The null bootstrap

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A stable physical system has an energy spectrum that is bounded from below. For quantum systems, the dangerous states of unboundedly low energies should decouple and become null. We propose the principle of nullness and apply it to the bootstrap study of Hermitian and non-Hermitian anharmonic oscillators.

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I. INTRODUCTION

The bootstrap is an ambitious program that attempts to classify and solve physics by basic principles and consistency relations of observables. The governing role of basic principles can be viewed as a reduction at the conceptual level, instead of the usual model-building level. On the other hand, the emphasis on the consistency of all observables, rather than a construction based on some smallest building blocks, is closer to the spirit of emergence.

The first concrete bootstrap program came from early investigations of the strong nuclear force. In the 1960’s, Chew proposed the S-matrix bootstrap to determine the scattering amplitudes by unitarity and some subtle analyticity conditions. This is not a simple program due to the strong coupling nature and became dormant later. In the 1970’s, the bootstrap idea also appeared in the study of conformal field theory (CFT) [1,2]. In two dimensions, the conformal bootstrap program has achieved considerable progress [3,4]. One of the most beautiful results is the complete classification of 2d minimal-model CFTs. They describe the critical behaviour of statistical physics models, such as the Ising, Yang-Lee and Potts models.

The conformal bootstrap in \( d > 2 \) is more challenging and thus not many results had been obtained until the modern numerical approach was proposed in [5]. In this seminal work, the unitarity assumption and the crossing equation are formulated as inequalities, so one can rule out the inconsistent theory space. One of the impressive results is the most precise determination of the 3d Ising critical exponents [6–9]. More recently, the positivity principle [1] has also been applied to the studies of matrix models and quantum mechanical systems [10–24].

In the 2d bootstrap studies, positivity does not play such a dominant role. In particular, since statistical physics models do not need to obey reflection positivity, they can be naturally related to non-unitary quantum systems. A notable example is the 2d bootstrap results of the Yang-Lee edge singularity [25,26]. To study the non-positive systems in \( d > 2 \), we are led to the following question:

Is there a bootstrap principle for non-positive systems?

II. THE PRINCIPLE OF NULLNESS

Let us start with the two-dimensional conformal field theories. They are among the most well-understood nonperturbative quantum systems at strong coupling. A particularly simple class of the 2d CFTs is the minimal models \( \mathcal{M}(p, p') \) [4]. Furthermore, the discrete series of unitary CFTs corresponds to the minimal models with \( |p - p'| = 1 \) [27]. In unitary theories, the norm of a state is either positive or zero. Positivity can be viewed as a basic principle for bootstrapping these unitary systems.

Why does the unitarity condition lead to a subset of minimal models? Since the minimal models are characterized by the existence of many null states, our interpretation is: The would-be dangerous states are removed as they become null and orthogonal to the positive norm states. In this way, the resulting truncated space of states is compatible with the unitarity condition.

However, unitarity is not a necessary property of physical systems. A celebrated example is the non-unitary \( i\phi^3 \) theory describing the critical behaviour of the Yang-Lee edge singularity [28]. In 2d, the corresponding CFT is the minimal model \( \mathcal{M}(5, 2) \). There are many more examples of non-unitary physical systems, such as polymer and percolation

\[ \begin{aligned}
^1 & \text{By positivity constraints, we mean some quantities should be either positive or zero.}
\end{aligned} \]
physics. The non-unitary CFTs can be associated with statistical physics models that are not reflection positive. More generally, the positivity condition can be violated in analytically continued or complex systems.

Is there a principle for these equally important non-positive systems? The 2d minimal models clearly suggest a principle based on nullness, i.e. there exist many null states. In 2d, the principle of nullness is more general than the principle of positivity because the unitary models are a subset of minimal models when the central charge is less than one. In 3d, the phenomena of operator decoupling have also been noticed in the positive bootstrap studies of the Ising CFT [6,7].

As concrete examples, we will consider some interacting quantum mechanical (QM) systems. As mentioned in the introduction, several Hermitian systems have been recently revisited using the positivity principle. In this work, we will use the null bootstrap to solve the anharmonic oscillator with a quartic term. To show its broader applicability, we will also consider a non-Hermitian Hamiltonian, which is closely related to the Yang-Lee edge singularity mentioned above.

For the harmonic oscillator, the null state conditions can be derived from a well-known fact: The vacuum state is annihilated by lowering operators. Should the null states present in more complicated problems? In an algebraic formulation of quantum mechanics, one can construct lowering operators that generate lower energy eigenstates from a given eigenstate. If the energy spectrum is unbounded from below due to these lowering operators, then the stability of the physical system is destroyed. In addition, one can extract an infinite amount of energies by repeatedly lowering the energy of the quantum system. Therefore, we need a termination mechanism for a physical space of quantum states. This can be naturally realized by null state conditions. If the lowering operators are associated with Lie algebra generators, the null state conditions reduce to the highest weight conditions in representation theory.

Roughly speaking, there are two types of null state conditions:

1. The first type is related to the model definitions, such as $\mathcal{O}(H - E)\psi_E = 0$ from the Schrödinger equation and "equations of motion" in field theory.

2. The second type removes the dangerous states in the spectra.

In this work, the first type will be called trivial as they follow directly from the definition of Hamiltonian. An unstable model may have unbounded spectra in all choices of state spaces, so the null state conditions of the first type also remove dangerous states and the division is not completely definite. The two types of null state conditions are unified into the principle of nullness.

III. THE NULL BOOTSTRAP FOR QUANTUM MECHANICS

In the bootstrap approach, we study the relationships of physical observables and extract the predictions from some basic principles. For a QM system, we will assume that the Hamiltonian is diagonalizable and consider the expectation values associated with an energy eigenstate $|\psi_E\rangle$. By definition, they satisfy the consistency relations

$$\langle \psi_E|\mathcal{O}|\psi_E\rangle = E\langle \psi_E|\mathcal{O}|\psi_E\rangle = \langle \psi_E|H\mathcal{O}|\psi_E\rangle, \quad (1)$$

where the Hamiltonian will be assumed to take the form $H = p^2 + V(x)$. We assume the position and momentum operators satisfy the canonical commutation relation $[x, p] = i\hbar$, where $\hbar = 1$. The choice of a state corresponds to a representation of the operator algebra. The space of states is generated by the action of operators on $|\psi_E\rangle$. Below we will also use a shorthand notation for the expectation values associated with the energy eigenstate $\psi_E$

$$\langle \mathcal{O}\rangle_E = \langle \psi_E|\mathcal{O}|\psi_E\rangle. \quad (2)$$

The first equality in (1) can be easily derived from the Schrödinger equation $H|\psi_E\rangle = E|\psi_E\rangle$. To obtain the second equality of (1), we need to specify an inner product. A natural choice will depend on the properties of the Hamiltonian $H$. If $H$ is Hermitian, we will use the standard Hermitian inner product, i.e. $\langle \psi_1|\psi_2\rangle = \int dx \psi_1^*(x)\psi_2(x)$ in the position representation. For a non-Hermitian Hamiltonian with space–time reflection and matrix transposition symmetries, we will instead use the $PT$ inner product, i.e. $\langle \psi_1|\psi_2\rangle_{PT} = \int dx \psi_1^*(x)\psi_2(x)$. In this way, we also have (1) and the orthogonality of eigenstates is preserved in the non-Hermitian case.

To make physical predictions using consistency relations, the bootstrap approach will follow some basic principles. For Hermitian systems, one can use the unitarity condition and the associated positivity constraints. For non-Hermitian systems, the norm of a state is allowed to be negative, then the principle of positivity can be violated. As proposed above, a more general principle is to impose the existence of many null states. They satisfy

$$\langle \psi_{test}|\psi_{null}\rangle = \langle \mathcal{O}_{test}\mathcal{O}_{null}|E = 0, \quad (3)$$

where the test and null states are generated by the action of operators $\mathcal{O}_{test},\mathcal{O}_{null}$ on the energy eigenstate $\psi_E$. For a given null state $|\psi_{null}\rangle = \mathcal{O}_{null}|\psi_E\rangle$, the inner product (3) should

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2 Some properties of non-minimal models are also constrained by the null state conditions.

3 In a more general context, the boundedness requirement on the real part of energies can be associated to different kinds of spectra, such as twists.

4 The state space is not necessarily a Hilbert space because the norm of a state is allowed to be negative.

5 Alternatively, they may be viewed as operator equations or properties of a concrete representation of an abstract operator algebra. Note that the first type is non-trivial in the conformal bootstrap.

6 The notion of null states should be independent of the choice of inner products, but it is useful to introduce inner products to detect them.
vanish for arbitrary test operator \( O_{\text{test}} \). We will not consider the dual version \( \langle \psi_{\text{null}} \rangle | \psi_{\text{test}} \rangle = 0 \), as it is not independent.

To show that the principle of nullness captures both positive and non-positive physics, we will apply it to some classic examples of Hermitian and non-Hermitian quantum systems in Sec. [IV]. Below we discuss the procedure of the null bootstrap in more detail.

### A. Complete vs reduced procedure

In the complete procedure of the null bootstrap, one first determines the spectrum and then imposes the null state conditions on the dangerous eigenstates. In terms of expectation values, the spectrum \( \{ E_k \} \) is defined by the consistent rewriting of (1) into the Schrödinger-like form

\[
\langle \psi_{\text{test}} | (H - E_k) | \psi_k \rangle = \langle O_{\text{test}} (H - E_k) L_k | E \rangle = 0 ,
\]

where the eigenstates are labelled by the abstract index \( k \). Note that \( O_{\text{test}} \) is again arbitrary and the ladder operator \( L_k \) connects different eigenstates, i.e., \( L_k | \psi_k \rangle = | \psi_{k+1} \rangle \). Then we impose the null state conditions on the dangerous states with unboundedly low energies.

As an explicit example, let us consider the simple harmonic oscillator \( H = \hat{x}^2 + \hat{p}^2 \). The rewriting of (1) into (4) leads to the spectrum \( E_k = E + 2k \) where \( k \) is an integer. The lowering operators can be represented by \( L_{-n} = (x + ip)^n \) with \( n = 1, 2, \cdots \). Since a null state has zero norm and \( \langle (x + ip)^n | x + ip \rangle \rangle = \langle 1 \rangle E \prod_{k=0}^{n-1} (E - 2k - 1) \), the null state conditions on the dangerous states with unboundedly low energies imply \( E_n = 2n + 1 \) with \( n = 0, 1, 2, \cdots \). In this way, we solve the exact energy spectrum of the harmonic oscillator using the complete procedure of the null bootstrap.

We can also consider a reduced procedure. In general, the lowering operators are expected to be much more complicated than polynomials of finite degrees, so it is hard to determine the precise spectrum in the first step of the complete procedure. Nevertheless, it is possible to impose the existence of many null states without specifying the explicit expressions of the dangerous states. A crucial point is that certain superposition of many null states can be well approximated by simple polynomials. Therefore, we can carry out the search for null states using a simple ansatz in the \( \eta \) minimization described below.

#### B. \( \eta \) minimization

In the numerical implementation, we will consider truncated null state conditions

\[
\langle \psi_{\text{test}} | \psi_{\text{null}} \rangle = 0 ,
\]

where the candidate null states are truncated to order \( K \) and the test states \( \langle \psi_{\text{test}} \rangle \) are truncated to order \( L \)

\[
\langle \psi_{\text{null}}^{(K)} \rangle = \sum_{m=0}^{K} \sum_{n=0}^{K-m} a_{mn} x^m (ip)^n | \psi_{E} \rangle ,
\]

\[
\langle \psi_{\text{test}}^{(L)} \rangle = \sum_{m=0}^{L} \sum_{n=0}^{L-m} b_{mn} | \psi_{E} \rangle x^m (ip)^n .
\]

Here we assume that the Hamiltonian is expressed in terms of the position and momentum operators \( x, p \), but it is straightforward to generalize (6), (7) to the case with more operators. The orders \( K, L \) are associated with the polynomials in \( x, p \) acting on the energy eigenstate \( \psi_E \).

To have a nontrivial system of constraints, the number of independent constraints should be greater than that of independent parameters. Usually we are not able to solve all truncated null state conditions at the same time, so we write \( \approx \) in (5). Now we introduce the \( \eta \) function

\[
\eta = \sqrt{ \sum_{m=0}^{L} \sum_{n=0}^{L-m} \frac{1}{m! n!} \left( \frac{\partial \langle \psi_{\text{test}}^{(L)} | \psi_{\text{null}}^{(K)} \rangle}{\partial b_{mn}} \right)^2 } ,
\]

which measures the violation of null constraints by summing the squares of different components. The null constraints of higher orders are suppressed by \((m! n!)^{-2} \). To avoid the trivial solution \( a_{mn} = 0 \), we will impose the constraint \( \sum_{m,n} a_{mn} = 1 \). For a given set of independent expectation values, the \( \eta \) minimization can be carried out easily by linear least squares methods.

As a simple example, let us consider again the harmonic oscillator \( H = \hat{x}^2 + \hat{p}^2 \). After solving (1), all the expectation values \( \langle x^m p^n \rangle \rangle \) are functions of \( E \). If we choose \( K = 1 \) and \( L \geq 1 \), the minimization of the \( \eta \) function has a unique zero at \( E = 1 \) and the null state is given by \( \langle \psi_{\text{null}}^{(1)} \rangle = (x + ip) | \psi_{E} \rangle \rangle \). For \( K = 2 \) and \( L \geq 2 \), we find two zeros of \( \eta_{\text{min}} \) and they are at \( E = 1, 3 \). One can see that the \( \eta \) minimization leads to exact results in this simple example. In addition, a larger \( K \) captures more excited states. For \( K = 2 \), we impose an additional constraint \( a_{00} = 0 \) to remove the trivial solution \( O_{\text{null}}^{\text{trivial}} = H - E \).

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\(^{10}\) One may consider more sophisticated weights.

\(^{11}\) In the conformal bootstrap, the \( \eta \) minimization was proposed in [29] as a more systematic formulation of the truncation approach initiated in [30]. The truncation of operator product expansion can be reinterpreted as an approximate null operators. As shown in [31], one can also rule out the inconsistent solutions using the \( \eta \) minimization. Recently, the \( \eta \) minimization was implemented with reinforcement-learning algorithms [32, 33].

\(^{12}\) We have imposed the constraint \( \sum_{m,n} a_{mn} = 1 \). In a different weighted sum, one may also find solutions with \( E = -1, -2, -3, \cdots \) which correspond to a spectrum bounded from above. In the dual case, the null conditions are for the positive energy eigenstates. From the perspective of wave functions, the boundary conditions are imposed at \( x \rightarrow \pm i \infty \). It would be fascinating to further study the relations between the choices of coordinates and null state conditions.
In general, an energy eigenstate can correspond to a non-trivial lowering operator, so it is much more complicated than finite degree polynomials. Can we still study the null state conditions using low order polynomials? The crucial point mentioned earlier is that the null states form a high-dimensional subspace for physical solutions. In the $\eta$ minimization, we are allowed to consider a linear combination of all of them and the polynomial ansatz becomes a good approximation. The minimized $\eta$ function is particularly small around the physical solutions, so we can extract the precise predictions using the reduced procedure of the null bootstrap.

Below we apply the simple $\eta$ minimization to Hermitian and non-Hermitian anharmonic oscillators. We obtain high precision results despite severe truncations.

IV. APPLICATION TO ANHARMONIC OSCILLATORS

In this section, we will study the Hamiltonian

$$H = p^2 + V(x)$$

(9)

using the null bootstrap. We will consider two different potentials $V(x)$.

A. $V(x) = x^2 + gx^4$

Let us add a quartic term to the harmonic oscillator. The Hamiltonian becomes

$$H = p^2 + x^2 + gx^4,$$

(10)

where we will set $g = 1$ in the concrete computation. This simple model can be viewed as the $0 + 1$ dimensional version of the $\phi^4$ quantum field theory and has long served as a testing problem for new field theory methods [13].

Using the consistency relations [1], we can express $\langle x^m p^n \rangle_E$ in terms of $\{ E, (x^2)_E, (1)_E \}$, where $m, n = 0, 1, 2, \cdots$ [13]. The expectation value $(1)_E$ determines the normalization of the inner product and we will set $(1)_E = 1$ for convenience. The null bootstrap is not sensitive to this normalization convention as long as $(1)_E \neq 0$. For instance, the results remain essentially the same even if we set $(1)_E = -1$.

In the null bootstrap, we minimize the $\eta$ function with $K = 1, 2, 3, 4$ and $L = K + 2$. [13] To remind the reader, the candidate null operator $O_{\text{null}}$ is approximated by a degree-$K$ polynomial in $x, p$. The degree of the test operator $O_{\text{test}}$ is given by $L$. As long as the system is over-constrained, we can remove some high order constraints and the results will be more precise, but we do not present them for the sake of simplicity. We can also consider more constraints by choosing a larger $L$. The results are qualitatively the same if $L$ is not significantly greater than $K$. In this way, we verify that the local minima are not coincidences and the null subspace is indeed of high dimensions.

In Fig. 1 we present the $K = 2$ results of $\log_{10} \eta_{\text{min}}$ as a function of the two independent parameters, i.e. $E$ and $(x^2)_E$. The two local minima correspond to the ground state and the first excited state. For a given $E$, the $\eta$ minimization with respect to $(x^2)_E$ is quite straightforward, so we present the $\eta_{\text{min}}$ at different $E$ with $K = 1, 2, 3$ in Fig. 2. As we increase $K$, more exited states are captured by the $\eta$ minimization. We further summarize the numerical predictions in Table I and II. One can see that the errors in the $E$, $(x^2)_E$ results decrease rapidly with $K$. Here the reference results are derived from the diagonalization of the Hamiltonian in a truncated basis of 60 harmonic oscillator eigenstates. As in the case of the harmonic oscillator, we set $\omega_0 = 0$ at $K = 4$ in order to avoid the trivial solution $C_{\text{null}}^{\text{trivial}} = H - E$.

![Image](https://via.placeholder.com/150)

**FIG. 1.** The $\eta_{\text{min}}$ landscape for the Hermitian Hamiltonian $H = p^2 + x^2 + x^4$ with $K = 2, L = 4$. The two local minima correspond to the ground state and the first excited state. We plot $\log_{10} \eta_{\text{min}}$ as a function of the energy $E$ and the expectation value $(x^2)_E$. As a height function, $\eta_{\text{min}}$ increases rapidly outside of the presented region. We use black color to denote the region with $\eta_{\text{min}} < 10^{-2}$.

| $\Delta E_{\text{min}}(K)$ | $K = 1$ | $K = 2$ | $K = 3$ | $K = 4$ |
|--------------------------|--------|--------|--------|--------|
| $n = 0$                  | $-1 \times 10^{-3}$ | $-2 \times 10^{-3}$ | $-4 \times 10^{-10}$ | $-7 \times 10^{-12}$ |
| $n = 1$                  | $3 \times 10^{-3}$   | $-3 \times 10^{-5}$ | $2 \times 10^{-11}$  | $6 \times 10^{-7}$   |
| $n = 2$                  | $5 \times 10^{-6}$   | $6 \times 10^{-7}$  | $1 \times 10^{-7}$   | $1 \times 10^{-7}$   |

**TABLE I.** The null bootstrap predictions for the low-lying energies of anharmonic oscillator with the Hermitian potential $V(x) = x^2 + x^4$ and $L = K + 2$. We present the first digit of the error $\Delta E_{\text{min}}(K) = E_{\text{min}}^{(K)} - E_n$. The reference energies $E_n = 1.39235164153029$, $E_1 = 4.648812704212$, $E_2^4 = 8.65504995776$ and $E_3^4 = 13.1568038980$ are computed from diagonalizing the Hamiltonian with size $60 \times 60$ in the basis of harmonic oscillator eigenfunctions.

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13 For parity symmetric solutions, we have $\langle x^m p^n \rangle_E = 0$ when $m + n$ is an odd number. It would also be interesting to consider the analytic continuation of $m, n$ to the non-integer or complex domain.

14 In this example, the system is under-constrained if $L = K + 1$. 
To be more precise, some matrices should be positive semidefinite as null states are $K = 1, 2, 3$. We carry out the minimization with respect to $\langle x^2 \rangle$ as well. As we increase $K$, the minima give sharper predictions and capture more excited states. Here we change the weight in Ref. [8] to one.

| $\Delta (x^2)^n(K)$ | $K = 1$ | $K = 2$ | $K = 3$ | $K = 4$ |
|----------------------|---------|---------|---------|---------|
| $n = 0$              | $-1 \times 10^{-2}$ | $-1 \times 10^{-4}$ | $2 \times 10^{-9}$ | $1 \times 10^{-11}$ |
| $n = 1$              | $1 \times 10^{-3}$ | $-1 \times 10^{-6}$ | $1 \times 10^{-11}$ |         |
| $n = 2$              | $-3 \times 10^{-6}$ | $6 \times 10^{-8}$ |         |         |
| $n = 3$              |         | $2 \times 10^{-8}$ |         |         |

TABLE II. The null bootstrap predictions for $\langle x^2 \rangle_n$ with the Hermitian potential $V(x) = x^4 + x^4$ and $L = K + 2$. We present the first digit of the error $\Delta (x^2)^n(K) = \langle x^2 \rangle_n^K - \langle x^2 \rangle_n^K$. The reference values $\langle x^2 \rangle_0^5 = 0.3058136507176$, $\langle x^2 \rangle_1^5 = 0.8012505955411$, $\langle x^2 \rangle_2^5 = 1.155440519200$ and $\langle x^2 \rangle_3^5 = 1.4675232154$ are computed using approximate wave functions from the diagonalization of the Hamiltonian with size $60 \times 60$ in the harmonic oscillator basis.

Instead of rigorous bounds in the positive bootstrap, the null bootstrap gives an $\eta_{\text{min}}$ landscape. At the price of rigorlessness, the null bootstrap results are more precise than those of the positive bootstrap. For instance, the size of the $E_0$ error bar from the rigorous island is $\delta E_0^K = \infty$, $3 \times 10^{-2}$, $1 \times 10^{-4}$, $5 \times 10^{-10}$ with $K = 1, 2, 3, 4$. In the positive bootstrap, one asks if the positivity condition is violated and there is no smooth change in the answer. Then it requires some care to determine the boundaries of the positive regions, especially when an isolated region has a tiny size. In the null bootstrap, it is relatively easy to find the local minima thanks to the smooth $\eta_{\text{min}}$ landscape.

Before moving to the non-Hermitian case, let us explain from a different perspective why the positive bootstrap results can be derived from the null bootstrap. In the positive boot-

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15 In Ref. [8], the corresponding results are more precise because they define $K$ as $\text{max}(m, n)$. The degrees of their polynomials are twice of ours.

16 To be more precise, some matrices should be positive semidefinite as null states are allowed.

17 But one might introduce a measure of positivity violation based on the dangerous states, such as the lowest eigenvalues of the should-be-positive matrices. See also the navigator function in the conformal bootstrap.

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Since we propose that nullness is a more general principle than positivity, we should apply the null bootstrap to a non-positive system. Here we consider a simple non-Hermitian Hamiltonian

$$H = p^2 + igx^3,$$

which is $\mathcal{PT}$ symmetric, i.e. invariant under the space–time reflection. We will set $g = 1$ in the concrete computation.

In the textbooks of quantum mechanics, a standard axiom is that a fundamental Hamiltonian is Hermitian. The Hermiticity assumption ensures that the energy spectrum is real and the Hermitian norm is non-negative. Furthermore, the eigenfunctions are guaranteed to be orthogonal and complete.

For non-Hermitian Hamiltonians, the Hermitian norm does not have a definite sign, but Bessis and Zinn-Justin noticed that the Hamiltonian [11] may have a real energy spectrum. Their unpublished numerical study was motivated by the properties of the Yang-Lee edge singularities, whose critical behaviour is described by the $i\phi^3$ quantum field theory. It was later shown by Bender and Boettcher that the energy spectrum is real for a large class of non-Hermitian Hamiltonians with unbroken $\mathcal{PT}$ symmetry [38]. Curiously, an imaginary cubic coupling constant also appeared in the study of Reggeon field theory [39].

When $g$ is small, the Hermitian case [10] can be viewed as a deformation of the harmonic oscillator, so it seems natural to find qualitatively similar results. By contrast, the non-Hermitian case [11] has no mass term and cannot be viewed...
As a deformed harmonic oscillator. Without knowing much about the lowering operators and spectra, it is a leap of faith to apply the null bootstrap to the non-Hermitian system \( \text{(11)}. \) Surprisingly, the results of the \( \eta \) minimization again exhibit a similar pattern.

After solving the consistency relations \( \text{(1)} \), we can express \( \langle x^{m}p^{n}\rangle_{E} \) in terms of \( \{E, \langle x \rangle_{E}, \langle 1 \rangle_{E}\} \). We again set \( \langle 1 \rangle_{E} = 1 \). Note that this normalisation does not guarantee the non-negativity of the \( \mathcal{PT} \) norm. As in the Hermitian example, the \( \eta \) minimization of the non-Hermitian Hamiltonian \( \text{(11)} \) also captures more excited states as \( K \) increases. In Table III we present the \( \eta \) minimization results of the low-lying energy spectrum. Again, the precision increases rapidly with the cutoff parameter \( K \). We set \( a_{00} = 0 \) at \( K = 3 \) to avoid the trivial solution \( \mathcal{O}_{\text{null}}^{\text{trivial}} = H - E \).

\[
\begin{array}{|c|c|c|c|}
\hline
\Delta E_{n}^{(K)} & K = 1 & K = 2 & K = 3 \\
\hline
n = 0 & 4 \times 10^{-4} & -1 \times 10^{-6} & 1 \times 10^{-11} \\
n = 1 & 2 \times 10^{-3} & -3 \times 10^{-9} \\
n = 2 & -1 \times 10^{-4} \\
\hline
\end{array}
\]

Table III. The null bootstrap predictions for the low-lying energies of anharmonic oscillator with the non-Hermitian Hamiltonian \( H = p^{2} + ix^{2} \) and \( L = K + 1 \). The errors are defined as \( \Delta E_{n}^{(K)} = E_{n}^{(K)} - E_{n}^{*} \) and we present only the first digits. The reference energies \( E_{n}^{0} = 1.156267072, E_{1}^{0} = 4.109228752, E_{2}^{0} = 7.562273854 \) were obtained with the Runge-Kutta method in [40]. A more precise value \( E_{0}^{0} = 1.156267071988 \) is computed by diagonalizing the Hamiltonian with size 60 \( \times 60 \).

Above, we successfully derived the real spectrum of the complex Hamiltonian \( \text{(11)} \) using the \( \eta \) minimization. A curious question is: Are there \( \eta \) minima with complex energies? It turns out that there indeed exist conjugate pairs of local minima with complex \( E \).24 These minima persist as we increase \( L - K \), so they are not numerical coincidences. They might be related to other boundary conditions for the wave functions, but we do not fully understand the physical implications of these complex energy solutions.

V. Ladder Operators and Matrix Elements

In the previous section, we have derived the precise values of low-lying spectra \( E_{n} \) and expectation values \( \langle x^{2} \rangle_{E} \). If we require the \( g \rightarrow 0 \) limit of \( \langle x^{m}p^{n}\rangle_{E} \) is regular, we obtain constraints on the small \( g \) expansion of \( E_{n} \) and \( \langle x^{2} \rangle_{E} \), so the number of independent parameters is reduced.

24 They are connected by analytic continuation in the power of \( ix \).

25 This suggests the existence of ladder operators but with the form \( \mathcal{H}_{a_{\pm}} = f_{\pm}(H) a_{\pm} \), where \( f_{\pm}(H) \) and \( f_{-}(H) \) are functions of the Hamiltonian, so the energy spacing is not constant. Furthermore, one may generate an operator algebra by \( a_{\pm} \) with \( [a_{-}, a_{+}] = \hbar \) and define the Hamiltonian \( \mathcal{H} \) in terms of \( a_{\pm} \).

26 This is also related to the fact that we do not introduce the complex conjugate of \( E \) in [4].

<ref>\langle x^{2} \rangle_{E} \end{ref>.

\[ \langle x^{2} \rangle_{E} \] using the null bootstrap. Can we also compute the non-diagonal matrix elements?28 To address this question, let us remind the reader that we have introduced the ladder operators in Sec. III, which generate different energy eigenstates from a given one

\[ |\psi_{E}'\rangle = L_{E'}E_{E}\langle \psi_{E} \rangle. \]

If one can determine the expressions of ladder operators, then the matrix elements are not independent parameters

\[ \langle \psi_{E'}|x^{m}p^{n}|\psi_{E}\rangle = \langle x^{m}p^{n}L_{E'}|E_{E}\rangle \]

We will again approximate the ladder operators by finite degree polynomials

\[ L_{E'}^{(M)} = \sum_{m=0}^{M} \sum_{n=0}^{M-m} c_{mn} x^{m}(ip)^{n}. \]

The coefficients \( c_{mn} \) are determined by the \( \eta \) minimization of the null constraints28

\[ \langle \psi_{\text{null}}^{(L)}|(H - E')L_{E'}^{(M)}|\psi_{\text{null}} \rangle \approx 0, \]

which is an approximate version of the Schrödinger-like equation [4]. We will choose a normalization such that the absolute norm of \( |\psi_{E}'\rangle \) is one and the signs of the lowest non-diagonal matrix elements are positive.

In Fig IV we present some results of the quartic potential based on the \( K = 4 \) results. Naturally, the low-lying observables are more precise than the high-lying ones. For \( M < K \), one can see that the results improve with the approximation order \( M \). If \( M > K \), the precision will decrease because the input parameters were based on the order-\( K \) truncation.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\Delta[|x^{m}|n]^{(M)} & m = 1 & m = 2 & m = 3 & m = 4 \\
\hline
n = 1, M = 1 & 4 \times 10^{-6} & 1 \times 10^{-5} & & \\
n = 2, M = 2 & 3 \times 10^{-7} & 1 \times 10^{-6} & & \\
n = 1, M = 3 & 1 \times 10^{-11} & 2 \times 10^{-11} & & \\
\hline
\end{array}
\]

Table IV. The null bootstrap predictions for the low-lying matrix elements of anharmonic oscillator with the Hermitian potential \( V(x) = x^{2} + x^{4} \). We use a shorthand notation \( |0|x^{m}|n\rangle_{(M)} = \langle \psi_{E_{n}}|x^{m}\psi_{E_{n}} \rangle. \) The differences are defined as \( \Delta[|x^{m}|n]^{(M)} = \langle 0|x^{m}|n\rangle^{(M)} - \langle 0|x^{m}|n\rangle^{*} \). The reference values from the Hamiltonian diagonalization are \( \langle 0|x^{2}|1\rangle^{*} = 0.552565959314, \langle 0|x^{2}|2\rangle^{*} = 0.456180405462, \langle 0|x^{2}|2\rangle^{*} = 0.406699817396, \langle 0|x^{2}|2\rangle^{*} = 0.6227044561. \) Some matrix elements should vanish due to the unbroken parity symmetry of the energy eigenstates. Their numerical values are of order \( 10^{-11} \) or smaller, which are not presented here.

Since the parity symmetry of (10) is unbroken, the ladder operators have numerically small coefficients for the parity

27 This question was inspired by the on-going bootstrap study by Hongbin Chen, Colin Nancarrow and Yuan Xin.

28 The inner product \( \langle \psi_{\text{null}}^{(L)}|(H - E')L_{E'}^{(M)}|\psi_{\text{null}} \rangle \) is replaced by \( \langle \psi_{\text{null}}^{(L)}|(H - E')L_{E'}^{(M)}|\psi_{\text{null}} \rangle \).
In 2d CFTs, they are related to the chiral algebra, such as the Virasoro symmetry. For free field theories, they are also associated with the higher spin symmetry. There may exist a nonlocal, higher generalization, closely related to the Regge continuation. See [43] for the categorical interpretation of $O(N)$ symmetry with non-integer $N$.

In the end, we list some questions that we find interesting:

- Can a physical operator algebra has a bounded-from-below representation without resorting to some null state conditions? It would be interesting to further study the algebraic properties of the null and ladder operators.
- Our bootstrap study is based on the operator-algebra formulation and we do not use the explicit expressions of the eigenfunctions. What are the algebraic counterparts of many important concepts in the wave function formulation, such as boundary conditions, Stokes sectors and WKB method?
- The non-Hermitian models can be extended to non-integer powers. Can we also study them from the bootstrap perspective? This will require the introduction of non-integer "power" to the operator algebra. This is closely related to the analyticity of the expectation values.
- Can a physical operator algebra has a bounded-from-below representation without resorting to some null state conditions? So the vacuum state is NOT annihilated by any lowering operators?
- Projection techniques play a significant role in both high energy physics and condensed matter physics, such as the GSO projection in superstring theories and projective construction of quantum spin-liquid states [45]. Can we view them as special cases of the null principle and make more general classifications?

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29 On may further impose stability under deformations of the Hamiltonian.
30 In [42], a connection was proposed between ordinary differential equations and integrable models, based on the similarity of functionals relations in the two contexts.
31 In 2d CFTs, they are related to the chiral algebra, such as the Virasoro symmetry. For free field theories, they are also associated with the higher spin symmetry. There may exist a nonlocal, higher generalization, closely related to the Regge continuation. See [43] for the categorical interpretation of $O(N)$ symmetry with non-integer $N$.
32 See [44] for a discussion about the semiclassical limit $\hbar \to 0$ in the bootstrap context.
33 A candidate Hamiltonian is $H = (a_- a_+)^2$ with $[a_- , a_+] = \hbar$.
34 For 2d minimal-model CFTs, the various global symmetries are consequences of the null classification.
[42] P. Dorey and R. Tateo, “Anharmonic oscillators, the thermodynamic Bethe ansatz, and nonlinear integral equations,” J. Phys. A 32, L419-L425 (1999) doi:10.1088/0305-4470/32/38/102 [arXiv:hep-th/9812211 [hep-th]].

[43] D. J. Binder and S. Rychkov, “Deligne Categories in Lattice Models and Quantum Field Theory, or Making Sense of O(N) Symmetry with Non-integer N,” JHEP 04, 117 (2020) doi:10.1007/JHEP04(2020)117 [arXiv:1911.07895 [hep-th]].

[44] F. Gliozzi, J. Scherk and D. I. Olive, “Supersymmetry, Supergravity Theories and the Dual Spinor Model,” Nucl. Phys. B 122, 253-290 (1977) doi:10.1016/0550-3213(77)90206-1

[45] X. G. Wen, “Quantum field theory of many-body systems: From the origin of sound to an origin of light and electrons.”