MALLIAVIN CALCULUS FOR DEGENERATE DIFFUSIONS

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Abstract: Let \((W, H, \mu)\) be the classical Wiener space on \(\mathbb{R}^d\). Assume that \(X = (X_t(x))\) is a diffusion process satisfying the stochastic differential equation with diffusion and drift coefficients \(\sigma : \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^d, b : \mathbb{R}^n \to \mathbb{R}^n\), \(B\) is an \(\mathbb{R}^d\)-valued Brownian motion. We suppose that \(b\) and \(\sigma\) are Lipschitz. Let \(P(x)\) be the orthogonal projection from \(\mathbb{R}^d\) to its closed subspace \(\sigma(x)^*(\mathbb{R}^n)\), assuming that \(x \to P(x)\) is continuously differentiable, we construct a covariant derivative \(\hat{\nabla}\) on the paths of the diffusion process, along the elements of the Cameron-Martin space and prove that this derivative is closable on \(L^p(\nu)\), where \(\nu\) represents the law of the above diffusion process, i.e., \(\nu = X(x)(\mu)\), the image of the Wiener measure under the function \(w \to X(w, x)\). We study the adjoint of this operator and we prove several results: representation theorem for \(L^2(\nu)\)-functionals, the logarithmic Sobolev inequality for \(\nu\). As applications of these results the proof of the Logarithmic Sobolev inequality on the path space of Dyson’s Brownian motion is given by using the covariant derivative. We then explain how to use this theory for deriving the functional inequalities for the measures defined by the semigroups of the diffusion process at the time \(t = 1\) and with fixed starting point.

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In a preceding work we have calculated a martingale representation theorem for the functionals of degenerate Itô process which is a unique weak solution of a stochastic differential equation with path depending coefficients under the hypothesis of the weak uniqueness, following the path breaking idea of C. Dellacherie, cf. \[9\]. In the calculations an important concept shows up, namely, the behaviour of the random projection operator which is defined as the orthogonal projection from \(\mathbb{R}^d\) to the closed subspace \(\sigma^*(t, X)(\mathbb{R}^n)\), where \(\sigma\) is the diffusion coefficient, this projection operator will be denoted as \(P_s(X)\) if the diffusion has path dependent coefficients (non-Markov case) and by \(P(X_s)\) in case the coefficients are Markov (i.e., the closed loop case, or the case where \(b(t, X(w)) = b(X_t(w))\) and \(\sigma(t, X(w)) = \sigma(X_t(w))\)). In particular in the Markov process case we consider only the homogeneous situation. This operator gives rise to the Itô representation theorem of the martingales adapted to the filtration of the diffusion process. In fact we have shown that, for any functional \(F(X)\) defined on the paths of the diffusion process \(X\), which is square integrable w.r.t. to its law, there exists an \(\mathbb{R}^d\)-valued process adapted to the filtration of \(X\), say \(\alpha(X)\), such that

\[
F(X) = E[F(X)] + \int_0^1 P_s(X)\alpha_s(X) \cdot dB_s, \\
\]

\(\mu\)-a.s. Although \(\alpha\) is not unique, the process \((P(X_s)\alpha_s(X), s \in [0, 1])\) is unique \(ds \times d\mu\)-a.s. In the case of strong solutions, we have also chaos representation à la N. Wiener (cf.\[12\]) for the functionals of \(X\) using the basic \(\mathcal{F}(X)\)-martingale \(m\) defined by

\[
m_t = \int_0^t P_s(X)dB_s.
\]

To calculate the integrands in this chaos development, as well as the process \(P(X)\alpha(X)\) above, we need to create some kind of differential calculus on the paths of the diffusion process \(X\). The first kind of derivative which comes to one’s mind is some kind of extension of the \(H\)-derivative of Leonard Gross (cf.\[9,13\]), where \(H\) denotes the Cameron-Martin space associated to the \(\mathbb{R}^d\)-valued Brownian motion which is at the origin of \(X\), as it has been done in the case of flat Malliavin Calculus, cf. \[16\]. However this approach fails since \(H\)-derivatives of diffusion’s functionals, even the simplest ones, fail to remain measurable with respect to the sigma algebra generated by the diffusion process, hence we can not express the representation results w.r.t. the probability governing the process under investigation. To circumvent this difficulty we propose a new derivation operator which consists of smoothing the \(H\)-derivative w.r.t. the final instant sigma algebra, namely \(\mathcal{F}_t(X) = \sigma(X_t, t \in [0, 1])\), of the diffusion process. This operator is denoted by \(\hat{\nabla}\) with a hat at the top to indicate the role of the conditional expectation in its definition. It is then easy to show that \(\hat{\nabla}\), restricted to cylindrical diffusion functionals \(S(X) = \{f(X_{t_1}, \ldots, X_{t_n}) : t_1 \leq t_2 \leq \ldots \leq t_n \leq 1, f \in \mathcal{S}(\mathbb{R}^n), n \geq 1\}\) is a derivation (under almost sure equality) and it is a closable operator in \(L^p(\nu)\) for \(p > 1\), where \(\nu\) denotes the law of \(X\). We construct a path space functional analysis based on this operator, its adjoint ( taken w.r.t. the law of \(X)\) and the composition of it with the former, which is a kind of Ornstein-Uhlenbeck operator with non-linear interactions. Let us note that this is the first work where one has a nonlinear frame (in the sense of Boson-Fermion situation) having a reasonable Wiener chaos representation. We then show several relations between the adjoint of this derivative operator and the martingale representation theorem; in particular we calculate explicitly the integrand for this representation theorem which extends the flat case, which is called in the flat
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Let us note nevertheless that in the case of degenerate diffusions the situation is quite different, for instance the derivative constructed above is not even torsion free. (As an immediate application of the theory in finance cf. [1].) We prove here the Poincaré inequality and the logarithmic Sobolev inequality (associated to the derivative \(\hat{\nabla}\)) with respect to \(\nu\), which is the law of \(X\). We also prove these equalities for the path measure of a very singular process, namely the Dyson Brownian motion. We mean by degeneracy here the fact that the probability law on the path space of this process is singular with respect to the Wiener measure. Afterwards we explain how to use all the theory developed in this document to derive the Poincaré and log-Sobolev inequalities for the semigroup measures of the diffusions for fixed instants. As a final application, we construct the “normal” process defined as the stochastic integral of the orthogonal of the projection operator \(P_s(X)\), namely \(I_{I_\mathbb{R}^d} - P_s(X)\) and prove that conditionally w.r.t. the final sigma algebra \(F_1(X)\) of the diffusion process \((X_t, t \in [0, 1])\), it is a non-homogeneous Gaussian process with independent increments. As an application of this fact, we prove a conditional integration by parts formula which implies a conditional Clark representation theorem and the associated conditional logarithmic Sobolev inequality for the paths of this normal process, which is new.

2. Preliminaries and Notations

\((W, H, \mu)\) represents the classical Wiener space, i.e., \(W = C([0, 1], \mathbb{R}^d)\), \(H = \{\int_0^1 \dot{h}(s) ds : \dot{h} \in L^2([0, 1], \mathbb{R}^d)\}\) is the Cameron-Martin space, with the norm defined by \(|h|_H = \|\dot{h}\|_{L^2([0, 1], \mathbb{R}^d)}\) and \(\mu\) is the Gauss measure on \(W\), defined, for any \(\alpha \in W^*\) (dual of \(W\) under the topology of uniform convergence),

\[
\int_W \exp\left(\sqrt{-1} \langle \alpha, w \rangle\right) \mu(dw) = \exp\left(-\frac{1}{2} |\tilde{\alpha}|_H^2\right),
\]

where \(\tilde{\alpha}\) denotes the image of \(\alpha\) under the injection \(W^* \to H\). We denote by \(\nabla F\) the Sobolev derivative of the Wiener function \(F\) defined as

\[
\nabla F = \sum_{i=1}^{\infty} \nabla e_i F,
\]

where \((e_i, i \geq 1)\) is an orthonormal basis in \(H\) as the \(L^p(\mu)\)-completion of the Gateaux derivative of cylindrical functions:

\[
\nabla h F(w) = \lim_{\lambda \to 0} \frac{F(w + \lambda h) - F(w)}{\lambda},
\]

for \(h \in H\). We denote by \(\mathbb{D}_{p,k}\) the space of Wiener functions \(F\) such that

\[
\|F\|_{p,k} = \sum_{i=0}^{k} \|\nabla^i F\|_{L^p(\mu, H^{\otimes i})} < \infty,
\]

\(\nabla^i\) denotes the \(i\)-th iterate of \(\nabla\) and \(\nabla^{\otimes i} F\) is regarded as an \(H^{\otimes i}\)-valued function. The operator \(\delta\) is the formal adjoint of \(\nabla\) w.r.t. the Wiener measure \(\mu\) and it coincides with the Itô integral on the vector fields \(\xi : W \to H\) such that \(s \to \xi(s)\) is adapted to the filtration of the Wiener space. We refer the reader to [18], [19] and the references there for further results about the linear case.

Let us recall now the representation theorem and its consequences, which have all been proven in [22]: Let \(X = (X_t, t \in [0, 1])\) be a weak solution of the following stochastic differential equation:

\[
\begin{equation}
\label{eq:2.1}
dX_t = b(t, X)dt + \sigma(t, X)dB_t, \quad X_0 = x,
\end{equation}
\]
where $B = (B_t, t ∈ [0, 1])$ is an $\mathbb{R}^d$-valued Brownian motion and $σ : [0, 1] × C([0, 1], \mathbb{R}^n) → L(\mathbb{R}^d, \mathbb{R}^n)$ and $b : [0, 1] × C([0, 1], \mathbb{R}^n) → \mathbb{R}^n$ are measurable maps, adapted to the natural filtration of $C([0, 1], \mathbb{R}^n)$ and of linear growth. Denote by $(\mathcal{F}_t(X), t ∈ [0, 1])$ the filtration of $X$ and let us denote by $K$ the set of $\mathbb{R}^n$-valued, $(\mathcal{F}_t(X), t ∈ [0, 1])$-adapted processes $α(X)$, s.t.

$$E \int_0^1 (a(s, X)α_s(X), α_s(X))ds < \infty,$$

where $a(s, w) = σ(s, w)σ^*(s, w)$, $s ∈ [0, 1]$, $w ∈ C([0, 1], \mathbb{R}^n)$.

**Theorem 1.** The set $Γ = \{N ∈ L^2(\mathcal{F}_1(X)) : N = E[N] + \int_0^1 (α_s(X), σ(s, X)dB_s), α ∈ K\}$ is dense in $L^2(\mathcal{F}_1(X))$.

Theorem 1 permits us to show the following extension of the martingale representation theorem to the functionals of degenerate diffusions:

**Theorem 2.** Denote by $P_s(X)$ a measurable version of the orthogonal projection from $\mathbb{R}^d$ onto $σ(s, X)^*(\mathbb{R}^n) ⊂ \mathbb{R}^d$ and let $F ∈ L^2(\mathcal{F}_1(X))$ be any random variable with zero expectation. Then there exists a process $ξ(X) ∈ L^2(dt × dP; \mathbb{R}^n)$, adapted to $(\mathcal{F}_t(X), t ∈ [0, 1])$, such that

$$F(X) = \int_0^1 (P_s(X)ξ_s(X), dB_s)_{\mathbb{R}^d} = \int_0^1 (ξ_s(X), P_s(X)dB_s)_{\mathbb{R}^n}$$

a.s.

Conversely, any stochastic integral of the form

$$\int_0^1 (P_s(X)ξ_s(X), dB_s)_{\mathbb{R}^d},$$

where $ξ(X)$ is an $(\mathcal{F}_t(X), t ∈ [0, 1])$-adapted, measurable process with $E \int_0^1 |P_s(X)ξ_s(X)|^2_{\mathbb{R}^d}ds < \infty$, gives rise to an $\mathcal{F}_1(X)$-measurable random variable.

**Remark:** Let $η$ be a process such that $η_h$ belongs to the orthogonal complement of $σ^*(\mathbb{R}^n)$ in $\mathbb{R}^d$ $ds × dP$-a.s. Then $η + ξ$ can also be used to represent $F(X)$. Hence $ξ(X)$ is not unique but $P(X)ξ(X)$ is always unique.

As a consequence of above theorems, we get

**Theorem 3.** Let $\hat{u} ∈ L^2(dt × dP; \mathbb{R}^d)$ be adapted to the Brownian filtration, then we have

$$E \left[ \int_0^1 (\hat{u}_s, dB_s)|\mathcal{F}_1(X) \right] = \int_0^1 (E[P_s(X)\hat{u}_s]|\mathcal{F}_s(X)], dB_s)$$

almost surely.

**Remark 1.** If $K$ is a process adapted to the Brownian filtration with values in $\mathbb{R}^m ⊗ \mathbb{R}^d$ which is in $L^2(dt × dμ, \mathbb{R}^m ⊗ \mathbb{R}^d)$, then we obtain from Theorem 3

$$E \left[ \int_0^1 K_sdB_s|\mathcal{F}_1(X) \right] = \int_0^1 E[K_s|\mathcal{F}_s(X)]P(X_s)dB_s$$

$μ$-almost surely.

**Corollary 1.** Let $h ∈ H^3([0, 1], \mathbb{R}^d)$ (i.e., the Cameron-Martin space), denote by $ρ(δh)$ the Wick exponential $\exp(\int_0^1 (\hat{h}_s, dB_s) - \frac{1}{2} \int_0^1 |\hat{h}_s|^2ds)$, then we have

$$E[ρ(δh)|\mathcal{F}_1(X)] = \exp \left( \int_0^1 (P_s(X)\hat{h}_s, dB_s) - \frac{1}{2} \int_0^1 |P_s(X)\hat{h}_s|^2ds \right).$$
Theorem 4. \( \bullet \) Assume that \( \mathcal{F}_t(X) \subset \mathcal{F}_t(B) \) for any \( t \in [0, 1] \), where \((\mathcal{F}_t(B), t \in [0, 1])\) represents the filtration of the Brownian motion. Define the martingale \( m = (m_t, t \in [0, 1]) \) as \( m_t = \int_0^t P_s(X)dB_s \), then the set
\[
K = \{ \rho(\delta_m(h)) : h \in H \}
\]
is total in \( L^2(\mathcal{F}_1(X)) \), where \( \rho(\delta_m(h)) = \exp \left( \int_0^1 \hat{h}_s dm_s - \frac{1}{2} \int_0^1 |P_s(X)\hat{h}_s|^2 ds \right) \). In particular, any element \( F \) of \( L^2(\mathcal{F}_1(X)) \) can be written in a unique way as the sum
\[
F = E[F] + \sum_{n=1}^{\infty} \int_{C_n} (f_n(s_1, \ldots, s_n), dm_{s_1} \otimes \ldots \otimes dm_{s_n})
\]
where \( C_n \) is the \( n \)-dimensional simplex in \([0, 1]^n\) and \( f_n \in L^2(C_n, ds^{\otimes n}) \otimes (\mathbb{R}^d)^{\otimes n} \).

\( \bullet \) More generally, without the hypothesis \( \mathcal{F}_t(X) \subset \mathcal{F}_t(B) \) for any \( t \in [0, 1] \) to assure the non-symmetric chaos representation (2.4). Without this hypothesis, although we have Theorem 4 and when we iterate it we have a similar representation, but, consisting of a finite number of terms. It is not possible to push this procedure up to infinity since we have no control at infinity.

Remark 3. Let us note that if there is no strong solution to the equation defining the process \( X \), the chaotic representation property may fail. For example, let \( U \) be a weak solution of
\[
dU_t = \alpha_t(U)dt + dB_t,
\]
with \( U_0 \) given. Assume that \( \mathbb{E}[\omega] \) has no strong solution, as it may happen in the famous example of Tsirelson (11), i.e., \( U \) is not measurable w.r.t. the sigma algebra generated by \( B \), then we have no chaotic representation property for the elements of \( L^2(\mathcal{F}_1(U)) \) in terms of the iterated stochastic integrals of deterministic functions on \( C_n, n \in \mathbb{N} \) w.r.t. \( B \); the contrary would imply the equality of \( \mathcal{F}_1(U) \) and of \( \mathcal{F}_1(B) \), which would contradict the non-existence of strong solutions.

3. Conditional and Covariant Sobolev derivatives for degenerate diffusions’ functionals

In this section, we shall work with the diffusion processes in the time homogeneous Markov case, i.e., \( b(t, w, x) = b(x) \) and \( \sigma(t, w, x) = \sigma(x) \), with Lipschitz hypothesis and of linear growth. To keep track of the nonlinearity due to the diffusion’s paths interactions, we need to develop a special derivation w.r.t. the Cameron-Martin space of the defining Brownian motion. The next result paves the road and it is of fundamental importance:

Theorem 5. For \( h \in H \), define \( \nabla_hX_t = E[\nabla_hX_t|\mathcal{F}_1(X)] \), then it satisfies the following relation:
\[
\nabla_hX_t = \int_0^t \partial\sigma(X_s)\nabla_hX_t P(X_s)dB_s + \int_0^t \partial b(X_s)\nabla_hX_s ds + \int_0^t \sigma(X_s)\hat{h}(s)ds
\]
\(\mu\)-almost surely. If \(\sigma\) and \(b\) are twice differentiable and bounded, and if the projection map is differentiable and of linear growth, for the iterated derivative \(\hat{\nabla}^2_h X_t\) we have the following expression:

\[
\hat{\nabla}^2_h X_t = \int_0^t \partial^2 \sigma(X_s) \hat{\nabla}_h X_s \otimes \hat{\nabla}_h X_s P(X_s) dB_s + \int_0^t \partial \sigma(X_s) \hat{\nabla}^2_h X_s P(X_s) dB_s \\
+ \int_0^t \partial \sigma(X_s) \hat{\nabla}_h X_s \partial P(X_s) \hat{\nabla}_h X_s P(X_s) dB_s + \int_0^t \partial \sigma(X_s) \hat{\nabla}_h X_s (P(X_s) + I_{\mathbb{R}^d}) \hat{h}(s) ds \\
+ \int_0^t \partial^2 b(X_s) (\hat{\nabla}_h X_s \otimes \hat{\nabla}_h X_s + \hat{\nabla}^2_h X_s) ds
\]

\(\mu\)-almost surely.

**Proof:** We have

\[(3.7) \quad \nabla_h X_t = \int_0^t \partial \sigma(X_s) \nabla_h X_s dB_s + \int_0^t \partial b(X_s) \nabla_h X_s ds + \int_0^t \sigma(X_s) \hat{h}(s) ds,
\]

taking the conditional expectation of both sides of this relation, the term with stochastic integral follows from Theorem 3 and the Lebesgue integral part follows from the fact that

\[E[\nabla_h X_t | \mathcal{F}_1(X)] = E[\nabla_h X_t | \mathcal{F}_1(X)]\]

due to the fact that \(B_t = \sigma(B_s, s \leq t)\) is independent of the future increments of the Brownian motion (after \(t\)). For the second derivative we just iterate the calculation of the first derivative.

\[\square\]

**Remark:** It is clear that \(\hat{\nabla}^2_h X_t \neq E[\nabla^2_h X_t | \mathcal{F}_1(X)]\).

**Corollary 2.** Let \(S(X)\) be the set of functions on \(W\) defined as

\[S(X) = \{ f(X_t, \ldots, X_{t_m}) : 0 \leq t_1 < \ldots < t_m, f \in S(\mathbb{R}^{nm}), m \geq 1 \},\]

where \(S(\mathbb{R}^m)\) denotes the space of rapidly decreasing smooth functions of Laurent Schwartz on \(\mathbb{R}^m\). Let \(h \in H\), assume that \((F_k(X), k \geq 1) \subset S(X)\) converges to zero in \(L^2(\mathcal{F}(X))\) and that \((\nabla_h F_k(X), k \geq 1)\) is Cauchy in \(L^2(\mathcal{F}(X))\), then

\[\lim_{k \to \infty} \nabla_h F_k(X) = 0\]

\(\mu\)-a.s. In other words \(\nabla_h\) is a closable operator on \(L^2(X(\mu))\), where \(X(\mu)\) denotes the law of the diffusion process, or the image of the Wiener measure \(\mu\) under the map \(X\) which is defined by the diffusion process.

**Proof:** Let \(\eta = \eta(X)\) be the limit of \((\nabla_h F_k(X), k \geq 1)\), then using Theorem 3 we have, for any cylindrical \(G(X) \in S(X)\),

\[E[\eta(X) G(X)] = \lim_k E[\nabla_h F_k(X) G(X)] = \lim_k E[\nabla_h F_k(X) G(X)]\]

\[= \lim_k E[F_k(X)(-\nabla_h G(X) + G(X) \delta h)]\]

\[= \lim_k E[F_k(X)(-\nabla_h G(X) + G(X) \delta (\mathbb{P}(X) \hat{h})))] = 0,\]

where \(\delta (\mathbb{P}(X) \hat{h}) = \int_0^1 P(X_s) \hat{h}(s) dB_s\). As the cylinder functions of the form \(G(X)\) are dense in \(L^p(\mathcal{F}(X))\), \(\eta(X) = 0 \mu\)-a.s., and this proves the closedness of \(\nabla_h\) hence that of \(\hat{\nabla}\).  \[\square\]
For \( F \in \mathcal{S}(X) \), we define \( \hat{\nabla} F \) as
\[
\hat{\nabla} F = \sum_{i=1}^{\infty} \hat{\nabla} e_i F,
\]
where \((e_i, i \geq 1)\) is an orthonormal basis in the Cameron-Martin space \( H \). The above theorem implies in particular that \( \hat{\nabla} \) is a closable operator from \( L^p(\mathcal{F}(X)) \) into \( L^p(\mathcal{F}(X), H) \). We shall denote its closure with the same notation. In particular, we denote by \( \mathbb{M}_{p,1} \) the completion of \( \mathcal{S}(X) \) w.r.t. the norm
\[
\| F(X) \|_{p,1} = \| \hat{\nabla} F(X) \|_{L^p(\mu)} + \| F(X) \|_{L^p(\mu)} .
\]
For \( p = 2 \) we use the following version:
\[
\| F(X) \|_{2,1}^2 = E \left[ |\hat{\nabla} F(X)|_H^2 + |F(X)|^2 \right].
\]
If \( K \) is a separable Hilbert space, we denote by \( \mathcal{S}(X, K) \), the set of \( K \)-valued cylindrical functions of the trajectories of \( X \) by replacing \( \mathcal{S}(\mathbb{R}^n) \) above by \( \mathcal{S}(\mathbb{R}^n) \otimes K \), where the latter denotes the (projective) tensor product.

**Definition 1.** Let \( \nu \) be the measure given as \( X(\mu) \), i.e., the law of the diffusion process as a probability measure on the path space. We denote the adjoint of \( \hat{\nabla} \) w.r.t. \( \nu \) as \( \delta_\nu \) (or sometimes as \( \hat{\nabla}^\ast \)). Namely, for \( \xi(X) \in \mathcal{S}(X) \otimes H \), i.e., the \( H \)-valued cylindrical, \( \mathcal{F}(X) \)-measurable functions and \( G(X) \in \mathcal{S}(X) \), we write
\[
E[\delta_\nu \xi(X) G(X)] = E_\nu[\delta_\nu \xi G] = E_\nu[(\xi, \hat{\nabla} G)_H] = E[(\xi(X), \hat{\nabla} G(X))_H].
\]

**Theorem 6.** Let \( \xi(X) \in L^2(\mathcal{F}(X), H) \) be of the form \( \sum_{i<\infty} \xi_i(X) e_i \), where \( \xi_i(X) \in \mathcal{S}(X) \) and \( e_i \in H \) for each \( i \in \mathbb{N} \).

1. Then
\[
\delta_\nu \xi(X) = \sum_i \left[ -\hat{\nabla} e_i \xi_i(X) + \xi_i(X) \delta(P(X)e_i) \right],
\]
where we define the action of \( P(X) \) on the Cameron-Martin space \( H \) as
\[
(P(X)h)(t) = \int_0^t P(X_s)h(s)ds,
\]
for \( h \in H \). In particular, we have
\[
E[\delta\xi(X)|\mathcal{F}_1(X)] = \delta_\nu \xi(X).
\]

2. If \( \eta(X) \in \mathcal{S}(X, H) \) is adapted to the filtration \( \mathcal{F}_t(X), t \in [0,1] \), where \( \mathcal{S}(X, H) \) denotes \( H \)-valued version of \( \mathcal{S}(X) \), then
\[
\delta_\nu \eta = \delta P(X) \eta \quad \mu\text{-a.s.}
\]
**Proof:** Let $g(X) \in S(X)$, then, using Definition 11 and the integration by parts formula for $\mu$, we get

\[
E[(\xi(X), \nabla g(X))H] = E \left[ \sum_i \xi_i(X) \nabla e_i g(X) \right] = E \left[ \sum_i \xi_i(X) \nabla e_i g(X) \right]
\]

\[
= \sum_i E [\nabla e_i \xi_i(X) g(X) + g(X) \xi_i(X) \delta e_i]
\]

\[
= \sum_i E [\nabla e_i \xi_i(X) g(X) + g(X) \xi_i(X) E[\delta e_i | F_i(X)]]
\]

\[
= \sum_i E \left[ -\nabla e_i \xi_i(X) g(X) + g(X) \xi_i(X) \delta (P(X)e_i) \right]
\]

\[
= E \left[ \delta e_i G(X) + \xi_i(X) \delta (P(X)e_i) \right] g(X)
\]

since $E[\delta e_i | F_i(X)] = \delta (P(X)e_i)$ as shown in Theorem 3. The relation (3.3) holds then true for the cylindrical case, the general case follows by passing to the limit in $\mathbb{M}_{2,1}(H)$ (i.e., the space $\mathbb{M}_{2,1} \otimes H$, completed Hilbert-Schmidt tensor product).

To prove the second part, it follows already from the relation (3.3) that $\delta P(X) \eta = E[\delta \eta | F_1(X)]$, let now $G = G(X) \in S(X)$, then

\[
E[\delta \eta(X) G(X)] = E \left[ (\eta(X), \nabla G(X))_H \right]
\]

\[
= E \left[ \delta \eta(X) G(X) \right] = E \left[ E[\delta \eta(X) | F_1(X)] G(X) \right]
\]

\[
= E \left[ \delta P(X) \eta G(X) \right],
\]

as, by Theorem 3, $\delta P(X) \eta$ is $F_1(X)$-measurable, the proof follows. \( \square \)

**Corollary 3.** Let $F(X), G(X) \in S(X)$ and let $h \in H$, then we have the integration by parts formula

\[
(3.10) \quad E[\nabla_h F(X) G(X)] = E \left[ F(X) (\nabla_h G(X) + G(X) \delta h) \right].
\]

**Proof:** As $E[\nabla_h F(X) G(X)] = E[\nabla_h F(X) G(X)]$, from the Gaussian integration by parts formula, we have

\[
E[\nabla_h F(X) G(X)] = E[F(X)(\nabla_h G(X) + G(X) \delta h)]
\]

\[
= E \left[ F(X) E \left[ (\nabla_h G(X) + G(X) \delta h) | F_1(X) \right] \right]
\]

\[
= E[F(X)(\nabla_h G(X) + G(X) \delta h)].
\]

\( \square \)

**Corollary 4.** For $a(X) \in \mathbb{M}_{2,1}$ and $\xi(X) \in S(X, H)$, we have

\[
(3.11) \quad \delta \nu(a(X)\xi(X)) = a(X) \delta \nu \xi(X) - (\nabla a(X), \xi(X))_H.
\]
Moreover, if $\dot{\xi}(X)$ is adapted to the filtration $(F_t(X), t \in [0, 1])$, then

$$\delta_{\nu}\xi(X) = \delta(\mathbb{P}(X)\xi(X)).$$

In particular, for any $h, k \in H$, we have

$$E[\delta(\mathbb{P}(X)\nabla h\mathbb{P}(X)k)|F_1(X)] = \delta_{\nu}(\mathbb{P}(X)\hat{\nabla}_h\mathbb{P}(X)k)$$

$\mu$-a.s.

**Proof:** The only claim which is not immediate is the last one and it follows from Theorem 3.

**Lemma 1.** For any $h, k \in H$, we have

$$\hat{\nabla}_k(\delta\mathbb{P}(X)h) = (\mathbb{P}(X)h, k)_H + \delta_{\nu}(\hat{\nabla}_k\mathbb{P}(X)h)$$

$\mu$-a.s.

**Proof:** By definition

$$\hat{\nabla}_k(\delta\mathbb{P}(X)h) = E[\nabla_k\delta\mathbb{P}(X)h|F_1(X)]$$

$$= (\mathbb{P}(X)h, k)_H + E[\delta\nabla_k\mathbb{P}(X)h|F_1(X)]$$

$$= (\mathbb{P}(X)h, k)_H + \delta\mathbb{P}(X)\hat{\nabla}_k\mathbb{P}(X)h$$

$$= (\mathbb{P}(X)h, k)_H + \delta_{\nu}\hat{\nabla}_k\mathbb{P}(X)h$$

by Theorem 3 and Corollary 4.

**Theorem 7.** Let $\xi = \xi(X), \eta = \eta(X)$ be in $S(X, H)$, then the following identity holds true:

$$E[\delta_{\nu}\xi \delta_{\nu}\eta] = E[(\mathbb{P}(X)\xi, \eta)_H + \text{trace}(\hat{\nabla}\xi \cdot \hat{\nabla}\eta)]$$

$$+ E[\text{trace}(\nabla\mathbb{P}(X)\xi \cdot \hat{\nabla}\eta) + \text{trace}(\hat{\nabla}\mathbb{P}(X)\eta \cdot \hat{\nabla}\xi)]$$

**Proof:** Writing $E[\delta_{\nu}\xi \delta_{\nu}\eta] = E[(\xi, \hat{\nabla}\delta_{\nu}\eta)_H]$, then we shall calculate $\hat{\nabla}\delta_{\nu}\eta$. Let $k \in H$ and let $(e_i, i \geq 1)$ be an orthonormal basis of $H$, then

$$\hat{\nabla}\delta_{\nu}\eta = \hat{\nabla}_k \left[ \sum_i \eta_i \delta\mathbb{P}(X)e_i - \hat{\nabla}_e_i \eta_i \right]$$

$$= \sum_i \left[ \hat{\nabla}_k \eta_i \delta\mathbb{P}(X)e_i + \eta_i \hat{\nabla}_k \delta\mathbb{P}(X)e_i - \hat{\nabla}_k \hat{\nabla}_e_i \eta_i \right]$$

$$= \sum_i \left[ \hat{\nabla}_k \eta_i \delta\mathbb{P}(X)e_i + \eta_i (\mathbb{P}(X)e_i, k)_H \right]$$

$$+ \sum_i \left[ \eta_i \delta_{\nu} \hat{\nabla}_k \mathbb{P}(X)e_i - \hat{\nabla}_k \hat{\nabla}_e_i \eta_i \right]$$

Taking the expectations after replacing the vector $k$ with $e_k$ and using the above equality, we get
\[ E[\delta \nu \delta \eta] = E[\langle \xi, \hat{\nabla} \delta \eta \rangle_H] = E \left[ \sum \eta_i \delta \nu \langle \delta \mathbb{P}(X) e_i \rangle + \langle \mathbb{P}(X) \xi, \eta \rangle_H \right] \]
\[ + E \left[ \sum_{k,i} \eta_k \delta \nu \langle \hat{\nabla}_e \mathbb{P}(X) e_i \rangle \xi_k - \xi_k \hat{\nabla}_e \eta_i, \hat{\nabla}_e \eta_i \right], \]

where \( \eta_i = (\eta, e_i)_H, \xi_i = (\xi, e_i)_H \). Using the relation
\[ \hat{\nabla}_\xi \hat{\nabla}_e \eta_i = \hat{\nabla}_e \eta_i - \hat{\nabla}_e \psi_{e_i} \eta_i, \]
we have
\[ E[\delta \nu \delta \eta] = E \left[ \sum \hat{\nabla}_\xi \eta_i \delta \mathbb{P}(X) e_i + \langle \mathbb{P}(X) \xi, \eta \rangle_H \right] \]
\[ + E \left[ \sum_{k,i} \eta_k \delta \nu \langle \hat{\nabla}_e \mathbb{P}(X) e_i \rangle \xi_k - \hat{\nabla}_e \hat{\nabla}_\xi \eta_i + \text{trace} \langle \hat{\nabla} \eta \cdot \hat{\nabla} \xi \rangle \right] \]
\[ = E \left[ \delta \nu \hat{\nabla} \xi \eta + \langle \mathbb{P}(X) \xi, \eta \rangle_H + \text{trace} \langle \hat{\nabla} \eta \cdot \hat{\nabla} \xi \rangle + \sum_{k,i} \eta_i \delta \nu \langle \hat{\nabla}_e \mathbb{P}(X) e_i \rangle \xi_k \right]. \]

Using the identity
\[ \delta \nu (g \tau) = g \delta \nu \tau - \langle \tau, \hat{\nabla} g \rangle_H, \]
for \( g \in \mathbb{S}(X), \tau \in \mathbb{S}(X, H) \), we have
\[ \sum k,i E \left[ \eta_k \delta \nu \langle \hat{\nabla}_e \mathbb{P}(X) e_i \rangle \xi_k \right] \]
\[ = \sum k,i E \left[ \eta_k \delta \nu \langle \xi_k \hat{\nabla}_e \mathbb{P}(X) e_i \rangle + \eta_k \langle \hat{\nabla}_e \xi_k, \hat{\nabla}_e \mathbb{P}(X) e_i \rangle_H \right] \]
\[ = E \left[ \sum_i \eta_i \delta \nu \langle \hat{\nabla} \xi \mathbb{P}(X) e_i \rangle + \sum_k \langle \hat{\nabla} \xi_k, \hat{\nabla}_e \mathbb{P}(X) \eta - \mathbb{P}(X) \hat{\nabla}_e \xi \rangle_H \right], \]

and since
\[ \eta \delta \nu \langle \hat{\nabla}_e \mathbb{P}(X) e_i \rangle = \delta \nu \langle \eta \hat{\nabla}_e \mathbb{P}(X) e_i \rangle + \langle \hat{\nabla}_e \mathbb{P}(X) e_i, \hat{\nabla} \eta_i \rangle_H, \]
we obtain by substituting (3.16) in (3.15)
\[ \sum k,i E \left[ \eta_k \delta \nu \langle \hat{\nabla}_e \mathbb{P}(X) e_i \rangle \xi_k \right] = I + II, \]
where
\[ I = \sum i E \left[ \delta \nu \langle \eta \hat{\nabla} \mathbb{P}(X) e_i \rangle + \langle \hat{\nabla} \mathbb{P}(X) e_i, \hat{\nabla} \eta_i \rangle_H \right] \]
\[ = E \left[ \text{trace} \langle \hat{\nabla} \xi \mathbb{P}(X) \cdot \hat{\nabla} \eta \rangle \right] \]
and

\[ II = \sum_{k,i} E[\eta_i (\hat{\nabla} \xi_k, \hat{\nabla} e_k \mathbb{P}(X)e_i)_H] \]

\[ = \sum_k E[(\hat{\nabla} \xi_k, (\hat{\nabla} \mathbb{P}(X))\eta)_H] \]

\[ = E[\text{trace}(\hat{\nabla} \xi \cdot (\hat{\nabla} \mathbb{P}(X))\eta)] \]

and this completes the proof. \qed

The next result shows the local character of the derivative operator \( \hat{\nabla} \), which is well-known in the flat case:

**Proposition 1.** Assume that \( F = F(X) \) is in \( \mathbb{M}_{2,1} \), then \( \hat{\nabla} F = 0 \) on the set \( \{ F = 0 \} \) \( \nu \)-almost surely.

**Proof:** Let \( \theta \) be a positive, smooth function of compact support on \( \mathbb{R} \) with \( \theta(0) = 1 \), denote by \( \zeta \) its primitive and let \( \zeta_\varepsilon(t) = \varepsilon \zeta(t/\varepsilon) \). Clearly \( \zeta_\varepsilon \circ F \) belongs to \( \mathbb{M}_{2,1}(X) \). If \( u \in S(X, H) \), from the definitions

\[ E[\zeta_\varepsilon(F) \delta_\nu u] = E[(\hat{\nabla} \zeta_\varepsilon(F), u)_H] \]

\[ = E[\theta(F/\varepsilon)(\hat{\nabla} F, u)_H] \to E[\mathbb{1}_{\{F=0\}}] \cdot \]

On the other hand, from the Dominated Convergence Theorem, we have \( \lim_{\varepsilon \to 0} E[\zeta_\varepsilon(F)\delta_\nu u] = 0 \) and as \( S(X, H) \) is dense in \( \mathbb{M}_{2,1}(X, H) \), the proof follows. \qed

The following result shows that \( \hat{\nabla} \) defines a Markovian Dirichlet form on the path space of the diffusion process:

**Proposition 2.** Assume that \( f, g \in \mathbb{M}_{2,1} \), then \( f \wedge g \in \mathbb{M}_{2,1} \), where \( \wedge \) denotes the minimum.

**Proof:** Let \( (f_n, n \geq 1) \) and \( (g_n, n \geq 1) \) be two sequences of cylindrical functions approximating \( f \) and \( g \) in \( \mathbb{M}_{2,1} \) respectively. We have

\[ \hat{\nabla}(f_n \wedge g_n) = E[\hat{\nabla} f_n \mathbb{1}_{\{f_n \geq g_n\}} + \hat{\nabla} g_n \mathbb{1}_{\{f_n < g_n\}}|\mathcal{F}_1(X)] \]

\[ = \hat{\nabla} f_n \mathbb{1}_{\{f_n \geq g_n\}} + \hat{\nabla} g_n \mathbb{1}_{\{f_n < g_n\}} \]

Hence

\[ |\hat{\nabla}(f_n \wedge g_n)|_H \leq |\hat{\nabla} f_n|_H + |\hat{\nabla} g_n|_H . \]

Consequently, \( (f_n \wedge g_n, n \geq 1) \) is bounded in \( \mathbb{M}_{2,1} \), then it has a subsequence which converges weakly. This implies that \( f \wedge g \in \mathbb{M}_{2,1} \) and that \( |\hat{\nabla}(f \wedge g)|_H \leq |\hat{\nabla} f|_H + |\hat{\nabla} g|_H \) almost surely. \qed

While making calculations we need also a second covariant derivation on the paths of the diffusion process, this time this will be the smoothed derivative w.r.t. the state space elements, namely we define

\[ \hat{\partial}_t X_t(\xi) = E[\partial_t X_t(\xi)|\mathcal{F}_1(X(\xi))] \]

where \( \partial_t \) denotes the derivative of the diffusion process at the instant \( t \in [0,1] \) with respect to its initial starting point \( \xi \in \mathbb{R}^n \), similarly, for smooth functions on \( \mathbb{R}^n \), we define

\[ \hat{\partial}_t f(X_t(\xi)) = E[\partial f(X_t(\xi))\partial_t X_t(\xi)|\mathcal{F}_1(X(\xi))] = \partial f(X_t(\xi))\hat{\partial}_t X_t(\xi), \]
in such a way that \( \hat{\partial} \) behaves as a derivative w.r.t. state space variable of the functionals of the paths of the diffusion process. We have the immediate result which follows from Theorem 3. The state space covariant derivative

\[
\hat{\partial}_x X_t(x) = J_t(x) = J_t \frac{\partial}{\partial x} X_t(x),
\]

(3.17)

satisfies the following stochastic differential equation:

\[
J_t = \mathbb{I}_{\mathbb{R}^n} + \int_0^t \partial \sigma(X_s(x)) J_s(x) P(X_s(x)) dB_s + \int_0^t \partial b(X_s(x)) J_s(x) ds,
\]

P-a.s., for any \( t \in [0, 1] \).

**Proof:** We have

\[
\partial X_t(x) = \mathbb{I}_{\mathbb{R}^n} + \int_0^t \partial \sigma(X_s(x)) \partial X_s(x) dB_s + \int_0^t \partial b(X_s(x)) \partial X_s(x) ds.
\]

(3.18)

Taking the conditional expectation of both sides of the equality (3.18), we get

\[
E \left[ \int_0^t \partial \sigma(X_s(x)) \partial X_s(x) dB_s | \mathcal{F}_t(X) \right] = \int_0^t E[\partial \sigma(X_s(x)) | \mathcal{F}_t(X)] P(X_s(x)) dB_s
\]

due to Theorem 3. We also have

\[
E \left[ \int_0^t \partial b(X_s(x)) \partial X_s(x) ds | \mathcal{F}_t(X) \right] = \int_0^t E[\partial b(X_s(x)) | \mathcal{F}_s(X)] ds
\]

from the Markov property of \((X_t(x))\) w.r.t. the Brownian filtration.

It follows from Theorem 8 and from the Itô formula that

**Proposition 3.** \((J_t, t \in [0, 1])\) is a semimartingale with values in \(GL(n, \mathbb{R})\) (non-singular matrices) and its inverse, denoted as \((K_t, t \in [0, 1])\) has the following representation:

\[
dK_t = - \sum_{i=1}^d K_t \partial \sigma^i(X_t(x)) dm^i_t + K_t \sum_{i,j=1}^d \partial \sigma^i(X_t(x)) \partial \sigma^j(X_t(x)) P_{i,j}(X_t(x)) - \partial b(X_t(x)) dt,
\]

(3.19)

where \( dm_t = P(X_t(x)) dB_t \), \( m_t = (m^1_t, \ldots, m^d_t) \), and \( \sigma^i \) is the \( i \)-th column vector of the \( n \times d \)-matrix \( \sigma \).

**Proof:** The equation (3.19) is a linear stochastic differential equation with bounded coefficients, hence it has a unique solution without explosion. To prove the claim it suffices to show that \( J_t K_t = K_t J_t = \mathbb{I}_{\mathbb{R}^n} \) almost surely, where \( \mathbb{I}_{\mathbb{R}^n} \) denotes the identity map of \( \mathbb{R}^n \), and this last equation follows from the Itô formula.

An immediate consequence of these calculations, combined with the variation of constants’ method, is the following

**Corollary 5.** For any \( t < \tau \in [0, 1] \), we have

\[
\hat{\partial}_t X_\tau(x) = J_t K_t \sigma(X_t(x))
\]
almost surely, where $\hat{D}_t X_t(x)$ is defined as
\[
(\hat{\nabla}_h X_t(x), z)_{\mathbb{R}^n} = \int_0^1 (\hat{D}_t X_t(x), z \otimes \hat{h}(t))_{\mathbb{R}^n \otimes \mathbb{R}^d} dt
\]
\[
= \int_0^T (\hat{D}_t X_t(x), z \otimes \hat{h}(t))_{\mathbb{R}^n \otimes \mathbb{R}^d} dt,
\]
for $h \in H, z \in \mathbb{R}^n$.

**Proof:** We can suppose without loss of generality $b = 0$, from the relation (3.6), using Euclidean coordinate representation we deduce the equation
\[
(3.20) \quad \hat{D}_t X_t^i = \sigma^i_j(X_t) + \int_t^T (\partial \sigma^j_{kl}(X_s) \hat{D}_t X_s^k) dm_s^l,
\]
where the sum is performed on repeated indices. We will show that $(J_t K_t \sigma(X_t(x)), \tau \in [t, 1])$ satisfies the equation (3.20). We have
\[
\sigma^i_j(X_t) = \sigma^i_j(X_t) + \int_t^T (\partial \sigma^j_{kl}(X_s) [J^e_{\alpha} (s) K^\beta_{\alpha} (t)] \sigma^\beta_j (X_t)) dm_s^l
\]
\[
= \sigma^i_j(X_t) + \int_t^T (\partial \sigma^j_{kl}(X_s) [J^e_{\alpha} (s) dK^\beta_{\alpha} (t)] \sigma^\beta_j (X_t))
\]
\[
= \sigma^i_j(X_t) + [J^e_{\alpha} (\tau) - J^e_{\alpha} (t)] K^\beta_{\alpha} (t) \sigma^\beta_j (X_t)
\]
\[
= J^e_{\alpha} (\tau) K^\beta_{\alpha} (t) \sigma^\beta_j (X_t) - \delta_i \sigma^\beta_j (X_t)
\]
\[
= J^e_{\alpha} (\tau) K^\beta_{\alpha} (t) \sigma^\beta_j (X_t).
\]
Hence by the uniqueness of the solutions of the equation (3.6) or (3.20), the claim follows. \[\square\]

4. Calculation of Integrands in Martingale Representation and its Applications

Here are some applications of the results of the preceding section. Let us begin with a non-trivial extension of Itô-Clark-Haussmann-Bismut-Ocone formula:

**Theorem 9.** Assume that $F(X) \in M_{2,1}$, then it can be represented as
\[
F(X) = E[F(X)] + \int_0^1 P(X_s) E[\hat{D}_s F(X)|\mathcal{F}_s(X)] dB_s,
\]
where $\hat{D}_s F(X)$ is defined as $\hat{\nabla} F(X)(\cdot) = \int_0^T \hat{D}_s F(X) ds$ and $P(X_s)$ is the orthogonal projection from $\mathbb{R}^d$ onto $\sigma(X_s)^*(\mathbb{R}^n)$.

**Proof:** We know from Theorem 2 that $F(X)$ can be represented as
\[
F(X) = E[F(X)] + \int_0^1 P(X_s) \alpha_s(X). dB_s,
\]
for some $\alpha(X) \in L^2(dt \times d\mu, \mathbb{R}^d)$, adapted to the filtration $(\mathcal{F}_t(X), t \in [0, 1])$. Moreover, if $F(X) \in S(X)$, then regarding $F(X)$ as a function of the Brownian path $B$, from classical representation theorem (cf. [14] [15] [16] [19]), we have
\[
(4.21) \quad F(X) = E[F(X)] + \int_0^1 E[D_s F(X)|B_s] dB_s,
\]
where \((B_s, s \in [0, 1])\) is the filtration of the Brownian motion. It follows from Theorem 9 taking the conditional expectations of both sides of the equality (4.21), that

\[
F(X) = E[F(X)] + \int_0^1 P(X_s)E[D_s F(X)|\mathcal{F}_s(X)]dB_s.
\]

Hence in this case \(P(X_s)\alpha_s(X) = P(X_s)E[D_s F(X)|\mathcal{F}_s(X)] = P(X_s)E[E[D_s F(X)|\mathcal{F}_1(X)|\mathcal{F}_s(X)] = P(X_s)E[\hat{D}_s F(X)|\mathcal{F}_s(X)]\) \(ds \times d\mu\)-a.s. Choose now a sequence \((F_n(X), n \geq 1)\) \(\subset S(X)\) which approximates \(F(X)\) in \(\mathbb{M}_{2,1}\). Then \((\hat{\nabla} F_n(X), n \geq 1)\) converges to \(\hat{\nabla} F(X)\) in \(L^2(\mu, H)\) (in other words \((\hat{\nabla} F_n)\) converges to \(\hat{\nabla} F\) in \(L^2(\nu, H)\)), consequently

\[
\lim_n E \int_0^1 |P(X_s)E[\hat{D}_s F_n(X) - \hat{D}_s F(X)|\mathcal{F}_s(X)]|^2 ds = 0,
\]

hence the result follows for any \(F(X) \in \mathbb{M}_{2,1}\). \(\square\)

**Corollary 6.** Assume that \(F = F(X) \in \mathbb{M}_{2,1}\) such that \(\hat{\nabla} F = 0 \ \nu\)-a.s. (or \(\hat{\nabla} F(X) = 0 \ \mu\)-a.s), then \(F\) is \(\nu\)-a.s. (or \(F(X)\) is \(\mu\)-a.s.) a constant.

**Proof:** The proof follows directly from Theorem 9 \(\square\)

We obtain the Poincaré inequality from Theorem 9 whose proof is trivial since the conditional expectation in the representation formula (4.21) operates as a contraction:

**Theorem 10.** For any \(F(X) \in \mathbb{M}_{2,1}\), we have

\[
E[|F(X) - E[F(X)]|^2] \leq E[|\mathbb{P}(X)\hat{\nabla} F(X)|^2_H].
\]

**Theorem 11.** For any \(F \in \mathbb{M}_{2,1}\), it holds true that

\[
E[F^2(X) \log F^2(X)] \leq E[F^2(X)] \log E[F^2(X)] + 2E[|\mathbb{P}(X)\hat{\nabla} F(X)|^2_H],
\]

where \(\mathbb{P}(X)\) is the action induced on \(H\) by \((P(X_s), s \in [0, 1])\), namely

\[
\mathbb{P}(X)h(t) = \int_0^t P(X_s)\hat{h}(s)ds,
\]

for \(h \in H\) and \(t \in [0, 1]\).

**Proof:** It suffices to prove the claim for any \(F(X) \in \mathbb{M}_{2,1}\) which is strictly positive and lower bounded by a positive constant. Taking \(f(X) = F^2(X)/E[F^2(X)]\), it suffices to show that

\[
E[f(X) \log f(X)] \leq \frac{1}{2} E \left[\frac{1}{f(X)} |\mathbb{P}(X)\hat{\nabla} f(X)|^2_H\right].
\]

Using Theorem 9 we can represent \(f(X)\) as

\[
f(X) = 1 + \int_0^1 P(X_s)E[\hat{D}_s(f(X))|\mathcal{F}_s(X)]dB_s
\]

\[
= 1 + \int_0^1 P(X_s)E[\hat{D}_s(f(X))|\mathcal{F}_s(X)]E[f(X)|\mathcal{F}_s(X)]dB_s.
\]

Hence \(f(X)\) can be represented as an exponential martingale’s final value:

\[
f(X) = \exp \left[\int_0^1 \frac{P(X_s)E[\hat{D}_s(f(X))|\mathcal{F}_s(X)]}{f_s(X)} dB_s - \frac{1}{2} \int_0^1 \left|\frac{P(X_s)E[\hat{D}_s(f(X))|\mathcal{F}_s(X)]}{f_s(X)}\right|^2 ds\right],
\]
where \( f_s(X) = E[f(X)|F_s(X)] \). Let \( \gamma \) be the probability measure on the path space defined by \( d\gamma = f(X)d\mu \), then the Girsanov Theorem and Bayes’ formula entail that the relative entropy of \( \gamma \) w.r.t. \( \mu \), denoted by \( H(\gamma|\mu) = \int_W \log \frac{d\gamma}{d\mu} d\gamma \) is equal to

\[
H(\gamma|\mu) = \int \log f(X)d\gamma
\]

\[
= \frac{1}{2} E_\gamma \int_0^1 \left| \frac{P(X_s)E[D_s(f(X))|F_s(X)]}{f_s(X)} \right|^2 ds
\]

\[
= \frac{1}{2} E_\gamma \int_0^1 \left| E[P(X_s)D_s(f(X))|F_s(X)] \right|^2 ds
\]

\[
= \frac{1}{2} E_\gamma \int_0^1 \left| E_\gamma[P(X_s)D_s(\log f(X))|F_s(X)] \right|^2 ds
\]

\[
\leq \frac{1}{2} E_\gamma \int_0^1 \left| P(X_s)D_s(\log f(X)) \right|^2 ds
\]

\[
= \frac{1}{2} E \left[ \frac{1}{f(X)} \| P(X)\nabla f(X) \|^2_H \right].
\]

**Remark:** Maybe it is pedagogically interesting to note this inequality in terms of the measure induced by the the diffusion process on the path space, namely \( \nu \):

\[
\int f^2 \log f^2 d\nu - \left( \int f^2 d\nu \right) \log \int f^2 d\nu \leq 2 \int \| P \nabla f \|^2_H d\nu
\]

for any \( f \in \mathbb{M}_{2,1} \).

5. **Examples and Further Applications**

We shall give applications of the analysis that we have developed to some specific cases beginning from the simplest to more complicated situations

5.1. **Classical Situation.** Let \((X_t, t \geq 0)\) be the solution of the equation

\[
X_t(x) = x + \int_0^t b(X_r(x)) dr + \Sigma B_t,
\]

where \( B = (B_t, t \geq 0) \) is an \( d \)-dimensional Brownian motion, \( b : \mathbb{R}^n \to \mathbb{R}^n \) is a Lipschitz-continuous vector field and \( \Sigma \in L(\mathbb{R}^n, \mathbb{R}^d) \). Regularizing \( b \) with a mollifier, we obtain a smooth and Lipschitz vector field \( b_\varepsilon \), let \( X_\varepsilon \) be the solution of SDE

\[
X_\varepsilon_t(x) = x + \int_0^t b_\varepsilon(X_\varepsilon_r(x)) dr + \Sigma B_t.
\]

For any \( h \in H \), where \( H = H^1(\mathbb{R}_+, \mathbb{R}^d) \) is the Cameron-Martin space of \( B \), we have

\[
\nabla_h X_\varepsilon_t = \int_0^t \partial b_\varepsilon(X_r) \nabla_h X_\varepsilon_r dr + \Sigma h(t)
\]

and the Gronwall lemma implies that, for any \( T > 0 \),

\[
\sup_{t \leq T} \| \nabla X_\varepsilon_t \|_{H \otimes \mathbb{R}^d} \leq \| \Sigma \| cTK,
\]
where $K$ is the Lipschitz constant of $b$ and $\|\Sigma\|$ is the operator norm of $\Sigma$. We can pass to the limit as $\varepsilon \to 0$ to obtain also
\[
\sup_{t \leq T} \|\nabla X_t\|_{H \otimes \mathbb{R}^m} \leq \|\Sigma\| e^{TK}.
\]
The following theorem is now a straight application of Theorem 11

**Theorem 12.** Let $T > 0$ be arbitrary and denote by $\nu^T$ the law of the process $(X_t, t \in [0, T])$. Then, for any smooth, cylindrical function on $W_T = C([0, T], \mathbb{R}^d)$, we have
\[
\int_{W_T} F^2(x) \log \frac{F^2(x)}{\nu^T(f^2)} \nu^T(dx) \leq 2\|\Sigma\| e^{TK} \int_{W_T} |\nabla F(x)|^2 \nu^T(dx),
\]
in particular, for any $t \in (0, T]$, denoting by $\nu_t$ the law of the random variable $X_t$, we have
\[
\int_{\mathbb{R}^m} f^2(y) \log \frac{f^2(y)}{\nu_t(f^2)} \nu_t(dy) \leq 2\|\Sigma\| e^{TK} \int_{\mathbb{R}^m} |\partial f(y)|^2 \nu_t(dy),
\]
for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$.

### 6. Log-Sobolev for Dyson’s Brownian Motion

In the example above, the drift coefficient has nice properties and the log-Sobolev inequality is almost trivial. Let us look at a more singular case, namely, let $(X_t = (X^1_t, \ldots, X^d_t), t \geq 0)$ be the Dyson Brownian motion, i.e., the strong solution of the following stochastic differential equation:

\[
dX^i_t = dB^i_t + \gamma \sum_{1 \leq j < i \leq d} \frac{1}{X^i_t - X^j_t} dt, \quad i = 1, \ldots, d; \quad \gamma > 0,
\]

with $X_0 = x_0$ such that $x_1 < \ldots < x_d$. We refer the reader to references [3, 4, 5] for the existence of the strong solutions, where they use the theory of dissipative mappings in their construction and to [8] for the origins of the equation. For notational convenience, we shall write above equation as

\[
dX_t = \gamma m(X_t) dt + dB_t, \quad X_0 = x_0 \in D(m),
\]

where $m$ is the mapping from a subset of $\mathbb{R}^d$, noted as $D(m)$ into $\mathbb{R}^d$ whose components are given in the equation (6.24). In fact $m$ is a set-valued, dissipative map. It is maximal and this implies that its values reduce to one-point subsets of $\mathbb{R}^d$ (cf. [24]). As a typical example for such a function, one can take the gradient of a concave function on $\mathbb{R}^d$. Equation (6.24) corresponds to the case $m = \partial f$, where $f(x) = \sum_{i<j} \log(x_j - x_i)$ if $x_1 < \ldots < x_d$ and $f(x) = -\infty$ otherwise. Let us announce first an important technical result:

**Lemma 2.** Let $X = (X_t, t \geq 0)$ be the unique strong solution of SDE (6.24) then, for any $T > 0$, $h, k \in H([0, T], \mathbb{R}^d)$ we have
\[
\sup_{s \leq T} |X_s(w + h) - X_s(w + k)| \leq 2\sqrt{t}\gamma |h - k|_{H([0, T], \mathbb{R}^d)}
\]
P-almost surely. Consequently, for any $t > 0$, $X_t \in \cap_p \mathbb{D}_{p, 1}(\mathbb{R}^d)$, in fact $|\nabla X_t|_{\mathbb{R}^d \otimes H([0, t], \mathbb{R}^d)} \in L^\infty(P)$ with the bound $2t\gamma$, where $H([0, t], \mathbb{R}^d)$ (respectively $H([0, T], \mathbb{R}^d)$) denotes the Cameron-Martin space for $\mathbb{R}^d$-valued functions on $[0, t]$ (respectively $[0, T]$).
Problem (cf. [17]) associated to the operator $K$ to the time horizon $[0, T]$ working on the canonical Wiener space (for practical reasons). From the equation 6.25 it comes

$$X_t(w + h) = B_t(w) + h(t) + \int_0^t m(X_s(w + h))ds,$$

hence

$$X_t(w + h) - X_t(w) = h(t) + \int_0^t [m(X_s(w + h)) - m(X_s(w))]ds.$$  

Let $y^h_t(w) = x_t(w + h) - X_t(w)$, then $(y^h_t, t \in [0, T])$ satisfies

$$y^h_t(w) = h(t) + \int_0^t [m(y^h_s(w) + X_s(w)) - m(X_s(w))]ds,$$

hence

$$\tag{6.26} y^h_t(w) - y^k_t(w) = h(t) - k(t) + \int_0^t [m(y^h_s(w) + X_s(w)) - m(y^k_s(w) + X_s(w))]ds.$$  

It follows from equation 6.26 and from the dissipativity of $m$ that

$$|y^h_t - y^k_t|^2 = \int_0^t \frac{d}{ds}|y^h_s - y^k_s|^2ds$$

$$= 2\int_0^t (y^h_s - y^k_s, \hat{h}(s) - \hat{k}(s))_{\mathbb{R}^d} + 2\int_0^t (y^h_s - y^k_s, m(y^h_s + X_s) - m(y^k_s + X_s))ds$$

$$\leq 2\sup_{s \leq t} |y^h_s - y^k_s| \int_0^t |\hat{h}(s) - \hat{k}(s)|ds$$

$$\leq 2\sqrt{T} \sup_{s \leq t} |y^h_s - y^k_s| \left( \int_0^t |\hat{h}(s) - \hat{k}(s)|^2ds \right)^{1/2}.$$  

Since the last line of above inequality is increasing w.r.t. $t \in \mathbb{R}_+$, the claim follows. The fact that $X_t \in \cap_{p > 1} D_{p,1}$ follows from [18], Lemma 2 at p.72. \qed

An immediate consequence of Lemma 2 is Fernique’s bound:

**Corollary 7.** For any $t > 0$, there exists some $\varepsilon = \varepsilon(\gamma, t)$, such that

$$E \left[ \exp \left( \varepsilon \sup_{s \leq t} |X_s|^2 \right) \right] < \infty.$$  

**Remark:** Let $T > 0$ be any fixed number and let us denote by $\nu$ the law of the process restricted to the time horizon $[0, T]$, starting from $x_0$. Evidently $\nu$ is the unique solution of the martingale problem (cf. [17]) associated to the operator $K$ defined as

$$Kf(x) = \sum_{i=1}^d \left[ \frac{1}{2} \frac{\partial^2 f}{\partial x_i^2}(x) + \gamma m_i(x) \frac{\partial f}{\partial x_i}(x) \right].$$  

It is immediate to see that under $\nu$, the coordinate map $x \to x(t)$ of $C_{x_0}([0, T], \mathbb{R}^d)$ can be represented as

$$dx_t = \gamma m(x_t)dt + d\beta_t,$$

where $(\beta_t, t \in [0, T])$ is a $\nu$-Brownian motion.

The following is now clear:
Proposition 4. The covariant derivative defined on the paths of Dyson process restricted to $[0, T]$, i.e., for $F(X) = f(X_t, \ldots, X_{ts})$,

$$\nabla_h F(X) = E[\nabla_h F(X)|\mathcal{F}_t(X)],$$

where $h \in H([0, T], \mathbb{R}^d)$ and $\hat{\nabla} F(X) = \sum_i \hat{\nabla}_{h_i} h_i$, such that $(h_i)$ is a basis in $H([0, T], \mathbb{R}^d)$, are closable operators on $L^p(\mathcal{F}_T(X))$ for any $p > 1$. Consequently, we have the following representation theorem:

$$F(X) = E[F(X)] + \int_0^T E[D_s F(X)|\mathcal{F}_s(X)] \cdot dB_s.$$

From Proposition 4 we obtain as explained in the general case, the logarithmic Sobolev inequality:

Proposition 5. For any $\mathcal{F}_T(X)$ measurable, cylindrical function $F(X)$, we have

$$E[F^2(X) \log(F^2(X)/E[F^2(X)])] \leq 2\gamma \sqrt{T} E[\hat{\nabla} F(X)^2]_{H([0, T], \mathbb{R}^d)}.$$

7. Intertwining Relations and Logarithmic Sobolev Inequality

Let us come back to the general Markovian situation studied in the paper: if $f$ be a smooth function on $\mathbb{R}^n$ and if $(Q_t, t \in [0, 1])$ is the semi-group associated to the Markov process induced by $(X_t, t \in [0, 1])$, then $(Q_{1-t} f(X_t(x)), t \in [0, 1])$ is an $(\mathcal{F}_t(X))$-martingale and, due to the Itô formula, it can be represented as

$$Q_{1-t} f(X_t(x)) = Q_1 f(x) + \int_0^t \partial_s Q_{1-s} f(X_s(x)) \cdot \sigma(X_s(x)) dB_s.$$  \hspace{1cm} (7.27)

From Theorem 3 we obtain a second representation:

$$Q_{1-t} f(X_t(x)) = Q_1 f(x) + \int_0^t P(X_s(x)) E[\hat{D}_s f(X_1(x))|\mathcal{F}_s(X)] \cdot dB_s,$$

where $\hat{D}_s f(X_1(x)) = E[D_s f(X_1(x))|\mathcal{F}_s(X)]$. Ito Isometry principle implies that

$$\partial_s Q_{1-s} f(X_s(x)) \cdot \sigma(X_s(x)) = P(X_s(x)) E[\hat{D}_s f(X_1(x))|\mathcal{F}_s(X)]$$

d$s \times dP$-almost surely, where

$$\hat{D}_t f(X_1) = E[D_t f(X_1)|\mathcal{F}_t(X)].$$

It will be tempting to compare in detail these two representations: As

$$\hat{D}_t f(X_1) = (\hat{D}_t X_1)^* \partial f(X_1) \in \mathbb{R}^d,$$

we obtain

Theorem 13. For any smooth function $f$ on $\mathbb{R}^n$, we have the following commutation-intertwining relation:

$$\sigma(X_t)^* \partial Q_{1-t} f(X_t) = P(X_t) E \left[(\hat{D}_t X_1)^* \partial f(X_1)|\mathcal{F}_t(X)\right].$$  \hspace{1cm} (7.29)

Corollary 5 gives that

$$(\hat{D}_t X_1)^* = \sigma^*(X_t) K_t^* J_1^*,$$
Lemma 3. expectation, it has an important byproduct that we write separately as a lemma:

\[ E[\hat{d}t f(X_t) | \mathcal{F}_t(X)] = E[\hat{d}t f(X_t)^* \hat{d}f(X_t) | \mathcal{F}_t(X)] = E[\sigma^* (X_t) K_t^* J_t^* \hat{d}f(X_t) | \mathcal{F}_t(X)] = \sigma^* (X_t) K_t^* E[J_t^* \hat{d}f(X_t) | \mathcal{F}_t(X)] . \]

To calculate the conditional expectation of the last line, we use the flow property of the diffusion process: For \( s < t \leq 1 \), let \( X^*_t(x, \omega) \) denote the solution of the SDE defining the diffusion process at the instant \( t \), starting from the point \( x \in \mathbb{R}^n \) at the instant \( s \). The flow property implies that \( X^*_1(x, \omega) = X^*_t(X_t(x, \omega), \tilde{\omega}) \), where \( \tilde{\omega} \) denotes a Brownian path independent of \( \{B_\tau : \tau \leq t\} \) Taking the derivative of both sides, from the chain rule, it comes

\[ \partial_x X^*_1(x, \omega) = \partial X^*_t(X_t(x, \omega), \tilde{\omega}) \partial X^*_t(x, \omega) \]

almost surely. Hence

\[ E[J_t^* \partial f(X_t) | \mathcal{F}_t(X)] = E[\partial X^*_t \partial f(X_t) | \mathcal{F}_t(X)] = E[\partial X^*_t \partial f(X_t) | \mathcal{F}_t(X)] = E [\partial X^*_t E[(\partial X^*_t)^* \circ (X_t, \tilde{\omega}) \partial f(X^*_t \circ X_t) | \mathcal{B}_t] | \mathcal{F}_t(X)] , \]

where \( (\mathcal{B}_t, t \geq 0) \) represents the filtration of the underlying Brownian motion. Due to the independence and the stationarity of the Brownian increments, the inner conditional expectation can be written as:

\[ E[(\partial X^*_t)^* \circ (X_t, \tilde{\omega}) \partial f(X^*_t \circ X_t) | \mathcal{B}_t] = E[(\partial X^*_1)^* (\xi) \partial f(X^*_1(X_t(x))) | \xi = X_t(x)] = E[(\partial X^*_1)^* (\xi) \partial f(X_1-x) | \xi = X_t(x)] \]

and in particular, it is \( \mathcal{F}_t(X) \)-measurable. Let us define the semigroup \( (M_t, t \geq 0) \) on the vector-valued functions as

\[ M_t \alpha (\xi) = E[(\partial X^*_t(\xi))^* \alpha (X_t(\xi))] , \]

\( \xi \in \mathbb{R}^n \). Using this semigroup we end up with the identity

\[ E[J_t^* \partial f(X_t) | \mathcal{F}_t(X)] = E[\partial X^*_t \mathcal{B}_t] = M_{t-1} \partial f(X_t(x)) \]

Finally we find that

\[ E[\hat{d}t f(X_t(x)) | \mathcal{F}_t(X)] = \sigma(X_t(x))^* M_{t-1} \partial f(X_t(x)) \]

almost surely. Although the expression \( E[\hat{d}t f(X_t(x)) | \mathcal{F}_t(X)] = \sigma(X_t(x))^* M_{t-1} \partial f(X_t(x)) \)

is what we can obtain by differentiation under the expectation, it has an important byproduct that we write separately as a lemma:

**Lemma 3.** For any smooth function \( f \) on \( \mathbb{R}^n \), the process \((\partial X^*_t(x) M_{t-1} \partial f(X_t(x)), t \in [0, 1])\) is an \( \mathcal{F}_t(X(x)) \)-martingale.

The following result may be called as a modified Poincaré inequality:
Proof: From the martingale representation theorem, from the relation (7.31) and from the Itô isometry, we obtain
\[
E[f(X_1(x)) - f(X_1(x))]^2 \leq E \left[ Z \left( \vartheta X_1(x) \cdot \vartheta X_1^*(x) \vartheta f(X_1(x)), \vartheta f(X_1(x)) \right) \right]_{\mathbb{R}^n}.
\]

Remark: Let operator-valued function \( \Gamma(x, y) \) be defined as
\[
\Gamma(x, y) = E[Z \vartheta X_1(x) \cdot \vartheta X_1^*(x)|X_1(x) = y],
\]
then we can read the same inequality in \( \mathbb{R}^n \) as
\[
\int_{\mathbb{R}^n} \left| f(y) - \int_{\mathbb{R}^n} f(z) q_1(x, z) dz \right|^2 \, q_1(x, y) \, dy \leq \int_{\mathbb{R}^n} \left( \Gamma(x, y) \vartheta f(y), \vartheta f(y) \right)_{\mathbb{R}^n} q_1(x, y) \, dy,
\]
where \( y \to q_1(x, y) \) is the density of the law of \( X_1(x) \).

Proof: From the martingale representation theorem, from the relation (7.31) and from the Itô isometry, we obtain
\[
E[f(X_1(x)) - f(X_1(x))]^2 = E \int_0^1 |\sigma^*(X_t(x)) M_{t - i} \vartheta f(X_t(x))|^2 \, dt
\]
\[
= E \int_0^1 |\sigma^*(X_t(x)) (\vartheta X_t^*)^{-1}(\vartheta X_t^*) M_{t - i} \vartheta f(X_t(x))|^2 \, dt
\]
\[
\leq E \int_0^1 |\sigma^*(X_t(x)) (\vartheta X_t^*)^{-1}|^2 |(\vartheta X_t^*) M_{t - i} \vartheta f(X_t(x)))|^2 \, dt
\]
\[
\leq E \int_0^1 |\sigma^*(X_t(x)) (\vartheta X_t^*)^{-1}|^2 |(\vartheta X_t^*) \vartheta f(X_t(x))|^2 \, dt,
\]
where the inequality at the last line follows from Lemma 3.

Same observation gives also an extension of the logarithmic Sobolev inequality for the measure \( q_1(x, y) \, dy \) which is the law of the random variable \( \omega \to X_1(\omega, x) \):

Theorem 15. Let \( f \in C_0^2(\mathbb{R}^n) \), then, we have
\[
E \left[ f^2(X_1) \log \frac{f^2(X_1)}{E[f^2(X_1)]} \right] \leq 2 \int_{\mathbb{R}^n} \left( \Gamma(x, y) \vartheta f(y), \vartheta f(y) \right)_{\mathbb{R}^n} q_1(x, y) \, dy,
\]
where \( \Gamma \) is defined with (7.33).

Proof: We proceed as in the proof of Theorem 11 with \( f \in C_0^2(\mathbb{R}^n) \) which is strictly positive. Moreover we can suppose that \( E[f(X_1(x))] = 1 \). In this case the claimed inequality takes the form
\[
E[f(X_1(x)) \log f(X_1)] \leq \int_{\mathbb{R}^n} \left( \Gamma(x, y) \vartheta f(y), \vartheta f(y) \right)_{\mathbb{R}^n} \frac{1}{f(y)} q_1(x, y) \, dy.
\]
Let \( d\gamma = f(X_1(x))dP \), similar calculations of entropy \( H(\gamma|P) \) as in the proof of Theorem 11 imply that

\[
H(\gamma|P) = \int \log f(X_1(x))d\gamma
\]

\[
= \frac{1}{2} E\gamma \int_0^1 \left| \frac{P(X_s)f(X_1)|\mathcal{F}_s(X)}{Q_{1-s}f(X_s)} \right|^2 ds
\]

\[
\leq \int_0^1 \left| \sigma(X_s)(\hat{\sigma}X_s)^{-1}\hat{\sigma}X_sM_{1-s}f(X_s) \right|^2 ds
\]

Note that, from Lemma 3, the process

\[
\left( \frac{\hat{\sigma}X_sM_{1-s}f(X_s)}{Q_{1-s}f(X_s)}, s \in [0, 1]\right)
\]

is a \( \gamma \)-martingale, consequently we have

\[
H(\gamma|P) \leq E\gamma \left[ \frac{1}{f(X_1)}|\hat{\sigma}X_1^*\hat{\sigma}f(X_1)|^2 \right]
\]

where \( Z \) is defined in the announcement of Theorem 14.

Left side of the equation (7.34) depends only on the law of the random variable \( X_1(x) \), but at the right hand side there is the term \( \Gamma(x, y) \) which is defined as

\[
\Gamma(x, y) = E[Z\hat{\sigma}X_1(x) \cdot \hat{\sigma}X_1^*(x)|X_1(x) = y],
\]

where

\[
Z(x) = \int_0^1 |\sigma^*(X_1(x))|\hat{\sigma}X_1^*(x)^{-1}|^2 dt.
\]

Consequently, the right hand side of the inequality (7.34) depends rather explicitly on \( \sigma \), while the left hand side depends only on \( a = \sigma\sigma^* \), hence only on the law of the process. Meanwhile, this inequality remains valid for any \( \sigma \) which is \( n \times d \)-matrix valued function satisfying the relation \( a = \sigma\sigma^* \). In particular the second dimension \( d \) is not fixed. Let us define

\[
\Sigma_a = \cup_{d \geq 1} \{ \sigma \in C^\infty_0(\mathbb{R}^n, M(n \times d)) : \sigma\sigma^* = a \},
\]

where \( M(n \times d) \) denotes \( n \times d \)-matrices.

**Corollary 8.** Let us define the functional \( Q(\partial f, \partial f) \) for \( f \in C^\infty_0(\mathbb{R}^n) \) as

\[
\inf_{\sigma \in \Sigma_a} \int_{\mathbb{R}^n} \Gamma_{\sigma}(x, y)\partial f(y), \partial f(y))_{\mathbb{R}^n}q_1(x, y)dy.
\]
Let \( \nu_x^\tau \) be the law of the random variable \( \omega \to X_1(x, \omega) \). Then it holds true that
\[
\int_{\mathbb{R}^d} f^2(y) \log \frac{f^2(y)}{\nu_x^\tau(dy)} \nu_x^\tau(dy) \leq Q(\partial f, \partial f).
\]

**Remark:** Assume that, for any smooth, upper and lower bounded function \( f \) on \( \mathbb{R}^n \), we have the following property: the process
\[
(t, w) \to \frac{\|\sigma(X_t)\partial Q_1 - t f(X_t)\|^2}{Q_1 - t f(X_t)}
\]
is a submartingale, then, replacing \( f \) by \( \sqrt{f} \), we have the following version of logarithmic Sobolev inequality, which follows from the Itô formula:
\[
E[f(X_1)^2 \log \frac{f(X_1)^2}{E[f(X_1)]^2}] \leq 2E[\|\sigma(X_1)\partial f(X_1)\|^2],
\]
for any smooth function \( f \).

### 8. Conditionally Orthogonal Functionals and Inequalities

Let us begin with the general case of the stochastic differential equation without Markov coefficients, i.e., the \( b \) and \( \sigma \) may depend in a causal manner on the Wiener path explicitly.

Let us define by \( R_s(X) \) the orthogonal projection defined as
\[
R_s(X) = I_{\mathbb{R}^d} - P_s(X),
\]
and let \( \mathbb{R}(X) \) be its action on the \( H \)-valued functions:
\[
\mathbb{R}(X)h(t) = \int_0^t R_s(X)\dot{h}(s)ds.
\]

This projection operator gives birth to a new concept of stochastic process which is intimately related to the degeneracy of the diffusion process \( X = (X_t, t \in [0,1]) \). Let us begin by

**Theorem 16.** Let \( u \in L_a^2(\mu, H) \), define
\[
L = \exp \left( \int_0^1 (R(X_s)\dot{u}_s, dB_s) - \frac{1}{2} \int_0^1 |R(X_s)\dot{u}_s|^2 ds \right)
\]
and assume that \( E_{\mu}[L] = 1 \). Then we have
\[
E_{\mu}[L|\mathcal{F}_1(X)] = 1
\]
\( \mu \)-almost surely. In particular, if \( u \) is \( \mathcal{F}_1(X) \)-measureable, then
\[
E\left[ \exp \left( \int_0^1 (R(X_s)\dot{u}_s, dB_s) \right) |\mathcal{F}_1(X) \right] = \exp \frac{1}{2} \int_0^1 |R(X_s)\dot{u}_s|^2 ds,
\]
hence under the conditional probability \( \mu^X \), defined by \( \mu^X(\cdot) = \mu(\cdot |\mathcal{F}_1(X)) \),
\[
\int_0^1 (R(X_s)\dot{u}_s, dB_s)
\]
is a Gaussian random variable with mean zero and variance \( \mathbb{E}(X)u_0^2_H \). In particular, the process
\[
N_t = \int_0^t R_s(X)dB_s
\]
is an additive (i.e., with independent increments) Gaussian martingale under the conditional probability \( \mu^X \).
Proof: Let $L_t = \exp \left( \int_0^t (R(X_s)\dot{u}_s, dB_s) - \frac{1}{2} \int_0^t |R(X_s)\dot{u}_s|^2 ds \right)$, from the Itô formula, we have
\[ L_t = 1 + \int_0^t L_s(R_s(X)\dot{u}(s), dB_s). \]
From Theorem 3, we have
\[ (8.38) \quad E[L_1|\mathcal{F}_1(X)] = 1 + \int_0^1 (P_s(X), R_s(X)E[\dot{u}(s)L_s|\mathcal{F}_s(X)], dB_s) = 1 \]
a.e.
almost surely. If $\dot{u}$ is adapted to the filtration $\mathcal{F}(X)$, from the relation $(8.38)$, we obtain
\[ E \left[ \exp \lambda \int_0^t (R(X_s)\dot{u}_s, dB_s)|\mathcal{F}(X) \right] = \exp \frac{\lambda^2}{2} \int_0^t |R_s(X)\dot{u}_s|^2 ds \]
a.e., for any $\lambda \in \mathbb{R}$. The last claim follows from the fact that, for any $h, k \in H$,
\[ E[\exp \int_0^1 (R_s(X)\dot{h}(s) + R_s(X)\dot{k}(s), dB_s)|\mathcal{F}_1(X)] = \exp \frac{1}{2} \int_0^1 (|R_s(X)\dot{h}(s)|^2 + |R_s(X)\dot{k}(s)|^2) ds \]
a.e. if $\dot{h}$ and $\dot{k}$ have disjoint supports. \hfill \Box

Proposition 6. Let $F = F(X, N)$ be in $L^2(\mu)$ be a cylindrical function, define $\nabla_h^{(2)} F(X, N)$ as
\[ \nabla_h^{(2)} F(X, N) = \lim_{\varepsilon \to 0} \frac{F(X, N + \varepsilon h) - F(X, N)}{\varepsilon}, \]
for $h \in H$ and $D_r^{(2)} F(X, N)$ as
\[ \nabla_h^{(2)} F(X, N) = \int_0^1 (D_r^{(2)} F(X, N), \dot{h}(\tau))_{\mathbb{R}^d} d\tau. \]
We have then
\[ (8.39) \quad F(X, N) = E[X][F] + \int_0^1 (R_\tau(X)E[X][D_r^{(2)} F(X, \cdot)|N_s], dB_s), \]
where $(N_s, s \in [0, 1])$ is the filtration generated by $N = (N_s, s \in [0, 1])$ and $E[X]$ is the conditional expectation corresponding to $\mu^X$. Consequently the following conditional log-Sobolev inequality holds true:
\[ E[X] \left[ F^2 \log \frac{F^2}{E[X]F^2} \right] \leq 2E[X][|\mathbb{R}\nabla^{(2)} F|_H^2]. \]
Proof: As the process $N$ is conditionally an additive Gaussian process it has martingale representation property under the law $\mu^X$. Besides, from Theorem 3, we obtain the following conditional integration by parts formula
\[ (8.40) \quad E[X] \left[ F(X, N) \int_0^1 (R_s(X)\dot{u}_s(X, N), dB_s) \right] = E[X] \left[ \int_0^1 (R_s(X)\dot{u}_s(X, N), D_s^{(2)} F(X, N)) ds \right], \]
for any bounded, and $\mathcal{N}$-adapted process $\dot{u}(X, \cdot)$. This implies the representation $(8.39)$ by conditioning. The rest of proof follows from the relation $(8.40)$ by using the usual techniques. \hfill \Box

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