Observable eigenstate overlap in a nonlinear mean-field quantum model

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Abstract

The soliton effect is defined in nonlinear physics by the transformation of a nonlinear time-dependent dynamical system into an equivalent linear spectral eigenproblem whose invariant eigenvalues unambiguously define all the dynamical properties of the original system. We point out the existence of such an effect in a non-relativistic isotropic two-electron mean-field quantum-dot model. It yields the prediction of observable modulation of radiation absorption between its two lowest-energy zero-angular-momentum nonlinear eigenstates (i.e. eigenstates which include electron-electron interaction: hence their non-orthogonality). Characteristic values for such a possible experiment are given in the case of GaAs. Furthermore it provides an intriguing nonlinear definition of the fine-structure constant solely in terms of these eigenstates.

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I. INTRODUCTION

Mean-field nonlinear quantum approaches have become fruitful in the description of many-body quantum systems [1]. The resulting nonlinear eigenstates are quite useful in the definition of their basic stationary properties [2] [3]. In some cases (e.g. eigenstates with same angular momenta), these nonlinear eigenstates are not orthogonal and thus not observable [4]. Therefore, they should rather be regarded as convenient mathematical tools (similar to trial functions in variational problems) whose interest lies in their accuracy to provide observable results. Surprisingly enough (since mean-field descriptions originally address many-particle quantum systems) they also appear quite useful in the physical description of a single couple of bound-state interacting electrons; e.g. highly-compressed astrophysical helium [5]; Schrödinger-Poisson theory for quantum-dot helium [6] and related nonlinear interference effects [7]; excitation of radial collective modes beyond linear response [8]. Note that a $S = 0$ oscillating electron pair was investigated by use of SP nonlinear eigenstates in an early attempt to provide a pioneering description of superconductivity prior to BCS theory [10].

Nevertheless, a formal link —if any— between linear and nonlinear quantum eigenstates is still missing. One possible way to establish it is by reference to the otherwise well-known soliton paradigm in nonlinear physics. It consists in transforming a nonlinear dynamical system into an equivalent linear spectral eigenproblem defined by a specific ad hoc Hamiltonian $H_{sol}$ whose time-independent eigenvalues unambiguously yield all the invariant physical properties of the original system [11] [12]. The crucial step in this so-called “inverse scattering transform” consists in defining $H_{sol}$ from the structural properties of the nonlinear system itself. In the pioneering $1+1$ d Korteweg-DeVries example [13], $H_{sol}$ is simply Schrödinger-like with a confining —or “soliton”— potential merely equal at each time $t$ to the original time-dependent nonlinear wave profile multiplied by $−1$. Although this soliton potential is thus time-dependent, its discrete eigenvalues are not: they are constant and each of them define a single stable (collision-proof: hence its name) asymptotic propagating soliton. Note that the linear eigenfunctions corresponding to these invariant eigenvalues are only technical —or virtual— mathematical intermediates without any physical meaning. Exactly like the interacting virtual-photons eigenstates of $H_{sol}$ shown further below (Section IV).

The aim of the present work is to point out the existence of such soliton transformation in isotropic quantum-dot helium [14] [15] and to draw observable consequences from it. The two-state nonlinear Schrödinger-Poisson (SP) quantum model is defined and discussed in Section II. Firstly, we introduce two real-valued SP eigenstates $|a\rangle$ and $|b\rangle$ defined by their respective eigenvalues $\mu_a$ and $\mu_b$. We use parenthesis instead of brackets in order to outline their non-orthogonality defined by the non-zero inner product—or eigenstate overlap— $(a|b) \neq 0$. Then we show in Section III by use of the lowest-order time-dependent Schrödinger propagator [16] that the transition amplitude induced by eigenstate overlap from ground state $|a\rangle$ to excited eigenstate $|b\rangle$ is equal, up to the mere phase factor $e^{-i\mu_b t/\hbar}$, to $(a|b)$. Therefore the corresponding leading-order transition probability $\Pi = |(a|b)e^{-i\mu_b t/\hbar}|^2$ from $|a\rangle$ to $|b\rangle$ is constant and equals $(a|b)^2$. This property resembles the above-mentioned soliton paradigm. Indeed we show that it can be recovered by the solution of a specific linear spectral eigenproblem defined by an appropriate electromagnetic Hamiltonian which we construct in Section IV by use of quantum electrodynamics (QED). Specifically, it is built up from the properties of the two-state nonlinear quantum model in a quite similar way as the soliton potential $H_{sol}$ is extracted from the structural properties of the original
nonlinear system. Then, we compare in Section V the quantum predictions of these two models—namely, the nonlinear SP differential system versus its corresponding soliton-like linear eigenproblem—within the unavoidable error bar $\sim 1\%$ due to self-consistency. Indeed our nonlinear SP quantum model makes an explicit use of classical Coulomb interaction defined by its electrostatic Poisson equation. On the other hand, only the lowest-order term of the QED perturbation series (which amounts to a $\sim 1\%$ error by considering one single electron-photon scattering per particle) corresponds to such a classical Coulomb potential \[17\]. Therefore only this leading term should be kept in order to construct from QED the soliton Hamiltonian $H_{\text{sol}}$. Then we check that the linear eigenproblem defined by $H_{\text{sol}}$ yields indeed mean quantum properties that are equivalent to those of the original SP nonlinear model at maximum eigenstate overlap. In Section VI, we provide some hints for experimental observation of such a soliton effect and conclude by a summary of the main results in Section VII.

II. NONLINEAR STATIONARY TWO-STATE QUANTUM SYSTEM

Consider a $S = 0$ couple of opposite-spin electrons of particle mass $m_e$ confined in an external isotropic harmonic potential of frequency $\omega$. Discard the center-of-mass motion which separates out anyway, due to the generalized Kohn theorem related to the assumption of parabolic confinement \[18\] \[19\] and approximate the main properties of the internal structure of the two-particle wave function by use of a single-particle two-state mean-field nonlinear differential model. Actually, such a simple model yields quite acceptable results \[6\]. Select further those two eigenstates—say ground-state $|a\rangle$ or excited eigenstate $|b\rangle$—where the two particles lie both in the same $s$ orbital state (no angular momentum) in agreement with Pauli exclusion principle. In this configuration there is neither exchange energy nor spin-orbit coupling. However correlation effects described by Density-Functional Theory \[18\] will not be taken into account in the present mean-field model. Therefore our two-state quantum system defines (in CGS units) the same single-particle internal eigenstate $\Psi_i$ ($i=a, b$) for both particles by use of the following SP nonlinear differential system \[6\]:

$$-\frac{\hbar^2}{2m_e} \left[ \nabla^2 + \frac{1}{2} m_e \omega^2 r^2 + e\Phi_i \right] \Psi_i = \mu_i \Psi_i,$$

$$\nabla^2 \Phi_i = -4\pi e |\Psi_i|^2,$$

where $r^2 = x^2 + y^2 + z^2$ and $\nabla^2 = d^2/dr^2 + (2/r)(d/dr)$ in 3d radial symmetry. Besides the external confining frequency $\omega$—the sole tunable parameter of the above differential system—the only fundamental parameters are $\hbar$ together with $e$ and $m_e$ (the electron’s charge and mass). In particular, there is no velocity of light $c$, due to the evidently non-relativistic description of both the Schrödinger wave function $\Psi_i$ by use of Eq. (1) and the classical Coulomb electrostatic interaction $\Phi_i$ by use of Eq. (2).

Defining the harmonic length $L = \sqrt{\hbar/(2m_e\omega)}$ and

$$N = \frac{e^2}{L} = \frac{e^2/L}{\hbar \omega} \propto \frac{1}{\sqrt{\omega}},$$

as the characteristic dimensionless nonlinearity—or Coulomb-interaction—parameter of our model (depending only on frequency $\omega$ of the external harmonic trap), Eqs (1) - (2) can be...
put in dimensionless form by using \( C_i = \tilde{\mu}_i - e\Phi_i \), \( u_i = \sqrt{4\pi NL^3} \Psi_i \) and \( X = r/L \) while the tilde superscript defines energies in units of \( \hbar \omega \). Equations (1 - 2) then become:

\[
\left[ \frac{d^2}{dX^2} + \frac{2}{X} \frac{d}{dX} + C_i - \frac{1}{4} X^2 \right] u_i = 0, \quad (4)
\]

\[
\left[ \frac{d^2}{dX^2} + \frac{2}{X} \frac{d}{dX} \right] C_i = u_i^2. \quad (5)
\]

We check that the non-interacting linear limit is obtained for \( N \to 0 \) which means \( \omega \to \infty \) and \( u^2 \propto N \sim 0 \) in the r.h.s. of Eq. (5). Then, Eqs. (4) and (5) become uncoupled and Poisson Eq. (5) yields vanishing particle-particle interaction \( \Phi_i \sim 0 \) while (Eq. 4) defines the standard 3d one-particle linear isotropic harmonic-oscillator eigenstate.

The single-particle normalization \( 4\pi \int_0^\infty |\Psi_i|^2 r^2 dr = 1 \) yields (\( i = a, b \)):

\[
\int_0^\infty u_i^2 X^2 dX = N. \quad (6)
\]

For a given value of nonlinearity parameter \( N \), the eigenstate overlap between \( |a\rangle \) and \( |b\rangle \) becomes in these dimensionless variables:

\[
(a|b) = \frac{1}{N} \int_0^\infty u_a u_b X^2 dX, \quad (7)
\]

where:

\[
\int_0^\infty u_a^2 X^2 dX = \int_0^\infty u_b^2 X^2 dX = N, \quad (8)
\]

in accordance with Eq. (6) (indeed the two eigenstates are located in the same harmonic trap defined by \( \omega \)). Given \( \omega \) and hence \( N \) from Eq. (3), the integro-differential system of coupled equations (4-8) is numerically solved by using the appropriate “no-cusp” initial conditions \( u_{a,b}(0) \sim \sqrt{N} \) (given), \( [du_{a,b}/dX]_{X=0} = 0 \) and \( C_{a,b}(0) \) (given), \( [dC_{a,b}/dX]_{X=0} = 0 \) in order to select, under equal-norm condition Eq. (8), the eigensolutions \( u_{a,b}(X) \) defined by their respective regular boundary conditions: \( \lim_{X \to \infty} u_{a,b}(X) = 0 \) and corresponding coupled potential solutions \( C_{a,b}(X) \). Then, eigenstate overlap \( (a|b) \) is obtained by Eq. (7).

It also reads by use of Hermiticity of the Laplacian operator \( \Box \):

\[
(a|b) = \frac{W_{ab}}{\mu_b - \mu_a}, \quad (9)
\]

where \( W_{ab} = (a|W|b) \) is the matrix element of Coulomb potential

\[
W = e(\Phi_b - \Phi_a), \quad (10)
\]

related to eigenstates \( |a, b\rangle \) and to their respective eigenvalues \( \mu_{a,b} \). Single-particle Coulomb-interaction potentials \( \Phi_{a,b} \) are defined in accordance with Eqs. (11 - 2). In the above dimensionless variables, the eigenvalue \( \tilde{\mu}_i = C_i(X) + e\Phi_i(X) \) \( (i = a, b) \) can be calculated by use of either the initial conditions or the boundary conditions (resp. \( X = 0 \) or \( X \to \infty \): this latter being analytically derived from Poisson equation (5)):

\[
\tilde{\mu}_i = C_i(0) + e\Phi_i(0) = \lim_{X \to \infty} \left[ C_i(X) + \frac{N}{X} \right], \quad (11)
\]

\( \Box \)

\( 4 \)
where
\[
e \tilde{\Phi}_i(0) = \int_0^\infty G(0, X) u_i^2 X^2 dX = \int_0^\infty \frac{1}{X} u_i^2 X^2 dX, = \int_0^\infty u_i^2 X dX,
\]
due to 3d Green function \( G(X', X) = \frac{1}{|X' - X|} \) of Eq. (11) at \( X' = 0 \). Equations (11-12) provide an excellent test for the accuracy of the numerical code (we actually obtained a \( 10^{-8} \) precision by use of MatLab’s ode45 integration code).

III. TIME-DEPENDENT DESCRIPTION OF NONLINEARLY-INDUCED TRANSITIONS

Let us first describe nonlinearly-induced (or “intrinsic”) transitions between both eigenstates \(|a, b\rangle\) of the above system due to eigenstate overlap defined by Eqs. (9-10). Assume that the system whose time-dependent wavefunction is \( \Psi(r, t) \) lies in ground-state \(|a)\) at some initial time \( t = 0 \). Add “perturbation potential” \( W \) given by Eq. (10) to the “unperturbed” Hamiltonian defined by the l.h.s. of (11) where \( i = a \) (ground state). Obviously, the effect of \( W \) is simply to interchange the respective Coulomb interactions. Therefore the system becomes time-dependent for \( t > 0 \). It is defined by the transition amplitude \( \langle b | \Psi(r, t) \rangle = \langle b | K_W(t, 0) | a \rangle \) from ground state \(|a)\) at initial time \( t = 0 \) to excited eigenstate \(|b)\) at some later time \( t \). The kernel \( K_W(t, 0) \) for this transition is given by the implicit

\[
|a\rangle | b) = |3/2\rangle (continuous) and the next s excited state |b) = |7/2\rangle (broken-dotted) when \( N = 0.1, 0.4, 0.8, 1.3, 1.9, 2.7, 3.9, 6.5 \) and 10 (in bold). The circle indicates the maximum \( (a|b)^2_{max} = 3.4698911... \times 10^{-3} \sim 0.4755 \alpha \) occurring at \( N = N_{max} = 6.3542 \). The broken-dotted line displays the analytical approximation \( (a|b)^2 \sim 0.4755 \sin^{3/2}[\pi/2(N/N_{max})] \) for \( 0 < N \leq N_{max} \).
integral equation:

\[ K_W(t, 0) = K_0(t, 0) - \frac{i}{\hbar} W \int_0^t K_0(t, t') K_W(t', 0) dt', \tag{13} \]

where the unperturbed kernel \( K_{W0} \), labelled \( K_0 \), becomes \( K_0(t, 0) = \exp[-(i/\hbar)\mu_a t] \) and \( K_0(t, t') = \exp[-(i/\hbar)\mu_b (t-t')] \) \[16\] \[21\]. The lowest-order transition amplitude in \( W \), defined by \( K_W(t', 0) \sim K_0(t', 0) \) in integral \( \tag{13} \), yields:

\[ (b|K_W(t, 0)|a) = e^{-i\mu_a t} (b|a) - \frac{i}{\hbar} W_{ab} e^{-i\mu_b t} \int_0^t e^{-i(\mu_b - \mu_a) t'} dt' + o(W_{ab}^2) + \ldots, \tag{14} \]

where \( (b|a) = (a|b) \), \( W_{ab} = (\mu_b - \mu_a)(a|b) \) in accordance with Eq. \( \tag{9} \) and \( (W_{ab}^2) = (a|W^2|b) = (b|W^2|a) \). Therefore:

\[ (b|\Psi(r, t)|) = (b|K_W(t, 0)|a) = (a|b) \exp[-i\frac{\mu_b}{\hbar} t] + o(W_{ab}^2) + \ldots, \tag{15} \]

and the transition probability from initial state \( |a) \) to final state \( |b) \neq |a) \) as a result of eigenstate-overlap \( \tag{9} - \tag{10} \) becomes time-independent at the lowest-order in \( (a|b) \sim 6 \times 10^{-2} \):

\[ \left| (b|\Psi(r, t)|) \right|^2 \sim (a|b)^2 = \text{constant}. \tag{16} \]

Property \( \tag{16} \) provides square eigenstate overlap \( (a|b)^2 \) with a clear leading-order physical meaning in terms of a time-independent transition probability between \( |a) \) and \( |b) \). Equation \( \tag{16} \) has been numerically checked up to \( t = 200/\omega \) \[22\]. It justifies the further use of time-independent perturbation theory. Figure 1 (right) displays \( (a|b)^2 \) as a function of \( N \) for the two first \( s \) (zero-angular-momentum) nonlinear eigenstates \( |a) = |3/2) \) and \( |b) = |7/2) \) illustrated by Fig. 1 (left). The half-integer labels echo their respective energy eigenvalues (in units of \( \hbar \omega \)) in the linear limit \( N \to 0 \). The maximum eigenstate overlap occurs at \( N = N_{\text{max}} = 6.3542 \) and yields \( (a|b)^2 = 3.4698911\ldots 10^{-3} \) (circle).

IV. BUILD-UP OF THE SOLITON HAMILTONIAN

In order to construct that soliton Hamiltonian \( H_{\text{sol}} \) which transforms the original nonlinear system Eqs \( \tag{4} - \tag{8} \) into an equivalent linear invariant eigenproblem \[11\], \[12\], \[13\], we follow the procedure explicitly given in Sections V.D.1 and V.D.2 of \[23\] and consider the following electromagnetic interaction Hamiltonian:

\[ H_{em} = \varepsilon \int j \mathbf{A} d^3x = \left( \frac{2\pi}{L} \right)^{3/2} e^{i \sqrt{2\pi \hbar c}} \int \left[ a_{k'}^{+} \rho_{k'} + a_{k'} \rho_{k'}^+ \right] \frac{d^3k'}{\sqrt{k'}} \tag{17} \]

This Hamiltonian is in agreement with the choice of the classical Coulomb interaction potential defined by Poisson Eq. \( \tag{2} \) \[17\] \[24\]. In Eq. \( \tag{17} \), \( \varepsilon \) is the velocity of light, \( j \) is the 4-vector particle current density and \( \mathbf{A} \) its related 4-potential operator while \( a_{k'}^+ \) and \( a_{k'} \) are respectively the creation and annihilation operators of scalar photons with wave vector \( k' \) and frequency \( \omega_{k'} = ck' \). Since our SP model is stationary at first order in accordance with Eq. \( \tag{16} \), \( j \) reduces to its 4th static charge-density component \( \rho \) whose Fourier component
is $p_k^{[23]}$. The choice of $\rho$ quite naturally occurs from Coulomb potential $^{[10]}$ defining eigenstate overlap $^{[9]}$. Indeed Poisson equation $\nabla^2 W = -4\pi \rho$ yields:

$$\rho(X) = e \left[ \Psi_b^2(X) - \Psi_a^2(X) \right] = \frac{e}{4\pi N L^3} \left[ u_b^2(X) - u_a^2(X) \right]. \quad (18)$$

Moreover, the soliton Hamiltonian $\mathcal{H}_{sol}$ should vanish in the linear limit $N \to 0$ where there is neither Coulomb interaction —hence no photon exchange defined by electromagnetic Hamiltonian Eq. $^{[17]}$— nor eigenstate overlap Eqs $^{[9]-[10]}$. Consequently we assume:

$$\mathcal{H}_{sol} = f(N) \mathcal{H}_{em}, \quad (19)$$

where $f(0) = 0$.

As a result, soliton Hamiltonian $\mathcal{H}_{sol}$ defined by Eqs $^{[17]-[19]}$ is indeed obtained from the structural properties of the original nonlinear SP system through both its characteristic nonlinear parameter $N$ and its square eigenstates $\Psi_{a,b}$ (or $u_{a,b}$ in their dimensionless form). It defines the “nonlinear radiation perturbation” of ground state $|a \rangle$ in the sense that it takes into account both the electron-electron electrostatic interaction energy in $|a \rangle$ defined by Eq. $^{[17]}$ and the scaling of this energy by eigenstate overlap Eqs $^{[9]-[10]}$ (or equivalently by nonlinearity $N$). For the sake of simplicity, we start our search of function $f$ with the easiest choice:

$$f(N) \equiv N. \quad (20)$$

We shall see that this choice is indeed acceptable in order to demonstrate the soliton property, but for that very peculiar value of the harmonic electron confinement $\omega_{max}$ which corresponds to $N = N_{max}$, i.e. to maximum eigenstate overlap. We believe that the soliton property for any other value $\omega$ of the electron confinement (i.e. for any other value of $N$) could indeed be demonstrated through an appropriate choice of function $f(N)$ such that $f(0) = 0$ but this conjecture lies out of the scope of the present work.

The eigenstates of above-defined $\mathcal{H}_{sol}$ are obviously the QED photon-exchange eigenstates of $\mathcal{H}_{em}$. Therefore the first-order amplitude (in $\mathcal{H}_{sol}$) to create from photon-vacuum ground-state $|a; 0 \rangle$ a photon of energy $\hbar \omega_k = \hbar c k$ is defined by Eqs $^{[17]-[20]}$ and reads $A_k = \langle k; a | \mathcal{H}_{sol} | a; 0 \rangle / (-\hbar \omega_k)$ in agreement with time-independent perturbation theory $^{[25]}$. Hence:

$$A_k = \frac{\langle k; a | \mathcal{H}_{sol} | a; 0 \rangle}{-\hbar c k} = - \left( \frac{2\pi}{L} \right)^{3/2} N \sqrt{2\pi \alpha} n_k \kappa^{3/2}, \quad (21)$$

where $n_k$ is the Fourier component of particle density $n = \rho / e$ given by $^{[18]}$. Amplitude $^{[21]}$ displays, as expected, QED’s single-photon leading-order amplitude per particle $\propto \sqrt{\alpha}$ $^{[21]-[26]}$ where $\alpha = e^2 / (\hbar c)$ is the fine-structure constant. Defining $\kappa = L k$ and calling $G_{a,b}(\kappa)$ the dimensionless isotropic radial Fourier integral

$$G_{a,b}(\kappa) = \int_0^\infty \left[ u_b^2(X) - u_a^2(X) \right] \frac{\sin \kappa X}{\kappa X} X^2 dX, \quad (22)$$

we have $n_k = n(\kappa) = G_{a,b}(\kappa) / N \sqrt{8\pi^3}$ and therefore:

$$A_k = - \sqrt{2\pi \alpha} \frac{G_{a,b}(\kappa)}{\kappa^{3/2}}. \quad (23)$$
V. SOLITON EQUIVALENCE AT MAXIMUM EIGENSTATE OVERLAP

Let us now show that Hamiltonian $H_{sol}$ defined by use of Eqs (17-20) transforms indeed the original nonlinear system Eqs (4-8) into the linear photon-exchange eigenproblem Eqs (17-23) for $N = N_{max}$. This means that any physical property of system Eqs (4-8) at $N = N_{max}$ — a priori non-observable due to non-orthogonality of the corresponding nonlinear eigenstates [4] — becomes actually measurable since it can be recovered by use of the QED-like linear eigenproblem Eqs (21-23). Specifically, the transition probability defined by Eq. (16) at maximum eigenstate overlap, as well as the corresponding energy gap between nonlinear excited eigenstate $|i\rangle$ and ground state $|a\rangle$ defined by Eqs (11-12), should be recovered by use of Eqs (21-23). We shall show that this is indeed the case within the present first-order description yielding a $\sim 1\%$ error.

The probability $P$ for all possible transitions from SP ground state $|a\rangle$ to the continuum of virtual-photon energy states $h\omega_k = hck = hck/L$ above $|a\rangle$ becomes:

$$P = \sum_k |A_k|^2 = \left(\frac{L}{2\pi}\right)^3 \int_0^\infty |A_k|^2 4\pi k^2 dk = \frac{c}{\alpha} \int_0^\infty G_{a,b}(\kappa) \frac{dk}{\kappa}. \quad (24)$$

Due to Eq. (8) which yields $G_{a,b}(0) = 0$, there is no $\kappa^{-1}$ divergence in Eq. (24), contrary to the well-known logarithmic divergences in standard QED [20]. Numerically calculating integrals (22) and (24) for values $0 < N \leq 10$ yields the broken line $P(N)$ in Fig. 2 (left). At $N = N_{max} = 6.3542$, we obtain $P = 3.440 \times 10^{-3} = P_{max}$ which agrees with $\Pi_{max} = (a|b|^2 = 3.4698911 \ldots 10^{-3}$ (circle) within the actual $\sim 1\%$ error.

Let us further show that the two probabilities $P_{max}$ and $\Pi_{max}$ do indeed define the same transition of the electron pair from nonlinear ground state $|a\rangle$ to its next nonlinear excited state $|b\rangle$ (remind: both electrons are assumed to be in the same “s” orbital state). Once the nonlinear eigenvalue $\tilde{\mu}_i \; (i = a, b)$ is obtained by Eq. (11) for a given value $N$, the quantum expectation value $e^{\langle \bar{\Phi}_i \rangle} = \tilde{\mu}_i - \langle C_i \rangle$ in eigenstate $|i\rangle$ yields the energy per particle

$$\tilde{E}_i = \tilde{\mu}_i - \frac{1}{2}e^{\langle \bar{\Phi}_i \rangle} = \frac{1}{2}(\tilde{\mu}_i + \langle C_i \rangle).$$

Therefore the total energy for the electron pair in $|i\rangle$

$$\tilde{E}_i = 2\tilde{E}_i = \tilde{\mu}_i + \langle C_i \rangle = \tilde{\mu}_i + \frac{1}{N} \int_0^\infty u_i^2(X)C_i(X)X^2dX. \quad (25)$$

When $0 < N \leq 10$, the energy gap $\Delta \tilde{E} = \tilde{E}_{7/2} - \tilde{E}_{3/2}$ corresponding to $a = 3/2$ and $b = 7/2$ is displayed by the continuous line in Fig. 2 (right). The value $\Delta \tilde{E} = 3.2145 = \Delta_{max}$ at $N = N_{max} = 6.3542$ is indicated by the circle. It must be compared with the quantum expectation value [23]:

$$\tilde{E} = \frac{1}{\hbar\omega} \sum_k \langle 0; a|H_{sol}|a; k\rangle \langle k; a|H_{sol}|a; 0\rangle \frac{1}{-\hbar ck} = \frac{1}{\hbar\omega} \sum_k |A_k|^2 \hbar ck = \frac{N}{\pi} \int_0^\infty G_{a,b}(\kappa) d\kappa, \quad (26)$$

given in units of $\hbar\omega$ and where the negative sign in the denominator of the series in Eq. (26) is cancelled by the property that the matrix elements of the creation and annihilation operators do have always opposite signs [23] while Fourier component $G_{a,b}(\kappa)$ is defined by Eq. (22). The mean energy $\tilde{E}$ of the photon continuum above the ground state is displayed in broken line for $0 < N < 10$ in Fig. 2 (right). At $N = N_{max}$, it yields $\tilde{E} = 3.2504 = \tilde{E}_{max}$, to be compared with SP’s energy gap $\Delta_{max} = 3.2145$ (circle). The fit of these two energy values within the estimated $\sim 1\%$ error, together with the similar fit of the two corresponding
FIG. 2: SP (continuous line) vs QED-like soliton (broken line). **Left:** Photon-exchange probability $P$ defined by (24) superimposed on the eigenstate overlap $(a|b)^2$ displayed by the r.h.s. of Fig. 1. At the eigenstate overlap maximum located at $N = N_{\text{max}} = 6.3542$ and $(a|b)^2_{\text{max}} = 3.4698911 \times 10^{-3}$ (circle), $P = 3.440 \times 10^{-3}$. **Right:** Quantum expectation value $\tilde{E}$ defined by (26) superimposed on SP energy gap $\Delta \tilde{E} = \tilde{E}_{7/2} - \tilde{E}_{3/2}$ defined by (25). The circle indicates $\Delta \tilde{E} = 3.2145$ at maximum eigenstate overlap $N_{\text{max}} = 6.3542$, to be compared with $\tilde{E} = 3.2504$ for the same nonlinearity.

transition probabilities displayed in Fig. 2 (left) at maximum eigenstate overlap $N = N_{\text{max}}$, justifies the soliton transformation defined by Eqs (17 - 23).

Note that the agreement $P_{\text{max}} \sim \Pi_{\text{max}}$ at $N = N_{\text{max}}$ actually yields (within the $\sim 1\%$ error bar) a nonlinear definition of $\alpha$ which solely depends on the nonlinear eigenstates $|a\rangle$ and $|b\rangle$ of the non-relativistic quantum-dot model defined by Eqs (1 - 2).

Indeed, Eq. (24) yields at $N = N_{\text{max}}$:

$$\alpha = \pi \frac{\langle a|b \rangle_{\text{max}}^2}{\int_0^\infty G_{a,b}^2(\kappa) d\kappa} = 7.36082 \times 10^{-3} = \frac{1}{135.85} \sim \frac{1}{137.0359...}.$$  \hspace{1cm} (27)

VI. POSSIBLE EXPERIMENTAL TEST FOR NONLINEAR EIGENSTATE OVERLAP

Far-infrared spectroscopy was from early on used to investigate the electronic structure of quantum dots of various types. Soon it was realized that, due to the extended Kohn theorem [19], such a dipole radiation whose wavelength is much larger than the dot size yields a pure center-of-mass motion of the “frozen” electrons, independent of their number and of the nature of their interaction, provided that the electron trap is harmonic [18]. Therefore one has to investigate quantum dots by use of resonant Raman scattering [27] [28] in order...
to analyze internal single-electron excitations and their collective modes with monopole, dipole, or quadrupole symmetry ($M = 0, 1, 2$ where $M$ is the quantum number for angular momentum). In particular, the $M = 0$ monopole collective oscillations are excitations that can be exclusively described by internal relative coordinates (e.g. the breathing mode that determines through its frequency several ground state properties of a parabolic quantum dot [29]). A numerical simulation of the excitation of such an internal breather mode has recently been performed [9]: the quantum dot is radiated by a short oscillatory THz pulse of frequency $\omega_{rad}$ whose spatial extend is of the order of the harmonic length $L$. By extending the duration of the pulse, we can select more specifically that radiation frequency $\omega_{rad}$ which is resonant (in units of trap harmonicity $\omega$) with the SP energy gap $\Delta \tilde{E} = \tilde{E}_{7/2} - \tilde{E}_{3/2}$ displayed as a continuous line in Fig. 2 (right). For the sake of simplicity, approximate this latter (in units of $\hbar \omega$) by:

$$\Delta \tilde{E} = 4 - 0.124 \mathcal{N}. \quad (28)$$

Now consider the particular GaAs isotropic quantum-dot helium where the effective “GaAs atomic unit” $\eta = 11.86$ meV is deduced from vacuum value $\eta_0 = 27.21$ eV by use of GaAs effective electron mass $m_e^{GaAs} = 0.067 m_e$ and dielectric constant 12.4. Thence the dependence of resonant $\omega_{rad} = \Delta E/\hbar$ with respect to the trap harmonic frequency $\omega$ (in meV/$\hbar$) by use of Eq. (28):

$$\hbar \omega_{rad} \bigg|_{\Delta E} \sim \hbar \omega \left( 4 - \frac{0.604}{\sqrt{\hbar \omega}} \right) \text{meV}, \quad (29)$$

since Eq. (3) yields:

$$\mathcal{N} = \sqrt{\frac{2 \eta}{\hbar \omega}}. \quad (30)$$

At maximum eigenstate overlap $\mathcal{N} = \mathcal{N}_{max} = 6.3542$, the corresponding confinement frequency $\omega = \omega_{max}$ becomes:

$$\hbar \omega_{max} = \frac{2 \eta}{\mathcal{N}_{max}^2} = 0.5873 \text{meV}, \quad (31)$$

Therefore a GaAs isotropic quantum-dot helium initially at rest in its ground state $|3/2\rangle$ and then radiated by an external pulse whose frequency is tuned according to Eq. (29) should display a minimum energy absorption — i.e. a minimum heating — when its confinement frequency $\omega$ is varied about frequency (31) because the population of its excited eigenstate $|7/2\rangle$ is maximum there in agreement with Eq. (16).

### VII. CONCLUSION

By building up the specific Hamiltonian $H_{sol}$ from both the original eigenstate properties of the two-electron isotropic mean-field nonlinear SP model and the single-photon electromagnetic Hamiltonian, we have shown a fundamental soliton equivalence between the former nonlinear differential description and the latter linear QED-like one. It is defined by the necessary self-consistency between the classical interacting Coulomb potential that is used in the nonlinear SP model on the one hand, and the corresponding selection of the sole single-photon leading term in the QED series on the other hand. Hence the unavoidable $\sim 1\%$ error of the present model. This soliton equivalence — which only occurs at maximum
eigenstate overlap in the present state of theory—allows to translate the nonlinear (and 
hence non-observable) quantum properties of the system into linear (and hence observable) 
one}s. Consequently we suggest modulated heating of radiated isotropic quantum-dot helium 
by experimentally varying its harmonic confinement and, thus, the population difference 
duced by eigenstate overlap between its two lowest-energy nonlinear eigenstates. Finally we 
point out an intriguing nonlinear definition of the fine-structure constant solely in terms of 
the non-relativistic SP eigenstates.

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