Calculation of a multi-span frame for stability

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Abstract. The loss of stability of the first kind as applied to the longitudinal bending of the
supports of a flat one-story free multispan frame under the action of vertical forces is
considered. The elements of the frame have arbitrary stiffnesses constant along the length of
the elements. The system of nonlinear equations of displacement method is solved. The matrix
of coefficients of the system of equations is presented in block form. Formulas of generalized
coefficients for each block of matrix are determined, and then one or another element of
general stability equation matrix. A slight correction of generalized coefficients formulas
makes it possible to extend the methodology to calculate the stability of non-free frames
(overpasses). Compact notation of generalized coefficients of stability equation matrix makes it
possible to considerably simplify development of algorithms and computer programs designed
for solution of such tasks. The algorithm is implemented in Excel. To verify the obtained
results, test calculations of stability frames for which the exact solution is known have been
performed. Stability calculation of plane frame of periodic structure has been done by the
suggested method. The diagram of the boundary curve has been made and the areas of frame
stability and instability under the action of the given system of forces have been determined.
Stability calculations of the model in the software package ANSYS, with subsequent
comparison of the calculation results, have been performed.

1. Introduction

For the load-bearing capacity of engineering structures consisting of flexible compressed and
compressed-curved elements, as a rule, the possibility of loss of stability of the structure as a whole or
of its individual elements is a decisive factor. Multispan single-story frames are one of the most
widestread of such structures. The stability of the compressed frame elements is reduced to the
determination of the critical value of the longitudinal force in them and then the calculated length and
flexibility of these elements. In the previously considered works, the exact way of calculating the
simplest frames and approximate ways of calculating the frames of symmetric contour with certain
restrictions on the applied load were used. In this paper, for the model of a free single-storey multispan
frame with rigid connection at the nodes, the derivation of expressions allowing to determine precisely
the critical value of an arbitrary system of nodal concentrated forces is considered. An algorithm and a
computer program have been developed to determine the critical load and analyze the factors that
change it.

2. Literature review

The question of determining the critical load at the loss of stability in multi-span frames has been
reflected in numerous works of domestic and foreign scientists.

A separate group includes works where, as a rule, the classical displacement method is used to
solve such problems [1], and the value of critical load is determined from the resulting transcendental
equation, the exact solution of which is a complex mathematical problem. In this connection,
approximate methods are used, in which the multi-span frame is represented by a simplified
calculation scheme, in which restrictions are introduced either on the configuration or on the applied
load. An approximate method for calculating free multi-span single-story frames in the case of loading
all nodes with the same load was proposed in [2]. This method is applicable only to the frames of symmetric contour having the same linear stiffness of ledgers of the same level and is reduced to the replacement of multi-span frame by equally stable one-span frame. In [3] there was used an approximate method of designing frames of symmetrical contour proposed earlier by N.V. Kornoukhov, but with asymmetrical distribution of stiffness of the columns of the given storey and with asymmetrical loading of its nodal load. A study of the stability [3] of multi-span non-free frames made it possible to develop an approximate method based on changing the design scheme of such frames by splitting them into separate T-frames.

A large series of works are devoted to the development of engineering methods of calculation for the stability of frame structures, the application of which is possible only with the use of modern software computing complexes. These are works [8-13] and many others. To determine the critical force and the reduced length of columns in one-story frames, the method of stiffness distribution [4] has been developed, but its contribution of adjacent elements is taken into account indirectly by summing the stiffnesses of elements in the upper and lower parts of the structure frame. The work [4] forms the theoretical basis for the calculation of stability loss in BS EN 1993 [5].

To assess the stability of multi-storey frames, the method [6] offers a modification of the reduced length ratio, which allows you to take into account the load in the longitudinal bending of neighboring columns.

The EBCS 2: 1995 documents introduce the concept of a replacement frame to determine the floor longitudinal bending load.

In the paper [7], when designing for the stability of thin reinforced concrete columns in free frames, correction of the moment of inertia of the column cross section is used. However, this approach gives a significant error in the design of irregular frame frames. In [8, 9], a method of engineering design of spatial frame frames for stability with regard to physical nonlinearity is given. In the paper [10] an engineering method of calculation of frame elements of variable height was developed, which allowed to simplify the test of such rods for stability without using for this purpose modern computing complexes. The study [11] presents an approximate method in which the entire frame structure is idealized as an equivalent multilayer beam that takes into account all deformations, while using the continuum approach and the transfer matrix method to analyze the lateral stability of buildings. In [12], an improved method for simplified stability analysis of free and non-free frames, taking into account the effects of vertical interaction of the columns, is proposed. The stability matrix for a one-story block is derived, and then the system stability matrix is obtained taking into account the compatibility of the floor blocks. In [13], the issue of determining the optimum ratio of frame frame columns for stability is considered and an engineering method for determining the critical force is proposed.

3. Research aim and objectives
The purpose of this paper is to develop an approach for determining the critical value of any system of concentrated forces acting on a flat free single-story multi-span frame with rigid nodes.

4. Research results
Let's consider the calculation for stability by displacement method of a flat one-story multispan frame, the model of which is shown in Figure 1. We consider the loss of stability of the first kind as applied to the longitudinal bending of the columns. A system of vertical forces acts on the frame in the nodes. In general case, the elements of the frame have arbitrary geometrical and physical-mechanical characteristics constant along the length of the elements.
Since the main unknowns of the displacement method are the angular displacements of the rigid nodes and their independent linear displacements, this frame model has the number of unknowns of the displacement method equal to n+1, where n – number of posts in the frame. The basic system of the model displacement method is shown in Figure 2.

The system of canonical equations of the displacement method in matrix form is:

\[ \begin{bmatrix} r_{1,1}(v) & r_{1,2}(v) & \cdots & r_{1,n}(v) & r_{1,n+1}(v) \\ r_{2,1}(v) & r_{2,2}(v) & \cdots & r_{2,n}(v) & r_{2,n+1}(v) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{n,1}(v) & r_{n,2}(v) & \cdots & r_{n,n}(v) & r_{n,n+1}(v) \\ r_{n+1,1}(v) & r_{n+1,2}(v) & \cdots & r_{n+1,n}(v) & r_{n+1,n+1}(v) \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \\ Z_{n+1} \end{bmatrix} = - \begin{bmatrix} R_1(v) \\ R_2(v) \\ \vdots \\ R_n(v) \end{bmatrix} \]

The applied forces are related to each other by a certain ratio and change in proportion to one parameter \( F_0 \):

\[ F_1 = f_1 \cdot F_0; \ldots \quad F_2 = f_2 \cdot F_0; \ldots \quad F_n = f_n \cdot F_0. \]

Nodal load application leads to the fact that the load coefficients \( R_i(v) \) are equal zero. Then the system of equations (1) has the form:
Determination of the critical load consists in finding the roots of the general stability equation, which we obtain by equating the determinant of the coefficient matrix (3) to zero, provided that $Z_i \neq 0$:

$$
D(v) = \begin{vmatrix}
    r_{1,1}(v) & r_{1,2}(v) & \ldots & r_{1,i}(v) & r_{1,i+1}(v) \\
    r_{2,1}(v) & r_{2,2}(v) & \ldots & r_{2,i}(v) & r_{2,i+1}(v) \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    r_{n,1}(v) & r_{n,2}(v) & \ldots & r_{n,i}(v) & r_{n,i+1}(v) \\
    r_{n+1,1}(v) & r_{n+1,2}(v) & \ldots & r_{n+1,i}(v) & r_{n+1,i+1}(v)
\end{vmatrix} = 0.
$$

(3)

By determining the parameters $v_{i,kp}$, let's find the values of the critical forces $F_{i,kp}$:

$$
F_{i,kp} = \frac{v_{i,kp}^2 EI}{H_i}.
$$

(4)

Let us construct the unit diagrams of the bending moments in the basic system of the displacement method (Figure 3 a, b). The diagrams are constructed taking into account the longitudinal bending from the action of the system of forces, the effect of compressive forces is taken into account using the corresponding nonlinear functions $\varphi_2(v_i), \varphi_3(v_i), \eta_2(v_i)$:

$$
\varphi_2(v_i) = \frac{v_i \cdot (tg v_i - v_i)}{8tg v_i} ; \quad \varphi_3(v_i) = \left(\frac{v_i}{2}\right)^2 \frac{tg v_i}{2} ; \quad \eta_2(v_i) = \frac{\left(\frac{v_i}{2}\right)^3}{3\left(\frac{tg v_i}{2} - \frac{v_i}{2}\right)} ,
$$

where $v_i = \alpha_i H_i = H_i \sqrt{\frac{F_i}{EI_{cl}}}$.

![Figure 3, a. Unit bending moment diagrams.](image-url)
For the considered frame model the symmetric matrix of coefficients of equation (4) with dimensions \((n+1) \times (n+1)\) can be represented in block form:

\[
D(v) = \begin{bmatrix}
    r_{1,1}(v) & r_{1,2}(v) & \cdots & r_{1,n}(v) \\
    r_{2,1}(v) & r_{2,2}(v) & \cdots & r_{2,n}(v) \\
    \cdots & \cdots & \cdots & \cdots \\
    r_{n,1}(v) & r_{n,2}(v) & \cdots & r_{n,n}(v) \\
    r_{n+1,1}(v) & r_{n+1,2}(v) & \cdots & r_{n+1,n}(v) \\
\end{bmatrix}
\]

where
- block I – symmetric ribbon matrix with dimensions \(n \times n\);
- block II – column with size \(n\);
- block III – transposed block II;
- block VI – consists of one element.

The physical meaning of elements of I and II blocks - reactive moments in superimposed rigid links, elements of III and VI blocks - reactions in superimposed linear links.

Let us introduce the following coefficients of variation of model characteristics:
- \(h_i\) – in the length of the columns; 
- \(s_i\) – on the value of the bending stiffness of the columns; 
- \(k_i\) – by the length of the ledgers; 
- \(p_i\) – on the value of the bending stiffness of the ledgers; 
- \(f_i\) – by force value.
With this in mind, we can write:

\[ H_i = H_i h_i; \quad EI_{C_i} = EI_{C_i} s_i; \quad L_i = L_i k_i; \quad EI_{P_i} = EI_{P_i} p_i; \quad F_i = F_i f_i. \]  

(7)

In accordance with the known method of static finding of the displacement method coefficients and introduced variation coefficients, let us determine the form of the record of the generalized coefficients of these blocks \( 1 \leq i \leq n \):

\[
 r_{ij} = \frac{4EI_{C_i}}{H_i} \varphi_2(v_i) + \frac{4EI_{P_{i-1}}}{L_i} + \frac{4EI_{P_i}}{L_i} \cdot \frac{s_i}{h_i} \varphi_2(v_i) + \frac{4EI_{P_{i-1}}}{L_i} \cdot \frac{p_{i-1}}{k_{i-1}} + \frac{4EI_{P_i}}{L_i} \cdot \frac{p_i}{k_i} = \frac{2EI_{P_{i-1}}}{L_i} + \frac{2EI_{P_i} p_{i-1}}{L_i k_i} + b \cdot \frac{p_{i-1}}{k_{i-1}} + b \cdot \frac{p_i}{k_i}. 
\]

(8)

\[
r_{i,i-1} = \frac{2EI_{P_i}}{L_i} = \frac{2EI_{P_{i-1} p_{i-1}}}{L_i k_i} = b \cdot \frac{p_{i-1}}{k_{i-1}}. 
\]

(9)

\[
r_{i,n+1} = -\frac{6EI_{C_i}}{H_i^2} \varphi_2(v_i) = -\frac{6EI_{C_0}}{H_0^2} \frac{s_i}{h_i^2} \varphi_2(v_i) = d \cdot \frac{s_i}{h_i^2} \varphi_2(v_i). 
\]

(10)

\[
r_{i,n+1} = \frac{12EI_{C_1}}{H_3^3} \eta_2(v_i) + \frac{12EI_{C_2}}{H_3^2} \eta_2(v_2) + \ldots + \frac{12EI_{C_n}}{H_3^3} \eta_2(v_n) = \frac{12EI_{C_0}}{H_3^3} \left[ s_1 \eta_2(v_1) + s_2 \frac{h_1}{h_3^2} \eta_2(v_2) + \ldots + s_n \frac{h_1}{h_3^n} \eta_2(v_n) \right] = c \cdot \left[ s_1 h_1^3 \eta_2(v_1) + s_2 \frac{h_1}{h_3^2} \eta_2(v_2) + \ldots + s_n \frac{h_1}{h_3^n} \eta_2(v_n) \right]. 
\]

(11)

In formulas (8) ÷ (11) the following symbols are used:

\[
a = \frac{4EI_{C_0}}{H_0}; \quad b = \frac{4EI_{P_0}}{L_0}; \quad c = \frac{12EI_{C_0}}{H_0}; \quad d = \frac{6EI_{C_0}}{H_0}. 
\]

(12)

Formulas of 4 generalized coefficients (8) ÷ (11), allow obtaining the entry of any of the elements of block matrices of the general stability equation.

A slight correction of the terms in formulas (8) ÷ (11) will allow to use the proposed expressions when forming the matrix of coefficients for non-free frames (Figure 4) (as a rule, these are overpass structures) which models are a particular case of free frame model (Figure 1).

Figure 4. Particular cases of the frame calculation model.

All nonlinear parameters \( v_i \) of functions \( \varphi_2(v_i), \varphi_3(v_i), \eta_2(v_i) \), included in the general stability equation (4) are reduced to a single argument \( v_0 \).
Then all coefficients \( r_{i,k}(v) \) of the determinant (4) will be functions of only one parameter \( v_0 \):

\[
r_{i,k}(v) = \Phi_{i,k}(v_0)
\]

and equation (4) will have the following form:

\[
D(v_0) = \left| \Phi_{i,k}(v_0) \right| = 0.
\]

The critical state equation of the system (15) is a transcendental equation. Such an equation can be solved only by fitting or the method of successive approximations. The smallest positive value \( v_0 \) determines the critical value of any of the concentrated forces of the system. The exact solution of the equation at \( n > 2 \) is a complex mathematical problem, the solution of which is possible only with the use of computer technology.

It is not always reasonable to use universal computational complexes, such as ANSYS, LIRA and others, for many reasons. The obtained compact form of generalized coefficients of the general stability equation matrix allows to facilitate significantly the creation of algorithms and writing of computer programs intended for solving such problems.

For the numerical realization of the obtained expressions a program was written in Excel tables, the choice of which is due to their availability.

The resulting program makes it easy to analyze the effect of changes in geometric and physical-mechanical parameters of model elements on the values of critical forces of the system.

To verify the calculation results, a number of test frame stability calculations with exact or approximate solution has been performed [2; 8].

Stability calculation of plane frame of periodic structure has also been done. Frames of such type are frameworks of widely spread building structures (or their fragments).

The model of the considered frame has a constant height at six identical spans (Figure 5), for it the number of unknown methods of displacements is eight.

\[
\begin{align*}
F_1 & \quad F_2 & \quad F_2 & \quad F_2 & \quad F_2 & \quad F_2 & \quad F_2 & \quad F_1 \\
1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7
\end{align*}
\]

**Figure 5.** Frame model with six equal spans.

For this frame the following initial data are taken: all columns (6 m height) and all beams (12 m span) are made of steel with the modulus of elasticity \( E = 2 \cdot 10^{11} \text{ Pa} \). Columns have a rectangular cross-section \( (a=0.2 \text{ m}; \ b=0.5 \text{ m}) \), the moment of inertia of the cross section of the ledger is 2.5 times greater than the moment of inertia of the cross section of the column. The following model parameters are set: \( H_1 = H_0 = 6 \text{ m}; \ EI_{c1} = EI_{c0}; \ L_4 = L_0 = 12 \text{ m}; \ EI_{m1} = 2.5 EI_{m0}; \ F_1 = F_0; \ F_2 = F_0 f_2. \)
Calculations to determine the critical force at values of the force variation coefficient \(f_2 = 0,5; 1,0; 2,0\). The partial values of the forces are also determined \(F_{1,kp}; F_{2,kp}\). The results of the calculations are summarized in the table 1.

**Table 1. Critical force values.**

| №  | \(F_{1,kp}\) | \(F_{2,kp}\) | \(F_1/EI_{c0}\) | \(F_2/EI_{c0}\) |
|----|--------------|--------------|----------------|----------------|
| 1  | 0,769        | 0            |                |                |
| 2  | 0,363        | 0,181        |                |                |
| 3  | 0,233        | 0,233        |                |                |
| 4  | 0,135        | 0,271        |                |                |
| 5  | 0            | 0,322        |                |                |

On the basis of the obtained results, the stability boundary curve plot (Figure 6) was plotted and the stability and instability regions of the frame under the action of a given system of forces were determined \(F_1\) and \(F_2\).

![Figure 6. Boundary curve plot: A – stability region; B – instability region; C – boundary curve.](image)

To verify the obtained results, the model was calculated for stability in the ANSYS software package [9].

The critical loads corresponding to the first two forms of stability loss were obtained (Figure 7). According to calculations, the first form of stability loss corresponds to the critical force \(F_1 = 5638\) kN; the second form of loss of stability - the critical force \(F_2 = 17402\) kN. From the author’s calculation the results obtained are practically the same.

![Figure 7. The first and second forms of frame stability loss](image)
5. Conclusions
The procedure for determining the critical forces in the problem of calculating the stability of a flat multi-span frame by the displacement method has been considered. It has been found that the structure of the matrix of coefficients of the general stability equation with n unknowns for the considered frames has a block structure. A technique for obtaining four generalized coefficients which allows one to easily obtain any of the elements of the block matrices of the general stability equation has been proposed. Numerical implementation of the developed algorithm has been performed in Excel spreadsheets. Verification of the algorithm was carried out by calculations of test examples. Critical forces for a six-span model of a frame of regular structure are determined. A boundary curve is plotted and the stability zone for this model is determined. Comparison of the results of calculations of the first two critical forces using the author's method with the results of numerical analysis in the ANSYS software package showed their complete coincidence (discrepancy is 0.1 %).

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