The second-order renormalization group flow for nonlinear sigma models in two dimensions

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Abstract
We show that for two-dimensional manifolds $M$ with negative Euler characteristics, there exist subsets of the space of smooth Riemannian metrics which are invariant and either parabolic or backwards-parabolic for the second-order RG flow. We also show that solutions exist globally on these sets. Finally, we establish the existence of an eternal solution that has both a UV and IR limit, and passes through regions where the flow is parabolic and backwards-parabolic.

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1. Introduction

The worldsheet nonlinear sigma model renormalization group flow arises from quantizing the classical action

$$S(x) = \frac{1}{4\pi \alpha'} \int_\Sigma \gamma^{\alpha\beta} g_{ij}(x) \partial_\alpha x^i \partial_\beta x^j dV(\gamma),$$

where $\alpha' > 0$ is the string coupling constant, $(\Sigma, \gamma)$ is a two-dimensional Riemannian manifold (i.e. worldsheet), $(M, g)$ is an $n$-dimensional Riemannian manifold (i.e. target space), and $x : \Sigma \to M; (\theta^1, \theta^2) \mapsto (x^1(\theta), \ldots, x^n(\theta))$ is a map. A perturbative quantization of this classical theory requires the introduction of a momentum cutoff $\Lambda > 0$, and gives rise to a one-parameter family of quantum field theories indexed by $\Lambda$. The requirement that the family of quantum field theories be equivalent on length scales $L \gg 1/\Lambda$ leads to the renormalization group (RG) flow equations

$$\partial_\Lambda g_{ij} = -\beta_{ij}^\Lambda,$$
In the regime where perturbation theory is valid \((\alpha' \ll 1)\), the \(\beta\)-functions \(\beta_{ij}^g\) can be expanded in powers of \(\alpha'\) [2, 7]:

\[
\beta_{ij}^g = \alpha' R_{ij} + \frac{\alpha'^2}{2} R_{iklm} R_{j}^{klm} + O(\alpha'^3).
\]

Defining a ‘time’ by \(t = -\ln(\Lambda)\), the RG flow equations become

\[
\partial_t g_{ij} = -\alpha' R_{ij} - \frac{\alpha'^2}{2} R_{iklm} R_{j}^{klm} + O(\alpha'^3). 
\tag{1.1}
\]

It is expected that in the perturbative regime, the first-order truncation

\[
\partial_t g_{ij} = -\alpha' R_{ij} 
\tag{1.2}
\]

should provide a ‘good approximation’ to the full flow. However, without an estimate of the error, the notion of a good approximation cannot be quantified. The problem of understanding the error is obstructed by the fact that a mathematically rigorous quantization of the nonlinear sigma model is presently unavailable. However, if it ultimately turns out that expansion (1.1) obtained using perturbation theory is valid for the RG flow, even as an asymptotic expansion, then it is not unreasonable to expect that the error between the full flow and (1.2) will be qualitatively described by the second-order truncation

\[
\partial_t g_{ij} = -\alpha' R_{ij} - \frac{\alpha'^2}{2} R_{iklm} R_{j}^{klm}, 
\tag{1.3}
\]

at least for situations where the curvature is not too large. We note that this expectation is borne out in other field theories where it has been established that it is enough to consider the second-order truncation of the RG flow to establish the existence of a continuum limit [3]. This reinforces the view that knowing the first-order flow is not always enough for applications where quantitative control on the error is required.

As has been noted now many times, the first-order RG flow (1.2) coincides with the Ricci flow. It is known that there are many solutions to the Ricci flow that become singular at a finite time. As shown by Hamilton [6], a singular time \(T\) of the Ricci flow is characterized by curvature blow up: \(\lim_{t \to T} \| R_{ijkl} R^{ijkl} \|_{L^\infty(M)} = \infty\). This suggests that near a singular time for the Ricci flow, the higher-order curvature corrections in (1.1) will dominate the behaviour of the flow even for \(\alpha' \ll 1\). Thus, it is natural from this viewpoint to consider the higher-order truncations of (1.1) to try and capture the effect of higher-order curvature terms.

With the above motivation in mind, the main aim of this paper is to continue the study initiated in [5] of the second-order RG flow (1.3) using techniques from geometric analysis. To facilitate comparisons with the Ricci flow, we rescale the time and metric to bring (1.3) into the form

\[
\partial_t g_{ij} = -2 R_{ij} - \frac{\alpha'}{2} R_{iklm} R_{j}^{klm}, 
\tag{1.4}
\]

which makes the leading term consistent with the standard presentation of the Ricci flow.

Although the Ricci flow may be recovered in the limit \(\alpha' \downarrow 0\), the second-order RG flow differs from the Ricci flow in two important respects: it is fully nonlinear and it is not parabolic for all choices of \(\alpha'\) and \(g_{ij}\). Therefore, in addition to curvature blow up, loss of parabolicity along the flow presents a possible new mechanism for singularity formation. To investigate this possibility, we restrict ourselves to the simplest possible setting of a closed two-dimensional target space \(M\). In this case, the curvature tensor is given by

\[
R_{ijkl} = \frac{1}{2} R (g_{il} g_{jk} - g_{ik} g_{jl})
\]

which implies that the second-order RG equations (1.4) reduce to

\[
\partial_t g_{ij} = -R g_{ij}. 
\tag{1.5}
\]
where
\[ R = R + \frac{\alpha'}{4} R^2. \] (1.6)

Following the Ricci flow terminology, we will call a solution \( g(t) \) to (1.4) (equivalently (1.5)) ancient, immortal or eternal, if the solution exists on a time interval of the form \( -\infty < t < t_0, t_0 < t < \infty \) or \( -\infty < t < \infty \), respectively. We will also use the physics terminology of a UV (IR) limit which refers to a fixed point of (1.4) from which an ancient (immortal) solution originates (terminates). From both a mathematical and physical viewpoint, the existence and classification of the ancient, immortal and eternal solutions are of fundamental interest. For physical applications, the UV limits are of particular importance as they correspond to cut-off removal for the quantum field theory. In other words, a UV limit identifies quantum field theories with a well-defined microscopic limit. The IR limits also have a physical interpretation and correspond to a well-defined macroscopic limit.

The main result of this paper is to show that if \( M \) has negative Euler characteristics, then there exist large regions in the space of smooth Riemannian metrics \( M \) on \( M \) for which the flow is either parabolic or backwards-parabolic, and that these regions remain invariant under the flow. Moreover, on these invariant regions, the flow has good long-term existence properties.

We also establish the existence of an eternal solution to (1.4) with both a UV and IR limit that passes from a region of backwards-parabolicity into a region of parabolicity. We find this solution particularly interesting as it is a consequence of the curvature correction \( \alpha' R_{iklm} R^{iklm} \) term to the Ricci flow. This solution shows that the lack of uniform parabolicity for all choices of \( \alpha' \) and \( g_{ij} \) should not necessarily be viewed as a defect. Instead, the notion of uniform parabolicity should be replaced with that of invariant parabolic or backwards-parabolic sets. As this paper demonstrates, the existence of neighbouring parabolic and backwards-parabolic regions opens up the possibility for constructing solutions by joining together two solutions at a degenerate parabolic point that joins a backwards-parabolic region to a parabolic one.

2. Parabolicity of the second-order RG equations

Due to the conformal nature of the second-order RG flow (1.5), it is consistent given initial data \( g_{ij}|_{t=0} = \tilde{g}_{ij} \) to write \( g_{ij}(t) = e^{u(t)} \tilde{g}_{ij} \) in which case the initial value problem
\[ \partial_t g_{ij} = -R g_{ij} : g_{ij}(0) = \tilde{g}_{ij} \] is equivalent to
\[ \partial_t u = -\mathcal{R}(u) : u(0) = 0, \] (2.1)
where
\[ \mathcal{R}(u) = R + \frac{\alpha'}{4} R^2 \quad \text{and} \quad R = e^{-u}(\Delta u + \tilde{R}). \] (2.2)

Here, \( \tilde{R} \) and \( \Delta \) denote the Ricci scalar and Laplacian of \( \tilde{g}_{ij} \), respectively.

The linearization
\[ D\mathcal{R}(u) \cdot v = \left( 1 + \frac{\alpha'}{2} R \right) (-e^{-u} \Delta v - Rv) \]
shows that (2.2) is parabolic when \( 1 + \frac{\alpha'}{2} R > 0 \) and backwards-parabolic when \( 1 + \frac{\alpha'}{2} R < 0 \). This and the properties of the evolution equation to be described in the following section motivate us to define the following subsets of the space of smooth Riemannian metrics \( \mathcal{M} \):
\[ \mathcal{M}_+ = \left\{ g \in \mathcal{M} \Bigg| -\frac{2}{\alpha'} < R < 0 \right\} \quad \text{and} \quad \mathcal{M}_- = \left\{ g \in \mathcal{M} \Bigg| -\frac{4}{\alpha'} < R < -\frac{2}{\alpha'} \right\}. \] (2.4)
To avoid the situation where these sets are empty, we will, as mentioned in section 1, restrict ourselves to closed manifolds with negative Euler characteristics.

3. Invariance of $\mathcal{M}_+\pm$ and global existence

**Theorem 3.1.** Suppose $\tilde{g} \in \mathcal{M}_+$. Then there exists a smooth one-parameter family of metrics $g(t)$ for $0 \leq t < \infty$ that satisfies the following:

(i) $g(t)$ solves the second-order RG equation (1.4) with $g(0) = \tilde{g}$;

(ii) $g(t) \in \mathcal{M}_+$ for all $t \geq 0$;

(iii) there exists a constant $C_{\tilde{R}}$ depending only on $C^+_R = \max_{x \in M} \bar{R}(x)$ and $C^-_R = \min_{x \in M} \bar{R}(x)$ such that

$$C^-_R \leq R(t, x) \quad \text{and} \quad |R(t, x)| \leq \frac{C_{\tilde{R}}}{1 + t}$$

for all $(t, x) \in [0, \infty) \times M$.

**Proof.** Since equation (2.2) is parabolic whenever $g = e^{\alpha} \tilde{g} \in \mathcal{M}_+$, it follows by standard local existence theorems for parabolic equations (see proposition 8.1, p 338 of [9]) that there exists a smooth solution $u(t)$ to the initial value problem (2.2) for $0 \leq t < T$. Given this solution, a short calculation using (1.5) shows that the Ricci scalar satisfies the equation

$$\partial_t R = \Delta R + RR,$$

or equivalently

$$\partial_t R = \left(1 + \frac{\alpha' R}{2}\right)\Delta R + \frac{\alpha'}{2} |\nabla R|^2 + RR. \quad \text{(3.2)}$$

To control the behaviour of $R(t)$, we will use the maximum principle. This requires us to analyse solutions of the ODE $\frac{dy}{dt} = y^2 + \frac{\alpha'}{4} y^3$.

**Lemma 3.2.** Suppose $y_0 \in \left(-\frac{4}{\alpha'}, 0\right)$. Then the unique solution $y(t)$ to the initial value problem

$$\frac{dy}{dt} = y^2 + \frac{\alpha'}{4} y^3 : y(0) = y_0$$

exists for all $t \geq 0$ and satisfies

$$y_0 \leq y(t) < 0 \quad \text{and} \quad |y(t)| \leq \frac{C_0}{1 + t} \quad \forall t \geq 0, \quad \text{(3.4)}$$

where $C_0$ is a constant that depends only on $y_0$.

**Proof.** Since $y = 0$ and $y = -\frac{4}{\alpha'}$ are the only fixed points of (3.3), $\frac{dy}{dt} > 0$ for $-\frac{4}{\alpha'} < y < 0$, and $y(0) = y_0 \in \left(-\frac{4}{\alpha'}, 0\right)$, it follows that the solution $y(t)$ exists for all $t \geq 0$ and satisfies

$$y_0 \leq y(t) < 0 \quad \forall t \geq 0$$

and

$$\lim_{t \to \infty} y(t) = 0. \quad \text{(3.6)}$$

Next, we observe that (3.3) can be integrated to get

$$\frac{\alpha'}{4} \ln \left(\frac{y_0 (1 + \frac{\alpha'}{4} y(t))}{y(t) (1 + \frac{\alpha'}{4} y_0)}\right) + \frac{1}{y_0} - \frac{1}{y(t)} = t. \quad \text{(3.7)}$$
Together, (3.5) and (3.7) imply that
\[ |y(t)| = \frac{1 - \frac{1}{\alpha} |y(t)| \ln(|y(t)|)}{t + \frac{1}{\ln|\alpha|} - \frac{4}{\alpha} \ln \left( \frac{|y_0(1 + \frac{1}{\alpha} t)|}{1 + \frac{1}{\alpha} t} \right)} \] (3.8)
But \( \lim_{t \to \infty} |y(t)| \ln(|y(t)|) = 0 \) by (3.6), and so it follows from (3.5) and (3.8) that
\[ |y(t)| \leq C_0(1 + t)^{-1} (t \geq 0) \] for some constant \( C_0 \) that depends only on \( y_0 \).

Now, set \( C^+_R = \max_{x \in M} \hat{R}(x) \) and \( C^-_R = \min_{x \in U} \hat{R}(x) \). Then from equation (3.2), lemma 3.2 and the maximum principle (see theorem 4.4, p 96 in [1]), there exists a constant \( C_k \) depending only on \( C^\pm_R \) such that
\[ C^-_R \leq R(t, x) \quad \text{and} \quad |R(t, x)| \leq \frac{C_k}{1 + t} \] (3.9)
for all \( (t, x) \in [0, T) \times M \). Integrating the evolution equation (1.5) in time and applying the inequality (3.9) then yields
\[ |u(t, x)| \leq \ln(1 + t) \quad \forall (t, x) \in [0, T) \times M. \] (3.10)
The inequalities (3.9)–(3.10) together with formulae (2.3) show that
\[ (1 + t)|\partial_t u(t, x)| + |\Delta u(t, x)| \lesssim 1 \quad \forall (t, x) \in [0, T) \times M. \] (3.11)
Clearly, the derivative \( \partial_t u \) satisfies the equation
\[ \partial_t (\partial_t u) = -D R(u) \cdot \partial_t u. \]
By the estimates (3.9)–(3.11), this equation is uniformly parabolic with bounded continuous coefficients. Consequently, we can apply the Krylov–Safonov estimates [8] to conclude the existence of constants \( C_T > 0 \) and \( 0 < \sigma < 1 \) such that
\[ \|\partial_t u(t)\|_{C^{\sigma}(M)} \leq C_T \quad \forall t \in [0, T). \] (3.12)
This implies, increasing \( C_T \) if necessary, that
\[ \|R(t)\|_{C^{\sigma}(M)} \leq C_T \quad \forall t \in [0, T). \] (3.13)
Viewing (2.2) as an elliptic equation for \( u \) with source term \( \partial_t u \), the estimates (3.9)–(3.11), (3.12) and (3.13) allow us to apply Schauder estimates (see lemma 6.16, p 103 of [4]) to conclude
\[ \|u(t)\|_{C^{\sigma}(M)} \leq C_T \quad \forall t \in [0, T). \]
Applying the parabolic continuation principle (see proposition 8.1, p 338 of [9]), the solution can be continued for at least a small time past \( T \). Thus we conclude that the solution \( u(t) \) exists for all \( t \geq 0 \).

**Theorem 3.3.** Suppose \( \hat{g} \in \mathcal{M} \). Then there exists a smooth one-parameter family of metrics \( g(t) \) for \( -\infty < t \leq 0 \) that satisfies the following:

(i) \( g(t) \) solves the second-order RG equation (1.4) with \( g(0) = \hat{g} \);
(ii) \( g(t) \in \mathcal{M} \) for all \( t \leq 0 \);
(iii) there exists a constant \( C_k \) depending only on \( C^+_R = \max_{x \in M} \hat{R}(x) \) and \( C^-_R = \min_{x \in M} \hat{R}(x) \) such that
\[ R(t, x) \leq C^+_R \quad \text{and} \quad \left| R(t, x) + \frac{4}{\alpha^2} \right| \leq C_k e^t \]
for all \( (t, x) \in (-\infty, 0) \times M \).
Proof. Since equation (2.2) is now backwards-parabolic, we instead consider the forwards equation obtained by replacing $t$ with $-t$:
\[ \partial_t u = \mathcal{R}(u). \]
As in the proof of theorem 3.1, we can control the curvature by analysing solutions to the ODE
\[ \frac{dy}{dt} = -y^2 - \frac{\alpha'}{4} y^3. \]

Lemma 3.4. Suppose $y_0 \in \left( -\frac{4}{\alpha'}, 0 \right)$. Then the unique solution $y(t)$ to the initial value problem
\[ \frac{dy}{dt} = -y^2 - \frac{\alpha'}{4} y^3 : y(0) = y_0 \] (3.14)
exists for all $t \geq 0$ and satisfies
\[ -\frac{4}{\alpha'} < y(t) \leq y_0 \quad \text{and} \quad \left| y(t) + \frac{4}{\alpha'} \right| \leq C_0 e^{-t} \quad \forall t \geq 0, \] (3.15)
where $C_0$ is a constant that depends only on $y_0$.

Proof. This proof is essentially the same as the proof of lemma 3.4 except for the asymptotics. To see the exponential convergence, we replace $t$ with $-t$ in formula (3.7) and rearrange to get
\[ \left| 1 + \frac{\alpha'}{4} y(t) \right| = \left| \frac{y(t)(1 + \frac{4}{\alpha'} y_0)}{|y_0|} \right| e^{\frac{4}{\alpha'} (1/|y_0| - 1/|y(t)|)} e^{-t}. \] (3.16)
Since $y(t)$ is bounded by $-\frac{4}{\alpha'} < y(t) \leq y_0 < 0$, the exponential convergence $\left| y(t) + \frac{4}{\alpha'} \right| \leq C_0 e^{-t}$ follows directly from (3.16). \hfill \Box

Using this lemma and the evolution equation for the scalar curvature
\[ \partial_t \mathcal{R} = -\Delta \mathcal{R} - \mathcal{R}, \]
the proof of theorem 3.3 follows from a simple adaptation of the arguments used in the proof of theorem 3.1. \hfill \Box

4. An eternal solution connecting $\mathcal{M}_- \to \mathcal{M}_+$

Directly from equation (1.5), it is clear that the flat metric $\bar{g}$, and the metric $\hat{g}$ with Ricci scalar equals to $-\frac{4}{\alpha'}$ are fixed points for the second-order RG flow (1.5). We now show that there exists an eternal solution that connects the UV fixed point $\hat{g}$ to the IR fixed point $\bar{g}$. Moreover, we show that this solution passes through both the sets $\mathcal{M}_k$.

To begin, let $\tilde{g}$ be a metric with constant negative curvature
\[ \tilde{R} = -1. \]
Next, define
\[ g(t, x) := -\frac{1}{y(t)} \tilde{g}(x), \] (4.1)
which implies that the Ricci scalar of $g$ is given by
\[ R(t) = y(t). \] (4.2)
The ansatz (4.1) is consistent for the second-order RG flow and leads to the same equation studied in lemma 3.2 (i.e. just substitute (4.2) into (3.1)), namely
\[ \frac{dy}{dt} = y^2 + \frac{\alpha'}{4} y^3. \] (4.3)
Choosing initial data

\[ y(0) = -\frac{2}{\alpha'}, \quad (4.4) \]

it follows immediately from lemmas 3.2 and 3.4 that the unique solution \( y(t) \) to the initial value problem (4.3) and (4.4) exists for all \( t \in (-\infty, \infty) \) and satisfies

\[ \frac{-4}{\alpha'} < y(t) < \frac{2}{\alpha'}, \quad \left| y(t) + \frac{4}{\alpha'} \right| \leq C - e^t \quad \forall \ t < 0 \]

and

\[ \frac{2}{\alpha'} < y(t) < 0, \quad |y(t)| \leq \frac{C_+}{1+t} \quad \forall \ t > 0, \]

for some constants \( C_{\pm} \). In particular, this implies that

\[ g(t) \in \mathcal{M}_- \quad \forall \ t < 0 \quad \text{and} \quad g(t) \in \mathcal{M}_+ \quad \forall \ t > 0. \]

Moreover, it is clear that

\[ \lim_{t \to -\infty} g(t) = \hat{g} \quad \text{and} \quad \lim_{t \to \infty} R(t) = 0. \]

In this sense, the solution \( g(t) \) connects the UV fixed point \( \hat{g} \) to the IR fixed point \( \bar{g} \).

5. Discussion

We have shown that the second-order RG flow admits an eternal solution that connects a constant negative curvature metric to the flat metric. Moreover, we have shown that the existence of the eternal solution is due to the curvature correction term \( \frac{2}{\alpha'} R_{iklm} R^{jklm} \) to the first-order flow (i.e. Ricci flow). More specifically, we demonstrated that the term \( \frac{2}{\alpha'} R_{iklm} R^{jklm} \) destroys the uniform parabolicity of the Ricci flow, and it is precisely this lack of uniform parabolicity that allows for the existence of the eternal solution.

The results of this paper show that the lack of uniform parabolicity of the second (and higher)-order RG flow is not necessarily a defect. Instead, the lack of parabolicity opens up the possibility of constructing solutions by matching together solutions from neighbouring regions of backwards and forwards parabolicity. We speculate that this construction could be useful for both physics and geometry as it suggests a new method for connecting different geometries (i.e. UV and IR fixed points) via a geometric flow.

It is important to note that the analysis contained in this paper is different from that used for the two-dimensional Ricci flow. The main difference is that control of the scalar curvature \( R \), and hence the full curvature tensor, does not immediately imply, through the use of maximum principles, control on the higher derivatives of \( R \). This is due to the fully nonlinear nature of the second-order flow. To gain control on the first derivative on the scalar curvature, we used the property that the second-order RG flow reduces to a scalar equation in two dimensions which allowed us to apply the Krylov–Safonov estimates to obtain the desired result. This prevents the analysis of this paper being extended to higher dimensions. Consequently, to extend the results of this paper to higher dimensions, a new method, preferably using standard maximum principles, of obtaining control on the higher-order derivatives of the curvature tensor is required. Even in two dimensions, this would represent a significant improvement over the analysis contained in this paper.
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