Instantons in six dimensions and twistors

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Abstract

We consider homogeneous spaces SU(4)/U(3) and Sp(2)/Sp(1)×U(1), both representing the complex projective space $\mathbb{CP}^3$, as well as the natural twistor spaces SU(4)/U(2)×U(1) and Sp(2)/U(1)×U(1) fibred over the above cosets with $\mathbb{CP}^2$ and $\mathbb{CP}^1$ fibres, respectively. We describe the relation between Hermitian Yang-Mills connections (instantons) on complex vector bundles over $\mathbb{CP}^3$ and holomorphic bundles over the associated twistor spaces.
1 Introduction and summary

Let us consider an oriented real four-manifold $X^4$ with a Riemannian metric $g$ and the principal bundle $P(X^4, SO(4))$ of orthonormal frames over $X^4$. The (metric) twistor space $\text{Tw}(X^4)$ of $X^4$ can be defined as an associated bundle [1]

$$\text{Tw}(X^4) = P \times_{SO(4)} SO(4)/U(2) \quad (1.1)$$

with the canonical projection $\text{Tw}(X^4) \to X^4$. This space parametrizes the almost complex structures on $X^4$ compatible with the metric $g$ (almost Hermitian structures). It was shown in [1, 2] that if the Weyl tensor of $(X^4, g)$ is anti-self-dual then the almost complex structure on the twistor space $\text{Tw}(X^4)$ is integrable. Furthermore, it was proven that the rank $r$ complex vector bundle $E$ over $X^4$ with an anti-self-dual gauge potential $A$ over such $X^4$ lifts to a holomorphic bundle $\hat{E}$ over complex twistor space $\text{Tw}(X^4)$ [1, 3].

The essence of the canonical twistor approach is to establish a correspondence between four-dimensional space $X^4$ (or its complex version) and complex twistor space $\text{Tw}(X^4)$ of $X^4$. Using this correspondence, one transfers data given on $X^4$ to data on $\text{Tw}(X^4)$ and vice versa. In twistor theory one considers holomorphic objects $h$ on $\text{Tw}(X^4)$ (Čech cohomology classes, holomorphic vector bundles etc.) and transforms them to objects $f$ on $X^4$ which are constrained by some differential equations [1]-[4]. Thus, the main idea of twistor theory is to encode solutions of some differential equations on $X^4$ in holomorphic data on the complex twistor space $\text{Tw}(X^4)$ of $X^4$.

The twistor approach was recently extended to maximally supersymmetric Yang-Mills theory on $\mathbb{C}^6$ [5]. It was also generalized to Abelian [6, 7] and non-Abelian [8] holomorphic principal 2-bundles over the twistor space $Q_6' \subset \mathbb{C}P^7 \setminus \mathbb{C}P^3$, corresponding to self-dual Lie-algebra-valued 3-forms on $\mathbb{C}^6$. These forms are the most important objects needed for constructing (2,0) superconformal field theories in six dimensions, which are believed to describe stacks of M5-branes in the low-energy limit of M-theory [9].

The papers [5]-[8] (see also references therein) show that the twistor methods can be useful in higher-dimensional Yang-Mills and superconformal field theories. However, there are some problems in generalizing the twistor approach to higher dimensions. Namely, let $X^{2n}$ be a Riemannian manifold of dimension $2n$. The metric twistor space of $X^{2n}$ is defined as the bundle $\text{Tw}(X^{2n}) \to X^{2n}$ of almost Hermitian structures on $X^{2n}$ associated with the principal bundle of orthonormal frames of $X^{2n}$, i.e.

$$\text{Tw}(X^{2n}) := P(X^{2n}, SO(2n)) \times_{SO(2n)} SO(2n)/U(n) \quad (1.2)$$

It is well known that $\text{Tw}(X^{2n})$ can be endowed with an almost complex structure $J$, which is integrable if and only if the Weyl tensor of $X^{2n}$ vanishes when $n > 2$ [10]. This is too strong a restriction. However, if the manifold $X^{2n}$ has a $G$-structure (not necessarily integrable), then one can often find a subbundle $\mathcal{Z}$ of $\text{Tw}(X^{2n})$ associated with the $G$-structure bundle $P(X^{2n}, G)$ for $G \subset SO(2n)$, such that an induced almost complex structure (also called $J$) on $\mathcal{Z}$ is integrable. Many examples were considered in [10]-[14]. Another problem is that, in higher dimensions, solutions of differential equations do not always correspond to holomorphic objects on the reduced twistor space $\mathcal{Z}$ (even if $\mathcal{Z}$ is a complex manifold). In [15] this was shown for the example of Yang-Mills instantons on the six-sphere $S^6$, which has the reduced complex twistor space $\mathcal{Z} = G_2/U(2)$. For the definition of instanton equations in dimensions higher than four and for some instanton solutions see e.g. [16]-[23].
In this paper we discuss instantons in gauge theory on the complex projective space $\mathbb{C}P^3$ by using twistor theory. Natural instanton-type equations in six dimensions are the Donaldson-Uhlenbeck-Yau (DUY) equations [17], which are SU(3) invariant but not invariant under the SO(6) Lorentz-type rotations of orthonormal frames. Hence, for their description one should consider reduced twistor spaces. The DUY equations are well defined on six-dimensional Kähler manifolds $M$ (as well as on nearly Kähler spaces [24, 25, 26]), and their solutions are natural connections $A$ on holomorphic vector bundles $E \to M$ [17]. On the example of $M = \mathbb{C}P^3$ we will show that such bundles $(E, A)$ are pulled back to holomorphic vector bundles $(\tilde{E}, \tilde{A})$ over the reduced twistor space $Z \subset \text{Tw}(\mathbb{C}P^3)$ trivial along the fibres of the fibration $Z \to \mathbb{C}P^3$ with $Z = SU(4)/U(2) \times U(1)$ or $Z = Sp(2)/U(1) \times U(1)$, depending on the chosen holonomy group. Note that this correspondence, valid for the reduced twistor spaces $Z \hookrightarrow \text{Tw}(\mathbb{C}P^3)$, does not hold for the metric twistor space $\text{Tw}(\mathbb{C}P^3)$.

2 Kähler and quasi-Kähler structure on $\mathbb{C}P^3$

Coset representation of $S^4$. Let us consider the group $Sp(2)$ fibred over $S^4 = Sp(2)/Sp(1) \times Sp(1)$,

$$Sp(2) \to S^4$$

i.e. consider $Sp(2)$ as the fibre bundle $P(S^4, Sp(1) \times Sp(1))$ with the structure group $Sp(1) \times Sp(1)$. Local sections of the fibrations (2.1) can be chosen as $4 \times 4$ matrices

$$Q := f^{-\frac{1}{2}} \begin{pmatrix} 1_2 & -x \\ x^\dagger & 1_2 \end{pmatrix} \quad \text{and} \quad Q^{-1} = Q^\dagger = f^{-\frac{1}{2}} \begin{pmatrix} 1_2 & x \\ -x^\dagger & 1_2 \end{pmatrix} \in Sp(2) \subset SU(4) ,$$

where

$$x = x^\mu \tau_\mu , \quad x^\dagger = x^\mu \tau^\dagger_\mu , \quad f := 1 + x^\dagger x = 1 + r^2 = 1 + \delta_{\mu\nu} x^\mu x^\nu ,$$

and matrices

$$(\tau_\mu) = (-i\sigma_i, 1_2) \quad \text{and} \quad (\tau^\dagger_\mu) = (i\sigma_i, 1_2)$$

obey

$$\tau^\dagger_\mu \tau_\nu = \delta_{\mu\nu} \cdot 1_2 + \eta^\dagger_{\mu\nu} i \sigma_i =: \delta_{\mu\nu} \cdot 1_2 + \eta_{\mu\nu} , \quad \{\eta^\dagger_{i\mu\nu}\} = \{-\eta^i_{\mu\nu}\} = \{\epsilon^i_{jk}, \mu = j, \nu = k; \ \delta^i_{j}, \mu = j, \nu = 4\} ,$$

$$\tau_\mu \tau^\dagger_\nu = \delta_{\mu\nu} \cdot 1_2 + \eta_{\mu\nu} i \sigma_i =: \delta_{\mu\nu} \cdot 1_2 + \eta_{\mu\nu} , \quad \{\eta_{i\mu\nu}\} = \{-\bar{\eta}^i_{i\mu\nu}\} = \{\epsilon^i_{jk}, \mu = j, \nu = k; \ \bar{\delta}^i_{j}, \mu = j, \nu = 4\} .$$

Here $\{x^\mu\}$ are local coordinates on an open set $U \subset S^4$. Matrices (2.2) are representative elements for the coset space $S^4 = Sp(2)/Sp(1) \times Sp(1)$.

Flat connection on $S^4$. Consider a flat connection $A_0$ on the trivial vector bundle $S^4 \times \mathbb{C}^4 \to S^4$ given by the one-form

$$A_0 = Q^{-1} dQ =: \begin{pmatrix} A^- \ 
\phi^+ & -\phi \\ \phi^\dagger & A^+ \end{pmatrix} ,$$

where from (2.2) we obtain

$$A^- = \frac{1}{2} \eta_{\mu\nu} x^\mu dx^\nu =: \begin{pmatrix} \alpha_- & -\bar{\beta}_- \\ \beta_- & -\alpha_- \end{pmatrix} \in su(2) ,$$

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\[ A^+ = \frac{1}{f} \eta_{\mu
u} x^\mu dx^\nu =: \begin{pmatrix} \alpha_+ & -\beta_+ \\ \beta_+ & -\alpha_+ \end{pmatrix} \in su(2), \] (2.8)

\[ \phi = \frac{1}{f} dx = -i \int \frac{dx^2 + idx^4}{dx^1 + idx^2} = -i \int \begin{pmatrix} dz & d\bar{y} \\ dy & -d\bar{z} \end{pmatrix} =: \begin{pmatrix} \theta_1 & \theta_1 \bar{r} \\ -\theta_1 & \theta_2 \bar{r} \end{pmatrix}, \] (2.9)

with

\[ \alpha_+ = \frac{1}{f}(\bar{y} dy + \bar{z} dz - y d\bar{y} - z d\bar{z}), \quad \beta_+ = \frac{1}{f}(y dz - z dy), \] (2.10)

\[ \alpha_- = \frac{1}{f}(\bar{y} dy + z d\bar{z} - y d\bar{y} - \bar{z} dz), \quad \beta_- = \frac{1}{f}(y dz - \bar{z} dy), \] (2.11)

\[ \theta_1 := \frac{idz}{1 + r^2}, \quad \theta_2 := -\frac{id\bar{y}}{1 + r^2}, \quad \theta_1^r := -\frac{id\bar{y}}{1 + r^2}, \quad \theta_2^r := \frac{id\bar{z}}{1 + r^2}. \] (2.12)

Here, the bar denotes complex conjugation.

**Coset representation of \( S^2 \).** Let us consider the Hopf bundle

\[ S^3 \to S^2 \] (2.13)

over the Riemann sphere \( S^2 \cong \mathbb{C}P^1 \) and the one-monopole connection \( a \) on the bundle (2.13) having in the local complex coordinate \( \zeta \in \mathbb{C}P^1 \) the form

\[ a = \frac{1}{2(1 + \zeta \bar{\zeta})} \left( \bar{\zeta} d\zeta - \zeta d\bar{\zeta} \right). \] (2.14)

Consider a local section of the bundle (2.13) given by the matrix

\[ g = \frac{1}{(1 + \zeta \bar{\zeta})^2} \begin{pmatrix} 1 & -\zeta \\ \zeta & 1 \end{pmatrix} \in SU(2) \cong S^3 \] (2.15)

and introduce the \( su(2) \)-valued one-form (flat connection)

\[ g^{-1} dg =: \begin{pmatrix} a & -\theta^3 \\ \theta^3 & -a \end{pmatrix} \] (2.16)

on the bundle \( S^2 \times \mathbb{C}^2 \to S^2 \), where

\[ \theta^3 = \frac{d\zeta}{1 + \zeta \bar{\zeta}} \quad \text{and} \quad \bar{\theta}^3 = \frac{d\bar{\zeta}}{1 + \zeta \bar{\zeta}} \] (2.17)

are the forms of type \((1,0)\) and \((0,1)\) on \( \mathbb{C}P^1 \) and \( a \) is the one-monopole gauge potential (2.14).

**Twistor space \( Tw(S^4) \).** Let us introduce \( 4 \times 4 \) matrices

\[ G = \begin{pmatrix} 1 & 0 \\ 0 & \hat{Q} \end{pmatrix} \quad \text{and} \quad \hat{Q} \in Sp(2) \subset SU(4), \] (2.18)

where \( Q \) and \( g \) are given in (2.2) and (2.15). The matrix \( \hat{Q} \) is a local section of the bundle

\[ \text{Sp}(2) \to \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1) =: \mathcal{M}. \] (2.19)
Let us consider a trivial complex vector bundle $\mathcal{M} \times \mathbb{C}^4 \to \mathcal{M}$ with the flat connection
\[
\hat{A}_0 = \hat{Q}^{-1} d\hat{Q} = G^{-1} A_0 G + G^{-1} dG =: \begin{pmatrix} \hat{A}^- & -\hat{\phi} \\ \hat{\phi} & \hat{A}^+ \end{pmatrix},
\]
where
\[
\hat{\phi} = \phi g =: \begin{pmatrix} \hat{\phi}^2 \\ -\hat{\phi}^1 \\ \hat{\phi}^3 \end{pmatrix}, \quad \hat{A}^- = A^- = \begin{pmatrix} \alpha_- \\ -\beta_- \\ \alpha_- \end{pmatrix}, \quad \hat{A}^+ = : \begin{pmatrix} \hat{\alpha}_+ \\ -\hat{\beta}_+ \\ -\hat{\alpha}_+ \end{pmatrix},
\]
with $\alpha_-, \beta_-$ given in (2.11) and
\[
\hat{\alpha}_+ := \frac{1}{1 + \zeta} \left\{ (1 - \zeta) \alpha_+ + \zeta \beta_+ - \zeta \beta_+ + \frac{1}{2} (\zeta d\zeta - \zeta d\zeta) \right\},
\]
\[
\hat{\beta}^1 := \frac{1}{(1 + \zeta)^{\frac{1}{2}}} (\theta^1 - \zeta \theta^2), \quad \hat{\beta}^2 := \frac{1}{(1 + \zeta)^{\frac{1}{2}}} (\theta^2 + \zeta \theta^1),
\]
\[
\hat{\beta}^3 := \frac{1}{(1 + \zeta)^{\frac{1}{2}}} (d\zeta + \beta_+ - 2\zeta \alpha_+ + \zeta^2 \beta_+).
\]

From flatness of the connection (2.20), $d\hat{A}_0 + \hat{A}_0 \wedge \hat{A}_0 = 0$, we obtain the equations
\[
\begin{pmatrix} \hat{\beta}^1 \\ \hat{\beta}^2 \\ \hat{\beta}^3 \end{pmatrix} + \begin{pmatrix} -\hat{\alpha}_+ - \alpha_- & \beta_- & \beta_- \\ -\hat{\beta}_- & 0 & 0 \\ -\hat{\beta}_- & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\beta}^1 \\ \hat{\beta}^2 \\ \hat{\beta}^3 \end{pmatrix} = 0,
\]
where we rescaled our one-forms $\hat{\beta}$'s as
\[
\hat{\beta}^1 \to \frac{1}{2\Lambda} \hat{\beta}^1, \quad \hat{\beta}^2 \to \frac{1}{2\Lambda} \hat{\beta}^2 \quad \text{and} \quad \hat{\beta}^3 \to \frac{1}{2R} \hat{\beta}^3.
\]

We see that (2.25) defines the Levi-Civita connection with $U(3)$ holonomy group (Kähler structure) on $\mathcal{M}$ if $R = \Lambda$, where $R$ is the radius of $S^2$ and $\Lambda$ is the radius of $S^4$.

Note that forms $\hat{\beta}^i$ define on $\mathcal{M}$ an integrable almost complex structure $\mathcal{J}_+$ [1] such that
\[
\mathcal{J}_+ \hat{\beta}^i = i \hat{\beta}^i
\]
with $i = 1, 2, 3$. In other words, $\hat{\beta}^i$'s are $(1,0)$-forms with respect to (w.r.t.) $\mathcal{J}_+$ and the manifold $\mathcal{M}$ with such a complex structure can be identified with the Kähler manifold $\mathbb{C}P^3 = SU(4)/U(3)$ with the Kähler form
\[
\hat{\omega} := \frac{i}{2} \left( \hat{\beta}^1 \wedge \hat{\beta}^1 + \hat{\beta}^2 \wedge \hat{\beta}^2 + \hat{\beta}^3 \wedge \hat{\beta}^3 \right).
\]

**Quasi-Kähler structure on $\mathcal{M}$.** Recall that on the same manifold $\mathcal{M}$ one can introduce the forms
\[
\Theta^1 := \hat{\beta}^1, \quad \Theta^2 := \hat{\beta}^2 \quad \text{and} \quad \Theta^3 := \hat{\beta}^3,
\]
which are forms of type (1,0) w.r.t. an almost complex structure $\mathcal{J}_-$ [27], $\mathcal{J}_- \Theta^i = i \Theta^i$, which is a never integrable almost complex structure. For $\Theta^i$ with the rescaling (2.26) we have

$$d \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} + \begin{pmatrix} -\hat{\alpha}_+ + \alpha_- & \beta_- & 0 \\ -\beta_- & -\hat{\alpha}_+ + \alpha_- & 0 \\ 0 & 0 & 2\hat{\alpha}_+ \end{pmatrix} \wedge \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} = \frac{1}{2R^2} \begin{pmatrix} \Theta^2 \wedge \Theta^3 \\ \Theta^3 \wedge \Theta^1 \\ \Theta^1 \wedge \Theta^2 \end{pmatrix}. \quad (2.30)$$

The manifold $(\mathcal{M}, \mathcal{J}_-)$ is a quasi-Kähler manifold. Recall that an almost Hermitian $2n$-manifold with the fundamental (1,1)-form $\omega$ is called quasi-Kähler if only $(3,0)+(0,3)$ components of $d\omega$ are non-vanishing [12, 25]. In our case

$$\omega := \frac{i}{2} (\Theta^1 \wedge \bar{\Theta}^1 + \Theta^2 \wedge \bar{\Theta}^2 + \Theta^3 \wedge \bar{\Theta}^3). \quad (2.31)$$

One can check that for arbitrary ratio $\Lambda/R$ the $(1,2)$ part of $d\omega$ vanishes and therefore $\mathcal{M}$ is quasi-Kähler [24, 27].

From (2.30) one sees that the manifold $\mathcal{M} = \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1)$ with an almost complex structure $\mathcal{J}_-$ becomes a nearly Kähler manifold if $\Lambda^2 = 2R^2$. Recall that a six-manifold is called nearly Kähler if [12, 24, 25]

$$d\omega = 3\rho \text{Im}\Omega \quad \text{for} \quad \Omega := \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \quad \text{and} \quad d\Omega = 2\rho \omega \wedge \omega, \quad (2.32)$$

where $\rho \in \mathbb{R}$ is proportional to the inverse “radius” $\Lambda = \sqrt{2R}$ of $\mathcal{M}$.

3 Twistors spaces of $\mathbb{C}P^3$

**Coset representation of $\mathbb{C}P^2$.** Let us consider the projection

$$\text{SU}(3) \to \text{SU}(3)/\text{U}(2) = \mathbb{C}P^2. \quad (3.1)$$

One can choose as a coset representative of $\mathbb{C}P^2$ a local section of the bundle (3.1) given by the matrix

$$V = \frac{1}{\gamma} \begin{pmatrix} 1 & Y^\dagger \\ \bar{Y} & W \end{pmatrix} := \frac{1}{\gamma} \begin{pmatrix} 1 & \bar{\lambda}^1 \\ -\lambda^1 & W_{11} \\ -\bar{\lambda}^2 & W_{21} \\ \lambda^2 & W_{12} \\ \bar{\lambda}^2 & W_{22} \end{pmatrix} \in \text{SU}(3), \quad (3.2)$$

where

$$\gamma^2 := 1 + Y^\dagger Y = 1 + \lambda^1 \bar{\lambda}^1 + \lambda^2 \bar{\lambda}^2 \quad \text{and} \quad W = V^\dagger = \gamma \cdot 1_2 - \frac{1}{\gamma + 1} YY^\dagger. \quad (3.3)$$

Here $\lambda^1$ and $\lambda^2$ are local complex coordinates on a patch of $\mathbb{C}P^2$. From (3.2) and (3.3) it is easy to see that

$$WY = Y \quad \text{and} \quad W^2 = \gamma^2 - YY^\dagger \quad \Leftrightarrow \quad V^\dagger V = 1_3 = VV^\dagger. \quad (3.4)$$

**Twistor space of $\text{SU}(4)/\text{U}(3)$.** Consider the coset space

$$\mathcal{Z} := \text{SU}(4)/\text{U}(2) \times \text{U}(1) \quad (3.5)$$
and the projection
\[ \pi : \text{SU}(4)/U(2) \times U(1) \to \text{SU}(4)/U(3) \cong \mathbb{C}P^3 \] (3.6)
with fibres $\mathbb{C}P^2$. Using the group element (3.2) to parametrize the typical $\mathbb{C}P^2$-fibre in (3.6), we introduce a flat connection $\tilde{A}_0$ on the trivial bundle $Z \times \mathbb{C}^4 \to Z$ as
\[ \tilde{A}_0 = \tilde{Q}^{-1}d\tilde{Q} = \tilde{V}^\dagger \tilde{A}_0 \tilde{V} + \tilde{V}^\dagger d\tilde{V}, \] (3.7)
where
\[ \tilde{Q} = Q\tilde{V} \in \text{SU}(4) \quad \text{and} \quad \tilde{V} := \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix} \text{ with } V \in \text{SU}(3). \] (3.8)

The flat connection $\tilde{A}_0$ is given in (2.20) but here we write it as
\[ \tilde{A}_0 = \begin{pmatrix} \alpha_- & -\bar{\beta}_- & -\tilde{\theta}^2 & -\tilde{\theta}^1 \\ \bar{\beta}_- & -\alpha_- & \tilde{\theta}^1 & -\tilde{\theta}^2 \\ \tilde{\theta}^2 & -\tilde{\theta}^1 & \bar{\alpha}_+ & -\tilde{\theta}^3 \\ \tilde{\theta}^1 & \tilde{\theta}^2 & \tilde{\theta}^3 & -\check{\alpha}_+ \end{pmatrix}, \] (3.9)
where
\[ B = \begin{pmatrix} \alpha_- & -\bar{\beta}_- & -\tilde{\theta}^2 \\ \bar{\beta}_- & -\alpha_- & \tilde{\theta}^1 \\ \tilde{\theta}^2 & -\tilde{\theta}^1 & \bar{\alpha}_+ \end{pmatrix}, \quad T := \begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \tilde{\theta}^3 \end{pmatrix} \text{ and } T^\dagger = (\tilde{\theta}^1 \tilde{\theta}^2 \tilde{\theta}^3). \] (3.10)

Using (3.7), we obtain the connection
\[ \tilde{\mathcal{A}}_0 = \begin{pmatrix} V^\dagger BV + V^\dagger dV & -V^\dagger T \\ T^\dagger V & -\check{\alpha}_+ \end{pmatrix} = \begin{pmatrix} \tilde{B} & -\tilde{T} \\ \tilde{T}^\dagger & -\check{\alpha}_+ \end{pmatrix} \text{ with } \tilde{T} = \begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \tilde{\theta}^3 \end{pmatrix} \] (3.11)
and for the curvature $\tilde{\mathcal{F}}_0 = d\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_0 \wedge \tilde{\mathcal{A}}_0$ we get
\[ \tilde{\mathcal{F}}_0 = \begin{pmatrix} dB + \tilde{B} \wedge \tilde{T} \wedge \tilde{T}^\dagger & -d\tilde{T} - (\tilde{B} + \check{\alpha}_+ \cdot 1_3) \wedge \tilde{T} \\ d\tilde{T}^\dagger + \tilde{T}^\dagger \wedge (\tilde{B} + \check{\alpha}_+ \cdot 1_3) & -d\check{\alpha}_+ - \tilde{T}^\dagger \wedge \tilde{T} \end{pmatrix}. \] (3.12)

We have
\[ \tilde{B} = V^\dagger BV + V^\dagger dV = \begin{pmatrix} \tilde{\alpha}_- \\ -\check{\gamma} \end{pmatrix} \Sigma \] (3.13)
with
\[ \Sigma = \begin{pmatrix} \tilde{a} - \check{\alpha}_- & -\tilde{b} \\ b & -\tilde{a} + \check{\alpha}_+ \end{pmatrix} \] and \[ \check{\gamma} = \begin{pmatrix} \tilde{\theta}^4 \\ \tilde{\theta}^5 \end{pmatrix}. \] (3.14)

Flatness $\tilde{\mathcal{F}}_0 = 0$ of the connection (3.11) yields
\[ d \begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \tilde{\theta}^3 \end{pmatrix} + \begin{pmatrix} \tilde{\alpha}_- - \check{\alpha}_+ & 0 & 0 \\ 0 & -\tilde{a} + \check{\alpha}_- - \check{\alpha}_+ & -\tilde{b} \\ 0 & \tilde{b} & \tilde{a} - 2\check{\alpha}_+ \end{pmatrix} \wedge \begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \tilde{\theta}^3 \end{pmatrix} = \begin{pmatrix} \tilde{\theta}^{24} + \tilde{\theta}^{35} \\ -\tilde{\theta}^{14} \\ -\tilde{\theta}^{15} \end{pmatrix}. \] (3.15)
From

\[ d\hat{B} + \hat{B} \wedge \hat{B} - \hat{T} \wedge \hat{T}^\dagger = 0 \]  

(3.16)

it follows that

\[ d\left( \frac{\hat{\theta}^4}{\hat{\theta}^5} \right) + \left( \frac{\tilde{a} - 2\tilde{\alpha}_-}{b} - \tilde{a} + \tilde{\alpha}_+ - \tilde{\alpha}_- \right) \wedge \left( \frac{\hat{\theta}^4}{\hat{\theta}^5} \right) = \left( \frac{\hat{\theta}^{12}}{\hat{\theta}^{13}} \right). \]  

(3.17)

We obtain

\[ \begin{pmatrix} \hat{\theta}^1 \\ \hat{\theta}^2 \\ \hat{\theta}^3 \\ \hat{\theta}^4 \\ \hat{\theta}^5 \end{pmatrix} = \begin{pmatrix} -\tilde{\alpha}_- - \tilde{\alpha}_+ & 0 & 0 & 0 & 0 \\ 0 & -\tilde{a} + \tilde{\alpha}_- - \tilde{\alpha}_+ & -b & 0 & 0 \\ 0 & b & \tilde{a} - 2\tilde{\alpha}_+ & 0 & 0 \\ 0 & 0 & 0 & \tilde{a} - 2\tilde{\alpha}_- & -\tilde{b} \\ 0 & 0 & 0 & b & -\tilde{a} - \tilde{\alpha}_+ + \tilde{\alpha}_+ \end{pmatrix} \wedge \begin{pmatrix} \hat{\theta}^1 \\ \hat{\theta}^2 \\ \hat{\theta}^3 \\ \hat{\theta}^4 \\ \hat{\theta}^5 \end{pmatrix} = \begin{pmatrix} \hat{\theta}^{24} + \frac{\Lambda}{R} \hat{\theta}^{35} \\ -\hat{\theta}^{14} \\ -\frac{R}{\Lambda} \hat{\theta}^{15} \\ \frac{1}{4\Lambda^2} \hat{\theta}^{12} \\ \frac{1}{4\Lambda R} \hat{\theta}^{13} \end{pmatrix}, \]  

(3.18)

where we rescaled our \( \hat{\theta}^a \) with \( a = 1, \ldots, 5 \) as in (2.26):

\[ \hat{\theta}^1 \rightarrow \frac{1}{2\Lambda} \hat{\theta}^1, \quad \hat{\theta}^2 \rightarrow \frac{1}{2\Lambda} \hat{\theta}^2, \quad \hat{\theta}^3 \rightarrow \frac{1}{2R} \hat{\theta}^3, \quad \hat{\theta}^4 \rightarrow \hat{\theta}^4 \quad \text{and} \quad \hat{\theta}^5 \rightarrow \hat{\theta}^5. \]  

(3.19)

The manifold \( \text{SU}(4)/\text{U}(2) \times \text{U}(1) \) is the twistor space for the Kähler space \( \mathbb{CP}^3 = \text{SU}(4)/\text{U}(3) \) for \( \Lambda^2 = R^2 \). Forms \( \hat{\theta}^a \) define on \( \text{SU}(4)/\text{U}(2) \times \text{U}(1) \) an integrable almost complex structure \( \hat{J}_+ \) such that

\[ \hat{J}_+ \hat{\theta}^a = i \hat{\theta}^a. \]  

(3.20)

In the Kähler case we choose \( \Lambda = R = \frac{1}{2} \).

**Twistor space of \( \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1) \).** Consider the coset space

\[ \mathcal{Z}' := \text{Sp}(2)/\text{U}(1) \times \text{U}(1) \]  

(3.21)

and the projection

\[ \pi' : \text{Sp}(2)/\text{U}(1) \times \text{U}(1) \rightarrow \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1) \cong \mathbb{CP}^3 \]  

(3.22)

with fibres \( \mathbb{CP}^1 \cong \text{Sp}(1)/\text{U}(1) \). We choose the group element

\[ \hat{g} = \frac{1}{(1 + \lambda \bar{\lambda})^{\frac{1}{2}}} \begin{pmatrix} 1 & -\bar{\lambda} \\ \lambda & 1 \end{pmatrix} \in \text{SU}(2) \cong \text{Sp}(1) \]  

(3.23)

to parametrize the typical \( \mathbb{CP}^1 \)-fibre in (3.22), where \( \lambda \) is a local complex coordinate on the Riemann sphere \( \mathbb{CP}^1 \). By formula

\[ \hat{g}^{-1} d\hat{g} = \begin{pmatrix} \hat{a} & -\theta^4 \\ \theta^4 & -\hat{a} \end{pmatrix} \]  

(3.24)

where

\[ \hat{a} := \frac{1}{2(1 + \lambda \bar{\lambda})} (\bar{\lambda} d\lambda - \lambda d\bar{\lambda}), \]  

(3.25)

we introduce on \( \mathbb{CP}^1 \) the forms

\[ \theta^4 = \frac{d\lambda}{1 + \lambda \bar{\lambda}} \quad \text{and} \quad \theta^4 = \frac{d\bar{\lambda}}{1 + \lambda \bar{\lambda}} \]  

(3.26)
of type \((1,0)\) and \((0,1)\), respectively.

Using the group element \((3.23)\), we introduce a flat connection \(A'_0\) on the trivial bundle \(Z' \times \mathbb{C}^4 \rightarrow Z'\) as

\[
A'_0 = \hat{Q}^{-1} d\hat{Q} = G^\dagger \hat{A}_0 \hat{G} + \hat{G}^\dagger d\hat{G} ,
\]

where

\[
\hat{Q} = \hat{Q} \hat{G} \in \text{Sp}(2) \quad \text{and} \quad \hat{G} := \begin{pmatrix} \hat{g} & 0 \\ 0 & 1_2 \end{pmatrix} \in \text{Sp}(1) \subset \text{Sp}(2) .
\]

The flat connection \(\hat{A}_0\) is given in \((2.20)\) and \((3.9)\). Using \((3.27)\), we obtain the connection

\[
A'_0 = \begin{pmatrix} \hat{g}^\dagger \hat{A}^- \hat{g} + \hat{g}^\dagger d\hat{g} & -\hat{g}^\dagger \hat{\phi} \\ \hat{\phi}^\dagger \hat{g} & \hat{A}^+ \end{pmatrix} = \begin{pmatrix} \hat{A}^- & 0 \\ 0 & \hat{A}^+ \end{pmatrix} ,
\]

with

\[
\hat{\phi} = \hat{g}^\dagger \hat{\phi} = \frac{1}{(1 + \lambda \lambda)^{1/2}} \begin{pmatrix} \hat{\theta}^2 - \bar{\lambda} \bar{\theta}_1 \\ -\hat{\theta}_1 - \lambda \bar{\theta}_2 \\ \hat{\theta}_2 - \lambda \bar{\theta}_1 \end{pmatrix} = \begin{pmatrix} \hat{\theta}^2 \\ -\hat{\theta}_1 \\ \hat{\theta}_2 \end{pmatrix} ,
\]

\[
\hat{\theta}^3 := \begin{pmatrix} \hat{\alpha}_+ - \overline{\hat{\theta}^3} \\ \hat{\theta}^3 - \overline{\hat{\theta}^3} \\ -\hat{\alpha}_+ \end{pmatrix} = \hat{\theta}^3 \quad \text{and} \quad \hat{\theta}^- := \begin{pmatrix} \hat{\alpha}_- - \overline{\hat{\theta}^4} \\ \hat{\theta}^4 - \overline{\hat{\theta}^4} \\ -\hat{\alpha}_- \end{pmatrix} ,
\]

where

\[
\hat{\alpha}_- = \frac{1}{1 + \lambda \lambda} \left\{ (1 - \lambda \lambda) \alpha_- + \bar{\lambda} \beta_- - \lambda \bar{\beta}_- + \frac{1}{2} (\bar{\lambda} d\lambda - \lambda d\bar{\lambda}) \right\} ,
\]

\[
\hat{\theta}^3 = \frac{1}{1 + \lambda \lambda} \left\{ d\lambda + \beta_- + 2 \lambda \alpha_- + \lambda^2 \bar{\beta}_- \right\} , \quad \overline{\hat{\theta}^4} := \hat{\theta}^4 .
\]

For the curvature \(F'_0 = dA'_0 + A'_0 \wedge A'_0\) we get

\[
F'_0 = \begin{pmatrix} d\hat{A}^- + \hat{A}^- \wedge \hat{A}^- & -d\bar{\phi} - \hat{\phi} \wedge \hat{\phi} \\ d\bar{\phi}^i + \hat{\phi}^i \wedge \hat{\phi}^i & d\hat{A}^+ + \hat{A}^+ \wedge \hat{A}^+ \end{pmatrix} .
\]

From the flatness \(F'_0 = 0\) of the connection \((3.29)\) we obtain the Maurer-Cartan equations

\[
d \begin{pmatrix} \hat{\theta}^1 \\ \hat{\theta}^2 \\ \hat{\theta}^3 \\ \hat{\theta}^4 \end{pmatrix} + \begin{pmatrix} -\hat{\alpha}_- - \overline{\hat{\theta}^3} \\ 0 \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} \hat{\theta}^1 \\ \hat{\theta}^2 \\ \hat{\theta}^3 \\ \hat{\theta}^4 \end{pmatrix} = \begin{pmatrix} -\hat{\theta}^{24} - \hat{\theta}^{32} \\ \hat{\theta}^{31} + \hat{\theta}^{14} \\ 2\hat{\theta}^{12} \\ -2\hat{\theta}^{12} \end{pmatrix} ,
\]

which define the \(u(1) \oplus u(1)\) torsionful connection on the twistor space \(Z' = \text{Sp}(2)/U(1) \times U(1)\). Forms \(\hat{\theta}^a\) in \((3.35)\) with \(a = 1, \ldots, 4\) define on \(Z'\) an integrable almost complex structure \(I'_+\) such that

\[
I'_+ \hat{\theta}^a = i \hat{\theta}^a .
\]

Its integrability follows from the vanishing \((0,2)\)-type components of the torsion on the right hand side of \((3.35)\).
4 Twistor description of instanton bundles over $\mathbb{C}P^3$

**Instanton bundles over $\mathbb{C}P^3$.** Consider a complex vector bundle $\mathcal{E}$ over $\mathbb{C}P^3$ with a connection one-form $\mathcal{A}$ having the curvature $\mathcal{F}$. Recall that $(\mathcal{E}, \mathcal{A})$ is called an instanton bundle if $\mathcal{A}$ satisfies the Hermitian Yang-Mills (HYM) equations,¹ which on $\mathbb{C}P^3$ can be written in the form

$$\mathcal{F}^{0,2} = 0 = \mathcal{F}^{2,0} \quad \Leftrightarrow \quad \hat{\Omega} \wedge \mathcal{F} = 0 \ ,$$

(4.1)

and

$$\hat{\omega}_\perp \mathcal{F} = 0 \quad \Leftrightarrow \quad \hat{\omega} \wedge \hat{\omega} \wedge \mathcal{F} = 0 \ ,$$

(4.2)

where the notation $\hat{\omega}_\perp$ exploits the underlying Riemannian metric $g = \delta_{\hat{a}\hat{b}} e^{\hat{a}}_\widehat{e}^{\hat{b}}$ on $\mathbb{C}P^3$, $\hat{a}, \hat{b}, \ldots = 1, \ldots, 6$. Here, $\hat{\omega}$ given in (2.28) is a $(1,1)$-form, and $\hat{\Omega} := \hat{\theta}^1 \wedge \hat{\theta}^2 \wedge \hat{\theta}^3$ is a locally defined $(3,0)$-form on $\mathbb{C}P^3$. Recall that, from the point of view of algebraic geometry, (4.1) means that the bundle $\mathcal{E} \to \mathbb{C}P^3$ is holomorphic and (4.2) means that $\mathcal{E}$ is a polystable vector bundle [17]. In fact, in the right hand side of (4.2) one can add the term $\beta \hat{\omega} \wedge \hat{\omega}$ with $\beta$ proportional to the first Chern number $c_1(\mathcal{E})$, but we assume $c_1(\mathcal{E}) = 0$ since for a bundle with field strength $\mathcal{F}$ of non-zero degree one can obtain a degree-zero bundle by considering $\hat{\mathcal{F}} = \mathcal{F} - \frac{1}{r} (\text{tr} \mathcal{F}) \cdot 1_r$, where $r = \text{rank} \mathcal{E}$.

**Pull-back to $\mathcal{Z}$.** Consider the twistor fibration (3.6). Let $(\hat{\mathcal{E}}, \hat{\mathcal{A}}) = (\pi^* \mathcal{E}, \pi^* \mathcal{A})$ be the pulled-back instanton bundle over $\mathcal{Z}$ with the curvature $\hat{\mathcal{F}} = d\hat{\mathcal{A}} + \hat{\mathcal{A}} \wedge \hat{\mathcal{A}}$. We have

$$\hat{\mathcal{F}} = \frac{1}{2} \hat{\mathcal{F}}_{ab} \hat{\theta}^a \wedge \hat{\theta}^b + \hat{\mathcal{F}}_{ab} \hat{\bar{\theta}}^a \wedge \hat{\bar{\theta}}^b + \frac{1}{2} \hat{\mathcal{F}}_{ab} \hat{\theta}^a \wedge \hat{\bar{\theta}}^b = \pi^* \mathcal{F}$$

(4.3)

with $a, b, \ldots = 1, \ldots, 5$. Using the relation between $\hat{\theta}^a$ and $\hat{\bar{\theta}}^a$ described in Section 3, we obtain

$$\hat{\mathcal{F}}_{ij} = C^k_i C^l_j \mathcal{F}_{kl} \quad \text{and} \quad \hat{\mathcal{F}}_{ij} = C^k_i C^l_j \mathcal{F}_{kl} \ ,$$

(4.4)

where $C = \hat{V}^\dagger$ with

$$C^1_1 = \frac{1}{\gamma} \ , \quad C^1_2 = -\frac{\lambda^1}{\gamma} \ , \quad C^1_3 = -\frac{\lambda^2}{\gamma} \ ,$$

$$C^2_1 = \frac{\bar{\lambda}^1}{\gamma} \ , \quad C^2_2 = \frac{\gamma + 1 + \lambda^2 \bar{\lambda}^1}{\gamma(\gamma + 1)} \ , \quad C^2_3 = -\frac{\lambda^2 \bar{\lambda}^1}{\gamma(\gamma + 1)} \ ,$$

$$C^3_1 = \frac{\bar{\lambda}^2}{\gamma} \ , \quad C^3_2 = -\frac{\lambda^1 \bar{\lambda}^2}{\gamma(\gamma + 1)} \ , \quad C^3_3 = \frac{\gamma + 1 + \lambda^1 \bar{\lambda}^1}{\gamma(\gamma + 1)} \ ,$$

(4.5)

and $\bar{C}$ is the complex conjugate matrix. Thus, more explicitly, we get

$$\hat{\mathcal{F}}_{12} = \frac{1}{\gamma} \left\{ \frac{\gamma + 1 + \lambda^1 \bar{\lambda}^1}{\gamma + 1} \mathcal{F}_{12} - \frac{\lambda^1 \bar{\lambda}^2}{\gamma + 1} \mathcal{F}_{31} - \bar{\lambda}^2 \mathcal{F}_{23} \right\} \ ,$$

(4.6)

$$\hat{\mathcal{F}}_{31} = \frac{1}{\gamma} \left\{ \frac{\gamma + 1 + \lambda^2 \bar{\lambda}^2}{\gamma + 1} \mathcal{F}_{31} - \frac{\lambda^2 \bar{\lambda}^1}{\gamma + 1} \mathcal{F}_{12} - \bar{\lambda}^1 \mathcal{F}_{23} \right\} \ ,$$

(4.7)

$$\hat{\mathcal{F}}_{23} = \frac{1}{\gamma} \left\{ \mathcal{F}_{23} + \lambda^1 \mathcal{F}_{31} + \lambda^2 \mathcal{F}_{12} \right\} \ ,$$

(4.8)

¹These equations are also called the Donaldson-Uhlenbeck-Yau equations.
\[
\tilde{F}_{i4} = \tilde{F}_{i5} = 0, \quad (4.9)
\]
\[
\tilde{F}_{11} + \tilde{F}_{22} + \tilde{F}_{33} + \tilde{F}_{44} + \tilde{F}_{55} = F_{11} + F_{22} + F_{33} . \quad (4.10)
\]

The vanishing of \( \tilde{F}_{23} \) for all values of \((\lambda^1, \lambda^2) \in CP^2 \) is equivalent to the holomorphicity equation (4.1). In homogeneous coordinates \( y' \) on \( CP^2 \) \((\lambda^1 = y^2 / y^1, \lambda^2 = y^3 / y^1, y^1 \neq 0) \), this condition can be written as
\[
\tilde{F}_{23} = 0 \iff y'^i \varepsilon_{ijk} F^{jk} = 0 , \quad (4.11)
\]
where the indices \( i, j, \ldots \) are raised with the metric \( \delta^{ij} \). From (4.6)-(4.9) we see that the bundle \( \tilde{E} \) is holomorphic for holomorphic \( E \) as well as polystable due to (4.2), (4.10) and it is holomorphically trivial after restricting to the fibres \( CP^2_x \rightarrow \mathcal{Z} \) of the projection \( \pi \) for each \( x \in CP^3 \).

**Pull-back to \( \mathcal{Z}' \).** Consider now the twistor fibration (3.22) and the pulled-back instanton bundle \((\mathcal{E}', \mathcal{A}') = (\pi'^* \mathcal{E}, \pi'^* \mathcal{A}) \) over \( \mathcal{Z}' \) with the curvature \( F' = d \mathcal{A}' + \mathcal{A}' \wedge \mathcal{A}' \). We again have the relation (4.3) with \( a, b, \ldots = 1, \ldots, 4 \). For the matrix \( C \) in (4.4) we now find
\[
C = \begin{pmatrix}
x & x \lambda & 0 \\
- \lambda \lambda & x & 0 \\
0 & 0 & 1
\end{pmatrix}
\text{with} \quad x = (1 + \lambda \bar{\lambda})^{-\frac{1}{2}} , \quad (4.12)
\]
where \( \lambda \) is a local complex coordinate on \( CP^1 \) used in (3.23)-(3.26).

Using (4.12), we obtain
\[
\begin{align*}
F'_{12} &= F_{12}, \\
F'_{31} &= x(F_{31} + \bar{\lambda} F_{23}), \\
F'_{23} &= x(F_{23} - \lambda F_{31}), \\
F'_{i4} &= 0 ,
\end{align*}
\quad (4.13)
\]
\[
\begin{align*}
F'_{11} + F'_{22} + F'_{33} + F'_{44} &= F_{11} + F_{22} + F_{33} .
\end{align*}
\quad (4.14)
\]

Therefore, instanton bundles \((\mathcal{E}, \mathcal{A}) \) over the nonsymmetric Kähler coset space \( Sp(2)/Sp(1) \times U(1) \cong CP^3 \) are pulled back to holomorphic polystable bundles \((\mathcal{E}', \mathcal{A}') \) over the complex twistor space \( \mathcal{Z}' = Sp(2)/U(1) \times U(1) \). Furthermore, \( \mathcal{E}' \) is flat along the fibres \( CP^1_x \) of the bundle (3.22), and one can set the components of \( \mathcal{A}' \) along the fibres equal to zero. Thus, restrictions of the vector bundle \( \mathcal{E}' \) to fibres \( CP^1_x \rightarrow \mathcal{Z}' \) of the projection \( \pi' \) are holomorphically trivial for each \( x \in Sp(2)/Sp(1) \times U(1) \cong CP^3 \). Note that (4.13) and (4.14) can be obtained from (4.6)-(4.10) by putting \( \lambda^1 = -\lambda \) and \( \lambda^2 = 0 \).

Then (3.11) will coincide with (3.29) after the substitution \( \bar{\theta}^4 \rightarrow -\bar{\theta}^4, \bar{\theta}^5 \rightarrow -\bar{\theta}^2, b \rightarrow -\bar{\theta}^1 \) etc. This correspondence follows from the fact that \( \mathcal{Z}' \) is a complex (codimension one) submanifold of the twistor space \( \mathcal{Z} \).

We have seen that solutions of the Hermitian Yang-Mills equations on the complex projective space \( CP^3 \), represented either as the symmetric space \( SU(4)/U(3) \) or as the nonsymmetric homogeneous space \( Sp(2)/Sp(1) \times U(1) \), have a twistor description similar to the four-dimensional case but valid for the reduced twistor spaces \( \mathcal{Z} \) and \( \mathcal{Z}' \) with fibres \( CP^2 \) and \( CP^1 \), respectively, instead of the metric twistor space \( Tw(CP^3) = P(CP^3, SO(6)) \times SO(6)/U(3) \) with fibres \( SO(6)/U(3) \cong CP^3 \).

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