In-domain control of a heat equation: an approach combining zero-dynamics inverse and differential flatness

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Abstract

This paper addresses the set-point control problem of a heat equation with in-domain actuation. The proposed scheme is based on the framework of zero dynamics inverse combined with flat system control. Moreover, the set-point control is cast into a motion planing problem of a multiple-input, multiple-out system, which is solved by a Green’s function-based reference trajectory decomposition. The validity of the proposed method is assessed through convergence and solvability analysis of the control algorithm. The performance of the developed control scheme and the viability of the proposed approach are confirmed by numerical simulation of a representative system.

Keywords: Distributed parameter systems; heat equation; zero-dynamics inverse; differential flatness.

1. Introduction

Control of parabolic partial differential equations (PDEs) is a long-standing problem in PDE control theory and practice. There exists a very rich literature devoted to this topic, and it is continuing to draw a great attention for both theoretical studies and practical applications. In the existing literature, the majority of work is dedicated to boundary control, which may be represented as a standard Cauchy problem to which functional analytic setting based on semigroup and other related tools can be applied.
It is interesting to note that in recent years, some methods that were originally developed for the control of finite-dimensional nonlinear systems have been successfully extended to the control of parabolic PDEs, such as backstepping (see, e.g., \cite{4, 5, 6}), flat systems (see, e.g., \cite{7, 8, 9, 10, 11, 12, 13}), as well as their variations (see, e.g., \cite{14, 15}).

This paper deals with the output regulation problem for set-point control of a heat equation via pointwise in-domain (or interior) actuation. Notice that due to the fact that the regularity of a pointwisely controlled inhomogeneous heat equation is qualitatively different from that of boundary controlled heat equations, the techniques developed for boundary control may not be directly applied to the former case. This constitutes a motivation for the present work. The control scheme developed in this paper is based-on the framework of zero-dynamics inverse (ZDI), which was introduced by Byrnes and his collaborators in \cite{16} and has been exploited and developed in a series of work (see, e.g., \cite{17} and the references therein). It is pointed out in \cite{18} that “for certain boundary control systems it is very easy to model the system's zero dynamics, which, in turn, provides a simple systematic methodology for solving certain problems of output regulation.” Indeed, the construction of zero dynamics for output regulation of certain interiorly controlled PDEs is also straightforward (see, e.g., \cite{19}) and hence, the control design can be carried out in a systematic manner. Nevertheless, a main issue related to the ZDI design is that the computation of dynamic control laws requires resolving the corresponding zero dynamics, which may be very difficult for generic regulation problems, such as the set-point control considered in the present work. To overcome this difficulty, we leverage one of the fundamental properties of flat systems, that is if a (lumped or distributed parameter) system is differentially flat (or flat for short), then its states and inputs can be explicitly expressed by the so-called flat output and its time-derivatives \cite{20, 9}. We show that in the context of ZDI design, the control can also be derived from flat output without explicitly solving the original dynamic equation. Moreover, in the framework of flat systems, set-point control can be cast into a problem of motion planning, which can also be carried out in a systematic manner. Note that it can be expected that the ZDI design is applicable to other systems, such as the interior control of beam and plate equations, as an alternative to the methods.
proposed in, e.g., [21, 22, 11].

The system model used in this work is taken from [19]. In order to perform control design based on the principle of superposition, we present the original system in a form of parallel connection. As the control with multiple actuators located in the domain leads to a multiple-input, multiple-output (MIMO) problem, we introduce a Green’s function-based reference trajectory decomposition scheme that enables a simple and computational tractable implementation of the proposed control algorithm.

The remainder of the paper is organized as follows. Section 2 describes the model of the considered system and its equivalent settings. Section 3 presents the detailed control design. Section 4 deals with motion planning and addresses the convergence and the solvability of the proposed control scheme. A simulation study is carried out in Section 5 and, finally, some concluding remarks are presented in Section 6.

2. Problem Setting

In the present work, we consider a scaler parabolic equation describing one-dimensional heat transfer with boundary and in-domain control, which is studied in [19]. Denote by \( z(x, t) \) the heat distribution over the one-dimensional space, \( x \), and the time, \( t \). The derivatives of \( z(x, t) \) with respect to its variables are denoted by \( z_x \) and \( z_t \), respectively. Consider \( m \) points \( x_j, j = 1, \ldots, m \), in the interval \((0, 1)\) and assume, without loss of generality, that \( 0 = x_0 < x_1 < x_2 < \cdots < x_m < x_{m+1} = 1 \). Let \( \Omega = \bigcup_{j=0}^{m} (x_j, x_{j+1}) \).

The considered heat equation with boundary and in-domain control in a normalized coordinate is of the form:

\[
\begin{align*}
  z_t(x, t) - z_{xx}(x, t) &= 0, \quad x \in \Omega, \ t > 0, \quad (1a) \\
  z(x, 0) &= \phi(x), \quad (1b) \\
  B_0 z &= z_x(0, t) - k_0 z(0, t) = 0, \ B_1 z &= z_x(1, t) + k_1 z(1, t) = 0, \quad (1c) \\
  z(x_j^+) &= z(x_j^-), \quad j = 1, \ldots, m, \quad (1d) \\
  B_{x_j} z &= [z']_{x_j} = u_j(t), \quad j = 1, 2, \ldots, m, \quad (1e)
\end{align*}
\]

where for a function \( f(\cdot) \) and a point \( x \in [0, 1] \) we define

\[
[f]_x = f(x^+) - f(x^-),
\]
with $x^-$ and $x^+$ denoting, respectively, the usual meaning of left and right hand limits to $x$. The initial condition is specified in (1b) with $\phi(x) \in L^2(0,1)$. It is assumed that in System (1), we can control the heat flow at the points $x_j$ for $j = 1, \ldots, m$, i.e.,

$$u_j(t) = [z_x]_{x_j} = z_x(x_j^+, t) - z_x(x_j^-, t).$$

Note that in (1), $B_{x_j}, x_j \in [0,1]$, represents the pointwise control located on the boundary or in the domain.

The space of weak solutions to System (1) is chosen to be $H^1(0,1)$. Note that System (1) is exponentially stable in $H^1(0,1)$ if the boundary controls $B_0$ and $B_1$ are chosen such that $k_0 > 0$ and $k_1 \geq 0$ [13].

Denote a set of reference signals by $\{z^D_i(x_i, t)\}_{i=1}^m$, where $z^D_i(x_i, t) \in C^\infty$, $i = 1, \ldots, m$, for all $t \in (0, T)$ and $T < \infty$. Let $e_i(t) = z(x_i, t) - z^D_i(x_i, t)$ be the regulation errors. Let $e(t) = \{e_i(t)\}_{i=1}^m$ and $u(t) = \{u_i(t)\}_{i=1}^m$.

Problem 1. The considered regulation problem for set-point control is to find a dynamic control $u(t)$ such that the regulation error satisfies $e(t) \to 0$ as $t \to \infty$.

For the purpose of control design, we introduce an equivalent formulation of the in-domain control problem described in (1) by replacing the jump conditions in (1e) by pointwise controls as source terms. The resulting system will be of the following form

$$z_t(x, t) - z_{xx}(x, t) = \sum_{j=1}^m \delta(x - x_j)\alpha_j(t), \quad 0 < x < 1, \quad t > 0, \quad (2a)$$

$$z(x, 0) = \phi(x), \quad (2b)$$

$$B_0z = z_x(0, t) - k_0z(0, t) = 0, \quad B_1z = z_x(1, t) + k_1z(1, t) = 0, \quad (2c)$$

where $\delta(x - x_j)$ is the Dirac delta function supported at the point $x_j$, denoting the position of control support, and $\alpha_j : t \mapsto \mathbb{R}, j = 1, \ldots, m$, are the in-domain control signals.

Remark 1. It is noticed that the approximate controllability of the heat equation with pointwise control may be lost if the support of the control is located on a nodal set of eigenfunctions (see, e.g., [23, 24]). Nevertheless, this situation will not happen to
the considered system. Indeed, the eigenfunctions of System (2a) with the boundary conditions (2c) are given by [25]

\[ \psi_n(x) = \cos(\mu_n x) + \frac{k_0}{\mu_n} \sin(\mu_n x), \quad n = 1, 2, \ldots \] (3)

where \( \mu_n \) are positive roots of the transcendental equation \( \tan(\mu) = \mu(k_0 + k_1)/(\mu^2 - k_0 k_1) \). Therefore, all the points \( x \in (0, 1) \) are strategic for any \( k_0 > 0 \) and \( k_1 \geq 0 \).

**Lemma 1.** Considering weak solutions in \( C^0(H^1(0, 1), [0, T]) \), \( T < \infty \), System (1) and System (2) are equivalent if

\[ \alpha_j(t) = -u_j(t) = -[z_j]_{x_j}, \quad j = 1, \ldots, m. \]

**Proof.** The proof follows the idea presented in [26]. Indeed, it suffices to prove “System (1) \( \Rightarrow \) System (2).” Let \( X = L^2(0, 1) \) be a Hilbert space equipped with the inner product \( \langle u, v \rangle = \int_0^1 u(x)v(x)dx \), for any \( u, v \in X \). Let the operator \( A \) be defined by \( Au = u_{xx} \), with domain \( \mathcal{D}(A) = \{ u \in H^2(0, 1); B_0 u = B_1 u = 0 \} \). It is easy to see that \( A^* \), the adjoint of \( A \), is equal to \( A \). Let \( \tilde{A} \) be an extension of \( A \) with the domain \( \mathcal{D}(\tilde{A}) = \{ u \in X; u \in H^2(\bigcup_{i=0}^m (x_i, x_{i+1})), B_0 u = B_1 u = 0, u(x_j^+) = u(x_j^-), \quad j = 1, \ldots, m \} \). Let \( u \in \mathcal{D}(\tilde{A}) \), \( v \in \mathcal{D}(A^*) = \mathcal{D}(A) \). Using integration by parts we obtain that

\[ \langle \tilde{A}u, v \rangle = \langle u, Av \rangle + \sum_{j=1}^m (u_x(x_j^-) - u_x(x_j^+))v(x_j). \] (4)

Let \( X_{-1} = (\mathcal{D}(A^*))' \), the dual space of \( \mathcal{D}(A) \). We need to define another extension for \( A \). Let \( \tilde{A} : H^1(0, 1) \to X_{-1} \) be defined by

\[ \langle \tilde{A}u, v \rangle = \langle u, A^* v \rangle \quad \text{for all } v \in \mathcal{D}(A^*), \] (5)

with \( \mathcal{D}(\tilde{A}) = H^1(0, 1) \). Note that \( \delta(\cdot - x_j) \) is not in \( X \), but in the larger space \( X_{-1} \). It follows from (4), (5), and \( A = A^* \) that

\[ \tilde{A}u = A\tilde{u} = \sum_{j=1}^m (u_x(x_j^-) - u_x(x_j^+))\delta(x - x_j) \] (6)

in \( X_{-1} \). If \( u \) satisfies System (1), then \( \dot{u}(t) = \tilde{A}u(t) \), which yields, considering (6),

\[ \dot{u}(t) = A\tilde{u}(t) + \sum_{j=1}^m (u_x(x_j^-) - u_x(x_j^+))\delta(x - x_j) \]

Finally, we can see that System (1)
becomes System (2) with \( \alpha_j(t) = -u_j(t) = -[z^i]_{x_j} \), \( j = 1, \ldots, m \), where we look for generalized solutions \( u(\cdot, t) \in D(\hat{A}) = H^1(0, 1) \) such that (6) is true in \( X_{-1} \). □

To establish in-domain control at every actuation point, we will proceed in the way of parallel connection, i.e., for every \( x_j \in (0, 1) \), consider the following two systems

\[
\begin{align*}
    z_t(x,t) - z_{xx}(x,t) &= 0, \quad x \in (0, x_j) \cup (x_j, 1), \quad t > 0, \\
    z(x,0) &= \phi_j(x), \\
    B_0z = z_x(0,t) - k_0z(0,t) &= 0, \quad B_1z = z_x(1,t) + k_1z(1,t) = 0, \\
    z(x_j^+) = z(x_j^-), \\
    B_{x_j}z = [z_x]_{x_j} = v_j(t).
\end{align*}
\]

and

\[
\begin{align*}
    z_t(x,t) - z_{xx}(x,t) &= \delta(x-x_j)\beta_j(t), \quad 0 < x < 1, \quad t > 0, \\
    z(x,0) &= \phi_j(x), \\
    B_0z = z_x(0,t) - k_0z(0,t) &= 0, \quad B_1z = z_x(1,t) + k_1z(1,t) = 0, \\
    z(x_j^+) = z(x_j^-), \\
    B_{x_j}z = [z_x]_{x_j} = v_j(t).
\end{align*}
\]

with \( \sum_{j=1}^{m} \phi_j(x) = \phi(x) \). Similarly, System (7) and (8) are equivalent provided \( z \in H^1(0, 1) \) and \( \beta_j = -v_j = -[z_x^i]_{x_j} \). Let \( \alpha_j = -u_j = \beta_j = -v_j = -[z_x^i]_{x_j} \), for all \( j = 1, 2, \ldots, m \), where \( z^i \) denotes the solution to System (8). One may directly check that \( z(x,t) = \sum_{j=1}^{m} z^i(x,t) \) is a solution to System (2). Moreover,

\[
[z_x^i]_{x_i} = \sum_{j=1}^{m} [z^i_{x_j}]_{x_i} = [z^i_x]_{x_i} = u_i,
\]

for all \( i = 1, 2, \ldots, m \). Hence \( z(x,t) = \sum_{j=1}^{m} z^i(x,t) \) is a solution to System (1). Therefore, throughout this paper, we assume \( \alpha_j = -u_j = \beta_j = -v_j = -[z_x^i]_{x_j} \), for all \( j = 1, 2, \ldots, m \). Due to the equivalences of System (1) and (2), and System (7) and (8), we may consider (2) and System (7) in the following parts.

3. Control Design Based on Zero-Dynamics Inverse and Differential Flatness

In the framework of zero-dynamics inverse, the in-domain control is derived from the so-called forced zero-dynamics. To work with the parallel connected system (7),
we first split the reference signal as:

\[ z^D(x, t) = \sum_{j=1}^{m} \gamma_j(x, x_j) z^d_j(x_j, t), \quad (9) \]

where \( \gamma_j(x, x_j) \) will be determined in Theorem 5 (see Section 4). Denoting by \( \varepsilon^j(t) = z^j(x_j, t) - z^d_j(x_j, t) \) the regulation error corresponding to System (7), the zero-dynamics can be obtained by replacing the input constraints in (7e) by the requirement that the regulation errors vanish identically, i.e., \( \varepsilon^j(t) = 0 \). Thus, we obtain for a fixed \( j \):

\[ \xi_t(x, t) = \xi_{xx}(x, t), \quad x \in (0, x_j) \cup (x_j, 1), \quad t > 0, \quad (10a) \]
\[ \xi(x, 0) = 0, \quad (10b) \]
\[ \xi_x(0, t) - k_0 \xi(0, t) = 0, \quad \xi_x(1, t) + k_1 \xi(1, t) = 0, \quad (10c) \]
\[ \xi(x, t) = z^d_j(x_j, t), \quad (10d) \]

where \( z^d_j(x_j, t) \in H^1(0, T) \) for any \( T < \infty \). The in-domain control signal of System (7) is then given by

\[ v_j = [z^d_j]_{x_j} = [\xi^d_j]_{x_j}. \quad (11) \]

Note that (10) and (11) form a dynamic control scheme. Note also that (11) implies that the in-domain control signals can be derived from either the system trajectory or the solution of the zero-dynamics. The convergence of regulation errors with ZDI-based control is given in following theorem.

**Theorem 2.** (19) The regulation error corresponding to System (7) \( \varepsilon^j(t) = z^j(x_j, t) - z^d_j(x_j, t) \) tends to 0 as \( t \) tends to \( \infty \).

The proof of Theorem 2 is detailed in [19]. Note that a key fact used in the proof of this theorem is that the system given in (7) with null interior control is exponentially stable for any initial data \( \phi_j(x) \in L^2(0, 1) \) if \( k_0 > 0 \) and \( k_1 \geq 0 \).

Obviously, to find the control signals, we need to solve the corresponding zero-dynamics (10). To this end, we leverage the technique of flat systems [27, 11, 9]. In particular, we apply a standard procedure of Laplace transform-based method to find the solution to (10). Henceforth, we denote by \( \hat{f}(x, s) \) the Laplace transform of a
function $f(x, t)$ with respect to the time variable. Then, for fixed $x_j \in (0, 1)$, the transformed equations of (10) in the Laplace domain read as

\[ s \hat{\xi}(x, s) = \hat{\xi}_{xx}(x, s), \ x \in (0, x_j) \cup (x_j, 1), \ s \in \mathbb{C}, \]  
\[ \hat{\xi}(x, 0) = 0, \]  
\[ \hat{\xi}_x(0, s) = k_0 \hat{\xi}(0, s) = 0, \]  
\[ \hat{\xi}_x(1, s) + k_1 \hat{\xi}(1, s) = 0, \]  
\[ \hat{\xi}(x_j, s) = z^d_j(x_j, s). \]  
\[ (12a) \]
\[ (12b) \]
\[ (12c) \]
\[ (12d) \]

We divide (12) into two sub-systems, i.e., for fixed $x_j \in (0, 1)$, considering

\[ s \hat{\xi}(x, s) = \hat{\xi}_{xx}(x, s), \ 0 < x < x_j, \ s \in \mathbb{C}, \]  
\[ \hat{\xi}(x, 0) = 0, \]  
\[ \hat{\xi}_x(0, s) = k_0 \hat{\xi}(0, s) = 0, \]  
\[ \hat{\xi}_x(x_j, s) = \hat{\xi}_x(x_j, s), \]  
\[ \hat{\xi}(x_j, s) = z^d_j(x_j, s), \]  
\[ (13a) \]
\[ (13b) \]
\[ (13c) \]
\[ (13d) \]

and

\[ s \hat{\xi}(x, s) = \hat{\xi}_{xx}(x, s), \ x_j < x < 1, \ s \in \mathbb{C}, \]  
\[ \hat{\xi}(x, 0) = 0, \]  
\[ \hat{\xi}_x(1, s) + k_1 \hat{\xi}(x, s) = 0, \]  
\[ \hat{\xi}(x_j, s) = z^d_j(x_j, s), \]  
\[ (14a) \]
\[ (14b) \]
\[ (14c) \]
\[ (14d) \]

Let $\hat{\xi}_-^j(x, s)$ and $\hat{\xi}_+^j(x, s)$ be the general solutions to (13) and (14), respectively, and denote their inverse Laplace transforms by $\xi_-^j(x, t)$ and $\xi_+^j(x, t)$. The solution to (10) can be written as

\[ \xi^j(x, t) = \xi_-^j(x, t) \chi((0, x_j)) + \xi_+^j(x, t) \chi([x_j, 1]), \]

where

\[ \chi(x)_{(\Omega_j)} = \begin{cases} 
1, & x \in \Omega_j \subseteq (0, 1); \\
0, & \text{otherwise.} 
\end{cases} \]

Then at each point $x_i \in (0, 1)$, by (11) and the argument of “parallel connection” (see Section 2), we have $[z_x]_{x_i} = \sum_{j=1}^{m} [z^j_x]_{x_i} = [z^i_x]_{x_i}, \ i = 1, \ldots, m$. Hence the
in-domain control signals of System (11) can be computed by

\[ u_i = [z_x]_{x_i} = [\xi_x^i]_{x_i}, \quad i = 1, \ldots, m. \]  

(15)

In the following steps, we present the computation of the solution to System (10), \( \xi^j \). Issues related to the generation reference trajectory \( z^D(x,t) \) for System (1) will be addressed in Section 4.

Note that \( \hat{\xi}^j_- (x,s) \) and \( \hat{\xi}^j_+ (x,s) \), the general solutions to (13) and (14), are given by

\[ \hat{\xi}^j_- (x,s) = C_1 \phi_1 (x,s) + C_2 \phi_2 (x,s), \]
\[ \hat{\xi}^j_+ (x,s) = C_3 \phi_1 (x,s) + C_4 \phi_2 (x,s), \]

with

\[ \phi_1 (x,s) = \frac{\sinh(\sqrt{s}x)}{\sqrt{s}}, \quad \phi_2 (x,s) = \cosh(\sqrt{s}x). \]

We obtain by applying (13c) and (13d)

\[ C_1 \phi_1 (x_j,s) + C_2 \phi_2 (x_j,s) = \hat{z}^d_j (x_j,s), \quad C_1 - k_0 C_2 = 0, \]

which can be written as

\[ \begin{pmatrix} \phi_1 (x_j,s) & \phi_2 (x_j,s) \\ 1 & -k_0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \hat{z}^d_j (x_j,s) \\ 0 \end{pmatrix}. \]

Let

\[ R^-_j = \begin{pmatrix} \phi_1 (x_j,s) & \phi_2 (x_j,s) \\ 1 & -k_0 \end{pmatrix} \]

and

\[ \hat{z}^d_j (x_j,s) = -\det(R^-_j) \hat{y}^j_-(x_j,s). \]  

(16)

We obtain

\[ \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{\text{adj}(R^-_j)}{\det(R^-_j)} \begin{pmatrix} \hat{z}^d_j (x_j,s) \\ 0 \end{pmatrix} = \begin{pmatrix} k_0 \hat{y}^j_-(x_j,s) \\ \hat{y}^j_-(x_j,s) \end{pmatrix}. \]

Therefore, the solution to (13) can be expressed as

\[ \hat{\xi}^j_- (x,s) = (k_0 \phi_1 (x) + \phi_2 (x)) \hat{y}^j_-(x_j,s). \]  

(17)
We may proceed in the same way to deal with (14). Indeed, letting
\[ R^j_+ = \begin{pmatrix} \phi_1(x_j, s) & \phi_2(x_j, s) \\ \phi_2(1, s) + k_1 \phi_1(1, s) & s \phi_1(1, s) + k_1 \phi_2(1, s) \end{pmatrix} \]
and
\[ \hat{z}^d_j(x_j, s) = \det(R^j_+) \hat{\gamma}^d_+(x_j, s), \tag{18} \]
we get from (14)
\[ \begin{pmatrix} C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} (s \phi_1(1, s) + k_1 \phi_2(1, s)) \hat{\gamma}^d_+(x_j, s) \\ -(\phi_2(1, s) + k_1 \phi_1(1, s)) \hat{\gamma}^d_+(x_j, s) \end{pmatrix}, \]
and
\[ \hat{\xi}^d_+(x, s) = ((s \phi_1(1, s) + k_1 \phi_2(1, s)) \phi_1(x) + (\phi_2(1, s) + k_1 \phi_1(1, s)) \phi_2(x)) \hat{\gamma}^d_+(x_j, s). \tag{19} \]

Applying modulus theory [28, 29] to (16) and (18), we may choose \( \hat{\gamma}_j(x_j, s) \) as the basic output such that
\[ \hat{\gamma}^d_+(x_j, s) = -\det(R^j_-) \hat{\gamma}_j(x_j, s), \tag{20} \]
\[ \hat{\gamma}^d_-(x_j, s) = \det(R^j_+) \hat{\gamma}_j(x_j, s). \tag{21} \]

Then, using the property of hyperbolic functions, we obtain from (17) and (19) that
\[ \hat{\xi}^d_-(x, s) = \left( k_1 \frac{\sinh(\sqrt{s}x_j) - \sqrt{s}}{\sqrt{s}} - \cosh(\sqrt{s}x_j - \sqrt{s}) \right) \left( k_0 \frac{\sinh(\sqrt{s}x_0)}{\sqrt{s}} + \cosh(\sqrt{s}x_0) \right) \hat{\gamma}_j(x_j, s), \tag{22} \]
\[ \hat{\xi}^d_+(x, s) = \left( k_1 \frac{\sinh(\sqrt{s}x_j) - \sqrt{s}}{\sqrt{s}} - \cosh(\sqrt{s}x_j - \sqrt{s}) \right) \left( k_0 \frac{\sinh(\sqrt{s}x_j)}{\sqrt{s}} + \cosh(\sqrt{s}x_j) \right) \hat{\gamma}_j(x_j, s). \tag{23} \]

Note that
\[ \hat{\xi}^d(x, s) = \hat{\xi}^d_-(x, s) \chi_{\{0, x_j\}} + \hat{\xi}^d_+(x, s) \chi_{\{x_j, 1\}} \tag{24} \]
is a solution to (12). Using the fact
\[
\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!},
\]
we obtain the time-domain solution to (10), which is given by
\[
\xi^j(x, t) = \left[ k_0 k_1 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^{2k+1} (x_j - 1)^{2(n-k)+1}}{(2k+1)!(2(n-k)+1)!} y_j^{(n)} \right] - k_0 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^{2k} (x_j - 1)^{2(n-k)}}{(2k)!(2(n-k))!} y_j^{(n)}
+ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^{2k} (x_j - 1)^{2(n-k)+1}}{(2k+1)!} y_j^{(n)} - k_0 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^{2k} (x_j - 1)^{2(n-k)}}{(2k)!(2(n-k))!} y_j^{(n)}
+ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^{2k} (x_j - 1)^{2(n-k)+1}}{(2k+1)!} y_j^{(n)} - k_0 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^{2k} (x_j - 1)^{2(n-k)}}{(2k)!(2(n-k))!} y_j^{(n)}.
\]
(25)

Furthermore, by a direct computation we get
\[
\left[ \frac{\partial \xi^j}{\partial x} \right]_{x_j} = \left[ k_0 k_1 + \sqrt{s} \right] \sinh(\sqrt{s}) + (k_0 + k_1) \cosh(\sqrt{s}) \tilde{y}_j(x_j, s)
= \left( k_0 k_1 \sum_{n=0}^{\infty} \frac{s^n}{(2n+1)!} + (k_0 + k_1) \sum_{n=0}^{\infty} \frac{s^n}{(2n)!} + \sum_{n=0}^{\infty} \frac{s^{n+1}}{(2n+1)!} \right) \tilde{y}_j(x_j, s).
\]
(26)

It follows from (15) that
\[
u_j(t) = \left[ \frac{\partial \xi^j}{\partial x} \right]_{x_j} = k_0 k_1 \sum_{n=0}^{\infty} \frac{y_j^{(n)}(x_j, t)}{(2n+1)!} + (k_0 + k_1) \sum_{n=0}^{\infty} \frac{y_j^{(n)}(x_j, t)}{(2n)!} + \sum_{n=0}^{\infty} \frac{y_j^{(n+1)}(x_j, t)}{(2n+1)!}.
\]
(27)

Finally, provided \( \xi^j(x_j, t) = \xi^j(x_j, t) \), for \( j = 1, \ldots, m \), the reference trajectory \( z^D(x, t) \) can be determined from (9) and (25).

4. Motion Planning

For control purpose, we have to choose appreciate reference trajectories, or equivalently the basic outputs. Denote now by \( \tilde{z}^D(x) \) the desired steady-state profile. Without
loss of generality, we consider a set of basic outputs of the form:

\[ y_j(t) = \mathcal{G}(x_j)\varphi_j(t), \quad j = 1, \ldots, m, \quad (28) \]

where \( \varphi_j(t) \) is a smooth function evolving from 0 to 1. Motion planning amounts then to deriving \( \mathcal{G}(x_j) \) from \( \bar{z}^D(x) \) and to determining appropriate functions \( \varphi_j(t) \), for \( j = 1, \ldots, m \).

To this aim and due to the equivalence of the systems (1) and (2), we consider the steady-state heat equation corresponding to System (2):

\[ \bar{z}_{xx}(x) = \sum_{j=1}^{m} \delta(x - x_j)\bar{\alpha}_j, \quad 0 < x < 1, \quad t > 0, \quad (29a) \]

\[ \bar{z}_x(0) - k_0\bar{z}(0) = 0, \quad \bar{z}_x(1) + k_1\bar{z}(1) = 0. \quad (29b) \]

Based on the principle of superposition for linear systems, the solution to the steady-state heat equation (29) can be expressed as:

\[ \bar{z}(x) = \int_0^1 \sum_{j=1}^{m} G(x, \zeta)\delta(\zeta - x_j)\bar{\alpha}_j d\zeta = \sum_{j=1}^{m} G(x, x_j)\bar{\alpha}_j. \quad (30) \]

where \( G(x, \zeta) \) is the Green’s function corresponding to (29), which is of the form

\[ G(x, \zeta) = \begin{cases} \frac{(k_1\zeta - k_1 - 1)(k_0x + 1)}{k_0 + k_1 + k_0k_1}, & 0 \leq x < \zeta; \\ \frac{k_0 + k_1 + k_0k_1}{(k_1x - k_1 - 1)(k_0\zeta + 1)}, & \zeta \leq x \leq 1. \end{cases} \quad (31) \]

Indeed, it is easy to check that \( G_{xx}(x, \zeta) = \delta(x - \zeta) \) and \( G(x, \zeta) \) satisfies the boundary conditions, \( G_x(0, \zeta) - k_0G(0, \zeta) = 0 \) and \( G_x(1, \zeta) + k_1G(1, \zeta) = 0 \), the joint condition, \( G(\zeta^+, \zeta) = G(\zeta^-, \zeta) \), and the jump condition, \( [G_x(x, \zeta)]_\zeta = 1 \).

Taking \( m \) distinguished points along the solution to (29), \( \bar{z}(x_1), \ldots, \bar{z}(x_m) \), we get

\[ \begin{pmatrix} \bar{z}(x_1) \\ \vdots \\ \bar{z}(x_m) \end{pmatrix} = \begin{pmatrix} G(x_1, x_1) & \cdots & G(x_1, x_m) \\ \vdots & \ddots & \vdots \\ G(x_m, x_1) & \cdots & G(x_m, x_m) \end{pmatrix} \begin{pmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_m \end{pmatrix}. \quad (32) \]

Note that in (32), the matrix formed by the Green’s function defined an input-output map in steady-state, which is also called the influence matrix.
Lemma 3. The influence matrix chosen as in (32) is invertible. Thus,
\[
\left( \begin{array}{c}
\tilde{\alpha}_1 \\
\vdots \\
\tilde{\alpha}_m 
\end{array} \right) = \left( \begin{array}{ccc}
G(x_1, x_1) & \cdots & G(x_1, x_m) \\
\vdots & \ddots & \vdots \\
G(x_m, x_1) & \cdots & G(x_m, x_m) 
\end{array} \right)^{-1} \left( \begin{array}{c}
\tilde{z}(x_1) \\
\vdots \\
\tilde{z}(x_m) 
\end{array} \right). \tag{33}
\]

PROOF. For \( m = 1 \), since \( k_0 > 0 \), \( k_1 > 0 \), and \( x_1 \in (0, 1) \), it follows that \( G(x_1, x_1) = \frac{(k_1 x_1 - k_1 - 1)(k_0 x_1 + 1)}{k_0 + k_1 + k_0 k_1} < 0 \). Hence it is invertible. We prove the claim for \( m > 1 \) by contradiction. Suppose that the influence matrix is not invertible, then it is of rank \( m - 1 \) or less. Without loss of generality, we may assume that for some \( x_n > x_i \) with \( i = 1, \ldots, n - 1 \), there exist \( n - 1 \) constants \( l_1, l_2, \ldots, l_{n-1} \) such that
\[
G(x_j, x_n) = \sum_{i=1}^{n-1} l_i G(x_1, x_i), \quad j = 1, \ldots, n, \tag{34}
\]
where \( 1 < n \leq m \) and \( \sum_{i=1}^{n-1} l_i^2 > 0 \). Let
\[
G(x) = G(x, x_n), \quad F(x) = \sum_{i=1}^{n-1} l_i G(x, x_i). \tag{35}
\]
(34) shows that \( F(x) = G(x) \) at every boundary point of \([x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n] \). Note that \( F(x) \) is a linear function in \([x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n] \), and that \( G(x) = (k_1 x_n - k_1 - 1)(k_0 x + 1)/(k_0 + k_1 + k_0 k_1) \) in \([x_1, x_n] \), i.e., \( G(x) \) is a linear function in \([x_1, x_n] \). Hence \( F(x) \equiv G(x) \) in \([x_1, x_n] \).

By \( F(x_1) = G(x_1) \), we get
\[
k_1 x_n - k_1 - 1 = \sum_{i=1}^{n-1} l_i (k_1 x_i - k_1 - 1). \tag{35}
\]
By \( F(x_n) = G(x_n) \), we get
\[
k_0 x_n + 1 = \sum_{i=1}^{n-1} l_i (k_0 x_i + 1). \tag{36}
\]
Therefore
\[
\sum_{i=1}^{n-1} l_i = 1. \tag{37}
\]
By $F_x(x_1^+) = G_x(x_1^+)$ and $F_x(x_n^-) = G_x(x_n^-)$, we get

$$k_0(k_1x_n - k_1 - 1) = k_0k_1 \sum_{i=1}^{n-1} l_ix_i - k_0(k_1 + 1) \sum_{i=2}^{n-1} l_i + l_1k_1,$$

$$= k_0k_1 \sum_{i=1}^{n-1} l_ix_i + k_1 \sum_{i=1}^{n-1} l_i. \quad (38)$$

It follows that $\sum_{i=2}^{n-1} l_i = 0$, which yields, considering (37), $l_1 = 1$. By $F_x(x_2^+) = G_x(x_2) = F_x(x_2^-)$, we deduce

$$l_1k_1(k_0x_1 + 1) + l_2k_1(k_0x_1 + 1) + k_0 \sum_{i=3}^{n-1} l_i(k_1x_i - k_1 - 1)$$

$$= l_1k_1(k_0x_1 + 1) + k_0 \sum_{i=2}^{n-1} l_i(k_1x_i - k_1 - 1),$$

which gives $l_2 = 0$. Similarly, by $F_x(x_j^+) = G_x(x_j) = F_x(x_j^-)$ ($j = 3, 4, ..., n - 2$), we obtain $l_3 = l_4 = ... = l_{n-2} = 0$. Hence $l_{n-1} = 0$. Then we deduce from (38) that

$$k_0(k_1x_n - k_1 - 1) = k_1(k_0x_1 + 1). \quad (39)$$

It follows from (36) that $k_0x_1 + 1 = k_0x_n + 1$. We conclude then by (39) that $k_0 + k_1 + k_0k_1 = 0$, which is a contradiction to $k_0 > 0$ and $k_1 \geq 0$. \hfill \square

In steady-state, we can obtain from (27) that

$$\bar{u}_j = (k_0k_1 + k_0 + k_1)\bar{y}(x_j) = -\bar{\alpha}_j. \quad (40)$$

Finally, $\bar{y}(x_j)$ can be computed by (33) and (40) for a given $\bar{z}^D(x)$.

It is worth noting that (33) provides a simple and straightforward way to compute the static control from the prescribed steady-state profile. Indeed, a direct computation can show that applying (33) will result in the same static control obtained in [19] where a serially connected model is used.

To ensure the convergence of (25) and (27), we choose the following smooth func-
tion as \( \varphi_j(t) \):

\[
\varphi_j(t) = \begin{cases} 
0, & \text{if } t \leq 0 \\
\int_0^t \exp(-1/(\tau(1-\tau)))^\varepsilon d\tau, & \text{if } t \in (0, T) \\
\int_0^T \exp(-1/(\tau(1-\tau)))^\varepsilon d\tau, & \text{if } t \geq T
\end{cases}
\]

(41)

which is known as Gevrey function of order \( \sigma = 1 + 1/\varepsilon, \varepsilon > 0 \) (see, e.g., [9]).

**Lemma 4.** If the basic outputs \( \varphi_j(t), j = 1, \ldots, m \), are chosen as Gevrey functions of order \( 1 < \sigma < 2 \), then the infinite series (25) and (27) are convergent.

**Proof.** We prove the convergence of the power series (25) and (27) using Cauchy-Hadamard Theorem. Indeed, it suffices to prove the convergence of

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(2k)!(2(n-k)!)^2} \varphi_j^{(n)}(t).
\]

(42)

Then the convergence of the series in (25) and (27) follows easily using the same argument.

We recall that the bounds of Gevrey functions of order \( \sigma \) are given by (43)

\[
\exists K, M > 0, \forall k \in \mathbb{Z} \geq 0, \forall t \in [t_0, T], \left| \varphi^{(k+1)}(t) \right| \leq M \frac{(k!)^\sigma}{K^k}.
\]

(43)

Denote in (42)

\[
b_n = \sum_{k=0}^{n} \frac{1}{(2k)!(2(n-k)!)^2} \varphi_j^{(n)}(t).
\]

Then, (42) converges if \( \limsup_{n \to \infty} \sqrt[n]{|b_n|} < 1 \). Now \( b_n \) can be estimated by (43)

\[
|b_n| \leq \sum_{k=0}^{n} \frac{M}{(2k)!(2(n-k)!)^2} \frac{(n!)^\sigma}{K^n} \leq M \frac{2^n}{(2n)!} \frac{(n!)^\sigma}{K^n}.
\]

Therefore

\[
\limsup_{n \to \infty} \sqrt[n]{|b_n|} \leq \limsup_{n \to \infty} \frac{2}{K} M^{1/n} \left( \frac{(n!)^{1/n}}{((2n)!)^{1/2n}} \right)^{\sigma} \leq \limsup_{n \to \infty} \frac{2}{K} \frac{(n/e)^\sigma}{(2n/e)^{2}}
\]

\[
= \frac{\varepsilon^{2-\sigma}}{2K} \limsup_{n \to \infty} n^{\sigma-2} = \begin{cases} 
0, & \sigma < 2, \\
\frac{1}{2K}, & \sigma = 2, \\
\infty, & \sigma > 2,
\end{cases}
\]

(44)

\[15\]
where in the second inequality we applied Stirling’s formula \( \sqrt[n]{n!} \approx (n/e) \). We can conclude by Cauchy-Hadamard Theorem that (42) converges for \( \sigma < 2 \), and for \( \sigma = 2 \) if \( 2K > 1 \). The series (42) diverges if \( \sigma > 2 \).

\[ \square \]

**Theorem 5.** Assume \( k_0 > 0 \) and \( k_1 \geq 0 \). Let the basic outputs \( \varphi_j(t) \), \( j = 1, \ldots, m \), be chosen as (41) with an order \( 1 < \sigma < 2 \). Let the reference trajectory of System (1) be given by (9) with

\[ \gamma_j(x, x_j) = -\frac{(k_0k_1 + k_0 + k_1)G(x, x_j)}{(k_0x_j + 1)(k_0(x_j - 1) - 1)} \quad j = 1, \ldots, m, \quad (45) \]

where \( G(x, \zeta) \) is the Green’s function defined in (31). Then the regulation error of System (1) with the control given in (27) tends to zero, i.e., \( e_i(t) = z(x_i, t) - z^D_i(x_i, t) \to 0 \) as \( t \to \infty \), for \( i = 1, 2, \ldots, m \).

**Proof.** By a direct computation we have

\[ |e_i(t)| = |z(x_i, t) - z^D_i(x_i, t)| \]

\[ = |z(x_i, t) + \sum_{j=1}^{m} \frac{(k_0k_1 + k_0 + k_1)G(x_i, x_j)\xi_j(x_j, t)}{(k_0x_j + 1)(k_0(x_j - 1) - 1)}| \]

\[ = |z(x_i, t) + \sum_{j=1}^{m} \frac{(k_0k_1 + k_0 + k_1)G(x_i, x_j)\xi_j(x_j, t)}{(k_0x_j + 1)(k_0(x_j - 1) - 1)}| \]

\[ \leq |z(x_i, t) - \bar{z}(x_i)| + |\bar{z}(x_i) + (k_0k_1 + k_0 + k_1)\sum_{j=1}^{m} G(x_i, x_j)\bar{y}(x_j)| \]

\[ + \left| (k_0k_1 + k_0 + k_1)\sum_{j=1}^{m} \frac{G(x_i, x_j)\xi_j(x_j, t)}{(k_0x_j + 1)(k_0(x_j - 1) - 1)} \right| - (k_0k_1 + k_0 + k_1)\sum_{j=1}^{m} G(x_i, x_j)\bar{y}(x_j) \] .

By (40) and (30), it follows \( \bar{z}(x_i) = -(k_0k_1 + k_0 + k_1)\sum_{j=1}^{m} G(x_i, x_j)\bar{y}(x_j) \). Based on (25), (28), and the property of \( \varphi_j(t) \) we have

\[ \frac{\xi_j(x_j, t)}{(k_0x_j + 1)(k_0(x_j - 1) - 1)} \to \bar{y}(x_j) \text{ as } t \to \infty. \]

Note that \( z(x_i, t) \to \bar{z}(x_i) \text{ as } t \to \infty \). Therefore \( |e_i(t)| \to 0 \) as \( t \to \infty \). \( \square \)
Remark 2. For any $x \in (0, 1)$, replace $x_i$ by $x$ in the proof of Theorem 5, we can get $|z(x, t) - z^D(x, t)| \to 0$ as $t \to \infty$, which shows that the solution $z(x, t)$ of System (1) converges to the reference trajectory $z^D(x, t)$ at every point $x \in (0, 1)$.

5. Simulation Study

In the simulation, we implement System (2) with 12 actuators evenly distributed in the domain at the spot points $\{1/13, 2/13, \ldots, 12/13\}$. The numerical implementation is based on a PDE solver, `pdepe`, in Matlab PDE Toolbox. In numerical simulation, 200 points in space and 100 points in time are used for the region $[0, 1] \times [0, 0.5]$. The basic outputs $\varphi_j(t)$ used in the simulation are Gevrey functions of the same order. In order to meet the convergence condition given in Lemma 4, the parameter of the Gevrey function is set to $\varepsilon = 1.1$. The feedback boundary control gains are chosen as $k_0 = k_1 = 10$. The initial condition in simulation is set to $z(x, 0) = \cos(\pi x)$.

The desired steady-state heat distribution is a piecewise linear curve, depicted in Fig. 1-(a), which is a solution to (29). The corresponding static controls, $\pi_1, \ldots, \pi_{12}$, are shown in Fig. 1-(b). Note that the dynamic control signals, $\alpha_i(t)$, are smooth functions connecting 0 to $\pi_i$ for $i = 1, \ldots, 12$. The evolution of heat distribution with static and dynamic control, as well as the corresponding regulation errors with respect to the static profile defined as $e(x, t) = z(x, t) - z^D(x)$, are depicted in Fig. 2. The simulation results show that the system performs well with the developed control scheme. It can also be seen that the dynamic control provides a faster response time compared to the static one.

![Figure 1](image-url)
6. Conclusion

This paper presented a solution to the problem of set-point control of heat distribution with in-domain actuation described by an inhomogeneous parabolic PDE. To apply the principle of superposition, the system is presented in a parallel connection form. The dynamic control problem introduced by the ZDI design is solved by using the technique of flat systems motion planning. As the control with multiple in-domain actuators results in a MIMO problem, a Green’s function-based reference trajectory decomposition is introduced, which considerably simplifies the control design and implementation. Convergence and solvability analysis confirms the validity of the control algorithm and the simulation results demonstrate the viability of the proposed approach. Finally, as both ZDI design and flatness-based control can be carried out in a systematic manner, we can expect that the approach developed in this work may be applicable to a broader class of distributed parameter systems.
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