A symmetry reduction technique for higher order Painlevé systems.

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ABSTRACT

The symmetry reduction of higher order Painlevé systems is formulated in terms of Dirac procedure. A set of canonical variables that admit Dirac reduction procedure is proposed for Hamiltonian structures governing the $A_{2M}^{(1)}$ and $A_{2M-1}^{(1)}$ Painlevé systems for $M = 2, 3, \ldots$. 
1. Introduction

Although first introduced in mathematics for their special singularity properties, Painlevé equations are now ubiquitous in physics. The list of their physical applications includes such topics as: the Ising and antiferromagnet models [7], the transport of particles across boundaries (Nernst-Planck equations) [3], dilute Bose-Einstein condensates in an external one dimensional field (Gross-Pitaevskii equation) [11], Hele-Shaw problems in viscous fluids [3] as well as various approaches to quantum field theory and topological field theory, supersymmetric gauge theories, random matrix theory, statistical mechanics, plasma physics, superconductivity, nonlinear optics and fiber optics, resonant oscillations in shallow water, and polymers. The $A^{(1)}_3$ Painlevé system, corresponding to the Painlevé V equation, appears in the context of the reduced density matrix of the impenetrable Bose gas model [6] and in connection of a unitary matrix model formulation of chiral Yang-Mills theory on the sphere [13]. Also, the Painlevé equations emerge as self-similarity reductions of well-known soliton equations. For instance, the Painleve system of $A^{(1)}_2$ type, corresponding to the Painlevé IV equation, can be obtained by reduction of the so-called AKNS soliton model.

In this note we will deal with generalization of the Painlevé equations of $A^{(1)}_2$ and $A^{(1)}_3$ types, mentioned above, to the higher order $A^{(1)}_n$ type of Painlevé system of ordinary differential equations for $n + 1$ functions $f_0, \ldots, f_n$ and variables $\alpha_0, \ldots, \alpha_n$ for $n = 2, 3, \ldots$. An affine Weyl group $A^{(1)}_n$ acts on a parameter space of system’s variables [8, 9, 10, 11, 12].

The higher order Painlevé equations are either quadratic or cubic depending on whether $n$ is even or odd. For $n = 2M$ ($M = 1, 2, \ldots$) equations are given by

$$A^{(1)}_{2M} : \quad f_i, x = f_i \Big( \sum_{r=1}^{M} f_{i+2r-1} - \sum_{r=1}^{M} f_{i+2r} \Big) + \alpha_i, \quad (1)$$

where $0 \leq i \leq 2M$ and conditions $f_0 + \cdots + f_n = -2x$ and $\alpha_0 + \cdots + \alpha_n = -2$ hold for system’s functions and parameters. For $n = 2M - 1$ ($M = 2, \ldots$) the system is described by:

$$A^{(1)}_{2M-1} : \quad 2xf_i, x = f_i \Big( \sum_{1 \leq r \leq s \leq M-1} f_{i+2r}f_{i+2s+1} - \sum_{1 \leq r \leq s \leq M-1} f_{i+2r-1}f_{i+2s} \Big) + (-1)^{i+1} f_i \Big( \sum_{r=0}^{M-1} \alpha_{2r+1} + 2 \Big) - \alpha_i \Big( \sum_{r=0}^{M-1} f_{i+2r} \Big), \quad (2)$$

with $0 \leq i \leq 2M - 1$ and with conditions $\sum_{r=0}^{M-1} f_{2r} = -2x$, $\sum_{r=0}^{M-1} f_{1+2r} = -2x$, $\sum_{r=0}^{2M-1} \alpha_r = -4$ for system’s functions and parameters. These conditions differ from that used in [12] by different normalization of system’s functions.

The above symmetric Painlevé equations are invariant under the following actions:

$$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i, \quad s_i(f_{i\pm 1}) = f_{i\pm 1} \pm \frac{\alpha_i}{f_i}, \quad \pi(f_j) = f_{j+1}, \quad \pi(\alpha_j) = \alpha_{j+1}, \quad (3)$$
of the extended affine Weyl group $A_n^{(1)}$ generators $\pi, s_i, i = 0, 1, \ldots, n$, where $a_{ij}$ are coefficients of the $A_n^{(1)}$ Cartan matrix $A = (a_{ij})_{0 \leq i,j \leq n}$ [8, 9, 10, 11].

The Painlevé equations (1) can be realized through a polynomial Hamiltonian system [9, 12] in functions $f_i$ with the underlying Poisson brackets:

$$\{f_i, f_{i+1}\} = 1, \quad \{f_i, f_{i-1}\} = -1, \quad i = 1, \ldots, 2M.$$  \hspace{1cm} (4)

It is convenient to express the relevant Hamiltonian formalism in terms of canonical coordinates defined through relations

$$p_i = f_{2i}, \quad q_i = \sum_{k=1}^{i} f_{2k-1}, \quad i = 1, \ldots, 2M,$$  \hspace{1cm} (5)

that map Poisson brackets (4) into the canonical brackets $\{q_i, p_j\} = \delta_{ij}$. In terms of canonical coordinates $p_i, q_i$ the $A_{2M}^{(1)}$ Hamiltonian is given by [9, 12, 2]:

$$H_{A_{2M}^{(1)}} = \sum_{j=1}^{M} p_j q_j (p_j + q_j + 2x) + 2 \sum_{1 \leq j < i \leq M} p_j q_j p_i - \sum_{j=1}^{M} \alpha_{2j} q_j + \sum_{j=1}^{M} p_j \left( \sum_{k=1}^{j} \alpha_{2k-1} \right)$$  \hspace{1cm} (6)

The corresponding Hamilton equations:

$$q_{i,x} = \frac{\partial H_M}{\partial p_i} = q_i \left( q_i + 2 \sum_{j=i}^{M} p_j + 2x \right) + 2 \sum_{j=1}^{i-1} p_j q_j + \sum_{j=1}^{i} \alpha_{2j-1}$$

$$p_{i,x} = -\frac{\partial H_M}{\partial q_i} = -p_i \left( 2q_i + p_i + 2 \sum_{j>i}^{M} p_j + 2x \right) + \alpha_{2i}$$  \hspace{1cm} (7)

are mapped through relations (5) into the $A_{2M}^{(1)}$ Painlevé equations (1).

In reference [2] we considered a specific integrable model, namely the 2M-Bose model from constrained KP hierarchy, to construct the $A_{2M}^{(1)}$ symmetric Painlevé equations. We studied their symmetries from the Lax point of view and obtained the corresponding Bäcklund transformations. In this paper by defining special combinations of coefficients of Lax operators from [2] as canonical variables we are able to show that Dirac reduction governs the symmetry reduction $A_{2M}^{(1)} \rightarrow A_{2M-1}^{(1)}$ of the respective Painlevé system by reducing the Hamiltonian $H_{A_{2M}^{(1)}}$ from the definition
to the Hamiltonian $\mathcal{H}_{2M-1}^{(1)}$ [9] [12] given by relation

$$2x\mathcal{H}_{2M-1}^{(1)} = \sum_{j=1}^{M-1} p_j (p_j + 2x) q_j (q_j + 2x) + 2 \sum_{1 \leq j < i \leq M-1} p_j q_j p_i (q_i + 2x)$$

$$+ \left( \sum_{k=0}^{M-1} \alpha_{2k+1} + 2 \right) \left( \sum_{j=1}^{M-1} p_j q_j \right)$$

$$- \sum_{j=1}^{M-1} \alpha_{2j} 2x q_j + \sum_{j=1}^{M-1} \left( \sum_{k=1}^{j} \alpha_{2k-1} \right) 2xp_j$$

with relations $p_i = f_{2i}, q_i = \sum_{k=1}^{i} f_{2k-1}$ for $i = 1, \ldots, M - 1$ between canonical variables $p_i, q_i, i = 1, \ldots, M - 1$ of the reduced phase space and the Painlevé functions $f_1, \ldots, f_{2M-2}$ from equation (2). In addition, functions $f_0$ and $f_{2M-1}$ appearing in equation (2) are defined through $f_0 = -2x - f_2 - \ldots - f_{2M-2}$ and $f_{2M-1} = -2x - f_1 - \ldots - f_{2M-3}$ resulting in the total of $n + 1 = (2M - 1) + 1 = 2M$ Painlevé functions. Note, that $\alpha_0$ does not enter expression in (8) and is determined from the definition $\alpha_0 = -4 - \alpha_1 - \ldots - \alpha_{2M-1}$.

2. Symplectic Map and New Canonical Variables

Before we present a set of new canonical variables to describe the higher order Painlevé system let us comment on a possibility of applying Dirac reduction procedure in the framework of canonical variables $p_i, q_i, i = 1, \ldots, 2M$ of the $\mathcal{H}_{2M}^{(1)}$ Hamiltonian given in [6]. A straightforward attempt that involves setting one of the canonical variables to zero inevitably leads to the dimension of the underlying phase space getting reduced by two degrees. For example, imposing a constraint $p_M = 0$ leads via the secondary constraint [4] $p_{M, x} = 0$ to $\alpha_{2M} = 0$ as seen from equation (7). That in turn eliminates a presence of $q_M$ from the Hamiltonian $\mathcal{H}_{2M}^{(1)}$ and effectively causes a reduction $\mathcal{H}_{2M}^{(1)} \rightarrow \mathcal{H}_{2M-2}^{(1)}$.

Below we describe a different reduction scheme leading to the advertised reduction $\mathcal{H}_{2M}^{(1)} \rightarrow \mathcal{H}_{2M-1}^{(1)}$. It employs a set of new canonical variables $\epsilon_i, Y_i, i = 1, \ldots, M$ obtained from $p_i, q_i$ through the the following definitions:

$$\epsilon_M = p_M + q_M + 2x, \quad \epsilon_{M-1} = -q_{M-1}$$

$$\epsilon_{M-2k} = -p_k - p_{k+1} - \ldots - p_{M-k-1}$$

$$\epsilon_{M-2k-1} = -q_{M-k-1} + q_k, \quad k = 1, 2, \ldots$$

(9)

and

$$Y_M = p_M + q_{M-1} + 2x, \quad Y_{M-1} = -q_M - p_{M-1}$$

$$Y_{M-2k} = -q_{k-1} + q_{M-k-1} + 2x$$

$$Y_{M-2k-1} = p_k - p_{M-k-1} + 2x, \quad k = 1, 2, 3, \ldots$$

(10)
where to simplify expressions we introduced auxiliary quantities:
\[ P_k^M \equiv p_k + p_{k+1} + \ldots + p_M, \quad P_{M-k}^M \equiv p_{M-k} + p_{M-k+1} + \ldots + p_M, \quad \text{(11)} \]
inter-connected via relation
\[ P_k^M = P_{M-k}^M - e_{M-2k}. \quad \text{(12)} \]
The above map, suggested by the study \cite{2} of Painlevé systems in the context of integrable Volterra-type lattices, is symplectic. The canonical brackets \( \{ q_i, p_j \} = \delta_{ij} \) are being transformed into the usual canonical Poisson brackets for \( e_i, Y_i \):
\[ \{ e_i, Y_j \} = \delta_{ij}, \quad \{ e_i, e_j \} = 0 = \{ Y_i, Y_j \}, \quad i, j = 1, \ldots, M \quad \text{(13)} \]
To rewrite the \( \mathcal{H}_{A_{2M}^{(1)}} \) Hamiltonian in (6) in terms of these new variables it is convenient to use inverse relations that give \( p_i, q_i \) in terms of \( e_i, Y_i \). The variables \( q_k \) are related to new variables as follows.
\[ q_k = -2kx + \sum_{i=1}^{k} Y_{M-2i} + \sum_{i=2}^{2k+1} e_{M-i}, \quad k = 1, 2, \ldots, \lfloor M/2 \rfloor - 1, \quad \text{(14)} \]
where \( \lfloor M/2 \rfloor = M/2 \) or \( \lfloor M/2 \rfloor = (M-1)/2 \), whichever is an integer. Furthermore,
\[ q_{M-k-1} = -2kx + \sum_{i=1}^{k} Y_{M-2i} + \sum_{i=2}^{2k} e_{M-i}, \quad \text{(15)} \]
for \( k = 1, 2, \ldots, \lfloor M/2 \rfloor - 1 \) for even \( M \) and \( k = 1, 2, \ldots, \lfloor M/2 \rfloor \) for odd \( M \). Finally,
\[ q_M = -Y_M - e_{M-1}, \quad q_{M-1} = -e_{M-1}. \quad \text{(16)} \]
Thus the relation to new variables becomes
\[ P_k^M = -q_{k-1} + \sum_{i=0}^{2k-1} (-1)^i (Y_{M-i} - e_{M-i}), \quad k = 1, 2, \ldots, \lfloor M/2 \rfloor - 1 \quad \text{(17)} \]
The relations for remaining indices are then obtained through (12):
\[ p_M = e_{M-1} + e_M + Y_M - 2x, \quad p_{M-1} = -e_M - Y_{M-1} + 2x. \quad \text{(18)} \]
For a specific example of \( M = 5 \) the above relations give:
\[
\begin{align*}
q_5 &= -Y_5 - e_4, & p_5 &= e_4 + e_5 + Y_5 - 2x \\
q_4 &= -e_4, & p_4 &= -e_5 - Y_4 + 2x \\
q_3 &= Y_3 - 2x, & p_3 &= -e_3 - Y_2 + 2x \\
q_2 &= Y_1 + Y_3 + e_2 - 4x, & p_2 &= -e_1 \\
q_1 &= e_2 + Y_3 - 2x, & p_1 &= e_1 + Y_2 - 2x
\end{align*}
\]

and it is easy to explicitly verify a symplectic nature of the above map.

In terms of \( M \) variables \( e_i, Y_i \) the Hamiltonian \( \mathcal{H}_{A_{2M}^{(1)}} \) becomes:

\[
\mathcal{H}_{A_{2M}^{(1)}} = -\sum_{j=1}^{M} e_j (Y_j - 2x) (Y_j - e_j) - 2 \sum_{1 \leq j < i \leq M} (-1)^{i+j} e_j (Y_j - 2x) (Y_i - e_i)
\]

\[
+ \sum_{j=1}^{M} \bar{k}_j Y_j - \sum_{j=1}^{M} k_j e_j
\]

with constants \( k_j, \bar{k}_j, j = 1, \ldots, M \) related to Painlevé parameters via:

\[
k_j = \frac{\alpha_{M-j-1} + \alpha_{M-j+1} + \cdots + \alpha_{M+j-3}}{j}, \quad j = 1, \ldots, M - 2
\]

\[
k_{M-1} = -\sum_{j=1}^{M} \alpha_{2j-1} - \alpha_{2M-2} - \alpha_{2M}, \quad k_M = \sum_{j=1}^{M-1} \alpha_{2j-1} + \alpha_{2M}
\]

\[
\bar{k}_l = -\alpha_{M-l} - \alpha_{M-l+2} - \cdots - \alpha_{M+l-2}, \quad l = 1, \ldots, M - 1,
\]

\[
\bar{k}_M = \sum_{j=1}^{M} \alpha_{2j-1} + \alpha_{2M}
\]

3. From \( A_{2M}^{(1)} \) to \( A_{2M-1}^{(1)} \) via Dirac Reduction

We proceed by imposing a constraint:

\[
Y_M = 0.
\]

The consistency requires that we also need to impose the secondary constraint (see e.g. [4]):

\[
0 = Y_M, x = -\frac{\partial}{\partial e_M} \mathcal{H}_M = 4x e_M - 2 \sum_{i=1}^{M-1} (-1)^{M-i} e_i (Y_i - 2x) + \kappa_M
\]

or

\[
e_M = \frac{1}{4x} \left( 2 \sum_{i=1}^{M-1} (-1)^{M-i} e_i (Y_i - 2x) - \kappa_M \right).
\]

Substituting the value of \( e_M \) from eq. (23) and \( Y_M = 0 \) into \( \mathcal{H}_M \) reduces it into

\[
\mathcal{H}_M = \mathcal{H}_M \big|_{Y_M=0, Y_M, x=0}
\]
given by

\[ 2x \tilde{H}_M = \sum_{j=1}^{M-1} e_j (Y_j - 2x) Y_j (e_j - 2x) \]

\[ + 2 \sum_{1 \leq i < j \leq M-1} (-1)^{i+j} e_i (Y_i - 2x) Y_j (e_j - 2x) \]

\[ - \kappa_M \sum_{i=1}^{M-1} (-1)^{M+i} e_i Y_i + \sum_{j=1}^{M-1} \bar{k}_j 2x Y_j \]

\[ - \sum_{j=1}^{M-1} (\kappa_j - (-1)^{M+j} \kappa_M) 2xe_j \]

Next, we proceed by inserting into the above Hamiltonian \( \tilde{H}_M \) expressions

\[ Y_{M-1} = -p_{M-1}, \quad e_{M-1} = -q_{M-1} \]

(25)

together with relations following from eqs. (9)-(10) for \( e_i, Y_i \) for the remaining indices \( i = 1, \ldots, M - 2 \):

\[ e_{M-2k} = -p_k - p_{k+1} - \cdots - p_{M-k-1} = -P_k^M + P_{M-k}^M \]

\[ e_{M-2k-1} = -q_{M-k-1} + q_k, \quad k = 1, 2, \ldots \]

\[ Y_{M-2k} = -q_{k-1} + q_{M-k-1} + 2x \]

\[ Y_{M-2k-1} = P_k^M - P_{M-k-1}^M + 2x, \quad k = 1, 2, 3, \ldots \]

(26)

This casts \( \tilde{H}_M \) into \( \mathcal{H}_{A_{2M-1}} \) from equation (8) provided that constants \( \kappa_j \) for \( j = 1, \ldots, M - 2 \) and \( \bar{k}_l \) for \( l = 1, \ldots, M - 1 \) agree with values given in relations (20) and in addition relations \( \kappa_{M-1} = -\sum_{j=1}^{M} \alpha_{2j-1} - \alpha_{2M-2} - 2 \) and \( \kappa_M = \sum_{j=1}^{M} \alpha_{2j-1} + 2 \) hold as well.

4. Example of reduction for \( M = 2 \). The Painlevé V Equation

We now study in detail an example of \( M = 2 \). Although this model and its reduction appeared in [2], here we employ new canonical variables to compactly rewrite the relevant Hamiltonian as

\[ \mathcal{H}_{A_{4(1)}} = -\sum_{j=1}^{2} e_j (Y_j - 2x) (Y_j - e_j) + 2e_1 (Y_1 - 2x) (Y_2 - e_2) + \sum_{j=1}^{2} \bar{k}_j Y_j - \sum_{j=1}^{2} \kappa_j e_j \]

and to directly implement reduction by substituting \( Y_2 = 0, e_2 = -(2e_1(Y_1 - 2x) - \kappa_2)/4x \)
to obtain the reduced Hamiltonian:

\[ 2x \mathcal{H}_2 = e_1 (Y_1 - 2x) Y_1 (e_1 - 2x) + \kappa_2 e_1 Y_1 + \bar{k}_1 2x Y_1 - (\kappa_1 + \kappa_2) 2xe_1 \]

(27)
The corresponding Hamilton equations are:

\[ e_1 x = 2x e_1 - e_1^2 - 2e_1 Y_1 + \bar{k}_1 + \frac{1}{2x} (2e_1^2 Y_1 + e_1 \kappa_2) \]

\[ Y_1 x = -2x Y_1 + 2e_1 Y_1 + Y_1^2 + \kappa_1 + \kappa_2 - \frac{1}{2x} (2Y_1^2 e_1 + Y_1 \kappa_2) \]  \hspace{1cm} (28)

The above equations are fully symmetric and invariant under the Bäcklund transformations:

\[ e_1 \xrightarrow{g} Y_1 \]

\[ Y_1 \xrightarrow{g} e_1 + Y_1 - \frac{Y_1 x}{Y_1} - \frac{1}{2x} (2e_1 Y_1 + \kappa_2) \]

\[ = -e_1 + 2x - \frac{\kappa_1 + \kappa_2}{Y_1} \]  \hspace{1cm} (29)

together with appropriate transformations of constants given in relation

\[ g(\kappa_1) = 2 - \kappa_1 - \kappa_2 - \bar{k}_1 \]

\[ g(\kappa_2) = 2\kappa_1 + \kappa_2 \]

\[ g(\bar{k}_1) = -\kappa_1 - \kappa_2 \]  \hspace{1cm} (30)

From eqs. (25) it follows that \( Y_1 = -p_1 = -f_2 \) and \( e_1 = -q_1 = -f_1 \). Subsequently \( f_0 = -2x + Y_1 \) and \( f_3 = -2x + e_1 \). In this notation one can cast equations of motion (28) into symmetric \( A^{(1)}_3 \) Painlevé V equations:

\[ 2x f_0 x = f_0 f_2 (f_3 - f_1) - (\alpha_1 + \alpha_3 + 2) f_0 - \alpha_0 (f_0 + f_2) \]

\[ 2x f_2 x = f_0 f_2 (f_1 - f_3) - (\alpha_1 + \alpha_3 + 2) f_2 - \alpha_2 (f_0 + f_2) \]

\[ 2x f_1 x = f_1 f_3 (f_0 - f_2) + (\alpha_1 + \alpha_3 + 2) f_1 - \alpha_1 (f_1 + f_3) \]

\[ 2x f_3 x = f_1 f_3 (f_2 - f_0) + (\alpha_1 + \alpha_3 + 2) f_3 - \alpha_3 (f_1 + f_3) \]  \hspace{1cm} (31)

with constants:

\[ \alpha_1 = -\bar{k}_1, \quad \alpha_2 = -\kappa_1 - \kappa_2, \quad \alpha_3 = -2 + \kappa_2 + \bar{k}_1 \]

and \( \alpha_0 = -4 - \alpha_1 - \alpha_2 - \alpha_3 = -2 + \kappa_1 \). In this setting the Bäcklund transformation \( g \) defined through relations (30) and (29) is identified with \( g = \pi s_1 \), where actions of \( \pi \) and \( s_1 \) are defined in (3) for \( n = 3 \).

5. Outlook

We have proposed a set of new canonical variables for Hamiltonian formalism for higher Painlevé equations that enables symmetry reduction by Dirac procedure. It would be interesting to extend such construction beyond the class of \( A^{(1)}_n \) type of equations treated in this paper and to use the formalism presented here to deal with questions of how to formulate Hamiltonian structures for various symmetry groups to study possible reduction mechanisms.
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