EXPONENTIAL CONVERGENCE TO A STEADY-STATE FOR A
POPULATION GENETICS MODEL WITH SEXUAL REPRODUCTION AND
SELECTION

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Abstract. We are interested in the dynamics of a population structured by a phenotypic trait. Individuals reproduce sexually, which is represented by a non-linear integral operator. This operator is combined to a multiplicative operator representing selection. When the strength of selection is small, we show that the dynamics of the population is governed by a simple macroscopic differential equation, and that solutions converge exponentially to steady-states that are locally unique. The analysis is based on Wasserstein distance inequalities using a uniform lower bound on distributions. These inequalities are coupled to tail estimates to show the stability of the steady-states.

1. Introduction

In this paper, we are interested in the dynamics of following kinetic equation that describes a population that is structured by a phenotypic trait and reproduces sexually:

\[
\partial_t n(t, x) = \int \int \Gamma_{\sigma^2} \left( x - \frac{y_s + y_{s'}}{2} \right) (1 + \alpha a(y_s)) n(t, y_s)(1 + \alpha a(y_{s'})) n(t, y_{s'}) dy_s dy_{s'},
\]

\[- \left( 1 + \alpha \int a(x) n(t, x) dx \right)^2 n(t, x).
\]

In this model, \( t \in [0, \infty) \) is the time variable, \( x \in \mathbb{R} \) is a phenotypic trait. We assume that \( \int n(t, x) dx = 1 \) for \( t \geq 0 \), so that \( n(t, \cdot) \) is the density of individuals along the phenotypic trait \( x \). The function \( a = a(x) \) represents selection: the trait \( x \) of an individual affects the number of reproductions it takes part in, and this is modulated by a parameter \( \alpha > 0 \) (the strength of selection) that we will assume small in this study. Throughout this manuscript, \( \Gamma_{\sigma^2} \) is a Gaussian distribution:

\[
\Gamma_{\sigma^2}(x) := \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}.
\]

The first term on the right hand side of (1) is the birth term: each individual of trait \( y \) contributes to the production of gametes at a rate \((1 + \alpha a(y))\). Two parents, with traits \( y_s \) and \( y_{s'} \), are necessary for a reproduction to occur, and the offspring’s trait is then drawn from a Gaussian distribution centred in \( \frac{y_s + y_{s'}}{2} \), following Fisher’s Infinitesimal Model [Fisher, 1919, Barton et al., 2017]. The last term of (1) is a death term, with a death rate that is independent of \( x \), and that keeps the population size constant. For a discussion of the biological aspects of this model, we refer to Section 1.3.

Selection models for asexual populations have been thoroughly studied in the last decades (see e.g. [Diekmann et al., 2005, Desvillettes et al., 2008]). A fruitful approach has been to use a Hopf-Cole transform to relate the asexual population models to constrained Hamilton-Jacobi equations when the mutation rate is small [Diekmann et al., 2005, Lorz et al., 2011]. In [Calvez et al., 2019], this idea was extended to a sexual selection model (very close to (1)), through \( U(t, x) := -\sigma \log n(t, x) \) and the asymptotics of \( \sigma > 0 \) is small. The authors are then able to show the local uniqueness...
of steady-states (the existence of which is proven in [Bourgeron et al., 2017]). The idea is then to take advantage of the contractive effect of the reproduction operator on derivatives of \(U\), that is \(\partial_i^2 U(t, x)\) for \(i \in \{1, 2, 3\}\). A combination of these derivatives is employed to build a functional spaces where the reproduction operator is contractive close to a steady-state of the equation. In this manuscript, we will try to describe the dynamics of (1), that is closely related to the model considered in [Calvez et al., 2019]. We will however use a different analysis approach, based on Wasserstein distances rather than regularity estimates.

The 2–Wasserstein distance is defined on \(\mathcal{P}_2(\mathbb{R})\), the set of probability measures on \(\mathbb{R}\) with a bounded second moment: for \(n, m \in \mathcal{P}_2(\mathbb{R})\),

\[
W_2(n, m) = \left( \inf_{\pi \in \Pi(n, m)} \int_{\mathbb{R}^2} |x - y|^2 \, d\pi(x, y) \right),
\]

where \(\Pi(n, m)\) is the set of measures on \(\mathbb{R}^2\) with marginals \(n\) and \(m\). Wasserstein distances have proven useful in probability and in analysis and we review below some of existing studies that are close to the analysis done in this manuscript.

Equation (1) is closely related to models that describe the alignment of either biological individuals (fish, birds, etc.) or physical rods [Bertin et al., 2006, Vicsek et al., 1995, Degond et al., 2014]. More broadly, it is related to inelastic Boltzmann equations [Villani, 2006]. In this context, the wasserstein distances have been used to describe the long time dynamics of solutions [Pulvirenti and Toscani, 2004, Bolley and Carrillo, 2007], following the seminal work of Tanaka [Tanaka, 1978]. Note that a related set of work uses Fourier-based distances that have related properties [Bobylev, 1976, Carrillo and Toscani, 2007].

Wasserstein distance methods have also been used to analyse Fokker-Planck equations with non-local interaction potentials. They have proven very useful to show the existence and stability of steady-states for these models. We refer to [Carrillo and Toscani, 2007] for dimension 1 and to [Malrieu et al., 2003, Carrillo et al., 2006] for the general case. An interesting development was proposed in [Carrillo et al., 2003, Carrillo et al., 2006], where the authors considered interaction potentials between particles that are (slightly) non-convex. The non-convexity of the interaction potentials is detrimental to the contraction of solutions towards the steady-state: the equation having a gradient-flow structure, the stability of steady-states is related to the convexity of an energy along some geodesics. It can however be shown that these detrimental effects can be balanced out by the contraction effects from other operators present in the equation, provided the non-convexity only occurs in parts of the domain where numerous particles are present. We refer to [Cattiaux et al., 2008, Bolley et al., 2012, Bolley et al., 2013] for the development of this idea into more quantitative results.

In this manuscript, we are interested in the stability of the steady-states of (1) when \(\alpha > 0\) is small. This hypothesis on \(\alpha\) is similar to the asymptotics considered in [Calvez et al., 2019] (but it is not exactly the same). We will however develop a different idea to describe the dynamics of solutions: we will rely on the Wasserstein contraction implied by the reproduction operator (see [13] and [Raoul, 2017]). The main difficulty is then to balance the detrimental effects of the selection term. We restrict our analysis to a situation where the selection function \(a(\cdot)\) is constant outside of a compact set, so that \(n(t, \cdot)\) can be bounded from below on that compact set. This allows us to use an argument inspired by the developments made on Fokker-Planck equations with non-convex interaction potentials and that have been discussed above.

In [Mirrahimi and Raoul, 2013], a model close to (1) with an additional spatial variable has been considered. Heuristic arguments and numerical simulations were used to describe the macroscopic quantities of that model and show that their dynamics can be approximated by the Kirkpatrick-Barton model [Kirkpatrick and Barton, 1997]. In [Raoul, 2017], this idea was made rigorous thanks to Wasserstein estimates combined to parabolic estimates. For the present study, this approach is not sufficient: we want to prove that the distribution \(n(t, \cdot)\) of the population converges to a unique
steady state. To do so, one needs to show that the contraction effect of the reproduction operator $T$ can dominate the influence of the multiplicative operator $n \mapsto \alpha a(\cdot) n$. We mention finally the on-going work [Poyato and Calvez] that could provide some interesting estimates on the effect of more general selection functions $a$, and could provide ideas to push the present asymptotic study further.

1.1. Main results and structure of the paper. Our goal is to prove the existence and uniqueness of a steady state of (1) in a neighbourhood of $\Gamma_{2\sigma^2}(-\bar{Z})$, provided $\alpha > 0$ is small enough. Furthermore, we show that this steady state is stable: under some conditions on the initial condition, solutions converge exponentially to that steady state. We may the following assumptions, on the coefficients of the model (Assumption 1) and the initial condition (Assumption 2):

**Assumption 1.** We assume that $a \in W^{2,\infty}(\mathbb{R}, \mathbb{R}_+)$ is compactly supported, and that for $\bar{Z} \in \mathbb{R}$, $$F(\bar{Z}) = 0, \quad F'(\bar{Z}) < 0,$$ where

$$F(Y) := \int x a(x) \Gamma_{2\sigma^2}(x - Y) \, dx - Y \int a(x) \Gamma_{2\sigma^2}(x - Y) \, dx.$$

**Assumption 2.** $n^0 \in \mathcal{P}_2(\mathbb{R}) \cap C^1(\mathbb{R})$, $R > 0$ and $\rho > 0$ satisfy

$$\forall x \in [R, +\infty), \quad \partial_x n^0(-x) > n^0(-x), \quad \partial_x n^0(x) < -n^0(x).$$

We assume that $W_2(n^0, \Gamma_{2\sigma^2}(- - Z^0)) \leq \rho$ and $n^0 \geq \frac{1}{R} \Gamma_{1R}$. Moreover, if $Z^0 := \int x n^0(x) \, dx$, we assume that $F(x) > 0$ on $[Z^0, \bar{Z}]$ if $Z^0 < \bar{Z}$ (resp. $F(x) < 0$ on $[\bar{Z}, Z^0]$ if $\bar{Z} < Z^0$).

We can now state our main result:

**Theorem 1.1.** Let $R > 0$ and $a \in W^{2,\infty}(\mathbb{R})$, $Z \in \mathbb{R}$ that satisfy Assumption 1.

There exists $\alpha > 0$, $\rho > 0$, $C > 0$ such that for $\alpha \in (0, \bar{\alpha})$, there exists a steady-state $\bar{n} \in \mathcal{P}_2(\mathbb{R})$ of (1) that satisfies $W_2(\bar{n}, \Gamma_{2\sigma^2}(- - \bar{Z})) \leq C \alpha$ and the following properties hold.

If $n^0 \in \mathcal{P}_2(\mathbb{R}) \cap C^1(\mathbb{R})$ satisfies Assumption 2 and $n \in L^\infty(\mathbb{R}_+, \mathcal{P}_2(\mathbb{R}))$ is the solution of (1) with initial data $n^0$, then for $t \in [0, \infty)$,

$$W_2(n(t, \cdot), \bar{n}) \leq C e^{(F'(\bar{Z}) + C \sqrt{\alpha}) t},$$

and

$$\left| \int x n(t, x) \, dx - Y(t) \right| \leq \frac{C}{-\ln \alpha} \quad W_2(n(t, \cdot), \Gamma_{2\sigma^2}(Y(t) - \cdot)) \leq \frac{C}{-\ln \alpha} + C e^{-t/8},$$

where $Y$ is the solution of

$$Y'(t) = F(Y(t)).$$

with initial data $Y(0) = \int x n^0(x) \, dx$.

The driving idea of this manuscript is contained in Section 3, inspired by [Bolley et al., 2012], we show that the fact that distributions are bounded away from 0 gives way to improved Wasserstein estimates on the effect of a multiplicative operator. Using explicit examples, we show in Section 3.4 that a lower bound on solutions is indeed necessary to obtain the estimates we introduce. The Wasserstein estimates requiring a lower bound on distributions can only be used on a compact set (since the distributions cannot be bounded away from 0 uniformly on $\mathbb{R}$). They should then be combined to tail estimates. This turned out to be a technical aspect of this study and to obtain these estimates, we had to assume that $a$ is compactly supported. Note that combining Wasserstein estimates to tail estimates is also present in the Fokker-Planck manuscripts that have inspired our approach (see [Bolley et al., 2012, Bolley et al., 2013]): the tail estimates are then based
on the convexity of potentials outside of a compact set. The analysis of the tails is however made
more complex here by the highly non-local effect of selection: the fact that the population size is
constant implies that each death corresponds to a birth, but these two events can occur for very
different traits $x \in \mathbb{R}$. These birth/death events can thus be seen as ”jumps” in the trait space, and
such events were not present in aforementioned works. The main difficulty raised by these events
concern the tails of the distributions, and we will need to develop technical estimates to control
them. In this manuscript, we only consider the one-dimensional case ($x \in \mathbb{R}$). One benefit is the
use of pseudo-inverse distributions (see (7)) which provide convenient tools to develop Wasserstein
estimates. It would be interesting to try to generalize these Wasserstein estimates to the case of a
multi-dimensional phenotypic trait, but new tail estimates would then also be necessary.

In Section 1.2 we introduce notations and basic properties, before discussing the biological
aspects of this study in Section 1.3, and illustrating the result with a few numerical simulations.
In Section 2, we show that the macroscopic quantity $Z(t) = \int x n(t, x) \, dx$ (the mean phenotypic
trait of the population) approximatively satisfies an ODE. We take advantage of this first result to
prove the existence of a steady-state and to obtain uniform tail estimates on $n(t, \cdot)$. In Section 3, we
develop Wasserstein distance estimates that will enable us to control the effects of the multiplicative
selection operator $n \mapsto \frac{(1 + \alpha a^n(y)) m(y) dy}{f(1 + \alpha a^n(y)) n(y) dy}$. We discuss the necessity of a lower bound assumption to
obtain these estimates in Section 3.4. Finally, in Section 4, we combine the results obtained so far
to show that the flow of (1) contracts solutions around the steady-state, concluding the proof of
Theorem 1.1.

1.2. Definitions and notations. Wasserstein distance and pseudo-inverse functions
Since we consider measures on $\mathbb{R}$, we can take advantage of the pseudo-inverse of the distributions
(see [Carrillo and Toscani, 2007] for a review on the one-dimensional Wasserstein distance): if
$u, v : [0, 1] \to \mathbb{R}$ are defined for $z \in [0, 1]$ by
\begin{equation}
    z = \int_{-\infty}^{u(z)} n(x) \, dx = \int_{-\infty}^{v(z)} m(x) \, dx,
\end{equation}
then $W_2(n, m)^2 = \int_0^1 |u(z) - v(z)|^2 \, dz$. If $n$ is continuous, the derivation of (7) provides the following
useful property:
\begin{equation}
    u'(z) = \frac{1}{n(u(z))}.
\end{equation}
In our analysis, we will use the following quantity to describe the distance between two distributions:
\begin{equation}
    w(n, m) = \min_{\zeta} W_2 (n, m(\cdot - \zeta)) .
\end{equation}
The idea behind this definition is to consider the $W_2$ distance parallelly to the macroscopic quantity
that is the center of mass of the distribution (in (15), we will see that the center of mass is an
invariant of reproduction operator). Using the pseudo-inverse of $n$ and $m$, the quantity $w(\cdot, \cdot)$ can be
written
\begin{equation}
    w(n, m)^2 = \min_{\zeta \in \mathbb{R}} \int_0^1 |u(z) - v(z) + \zeta|^2 \, dz.
\end{equation}
If we differentiate this expression, we get that $\zeta = \int_0^1 v(z) - u(z) \, dz$. This provides a more explicit
expression of $w(\cdot, \cdot)$:
\begin{equation}
    w(n, m) = W_2 (n (\cdot - Z_n), m (\cdot - Z_m)) ,
\end{equation}
where
\begin{equation}
    Z_n = \int x n(x) \, dx, \quad Z_m = \int x m(x) \, dx.
\end{equation}
Finally, we will use the following coupling inequality satisfied by \( W_2 \): for \( n, n_2, m, m_2 \in P_2(\mathbb{R}) \) and \( \theta \in (0, 1) \),
\[
W_2(\theta n_1 + (1 - \theta)n_2, \theta m_1 + (1 - \theta)m_2) \leq \theta W_2(n_1, m_1) + (1 - \theta)W_2(n_2, m_2).
\]

**Reproduction operator**

For \( n, m \in P_2(\mathbb{R}) \) and \( a \in W^{1, \infty}(\mathbb{R}) \), we define
\[
I_n = \int a(x)n(x) \, dx, \quad I_m = \int a(x)m(x) \, dx.
\]
If \( n, m \in P_2(\mathbb{R}) \) have the same center of mass, i.e., such that \( \int xn(x) \, dx = \int xm(x) \, dx \), then (see Raoul, 2017),
\[
W_2(T(n), T(m)) \leq \frac{1}{\sqrt{2}} W_2(n, m).
\]

We define the operator \( T(n, m) \), for \( n, m \in P_2(\mathbb{R}) \) as follows:
\[
T(n, m)(x) = \int \int \Gamma \left( x - \frac{y_s + y'_s}{2} \right) n(y_s)m(y'_s) \, dy_s \, dy'_s,
\]
and use the notation \( T(n) := T(n, n) \). Note that the center of mass is conserved by \( T \): if \( n \in P_2(\mathbb{R}) \), then
\[
\int xT(n)(x) \, dx = \int xn(x) \, dx.
\]

Another property of \( T \) that will be important for the analysis is that for any \( Z \in \mathbb{R} \),
\[
\Gamma_{2\sigma^2}(x - Z) = \int \int \Gamma_{\sigma^2} \left( x - \frac{y_s + y'_s}{2} \right) \Gamma_{2\sigma^2}(y_s - Z)\Gamma_{2\sigma^2}(y'_s - Z) \, dy_s \, dy'_s,
\]
that is \( T(\Gamma_{2\sigma^2}(\cdot - Z)) = \Gamma_{2\sigma^2}(\cdot - Z) \) (see Turelli and Barton, 1994, Raoul, 2017). We introduce the notation \( I_n(t) := \int a(x)n(t, x) \, dx \), for \( n \in L^{\infty}(\mathbb{R}_+, L^1(\mathbb{R})) \), and notice that the solutions of (1) can be written in the following integral form
\[
n(t, x) = n(0, x)e^{-\int_0^t (1 + \alpha I_n(s))^2 \, ds} + \int_0^t (1 + \alpha I_n(s))^2 T \left( \frac{(1 + \alpha a)n(s, \cdot)}{1 + \alpha I_n(s)} \right)(x)e^{-\int_s^t (1 + \alpha I_n(\tau))^2 \, d\tau} \, ds.
\]

**1.3. Biological setting and numerical simulations.** In natural populations, phenotypic traits can be measured on individuals of a given species: size of a bacteria, number of eggs laid by a geese, etc. From these observations, biologists have noticed that the phenotypic traits of offspring are often normally distributed around a mean. A theoretical framework has been developed for such populations by Fisher in 1919: the Infinitesimal Model (see Fisher, 1919, Turelli and Barton, 1994, Barton et al., 2017). This model considers a situation where the phenotype is obtained as a sum of the allelic effects of a large number of loci. Under a weak selection assumption (see Hartl et al., 1997, Bürger, 2000), that is if the phenotype has a limited impact on the survival of the species (this is similar to the assumption that \( \alpha > 0 \) is small in our study), the allelic values on different loci decorrelate. This decorrelation property is called Linkage Equilibrium and it leads to the Gaussian kernel appearing in (1). It also implies that the phenotypic traits of the population are normally distributed (this property is related to the relation (16)), and the variance of that Gaussian is the Variance at Linkage Equilibrium (denoted by \( V_{LE} \)). This quantity is related to our parameter \( \sigma^2 \) (\( \sigma^2 \) which is sometimes called segregation variance): \( V_{LE} = 2\sigma^2 \).

The fact that the phenotypic traits of a population are normally distributed is a central assumption in Population Genetics (see e.g. Haldane, 1990, Wright, 1935). Note that normal distribution
can also be observed in asexual populations for very different reasons (see Kimura, 1965), so that the assumption that a population is normally distributed is not directly linked to the infinitesimal model. Typically, a biological problem as depicted by \(1\) would be analysed as follows: provided the selection is weak (that is \(\alpha > 0\) is small), one may assume that the population is a Gaussian distribution around its mean, that is \(n(t,x) \sim \Gamma_{2\sigma^2}(x-Z(t))\). The dynamics of \(Z\) can then be approximated by

\[
Z'(t) = \frac{d}{dt} \int xn(t,x) \, dx = \alpha I_n(t) (1 + \alpha I_n(t)) \left( \int xa(x)n(t,x) \, dx - I_n(t) \int xn(t,x) \, dx \right)
\]

(18)

\[
\sim \alpha F(Z(t)),
\]

where \(F\) is defined by \((3)\). This argument is directly related to the so-called Fundamental Theorem of natural selection introduced by Fisher [Fisher, 1930]. It provides a convenient description of the evolutionary dynamics of populations. One outcome of this argument would be that the mean phenotypic trait of the population converges to a steady-state \(\bar{Z}\), and the population would then be normally distributed around \(\bar{Z}\) with a variance \(V_{LE} = 2\sigma^2\). An interesting aspect of our analysis is that it is connected on this approximation argument: The proof of our main result relies on Wasserstein distances is an convenient quantity to obtain these quantitative estimates.

The model \(1\) describes the dynamics of a constant size population: \(\int n(t,x) \, dx = 1\). This assumption is classical in Population genetics. Our analysis can however be used for models that do not satisfy this assumption. We may for instance consider a more ecological model, with a logistic regulation of the population size:

\[
\partial_t f(t,x) = \int \Gamma_{\sigma^2} \left( x - \frac{y_s + y'_s}{2} \right) \left( 1 + \alpha a(y_s) \right) f(t,y_s)(1 + \alpha a(y'_s)) \frac{f(t,y'_s)}{\int f(t,y) \, dy} \, dy_s \, dy'_s
\]

(19)

\[
- \left( \int K(y)f(t,y) \, dy \right) f(t,x).
\]

Then \(n(t,x) := \frac{f(t,x)}{\int f(t,y) \, dy} \) satisfies \((1)\), and the convergence of \(f\) to a steady-state \(\bar{f}\) can be deduced from Theorem \([1]\).

To illustrate the results of this manuscript, we represent numerical simulations of \((1)\) and \((6)\) in figure 1. For \((1)\), we use an explicit Euler scheme in time. To compute efficiently the birth term, we notice that it can be written as a double convolution (see [Turelli and Barton, 1994, Mirrahimi and Raoul, 2013]):

\[
T(n,m)(x) = 2\Gamma_{4\sigma^2} * n * m(2x).
\]

Thanks to this relation, it is possible to use a spectral method, leading to rapid simulations. We have chosen the coefficients \(\alpha = 1\) and \(a(x) := 2e^{-\frac{(x-5)^2}{4}} + e^{-\frac{(x+5)^2}{4}}\). In Figure 1a, we represent \(x \mapsto 1 + \alpha a(x)\), that characterizes the number of offspring produced by an individual of trait \(x\). In Figure 1b, we plot \(x \mapsto F(x)\) (see \((3)\)), that is the speed field for the ODE \((6)\) that corresponds to the macroscopic model \((18)\). We notice that two stable steady-states (\(\bar{Z}_1 \sim -5\) and \(\bar{Z}_2 \sim 5\)) and one unstable steady state (\(Z_u \sim 0\)) exist for this macroscopic model.

In Figure 1c, we represent the function \((t,x) \mapsto n(t,x)\) and the mean phenotypic trait \(Z(t) = \int xn(t,x) \, dx\) (continuous black line). The black dashed line is \(Y(t/\alpha)\), that is the approximation of \(Z(t)\) that is provided by \((6)\). As indicated by Theorem \([1.1]\), \(n(t,\cdot)\) appears close to a Gaussian distribution centred around \(Z(t)\).
Function \( x \mapsto 1 + \alpha a(x) \) used for the simulations. It is the fitness of trait \( x \), and we use the values \( \alpha = 1, a(x) := 2e^{-\frac{(x-5)^2}{4}} + e^{-\frac{(x+5)^2}{4}} \).

Function \( F \) as defined by (3).

The colors correspond to the density of the solution \((t, x) \mapsto n(t, x)\) of (1): the red color indicates a high density of individuals. The continuous black line represents the mean phenotypic trait \( t \mapsto Z(t) = \int xn(t, x) \, dx \), while the dashed line is \( t \mapsto Y(t/\alpha) \) defined by (6).

Each color corresponds to a different initial condition, with \( Z_0 = -10, -7.5, -5, -2.5, -0.25, 0, 0.25, 2.5, 5, 7.5, 10 \) respectively. For each initial condition, we represent \( Z(t) = \int x n(t, x) \, dx \) (continuous line) and \( t \mapsto Y(t/\alpha) \) (dashed line), where \( n \) is the solution of (1) and \( Y \) is the solution of (6).

**Figure 1.** Numerical simulations of (1) and comparison with the dynamics of (6).

In Figure 1d, we represent \( Z(t) = \int x n(t, x) \, dx \) (continuous lines) and \( Y(t/\alpha) \) (dashed line), for different initial conditions \( Z^0 = -10, -7.5, -5, -2.5, -0.25, 0, 0.25, 2.5, 5, 7.5, 10 \). We see that the functions \( Z(t) \), just as \( Y(t) \), converge to one of two steady-states that correspond to \( Z_1 \sim -5 \) and \( Z_2 \sim 5 \), the two stable steady-states of (6), that are also the two traits that satisfy Assumption 1. Notice also that for the initial values chosen here, \( Z(t) \) seems well approximated by \( Y(t/\alpha) \), in accordance with Theorem 1.1 (see also Lemma 2.1), even if we have chosen a selection strength \( \alpha = 1 \) that is not very small. The relevance of the approximation \( Y(t/\alpha) \), that corresponds to the approximation described in (18), pleads for the use of this macroscopic approximation. Theorem 1.1 shows that this efficiency of the macroscopic model extends to the asymptotic distributions: \( n(t, \cdot) \) converges to either of two steady-states close to \( \Gamma_{2\sigma^2}(\cdot - Z_1) \) and \( \Gamma_{2\sigma^2}(\cdot + Z_2) \) for all the initial conditions we have tested.
2. Preliminary results

2.1. Wasserstein distance to a Gaussian distribution. In this section, we show that the solutions of (1) are close to Gaussian distributions in $x$.

Lemma 2.1. Let $R > 0$, $\rho > 0$ and $a \in W^{2,\infty}(\mathbb{R})$, $\bar{Z} \in \mathbb{R}$ satisfying Assumption 1.

There exists $\bar{\alpha} > 0$, and $C > 0$ such that for $\alpha \in (0, \bar{\alpha})$ and $n^0 \in \mathcal{P}_2(\mathbb{R}) \cap C^1(\mathbb{R})$ satisfying Assumption 2, the solution $n \in L^\infty(\mathbb{R}_+, \mathcal{P}_2(\mathbb{R}))$ of (1) with initial data $n^0$ satisfies

$$\forall t \in [-C \ln \alpha/\alpha, \infty), \quad W_2(n(t, \cdot), \Gamma_{2\alpha^2}) \leq C\alpha,$$

and

$$\forall t \in [0, \infty), \quad |Z_n(t) - Y(t/\alpha)| \leq \frac{C}{-\ln \alpha},$$

where $Y$ is the solution of (6) with initial data $Z^0 = \int x n^0(x) \, dx$. Moreover,

$$\forall t \in [0, \infty), \quad \int x^2 n(t, x) \, dx \leq C.$$

Proof of Lemma 2.1. Step 1: We derive uniform estimates on moments of $n(t, \cdot)$

Let us first notice that Assumption 2 implies a bound on the second moment of $n^0$. Indeed, $n^0$ is decreasing on $[R, R + 1]$, thus $n(R + 1) \leq \int_R^{R+1} n \leq 1$. Since $\partial_x n^0 < -n^0$ on $[R + 1, \infty)$, $\int_{R+1, \infty} x^2 n^0(x) \, dx \leq \int_{R+1, \infty} x^2 n^0(R + 1)e^{(R+1)-x} \, dx \leq C$ for some constant $C > 0$. The same argument can be repeated for $x \leq -(R + 1)$, and proves the existence of an upper bound on $\int x^2 n^0(x) \, dx$ that depends on $R$ only. This bound also implies a bound on $Z^0$: $|Z^0| \leq \int \frac{1 + x^2}{2} n(x) \, dx \leq C$.

We consider now $Z_n(t) = \int x n(t, x) \, dx$ and notice it satisfies

$$Z'_n(t) = (1 + \alpha I_n(t)) \int y(1 + \alpha a(y)) n(t, y) \, dy - (1 + \alpha I_n(t))^2 Z_n(t)$$

$$\quad \quad = \alpha I_n(t)(1 + \alpha I_n(t)) \left[ \int y \frac{a(y)n(t, y)}{I_n(t)} \, dy - Z_n(t) \right].$$

Since $a(\cdot) \geq 0$, the quantity $\int y \frac{a(y)}{I_n(t)} n(t, y) \, dy$ belongs to the convex envelop of the support of $a$, and (25) then implies the uniform bound (23). If we multiply (1) by $x^2$ and integrate, we obtain

$$\frac{d}{dt} \int x^2 n(t, x) \, dx \leq (1 + \alpha I_n(t))^2 \sigma^2 + \alpha \frac{1 + \alpha I_n(t)}{2} \int x^2 a(x)n(t, x) \, dx$$

$$\quad + \frac{1}{2} \left( Z_n(t) + \alpha \int x a(x)n(t, x) \, dx \right)^2 - \frac{1 + 2\alpha I_n(t)}{2} (1 + \alpha I_n(t)) \int x^2 n(t, x) \, dx$$

$$\leq C - \left( \frac{1}{2} + O(\alpha) \right) \int x^2 n(t, x) \, dx,$$

which, combined to the upper bound already established on $\int x^2 n^0(x) \, dx$, implies (22).

Step 2: We show that $n(t, \cdot)$ is close to $\Gamma_{2\alpha^2}(-Z_n(t))$ when $\alpha > 0$ is small and $t \geq -\ln \alpha$. 

We use (17), which, thanks to (14), implies
\[
W_2(n(t, \cdot), \Gamma_{2\sigma^2}(-Z_n(t))) \leq e^{-\int_0^t (1 + \alpha I_n(s))^2 \, ds} W_2(n(0, \cdot), \Gamma_{2\sigma^2}(-Z_n(t))) \\
+ \int_0^t e^{-\int_0^t (1 + \alpha I_n(\tau))^2 \, d\tau} (1 + \alpha I_n(s))^2 W_2 \left( T \left( \frac{(1 + \alpha a)n(s, \cdot)}{1 + \alpha I_n(s)} \right), \Gamma_{2\sigma^2}(-Z_n(t)) \right) \, ds.
\]
Thanks to (16) and if \( \alpha \|a\|_\infty \leq 1 - \frac{31/4}{\sqrt{2}}, \)
\[
W_2(n(t, \cdot), \Gamma_{2\sigma^2}(-Z_n(t))) \leq e^{-\int_0^t (1 + \alpha I_n(s))^2 \, ds} W_2(n(0, \cdot), \Gamma_{2\sigma^2}(-Z_n(t))) \\
+ \int_0^t e^{-\int_0^t (1 + \alpha I_n(\tau))^2 \, d\tau} (1 + \alpha I_n(s))^2 W_2 \left( T \left( \frac{(1 + \alpha a)n(s, \cdot)}{1 + \alpha I_n(s)} \right), T \left( \Gamma_{2\sigma^2}(-Z_n(t)) \right) \right) \, ds \\
\leq W_2(n(0, \cdot), \Gamma_{2\sigma^2}(-Z_n(t))) e^{-\frac{31/4}{\sqrt{2}} t} \\
+ (1 + \alpha \|a\|_\infty)^2 \int_0^t e^{-\frac{31/4}{\sqrt{2}} (t-s)} \left[ W_2 \left( T \left( \frac{(1 + \alpha a)n(s, \cdot)}{1 + \alpha I_n(s)} \right), T \left( \Gamma_{2\sigma^2}(-\hat{Z}_n(s)) \right) \right) \\
+ W_2 \left( T \left( \Gamma_{2\sigma^2}(-\hat{Z}_n(s)) \right), T \left( \Gamma_{2\sigma^2}(-Z_n(t)) \right) \right) \right] ds \\
\leq W_2(n(0, \cdot), \Gamma_{2\sigma^2}(-Z_n(t))) e^{-\frac{31/4}{\sqrt{2}} t} \\
+ (1 + \alpha \|a\|_\infty)^2 \int_0^t e^{-\frac{31/4}{\sqrt{2}} (t-s)} \left[ \frac{1}{\sqrt{2}} W_2 \left( T \left( \frac{(1 + \alpha a)n(s, \cdot)}{1 + \alpha I_n(s)} \right), \Gamma_{2\sigma^2} \left( -\hat{Z}_n(s) \right) \right) + \left| Z_n(t) - \hat{Z}_n(s) \right| \right] ds,
\]
where we have used the notation \( \hat{Z}_n(s) := \int x^{(1 + \alpha a)n(s, \cdot)} \, dx \) and the \( W_2 \)-contraction implied by \( T \) on probability measures with the same center of mass (see (13)). This definition of \( \hat{Z}_n \) implies in particular that \( \hat{Z}_n(s) = Z_n(s) + \mathcal{O}(\alpha) \), and since \( \|Z_n\|_\infty = \mathcal{O}(\alpha) \) (see (25)),
\[
\hat{Z}_n(s) = Z_n(s) + \mathcal{O}(\alpha) = Z_n(t) + \mathcal{O}(\alpha)(1 + |t - s|).
\]
We may moreover use the following convex decomposition:
\[
\frac{(1 + \alpha a)n(s, \cdot)}{1 + \alpha I_n(s)} = \frac{1}{1 + \alpha I_n(s)} n(s, \cdot) + \frac{\alpha I_n(s)}{1 + \alpha I_n(s)} an(s, \cdot)
\]
to show that
\[
W_2(n(t, \cdot), \Gamma_{2\sigma^2}(-Z_n(t))) \leq W_2(n(0, \cdot), \Gamma_{2\sigma^2}(-Z_n(t))) e^{-\frac{31/4}{\sqrt{2}} t} \\
+ \frac{1 + \mathcal{O}(\alpha)}{\sqrt{2}} \int_0^t e^{-\frac{31/4}{\sqrt{2}} (t-s)} \left[ W_2(n(s, \cdot), \Gamma_{2\sigma^2}(-Z_n(s))) + \mathcal{O}(\alpha)(1 + |t - s|) \right] ds \\
\leq W_2(n(0, \cdot), \Gamma_{2\sigma^2}(-Z_n(t))) e^{-\frac{31/4}{\sqrt{2}} t} \\
+ \frac{1 + \mathcal{O}(\alpha)}{\sqrt{2}} \int_0^t e^{-\frac{31/4}{\sqrt{2}} (t-s)} W_2(n(s, \cdot), \Gamma_{2\sigma^2}(-Z_n(s))) ds + \mathcal{O}(\alpha).
\]
The quantity \( \phi(t) = e^{\frac{31/4}{\sqrt{2}} t} W_2(n(t, \cdot), \Gamma_{2\sigma^2}(-Z_n(t))) \) then satisfies
\[
\phi(t) \leq W_2(n(0, \cdot), \Gamma_{2\sigma^2}(-Z_n(t))) + \mathcal{O}(\alpha) e^{\frac{31/4}{\sqrt{2}} t} + \frac{1 + \mathcal{O}(\alpha)}{\sqrt{2}} \int_0^t \phi(s) ds.
\]
Thanks to a Grönwall’s inequality,
\[
\phi(t) \leq W_2(n(0, \cdot), \Gamma_{2\sigma^2}(\cdot - Z_n(t))) + O(\alpha)e^{\frac{\sqrt{\alpha}}{2}t} \\
+ \int_0^t \left( W_2(n(0, \cdot), \Gamma_{2\sigma^2}(\cdot - Z_n(t))) + O(\alpha)e^{\frac{\sqrt{\alpha}}{2}s} \right) \frac{1 + O(\alpha)}{\sqrt{2}} e^{\frac{1 + O(\alpha)}{\sqrt{2}}(t-s)} ds \\
\leq CW_2(n(0, \cdot), \Gamma_{2\sigma^2}(\cdot - Z_n(t)))e^{\frac{1 + O(\alpha)}{\sqrt{2}}t} + O(\alpha)e^{\frac{\sqrt{\alpha}}{2}t} + O(\alpha) \int_0^t e^{\frac{\sqrt{\alpha}}{2}s} e^{\frac{1 + O(\alpha)}{\sqrt{2}}(t-s)} ds \\
\leq CW_2(n(0, \cdot), \Gamma_{2\sigma^2}(\cdot - Z_n(t)))e^{\frac{1 + O(\alpha)}{\sqrt{2}}t} + O(\alpha)e^{\frac{\sqrt{\alpha}}{2}t}.
\]

Then
\[
W_2(n(t, \cdot), \Gamma_{2\sigma^2}(Z_n(t) - \cdot)) \leq CW_2(n(0, \cdot), \Gamma_{2\sigma^2}(\cdot - Z_n(t)))e^{-\left(\frac{\sqrt{\alpha}}{2} - \frac{1 + O(\alpha)}{\sqrt{2}}\right)t} + O(\alpha),
\]
which proves (24), and in particular, if \( \alpha > 0 \) is small enough,
\[
(27) \quad \forall t \geq -8\ln \alpha, \quad W_2(n(t, \cdot), \Gamma_{2\sigma^2}(Z_n(t) - \cdot)) \leq C\alpha.
\]

**Step 3: We show that \( Z_n(t) \sim \bar{Z} \) for \( t \geq -\frac{C\ln \alpha}{\alpha} \), provided \( \alpha > 0 \) is small enough.**

Thanks to the Kantorovich-Rubinstein formula, (25) implies
\[
Z'_n(t) = \alpha I_n(t)(1 + \alpha I_n(t)) \left[ \int y \frac{a(y)}{I_n(t)} \Gamma_{2\sigma^2}(Z_n(t) - y) dy - Z_n(t) + \frac{O(1)}{I_n(t)} W_1(n(t, \cdot), \Gamma_{2\sigma^2}(Z_n(t) - \cdot)) \right].
\]

Thanks to (27) (note that the 1–Wasserstein distance is dominated by the 2–Wasserstein distance), for \( \alpha > 0 \) small enough and \( t \geq -8\ln \alpha \),
\[
(28) \quad Z'_n(t) = \alpha(1 + \alpha I_n(t)) \left[ F(Z_n(t)) + O(\alpha) \right].
\]

We notice that for \( t \in [0, -8\ln \alpha] \), \( Z_n(t) \) remains close to \( Z^0 \):
\[
(29) \quad |Z_n(t) - Z^0| \leq ||Z'||_{\infty}(-8\ln \alpha) = O(\alpha)(-8\ln \alpha) \xrightarrow{\alpha \to 0} 0.
\]

We recall Assumption 1, Assumption 2, and consider three cases:

**Case 1: \( Z^0 \) close to \( \bar{Z} \)**
Since \( F \) is continuously differentiable, there exists \( \eta > 0 \) such that \( F'(Z) \leq \frac{F'(Z)}{2} < 0 \) for \( Z \in [\bar{Z} - \eta, \bar{Z} + \eta] \). If \( |Z^0 - \bar{Z}| \leq \frac{\eta}{2} \), then (29) implies \( |Z_n(-8\ln \alpha) - \bar{Z}| \leq \eta \) provided \( \alpha > 0 \) is small enough. For \( t \geq -8\ln \alpha \), (28) implies
\[
\frac{d}{dt} |Z_n - \bar{Z}|(t) = \alpha(1 + \alpha I_n(t)) \left[ \left| F(Z_n(t)) - F(\bar{Z}) \right| \text{sgn} \left( Z_n(t) - \bar{Z} \right) + O(\alpha) \right] \\
= \alpha \left| F'(\theta) \right| |Z_n(t) - \bar{Z}| + O(\alpha),
\]
for some \( \theta \in [\bar{Z} - \eta, \bar{Z} + \eta] \). Then if \( y(t) := |Z_n(t) - \bar{Z}| - C\alpha \) (for some constant \( C > 0 \)), we have \( y' \leq \alpha F'(\theta)y \) when \( y \geq 0 \). Then \( y(t) \leq \eta e^{\alpha t\frac{F'(\bar{Z})}{2}} \) and as soon as \( \alpha > 0 \) is small enough, for some constant \( C > 0 \),
\[
(31) \quad \forall t \geq \frac{C\ln(-\ln \alpha)}{\alpha}, \quad |Z_n(t) - \bar{Z}| \leq \frac{C}{-\ln \alpha},
\]
and for longer times, we have the improved estimate
\[
(32) \quad \forall t \geq -\frac{C\ln \alpha}{\alpha}, \quad |Z_n(t) - \bar{Z}| \leq C\alpha.
\]
Case 2: $Z^0 < \bar{Z}$
Since $F$ is a continuous function, Assumption 1 implies that for some $\varepsilon > 0$,
\[
\forall Z \in (Z^0 - \varepsilon, \bar{Z} - \eta/2), \quad \int a(x) \Gamma_{2\sigma^2}^\alpha(Z - x) \, dx - Z \int a(x) \Gamma_{2\sigma^2}^\alpha(Z - x) \, dx \geq \varepsilon.
\]
Then, provided $\alpha > 0$ is small enough, (28) implies $Z_n'(t) > \frac{\varepsilon}{2}$ when $Z_n(t) \in (Z^0 - \varepsilon, \bar{Z} - \eta)$ and $t \geq -8\ln \alpha$. This estimate and (29) imply $Z_n(t) \geq \bar{Z} - \eta/2$ for some $t \geq -8\ln \alpha + \frac{|\bar{Z} - Z^0| + \varepsilon}{\alpha \varepsilon}$. We can then apply the argument for Case 1 for $t \geq -8\ln \alpha + \frac{|\bar{Z} - Z^0| + \varepsilon}{\alpha \varepsilon}$ and prove (31) and (32), for $\alpha > 0$ small enough, in this case also.

Case 3: $Z^0 < \bar{Z}$
This case is similar to Case 2, and we have thus proven (31) and (32) (for $\alpha > 0$ small enough) in all three cases.

Step 4: We prove that $Z_n(t)$ is close to $Y(\alpha t)$ when $\alpha > 0$ is small.
Thanks to (29) and the Lipschitz continuity of $Y$,
\[
\forall t \in [0, -8\ln \alpha], \quad |Z_n(t) - Y(\alpha t)| \leq |Z_n(t) - Z^0| + |Y(\alpha t) - Y(0)| \leq O(\alpha)(-\ln \alpha).
\]
We can then use (28) to show that for $t \geq -8\ln \alpha$,
\[
\frac{d}{dt} |Z_n(t) - Y(\alpha t)| \leq |Z_n(t) - \alpha F(Z_n(t))| + |\alpha F(Z_n(t)) - \alpha Y'(\alpha t)|
\leq O(\alpha^2) + O(\alpha)|Z_n(t) - Y(\alpha t)| \leq C(\alpha + |Z_n(t) - Y(\alpha t)|),
\]
where $C > 0$ is a constant. Then, thanks to a Grönwall inequality,
\[
|Z_n(t) - Y(\alpha t)| \leq (|Z_n(-8\ln \alpha) - Y(-8\alpha \ln \alpha)| + \alpha) e^{Ca(t + 8\alpha \ln \alpha)} - \alpha.
\]
For $t \in [-8\ln \alpha, C \ln(-\ln \alpha)/\alpha]$, we use (29) to show
\[
|Z_n(t) - Y(\alpha t)| \leq C\alpha(-\ln \alpha)e^{-C\ln \alpha} = C\alpha(-\ln \alpha)^{C+1}.
\]
Combining (29), (31) and (33), we obtain (21). Finally, (27) and (32) imply (20), which concludes the proof of Lemma 2.1.

\begin{proof}
\end{proof}

2.2. Existence of a steady-state.

Lemma 2.2. Let $R > 0$ and $a \in W^{2,\infty}(\mathbb{R})$, $\bar{Z} \in \mathbb{R}$ satisfying Assumption 1. Let $\rho > 0$.
There exists $\bar{\alpha} > 0$ and $C > 0$ such that for $0 < \alpha < \bar{\alpha}$, there is a steady-state $\bar{n} \in \mathcal{P}_2(\mathbb{R})$ for (1) that satisfies
\[
W_2(\bar{n}, \Gamma_{2\sigma^2}(\cdot - \bar{Z})) \leq C\alpha.
\]

Proof of Lemma 2.2. Let $n \in \mathcal{P}_2(\mathbb{R})$ satisfying $W_2(n, \Gamma_{2\sigma^2}(\cdot - \bar{Z})) \leq 1$. We recall the definition (14) of $T$ and let $\mathcal{T}_n := T \left(\frac{1 + a(\cdot) n}{1 + a I_n}\right)$, where $I_n$ is defined from $n$ through (12). We also define $Z_n$ from $n$ thanks to (10). Thanks to (15),
\[
\bar{Z} := \int y \mathcal{T}_n(y) \, dy = \int y \frac{(1 + a(y) n(y))}{1 + a I_n} \, dy = \frac{Z_n(1 + \alpha I_n) - \alpha I_n Z_n + \alpha \int a(y)n(y) \, dy}{1 + \alpha I_n}
\]
\[
= Z_n + \alpha F(Z_n) + \left(\int a(y)n(y) \, dy - \int a(y)\Gamma_{2\sigma^2}(y - Z_n) \, dy\right) - \left(\int a(y)n(y) \, dy - \int a(y)\Gamma_{2\sigma^2}(y - Z_n) \, dy\right) Z_n
\]
\[
= Z_n + \alpha F'(\theta)(Z_n - \bar{Z}) + O(\left(W_2(n, \Gamma_{2\sigma^2}(\cdot - Z_n))\right))
\]
\[\frac{1}{1 + \alpha I_n},\]

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with $\theta \in [\bar{Z}, Z_n]$ (or $\theta \in [Z_n, \bar{Z}]$ if $Z_n < \bar{Z}$), and thanks to the Kantorovich-Rubinstein formula. We use the contraction $[13]$ implied by $T$ and inequality $[16]$ to show

$$W_2 \left( T_\alpha(n), \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right) \leq \frac{1}{\sqrt{2}} W_2 \left( \left( 1 + a_{\alpha} \right) n \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right)$$

$$\leq \frac{1}{\sqrt{2}} \left( 1 - 2\frac{a}{1 + a_{\alpha}} \right) W_2 \left( n, \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right) + \frac{a}{2} \left| \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right| + \frac{a}{2} \left| \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right|$$

$$\leq \frac{1 + O(\alpha)}{\sqrt{2}} \left( W_2 \left( n, \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right) + \left| \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right| \right)$$

$$+ O(\alpha) \left( W_2 \left( \frac{2}{2} \left| \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right| + \frac{a}{2} \left| \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right| \right) \right)$$

(35)

$$\leq \frac{1 + O(\alpha)}{\sqrt{2}} W_2 \left( n, \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right) + O(\alpha) \left( \left| \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right| + W_2 \left( n, \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right) \right) + O(\alpha).$$

We use this inequality and $[34]$ to estimate

$$\left| \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right| + \sqrt{\alpha} W_2 \left( T_\alpha(n), \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right)$$

(36)

$$\leq \left( 1 + a F'(\theta) + O \left( \alpha^{3/2} \right) \right) \left| \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right| + \left( \frac{1}{\sqrt{2}} + O \left( \sqrt{\alpha} \right) \right) \sqrt{\alpha} W_2 \left( n, \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right) + O(\alpha^{3/2}).$$

Let $L(m) := \left| \int y m(y) dy - \bar{Z} \right| + \sqrt{\alpha} W_2 \left( m, \Gamma_{2\sigma^2} \left( \cdot - \int y m(y) dy \right) \right)$ for $m \in P_2(\mathbb{R})$. There exist $\delta > 0$ such that if $\left| \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right| \leq \delta$ (which is implied by $L(n \leq \delta)$) and $\alpha > 0$, then

$$L(T_\alpha(n)) \leq L(n) + \left( a F'(\bar{Z}) + O \left( \alpha^{3/2} \right) \right) \left| \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right| + \left( \frac{1}{\sqrt{2}} - 1 + O \left( \sqrt{\alpha} \right) \right) \sqrt{\alpha} W_2 \left( n, \Gamma_{2\sigma^2} \left( \cdot - \bar{Z} \right) \right) + O(\alpha^{3/2})$$

$$\leq L(n) + \alpha F'(\bar{Z}) \left( \frac{1}{2} - L(n) + O(\alpha^{3/2}). \right)$$

Notice that for $\alpha > 0$ small, $1 + a F'(\bar{Z}) \left( \frac{1}{2} - L(n) + O(\alpha^{3/2}) \right) \leq 0$, and $L(n) \leq \alpha^{1/3}$ then implies $L(T_\alpha(n)) \leq \alpha^{1/3} + F'(\bar{Z}) \alpha^{1/3} + \tilde{\sigma} \alpha^{3/2} < \alpha^{1/3}$. The operator $T_\alpha$ then satisfies $T_\alpha(\Omega) \subset \Omega$, where

$$\Omega := \left\{ n \in P_2(\mathbb{R}); L(n) \leq \alpha^{1/3} \right\} \cap L^1(\mathbb{R}).$$

The set $\Omega$ is a non-empty convex set and $T_\alpha$ is a continuous and compact operator on $\Omega$: the compactness is a consequence of the Ascoli Theorem, since $T(n) \cdot$ uniformly Lipschitz continuous for $n \in \Omega$. We can thus apply the Schauder fixed point Theorem to show that there exists a steady-state $\tilde{n}$ of $T_\alpha$ within $\Omega$. This steady-state satisfies $0 \leq \alpha F'(\bar{Z}) \left( \frac{1}{2} - L(n) + O(\alpha^{3/2}) \right)$, that is $L(n) \leq C \sqrt{\alpha}$ and thus $\left| \bar{Z} - \int x \tilde{n}(x) dx \right| \leq C \sqrt{\alpha}$. One can check that $\tilde{n}$ and $\bar{R} := \max\{x; x \in \operatorname{supp}\tilde{n} \} + 1$ satisfy Assumption 2 if $\alpha > 0$ is small, and we may thus apply Lemma $[21]$ to obtain a more precise estimate: $W_2(\tilde{n}, \Gamma_{\sigma^2} \left( \cdot - \bar{Z} \right)) \leq C \alpha$ for some constant $C > 0$. \[ \square \]

2.3. Tail estimates. In this section, derive properties of solutions of $[1]$ from estimate $[20]$. The first property we are interested in is a lower bound of solutions around $x = \bar{Z}$.

Lemma 2.3. Let $R > 0$ and $a \in W^{2,\infty}(\mathbb{R})$, $\bar{Z} \in \mathbb{R}$ satisfying Assumption 1. Let $\rho > 0$.

There exists $\bar{\alpha} > 0$ and $C > 0$ such that for $\alpha \in (0, \bar{\alpha})$ and $n^0 \in P_2(\mathbb{R}) \cap C^1(\mathbb{R})$ satisfying Assumption 2, the solution $n \in L^\infty(\mathbb{R}_+, P_2(\mathbb{R}))$ of $[1]$ with initial data $n^0$ satisfies

$$\forall (t, x) \in [0, \infty) \times \mathbb{R}, \quad n(t, x) \geq C \frac{1}{R + \alpha x^2 / 2}(\bar{Z} - x).$$

(37)
Proof of Lemma 2.3. Thanks to Lemma 2.1 (see (22)), for \( x_0 > 0 \),
\[
\int_{-x_0}^{x_0} n(t, x) \, dx \geq 1 - \int_{(x_0, x_0)^c} n(t, x) \, dx \geq 1 - \frac{1}{x_0} \int x^2 n(t, x) \, dx \geq \frac{1}{2},
\]
provided \( x_0 \) is chosen large enough. For \( \alpha > 0 \) small enough, thanks to (17),
\[
n(t, x) \geq n^0(x) e^{-2t} + \int_{(t-1) \cap 0}^t \int \int \Gamma_{\sigma^2} \left( x - \frac{y_s + y'_s}{2} \right) n(t, y_s) n(t, y'_s) \, dy_s \, dy'_s e^{-2(t-s)} \, ds
\]
\[
\geq \frac{1}{R} \Gamma_{1/ R}(x) e^{-2t} + \frac{1}{C} \min_{y_s, y'_s \in [-x_0, x_0]} \Gamma_{\sigma^2} \left( x - \frac{y_s + y'_s}{2} \right) \int_{(t-1) \cap 0}^t e^{-2(t-s)} \, ds
\]
\[
\geq 1 + \epsilon \frac{1}{R} \Gamma_{1/ R}(x) + 1 \cdot \epsilon \frac{1}{1-1} \Gamma_{\sigma^2} \left( |x - \bar{Z}| + 2x_0 + |\bar{Z}| \right),
\]
and this estimate concludes the proof of (37). \( \square \)

We now want to show that Assumption 2 implies that the solution \( n(t, \cdot) \) of (1) is uniformly exponentially decreasing far from \( \bar{Z} \).

Lemma 2.4. Let \( R > 0 \) and \( a \in W^{2, \infty}(\mathbb{R}) \), \( \bar{Z} \in \mathbb{R} \) satisfying Assumption 1.

There exist \( \bar{\alpha} > 0 \), \( \rho > 0 \) and \( R_0 > 0 \) such that if \( \alpha \in (0, \bar{\alpha}) \) and \( n^0 \in P_2(\mathbb{R}) \cap C^1(\mathbb{R}) \) verifies Assumption 2, then the solution \( n \in L^\infty(\mathbb{R}_+, P_2(\mathbb{R})) \) of (1) with initial data \( n^0 \) satisfies
\[
\forall (t, x) \in [0, \infty) \times [R_0, +\infty), \quad \partial_x n(t, -x) > n(t, -x), \quad \partial_x n(t, x) < -n(t, x).
\]

Proof of Lemma 2.4. Thanks to Lemma 2.1 (see (22)), we have a bound on \( ||Z||_\infty \) that is uniform in \( \alpha \in (0, \bar{\alpha}) \). We use this bound to define \( R_0 \):
\[
R_0 = \max \{ R, Z(0), \max \{ |x|; x \in \text{supp} a \} \} + 4\sigma^2,
\]
and we define a second quantity \( R_1 \): see (22). We assume that \( n(t, \cdot) \) satisfies
\[
\forall x \in [R_0, +\infty), \quad \partial_x n(t, -x) \geq n(t, -x), \quad \partial_x n(t, x) \leq -n(t, x),
\]
and we will show that this assumption on \( n(t, \cdot) \) implies
\[
\forall x \in [R_0, +\infty), \quad \partial_x T(t, -x) \geq \frac{3}{2} T(t, -x), \quad \partial_x T(t, x) \leq -\frac{3}{2} T(t, x),
\]
where
\[
T(t, x) := \int \int \Gamma_{\sigma^2} \left( x - \frac{y_s + y'_s}{2} \right) (1 + \alpha a(y_s)) n(t, y_s) (1 + \alpha a(y'_s)) n(t, y'_s) \, dy_s \, dy'_s.
\]

Thanks to the symmetry of the problem, we can focus on the second part of (39), that is for \( x \geq R_0 \). We introduce \( \varphi \), the piecewise linear function such that \( \varphi(x) = 0 \) on \((-\infty, R_0], \varphi(x) = 1 \) on \([R_0 + R_1, \infty) \) and \( ||\partial_x \varphi||_{L^\infty} \leq \frac{1}{R_1} \). We use this function \( \varphi \) to decompose the following integral expression:
\[
\partial_x T(t, x) = \int \int \partial_x \Gamma_{\sigma^2} \left( x - \frac{y_s + y'_s}{2} \right) (1 + \alpha a(y_s)) n(t, y_s) (1 + \alpha a(y'_s)) n(t, y'_s) \, dy_s \, dy'_s
\]
\[
= \int \int (1 - \varphi(y_s)) (1 - \varphi(y'_s)) \partial_x \Gamma_{\sigma^2} \left( x - \frac{y_s + y'_s}{2} \right) (1 + \alpha a(y_s)) n(t, y_s) (1 + \alpha a(y'_s)) n(t, y'_s) \, dy_s \, dy'_s
\]
\[
+ \int \int \varphi(y_s) \partial_x \Gamma_{\sigma^2} \left( x - \frac{y_s + y'_s}{2} \right) (1 + \alpha a(y_s)) n(t, y_s) (1 + \alpha a(y'_s)) n(t, y'_s) \, dy_s \, dy'_s
\]
\[
+ \int \int \varphi(y'_s) (1 - \varphi(y_s)) \partial_x \Gamma_{\sigma^2} \left( x - \frac{y_s + y'_s}{2} \right) (1 + \alpha a(y_s)) n(t, y_s) (1 + \alpha a(y'_s)) n(t, y'_s) \, dy_s \, dy'_s
\]
\[= I_1(x) + I_2(x) + I_3(x).\]
To estimate $I_2$, we use an integration by parts in $y_*$ and the assumption that $\partial_x n(t, x) \leq -n(t, x)$ for $x \in [R_0, \infty) \supset \text{supp} \varphi$:

$$I_2(x) = -2 \int \int \varphi(y_*) \frac{d}{dy_*} \left[ \Gamma_{\sigma^2} \left( x - \frac{y_* + y'_*}{2} \right) \right] (1 + \alpha a(y_*)) n(t, y_*) (1 + \alpha a(y'_*)) n(t, y'_*) dy_* dy'_*$$

$$= 2 \int \int \left[ \varphi'(y_*) (1 + \alpha a(y_*)) n(t, y_*) + \varphi(y_*) ((1 + \alpha a(y_*)) \partial_x n(t, y_*) + \alpha'(y_*) n(t, y_*)) \right]$$

$$\Gamma_{\sigma^2} \left( x - \frac{y_* + y'_*}{2} \right) (1 + \alpha a(y'_*)) n(t, y'_*) dy_* dy'_*$$

$$\leq \int \int \left[ O \left( \frac{1}{R_1} + \alpha \right) - 2 \varphi(y_*) \right] \Gamma_{\sigma^2} \left( x - \frac{y_* + y'_*}{2} \right) (1 + \alpha a(y_*)) n(t, y_*) (1 + \alpha a(y'_*)) n(t, y'_*) dy_* dy'_*.$$ 

where $\left| O \left( \frac{1}{R_1} + \alpha \right) \right| \leq \frac{1 + \alpha \|a\|_{\infty}}{R_1} + \alpha(\|a\|_{\infty} + \|a'\|_{\infty}) \leq \frac{1}{4}$ if $\alpha \leq \frac{1}{8(\|a\|_{\infty} + \|a'\|_{\infty}) + 1}$. $I_3(x)$ can be estimated in a similar manner (through an integration by parts in $y'_*$).

To estimate $I_1(x)$, we notice that $1 - \varphi$ is supported on $(-\infty, R_0 + R_1]$, and then, for $x \geq R_0 + R_1 + 2\sigma^2$ and $y_*, y'_* \leq R_0 + R_1$, $\partial_x \Gamma_{\sigma^2} \left( x - \frac{y_* + y'_*}{2} \right) \leq -2 \Gamma_{\sigma^2} \left( x - \frac{y_* + y'_*}{2} \right)$, and then

$$I_1(x) \leq -2 \int \int (1 - \varphi(y_*)) (1 - \varphi(y'_*)) \Gamma_{\sigma^2} \left( x - \frac{y_* + y'_*}{2} \right) (1 + \alpha a(y_*)) n(t, y_*) (1 + \alpha a(y'_*)) n(t, y'_*) dy_* dy'_*.$$ 

Our original estimate then becomes, for $x \geq R_0 + R_1 + 2\sigma^2$:

$$\partial_x T(t, x) \leq \int \int \left[ O \left( \frac{1 + \alpha}{R_1} + \alpha \right) - 2 \right] \Gamma_{\sigma^2} \left( x - \frac{y_* + y'_*}{2} \right) (1 + \alpha a(y_*)) n(t, y_*) (1 + \alpha a(y'_*)) n(t, y'_*) dy_* dy'_*,$$

where $\left| O \left( \frac{1 + \alpha}{R_1} + \alpha \right) \right| \leq \frac{1}{4}$. Thus,

$$\forall x \in [R_0 + R_1 + 2\sigma^2, \infty), \quad \partial_x T(t, x) \leq -\frac{3}{2} T(t, x). \tag{40}$$

To obtain a similar estimate for $x \in [R_0, R_0 + R_1 + 2\sigma^2]$, we introduce a different estimate, using the Lipschitz continuity of $(y_*, y'_*) \mapsto \Gamma_{\sigma^2} \left( x - \frac{y_* + y'_*}{2} \right)$ and the Kantorovich-Rubinstein formula:

$$\partial_x T(t, x) = \int \int \partial_x \Gamma_{\sigma^2} \left( x - \frac{y_* + y'_*}{2} \right) n(t, y_*) n(t, y'_*) dy_* dy'_*$$

$$= \frac{d}{dx} \int \int \Gamma_{\sigma^2} \left( x - \frac{y_* + y'_*}{2} \right) \Gamma_{2\sigma^2}(y_* - Z_n(t)) \Gamma_{2\sigma^2}(y'_* - Z_n(t)) dy_* dy'_*$$

$$+ O \left( W_1(n(t, \cdot), \Gamma_{2\sigma^2}(\cdot - Z_n(t))) \right) = \Gamma'_{2\sigma^2}(x - Z_n(t)) + O(\alpha) + O(\rho)e^{-t/8},$$

$$14$$
thanks to Lemma 2.1 (see (24)). Then,
\[ \partial_x T(t, x) = -\frac{x - Z_n(t)}{2\sigma^2} \\Gamma_{2\sigma^2}(x - Z_n(t)) + O(\alpha)C\alpha + O(\alpha) + O(p)e^{-t/8} \]
\[ = -\frac{x - Z_n(t)}{2\sigma^2} \int \int \Gamma_{\sigma^2} \left( x - \frac{y_s + y_s'}{2} \right) \Gamma_{2\sigma^2}(y_s - Z(t))\Gamma_{2\sigma^2}(y_s' - Z_n(t)) dy_s dy_s' \]
\[ + O(\alpha) + O(p)e^{-t/8} \]
\[ = -\frac{x - Z_n(t)}{2\sigma^2} T(t, x) + O(\alpha) + O(1)e^{-t/8} \leq -2T(t, x) + O(\alpha) + O(p)e^{-t/8}, \]

since the definition of \( R_0 \) and (23) imply \( x - Z_n(t) \geq 4\sigma^2 \). Thanks to Lemma 2.3 \( n(t, \cdot) \) is uniformly bounded from below by a Gaussian function, and then \( T \) is also uniformly bounded for \( (t, x) \in [0, \infty) \times [R_0, R_0 + R_1 + 2\sigma^2] \). Provided \( \rho > 0 \) and \( \alpha > 0 \) are small enough, we have therefore

\[ \forall x \in [R_0, R_0 + R_1 + 2\sigma^2], \quad \partial_x T(t, x) \leq \frac{-3}{2} T(t, x). \]

This estimate and (40) prove that (38) implies (39).

To conclude the proof, we introduce a modified equation: we consider solutions of (1) over a compact interval, that is, for \( \varepsilon > 0 \),
\[
\begin{cases}
  u_\varepsilon(0, x) = \frac{n(0, x)1_{|x| \leq \frac{1}{\varepsilon}}}{\int_{1/\varepsilon}^{1} n(0, y) dy}, \\
  \partial_t u_\varepsilon(t, x) = T\left((1 + \alpha a)u_\varepsilon(t, \cdot), (1 + \alpha a)u_\varepsilon(t, \cdot)\right)_{|x| \leq \frac{1}{\varepsilon}} - \left(\int_{1/\varepsilon}^{1} T\left(\frac{(1 + \alpha a)u_\varepsilon(t, \cdot)}{1 + \alpha a}\right) dy\right)_{|x| \leq \frac{1}{\varepsilon}} u_\varepsilon(t, x).
\end{cases}
\]

Note that \( u_\varepsilon \) satisfies Assumption 2 if \( \varepsilon > 0 \) is small enough, except for the lower bound \( n^0 \geq 1/R_1 \), that has actually only be used on the finite interval \([R_0, R_0 + R_1 + 2\sigma^2]\), where this lower bound is also satisfied by \( u_\varepsilon \). The estimates above then apply to \( u_\varepsilon \) as well: if \( u_\varepsilon(t, \cdot) \) satisfies (38), then \( T((1 + \alpha a)u_\varepsilon(t, \cdot), (1 + \alpha a)u_\varepsilon(t, \cdot))_{|x| \leq \frac{1}{\varepsilon}} \) satisfies (39). Moreover, one may check that \( u_\varepsilon \) and \( \partial_x u_\varepsilon \) are continuous on \([0, \infty) \times [-1/\varepsilon, 1/\varepsilon]\).

Let \( t_0 \geq 0 \), the infimum of the times when (38) does not hold. Thanks to our assumption on the initial condition and the continuity of \( u_\varepsilon \) and \( \partial_x u_\varepsilon \), \( t_0 > 0 \) and \( u_\varepsilon(t_0, \cdot) \) does satisfy (38) for \( |x| < 1/\varepsilon \). Then \( T((1 + \alpha a)u_\varepsilon(t, \cdot), (1 + \alpha a)u_\varepsilon(t, \cdot))_{|x| \leq \frac{1}{\varepsilon}} \) satisfies (39) for \( t \in [0, t_0] \) and \( |x| \leq 1/\varepsilon \), and, thanks to the regularity of \( u_\varepsilon \), this is actually also true for \( t \in [t_0, t_0 + \chi] \), for some small \( \chi > 0 \). Then, thanks to (17), for \( R_0 \leq x \leq 1/\varepsilon \),
\[
\partial_x u_\varepsilon(t, x) = \partial_x n(0, x)1_{|x| \leq \frac{1}{\varepsilon}}e^{-\int_0^t (1 + \alpha I_n(s))^2 ds} \\
+ \int_0^t (1 + \alpha I_n(s))^2 \partial_x T\left(\frac{(1 + \alpha a)u_\varepsilon(t, \cdot)}{1 + \alpha a}\right) dy (x)1_{|x| \leq \frac{1}{\varepsilon}}e^{-\int_0^t (1 + \alpha I_n(s))^2 ds} \\
- \frac{3}{2} \int_0^t (1 + \alpha I_n(s))^2 T\left(\frac{(1 + \alpha a)u_\varepsilon(t, \cdot)}{1 + \alpha a}\right) dy (x)1_{|x| \leq \frac{1}{\varepsilon}}e^{-\int_0^t (1 + \alpha I_n(s))^2 ds} \\
< -u_\varepsilon(t, x).
\]

This, combined to the fact that \( u_\varepsilon(t, \cdot) = 0 \) on \((1/\varepsilon, \infty)\), implies that (38) is satisfied for \( t \in [0, t_0 + \chi] \) \( u_\varepsilon(t, \cdot) \) is not differentiable in \( x = \pm 1/\varepsilon \), but the week derivative of \( u_\varepsilon(t, \cdot) \) satisfies this inequality, contradicting the definition of \( t_0 \), so that \( u_\varepsilon \) satisfies (38) for \( t \in [0, \infty) \). We may finally use the convergence of \( u_\varepsilon \) towards \( n \) when \( \varepsilon \to 0 \) to conclude the proof.
3. Wasserstein estimates

3.1. Effect of a multiplicative operator acting on the bulk of the distribution.

Lemma 3.1. Let $a \in W^{1,\infty}(\mathbb{R},\mathbb{R}_+)$ compactly supported and $\nu > 0$. There exists $\bar{\alpha} > 0$ and $C > 0$ such that, if $n, m \in \mathcal{P}_2(\mathbb{R}) \cap C^0(\mathbb{R})$ satisfy

$$\begin{align*}
n &\geq \nu \Gamma_\nu(\bar{Z} - \cdot), \\
m &\geq \nu \Gamma_\nu(\bar{Z} - \cdot),
\end{align*}$$

then, for $\alpha \in (0, \bar{\alpha})$,

$$W_2\left(\left(1 - \alpha + \frac{a}{\int a(x) n(x) dx}\right) n, \left(1 - \alpha + \frac{a}{\int a(x) m(x) dx}\right) m\right) \leq (1 + C \sqrt{\alpha}) W_2(n, m).$$

Proof of Lemma 3.1

Step 1: Preliminaries

Let $\tilde{n}, \tilde{m} \in \mathcal{P}_2(\mathbb{R})$ defined by

$$\tilde{n}(x) = \left(1 - \alpha + \frac{a}{I_n}\right) n(x), \quad \tilde{m}(x) = \left(1 - \alpha + \frac{a}{I_m}\right) m(x),$$

We denote by $u, v : (0, 1) \to \mathbb{R}$ the pseudo-inverse of the cumulative distributions of $n, m$ and $\hat{u}, \hat{v}$ the ones corresponding to $\tilde{n}, \tilde{m}$ (see [7] for the definition of pseudo-inverse of cumulative distributions). We define $I_n, I_m$ as in [12]. Let also $R := \max\{|x|; x \in \text{supp} a\}$, and

$$r := \frac{\nu}{2} \int_{|x|+R}^{\infty} \Gamma_\nu.$$

Then, thanks to (41),

$$r < \int_{-\infty}^{-R} n, \quad r < \int_{-\infty}^{-R} m,$$

and

$$(1 - \alpha) r < \int_{R}^{\infty} (1 - \alpha) n = \int_{R}^{\infty} \tilde{n}, \quad (1 - \alpha) r < \int_{R}^{\infty} (1 - \alpha) m = \int_{R}^{\infty} \tilde{m},$$

which implies

$$\max(u(r), v(r)) < -R, \quad \min(\hat{u}(1 - (1 - \alpha)r), \hat{v}(1 - (1 - \alpha)r)) > R.$$

Step 2: Definition and properties of $\phi_u, \phi_v$

We define $\phi_u$ by

$$\phi_u((1 - \alpha)r) = r, \quad \phi_u(s) = \frac{1}{1 - \alpha + \frac{a(u(\phi_u(s)))}{I_n}}.$$

Thanks to (41), $u$ is Lipschitz continuous (we recall [8]). $a$ is also Lipschitz continuous so that the Cauchy-Lipschitz Theorem provides the local existence and uniqueness of $\phi_u$ around $s = (1 - \alpha)r$. If $\bar{\alpha} > 0$ is chosen small enough, namely $\bar{\alpha} \leq \frac{1}{2} \nu \int a(x) \Gamma_\nu(\bar{Z} - x) dx$, then $\frac{2}{3} \leq \phi'_u(s) \leq 2$ and $\phi_u : [0, 1] \to \mathbb{R}$ is a well defined increasing function.

For $s \in [0, (1 - \alpha)r]$, since both $u$ and $\phi_u$ are non-decreasing, (44) implies $u(\phi_u(s)) \leq -R$ and thus $a(u(\phi_u(s))) = 0$. This implies, firstly, that for $s \in [0, (1 - \alpha)r]$, we have $\phi'_u(s) = \frac{1}{1 - \alpha}$ and then

$$\forall s \in [0, (1 - \alpha)r], \quad \phi_u = \frac{s}{1 - \alpha}.$$

Secondly, it provides the following relation for $s \in [0, (1 - \alpha)r]$

$$s = \int_{-\infty}^{\hat{u}(s)} \tilde{n}(x) dx = (1 - \alpha) \int_{-\infty}^{\hat{u}(s)} n(x) dx = (1 - \alpha) u^{-1}(\hat{u}(s)).$$
This implies \( \phi \) and then

\[
\frac{d}{ds}(u \circ \phi_u)(s) = \phi_u'(s)u'(\phi_u(s)) = \frac{1}{1 - \alpha + \alpha \frac{a(u(\phi_u(s)))}{I_n}} \frac{1}{n(u(\phi_u(s)))},
\]

\[
\frac{d}{ds}(\tilde{u})(s) = \frac{1}{\tilde{n}(\tilde{u}(s))} = \frac{1}{1 - \alpha + \alpha \frac{a(\tilde{u}(s))}{I_n}} \frac{1}{n(\tilde{u}(s))},
\]

so that, thanks to the uniqueness given by the Cauchy-Lipschitz Theorem,

(47) \( \forall s \in [0, 1], \ u \circ \phi_u(s) = \tilde{u}(s), \)

The argument we have made can be repeated for \( \phi_v \) and we then get

\( \forall s \in [0, 1], \ \tilde{u}(s) = u \circ \phi_u(s) \) and \( \tilde{v}(s) = v \circ \phi_v(s), \)

with \( \phi_v \) defined by \( \phi_v((1 - \alpha)r) = r \) and

\[
\phi_v'(s) = \frac{1}{1 - \alpha + \alpha \frac{a(v(\phi_v(s)))}{I_m}}.
\]

Thanks to (46), \( \phi_u(s) = \frac{s}{1 - \alpha} = \phi_v(s) \) for \( s \in [0, (1 - \alpha)r] \). Moreover, (44) implies

\[
R < \tilde{u}(1 - (1 - \alpha)r) = u(\phi_u(1 - (1 - \alpha)r)),
\]

and since both \( u \) and \( \phi_u \) are non decreasing, the fact that \( a(x) = 0 \) for \( x \geq R \) implies that \( a(u(\phi_u(s))) = a(\tilde{u}(s)) = 0 \) for \( s \in (1 - (1 - \alpha)r, 1) \). (45) then writes \( \phi_u'(s) = \frac{1}{1 - \alpha}, \) and thus \( \phi_u(s) = C_0 + \frac{s}{1 - \alpha}. \) To determine \( C_0, \) we notice that \( \tilde{u}(s) = u(\phi_u(s)) \to s \to 1 \). Then, \( \phi_u(s) \to s \to 1 \), which leads to \( \phi_u(s) = \frac{s - \alpha}{1 - \alpha} \) for \( s \geq 1 - (1 - \alpha)r. \) This argument can be repeated for \( \phi_v \) and we get

(48) \( \forall s \in (0, (1 - \alpha)r), \ \phi_u(s) = \frac{s}{1 - \alpha} = \phi_v(s), \ \forall s \in (1 - (1 - \alpha)r, 1), \ \phi_u(s) = \frac{s - \alpha}{1 - \alpha} = \phi_v(s). \)

**Step 3: Estimate on \( W_2(\tilde{n}, \tilde{m}) \)**

For \( s \in (0, 1), \) we have

\[
\phi_v(s) - \phi_u(s) = \int_r^s \frac{1}{1 - \alpha + \alpha \frac{a(u(\phi_u(\tau)))}{I_n}} - \frac{1}{1 - \alpha + \alpha \frac{a(v(\phi_v(\tau)))}{I_m}} \ d\tau
\]

\[
= \int_r^s O(\alpha) \left( \frac{|I_n - I_m| + \|a'\|_\infty |u(\phi_u(\tau)) - v(\phi_v(\tau))|}{1 - \alpha + \alpha \frac{a(u(\phi_u(\tau)))}{I_n}} \right) \left( 1 - \alpha + \alpha \frac{a(v(\phi_v(\tau)))}{I_m} \right) d\tau,
\]

and then

\[
|\phi_v(s) - \phi_u(s)| = O(\alpha)|I_n - I_m| + O(\alpha) \int_r^s |u(\phi_u(\tau)) - v(\phi_v(\tau))| d\tau
\]

\[
= O(\alpha)|I_n - I_m| + O(\alpha) \int_r^s |\tilde{u}(\tau) - \tilde{v}(\tau)| d\tau.
\]

This implies

(49) \( |\phi_v(s) - \phi_u(s)| \leq O(\alpha)\|a'\|_\infty W_1(n, m) + O(\alpha)W_1(\tilde{n}, \tilde{m}). \)
Thanks to this estimate, we get:

$$
\int_0^1 |\tilde{u}(s) - \tilde{v}(s)|^2 \, ds = \int_0^1 |u(\phi_u(s)) - v(\phi_v(s))|^2 \, ds
$$

$$
= \int_0^1 |(u(\phi_u(s)) - v(\phi_v(s))) + (v(\phi_u(s)) - v(\phi_v(s)))|^2 \, ds
$$

$$
\leq (1 + \varepsilon) \int_0^1 |u(\phi_u(s)) - v(\phi_v(s))|^2 \, ds + \left(1 + \frac{1}{\varepsilon}\right) \int_0^1 |v(\phi_u(s)) - v(\phi_v(s))|^2 \, ds,
$$

thanks to a Young inequality. The definition of $\phi_u$ implies $\phi_u'(s) = 1 + O(\alpha)$, and then

$$
\int_0^1 |u(\phi_u(s)) - v(\phi_v(s))|^2 \, ds \leq (1 + O(\alpha)) \int_0^1 |u(s) - v(s)|^2 \, ds.
$$

To estimate the last term of (50), we start by estimating the integral over $[r, 1 - r]$. Since $\phi_u(s) = (1 + O(\alpha)) s$, we have $\phi_u([r, 1 - r]) \subset [r/2, 1 - r/2]$ (and, with a similar argument, $\phi_v([r, 1 - r]) \subset [r/2, 1 - r/2]$) as soon as $\alpha > 0$ is small enough. Then, thanks to (41), we have a lower bound

$$
m(x) \geq \min_{v(r/2), x(1 - r/2)} \nu \Gamma_{\sigma^2/2} (\bar{Z} - \cdot) > 0
$$

on $[v(r) - 1, v(1 - r) + 1]$, which implies (see (38)) an upper bound $|v'(s)| = \left|\frac{1}{m(v(s))}\right| \leq C$ that is uniform for $\alpha > 0$ small enough. Using this estimate and (49),

$$
\int_r^{1-r} |v(\phi_u(s)) - v(\phi_v(s))|^2 \, ds \leq \|v''\|_{L^\infty([r/2, 1 - r/2])} \int_r^{1-r} |\phi_u(s) - \phi_v(s)|^2 \, ds
$$

$$
\leq O(\alpha) (W_1(n, m) + W_1(\bar{n}, \bar{m}))^2
$$

$$
\leq O(\alpha) (W_2^2(n, m) + W_2^2(\bar{n}, \bar{m})).
$$

For $s \in (0, r) \cup (1 - r, 1]$, $\phi_u(s) = \phi_v(s)$ thanks to (48), and then

$$
\int_0^1 |v(\phi_u(s)) - v(\phi_v(s))|^2 \, ds \leq O(\alpha) (W_2^2(n, m) + W_2^2(\bar{n}, \bar{m})).
$$

Thanks to estimates above, (50) becomes

$$
\int_0^1 |\tilde{u}(s) - \tilde{v}(s)|^2 \, ds \leq (1 + \varepsilon)(1 + O(\alpha)) \int_0^1 |u(s) - v(s)|^2 \, ds
$$

$$
+ \left(1 + \frac{1}{\varepsilon}\right) O(\alpha) (W_2^2(n, m) + W_2^2(\bar{n}, \bar{m})).
$$

If we choose $\varepsilon := \sqrt{\alpha}$, we get

$$
W_2^2(\bar{n}, \bar{m}) \leq (1 + O(\sqrt{\alpha}))W_2^2(n, m) + O(\sqrt{\alpha}) W_2^2(\bar{n}, \bar{m}),
$$

and then

$$
W_2(\bar{n}, \bar{m}) \leq (1 + O(\sqrt{\alpha}))W_2(n, m),
$$

provided $\alpha > 0$ is small enough.

\[\square\]

3.2. Effect of a multiplicative factor on the tails of a distribution.

**Lemma 3.2.** Let $\nu > 0$, $R > 0$, $\tilde{Z}$ and $M > 0$. There exists $C > 0$ such that if $n \in \mathcal{P}_2(\mathbb{R})$ satisfies

$$
\int x^2 n(x) \, dx \leq M,
$$

(51)

$$
\forall x \in (-\infty, -R], \quad n'(x) > n(x), \quad \forall x \in [R, \infty), \quad n'(x) < -n(x),
$$

(52)

$$
\forall x \in \mathbb{R}, \quad n \geq \nu \Gamma_{\nu}(\tilde{Z} - \cdot),
$$

\[\forall x \in (-\infty, -R], \quad n'(x) > n(x), \quad \forall x \in [R, \infty), \quad n'(x) < -n(x),
$$

(52)

$$
\forall x \in \mathbb{R}, \quad n \geq \nu \Gamma_{\nu}(\tilde{Z} - \cdot),
$$
and if $p \in \mathcal{P}_2(\mathbb{R})$ satisfies $\text{supp } p \subset [-R, R]$, then, for any $\alpha, \tilde{\alpha} \in [0, 1/4]$, we have:

$$W_2((1 - \alpha)n + \alpha p, (1 - \tilde{\alpha})n + \tilde{\alpha} p) \leq C|\alpha - \tilde{\alpha}|.$$ 

**Remark 3.3.** Notice that the convexity of $W_2$ provides a related estimate that holds for any $n, p \in \mathcal{P}_2(\mathbb{R})$:

$$W_2((1 - \alpha)n + \alpha p, (1 - \alpha')n + \alpha' p) \leq C\sqrt{|\alpha - \alpha'|},$$

but we need a stronger estimate, linear in $\alpha - \alpha'$. This improved estimate derives from the assumptions made on the tails of the distribution $n$.

**Proof of Lemma 3.2.** W.l.o.g., $\tilde{\alpha} \leq \alpha$. Let $u$ the pseudo-inverse of the cumulative distribution of $n$, and $u_\alpha$ the one of $(1 - \alpha)n + \alpha p$. We define $r := \frac{\nu}{2} \int_{|Z|+R+1} \Gamma_r$, which implies in particular that $u_\alpha(s) < -R - 1$ for $s \in (0, r)$. For $s \in (0, r]$ and $h \in (0, 1)$,

$$\int_{-\infty}^{u_\alpha(s) + h} (1 - \alpha)n(x) + \alpha p(x) \, dx = \int_{-\infty}^{u_\alpha(s) + h} (1 - \alpha)n(x) \, dx,$$

thanks to (51). Then,

$$\int_{-\infty}^{u_\alpha(s) + h} (1 - \alpha)n(x) + \alpha p(x) \, dx \geq (1 + h)\int_{-\infty}^{u_\alpha(s)} (1 - \alpha)n(x) \, dx$$

$$\geq (1 + h) \frac{1 - \alpha}{1 - \tilde{\alpha}} \int_{-\infty}^{u_\alpha(s)} (1 - \tilde{\alpha})n(x) \, dx = (1 + h) \frac{1 - \alpha}{1 - \tilde{\alpha}} s,$$

and if we choose $h := \frac{\alpha - \tilde{\alpha}}{1 - \alpha}$, we get

$$\int_{-\infty}^{u_\alpha(s) + \frac{\alpha - \tilde{\alpha}}{1 - \alpha}} (1 - \alpha)n(x) + \alpha p(x) \, dx \geq s,$$

and then $u_\alpha(s) \leq u_{\tilde{\alpha}}(s) + \frac{\alpha - \tilde{\alpha}}{1 - \alpha}$.

The opposite relationship between $u_\alpha(s)$ and $u_{\tilde{\alpha}}(s)$ (we still consider $s \in (0, r)$ here) is easier to obtain: since $\tilde{\alpha} \leq \alpha$ and $p(x) = 0$ for $x \in (-\infty, u_{\tilde{\alpha}}(s)]$,

$$\int_{-\infty}^{u_\alpha(s)} (1 - \alpha)n(x) + \alpha p(x) \, dx \leq \int_{-\infty}^{u_{\tilde{\alpha}}(s)} (1 - \tilde{\alpha})n(x) + \tilde{\alpha} p(x) \, dx = s,$$

and then $u_{\tilde{\alpha}}(s) \leq u_\alpha(s)$. This argument for $s \in (0, r)$ can be repeated for $s \in (1 - r, 1)$ and we get

$$\forall s \in (0, r) \cup (1 - r, 1), \quad |u_{\tilde{\alpha}}(s) - u_\alpha(s)| \leq \frac{\alpha - \tilde{\alpha}}{1 - \alpha}.$$

For $s \in (r, 1 - r)$, we notice that

$$\int_{-\infty}^{u_{\tilde{\alpha}}(s)} (1 - \alpha)n(x) + \alpha p(x) \, dx = s = \int_{-\infty}^{u_\alpha(s)} (1 - \tilde{\alpha})n(x) + \tilde{\alpha} p(x) \, dx,$$

and thus

$$\int_{-\infty}^{u_{\tilde{\alpha}}(s)} (\tilde{\alpha} - \alpha)n(x) + (\alpha - \tilde{\alpha})p(x) \, dx \geq (1 - \tilde{\alpha})|u_\alpha(s) - u_{\tilde{\alpha}}(s)| \inf_{[-\chi, \chi]} n,$$

where $\chi = \max \{|u_\alpha(r)|, |u_{\tilde{\alpha}}(r)|, |u_\alpha(1 - r)|, |u_{\tilde{\alpha}}(1 - r)|\}$. Since $u_\alpha(r) < -R$, we have

$$r = \int_{-\infty}^{u_{\tilde{\alpha}}(r)} (1 - \alpha)n + \alpha p = \int_{-\infty}^{u_{\tilde{\alpha}}(r)} (1 - \alpha)n \leq \frac{1 - \alpha}{u_\alpha(r)^2} \int_{-\infty}^{u_\alpha(r)} x^2 n(x) \, dx \leq \frac{M}{u_\alpha(r)^2},$$
and then $|u_\alpha(r)| \leq \sqrt{\frac{M}{r}}$. We may repeat the argument to show that $\chi \leq \sqrt{\frac{M}{r}}$, and then $\inf_{[-\chi, \chi]} n \geq \nu \Gamma_\nu \left( |\tilde{Z}| + \sqrt{M/r} \right)$, so that (54) implies

$$\tag{56} |u_\alpha(s) - u_\alpha'(s)| \leq \frac{3}{\nu \Gamma_{\alpha^2/2}} \left( |\tilde{Z}| + \sqrt{M/r} \right) |\alpha - \bar{\alpha}|.$$

This estimate for $s \in [r, 1 - r]$ and (53) for $(0, r) \cup (1 - r, 1)$ can be used to conclude the proof:

$$W_2^2 \left((1 - \alpha)n + \alpha p, (1 - \bar{\alpha})n + \bar{\alpha}p\right) = \int_0^1 |u_\alpha'(s) - u_\alpha(s)|^2 ds \leq C|\alpha - \bar{\alpha}|^2.$$

$$\square$$

3.3. Effect of a translation.

**Lemma 3.4.** Let $\nu > 0$, $\tilde{Z}$, $M > 0$, and $a \in W^{1,\infty}(\mathbb{R}, \mathbb{R}_+)$ with compact support. There exists $\alpha > 0$ and $C > 0$ such that if $n \in P_2(\mathbb{R})$ satisfies $\int x^2n(x) \, dx \leq M$ and

$$\forall x \in \mathbb{R}, \quad n \geq \nu \Gamma_\nu (\tilde{Z} - \cdot),$$

then, for any $Z \in [-1, 1]$ and $\alpha \in (0, \bar{\alpha})$,

$$W_2 \left(1 - \alpha + \alpha \int \frac{a(y)n(y) \, dy}{\int a(y) \, dy}, 1 - \alpha + \alpha \int \frac{a(y-Z)n(y) \, dy}{\int a(y-Z) \, dy} \right) \leq C\alpha|Z|.$$

**Proof of Lemma 3.4:** Let $n_Z := \left(1 - \alpha + \alpha \int \frac{a(y-Z)n(y) \, dy}{\int a(y-Z) \, dy} \right) n$ and $u_Z$ the corresponding pseudo-inverse. Let also $R := \max\{|x|; x \in \text{supp } a\} + 1$ and $r := \frac{\nu}{2} \int_{|Z|+R} \Gamma_\nu$. Then $u_Z(s) = u_0(s)$ for $s \in (0, r) \cup (1 - r, 1)$, while for $s \in (r, 1 - r)$, we will use an argument similar to what was used in the proof of Lemma 3.2, we notice that $\int_{-\infty}^{u_0(s)} n_0 = s = \int_{-\infty}^{u_Z(s)} n_Z$, and thus

$$\alpha \int_{-\infty}^{u_0(s)} \left(\frac{a(x) - a(y-Z)}{\int a(y)n(y) \, dy - \int a(y-Z)n(y) \, dy} \right) n(x) \, dx = \int_{-\infty}^{u_Z(s)} n_Z,$$

which implies

$$C\alpha ||a'||_\infty |Z| \geq (1 - \alpha)|u_0(s) - u_Z(s)| \inf_{[-\chi, \chi]} n,$$

where $\chi = \max(|u_0(r)|, |u_Z(r)|, |u_0(1 - r)|, |u_Z(1 - r)|)$ and $s \in (0, r) \cup (1 - r, 1)$. We can use the bound $M$ on the second moment of $n$ to control $\xi$, just as it was done in (55) and (56). This argument and assumption (57) show that for some constant $C > 0$, $|u_0(s) - u_Z(s)| \leq C\alpha|Z|$, which is enough to conclude the proof:

$$W_2^2 (n_0, n_Z) = \int_0^1 |u_0(s) - u_Z(s)|^2 ds = \int_r^{1-r} |u_0(s) - u_Z(s)|^2 ds \leq C\alpha^2 |Z|^2.$$

$$\square$$

3.4. Remarks on the Wasserstein estimates of Section 3. In this section, we discuss the necessity of a lower bound assumption for the estimates developed in Lemma 3.4 and Lemma 3.1. Note that solutions of (1) do satisfy a lower bound assumption thanks to Lemma 2.3.

The necessity of assumption (57) for Lemma 3.4 can be checked with the following example: let $n := \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1$, and $a(x) = 1 + \frac{x}{2}$ on $[-1, 1]$. Then, for $Z \in \mathbb{R}$,

$$\left(1 - \alpha + \alpha \int \frac{a(x-Z)}{\int a(y-Z)n(y) \, dy} \right) n(x) = \frac{1}{2} \left(1 - \alpha + \frac{1}{2 - 1/Z/2}\right) \delta_{-1} + \frac{1}{2} \left(1 + \alpha + \frac{1}{2 - 1/Z/2}\right) \delta_1,$$
and thus
\[ W_2 \left( \left( 1 - \alpha + \alpha \frac{a}{\int a(y)n(y)\,dy} \right)n, \left( 1 - \alpha + \alpha \frac{a(-z)}{\int a(y-Z)n(y)\,dy} \right)n \right) \]
\[ = \frac{\sqrt{\alpha}}{2} \sqrt{1 - \frac{1}{1 - Z/2}} \sim \frac{\sqrt{\alpha}}{2\sqrt{2}} \sqrt{|Z|}, \]
so that \( n \) does not satisfy (58). This shows the lower bound assumption is necessary for the estimate provided by Lemma 3.4 to hold.

The necessity of (41) for Lemma 3.1 requires a slightly more complex distribution. We still consider two initial conditions \( n \) such that (58). This shows the lower bound assumption is necessary for the estimate provided by Lemma 3.4 to hold.

The necessity of (41) for Lemma 3.1 requires a slightly more complex distribution. We still consider two initial conditions \( n \) such that (58). This shows the lower bound assumption is necessary for the estimate provided by Lemma 3.4 to hold.

Then the pseudo-inverse distribution of \( n_\rho \) (see (7)), that we denote \( u_\rho \), is
\[ u_\rho(z) = \begin{cases} 
-1 & \text{if } z \in [0, 1/2), \\
0 & \text{if } z \in [1/2, 1/2 + \rho/4), \\
4(z - (1/2 + \rho/4)) & \text{if } z \in [1/2 + \rho/4, 3/4 + \rho/4), \\
1 & \text{if } z \in [3/4 + \rho/4, 1],
\end{cases} \]
so that
\[ W_2(n_\rho, n_{\rho'}) = \left( \int_0^1 |u_\rho(z) - u_{\rho'}(z)|^2 \, dz \right)^{1/2} \leq \left( \int_{1/2}^1 |\rho - \rho'|^2 \, dz \right)^{1/2} \leq \frac{|\rho - \rho'|}{\sqrt{2}}. \]

We compute \( \int a(x)n_\rho(x)\,dx = \frac{15}{16} - \frac{\rho}{8} \), and notice that
\[ \left( 1 - \alpha + \alpha \frac{a(x)}{\int a(y)n_\rho(y)\,dy} \right)n_\rho(x) = \left( 1 - \alpha + \frac{8\alpha}{15 - \rho} \right) \frac{\delta_{-1}}{2}, \text{ for } x \in (-\infty, 0). \]

The pseudo-inverse \( \tilde{u}_\rho \) of this distribution then satisfies \( \tilde{u}_\rho(z) = -1 \) for \( z < \frac{1}{2} \left( 1 - \alpha + \frac{8\alpha}{15 - 2\rho} \right) \), while \( \tilde{u}_\rho(z) \geq 0 \) for \( z > \frac{1}{2} \left( 1 - \alpha + \frac{8\alpha}{15 - 2\rho} \right) \). It follows that
\[ W_2 \left( \left( 1 - \alpha + \alpha \frac{a(x)}{\int a(y)n_\rho(y)\,dy} \right)n_\rho, \left( 1 - \alpha + \alpha \frac{a(x)}{\int a(y)n_{\rho'}(y)\,dy} \right)n_{\rho'} \right) \]
\[ \geq \sqrt{\frac{1}{2} \left( 1 - \alpha + \frac{8\alpha}{15 - 2\rho} \right) - \frac{1}{2} \left( 1 - \alpha + \frac{8\alpha}{15 - 2\rho} \right)} = \frac{\sqrt{2\alpha}}{(15 - 2\rho)(15 - 2\rho)\sqrt{|\rho - \rho'|}} \]
\[ \geq \frac{C\sqrt{\alpha}}{\sqrt{|\rho - \rho'|}} W_2(n_\rho, n_{\rho'}), \]
for \( \rho, \rho' \sim \frac{1}{2} \), thanks to (59). It is then impossible to have a constant \( C > 0 \) such that (42) holds for any \( \rho, \rho' \sim \frac{1}{2} \), even if \( \alpha > 0 \) is small. We have shown that a lower bound assumption on \( n \) and \( m \) is necessary for the conclusion of Lemma 3.1 to hold.

4. Proof of Theorem 1.1

This section is devoted to the proof of the main result of this manuscript, Theorem 1.1. We consider two initial conditions \( n^0 \) and \( m^0 \) satisfying Assumption 1. Let \( n, m \) the corresponding solutions of (1).
4.1. Contraction estimate for the microscopic distribution. Let \( \varphi_n, \varphi_m : \mathbb{R}_+ \to \mathbb{R}_+ \) the solutions of \( \varphi_n(0) = \varphi_m(0) = 0 \), and

\[
\frac{d\varphi_n}{dt}(t) = \frac{1}{(1 + \alpha I_n(\varphi_n(t)))^2}, \quad \frac{d\varphi_m}{dt}(t) = \frac{1}{(1 + \alpha I_m(\varphi_m(t)))^2}.
\]

Notice that \( 0 \leq I_n, I_m \leq \|a\|_{\infty} \), so that there exists \( C > 0 \) such that

\[
|\varphi_n(t) - t| \leq C \alpha t, \quad |\varphi_m(t) - t| \leq C \alpha t.
\]

Notice also that \( \mathcal{N}(t, x) := n(\varphi_n(t), x) \) satisfies

\[
\partial_t \mathcal{N}(t, x) = T \left( \frac{1 + \alpha a(y_*)}{1 + \alpha I_n(\varphi_n(t))} \mathcal{N}(t, y_*) \right) - \mathcal{N}(t, x),
\]

and for \( 0 \leq t_0 \leq t \),

\[
\mathcal{N}(t, x) = \mathcal{N}(t_0, x)e^{-(t-t_0)} + \int_{t_0}^t T \left( \frac{1 + \alpha a(y_*)}{1 + \alpha I_n(\varphi_n(s))} \mathcal{N}(s, y_*) \right) e^{-(t-s)} ds
\]

and similar properties are satisfied by \( \mathcal{M}(t, x) := m(\varphi_m(t), x) \). We introduce the notation

\[
\beta_n(t) := \frac{\alpha I_n \circ \varphi_n(t)}{1 + \alpha I_n \circ \varphi_n(t)}, \quad \beta_n(t) := \frac{\alpha I_n \circ \varphi_n(t)}{1 + \alpha I_n \circ \varphi_n(t)}.
\]

The notation \( \beta_n \) (and similarly for \( \beta_m \)) is convenient to write the following quantity as a convex combination of two probability measures:

\[
\frac{1 + \alpha a(x)}{1 + \alpha I_n \circ \varphi_n(t)} \mathcal{N}(t, x) = (1 - \beta_n(t)) \mathcal{N}(t, x) + \beta_n(t) \frac{a(x)}{I_n \circ \varphi_n(t)} \mathcal{N}(t, x).
\]

Thanks to [11], we have, for \( t \geq 0 \):

\[
w(\mathcal{N}(t, \cdot), \mathcal{M}(t, \cdot)) = W_2 [\mathcal{N}(t, \cdot), \mathcal{M}(t, \cdot - (Z_m \circ \varphi_m(t) - Z_n \circ \varphi_n(t)))]
\]

\[
\leq W_2 [\mathcal{N}(0, \cdot), \mathcal{M}(0, \cdot - (Z_m \circ \varphi_m(t) - Z_n \circ \varphi_n(t)))] e^{-t}
\]

\[
+ \int_0^t e^{-(t-s)} W_2 \left[ T \left( \left( 1 - \beta_n(s) + \beta_n(s) \frac{a(\cdot)}{I_n \circ \varphi_n(s)} \right) \mathcal{N}(s, \cdot) \right),
\right.
\]

\[
\left. T \left( \left( 1 - \beta_m(s) + \beta_m(s) \frac{a(\cdot - (Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s)))}{I_m \circ \varphi_m(s)} \right) \mathcal{M}(s, \cdot - (Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s))) \right) \right] ds.
\]

To estimate the last term of (66), we notice that

\[
J_1 := W_2 \left[ T \left( \left( 1 - \beta_n + \beta_n(s) \frac{a(\cdot)}{I_n \circ \varphi_n(s)} \right) \mathcal{N}(s, \cdot) \right),
\right.
\]

\[
T \left( \left( 1 - \beta_m(s) + \beta_m(s) \frac{a(\cdot - (Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s)))}{I_m \circ \varphi_m(s)} \right) \mathcal{M}(s, \cdot - (Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s))) \right) \right] \]

\[
\leq \left| (Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s)) - (\tilde{Z}_m \circ \varphi_m(s) - \tilde{Z}_n \circ \varphi_n(s)) \right|
\]

\[
+ W_2 \left[ T \left( \left( 1 - \beta_n(s) + \beta_n(s) \frac{a(\cdot)}{I_n \circ \varphi_n(s)} \right) \mathcal{N}(s, \cdot) \right),
\right.
\]

\[
T \left( \left( 1 - \beta_m(s) + \beta_m(s) \frac{a(\cdot - (\tilde{Z}_m \circ \varphi_m(s) - \tilde{Z}_n \circ \varphi_n(s)))}{I_m \circ \varphi_m(s)} \right) \mathcal{M}(s, \cdot - (\tilde{Z}_m \circ \varphi_m(s) - \tilde{Z}_n \circ \varphi_n(s))) \right) \right],
\]
where \( \tilde{Z}_m \circ \varphi_m(s) \) and \( \tilde{Z}_n \circ \varphi_n(s) \), are defined from \( m(\varphi_m(s), \cdot), n(\varphi_n(s), \cdot) \) by (76). We may then apply the contraction estimate (13) satisfied by the operator \( T \):

\[
J_1 \leq \left| (Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s)) - (\tilde{Z}_m \circ \varphi_m(s) - \tilde{Z}_n \circ \varphi_n(s)) \right|
\]

\[
+ \frac{1}{\sqrt{2}} W_2 \left[ \left( 1 - \beta_n(s) + \beta_n(s) \frac{a(\cdot)}{I_n \circ \varphi_n(s)} \right) \mathcal{N}(s, \cdot), \right.
\]

\[
\left. \left( 1 - \beta_n(s) + \beta_n(s) \frac{a(\cdot) - (\tilde{Z}_m \circ \varphi_m(s) - \tilde{Z}_n \circ \varphi_n(s)))}{I_m \circ \varphi_m(s)} \right) \mathcal{M}(s, \cdot) - (\tilde{Z}_m \circ \varphi_m(s) - \tilde{Z}_n \circ \varphi_n(s)) \right]
\]

(67)

\[
= \left| (Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s)) - (\tilde{Z}_m \circ \varphi_m(s) - \tilde{Z}_n \circ \varphi_n(s)) \right| + \frac{1}{\sqrt{2}} J_2,
\]

We have introduced the notation \( J_2 \) for commodity, and we can decompose it as follows:

\[
J_2 \leq W_2 \left[ \left( 1 - \beta_n(s) + \beta_n(s) \frac{a(\cdot)}{I_n \circ \varphi_n(s)} \right) \mathcal{N}(s, \cdot), \right.
\]

\[
\left. \left( 1 - \beta_n(s) + \beta_n(s) \frac{a(\cdot) - (\tilde{Z}_m \circ \varphi_m(s) - \tilde{Z}_n \circ \varphi_n(s)))}{I_m \circ \varphi_m(s)} \right) \mathcal{M}(s, \cdot) - (\tilde{Z}_m \circ \varphi_m(s) - \tilde{Z}_n \circ \varphi_n(s)) \right]
\]

(68)

Thanks to Lemma 2.3, \( n(t, \cdot) \) and \( m(t, \cdot) \) satisfy the lower bound (37), and thanks to the definition of \( \varphi_n, \varphi_m \) (see (62)), \( \mathcal{N}(t, \cdot) \) and \( \mathcal{M}(t, \cdot) \) satisfy the same estimate. Since additionally \( \mathcal{N}, \mathcal{M} \) satisfy (22), we may use Lemma 3.4 to estimate the first term on the right hand side of (68). We then use Lemma 3.1 to estimate the second term in (68). Since, additionally, \( n(t, \cdot) \) and \( m(t, \cdot) \) satisfy Lemma 2.4, we apply Lemma 3.2 to control the third term of (68). Then, for \( t \geq 0 \),

\[
J_2 \leq \mathcal{O} \left( \mathcal{O}(\alpha |\tilde{Z}_m \circ \varphi_m(s) - \tilde{Z}_n \circ \varphi_n(s)|) + (1 + \mathcal{O}(\sqrt{\alpha})) W_2 \left( \mathcal{N}(s, \cdot), \mathcal{M}(s, \cdot) - (\tilde{Z}_m \circ \varphi_m(s) - \tilde{Z}_n \circ \varphi_n(s)) \right) \right)
\]

\[
+ \mathcal{O}(|\beta_m(s) - \beta_n(s)|).
\]

We can estimate further, thanks to the definition of \( \beta_n, \beta_m \) (see (65)), and the estimate (74) provided by Lemma A.1 (where the assumption \( I_m, I_n \geq \kappa > 0 \) holds thanks to (37)):

\[
|\beta_m(s) - \beta_n(s)| \leq \mathcal{O}(\alpha |I_m \circ \varphi_m(s) - I_n \circ \varphi_n(s)|) \leq \mathcal{O}(\alpha \left( |Z_n \circ \varphi_n(s) - Z_m \circ \varphi_m(s)| + \mathcal{W}(\mathcal{N}(s, \cdot), \mathcal{M}(s, \cdot)) \right).
\]
and the definition of \( w(\cdot, \cdot) \) (see (9)) implies
\[
W_2\left( \mathcal{N}(s, \cdot), \mathcal{M}(s, \cdot - (\tilde{Z}_m(s) - \tilde{Z}_n(s))) \right) = w(\mathcal{N}(s, \cdot), \mathcal{M}(s, \cdot))
+ \left| (Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s)) - (\tilde{Z}_m \circ \varphi_m(s) - \tilde{Z}_n \circ \varphi_n(s)) \right|.
\]

These estimates and (75) lead to
\[
J_2 \leq \mathcal{O}(\alpha) |Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s)| + (1 + \mathcal{O}(\sqrt{\alpha})) w(\mathcal{N}(s, \cdot), \mathcal{M}(s, \cdot)),
\]
and then, using again (75), estimate (67) becomes
\[
J_1 \leq \mathcal{O}(\alpha) |Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s)| + \left( \frac{1}{\sqrt{2}} + \mathcal{O}(\sqrt{\alpha}) \right) w(\mathcal{N}(s, \cdot), \mathcal{M}(s, \cdot)).
\]

Thanks to Assumption 2, the second moments of both \( n(0, \cdot) \) and \( m(0, \cdot) \) are bounded and since \( Z_m \circ \varphi_m - Z_n \circ \varphi_n \) is also bounded thanks to Lemma 2.1,
\[
w(\mathcal{N}(t, \cdot), \mathcal{M}(t, \cdot)) \leq C e^{-t}
+ \int_0^t e^{-(t-s)} \left( \mathcal{O}(\alpha)|Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s)| + \left( \frac{1}{\sqrt{2}} + \mathcal{O}(\sqrt{\alpha}) \right) w(\mathcal{N}(s, \cdot), \mathcal{M}(s, \cdot)) \right) ds,
\]
that is
\[
w(\mathcal{N}(t, \cdot), \mathcal{M}(t, \cdot)) \leq C e^{-t}
+ \int_0^t e^{-(t-s)} \left( C\alpha|Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s)| + \left( \frac{1}{\sqrt{2}} + C\sqrt{\alpha} \right) w(\mathcal{N}(s, \cdot), \mathcal{M}(s, \cdot)) \right) ds,
\]
for some \( C > 0 \). We define \( X : t \in [0, \infty) \mapsto X(t) \) by \( X(0) = 2C \), and
\[
X'(t) = - \left( 1 - \frac{1}{\sqrt{2}} + C\sqrt{\alpha} \right) X(t) + C\alpha|Z_m \circ \varphi_m(t) - Z_n \circ \varphi_n(t)|.
\]

Then \( X \) satisfies
\[
X(t) = C e^{-t} + \int_0^t e^{-(t-s)} C\alpha|Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s)| + \left( \frac{1}{\sqrt{2}} + C\sqrt{\alpha} X(s) \right) ds,
\]
and thanks to the comparison principle,
\[
\forall t \in [0, \infty), \quad w(\mathcal{N}(t, \cdot), \mathcal{M}(t, \cdot)) \leq X(t).
\]

We notice finally that \( |Z_m \circ \varphi_m - Z_n \circ \varphi_n| \) is uniformly bounded thanks to (23), and then \( X'(t) \leq 0 \) as soon as \( X(t) \geq C'\alpha \) (for some constant \( C' > 0 \)). We have then the following rough estimate:
\[
X(t) \leq 2C + C\alpha.
\]

4.2. **Contraction estimate for the macroscopic quantity** \( Z(t) \). We recall the notations (62) and notations \( \mathcal{N}, \mathcal{M} \) introduced in Section 4.1. The goal of this section is to prove a contraction result on the difference \( Z_n \circ \varphi_n - Z_m \circ \varphi_m \). If we multiply (1) by \( x \) and integrate it, we obtain (see (25) for a similar calculation):
\[
(Z_n \circ \varphi_n - Z_m \circ \varphi_m)'(t) = \alpha \left( \frac{1}{1 + \alpha I_n \circ \varphi_n(t)} - \frac{1}{1 + \alpha I_m \circ \varphi_m(t)} \right) \left( \int y a(y) N(t, y) dy - I_n(t) Z_n \circ \varphi_n(t) \right)
+ \frac{\alpha}{1 + \alpha I_m \circ \varphi_m(t)} \left( \int y a(y) N(t, y) dy - \int y a(y) M(t, y) dy \right)
+ \frac{\alpha}{1 + \alpha I_m \circ \varphi_m(t)} \left( I_m \circ \varphi_m(t) Z_m \circ \varphi_m(t) - I_n \circ \varphi_n(t) Z_n \circ \varphi_n(t) \right).
\]
The first term can be estimated as follows, thanks to (74):

\[
\alpha \left( \frac{1}{1 + \alpha I_n \circ \varphi_n(t)} - \frac{1}{1 + \alpha I_n \circ \varphi_m(t)} \right) \left( \int ya(y)n(t, y) \, dy - I_n(t)Z_n \circ \varphi_n(t) \right)
\]

\[\leq O(\alpha^2) \left| I_n \circ \varphi_n(t) - I_m \circ \varphi_m(t) \right| = O(\alpha^2)\left| Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t) \right| + O(\alpha^2)w(N(t, \cdot), M(t, \cdot)).\]

Next we decompose the following factor of the second term on the right hand side of (72):

\[
\int ya(y)N(t, y) \, dy - \int ya(y)M(t, y) \, dy
\]

\[= \int (y - Z_m \circ \varphi_m(t))a(y - Z_m \circ \varphi_m(t))(N(t, y - Z_n \circ \varphi_n(t)) - M(t, y - Z_m \circ \varphi_m(t))) \, dy
\]

\[+ \int ((y - Z_n \circ \varphi_n(t))a(y - Z_n \circ \varphi_n(t)) - (y - Z_m \circ \varphi_m(t))a(y - Z_m \circ \varphi_m(t)))
\]

\[\left( N(t, y - Z_n \circ \varphi_n(t)) - \Gamma_{2\sigma^2}(y) \right) \, dy
\]

\[+ \int ((y - Z_n \circ \varphi_n(t))a(y - Z_n \circ \varphi_n(t)) - (y - Z_m \circ \varphi_m(t))a(y - Z_m \circ \varphi_m(t)))\Gamma_{2\sigma^2}(y) \, dy.
\]

We notice that 
\[g(y) := (y - Z_n \circ \varphi_n(t))a(y - Z_n \circ \varphi_n(t)) - (y - Z_m \circ \varphi_m(t))a(y - Z_m \circ \varphi_m(t))\]

satisfies 
\[g'(y) = a(y-Z_n\circ \varphi_n(t))+(y-Z_n\circ \varphi_n(t))a'(y-Z_n\circ \varphi_n(t)) - a(y-Z_m\circ \varphi_m(t)) - (y-Z_m\circ \varphi_m(t))a'(y-Z_m\circ \varphi_m(t)),\]

and then 
\[\|g'\|_\infty \leq O\left( |Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)| (\|a'\|_\infty + \|a''\|_\infty) \right) \text{ (note that } a \in W^{2,\infty}(\mathbb{R}) \text{ thanks to Assumption 1). If } \pi \text{ is a measure on } \mathbb{R}^2 \text{ with marginals } N(t, \cdot - Z_n \circ \varphi_n(t)) \text{ and } \Gamma_{2\sigma^2}, \text{ then}
\]

\[
\int g(y)(N(t, y - Z_n \circ \varphi_n(t)) - \Gamma_{2\sigma^2}(y)) \, dy = \int g(x) - g(y) \, d\pi(x, y)
\]

\[\leq \|g'\|_\infty \int |x - y| \, d\pi(x, y) \leq \|g'\|_\infty W_2(N(t, \cdot - Z_n \circ \varphi_n(t)), \Gamma_{2\sigma^2}).
\]

We can use this estimate and (9) to obtain

\[
\int ya(y)N(t, y) \, dy - \int ya(y)M(t, y) \, dy = O\left( w(N(t, \cdot), M(t, \cdot)) \right)
\]

\[+ O\left( |Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)| W_2(N(t, \cdot - Z_n \circ \varphi_n(t)), \Gamma_{2\sigma^2}) \right)
\]

\[+ \int \left[ (y - Z_n \circ \varphi_n(t))a(y - Z_n \circ \varphi_n(t)) - (y - Z_m \circ \varphi_m(t))a(y - Z_m \circ \varphi_m(t)) \right] \Gamma_{2\sigma^2}(y) \, dy.
\]

A similar argument can be used to estimate the last term of (72):

\[
I_m \circ \varphi_m(t)Z_m \circ \varphi_m(t) - I_n \circ \varphi_n(t)Z_n \circ \varphi_n(t) = \int Z_m \circ \varphi_m(t)a(y - Z_m \circ \varphi_m(t))M(t, y - Z_m \circ \varphi_m(t)) \, dy
\]

\[= \int Z_m \circ \varphi_m(t)a(y - Z_n \circ \varphi_n(t))N(t, y - Z_n \circ \varphi_n(t)) \, dy
\]

\[= \int Z_m \circ \varphi_m(t)a(y - Z_m \circ \varphi_m(t))(M(t, y - Z_m \circ \varphi_m(t)) - N(t, y - Z_n \circ \varphi_n(t))) \, dy
\]

\[+ \int (Z_m \circ \varphi_m(t)a(y - Z_m \circ \varphi_m(t)) - Z_n \circ \varphi_n(t)a(y - Z_n \circ \varphi_n(t))) \left( N(t, y - Z_n \circ \varphi_n(t)) - \Gamma_{2\sigma^2}(y) \right) \, dy
\]

\[+ \int (Z_m \circ \varphi_m(t)a(y - Z_m \circ \varphi_m(t)) - Z_n \circ \varphi_n(t)a(y - Z_n \circ \varphi_n(t)))\Gamma_{2\sigma^2}(y) \, dy
\]

and if we define 
\[h(y) := Z_m \circ \varphi_m(t)a(y - Z_m \circ \varphi_m(t)) - Z_n \circ \varphi_n(t)a(y - Z_n \circ \varphi_n(t)), \text{ then } h'(y) = Z_m \circ \varphi_m(t)a'(y - Z_m \circ \varphi_m(t)) - Z_n \circ \varphi_n(t)a'(y - Z_n \circ \varphi_n(t)) \text{ and thus } \|h'\|_\infty \leq (\|a'\|_\infty + \|a''\|_\infty) O(|Z_m \circ \varphi_n(t)|,
\]

\[+ O\left( |Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)| W_2(N(t, \cdot - Z_n \circ \varphi_n(t)), \Gamma_{2\sigma^2}) \right)
\]

\[+ \int \left[ (y - Z_n \circ \varphi_n(t))a(y - Z_n \circ \varphi_n(t)) - (y - Z_m \circ \varphi_m(t))a(y - Z_m \circ \varphi_m(t)) \right] \Gamma_{2\sigma^2}(y) \, dy.
\]
\[ \varphi_m(t) - Z_n \circ \varphi_n(t) \]. We have therefore
\[
I_m \circ \varphi(t) Z_m \circ \varphi(t) - I_n \circ \varphi_n(t) Z_n \circ \varphi_n(t) = \mathcal{O}(w(N(t, \cdot), M(t, \cdot)))
\]
\[ + \mathcal{O}(|Z_m \circ \varphi_m(t) - Z_n \circ \varphi_n(t)|) W_2(N(t, \cdot) - Z_n \circ \varphi_n(t), \Gamma_{2\sigma^2}) \]
\[ + \int [Z_m \circ \varphi_m(t) a(y - Z_m \circ \varphi_m(t)) - Z_n \circ \varphi_n(t) a(y - Z_n \circ \varphi_n(t))] \Gamma_{2\sigma^2}(y) \, dy. \]

Thanks to the estimates we have established, \((72)\) becomes
\[
(Z_n \circ \varphi_n - Z_m \circ \varphi_m)'(t) = \alpha^2 \mathcal{O}(|Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)|)
\]
\[ + \alpha \mathcal{O}(w(N(t, \cdot), M(t, \cdot))) + \alpha W_2(N(t, \cdot) - Z_n \circ \varphi_n(t), \Gamma_{2\sigma^2}) \mathcal{O}(|Z_m \circ \varphi_m(t) - Z_n \circ \varphi_n(t)|) \]
\[ + \frac{\alpha}{1 + \alpha I_m \circ \varphi_m(t)} \left[ \int y(a(y)) \Gamma_{2\sigma^2}(y + Z_m \circ \varphi_m(t)) \, dy - Z_n \circ \varphi_n(t) \int a(y) \Gamma_{2\sigma^2}(y + Z_n \circ \varphi_n(t)) \, dy \right] \]
\[ - \frac{\alpha}{1 + \alpha I_m \circ \varphi_m(t)} \left[ \int y(a(y)) \Gamma_{2\sigma^2}(y + Z_m \circ \varphi_m(t)) \, dy - Z_m \circ \varphi_m(t) \int a(y) \Gamma_{2\sigma^2}(y + Z_m \circ \varphi_m(t)) \, dy \right] \]
\[ = \alpha^2 \mathcal{O}(|Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)|) \]
\[ + \mathcal{O}(\alpha) X(t) + \alpha W_2(N(t, \cdot) - Z_n \circ \varphi_n(t), \Gamma_{2\sigma^2}) \mathcal{O}(|Z_m \circ \varphi_m(t) - Z_n \circ \varphi_n(t)|) \]
\[ + \frac{\alpha}{1 + \alpha I_m \circ \varphi_m(t)} [F(Z_n \circ \varphi_n(t)) - F(Z_m \circ \varphi_m(t))], \]

thanks to \((70)\). For \( t \geq t_0 = -C \ln \alpha/\alpha, \) Lemma \(2,1\) implies \( |Z_n \circ \varphi_n(t) - \bar{Z}| \leq C\alpha, \) \( |Z_m \circ \varphi_m(t) - \bar{Z}| \leq C\alpha, \) and \( W_2(N(t, \cdot) - Z_n \circ \varphi_n(t), \Gamma_{2\sigma^2}) \leq C\alpha. \) For \( \alpha > 0 \) small enough, since \( F'(\bar{Z}) < 0 \) (see Assumption 1), we then get
\[
(73) \quad \frac{d}{dt} |Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)| \leq (F'(\bar{Z}) + \mathcal{O}(\alpha)) \alpha |Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)| + \mathcal{O}(\alpha) X(t),
\]
for \( t \geq t_0 = -C \ln \alpha/\alpha. \)

4.3. Conclusion of the proof of Theorem 1.1. Thanks to \((69)\) and \((73)\), for \( t \geq t_0 = -C \ln \alpha/\alpha, \)
\[
\frac{d}{dt} (\sqrt{\alpha} X(t) + |Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)|) \]
\[ \leq -\sqrt{\alpha} \left( 1 - \frac{1}{\sqrt{2}} + C \sqrt{\alpha} \right) X(t) + C \alpha^{3/2} |Z_m \circ \varphi_m(s) - Z_n \circ \varphi_n(s)| \]
\[ + (F'(\bar{Z}) + \mathcal{O}(\alpha)) \alpha |Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)| + \mathcal{O}(\alpha) X(t) \]
\[ \leq (F'(\bar{Z}) + \mathcal{O}(\sqrt{\alpha})) \alpha (\sqrt{\alpha} X(t) + |Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)|). \]

Then \((70)\) and \((71)\) imply
\[
\sqrt{\alpha} w(N(t, \cdot), M(t, \cdot)) + |Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)| \leq \sqrt{\alpha} X(t) + |Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)| \]
\[ \leq C e^{(F'(\bar{Z}) + \mathcal{O}(\sqrt{\alpha})) \alpha (t-t_0)}, \]
and thanks to the definition of \( w \) (see \((9)\)), for \( t \geq t_0, \)
\[
W_2(N(t, \cdot), M(t, \cdot)) \leq w(N(t, \cdot), M(t, \cdot)) + |Z_n \circ \varphi_n(t) - Z_m \circ \varphi_m(t)| \]
\[ \leq \frac{C}{\sqrt{\alpha}} e^{(F'(\bar{Z}) + \mathcal{O}(\sqrt{\alpha})) \alpha (t-t_0)}, \]
and thus, for some constant \( C > 0, \)
\[
W_2(N(t, \cdot), M(t, \cdot)) \leq C e^{(F'(\bar{Z}) + C \sqrt{\alpha}) \alpha t}. \]
If we apply this estimate with \( m(t, \cdot) \equiv \bar{n} \), where \( \bar{n} \) is the steady-state provided by Lemma 2.2, then (63) implies (4). The first inequality in (5) is provided by (21), while the second is implied by (21) and (24). We have then completed the proof of Theorem 1.1.

**Appendix A. Appendix: Estimates on macroscopic quantities**

We derive here technical estimates on \( I_n - I_m \) and the centres of mass of \( Z_n \) and \( Z_m \).

**Lemma A.1.** Let \( a \in W^{1,\infty}(\mathbb{R}) \) compactly supported, and \( \kappa > 0 \). There exists \( C > 0 \) such that if \( m, n \in \mathcal{P}_2(\mathbb{R}) \) satisfy \( I_m, I_n \geq \kappa > 0 \) (where \( I_m, I_n \) are defined from \( m, n \) thanks to (12)), then for \( \alpha \in (0, 1) \),

\[
|I_n - I_m| \leq C \left( |Z_n - Z_m| + w(n, m) \right),
\]

\[
|\tilde{Z}_n - \tilde{Z}_m| - (Z_n - Z_m)| \leq C \alpha \left( |Z_n - Z_m| + w(n, m) \right),
\]

where \( Z_n, Z_m \) defined from \( m, n \) by (10) and \( \tilde{Z}_n, \tilde{Z}_m \) are

\[
\tilde{Z}_n := \int y \left( 1 - \alpha + \frac{a(y)}{I_n} \right) n(y) \, dy,
\]

\[
\tilde{Z}_m := \int y \left( 1 - \alpha + \frac{a(y)}{I_m} \right) m(y) \, dy.
\]

**Proof of Lemma A.1.** We estimate:

\[
I_n - I_m = \int a(y)n(y) \, dy - \int a(y)m(y) \, dy
\]

\[
= \int a(y) - Z_n)n(y - Z_n) \, dy - \int a(y - Z_m)n(y - Z_n) \, dy
\]

\[
+ \int a(y - Z_m)(n(y - Z_n) - m(y - Z_m)) \, dy
\]

\[
\leq \|a'\|_\infty O(|Z_n - Z_m|) + \|a'\|_\infty W_2(n \cdot - Z_n), m(\cdot - Z_m))
\]

\[
\leq O(|Z_n - Z_m|) + O(1)w(n, m),
\]

where we have the property (9) satisfied by \( w(\cdot, \cdot) \). To obtain the second inequality, we proceed as follows:

\[
|\tilde{Z}_n - \tilde{Z}_m| - (Z_n - Z_m)|
\]

\[
\leq \alpha |Z_n - Z_m| + \alpha \left| \int a(y) \left( \frac{n(y)}{I_n} - \frac{m(y)}{I_m} \right) \, dy \right|
\]

\[
\leq \alpha |Z_n - Z_m| + \alpha \left| \int a(y - Z_m) \left( \frac{n(y - Z_n)}{I_n} - \frac{m(y - Z_m)}{I_m} \right) \, dy \right|
\]

\[
+ \alpha \left| \int a(y) \left( \frac{n(y)}{I_n} - \frac{n(y - Z_n + Z_m)}{I_m} \right) \, dy \right|
\]

\[
\leq \alpha |Z_n - Z_m| + O(\alpha)w(n, m) + O(\alpha) \left( \frac{1}{I_n} - \frac{1}{I_m} \right) + \alpha \left| \int (a(y) - a(y + Z_n - Z_m)) \frac{n(y)}{I_n} \, dy \right|
\]

\[
\leq O(\alpha) \left( |Z_n - Z_m| + w(n, m) \right),
\]

which concludes the proof of the lemma.

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