Quartic Anharmonicity in Different Spatial Dimensions

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Abstract

A path-integral method effective beyond the perturbation expansion approach is suggested to consider the quartic anharmonicity in different spatial dimensions. Due to an optimal representation of the partition function, the leading term has already taken into account the correct strong-coupling behaviour. In the simplest cases of zero and one dimension we have obtained reasonable results in a simple way. Then, this technique is applied to the superrenormalizable scalar theory \( \phi_4^2 \) in two dimensions. This results in an accurate estimation of the ground-state energy that provides exact weak- and strong-coupling behaviour already in the leading-order approximation. The next-to-leading terms give rise in insignificant corrections.

I. INTRODUCTION

The anharmonic oscillator plays an important role by providing a theoretical laboratory for the examination of new calculational schemes and approximation techniques (see, e.g. [1,2]) in quantum mechanics. Besides, it describes more or less real self interaction disturbing the idealized picture of the free harmonic oscillators. The most known version is the quartic oscillator [3,4]. On the other hand, scalar fields play a fundamental role in the unified theories of strong, electromagnetic and weak interactions. The mechanism of symmetry rearrangement specific for scalar fields occurs in realistic scalar field theories [5]. At the same time, the mathematical structure of scalar field theory is simpler than that of the vector and spinor cases. A native and well-known generalization of the quartic oscillator is the model of scalar self-interaction \( \phi^4 \) in quantum field theory. Particularly, the quantum mechanical quartic oscillator with a Hamiltonian

\[
H = \frac{1}{2}(p^2 + m^2 q^2 + g^2 q^4)
\]

(1)

can be considered [3] a theory of a scalar field in one-dimensional \((q(t), \ t \in \mathbb{R}^1)\) Euclidean space-time described by the Lagrangian:

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\[ L[q, \dot{q}] = \frac{1}{2} \left[ q^2(t) - m^2 q^2(t) - g^2 q^4(t) \right]. \tag{2} \]

Thus, the scalar field model (2) represents an attractive, simple but nontrivial object for research. The problem of finding the energy spectrum \( \{ E_n \} \) and the proper wave functions \( \{ \Phi_n \} \) of the Hamiltonian can be solved by using the partition function and path-integral formalism. Note, the corresponding Schrödinger equation cannot be solved explicitly for general case, except the weak- and strong-coupling limits.

Further extension of the model (2) can be done by considering the quartic self interaction \( \phi^4(x) \) in higher spatial dimensions \( x \in \mathbb{R}^d, \ d \geq 1 \). Earlier investigations of the triviality problem have shown that for dimensions \( d > 4 \) the scalar theory of self interaction behaved either unstable or trivial. Much interest is attracted to the cases \( d = 2 \) and \( d = 3 \) demonstrating nontrivial phase restructure.

Nowadays, the anharmonic oscillator is a system well understood in both perturbative and non-perturbative aspects [3,4].

Typically, the problem is treated by using various perturbative expansion scheme. Particularly, the conventional loop expansion [8] gives asymptotic Rayleigh-Schrödinger series for energy levels which numerically converges only for \( g \ll 1 \). The loop expansion for composite operators [4] and optimized expansion [10,11] provide better convergence for numerical series; however, in this case the approximation becomes worse as \( g \) increases. To obtain a reasonable result, one needs in various resummation techniques for these divergent series [12]. Particularly, the Heaviside transformation of the mass parameter [13] and a modified Laplace transformation method [14] can result in good approximations to the ground state energy, but the perturbation series depends on cut-off parameter \( x^* \) and converges very slowly [14]. The Pàde approximants and the Borel transform can be used to summarize the formally diverging series [15]. But, rigorous proofs of the convergence can be made only in particular cases [16] and the convergence is nonacceptably slow.

Going beyond the perturbation formalism, a variational interpolational method [17] and the strong-coupling expansion [18] are used to estimate the energy spectrum of the AHO. Also, by applying the renorm-group equation, one obtains a resummed perturbation series [13]. The renorm-group methods may improve the perturbative expansions [20,21]. However, the agreement with the WKB result becomes worse in the higher orders than the fourth at which the agreement is the best. Recently, it is have shown that multiple-scale [22] and reductive perturbation theory [12] can be successfully applied to the quantum AHO to construct the asymptotic behaviour of the wave function for large \( x \). A similar method had been proposed in [23]. But, it is not trivial to identify the secular terms for the wave functions which can be made to vanish at a renormalization point for general case.

Within the method of orthogonal polynomials it has been established [25] that the solutions of the bound state equation for the quartic anharmonic oscillator are related to a sequence where the potential is a even polynomial of degree six (a quasi-exactly solvable model). A regular iterative method for the ground state energy of quartic oscillator as an analytic function of coupling constant [23] is formulated. But this method introduces a PI averaging of the complicated many-point potential of the logarithmic type that makes it hard to obtain explicit solutions. By using an extension of quasi-exactly solvable models [20] to Bose systems, one obtains (in some particular cases) simple exact expressions for several energy levels of an anharmonic Bose oscillator [27] beyond the exploit perturbation theory. A relationship between the \( x^{2M} \) AHOs and the \( A_{2M-1} \) TBA systems have been conjectured [28] by using an alternative integral expression for the spectral determinants of the quartic oscillator.
Therefore, it is important to build a method which is effective in the strong-coupling. On the other hand, it has to work even in the case of renormalizable theories (e.g., $\phi^4_2(x)$) with formally divergent or complex functionals, unlike the variational methods. Besides, the desired method should be relatively plain and has to catch the main strong-coupling contribution within the leading-order approximation. Below we represent a path-integral technique obeying these features.

In our earlier works [29,30] the problem of strong coupling regime and phase structure of the $\phi^4_2$ quantum field theory has been investigated. We have found that an approximate solution of the problem can be obtained by combining the canonical transformations of field variables and normal ordering of the creation and annihilation operators. In the lowest approximation, the solution obtained there is identical to the result of the variational method of the Gaussian effective potential [31]. Meanwhile, in contrast with the variational method our suggested technique ensures a canonical structure of the theory and, hence, provides calculation of perturbation corrections over effective coupling constants which can also be used to control an accuracy of approximation. Later on, the method was generalized and applied to renormalizable quantum field theories, path integrals and quantum mechanics of bound states [32].

In the present paper we demonstrate a significant modification of our method suitable to investigate the $\phi^4$ model effectively even for large coupling. Below we consider the case of low dimensions, namely, $d = 0, 1, 2$ to avoid nonprincipal but lengthy renormalization procedures occurring for $d = 4-\epsilon$, $\epsilon \to 0$. In fact, an extension of our method from $d = 2$ to $d = 4-\epsilon$ does not meet any principal troubles, because only renormalization of the mass is sufficient to remove all divergencies in both models. Therefore, upgrading the dimensionality to $d = 4-\epsilon$ will affect only some coefficients and factors, not changing the very form of our final expressions. However, we should note that the consideration of the theory $\phi^4_4$ requires a serious modification of our method because one has to workout first an effective technique to perform the renormalization of the coupling constant.

Our basic idea is to find an optimal representation of the partition function (e.g., see (17)) in which the main contribution is taken into account correctly for large $g^2$ already in the lowest approximation. For this purpose we will use the Gaussian representation of the initial quartic interaction in the form of an integrable quadratic one by involving an auxiliary integration as follows:

$$
\int \delta \phi \exp \left\{ -\frac{1}{2} (gD_0^{-1}q) - \frac{g^2}{2} q^4 \right\} \propto \int A \exp \left\{ -\frac{A^2}{2} - \frac{1}{2} \text{tr} \ln (1 + 2igqD_0) \right\}.
$$

Then, the inner path integral over $\phi$ can be taken out explicitly and the remaining integral over auxiliary variable (field) $A$ can be evaluated effectively, in particular, by isolating the total Gaussian contribution.

**II. A NON-GAUSSIAN INTEGRAL: $D = 0$**

Below we demonstrate the basic idea of our approach on the simplest case of a plain non-Gaussian integral.

First, we introduce a notation indicating an averaging with respect to a normalized Gaussian measure as follows:
\[ \langle \bullet \rangle_x = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2}(\bullet), \quad \langle 1 \rangle_x = 1. \] 

Let us define a "partition function":

\[ Z[g, J] = \left\langle \exp \left\{ -\frac{g^2}{2} x^4 + ixJ \right\} \right\rangle_x. \] 

Then, we can consider the zero-dimensional analogues of the Quantum Mechanical ground-state energy and Green function as follows:

\[ \exp\{-E(g)\} = Z[g, 0], \quad G(g) = -\frac{1}{Z[g, 0]} \frac{d^2Z[g, J]}{dJ^2}\bigg|_{J=0}. \] 

As is known, these integrals are not solvable exactly. According to our approach we write

\[ Z[g, J] = \left\langle \int_{-\infty}^{\infty} dx \exp \left\{ -\frac{x^2}{2} - igx^2A + ixJ \right\} \right\rangle_A \]
\[ = \left\langle \exp \left\{ -\frac{1}{2} \left[ \ln (1 + 2igA) + \frac{J^2}{1 + 2igA} \right] \right\} \right\rangle_A. \] 

To build the optimal oscillator basis, we have to eliminate the linear \( A \) terms out of the exponent as follows:

\[-\frac{aA}{2ig} - \frac{igA}{1+a} = 0 \quad \implies \quad g^2 = a(1+a), \quad h = \frac{g^2}{2} = \frac{a(1+a)}{4}. \] 

After a shift \( A \rightarrow A + a/2ig \) with a parameter \( a > 0 \) we obtain

\[ Z[g, J] = \left\langle \exp \left\{ -\frac{1}{2} \left[ \frac{aA}{ig} - \frac{a^2}{4g^2} + \ln (1 + a + 2igA) + \frac{J^2}{1 + a + 2igA} \right] \right\} \right\rangle_A. \] 

First we rewrite the logarithm in the exponent as

\[ \ln (1 + a + 2igA) = \ln (1 + a) + \frac{2igA}{1 + a} + \frac{2g^2A^2}{(1 + a)^2} + \ln_2 (1 + a + 2igA), \] 

where

\[ \ln_2 (1 + z) = \ln (1 + z) - z + \frac{z^2}{2}, \quad \forall \text{Re } z \geq -1. \] 

Define a function

\[ W(s) = -\frac{1}{2} \ln_2 (1 + is\lambda), \quad \lambda = \sqrt{\frac{2a}{1+2a}} < 1. \] 

Then, we obtain

\[ Z[g, J] = \exp \left\{ \frac{a^2}{8g^2} - \frac{\ln(1+a)}{2} \right\} \cdot \left\langle \exp \left\{ W(s) - \frac{J^2}{2(1+a)(1+is\lambda)} \right\} \right\rangle_s. \]
Accordingly,
\[
e^{-E(g)} = e^{-E_0(g)} \langle e^{W(s)} \rangle_s, \quad E_0(g) = -\frac{a}{4(1+a)} + \frac{\ln(1+a)}{2},
\]

\[
G(g) = \frac{1}{1+a} \cdot \frac{\frac{1}{1+is\lambda}e^{W(s)}}{\langle e^{W(s)} \rangle_s}.
\]

Systematically expanding the exponentials in (14) we estimate the "ground-state energy"

\[
E_N(g) = E_0(g) + \sum_{j=1}^{N} \Delta E_j(g), \quad \Delta E_1(g) = -\langle W(s) \rangle_s
\]

and the "Green function"

\[
G_N(g) = G_0(g) + \sum_{j=1}^{N} \Delta G_j(g), \quad G_0(g) = \frac{1}{1+a} \langle \frac{1}{1+is\lambda} \rangle_s,
\]

\[
\Delta G_1(g) = \frac{1}{1+a} \left\{ \frac{\langle W(s) \rangle_s}{1+is\lambda} - \frac{\langle W(s) \rangle_s}{1+is\lambda} \langle W(s) \rangle_s \right\},
\]

where \(E_0(g)\) and \(G_0(g)\) are the leading-order approximations.

In Table 1 we represent our estimates due to (15) and (16) compared with the exact numerical solutions to (3). We see that our approach describes well these functions even within the zero-order approximation. The next-to-leading corrections systematically improve the obtained results approaching close to the exact solutions. Our technique works well at least in zero dimension.

### III. ANHARMONIC OSCILLATOR: \(D = 1\)

The most known example of the quartic interaction is the anharmonic oscillator \(g^2q^4/2\) in Quantum Mechanics. The main problem is to calculate its energy spectrum and to find the corresponding wave functions.

Conventionally, the problem is treated by using various perturbative methods (e.g., \([8,9,11]\)). However, the ground-state energy in the naive perturbation series \([3]\) reads,

\[
E = m \sum_{n=0}^{\infty} A_n \cdot (g^2/m^3)^n,
\]

where the coefficient \(A_n\) grows as \(A_n \sim (-3/2)^n \Gamma(n)\) for large \(n\) \([33]\). Thus, the series formally diverges for any finite coupling. Only for sufficiently small coupling constant, \(g^2/m^3 \sim 0.1\), the series becomes numerically useful by the appropriate truncation. Beyond the perturbative approach, variational \([17]\) and strong-coupling expansion methods \([18]\) can be applied to this problem.

Our effective nonvariational approach described in previous section can be extended to the quantum mechanical anharmonic oscillator as follows.

Consider a normalized partition function

\[
Z_V(g) = \int Dq \ e^{-\frac{1}{2}(qD_0^{-1}q) - \frac{g^2}{4}(q^4)}, \quad Z_V(0) = 1,
\]

where \(q(t)\) is the position vector and \(t \in [-T, T], \int dt = V \rightarrow \infty\).

The differential operator \(D_0^{-1}\) and its Green function \(D\) obeying the conventional boundary conditions are:

\[
D_0^{-1}(t, s) = \left( -\frac{d^2}{dt^2} + 1 \right) \delta(t - s), \quad D_0(t) = \frac{e^{-|t|}}{2}, \quad \tilde{D}(k) = \frac{1}{k^2 + 1}.
\]
For further convenience we use the unit mass $m = 1$.

First, we introduce an additional integration variable (an auxiliary trajectory) $\phi$ to convert the initial quartic interaction to an integrable quadratic one by using the following Feynman representation

$$ Z_V(g) = \int D\phi \ e^{-\frac{1}{2}(\phi\phi)} \int Dq \ e^{-\frac{1}{2}(qD_0^{-1}q) - ig(q^2)} = \int D\phi \ e^{-\frac{1}{2}(-\phi\phi)} - \frac{1}{2}\text{Tr} \ln[1 + 2ig\phi D_0]. \quad (17) $$

After this transformation, the interactional functional reaches its minimum distinct from the origin $\phi = 0$. Therefore, a shift of the variable may be suggested

$$ \phi \rightarrow \phi + \frac{a}{2ig}. $$

Then, one rewrite

$$ Z_V(g) = \exp \left\{ \frac{1}{8g^2}(aa) - \frac{1}{2}\text{Tr} \ln[1 + aD_0] \right\} \cdot \int D\phi \ \exp \left\{ -\frac{1}{2}(\phi\phi) - \frac{1}{2ig}(a\phi) - \frac{1}{2}\text{Tr} \ln[1 + 2ig\phi D] \right\}, $$

where

$$ D = \frac{1}{1 + aD_0}D_0, \quad D(t) = \int \frac{dk}{2\pi} \ \frac{e^{ikt}}{k^2 + \mu^2} = \frac{e^{-\mu|t|}}{2\mu}, \quad \mu = \sqrt{1 + a}. $$

The optimal value of the shift parameter $a$ obeys the equation:

$$ -\frac{1}{2ig}(a\phi) - \frac{2ig}{2}\text{Tr}[\phi D] = 0 \quad \Rightarrow \quad a = 2g^2 \int \frac{dk}{2\pi} \ \frac{1}{k^2 + \mu^2}, \quad g^2 = a\sqrt{1 + a}. $$

In particular, $a = g^4$ for $g \rightarrow \infty$. By using the optimal shift we rewrite

$$ Z_V(g) = \exp \left\{ \frac{1}{8g^2}(aa) - \frac{1}{2}\text{Tr} \ln[1 + aD_0] \right\} \cdot \int D\phi \ \exp \left\{ -\frac{1}{2}(\phi D^{-1}\phi) - \frac{1}{2}\text{Tr} \ln[1 + 2ig\phi D] \right\}, $$

where

$$ D^{-1}(t) = \delta(t) + 2g^2D^2(t), \quad \tilde{D}^{-1}(k^2) = 1 + \frac{2a}{k^2 + 4(1 + a)}; $$

$$ D(t) = \delta(t) - \frac{a}{\nu} e^{-\nu|t|}, \quad \tilde{D}(k^2) = 1 - \frac{2a}{k^2 + \nu^2}, \quad \nu = \sqrt{4 + 6a}. $$

By introducing a normalized Gaussian measure

$$ d\Sigma_D = \frac{D\phi}{\sqrt{\det D}} \ e^{-\frac{1}{2}(\phi D^{-1}\phi)}, \quad \int d\Sigma_D = 1 $$

we obtain
\[ Z_V(g) = e^{-V(E_0(g)+\Delta E(g))} , \]
\[ E_0(g) = \frac{1}{V} \left\{ -\frac{(a \cdot a)}{8g^2} + \frac{1}{2} \text{Tr} \ln[1+aD_0] + \frac{1}{2} \text{Tr} \ln[1+2g^2D^2] \right\} \]
\[ = -\frac{1}{2} - \frac{a}{8\sqrt{1+a}} - \frac{1}{2}\sqrt{1+a} + \sqrt{1+\frac{3}{2}a} , \]
\[ \Delta E(g) = -\frac{1}{V} \ln \int d\Sigma D \exp \left\{ \frac{1}{2} \text{Tr} \ln[1+2ig\phi D] \right\} . \] (18)

We consider \( E_0(g) \) the leading-order approximation to \( E(g) \). It can be compared (in Table 2) with our earlier result obtained within the oscillator approximation method (32):
\[ E_{osc}(g) = -\frac{1}{2} + \min_{\xi, \rho} \left\{ \frac{\xi}{8\rho} \frac{\Gamma(2-\rho)}{\Gamma(1+\rho)} + \frac{1}{6\xi} \frac{\Gamma(1+3\rho)}{\Gamma(1+\rho)} + \frac{a\sqrt{1+a}}{10\xi^2} \frac{\Gamma(1+5\rho)}{\Gamma(1+\rho)} \right\} \] (19)
and with a direct numerical calculation \( E_{num}(g) \) (see, e.g. [7]).

In the weak-coupling limit \( (g \ll 1) \) we have:
\[ E_0(g) = \frac{3}{8} g^2 - \frac{11}{32} g^4 + O(g^6) , \quad E_{osc}(g) = \frac{3}{8} g^2 - \frac{9\pi^2 - 70}{16(\pi^2 - 8)} g^4 + O(g^6) , \]
\[ E(g) = \frac{3}{8} g^2 - \frac{21}{32} g^4 + O(g^6) . \] (20)

For \( g \to \infty \) one obtains
\[ E_0(g) \to g^2 \left( \sqrt{\frac{3}{2}} - \frac{5}{8} \right) = g^2 \cdot 0.599745 , \quad E_{osc}(g) \to g^2 \cdot 0.531248 , \]
\[ E(g) \to g^2 \cdot 0.530181 . \]

We observe that our leading term \( E_0(g) \) provides result about 15 percent worse than \( E_{osc}(g) \), but our present technique with only parameter \( a \) is much easier than the oscillator approximation with two variational parameters and complicated transformations.

A. Next-to-leading Correction

Unfortunately, we cannot calculate explicitly the last path integral in (18). On the other hand, we have already factorized out the main (generalized Gaussian) contribution \( e^{-V E_0(g)} \) and therefore, the remaining part \( e^{-V \Delta E(g)} \) should not result in a relatively strong correction. Therefore, we develop a systematic scheme to estimate the non-Gaussian correction.

We define the next-to-leading non-Gaussian term as follows:
\[ \Delta E_1(g) = \frac{1}{2V} \int d\Sigma D \ \text{Tr} \left\{ \ln[1+2ig\phi D] - 2g^2[\phi D\phi D] \right\} . \]

Taking into account the symmetry \( \phi \leftrightarrow -\phi \) we rewrite
\[ \Delta E_1(g) = \frac{1}{4V} \int d\Sigma D \ \text{Tr} \left\{ \ln[1+4g^2\phi D\phi D] - 4g^2[\phi D\phi D] \right\} . \]
Then, going to a new scale
\[ t \to \frac{1}{\sqrt{a}} t, \quad k \to \sqrt{a} k, \quad \tilde{D}(k) \to \frac{1}{a(k^2 + 1 + 1/a)}, \quad \tilde{D}(k) \to 1 - \frac{2}{k^2 + 6 + 4/a} \]
we rewrite
\[
\triangle E_1(g) = \frac{g^2}{4Va} \text{Tr} \int d\Sigma_D \ln(1 + \Theta - \Theta), \quad \Theta = 4\phi D\phi D.
\]

In general, any direct evaluation of \( \triangle E_1(g) \) represents a heavy and lengthy procedure. But we can easily estimate it as follows. First, we note that function \( \ln(1 + x) - x \) is concave while \( \ln(1 + x) - x + x^2/2 \) is a convex one. Then, by using these properties, we can easily find a lower and an upper bound to \( \triangle E_1(g) \) as follows:

\[
\triangle E_1^{-}(g) \leq \triangle E_1(g) \leq \triangle E_1^{+}(g),
\]

\[
\triangle E_1^{-}(g) = \frac{g^2}{4Va} \left\{ \ln[1 + \langle \Theta \rangle] - \langle \Theta \rangle - \frac{\langle \Theta^2 \rangle - \langle \Theta \rangle^2}{2} \right\},
\]

\[
\triangle E_1^{+}(g) = \frac{g^2}{4Va} \left\{ \ln[1 + \langle \Theta \rangle] - \langle \Theta \rangle \right\}, \quad \langle (...) \rangle = \frac{1}{V} \int d\Sigma_D \text{Tr}(...).
\]

We calculate
\[
\langle \Theta \rangle = 4 \int \frac{dp}{2\pi} \left(1 - \frac{2}{p^2 + 6 + 4/a}\right) \int \frac{dk}{2\pi} \frac{1}{k^2 + 1 + 1/a (k + p)^2 + 1 + 1/a} = 4Q(a),
\]

\[
Q(a) = a\frac{2\sqrt{1 + a + \sqrt{4 + 6a}}}{4(1 + a)^{1/2}(2 + 3a + \sqrt{(1 + a)(4 + 6a)})},
\]

\[
\langle \Theta^2 \rangle = 16 \left[ F(a) + 2R(a) \right],
\]

\[
F(a) = a^{3/2} \frac{\sqrt{1 + a} (24 + 31a) - \sqrt{4 + 6a} (12 + 11a)}{16(1 + a)(2 + 3a)(3 + 5a)},
\]

\[
R(a) = a^{3/2} \frac{9\sqrt{(1 + a)(4 + 6a)} - 4 - 6a}{32(1 + a)^{3/2}(2 + 3a)(3 + 5a)}.
\]

For strong-coupling regime \( g \gg 1 \) we obtain
\[
\langle \Theta \rangle = \frac{2}{\sqrt{3}} = 0.816497, \quad \langle \Theta^2 \rangle = \frac{5}{3} - \frac{2\sqrt{6}}{115} = 1.62407
\]
so
\[
-g^{2/3} \cdot 0.139072 \leq \triangle E_1(g) \leq -g^{2/3} \cdot 0.054897
\]
and
\[
g^{2/3} \cdot 0.460673 \leq E_1(g) \leq g^{2/3} \cdot 0.544848
\]
while
\[
E_0(g) = g^{2/3} \cdot 0.599745, \quad E(g) = g^{2/3} \cdot 0.530181.
\]

We see that the next-to-leading correction \( \triangle E_1(g) \) is negative and it may lower the energy \( E_0(g) \) about 10 percent approaching close to the exact numerical result.
IV. SCALAR FIELD MODEL: $D = 2$

The scalar $\varphi^4$ theory in two spatial dimensions has been intensively investigated as a simple, but nontrivial example, within which the vacuum exhibits a nontrivial structure. Within the framework of constructive QFT a set of general theorems has been proven to establish the existence of nontrivial two-dimensional theories of self-coupling scalar field. Unfortunately constructive quantum field theory gave no effective instrument (like Feynman diagrams) for the calculation of important physical characteristics of the QFT models.

An attractive approach to the problem under discussion is the variational method of the Gaussian effective potential. Original investigations in the same direction were made in. An attempt to go beyond the Gaussian approximation has been made in. However, specific features of the variational approach in QFT make their results unreliable if the theory has divergencies (for $d > 1$) in the highest perturbation orders (see, e.g.).

In this paper we calculate the ground-state (vacuum) energy. We demonstrate an effective scheme resulting in the explicit and exact asymptotics of the self energy for strong coupling.

Consider a superrenormalizable scalar model $\phi^4$ in two-dimensional Euclidean space-time $x \in V \subset \mathbb{R}^2$. The Lagrangian is given:

$$L = \frac{1}{2}(\phi[\Box - m^2]\phi) - \frac{g^2}{2} \phi^4, \quad (21)$$

The kernel $\Box - m^2$ and its Green function are

$$D^{-1}(x - x') = (-\Box + m^2)\delta(x - x'), \quad D(x) = \int \frac{dk}{(2\pi)^2} \frac{e^{ikx}}{k^2 + m^2} = \int_0^{\infty} d\beta \frac{e^{-\frac{\beta}{2}m^2 - \frac{x^2}{2\beta}}}{4\pi\beta}, \quad D_0 = D(0) = \int_0^{\infty} d\beta \frac{e^{-\frac{\beta}{2}m^2}}{4\pi\beta}.$$

The super-renormalizable theory $\phi^4$ contains divergences which can be removed by the renormalization of mass and vacuum energy. However both these divergences can be removed if the interaction Lagrangian in (21) is written in the normal form:

$$: \phi^4 := \phi^4 - 6D_0\phi^2 + 3D_0^2 = (\phi^2 - 3D_0)^2 - 6D_0^2. \quad (22)$$

This form of interaction removes all the divergences in this theory. For intermediate calculations we shall use the dimensional regularization, i.e. we consider the theory in the space $\mathbb{R}^d (d < 2)$

$$D_{\text{reg}}(x) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{ikx}}{k^2 + m^2} = \frac{1}{2} \int_0^{\infty} d\beta \frac{e^{-\frac{\beta}{2}m^2 - \frac{x^2}{2\beta}}}{(2\pi\beta)^{d/2}}, \quad D_0 = D_{\text{reg}}(0) = \frac{1}{2} \int_0^{\infty} d\beta \frac{e^{-\frac{\beta}{2}m^2}}{2(2\pi)^{d/2}} = \left(\frac{2}{m^2}\right)^{1-\frac{d}{2}}.$$

All divergences are explicitly removed in final formulae and then, we put the true spatial dimension $d = 2$. For simplicity we omit the index "reg" throughout the text having in mind that the regularization has been introduced.

It is convenient to rewrite the Lagrangian as follows:
\[ L = \frac{1}{2}(\phi[\Box - m^2] \phi) - \frac{g^2}{2} \left( \phi^2 - 3D_0 \right)^2 + 3g^2D_0^2. \]  

(23)

In this paper we shall calculate the vacuum energy. Let us consider the partition function and use the Gaussian representation:

\[
Z_V[g] = \det(-\Box + m^2) \int \delta \phi \ e^{-\frac{1}{2}(\phi[\Box + m^2] \phi) - \frac{g^2}{2} \left( \phi^2 - 3D_0 \right)^2 + 3g^2D_0^2V} = \int D\Phi \ \det(-\Box + m^2) \int \delta \phi \ e^{-\frac{1}{2}(\phi[\Box + m^2] \phi) - ig(\phi[\Box - 3D_0] \phi) + 3g^2D_0^2V} = \int D\Phi \ e^{igD_0(\Phi) - \frac{1}{2}Tr \ln(1 + i2g\Phi D) + 3g^2D_0^2V}, \quad D\Phi = \delta \Phi \ e^{-\frac{1}{2}(\Phi\Phi)}
\]

(24)

with normalization \( Z_V[0] = 1 \).

Let us consider the interaction part

\[
U[\Phi] = Tr \ln \left[ 1 - 4g^2D_0D + i2g\Phi D \right] = Tr \ln \left[ \frac{D^{-1} - 4g^2D_0 + i2g\Phi}{D^{-1}} \right] = \int dx \int_0^\infty \frac{d\alpha}{\alpha} \left[ e^{-\frac{\alpha}{2}(-\Box + m^2)} - e^{-\frac{\alpha}{2}(-\Box + m^2 - 4g^2D_0+ i2g\Phi(x))} \right] \delta^d(x - x') \bigg|_{x = x'}
\]

\[
= \int dx \left[ W_m^2(x, x|0) - W_m^2-4g^2D_0(x, x|\Phi) \right].
\]

where the function \( W \) is defined as:

\[
W_m^2(x, x'|\Phi) = \int_0^\infty \frac{d\alpha}{\alpha} \ e^{-\frac{\alpha}{2}(-\Box + m^2 + i2g\Phi(x))} \cdot \delta^d(x - x').
\]

It satisfies the condition

\[
W_m^2(x, x'|\Phi + igA) = W_m^2-2g^2A(x, x'|\Phi).
\]

The function \( W \) can be represented in the form of a functional integral (see Appendix A):

\[
W_m^2(x, x'|\Phi) = \int_0^\infty \frac{d\alpha}{(2\pi)^{d/2}\alpha^{1+\frac{d}{2}}} \ e^{-\frac{\alpha m^2}{2} - \frac{(x - x')^2}{2\alpha}} \int d\sigma[\xi] \ e^{-i\int_0^\alpha \Phi(x(\frac{\alpha}{2}) + x'(-\frac{\alpha}{2}) + \xi(\beta))},
\]

(25)

where \( \xi(\beta) \in \mathbb{R}^d \) and

\[
d\sigma[\xi] = \delta\xi \cdot e^{-\frac{1}{2} \int d\beta \ \xi^2(\beta)} = \delta\xi \cdot e^{-\frac{1}{2}(\xi K^{-1}\xi)}; \quad \int d\sigma[\xi] = 1.
\]

with the boundary conditions \( \xi(0) = \xi(\alpha) = 0 \).

The Green function of the kernel

\[
K^{-1}(\beta, \beta') = \frac{-d^2}{d\beta^2} \delta(\beta - \beta').
\]

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reads
\[
K(\beta, \beta') = -\frac{1}{2}|\beta - \beta'| + \frac{1}{2}(\beta + \beta') - \frac{\beta\beta'}{\alpha}
\]
and satisfies the boundary conditions
\[
K(\beta, \beta') = K(\beta', \beta), \quad K(\beta, 0) = K(\beta, \alpha) = 0.
\]

Then,
\[
W_{m^2}(x, x|\Phi) = \int D\alpha \int d\sigma[\xi] e^{-ig(\Phi B\xi)} , \quad \int D\alpha \cdot (\ldots) = \int_0^\infty \frac{d\alpha}{(2\pi\alpha)^{d/2}} e^{-\frac{\alpha m^2}{2}} \cdot (\ldots). \quad (26)
\]
where
\[
B\xi(y) = \int_0^\alpha d\beta \delta(y - x - \xi(\beta)), \quad \int dy B\xi(y) = \alpha.
\]
so that
\[
\int_0^\alpha d\beta \Phi(x + \xi(\beta)) = \int dy \Phi(y) B\xi(y) = (\Phi B\xi), \quad \int dx (\Phi B\xi) = \alpha(\Phi).
\]
Finally we get
\[
U[\Phi] = \int dx \int D\alpha \int d\sigma[\xi] \left[ 1 - e^{2\alpha g^2 D_0 - ig(\Phi B\xi)} \right].
\]
Following the idea mentioned in the previous sections, we go to a shift
\[
\Phi(x) \to \Phi(x) - igA
\]
where \(A\) is a constant depending on \(g\).

We rewrite
\[
Z_V[g] = e^{g^2[\frac{1}{2} A^2 + D_0 A - D_0^2]V} \int D\Phi e^{ig(D_0 + A)(\Phi) - \frac{1}{2} U[\Phi - igA]}
\]
and
\[
U[\Phi - igA] = \int dx \int_0^\infty \frac{d\alpha}{(2\pi\alpha)^{d/2}} e^{-\frac{\alpha m^2}{2}} \cdot \int d\sigma[\xi] \cdot \left[ 1 - e^{g^2\alpha(2D_0 - A) - ig(\Phi B\xi)} \right]. \quad (27)
\]
Now we introduce the normal-ordered form:
\[
e^{-ig(\Phi B\xi)} =: e^{-ig(\Phi B\xi)} :\Phi e^{-\frac{g^2}{2}(B\xi B\xi)} ,
\]
\[
e^{ik(\xi(\beta_1) - \xi(\beta_2))} =: e^{ik(\xi(\beta_1) - \xi(\beta_2))} :\xi e^{-\frac{k^2}{4} F_\alpha(|\beta_1 - \beta_2|)} , \quad F_\alpha(|\beta_1 - \beta_2|) = |\beta_1 - \beta_2| - \frac{(\beta_1 - \beta_2)^2}{\alpha}
\]
obeying the following relations
\[
\int D\Phi : e^{-ig(\Phi B\xi)} :\Phi = 1, \quad \int d\sigma[\xi] : e^{ik(\xi(\beta_1) - \xi(\beta_2))} :\xi = 1.
\]
One gets
\[(B_\xi B_\xi) = \int dy B_\xi(y) B_\xi(y) = \int_0^\alpha d\beta_1 d\beta_2 \delta^d(\xi(\beta_1) - \xi(\beta_2))\]

\[
= \int_0^\alpha d\beta_1 d\beta_2 \int \frac{d^d k}{(2\pi)^d} : e^{ik(\xi(\beta_1) - \xi(\beta_2))} e^{-\frac{k^2}{2} F_\alpha(\beta_1 - \beta_2)} : = \langle (B_\xi B_\xi) \rangle_\xi + : W_\alpha[\xi] :_\xi ,
\]

where

\[
\langle (B_\xi B_\xi) \rangle_\xi = \int d\sigma[\xi] \langle B_\xi B_\xi \rangle_\xi = \int d\beta_1 d\beta_2 \int \frac{d^d k}{(2\pi)^d} e^{ik(\xi(\beta_1) - \xi(\beta_2))} e^{-\frac{k^2}{2} F_\alpha(\beta_1 - \beta_2)}
\]

\[
= \frac{2}{(2\pi)^{d/2}} \int_0^\alpha d\beta(\alpha - \beta)^{1-\frac{d}{2}} \left( \frac{\alpha}{\beta} \right)^{\frac{d}{2}} = \frac{2\alpha^{2-\frac{d}{2}}}{(2\pi)^{d/2}} B\left(2 - \frac{d}{2}, 1 - \frac{d}{2}\right)
\]

and the interaction functional is introduced

\[
: W_\alpha[\xi] :_\xi = \int_0^\alpha d\beta_1 d\beta_2 \int \frac{d^d k}{(2\pi)^d} : e^{ik(\xi(\beta_1) - \xi(\beta_2))} :_\sigma e^{-\frac{k^2}{2} F_\alpha(\beta_1 - \beta_2)} ,
\]

\[
e^\xi_k = e^\xi - \sum_{s=0}^k \frac{z^s}{s!} , \int d\sigma[\xi] \cdot : W_\alpha[\xi] :_\xi = 0 .
\]

It is easy to check that

\[
e^{g^2\alpha(2D_0 - A) - ig(\Phi B_\xi)} = e^{-g^2\alpha A + 2g^2\alpha D_0 - \frac{g^2}{2} (B_\xi B_\xi)} : e^{-ig(\Phi B_\xi)} , \Phi
\]

\[
e^{-g^2\alpha A - g^2 N(\alpha)} \cdot e^{-\frac{g^2}{2} : W_\alpha[\xi] :_\xi} \cdot : e^{-ig(\Phi B_\xi)} :_\Phi + 1 - ig(\Phi B_\xi) ,
\]

where

\[
-g^2 N(\alpha) = 2\alpha g^2 D_0 - \frac{g^2}{2} \langle (B_\xi B_\xi) \rangle_\xi
\]

\[
= g^2 \left( 2\alpha \cdot \frac{\Gamma \left(1 - \frac{d}{2}\right)}{2(2\pi)^{d/2}} \cdot \frac{1}{m^2} \right)^{1-\frac{d}{2}} - \frac{1}{2} \cdot \frac{2\alpha^{2-\frac{d}{2}}}{(2\pi)^{d/2}} \cdot \frac{\Gamma \left(2 - \frac{d}{2}\right) \Gamma \left(1 - \frac{d}{2}\right)}{\Gamma (3 - d)}
\]

\[
= \frac{g^2\alpha}{(2\pi)^{d/2}} \Gamma \left(1 - \frac{d}{2}\right) \left( \frac{2}{m^2} \right)^{1-\frac{d}{2}} \left\{ 1 - \left( \frac{m^2\alpha}{2} \right)^{1-\frac{d}{2}} \cdot \frac{\Gamma \left(2 - \frac{d}{2}\right) \Gamma \left(1 - \frac{d}{2}\right)}{\Gamma (3 - d)} \right\} = (d \to 2)
\]

\[
= -\frac{\alpha g^2}{2\pi} \left[ C + \ln \left( \frac{\alpha m^2}{2} \right) \right] , \quad C = 0.577215...
\]

Then, the functional \( U \) in (21) can be represented in the form

\[
U [\Phi - igA] + \int dx \int D\alpha \int d\sigma[\xi] \left[ 1 - e^{Q(g^2\alpha, A)} e^{-\frac{g^2}{2} W_\alpha[\xi] :_\xi} \cdot : e^{-ig(\Phi B_\xi)} :_\Phi + 1 - ig(\Phi B_\xi) \right]
\]

\[
= V U_0 + ig(\Phi) U_1 + U_1[\Phi] ,
\]

where

\[
Q(g^2\alpha, A) = -g^2 N(\alpha) - g^2\alpha A , \quad R(g^2\alpha) = \int d\sigma[\xi] e^{-\frac{g^2}{2} W_\alpha[\xi] :_\xi} ,
\]

\[
U_0 = \int D\alpha \left[ 1 - e^{Q(g^2\alpha, A)} R(g^2\alpha) \right] , \quad U_1 = \int D\alpha \alpha e^{Q(g^2\alpha, A)} R(g^2\alpha) ,
\]

\[
U_1[\Phi] = -\int dx \int D\alpha e^{Q(g^2\alpha, A)} \int d\sigma[\xi] e^{-\frac{g^2}{2} W_\alpha[\xi] :_\xi} : e^{-ig(\Phi B_\xi)} :_\Phi .
\]
The optimal value of parameter $A$ should be obtained from the condition of elimination of the total linear terms over $\Phi$:

$$ig(\Phi)\left[D_0 + A - \frac{1}{2}U_1\right] = 0,$$

i.e. this equation reads

$$A = -\frac{1}{2} \int D\alpha \alpha \left[1 - e^{Q(g^2\alpha, A)}R\left(g^2\alpha\right)\right].$$

The vacuum energy in the lowest approximation looks

$$E_0 = -g^2 \left(\frac{1}{2}A^2 + D_0A - D_0^2\right) + \frac{1}{2}U_0 = -\frac{g^2}{2}A^2 + W_0,$$

$$W_0 = \frac{1}{2} \int D\alpha \left\{e^{Q(g^2\alpha, A)} - 1 - Q(g^2\alpha, A) + e^{Q(g^2\alpha, A)}[1 - R(g^2\alpha)]\right\},$$

where divergences are completely removed as follows:

$$-g^2D_0A + g^2D_0^2 - \frac{1}{2} \int D\alpha Q(g^2\alpha, A) = 0.$$

After complete removal of divergences we can put $d = 2$. Then,

$$A = -\int_0^\infty \frac{d\alpha}{4\pi\alpha} e^{-\frac{m^2}{2\alpha^2}} \left[1 - e^{Q(g^2\alpha, A)}R\left(g^2\alpha\right)\right],$$

$$W_0 = \int_0^\infty \frac{d\alpha}{4\pi\alpha^2} e^{-\frac{m^2}{\alpha^2}} \left[e^{Q(g^2\alpha, A)} - 1 - Q(g^2\alpha, A) + e^{Q(g^2\alpha, A)}\right] \left(1 - R(g^2\alpha)\right).$$

Going to new re-scaled variables:

$$\beta = \alpha\tau, \quad \xi(\beta) = \sqrt{\alpha} \eta(\tau), \quad k = \frac{q}{\sqrt{\alpha}}, \quad s = \frac{g^2\alpha}{2} = ht, \quad B = \frac{A}{h}$$

with

$$N(t) = t [C + \ln(t)], \quad \alpha = \frac{2}{m^2} t, \quad h = \frac{g^2}{\pi m^2},$$

we obtain

$$Z[g] = e^{-VE} = e^{-VE_0} \int D\Phi e^{-\frac{1}{2}U_1[\Phi]} = e^{-VE_0} e^{-VE_{corr}},$$

where the leading-order term for the vacuum energy is

$$\left(\frac{m^2}{8\pi}\right)^{-1} E_0 = E_0 = -h \left\{B^2 + \int_0^\infty \frac{dt}{t} e^{-t} \frac{1}{ht} \left[e^{Q(ht)} - 1 - Q(ht) + e^{Q(ht)} \left(R(ht) - 1\right)\right]\right\}$$

and the remaining higher-order correction reads

$$E_{corr} = -\frac{1}{V} \ln \int D\Phi e^{-\frac{1}{2}U_1[\Phi]}.$$
The shift parameter $B$ is governed by the equation:

$$B = \frac{1}{2} \int_0^\infty \frac{dt}{t} e^{-t} \left[ e^{Q(ht)} R(ht) - 1 \right], \quad (31)$$

where

$$Q(ht) = -h t [B + C + \ln(t)], \quad R(ht) = \int d\sigma[\eta] e^{-ht W[\eta] : \sigma},$$

$$d\sigma[\eta] = \delta \eta \cdot e^{-\frac{1}{2} \int_0^1 d\tau \; \eta^2(\tau)}, \quad \int d\sigma[\eta] : W[\eta] : \sigma = 0,$$

$$: W[\eta] : \sigma = \int_0^1 d\tau_1 d\tau_2 \int \frac{dq}{(2\pi)^2} : e^{i\eta(\tau_1 - \eta(\tau_2))} : \sigma e^{-\frac{1}{2} (|\tau_1 - \tau_2| - (\tau_1 - \tau_2)^2)}.$$

The Green function for the kernel of $d\sigma[\eta]$ is

$$K(t, s) = -\frac{|t - s|}{2} + \frac{t + s}{2} - ts.$$

Eqs. (28), (29), (30) and (31) are basic in our consideration and they completely define the ground-state energy of the system at any given coupling. For general coupling these equations should be evaluated by numerical means, but we are able to solve them explicitly in the weak- and strong-coupling regimes.

**A. Explicit weak-coupling solutions**

i. First, we calculate the exact perturbation solution up to the order $O(g^6)$ as follows:

$$e^{-VE(g)} = 1 + \frac{1}{2} \frac{g^4}{4!} V \int d^2x D^4(x) + O(g^6)$$

$$= 1 + \frac{3g^4 V}{(2\pi)^3 m^2} \int_0^\infty du u K_0^4(u) + O(g^6) = 1 + \frac{3g^4 V}{(2\pi)^3 m^2} \frac{7}{8} \zeta(3) + O(g^6),$$

$$E = -\frac{g^4}{(2\pi)^3 m^2} \frac{21}{8} \zeta(3) + O(g^6) = -h^2 \frac{m^2}{8\pi} \frac{21}{8} \zeta(3) + O(h^3),$$

where $K_0(u)$ and $\zeta(x)$ are the Bessel and the Riemann function, correspondingly.

ii. Now we evaluate $E_{corr}(g)$. We obtain the explicit weak-coupling solution to the higher-order total correction as follows (see Appendix C):

$$E_{corr}(g) = -\frac{g^4}{(2\pi)^3 m^2} \frac{7}{8} \zeta(3) + O(g^6) = -h^2 \frac{m^2}{8\pi} \frac{7}{8} \zeta(3) + O(h^3). \quad (32)$$

iii. We have $E = E_0 + E_{corr}$ by the very definition (28). Therefore, the explicit weak-coupling solution to the leading-order energy reads:

$$E_0 = -\frac{g^4}{(2\pi)^3 m^2} \frac{7}{4} \zeta(3) + O(g^6) = -h^2 \frac{m^2}{8\pi} \frac{7}{4} \zeta(3) + O(h^3).$$
Therefore, the following relations take place (within the given accuracy)

\[ E_{\text{corr}}(g) = \frac{1}{2} E_0(g) = \frac{1}{3} E(g) \, . \]

One can note that our leading-order energy \( E_0 \) underestimates (i.e., \( 2/3 \) of) the exact energy \( E \). The gap of one third is compensated by taking into account the higher-order energy term \( (32) \). As we will show below, our leading-order approximation becomes better as \( g \) grows and, approaches the exact solution for \( g \to \infty \).

**B. Exact strong-coupling solutions**

However, our method is designed to investigate the strong-coupling behaviour of the considered system. Below we demonstrate explicit and analytic solutions of Eqs. (28), (29) and (31) that leads to the exact strong-coupling answer.

We note that for \( h \to \infty \) the factor \( R(ht) \) gives correction \( \sim e^{O(\ln(h)/h)} \) with respect to the main exponentials in (29) and (31). Therefore, one can neglect it considering the main asymptotical values for \( B \) and \( E_0 \) (for more details see Appendix B).

For \( h \to \infty \) the integrand in (31) behaves a very sharply expressed Gaussian exponential centered at point \( t_m \). The exponent function is

\[ f(t) = -t - \ln(t) - ht[B + C + \ln(t)] \, . \]

Then, with the use of the ”saddle-point” method we estimate

\[ B = \frac{1}{2} \int_{-\infty}^{\infty} dt \exp \left\{ f(t_m) - \frac{1}{2} t^2 [-f''(t_m)] \right\} = \frac{1}{2} \exp \{ f(t_m) \} \sqrt{\frac{2\pi}{-f''(t_m)}} , \tag{33} \]

where the maximum point \( t_m \) obeys the conditions

\[ f'(t_m) = 0 \, , \quad f''(t_m) < 0 \, . \]

By resolving these equations we obtain (see Appendix D):

\[ B = \ln(h) - C - 2 + O(1/\ln^4(h)) \, . \tag{34} \]

Note, this asymptotics is reached very slowly, e.g., for \( h = 10^{20} \) we obtain \( B = 0.93265 \cdot \ln(h) \) instead of \( B = \ln(h) \). Substituting (34) into (29) and by neglecting terms vanishing as \( O(1/h) \) we obtain (Appendix D):

\[ E_0 = -\frac{3}{2} h \ln^2(h) + 3(C + 2)h \ln(h) + O(h) \, . \tag{35} \]

Particularly, for \( h = 10^{20} \) this asymptotics provides \( E_0 = -2.82 \cdot 10^{23} \) while the numerical result is \( E_0 = -2.81 \cdot 10^{23} \). We see that this asymptotical behaviour becomes exact for very large coupling.
V. CONCLUSION

We have represented a path-integral technique suitable to evaluate the quartic self interaction in the strong-coupling regime. The leading-order approximation is obtained in a relatively simple way and the remaining corrections can be estimated. A simple version of our method has been tested in the examples of a plain quartic-exponential function and the well-investigated anharmonic oscillator in Quantum Mechanics. We see that our technique effectively isolates the main contribution for arbitrary coupling. Then, we have applied this technique to the solution of the ground-state energy in the superrenormalizable scalar theory $\phi^4(x)$ in two dimensions. Hereby, the auxiliary appearing complex functional does not represent any difficulties for this technique. Our leading-order approximation becomes much accurate as the coupling $g$ increases and for $g \to \infty$ it coincides with the exact asymptotics. We can conclude that our technique may effectively isolate the main contribution of the strong-coupling regime even in theories with divergencies and complex functionals.
### Table 1. Approximate and exact numerical results for $E(g)$ and $G(g)$ in zero dimension.

| $g^2/2$ | $E_0(g)$ | $E_1(g)$ | $E(g)$  | $G_0(g)$ | $G_1(g)$ | $G(g)$  |
|---------|---------|---------|---------|---------|---------|---------|
| 0.01    | 0.02783 | 0.02626 | 0.02629 | 0.90518 | 0.90650 | 0.90653 |
| 0.1     | 0.18027 | 0.15267 | 0.15361 | 0.60786 | 0.61378 | 0.61553 |
| 0.2     | 0.27274 | 0.22892 | 0.23000 | 0.49492 | 0.50023 | 0.50312 |
| 0.5     | 0.42431 | 0.35925 | 0.35993 | 0.35854 | 0.36211 | 0.36596 |
| 1.0     | 0.55590 | 0.47747 | 0.47758 | 0.27256 | 0.27489 | 0.27884 |
| 2.0     | 0.69826 | 0.60939 | 0.60890 | 0.20316 | 0.20461 | 0.20823 |
| 5.0     | 0.89861 | 0.79988 | 0.79874 | 0.13478 | 0.13553 | 0.13840 |
| 10.0    | 1.05688 | 0.95301 | 0.95150 | 0.09766 | 0.09811 | 0.10039 |
| 100.0   | 1.60679 | 1.49423 | 1.49209 | 0.03217 | 0.03226 | 0.03313 |

### Table 2. Comparison of results for $E_0(g)$, $E_{osc}(g)$ and $E_{num}(g)$ in one dimension.

| $g^2/2$ | $E_0(g)$ | $E_{osc}(g)$ | $E_{num}(g)$ |
|---------|---------|--------------|--------------|
| 0.1     | 0.06434 | 0.05938      | 0.05915      |
| 0.5     | 0.22666 | 0.19697      | 0.19618      |
| 1.0     | 0.35522 | 0.30490      | 0.30377      |
| 10.0    | 1.17291 | 1.00778      | 1.00497      |
| 50.0    | 2.30990 | 2.00461      | 1.99971      |
| 100.0   | 3.02802 | 2.63759      | 2.63138      |
Appendix A.

The Green function in the presence of an external field reads:

\[ G_m(t, t'|\Phi) = \frac{1}{D^{-1} + m^2 + 2ig\Phi(x)}\delta(t - t') \]

\[ = \int_0^\infty ds\ e^{-s(1+a+m^2)}\cdot T\ e^0 \cdot \int_0^s dr\ (\frac{d}{dr}\Phi)^2 - 2ig\int_0^s dr\ \Phi(t(\tau))\ \delta(t-t'). \]

Then,

\[ T\ e^0 \cdot \int_0^s dr\ (\frac{d}{dr}\Phi)^2 - 2ig\int_0^s dr\ \Phi(t(\tau))\ \delta(t-t') \]

\[ = \int D\nu\ e^{-s\nu^2(\tau)+2\int_0^s dr\nu(t)\frac{d}{dr}\Phi(t(\tau))}\ \delta(t-t') \]

\[ = \int D\nu\ e^{-s\nu^2(\tau)-2ig\int_0^s dr\ \Phi(t+2(s-\tau)\nu_0-2\xi(\tau))}\ \delta(t-t' + 2\int_0^s d\tau\nu(\tau)) \]

\[ = \int D\xi\ \int_{-\infty}^{\infty} d\nu_0\ e^{-\frac{s\nu_0^2}{2} - \int_0^s dr\ \xi^2(\tau)-2ig\int_0^s dr\ \Phi(t+2(s-\tau)\nu_0-2\xi(\tau))}\ \delta(t-t' + 2s\nu_0) \]

\[ = \frac{1}{\sqrt{4\pi s}} e^{-\frac{(t-t')^2}{4s}} \cdot \int_{\xi(0)=\xi(s)=0} D\xi\ e^{-\int_0^s dr\ \xi^2(\tau)-2ig\int_0^s dr\ \Phi(t+2(s-\tau)\xi(\tau))}, \]

where

\[ \nu(\tau) = \nu_0 + \xi(\tau), \quad \xi(0) = \xi(s) = 0. \]

Appendix B.

The following effective approximation takes place:

\[ R(s) = \int d\sigma[\eta]\ e^{-sW} = \langle e^{-sW} \rangle \approx e^{-s\langle W \rangle} \cdot \cosh\left(s\sqrt{\langle W^2 \rangle - \langle W \rangle^2}\right), \quad W = :W[\eta]:_0. \]

Due to normal-ordered form of \( W \), one finds \( \langle W \rangle = 0 \). Then,

\[ R(s) \approx \cosh(sw), \quad w^2 = \int d\sigma[\eta] \cdot (:W[\eta]:_0)^2. \]

For \( h \to \infty \) its correction to the shift parameter \( B = \ln(h) \) is \( \sim O(1) \) and to the leading-order energy \( E_0 = -(3/2)h\ln^2(h) \sim O(h) \). Therefore, its influence is negligible and we just drop the factor \( R(ht) \) by considering the large \( h \) asymptotics.

Appendix C.

By expanding the interaction functional

\[ U_I[\Phi] = g^2U_2[\Phi] + g^4U_4[\Phi] + O(g^6) \]
and taking into account its normal-ordered form, we obtain
\[ E_{corr}(g) = -\frac{g^4}{32V} \int D\alpha D\beta \int d\sigma[\xi]d\sigma[\eta] \int dxdy \int D\Phi : (\Phi B_\xi)_{x\alpha} : (\Phi B_\eta)_{y\beta} : + O(g^6). \]
The functional averaging over field \( \Phi \) results in
\[ \int D\Phi : (\Phi B_\xi)_{x\alpha} :: (\Phi B_\eta)_{y\beta} : = 2 \int_0^\alpha dt du \int_0^\beta ds dw \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2p}{(2\pi)^2} e^{i(k+p)(x-y)+ik[\xi(t)-\eta(s)]+ip[\xi(u)-\eta(w)]}. \]
Further we use the following plain relations:
\[ \int dx \int dy e^{i(k+p)(x-y)} = V \cdot (2\pi)^2 \cdot \delta^2(k+p), \]
\[ \int d\sigma[\xi] e^{ik[\xi(t)-\xi(s)]} = \exp \left[ -\frac{k^2}{2} F_\alpha(|t-s|) \right], \]
\[ \int_0^\alpha dt \int ds e^{-\frac{k^2}{2} F_\alpha(|t-s|)} = 2\alpha \int_0^\alpha dt e^{-\frac{k^2}{2} F_\alpha(t)} \left( 1 - \frac{t}{\alpha} \right), \]
\[ \int \frac{d^2k}{(2\pi)^2} e^{-\frac{k^2}{2} [F_\alpha(t)+F_\beta(s)]} = \frac{1}{4\pi} \int_0^\infty du e^{-\frac{u}{2} [F_\alpha(t)+F_\beta(s)]}. \]
Then,
\[ E_{corr}(g) = -\frac{g^4}{8(2\pi)^3} \int_0^1 dxdy(1-x)(1-y) \int_0^\infty du \int d\alpha d\beta e^{-\frac{u}{2} [m^2 + ux(1-x)] - \frac{\alpha}{2} [m^2 + uy(1-y)]} \]
\[ = -\frac{g^4}{16\pi^3 m^2} \int_0^1 dxdy(1-x)(1-y) \frac{\ln[x(1-x)] - \ln[y(1-y)]}{x(1-x) - y(1-y)} \]
\[ = -\frac{g^4}{(2\pi)^3 m^2} \frac{7}{8} \zeta(3). \]

**Appendix D.**

The maximum point \( t_m \) of the sharp exponential function
\[ f(t) = -t - \ln(t) - ht[B + C + \ln(t)] \]
is dictated by the conditions
\[ f'(t_m) = -1 - \frac{1}{t_m} - h[B + C + 1 + \ln(t_m)] = 0, \quad f''(t_m) = \frac{1 - ht_m}{t_m^2} < 0. \]
This results in
\[ B = \ln(h) - \ln(\xi) - C - 1 - \frac{1}{\xi} - \frac{1}{h} = \sqrt{\frac{\pi}{2(\xi - 1)}} e^{\xi+1}, \quad \xi = ht_m > 1. \]
Hereby,

\[ \lim_{h \to \infty} \xi(h) = +1. \]

The solution is

\[ B = \ln(h) - C - 2 + O(1/\ln^4(h)). \]

For the energy, we rewrite it in a more convenient form

\[ \mathcal{E}_0 = -h \left\{ B^2 - \int_0^\infty \frac{dt}{t} e^{-t} \left[ F(t) - \Phi(ht) \right] \right\}, \]

where

\[ F(t) = \int_t^\infty \frac{ds}{s} e^{-s} + \ln(t) + C = \sum_{n=1}^\infty (-1)^{n+1} \frac{t^n}{n!} \]

and

\[ \Phi(ht) = \frac{1}{ht} \left\{ e^{-ht[B+C-\ln(h)+\ln(ht)]} - 1 + ht[B + C - \ln(h) + \ln(ht)] \right\}. \]

Therefore,

\[ \int_0^\infty \frac{dt}{t} e^{-t} F(t) = \frac{\pi^2}{12} \sim O(1) \]

and

\[ \int_0^\infty \frac{dt}{t} e^{-t} \Phi(ht) = \frac{1}{2} \ln^2(h) + O(\ln(h)). \]

Substituting these solutions to the energy, one obtains

\[ \mathcal{E}_0 = -\frac{3}{2} h \ln^2(h) + 3(C + 2)h \ln(h) + O(h). \]
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