How to Construct the Lattice of Submodules of a Multiplicity free Module from Partial Information.

Ian M. Musson
Department of Mathematical Sciences
University of Wisconsin-Milwaukee
email: musson@uwm.edu

July 19, 2022

Abstract

In general it is a difficult problem to construct the lattice of submodules $L(M)$ of a given module $M$. In [Sta12] R. P. Stanley outlined a method for constructing a distributive lattice from a knowledge of its join irreducibles. However it is not an easy task to identify all join irreducible submodules of a given module. In the case of a multiplicity free module $M$ we present a modification of Stanley’s method based on the composition factors of $M$. As input we require a set of submodules $A_1, \ldots, A_n$ whose submodule lattices are known and which contain all composition factors of $M$. From this we can reconstruct $L(M)$. We illustrate the process for a family of Verma modules $M(\lambda_n)$, with $n$ a positive integer, for the Lie superalgebra $osp(3,2)$. We show that for $n \geq 2$, $L(M(\lambda_n))$ is isomorphic to the (extended) free distributive lattice of rank 3.

1 Introduction.

2 Lie Superalgebras.

3 A Family of Elusive Cases.

4 The Cases $n \geq 2$.

5 Application: Building the Lattice of Submodules of the Verma Modules $M(\lambda_n)$.
See [Mus12] Exercise 10.5.2 for the case of $\mathfrak{sl}(3)$. To use the method suggested in the abstract effectively, we need to find submodules $A_i$ having a simpler submodule structure than $M$. This does not help in Jantzen’s examples. The examples from Section 5 suggest that when the method can be successfully applied, the submodule structure of $M$ turns out to be much more complex. Hopefully the result mentioned in the abstract can be applied to other cases apart from those described there.

In her thesis America Masaros, [Mas13] studied Verma modules $M(\lambda)$ for the simple Lie superalgebra $g = \mathfrak{osp}(3,2)$ and determined their lattice of submodules in many cases. However for the Vermas $M(\lambda_n)$, the lattice structure remained elusive. Some partial information was obtained, including the determination of all composition factors. These modules are multiplicity free of length 8.

Since lattices and Lie superalgebras might be considered as distantly related areas, we give more detailed background on both subjects than we would otherwise. This is done in the rest of this Section and the next. After that we work towards finding a set of submodules $A_1, \ldots, A_n$ satisfying the conditions in the abstract. By combining results from lattice theory and [Mas13] we arrive at Corollary 3.11 which gives us two possibilities.

To make further progress, we require a new ingredient which may be of independent interest. Namely we use an important property of Sapovalov elements: if $\lambda \in \mathcal{H}_\gamma$, then $\theta^2_\gamma M(\lambda) = 0$ (the notation is defined before Lemma 3.1). This leads to a complex of Verma modules with highest weights in $\mathcal{H}_\gamma$, where every map is multiplication by $\theta_\gamma$. In the present context this complex is

\[ B^\bullet : M(\lambda_1) \rightarrow M(\lambda_2) \rightarrow M(\lambda_2) \ldots \]

from (4.1), and a key result Corollary 4.9 is that the complex is exact at $M(\lambda_n)$ for $n \geq 2$. This suggests a connection between exactness of the more general complex from [Mus17b] (5.1) and representation theory. Based on results from Sections 3 and 4 the main result for the $\mathfrak{osp}(3,2)$ Verma modules $M(\lambda_n)$ is proved in Section 5 by building the Hasse diagram in a series of steps.

### 1.1 Basic Notions of Lattice Theory.

Throughout $[n]$ denotes the set of the first $n$ positive integers. In this paper all lattices and posets are finite. In a poset $P$, if $B < A$ and $B \leq C \leq A$ implies that $C = A$ or $C = B$ we say that $A$ covers $B$. The Hasse diagram of $P$ is a graph whose vertices are the elements of $P$, with an edge drawn upwards from $B$ to $A$ whenever $A$ covers $B$. A lattice is a poset $(\Lambda, \leq)$ in which any two elements $A, B$ have a least upper bound (lub) $A \lor B$, and a greatest lower bound (glb) $A \land B$ called the join and meet of $A, B$ respectively. These are necessarily unique. Furthermore the conditions $B \leq A, B = A \land B$ and $A = A \lor B$ are equivalent. In this case we say that $[B, A]$ is an interval. The dual lattice to $\Lambda$ has as underlying set a copy $\overline{\Lambda}$ of $\Lambda$, with a bijection $X \rightarrow \overline{X}$ from $\Lambda$ to $\overline{\Lambda}$, and then the partial order, join
and meet in $\Lambda$ are defined by $X \leq Y$, iff $X \geq Y$, $X \lor Y = X \land Y$ and $X \land Y = X \lor Y$.

We recall some terminology from [Sta12]. We say that $A$ is \textit{join irreducible} if $A = X \lor Y$ implies that either $A = X$ or $A = Y$. The glb of all elements in the lattice $\Lambda$, denoted by 0, is clearly join irreducible. Similarly the lub of all elements in $\Lambda$ is denoted 1. An element of that covers 0 is called an \textit{atom}. Dually an element covered by 1 is called a \textit{coatom}. A lattice $\Lambda$ is \textit{graded} if any two maximal (saturated) chains in $\Lambda$ have the same length $n$, and in this case we say $\Lambda$ is \textit{graded of degree} $n$. For such a lattice, define a \textit{degree function} $\rho : \Lambda \rightarrow \{0, 1, \ldots, n\}$ such that $\rho(0) = 0$, and $\rho(A) = \rho(B) + 1$ if $A$ covers $B$. We say $\Lambda$ is \textit{upper semimodular} if $\Lambda$ is graded and

$$\rho(B) + \rho(A) \geq \rho(B \lor A) + \rho(B \land A),$$

for all $B, A \in \Lambda$. There is an equivalent condition in [Sta12] Proposition 3.3.2. If the dual lattice $\tilde{\Lambda}$ is also upper semimodular we say $\Lambda$ is \textit{modular}. (Thus if a lattice is modular, so is its dual.) In this case we have

$$\rho(B) + \rho(A) = \rho(A \lor B) + \rho(A \land B).$$

Alternatively

$$\rho(B) - \rho(A \land B) = \rho(A \lor B) - \rho(A).$$

The interval $[A, B]$ has length $\rho(B) - \rho(A)$. According to [Bir79] Corollary to Theorem 15, page 41, modularity is equivalent to the condition

$$B \lor (A \land C) = (B \lor A) \land C,$$

for all $A, B, C \in \Lambda$ with $B \leq C$. We say $\Lambda$ is \textit{distributive} if

$$B \lor (A \land C) = (B \lor A) \land (B \lor C),$$

for all $A, B, C \in \Lambda$. This is equivalent to the dual condition.

$$B \land (A \lor C) = (B \land A) \lor (B \land C),$$

for all $A, B, C \in \Lambda$, [DH78] Theorem 6.5.3. Any distributive lattice is modular, [DH78] Lemma 6.5.1. A sublattice of a lattice $\Lambda$ is a subset which is closed under the meet and join operations.

\textbf{Example 1.1.} Here are the Hasse diagrams for two important lattices. Each is minimal with respect to not satisfying certain desirable properties of lattices.

\begin{center}
\begin{tikzpicture}
\node at (0,0) (a) {$0$};
\node at (1,1) (b) {$b$};
\node at (-1,1) (c) {$c$};
\node at (0,-1) (d) {$a$};
\node at (2,2) (e) {$1$};
\node at (1,0) (f) {$a$};
\node at (2,0) (g) {$b$};
\node at (3,0) (h) {$c$};
\node at (1,-2) (i) {$a$};
\draw (a) -- (b);
\draw (a) -- (c);
\draw (b) -- (e);
\draw (c) -- (f);
\draw (f) -- (g);
\draw (g) -- (h);
\end{tikzpicture}
\end{center}

\[\text{In [Sta12] such a lattice is called \textit{graded of rank} } n. \text{ But the extended free distributive lattice of rank } n \text{ has degree } 2^n. \text{ So to avoid a clash of notation, we depart from the usual terminology here.}\]
Indeed a lattice is modular (resp. distributive) iff it contains no sublattice with Hasse diagram as in the first (resp. both) diagrams above, \([LP84]\) Theorem 1.24 (resp. Theorem 1.29). The second diagram above is the lattice of subgroups of the Klein four group \(V\). As a \(\mathbb{Z}\)-module \(V\) is semisimple of length two, and has a unique composition factor up to isomorphism.

### 1.2 Modular Lattices.

We sometimes refer to elements of a lattice \(\Lambda\) as submodules. First we recall some definitions from \([Mus17]\). Let \(\equiv\) be the smallest equivalence relation on the set of intervals such that

\[
[A \land B, B] \equiv [A, A \lor B]
\]  

(1.2)

We sometimes write the interval \([X, Y]\) as \(Y/X\). Thus (1.2) is motivated by the second isomorphism theorem: \(B/(A \cap B) \cong (A + B)/A\). The equivalence relation \(\equiv\) is not well behaved for non-modular lattices. Indeed from the first diagram in Example 1.1 we have since \(1 = a \lor b\) and \(0 = a \land b\), that \(1/a \equiv b/0\). Thus an interval of length 1 can be equivalent to an interval of length 2. However for modular lattices, equivalent intervals have the same length by (1.1). For a module \(M\) the lattice of submodules \(L(M)\) of \(M\) is modular by Dedekind’s modular law, see for example \([Mus12]\) Lemma 1.2.6. From now on we assume that all lattices are modular. We call an interval of length one \([X, Y]\) or \(Y/X\), a simple lattice factor of \(\Lambda\). The equivalence classes under \(\equiv\) restricted to the set of intervals of length one will be called lattice composition factors of \(\Lambda\). If \(J \neq 0\) is join irreducible it has a unique maximal submodule, denoted by \(J^0\). Define \(L_J = J/J^0\).

**Lemma 1.2.** Let \(\Lambda\) be a modular lattice. If \(M/N\) is a simple lattice factor of \(\Lambda\), then \(M/N \equiv J/J^0\) where \(J \neq 0\) is join irreducible in \(\Lambda\) and \(M \geq J\).

**Proof.** If the result is false choose a counterexample with \(M\) minimal under the partial order \(\geq\). Then \(M\) is not join irreducible, so \(N\) is not the only maximal submodule of \(M\). Let \(P\) be another maximal submodule. Then \(M = N \lor P\), so \(M/N \equiv P/(P \land N)\) with \(M > P\). By the minimality of \(M\), \(P/(P \land N) \equiv J/J^0\) with \(J\) join irreducible. This contradiction yields the result. \(\square\)

Consider a maximal chain in the lattice \(\Lambda\):

\[
0 = X_0 < X_1 < \ldots < X_n = 1.
\]  

(1.3)

We say the lattice composition factor \(a\) has multiplicity \(k\) in (1.3) if exactly \(k\) of the intervals \([X_{i-1}, X_i]\), \(1 \leq i \leq n\) belong to the equivalence class \(a\). There is Jordan-Hölder Theorem for modular lattices.

**Theorem 1.3.** In a modular lattice \(\Lambda\), the multiplicities of a lattice composition factor in any two maximal chains are equal.
Proof. If the result is false, let \( \Lambda \) be a counterexample of minimal degree \( n \). Since \( \Lambda \) is modular, all maximal chains in \( \Lambda \) have length \( n \). If \( \Lambda \) had a unique atom \( A \), then the result holds for the interval \([A, 1]\), and any maximal chain would start with \( 0 < A \). Thus \( \Lambda \) would not be a counterexample. So suppose \( A, B \) are distinct atoms. Then \( A \land B = 0 \), so by modularity \( A \lor B \) covers both \( A \) and \( B \). Let \( A \lor B = Y_2 < \ldots < Y_n = 1 \) be a maximal chain in the interval \([A \lor B, 1]\). Since the interval \([A, 1]\) is not a counterexample, any maximal chain in \([A, 1]\) has the same multiplicities as the chain

\[
A < A \lor B = Y_2 < \ldots < Y_n = 1.
\]

Thus any maximal chain in \( \Lambda \) that begins with \( 0 < A \) has the same multiplicities as the chain

\[
0 < A < A \lor B = Y_2 < \ldots < Y_n = 1. \tag{1.4}
\]

But then reversing the roles of \( A \) and \( B \) and using the definition of equivalence, shows that any maximal chain that begins with \( 0 < B \) has the same multiplicities as the chain in (1.4). Since any maximal chain must start with an atom, and \( A, B \) are arbitrary, \( \Lambda \) is not in fact a counterexample.

We say that \( \Lambda \) is \textit{multiplicity-free} if the multiplicity of any lattice composition factor in a maximal chain equals 1. Similarly a module \( M \) is \textit{multiplicity-free} if the multiplicity of any composition factor equals 1.

1.3 Distributive Lattices.

For a module \( M \) the multiplicity-free condition on \( M \) is closely related to distributivity of \( L(M) \). It is well known that \( L(M) \) is distributive iff every semisimple subfactor of \( M \) is multiplicity-free. It is hard to track down the first citation, but a short proof is given in [Mus17] Proposition 2.1. According to Stephenson [Ste74] Proposition 1.1, \( L(M) \) is distributive iff for every subfactor of \( M \) which is a direct sum \( A \oplus B \) of two non-zero submodules, we have \( \text{Hom}(A, B) = 0 \). It is not hard to deduce the result from this.

We say a lattice is \textit{restricted} if it has at least 2 atoms and 2 coatoms. A restricted lattice \( \Lambda \) has \textit{rank} if \( n \) is minimal such that \( \Lambda \) can be generated by \( n \) join irreducibles \( J_1, \ldots, J_n \) using the meet and join operations. For \( \emptyset \neq I \subseteq [n] = \{1, \ldots, n\} \), set\(^2\)

\[
\bigvee_{I} J = \bigvee_{i \in I} J_i \text{ and } \bigwedge_{I} J = \bigwedge_{i \in I} J_i. \tag{1.5}
\]

Lemma 1.4. Each \( \bigwedge_{I} J \) is join irreducible.

\(^2\)If \( X, Y \) are disjoint subsets (of some multiplicative abelian group), and \( \prod_X \) denotes the product of elements in \( X \), then \( \prod_{X \cup Y} = \prod_X \prod_Y \). Taking \( X \) to be empty tells us that the empty product equals 1, and similarly the empty sum is zero. But the same argument applied to \( \bigvee_I J \) and \( \bigwedge_I J \) tells us only that \( \bigwedge_I J \leq \bigwedge_{i \in I} J_i \) and \( \bigvee_I J \leq \bigvee_{\emptyset} J \) for all \( I \subseteq [n] \). For extended \( \Lambda_n \) (defined in Subsection 1.4) this gives two possibilities for \( \bigwedge_{\emptyset} J \) and \( \bigvee_{\emptyset} J \). For this reason we prefer not to think about empty meets and joins.
Proof. By induction on \( n \). For \( n > 1 \), the previous step takes care of all cases where \(|I| \leq n - 1\). In the remaining case \( \bigwedge_{[n]} J \) is the unique minimal element of the lattice.

**Example 1.5.** The power set \( \mathbb{B}_n \) on \([n]\) is a distributive lattice with union as join, and intersection as meet. We call \( \mathbb{B}_n \) the Boolean lattice (or poset) of rank \( n \). We also call \( \mathbb{B}_3 \) the box lattice \( \mathbb{B} \) because of its Hasse diagram, \[\text{Lau84}\] Figure 5.2. The lattice \( \mathbb{B}_n \) is generated by the join irreducibles \( \{1\}, \{2\}, \ldots, \{n\} \) and has \( n \) composition factors up to isomorphism. For example \( \mathbb{B} \) has 3 composition factors up to isomorphism, because parallel edges of the box are equivalent.

If \( P \) is a poset, a subset \( I \) of \( P \) is called a down-set (or order ideal) if \( x \in I, y \in P \) and \( y \leq x \) implies \( y \in I \). If \( x \in P \), the down-set \( \Lambda_x = \{ y \in P | y \leq x \} \) is known as a principal down-set. Denote the set of down-sets of \( P \) by \( J(P) \). Then \( J(P) \) is a distributive lattice taking union and intersection as \( \lor, \land \) respectively. Every finite distributive lattice has this form. This is the content of the next result known as the fundamental theorem on finite distributive lattices.

**Theorem 1.6.** Let \( L \) be a finite distributive lattice. Then there is a unique, up to isomorphism, finite poset \( P \) such that \( L \cong J(P) \)

**Proof.** Given a lattice \( L \) as above, let \( P \) be the set of join irreducible elements of \( L \). This is a poset with the order induced from that on \( L \). It is shown in \[\text{Sta12}\] Theorem 3.4.1 that \( L \cong J(P) \) as lattices. See also \[\text{Bir79}\] Theorem 3, page 59.

When \( P = \mathbb{B}_n \), \( J(P) \) is isomorphic to extended \( \Lambda_n \) which has \( 2^n \) composition factors up to isomorphism. This could be an argument in favor of defining the free distributive lattice of rank \( n \) to be extended \( \Lambda_n \).

A method for drawing the Hasse diagram of \( J(P) \) given \( P \), is given in \[\text{Sta12}\] pages 108-109, followed by an example where \( P \) is a zig-zag poset. We use this method in Section 5 to determine the lattice of submodules of our Verma modules. We briefly recall the details of Stanley’s method. The idea is that \( J(P) \) can be constructed using \( P \) by gluing together Boolean lattices. Begin with the set \( I \) of minimal elements of \( P \). If \( |I| = m \) form the lattice \( J(I) \cong \mathbb{B}_m \). If \( I \neq P \) choose a minimal element \( x \) of \( P \setminus I \). Adjoin a join irreducible to \( J(I) \) covering the down set \( \Lambda_x \setminus \{x\} \). The set of joins of elements covering \( \Lambda_x \setminus \{x\} \) forms a Boolean algebra, draw in any new joins required to show this. (An example occurs passing from Fig. 3-13 to Fig. 3-14 in \[\text{Sta12}\].) There still may be elements that do not yet have joins. Add in these joins and repeat until we obtain \( J(P) \). (An example occurs passing from Fig. 3-19 to Fig. 3-21 in \[\text{Sta12}\].)

1.4 The Free Distributive Lattice of Rank \( n \).

A proposition (or Boolean variable) is a variable that can take on the values T or F. Propositions \( P_1, \ldots, P_n \) are independent if the values of the \( P_i \) can be assigned independently of each other. For propositions \( X, Y \), define \( X \land Y \), (resp. \( X \lor Y \)) by
requiring that \( X \land Y = T \) iff \( X = T \) and \( Y = T \) (resp. \( X \lor Y = T \) iff \( X = T \) or \( Y = T \)). Then define \( X \leq Y \) to mean \( X \land Y = X \). Thus \( X \leq Y \) means that \( X \) implies \( Y \).

The free distributive lattice \( \Lambda_n \) of rank \( n \) is the lattice of propositions generated from \( P_1, \ldots, P_n \) by using the join \( \lor \) and meet \( \land \) operations. We define \( \bigvee I P \) and \( \bigwedge I P \), by analogy with (1.5). There is no universal agreement on the definition of \( \Lambda_n \). Our definition is equivalent to that given in [DH78] Section 6.8, where a labelled diagram of the lattice \( \Lambda_3 \) can be found, see also Section [5] and [Bir79] Figure 8, page 33. We obtain extended \( \Lambda_n \) from \( \Lambda_n \) by adjoining additional elements \( \hat{0} \) and \( \hat{1} \), such that \( \hat{0} < X \) and \( X < \hat{1} \) for all \( X \in \Lambda_n \). Often the free distributive lattice of rank \( n \) is defined as extended \( \Lambda_n \). For additional clarity we sometimes refer to \( \Lambda_n \) as restricted \( \Lambda_n \).

Remark 1.7. The lattice \( \Lambda_n \) is self dual. Indeed, if \( P_1, \ldots, P_n \) are independent propositions, then so too are their negations. So the statement follows from DeMorgan’s Laws.

The Wikipedia article on Dedekind numbers gives the Hasse diagram of extended \( \Lambda_n \). The diagram is labelled using independent Boolean variables \( A, B, C \). The number of elements of extended \( \Lambda_n \) is called the \( n \)th Dedekind number \( M(n) \), and the values of \( M(n) \) are only known for \( n \leq 8 \). For example \( M(3) = 20 \) and \( M(8) \) is approximately \( 5.61 \times 10^{22} \). We could define a function \( f \) defined on \( \mathbb{N} \) to have exponential growth if asymptotically \( \log_2 f(n) \) grows like a linear polynomial in \( n \). According to Wikipedia,

\[
\left( \frac{n}{\lfloor n/2 \rfloor} \right) \leq \log_2 M(n) \leq \left( \frac{n}{\lfloor n/2 \rfloor} \right) (1 + O \left( \frac{\log n}{n} \right)).
\]

Now \( \left( \frac{n}{\lfloor n/2 \rfloor} \right) \) is a polynomial in \( n \) of degree \( \lfloor n/2 \rfloor \). As \( n \) increases, this means that \( \log_2 M(n) \) eventually grows faster than any polynomial in \( n \). Thus the growth of \( M(n) \) is much faster than exponential.

In the Wikipedia article several equivalent definitions are given for extended \( \Lambda_n \), one of which involves involves Boolean variables. In the diagram for \( \Lambda_3 \), \( 0 \) an \( 1 \) are labelled as “contradiction” and “tautology” respectively. However we don’t regard these as genuine propositions, since they can only take on a single value. Also \( P_1, \ldots, P_n \) already have a lub and glb of \( \bigvee \{P_i\} P \) and \( \bigwedge \{P_i\} P \) respectively.

We briefly explain the theory of Disjunctive Normal Form (DNF), following [Lau84]. Begin with independent Boolean variables \( P_1, \ldots, P_n \) as above. A literal is either one of the \( P_i \) or its negation. A Boolean function is a proposition that can be formed from the literals using meets and joins. A miniterm is a conjunction, that is a meet of literals. Any Boolean function has an essentially unique expression (called the DNF) as an irredundant disjunction or join of miniterms [Lau84] Proposition 5.9 and Theorem 5.10. More precisely, the uniqueness is up to changing the order of

\[3\text{https://en.wikipedia.org/wiki/Dedekind_number}\]
the literals in miniterms, and changing the order of the miniterms in the disjunction. DNF is useful in efficient circuit design. If we use and gates and or gates that admit multiple inputs, then we need only a single or gate, and an and gate for each miniterm, as well as a number or not gates. DNF applies to the lattice $\Lambda_n$ provided we redefine a literal to be one of the $P_i$. Negation does not preserve $\Lambda_n$. Thus for a Boolean function in $\Lambda_n$, no not gates are needed.

**Lemma 1.8.** (a) $\bigwedge_I P = \bigwedge_J P$ iff $I = J$.

(b) Any join irreducible element of $\Lambda_n$ is equal to $\bigwedge_I P$ for some $I$.

*Proof.* Both parts follow from the uniqueness of disjunctive normal form. An easier proof of (a) is as follows. Suppose $i \in I$, but $i \notin J$. Assign values $P_i := F$, and $P_j := T$ for all $j \in J$. Then $\bigwedge_I P = F$ and $\bigwedge_J P = T$.

**Remark 1.9.** The free modular lattice $M_n$ of rank $n$ was first considered by Dedekind [Ded00], who showed that $M_3$ has 28 elements. A proof can also be found in [Bir79] Section III.6, where it is also shown that $M_4$ is infinite. In [Fre80] it is shown that $M_5$ has unsolvable word problem.

**Theorem 1.10.** Suppose $\Lambda$ is a restricted distributive lattice of rank $n$ generated by join irreducibles $J_1, \ldots, J_n$. Then there is a surjective map of lattices restricted $\Lambda_n \rightarrow \Lambda$ sending $P_i$ to $J_i$.

*Proof.* See [Bir79] Ch III.4.

**Remark 1.11.** The analog of Lemma 1.8 (a) fails for the box lattice $\mathbb{B}$ because $\{1\} \cap \{2\} = \{1\} \cap \{3\} = \{2\} \cap \{3\}$. It is an interesting exercise to construct a surjective map of lattices from $\Lambda_3$ to $\mathbb{B}$, and check that it is well defined.

**Example 1.12.** Below is the lattice of subgroups of the cyclic group $\mathbb{Z}/(12)$ of order 12. Note that $\mathbb{Z}/(12)$ is not multiplicity free but $L(\mathbb{Z}/(12))$ is distributive. The smallest module with this property is the cyclic group of order 4, but $L(\mathbb{Z}/(12))$ will reappear in Lemma 1.15.

The module $\mathbb{Z}/(12)$ contains isomorphic simple subfactors which do not isomorphic correspond to simple lattice factors. Their isomorphism is not a consequence of the second isomorphism theorem. We call such isomorphisms *accidental*.
**Lemma 1.13.** If the module $M$ is multiplicity free then $L(M)$ has no accidental isomorphisms. In other words, if two simple subquotients of $M$ are isomorphic as modules, then the corresponding intervals are equivalent in $L(M)$.

**Proof.** If the result is false, then by Lemma 1.12 $M$ contains distinct join irreducible submodules $J, K$ such that $J/J^0 \cong K/K^0$. If $K \subseteq J$, then $M$ contains the chain of submodules

$$K^0 \subset K \subseteq J^0 \subset J,$$

with two isomorphic composition factors, a contradiction. Similarly $K$ cannot contain $J$. Thus $J \cap K$ is a proper submodule of both $J$ and $K$. Since $J^0, K^0$ are the unique maximal submodules of $J, K$ it follows that $J \cap K \subseteq J^0 \cap K^0$. Thus $J + K/J \cap K$ is a subfactor of $M$ that is also a counterexample. Starting again with $M = J + K$ and $J \cap K = 0$ we see that $M = J \oplus K$. But then $M$ is not multiplicity free.

**Corollary 1.14.** If the module $M$ is multiplicity free, then the map $J \rightarrow L_J$ induces a bijection between join irreducibles $J$ (with $J \neq 0$) in $L(M)$ and isomorphism classes of composition factors of $M$.

**Proof.** Combine Lemma 1.12 and Lemma 1.13.

We record some well known examples of lattices from [Sta12] Example 3.1.1. The lattice $[n]$ is the set $[n]$ with its usual order. This lattice is uniserial, that is any two elements are comparable. If $n$ is a positive integer, then the lattice of subgroups of $\mathbb{Z}/(n)$ is denoted $D_n$. The following result is well known.

**Lemma 1.15.** The list of distributive lattices of length 3 is given as follows. If the lattice does not have a name, we give its Hasse diagram.

![Hasse diagram of distributive lattices of length 3](image)

**Proof.** We work throughout up to lattice isomorphism. Let $\Lambda$ be a lattice as in the statement of the Lemma, and denote the lub of its elements by 1. If 1 is join irreducible, it is easy to see we have Case 1 or 4. Now suppose $\Lambda$ has two (distinct) coatoms $b, d$. Then $1 = b \lor d$. Clearly $b \land d$ covers 0. If both $b, d$ are join irreducible,
we have Case 5. If exactly one is join irreducible, we have Case 2. If neither is join
irreducible, we obtain the lattice with Hasse diagram

```
1
  d
 /  \\  
 a   b ∧ d
  \\
\  
0
```

But then the sublattice consisting of the elements 0, 1, a, b, c is the 5 element non-
modular lattice from Example \[.1\] So \(\Lambda\) is not distributive. If \(\Lambda\) has \(n \geq 3\) coatoms,
then the dual lattice would contain \(n\) atoms, and hence contain the self-dual sub-
lattice \(B_n\) which has length \(n\). Thus the only possibility is \(n = 3\) and we have Case
3.

Remarks 1.16. The lattices listed in the Lemma have the form \(J(P)\) for the five
posets \(P\) of size three listed in \[Sta12\] Figure 3-1, but it will be useful to know their
Hasse diagrams. Clearly there is a bijection between the lattices \(\Lambda\) in the Lemma
and distributive lattices of length 4 with a unique atom. If \(\Lambda\) is as in the Lemma,
we call the corresponding length 4 lattice augmented \(\Lambda\).

Remark 1.17. We outline how to prove the statement in the abstract, that we can
reconstruct \(L(M)\) from a set of submodules \(A_1, \ldots, A_n\) which contain all composition
factors of \(M\), and such that each \(L(A_i)\) is known. Using Lemma \[.12\] we can assume
the \(A_i\) are join irreducible. This may entail changing the original set of submodules,
see Subsection 5.1. From this information, by Corollary \[.14\] we know the structure
of the poset of join irreducibles, and then we can use Stanley’s method.

2 Lie Superalgebras.

2.1 Introduction.

We use \[Hum72\], \[Kac83\] and \[Kac77\], \[Mus12\] as general references for Lie algebras
and Lie superalgebras respectively. A Lie superalgebra is a \(\mathbb{Z}_2\)-graded vector space
\(g = g_0 \oplus g_1\) together with a bilinear map \([\cdot, \cdot] : g \times g \to g\) satisfying a \(\mathbb{Z}_2\)-graded
version of the Jacobi identity and skew-symmetry, \[Mus12\] Section 1.1. The \(\mathbb{Z}_2\)-grading
on \(g\) means that \([g_i, g_j] \subseteq g_{i+j}\) for \(i, j \in \mathbb{Z}_2\). This means that \(g_0\) is a Lie algebra and
\(g_1\) is a \(g_0\)-module via \([\cdot, \cdot]\). Finite dimensional simple Lie superalgebras over an algebra-
ically closed field \(k\) of characteristic zero were classified by Kac. Several special
cases of the classification were obtained by other authors, see \[Kac77\] pages 47-48
for details. Such an algebra \(g\) is called classical if \(g_0\) is reductive. The remainder
are called of Cartan type. From now on we consider only finite dimensional classical
simple Lie superalgebras \(g\) over \(k\). The classification of such Lie superalgebras can

\[^4\] A Lie algebra \(\mathfrak{t}\) is reductive if \([\mathfrak{t}, \mathfrak{t}]\) is semisimple, that is a direct sum of simple Lie algebras. If
\(\mathfrak{t}\) is reductive then \(\mathfrak{t} = \mathfrak{z}(\mathfrak{t}) \oplus [\mathfrak{t}, \mathfrak{t}]\), where \(\mathfrak{z}(\mathfrak{t}) = \{x \in \mathfrak{t} | [x, y] = 0 \text{ for all } y \in \mathfrak{t}\}\) is the center of \(\mathfrak{t}\).
be found in [Mus12] Theorem 1.3.1.

For simplicity we assume that $\mathfrak{g} \neq \mathfrak{p}(n), \mathfrak{q}(n)$, two infinite families of algebras in the Theorem just cited. The remaining Lie superalgebras are called basic classical simple. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$. This is an abelian subalgebra (meaning that $[\mathfrak{h}, \mathfrak{h}] = 0$) of $\mathfrak{g}_0$ such that (2.2) below holds. For $\alpha \in \mathfrak{h}^*$, set

$$g^\alpha = \{ x \in \mathfrak{g} | [h, x] = \alpha(h)x \}. \quad (2.1)$$

We say that $\alpha$ is a root of $\mathfrak{g}$ if $\alpha \neq 0$ and $g^\alpha \neq 0$. Denote by the set of roots of $\mathfrak{g}$ by $\Delta$. The adjoint action $(\mathfrak{h}, \mathfrak{g}) \rightarrow \mathfrak{g}$ of $\mathfrak{h}$ on $\mathfrak{g}$, defined by $(h, x) \rightarrow [h, x]$, is diagonalizable and clearly each $g^\alpha$ is an eigenspace for this action. We have a root space decomposition, [Hum72] Chapter 8, [Mus12] 2.1, 8.1,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} g^\alpha. \quad (2.2)$$

Moreover if $\mathfrak{g} \neq \mathfrak{psl}(2,2)$ (which we will assume for simplicity), then each root space in (2.2) has dimension 1 over $k$ and we choose a basis element $e_\alpha$ for $g^\alpha$. A root $\alpha$ is called even (resp. odd) if $g^\alpha \subset \mathfrak{g}_0$ (resp. $g^\alpha \subset \mathfrak{g}_1$). Denote the set of even (resp. odd) roots by $\Delta_0$ (resp. $\Delta_1$). We have a disjoint union $\Delta = \Delta_0 \cup \Delta_1$. This fails for $\mathfrak{q}(n)$. There is a similar root space decomposition for $\mathfrak{g}_0$,

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0} g^\alpha. \quad (2.3)$$

In addition there is a disjoint union $\Delta = \Delta_+ \cup -\Delta_+$, with the property that $\alpha, \beta \in \Delta_+, \alpha + \beta \in \Delta$ implies $\alpha + \beta \in \Delta_+$. This fails for $\mathfrak{p}(n)$. For $i = 0, 1$, set $\Delta_0^\pm = \Delta_i \cap \Delta_\pm$. Define $n_\pm = \bigoplus_{\alpha \in \Delta_\pm} g^\alpha$. From the defining properties of $\Delta_\pm$ and (2.2), it follows that $n_\pm$ are subalgebras of $\mathfrak{g}$ and we have a triangular decomposition of $\mathfrak{g}$.

$$\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+. \quad (2.4)$$

There is a similar triangular decomposition of $\mathfrak{g}_0$ with $n_0^\pm = \bigoplus_{\alpha \in \Delta_0^\pm} g^\alpha$,

$$\mathfrak{g}_0 = n_0^- \oplus \mathfrak{h} \oplus n_0^+. \quad (2.5)$$

A root in $\Delta_+$ is simple if it cannot be written as a sum of two other roots in $\Delta_+$. Denote the set of simple roots by $\Pi$. In [Hum72] Chapter 10, $\Delta_+$ and $\Pi$ are denoted by $\Phi^+$ and $\Delta$ respectively. For the Lie superalgebra versions, see [Mus12] 3.4 and 8.1. We use the Borel subalgebras $\mathfrak{b} = \mathfrak{h} \oplus n^+$ and $\mathfrak{b}^- = n^- \oplus \mathfrak{h}$. Equation (2.4) leads to some other useful decompositions

$$\mathfrak{g} = n^- \oplus \mathfrak{b}, \quad \mathfrak{b} = \mathfrak{h} \oplus n^+. \quad (2.6)$$

Since $n^\pm$ is defined as a sum of root spaces, it follows that $[\mathfrak{h}, n^\pm] \subseteq n^\pm$ (in fact equality holds). Then from the second equation in (2.6), we have

$$[\mathfrak{b}, n^+] \subseteq n^+. \quad (2.7)$$
This says that \( n^+ \) is an ideal in \( \mathfrak{b} \). Thus there is a natural Lie superalgebra structure on \( \mathfrak{b}/n^+ \), and we have \( \mathfrak{b}/n^+ \cong \mathfrak{h} \) as algebras. Define

\[
\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha, \quad \rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta^-} \alpha, \quad \rho = \rho_0 - \rho_1. \tag{2.8}
\]

There is a uniform construction (due to Kac) of basic classical simple Lie superalgebras using contragredient Lie superalgebras, often called Kac-Moody (KM) Lie superalgebras, beginning with a generalized Cartan matrix \( A \), [Mus12] Chapter 5. The KM Lie superalgebras of finite type are exactly the finite dimensional simple Lie superalgebras, [Kac83], and the simple KM Lie superalgebras of finite type are exactly the finite dimensional basic classical simple Lie superalgebras. Replacing \( A \) by a \( DA \) for a non-singular diagonal matrix \( D \) results in an isomorphic algebra. In the finite type case this can always be done with \( DA \) symmetric, and then the symmetrized Cartan matrix can be used to define an even non-degenerate invariant form on the simple algebras, [Kac83] Theorem 2.2, [Mus12] Section 5.4 whose restriction \( < , > \) to the Cartan subalgebra \( \mathfrak{h} \) is also non-degenerate. If \( A \) is non-singular, which in finite type is always true in the Lie algebra case, as it is for all basic classical simple superalgebras different from \( \mathfrak{psl}(n,n) \) for \( n \geq 2 \), then the simple roots form a basis for \( \mathfrak{h} \). For simplicity we assume \( \mathfrak{g} \neq \mathfrak{psl}(n,n) \). We can use the non-degeneracy of the bilinear form to define a linear isomorphism \( \mathfrak{h} \rightarrow \mathfrak{h}^* \). The inverse isomorphism \( \alpha \mapsto h_\alpha \) is defined by \( < h_\alpha , h > = \alpha(h) \) for all \( h \in \mathfrak{h} \). Then we can define a bilinear form \( ( , ) \) on \( \mathfrak{h}^* \) by

\[
(\alpha, \beta) = < h_\alpha , h_\beta > \quad (= \alpha(h_\beta) = h_\beta(\alpha)). \tag{2.9}
\]

We say that a root \( \alpha \) is isotropic if \( < \alpha, \alpha > = 0 \). Such roots do not exist for simple Lie algebras and any isotropic root is odd. If \( \alpha \) is a non-isotropic, set \( \alpha^\vee = 2\alpha/(\alpha,\alpha) \).

The reflection \( s_\alpha : \mathfrak{h}^* \rightarrow \mathfrak{h}^* \) is defined by \( s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha \). Note that \( s_\alpha \) sends \( \alpha \) to \( -\alpha \) and fixes the hyperplane, \( \mathcal{H} = \{ \lambda | (\lambda, \alpha) = 0 \} \). Thus geometrically \( s_\alpha \) is the reflection in the hyperplane orthogonal to \( \alpha \). The subgroup, \( W \) of \( \text{End}(\mathfrak{h}^*) \) generated by all reflections is called the Weyl group. The dot action of the Weyl group on \( \mathfrak{h}^* \) is defined by \( (w, \lambda) \rightarrow w \cdot \lambda = w(\lambda + \rho) - \rho \). This can be thought of as a version of the usual \( W \)-action, where the origin is shifted to \( -\rho \). All reflecting hyperplanes for the dot action pass through \( -\rho \).

### 2.2 Enveloping Algebras.

The enveloping algebra of a Lie superalgebra \( \mathfrak{k} \) is a \( \mathbb{Z}_2 \)-graded associative \( \mathfrak{k} \)-algebra \( U(\mathfrak{k}) \) that has the same representation theory as \( \mathfrak{k} \), (a \( U(\mathfrak{k}) \)-module is the same thing as a \( \mathfrak{k} \)-module) see [Mus12] Chapter 6. This allows methods from ring theory to be use to study modules for \( \mathfrak{k} \). The Poincare-Birkhoff-Witt theorem (PBW) gives a vector space basis for \( U(\mathfrak{k}) \). From PBW it follows that \( \mathfrak{k} \) embeds as a subspace of \( U(\mathfrak{k}) \), and the image of \( \mathfrak{k} \) generates \( U(\mathfrak{k}) \) as an algebra. As another consequence of PBW and (2.6), we have as vector spaces

\[
U(\mathfrak{b}) = U(\mathfrak{h}) \otimes U(n^+), \quad \text{and} \quad U(\mathfrak{g}) = U(n^-) \otimes U(\mathfrak{b}), \tag{2.10}
\]
Lemma 6.1.4. Equation (2.10) has some important consequences. First there is an \( \kappa \)-algebra map \( \varepsilon : U(n^+) \longrightarrow \kappa \) sending each \( x \in n^+ \) to zero, and \( \ker(\varepsilon) = U(n^+)n^+ = n^+U(n^+) \). From (2.7) we see that \( U(b)n^+ = n^+U(b) \) is a two sided ideal of \( U(b) \). Tensoring the exact sequence

\[
0 \longrightarrow U(n^+)n^+ \longrightarrow U(n^+) \overset{\varepsilon}{\longrightarrow} \kappa \longrightarrow 0
\]

with \( U(h) \) and using (2.10), we see that \( U(b)/U(b)n^+ \cong U(h) \) as \( \kappa \)-algebras. Thus any \( U(h) \)-module can be regarded as a \( U(b) \)-module which is annihilated by \( U(b)n^+ \). The Verma module \( M(\lambda) \) with highest weight \( \lambda \in \mathfrak{h}^* \), and highest weight vector \( v_\lambda \) is defined as follows. Start with a one dimensional \( U(h) \)-module \( kv_\lambda \) with weight \( \lambda \). By the above remarks, \( kv_\lambda \) can be regarded as a \( U(b) \)-module with \( n^+v_\lambda = 0 \). Then define \( M(\lambda) = U(\mathfrak{g}) \otimes_{U(b)} kv_\lambda \) as an induced module. There is a unique simple factor of \( M(\lambda) \), which we denote by \( L(\lambda) \) \[\text{Mus12}\] 8.2. From (2.10), we have

\[
M(\lambda) = U(n^-) \otimes U(b) \otimes kv_\lambda = U(n^-) \otimes kv_\lambda,
\]

so \( M(\lambda) \) is a free \( U(n^-) \)-module of rank 1, generated by \( v_\lambda \). In \[\text{Hum72}\] Chapters 20 and 21, \( M(\lambda) \) and \( L(\lambda) \) are denoted by \( Z(\lambda) \) (called a standard cyclic module) and \( V(\lambda) \) respectively.

### 2.3 Uniqueness up to Conjugacy of some Constructions.

Let \( \mathfrak{g} \) be a classical simple Lie superalgebra with \( \mathfrak{g} \neq \mathfrak{p}(n), \mathfrak{q}(n) \) and set \( \mathfrak{t} = \mathfrak{g}_0 \). From the definitions we see that any one of \( n^+, \Delta^+ \) or \( \Pi \) determines the other two. A similar statement holds for \( \mathfrak{t} \). By \[\text{Hum72}\] Theorem 16.4 and Corollary 16.4 any two Borel or Cartan subalgebras of \( \mathfrak{t} \) are conjugate under the group \( \mathcal{E}(\mathfrak{t}) \) of inner automorphisms. Also by \[\text{Hum72}\] Theorem 10.3, any two sets of simple roots of \( \mathfrak{t} \) are conjugate under \( \mathcal{W} \). Hence the definitions of \( \mathfrak{h}, \mathfrak{b}, n^+, \Delta^+, \Pi, \rho, \) the dot action and Verma modules could be regarded as canonical in the reductive case.

For \( \mathfrak{g} \) the situation is more complicated, though well understood. Any element of \( \mathcal{E}(\mathfrak{t}) \) extends uniquely to an automorphism of \( \mathfrak{g} \). Thus concerning Borel subalgebras, we can fix \( \mathfrak{h} \) and a Borel \( \mathfrak{b} \) in \( \mathfrak{t} \), and then any Borel in \( \mathfrak{g} \) is conjugate to another Borel \( \mathfrak{b} \) with \( \mathfrak{b}_0 = \mathfrak{b} \). Now by the proof of [Mus12] Theorem 3.1.2, there are only finitely many Borels \( \mathfrak{b} \) of \( \mathfrak{g} \) with \( \mathfrak{b}_0 = \mathfrak{b} \). For \( \mathfrak{osp}(3, 2) \) there are exactly two such. This means that in the next Subsection, we will need two versions of \( \Pi, \Delta^+, \mathfrak{b}, \rho, M(\lambda) \) and the dot action. The relation between the two versions of \( \Delta^+ \) is easily explained: we can obtain one from the other by changing the sign of an isotropic simple root. Apart from this the two versions have the same positive roots. This operation or its extension to \( \mathfrak{b} \) is sometimes called an odd reflection. More generally, by [Mus12] Theorem 3.1.3, any two Borels in \( \mathfrak{g} \) with the same even part are related by a sequence of odd reflections.

---

5\( \varepsilon \) is the counit in the bialgebra structure of \( U(n^+) \) and \( \ker(\varepsilon) \) is called the augmentation ideal of \( U(n^+) \).
2.4 The Lie Superalgebra $\mathfrak{osp}(3,2)$.

Now suppose $\mathfrak{g}$ is the classical simple Lie superalgebra $\mathfrak{osp}(3,2)$. In this case $\mathfrak{g}_0 \cong \mathfrak{sl}(2) \times \mathfrak{sl}(2)$, and as positive roots for $\mathfrak{g}_0$ we take $\Delta^+_0 = \{\epsilon, 2\delta\}$. Then there are 2 choices of roots $\Delta^+_1$ of $\mathfrak{g}_1$, such that $\Delta^+ = \Delta^+_0 \cup \Delta^+_1$ is a set of positive roots for $\mathfrak{g}$. These sets and the corresponding set of simple roots $\Pi$ for $\mathfrak{g}$ are given by

$$\Pi^d = \{\epsilon, \delta - \epsilon\}, \quad \Delta^+_{1,d} = \{\delta, \delta \pm \epsilon\},$$

$$\Pi^a = \{\delta, \epsilon - \delta\}, \quad \Delta^+_{1,a} = \{\delta, \epsilon \pm \delta\}.$$

Each of these corresponds to a Borel subalgebras $\mathfrak{b}^b$ and a version $\rho^b$ of $\rho$ (for $b = a, d$) as defined in (2.8). The latter are, by computation

$$\rho^d = (\epsilon - \delta)/2, \quad \rho^a = (\delta - \epsilon)/2.$$

Note that $\delta - \epsilon$ is a root of $\mathfrak{b}^d$, but instead $\epsilon - \delta$ is a root of $\mathfrak{b}^a$. Apart from this $\mathfrak{b}^d$ and $\mathfrak{b}^a$ have the same roots. Thus $\mathfrak{b}^d$ and $\mathfrak{b}^a$ are related by an odd reflection. We define the symmetric bilinear form $(,)$ on $\mathfrak{h}^*$ by $(\epsilon, \epsilon) = -(\delta, \delta) = 1$ and $(\delta, \epsilon) = 0$. Up to a non-zero scalar multiple, this bilinear form agrees with the one defined in (2.9) by a general construction. We denote the Verma module with highest weight $\lambda$ induced from $\mathfrak{b}^b$ by $M^b(\lambda)$. There are two versions of the dot action, which we denote by $(w, \lambda) \rightarrow w \cdot^b \lambda = w(\lambda + \rho^b) - \rho^b$. The Borel subalgebra $\mathfrak{b}^d$ is called distinguished in [Kac78]. We call $\mathfrak{b}^a$ anti-distinguished.

3 A Family of Elusive Cases.

Consider the isotropic root $\gamma = \epsilon + \delta$, and for a positive integer $n$, define $\lambda_n, \mu_n \in \mathfrak{h}^*$ by

$$\lambda^d_n + \rho^d = (2n - 1)\gamma/2, \quad \mu^d_n + \rho^d = (2n - 1)(\delta - \epsilon)/2.$$

We have

$$(\lambda^d_n + \rho^d, \delta) = (\lambda^d_n + \rho^d, \epsilon) = (\lambda^d_n + \rho^d, \epsilon - \delta) = 2n - 1, \quad (\lambda^d_n + \rho^d, \gamma) = 0. \quad (3.1)$$

Thus $s_\epsilon(\lambda^d_n + \rho^d) = \mu^d_n + \rho^d$. In this Section we fix $n$ and set $\lambda^d = \lambda^d_n$. Now if $v_{\lambda^d}$ is the highest weight vector in $M^d(\lambda^d)$, then $v_{\lambda^a} := e_{\epsilon - \delta} v_{\lambda^d}$ is a highest weight vector for $\mathfrak{b}^a$ with weight $\lambda^a = \lambda^d + \epsilon - \delta$, which generates a copy of the Verma module $M^a(\lambda^a)$. Since $(\lambda^d + \rho^d, \delta - \epsilon) \neq 0$, $e_{\delta - \epsilon} v_{\lambda^a}$ is a non-zero multiple of $v_{\lambda^d}$, and we have $U(\mathfrak{g}) v_{\lambda^a} = M^d(\lambda^d)$. Thus we may write

$$M(\lambda) := M^d(\lambda^d) = M^a(\lambda^a). \quad (3.2)$$

Moreover we have $\lambda^d + \rho^d = \lambda^a + \rho^a$, compare [Mus12] Corollaries 8.6.2 and 8.6.3. In this case $M(\lambda)$ has length 8, and is multiplicity free. For ease of comparison with [Mus13], we use the same notation for simple modules. Thus the composition factors of $M(\lambda)$ have the form $L_x$ for $x \in \{a, b, c, d, e, f, g, h\}$, and in the diagrams below we label a length one interval by $x$ if the corresponding subfactor is isomorphic to $L_x$.  

14
For a positive root \( \eta \), let \( \mathcal{H}_{\eta,r} = \{ \nu \in \mathfrak{h}^* | (\nu + \eta, \eta) = r(\eta, \eta)/2 \} \). Let \( r \) be a positive integer. If \( \eta \) is isotropic assume that \( r = 1 \), and if \( \eta \) is odd non-isotropic assume that \( r \) is odd. A Sapovalov element \( \theta_{\eta,r} \in U(\mathfrak{b}^-) \), for the pair \( (\eta, r) \) has the property that for \( \nu \in \mathcal{H}_{\eta,r} \), \( \theta_{\eta,r}v_\nu \in M(\nu) \) is a non-zero highest weight vector. A Sapovalov element \( \theta_{\eta,r} \) is only determined modulo a left ideal in \( U(\mathfrak{b}^-) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \). However for \( \nu \in \mathcal{H}_{\eta,r} \), the evaluation \( \theta_{\eta,r}(\nu) \in U(\mathfrak{n}^-) \) is uniquely determined, and we have \( \theta_{\eta,r}(\nu)v_\nu = \theta_{\eta,r}v_\nu \). For a simple root \( \eta \), we have \( \theta_{\eta,e} = e_{-\eta} \). If \( \eta \) is isotropic set \( \theta_\eta = \theta_{\eta,1} \) and \( \mathcal{H}_\eta = \mathcal{H}_{\eta,1} \).

**Lemma 3.1.** Let \( \gamma \) be a positive isotropic root and suppose \( \nu \in \mathcal{H}_\gamma \). Then

(a) There is a factor module \( M^\gamma(\nu) \) of \( M(\nu) \) such that in \( K(\mathcal{O}) \) we have \( [M(\nu)] = [M^\gamma(\nu)] + [M^\gamma(\nu - \gamma)] \).

(b) If \( K(\nu) \) is the kernel of the natural map \( M(\nu) \to M^\gamma(\nu) \), then \( [K(\nu)] = [M^\gamma(\nu - \gamma)] \).

(c) \( \theta_\gamma v_\nu \in K(\nu) \).

**Proof.** (a) is shown in [Mus17b] Section 6 using Sapovalov elements, (b) is an immediate consequence and (c) is proved in [Mus17b] Corollary 6.8. \( \square \)

For \( \nu^b \in \mathfrak{h}^* \) define

\[
A(\nu^b)_0 = \{ \alpha \in \Delta_0^+ | \alpha/2 \text{ is not a root and } (\nu^b + \rho^b, \alpha^\vee) \in \mathbb{N} \setminus \{0\} \}
\]

\[
A(\nu^b)_1 = \{ \alpha \in \Delta_1^+ | \alpha \text{ is non-isotropic and } (\nu^b + \rho^b, \alpha^\vee) \in 2\mathbb{N} + 1 \} \tag{3.3}
\]

\[
A(\nu^b) = A(\nu^b)_0 \cup A(\nu^b)_1.
\]

\[
B(\nu^b) = \{ \alpha \in \Delta_1^+ | \alpha \text{ is isotropic and } (\nu^b + \rho^b, \alpha) = 0 \}.
\]

**Lemma 3.2.** Suppose \( \alpha \in A(\nu^b) \) is a simple non-isotropic root of the Borel subalgebra \( \mathfrak{b}^b \), and \( (\nu + \rho, \alpha^\vee) = m \). Set \( \mu^b = s_\alpha \cdot \nu^b \). Then

(a) \( e_m^\alpha v_{\nu^b} \) generates a submodule of \( M(\nu^b) \) which is isomorphic to \( M^b(\mu^b) \).

(b) We have \( \dim \text{Hom}(M^b(\mu^b), M^b(\nu^b)) = 1 \).

**Proof.** Combine Lemma 9.2.1 and Theorem 9.3.2 from [Mus12]. \( \square \)

A key tool is the Jantzen sum formula, which is best expressed in the Grothendieck group \( K(\mathcal{O}) \) of the module category \( \mathcal{O} \). \footnote{This is defined as the category of \( \mathbb{Z}_2 \)-graded modules which belong to the usual BGG category \( \mathcal{O} \) when regarded as \( \mathfrak{g}_0 \)-modules, see [Hum08].} Objects in this category have finite length, so we can define \( K(\mathcal{O}) \) to be the free abelian group on the symbols \( [L] \), where \( L \) is a simple module. If \( M \) is any object of \( \mathcal{O} \) we define \( [M] \) to be \( \sum_L [M:L][L] \) where \( [M:L] \) is the multiplicity of \( L \) as a composition factor of \( M \). We note that \( K(\mathcal{O}) \) has a natural partial order. For \( A,B \in \mathcal{O} \) we write \( [A] \geq [B] \) if \( [A] - [B] \) is a linear combination of classes of simple modules with non-negative integer coefficients.
Now in a general Verma module $M(\nu)$ has a Jantzen filtration $M_1(\nu) \supseteq M_2(\nu) \supseteq \ldots$ where $M_1(\nu)$ is the unique maximal submodule of $M(\nu)$. The following is the Jantzen sum formula (for the distinguished Borel subalgebra).

**Theorem 3.3.** For all $\nu^d \in \mathfrak{h}^*$

$$
\sum_{i \geq 1} [M_i(\nu^d)] = \sum_{\alpha \in A(\nu^d)} [M(\sigma_\alpha \cdot^d \nu^d)] + \sum_{\gamma \in B(\nu^d)} [M^\gamma(\nu^d - \gamma)].
$$

(3.4)

**Proof.** See [Mus12] Theorem 10.3.1 for a preliminary version and [Mus17b] Theorem 1.11 for the statement in this form. 

In the case of $g = \mathfrak{osp}(3,2)$, set $\gamma = \epsilon + \delta$. Then for $\lambda = \lambda^d$ as above we have by (3.1), $A(\lambda^d) = \{\epsilon, \delta\}$ and $B(\lambda^d) = \{\gamma\}$, so (3.4) takes the form

$$
\sum_{i \geq 1} [M_i(\lambda)] = [A] + [M^d(\sigma_\delta \cdot^d \lambda^d)] + [M^\gamma(\lambda - \gamma)]
$$

$$
= [L_b] + [L_c] + [L_e] + 2[L_g] + 2[L_d] + 2[L_f] + 3[L_h],
$$

(3.5)

see [Mas13] Equations (5.2.7) and (5.2.8), where $A = M^d(\sigma_\epsilon \cdot^d \lambda^d)$ and $M(\lambda)/M_1(\lambda) \cong L_\alpha := L(\lambda)$. By Lemma 3.2, $A$ is a join irreducible submodule of $M(\lambda)$, but it is not clear that the other terms on the right of (3.5) are even submodules. However we can improve the formula using Lemma 3.1 (b) and Corollary 3.6.

**Lemma 3.4.** The composition factors of each $M_i(\lambda)$ are given as follows

$$
M_i(\lambda) = 0, \quad [M_3(\lambda)] = [L_h], \quad [M_2(\lambda)] = [L_g] + [L_d] + [L_f] + [L_h]
$$

(3.6)

and

$$
[M_1(\lambda)] = [L_b] + [L_c] + [L_e] + [L_g] + [L_d] + [L_f] + [L_h].
$$

(3.7)

**Proof.** See [Mas13] Theorem 5.2.17. We can also argue directly from (3.5) as follows. Since no composition factor has multiplicity greater than 3, we have $M_4(\lambda) = 0$. Then $[L_h]$ is the only composition factor with multiplicity 3 and so $[M_3(\lambda)] = [L_h]$. More generally for any $j \geq 1$, $[M_j(\lambda)] = \sum_{x: \sum_{i \geq 1} |M_i(\lambda)| \geq j} |L_x|$ and this gives the result. 

**Lemma 3.5.** (a) $[A] = [L_e] + [L_g] + [L_f] + [L_h]$,

(b) $[M^d(\sigma_\delta \cdot^d \lambda^d)] = [L_e] + [L_g] + [L_d] + [L_h]$,

(c) $[M^\gamma(\lambda - \gamma)] = [L_b] + [L_d] + [L_f] + [L_h]$.

**Proof.** See [Mas13] page 73.

**Corollary 3.6.**

$$
[M^d(\sigma_\delta \cdot^d \lambda^d)] = [M^a(\sigma_\delta \cdot^a \lambda^a)]
$$
Proof. The composition factors of $M^d(\sigma \cdot \lambda^d)$ are given by the Lemma. We use Corollary 4.8.6 with $\lambda$ replaced by $\sigma \cdot \alpha \lambda$ from [Mas13] to find the composition factors of $M^a(\sigma \cdot \alpha \lambda)$. The Corollary gives the highest weights of the composition with respect to the Borel subalgebra $b^a$. To complete the calculation we use [Mas13] Theorem 5.2.12 which gives the highest weight of each $L_\alpha$ for both Borels. Thus from the Corollary we know that the composition factors of $M^a(\sigma \cdot \alpha \lambda)$ are as follows:

(a) $L^a(\sigma \cdot \alpha \lambda) \cong L_{c}$
(b) $L^a(\sigma \cdot \alpha \lambda - (\epsilon - \delta)) \cong L^a(\sigma \cdot \alpha \lambda - (\epsilon + \delta)) \cong L_d$
(c) $L^a(\sigma e \cdot \alpha \lambda - (\epsilon + \delta)) \cong L^a(\sigma e \cdot \alpha \lambda + (\epsilon + \delta)) \cong L_g$
(d) $L^a(\sigma \cdot \alpha \lambda) \cong L_h$.

Now we state an improved version of the Jantzen sum formula. Set $A = M^d(\sigma \cdot d \lambda)$, $B = M^a(\sigma \cdot a \lambda)$ and $C = K(\lambda)$. Then $A, B, C$ are submodules of $M(\lambda)$, with $A, B$ join irreducible, and we have

$$\sum_{i \geq 1} [M_i(\lambda)] = [A] + [B] + [C]$$

$$= [L_h] + [L_c] + [L_e] + 2[L_d] + 2[L_g] + 2[L_f] + 3[L_h], \quad (3.8)$$

where the $L_i$ are as before.

By Lemma 3.4, $M(\lambda)$ has simple socle $L_h$, and thus $L_h$ is also the socle of $A, B$ and $C$. Now $A$ and $B$ are also join irreducible, with distinct simple factors, which are also different from $L_d, L_f, L_g$ and $L_h$. Thus we can choose the notation so that $A$ and $B$ have unique simple factors $L_e$ and $L_c$ respectively. This is consistent with Lemma 3.5.

Lemma 3.7. The lattice of submodules of $A$ and $B$ have Hasse diagrams

- Diagram for $A$
- Diagram for $B$

By Lemma 3.4, $M(\lambda)$ has simple socle $L_h$, and thus $L_h$ is also the socle of $A, B$ and $C$. Now $A$ and $B$ are also join irreducible, with distinct simple factors, which are also different from $L_d, L_f, L_g$ and $L_h$. Thus we can choose the notation so that $A$ and $B$ have unique simple factors $L_e$ and $L_c$ respectively. This is consistent with Lemma 3.5.

Lemma 3.7. The lattice of submodules of $A$ and $B$ have Hasse diagrams

- Diagram for $A$
- Diagram for $B$
Proof. This follows from the following Lemma.

For $X$ a non-zero submodule of $M$ containing $\text{soc}(M)$, set $\bar{X} = X/\text{soc}(M)$.

Lemma 3.8. We have

(a) $[A \cap B] = [L_g] + [L_h]$.
(b) $[A \cap C] = [L_f] + [L_h]$.
(c) $[B \cap C] = [L_d] + [L_h]$.

Proof. The common composition factors of $A$ an $B$ are $L_g$ and $L_h$. If (a) is false, then $A \cap B = \text{soc}(M)$. But then $A + B = A \oplus B$ would have $L_g$ as a composition factor of multiplicity 2. The proofs of (b), (c) are similar, since we know the composition factors of $C$.

Concerning $L(C)$ we know that $C$ has simple socle $\text{soc}(M)$, and the composition factors of $C$ from Lemma 3.5. Thus $L(\bar{C})$ is isomorphic to one of the lattices from Lemma 1.15.

Theorem 3.9. The lattice of submodules $L(\bar{C})$ is isomorphic to the lattice $\Lambda$ in Case 2 or Case 4 of Lemma 1.15. Thus $L(C)$ is isomorphic to the lattices augmented $\Lambda$.

Proof. In Cases 1 and 5 $L(\bar{C})$ has a unique atom, which by Lemma 3.8 would have to equal both $(\bar{A} \cap \bar{C})$ and $(\bar{B} \cap \bar{C})$. This would contradict the Lemma. We still have to rule out Case 3, where $\Lambda$ is isomorphic to $\mathcal{B}$. We label $L(\bar{C})$ with the join irreducibles we already know about. There is an extra join irreducible, which we label $\bar{D}$. From the information on composition factors in Lemmas 3.5 and 3.8 we see that $\bar{D} \neq A, B, C$. If $\bar{D} = B \cap \bar{C}$, then $\bar{C} = (A \cap \bar{B}) + (A \cap \bar{C}) + (B \cap \bar{C})$ has composition factors $L_d, L_f$ and $L_h$. This contradicts Lemma 3.5. Thus $\bar{D}$ is a new join irreducible, and it follows that $M/\text{soc}(M)$ has at least 8 composition factors, a contradiction.

Lemma 3.10. (a) If the Hasse diagram for $L(\bar{C})$ is as in Case 4 of Lemma 1.15 then the Hasse diagram for $L(C)$ can be labelled as below on the left.
(b) If the Hasse diagram for $L(C)$ is as in Case 2 of Lemma 1.15, then there is a join irreducible submodule $D$ of $C$ such that the Hasse diagram for $L(C)$ can be labelled as below on the right, possibly with the labels $A \cap C$ and $B \cap C$ interchanged.

\[ \begin{array}{c}
A \cap C \\
\text{soc}(M) \\
0
\end{array} \begin{array}{c}
A \cap C \\
\text{soc}(M) \\
0
\end{array} \]

Proof. (a) follows from Lemma 3.8. If (b) is false then $D$ would not be join irreducible, and it is easy to show that $C$ would not have simple socle. □

Corollary 3.11. There are only two ways the Hasse diagrams for the lattice of submodules for $A, B, C$ can fit together as a subdiagram of the Hasse diagram for $N = M_1(\lambda)/\text{soc}(M)$.

(a) If (a) holds in Lemma 3.10, then the Hasse diagrams fit together as in Step 1 of Subsection 5.2

(b) If (b) holds in Lemma 3.10, then the Hasse diagrams fit together as in Step 1 of Subsection 5.1

Proof. In each case, we know the Hasse diagrams for the lattice of submodules of $A, B, C$, and we know how these submodules intersect. □

Henceforth, unless otherwise stated, all highest weights are highest weights for the distinguished Borel subalgebra $\mathfrak{b}$ and all Verma modules are induced from $\mathfrak{b}$. We have $\rho = (\epsilon - \delta)/2$. Set $N_n = M_1(\lambda_n)/\text{soc}(M(\lambda_n))$ and $C = K_n$ is the kernel of the natural map $M(\lambda_n) \rightarrow M^\gamma(\lambda_n)$. The case $n = 1$ is easily dealt with.

Theorem 3.12. If $n = 1$, and $C = K_1$, then we have $C = D + (B \cap C)$ in Lemma 3.10, where $D = \theta, v_{\lambda_1}$ and the Hasse diagram for $N_1$ has a subdiagram as in Step 1 of Subsection 5.1.

\[ \text{By Theorem 3.12 in (b) we have } C = D + (B \cap C). \text{ Thus there is no need to interchange the labels, and the intervals are correctly labelled.} \]
Proof. By Lemma \ref{lem:3.1}, \( D \subseteq K_1 \). Note that \( \theta_\gamma v_{\lambda_1} \) is a highest weight vector with weight \( \lambda_1 - \gamma = -\epsilon \). Thus any weight of \( D \) belongs to the set \( -\epsilon - \sum_{\alpha \in \Pi} n_\alpha \). But by Lemma \ref{lem:3.4}, \( (B \cap C)/\text{soc}(M) \cong L_d \), and by \cite{Mas13} Theorem 5.2.12, \( L_d = L(s_\delta \cdot \lambda_1) \). Now \( \lambda_1 = (\epsilon + \delta)/2 - (\epsilon - \delta)/2 = \delta \), and \( s_\delta \cdot \lambda_1 = 0 \). This is not a weight of \( D \) by the above remark. Thus \( L_d \) cannot be a composition factor of \( D \), and it follows that \( C = D + (B \cap C) \) is not join irreducible as in (b) of Lemma \ref{lem:3.10}. The last statement follows from Corollary \ref{cor:3.11}.

Since both cases actually occur in in Corollary \ref{cor:3.11} this is as far as we can expect lattice theory and the results from \cite{Mas13} alone to take us.

4 The Cases \( n \geq 2 \).

Theorem 4.1. For \( n \geq 2 \), \( K_n \) is a highest weight submodule of \( M(\lambda_n) \) and the Hasse diagram for \( N_n \) has a subdiagram as in Step 1 of Subsection 5.2.

Now set \( B^n = M(\lambda_n) \), \( A^n = M(\mu_n) \). Note that \( s_\epsilon \cdot \lambda_n = \mu_n \). Also by Lemma \ref{lem:3.2} \( M(\mu_n) \) is isomorphic to the submodule of \( M(\lambda_n) \), generated by \( e^{-2\epsilon - 1} v_{\lambda_n} \). Define a map \( \sigma_n : A^n \rightarrow B^n \) by \( \sigma_n(xv_{\mu_n}) = xe^{-2\epsilon - 1} v_{\lambda_n} \) and set \( Q^n = B^n / \sigma_n(A^n) \).

Lemma 4.2. We have an exact sequence of complexes.

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
A^* : & A^1 & \rightarrow & A^2 & \rightarrow & A^3 & \rightarrow \\
\downarrow & \downarrow & \downarrow & \\
B^* : & B^1 & \rightarrow & B^2 & \rightarrow & B^3 & \rightarrow \\
\downarrow & \downarrow & \downarrow & \\
Q^* : & Q^1 & \rightarrow & Q^2 & \rightarrow & Q^3 & \rightarrow \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
\]

Proof. The maps in the complexes \( A^*, B^* \) are given by

\[
\tau_n : A^n \rightarrow A^{n+1}, \ xv_{\mu_n} \rightarrow x e^{-\delta} v_{\mu_{n+1}}, \ \text{and} \ \psi_n : B^n \rightarrow B^{n+1}, \ xv_{\lambda_n} \rightarrow x \theta_\gamma v_{\lambda_{n+1}}.
\]

To check that the square

\[
\begin{array}{cccc}
A^n & \rightarrow & A^{n+1} \\
\downarrow & & \downarrow \\
B^n & \rightarrow & B^{n+1}
\end{array}
\]

commutes, we show that \( \psi_n \sigma_n(v_{\mu_n}) = \sigma_{n+1} \tau_n(v_{\mu_n}) \). Now \( \psi_n \sigma_n(v_{\mu_n}) = e^{-2\epsilon} \theta_\gamma v_{\lambda_{n+1}} \) and \( \sigma_{n+1} \tau_n(v_{\mu_n}) = e^{-\delta} e^{-2\epsilon} v_{\lambda_{n+1}} \). These are equal by \cite{Mus17b} (3.7). The map \( \tilde{\psi}_n : Q^n \rightarrow Q^{n+1} \) is induced by \( \psi_n \). By \cite{Mus17b} Theorem 5.1, if \( \eta \) is an isotropic root and \( \nu \in H_\eta \), then \( \theta_\eta^2 v_\nu = 0 \). Thus the rows in the diagram are complexes.

Lemma 4.3. The complex \( A^* \) is exact at \( A^n \) for all \( n \geq 2 \).
Proof. Since $\epsilon - \delta$ is the negative of a simple root of $b$ this follows easily from the PBW theorem.

Now to study the complex $Q^*$, define

$$\Delta^q = \{ -\epsilon, -\epsilon + \delta, \delta, 2\delta, \epsilon + \delta, \epsilon \}, \quad \Delta^m = \{ \epsilon - \delta, -\delta, -2\delta, -\epsilon - \delta \}.$$ 

$$q = h \oplus \bigoplus_{\eta \in \Delta^q} g^\eta, \quad m = \bigoplus_{\eta \in \Delta^m} g^\eta.$$ 

Then $q$ is a parabolic subalgebra of $g$, whose Levi part $t$ satisfies

$$\mathfrak{t} := [\mathfrak{t}, \mathfrak{t}] = \text{span}\{ h_\epsilon, e_\epsilon, e_{-\epsilon} \}.$$ 

Also $g = m \oplus q$ and $q = k e_{-\epsilon} \oplus b$. Now, since $(\lambda_n, \epsilon) = n - 1$, $U(q) \otimes_{U(b)} k v_{\lambda_n}$ maps onto the simple $U(q)$-module, $L_n$ with character $e^{n\delta}(e^{\epsilon} - e^{-\epsilon})/(e^{\epsilon} - e^{-\epsilon})$. In particular $L_n$ has highest weight $\lambda_n = (n - 1)\epsilon + n\delta$ with highest weight vector $w_{\lambda_n}$.

**Lemma 4.4.** $Q^n \cong \text{Ind}_q^n L_n$.

**Proof.** [Mas13] Lemma 5.2.3. For convenience we sketch the proof. First there is a surjection from $B^n = M(\lambda_n)$ onto $\text{Ind}_q^n L_n$ sending $v_{\lambda_n}$ to $w_{\lambda_n}$. But the image of $A^n$ in $B^n$ is generated by $e^{2n-1}\delta_{\lambda_n}$ which maps to $e^{2n-1}\epsilon_{\lambda_n} = 0$ in $\text{Ind}_q^n L_n$. Thus $Q^n \cong B^n/A^n$ maps onto $\text{Ind}_q^n L_n$. On the other hand $B^n, A^n$ and $\text{Ind}_q^n L_n$ are all induced from modules with known characters, so it is a simple matter to compute their characters, and show that $Q^n$ and $\text{Ind}_q^n L_n$ have the same characters. \qed

**Lemma 4.5.** Let $t = \text{span}\{ h_\delta, e_{\pm \delta}, e_{\pm 2\delta} \} \cong \mathfrak{osp}(1, 2)$.

(a) As a $U(t)$-module, $Q^n$ has a Verma flag.

(b) Any non-zero submodule of $Q^n$ has Gelfand-Kirillov- dimension one.

**Proof.** Since $g = m \oplus q$, $Q^n = U(m) \otimes L_n$ is a free $U(m)$-module. Set $r = \text{span}\{ e_\delta, e_{-2\delta} \}$. Then $r \subset m$, and $U(m) = \bigoplus_{i=1}^4 U(r) z_i$ is a free $U(r)$-module of rank 4, where

$$z_1 = 1, \quad z_2 = e_{\epsilon - \delta}, \quad z_3 = e_{-\delta - \epsilon} \quad \text{and} \quad z_4 = e_{-\delta - \epsilon} e_{\epsilon - \delta}. $$

On the other hand $L_n = \text{span}\{ e^j_{-\epsilon} w_{\lambda_n} \}_{j=0}^{2n-2}$, and it follows that

$$Q^n = \bigoplus_{i=1}^4 \bigoplus_{j=0}^{2n-2} U(r) z_i e^j_{-\epsilon} w_{\lambda_n}. \quad (4.2)$$

Now define for $k \geq 1$,

$$T_k = \{(i, j) | e^k_\delta z_i e^j_{-\epsilon} w_{\lambda_n} = 0 \} \quad \text{and} \quad M_k = \bigoplus_{(i, j) \in T_k} U(r) z_i e^j_{-\epsilon} w_{\lambda_n}. $$

\[^8\text{Gelfand-Kirillov, see KL00}\]
Since $e_\delta$ acts nilpotently on $Q^n$, there is a least integer $m$ such that $M_m = Q^n$. Thus we have a filtration
\[ 0 = M_0 \subset M_1 \subset \ldots \subset M_m = Q^n. \]
By construction for $1 \leq k \leq m$, $M_k/M_{k-1}$ is a direct sum of $U(\mathfrak{t})$-Verma modules, and this gives the result.

(b) It is well-known that any non-zero submodule of a $U(\mathfrak{t})$-Verma module GK-dimension one and the result follows from (a) and an easy argument, see [Mus22]. □

**Lemma 4.6.** For $n \geq 2$, the lattice of submodules of $Q^n$ has the form

\[
Q^n = U(\mathfrak{g})w_{\lambda_n}
\]

\[
\begin{array}{c}
V_1 + V_2 \\
V_1 \\
V_3 = V_1 \cap V_2 \\
0
\end{array}
\]

with $V_i = U(\mathfrak{g})v_i$ where
\[
v_1 = e^{2n-1}_{-\delta} e_{\epsilon-\delta} w_{\lambda_n}, \quad v_2 = \theta_{\gamma} w_{\lambda_n} \text{ and } v_3 = e_{\delta-\epsilon} e^{2n-1}_{-\delta} e_{\epsilon-\delta} w_{\lambda_n}.
\]

In particular the highest weights of the composition factors of $Q^n$ are
\[
\lambda_n, \quad \sigma_\delta \cdot \lambda_n - (\epsilon - \delta), \quad \lambda_n - \gamma \text{ and } \quad \sigma_\delta \cdot \lambda_n.
\]

**Proof.** First we show that the $v_i$, $i = 1, 2, 3$ generate proper submodules of $Q^n$. For $v_2$ this follows from the general remarks about Šapovalov elements and the observation that $\theta_{\gamma} v_{\lambda_n}$ cannot belong to the submodule of $M(\lambda_n)$ generated by $e_{\epsilon-\delta}^{-1} v_{\lambda_n}$ by weight considerations. Next note that $e_{\epsilon-\delta} w_{\lambda_n}$ is a highest weight vector for $\mathfrak{b}^a$ which also generates $Q^n$, compare (3.2). Since $\delta$ is a simple root of $\mathfrak{b}^a$, $v_1$ is a highest weight vector for $\mathfrak{b}^a$ compare Lemma 3.2. It is a singular vector for $\mathfrak{b}^d$. Then changing Borels again, we see that $v_3 = e_{\delta-\epsilon} v_1$ is a highest weight vector for $\mathfrak{b}^d$. Note that the weights of $v_1, v_2, v_3$ are the last 3 weights listed in (4.3). Because the quadratic Casimir element $\Omega$ acts on $Q^n$ as the scalar $(\lambda_n + 2\rho, \lambda_n)$, [Mus12] Lemma 8.5.3, it follows that if $\lambda_n - \zeta$ is the highest weight of a composition factor of $Q^n$ we have $(\zeta, \zeta) = 2(\lambda_n + \rho, \zeta)$, and $\zeta$ is a weight of $U(\mathfrak{m})$. Using these facts, it is not hard to show by a direct computation that the only possible highest weights of the composition factors are those listed in (4.3), see [Mas13] Theorem 4.31. We remark that by (4.2), $Q^n$ is a finitely generated free module over the polynomial ring $k[e_{-\delta}]$, so there are not many cases to check. In addition $Q^n$ is multiplicity free, since $B^n$ is multiplicity free by (3.5) and Lemma 3.4. The only thing left to show is that $V_3 \subset V_2$. If this is not the case then $V_2$ is simple. The highest weight of $V_2$ with respect to $\mathfrak{b}^d$ is $\nu^d = \lambda_n - \gamma$ and
\[
\nu^d + \rho^d = (2n - 3)\gamma/2.
\]
Since \((\gamma, \epsilon - \delta) \neq 0\), \(e_{\epsilon - \delta} v_2\) is a highest weight vector for \(\mathfrak{b}^a\) with weight \(\nu^a\) which also generates \(V_2\). By [Mus12] Corollary 8.6.3, \(\nu^d + \rho^d = \nu^a + \rho^a\), so \((\nu^a + \rho^a, \delta^V) = 2n - 3\) is odd and positive. Since \(e_\delta\) is a simple root for \(\mathfrak{b}^a\), and \(V_2\) is a submodule of a free \(k[e_\delta]\)-module, by Lemma 4.5 (and thus also free), it follows that \(e_{2n-3} e_{\epsilon - \delta} v_2\) is a non-zero highest weight vector in \(V_2\). Thus \(V_2\) cannot be simple. In fact by a long and tedious computation \(e_{2n-3} e_{\epsilon - \delta} v_2\) is a non-zero scalar multiple of \(v_3\).

**Corollary 4.7.** For \(n \geq 1\), the maps \(\tilde{\psi}_n : Q^n \rightarrow Q^{n+1}\) in the complex \(Q^*\) are non-zero.

**Proof.** This follows since \(\psi_n(v_{\lambda_n}) = \theta \gamma v_{\lambda_{n+1}}\) and by the proof of the Lemma, this cannot belong to the submodule of \(M(\lambda_{n+1})\) generated by \(e_{\epsilon - \delta} v_{\lambda_{n+1}}\).

**Lemma 4.8.** The complex \(Q^*\) is exact at \(Q^n\) for all \(n \geq 2\).

**Proof.** Consider the diagram from Lemma 4.6. Clearly \(\text{Im} \tilde{\psi}_{n-1} = V_2\). Since \(Q^*\) is a complex, \(\text{Im} \tilde{\psi}_{n-1} \subseteq \ker \tilde{\psi}_n\). If the inclusion is strict, then by Corollary 4.7 and Lemma 4.6 \(Q^n / \ker \tilde{\psi}_n\) is the unique simple factor of \(Q^n\), which has finite dimension. But then \(\text{Im} \tilde{\psi}_n\) is a finite dimensional submodule of \(Q^{n+1}\) contradicting Lemma 4.5.

**Corollary 4.9.** The complex \(B^*\) is exact at \(B^n\) for all \(n \geq 2\).

**Proof.** For \(X = B\) or \(Q\), denote the homology of the complex \(X^*\) is exact at \(X^n\) by \(H(X^n)\). From the long exact homology sequence and Lemma 4.3, we see that \(H(B^n) = H(Q^n)\). Thus the result follows from Lemma 4.8.

**Proof of Theorem 4.7** First suppose \(n \geq 1\). In the complex \(B^*\) from (4.1) we have \(B^n = M(\lambda_n)\), and the map \(\psi_n : M(\lambda_n) \rightarrow M(\lambda_{n+1})\) is given by \(\psi_n(xv_{\lambda_n}) = x\theta \gamma v_{\lambda_{n+1}}\). Thus \(\text{Im} \psi_n = U(\mathfrak{g}) \theta \gamma v_{\lambda_{n+1}}\). Also by Lemma 3.1, we have \(\text{Im} \psi_{n-1} \subseteq K_n\). Now if \(n \geq 2\), then by Corollary 4.9 we have an exact sequence

\[
0 \longrightarrow \text{Im} \psi_{n-1} \longrightarrow M(\lambda_n) \longrightarrow \text{Im} \psi_n \longrightarrow 0.
\]

Thus in \(K(O)\) we have, using Lemma 3.1

\[
[M(\lambda_n)] = [\text{Im} \psi_{n-1}] + [\text{Im} \psi_n] \leq [K_n] + [K_{n+1}] = [M(\lambda_n)].
\]

Hence \(K_n = \text{Im} \psi_{n-1}\) is an image of a Verma module, and this gives the result.

**5 Application: Building the Lattice of Submodules of the Verma Module \(M(\lambda_n)\).**

The set of submodules of a module \(N\) becomes a lattice \(L(N)\) when \(\lor\) is taken to mean +, and \(\land\) to mean \(\cap\). We want to construct the lattice of submodules of the Verma module \(M(\lambda_n)\) as in (3.2). Since \(M(\lambda_n)\) has a unique maximal submodule \(M_1(\lambda_n)\) and a simple socle \(M_2(\lambda_n)\), an equivalent problem is to determine the lattice of submodules of \(N_n = M_1(\lambda_n) / M_2(\lambda_n)\). There are two possibilities, depending on which case holds in Corollary 3.11.
5.1 The Case $n = 1$.

Suppose that $n = 1$ and set $N = N_1$. We construct the Hasse diagram for $L(N)$, and show that $L(N)$ has 21 elements. Thus $M(\lambda_1)$ has 23 submodules. There are several unusual features in this case. The submodules $A, B, C$ generate $L(N)$ but $C$ is not join irreducible. The maximal submodule $D$ of $C$ is join irreducible, but the sublattice of $L(N)$ generated by $A, B, D$ does not contain $B \cap C$. The submodules $A, B, D$ and $B \cap C$ form a minimal generating set of join irreducibles. Thus $L(N)$ has rank 4. All this is apparent from Step1.

Step 1. From Theorem 3.12, we know that $N$ contains at least the following submodules, where $A, B, D$ are join irreducible. Furthermore any composition factor of $N$ is also a composition factor of $A, B$ or $C$.

The diagrams can be drawn more compactly using the abbreviations

$$G = (B \cap C) + (A \cap B), \quad F = (A \cap B) + (A \cap C)$$

$$E = (A \cap C) + (B \cap C), \quad S = (A \cap B) + (A \cap C) + (B \cap C).$$

Thus the Hasse diagram for $L(N)$ contains the following subdiagram.
Step 2. In Step 1, we have a box with one corner missing. Complete the box by adding $S = E + F = E + G = F + G$. Also add $D + F$.

Remarks 5.1. (a) In Step 2, all modules in the row starting with $A$ have length 3. This will help with the rest of the construction. We add some notes to show that we have not missed any sums of submodules from previous steps. At the end we list the composition factors of all non-simple submodules, to ensure that nothing has been repeated.

(b) It is easy to see how to proceed based on a consideration of composition factors. For example consider the quadrilateral in Step 3, with base $F$. This submodule has composition factors $L_f$ and $L_g$, and is covered by two submodules $A$ and $S$, that also have $L_e$ or $L_d$ as a composition factor. This can be seen already in Step 2. By completing the quadrilateral in Step 3, we obtain the length 4 submodule $A + S$. The length 4 submodule $A + D$ is constructed similarly. Opposite sides of a quadrilateral have the same labels by the second isomorphism theorem.

Step 3. Next add all submodules of length 4.
(a) The top row contains sums of 4 of the 5 length 3 submodules. However

(b) $F \subset A$, so $A + D + F = A + D$. We give this submodule a shorter name.

(c) $F \subset S$, so $C + F \leq C + S$, but $C + F \supset S$, so $C + F = C + S = C + D + F$.

(d) $C + S = D + F + S$

(e) $B + C = B + D + F$, $A + C$ and $A + B$ have length 5.

(f) Together (a), (c), (d), (f) account for all 10 sums of the length 3 submodules.

**Step 4.** Next add all submodules of length 5.

\[
\begin{align*}
B + C & \quad A + B \quad A + C \\
B + S & \quad C + S \quad A + D \\
& \quad A + S \\
& \quad B \quad S \quad C \quad D + F \quad A \\
& \quad B \cap C \quad A \cap B \quad A \cap C \\
& \quad B \cap B \cap C
\end{align*}
\]

(a) The top row contains sums of 3 of the 4 length 4 submodules. However

(b) $B + C \supset S = (A \cap B) + (A \cap C) + (B \cap C)$. So $B + C + S = B + C$. We give this submodule the shorter name.

(c) We do likewise with $A + C = A + D + S$.

(d) $A + C = (A + D) + (C + S) = (A + S) + (C + S)$.

(e) $(B + S) + (A + S) = N$, since it contains all composition factors of $N$.

(f) Together (a), (d), (e) account for all 6 sums of the length 4 submodules.
**Final Step.** Add the sum of the maximal submodules $A + B + C$ from Step 4.

**Composition Factors.** To show that all submodules in the diagram are distinct we list the composition factors of non-simple submodules. The equations below hold in the Grothendieck group $K(O)$, but we use a more compact notation than before. Thus if $X$ is a submodule of $N$, we write $X = \sum_{|X:L_x|=1} x$. Modules in the same row have the same length.

$$
G = d + g, \quad E = d + f, \quad D = b + f, \quad F = f + g
$$

$$
B = c + d + g, \quad S = d + f + g, \quad C = b + d + f, \quad D + F = b + f + g, \quad A = e + f + g.
$$

$$
B + S = c + d + f + g, \quad C + S = b + d + f + g, \quad A + D = b + e + f + g, \quad A + S = d + e + f + g,
$$

$$
B + C = b + d + c + f + g, \quad A + C = b + d + e + f + g, \quad A + B = c + d + e + f + g.
$$

The submodule $S$ is the socle of $N$, and the Hasse diagram for $L(S)$ is the same as that of the box lattice $\mathbb{B}$. To determine the Hasse diagram for $L(N/S)$ consider all submodules not contained in $S$, and identify two submodules that are joined by an edge labelled by a submodule of $S$. We find that $L(N/S) \cong \mathbb{B}$ as lattices.

### 5.2 The Case $n \geq 2$.

Suppose that $n \geq 2$. We show that $L(N_n)$ is isomorphic to the restricted free distributive lattice of rank 3. Thus $L(M(\lambda_n))$ is isomorphic to extended $\Lambda_3$. 

27
Step 1. From Theorem 4.1 we know that $N = N_n$ contains at least the following submodules, where $A, B, C$ are join irreducible

We omit the remaining details, because as remarked in the first version of this paper, the result is predictable in advance, since from Step 1, we see that the poset of join irreducible subsets of $M(\lambda_n)$ is isomorphic to to $\mathbb{B}$, and $J(\mathbb{B})$ is isomorphic to extended $\Lambda_3$. For full details on the stepwise construction of $L(N_n)$ see [Mus21].

References

[Bir79] G. Birkhoff, *Lattice theory*, 3rd ed., American Mathematical Society Colloquium Publications, Vol. 25, American Mathematical Society, Providence, R.I., 1979. MR598630

[Ded00] R. Dedekind, *Ueber die von drei Moduln erzeugte Dualgruppe*, Math. Ann. 53 (1900), no. 3, 371–403, DOI 10.1007/BF01448979 (German). MR1511094

[DH78] L. L. Dornhoff and F. E. Hohn, *Applied modern algebra*, Macmillan Publishing Co., Inc., New York; Collier Macmillan Publishers, London, 1978. MR0460006

[Fre80] R. Freese, *Free modular lattices*, Trans. Amer. Math. Soc. 261 (1980), no. 1, 81–91, DOI 10.2307/1998318. MR576864

[Hum72] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972. Graduate Texts in Mathematics, Vol. 9. MR0323842 (48 #2197)

[Hum08], *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, Graduate Studies in Mathematics, vol. 94, American Mathematical Society, Providence, RI, 2008. MR2428237

[Jan79] J. C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Mathematics, vol. 750, Springer, Berlin, 1979 (German). MR552943 (81m:17011)

[Kac77] V. G. Kac, *Lie superalgebras*, Advances in Math. 26 (1977), no. 1, 8–96. MR0486011 (58 #5803)

[Kac78], *Representations of classical Lie superalgebras*, Differential geometrical methods in mathematical physics, II (Proc. Conf., Univ. Bonn, Bonn, 1977), Lecture Notes in Math., vol. 676, Springer, Berlin, 1978, pp. 597–626. MR519631 (80f:17006)

[Kac83] V. G. Kac, *Infinite-dimensional Lie algebras*, Progress in Mathematics, vol. 44, Birkhäuser Boston Inc., Boston, MA, 1983. An introduction. MR739850

[KL00] G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Revised edition, Graduate Studies in Mathematics, vol. 22, American Mathematical Society, Providence, RI, 2000. MR1721834 (2000j:16035)

[Lau84] H. B. Laufer, *Discrete Mathematics and Applied Modern Algebra*, Prindle, Weber and Schmidt, Boston, MA, 1984.
[LP84] R. Lidl and G. Pilz, *Applied abstract algebra*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1984. MR765220

[Mas13] A. Masaros, *Category O Representations of the Lie Superalgebra osp(3, 2)*, Thesis, University of Wisconsin-Milwaukee, 2013.

[Mus12] I. M. Musson, *Lie Superalgebras and Enveloping Algebras*, Graduate Studies in Mathematics, vol. 131, American Mathematical Society, Providence, RI, 2012.

[Mus17a] I. M. Musson, *The lattice of submodules of a multiplicity-free module*, Groups, rings, group rings, and Hopf algebras, Contemp. Math., vol. 688, Amer. Math. Soc., Providence, RI, 2017, pp. 237–247, DOI 10.1090/conm/688. MR3649178

[Mus17b] I. M. Musson, *Sapovalov elements and the Jantzen filtration for contragredient Lie superalgebras*, arXiv:math/1710.10528. (2017).

[Mus21] ———, *How to Construct the Lattice of Submodules of a Multiplicity free Module from Partial Information*, arXiv:2112.15142 v1. (2021).

[Mus22] ———, *Sapovalov elements and the construction of modules with prescribed characters for contragredient Lie superalgebras*, in preparation, 2022.

[Sta12] R. P. Stanley, *Enumerative combinatorics. Volume 1*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR2868112

[Ste74] W. Stephenson, *Modules whose lattice of submodules is distributive*, Proc. London Math. Soc. (3) 28 (1974), 291–310, DOI 10.1112/plms/s3-28.2.291. MR338082

\textsuperscript{9}Available at https://dc.uwm.edu/etd/137/