The Action of Hesienberg Group on Finite Dimention Manifold

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Abstract

Our goal in this paper is introduce the action of hesienberg group on finite dimention manifold M. the interrelation ship between hesienberg group and the finite dimention manifold M is an old and vast subject. To simplify this treatment we work with lie algebra defined on a finite dimention manifold M, the heisenberg group forned an action over these lie algebra on finite manifold M.

No doubt, anotion on the heisnberg group can constitute avery important situation in the a differential finite manifold M, therefore, our work presents a key role mainly in some properties and characteriztions of the action of heinberg group on finite dimention manifold M, and also we study characterization on the relation between heisnberg group and finite dimention manifold M, then we introduce an an action of heisnberg group by the tensor product of the two repersentation which are (Acolyte groups) on hom (V2 , V1) be the tensor product of two representations of heisberg group and construct the definition of AC-heisnberg group, also study the properties of this action.

Key words: Heisenberg group, Lie algebra, Finite dimention manifold, Representation and bilinear map.

1. Introduction

The Heis (R) group is the group of 3*3 upper triangular matrices of the form

\[
\begin{bmatrix}
1 & x & x''
0 & 1 & x'
0 & 0 & 1
\end{bmatrix}
\]

[1], [2], under the operation of matrix multiplication elements x,x’ and x” can be taken from any commutative ring with identity [3], [4].
In the three-dimensional case, the product of two Heisenberg matrices is given by:

\[
\begin{bmatrix}
1 & x & x' \\
0 & 1 & x' \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & y & y' \\
0 & 1 & y' \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & x + y & x' + y' + xy' \\
0 & 1 & x' + y' \\
0 & 0 & 1
\end{bmatrix}
\]

Since the multiplication is not commutative, the group is non-abelian [5], [6].

The neutral element of \( \text{Heis}(\mathbb{R}) \) group is the identity matrix and inverse given by:

\[
\begin{bmatrix}
1 & x & x' \\
0 & 1 & x' \\
0 & 0 & 1
\end{bmatrix}
^{-1}
= \begin{bmatrix}
1 & -x & xx' - x'' \\
0 & 1 & -x' \\
0 & 0 & 1
\end{bmatrix}
\]

The author in [7] the pursuit examples

- Higher dimensions.
- Discrete \( \text{Heis}(\mathbb{R}) \) group.
- Lie algebra.
- \( \text{Heis}(\mathbb{R}) \) group modulo 2.
- \( \text{Heis}(\mathbb{R}) \) group modulo an odd prime \( P \).
- Continuous \( \text{Heis}(\mathbb{R}) \) group.

The \( \text{Heis}(\mathbb{R}) \) group is not an object, [8], [9].

A differential manifold \( M \) can be described by a collection of charts, named an atlas [10].

In differential geometry, the \( \text{Heis}(\mathbb{R}) \) group action on a finite dimension manifold \( M \) is a group action by lie algebra group \( G \) on \( M \) that is a differentiable map action with lie-algebra group action \( G \) [11].

From [10] [12] and [13], \( M \) is called a \( G \)-Manifold

Let \( \gamma : G \times M \rightarrow M, (g, x) \rightarrow g.x \).

Be a group action, it is a lie algebra group action if it is differentiable, thus:

\( \gamma_x : G \rightarrow M, g \rightarrow g.x. \)

and we can compute its differentiable at the identity element of \( G \) the action of \( \text{Heis}(\mathbb{R}) \) group on hom space.

From [5], Schur’s Lemma

suppose that \( \gamma'_1, \gamma'_2 \) are two representation of lie algebra action on finite – dimensional space \( V_1, V_2 \) respectively. Define an action of \( g \) on hom \( (V_2, V_1) \):

\( \gamma' : g \rightarrow gL (\text{Hom}(V_2, V_1)) \)

\( \gamma'_x = \gamma'_1 F - F \gamma'_2, \) for all \( x \in g \) and \( f \in (V_2, V_1) \) and \( \text{Hom}(V_2, V_1) \cong V'_2 \times V_1 \) as equivalence of representation.
2. Results

Proposition (2.1):

Let $\gamma_i : \text{Heis}(R) \to GL(V_i)$ be representation of Heis(R) group on $M$ – finite dimensional manifold, space $V_i$, $i = 1,2$ respectively and 

$\gamma'_i : \text{Heis}(R) \to GL(V'_i)$ the dual representation on $V'_i$ for $i = 1,2$ then $\gamma'_i(x) = F_i \cdot \gamma_i(x)$ for all $x \in \text{Heis}(R)$, where $F_i : V_i \to M$ and $M$ is a field.

Proof: assume Hom ($V_2$, $V'_1$) where $V'_1 = V_1 \to M$

We will define an action of heis(R) group on hom ($V_2$, $V'_i$) by:

$\gamma : \text{Heis}(R) \to GL(\text{Hom}(V_2, V'_1))$ defined by:

$\gamma(x) = \gamma'_1(x) \circ \gamma_2(x)^{-1}$ for all $x \in \text{Heis}(R), F = F_2 \circ F_1 \in \text{Hom}(V_2, V'_1)$

$F_1 \in \text{Hom}(V_2, V_1), F_2 \in \text{Hom}(V_1, V'_i), \gamma(x) = \gamma'_1(x) \circ (\gamma_2(x)^{-1})$, $w \in V_2$

Proposition (2.2):

Let $\gamma_i \ i = 1,2$ be two representation of Heis(R) group acting on $V_2$ and $V'_i$ then:

$\gamma : \text{Heis}(R) \to GL(\text{Hom}(V_2, V'_1))$ is a representation of Heis(R) group acting on Hom ($V_2$, $V'_i$) which $\gamma$ called AC – Heis (R) group on Hom ($V_2$, $V'_i$)

Proof:
\( \gamma(xy) = \gamma_1'(xy) \circ \gamma_2(xy)^{-1} \)
\[ = (\gamma_1'(x) \circ \gamma_1'(y)) \circ \gamma_2(xy)^{-1} \gamma_2(x)^{-1} \]
\[ = (\gamma_1'(y) \circ \gamma_1'(x) \circ \gamma_2(xy)^{-1}) \circ \gamma_2(y)^{-1} \] .................(1)

and \( \gamma(x)\gamma(y) = \gamma(y) \circ \gamma(x) \)
\[ = \gamma(y) \circ (\gamma_1'(x) \circ \gamma_2(x)^{-1}) \]
\[ = \gamma_1'(y) \circ (\gamma_1'(x) \circ \gamma_2(x)^{-1}) \circ \gamma_2(y)^{-1} \] .................(2)

since \( \gamma(xy) = \gamma(x)\gamma(y) \) thus by (1) and (2) \( \gamma(x) \) is a representation of Heis (R) group.

From the following diagram we show that \( \gamma \) is a group homomorphism:

\[ F = \gamma_1(x) \circ \circ \gamma_2(x)^{-1} \in Hom (V_2, V_1) \]

Example (2.3):

Suppose that \( \gamma_1 \) is a representation from Lie algebra in to Heis(R) group, such that:
\[ \gamma_1(x) = \begin{bmatrix} 1 & 0 & 0 & a \\ b & 1 & a & c \\ 0 & 0 & 1 & -b \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ a, b, c \in R \] and \( \gamma_2 \) is a representation from Lie algebra in to SL(2, R)
\[ \gamma_2(x) = \begin{bmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix} \] then the AC – Heis(R) group on Hom is:

\[ \gamma(x) = \gamma_1'(x) \circ \gamma_2(x)^{-1} \] for all \( x \in \text{Lie algebra group} \)

\[
\begin{bmatrix}
1 & 0 & 0 & a \\
b & 1 & a & c \\
0 & 0 & 1 & -b \\
0 & 0 & 0 & 1
\end{bmatrix}
\circ
\begin{bmatrix}
\sin \theta & \cos \theta \\
-cos \theta & \sin \theta
\end{bmatrix}
\]

Proposition (2.4):

Let \( \gamma_i, i = 1, 2 \) be a representation of Heis group acting on \( V_2 \) and \( V_1' \), then there exist \( \gamma: \text{Heis}(R) \rightarrow \text{GL}(V_2' \ast V_1') \) which is a representation of Heis(R) group acting on the vector space \( V_2' \ast V_1' \)

such that:

\[ \gamma(x) = \gamma_2(x)^{-1} \ast \gamma_1'(x), \text{ for all } x \in \text{Heis}(R) \]

Proof:

\[ \gamma(xy) = \gamma_2(xy)^{-1} \ast \gamma_1'(xy) \]
\[ = (\gamma_2(y)^{-1} \gamma_2(x)^{-1}) \ast (\gamma_1'(x) \gamma_1'(y)) \]
\[ = \gamma_2(y)^{-1} \left( \gamma_2(x)^{-1} \ast \gamma_1'(x) \right) \gamma_1'(y) \]

Since \( \gamma(y) \gamma(x) = \gamma(y)(\gamma_2(x)^{-1} \ast \gamma_1'(x)) \)

Then:

\[ -\gamma_2(y)^{-1} \left( \gamma_2(x)^{-1} \ast \gamma_1'(x) \right) \gamma_1'(y) \]

Thus:

\[ \gamma(xy) = \gamma(y) \gamma(x) = \gamma(x) \circ \gamma(y) \]
From the diagram $\gamma$ is a group homomorphism of Heis(R) group on $GL(V_2' \ast V_1')$ and it is smooth map.

Note: A diagonal map $\Delta$, $\gamma_1$, $\gamma_2$, in clusin map $\pi$ and bilinear maps are smooth.

Hence $\gamma$ is a representation of Heis(R) group

**Proposition (2.5):**

Let $\gamma_1$ and $\gamma_2$ be two representation of Heis(R) group acting on k-finite dimensional vector space $V_1'$, $V_2$ respectively then the AC-Heis(R) group of G on Hom ($V_2'$, $V_1'$) are equivalent to the AC-Heis(R) group on $V_2' \ast V_1'$

**Proof:**

To explain that $y : V_2' \ast V_1' \to Hom(V_2', V_1')$ is abilinear map, defined by:

$y : V_2' \ast V_1' = S$, for all $V_2' \in V_2', V_1' \in V_1'$
Where S: \( V_2 \rightarrow V_1 \) is a linear map, defined by \( S(V) = V_2'(V) V_1' \)

For all \( V_2', V_2'' \in V_2', V \in V_2, V_1' \in V_1', X, Y \in K \)

\[
y(xV_2' + yV_2'', V_1') = (xV_2' + yV_2'')_V V_1'
\]

\[
= (xV_2' (v)V_1' + yV_2'' (v)V_1')
\]

\[
= xy(V_2', V_1') + yy(V_2'', V_1')
\]

For all \( V_1', V_2' \in V_1', V_2'' \in V_2'' \)

\[
y(V_2', xV_2' + yV_2'', V_1') = V_2'(v)(xV_1' + yV_1'')
\]

\[
= V_2'(v)(xV_2' + yV_2'')
\]

\[
= xy(V_2', V_1') + yy(V_2'', V_2')
\]

That is:

\[
y: (V_2', V_1') \rightarrow Hom (V_2, V_1')
\]

is a bilinear map, thus by tensor product

\[
y: (V_2' \ast V_1') \rightarrow Hom (V_2, V_1')
\]

then there exists a unique linear map

\[
y: (V_2' \ast V_1') \rightarrow Hom (V_2, V_1')
\]

that makes the above diagram commutative.

3. The Co-Action of Heis(R) Group on Hom and Tensor Product

Proposition (3.1):

Let \( \gamma_i: \text{Heis}(R) \rightarrow GL (V_i) \), for \( i = 1,2 \) be a dual representation of \( \gamma_i \) then the co-action of Heis(R) on \( (\text{Hom} (V_2, V_1'))' \) are given by a dual representation of Heis (R) such that:

\[
y^*(x) = \gamma_2(x)^{-1} o o \gamma_1 (x)\] for all \( x \in \text{Heis}(R) \)

Proof:
\[ \gamma'(x) = (\gamma(x))' = (\gamma_1(x) o f o \gamma_2(x)^{-1})' \]
\[ = \gamma_2(x)^{-1} o f o \gamma_2''(x) \]
\[ = \gamma_2(x)^{-1} o f o \gamma_1(x) \]

Where: \( F: V_1'' \to V_2 \) and \( \gamma'(xy) = (\gamma(xy))' \)
\[ = (\gamma(Y) o \gamma(X))' = \gamma'(x) o \gamma'(y) \]

Thus the co-action of Heis(R) is a group homomorphism.

**Corollary (3.2):**

Let \( \gamma_i : Heis(R) \to GL(n, k) \cong GL(V_i) \), \( i=1,2 \) be a matrix representation, thus the co-action of Heis(R) on \( GL(n, k) \) is a matrix representation defined by:

\[ \gamma(x) = (\gamma_1(x)^{-1tr} o f o \gamma_2(x)^{-1} \text{ with duality} \]

\[ \gamma' : G \to GL(Hom(V_2, V_1)) \]

Such that \( \gamma'(x) = (\gamma_2(x))^{tr} o F \gamma_1(x) \text{ for all } x \in Heis(R) \)

Is a matrix representation of Heis(R) then the c0-action is given by:

\[ \gamma'(x) = (\gamma_2(x))^{tr} o F \gamma_1(x) \]

\[
\begin{bmatrix}
\cos\theta & -\sin\theta & -\cos\theta \\
\sin\theta & \cos\theta & -\sin\theta \\
-\cos\theta & \sin\theta & \cos\theta
\end{bmatrix}
\begin{bmatrix}
e^{i\theta} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-i\theta}
\end{bmatrix}
\begin{bmatrix}
1 & e^{i\theta} & e^{i\theta} \\
e^{i\theta} & 0 & 0 \\
e^{i\theta} & 0 & 1
\end{bmatrix}
\]
Proposition (3.3):

If \( \gamma \) is an action of Heis(R) group on \((V_2^* \times V_1^*)\)

Such that: \( \gamma(x) = \gamma_2^*(x)^{-1} \ast \gamma_1(x) \) for all \( x \in Heis(R) \)

Then the co-action of Heis(R) on \((V_2^* \times V_1^*)\) ia adual representation of Heis(R) such that:

\[ \gamma^*: \text{Heis}(R) \rightarrow \text{GL}(V_2^* \times V_1^*) \] defined by \( \gamma^*(x) = \gamma_2^*(x)^{-1} \ast \gamma_1^*(x) \) for all \( x \in Heis(R) \)

Proof:

\[ \gamma(x) = \gamma_2^*(x)^{-1} \ast \gamma_1^*(x) \text{ that is } \gamma^*(x) = (\gamma(x))^* \]

\[ = (\gamma_2^*(x)^{-1} \ast \gamma_1^*(x))^* \]

\[ = \gamma_2^*(x)^{-1} \ast \gamma_1^*(x) \text{ for all } x \in Heis(R) \]

And \( \gamma^*(xy) = \gamma_2^*(xy)^{-1} \ast \gamma_1^*(xy) \)

\[ = \gamma_2^*(y)^{-1} (\gamma_2^*(x)^{-1} \ast \gamma_1^*(x)) \gamma_1^*(y) \] .................................(1)

\[ \gamma^*(x)\gamma^*(y) = \gamma^*(y) \circ (\gamma_2^*(x)^{-1} \ast \gamma_1^*(x)) \]

\[ = \gamma_2^*(y)^{-1} (\gamma_2^*(x)^{-1} \ast \gamma_1^*(x)) \gamma_1^*(y) \] .................................(2)

Where (1) and (2) are equal

Then the co-action is a dual representation of Heis(R) on \((V_2^* \times V_1^*)^*\)

Proposition (3.4):

If the ac-Heis(R) on \( GL(n,k) \cong (V_2 \times V_1) \) is a matrix representation then ac-Heis(R) on \((V_2^* \times V_1^*)^*\) is a matrix representation defined by:

\[ \gamma^*(x) = \gamma_2^*(x)^{-1} \ast \gamma_1^*(x) \text{ for all } x \in Heis(R) \]

Proof:
Suppose $\gamma_1(x) = \gamma_i(x)$

Then $\gamma_1(x) = \gamma_2(x)^{-1} * \gamma_1(x)$
\[ = \gamma_2(x)^{-1} * \gamma_1(x) \text{ for all } x \in \text{Heis}(R) \]

And $\gamma'(x) = (\gamma(xy)^{-1})^\text{tr}$
\[ = (\gamma(y)^{-1} \gamma(x)^{-1})^\text{tr} \]
\[ = (\gamma(x)^{-1} o \gamma(y)^{-1})^\text{tr} \]
\[ = (\gamma(x)^{-1})^\text{tr} o (\gamma(y)^{-1})^\text{tr} \]
\[ = \gamma'(x) o \gamma'(y) \]

Example (3.5):

Let $\gamma_1 : \text{Heis}(R) \to \text{so}(3)$

And $\gamma_2 : \text{Heis}(R) \to \text{su}(T(R))$ such that $\text{su}(T(R)) = \text{GL}(n,R) \cap O(n,R)$

$\gamma_2(g) = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, $b \in z$ be a matrix representation of Heis (R) group then the co-action are given by:

$\gamma'(g) = \gamma_2(g)^{-1} * \gamma_1(g)$
\[ = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^{-1} * \begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
\[ = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
\[ = \begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 & -b \cos 2\theta & b \sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 & -b \sin 2\theta & -b \cos 2\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & -b \\ 0 & 0 & 0 & \cos 2\theta & -\sin 2\theta & 0 \\ 0 & 0 & 0 & \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6 \times 6} \]
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