Enumeration of $k$-Fibonacci Paths Using Infinite Weighted Automata

Rodrigo De Castro¹ and José L. Ramírez¹,²

1. Departamento de Matemáticas, Universidad Nacional de Colombia, AA 14490, Bogotá, Colombia
2. Instituto de Matemáticas y sus Aplicaciones, Universidad Sergio Arboleda, Bogotá, Colombia

E-mail: rdcastrok@unal.edu.co, jlramirezr@unal.edu.co

Abstract: In this paper, we introduce a new family of generalized colored Motzkin paths, where horizontal steps are colored by means of $F_{k,l}$ colors, where $F_{k,l}$ is the $l$-th $k$-Fibonacci number. We study the enumeration of this family according to the length. For this, we use infinite weighted automata.

Key Words: Fibonacci sequence, Generalized colored Motzkin path, $k$-Fibonacci path, infinite weighted automata, generating function.

AMS(2010): 52B05, 11B39, 05A15

§1. Introduction

A lattice path of length $n$ is a sequence of points $P_1, P_2, \ldots, P_n$ with $n \geq 1$ such that each point $P_i$ belongs to the plane integer lattice and each two consecutive points $P_i$ and $P_{i+1}$ connect by a line segment. We will consider lattice paths in $\mathbb{Z} \times \mathbb{Z}$ using three step types: a rise step $U = (1, 1)$, a fall step $D = (1, -1)$ and a $F_{k,l}$-colored length horizontal step $H_l = (l, 0)$ for every positive integer $l$, such that $H_l$ is colored by means of $F_{k,l}$ colors, where $F_{k,l}$ is the $l$-th $k$-Fibonacci number.

Many kinds of generalizations of the Fibonacci numbers have been presented in the literature [10,11] and the corresponding references. Such as those of $k$-Fibonacci numbers $F_{k,n}$ and the $k$-Smarandache-Fibonacci numbers $S_{k,n}$. For any positive integer number $k$, the $k$-Fibonacci sequence, say $\{F_{k,n}\}_{n \in \mathbb{N}}$, is defined recurrently by

$$F_{k,0} = 0, \quad F_{k,1} = 1, \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad \text{for } n \geq 1.$$ 

The generating function of the $k$-Fibonacci numbers is $f_k(x) = \frac{x}{1 - kx - x^2}$, [4,6]. This sequence was studied by Horadam in [9]. Recently, Falcón and Plaza [6] found the $k$-Fibonacci numbers by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge ($4TLE$) partition. The interested reader is also referred to [1, 3, 4, 5, 6, 12, 13, 16] for further information about this.

¹Received November 14, 2013, Accepted May 20, 2014.
A generalized $F_{k,l}$-colored Motzkin path or simply $k$-Fibonacci path is a sequence of rise, fall and $F_{k,l}$-colored length horizontal steps ($l = 1, 2, \cdots$) running from $(0, 0)$ to $(n, 0)$ that never pass below the $x$-axis. We denote by $\mathcal{M}_{F_{k,n}}$ the set of all $k$-Fibonacci paths of length $n$ and $\mathcal{M}_k = \bigcup_{n=0}^{\infty} \mathcal{M}_{F_{k,n}}$. In Figure 1 we show the set $\mathcal{M}_{F_{2,3}}$.

Figure 1 $k$-Fibonacci Paths of length 3, $|\mathcal{M}_{F_{2,3}}| = 13$

A grand $k$-Fibonacci path is a $k$-Fibonacci path without the condition that never going below the $x$-axis. We denote by $\mathcal{M}^*_F_{k,n}$ the set of all grand $k$-Fibonacci paths of length $n$ and $\mathcal{M}^*_k = \bigcup_{n=0}^{\infty} \mathcal{M}^*_F_{k,n}$. Analogously, we have the family of prefix grand $k$-Fibonacci paths. We denote by $\mathcal{P}M^*_F_{k,n}$ the set of all prefix grand $k$-Fibonacci paths of length $n$ and $\mathcal{P}M^*_k = \bigcup_{n=0}^{\infty} \mathcal{P}M^*_F_{k,n}$.

In this paper, we study the generating function for the $k$-Fibonacci paths, grand $k$-Fibonacci paths, prefix $k$-Fibonacci paths, and prefix grand $k$-Fibonacci paths, according to the length. We use Counting Automata Methodology (CAM) [2], which is a variation of the methodology developed by Rutten [14] called Coinductive Counting. Counting Automata Methodology uses infinite weighted automata, weighted graphs and continued fractions. The main idea of this methodology is find a counting automaton such that there exist a bijection between all words recognized by an automaton $\mathcal{M}$ and the family of combinatorial objects. From the counting automaton $\mathcal{M}$ is possible find the ordinary generating function (GF) of the family of combinatorial objects [4].

§2. Counting Automata Methodology

The terminology and notation are mainly those of Sakarovitch [13]. An automaton $\mathcal{M}$ is a 5-tuple $\mathcal{M} = (\Sigma, Q, q_0, F, E)$, where $\Sigma$ is a nonempty input alphabet, $Q$ is a nonempty set of states of $\mathcal{M}$, $q_0 \in Q$ is the initial state of $\mathcal{M}$, $\emptyset \neq F \subseteq Q$ is the set of final states of $\mathcal{M}$ and $E \subseteq Q \times \Sigma \times Q$ is the set of transitions of $\mathcal{M}$. The language recognized by an automaton $\mathcal{M}$ is denoted by $L(\mathcal{M})$. If $Q, \Sigma$ and $E$ are finite sets, we say that $\mathcal{M}$ is a finite automaton [15].

Example 2.1 Consider the finite automaton $\mathcal{M} = (\Sigma, Q, q_0, F, E)$ where $\Sigma = \{a, b\}$, $Q = \{q_0, q_1\}$, $F = \{q_0\}$ and $E = \{(q_0, a, q_1), (q_0, b, q_0), (q_1, a, q_0)\}$. The transition diagram of $\mathcal{M}$ is as shown in Figure 2. It is easy to verify that $L(\mathcal{M}) = (b \cup aa)^*$. 
Example 2.2 Consider the infinite automaton $M_D = (\Sigma, Q, q_0, F, E)$, where $\Sigma = \{a, b\}$, $Q = \{q_0, q_1, \cdots\}$, $F = \{q_0\}$ and $E = \{(q_i, a, q_{i+1}), (q_{i+1}, b, q_i) : i \in \mathbb{N}\}$. The transition diagram of $M_D$ is as shown in Figure 3.

![Figure 3](transition_diagram_md.png)

The language accepted by $M_D$ is

$L(M_D) = \{w \in \Sigma^* : |w|_a = |w|_b \text{ and for all prefix } v \text{ of } w, |v|_b \leq |v|_a\}$.

An ordinary generating function $F = \sum_{n=0}^{\infty} f_n z^n$ corresponds to a formal language $L$ if $f_n = |\{w \in L : |w| = n\}|$, i.e., if the $n$-th coefficient $f_n$ gives the number of words in $L$ with length $n$.

Given an alphabet $\Sigma$ and a semiring $\mathbb{K}$. A formal power series or formal series $S$ is a function $S : \Sigma^* \rightarrow \mathbb{K}$. The image of a word $w$ under $S$ is called the coefficient of $w$ in $S$ and is denoted by $s_w$. The series $S$ is written as a formal sum $S = \sum_{w \in \Sigma^*} s_w w$. The set of formal power series over $\Sigma$ with coefficients in $\mathbb{K}$ is denoted by $\mathbb{K}\langle\langle \Sigma^*\rangle\rangle$.

An automaton over $\Sigma^*$ with weights in $\mathbb{K}$, or $\mathbb{K}$-automaton over $\Sigma^*$, is a graph labelled with elements of $\mathbb{K}\langle\langle \Sigma^*\rangle\rangle$, associated with two maps from the set of vertices to $\mathbb{K}\langle\langle \Sigma^*\rangle\rangle$. Specifically, a weighted automaton $M$ over $\Sigma^*$ with weights in $\mathbb{K}$ is a 4-tuple $M = (Q, I, E, F)$ where $Q$ is a nonempty set of states of $M$, $E$ is an element of $\mathbb{K}\langle\langle \Sigma^*\rangle\rangle^{Q \times Q}$ called transition matrix. $I$ is an element of $\mathbb{K}\langle\langle \Sigma^*\rangle\rangle^Q$, i.e., $I$ is a function from $Q$ to $\mathbb{K}\langle\langle \Sigma^*\rangle\rangle$. $I$ is the initial function of $M$ and can also be seen as a row vector of dimension $Q$, called initial vector of $M$ and $F$ is an element of $\mathbb{K}\langle\langle \Sigma^*\rangle\rangle^Q$. $F$ is the final function of $M$ and can also be seen as a column vector of dimension $Q$, called final vector of $M$.

We say that $M$ is a counting automaton if $\mathbb{K} = \mathbb{Z}$ and $\Sigma^* = \{z\}^*$. With each automaton, we can associate a counting automaton. It can be obtained from a given automaton replacing every transition labelled with a symbol $a$, $a \in \Sigma$, by a transition labelled with $z$. This transition is called a counting transition and the graph is called a counting automaton of $M$. Each transition
from $p$ to $q$ yields an equation

\[ L(p)(z) = zL(q)(z) + [p \in F] + \cdots. \]

We use $L_p$ to denote $L(p)(z)$. We also use Iverson’s notation, $[P] = 1$ if the proposition $P$ is true and $[P] = 0$ if $P$ is false.

### 2.1 Convergent Automata and Convergent Theorems

We denote by $L^{(n)}(M)$ the number of words of length $n$ recognized by the automaton $M$, including repetitions.

**Definition 2.3** We say that an automaton $M$ is convergent if for all integer $n \geq 0$, $L^{(n)}(M)$ is finite.

The proof of following theorems and propositions can be found in [2].

**Theorem 2.4** (First Convergence Theorem) Let $M$ be an automaton such that each vertex (state) of the counting automaton of $M$ has finite degree. Then $M$ is convergent.

**Example 2.5** The counting automaton of the automaton $M_D$ in Example 2 is convergent.

The following definition plays an important role in the development of applications because it allows to simplify counting automata whose transitions are formal series.

**Definition 2.6** Let $M$ be an automaton, and let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be a formal power series with $f_n \in \mathbb{N}$ for all $n \geq 0$ and $f_0 = 0$. In a counting automaton of $M$ the set of counting transitions from state $p$ to state $q$, without intermediate final states, see Figure 4 (left), is represented by a graph with a single edge labeled by $f(z)$, see Figure 4 (right).

![Transition Graph](image)

**Figure 4** Transitions from the state $p$ to $q$ and its transition in parallel
This kind of transition is called a transition in parallel. The states $p$ and $q$ are called visible states and the intermediate states are called hidden states.

**Example 2.7** In Figure 5 (left) we display a counting automaton $M_1$ without transitions in parallel, i.e., every transition is label by $z$. The transitions from state $q_1$ to $q_2$ correspond to the series \( \frac{1 - \sqrt{1 - 4z}}{2} = z + z^2 + 2z^3 + 5z^4 + 14z^5 + \cdots \). However, this automaton can also be represented using transitions in parallel. Figure 5 (right) displays two examples.

**Figure 5** Counting automata with transitions in parallel

**Theorem 2.8 (Second Convergence Theorem)** Let $M$ be an automaton, and let $f_{q_1}^q(z), f_{q_2}^q(z), \cdots$, be transitions in parallel from state $q \in Q$ in a counting automaton of $M$. Then $M$ is convergent if the series

\[
F^q(z) = \sum_{k=1}^{\infty} f_k^q(z)
\]

is a convergent series for each visible state $q \in Q$ of the counting automaton.

**Proposition 2.9** If $f(z)$ is a polynomial transition in parallel from state $p$ to $q$ in a finite counting automaton $M$, then this gives rise to an equation in the system of GFs equations of $M$

\[
L_p = f(z)L_q + [p \in F] + \cdots
\]

**Proposition 2.10** Let $M$ be a convergent automaton such that a counting automaton of $M$ has a finite number of visible states $q_0, q_1, \cdots, q_r$, in which the number of transitions in parallel starting from each state is finite. Let $f_{q_0}^{q_1}(z), f_{q_0}^{q_2}(z), \cdots, f_{q_0}^{q_r}(z)$ be the transitions in parallel from the state $q_i \in Q$. Then the GF for the language $L(M)$ is $L_{q_0}(z)$. It is obtained by solving
the system of $r + 1$ GFs equations

$$L(q_t)(z) = f^q_t(z)L(q_{t_1})(z) + f^q_t(z)L(q_{t_2})(z) + \cdots + f^q_{t(t)}(z)L(q_{t_{t(t)}})(z) + [q_t \in F]$$

with $0 \leq t \leq r$, where $q_{t_k}$ is the visible state joined with $q_t$ through the transition in parallel $f^q_t$, and $L(q_{t_k})$ is the GF for the language accepted by $M$ if $q_{t_k}$ is the initial state.

**Example 2.11** The system of GFs equations associated with $M_2$, see Example 2.7, is

$$\begin{align*}
L_0 &= (2z + z^2)L_1 + 1 \\
L_1 &= \frac{1 - \sqrt{1 - 4z}}{2}L_2 \\
L_2 &= 2zL_0.
\end{align*}$$

Solving the system for $L_0$, we find the GF for the language $M_2$ and therefore of $M_1$ and $M_3$

$$L_0 = \frac{1}{1 - (2z^2 + z^3)(1 - \sqrt{1 - 4z})} = 1 + 4z^3 + 6z^4 + 10z^5 + 40z^6 + 114z^7 + \cdots .$$

### 2.2 An Example of the Counting Automata Methodology (CAM)

A counting automaton associated with an automaton $M$ can be used to model combinatorial objects if there is a bijection between all words recognized by the automaton $M$ and the combinatorial objects. Such method, along with the previous theorems and propositions constitute the **Counting Automata Methodology (CAM)**, see [2].

We distinguish three phases in the CAM:

1. Given a problem of enumerative combinatorics, we have to find a convergent automaton $M$ (see Theorems 2.4 and 2.8), whose GF is the solution of the problem.
2. Find a general formula for the GF of $M'$, where $M'$ is an automaton obtained from $M$ truncating a set of states or edges see Propositions 2.9 and 2.10. Sometimes we find a relation of iterative type, such as a continued fraction.
3. Find the GF $f(z)$ to which converge the GFs associated to each $M'$, which is guaranteed by the convergences theorems.

**Example 2.12** A Motzkin path of length $n$ is a lattice path of $\mathbb{Z} \times \mathbb{Z}$ running from $(0, 0)$ to $(n, 0)$ that never passes below the $x$-axis and whose permitted steps are the up diagonal step $U = (1, 1)$, the down diagonal step $D = (1, -1)$ and the horizontal step $H = (1, 0)$. The number of Motzkin paths of length $n$ is the $n$-th **Motzkin number** $m_n$, sequence A001006\(^1\). The number of words of length $n$ recognized by the convergent automaton $M_{Mot}$, see Figure 6, is the $n$th Motzkin number and its GF is

$$M(z) = \sum_{i=0}^{\infty} m_i z^i = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}.$$  

\(^1\)Many integer sequences and their properties are found electronically on the On-Line Encyclopedia of Sequences [17].
Figure 6 Convergent automaton associated with Motzkin paths

In this case the edge from state $q_i$ to state $q_{i+1}$ represents a rise, the edge from the state $q_{i+1}$ to $q_i$ represents a fall and the loops represent the level steps, see Table 1.

| (qi, z, qi+1) ∈ E ⇔ | (qi+1, z, qi) ∈ E ⇔ | (qi, z, qi) ∈ E ⇔ |

| Table 1 Bijection between $\mathcal{M}_{\text{Mot}}$ and Motzkin paths |

Moreover, it is clear that a word is recognized by $\mathcal{M}_{\text{Mot}}$ if and only if the number of steps to the right and to the left coincide, which ensures that the path is well formed. Then

$m_n = |\{w \in L(\mathcal{M}_{\text{Mot}}) : |w| = n\}| = L^{(n)}(\mathcal{M}_{\text{Mot}})$.

Let $\mathcal{M}_{\text{Mot}_s}$, $s \geq 1$ be the automaton obtained from $\mathcal{M}_{\text{Mot}}$, by deleting the states $q_{s+1}, q_{s+2}, \ldots$. Therefore the system of GFs equations of $\mathcal{M}_{\text{Mot}_s}$ is

\[
\begin{align*}
L_0 &= zL_0 + zL_1 + 1, \\
L_i &= zL_{i-1} + zL_i + zL_{i+1}, \quad 1 \leq i \leq s - 1, \\
L_s &= zL_{s-1} + zL_s.
\end{align*}
\]

Substituting repeatedly into each equation $L_i$, we have

\[
L_0 = \left. \frac{H}{1 - \frac{F^2}{1 - \frac{F^2}{\ddots}} - \frac{1}{1 - F^2}} \right|_{s \text{ times}},
\]

where $F = \frac{z}{1 - z}$ and $H = \frac{1}{1 - \frac{1}{z}}$. Since $\mathcal{M}_{\text{Mot}}$ is convergent, then as $s \to \infty$ we obtain a convergent continued fraction $M$ of the GF of $\mathcal{M}_{\text{Mot}}$. Moreover,

\[
M = \frac{H}{1 - F^2 \left( \frac{4H}{H} \right)}
\]
Hence $z^2M^2 - (1 - z)M + 1 = 0$ and

$$M(z) = \frac{1 - z \pm \sqrt{1 - 2z - 3z^2}}{2z^2}.$$ 

Since $\epsilon \in L(\mathcal{M}_{\text{Mot}})$, $M \to 0$ as $z \to 0$. Hence, we take the negative sign for the radical in $M(z)$.

§3. Generating Function for the $k$-Fibonacci Paths

In this section we find the generating function for $k$-Fibonacci paths, grand $k$-Fibonacci paths, prefix $k$-Fibonacci paths and prefix grand $k$-Fibonacci paths, according to the length.

**Lemma 3.1** (2) The GF of the automaton $\mathcal{M}_{\text{Lin}}$, see Figure 7, is

$$E(z) = \frac{1}{1 - h_0(z) - \frac{f_0(z) g_0(z)}{1 - h_1(z) - \frac{f_1(z) g_1(z)}{\ddots}}}$$

where $f_i(z), g_i(z)$ and $h_i(z)$ are transitions in parallel for all integer $i \geq 0$.

![Figure 7 Linear infinite counting automaton $\mathcal{M}_{\text{Lin}}$](image)

The last lemma coincides with Theorem 1 in [7] and Theorem 9.1 in [14]. However, this presentation extends their applications, taking into account that $f_i(z), g_i(z)$ and $h_i(z)$ are GFs, which can be GFs of several variables.

**Corollary 3.2** If for all integers $i \geq 0$, $f_i(z) = f(z), g_i(z) = g(z)$ and $h_i(z) = h(z)$ in $\mathcal{M}_{\text{Lin}}$, then the GF is

$$B(z) = \frac{1 - h(z) - \sqrt{(1 - h(z))^2 - 4f(z)g(z)}}{2f(z)g(z)}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n \binom{m + 2n}{m} (f(z)g(z))^n (h(z))^m$$

$$= \frac{1}{1 - h(z) - \frac{f(z) g(z)}{1 - h(z) - \frac{f(z) g(z)}{\ddots}}}.$$
where $C_n$ is the $n$th Catalan number, sequence A000108.

**Theorem 3.3** The generating function for the $k$-Fibonacci paths according to their length is

\[
T_k(z) = \sum_{i=0}^{\infty} |M_{F_{k,i}}| z^i
\]

\[
= 1 - (k + 1)z - z^2 - \sqrt{(1 - (k + 1)z - z^2)^2 - 4z^2(1 - k z - z^2)^2}
\]

\[
= \frac{1 - \frac{z}{1-kz-z^2} - \frac{z^2}{1-kz-z^2}}{1 - \frac{z}{1-kz-z^2} - \frac{z^2}{1-kz-z^2}}
\]

and

\[
[z^t] T_k(z) = \sum_{n=0}^{t-2n} \sum_{m=0}^{t} \binom{m + 2n}{m} C_n F_{k,t-2n-m+1},
\]

where $C_n$ is the $n$-th Catalan number and $F_{k,j}^{(r)}$ is a convolved $k$-Fibonacci number.

Convolved $k$-Fibonacci numbers $F_{k,j}^{(r)}$ are defined by

\[
f_k^{(r)}(x) = (1 - kx - x^2)^{-r} = \sum_{j=0}^{\infty} F_{k,j+1}^{(r)} x^j, \quad r \in \mathbb{Z}^+.
\]

Note that

\[
F_{k,m+1}^{(r)} = \sum_{j_1 + j_2 + \cdots + j_r = m} F_{k,j_1+1} F_{k,j_2+1} \cdots F_{k,j_r+1}.
\]

Moreover, using a result of Gould[8, p.699] on Humbert polynomials (with $n = j, m = 2, x = k/2, y = -1, p = -r$ and $C = 1$), we have

\[
F_{k,j+1}^{(r)} = \sum_{l=0}^{[j/2]} \binom{j + r - l - 1}{j - l} \binom{j - l}{l} k^{j-2l}.
\]

Ramírez [13] studied some properties of convolved $k$-Fibonacci numbers.

**Proof** Equations (5) and (6) are clear from Corollary 3.2 taking $f(z) = z = g(z)$ and $h(z) = \frac{z}{1-kz-z^2}$. Note that $h(z)$ is the GF of $k$-Fibonacci numbers. In this case the edge from state $q_i$ to state $q_{i+1}$ represents a rise, the edge from the state $q_{i+1}$ to $q_i$ represents a fall and the loops represent the $F_{k,l}$-colored length horizontal steps ($l = 1, 2, \cdots$). Moreover, from
Equation (2), we obtain
\[
T_k(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n \binom{m+2n}{m} z^{2n} \left( \frac{z}{1-kz-z^2} \right)^m
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n \binom{m+2n}{m} z^{2n+m} \left( \frac{1}{1-kz-z^2} \right)^m
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n \binom{m+2n}{m} z^{2n+m} \sum_{i=0}^{\infty} F_{k,i+1}^{(m)} z^i
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} C_n F_{k,i+1}^{(m)} \binom{m+2n}{m} z^{2n+m+i},
\]

taking \( s = 2n + m + i \)
\[
T_k(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=2n+m}^{\infty} C_n F_{k,s-2n-m+1}^{(m)} \binom{m+2n}{m} z^s.
\]

Hence
\[
[z^t] T_k(z) = \sum_{n=0}^{t} \sum_{m=0}^{t-2m} C_n F_{k,t-2n-m+1}^{(m)} \binom{m+2n}{m}.
\]

In Table 2 we show the first terms of the sequence \(|\mathcal{M}_{F_k}|\) for \( k = 1, 2, 3, 4 \).

| \( k \) | Sequence |
|---|---|
| 1 | 1, 1, 3, 8, 23, 67, 199, 600, 1834, 5674, 17743, \ldots |
| 2 | 1, 1, 4, 13, 47, 168, 610, 2226, 8185, 30283, 112736, \ldots |
| 3 | 1, 1, 5, 20, 89, 391, 1735, 7712, 34402, 153898, 690499, \ldots |
| 4 | 1, 1, 6, 29, 155, 820, 4366, 23262, 124153, 663523, 3551158, \ldots |

**Table 2** Sequences \(|\mathcal{M}_{F_k}|\) for \( k = 1, 2, 3, 4 \)

**Definition 3.4** For all integers \( i \geq 0 \) we define the continued fraction \( E_i(z) \) by:
\[
E_i(z) = \frac{1}{1 - h_i(z) - \frac{f_i(z) g_i(z)}{1 - h_{i+1}(z) - \frac{f_{i+1}(z) g_{i+1}(z)}{1 - h_{i+2}(z) - \ddots}}},
\]
where \( f_i(z), g_i(z), h_i(z) \) are transitions in parallel for all integers positive \( i \).
**Lemma 3.5** ([2]) The GF of the automaton \( M_{BLin} \), see Figure 8, is

\[
E_b(z) = \frac{1}{1 - h_0(z) - f_0(z)g_0(z)E_1(z) - f_0'(z)g_0'(z)E'_1(z)},
\]

where \( f_i(z), f'_i(z), g_i(z), g'_i(z), h_i(z) \) and \( h'_i(z) \) are transitions in parallel for all \( i \in \mathbb{Z} \).

\[ \begin{array}{c}
\cdot \cdot \cdot \\
, \\
\end{array} \]

**Figure 8** Linear infinite counting automaton \( M_{BLin} \)

**Corollary 3.6** If for all integers \( i \), \( f_i(z) = f(z) = f'_i(z), g_i(z) = g(z) = g'_i(z) \) and \( h_i(z) = h(z) = h'_i(z) \) in \( M_{BLin} \), then the GF

\[
B_b(z) = \frac{1}{\sqrt{(1 - h(z))^2 - 4f(z)g(z)}}
\]

\[
= \frac{1}{1 - h(z) - \frac{2f(z)g(z)}{1 - h(z) - \frac{f(z)g(z)}{1 - h(z) - \frac{f(z)g(z)}{\ddots}}}}.
\]

where \( f(z), g(z) \) and \( h(z) \) are transitions in parallel. Moreover, if \( f(z) = g(z) \), then the GF

\[
B_b(z) = \frac{1}{1 - h(z)} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{2^n}{n + 2k} \binom{n + 2k}{k} \left( \frac{l + 2n + 2k}{l} \right) f(z)^{2n+2k} h(z)^l.
\]

**Theorem 3.7** The generating function for the grand \( k \)-Fibonacci paths according to their length is

\[
T^*_k(z) = \sum_{i=0}^{\infty} |M^*_{F_{k,i}}| z^i = \frac{1 - k z - z^2}{\sqrt{(1 - (k + 1) z - z^2)^2 - 4z^2(1 - k z - z^2)^2}}
\]

\[
= \frac{1}{1 - \frac{z}{1 - k z - z^2} - \frac{2z^2}{1 - \frac{z}{1 - k z - z^2} - \frac{z^2}{1 - \frac{z}{1 - k z - z^2} - \frac{z^2}{\ddots}}}}.
\]
and

\[
[z^t] T_k^* (z) = F_{k+1, t}^{(1)} + \sum_{n=1}^{t} \sum_{m=0}^{t-2} \sum_{l=0}^{2m} 2^n \frac{n}{n+2m} \binom{n+2m}{m} \binom{l+2m+2}{l} F_{k, t-2m-2m-l+1}^{(l)} ,
\]

(12)

with \( t \geq 1 \).

**Proof** Equations (10) and (11) are clear from Corollary 3.6, taking \( f(z) = z = g(z) \) and \( h(z) = \frac{z}{1-kz-z^2} \). Moreover, from Equation (9), we obtain

\[
T_k^* (z) = \frac{1}{1-kz-z^2} + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} 2^n \frac{n}{n+2m} \binom{n+2m}{m} \binom{l+2m+2}{l} z^{2n+2m} \left( \frac{z}{1-kz-z^2} \right)^{l},
\]

\[
= 1 + \sum_{j=0}^{\infty} F_{k+1, j}^{(1)} z^j + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} 2^n \frac{n}{n+2m} \binom{n+2m}{m} \binom{l+2m+2}{l} F_{k, u}^{(l)} z^{2n+2m+u+1},
\]

taking \( s = 2n + 2m + l + u \)

\[
T_k^* (z) = 1 + \sum_{j=0}^{\infty} F_{k+1, j}^{(1)} z^j + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} 2^n \frac{n}{n+2m} \binom{n+2m}{m} \binom{l+2m+2}{l} F_{k, s}^{(l)} z^{2n+2m+2m-l}.
\]

Therefore, Equation (12) is clear. \( \square \)

In Table 3 we show the first terms of the sequence \( |M^{*}_{F_k, i}| \) for \( k = 1, 2, 3, 4 \).

| \( k \) | Sequence |
|---|---|
| 1 | 1, 4, 11, 36, 115, 378, 1251, 4182, 14073, 47634, \cdots |
| 2 | 1, 5, 16, 63, 237, 920, 3573, 14005, 55156, 218359, \cdots |
| 3 | 1, 6, 23, 108, 487, 2248, 10371, 48122, 223977, 1046120, \cdots |
| 4 | 1, 7, 32, 177, 949, 5172, 28173, 153963, 842940, 4624581, \cdots |

**Table 3** Sequences \( |M^{*}_{F_k, i}| \) for \( k = 1, 2, 3, 4 \) and \( i \geq 1 \)

In Figure 9 we show the set \( M^{*}_{F_{2,3}} \).

![Figure 9](image_url)

**Figure 9** Grand \( k \)-Fibonacci Paths of length 3, \( |M^{*}_{F_{2,3}}| = 16 \)
Lemma 3.8([2]) The GF of the automaton FIN\(_n\)(\(\mathcal{M}_{\text{Lin}}\)), see Figure 10, is

\[
G(z) = E(z) + \sum_{j=1}^{\infty} \left( \prod_{i=0}^{j-1} (f_i(z)E_i(z))E_j(z) \right),
\]

where \(E(z)\) is the GF in Lemma 3.1.

\[\text{Figure 10} \quad \text{Linear infinite counting automaton } \text{FIN}_n(\mathcal{M}_{\text{Lin}})\]

Corollary 3.9 If for all integer \(i \geq 0\), \(f_i(z) = f(z)\), \(g_i(z) = g(z)\) and \(h_i(z) = h(z)\) in \(\text{FIN}_n(\mathcal{M}_{\text{Lin}})\), then the GF is:

\[
G(z) = \frac{1 - 2f(z) - h(z) - \sqrt{(1-h(z))^2 - 4f(z)g(z)}}{2f(z)(f(z) + g(z) + h(z) - 1)} \quad (13)
\]

\[
= \frac{1}{1 - f(z) - h(z) - \frac{f(z)g(z)}{1 - h(z)} - \frac{f(z)g(z)}{1 - h(z)} - \frac{f(z)g(z)}{1 - h(z)} - \cdots} \quad (14)
\]

where \(f(z)\), \(g(z)\) and \(h(z)\) are transitions in parallel and \(B(z)\) is the GF in Corollary 3.2. Moreover, if \(f(z) = g(z)\) and \(h(z) \neq 0\), then we obtain the GF

\[
G(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{n + 2k + 1}{n + k + 1} \binom{n + 2k + l}{k, l, k + n} f^{2k+n}(z)h^l(z). \quad (15)
\]

Theorem 3.10 The generating function for the prefix \(k\)-Fibonacci paths according to the their length is

\[
PT_k(z) = \sum_{i=0}^{\infty} |\mathcal{P}\mathcal{M}_{F_{k,i}}|z^i
\]

\[= \frac{(1 - 2z)(1 - kz - z^2) - z - \sqrt{(1 - z(k + 1) - z^2)^2 + 4z^2(1 - kz - z^2)^2}}{2z((1 - kz - z^2)(2z - 1) + z)}
\]
and

\[ [z^t] PT_k(z) = \sum_{n=0}^{t} \sum_{m=0}^{t} \sum_{l=0}^{t-2m-n} \frac{n+1}{n+m+1} \binom{n+2m+l}{m,l,m+n} F_{k,t-2m-n-l+1}^{(l)} \quad t \geq 0. \]

**Proof** The proof is analogous to the proof of Theorem 3.3 and 3.7. \qed

In Table 4 we show the first terms of the sequence \(|PM_{F_k,i}|\) for \(k = 1, 2, 3, 4\).

| \(k\) | Sequence |
|------|---------|
| 1    | 1, 2, 6, 19, 62, 205, 684, 2298, 7764, 26355, 89820, \cdots |
| 2    | 1, 2, 7, 26, 101, 396, 1564, 6203, 24693, 98605, 394853, \cdots |
| 3    | 1, 2, 8, 35, 162, 757, 3558, 16766, 79176, 374579, 1775082, \cdots |
| 4    | 1, 2, 9, 46, 251, 1384, 7668, 42555, 236463, 1315281, 7322967, \cdots |

**Table 4** Sequences \(|PM_{F_k,i}|\) for \(k = 1, 2, 3, 4\)

In Figure 11 we show the set \(\mathcal{PM}_{F_2,3}\).

**Figure 11** Prefix \(k\)-Fibonacci paths of length 3, \(|\mathcal{PM}_{F_2,3}| = 26\)

**Lemma 3.11** The GF of the automaton \(\text{FIN}_{Z}(\mathcal{M}_{BLin})\), see Figure 12, is

\[
H(z) = \frac{E E'}{E + E' - EE'(1 - h_0)} \left(1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j-1} E_k f_k f_0 E_j + \sum_{j=1}^{\infty} \prod_{k=1}^{j-1} g_k' E_k g_0' E_j'\right) \quad \frac{E'(z) G(z) + E(z) G'(z) - E(z) E'(z)}{E(z) + E'(z) - E(z) E'(z)(1 - h_0(z))},
\]

where \(G(z)\) is the GF in Lemma 3.8 and \(G'(z), E'(z)\) are the GFs obtained from \(G(z)\) and \(E(z)\) changing \(f(z)\) to \(g'(z)\) and \(g(z)\) to \(f'(z)\).
Moreover, if for all integer $i \geq 0$, $f_i(z) = f(z) = f'_i(z)$, $g_i(z) = g(z) = g'_i(z)$ and $h_i(z) = h(z) = h'_i(z)$ in $\text{FIN}_Z(\mathcal{M}_{BLin})$, then the GF is

$$H(z) = \frac{1}{1 - f(z) - g(z) - h(z)}. \quad (16)$$

**Theorem 3.12** The generating function for the prefix grand $k$-Fibonacci paths according to their length is

$$PT_k(z) = \sum_{i=0}^{\infty} |\mathcal{PM}_{F_k,i}| z^i = \frac{1 - kz - z^2}{1 - (k + 3)z - (1 - 2k)z^2 + 2z^3}.$$

**Proof** The proof is analogous to the proof of Theorem 3.3 and 3.7. \qed

In Table 5 we show the first terms of the sequence $|\mathcal{PM}_{F_k,i}|$ for $k = 1, 2, 3, 4$.

| $k$ | Sequence |
|-----|----------|
| 1   | 1, 3, 10, 35, 124, 441, 1570, 5591, 19912, 70917, 252574, ... |
| 2   | 1, 3, 11, 44, 181, 751, 3124, 13005, 54151, 225492, 938997, ... |
| 3   | 1, 3, 12, 55, 264, 1285, 6280, 30727, 150392, 736157, 3603528, ... |
| 4   | 1, 3, 13, 68, 379, 2151, 12268, 70061, 400249, 2286780, 13065595 ... |

**Table 4** Sequences $|\mathcal{PM}_{F_k,i}|$ for $k = 1, 2, 3, 4$

Acknowledgments

The second author was partially supported by Universidad Sergio Arboleda under Grant No. DII-262.

**References**

[1] C. Bolat and H. Köse, On the properties of $k$-Fibonacci numbers, *International Journal of Contemporary Mathematical Sciences*, 5(22)(2010), 1097–1105.

[2] R. De Castro, A. Ramírez and J. Ramírez, Applications in enumerative combinatorics of infinite weighted automata and graphs, *Scientific Annals of Computer Science*, accepted, (2014).
[3] S. Falcón, The $k$-Fibonacci matrix and the Pascal matrix, *Central European Journal of Mathematics*, 9(6)(2011), 1403–1410.

[4] S. Falcón and A. Plaza, The $k$-Fibonacci sequence and the Pascal 2-triangle, *Chaos Solitons and Fractals*, 33(1)(2007), 38–49.

[5] S. Falcón and A. Plaza, On $k$-Fibonacci sequences and polynomials and their derivatives, *Chaos Solitons and Fractals*, 39(3)(2009), 1005–1019.

[6] S. Falcón and A. Plaza, On the Fibonacci $k$-numbers, *Chaos Solitons and Fractals*, 32(5)(2007), 1615–1624.

[7] P. Flajolet, Combinatorial aspects of continued fractions, *Discrete Mathematics*, 32(2)(1980), 125–161.

[8] H. W. Gould, Inverse series relations and other expansions involving Humbert polynomials, *Duke Mathematical Journal*, 32(4)(1965), 697–711.

[9] A. F. Horadam, A generalized Fibonacci sequence, *American Mathematical Monthly*, 68(1961), 455–459.

[10] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, A Wiley-Interscience Publication, 2001.

[11] P. Larcombe, O. Bagdasar and E. Fennessey, Horadam sequences: a survey, *Bulletin of the Institute of Combinatorics and its Applications*, 67(2013), 49–72.

[12] J. Ramírez, Incomplete $k$-Fibonacci and $k$-Lucas numbers, *Chinese Journal of Mathematics*, (2013).

[13] J. Ramírez, Some properties of convolved $k$-Fibonacci numbers, *ISRN Combinatorics*, 2013.

[14] J. Rutten, Coinductive counting with weighted automata, *Journal of Automata, Languages and Combinatorics*, 8(2)(2003), 319–352.

[15] J. Sakarovitch, *Elements of Automata Theory*, Cambridge University Press, Cambridge, 2009.

[16] A. Salas, About $k$-Fibonacci numbers and their associated numbers, *International Mathematical Forum*, 50(6)(2011), 2473–2479.

[17] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. [https://oeis.org/](https://oeis.org/).