_{(2,1)}) = M_2(1) \wedge M_2(1) \),

(iii) \( \mathcal{Y}^{C_2}_{(h,0)} := C^{C_2}(h) \wedge C^{C_2}(\eta_{1,0}) \), with \( \Phi^{C_2}(\mathcal{Y}^{C_2}_{(h,0)}) = \Sigma C(\eta) \vee C(\eta) \), and,

(iv) \( \mathcal{Y}^{C_2}_{(h,1)} := C^{C_2}(h) \wedge C^{C_2}(\eta_{1,1}) \), with \( \Phi^{C_2}(\mathcal{Y}^{C_2}_{(h,1)}) = \Sigma M_2(1) \vee M_2(1) \).

The \( C_2 \)-spectra \( \mathcal{Y}^{C_2}_{\text{triv}}, \mathcal{Y}^{C_2}_{(2,1)} \) and \( \mathcal{Y}^{C_2}_{(h,0)} \) are of type \((1,1)\), and \( \mathcal{Y}^{C_2}_{(h,0)} \) is of type \((1,0)\). There are unique \( \mathbb{R} \)-motivic lifts of the classes \( 2, h, \eta_{1,0}, \) and \( \eta_{1,1}, \) and therefore we have unique \( \mathbb{R} \)-motivic lifts of \( \mathcal{Y}^{C_2}_{\text{triv}}, \mathcal{Y}^{C_2}_{(2,1)}, \mathcal{Y}^{C_2}_{(h,0)}, \) and \( \mathcal{Y}^{C_2}_{(h,1)} \) which we will simply denote by \( \mathcal{Y}^{\mathbb{R}}_{\text{triv}}, \mathcal{Y}^{\mathbb{R}}_{(2,1)}, \mathcal{Y}^{\mathbb{R}}_{(h,0)}, \) and \( \mathcal{Y}^{\mathbb{R}}_{(h,1)} \), respectively. In this paper we prove:

**Theorem 1.11.** The \( \mathbb{R} \)-motivic spectrum \( \mathcal{Y}^{\mathbb{R}}_{(h,1)} \) admits a \( v_{1,\text{nil}} \)-self-map

\[
v : \Sigma^{2,1} \mathcal{Y}^{\mathbb{R}}_{(h,1)} \longrightarrow \mathcal{Y}^{\mathbb{R}}_{(h,1)}
\]

of periodicity 1.

By applying the Betti realization functor we get:

**Corollary 1.12.** The \( C_2 \)-equivariant spectrum \( \mathcal{Y}^{C_2}_{(h,1)} \) admits a 1-periodic \( v_{1,\text{nil}} \)-self-map

\[
\beta(v) : \Sigma^{2,1} \mathcal{Y}^{C_2}_{(h,1)} \longrightarrow \mathcal{Y}^{C_2}_{(h,1)}.
\]

**Corollary 1.13.** The geometric fixed-point spectrum of the telescope

\[
\beta(v)^{-1} \mathcal{Y}^{C_2}_{(h,1)}
\]

is contractible.

Classically, the cofiber of a \( v_1 \)-self-map on \( \mathcal{Y} \) is a realization of the finite subalgebra \( \mathcal{A}(1) \) of the Steenrod algebra \( \mathcal{A} \). We see a very similar phenomenon in the \( \mathbb{R} \)-motivic as well as in the \( C_2 \)-equivariant cases. The \( C_2 \)-equivariant Steenrod algebra \( \mathcal{A}^{C_2} \) as well as the \( \mathbb{R} \)-motivic Steenrod algebra \( \mathcal{A}^{\mathbb{R}} \) admit subalgebras analogous to \( \mathcal{A}(1) \) (generated by \( \text{Sq}^1 \) and \( \text{Sq}^2 \)) [H, R2], which we denote by \( \mathcal{A}^{C_2}(1) \) and \( \mathcal{A}^{\mathbb{R}}(1) \), respectively. We observe that:
Theorem 1.14. The spectrum $C^R(v) := \text{Cof}(v : \Sigma^{2,1}Y^R_{(h,1)} \to Y^R_{(h,1)})$ is a type $(2,1)$ complex whose bigraded cohomology is a free $A^R(1)$-module on one generator.

Corollary 1.15. The bigraded cohomology of the $C_2$-equivariant spectrum $C^{C_2}(\beta(v)) \cong \beta(C^R(v))$ is a free $A^{C_2}(1)$-module on one generator.

Our last main result in this paper is the following.

Theorem 1.16. The spectrum $Y^R_{(h,0)}$ does not admit a $v_{(1,0)}$-self-map.

The above results immediately raise some obvious questions. Pertaining to self-maps one may ask: Does $Y^R_{(2,1)}$ admit a $v_{1,nil}$-self-map? Does $Y^R_{(2,1)}$ or $Y^R_{(h,1)}$ admit a $v_{(1,1)}$-self-map? Does $Y^R_{(2,1)}$, $Y^R_{(3,1)}$ or $Y^R_{(h,1)}$ admit $v_{nil,1}$-self-map? Or more generally, how many different homotopy types of each kind of periodic self-maps exist? Related to $A^R(1)$, one may inquire: How many different $A^R$-module structures can be given to $A^R(1)$? Can those $A^R$-modules be realized as a spectrum? Are the realizations of $A^R(1)$ equivalent to cofibers of periodic self-maps of $Y^R_{(i,j)}$? We hope to address most, if not all, of the above questions in our upcoming work (see Remark 3.21, Remark 4.18 and Remark 5.6).

1.1. Outline of our method. We first construct a spectrum $A^R_1$ which realizes the algebra $A^R(1)$ using a method of Smith (outlined in [R1, Appendix C]) which constructs new finite spectra (potentially with larger number of cells) from known ones. The idea is as follows. If $X$ is a $p$-local finite spectrum then the permutation group $\Sigma_n$ acts on $X^{\wedge n}$. One may then use an idempotent $e \in \mathbb{Z}_p[\Sigma_n]$ to obtain a split summand of the spectrum $X^{\wedge n}$. As explained in [R1, Appendix C], Young tableaux provide a rich source of such idempotents. For a judicious choice of $e$ and $X$, the spectrum $e(X^{\wedge n})$ can be interesting.

We exploit the relation that $h \cdot \eta_{1,1} = 0$ in $\pi_*(S^R)$ [M2] to construct an $R$-motivic analogue of the question mark complex. The cell-diagram of the question mark complex is as described in the picture below. For a choice of idempotent element $e$

\[
\begin{align*}
Q_R &= \begin{tikzpicture}
    \node (h) at (0,0) {$h$};
    \node (eta) at (1,1) {$\eta_{1,1}$};
    \draw (h) edge (eta);
\end{tikzpicture}
\end{align*}
\]

Figure 1.17. Cell-diagram of the $R$-motivic question mark complex

in the group ring $\mathbb{Z}_2[\Sigma_3]$, we observe that $e(H^{*,*}(Q_R)^{\otimes 3})$ is a free $A^R(1)$-module. This is the cohomology of an $R$-motivic spectrum $e(Q_R^{\otimes 3})$, which we call $\Sigma^{1,0}A^R_1$ (see (3.4) for details). The observation requires us to develop a criterion that will detect freeness for modules over certain subalgebras of $A^R$. Writing $M_2^R$ for the $R$-motivic cohomology of a point, we prove:
Theorem 1.18. A finitely generated $\mathbb{A}^R(1)$-module $M$ is free if and only if

1. $M$ is free as an $\mathbb{M}^R_2$-module, and
2. $F_2 \otimes \mathbb{M}^R_2 M$ is a free $F_2 \otimes \mathbb{M}^R_2 \mathbb{A}^R(1)$-module.

The cohomology of $\mathbb{A}^R_1$ provides an $\mathbb{A}^R$-module structure on $\mathbb{A}^R(1)$, which immediately gives us a short exact sequence

$$0 \to H^\ast,\ast(\Sigma^3 \mathbb{A}^R(1)) \to H^\ast,\ast(\mathbb{A}^R_1) \to H^\ast,\ast(\mathbb{A}^R(1)) \to 0$$

of $\mathbb{A}^R$-modules. Thus, we get a candidate for a $v_{1,\text{nil}}$-self-map in the $\mathbb{R}$-motivic Adams spectral sequence

$$\varpi \in \text{Ext}^\ast,\ast(\mathbb{A}^R_1, \mathbb{A}^R(1)) \Rightarrow [\mathbb{A}^R(1), \mathbb{A}^R(1)]$$

which survives as there is no potential target for a differential supported by $\varpi$.

Organization of the paper. In Section 2, we review the $\mathbb{R}$-motivic Steenrod algebra $\mathbb{A}^R$, discuss the structure of its subalgebra $\mathbb{A}^R(n)$, and prove Theorem 1.18. In Section 3, we construct the spectrum $\mathbb{A}^R_1$ that realizes the subalgebra $\mathbb{A}^R(1)$ and prove that it is of type $(2,1)$. In Section 4, we prove Theorem 1.11 and Theorem 1.14; i.e., we show that $\mathbb{Y}^R_{(h,1)}$ admits a $v_{1,\text{nil}}$-self-map and that its cofiber has the same $\mathbb{A}^R$-module structure as that of $H^\ast,\ast(\mathbb{A}^R_1)$. In Section 5, we show the non-existence of a $v_{(1,0)}$-self-map on $C^R(h)$ and $\mathbb{Y}^R_{(h,0)}$; i.e., we prove Theorem 1.10 and Theorem 1.16.

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2. The $\mathbb{R}$-motivic Steenrod algebra and a freeness criterion

We begin by reviewing the $\mathbb{R}$-motivic Steenrod algebra $\mathbb{A}^R$ following Voevodsky [V]. The algebra $\mathbb{A}^R$ is the bigraded homotopy classes of self-maps of the $\mathbb{R}$-motivic Eilenberg-Mac Lane spectrum $\mathbb{H}^R_2$:

$$\mathbb{A}^R = [\mathbb{H}^R_2, \mathbb{H}^R_2].$$

The unit map $\mathbb{S}_R \to \mathbb{H}^R_2$ induces a canonical projection map

$$\epsilon : \mathbb{A}^R \to \mathbb{M}^R_2 := [\mathbb{S}_R, \mathbb{H}^R_2].$$

where $|\tau| = (0, -1)$ and $|\rho| = (-1, -1)$. Further, using the multiplication map $\mathbb{H}^R_2 \wedge \mathbb{H}^R_2 \to \mathbb{H}^R_2$ one can give $\mathbb{A}^R$ a left $\mathbb{M}^R_2$-module structure as well as a right $\mathbb{M}^R_2$-module structure. Voevodsky shows that $\mathbb{A}^R$ is a free left $\mathbb{M}^R_2$-module. There is an analogue of the classical Adem basis in the motivic setting, and Voevodsky established motivic Adem relations, thereby completely describing the multiplicative structure of $\mathbb{A}^R$. 
The motivic Steenrod algebra $A^R$ also admits a diagonal map, so that its left $M^R_{2}$-linear dual is an algebra over $F_2$. Note that $A^R$ is an $F_2$-algebra but not an $M^R_{2}$-algebra as $\tau$ is not a central element since

\begin{equation}
Sq^1(\tau) = \rho \neq \tau Sq^1.
\end{equation}

This complication is also reflected in the fact that the pair $(M^R_{2}, \text{hom}_{M^R_{2}}(A^R, M^R_{2}))$ is a Hopf-algebroid, and not a Hopf-algebra like its complex counterpart. The underlying algebra of the dual $R$-motivic Steenrod algebra is given by

\[ A^R_\ast \cong M^R_{2}[\xi, \tau, \rho] / (\tau^2 = \tau \xi + \rho \tau + \rho \xi) \]

where $\xi$ and $\tau$ live in bidegree $(2i + 1, 2i - 1)$ and $(2i + 1, 2i - 1)$, respectively.

The complete description of the Hopf-algebroid structure can be found in [V]. Ricka\footnote{Ricka actually identified the quotient Hopf-algebroids of the $C_2$-equivariant dual Steenrod algebra. However, the difference between the $R$-motivic Steenrod algebra and the $C_2$-equivariant Steenrod algebra lies only in the coefficient rings and results of Ricka easily identifies the quotient Hopf-algebroids of the $R$-motivic Steenrod algebra.} [R2] identified the quotient Hopf-algebroids of $A^R_\ast$ (see also [H]). In particular, there are quotient Hopf-algebroids

\[ A^R(n)_\ast = A^R_\ast / (\xi_1^{2^n}, \ldots, \xi_i^2, \xi_{i+1}, \ldots, \tau_0^{2^n+1}, \ldots, \tau_i^2, \tau_{i+1}, \ldots) \]

which can be thought of as analogues of the quotient Hopf-algebra

\[ A(n)_\ast = A_\ast / (\xi_1^{2^n+1}, \ldots, \xi_{n+1}^2, \xi_{n+2}, \ldots) \]

of the classical dual Steenrod algebra $A_\ast$. It is an algebraic fact that

\[ \tau^{-1}(A^R(n)_\ast / (\rho)) \cong F_2[\tau^{\pm 1}] \otimes A(n)_\ast \]

as Hopf algebras. The quotient Hopf-algebroid $A^R(n)_\ast$ is the $M^R_{2}$-linear dual of the subalgebra $A^R(n)$ of $A^R$, which is generated by $\{\tau, \rho, Sq^1, Sq^2, \ldots, Sq^{2n}\}$.

Even though $\tau$ is not in the center (2.1), $\rho$ is in the center. We make use of this fact to prove the following result.

**Lemma 2.2.** A finitely-generated $A^R(n)$-module $M$ is free if and only if

1. $M$ is free as an $F_2[\rho]$-module, and,
2. $M / (\rho)$ is a free $A^R(n) / (\rho)$-module.

**Proof.** The ‘only if’ part is trivial. For the ‘if’ part, choose a basis $B = \{b_1, \ldots, b_n\}$ of $M / (\rho)$ and let $b_i \in M$ be any lift of $b_i$. Let $F$ denote the free $A^R(n)$-module generated by $B$ and consider the map

\[ f : F \to M \]

which sends $b_i \mapsto \hat{b}_i$. We show that $f$ is an isomorphism by inductively proving that $f$ induces an isomorphism $F / (\rho^n) \cong M / (\rho^n)$ for all $n \geq 1$. The case of $n = 1$ is true by assumption.
For the inductive argument, first note that the diagram
\[
\begin{array}{c}
0 \\
\downarrow f_{n-1} \\
0 \\
\end{array}
\begin{array}{cccc}
F/(\rho^{n-1}) & \xrightarrow{f} & F/(\rho^n) & \xrightarrow{f} & F/(\rho) & \xrightarrow{f_0} & 0 \\
M/(\rho^{n-1}) & \xrightarrow{f} & M/(\rho^n) & \xrightarrow{f_0} & M/(\rho) & \xrightarrow{f_0} & 0 \\
\end{array}
\]
is a diagram of $A^R(n)$-modules (since $\rho$ is in the center) where the horizontal rows are exact. The map $f_0$ is an isomorphism by assumption (2), and $f_{n-1}$ is an isomorphism by the inductive hypothesis; hence, $f_n$ is an isomorphism by the five lemma.

\[\square\]

**Proof of Theorem 1.18.** The result follows immediately from Lemma 2.2 combined with [HK, Theorem B] and the fact that \(A^C(n) = A^R(n)/(\rho)\).

The work of Adams and Margolis [AM] provides a freeness criterion for modules over finite-dimensional subalgebras of the classical Steenrod algebra. For an $A(n)$-module $M$ and element $x \in A(n)$ such that $x^2 = 0$, one can define the Margolis homology of $M$ with respect to $x$ as
\[
\mathcal{M}(M, x) = \frac{\ker(x : M \to M)}{\text{img}(x : M \to M)}.
\]

**Theorem 2.3.** [AM, Theorem 4.4] A finitely generated $A(n)$-module $M$ is free if and only if $\mathcal{M}(M, P^s_t) = 0$ for $0 < s < t$ with $s + t \leq n$.

**Remark 2.4.** In the classical Steenrod algebra, $P^s_t$ is the element dual to $\xi^s_{t-1}$. In terms of the Milnor basis, this is $\text{Sq}(0, \ldots, 0, 2^s)$. The element $P^0_t$ is often denoted by $Q_{t-1}$.

Note that
\[
A^R(n)/\langle\rho, \tau\rangle = F_2[\xi_1, \ldots, \xi_n] / (\xi_1^{r_1}, \ldots, \xi_n^{r_n}) \otimes \Lambda(\tau_0, \ldots, \tau_n)
\]
as a Hopf-algebra. Further, if we forget the motivic grading, we have an isomorphism
\[
(2.5) \hspace{1cm} A^R(n)/\langle\rho, \tau\rangle \cong \varphi A(n-1) \otimes \Lambda(P^0_1, \ldots, P^0_n),
\]
where $\varphi A(n-1)$ denotes the ‘double’ (see [M1, Chapter 15, Proposition 11]) of $A(n-1)$. Let
\[
P^s_t = (\xi^s_{t-1})^* \in A^R(n).
\]
It can be shown that
\[
(P^s_t)^2 \equiv 0 \mod \langle\rho, \tau\rangle
\]
for $s \leq t$. Combining (2.5), Theorem 2.3 and a similar result for primitively generated exterior Hopf-algebras [AM, Theorem 2.2], we deduce:

**Lemma 2.6.** A finitely generated $A^R(n)/\langle\rho, \tau\rangle$-module $\mathcal{M}$ is free if and only if $\mathcal{M}(\mathcal{M}, P^s_t) = 0$ whenever $0 \leq s \leq t$ and $1 \leq s + t \leq n + 1$. 

We end this section by recording the following corollary, which is immediate from Theorem 1.18 and Lemma 2.6.

**Corollary 2.7.** A finitely generated $\mathcal{A}^R(n)$ module $M$ is free if and only if

1. $M$ is free as an $\mathcal{M}_2^n$-module, and,
2. $\mathcal{M}(M \otimes_{\mathcal{M}_2^n} \mathbb{F}_2, P_t) = 0$ for $0 \leq t \leq s$ and $s + t = n + 1$.

3. **A REALIZATION OF $\mathcal{A}^R(1)$**

Consider the $\mathcal{R}$-motivic question mark complex $Q_\mathcal{R}$, as introduced in Subsection 1.1. Let $\Sigma_n$ act on $Q_\mathcal{R}^n$ by permutation. Any element $e \in \mathbb{Z}_2[\Sigma_n]$ produces a canonical map

$$\tilde{e} : Q_\mathcal{R}^n \longrightarrow Q_\mathcal{R}^n.$$ 

Now let $e$ be the idempotent

$$e = \frac{1 + (1 2) - (1 3) - (1 3 2)}{3}$$

in $\mathbb{Z}_2[\Sigma_3]$, and denote by $\overline{e}$ the resulting idempotent of $\mathbb{F}_2[\Sigma_3]$. We record the following important property of $\overline{e}$ which is a special case of [R1, Theorem C.1.5].

**Lemma 3.1.** If $V$ is a finite-dimensional $\mathbb{F}_2$-vector space, then $\overline{e}(V \otimes^3) = 0$ if and only if $\dim V \leq 1$.

The following result, which gives the values of $\overline{e}$ on induced representations, is also straightforward to verify:

**Lemma 3.2.** Suppose that $W = \text{Ind}_{C_2}^{\Sigma_3} \mathbb{F}_2$ is induced up from the trivial representation of a cyclic 2-subgroup. Then $\overline{e}(W) \cong \mathbb{F}_2$. Moreover, for the regular representation $\mathbb{F}_2[\Sigma_3] = \text{Ind}_{C_2}^{\Sigma_3} \mathbb{F}_2$, we have $\dim \overline{e}(\mathbb{F}_2[\Sigma_3]) = 2$.

We also record the fact that when $\dim_{\mathbb{F}_2} V = 2$ and $\dim_{\mathbb{F}_2} W = 3$ then

$$\dim_{\mathbb{F}_2} \overline{e}(V \otimes^3) = 2 \quad \text{and} \quad \dim_{\mathbb{F}_2} \overline{e}(W \otimes^3) = 8,$$

as we will often use this.

The bottom cell of $\tilde{e}(Q_\mathcal{R}^{\lambda^3})$ is in degree $(1, 0)$, and we define

$$\mathcal{A}^R_1 := \Sigma^{-1, 0} \tilde{e}(Q_\mathcal{R}^{\lambda^3}) = \Sigma^{-1, 0} \text{hocolim}(Q_\mathcal{R}^{\lambda^3} \rightarrow Q_\mathcal{R}^{\lambda^3} \rightarrow \ldots).$$

The purpose of this section is to prove the following theorem.

**Theorem 3.5.** The spectrum $\mathcal{A}^R_1$ is a type $(2, 1)$ complex whose bi-graded cohomology $H^{*, *}(\mathcal{A}^R_1)$ is a free $\mathcal{A}^R(1)$-module on one generator.
3.1. \(\mathcal{A}^R\) is of type \((2,1)\). Let \(\mathcal{A}^C_2 := \beta(\mathcal{A}^R_1)\) and \(Q_{C_2} := \beta(Q_\mathbb{R})\). Note that we have a \(C_2\)-equivariant splitting
\[
Q_{C_2}^{\wedge 3} \simeq \tilde{e}(Q_{C_2}^{\wedge 3}) \vee (1 - \tilde{e})(Q_{C_2}^{\wedge 3})
\]
which splits the underlying spectra as well as the geometric fixed-points, as both \(\Phi^e\) and \(\Phi^{C_2}\) are additive functors.

We will identify the underlying spectrum \(\Phi^e(\mathcal{A}^C_1)\) by studying the \(A\)-module structure of its cohomology with \(F_2\)-coefficients. Firstly, note that
\[
\Phi^e(\mathcal{A}^C_1) \simeq \Sigma^{-1} \tilde{e}(\Phi^e(\mathcal{Q}^{\wedge 3}_{C_2})) \simeq \Sigma^{-1} \tilde{e}(Q^{\wedge 3}_C),
\]
where \(Q\) is the classical question mark complex, whose \(HF_2\)-cohomology as an \(A\)-module is well understood. It consists of three \(F_2\)-generators \(a, b, c\) in internal degrees 0, 1, and 3, such that \(Sq^1(a) = b\) and \(Sq^2(b) = c\) are the only nontrivial relations, as displayed in Figure 3.6.

![Figure 3.6](image)

**Figure 3.6.** We depict the \(A\)-structure of \(H^*(\mathcal{Q}; F_2)\) by marking \(Sq^1\)-action by black straight lines and \(Sq^2\)-action by blue curved lines between the \(F_2\)-generators.

Because of the Kunneth isomorphism and the fact that the Steenrod algebra is cocommutative, we have an isomorphism of \(A\)-modules
\[
H^{*+1}(\Phi^e(\mathcal{A}^C_1); F_2) \cong H^*(\tilde{e}(Q^{\wedge 3}_C); F_2) \cong \tau(H^*(\mathcal{Q}; F_2)^{\wedge 3}).
\]

**Lemma 3.7.** The underlying \(A(1)\)-module structure of \(H^*(\Phi^e(\mathcal{A}^C_1); F_2)\) is free on a single generator.

**Proof.** Let us denote the \(A\)-module \(H^*(\mathcal{Q}; F_2)\) by \(V\). Since \(\dim \mathcal{M}(V, Q_i) = 1\) for \(i \in \{0, 1\}\), it follows from the Kunneth isomorphism of \(Q_i\)-Margolis homology groups, cocommutativity of the Steenrod algebra, and Lemma 3.1 that
\[
\mathcal{M}(\tau(V^{\wedge 3}), Q_i) = \tau(\mathcal{M}(V, Q_i)^{\wedge 3}) = 0
\]
for \(i = \{1, 2\}\). It follows from [AM, Theorem 3.1] that \(H^*(\Phi^e(\mathcal{A}^R_1); F_2)\) is free as an \(A(1)\)-module. It is singly generated because of (3.3). \(\square\)

We explicitly identify the image of \(\tau : H^*(\mathcal{Q}; F_2)^{\wedge 3} \longrightarrow H^*(\mathcal{Q}; F_2)^{\wedge 3}\) in Figure 3.8.

**Remark 3.9.** Using the Cartan formula, we can identify the action of \(Sq^4\) on \(\Phi^e(\mathcal{A}^C_1)\). We notice that its \(A\)-module structure is isomorphic to \(A_1[00]\) of [BEM]. Since such an \(A\)-module is realized by a unique 2-local finite spectrum, we conclude
\[
\Phi^e(\mathcal{A}^C_1) \simeq A_1[00]
\]
and is of type 2.

Our next goal is to understand the homotopy type of the geometric fixed-point spectrum $\Phi_{C_2}(\mathcal{A}_1^{C_2})$. First observe that the geometric fixed-points of the $C_2$-equivariant question mark complex $Q_{C_2}$ is the exclamation mark complex

$$E := \bigcirc \simeq S^0 \vee \Sigma M_2(1)!$$

This is because $\Phi_{C_2}(h) = 0$ and $\Phi_{C_2}(\eta_{1,1}) = 2$. Secondly,

$$H^{*+1}(\Phi_{C_2}(\mathcal{A}_1^{C_2}); \mathbb{F}_2) \cong H^*(\tilde{e}(E^3); \mathbb{F}_2) \cong \tau(H^*(E; \mathbb{F}_2)^{\otimes 3})$$

is an isomorphism of $A$-modules. We explicitly calculate the $A$-module structure of $H^*(\Phi_{C_2}(\mathcal{A}_1^{C_2}); \mathbb{F}_2)$ from the above isomorphism and record it in Figure 3.10.



**Figure 3.10.** The $A$-module structure of $H^*(\Phi_{C_2}(\mathcal{A}_1^{C_2}); \mathbb{F}_2)$.

**Lemma 3.11.** The finite spectrum $\Phi_{C_2}(\mathcal{A}_1^{C_2})$ is a type 1 spectrum and equivalent to $\Phi_{C_2}(\mathcal{A}_1^{C_2}) \simeq M_2(1) \vee \Sigma(M_2(1) \wedge M_2(1)) \vee \Sigma^3 M_2(1)$. 
Proof. From Figure 3.10, it is clear that we have an isomorphism of \( \mathcal{A} \)-modules

\[
H^*(\Phi G^2(\mathcal{A}_1^2); \mathbb{F}_2) \cong H^*(M_2(1) \vee \Sigma(M_2(1) \wedge M_2(1)) \vee \Sigma^2M_2(1); \mathbb{F}_2).
\]

It is possible that the \( \mathcal{A} \)-module \( H^*(\Phi G^2(\mathcal{A}_1^2); \mathbb{F}_2) \) may not realize to a unique finite spectrum (up to weak equivalence). However, other possibilities can be eliminated from the fact that \( \mathcal{E}^\wedge 3 \) splits \( \Sigma_3 \)-equivariantly into four components:

\[
\mathcal{E}^\wedge 3 \cong S \vee \left( \bigvee_{i=1}^3 \Sigma M_2(1) \right) \vee \left( \bigvee_{i=1}^3 \Sigma^2M_2(1) \wedge 2 \right) \vee \Sigma^3M_2(1) \wedge 3.
\]

The idempotent \( \tilde{e} \) annihilates \( S \cong \mathcal{E}^\wedge 3 \), and Lemma 3.2 implies that

\[
\tilde{e} \left( \bigvee_{i=1}^3 \Sigma M_2(1) \right) \cong \Sigma M_2(1) \quad \text{and}
\]

\[
\tilde{e} \left( \bigvee_{i=1}^3 \Sigma^2M_2(1) \wedge M_2(1) \right) \cong \Sigma^2M_2(1) \wedge M_2(1).
\]

Similarly, we see using (3.3) that

\[
H^* \left( \tilde{e} (\Sigma^3M_2(1) \wedge 3) \right) \cong \tilde{e} \left( H^* (\Sigma M_2(1)) \wedge 3 \right) \cong H^* (\Sigma^3M_2(1)).
\]

Hence, the result. \( \square \)

3.2. The cohomology of \( \mathcal{A}_1^R \) is free over \( \mathcal{A}^R(1) \). Next, we analyze the \( \mathcal{A}^R \)-module structure of \( H^{*,*}(\mathcal{A}_1^R) \). We begin by recalling some general properties of the cohomology of motivic spectra.

If \( X, Y \in \text{Sp}_R \) such that \( H^{*,*}(X) \) is free as a left \( \mathbb{M}_R \)-module, then we have a Kunneth isomorphism [DI2, Proposition 7.7]

\[
(3.12) \quad H^{*,*}(X \wedge Y) \cong H^{*,*}(X) \otimes_{\mathbb{M}_R} H^{*,*}(Y)
\]

as the relevant Kunneth spectral sequence collapses. Further, if \( H^{*,*}(X) \) is free as a left \( \mathbb{M}_R \)-module, then so is \( H^{*,*}(X \wedge Y) \). The \( \mathcal{A}^R \)-module structure of \( H^{*,*}(X \wedge Y) \) can then be computed using the Cartan formula. The comultiplication map of \( \mathcal{A}^R \) is left \( \mathbb{M}_R \)-linear, coassociative and cocommutative [V, Lemma 11.9], which is also reflected in the fact that its \( \mathbb{M}_R \)-linear dual is a commutative and associative algebra. Thus, when \( H^{*,*}(X) \) is a free left \( \mathbb{M}_R \)-module, the elements of \( \mathbb{F}_2[\Sigma_n] \) acts on

\[
H^{*,*}(X^\wedge n) \cong H^{*,*}(X) \otimes_{\mathbb{M}_R} \cdots \otimes_{\mathbb{M}_R} H^{*,*}(X)
\]

via permutation and commutes with the action of \( \mathcal{A}^R \). This also implies that \( \mathbb{F}_2[\Sigma_n] \) also acts on

\[
H^{*,*}(X^\wedge n)/\langle \rho, \tau \rangle \cong (H^{*,*}(X)/\langle \rho, \tau \rangle) \otimes \cdots \otimes H^{*,*}(X)/\langle \rho, \tau \rangle
\]

and commutes with the action of \( \mathcal{A}^R/\mathbb{M}_R \). From the above discussion we may conclude that

\[
(3.13) \quad H^{*,*}(\mathcal{A}_1^R) \cong \Sigma^{-1} \tau(H^{*,*}(\mathbb{Q}_R) \wedge 3)
\]

is an isomorphism of \( \mathcal{A}^R \)-module.
We will also rely upon the following important property of the action of the motivic Steenrod algebra on the cohomology of a motivic space (as opposed to a motivic spectrum):

**Remark 3.14** (Instability condition for \(R\)-motivic cohomology). If \(X\) is an \(R\)-motivic space then \(H^*,_* (X)\) admits a ring structure, and, for any \(u \in H^{n,i} (X)\), the \(R\)-motivic squaring operations obey the rule

\[
\text{Sq}^2_i (u) = \begin{cases} 
0 & \text{if } n < 2i \\
u^2 & \text{if } n = 2i.
\end{cases}
\]

This is often referred to as the *instability condition*.

To understand the \(A^R\)-module structure of \(H^*,_* (Q_R)\), we first make the following observation regarding \(H^*,_* (C^R (h))\) (as \(C^R (h)\) is a sub-complex of \(Q_R\)) using an argument very similar to [DI1, Lemma 7.4].

**Proposition 3.15.** There are two extensions of \(A^R (0)\) to an \(A^R\)-module, and these \(A^R\)-modules are realized as the cohomology of \(C^R (h)\) and \(C^R (2)\).

**Proof.** For degree reasons, the only choice in extending \(A^R (0)\) to an \(A^R\)-module is the action of \(\text{Sq}^2\) on the generator in bidegree \((0, 0)\). Writing \(y_{0,0}\) for the generator in degree \((0, 0)\) and \(y_{1,0}\) for \(\text{Sq}^1 (y_{0,0})\) in (cohomological) bidegree \((1, 0)\). The two possible choices are

- \(\text{Sq}^2 (y_{0,0}) = 0\) and
- \(\text{Sq}^2 (y_{0,0}) = \rho \cdot y_{1,0}\).

We can realize the degree 2 map as an unstable map \(S^1,0 \to S^1,0\), and we will write \(C^R (2)\) for the cofiber. We deduce information about the \(A^R\)-module structure of...
$H^\ast\ast(C^R(2))$ by analyzing the cohomology ring of $S^{1,1} \wedge C^R(2)_u$ using the instability condition of Remark 3.14. First, note that in

$$H^\ast\ast(S^{1,1}) \cong M^R_2 \cdot \iota_{1,1}$$

we have the relation $\iota^2_{1,1} = \rho \cdot \iota_{1,1}$ [V, Lemma 6.8]. Also note that

$$H^\ast\ast((C^R(2)_u) \cup) \cong M^R_2[\chi]/(\chi^3)$$

where $\chi$ is in cohomological degrees $(1,0)$. Therefore, in

$$H^\ast\ast(S^{1,1} \wedge C^R(2)_u) = M^R_2 \cdot \iota_{1,1} \otimes M^R_2 \{x, x^2\}$$

the instability condition implies

$$Sq^2(\iota_{1,1} \otimes x) = \iota^2_{1,1} \otimes x^2 = \rho \cdot \iota_{1,1} \otimes x^2.$$ 

Here the space-level cohomology class $x^2$ corresponds to the spectrum-level class $y_{1,0}$. Therefore, $Sq^2(y_{0,0}) = \rho \cdot y_{1,0}$ in $H^\ast\ast(C^R(2))$. This is also reflected in the fact that multiplication by 2 is detected by $h_0 + \rho h_1$ in the $\mathbb{R}$-motivic Adams spectral sequence [DI1, §8].

On the other hand $h_0$ is the ‘zeroth $\mathbb{R}$-motivic Hopf-map’ detected by the element $h_0$ in the motivic Adams spectral sequence. It follows that $Sq^2(y_{0,0}) = 0$. □

In order to express the $A^R$-module structure on $H^\ast\ast(X)$ for a finite spectrum $X$, it is enough to specify the action of $A^R$ on its left $M^R_2$-generators as the action of $\tau$ and $\rho$ multiples are determined by the Cartan formula.

**Example 3.17.** Let $\{y_{0,0}, y_{1,0}\} \subset H^\ast\ast(C^R(h))$ denote a left $M^R_2$-basis of $H^\ast\ast(C^R(h))$. The data that

- $Sq^1(y_{0,0}) = y_{1,0}$
- $Sq^2(y_{0,0}) = 0$

completely determines the $A^R$-module structure of $H^\ast\ast(C^R(h))$.

**Proposition 3.18.** $H^\ast\ast(Q_\mathbb{R})$ is a free $M^R_2$-module generated by $a, b$ and $c$ in cohomological bidegrees $(0,0), (1,0)$ and $(3,1)$, and the relations

1. $Sq^1(a) = b$,
2. $Sq^2(b) = c$,
3. $Sq^4(a) = 0$.

completely determine the $A^R$-module structure of $H^\ast\ast(Q_\mathbb{R})$.

**Proof.** $H^\ast\ast(Q_\mathbb{R})$ is a free $M^R_2$-module because the attaching maps of $Q_\mathbb{R}$ induce trivial maps in $H^\ast\ast(-)$. The first two relations can be deduced from the obvious maps

1. $C^R(h) \to Q_\mathbb{R}$
2. $Q_\mathbb{R} \to \Sigma^{1,0} C^R(\eta_{1,1})$
which are respectively surjective and injective in cohomology.

Let $h^u : S^{3,2} \to S^{3,2}$ and $\eta^u : S^{3,2} \to S^{2,1}$ denote the unstable maps that stabilize to $h$ and $\eta^u$, respectively. The unstable $\mathbb{R}$-motivic space $\mathbb{Q}_{\mathbb{R}}^u$ (which stabilizes to $\mathbb{Q}_{\mathbb{R}}$) can be constructed using the fact that the composite of the unstable maps

$$S^{3,2} \xrightarrow{\Sigma^1 h^u} S^{3,2} \xrightarrow{h^u} S^{3,2}$$

is null. Thus $H^\bullet(\mathbb{Q}_{\mathbb{R}}^u)$ consists of three generators $a_u, b_u$ and $c_u$ in bidegrees (3,2), (4,2) and (6,3). It follows from the instability condition that $Sq^4(a_u) = 0$. \hfill $\Box$

**Proof of Theorem 3.5.** From Remark 3.9 and Lemma 3.11, we deduce that $\mathcal{A}_1^\mathbb{R}$ is a type $(2,1)$ complex. To show that the bi-graded $\mathbb{R}$-motivic cohomology of $\mathcal{A}_1^\mathbb{R}$ is free as an $\mathcal{A}^\mathbb{R}(1)$, we make use of Corollary 2.7.

Since $H^\bullet(\mathcal{A}_1^\mathbb{R})$ is a summand of a free $\mathbb{M}_2^\mathbb{R}$-module, it is projective as an $\mathbb{M}_2^\mathbb{R}$-module. In fact, $H^\bullet(\mathcal{A}_1^\mathbb{R})$ is free, as projective modules over (graded) local rings are free. Also note that the elements

$$\mathfrak{F}_1^0, \mathfrak{F}_1^1, \mathfrak{F}_2^d \in \mathcal{A}_{\mathbb{R}}(1)/(\rho, \tau) \cong \Lambda(\mathfrak{F}_1^0, \mathfrak{F}_1^1, \mathfrak{F}_2^d)$$

are primitive. Hence we have a Künneth isomorphism in the respective Margolis homologies, in particular we have,

$$H^\bullet(\mathcal{A}_1^\mathbb{R})/(\rho, \tau, \mathfrak{F}_t^i) = \tau(H^\bullet(\mathbb{Q}_{\mathbb{R}})/(\rho, \tau, \mathfrak{F}_t^i) \otimes^\mathbb{R} \mathbb{Q})$$

for $(s, t) \in \{(0, 1), (1, 1), (0, 2)\}$. Since $\dim_{\mathbb{R}} H^\bullet(\mathbb{Q}_{\mathbb{R}})/(\rho, \tau, \mathfrak{F}_t^i) = 1$, by Lemma 3.1

$$\dim_{\mathbb{R}} H^\bullet(\mathcal{A}_1^\mathbb{R})/(\rho, \tau, \mathfrak{F}_t^i) = 0$$

for $(s, t) \in \{(0, 1), (1, 1), (0, 2)\}$. Thus, by Corollary 2.7 we conclude that $H^\bullet(\mathcal{A}_1^\mathbb{R})$ is a free $\mathcal{A}^\mathbb{R}(1)$-module. A direct computation shows that

$$\dim_{\mathbb{R}} H^\bullet(\mathcal{A}_1^\mathbb{R})/(\rho, \tau) = 8,$$

hence $H^\bullet(\mathcal{A}_1^\mathbb{R})$ is $\mathcal{A}^\mathbb{R}(1)$-free of rank one. \hfill $\Box$

### 3.3. The $\mathcal{A}^\mathbb{R}$-module structure.

Using the description (3.13) and Cartan formula we make a complete calculation of the $\mathcal{A}^\mathbb{R}$-module structure of $H^\bullet(\mathcal{A}_1^\mathbb{R})$. Let $a, b, c \in H^\bullet(\mathbb{Q}_{\mathbb{R}})$ as in Proposition 3.18. In Figure 3.20 we provide a pictorial representation with the names of the generators that are in the image of the idempotent $\tau$. For convenience we relabel the generators in Figure 3.20, where the indexing on a new label records the cohomological bidegrees of the corresponding generator. The following result is straightforward, and we leave it to the reader to verify.

**Lemma 3.19.** In $H^\bullet(\mathcal{A}_1^\mathbb{R})$, the underlying $\mathcal{A}^\mathbb{R}(1)$-module structure, along with the relations

1. $Sq^4(v_{0,0}) = 0$,
2. $Sq^4(v_{1,0}) = \tau \cdot w_{5,2}$,
3. $Sq^4(v_{2,1}) = 0$,
4. $Sq^4(v_{3,1}) = 0 = Sq^4(w_{3,1})$, 

$\mathbb{R}$-motivic $v_1$-self-map of periodicity 1
(5) \( Sq^8(v_{0,0}) = 0, \) completely determine the \( \mathcal{A}^R \)-module structure.

![Diagram](image)

**Figure 3.20.** We depict the \( \mathcal{A}^R \)-module structure of \( H^{*,*}(A_1) \). The black, blue, and red lines represent the action of motivic \( Sq^1, Sq^2, \) and \( Sq^4 \), respectively. Black dots represent \( MR \)-generators, and a dotted line represents that the action hits the \( \tau \)-multiple of the given \( MR \)-generator.

**Remark 3.21.** In upcoming work, we show that \( \mathcal{A}^R(1) \) admits 128 different \( \mathcal{A}^R \)-module structures. Whether all of the 128 \( \mathcal{A}^R \)-module structures can be realized by \( \mathbb{R} \)-motivic spectra, or not, is currently under investigation.

4. **An \( \mathbb{R} \)-Motivic \( v_1 \)-Self-Map**

With the construction of \( \mathcal{A}^R_1 \), we hope that any one of \( \mathcal{Y}^R_{(i,j)} \) fits into an exact triangle

\[
\Sigma^{2,1} \mathcal{Y}^R_{(i,j)} \xrightarrow{v} \mathcal{Y}^R_{(i,j)} \xrightarrow{} \mathcal{A}^R_1 \xrightarrow{} \Sigma^{3,1} \mathcal{Y}^R_{(i,j)} \xrightarrow{\Sigma v} \ldots
\]

in \( Ho(\mathcal{Sp}^R_{2,fin}) \). The motivic weights prohibit \( \mathcal{A}^R_1 \) from being the cofiber of a self-map on \( \mathcal{Y}_{triv} \) or \( \mathcal{Y}_{(h,0)} \). We will also see that the spectrum \( \mathcal{Y}^R_{(2,1)} \) cannot be a part of (4.1) because of its \( \mathcal{A}^R \)-module structure (see Lemma 4.5). If \( \mathcal{Y}^R_{(i,j)} = \mathcal{Y}^R_{(h,1)} \) in (4.1), then the map \( v \) will necessarily be a \( v_{1,nil} \)-self-map because \( \mathcal{Y}^R_{(h,1)} \) is of type \( (1,1) \) and \( \mathcal{A}^R_1 \) is of type \( (2,1) \). The main purpose of this section is to prove Theorem 1.11 and Theorem 1.14 by showing that \( \mathcal{Y}^R_{(h,1)} \) does fit into an exact triangle very similar to (4.1)

\[
\Sigma^{2,1} \mathcal{Y}^R_{(i,j)} \xrightarrow{v} \mathcal{Y}^R_{(i,j)} \xrightarrow{} C^R(v) \xrightarrow{} \Sigma^{3,1} \mathcal{Y}^R_{(i,j)} \xrightarrow{\Sigma v} \ldots
\]

where \( C^R(v) \) is of type \( (2,1) \) and \( H^{*,*}(C^R(v)) \cong H^{*,*}(A^R_1) \) as \( \mathcal{A}^R \)-modules but potentially may have a homotopy type different than that of \( A^R_1 \). We begin by discussing the \( \mathcal{A}^R \)-module structures of \( H^{*,*}(\mathcal{Y}^R_{(h,1)}) \).
Using Adem relations, one can show that the element 

$$Q_1 := Sq^1 Sq^2 + Sq^2 Sq^1 \in A^R(1)$$

squares to zero. Let \( A(Q_1) \) denote the exterior subalgebra \( M_2^R/Q_1 \) of \( A^R(1) \). Let \( B^R(1) \) denote the \( A^R(1) \)-module

$$B^R(1) := A^R(1) \otimes \Lambda(Q_1) M_2^R.$$

Both \( Y^R_{(2,1)} \) and \( Y^R_{(h,1)} \) are realizations of \( B^R(1) \). In other words:

**Proposition 4.2.** There is an isomorphism of \( A^R(1) \)-modules

$$H^{*,*}(Y^R_{(i,j)}) \cong B^R(1)$$

for \((i, j) \in \{(2, 1), (h, 1)\} \).

**Proof.** Note that \( H^{*,*}(Y^R_{(i,j)}) \) is cyclic as an \( A^R(1) \)-module for \((i, j) \in \{(2, 1), (h, 1)\} \). Thus we have a map

$$f_1 : A^R(1) \to H^{*,*}(Y^R_{(i,j)}).$$

The result follows from the fact that \( Q_1 \) acts trivially on \( H^{*,*}(Y^R_{(i,j)}) \) and a dimension counting argument.

**Remark 4.4.** Let \( \{y_{0,0}, y_{1,0}\} \) be the \( M_2^R \)-basis of \( H^{*,*}(C^R(h)) \) or \( H^{*,*}(C^R(2)) \), so that \( Sq^1(y_{0,0}) = y_{1,0} \), and let \( \{x_{0,0}, x_{2,1}\} \) a basis of \( C^R(\eta_{1,1}) \), so that \( Sq^1(x_{0,0}) = x_{2,1} \). If we consider the \( M_2^R \)-basis \( \{v_{0,0}, v_{1,0}, v_{2,1}, v_{3,1}, v_{5,2}, v_{4,2}, w_{5,3}, w_{6,3}\} \) of \( A^R(1) \) from Subsection 3.3, then the maps \( f_1 \) of (4.3) are given as in Table 1.

| \( x \)    | \( f_2(x) \)   | \( f_1(x) \) |
|-----------|---------------|------------|
| \( v_{0,0} \) | \( y_{0,0}x_{0,0} \) | \( y_{0,0}x_{0,0} \) |
| \( v_{1,0} \) | \( y_{1,0}x_{0,0} \) | \( y_{1,0}x_{0,0} \) |
| \( v_{2,1} \) | \( y_{0,0}x_{2,0} + \rho \cdot y_{1,0}x_{0,0} \) | \( y_{0,0}x_{2,0} \) |
| \( v_{3,1} \) | \( y_{1,0}x_{2,0} \) | \( y_{1,0}x_{2,0} \) |
| \( w_{5,3} \) | \( 0 \) | \( 0 \) |
| \( w_{6,3} \) | \( 0 \) | \( 0 \) |

**Lemma 4.5.** The \( A^R \)-module structures on \( H^{*,*}(Y^R_{(2,1)}) \) and \( H^{*,*}(Y^R_{(h,1)}) \) are given as in Figure 4.6.

**Proof.** The result is an easy consequence of a calculation using the Cartan formula

$$Sq^i(xy) = Sq^i(x)y + \tau Sq^i(x)Sq^i(y) + Sq^2(x)Sq^2(y) + \tau Sq^1(x)Sq^1(y) + x Sq^4(y)$$

and the fact that \( Sq^2(y_{0,0}) = \rho y_{1,0} \) in \( H^{*,*}(C^R(2)) \), whereas \( Sq^2(y_{0,0}) = 0 \) in \( H^{*,*}(C^R(h)) \) (see Proposition 3.15).

**Remark 4.7.** Comparing Lemma 4.5 and Lemma 3.19, we see that the \( A^R(1) \)-module map \( f_2 \), as in Remark 4.4, cannot be extended to a map of \( A^R \)-modules.
For $f \geq 3$, $\text{Ext}^{1,f-1,\bullet}_A(H^{*,*}(\mathcal{Y}^R_{(h,1)}), H^{*,*}(\mathcal{Y}^R_{(h,1)})) = 0$. 

**Proposition 4.13.**
In order to calculate $\text{Ext}^{*,*,*}_{A^2}(H^{*,*}(Y_{R(h,1)}), H^{*,*}(Y_{R(h,1)}))$, we filter the spectrum $Y_{R(h,1)}$ via the evident maps

\[
\begin{array}{cccc}
Y_0 & \rightarrow & Y_1 & \rightarrow & Y_2 & \rightarrow & Y_3. \\
\mathbb{S}_R & \rightarrow & C^R(h) & \rightarrow & C^R(h) \cup_{S_R} C^R(\eta_{1,1}) & \rightarrow & Y_{R(h,1)}
\end{array}
\]

Note that $H^{*,*}(Y_j)$ are free $M^R_{2}$-modules. The above filtration results in cofiber sequences

\[
\begin{array}{c}
Y_0 \rightarrow Y_1 \rightarrow \Sigma^{1,0}S_R, \\
Y_1 \rightarrow Y_2 \rightarrow \Sigma^{2,1}S_R, \\
Y_2 \rightarrow Y_3 \rightarrow \Sigma^{3,1}S_R,
\end{array}
\]

which induce short exact sequences of $A^2$-modules as the connecting map

\[C^R(Y_j \rightarrow Y_{j+1}) \rightarrow \Sigma Y_j\]

induces the zero map in $H^{*,*}(-)$. Thus, applying the functor $\text{Ext}^{*,*,*}_{A^2}(H^{*,*}(Y_{R(h,1)}), -)$ to these short-exact sequences, we get long exact sequences, which can be spliced together to obtain an Atiyah-Hirzebruch like spectral sequence

\[
E_1^{i,j} = \text{Ext}^{*,*,*}_{A^2}(H^{*,*}(Y_{R(h,1)}), M^R_{2}) \{\eta_{0,0}, \eta_{1,0}, \eta_{2,1}, \eta_{3,1}\} \rightarrow \text{Ext}^{*,*,*}_{A^2}(H^{*,*}(Y_{R(h,1)}), H^{*,*}(Y_{R(h,1)})).
\]

An element $x \cdot \eta_{i,j}$ in the $E_2$-page contributes to the degree $|x| - (i, 0, j)$ of the abutment. Thus, Proposition 4.13 is a straightforward consequence of the following Proposition 4.15.

**Remark 4.14.** Because, $H^{*,*}(Y_{R(h,1)})$ is $M^R_{2}$-free and finite, we have

\[H_{*,*}(Y_{R(h,1)}) \cong \text{hom}_{M^R_{2}}(H^{*,*}(Y_{R(h,1)}), M^R_{2}),\]

and therefore, $\text{Ext}^{*,f,w}_{A^2}(H^{*,*}(Y_{R(h,1)}), M^R_{2}) \cong \text{Ext}^{*,f,w}_{A^2}(M^R_{2}, H_{*,*}(Y_{R(h,1)}))$.

**Proposition 4.15.** For $f \geq 3$ and $(i, j) \in \{(0, 0), (1, 0), (2, 1), (3, 1)\}$, we have that

\[\text{Ext}^{1+i,f,1+j}_{A^2}(M^R_{2}, H_{*,*}(Y_{R(h,1)})) = 0.\]

**Proof.** Our desired vanishing concerns only the groups $\text{Ext}^{*,*,*}_{A^2}(M^R_{2}, H_{*,*}(Y_{R(h,1)}))$ in coweights 0, 1 and 2. These groups can be easily calculated starting from the computations of $\text{Ext}^{*,*,*}_{M^R_{2}}(M^R_{2}, M^R_{2})$ in [D1] and [B1] and by using the short exact sequences in $\text{Ext}_{A^2}^{*,*}$ arising from the cofiber sequences

\[
\begin{array}{ccc}
\Sigma^{1,1}S_R & \rightarrow & \eta_{1,1} \rightarrow \mathbb{S}_R \rightarrow C^R(\eta_{1,1}) \\
C^R(\eta_{1,1}) & \rightarrow & C^R(N_{1,1}) \rightarrow C^R(h) \land C^R(\eta_{1,1}) = Y_{R(h,1)}.
\end{array}
\]

We display $\text{Ext}^{*,*}_{A^2}(M^R_{2}, H_{*,*}(C^R(\eta_{1,1})))$ in coweights 0, 1 and 2 in the charts below.
We find that $\text{Ext}_{\mathcal{A}^e}(M_2, H_\ast, \ast(C^R_{(\eta_1, 1)}))$ is, in coweights zero, one and two, also given by the charts below.

The result follows from the above charts. □

Remark 4.16. One can also resolve Proposition 4.15 directly using the $\rho$-Bockstein spectral sequence

$$E_1 := \text{Ext}_{\mathcal{A}^C_2}(\mathbb{F}_2[\tau], H_\ast, \ast(C^C_{(\eta_1, 1)})) \otimes \mathbb{F}_2[\rho]$$

(4.17)

and identifying a vanishing region for $\text{Ext}_{\mathcal{A}^C_2}^{s,f,w}(\mathbb{F}_2[\tau], H_\ast, \ast(C^C_{(\eta_1, 1)}))$. Even a rough estimate of the vanishing region using the $E_1$-page of the $\mathcal{C}$-motivic May spectral sequence leads to Proposition 4.15. Such an approach would avoid explicit calculations of $\text{Ext}_{\mathcal{A}^e}$ as in [DI1] and [BI].
Proof of Theorem 1.11. Since Proposition 4.15 $\implies$ Proposition 4.13, every map
\[ v : \Sigma^{2,1}Y \to Y \]
detected by $\pi$ of (4.10) is a nonzero permanent cycle. In order to finish the proof of Theorem 1.11 we must show that $v$ is necessarily $v_{(1,\text{nil})}$–self-map of periodicity 1. It is easy to see that the underlying map
\[ \Phi^e(\beta(v)) : \Sigma^2Y \to Y \]
is a $v_1$–self-map of periodicity 1 as
\[ C(\Phi^e(\beta(v))) \simeq \Phi^e(\beta(C^R(v))) \simeq A_1[00] \]
is of type 1 (see Remark 3.9). On the other hand,
\[ \Phi^e(\beta(v)) : \Sigma^2(M_2(1) \vee M_2(1)) \to \Sigma M_2(1) \vee M_2(1) \]
is necessarily a nilpotent map because of [HS, Theorem 3(ii)] and the fact that a $v_1$–self-map of $M_2(1)$ has periodicity at least 4 (see [DM] for details) which lives in $[M_2(1), M_2(1)]_{sk}$ for $k \geq 1$. □

Proof of Theorem 1.14. Since $v$ is a $v_{(1,\text{nil})}$–self-map and $Y_{(h,1)}$ is of type $(1,1)$, it follows that $C^R(v)$ is of type $(2, 1)$. Moreover,
\[ H^{*,*}(C^R(v)) \cong H^{*,*}(A^R_1) \]
as $v$ is detected by $\pi$ of (4.10) in the $E_2$-page of the Adams spectral sequence. Thus, $H^{*,*}(C^R(v))$ is a free $A^R_{1}$-module on single generator. □

Remark 4.18. It is likely that realizing a different $A^R$-module structure on $A^R_{1}$ as a spectrum (see also Remark 3.21) may lead to a 1-periodic $v_1$–self-map on $Y_{(2,1)}$ as well as on $Y_{(2,1)}^{C^2}$. We explore such possibilities in upcoming work.

5. Nonexistence of $v_{1,0}$–self-map on $C^R(h)$ and $Y_{(h,0)}^{R}$

Let $X$ be a finite $R$-motivic spectrum and let $f : \Sigma^{i-j}X \to X$ be a map such that
\[ \Phi^e(\beta(f)) : \Sigma^{i-j} \Phi^e(\beta(X)) \to \Phi^e(\beta(X)) \]
is a $v_0$–self-map. Then it must be the case that $i = j$, as $v_0$–self-maps preserve dimension. Note that both $C^R(h)$ and $Y_{(h,0)}^{R}$ are of type $(1,0)$.

Proposition 5.1. The $v_1$–self-maps of $M_2(1)$ are not in the image of the underlying homomorphism
\[ \Phi^e \circ \beta : [\Sigma^{8k} C^R(h), C^R(h)] \to [\Sigma^{8k} M_2(1), M_2(1)]. \]

Proof. The minimal periodicity of a $v_1$–self-map of $M_2(1)$ is 4. Let $v : \Sigma^{8k} M_2(1) \to M_2(1)$ be a 4k-periodic $v_1$–self-map. It is well-known that the composite
\[ (5.2) \quad \Sigma^{8k}S \to \Sigma^{8k} M_2(1) \to M_2(1) \to \Sigma^1S \]
is not null (and equals $P^{k-1}(8 \sigma)$ where $P$ is a periodic operator given by the Toda bracket $(\sigma, 16, -)$.)

Suppose there exists $f : \Sigma^{8k,8k}C^R(h) \to C^R(h)$ such that $\Phi^e \circ \beta(f) = \nu$. Then (5.2) implies that the composition

\begin{equation}
\Sigma^{8k,8k}C^R \xrightarrow{\nu} \Sigma^{8k,8k}C^R(h) \xrightarrow{\nu} C^R(h) \xrightarrow{} \Sigma^{1,0}S
\end{equation}

is nonzero as the functor $\Phi^e \circ \beta$ is additive. The composite of the maps in (5.3) is a nonzero element of $\pi_* (S_R)$ in negative co-weight. This contradicts the fact that $\pi_*(S_R)$ is trivial in negative co-weights [DI1].

**Proposition 5.4.** The $v_1$--self-maps of $\mathcal{Y}$ are not in the image of the underlying homomorphism

$$\Phi^e \circ \beta : [\Sigma^{2k,2k}Y_{[h,0]}, Y_{[h,0]}]^R \to [\Sigma^{8k}Y, \mathcal{Y}].$$

**Proof.** Let $\nu : \Sigma^{2k}Y \to Y$ denote a $v_1$--self-map of periodicity $k$. Notice that the composite

\begin{equation}
\Sigma^{2k} \xrightarrow{\nu} \Sigma^{2k}Y \xrightarrow{\nu} Y \xrightarrow{} Y_{\geq 1}
\end{equation}

where $Y_{\geq 1}$ is the first coskeleton, must be nonzero. If not, then $\nu$ factors through the bottom cell resulting in a map $S^{2k} \to \Sigma^{2k}Y \to S$ which induces an isomorphism in $K(1)$-homology, contradicting the fact that $S$ is of type 0.

If $f : \Sigma^{2k,2k}Y_{[h,0]} \to Y_{[h,0]}$ were a map such that $\Phi^e \circ \beta(f) = \nu$, then (5.5) would force one among the hypothetical composites (A), (B) or (C) in the diagram

$$\begin{array}{ccc}
\Sigma^{2k,2k}S_R & \xrightarrow{\nu} & \Sigma^{2k,2k}Y_{[h,0]} \\
& \downarrow & \\
& \text{Fib}(p_3) & \xrightarrow{p_3} \Sigma^{1,0}S_R
\end{array} \quad \begin{array}{ccc}
\Sigma^{2k,2k}Y_{[h,0]} & \xrightarrow{\nu} & Y_{[h,0]} \\
& \downarrow & \\
& \text{Fib}(p_2) & \xrightarrow{p_2} \Sigma^{2,0}S_R
\end{array} \quad \begin{array}{ccc}
\Sigma^{2k,2k}Y_{[h,0]} & \xrightarrow{\nu} & \Sigma^{3,0}S_R \\
& \downarrow & \\
& \text{Fib}(p_1) & \xrightarrow{p_1} \Sigma^{1,0}S_R
\end{array}
$$

(A) (B) (C)

to exist as a nonzero map, thereby contradicting the fact that $\pi_*(S_R)$ is trivial in negative co-weights. \hfill \Box

**Remark 5.6.** The above results do not preclude the existence of a $v_{1,0}$--self-map on $C^{C_2}(h)$ and $Y_{(h,0)}^{C_2}$. Forthcoming work [GI2] of the second author and Isaksen shows that $8 \sigma$ is in the image of $\Phi^e : \pi_7(S_{C_2}) \to \pi_7(S)$ and suggests that $C^{C_2}(h)$ supports a $v_{1,0}$--self-map.

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