RELATIVE DIMENSION OF MORPHISMS AND DIMENSION FOR ALGEBRAIC STACKS

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ABSTRACT. Motivated by applications in moduli theory, we introduce a flexible and powerful language for expressing lower bounds on relative dimension of morphisms of schemes, and more generally of algebraic stacks. We show that the theory is robust and applies to a wide range of situations. Consequently, we obtain simple tools for making dimension-based deformation arguments on moduli spaces. Additionally, in a complementary direction we develop the basic properties of codimension for algebraic stacks. One of our goals is to provide a comprehensive toolkit for working transparently with dimension statements in the context of algebraic stacks.

1. Introduction

The notion of dimension for schemes is poorly behaved, even for relatively basic examples such as schemes smooth over the spectrum of a discrete valuation ring. Related concepts which are better behaved are codimension and dimension of local rings. Thus, if one wants to translate naive dimension-based arguments from the context of schemes of finite type over a field to a more general setting, a standard approach is to rephrase results using one of these two alternatives. The purpose of the present paper is twofold: first, to introduce a more natural way of working with relative dimension of morphisms, and second, to generalize this – as well as basic properties of codimension – to algebraic stacks.

More specifically, we introduce in Definition 3.1 below a precise formulation of what it should mean for a morphism to have relative dimension at least a given number $n$. Our immediate motivation is that in certain moduli space constructions in Brill-Noether theory (particularly the limit linear series spaces introduced by Eisenbud and Harris), a key property of the moduli space it that it has at least a certain dimension relative to the base. This allows the use of deformation arguments based purely on dimension counts. However, the existing language to express these ideas is notably lacking when the base is not of finite type over a field. Our definition gives very natural language to capture what is going on, and we show that it is formally well behaved, occurs frequently, and has strong consequences of the sort that one wants for moduli theory.

In generalizing limit linear series to higher-rank vector bundles, it is natural to work not with schemes but with algebraic stacks, and in this context, it is even more difficult to give transparent relative dimension statements using the usual tools. In particular, there is no good notion of dimension of local rings of stacks (see Example 6.9 below). Thus, the fact that our language generalizes readily to

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the stack context makes it especially useful, and it is incorporated accordingly into [Oss] and [OT].

Finally, as a complement to the theory of relative dimension, we also develop the theory of codimension in the context of stacks. Here, the statements are as expected, but we emphasize that the context of stacks introduces certain subtleties which demand a careful treatment. These arise in large part because of the tendency of smooth (and even etale) covers to break irreducible spaces into reducible ones. For instance, it is due to these phenomena that the condition of being universally catenary does not descend under etale morphisms, in general.

The paper is organized as follows: in §2, we recall some background definitions and results of particular relevance. In §3, we give the definition of a morphism having relative dimension at least \( n \), which forms the heart of the paper. We analyze the basic properties of this definition, as well as consequences of the defined properties, with an eye towards moduli theory. In §4, we analyze the formal behavior of our theory, particularly with respect to compositions of morphisms, and in §5 we give a number of examples as well as further discussion of the theory. In §6 we develop the theory of codimension of stacks. Finally, in §7 we generalize the notion of relative dimension to the context of stacks.

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2. Preliminaries

Recall that if \( Z \subseteq X \) is a closed subscheme, then we define

\[
\text{codim}_X Z = \min_{Z' \subseteq Z} \text{codim}_X Z'
\]

\[
= \min_{Z' \subseteq Z} \left( \max_{X' \supseteq Z'} \text{codim}_{X'} Z' \right),
\]

where we can take \( Z' \) and \( X' \) to run over irreducible components of \( Z \) and \( X \) respectively, or over all irreducible closed subschemes. If \( X \) is locally Noetherian, this is always finite.

We will work extensively with universally catenary schemes. Recall that schemes are only required to be locally Noetherian in order to be universally catenary, and a scheme locally of finite type over a universally catenary scheme remains universally catenary (see Remark 5.6.3 (ii) and (iv) of [GD65]). The distinction between finite type and locally finite type is particularly important in the context of stacks, where naturally arising irreducible stacks may be locally of finite type without being of finite type.

One of the primary ways in which the universally catenary hypothesis will arise is the “dimension formula,” which we now recall in the particular form which we will use.

Proposition 2.1. Let \( Y \) be a universally catenary scheme, and \( f : X \to Y \) a morphism locally of finite type. Let \( Z' \subseteq Z \) be irreducible closed subschemes of \( X \), and let \( \eta' \) and \( \eta \) be their respective generic points. Then

\[
\text{codim}_Z Z' - \text{codim}_{f(Z') f(Z')} = \dim Z_{f(\eta)} - \dim Z'_{f(\eta')}.
\]
In the above, and throughout the paper, the subscripts are used to denote the relevant fibers.

Proposition 2.1 is just a rephrasing of Proposition 5.6.5 of [GD65].

We will also make frequent use of Chevalley’s theorem on upper semicontinuity of fiber dimension for morphisms locally of finite type; this is Theorem 13.1.3 of [GD66], and holds without any further hypotheses.

We use the term “algebraic stack” in the sense of Artin, following the conventions of [LMB00], Chapter 4; in particular, their quasiseparation hypotheses imply that the underlying topological spaces of arbitrary algebraic stacks still have the property that every irreducible closed subset has a unique generic point (see Corollary 5.7.2 of [LMB00]).

A **smooth presentation** of an algebraic stack $\mathcal{X}$ consists of a scheme $X$ together with a smooth, surjective morphism $P : X \to \mathcal{X}$. A **smooth presentation** of a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks consists of schemes $X,Y$ together with smooth surjective morphisms $P : Y \to \mathcal{Y}$ and $P' : X \to \mathcal{X} \times_{\mathcal{Y}} Y$. This thus induces a morphism $X \to Y$.

We will frequently use the fact that smooth morphisms of stacks (and in particular smooth presentations) are open, and hence allow for lifting of generizations; see Proposition 5.6 and Corollary 5.7.1 of [LMB00].

**Notation 2.2.** If $f$ is a smooth morphism of schemes or algebraic stacks, we write $\text{reldim}_x f$ for the relative dimension of $f$ at a point $x$ of the source.

An example of de Jong (see Tag 0355 of [Sta13]) shows that the universally catenary condition does not descend under etale morphisms, so it does not make sense to say that an algebraic stack is universally catenary. However, a natural substitute is to work with algebraic stacks locally of finite type over a universally catenary base scheme. We will show that such stacks behave much like universally catenary schemes.

Finally, we make some definitions and basic observations relating to fibers of morphisms of stacks. Even though the fibers themselves are not in general well defined, properties such as dimension and geometric reducedness behave well.

**Definition 2.3.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. For $x \in X$, let $y$ be a geometric point representing $f(x)$, and $\bar{x}$ a point of $X \times_{\mathcal{Y}} Y$ lying over $x$. We say that $f$ is **reduced** at $x$ if $X \times_{\mathcal{Y}} Y$ is reduced at $\bar{x}$.

In general, if $y'$ and $\bar{x}'$ are different choices of a geometric realization of the fiber as in the definition, then there exists a common realization simultaneously extending $y$ and $y'$, and with a point mapping to $\bar{x}$ and $\bar{x}'$. In particular, because geometric reducedness is invariant under field extension, we have:

**Proposition 2.4.** The definition of $f$ being reduced at $x$ is independent of the choices of $y$ and $\bar{x}$.

We now treat fiber dimension.

**Definition 2.5.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a locally finite type morphism of algebraic stacks. For $x \in X$, let $y$ be a geometric point representing $f(x)$, and $\bar{x}$ a point of $X \times_{\mathcal{Y}} Y$ lying over $x$, and define $\delta_x f$ to be the dimension at $\bar{x}$ of $X \times_{\mathcal{Y}} Y$.

We then conclude:
Proposition 2.6. The definition of $\delta_x f$ is independent of the choices of $y$ and $\bar{x}$, and can be expressed equivalently as follows: if $P : Y \to \mathcal{Y}$ and $P' : X \to \mathcal{X} \times \mathcal{Y}$ give a smooth presentation of $f$, and we have $\bar{x} \in X$ and $\bar{y} \in Y$ mapping to $x$ and $f(x)$ respectively, then

$$\delta_x f = \dim_{\bar{x}} X_{\bar{y}} - \text{reldim}_{\bar{x}} P'.$$

In addition, the function $x \mapsto \delta_x f$ is upper semicontinuous.

Proof. The first statement follows just as for reducedness, since dimension is invariant under field extension. Next, let $P$ and $P'$ be a smooth presentation of $f$, and let $y$ be a geometric point extending $\bar{y} \in Y$. Then $P'$ induces a smooth cover of $X \times \mathcal{Y}$ as well, so we conclude the desired identity directly from the definition of dimension of an algebraic stack.

Finally, for semicontinuity, we wish to show that for all $n$, the subset of $X$ on which $\delta_x f \leq n$ is open. Let $g : X \to Y$ be the morphism induced by $P, P'$; this is locally of finite type, so by semicontinuity of fiber dimension, the locus on $X$ where the fiber dimension of $g$ is at most $m$ is open for all $m$. But by the previous statement of the proposition, $\delta_x f$ is obtained as

$$\dim_{\bar{x}} X_{g(\bar{x})} - \text{reldim}_{\bar{x}} P'$$

for some $\bar{x} \in X$ lying over $x$, and since the relative dimension of $P'$ is locally constant, we find that the preimage in $X$ of the locus of $X$ on which $\delta_x f \leq n$ is open, and hence the locus itself is open. $\square$

3. Relative dimension for morphisms

In this section, we define relative dimension, give some examples, and investigate some basic properties, including the “smoothing” results (Propositions 3.7 and 3.8) which constitute the primary application of the theory.

The definition is the following:

Definition 3.1. Let $f : X \to Y$ be a morphism locally of finite type of universally catenary schemes. We say that $f$ has relative dimension at least $n$ if for any irreducible closed subscheme $Y'$ of $Y$, and any irreducible component $X'$ of $X|_{Y'}$, with generic point $\eta$, we have

$$\dim_{\eta} X'_{f(\eta)} - \text{codim}_{Y'} f(X') \geq n. \tag{3.1}$$

We say that $f$ has universal relative dimension at least $n$ if for all universally catenary $Y$-schemes $S$, the base change $X \times_Y S \to S$ has relative dimension at least $n$.

We say that $f$ has relative dimension (respectively, universal relative dimension) at least $n$ at a point $x \in X$ if there is an open neighborhood $U$ of $x$ such that the induced morphism $U \to Y$ has relative dimension (respectively, universal relative dimension) at least $n$.

Remarks 3.2. (i) The terminology is justified by the observation that if $Y$ is of finite type over a field, then (3.1) is equivalent to $\dim X' - \dim Y' \geq n$. Although the latter expression is independent of $f$, the property of having relative dimension at least $n$ may still depend on $f$ in general; see Remark 5.5.

(ii) If $f : X \to Y$ has relative dimension at least $n$, then we see immediately from the definition that every irreducible component of every fiber of $f$ must have
dimension at least $n$; see Proposition 3.5 below for a stronger version of this statement.

(iii) Note that here we allow $n$ to be negative, and even then the condition is not vacuous; see Example 3.4.

(iv) Because dimension theory only works well for universally catenary schemes, it is natural to restrict to this case. We will use the hypothesis primarily to analyze composition of morphisms.

(v) Universally catenary schemes are by definition closed under morphisms which are locally of finite type, so we have in particular that if a morphism has universal relative dimension at least $n$, then every locally finite type base change has relative dimension at least $n$.

(vi) Although fiber dimension is upper semicontinuous, the same is not true for our definition of relative dimension. See Example 5.3 below.

(vii) We are primarily interested in the global case; we use the local concept of relative dimension mainly to simplify the treatment of stacks, where smooth presentations may have different relative dimensions at different points.

We begin with two classes of examples of morphisms satisfying Definition 3.1.

**Example 3.3.** If $f$ is universally open with every generic fiber having every component of dimension at least $n$ (in particular, if $f$ is smooth of relative dimension $n$ or larger), then $f$ has universal relative dimension at least $n$. Indeed, because of semicontinuity of fiber dimension, the condition is preserved under base change (including under restriction to irreducible subschemes of the base), so this follows immediately from the fact that open morphisms map generic points to generic points.

**Example 3.4.** For closed immersions, the first difference between relative dimension and codimension is that codimension is defined to be $c$ if the minimum codimension over all irreducible components of the source is equal to $c$, while the relative dimension is at least $-c$ only if the maximum codimension over all irreducible components of the course is (at most) $c$. More generally, for closed immersions relative dimension measures intersection codimension behavior.

In particular, if $f : X \to Y$ is a closed immersion, $Y$ is regular, and every component of $X$ has codimension at most $c$ in $Y$, it is a theorem of Serre (Theorem V.3 of [Ser65]) that $f$ has relative dimension at least $-c$, and building on Serre’s theorem Hochster proved in Theorem 7.1 of [Hoc75] that $f$ has universal relative dimension at least $-c$.

In a different direction, if $Y$ is universally catenary, and everywhere locally $X$ is cut out by at most $c$ equations in $Y$, then $f$ has universal relative dimension at least $-c$ by Krull’s principal ideal theorem. More generally, if $X$ is everywhere locally cut out by the vanishing of the $(k + 1) \times (k + 1)$ minors of an $n \times m$ matrix, then $f$ has universal relative dimension at least $-(n - k)(m - k)$ (see Exercise 10.4 of [Eis95]). Further, because $Y$ is catenary, and such descriptions are preserved under restriction, we see that the same also holds for intersections of closed subschemes of the above forms, if we sum the corresponding relative dimensions.

Finally, we note that since our definitions are blind to non-reduced structure, it is in fact enough for the above descriptions of $X$ to hold up to taking reduced structures. In particular, if $Y$ has pure dimension $c$ at a closed point $x$, then setting $X = \{x\}$ we find that $x \mapsto Y$ has universal relative dimension at least $-c$. 
We next make some preliminary observations:

**Proposition 3.5.** If \( f : X \to Y \) has relative dimension at least \( n \), then for any irreducible closed subset \( Y' \) of \( Y \), and any irreducible component \( X' \) of \( f^{-1}(Y') \), for every \( x \in X' \) we have

\[
\dim_x X'_f(\eta) - \text{codim}_{Y'} f(X') \geq n.
\]

*Proof.* Observing that in Definition 3.1, we have \( X'_f(\eta) \) irreducible and hence equidimensional, the statement follows immediately from semicontinuity of fiber dimension. \( \square \)

**Proposition 3.6.** If \( f : X \to Y \) has relative dimension at least \( n \), and \( U \subseteq X \) is a nonempty open subset, then the induced morphism \( U \to Y \) has relative dimension at least \( n \). Conversely, if \( \{U_i\} \) is an open cover of \( X \) and each morphism \( U_i \to Y \) has relative dimension at least \( n \), then \( X \to Y \) has relative dimension at least \( n \).

The same statements hold for universal relative dimension.

Consequently, \( f \) has (universal) relative dimension at least \( n \) if and only if it has (universal) relative dimension at least \( n \) at every point of \( X \).

*Proof.* The first statement is clear from the definition, since \( X'_f(\eta) \) is irreducible and of finite type over a field, so its dimension doesn’t change when restricting to open subsets.

The second statement follows similarly: if \( \{U_i\} \) is an open cover of \( X \), then some \( U_i \) contains the generic point \( \eta \) of \( X' \), and both the dimension and codimension in (3.1) are unchanged by restriction to \( U_i \).

The universal case follows immediately, since open subsets (respectively, open covers) are preserved under base change. \( \square \)

We conclude this section with two propositions describing the main applications of Definition 3.1; in the context of moduli spaces, they constitute “smoothing theorems” based on dimension counts and analysis of special fibers. The strongest statements occur in the case of nonnegative relative dimension.

**Proposition 3.7.** Given \( f : X \to Y \), suppose there exists \( x \in X \) such that \( f \) has relative dimension at least \( n \) at \( x \), and the fiber \( X'_f(x) \) has dimension \( n \) at \( x \). Then there exists a neighborhood \( U \) of \( x \) on which \( f \) has relative dimension at least \( n \) and pure fiber dimension \( n \), and on any such neighborhood \( f \) is open.

If further \( f \) has universal relative dimension at least \( n \) at \( x \), then \( f \) is universally open on \( U \).

If further \( Y \) is reduced and the fiber of \( f \) is geometrically reduced at \( x \), then \( f \) is flat at \( x \).

Note that this proposition can also be viewed as a complement to the standard criterion for flatness in terms of fiber dimension in the case of a Cohen-Macaulay scheme over a regular scheme.

*Proof.* By definition, there exists an open neighborhood of \( x \) on which \( f \) has relative dimension at least \( n \). Applying Proposition 3.5 and semicontinuity of fiber dimension, there exists a possibly smaller open neighborhood \( U \) of \( x \) on which \( f \) also has pure fiber dimension \( n \). We claim that \( f \) is necessarily open on \( U \). Let \( U' \) be any non-empty open neighborhood of \( U \); by Proposition 3.6, we have that
f has relative dimension at least n and pure fiber dimension n, so it is enough to show that the image of such a morphism is open. Since the morphism is locally of finite type, by Corollary 1.10.2 of [GD64] it is enough to show that the image is closed under generization. Let y specialize to y' in Y, with y' in the image of f, say y' = f(x'). Set Y' to be the closure of y, and X' any irreducible component of f⁻¹(Y') containing x'. Then from the definition of having relative dimension at least n, together with the hypothesis that the fibers have dimension n, we conclude that X' dominates Y', and in particular y is in the image of f, as desired.

Next, if f has universal relative dimension at least n, we conclude that f is universally open on U because according to Corollary 8.10.2 of [GD66] we may check that a morphism is universally open after finite type base change.

Finally, if the fiber of f is geometrically reduced at x, and Y is reduced, flatness of f follows from Theorem 15.2.2 of [GD66]. □

It is clear that Proposition 3.7 will never apply in the case of negative relative dimension. Because it is sometimes important for applications, we also state a weaker version which works even when the relative dimension is negative.

** Proposition 3.8.** Given f : X → Y, suppose that Y is irreducible, and there exist x ∈ X and Y' ⊆ Y closed and irreducible containing f(x) and with support strictly smaller than Y, such that

(I) f has relative dimension at least n at x;
(II) every irreducible component X' of f⁻¹(Y') containing x has
\[ \dim X'_{f(\eta)} - \operatorname{codim}_Y f(X') = n, \]
where \( \eta \) is the generic point of X'.

Then for every irreducible component X'' of X containing x, we have
\[ f(X'') \not\subseteq Y'. \]

If further we have

(III) Y' has codimension c in Y, and the inclusion Y' ↪ Y has universal relative dimension at least −c,

then we have
\[ \dim X''_{f(\eta')} - \operatorname{codim}_Y f(X'') = n, \]
where \( \eta' \) is the generic point of X''.

**Proof.** Given X'' an irreducible component of X containing x, let \( \eta' \) be its generic point. By (I), we have
\[ \dim X''_{f(\eta')} - \operatorname{codim}_Y f(X'') \geq n. \]

If we had f(X'') ⊆ Y', then X'' would also be an irreducible component of f⁻¹(Y'), so by (II) we would have
\[ \dim X''_{f(\eta')} - \operatorname{codim}_Y f(X'') = n. \]

But (in light of our running catenary hypotheses) this contradicts the hypothesis that \( \operatorname{codim}_Y Y' > 0 \). Thus the first assertion holds.
Next suppose that (III) is also satisfied, and let $X'$ be an irreducible component of $f^{-1}(Y')$ containing $x$ and contained in $X''$, and $\eta$ its generic point. Then, since we already have the opposite inequality, we want to show that
\[
\dim X''_{f(\eta)} - \operatorname{codim}_Y f(X'') \leq n = \dim X'_{f(\eta)} - \operatorname{codim}_Y f(X').
\]

But applying the catenary hypothesis together with the hypothesis on the universal relative dimension of $Y' \hookrightarrow Y$, we have
\[
\operatorname{codim}_{Y'} f(X') - \operatorname{codim}_Y f(X'') = \operatorname{codim}_{Y'} f(X') - \operatorname{codim}_{Y'} Y' - \operatorname{codim}_Y f(X'') \\
\leq \operatorname{codim}_Y f(X') - \operatorname{codim}_{X''} X' - \operatorname{codim}_{X''} f(X'') \\
= \operatorname{codim}_Y f(X'') - \operatorname{codim}_{X''} X'.
\]

But the catenary hypothesis also allows us to use the dimension formula, Proposition 2.1 to conclude that the righthand side above is equal to $\dim X'_{f(\eta)} - \dim X''_{f(\eta)}$, yielding the desired inequality. \[\square\]

4. Formal properties of relative dimension

In this section, we investigate the formal behavior of relative dimension, including behavior with respect to compositions in general, and with respect to composition with smooth morphisms.

Our first observation is the following, which will allow us to generalize Definition 3.1 to morphisms of stacks.

**Proposition 4.1.** Suppose that $f : X \rightarrow Y$ is smooth, and $g : Y \rightarrow Z$ a morphism with $Z$ universally catenary. Given $x \in X$, we have that $g$ has relative dimension at least $n$ at $f(x)$ if and only if $g \circ f$ has relative dimension at least $n + \operatorname{rel.dim}_x f$ at $x$. Additionally, the same statement holds for universal relative dimension.

**Proof.** The statements being local at $x$, we may assume that $f$ has pure relative dimension $m := \operatorname{rel.dim}_x f$. Since smooth morphisms are open, we reduce immediately to showing that if $f$ is surjective, then $g$ has (universal) relative dimension at least $n$ if and only if $g \circ f$ has (universal) relative dimension at least $n + m$. Because smoothness is preserved under base change, and composition commutes with base change, the universal statement follows immediately from the non-universal statement. Next, observe that if $g \circ f$ is locally of finite type then $g$ is locally of finite type by Lemma 17.7.5 of [GD67].

Now, if $Z'$ is an irreducible closed subscheme of $Z$, then by smoothness, every component $X'$ of $X|_{Z'}$ dominates a component $Y'$ of $Y|_{Z'}$, and by surjectivity of $f$, every component $Y''$ of $Y|_{Z'}$ is dominated by a component $X'$ of $X|_{Z'}$. Thus, given such $X'$ and $Y''$, having generic points $\eta$ and $\xi$ respectively, we have $g(f(\eta)) = g(\xi)$, and by smoothness $\dim X'_{g(f(\eta))} = \dim Y''_{g(\xi)} + m$; the desired statement follows. \[\square\]

Now we come to the more substantial statement that Definition 3.1 behaves well with respect to composition.

**Lemma 4.2.** Suppose that $f : X \rightarrow Y$ has relative dimension at least $m$ at $x$, and $g : Y \rightarrow Z$ has relative dimension at least $n$ at $f(x)$. Then $g \circ f$ has relative dimension at least $m + n$ at $x$.

The same holds for universal relative dimension.
Proof. By restricting to suitable neighborhoods of $x$ and $f(x)$, we may assume that $f$ and $g$ have (universal) relative dimension at least $m$ and $n$ everywhere, and we show that the composition has (universal) relative at least $m + n$ everywhere. The universal statement follows immediately from the non-universal statement, since composition commutes with base change.

For the non-universal statement, let $Z'$ be an irreducible component of $Z$, and $X'$ an irreducible component of $X|_{Z'}$, with generic point $\eta$; then we wish to show that

$$\dim X'_{g(f(\eta))} - \operatorname{codim}_{Z'} g(f(X')) \geq m + n.$$ 

Let $Y'$ be an irreducible component of $Y|_{Z'}$ containing $f(\eta)$, with generic point $\xi$.

Then $X'$ is an irreducible component of $X|_{Y'}$, so we have

$$\dim X'_{f(\eta)} - \operatorname{codim}_{Y'} f(X') \geq m, \quad \text{and} \quad \dim Y'_{g(\xi)} - \operatorname{codim}_{Z'} g(Y') \geq n.$$ 

Now, if $\zeta$ is any generic point of an irreducible component $Y''$ of the fiber $Y'_{g(f(\eta))}$ with $f(\eta) \in Y''$, note that $g(\zeta) = g(f(\eta))$.

The relationship between $\dim X'_{g(f(\eta))}$ and $\dim X'_{f(\eta)}$ is given by

$$\dim X'_{g(f(\eta))} = \dim X'_{f(\eta)} + \dim_{X''_{g(f(\eta))}} f(X') - \operatorname{codim}_{Y''} f(X').$$

Adding the two inequalities and using the above equation, we find that

$$\dim X'_{g(f(\eta))} - \dim_{Y''_{g(f(\eta))}} f(X') + \operatorname{codim}_{Y''} f(X') - \operatorname{codim}_{Y'} f(X') + \dim Y'_{g(\xi)} - \operatorname{codim}_{Z'} g(Y') \geq m + n.$$ 

We now apply our catenary hypothesis, first to conclude that

$$\operatorname{codim}_{Y''} f(X') = \operatorname{codim}_{Y'} f(X') - \operatorname{codim}_{Y'} Y''$$

and

$$\operatorname{codim}_{g(Y')} g(f(X')) = \operatorname{codim}_{Z'} g(f(X')) - \operatorname{codim}_{Z'} g(Y').$$

and second, to apply the dimension formula, Proposition 2.1, to the morphism $Y' \to g(Y')$ at the point $\zeta$, concluding that

$$\operatorname{codim}_{Y'} Y'' = \operatorname{codim}_{Y''_{g(f(\eta))}} g(Y') + \dim Y'_{g(\xi)} - \dim_{Y'_{g(f(\eta))}} g(Y').$$

Putting these three equations together with the previous inequality gives the desired inequality. □
We conclude that Definition 3.1 behaves well with respect to smooth covers.

**Corollary 4.3.** Suppose that $f : X \to Y$ is a morphism, with $Y$ universally catenary. Let $g : Y' \to Y$ be a smooth cover, and let $h : X'' \to X'$ be a smooth cover, where $X' = X \times_Y Y'$. Denote by $f''$ the induced morphism $X'' \to Y'$. Then $f$ has universal relative dimension at least $n$ at a point $x$ if and only if $f''$ has universal relative dimension at least $n + \text{rel.dim}_{x''} h$ at some (equivalently, at every) $x'' \in X''$ lying over $x$.

We remark that without universality, relative dimension at least $n$ need not be preserved under smooth base change; see Remark 5.9.

**Proof.** Denote by $f'$ the morphism $X' \to Y'$, so that $f'' = f' \circ h$.

If $f$ has universal relative dimension at least $n$ at $x$, then by definition $f'$ has universal relative dimension at $x'$ for any $x'$ lying over $x$, and in particular for $x' = h(x'')$. We conclude that $f''$ has universal relative dimension at least $n + \text{rel.dim}_{x''} h$ from Proposition 4.1. Conversely, if $f''$ has universal relative dimension at least $n + \text{rel.dim}_{x''} h$ at $x''$, then the other direction of Proposition 4.1 implies that $f'$ has universal relative dimension at least $n$ at $h(x'')$. Then by Lemma 4.2, we have that $g \circ f'$ has universal relative dimension at least $n + \text{rel.dim}_{f'(h(x''))} g$ at $h(x'')$. Write $g' : X' \to X$ for the base change of $g$; then $g'$ is a smooth cover with $\text{rel.dim}_{h(x'')} g' = \text{rel.dim}_{f'(h(x''))} g$, and $g \circ f' = f \circ g'$, so again applying Proposition 4.1, we have that $f$ has universal relative dimension at least $n$ at $g'(h(x'')) = x$, as desired.

Using Lemma 4.2 and the standard Grothendieck six conditions argument (see for instance Remark 5.12 of [GD60]), we see that Definition 3.1 also behaves well under post-composition by a smooth morphism.

**Corollary 4.4.** Given morphisms $f : X \to Y$ and $g : Y \to Z$, suppose $g$ is smooth of relative dimension $n$, with $Z$ universally catenary. Then $f$ has universal relative dimension at least $m$ if and only if $g \circ f$ has universal relative dimension at least $m + n$.

**Proof.** The “only if” direction is immediate from Lemma 4.2. For the converse, first observe that $Y$ is universally catenary, and the smoothness of $g$ implies the diagonal $\Delta_g : Y \to Y \times_Z Y$ has universal relative dimension at least $-n$, because if we factor $\Delta_g$ as a closed immersion followed by an open immersion, the closed immersion is locally cut out by $n$ equations (See Proposition 2.2.7 of [BLR91]). Now, if $g \circ f$ has universal relative dimension at least $m + n$, then the second projection $p_2 : X \times_Z Y \to Y$ is the base change of $g \circ f$ to $Y$, so likewise has universal relative dimension at least $m + n$. On the other hand, the graph morphism $\Gamma_f : X \to X \times_Z Y$ is the base change of $\Delta_g$ under $f \times \text{id}$, so it has universal relative dimension at least $-n$. Since $f = p_2 \circ \Gamma_f$, we conclude that $f$ has universal relative dimension at least $m$ from Lemma 4.2.
5. Examples and further discussion

We conclude our examination of relative dimension with positive and negative examples, and a number of remarks on various aspects of the subject.

First, as another consequence of Lemma 4.2, we obtain the following wide classes of examples.

**Corollary 5.1.** Suppose that 
\[ f : X \to Y \] is a closed immersion, and \[ g : Y \to Z \] is smooth of relative dimension \( n \), with \( Z \) universally catenary. If either \( Z \) is regular and every component of \( X \) has codimension at most \( c \) in \( Y \), or \( X \) may be expressed locally as an intersection of determinantal conditions with expected codimensions adding up to \( c \), then \( g \circ f \) has universal relative dimension at least \( n - c \).

Alternatively, suppose that \[ f : X \to Y \] is a morphism of smooth \( S \)-schemes, with \( S \) universally catenary, and \( m \) and \( n \) the relative dimensions of \( X \) and \( Y \) over \( S \), respectively. Then \( f \) has universal relative dimension at least \( m - n \).

Again, we mention that universal relative dimension is insensitive to non-reduced structures, so in fact the descriptions of the corollary only need to hold up to taking reduced structures in order to conclude the desired statements.

**Proof.** The first statement is a direct consequence of Lemma 4.2, together with Examples 3.3 and 3.4. For the second, we observe that \( f \) is necessarily locally of finite type, so \( X \) can locally be written as a closed subscheme of \( \mathbb{A}^N_Y \). Then, because \( X \) is smooth over \( S \) of relative dimension \( m \), and \( \mathbb{A}^N_Y \) is smooth over \( S \) of relative dimension \( N + n \), we have that \( X \) is everywhere locally cut out by \( N + n - m \) equations inside \( \mathbb{A}^N_Y \) by Proposition 2.2.7 of [BLR91], so we have reduced the second statement to the first. \( \square \)

**Example 5.2.** If \( Y \) is a smooth variety over a field, and \( f : X \to Y \) is a blowup with smooth center, the second part of Corollary 5.1 implies that \( f \) has strong relative dimension at least 0.

We next provide some negative examples.

**Example 5.3.** As usual, the normalization \( \tilde{C} \) of an irreducible nodal curve \( C \) provides an interesting example to consider. It has relative dimension at least 0, but not universal relative dimension at least 0. Indeed, if we take \( \tilde{C} \times \tilde{C} \), we obtain \( \tilde{C} \) together with two isolated points, each of which maps to one of the preimages of the node under projection to \( C \).

Another base change which does not have relative dimension at least 0 is obtained by taking the product with \( \tilde{C} \), considered over the base field. In this case, if we let \( \Delta \subset \tilde{C} \times \tilde{C} \) be the diagonal, and \( Z \subset C \times \tilde{C} \) its image, then the restriction of \( \tilde{C} \times \tilde{C} \) to (the preimage of) \( Z \) likewise consists of a copy of \( \Delta \cong \tilde{C} \) together with two isolated points.

Notice that this means that relative dimension does not have a semicontinuity property: the map \( \tilde{C} \times \tilde{C} \to C \times \tilde{C} \) has relative dimension at least 0 over the nonsingular locus of \( C \times \tilde{C} \), but not over all of \( C \times \tilde{C} \).

**Example 5.4.** An example of a closed immersion of codimension \( c \) which does not have relative dimension at least \( -c \) is given by the standard example of failure of subadditivity of codimension for intersections: let \( X \) be a cone over a quadric surface, and \( Z \) the cone over a line in the surface. Then \( Z \) has codimension 1, but
we claim that the inclusion \( Z \to X \) does not have relative dimension at least \(-1\). Indeed, if \( Z' \) is the cone over any other line in the same ruling, then \( Z' \cap Z \) is equal to the cone point, which has codimension 2 in \( Z' \).

**Remark 5.5.** According to Corollary 5.1, relative dimension does not depend on the map \( f \) when \( X \) and \( Y \) are both smooth over a common base. However, in the general case, we see from Example 5.3 that there is a nontrivial dependence on \( f \). Indeed, the morphism \( \tilde{C} \times \tilde{C} \to C \times \tilde{C} \) considered in that example does not have relative dimension at least 0, but there are many other morphisms with the same source and target which do. For instance, if we choose a point \( P \in \tilde{C} \) and consider the composition

\[
\tilde{C} \times \tilde{C} \xrightarrow{P_1} \tilde{C} \to C \times \{P\} \subseteq C \times \tilde{C},
\]

where the second morphism is the normalization, then we have a composition of morphisms of relative dimension at least 0 and \(-1\) respectively, so obtain relative dimension at least 0, as asserted.

**Remark 5.6.** We elaborate slightly on the hypotheses and conclusions of Proposition 3.8. First, we observe that under the hypotheses of the proposition, although \( \dim X'_{f(\eta)} - \operatorname{codim}_Y f(X') \) remains unchanged when we replace \( X' \) by \( X'' \), the separate terms need not. Indeed, the map \( \mathbb{A}^2 \to \mathbb{A}^2 \) given by \((x, y) \mapsto (x, xy)\) has dimension dimension at least 0, and if we restrict to the line \( x = 0 \) in the target, we get that the \( \dim X'_{f(\eta)} = \operatorname{codim}_Y f(X') = 1 \), so the proposition applies. However, without restricting (and letting \( \eta' \) be the generic point of \( X \)), we have \( \dim X'_{f(\eta')} = \operatorname{codim}_Y f(X) = 0 \). This behavior is discussed further in Remark 5.7.

The final conclusion of Proposition 3.8 may seem superfluous in the context of smoothing arguments, but in fact it plays an important role in inductive arguments. We see from the second part of Example 5.3 that for this final conclusion, hypothesis (III) is indeed necessary. In the notation of that example, consider the map

\[
\tilde{C} \times \tilde{C} \setminus \Delta \to C \times \tilde{C}.
\]

This has relative dimension at least \(-1\), and if we restrict to \( Z \), we find that the conditions of Proposition 3.8 are satisfied with \( n = -1 \), except that \( Z \) does not have relative dimension at least \(-1\) in \( C \times \tilde{C} \). However, in this case (again setting \( \eta' \) to be the generic point of \( X \)) we have \( \dim X'_{f(\eta')} = \operatorname{codim}_Y f(X) = 0 \).

**Remark 5.7.** Relative dimension of morphisms can also shed light on the behavior of constructibility of images, in the sense of understanding when a dominant morphism fails to have open image. The first example of this is always the map \( \mathbb{A}^2 \to \mathbb{A}^2 \) given by \((x, y) \mapsto (x, xy)\), and it is natural to notice that the failure of openness occurs at a jump in fiber dimension, and to wonder whether this phenomenon is general.

In fact, Example 5.3 yields an example of a dominant morphism of varieties for which the image is not open and the fiber dimension does not jump. Namely, in the notation of that example, consider the map

\[
\tilde{C} \times \tilde{C} \setminus \Delta \to C \times \tilde{C}.
\]

This is a dominant morphism of varieties, with all fibers 0-dimensional, but its image is \((C \times \tilde{C} \setminus Z) \cup \{P_1, P_2\}\), where \( P_1 \) and \( P_2 \) are the intersection of \( Z \) with the singular locus of \( C \times \tilde{C} \).
However, we see immediately from Proposition 3.7 that if $f: X \to Y$ has relative dimension at least $n$, and there is no jumping of fiber dimension (i.e., all fibers have dimension $n$), then $f$ necessarily has open image. By Corollary 5.1, the relative dimension hypothesis applies in particular whenever $X$ and $Y$ are smooth and $\dim X - \dim Y = n$. Thus, the behavior of the first example is in fact rather general.

**Remark 5.8.** Suppose that we have $f: X \to Y$ and $g: Y \to Z$, where $g$ is a closed immersion, and $g \circ f$ has relative dimension at least $n$. Then it is immediate from the definition that $f$ has relative dimension at least $n$ as well, but it is natural to wonder, if $g(Y)$ has codimension $c$ in $Z$, whether in fact $f$ has relative dimension at least $n + c$. However, this is not the case. For instance, if $f$ is the inclusion of the cone over a line into the cone over a quadric surface, as in Example 5.4, then we know that $f$ does not have relative dimension at least $-1$. However, if we compose with the inclusion of $Y$ into $\mathbb{A}^3$, then because $\mathbb{A}^3$ is regular, we have that the composition has relative dimension at least $-2$, in fact universally.

**Remark 5.9.** Example 5.3 also demonstrates that the statement of Corollary 4.4 fails if we replace universal relative dimension with relative dimension. Indeed, in the notation of the example, the composed morphism

$$\tilde{C} \times \tilde{C} \to C \times \tilde{C} \to C$$

has relative dimension at least 1, and $C \times \tilde{C} \to C$ is smooth of relative dimension 1, but $\tilde{C} \times \tilde{C} \to C \times \tilde{C}$ does not have relative dimension at least 0.

### 6. Codimension for stacks

We now give a treatment of the theory of codimension for algebraic stacks. Although the definitions and statements are as expected, there are some non-trivial aspects to the proofs, largely coming from the fact that basic statements on codimension for schemes require irreducibility, and irreducibility is not preserved under smooth covers.

The definition is as one would expect.

**Definition 6.1.** Let $\mathcal{X}$ be a locally Noetherian algebraic stack, and $Z \subseteq \mathcal{X}$ a closed substack. The **codimension** of $Z$ in $\mathcal{X}$ is defined to be the codimension in $X$ of $X \times_{\mathcal{X}} Z$, where $X \to \mathcal{X}$ is a smooth presentation of $\mathcal{X}$.

We remark that although the Zariski topology on $\mathcal{X}$ is sober, topological codimension does not yield the correct definition – see Example 6.10.

**Proposition 6.2.** Codimension is well defined, and finite.

**Proof.** We first verify that codimension is well defined. If $X \to \mathcal{X}$, $X' \to \mathcal{X}$ are both smooth presentations of $\mathcal{X}$, then $X \times_{\mathcal{X}} X'$ is a smooth cover of both $X$ and $X'$; in principle, it may be that $X \times_{\mathcal{X}} X'$ is an algebraic space, but even so we can take a further smooth cover to obtain a common smooth cover of $X$ and $X'$ be a scheme. Thus, it is enough to see that codimension is preserved under passage to a smooth cover. But this is true more generally for flat covers, see Corollary 6.1.4 of [GD65].

We next verify finiteness. By definition, any smooth presentation $P: X \to \mathcal{X}$ has $X$ locally Noetherian. Now, $\text{codim}_X Z := \text{codim}_X P^{-1}Z$, and in order to establish finiteness of $\text{codim}_X P^{-1}Z$, it is enough to show that $\text{codim}_X Z'$ is finite for some
irreducible component $Z'$ of $P^{-1}Z$. But let $Z'$ be any irreducible component; since $X$ is locally Noetherian, there exists an open subscheme $U \subseteq X$ which meets $Z'$ and is Noetherian. Then $\text{codim}_X Z' = \text{codim}_U Z'$, and $\text{codim}_U Z'$ is finite, as desired. □

**Proposition 6.3.** Let $Z \subseteq \mathcal{X}$ be a closed substack. Then $\text{codim}_X Z = 0$ if and only if $Z$ contains an irreducible component of $\mathcal{X}$.

**Proof.** The statement is obvious for schemes. Thus, if $P : X \to \mathcal{X}$ is a smooth presentation, and $Z = P^{-1}(Z)$ it suffices to prove that $Z$ contains an irreducible component of $\mathcal{X}$ if and only if $Z$ contains an irreducible component of $X$. But this is clear from the fact that $P$ maps generic points to generic points. □

We now state our main intrinsically stack-theoretic results, deferring for the moment the supporting results needed to prove them. We begin with a statement to the effect that, in the case of stacks, codimension still behaves the same with respect to irreducible components.

**Proposition 6.4.** Let $Z \subseteq \mathcal{X}$ be a closed substack. Then

$$\text{codim}_X Z = \min_{Z' \subseteq Z} \text{codim}_X Z'$$

$$= \min_{Z' \subseteq Z} \left( \max_{X' \subseteq \mathcal{X}} \text{codim}_{X'} Z' \right),$$

where $Z'$ and $X'$ range over irreducible components (or equivalently, irreducible closed substacks) of $Z$ and $\mathcal{X}$ respectively.

The next two results are generalizations to stacks of standard statements in the scheme setting, both of which require catenary hypotheses. The first is additivity of codimension.

**Proposition 6.5.** Let $\mathcal{X}$ be an irreducible algebraic stack, locally of finite type over a universally catenary base scheme, and $Z' \subseteq Z \subseteq \mathcal{X}$ irreducible closed substacks. Then

$$\text{codim}_X Z' = \text{codim}_X Z + \text{codim}_Z Z'.$$

The second is the generalization of the “dimension formula” to stacks.

**Proposition 6.6.** Let $Y$ be an algebraic stack, of finite type over a universally catenary scheme, and $f : X \to Y$ a morphism locally of finite type. Let $Z' \subseteq Z$ be irreducible closed subschemes of $\mathcal{X}$, and let $\eta'$ and $\eta$ be their respective generic points. Then

$$\text{codim}_X Z' - \text{codim}_{f(X)} f(Z') = \dim Z_{f(\eta)} - \dim Z'_{f(\eta')}.$$ 

Propositions 6.4, 6.5 and 6.6 will be easy consequences of the following.

**Lemma 6.7.** Let $\mathcal{X}$ be a locally Noetherian irreducible algebraic stack, and $Z \subseteq \mathcal{X}$ an irreducible closed substack. Let $P : X \to \mathcal{X}$ be a smooth presentation of $\mathcal{X}$, and $P' : Z \to Z$ the induced presentation of $Z$. Then for every irreducible component $Z'$ of $Z$, we have

$$\text{codim}_X Z = \text{codim}_X Z'.$$

If further $\mathcal{X}$ is locally of finite type over a universally catenary base scheme, then for every irreducible component $Z'$ of $Z$, and every irreducible component $X'$ of $X$ containing $Z'$, we have

$$\text{codim}_X Z = \text{codim}_{X'} Z'.$$
As usual, the proof of the lemma reduces to a related result on schemes, which we state separately.

**Proposition 6.8.** Let \( f : X \to Y \) be a smooth morphism of schemes, with \( Y \) irreducible and locally Noetherian. Let \( Z \subseteq Y \) be an irreducible closed subscheme. Then if \( Z' \) is any irreducible component of \( f^{-1}(Z) \), we have

\[
\text{codim}_Y Z = \text{codim}_X Z'.
\]

If further \( Y \) is universally catenary, then for any irreducible component \( Z' \) of \( f^{-1}(Z) \), and any irreducible component \( X' \) of \( X \) containing \( Z' \), we have

\[
\text{codim}_Y Z = \text{codim}_{X'} Z'.
\]

**Proof.** Using openness of smooth morphisms, we may replace \( X \) and \( Y \) by open subsets on which \( f \) is surjective, and \( f^{-1}(Z) = Z' \). The first statement then follows from Corollary 6.1.4 of [GD65].

In the universally catenary case, since the relative dimension of a smooth morphism is locally constant, and smoothness also ensures that \( X' \) dominates \( Y \) and \( Z' \) dominates \( Z \), this is an immediate application of the dimension formula, Proposition 2.1. □

**Proof of Lemma 6.7.** Our first claim is that if \( X'' \supseteq Z'' \) are different choices of irreducible components of \( X \) and \( Z \) respectively, then exists a scheme \( Y \) with irreducible closed subschemes \( \bar{Z} \subseteq \bar{X} \) such that \( Y \) is a smooth cover of \( X \) in two different ways, and under these maps, \( \bar{Z} \) is an irreducible component of the preimage of \( Z' \) and \( Z'' \) respectively, and \( \bar{X} \) is an irreducible component of the preimage of \( X' \) and \( X'' \) respectively. Indeed, by considering induced mappings on generic points, we can find an irreducible component \( \bar{Z} \) of \( Z \times_X Z \) which dominates \( Z' \) and \( Z'' \) respectively under the projection morphisms, as well as an irreducible component \( \bar{X} \) of \( X \times_X X \) which contains \( \bar{Z} \) and likewise dominates \( X' \) and \( X'' \) respectively under the projection morphisms. Note also that

\[
Z \times_X X = Z \times_X (X \times_X X) = X \times_X Z = (X \times_X X) \times_X Z,
\]

so we have also that \( \bar{Z} \) is simultaneously an irreducible component of the preimage in \( X \times_X X \) of \( Z' \) via one projection, and of \( Z'' \) via the other. Now, it may be that \( X \times_X X \) is an algebraic space rather than a scheme, but if so we can pass to a further smooth cover \( Y \to X \times_X X \) and let \( \bar{X}' \) and \( \bar{Z}' \) be components of the preimages of \( \bar{X} \) and \( \bar{Z} \). This proves the desired claim.

Now, for the first statement, it suffices to show that

\[
\text{codim}_X Z' = \text{codim}_X Z''.
\]

Choosing \( Y \) and \( \bar{Z}' \) as above (with the choice of \( X'' \) and \( \bar{X}' \) being irrelevant), we apply the first part of Proposition 6.8 to conclude that

\[
\text{codim}_X Z' = \text{codim}_Y \bar{Z}' = \text{codim}_X Z''.
\]

For the second statement, it suffices to show that

\[
\text{codim}_{X'} Z' = \text{codim}_{X''} Z''.
\]

Choosing \( \bar{X}' \) and \( \bar{Z}' \) as above, we apply the second part of Proposition 6.8 to deduce that

\[
\text{codim}_{X'} Z' = \text{codim}_{\bar{X}, \bar{Z}'} = \text{codim}_{X''} Z'',
\]

as desired. □
We now give the proofs of our main propositions.

Proof of Proposition 6.4. The first equality is an immediate consequence of the definitions of codimension for subschemes and substacks. The second equality reduces immediately to the case that $Z = Z'$, which is to say that we need to verify that, when $Z$ is irreducible, we have

$$\text{codim}_X Z = \max_{X' \subseteq X; Z \subseteq X'} \text{codim}_{X'} Z.$$  

Letting $P : X \to \mathcal{X}$ be a smooth presentation, the above equation gives us that

$$\min_{Z \subseteq P^{-1}(Z)} \text{codim}_X Z = \max_{X' \subseteq \mathcal{X}; Z \subseteq X'} \min_{Z' \subseteq P^{-1}(Z)} \text{codim}_{X'} Z,$$

where $X' = P^{-1}(X')$, and $Z$ ranges over irreducible components of $P^{-1}(Z)$. However, according to Lemma 6.7, the values of $\text{codim}_X Z$ and $\text{codim}_{X'} Z$ are each independent of the choice of $Z$, so the desired identity reduces to the standard identity

$$\text{codim}_X Z = \max_{X' \subseteq \mathcal{X}; Z \subseteq X'} \text{codim}_{X'} Z.$$  

□

Proof of Proposition 6.5. The issue here is that even when everything is of finite type over a field, the statement can fail without irreducibility hypotheses, and since irreducibility is not preserved under smooth covers, we again have to exercise care in reducing to the scheme case. However, according to Lemma 6.7 we can compute all the codimensions in question in terms of a fixed chain of irreducible closed subschemes in a smooth presentation of $\mathcal{X}$, so the statement reduces to the usual statement for schemes, which is immediate from the definition of catenary. □

Proof of Proposition 6.6. Let $P : Y \to \mathcal{Y}$ and $P' : X \to \mathcal{X} \times_\mathcal{Y} Y$ be a smooth presentation of $f$, with $f' : X \to Y$ the induced morphism. By hypothesis, $f'$ is locally of finite type, and $Y$ is universally catenary. Let $Z$ and $Z'$ be the preimages in $X$ of $Z$ and $Z'$ respectively, and let $W'$ be an irreducible component of $Z'$, and $W$ an irreducible component of $Z$ containing $W'$. Then $f'(W')$ and $f'(Z')$ are irreducible components of $P^{-1}(f(Z))$ and $P^{-1}(f(Z'))$, respectively. It follows from Lemma 6.7 that we have

$$\text{codim}_{Z'} Z' = \text{codim}_W W' \quad \text{and} \quad \text{codim}_{f'(Z)} f'(Z') = \text{codim}_{f'(W)} f'(W').$$

On the other hand, the relative dimension of $P'$, being locally constant, is the same at all points of $W$ and in particular on $W'$, so we see that

$$\dim Z'_{f'(w)} - \dim Z'_{f'(w')} = \dim W'_{f'(\xi)} - \dim W'_{f'(\xi')},$$

where $\xi$ and $\xi'$ are the generic points of $W$ and $W'$ respectively. The proposition then follows from Proposition 2.1. □

We conclude with examples illustrating the obstruction to defining a notion of “dimension of local ring” for stacks, and the difference between codimension and topological codimension for Artin stacks.

Example 6.9. Let $k$ be a field, and $\mathcal{X}$ the stack given by $\mathbb{A}^1_k/G_m$. Then $\mathcal{X}$ is smooth of dimension 0 over $\text{Spec} \ k$. It has two points $x_1$ and $x_0$, with $x_1$ specializing to $x_0$, and via the smooth presentation $\mathbb{A}^1_k \to \mathcal{X}$, we see that $x_0$ has codimension 1 in $\mathcal{X}$. This example demonstrates that there is no theory of “dimension of local rings"
for algebraic stacks – even those of finite type over a field – which simultaneously satisfies the following three conditions:

(i) If \( \delta(x) \) denotes the “dimension of the local ring at \( x \),” and \( x_1 \) specializes to \( x_0 \), then
\[
\delta(x_0) = \delta(x_1) + \text{codim}_{x_1} x_0.
\]

(ii) If \( X \) is smooth over a field, of pure dimension \( n \), and \( x \) is a closed point of \( X \), then \( \delta(x) = n \).

(iii) \( \delta(x) \) can be computed on any open neighborhood of \( x \).

Indeed, conditions (ii) and (iii) together imply that in our example, we must have \( \delta(x_0) = 0 = \delta(x_1) \), but this contradicts (i).

**Example 6.10.** Identify \( A^4_k \) with \( 2 \times 2 \) matrices over \( k \), and let \( \text{GL}_2(k) \) act by left multiplication. The orbits of this action are in bijection with the kernels of the associated linear map. Thus, if \( X \) is the quotient stack \( [A^4_k/\text{GL}_4(k)] \), we see that the point corresponding to the zero matrix has codimension 2 in the underlying topological space of \( X \), but codimension 4 in \( X \) itself.

### 7. Relative dimension and stacks

Finally, we generalize our results on relative dimension of morphisms to the setting of algebraic stacks. First, following the standard procedures, we will obtain from Corollary 4.3 that our definition of relative dimension generalizes to stacks.

**Definition 7.1.** Suppose that \( \mathcal{X}, \mathcal{Y} \) are algebraic stacks, locally of finite type over a universally catenary base scheme \( S \), and \( f : \mathcal{X} \to \mathcal{Y} \) a morphism. We say that \( f \) has **universal relative dimension at least** \( n \) if there exists a smooth presentation \( P : Y \to Y \) and \( P' : X \to X \times_Y Y \) of \( f \) such that for all \( x \in X \), the induced morphism \( X \to Y \) has universal relative dimension at least \( n + \text{reldim}_x P' \) at \( x \).

The following proposition says in essence that universal relative dimension is well defined for stacks.

**Proposition 7.2.** Suppose that \( \mathcal{X}, \mathcal{Y} \) are algebraic stacks, locally of finite type over a universally catenary base scheme \( S \), and \( f : \mathcal{X} \to \mathcal{Y} \) a morphism of universal relative dimension at least \( n \). Then for all smooth presentations \( P : Y \to \mathcal{Y} \) and \( P' : X \to \mathcal{X} \times \mathcal{Y} \), we have that for all \( x \in X \), the induced morphism \( X \to Y \) has universal relative dimension at least \( n + \text{reldim}_x P' \) at \( x \).

The proof of this proposition is completely standard, and underlies the fundamental Definition 4.14 of [LMB00]. However, since the proof is omitted in loc. cit., we sketch it here for the convenience of the reader.

**Proof.** Suppose that \( P \) and \( P' \) satisfy the condition of Definition 7.1, and we are given \( Y' \to \mathcal{Y} \) and \( X' \to \mathcal{X} \times \mathcal{Y} \) yielding another smooth presentation of \( f \). Let \( Y'' \to Y \times_Y Y' \) and \( X'' \to (X \times_X X') \times_{Y' \times \mathcal{Y}} Y'' \) be smooth presentations of the relevant algebraic spaces. Then the morphism \( X'' \to Y'' \) factors through both
\[ X \times_Y Y'' = (X \times_Y Y') \times_{Y \times_Y Y'} Y'' \] and \[ X' \times_Y Y'' = (Y \times_Y X') \times_{Y \times_Y Y'} Y''. \]

\begin{center}
\begin{tikzcd}
X \times_Y Y'' & (X \times_X X') \times_{Y \times_Y Y'} Y'' \\
X' \times_Y Y'' & Y'' \arrow[swap]{u}{X''} \arrow[swap]{d}{Y''} \\
\end{tikzcd}
\end{center}

Moreover, because \( X \times_Y Y' = X \times_X (X \times_Y Y') \), we see that in fact \( X'' \) is a smooth cover of \( X \times_Y Y'' \), and similarly of \( X' \times_Y Y'' \). Applying Corollary 4.3 twice, we conclude that the hypothesis on the universal relative dimension of \( X \to Y \) implies the desired statement on \( X' \to Y' \). □

We next use the properties of stack codimension which we have developed to verify that universal relative dimension for stacks could be defined without reference to smooth presentations.

**Proposition 7.3.** Suppose that \( X, Y \) are algebraic stacks, locally of finite type over a universally catenary base scheme \( S \), and \( f : X \to Y \) a morphism. Then \( f \) has universal relative dimension at least \( n \) if and only if for every irreducible algebraic stack \( Z \) over \( Y \), locally of finite type over a universally catenary scheme (possibly distinct from \( S \)), and every irreducible component \( Z' \) of \( X \times_Y Z \), we have

\[ \dim Z'_{(\eta)} - \text{codim}_Z f'(\overline{Z'}) \geq n, \]

where \( \eta \) is the generic point of \( Z' \), and \( f' : X \times_Y Z \to Z \) is the second projection.

**Proof.** Fix \( P : Y \to \mathcal{Y} \) and \( P' : X \to X \times_Y Y \) a smooth presentation of \( f \).

First suppose that \( f \) has universal relative dimension at least \( n \), and let \( Z \to \mathcal{Y} \) be as in the statement of the proposition. Fix a smooth cover \( Z \to Z \times_Y Y \), and let \( g : X \times_Y Z \to Z \) be the second projection. We observe that \( X \times_Y Z \) is a smooth cover of \( X \times_Y Z \): indeed, we obtain this as the composition

\[ X \times_Y Z \to (X \times_Y Y) \times_Y Z = X \times_Y Z \to X \times_Y Z. \]

Accordingly, we can let \( Z' \subseteq X \times_Y Z \) be an irreducible component of the preimage of \( Z' \subseteq X \times_Y Z \), and \( Z_0 \) an irreducible component of \( Z \) containing the image of \( g(Z') \).

\begin{center}
\begin{tikzcd}
Z' & Z' \\
X \times_Y Z & X \times_Y Z \\
\phantom{Z_0} Z_0 & Z \\
Z_0 & Z \\
\end{tikzcd}
\end{center}

Then \( Z' \) is an irreducible component of \( X \times_Y Z \), and hence also of the preimage of \( Z_0 \) in \( X \times_Y Z \). Let \( \eta' \) be the generic point of \( Z' \), and \( \xi \) its image in \( X \). Then by
hypothesis, we have
\[ \dim Z'_{g(\eta')} - \operatorname{codim}_Z g(Z') \geq n + \operatorname{reldim}_P P'. \]

We claim that we have
\[ \dim Z'_{g(\eta')} = \dim Z'_{f(\eta')} + \operatorname{reldim}_P P', \quad \text{and} \quad \operatorname{codim}_Z g(Z') = \operatorname{codim}_Z f'(Z'); \]

together, these claims yield the desired inequality. For the latter claim, according to Lemma 6.7, it is enough to show that \( g(Z') \) is an irreducible component of the preimage of \( f'(Z') \). But because \( X \times_Y Z \to X \times_Y Z \) is smooth, we have that \( Z' \)

\[ \text{dominates} \quad Z, \]

so this is clear. Similarly, if we let \( Z' \) be an irreducible component of \( X \times_Y Z \) contained in the preimage of \( Z' \)

\[ \text{and containing the image of} \quad Z', \]

then we see that \( \dim Z'_{f(\eta')} = \dim Z'_{g(\eta')} \) since the latter is (an irreducible component of) a base extension of the former, and \( \dim Z'_{g(\eta')} = \dim Z'_{g(\eta')} + \operatorname{reldim}_P P', \) since \( Z' \)

\[ \text{is (an irreducible component of) a smooth cover of} \quad Z', \]

obtained as a base change of \( P'. \) We thus conclude one direction of the proposition.

For the converse, given \( x \in X \), restricting to the connected component of \( X \)

\[ \text{containing} \quad x, \]

we may assume that \( X \to X \times_Y Y \) has constant relative dimension \( m, \) and we want to prove that \( X \to Y \)

\[ \text{has universal relative dimension} \quad n + m. \]

Suppose we have \( Z \to Y, \) with \( Z \) irreducible and universally catenary, and let \( Z' \)

\[ \text{be an irreducible component of} \quad X \times_Y Z, \]

with generic point \( \eta; \) if \( g : X \times_Y Z \to Z \)

\[ \text{is the second projection, we wish to show that} \]

\[ \dim Z'_{g(\eta)} - \operatorname{codim}_Z g(Z') \geq n + m. \]

Now, let \( Z' \) be an irreducible component of \( X \times_Y Z \)

\[ \text{containing the image of} \quad Z', \]

under the induced map \( X \times_Y Z \to X \times_Y Z, \) and let \( \eta' \) be the generic point of \( Z'. \)

By hypothesis, we have
\[ \dim Z'_{g(\eta')} - \operatorname{codim}_Z g(Z') \geq n, \]

where \( g' : X' \times_Y Z \to Z \)

\[ \text{is the second projection. We now observe that} \]

\[ X \times_Y Z = X \times_{X' \times_Y Y} X' \times_Y Z, \]

so \( Z' \)

\[ \text{is an irreducible component of a scheme smooth of relative dimension} \quad m \]

\[ \text{over} \quad Z', \]

and in particular also dominates \( Z'. \) The desired inequality follows. \( \square \)

We can now state the main application of relative dimension of morphisms in the context of stacks. Recall that we have defined the notions of reducedness of fibers and of the fiber dimension function \( \delta_x f \) in Definitions 2.3 and 2.5.

**Corollary 7.4.** If \( f : X \to Y \)

\[ \text{has universal relative dimension at least} \quad n, \]

then \( \delta_x f \geq n \) for all \( x \in X. \) If for some \( x \in X, \) we have \( \delta_x f = n, \) then there exists a neighborhood \( U \)

\[ \text{of} \quad x \]

\[ \text{with} \quad \delta_x f = n \quad \text{for all} \quad x' \in U, \]

and on any such neighborhood we have that \( f \)

\[ \text{is universally open.} \]

If further \( Y \)

\[ \text{is reduced and} \quad f \]

\[ \text{is reduced at} \quad x, \]

then \( f \)

\[ \text{is flat at} \quad x. \]

**Proof.** It is immediate from the definitions and from Proposition 3.5 that \( \delta_x f \geq n \)

for all \( x \in X, \) and it then follows from Proposition 2.6 that if \( \delta_x f = n, \) there exists an open neighborhood \( U \)

\[ \text{with} \quad \delta_x f = n \quad \text{for all} \quad x' \in U. \]

The universal openness and flatness assertions then follow from the corresponding statements of Proposition 3.7. \( \square \)
Corollary 7.5. Suppose that $\mathcal{Y}$ is irreducible, and there exist $x \in \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$ closed and irreducible containing $f(x)$ and with support strictly smaller than $\mathcal{Y}$, such that

(I) $f$ has universal relative dimension at least $n$ at $x$;

(II) every irreducible component $\mathcal{X}'$ of $f^{-1}(\mathcal{Y}')$ containing $x$ has
\[ \dim \mathcal{X}' f(\eta) - \operatorname{codim}_{\mathcal{Y}} f(\mathcal{X}') = n, \]
where $\eta$ is the generic point of $\mathcal{X}'$.

Then for every irreducible component $\mathcal{X}''$ of $\mathcal{X}$ containing $x$, we have
\[ f(\mathcal{X}'') \not\subseteq \mathcal{Y}'. \]

If further we have

(III) $\mathcal{Y}'$ has codimension $c$ in $\mathcal{Y}$, and the inclusion $\mathcal{Y}' \hookrightarrow \mathcal{Y}$ has universal relative dimension at least $-c$,

then
\[ \dim \mathcal{X}'' f(\eta') - \operatorname{codim}_{\mathcal{Y}} f(\mathcal{X}'') = n, \]
where $\eta'$ is the generic point of $\mathcal{X}''$.

Proof. In light of Proposition 7.3, the proof of Proposition 3.8 goes through verbatim in the stack setting, using Proposition 6.3 for the first part, and Propositions 6.5 and 6.6 for the second.

We next observe that composition of morphisms still behaves well in the stack setting.

Corollary 7.6. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ has universal relative dimension at least $m$, and $g : \mathcal{Y} \to \mathcal{Z}$ has universal relative dimension at least $n$. Then $g \circ f$ has universal relative dimension at least $m + n$.

Proof. Let $P : \mathcal{Z} \to \mathcal{Z}$, $P' : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Z}$ and $P'' : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}$ give smooth presentations of $g$ and $f$. Then using that
\[ \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y} = (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Z}) \times_{\mathcal{Y} \times_{\mathcal{Z}} \mathcal{Z}} \mathcal{Y}, \]
we have a morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Z}$ which is likewise a smooth cover, and thus yields a smooth presentation of $g \circ f$ as well.

The desired statement is then immediate from Lemma 4.2.

We conclude by giving the obvious generalization of Corollary 5.1 to the stack setting.
Corollary 7.7. Suppose that $f : X \to Y$ is a closed immersion, and $g : Y \to Z$ is smooth of relative dimension $n$, with $Z$ locally of finite type over a universally catenary scheme. If either $Z$ is regular and every component of $X$ has codimension at most $c$ in $Y$, or $X$ may be expressed locally as an intersection of determinantal conditions with expected codimensions adding up to $c$, then $g \circ f$ has universal relative dimension at least $n - c$.

Alternatively, suppose that $f : X \to Y$ is a morphism of smooth algebraic stacks locally of finite type over a scheme $S$, with $S$ universally catenary, and let $m$ and $n$ be the relative dimensions of $X$ and $Y$ over $S$, respectively. Then $f$ has universal relative dimension at least $m - n$.

Here, when we say that $X$ may be expressed locally as an intersection of determinantal conditions with expected codimension adding up to $c$, we mean simply that there exists some smooth presentation of $Y$ for which this is the case.

Proof. For the first statement, it follows immediately from the scheme case that $f$ has universal relative dimension at least $-c$, and $g$ has universal relative dimension at least $n$, so the desired assertion follows from Corollary 7.6.

The second statement can be reduced immediately to the scheme statement given in Corollary 5.1.

\section*{References}

[BLR91] Siegfried Bosch, Werner Lutkebohmert, and Michel Raynaud, \textit{Neron models}, Springer-Verlag, 1991.

[Eis95] David Eisenbud, \textit{Commutative algebra with a view toward algebraic geometry}, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, 1995.

[GD60] Alexander Grothendieck and Jean Dieudonné, \textit{Éléments de géométrie algébrique: I. Le langage des schémas}, Publications mathématiques de l'I.H.É.S., vol. 4, Institut des Hautes Études Scientifiques, 1960.

[GD64] \textit{Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, première partie}, Publications mathématiques de l'I.H.É.S., vol. 20, Institut des Hautes Études Scientifiques, 1964.

[GD65] \textit{Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, seconde partie}, Publications mathématiques de l'I.H.É.S., vol. 24, Institut des Hautes Études Scientifiques, 1965.

[GD66] \textit{Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, troisième partie}, Publications mathématiques de l'I.H.É.S., vol. 28, Institut des Hautes Études Scientifiques, 1966.

[GD67] \textit{Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, quatrième partie}, Publications mathématiques de l'I.H.É.S., vol. 32, Institut des Hautes Études Scientifiques, 1967.

[Hoc75] Melvin Hochster, \textit{Big Cohen-Macaulay modules and algebras and embeddability in rings of Witt vectors}, Queen’s Papers on Pure and Applied Math 42 (1975), 106–195.

[LMB00] Gérard Laumon and Laurent Moret-Bailly, \textit{Champs algébriques}, Springer-Verlag, 2000.

[Oss] Brian Osserman, \textit{Limit linear series moduli stacks in higher rank}, in preparation.

[OT] Brian Osserman and Montserrat Teixidor i Bigas, \textit{Linked symplectic forms and limit linear series in rank 2 with special determinant}, in preparation.

[Ser65] Jean-Pierre Serre, \textit{Algèbre locale. Multiplicités}, Lecture Notes in Mathematics, no. 11, Springer-Verlag, 1965.

[Sta13] The Stacks Project Authors, \textit{Stacks project}, http://stacks.math.columbia.edu, 2013.