Symplectic Resolutions for Quotient Singularities

Baohua FU

June 8, 2021

Abstract

Let $X$ be a smooth irreducible complex variety and $G$ a finite subgroup of $Aut(X)$. There is a natural action of $G$ on $T^*X$ preserving the canonical symplectic form. We show that if $T^*X/G$ admits a symplectic resolution $\pi : Z \to T^*X/G$, then $X/G$ is smooth and $Z$ contains an open set isomorphic to $T^*(X/G)$. In the case of $X = \mathbb{P}^n$ and $G \subset SL(n + 1, \mathbb{C})$, we give an McKay correspondence type description for the Hodge numbers of $Z$.

1 Introduction

Recall that a regular 2-form on a smooth complex algebraic variety is symplectic if it is closed and non-degenerate at every point. A resolution is a projective morphism from a smooth variety onto an algebraic variety which is an isomorphism outside of the singular locus of $X$. In the pioneering paper [Be], A. Beauville has proposed and initiated the following notion of symplectic singularities.

Definition 1. A normal complex algebraic variety $X$ is said to have symplectic singularities if there exists a regular symplectic 2-form $\omega$ on $X_{\text{reg}}$ such that for any resolution of singularities $\pi : \tilde{X} \to X$, the 2-form $\pi^*\omega$ defined a priori on $\pi^{-1}(X_{\text{reg}})$ can be extended to a regular 2-form on $\tilde{X}$. If furthermore the 2-form $\pi^*\omega$ extends to a symplectic 2-form on the whole of $\tilde{X}$ for some resolution of $X$, then we say that $X$ admits a symplectic resolution.
As shown in [Fu] (see also [Ka1] and [Ve]), a resolution of symplectic singularities is symplectic if and only if it is crepant. An important class of examples of symplectic singularities is the normalization of the closure of a nilpotent orbit in a semi-simple Lie algebra. For these singularities, the problem of the existence of symplectic resolutions has been completely solved in [Fu].

**Convention:** throughout this note, all varieties are supposed to be quasi-projective, which guarantees the existence of the quotient by a finite group.

Another class of examples of symplectic singularities comes from the following:

**Proposition 1.1 ([Be], Prop. 2.4).** Let $V$ be a variety with symplectic singularities, $G$ a finite group of automorphisms of $V$, preserving a symplectic 2-form on $V_{\text{reg}}$. Then the variety $V/G$ has symplectic singularities.

A very particular case is the quotient of a complex vector space $\mathbb{C}^{2n}$ by a finite group $G$ of symplectic automorphisms. The problem of the existence of symplectic resolutions for such quotient singularities has been studied by D.Kaledin in [Ka1] and M.Verbitsky in [Ve]. In particular, they have shown that the existence of a symplectic resolution for $\mathbb{C}^{2n}/G$ implies that $G$ is generated by symplectic reflections, i.e. by the elements $g \in G$ such that the fixed point set by $g$ in $\mathbb{C}^{2n}$ is of co-dimension 2. However the complete classification problem of $G$ such that $\mathbb{C}^{2n}/G$ admits a symplectic resolution is far from being solved.

The importance of this problem comes from the generalized McKay correspondence. If a symplectic resolution $\pi : Z \to \mathbb{C}^{2n}/G$ exists, then the McKay correspondence relates the geometry of $Z$ to some properties of $G$. An example is the correspondence between a “natural” basis of $H_{BM}^*(Z, \mathbb{Q})$ and the conjugacy classes of $G$ (see [Ka2] and [Br]), here $H_{BM}^*(Z, \mathbb{Q})$ is the Borel-Moore homology groups.

In this note, we study the existence of symplectic resolutions for general quotient symplectic singularities. We prove in section 2 the following:

**Theorem 1.2.** Let $V$ be a smooth irreducible symplectic variety and $G$ a finite subgroup of $\text{Aut}(V)$ which preserves the symplectic form on $V$. Suppose that $V/G$ admits a symplectic resolution. Then the closed subvariety $\bigcup_{g \neq 1} \text{Fix}(g)$ is either empty or of pure codimension 2 in $V$.

As an easy corollary, an isolated quotient symplectic singularity admits a symplectic resolution if and only if it is of dimension 2. In section 3, we
restrict to the particular case of \( V = T^*X \), where \( X \) is a smooth irreducible complex algebraic variety, and \( G \) a finite subgroup of \( \text{Aut}(X) \) which acts naturally on \( T^*X \) preserving the canonical symplectic structure on \( V \). Then using some ideas of Kaledin [Ka1], we prove

**Theorem 1.3.** If the quotient \( T^*X/G \) admits a symplectic resolution \( \pi : Z \to T^*X/G \), then \( X/G \) is smooth and there exists a Zariski open set \( U \) of \( Z \) which is isomorphic to the total space of the cotangent bundle \( T^*(X/G) \) of \( X/G \).

The idea of the proof is to study the natural \( \mathbb{C}^* \)-action on \( T^*X/G \), which lifts on \( Z \). The symplectic form \( \Omega \) on \( Z \) satisfies \( \lambda^* \Omega = \lambda \Omega \), which makes the arguments in [Fu] (see also [Ka] and [Nak]) possible.

Applying this to the case of \( X = \mathbb{C}^n \), we get

**Corollary 1.4.** If \( \pi : Z \to \mathbb{C}^n \oplus \mathbb{C}^n/G \) is a symplectic resolution, then \( Z \) is simply connected and rational.

In section 4, we study McKay correspondence for symplectic resolutions of \( T^*X/G \). The key tool is the orbit E-functions (see [Ba]), which is shown to be equal to the string E-function in loc.cit.. In the particular case of \( X = \mathbb{P}^n \), we prove the following:

**Theorem 1.5.** Let \( X = \mathbb{P}^n \) be the projective space and \( G \) a finite group in \( \text{SL}(n+1, \mathbb{C}) \). Suppose that \( \pi : Z \to T^*X/G \) is a symplectic resolution. Then the Hodge numbers of \( Z \) can be calculated by the following

\[
\sum_{p+q} (-1)^p h^{p,q}(Z) u^p v^q = (uv)^n \sum_{g} \sum_{i} \frac{(uv)^{k_i(g)} - 1}{uv - 1},
\]

where \( k_i(g) \) are multiplicities of distinct eigenvalues of \( g \in \text{SL}(n+1, \mathbb{C}) \) and \( \{g\} \) runs over conjugacy classes in \( G \) considered as a subgroup in \( \text{Aut}(\mathbb{P}^n) \).

**Corollary 1.6.** The Euler number of \( Z \) is given by \( e(Z) = (n+1)c(G) \), where \( c(G) \) is the number of conjugacy classes of elements in \( G \) considered as a subgroup in \( \text{Aut}(\mathbb{P}^n) \).

In the last section, we prove Göttessche’s formula for Hilbert schemes of the cotangent bundle of a curve, which has also been proved by H.Nakajima [Nak] using another approach.
Acknowledgments: I want to thank D.Kaledin, C.Walter and J.Wierzba for some helpful discussions. I am especially grateful to A.Beauville and A.Hirschowitz for many interesting discussions and suggestions. Without their help, this work could never have been done.

2 General case

Recall that a variety is \( \mathbb{Q} \)-factorial if any Weil divisor has a multiple that is a Cartier divisor. A smooth variety is \( \mathbb{Q} \)-factorial.

**Theorem 2.1.** Let \( V \) be a smooth irreducible symplectic variety and \( G \) a finite group of symplectic automorphisms. Suppose that \( V/G \) admits a symplectic resolution. Then the closed subvariety \( F = \bigcup_{g \neq 1} Fix(g) \) is either empty or of pure codimension 2 in \( V \).

**Proof.** Being a quotient of a \( \mathbb{Q} \)-factorial normal variety by a finite group, \( V/G \) is again \( \mathbb{Q} \)-factorial and normal. This gives that any component \( W \) of the exceptional set of a resolution \( \pi: Z \to V/G \) is of pure codimension one. On the other hand, by a result of D. Kaledin ([Ka3] proposition 1.2, see also [Nam] corollary 1.5 for the same assertion without assuming projectivity) any symplectic resolution is semismall, i.e. \( 2 \text{codim}(W) \geq \text{codim}(\pi(W)) \), thus \( 2 = 2 \text{codim}(W) \geq \text{codim}(\pi(W)) \). Suppose that \( V/G \) is not smooth, then the singular locus of \( V/G \) is of codim \( \geq 2 \), hence \( \pi(W) \) is of codimension 2.

However, the singular set of \( V/G \) is contained in \( p(F) \) with \( p: V \to V/G \) and hence \( \text{codim}(F) \geq 2 \). It thus suffices to show that \( Fix(g) \) is not of codimension one for any \( g \in G \). But this can be excluded as any element \( g \in G \) is symplectic. \qed

**Remark 1.** The above proof shows that the same assertion holds if \( V \) is only \( \mathbb{Q} \)-factorial with singularities in codimension \( \geq 3 \).

As in dimension 2, quotient singularities are A-D-E singularities which admit crepant resolutions, the above theorem shows:

**Corollary 2.2.** Let \( V \) be a smooth irreducible symplectic variety and \( G \) a finite group of symplectic automorphisms of \( V \). Suppose that \( V/G \) has only isolated singularities. Then \( V/G \) admits a symplectic resolution if and only if \( \dim(V) = 2 \).
Remark 2. In [SB], N. Shepherd-Barron has proved that under some extra
conditions, any symplectic resolution of an isolated symplectic singularity
is isomorphic to the collapsing of the zero-section in the cotangent bundle
\( T^* \mathbb{P}^n \to \overline{O}_{\text{min}}, \) where \( \overline{O}_{\text{min}} \) is the minimal nilpotent orbit in \( \mathfrak{sl}(n+1, \mathbb{C}) \).

3 Case of \( T^* X/G \)

In this section, we will restrict to the following case. Let \( X \) be an \( n \)-
dimensional (quasi-projective) smooth irreducible complex variety and \( G \) a
finite group of automorphisms of \( X \). Let \( T^* X \) be the total space of cotangent
bundle of \( X \) and \( \omega \) the canonical symplectic form on \( T^* X \). The group \( G \) acts
naturally on \( T^* X \) preserving the symplectic form \( \omega \), thus \( T^* X/G \) is a variety
with symplectic singularities. Our purpose here is to give some necessary
conditions on \( G \) for the existence of a symplectic resolution for \( T^* X/G \).

Theorem 3.1. If \( T^* X/G \) admits a symplectic resolution, then \( X/G \) is smooth.

This theorem is essentially proved by Kaledin in [Ka1] (and some of his
other results extend also to our case), where he considered the special case
of \( X = \mathbb{C}^n \). Here we just give an outline of the proof.

Consider the natural \( \mathbb{C}^* \)-action on the vector bundle \( T^* X = T^* X \), which
commutes with the \( G \)-action, so we get a \( \mathbb{C}^* \)-action on \( T^* X/G \). For this
\( \mathbb{C}^* \)-action, we have \( \lambda^* \omega = \lambda \omega \). The fixed points of this action are identified
with the subvariety \( X/G \subset T^* X/G \). Let \( \pi : Z \to T^* X/G \) be a symplectic
resolution and \( \Omega \) the corresponding symplectic form on \( Z \). As shown in [Ka1]
(see also section 3.1 of [Fu]), the \( \mathbb{C}^* \)-action on \( T^* X/G \) lifts to \( Z \) in such a
way that \( \pi \) is \( \mathbb{C}^* \)-equivariant. For this \( \mathbb{C}^* \)-action, we have \( \lambda^* \Omega = \lambda \Omega \). The
key lemma is

Lemma 3.2. For every \( x \in X/G \subset T^* X/G \), there exists at most finitely
many points in \( \pi^{-1}(x) \) which are fixed by the \( \mathbb{C}^* \)-action on \( Z \).

The proof is based on the equation \( \lambda^* \Omega = \lambda \Omega \) and the semismallness of
the map \( \pi \). For details, see prop. 6.3 of [Ka1]. Since a generic point on
\( X/G \) is smooth, the map \( \pi : \pi^{-1}(X/G) \to X/G \) is generically one-to-one and
surjective, thus there exists a connected component \( Y \) of fixed points \( Z^{\mathbb{C}^*} \)
such that \( \pi : Y \to X/G \) is dominant and generically one-to-one. Now by the
above lemma, this map is also finite. Now that \( X/G \) is normal implies that
\( \pi : Y \to X/G \) is in fact an isomorphism. Since \( Z \) is smooth, \( Z^{\mathbb{C}^*} \) is a union
of smooth components, so \( Y \) is smooth, thus \( X/G \) is smooth.
Theorem 3.3. If $T^*X/G$ admits a symplectic resolution $\pi : Z \to T^*X/G$, then $Z$ contains an open set $U$ which is isomorphic to $T^*(X/G)$.

Proof. By the proof of above theorem, there exists a connected component $Y$ of $Z_{\mathbb{C}^*}$ such that $\pi : Y \to X/G$ is an isomorphism. In particular we have $dim(Y) = n$.

For any fixed point $y \in Y$, the action of $\mathbb{C}^*$ on $Z$ induces a weight decomposition $T_yZ = \bigoplus_{p \in \mathbb{Z}} T^p_yZ,$ where $T^p_yZ = \{ v \in T_yZ | \lambda_* v = \lambda^p v \}$, and $T_yY$ is identified to $T^0_yZ$. The equation $\lambda^*\Omega = \Omega$ gives a duality between $T^p_y(Z)$ and $T^{1-p}_y(Z)$. In particular, $dim(T^0_yZ) = dim(T^0_yZ) = dimY = n$, so $T^p_yZ = 0$ for all $p \neq 0, 1$, which gives a decomposition $T_yZ = T_yY \oplus T^1_yZ$ and that $Y$ is Lagrangian with respect to $\Omega$.

Let $U$ be the attraction subvariety of $Y$, i.e. $U = \{ z \in Z | \lim_{\lambda \to 0} \lambda \cdot z \in Y \}$, and $p : U \to Y$ the attraction map. By the work of A.Bialynicki-Birula [BB], the decomposition $T_yZ = T_yY \oplus T^1_yZ$ gives that $U$ a vector bundle of rank $n$ over $Y$, so $U$ is identified with the total space of the normal bundle $N$ of $Y$ in $Z$. Now we establish an isomorphism between $N$ and $T^*Y$ as follows. Denote by $\Omega_{can}$ the canonical symplectic structure on $T^*Y$. Take a point $y \in Y$, and a vector $v \in N_y$. Since $Y$ is Lagrangian in the both symplectic spaces, there exists a unique vector $w \in T^*_yY$ such that $\Omega_y(v, u) = \Omega_{can,y}(w, u)$ for any $u \in T^*_yY$. We define the map $i : N \to T^*Y$ to be $i(v) = w$. It is clear that $i$ is a $\mathbb{C}^*$-equivariant isomorphism.

Here are some immediate corollaries.

Corollary 3.4. If $X/G$ is simply connected, then $Z$ is also simply connected.

Proof. If $X/G$ is simply connected, so is $T^*(X/G)$. Notice that the open map $i : U \simeq T^*(X/G)$ induces a surjective map $\pi_1(U) \to \pi_1(Z)$, so $\pi_1(Z) = 0$. □

Now we will consider the case of $X = \mathbb{C}^n$ and $G$ a finite group in $GL(n, \mathbb{C})$.

Corollary 3.5. Let $\pi : Z \to \mathbb{C}^n \oplus \mathbb{C}^n/G$ be a symplectic resolution, then $Z$ is simply connected and rational.

Proof. By our theorem, $\mathbb{C}^n/G$ is smooth. A classical theorem (§5 Chap. V of [Bo]) implies then $\mathbb{C}^n/G$ is isomorphic to $\mathbb{C}^n$, thus $T^*(\mathbb{C}^n/G)$ is simply connected and rational, which gives the corollary. □
That $Z$ is simply connected has been proved by M.Verbitsky in [Ve].

**Corollary 3.6.** Let $\pi : Z \to \mathbb{C}^n \oplus \mathbb{C}^n / G$ be a symplectic resolution and $F = \pi^{-1}(0)$ the exceptional set over $0$. If $F$ is either $n$-dimensional irreducible or smooth, then $F$ is rational.

**Proof.** Denote by $y_0 \in Y$ the point mapped to $0 \in V \oplus V^*/G$. In both cases, $F$ is irreducible. Since $\pi$ is $\mathbb{C}^*$-equivariant, $F \cap U$ is contained in $T^*_y Y$, so $\mathbb{C}^n \simeq T^*_y Y \subset F$, i.e. $F$ is rational. If $F$ is smooth, then $F \cap U$ is also smooth. Note that $F \cap U$ is a smooth closed cone in $\mathbb{C}^n$, so $F \cap U$ is a linear space $\mathbb{C}^k$, with $k = \text{dim}(F)$, which gives that $F$ is rational. \qed

The sub-variety $F$ is the essential difficulty to understand $Z$. It looks like that every irreducible component of $F$ is rational, but we could not prove this in general. For the 4-dimensional case, this has been proved by J.Wierzba [Wi].

## 4 McKay correspondence

Let $X$ be an algebraic complex variety. The cohomology groups with compact supports $H^*_c(X, \mathbb{C})$ admit a canonical mixed Hodge structure. We will denote by $h^{p,q}(H^k_c(X, \mathbb{C}))$ the dimension of the $(p, q)$-Hodge component of the $k$-th cohomology.

**Definition 2.** We define $e^{p,q} := \sum_{k \geq 0} (-1)^k h^{p,q}(H^k_c(X, \mathbb{C}))$, and the polynomial $E(X; u, v) := \sum_{p,q} e^{p,q}(X) u^p v^q$ is called the E-polynomial of $X$.

Here are some basic properties of $E$-polynomials, for proofs see [BD].

- If the Hodge structure is pure, i.e. $h^{p,q}(H^k_c(X, \mathbb{C})) = 0$ when $p + q \neq k$, then $e^{p,q} = (-1)^{p+q} h^{p,q}(X)$.
- The Euler characteristic of $X$ is equal to $E(X; -1, -1)$.
- If $X$ is a disjoint union of locally closed subvarieties $X_i$, then $E(X; u, v) = \sum_i E(X_i; u, v)$.
- If $X \to Y$ is a locally trivial fibration with fiber $F$, then $E(X; u, v) = E(Y; u, v) \cdot E(F; u, v)$.

Consider a smooth irreducible complex variety $V$. Let $G$ be a finite group in $\text{Aut}(V)$ and $g \in G$ an arbitrary element. To simplify the exposition, we suppose that $G$ preserves a volume form on $V$. Take a connected component
of $V^g$, fixed points by $g$. For a point $x \in W$, the differential $dg$ is an automorphism of $T_x V$. Denote by $e^{2\pi \sqrt{-1}a_i}$ the eigenvalues of $dg$ with $a_i \in \mathbb{Q} \cap [0,1]$. The weight $wt(g, W)$ is defined to be $\sum a_i$. Since $G$ preserves a volume form, $dg$ has determinant 1, this gives that the weight is an integer, thus it is independent of the choice of the point $x$ in the connected component $W$. Moreover, if $h \in G$ commutes with $g$, and $W' = h \cdot W$ another component of $X^g$, then $wt(g, W') = wt(g, W)$. Denote by $C(g)$ the centralizer of $g$ in $G$ and by $C(g, W)$ the subgroup of all elements in $C(g)$ which leaves the component $W$ invariant.

**Definition 3.** The orbifold E-function of a $G$-manifold $V$ is defined to be

$$E_{orb}(V; G; u, v) := \sum_{\{g\}} \sum_{\{W\}} (uv)^{wt(g, W)} E(W/C(g, W); u, v),$$

where $\{g\}$ runs over all conjugacy classes in $G$, and $\{W\}$ runs over the set of representatives of all $C(g)$-orbits in the set of connected components of $V^g$.

In the paper [Ba], V.Batyrev has proved that the orbifold E-function is equal to the string E-function. Applying this to our case, we get

**Theorem 4.1.** Let $X$ be an $n$-dimensional smooth projective variety and $G$ a finite group in $\text{Aut}(X)$. Suppose there exists a symplectic resolution $\pi : Z \to T^*X/G$. Then: (1) the Hodge structure on $Z$ is pure; (2) the Hodge numbers of $Z$ can be calculated by the following:

$$\sum_{p,q} (-1)^{p+q} h^{p,q}(Z) u^p v^q = (uv)^n \sum_{\{g\}} \sum_{\{W\}} E(W/C(g, W); u, v),$$

where $\{g\}$ runs over all conjugacy classes in $G$ and $\{W\}$ runs over the set of representatives of all $C(g)$-orbits in the set of connected components of $X^g$.

**Proof.** Consider the natural $\mathbb{C}^*$-action on $T^*X/G$, which lifts to $Z$. We identify $X/G$ with the fixed points by the $\mathbb{C}$-action in $T^*X/G$. Since $X/G$ is projective and the map $\pi : \pi^{-1}(X/G) \to X/G$ is projective, the variety $\pi^{-1}(X/G)$ is projective. Notice that the fixed points $Z^{\mathbb{C}^*}$ by the $\mathbb{C}^*$-action on $Z$ is contained in $\pi^{-1}(X/G)$ and $Z$ is smooth, so $Z^{\mathbb{C}^*}$ is a union of smooth projective varieties. In particular, the Hodge structure on $Z^{\mathbb{C}^*}$ is pure. Now the same argument of the proof of theorem 8.4 in [Ba] gives the affirmation (1).
For the second affirmation, theorem 7.5 of loc. cit. gives that \( E(Z; u, v) = E_{\text{orb}}(T^*X; G; u, v) \). Since the Hodge structure on \( Z \) is pure, \( E(Z; u, v) = \sum_p (-1)^{p+q} h^{p,q}(Z) u^p v^q \), thus we need to calculate \( E_{\text{orb}}(T^*X; G; u, v) \). The key point here is that every fixed component \( F \) in \( T^*X \) is of the form \( T^*W \), for some fixed component \( W \) in \( X \) and \( C(g, F) = C(g, W) \). The map \( F/C(g, F) \to W/C(g, W) \) is a fibration with fiber \( \mathbb{C}^{\dim(W)} \), so

\[
E(F/C(g, F); u, v) = (uv)^{\dim(W)} E(W/C(g, W); u, v).
\]

Now what we need is to calculate the weight of \( g \) at \( W \). Since \( g \) acts on \( T^*X \) as a symplectic automorphism, lemma 2.6 of [Ka2] gives that \( \text{wt}(g, W) = \frac{1}{2} \text{codim}(T^*W) = \text{codim}(W) \). Now our theorem follow immediately. \( \square \)

A particular simple situation is the following:

**Theorem 4.2.** Let \( X = \mathbb{P}^n \) be the projective space and \( G \) a finite group in \( SL(n+1, \mathbb{C}) \). Suppose that \( \pi : Z \to T^*X/G \) is a symplectic resolution. Then the Hodge numbers of \( Z \) can be calculated by the following

\[
\sum p_q (-1)^{p+q} h^{p,q}(Z) u^p v^q = (uv)^n \sum_{\{g\}} \sum_i (uv)^{k_i(g)} - 1 \frac{1}{uv - 1},
\]

where \( k_i(g) \) are multiplicities of distinct eigenvalues of \( g \in SL(n+1, \mathbb{C}) \) and \( \{g\} \) runs over conjugacy classes in \( G \), which is considered as a subgroup in \( \text{Aut} (\mathbb{P}^n) \).

**Proof.** Let \( p : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n \) be the natural projection. For any \( g \in SL(n+1, \mathbb{C}) \), the point \( p(x) \in \mathbb{P}^n \) is fixed by \( g \) if and only if \( g(x) = \lambda x \) for some \( \lambda \). In particular, \( x \) should be an eigenvector of \( g \). If we denote by \( L_i \) the eigenspaces of \( g \), then the fixed points \( X^g \) are \( X^g_i = p(L_i) \). Notice that if \( h \in G \) commutes with \( g \), then \( g(h(x)) = \lambda h(x) \), i.e. \( h(x) \) and \( x \) are in the same eigenspace of \( g \), which shows that the \( C(g) \)-action on the set of connected components of \( X^g \) is trivial. So the index set \( \{W\} \) in the sum of the orbit E-function is the same as \( \{X^g_i\} \).

Consider an eigenspace \( L_i \), whose dimension is denoted by \( k_i(g) \). Then there is a fibration \( L_i - \{0\}/C(g) \to X^g_i/C(g) \) with fiber \( \mathbb{C}^* \). Now use properties of E-polynomials, we have \( E(L_i - \{0\}/C(g); u, v) = E(L_i/C(g); u, v) - E(\{0\}; u, v) = (uv)^{k_i(g)} - 1 \) and \( E(\mathbb{C}^*; u, v) = uv - 1 \). This gives that

\[
E(X^g_i/C(g); u, v) = (uv)^{k_i(g)} - 1 \frac{1}{uv - 1}.
\]

Now the theorem follows directly from theorem 4.1. \( \square \)
Corollary 4.3. The Euler number of $Z$ is given by $c(Z) = (n + 1)c(G)$, where $c(G)$ is the number of conjugacy classes of elements in $G$, which is considered as a subgroup of $\text{Aut}(\mathbb{P}^n)$.

Corollary 4.4. $H^{2j+1}(Z, \mathbb{C}) = 0$ for all $j$ and $H^{2i}(Z, \mathbb{C}) = 0$ for $i \leq n − 1$. Furthermore $H^{2i}(Z, \mathbb{C})$ has the Hodge type $(i, i)$ for all $i$.

5 One application

Let $\Sigma$ be a smooth projective curve. There is a natural action of the permutation group $S_n$ on the smooth projective variety $\Sigma^n$. This action gives an action of $S_n$ on $T^*(\Sigma^n) ≃ (T^*\Sigma)^n$. As easily seen, the action of $S_n$ on $(T^*\Sigma)^n$ is exactly the permutation action, so $T^*(\Sigma^n)/S_n ≃ S^n(T^*\Sigma)$. The well-known resolution $\text{Hilb}^n(T^*\Sigma) → S^n(T^*\Sigma)$ gives a symplectic resolution for the quotient $T^*(\Sigma^n)/S_n$. Now we will use the McKay correspondence to deduce the cohomology of $\text{Hilb}^n(T^*\Sigma)$.

Notice that the conjugacy classes of the permutation group $S_n$ correspond to partitions of $n$. Let $\nu = (1^{a_1}2^{a_2} \cdots n^{a_n})$ be a partition of $n$, i.e. $a_i \geq 0$ and $\sum_j j a_j = n$. Here $a_i$ is the number of length $i$-cycles in the permutation. Then the fixed points subvariety $\text{Fix}(\nu)$ in $\Sigma^n$ by $\nu$ is isomorphic to $\prod_j \Sigma^{a_j}$. This gives that $\text{Fix}(\nu)/C(\nu) ≃ S^\nu\Sigma := \prod_j S^{a_j}\Sigma$. By our theorem 4.1, we have

$$\sum_k h^{2k}(\text{Hilb}^n(T^*\Sigma))t^{2k} = t^{2n} \sum_{\nu} \sum_j h^{2j}(S^\nu\Sigma)t^{2j}.\$$

By the Poicaré duality and change $t$ to $1/t$, this gives

**Lemma 5.1.**

$$P_t(\text{Hilb}^n(T^*\Sigma)) = \sum_{\nu} t^{2n−2d(\nu)} P_t(S^\nu\Sigma),$$

where $d(\nu) = \sum_j a_j$ is the complex dimension of $S^\nu\Sigma$.

Now using Macdonald’s formula (see [Ma]) for cohomology of symmetric products of algebraic curves, we can deduce Göttsche’s formula in the case of $T^*\Sigma$:

**Proposition 5.2.**

$$\sum_{n=0}^{\infty} P_t(\text{Hilb}^n(T^*\Sigma))q^n = \prod_{d=1}^{\infty} \frac{(1 + t^{2d−1}q^d)b_1(\Sigma)}{(1 − t^{2d−2}q^d)b_0(\Sigma)(1 − t^{2d}q^d)b_2(\Sigma)}.$$
Remark 3. In chap.7 [Nak], H.Nakajima has deduced this formula via an elementary argument. He firstly proved that the fixed points subvariety of the $\mathbb{C}^*$-action in $\text{Hilb}^n(T^*\Sigma)$ is the disjoint union $\bigsqcup_{\nu} S^\nu \Sigma$, then using Morse theory, he proved lemma 7.1.

References

[Ba] V. Batyrev, Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs, J. Eur. Math. Soc. 1, no. 1, 5-33(1999).

[BD] V. Batyrev and D. Dais, Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry, Topology 35, No.4, 901-929 (1996).

[Be] A. Beauville, Symplectic singularities, Invent. Math. 139, 541-549(2000).

[BB] A. Bialynicki-Birula, Some theorems on actions of algebraic groups, Ann. of Math., II. Ser. 98, 480-497 (1973).

[Bo] N. Bourbaki, Groupes et algèbres de Lie, Chap.IV–VI, Hermann(Paris) 1968.

[Fu] B. Fu, Symplectic resolutions for nilpotent orbits, preprint math.AG/0205048, to appear in Invent. Math.

[Ka1] D. Kaledin, Dynkin diagrams and crepant resolutions of quotient singularities, preprint math.AG/9903157.

[Ka2] D. Kaledin, McKay correspondence for symplectic quotient singularities, Invent. Math. 148 1, 150-175(2002).

[Ka3] D. Kaledin, Symplectic resolutions: deformations and birational maps, preprint math.AG/0012008.

[Ma] I. Macdonald, The Poicaré polynomial of a symmetric product, Proc. Camb. Phil. Soc. 58, 563-568 (1962).

[Nak] H. Nakajima, Lectures on Hilbert schemes of points on surfaces, University Lecture Series 18, Providence AMS, 1999.
[Nam] Y. Namikawa, *Deformation theory of singular symplectic n-folds*, Math. Ann. **319**, no. 3, 597-623 (2001).

[Re] M. Reid, *La correspondance de McKay*, Séminaire Bourbaki n.867, preprint math.AG/9911165.

[SB] N. Shepherd-Barron, *Long extremal rays and symplectic resolutions*, preprint.

[Ve] M. Verbitsky, *Holomorphic symplectic geometry and orbifold singularities*, Asian J. Math. **4**, no. 3, 553-563 (2000).

[Wi] J. Wierzba, *Symplectic Singularities*, Ph.D. thesis, Trinity college, Cambridge University, September 2000.

Labortoire J.A.Dieudonné, Parc Valrose
06108 Nice cedex 02, FRANCE
baohua.fu@polytechnique.org