On the implicit equations of a mechanical systems motion

V V Lapshin

Bauman Moscow State Technical University, 2-ya Baumanskaya st., 5, Moscow, Russia

E-mail: lapshin032@bmstu.ru

Abstract. It is shown that the forces acting on the points of the mechanical system may depend on their accelerations. The differential equations of a mechanical system motion prove to be implicit. It is not allowed with respect to higher derivatives. There are fundamental mathematical problems related to the possibility and the only solution of these equations with respect to higher derivatives. Implicit equations of motion are typical for mechanical systems with dry friction sliding and rolling. In the dynamics of material point such problems do not arise. But in more complex mechanical systems, including the study of the motion of a solid whole mass is concentrated at one point, as well as in systems with one degree of freedom, such a situation is very characteristic. The paper discusses four fairly simple examples of mechanical systems movement, which is described by implicit differential equations of motion.

1. Introduction
The axioms of dynamics are based on the principle of determinism [1], which states that the initial state of a mechanical system uniquely determines the further behavior of the system under the action of given forces. The principle of determinism is a special case of the principle of the repetition of experience in physics. If you conduct the same experiment under the same conditions, you will get the same result.

The principle of determinism served as the basis for the formulation of Newton’s second law, according to which the differential equation of motion of a material point under the action of force has the form

\[ m \ddot{r} = \vec{F}(\vec{r}, \vec{v}, t) \]

where \( m \) – the mass of the point, \( \vec{r} \) – its radius vector, \( \vec{v} = \dot{\vec{r}} \) – the speed of the point, and the force \( \vec{F} \) is a function of the position of the point, its velocity and time. If the initial state of the point is specified

\[ \vec{r}(t_0) = \vec{r}_0, \quad \vec{v}(t_0) = \vec{v}_0 \]

then the main problem of dynamics on determining the further motion of a point is the Cauchy problem and has a unique solution if the conditions of the existence and uniqueness theorem of the solution of differential equations are satisfied [1-5].
Note that while the default in mechanics assumes that all mechanical systems are deterministic, non-deterministic systems are existing too. Examples of such systems for the simplest case of the rectilinear motion of a material point are given in [4-6]. In these systems, the same initial state (2) can correspond to several different solutions of the equations of motion (1). Examples of non-determinism of the behavior of a mechanical system in impact theory are given in [7].

2. Statement of the problem

The differential equation of motion of the point (1) is transferred to the mechanical system. For a system of material points, the differential equations of motion have the form [1-5]

\[ m_k \ddot{r}_k = \bar{F}_k, \quad k = 1,2,\ldots,n, \]

where \( m_k \) – the mass of the \( k \)-th point of the system, \( \bar{r}_k \) – its radius vector. The force \( \bar{F}_k \) acting on the \( k \)-th point of the system is a function of the position and velocities of all points of the system and time

\[ \bar{F}_k = \bar{F}_k (\bar{r}, \bar{r}_1, \ldots, \bar{r}_n, \dot{\bar{r}}, \ddot{\bar{r}}, \ldots, \dddot{\bar{r}}, t), \quad k = 1,2,\ldots,n, \]

Consequently, the equations of motion of a mechanical system form a system of differential equations resolved with respect to the highest derivatives.

However, situations are possible when the forces acting on the points of the system also depend on the accelerations of the points of the system

\[ \bar{F}_k = \bar{F}_k (\bar{r}, \bar{r}_1, \ldots, \bar{r}_n, \dot{\bar{r}}, \ddot{\bar{r}}, \ldots, \dddot{\bar{r}}, \dddot{\bar{r}}, t), \quad k = 1,2,\ldots,n, \]

The system of differential equations of motion of a mechanical system is implicit. It is not resolved with respect to higher derivatives. Fundamental mathematical problems arise related to the possibility and uniqueness of the solution of these equations with respect to higher derivatives.

The naturally implicit form of the equations of motion is preserved when passing to generalized coordinates using general dynamics theorems or Lagrange equations of the second kind.

Similar situations are characteristic of mechanical systems with dry friction [8–11]. In the dynamics of the point, such problems do not arise. But in more complex mechanical systems, including investigation of a rigid body motion, which whole mass is concentrated at one point, as well as in systems with one degree of freedom, this situation is very characteristic.

This is explained by the fact that in accordance with the Coulomb law during sliding, the dry friction force

\[ \bar{F}_{np} = - f |\bar{N}| \frac{\bar{v}}{|\bar{v}|}, \]

where \( f \) – the coefficient of sliding friction, \( \bar{N} \) – the normal reaction, \( \bar{v} \) – the relative velocity of sliding. The normal reaction may depend on the accelerations of the points of the system, and then the equations of motion are implicit (3) - (4). In problems with dry friction, another complication arises due to the fact that, when \( \bar{v} = 0 \), the dry friction force can take any value by modulo not exceeding the maximum \(|\bar{F}_{np}| \leq f |\bar{N}|\), and is directed in the direction opposite to the direction of possible slip. This can lead to the existence of areas of stagnation, hit in which at zero speed leads to the cessation of sliding.
If bodies can not only slide relative to each other, but also roll, then in addition to the sliding friction force, a sliding friction moment arises, which is determined by similar relationships, and it is possible to alternate the phases of sliding, rolling with slipping and rolling without slipping.

Such systems were the subject of discussion in connection with the Painleve paradoxes [8–11], related to the fact that the equations of motion in some cases turn out to be insoluble with respect to higher derivatives or have several solutions.

The non-deterministic behavior of a mechanical system, as noted above, is also possible in systems without friction with equations of motion resolved with respect to higher derivatives.

The mathematical conditions for the existence and uniqueness of the solution of implicit differential equations of motion of mechanical systems in a general form are devoted to [12–13].

Let us dwell on some examples of implicit equations of motion of mechanical systems with dry sliding and rolling friction.

This article considers a slightly modified Painleve example and three more new examples of mechanical systems with friction, which lead to implicit differential equations of motion of the system. In the first two examples, the differential equations of motion are uniquely resolved with respect to the highest derivative. In the last example (mathematical pendulum), situations arise similar to the Painleve paradoxes. With a sufficiently large coefficient of friction, the equations of motion cannot be solved with respect to the highest derivative.

3. Uniform rod (ladder)
A uniform rod (ladder), (Figure 1), is in contact with a horizontal floor and a vertical wall. The mass of the rod is equal \( m \), the length of the rod – \( AB = 2l \). Point \( C \) is the center of mass of the rod, \( AC = CB = l \). The sliding friction coefficients at points \( A \) and \( B \) are equal \( f \). The position of the rod is determined by the angle that it forms with the vertical.

\[
\phi(0) = \varphi_0 , \quad \dot{\varphi}(0) = \omega_0 > 0 \quad .
\]  

Denote by \( x, y \) the coordinates of the rod center of mass \( C \), then
\[ x = l \sin \varphi , \quad y = l \cos \varphi . \]

Differentiating these link equations twice, we obtain

\[ \ddot{x} = l \cos \varphi \dot{\varphi} - l \sin \varphi \dot{\varphi}^2 , \quad \ddot{y} = -l \sin \varphi \dot{\varphi} - l \cos \varphi \dot{\varphi}^2 . \tag{7} \]

We denote by \( N_1, N_2 \) normal reactions, and a \( F_1, F_2 \) – friction forces (Figure 1). Consider the stage of the movement of the rod, while maintaining contact with the floor and wall, i.e. \( N_1 \geq 0 \) and \( N_2 \geq 0 \). According to the law of Coulomb (5)

\[ F_1 = fN_1 , \quad F_2 = fN_2 . \]

The theorem on the center of mass motion has the form

\[ m\ddot{x} = N_2 - fN_1 , \quad m\ddot{y} = -mg + N_1 + fN_2 . \]

Solving these relations relatively, we obtain

\[ N_1 = -\frac{f\ddot{x} + g + \ddot{y}}{1 + f^2} , \quad N_2 = \frac{\ddot{x} + f(g + \ddot{y})}{1 + f^2} . \]

In accordance with the theorem on the change in the angular momentum relative to the center of mass

\[ \frac{ml^2}{3} \dot{\varphi} = (N_1 - fN_2)l \sin \varphi - (N_2 + fN_1)l \cos \varphi . \]

From the last two relations it follows

\[ \frac{l(1 + f^2)}{3} \dot{\varphi} = \left[ -2f\ddot{x} + (1 - f^2)\ddot{y} \right] \sin \varphi - \left[ (1 - f^2)\ddot{x} + 2\ddot{y} \right] \cos \varphi - 2fg \cos \varphi + (1 - f^2)g \sin \varphi . \]

Given (7), we obtain the implicit differential equation of motion

\[ \frac{l(1 + f^2)}{3} \dot{\varphi} = \left[ -2f\ddot{x} + (1 - f^2)\ddot{y} \right] \sin \varphi - \left[ (1 - f^2)\ddot{x} + 2\ddot{y} \right] \cos \varphi - 2fg \cos \varphi + (1 - f^2)g \sin \varphi . \]

which is easily resolved with respect to the highest derivative
The differential equation of motion (8) has a first integral. We denote by $\omega = \dot{\varphi}$ the angular velocity, and using the replacement

$$\dot{\varphi} = \frac{d\omega}{dt} = \frac{d\omega}{d\varphi} \frac{d\varphi}{dt} = \omega \frac{d\omega}{d\varphi} = \frac{d}{d\varphi} \frac{d\omega}{dt}$$

(9)

exclude time from equation (8). Then we get an inhomogeneous linear differential equation with constant coefficients for the dependence of the square of the angular velocity on the angle of rotation

$$\frac{2 - f^2}{3} \frac{d^2 \omega}{d\varphi^2} - 2f \omega^2 = -2fg \cos \varphi + (1 - f^2)g \sin \varphi ,$$

which has a solution

$$\omega^2 = ce^{\lambda \varphi} + a \sin \varphi + b \cos \varphi ,$$

where

$$\lambda = \frac{6f}{2 - f^2} , \quad a = \frac{-6fg(5 - 4f^2)}{4 + 32f^2 + f^4} , \quad b = \frac{-3g(2 - 7f^2 + f^4)}{4 + 32f^2 + f^4} .$$

The integration constant $c$ is determined from the initial conditions (6).

### 4. Wheel with a displaced center of mass

A non-uniform disk (Figure 2) rolls without slipping along a horizontal guide. The disk is in a vertical plane. The disk center of mass $C$ does not coincide with its geometric center $A$. The mass of the disk is $m$, radius $r$, the distance from the center of the disk to its center of mass $AC = l$, the moment of inertia of the disk relative to the center of mass $J_c = mp^2$. The position of the disk is determined by the generalized coordinate $\varphi$. The rolling friction coefficient is $\delta$. The disk rolls by inertia and does not jump above the supporting surface.
We denote by $x, y$ the coordinates of the point $A$ — geometrical center of the wheel, and by $x_c, y_c$ — the coordinates of the disk center of mass. Then

$$x_c = x - l \sin \varphi = r \varphi - l \sin \varphi, \quad y_c = y - l \cos \varphi = r - l \cos \varphi.$$  

Differentiating these ratios twice, we obtain

$$\ddot{x}_c = r \ddot{\varphi} - l \cos \varphi \dot{\varphi} + l \sin \varphi \dot{\varphi}^2, \quad \ddot{y}_c = l \sin \varphi \ddot{\varphi} + l \cos \varphi \dot{\varphi}^2. \quad (10)$$

In accordance with the theorem on the center of mass motion and the theorem on the change in the angular momentum relative to the center of mass, we have

$$m \ddot{x}_c = -F_{tr}, \quad m \ddot{y}_c = N - mg, \quad mp^2 \ddot{\varphi} = -M_{tr} + F_{tr} (r - l \cos \varphi) - N l \sin \varphi. \quad (11)$$

Then, taking into account (10), from the first two of equations (11) we obtain

$$F_{tr} = -m(r \ddot{\varphi} - l \cos \varphi \dot{\varphi} + l \sin \varphi \dot{\varphi}^2), \quad N = mg + ml(sin \varphi \ddot{\varphi} + cos \varphi \dot{\varphi}^2).$$

Substituting these relations into the third of equations (11), we obtain

$$m[(\dot{\varphi}^2 + r^2 + l^2 - 2rl \cos \varphi) \ddot{\varphi} + rl \sin \varphi \dot{\varphi}^2] = -mg l \sin \varphi - M_{tr}.$$  

Assume that the disc does not jump above the supporting surface, i.e. $N \geq 0$. For this, it is necessary and sufficient that $\dot{\varphi}$ and $\ddot{\varphi}$ do not exceed some critical values

$$\min_{t} (\sin \varphi \ddot{\varphi} + \cos \varphi \dot{\varphi}^2) = \min_{t} \frac{d}{dt}(\sin \varphi \ddot{\varphi}) \geq -\frac{g}{l}. \quad (12)$$

**Figure 2.** Wheel with a displaced center of mass
The rolling friction moment is

\[ M_{\varphi} = \delta N \sgn \varphi = \delta m [g + l (\sin \varphi \dot{\varphi} + \cos \varphi \dot{\varphi}^2)] \sgn \varphi \ . \]

The implicit differential equation of wheel motion has the form

\[ m [(\rho^2 + r^2 + l^2 - 2rl \cos \varphi) \dot{\varphi} + rl \sin \varphi \dot{\varphi}^2] = -mg l \sin \varphi - \delta m [g + l (\sin \varphi \dot{\varphi} + \cos \varphi \dot{\varphi})^2] \sgn \varphi \ , \]

and easily resolved with respect to the highest derivative. As a result, we obtain

\[ (\rho^2 + r^2 + l^2 - 2rl \cos \varphi + \delta l \sin \varphi \sgn \dot{\varphi}) \ddot{\varphi} = -l (r \sin \varphi + \delta \cos \varphi \sgn \dot{\varphi}) \dot{\varphi}^2 - gl \sin \varphi - \delta g \sgn \varphi \ . \]

(13)

Note that the condition for not jumping the disk above the supporting surface (12) must be substituted \( \dot{\varphi} \) from equation (13), and then this condition has the form

\[ \min \left[ -\frac{l (r \sin \varphi + \delta \cos \varphi \sgn \dot{\varphi}) \dot{\varphi}^2 + gl \sin \varphi + \delta g \sgn \dot{\varphi}}{\rho^2 + r^2 + l^2 - 2rl \cos \varphi + \delta l \sin \varphi \sgn \dot{\varphi}} \sin \varphi + \cos \varphi \dot{\varphi}^2 \right] \geq -\frac{g}{l} \ . \]

According to function \( \sgn x \) definition

\[ \sgn 0 \in [-1, 1] \ , \]

(14)

those can take any value modulo not exceeding 1.

From (14) it follows that the equation of motion (13) has stagnation zones. If the disk enters with a zero angular velocity \( \dot{\varphi} = 0 \) in the region in which

\[ |\sin \varphi| \leq \frac{\delta}{l} \ . \]

(15)

then it follows from (14) that \( \dot{\varphi} = 0 \), and, therefore, the disk stops.

If the friction is large \( \delta \geq l \), then the stagnation zone (15) covers all possible positions of the disk. The disk rolls in one direction (without changing the sign of the angular velocity).

If the friction is not large \( \delta < l \), then the stagnation zones (15) form symmetrical regions in the neighborhood of the positions when the center of mass is on the same vertical as the geometric center of the disk, i.e. extreme upper and lower position of the center of mass. If the first time when the disk moves, the zero value of the angular velocity is reached outside the stagnation zone, then at the final stage the disk movement has the form of damped oscillations and ends with a stop in the stagnation zone near the extreme lower position of the center of mass.
Using the replacement (9), the equation of motion (13) is reduced to an inhomogeneous linear differential equation with variable coefficients for the dependence of the square of the angular velocity $\omega^2$ on the angle of rotation $\varphi$

$$
(\rho^2 + r^2 + l^2 - 2rl \cos \varphi + \delta l \sin \varphi \text{sgn } \dot{\varphi}) \frac{d\omega^2}{d\varphi} + \\
+ 2l(r \sin \varphi + \delta \cos \varphi \text{sgn } \dot{\varphi}) \omega^2 = -2gl \sin \varphi - 2\delta g \text{sgn } \dot{\varphi},
$$

which determines the phase trajectories of the system.

5. Elliptical pendulum. Painleve paradoxes

The elliptical pendulum (Figure 3) moves in a vertical plane. Slider $A$ moves along a rough horizontal guide. The sliding friction coefficient is equal to $f$. A weightless rod $AB$ with a length $2l$ at the end of which is a material point is pivotally attached to slider. The size of the slider can be neglected. The masses of the slider and the material point are the same and equal. The friction in the hinge can be neglected. We denote by both the normal reaction and the sliding friction force acting on the slider.

![Figure 3. Elliptical pendulum.](image)

We introduce the generalized coordinates: $x$ – the position of the slider $A$ and $\varphi$ – the angle of deviation of the rod from the vertical. The center of mass of the system $C$ is located in the middle of the rod $AC = CB = l$ and its coordinates are equal

$$
x_c = x + l \sin \varphi, \quad y_c = -l \cos \varphi.
$$

Differentiating these ratios twice, we obtain

$$
\ddot{x}_c = \ddot{x} + l \cos \varphi \ddot{\varphi} - l \sin \varphi \dot{\varphi}^2, \quad \ddot{y}_c = l \sin \varphi \ddot{\varphi} + l \cos \varphi \dot{\varphi}^2.
$$

In accordance with the theorem on the center of mass, we have
From the theorem on the change in the angular momentum relative to the center of mass, we obtain
\[ 2m\ddot{\varphi}_c = 2m(l\cos\varphi\ddot{x} - l\sin\varphi\dot{x}^2) = F_{\text{fr}} \, , \]  
\[ 2m\ddot{\varphi}_c = 2ml(\sin\varphi\ddot{\psi} + \cos\varphi\dot{\psi}^2) = N - 2mg \, . \]

From the law of Coulomb follows (5)
\[ F_{\text{fr}} = -f|N|\text{sgn}\dot{x} = -kfN \, , \]

where
\[ k = \text{sgn}(N\dot{x}) \, . \]

When \( N\dot{x} \neq 0 \), \( k = \pm 1 \) and
\[ kN\dot{x} > 0 \, . \]

We substitute (20) into (16), and solve (16), (17) and (19) with respect to \( \dot{x}, \dot{\varphi}, x \) as a system of three linear algebraic equations. As a result, we obtain
\[ \lambda N = 2(g + l\cos\varphi\dot{\psi}^2) \, , \]
\[ \dot{x} = 2l\dot{\psi}^2(\sin\varphi - kf\cos\varphi) - kfg(1 + \cos^2\varphi)\sin\varphi + g\sin\varphi\cos\varphi \, , \]
\[ \lambda\ddot{\varphi} = (g + l\cos\varphi\dot{\psi}^2)(kf\cos\varphi - \sin\varphi) \, , \]

where
\[ \lambda = 1 + \sin^2\varphi - kf\sin\varphi\cos\varphi \, . \]

Equations (24)–(25) form a system of differential equations of system motion, resolved with respect to the highest derivatives. Equations (21)–(23) and (26) allow us to determine the values of \( k \) and \( \lambda \). Wherein \( \lambda \). It is uniquely determined by the value of \( k \). The question remains about the uniqueness of their decision relatively to \( k \).

Let the coefficient of friction be sufficiently small, so that for any values
\[ 1 + \sin^2\varphi - f\sin\varphi\cos\varphi > 0 \quad \text{and} \quad 1 + \sin^2\varphi + f\sin\varphi\cos\varphi > 0 \, . \]

Then \( \lambda \) is always greater than zero, and in any current state \( x, \varphi, \dot{x}, \dot{\varphi} \), relations (21)–(23) and (26) uniquely determine the values of \( k \) and \( \lambda \).

Note that conditions (27) are satisfied if \( f \leq 2 \). Valid in this case
\[ 1 + \sin^2\varphi \pm f\sin\varphi\cos\varphi \geq 1 + \sin^2\varphi - f|\sin\varphi\cos\varphi| > 1 + \sin^2\varphi - 2|\sin\varphi\cos\varphi| = \]
\[ = 1 + (|\sin\varphi| - |\cos\varphi|)^2 - \cos^2\varphi > 1 - \cos^2\varphi \geq 0 \]

Let the friction coefficient be large enough and conditions (27) are not satisfied. For definiteness, we assume that in the current state of the system \( \varphi \in (0, \frac{1}{2}\pi) \), then
1 + \sin^2 \varphi - f \sin \varphi \cos \varphi < 0 \quad \text{and} \quad 1 + \sin^2 \varphi + f \sin \varphi \cos \varphi > 0 . \quad (28)

It follows that when \( \dot{x} > 0 \) it is impossible to satisfy conditions (21)-(23), (26). When \( k = 1 \) from (26), (28) follows \( \lambda < 0 \), and from (23) follows that \( N < 0 \), and then condition (22) is not satisfied. If \( k = -1 \) we have \( \lambda > 0 \) and \( N > 0 \), then condition (22) is not satisfied. The equations of motion are not solvable with respect to the highest derivatives; there is no solution of the form (24)-(25).

On the contrary, if \( \dot{x} < 0 \), both solutions \( k = \pm 1 \) satisfy to conditions (21)-(26) and there are two different types of equations of motion (24)-(25). Therefore, the principle of determinism is not fulfilled.

These contradictions to the basic principles of mechanics published by Painleve in a monograph [8] are called Painleve paradoxes [9–11].

6. Pendulum
Consider the mathematical pendulum (Figure 4), which rotates around the horizontal axis of rotation, i.e. moves in a vertical plane around the hinge \( O \). The pendulum consists of a weightless rod with length \( OA = l \). At the end of the rod is a material point \( A \) whose mass is equal \( m \). The position of the pendulum is determined by its angle \( \varphi \) of deviation of the rod from the vertical.

![Figure 4. Pendulum.](image)

Suppose that there is dry friction in the hinge \( O \), which reduces to the moment of friction, which, similar to Coulomb’s law (5), is proportional to the reaction modulus \( R \) in the hinge \( O \)

\[
M_{\varphi} = -\delta |R| \text{sign} \varphi ,
\]

(29)

where \( \delta \) is the coefficient of friction, which has a dimension of length.

By virtue of the theorem on the change of the angular momentum relative to the point \( O \), we have

\[
ml^2 \ddot{\varphi} = -mg \sin \varphi + M_{\varphi} .
\]

(30)

From the theorem of motion of the center of mass, in the projection on the axis of the natural trihedron, follows

\[
a_n = ml \dot{\varphi}^2 = -mg \cos \varphi + R_n , \quad ma_\varphi = ml \dot{\varphi} = -mg \sin \varphi + R_\varphi .
\]
Then
\[ |R| = \sqrt{R_x^2 + R_y^2} = m\sqrt{(l\dot{\phi}^2 + g \cos \phi)^2 + (l\ddot{\phi} + g \sin \phi)^2}. \] (31)

It follows from (29)-(31) that the differential equation of motion of the pendulum in an implicit form is an irrational equation for the highest derivative and has the form
\[ l(l\dot{\phi} + g \sin \phi) = -\delta \sqrt{(l\dot{\phi}^2 + g \cos \phi)^2 + (l\ddot{\phi} + g \sin \phi)^2} \text{sgn } \phi. \] (32)

Getting rid of irrationality by squaring, we have
\[ (l^2 - \delta^2)(l\dot{\phi} + g \sin \phi)^2 = \delta^2 (l\dot{\phi}^2 + g \cos \phi)^2. \] (33)

For large values of the coefficient of friction $\delta \geq l$, this equation has no solutions. We get a paradox similar to the Painleve paradoxes. Differential equation of motion doesn’t exist. Therefore, in this case, the movement of the system is impossible.

When $\delta < l$ equation (33) has two solutions
\[ l\ddot{\phi} = -g \sin \phi \pm \frac{\delta}{\sqrt{l^2 - \delta^2}} |l\dot{\phi}^2 + g \cos \phi|. \]

One of them is an outsider obtained by squaring equation (32). By virtue of (32), only one of these solutions is suitable
\[ l\ddot{\phi} = -g \sin \phi - \frac{\delta}{\sqrt{l^2 - \delta^2}} |l\dot{\phi}^2 + g \cos \phi| \text{sgn } \phi. \] (34)

This equation has stagnation zones. If the pendulum enters at zero angular velocity $\dot{\phi} = 0$ in the region in which
\[ |g \phi| \leq \frac{\delta}{\sqrt{l^2 - \delta^2}}, \] (35)
then from (14) it follows that the right-hand side in the equation of motion (34) is equal to zero and the pendulum stops. Condition (35) is fulfilled in the neighborhood of the extreme lower and extreme upper positions of the pendulum.

For a pendulum that is in zero gravity (i.e. $g = 0$), the differential equation of motion (34) has the form
\[ \ddot{\phi} = -\frac{\delta}{\sqrt{l^2 - \delta^2}} \dot{\phi}^2 \text{sgn } \phi. \] (36)

Dry friction manifests itself as viscous (in this case, proportional to the square of the angular velocity). Equation (36) is solved analytically. The direction of rotation does not change. For definiteness, we assume that, $\phi(0) = \phi_0$, then $\omega(0) = \omega_0 > 0$

\[ \omega = \frac{\omega_0}{1 + \omega_0 kt}, \quad \phi = \phi_0 + \frac{1}{k} \ln(1 + \omega_0 kt), \text{ where } k = \frac{\delta}{\sqrt{l^2 - \delta^2}}. \]
7. Conclusion

In mechanical systems with dry friction, situations are very characteristic when the motion of the system is described by implicit differential equations of motion, unresolved with respect to higher derivatives. These equations can be either unsolvable with respect to higher derivatives, or have several solutions. Similar situations were called Painleve paradoxes.

We also note that in all the considered examples, after resolving the implicit equations of motion with respect to the highest derivatives, dry friction leads to the appearance of resistance forces (moments) proportional to the square of the velocity, i.e. dry friction also manifests itself as viscous. A similar effect occurs in systems with transformed dry friction and in multicomponent models of dry friction [11].

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