Brane Solitons for $G_2$ Structures in Eleven-Dimensional Supergravity
Re-Visited

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Abstract

We investigate the four-dimensional supergravity theory obtained from the compactification of eleven-dimensional supergravity on a smooth manifold of $G_2$ holonomy. We give a new derivation for the Kähler potential associated with the scalar kinetic term of the $N = 1$ four-dimensional theory. We then examine some solutions of the four-dimensional theory which arise from wrapped M-branes.

1 Introduction

Considerable interest has been shown recently in compactifications of M-theory on manifolds which have holonomy group $G_2$. This is primarily because the compactified theory exhibits a realistic amount of supersymmetry in four dimensions. However, it has been known for some time that for compactifications of eleven-dimensional supergravity on smooth $G_2$ manifolds the four-dimensional effective theory does not have charged particles. Recently charged particles have been introduced by considering compactifications on singular $G_2$ manifolds. This has led to renewed interest in the properties of $G_2$ compactifications. This has also motivated the investigation of $G_2$ singularities and has led to the construction of several new non-compact $G_2$ manifolds and references within.
In this paper we shall re-derive the Kähler potential of the N=1 four-dimensional supergravity action obtained from the compactification of eleven-dimensional supergravity on a smooth $G_2$ manifold. Geometrically this gives a Kähler metric on the tangent space of the moduli space of $G_2$ structures. This Kähler potential has been obtained previously in [1], [7], [8] and [9]. This was done by computing first the terms quadratic in the 4-form field strength obtained on making a Kaluza-Klein ansatz. Then $N=1$ supersymmetry together with the results of [10] were used to determine the Kähler metric. In this paper we explicitly derive an expression for the Kähler potential of the four-dimensional supergravity action from the eleven-dimensional Einstein-Hilbert term, using some results of [11]. This new derivation is similar in spirit to the calculation of the moduli metric of supersymmetric multi-black hole solutions; see for example [12], [13], [14]. One of the advantages of this new approach is that the Kähler structure on the tangent bundle of the moduli space of $G_2$ structures is obtained directly without making an explicit use of supersymmetry in four dimensions. Our computation will be used to compare a normalization factor in the definition of the Kähler potential.

The plan of this paper is as follows. In section two we present the computation of the Kähler potential of the four-dimensional theory obtained on compactification by making use of various symmetries of the $G_2$ manifold. In section three we examine some black hole and domain wall solutions of the resulting four dimensional effective theory. Some useful $G_2$ identities are presented in the Appendix.

2 Compactifications of Eleven-Dimensional Supergravity

The bosonic part of the eleven-dimensional supergravity Lagrangian is

$$
\mathcal{L} = \sqrt{h} R - \frac{1}{2} F \wedge \star F + \frac{1}{6} C \wedge F \wedge F ,
$$

(2.1)

where $h$ is the eleven dimensional spacetime metric, $R$ is the eleven dimensional Ricci scalar, $C$ is a 3-form gauge potential with associated field strength $F = dC$. The compactification ansatz that we shall consider is

$$
d s^2 = d s^2 (\mathbb{R}^{1,3}) + d s_N^2 ,
$$

(2.2)

with $F = 0$, where $N$ is a compact smooth 7-manifold with $G_2$ holonomy. In particular, $N$ is Ricci-flat, and there exists a covariantly constant 3-form $\phi$ on $N$. Some algebraic properties of this 3-form are presented in the Appendix. The dimension of the moduli space of $G_2$ structures of the manifold $N$ is $b_3(N)$. The solution (2.2) preserves $\frac{1}{8}$ of the 32 supersymmetries of eleven-dimensional supergravity. Hence, we expect that the low energy four-dimensional theory is a $N=1$, four-dimensional supergravity.

In order to probe the low energy dynamics of this solution, we shall allow the moduli $s^i$ to depend on the spacetime co-ordinates $x^M$ as $s^i \rightarrow s^i(x^M)$. Specifically, we shall take the perturbed spacetime metric to be

$$
d s^2 = g_{MN} d x^M d x^N + g_{ab} d y^a d y^b ,
$$

(2.3)

where we have split the spacetime co-ordinates in a 4+7 manner according to $x^\mu = \{ x^M , y^a \}$ for $M, N = 0, \ldots, 3 \text{ and } a, b = 1, \ldots, 7$; and $g_{ab} = g_{ab}(y, s^i(x^M))$, $g_{MN} = g_{MN}(x^L)$. Here $g_{ab}$ depends on $x^M$ only via the moduli $s^i$, so that at fixed $x^M$, $g_{ab} d y^a d y^b$ is a metric with $G_2$
holonomy. We shall assume that all components of the metric vary slowly with regard to \( x^M \), and we shall refer to quantities being of first or second order with respect to derivatives of \( x^M \).

To proceed, it is necessary to solve the field equations up to first order, and to determine the effective action up to second order. Solving the field equations up to first order ensures that the back reaction of the geometry to the varying moduli is taken into account. In what follows, we shall concentrate on the Einstein equations. We remark that perturbing the moduli in the 3-form potential \( C \) produces a first order 4-form field strength, and hence, in order to solve the first order Einstein equations, it suffices to solve the Ricci-flatness condition

\[
R_{\mu \nu} = 0 \tag{2.4}
\]

up to first order. The only first order terms contributing to the Ricci tensor arise in the \( R_{aM} \) components. This implies that

\[
\nabla_a W_{Ma}^d - \nabla_a W_{M}^d d = 0 \tag{2.5}
\]

where \( \nabla_a \) is the covariant derivative with respect to the (perturbed) metric \( g_{ab} \) on \( N \) and

\[
W_{Ma} = \mathcal{L}_{\nabla_M} g_{ab}; \tag{2.6}
\]

\( \mathcal{L} \) is the Lie derivative.

To investigate the consequences of (2.5), it is convenient to express \( W_{Ma} \) in terms of the 3-forms \( \psi_M = \mathcal{L}_{\nabla_M} \phi \) and \( \chi_M = \nabla_M \phi \), where here \( \nabla_M \) denotes the covariant derivative with respect to the (perturbed) eleven-dimensional metric. Note that up to first order, the only non-vanishing components of \( \psi_M \) and \( \chi_M \) are \( \psi_{Mabc} \) and \( \chi_{Mabc} \). Next, taking the Lie derivative of (a.4) in the appendix, we find that

\[
W_{Ma} = \frac{1}{6} \psi_{Ma} \phi_{abc} + \frac{1}{6} \phi_{a} \psi_{Mbc} + \frac{1}{3} \partial_M g^{ef} \phi_{ae} \phi_{bf} \tag{2.7}
\]

For use later observe that contracting (2.7) with \( g^{ab} \), one obtains

\[
\partial_M \log \det g^7 = \frac{1}{9} \psi_{Mde} \phi_{def} \tag{2.8}
\]

Furthermore, using (a.7), the last term in (2.7) can be rewritten as

\[
\partial_M g^{ef} \phi_{ac} \phi_{bf} = W_{Ma} - g_{ab} \partial_M \log \det g^7. \tag{2.9}
\]

Substituting this expression back into (2.7), one obtains

\[
W_{Ma} = \frac{1}{4} (\psi_{Ma} \phi_{abc} + \phi_{a} \psi_{Mbc}) - \frac{1}{2} g_{ab} \partial_M \log \det g^7. \tag{2.10}
\]

It follows that

\[
\nabla_b W_{Ma} - \nabla_a W_{M}^b b = \frac{1}{4} \nabla_b (\phi_{abc} \psi_{Mbc} - \psi_{Mae} \phi_{bc} \phi_{de}). \tag{2.11}
\]

Using the relation

\[
\chi_{Mabc} = \frac{1}{4} \psi_{Mabc} - \frac{1}{24} \phi_{abc} \psi_{Mde} \phi_{de} - \frac{3}{8} \psi_{Mde} [a \phi_{bc}] \tag{2.12}
\]
between $\chi_{Mabc}$ and $\psi_{Mabc}$, the condition (2.5) can be written as

$$^*\phi \land \delta \chi = 0,$$

(2.13)

where $\delta$ is the adjoint of $d$ defined on $\Lambda^*(N)$.

To continue, recall that $\Lambda^2(\mathbb{R}^7)$ decomposes into two irreducible $G_2$ representations, $\Lambda^2(\mathbb{R}^7) = \Lambda^2_7 \oplus \Lambda^2_{14}$. The seven-dimensional representation $\Lambda^2_7$ arises by contracting a two-form with the parallel three-form $\phi$ to yield a vector. Then the condition (2.13) implies that $\pi_7 \delta \chi_M = 0$, where $\pi_7$ is the projection onto $\Lambda^2_7$.

Now we shall show that the field equations (2.5) are solved if $\psi_M$ is harmonic. For this we choose the gauge fixing condition $\pi_7 \delta \psi_M = 0$. This is a good gauge fixing condition because it can be shown that it is equivalent to requiring that $L_{\omega_{\phi'}} \phi$ is $L^2$-orthogonal to $L_{\omega_{\phi'}} \phi$, i.e., the deformations along the moduli directions are orthogonal to the deformations induced by the diffeomorphisms of $N$. It turns out that the closure of $\psi_M$ together with the condition $\pi_7 \delta \psi_M = 0$ imply that $\psi_M$ is harmonic. In particular $\psi_M$ is coclosed and so the first term in the right-hand side of (2.11) vanishes. Using the fact that $\psi_M$ is harmonic, it can be shown that $\psi_{Mabc} \phi^{abc}$ is independent of $y^a$ and so the second term in the right-hand side of (2.11) vanishes as well.

It remains to compute the second order effective action from (2.1). We shall concentrate on the Einstein-Hilbert part of the action, as this is sufficient to determ ine the Kähler potential of the four-dimensional $N = 1$ theory. The zeroth and first order parts in four-dimensional spacetime derivatives of the eleven-dimensional action vanish. The second order part simplifies to

$$S_{EH} = \int d^{11}x \sqrt{-\det g_4} \sqrt{\det g_7} [R^{(4)} - \frac{1}{4} g^{MN} (W_M^{ab} W_{Nab} - W_M^{a} W_N^{b})].$$

(2.14)

Using the expression for $W_{Mab}$ in terms of $\psi_M$ in (2.10), we find that the eleven-dimensional Einstein-Hilbert term of the action can be written as

$$S_{EH} = \int d^{11}x \sqrt{-\det g_4} \sqrt{\det g_7} [R^{(4)} - \frac{1}{16} g^{MN} \psi_{Mabc} \psi_N^{abc} - \frac{1}{32} g^{MN} \phi^{cdeh} \psi_{Mabcd} \psi_N^{aeh} + \frac{1}{864} g^{MN} (\phi^{abc} \psi_{Mabc}) (\phi^{deh} \psi_{Ndeh})].$$

(2.15)

This expression simplifies even further when we recall that the 3-forms on $N$ decompose into three $G_2$ irreducibles: $\Lambda^3(N) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$ and the harmonic 3-forms decompose as $H^3(N, \mathbb{R}) = H^3_1(N, \mathbb{R}) \oplus H^3_{27}(N, \mathbb{R})$. From this it follows that $\pi_7 \psi_M = 0$, which implies that $\phi \land \psi_M = 0$. Hence we obtain

$$S_{EH} = \int d^{11}x \sqrt{-\det g_4} \sqrt{\det g_7} [R^{(4)} - \frac{1}{12} g^{MN} \psi_{Mabc} \psi_N^{abc} + \frac{1}{216} g^{MN} (\phi^{abc} \psi_{Mabc}) (\phi^{deh} \psi_{Ndeh})].$$

(2.16)

To continue, it is convenient to define $\Theta = \int d^7y \sqrt{\det g_7}$, so that

$$\psi_{Mabc} \phi^{abc} = 18 \Theta^{-1} \partial_M \Theta.$$

(2.17)
To put the four-dimensional action into the Einstein frame, we make the conformal re-scaling
\[
\hat{g}_{MN} = \Theta g_{MN}
\] (2.18)
of the four-dimensional metric. Then the Einstein-Hilbert action is written as
\[
S_{EH} = \int d^4x \sqrt{-\det g_4} \left[ R^{(4)} - \frac{1}{2} \Theta^{-1} g_{MN} (\psi_M, \psi_N) \right],
\] (2.19)
where we have dropped the ^ and
\[
(\tau, \kappa) \equiv \frac{1}{6} \int d^7y \sqrt{-\det g_7} \tau_{abc} \kappa^{abc}
\] (2.20)
for 3-forms \(\tau, \kappa \in \Lambda^3(N)\). After a calculation similar to the one presented in [8] and [9], we obtain
the Kähler potential
\[
K = -3 \log \Theta,
\] (2.21)
of the sigma model matter in the effective \(N = 1\), four-dimensional supergravity theory. In fact (2.21) should be thought of as the Kähler potential on the tangent bundle of the moduli space of \(G_2\) structures on the compact manifold \(N\). Note that the normalization factor of the Kähler potential is that of [9] and it differs from that of [8] by a factor of seven.

The remaining couplings of the four-dimensional effective supergravity have been described extensively in the literature [1]. The bosonic fields are \(b_3(N)\) complex scalars \(z_i = \frac{1}{2} (s_i - ip_i)\), \(i = 1, \ldots, b_3(N)\) and \(b_2(N)\) Maxwell fields \(F^a = dA^a\), \(a = 1, \ldots, b_2(N)\), where the real scalars \(p_i\) and the Maxwell fields arise from the Kaluza-Klein reduction of the three-form gauge potential of eleven-dimensional supergravity given by
\[
C = p^i \phi_i + A^a \wedge \omega_a
\] (2.22)
where \(\{\phi_i\}\) is a basis for \(H^3(N, \mathbb{R})\) and \(\{\omega_a\}\) is a basis for \(H^2(N, \mathbb{R})\). The total effective action obtained via this reduction combined with the calculation given above is then embedded within the \(N = 1\) four-dimensional supergravity as
\[
S_{G_2} = \int d^4x \sqrt{-g} \left[ R - 2 \gamma_{ij} \partial_M z^i \partial^M \bar{z}^j - \frac{1}{2} \text{Re} h_{ab} F^a_{MN} F^{bMN} + \frac{1}{2} \text{Im} h_{ab} F^a_{MN} \ast F^{bMN} \right]
\] (2.23)
where \(\gamma_{ij} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^j}\) and \(h_{ab} = h_{ab}(z^i)\) is holomorphic in \(z^i\). In particular, from the Kaluza-Klein reduction of the term \(-\frac{1}{4} F \wedge \ast F\) in the eleven-dimensional supergravity action we obtain the following four-dimensional terms
\[
\int d^4x \sqrt{-g} \left[ - \frac{1}{2} \Theta^{-1} g^{MN} \partial_M p^i \partial_N p^j (\phi_i, \phi_j) - \frac{1}{4} s^i C_{iab} F^a_{MN} F^{bMN} \right]
\] (2.24)
where \(C_{iab}\) are real topological intersection numbers given by
\[
C_{iab} = \int_N \phi_i \wedge \omega_a \wedge \omega_b,
\] (2.25)
and we have made use of the identity $\omega_a \wedge^* \omega_b = \omega_a \wedge \omega_b \wedge \phi$ which follows from (a.10). From the eleven dimensional term $\frac{1}{6} F \wedge F \wedge C$ we obtain the four-dimensional term

$$\int d^4 x \sqrt{-g} \left[ - \frac{1}{4} p^i C_{iab} F^a_{MN} * F^{bMN} \right].$$

(2.26)

Hence, comparing (2.24) and (2.26) with (2.23) we find that $\text{Re} h_{ab} = \frac{1}{2} s C_{iab}$ and $\text{Im} h_{ab} = -\frac{1}{2} p_i C_{iab}$; and so $h_{ab} = z^i C_{iab}$ where $z^i = \frac{1}{2} (s^i - i p^i)$.

We remark that a potential term of the form

$$S_{\text{pot}} = - \int d^4 x \sqrt{-g} \left( e^K [\gamma^{i\bar{j}} D_i f D_{\bar{j}} \bar{f} - 3|f|^2] + \frac{1}{2} D_a D^a \right)$$

(2.27)

may in principle be added to the $G_2$ action given in (2.23), where $f = f(z^i)$ is a holomorphic superpotential, $D_i f = f K_i + \frac{\partial f}{\partial z^i}$, $K_i = \frac{\partial K}{\partial z^i}$, $D_a$ is a Fayet-Iliopoulos term and $D^a = \text{Re} h_{ab} D_b$ where $\text{Re} h^{ab}$ is the inverse of $\text{Re} h_{ab}$. The action given by $S_{G_2} + S_{\text{pot}}$ possesses $N = 1$ supersymmetry in four dimensions; however, it is not generically possible to obtain the potential term given above through the $G_2$ compactification described here.

3 Solutions of the Effective Theory

3.1 Dilatonic Black Holes

The simplest possible black-hole type of solution with active scalars is that for which the only non-vanishing modulus field is the volume of the compact space. The ansatz for such a black hole have been given in [8]. In particular one takes $ds^2 = -A^2 dt^2 + B^2 (dr^2 + r^2 ds^2 (S^2))$, $s^i = sc^i$ with $c^i$ constant; $p^i = 0$ and $A^a = C^a dt$. The potential term is set to vanish. The solution can be found by a direct substitution into the field equations of the truncated action

$$S_{G_2} = \int d^4 x \sqrt{-g} \left[ R - 2 \gamma^{i\bar{j}} \partial_M z^i \partial^M \bar{z}^j - \frac{1}{2} \text{Re} h_{ab} F^a_{MN} F^{bMN} \right],$$

(3.28)

where the $pF \wedge F$ terms have been truncated. The spacetime metric, gauge field strengths and scalars are given by

$$ds^2 = -L^{-\frac{4}{5}} F dt^2 + L^{-\frac{1}{5}} [F^{-1} dr^2 + r^2 ds^2 (S^2)]$$

$$F^a = H^a dL \wedge dt$$

$$s = \rho L^{-\frac{4}{5}}$$

(3.29)

where $H^a$ and $\rho$ are constants, and

$$L^{-1} = 1 + \frac{4 \kappa^2 \mu}{r}$$

$$F = 1 - \frac{4 \mu}{r},$$

(3.30)

for constant $\mu, \kappa$. It is also convenient to define the constants $H_{ab} = c^i C_{iab}$, and set $K_i = s^{-1} \lambda_i$ where $\lambda_i$ are constants such that $c^i \lambda_i = -7$, as a consequence of the homogeneity properties of
the volume of the $G_2$ manifold. It is straightforward to show that in order for the scalar equations to be satisfied, the $\lambda_\ell$ must be constrained as

$$C_{\ell ab} H^a H^b = -\frac{(\kappa^2 + 1)}{2 \rho \kappa^2} \lambda_\ell ,$$

(3.31)

and given this constraint, the Einstein equations are also satisfied. Note that $\mu$ is the non-extremality parameter of the black hole. Taking the limit $\mu \to 0$ with $4\kappa^2 \mu = Q$ finite, the resulting solution is an extremal black hole. In fact it is one of the dilatonic black holes found in [12]. To see this, observe that a consistent truncation of the four-dimensional action (3.28) associated with this ansatz, after an appropriate rescaling of the Maxwell field, is

$$S'_{G_2} = \int d^4 x \sqrt{-\det g_4} \left[ R^{(4)} - \frac{7}{2} s^{-2} |\nabla s|^2 - s F^2 \right] ,$$

(3.32)

where $F = dA$ is the 2-form Maxwell field strength. This action is equivalent to the Einstein-Maxwell action for dilatonic black holes given in [12]

$$S = \int d^4 x \sqrt{-\det g_4} \left[ R^{(4)} - 2 |\nabla \sigma|^2 - e^{-2a \sigma} F^2 \right] ,$$

(3.33)

on setting $s = e^{-\frac{2}{\sqrt{7}} \sigma}$ together with $a = \frac{1}{\sqrt{7}}$. The multi-black hole extreme solutions of (3.32) are given as

$$ds_4^2 = H^{-\frac{2}{7}} dt^2 + H^{\frac{2}{7}} dx^2$$

(3.34)

together with $s = H^{\frac{1}{7}}$ and $A = \sqrt{\frac{2}{7}} H^{-1} dt$ where

$$H = 1 + \sum_{A=1}^N \frac{\mu_A}{|x - z_A|}$$

(3.35)

is a harmonic function†. Note that the volume of the $G_2$ manifold blows up at the poles of $H$, which are curvature singularities of the four-dimensional spacetime solution.

There are also magnetic black holes associated with $G_2$ compactifications and their relation to wrapped branes has been described in [8]. The metric for these black holes is the same as that of the electric black holes above, but the modulus scalar for the magnetic black holes is the inverse of the electric black hole modulus scalar. Both the electrically and magnetically charged black holes break all of the $N = 1$ four-dimensional supersymmetry.

### 3.2 Supersymmetric Domain Walls

In order to find domain wall solutions preserving half of the $N = 1$ supersymmetry, it is necessary to introduce a potential term into the action [16]. Although such a term cannot be obtained from the classical compactification described here, one way of obtaining this potential is to add additional fluxes to the $G_2$ manifold [17]. However, it has been argued that this is inconsistent with the assumption that the $G_2$ manifold is compact [18]. Alternatively, one may include quantum

†The power of the harmonic function in these solutions differs from that given in [8] because of the change of normalization of the Kähler potential.
corrections which arise from the wrapping of Euclidean M2-brane instantons on associative cycles \( \mathcal{X}, \mathcal{Y} \). A superpotential for this configuration has been conjectured in \([20]\) which is of the form

\[
f = \mu \sum_{k=1}^{\infty} \frac{1}{k^2} e^{k\lambda_i z^i}
\]  

(3.36)

for constants \( \mu \) and \( \lambda_i \).

To describe domain wall type solutions we set the gauge fields and the Fayet-Iliopoulos terms to vanish, and take the spacetime metric to be

\[
ds^2 = B(y)^2 \left[ ds^2(\mathbb{R}^{1,2}) + dy^2 \right]
\]  

(3.37)

with \( z^i = z^i(y) \). The generic constraint equations for such a solution which preserves half of the supersymmetry have been given in \([21]\); they are

\[
\partial_y B = -e^{i\phi} B^{-2} e^{\frac{i}{2} F} \tilde{f} \\
\partial_y z^i = -B^{-1} \partial_y B \tilde{f}^{-1} \gamma^i j D_j \tilde{f} \\
\partial_y \left[ 2\Psi + iK \right] = 2i K_i \partial_y z^i,
\]  

(3.38)

where \( \Psi \) is an angular variable obtained from solving the Killing spinor equations. To solve these equations we shall set \( p^i = 0 \). The third equation in (3.38) implies that \( \partial_y \Psi = 0 \). Hence it is convenient to rescale \( f = e^{i\Psi_0} F \) where \( \Psi = \Psi_0 = \text{const.} \) Then (3.38) simplifies to

\[
\partial_y B = -B^{-2} e^{\frac{i}{2} \Psi} \tilde{F} \\
\partial_y z^i = B e^{\frac{i}{2} \gamma^i j} D_j \tilde{F}
\]  

(3.39)

To proceed we set \( s^i = s(y) c^i \) for constant \( c^i \). As a consequence of the homogeneity of the \( G_2 \) volume, we also have \( K_i = \lambda_i s^{-1} \) for constant \( \lambda_i \), where \( e^\lambda i = -7 \). Furthermore, \( \Theta = \mu s^\frac{2}{7} \) for constant \( \mu > 0 \). If we set \( F = F(-\frac{2}{7} z^i \lambda_i) \), so that \( F|_{p^i = 0, s^i = sc^i} = P(s) \) then (3.38) simplifies to

\[
\partial_y B^{-1} = -\mu^{-\frac{2}{7}} s^{-\frac{2}{7}} P \\
\partial_y s = 4 \mu^{-\frac{2}{7}} s^2 (s^{-\frac{2}{7}} P)'
\]  

(3.40)

where \( ' = \frac{d}{ds} \). Hence, defining \( W = s^{-\frac{2}{7}} P \), we note that we can write \( B \) in terms of \( s \) as

\[
B = e^{-\frac{2}{7} f} \frac{W}{s^{-\frac{2}{7}} P} ds
\]  

(3.41)

and given such a solution, the behaviour of \( s \) is determined by the equation

\[
\partial_y B^{-1} = -\mu^{-\frac{2}{7}} W.
\]  

(3.42)

We note that just as for the case of the black holes, the power of the exponential in (3.41) is altered due to the change of normalization of the Kähler potential.

We have proposed a new method to compute the Kähler potential of scalars of four-dimensional effective theory associated with D=11 supergravity compactifications on compact smooth \( G_2 \).
manifolds. This method may be proved useful to compute the K"ahler potential of the effective theory when the underlying $G_2$ manifold in not compact.

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**Appendix : Some Useful $G_2$ Identities**

If $N$ is an integrable $G_2$ manifold, then there exists a covariantly constant 3-form $\phi$, which, in an appropriate orthonormal frame $\{e^a : a = 1, \ldots, 7\}$ may be written as

$$
\phi = e^1 \wedge e^2 \wedge e^3 + e^1 \wedge (e^4 \wedge e^5 - e^6 \wedge e^7) + e^2 \wedge (e^4 \wedge e^6 + e^5 \wedge e^7) + e^3 \wedge (e^4 \wedge e^7 - e^5 \wedge e^6), \tag{a.1}
$$

and the dual 4-form is

$$
*\phi = e^4 \wedge e^5 \wedge e^6 \wedge e^7 + e^2 \wedge e^3 \wedge (e^6 \wedge e^7 - e^4 \wedge e^5) + e^1 \wedge e^3 \wedge (e^4 \wedge e^6 + e^5 \wedge e^7) + e^1 \wedge e^2 \wedge (e^5 \wedge e^6 - e^4 \wedge e^7), \tag{a.2}
$$

where * is the Hodge dual defined with respect to $g_7$ and we fix $\epsilon^{1234567} = +1$. The $G_2$ metric is obtained from the 3-form $\phi$ via the non-linear relation

$$
g_{ab} = -\frac{1}{24} \phi_{acdf} \phi^c_{bef} \phi^{dfe}, \tag{a.3}
$$

In addition, we have

$$
g_{ab} = \frac{1}{6} \phi_{acdf} \phi^c_{bef}. \tag{a.4}
$$

Another useful identity is

$$
\phi_{acdf} \phi^d_{hbf} = \left[ g_{cb} \phi_{dhf} + g_{ch} \phi_{dfb} + g_{cf} \phi_{dbh} - g_{db} \phi_{chf} - g_{dh} \phi_{cfb} - g_{df} \phi_{cbf} \right], \tag{a.5}
$$

which implies

$$
\phi_{acdf} \phi^d_{hbf} = -4 \phi_{dhf}. \tag{a.6}
$$

We also have the identity

$$
\phi_{dab} \phi^d_{cf} = g_{acbf} - g_{afbc} - \phi_{abc} + \phi_{abc}, \tag{a.7}
$$

and for any 3-form $\theta$,

$$
-\frac{1}{3} \phi_{abce} \theta_{dce} \phi^{deab} - 2 \theta_{dca} \phi_{bce} + \theta_{dce} \phi^{bce}_a \phi_{bac} = 0. \tag{a.8}
$$

Note also that if $\omega \in \Lambda^2(N)$ is a 2-form with $\pi_7 \omega = 0$ then the constraint

$$
\phi_a^{bc} \omega_{bc} = 0, \tag{a.9}
$$
which is satisfied if \( \omega \) is harmonic, is equivalent to

\[
\omega_{ab} = \frac{1}{2} \phi_{ab}^{cd} \omega_{cd}
\]

as a consequence of (a.7).

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