Revisiting renormalization schemes in a differential equation approach

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We reconsider the choice of renormalization schemes in a differential-equation approach to aid the discussion of the renormalization of the unstable particles and the CKM matrix in the Standard Model. Certain mass dependent schemes do not satisfy these natural differential equations modulo trace anomaly. By the way, the Callan-Symanzik equations were employed to show that both mass dependent and independent schemes realize fermion decoupling in the same way.

Theoretically, different renormalization schemes are perfectly equivalent provided the perturbation is completed. The perturbative scheme dependence is due to the truncation of the perturbation series [1]. Recently, the issue of renormalization scheme has been intensively investigated in Standard Model (SM) and its extensions with respect to the unstable sector (like W±, Z0 and Higgs particles) [2,3] and that of the Cabibbo-Kobayashi-Maskawa (CKM) matrix [4,5], where certain mass dependent (like MOM and on-shell) schemes become problematic in these contexts: the on-shell schemes (mass dependent) would bring in severe gauge dependence and IR singularities or ambiguities. The lesson that could be drawn from these progresses is that the freedom in choosing renormalization prescriptions deserves closer investigation in applications. In general, a gauge invariant definition is preferred while gauge dependent ones should be avoided at all [3,6].

In this brief report, we wish to draw attention to other aspects of the same issue. We start with the standard point of view that all the known QFT’s are effective theories for the low energy sectors in a completely well-defined quantum theory containing ‘correct’ high energy details [7]. Then we can derive from this postulate a set of natural differential equations for the loop amplitudes due to the well-known fact that differentiating the loop amplitudes with respect to external parameters like momenta and mass(es) reduces the divergence degree: the divergent loop amplitudes are therefore obtained as solutions to these well-defined differential equations, which are generally finite nonlocal terms plus certain a polynomial in terms of external momenta, masses and couplings with arbitrary coefficients to be fixed through physical ‘boundary conditions’ [8] as we usually do in quantum mechanics and electrodynamics. We have demonstrated this simple approach in a recent study of dynamical symmetry breaking in the two loop effective potential of massless λφ⁴ [9]. Obviously, for convergent diagrams such differential equations are naturally satisfied.

The key point is that, the coefficients in the polynomial, to be fixed through physical ‘boundary conditions’, should be independent of the differentiation parameters like masses. If one affords some intermediate unphysical definition to these constants, one could arrive at any renormalization prescription, like MOM, on-shell and MS. However, some prescription might be inconsistent: the original differential equations might be violated by such prescriptions. Since such differential equations are well defined, the physical amplitudes must satisfy these equations and any renormalization prescriptions that solve these equations could be used. However, it is unclear how those prescriptions that fail to satisfy such equations could finally lead to physical amplitudes that satisfy the same equations, as a practical complete summation of the perturbations series is impossible.

Now let us demonstrate the differential equation solution of a loop amplitude with a simple vertex function at the lowest order in QED, say, the 1-loop photon vacuum polarization tensor

\[ \Pi^{\mu\nu}(p, -p, m) \equiv -ie^2 \int d^4k \, tr \left\{ \gamma^{\mu} \frac{1}{p + k - m} \gamma^{\nu} \frac{1}{k - m} \right\} \]

(1)

in QED for simplicity. Here m refers to the mass of any fermion with electric charge. We are mainly concerned with the mass dependence and the differential equation is explicitly given in terms of mass. In the more interesting case of symmetry breaking masses, similar analysis should also be feasible and will be studied in the future.

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1We note that differentiating a Feynman amplitude with respect to external parameters amounts to physically probing the quantum process defined by this amplitude, as such operation (∂pμ) automatically leads to the insertion of elementary vertices. Therefore, in this approach a superficially ill-defined amplitude is determined through its response to physical probes or 'measurements' plus physical boundary conditions. In conventional renormalization programs, such kind of procedure is effected through confronting the intermediately renormalized amplitudes with experiments to obtain physical parametrization [10].
This amplitude satisfies the following well-defined inhomogeneous differential equation in any gauge invariant regularization (GI) or in the complete underlying theory,

$$\partial_m \Pi^{\mu\nu}(p,-p,m) = -ie^2 \int (d^4 k) GI tr \{ \gamma^\mu \left( \frac{1}{p + k - m} \right)^2 \gamma^\nu \left( \frac{1}{k - m} \right) + \gamma^\mu \left( \frac{1}{p + k - m} \right)^4 \gamma^\nu \left( \frac{1}{k - m} \right)^2 \}. \quad (2)$$

Moreover it satisfies the following well defined equation without regularization at all

$$(\partial_m)^3 \Pi^{\mu\nu}(p,-p,m) = -ie^2 \sum_{l=0}^3 C^3_l \int d^4 k \ tr \{ \gamma^\mu \left( \frac{1}{p + k - m} \right)^{l+1} \gamma^\nu \left( \frac{1}{k - m} \right)^{4-l} \} \quad (3)$$

where $C^3_l$ is the combinatorial factor arising from the differentiation operation.

It suffices to demonstrate our points with Eq. (2). Factorizing out the gauge invariant projector $(g^\mu\nu p^2 - p^\mu p^\nu)$ we arrive at the following natural equation

$$\partial_m \Pi(p^2,m) = \frac{e^2}{2\pi^2} \int_0^1 dx \frac{x(1-x)m}{m^2 - x(1-x)p^2} \quad (4)$$

and the solution to this equation reads

$$\Pi(p^2,m;C) = \frac{e^2}{2\pi^2} \int_0^1 dx (1-x) \ln \frac{m^2 - x(1-x)p^2}{C} \quad (5)$$

with $C$ being the arbitrary integration constant to be fixed somehow. Then we have

$$\Pi^{\mu\nu}(p,-p,m;C) = \frac{e^2}{2\pi^2} (g^{\mu\nu} p^2 - p^\mu p^\nu) \int_0^1 dx (x-x^2) \ln \frac{m^2 - (x-x^2)p^2}{C}. \quad (6)$$

Here the main conclusion we can draw about $C$ is that it must be gauge invariant and mass independent.

Now we observe that not all renormalization prescriptions preserve Eq. (4). In all mass independent prescriptions, $C$ is fixed so that the renormalized amplitude still satisfies Eq. (4). However, in the MOM schemes, $C$ is defined in the following way,

$$\Pi(p^2,m;C_{MOM})|_{p^2=-m^2} = 0 \implies C_{MOM} = m^2 + x(1-x)\mu^2, \quad (7)$$

which leads to the violation of Eq. (4),

$$\partial_m \Pi(p^2,m;C_{MOM}) = \frac{e^2}{\pi^2} \int_0^1 dx \frac{x(1-x)m}{m^2 - x(1-x)p^2} + \delta(\mu^2,m), \quad (8)$$

with $\delta(\mu^2,m) \equiv -\frac{e^2}{2\pi^2} \int_0^1 dx \frac{x(1-x)m}{m^2 + x(1-x)p^2}$. This is the main point of our concern, as such violation might lead to problems in practical applications, especially in the complicated case like SM and its extensions. The spirit followed here is the same as is followed in [3,5,6] where physical or novel requirement like gauge invariance are guidelines. In these literature the pole mass renormalization for unstable particles is therefore advocated.

There is a subtle point concerned with the pole mass renormalization for the massless photon, which is just the on-shell scheme, as $\Pi(p^2,m;C)|_{p^2=0} = 0 \rightarrow C = m^2$. Thus it seems that the pole mass definition (or the on-shell definition) for photons also violates Eq. (4). To this end we note that in this case the physical scale or trace anomaly [11] is defined in the following way

$$\{p^2 \partial_{p^2} + m^2 \partial_{m^2} - \gamma_A \}[1 + \Pi(p^2,m;C)] = 0, \implies \beta(\alpha) = \gamma_A = -2C \partial_C \Pi(p^2,m;C)|_{C=m^2} = \frac{e^2}{6\pi^2} = \frac{2\alpha}{3\pi}. \quad (9)$$

That is, the mass also play the role of the running scale and the extra term in Eq. (8) is now just the gauge invariant trace anomaly with physical significance—the quantum mechanical violation of naive scaling law. Then we see that the differential equations could be modified by physical trace anomaly. From now on we consider the violation of the differential equations modulo trace anomaly.

In the more complicated gauge theories such as SM, due to the absorptive (threshold effect) part of the self-energy of the unstable particles, mass dependent schemes could lead to gauge dependence and inconsistency [4], i.e., in such theories, the mass dependent schemes were contaminated with severe gauge dependence. We expect that in such
theories the novel differential equations proposed here would also be severely violated modulo gauge invariant trace anomalies. Thus from the differential equation point of view, combining with the recent studies [2–6] severe gauge dependence and other defects, the mass dependent schemes should be avoided in computing SM radiative corrections. Theoretically we could not perform a complete sum of the perturbation series to test if such defects automatically disappear at all and practically we have to employ ‘good’ schemes that are free from the above mentioned defects before going to experimental data.

In addition, we should note that the pole mass renormalization for heavy quarks does satisfy the differential equations in terms of lagrangian masses modulo trace anomalies, as it is gauge invariant and IR finite [6,12] according to the definition \( m_{\text{eff}}(p^2, \bar{m}) |_{p^2=M_{\text{pol}}^2} = M_{\text{pol}} \) [12]. To see this point we simply note that \( m_{\text{eff}} \) is a physical 'form factor' that is scheme independent [13], thus its dependence upon Lagrangian mass is the same as that in the mass independent schemes. This argument also applies to the case of massive bosons, see [2].

We can understand the above issues from the underlying theory point of view. Suppose we could compute everything from the complete underlying theory, including the 'low energy' amplitudes. Then in the 'low energy' limits, the 'low energy' amplitudes must be definite functions (including definite pieces of local terms that arise from the this limit) in terms of the 'low energy' physical parameters. There must also be a reference scale for specifying the relevant ‘low energy’ processes. Then the one loop photon polarization should take the following form

\[
\Pi^{\mu\nu}(p^2, m_{u.t.}; \{\sigma\})|_{\{\sigma\} = 0} = e^{\gamma_{\text{E}} \frac{1}{2}} (g^{\mu\nu} p^2 - p^\mu p^\nu) \int_0^1 dx (1 - x) \ln \frac{m_{u.t.}^2 - x(1 - x)p^2}{p_{u.t.}^2},
\]

with the subscript 'u.t.' referring to the underlying theory and \( \{\sigma\} \) denoting the fundamental constants of the underlying theory. It is natural to identify the 'low energy' parameters like \( m_{u.t.} \) with the Lagrangian parameters of QFT (bare but finite). The scale \( \mu_{u.t.} \), being process specific and physical and hence renormalization group (RG) and renormalization scheme (RS) invariant, is not the running scale though it appears in the same place. Then any renormalization prescriptions that are equivalent to such physical parametrization is acceptable. However, the difficulty is, we do not know how to identify such process dependent scale theoretically before the complete underlying theory is known, therefore we must resort to other means like experimental data and some reasonable or physical requirements. Before the complete theory is found we could not exclude the possibility that there might be schemes that are physically inequivalent to the complete theory definitions at all.

Moreover, from the differential equation point of view, the scheme invariance of certain quantities in QFT originally established within the mass independent schemes [1] could be promoted to a full 'invariance' for all the schemes that satisfy the differential equations in terms of all physical parameters. Otherwise, if the mass dependent schemes also qualified for applications, the significance of such 'invariance' established within the mass independent schemes is diminished in the cases where masses could not be neglected. In a sense we put forward an argument for advocating mass independent schemes, and this point of view could draw supports from the recent literature on renormalization schemes, though the relation between our approach here and those in the literature is unclear at this moment.

Before closing the presentation we would like to discuss the issue of heavy particles decoupling where the MOM and other mass dependent schemes are thought to be advantageous over the mass independent schemes due to its good decoupling behavior. We first note that this advantage is only in the context of RGE. Then we also remind that, as we will demonstrate below, in the context of Callan-Symanzik equations [14] (CSE), which describe the full scale behavior, the decoupling of heavy fields [15] is achieved in the same way in all schemes.

Again we illustrate it with a simple model, QED with a massive fermion in addition to \( n_l \) massless fermions. In a mass independent scheme the Callan-Symanzik equation reads,

\[
\{\lambda \partial \lambda - \beta \alpha \partial \alpha + \gamma \tau - D \tau\} \Gamma((\lambda p), m, \alpha, \mu) = -i \Gamma^\Theta((\lambda p), m, \alpha, \mu)
\]

where \( \Theta \equiv [1 + \gamma_m] m \bar{\psi} \psi \), \( \beta, \gamma \) and \( \gamma_m \) are mass independent functions of the renormalized coupling \( \alpha \) and all quantities are renormalized ones. At lowest order, \( \beta = \frac{3n_l}{16\pi} (n_l + 1) \).

When the mass goes to infinity, \( \beta, \gamma \) on the left hand side of CSE does not change at all, but in the meantime, the inhomogeneous term of CSE does not vanish,

\[
-i \Gamma^\Theta((p), m, \alpha, \mu)|_{m \to \infty} = (\Delta \beta \alpha \partial \alpha - \Delta \gamma \tau) \Gamma_0((p), \alpha, \mu),
\]

\[
\Rightarrow \{\lambda \partial \lambda - \beta_0 \alpha \partial \alpha + \gamma \tau; 0 - D \tau\} \Gamma_0((\lambda p), \alpha, \mu) = 0.
\]

Here \( \beta_0 \equiv \beta + \Delta \beta, \gamma \tau; 0 \equiv \gamma \tau + \Delta \gamma \tau \) with the delta contributions coming from the mass insertion part in the infinite mass limit which will cancel the heavy field’s contribution to \( \beta \) and \( \gamma \). The subscript 0 means that the heavy particle
is removed. The generalization to other theories with boson masses is an easy exercise. From Eq. (13) we see that the decoupling of heavy particles is realized in a natural way in the contexts of Callan-Symanzik equations.

To verify the above deduction it is enough to demonstrate Eq. (12) at the lowest order which is closely related to the observation that heavy particle limit provides a convenient algorithm for calculating trace anomalies [16]:

$$-i\langle m\bar{\psi}\gamma^{\mu}\psi J^\mu\rangle|_{m\to\infty} = \frac{2\alpha}{3\pi} (p^2 g^{\mu\nu} - p^\mu p^\nu) \Rightarrow m(1 + \gamma m)\bar{\psi}\psi|_{m\to\infty} = \frac{1}{4} \Delta\beta F^{\mu\nu} F_{\mu\nu},$$

(14)

with $J^\mu = -ie\bar{\psi}\gamma^{\mu}\psi$ and $\Delta\beta = -\frac{2\alpha}{3\pi} = \Delta\gamma A$. When translated into Callan-Symanzik equations Eq. (14) is just Eq. (12). The cancellation of the heavy particle contributions is obvious since $\beta + \Delta\beta = \frac{2\alpha}{3\pi}[n_1 + 1] - \frac{2\alpha}{3\pi} = \frac{2\alpha}{3\pi} n_1$.

While in the MOM like schemes the Callan-Symanzik equation reads,

$$\{\lambda\partial_\lambda - \beta_{\text{MOM}}\partial_\alpha + \gamma_\lambda;\text{MOM} - D_\lambda\}\Gamma_{\text{MOM}}((\lambda p), m, \alpha, \mu) = -i\Gamma_{\text{MOM}}^0((\lambda p), m, \alpha, \mu)$$

(15)

with the beta function etc. being defined as $\beta_{\text{MOM}} \equiv [\mu\partial_\mu + m(1 + \gamma m;\text{MOM})\partial_m]\alpha, \ldots$, in contrast to the RGE definition: $\beta_{\text{RGE}}^\text{MOM} \equiv \mu\partial_\mu\alpha, \ldots$, due to the mass dependence of the renormalization constants. The resulting $\beta_{\text{MOM}}$ also exhibits non-decoupling feature as in the mass independent schemes. For example, at the lowest order, from the definition given above, a heavy particle’s contribution to $\beta$ at lowest order is mass independent

$$\beta_{\text{MOM}}:M = (\mu\partial_\mu + m(1 + \gamma m;\text{MOM})\partial_m) \frac{e^2}{2\pi^2} \int_0^1 dx x(1 - x) \ln C_{\text{MOM}} = \frac{2\alpha}{3\pi}.$$  

(16)

Without any doubt such ‘non-decoupling’ term is cancelled by the contribution from the decoupling limit of the inhomogeneous term in CSE, that is, just like in Eq. (12), we have

$$-i\Gamma_{\text{MOM}}^0((p), m, \alpha, \mu)|_{m\to\infty} = (\Delta\beta_{\text{MOM}}\partial_\alpha - \Delta\gamma_\lambda;\text{MOM})\Gamma_{\text{MOM},0}((p), \alpha, \mu),$$

(17)

with $\Delta\beta_{\text{MOM}} = -\frac{2\alpha}{3\pi}$, which exactly cancels the heavy particle’s contribution to the $\beta, \gamma$ etc. Thus we proved that Eq. (12) is also true in the MOM schemes at the lowest order. It is well known that the first loop order beta of RGE in mass independent schemes differs from that in the MOM schemes [13]. While in the context of Callan-Symanzik equation the beta function is the same in all schemes at one loop order (C.f. Eq. (16)), which in turn implies that the same decoupling mechanism works in both mass independent and mass dependent schemes in the context of Callan-Symanzik equation. Of course, in the preceding derivation it is assumed that there is no such trouble as gauge dependence and/or any form of inconsistency for the mass dependent schemes, i.e., we consider the theories that are not beset with the troubles that afflicts the electroweak sector of SM.

In summary, we revisited the issue of renormalization schemes that is intensely discussed in the literature of Standard Model. The analysis, basing on a set of natural differential equations, seems to favor the mass independent schemes against the mass dependent ones, supporting the point of view of the recent literature concerning the renormalization of CKM matrix and the unstable sectors of SM. It is also demonstrated that in the context of Callan-Symanzik equation all the schemes facilitate the decoupling of heavy particles in the same way. Further investigation in this direction seems worthwhile.

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2 In the standard inhomogeneous form of Callan-Symanzik equation, the mass operators inserted vertex functions appear in the other side of the equation and the definitions of $\beta, \gamma$ in the MOM-like mass dependent schemes must include $m(1 + \gamma m;\text{MOM})\partial_m$ to account for the ‘full running’ of the renormalization constants. But in the alternative homogeneous form of Callan-Symanzik equation (CSE), i.e., \{\lambda\partial_\lambda - \beta_\alpha\partial_\alpha + m(1 + \gamma m)\partial_m + \gamma_\lambda - D_\lambda\}\Gamma((\lambda p), m, \alpha, \mu) = 0, the definitions of the $\beta, \gamma$ etc. are the same as in RGE.
[1] P. M. Stevenson, Phys. Rev. D23, 2916 (1981); G. Grunberg, Phys. Rev. D29, 2315 (1984); S. J. Brodsky, G. P. Lepage and P. B. Mackenzie, Phys. Rev. D28, 228 (1983); D. T. Barclay, C. J. Maxwell and M. T. Reader, Phys. Rev. D49, 3480 (1994).

[2] See, e.g., S. Willenbrock and G. Valencia, Phys. Lett. B259, 373 (1991); R. G. Stuart, Phys. Lett. B262, 113 (1991); A. Sirlin, Phys. Rev. Lett. 67, 2127 (1991), Phys. Lett. B267, 240 (1991); T. Bhattacharya and S. Willenbrock, Phys. Rev. D47, 4022 (1993); H. Veltman, Z. Phys. C62, 35 (1994); M. Passera and A. Sirlin, Phys. Rev. Lett. 77, 4146 (1996), Phys. Rev. D58, 113010 (1998); B. A. Kniehl and A. Sirlin, Phys. Rev. Lett. 81, 1373 (1998) and references therein.

[3] P. Gambino and P. A. Grassi, Phys. Rev. D62, 076002 (2000).

[4] See, e.g., A. Denner and T. Sack, Nucl. Phys. B347, 203 (1990); B. A. Kniehl, F. Madricardo, M. Steinhauser, Phys. Rev. D62, 073010 (2000); A. Barroso, L. Brucher and R. Santos, Phys. Rev. D62, 096003 (2000); Y. Yamada, Phys. Rev. D64, 036008 (2001); K. P. Diener and B. A. Kniehl, Nucl. Phys. B617, 291 (2001) and references therein.

[5] P. Gambino, P. A. Grassi and F. Madricardo, Phys. Lett. B454, 98 (1999); D. Espriu, J. Manzano and P. Talavera, Phys. Rev. D66, 076002 (2002) and references therein.

[6] A. S. Kronfeld, Phys. Rev. D58, 051501 (1998).

[7] See, e.g., M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory, (Addison-Wesley, 1995), Chapter 8 and S. Weinberg, The Quantum Theory of Fields, Volume I, (Cambridge University Press, Cambridge, 1995), Chapter 12.

[8] Ji-Feng Yang, arXiv: hep-th/9708104; invited talk, in Proceedings of the 11th International Conference: 'PQFT98', Ed. by B. M. Barbashov, et al, (Publishing Department of JINR, Dubna, 1999), 202-206[arXiv: hep-th/9901138]; arXiv: hep-th/9904055; arXiv: hep-ph/0212298.

[9] Ji-Feng Yang and Jian-Hong Ruan, Phys. Rev. D65, 125009 (2002).

[10] G. Sterman, An Introduction to Quantum Field Theory, (Cambridge University Press, 1993), Chapter 10.

[11] S. L. Adler, J. C. Collins and A. Duncan, Phys. Rev. D15, 1712 (1977); J. C. Collins, A. Duncan and S. D. Joglekar, Phys. Rev. D16, 438 (1977).

[12] R. Tarrach, Nucl. Phys. B183, 384 (1981); N. Gray, et al, Z. Phys. C48, 673 (1990).

[13] See, e.g., R. Coquereaux, Ann. Phys. 125, 401 (1980).

[14] C. G. Callan, Jr., Phys. Rev. D2, 1541 (1970); K. Symanzik, Comm. Math. Phys. 18, 227 (1970).

[15] T. Appelquist and J. Carazzone, Phys. Rev. D11, 2856 (1975).

[16] Ji-Feng Yang, Ph D Thesis, Fudan University, unpublished, (1994); G.-j. Ni and Ji-Feng Yang, Phys. Lett. B393, 79 (1997).