Multiparticle distributions and intercepts of $r$-particle correlation functions in the symmetric Tamm-Dancoff type $q$-Bose gas model

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Abstract

Symmetric Tamm-Dancoff (STD) type $q$-deformed quantum oscillators are used as the base for constructing the respective STD type $q$-Bose gas model. In this letter, within the STD $q$-deformed Bose gas model we derive explicit analytic expressions for the $r$-particle momentum distribution functions, and for the (momentum) correlation function intercepts of 2nd, 3rd, and any $r$th order. Besides, we obtain large-momentum asymptotic formulas for the $r$th order intercepts which show dependence on the $q$-parameter only. The obtained formulas provide new example, in addition to already known two cases, of exact results for $r$-particle distributions and the correlation function intercepts in deformed analogs of Bose gas model.

Keywords. Symmetric Tamm-Dancoff deformation, deformed Bose gas model, deformed oscillators, correlation functions, intercepts, multiparticle distributions.

1 Introduction

The statistical (two-dimensional, lattice) models constitute large class of exactly solvable models [1] whose development is witnessed even in present days. The solvability in those models implies the possibility of finding exact results for the partition function, correlation functions etc. On the other hand, there exists another important branch of modern statistical physics, namely the study of deformed analogs of Bose (and Fermi) gas model in two, three, etc. dimensions. Its development, from the earliest papers on the subject [2–6] and till some most recent ones [7–9] demonstrates that this still remains a hot topic (see also [10, 11] for some, by no means exhaustive, list of works published during the last two decades), with many interesting problems awaiting for their resolving. Among these, it is worth to point out the task of deriving exact expressions for statistical quantities, in particular the multi-particle distribution functions and respective (momentum) correlation functions or their intercepts. At present, to the best of our knowledge, for only two essentially differing deformed analogs of the Bose gas model that task has been performed. Namely, the $r$-particle ($r \geq 1$) distribution functions and the respective $r$-particle correlation function intercepts have been obtained (i) for the $p,q$-deformed Bose gas...
model, see [12], and (ii) for the recently proposed \( \mu \)-deformed analog of Bose gas model [13]. The two models principally differ: while the first one belongs to Fibonacci class [14], the second one is not Fibonacci, but is a typical representative of the class of quasi-Fibonacci models [15].

Recently, a new symmetric Tamm-Dancoff (STD) \( q \)-deformed oscillator has been introduced [16] and some of its major properties studied. Note that usual Tamm-Dancoff \( q \)-oscillator [17, 18] involves only real values of the parameter \( q \), and possesses diverse cases of accidental degeneracies of energy levels [19]. The STD \( q \)-oscillator admits, besides real, also the complex phase-like values of deformation parameter \( q \) and, moreover, there exist definite values of \( q = \exp(i\theta) \) for which accidental degeneracies of energy levels do occur [16]. The possibility to deal with the complex-valued deformation parameter, which we have in particular in the STD type \( q \)-deformed model, gives some important advantages, as it was discussed in (the last paragraph of) ref. [16].

Let us emphasize that whereas usual Tamm-Dancoff \( q \)-oscillator belongs to the class of Fibonacci oscillators, the STD \( q \)-deformed oscillator does not. Namely, as shown in [16], it is a quasi-Fibonacci one (the notion of quasi-Fibonacci oscillators was introduced in [15]).

In this letter our goal is to derive, for the STD type \( q \)-deformed Bose gas model based on the set of STD type \( q \)-deformed oscillators, the exact expressions for the \( r \)-particle \((r \geq 1)\) distribution functions and the respective \( r \)-th order correlation function intercepts. By deriving this result we demonstrate that there is one more, the third, \( q \)-deformed family of modified Bose-gas like models for which it proves possible to obtain the exact analytical formulas for the \( r \)th order distribution functions and \( r \)-particle correlation intercepts.

## 2 Basics of the symmetric Tamm-Dancoff \( q \)-oscillator [16]

The symmetric \( q \)-deformed bosonic Tamm-Dancoff oscillator algebra is defined as

\[
aa^\dagger - a^\dagger a = \{N + 1\}_q - \{N\}_q = \frac{1}{2} (1 + (1 - q^{-1})N))q^N + \frac{1}{2} (1 + (1 - q)N)q^{-N},
\]

\[
[N, a^\dagger] = a^\dagger, \quad [N, a] = -a,
\]

where

\[
a^\dagger a = \{N\}_q \equiv \varphi_{\text{STD}}(N) \equiv \frac{N}{2} (q^{N-1} + q^{-N+1}) = \frac{N}{2} q^{N+1} (1 + (q^2)^{N-1})
\]

with \( \{N\}_q \) denoting the STD type \( q \)-number \((q\)-bracket\) or structure function, and \( q \) is either real, \( 0 < q \leq \infty \), or complex phase-like: \( q = \exp(i\theta), \quad -\pi \leq \theta \leq \pi \).

One can easily show that the \( q \)-analog Fock-type representation of the algebra [1] is valid:

\[
N|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \ldots,
\]

\[
a|n\rangle = \sqrt{n} \frac{1}{2} (q^n + q^{-n+1})|n-1\rangle = \sqrt{\{n\}_q}|n-1\rangle,
\]

\[
a^\dagger|n\rangle = \sqrt{n+1} \frac{1}{2} (q^n + q^{-n})|n+1\rangle = \sqrt{\{n+1\}_q}|n+1\rangle.
\]

Remark that this STD \( q \)-bracket can be also written as a “multiplicative hybrid” of the structure function \( \varphi(n) = n \) of usual oscillator and that of the Biedenharn - Macfarlane \( q \)-deformed oscillator [20, 21] i.e. the (obviously symmetric under \( q \leftrightarrow q^{-1} \)) structure function \( \varphi_{BM}(n) \equiv [n]_q = \frac{1}{2} (q^n + q^{-n}) \).
\[
\frac{q^n - q^{-n}}{q - q^{-1}},
\]
as follows:
\[
\{n\}_q = n \left[ n \frac{[n]_q - [n-2]_q}{2} \right] \quad \text{or} \quad \{n\}_q = n \frac{2(n-1)[n]_q}{2[n-1]_q}.
\]

### 3 Intercepts of 2nd and 3rd order correlation functions

The deformed Bose gas model constructed from the set of independent modes of deformed oscillators with STD type structure function of deformation \(\varphi_{\text{STD}}(N)\), see eq. (2), is studied here. We start with the following defining expression for the intercept (see e.g. [22] for a nondeformed case, and [12] for deformed one) of \(r\)th order momentum correlation function, with fixed momentum \(\mathbf{k}\):
\[
\lambda^{(r)}(\mathbf{k}) = \frac{\langle (a^\dagger_{\mathbf{k}})^r a_{\mathbf{k}} \rangle}{\langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \rangle} - 1 = \frac{\varphi(N_{\mathbf{k}}) \varphi(N_{\mathbf{k}} - 1) \cdots \varphi(N_{\mathbf{k}} - r + 1)}{\langle \varphi(N_{\mathbf{k}}) \rangle^r} - 1. \tag{4}
\]

The bracket \(\langle \ldots \rangle\) denotes statistical (thermal) average. As seen, to find the intercepts \(\lambda^{(2)}(\mathbf{k})\) and \(\lambda^{(3)}(\mathbf{k})\) we have to calculate the averages \(\langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \rangle\), \(\langle (a^\dagger_{\mathbf{k}})^2 (a_{\mathbf{k}})^2 \rangle\) and \(\langle (a^\dagger_{\mathbf{k}})^3 (a_{\mathbf{k}})^3 \rangle\). In what follows, when performing the calculations that involve the quantities for a fixed mode, the index \(\mathbf{k}\) will be omitted for the sake of simplicity.

Taking the Hamiltonian in the simplest linear (additive) form
\[
H = \sum_{\mathbf{k}} \varepsilon(\mathbf{k}) N_{\mathbf{k}} = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} N_{\mathbf{k}}, \tag{5}
\]
for the average \(\langle a^\dagger a \rangle\) we find:
\[
\langle a^\dagger a \rangle = \langle \varphi(N) \rangle = \sum_{n=0}^{\infty} \frac{n}{2} (q^{n+1} + q^{-n+1}) e^{-\beta \hbar \omega_n} / \sum_{n=0}^{\infty} e^{-\beta \hbar \omega_n} = \frac{1}{2} (1 - e^{-x}) \left( -\frac{\partial}{\partial x} \right) \sum_{n=0}^{\infty} (q^{n+1} + q^{-n+1}) e^{-nx} = \frac{1 - e^{-x}}{2} \left( -\frac{\partial}{\partial x} \right) \left( \frac{q^{-1}}{1 - qe^{-x}} + \frac{q}{1 - q^{-1} e^{-x}} \right) = \frac{e^{-x}(1 - e^{-x})}{2} \left( \frac{1}{1 - qe^{-x}} + \frac{1}{1 - q^{-1} e^{-x}} \right) \tag{6}
\]
where \(x = \beta \hbar \omega_k, \beta = \frac{1}{k_B T} \), \(k_B\) is Boltzmann’s constant. In a similar way we calculate \(\langle (a^\dagger)^2 a^2 \rangle\):
\[
\langle (a^\dagger)^2 a^2 \rangle = \langle \varphi(N) \varphi(N - 1) \rangle = \frac{1}{4} (N(N - 1)(q^{N-1} + q^{-N+1})(q^{N-2} + q^{-N+2})) = \frac{1}{4} (1 - e^{-x}) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} \right) \sum_{n=0}^{\infty} (q^{n+1} + q^{-n+1})(q^{n-2} + q^{-n+2}) e^{-nx} = \frac{1}{4} (1 - e^{-x}) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} \right) \sum_{n=0}^{\infty} (q^{2n-3} + q^{-2n+3} + q^{-1} + q) e^{-nx} = \frac{1}{4} (1 - e^{-x}) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} \right) \left( \frac{q^{-3}}{1 - q^{-2} e^{-x}} + \frac{q^3}{1 - q^2 e^{-x}} + \frac{q + q^{-1}}{1 - e^{-x}} \right).\]
Utilizing the equality \( \left( \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} \right) \frac{1}{1-ae^{-x}} = 2 \frac{a^2e^{-2x}}{(1-ae^{-x})^2} \) we arrive at the desired expressions:

\[
\langle (a^\dagger)^2 a^2 \rangle = \frac{1}{2} e^{-2x}(1-e^{-x}) \left[ \frac{q}{(1-q^2e^{-x})^3} + \frac{q^{-1}}{(1-q^{-2}e^{-x})^3} + \frac{q+q^{-1}}{(1-e^{-x})^3} \right],
\]

\[
\lambda^{(2)}(k) = \frac{2\left[ \frac{q}{(1-q^2e^{-x})^3} + \frac{q^{-1}}{(1-q^{-2}e^{-x})^3} + \frac{q+q^{-1}}{(1-e^{-x})^3} \right]}{(1-e^{-x})((1-qe^{-x})^2 + (1-q^{-1}e^{-x})^2)^2} - 1.
\]

Analogously, with account of equality \( \left( \frac{\partial}{\partial x} + 2 \right) \left( \frac{\partial}{\partial x} + 1 \right) \frac{1}{1-ae^{-x}} = -6 \frac{a^3e^{-3x}}{(1-ae^{-x})^3} \), for \( \langle (a^\dagger)^3 a^3 \rangle \) we obtain the third order distribution function and same order correlation function intercept:

\[
\langle (a^\dagger)^3 a^3 \rangle = \frac{3}{4} e^{-3x}(1-e^{-x}) \left[ \frac{q^3}{(1-q^3e^{-x})^4} + \frac{q^{-3}}{(1-q^{-3}e^{-x})^4} + \frac{q^3(1+q^{-2}+q^{-4})}{(1-qe^{-x})^4} + \frac{q^{-3}(1+q^2+q^4)}{(1-q^{-1}e^{-x})^4} \right],
\]

\[
\lambda^{(3)}(k) = \frac{6\left[ \frac{q^3}{(1-q^3e^{-x})^4} + \frac{q^{-3}}{(1-q^{-3}e^{-x})^4} + \frac{q^3(1+q^{-2}+q^{-4})}{(1-qe^{-x})^4} + \frac{q^{-3}(1+q^2+q^4)}{(1-q^{-1}e^{-x})^4} \right]}{(1-e^{-x})^2((1-qe^{-x})^2 + (1-q^{-1}e^{-x})^2)^3} - 1.
\]

### 4 Distributions and correlation function intercepts of rth order

The definition \( \lambda^{(r)} \) of rth order intercepts \( \lambda^{(r)} \) involves the averages (deformed distributions) \( \langle (a^\dagger)^r a^r \rangle \), that in the STD q-Bose gas case gives

\[
\langle (a^\dagger)^r a^r \rangle = \langle \varphi(N) \cdot \ldots \cdot \varphi(N-r+1) \rangle = \langle N(N-1)\ldots(N-r+1) \prod_{j=1}^{r} (q^{N-j} + q^{-N+j}) \rangle. \tag{9}
\]

For the product appearing in this equality we obtain

\[
\prod_{j=1}^{r} (q^{N-j} + q^{-N+j}) = \prod_{j=1}^{r} q^{N-j} \prod_{j=1}^{r} (1 + (q^2)^{-N+j}) = q^{rN-r(r+1)/2} \sum_{k=0}^{r} (q^2)^{-kN} \sum_{1 \leq j_1 < \ldots < j_k \leq r} (q^2)_{j_1+\ldots+j_k} =
\]

\[
= q^{rN-r(r+1)/2} \sum_{k=0}^{r} (q^2)^{-kN} \sum_{s=k(k+1)/2}^{(2r-k+1)k/2} p(k, r, s) q^{2s} \tag{10}
\]

where \( p(k, r, s) \) is the number of partitions of \( s \) into \( k \) distinct summands each of which being not greater than \( r \). Remark also that the product in \( \prod_{j=1}^{r} (1 + (q^2)^{-N+j}) \) can be expressed through the Gaussian \( q \)-binomial coefficients \( \binom{n}{k}_q \) (see e.g. \cite{23} for definition) by means of the identity

\[
\prod_{j=1}^{r} (1 + (q^2)^{-N+j}) = \prod_{j=0}^{r-1} (1 + q^{-2N+2} \cdot q^{2j}) = \sum_{k=0}^{r} q^{k(k+1)} \binom{r}{k}_q (q^2)^{-kN}. \tag{11}
\]

Now perform the calculation that generalizes the case of \( \langle (a^\dagger)^2 a^2 \rangle \) or \( \langle (a^\dagger)^3 a^3 \rangle \) to any rth order:
\[
\langle (a^\dagger)^r a^r \rangle = \langle N(N-1)...(N-r+1) \prod_{j=1}^r (q^{N-j} + q^{-N+j}) \rangle = (1 - e^{-x}) \sum_{n=0}^\infty \frac{1}{2^r n(n-1)...(n-r+1)} \prod_{j=1}^r (q^{N-j} + q^{-N+j})e^{-nx} = (1 - e^{-x}) (\partial \frac{\partial}{\partial x} + r - 1) \prod_{n=0}^\infty \prod_{j=1}^r (q^{N-j} + q^{-N+j})e^{-nx} = \]
\[
\frac{(-1)^r}{2^r} (1 - e^{-x}) (\partial \frac{\partial}{\partial x} + r - 1) \prod_{n=0}^\infty \sum_{q^{rn-r(r+1)/2}} \sum_{k=0}^r (q^2 - k) \sum_{s=s(k+1)/2} p(k, r, s) q^{2s-r(r+1)/2} (\partial \frac{\partial}{\partial x} + r - 1) \prod_{n=0}^\infty \prod_{j=1}^r (q^{N-j} + q^{-N+j})e^{-nx} = \]
\[
= \frac{(-1)^r}{2^r} (1 - e^{-x}) \sum_{k=0}^r (2k+1)/2 \sum_{s=k+1/2} p(k, r, s) q^{2s-r(r+1)/2} (\partial \frac{\partial}{\partial x} + r - 1) \prod_{n=0}^\infty \prod_{j=1}^r (q^{N-j} + q^{-N+j})e^{-nx}. \quad (12)
\]

By induction, it can be verified that
\[
(\partial \frac{\partial}{\partial x} + r - 1) \prod_{n=0}^\infty \prod_{j=1}^r (q^{N-j} + q^{-N+j})e^{-nx} = (-1)^r r! q^{-rx} e^{-rx} (1 - qe^{-x})^{r+1}. \quad (13)
\]

Indeed,
\[
(\partial \frac{\partial}{\partial x} + r - 1) \prod_{n=0}^\infty \prod_{j=1}^r (q^{N-j} + q^{-N+j})e^{-nx} = (-1)^r r! q^{-rx} e^{-rx} (1 - qe^{-x})^{r+1} = (-1)^r r! q^{-rx} e^{-rx} (1 - qe^{-x})^{r+1}.
\]

From (13) and (12) we derive one of our major formulas, namely
\[
\langle (a^\dagger)^r a^r \rangle = \frac{r!}{2^r} (1 - e^{-x}) e^{-rx} \sum_{k=0}^r \sum_{s=s(k+1)/2} p(k, r, s) q^{2s-r(r+1)/2} \sum_{s=s(k+1)/2} p(k, r, s) q^{2s-r(r+1)/2} = \frac{r!}{2^r} (1 - e^{-x}) e^{-rx} \sum_{k=0}^r \sum_{s=s(k+1)/2} p(k, r, s) q^{2s-r(r+1)/2} \sum_{s=s(k+1)/2} p(k, r, s) q^{2s-r(r+1)/2} \quad (15)
\]

which in terms of well-known \(q\)-binomials takes the form
\[
\langle (a^\dagger)^r a^r \rangle = \frac{r!}{2^r} (1 - e^{-x}) e^{-rx} \sum_{k=0}^r \sum_{s=s(k+1)/2} q^{k(k+1)-r(r+1)/2} \sum_{s=s(k+1)/2} p(k, r, s) q^{2s-r(r+1)/2} = \frac{r!}{2^r} (1 - e^{-x}) e^{-rx} \sum_{k=0}^r \sum_{s=s(k+1)/2} q^{k(k+1)-r(r+1)/2} \sum_{s=s(k+1)/2} p(k, r, s) q^{2s-r(r+1)/2} \quad (16)
\]

Using (6), (13) and definition (4) we write out the expression for the \(r\)th order intercept as our final result:
\[
\lambda^{(r)}(k) = \frac{r! q^{-r(r+1)/2} \sum_{k=0}^r (r) \sum_{s=s(k+1)/2} q^{k(k+1)-r(r+1)/2} \sum_{s=s(k+1)/2} q^{k(k+1)-r(r+1)/2} (1-q^{-2k})^{r+1}}{(1-e^{-x})^{r+1}((1-qe^{-x})^2 + (1-q^{-1}e^{-x})^2)^r} - 1, \quad x = \beta \hbar \omega_k. \quad (17)
\]

Its \(\beta \hbar \omega_k \to \infty \) (large momentum or low temperature) asymptotics takes the form:
\[
\lambda^{(r)}_{as} = \sum_{k=0}^r (r) q^{-1} = \frac{r!}{2r} \prod_{k=1}^r (q^{k-1} + q^{-k+1}) - 1. \quad (18)
\]
Let us stress that the obtained asymptotics depends, besides the order \( r \), on the deformation parameter \( q \) only; neither mass of particle nor the temperature of \( q \)-Bose gas survive in the asymptotics. In the particular case of \( r = 4 \) we have:

\[
\langle (a_\uparrow a_\downarrow)^4 \rangle = \frac{3}{2}(1 - e^{-x})e^{-4x}\left[ \frac{q^6}{(1 - q^4 e^{-x})^5} + \frac{1 + q^2 + q^4 + q^6}{(1 - q^2 e^{-x})^5} + \frac{q^{-4} + q^{-2} + 2 + q^2 + q^4}{(1 - e^{-x})^5} + \frac{1 + q^{-2} + q^{-4} + q^{-6}}{(1 - q^{-2} e^{-x})^5} + \frac{q^{-6}}{(1 - q^{-4} e^{-x})^5} \right],
\]

and

\[
\lambda^{(4)}(k) = \frac{24 \left[ \frac{q^6}{(1 - q^4 e^{-x})^5} + \frac{1 + q^2 + q^4 + q^6}{(1 - q^2 e^{-x})^5} + \frac{q^{-4} + q^{-2} + 2 + q^2 + q^4}{(1 - e^{-x})^5} + \frac{1 + q^{-2} + q^{-4} + q^{-6}}{(1 - q^{-2} e^{-x})^5} + \frac{q^{-6}}{(1 - q^{-4} e^{-x})^5} \right]}{(1 - e^{-x})^3((1 - q e^{-x})^{-2} + (1 - q^{-1} e^{-x})^{-2})^4} - 1.
\]

It is worth noting that the symmetry under \( q \rightarrow q^{-1} \), though certainly valid, is not so obvious in the obtained general result (19), but, it is easily seen for each of the particular results (7), (8), (19) for \( r = 2, 3, 4 \) respectively.

As a kind of consistency check, we take the \( q \rightarrow 1 \) limit of general formula (16) and obtain

\[
\langle (a_\uparrow a_\downarrow)^r \rangle |_{q \rightarrow 1} = \frac{e^{-rx}}{2^r} \sum_{k=0}^{r} \binom{r}{k} \frac{1}{(1 - e^{-x})^r} = \frac{r! e^{-rx}}{(1 - e^{-x})^r}, \quad \lambda^{(r)}_{\text{as}} |_{q \rightarrow 1} = r! - 1,
\]

as it should be in the usual case of standard (ideal) Bose gas model.

Thus, the expressions (16)-(18) constitute our exact results for the \( r \)-particle momentum distributions and \( r \)-th order \( (r \geq 2) \) correlation function intercepts, along with their large momentum asymptotics, established in the symmetric TD type \( q \)-Bose gas model.

## 5 Concluding remarks

The results obtained in this letter (presented in Eqs. (6), (16), (17), (18)) represent the third particular case from among different deformed analogs of Bose gas model wherein the exact expressions for the \( r \)-particle distribution functions and for the respective \( r \)th order correlation functions intercepts have been derived. It is of interest to compare (the form of) these formulas with the two analogous previous results for the \( p, q \)-Bose gas model and for the \( \mu \)-deformed analog of Bose gas model. Whereas in ref. [12] the \( r \)th order correlation intercept \( \lambda^{(r)}_{\mu, p, q} + 1 \) appears as fully factorized one-term expression depending on \( e^x \equiv e^{\beta \hbar \omega} \) and \( p, q \), the analogous results for \( \lambda^{(r)}_{\mu} + 1 \) in [13] and \( \lambda^{(r)}_{q} + 1 \) in the present work are obtained as non-factorized expressions consisting of \( r + 1 \) terms, each of which formed from elementary functions of \( e^{\beta \hbar \omega} \) in this letter (see eq. (17) above), but, each one containing special function (Lerch transcendent) in ref. [13]. Anyway, with these particular three models of deformed Bose gas at hands, we have to mention in conclusion that now an interesting problem arises of describing the whole class of deformed Bose gas models which admit the obtaining of similar exact results.
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References

1. Baxter, R. J.: Exactly Solved Models in Statistical Mechanics, Dover Publications, Mineola, N.Y. (2007).
2. Lee, C.R., Yu, J.-P.: On $q$-analogues of the statistical distribution, Phys. Lett. A 150, 63-66 (1990).
3. Ge, M.-L., Su, G.: The statistical distribution function of the $q$-deformed harmonic oscillator, J. Phys. A: Math. Gen. 24, L721-L723 (1991).
4. Martin-Delgado, M. A.: Planck distribution for a $q$-boson gas, J. Phys. A: Math. Gen. 24, L1285-L1291 (1991).
5. Chaichian, M., Felipe, R.G., Montonen, C.: Statistics of $q$-oscillators, quons and relations to fractional statistics, J. Phys. A: Math. Gen. 26, 4017-4034 (1993).
6. Man’ko V. I. et al: Correlation functions of quantum $q$-oscillators, Phys. Lett. A 176, 173-175 (1993).
7. Algin, A., Ilik, E.: Low-temperature thermostatistics of Tamm-Dancoff deformed boson oscillators, Phys. Lett. A 377, 1797-1803 (2013).
8. Gavrilik, A.M., Mishchenko, Yu.A.: Deformed Bose gas models aimed at taking into account both compositeness of particles and their interaction, Ukr. J. Phys. 58, 1171-1177 (2013).
9. Rovenchak, A.: Complex-valued fractional statistics for $D$-dimensional harmonic oscillators, Phys. Lett. A 378, 100-108 (2014).
10. Dai, W.-S., Xie, M.: Calculating statistical distributions from operator relations: The statistical distributions of various intermediate statistics, Ann. Phys. 332, 166-179 (2013).
11. Algin, A., Senay, M.: High-temperature behavior of a deformed Fermi gas obeying interpolating statistics, Phys. Rev. E 85, 041123-(1-10) (2012).
12. Adamska, L.V., Gavrilik, A.M.: Multi-particle correlations in $q_p$-Bose gas model, J. Phys. A: Math. Gen. 37 4787-4796 (2004).
13. Gavrilik, A.M., Mishchenko, Yu.A.: Exact expressions for the intercepts of $r$-particle momentum correlation functions in $\mu$-Bose gas model, Phys. Lett. A 376, 2484-2489 (2012).
14. Arik, M. et al: Fibonacci oscillators, Z. Phys. C 55, 89-95 (1992).
15. Gavrilik, A.M., Kachurik, I.I., Rebesh, A.P.: Quasi-Fibonacci oscillators, J. Phys. A: Math. Theor. 43 245204 (1-16) (2010).

16. Chung, W.S., Gavrilik, A.M., Kachurik, I.I., Rebesh, A.P.: The symmetric Tamm-Dancoff \( q \)-oscillator: the representation, quasi-Fibonacci nature, accidental degeneracy and coherent states, J. Phys. A: Math. Theor. 47, 305304 (pp.1-14) (2014).

17. Odaka, K., Kishi, T., Kamefuchi, S.: On quantization of simple harmonic oscillators, J. Phys. A: Math. Gen. 24, L591-L596 (1991).

18. Chaturvedi, S., Srinivasan, V. and Jagannathan, R.: Tamm-Dancoff deformation of bosonic oscillator algebras, Mod. Phys. Lett. A 8, 3727-3734 (1993).

19. Gavrilik, A.M., Rebesh, A.P.: A \( q \)-oscillator with “accidental” degeneracy of energy levels, Mod. Phys. Lett. A 22, 949-960 (2007).

20. Biedenharn L.C.: The quantum group \( SU_q(2) \) and a \( q \)-analog of the boson operators, J. Phys. A: Math. Gen. 22, L873-L878 (1989).

21. Macfarlane A.J.: On \( q \)-analogues of the quantum harmonic oscillator and the quantum group \( SU_q(2) \), J. Phys. A: Math. Gen. 22, 4581-4585 (1989).

22. Chapman, S., Heinz, U.: HBT correlators - current formalism vs. Wigner function formulation, Phys. Lett. B 340, 250-253 (1994).

23. Kac, V., Cheung, P.: Quantum Calculus, Springer, Berlin (2002).