GRAVITATIONAL RADIATION FROM MONOPOLES CONNECTED BY STRINGS

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Monopole-antimonopole pairs connected by strings can be formed as topological defects in a sequence of cosmological phase transitions. Such hybrid defects typically decay early in the history of the universe but can still generate an observable background of gravitational waves. We study the spectrum of gravitational radiation from these objects both analytically and numerically, concentrating on the simplest case of an oscillating pair connected by a straight string.

I. INTRODUCTION

Monopoles connected by strings can be formed in a sequence of symmetry breaking phase transitions in the early universe. The simplest sequence of this sort is

\[ G \rightarrow H \times U(1) \rightarrow H. \] (1)

For a semi-simple group \( G \), the first of these phase transitions gives rise to monopoles which get connected by strings at the second phase transition. If both of these phase transitions occur during the radiation era, then the average monopole separation is always smaller than the Hubble radius, and when monopole-antimonopole (MM) pairs get connected by strings and begin oscillating, they typically dissipate the bulk of their energy to friction in less than a Hubble time.

A more interesting possibility arises in the context of inflationary scenario, when monopoles are formed during inflation but are not completely inflated away. Strings can
either be formed later during inflation, or in the post-inflationary epoch. In this case, the strings connecting $M\bar{M}$ pairs can be very long. The correlation length of strings, $\xi$, can initially be much smaller than the average monopole separation, $d$; then the strings connecting monopoles have Brownian shapes. But in the course of the evolution, $\xi$ grows faster than $d$, due to small loop production, and to the damping force acting on the strings. Eventually, $\xi$ becomes comparable to the monopole separation, and we are left with $M\bar{M}$ pairs connected by more or less straight strings. At later times, the pairs oscillate and gradually lose their energy by gravitational radiation and by radiation of light gauge bosons (if the monopoles have unconfined magnetic charges). When the energy of a string connecting a pair is dissipated, the monopole and antimonopole annihilate into relativistic particles.

The gravitational waves emitted by oscillating $M\bar{M}$ pairs add up to a stochastic background which can have an observable intensity $[^4]$. In order to calculate the spectrum of this background, one first needs to find the radiation spectrum produced by an individual oscillating pair. This is our main goal in the present paper.

In contrast to the case of cosmic string loops, the dynamics of $M\bar{M}$ pairs connected by strings has not been studied in any detail. We shall therefore concentrate on the simplest case of an oscillating pair connected by a straight string, for which the equations of motion can be solved exactly. Apart from its simplicity, this system has the advantage of being close to monopole-string configurations one would expect to find in the early universe.

After reviewing the dynamics of monopoles connected by strings in the next Section, we calculate the gravitational radiation spectrum from an oscillating pair in Section $[^3]$, then sum up and discuss our results in Section $[^4]$. Some technical details are given in the Appendices.
II. EQUATIONS OF MOTION

The characteristic monopole radius $\delta_m$ and string thickness $\delta_s$ are determined primarily by the corresponding symmetry breaking scales, $\eta_m$ and $\eta_s$. Typically, $\delta_m \sim \eta_m^{-1}$ and $\delta_s \sim \eta_s^{-1}$. In order to have monopoles connected by long strings, the two symmetry breaking scales should be well separated, $\eta_s \ll \eta_m$, and thus the monopole radius is much smaller than the string thickness, $\delta_m \ll \delta_s$.

Assuming that the string length is much greater than its thickness, we can treat monopoles as point particles and strings as infinitely thin lines. The dynamics of a $\bar{M}M$ pair connected by a string can then be described by the action

$$I = -m \int ds_1 - m \int ds_2 - \mu \int dS. \quad (2)$$

Here, $\mu$ is the string tension (which is equal to the mass of string per unit length), the first two integrations are over the monopole and antimonopole world lines and the third is over the string worldsheet.

The last term in Eq. (2) is the Goto-Nambu action for the string. Its variation gives the standard string equations of motion

$$\delta I_{\text{string}} = \mu \int \partial_a (\sqrt{-\gamma} \gamma^{ab} x^\mu_b) \delta x_\mu d\zeta_0 d\zeta_1 - \mu \int \partial_a (\sqrt{-\gamma} \gamma^{ab} x^\mu_b \delta x_\mu) d\zeta_0 d\zeta_1 \quad (3)$$

where $a = 0, 1$, $\zeta_a$ are a set of internal coordinates for the string worldsheet, $x_\mu^a = \partial_a (x^\mu)$, and $\gamma_{ab} = g_{\mu\nu} x^\mu_a x^\nu_b$ is the two dimensional worldsheet metric. The first term gives the equations of motion for the string which take a particularly simple form in the transverse traceless gauge

$$\dot{x}^2 + x^{'2} = 1 , \quad \dot{x} \cdot x' = 0, \quad (4)$$

where the set of internal coordinates was taken to be $(t, \sigma)$, and primes and dots refer to derivatives with respect to $\sigma$ and $t$, respectively. In this gauge, the dynamical equations are simply
\[ \ddot{x} = x'' \]  

and can be solved exactly as

\[ x(t, \sigma) = \frac{1}{2} [a(\sigma - t) + b(\sigma + t)]/2, \]  

with the gauge conditions taking the form

\[ a^2 = b^2 = 1. \]  

Since the string has two end points (one at each monopole), the spatial parameter must be restricted to an interval \([\sigma_1(t), \sigma_2(t)]\) where

\[ x_i(t) = x(t, \sigma_i(t)) \]  

are the positions of the monopoles.

The second integral in Eq. (3) can be turned into a boundary term and thus contributes to the variation of the monopole and antimonopole worldlines

\[
\int \partial_a(\sqrt{-\gamma^{\alpha\beta}x_i^\mu\delta x_\mu})d\zeta_0 d\zeta_1 = - \int \lambda_{1a} \gamma^{\alpha\beta}x_i^\mu\delta x_\mu ds_1 - \int \lambda_{2a} \gamma^{\alpha\beta}x_i^\mu\delta x_\mu ds_2,
\]  

where \(\lambda_{1a}\) and \(\lambda_{2a}\) are unit vectors orthogonal to their respective worldlines and oriented into the string worldsheet, and \(x_i^\mu\) are evaluated on the monopoles worldline at \((t, \sigma_i(t))\).

By definition, the same vector expressed in external coordinates is

\[ \lambda_0^\mu(t) = \lambda_{ia} \gamma^{\alpha\beta}x_i^\mu = \pm \gamma_i(t) [\dot{x}_i^\mu(t, \sigma_i(t)) + \ddot{x}_i^\mu(t, \sigma_i(t))], \]  

where \(\gamma_i = (1 - \dot{x}_i^2)^{-1/2}\) is the Lorentz factor of the monopoles. If we now add the terms coming from the variation of the monopole and antimonopole actions, we get the equations of motion for them in the form

\[ \frac{d^2 x_i^\nu}{ds^2} = \frac{\mu}{m} \lambda_i^\nu, \]  

\[ 4 \]
where \( \mu \) is the mass per unit length (and tension) of the string and \( m \) is the monopole mass. Since \( \lambda_i^\nu \) are unit vectors, it follows from (11) that

\[
a = \frac{\mu}{m},
\]

(12)
is the proper acceleration of the monopoles. By multiplying equations (11) by \( \dot{x}_{i\nu} \), it can be seen that only three of the four equations are independent. The time component equation of the system (11), expressing the exchange of energy between the length of string created or destroyed at the monopole end, and the kinetic energy of the monopole, takes the very simple integrable form

\[
a \dot{\sigma}_i = \pm \dot{\gamma}_i,
\]

(13)
while the spatial equations can be put in the form

\[
\gamma_i^3 \ddot{x}_i(t) = \pm \alpha \left( \frac{x'}{(x')^2} \right) (t, \sigma_i(t)) = a \gamma_i^{(s)} n_i.
\]

(14)
Here, \( \gamma_i^{(s)} = |x'|^{-1} \) is the Lorentz factor of the string at the location of the monopole, and \( n_i \) is a unit vector pointing from the monopole in the direction of the string.

The complete set of dynamical equations for the system of two monopoles connected by a string is thus given by the systems of equations (5) and (14) with the constraint (4). In general, the solutions of these equations are not periodic.

Though the string part (4-5) can be solved immediately in the standard form (6-7), the motion of the monopoles is impossible to solve in general because of the presence of \( \sigma_i(t) \), which is one of the unknowns, as a parameter of \( x' \) in equation (14). However, two sets of particular exact solutions can be found by assuming either that \( \sigma(t) \) is constant or that \( x' \) does not depend on its spatial parameter, thereby removing the problem.

In the first solution, the string has the form of a rotating rod of length \( l \) with the centrifugal force acting on the monopoles balanced by the string tension:
\[ \mathbf{x}(t, \sigma) = R \sin(\sigma/R) \mathbf{e}(t/R), \quad (15a) \]
\[ \mathbf{x}_i(t) = \pm (l/2) \mathbf{e}(t/R), \quad (15b) \]
\[ \sigma_i(t) = \pm R \arcsin(l/2R), \quad (15c) \]

where \( \mathbf{e}(\theta) = (\cos \theta, \sin \theta) \) is the radial unit vector associated with the angle \( \theta \) and
\[ l = ((1 + 4a^2 R^2)^{1/2} - 1)/a. \]

We note that in the case when the monopoles are relativistic, that is when \( aR \gg 1 \), the energy of the string is much larger than the energy of the monopoles, \( E_s / E_m \sim aR \). We will make use of this property to approximate the gravitational radiation spectrum of this solution by that of a simple cosmic string loop (rotating double line).

The second solution describes an oscillating pair of monopoles connected by a straight string:

\[ \mathbf{x}(t, \sigma) = \sigma \mathbf{e}, \quad (16a) \]
\[ \mathbf{x}_i(t) = \pm \frac{\text{sgn}(t)}{a} \left( \gamma_0 - \sqrt{1 + (\gamma_0 v_0 - a|t|)^2} \right) \mathbf{e} \quad (16b) \]
\[ \sigma_i(t) = x_i(t). \quad (16c) \]

Here \( \mathbf{e} \) is a unit vector along the string, which we choose to be directed along the \( x \)-axis, \( a \) is the monopole proper acceleration defined in [12], \( v_0 \) and \( \gamma_0 = (1 - v_0^2)^{-1/2} \) are respectively the maximum velocity and Lorentz factor of the monopoles, reached at \( t = 0 \). The monopole and antimonopole meet at \( t = 0 \) and could be expected to annihilate. However, this solution is considered as an approximation for an almost straight configuration where the monopole and antimonopole would merely come close to each other and would not collide. Besides, as we already mentioned, the monopole radii are much smaller than the string thickness, thus the monopoles are not likely to collide, even for a straight string. A peculiar feature of the solution [14] is that the monopole accelerations abruptly change direction when the monopoles meet and pass one another.

The solution [14] is valid for \( |t| \leq \gamma_0 v_0 / a \). At \( t = -\gamma_0 v_0 / a \), the monopoles are at
rest, with the string having its maximum length, \( L = 2(\gamma_0 - 1)/a \). At \( t = +\gamma_0 v_0/a \), the monopoles come to rest again, with their positions interchanged. As far as gravitational effects are concerned, since the monopole and antimonopole have the same mass, they are identical, and the full period of motion is \( T = 2\gamma_0 v_0/a \). On the other hand, if electromagnetic effects are considered, the monopole and antimonopole have opposite charges, and thus equation (16) describes only half a period, \( T = 4\gamma_0 v_0/a \), the other half period being obtained by exchanging the positions of the monopole and antimonopole:

\[
x_1(t + T/2) = x_2(t) = -x_1(t).
\] (17)

In the following section, we shall study the gravitational radiation from an oscillating pair described by the solution (16). Radiation from the rotating rod configuration will be discussed in Appendix A.

III. GRAVITATIONAL RADIATION

The power in gravitational radiation from a weak, isolated, periodic source to lowest order in \( G \), can be found from the following equations, without any further assumptions about the source [5]:

\[
P = \sum_n P_n = \sum_n \int d\Omega \frac{dP_n}{d\Omega},
\] (18a)

\[
\frac{dP_n}{d\Omega} = \frac{G\omega_n^2}{\pi} (T_{n\nu}(\omega_n, k)T^{\mu\nu}(\omega_n, k) - \frac{1}{2}|T_{\mu}^{\nu}(\omega_n, k)|^2).
\] (18b)

Here, \( dP_n/d\Omega \) is the radiation power at frequency \( \omega_n = 2\pi n/T \) per unit solid angle in the direction of \( k \), \( |k| = \omega_n \), \( T \) is the period of the oscillation, and

\[
T^{\mu\nu}(\omega_n, k) = \frac{1}{T} \int_0^T dt \exp(i\omega_n t) \int d^3x \exp(-i k \cdot x) T^{\mu\nu}(x, t)
\] (19)

is the Fourier transform of the energy-momentum tensor.
For the solution (16) considered in this section, since the system is one dimensional, the energy-momentum tensor has only three non zero components: $T^{00}$, $T^{01}$ and $T^{11}$. It also satisfies the conservation equations $\nabla_\mu T^{\mu\nu} = 0$ which in Fourier space can be written simply

$$\omega_n T^{0\nu} = k^i T^{i\nu}. \quad (20)$$

This means that $T^{\mu\nu}$ has in fact only one independent component; the simplest choice is

$$T^{01}(t, x) = m(\gamma_0 v_0 - a|t|)[\delta(x - x_1(t)e) - \delta(x + x_1(t)e)], \quad (21)$$

which has no contribution from the string part of the system. Its Fourier transform as defined by (19) can be simplified to

$$T^{01}(\omega_n, k) = m\gamma_0 v_0 I_n(u), \quad (22a)$$

$$I_n(u) = \int_0^1 \xi d\xi [\cos(n\pi(1 - \xi - \frac{u}{v_0} + u\sqrt{\xi^2 + 1/(\gamma_0 v_0)^2}))$$

$$- \cos(n\pi(1 - \xi + \frac{u}{v_0} - u\sqrt{\xi^2 + 1/(\gamma_0 v_0)^2}))], \quad (22b)$$

where $\omega_n = n\pi a/(\gamma_0 v_0)$ is the angular frequency of the $n$-th mode and we have introduced the notation $u = k_x/\omega_n$. The two other non-zero components of the Fourier transform of the energy-momentum tensor can then be deduced from the conservation equations (20) as

$$T^{00}(\omega_n, k) = uT^{01}(\omega_n, k), \quad (23a)$$

$$T^{11}(\omega_n, k) = \frac{1}{u}T^{01}(\omega_n, k). \quad (23b)$$

The gravitational energy radiated in the mode $n$ can then be expressed from (18b) as

$$P_n = 2G(n\pi\mu)^2 \int_0^1 du(\frac{1}{u} - u)^2|I_n(u)|^2, \quad (24)$$

where $I_n(u)$ is given by (22a) and depends only on $u = k_x/\omega_n$. This power spectrum can not be integrated in a closed form. However, it is possible to get analytic approximations at low and high frequency.
At high frequency, an expansion of (22) in \( 1/n \) can be made as shown in Appendix B. First, a change of variable

\[
\zeta = \pm \xi - u \sqrt{\xi^2 + 1/(\gamma_0 v_0)^2}
\]  

(25)

enables us to rewrite (22b) in the standard form (B1)

\[
I_n(u) = \frac{u}{(1-u^2)^2} \int_{-1-\frac{u}{v_0}}^{1-\frac{u}{v_0}} f(u, \zeta) \cos[n\pi(1 + \frac{u}{v_0} + \zeta)] d\zeta,
\]  

(26a)

\[
f(u, \zeta) = 2(\zeta^2 + \frac{1-u^2}{\gamma_0^2 v_0^2})^{1/2} - \frac{1-u^2}{\gamma_0^2 v_0^2} (\zeta^2 + \frac{1-u^2}{\gamma_0^2 v_0^2})^{-1/2}.
\]  

(26b)

The expansion is obtained by repeatedly integrating by parts in Eq. (26a). The dominant term is

\[
I_n(u) \approx \frac{uv_0}{(1-u^2)} \left( \frac{4}{1-u^2v_0^2} - \frac{1-v_0^2}{(1+uv_0)^3} - \frac{1-v_0^2}{(1-uv_0)^3} \right).
\]  

(27)

To get the interval of validity of this expansion, this term must be compared with the next as shown in Eq. (B5). For simplicity and because it is the most interesting case, we shall assume in the following that the monopoles are ultra-relativistic, \( \gamma_0 \gg 1 \). Then, gravitational radiation is beamed in the monopole’s direction of motion, into a cone of a small opening angle \( \theta \sim 1/\gamma_0 \). Thus, the main contribution to the total power (18) comes from values around \( u \simeq 1 \) and the comparison of the two terms can be performed there. This gives

\[
[f']_{-1-\frac{u}{v_0}}^{1-\frac{u}{v_0}} \approx (1-u^2)\gamma_0^4,
\]  

(28a)

\[
[f''']_{-1-\frac{u}{v_0}}^{1-\frac{u}{v_0}} \approx (1-u^2)\gamma_0^8,
\]  

(28b)

so that the expansion is valid as long as

\[
\frac{4}{\gamma_0^2}.
\]  

(29)

The dominant term of the power spectrum can be expressed as
\[ P_n \approx \frac{2G\mu^2}{n^2 \pi^2 v_0^2} \int_{1-v_0}^{1} dw \left[ \frac{4v_0^2}{w(2-w)} + \frac{\gamma_0^2 w - 1}{\gamma_0^4 w^4} + \frac{1 + v_0^2 - w}{\gamma_0^4 (2-w)^4} \right]^2. \] (30)

The $1/n^2$ behavior at infinity could be expected since the second derivative of the stress energy tensor has a discontinuity at $t = 0$. When $\gamma_0$ is also assumed to be large, that is the monopoles reach ultrarelativistic velocities, the integral (30) can be simplified to

\[ P_n \approx \frac{64}{5\pi^2} \left( \frac{\gamma_0}{n} \right)^2 G\mu^2. \] (31)

For $n \ll \gamma_0^2$ with $\gamma_0 \gg 1$, it is legitimate to neglect the terms $(\gamma_0 v_0)^{-2}$ in the square roots of equation (22). This gives

\[ T^{01}(\omega_n, k) \approx 8m\gamma_0 u \left( \frac{\sin(n\pi(1-u)/2)}{n\pi(1-u^2)} \right)^2, \] (32)

and the power in the $n$-th mode can be simplified to

\[ P_n \approx \frac{16}{n\pi} G\mu^2 \int_{0}^{n\pi/2} du \frac{\sin^4 u}{u^2(1 - \frac{u}{n\pi})^2}. \] (33)

It is interesting to note that this low frequency behavior is independent of the maximum Lorentz factor of the monopole $\gamma_0$, as long as it remains large. Furthermore, for large $n$, it is possible to neglect $u/n\pi$ and extend the integration to infinity so that we get

\[ P_n \approx \frac{16}{n\pi} G\mu^2 \int_{0}^{\infty} du \frac{\sin^4 u}{u^2} = \frac{4G\mu^2}{n}. \] (34)

The low frequency behavior (33) has been plotted in Figure [I] and is indeed well approximated by $4G\mu^2/n$ for $n \gtrsim 30$. From a cosmological point of view, the quantity of interest is the gravitational energy per logarithmic interval $nP_n$ which is therefore quasi-constant in the frequency interval $\gamma_0^2 \gg n \gtrsim 30$. Curiously, the low frequency spectrum for the other solution, the rotating rod (15), though it is very different from the straight string solution, also behaves like $1/n$ (see Appendix [A]).

The power spectrum can be computed numerically by integrating successively (22) and (24). The evaluation was done for various values of $\gamma_0$ and mostly exhibits a smooth
evolution from the low frequency ($33$) to the high frequency ($31$) behaviors. An example with $\gamma_0 = 25$ is shown in Figure 1.

It is also useful to know the behavior of the total gravitational energy loss rate $P$ as a function of $\gamma_0$. Once again only a numerical solution is possible. However, instead of adding up the power in different modes, it is much faster to compute $P$ directly without going to Fourier space. The calculation is outlined in Appendix C, and the resulting radiation rate is plotted in Figure 2 as a function of $\gamma_0$. Empirically, it can be closely approximated by

$$P(\gamma_0) \simeq (8 \ln(\gamma_0) + 2.2)G\mu^2,$$

which is consistent with a spectrum behaving first like $4G\mu^2/n$ up to $n \sim \gamma_0^2$ and then like $(64/5\pi^2)(\gamma_0/n)^2G\mu^2$. For large values of $\gamma_0$, the flat low frequency part of the spectrum makes the dominant contribution to the total gravitational power emitted. In the case $\gamma_0 = 25$ considered above, the power computed numerically is $P = 28.0G\mu^2$ (in agreement with a summation of all the modes of the spectrum found in Figure 1) while the algebraic approximation gives $P \approx 28.3G\mu^2$ with a relative error of only 1%.

IV. CONCLUSIONS

We have analyzed the gravitational radiation from an oscillating monopole-antimonopole pair connected by a straight string. The motion of the pair is described by Eq. (16). The gravitational radiation is emitted at a discrete set of frequencies $\omega_n = n\omega_1$, where $\omega_1 = \pi a/\gamma_0 v_0$, $a = \mu/m$ is the proper acceleration of the monopoles, $\mu$ is the string tension, $m$ is the monopole mass, $v_0$ is the highest velocity reached by the monopoles, and $\gamma_0 = (1 - v_0^2)^{-1/2}$ is the corresponding Lorentz factor. In the most interesting case of ultra-relativistic motion, $\gamma_0 \gg 1$, most of the radiation is emitted in the range $1 \lesssim n \lesssim \gamma_0^2$, with a spectrum $P_n \approx 4G\mu^2/n$. For $n \gg 1$, the spectrum can be approximated as continuous
with

\[ \frac{dP}{d\omega} \approx 4G\mu^2/\omega. \]  

(36)

At higher frequencies, \( dP/d\omega \propto \omega^{-2} \). The total radiation power is

\[ P \approx 8G\mu^2 \ln(\gamma_0). \]  

(37)

The one dimensional solution (16) should be regarded as an approximation for a more general configuration of monopoles connected by a nearly straight string. In a more realistic case, the \( 1/\omega \) and \( 1/\omega^2 \) behavior is expected to be modified for sufficiently large \( n \gg n_c \). The characteristic value \( n_c \) and the corresponding frequency \( \omega_c \) can be estimated from \( \omega_c \sim (\Delta l)^{-1}, n_c \sim l/\Delta l \), where \( \Delta l \) is the monopole separation at which deviations from the straight-line shape become important and \( l \) is the maximum extent of the string. Typically, \( \Delta l \) is comparable to the minimum distance between the monopoles as they pass one another.

We cannot tell from our analysis how the spectrum is modified at \( \omega > \omega_c \). This remains a problem for future research. We expect that eventually \( P_n \) will fall off exponentially at \( n \to \infty \), but there may also be some intermediate regime. Since no solutions of the equations of motion are known in which the string would deviate from a straight-line shape, this problem will probably have to be tackled numerically. In particular, one could employ the numerical simulations of the monopole-string system that are currently being developed [8].

Though in the simplest models of symmetry breaking all the magnetic flux of the monopoles is confined in the strings, in more realistic models, stable monopoles can have unconfined magnetic charges. In such a case, monopoles can lose energy by radiating gauge quanta. The gauge fields associated with the magnetic charge may include electromagnetic or gluon fields, but may also correspond to broken gauge symmetries and have non-zero masses. The gauge boson radiation is important since it can greatly affect the lifetime of
the pair and thus the total output of gravitational waves. An evaluation of the radiation of massless or massive gauge bosons by monopoles will be given elsewhere [3].

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APPENDIX A: GRAVITATIONAL RADIATION SPECTRUM FOR A “ROTATING ROD” SOLUTION

The rotating rod solution (15) does not seem likely to arise naturally in the early universe but could give some insight into what happens for a configuration very different from the one-dimensional solution (16). We shall concentrate on the case when the monopoles are relativistic, that is when $aR \gg 1$. In that case, it is easy to check that the energy of the string is much larger than the energy of the monopoles, $E_s/E_m \sim aR \gg 1$. This means that the monopoles can be ignored in calculating the gravitational radiation of the system. As for the string part of the system, it should be well approximated by the straight rotating double line solution

$$x(t, \sigma) = R \sin(\sigma/R)e(t/R),$$

(A1)

with $|\sigma| \leq \pi R$, as long as the wavelengths of the gravitational waves remain large compared to the length difference of the rotating string segments in the solution (15) and its approximation (A1): $n \ll aR$. Since the loop (A1) is made of two straight strings, its mass per unit length $\tilde{\mu}$ should be half that of the straight rod solution:

$$\tilde{\mu} = \mu/2.$$  

(A2)

The straight loop solution (A1) is singular at its “end points” $\sigma = \pm \pi R/2$, which move at the speed of light. This singularity results in a $1/n$ decay of its gravitational radiation
spectrum and thus in a divergent total power. This is not really a problem since our approximation is only valid at low frequencies, \( n \ll aR \). However, it is for this reason that, though the gravitational radiation of similar loop solutions has been studied in the past \([10,11]\), the calculation for this particular solution do not appear to have ever been done.

Using the standard formula for gravitational radiation from a string in the direction of the unit vector \( \mathbf{k} \), \([12]\)

\[
\frac{dP_n}{d\Omega}(\mathbf{k}) = 8\pi G\tilde{\mu}^2 n^2 \left\{ |I_n(\mathbf{n}_1)J_n(\mathbf{n}_1) - I_n(\mathbf{n}_2)J_n(\mathbf{n}_2)|^2 + |I_n(\mathbf{n}_1)J_n(\mathbf{n}_2) + I_n(\mathbf{n}_2)J_n(\mathbf{n}_1)|^2 \right\},
\]

where \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) are unit vectors orthogonal to \( \mathbf{k} \) and to one another, \( I_n(\mathbf{n}) \) and \( J_n(\mathbf{n}) \) are defined as

\[
I_n(\mathbf{n}) = \frac{1}{L} \int_0^L d\zeta \mathbf{a}'(\zeta) \cdot \mathbf{n} \exp\left[ -\frac{i\omega}{2} (\zeta + \mathbf{k} \cdot \mathbf{a}(\zeta)) \right],
\]

\[
J_n(\mathbf{n}) = \frac{1}{L} \int_0^L d\zeta \mathbf{b}'(\zeta) \cdot \mathbf{n} \exp\left[ i\omega \frac{2}{2} (\zeta - \mathbf{k} \cdot \mathbf{b}(\zeta)) \right],
\]

\( \mathbf{a}(\zeta) \) and \( \mathbf{b}(\zeta) \) are the generators of the string worldsheet as defined in equation (6), \( L \) is their period and \( \omega \) is the angular frequency of the loop. For the loop solution (A1), after integrating the gravitational power over all directions and replacing \( \tilde{\mu} \) by its value (A2), we have

\[
P_n = 8\pi^2 G\mu^2 n^2 \int_0^{\pi/2} \sin x dx \left[ \frac{\cos^4 x}{\sin^4 x} J_n'(n \sin x) + 6 \frac{\cos^2 x}{\sin^2 x} \right] \]

\[
- J_n^2(n \sin x) J_n'(n \sin x) + J_n^4(n \sin x) J_n'^2(n \sin x),
\]

where \( J_n(x) \) are Bessel functions of the first kind. This integral can be computed numerically and gives a power spectrum which behaves almost exactly like \( 5.75G\mu^2/n \), even at low frequencies. The spectrum of the rotating rod solution \([15]\) should therefore behave like \( 1/n \) for low-frequency modes \( n \ll aR \), but is expected to fall exponentially at
high frequencies (since the stress energy tensor of the system, and all its derivatives, are regular).

APPENDIX B: LARGE-n EXPANSION

In this Appendix, we develop a method for deriving large-$n$ asymptotic expansions for integrals of the form

$$I_n = \int_a^{a+2} f(u) \cos(n\pi(u - a))du,$$  \hspace{1cm} (B1)

where $f$ is a regular function which does not depend on $n$. Examples of such expressions are found in this paper in equation (26), and more generally in problems involving Fourier transforms. The trick is to integrate by parts the cosinus

$$I_n = \frac{1}{n\pi}[f(u) \sin(n\pi(u - a))]_{a}^{a+2} - \frac{1}{n\pi} \int_{a}^{a+2} f'(u) \sin(n\pi(u - a))du$$

$$= \frac{1}{n^2\pi^2}[f'(u) \cos(n\pi(u - a))]_{a}^{a+2} - \int_{a}^{a+2} f''(u) \cos(n\pi(u - a))du.$$  \hspace{1cm} (B2)

The last integral is of the same general form as the initial expression (B1) so that this expansion can be iterated to get

$$I_n = \sum_{l=1}^{\infty} \left( f^{(2l-1)}(a + 2) - f^{(2l-1)}(a) \right) \left( \frac{-1}{(n\pi)^{2l}} \right).$$  \hspace{1cm} (B3)

In this paper, we are only concerned with an expansion for large $n$ to the first significant order

$$I_n = \frac{1}{n^2\pi^2}[f'(u)]_{a}^{a+2} + O\left( \frac{1}{(n\pi)^{2}} \right).$$  \hspace{1cm} (B4)

The interval of validity of this expansion can be evaluated by requiring that the first neglected term in the expansion be much smaller than the one we keep. This gives

$$n^2 \gg \frac{[f''(u)]_{a}^{a+2}}{[f'(u)]_{a}^{a+2}}.$$  \hspace{1cm} (B5)
APPENDIX C: DIRECT CALCULATION OF THE TOTAL GRAVITATIONAL POWER

Here, we derive an expression for the total gravitational power radiated by the one-dimensional solution (16). The metric perturbation \( h^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu} \) induced by a weak source at a large distance \( r \) in the direction of the unit vector \( n \) can be expressed as

\[
\bar{h}^{\mu\nu}(t, r, n) = \frac{4G}{r} \int T^{\mu\nu}(t', r', x') d^3x',
\]

where \( \eta^{\mu\nu} \) is the Minkowski metric. The gravitational power emitted by the source can then be expressed as

\[
dP \frac{d}{d\Omega}(t) = \frac{r^2}{32\pi G} (\bar{h}^{\mu\nu} \bar{h}_{\mu\nu} - \frac{1}{2} (\bar{h}_\mu)^2),
\]

which no longer depends on the distance to the source \( r \). In the present case, there are only three non zero terms \( h^{\mu\nu} \) given by

\[
h^{00} = h_0 + \frac{m n_x}{2} \left( \frac{d u_{1r}^2}{du} + \frac{d u_{2r}^2}{du} \right),
\]

\[
h^{01} = m \left( \frac{d u_{1r}^2}{du} + \frac{d u_{2r}^2}{du} \right),
\]

\[
h^{11} = h_1 - \frac{m n_x}{2} \left( \frac{d u_{1r}^2}{du} + \frac{d u_{2r}^2}{du} \right) + m \left( \frac{1 + 2u_{2r}}{\gamma_2 - n_x u_{2r}} + \frac{1 + 2u_{1r}}{\gamma_1 + n_x u_{1r}} \right),
\]

where \( h_0 \) and \( h_1 \) are constants which do not contribute to the total gravitational power, \( u_{1r} = \gamma_0 v_0 - a|t_{1r}| \) and \( u_{2r} = \gamma_0 v_0 - a|t_{2r}| \) are functions of the retarded time for each monopole which can be expressed as functions of \( u = \gamma_0 v_0 - a|t| \) by

\[
u_i = (-1)^i n_x \gamma_0 - u.
\]
What we are really interested in is the gravitational radiation power averaged over a period. This is obtained by integrating equation (C2) over all directions and then averaging over time. This gives

$$P = \frac{r^2}{16G\gamma_0 v_0} \int_{0}^{\gamma_0 v_0} du \int_{0}^{1} dn_x (\dot{h}^{00} + \dot{h}^{11} + \dot{h}^{01}) (\dot{h}^{00} + \dot{h}^{11} - \dot{h}^{01}).$$

(C3f)

Though the expressions (C3) are complicated, finding the average total emitted power $P(\gamma_0)$ only requires a double numerical integration. Since the evaluation of each point of the spectrum also required a double integration, this method is much more efficient than simply summing the power of all the modes.

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FIG. 1. A log-log plot of the gravitational radiation spectrum $nP_n/(G\mu^2)$ in the case $\gamma_0 = 25$ (solid line) with its high frequency approximation (31) (dashed line) and low frequency approximation (33) (dotted line).

FIG. 2. Total gravitational power emitted (solid line) and its empirical algebraic approximation (35) (dashed line) as functions of $\ln(\gamma_0)$. 


$\ln(nP_n/G\mu^2)$ vs. $\ln(n)$
Figure 2