Yet another proof of Hawking and Ellis’s Lemma 8.5.5

S Krasnikov
Central Astronomical Observatory at Pulkovo, St Petersburg, 196140, Russia
E-mail: S.V.Krasnikov@mail.ru

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Abstract
The fact that the null generators of a future Cauchy horizon are past-complete was first proved by Hawking and Ellis (1973 *The Large Scale Structure of Spacetime* (Cambridge: Cambridge University Press)). Then, Budzyński, Kondracki and Królak outlined a proof free from the error found in the original one (2000 New properties of Cauchy and event horizons arXiv:gr-qc/0011033). Now, Minguzzi has published his version of the proof (2014 *J. Math. Phys.* 55 082503), patching a previously unnoticed hole in the preceding two. I am not aware of any flaws in that last proof, but it is quite difficult. In this note, I present a simpler one.

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1. Introduction

Let $H^+$ denote a future Cauchy horizon. A lemma by Hawking and Ellis says

**Lemma 8.5.5 of [1].** If $H^+(Q)$ is compact for a partial Cauchy surface $Q$, then the null geodesic generating segments of $H^+(Q)$ are geodesically complete in the past direction.

The lemma itself has never been doubted (to my knowledge), but the proof offered in [1] was found to be flawed; see [2] and references therein. To improve this situation—which is important because the lemma is a popular tool in mathematical relativity—Minguzzi recently published a new, more accurate proof of the lemma (or, to be precise, of some strengthening of it).

In this note, I present yet another proof of the same fact. The reason for doing this is that my version is much simpler (partly because its major part is replaced by a reference to a lemma proved elsewhere).
2. The proposition and its proof

In a spacetime \( M \), consider a past inextendible null curve \( \gamma \) totally imprisoned in a compact set \( \mathcal{K} \). Then pick a smooth, unit timelike future-directed vector field \( \tau \) on \( M \) and define (uniquely up to an additive constant) the ‘arc length parameter’ \( l \) on \( \gamma \) by the requirement

\[
g(\partial_l, \tau) = -1.
\]  

(1)

In addition to \( l \), define on \( \gamma \) an affine parameter \( s \) so that \( \partial_s \) is future-directed and \( s = 0 \) at \( l = 0 \). Then, \( \gamma \) is characterized by the (evidently negative) function

\[
h \equiv g(\partial_s, \tau),
\]

which relates \( l \) to \( s \):

\[
h = -\frac{dl}{ds}, \quad s(l) = \int_{l=0}^{l=0} \frac{dl}{h(l)}.
\]

(2)

As is proven in [3]

\[
h'/h \quad \text{is bounded on} \quad \gamma.
\]

(3)

Further, a few minor changes—a past inextendible \( \gamma(l) \) with \( l \in (-\infty, 0] \) instead of the future inextendible \( \gamma(l) \) with \( l \in [0, \infty) \) and an arbitrary compact \( \mathcal{K} \) instead of some specific \( \mathcal{E} \)—leave [3, lemma 8] valid while bringing it to the following form.

**Lemma 8 of [4].** Assume \( h(l) \) is such that there exists a smooth function \( f(l) \) defined at non-positive \( l \) and obeying for some non-negative constants \( c_1, \; f \), \( \overline{f} \) the inequalities

\[
f \leq f \leq \overline{f}, \quad |f'/f| < \infty
\]

and

\[
h'/h < -f'/f - cf, \quad \forall l \leq 0.
\]

(4)

Then there is a timelike past inextendible curve \( \gamma_{\infty} \) that is obtained by moving each point of \( \gamma \) to the past along the integral curves of \( \tau \) and which is totally imprisoned in a compact set \( \mathcal{O} \).

**Proposition.** If \( \mathcal{K} \) is a subset of the boundary of a globally hyperbolic past set \( M^{\text{in}} \), then \( \gamma \) is past-complete.

**Proof.** Suppose the lemma is false and \( \gamma \) is past-incomplete. This would mean that the affine parameter \( s \) is bounded from below and, correspondingly, the integral (2) converges at \( l \to -\infty \). This allows one to define the following smooth positive function on \( (-\infty, 0] \)

\[
f(l) \equiv \frac{1}{h} \left[ -\int_{l}^{0} \frac{dl}{h(l)} + 2 \int_{-\infty}^{0} \frac{dl}{h(l)} \right].
\]

(5)

\( f \) so defined satisfies the equation

\[
f'/f + h'/h = -f
\]

(6)

and consequently, condition (4) holds.
As $h$ is negative, the boundedness of the integral (2) provides a simple estimate
\[ \int_{-\infty}^{\infty} -\frac{1}{h(l)} \frac{dI}{I} > -\frac{1}{h(l)} \frac{1}{(h''/h)_{\text{max}}}, \quad \forall l \in (-\infty, 0], \]
which implies, due to (3), that $1/h$ is bounded. It follows then from (5) that $f$ is bounded too. Finally, the just proven boundedness of $f$ combined with (6) and (3) implies the boundedness of $f'/f$. Thus all the conditions of lemma 2 are fulfilled and the corresponding variation transforms $\gamma$ into a past inextendible timelike curve $\gamma_{\kappa_0}$. The latter, being timelike, lies entirely in the closed (due to the global hyperbolicity of $M^{in}$, to which $\gamma(0)$ belongs) set $J^-(\gamma_{\kappa_0}(0)) \subset M^{in}$ (the inclusion follows from the fact that $M^{in}$ is a past set). Thus, $\gamma_{\kappa_0}$ is totally imprisoned in the compact subset $\mathcal{O} \cap J^-(\gamma_{\kappa_0})$ of the globally hyperbolic spacetime $M^{in}$, which is forbidden by [1, proposition 6.4.7].

References

[1] Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Spacetime* (Cambridge: Cambridge University Press)
[2] Minguzzi E 2014 *J. Math. Phys.* 55 082503
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