PSEUDO-POLYNOMIAL TIME ALGORITHM FOR COMPUTING MOMENTS OF POLYNOMIALS IN FREE SEMICIRCULAR ELEMENTS

REI MIZUTA

Abstract. We consider about calculating $M$th moments of a given polynomial in free independent semicircular elements in free probability theory. By a naive approach, this calculation requires exponential time with respect to $M$. We explicitly give an algorithm for calculating them in polynomial time by rearranging Schützenberger’s algorithm.

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1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $X, Y \in L^1(\Omega)$ be two independent $\mathbb{R}$-valued random variables whose means are zero. Cramér’s theorem [4] states that the sum of these two random variables follows a normal distribution if and only if both of $X$ and $Y$ are so.

On the other hand, it is known that this theorem does not have counterparts in free probability theory [1] and there is an attempt to obtain the same result as this theorem in free probability in fixed Wigner chaos [3]. We continue this attempt for polynomials in free independent semicircular elements, so our setting is as follows.

Question 1.1. Let $s_1, s_2$ be two free independent standard semicircular elements, and $p(X), q(X)$ be two polynomials of one variable such that both of them are not constant. Does $p(s_1) + q(s_2) \sim S(0, 1)$ imply $p(s_1) \sim S(0, \sigma_2^2)$ and $q(s_1) \sim S(0, \sigma_1^2)$ for some $\sigma_1, \sigma_2 > 0$?

Here $a \sim S(0, 1)$ means that the spectral density of $a$ is equivalent to the semicircular density (3.1) for any operator $a$ and $a \sim S(0, \sigma^2)$ means $a/\sigma \sim S(0, 1)$ for any positive real number $\sigma$. The above problem is generalized as follows.

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**Question 1.2.** Let $s_1, s_2, \ldots, s_n$ be free independent standard semicircular elements and $p(X_1, X_2, \ldots, X_n)$ be a non-commutative polynomial of $n$ variables. If $p(s_1, s_2, \ldots, s_n) \sim S(0, 1)$, is $p$ $\mathbb{R}$-linear?

When we try to solve this problem for a given $p(X_1, X_2, \ldots, X_n)$, the following strategy is available: calculating $p(s_1, s_2, \ldots, s_n)$’s moments and comparing them with that of a standard semicircular element. Therefore the following subtask is important.

**Question 1.3.** In the setting of Question 1.2, can we calculate $M$-th moment of $p(s_1, s_2, \ldots, s_n)$ in practical time?

While doing a naive calculation, expanding $M$-th power to $(m_p)^M$ monomials and taking summation of expectation of them, where $m_p$ is the number of monomials which appear in $p$, the computational time costs exponential time with respect to $M$. We give a practical tool for this subtask, Question 1.3, by giving an algorithm which calculates the output of Question 1.3 in polynomial time with respect to $M$ by using Schützenberger’s algorithm [8].

In this paper, we introduce some related work about Question 1.2 and operator algebraic research which uses Jungen’s theorem in Chapter 2. In Chapter 3, we prepare some preliminaries about free probability theory and Jungen’s theorem. Finally, we show our algorithm in Chapter 4.

2. Related Work

In this section, we firstly introduce some related work about Question 1.2 in the previous chapter, secondly introduce some conventional work which uses Schützenberger’s work[8] for operator algebras.

2.1. Polynomial Identification Problem. In this subsection, we introduce some conventional work which gives a partial solution for the Problem 1.2 which is defined in the previous chapter. We mention a result in the setting of fixed Wigner chaos [5].

**Theorem 2.1 ([5, Corollary 1.7]).** Let $m \geq 2$ be a positive integer and $f \in L^2(\mathbb{R}^m)$ be a mirror-symmetric function. Then the fourth cumulant of Wigner integral $I_m(f)$ is positive unless $f = 0$ a.e.

The Wigner integral $I_m(f)$ is defined in [2, Definition 5.3.1], also mirror-symmetric is defined in [5, Definition 1.19] where this condition is equivalent to self-adjointness of $I_m(f)$.

Let $(T_k)_{k \geq 0} \subseteq \mathbb{R}[X]$ be the Chebyshev polynomial of second type [2, Chapter 5.1] and we define an operator $s_k$ as $s_k := I_1(1_{[k-1,k]})$ for each positive integer $k$. Since $T_m(s_k) = I_m(1_{[k-1,k]}^{[m]})$ holds by an argument in the proof of [2, Theorem 5.3.4],

$$T_{k_1}(s_{i_1}) \cdots T_{k_N}(s_{i_N}) = I_{k_1}(1_{[i_1-1,i_1]}^{[k_1]}) \cdots I_{k_N}(1_{[i_N-1,i_N]}^{[k_N]}) = I_m(1_{[i_1-1,i_1]}^{[k_1]} \otimes 1_{[i_2-1,i_2]}^{[k_2]} \otimes \cdots \otimes 1_{[i_N-1,i_N]}^{[k_N]})$$

holds for any positive integer $m$ and $k$ by the product formula of Wigner chaszes [2, Proposition 5.3.3]. Then following Corollary holds.

**Corollary 2.2.** Let $\mathbb{C}(X_1, X_2, \ldots, X_n)_{s.a.}$ is the collection of all self-adjoint non-commutative polynomials and $m \geq 2$ be a positive integer and

$$p \in \text{span}_\mathbb{R}\{T_{k_1}(X_{i_1}) \cdots T_{k_N}(X_{i_N}) \mid N \in \mathbb{N}, 1 \leq i_1, i_2, \ldots, i_N, k_1, k_2, \ldots, k_N \leq n$$

such that $k_1 + k_2 + \ldots + k_N = m, i_j \neq i_{j+1}$ for $j = 1, 2, \ldots, N - 1 \} \cap \mathbb{C}(X_1, X_2, \ldots, X_n)_{s.a.}$ be a non-commutative polynomial. Then $p(s_1, s_2, \ldots, s_n)$ is not a standard semicircular element.
However, the positivity of the fourth cumulant fails for linear combination of different chaoses. For example
\[
\kappa_4(I_3(1_{[0,1]}^3) - 2I_1(1_{[0,1]})) = \kappa_4(s_1^3 - 3s_1) = -2 < 0
\]
holds. So we cannot extend this argument to a general polynomial for solving Question 1.2.

2.2. Schützenberger’s Work in Operator Algebra. We also remark on the work of conventional work which uses Schützenberger’s work\[8\] for the region of operator algebras. In [7], Sauer proves the rationality of Novikov-Shubin invariants in $ZG$ if $G$ is a virtually free group. In [9], Shlyakhtenko and Skoufranis prove the non-atomicness of spectral distribution of polynomials in free independent semicircular elements.

We remark that these pieces of conventional research use only the existence of a proper algebraic system of a certain operator, which is defined in Definition 3.6, so they do not focus on the algorithm for obtaining a proper algebraic system which is suggested in [8].

3. Preliminaries

We begin brief preliminaries on free probability theory and Jungen’s theorem.

3.1. Free Probability. In this subsection, we prepare a background about free probability theory. For any von Neumann algebra $M$, we denote the collection of all self-adjoint operators in $M$ by $M_{s.a.}$.

**Definition 3.1.** Let $M$ be a von Neumann algebra and $\tau : M \to \mathbb{C}$ be a faithful normal tracial state. The pair $(M, \tau)$ is called $W^*$-probability space.

For any $a \in M_{s.a.}$, we define spectral distribution of $a$ as the unique probability distribution $\mu$ such that
\[
\tau(a^m) = \int_{\mathbb{R}} x^m d\mu(x)
\]
for any positive integer $m$, and denote it by $\mu_a$.

We call operators $x_1, x_2, \ldots, x_n \in M_{s.a.}$ are free independent if $\tau(p_1(x_{i_1})p_2(x_{i_2})\ldots p_N(x_{i_N})) = 0$ for all positive integer $N, 1 \leq i_1, i_2, \ldots, i_N \leq n$ and $p_1, p_2, \ldots, p_N \in \mathbb{C}[X]$ such that $\tau(p_j(x_{i_j})) = 0$ for any $1 \leq j \leq N$ and $i_k \neq i_{k+1}$ for all $1 \leq k \leq N - 1$.

**Definition 3.2.** Let $(M, \tau)$ be a $W^*$-probability space, an operator $s \in M_{s.a.}$ is called a standard semicircular element if its moments are given by
\[
\tau(s^m) = \begin{cases} 0 & m \text{ : odd} \\ \frac{1}{m/2+1} p_m C_m/2 & m \text{ : even}. \end{cases}
\]

**Remark 3.3.** In the setting of Definition 3.1, $s \in M_{s.a.}$ is a standard semicircular element if and only if
\[
d\mu_s = \frac{\sqrt{4 - x^2}}{2\pi} 1_{[-2,2]}(x)dx.
\]

We denote the above condition by $s \sim S(0,1)$, and we call the probability density function in right hand side of (3.1) the semicircular density.
Remark 3.4. By above Definition 3.1, \( \tau(p(x_1, x_2, ..., x_n)) \) is uniquely determined by the moments of \( (x_i)_{1 \leq i \leq n} \). In particular, by the arguments in [6, Chapter 1],

\[
\tau(x^M) = \sum_{\pi \in NC(M)} \kappa_\pi(x) \tag{3.2}
\]

holds for any \( x \in M_{s.a.} \), where \( NC(M) \) means the collection of all non-crossing partition of \( \{1, 2, ..., M\} \) [6, Chapter 1.8] and \( \kappa_\pi \) is multiplication of cumulants of \( x \) which is defined in [6, Chapter 2.2, Definition 8].

However, if \( x \) in the left hand side of (3.2) takes a polynomial in free independent operators, a summand of the right hand side of (3.2) becomes \( m^M \) numbers of multiplications of cumulants for a fixed \( \pi \in NC(M) \) where \( m := \#\pi \) is the number of block in \( \pi \). So it takes exponential time complexity with respect to \( M \) for computing (3.2), while we expand the right hand side of (3.2) as multiplications of cumulants of monomial appeared in powers of \( x \) and take summation of them.

Remark 3.5. For any positive integer \( n \geq 2 \), there is a \( W^* \)-probability space which has \( n \) free independent standard semicircular elements. Let \( F_n \) be the free group of rank \( n \). Then the free group factor \( \mathcal{L}(F_n) \) is defined as the weak closure of the image of the left regular representation in \( B(l_2(F_n)) \) and has the unique faithful normal trace \( \tau \). Then for each \( 1 \leq i \leq n \), there exists a standard semicircular elements \( s_i \in (\lambda_n)^\tau \subseteq \mathcal{L}(F_n) \) (6, Chapter 6) and hence \( s_1, s_2, ..., s_n \) are free independent, where \( a_1, ..., a_n \in F_n \) are the generators of free group and \( \lambda_g \) is the left regular representation of \( g \in F_n \).

3.2. Junger’s Theorem. In this subsection, we give a preliminary on Schützenberger’s work about Junger’s theorem [8].

Let \( R \) be a unital ring (possibly non-commutative), \( X = \{X_1, X_2, ..., X_n\} \) be a finite set and \( F(X) \) be the free monoid generated by \( X \). We denote the free \( R \)-algebra generated by \( F(X) \) by \( R(X) \). We also denote the \( R \)-coefficients formal power series generated by \( F(X) \) by \( R(\langle \langle X \rangle \rangle) \).

We consider \( R(X) \) as a subring of \( R(\langle \langle X \rangle \rangle) \) by the natural inclusion. For any \( F \in F(X) \) and \( p \in R(\langle \langle X \rangle \rangle) \), we denote the coefficient of \( F \) in \( p \) by \( \text{cf}(p; F) \). For any \( p \in R(\langle X \rangle) \) and \( r_1, r_2, ..., r_n \in R \), \( p(r_1, r_2, ..., r_n) \in R \) means the substitution of \( X_1, X_2, ..., X_n \) respectively for \( r_1, r_2, ..., r_n \).

Definition 3.6. We define the rational closure \( R_{\text{rat}}(\langle \langle X \rangle \rangle) \subseteq R(\langle \langle X \rangle \rangle) \) as the smallest subring of \( R(\langle \langle X \rangle \rangle) \) which contains \( \{p^{-1} \in R(\langle \langle X \rangle \rangle) \mid p \in R(\langle X \rangle) \} \) and invertible in \( R(\langle \langle X \rangle \rangle) \).

We also define the algebraic closure \( R_{\text{alg}}(\langle \langle X \rangle \rangle) \subseteq R(\langle \langle X \rangle \rangle) \) as the all collection of \( p \in R(\langle \langle X \rangle \rangle) \) which has a proper algebraic system, where \( p \) has a proper algebraic system if

1. There are \( Q_1, ..., Q_L \in R(X \coprod Y) \) with a finite set \( Y = \{Y_1, ..., Y_L\} \) such that \( \text{cf}(Q_i; Y_j) = 0 \) is satisfied for all \( 1 \leq i, j \leq L \) for some \( L \in \mathbb{N} \).
2. There are \( p_1, ..., p_L \in R(\langle \langle X \rangle \rangle) \) such that \( p = p_1 \) and \( p_i = Q_i(X_1, ..., X_n, p_1, ..., p_L) \) are satisfied for each \( 1 \leq i \leq L \).

Let \( \mathcal{I} : R(\langle X \rangle) \rightarrow (\langle X \rangle) \) is a homomorphism of \( R \)-module such that it sends \( p \in R(\langle X \rangle) \) to \( p - \text{cf}(p; e) e \). We also define \( R^*(\langle X \rangle), R^*_{\text{alg}}(\langle X \rangle), R^*_{\text{rat}}(\langle X \rangle) \) and \( R^*_{\text{alg}}(\langle X \rangle) \) respectively as the \( \mathcal{I} \)’s range of \( R(\langle X \rangle), R_{\text{rat}}(\langle X \rangle) \) and \( R_{\text{alg}}(\langle X \rangle) \).

Remark 3.7. All element \( p \in R^*_{\text{rat}}(\langle X \rangle) \) can be obtained from \( X_1, X_2, ..., X_n \in R^*_{\text{rat}}(\langle X \rangle) \) via finite composition of following procedures [8].

- (pseudo-inverse) \( a \in R^*_{\text{rat}}(\langle X \rangle) \mapsto a^* := \sum_{k=1}^{\infty} a^k \)
- (linear combination) \( r_1, r_2 \in R, a, b \in R^*_{\text{rat}}(\langle X \rangle) \mapsto r_1 a + r_2 b \)
- (multiplication) \( a, b \in R^*_{\text{rat}}(\langle X \rangle) \mapsto ab \)

In addition, the algebraic closure \( R_{\text{alg}}(\langle X \rangle) \) is a subring of \( R(\langle X \rangle) \) [8].

Remark 3.8. so we also say \( p \in R(\langle X \rangle) \) has a proper algebraic system if there are \( Q_1, ..., Q_L \) such that their unique solution \( p_1, ..., p_L \) satisfies the condition 2 in Definition 3.6.
Next theorem is prepared for proving an analytic property of Cauchy transforms of polynomial in free independent semicircular elements [9].

**Theorem 3.9** ([9, Lemma 5.12]). We define \( P_{semi} \in \mathbb{C}^*\langle\langle X \rangle\rangle \) as

\[
P_{semi} := \sum_{F \in F(X), F \neq e} \tau(F(s_1, s_2, ..., s_n))F.
\]

Then \( P_{semi} \) is an element of \( \mathbb{C}^*_{alg}\langle\langle X \rangle\rangle \) and whose proper algebraic system can be taken as \( Q_1 := \sum_{i=1}^{n}(X_i(Y_i + 1))^2 \) with \( L = 1 \).

We remark that our definition of \( P_{semi} \) is slightly different from [9]. We defined \( P_{semi} \) as an element of \( \mathbb{C}^*\langle\langle X \rangle\rangle \) and so their difference is only coefficients of the unit of \( F(X) \).

**Definition 3.10.** Let \( a, b \) are elements in \( R\langle\langle X \rangle\rangle \), we denote the Hadamard product of \( a \) and \( b \) by \( a \odot b \) which is the unique element in \( R\langle\langle X \rangle\rangle \) defined as \( cf(a \odot b; F) = cf(a; F)cf(b; F) \) for any \( F \in F(X) \).

We introduce next Jungen’s theorem which are rearranged by Schützenberger in [8].

**Theorem 3.11** ([8, Property 2.2]). Let \( R', R'' \subseteq R \) be commuting subalgebras of \( R \) and \( a \in R'_{rad}\langle\langle X \rangle\rangle, b \in R''_{alg}\langle\langle X \rangle\rangle \) be two elements of \( R'\langle\langle X \rangle\rangle \), then \( a \odot b \) is an element of \( R''_{alg}\langle\langle X \rangle\rangle \).

4. ALGORITHM

Let \( X = \{X_1, X_2, ..., X_n\} \) be a finite set, \( p \in \mathbb{C}(X_1, X_2, ..., X_n) \) be a non-commutative polynomial and \( M \) be a positive integer. In this chapter, we give an algorithm which calculates the \( M \)-th moment of \( p(s_1, s_2, ..., s_n) \) where \( s_1, s_2, ..., s_n \) are free independent standard semicircular elements.

A sketch of our algorithm is the following. Assume \( p \) is an element of \( \mathbb{C}^*\langle\langle X \rangle\rangle \). Since an element \( \sum_{m \geq 1} z^m p(X_1, X_2, ..., X_n)^m = (zp(X_1, X_2, ..., X_n))^\ast \) is in \( \mathbb{C}^*\langle\langle X \rangle\rangle \) by an argument in Remark 3.7, we can apply Jungen’s Theorem (Theorem 3.11) in previous chapter for \( R = R' = \mathbb{C}[z], R'' = \mathbb{C}, a = (zp(X_1, X_2, ..., X_n))^\ast \) and \( b = P_{semi} \) which is defined in Theorem 3.9. We then substitute each \( X_1, X_2, ..., X_n \) for 1 and obtain \( a \odot b(1, 1, ..., 1) =: A(z) \in \mathbb{C}^*[z] \) which is well-defined. Since \( cf(A(z); z^m) = \tau(p(s_1, ..., s_n)^m) \) for any \( m \geq 1 \), all we have to do is calculating \( cf(A(z); z^M) \), but this can be done by iterating a proper algebraic system of \( a \odot b \in \mathbb{C}[z]/(z^{M+1}) \) instead of \( \mathbb{C}[z]\langle\langle X \rangle\rangle \) by sending elements as \( \mathbb{C}[z]\langle\langle X \rangle\rangle \ni f \mapsto f(1, 1, ..., 1)/(z^{M+1}) \in \mathbb{C}[z]/(z^{M+1}) \).

We explicitly give the procedures of the algorithm as follows.

**Step 1** (Split \( p \) into \( \mathbb{C}^*\langle\langle X \rangle\rangle \) and \( \mathbb{C} \)). Let \( c := p(0, 0, ..., 0) \in \mathbb{C} \) be the constant part of \( p \). Then we denote the reminder part by \( q := p - c \in \mathbb{C}^*\langle\langle X \rangle\rangle \).

**Step 2** (Encode \( (zq)^\ast \) as a tuple of matrices). For obtaining a proper algebraic system of \( (zq)^\ast \odot P_{semi} \) by using Jungen’s theorem, we encode \( (zq)^\ast \) as a monoid homomorphism by the argument in [8]. Let \( M_N(R) \) be the \( N \times N \)-matrix algebra over \( R \).

**Proposition 4.1** ([8, Property 2.1]). Assume \( a \in R^*\langle\langle X \rangle\rangle \), the following are equivalent.

1. \( a \in R'_{rad}\langle\langle X \rangle\rangle \)
2. There are a positive integer \( N \geq 2 \) and a monoid homomorphism \( \mu : F(X) \to M_N(R) \) such that \( cf(a; F) = \mu(F)_{1,N} \) for any \( F \in F(X) \).

We review on the constructive part of a proof in [8, Property 2.1] for evaluating the time complexity of our algorithm.
Proof in [8, Property 2.1]. Assume \( a \in R^*_{\text{rat}}(\langle X \rangle) \). All we have to do is constructing associated monoid homomorphism which satisfies 2 by induction on the structure of \( a \) in Remark 3.7. Since \( F(X) \) is a free monoid, a monoid homomorphism \( \mu : F(X) \to M_N(R) \) is uniquely determined by ranges of generators \( X_1, X_2, \ldots, X_n \).

If \( a \) is given by \( a = X_i \in R^*_{\text{rat}}(\langle X \rangle) \) for some \( 1 \leq i \leq n \), a monoid homomorphism \( \mu \) which satisfies 2 can be obtained with \( N = 2 \) as

\[
\begin{align*}
\mu(X_i) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
\mu(X_j) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (i \neq j).
\end{align*}
\]

Then we assume there is a \( a' \in R^*_{\text{rat}}(\langle X \rangle) \) with a monoid homomorphism \( \mu' : F(X) \to M_{N'}(R) \) which satisfies 2. Then \( a := (a')^* \), the pseudo-inverse of \( a' \), is in \( R^*_{\text{rat}}(\langle X \rangle) \), and a monoid homomorphism \( \mu \) which satisfies 2 can be obtained with \( N = N' \) as

\[
\mu(X_i) = \begin{pmatrix} \mu'(X_{i,1},N) & \mu'(X_{i,2},N) & \mu'(X_{i,3},N) & \cdots & \mu'(X_{i,N},N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu'(X_{i,N,N}) & \mu'(X_{i,N,2}) & \mu'(X_{i,N,3}) & \cdots & \mu'(X_{i,N,N}) \end{pmatrix}
\quad \text{for any } 1 \leq i \leq n. \quad (4.1)
\]

Finally, we assume there are two rational elements \( a', b' \in R^*_{\text{rat}}(\langle X \rangle) \) with two monoid homomorphism \( \mu'_1 : F(X) \to M_{N'_1}(R), \mu'_2 : F(X) \to M_{N'_2}(R) \) which satisfy 2 respectively.

Let \( r_1, r_2 \in R \) be two elements. Then the linear combination \( a := r_1a' + r_2b' \) is in \( R^*_{\text{rat}}(\langle X \rangle) \) and a monoid homomorphism \( \mu \) which satisfies 2 can be obtained with \( N = N_1 + N_2 + 2 \) as

\[
\mu(X_i) = \begin{pmatrix} 0 & Z_{1,1}^1 & \cdots & Z_{1,N_1}^1 & W_{1,1}^1 & \cdots & W_{1,N_2}^1 & Z_{1,N_1}^1 + W_{1,N_2}^1 \\ 0 & Z_{1,1}^1 & \cdots & Z_{1,N_1}^1 & 0 & \cdots & 0 & Z_{1,N_1}^1 \\ 0 & Z_{2,1}^1 & \cdots & Z_{2,N_1}^1 & 0 & \cdots & 0 & Z_{2,N_1}^1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & Z_{N_1,1}^1 & \cdots & Z_{N_1,N_1}^1 & 0 & \cdots & 0 & Z_{N_1,N_1}^1 \\ 0 & 0 & \cdots & 0 & W_{1,1}^2 & \cdots & W_{1,N_2}^2 & W_{1,N_2}^2 \\ 0 & 0 & \cdots & 0 & W_{2,1}^2 & \cdots & W_{2,N_2}^2 & W_{2,N_2}^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & W_{N_2,1}^2 & \cdots & W_{N_2,N_2}^2 & W_{N_2,N_2}^2 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},
\]

for any \( 1 \leq i \leq n \), where \( Z^i := r_1\mu'_1(X_i) \) and \( W^i := r_2\mu'_2(X_i) \).

The multiplication \( a := a'b' \) is in \( R^*_{\text{rat}}(\langle X \rangle) \) and a monoid homomorphism \( \mu \) which satisfies 2 can be obtained with \( N = N_1 + N_2 \) as
for any $1 \leq i \leq n$, where $Z_i := \mu'_i(X_i)$ and $W^i := \mu'_2(X_i)$.

Therefore we can obtain a monoid homomorphism associated with $(zq)^\ast$ by constructing that of $zq$ and take pseudo-inverse via (4.1).

Remark 4.2. By the above construction, the size $N$ of an associated monoid homomorphism of $(zq)^\ast$ is estimated to be less than equal $2m_q \cdot (d + q) + 2m_q$ for given $q$ in Step 1, where $m_q$ is defined in Chapter 1.

Step 3 (Make a proper algebraic system of $(zq)^\ast \circ P_{\text{semi}}$). We review the proof of [8, Property 2.2] which gives a construction of a proper algebraic system of the Hadamard product in Theorem 3.11. The following theorem is a combination of [8, Property 2.2] and [9, Lemma 5.12].

Theorem 4.3 ([8, Property 2.2],[9, Lemma 5.12]). Assume $a \in R_{\text{rat}}^\ast ((X))$ and a monoid homomorphism $\mu : F(X) \to M_N(R^\ast)$ satisfies 2 in Proposition 4.1. Then $a \circ P_{\text{semi}} \in R_{\text{alg}}^\ast ((X))$ and its proper algebraic system can be taken with $L = N^2$ as

$$(Q_1, \ldots, Q_{N^2}) := (Q'_{1,N}, Q'_{\sigma_1,1}, \ldots, Q'_{\sigma_{N^2-1},1})$$

where $\sigma_1, \ldots, \sigma_{N^2-1}$ is any permutation of $\{(i,j) \mid 1 \leq i, j \leq N\} \setminus \{(1,N)\}$, and $Q' \in M_N(R(X \coprod Y))$ is defined as

$$Q' = \sum_{i=1}^{n} (\mu(X_i)(Y + I_{N \times N}))^2,$$

where $Y \in M_N(R(X \coprod Y))$ is defined as $Y_{i,j} = Y_{i+(j-1)N}$ for each $1 \leq i, j \leq N$ and $I_{N \times N}$ is the multiplication unit of $M_N(R(X \coprod Y))$.

So we can obtain a proper algebraic system of $(zq)^\ast \circ P_{\text{semi}}$ with size $L = N^2$ and it can be written as (4.2).

Step 4 (Iterate the proper algebraic system in Step 3 for $(d + p)M$ times). In this step, we calculate $\text{cf}((zq)^\ast \circ P_{\text{semi}}(1,1,\ldots,1); z^m) = \tau(q(s_1, s_2, \ldots, s_n)^m)$ for each positive integer $m \leq M$ by using the proper algebraic system in Step 3. Let $\pi : \mathbb{C}[z] \to \mathbb{C}[z]/(z^{M+1})$ be the quotient map.

Definition 4.4. We say $a \in \mathbb{C}[z]((X))$ is good if

$$\{F \in F(X) \mid \text{cf}((a;F); z^m) \neq 0\}$$

is finite set for any non-negative integer $m$. We denote the collection of all good elements in $\mathbb{C}[z]((X))$ by $\mathbb{C}[z]_{\text{good}}((X))$. 
We define $\varphi : \mathbb{C}[z]_{\text{good}}(\langle X \rangle) \to \mathbb{C}[z]$ by defining $\varphi(a)$ for each good $a$ as a unique element such that

$$\operatorname{cf}(\varphi(a); z^m) = \sum_{F \in F(X)} \operatorname{cf}(\varphi(a); z^m)$$

holds for any non-negative integer $m$.

**Remark 4.5.** All $a \in R(X) \subseteq R(\langle X \rangle)$ are good and $a(1, 1, \ldots, 1) = \varphi(a)$ holds. For any good $a$ and non-negative integer $m$, summands of the right hand side of (4.3) are zero except for finite monomials $F$.

We also remark that $(zq)^* \circ P_{\text{semi}} \in \mathbb{C}[z]_{\text{alg}}^*(\langle X \rangle)$ is good since summands of the right hand side of (4.3) for $a = (zq)^* \circ P_{\text{semi}}$ are zero except for monomials $F$ such that $\operatorname{cf}(q^m; F) \neq 0$. Then

$$\operatorname{cf}(\varphi((zq)^* \circ P_{\text{semi}}); z^m) = \sum_{F \in F(X)} \operatorname{cf}((zq)^* \circ P_{\text{semi}}; z^m)$$

for any positive integer $m$ since $\operatorname{cf}(q^m; e) = 0$ holds by $\operatorname{cf}(q; e) = 0$.

Assume $a \in \mathbb{C}[z]_{\text{alg}}^*(\langle X \rangle)$ and a proper algebraic system of $a$ is given as $Q_1, \ldots, Q_L \in \mathbb{C}[z] \langle X \coprod Y \rangle$. We define $((a_1^m, \ldots, a_L^m))_{m \geq 0} \subseteq (\mathbb{C}[z] \langle X \rangle)^L$ as

$$(a_1^0, \ldots, a_L^0) = (0, \ldots, 0),$$

$$(a_1^{m+1}, \ldots, a_L^{m+1}) = (Q_1(X_1, \ldots, X_n, a_1^m, \ldots, a_L^m), \ldots, Q_L(X_1, \ldots, X_n, a_1^m, \ldots, a_L^m))$$

for $m \geq 0$.

**Theorem 4.6.** Let $M$ be a positive integer and $a \in \mathbb{C}[z]_{\text{alg}}^*(\langle X \rangle)$ be a good element as above. Then $\operatorname{cf}(\varphi(a); z^m) = \operatorname{cf}(\varphi(a_1^m); z^m)$ holds for any $m \leq M$ such that

$$\max_{F \in F(X)} \{\deg F \mid 0 \leq m \leq M, \operatorname{cf}(a; F); z^m \neq 0\} \leq M'.$$  \hspace{1cm} (4.4)

Moreover, the left hand side of (4.4) is finite for any positive integer $M$.

**Proof.** Assume $a \in \mathbb{C}[z]_{\text{alg}}^*(\langle X \rangle)$ is good and $((a_1^m, \ldots, a_L^m))_{m \geq 0} \subseteq (\mathbb{C}[z] \langle X \rangle)^L$ is defined as above. By an argument in [8], $\operatorname{cf}(a; F) = \operatorname{cf}(a_1^m; F)$ holds for any positive integer $m'$ and $F \in F(X)$ such that $\deg F \leq m'$. So
\[
\text{cf}(\phi(a); z^m) = \sum_{F \in F(X)} \text{cf}(\text{cf}(a; F); z^m)
\]

\[
= \sum_{F \in F(X), \text{cf}(\text{cf}(a; F); z^m) \neq 0} \text{cf}(a; F); z^m
\]

\[
= \sum_{F \in F(X), \deg F \leq M'} \text{cf}(a; F); z^m
\]

\[
= \sum_{F \in F(X), \deg F \leq M'} \text{cf}(a^{M'}_i; F); z^m
\]

\[
= \text{cf}(\phi(a^{M'}_i); z^m)
\]

holds for any non-negative integer \(m \leq M\). Therefore the former statement holds.

The latter statement holds since \(\text{cf}(\text{cf}(a; F); z^m) \neq 0\) holds only for finite monomials \(F \in F(X)\) for any non-negative integer \(m\) by goodness of \(a\). \(\square\)

Since the left hand side of (4.4) is less than equal \((\deg p)M\) for \(a = (zq)^* \circ P_{\text{semi}},\)

\[
\text{cf}(\phi(a); z^m) = \text{cf}(\phi(\text{cf}(a^{(\deg p)M}_1); z^m)
\]

is satisfied for any positive integer \(m \leq M\) and proper algebraic system of \(a\) by Theorem 4.6. However, since

\[
a_i^{m+1}(1, 1, ..., 1) = Q_i(1, 1, ..., 1, a_i^m(1, 1, ..., 1), ..., a_L^m(1, 1, ..., 1))
\]

holds,

\[
\pi(a_i^{m+1}(1, 1, ..., 1)) = \pi(Q_i(1, 1, ..., 1, a_i^m(1, 1, ..., 1), ..., a_L^m(1, 1, ..., 1))
\]

\[
= \tilde{Q}_i(1, 1, ..., 1, \pi(a_i^m(1, 1, ..., 1)), ..., \pi(a_L^m(1, 1, ..., 1)))
\]

holds for all \(1 \leq i \leq L\) and \(m \geq 0\) where \(\tilde{Q}_i \in \mathbb{C}[z]/(z^{M+1})(X \coprod Y)\) is given by

\[
\text{cf}(\tilde{Q}_i; F) = \pi(\text{cf}(Q_i; F)).
\]

Therefore \(\text{cf}(\phi((zq)^* \circ P_{\text{semi}}); z^m) = \text{cf}(P^{(\deg p)M}_{1, N}; z^m)\) holds for any \(1 \leq m \leq M\), where \((P^m)_{m \geq 0} \subseteq M_N(\mathbb{C}[z]/(z^{M+1}))\) is defined as

\[
P^0_{j, k} = 0 \text{ for each } 1 \leq j, k \leq N;
\]

\[
P^{m+1}_{j, k} = \sum_{i=1}^n (\mu_i(P^m + \tilde{I}_{N \times N}))^2 \text{ for } m \geq 0,
\]

(4.5)

where \(\mu_i \in M_N(\mathbb{C}[z]/(z^{M+1}))\) is defined for any \(1 \leq i \leq n\) as \((\mu_i)_{j, k} = \pi(\mu(X_i)_{j, k})\) for each \(1 \leq j, k \leq N\) and \(\tilde{I}_{N \times N} \in M_N(\mathbb{C}[z]/(z^{M+1}))\) is the multiplication unit of \(M_N(\mathbb{C}[z]/(z^{M+1}))\).

So we obtain \(\tau(q(s_1, s_2, ..., s_n)^m) = \text{cf}(P^{(\deg p)M}_{1, N}; z^m)\) for each \(1 \leq m \leq M\) by iterating (4.5) for \((\deg p)M\) times. Since once iteration of (4.5) requires \(n\) summation of matrices where each summand are produced by multiplication of size \(N\) matrices, this takes \(O(nN^3M^2)\) time because ordinal multiplication of two polynomials \(p', q' \in \mathbb{C}[z]/(z^{M+1})\) takes \(O(M^2)\) time. Totally, the complexity of this step is \(O((\deg p)nN^3M^3)\) time since we iterate (4.2) for \((\deg p)M\) times.
**Step 5** (Calculate the output by binomial theorem). Finally we can calculate the output $	au(p(s_1, s_2, ..., s_n)^M)$ since this is equivalent to

$$\sum_{k=0}^{M} M C_k c^k \tau(q(s_1, s_2, ..., s_n)^{M-k}),$$

while we have already calculated $(\tau(q(s_1, s_2, ..., s_n)^k))_{1 \leq k \leq M}$ in Step 4 and $c \in \mathbb{C}$ is obtained in Step 1.

The computational bottleneck of the overall procedures is Step 4. Therefore we can calculate the output in $O((\deg p)nL^3M^3)$ time, where $L$ is the minimum size $N$ of monoid homomorphism in 2 of Proposition 4.1 associated with $(zq)^* \in \mathbb{C}[z]_{\text{rat}}(\langle X \rangle)$ where $q$ is obtained in Step 1. Since $L$ has an estimation $L \leq am_p(\deg p) + b$ for some $a, b > 0$ in Remark 4.2, the time complexity of this algorithm is $O((\deg p)^3m_p^3nM^3)$.

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