A New Version of a Posteriori Choosing Regularization Parameter in Ill-Posed Problems

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Abstract. The new version of a posteriori choice (NVAC) of the regularization parameter $\alpha$ in the classical Tikhonov regularization method is considered. Lemmas and theorems on the error and the asymptotic convergence rate of the regularized solution are proved. A numerical example is given.

Key words. The classical Tikhonov regularization method; Choice of the regularization parameter $\alpha$; Estimates for $\alpha$ and for the regularized solution error.

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1. Introduction

Consider an operator equation of the first kind

$$Ay = f, \quad y \in H_1, \quad f \in H_2,$$

where $H_1$ and $H_2$ are Hilbert spaces and $A : H_1 \to H_2$ is a linear bounded operator. Suppose that the exact solution $\tilde{y}$ is the normal pseudosolution [1, 2]. Let, instead of the exact $f$ and $A$, we have $\tilde{f}$ and $\tilde{A}$ such that $\|\tilde{f} - f\| \leq \delta$, $\|\tilde{A} - A\| \leq \theta$, $\theta \geq 0$. Denote by $\gamma \equiv (\delta, \theta)$. Given $\tilde{f}$, $\tilde{A}$, $\delta$, and $\theta$, the problem is to find an element $y_\gamma \in H_1$ that is a stable approximation of $\tilde{y}$ such that $\|y_\gamma - \tilde{y}\| \to 0$ as $\gamma \to 0$.

In the classical Tikhonov regularization method (using stabilizers of the type $\|y\|^2_{L_2}$ or $\|y\|^2_{W_n}$), one solves the equation [1][10]

$$\alpha y_\alpha + \tilde{A}^* \tilde{A} y_\alpha = \tilde{A}^* \tilde{f},$$

where $\alpha > 0$ is the regularization parameter.

Well-known ways for choosing the regularization parameter $\alpha$ were developed, namely, the discrepancy principle [11], the generalized discrepancy principle (GDP) [7], the modified discrepancy principle (MDP) [12][17], the cross-validation method [18], the iteration stopping rule by discrepancy [5][6], the local regularising algorithm [19], the adaptive specialized generalized discrepancy principle (SGDP) [1], etc. Estimates of the error $\|y_\alpha - \tilde{y}\|$ for the regularized solution $y_\alpha$ were obtained, among them, with use of an a priori information about the solution $\tilde{y}$ (the sourcewise representability, etc.) [1][3][5][11][17][20][21].


However, solving a number of model examples shows the following (see [4, 7], et al.). For finite \( \delta \) and \( \theta \), the principles can overstate the value of \( \alpha \) in comparison with \( \alpha_{\text{opt}} \). As a result, the error \( \| y_\alpha - \bar{y} \| \) is overstated in comparison with \( \| y_{\alpha_{\text{opt}}} - \bar{y} \| \), and the solution \( y_\alpha \) becomes more smooth than \( y_{\alpha_{\text{opt}}} \), and “the fine structure” of the solution \( y_\alpha \) is lost (cf. [22]). Here, \( \alpha_{\text{opt}} \) is the value of \( \alpha \) for which \( \| y_\alpha - \bar{y} \| = \min_{\alpha} \) (the value of \( \alpha_{\text{opt}} \) can be determined without strong a priori suppositions about the solution only in solving model examples). This effect usually appears when the relative errors \( \delta_{\text{rel}} \) and \( \theta_{\text{rel}} \gtrsim 1\% \) [4, p. 283], [7].

The aim of this paper is the further development of the new version of a posteriori choice of \( \alpha \) (NVAC) [2] concentrating attention on the question about closeness of \( \alpha \) to \( \alpha_{\text{opt}} \) and, as a result, of \( \| y_\alpha - \bar{y} \| \) to \( \| y_{\alpha_{\text{opt}}} - \bar{y} \| \), furthermore, not so much in asymptotics for \( \delta, \theta \to 0 \), as for finite \( \delta \) and \( \theta \). In this paper, the modified formulations of the NVAC’s statements are given, moreover, as far as possible without using the sourcewise representability of \( \bar{y} \). In this case, the solution error estimates for finite \( \delta, \theta, \alpha \) depend on the exact solution \( \bar{y} \) that is known only in model examples. And in asymptotics (for \( \delta, \theta, \alpha \to 0 \)), the order of convergence of \( y_\alpha \) to \( \bar{y} \) will be obtained.

**Remark 1.** Since \( \alpha_{\text{opt}} \) and \( y_{\alpha_{\text{opt}}} \) are known only in model examples but are unknown in real problems, so the efficiency of the new version must be verified for model examples.

2. The idea of the NVAC

Let us write Eq. (2) in the form

\[
\alpha y_\alpha + \tilde{R} y_\alpha = \tilde{F} ,
\]

where \( \tilde{R} = \widetilde{A}^* \widetilde{A} \), \( \tilde{F} = \widetilde{A}^* \tilde{f} \).

Along with the operator equation (1), consider the Fredholm integral equation of the first kind

\[
Ay \equiv \int_a^b K(x, s) y(s) \, ds = f(x) , \quad c \leq x \leq d .
\]

In the Tikhonov regularization method, instead of Eq. (4), one solves the equation (for \( H_1 = W_2^1, H_2 = L_2^1 \)) [4, p. 24], [23]

\[
\alpha [y_\alpha(t) - \tau y_\alpha''(t)] + \int_a^b \tilde{R}(t, s) y_\alpha(s) \, ds = \tilde{F}(t) , \quad a \leq t \leq b , \quad \tau \geq 0 ,
\]

\[
y_\alpha(a) = y_\alpha(b) = 0 ,
\]

\[
\tilde{R}(t, s) = \tilde{R}(s, t) = \int_c^d \widetilde{K}(x, t) \widetilde{K}(x, s) \, dx ,
\]

\[
\tilde{F}(t) = \int_c^d \widetilde{K}(x, t) \tilde{f}(x) \, dx .
\]

Actually, the original equation in the Tikhonov regularization method is the equation \( \widetilde{A}^* \widetilde{A} y = \widetilde{A}^* \tilde{f} \) rather than \( \widetilde{A} y = \tilde{f} \). Different variants of the

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discrepancy principle \[3, 9, 11, 17, 20, 21\] use the error \( \delta \) of the right-hand side \( f \). However, the function \( f(x) \) does not appear explicitly as a right-hand side in the Tikhonov method. The right-hand side is the function \( \tilde{F}(t) \) (see \[4\] and \[5\]). The function \( f(x) \) comes under the integral sign in the expression for \( \tilde{F}(t) \) (see \[7\]), while the integration operation is a smoothing filter with respect to \( \tilde{f} \). As a result, random errors in \( f(x) \) will be smoothed to a certain extent. In this case, the relative error in \( \tilde{F}(t) \) can become considerably less than the relative error in \( \tilde{f} \) \[2\].

Concerning the error \( \theta \) of the operator \( \tilde{A} \), the factual operator in the Tikhonov method is the operator \( R \equiv \tilde{A}^* \tilde{A} \) rather than \( \tilde{A} \). Therefore, in choosing \( \alpha \) from a discrepancy, it is more appropriately to use the errors of the elements \( \tilde{F} \) and \( R \) rather than \( \delta \) and \( \theta \) (the errors of \( \tilde{f} \) and \( \tilde{A} \)). However, on deriving asymptotic estimates for \( \alpha \) and for an error of the solution \( y_\alpha \), one should use the errors of both the elements \( \tilde{F} \) and \( R \) and ones \( \tilde{f} \) and \( \tilde{A} \).

In the generalized discrepancy principle (GDP) \[4\], \( \alpha = \alpha_d \) (from discrepancy) is chosen to be a root of the equation \( \| \tilde{A} y_\alpha - f \|^2 = (\delta + \theta \| y_\alpha \|)^2 + \tilde{\mu}^2 \), where \( \tilde{\mu} = \inf_y \| \tilde{A} y - \tilde{f} \| \) is the incompatibility measure of the equation \( \tilde{A} y = \tilde{f} \).

According to the Kojdecki way \[9\], \( \alpha \) is a root of the equation

\[
\alpha^q \| \tilde{A}^* \tilde{A} y_\alpha - \tilde{A}^* \tilde{f} \| = \beta \| \tilde{A} \| (\delta + \theta \| y_\alpha \|) \tag{8}
\]

or, with regard to \[2\],

\[
\alpha^{q+1} \| y_\alpha \| = \beta \| \tilde{A} \| (\delta + \theta \| y_\alpha \|),
\]

where \( q \geq 0 \) and \( \beta > 0 \) are some numbers. One has proved \[2\] the following lemma.

**Lemma 1.** The incompatibility measure \( \tilde{\nu} = \inf_y \| R y - \tilde{F} \| \) of the equation \( R y = \tilde{F} \) is equal to zero.

Now, we formulate again the new version of the a posteriori choice of \( \alpha \) (NVAC), moreover, the results obtained in \[2\] will be given without proofs. According to the NVAC, with regard to Lemma 1, the regularization parameter \( \alpha \) is chosen to be a root of the equation \[2\]

\[
\alpha^q \| R y_\alpha - \tilde{F} \| = \beta (\Delta + \Theta \| y_\alpha \|), \quad q \geq 0, \quad \beta > 0, \tag{9}
\]

or a root of the equivalent equation

\[
\alpha^{q+1} \| y_\alpha \| = \beta (\Delta + \Theta \| y_\alpha \|), \quad q \geq 0, \quad \beta > 0, \tag{10}
\]

furthermore, \( \| \tilde{F} - F \| \leq \Delta \) and \( \| \tilde{R} - R \| \leq \Theta \), where \( \Delta = \Delta(\delta, \theta) > 0 \) is an upper estimate for the error of the right-hand side \( \tilde{F} \) and \( \Theta = \Theta(\theta) \geq 0 \) is an upper estimate for the error of the operator \( \tilde{R} \). Denote by \( \Gamma \equiv (\Delta, \Theta) \) and by \( \alpha_n \) a root of \[9\] or \[10\] (the symbol “n” denotes “new”).

**Remark 2.** Equation \[9\] is rather like the equation \[8\]. However, these equations have the difference of principle, namely, in Eq. \[8\], the errors \( \delta \) and \( \theta \) are used and the factor \( \| \tilde{A} \| \) is separated from \( \delta \) and \( \theta \), whereas in Eq. \[9\], \( \Delta \) and \( \Theta \) are used. Meanwhile, the value of \( \| \tilde{A} \| (\delta + \theta \| y_\alpha \|) \) can be considerably

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greater than $\Delta + \Theta \|y\|$. This difference can lead to overstated values of $\alpha$ and $\|y - \tilde{y}\|$.

3. Justification of the New Version of a Posteriori Choosing $\alpha$

Denote the left-hand side of (9) or (10) as
$$
\psi(\alpha) \equiv \alpha^q \|\tilde{R} y - \tilde{F}\| = \alpha^{q+1} \|y\|
$$
and the right-hand side of (9) or (10) as
$$
\xi(\alpha) \equiv \beta(\Delta + \Theta \|y\|).
$$

Then Eq. (9) or (10) can be written in the form of the equation
$$
\psi(\alpha) = \xi(\alpha). \tag{11}
$$

Lemma 2 [2]. Under the condition
$$
\|\tilde{F}\| > \beta \Delta, \quad q = 0, \quad q > 0 \tag{12}
$$
the function $\psi(\alpha)$ is continuous and strictly monotonically increasing, moreover,
$$
\lim_{\alpha \to 0^+} \psi(\alpha) = 0,
$$
and
$$
\lim_{\alpha \to +\infty} \psi(\alpha) = \begin{cases} 
\|\tilde{F}\|, & q = 0, \\
0, & q > 0 \text{ and } \|\tilde{F}\| = 0, \\
\infty, & q > 0 \text{ and } \|\tilde{F}\| > 0,
\end{cases}
$$
and function $\xi(\alpha)$ is continuous and strictly monotonically decreasing, moreover,
$$
\lim_{\alpha \to 0^+} \xi(\alpha) > \beta \Delta > 0, \quad \lim_{\alpha \to +\infty} \xi(\alpha) = \beta \Delta > 0.
$$

Now, the NVAC can be formulated as the following theorem.

**Theorem 1.** Let the equation $\tilde{A} y = \tilde{f}$, $y \in H_1$, $\tilde{f} \in H_2$, be solved by the Tikhonov regularization method according to (2) or (3), where $\|\tilde{f} - \tilde{f}\| \leq \delta$, $\delta > 0$, $\|\tilde{A} - A\| \leq \theta$, $\theta \geq 0$. Suppose that the regularization parameter $\alpha$ is chosen to be a root of Eq. (9), (10) or (11), furthermore, $\|\tilde{F} - F\| \leq \Delta$, $\|\tilde{R} - R\| \leq \Theta$, where $\Delta = \Delta(\delta, \theta) > 0$, $\Theta = \Theta(\theta) \geq 0$. Then, under condition (12), a root $\alpha = \alpha_n$ of Eq. (11) exists and is unique, and the solution $y_{\alpha_n}$ can be found by solving Eq. (3) with $\alpha = \alpha_n$. If condition (12) is not fulfilled, then $y_{\alpha_n} = 0$.

4. Some dependences

Let us establish the dependences $\Delta = \Delta(\delta, \theta)$ and $\Theta = \Theta(\theta)$. The estimate for the error $\Delta$ of the right-hand side $\tilde{F}$ has the form [2]
$$
\Delta \leq \|\tilde{A}\| \delta + \|\tilde{f}\| \theta, \tag{13}
$$
and the estimate for the error \( \Theta \) of the operator \( \tilde{R} \) has the form \[2\]

\[\Theta \leq 2 \| \tilde{A} \| \theta. \quad (14)\]

**Remark 3.** The estimates \((13)\) and \((14)\) are necessary for justifying the convergence of the NVAC. However, in practice for a finite \( \delta \) and \( \theta \), the formulas \((13)\) and \((14)\) may give an overstatement of \( \Delta \) and \( \Theta \) (see example in the end of the present paper) and, hence, of \( \alpha_n \) if one uses the upper estimates: \( \Delta = \| \tilde{A} \| \delta + \| \tilde{f} \| \theta \) and \( \Theta = 2 \| \tilde{A} \| \theta. \) This overstatement is caused by that the factor \( \| \tilde{A} \| \) is separated from \( \delta \) and \( \theta \) in the estimates \((13)\) and \((14)\). To obtain more exact estimates of \( \Delta \) and \( \Theta \), one can use, for example, the algorithms II, III and V from the paper \[2\].

### 5. Estimates for \( \alpha_n \)

We give two upper estimates for \( \alpha_n \) in the NVAC. Define \[2, \, [9, \, p. \, 78]\]

\[\alpha_0 = \| \tilde{R} \| = \| \tilde{A} \|^2 = \| \tilde{A}^* \|^2. \quad (15)\]

The condition \((12)\) for \( q = 0 \) can be written as

\[\frac{\Delta}{\| F \|} < \frac{1}{\beta}. \quad (16)\]

Let us introduce as an extended variant of condition \((16)\) the following condition \[2\]

\[\frac{\Delta}{\| F \|} + \frac{\Theta}{\| R \|} \leq \frac{1}{\beta} \| \tilde{R} \|^q \quad (17)\]

Condition \((17)\) can also be considered as a modification of condition \((53)\) in \[9\]. It is proved \[2\]

**Lemma 3.** Under condition \((17)\), one has the inequality

\[\psi(\alpha_0) \geq \xi(\alpha_0). \quad (18)\]

**Corollary 1** \[2\]. Since the functions \( \psi(\alpha) \) and \( \xi(\alpha) \) are increasing and decreasing, respectively, relations \((15), \, (17), \, (18)\) imply that

\[\alpha_n \leq \alpha_0 = \| \tilde{R} \|. \quad (19)\]

Inequality \((19)\) gives an upper estimate for \( \alpha_n \) in terms of the norm of the operator. It is also proved \[2\]

**Lemma 4.** Under condition \((12)\), it holds that

\[\alpha_n \leq \left( \beta \left( \frac{2 \| \tilde{R} \|}{\| F \|} \Delta + \Theta \right) \right)^{1/(q+1)}. \quad (20)\]

Inequality \((20)\) gives another upper estimate for \( \alpha_n \) (in terms of the errors in the original data).
Corollary 2. Since
\[
\frac{2 \| \tilde{R} \|}{\| F \|} \Delta + \Theta \leq \max \left\{ \frac{2 \| \tilde{R} \|}{\| F \|}, 1 \right\} (\Delta + \Theta),
\]
the estimate (20) can be written as
\[
\alpha_n \leq c_1 (\Delta + \Theta)^{1/(q+1)}, \tag{21}
\]
where
\[
c_1 = \left[ \beta \cdot \max \left\{ \frac{2 \| \tilde{R} \|}{\| \tilde{F} \|}, 1 \right\} \right]^{1/(q+1)} > 0. \tag{22}
\]
Corollary 3. Inequality (21) generates the asymptotic estimate
\[
\alpha_n = O \left( (\Delta + \Theta)^{1/(q+1)} \right), \quad \Delta, \Theta \to 0. \tag{23}
\]
Using (13) and (14), we can write the estimates (21) and (22) also as
\[
\alpha_n \leq c_2 (\delta + \theta)^{1/(q+1)}, \tag{24}
\]
where
\[
c_2 = \left[ 2 \beta \| \tilde{A} \| \cdot \max \left\{ \| \tilde{R} \| / \| \tilde{F} \|, \| \tilde{A} \| \cdot \| \tilde{f} \| / \| \tilde{F} \| + 1 \right\} \right]^{1/(q+1)} > 0, \tag{25}
\]
and
\[
\alpha_n = O \left( (\delta + \theta)^{1/(q+1)} \right), \quad \delta, \theta \to 0. \tag{26}
\]
The relations (21), (22), (24), (25) show that the estimate for \( \alpha_n \) decreases with decrease of \( \beta \).

6. Error Estimate for the Regularized Solution

We give a new, more precise, estimate for the error \( \| y_{\alpha_n} - \bar{y} \| \) of the regularized solution \( y_{\alpha_n} \) in the NVAC. In the papers [3, 5, 6, 9, 16, 21] et al., it was shown that in the Tikhonov regularization method there holds the following error estimate for the regularized solution (on the assumption that the exact solution \( \bar{y} \) is sourcewise representable with index 1, i.e. \( \bar{y} = A^* A w, w \in H_1 \)):
\[
\| y_n - y \| \leq c_3 \frac{\delta + \theta}{\sqrt{\alpha}} + c_4 \alpha, \tag{27}
\]
where \( c_3, c_4 > 0 \) are some constants.

Let us use the estimate (27). For \( \alpha_n = O \left( (\delta + \theta)^{1/(q+1)} \right) \) (see (20)) there exist such positive constants \( a_1 \) and \( a_2 \) that (cf. [9, p. 65])
\[
a_1 (\delta + \theta)^{1/(q+1)} < \alpha_n < a_2 (\delta + \theta)^{1/(q+1)}. \tag{28}
\]
Hence,
\[
\| y_{\alpha_n} - \bar{y} \| \leq \frac{c_3}{a_1} (\delta + \theta)^{(q+0.5)/(q+1)} + c_4 a_2 (\delta + \theta)^{1/(q+1)}. \tag{29}
\]
The estimate (29) makes possible to obtain the following asymptotic estimates.

For sufficiently small $\delta$ and $\theta$, we have:

$$\|y_{\alpha_n} - \bar{y}\| \leq c (\delta + \theta)^{\tilde{q}}, \quad c > 0,$$

(30)

where

$$\tilde{q} = \min\left\{q + 0.5, 1\right\} = \begin{cases} (q + 0.5)/(q + 1), & q \in [0, 0.5], \\ 1/(q + 1), & q \geq 0.5 \end{cases}.$$

(31)

As $\delta, \theta \to 0$, we obtain the asymptotic estimate for the convergence rate of $y_{\alpha_n}$ to $\bar{y}$:

$$\|y_{\alpha_n} - \bar{y}\| = O\left((\delta + \theta)^{\tilde{q}}\right),$$

(32)

as well as (we write again the estimate for $y_{\alpha_n}$)

$$\alpha_n = O\left((\delta + \theta)^{1/(q+1)}\right).$$

(33)

The best asymptotic estimates are obtained for $q = 0.5$:

$$\|y_{\alpha_n} - \bar{y}\| = O\left((\delta + \theta)^{2/3}\right), \quad \alpha_n = O\left((\delta + \theta)^{2/3}\right),$$

(34)

i.e. the optimal order of convergence is obtained. This is conform to results of the papers [12, 16, 21] et al., in which the optimal order of convergence has also been obtained, but for other ways for choosing $\alpha$ (the modified discrepancy principle, etc.).

If, e.g., $q = 0$ then $\|y_{\alpha_n} - \bar{y}\| = O\left((\delta + \theta)^{1/2}\right)$ - the suboptimal order of convergence as in the GDP [17].

7. Final Theorem

In conclusion, we prove the summarizing theorem.

**Theorem 2.** Let the equation (2) be solved. Furthermore, the regularization parameter $\alpha$ is chosen with the help of the NVAC according to (11) by equal $\alpha = \alpha_n$. In this case, the estimates (19) - (26) for $\alpha_n$ and the estimates (29) - (32) for the error $\|y_{\alpha_n} - \bar{y}\|$ of the regularized solution $y_{\alpha_n}$ are valid. One has a convergence of the regularized solution $y_{\alpha_n}$ to the exact solution $\bar{y}$ as $\delta, \theta \to 0$, i.e. the NVAC generates a regularizing algorithm.

**Proof.** According to (30), (32), $\|y_{\alpha_n} - \bar{y}\| \to 0$ as $\delta, \theta \to 0$. This means that $y_{\alpha_n} \xrightarrow{\delta,\theta\to0} \bar{y}$. Theorem 2 is proved.

8. Numerical example

To realize the new version of the a posteriori choice of $\alpha$, we have developed the program package NVAC using Fortran PowerStation 4.0. The following model example (cf. [10, p. 162]) was solved with the help of this package.

The exact solution was set as a superposition of five gaussians (the solution with variations):

$$\bar{y}(s) = 6.5 e^{-[(s+0.66)/0.085]^2} + 9 e^{-[(s+0.41)/0.075]^2} + 12 e^{-[(s+0.14)/0.084]^2} + 14 e^{-[(s-0.41)/0.093]^2} + 9 e^{-[(s-0.67)/0.065]^2},$$
\[ a = -0.85, \ b = 0.85, \ c = -1, \ d = 1, \] the kernel
\[ K(x, s) = \sqrt{\frac{r}{\pi}} e^{-r(x-s)^2/(1+x^2)}, \]
where the exact value \( r \) is \( r = 59.924 \). The numbers of discretization nodes are \( l = 161 \) (on \( x \)) and \( n = 137 \) (on \( s \) and \( t \)). The discretization steps are \( \Delta x = \Delta s = \Delta t = \text{const} = 0.0125 \). In this example, \( \|\bar{y}\| = 7.606, \|f\| = 6.907, \|A\| = 2.419, \|F\| = 7.216, \|R\| = 2.196 \). Figure 1 shows the exact solution \( \bar{y}(s) \), the right-hand side \( f(x) \) (considerably more smooth than \( \bar{y}(s) \)), and the new right-hand side \( F(t) \) (still more smooth than \( f(x) \)).

![Figure 1](image)

At first, the direct problem was solved. The values \( f_i, i = 1, \ldots, l \), were calculated. The errors \( \delta f_i \) distributed by the normal law with zero expectation and with the mean square deviation \( \delta = 0.0001, 0.15 \) and 0.5 were added to the values \( f_i \). The values \( \tilde{r} = 59.920, 60 \) and 65 were used instead of the exact value of \( r \). Table 1 shows, as an instance, the values of \( \delta, \delta/\|f\|, \Delta = \|\Delta F\|, \Delta/\|F\| \) and (for comparison) \( \|A\|\delta + \|f\|\theta \) for \( \tilde{r} = 60 \). Such value of \( \tilde{r} \) corresponds to the following parameters: \( \theta = \|\Delta A\| = 1.321 \cdot 10^{-3}, \theta/\|A\| = 5.46 \cdot 10^{-4} = 0.0546\% \), \( \Theta = \|\Delta R\| = 1.194 \cdot 10^{-3}, \Theta/\|R\| = 5.44 \cdot 10^{-4} = 0.0544\% \), \( 2\|\tilde{A}\|\theta = 6.392 \cdot 10^{-3} \).

| \( \delta \) | \( \delta/\|f\| \) | \( \Delta = \|\Delta F\| \) | \( \Delta/\|F\| \) | \( \|A\|\delta + \|f\|\theta \) |
|---|---|---|---|---|
| 0.0001 | \( 1.448 \cdot 10^{-3} \approx 1.4 \cdot 10^{-3}\% \) | \( 0.6691 \cdot 10^{-3} \approx 0.93 \cdot 10^{-2}\% \) | \( 0.927 \cdot 10^{-4} \approx 0.26\% \) | \( 9.4 \cdot 10^{-3} \) |
| 0.15 | \( 2.172 \cdot 10^{-2} \approx 2.2\% \) | \( 0.01878 \) | \( 0.259 \cdot 10^{-2} \approx 0.26\% \) | \( 0.3721 \) |
| 0.5 | \( 7.239 \cdot 10^{-2} \approx 7.2\% \) | \( 0.06256 \) | \( 0.867 \cdot 10^{-2} \approx 0.87\% \) | \( 1.219 \) |

Furthermore, the operator norms \( \|A\|, \theta = \|\tilde{A} - A\|, \|R\|, \) and \( \Theta = \|\tilde{R} - R\| \)
were calculated by means of the Hilbert–Schmidt norm, e.g.,

$$\|A\| = \left\{ \int_a^b \int_c^d K^2(x,s) \, dx \, ds \right\}^{1/2}.$$  

Comparing the values of \(\Delta\) and \(\|\tilde{A}\|\delta + \|\tilde{f}\|\theta\) as well as \(\Theta\) and \(2\|\tilde{A}\|\theta\) (see (13) and (14)) we see that the upper estimates \(\|\tilde{A}\|\delta + \|\tilde{f}\|\theta\) overstate by one order the values of \(\Delta\) and \(\Theta\), and comparison of \(\delta/\|f\|\) and \(\Delta/\|F\|\) shows that \(\Delta/\|F\|\) less by one order than \(\delta/\|f\|\) for \(\delta/\|f\|\approx 1\%\). About this, one says already above.

Afterwards, the inverse problem was solved. Equation (5) was solved by the quadrature method at \(\tau = 1\) [4, pp. 249–251]. Figure 2 shows some curves of the relative solution error \(\|y_{\alpha} - \bar{y}\|/\|\bar{y}\|\) (it can be calculated only in solving a model example with known \(\bar{y}\)).

Figure 2: The relative solution error \(\|y_{\alpha} - \bar{y}\|/\|\bar{y}\|\) at \(\tau = 1\) 1 — \(\delta = 0.0001, 0.15\); 2 — \(\delta = 0.15, 0.5\); 3 — \(\delta = 0.5, \tilde{r} = 65\)

Table 2 shows, as an instance, the values of \(\alpha_{\text{opt}}, \alpha_n\) and the relative errors of the solutions \(y_{\alpha_{\text{opt}}} \) and \(y_{\alpha_n}\) for \(\tilde{r} = 60, q = 0, \tau = 1, \beta = 1\) and \(\beta = 0.1\).

| \(\delta\) | \(\lg \alpha_{\text{opt}}\) | \(\|y_{\alpha_{\text{opt}}} - \bar{y}\|/\|\bar{y}\|\) | \(\lg \alpha_n\) | \(\|y_{\alpha_n} - \bar{y}\|/\|\bar{y}\|\) |
|---|---|---|---|---|
| 0.0001 | -8.7 | 0.0385 | -5.1 | 0.2107 | 0.1099 |
| 0.15 | -5.8 | 0.1848 | -4.3 | 0.3466 | 0.1858 |
| 0.5 | -5.2 | 0.2644 | -3.6 | 0.4311 | 0.2644 |

Figure 3 shows the logarithms of the functions \(\psi(\alpha) = \alpha^q\|\hat{R} y_{\alpha} - \tilde{F}\|\) and \(\xi(\alpha) = \beta (\Delta + \Theta \|y_{\alpha}\|)\).

Figure 4 shows the exact solution \(\hat{y}(s)\) and the regularized solutions \(y_{\alpha}(s)\) at \(\alpha = \alpha_{\text{opt}} = 10^{-5.8}, \alpha = \alpha_n = 10^{-5.7} (\beta = 0.1)\) and \(\alpha = \alpha_n = 10^{-4.3} (\beta = 1)\) for \(\delta = 0.15, \tilde{r} = 60, q = 0, \tau = 1\).

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Figure 3: $1 - \lg \psi(\alpha); 2 - \lg \xi(\alpha), \beta = 1; 3 - \lg \xi(\alpha), \beta = 0.1$

Figure 4: $1 - \bar{y}(s); 2 - y_\alpha(s), \alpha = \alpha_{\text{opt}} = 10^{-5.8}; 3 - y_\alpha(s), \alpha = \alpha_n = 10^{-5.7}; \beta = 0.1; 4 - y_\alpha(s), \alpha = \alpha_n = 10^{-4.3}; \beta = 1$
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