The extended, relativistic hyperon star model.

I. Bednarek, M. Keska, and R. Manka
Department of Astrophysics and Cosmology, Institute of Physics, University of Silesia, Uniwersytecka 4, PL-40-007 Katowice, Poland
(Dated: March 30, 2022)

In this paper an equation of state of neutron star matter which includes strange baryons in the framework of Zimanyi and Moszkowski (ZM) model has been obtained. We concentrate on the effects of the isospin dependence of the equation of state constructing for the appropriate choices of parameters the hyperons star model. Numerous neutron star models show that the appearance of hyperons is connected with the increasing density in neutron star interiors. Various studies have indicated that the inclusion of $\delta$ meson mainly affects the symmetry energy and through this the chemical composition of a neutron star. As the effective nucleon mass contributes to hadron chemical potentials it alters the chemical composition of the star. In the result the obtained model of the star not only excludes large population of hadrons but also does not reduce significantly lepton contents in the star interior.

PACS numbers: 24.10.Jv, 26.60.+c

I. INTRODUCTION

Compact stars which are known from observations can be classified into two distinctive groups. The first one is exemplified by white dwarfs and the second by neutron stars. Neutron stars themselves are identify with pulsars and compact X-ray sources. At the core of a neutron star the density of matter ranges from a few times of the density of normal nuclear matter to about an order of magnitude higher. Thus various exotic forms of matter such as hyperons or quark-hadron mix phase are expected to emerge in the interior of a neutron star. The appearance of these additional degrees of freedom and their impact not only on neutron stars but on proto-neutron stars structure and evolution as well has been the subject of extensive studies. Properties of matter at such extreme densities are of particular importance in determining forms of equations of state relevant for neutron stars and in examining their global parameters. Theoretical description of hadronic systems should be performed with the use of quantum chromodynamics (QCD) as it is the fundamental theory of strong interactions. However, at the hadronic energy scale where the observed experimentally degrees of freedom are not quarks but hadrons the direct description of nuclei in terms of QCD become inadequate. Other alternative approach has to be formulated one of which is quantum hadrodynamics (QHD) giving quantitative description of the nuclear many body problem. QHD is a relativistic quantum field theory in which nuclear matter description in terms of baryons and mesons is provided. The original model (QHD-I) contains nucleons interacting through the exchange of simulating medium range attraction $\sigma$ meson and $\omega$ meson responsible for short range repulsion. Extension (QHD-II) of this theory includes the isovector meson $\rho$. Nonlinear terms in the scalar and vector fields were added in order to get the correct value of the compressibility of nuclear matter and the proper density dependence in the vector self-energy. The variation of nucleon properties in nuclear medium is the key problem in nuclear physics. In order to incorporate quark degrees of freedom in the analysis of nuclear many-body system Guichon provides a quark-meson coupling model (QMC). The extension of the QMC theory namely the quark mean field model (QMF), describing a nucleon with the use of the constituent quark model, has been successfully applied to study the properties of both nuclear matter and finite nuclei. The model considered in this paper is an alternative version of the Walecka approach with enlarged meson sector. In the interior of neutron stars the density of matter could exceed normal nuclear matter density up to a few times, in such high density regime nucleon Fermi energies exceed the value of hyperon masses and thus the new hadronic degrees of freedom are expected to emerge. The higher the density the more various hadronic species are expected to populate. The onset of hyperon formation depends on the hyperon-nucleon and hyperon-hyperon interactions. Hyperons can be formed both in leptonic and baryonic processes. Several relevant strong interaction processes proceed and establish the hyperon population in the neutron star matter. Neutron star models are constructed at different levels of complexity starting from the most elementary one which assumes that neutrons

*Electronic address: bednarek@us.edu.pl
†Electronic address: markeslio@pf.pl
‡Electronic address: manka@us.edu.pl; URL: http://www.cto.us.edu.pl/~manka
are the only component. The more sophisticated version is formulated under the assumption that the neutron star matter has to obey the constrains of charge neutrality and $\beta$ equilibrium. Thus the model considered describes high isospin asymmetric matter and it has to be extended by the inclusion of isovector-scalar meson $g_0(980)$ ($\delta$ meson)\(^1\). For the sake of completeness additional nonlinear vector meson interactions are included. When strange hadrons are taken into account uncertainties which are present in the description of nuclear matter are intensified due to the incompleteness of the available experimental data. The standard approach does not reproduce the strongly attractive hyperon-hyperon interaction seen in double $\Lambda$ hypernuclei. In order to construct a proper model which do include hyperons the effects of hyperon-hyperon interactions have to be taken into account. These interactions are simulated via (hidden) strange meson exchange: scalar meson $f_0(975)$ ($\sigma^*$ meson) and vector meson $\phi(1020)$ ($\phi$ meson) and influence the form of the equation of state and neutron stars properties.

The solution of the presented model is gained with the mean field approximation in which meson fields are replaced by their expectation values. The parameters used are adjusted in the limiting density range around saturation density $\rho_0$ and in this density range give very good description in finite nuclei. However, incorporation of this theory to higher density require an extrapolation which in turn leads to some uncertainties and suffers of several shortcomings. The standard TM1 parameter set for high density range reveals an instability of neutron star matter which is connected with the appearance of negative nucleon effective mass due to the presence of hyperons. The Zimanyi-Moszkowski (ZM)\(^2\) model in which the Yukawa type interaction $g_{sN}\varphi$ is replaced by the derivative one $(g_{sN}\varphi/M_N)\bar{\psi}_N\gamma_\mu\partial^\mu\psi$ exemplifies an alternative version of the Walecka model which improves the behaviour of the nucleon effective masses. It also influences the value of the incompressibility $K$ of neutron star matter. The derivative coupling effectively introduces the density dependence of the scalar and vector coupling constants. Knowing the form of the equation of state (EOS) is the decisive factor in determining properties of neutron stars such as: central density, mass-radius relation, crust extent or the moment of inertia.

The essential goal of this paper is to obtain within the described above model the equation of state for the neutron star matter on the basis of calculations carried out for asymmetric nuclear matter in the relativistic mean field approach (RMF) and to compare the obtained results with ones that are relevant for high density calculations - namely with a quark star. The inclusion of $\delta$ meson affects the neutron stars chemical composition changing the proton fraction which in turn affects the cooling mechanism. If the proton fraction is higher than the critical value of about $Y_p \sim 0.11$ the direct URCA processes can proceed and this enhances the rate of neutron star cooling. Whether the proton fraction can exceed the critical value and at what density it occurs depends on the model. The proton fraction is almost entirely determined by the isospin-dependent part of the EOS thus the inclusion of $\delta$ meson and nonlinear vector meson interactions influence the neutron star structure and properties.

This paper is organized as follows. Section 2 outlines the extended model with derivative coupling including hyperons and additional mesons, together with the collected equations of motions. Their solutions enable the construction of the equation of state. In section 3 the equilibrium conditions leading to the relevant hyperon star composition are presented together with the chosen values of parameters. Section 4 contains numerical results and the discussion of their influence on neutron stars properties.

## II. THE MODEL

For the description of properties of the infinite nuclear matter with nonzero strangeness the relevant parts of the SU(3) structure have been involved. Due to the parity conservation the appearance of pseudoscalar mesons is forbidden. The Lagrangian function for the system can be written as a sum of a baryonic part including the full octet of baryons together with baryon-meson interaction terms, a mesonic part containing additional interactions between mesons which mathematically express themselves as supplementary, nonlinear terms in the Lagrangian function, and a free leptonic part

$$\mathcal{L} = \mathcal{L}_{BM} + \mathcal{L}_M + \mathcal{L}_L.$$  \hspace{1cm} (1)

The interacting baryons are described by the Lagrangian function $\mathcal{L}_{BM}$ which is given by

$$\mathcal{L}_{BM} = \sum_B \left(1 + \frac{g_{sB}\sigma + g_{\sigma^*}\sigma^{++}}{M_B} + \frac{I_{AB}\delta^{aB}}{M_B} \right)\bar{\psi}_B\gamma^\mu D_\mu\psi_B - \sum_B \bar{\psi}_BM_B\psi_B.$$  \hspace{1cm} (2)
where the spinor $\Psi_B^T = (\psi_N, \psi_\Lambda, \psi_\Sigma, \psi_\Xi)$ is composed of the following isomultiplets \[1, 3\]:

$$
\Psi_N = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}, \quad \Psi_\Lambda = \psi_\Lambda,
$$

$$
\Psi_\Sigma = \begin{pmatrix} \psi_\Sigma^+ \\ \psi_\Sigma^- \end{pmatrix}, \quad \Psi_\Xi = \begin{pmatrix} \psi_\Xi^0 \\ \psi_\Xi^- \end{pmatrix}.
$$

The covariant derivative $D_\mu$ is defined as

$$
D_\mu = \partial_\mu + i g \omega_\mu \omega - i g_\phi \phi_\mu + i g_\rho B \tau^a v^a_\mu.
$$

The model considered represents an alternative version of the Walecka model in which the Yukawa interaction term is replaced by the derivative coupling one. This Zimanyi-Moszkowski \[12\] method allows to solve the problem of too low effective nucleon masses achieved in the original approach. Rescaling the baryon field in a way proposed by Zimanyi and Moszkowski \[12\] the modified Lagrange function for interacting baryons is obtained

$$
\mathcal{L}_B = -\sum_B \bar{\psi}_B i\gamma^\mu D_\mu \psi - \sum_B \left(1 + \frac{g_\rho B \sigma + g_\phi B \sigma^* + \bar{\tau}\sigma B \tau^a \delta^a}{M_B} \right) - \bar{\psi}_B M_B \psi_B.
$$

The mesonic part of the Lagrangian function \[9\] is given by

$$
\mathcal{L}_M = \frac{1}{2} \partial_{\mu} \sigma \partial^{\mu} \sigma - U(\sigma) + \frac{1}{2} \partial_{\mu} \sigma^* \partial^{\mu} \sigma^* - \frac{1}{2} m_\sigma^2 \sigma^2 - \frac{1}{2} \partial_{\mu} \delta^a \partial^{\mu} \delta^a - \frac{1}{2} m_\delta^a \delta^a + \frac{1}{2} \Omega_{\mu\nu} \Omega^{\mu\nu}
$$

$$
+ \frac{1}{2} m_{\phi}^2 \phi_{\mu} \phi_{\mu} - \frac{1}{4} \partial_{\mu} \partial_{\nu} \phi_{\mu} \phi_{\nu} + \frac{1}{4} \Omega_{\mu\nu} \Omega^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_{\mu} \omega_{\mu} + (g_\rho g_\omega) A_\mu b^a_\mu g_\omega + (g_\rho g_\omega) A_\mu b^a_\mu g_\omega + \frac{1}{4} c_3 (\omega_{\mu} \omega_{\mu})^2 + \frac{1}{4} c_4 (b^a_\mu b^a_\mu)^2.
$$

The field tensors $R_{\mu\nu}^a, \Omega_{\mu\nu}, \phi_{\mu\nu}$ are defined as

$$
R_{\mu\nu}^a = \partial_{\mu} b^a_{\nu} - \partial_{\nu} b^a_{\mu} + g_\rho \varepsilon_{abc} b^b_\mu b^c_\nu,
$$

$$
\Omega_{\mu\nu} = \partial_{\mu} \omega_{\nu} - \partial_{\nu} \omega_{\mu}, \quad \phi_{\mu\nu} = \partial_{\mu} \phi_{\nu} - \partial_{\nu} \phi_{\mu}.
$$

The potential function $U(\sigma)$ possesses the well-known polynomial form introduced by Boguta and Bodmer \[9\]

$$
U(\sigma) = \frac{1}{2} m_\sigma^2 \sigma^2 + \frac{1}{3} g_3 \sigma^3 + \frac{1}{4} g_4 \sigma^4.
$$

The baryon mass is denoted by $M_B$ whereas $m_i$ ($i = \sigma, \omega, \rho, \phi, \delta$) are masses assigned to the meson fields and they are taken at their physical values. The parameters entering the Lagrangian function are the coupling constants $g_\omega B, g_\rho B, g_\phi B, g_\rho B, g_\phi B, g_\rho B, g_\phi B$ for meson fields and the self-interacting coupling constants $g_3, g_4, c_3, c_4$, $\zeta$. These parameters are adjusted to reproduce the bulk properties of the symmetric nuclear matter at equilibrium. They are collected in Table 2. As the $\delta$ meson field and nonlinear vector meson interactions carry isospin they contribute to the symmetry energy $E_s$ of the system. This imposes constrains on $g_\rho$ and $g_3$ $\Lambda_\sigma$ and $\zeta$ parameters. They are adjusted to get the experimental value of the symmetry energy $E_s$ which is equal $32 \pm 4$ MeV. Thus the parameter $g_\rho$ has been redefined by comparison with the one in the original TM1 parameter set.

The presence of hyperons demands additional coupling constants (they are collected in Table 3) which have been established with the use of experimental data and theoretical analysis.

The inclusion of hyperons improve our understanding of neutron stars at higher densities. There is common belief that at suitable densities a quark star is more stable than the neutron one. Thus it is interesting to compare the properties of hyperon stars with those obtained in the framework of QMF model for a quark star \[6\]. Now the Lagrangian function possesses the following form

$$
\mathcal{L}_{QMF} = \mathcal{L}(i\gamma^\mu D_\mu - m_e) q + \frac{1}{2} \partial_{\mu} \varphi_a \partial^{\mu} \varphi_a - \frac{1}{2} m_\varphi^2 \varphi_a \varphi_a
$$

$$
- \frac{1}{4} F_{\mu\nu}^a F^{a, \mu\nu} + \frac{1}{2} m_\varphi^2 \varphi_a \varphi_a.
$$
where $q$ denotes a quark field with three flavors, $u$, $d$ and $s$, and three colors. The construction of the model mimics the relativistic mean field theory, where the scalar $\sigma$ and the vector meson $\omega$ fields do not couple with nucleons but directly with quarks. The quark mass has to change from its bare current mass due to the coupling to the $\sigma$ meson. In the framework of QMF model with the use of the TM1 parameter set quarks constituent masses are $m_{c,u} = m_{c,d} = 367.61$ MeV and $m_{c,s} = 504.1$ MeV whereas the bag constant is chosen at the level of $B^{1/4} = 154.5$ MeV. There are no significant differences between the presented above Lagrangian function (10) and the Lagrangian function (4). They differ in the baryonic sectors whereas the mesonic sectors for both models are nearly the same. There are only differences in coupling constants: $g_{\sigma q} = g_{\sigma B}/3, g_{\omega q} = g_{\omega B}/3, g_{\rho q} = g_{\rho B}, g_{\delta q} = g_{\delta B}$. We restrict ourselves to the isospin SU(2) unbroken symmetric case, $m_{c,u} = m_{c,d}$, so

$$m_c = m_{c,f} \delta_{f,f'} = \begin{pmatrix} m_{c,u} & m_{c,d} \\ m_{c,d} & m_{c,s} \end{pmatrix}$$

Generators of the U(3) algebra $\lambda^a = \{\lambda^0 = \sqrt{2/3}I, \lambda^i\}$ (where $I$ is an identity matrix, $\lambda^i$ are Gell-Mann matrices SU(3) algebra) obeying $Tr(\lambda^a \lambda^b) = 2\delta_{ab}$. Restricting only to $U(2) \times U(1)$ subalgebra ($a = \{0, 1, 2, 3, 8\}$) case the simplest version of the QMF theory can be obtained. Defining the new base with $\tau^a, a = \{0, 1, 2, 3, 4\}$ as

$$\tau^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau^i = \begin{pmatrix} \sigma^i \ 0 \\ 0 \ 0 \end{pmatrix} \quad \text{for} \quad i = \{1, 2, 3\}, \quad \tau^4 = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the meson fields may be decomposed as follows

$$\varphi = \varphi_\alpha \tau^\alpha = \sigma \tau^0 + \delta_i \tau^i + \sigma^* \tau^4$$

and

$$V_{\mu} = V_{\mu}^a \lambda^a = \omega_{\mu} \tau^0 + \phi_{\mu} \tau^4 + b_i \tau^i$$

Here the QMF model is enlarged by the inclusion of the isovector $\delta$ meson. Similarly to nucleons it splits $u$ and $d$ masses. Both $\delta$ and $\rho$ mesons may be neglected in the case of the symmetric nuclear matter. Baryon and meson parameters

| Particle | $m$ [MeV] | J I | I S Y | $I_3$ | Q |
|----------|-----------|-----|------|-------|---|
| n        | 939.566   | 1/2 | 1/2  | 1     | -1/2 | 0 |
| p        | 938.272   | 1/2 | 1/2  | 1     | 1    | 1 |
| $\Lambda$| 1115.63   | 1/2 | 0    | -1    | 0    | 1 |
| $\Sigma^-$| 1197.43  | 1/2 | 1    | -1    | 0    | -1 |
| $\Sigma^0$| 1192.55  | 1/2 | 1    | -1    | 0    | 0 |
| $\Sigma^+$| 1189.37  | 1/2 | 1    | -1    | 0    | 1 |
| $\Xi^0$   | 1314.9    | 1/2 | 1/2  | -2    | 1    | 0 |
| $\Xi^-$   | 1321.3    | 1/2 | 1/2  | -2    | -1   | -1 |
| $\sigma$  | 550       | 0   | 0    | 0     | 0    | 0 |
| $\omega$  | 783       | 1   | 0    | 0     | 0    | 0 |
| $\rho$    | 770       | 1   | 0    | 0     | 0    | 0 |
| $\delta$  | 980       | 0   | 1    | 0     | 0    | 0 |
| $\sigma^*$| 975       | 0   | 0    | 0     | 0    | 0 |
| $\phi$    | 1020      | 1   | 0    | 0     | 0    | 0 |
Table 2
Parameter sets used in this paper

| Parameter | TM1      | ZM      | ZM + $\delta$ | ZM + $\delta$ + nonlinear terms |
|-----------|----------|---------|---------------|----------------------------------|
| M         | 938 MeV  | 938 MeV | 938 MeV       | 938 MeV                          |
| $c_3$     | 71.3075  | 0       | 0             | 0                                |
| $g_3$     | 7.23 fm$^{-1}$ | 0 | 0             | 0                                |
| $g_4$     | 0.6183   | 0       | 0             | 0                                |
| $g_{\omega N}$ | 12.6239  | 6.671   | 6.671         | 6.671                            |
| $g_{\sigma N}$ | 10.0289  | 7.8449  | 7.8449        | 7.8449                           |
| $g_{\rho N}$ | 9.2644   | 8.9     | 9.5           | 9.5                              |
| $g_{\delta N}$ | 0        | 0       | 3.1           | 3.1                              |
| $g_{\sigma^* N}$ | 0        | 0       | 0             | 0                                |
| $g_{\phi N}$ | 0        | 0       | 0             | 0                                |
| $\Lambda_v$ | 0        | 0       | 0             | 0.008                            |
| $\Lambda_4$ | 0        | 0       | 0             | 0.001                            |
| $\zeta$   | 0        | 0       | 0             | 0.5                              |

Table 3
Hyperon-meson couplings

| Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|
| $g_{\Lambda \sigma}$ | 0.5207 $g_{\sigma N}$ | $g_{\Lambda \sigma^*}$ | 0.5815 $g_{\sigma N}$ |
| $g_{\Lambda \rho}$ | 0 | $g_{\Lambda \delta}$ | 0 |
| $g_{\Lambda \omega}$ | $\frac{2}{3} g_{\omega N}$ | $g_{\Lambda \phi}$ | $-\frac{\sqrt{2}}{3} g_{\omega N}$ |
| $g_{\Sigma \sigma}$ | 0.1565 $g_{\sigma N}$ | $g_{\Sigma \sigma^*}$ | 0 |
| $g_{\Sigma \rho}$ | 2 $g_{\rho N}$ | $g_{\Sigma \delta}$ | $2 g_{\delta N}$ |
| $g_{\Sigma \omega}$ | $\frac{2}{3} g_{\omega N}$ | $g_{\Sigma \phi}$ | $-\frac{\sqrt{2}}{3} g_{\omega N}$ |
| $g_{\Xi \sigma}$ | 0.2786 $g_{\sigma N}$ | $g_{\Xi \sigma^*}$ | 0.5815 $g_{\sigma N}$ |
| $g_{\Xi \rho}$ | $g_{\rho N}$ | $g_{\Xi \delta}$ | $g_{\delta N}$ |
| $g_{\Xi \omega}$ | $\frac{1}{3} g_{\omega N}$ | $g_{\Xi \phi}$ | $-\frac{\sqrt{3}}{3} g_{\omega N}$ |
As it is usually assumed in quantum hadrodynamics the mean field approximation is adopted and for the ground state of homogeneous infinite matter quantum fields operators are replaced by their classical expectation values. Thus one can separated mesonic fields into classical mean field values and quantum fluctuations which are not included in the ground state:

\[
\begin{align*}
\sigma &= \overline{\sigma} + s \\
\sigma^* &= \overline{\sigma^*} + s^* \\
\delta^a &= \overline{\delta^a} + d^a \\
\phi_\mu &= \overline{\phi_\mu} + f_0 \delta_{\mu 0} \\
\omega_\mu &= \overline{\omega_\mu} + w_0 \delta_{\mu 0} \\
b^a_\mu &= \overline{b^a_\mu} + r_0 \delta_{\mu 0} \delta^3 a
\end{align*}
\]

The derivative terms are neglected and if one assume homogenous and isotropic infinite matter only time-like components of the vector mesons will survive. The field equations derived from the Lagrange function at the mean field level are

\[
\begin{align*}
m^2_s s + g_3 s^2 + g_4 s^2 - 2(g_\rho B g_s B)^2 A r_0^2 s &= \sum_B g_s B M^2_{\text{eff}, k_F} S(M_{\text{eff}, B}, k_{F, B}) \\
m^2_\omega w_0 + c^3 w_0^3 + 2(g_\rho B g_\omega B)^2 A r_0^2 w_0 &= \sum_B g_\omega B n_B \\
m^2_r r_0 + 2(g_\rho B g_s B)^2 A r_0 s^2 + 2(g_\rho B g_\omega B)^2 A v r_0 w_0^2 &= \sum_B g_\rho B I_3 B n_B \\
m^2_d^3 &= \sum_B g_\delta B I_3 B S(M_{\text{eff}, B}, k_{F, B}) \\
m^2_{\sigma^*} s^* &= \sum_B g_{\sigma^*} B M^2_{\text{eff}, k_F} S(M_{\text{eff}, B}, k_{F, B}) \\
m^2_\phi f_0 &= \sum_B g_\phi B n_B.
\end{align*}
\]

The function \( S(M_{\text{eff}, B}, k_{F, B}) \) is expressed with the use of an integral

\[
S(M_{\text{eff}, B}, k_{F, B}) = \frac{2 J_B + 1}{2 \pi^2} \int_0^{k_{F, B}} \frac{M_{B, \text{eff}}}{\sqrt{k^2 + M_{B, \text{eff}}}} k^2 dk
\]

where \( J_B \) and \( I_3 B \) are the spin and isospin projection of baryon \( B \), \( k_{F, B} \) is the Fermi momentum of species \( B \), \( n_B \) denote the baryon number density. The Dirac equation for baryons that is obtained from the Lagrangian function has the following form

\[
(\gamma^\mu \partial_\mu - M_{B, \text{eff}} - g_\omega B \gamma^0 \omega_0 - g_\phi B \gamma^0 f_0 - \frac{1}{2} g_\rho B \gamma^0 r^3 r_0) \psi = 0
\]

with \( M_{B, \text{eff}} \) being the effective nucleon mass generated by the nucleon and scalar fields interactions and is defined as

\[
M_{B, \text{eff}} = \frac{M_B}{1 + (g_s B s + g_{\sigma^*} B s^* + I_3 B g_\delta B d)/M_B}
\]
III. THE EQUILIBRIUM CONDITIONS AND COMPOSITION OF MATTER

In the high density regime in neutron star interiors when the Fermi energy of nucleons exceeds the hyperon masses additional hadronic states are produced. The onset of hyperon formation depends on the attractive hyperon-nucleon interaction. The higher the density the more various hadronic species are expected to populate. They can be formed both in leptonic and baryonic processes. In the last one the strong interaction process such as

\[ n + n \rightarrow n + \Lambda \]  

proceeds. There are other relevant strong reactions that establish the hadron population in neutron star matter e.g.:

\[ \Lambda + n \rightarrow \Sigma^- + p \quad \Lambda + \Lambda \rightarrow \Xi^- + p \]  

The comparison of weak interaction time scales \((10^{-10} \text{ s})\) and the time scale connected with the lifetime of a relevant star indicate that there is a difference between the matter in high energy collisions which is constrained by the isospin symmetry and the strangeness conservation whereas neutron star matter by the charge neutrality and the generalized \(\beta\)-equilibrium. Thus realistic neutron star models describe electrically neutral high density matter with no strangeness conservation being in \(\beta\) equilibrium. The last condition implies the presence of leptons. Mathematically it is expressed by adding the Lagrangian of free leptons to the lagrangian function

\[ L_L = \sum_{l=e,\mu} \overline{\psi}_l(i\gamma^\mu \partial_\mu - m_l)\psi_l. \]  

Neutrinos are neglected here since they leak out from a neutron star, whose energy diminishes at the same time. After electron chemical potential \(\mu_e\) reaches the value equal to the muon mass, muons start to appear. Equilibrium with respect to the reaction

\[ e^- \leftrightarrow \mu^- + \nu_e + \bar{\nu}_\mu \]  

is assured when \(\mu_\mu = \mu_e\) (setting \(\mu_\nu_e = \mu_\bar{\nu}_\mu = 0\)). The appearance of muons reduces number of electrons and affects also the proton fraction. The introduction of the asymmetry parameter \(f_a\) which describes the relative neutron excess defined as

\[ f_a = \frac{n_n - n_p}{n_N} \]  

allows to study the symmetry properties of the system. The equilibrium conditions between baryonic and leptonic species which are present in the neutron star matter lead to the following relations between their chemical potentials and constrain the species fraction in the star interior

\[ \begin{align*} 
\mu_p &= \mu_{\Sigma^+} = \mu_n - \mu_e \\
\mu_\Lambda &= \mu_{\Sigma^0} = \mu_{\Xi^-} = \mu_n + \mu_e \\
\mu_\mu &= \mu_e 
\end{align*} \]  

where the baryonic chemical potential is given by the relation

\[ \mu_i = \sqrt{k_{F,B}^2 + M_{B,s}^2} + g_{\omega B}\omega_0 + g_{\phi B}\phi_0 + I_{3B}g_{\rho B}r_0 \]  

Similarly to the asymmetry parameter \(f_a\) a parameter \(f_s\) which specify the strangeness content in the system and is strictly connected with the appearance of particular hyperon species in the model has been introduced.

\[ f_s = \frac{n_\Lambda + n_\Sigma + 2n_\Xi}{n_\Lambda + n_\Sigma + n_\Xi + n_N}. \]  

The RMF theory as an effective one requires the knowledge of coupling constants. There are several selected parameterizations of the theory strictly connected with the assumption that the exchange of scalar, isoscalar-vector, vector mesons and two hidden-strangeness scalar and vector mesons are responsible for interactions between specific constituents of the matter. Particular sets of coupling constants are associated with the description of nucleon-nucleon, hyperon-nucleon and hyperon-hyperon interactions and are indispensable for the the construction of the equation of state which in turn is applied for determining a neutron star properties. The enlarged Walecka model uses the TM1 parameter set (Table 2)[24]. More realistic description of a neutron star requires taking into consideration not only the interior region of a neutron star but also remaining layers, namely the inner and outer crust and surface layers.
The composite EOS constructed by adding Bonn \cite{14} and Negele-Vautherin (NV) \cite{15} equations of state (describing the inner crust) to the TM1 one allows to calculate the neutron star structure for the entire neutron star density span. The parameters describing the nucleon-nucleon interactions are created in order to reproduced the properties of the symmetric nuclear matter at saturation such as the binding energy, symmetry energy and the incompressibility. Other parameters are related to the hyperon-nucleon interactions. The scalar meson coupling to hyperons can be calculated from the potential depth of the hyperon in the saturated nuclear matter

$$U_N^\Sigma(\rho_0) = -g_{\Sigma N}^\Sigma + g_{\Sigma Y} \varphi. \quad (31)$$

The vector coupling constants for hyperons are determined from $SU(3)$ symmetry as \cite{19}

$$\frac{1}{2} g_{\rho \Sigma} = \frac{1}{2} g_{\rho N} = \frac{1}{3} g_{\rho N}$$

$$\frac{1}{2} g_{\rho \Sigma} = g_{\rho \Sigma} = g_{\rho \Sigma} = \frac{2 \sqrt{2}}{3} g_{\rho N}. \quad (32)$$

The single $\Lambda$ potential in nuclear matter is well determined as \cite{22} $-U_N^\Lambda(\rho_0) = 27 - 28$ MeV. Recent analysis of atomic data \cite{24, 25} indicate for repulsive $\Sigma$ potential in the interior of nuclei $U_N^\Sigma(\rho_0) = 30$ MeV thus $\Sigma$ hyperons do not appear in the bulk matter calculations. The interpretation of the $\Xi$ hyperons data gives the value of the potential well depth $-U_N^\Xi(\rho_0) = 18$ MeV. The experimental data concerning hyperon-hyperon \cite{18, 23} interactions are extremely scare. Analysis of events which can be interpreted as the creation of $\Lambda\Lambda$ hypernuclei allows to determine the well depths of hyperon in hyperon matter

$$-U_{hh}(\rho_0) = 40 \text{MeV} \quad (33)$$

Having specified parameters of the model the equation of state can be calculated. This has been done with the use of the energy-momentum tensor $T_{\mu \nu}$ defined as

$$T_{\mu \nu} = 2 \frac{\partial L}{\partial g_{\mu \nu}} - g_{\mu \nu} L. \quad (34)$$

It allows to calculate the pressure $P$ and energy density $\varepsilon$ of the system. The pressure $P$ is related to the statistical average of the trace of the spatial component $T_{ij}$ of the energy-momentum tensor $P = \frac{1}{3} < T_{ii} >$, whereas the energy density $\varepsilon$ equals $< T_{00} >$. Thus the complete form of the equation of state includes contributions coming from meson, fermion and baryon fields and finally one can get the following equations for the energy density $\varepsilon$ and pressure $P$

$$\varepsilon = \frac{1}{2} m^2 p^2 (r_0)^2 + \frac{1}{2} m^2 d^2 + \frac{1}{2} m^2 f_0^2 + \frac{1}{2} m^2 (s^*)^2 \quad (35)$$

$$+ \frac{1}{2} m^2 w_0^2 + \frac{3}{4} c_3 w_0^4 + U(s) + 3 \Lambda \phi \phi_0^2 \phi_0^2 + \Lambda_4 \phi \phi_0^2 \phi_0^2 + \frac{1}{8} \zeta \phi_0^4 + \varepsilon_B + \varepsilon_L$$

with $\varepsilon_B$ and $\varepsilon_L$ given by

$$\varepsilon_B = \sum_B \frac{1}{3 \pi^2} \int^{k_{F,B}}_0 k^2 dk \sqrt{k^2 + M^2_{\epsilon f,B}} \quad (36)$$

$$\varepsilon_L = \sum_B \frac{1}{3 \pi^2} \int^{k_{F,B}}_0 k^2 dk \sqrt{k^2 + m_i^2} \quad (37)$$

$$P = \frac{1}{2} m^2 p^2 + \frac{1}{2} m^2 d^2 + \frac{1}{4} c_3 w_0^4 - \frac{1}{2} m^2 f_0^2 + \frac{1}{2} m^2 (s^*)^2 - U(s) + 3 \Lambda \phi \phi_0^2 \phi_0^2 + \Lambda_4 \phi \phi_0^2 \phi_0^2 + \frac{1}{2} \zeta \phi_0^4 + P_B + P_L \quad (38)$$

$$P_B = \sum_B \frac{1}{3 \pi^2} \int^{k_{F,B}}_0 k^4 dk \sqrt{k^2 + M^2_{\epsilon f,B}} \quad (39)$$
\[ P_L = \sum_i \frac{1}{3\pi^2} \int_0^{k_{F,i}} \frac{k^4 dk}{\sqrt{k^2 + m_i^2}} \]  

(40)

IV. NEUTRON STAR PARAMETERS

Besides other properties of neutron stars the value of their masses and radii are very sensitive to the chosen model of strong interactions which in turn lead to the significant constrains on the form of the equation of state of the neutron star matter. In this paper three different groups of parameters have been applied in order to examined neutron star properties. In all of them the nucleon-hyperon \( \Sigma \) interaction is assumed to be repulsive. The first case marked as set I does not contain \( \delta \) meson, the second one (set II) do include \( \delta \) meson, in the third both \( \delta \) meson and nonlinear vector meson interactions are taken into account (set III). As the neutron star matter is of sizable asymmetry the last case seems to be the most adequate for the complete description of the asymmetric neutron star matter. The main effect of such an extension of the theory becomes evident studying properties of the neutron star matter especially baryon masses splitting and the form of the equation of state. Fig.1 depicts the effective baryon masses obtained for the three mentioned above cases as functions of the baryon number density \( n_B \). For the second and third cases there are noticeable in-medium baryon masses splitting for each isomultiplet. This effect reaches the maximum value for the sign of the third component of particular baryon isospin the effective masses and decreases masses of neutron, \( \Sigma^- \) and \( \Xi^- \) as functions of baryon number density. Depending on the sign of the third component of particular baryon isospin the \( \delta \) meson interaction increases the proton and \( \Xi^0 \) effective masses and decreases masses of neutron, \( \Sigma^- \) and \( \Xi^- \). Contrary to this situation, for parameters marked as the set I, the baryonic masses for the given isomultiplet remain degenerate as it is shown in the left panel of Fig.1(a). Having obtained the effective baryonic masses one can compare them with those obtained in the Walecka model. In the original Walecka approach the nucleon effective mass rapidly diminishes its value passing through zero and even become negative for higher densities. Throughout the effective baryonic masses the asymmetry of the system alters baryon chemical potentials what realizes in characteristic modification of the appearance, abundance and distributions of the individual flavors. This is evident comparing the results obtained for the three mentioned above parameter sets. Fig.2 presents fractions of particular baryon species \( Y_B \) as functions of baryon number density \( n_B \). Starting the analysis of these graphs from low and moderate densities it is evident that at very low density neutrons are the most abundant baryon and a star resembles the pure neutron one. For higher densities protons and electrons and then muons emerge. Fig.2 points that the first strange baryon that appears at \( \rho = 2.5 \times \rho_0 \) is the \( \Lambda \), it is followed by \( \Xi^- \) and \( \Xi^0 \). In the presence of \( \delta \) meson and in the case when nonlinear vector meson interactions are included the sequence of appearance of hyperons is the same as in the first case, however shifted towards higher densities. Due to the repulsive potential of \( \Sigma \) hyperons their onset points are possible at very high densities which are not relevant for neutron stars. Taking into account results obtained when the attractive nucleon-hyperon \( \Sigma \) potential is assumed \( \Sigma^0 \) appear as the first strange baryon and than \( \Lambda \) and \( \Sigma^- \). \( \Xi^0 \) emerge at the density above \( 7 \rho_0 \), whereas other hyperons at even higher densities. The appearance of charged hyperons permits the lower lepton contents and thus charge neutrality tends to be guaranteed without lepton contribution. This kind of deleptonization in the case of the set II and III parameterizations similarly to the baryon distributions takes place at higher densities. Larger effective baryon masses cause the shift of the given hyperon onset point especially for the charged ones towards higher densities. The emergence of \( \Xi^- \) hyperons through the condition of charge neutrality affects the lepton fraction and causes a drop in their contents. The proton fraction \( Y_p \) has a crucial role in neutron stars cooling history. At a certain critical value of \( Y_p \) the direct URCA processes for neutrino emissions are allowed. The condition which has to be satisfied for the URCA process is given by the relation between Fermi momenta of nucleons and electrons

\[ k_{Fp} + k_{Fe} \geq k_{Fn} \]

(41)

The threshold proton fraction for this process is about 0.11 in the case when the neutron star matter consists only of nucleons and electrons. Inclusion of muons increases \( Y_p \) to the value \( \sim 0.14 \). The proton fractions which are obtained for the presented parameter groups are shown in Fig.3. The third parameter set gives the lower value of \( Y_p \).
FIG. 1: Effective baryon masses as a function of baryon number density $n_B$. Fig.1a shows effective baryon masses obtained in the model without $\delta$ meson (the parameter set I). In Fig.1b and Fig.1c the effective baryon masses are splitted due to the presence of $\delta$ meson. This effect is even more significant in the case of more asymmetric matter (set III).
FIG. 2: The equilibrium compositions as functions of baryon number density $n_B$ for given parameter sets. Fig. 2a shows results obtained for the parameter set I. Fig. 2a and 2b depict equilibrium compositions of matter which contain $\delta$ meson (set II) and additional vector meson interactions (set III).
FIG. 3: Proton fractions $Y_p$, as functions of baryon number density $n_B$. Solid line represents the proton fraction for the most asymmetric matter. Dotted and dashed lines are obtained for parameter set I and II respectively.
Baryon distributions are strictly connected with the behavior of meson fields. The values of scalar and vector meson fields influence the onset points of individual hyperon species. Fig. 4 shows meson fields as functions of the baryon number density.

The calculated form of the equation of state (EOS) determines the physical state and composition of matter at high densities and is presented in Fig. 5. The relative hadron-lepton composition calculated for all parameter groups can be analyzed through the density dependence of the asymmetry parameter $f_a$ and the parameter $f_s$ which is connected with the strangeness contents. Fig. 6 presents both parameters as functions of the baryon number density $n_B$. As the density increases the asymmetry parameter decreases. The third parameter group gives the higher value of the asymmetry parameter. For parameter sets I and II the asymmetry is comparable. The strangeness contents increases with the density. The inclusion of $\delta$ meson and nonlinear vector meson interactions results in the lowest hyperon contents.

The obtained form of the equation of state serves as an input to the Oppenheimer-Volkoff equations and determines the structure of spherically symmetric stars.

\[
\frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \left(1 + \frac{P(r)}{\rho(r)} \right) (1 + \frac{4\pi r^3 P(r)}{m(r)}) \frac{1 - \frac{2Gm(r)}{r}}{1 - 2\frac{Gm(r)}{r}},
\]

\[
\frac{dm(r)}{dr} = 4\pi r^2 \rho(r).
\]

Numerical solutions of these equations allow to construct the mass-radius relations. These relations for the chosen
FIG. 5: The equation of state (EOS). Straight, dashed and dotted lines represent models constructed with the use of parameter sets I, II and III respectively.

FIG. 6: The asymmetry parameter $f_a$ and the strangeness parameter $f_s$ as functions of baryon number density $n_B$. 
form of the equations of state are presented in Fig. 7. Arrows represent points at which hyperons emerge whereas asterisks the configurations for which URCA processes start to proceed. For the parameter set III these two marks are in the same position. This figure also shows that $\delta$ meson itself and the additional nonlinear vector meson interactions change the maximal mass but not in a significant way. However, there are visible differences in the maximal radius configurations. For comparison, the mass-radius relation for a quark star is enclosed. In Fig. 8 the obtained neutron star masses as functions of central densities are presented.
FIG. 8: The neutron star masses as functions of the central density of neutron stars.
White dwarfs and neutron stars are purely gravitationally bound compact stars. The gravitational binding energy of a relativistic star is defined as a difference between its gravitational and baryon masses.

\[ E_{b,g} = (M_p - m(R))c^2 \]  

(43)

where

\[ M_p = 4\pi \int_0^R dr r^2 (1 - \frac{2Gm(r)}{c^2r})^{-\frac{3}{2}} \rho(r) \]  

(44)

Of considerable relevance is the numerical solution of the above equation for the selected EOS. Results are shown in Fig. 9. For the sake of completeness calculations have been done not only for the presented parameter sets but also for the original TM1 parameter set and for a quark star. From this figure it is evident that at moderate densities the standard neutron star model calculated with the use of the TM1 parameter set is the most energetically favorable configuration. At higher densities this standard RMF approach becomes unsatisfactory. There is room for the improvement of this theory by the introduction of other degrees of freedom. The comparison of the models which contain hyperons with the Walecka one is performed. At high densities the models with hyperons are more favorable than the model calculated with the use of the TM1 parameter set. This is directly connected with the value of effective baryon masses, obtained with the use of the TM1 parameters, which diminish considerably their values which after passing through zero can take even the negative value. This can be interpreted as a break down of the theory and has its confirmation in the form of the gravitational binding energy. Fig. 9 depicts the gravitational binding energies as functions of the density. From this figure it is evident that the models without hyperons are energetically favorable at the density range relevant for neutron stars. The situation changes for higher densities and now models with hyperons become energetically favorable. There are theoretical suggestions about the existence of a quark star but Fig. 9 exhibits that a quark star configuration should appear at very high densities.

The gained solutions of the structure equations allow us to carry out a similar analysis of the onset point, abundance and distributions of the individual hadron and lepton species but now as functions of the star radius \( R \). Comparing results obtained for the three presented above cases (parameter set I, II and III) one can come to the conclusion that in all the cases the assumption of the repulsive \( \Sigma \) interaction shifts the onset point of \( \Sigma \) hyperons to very high densities and they do not appear in neutron star interiors calculated in this model. Two characteristic configurations have
FIG. 10: The equilibrium compositions for the maximum mass configuration as a function of the star radius. Fig.10a is constructed for the parameter set I, Fig.10b for the parameter set II. Panel (c) represents results obtained for the parameter set III.

been considered. Namely the one connected with the maximum mass configuration and the second with that of the maximum radius. The very compact hyperon core which emerges in the interior of the maximum mass configuration consist of $\Xi^0$, $\Xi^-$ and $\Lambda$ hyperons only for the second parameter set. The hyperon population is reduced to $\Lambda$ and $\Xi^-$ for the parameter sets I and III. This can be seen in Fig.10 and confirm by the behavior of meson fields in the neutron star. Especially by the hidden strange $\sigma^*$ and $\phi$ mesons which appear in the vicinity of the neutron star center (Fig.12 and Fig.13). For the maximum radius configuration for all groups of parameters hyperons do not emerge in the interior of neutron stars.
FIG. 11: The equilibrium compositions for the maximum radius configuration as a function of the star radius. Panel (a) is for the parameter set I whereas (b) and (c) for parameter sets II and III.
FIG. 12: The behavior of meson fields in the configuration with maximum mass as functions of the star radius. Panels (a), (b) and (c) are for parameter sets I, II and III respectively.
FIG. 13: The meson fields in the neutron star with maximum radius as a function of star radius. The panel (a) is for parameter set I, the panel (b) for parameter set II the panel (c) for parameter set III.

V. SUMMARY AND CONCLUSIONS

In this paper the complete form of the equation of state of hyperon matter has been obtained with the use of the derivative coupling model in the framework of an extended RMF theory which besides hyperons and leptons includes the extended meson sector with additional $\delta$ meson and hidden strange mesons $\sigma^*$ and $\phi$. The model considered is also supplemented with nonlinear vector meson interactions. This enlargement alters the symmetry properties of neutron star matter and through this neutron stars parameters. The value of baryon effective masses depend on the scalar meson condensates and at high densities when hyperon species appear the possibility of negative nucleon masses emerges. The derivative coupling model allows to avoid this difficulty reproducing reasonable value of baryon effective masses for densities relevant for neutron stars. The inclusion of $\delta$ meson and nonlinear vector meson interactions influences the chemical composition of a neutron star. This is especially evident comparing the effective baryon masses in the density span $(3 - 4) \times \rho_0$ and equilibrium compositions of the star. For the third group of parameters the higher value of asymmetry has been obtained. This changes the properties of a neutron star diminishing the hyperon core extent. The asymmetry of the system also influence the star radius, for the third group of parameters the one can obtain the lower value of the star radius. There is also possible a configuration representing a star with lower densities (a maximum radius configuration) which excluded the existence of a hyperon core. The model considered excludes a large hyperon fraction which can be connected with thermal properties of a hot neutron star. As the most populated strange baryon is the $\Lambda$ hyperon it does not reduce significantly number of leptons in the star interior and thus the models calculated with the use of the parameter set I and II do not exclude rapid cooling rate of
the star. This is even more evident analyzing the proton fractions obtained for all parameter groups. The equilibrium proton fraction is also determined by the nuclear symmetry energy. The parameter sets I and II permit higher values of proton fraction which is indispensable for URCA processes to proceed. However, the equilibrium proton fraction $Y_p$ is significantly reduced for the third group of parameter.

Analyzing the gravitational binding energy one can come to the conclusion that configurations with hyperons are energetically favorable than the one obtained with the use of TM1 parameter set for higher densities.

All assumption which have been made namely: the derivative coupling model being connected with the higher effective baryon masses, the inclusion of $\delta$ meson and nonlinear vector meson interactions, and the repulsive nucleon-hyperon $\Sigma$ interaction lead to the neutron star model with the value of maximum mass close to $1.5 M_{\odot}$ with the reduced value of proton fraction and very compact hyperon core. The calculation of the quark matter equation of state allows to construct the mass-radius relation for the quark star. Comparing gravitational binding energies one can come to the conclusion that the addition of hyperons to the model shifts the stable hyperon matter configuration towards higher densities even to the density range which is relevant for a quark star and at the same time makes the existence of a pure quark star more problematic.

[1] Weber F. *Pulsars as Astrophysical Laboratories for Nuclear and Particle Physics*, IOP Publishing, Philadelphia, 1999
[2] Glendenning N.K. 1985 *Astrophys.J.*, 293 470; Also see in *Compact Stars* by N. K Glendenning Springer-Verlag, New York, 1997
[3] Bednarek I., Manka R. 2001 *Int. Journal Mod. Phys.* D10, 607
[4] Lattimer J.M., Pethick C.J., Prakash M., Haensel P. 1991, *Phys. Rev. Lett* 66, 2701
[5] Guichon P.A.M., *Phys. Lett.* B 200 235; Guichon P.A.M., Saito K., Rodionov E., Thomas A.W. *Nucl. Phys.* A 601 394; Saito K., Tsushima K., Thomas A.W. *Phys. Rev. C* 55 2637;
[6] Manka R., Bednarek I. 2002, *New Journal of Physics* 4, 1
[7] Serot B.D., Walecka J.D. 1986 *Adv. Nucl. Phys.* 16 1; 1997 *Int. J. Mod. Phys.* E6 515
[8] Boguta J. and Bodmer A.R. 1977 *Nucl. Phys.* A292 413 *Int. J. Mod. Phys.* E6 515
[9] Bodmer A.R. 1991 *Nucl. Phys.* A526 703
[10] Gmuca S.J. 1991 *J. Phys.* G17 1115
[11] Kubis S., Kutschera M., 1997, *Phys. Lett.* B399 191
[12] Zimanyi J., Moszkowski S.A., 1990, *Phys. Rev.* C42 1416
[13] Kiel C., Hofmann F., Lenske H., 2000, *Phys. Rev.* C61 064309
[14] Machleidt R., Holinde K., Elster C., 1987, *Phys. Rep.* 149 1
[15] Negele J.W., Vautherin D., 1973, *Nucl. Phys.* A207 298
[16] Aguirre R., De Paoli A.L. 2002 *Eur. Phys. J.* A13, 501
[17] Schaffner J. Mishustin I.N. 1996 *Phys. Rev.* C53, 1416
[18] Millener D.J., Dover C.B., Gal A. 1988 *Phys. Rev.* C38, 2700
[19] Dover C.B., Gal A. 1983 *Ann. Phys.* 146, 209
[20] Bart S. et al.,1999, *Phys. Rev. Lett.* 83, 5238
[21] Mares J., Friedman E., Gal A., Jennings B.K. 1995, *Nucl. Phys.* A584, 311
[22] Batty C.J., Friedman E., Gal A., 1994 *Phys. Lett.* 335, 273; 1994, *Prog. Theo. Phys. Suppl.* 117 145
[23] Schaffner J. Dover C.B., Gal A., Greiner C., Millner D.J., Stocker H. 1994 *Ann. Phys.*, 235, 35 C55, 540; P.-G.Reinhard, Rufa M., Maruhn J., Greiner W. and Friedrich J. 1986 Z. Phys. A - Atomic Nuclei 323 13
[24] Sugahara Y. and Toki H. 1994 *Prog. Theo. Phys* 92 803 *Nucl. Phys.* A616 498c
[25] Schaffner-Bielich J., Gal A. 1999 *Phys. Rev.* C62, 034311
$E_b (M/s) \quad \rho_c (10^{14} \text{g/cm}^3)$

- ZM1 + hiperons + $\delta + n$
- ZM1 + hiperons + $\delta$
- ZM1 + hiperons
- Quark star
$\mu_e$ and $\mu_n$ as a function of $R$ (km).
$ZM1 + \text{hiperons} + \delta + \text{nonlinear terms}$

$ZM1 + \text{hiperons} + \delta$

$ZM1 + \text{hiperons}$

$f_s$ vs $\rho_c \left(10^{14} \text{ g/cm}^3\right)$