Deconfinement transition in 2+1-dimensional SU(4) lattice gauge theory

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A missing piece is added to the Svetitsky-Yaffe conjecture. The spin model in the same universality class as the (2 + 1)d SU(4) theory, the 2d Ashkin-Teller model, has a line of continuously varying critical exponents. The exponents measured in the gauge theory correspond best to the Potts point on the Ashkin-Teller line.

1. FRAMEWORK

The SU(4) Yang-Mills theory in (2 + 1) dimensions has a special status: its critical exponents at the finite temperature deconfinement transition are not specified by the Svetitsky-Yaffe (S-Y) conjecture \cite{1}. By measuring them, we can learn about the couplings of the equivalent spin model, i.e. of the effective Polyakov loop model.

The S-Y conjecture says that, if the deconfinement transition of the gauge theory is second-order, then it is in the same universality class as the spin model having the symmetry-breaking pattern of the Polyakov loop.

Assume that the (2 + 1)d SU(4) deconfinement transition is second-order – we will investigate this issue numerically next. Then the equivalent 2d spin model has $Z(4)$ spins, with orientation $\{0, \pm \pi/2, \pi\}$ corresponding to the 4 Polyakov loop sectors and the 4 perturbative vacua of the gauge theory. A nearest-neighbour Hamiltonian, which gives the same critical properties as the SU(4) theory according to S-Y, will have 3 possible energy levels for each link: $E_0$ for parallel spins $\uparrow\uparrow$, $E_1$ for perpendicular spins $\uparrow\rightarrow$, and $E_2$ for anti-parallel spins $\uparrow\downarrow$. Remarkably, the critical exponents of this spin system are known to vary continuously with the ratio $\rho \equiv \frac{E_2 - E_1}{E_1 - E_0}$ and the corresponding couplings of the Hamiltonian. Therefore, the S-Y conjecture does not tell us what the SU(4) critical exponents are. Measuring them fixes the couplings of the effective Hamiltonian.

A $Z(4)$ spin model with 3 energy levels per link is called a symmetric Ashkin-Teller model \cite{2}. It is the symmetric case $J = J'$ of the Ashkin-Teller model, which describes 2 coupled Ising systems $\{\sigma_i, \tau_i\}$ on a square lattice, with Hamiltonian:

$$H = -J \sum_{<ij>} \sigma_i \sigma_j - J' \sum_{<ij>} \tau_i \tau_j - K \sum_{<ij>} \sigma_i \tau_i \sigma_j \tau_j$$

(1)

When $J = J'$, it can be rewritten

$$H = -2J \sum_{<ij>} \cos(\theta_i - \theta_j) - K \sum_{<ij>} \cos(2(\theta_i - \theta_j))$$

(2)

where now $\theta_i = \{0, \pm \pi/2, \pi\}$ represents the orientation of an SU(4) Polyakov loop. The ratio of excitation energies mentioned above, $\rho \equiv \frac{E_2 - E_1}{E_1 - E_0}$, is then equal to $\frac{J-K}{J+K}$. Two special cases arise:

- the 4-state Potts case: $K = J \Rightarrow \rho = 0$. No additional energy is needed to make an orthogonal spin pair anti-parallel.
- the Ising$^2$ case: $K = 0 \Rightarrow \rho = 1$. It takes twice as much energy to flip one spin in an aligned pair, as to make it orthogonal. The name comes from the equivalent description as two decoupled Ising systems in Eq. (1).

On physical grounds, one expects $0 \leq \rho \leq 1$, so that the two cases above are limiting cases. As $\rho$ varies in the interval, the critical exponents also do, between the following limits:

|        | $\gamma^*$ | $\gamma'\gamma^*$ | $\beta^*\beta^*$ | $\beta^*\gamma^*$ | $\alpha\gamma^*$ | $\nu$ |
|--------|------------|--------------------|------------------|------------------|------------------|------|
| Potts  | 7/4        | 7/4                | 1/8              | 1/8              | 1                | 2/3  |
| Ising$^2$ | 7/4        | 3/2                | 1/8              | 1/4              | 0                | 1    |
where $\gamma'$ and $\beta'$ are susceptibility and magnetization exponents for spins $e^{2i\theta}$. Since $\gamma/\nu$ and $\beta/\nu$ are independent of $\rho$, these known exponents provide crosschecks on our systematic errors. This is particularly valuable here, because of large, logarithmic finite-size corrections known to make the numerical extraction of Ashkin-Teller exponents quite challenging.

2. NUMERICAL STUDY

We simulate the $(2 + 1)d$ $SU(4)$ gauge theory using the Wilson plaquette action on a cubic grid of size $L^2 \times N_t$, with $N_t = 2, 3, 4$ and $L$ up to 40. To accelerate Monte Carlo evolution, we use as elementary update a mixture of pseudo-heatbath (in all 6 $SU(2)$ subgroups) and overrelaxation in the full $SU(4)$ group. The latter requires a similar amount of work to 6 $SU(2)$ overrelaxation steps, but gives a larger step size. As a result, the number of sweeps needed to decorrelate the Polyakov loop is reduced by a factor $\sim 3$. Up to $10^6$ sweeps are performed on each volume. We analyze results for various couplings $\beta$ near criticality together with multihistogram reweighting.

This model gives a clear first-order transition on $N_t = 1$ lattices. On $N_t = 2$, an old Monte Carlo study found a second-order deconfinement transition. A likely explanation of these first-order transitions is the vicinity of a sharp, bulk crossover at $\beta \sim 13.5$, whereas the $N_t = 1$ and 2 transitions occur at $\beta_c \approx 8.67$ and 14.87 respectively.

This forced us to consider $N_t = 3$ lattices, with correspondingly higher $\beta_c$. No double-peak structure is visible on the largest size considered ($L = 32$). However, the $L \rightarrow \infty$ extrapolation of the Binder cumulant still misses 2/3 by a small but somewhat significant amount. It may well be that the transition is still weakly first-order. Our current $N_t = 4$ results are consistent with a second-order transition, but do not reach as large volumes yet. Therefore, we present the critical exponents analyzed from the $N_t = 3$ data.

The effect of logarithmic corrections to scaling can be seen in Fig. 2. They limit the usefulness of accurate, small-size data.

The bulk crossover at $\beta \sim 13.5$ has a more pernicious effect. It dominates the behaviour of the

\[ C_{\text{max}}(L) \text{ vs } L^{\frac{2}{\nu}} \]

Figure 2. Check of magnetic susceptibility exponent $\frac{2}{\nu} = \frac{7}{4}$, without (left) and with (right) logarithmic corrections to scaling.

Binder cumulant $1 - \frac{\langle O^4 \rangle}{3\langle O^2 \rangle^2}$ for the plaquette extrapolates as $L \rightarrow \infty$ to a value below 2/3.

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Figure 3. $\frac{2}{\nu}$ determined from the scaling of the specific heat, without (left) and with (right) subtraction of smooth background caused by nearby crossover.
specific heat over the singular piece, leading to an exponent $\alpha \approx 0$ (Fig. 3 left). We subtracted the bulk specific heat, measured on a $4^4$ lattice, to isolate the singular contribution. The exponent then becomes consistent with the Potts case (Fig. 3 right).

All other exponents also favor the Potts case. Fig. 4 shows the scaling of $dU/d\beta$, where $U$ is the Polyakov loop Binder cumulant. $\nu$ remains consistent with the Potts value 2/3, even allowing for a large ($> 2\sigma$) variation of $\beta_c$ by $\pm 0.01$ and logarithmic finite-size corrections.

Agreement among all observables is shown in Fig. 5. The pseudo-critical $\beta$’s obtained on various volumes all extrapolate to a common thermodynamic value $\beta_c = 20.4356(41)$, with corrections of the form $aL^{-\frac{2}{3}} (1+b/L)$, and $\nu = 2/3$, the Potts value.

3. CONCLUSION

The unexpected $N_t = 2$ first-order transition, together with expected but unwelcome logarithmic finite-size corrections, have turned this simple problem into a numerically challenging one.

Our results need to be made more precise, by simulating larger volumes. They also must be confirmed on a finer lattice, $N_t \geq 4$, where the transition presumably is second-order. In practice, we can never exclude a weak first-order transition. What we want is to reach physical volumes large enough to reliably determine effective critical exponents, but small enough compared to the possibly finite correlation length at criticality.

At this stage, the set of measured exponents favors the Potts case, which would be perhaps the most naive guess.

Obviously, much remains to be done. The effort is worthwhile because of the insight gained from this simple case. Interest in Polyakov loop models has been growing, because they can supplement dimensional reduction and provide an effective description at temperatures nearer to $T_c$ than the latter. The difficulty is to determine their many couplings, especially for large $-N$ $SU(N)$ theories. Lessons learnt here for $SU(4)$ may help choose among the simplest $SU(N)$ generalizations, like $N-$states Potts or $Z(N)$ clock models. In that respect, our preliminary result is perhaps surprising, because an effective $N$-state Potts model for $(3 + 1)d$ $SU(N)$ would lead to $k$-string tensions independent of $k$ at $T_c$.

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