The Second Main Theorem for Holomorphic Curves into Semi-Abelian Varieties II *

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Abstract

We establish the second main theorem with the best truncation level one

\[ T(r; \omega_Z, J_k(f)) \leq N_1(r; J_k(f)^*Z) + \epsilon T_f(r)||\epsilon \]

for the \( k \)-jet lift \( J_k(f) : C \to J_k(A) \) of an algebraically non-degenerate entire holomorphic curve \( f : C \to A \) into a semi-abelian variety \( A \) and an arbitrary algebraic reduced subvariety \( Z \) of \( J_k(A) \); the low truncation level is important for applications. Finally we give some applications, including the solution of a problem posed by Mark Green (1974).

1 Introduction and main result

Let \( f : C \to V \) be a holomorphic curve into a complex projective manifold \( V \) with Zariski dense image and let \( D \) be an effective reduced divisor on \( V \). Under some ampleness condition for the space \( H^0(V, \Omega^1_V((\log D))) \) of logarithmic 1-forms along \( D \) we proved in [N77], [N81] the following inequalities of the second main theorem type,

\[
\kappa T_f(r) \leq N(r; f^*D) + O(\log r) + O(\log T_f(r))||, \\
\kappa' T_f(r) \leq N_1(r; f^*D) + O(\log r) + O(\log T_f(r))||,
\]

where \( T_f(r) \) denotes the order function of \( f \), \( N(r; f^*D) \) (resp. \( N_1(r; f^*D) \)) the counting function (resp. truncated to level \( l \)) of the pull-backed divisor \( f^*D \), and \( \kappa \) and \( \kappa' \) are positive constants (cf. §2). It is an interesting and fundamental problem to determine the constant \( \kappa \) or \( \kappa' \). In the case where \( V \) is the compactification of a semi-abelian variety \( A \) this problem is related to what kind of compactification \( V \) of \( A \) we take. In our former

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paper [NWY02] we proved that for a holomorphic curve $f : \mathbb{C} \to A$ into a semi-abelian variety $A$ and an algebraic divisor $D$ on $A$,

$$T_f(r; L(D)) \leq N_1(r; f^*D) + O(\log r) + O(\log T_f(r; L(D)))|.|.$$

Here we used a compactification $\bar{A}$ of $A$ such that the maximal affine subgroup $(\mathbb{C}^*)^t$ of $A$ was compactified by $(\mathbb{P}^1(\mathbb{C}))^t$, and we assumed a boundary condition (Condition 4.11 in [NWY02]) for the closure $\bar{D}$ of $D$ in $\bar{A}$; this roughly meant the divisor $\bar{D} + (\bar{A} \setminus A)$ to be in general position and has been expected to be removed by a suitable choice of a compactification of $A$. It is an important and very interesting problem to take the truncation level $l$ as small as possible.

Let $X_k(f)$ denote the Zariski closure of the image of the $k$-jet lift of $f$ in the $k$-jet space $J_k(A)$ over $A$. The purpose of this paper is to prove (cf. §§2, 3 for notation)

**Main Theorem.** Let $A$ be a semi-abelian variety. Let $f : \mathbb{C} \to A$ be a holomorphic curve with Zariski dense image.

(i) Let $Z$ be an algebraic reduced subvariety of $X_k(f)$ ($k \geq 0$). Then there exists a compactification $\bar{X}_k(f)$ of $X_k(f)$ such that

$$T(r; \omega_{\bar{Z}, J_k(f)}) \leq N_1(r; J_k(f)^*Z) + \epsilon T_f(r)||_{\epsilon}, \quad \forall \epsilon > 0,$$

where $\bar{Z}$ is the closure of $Z$ in $\bar{X}_k(f)$.

(ii) Moreover, if $\text{codim}_{X_k(f)}Z \geq 2$, then

$$T(r; \omega_{Z, J_k(f)}) \leq \epsilon T_f(r)||_{\epsilon}, \quad \forall \epsilon > 0.$$

(iii) In the case when $k = 0$ and $Z$ is an effective divisor $D$ on $A$, the compactification $\bar{A}$ of $A$ can be chosen as smooth, equivariant with respect to the $A$-action, and independent of $f$; furthermore, (1.2) takes the form

$$T_f(r; L(\bar{D})) \leq N_1(r; f^*D) + \epsilon T_f(r; L(\bar{D})))||_{\epsilon}, \quad \forall \epsilon > 0.$$

Note that in the above estimate (1.2), (1.3) or (1.4) the small error term “$\epsilon T_f(r)$” cannot be replaced by “$O(\log r) + O(\log T_f(r))$” (see [NWY02] Example (5.36)).

The Main Theorem is an advancement of [NWY02] and [Y04]. When $A$ is an abelian variety, (1.4) was proved by Yamanoi [Y04] (cf. [Y04] (3.1.8)). A key of the proof of (1.1) in [NWY02] was Lemma 5.6 at p. 147, and here we will again use the same idea for jets of jets (see Claim 4.13).
There is a related result due to Siu-Yeung [SY03], where they obtained with an improved truncation level \( l = l(D) \) dependent only on the Chern numbers of \( D \). In their proof the key was Claim 1 at p. 443 which was the same as [NYW02] Lemma 5.6 restricted to the abelian case with a computation of intersection numbers.

It is interesting to observe that the error term being “\( O(\log r) + O(\log T_f(r; L(\bar{D}))) \)” \( \| \), the truncation level \( l \) in (1.1) has to depend on \( D \), but the error term being allowed to be a little bit large, “\( \epsilon T_f(r; L(\bar{D})) \|_r \)”, \( l \) can be one, the smallest possible. In applications, the truncation of level one is very definite.

To deal with semi-abelian varieties the main difficulties are caused by the following two points:

(i) Semi-abelian varieties are not compact and need some good compactifications.

(ii) There is no Poincaré reducibility theorem for semi-abelian varieties.

It is also noted that a part of the proof of the Main Theorem for abelian varieties in [Y04] does not hold for semi-abelian varieties ([Y04] §3 Claim), and that a different and considerably simpler proof for that part will be provided (see Lemma 6.1).

In §7 we will give two applications of the Main Theorem. The first is a complete affirmative answer to a conjecture of M. Green [G74] pp. 229–230 (cf. Theorem 7.2). The second is a non-existence theorem for some differential equations defined over semi-abelian varieties (cf. Theorem 7.6).

More applications will be obtained in [NWY05].

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2 Notation

The notation here follows that of [NWY02]. For a general reference of this section, cf. [NO8490]. For convenience we recall some of definitions. Let \( M \) be a compact complex manifold and let \( \omega \) be a smooth (1,1)-form on \( M \). Let \( f : C \to M \) be a holomorphic curve into \( M \). We define the order function of \( f \) with respect to \( \omega \) by

\[
T_f(r; \omega) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^*\omega \quad (r > 1).
\]

If \( M \) is Kähler and \( d\omega = 0 \),

\[
T_f(r; \omega) = T_f(r; \omega') + O(1)
\]
for a $d$-closed $(1,1)$-form $\omega'$ in the same cohomology class $[\omega] \in H^2(M, \mathbb{R})$. Therefore we set, up to $O(1)$-term,

\begin{equation}
T_f(r; [\omega]) = T_f(r; \omega).
\end{equation}

Let $L \to M$ be a hermitian line bundle with Chern class $c_1(L)$. Then we set

\[ T_f(r; L) = T_f(r; c_1(L)), \]

which is defined again up to $O(1)$-term.

For a divisor $D$ on $M$ we denote by $L(D)$ the line bundle determined by $D$.

Let $E = \sum_{\mu=1}^\infty \nu_\mu z_\mu$ be a divisor on $\mathbb{C}$ with distinct $z_\mu \in \mathbb{C}$. Then we set

\[ \text{ord}_z E = \begin{cases} \nu_\mu, & z = z_\mu, \\ 0, & z \not\in \{z_\mu\}. \end{cases} \]

We define the counting functions of $E$ truncated to $l \leq \infty$ by

\[ n_l(t; E) = \sum_{\{z_\mu \mid |z_\mu| < t\}} \min\{\nu_\mu, l\}, \]

\[ N_l(r; E) = \int_1^r \frac{n_l(t; E)}{t} dt. \]

We define the counting functions of $E$ by

\[ n(t; E) = n_\infty(t; E), \quad N(r; E) = N_\infty(r; E). \]

**Definition of small terms.** (i) For a line bundle $L \to M$ and a holomorphic curve $f : \mathbb{C} \to M$ we denote by $S_f(r; L)$ such a small term as

\[ S_f(r; L) = O(\log r) + O(\log^+ T_f(r; L)) ||, \]

where “$||$” stands for the inequality to hold for every $r > 1$ outside a Borel set of finite Lebesgue measure.

(ii) Let $h(r)$ ($r > 1$) be a real valued function. We write

\[ h(r) \leq \varepsilon T_f(r; L)||_\varepsilon, \quad \forall \varepsilon > 0, \]

if the stated inequality holds for every $r > 1$ outside a Borel set of finite Lebesgue measure, dependent on an arbitrarily given $\varepsilon > 0$.

**Definition.** When $M$ is an algebraic variety, we say that $f : \mathbb{C} \to M$ is *algebraically (resp. non-) degenerate* if the image $f(\mathbb{C})$ is (resp. not) contained in a proper algebraic subset of $M$.

The following follows from general properties of order functions ([NO84]).
Lemma 2.3 Let $f : C \to M$ be a holomorphic curve into a complex projective manifold $M$ and $H$ a line bundle on $M$. Assume that $H$ is big, and that $f$ is algebraically non-degenerate. Then

$$T_f(r, L) = O(T_f(r, H))$$

for every line bundle $L$ on $M$.

If $f : C \to M$ is algebraically degenerate, we may consider the Zariski closure $N$ of $f(C)$ and a desingularization $\tau : \tilde{N} \to N$. Then $f$ lifts to a map to $\tilde{N}$ and $\tau^*(H|_N)$ is big on $\tilde{N}$ for every ample line bundle $H$ on $M$. As a consequence we obtain:

Lemma 2.4 Let $f : C \to M$ be a holomorphic curve into a complex projective manifold $M$. Let $h(r)$ be a non-negative valued function in $r > 1$. Then $h(r) = S_f(r; H)$ holds for every ample line bundle if and only if it holds for at least one ample line bundle.

Similarly the statement $h(r) \leq c T_f(r; H)$, $\forall c > 0$, respectively $h(r) = O(T_f(r; H))$ holds for every ample line bundle $H$ if and only if it holds for at least one ample line bundle.

If one of these conditions holds for one and therefore for all ample line bundles $H$, we simply write $h(r) = S_f(r)$ (resp. $h(r) \leq c T_f(r)$, $h(r) = O(T_f(r))$).

For a quasi-projective manifold $V$ and for a holomorphic curve $f : C \to V$ we write simply $T_f(r) = T_f(r; H)$ for the order function with respect to an ample line bundle $H$ over a projective compactification $\bar{M}$ of $M$ if the choice of $\bar{M}$ and $H$ do not matter.

The following related property of order functions will be frequently used ([NO8490] Lemma (6.1.5)).

Lemma 2.5 Let $\eta : V \to W$ be a rational mapping between quasi-projective manifolds $V$ and $W$. Then for an algebraically non-degenerate holomorphic curve $f : C \to V$

$$T_{\eta \circ f}(r) = O(T_f(r)).$$

Moreover, if $\eta$ is generically finite, then

$$T_f(r) = O(T_{\eta \circ f}(r)).$$

We define the proximity function $m_f(r; \mathcal{I})$ not only for divisors but also for a coherent ideal sheaf $\mathcal{I}$ of the structure sheaf $\mathcal{O}_M$ over $M$. Let $\{U_j\}$ be a finite open covering of $M$ such that

(i) there is a partition of unity $\{c_j\}$ associated with $\{U_j\}$,
(ii) there are finitely many sections \( \sigma_{jk} \in \Gamma(U_j, \mathcal{I}), k = 1, 2, \ldots \), generating every fiber \( \mathcal{I}_x \) over \( x \in U_j \).

Setting \( \rho_{\mathcal{I}}(x) = \left( \sum_j c_j(x) \sum_k |\sigma_{jk}(x)|^2 \right)^{1/2} \), we take a positive constant \( C \) so that

\[
C \rho_{\mathcal{I}}(x) \leq 1, \quad x \in M.
\]

Using the compactness of \( M \), one easily verifies that, up to addition by a bounded continuous function on \( M \), \( \log \rho_{\mathcal{I}} \) is independent of the choices of the open covering, the partition of unity, the local generators of the ideal sheaf \( \mathcal{I} \), and the constant \( C \).

We define the proximity function of \( f \) for \( \mathcal{I} \) or for the subspace (may be non-reduced) \( Y = (\text{Supp } \mathcal{O}_M/\mathcal{I}, \mathcal{O}/\mathcal{I}) \) by

\[
m_f(r; Y) = m_f(r; \mathcal{I}) = \int_{|z|=r} \log \frac{1}{C \rho_{\mathcal{I}}(f(re^{i\theta}))} \frac{d\theta}{2\pi} \quad (\geq 0),
\]

provided that \( f(C) \not\subseteq \text{Supp } Y \). Note that if \( \mathcal{I} \) is the ideal sheaf defined by an effective divisor \( D \) on \( M \), \( m_f(r; \mathcal{I}) \) coincides \( m_f(r; D) \) defined in [NWY02] up to \( O(1) \)-term. The function \( \rho_{\mathcal{I}} \circ f(z) \) is smooth over \( C \setminus f^{-1} (\text{Supp } Y) \). For \( z_0 \in f^{-1}(\text{Supp } Y) \) choose an open neighborhood \( U \) of \( z_0 \) and a positive integer \( \nu \) such that \( f^*\mathcal{I} = (z - z_0)^\nu \). Then

\[
\log \rho_{\mathcal{I}} \circ f(z) = \nu \log |z - z_0| + \psi(z), \quad z \in U.
\]

for some smooth function \( \psi(z) \) defined on \( U \). We define the counting function \( N(r; f^*\mathcal{I}) \) and \( N_t(r; f^*\mathcal{I}) \) by using \( \nu \) in the same way as using \( \text{ord}_{z_0}(E) \) in the definition of \( N(r; E) \) and \( N_t(r; E) \). Moreover we define

\[
\omega_{\mathcal{I}, f} = \omega_{Y,f} = -dd^c \psi(z) = -\frac{i}{2\pi} \partial \bar{\partial} \psi(z)
\]

\[
= dd^c \log \frac{1}{\rho_{\mathcal{I}} \circ f(z)} \quad (z \in U),
\]

which is well-defined on \( C \) as a smooth \((1,1)\)-form. The order function of \( f \) for \( \mathcal{I} \) or \( Y \) is defined by

\[
T(r; \omega_{\mathcal{I}, f}) = T(r; \omega_{Y,f}) = \int_1^r \frac{dt}{t} \int_{|z|<t} \omega_{\mathcal{I}, f}.
\]

When \( \mathcal{I} \) defines a divisor \( D \) on \( M \), we see that

\[
T(r; \omega_{\mathcal{I}, f}) = T_f(r; L(D)) + O(1).
\]

Let \( \mathcal{I}_i (i = 1, 2) \) be coherent ideal sheaves of \( \mathcal{O}_M \) and let \( Y_i \) be the subspace defined by \( \mathcal{I}_i \). We write \( Y_1 \supset Y_2 \) if \( \mathcal{I}_1 \subset \mathcal{I}_2 \).
Theorem 2.9  Let \( f : C \to M \) and \( \mathcal{I} \) be as above. Then we have the following:

(i) (First Main Theorem)

\[
T(r; \omega_{\mathcal{I}, f}) = N(r; f^* \mathcal{I}) + m_f(r; \mathcal{I}) - m_f(1; \mathcal{I}).
\]

(ii) If \( M \) is projective, \( m_f(r; \mathcal{I}) = O(T_f(r)). \)

(iii) Let \( \mathcal{I}_i \) \((i = 1, 2)\) be coherent ideal sheaves of \( \mathcal{O}_M \) and let \( Y_i \) be the subspace defined by \( \mathcal{I}_i \). If \( \mathcal{I}_1 \subset \mathcal{I}_2 \) or equivalently \( Y_1 \supset Y_2 \), then

\[
m_f(r; \mathcal{I}_2) \leq m_f(r; \mathcal{I}_1) + O(1),
\]

or equivalently,

\[
m_f(r; Y_2) \leq m_f(r; Y_1) + O(1).
\]

(iv) Let \( \phi : M_1 \to M_2 \) be a holomorphic mappings between compact complex manifolds. Let \( \mathcal{I}_2 \subset \mathcal{O}_{M_2} \) be a coherent ideal sheaf and let \( \mathcal{I}_1 \subset \mathcal{O}_{M_1} \) be the coherent ideal sheaf generated by \( \phi^* \mathcal{I}_2 \). Then

\[
m_f(r; \mathcal{I}_1) = m_{\phi \circ f}(r; \mathcal{I}_2) + O(1).
\]

(v) Let \( \mathcal{I}_i \), \( i = 1, 2 \) be two coherent ideal sheaves of \( \mathcal{O}_M \). Suppose that \( f(C) \not\subset \operatorname{Supp}(\mathcal{O}_M/\mathcal{I}_1 \otimes \mathcal{I}_2) \). Then we have

\[
T(r; \omega_{\mathcal{I}_1 \otimes \mathcal{I}_2, f}) = T(r; \omega_{\mathcal{I}_1, f}) + T(r; \omega_{\mathcal{I}_2, f}) + O(1).
\]

Proof.  (i) This immediately follows from the well-known Jensen formula (cf. [NO84] Theorem (5.2.15)).

(ii) Let \( Y \) be the subvariety defined by \( \mathcal{I} \). There is an ample divisor \( D \) on \( M \) such that \( D \supset Y \) (counting multiplicities). It follows from Theorem (2.9) (iii) that

\[
m_f(r; Y) \leq m_f(r; D) \leq T_f(r; L(D)) = O(T_f(r)).
\]

(iii) (iv) (v) These are immediate by definition. Q.E.D.

3  General position

Convention 3.1  Unless explicitly stated otherwise, all varieties, morphisms, group actions, compactifications, divisors etc. are assumed to be algebraic.
3.1 General position

Let $A$ be a semi-abelian variety and let $X$ be a complex algebraic variety (possibly singular) on which $A$ acts:

$$(a, x) \in A \times X \rightarrow a \cdot x \in X.$$ 

Let $Y$ be a subvariety embedded into a Zariski open subset of $X$.

**Definition 3.2** We say that $Y$ is generally positioned in $X$ if the closure $\bar{Y}$ of $Y$ in $X$ contains no $A$-orbit. If the support of a divisor $E$ on a Zariski open subset of $X$ is generally positioned in $X$, then $E$ is said to be generally positioned in $X$.

Let $\pi : X_1 \rightarrow X$ be a blow-up of smooth projective manifolds on which $A$ acts. Let $D$ be a divisor on $X$ and let $D_1$ be its strict transform. Then $D_1 \sim \pi^*D - E$, where $E$ is an effective divisor with support contained in the exceptional locus of the blow-up. If $\pi$ is the blow-up along a smooth connected submanifold $C \subset X$, then $E$ is empty unless $C \subset D$.

**Lemma 3.3** Assume that $D$ is generally positioned in $X$. Let $\pi : X_1 \rightarrow X$ be an equivariant blow-up. Then $D_1 = \pi^*D$, i.e., $E$ is empty.

**Proof.** Since the blow-up is assumed to be equivariant, its center $C$ must be an invariant subset, i.e., $C$ is a union of $A$-orbits. Now $D$ is assumed to be generally positioned in $X$. This implies that $D$ contains no $A$-orbit. Therefore no irreducible component of $C$ is contained in $D$. Q.E.D.

**Corollary 3.4** Assume that $D$ is big and generally positioned in $X$. Then $D_1$ is big, too.

**Proof.** This is immediate from $D_1 = \pi^*D$. Q.E.D.

Unfortunately the assumption of being generally positioned can not be dropped. For example, let us consider $X = \mathbb{P}^2(C)$. Let $D$ be a line and let $X_1 \rightarrow X$ be the blow-up of a point $p$ on the line $D$. Then $X_1$ is a ruled surface. It admits a fibration $\tau : X_1 \rightarrow \mathbb{P}^1(C)$ which arises as follows: We may identify $\mathbb{P}^1(C)$ with $\mathbb{P}(T_p\mathbb{P}^2(C))$. Then for $x \in \mathbb{P}^2(C) \setminus \{p\}$ we set $\tau(x)$ to be the tangent line at $p$ of the unique line in $\mathbb{P}^2(C)$ connecting $p$ and $x$. Now the strict transform $D_1$ of $D$ turns out to be a fiber of $\tau$. As a fiber of a holomorphic map, it can not be big. However, $D$, as an effective divisor on $\mathbb{P}^2(C)$, is big.

To give another example, consider the blow-up of $\mathbb{P}^2(C)$ in two points $p, q \in D$. A blow-up decreases the self-intersection number of a curve by 1. Therefore the self-intersection
number of the strict transform $D_2$ of $D$ under this blow-up $X_2 \to X$ is a curve with self-intersection number $-1$. As a consequence we have $\dim H^0(X_2, L(nD_2)) = 1$ for all $n \in \mathbb{N}$.

Note that these examples are equivariant for a suitably chosen action of $A = (\mathbb{C}^*)^2$, but $D$ is not generally positioned in $\mathbb{P}^2(\mathbb{C})$.

On the other hand, bigness can only be destroyed, not created via blow-up. This follows from the following fact: $D_1 = \pi^*D - E$ where $E$ is effective. Thus fixing a section $\sigma \in H^0(X_1, E)$ we obtain an injection

$$H^0(X_1, L(nD_1)) \hookrightarrow H^0(X_1, L(n\pi^*D)) \cong H^0(X, L(nD)) \quad (\forall n \in \mathbb{N})$$

given by mapping a section to its tensor product with $\sigma^n$. Therefore the Iitaka $D$-dimension can only decrease ([I71]).

**Lemma 3.5** Let $\pi : X_1 \to X$ be an equivariant blow-up, let $D$ be a divisor on $X$ which is generally positioned in $X$, and let $D_1$ be its strict transform. Then $D_1$ is generally positioned in $X_1$, too.

**Proof.** If $D_1$ would contain an $A$-orbit $\Omega$, we could infer that $\pi(\Omega) \subset \pi(D_1) = D$. Since $\pi$ is assumed to be equivariant, this would imply that $D$ contains an $A$-orbit, namely $\pi(\Omega)$. Q.E.D.

### 3.2 Stabilizer

Let $A$ be a semi-abelian variety such that

$$(3.6) \quad 0 \to T \to A \xrightarrow{\pi} A_0 \to 0,$$

where $T \cong (\mathbb{C}^*)^t$ and $A_0$ is an abelian variety. Let $D$ be a divisor on $A$. The stabilizer of $D$ is defined by

$$(3.7) \quad \text{St}(D) = \{a \in A : a + D = D\}^0,$$

where $\{\cdot\}^0$ denotes the identity component.

**Lemma 3.8** Let $D$ be an effective divisor on $A$ and let $\bar{D}$ be its closure in an equivariant compactification $\bar{A}$ of $A$. Let $L_0 \in \text{Pic}(A_0)$ and let $E$ be an $A$-invariant divisor on $\bar{A}$ such that $L(\bar{D}) \cong L(E) \otimes \pi^* L_0$. Assume that $\text{St}(D)$ is contained in $T$. Then $L_0$ is ample on $A_0$. 

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Proof. By [NW04] Lemma 5.2 we obtain \( c_1(L_0) \geq 0 \). We may regard \( c_1(L_0) \) as a bilinear form on a vector space \( V \) which can be interpreted as the Lie algebra \( \text{Lie}(A_0) \) or the dual of cotangent bundle \( \Omega^1(A_0)^* \) over \( A_0 \). Assume that \( L_0 \) is not ample. Then there is a vector \( v \in V \setminus \{0\} \) such that \( c_1(L_0)|_C v \equiv 0 \). Choose a direct sum decomposition (orthogonal with respect to \( c_1(L_0) \)) \( V = Cv \oplus V' \) and let \( \omega \) be a \((1,1)\)-form which is positive on \( V' \), but annihilates \( Cv \). Then \( c_1(L) \wedge \omega^{g-1} = 0 \) where \( g = \dim A_0 = \dim V \). Let \( \Omega \) be a \((1,1)\)-form on \( \bar{A} \) which is positive along the fibers of \( \bar{A} \to A_0 \) as constructed in [NW03] Lemma 5.1. Then

\[
0 = \int_{\bar{A}} \Omega^s \wedge \pi^* (c_1(L_0) \wedge \omega^{g-1}) = \int_D \Omega^s \wedge \pi^* (\omega^{g-1})
\]

By construction of \( \omega \) this implies that \( v \) is everywhere tangent to \( D \). But in this case \( v \in \text{Lie}(A_0) \) is in the Lie algebra of the stabilizer \( \text{St}(D) \). This is a contradiction. Q.E.D.

**Proposition 3.9** Let \( \bar{A} \) be a smooth equivariant compactification of a semi-abelian variety \( A \). Let \( D \) be an effective divisor on \( A \) and let \( \bar{D} \) be its closure in \( \bar{A} \). Then the following properties hold.

(i) \( \bar{A} \setminus A \) is a divisor with only simple normal crossings.

(ii) If \( \text{St}(D) = \{0\} \), then \( \bar{D} \) is big on \( \bar{A} \).

Proof. (i) This is [NW04] Lemma 3.4.

(ii) Due to [NW04] there is a line bundle \( L_0 \) on \( A_0 \) and an \( A \)-invariant divisor \( E \) on \( \bar{A} \) such that \( L(\bar{D}) \cong L(E) \otimes \pi^* L_0 \). By Lemma 3.8 the triviality of \( \text{St}(D) \) implies the ampleness of \( L_0 \).

Now consider the \( T \)-action. Evidently \( E \) is \( T \)-invariant. Since \( T \) acts only along the fibers of \( \pi : \bar{A} \to A_0 \), the line bundle \( \pi^* L_0 \) is also \( T \)-invariant. It follows that for every \( g \in T \) the pull-back \( g^* D \) is linearly equivalent to \( D \).\(^1\) Next we define sets \( S_x \) for \( x \in A \) as follows:

\[
S_x = \cap_{g \in T; g(x) \in D} g^* D.
\]

By this definition we know that for every \( y \notin S_x \) there is a section \( \sigma \) in \( L(D) \) such that \( \sigma(x) = 0 \neq \sigma(y) \). From the definition it follows furthermore that \( S_x \) is an algebraic subvariety of \( A \). Using the \( A \)-invariant trivialization of the tangent bundle \( TA \cong A \times \)

\(^1\)Actually \( g^* D \sim D \) holds for every \( g \in T \) and every \( T \cong (\mathbb{C}^*)^n \)-action on a projective manifold. This can be deduced from the fact that the Picard variety of a projective manifold contains no rational curves.
Lie(A) we can identify $T_x(S_x)$ with a vector subspace of Lie(A). In this identification we obtain

$$T_x(S_x) = \cap_{g \in T : g(x) \in D} g^*D = \cap_{g \in T : g(x) \in D} T_g(x)D = \cap_{y \in \pi^{-1}(\pi(x)) \cap D} T_y(D).$$

Thus $T_x(S_x)$ depends only on $\pi(x)$. Let $F_x = \pi^{-1}(\pi(x))$. Then all the points in $F_x \cap S_x$ have the same tangent space. It follows that $F_x \cap S_x$ is an orbit under a Lie subgroup of $T$. On the other hand, $F_x \cap S_x$ is an algebraic subvariety. Therefore $F_x \cap S_x$ is an orbit under an algebraic subgroup of $T$. A priori this subgroup may depend on the point $x$. However, $T \cong (\mathbb{C}^*)^s$ contains only countably many algebraic subgroups. For this reason it follows that this algebraic subgroup must be the same for almost all points $x \in A$. Thus there is an algebraic subgroup $H \subset T$ such that each connected component of $S_x \cap F_x$ is a $H$-orbit for almost all $x \in A$. But this implies that $D$ is invariant under $H$. Since $\text{St}(D) = \{0\}$, $H$ is finite. Thus $S_x \to A_0$ is generically finite for almost all $x \in A$. Combined with the ampleness of $L_0$ this implies that $D$ is big. \textit{Q.E.D.}

\textbf{Proposition 3.10} Let $Z$ be a reduced subvariety of $A$ and let $\tilde{Z}$ be its closure in a smooth equivariant compactification $\tilde{A}$ of $A$. If $\text{St}(Z) = \{0\}$, then there is an equivariant blow-up $\tilde{A}^\dagger \to \tilde{A}$ such that the strict transform of $\tilde{Z}$ is generally positioned in $\tilde{A}^\dagger$.

In particular, there exists a smooth equivariant compactification of $A$ in which $Z$ is generally positioned.

\textit{Proof.} We can find an effective reduced divisor $D$ on $A$ such that $D \supset Z$ and $\text{St}(D) = \{0\}$. Thus it suffices to assume that $Z = D$, a divisor. Using a result of Vojta ([V99] Theorem 2.4 (2)) we obtain a (possibly singular) equivariant completion $i : A \hookrightarrow \tilde{A}$ such that $D$ is generally positioned in $\tilde{A}$. Consider the diagonal embedding $j : A \hookrightarrow \tilde{A} \times \tilde{A}$ given by $j = (i, i)$ and let $\tilde{A}'$ denote the closure of the image $j(A)$. Let $\tilde{A}^\dagger \to \tilde{A}'$ be an equivariant desingularization (cf. [Hi64], [BM97]). Then the composed map $\tilde{A}^\dagger \to \tilde{A}$ is a blow-up of $\tilde{A}$. Considering the natural projection $\tilde{A}^\dagger \to \tilde{A}$, we conclude, as in Lemma 3.10, that $D$ is generally positioned in $\tilde{A}^\dagger$. \textit{Q.E.D.}

\textbf{Proposition 3.11} Let $A$ be a semi-abelian variety, let $A \to \tilde{A}$ be an equivariant compactification and let $Z$ be a subvariety of $A$. Then there is an equivariant blow-up $\tilde{A} \to \tilde{A}$ such that the quotient $\tilde{A}/\text{St}(Z)$ exists.

\textit{Proof.} $\text{St}(Z)$ is an algebraic subgroup of $A$. Hence there is a quotient morphism $q : A \to A/\text{St}(Z)$. Let $A/\text{St}(Z) \hookrightarrow Z$ be an $A$-equivariant smooth compactification. Then $q$ is a morphism from an Zariski open subset of $\tilde{A}$ to $Z$ and thus defines a rational
map from $\bar{A}$ to $Z$. Now we just blow up $\bar{A}$ and $Z$ to remove the indeterminacies and obtain a regular morphism. Since $q: A \to A/\text{St}(Z)$ is equivariant, it is clear that the indeterminacies on $\bar{A}$ are $A$-invariant subvarieties. Therefore the blow-up can be done equivariantly. Q.E.D.

### 3.3 Finitely many orbits

We will need the following auxiliary result.

**Lemma 3.12** Let $A$ be a semi-abelian variety and $A \hookrightarrow \bar{A}$ a smooth equivariant algebraic compactification. Then there are only finitely many $A$-orbits in $\bar{A}$.

**Proof.** Let $\tau : \mathbb{C}^n \to A$ denote the universal covering. Then $A = \mathbb{C}^n/\Gamma$, where $\Gamma = \tau^{-1}\{0\}$. Note that $\Gamma$ generates $\mathbb{C}^n$ as complex vector space.

Let $H$ be an algebraic subgroup of $A$. Then $H$ is a semi-abelian variety, too. It follows that the connected component $\hat{H}$ of $\tau^{-1}(H)$ coincides with the complex vector subspace of $\mathbb{C}^n$ generated by $\hat{H} \cap \Gamma$. Evidently there are only countably many finitely generated subgroups of $\Gamma$. It follows that there are only countably many algebraic subgroups $H$ of $A$.

Let $p$ be a point in $\bar{A}$ and let $H = A_p$ be its isotropy group. Let $A_p$ denote the $A$-orbit through $p$. Let $\bar{A}^H$ denote the fixed point set of $H$-action, i.e., $\bar{A}^H = \{x \in \bar{A} : ax = x, \forall a \in H\}$. Then $\bar{A}^H$ is a closed algebraic subvariety of $\bar{A}$. Let $T_p(\bar{A}^H)$ be its Zariski tangent space at $p$. Because $H$ is reductive, the $H$-action on $T_p(\bar{A})$ is almost effective. On the other hand, because $H$ acts trivially on $\bar{A}^H$, the action on $T_p(\bar{A}^H)$ is likewise trivial. Therefore there is an almost effective $H$-action on the quotient vector space $T_p(\bar{A})/T_p(\bar{A}^H)$. Since $H$ is abelian, this implies $\dim H \leq \dim (T_p(\bar{A})/T_p(\bar{A}^H))$. From this we deduce

$$\dim(A_p) = \dim A - \dim H \geq \dim X - \dim (T_p(\bar{A})/T_p(\bar{A}^H)) = \dim T_p(\bar{A}^H)$$

Since $A_p \subset \bar{A}^H$, it follows that $\bar{A}^H$ is smooth at $p$ and $A_p$ is open in $\bar{A}^H$. In particular, there is an open neighborhood $W$ of $p$ in $\bar{A}$ such that $A_p$ is the only $A$-orbit in $W$ with $H$ as isotropy group. By virtue of algebraicity it follows that there are only finitely many $A$-orbits in $\bar{A}$ with $H$ as isotropy group.

Since there are only countably many algebraic subgroups of $A$, we obtain as a consequence that there are only countably many $A$-orbits in $\bar{A}$.

Thus $A$ is an algebraic group acting on an algebraic variety $\bar{A}$ with only countably many orbits. This implies that there are actually only finitely many orbits. Q.E.D.
3.4 Action

Let $A$ be a semi-abelian variety and let $\mathbb{P}^N(C)$ be the complex projective $N$-space. Then $A$ acts on the product $A \times \mathbb{P}^N(C)$ by the group action of the first factor:

$$(a, (b, x)) \in A \times (A \times \mathbb{P}^N(C)) \to a \cdot (b, x) = (a + b, x) \in A \times \mathbb{P}^N(C).$$

Let $p : A \times \mathbb{P}^N(C) \to A$ be the first projection. Let $X$ be an irreducible algebraic subset of $A \times \mathbb{P}^N(C)$ such that $p(X) = A$. We set

$$B = \text{St}(X) = \{a \in A; a \cdot X = X\}^0,$$

and assume that $\dim B > 0$. Set $C = A/B$.

Taking direct products with $\mathbb{P}^N(C)$, one extends the projection $A \to C$ to $\tau : A \times \mathbb{P}^N(C) \to C \times \mathbb{P}^N(C)$. This is a $B$-principal bundle. The subvariety $X$ of $A \times \mathbb{P}^N(C)$ is $B$-invariant; therefore $X = \tau^{-1}(\tau(X))$. It follows that $\tau(X)$ is a closed subvariety of $C \times \mathbb{P}^N(C)$ which we can regard as the quotient $X/B$ of $X$ with respect to the $B$-action. In particular $\pi = \tau|_X : X \to Y = \tau(X)$ is a $B$-principal bundle such that the $B$-action on $X$ is simply the principal right action of $B$ for this bundle structure.

Let $\hat{B}$ be a smooth equivariant compactification of $B$. Then we have a relative compactification $\hat{A} \to C$ of $A \to C$ arising as the $\hat{B}$-bundle associated to the $B$-principal bundle $A \to C$. In other words: $\hat{A} = A \times_B \hat{B}$ where $A \times_B \hat{B}$ denotes the quotient of $A \times \hat{B}$ with respect to the equivalence relation for which $(a, b) \sim (a', b')$ if and only if there exists an element $g \in B$ such that $ag = a'$ and $b = gb'$. The projection map $p$ extends to $\hat{p} : \hat{A} \times \mathbb{P}^N(C) \to \hat{A}$. Let $\hat{X}$ be the closure of $X$ in $\hat{A}$. Then $\hat{X} = X \times_B \hat{B}$. The compactness of $\hat{B}$ implies that the projection map $\hat{\pi} : \hat{X} \to Y$ is proper.

Let $E \subset X$ be an irreducible algebraic subset such that

$$(3.13) \quad B \cap \text{St}(E) = \{0\}.$$ 

**Proposition 3.14** Let $\hat{X}$, $X$, $E$, etc. be as above. Assume in addition that $E$ is of codimension one, i.e., a divisor. Then there is a $B$-equivariant blow-up

$$\psi : X^\dagger \to \hat{X}$$

with center in $\hat{X} \setminus X$ such that $X^\dagger$ has a stratification by $B$-invariant strata

$$X^\dagger = \bigcup \Gamma_\lambda$$

satisfying the following properties:
(i) \( \Gamma_x \cong X/B_x \) \( (x \in \Gamma_x) \) where \( B_x = \{ b \in B : b \cdot x = x \} \) is the isotropy group at \( x \).

(ii) The closure of \( E \) in \( X^\dagger \) contains none of the strata \( \Gamma_\lambda \).

(iii) The open subset \( X \) of \( X^\dagger \) coincides with one of the strata \( \Gamma_\lambda \).

Proof. Before starting the proof we make a remark: Since \( X \to Y \) is a \( B \)-principal bundle, we can define quotient varieties \( X/H \) for all algebraic subgroups \( H \) of \( B \). Therefore statement (i) of the proposition makes sense.

Now we start the proof. We will only consider blow-ups \( X^\dagger \to \hat{X} \) which arise in the following way: We take an equivariant blow-up \( \hat{B} \to \hat{B} \) and define \( X^\dagger = X \times_B \hat{B} \). We recall that there are only finitely many \( B \)-orbits in \( \hat{B} \) (Lemma 3.12) and that \( X \times_B \hat{B} \) is defined as a quotient of \( X \times B \). Let \( \{ \Omega_\lambda \}_\lambda \) be the family of \( B \)-orbits in \( \hat{B} \). Then a stratification \( \{ \Gamma_\lambda \}_\lambda \) of \( X^\dagger \) is induced as follows: For each \( \lambda \) we define \( \Gamma_\lambda \) is the image of \( X \times \Omega_\lambda \) under the projection \( X \times B \to X \times_B \hat{B} = X^\dagger \). Each of these \( B \)-orbits \( \Omega_\lambda \) can be written as quotient of \( B \) by some closed algebraic subgroup \( H_\lambda \):

\[ \Omega_\lambda \cong B/H_\lambda. \]

Then \( H_\lambda \) is the isotropy group of the \( B \)-action on \( \Gamma_\lambda \) at any point \( x \in \Gamma_\lambda \) and \( \Gamma_\lambda = X/H_\lambda \). Thus the stratification \( \{ \Gamma_\lambda \}_\lambda \) of \( X^\dagger \) has the properties required by (i), for every choice of an equivariant blow-up \( B^\dagger \to \hat{B} \).

By construction, the open subset \( X \) of \( X^\dagger \) coincides with the open \( B \)-orbit in \( B^\dagger \), hence (iii) follows.

Let us now verify that \( B^\dagger \to B \) can be chosen in such a way that property (ii) holds, too. For \( y \in Y \) let \( E_y \) be defined as \( E_y = \{ p \in E : \pi(p) = y \} \). We observe that \( \tilde{E}_y = \pi^{-1}(y) \cap \tilde{E} \) for almost all \( y \in \pi(E) \). Using [N81], Lemma 4.1., we infer from [3.13] that for a generic point \( y \in \pi(E) \) the fiber \( E_y \) has a discrete stabiliser with respect to the \( B \)-action on \( X \). Thus we may invoke Proposition 3.10 and deduce that there exists an equivariant blow-up \( B^\dagger \to \hat{B} \) such that \( E_y \) is generally positioned in \( B^\dagger \). Let \( X^\dagger \to \hat{X} \) be the associated blow-up of \( \hat{X} \). Now \( E_y \) being generally positioned in \( B^\dagger \) implies that the closure of \( E \) in \( X^\dagger \) contains none of the strata \( \Gamma_\lambda \). Q.E.D.

4 Second main theorem for jet lifts

Let \( A \) be a semi-abelian variety of dimension \( n \) and let \( T \) be the maximal affine subgroup of \( A \). Then \( T \cong (\mathbb{C}^*)^t \) and there is an exact sequence of rational homomorphisms

\[ 0 \to T \to A \to A_0 \to 0, \]
where $A_0$ is an abelian variety. Let $\bar{A}$ be a smooth equivariant compactification of $A$. Set $\partial A = \bar{A} \setminus A$ and let $J_k(\bar{A}, \log \partial A)$ be the logarithmic $k$-jet bundle along $\partial A$ (cf. [N86]). Then $A$ acts on $J_k(\bar{A}, \log \partial A)$ and there is an equivariant trivialization

$$J_k(\bar{A}, \log \partial A) \cong \bar{A} \times J_{k,A},$$

where $A$ acts trivially on the second factor $J_{k,A} = C^{kn}$. Let $\bar{J}_{k,A}$ be a projective compactification of $J_{k,A}$. With the trivial action of $A$ on $\bar{J}_{k,A}$ and the usual action on $A$ (by translations) and $\bar{A}$ this yields an $A$-equivariant compactification

$$\bar{J}_k(\bar{A}, \log \partial A) = \bar{A} \times \bar{J}_{k,A},$$

of $J_k(A)$ with an open $A$-invariant subset

$$\bar{J}_k(A) = A \times \bar{J}_{k,A}.$$

For example, we may set $\bar{J}_{k,A} = \mathbb{P}^{nk}(C)$ or $\bar{J}_{k,A} = (\mathbb{P}^n(C))^k$. Then $J_k(A) = J_k(\bar{A}, \log \partial A)|_A$ is a Zariski open subset of $\bar{J}_k(\bar{A}, \log \partial A)$ and

$$J_k(A) \cong A \times J_{k,A}.$$

We set

$$J_k^{reg}(\bar{A}, \log \partial A) = \{ j_k(g) \in J_k(\bar{A}, \log \partial A); \ j_1(g) \neq 0 \} \cong \bar{A} \times J_{k,A}^{reg},$$

$$J_k^{reg}(A) = J_k^{reg}(\bar{A}, \log \partial A)|_A \cong A \times J_{k,A}^{reg},$$

of which elements are called regular jets.

Let $f : C \to A$ be a holomorphic curve and $J_k(f) : C \to J_k(A)$ be the $k$-jet lift of $f$. We denote by $X_k(f)$ (resp. $\bar{X}_k(f)$) the Zariski closure of the image $J_k(f)(C)$ in $J_k(A)$ (resp. $\bar{J}_k(A)$):

(4.1) $X_k(f) \subset J_k(A), \quad \bar{X}_k(f) \subset \bar{J}_k(A).$

**Theorem 4.2** (Second Main Theorem) Let $f : C \to A$ be an algebraically non-degenerate holomorphic curve. Let $Z$ be a reduced subvariety of $X_k(f)$. Then there exists a natural number $l_0$ and a compactification $\bar{X}_k(f)$ of $X_k(f)$ such that for the closure $\bar{Z}$ of $Z$ in $\bar{X}_k(f)$

(4.3) $m_{J_k(f)}(r; \bar{Z}) = S_f(r),$

(4.4) $T(r; \omega_{Z,J_k(f)}) \leq N_{l_0}(r; J_k(f)^*Z) + S_f(r).$
In the case of $k = 0$ the compactification $\bar{A}$ of $A$ can be chosen smooth, equivariant, and independent of $f$; moreover, if $Z$ is a divisor $D$, (4.3) and (4.4) take the following forms, respectively:

\begin{align*}
(4.5) & \quad m_f(r; \bar{D}) = S_f(r; L(\bar{D})), \\
(4.6) & \quad T_f(r; L(\bar{D})) \leq N_{0_0}(r; f^*D) + S_f(r; L(\bar{D})).
\end{align*}

Proof. Since the very basic idea of the proof is the same as that of the Main Theorem of [NWY03], it will be helpful to confer it.

We extend the subvariety $Z$ to the closure in $\tilde{X}_k(f)$ which is denoted by the same $Z$.

We first prove (4.3) and (4.5). Set $B = \text{St}(X_k(f))$. Then we have the quotient maps:

\begin{align*}
q^B : A \to A/B &= C, \\
q^B_k : J_k(A) \to J_k(A)/B \cong C \times J_{k,A}, \\
\hat{q}^B_k : \hat{J}_k(A) \to C \times \hat{J}_{k,A}.
\end{align*}

By [N98] and [NW03] Lemma 2.3

\begin{equation}
(4.7) \quad \dim B > 0, \quad T_{q^B_0 J_k(f)}(r) = S_f(r).
\end{equation}

Setting $\bar{Y}_k = \bar{X}_k(f)/B$, we have a quotient map:

$$\bar{\pi}_k : \bar{X}_k(f) \to \bar{Y}_k \subset C \times \bar{J}_{k,A}.$$ 

Let $\bar{B}$ be a smooth equivariant compactification of $B$. Define $\hat{A}$, $\hat{X}_k(f)$, $\hat{Z}$, etc. as the partial compactifications of $A$, $\bar{X}_k(f)$, $Z$, etc. as in subsection 3.4. We then have proper maps,

$$\hat{q}^B_k : \hat{A} \times \hat{J}_{k,A} \to C \times \hat{J}_{k,A},$$

$$\hat{\pi}_k = \hat{q}^B_k|_{\hat{X}_k(f)} : \hat{X}_k(f) \to \bar{Y}_k \subset C \times \bar{J}_{k,A},$$

whose fibers are isomorphic to $\bar{B}$.

There are two cases, $B \subset \text{St}(Z)$ and $B \not\subset \text{St}(Z)$, which we consider separately.

(a) Suppose that $B \subset \text{St}(Z)$. Set $\hat{W} = \hat{\pi}_k(\hat{Z}) = \hat{Z}/B$. Then $\hat{W}$ has at least codimension one in $\tilde{Y}_k$. Let $T \cong (\mathbb{C}^*)^t$ be the maximal affine subgroup of $A$ and let $S$ be that of $B$. Then $S$ is a subgroup of $T$ and there is a splitting, $T \cong S \times S'$. Take an equivariant compactification $\bar{S}'$ of $S'$ and set $\bar{A} = \hat{A} \times_{S'} \bar{S}'$. 

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Then $\hat{A}$ is an equivariant compactification of $A$ and $\hat{A}$. We have an algebraic exact sequence
\[
0 \to S' \to C \to C_0 \to 0,
\]
where $C_0$ is an abelian variety, and an equivariant compactification $\hat{C} = C \times_{S'} \hat{S}'$. Thus $\hat{q}_k^B$ extends to
\[
\hat{q}_k^B : \hat{A} \times J_{k,A} \to \hat{C} \times J_{k,A},
\]
Let $\hat{X}_k(f)$ (resp. $\hat{Y}_k$, $\hat{W}$) be the closure of $X_k(f)$ (resp. $\check{Y}_k$, $\check{W}$) in $\hat{A} \times \check{J}_{k,A}$ (resp. $\hat{C} \times \check{J}_{k,A}$). Thus we have the restriction
\[
\bar{\pi}_k = \hat{q}_k^B|_{\hat{X}_k(f)} : \hat{X}_k(f) \to \check{Y}_k.
\]
Note that $\bar{\pi}_k$ is surjective and
\[
\check{W} \neq \check{Y}_k.
\]
It follows from Theorem 2.9 (ii) and (4.7) that
\[
m_{J_k(f)}(r; \bar{Z}) \leq m_{\hat{q}_k \circ J_k(f)}(r; \check{W}) + O(1)
= O(T_{\hat{q}_k \circ J_k(f)}(r)) = S_f(r).
\]
(b) Suppose that $B \not\subseteq \text{St}(Z)$. We set
\[
B' = B \cap \text{St}(Z), \quad Z' = Z/B', \quad \tilde{X}_k'(f) = \tilde{X}_k(f)/B', \quad A' = A/B', \quad B'' = B/B'.
\]
Moreover, we define $W$ as the image of $Z$ under the quotient $\tilde{X}_k'(f) \to \tilde{Y}_k'(f)/B'' = \check{Y}_k$. We have the following commutative diagram and quotient maps:

\[
\begin{array}{cccc}
Z & \subseteq & \tilde{X}_k(f) & \subset A \times \check{J}_{k,A} \\
\downarrow & & \downarrow & \downarrow \hat{q}_k^{B'} \\
Z' & \subseteq & \tilde{X}_k'(f) & \subset A' \times \check{J}_{k,A} \\
\downarrow \bar{\pi}_k|_{Z'} & & \downarrow \bar{\pi}_k' & \downarrow \hat{q}_k^{B''} \\
W & \subset & \check{Y}_k & \subset C \times \check{J}_{k,A}
\end{array}
\]

Note that
\[
\text{St}(X_k'(f)) = B'', \quad \text{St}(Z') \cap B'' = \{0\}.
\]
Let $\tilde{B}''$ be a smooth equivariant compactification of $B''$. We have

$$
\hat{A}' = A' \times_{B''} B'',
\hat{\partial}A' = \hat{A}' \setminus A',
\hat{X}'_k(f) = \hat{X}'_k(f) \times_{B''} B'',
\hat{Z}' = \hat{Z}' \text{ (the closure of } Z' \text{ in } \hat{X}'_k(f)),
\hat{\partial}X'_k(f) = \hat{X}'_k(f) \setminus \hat{X}'_k(f).
$$

Note that the boundary divisor $\hat{\partial}A'$ has only normal crossings (Proposition 3.9 (i)). We obtain proper maps

$$
\hat{Z}' \subset \hat{X}'_k(f) \subset \hat{A}' \times \hat{J}_{k,A}
$$

where $\hat{W} = \hat{\pi}'_k(\hat{Z}')$. By Proposition 3.14 we have a blow-up

$$
\psi : \hat{X}^{\dagger}_k(f) \to \hat{X}'_k(f)
$$

with center in $\hat{\partial}X'_k(f)$, the strict transform $\hat{Z}'^\dagger$ of $\hat{Z}'$ and the boundary

$$
\Gamma = \hat{X}^{\dagger}_k(f) \setminus \hat{X}'_k(f)
$$

with stratification $\Gamma = \cup \lambda \Gamma_\lambda$ such that

$$
\begin{align*}
(4.11) & \quad \Gamma_\lambda \cong \hat{X}'_k(f)/\text{Iso}_x(B'') \quad (x \in \Gamma_\lambda), \\
(4.12) & \quad \Gamma_\lambda \cap \hat{Z}'^{\dagger} \neq \Gamma_\lambda.
\end{align*}
$$

Here, if $k = 0$, we use Proposition 3.10 in place of Proposition 3.14 and deduce the stated property for $A$.

Let $\psi_{st} : J_l(\hat{X}^{\dagger}_k(f), \log \Gamma) \to J_l(\hat{X}'_k(f), \log \hat{\partial}X'_k(f))$ be the morphism naturally induced by $\psi$. We consider a sequence of morphisms

$$
J_l(\hat{Z}'^{\dagger}, \log \Gamma) \subset J_l(\hat{X}^{\dagger}_k(f), \log \Gamma) \xrightarrow{\psi_{st}} J_l(\hat{X}'_k(f), \log \hat{\partial}X'_k(f)) \xrightarrow{\text{proj.}} J_l(\hat{J}_{k,A'}) \times J_l(\hat{J}_{k,A}).
$$
Thus we have a morphism

$$\beta_l : J_l(\hat{X}_k^{\prime l}(f), \log \Gamma) \to J_l(J_k(A') \times J_l(J_k(A)).$$

Let \( p_l : J_l(\hat{X}_k^{\prime l}(f)) \to \hat{X}_k^{\prime l}(f) \) be the projection to the base space. Henceforth we obtain a proper morphism

$$\gamma_l = (\hat{\pi}_l^l \circ \psi \circ p_l) \times \beta_l : J_l(\hat{X}_k^{\prime l}(f), \log \Gamma) \to \hat{Y}_l \times J_l(J_k(A') \times J_l(J_k(A)).$$

We claim that for some \( l_0 \geq 1 \)

**Claim 4.13** \( \gamma_{l_0}(J_{l_0}(\hat{Z}')) \neq \gamma_{l_0}(J_{l_0}(\hat{X}_l^{\prime l}(f))). \)

Assume contrarily that \( \gamma_l(J_l(\hat{Z}')) = \gamma_l(J_l(\hat{X}_l^{\prime l}(f))) \) for all \( l \geq 1 \). Then for an arbitrary \( z \in \mathbb{C} \)

\[(4.14) \quad J_l(q_l^{B'} \circ J_k(f))(z) \in \gamma_l(J_l(\hat{Z}^{\prime l}, \log \Gamma)). \]

Fix \( z_0 \in \mathbb{C} \). Then \( \hat{\pi}_l \circ J_k(f)(z_0) \in \hat{Y}_l \) and we set

$$\xi_l = J_l(q_l^{B'} \circ J_k(f))(z_0) \in \gamma_l(J_l(\hat{Z}^{\prime l}, \log \Gamma)), \quad l \geq 1.$$

Set \( \Xi_l = \gamma_l^{-1}(\xi_l) \) for \( l \geq 0 \). Then the restriction \( p_l|_{\Xi_l} \) is proper and \( p_l|_{\Xi_l} : \Xi_l \to p_l(\Xi_l) \) is an isomorphism. We set

$$\Lambda_l = p_l(\Xi_l), \quad l = 1, 2, \ldots.$$

The sequence of \( \Lambda_l \supset \Lambda_{l+1}, l = 1, 2, \ldots \) terminates to \( \Lambda_\infty = \Lambda_{l_0} = \Lambda_{l_0+1} = \cdots \subset \hat{X}_k^{\prime l}(f) \) for some \( l_0 \). Then \( \Lambda_\infty \neq \emptyset \). If \( \Lambda_\infty \cap \hat{X}_l^{\prime l}(f) = \emptyset \), there is an element \( a \in A' \) such that

$$a \cdot (J_l(q_l^{B'} \circ J_k(f))(z_0)) \in J_l(\hat{Z}'), \quad \forall l \geq 0.$$

By the identity principle we deduce that \( a \cdot \hat{X}_k^{l}(f) \subset Z' \); this is absurd.

Now assume that \( \Lambda_\infty \cap \Gamma \neq \emptyset \). There is a point \( x_0 \in \Lambda_\infty \cap \Gamma \) such that

\[(x_0, \xi_l) \in J_l(\hat{Z}^{\prime l})_{x_0}, \quad l \geq 1. \]

Let \( \Gamma_{x_0} \) be the boundary stratum containing \( x_0 \). Let \( \alpha : \hat{X}_k^{l}(f) \to \hat{X}_k^{l}(f)/\text{Iso}_{x_0}(B'^l) \cong \Gamma_{x_0} \) be the quotient map. Then there exists an element \( a_0 \in A \) such that

$$a \cdot (\alpha \circ q_l^{B'} \circ J_k(f)(z)) \in \Gamma_{x_0} \cap \hat{Z}^{\prime l}$$

in a neighborhood of \( z_0 \) and hence for all \( z \in \mathbb{C} \). Henceforth a contradiction follows from this, \((4.12)\) and the image \( J_k(f)(\mathbb{C}) \) being Zariski dense in \( X_k(f) \).
This proves Claim 4.13.

We infer (4.4) and (4.6) as in the proof of the Main Theorem of [NWY02] p. 152 (cf. [NWY02] (5.12)) with a modification as follows. Let $\bar{X}_k(f) = \bigcup_{\alpha} U_{\alpha}$ be a finite affine covering, and let $\sigma_{\alpha\nu}$ be the defining functions of $Z \cap U_{\alpha}$. Then by Claim 4.13 there is a rational function $\eta$ on $\tilde{Y}_k \times J_{l_0}(J_{k,A') \times J_{l_0}(\bar{J}_{k,A}))$, regarded as a rational function on $J_{l_0}(X_k(f))$ such that

$$\eta \circ J_{l_0}(q_{1''} \circ J_k(f))(z) \neq 0, \quad z \in \mathbb{C},$$

$$\eta|_{U_{\alpha}} = \sum_{\nu} \sum_{0 \leq j \leq l_0} a_{\alpha\nu j} d^j \sigma_{\alpha\nu},$$

where the coefficients $a_{\alpha\nu j}$ are jet differentials on $U_{\alpha}$. Then we have

$$|\eta|_{U_{\alpha}} \leq \left( \sum_{\nu} |\sigma_{\alpha\nu}|^2 \right)^{1/2} \left( \sum_{\nu} \left( \sum_{0 \leq j \leq l_0} \left| a_{\alpha\nu j} \right| \left| d^j \sigma_{\alpha\nu} \right| \right)^2 \right)^{1/2},$$

$$\frac{1}{(\sum_{\nu} |\sigma_{\alpha\nu}|^2)^{1/2}} \leq \frac{1}{|\eta|_{U_{\alpha}}} \left( \sum_{\nu} \left( \sum_{0 \leq j \leq l_0} \left| a_{\alpha\nu j} \right| \left| d^j \sigma_{\alpha\nu} \right| \right)^2 \right)^{1/2}.$$ 

Therefore we deduce from (4.7) that

$$m_{J_k(f)}(r; \bar{Z}) = S_f(r),$$

$$N(r; J_k(f)^*Z) = N_{l_0}(r; J_k(f)^*Z) + S_f(r).$$

Combining these with the First Main Theorem 2.9 we obtain (4.3) and (4.4).

Let us now prove the additional statements for the case $k = 0$. In this case we take the quotient, $q : A \to A/\text{St}(Z)$ and we deal with the holomorphic curve $q \circ f : \mathbb{C} \to A/\text{St}(Z)$ and the subvariety $Z/\text{St}(Z)$. In this way it is reduced to the case when $\text{St}(Z) = \{0\}$. Then the compactification of $A$ due to Proposition 3.10 works for an arbitrary algebraically nondegenerate $f : \mathbb{C} \to A$.

If $Z$ is a divisor $D$ on $A$, then Proposition 3.9 (ii) implies that $D$ is big and we can deduce 4.5 with the help of Lemma 2.4. Q.E.D.

5 Higher codimensional subvarieties of $X_k(f)$

Let $f : \mathbb{C} \to A$ be a holomorphic curve in a semi-abelian variety $A$. We use the same notation, $X_k(f)$, $\text{St}(X_k(f))$, etc. as in the previous section.
The purpose of this section is to prove the following.

**Theorem 5.1** Let \( f : C \to A \) be a holomorphic curve and let \( Z \subset X_k(f) \) be a subvariety of \( \text{codim}_{X_k(f)}Z \geq 2 \). Then there is a compactification \( \bar{X}_k(f) \) such that for the closure \( \bar{Z} \) of \( Z \) in \( \bar{X}_k(f) \)

\[
T(r; \omega_{Z;k(f)}) \leq \epsilon T_f(r) ||r||, \quad \forall \epsilon > 0.
\]

In particular,

\[
(5.2)
N(r; J_k(f)^* Z) \leq \epsilon T_f(r) ||r||, \quad \forall \epsilon > 0.
\]

**Remark.** (i) For an abelian variety \( A \) this was proved by [Y04].

(ii) As a consequence, estimate (5.2) is independent of the choice of the compactification \( \bar{X}_k(f) \).

It suffices to prove Theorem 5.1 for irreducible \( Z \). Hence, we assume throughout this section that \( Z \) is irreducible.

Our proof naturally divides into three steps (a)∼(c). Before going to discuss the details, we give an outline of the proof.

(a) First, we reduce the case to the one that \( A \) admits a splitting \( A = B \times C \) where \( B \) and \( C \) are semi-abelian varieties such that

\[
(5.3)
B \subset \text{St}(X_l(f)) \quad \text{for all } l \geq 0
\]

and the composition of \( f \) and the second projection \( q^B : A \to A/B = C \) satisfies

\[
(5.4)
T_{q^B \circ f}(r) = S_f(r).
\]

By this reduction, we may assume that the variety \( X_l(f) \) has splitting \( X_l(f) = B \times (X_l(f)/B) \) for all \( l \geq 0 \).

We also make a reduction such that the image of \( Z \) under the second projection \( \pi_k : X_k(f) \to X_k(f)/B \) has a Zariski dense image. Hence by the assumption \( \text{codim}_{X_k(f)}Z \geq 2 \), we may assume \( \text{codim}_{\pi_k^{-1}(x)}Z \cap \pi_k^{-1}(x) \geq 2 \) for general \( x \in X_k(f)/B \).

(b) The second step is the main part of the proof. Using the above reduction, we shall construct auxiliary divisors \( F_l \subset \bar{B} \times (X_{k+l}(f)/B) \) for all \( l \geq 0 \) with the following properties:

(i) \((l+1)N_1(r; J_k(f)^* Z) \leq N(r; J_{k+l}(f)^* F_l) + \epsilon T_f(r) ||r||, \forall \epsilon > 0:

(ii) \( T_{J_{k+l}(f)}(r; L(F_l)) \leq n(l)T_{\gamma \circ f}(r; D_B) + \epsilon T_f(r; D)||r||, \forall \epsilon > 0,

where \( \gamma : A \to B \) is the first projection, \( D \) is an ample line bundle over \( \bar{A} \), \( D_B \) is an ample line bundle over \( \bar{B} \) and \( n(l) \) is a positive integer such that \( \lim_{l \to \infty} n(l)/l = 0 \).
Finally, by (i) and (ii) above we have
\[ N_1(r; J_k(f)^*Z) \leq \frac{1}{l+1} N(r; J_{k+l}(f)^*F_i) + \frac{\epsilon}{l+1} T_f(r; D)||\epsilon \]
\[ \leq \frac{n(l)}{l+1} T_{\gamma f}(r; D_B) + \frac{\epsilon}{l+1} T_f(r; D)||\epsilon \]
for all \( \epsilon > 0 \) and all integer \( l \geq 0 \). Since \( n(l)/l \to 0 (l \to \infty) \), we have
\[ N_1(r; J_k(f)^*Z) \leq \epsilon (T_{\gamma f}(r; D_B) + T_f(r; D)||\epsilon), \quad \forall \epsilon > 0. \]

Since \( T_{\gamma f}(r; D_B) = O(T_f(r; D)) \), the proof is completed.

(a) Reduction. Let \( f : C \to A \) be as above. Let \( I_k : \hat{X}_k(f) \to A \times J_{k,A} \) be the jet projection. It follows from [N77] (or [NWY02] Lemma 3.8) that
\[ T_{I_k \circ J_k(f)}(r) = S_f(r). \]

We need the following.

**Lemma 5.6** Let the notation be as above. Let \( G = \cap_{l \geq 0} \text{St}(X_l(f)) \) and let \( q^G : A \to A/G \) be the quotient map. Then

\[ T_{q^G \circ f}(r) = O(T_{I_k \circ J_k(f)}(r)) = S_f(r). \]

**Proof.** This is essentially the same as [47] and follows from the jet projection method; cf. [NW03] Lemma 2.4, [NWY02] Lemma 3.8 and their proofs. Q.E.D.

**Lemma 5.7** Let \( B \subset A \) be a semi-abelian subvariety. Put \( B' = B \cap (\cap_{l \geq 0} \text{St}(X_l(f))) \). Let \( q^B : A \to A/B \) and \( q^{B'} : A \to A/B' \) be quotient mappings. Then we have

\[ T_{q^{B'} \circ f}(r) = O(T_{q^B \circ f}(r)) + S_f(r). \]

**Proof.** We write \( G = \cap_{l \geq 0} \text{St}(X_l(f)) \). Taking the natural embedding \( A/B' \to (A/B) \times (A/G) \), we see that

\[ T_{q^{B'} \circ f}(r) = O(T_{q^B \circ f}(r) + T_{q^{B'} \circ f}(r)). \]

Thus the claim follows from Lemma 5.6. Q.E.D.

**Lemma 5.8** Let \( A \) and \( A' \) be semi-abelian varieties with a surjective homomorphism \( p : A \to A' \). Let \( g : C \to A' \) be a holomorphic curve. Then we have a holomorphic curve \( \hat{g} : C \to A \) such that \( p \circ \hat{g} = g \) and

\[ T_{\hat{g}}(r) = O(T_g(r)). \]
Proof. Set $n = \dim A$ and $n' = \dim A'$. Let $\varpi: \tilde{A} \cong C^n \to A$ and $\tilde{A}' \cong C^n' \to A'$ be the universal covering. Then there is a surjective linear homomorphism $\tilde{\rho}: \tilde{A} \to \tilde{A}'$. Let $\tilde{g}: C \to \tilde{A}'$ be the lifting of $g$. Let $g(z) = \sum_{j=1}^{n'} g_j(z) e_j'$ with basis $\{e_j'\}$ of $\tilde{A}'$. Take a basis $\{e_j\}$ of $\tilde{A}$ such that $\tilde{\rho}(e_j) = e_j'$, $1 \leq j \leq n'$. Then we set $\tilde{g}(z) = \varpi(\sum_{j=1}^{n'} g_j(z) e_j)$. It immediately follows from the definition of order functions (see [NWY02] §3) that $\tilde{g}$ satisfies the requirement. Q.E.D.

Now we are going to reduce our proof to the case such that $A = B \times C$ and that $B$ and $C$ are semi-abelian subvarieties satisfying (5.3) and (5.4). Let $\mathcal{B}$ be the set of all semi-abelian subvarieties $B \subset A$ such that

$$T_{q^B \circ f}(r) = S_f(r).$$

Then since $\cap_{l \geq 0} \text{St}(X_l(f)) \in \mathcal{B}$, we have $\mathcal{B} \neq \emptyset$. Let $B \in \mathcal{B}$ be a minimal element of $\mathcal{B}$; i.e., if $B' \subset B$ and $B' \in \mathcal{B}$, then $B' = B$. If $B_i \in \mathcal{B}$, $i = 1, 2$, we deduce from Lemma 5.7 that $B_1 \cap B_2 \in \mathcal{B}$. Thus we get

$$B \subset \cap_{l \geq 0} \text{St}(X_l(f)).$$

Put $C = A/B$ and let $q^B: A \to C$ be the quotient map. By Lemma 5.8 we may take a holomorphic curve $g: C \to A$ such that $q^B \circ g = q^B \circ f$ and

$$T_g(r) = S_f(r).$$

(5.9)

We may assume that the Zariski closure of the image $g(C)$ is a semi-abelian subvariety $C' \subset A$ ([N77], [N81]). Define the semi-abelian variety $\tilde{A}$ by the following pull-back.

$$\begin{array}{ccc}
\tilde{A} & \xrightarrow{p_2} & A \\
\downarrow{p_1} & \downarrow{q^B} \\
C' & \xrightarrow{q^B|_{C'}} & C
\end{array}$$

Then $\tilde{A} = \{(c, a) \in C' \times A : q^B(c) = q^B(a)\}$. The inclusion map $i: C' \to A$ yields a map $\tau: C' \to \tilde{A}$ defined by $\tau(x) = (x, i(x))$. Note that this morphism $\tau$ is a section for $p_1: \tilde{A} \to C'$. Hence this bundle is trivial, i.e. $\tilde{A} \cong B \times C'$ and $\tilde{A}/B = C'$.

Put $\tilde{f} = g \times f: C \to \tilde{A}$. Then by (5.9) we have

$$T_{\tilde{f}}(r) = O(T_{\tilde{f}}(r)), \quad T_{\tilde{f}}(r) = O(T_f(r)), \quad T_{p_1 \circ f}(r) = S_{\tilde{f}}(r).$$

(5.10)

(5.11)

(5.12)

$$B' = B \cap \left(\cap_{l \geq 0} \text{St}(X_l(\tilde{f}))\right)$$

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and $p'_1 : \tilde{A} \to \tilde{A}/B'$ be the quotient map. By Lemma 5.7 and (5.11), we have
\[(5.13) \quad T_{p'_1 \circ \tilde{f}}(r) = S_f(r).\]

Put $q' : A \to A/B'$ be the quotient map. Then we have
\[(5.14) \quad T_{q' \circ f}(r) = O(T_{p'_1 \circ \tilde{f}}(r)).\]

Hence by (5.10), (5.13) and (5.14) we conclude $B' \in B$. Since $B$ is minimal in $B$, we get $B' = B$. By (5.12) we have $B \subset \cap_{l \geq 0} \text{St}(X_l(\tilde{f}))$. Let $p_{2,k} : X_k(\tilde{f}) \to X_k(f)$ be the morphism induced from $p_2 : \tilde{A} \to A$. Set
\[\tilde{Z} = p_{2,k}^{-1}(Z) \subset X_k(\tilde{f}).\]

Note that
\[N_1(r; J_k(f)^*Z) = N_1(r; J_k(\tilde{f})^*\tilde{Z})\]
and that (5.10) holds.

For the reduction we need $\text{codim}_{X_k(\tilde{f})} \tilde{Z} \geq 2$. By Lemma 5.7 we see that
\[B \subset \left( \cap_{l \geq 0} \text{St}(X_l(f)) \right) \cap \left( \cap_{l \geq 0} \text{St}(X_l(\tilde{f})) \right).\]

Thus $p_{2,l} : X_l(\tilde{f}) \to X_l(f)$ is $B$-equivariant, and induces a morphism
\[p_{2,l}^B : X_l(\tilde{f})/B \to X_l(f)/B.\]

Let $\pi_l : X_l(f) \to X_l(f)/B$ be the quotient map. Then it follows from (5.1) and (5.3) that
\[(5.15) \quad T_{\pi_l \circ J_l(f)}(r) = S_f(r).\]

If the image $\pi_k(Z)$ is not Zariski dense in $X_k(f)/B$, there is a Cartier divisor $H$ on $X_k(f)/B$ containing $\pi_k(Z)$. Then, making use of (5.15) and the natural embedding $X_k(f)/B \hookrightarrow (A/B) \times J_{k,A}$ we get
\[(5.16) \quad N_1(r; J_k(f)^*Z) \leq N(r; (\pi_k \circ J_k(f))^*H) = O(T_{\pi_k \circ J_k(f)}(r)) = S_f(r).\]

Therefore the proof of Theorem 5.1 is finished in this case.

We assume that $\pi_k(Z)$ is Zariski dense in $X_k(f)$, and has a relative dimension at most $\dim B - 2$. Therefore the relative dimension of $\tilde{Z} \to X_k(\tilde{f})/B$ is at most $\dim B - 2$, and hence $\text{codim}_{X_k(\tilde{f})} \tilde{Z} \geq 2$. 

[24]
Hence, by replacing $A$ by $\tilde{A}$, $C$ by $C'$, $f$ by $\tilde{f}$ and $Z$ by $p_2^{-1}(Z)$, we may reduce our problem to the desired situation (5.3) and (5.4). Therefore we assume the following in the sequel:

(i) Let $B \subset A$ be a semi-abelian subvariety satisfying

\[ B \subset \cap_{l \geq 0} \text{St}(X_l(f)), \quad (5.17) \]
\[ T_{qB \circ f}(r) = S_f(r), \quad (5.18) \]
\[ A \cong B \times (A/B), \quad (5.19) \]

where $q^B : A \to A/B$ is the quotient map.

(ii) $\pi_k(Z)$ is Zariski dense in $X_k(f)/B$.

(b) **Auxiliary divisor.** Let the notation and the assumption be as above. Set $C = A/B$. We have

\[ A \cong B \times C, \quad (5.20) \]

Then it naturally induces

\[ X_l(f) \cong B \times (X_l(f)/B) \quad (l \geq 0). \]

Let $\bar{B}$ be an equivariant compactification of $B$ and set $\hat{X}_l(f) = \bar{B} \times (X_l(f)/B)$. Let

\[ \hat{\gamma}_l : \hat{X}_l(f) \to \bar{B}, \]
\[ \hat{\pi}_l : \hat{X}_l(f) \to X_l(f)/B \]

be the natural projections.

We denote by $Z^{ns}$ the set of non-singular points of $Z$.

**Lemma 5.21** Let $L \to \bar{B}$ be an ample line bundle. Then there is a sequence of natural numbers $n(1), n(2), n(3), \ldots$ satisfying the following:

(i) $\lim_{l \to \infty} \frac{n(l)}{l} = 0$.

(ii) There exist effective Cartier divisors $F_l \subset \hat{X}_{k+l}(f)$ and line bundles $M_l$ on $X_{k+l}(f)/B$ such that $F_l$ is defined by a non-zero element of

\[ H^0(\hat{X}_{k+l}(f), \gamma_{k+l}^* L^{\otimes n(l)} \otimes (\hat{\pi}_{k+l})^* M_l) \]

and that for every point $a \in C$ with $J_k(f)(a) \in Z^{ns}$

\[ \text{ord}_a J_{k+l}(f)^* F_l \geq l + 1. \]
**Proof.** Let \( f_B : C \to B \) be the holomorphic curve defined by the composition of \( f \) and the first projection \( A \to B \). Let \( f_C : C \to C \) be the holomorphic curve defined by the composition of \( f \) and the second projection \( A \to C \). Then \( f_B \) and \( f_C \) have Zariski-dense images. Let \( l \geq 0 \) be an integer, let \( p_{k+l,k} : J_{k+l,A} \to J_{k,A} \) be the natural projection, and let

\[
T \subset J_{k+l}(A) \times C \times J_{k,A} \cong B \times C \times J_{k+l,A} \times C \times J_{k,A}
\]

be the Zariski closed subset defined by

\[
T = \{(b, c, v, c', v') \in B \times C \times J_{k+l,A} \times C \times J_{k,A}; b = 0, c = c', v' = p_{k+l,k}(v)\}.
\]

Let \( \lambda : B \times C \times J_{k+l,A} \times C \times J_{k,A} \to C \times J_{k,A} \) be the product of the second projection and the third projection. We recall the following from [Y04] Proposition 2.1.1.

**Lemma 5.22** There exists a closed subscheme \( T \subset J_{k+l}(A) \times C \times J_{k,A} \) with the following properties:

(i) \( \text{Supp} \, T = T \).

(ii) The restriction \( \lambda' = \lambda|_T : T \to C \times J_{k+l,A} \) is a finite morphism. Furthermore the restriction of the direct image sheaf \( \lambda'_*(\mathcal{O}_T) \) to \( C \times J_{k+l,A}^{\text{reg}} \) is a rank \( l + 1 \) locally free \( \mathcal{O}_{C \times J_{k+l,A}^{\text{reg}}} \)-module.

(iii) Let \( f : C \to A \) be a holomorphic curve such that \( f_B(a) = 0 \). Then

\[
\text{ord}_a J_{k+l}(f)^* T_{p_0 J_{k}(f)(a))} \geq l + 1.
\]

Let \( r_1 : Z^\dagger \to \tilde{Z} \) be a desingularization of \( \tilde{Z} \) such that \( r_1 \) gives an isomorphism over \( Z^{\text{ns}} \). Put \( Y_k = X_k(f)/B \). Consider the sequence of morphisms

\[
(Z^{\text{ns}} \xrightarrow{r_0} Z^\dagger \xrightarrow{r_1} \tilde{Z} \xrightarrow{r_2} \hat{X}_k(f) \xrightarrow{\pi_k} Y_k).
\]

Here \( r_0, r_1 \circ r_0 \) are open immersions and \( r_2 \) is a closed immersion. Put the composition of morphisms to be \( r = \pi_k \circ r_2 \circ r_1 : Z^\dagger \to Y_k \). Let \( Y_k^{\text{fl}} \) be a Zariski open subset of \( Y_k \) such that \( Y_k^{\text{fl}} \) is non-singular and the fibers of \( r : Z^\dagger \to Y_k \) over \( Y_k^{\text{fl}} \) are all of the same dimension \( \dim Z^\dagger - \dim Y_k \). Then the restriction of the family \( r : Z^\dagger \to Y_k \) to \( Y_k^{\text{fl}} \) is a flat family.

Consider the pull back of the sequence of morphisms \((5.23)\) by the natural projection \( B \times Y_k \to Y_k \):

\[
B \times Z^{\text{ns}} \xrightarrow{s_0} B \times Z^\dagger \xrightarrow{s_1} B \times \tilde{Z} \xrightarrow{s_2} B \times \hat{X}_k(f) \xrightarrow{s_3} B \times Y_k.
\]
Again put the composition of these morphisms to be 
$s = s_3 \circ s_2 \circ s_1 : B \times Z^\dagger \to B \times Y_k$.
Then $s$ maps as

$$
s : (a, z) \in B \times Z^\dagger \to (a, r(z)) \in B \times Y_k.
$$

Let $L$ be an ample line bundle on $\bar{B}$ and set

$$
(5.24) \quad \phi : (a, w) \in B \times \hat{X}_k(f) \to a + \gamma_k(w) \in \bar{B}.
$$

Let $L^\dagger_1$ be the line bundle on $B \times Z^\dagger$ which is the pull back of $L$ by the composition of morphisms

$$
B \times Z^\dagger \xrightarrow{s_2 \circ s_1} B \times \hat{X}_k(f) \xrightarrow{\phi} \bar{B}.
$$

Since the restriction of $s$ over $B \times Y_k^\dagger$ (i.e., $s|_{B \times Y_k^\dagger} : B \times (Z^\dagger|_{Y_k^\dagger}) \to B \times Y_k^\dagger$) is a flat family, the semi-continuity theorem [H77] p. 288 implies that there is a Zariski open subset $U_n \subset B \times Y_k^\dagger$ ($n > 0$) such that $H^0((B \times Z^\dagger)|_P, L^\dagger_{1,P})$ are all the same dimensional $C$-vector spaces for $P \in U_n$. Put this dimension as $G_n$. Here $(B \times Z^\dagger)|_P$ denotes the fiber of the morphism $s : B \times Z^\dagger \to B \times Y_k$ over $P \in B \times Y_k$, and $L^\dagger_{1,P}$ is the induced line bundle. Since the intersection $\cap_{n \geq 1} U_n$ is non-empty, put $(a, w) \in \cap_{n \geq 1} U_n$ and replacing $L$ by the pull back by the morphism

$$
B \ni x \mapsto x + a \in B
$$

we may assume $a = 0 \in B$.

Now for a positive integer $l > 0$, let $T^\dagger_l \subset A \times J_{k+l,A} \times C \times J_{k,A}$ be the closed subscheme, and let $\lambda : T^\dagger_l \to C \times J_{k+l,A}$ be the morphism obtained in Lemma 5.22. Then $\lambda$ has the following properties;

(i) $\lambda$ is finite,

(ii) the direct image sheaf $\lambda_* \mathcal{O}_{T^\dagger_l}$ is locally generated by $l + 1$ elements as $\mathcal{O}_{C \times J_{k+l,A}}$ module on $C \times J_{k+l,A}^{\text{reg}}$,

(iii) $\lambda$ induces an isomorphism of the underlying topological spaces of $T^\dagger_l$ and $C \times J_{k+l,A}$.

Since $Y_{k+l}$ is a Zariski closed subset of $C \times J_{k+l,A}$, we denote $\sigma_{k+l} : Y_{k+l} \to C$ for the composition with the first projection $C \times J_{k+l,A} \to C$ and denote $\eta_{k+l} : Y_{k+l} \to J_{k+l,A}$ for the composition with the second projection. We have the closed immersion

$$
(5.25) \quad B \times Y_{k+l} \times Y_k \subset B \times C \times J_{k+l,A} \times C \times J_{k,A} \cong A \times J_{k+l,A} \times C \times J_{k,A},
$$

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where the first inclusion is given by
\[ B \times Y_{k+l} \times Y_k \ni (b, v, v') \mapsto (b, \sigma_{k+l}(v), \eta_{k+l}(v), \sigma_k(v'), \eta_k(v')) \in B \times C \times J_{k+l,A} \times C \times J_{k,A} \]
and the second identification is given by
\[ B \times C \times J_{k+l,A} \times C \times J_{k,A} \ni (b, c, u, c', u') \mapsto ((b, c), u, c', u') \in A \times J_{k+l,A} \times C \times J_{k,A}. \]

Let \( S_t \subset B \times Y_{k+l} \times Y_k \) be the closed subscheme obtained by the pull-back of \( T_t^1 \) by (5.26).

Let \( q : S_t \to Y_{k+l} \) be the composition with the second projection \( B \times Y_{k+l} \times Y_k \to Y_{k+l} \).

We put
\[ Y^\text{reg}_{k+l} = Y_{k+l} \cap (C \times J^\text{reg}_{k+l,A}), \]
which is the Zariski open subset of \( Y_{k+l} \). Then by the above properties of \( \lambda \), we have the corresponding properties for \( q \):

(i) \( q \) is finite,

(ii) the direct image sheaf \( q_* \mathcal{O}_{S_t} \) is locally generated by \( l + 1 \) elements as \( \mathcal{O}_{Y_{k+l}} \)-module on \( Y^\text{reg}_{k+l} \),

(iii) \( q \) gives the isomorphism of underlying topological spaces of \( S_t \) and \( Y_{k+l} \).

We consider the following commutative diagram (5.26) obtained by the base change of (5.25) with a sequence of morphisms
\[ S_t \hookrightarrow B \times Y_{k+l} \times Y_k \twoheadrightarrow B \times Y_k \to Y_k. \]

Here \( B \times Y_{k+l} \times Y_k \to B \times Y_k \) is the natural projection:
\[ B \times Y_{k+l} \times Y_k \ni (a, w, w') \mapsto (a, w') \in B \times Y_k. \]

\[
\begin{array}{cccccc}
Z_{l}^{\text{ns}} & \longrightarrow & B \times Y_{k+l} \times Z^{\text{ns}} & \longrightarrow & B \times Z^{\text{ns}} & \longrightarrow & Z^{\text{ns}} \\
\downarrow u_0 & & \downarrow t_0 & & \downarrow s_0 & & \downarrow r_0 \\
Z_{l}^1 & \longrightarrow & B \times Y_{k+l} \times Z^1 & \longrightarrow & B \times Z^1 & \longrightarrow & Z^1 \\
\downarrow u_1 & & \downarrow t_1 & & \downarrow s_1 & & \downarrow r_1 \\
Z_{l} & \xrightarrow{v'} & B \times Y_{k+l} \times \bar{Z} & \longrightarrow & B \times \bar{Z} & \longrightarrow & \bar{Z} \\
\downarrow u_2 & & \downarrow t_2 & & \downarrow s_2 & & \downarrow r_2 \\
& \longrightarrow & B \times Y_{k+l} \times \hat{X}_k(f) & \longrightarrow & B \times \hat{X}_k(f) & \longrightarrow & \hat{X}_k(f) \\
\downarrow u_3 & & \downarrow t_3 & & \downarrow s_3 & & \downarrow \hat{z}_k \\
S_t & \xrightarrow{v} & B \times Y_{k+l} \times Y_k & \longrightarrow & B \times Y_k & \longrightarrow & Y_k
\end{array}
\]
Let $\mathcal{L}_l^\dagger$ be the line bundle on $Z_l^\dagger$ obtained by the pull back of $L_1^\dagger$ by the morphisms in the above diagram (5.26). Let $S_{l,n}$ be the non-empty Zariski open subset of $S_l$ obtained by the inverse image of $U_n$. Since $\dim H^0((B \times Z_l^\dagger)|_P, L_1^\dagger) = G_n$ for $P \in U_n$, the direct image sheaf $s_*L_1^\dagger$ is a locally free sheaf of rank $G_n$ on $U_n$ and the natural map

$$s_*L_1^\dagger \otimes \mathcal{C}(P) \to H^0((B \times Z_1)|_P, L_1^\dagger)$$

is an isomorphism for $P \in U_n$. This follows by the Theorem of Grauert [H77] p.288, since $U_n$ is reduced and irreducible. Here $s : B \times Z \to B \times Y_k$ is the natural map; i.e., $s = s_3 \circ s_2 \circ s_1$. Let $u$ be the morphism $u : Z_l^\dagger \to S_l$ obtained by the composition $u = u_3 \circ u_2 \circ u_1$, where $u_1,u_2,u_3$ are the morphisms in the above diagram (5.26). Then the natural map

$$u_*L_1^\dagger \otimes \mathcal{C}(P) \to H^0(Z_l^\dagger|_P, L_1^\dagger)$$

is also surjective, so an isomorphism on $P \in S_{l,n}$. This follows by the Theorem of Cohomology and Base Change [H77] p. 290. Hence $u_*L_1^\dagger$ is locally generated by $G_n$ elements as an $\mathcal{O}_{S_l}$-module on $S_{l,n} \subset S_l$. Let $Y_{k+l,n} = q(S_{l,n})$ be a non-empty Zariski open subset of $Y_{k+1}$ (note that the under lying topological spaces of $S_l$ and $Y_{k+1}$ are the same). Then by the above properties of $q$, the direct image sheaf $(q \circ u)_*L_1^\dagger$ is locally generated by $(l+1)G_n$ elements as an $\mathcal{O}_{Y_{k+1}}$-module on $Y_{k+l,n} \cap Y_{k+1}^{\mathrm{reg}}$. Here, note that $Y_{k+l}^{\mathrm{reg}}$ is non-empty (otherwise $f$ must be constant) and $Y_{k+1}$ is irreducible. Hence $Y_{k+l,n} \cap Y_{k+1}^{\mathrm{reg}}$ is also non-empty.

Now look at the following commutative diagram

$$\begin{array}{ccc}
Z_l^\dagger & \xrightarrow{t_2 \circ u \circ u_1} & B \times Y_{k+l} \times \hat{X}_k(f) \\
\downarrow q \circ u & & \downarrow \psi & \downarrow \rho \\
Y_{k+l} & \xrightarrow{2 \mathrm{nd \ proj}} & Y_{k+l} & \xrightarrow{\tau} Y_{k+l}
\end{array}$$

where $\rho$ is the first projection, $\tau$ is the second projection and $\psi$ is the morphism

$$\psi : B \times Y_{k+l} \times \hat{X}_k(f) \ni (a,v,w) \mapsto (a + \gamma_k(w),v) \in \tilde{B} \times Y_{k+l}.$$ 

Since $(\rho \circ \psi \circ t_2 \circ u' \circ u_1)^*L = L_1^\dagger$, we have a natural morphism

$$\tau_*\rho^*L^{\otimes n} = H^0(\tilde{B}, L^{\otimes n}) \otimes \mathcal{O}_{Y_{k+l}} \to (q \circ u)_*L_1^\dagger \otimes n.$$

Here, note that $\rho \circ \psi = \phi \circ \beta$ where $\beta : B \times Y_{k+l} \times \hat{X}_k(f) \to B \times \hat{X}_k(f)$ is the morphism in the diagram (5.26) and $\phi$ was defined by (5.24).
Now put \( I_n = \dim_{\mathbb{C}} H^0(\bar{B}, L_{\mathbb{C}}^n) \). Then there is a positive integer \( n_0 \) and positive constants \( C_1, C_2 \) such that

\[
I_n > C_1 n^{\dim \bar{B}}, \quad G_n < C_2 n^{\dim \bar{B} - 2} \quad \text{for} \ n > n_0.
\]

Here note that \( G_n = \dim_{\mathbb{C}} H^0(\bar{B} \times Z^t|_P, L_{I_{1/P}}^{1+n}) \) for \( P \in \bigcap_{n \geq 1} U_n \), and \( \bar{B} \times Z^t|_P = s^{-1}(P) \) has dimension \( \leq \dim \bar{B} - 2 \), for \( \text{codim}_{\tilde{X}_k(f)} Z \geq 2 \) and \( \pi_k \circ r_2 : Z \to Y_k \) is dominant. Hence for a positive integer \( l \), we can take a positive integer \( n(l) \) (e.g. \( \sim l^{3/4} \)) such that

\[
I_{n(l)} > (l + 1)G_{n(l)}; \quad \lim_{l \to \infty} \frac{n(l)}{l} = 0.
\]

Let \( \mathcal{F} \) be the kernel of \((5.27)\) for \( n = n(l) \);

\[
0 \to \mathcal{F} \to \tau_* \rho^* L_{\mathbb{C}}^{\otimes n(l)} \to (q \circ u)_* L_{l}^{1+n(l)} \quad \text{(exact)}.
\]

Then we have \( \mathcal{F} \neq 0 \). By taking the tensor of a sufficiently ample line bundle \( M_l \) on \( Y_{k+l} \) with \( \mathcal{F} \), we may assume that \( H^0(Y_{k+l}, \mathcal{F} \otimes M_l) \neq 0 \). Since we have

\[
H^0(Y_{k+l}, \mathcal{F} \otimes M_l) \subset H^0(Y_{k+l}, (\tau_* \rho^* L_{\mathbb{C}}^{\otimes n(l)}) \otimes M_l)
\]

\[
= H^0(Y_{k+l}, (\tau_* (\rho^* L_{\mathbb{C}}^{\otimes n(l)} \otimes \tau^* M_l))
\]

\[
= H^0(\bar{B} \times Y_{k+l}, \rho^* L_{\mathbb{C}}^{\otimes n(l)} \otimes \tau^* M_l),
\]

we may take a divisor \( F_l \subset \bar{B} \times Y_{k+l} \) which is defined by a non-zero global section of \( H^0(Y_{k+l}, \mathcal{F} \otimes M_l) \). Then we have

\[
\mathcal{Z}_l^{\text{ns}} \subset \psi^* F_l.
\]

Here note that \( \mathcal{Z}_l^{\text{ns}} \subset \mathcal{Z}_l \) is an open immersion and \( \mathcal{Z}_l^{1+2n'} \hookrightarrow B \times Y_{k+l} \times \hat{X}_k(f) \) is a closed subscheme.

Using the decomposition \( A = B \times C \), we let \( f_B : C \to B \) be the holomorphic curve obtained by the composition of \( f \) and the first projection \( A \to B \), and let \( f_C : C \to C \) be the holomorphic curve obtained by the composition of \( f \) and the second projection \( A \to C \). Now let \( a \in C \) be a point such that \( J_k(f)(a) \in \mathcal{Z}_{l}^{\text{ns}} \). Put \( \tilde{f} : C \to B \times Y_{k+l} \) \( \times \hat{X}_k(f) \) as

\[
\tilde{f}(z) = (f_B(z) - f_B(a), \tilde{\pi}_{k+l} \circ J_{k+l}(f)(z), J_k(f)(a)).
\]

Then we have

\[
\tilde{f}(C) \subset B \times Y_{k+l} \times Z, \quad \tilde{f}(a) \in \text{Supp} \mathcal{Z}_l^{\text{ns}}, \quad \psi \circ \tilde{f} = J_{k+l}(f),
\]

where the last equality holds under the identification \( \bar{B} \times Y_{k+l} = \hat{X}_{k+l}(f) \).
Since \( v' \) is the base change of \( v \) in (5.26) and \( \tilde{f} \) factors through \( t_2 \), we have

\[
\text{ord}_a \tilde{f}^* Z_l = \text{ord}_a (t_3 \circ \tilde{f})^* S_l,
\]

hence by the construction of \( S_l \) and Lemma 5.22 we have

\[
\text{ord}_a \tilde{f}^* Z_l = \text{ord}_a (J_{k+l}(f) - f(a))^* T_{l,(J_k(f)-f(a))(a)} \geq l + 1.
\]

Hence we have

\[
\text{ord}_a J_{k+l}(f)^* F_l = \text{ord}_a \tilde{f}^* \psi^* F_l \geq \text{ord}_a \tilde{f}^* Z_l^{ns} = \text{ord}_a \tilde{f}^* Z_l \geq l + 1.
\]

Here note that we consider \( F_l \) as the divisor on \( \hat{X}_{k+l}(f) \) by the identification \( B \times Y_{k+l} \cong \hat{X}_{k+l}(f) \), and \( \tau \) correspond to \( \pi_{k+l} \) by this identification. \( Q.E.D. \)

(c) The end of the proof. By (4.3) and (4.4) it suffices to show

\[
N_{l_0}(r; J_k(f)^* Z) \leq \epsilon T_f(r)||_{\epsilon}, \quad \forall \epsilon > 0.
\]

Note that

\[
N_{l_0}(r; J_k(f)^* Z) \leq l_0 N_1(r; J_k(f)^* Z).
\]

Furthermore, it suffices to prove

\[
N_1(r; J_k(f)^* Z^{ns}) \leq \epsilon T_f(r)||_{\epsilon}, \quad \forall \epsilon > 0.
\]

For we have

\[
N_1(r; J_k(f)^* Z) = N_1(r; J_k(f)^* Z^{ns}) + N_1(r; J_k(f)^* (Z \setminus Z^{ns}))
\]

and the second term of the right hand side is estimated to be at most “\( \epsilon T_f(r)||_{\epsilon} \)” by induction on dimension of \( Z \). Here note that \( \dim Z > \dim(Z \setminus Z^{ns}) \).

It follows from Lemma 5.21 and (5.15) that

\[
(l + 1)N_1(r; J_k(f)^* Z^{ns}) \leq N(r; J_{k+l}(f)^* F_l) \leq T_{J_{k+l}(f)}(r; L(F_l))
\]

\[
= n(l)T_{\gamma_{k+l} \circ J_{k+l}(f)}(r; L) + T_{\pi_{k+l} \circ J_{k+l}(f)}(r; M_l)
\]

\[
\leq n(l)T_{fB}(r; L) + S_f(r).
\]

Using \( \lim_{l \to \infty} n(l)/(l + 1) = 0 \) and \( T_{fB}(r; L) = O(T_f(r; D)) \), we obtain (5.28) and our Theorem 5.1.
6 Proof of Main Theorem

(a) Let the notation be as in the Main Theorem. The case of \( \text{codim}_{X_k(f)} Z \geq 2 \) was finished by Theorem 5.1. Therefore we assume in the rest of this section that \( Z \) is a reduced Weil divisor \( D \) on \( A \).

Set \( B = \text{St}(X_{k+1}(f)) \), which has a positive dimension (cf. (4.7)).

**Lemma 6.1** Assume that \( D \) is irreducible and \( B \not\subset \text{St}(D) \). Taking an embedding \( X_{k+1}(f) \hookrightarrow J_1(X_k(f)) \), we have

\[
\text{codim}_{X_{k+1}(f)}(X_{k+1}(f) \cap J_1(D)) \geq 2.
\]

**Proof.** Let \( k = 0 \). It is first noted that \( J_1(A) \) is the holomorphic tangent bundle \( T(A) \) over \( A \), and \( X_1(f) \subset T(A) \).

Assume that \( \text{codim}_{X_1(f)}(X_1(f) \cap J_1(D)) = 1 \). Let \( E \) be an irreducible component of codimension 1 of \( X_1(f) \cap J_1(D) \). Let \( \pi_1 : X_1(f) \to A \) be the natural projection. Then \( E \) is an irreducible component of \( X_1(f) \cap \pi_1^{-1}(D) \) and \( \pi_1(E) = D \).

Now \( \pi_1(E) = D \) combined with \( B \not\subset \text{St}(D) \) implies that \( B \) can not stabilize \( E \). Therefore \( B \cdot E \) (resp. \( B \cdot D \)) contains an open subset of \( X_1(f) \) (resp. \( A \)). In fact, since \( B \) and \( E \) are algebraic, \( B \cdot E \) contains a \( B \)-invariant Zariski open subset \( \Omega \) of \( X_1(f) \).

Let \( p = f(z_0) \in f(C) \) be a point with the properties:

(i) The orbit \( B \cdot p \) intersects \( D \setminus \text{Sing}(D) \) transversely in a point \( q \);

(ii) \( J_1(f)(z_0) \in \Omega \).

Then we choose an analytic 1-dimensional disk \( \Delta \subset B \) which contains the unit element \( e_B \) of \( B \) and we choose a non-empty open subset \( U \) of the non-singular part \( D^{\text{ns}} \) of \( D \) containing \( q \) such that

(i) the map \( \phi : \Delta \times U \hookrightarrow A \) given by \( \phi(b, u) = b \cdot u \) is an open embedding,

(ii) the subbundle \( \cup_{\zeta \in \Delta} T(\{\zeta\} \times U) \subset T(\Delta \times U) \) with \( T(\{\zeta\} \times U) \cong T(U) \) gives rise to a holomorphic foliation on \( \phi(\Delta \times U) \subset A \).

Consider \( \hat{f}(z) = b \cdot f(z + z_0) \) with \( b \in B \) such that \( b \cdot p = q \) and \( p = f(z_0) \). Note that \( \hat{f}(0) = b \cdot p = q \) and \( J_1(\hat{f})(0) = b \cdot J_1(f)(z_0) \in \Omega \). Then there is an open neighbourhood \( W \) of 0 in \( C \) such that \( J_1(\hat{f})(z) \in \Omega \) for all \( z \in W \). Since \( \Omega \subset B \cdot E \subset B \cdot J_1(D) \), it follows that \( \hat{f}'(z) \) is tangent to the leaves of the above defined foliation for all \( z \in W \). By the identity principle this implies \( \hat{f}(C) = b \cdot f(C) \subset D \) which is absurd, since \( f \) is algebraically non-degenerate.
The proof for \( k \geq 1 \) is similar to the above. \textit{Q.E.D.}

(b) \textit{Proof of the Main Theorem.} Let \( D = \sum_i D_i \) be the irreducible decomposition. By making use of Theorem 4.2 we have

\[
T(r; \omega_{D,k(f)}) \leq N_k(r; J_{k(f)}D) + S_f(r)
\leq N_1(r; J_{k(f)}D) + k_0 \sum_{i<j} N_1(r; J_{k(f)}(D_i \cap D_j))
+ k_0 \sum_i N_1(r; J_{k+1}(f) \cdot J_1(D_i)) + S_f(r).
\]

Since \( \operatorname{codim}_{X_k(f)} D_i \cap D_j \geq 2 \) for \( i \neq j \), it follows from Theorem 5.1 that

\[
k_0 \sum_{i<j} N_1(r; J_{k(f)}(D_i \cap D_j)) \leq \epsilon T_f(r)\|_\epsilon, \quad \forall \epsilon > 0.
\]

Note that \( J_{k+1}(f) \cdot J_1(D_i) = J_{k+1}(f) \cdot (X_{k+1}(f) \cap J_1(D_i)) \). If \( B \subset \text{St}(D_i) \), then the image of \( D_i \) by \( X_k(f) \rightarrow X_k(f)/B \) is contained in a divisor on \( X_k(f)/B \). Then as in (4.9) we infer that

\[
N_1(r; J_{k+1}(f) \cdot J_1(D_i)) \leq N(r; J_{k(f)}D_i) \leq S_f(r).
\]

Suppose that \( B \not\subset \text{St}(D_i) \). It follows from Lemma 6.1 and Theorem 5.1 that

\[
N_1(r; J_{k+1}(f) \cdot J_1(D)) \leq N_1(r; J_{k+1}(f) \cdot (X_{k+1}(f) \cap J_1(D_i))) \leq \epsilon T_f(r)\|_\epsilon, \quad \forall \epsilon > 0.
\]

Combining these with (6.2), we obtain

\[
T(r; \omega_{D,k(f)}) \leq N_1(r; f^*D) + \epsilon CT_f(r)\|_\epsilon, \quad \forall \epsilon > 0,
\]

where \( C \) is a positive constant independent of \( \epsilon \). Now the proof of the Main Theorem is completed. \textit{Q.E.D.}

7 Applications

(a) In [G74] M. Green discussed the algebraic degeneracy of a holomorphic curve \( f : C \rightarrow \mathbb{P}^n(C) \) omitting an effective reduced divisor \( D \) on \( \mathbb{P}^n(C) \) with normal crossings and of degree \( \geq n + 2 \). He proved the following theorem and conjectured that it would hold without the condition of finite order for \( f \):

\textbf{Theorem 7.1} (M. Green [G74]) Let \( f : C \rightarrow \mathbb{P}^2(C) \) be a holomorphic curve of finite order and let \( [x_0, x_1, x_2] \) be the homogeneous coordinate system of \( \mathbb{P}^2(C) \). Assume that \( f \) omits two lines \( \{ x_i = 0 \}, i = 1, 2 \), and the conic \( \{ x_0^2 + x_1^2 + x_2^2 = 0 \} \). Then the image \( f(C) \) lies in a line or a conic.
Here we answer his conjecture in more general form:

**Theorem 7.2** Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve and let $[x_0, \ldots, x_n]$ be the homogeneous coordinate system of $\mathbb{P}^n(\mathbb{C})$. Assume that $f$ omits hyperplanes given by

(7.3) \hspace{1cm} x_i = 0, \quad 1 \leq i \leq n,

and a hypersurface defined by

\[ x_0^q + \cdots + x_n^q = 0, \quad q \geq 2. \]

Then $f$ is algebraically degenerate.

**Proof.** Let $f(z) = [f_0(z), \ldots, f_n(z)]$ be a reduced representation of $f$. Then $f_i(z)$ have no zero for $1 \leq i \leq n$. The assumption implies the existence of an entire function $h(z)$ such that

\[ f_0^q(z) + \cdots + f_n^q(z) = e^{h(z)}. \]

Write the above equation as

\[ (f_0(z)e^{-h(z)/q})^q + \cdots + (f_n(z)e^{-h(z)/q})^q = 1. \]

Changing the reduced representation of $f$, we may have that

(7.4) \hspace{1cm} f_1^q(z) + \cdots + f_n^q(z) - 1 = -f_0^q(z).

Now we take a holomorphic curve into a semi-abelian variety $A = (\mathbb{C}^*)^n$ with the natural coordinate system $(x_1, \ldots, x_n)$ defined by

\[ g : z \in \mathbb{C} \to (f_1(z), \ldots, f_n(z)) \in A. \]

Define a divisor $D$ on $A$ by

\[ x_1^q + \cdots + x_n^q - 1 = 0. \]

Let $\tilde{A}$ be a equivariant compactification in which $D$ is generally positioned. Let $\tilde{D}$ be the closure of $D$ in $\tilde{A}$. Note that $\text{St}(D) = \{0\}$ and that $\text{ord}_z g^*D \geq 2$ for all $z \in g^{-1}(D)$ by (7.3). Combining this with the Main Theorem ($k = 0$), we see that for arbitrary $\epsilon > 0$

\[
T_g(r; L(\tilde{D})) \leq N_1(r; g^*D) + \epsilon T_g(r; L(\tilde{D})) \|_{\epsilon} \\
\leq q N(r; g^*D) + \epsilon T_g(r; L(\tilde{D})) \|_{\epsilon} \\
\leq \frac{1 + q \epsilon}{q} T_g(r; L(\tilde{D})) \|_{\epsilon}.
\]
This leads to a contradiction for $\epsilon < (q - 1)/q$. \textit{Q.E.D.}

\textit{Remark.} The Zariski closure of the image $f(C)$ can be more specified in terms of $g$ defined in the above proof. It follows from [N98] that the Zariski closure of $g(C)$ is a translate $X$ of a proper semi-abelian subvariety of $A$ such that $X \cap D = \emptyset$.

(b) Let $A$ be a semi-abelian variety as above and let $X \subset J_k(A)$ be an irreducible algebraic subvariety. We consider the existence problem of an algebraically nondegenerate entire holomorphic curve $f : C \to A$ such that $J_k(f)(C) \subset X$ and $J_k(f)(C)$ is Zariski dense in $X$. This is a problem of a system of algebraic differential equations described by the equations defining the subvariety $X$.

The first necessary condition for the existence of such solution $f$ is that $\text{St}(X) \neq \{0\}$ (cf. (4.7)). Now we assume the existence of such $f$. Then we take a big line bundle $L \to X$ and a section $\sigma \in H^0(X, L)$ which defines a reduced divisor on $X$. We arbitrarily fix a trivialization

$$(7.5) \quad J_k(f)^*L \cong C \times C,$$

and regard $J_k(f)^*\sigma$ as an entire function.

\textbf{Theorem 7.6} Let the notation be as above. Then there is no entire function $\psi(z)$ such that every zero of $\psi(z)$ has degree $\geq 2$ and

$$(7.7) \quad J_k(f)^*\sigma(z) = \psi(z), \quad z \in \mathbb{C}.$$

In particular, there is no entire function $\psi(z)$ satisfying

$$(7.8) \quad J_k(f)^*\sigma(z) = (\psi(z))^q, \quad z \in \mathbb{C},$$

where $q \geq 2$ is an integer.

\textit{Remark.} The property given by (7.7) or (7.8) is independent of the choice of the trivialization (7.5).

\textit{Proof.} Suppose that there is an entire function $\psi(z)$ satisfying (7.7) or (7.8). Then it follows that

$$N_1(r; J_k(f)^*D) \leq \frac{1}{2} N(r; J_k(f)^*D).$$

Combining this with the Main Theorem, we infer the following contradiction:

$$T_{J_k(f)}(r; L) \leq \frac{1}{2} T_{J_k(f)}(r; L) + \epsilon T_{J_k(f)}(r; L) ||_{\epsilon}.$$

\textit{Q.E.D.}
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