Solar system tests of Hořava-Lifshitz gravity

Tiberiu Harko† and Zoltan Kovács‡

Department of Physics and Center for Theoretical and Computational Physics,
The University of Hong Kong, Pok Fu Lam Road, Hong Kong

Francisco S. N. Lobo§

Centro de Física Teórica e Computacional, Faculdade de Ciências da Universidade de Lisboa,
Avenida Professor Gama Pinto 2, P-1649-003 Lisboa, Portugal

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Recently, a renormalizable gravity theory with higher spatial derivatives in four dimensions was proposed by Hořava. The theory reduces to Einstein gravity with a non-vanishing cosmological constant in IR, but it has improved UV behaviors. The spherically symmetric black hole solutions for an arbitrary cosmological constant, which represent the generalization of the standard Schwarzschild-(A)dS solution, has also been obtained for the Hořava-Lifshitz theory. The exact asymptotically flat Schwarzschild type solution of the gravitational field equations in Hořava gravity contains a quadratic increasing term, as well as the square root of a fourth order polynomial in the radial coordinate, and it depends on one arbitrary integration constant. The IR modified Hořava gravity seems to be consistent with the current observational data, but in order to test its viability more observational constraints are necessary. In the present paper we consider the possibility of observationally testing Hořava gravity at the scale of the Solar System, by considering the classical tests of general relativity (perihelion precession of the planet Mercury, deflection of light by the Sun and the radar echo delay) for the spherically symmetric black hole solution of Hořava-Lifshitz gravity. All these gravitational effects can be fully explained in the framework of the vacuum solution of the gravity. Moreover, the study of the classical general relativistic tests also constrain the free parameter of the solution.

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I. INTRODUCTION

Recently, a renormalizable gravity theory in four dimensions which reduces to Einstein gravity with a non-vanishing cosmological constant in IR but with improved UV behaviors was proposed by Hořava [1, 2]. The latter theory admits a Lifshitz scale-invariance in time and space, exhibiting a broken Lorentz symmetry at short scales, while at large distances higher derivative terms do not contribute, and the theory reduces to standard general relativity (GR). The Hořava theory has received a great deal of attention and since its formulation various properties and characteristics have been extensively analyzed, ranging from formal developments [3], cosmology [4], dark energy [5] and dark matter [6], and spherically symmetric solutions [7, 8, 9, 10]. Although a generic vacuum of the theory is anti-de Sitter one, particular limits of the theory allow for the Minkowski vacuum. In this limit post-Newtonian coefficients coincide with those of the pure GR. Thus, the deviations from the conventional GR can be tested only beyond the post-Newtonian corrections, that is for a system with strong gravity at astrophysical scales.

In this context, IR-modified Hořava gravity seems to be consistent with the current observational data, but in order to test its viability more observational constraints are necessary. In Ref. [11], potentially observable properties of black holes in the Hořava-Lifshitz gravity with Minkowski vacuum were considered, namely, the gravitational lensing and quasinormal modes. It was shown that the bending angle is seemingly smaller in the considered Hořava-Lifshitz gravity than in GR, and the quasinormal modes of black holes are longer lived, and have larger real oscillation frequency in the Hořava-Lifshitz gravity than in GR. In Ref. [12], by adopting the strong field limit approach, the properties of strong field gravitational lensing in the deformed Hořava-Lifshitz black hole were considered, and the angular position and magnification of the relativistic images were obtained. Compared with the Reissner-Nordström black hole, a significant difference in the parameters was found. Thus, it was argued this may offer a way to distinguish a deformed Hořava-Lifshitz black hole from a Reissner-Nordström black hole. In Ref. [13], the behavior of the effective potential was analyzed, and the timelike geodesic motion in the Hořava-Lifshitz spacetime was also explored. In Ref. [14], the basic physical properties of matter forming a thin accretion disk in the vacuum spacetime metric of the Hořava-Lifshitz gravity models were considered. It was shown that significant differences as compared to the general relativistic case exist, and that the determination of these observational quantities could discriminate, at least in principle, between standard general relativity and Hořava-Lifshitz theory, and constrain the parameter of the model.

It is the purpose of the present paper to consider the
classical tests (perihelion precession, light bending and radar echo delay) of general relativity for static gravitational fields in the framework of Hořava-Lifshitz gravity. To do this we shall adopt for the geometry outside a compact, stellar type object (the Sun), the spherically symmetric, static metric obtained by Kehagias and Sfetsos’s \cite{10}. As a first step in our study we consider the classical tests of general relativity in arbitrary spherically symmetric spacetimes, and develop a general formalism that can be used for any given metric. For the Kehagias and Sfetsos’s metric, we first consider the motion of a particle (planet), and analyze the perihelion precession. In addition to this, by considering the motion of a photon in the static Kehagias and Sfetsos’s field, we study the bending of light by massive astrophysical objects and the radar echo delay, respectively. All these gravitational effects can be explained in the framework of Kehagias and Sfetsos’s geometry. Existing data on light-bending effects can be explained in the framework of Kehagias and Sfetsos’s \cite{10}. As a first step in our study we consider the \textit{IR} modified Hořava action is given by

\begin{equation}
S = \int dt d^3x \sqrt{g} N \left[ \frac{\kappa^2}{2 \nu g} \left( K_{ij} K^{ij} - \lambda g K^2 - \frac{\kappa^2}{2 \nu g} C_{ij} C^{ij} \right) \right] + \int_{\Sigma} d^3x \sqrt{h} \left( \frac{1}{16 \pi G} \kappa^2 \rho - \frac{\kappa^2}{4} \frac{\lambda g_{ij} \epsilon^{ij}}{\lambda} \right),
\end{equation}

where \( G \) is Newton’s constant, \( R^{(3)} \) is the three-dimensional curvature scalar for \( g_{ij} \), and \( K_{ij} \) is the extrinsic curvature, defined as

\begin{equation}
K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right),
\end{equation}

where the dot denotes a derivative with respect to \( t \).

The IR-modified Hořava action is given by

\begin{equation}
S = \int dt d^3x \sqrt{g} N \left[ \frac{\kappa^2}{2 \nu g} \left( K_{ij} K^{ij} - \lambda g K^2 - \frac{\kappa^2}{2 \nu g} C_{ij} C^{ij} \right) \right] + \int_{\Sigma} d^3x \sqrt{h} \left( \frac{1}{16 \pi G} \kappa^2 \rho - \frac{\kappa^2}{4} \frac{\lambda g_{ij} \epsilon^{ij}}{\lambda} \right),
\end{equation}

where \( \kappa, \lambda g, \nu_1, \mu, \omega \) and \( \Lambda W \) are constant parameters. \( C^{ij} \) is the Cotton tensor, defined as

\begin{equation}
C^{ij} = \epsilon^{ikl} \nabla_k \left( R^{(3)ij} \right) - \frac{1}{4} R^{(3)} \delta_i^j.
\end{equation}

Note that the last term in Eq. (3) represents a ‘soft’ violation of the ‘detailed balance’ condition, which modifies the IR behavior. This IR modification term, \( \mu^2 R^{(3)} \) (we have used the notation of Ref. \cite{8}), with an arbitrary cosmological constant, represent the analogs of the standard Schwarzschild-(\Lambda)dS solutions, which were absent in the original Hořava model.

The fundamental constants of the speed of light \( c \), Newton’s constant \( G \), and the cosmological constant \( \Lambda \) are defined as

\begin{equation}
c^2 = \frac{\kappa^2 \mu^2}{8(3 \lambda g - 1)^2}, \quad G = \frac{\kappa^2 c^2}{16 \pi (3 \lambda g - 1)} \quad \Lambda = \frac{3}{2} \Lambda W c^2.
\end{equation}

Throughout this work, we consider the static and spherically symmetric metric given by

\begin{equation}
ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\end{equation}

where \( e^{\nu(r)} \) and \( e^{\lambda(r)} \) are arbitrary functions of the radial coordinate \( r \).

Imposing the specific case of \( \lambda_g = 1 \), which reduces to the Einstein-Hilbert action in the IR limit, one obtains the following solution of the field equations in Hořava gravity,

\begin{equation}
e^{\nu(r)} = e^{-\lambda(r)} = 1 + (\omega - \Lambda W) r^2 - \sqrt{r^2 - (\omega - 2 \Lambda W) r^3 + \beta},
\end{equation}

where \( \beta \) is an integration constant \cite{8}.

By considering \( \beta = -\alpha^2/\Lambda W \) and \( \omega = 0 \) the solution given by Eq. (5) reduces to the Lu, Mei and Pope (LMP) solution \cite{9}, given by

\begin{equation}
e^{\nu(r)} = 1 - \Lambda W r^2 - \frac{\alpha^2}{\sqrt{-\Lambda W}} \sqrt{r}.
\end{equation}
Alternatively, considering now $\beta = 4\omega M$ and $\Lambda W = 0$, one obtains the Kehagias and Sfetsos’s (KS) asymptotically flat solution\(^{10}\), given by
\[
e^{\nu(r)} = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}.
\] (10)

We shall use the Kehagias-Sfetsos solution for analyzing the Solar System constraints of the theory. Note that there are two event horizons at
\[
r_{\pm} = M \left[ 1 \pm \sqrt{1 - 1/(2\omega M^2)} \right].
\] (11)

To avoid a naked singularity at the origin, one also needs to impose the condition
\[
\omega M^2 \geq \frac{1}{2}.
\] (12)

Note that in the GR regime, i.e., $\omega M^2 \gg 1$, the outer horizon approaches the Schwarzschild horizon, $r_+ \approx 2M$, and the inner horizon approaches the central singularity, $r_- \approx 0$.

### III. CLASSICAL TESTS OF GENERAL RELATIVITY IN ARBITRARY SPHERICALLY SYMMETRIC STATIC SPACE-TIMES

At the level of the Solar System there are three fundamental tests, which can provide important observational evidence for general relativity and its generalizations, and for alternative theories of gravitation in flat space. These tests are the precession of the perihelion of Mercury, the deflection of light by the Sun and the radar echo delay observations, and have been used to successfully test the Schwarzschild solution of general relativity and some of its generalizations. In order to constrain the Hořava-Lifshitz gravity at the level of the Solar System, we first have to study these effects in spherically symmetric space-times with arbitrary metrics. Throughout the next Sections we use the natural system of units with $G = c = 1$.

#### A. The perihelion precession

The motion of a test particle in the gravitational field of the metric given by Eq. (7) can be derived from the variational principle
\[
\delta \int \left( -e^{\nu} \dot{t}^2 + e^{\lambda} \dot{r}^2 + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right) ds = 0,
\] (13)

where the dot denotes $d/ds$. It may be verified that the orbit is planar, and hence without any loss of generality we can set $\theta = \pi/2$. Therefore we will use $\phi$ as the angular coordinate. Since neither $t$ nor $\phi$ appear explicitly in Eq. (13), their conjugate momenta are constant,
\[
e^{\nu} \dot{t} = E = \text{constant}, \quad r^2 \dot{\phi} = L = \text{constant}. \quad (14)
\]

The line element Eq. (7) provides the following equation of motion for $r$
\[
\dot{r}^2 + e^{-\lambda} r^2 \dot{\phi}^2 = e^{-\lambda} (e^{\nu} \dot{t}^2 - 1).
\] (15)

Substitution of $\dot{t}$ and $\dot{\phi}$ from Eqs. (14) gives
\[
\dot{r}^2 + e^{-\lambda} \frac{L^2}{r^2} = e^{-\lambda} (E^2 e^{-\nu} - 1).
\] (16)

The change of variable $r = 1/u$ and the substitution $d/u = Lu^2 d/\phi$ transforms Eq. (16) into the form
\[
\left( \frac{du}{d\phi} \right)^2 + e^{-\lambda} u^2 = \frac{1}{L^2} e^{-\lambda} (L^2 e^{-\nu} - 1).
\] (17)

By formally representing $e^{-\lambda} = 1 - f(u)$, we obtain
\[
\left( \frac{du}{d\phi} \right)^2 + u^2 = f(u)u^2 + \frac{E^2}{L^2} e^{-\nu - \lambda} - \frac{1}{L^2} e^{-\lambda} \equiv G(u).
\] (18)

By taking the derivative of the previous equation with respect to $\phi$ we find
\[
\frac{d^2 u}{d\phi^2} + u = F(u),
\] (19)

where
\[
F(u) = \frac{1}{2} \frac{dG(u)}{du}.
\] (20)

A circular orbit $u = u_0$ is given by the root of the equation $u_0 = F(u_0)$. Any deviation $\delta = u - u_0$ from a circular orbit must satisfy the equation
\[
\frac{d^2 \delta}{d\phi^2} + \left[ 1 - \left( \frac{dF}{du} \right)_{u=u_0} \right] \delta = O(\delta^2),
\] (21)

which is obtained by substituting $u = u_0 + \delta$ into Eq. (19). Therefore, in the first order in $\delta$, the trajectory is given by
\[
\delta = \delta_0 \cos \left( \sqrt{1 - \left( \frac{dF}{du} \right)_{u=u_0}} \phi + \beta \right),
\] (22)

where $\delta_0$ and $\beta$ are constants of integration. The angles of the perihelia of the orbit are the angles for which $r$ is minimum, and hence $u$ or $\delta$ is maximum. Therefore, the variation of the orbital angle from one perihelion to the next is
\[
\phi = \frac{2\pi}{\sqrt{1 - \left( \frac{dF}{du} \right)_{u=u_0}}} = \frac{2\pi}{1 - \sigma}.
\] (23)

The quantity $\sigma$ defined by the above equation is called the perihelion advance. It represents the rate of advance
of the perihelion. As the planet advances through \( \phi \) radians in its orbit, its perihelion advances through \( \sigma \phi \) radians. From Eq. (23), \( \sigma \) is given by

\[
\sigma = 1 - \sqrt{1 - \frac{dF}{du}_{u=u_0}},
\]

(24)

or, for small \( \frac{dF}{du}_{u=u_0} \), by

\[
\sigma = \frac{1}{2} \left( \frac{dF}{du} \right)_{u=u_0}.
\]

(25)

For a complete rotation we have \( \phi \approx 2\pi(1 + \sigma) \), and the advance of the perihelion is \( \delta \phi = \phi - 2\pi \approx 2\pi \sigma \). In order to be able to perform effective calculations of the perihelion precession we need to know the expression of \( L \) as a function of the orbit parameters. Let's consider the motion of a planet on a Keplerian ellipse with semi-axis \( a \) and \( b \), where \( b = a\sqrt{1 - e^2} \), and \( e \) is the eccentricity of the orbit. The surface area of the ellipse is \( \pi ab \).

Since the elementary oriented surface area of the ellipse is \( d\sigma = \left( \frac{r^2}{b^2} \right) d\phi \), the areal velocity of the planet is \( |d\sigma/dt| = |\vec{r} \times d\vec{r}|/2 = r^2 (d\phi/dt)/2 = \pi a^2 \sqrt{1 - e^2}/T \), where \( T \) is the period of the motion, which can be obtained from Kepler's third law as \( T^2 = 4\pi^2 a^3/M \). In the small velocity limit \( ds \approx dt \), and the conservation of the relativistic angular momentum gives \( r^2 d\phi/dt = L \).

Therefore we obtain \( L = 2\pi a^2 \sqrt{1 - e^2}/T \) and \( 1/L^2 = 1/2Ma(1 - e^2) \).

As a first application of the present formalism we consider the precession of the perihelion of a planet in the Schwarzschild geometry, with \( e^\nu = e^{-\lambda} = 1 - 2M/r = 1 - 2Mu \). Hence \( f(u) = 2Mu \). Since for this geometry \( \nu + \lambda = 0 \), we obtain first

\[
G(u) = 2Mu^2 + \frac{1}{L^2} (E^2 - 1) + 2M/L^2 u,
\]

(26)

and then

\[
F(u) = 3Mu^2 + M/L^2.
\]

(27)

The radius of the circular orbit \( u_0 \) is obtained as the solution of the equation

\[
u_0 = 3Mu_0^2 + M/L^2.
\]

(28)

with the only physical solution given by

\[
u_0 = \frac{1}{2} \left( 1 - \frac{12M^2}{L^2} \right) = \frac{M}{L^2}.
\]

(29)

Therefore

\[
\sigma = \frac{1}{2} \left( \frac{dF}{du} \right)_{u=u_0} = \frac{6\pi M}{a(1 - e^2)},
\]

(30)

which is the standard general relativistic result.

\section{The deflection of light}

In the absence of external forces a photon follows a null geodesic, \( ds^2 = 0 \). The affine parameter along the photon's path can be taken as an arbitrary quantity, and we denote again by \( \nu \) the dot the derivatives with respect to the arbitrary affine parameter. There are two constants of motion, the energy \( E \) and the angular momentum \( L \), given by

\[
u^i = E = \text{constant}, \quad L = \text{constant}.
\]

(31)

The equation of motion of the photon is

\[
\nu^2 + e^{-\lambda} r^2 \nu^2 = e^{-\nu - \lambda} f^2,
\]

(32)

which, with the use of the constants of the motion can be written as

\[
\nu^2 + e^{-\lambda} \nu^2 = E^2 e^{-\nu - \lambda},
\]

(33)

By introducing the variable \( \nu \), and taking into account Eqs. (31), then Eq. (33) can be expressed as

\[
\frac{L^2}{r^4} \left( \frac{dr}{d\nu} \right)^2 + e^{-\lambda} \frac{L^2}{r^2} = E^2 e^{-\nu - \lambda},
\]

(34)

which may be rewritten as

\[
\frac{d\nu}{\sqrt{(E^2/L^2) r^2 - e^\nu}} = \frac{dr}{r}.
\]

(35)

At the point of the light ray’s trajectory closest to the Sun we have \( dr/d\nu|_{r=r_0} = 0 \), where \( r_0 \) is the distance of the closest approach. From this condition we find \( (E^2/L^2) = \exp[\nu/(r_0)]/r_0^2 \) and therefore, after integrating Eq. (35), we obtain

\[
\phi(r) = \phi(\infty) + \int_0^r \frac{e^{\nu/2}}{\sqrt{e^{\nu}(r_0 - \nu)(r/r_0)^2 - 1}} dr.
\]

(36)

The deflection angle of a light ray is

\[
\Delta \phi = 2 |\phi(r_0) - \phi(\infty)| - \pi.
\]

(37)

Here we have taken into account that in the absence of a gravitational field a light ray propagates along a straight line, and precisely for this reason \( \pi \) has appeared in Eq. (37).

In the case of the Schwarzschild metric we have \( e^\nu = e^{-\lambda} = 1 - 2M/r \), which gives
\[ \phi(r) = \phi(\infty) + \int_{r_0}^{\infty} \frac{1}{\sqrt{1 - 2M/r\sqrt{(1 - 2M/r_0)/(1 - 2M/r)(r/r_0)^2 - 1}}} \, dr, \]  

(38)

By introducing a new variable \( r = r_0 x \), we obtain

\[ \phi(x) = \phi(\infty) + \int_{1}^{\infty} \frac{1}{\sqrt{1 - 2R_g/x\sqrt{(1 - 2R_g)/(1 - 2R_g/x)x^2 - 1}}} \, dx, \]

(39)

where \( R_g = M/r_0 \). In the case of the Sun, by taking \( M = M_\odot = 1.989 \times 10^{33} \) g, \( c = 2.998 \times 10^{10} \) cm/s, \( G = 6.67 \times 10^{-8} \) cm\(^3\)g\(^{-1}\)s\(^{-2}\) and \( r_0 = R_\odot = 6.955 \times 10^{10} \) cm, respectively [17], we obtain \( \phi(x) - \phi(\infty) = 1.5708 \). Thus the deflection angle of a light ray by the Sun is \( \Delta \phi = 1.72752^\circ \).

C. Radar echo delay

A third Solar System test of general relativity is the radar echo delay [17]. The idea of this test is to measure the time required for radar signals to travel to an inner planet or satellite in two circumstances: a) when the signal passes very near the Sun and b) when the ray does not go near the Sun. The time of travel of light \( t_0 \) between two planets, situated far away from the Sun, is given by

\[ t_0 = \int_{l_1}^{l_2} dy, \]

(40)

where \( l_1 \) and \( l_2 \) are the distances of the planets to the Sun. If the light travels close to the Sun, the time travel is

\[ t = \int_{-l_1}^{l_2} dy/v = \int_{-l_1}^{l_2} e^{[\lambda(r) - \nu(r)]/2} dy, \]

(41)

where \( v = e^{(\nu - \lambda)/2} \) is the speed of light in the presence of the gravitational field. The time difference is

\[ \Delta t = t - t_0 = \int_{-l_1}^{l_2} \left( e^{[\lambda(r) - \nu(r)]/2} - 1 \right) dy. \]

(42)

Since \( r = \sqrt{y^2 + R^2} \), where \( R \) is the radius of the Sun, we have

\[ \Delta t = \int_{-l_1}^{l_2} \left( e^{[\lambda(\sqrt{y^2 + R^2}) - \nu(\sqrt{y^2 + R^2})]/2} - 1 \right) dy. \]

(43)

In the case of the Schwarzschild metric \( \lambda = -\nu \), \( \exp(\lambda/2 - \nu/2) = \exp(-\nu) = (1 - 2M/r)^{-1} \approx 1 + 2M/r \), and therefore

\[ \Delta t = 2M \int_{-l_1}^{l_2} \frac{dy}{\sqrt{y^2 + R^2}} = 2M \ln \frac{\sqrt{R^2 + l_2^2} + l_1}{\sqrt{R^2 + l_1^2} - l_1}. \]

(44)

Since

\[ \ln \frac{\sqrt{R^2 + l_2^2} + l_1}{\sqrt{R^2 + l_1^2} - l_1} \approx \ln \frac{4l_1l_2}{R^2}, \]

(45)

where we have used the conditions \( R^2/l_1^2 \ll 1 \) and \( R^2/l_2^2 \ll 1 \), respectively, the time delay for a single passage of the light ray near the Sun is given by [17]

\[ \Delta t = 2M \ln \frac{4l_1l_2}{R^2}. \]

(46)

IV. SOLAR SYSTEM TESTS FOR HOŘAVA-LIFSHITZ GRAVITY BLACK HOLES

In this Section we consider the standard Solar System tests of general relativity (perihelion precession, deflection of light by the Sun and the radar echo delay) in the case of the Kehagias and Sfetsos’s (KS) asymptotically flat solution [10] of Hořava-Lifshitz gravity.

A. The perihelion precession of the planet Mercury

The observed value of the perihelion precession of the planet Mercury is \( \delta \varphi_{\text{obs}} = 43.11 \pm 0.21 \) arcsec per century [17]. As a first step in the study of the perihelion precession in Hořava-Lifshitz gravity, we introduce the variable \( u = 1/r \), and write the metric of the Hořava-Lifshitz theory black hole solution as

\[ e^\nu = e^{-\lambda} = 1 + \frac{\omega}{u^2} - \frac{\omega}{u^2} \sqrt{1 + \frac{4M}{\omega}u^3}. \]

(47)

The function \( f(u) \) is given by \( f(u) = -\omega/u^2 + \omega\sqrt{1 + 4M u^3}/u^2 \). The function \( G(u) \), defined as \( G(u) = f(u)u^2 + E^2/L^2 - e^{-\lambda}/L^2 \) is obtained as

\[ G(u) = -\omega \left( 1 + \frac{1}{L^2 u^2} \right) \left( 1 - \sqrt{1 + \frac{4M}{\omega}u^3} \right) + \frac{1}{L^2} (E^2 - 1). \]

(48)

For \( F(u) = (1/2)dG(u)/du \) we find

\[ F(u) = \frac{\omega}{L^2 u^3} \left( 1 - \sqrt{1 + \frac{4M}{\omega}u^3} \right) + \]
The value of the precession angle is 43 arcsec per century.

In order to solve Eq. (50) we represent the parameter \( \omega \) as \( \omega = \omega_0/M^2 \), and \( u_0 \) as \( u_0 = x_0/M \), where \( \omega_0 \) and \( x_0 \) are dimensionless parameters, respectively. Then Eq. (50) can be written as

\[
3x_0^2 - b^2 = \frac{\omega_0 b^2}{x_0} + \sqrt{1 + \frac{4}{\omega_0} x_0^3 \left( x_0 - \frac{\omega_0 b^2}{x_0^3} \right)},
\]

where \( b^2 = M^2/L^2 \).

In the case of the planet Mercury we have \( a = 57.91 \times 10^{11} \) cm and \( e = 0.205615 \), while for the values of the mass of the Sun and of the physical constants we take \( M = M_\odot = 1.989 \times 10^{33} \) g, \( c = 2.998 \times 10^{10} \) cm/s, and \( G = 6.67 \times 10^{-8} \) cm\(^3\)g\(^{-1}\)s\(^{-2}\), respectively [17]. Mercury also completes 415.2 revolutions each century. With the use of these numerical values we first obtain \( b^2 = M/a (1 - e^2) = 2.66136 \times 10^{-8} \). By performing a first order series expansion of the square root in Eq. (51), we obtain the standard general relativistic equation \( 3x_0^2 - x_0 + b^2 = 0 \), with the physical solution \( x_0^{(GR)} \approx b^2 \). In the general case, the value of \( x_0 \) also depends on the numerical value of \( \omega_0 \), and, for a given \( \omega_0 \), \( x_0 \) must be obtained by numerically solving the nonlinear algebraic equation Eq. (51). The variation of \( x_0 \) as a function of \( \omega_0 \) is represented in Fig. 1.

The perihelion precession is given by \( \delta \phi = \pi (dF(u)/du)|_{u=u_0} \), and in the variables \( x_0 \) and \( \omega_0 \) can be written as

\[
\delta \phi = 3 \sqrt{\omega_0} \left\{ 2 \left( x_0^3 + \omega_0 \right) x_0^3 + b^2 \left[ 2x_0^6 + \left( 6\omega_0 - 4\sqrt{\omega_0} \left( 4x_0^3 + \omega_0 \right) \right) x_0^3 + \omega_0^2 - \sqrt{\omega_0} \left( 4x_0^3 + \omega_0 \right) \right] \right\} / x_0^3 \left( 4x_0^3 + \omega_0 \right)^{3/2}.
\]

The variation of the perihelion precession angle as a function of \( \omega_0 \) is represented in Fig. 2. Thus, for \( \omega_0 \approx 7 \times 10^{-16} \), the value of the precession angle is 43 arcsec per century.

### B. Light deflection by the Sun

The deflection angle of light rays passing nearby the Sun in the Kehagias and Sfetsos’s geometry is given by

\[
\phi(r) = \phi(\infty) + \int_r^\infty \frac{1 + \omega r^2 - \sqrt{r \left( \omega^2 r^3 + 4\omega M \right)}}{\sqrt{1 + \omega r_0^2 - \sqrt{r_0 \left( \omega^2 r_0^3 + 4\omega M \right) \left( r/r_0 \right)^2 / \left[ 1 + \omega r^2 - \sqrt{r \left( \omega^2 r^3 + 4\omega M \right) - 1 \right]}}} \, dx.
\]

By introducing a new variable \( x \) by means of the transformation \( r = r_0 x \), Eq. (53) can be written as

\[
\phi(r_0) = \phi(\infty) + \int_1^\infty \frac{1 + \omega r_0^2 x^2 - \sqrt{r_0 x \left( \omega^2 r_0^3 x^3 + 4\omega M \right)}}{\sqrt{1 + \omega r_0^2 - \sqrt{r_0 \left( \omega^2 r_0^3 + 4\omega M \right) x^2 / \left[ 1 + \omega r_0^2 x^2 - \sqrt{r_0 x \left( \omega^2 r_0^3 x^3 + 4\omega M \right) - 1 \right]}}} \, dx.
\]
By representing $\omega$ as $\omega = \omega_0/M^2$, and $r_0$ as $r_0 = x_0 M$, we obtain

$$\phi(x_0) = \phi(\infty) + \int_1^{\infty} \frac{\left[1 + \omega_0 x_0^2 x^2 - \sqrt{x_0 x(x_0^2 x^2 + 4 \omega_0)}\right]^{-1/2}}{\sqrt{1 + \omega_0 x_0^2 - \sqrt{x_0(x_0^2 x^2 + 4 \omega_0)}} x^2 / \left[1 + \omega_0 x_0^2 x^2 - \sqrt{x_0 x(x_0^2 x^2 + 4 \omega_0)}\right]} \frac{dx}{x}. \quad (55)$$

For the Sun, by taking $r_0 = R_\odot = 6.955 \times 10^{10}$ cm, we find for $x_0$ the value $x_0 = 4.71194 \times 10^5$. The variation of the deflection angle $\Delta \phi = 2 |\phi(x_0) - \phi(\infty)| - \pi$ is represented, as a function of $\omega_0$, in Fig. 3. The observational value of the deflection angle of $\Delta \phi = 1.7275$ arcsec corresponds to $\omega_0 = 10^{-15}$.

![FIG. 3: The light deflection angle $\Delta \phi$ (in arcseconds) as a function of the parameter $\omega_0$.](image)

C. Radar echo delay

The best experimental Solar System constraints on time delay so far have come from the Viking lander on Mars [18]. In the Viking mission two transponders landed on Mars and two others continued to orbit around it. The latter two transmitted two distinct bands of frequencies, and thus the Solar coronal effect could be corrected for. For the time delay of the signals emitted on Earth, and which graze the Sun, one obtains $\Delta t_{RD} = \Delta t_{RD}^{(GR)} (1 + \Delta_{RD})$, with $\Delta_{RD} \leq 0.002$ [18].

For the case of the Earth-Mars-Sun system we have $R_E = l_1 = 1.525 \times 10^{13}$ cm (the distance Earth-Sun) and $R_P = l_2 = 2.491 \times 10^{13}$ cm (the distance Mars-Sun). The radius of the Sun is $R_\odot = 6.955 \times 10^{10}$ cm. With these values the standard general relativistic radar echo delay has the value $\Delta t_{RD}^{(GR)} \approx 4M \ln \left(4l_1 l_2 / R_\odot^2\right) \approx 2.4927 \times 10^{-4}$ s. With the use of Eq. (53), it follows that the time delay for the Kehagias and Sfetsos’s black hole solution of Hořava-Lifshitz gravity can be represented as

$$\Delta t_{RD} = \frac{2}{c} \int_{-l_1}^{l_2} \frac{\omega \left(y^2 + R^2\right)}{1 - \omega \left(y^2 + R^2\right)} \frac{\left[\sqrt{1 + (4M/\omega) (y^2 + R^2)^{-3/2}} - 1\right]}{\sqrt{1 + (4M/\omega) (y^2 + R^2)^{-3/2} - 1}} dy. \quad (56)$$
By introducing a new variable $\xi$ defined as $y = 2 \xi M$, and by representing again $\omega$ as $\omega = \omega_0/M^2$, we obtain for the time delay the expression

$$\Delta t_{RD} = \frac{16M}{c} \omega_0 \int_{\xi_2}^{\xi_1} \frac{(\xi^2 + a^2) \left[ \sqrt{1 + (1/2\omega_0)(\xi^2 + a^2)^{-3/2}} - 1 \right]}{1 - 4\omega_0(\xi^2 + a^2) \left[ \sqrt{1 + (1/2\omega_0)(\xi^2 + a^2)^{-3/2}} - 1 \right]} d\xi, \quad (57)$$

where $a^2 = R_2^2/4M_0^2$, $\xi_1 = l_1/2M_0$, and $\xi_2 = l_2/2M_0$, respectively. The variation of the time delay as a function of $\omega_0$ is represented in Fig. 4.

The general relativistic value $\Delta t_{RD} = 2.4927 \times 10^{-4}$ is obtained for $\omega_0 \approx 4 \times 10^{-15}$. For $\omega_0 = 7 \times 10^{-16}$, the numerical value of the time delay is $\Delta t_{RD} = 2.4885 \times 10^{-4}$ s.

V. DISCUSSIONS AND FINAL REMARKS

In the present paper we have considered the observational and experimental possibilities for testing, at the level of the Solar System, the Kehagias and Sfetsos’s solution of the vacuum field equations in Hořava-Lifshitz gravity. We have found that this solution can give a very satisfactory description of all the gravitational phenomena in the Solar System. The classical tests of general relativity (perihelion precession, light deflection and radar echo delay) give strong constraints on the numerical value of the parameter $\omega$ of the model. The parameter $\omega$, having the physical dimensions of length$^{-2}$, is constrained by the perihelion precession of the planet Mercury to a value of $\omega = 7 \times 10^{-16}/M_0^2 = 3.212 \times 10^{-26}$ cm$^{-2}$, while the radar echo delay experiment suggests a value of $\omega = 4 \times 10^{-15}/M_0^2 = 1.835 \times 10^{-25}$ cm$^{-2}$. The deflection angle of the light rays by the Sun can be fully explained in Hořava-Lifshitz gravity with the parameter $\omega$ having the value $\omega = 10^{-15}/M_0^2 = 4.5899 \times 10^{-26}$. It is interesting to note that the values of $\omega$ obtained from the study of the light deflection by the Sun and of the perihelion precession of the planet Mercury are extremely close, while the radar echo delay experiments provide a larger value.

Thus, the study of the classical tests of general relativity provide a very powerful method for constraining the allowed parameter space of the Hořava-Lifshitz gravity solutions, and to provide a deeper insight into the physical nature and properties of the corresponding spacetime metrics. Therefore, this opens the possibility of testing Hořava-Lifshitz gravity by using astronomical and astrophysical observations at the Solar System scale. Of course, this analysis requires developing general methods for the high precision study of the classical tests in arbitrary spherically symmetric spacetimes. In the present paper we have provided some basic theoretical tools necessary for the in depth comparison of the predictions of the Hořava-Lifshitz gravity model with the observational/experimental results.

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