Holographic Kondo Model in Various Dimensions

Paolo Benincasa†, Alfonso V. Ramallo‡

Departamento de Física de Partículas,
Universidade de Santiago de Compostela
E-15782 Santiago de Compostela, Spain

†paolo.benincasa@usc.es, ‡alfonso@fpaxp1.usc.es

Abstract

We study the addition of localised impurities to $U(N)$ Supersymmetric Yang-Mills theories in $(p+1)$-dimensions by using the gauge/gravity correspondence. From the gravity side, the impurities are introduced by considering probe $D(8-p)$-branes extending along the time and radial directions and wrapping an $(7-p)$-dimensional submanifold of the internal $(8-p)$-sphere, so that the degrees of freedom are point-like from the gauge theory perspective. We analyse both the configuration in which the branes generate straight flux tubes – corresponding to actual single impurities – and the one in which connected flux tubes are created – corresponding to dimers. We discuss the thermodynamics of both the configurations and the related phase transition. In particular, the specific heat of the straight flux-tube configuration is negative for $p < 3$, while it is never the case for the connected one. We study the stability of the system by looking at the impurity fluctuations. Finally, we characterise the theory by computing one- and two-point correlators of the gauge theory operators dual to the impurity fluctuations. Because of the underlying generalised conformal structure, such correlators can be expressed in terms of an effective coupling constant (which runs because of its dimensionality) and a generalised conformal dimension.

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1 Introduction

The gauge/gravity correspondence [1–4] provides a powerful handle on the strongly coupled regime of gauge theories. As originally formulated [1], it conjectures a relation between the $AdS_5 \times S^5$ “near-horizon” geometry generated by a stack of $N$ coincident D3-branes and the four-dimensional $\mathcal{N} = 4 SU(N)$ Supersymmetric Yang-Mills theory living on the boundary of $AdS_5$. Such an equivalence between supergravity on $AdS_5 \times S^5$ on one side and $\mathcal{N} = 4$ SYM theory on the other, can be made precise by identifying the supergravity partition function with the generating function for the gauge theory correlators, with the boundary value of the bulk modes acting as a source of the corresponding gauge theory operator [3].

The correspondence can be extended to the case of generic D$p$-branes ($p \neq 3$), whose “near-horizon” geometry describes the adjoint degrees of freedom of $U(N)$ Supersymmetric Yang-Mills theory in $(p+1)$-dimensions [5]. One main difference with the $p = 3$ case is that, while the latter is characterised by a dimensionless coupling constant, the coupling constant turns out to be dimensionful for $p \neq 3$ and the effective coupling constant runs with the energy scale, making the theory non-conformal. However, there exists a frame [6] – named dual frame – in which the background induced by the stack of $N$ D$p$-branes is conformally $AdS_{p+1} \times S^{8-p}$ with the conformal factor depending on a non-trivial dilaton [7–9]. This frame is further characterised by a manifest generalised conformal symmetry [10–12]. Specifically, the generalised conformal structure emerges by allowing the string/Yang-Mills coupling to transform as a background field under conformal transformation, providing, on the other hand, diffeomorphism and trace Ward identities. Moreover, in this frame the radial direction plays the role of the energy scale of the dual gauge-theory [9,13]. The holographic RG flow is, in any case, trivial since the theory flows just because of the dimensionality of the coupling constant. The dual frame has been crucial to make holography for D$p$-branes precise and to extend the holographic renormalisation procedure¹ [15–17]. In particular, it has been observed [17] that the $(p+2)$-effective supergravity action, obtained by the Kaluza-Klein reduction on the $(8-p)$-dimensional sphere, can be recovered by dimensional reduction of the theory on pure asymptotically $AdS_{2\sigma+1}$ on a torus, where $\sigma$ is a parameter related to the power of the radial direction in the dilaton and, generically, takes fractional values for generic values of $p$. Such a parameter $\sigma$ can be considered as an integer and, after the reduction on the torus, can be analytically continued to take its actual $p$-dependent value. One can therefore map the problem to a pure $AdS_{2\sigma+1}$ theory and then obtain the $(p+2)$-dimensional answer through Kaluza-Klein reduction. This way to rephrase the problem allows to drastically simplify the computation of the counterterms needed for holographic renormalisation and of quantities such as, for example, the correlators of the stress-energy tensor.

¹For more details about holographic renormalisation, see [14].
As we mentioned earlier, the backgrounds generated by a stack of Dp-branes describe the adjoint degrees of freedom of the dual gauge theory. The correspondence can however be generalised by inserting extra degrees of freedom in the theory. In particular, one can add a certain number of branes as probes, so that they do not backreact on the geometry, introducing a fundamental hyper-multiplet and partially or completely breaking the initial supersymmetries [18].

In this paper we are concerned with the introduction of D(8−p)-branes in the conformally AdS$_{p+1} \times S^{8−p}$ background generated by a stack of Dp-branes, in such a way that they wrap an $S^{7−p} \subset S^{8−p}$ and the induced metric on their world-volume turns out to be conformally AdS$_2 \times S^{7−p}$. Such configurations were analysed in [19, 20] and they describe bound states of fundamental strings stretching along the radial direction. From the point of view of the dual SYM theory, this is equivalent to introducing point like objects acting as impurities in an interacting (trivially) non-conformal field theory. The conformal case $p = 3$ has been recently studied to holographically realise Kondo models [21–25], and such idea has been extended to the case of ABJM theory for which the impurities have been introduced via D6-branes extended along $AdS_2 \subset AdS_4$ and wrapping a squashed $T^{1,1}$ space inside the internal $\mathbb{C}P^3$ [26].

We mainly study two classes of configurations: one in which a straight flux tube is formed and the other in which two flux tubes ending on two different points at the boundary are connected in the bulk. In order to establish the stability of the brane configurations, we analyse the fluctuations of the probe branes at zero temperature. After decoupling the modes, we can characterise the operators dual to such fluctuations through the effective coupling constant and the generalised scaling dimension. In particular, we can read them off by computing one-point and two-point correlation functions in the coordinate space. The calculation is made straightforward by observing that such fluctuations satisfy the equation of motion of a free massive scalar propagating in a higher dimensional $AdS_{q+1}$ space, where the dimension $q$ needs to be analytically continued to a (generically) fractional value$^2$. The dual operators are irrelevant, so we can make sense of the correlation functions by using the observation made in [29] stating that it is possible to holographically renormalise the bulk fields in a perturbative way up to some fixed order $n$. Given that we are interested into one-point and two-point correlators, it is enough to holographically renormalise the fluctuation action up to quadratic order, where it can always be written as the action of a free massive scalar in $AdS_{q+1}$. The results we find for the generalised scaling dimensions coincide with the known conformal dimensions for such operators in the case $p = 3$.

$^2$This has already been observed in the case of probe flavour branes [27] in these non-conformal backgrounds, for which mainly the embedding functions have been considered. For a discussion about the holographic renormalisation of probe flavour branes in the conformal case, see [28].
We further analyse the basic thermodynamic properties of the impurities, such as their free energy and entropy. Interestingly, we find that, in the class of systems of interest, the impurity entropy is generically non-analytic in the filling fraction, except for the D4/D4-system for which it is possible to obtain a simple closed expression. This is actually the only system in which the impurity specific heat turns out to be positive, while it is zero for $p = 3$ and it is negative for $p < 3$.

The paper is organised as follows. In Section 2 we review the basic features of the Kondo models. In particular, we discuss the effect of a single impurity-spin in a gas of free fermions in (3+1)-dimensions with one or $k$ identical channels and with an $SU(2)$ or $SU(N)$ spin-symmetry group. We also review the case in which the impurity spin is inserted in (1+1)-dimensional systems, in which the interactions cannot be generally neglected. In Section 3 we discuss the holographic setup for the ambient non-conformal theory, emphasising the existence of a frame in which the bulk is conformally $AdS$, and we also introduce the impurities as probe D$(8-p)$-branes wrapping a conformally $AdS_2 \times S^{7-p}$ submanifold of the bulk geometry. We also comment on the underlying generalised conformal structure in the systems we are considering. In Sections 4 and 5 we discuss the two classes of configurations of interests, respectively the one in which the probe branes generate straight flux tubes and the one in which two flux tubes ending on two different points at the boundary are connected in the bulk. In particular, Section 4 is dedicated to the study of the thermodynamics of the straight-flux tube/impurity by computing explicitly the free energy, the entropy, the internal energy, the specific heat and the susceptibility as functions of the temperature and of the filling fraction. Section 5 is instead dedicated to a preliminary study of the hanging flux tube configurations, for which the free energy and the one-point function of the operator dual to the embedding function are computed. In Section 6 we study the thermodynamics of such connected configurations. We again compute the free energy, entropy, internal energy, and specific heat and we study the competition between the dimer configurations and the disconnected impurities. We identify a first order phase transition and the related temperature at which it occurs. Section 7 is devoted to the detailed analysis of the fluctuations for the probe branes. We identify two decoupled channels and we compute the one-point and two-point correlators of the operators dual to such fluctuations, emphasising their generalised conformal structure. Finally, Section 8 contains our conclusion.

2 Generalities of the Kondo Model

The Kondo model [30–33] is one of the examples of quantum impurity problems [33], where by impurity one refers to point-like degrees of freedom inserted in and interacting with a (generally non-interacting) gas. This class of systems is characterised by two main features:
the existence of gapless excitations far away from the impurity and the localisation at one point in position space of the interaction with the impurity. Furthermore, even if the impurities are introduced in (generally) \((3 + 1)\)-dimensional theories, such systems can be mapped to \((1 + 1)\)-dimensional one – except for the case in which the bulk theory is already \((1 + 1)\)-dimensional –, and the impurities appear located at the origin of the one-dimensional position space.

Typically, one talks about Kondo problem when a free fermion gas is coupled to an impurity with a spin degree of freedom. The interesting feature of such a system is that the resistivity shows a minimum as a function of the temperature: when the temperature approaches the so-called Kondo temperature, the interaction of the bulk electrons with the impurity starts to compensate the effect of the interaction with the lattice phonons (which leads to a decrease of the resistivity as the temperature is lowered) so that, as the temperature becomes lower than the Kondo temperature, the effect of the impurity dominates.

### 2.1 Single-Channel Kondo Model

The Hamiltonian density describing the Kondo model with just one impurity spin is

\[
\mathcal{H}_{(3+1)} = \psi_{\alpha}^\dagger \left[ -\frac{\nabla^2}{2m} - \epsilon_F \right] \psi_{\alpha} + J \delta^{(3)}(\mathbf{x}) \psi_{\alpha}^\dagger \frac{\mathbf{\sigma}}{2} \psi_{\alpha} \cdot \mathbf{S},
\]

(2.1)

where \(\psi_{\alpha}\) is the bulk electron field with spin-index \(\alpha\), \(\epsilon_F\) is the Fermi energy, \(\mathbf{\sigma}\) are the Pauli matrices, \(\mathbf{S}\) is the spin-1/2 impurity operator and \(J\) is the coupling for a Heisenberg-type exchange interaction between the impurity spin and the electron spin density. Moreover, a sum over the spin-index \(\alpha\) is understood.

The coupling constant \(J\) is dimensionful. Its dimensionless version can be defined as \(\lambda \overset{\text{def}}{=} J \rho\), \(\rho\) being the density of states per unit of energy, per unity volume and per spin. In the case of a free fermion gas \(\rho = m k_F/\pi^2\), with \(k_F\) being the Fermi momentum.

The delta function in the coupling selects just the s-wave to interact with impurity. This allows the \((3+1)\)-dimensional model described by the Hamiltonian density (2.1) to be reduced to an effective \((1 + 1)\)-dimensional model defined on half-line with the impurity located at the origin:

\[
\mathcal{H}_{(1+1)} = \psi_{\alpha}^\dagger \frac{d}{dx} \psi_{\alpha} + \lambda \delta(x) \psi_{\alpha}^\dagger \frac{\mathbf{\sigma}}{2} \psi_{\alpha} \cdot \mathbf{S}.
\]

(2.2)

This is the Hamiltonian density for a massless Dirac fermion in \((1 + 1)\)-dimensions and therefore the theory is conformal, with a boundary. A characteristic of this system is that the renormalised coupling constant \(\lambda(E)\) increases as the energy scale \(E\) is lowered, so that at low enough energies perturbation theory breaks down. The \(\beta\)-function turns out to be

\[
\frac{d\lambda}{d\ln \Lambda} = -\lambda^2 + \ldots, \quad \Rightarrow \quad \lambda(\Lambda) \sim \frac{\lambda_0}{1 - \lambda_0 \ln (\Lambda_0/\Lambda)},
\]

(2.3)
Λ being a scale, and Λ₀ is the scale at which the Kondo coupling constant acquires the bare value λ₀. In the case in which the bare coupling λ₀ is negative (ferromagnetic behaviour), the coupling λ decreases as the energy scale decreases, being therefore well-behaved. If instead λ₀ is positive, there is a value for the energy scale at which perturbation theory breaks down and it is represented by the so-called Kondo temperature Tₖ

\[ Tₖ \sim Λ₀ e^{-1/λ₀}. \]  

(2.4)

In this case, the Kondo coupling constant can be thought to renormalise to infinity [34–36]. This can be understood by looking at the lattice version of the (1+1)-dimensional Hamiltonian density (2.2)

\[ H_{\text{Lat}}^{(1+1)} = -t \sum_i \left[ \psi_i^+ \psi_{i+1} + \text{h.c.} \right] + J \psi_0^+ \frac{σ}{2} \psi_0 \cdot \vec{S}, \]  

(2.5)

where the strong coupling regime is at \( J \gg t \). For \( t = 0 \), the electron configuration in a general site is arbitrary, except at the origin where the electron form a singlet with the impurity. For relatively small \( t \) compared with the Kondo coupling \( λ \), the electrons in a general site are in a Bloch state whose single-particle wave-function vanishes at the origin in order to preserve the singlet-condition.

The interesting feature of this system is that, in the even-parity sector, the wave-functions at zero and infinity Kondo couplings differ from each other, while in the odd-parity sector they do not. The strong coupling fixed point is the same as the weak coupling one, with the difference that the impurity is screened and substituted by the boundary condition \( ψ(0) = 0 \). For finite/small Kondo coupling this is still true but only at low energies and long distances. The boundary condition is a fixed point and this is a feature of all quantum impurity systems.

The flow to the low energy fixed point is controlled by the leading irrelevant operator, which is constructed out of the fermion fields and it is \( SU(2) \)-invariant. Such a symmetry allows for two dimension-2 operators, \( (ψ^+α(0)ψ_α(0))^2 \) and \( \left( (ψ^+α(0)σ_α βψ_α(0))/2 \right)^2 \). However, the first one is suppressed given that its coefficient turns out to be of order \( 1/Λ₀ \), while the coefficient of the latter operator of order \( 1/Tₖ \), with \( Λ₀ \gg Tₖ \).

Using perturbation theory in \( 1/Tₖ \), one finds that, at low temperature, the impurity susceptibility – defined as the difference between susceptibility with and without impurity – turns out to behave as a constant. Furthermore, at high temperature and in the scaling limit of small bare Kondo coupling, it is proportional to the inverse of the temperature. Actually, the impurity susceptibility can be written as

\[ \chi^{\text{imp}}(T) = \frac{1}{4Tₖ} f \left( \frac{T}{Tₖ} \right), \]

\[ \left\{ \begin{array}{l} \chi^{\text{imp}}(T) \rightarrow (4Tₖ)^{-1}, \text{ as } T \rightarrow 0, \\ \chi^{\text{imp}}(T) \rightarrow (4T)^{-1}, \text{ for } T \gg Tₖ. \end{array} \right. \]  

(2.6)
2.2 Multi-Channel Kondo Model

The single-impurity model discussed in the previous section can be generalised by considering $k$ conduction electron channels interacting with the impurity spin operator $\vec{S}$ [37,38]

$$\mathcal{H}^{(k)}_{(1+i)} = \psi^{\dagger j\alpha} i \frac{d}{dx} \psi_{j\alpha} + \lambda \psi^{\dagger j\alpha}(0) \frac{\sigma^j}{2} \psi_{j\alpha}(0) \cdot \vec{S},$$

(2.7)

where $\alpha$ is a spin-index (for the moment considered to run on two possible spin-states), the sum over the index $j$ is understood and $j = 1, \ldots, k$. In this Hamiltonian density, the channels are assumed to be identical, preserving a $SU(k)$-symmetry.

The $\beta$-function turns out to be

$$\frac{d\lambda}{d\ln \Lambda} = -\lambda^2 + \frac{k}{2} \lambda^3 + \ldots$$

(2.8)

As in the single-channel case, the Kondo coupling $\lambda$ renormalises to infinity for an antiferromagnetic bare coupling $\lambda_0$, as long as $S \geq k/2$. More precisely, one electron per channel is expected to go into a symmetric state near the origin forming a total spin $k/2$ and the ground state has size $|S - k/2|$. The low temperature features in the under-screened case $S > k/2$ are similar to the ones of the single-channel Kondo model, where the strong coupling fixed point is stable. The same occurs in the exactly screened case $S = k/2$. In the over-screened case $S < k/2$, the RG-flow leads to a non-trivial finite coupling fixed point, for which the free energy is non-analytic as function of temperature and magnetic field. Let us look more in detail to the thermodynamic properties. In particular, the impurity specific heat and the impurity magnetic susceptibility in the over-screened case turn out to be

$$C^{\text{imp}}(T) \propto \begin{cases} T^{4/(k+2)}, & k \neq 2 \\ \frac{T}{T_K} \ln \frac{T}{T_K}, & k = 2 \end{cases}, \quad \chi^{\text{imp}}(T) \propto \begin{cases} T^{-\frac{k-2}{k+2}}, & k \neq 2 \\ \ln \frac{T}{T_K}, & k = 2 \end{cases}.$$  

(2.9)

In the under-screened case, the impurity magnetic susceptibility diverges as the temperature $T$ is lowered, leading to a non-Fermi-liquid behaviour. In the large-$k$ case, instead, one has

$$C^{\text{imp}}(T) \propto 1 + \frac{4}{k} \ln T + \mathcal{O}(k^{-2}), \quad \chi^{\text{imp}}(T) \propto \frac{1}{T} \left[ 1 + \frac{4}{k} \ln T + \mathcal{O}(k^{-2}) \right].$$

(2.10)

The magnetic susceptibility diverges as $T^{-1}$ as the temperature is lowered, while a low-temperature logarithmic divergence in the impurity specific heat appears as a $1/k$-effect.

Let us now consider the case in which the spin-symmetry group is $SU(N)$, so that the spin-index $\alpha$ in the Hamiltonian density (2.7) run from 1 to $N$ [39]. The bulk fermions transform under the fundamental representation of $SU(N)$, while the impurity-spin transforms under an anti-symmetric representation with $Q$ indices. As in the $SU(2)$ multichannel model...
discussed above, also its $SU(N)$ generalisation presents a fixed point at intermediate Kondo coupling. In this fixed point, the local impurity spin two-point function has the following zero-temperature behaviour

$$\langle S(t)S(0) \rangle \sim \frac{1}{t^{2\Delta_{\text{imp}}}}, \quad \Delta_{\text{imp}} = \frac{N}{N+k},$$

and the impurity susceptibility behaves as the local susceptibility (which is obtained integrating the above correlation function)

$$\chi_{\text{imp}}(T) \sim \chi_{\text{loc}}(T) \sim \begin{cases} 
T^{-\frac{k+N}{k+N}}, & k > N, \\
\ln T^{-1}, & k = N, \\
\text{const.}, & k < N
\end{cases}$$

Notice that the expression (2.12) is the natural generalisation to the $SU(N)$-spin symmetry of Eq. (2.9), where the spin symmetry is $SU(2)$ – the impurity susceptibility acquires a logarithmic behaviour when the number of channels is equal to the rank of the spin symmetry group.

It is interesting to consider the large-$N$ limit of this class of systems. There are two ways to take such a limit: one can take just the rank of the spin-symmetry group to be large (with $k \ll N$) or taking it to be large and keeping the ratio $\gamma \equiv k/N$ between the number of channels and the rank of the spin-symmetry group to be fixed. In the latter case, the expression for the scaling dimension of the local impurity spin and the behaviour of the impurity susceptibility with the temperature do not change – one can just conveniently rewrite them in terms of the parameter $\gamma$ as $\Delta_{\text{imp}} = (1+\gamma)^{-1}$ and $\chi_{\text{imp}} \sim T^{-(\gamma-1)/(\gamma+1)}$ for $\gamma > 1$. In case $k$ is instead not taken to be large (and of the same order of $N$), the large-$N$ limit of the local impurity spin scaling dimension and of the impurity susceptibility become

$$\Delta_{\text{imp}} = 1 - \frac{k}{N} + O \left( (k/N)^2 \right), \quad \chi_{\text{imp}}(T) \sim \text{const.}, \quad k \ll N.$$ 

Finally, it is interesting to write down the explicit expression for the impurity entropy

$$S_{\text{imp}} = \ln \prod_{v=1}^{Q} \frac{\sin[\pi(v-1+v+N)/(k+N)]}{\sin[\pi v/(k+N)]},$$

which vanishes in the single-channel case.

### 2.3 Impurities in Luttinger Liquids

The impurity systems reviewed in the previous two sections are characterised by the fact that the bulk fermions are non-interacting and the localisation of the impurity at a point allows to
reduce the original \((3 + 1)\)-dimensional problem to a \((1 + 1)\)-dimensional one. However, one can think to introduce an impurity spin directly in a \((1+1)\)-dimensional system. In this case, the bulk degrees of freedom cannot always be considered as non-interacting and, moreover, they cannot behave neither as a Bose liquid nor as a Fermi liquid. Rather, they are thought to be described by Tomonaga-Luttinger liquids [40–45].

This class of systems is characterised by the absence of quasi-particle excitations and the presence of plasmons and spin density waves, which are independent of each other. Furthermore, while for Bose and Fermi liquids the specific heat at low temperature scales as \(T^d\), \(d > 1\) being the number of spatial dimensions, and \(T\) respectively, in \((1 + 1)\)-dimensions it can scale either as \(\sim T^{\alpha(\lambda)}\) or linearly with the temperature, depending on whether the system is interacting or not – \(\alpha(\lambda)\) is a function of the Luttinger parameter \(\lambda\) and its form depends on the attractive or repulsive nature of the interaction.

The Hamiltonian density for a single impurity in a Luttinger liquid can be written as

\[
\mathcal{H}_{\text{Lut}} = -v_F \sum_j \left[ (\psi_j^\dagger \psi_{j+1} + \text{h.c.}) + U \hat{n}_j^2 \right] + \mathcal{H}_{\text{bulk}} + \mathcal{H}_{\text{imp}},
\]

where the first term is the Hubbard model describing the bulk degrees of freedom, with \(\hat{n}_j\) being the total-number operator at site \(j\), \(v_F = 2at \sin (ak_F)\), \(U\) and \(t\) are the usual Hubbard parameters, \(\mathcal{H}_{\text{bulk}}\) is the bulk interaction term, and \(\mathcal{H}_{\text{imp}}\) is the bulk-impurity interaction. Several bulk interactions are possible. Examples are the operators

\[
\hat{O}_1 \equiv \lambda(\lambda) J_L J_R, \quad \hat{O}_2 \equiv \lambda(\lambda) \psi_{L,\alpha}^\dagger \psi_{R,-\alpha}, \quad \hat{O}_3 \equiv - (\lambda(\lambda)/2\pi) \hat{J}_L \cdot \hat{J}_R,
\]

as well as spin anisotropic interactions of zero conformal spin. In the case \(\mathcal{H}_{\text{bulk}} \sim \hat{O}_1\), through a field redefinition, the Luttinger-liquid Hamiltonian density acquires the non-interacting form, at the price that the scaling dimensions of various operators change. If \(\mathcal{H}_{\text{bulk}} \sim \hat{O}_2\) is considered, its effect is not-negligible just in the half-filling case and produces a charge gap for values of the Luttinger parameter for which the interaction is repulsive.

Let us now turn to the bulk-impurity term of \(\mathcal{H}_{\text{Lut}}\), which can be taken as

\[
\mathcal{H}_{\text{imp}} = \lambda_{kl} \psi_{k,\alpha}^\dagger \frac{\sigma^\beta}{2} \psi_{l,\beta}^\dagger,
\]

where the indices \(k, l\) run over \(L, R\), which indicates left- and right-movers, while the indices \(\alpha, \beta\) run over the spin values – in the case of \(SU(2)\) spin-symmetry, they can take just two values. For \(\lambda_{LL} = \lambda_{RR}\) and \(\lambda_{LR} = \lambda_{RL}\) one gets respectively the amplitudes for forward and backward scattering of the bulk degrees of freedom with the impurity. The Kondo interaction is obtained when all these couplings are equal.

In a model with such a bulk-impurity interaction with the bulk-interaction switched off, the Hamiltonian density can be mapped to the one of a two-channel model with the impurity coupled to the bulk fermions just in one of the channels. Like in the \((3+1)\)-dimensional
Kondo problem, the model renormalises to a local Fermi liquid and the response function scales analytically with the temperature \([37]\).

When a bulk interaction such as \(\hat{O}_2\) is switched on, any impurity interaction is expected to be substituted by a renormalised boundary condition on the critical bulk theory. The theory turns out to be characterised by the impurity specific heat and the impurity susceptibility at low temperature given by \([46]\)

\[
C^{\text{imp}}(T) = c_1 \left[ \frac{1}{\alpha (\lambda^{(L)})} - 1 \right]^2 \frac{1}{T^\alpha (\lambda^{(L)}) - 1} + c_2 T, \quad \chi^{\text{imp}}(T) = c_3 T^{0}, \tag{2.17}
\]

with \(\alpha (\lambda^{(L)}) = (1 + 2\lambda^{(L)}/v_p)^{-1/2}\).

### 3 The Holographic Set-up

Let us start with a brief review of the holographic realisation of the “ambient” theory where the impurities will be later introduced. Consider the background generated by a stack of \(N\) \(D_p\)-branes in the string frame:

\[
ds_{10}^2 = \left( 1 + \frac{r_p^{7-p}}{r^{7-p}} \right)^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left( 1 + \frac{r_p^{7-p}}{r^{7-p}} \right)^{1/2} ds_T^2, \tag{3.1}
\]

where \(\mu, \nu = 0, \ldots, p\), \(\eta_{\mu\nu}\) is the flat Minkowski metric in \(p+1\) dimensions, \(r\) is the radial coordinate of the transverse space (with \(ds_T^2\) being its line element) and the constant \(r_p\) is defined through

\[
r_p^{7-p} \equiv (2\sqrt{\pi})^{5-p} \Gamma \left( \frac{7-p}{2} \right) g_s N \left( \alpha' \right)^{(7-p)/2} \equiv d_p g_s N \left( \alpha' \right)^{(7-p)/2}, \tag{3.2}
\]

where, in the last step, we have introduced the numerical constant \(d_p\), defined as:

\[
d_p = (2\sqrt{\pi})^{5-p} \Gamma \left( \frac{7-p}{2} \right). \tag{3.3}
\]

The decoupling limit

\[
g_s \to 0, \quad \alpha' \to 0, \quad U \equiv \frac{r}{\alpha'} \equiv \text{fixed}, \quad g_{YM}^2 N \equiv \text{fixed}, \tag{3.4}
\]

where the coupling constant \(g_{YM}\) is dimensionful and defined by

\[
g_{YM}^2 \equiv 2 \left( 2\pi \right)^{p-2} g_s \left( \alpha' \right)^{(p-3)/2}, \tag{3.5}
\]

corresponds to the “near-horizon” geometry for \(D_p\)-branes. In this limit the metric becomes

\[
ds_{10}^2 = g_{MN} dx^M dx^N =
\]

\[
= \alpha' \left\{ \left( \frac{U}{U_p} \right)^{(7-p)/2} \eta_{\mu\nu} dx^\mu dx^\nu + \left( \frac{U_p}{U} \right)^{(7-p)/2} \left[ dU^2 + U^2 d\Omega_8^{2-p} \right] \right\}. \tag{3.6}
\]
In (3.6) $d\Omega_{8-p}^2$ is the line element of a $S^{8-p}$ sphere and we have introduced the constant $U_p$, which is defined as:

$$U_p^{7-p} \overset{\text{def}}{=} \frac{d_p}{2(2\pi)^{p-2} g_{YM}^2 N}.$$ (3.7)

The D$p$-brane background is also endowed with a non-trivial dilaton $\phi$ and a RR $(p+1)$-form potential $C^{(p+1)}$, which are given by:

$$e^{\phi} = \frac{g_{YM}^2 N}{2(2\pi)^{p-2} N} \left( \frac{U}{U_p} \right)^{\frac{7-p}{2}} \frac{1}{(2\pi)^{p-2}} (\alpha')^{(p+1)/2} N \left( \frac{U}{U_p} \right)^{7-p}.$$ (3.8)

As it has just been mentioned, for $p \neq 3$ the coupling constant is dimensionful and, therefore, the effective coupling runs with energy scale as:

$$g_{\text{eff}}^2 = g_{YM}^2 N U_p^{p-3}.$$ (3.9)

It turns out that the background metric (3.6) is conformal to an $AdS_{p+2} \times S^{8-p}$ space for $p \neq 5$ [6, 9, 13]. This feature can be seen explicitly by redefining the radial coordinate as follows

$$u_p^{2} \overset{\text{def}}{=} \left( \frac{d_p}{2(2\pi)^{p-2} g_{YM}^2 N} \right)^{1} U^{5-p}, \quad u_p = \frac{5-p}{2},$$ (3.10)

and rewriting the line element (3.6) as

$$ds_{10}^2 = \left( N e^{\phi} \right)^{2/(7-p)} B_p d\tilde{s}_{10}^2, \quad B_p \overset{\text{def}}{=} \alpha' \frac{d_p^2}{u_p^2} d(7-p).$$ (3.11)

so that the line element $d\tilde{s}_{10}^2$ describes an $AdS_{p+2} \times S^{8-p}$ geometry in the Poincaré patch

$$d\tilde{s}_{10}^2 = \tilde{g}_{MN} d^M d^N = u^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2} + u_p^2 d\Omega_{8-p}^2.$$ (3.12)

In such a frame, the radial coordinate $u$ plays the role of the energy scale of the boundary theory and its rescalings are just the dilatations in the boundary theory [13]. One can also use the Fefferman-Graham coordinates by introducing a new radial coordinate $\rho$ as the inverse of the current one, namely

$$\rho = u^{-1}.$$ (3.13)

In terms of $\rho$ the metric $d\tilde{s}_{10}^2$ of (3.12) can be written as

$$d\tilde{s}_{10}^2 = \frac{d\rho^2}{\rho^2} + u_p^2 d\Omega_{8-p}^2,$$ (3.14)

with dilaton and $(8-p)$-form field strength $F^{(8-p)} = * C^{(p+1)}$ which become

$$e^{\phi} = \frac{g_{YM}^2 N}{2(2\pi)^{p-2} N} \left( \frac{u_p}{U_p} \right)^{\frac{3-p}{2(8-p)}} d\rho^2 \left( \frac{u_p}{U_p} \right)^{\frac{7-p}{2(8-p)}}$$

$$F^{(8-p)} = (7-p) d_p N (\alpha')^{\frac{7-p}{2}} \text{Vol} (S^{8-p}) = dC^{(7-p)}.$$ (3.15)
where Vol ($S^q$) denotes the volume form of a $S^q$ round sphere. In order to write explicitly the form of the $(7-p)$-form potential $C^{(7-p)}$, let us represent the line element of the $S^{(8-p)}$ sphere in terms of a polar angle $\theta$ and the line element of a $S^{(7-p)}$ sphere, namely

$$d\Omega_{8-p}^2 = d\theta^2 + \sin^2 \theta d\Omega_{7-p}^2,$$

(3.16)

with $\theta$ taking values in the range $0 \leq \theta \leq \pi$ (see figure 1). Then, the $(7-p)$-form potential can be taken to be

$$C_{7-p} = -Nd_p(\alpha')^{(7-p)/2}C_p(\theta) \text{Vol}(S^{7-p}),$$

(3.17)

where $C_p(\theta)$ is the function uniquely defined by the conditions:

$$\frac{dC_p(\theta)}{d\theta} = -(7-p)\sin^{7-p} \theta, \quad C_p(0) = 0.$$  

(3.18)

In the case of black hole geometries, the metric (3.14) becomes

$$ds^2_{10} = \frac{-h_p(\rho)d\tau^2 + d\Omega^2}{\rho^2} + \left[h_p(\rho)\right]^{-1}\frac{1}{\rho^2}d\rho^2 + u_p^2d\Omega_{8-p}^2,$$

(3.19)

with

$$h_p(\rho) = 1 - \left(\frac{\rho}{\rho_h}\right)^{2\frac{7-p}{5-p}}.$$  

(3.20)

Parametrising the position of the event horizon by $\rho_h$, the background temperature takes the form

$$T = \frac{7-p}{2\pi(5-p)\rho_h}.$$  

(3.21)

### 3.1 Generalised Conformal Symmetry

The “near-horizon” geometry generated by a stack of $N$ D$p$-branes discussed in the previous section is dual to an $SU(N)$ supersymmetric Yang-Mills theory in $(p + 1)$-dimensions, with
coupling constant $g_{YM}$ which is generally dimensionful, except for $p = 3$, and with a trivial RG flow which is just due to the dimensionfulness of the coupling constant.

Following [16], let us consider a generalised version of the $(p + 1)$-dimensional Supersymmetric Yang-Mills Euclidean action

$$S_{p+1} = - \int d^{p+1}x \sqrt{g_{(0)}} \left\{ - \Phi_{(0)} \frac{1}{4} \text{Tr} \left\{ F_{\mu\nu} F^{\mu\nu} \right\} + \frac{1}{2} \text{Tr} \left\{ X \left[ D^2 - \frac{p-1}{4p} R_{(0)}^{(p+1)} \right] X \right\} + \frac{\Phi_{(0)}^{-1}}{4} \text{Tr} \left\{ [X, X]^2 \right\} \right\} ,$$

(3.22)

where $X^I$ are scalar fields, with $I = 1, \ldots, 9 - p$, $g_{\mu\nu}^{(0)}$ is the background metric and $\Phi_{(0)}$ is a background scalar field. The usual Supersymmetric Yang-Mills action is recovered for $g_{\mu\nu}^{(0)} = \delta_{\mu\nu}$ and $\Phi_{(0)} = g_{YM}^{-2}$. The action (3.22) turns out to be invariant under a Weyl transformation of the following form

$$g_{\mu\nu}^{(0)} \rightarrow e^{2\omega} g_{\mu\nu}^{(0)}, \quad X \rightarrow e^{-\frac{p+1}{2p} \omega} X, \quad A_\mu \rightarrow A_\mu, \quad \Phi_{(0)} \rightarrow e^{-(p-3)\omega} \Phi_{(0)}. \quad (3.23)$$

Defining the stress-energy tensor $T_{\mu\nu}$ and the scalar operator $O_\Phi$ as

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{g_{(0)}}} \frac{\delta S_{p+1}}{\delta g_{\mu\nu}^{(0)}}, \quad O_\Phi \equiv \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{p+1}}{\delta \Phi_{(0)}}, \quad (3.24)$$

one can obtain diffeomorphism and Ward identities

$$\nabla^\nu \langle T_{\mu\nu} \rangle_J + \langle O_\Phi \rangle_J \partial_\mu \Phi_{(0)} = 0, \quad \langle T_{\mu\nu} \rangle_J + (p-3) \Phi_{(0)} \langle O_\Phi \rangle_J = 0, \quad (3.25)$$

with $\langle \cdot \rangle_J$ indicates the expectation value with respect to a source $J$. For $g_{\mu\nu}^{(0)} = \delta_{\mu\nu}$ and $\Phi_{(0)} = g_{YM}^{-2}$ the first equation in (3.25) provides the stress-energy conservation, while the second one the tracelessness of the stress-energy tensor and therefore the restoration of conformal symmetry. In particular, notice from the second equation of (3.25) that conformal symmetry is broken because of the dimensionfulness of the coupling constant.

In a theory of this type, the entropy $S$ at finite temperature $T$ has to scale as

$$S = \mathcal{C} \left( g_{\text{eff}}^2(T), N, \ldots \right) \Omega_p T^p, \quad (3.26)$$

with $\Omega_p$ being the spacial volume, $g_{\text{eff}}^2(T)$ as defined in (3.9) and $\mathcal{C} \left( g_{\text{eff}}^2(T), N, \ldots \right)$ is a generic function of dimensionless parameters.

Furthermore, the two-point functions of an operator $O$ need to have the following form

$$\langle O(x)O(0) \rangle = \mathcal{R} \left( f \left( g_{\text{eff}}^2(x), N, \ldots \right) \frac{1}{|x|^{2\Delta}} \right), \quad (3.27)$$

with $g_{\text{eff}}^2(x) = g_{YM}^2 N x^{3-p}$, $f \left( g_{\text{eff}}^2(x), N, \ldots \right)$ being a function of dimensionless parameters, and $\mathcal{R}$ provides the renormalised version of its argument. More precisely, these two objects are the same for $x \neq 0$, while at $x = 0$ they differ by infinite renormalisation.
3.2 Holographic impurities as probe D(8 − p)-branes

We would like now to introduce localised degrees of freedom in the (p + 1)-dimensional $U(N)$ SYM theory. Before looking at general values of $p < 5$, let us review the $p = 3$ case following [47].

3.2.1 Wilson loops and D5-branes

Let us start with considering a single probe D5-brane in $AdS_5 \times S^5$ in such a way that it wraps an $S^3 \subset S^5$ and extends along the time and radial direction, so that the induced metric on the world-volume is $AdS_2 \times S^3$. This system is the decoupling limit of a system with $N$ D3-branes in flat space and a single probe D5-brane, whose degrees of freedom are carried by the 3 − 3 strings (the “ambient” gauge theory), the 3 − 5/5 − 3 strings which are localised in the codimension-3 defect identified by the intersection of the two types of D-branes, and finally the non-dynamical 5 − 5 strings. The action for the defect (point-like) degrees of freedom can be obtained by performed T-duality on the D0/D8 system, obtaining

$$S_{\text{imp}} = \int dt \left[ i \Psi^\dagger \partial_t \Psi + \Psi^\dagger (A_t + X_I v^I + a_t) \Psi - n a_t \right], \tag{3.28}$$

where $A_t$ and $X_I$ are respectively the time-component of the gauge field and the scalars in $\mathcal{N} = 4$ SYM, $v^I$ is a unit vector, $\Psi$ is the resulting fermionic impurity field, $a_t$ is the non-dynamical gauge field on the D5-brane, and $n$ is the unit of background gauge field localised on the point-like defect.

Integrating out the degrees of freedom associated to the D5-branes in the decoupling limit, it has been shown [47] that the introduction of the D5-branes corresponds to a 1/2-BPS Wilson loop operator in $\mathcal{N} = 4$ SYM in the anti-symmetric representation of the gauge group $U(N)$.

3.2.2 General case

As a natural generalisation of the D-brane construction reviewed above\(^3\), one may think to introduce extra degrees of freedom in the boundary theory by considering in the bulk probe D(8 − p)-branes wrapping an $S^{7−p} \subset S^{8−p}$ and extending along the radial direction. The embedding of such branes in the Dp-brane background can be described by two functions $x^p \equiv z(\rho)$ and $\theta \equiv \theta(\rho)$, where $\theta$ is the angular coordinate introduced in (3.16). For the time being, let us keep both the two embedding functions. The induced metric on the

\(^3\)A comment is now in order. While indeed the construction we are going to consider introduces point-like degrees of freedom, a neat correspondence between the D(8 − p)-branes and Wilson-loop operators has not been shown yet.
world-volume of the D\((8 - p)\)-brane is

\[
\begin{align*}
\mathcal{A}_{9-p}^2 &= g_{\alpha\beta} d\zeta^\alpha d\zeta^\beta = \\
&= \frac{-h_p(\rho) \, dt^2}{\rho^2} + \left[1 + h_p(\rho)(z')^2 + u_p^2 \rho^2 h_p(\rho)(\theta')^2\right] \frac{d\rho^2}{h_p(\rho)\rho^2} + u_p^2 \sin^2 \theta \, d\Omega_{7-p}^2,
\end{align*}
\]

which is substantially the \(AdS_2 \times S^{7-p}\) geometry with the two functions \(z(\rho)\) and \(\theta(\rho)\) controlling, respectively, the embedding of branes in the \((p + 2)\)-dimensional (conformal)-\(AdS\) manifold, where they wrap a (conformally)-\(AdS_2\) subspace, and the embedding in the transverse space. The action for the probe branes is the sum of a Dirac-Born-Infeld and a Wess-Zumino term

\[
S_{D(8 - p)} = -T_{D(8 - p)} \int d^{9-p} \zeta e^{-\phi} \sqrt{-\det \{g_{\alpha\beta} + (2\pi\alpha') F_{\alpha\beta}\}} + T_{D(8 - p)}(2\pi\alpha') \int F \wedge C_{7-p},
\]

where \(g_{\alpha\beta}\) is the pullback of the near-horizon string frame metric and \(F\) is the world-volume abelian gauge field strength. The Wess-Zumino term in the action (3.30) acts as a source of the electric component \(F_{t\rho}\) of the world-volume gauge field and, therefore, one cannot put consistently \(F_{t\rho} = 0\). Let us take \(\zeta^\alpha = (t, \rho, \theta^i)\) as world-volume coordinates, where the \(\theta^i\) \((i = 1, \cdots, 7 - p)\) parametrise the \(S^{7-p}\) sphere in (3.16). Assuming that the electric field is independent of the angles \(\theta^i\), one can write the action as:

\[
S_{D(8 - p)} = -T_{D(8 - p)} N B_{p}^{\frac{9-p}{2}} \frac{1}{\Omega_{7-p}} u_{p}^{\frac{7-p}{2}} \int dt \, d\rho \left\{ \left(N e^{\phi}\right)^{\frac{7-p}{7-p}} \rho^{-2} \sin^{7-p} \theta \times \right. \\
\left. \left[1 + h_p(z')^2 + u_p^2 \rho^2 h_p(\theta')^2 - \frac{(2\pi\alpha')^2}{B_p^2} \frac{\rho^4 F_{t\rho}^2}{(N e^{\phi})^{\frac{7-p}{7-p}}} \right]^{1/2} + \frac{2\pi\alpha'}{B_p} F_{t\rho} C_{p}(\theta) \right\},
\]

where \(\Omega_{7-p} = 2\pi^{\frac{8-p}{2}}/\Gamma(\frac{8-p}{2})\) is the volume of the \(S^{7-p}\). One notices that \(S_{D(8 - p)}\) depends on the embedding function \(z\) and on the world-volume gauge field just through their derivatives. Therefore, there are two first integrals of motion

\[
\frac{\partial L_{D(8 - p)}}{\partial F_{t\rho}} = \text{const.}, \quad \frac{\partial L_{D(8 - p)}}{\partial z'} = c_z,
\]

where \(L_{D(8 - p)}\) is the Lagrangian density of the D\((8 - p)\)-brane, whose expression can be read from (3.31). The first constant can be fixed through the quantisation condition \([20]\)

\[
\int_{S^{7-p}} d^{7-p} \theta \frac{\partial L_{D(8 - p)}}{\partial F_{t\rho}} = n (2\pi\alpha') T_{F_1}, \quad n \in \mathbb{Z}.
\]
The integer \( n \) in (3.33) represents the number of fundamental strings (quarks) dissolved in the probe brane. Computing the left-hand-side of (3.33), the first integral of motion related to the world-volume gauge-field becomes

\[
\frac{c_f}{N} n = \frac{\sin^{7-p} \theta \, F_{\ell \rho}}{\left[ 1 + h_p(z')^2 + u_p^2 \rho^2 h_p(\theta')^2 - F_{\ell \rho} \right]^{1/2}} - C_p(\theta),
\]

(3.34)

where we have introduced the rescaled world-volume gauge field \( F_{\alpha \beta} \), defined as

\[
F_{\alpha \beta} \overset{\text{def}}{=} \frac{2 \pi \alpha'}{B_p} \frac{\rho^2 F_{\alpha \beta}}{(N e^\phi)^{7-p}},
\]

(3.35)

and the constant \( c_f \) is defined as

\[
c_f \overset{\text{def}}{=} \frac{T_{\ell \rho} B_p}{T_{D(8-p)} N \Omega_{7-p} u_p^{7-p}} = 2 \sqrt{\pi} \frac{\Gamma \left( \frac{8-p}{2} \right)}{\Gamma \left( \frac{7-p}{2} \right)}.
\]

(3.36)

The equation of motion (3.34) can be easily solved for the gauge field-strength \( F_{\ell \rho} \) to get

\[
F_{\ell \rho} = C^{(p,n)}(\theta) \left[ \frac{1 + h_p(z')^2 + u_p^2 \rho^2 h_p(\theta')^2}{\sin^{2(7-p)} \theta + C_{(p,n)}^2(\theta)} \right]^{1/2},
\]

(3.37)

where the function \( C^{(p,n)}(\theta) \) has been defined as

\[
C^{(p,n)}(\theta) \overset{\text{def}}{=} C_p(\theta) + c_f \frac{n}{N}.
\]

(3.38)

As far as the first integral of motion related to the embedding function \( z(\rho) \) is concerned, it acquires the form

\[
\tilde{c}_z = - \frac{\rho^{-2} \sin^{7-p} \theta (N e^\phi)^{7-p} h_p z'}{\left[ 1 + h_p(z')^2 + u_p^2 \rho^2 h_p(\theta')^2 - F_{\ell \rho} \right]^{1/2}},
\]

(3.39)

where \( \tilde{c}_z \) is a new constant related to \( c_z \) by a rescaling

\[
\tilde{c}_z \overset{\text{def}}{=} \frac{c_z}{T_{D(8-p)} N B_p^{9-p} \Omega_{7-p} u_p^{7-p}} = 2 \sqrt{\pi} \frac{\Gamma \left( \frac{8-p}{2} \right)}{N T_{F_1} \Gamma \left( \frac{7-p}{2} \right) B_p} \, c_z.
\]

(3.40)

Let us now evaluate the energy of the system. By performing a Legendre transform, the Hamiltonian \( H \) of the D(8–p)-brane is given by

\[
H = \int_{\Sigma_{7-p}} \sqrt{\gamma_{7-p}} \int d\rho \left[ F_{\ell \rho} \frac{\partial L_{D(8-p)}}{\partial F_{\ell \rho}} - L_{D(8-p)} \right].
\]

(3.41)

By using the explicit expression of the Lagrange density written in (3.31), one gets

\[
H = T_{D(8-p)} N B_p^{9-p} \Omega_{7-p} u_p^{7-p} \int d\rho \left( N e^\phi \right)^{7-p} \rho^{-2} \left[ 1 + h_p(z')^2 + \rho^2 h_p(\theta')^2 \right]^{1/2} \times \left[ \sin^{2(7-p)} \theta + C_{(p,n)}^2(\theta) \right]^{1/2}.
\]

(3.42)
Let us now discuss some possible configurations for the D(8 − p)-brane. In particular, we will focus on the case in which the position of the probe branes in both the (conformally) AdS space and in the transverse space is fixed (straight flux tubes with \( \theta' = z' = 0 \)), and on the configurations in which the probe branes are allowed to bend either in the extended directions or in the transverse space.

## 4 Straight Flux Tube Configurations

Let us start with the analysis of the configurations in which both the angular and linear coordinates \( \theta \) and \( z \) are constants. Notice that these configurations introduce a localised defect on the boundary theory. Following [20], the stable configurations are the ones minimising the energy (3.42), i.e. the ones that satisfy the condition:

\[
0 = \frac{dH}{d\theta} \bigg|_{\{ \theta = \text{const} \}} = T_{D(8-p)} N B_{p}^{\frac{9-p}{2}} \Omega_{7-p} u_{p}^{7-p} \int d\rho \left( N e^{\phi} \right)^{\frac{2}{7-p}} \rho^{-2} \times \nabla^{2} \sin^{7-p} \theta \left[ \sin^{2}(7-p) \theta + C_{p,n}^{2}(\theta) \right]^{1/2},
\]

where the function \( \Lambda_{p,n}(\theta) \) is defined as

\[
\Lambda_{p,n}(\theta) \overset{\text{def}}{=} \sin^{6-p} \theta \cos \theta - C_{p,n}(\theta).
\]

The least energy condition (4.1) can be satisfied if and only if

\[
\sin \theta = 0, \quad \text{or} \quad \Lambda_{p,n}(\theta) = 0.
\]

In the first case, such configurations occur at \( \theta = 0, \pi \) which are points in which the sphere \( S^{7-p} \) shrinks to zero size. In the second case, instead, the branes are located at \( \theta = \bar{\theta}_{(p,n)} \) with \( \bar{\theta}_{(p,n)} \) defined by the condition itself: \( \Lambda_{p,n}(\bar{\theta}_{(p,n)}) = 0 \). The functions \( \Lambda_{p,n}(\theta) \) for different values of \( p \) are listed in appendix A. As it is clear from (3.38) they depend on the quantisation integer \( n \) and on the rank of the gauge group through the combination

\[
\nu = \frac{n}{N},
\]

which we will refer to as the filling fraction. The reason for this name is the fact that, for a given value of \( p \), only for \( 1 < n < N \) there exists a unique solution for the angles \( \bar{\theta}_{(p,n)} \) in the range \( 0 < \bar{\theta}_{(p,n)} < \pi \). The integer \( n \) represents the number of impurity fermions introduced in the boundary theory and the ratio \( \nu \) takes values in the range \( 0 < \nu < 1 \). The upper bound of \( n \) is a manifestation of the so-called stringy exclusion principle and is a piece of evidence supporting the identification of these brane configurations as the holographic duals of Wilson
lines in the antisymmetric representation of the gauge group. Indeed, in the conformal case \( p = 3 \) this identification was explicitly checked in refs. [47, 48].

From the explicit formulas of the \( \Lambda_{p,n}(\theta) \) functions displayed in appendix A, one can verify that the angles \( \tilde{\theta}_{(p,n)} \) satisfy the relation:

\[
\tilde{\theta}_{(p,N-n)} = \pi - \tilde{\theta}_{(p,n)} .
\]  (4.5)

It follows that changing the polar angle of the embedding as \( \theta \to \pi - \theta \) is equivalent to the particle-hole transformation \( \nu \to 1 - \nu \). The energy density (tension) of these configurations was derived in [20] and it turns out to be

\[
E_{(p,n)} = \frac{N T_{F1}}{2\sqrt{\pi}} \frac{\Gamma \left( \frac{7-p}{2} \right)}{\Gamma \left( \frac{8-p}{2} \right)} \sin^{6-p} \tilde{\theta}_{(p,n)} .
\]  (4.6)

It follows from (4.5) and (4.6) that the tension satisfies the relation:

\[
E_{(p,n)} = E_{(p,N-n)} ,
\]  (4.7)

i.e. it is invariant under the transformation \( \nu \to 1 - \nu \).

The electric world-volume field \( \tilde{F}_{t\rho} \) for the flux-tube configuration \( \theta = \tilde{\theta}_{(p,n)} \) can be obtained from (3.37) by using that \( C_{(p,n)}(\tilde{\theta}_{(p,n)}) = (\sin \tilde{\theta}_{(p,n)})^{6-p} \cos \tilde{\theta}_{(p,n)} \). One gets:

\[
\tilde{F}_{t\rho} = \cos \tilde{\theta}_{(p,n)} .
\]  (4.8)

Notice that \( \tilde{F}_{t\rho} \) changes its sign under the transformation \( \nu \to 1 - \nu \).

As was proven in [20], in the case of the zero temperature background, the flux-tube configurations described above are the solution of a first-order BPS equation for the embedding and the world-volume gauge field of the probe brane. One can show that this equation is the one that is obtained by imposing kappa symmetry to the D\((8-p)\)-brane in such a way that it preserves 1/4 of the supersymmetry.

### 4.1 Impurity Entropy

Let us investigate some aspects of the thermodynamics of this class of systems. First of all, for \( p < 5 \) the relation (3.21) between the temperature \( T \) and the position of the horizon \( \rho \) can be inverted to obtain

\[
\rho_h = \frac{7 - p}{2\pi(5 - p)T} .
\]  (4.9)

In order to compute the free energy and entropy of the flux tube configuration, let us evaluate the Euclidean action of one of such configurations that extends from the horizon at \( \rho = \rho_h \) until a cutoff value of the radial coordinate \( \rho = \epsilon \). If we define \( \beta_p \) as

\[
\beta_p \overset{\text{def}}{=} \rho_h^{-\frac{2}{5-p}} \left( \frac{g_{YM}^2 N}{2(2\pi)^{p-2}} \right)^{\frac{2}{p-2}} \left( \frac{U_p}{u_p} \right)^{\frac{p-3}{p-2}} ,
\]  (4.10)
we get
\[ I^{\text{Eucl}}_{D(8-p)} \bigg|_{\text{on-shell}} = \frac{\mathcal{E}_{(p,n)}}{T} \beta_p \rho_\phi \frac{2}{(8-p)} \int_\epsilon^{(8-p)} d\rho \rho^{\frac{7-p}{(8-p)}}. \tag{4.11} \]

To arrive at the expression (4.11) we have integrated over a periodic Euclidean time circle of period 1/T. Notice that the integral (4.11) diverges as \( \epsilon \to 0 \) and therefore it needs to get renormalised in a neighbourhood of the boundary. It easy to see that it is renormalised just by a term proportional to the volume of the boundary, regularised with \( \rho = \epsilon \)

\[ I^{\text{Eucl}}_{D(8-p)} \bigg|_{ct} = -\frac{5-p}{2} \frac{\mathcal{E}_{(p,n)}}{T} B_p \left( Ne^{\phi}|_{\epsilon} \right)^{\frac{2}{(8-p)}} \sqrt{g_{tt}|_{\epsilon}}, \tag{4.12} \]

where \( g_{tt}|_{\epsilon} = \epsilon^2 \) is the tt component of the induced metric (3.29) at \( \rho = \epsilon \). Therefore, the renormalised action is given by

\[ I^{\text{Eucl}}_{D(8-p)} \bigg|_{\text{ren}} = -\frac{5-p}{2} \frac{\mathcal{E}_{(p,n)}}{T} \beta_p B_p. \tag{4.13} \]

Let us rewrite (4.13) in a more convenient form. By using the definitions of \( \beta_p \) and \( B_p \) written in (4.10) and (3.11), one easily proves that:

\[ \beta_p B_p = \frac{2\sqrt{\pi}}{(5-p) T_{F1}} \left[ \frac{\Gamma\left(\frac{7-p}{2}\right)}{(7-p)^2 \pi} \right]^{\frac{1}{5-p}} \left[ g^2_{\text{eff}}(T) \right]^{\frac{1}{5-p}} T, \tag{4.14} \]

where \( g^2_{\text{eff}}(T) \) is the gauge theory effective coupling constant at the temperature \( T \), defined as

\[ g^2_{\text{eff}}(T) = g^2_{\text{YM}} N T^{p-3}. \tag{4.15} \]

By using the explicit expression of \( \mathcal{E}_{(p,n)} \) written in (4.6) one can recast the renormalised action (4.13) as

\[ I^{\text{Eucl}}_{D(8-p)} \bigg|_{\text{ren}} = -c_p N \left( \sin \bar{\theta}_{(p,n)} \right)^{6-p} \left[ g^2_{\text{eff}}(T) \right]^{\frac{1}{5-p}} T, \tag{4.16} \]

where the coefficient \( c_p \) is given by:

\[ c_p = \frac{\left[ \frac{\Gamma\left(\frac{7-p}{2}\right)}{(7-p)^2 \pi} \right]^{\frac{6-p}{5-p}}}{\Gamma\left(\frac{8-p}{2}\right)} \left( \frac{(7-p)^2 \pi}{(8-p)^2 \pi} \right)^{\frac{1}{5-p}}. \tag{4.17} \]

From the renormalised action (4.16) it is immediate to extract the free energy for the straight flux tubes, namely

\[ F_{\text{str}}^{D(8-p)} = T I^{\text{Eucl}}_{D(8-p)} \bigg|_{\text{ren}} = -c_p N \left( \sin \bar{\theta}_{(p,n)} \right)^{6-p} \left[ g^2_{\text{eff}}(T) \right]^{\frac{1}{5-p}} T. \tag{4.18} \]

If we ignore non-abelian interactions among the D(8 – p)-branes, the impurity entropy generated by M D(8 – p)-branes is just M times the result obtained for M = 1, namely

\[ S_{\text{str}}^{D(8-p)} = -M \frac{\partial F_{\text{str}}^{D(8-p)}}{\partial T} = \frac{2c_p}{5-p} M N \left( \sin \bar{\theta}_{(p,n)} \right)^{6-p} \left[ g^2_{\text{eff}}(T) \right]^{\frac{1}{5-p}}. \tag{4.19} \]
It is interesting to study what happens in case of small filling-fractions $\nu = n/N$. The impurity entropy (4.19) depends on the filling-fraction $\nu$ through the factor $\sin^{6-p} \bar{\theta}_{(p,n)}$ contained in the energy density $\mathcal{E}_{(p,n)}$. In the small filling-fraction limit the angle $\bar{\theta}_{(p,n)}$ is also small and one can check [20] that, at leading order in $n/N$, one has:

$$
\left( \bar{\theta}_{(p,n)} \right)^{6-p} \approx 2\sqrt{\pi} \frac{\Gamma \left( \frac{8-p}{2} \right)}{\Gamma \left( \frac{7-p}{2} \right)} \frac{n}{N} + \cdots .
$$

Then, the expansion of $\mathcal{E}_{(p,n)}$ is given by

$$
\lim_{\frac{n}{N} \to 0} \mathcal{E}_{(p,n)} = n T_{F1} \left[ 1 - b_p \left( \frac{n}{N} \right)^{\frac{2}{6-p}} + \ldots \right], \quad b_p \overset{\text{def}}{=} \frac{6-p}{2(8-p)} \left[ 2\sqrt{\pi} \frac{\Gamma \left( \frac{8-p}{2} \right)}{\Gamma \left( \frac{7-p}{2} \right)} \right]^{\frac{2}{6-p}},
$$

and, as a consequence, the impurity entropy can be written as

$$
\lim_{\frac{n}{N} \to 0} S_{\text{str}}^{\text{st}}_{D(8-p)} = n M a_p \left[ g_{\text{eff}}^2(T) \right]^{\frac{1}{6-p}} \left[ 1 - b_p \left( \frac{n}{N} \right)^{\frac{2}{6-p}} + \ldots \right],
$$

with

$$
a_p \overset{\text{def}}{=} 4\sqrt{\pi} \left[ \frac{\Gamma \left( \frac{7-p}{2} \right)}{(7-p)^{2\pi}} \right]^{\frac{1}{6-p}},
$$

One can verify that (4.19) and (4.22) for $p = 3$ coincide with the impurity entropy computed in [23,24] for the holographic dual of the maximally supersymmetric Kondo model. Our results generalise those in [23,24] for non-conformal D$p$-brane backgrounds.

From (4.22) we notice that the dependence of $\mathcal{S}$ on the filling fraction is, in general, non-analytic. However, in the particular case of the D4/D4 system, the impurity entropy turns out to be analytic in the filling fraction. Actually, it is possible to obtain a simple closed expression of $S_{\text{str}}^{\text{st}}_{D4}$ as a function of $\nu$. In particular, for such a case the system is in the minimal-energy configuration if the probe D4-brane is located at $\theta = \bar{\theta}_{(4,n)}$ such that

$$
\cos \bar{\theta}_{(4,n)} = 1 - 2\nu,
$$

satisfying the condition (4.3). Using (4.23) the expression for the entropy density for $p = 4$ can be written as

$$
S_{\text{str}}^{\text{st}}_{D4} = \frac{2n M}{9} g_{\text{eff}}^2(T) (1 - \nu) .
$$

where $g_{\text{eff}}^2(T) = g_{YM}^2 N T$ in this $p = 4$ case.

The internal energy $\mathcal{E}$ is obtained from the free energy $F$ by means of the standard thermodynamic formula

$$
\mathcal{E} = F + TS .
$$
For the straight brane configuration one can immediately obtain $\mathcal{E}$ by combining (4.18) and (4.19), namely
\[
\mathcal{E}^\text{str}_{D(8-p)} = \frac{p-3}{2} T \mathcal{S}^\text{str}_{D(8-p)}.
\] (4.26)

Notice that $\mathcal{E}^\text{str}_{D5}$ vanishes, while $\mathcal{E}^\text{str}_{D(8-p)}$ becomes negative for $p < 3$.

In order to complete our thermodynamics analysis, we can compute the impurity specific heat from the impurity entropy, namely (4.19)
\[
C^\text{str}_{D(8-p)} = T \frac{\partial \mathcal{S}^\text{str}_{D(8-p)}}{\partial T} = \frac{2(p-3) c_p}{(5-p)^2} M N (\sin \bar{\theta}(p,n))^{6-p} \left[ g^2_{\text{eff}}(T) \right]^{\frac{1}{7-p}},
\] (4.27)

which vanishes for $p = 3$ and is negative for $p < 3$, giving a signature of a thermodynamic instability. From (4.27), it is easy to compute the impurity susceptibility
\[
\chi^\text{str}_{D(8-p)} \propto \frac{\partial C^\text{str}_{D(8-p)}}{\partial T} \propto (p-3)^2 \left[ g^2_{\text{YM}} N \right]^{\frac{1}{7-p}} T^2 \frac{c_{\text{eff}}}{5-p},
\] (4.28)
which is constant for $p = 4$.

5 Hanging Flux Tube Configurations

Let us now consider the configuration in which $\theta$ is a constant and the embedding of the probe brane in the $(p+2)$-dimensional (conformally)-$AdS$ manifold is described through the scalar $z(\rho)$. The probe branes are located at $\theta = \bar{\theta}(p,n)$.

Analysing the form of the first integral of motion (3.39), one can notice that, in the case of black hole embedding phase, the regularity condition at the horizon fixes the (rescaled) first integral of motion $\tilde{c}_z$ to be zero, and, as a consequence, the embedding function $z$ must have a trivial profile.

Focusing instead on the configuration in which the probe brane lies completely outside the black hole, (3.39) implies the existence of a turning point. Inverting (3.39) with respect to $z'$ and integrating the obtained equation, one gets
\[
z(\rho) = \pm \int_{\rho_t}^{\rho} \frac{c_{z_{(p,n)}}(\rho') d\rho'}{\sqrt{h_p \left[ \left( N c^2 e^{-\frac{2\pi}{7-p}} (\rho')^{-4} h_p - (c_{z_{(p,n)}})^2 \right) \right]}}, \quad c_{z_{(p,n)}} \overset{\text{def}}{=} \frac{\tilde{c}_z}{\sin^{6-p} \theta_{p,n}}.
\] (5.1)

It is easy to see from (5.1) and (3.40) that the constant $c_{z_{(p,n)}}$ is actually related to the energy density (4.6)
\[
c_{z_{(p,n)}} = \frac{c_z}{\mathcal{E}_{(p,n)} B_p},
\] (5.2)
where $c_z$ is the constant introduced in (3.32) and the position $\rho_t$ of the turning point is given by
\[
\rho_t = \frac{\rho_n}{\sqrt{1 + \left( \frac{\rho_n}{\rho_p} \right)^2 \left( c_{z_{(p,n)}} \right)^2}},
\] (5.3)
Let us next introduce a new radial variable $\sigma$, related to $\rho$ by means of the expression

$$\sigma = \frac{\rho}{\rho_h}. \quad (5.4)$$

Clearly, the turning point in this new variable is just

$$\sigma_t = \left[ 1 + \left( \frac{\rho_h}{\beta_p} \right)^2 (c_z^{(p, n)})^2 \right]^{\frac{p-5}{2(7-p)}}. \quad (5.5)$$

Moreover, the separation on the boundary between the $n$-quark and the $n$-antiquark is given by

$$L = 2\rho_h \frac{c_z^{(p, n)}}{\beta_p} \frac{2^{\frac{7-p}{5-p}}}{\sigma_t} I(\sigma_t), \quad (5.6)$$

where $I(\sigma_t)$ is the integral

$$I(\sigma_t) = \int_0^{\sigma_t} d\sigma \frac{\sigma^{\frac{7-p}{5-p}}}{\left[ \left( 1 - \sigma^{2^{\frac{7-p}{5-p}}} \right) \left( \frac{2^{\frac{7-p}{5-p}}}{\sigma_t} - \sigma^{2^{\frac{7-p}{5-p}}} \right) \right]^{1/2}}. \quad (5.7)$$

By using (5.5) one can eliminate in (5.6) the constant $c_z^{(p, n)}$ in favour of $\sigma_t$, with the result

$$L = 2\rho_h \sqrt{1 - (\sigma_t)^{2^{\frac{7-p}{5-p}}}} I(\sigma_t). \quad (5.8)$$

By applying the same techniques as in appendix B, the integral (5.7) can be obtained in terms of the hypergeometric function. One gets:

$$I(\sigma_t) = \frac{\Gamma\left(\frac{6-p}{7-p}\right)}{\Gamma\left(\frac{5-p}{2(7-p)}\right)} \sqrt{\pi} \sigma_t F\left(\frac{1}{2}, 1 - \frac{1}{7-p}; \frac{3}{2} - \frac{1}{7-p}; \frac{2^{\frac{7-p}{5-p}}}{\sigma_t}\right). \quad (5.9)$$

Let us use these results to write $L$ in a closed form. Actually, it is very convenient to define the parameter $\gamma$ as

$$\gamma \equiv \frac{2^{\frac{7-p}{5-p}}}{\sigma_t}. \quad (5.10)$$

It follows from this definition that $0 \leq \gamma \leq 1$. Moreover, $L$ can be written as

$$L = \frac{7-p}{(5-p)\sqrt{\pi} T} \frac{\Gamma\left(\frac{6-p}{7-p}\right)}{\Gamma\left(\frac{5-p}{2(7-p)}\right)} \gamma^{\frac{5-p}{2(7-p)}} \sqrt{1 - \gamma} F\left(\frac{1}{2}, 1 - \frac{1}{7-p}; \frac{3}{2} - \frac{1}{7-p}; \gamma\right), \quad (5.11)$$

where we used (3.21) to eliminate $\rho_h$ and we wrote the result in terms of the temperature $T$.

The energy for this hanging configuration is

$$E = 2\mathcal{E}_{(p, n)} \beta_p B_p \sigma_t^{\frac{2^{\frac{7-p}{5-p}}}{\sigma_t}} \int_\epsilon^{\sigma_t} \frac{d\sigma}{\sigma^{\frac{7-p}{5-p}}} \frac{\sqrt{1 - \sigma^{2^{\frac{7-p}{5-p}}}}}{\sqrt{\frac{2^{\frac{7-p}{5-p}}}{\sigma_t} - \sigma^{2^{\frac{7-p}{5-p}}}}}, \quad (5.12)$$
where the cut-off $\epsilon$ has been introduced to regularise the Hamiltonian near the boundary $\sigma = 0$, where the integral diverges. It can be regulated by minimal subtraction, i.e. by the energy corresponding to a disconnected configuration. Therefore, let us look at the divergences near the boundary $\sigma = 0$, where the embedding function $z(\rho)$ and its first derivative have the following asymptotic expansion

$$z'(\sigma) = \pm \frac{\beta^2}{\beta_p} c_{z}^{(p,n)} \sigma^{\frac{7-p}{5-p}} \left[ 1 + \frac{1}{2} \left( 1 + \sigma_i^{-2 \frac{7-p}{5-p}} \right) \sigma^2 \sigma^{\frac{7-p}{5-p}} + O \left( \sigma^{4 \frac{7-p}{5-p}} \right) \right],$$

$$z(\sigma) = \pm \frac{L}{2} \pm \frac{5-p}{2(6-p)} \beta^2 \beta_p \sigma^{\frac{9-p}{5-p}} + O \left( \sigma^{2 \frac{13-2p}{5-p}} \right).$$

(5.13)

It is easy to read off the divergent boundary Hamiltonian

$$H_{\text{div}} = (5-p) \mathcal{E}_{(p,n)} \beta_p B_p \epsilon^{-\frac{2}{5-p}}.$$

(5.14)

As a consequence, one needs to add just a counterterm which is proportional to the volume of the boundary, regularised with $\sigma = \epsilon$

$$H_{\text{ct}} = -(5-p) \mathcal{E}_{(p,n)} B_p \left( N e^{\phi} \right)^{\frac{2}{5-p}} \sqrt{-g_{tt}} \epsilon.$$

(5.15)

As noticed in [27], the divergent Hamiltonian (5.14) is exactly the divergent boundary Hamiltonian that one would obtain from probe D-branes in an $AdS$-space of dimensions $2/(5-p)+1$, and the counterterm (5.15) is actually given in terms of the volume of the boundary of this $AdS$-space.

Therefore, the renormalised Hamiltonian is given by

$$H_{\text{ren}} = \lim_{\epsilon \to 0} \left[ H_{\text{on-shell}} + H_{\text{ct}} \right].$$

(5.16)

and the free energy can be written as

$$F_{D(8-p)}^{(\gamma)} = 2 \mathcal{E}_{(p,n)} \beta_p B_p \left[ \int_0^{\sigma_i} d\sigma \left( \frac{\sigma^{-p}}{\sigma^{5-p}} \sqrt{1 - \sigma^{\frac{7-p}{5-p}}} - 1 \right) - \frac{5-p}{2} \sigma_i^{-\frac{7-p}{5-p}} \right].$$

(5.17)

Let us now change to a new variable $\xi$ in the integral (5.17), defined as

$$\xi = \left( \frac{\sigma_i}{\sigma} \right)^{\frac{2}{5-p}}.$$

(5.18)

After this change of variable, it is easy to verify that $F_{D(8-p)}^{(\gamma)}$ becomes

$$F_{D(8-p)}^{(\gamma)} = (5-p) \mathcal{E}_{(p,n)} \beta_p B_p \gamma^{-\frac{1}{5-p}} \left[ \int_1^{\infty} d\xi \left( \frac{\sqrt{\xi^{7-p} - \gamma}}{\sqrt{\xi^{7-p} - 1}} - 1 \right) - 1 \right].$$

(5.19)
where $\gamma$ is related to $\sigma_i$ as in (5.10). The term inside the square brackets in (5.19) is just the integral $J(7 - p, \gamma)$ calculated in appendix B. Thus, we can write

$$F_{D(8-p)}^{\cup} = (5 - p) \mathcal{E}_{(p, n)} \beta_p B_p \gamma^{-1 - p} J(7 - p, \gamma). \tag{5.20}$$

The prefactor in (5.20) can be related to the free energy of the straight brane. Indeed, according to (4.13) and (4.18) one has $(5 - p) \mathcal{E}_{(p, n)} \beta_p B_p = F_{D(8-p)}^{str}$. Moreover, by using the explicit value of the integral $J(7 - p, \gamma)$ (see eq. (B.6)), one gets

$$F_{D(8-p)}^{\cup} = 2 F_{D(8-p)}^{str} \gamma^{-1 - p} \frac{\Gamma(\frac{6 - p}{7 - p})}{\Gamma(\frac{5 - p}{2(7 - p)})} \sqrt{\pi} F\left(- \frac{1}{2}, - \frac{1}{7 - p}; \frac{5 - p}{2(7 - p)}; \gamma\right). \tag{5.21}$$

The constant $\gamma$ appearing on the expressions of the length $L$ and free energy $F_{D(8-p)}^{\cup}$ (eqs. (5.11) and (5.21)) parametrises the turning point of the hanging brane configuration, i.e. how much the brane penetrates into the bulk. The configuration with $\gamma \to 0$ corresponds to $\rho_i \to 0$, which means that the brane lies completely on the boundary in this limit. On the contrary, when $\gamma = 1$ one has $\rho_i = \rho_h$ and, thus, this configuration corresponds to two disconnected straight branes ending on the black hole horizon. It is easy to check that our formula (5.21) is consistent with this interpretation. Indeed, as

$$F\left(- \frac{1}{2}, - \frac{1}{7 - p}; \frac{5 - p}{2(7 - p)}; 1\right) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{5 - p}{2(7 - p)})}{\Gamma\left(\frac{6 - p}{7 - p}\right)}, \tag{5.22}$$

one gets

$$\lim_{\gamma \to 1} F_{D(8-p)}^{\cup} = 2 F_{D(8-p)}^{str}. \tag{5.23}$$

Finally, we can now compute the one-point correlator of the boundary operator $\mathcal{O}_z$ associated with the scalar function $z$. In order to perform this computation, we consider the Euclidean version of the action (3.31) at $\theta = \bar{\theta}_{(p, n)}$

$$I_{D(8-p)}^{\text{Eisc}} = T_{D(8-p)} N B_p \frac{\theta_{D(p, n)}}{2(2\pi)^{p-2}} \int dt d\rho \left(N e^{\phi}\right)^{-1} \rho^{-2} \left\{\sin^{-p} \bar{\theta}_{(p, n)} \times \right.
$$

$$\times \left[1 + h_p (z')^2 - \mathcal{F}_{tp}^2\right]^{1/2} + \mathcal{F}_{tp} C_p \left(\bar{\theta}_{(p, n)}\right) \right\} = \right.$$

$$= \left[1 + h_p (z')^2 - \mathcal{F}_{tp}^2\right]^{1/2} + \mathcal{F}_{tp} C_p \left(\bar{\theta}_{(p, n)}\right), \tag{5.24}$$

where the second equality has been written using the explicit expression for the dilaton (3.15) and $U_p$ (3.7) as well as the relation (3.36) and the expression for the energy density
It is interesting to notice that the overall term $\rho^{-\frac{5-p}{7-p}}$ in (5.24) can be seen as the determinant of an AdS black-hole metric from two-dimension to $1+q$, with $q = 2/(5 - p)$. Therefore, we can think that our probe $D(8-p)$-brane configuration is actually in a higher dimensional AdS black-hole background (times $S^{8-p}$) and that the induced metric on the brane has an (asymptotically) AdS$_{1+q}$ factor, with $q$ integer – then we can perform the analytic continuation to the value $2/(5 - p)$. This action is renormalised exactly by the same term (5.15) used to renormalised the Hamiltonian – notice that if one writes the explicit expression for the dilaton in (5.15), the counterterm can be seen as proportional to the volume of the boundary of AdS$_{1+q}$. In this way, we can compute the one-point function for the operator $O_z$ dual to the embedding function $z(\rho)$ as

$$\langle O_z \rangle = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{\frac{5-p}{2}} \sqrt{g_{0\text{AdS}_{1+q}}|_{\epsilon}}} \frac{1}{\epsilon^{\frac{5-p}{2}}} \frac{\delta F_{\text{Eucl}}^{D(8-p)}|_{\text{ren}}}{\delta z(\epsilon)}.$$

(5.25)

where $g_{0\text{AdS}_{1+q}}|_{\epsilon}$ is the determinant of the metric at the boundary of AdS$_{1+q}$ regularized at $\rho = \epsilon$, and $\Delta_z = 2/(5 - p)$. Using equation (5.25), the explicit expression for the one-point correlator is

$$\langle O_z \rangle = \frac{\mathcal{E}_{(p,n)} B_p}{\sin^{5-p} \theta_{(p,n)}} \left[ \frac{g_{YM}^2 N}{2(2\pi)^{p-2}} \right]^{\frac{1}{p-7}} \left( \frac{d_p}{u_p} \right)^{\frac{p-3}{(5-p)(5-p)}} \times$$

$$\times \lim_{\epsilon \to 0} \frac{1}{\epsilon^{\frac{5-p}{2}}} \frac{1}{\sqrt{g_{0\text{AdS}_{1+q}}|_{\epsilon}}} \sin \theta_{(p,n)} \left[ 1 + h_p (z')^2 \right]^{1/2} \bigg|_{\epsilon} = -c_z,$$

(5.26)

where the last equality has been obtained by using the equation of motion (3.39) for $z^4$, with $c_z$ has been defined in eq (3.32).

### 6 Dimer Thermodynamics

In this section we will study in detail the thermodynamic properties of the connected configuration of the D$(8-p)$-brane. First of all, we will rewrite the free energy (5.21) in a more convenient way. With this purpose, let us define two functions $h(\gamma)$ and $g(\gamma)$ as follows:

$$h(\gamma) \overset{\text{def}}{=} \gamma^{-\frac{1}{7-p}} F \left( -\frac{1}{2}, -\frac{1}{7-p}; \frac{1}{2} - \frac{1}{7-p}; \gamma \right),$$

$$g(\gamma) \overset{\text{def}}{=} \gamma^{\frac{3}{2} - \frac{1}{5-p}} \sqrt{1-\gamma} F \left( \frac{1}{2}, 1 - \frac{1}{7-p}; \frac{3}{2} - \frac{1}{7-p}; \gamma \right),$$

(6.1)

One has to remember that this computation has been performed by using the Euclidean action, as usual. If one repeats the same calculation using the Minkowski signature, the result would be the same up to a sign.
as well as the constant $\Delta_p$, given by

$$\Delta_p = \frac{\Gamma\left(\frac{6-p}{7-p}\right)}{\Gamma\left(\frac{5-p}{2(7-p)}\right)} \sqrt{\pi}.$$  

(6.2)

Then, the free energy (5.21) can be written as

$$F_{D(8-p)}^U(T) = 2 \Delta_p F_{D(8-p)}^{str}(T) \ h(\gamma),$$

(6.3)

where we have stressed the fact that both $F_{D(8-p)}^U$ and $F_{D(8-p)}^{str}$ depend on $T$. The dependence of $F_{D(8-p)}^{str}$ on $T$ is shown explicitly in (4.18). Moreover, $F_{D(8-p)}^U$ depends implicitly on $T$ through its dependence on the parameter $\gamma$. Actually, by rewriting (5.11) as

$$T = \frac{7-p}{(5-p) \pi L} \Delta_p g(\gamma),$$

(6.4)

we get the explicit relation between the temperature $T$ and the parameter $\gamma$ (for fixed length $L$). In order to study the dependence of the temperature on $\gamma$, let us define the dimensionless reduced temperature $T$ as

$$T = \frac{(5-p)\pi}{(7-p)\Delta_p} \ L \ T.$$  

(6.5)

It follows from (6.4) that $T(\gamma)$ is just given by the function $g(\gamma)$, namely

$$T(\gamma) = g(\gamma).$$

(6.6)

Let us now analyse the behaviour of $T(\gamma)$ near $\gamma \approx 0, 1$. From the explicit expression of $g(\gamma)$ in terms of the hypergeometric function (see the second equation in (6.1)), one can verify that

$$T(\gamma) \approx \gamma^{\frac{1}{2}} - \frac{1}{\gamma^{1-p}}, \quad \gamma \approx 0,$$

$$T(\gamma) \approx - \frac{\Gamma\left(\frac{3}{2} - \frac{1}{7-p}\right)}{\sqrt{\pi} \Gamma\left(1 - \frac{1}{7-p}\right)} \ \sqrt{1-\gamma} \ \log(1-\gamma), \quad \gamma \approx 1,$$

(6.7)

and, therefore, $T(\gamma) \to 0$ as $\gamma \to 0, 1$. Actually, by plotting the function $T(\gamma)$ for different values of $p$ one discovers that it reaches a maximum at some value $\gamma = \gamma_p^{\max}$ with $0 < \gamma_p^{\max} < 1$. These plots are shown in figure 2 for $0 \leq p \leq 4$.

The fact that $T$ has an upper bound implies that above a certain temperature $T_p^{\max} = T(\gamma = \gamma_p^{\max})$ (which depends on $p$), only the disconnected solution exists. The values of $\gamma_p^{\max}$ and $T_p^{\max}$ can be obtained by requiring the vanishing of the first derivative of $g(\gamma)$. We can calculate $g'(\gamma)$ by differentiating the right-hand-side of the second equation in (6.1). By
Figure 2: On the left we plot the reduced temperature $T$ versus the parameter $\gamma$ for $0 \leq p \leq 4$. As $p$ is increased the maximum of the curve is shifted towards lower values of $\gamma$. On the right we plot $\Delta f_p$ versus $\gamma$ for $0 \leq p \leq 4$. The points where the different curves cut the horizontal axis define the $\gamma$ parameter $\gamma_p^*$ of the dimerisation transition.

doing so we get a term which contains the derivative on the hypergeometric function, which we simplify by using the relation

$$
\gamma (1 - \gamma) \frac{d}{d\gamma} F(a, b; c; \gamma) = (c - a) F(a - 1, b; c; \gamma) + (a - c + b\gamma) F(a, b; c; \gamma).
$$

(6.8)

Then, $g'(\gamma)$ can be written as a combination of two hypergeometric functions which, remarkably, can be simplified to be:

$$
g'(\gamma) = \frac{5 - p}{2(7 - p)} \gamma^{-\frac{3}{2} + \frac{1}{7 - p}} (1 - \gamma)^{-\frac{1}{2}} F\left(-\frac{1}{2}, 1 - \frac{1}{7 - p}; \frac{1}{2} - \frac{1}{7 - p}; \gamma\right).
$$

(6.9)

Therefore, the maximum of the reduced temperature $T(\gamma)$ occurs at $\gamma = \gamma_p^*$, where $\gamma_p^*$ is the solution of the equation:

$$
F\left(-\frac{1}{2}, 1 - \frac{1}{7 - p}; \frac{1}{2} - \frac{1}{7 - p}; \gamma_p^*\right) = 0.
$$

(6.10)

The numerical values of $\gamma_p^*$ for $0 \leq p \leq 4$ and of the corresponding reduced temperatures $T_p^*$ have been written in table 1. One notices that $\gamma_p^*$ becomes smaller as $p$ is increased, while $T_p^*$ grows with $p$.

For $T < T_p^*$ it is clear from figure 2 that there are two connected solutions with different $\gamma$ for a given value of $T$. Recall that $\gamma$ parametrizes the holographic coordinate of the turning point of the connected configurations. In addition, we have the disconnected configuration, which is the only one that exists for $T > T_p^*$ and is competing with the two connected ones when the temperature is lowered below $T_p^*$. In order to determine which one of these three configurations is thermodynamically more stable we have to find out which one has the lowest free energy. One can establish numerically that the configuration with higher $\gamma$ (i.e.
Table 1: Numerical values of $\gamma_p^{\text{max}}$, $T_p^{\text{max}}$, $\gamma_p^*$ and $T_p^*$ for $0 \leq p \leq 4$. On the last column we have displayed the reduced latent heat of the dimerisation transition, as defined in (6.38).

| $p$ | $\gamma_p^{\text{max}}$ | $T_p^{\text{max}}$ | $\gamma_p^*$ | $T_p^*$ | $\Delta \epsilon_p^*$ |
|-----|-----------------|-----------------|-------|-------|----------------|
| 0   | 0.6141          | 0.6916          | 0.2745 | 0.5927 | 1.5150       |
| 1   | 0.5965          | 0.6979          | 0.2557 | 0.5995 | 1.5274       |
| 2   | 0.5693          | 0.7078          | 0.2285 | 0.6101 | 1.5465       |
| 3   | 0.5215          | 0.7254          | 0.1855 | 0.6293 | 1.5799       |
| 4   | 0.4155          | 0.7658          | 0.1102 | 0.6752 | 1.6541       |

extending deeper in the bulk) has higher free energy than the connected one with lower $\gamma$. To compare the latter with the disconnected one, let us define the reduced difference of free energies $\Delta f_p$ as:

$$\Delta f_p \overset{\text{def}}{=} \frac{F_{\text{str}}^{D(8-p)} - 2 F_{\text{str}}^{D(8-p)}}{2 F_{\text{str}}^{D(8-p)}}. \quad (6.11)$$

From (6.3) we get that, as a function of $\gamma$, $\Delta f_p$ is given by:

$$\Delta f_p = 1 - \Delta_p h(\gamma). \quad (6.12)$$

Clearly, by its definition (6.11) and by (5.23) it follows that $\Delta f_p \to 0$ as $\gamma \to 1$. Moreover, near $\gamma \approx 0$ one can check that $\Delta f_p$ diverges as $\Delta f_p \approx -\Delta_p \gamma^{-\frac{1}{p}} \to -\infty$. The functions $\Delta f_p$ have been plotted in figure 2. It turns out that $\Delta f_p(\gamma)$ vanishes for an intermediate value $\gamma = \gamma_p^*$ which, according to (6.12), satisfies:

$$h(\gamma_p^*) = \frac{1}{\Delta_p}, \quad \gamma_p^* < 1. \quad (6.13)$$

The different numerical values of $\gamma_p^*$ for $0 \leq p \leq 4$ have been collected in table 1, together with the corresponding values of the temperature $T_p^* = T(\gamma_p^*)$. As shown in table 1, $\gamma_p^* < \gamma_p^{\text{max}}$, which means that the connected configuration with lower $\gamma$ is the more stable one when $T < T_p^*$. At $T = T_p^*$ it takes place a dimerisation transition of the type advocated in [21], in which two disconnected branes are recombined to form a hanging brane and the impurities of the boundary theory condense to form bonds or dimers. In order to complete the thermodynamic description of this dimerised phase, let us calculate the entropy of the connected configuration. By computing the derivative of (6.3) with respect to $T$, we obtain:

$$S_{\text{D}(8-p)}^J(T) = -\frac{\partial}{\partial T} F_{\text{D}(8-p)}^J(T) = 2\Delta_p \left[ S_{\text{D}(8-p)}^\text{str} h(\gamma) - F_{\text{str}}^{D(8-p)} \frac{dh}{d\gamma} \left( \frac{dT}{d\gamma} \right)^{-1} \right]. \quad (6.14)$$

By using that (see (4.18) and (4.19))

$$F_{\text{str}}^{D(8-p)} = -\frac{5 - p}{2} T S_{\text{D}(8-p)}^\text{str}, \quad (6.15)$$

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together with (6.4), the relation (6.14) can be written as

\[
S_{D(8-p)}^{\cup}(T) = 2\Delta_p \, S_{D(8-p)}^{str} \left[ h(\gamma) + \frac{5-p}{2} \, g(\gamma) \, \frac{h'(\gamma)}{g'(\gamma)} \right].
\] (6.16)

The derivative of \(g(\gamma)\) has been calculated above (see (6.9)). We now compute \(h'(\gamma)\). To carry out this calculation we will differentiate the right-hand-side of the first equation in (6.1) and we will use the relation:

\[
\frac{d}{d\gamma} F(a,b;c;\gamma) = \frac{ab}{c} \, F(a+1,b+1;c+1;\gamma).
\] (6.17)

Proceeding in this way, one can represent \(h'(\gamma)\) as a combination of two hypergeometric functions which, remarkably, can be written as a single hypergeometric function. One gets

\[
h'(\gamma) = -\frac{1}{7-p} \, \gamma^{-1-\frac{1}{7-p}} \, F\left(-\frac{1}{2},1-\frac{1}{7-p};\frac{1}{2}-\frac{1}{7-p};\gamma\right).
\] (6.18)

Amazingly, the hypergeometric functions on the right-hand-sides of (6.18) and (6.9) are the same and, therefore, the ratio \(h'(\gamma)/g'(\gamma)\) is remarkably simple, namely

\[
\frac{h'(\gamma)}{g'(\gamma)} = -\frac{2}{5-p} \, \sqrt{\frac{1-\gamma}{\gamma}}.
\] (6.19)

Therefore, the entropy of the connected configuration can be written as

\[
S_{D(8-p)}^{\cup}(T) = 2\Delta_p \, S_{D(8-p)}^{str} \left[ h(\gamma) - \gamma \, g(\gamma) \right].
\] (6.20)

Again, the magic of the hypergeometric functions allows us to write the term in brackets in (6.20) in a simplified form. One can actually check that

\[
h(\gamma) - \sqrt{\frac{1-\gamma}{\gamma}} \, g(\gamma) = \frac{2(6-p)(7-p)}{(19-3p)(5-p)} \, \sigma(\gamma),
\] (6.21)

where \(\sigma(\gamma)\) is a new function defined as

\[
\sigma(\gamma) \overset{\text{def}}{=} \gamma^{1-\frac{1}{7-p}} \, F\left(\frac{1}{2},1-\frac{1}{7-p};\frac{1}{2}-\frac{1}{7-p};\gamma\right).
\] (6.22)

By using this result, one can easily show that the entropy can be written as

\[
S_{D(8-p)}^{\cup}(T) = \hat{\Delta}_p \, S_{D(8-p)}^{str}(T) \, \sigma(\gamma),
\] (6.23)

where the coefficient \(\hat{\Delta}_p\) is given by:

\[
\hat{\Delta}_p \overset{\text{def}}{=} \frac{\Gamma\left(\frac{2}{2}-\frac{1}{7-p}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{7-p}\right)} \, \sqrt{\pi}.
\] (6.24)
As a check of the equation (6.23), let us take the limit \( \gamma \to 1 \). One can easily show that

\[
\sigma(\gamma = 1) = \frac{2}{\Delta_p},
\]

and, therefore, one has

\[
\lim_{\gamma \to 1} S_{D(8-p)}^J(T) = 2 S_{D(8-p)}^{\text{str}}(T),
\]

(6.26)
as expected. In order to study in more detail the entropy of the dimers, let us define the reduced entropy \( s_p(T) \) as the ratio

\[
s_p(T) \overset{\text{def}}{=} \frac{S_{D(8-p)}^J(T)}{2 S_{D(8-p)}^{\text{str}}(T)}.
\]

(6.27)

It follows from (6.23) that \( s_p(T) \) is given by

\[
s_p(T) = \frac{\Delta_p}{2} \sigma(\gamma).
\]

(6.28)

By plotting the right-hand-side of (6.28) versus \( \gamma \) one realises that \( s_p \) is a monotonic function of \( \gamma \) which grows from zero to one as \( \gamma \) varies in the interval \([0, 1]\). Let us study the behaviour of the entropy of the dimer near \( T \sim 0 \), which corresponds to \( \gamma \approx 0 \). From the definition of \( \sigma(\gamma) \) in (6.22) it follows that \( \sigma(\gamma) \approx \gamma^{1-p} \) for small \( \gamma \). Moreover, it is straightforward to prove from the expression of the reduced temperature in (6.6) that \( \gamma \approx \left( T \right)^{\frac{2(7-p)}{8-p}} \) for small \( \gamma \). Thus, we have for small \( T \) the reduced entropy of the dimer is given by

\[
s_p \approx \frac{\Delta_p}{2} \left( T \right)^{\frac{2(6-p)}{8-p}}.
\]

(6.29)

Let us now use this result to obtain the low temperature behaviour of the entropy \( S_{D(8-p)}^J \). By using (4.19) for \( M = 1 \) and (6.5), it follows that, for low \( T \), the entropy of the dimer of length \( L \) behaves as

\[
S_{D(8-p)}^J \approx \alpha_p N \left( \sin \theta_{\langle p,n \rangle} \right)^{6-p} \left( g_{YM}^2 N L^{3-p} \right)^{\frac{1}{8-p}} \left( LT \right)^{\frac{9-p}{8-p}},
\]

(6.30)

where \( \alpha_p \) is a numerical coefficient given by:

\[
\alpha_p = 2\pi^2 \left[ \frac{(5-p)\sqrt{\pi}}{(7-p)^2} \right]^{\frac{7-p}{8-p}} \left[ \frac{\Gamma\left( \frac{7-p}{2} \right)}{\Gamma\left( \frac{8-p}{2} \right)} \right]^{\frac{6-p}{8-p}} \frac{\Gamma\left( 2 - \frac{1}{7-p} \right)}{\Gamma\left( \frac{5}{2} - \frac{1}{7-p} \right)}.
\]

(6.31)

Notice that we arranged the right-hand-side of (6.30) in such a way that all factors are dimensionless. The first two factors in parenthesis contain the dependence on the filling faction and on the Yang-Mills coupling respectively, while the last factor contains the dependence on the temperature. Notice that the power of \( T \) in (6.30) is positive for all values of \( p \) (contrary
to what happens with the straight-brane configuration) which means that the entropy of the dimer vanishes at $T = 0$. In particular, for $p = 3$, the entropy of the dimer configuration of D5-branes in the $AdS_5 \times S^5$ background is given by

$$S_{D5}^D \sim N \left( \sin \bar{\theta}_{(3,n)} \right)^3 \sqrt{\lambda} (LT)^3 ,$$

(6.32)

where the relation between the angle $\bar{\theta}_{(3,n)}$ and the filling fraction has been written in (A.4), $\lambda$ is the ’t Hooft coupling of the bulk $\mathcal{N} = 4$ theory and the numerical coefficient is just $\alpha_3$.

Let us next compute the internal energy for the connected configuration, which is given by:

$$\mathcal{E}_{D(8-p)}^{\cup} = F_{D(8-p)}^{\cup} + T S_{D(8-p)}^{\cup} .$$

(6.33)

By combining (6.3), (6.15) and (6.20), one gets

$$\mathcal{E}_{D(8-p)}^{\cup} = \Delta_p T S_{D(8-p)}^{\text{str}}(T) \left[ (p-3) h(\gamma) - 2 \sqrt{\frac{1-\gamma}{\gamma}} g(\gamma) \right] .$$

(6.34)

Let us rewrite (6.34) by using (6.21) to eliminate $h$. After some simplifications one has

$$\mathcal{E}_{D(8-p)}^{\cup}(T) = T S_{D(8-p)}^{\text{str}}(T) \left[ \frac{p-3}{2} \hat{\Delta}_p \sigma(\gamma) + (p-5) \Delta_p \sqrt{\frac{1-\gamma}{\gamma}} g(\gamma) \right] .$$

(6.35)

Notice that, due to (6.25), the right-hand-side of (6.35) reduces to $2 \mathcal{E}_{D(8-p)}^{\text{str}}(T)$ as $\gamma \to 1$, as it should. In order to characterise the dimerisation transition, let us define the reduced latent heat $\Delta \epsilon_p$ as

$$\Delta \epsilon_p \overset{\text{def}}{=} \frac{2 \mathcal{E}_{D(8-p)}^{\text{str}} - \mathcal{E}_{D(8-p)}^{\cup}}{T S_{D(8-p)}^{\text{str}}} .$$

(6.36)

Taking into account that $2 \mathcal{E}_{D(8-p)}^{\text{str}} = (p-3) T S_{D(8-p)}^{\text{str}}$, it follows that

$$\Delta \epsilon_p(\gamma) = (3-p) \left[ \frac{\hat{\Delta}_p}{2} \sigma(\gamma) - 1 \right] + (5-p) \Delta_p \sqrt{\frac{1-\gamma}{\gamma}} g(\gamma) .$$

(6.37)

By construction $\Delta \epsilon_p(\gamma) \to 1$ when $\gamma \to 1$. Moreover, $\Delta \epsilon_p(\gamma) \to +\infty$ as $\gamma \to 0$ since $\Delta \epsilon_p(\gamma) \approx (5-p) \Delta_p \sqrt{\frac{1-\gamma}{\gamma}}$ for small $\gamma$. By plotting $\Delta \epsilon_p(\gamma)$ as a function of $\gamma$ for different values of $p$ one can verify that $\Delta \epsilon_p(\gamma) \geq 0$ for $\gamma \in [0,1]$. It follows that the reduced latent heat at the dimerisation transition, given by

$$\Delta \epsilon_p^* = \Delta \epsilon_p(\gamma = \gamma_p^*),$$

(6.38)

is always positive and therefore the dimerisation transition is a first-order phase transition. The values of $\Delta \epsilon_p^*$ for different $p$'s have been written in the last column of table 1.

We now compute the specific heat for the dimer configuration. Proceeding as in the calculation of the entropy, we arrive at

$$C_{D(8-p)}^{\cup}(T) = T \frac{\partial}{\partial T} S_{D(8-p)}^{\cup}(T) = \hat{\Delta}_p S_{D(8-p)}^{\text{str}}(T) \left[ \frac{p-3}{5-p} \sigma(\gamma) + T \frac{d \sigma}{d \gamma} \left( \frac{d T}{d \gamma} \right)^{-1} \right] .$$

(6.39)

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Figure 3: Reduced specific heats $c_p(\gamma)$ for $0 \leq p \leq 4$. Only a portion of the region $0 \leq \gamma \leq \gamma_p^{\max}$, where $c_p \geq 0$, has been represented in the plot. The curve that grows faster (slower) corresponds to $p = 4$ ($p = 0$).

By using identities satisfied by the hypergeometric functions, the derivative of $\sigma(\gamma)$ with respect to $\gamma$ can be simply written as

$$
\sigma'(\gamma) = \frac{6 - p}{7 - p} \frac{1}{\gamma^\frac{1}{2-p}} F\left(\frac{1}{2}, 2 - \frac{1}{7 - p}; \frac{5}{2} - \frac{1}{7 - p}; \gamma\right),
$$

(6.40)

Moreover, since $T$ can be written in terms of the function $g$ (see (6.4)), equation (6.39) can be recast as

$$
C_{D(8-p)}^{(1)}(T) = \frac{\Delta_p S_{D(8-p)}^{\text{str}}(T)}{\hat{\Delta}_p S_{D(8-p)}^{\text{str}}(T)} \left[ \frac{p - 3}{5 - p} \sigma(\gamma) + g(\gamma) \frac{\sigma'(\gamma)}{g'(\gamma)} \right],
$$

(6.41)

where $g'(\gamma)$ and $\sigma'(\gamma)$ are written in (6.9) and (6.40) respectively. In figure 3 we plot the reduced specific heat, defined as

$$
c_p = \frac{C_{D(8-p)}^{(1)}}{\Delta_p S_{D(8-p)}^{\text{str}}} = \frac{p - 3}{5 - p} \sigma(\gamma) + g(\gamma) \frac{\sigma'(\gamma)}{g'(\gamma)},
$$

(6.42)

as a function of $\gamma$ for different values of $p$. Notice that $c_p \to \infty$ as $\gamma \to \gamma_p^{\max}$, since the derivative of $g(\gamma)$ vanishes at this point. Actually, one can check that $c_p(\gamma)$ is positive in the region $0 \leq \gamma \leq \gamma_p^{\max}$ for all values of $p \leq 4$. Notice that the denominator $\hat{\Delta}_p S_{D(8-p)}^{\text{str}}$ in (6.42) is always positive and, thus, $C_{D(8-p)}^{(1)} \geq 0$ for $\gamma \in [0, \gamma_p^{\max}]$. The behaviour of $C_{D(8-p)}^{(1)}(T)$ for low $T$ can be obtained directly from (6.30). It follows that $C_{D(8-p)}^{(1)}(T) \sim T^\frac{9-p}{5-p} \to 0$ as $T \to 0$.

7 Impurity Fluctuations

In this section, following [20, 25] we discuss the fluctuations on the probe branes around the straight flux tube configurations in the zero temperature background. The goal is to check the stability of these configurations and to carry out the holographic renormalisation program.
for the corresponding action of the fluctuations and to compute the one-point and two-point functions for the fluctuation operators.

The action that governs the fluctuations of the D(8 - p)-brane probe can be obtained by expanding its DBI action around the configuration with \( z' = 0 \) and \( \theta = \bar{\theta}_{(p,n)} \). The induced metric on the world-volume for this configuration at zero temperature is just obtained by taking \( z' = \theta' = 0 \) and \( h_p = 1 \) in (3.29), namely

\[
\tilde{g}^{(0)}_{\alpha\beta} d\zeta^\alpha d\zeta^\beta = \frac{-dt^2 + d\rho^2}{\rho^2} + u_p^2 \sin^2 \bar{\theta}_{(p,n)} d\Omega_7^{2} . \tag{7.1}
\]

Notice that this metric is of the type \( AdS_2 \times S^{7-p} \), with \( u_p^2 \sin^2 \bar{\theta}_{(p,n)} \) being the radius of the \( S^{7-p} \) sphere.

Recall that our probe branes have a non-trivial world-volume gauge field \( \bar{F}_{\ell p} = \cos \bar{\theta}_{(p,n)} \) (see (4.8)). Due to this, the metric that is relevant for the fluctuations is not the induced metric \( \tilde{g}^{(0)} \) but the so-called open string metric, which includes the effects of the world-volume gauge field. In order to define this metric let us consider the matrix \( \left( \tilde{g}^{(0)} + \bar{F} \right)^{-1} \), which is the inverse of the matrix appearing in the DBI Lagrangian for the unperturbed configuration. Let us write this inverse as the sum of a symmetric and an antisymmetric matrix

\[
\left( \tilde{g}^{(0)} + \bar{F} \right)^{-1} = G^{-1} + J , \tag{7.2}
\]

with \( J \) being the antisymmetric component. Then, the symmetric matrix \( G_{\alpha\beta} \) is defined as the open string metric. In our case it is straightforward to obtain the expression of \( G_{\alpha\beta} \). One gets

\[
G_{\alpha\beta} d\zeta^\alpha d\zeta^\beta = \sin^2 \bar{\theta}_{(p,n)} \left[ \frac{-dt^2 + d\rho^2}{\rho^2} + u_p^2 d\Omega_7^{2} \right] . \tag{7.3}
\]

By analysing the possible fluctuations of the D(8 - p)-brane probe one can identify two decoupled channels:

- fluctuations \( \chi \) of the Cartesian coordinates;
- fluctuations \( (\xi, f) \) of the embedding function \( \theta \) and of the world-volume gauge field.

These two channels will be studied separately in the next two subsections.

### 7.1 Fluctuations of the Cartesian Coordinates

Let us assume that the Cartesian coordinates are not fixed but, instead, they fluctuate, and let \( \chi \) be the corresponding fluctuation. In this fluctuating configuration, the induced metric on the probe D(8 - p)-branes becomes

\[
d\tilde{s}_{g_{-p}}^2 = \tilde{g}^{(0)}_{\alpha\beta} d\zeta^\alpha d\zeta^\beta + \tilde{g}^{(1)}_{\alpha\beta} d\zeta^\alpha d\zeta^\beta , \quad \tilde{g}^{(1)}_{\alpha\beta} \equiv \rho^{-2} \left( \partial_\alpha \chi \right) \left( \partial_\beta \chi \right) . \tag{7.4}
\]
with $g_{\alpha\beta}^{(0)}$ being the $AdS_2 \times S^{7-p}$ zero-order metric (7.1). By plugging the metric (7.4) into the DBI action and by expanding to second order in the fluctuation, one gets that the action for $\chi$ can be written as

$$S^{(\chi)} = -\frac{E_{(p,n)} B_p \sin^2 \tilde{\theta}_{(p,n)}}{\Omega_{7-p}} \int d^{9-p} \zeta \left( N e^\phi \right)^{\frac{7-p}{2}} \sqrt{-g_{AdS_2} \sqrt{g_{S^{7-p}}}} \times \left[ \frac{1}{2} \rho^{-2} G^{\alpha\beta} (\partial_\alpha \chi) (\partial_\beta \chi) + O(\chi^4) \right],$$

(7.5)

where $g_{AdS_2}$ and $g_{S^{7-p}}$ are respectively the determinant of the $AdS_2$ metric and of the unit sphere $S^{7-p}$, while $G_{\alpha\beta}$ is the open string metric (7.3). In (7.5) we have written the global coefficient in terms of the flux-tube tension $E_{(p,n)}$ and of the constant $B_p$ defined in (3.11).

It is convenient to rewrite the dilaton by factorising the constant contained in its definition (3.15) and to introduce a rescaled dilaton $\varphi$ as

$$N e^\phi = \left( \frac{g_{YM}^2 N}{2 (2\pi)^{p-2}} \right)^{\frac{7-p}{2}} \left( \frac{d_p}{\Omega_p} \right)^{\frac{p-3}{2}} e^{\frac{(p-3)(7-p)}{4d_p}} \varphi = N e^{\phi^*} e^{\frac{(p-3)(7-p)}{4d_p}} \varphi,$$

(7.6)

so that the new dilaton $\varphi$ is just $\varphi = \log \rho^{-1}$ and, in the second step of (7.6), we have defined a new constant $\phi^*$. In terms of $\varphi$ the action acquires the form

$$S^{(\chi)} = -\tau_{(p,n)} \int d^{9-p} \zeta e^{\frac{p-3}{2}} \sqrt{-g_{AdS_2} \sqrt{g_{S^{7-p}}}} \left[ \frac{1}{2} \rho^{-2} G^{\alpha\beta} (\partial_\alpha \chi) (\partial_\beta \chi) + O(\chi^4) \right],$$

(7.7)

where the effective tension $\tau_{(p,n)}$ is given by:

$$\tau_{(p,n)} \overset{\text{def}}{=} \frac{B_p \left( N e^{\phi^*} \right)^{\frac{7-p}{2}}}{\Omega_{7-p}} E_{(p,n)} \sin^2 \tilde{\theta}_{(p,n)}.$$

(7.8)

By using the explicit values of $B_p$, $\phi^*$ and $E_{(p,n)}$ it is straightforward to verify that $\tau_{(p,n)}$ can be written in terms of field theory quantities as

$$\tau_{(p,n)} = \frac{\Gamma \left( \frac{5-p}{2} \right)}{4 \pi \frac{10-p}{2}} \left[ \frac{\Gamma \left( \frac{5-p}{2} \right)}{(5-p) (2\pi)^{p-2}} g_{YM}^2 N \right]^{\frac{1}{5-p}} \frac{1}{\sin^8 \tilde{\theta}_{(p,n)}} \overset{\text{def}}{=} \tilde{\tau}_{(p,n)} \left[ g_{YM}^2 N \right]^{\frac{1}{5-p}}.$$

(7.9)

It is interesting to notice that the action (7.7) is equivalent to the action of the Cartesian fluctuations for a brane wrapping an $AdS_1+\sigma$-space rather than $AdS_2$

$$S^{(\chi)}_{1+\sigma} = -T_{D(7+\sigma-p)} \int d^{8+\sigma-p} \zeta \sqrt{-g_{AdS_1+\sigma} \sqrt{g_{S^{7-p}}}} \left[ \frac{1}{2} \rho^{-2} G^{\alpha\beta} (\partial_\alpha \chi) (\partial_\beta \chi) + O(\chi^4) \right],$$

(7.10)

up to a compactification on a $T^{\sigma-1}$ torus in such a way that the metric is

$$ds^2_{AdS_1+\sigma} = -dt^2 + \frac{dp^2}{\rho^2} + e^{2 \frac{p-3}{(5-p)(\sigma-1)} \varphi} \delta_{ab} d\xi^a d\xi^b,$$

(7.11)

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where \( \bar{a}, \bar{b} \) run on the \( \sigma - 1 \) extra directions, and the the extra coordinates \( \zeta^a \) take values in the interval \([0, 2\pi R_T]\). Notice that the metric (7.11) corresponds to the one of the \( AdS_{1+\sigma} \) space only when the prefactor multiplying the line element of the extra coordinates is \( e^{2\varphi} = \rho^{-2} \).

This happens when \( \sigma \) is analytically continued to take the factional value \( \sigma = \frac{2}{(5 - p)} \).

Moreover, the compactification radius \( R_T \) is related to the tensions \( T_{D(7+\sigma-p)} \) and \( \tau(p,n) \) by means of the equation

\[
T_{D(7+\sigma-p)} (2\pi R_T)^{\sigma-1} = \tau(p,n). \tag{7.12}
\]

This action can be considered as the action of a \( D(7+\sigma-p) \)-brane in an \( AdS_{1+\sigma} \times S^{8-p} \times S^{7-p} \) submanifold.

In order to easily compute the asymptotic behaviour of the fluctuating field \( \chi \), one can consider the factor \( \rho^{-2} \) in the square brackets of (7.10) as a further dilaton factor and then the action (7.10) can be conveniently written as

\[
S^{(\chi)}_{\tau+\sigma-p} = -T_{D(7+\sigma-p)} \int d^{8+p} \zeta \ e^{2\varphi} \sqrt{-g_{AdS_{1+\sigma}}} \sqrt{g_{S^{7-p}}} \left[ \frac{1}{2} G^{\alpha\beta} (\partial_\alpha \chi) (\partial_\beta \chi) + O(\chi^4) \right]. \tag{7.13}
\]

As before, one can consider the dilaton factor as further extra-dimensions in the world-volume

\[
S^{(\chi)}_{\tau+q-p} = -T_{D(7+q-p)} \int d^{8+q-p} \zeta \sqrt{-g_{AdS_{1+q}}} \sqrt{g_{S^{7-p}}} \left[ \frac{1}{2} G^{\alpha\beta} (\partial_\alpha \chi) (\partial_\beta \chi) + O(\chi^4) \right], \tag{7.14}
\]

where \( q \) is given by

\[
q = 2(6 - p)/(5 - p), \tag{7.15}
\]

and the compactification on the \( T^{q-p} \) torus is determined by the relation between the tensions of the \( D(7+q-p) \)-brane and \( \tau(p,n) \), namely

\[
T_{D(7+q-p)} (2\pi R_T)^{q-1} = \tau(p,n). \tag{7.16}
\]

Let us next consider an ansatz for the fluctuation field \( \chi \) such that is factorises as the product of functions of the AdS space and of the \( S^{7-p} \) sphere, namely:

\[
\chi = Y_l(S^{7-p}) \hat{\chi}(t, \rho) = e^{iEt} Y_l(S^{7-p}) \hat{\chi}(E, \rho). \tag{7.17}
\]

Notice that we have assumed that \( \chi \) does not depend on the coordinates of the extra dimensions. Moreover, in (7.17) the functions \( Y_l(S^{7-p}) \) are the scalar spherical harmonics on the \( S^{7-p} \) sphere which, among other quantum numbers, depend on an integer \( l \). The \( Y_l(S^{7-p}) \) are eigenfunctions of the Laplacian operator on \( S^{7-p} \), with an eigenvalue which depends on \( l \) and \( p \):

\[
\square_{S^{7-p}} Y_l = -l(l+6-p)Y_l. \tag{7.18}
\]
By plugging this ansatz for $\chi$, the action (7.14) becomes the action of a free massive particle propagating in $AdS_{1+q}$

$$S_{r+q-p}^{(l)} = -T_{D(7+q-p)} N_l \int dt d^{q-1}\zeta d\rho \sqrt{-g_{AdS_{1+q}}} \left[ \frac{1}{2} (\partial \tilde{\chi})^2 + \frac{M^2}{2} \chi^2 \right], \quad (7.19)$$

where $N_l$ is the integral

$$N_l = \int d^{7-p} \zeta \sqrt{g_{S^7-p}} \left( Y_l(S^7-p) \right)^2, \quad (7.20)$$

and the mass $M^2$ of the $l$th Kaluza-Klein mode is given by

$$M^2 = \frac{l(l+6-p)}{u_p^2} = \frac{4l(l+6-p)}{(5-p)^2}. \quad (7.21)$$

The equation of motion of $\tilde{\chi}$ can be simply written as the equation of motion of a free massive particle in $AdS_{q+1}$

$$(-\Box_{q+1} + M^2) \tilde{\chi} = 0, \quad (7.22)$$

with $\Box_{q+1}$ being the d’Alembertian in $AdS_{q+1}$. In Euclidean signature, the regular solution of (7.22) can be written in terms of the modified Bessel function of the second type

$$\tilde{\chi}(t, \rho) = e^{iEt} \tilde{\chi}(E, \rho) = e^{iEt} (E\rho)^{q/2} K_\tilde{\alpha}(E\rho), \quad \tilde{\alpha} \overset{def}{=} \sqrt{M^2 + \frac{q^2}{4}} = \frac{2l+6-p}{5-p}, \quad (7.23)$$

and the asymptotic behaviour of $\tilde{\chi}$ as the boundary is approached turns out to be

$$\tilde{\chi} \sim \rho^{2\alpha_-} (\tilde{\chi}^{(0)}_- + \ldots) + \rho^{2\alpha_+} (\tilde{\chi}^{(0)}_+ + \ldots), \quad (7.24)$$

with the exponents $\alpha_{\pm}$ obtained by solving

$$M^2 = 2\alpha (2\alpha - q), \quad \Longrightarrow \quad \begin{cases} 
\alpha_- = -\frac{l}{5-p}, \\
\alpha_+ = \frac{l+6-p}{5-p}.
\end{cases} \quad (7.25)$$

Notice that the parameter $\tilde{\alpha}$ in (7.23) is nothing but the difference between $\alpha_+$ and $\alpha_-$ (i.e. $\tilde{\alpha} = \alpha_+ - \alpha_-$).

In the framework of holographic renormalisation, the non-normalisable fluctuation $\tilde{\chi}^{(0)}_-$ acts as a source of the operator $\hat{O}_\chi$ dual to the fluctuation $\chi$, whereas $\tilde{\chi}^{(0)}_+$ as a vev. In this respect some comments are now in order. Sourcing the mode $\chi$ in the bulk theory is equivalent to sourcing an irrelevant operator $\hat{O}_\chi$ at the boundary. We are working perturbatively in the mode $\chi$, $\chi$ being a fluctuation. This is equivalent to consider parametrically small sources of the irrelevant operator $\hat{O}_\chi$. As first noticed in [29] (see [49] for a further
reference on holographic renormalisation of irrelevant operators), one can consistently holographic renormalise the theory order by order in \( \chi \), making the \( n \)-point correlator of the dual irrelevant operator normalised up to some fixed \( n \). We are interested in computing the two point function, and therefore it will be enough to consider and renormalise the part of the action which is quadratic in the fluctuations.

### 7.1.1 Holographic Renormalisation

As a first step, let us consider the asymptotic expansion of \( \tilde{\chi} \) in a neighbourhood of the boundary \( \rho = 0 \). This expression can be written as

\[
\tilde{\chi} \sim \rho^{2\alpha - 2} \left[ \sum_{k=0}^{\infty} \tilde{\chi}_+^{(2k)} \rho^{2k} + \rho^{2\tilde{\alpha}} \left( \sum_{k=0}^{\infty} \tilde{\chi}_+^{(2k)} \rho^{2k} \delta_{\beta, \tilde{\alpha} - 1} \right) \sum_{k=0}^{\infty} \tilde{\vartheta}_+^{(2k)} \rho^{2k} \log \rho + \delta_{\beta, \tilde{\alpha} - 1} \sum_{k=0}^{\infty} \sum_{l=2}^{s} \sigma_+^{(2k,l)} \rho^{2k} \log^l(\rho) \right]\]  \quad (7.26)

where \( [a] \) denotes the integer part of \( a \) and the logarithmic terms can be present if and only if \( \tilde{\alpha} \) is an integer (implying that \( p \) must be even), which can never occur at leading order as it can be easily seen from the explicit expression for \( \tilde{\alpha} \) in (7.23). More precisely, the logarithmic terms can start to appear at order \( O(\rho^{2\alpha - n}) \), with \( n \) being a (non-zero) multiple of 4. Therefore, the modes characterised by logarithmic terms in the asymptotic expansion close to the boundary have quantum number \( l \) given by

\[
l = (5 - 2\tilde{p})\tilde{n} - (3 - \tilde{p}), \quad \tilde{n} \geq \frac{3 - \tilde{p}}{5 - 2\tilde{p}} \in \mathbb{Z}_+, \quad \tilde{p} = 0, 1, 2, \quad (7.27)
\]

where \( \tilde{p} \) and \( \tilde{n} \) are respectively related to \( p \) and \( n \) by \( n = 4\tilde{n} \) and \( p = 2\tilde{p} \). Furthermore, in absence of the logarithmic terms, the coefficients in such an expansion can be obtained recursively to be

\[
\tilde{\chi}_+^{(2k)} = \frac{\Box_g \tilde{\chi}_+^{(2k-2)}}{4k(\alpha - k)} = \frac{\Gamma(\alpha - k)}{4k(\alpha + 1)\Gamma(\alpha)} \Box_g \tilde{\chi}_+^{(0)}, \quad k \in [1, +\infty], \quad (7.28)
\]

with \( \Box_g \) being the d’Alembertian with respect to the boundary metric \( g \). In presence of the logarithmic terms, the recursive relation (7.28) still holds for the coefficients \( \tilde{\chi}_+^{(2k)} \), \( k < \tilde{\alpha} \), and the coefficient \( \tilde{\vartheta}_+^{(2)} \) of the first logarithmic term is

\[
\tilde{\vartheta}_+^{(2)} = \frac{\Box_g \tilde{\chi}_+^{(2k-2)}}{2\tilde{\alpha}} = \frac{2\Box_g \tilde{\chi}_+^{(0)}}{4\alpha \Gamma(\alpha + 1)\Gamma(\alpha)}. \quad (7.29)
\]

The on-shell action for the fluctuation is

\[
S^{(x)} \sim \frac{1}{2} \int_{\mathcal{M}_t} dt d\tilde{\rho} \left[ \sqrt{g} \tilde{\chi} \rho \partial_\rho \tilde{\chi} \right], \quad (7.30)
\]

\(^5\)We thank Balt van Rees for discussion about this point.
with \( \varepsilon \) being a cut-off for small \( \rho \) and \( \mathcal{M}_\varepsilon \) is the boundary of \( AdS_{1+q} \) at \( \rho = \varepsilon \). Using the asymptotic expansion (7.26), the divergent part of the action turns out to be

\[
S^{(x)} \big|_{\text{div}} \sim \frac{1}{2} \int_{\mathcal{M}_\varepsilon} dt d^{d-1} \zeta \varepsilon^{-2\tilde{\alpha}} \left[ 2 \sum_{k=0}^{\beta} \varepsilon^{2k} \sum_{m=0}^{\alpha_-} (\alpha_- + k - m) \chi^{(2m)} \tilde{\chi}^{(2k-2m)} + 2(\alpha_+ + \alpha_-) \chi^{(0)} \tilde{\chi}^{(0)} \varepsilon^{2\tilde{\alpha}} \log \varepsilon \right],
\]

and by re-expressing the action in terms of the original field \( \tilde{\chi} \), one gets the following counterterm action

\[
S^{(x)}_{cT} \sim \int_{\mathcal{M}_\varepsilon} dt d^{d-1} \zeta \sqrt{g} \left[ \sum_{k=0}^{\beta} a^{(2k)} \tilde{\chi}^{(k)} \tilde{\chi}^{(2k)} + a^{(2\tilde{\alpha})} \tilde{\chi}^{(2\tilde{\alpha})} \tilde{\chi} \log \varepsilon \right],
\]

with \( a^{(0)} = 2\alpha_- \). Thus, the renormalised action becomes

\[
S^{(x)} \big|_{\text{ren}} = \lim_{\varepsilon \to 0} \left[ S^{(x)} \big|_{\varepsilon} + S^{(x)}_{cT} \right] = \lim_{\varepsilon \to 0} \left[ \frac{T_{D(7-q+p)}}{2} \mathcal{N}_i \int_{\mathcal{M}_\varepsilon} dt d^{d-1} \zeta \sqrt{g} \left( \tilde{\chi} \varepsilon \partial_\varepsilon \tilde{\chi} \right) \big|_{\varepsilon} + \frac{T_{D(7-q+p)}}{2} \mathcal{N}_i \int_{\mathcal{M}_\varepsilon} dt d^{d-1} \zeta \sqrt{g} \left[ \sum_{k=0}^{\beta} a^{(2k)} \tilde{\chi}^{(k)} \tilde{\chi}^{(2k)} + a^{(2\tilde{\alpha})} \tilde{\chi}^{(2\tilde{\alpha})} \tilde{\chi} \log \varepsilon \right] \big|_{\varepsilon} \right],
\]

where \( \mathcal{N}_i \) has been defined as the integral of the spherical harmonics in (7.19).

### 7.1.2 One-Point and Two-Point Correlation Function

Let us now use the renormalised action to compute the one-point correlation function, which is defined as

\[
\langle \hat{O}_\chi \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{2\alpha_+}} \frac{1}{\sqrt{g}_\varepsilon} \frac{\delta S^{(x)} \big|_{\text{ren}}}{\delta \tilde{\chi}(t, \varepsilon)}. \tag{7.34}
\]

The first variation of the renormalised action turns out to be

\[
\delta S^{(x)} \big|_{\text{ren}} = T_{D(7-q+p)} \mathcal{N}_i \int_{\mathcal{M}_\varepsilon} dt d^{d-1} \zeta \varepsilon^{-2\alpha_-} \left[ 2(\alpha_- - \alpha_+) \chi_+^{(0)} \tilde{\chi}_+^{(0)} + \delta_+^{(0)} \right] \delta \tilde{\chi}(t, \varepsilon), \tag{7.35}
\]

and therefore the one-point function is given by

\[
\langle \hat{O}_\chi \rangle = \tau_{(p, n)} \mathcal{N}_i \left[ 2(\alpha_- - \alpha_+) \chi_+^{(0)} \tilde{\chi}_+^{(0)} + \delta_{\beta, \tilde{\alpha}-1}^{(0)} \right] = \tau_{(p, n)} \mathcal{N}_i \left[ -2 \frac{2l + 6 - p}{5 - p} \chi_+^{(0)} \tilde{\chi}_+^{(0)} + \delta_{\beta, \tilde{\alpha}-1}^{(0)} \frac{2}{4 \tilde{\alpha} \Gamma(\tilde{\alpha})} \right]. \tag{7.36}
\]

Typically, the second term in (7.36) can be removed completely by adding a finite counter-term, corresponding to the matter conformal anomaly in \( AdS_{q+1} \) [50].

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The coefficient $\tilde{\chi}_+^{(0)}$ can be read off from the full-solution (7.22). In order to perform this calculation, let us extract the time dependence of $\tilde{\chi}(t, \rho)$ and work with $\tilde{\chi}(E, \rho)$ (see eq. (7.17)). Expanding the latter in a neighbourhood of the boundary we have to distinguish two cases, namely
\[ \tilde{\alpha} \text{ non-integer:} \]
\[
\tilde{\chi}(E, \rho) = \rho^{2\tilde{\alpha}} - \tilde{\chi}_+^{(0)}(E) \left[ 1 + \frac{(E\rho)^2}{4(\tilde{\alpha} + 1)} + \ldots - \frac{(E\rho)^{2\tilde{\alpha}}}{2^{2\tilde{\alpha}}} \Gamma(1 - \tilde{\alpha}) \frac{\Gamma(1 + \tilde{\alpha})}{2\tilde{\alpha}} \right],
\]
\[ \tilde{\alpha} \text{ integer:} \]
\[
\tilde{\chi}(E, \rho) = \rho^{2\tilde{\alpha}} - \tilde{\chi}_+^{(0)}(E) \left[ 1 + \frac{(E\rho)^2}{4(1 - \tilde{\alpha})} + \ldots + \frac{(-1)^{\tilde{\alpha}} (E\rho)^{2\tilde{\alpha}}}{2^{2\tilde{\alpha}} \Gamma(\tilde{\alpha}) \Gamma(\tilde{\alpha} + 1)} \times \right.
\]
\[
\times \left. \left[ H_{\tilde{\alpha}} - 2\gamma_{EM} - \frac{2}{\tilde{\alpha}(\tilde{\alpha} - 1)} \log \frac{E}{2} \right] \right] + \frac{(-1)^{\tilde{\alpha} + 1/2} (E\rho)^{2\tilde{\alpha}}}{2^{2\tilde{\alpha}} \Gamma(\tilde{\alpha} + 1) \Gamma(\tilde{\alpha} + 1)} \log \rho + \ldots \right],
\]
where $\gamma_{EM} = 0.577$ is the Euler-Mascheroni constant and, for a given integer $n$, $H_n$ is the harmonic number ($H_n = \sum_{k=1}^{n} \frac{1}{k}$). By using these results the coefficient $\tilde{\chi}_+^{(0)}$ turns out to be
\[
\tilde{\chi}_+^{(0)} = \begin{cases} 
\frac{-E^{2\tilde{\alpha}} \Gamma(1 - \tilde{\alpha})}{2^{2\tilde{\alpha}} \Gamma(1 + \tilde{\alpha})} \tilde{\chi}_+^{(0)}(E), & \tilde{\alpha} \text{ non-integer} \\
\frac{(-1)^{\tilde{\alpha}} E^{2\tilde{\alpha}}}{2^{2\tilde{\alpha}} \Gamma(1 + \tilde{\alpha})} \left[ H_{\tilde{\alpha}} - 2\gamma - \frac{2}{\tilde{\alpha}(\tilde{\alpha} - 1)} \log \frac{E}{2} \right] \tilde{\chi}_+^{(0)}(E), & \tilde{\alpha} \text{ integer} 
\end{cases}
\]
(7.38)

The correlator (7.36) therefore acquires the following form
\[
\langle \hat{\mathcal{O}}_\chi(E) \rangle = -N_i^{(p, n)} \frac{2l + 6 - p}{5 - p} \left[ g_{EM}^2(E) \right]^{\frac{1}{5-p}} E^{\frac{4l + 3(5-p)}{5-p}} \Theta(E),
\]
(7.39)

where $N_i^{(p, n)} = N_i \left[ \tau_{(p, n)} \left( \tau_{(p, n)} \right) \right.$ (\tau_{(p, n)}$ has been defined in (7.9)), $g_{EM}^2(E) \equiv g_{EM}^2 E^{p-3}$ and $\Theta(E) = E^{-2\tilde{\alpha}} \tilde{\chi}_+^{(0)}$. Notice that $\Theta(E)$ can be written in terms of the source $\tilde{\chi}_+^{(0)}(E)$ by using (7.38).

This relation depends on whether $\tilde{\alpha}$ is integer or not. As $\tilde{\alpha} = 1 + \frac{2l + 1}{5 - p}$, one has that $\tilde{\alpha}$ is integer iff $2l + 1 = 0 \pmod{5 - p})$. This never occurs if $p$ is odd and it happens in all cases if $p = 4$. In the remaining cases $p = 0, 2$ it only occurs for some particular values of the Kaluza-Klein mode $l$. Furthermore, for $\tilde{\alpha}$ integer, the only relevant term is the one containing $\log E$, while the others are scheme dependent and we will omit them.

Finally, the two-point correlator can be obtained from (7.39) by differentiating with respect to the source $\tilde{\chi}_+^{(0)}$. It is easy to see that for $\tilde{\alpha}$ non integer, it is just given by contact terms
\[
\langle \hat{\mathcal{O}}_\chi \hat{\mathcal{O}}_\chi \rangle = -N_i^{(p, n)} \frac{2\tilde{\alpha}}{4\tilde{\alpha}} \left[ g_{EM}^2(E) \right]^{\frac{1}{5-p}} E^{\frac{4l + 3(5-p)}{5-p}} \frac{\Gamma(1 - \tilde{\alpha})}{\Gamma(1 + \tilde{\alpha})},
\]
(7.40)

while for $\tilde{\alpha}$ integer it has a logarithmic dependence
\[
\langle \hat{\mathcal{O}}_\chi \hat{\mathcal{O}}_\chi \rangle = \frac{2(-1)^{\tilde{\alpha}} N_i^{(p, n)}}{4\tilde{\alpha} \Gamma(\tilde{\alpha}) \Gamma(\tilde{\alpha} - 1)} \left[ g_{EM}^2(E) \right]^{\frac{1}{5-p}} E^{\frac{4l + 3(5-p)}{5-p}} \log \frac{E^{2\tilde{\alpha}}}{\mu^{2\tilde{\alpha}}},
\]
(7.41)
The two-point correlator in position space can be obtained by Fourier transforming (7.40) and (7.41). With this purpose, we will use the relations

$$I_a(t) \equiv \int dE e^{-iEt} E^{2\alpha} \Gamma \frac{1}{2} \frac{1}{|t|^{1+2\alpha}}, \quad \alpha \text{ non-integer},$$

$$I_b(t) \equiv \int dE e^{-iEt} E^{2\alpha} \log E = (-1)^{1+\alpha} \frac{(2\alpha)!}{2} \mathcal{R} \left( \frac{1}{|t|^{1+2\alpha}} \right), \quad \alpha \text{ integer},$$

(7.42)

where \(\mathcal{R}(\cdot)\) indicates the renormalised version of its argument and is defined as

$$\mathcal{R} \left( \frac{1}{|t|^{1+2\alpha}} \right) \equiv \frac{1}{|t|^{1+2\alpha}} - \frac{2\gamma_{\text{EM}}}{(2\alpha)!} \delta^{(2\alpha)}(t),$$

(7.43)

with \(\delta^{(m)}(t)\) denoting the \(m\)th derivative of the Dirac \(\delta\)-function. The two-point correlator in position space becomes

$$\langle \hat{O}_\chi \hat{O}_\chi \rangle = -N^{(p,n)}_{\text{eff}} \frac{\Gamma(1/2 - \alpha) \Gamma(1 - \alpha)}{\Gamma(-\alpha) \Gamma(1 + \alpha)} \frac{1}{|t|^{2\Delta_\chi}} g_{\text{eff}}^2(t) \bigg|_{1^2}^{1+2\alpha}, \quad \alpha \text{ non-integer},$$

$$\langle \hat{O}_\chi \hat{O}_\chi \rangle = -\frac{(2\alpha)! N^{(p,n)}_{\text{eff}}}{4^\alpha \Gamma(\alpha) \Gamma(\alpha - 1)} \mathcal{R} \left( \frac{g_{\text{eff}}^2(t)^{1/2}}{|t|^{2\Delta_\chi}} \right), \quad \alpha \text{ integer},$$

(7.44)

where the effective coupling constant is defined as \(g_{\text{eff}}^2(t) \equiv g_{\text{YM}}^2 N |t|^{3-p}\) and \(N^{(p,n)}_{\text{eff}} \sim N\). Notice that the exponent of \(|t|\) in (7.44) provides twice the generalised conformal dimension \(\Delta_\chi\) of the operator \(\hat{O}_\chi\), which is given by

$$\Delta_\chi = \frac{2}{5-p} l + 2.$$  

(7.45)

For \(p = 3\) this result coincides with the correct value \(\Delta_\chi = l + 2\) for the truly conformal case [20, 24, 25]. Notice that \(\Delta_\chi\) is fractional for \(p < 3\).

### 7.2 Fluctuations of the Angular Embedding Function and World-volume Gauge Field

Let us move on to the angular embedding, whose fluctuations are coupled to the ones of the world-volume gauge field. We analyse the fluctuations around the configuration \(\theta = \bar{\theta}_{(p,n)}\), namely

$$\theta = \bar{\theta}_{(p,n)} + \xi, \quad \mathcal{F}_{\alpha\beta} = \mathcal{F}^{(0)}_{\alpha\beta} + \mathcal{F}^{(1)}_{\alpha\beta},$$

(7.46)

where the only non-zero component of \(\mathcal{F}^{(0)}_{\alpha\beta}\) is \(\mathcal{F}^{(0)}_{4p} = \cos \bar{\theta}_{(p,n)}\) and, in terms of the rescaled dilaton \(\varphi\) defined in (7.6), the fluctuation field can be rewritten as

$$\mathcal{F}^{(1)}_{\alpha\beta} = e^{\frac{3-p}{p}} \varphi f_{\alpha\beta}.$$  

(7.47)
When these fluctuations are switched on the induced metric on the probe D(8 − p)-branes acquires a further term given by

\[
\tilde{g}^{(1)}_{\alpha\beta} = u_p^2 \left[ \left( \partial_\alpha \xi \right) \left( \partial_\beta \xi \right) + \left( 2 \sin \theta_{(p,n)} \cos \theta_{(p,n)} \xi + \left( \cos^2 \theta_{(p,n)} - \sin^2 \theta_{(p,n)} \right) \right) \xi^2 \delta_\alpha^a \delta_\beta^b g_{ab} \right],
\]

(7.48)

where \( a, b \) are indices along the \( S^{7-p} \) sphere and \( g_{ab} \) denotes its round metric. The D(8 − p)-brane action for these fluctuations acquires the following form

\[
S_{D(8-p)} = -\tau_{(p,n)} \int d^{9-p} \xi e^{\frac{p-3}{5-p} \varphi} \sqrt{\tilde{g}_{\text{AdS}_2}} \sqrt{g_{S^{7-p}}} \times \left[ \sqrt{\det \left\{ I + X \right\}} + \frac{\left( F_{\ell p}^{(0)} + F_{\ell p}^{(1)} \right) C_p \left( \tilde{g}_{(p,n)} + \xi \right)}{\sin^{3-p} \theta_{(p,n)}} \right],
\]

(7.49)

where \( C_p(\theta) \) is the function defined in (3.18) and \( X \) is a matrix defined as

\[
X \overset{\text{def}}{=} \left( \tilde{g}^{(0)} + F^{(0)} \right)^{-1} \left( \tilde{g}^{(1)} + F^{(1)} \right).
\]

(7.50)

Expanding the action (7.49) and keeping the terms up to the second order in the fluctuations, one gets

\[
S^{(\xi, f)} = -\tau_{(p,n)} \int d^{9-p} \xi e^{\frac{p-3}{5-p} \varphi} \sqrt{\tilde{g}_{\text{AdS}_2}} \sqrt{g_{S^{7-p}}} \times \left[ \frac{u_p^2}{2} G^{\alpha\beta} \left( \partial_\alpha \xi \right) \left( \partial_\beta \xi \right) - \frac{7 - p}{2} \xi^2 \frac{\sin^2 \theta_{(p,n)}}{\sin^2 \theta_{(p,n)}} \right] + \frac{1}{4} G^{\alpha\gamma} G^{\beta\delta} e^{\frac{7-p}{5-p} \varphi} f_{\alpha\beta} f_{\gamma\delta} - \frac{7 - p}{\sqrt{\tilde{g}_{\text{AdS}_2}}} \frac{\xi e^{\frac{3-p}{5-p} \varphi} f_{\ell p}}{\sin^3 \theta_{(p,n)}},
\]

(7.51)

where the open-string metric \( G_{\alpha\beta} \) is defined as in (7.3). Such a Lagrangian coincides with eq (2.61) in [20] if the components \( f_{ab} \) along the \( S^{7-p} \) directions are taken to be zero. Let us represent \( f_{\alpha\beta} \) in terms of a potential \( a_\alpha \) as \( f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha \). Then, the equations of motion for the fluctuations \( (\xi, a_\alpha) \) have the following form

\[
0 = \frac{1}{\sqrt{-g}} \partial_\alpha \left[ \sqrt{-g} G^{\alpha\beta} \partial_\beta \xi \right] + \frac{7 - p}{u_p^2 \sin^2 \theta_{(p,n)}} \left[ \xi + \frac{e^{\frac{7-p}{5-p} \varphi} f_{\ell p}}{\sqrt{\tilde{g}_{\text{AdS}_2} \sin \theta_{(p,n)}}} \right],
\]

\[
0 = \frac{1}{\sqrt{-g}} \partial_\alpha \left[ \sqrt{-g} G^{\alpha\gamma} G^{\beta\delta} e^{\frac{7-p}{5-p} \varphi} f_{\gamma\delta} \right] + \frac{7 - p}{\sin^2 \theta_{(p,n)}} \xi \left[ \delta_\alpha^\rho \delta_\beta^\beta - \delta_\alpha^\beta \delta_\beta^\rho \right],
\]

(7.52)

where \( \sqrt{-g} \overset{\text{def}}{=} e^{\frac{7-p}{5-p} \varphi} \sqrt{-g_{\text{AdS}_2} \sqrt{g_{S^{7-p}}}} \). Similarly to the case analysed in Section 7.1, the equations of motion (7.52) are equivalent to the ones in \( AdS_{\hat{q}+1} \) compactified on a \( T^{\hat{q}-1} \) and with \( \hat{q} \) analytically continued to take the value

\[
\hat{q} = \frac{2}{5 - p}.
\]

(7.53)
Indeed, it is straightforward to check that one can rewrite the system (7.52) as

\[ 0 = \Box_{q+1} \xi + u_p^{-2} \Box_{S^7-p} \xi + \frac{7 - p}{u_p^2} \left[ \xi + \frac{f_t}{\sqrt{-g_{AdS^q+1}} \sin \theta_{(p,n)}} \right], \]

\[ 0 = \frac{1}{\sqrt{-g_{AdS^q+1}}} \partial_a \left[ \sqrt{-g_{AdS^q+1}} g^{\hat{A} \hat{C}} g^{BD} e^{\frac{2}{3} \frac{p}{5-p} \varphi} f_{CD} + \frac{7 - p}{\sin \theta_{(p,n)}} \xi \left[ \delta^A \delta^B - \delta^A \delta^B \right] \right] + (7.54) \]

\[ + \frac{u_p^{-2}}{\sqrt{g_{S^7-p}}} \partial_a \left[ \sqrt{g_{S^7-p}} g^{ac} g^{BD} e^{\frac{2}{3} \frac{p}{5-p} \varphi} f_{CD} \right], \]

Let us now analyse in details the different modes, following the classification in [51].

7.2.1 Coupled Modes

Let us now choose the following ansatz for the scalar \( \xi \) and the gauge field

\[ \xi = e^{iEt} Y_t(S^{7-p}) \hat{\xi}(\rho), \quad a_a = e^{iEt} \nabla_a Y_t(S^{7-p}) \hat{a}(\rho), \quad a_\rho = e^{iEt} Y_t(S^{7-p}) \hat{a}_\rho(\rho), \quad (7.55) \]

with all the other components of the gauge field set to zero. Notice that the field component \( a_a \) is a gradient in the \( S^{7-p} \) sphere. Therefore, it is always possible to perform a gauge transformation such that the components of the gauge field along the \( S^{7-p} \) are set to zero while the non-zero components are the ones along the \( AdS_2 \) directions. Accordingly, in the following we will adopt the following ansatz for the components of the gauge field

\[ a_t = -i E e^{iEt} Y_t(S^{7-p}) \hat{a}_t(\rho), \quad a_\rho = e^{iEt} Y_t(S^{7-p}) (\hat{a}_\rho - \partial_\rho \hat{a}_t). \quad (7.56) \]

With such an ansatz, the equations of motion (7.54) acquire the following form

\[ 0 = \Box_{q+1} \hat{\xi} - E^2 g^{tt} \hat{\xi} - \frac{4l(l + 6 - p)}{(5 - p)^2} \hat{\xi} + \frac{4(7 - p)}{(5 - p)^2} \left[ \hat{\xi} + i E \frac{\hat{a}_\rho}{\sqrt{-g_{AdS^q+1}} \sin \theta_{(p,n)}} \right], \]

\[ 0 = \partial_\rho \left[ \frac{\hat{a}_\rho}{\sqrt{-g_{AdS^q+1}} \sin \theta_{(p,n)}} - \frac{7 - p}{E} \hat{\xi} \right] - \frac{4l(l + 6 - p)}{(5 - p)^2} \frac{g_{\rho \rho} \hat{a}_t}{\sqrt{-g_{AdS^q+1}} \sin \theta_{(p,n)}}, \]

\[ 0 = E^2 \left[ \frac{g_{\rho \rho} \hat{a}_t}{\sqrt{-g_{AdS^q+1}} \sin \theta_{(p,n)}} + \partial_\rho \left[ \frac{g_{tt} (\hat{a}_\rho - \partial_\rho \hat{a}_t)}{\sqrt{-g_{AdS^q+1}} \sin \theta_{(p,n)}} \right] \right] \]

where \( \Box_{q+1} \) is the d’Alembertian operator for \( AdS^q+1 \). This is a system of four equations in three unknowns \( (\hat{\xi}, \hat{a}_t, \hat{a}_\rho) \), and therefore one of the equations must be redundant. It is easy to see that solving the second and third equations in (7.57) for their last terms and inserting such solutions in the last equation, the latter turns out to be an identity. So, we can consider
the last equation in (7.57) as redundant and focus on the first three. Their form suggests that it is convenient to define two new fields \((\tilde{\xi}, \tilde{\eta})\)

\[
\tilde{\xi} \overset{\text{def}}{=} -\frac{i}{E} \hat{\xi} e^{iEt}, \quad \tilde{\eta} \overset{\text{def}}{=} \left( \frac{\hat{\alpha}_\rho}{\sqrt{-g_{AdS_{q+1}}} \sin \theta} - \frac{7 - p}{E} \hat{\xi} \right) e^{iEt}. \tag{7.58}
\]

Now, eliminating the field \(\hat{a}_t\) by using the second equation in (7.57), the equations of motion can be reduced to the following system of two equations

\[
0 = \Box_{q+1} \tilde{\xi} - \frac{4}{(5 - p)^2} [l(l + 6 - p) + (7 - p)(6 - p)] \tilde{\xi} + \frac{4(7 - p)}{(5 - p)^2} \tilde{\eta},
\]

\[
0 = \Box_{q+1} \tilde{\eta} - \frac{4(l + 6 - p)}{(5 - p)^2} \tilde{\eta} + \frac{4(7 - p)l(l + 6 - p)}{(5 - p)^2} \tilde{\xi}. \tag{7.59}
\]

It is convenient to rewrite the above system of two equations in matrix form

\[
\Box_{q+1} \psi = M_{(l,p)} \psi, \quad \psi = \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix}, \tag{7.60}
\]

and the mass matrix \(M_{(l,p)}\), as in [20], is

\[
M_{(l,p)} \overset{\text{def}}{=} \frac{4}{(5 - p)^2} \begin{pmatrix} l(l + 6 - p) + (7 - p)(6 - p) & p - 7 \\ (p - 7)l(l + 6 - p) & l(l + 6 - p) \end{pmatrix}. \tag{7.61}
\]

The eigenvalues \(\lambda^{(i)}\) and eigenvectors \(\psi^{(i)}\) of \(M_{(l,p)}\) are

\[
\lambda^{(1)} = \frac{4(l + 6 - p)(l + 7 - p)}{(5 - p)^2}, \quad \psi^{(1)} = (l + 6 - p) \tilde{\xi} - \tilde{\eta}, \quad l \geq 0,
\]

\[
\lambda^{(2)} = \frac{4l(l - 1)}{(5 - p)^2}, \quad \psi^{(2)} = \tilde{\xi} + \tilde{\eta}, \quad l \geq 1, \tag{7.62}
\]

and they satisfy the differential equation

\[
\Box_{q+1} \psi^{(i)} = \lambda^{(i)} \psi^{(i)}. \tag{7.63}
\]

It is now straightforward to obtain the behaviour of the normalizable and non-normalizable modes for \(\psi^{(i)}\) in a neighbourhood of the boundary \(\rho \to 0\). One gets

\[
\psi^{(1)} \underset{\rho \to 0}{\sim} \psi^{(1)}_+ \rho^{2\alpha^{(1)}_+} + \psi^{(1)}_- \rho^{2\alpha^{(1)}_-}, \quad \begin{cases} \alpha^{(1)}_- = \frac{-(l+6-p)}{5-p}, \\
\alpha^{(1)}_+ = \frac{-(l+7-p)}{5-p}. \end{cases}
\]

\[
\psi^{(2)} \underset{\rho \to 0}{\sim} \psi^{(2)}_+ \rho^{2\alpha^{(2)}_+} + \psi^{(2)}_- \rho^{2\alpha^{(2)}_-}, \quad \begin{cases} \alpha^{(2)}_- = \frac{-(l-1)}{5-p}, \\
\alpha^{(2)}_+ = \frac{l}{5-p}. \end{cases} \tag{7.64}
\]

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In order to compute the correlation functions for the decoupled modes $\psi^{(i)}$, one can notice that their equations of motion (7.63) are just the equation of motion for free massive scalars in $AdS_{q+1}$ and, therefore, the dynamics of the fluctuations – up to quadratic order – can be described by the following effective Euclidean action

$$S_{\text{eff}}^{(\psi^{(i)})} = T_{D(7+q-p)} N_i \int dt d^{q-1} \xi d\rho \sqrt{g_{AdS_{q+1}}} \frac{1}{2} \left[ \left( \partial \psi^{(i)} \right)^2 + \lambda^{(i)} (\psi^{(i)})^2 \right].$$

(7.65)

The holographic renormalisation of this action goes exactly as in 7.1.1, so that the one-point function can be written as

$$\langle \hat{O}_{\psi^{(i)}} \rangle = -2\tilde{\alpha}^{(i)} N_{\psi^{p,n}} \left[ g_{\text{eff}}^2 (E) \right]^{\frac{1}{b-p}} E^{2\tilde{\alpha}^{(i)} + \frac{3-p}{b-p} } \Theta^{(i)} (E),$$

(7.66)

with $\tilde{\alpha}^{(i)} \equiv \alpha_+^{(i)} - \alpha_-^{(i)}$ and $\Theta^{(i)}$ given by

$$\Theta^{(i)} = \left\{ \begin{array}{ll}
- \frac{\Gamma(1-\tilde{\alpha}^{(i)})}{\Gamma(1+\tilde{\alpha}^{(i)})} \psi_{\tilde{\alpha}^{(i)}}, & \tilde{\alpha}^{(i)} \text{ non-integer} \\
\frac{(-1)^{\tilde{\alpha}^{(i)}}}{4^{\tilde{\alpha}^{(i)}} \Gamma(\tilde{\alpha}^{(i)}) \Gamma(\tilde{\alpha}^{(i)} - 1)} \left[ H_{\tilde{\alpha}^{(i)}} - 2E_{\text{EM}} - \frac{2}{\alpha^{(p,n)}(\tilde{\alpha}^{(i)} - 1)} \log \frac{E}{2} \right] \psi_{\tilde{\alpha}^{(i)}}, & \tilde{\alpha}^{(i)} \text{ integer.}
\end{array} \right.$$

(7.67)

From the values of $\alpha_{\pm}^{(i)}$ written in (7.64), one immediately determines the values of $\tilde{\alpha}^{(i)}$, namely

$$\tilde{\alpha}^{(1)} = \frac{2l + 3}{5-p} + 2, \quad \tilde{\alpha}^{(2)} = \frac{2l - 1}{5-p}.$$  

(7.68)

Then, it follows that $\tilde{\alpha}^{(1)}$ is integer iff $2l + 3 = 0 \text{ (mod} (5-p))$, while $\tilde{\alpha}^{(1)} \in \mathbb{Z}$ only when $2l - 1 = 0 \text{ (mod} (5-p))$. As in the case of the $\chi$ fluctuation, this can only happen when $p$ is even and always happens if $p = 4$, whereas for $p = 0, 2$ it occurs for some values of $l$.

Similarly to the case of the fluctuations of the Cartesian coordinates, the two point function turns out to have the following functional form

$$\langle \hat{O}_{\psi^{(i)}} \hat{O}_{\psi^{(i)}} \rangle \sim \left[ g_{\text{eff}}^2 (E) \right]^{\frac{1}{b-p}} E^{2\tilde{\alpha}^{(i)} + \frac{3-p}{b-p} }, \quad \tilde{\alpha}^{(i)} \text{ non-integer},$$

$$\langle \hat{O}_{\psi^{(i)}} \hat{O}_{\psi^{(i)}} \rangle \sim \left[ g_{\text{eff}}^2 (E) \right]^{\frac{1}{b-p}} E^{2\tilde{\alpha}^{(i)} + \frac{3-p}{b-p} } \log \frac{E^{\tilde{\alpha}^{(i)}}}{\mu^{2\tilde{\alpha}^{(i)}}}, \quad \tilde{\alpha}^{(i)} \text{ integer,}$$

(7.69)

which in the coordinate space becomes

$$\langle \hat{O}_{\psi^{(i)}} \hat{O}_{\psi^{(i)}} \rangle \sim \frac{\left[ g_{\text{eff}}^2 (t) \right]^{\frac{1}{b-p}}}{|t|^{2\Delta_{\psi^{(i)}}}}, \quad \tilde{\alpha}^{(i)} \text{ non-integer}$$

$$\langle \hat{O}_{\psi^{(i)}} \hat{O}_{\psi^{(i)}} \rangle \sim \mathcal{R} \left( \frac{\left[ g_{\text{eff}}^2 (t) \right]^{\frac{1}{b-p}}}{|t|^{2\Delta_{\psi^{(i)}}}} \right), \quad \tilde{\alpha}^{(i)} \text{ integer.}$$

(7.70)
with the exponent $\Delta^{(i)}_\psi$ given by

$$
\Delta^{(i)}_\psi = \alpha^{(i)} + 1 - \frac{1}{5-p} = \begin{cases} 
\frac{2}{5-p} (l + 1) + 3, & i = 1, \\
\frac{2}{5-p} (l - 1) + 1, & i = 2.
\end{cases} 
$$

(7.71)

Notice that the exponents $\Delta^{(i)}_\psi$ correctly reproduce the conformal dimensions in the case $p = 3$, where $\Delta^{(1)}_\psi = l + 4$ and $\Delta^{(2)}_\psi = l$. Again the conformal dimensions are fractional for $p < 3$.

### 7.2.2 Internal Gauge Field Modes

Finally, let us consider an ansatz for which the scalar $\xi$ is set to zero and the only non-zero component for the gauge field are the ones along the $S^{7-p}$-directions

$$
\xi = 0, \quad f_{a} = \partial_{a} a, \quad f_{\rho a} = \partial_{\rho} a_{a}, \quad f_{ab} = \partial_{a} a_{b} - \partial_{b} a_{a},
$$

(7.72)

with

$$a_{a} = Y^{l}_{a} (S^{7-p}) \t a(t, \rho) \equiv Y^{l}_{a} (S^{7-p}) e^{E_{l} \t a}(\rho),
$$

(7.73)

where $Y^{l}_{a} (S^{7-p})$ is a vector spherical harmonic on the $S^{7-p}$ sphere. With such an ansatz, the equations of motion (7.54) can be written as

$$
0 = \partial_{t} \left( \nabla_{a} a^{a} \right), \\
0 = \partial_{\rho} \left( \nabla_{a} a^{a} \right), \\
0 = \frac{g_{\rho \rho}}{\sqrt{-g_{AdS_{q+1}}} \partial_{t} a^{b} - \partial_{\rho} \left[ \frac{g_{tt}}{\sqrt{-g_{AdS_{q+1}}} \partial_{t} a^{b}} \right] - \frac{g_{tt} g_{\rho \rho}}{\sqrt{-g_{AdS_{q+1}}}} u_{\rho}^{-2} \left( \Delta a^{b} - \nabla^{b} \nabla_{c} a^{c} \right)},
$$

(7.74)

where the operators $\Delta$ (the Hodge-de Rham operator for one-forms on the $S^{7-p}$) and $\nabla_{b}$ are defined as

$$
\Delta a^{b} = \Box_{S^{7-p}} a^{b} - R_{c}^{b} a^{c}, \quad \nabla_{b} a^{b} = \frac{1}{\sqrt{g_{S^{7-p}}}} \partial_{b} \left[ \sqrt{g_{S^{7-p}}} g^{bc} a^{c} \right],
$$

(7.75)

with $R_{c}^{b}$ being the Ricci tensor on the $S^{7-p}$-sphere. From the first two equations (7.74), one can deduce the requirement

$$
\nabla_{b} a^{b} = 0,
$$

(7.76)

which implies that the last term in the third equation (7.74) vanishes. Furthermore, from the decomposition (7.73) the condition (7.76) implies that the vector spherical harmonic $Y^{l}_{a} (S^{7-p})$ has to be such that

$$
\nabla^{b} Y^{l}_{b} (S^{7-p}) = 0.
$$

(7.77)
Considering also that the vector harmonic $Y^l_a(S^{7-p})$ is an eigenfunction of the Hodge-de Rham operator $\Delta$ defined in (7.75)

$$\Delta Y^l_a(S^{7-p}) = -(l + 1)(l + 5 - p)Y^l_a(S^{7-p}) ,$$

(7.78)

the third equation in (7.74) takes the form

$$0 = (\Box_{s+1} - M^2_{\tilde{a}}) \tilde{a},$$

(7.79)

where $\Box_{s+1}$ is the Dalambertian operator in $AdS_{s+1}$ with $s = 2 - \hat{q} = 2(4 - p)/(5 - p)$ and the mass $M^2_{\tilde{a}}$ is

$$M^2_{\tilde{a}} = \frac{4}{(5 - p)^2} [(l + 1)(l + 5 - p)].$$

(7.80)

The behaviour of the normalizable and non-normalizable modes $\tilde{a} \sim \tilde{a}_- \rho^{2\alpha_-} + \tilde{a}_+ \rho^{2\alpha_+}$ in a neighbourhood of the boundary is given by

$$M^2_{\tilde{a}} = 2\alpha (2\alpha - s) \implies \begin{cases} \alpha_- = -\frac{1}{5 - p} (l + 1), \\ \alpha_+ = \frac{1}{5 - p} (l + 5 - p) \end{cases}$$

(7.81)

In this case

$$\tilde{\alpha}_a = \alpha_+ - \alpha_- = \frac{2l + 1}{5 - p} + 1 .$$

(7.82)

Proceeding in a similar fashion to the analysis in the previous cases, it is easy to find that the two-point function of the operator dual to these fluctuations has the following form

$$\langle \hat{O}_a \hat{O}_a \rangle \sim \frac{[g^2_{\text{eff}}(t)]^{\frac{1}{5 - p}}}{|t|^{2\Delta_a}}, \quad \tilde{\alpha}_a \text{ non-integer}$$

$$\langle \hat{O}_a \hat{O}_a \rangle \sim \mathcal{R} \left( \frac{[g^2_{\text{eff}}(t)]^{\frac{1}{5 - p}}}{|t|^{2\Delta_a}} \right), \quad \tilde{\alpha}_a \text{ integer},$$

(7.83)

with $\Delta_a$ being given by

$$\Delta_a = \tilde{\alpha}_a + 1 - \frac{1}{5 - p} = \frac{2}{5 - p} l + 1$$

(7.84)

Notice that $\Delta_a = \Delta_\chi$ and for $p = 3$ the exponent $\Delta_a$ coincides with the conformal dimension of the operator $\hat{O}_a$ in the $AdS_5 \times S^5$ background, which is $\Delta_a = l + 2$, as it should $[20,24,25]$.

8 Conclusion

In this paper we investigated the insertion of impurities in $(p+1)$-dimensional Supersymmetric Yang-Mills theories, which are (trivially) non-conformal. The ambient theory is holographically described by the near-horizon geometry generated by a stack of $N$ Dp-branes, while
the impurities are added by introducing probe D(8 − p)-branes in this background in such a way that the induced world-volume metric is conformally $AdS_2 \times S^{7-p}$. The background RR $(7-p)$-form potential induces an electric gauge-field on the world-volume of the probe branes, giving rise to a bundle of strings stretching in the radial direction and forming a flux tube.

We analysed in some detail two possible classes of configurations for such systems. In the first one, the flux tube is straight – which corresponds to a constant embedding function for the probe branes –, while in the second one two flux tubes get connected in the bulk. From the gauge theory point of view such configurations respectively correspond to single impurities and to dimers.

For these systems we studied the basic thermodynamic properties such as the free energy, entropy and specific heat. Interestingly, the impurity entropy turns out to be in general non-analytic in the filling fraction, except for the case of the D4/D4 systems for which we had been able to find a closed form. This latter system is actually the only one showing a positive specific heat: for $p = 3$, the specific heat is zero, while for $p < 3$ it is negative. This is a signature of a thermodynamic instability. We also computed the impurity susceptibility, which is constant for $p = 4$.

We then analysed the case of the hanging flux tubes, for which the position of the probe branes is fixed in the transverse direction, while their embedding in the conformal-$AdS$ manifold is controlled by a scalar. At finite temperature, there are two possible configurations: in one the two flux tubes are connected in the bulk and lie outside the black hole, while in the second one the two flux tubes end into the black hole. The transition from the second configuration to the first one is of first order and it corresponds to the dimerisation transition. We analysed the thermodynamics of these connected configurations. All the thermodynamic functions (free energy, entropy, internal energy, specific heat and latent heat) can be written in terms of hypergeometric functions. We studied the competition between the two configurations and determined the temperatures at which the dimerisation occurs. This phase transition turns out to be of first order. Furthermore, the specific heat for the dimer configuration is always positive for any $p < 3$ and vanishes as $T \to 0$. As mentioned before, the specific heat for the straight flux-tube configurations is negative for $p < 3$. One possible interpretation of this result can be that such a configuration is just not allowed for $p < 3$ and that the flux-tubes are forced to reconnect and form a dimer.

We further studied the stability of the systems at zero temperature by analysing the fluctuations of the probe branes in the straight configuration (impurity fluctuations). The fluctuations decouple into two channels: one contains just the fluctuations of the Cartesian coordinates, the other channel instead contains the coupled fluctuations of the angular em-
bedding function and of the world-volume gauge field. The modes in the second channel can be conveniently decoupled. Interestingly, all these modes satisfy the equations of motion of free massive scalars in a higher-dimensional $AdS$-space, where the enhancement of dimensions is due to the presence of a dilaton with a non-trivial profile. These modes are dual to irrelevant operators. In order to make sense of the correlators of such operators, one needs to holographic renormalise the action for these modes perturbatively up to order $n$, if one needs to compute the $n$-point correlators. For the aim of characterising the theory, it may be enough just to compute the one- and two-point correlators. Therefore we needed to renormalise the quadratic action which, as we have mentioned, is just the action for a free massive scalar in a higher-dimensional $AdS$-space. Then, we computed the correlators and wrote them in such a way that the underlying generalised conformal symmetry is manifest. From such an expression we can extract the generalised scaling dimension for all the operators we are interested in. Remarkably, such generalised scaling dimensions are fractional for $p < 3$ since the term which depends on the KK quantum number $l$ gets multiplied by the factor $2/(5 - p)$, which is just $1$ for $p = 3$. It would be interesting to understand this scaling from a purely field theoretical point of view. From the discussion about the generalised conformal structure, the most naive expectation is that the impurity action has a factor with a suitable power of the dimensionful coupling constant.

It would be interesting to repeat this fluctuation analysis for the case of the dimer configuration. One would expect that some of the dimer fluctuations satisfy the Heun equation, as it happens in the ABJM case [26], which is connected to integrable models. If on one side the emergence of this integrable structure could be understandable for the case $p = 3$, it might be a bit counter-intuitive in the general $p < 5$ system. However, as we discussed, the modes for $p \neq 3$ can be generally seen as modes propagating in a higher-dimensional $AdS$, so that the features of these systems may be deduced from conformal systems.

As we mentioned in this paper, while it has been showed [47] that the introduction of D5-branes in a D3-brane background such that the world-volume induced metric is $AdS_2 \times S^4$ is dual to a 1/2-BPS Wilson loop operator in $\mathcal{N} = 4$ SYM in the anti-symmetric representation of the gauge group $U(N)$, it is not yet clear whether a similar interpretation holds in the case of D$(8 - p)$-branes in the $Dp$-brane background with the same type of embedding. Therefore, a natural thing to do would be to clarify this issue. In this spirit, one can also study fermionic excitations in the $Dp/D(8 - p)$-system as done in [25] for the conformal case $p = 3$.

The analysis we carried out in this paper can be generalised to the case in which the probe D$(8 - p)$-branes wrap different submanifold of $S^{(8-p)}$, as in [52]. More interestingly, one can think to introduce chemical potential to the system by considering Reissner-Nordström type of backgrounds and analyse the configuration allowed and the phase structure. Importantly,
one could compute the resistivity as a function of the temperature and to study whether the holographic impurities induce a minimum similar to the one that occurs in the condensed matter models reviewed in section 2.

A final – and longer term – direction concerns the effect of the backreaction of the D(8−p)-branes. Presumably, this would require dealing with a bubbling geometry, generalizing the one found in [24] for the conformal p = 3 case.

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A Polar angles for stable embeddings

In this appendix we analyze the main properties of the polar angles of the flux tube configurations. We begin by listing the different functions \( \Lambda_{p,n}(\theta) \) for \( 0 \leq p \leq 5 \), namely

\[
\begin{align*}
\Lambda_{0,n}(\theta) &= -\frac{2}{5} \left[ \cos \theta \left( 3 \sin^4 \theta + 4 \sin^2 \theta + 8 \right) + 8 \left( \frac{2n}{N} - 1 \right) \right], \\
\Lambda_{1,n}(\theta) &= -\frac{5}{4} \left[ \cos \theta \left( \sin^3 \theta + \frac{3}{2} \sin \theta \right) + \frac{3}{2} \left( \frac{n}{N} \pi - \theta \right) \right], \\
\Lambda_{2,n}(\theta) &= -\frac{4}{3} \left[ \cos \theta \left( \sin^2 \theta + 2 \right) + 2 \left( \frac{2n}{N} - 1 \right) \right], \\
\Lambda_{3,n}(\theta) &= -\frac{3}{2} \left[ \cos \theta \sin \theta + \frac{n}{N} \pi - \theta \right], \\
\Lambda_{4,n}(\theta) &= -2 \left[ \cos \theta + \frac{n}{N} - 1 \right], \\
\Lambda_{5,n}(\theta) &= \theta - \frac{n}{N} \pi.
\end{align*}
\]  

(A.1)

As mentioned in the main text the functions \( \Lambda_{p,n}(\theta) \) depend on the quantization number \( n \) and on the number of colors \( N \) through their ratio \( \nu = n/N \) (the filling fraction). Moreover, one can check explicitly from (A.1) that these functions satisfy

\[
\Lambda_{p,n}(\theta) = -\Lambda_{p,N-n}(\pi - \theta),
\]

(A.2)
from which (4.5) follows immediately. As shown in [20], the $\Lambda_{p,n}(\theta)$ are monotonically increasing functions of $\theta$ in the interval $0 < \theta < \pi$. Moreover, one can check that:

$$\Lambda_{p,n}(0) = -2\sqrt{\pi} \frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)} \frac{n}{N}, \quad \Lambda_{p,n}(\pi) = 2\sqrt{\pi} \frac{\Gamma\left(\frac{8-p}{2}\right)}{\Gamma\left(\frac{7-p}{2}\right)} \left(1 - \frac{n}{N}\right). \quad (A.3)$$

and, therefore, $\Lambda_{p,n}(0) < 0$ if $n > 0$ and $\Lambda_{p,n}(\pi) > 0$ if $n < N$. Thus, it follows that there exists only one solution $\tilde{\theta}(p,n) \in (0, \pi)$ of the equation $\Lambda_{p,n}(\tilde{\theta}(p,n)) = 0$ for each $n$ in the interval $0 < n < N$, i.e. there are exactly $N - 1$ angles which correspond to non-singular wrappings of the D$(8-p)$-brane probe on the $S^{7-p}$ sphere. Let us work out the expressions of some of them for different values of $p$. In the conformal case $p = 3$, the angles $\tilde{\theta}(3,n)$ are the solutions of the equation:

$$\tilde{\theta}(3,n) - \cos \tilde{\theta}(3,n) \sin \tilde{\theta}(3,n) = \frac{n}{N} \pi. \quad (A.4)$$

For $p = 4$ the angles $\tilde{\theta}(4,n)$ can be immediately obtained from (A.1), with the result

$$\cos \tilde{\theta}(4,n) = 1 - 2 \frac{n}{N}. \quad (A.5)$$

The analytic expression of the angles $\tilde{\theta}(p,n)$ for $p \leq 2$ is more difficult to obtain. Let us analyse in detail the equation which determines the angles for $p = 2$, namely the solutions of the equation $\Lambda_{2,n}(\theta) = 0$. In terms of the filling fraction $\nu$, after using the third expression in (A.1), this equation becomes

$$\cos^3 \tilde{\theta}(2,n) - 3 \cos \tilde{\theta}(2,n) = 4\nu - 2, \quad (A.6)$$

which is a cubic equation in $\cos \tilde{\theta}(2,n)$. In general, an equation of the type

$$x^3 + rx = s, \quad (A.7)$$

can be solved for $x$ by means of the so-called Vieta’s substitution, namely:

$$x = w - \frac{r}{3w}. \quad (A.8)$$

Indeed, by substituting (A.8)) into the cubic equation (A.7), one obtains the following quadratic equation for $w^3$:

$$(w^3)^2 - s(w^3) - \frac{r^3}{27} = 0. \quad (A.9)$$

In our case $x = \cos \tilde{\theta}(2,n)$, with $r = -3$ and $s = 4\nu - 2$. The two solutions for $w^3$ are just

$$w^3 = 2\nu - 1 \pm 2i \sqrt{\nu(1-\nu)}. \quad (A.10)$$
Remarkably, the right-hand side of (A.10) is a complex number of modulus one and, therefore, can be represented as

\[ w^3 = e^{i\alpha} \implies w = e^{i\frac{\alpha}{3}}, \quad (A.11) \]

where \( \alpha \) is an angle such that

\[ \cos \alpha = 2\nu - 1, \quad \sin \alpha = \pm 2\sqrt{\nu(1 - \nu)}. \quad (A.12) \]

Notice that the relation between \( x = \cos \theta \) and \( w \) is, in our case, given by (A.8) with \( r = -3 \), namely

\[ x = w + w^{-1} = \cos \bar{\theta}_{(2,n)}. \quad (A.13) \]

It follows that \( \bar{\theta}_{(2,n)} \) is given by

\[ \cos \bar{\theta}_{(2,n)} = 2\cos \left( \frac{\alpha}{3} \right), \quad \cos \alpha = 2\nu - 1 = \frac{2n - N}{N}. \quad (A.14) \]

As a further example, let us point out that the angles for \( p = 1 \) can be obtained by solving the following transcendental equation

\[ \bar{\theta}_{(1,n)} - \frac{2}{3} \sin(2\bar{\theta}_{(1,n)}) \left[ 1 - \frac{1}{4} \sin(2\bar{\theta}_{(2,n)}) \right] = \frac{n}{N} \pi. \quad (A.15) \]

### B Dimer integrals

The purpose of this appendix is to derive the integrals needed in the calculations of section 5 of the thermodynamic properties of the dimer configurations. First of all, we define the following two integrals \( I_1(\alpha, \gamma) \) and \( I_2(\alpha, \gamma) \) as

\[ I_1(\alpha, \gamma) \overset{\text{def}}{=} \int_1^{\infty} \frac{dz}{\sqrt{(z^\alpha - 1)(z^\alpha - \gamma)}}, \]

\[ I_2(\alpha, \gamma) \overset{\text{def}}{=} \int_1^{\infty} \frac{z^\alpha dz}{\sqrt{(z^\alpha - 1)(z^\alpha - \gamma)^3}}. \quad (B.1) \]

where \( \alpha \) and \( \gamma \) are real numbers such that \( \alpha > 2 \) and \( |\gamma| < 1 \). These integrals can be performed in terms of hypergeometric functions, namely:

\[ I_1(\alpha, \gamma) = \frac{1}{\alpha} B \left( 1 - \frac{1}{\alpha}, \frac{1}{2} \right) F \left( \frac{1}{2}, 1 - \frac{1}{\alpha}; \frac{3}{2} - \frac{1}{\alpha}; \gamma \right), \]

\[ I_2(\alpha, \gamma) = \frac{1}{\alpha} B \left( 1 - \frac{1}{\alpha}, \frac{1}{2} \right) F \left( \frac{3}{2}, 1 - \frac{1}{\alpha}; \frac{3}{2} - \frac{1}{\alpha}; \gamma \right), \quad (B.2) \]

where \( B(x, y) \) is the Euler Gamma function: \( B(x, y) = \Gamma(x) \Gamma(y)/\Gamma(x + y) \). Let us next consider the integral:

\[ J(\alpha, \gamma) \overset{\text{def}}{=} \lim_{R \to \infty} \left[ \int_1^R \sqrt{\frac{z^\alpha - \gamma}{z^\alpha - 1}} dz - R \right]. \quad (B.3) \]
To evaluate this integral we shall proceed as follows. First of all, let us rewrite the integrand in (B.3) as
\[
\frac{\sqrt{z^\alpha - \gamma}}{z^\alpha - 1} = \frac{d}{dz} \left[ z \sqrt{z^\alpha - 1} \right] - \frac{\alpha(1 - \gamma)}{2} \frac{z^\alpha}{\sqrt{(z^\alpha - 1)(z^\alpha - \gamma)^3}} - \frac{1 - \gamma}{\sqrt{(z^\alpha - 1)(z^\alpha - \gamma)}}
\]  (B.4)

Using this result it is straightforward to relate \( J(\alpha, \gamma) \) to the following combination of the integrals \( I_1(\alpha, \gamma) \) and \( I_2(\alpha, \gamma) \) defined above
\[
J(\alpha, \gamma) = (1 - \gamma) \left[ I_1(\alpha, \gamma) - \frac{\alpha}{2} I_2(\alpha, \gamma) \right].
\]  (B.5)

Moreover, by using identities satisfied by the hypergeometric functions, one can show that (for \( \alpha > 2 \))
\[
J(\alpha, \gamma) = -\frac{\alpha - 2}{2\alpha} B \left( 1 - \frac{1}{\alpha}, \frac{1}{2} \right) F \left( -\frac{1}{2}, -\frac{1}{\alpha}; \frac{1}{2} - \frac{1}{\alpha}; \gamma \right).
\]  (B.6)

References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231–252, arXiv:hep-th/9711200.

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B428 (1998) 105–114, arXiv:hep-th/9802109.

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253–291, arXiv:hep-th/9802150.

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323 (2000) 183–386, arXiv:hep-th/9905111.

[5] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, “Supergravity and the large N limit of theories with sixteen supercharges,” Phys. Rev. D58 (1998) 046004, arXiv:hep-th/9802042.

[6] M. J. Duff, G. W. Gibbons, and P. K. Townsend, “Macroscopic superstrings as interpolating solitons,” Phys. Lett. B332 (1994) 321–328, arXiv:hep-th/9405124.

[7] H. J. Boonstra, B. Peeters, and K. Skenderis, “Duality and asymptotic geometries,” Phys. Lett. B411 (1997) 59–67, arXiv:hep-th/9706192.

\[6\] We are grateful to Wolfgang Mück for correspondence about this method to compute the integral B.3.
[8] H. J. Boonstra, B. Peeters, and K. Skenderis, “Brane intersections, anti-de Sitter spacetimes and dual superconformal theories,” *Nucl. Phys.* **B533** (1998) 127–162, arXiv:hep-th/9803231.

[9] H. J. Boonstra, K. Skenderis, and P. K. Townsend, “The domain wall/QFT correspondence,” *JHEP* **01** (1999) 003, arXiv:hep-th/9807137.

[10] A. Jevicki and T. Yoneya, “Space-time uncertainty principle and conformal symmetry in D particle dynamics,” *Nucl.Phys.* **B535** (1998) 335–348, arXiv:hep-th/9805069 [hep-th].

[11] A. Jevicki, Y. Kazama, and T. Yoneya, “Quantum metamorphosis of conformal transformation in D3-brane Yang-Mills theory,” *Phys.Rev.Lett.* **81** (1998) 5072–5075, arXiv:hep-th/9808039 [hep-th].

[12] A. Jevicki, Y. Kazama, and T. Yoneya, “Generalized conformal symmetry in D-brane matrix models,” *Phys. Rev.* **D59** (1999) 066001, arXiv:hep-th/9810146.

[13] K. Skenderis, “Field theory limit of branes and gauged supergravities,” *Fortsch. Phys.* **48** (2000) 205–208, arXiv:hep-th/9903003.

[14] K. Skenderis, “Lecture notes on holographic renormalization,” *Class. Quant. Grav.* **19** (2002) 5849–5876, arXiv:hep-th/0209067.

[15] T. Wiseman and B. Withers, “Holographic renormalization for coincident Dp-branes,” *JHEP* **10** (2008) 037, arXiv:0807.0755 [hep-th].

[16] I. Kanitscheider, K. Skenderis, and M. Taylor, “Precision holography for non-conformal branes,” *JHEP* **09** (2008) 094, arXiv:0807.3324 [hep-th].

[17] I. Kanitscheider and K. Skenderis, “Universal hydrodynamics of non-conformal branes,” *JHEP* **0904** (2009) 062, arXiv:0901.1487 [hep-th].

[18] A. Karch and E. Katz, “Adding flavor to AdS/CFT,” *JHEP* **06** (2002) 043, arXiv:hep-th/0205236.

[19] J. Pawelczyk and S.-J. Rey, “Ramond-Ramond flux stabilization of D-branes,” *Phys. Lett.* **B493** (2000) 395–401, arXiv:hep-th/0007154.

[20] J. Camino, A. Paredes, and A. Ramallo, “Stable wrapped branes,” *JHEP* **0105** (2001) 011, arXiv:hep-th/0104082 [hep-th].
[21] S. Kachru, A. Karch, and S. Yaida, “Holographic Lattices, Dimers, and Glasses,” Phys. Rev. D81 (2010) 026007, arXiv:0909.2639 [hep-th].

[22] S. Kachru, A. Karch, and S. Yaida, “Adventures in Holographic Dimer Models,” New J. Phys. 13 (2011) 035004, arXiv:1009.3268 [hep-th].

[23] W. Mück, “The Polyakov Loop of Anti-symmetric Representations as a Quantum Impurity Model,” Phys. Rev. D83 (2011) 066006, arXiv:1012.1973 [hep-th].

[24] S. Harrison, S. Kachru, and G. Torroba, “A maximally supersymmetric Kondo model,” arXiv:1110.5325 [hep-th].

[25] A. Faraggi, W. Mück, and L. A. Pando Zayas, “One-loop Effective Action of the Holographic Antisymmetric Wilson Loop,” arXiv:1112.5028 [hep-th].

[26] P. Benincasa and A. V. Ramallo, “Fermionic Impurities in Chern-Simons-Matter Theories,” JHEP 1202 (2012) 076, arXiv:1112.4669 [hep-th].

[27] P. Benincasa, “A Note on Holographic Renormalization of Probe D-Branes,” arXiv:0903.4356 [hep-th].

[28] A. Karch, A. O’Bannon, and K. Skenderis, “Holographic renormalization of probe D-branes in AdS/CFT,” JHEP 04 (2006) 015, arXiv:hep-th/0512125.

[29] B. C. van Rees, “Holographic renormalization for irrelevant operators and multi-trace counterterms,” JHEP 1108 (2011) 093, arXiv:1102.2239 [hep-th].

[30] J. Kondo, “Resistance minimum in dilute magnetic alloys,” Prog. Theor. Phys. 32 (1964) 37.

[31] I. Affleck, “Conformal field theory approach to the kondo effect,” Acta Phys. Polon. B26 (1995) 1869, arXiv:cond-mat/9512099.

[32] A. C. Hewson, “The Kondo Model to Heavy Fermions,” Cambridge University Press (1993).

[33] I. Affleck, “Quantum Impurity Problems in Condensed Matter Physics,” arXiv:0809.3474 [cond-mat].

[34] P. W. Anderson, “ A poor man’s derivation of scaling laws for the Kondo problem,” J. Phys. C3 (1970) 2346.

[35] K. G. Wilson, “ The renormalization group: Critical phenomena and the Kondo problem,” Rev. Mod. Phys. 47 (1975) 773.
[36] P. Nozières, Proc. of 14th Int. Conf. on Low Temp. Phys. 5 (1975) 339.

[37] P. Nozières and A. Blandin, “Kondo Effect in Real Materials,” J. Phys. (Paris) 41 (1980) 193.

[38] D. Cox and A. Zawadowski, “Exotic Kondo Effects in Metals: Magnetic Ions in a Crystalline Electric Field and Tunneling Centers,” Adv. Phys. 47 (1998) 599, arXiv:cond-mat/9704103.

[39] O. Parcollet, A. Georges, G. Kotliar, and A. Sengupta, “Overscreened Multichannel SU(N) Kondo Model: Large-N Solution and Conformal Field Theory,” Phys. Rev. B 58 (1998) 3794–3813, arXiv:cond-mat/9711192.

[40] S.-i. Tomonaga, “Remarks on bloch’s method of sound waves applied to many-fermion problems,” Progress of Theoretical Physics 5 (1950) no. 4, 544–569. http://ptp.ipap.jp/link?PTP/5/544/.

[41] J. Luttinger, “An exactly soluble model of a many-fermion system,” J. Math. Phys. 4 (1963) 1154.

[42] D. C. Mattis and E. H. Lieb, “Exact Solution of a Many-Fermion System and Its Associated Boson Field,” J. Math. Phys. 6 (1965) 304–312.

[43] F. D. M. Haldane, “‘Luttinger Liquid Theory’ of One-Dimensional Quantum Liquid: I. Properties of the Luttinger Model and Their Extension to the General 1D Interacting Spinless Fermi Gas,” J. Phys. C 14 (1981) 2585–2609.

[44] H. J. Schulz, G. Cuniberti, and P. Pieri, “Fermi Liquids and Luttinger Liquids,” G. Morandi et al eds, Springer (2000), arXiv:cond-mat/9807366.

[45] T. Giamarchi, “Quantum physics in one-dimensions,” Oxford University Press (2003).

[46] P. Frojdh and H. Johanneson, “Kondo effect in a luttinger liquid: Exact results from conformal field theory,” Phys. Rev. Lett. 75 (1995) 300, arXiv:cond-mat/9505100.

[47] J. Gomis and F. Passerini, “Holographic Wilson loops,” JHEP 08 (2006) 074, arXiv:hep-th/0604007.

[48] S. Yamaguchi, “Wilson loops of anti-symmetric representation and D5- branes,” JHEP 05 (2006) 037, arXiv:hep-th/0603208.

[49] B. C. van Rees, “Irrelevant deformations and the holographic Callan-Symanzik equation,” JHEP 1110 (2011) 067, arXiv:1105.5396 [hep-th].
[50] A. Petkou and K. Skenderis, “A Nonrenormalization theorem for conformal anomalies,” *Nucl. Phys. B* **561** (1999) 100–116, [arXiv:hep-th/9906030 [hep-th]].

[51] M. Kruczenski, D. Mateos, R. C. Myers, and D. J. Winters, “Meson spectroscopy in AdS / CFT with flavor,” *JHEP* **0307** (2003) 049, [arXiv:hep-th/0304032 [hep-th]].

[52] N. Karaískos, K. Sfetsos, and E. Tsatis, “Brane embeddings in sphere submanifolds,” *Class. Quant. Grav.* **29** (2012) 025011, [arXiv:1106.1200 [hep-th]].

[53] W. Stein *et al.*, *Sage Mathematics Software (Version 4.6.1)*. The Sage Development Team, 2011. [http://www.sagemath.org](http://www.sagemath.org).

[54] Maxima, “Maxima, a computer algebra system. version 5.25.1,” 2011. [http://maxima.sourceforge.net/](http://maxima.sourceforge.net/).

[55] E. Jones, T. Oliphant, P. Peterson, *et al.*, “SciPy: Open source scientific tools for Python,” 2001–. [http://www.scipy.org/](http://www.scipy.org/).