H_T-Vertex Algebras

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Abstract. The usual vertex algebras have as underlying symmetry the Hopf algebra $H_D = \mathbb{C}[D]$ of infinitesimal translations. We show that it is possible to replace $H_D$ by another symmetry algebra $H_T = \mathbb{C}[T, T^{-1}]$, the group algebra of the Abelian group generated by $T$. $H_T$ is the algebra of symmetries of a lattice of rank 1, and the construction gives a class of vertex algebras related to the Infinite Toda Lattice in the same way as the usual $H_D$-vertex algebras are related to Korteweg-de Vries hierarchies.

1. Introduction

A vertex algebra is, roughly speaking, a singular, commutative, associative, unital algebra with symmetry, see for instance [Bor01] for a proposal to make this vague statement precise.

The usual vertex algebras ([Bor86], [FLM88], [Kac98]) have as symmetry algebra the Hopf algebra $H_D = \mathbb{C}[D]$ of infinitesimal translations ($D = \frac{d}{dz}$, say). The singularities are obtained by localizing the dual $H_D^* = \mathbb{C}[[z]]$ by inverting $z$, to obtain the Laurent series algebra $K_D = \mathbb{C}[[z]][z^{-1}]$. To emphasize the role of $H_D$ as symmetry algebra we will refer to the usual vertex algebras as $H_D$-vertex algebras.

We can think of $H_D$ as the universal enveloping algebra of the 1 dimensional (Abelian) Lie algebra generated by $D$. In this paper I will sketch what a vertex algebra looks like when we take not $H_D$ but $H_T$ as symmetry algebra, where $H_T = \mathbb{C}[T, T^{-1}]$ is the group algebra of the free Abelian group generated by $T$. More details can be found in [Ber].

A motivation to study $H_T$-vertex algebras comes from the theory of integrable systems. There are many similarities between Korteweg-de Vries type hierarchies on the one hand, and infinite Toda lattice hierarchies on the other. For instance, both have infinitely many conservation laws, multiple Hamiltonian structures, tau-functions and connections with infinite dimensional Grassmann manifold, and Miura transformations. (See [Kup85] for an overview of lattice hierarchies). It is well known that Korteweg-de Vries type hierarchies are intimately related to $H_D$-vertex algebras, see for instance [FBZ04]. Now the infinite Toda lattice hierarchy has $H_T$-symmetry, with the generator $T$ acting as a shift by one step on

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the lattice. So it is natural to look for a type of vertex algebras that are related to the infinite Toda lattice in the same way as $H_D$-vertex algebras are related to Korteweg-de Vries hierarchies.

2. Review of some ingredients for $H_D$-vertex algebras

We will review the ingredients that go into the definition of $H_D$-vertex algebras from a slightly unorthodox point of view, in order to prepare the way for replacing $H_D$ as the symmetry algebra by $H_T$. See also [Bor98] or [Sny].

Let $W_1, W_2$ be vector spaces. A $W_2$-valued distribution on $W_1$ is just a linear map $D: W_1 \to W_2$. We will write $\langle D, w_1 \rangle$ for the value of $D$ on $w_1 \in W_1$. Let $M$ be an $H_D$-module. For all $m \in M$ we get an $M$-valued distribution on $H_D$:

$$Y(m): h \mapsto h.m, \ h \in H_D.$$ 

We can represent any distribution on $H_D$ by a generating series (or kernel):

$$\lambda(z) = \sum_{n=0}^{\infty} \langle \lambda, D^{(n)} \rangle z^n \in M[[z]], \ D^{(n)} = D^n/n!,$$

and we have

$$\lambda(h) = \epsilon(h.\lambda(z)) = h.\lambda(z)|_{z=0},$$

where $\epsilon: H_D^* \to \mathbb{C}$ is the counit and where $h.$ indicates the action of $H_D$ on its dual $H_D^*$. In particular we have

$$Y(m)(z) = e^{zD}m.$$ 

In the theory of vertex algebras one usually writes $Y(m, z)$ as $Y(m, z)$.

It is easy to write down a system of axioms for the holomorphic vertex operators $Y(m, z)$ that is equivalent to the statement that $M$ is a commutative, associative, unital algebra with a derivation. In fact, one can easily define holomorphic vertex algebras for any commutative and cocommutative Hopf-algebra $H$. These correspond to commutative (etc.) rings with compatible $H$-action. In case $H$ is not longer cocommutative a holomorphic vertex algebra corresponds to a more general ring (braided for instance) with $H$-action. In this paper we restrict ourselves to the commutative and cocommutative Hopf algebras $H_D$ and $H_T$.

We call $Y(m, z)$ holomorphic since it has no singularities, it is a power series. The idea is now to define more general vertex operators, that might contain singularities, while preserving the axioms as much as is possible.

Introduce a singularization of the dual $H_D^*$:

$$K_D = \mathbb{C}((z)) = K_D^{\text{Hol}} \oplus K_D^{\text{Sing}}, \ K_D^{\text{Hol}} = \mathbb{C}[[z]], \ K_D^{\text{Sing}} = \mathbb{C}[z^{-1}]z^{-1}.$$ 

Then we have an isomorphism of $H_D$-modules:

$$\alpha: H_D \to K_D^{\text{Sing}}, \ h \mapsto S(h)\frac{1}{z}.$$ 

Here $S: H_D \to H_D$ is the antipode, where $D \mapsto -D$. Let $\epsilon: H_D \to \mathbb{C}$ be the counit of $H_D$, the multiplicative map such that $D \mapsto 0$. Observe then that $\epsilon$ corresponds under $\alpha$ to the residue on $K_D^{\text{Sing}}$.

We defined holomorphic vertex operators as distributions on $H_D$. Using $\alpha$ we can think of them also as distributions on $K_D^{\text{Sing}}$. Then it is obvious how one can introduce singular vertex operators: they should be distributions on all of $K_D$, not just on $K_D^{\text{Sing}}$. 
Another ingredient in the theory of vertex algebras, besides vertex operators and the residue, is the Dirac $\delta$-distribution, which appears in the commutator of vertex operators. It can be described in terms of the singularization $K_D$ of $H^*_D$, as follows. First of all we have on $H^*_D$ the twisted coproduct $f(z) \mapsto f(z_1 - z_2)$, dual to the $S$-twisted multiplication $h_1 \otimes h_2 \mapsto h_1 S(h_2)$ on $H_D$. We can calculate this twisted coproduct in two ways, using $S$-twisted exponentials:

\[
(2.2) \quad f(z_1 - z_2) = \mathcal{L}_S f(z_1) = \mathcal{R}_S f(z_2), \quad \mathcal{L}_S = e^{-z_2 \partial_1}, \quad \mathcal{R}_S = S e^{z_1 \partial_2}.
\]

If we try to extend the definition of the twisted coproduct from $H^*_D$ to its singularization $K_D$ we see that the actions of $\mathcal{L}_S$ and $\mathcal{R}_S$ no longer give the same answer. We then define the Dirac distribution for $p \in K_D$ as the obstruction to the extension of the twisted coproduct:

\[
(2.3) \quad \delta(p(z)) = \mathcal{L}_S(p(z_1)) - \mathcal{R}_S(p(z_2)).
\]

For instance,

\[
\delta\left(\frac{1}{z}\right) = \delta(z_1, z_2) = \sum_{k \in \mathbb{Z}} z_1^k z_2^{-k-1}
\]

is the usual Dirac $\delta$-distribution.

Now we have formulated some ingredients of $H_D$-vertex algebras in terms of the symmetry Hopf-algebra $H_D$ and the localization of the dual, we are ready to try to extend the theory to other symmetry algebras.

### 3. $H_T$-symmetry and singularities.

Let $H_T = \mathbb{C}[T, T^{-1}]$, and give it a Hopf algebra structure by thinking of it as the group algebra of the free Abelian group generated by $T$. This group is of course just the (additive) group of integers $\mathbb{Z}$, and the dual of $H_T$ is the space

\[
H^*_T = \mathbb{C}_Z = \{s: \mathbb{Z} \to \mathbb{C}\}
\]

of arbitrary complex valued functions on $\mathbb{Z}$, or, equivalently, the space of two-sided infinite sequences $s = (s_n)_{n \in \mathbb{Z}}$. We can expand any element $s \in \mathbb{C}_Z$ as an infinite sum

\[
s = \sum_{n \in \mathbb{Z}} s_n \delta_n,
\]

where $\delta_n$ is the Kronecker sequence; as a function on $\mathbb{Z}$ we have $\delta_n: k \mapsto \delta_{kn}$. The natural action of $H_T$ on its dual is given by $T \delta_n = \delta_{n-1}$.

Now we want to find a singularization of $H^*_T$. Note that it would not be useful to invert some of the Kronecker sequences, as they are zero divisors: $\delta_n \delta_k = \delta_{nk} \delta_n$. To find suitable elements of $H^*_T$ to invert we observe that in the $H_D$ case the powers $z^k$ we invert are the unique solutions of a system of differential equations: if $f_\ell(z) \in H_D^*$ is a solution of

\[
\partial_z f_\ell(z) = \ell f_{\ell-1}(z), \quad f_\ell(z)|_{z=0} = 0, \quad \ell \geq 0,
\]

with $f_0(z) = 1$, then $f_\ell(z) = z^\ell$. Similarly, introduce in $H_T$ the difference operator $\Delta = T - 1$ and consider the system of difference equations for $\tau(\ell) \in H^*_T$, $\ell \geq 1$

\[
\Delta \tau(\ell) = \ell \tau(\ell - 1), \quad \tau(0)|_0 = 0,
\]

where $\tau(0) = 1 = \sum_{n \in \mathbb{Z}} \delta_n$. Then the $\tau(\ell)$s are uniquely determined. We have $\tau = \tau(1) = \sum_{n \in \mathbb{Z}} n \delta_n$, and $\tau(\ell) = \tau(\tau - 1) \ldots (\tau - \ell + 1)$. The $\tau(\ell)$s are the restriction of polynomial functions on $\mathbb{C}$ to the integers.
Now let $C^\text{pol}_\mathbb{Z} = \mathbb{C}[\tau] \subset H^*_\tau$, the space of polynomial functions on $\mathbb{Z}$. Let then $M \subset C^\text{pol}_\mathbb{Z}$ be the multiplicative set generated by the translates $T^k\tau = \tau + k$ of $\tau$. Then we define the following singularization

$$K_T = M^{-1}C^\text{pol}_\mathbb{Z}.$$ 

We could have localized all of $H^*_\tau$, but then there would have been no clear distinction between singular and nonsingular elements in the localization, see [Ber] for details.

Let $S : H_T \to H_T$ be the antipode, $T^k \mapsto T^{-k}$. We have, as in the case of $H_D$, see (2.1), a map

$$\alpha : H_T \to K_T^{\text{Sing}}, \quad h \mapsto S(h)\frac{1}{\tau},$$

but it is not longer an isomorphism, for instance $\frac{1}{\tau}^2$ is not in the image of $\alpha$. However, when we complete $H_T$ by adjoining to $H_T$ the infinite sum

$$\partial_\tau = \log(T) = \log(1 + \Delta) = \sum_{n=1}^{\infty} (-\Delta)n/n,$$

to define $\hat{H}_T = \mathbb{C}[T, T^{-1}, \partial_\tau]$, then

$$\alpha : \hat{H}_T \to K_T^{\text{Sing}},$$

defined as before, is an isomorphism. The counit $\epsilon : H_T \to \mathbb{C}$ corresponds, via $\alpha$, to the map, called the trace,

$$\text{Tr} : K_T \to \mathbb{C}, \quad f(\tau) \mapsto \sum_{n \in \mathbb{Z}} \text{Res}_n(f(\tau)d\tau).$$

Associated to $H_T$ we have twisted exponential operators $L_S, R_S$, analogous to those of (2.2), and we can define Dirac distributions as before. In particular we have the Dirac $\delta$-distribution defined by

$$\delta(\frac{1}{\tau}) = L_S(\frac{1}{\tau_1}) - R_S(\frac{1}{\tau_2}) = \sum_{n \in \mathbb{Z}} \tau(n) \otimes \tau(-n - 1), \quad \tau(-|k|) = \frac{1}{\tau(|k|)}.$$

We have an action of $H_T \otimes H_T$ on such two-variable distributions, and we write $h_1 = h \otimes 1$, $h_2 = 1 \otimes h$. Similarly we write $\tau_1(\ell) = \tau(\ell) \otimes 1$ and $\tau_2(\ell) = 1 \otimes \tau(\ell)$ and we denote the distribution $\delta(\frac{1}{\tau})$ also by $\delta(\tau_1, \tau_2)$. It has the usual properties:

- $h_1 h_2 = S(h_2)h_1$,
- $f(\tau_1)\delta(\tau_1, \tau_2) = f(\tau_2)\delta(\tau_1, \tau_2)$, $f(\tau) \in K_T$.
- $\text{Tr}_{\tau_1}(f) = f(\tau_2)$.
- If a distribution $a(\tau_1, \tau_2)$ is killed by the twisted coproduct of an element $p \in C^\text{pol}_\mathbb{Z}$ then it is a finite sum

\begin{equation}
(3.1) \quad a(\tau_1, \tau_2) = \sum_{n, k} a_{n, k}(\tau_2)e_{n, k}\delta(\tau_1, \tau_2),
\end{equation}

where $\{e_{n, k}\}$ is a basis for $\hat{H}_T$. 

4. Distributions and State-Field correspondence

Let $W$ be a vector space, and let $D$ be a $W$-valued distribution on $K_T$. Every such distribution $D$ has a formal expansion:

$$D(\tau) = \sum_{n \in \mathbb{Z}} \langle D, \tau(n) \rangle \tau(-n - 1).$$

If $F \in K_T$ then the value of $D$ on $F$ is a trace:

$$\langle D, F \rangle = \text{Tr}(D(\tau)F).$$

We will often identify a distribution $D$ with its kernel $D(\tau)$. Any distribution has a decomposition in holomorphic and singular part:

$$D = D_{\text{hol}} + D_{\text{sing}},$$

where the kernels of the holomorphic and singular parts have expansion in $\tau(n)$ for $n$ nonnegative, respectively negative. A distribution $D$ is called rational if there is $\phi \in W \otimes K_T$ such that $\langle D, F \rangle = \text{Tr}(\phi F)$ for all $F \in K_T$.

Now let $V$ be a vector space. A field on $V$ is then an $\text{End}(V)$-valued distribution $f = f(\tau)$ such that for all $v \in V$ the $V$-valued distribution $f(\tau)v$ has rational singular part: $f_{\text{sing}}(\tau)v \in V \otimes K_T^{\text{sing}}$. Denote by $V(\tau)$ the space of fields on $V$. Then a State-Field Correspondence is a linear map $Y: V \rightarrow V(\tau)$. We write, if $a \in V$ and $Y$ is a state-field correspondence, $Y(a)(\tau) = Y(a, \tau)$ and we call $Y(a, \tau)$ the vertex operator of $a$, as usual.

Now let $1 = 1_V \in V$ be a distinguished vector, called the vacuum. We say that a state-field correspondence $Y$ satisfies the vacuum axioms (for $1 \in V$) in case

$$Y(1, \tau) = 1_{\text{End}(V)}, \quad Y(f, \tau)1 = f_{\text{hol}}(\tau)1,$$

where $f_{\text{hol}}$ is a holomorphic distribution such that acting on the vacuum the constant term is $f$: $f_{\text{hol}}(\tau)1|_{\tau=0} = f$.

Let $h \in H_T$. Then we define, given a state-field correspondence $Y$ satisfying the vacuum axioms, a linear map $h_Y: V \rightarrow V$ by

$$h_Y f = \langle h, f_{\text{hol}}(\tau) \rangle 1,$$

where $\langle , \rangle$ is the pairing between $H_T$ and $H_T^*$, extended in the obvious way to an $\text{End}(V)$-valued pairing between $H_T$ and holomorphic distributions. At this point we don’t know that the map $h \mapsto h_Y$ gives an $H_T$-module structure to $V$, this is an extra condition on the state-field correspondence.

If $a(\tau), b(\tau)$ are fields on $V$, they are in particular $\text{End}(V)$-valued distributions, and we can calculate their commutator distribution: this is the distribution (on $K_T \otimes K_T$) that acts on $v \in V$ by

$$[a(\tau_1), b(\tau_2)](F \otimes G)v = \left(a(F)b(G) - b(G)a(F)\right)v, \quad F, G \in K_T.$$

We say that these fields are mutually rational if the commutator distribution $[a(\tau_1), b(\tau_2)]$ has rational singularities, i.e., is killed by some element $m^*_\tau(F)$, for $F \in H_T^*$ and $m^*_\tau$ the twisted coproduct on $H_T^*$. In this case the commutator is a finite sum of differential-differences of the delta distribution, see [3.1].
5. Definition and some properties of $H_T$-vertex algebras

An $H_T$-vertex algebra is an $H_T$-module $V$ with a vacuum vector $1_V \in V$ and a state-field correspondence $f \mapsto f(\tau) = Y(f, \tau)$, satisfying the vacuum axioms and furthermore

- (Compatibility) The action of $H_T$ on $V$ is compatible with the state-field correspondence:
  $$h.f = h_V f,$$
  where the left hand side is the action of $H_T$ on $V$ and the right hand side is defined in (3.1).
- (ad-covariance) For all $f \in V$ and $h \in H_T$
  $$\text{ad}_h^V Y(f, \tau) = h_{K_T} Y(f, \tau).$$
  Here $\text{ad}_h^V (X) = \sum h' XS(h'')$, for $X \in \text{End}(V)$, $h \in H_T$ with coproduct $\Delta(h) = \sum h' \otimes h''$.
- (Mutual Rationality) The vertex operators $Y(f, \tau_1)$ and $Y(g, \tau_2)$ are for all $f, g \in V$ mutually rational.

From these axioms one derives easily properties similar to those of $H_D$-vertex algebras. For instance we have covariance of the state-field correspondence:

$$Y(h.f, \tau) = h_{K_T} Y(f, \tau),$$

and skew-symmetry of vertex operators:

$$Y(f, \tau) g = R_V(\tau) Y^S(g, \tau) f,$$

where $R_V(\tau)$ is the exponential operator corresponding to the $H_T$-action on $V$ and $Y^S$ is the antipodal vertex operator: more generally if $D$ is a distribution on $K_T$ then we define its antipode by $\langle D^S, F \rangle = \langle D, S(F) \rangle$.

We can define for $F \in K_T$ the $F$-product of $f, g$ in an $H_T$-vertex algebra:

$$f \{ F \} g = \text{Tr} \left( Y(f, \tau) g F(\tau) \right) \in V.$$

Also we can define the $F$-product of fields by

$$f(\tau_2) \{ F \} g(\tau_2) = \text{Tr}_{\tau_1} \left( f(\tau_1) g(\tau_2) R_S(\tau_2) F(\tau_1) - g(\tau_2) f(\tau_1) L_S(\tau_1) F(\tau_2) \right).$$

Here $R_S, L_S$ are the twisted exponentials already used in (2.3). In particular, for $F = \frac{1}{\tau}$ we obtain the normal ordered product of fields:

$$f(\tau) \{ \frac{1}{\tau} \} g(\tau) =: f(\tau) g(\tau) := f_{\text{hol}}(\tau) g(\tau) - g(\tau) f_{\text{sing}}(\tau).$$

Then we have the fundamental fact that the state-field correspondence is a homomorphism of $F$-products:

$$Y(f \{ F \} g, \tau) = f(\tau) \{ F \} g(\tau).$$

6. $H_T$-conformal algebras

For an $H_D$-vertex algebra $V_D$ we can concentrate on the singular part of the operator expansion to obtain on $V_D$ the structure of conformal algebra (see [Kac98], or [Pr99], where conformal algebras are called vertex Lie algebras). In the same way we can ignore in an $H_T$-vertex algebra $V$ all $F$-products $f \{ F \} g$, except for those with $F \in \mathbb{C}_2^{\text{pol}}$, or more generally $F \in H_T^*$. This defines the notion of an $H_T$-conformal algebra structure on $V$. 

More generally, one can start with an $H_T$-module $C$ and define an $H_T$-conformal structure on $C$ as a collection of conformal products $f\{F\}g$, $F \in H_T^*$ for $f, g \in C$, satisfying a number of axioms that we don’t want write down here\(^1\). In particular there is in an $H_T$-conformal algebra the distinguished product corresponding to the element $F = 1 \in H_T^*$. This product satisfies

\[(Tf)(1)g = f(1)g, \quad f(1)(Tg) = T(f(1)g), \quad f(1)g - g(1)f \in \frak{m}_T C, \quad [f(1), g(1)] = (f(1)g)(1).
\]

Here $\frak{m}_T \subset H_T$ is the augmentation ideal, the kernel of the counit on $H_T$. It is the ideal generated by $\Delta = T - 1$. We see that the 1-product induces a Lie algebra structure on $C/\frak{m}_T C$, for any $H_T$-conformal algebra $C$.

7. Affinization

If $C$ is an $H_T$-conformal algebra and $L$ is a commutative $H_T$-Leibniz algebra (i.e., a commutative algebra in the category of $H_T$-modules), then one can show that also the affinization $LC = C \otimes L$ is canonically an $H_T$-conformal algebra, and hence we obtain on $\mathcal{LC} = LC/\frak{m}_T LC$ the structure of Lie algebra. In particular we can take $L = K_T$, and we will restrict ourselves to this case. Denote by

\[
\text{Tr}: C \otimes K_T \to \mathcal{LC}
\]

the canonical projection and write, for $p \in K_T$ and $f \in C$,

\[
f(p) = \text{Tr}(f \otimes p) \in \mathcal{LC}.
\]

For an explanation of using the same term for both this map and the trace on $K_T$, see [Ber]. Then the commutator in $\mathcal{LC}$ is given by

\[
[f(p), g(q)] = \sum (f(\epsilon_i p) g(\epsilon_i q)), \quad (7.1)
\]

where $\{\epsilon_i\}$ is a basis for $H_T$ and $\{\epsilon^*_i\}$ a dual basis for $H^*_T$. We can define generating series of elements of $\mathcal{LC}$, called currents, for each $f \in C$ by

\[
f(\tau_2) = \text{Tr}_{\tau_1} (f \otimes \delta(\tau_1, \tau_2)).
\]

Then the commutator of currents is given by

\[
[f(\tau_1), g(\tau_2)] = \sum f(\epsilon^*_i \tau_2) g(\tau_2) \epsilon_{i, 2} \delta(\tau_1, \tau_2).
\]

8. The Toda Vertex algebra

If $L$ is a commutative $H_T$-Leibniz algebra, then $L$ is automatically an $H_T$-vertex algebra, giving the simplest examples of them. In this case the vertex operators are holomorphic, see the discussion in Section B so this is not really interesting.

To get a more interesting example we start with the Toda conformal algebra $\mathcal{CToda}$. This is the free $H_T$-module generated by $B$ and $C$, with conformal products

\[
B_\{F\} B = C_\{F\} C = 0, \quad F \in H_T^*, \quad B_\{\delta_n\} C = C_\{\delta_n\} = C(\delta_{n-1} - \delta_{n, 0}), \quad C_\{\delta_n\} B = C\delta_{n, 0} - TC\delta_{n, 1}.
\]

Here the $\delta_n \in H_T^*$ are the Kronecker sequences, see Section B. $\mathcal{CToda}$ is the $H_T$-conformal algebra corresponding to the first Hamiltonian structure of the infinite algebra, along these lines, see the notion of Lie pseudo algebra in Bakalov, D’Andrea and Kac, [BDK01].
Toda lattice, see [Kup85]. We write $\mathcal{L}$Toda for the Lie algebra $\mathcal{L}C$Toda associated to the Toda conformal algebra. The current commutator in $\mathcal{L}$Toda is
\begin{equation}
[B(\tau_1), C(\tau_2)] = C(\tau_2)(T_2^{-1} - 1)\delta(\tau_1, \tau_2).
\end{equation}
We have a decomposition
\begin{equation}
\mathcal{L}$Toda = $\mathcal{L}$Toda$_{\text{Hol}} \oplus $\mathcal{L}$Toda$_{\text{Sing}},
\end{equation}
where $\mathcal{L}$Toda$_{\text{Hol}}$ (respectively $\mathcal{L}$Toda$_{\text{Sing}}$) is spanned by elements $f_{(p)}$, for $p \in K^T_{\text{Hol}}$ (respectively $p \in K^T_{\text{Sing}}$). By (8.1) each summand in (8.2) is a Lie subalgebra. Let $\mathbb{C}$ be the trivial 1-dimensional $\mathcal{L}$Toda$_{\text{Hol}}$-module, and let VToda be the induced $\mathcal{L}$Toda-module:
\begin{equation}
\text{VToda} = \mathcal{U}(\mathcal{L}$Toda) $\otimes \mathcal{U}(\mathcal{L}$Toda$_{\text{Hol}}) \mathbb{C}.
\end{equation}
Then one proves that VToda has an $\hat{H}_T$-vertex algebra structure (with $1_{\text{VToda}} = 1 \otimes 1$ as vacuum) such that if we write $B$ and $C$ for the elements $B(\frac{1}{k})1_{\text{VToda}}$ and $C(\frac{1}{k})1_{\text{VToda}}$, then we have
\begin{equation}
Y(B(\tau_1), Y(C(\tau_2) = Y(C(\tau_2)(T_2^{-1} - 1)\delta(\tau_1, \tau_2),
\end{equation}
compare with (8.1). We call this the Toda $\hat{H}_T$-vertex algebra structure on VToda.

Remark 8.1. Note that, since $K_T$ is in fact an $\hat{H}_T$-module, also VToda will be not just an $\hat{H}_T$-module, but an $\hat{H}_T$-module. In general, all non holomorphic $\hat{H}_T$-vertex algebras seem to be $\hat{H}_T$-modules. This is in contrast with $H_T$-conformal algebras, which don’t need to have an $\hat{H}_T$-modules structure. Now $H_T = H_T[\partial_x]$, if we take $D = \partial_x$, and so we can think of nontrivial $\hat{H}_T$-vertex algebras as a kind of extension of $H_D$-vertex algebras, where we allow in the operator product expansion not only singularities in $\tau_1 - \tau_2$ but also at arbitrary shifts $T_2^k(\tau_1 - \tau_2) = \tau_1 - \tau_2 - k$. Currents with such operator products expansions occur in the theory of Yangians, see e.g., [Kho97].

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