Entrenchment Relations: A Uniform Approach to Nonmonotonicity

Konstantinos Georgatos
Dipartimento di Informatica e Sistemistica
Università di Roma “La Sapienza”
Via Salaria 113, Roma 00198
Italy

Abstract

We show that Gabbay’s nonmonotonic consequence relations can be reduced to a new family of relations, called entrenchment relations. Entrenchment relations provide a direct generalization of epistemic entrenchment and expectation ordering introduced by Gärdenfors and Makinson for the study of belief revision and expectation inference, respectively.

1 Introduction

Nonmonotonicity has offered great promise as a logical foundation for knowledge representation formalisms. The reason for such a promise is that nonmonotonic logic allows “jumping” to conclusions, completes in a reasonable way our (incomplete) knowledge and withdraws conclusions in the light of new information. Therefore, most approaches to central problems of Artificial Intelligence, such as belief revision, database updating, abduction and action planning, seem to rely on one way or another to some form of nonmonotonic reasoning.

There are several proposals of logical systems performing nonmonotonic inference. Among the most popular of them are: circumscription, negation as failure, default logic, (fixed points of) various modal logics and inheritance systems. However, and despite the numerous results intertranslating one of the above systems to the other, none of the above formalisms emerged as a dominant logical framework under which all other nonmonotonic formalisms can be classified, compared and reveal their logical content. This fact signifies that our intuitions on the process of nonmonotonic inference are fragmented. Although, all the above mentioned logics are worth be studied and employed as a central inference mechanism they cannot serve as the place where finally our basic intuitions about nonmonotonicity can rest.

*Work supported by Training through Research Contract No. ERBFMBICT950324 between the European Community and Università degli Studi di Roma “La Sapienza”.*
Addressing this problem, Gabbay in [Gab85] proposed to study nonmonotonic inference through Gentzen-like context sensitive sequents. Following this proposal, a new line of research flourished by studying properties of the so-called nonmonotonic consequence relations leading to a semantic characterization through (a generalization of) Shoham’s preferential models. This line of research led to classification of several nonmonotonic formalisms and recognized several logical properties properties that a nonmonotonic system should desirably satisfy such as cumulativity or distributivity. However, there are two disadvantages of this framework:

- nonmonotonic consequence relations express the sceptical inference of a nonmonotonic proof system and therefore fail to describe nonmonotonicity in its full generality, that is, the existence of multiple extensions.

- it does not seem that there is a straightforward way to design a nonmonotonic consequence relation from existing data unless they already encode some short of conditional information (see [LM92]).

These two disadvantages suggest that a nonmonotonic consequence relation is not a primitive notion but derived from a more basic inference mechanism.

In this paper, we shall introduce a novel framework for generating nonmonotonic inference, through a class of relations, called entrenchment relations. We shall see that the framework of entrenchment relations is at least as expressive than that of nonmonotonic consequence relations. In particular, nonmonotonic consequence relations can be reduced to entrenchment relations (in the classical case) while the inference defined through entrenchment relations admits and identifies the existence of multiple extensions. On the other hand, entrenchment relations seem to build inference easily and from the bottom up. Simple frequency data, from example generates easily at least one class of them (rational orderings — see [AG96]).

Entrenchment relations are relations and will be denoted by $\preceq$. $\alpha \preceq \beta$ will be read as

$\beta$ is at least as entrenched as $\alpha$

in the sense that “$\alpha$ is more defeasible than $\beta$”. In other words, “if $\alpha$ is accepted then so is $\beta$”. For example, consider the partial description of a (transitive) entrenchment relation in Figure 1.

In that figure, a path upwards from $\alpha$ to $\beta$ indicates that $\alpha \preceq \beta$, where $\preceq$ denotes the entrenchment relation. The entrenchment relation of Figure 1 says for example that $\perp$ is less entrenched than all formulas, $\neg f$ is less entrenched than $b$, $p$ and $p \land f$, and $f$ is less entrenched than $p$.

How will an entrenchment relation be used for inference? The idea is simple. We shall use entrenchment for excluding sentences.

A sentence $\alpha$ will infer (in a nonmonotonic way) another sentence $\beta$ if $\alpha$ together with a sentence $\gamma$, that is not less entrenched than $\neg \alpha$, (classically) imply $\beta$. 

2
The reason we exclude sentences less entrenched than \( \lnot \alpha \) is that if we allow such a sentence then we should also allow \( \lnot \alpha \). However this will bring inconsistency. For instance, using the above example and assuming \( p \) we should exclude \( \lnot p \), \( \lnot b \) and \( \lnot f \). We remain with \( p \rightarrow \lnot f \). Adding \( p \rightarrow \lnot f \) to the classical theory of \( p \) we have that \( p \) nonmonotonically implies \( \lnot f \). Similarly, assuming \( b \) we exclude \( \lnot b \) and \( \lnot f \). We remain with \( \lnot p \) and \( f \). So, \( b \) nonmonotonically implies \( f \) and \( \lnot p \). With no assumptions we have two consistent sets of sentences that remain after excluding \( \perp \): \( \{ p \rightarrow \lnot f, \lnot p, f \} \) and \( \{ p \rightarrow \lnot f, \lnot p, \lnot b, \lnot f \} \). Therefore, it is possible to have more than one possibilities for extending the theory of our assumptions and that leads to the well-known phenomenon of multiple extensions.

It is clear that our framework separates nonmonotonic inference to two different monotonic proof procedures: one positive and the other negative. The claim that entrenchment relations is a useful concept towards our understanding of nonmonotonicity will be substantiated by a series of representation results. We shall show that Gabbay’s nonmonotonic consequences relations can be expressed through entrenchment relations, and identify those classes of entrenchment relations which correspond to the classes of nonmonotonic consequence relations that have attracted special interest in the literature. In addition our framework provides more:

- **Uniformity.** Inference defined through an entrenchment relation remains the same throughout the above characterization.

- **Monotonicity.** Any strong cumulative nonmonotonic inference relies on a monotonic (on both sides) entrenchment relation.

- **Identification of multiple extensions.** The way we define inference allows the identification of multiple extensions. Therefore, both the sceptical and credulous approach towards nonmonotonicity are expressible in our framework.
• A conceptually primitive view of nonmonotonicity. In our framework, a nonmonotonic formalism separates into two logical mechanisms handling positive and negative information.

Entrenchment relations provide a generalization of Gärdenfors-Makinson’s expectation orderings introduced for the characterization of expectation nonmonotonic consequence relations ([GM94]). This result was later extended to rational consequence relations in [Geo96]. In [FadCHL94], incompletely specified expectation orderings were studied. But, to our knowledge, there is no study of such relations outside the non-Horn classes of nonmonotonic consequence relations. This paper fills exactly this gap by showing that all nonmonotonic consequence relations can be represented through entrenchment relations.

However, entrenchment relations have a close relative in the study of belief revision, called epistemic entrenchments ([GM88]). Epistemic entrenchments proved to be very useful for belief revision and became the standard tool ([]) for studying the AGM postulates ([AGM85]). Moreover, generalizations of epistemic entrenchment have been proposed by Lindström-Rabinowicz ([LR91]) and Rott ([Rot92]). They both proposed to drop linearity from epistemic entrenchment. Lindström-Rabinowicz used such a partial ordering for the study of relational belief revision. On the other hand, Rott’s generalized epistemic entrenchments, use the original Gärdenfors-Makinson syntactic translation for generating belief revision functions. As a consequence, Rott characterizes non-Horn belief revision functions with Horn epistemic entrenchments and vice versa. Therefore, our results cannot be derived, even through a suitable translation, by the above works, although intuition and motivation should be credited on both of them.

Relations of expectation orderings with other systems performing some sort of nonmonotonic reasoning are abundant ([Bou92a],[Bou92b],[Lam91],[Lam92],[Wob92],[DP91]), such as Pearl’s system $Z$ ([Pea90]), conditional logic ([Sta68],[Lew73]), and possibilistic logic ([DP88]). It is worth mentioning that orderings appear abundantly in the literature of nonmonotonic logic. Orderings of models lead to the preferential model framework ([Sho88],[KLM90],[KS91],[Mak94]), while ordering of sentences lead to prioritization. Most nonmonotonic formalisms have been enriched with priority handling. However, entrenchment relations are not priorities but rather rules for extending a special form of priority statements. The connection of priority statements with entrenchment relations are similar to that of sequents with proof rules.

The further contents of this paper are as follows. In Section 2, we present entrenchment relations. We discuss their informal meaning and present various properties of them. Then, in Section 3, we define the notion of maxiconsistent and weak maxiconsistent inference as derived from a pair of relations. Both inference schemes can generate all nonmonotonic consequence relations. In Section 4, we review nonmonotonic consequence relations and present our representation results. In Section 5, we summarize.
Gärdenfors and Makinson recently showed (GM94) that the study of a strong non-Horn class of nonmonotonic consequence relations, called *entrenchment inference relations*, can be reduced to the study of a particular class of linear preorders among sentences called *entrenchment orderings*. Subsequently, in Geo96, the author extended this result to the well-known Lehmann and Magidor’s class of rational inference relations (LM92). The purpose of this paper is to show that the study of *all* nonmonotonic consequence relations can be reduced to the study of relations among sentences which generalize the class of above mentioned orderings.

The interpretation of entrenchment orderings which Gärdenfors and Makinson proved equivalent to entrenchment inference relations is the following. Assume there is an ordering \(\preceq\) of the sentences of a propositional language \(\mathcal{L}\), where \(\alpha \preceq \beta\) means “\(\beta\) is at least as entrenched as \(\alpha\)” or “\(\beta\) is at least as surprising as \(\alpha\)”. Therefore, \(\preceq\) is a relation comparing degrees of defeasibility among sentences.

This interpretation of \(\preceq\), as well as a similar one based on possibility given in FadCHL94, although seems fit for the particular class of nonmonotonic inferences it characterizes, has in our opinion the following disadvantages. First, it has a complicated flavor by relying on notions such as expectation, defeasibility, and surprise that are far from primitive. Second, it points to a semantical interpretation by committing to a subjective evaluation of sentences and therefore is lacking the proof-theoretic interpretation meant for relations generating inference. Finally, this interpretation loses its plausibility once we weaken one of its defining properties (for example linearity or transitivity) and restricts us to a unique class of orderings.

Entrenchment relations are nothing more than a generalization of the above ordering. We will drop first linearity of the preorder, for characterizing preferential inference, and subsequently transitivity. Note that the entrenchment interpretation is weakened once we drop transitivity: if a sentence \(\alpha\) is less entrenched than \(\beta\) and \(\beta\) less entrenched than \(\gamma\), then \(\alpha\) should be less entrenched than \(\gamma\). However, inference through an entrenchment relation remains the same, that is we still exclude sentences that relate to \(\neg\alpha\), i.e. \(\beta \preceq \neg\alpha\). Therefore the notion of entrenchment becomes contextual. The situation is similar to that of a consequence relation that it is not necessarily monotonic, that is, just as \(\alpha \vdash \gamma\) but not necessarily \(\beta \vdash \gamma\) whenever \(\beta \vdash \alpha\). However, our representation remains useful as we can express multiple extensions.

Here are our assumptions on the language. We assume a language \(\mathcal{L}\) of propositional constants closed under the boolean connectives \(\lor\) (disjunction), \(\land\) (conjunction), \(\neg\) (negation) and \(\rightarrow\) (implication). We use greek letters \(\alpha, \beta, \gamma\), etc. for propositional variables. We also assume an underlying consequence relation that it will act as the underlying proof-theoretic mechanism. For all practical purposes, it can be thought as classical propositional calculus, but all following definitions and theorems can be carried out in any consequence relation \(\vdash \subseteq 2^\mathcal{L} \times \mathcal{L}\) that includes classical propositional logic, satisfies compactness (i.e.,
if \( X \vdash \beta \) then there exists a finite subset \( Y \) of \( X \) such that \( Y \vdash \beta \).\(^1\) the deduction theorem (i.e., \( X, \alpha \vdash \beta \) if and only if \( X \vdash \alpha \rightarrow \beta \)) and disjunction in premises (i.e., if \( X, \alpha \vdash \beta \) and \( X, \gamma \vdash \beta \) then \( X, \alpha \lor \gamma \vdash \beta \)). We denote the consequences of \( \alpha \) with \( \text{Cn}(\alpha) \). We should add that nonmonotonic inference which does not contain classical tautologies is a rather rare exemption.

Now, let us assume a relation \( \preceq \) between sentences of \( \mathcal{L} \). \( \alpha \preceq \beta \) should be interpreted as

\[ \beta \text{ is (at least) as entrenched as } \alpha. \]

Now, read \( \preceq \) as depending on \( \beta \), that is as a unary predicate indexed by \( \beta \). Therefore, if we strengthen the left part we expect this relation to hold. On the other hand, sentences on the right of \( \preceq \) express context, so properties imposed on that part translate to our conception of context. We can be either monotonic or non-monotonic on context. We will see that either way can still generate nonmonotonic inference. What then would the properties of \( \preceq \) be? We shall assume the following three basic properties:

1. \( \alpha \preceq \alpha \) (Reflexivity)
2. If \( \alpha \vdash \beta \) and \( \beta \preceq \gamma \) then \( \alpha \preceq \gamma \). (Left Monotonicity)
3. If \( \alpha \vdash \beta \) and \( \beta \vdash \alpha \) then \( \gamma \preceq \alpha \) iff \( \gamma \preceq \beta \). (Logical Equivalence)

The meaning of Reflexivity is straightforward.

Left Monotonicity says that if \( \beta \) is less entrenched then \( \gamma \) so is any sentence stronger than \( \beta \).

Finally, Logical Equivalence says that two logically equivalent sentences (under \( \vdash \)) construct the same context and therefore if a sentence is less entrenched than one of them must be less entrenched than the other as well.

We summarize the above in the following definition of entrenchment frame.

**Definition 1** An entrenchment frame is a pair \( \langle \mathcal{L}, \vdash, \preceq \rangle \), where \( \preceq \) is a relation on \( \mathcal{L} \times \mathcal{L} \), called entrenchment relation, that satisfies the above properties, that is Reflexivity, Left Monotonicity and Logical Equivalence.

All properties of entrenchment relations mentioned in the subsequent appear on Table \[.\]

The following property has been considered in the framework of entrenchment orderings ([GM94])

\[ \text{If } \alpha \vdash \beta \text{ then } \alpha \preceq \beta. \] (Dominance)

In view of Reflexivity, Left Monotonicity implies Dominance. Given Dominance and Reflexivity of \( \vdash \), Reflexivity of \( \preceq \) follows. Dominance is a very useful property that is used abundantly in the subsequent and was in fact one of the defining properties of Gärdenfors and Makinson’s entrenchment ordering and epistemic entrenchment.

The following property is derived by Left Monotonicity

\[ ^1\text{We write } X, \alpha \vdash \beta \text{ for } X \cup \{\alpha\} \vdash \beta. \]
\( \alpha \preceq \alpha \)  
(Reflexivity)

\[
\frac{\alpha \vdash \beta \quad \beta \preceq \gamma}{\alpha \preceq \gamma}
\]
(Left Monotonicity)

\[
\frac{\alpha \vdash \beta}{\alpha \preceq \beta}
\]
(Dominance)

\[
\frac{\alpha \vdash \beta \quad \beta \preceq \alpha}{\gamma \preceq \alpha}
\]
(Logical Equivalence)

\[
\frac{\alpha \lor \beta \preceq \beta \quad \alpha \lor \beta \preceq \alpha \quad \alpha \lor \gamma \preceq \alpha}{\beta \lor \gamma \preceq \alpha}
\]
(Weak Equivalence)

\[
\frac{\alpha \lor \beta \preceq \alpha \quad \alpha \lor \gamma \preceq \alpha}{\beta \lor \gamma \preceq \alpha}
\]
(Equivalence)

\[
\frac{\alpha \lor \beta \preceq \alpha \lor \gamma \preceq \alpha}{\alpha \lor \gamma \preceq \alpha}
\]
(Weak Left Disjunction)

\[
\frac{\beta \preceq \alpha \quad \gamma \preceq \alpha}{\beta \lor \gamma \preceq \alpha}
\]
(Left Disjunction)

\[
\frac{\alpha \lor \beta \preceq \alpha \lor \beta \quad \alpha \lor \beta \preceq \alpha}{\alpha \lor \gamma \preceq \alpha}
\]
(Weak Bounded Cut)

\[
\frac{\gamma \preceq \alpha \lor \beta \quad \beta \preceq \alpha}{\gamma \preceq \alpha \lor \beta}
\]
(Bounded Cut)

\[
\frac{\alpha \lor \gamma \preceq \alpha \lor \beta \preceq \alpha}{\alpha \lor \beta \preceq \alpha \lor \beta}
\]
(Weak Bounded Right Monotonicity)

\[
\frac{\gamma \preceq \alpha \lor \beta \quad \beta \preceq \alpha}{\gamma \preceq \alpha \lor \beta}
\]
(Bounded Right Monotonicity)

\[
\frac{a_0 \preceq a_n \quad a_n \preceq a_{n-1} \quad \cdots \quad a_1 \preceq a_0}{a_n \preceq a_0}
\]
(Acyclicity)

\[
\frac{a_0 \lor a_1 \preceq a_0 \quad a_1 \lor a_2 \preceq a_1 \quad \cdots \quad a_n \lor a_0 \preceq a_0}{a_0 \lor a_n \preceq a_0}
\]
(Weak Acyclicity)

\[
\frac{\alpha \preceq \beta \quad \beta \vdash \gamma}{\alpha \preceq \gamma}
\]
(Right Monotonicity)

\[
\frac{\gamma \preceq \alpha \quad \gamma \preceq \beta}{\gamma \preceq \alpha \land \beta}
\]
(Right Conjunction)

\[
\frac{\alpha \preceq \beta \quad \beta \preceq \gamma}{\alpha \preceq \gamma}
\]
(Transitivity)

\[
\frac{\alpha \preceq \beta \quad \beta \preceq \alpha}{\alpha \lor \beta \preceq \alpha}
\]
(Connectivity)

Table 1: Properties for entrenchment relations
\[ \alpha \vdash \beta \text{ and } \beta \vdash \alpha \text{ implies } \alpha \preceq \gamma \text{ iff } \beta \preceq \gamma. \]

While Left Monotonicity allows us to strengthen arbitrarily sentences on the left, Bounded Left Disjunction and Left Disjunction allow us to weaken them. These properties amount to a disjunction property. An entrenchment relation will be called disjunctive (weak disjunctive) if it satisfies Left Disjunction (Weak Left Disjunction). Similarly, for entrenchment frames.

The definition of entrenchment frame says nothing about how one should go combining sentences on the right, i.e., combining contexts. Bounded Cut and Right Conjunction express our ability to strengthen the right part so strengthen the context. Bounded Right Monotonicity, Right Monotonicity and Weak Right Monotonicity weaken the right part so weaken the context. It is worth noting that Right Conjunction makes a sentence, less entrenched than another sentence and its negation, less entrenched than all sentences. Right conjunction allow us to combine contexts using conjunction. Bounded Right Monotonicity follows from Right Monotonicity. Weak Bounded Right Monotonicity together with Bounded Cut implies Weak Left Disjunction.

Weak Bounded Right Monotonicity and Weak Bounded Cut together are equivalent to Weak Equivalence.

Bounded Cut and Bounded Right Monotonicity together imply Equivalence. While given Left Disjunction, Equivalence implies Bounded Cut and Bounded Right Monotonicity.

Observe that Transitivity implies Right Monotonicity, and thus Bounded Right Monotonicity. Transitivity is equivalent to Right Monotonicity given Bounded Cut. Transitivity and Dominance implies Left Monotonicity.

Connectivity is the only non-Horn property among the above properties. Therefore, any class of entrenchment connectivity relations satisfying the above properties except Connectivity is closed under intersections. Gärdensfors and Makinson merged Connectivity and Right Conjunction into

\[ \alpha \preceq \alpha \land \beta \text{ or } \beta \preceq \alpha \land \beta. \] (Conjunctiveness)

and along with Dominance and Transitivity make the defining set of properties of Gärdensfors and Makinson’s entrenchment orderings which is the notion we generalize.

In this paper, we will only study Horn properties. To our knowledge previous results concern only non-Horn entrenchment relations satisfying connectivity: entrenchment and rational ordering in [GM94] and [Geo96], respectively.

### 3 Maxiconsistent and Weak Maxiconsistent Inference

We shall now describe an inference scheme based on an entrenchment relation \( \preceq \). We will define two finitary consequence relation, that is subsets of \( \mathcal{L} \times \mathcal{L} \), called maxiconsistent \((\vdash_{\preceq})\) and weak maxiconsistent inference \((\vdash_{\preceq}^w)\).
Our notion of inference is based on maxiconsistency. The idea of using maximal consistent sets for inference is not new. Maximal consistent sets have been used in databases ([FUV83]), conditional logic ([Res64], [Vel76], [Kra81], [Gin86]), and belief revision ([AGM85]). However, the notion of maximal consistency is already present in classical entailment. In order to compute the inferences of a formula \( \alpha \), one can find all maximal consistent sets that do not contain \( \neg \alpha \), that is all prime filters containing \( \alpha \), and take their intersection. This is the filter that contains all theorems of \( \alpha \). Our definition of inference is similar. First, we find all maximal consistent sets whose elements do not have lower entrenchment than \( \neg \alpha \). These sets do not necessarily contain \( \alpha \), as opposed, say, to classical logic. Next, we consider their intersection. If \( \alpha \rightarrow \beta \) is contained in this intersection then \( \alpha \) entails maxiconsistently \( \beta \), that is \( \alpha \vdash ^* \beta \).

**Definition 2** Let \( U \) be a set of formulas and \( \alpha \in \mathcal{L} \). Then the \( \alpha \)-conditionalization \( U^\alpha \) of \( U \) is the set

\[
U^\alpha = \{ \alpha \rightarrow \beta \mid \beta \in U \}.
\]

**Lemma 3** Let \( U, V \) be deductively closed (under \( \vdash \)). We have the following

1. \( U = V \) iff \( U^\alpha = V^\alpha \).
2. \( \text{Cn}(U^\alpha, \alpha) = \text{Cn}(U, \alpha) \).

**Definition 4** Let \( \langle \mathcal{L}, \vdash, \preceq \rangle \) be an entrenchment frame. The set of coherent sentences for a formula \( \alpha \in \mathcal{L} \) is the set

\[
\text{Coh}(\alpha) = \{ \beta \mid \beta \not\preceq \neg \alpha \}.
\]

The base of \( \alpha \) is the set

\[
\mathcal{B}(\alpha) = \{ U \mid U = \text{Cn}(U), U \subseteq \text{Coh}(\alpha) \}.
\]

The weak base of \( \alpha \) is the set

\[
\mathcal{B}^w(\alpha) = \{ U \mid U = \text{Cn}(U), U^\alpha \subseteq \text{Coh}(\alpha) \}.
\]

The maximal base of \( \alpha \) is the set

\[
\mathcal{B}_{\text{max}}(\alpha) = \{ U \mid U \in \mathcal{B}(\alpha) \text{ and if } U' = \text{Cn}(U') \text{ with } U \subset U' \text{ then } U' \not\in \mathcal{B}(\alpha) \}.
\]

The maximal weak base of \( \alpha \) is the set

\[
\mathcal{B}^w_{\text{max}}(\alpha) = \{ U \mid U^\alpha \in \mathcal{B}^w(\alpha) \text{ and if } V = \text{Cn}(V) \text{ with } U^\alpha \subset V^\alpha \text{ then } V^\alpha \not\in \mathcal{B}^w(\alpha) \}.
\]

The extension set of \( \alpha \) is the set

\[
e(\alpha) = \{ \text{Cn}(U, \alpha) \mid U \in \mathcal{B}_{\text{max}}(\alpha) \}.
\]
while the weak extension set of $\alpha$ is the set
\[ e^w(\alpha) = \{ U \mid U \in B^w_{\text{max}}(\alpha) \}, \]

The sceptical extension of $\alpha$ is the set
\[ E(\alpha) = \bigcap e(\alpha), \]
and the sceptical weak extension of $\alpha$ is the set
\[ E^w(\alpha) = \bigcap e^w(\alpha). \]

Now define
\[ \alpha \vdash \preceq \beta \iff \beta \in E(\alpha), \]
and say that $\alpha$ maxiconsequently infers $\beta$ in the entrenchment frame $\langle L, \vdash, \preceq \rangle$.

Also, define
\[ \alpha \vdash^w \preceq \beta \iff \beta \in E^w(\alpha), \]
and say that $\alpha$ weak maxiconsequently infers $\beta$ in the entrenchment frame $\langle L, \vdash, \preceq \rangle$.

Since $L$ and $\vdash$ remain fixed throughout the following we shall usually drop $\langle L, \vdash, \preceq \rangle$ and refer to maxiconsequent inference on an entrenchment consequence relation $\preceq$.

Note, that if $\preceq$ is $\vdash$, that is, if we equate an entrenchment relation with classical provability then both $\vdash \preceq$ and $\vdash^w \preceq$ collapse to classical $\vdash$.

The following lemma deals with inconsistency. In fact, an entrenchment frame is defined in such a way so that it isolates inconsistency. Also, this lemma ensures that whatever theory remains after excluding sets of sentences is consistent. Therefore, bases and weak bases of a sentence $\alpha$ contain only consistent sets with $\alpha$.

**Lemma 5** Given an entrenchment frame $\langle L, \vdash, \preceq \rangle$, the following hold

1. If $\beta \in \text{Coh}(\alpha)$ then $\beta \not\vdash \neg \alpha$.
2. If $U \subseteq \text{Coh}(\alpha)$ and $U = \text{Cn}(U)$ then $U, \alpha \not\vdash \bot$, i.e. $U$ is consistent with $\alpha$.
3. If $U^\alpha \subseteq \text{Coh}(\alpha)$ then $U, \alpha \not\vdash \bot$, i.e. $U$ is consistent with $\alpha$.
4. If $U \in B^w_{\text{max}}(\alpha)$ then $\alpha \in U$.

In the following, we give conditions under which inconsistency is maxiconsequently derivable.

**Lemma 6** Given an entrenchment frame $\langle L, \vdash, \preceq \rangle$, then
\[ \alpha \vdash \preceq \bot \iff \text{Coh}(\alpha) = \emptyset \quad \text{iff} \quad \top \preceq \neg \alpha \quad \text{iff} \quad \beta \preceq \neg \alpha, \text{ for all } \beta \in L. \]
Note that the above Lemma still holds if we replace $\vdash \subseteq$ with $\lhd \subseteq$.

Next we state several properties that maxiconsistent inference entails in an entrenchment frame which will be very useful in the following.

**Lemma 7** Given an entrenchment frame $\langle \mathbb{L}, \vdash, \subseteq \rangle$, the following hold

1. If $\beta \in \text{Coh}(\alpha)$ then $\text{Cn}(\beta) \subseteq \text{Coh}(\alpha)$.
2. If $\alpha \vdash \beta$ then $\alpha \rightarrow \neg \beta \leq \neg \alpha$.
3. If $\alpha \vdash \beta$ then $\neg \beta \leq \neg \alpha$.
4. If $\leq$ is disjunctive then $\alpha \leq \beta$ is equivalent to $\alpha \lor \beta \leq \beta$.

The corresponding lemma to the above lemma for weak maxiconsistent inference is the following.

**Lemma 8** Given an entrenchment frame $\langle \mathbb{L}, \vdash, \subseteq \rangle$, the following hold

1. If $\alpha \rightarrow \beta \in \text{Coh}(\alpha)$ then $\text{Cn}(\beta) \in B_w(\alpha)$.
2. If $\alpha \vdash \beta \leq \beta$ then $\alpha \rightarrow \neg \beta \leq \neg \alpha$.
3. If $\alpha \vdash \beta \leq \beta$ then $\neg \beta \leq \neg \alpha$.

Bases and weak bases relate to each other through the following lemma.

**Lemma 9** Given an entrenchment frame $\langle \mathbb{L}, \vdash, \subseteq \rangle$, the following hold

1. If $U \in B_{\text{max}}(\alpha)$ then $\text{Cn}(U, \alpha) \in B_w(\alpha)$.
2. If $U \in B_{\text{max}}(\alpha)$ then $\text{Cn}(U, \alpha) \subseteq \text{Coh}(\alpha)$.

Note that in Part 3 we do not have $\text{Cn}(U, \alpha) \in B_{\text{max}}(\alpha)$. Otherwise, the two notions of maxiconsistent inference would collapse to each other.

The following lemma shows how different properties of an entrenchment relation translate to corresponding properties of bases in an entrenchment frame.

**Lemma 10** Given an entrenchment frame $\langle \mathbb{L}, \vdash, \subseteq \rangle$, the following hold

1. If $\alpha \vdash \beta$ and $\beta \vdash \alpha$ imply $\text{Coh}(\alpha) = \text{Coh}(\beta) = \text{Coh}(\alpha \land \beta)$.
2. If $\leq$ satisfies Bounded Cut then $\neg \beta \leq \neg \alpha$ implies $\text{Coh}(\alpha) \subseteq \text{Coh}(\alpha \land \beta)$ (so $B(\alpha) \subseteq B(\alpha \land \beta)$).
3. If $\leq$ satisfies Bounded Right Monotonicity then $\neg \beta \leq \neg \alpha$ implies $\text{Coh}(\alpha \land \beta) \subseteq \text{Coh}(\alpha)$ (so $B(\alpha \land \beta) \subseteq B(\alpha)$).
4. If $\leq$ satisfies Bounded Cut and Bounded Right Monotonicity then $\neg \beta \leq \neg \alpha$ implies $\text{Coh}(\alpha \land \beta) = \text{Coh}(\alpha)$ (so $B(\alpha) = B(\alpha \land \beta)$).
5. If $\leq$ satisfies Right Monotonicity then $\alpha \vdash \beta$ implies $\text{Coh}(\alpha) \subseteq \text{Coh}(\beta)$ (so $B(\alpha) \subseteq B(\beta)$).
The corresponding lemma for weak bases is the following.

**Lemma 11** Given an entrenchment frame \((\mathcal{L}, \vdash, \preceq)\), the following hold

1. If \(\alpha \vdash \beta\) and \(\beta \vdash \alpha\) imply \(B^w(\alpha) = B^w(\beta) = B^w(\alpha \land \beta)\).

2. If \(\preceq\) satisfies Weak Bounded Cut then \(-\alpha \lor -\beta \preceq -\alpha\) implies \(B^w(\alpha) \subseteq B^w(\alpha \land \beta)\).

3. If \(\preceq\) satisfies Weak Bounded Right Monotonicity then \(-\alpha \lor -\beta \preceq -\alpha\) implies \(B^w(\alpha \land \beta) \subseteq B^w(\alpha)\).

4. If \(\preceq\) satisfies Weak Bounded Cut and Weak Bounded Right Monotonicity then \(-\alpha \lor -\beta \preceq -\alpha\) implies \(B^w(\alpha \land \beta) = B^w(\alpha)\).

5. If \(\preceq\) satisfies Weak Right Monotonicity then \(\alpha \vdash \beta\) implies \(B^w(\alpha) \subseteq B^w(\beta)\).

The following lemmas and theorems are the most important of this section. They provide us with the converse of Lemma 7 and 8. Through these results we are able to reduce the problem of deciding maxiconsistent inference to a problem of deciding an entrenchment relation. For that we assume that the entrenchment frame is either disjunctive or weak disjunctive.

**Lemma 12** Let \(\preceq\) be a weak disjunctive entrenchment relation. Then

1. \(-\alpha \lor -\beta \preceq -\alpha\) iff \(\alpha \vdash^w_\preceq \beta\).

2. If \(\preceq\) satisfies, in addition, Transitivity and Right Conjunction then \(-\alpha \lor -\beta \preceq -\alpha\) implies \(\alpha \vdash^w_\preceq \beta\).

Given the above lemma, we can state the connection between maxiconsistent and weak maxiconsistent inference on a weak disjunctive entrenchment frame.

**Theorem 13** Let \(\preceq\) be a weak disjunctive entrenchment relation. Then

1. \(\alpha \vdash^w_\preceq \beta\) implies \(\alpha \vdash^{w\prime}_\preceq \beta\).

2. If \(\preceq\) satisfies, in addition, Transitivity and Right Conjunction then \(\alpha \vdash^{w\prime}_\preceq \beta\) implies \(\alpha \vdash^{w\prime}_\preceq \beta\).

The following theorem shows that, for disjunctive entrenchment relations, maxiconsistent inference is decided by a kind of contraposition. We have that \(\alpha\) maxiconsistently infers \(\beta\) if "\(\neg \beta\) is less entrenched than \(\neg \alpha\)".

**Theorem 14** Let \(\preceq\) be a disjunctive entrenchment relation. Then

\(-\beta \preceq \neg \alpha\) if and only if \(\alpha \vdash^{w\prime}_\preceq \beta\).

The following corollary says that maxiconsistent and weak maxiconsistent inference coincide on disjunctive entrenchment frames.

**Corollary 15** Let \(\preceq\) be a disjunctive entrenchment relation. Then

\(\alpha \vdash^{w\prime}_\preceq \beta\) if and only if \(\alpha \vdash^{w\prime}_\preceq \beta\).
4 Nonmonotonic Consequence Relations and their Representations

A recent breakthrough in nonmonotonic logic is the beginning of study of nonmonotonic consequence through postulates for abstract nonmonotonic consequence relations, using Gentzen-like context-sensitive sequents ([Gab85], [Mak89], [KLM90]). The outcome of this research turns out to be valuable in at least two ways

- it provides a sufficiently general axiomatic framework for comparing and classifying nonmonotonic formalisms, and
- it gave rise to new, simpler, and better behaved systems for nonmonotonic reasoning, such as cumulative ([Gab85]), preferential ([KLM90]), and rational ([LM92]) inference relations.

In this paper, we shall present a variety of representations results for nonmonotonic consequence relations through maxiconsistent inference on entrenchment frames.

Before presenting the results of this section (and main results of this paper), we shall define a variety of classes of nonmonotonic consequence relations. The rules mentioned in the following are presented in Table 2. For a motivation of these rules see [KLM90] and [Mak94]. (The latter serves as an excellent introduction to nonmonotonic consequence relations.)

**Definition 16** Following ([KLM90], [LM92], [GM94]), we shall say that a relation $\sim$ on $\mathcal{L}$ is a nonmonotonic consequence relation (based on $\vdash$) if it satisfies Supraclassicality, Left Logical Equivalence, Right Weakening, and And. We call a nonmonotonic consequence relation $\sim$ cumulative if it satisfies, in addition, Cut and Cautious Monotonicity, strongly cumulative if it is cumulative and satisfies, in addition, Loop, preferential if it is cumulative and satisfies, in addition, Or, and rational if it is preferential and satisfies, in addition, Rational Monotonicity.

The most controversial of these rules is Rational Monotony, which, moreover, is non-Horn. For a plausible counterexample, see [Sta94].

The class of nonmonotonic consequence relations is too general and therefore very weak. The class of default inference relations contains sceptical inference of Reiter’s default systems ([Rei80]). Poole systems with constraints ([Poo88]) and cumulative default systems such as the one appeared in [Bre91] belong to the class of cumulative inference relations. Strong cumulativity has no concrete formalism, as far as we know. Inference defined on Poole systems without constraints as well as entailment on classical preferential models belong to the class of preferential inference relations. Finally, ranked operators ([Geo95]), as well as, the AGM belief revision operator belong to the class of rational inference relations.
The first theorem of this section shows that maxiconsistent inference in an arbitrary entrenchment frame is a nonmonotonic consequence relation. All subsequent results assume that the entrenchment frame is either disjunctive or weak disjunctive.

**Theorem 17** Let \((\mathcal{L}, \vdash, \preceq)\) be a entrenchment frame. Then its maxiconsistent inference \(\models \preceq\) is a nonmonotonic consequence relation. Moreover,

1. If \(\preceq\) satisfies Bounded Cut and Bounded Right Monotonicity then \(\models \preceq\) is a cumulative inference relation.

2. If \(\preceq\) satisfies Bounded Cut and Right Monotonicity then \(\models \preceq\) is a strong cumulative inference relation.

3. If \(\preceq\) satisfies Transitivity and Right Conjunction then \(\models \preceq\) is a preferential inference relation.

From now on, we will assume a disjunctive entrenchment relation. The maxiconsistent inference of a disjunctive entrenchment relation will give a canonical

---

**Table 2: Rules for Nonmonotonic Inference**

4.1 Maxiconsistent Inference

The first theorem of this section shows that maxiconsistent inference in an arbitrary entrenchment frame is a nonmonotonic consequence relation. All subsequent results assume that the entrenchment frame is either disjunctive or weak disjunctive.

**Theorem 17** Let \((\mathcal{L}, \vdash, \preceq)\) be a entrenchment frame. Then its maxiconsistent inference \(\models \preceq\) is a nonmonotonic consequence relation. Moreover,

1. If \(\preceq\) satisfies Bounded Cut and Bounded Right Monotonicity then \(\models \preceq\) is a cumulative inference relation.

2. If \(\preceq\) satisfies Bounded Cut and Right Monotonicity then \(\models \preceq\) is a strong cumulative inference relation.

3. If \(\preceq\) satisfies Transitivity and Right Conjunction then \(\models \preceq\) is a preferential inference relation.

From now on, we will assume a disjunctive entrenchment relation. The maxiconsistent inference of a disjunctive entrenchment relation will give a canonical
representation of nonmonotonic consequence relation. The following definitions provide, for each nonmonotonic consequence relation, an entrenchment relation with the same maxiconsistent inference, and conversely.

**Definition 18** Given an entrenchment relation $\preceq$ and a nonmonotonic inference relation $\vdash$, then define a consequence relation $\vdash'$ and a relation $\preceq'$ as follows

(N) $\alpha \vdash' \beta$ iff $\neg \beta \preceq \neg \alpha$

(P) $\alpha \preceq' \beta$ iff $\neg \beta \vdash \neg \alpha$.

We shall also denote $\vdash'$ and $\preceq'$ with $N(\preceq)$ or and $P(\vdash)$, respectively.

Given the above definition one can prove the following lemma

**Lemma 19** Let $\preceq$ and $\vdash$ be an entrenchment and a nonmonotonic consequence relation, respectively. Then

1. $P(N(\preceq)) = \preceq$, and
2. $N(P(\vdash)) = \vdash$.

**Corollary 20** Let $\preceq$ be a disjunctive entrenchment relation. Then

$$N(\preceq) = \vdash_{\preceq},$$

where $\vdash_{\preceq}$ is the maxiconsistent inference of $\preceq$.

We have the following

**Theorem 21** Let $\langle \mathcal{L}, \vdash, \preceq \rangle$ be a disjunctive entrenchment frame. Then the inference relation $\vdash$ defined by $N$ is a nonmonotonic consequence relation such that, for all $\alpha, \beta$ in $\mathcal{L}$,

$$\alpha \vdash \beta \quad \text{iff} \quad \alpha \vdash_{\preceq} \beta.$$  

Moreover, if $\preceq$ satisfies Bounded Cut, Bounded Right Monotonicity, Acyclicity and Conjunction then $\vdash_{\preceq}$ satisfies Cut, Cautious Monotonicity, Loop and Or, respectively.

Going from nonmonotonic consequence relations to disjunctive entrenchment relations, we have the following theorem.

**Theorem 22** Let $\vdash$ be a nonmonotonic inference relation, then the relation $\preceq$ defined by $(P)$ is a disjunctive entrenchment relation such that, for all $\alpha, \beta$ in $\mathcal{L}$,

$$\alpha \vdash \beta \quad \text{iff} \quad \alpha \vdash_{\preceq} \beta.$$  

Moreover, if $\vdash$ satisfies Cut, Cautious Monotonicity, Loop, and Or then $\preceq$ satisfies Bounded Cut, Bounded Right Monotonicity, Acyclicity, and Conjunction, respectively.
4.2 Weak Maxiconsistent Inference

In this section, we will study weak maxiconsistent inference on weak disjunctive entrenchment frames. This will allow us to find better behaved entrenchment relations equivalent with a given nonmonotonic consequence relation. First, a theorem analogous to Theorem 17 which shows that weak maxiconsistent inference is nonmonotonic.

**Theorem 23** Let \( (\mathcal{L}, \vdash, \preceq) \) be an entrenchment frame. Then its weak maxiconsistent inference \( \vdash'_{w\preceq} \) is a nonmonotonic consequence relation.

As in the previous section, we will define maps between the classes of nonmonotonic consequence relations and weak disjunctive entrenchment relations, and conversely.

**Definition 24** Given an entrenchment relation \( \preceq \) and a nonmonotonic inference relation \( \vdash \), then define a consequence relation \( \vdash' \) and a relation \( \preceq' \) as follows

\[
\text{(N→)} \quad \alpha \vdash' \beta \quad \text{iff} \quad \neg \alpha \lor \neg \beta \preceq \neg \alpha
\]

\[
\text{(P→)} \quad \alpha \preceq' \beta \quad \text{iff} \quad \neg \alpha \lor \neg \beta \vdash \neg \alpha
\]

\[
\text{(Ptr)} \quad \alpha \preceq'' \beta \quad \text{iff there exist } \delta_1, \ldots, \delta_n \in \mathcal{L} \text{ such that } \\
\neg \beta \vdash \delta_1, \delta_1 \vdash \delta_2, \ldots, \delta_n \vdash \neg \alpha.
\]

We shall also denote \( \vdash' \), \( \preceq' \) and \( \preceq'' \) with \( N→(\vdash) \), \( P→(\vdash) \) and \( Ptr(\vdash) \), respectively.

Given the above definition one can prove the following lemma

**Lemma 25** Let \( \preceq \) be an entrenchment relation and \( \vdash \) a nonmonotonic consequence relation. Then

1. if \( \preceq \) satisfies Right Monotonicity and Right Conjunction then
   \[ P→(N→(\preceq)) = \preceq, \]
2. \( N→(P→(\vdash)) = \vdash \),
3. if \( \preceq \) is transitive then \( Ptr(N→(\preceq)) = \preceq, \) and
4. if \( \vdash \) satisfies Loop then \( N→(Ptr(\vdash)) = \vdash \).

**Corollary 26** Let \( \preceq \) be a weak disjunctive entrenchment relation. Then

\[ N→(\preceq) = \vdash'_{w\preceq}, \]

where \( \vdash'_{w\preceq} \) is the weak maxiconsistent inference of \( \preceq \).

We now have the following
Theorem 27 Let \( \langle \mathcal{L}, \vdash, \preceq \rangle \) be a weak disjunctive entrenchment frame. Then the inference relation \( \vdash \) defined by \( N_\vdash \) is a nonmonotonic consequence relation such that, for all \( \alpha, \beta \) in \( \mathcal{L} \),

\[
\alpha \vdash \beta \quad \text{iff} \quad \alpha \vdash_{\preceq} \beta.
\]

Moreover, if \( \preceq \) satisfies Weak Bounded Cut, Weak Bounded Right Monotonicity, Weak Acyclicity and Right Conjunction then \( \vdash_{\preceq} \) satisfies Cut, Cautious Monotonicity, Loop and Or, respectively.

We do not have a similar theorem to Theorem 22 because an arbitrary nonmonotonic consequence relation does not define an entrenchment relation through \( (P_\vdash) \). However, it does so if we assume that it is preferential.

Theorem 28 Let \( \vdash \) be a preferential inference relation, then the relation \( \preceq \) defined by \( (P_\vdash) \) is a weak disjunctive and transitive entrenchment relation satisfying Conjunction such that, for all \( \alpha, \beta \) in \( \mathcal{L} \),

\[
\alpha \vdash \beta \quad \text{iff} \quad \alpha \vdash_{\preceq} \beta \quad \text{iff} \quad \alpha \vdash_{\preceq} \beta.
\]

We can characterize strong cumulative inference relations through weak maxiconsistent inference, if we employ \( (P_{tr}) \).

Theorem 29 Let \( \vdash \) be a nonmonotonic consequence relation satisfying Loop, then the relation \( \preceq \) defined by \( (P_{tr}) \) is a weak disjunctive transitive entrenchment relation such that, for all \( \alpha, \beta \) in \( \mathcal{L} \),

\[
\alpha \vdash \beta \quad \text{iff} \quad \alpha \vdash_{\preceq} \beta.
\]

5 Conclusion

In this section, we will give a summary of the correspondence between classes of entrenchment and nonmonotonic consequence relations.

Let \( \mathcal{A} \) be a class of nonmonotonic consequence relations and \( \mathcal{B} \) a class of entrenchment relations. Let \( \mathcal{C}, \mathcal{C}^w \) be maps from \( \mathcal{B} \) to \( \mathcal{A} \) with \( \mathcal{C}(\preceq) = \vdash_{\preceq} \) and \( \mathcal{C}^w(\preceq) = \vdash_{\preceq}^w \), respectively, where \( \vdash_{\preceq} \) and \( \vdash_{\preceq}^w \) are the maxiconsistent and weak maxiconsistent inference on \( \preceq \).

We will say that a class \( \mathcal{A} \) of nonmonotonic consequence relations is dual to a class \( \mathcal{B} \) of entrenchment relations and denote it with \( \mathcal{A} \equiv \mathcal{B} \) if there exists a map \( N : \mathcal{B} \rightarrow \mathcal{A} \) with \( \mathcal{C}(\preceq) = \vdash_{\preceq} \) and \( \mathcal{C}^w(\preceq) = \vdash_{\preceq}^w \), respectively, where \( \vdash_{\preceq} \) and \( \vdash_{\preceq}^w \) are the maxiconsistent and weak maxiconsistent inference on \( \preceq \).

We will say that a class \( \mathcal{A} \) of nonmonotonic consequence relations is a retract of a class \( \mathcal{B} \) of entrenchment relations and denote it with \( \mathcal{A} \models \mathcal{B} \) if there exists a map \( N : \mathcal{B} \rightarrow \mathcal{A} \) and \( \mathcal{C} \circ N = \text{Id}_\mathcal{A} \), where \( \text{Id} \) is the identity map. Similarly, \( \mathcal{A} \) and \( \mathcal{B} \) will be weakly dual and we denote it with \( \mathcal{A} \equiv \mathcal{B} \) if there exists a map \( N : \mathcal{B} \rightarrow \mathcal{A} \) with \( \mathcal{C}(\preceq) = \vdash_{\preceq} \) and \( \mathcal{C}^w(\preceq) = \vdash_{\preceq}^w \).
Table 3: Classes of nonmonotonic and entrenchment relations

A list of all classes of nonmonotonic and entrenchment relations mentioned in the following appear on Table 3.

The classes of nonmonotonic consequence relations relate to each other through the following scheme (right to left direction denotes inclusion).

\[
\begin{array}{c}
\text{NM} \\
\text{D} \\
\text{CM} \\
\text{C} \\
\text{SC} \\
\text{P} \\
\end{array}
\]

Similarly, the classes of entrenchment relations relate to each other as follows.

\[
\begin{array}{c}
\text{E} \\
\text{BC} \\
\text{BCR} \\
\text{BA} \\
\text{TC} \\
\text{BR} \\
\end{array}
\]

Moreover, If \( B \) is any entrenchment relation class then \( d\cdot B \) and \( wd\cdot B \) are \( B \) augmented with Left Disjunction and Weak Left Disjunction, respectively.

Clearly, \( d\cdot B \subseteq wd\cdot B \subseteq B \).

We now have the following corollary

**Corollary 30** The following hold

1. \( \text{NM} \equiv d\cdot \text{E}, \text{NM} \equiv d\cdot \text{E}, \text{NM} \models \text{E} \), \( \text{NM} \models \text{E} \), and \( \text{NM} \models \text{wd}\cdot \text{E} \).

2. \( \text{D} \equiv d\cdot \text{BC} \) and \( \text{D} \equiv d\cdot \text{BC} \).

18
3. CM ≡ d-BR and CM \equiv d-BR.

4. C ≡ d-BCR, C \equiv d-BCR, C \models BCR and C \models BCR.

5. SC ≡ d-BA, SC \equiv d-BA and SC \equiv wd-T.

6. P \models d-TC, P \equiv wd-TC and P \models wd-TC.
References

[AG96] Gianni Amati and Konstantinos Georgatos. Relevance as Deduction: A Logical View of Information Retrieval. In 2nd International Workshop on Information Retrieval, Uncertainty and Logics, pages 21–27, Glasgow, UK, July 1996.

[AGM85] C. E. Alchourrón, Peter Gärdenfors, and David Makinson. On the logic of theory change: partial meet contraction and revision functions. Journal of Symbolic Logic, 50:510–530, 1985.

[Bou92a] Craig Boutilier. Conditional logics for default reasoning. Ph.D. Thesis, 1992.

[Bou92b] Craig Boutilier. Modal logics for qualitative possibility and beliefs. In Proceedings of the 8th Workshop on Uncertainty in Artificial Intelligence, pages 17–24, Stanford, CA, 1992.

[Bre91] G. Brewka. Cumulative default logic: in defence of nonmonotonic inference rules. Artificial Intelligence, 50:183–205, 1991.

[DP88] D. Dubois and H. Prade. Possibility Theory: An Approach to Computerized Processing of Uncertainty. Plenum, New York, 1988.

[DP91] D. Dubois and H. Prade. Epistemic entrenchment and possibilistic logic. Artificial Intelligence, 50:223–239, 1991.

[FadCHL94] Luis Farinas del Cerro, Andreas Herzig, and Jerôme Lang. From ordering-based nonmonotonic reasoning to conditional logics. Artificial Intelligence, 66:375–393, 1994.

[FUV83] Ronald Fagin, J. D. Ullman, and Moshe Y. Vardi. On the semantics of updates in databases. In Proceedings of the Second ACM SIGACT-SIGMOD, pages 352–365, 1983.

[Gab85] Dov Gabbay. Theoretical foundations for nonmonotonic reasoning in expert systems. In K. Apt, editor, Logics and Models of Concurrent Systems. Springer-Verlag, Berlin, 1985.

[Geo95] Konstantinos Georgatos. Default logic and nonmonotonic consequence relations (abstract). In 10th International Congress of Logic, Methodology and Philosophy of Science, page 97, Florence, Italy, 1995.

[Geo96] Konstantinos Georgatos. Ordering-based representations of rational inference. In José Júlio Alferes, Luís Moniz Pereira, and Ewa Orłowska, editors, Logics in Artificial Intelligence (JELIA ’96), number 1126 in Lecture Notes in Artificial Intelligence, pages 176–191, Berlin, 1996. Springer-Verlag.
[Gin86] Matthew L. Ginsberg. Counterfactuals. *Artificial Intelligence*, 30:35–79, 1986.

[GM88] Peter Gärdenfors and David Makinson. Revisions of knowledge systems using epistemic entrenchment. In *Proceedings of the Second Conference on Theoretical Aspects of Reasoning about Knowledge*, pages 661–672, 1988.

[GM94] Peter Gärdenfors and David Makinson. Nonmonotonic inference based on expectations. *Artificial Intelligence*, 65:197–245, 1994.

[KLM90] S. Kraus, Daniel Lehmann, and Menachem Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44:167–207, 1990.

[Kra81] A. Kratzer. Partition and revision: The semantics of conditionals. *Journal of Philosophical Logic*, 10:201–216, 1981.

[KS91] Hirofumi Katsuno and D. Satoh. A unified view of consequence relation, belief revision and conditional logic. In *Proceedings of IJCAI-91*, pages 406–412, Sydney, Australia, 1991.

[Lam91] P. Lamarre. S4 as the conditional logic of nonmonotonicity. In *Proceedings of KR91*, pages 357–367, Cambridge, MA, 1991.

[Lam92] P. Lamarre. From monotonicity to nonmonotonicity via a theorem prover. In *Proceedings of KR92*, pages 572–580, Cambridge, MA, 1992.

[Lew73] D. Lewis. *Counterfactuals*. Harvard University Press, Cambridge, MA, 1973.

[LM92] Daniel Lehmann and Menachem Magidor. What does a conditional knowledge base entail? *Artificial Intelligence*, 55:1–60, 1992.

[LR91] S. Lindström and W. Rabinowicz. Epistemic entrenchment with incomparabilities and relational belief revision. In A. Fuhrmann and M. Morreau, editors, *The Logic of Theory Change*, number 465 in Lecture Notes in Artificial Intelligence, pages 93–126, Berlin, 1991. Springer-Verlag.

[Mak89] David Makinson. General theory of cumulative inference. In M. Reinfranck, editor, *Non-Monotonic Reasoning*, number 346 in Lecture Notes in Artificial Intelligence, pages 1–18. Springer-Verlag, Berlin, 1989.

[Mak94] David Makinson. General patterns in nonmonotonic reasoning. In Dov Gabbay, C. Hogger, and Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming*, volume III, pages 35–110. Oxford University Press, 1994.
[Pea90] Judea Pearl. System Z: a natural ordering of defaults with tractable applications to default reasoning. In Proceedings of the 3rd Conference on Theoretical Aspects of Reasoning about Knowledge, pages 121–135, Pacific Grove, CA, 1990.

[Poo88] D. Poole. A logical framework for default reasoning. Artificial Intelligence, 36:27–47, 1988.

[Rei80] R. Reiter. A logic for default reasoning. Artificial Intelligence, 2:147–187, 1980.

[Res64] N. Rescher. Hypothetical Reasoning. North-Holland, Amsterdam, 1964.

[Rot92] H. Rott. Preferential belief change using generalized epistemic entrenchment. Journal of Logic, Language and Information, 1:45–78, 1992.

[Sho88] Y. Shoham. Reasoning about Change. MIT Press, Cambridge, 1988.

[Sta68] Robert Stalnaker. A theory of conditionals. In N. Rescher, editor, Studies in Logical Theory. Oxford University Press, Oxford, 1968.

[Sta94] Robert Stalnaker. What is a nonmonotonic consequence relation? Fundamenta Informaticae, 21:7–21, 1994.

[Vel76] F. Veltman. Prejudices, presuppositions and the theory of counterfactuals. In J. Groenendijk and M. Stokhof, editors, Amsterdam Papers in Formal Grammar, volume I. University of Amsterdam, 1976.

[Wob92] W. Wobcke. On the use of epistemic entrenchment in nonmonotonic reasoning. In Proceedings of ECAI-92, pages 324–328, Vienna, Austria, 1992.