On Hilbert-Schmidt operator formulation of noncommutative quantum mechanics

Isiaka Aremua \textsuperscript{a,b}, Ezinvi Baloïtcha \textsuperscript{b}, Mahouton Norbert Hounkonnou \textsuperscript{b} and Komi Sodoga \textsuperscript{a,b}

\textsuperscript{a} Université de Lomé, Faculté des Sciences, Département de Physique, Laboratoire de Physique des Matériaux et de Mécanique Appliquée, 02 BP 1515 Lomé, Togo
E-mail: iaremua@univ-lome.tg, ksodoga@univ-lome.tg

\textsuperscript{b} University of Abomey-Calavi, International Chair in Mathematical Physics and Applications (ICMPA-UNESCO Chair), 072 B.P. 050 Cotonou, Benin
E-mail: ezinvi.baloitcha@cipma.uac.bj, norbert.hounkonnou@cipma.uac.bj

This work gives value to the importance of Hilbert-Schmidt operators in the formulation of a noncommutative quantum theory. A system of charged particle in a constant magnetic field is investigated in this framework.

Introduction

The theory of Hilbert-Schmidt operators plays a key role in the formulation of the noncommutative quantum mechanics. In the past three decades, the von Neumann algebras \cite{19, 27} underwent a vigorous growth after the discovery of a natural infinite family of pairwise nonisomorphic factors and the advent of Tomita-Takesaki theory \cite{25}, as well as Connes noncommutative geometry \cite{10} initiated by the classification theorems for von Neumann algebras and structure of extensions of \( C^\ast \)-algebras \cite{11}. The modular theory of von Neumann algebras has been created by M. Tomita \cite{26} in 1967 and perfected by M. Takesaki around 1970.

From physical point of view, a charged particle interacting with a constant magnetic field is one of the important problems in quantum mechanics described by the Hamiltonian

\[ H = \frac{1}{2M} (p + eA)^2, \]

inspired by condensed matter physics, quantum optics, etc. The Landau problem \cite{18} is related to the motion of a charged particle on the flat plane \( xy \) in the presence of a constant magnetic field along the \( z \)-axis. In metals, the electrons occupy many Landau levels \cite{13} \( E_n = \hbar \omega_c(n + \frac{1}{2}), \) each level being infinitely degenerate, with \( \omega_c = eB/Mc, \) the cyclotron frequency, which correspond to the kinetic energy levels of electrons, and are those of the one-dimensional harmonic oscillator.

This physical model represents an interesting application \cite{1} of the Tomita-Takesaki modular theory \cite{25, 26}. Taking into account the sense of the magnetic field, one obtains a pair of commuting Hamiltonians. Both these Hamiltonians can be written in terms of two pairs of mutually commuting oscillator-type creation and annihilation operators, which then generate two von Neumann algebras which mutually commute, and in fact are commutants of each other. The associated von Neumann algebra of observables displays a modular structure in the sense of the Tomita-Takesaki theory, with the algebra and its commutant referring to the two orientations of the magnetic field.

Hilbert spaces, at the mathematical side, realize the skeleton of quantum theories. Coherent states (CS), defined as a specific overcomplete family of vectors in the Hilbert space of the problem that describes quantum phenomena \cite{3, 12, 17, 20, 23}, constitute an important framework. In the studies and understanding of noncommutative geometry, CS have been proved to be useful tools \cite{14}. Based on the approach developed in \cite{22}, Gazeau-Klauder CS have been constructed in noncommutative quantum mechanics \cite{6}. Besides, in studying, in the noncommutative plane \cite{15}, the behavior of an electron in an external uniform...
electromagnetic background coupled to a harmonic potential, matrix vector coherent states (MVCS) as well as quaternionic vector coherent states (QVCS) have been constructed and discussed.

Our present contribution paper is organized as follows:

- First, we recast the Hilbert-Schmidt operators and the Tomita-Takesaki modular theories in the framework of noncommutative quantum mechanics by giving basic preliminaries and results.
- Detailed proofs are given for main frequently used statements in the study of the modular theory and Hilbert-Schmidt operators. As application, a construction of CS from the thermal state is achieved as in a previous work [1] with relevant properties discussed. Then, a light is put on the Wigner map as an interplay between the noncommutative quantum mechanics formalism [22] and the modular theory based on Hilbert-Schmidt operators.
- Finally, as illustrative example, the motion of a charged particle on the flat plane $xy$ in the presence of a constant magnetic field along the $z$-axis is studied.

1 von Neumann algebras: modular theory, Hilbert-Schmidt operators and coherent states

This section recapitulates fundamental notions and main ingredients of the modular theory used in the sequel. More details on these mathematical structures and their applications may be found in a series of works [1, 2, 7, 8, 11, 19, 21, 25–27] (and references therein), widely exploited to write this review section.

1.1 Basics on von Neumann algebras

In this paragraph, $H$ denotes a Hilbert space over $\mathbb{C}$. $H$ is assumed to be separable, of dimension $N$, which could be finite or infinite. Denote by $\mathcal{L}(H)$ the $C^*$-algebra of all bounded operators on $H$.

The following definitions are in order:

**Definition 1.1**

Let $\mathfrak{G}$ be an algebra. A mapping $A \in \mathfrak{G} \mapsto A^* \in \mathfrak{G}$ is called an *involution*, or *adjoint operation*, of the algebra $\mathfrak{G}$, if it has the following properties:

1. $A^{**} = A$
2. $(AB)^* = B^*A^*$, with $A, B \in \mathfrak{G}$, $A^*, B^* \in \mathfrak{G}$
3. $(\alpha A + \beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^*$, $\alpha, \beta \in \mathbb{C}$.

($\bar{\alpha}$ is the complex conjugate of $\alpha$.)

**Definition 1.2** $^*$-algebra

An algebra with an involution is called $^*$-algebra and a subset $\mathfrak{B}$ of $\mathfrak{G}$ is called self-adjoint if $A \in \mathfrak{B}$ implies that $A^* \in \mathfrak{B}$.

The algebra $\mathfrak{G}$ is a normed algebra if to each $A \in \mathfrak{G}$ there is associated a real number $||A||$, the norm of $A$, satisfying the requirements

1. $||A|| \geq 0$ and $||A|| = 0$ if, and only if, $||A|| = 0$,
2. $||\alpha A|| = |\alpha|||A||$,
3. $||A + B|| \leq ||A|| + ||B||$,
4. $||AB|| \leq ||A|| ||B||$. 


The third of these conditions is called the triangle inequality and the fourth product inequality. The norm defines a metric topology on \( \mathfrak{G} \) which is referred to as the uniform topology. The neighborhoods of an element \( A \in \mathfrak{G} \) in this topology are given by
\[
U(A; \varepsilon) = \{ B; B \in \mathfrak{G}, \| B - A \| < \varepsilon \},
\]
where \( \varepsilon > 0 \). If \( \mathfrak{A} \) is complete with respect to the uniform topology then it is called a Banach algebra. A normed algebra with involution which is complete and has the property \( \| A \| = \| A^* \| \) is called a Banach \(*\)-algebra. Then, follows the definition:

**Definition 1.3**
A \( C^* \)-algebra is a Banach \(*\)-algebra \( \mathfrak{G} \) with the property
\[
\| A^* A \| = \| A \|^2
\]
for all \( A \in \mathfrak{G} \).

Before going further, let us deal, in the following, with some notions about representations and states.

**Definition 1.4** \(*\)-Morphism between two \(*\)-algebras
Let \( \mathfrak{G} \) and \( \mathfrak{B} \) be two \(*\)-algebras. The \(*\)-morphism between \( \mathfrak{G} \) and \( \mathfrak{B} \) is given by the mapping \( \pi : A \in \mathfrak{G} \mapsto \pi(A) \in \mathfrak{B} \), satisfying:

1. \( \pi(\alpha A + \beta B) = \alpha \pi(A) + \beta \pi(B) \)
2. \( \pi(AB) = \pi(A)\pi(B) \)
3. \( \pi(A^*) = \pi(A)^* \)

for all \( A, B \in \mathfrak{G}, \alpha \in \mathbb{C} \).

**Remark 1.5** Each \(*\)-automorphism \( \pi \) between two \(*\)-algebras \( \mathfrak{G} \) and \( \mathfrak{B} \) is positive because if \( A \geq 0 \), then \( A = B^* B \) for some \( B \in \mathfrak{G} \). Hence,
\[
\pi(A) = \pi(B^* B) = \pi(B)^* \pi(B) \geq 0.
\]

**Definition 1.6** Representation of a \( C^* \)-algebra
A representation of a \( C^* \)-algebra \( \mathfrak{G} \) is defined to be a pair \((\mathcal{H}, \pi)\), where \( \mathcal{H} \) is a complex Hilbert space and \( \pi \) is a \(*\)-morphism of \( \mathfrak{G} \) into \( \mathcal{L}(\mathcal{H}) \). The representation \((\mathcal{H}, \pi)\) is said to be faithful if, and only if, \( \pi \) is a \(*\)-isomorphism between \( \mathfrak{G} \) and \( \pi(\mathfrak{G}) \), i.e., if, and only if, \( \ker \pi = \{0\} \).

Each representation \((\mathcal{H}, \pi)\) of a \( C^* \)-algebra \( \mathfrak{G} \) defines a faithful representation of the quotient algebra \( \mathfrak{G}_\pi = \mathfrak{G}/\ker \pi \).

Then, follows the proposition on the criteria for faithfulness:

**Proposition 1.7** (\cite{B}, p.44)
Let \((\mathcal{H}, \pi)\) be a representation of the \( C^* \)-algebra \( \mathfrak{G} \). The representation is faithful if, and only if, it satisfies each of the following equivalent conditions:

1. \( \ker \pi = \{0\} \);
2. \( \| \pi(A) \| = \| A \| \) for all \( A \in \mathfrak{G} \);
3. \( \pi(A) > 0 \) for all \( A > 0 \).
The Proof of this proposition is achieved by the following proposition:

**Proposition 1.8** ([8], pp.42-43)

Let $\mathfrak{G}$ be a Banach $^\ast$-algebra with identity, $\mathfrak{B}$ a $C^\ast$-algebra, and $\pi$ a $^\ast$-morphism of $\mathfrak{G}$ into $\mathfrak{B}$. Then $\pi$ is continuous and

$$\|\pi(A)\| \leq \|A\|$$

(1.4)

for all $A \in \mathfrak{G}$. Moreover, if $\mathfrak{G}$ is a $C^\ast$-algebra then the range $\mathfrak{B}_\pi = \{\pi(A); A \in \mathfrak{G}\}$ of $\pi$ is a $C^\ast$-subalgebra of $\mathfrak{B}$.

**Proof.** (see [8] p.43)

First assume $A = A^\ast$. Then since $\mathfrak{B}$ is a $C^\ast$-algebra and $\pi(A) \in \mathfrak{B}$, one has

$$\|\pi(A)\| = \sup \{\lambda; \lambda \in \sigma(\pi(A))\}$$

(1.5)

by Theorem 2.2.5(a) (see [8] p.29). Next, define $P = \pi(\mathbb{1}_\mathfrak{G})$ where $\mathbb{1}_\mathfrak{G}$ denotes the identity of $\mathfrak{G}$. It follows from the definition of $\pi$ that $P$ is a projection in $\mathfrak{B}$.

Hence replacing $\mathfrak{B}$ by the $C^\ast$-algebra $P\mathfrak{B}P$ the projection $P$ becomes the identity $\mathbb{1}_\mathfrak{B}$ of the new algebra $\mathfrak{B}$. Moreover, $\pi(\mathfrak{G}) \subseteq \mathfrak{B}$. Now it follows from the definitions of a morphism and of the spectrum that $\sigma_{\mathfrak{B}}(\pi(A)) \subseteq \sigma_{\mathfrak{G}}(A)$. Therefore,

$$\|\pi(A)\| \leq \sup \{\lambda; \lambda \in \sigma_{\mathfrak{G}}(A)\} \leq \|A\|$$

(1.6)

by the following Proposition:

**Proposition 1.9** ([8] p.26)

Let $A$ be an element of a Banach algebra with identity and define the spectral radius $\rho(A)$ of $A$ by

$$\rho(A) = \sup \{\lambda; \lambda \in \sigma_{\mathfrak{G}}(A)\}.$$ 

(1.7)

It follows that

$$\rho(A) = \lim_{n \to \infty} \|A^n\|^{1/n} = \inf_n \|A^n\|^{1/n} \leq \|A\|.$$ 

(1.8)

In particular, the limit exists. Thus the spectrum of $A$ is a nonempty compact set.

**Proof.** (see [8] p.26)

Let $|\lambda|^n > \|A^n\|$ for some $n > 0$. As each $m \in \mathbb{Z}$ can be decomposed as $m = pn + q$ with $p, q \in \mathbb{Z}$ and $0 \leq q < n$ one again establishes that the series

$$\lambda^{-1} \sum_{m \geq 0} \left(\frac{A}{\lambda}\right)^m$$

(1.9)

is Cauchy in the uniform topology and defines $(\lambda \mathbb{1} - A)^{-1}$. Therefore,

$$\rho(A) \leq \|A^n\|^{1/n}$$

(1.10)

for all $n > 0$, and consequently

$$\rho(A) \leq \inf_n \|A^n\|^{1/n} \leq \lim_{n \to \infty} \inf_{n} \|A^n\|^{1/n}.$$ 

(1.11)

Thus to complete the proof it suffices to establish that $\rho(A) \geq r_A$, where

$$r_A = \lim_{n \to \infty} \sup \|A^n\|^{1/n}.$$ 

(1.12)
There are two cases.
Firstly, assume 0 ∈ rφ(A), i.e., A is invertible. Then 1 = ||A^n A^{-n}|| ≤ ||A^n|| ||A^{-n}|| and hence 1 ≤ r_A r_{A^{-1}}. This implies r_A > 0. Consequently, if r_A = 0 one must have 0 ∈ rφ(A) and ρ(A) ≥ r_A.

Secondly, we may assume r_A > 0. We will need the following observation. If A_n is any sequence of elements such that R_n = (I - A_n)^{-1} exists then I - R_n = -A_n(I - A_n)^{-1} and A_n = -(I - R_n)(I - (I - R_n))^{-1}. Therefore, ||I - R_n|| → 0 is equivalent to ||A_n|| → 0 by power series expansion.

Define S_A = {λ; λ ∈ C, |λ| ≥ r_A}. We assume that S_A ⊆ rφ(A) and obtain a contradiction. Let ω be a primitive nth root of unity. By assumption

\[ R_n(A; λ) = n^{-1} \sum_{k=1}^{n} \left( I - \frac{ω^k A}{λ} \right)^{-1} \]  

(1.13)
is well defined for all λ ∈ S_A. But an elementary calculation shows that

\[ R_n(A; λ) = \left( I - \frac{A^n}{λ^n} \right)^{-1}. \]  

(1.14)Next one has the continuity estimate

\[ \left\| \left( I - \frac{ω^k A}{r_A} \right)^{-1} - \left( I - \frac{ω^k A}{λ} \right)^{-1} \right\| = \left\| \left( I - \frac{ω^k A}{r_A} \right)^{-1} \left( I - \frac{ω^k A}{λ} \right) \right\| \leq |λ - r_A| ||A|| \sup_{γ ∈ S_A} ||(I - γ I)^{-1}||^2, \]  

(1.15)which is uniform in k. The supremum is finite since λ ↦ ||(λ I - A)^{-1}|| is continuous on rφ(A) and for |λ| > ||A|| one has

\[ ||(λ I - A)^{-1}|| ≤ |λ|^{-1} \sum_{n≥0} ||A||^n / |λ|^n = (|λ| - ||A||)^{-1}. \]  

(1.16)It follows then that for each ε > 0 there is a λ > r_A such that

\[ \left\| \left( I - \frac{A^n}{r_A^n} \right)^{-1} - \left( I - \frac{A^n}{λ^n} \right)^{-1} \right\| < ε \]  

(1.17)uniformly in n. But ||A^n||/λ^n → 0 and by the above observation ||(I - A^n/λ^n)^{-1} - I|| → 0. This implies that ||(I - A^n/r_A^n)^{-1} - I|| → 0 and ||A^n||/r_A^n → 0 by another application of the same observation. This last statement contradicts, however, the definition of r_A and hence the proof is complete.

□

Finally, if A is not selfadjoint one can combine this inequality with the C*-norm property and the product inequality to deduce that

\[ ||π(A)||^2 = ||π(A^* A)|| ≤ ||A^* A|| ≤ ||A||^2. \]  

(1.18)Thus ||π(A)|| ≤ ||A|| for all A ∈ φ and π is continuous.

The range Φ is a *-subalgebra of Φ by definition and to deduce that it is a C*-subalgebra we must prove that it is closed, under the assumption that Φ is a C*-algebra.

Now introduce the kernel ker π of π by

\[ \ker π = \{ A ∈ φ; π(A) = 0 \} \]  

(1.19)then ker π is closed two-sided *-ideal. Given A ∈ φ and B ∈ ker π then π(AB) = π(A)π(B) = 0, π(BA) = π(B)π(A) = 0, and π(B^*) = π(B) = 0. The closedness follows from the estimate ||π(A)|| ≤ ||A||. Thus we can form the quotient algebra Φ/ker π and Φ is a C*-algebra. The elements of Φ are the
classes \( \hat{A} = \{ A + I; I \in \ker \pi \} \) and the morphism \( \pi \) induces a morphism \( \hat{\pi} \) from \( \mathfrak{G}_\pi \) onto \( \mathfrak{B}_\pi \) by the definition \( \hat{\pi}(\hat{A}) = \pi(A) \). The kernel of \( \hat{\pi} \) is zero by construction and hence \( \hat{\pi} \) is an isomorphism between \( \mathfrak{G}_\pi \) and \( \mathfrak{B}_\pi \). Therefore, one can define a morphism \( \hat{\pi}^{-1} \) from the *-algebra \( \mathfrak{B}_\pi \) onto the \( C^* \)-algebra \( \mathfrak{G}_\pi \) by \( \hat{\pi}^{-1}(\hat{\pi}(\hat{A})) = \hat{A} \) and then applying Proposition 1.8 to Remark 1.12.

Thus \( ||\hat{A}|| = ||\hat{\pi}^{-1}(\hat{\pi}(\hat{A}))|| \leq ||\hat{\pi}(\hat{A})|| \leq ||\hat{A}|| \). (1.20)

Consequently, if \( \pi(A_\alpha) \) converges uniformly in \( \mathfrak{B} \) to an element \( A_\pi \) then \( A_\pi \) converges in \( \mathfrak{G}_\pi \) to an element \( \hat{A} \) and \( A_\pi = \hat{\pi}(\hat{A}) = \pi(A) \) where \( A \) is any element of the equivalence class \( \hat{A} \). Thus \( A_\pi \in \mathfrak{B}_\pi \) and \( \mathfrak{B}_\pi \) is closed.

**Proof.** of Proposition 1.7 (see [8] p.44).

The equivalence of condition (1) and faithfulness is by definition. Prove that (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (1).

(1) \( \Rightarrow \) (2) Since \( \ker \pi = \{0\} \), we can define a morphism \( \pi^{-1} \) from the range of \( \pi \) into \( \mathfrak{G} \) by \( \pi^{-1}(\pi(A)) = A \) and then applying Proposition 1.8 to \( \pi^{-1} \) and \( \pi \) successively one has

\[
||A|| = ||\pi^{-1}(\pi(A))|| \leq ||\pi(A)|| \leq ||A||.
\] (1.21)

(2) \( \Rightarrow \) (3) If \( A > 0 \) then \( ||A|| > 0 \) and hence \( ||\pi(A)|| > 0 \), or \( \pi(A) \neq 0 \). But \( \pi(A) \geq 0 \) by Proposition 1.8 and therefore \( \pi(A) > 0 \).

(3) \( \Rightarrow \) (1) If condition (1) is false then there is a \( B \in \ker \pi \) with \( B \neq 0 \) and \( \pi(B^*B) = 0 \). But \( ||B^*B|| > 0 \) and as \( ||B^*B|| = ||B||^2 \) one has \( B^*B > 0 \). Thus condition (3) is false.

**Definition 1.10** Cyclic representation of a \( C^* \)-algebra

A cyclic representation of a \( C^* \)-algebra \( \mathfrak{G} \) is defined to be a triplet \((\mathfrak{H}, \pi, \Omega)\), where \((\mathfrak{H}, \pi)\) is a representation of \( \mathfrak{G} \) and \( \Omega \) is a vector in \( \mathfrak{H} \) which is cyclic for \( \pi \), in \( \mathfrak{H} \).

\( \Omega \) is called cyclic vector or cyclic vector for \( \pi \).

If \( \mathfrak{K} \) is a closed subspace of \( \mathfrak{H} \) then \( \mathfrak{K} \) is called a cyclic subspace for \( \mathfrak{H} \) whenever the set

\[
\left\{ \sum_i \pi(A_i)\psi_i; A_i \in \mathfrak{G}, \psi_i \in \mathfrak{K} \right\}
\] (1.22)

is dense in \( \mathfrak{H} \).

**Definition 1.11** State over a \( C^* \)-algebra

A linear functional \( \omega \) over the \( C^* \)-algebra \( \mathfrak{G} \) is defined to be positive if

\[
\omega(A^*A) > 0
\] (1.23)

for all \( A \in \mathfrak{G} \). A positive linear functional \( \omega \) over a \( C^* \)-algebra \( \mathfrak{G} \) with \( ||\omega|| = 1 \) is called a state.

**Remark 1.12**

1. Every positive element of a \( C^* \)-algebra is of the form \( A^*A \) and hence positivity of \( \omega \) is equivalent to \( \omega \) being positive on positive elements.

2. Considering a representation \((\mathfrak{H}, \pi)\) of the \( C^* \)-algebra \( \mathfrak{G} \), taking \( \Omega \in \mathfrak{H} \) being a nonzero vector and define \( \omega_\Omega \) by

\[
\omega_\Omega(A) = (\Omega, \pi(A)\Omega)
\] (1.24)

for all \( A \in \mathfrak{G} \). It follows that \( \omega_\Omega \) is a linear function over \( \mathfrak{G} \), it is also positive since

\[
\omega_\Omega(A^*A) = ||\pi(A)\Omega||^2 \geq 0.
\] (1.25)
\[|\omega_{\Omega}| = 1 \text{ whenever } |\Omega| = 1 \text{ and then, } \pi \text{ is nondegenerate. In this case } \omega_{\Omega} \text{ is a state, and is usually called vector state for the representation } (\mathcal{H}, \pi).\]

**Definition 1.13**

The cyclic representation \((\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)\), constructed from the state \(\omega\) over the \(C^*\)-algebra \(\mathcal{G}\), is defined as the canonical cyclic representation of \(\mathcal{G}\) associated with \(\omega\).

Next it will be demonstrated that the notions of purity of a state \(\omega\) and irreducibility of the representation associated with \(\omega\) are intimately related.

**Theorem 1.14** (\cite{8} p.57)

Let \(\omega\) be a state over the \(C^*\)-algebra \(\mathcal{G}\) and \((\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)\) the associated cyclic representation. The following conditions are equivalent:

1. \((\mathcal{H}_\omega, \pi_\omega)\) is irreducible;
2. \(\omega\) is pure;
3. \(\omega\) is an extremal point of the set \(E_{\mathcal{G}}\) of states over \(\mathcal{G}\).

Furthermore, there is one-to-one correspondence

\[\omega_T(A) = (T\Omega_\omega, \pi_\omega(A)\Omega_\omega) \quad (1.26)\]

between positive functionals \(\omega_T\), over \(\mathcal{G}\), majorized by \(\omega\) and positive operators \(T\) in the commutant \(\pi'_\omega\) of \(\pi_\omega\), with \(|T| \leq 1\).

**Proof.** (see \cite{8} pp.57-58)

(1) \(\Rightarrow\) (2) Assume that (2) is false. Thus there exists a positive functional \(\rho\) such that \(\rho(A^*A) \leq \omega(A^*A)\) for all \(A \in \mathcal{G}\). But applying the Cauchy-Schwarz inequality one then has

\[|\rho(B^*A)|^2 \leq \rho(B^*B)\rho(A^*A) \leq \omega(B^*B)\omega(A^*A) = ||\pi_\omega(B)\Omega_\omega||^2||\pi_\omega(A)\Omega_\omega||^2.\]

Thus \(\pi_\omega(B)\Omega_\omega \times \pi_\omega(A)\Omega_\omega \rightarrow \rho(B^*A)\) is a densely defined, bounded, sesquilinear functional, over \(\mathcal{H}_\omega \times \mathcal{H}_\omega\), and there exists a unique bounded operator \(T\), on \(\mathcal{H}_\omega\), such that

\[(\pi_\omega(B)\Omega_\omega, T\pi_\omega(A)\Omega_\omega) = \rho(B^*A).\]

As \(\rho\) is not a multiple of \(\omega\) operator \(T\) is not a multiple of the identity. Moreover,

\[0 \leq \rho(A^*A) = (\pi_\omega(A)\Omega_\omega, T\pi_\omega(A)\Omega_\omega) \leq \omega(A^*A) = (\pi_\omega(A)\Omega_\omega, \pi_\omega(A)\Omega_\omega)\]

and hence \(0 \leq T \leq \mathbb{I}\). But

\[(\pi_\omega(B)\Omega_\omega, T\pi_\omega(C)\pi_\omega(A)\Omega_\omega) = \rho(B^*CA) = \rho((C^*B)^*A) = (\pi_\omega(B)\Omega_\omega, \pi_\omega(C)T\pi_\omega(A)\Omega_\omega)\]

and therefore \(T \in \pi'_\omega\). Thus condition (1) is false.

(2) \(\Rightarrow\) (1) Assume that (1) is false. If \(T \in \pi'_\omega\) then \(T^* \in \pi'_\omega\) and \(T + T^*\), \((T - T^*)/i\) are also elements of the commutant. Thus there exists a selfadjoint element \(S\) of \(\pi'_\omega\) which is not a multiple of the identity. Therefore there exists a spectral projector \(P\) of \(S\) such that \(0 < P < \mathbb{I}\) and \(P \in \pi'_\omega\). Consider the functional

\[\rho(A) = (P\Omega_\omega, \pi_\omega(A)\Omega_\omega).\]
This is certainly positive since
\[ \rho(A^*A) = (P\pi_\omega(A)\Omega_\omega, P\pi_\omega(A)\Omega_\omega) \geq 0. \]
Moreover,
\[ \omega(A^*A) - \rho(A^*A) = (\pi_\omega(A)\Omega_\omega, (I - P)\pi_\omega(A)\Omega_\omega) \geq 0. \]
Thus \( \omega \) majorizes \( \rho \). It is verified that \( \rho \) is not a multiple of \( \omega \) and hence (2) is false.

This proves the equivalence of the first two conditions stated in the theorem and simultaneously establishes the correspondence described by the last statement.

The equivalence of conditions (2) and (3) is performed as follows. Suppose that \( \omega \) is an extremal point of \( E_\mathcal{A} \) and \( \omega \neq 0 \). Then, we must have \( ||\omega|| = 1 \). Thus \( \omega \) is a state and we must deduce that it is pure. Suppose the contrary; then there is a state \( \omega_1 \neq \omega \) and a \( \lambda \) with \( 0 < \lambda < 1 \) such that \( \omega \geq \lambda\omega_1 \). Define \( \omega_2 \) by \( \omega_2 = (\omega - \lambda\omega_1)/(1 - \lambda) \); then \( ||\omega_2|| = (||\omega|| - \lambda||\omega_1||)/(1 - \lambda) = 1 \) and \( \omega_2 \) is also a state. But \( \omega = \lambda\omega_1 + (1 - \lambda)\omega_2 \) and \( \omega \) is not extremal, which is a contradiction.

\[ \square \]

In the following, some notions on von Neumann algebra are provided. To specify the Hilbert space upon which a von Neumann algebra \( \mathcal{A} \) acts, one often uses the notation \( \{\mathcal{A}, \mathcal{H}\} \) to denote the von Neumann algebra \( \mathcal{A} \).

**Definition 1.15 von Neumann algebra**

Let \( \mathcal{H} \) be a Hilbert space. For each subset \( \mathcal{A} \) of \( \mathcal{L}(\mathcal{H}) \), let \( \mathcal{A}' \) denote the set of all bounded operators on \( \mathcal{H} \) commuting with every operator in \( \mathcal{A} \). Clearly, \( \mathcal{A}' \) is a Banach algebra of operators containing the identity operator \( I_\mathcal{H} \) on \( \mathcal{H} \).

(i) A von Neumann algebra is a \( * \)-subalgebra \( \mathcal{A} \) of \( \mathcal{L}(\mathcal{H}) \) such that \( \mathcal{A} = \mathcal{A}' \).

(ii) \( \mathcal{A}' \) denotes the commutant of \( \mathcal{A} \), the set of all elements in \( \mathcal{L}(\mathcal{H}) \) which commute with every element of \( \mathcal{A} \).

(iii) A von Neumann algebra always contains the identity operator \( I_\mathcal{H} \) on \( \mathcal{H} \). It is called a factor if \( \mathcal{A} \cap \mathcal{A}' = \mathbb{C}I_\mathcal{H} \).

(iv) If a subset \( \mathcal{S} \) of \( \mathcal{L}(\mathcal{H}) \) is invariant under the \( * \)-operation, then \( \mathcal{S}' \), the double commutant of \( \mathcal{S} \), is the smallest von Neumann algebra containing \( \mathcal{S} \), and it is called the von Neumann algebra generated by \( \mathcal{S} \).

We also have the following definition:

**Definition 1.16**

A von Neumann algebra \( \mathcal{A} \subset \mathcal{L}(\mathcal{H}) \) is a \( C^* \)-algebra acting on the Hilbert space \( \mathcal{H} \) that is closed under the weak-operator topology: \( A_n \xrightarrow{n \to \infty} A \) iff \( \langle \xi | A_n \eta \rangle \xrightarrow{n \to \infty} \langle \xi | A \eta \rangle \), \( \forall \xi, \eta \in \mathcal{H} \), or equivalently under the \( \sigma \)-weak topology: \( A_n \xrightarrow{n \to \infty} A \) iff for all sequences \( (\xi_k), (\zeta_k) \) in \( \mathcal{H} \) such that \( \sum_{k=1}^{+\infty} |\xi_k|^2 < +\infty \) and \( \sum_{k=1}^{+\infty} |\zeta_k|^2 < +\infty \) we have \( \sum_{k=1}^{+\infty} \langle \xi_k | A_n \zeta_k \rangle \xrightarrow{n \to \infty} \sum_{k=1}^{+\infty} \langle \xi_k | A \zeta_k \rangle \).

Case of \( \mathcal{L}(\mathcal{H}) \)

(i) \( \mathcal{L}(\mathcal{H}) \) is a von Neumann algebra and even a factor since \( \mathcal{L}(\mathcal{H})' = \mathbb{C}I_\mathcal{H} \) (\( I_\mathcal{H} = I_\mathcal{H} \)).

(ii) The Hilbert space adjoint operation defines an involution on \( \mathcal{L}(\mathcal{H}) \) and with respect to these operations and this norm, \( \mathcal{L}(\mathcal{H}) \) is a \( C^* \)-algebra. In particular, the \( C^* \)-norm property follows from \( ||A||^2 \leq ||A^*|| ||A|| = ||A||^2 \).
(iii) Any uniformly closed subalgebra \( M \) of \( \mathcal{L}(\mathcal{H}) \) which is self-adjoint is also a \( C^* \)-algebra.

Next, it comes the following definition:

**Definition 1.17** Closure-Orthogonal projection

If \( M \) is a subset of \( \mathcal{L}(\mathcal{H}) \) and \( K \) is a subset of \( \mathcal{H} \), \([MK]\) denotes the closure of the linear span of elements of the form \( A\xi \), where \( A \in M \) and \( \xi \in K \). \([MK]\) also denotes the orthogonal projection onto \([MK]\).

(iv) A *-subalgebra \( M \subseteq \mathcal{L}(\mathcal{H}) \) is said to be nondegenerate if \([MH] = \mathcal{H}\).

(v) If \( M \subseteq \mathcal{L}(\mathcal{H}) \) contains the identity operator, then, it is automatically nondegenerate.

A nondegenerate *-algebra contains the identity operator; If a subalgebra of \( \mathcal{L}(\mathcal{H}) \) is invariant under the *-operation, then it is called a *-subalgebra of \( \mathcal{L}(\mathcal{H}) \) or a *-algebra of operators on \( \mathcal{H} \).

We have the following proposition (see [25] pp.72-73):

**Proposition 1.18** ( [25], p.72)

*The subset* \( M \)* of \( \mathcal{L}(\mathcal{H}) \)* is a von Neumann algebra on \( \mathcal{H} \).*

**Proof.** (see [25], p.72)

Let \( \{M_i, \mathcal{H}_i\}_{i \in I} \) be a family of von Neumann algebras. Let \( \mathcal{H} \) denote the direct sum \( \bigoplus_{i \in I} \mathcal{H}_i \) of Hilbert spaces \( \{\mathcal{H}_i\}_{i \in I} \). Each vector \( \xi = \{\xi_i\}_{i \in I} \) in \( \mathcal{H} \) is denoted by \( \sum_{i \in I} \xi_i \). For each bounded sequence \( \{x_i\}_{i \in I} \) in \( \prod_{i \in I} M_i \), one defines an operator \( x \) on \( \mathcal{H} \) by

\[
x \sum_{i \in I} x_i \xi_i = \sum_{i \in I} x_i \xi_i
\]

Then, \( x \) is a bounded operator on \( \mathcal{H} \) denoted by \( \sum_{i \in I} x_i \). Let \( M \) be the set of all such \( x \). Particularly, taking \( M \) as a subset of \( \mathcal{L}(\mathcal{H}) \), the proof is completed.

\( \square \)

**Definition 1.19** Cyclic and separating vector

The modular theory of von Neumann algebras is such that to every von Neumann algebra \( M \subseteq \mathcal{L}(\mathcal{H}) \), and to every vector \( \xi \in \mathcal{H} \) that is cyclic

\[
([M\xi]) = \mathcal{H}
\]

i.e. the set \( \{A\Phi; A \in M\} \) (\( M \) denoting a set of bounded operators on \( \mathcal{H} \)) is dense in \( \mathcal{H} \); and separating i.e. for \( A \in M \),

\[
A \xi = 0 \Rightarrow A = 0.
\]

Moreover, a vector \( \psi \in \mathcal{H} \) is said separating for a von Neumann algebra \( \mathfrak{A} \) if \( A \psi = B \psi \), \( A, B \in \mathfrak{A} \), if and only if \( A = B \).

We have the following definitions:

**Definition 1.20** Separating subset

Let \( \mathfrak{A} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \). A subset \( \mathcal{K} \subseteq \mathcal{H} \) is separating for \( \mathfrak{A} \) if for any \( A \in \mathfrak{A} \), \( A \xi = 0 \) for all \( \xi \in \mathcal{K} \) implies \( A = 0 \).
**Definition 1.21** Cyclic and separating subset of a von Neumann algebra
Let \( \{ A, H \} \) be a von Neumann algebra. A subset \( M \) of \( H \) is called *separating* (resp. *cyclic*) for \( A \) if \( a\xi = 0, a \in A \), for every \( \xi \in M \) implies \( a = 0 \) (resp. the smallest invariant subspace \([AM]\) under \( A \) containing \( M \) in the whole space \( H \)).

Recall that a subset \( K \subseteq H \) is cyclic for \( A \) if \( [MK] = H \). There is a dual relation between the properties of cyclic for the algebra and separating for the commutant.

We have the following propositions:

**Proposition 1.22** (\( \mathbb{R} \) p.85)
Let \( A \) be a von Neumann algebra on \( H \) and \( K \subseteq H \) a subset. The following conditions are equivalent:

1. \( K \) is cyclic for \( A \);
2. \( K \) is separating for \( A' \).

**Proof.** (see \( \mathbb{R} \) p.85)

1. \( \Rightarrow \) (2) Assume that \( K \) is cyclic for \( A \) and choose \( A' \in A' \) such that \( A'K = \{0\} \). Then, for any \( B \in A \) and \( \xi \in K \), \( A'B\xi = BA'\xi = 0 \), hence \( A'[AK] = 0 \) and \( A' = 0 \).

2. \( \Rightarrow \) (1) Suppose that \( K \) is separating for \( A' \) and set \( P' = [AK] \). \( P' \) is then a projection in \( A' \) and \( (1 - P')K = \{0\} \). Hence \( 1 - P' = 0 \) and \([AK]\) = \( H \).

\( \square \)

**Definition 1.23** The weak and \( \sigma \)-weak topologies
If \( \xi, \eta \in H \), then \( A \mapsto |(\xi, A\eta)| \) is a seminorm on \( \mathcal{L}(H) \). The locally convex topology on \( \mathcal{L}(H) \) defined by these seminorms is called the *weak topology*. The seminorms defined by the vector states \( A \mapsto |(\xi, A\xi)| \) suffice to define this topology because \( H \) is complex and one has the polarization identity

\[
4(\xi, A\eta) = \sum_{n=0}^{3} i^{-n}(\xi + i^n\eta, A(\xi + i^n\eta)).
\] (1.30)

Let \( \{\xi_n\}, \{\eta_n\} \) be two sequences from \( H \) such that

\[
\sum_n ||\xi_n||^2 < \infty, \quad \sum_n ||\eta_n||^2 < \infty.
\] (1.31)

Then for \( A \in \mathcal{L}(H) \)

\[
|\sum_n (\xi_n, A\eta_n)| \leq \sum_n ||\xi_n|| ||A|| ||\eta_n||
\leq ||A|| \left( \sum_n ||\xi_n||^2 \right)^{1/2} \left( \sum_n ||\eta_n||^2 \right)^{1/2}
\leq \infty.
\] (1.32)

Hence \( A \mapsto |\sum_n (\xi_n, A\eta_n)| \) is a seminorm on \( \mathcal{L}(H) \). The locally convex topology on \( \mathcal{L}(H) \) induced by these seminorms is called the \( \sigma \)-weak topology.

**Notations:**

In the sequel,
• \( \mathfrak{A}_+ \) denotes the *positive part* of the von Neumann algebra \( \mathfrak{A} \) or the set of positive elements of the von Neumann algebra \( \mathfrak{A} \);

• \( \mathfrak{A}_* \) denotes the *predual of a von Neumann algebra*. It is the space of all \( \sigma \)-weakly continuous linear functionals on \( \mathfrak{A} \);

• \( \mathcal{L}(\mathfrak{H})_1 \) denotes the unit ball of \( \mathcal{L}(\mathfrak{H}) \). \( \mathcal{L}(\mathfrak{H})_1 \) is norm dense in the unit ball of the norm closure of \( \mathcal{L}(\mathfrak{H}) \), and it is taken as a \( C^* \)-algebra (see [8] p.74).

**Definition 1.24**

Let \( \varphi : \mathfrak{A} \to \mathbb{C} \) be a bounded linear functional on \( \mathfrak{A} \), which is denoted by \( \langle \varphi; A \rangle \), \( A \in \mathfrak{A} \).

\( \varphi \) is called a state on this algebra if it also satisfies the two conditions:

(a) \( \langle \varphi; A^*A \rangle \geq 0, \quad \forall A \in \mathfrak{A} \)

(b) \( \langle \varphi; I\mathfrak{H} \rangle = 1 \).

The state \( \varphi \) is called a *vector state* if there exists a vector \( \phi \in \mathfrak{H} \) such that

\[ \langle \varphi; A \rangle = \langle \phi|A\phi \rangle, \quad \forall A \in \mathfrak{A}. \]  

(1.33)

Such a state is also *normal*.

**Definition 1.25**

A state \( \omega \) on a von Neumann algebra \( \mathfrak{A} \) is *faithful* if \( \omega(A) > 0 \) for all nonzero \( A \in \mathfrak{A}_+ \).

**Remark 1.26** (see [8], Example 2.5.5 p.85) Let \( \mathfrak{A} = \mathcal{L}(\mathfrak{H}) \) with \( \mathfrak{H} \) separable. Every normal state \( \omega \) over \( \mathfrak{A} \) is of the form

\[ \omega(A) = \text{Tr}(\rho A), \]  

(1.34)

where \( \rho \) is a density matrix. If \( \omega \) is faithful then \( \omega(E) > 0 \) for each rank one projector, i.e., \( ||\rho^{1/2}\psi|| > 0 \) for each \( \psi \in \mathfrak{H} \setminus \{0\} \). Thus \( \rho \) is invertible (in the densely defined self-adjoint operators on \( \mathfrak{H} \)). Conversely, if \( \omega \) is not faithful then \( \omega(A^*A) = 0 \) for some nonzero \( A \) and hence \( ||\rho^{1/2}A^*\psi|| = 0 \) for all \( \psi \in \mathfrak{H} \), i.e., \( \rho \) is not invertible. This establishes that \( \omega \) is faithful if, and only if, \( \rho \) is invertible.

**Lemma 1.27** ([8] p.76)

Let \( \{A_\alpha\} \) be an increasing set in \( \mathcal{L}(\mathfrak{H})_+ \) with an upper bound in \( \mathcal{L}(\mathfrak{H})_+ \). Then \( \{A_\alpha\} \) has a least upper bound (l.u.b.) \( A \), and the net converges \( \sigma \)-strongly to \( A \).

**Proof.** (see [8] p.76)

Let \( \mathfrak{K}_\alpha \) be the weak closure of the set of \( A_\beta \) with \( \beta > \alpha \). Since \( \mathcal{L}(\mathfrak{H})_1 \) is weakly compact, there exists an element \( A \) in \( \bigcap_\alpha \mathfrak{K}_\alpha \). For all \( A_\alpha \) the set of \( B \in \mathcal{L}(\mathfrak{H})_+ \) such that \( B \geq A_\alpha \) is \( \sigma \)-weakly closed and contains \( \mathfrak{K}_\alpha \), hence \( A \geq A_\alpha \). Thus, \( A \) majorizes \( \{A_\alpha\} \) and lies in the weak closure of \( \{A_\alpha\} \). If \( B \) is another operator majorizing \( \{A_\alpha\} \), then it majorizes its weak closure; thus \( B \geq A \) and \( A \) is the least upper bound of \( \{A_\alpha\} \). Finally, if \( \xi \in \mathfrak{H} \) then

\[
\|(A - A_\alpha)\xi\|^2 \leq \|A - A_\alpha\| \|A - A_\alpha\|^2 \xi\|^2 \leq \|A\| \|(\xi, (A - A_\alpha)\xi)\| \rightarrow_\alpha 0. \]  

(1.35)

Since the strong and \( \sigma \)-strong topology coincide on \( \mathcal{L}(\mathfrak{H})_1 \), this ends the proof.
Proposition 1.28 (§ p.68)

Let $\text{Tr}$ be the usual trace on $\mathcal{L}(\mathcal{H})$, and let $\mathcal{T}(\mathcal{H})$ be the Banach space of trace-class operators on $\mathcal{H}$ equipped with the trace norm $T \mapsto \text{Tr}(|T|) = ||T||_\tau$. Then it follows that $\mathcal{L}(\mathcal{H})$ is the dual $\mathcal{T}(\mathcal{H})^*$ of $\mathcal{T}(\mathcal{H})$ by the duality

$$A \times T \in \mathcal{L}(\mathcal{H}) \times \mathcal{T}(\mathcal{H}) \mapsto \text{Tr}(AT).$$

(1.36)

The weak* topology on $\mathcal{L}(\mathcal{H})$ arising from this duality is just the $\sigma$-weak topology.

Proof. (see § pp.68-69)

Due to the inequality $|\text{Tr}(AE_{\varphi,\psi})| \leq ||A|| ||\varphi|| ||\psi||$, $\mathcal{L}(\mathcal{H})$ is the subspace of $\mathcal{T}(\mathcal{H})^*$ by the duality described in the proposition. Conversely, assume $\omega \in \mathcal{T}(\mathcal{H})^*$ and consider a rank one operator $E_{\varphi,\psi}$ defined for $\varphi, \psi \in \mathcal{H}$ by

$$E_{\varphi,\psi} = \varphi(\psi, \chi).$$

(1.37)

One has $E_{\varphi,\psi}^* = E_{\psi,\varphi}$ and $E_{\varphi,\psi}E_{\psi,\varphi} = ||\psi||^2 E_{\varphi,\varphi}$. Hence

$$||E_{\varphi,\psi}||_\tau = ||\psi|| \text{Tr}(E_{\varphi,\varphi})^{1/2} = ||\psi|| ||\varphi||.$$

(1.38)

It follows that

$$|\omega(E_{\varphi,\psi})| \leq ||\omega|| ||\varphi|| ||\psi||.$$

(1.39)

Hence there exists, by the Riesz representation theorem, an $A \in \mathcal{L}(\mathcal{H})$ with $||A|| \leq ||\omega||$ such that

$$\omega(E_{\varphi,\psi}) = \langle \psi, A\varphi \rangle.$$

(1.40)

Consider $\omega_0 \in \mathcal{T}(\mathcal{H})^*$ defined by

$$\omega_0(T) = \text{Tr}(AT)$$

(1.41)

then

$$\omega_0(E_{\varphi,\psi}) = \text{Tr}(AE_{\varphi,\psi}) = \langle \psi, A\varphi \rangle = \omega_0(E_{\varphi,\psi}).$$

(1.42)

Now for any $T \in \mathcal{T}(\mathcal{H})$ there exist bounded sequences $\{\psi_n\}$ and $\{\varphi_n\}$ and a sequence $\{\alpha_n\}$ of complex numbers such that

$$\sum_n |\alpha_n| < \infty$$

(1.43)

and

$$T = \sum_n \alpha_n E_{\varphi_n,\psi_n}.$$

(1.44)

The latter series converges with respect to the trace norm and hence

$$\omega(T) = \sum_n \alpha_n \omega(E_{\varphi_n,\psi_n}) = \sum_n \alpha_n \omega_0(E_{\varphi_n,\psi_n}) = \omega_0(T) = \text{Tr}(AT).$$

(1.45)

Thus $\mathcal{L}(\mathcal{H})$ is just the dual of $\mathcal{T}(\mathcal{H})$. 
The weak* topology on $\mathcal{L}(\mathcal{H})$ arising from this duality is given by the seminorms
\[ A \in \mathcal{L}(\mathcal{H}) \mapsto |\text{Tr}(AT)|. \] (1.46)

Now, for $T$ as in (1.44), one has
\[
\text{Tr}(AT) = \sum_n \alpha_n \text{Tr}(E_{\varphi_n, \psi_n} A) = \sum_n \alpha_n (\psi_n, A\varphi_n). \] (1.47)

Thus the seminorms are equivalent to the seminorms defining the $\sigma$-weak topology.

It follows the theorem below:

**Theorem 1.29** ([8], p.76)

Let $\omega$ be a state on a von Neumann algebra $\mathfrak{A}$ acting on a Hilbert space $\mathcal{H}$. The following conditions are equivalent:

1. $\omega$ is normal;
2. $\omega$ is $\sigma$-weakly continuous;
3. there exists a density matrix $\rho$, i.e., a positive trace-class operator $\rho$ on $\mathcal{H}$ with $\text{Tr}(\rho) = 1$, such that
   \[ \omega(A) = \text{Tr}(\rho A). \] (1.48)

**Proof.** (See [8] pp.76 to 78).

$(3) \Rightarrow (2)$ follows from Proposition 1.28 and $(2) \Rightarrow (1)$ from Lemma 1.27. Next show $(2) \Rightarrow (3)$. If $\omega$ is $\sigma$-weakly continuous there exist sequences $\{\xi_n\}, \{\eta_n\}$ of vectors such that $\sum_n ||\xi_n||^2 < \infty, \sum_n ||\eta_n||^2 < \infty$, and $\omega(A) = \sum_n \langle \xi_n, A\eta_n \rangle$. Define $\tilde{\mathcal{H}} = \bigoplus_{n=1}^\infty \mathcal{H}_n$ and introduce a representation $\pi$ of $\mathfrak{A}$ on $\tilde{\mathcal{H}}$ by $\pi(A)(\bigoplus_n \psi_n) = \bigoplus_n (A\psi_n)$. Let $\xi = \bigoplus_n \xi_n, \eta = \bigoplus_n \eta_n$ and then $\omega(A) = \langle \xi, \pi(A)\eta \rangle$.

Since $\omega(A)$ is real for $A \in \mathfrak{A}_+$ (with $\mathfrak{A}_+$ denoting the positive part of the von Neumann algebra $\mathfrak{A}$ or the set of positive elements of the von Neumann algebra $\mathfrak{A}$), we have
\[
4\omega(A) = 2(\xi, \pi(A)\eta) + 2(\xi, \pi(A^*)\eta) = 2(\xi, \pi(A)\eta) + 2(\eta, \pi(A)\xi) = (\xi + \eta, \pi(A)(\xi + \eta)) - (\xi - \eta, \pi(A)(\xi - \eta)) \leq (\xi + \eta, \pi(A)(\xi + \eta)). \] (1.49)

Hence, by Theorem 1.14 there exists a positive $T \in \pi(\mathfrak{A})'$ with $0 \leq T \leq 1/2$ such that
\[
(\xi, \pi(A)\eta) = (T(\xi + \eta), \pi(A)T(\xi + \eta)) = (\psi, \pi(A)\psi). \] (1.50)

Now $\psi \in \tilde{\mathcal{H}}$ has the form $\psi = \bigoplus_n \psi_n$, and therefore
\[ \omega(A) = \sum_n \langle \psi_n, A\psi_n \rangle. \] (1.51)

The right side of this relation can be used to extend $\omega$ to a $\sigma$-weakly continuous positive linear functional $\tilde{\omega}$ on $\mathcal{L}(\mathcal{H})$. Since $\tilde{\omega}(1) = 1$, it is a state. Thus, by Proposition 1.28 there exists a trace-class operator $\rho$ with $\text{Tr}(\rho) = 1$ such that
\[ \tilde{\omega}(A) = \text{Tr}(\rho A). \] (1.52)
Let \( P \) be the rank one projector with range \( \xi \); then

\[
(\xi, \rho \xi) = Tr(P \rho P) = Tr(\rho P) = \tilde{\omega}(P) \geq 0. \tag{1.53}
\]

Thus \( \rho \) is positive.

Turn now to the proof of (1) \( \Rightarrow \) 2. Assume that \( \omega \) is a normal state on \( \mathfrak{A} \). Let \( \{B_\alpha\} \) be an increasing net of elements in \( \mathfrak{A}_+ \) such that \( ||B_\alpha|| \leq 1 \) for all \( \alpha \) and such that \( A \mapsto \omega(AB_\alpha) \) is \( \sigma \)-strongly continuous for all \( \alpha \). One can use Lemma 1.27 to define \( B \) by

\[
B = \ell.u.b. B_\alpha = \sigma\text{-strong lim}_\alpha B_\alpha. \tag{1.54}
\]

Then 0 \( \leq B \leq 1 \) and \( B \in \mathfrak{A} \). But for all \( A \in \mathfrak{A} \) we have

\[
||\omega(AB - AB_\alpha)||^2 = ||\omega(A(B - B_\alpha)^{1/2}(B - B_\alpha)^{1/2})||^2 \leq ||A||^2 \omega(B - B_\alpha). \tag{1.55}
\]

Hence

\[
||\omega(B) - \omega(B_\alpha)|| \leq (\omega(B) - \omega(B_\alpha))^{1/2}. \tag{1.56}
\]

But \( \omega \) is normal. Therefore \( \omega(B - B_\alpha) \to 0 \) and \( \omega(B_\alpha) \) tends to \( \omega(B) \) in norm. As \( \mathfrak{A}_+ \) is a Banach space, \( \omega(\cdot B) \in \mathfrak{A}_+ \). Now, applying Zorn’s lemma, we can find a maximal element \( P \in \mathfrak{A}_+ \cap \mathfrak{A}_1 \) such that \( A \mapsto \omega(AP) \) is \( \sigma \)-strongly continuous. If \( P = 1 \) the theorem is proved. Assume \textit{ad absurdum} that \( P \neq 1 \).

Put \( P' = 1 - P \) and choose \( \xi \in \mathcal{H} \) such that \( \omega(P') < (\xi, P' \xi) \). If \( \{B_\alpha\} \) is an increasing net in \( \mathfrak{A}_+ \), such that \( B_\alpha \leq P' \), \( \omega(B_\alpha) \geq (\xi, B_\alpha \xi) \), and \( B = \ell.u.b. B_\alpha = \sigma\text{-strong lim}_\alpha B_\alpha \), then \( B \in \mathfrak{A}_+ \), \( B \leq P' \). Since \( \omega(B) = 1 \) = \( \sup \omega(B_\alpha) \geq \sup (\xi, B_\alpha \xi) = (\xi, B_\xi) \). Hence, by Zorn’s lemma, there exists a maximal \( B \in \mathfrak{A}_+ \), such that \( B \leq P' \) and \( \omega(B) \geq (\xi, B_\xi) \). Take \( Q = P' - B \). Then, \( Q \in \mathfrak{A}_+ \), \( Q \neq 0 \), since \( \omega(P') < (\xi, P' \xi) \), and if \( A \in \mathfrak{A}_+ \), \( A \leq Q \), \( A \neq 0 \), then \( \omega(A) < (\xi, A \xi) \) by the maximality of \( B \).

For any \( A \in \mathfrak{A} \) one has

\[
QA^* AQ \leq ||A||^2 Q^2 \leq ||A||^2 ||Q||^2. \tag{1.57}
\]

Hence \( (QA^* AQ)/||A||^2 ||Q|| \leq Q \) and \( \omega(QA^* AQ) < (\xi, QA^* AQ \xi) \). Combining this with the Cauchy-Schwarz inequality one finds

\[
|\omega(AQ)|^2 \leq \omega(\mathbb{1}) \omega(QA^* AQ) < (\xi, QA^* AQ \xi) = ||AQ \xi||^2. \tag{1.58}
\]

Thus both \( A \mapsto \omega(AQ) \) and \( A \mapsto \omega(A(P + Q)) \) are \( \sigma \)-strongly continuous. Since \( P + Q \leq 1 \), this contradicts the maximality of \( P \).

\[\square\]

**Proposition 1.30** \( \square \) p. 86

Let \( \mathfrak{A} \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \).

Then the following four conditions are equivalent:

1. \( \mathfrak{A} \) is \( \sigma \)-finite;
2. there exists a countable subset of \( \mathcal{H} \) which is separating for \( \mathfrak{A} \);
3. there exists a faithful normal state on \( \mathfrak{A} \);
(4) $\mathcal{A}$ is isomorphic with a von Neumann algebra $\pi(\mathcal{A})$ which admits a separating and cyclic vector.

**Proof.** (see [8] p.86)

(1) $\Rightarrow$ (2) Let \( \{\xi_\alpha\} \) be a maximal family of vectors in $\mathcal{H}$ such that $[\mathcal{A}'\xi_\alpha]$ and $[\mathcal{A}'\xi_{\alpha'}]$ are orthogonal whenever $\alpha \neq \alpha'$. Since $[\mathcal{A}'\xi_\alpha]$ is a projection in $\mathcal{A}$ (in fact the smallest projection in $\mathcal{A}$ containing $\xi_\alpha$), $\{\xi_\alpha\}$ is countable. But by the maximality,

$$\sum_\alpha [\mathcal{A}'\xi_\alpha] = 1.$$  \hspace{1cm} (1.59)

Thus $\{\xi_\alpha\}$ is cyclic for $\mathcal{A}'$. Hence $\{\xi_\alpha\}$ is separating for $\mathcal{A}$ by Proposition 1.22.

(2) $\Rightarrow$ (3) Choose a sequence $\xi_n$ such that the set $\{\xi_n\}$ is separating for $\mathcal{A}$ and such that $\sum_n ||\xi_n||^2 = 1$.

Define $\omega$ by

$$\omega(A) = \sum_n (\xi_n, A\xi_n).$$  \hspace{1cm} (1.60)

$\omega$ is $\sigma$-weakly continuous, hence normal, by using Theorem 1.29. If $\omega(A^*A) = 0$ then $0 = (\xi_n, A^*A\xi_n) = ||A\xi_n||^2$ for all $n$, hence $A = 0$.

(3) $\Rightarrow$ (4) Let $\omega$ be a faithful normal state on $\mathcal{A}$ and $(\mathcal{H}, \pi, \Omega)$ the corresponding cyclic representation. Since $\pi(\mathcal{A})$ is a von Neumann algebra, if $\pi(A)\Omega = 0$ for an $A \in \mathcal{A}$ then $\omega(A^*A) = ||\pi(A)\Omega||^2 = 0$, hence $A^*A = 0$ and $A = 0$. This proves that $\pi$ is faithful and $\Omega$ separating for $\pi(\mathcal{A})$.

(4) $\Rightarrow$ (1) Let $\Omega$ be the separating (and cyclic) vector for $\pi(\mathcal{A})$, and let $\{E_\alpha\}$ be a family of mutually orthogonal projections in $\mathcal{A}$. Set $E = \sum_\alpha E_\alpha$. Then

$$||\pi(E)\Omega||^2 = (\pi(E)\Omega, \pi(E)\Omega) = \sum_{\alpha, \alpha'} (\pi(E_\alpha)\Omega, \pi(E_{\alpha'})\Omega) = \sum_\alpha ||\pi(E_\alpha)\Omega||^2$$  \hspace{1cm} (1.61)

by Lemma 1.27.

Since $\sum_\alpha ||\pi(E_\alpha)\Omega||^2 < +\infty$, only a countable number of the $\pi(E_\alpha)\Omega$ is nonzero, and thus the same is true for the $E_\alpha$.

\[ \square \]

### 1.2 Hilbert space of Hilbert-Schmidt operators

Here we recall some definitions provided in [1][2][21] (and references therein).

**Definition 1.31** The trace of a linear operator

A linear operator $A$ defined on the separable Hilbert space $\mathcal{H}$ is said to be of trace class if the series $\sum_k \langle e_k | Ae_k \rangle$ converges and has the same value in any orthonormal basis $\{e_k\}$ of $\mathcal{H}$. The sum

$$\text{Tr}A = \sum_k \langle e_k | Ae_k \rangle$$  \hspace{1cm} (1.62)

is called the trace of $A$.

**Definition 1.32** Trace norm

Consider the class of Hilbert-Schmidt operators. For every such operator $A$, the trace norm is given by

$$\text{Tr}[\sqrt{A^*A}] = \text{Tr}[\sum_k |e_k\rangle \lambda_k \langle e_k|] = \sum_k \lambda_k < +\infty.$$  \hspace{1cm} (1.63)
Remark 1.33 If the series (1.63) is infinite, its convergence implies that \( \lambda_k \to 0 \) when \( k \to +\infty \). Consequently, \( \lambda_k^2 \geq \lambda_k \) for sufficiently large value of \( k \). Hence, \( \sum_k \lambda_k^2 \) converges when \( \sum_k \lambda_k \) converges. This shows that any completely continuous operator \( A \) satisfying (1.63) is a Hilbert-Schmidt operator.

Definition 1.34 Hilbert-Schmidt operator
Given a bounded operator, having the decomposition \( A = \sum_k |\phi_k\rangle \lambda_k \langle \phi_k| \), where \{\( \phi_k \)\} is an orthonormal basis of \( \mathcal{H} \), and \( \lambda_1, \lambda_2, \ldots \) positive numbers, \( A \) is called a *Hilbert-Schmidt operator* if
\[
\text{Tr}[AA^*] = \sum_k \langle \phi_k | A^* A | \phi_k \rangle = \sum_k \lambda_k^2 < +\infty.
\] (1.64)

Remark 1.35 If \( A \) is any operator of trace class, then \( A^* \) is also of trace class:
\[
\text{Tr}A^* = \sum_k \langle e_k | A^* e_k \rangle = \sum_k \langle e_k | A e_k \rangle^* = (\text{Tr}A)^*.
\] (1.65)

Definition 1.36 Hilbert-Schmidt norm
For any Hilbert-Schmidt operator \( A \), the quantity
\[
||A||_2 = \sqrt{\text{Tr}[A^* A]}
\] (1.66)
exists, and is called *Hilbert-Schmidt norm* of \( A \).

Definition 1.37
Let \( B_2(\mathcal{H}) \), \( B_2(\mathcal{H}) \subset \mathcal{L}(\mathcal{H}) \) the set of all bounded operators on \( \mathcal{H} \), be the Hilbert space of Hilbert-Schmidt operators on \( \mathcal{H} = L^2(\mathbb{R}) \), with the scalar product
\[
\langle X|Y \rangle_2 = \text{Tr}[X^* Y] = \sum_k \langle \Phi_k | X^* Y \Phi_k \rangle,
\] (1.67)
where \{\( \Phi_k \)\}_{k=0}^\infty is an orthonormal basis of \( \mathcal{H} \).

\( B_2(\mathcal{H}) \simeq \mathcal{H} \otimes \mathcal{H} \) (where \( \mathcal{H} \) denotes the dual of \( \mathcal{H} \)) and basis vectors of \( B_2(\mathcal{H}) \) are given by
\[
\Phi_{n,l} := |\Phi_n\rangle \langle \Phi_l|, \quad n, l = 0, 1, 2, \ldots, \infty.
\] (1.68)

Remark 1.38 In the notation \( B_2(\mathcal{H}) \simeq \mathcal{H} \otimes \mathcal{H} \), \( \mathcal{H} \otimes \mathcal{H} \) is taken here as the completude of the algebraic tensor product of \( \mathcal{H} \) by \( \mathcal{H} \) which is a prehilbert space containing finite sums of the type \( \sum_{j,k=0}^n \lambda_{jk} |\phi_j\rangle \otimes |\phi_k\rangle \), where a basis of \( \mathcal{H} \otimes \mathcal{H} \) is \( \{|\phi_j\rangle \otimes |\phi_k\rangle\}_{j,k=0}^\infty \). Then, \( B_2(\mathcal{H}) \), being the Hilbert space of Hilbert-Schmidt operators on \( \mathcal{H} \) is isomorphic to \( \mathcal{H} \otimes \mathcal{H} \), since the separable Hilbert spaces are taken two by two isomorphic each other. Setting \( |\phi_j\rangle \otimes |\phi_k\rangle = |\phi_j\rangle \langle \phi_k| \), \( B_2(\mathcal{H}) \) admits for orthonormal basis \( \{\phi_{jk}\}_{j,k=0}^\infty \) such that \( \phi_{jk} := |\phi_j\rangle \langle \phi_k| \).

Definition 1.39
Let \( A \) and \( B \) two operators on \( \mathcal{H} \). The operator \( A \vee B \), is such that
\[
A \vee B(X) = AXB^*, \; X \in B_2(\mathcal{H}).
\] (1.69)
For \( A \) and \( B \), both bounded operators, \( A \vee B \) defines a linear operator on \( B_2(\mathcal{H}) \).

Indeed, \( \forall A, B \in \mathcal{L}(\mathcal{H}) \) (the space of bounded linear operators on \( \mathcal{H} \)), since \( B_2(\mathcal{H}) \subset \mathcal{L}(\mathcal{H}) \) and from (1.69), we get \( \forall X \in B_2(\mathcal{H}) \) satisfying the relation, \( X \in \mathcal{L}(\mathcal{H}) \). Then, \( AXB^* \in \mathcal{L}(\mathcal{H}) \), i.e. \( (A \vee B) \in \mathcal{L}(\mathcal{H}) \). Thus, \( A \vee B \) defines a bounded linear operator on \( B_2(\mathcal{H}) \).

If \( A, B \in \mathcal{L}(\mathcal{H}) \), the operator \( A \vee B \) on \( B_2(\mathcal{H}) \) acts on a vector \( X \) on \( B_2(\mathcal{H}) \) as follows
\[
A \vee B(X) = AXB^*, \; X \in B_2(\mathcal{H}).
\] (1.70)
Since, $\mathcal{B}_2(\mathfrak{F}) \subset \mathcal{L}(\mathfrak{F})$ and from (1.70), it follows that $AXB^* \in \mathcal{L}(\mathfrak{F})$. Then, $(A \lor B) \in \mathcal{L}(\mathfrak{F})$.

From the scalar product in $\mathcal{B}_2(\mathfrak{F})$

$$\langle X|Y\rangle_2 = Tr[X^*Y], \quad X,Y \in \mathcal{B}_2(\mathfrak{F})$$

(1.71)

it comes

$$Tr[X^*(AYB^*)] = Tr[(A^*XB)^*Y] \Rightarrow (A \lor B)^* = A^* \lor B^*$$

(1.72)

and since for any $X \in \mathcal{B}_2(\mathfrak{F})$

$$(A_1 \lor B_1)(A_2 \lor B_2)(X) = A_1[(A_2 \lor B_2)(X)]B_1^* = A_1A_2XB_2^*B_1^*$$

(1.73)

giving

$$(A_1 \lor B_1)(A_2 \lor B_2) = (A_1A_2) \lor (B_1B_2).$$

(1.74)

Before introducing the von Neumann algebra generated by the unitary operators, let us consider the following:

**Definition 1.40**

Consider the unitary operator $U(x,y)$ on $\mathfrak{F}$ given by

$$(U(x,y)\Phi)(\xi) = e^{-ix(\xi - y/2)}\Phi(\xi - y),$$

(1.75)

$x,y,\xi \in \mathbb{R}$, where $U(x,y) = e^{-i(xQ+yP)}$, with $[Q,P] = i\hbar\mathfrak{F}$ and the Wigner transform, given by

$$\mathcal{W} : \mathcal{B}_2(\mathfrak{F}) \rightarrow L^2(\mathbb{R}^2,dxdy)$$

$$\langle WXX(x,y) = \frac{1}{(2\pi)^{1/2}}Tr[(U(x,y))^*X],$$

(1.76)

where $X \in \mathcal{B}_2(\mathfrak{F})$, $x,y \in \mathbb{R}$. $\mathcal{W}$ is unitary.

Indeed, given $X_1, X_2 \in \mathcal{B}_2(\mathfrak{F})$,

$$\int_{\mathbb{R}^2} \langle WX_2(x,y)(WX_1(x,y))dxdy = \langle X_2|X_1\rangle_2 = \langle X_2|X_1\rangle_{\mathcal{B}_2(\mathfrak{F})}.$$

(1.77)

### 1.3 Modular theory and Hilbert-Schmidt operators

This paragraph is devoted to Tomita-Takesaki modular theory of von Neumann algebras. Recall that the origins of Tomita-Takesaki modular theory lie in two unpublished papers of M. Tomita in 1967 and a slim volume by M. Takesaki. It has developed into one of the most important tools in the theory of operator algebras and has found many applications in mathematical physics.

We provide some key ingredients from as needed for this section. First, let us deal with some notions from \[1,2,7,8,25\]:

1. Let $\mathfrak{A}$ be a von Neumann algebra on a Hilbert space $\mathfrak{A}$ and $\mathfrak{A}'$ its commutant. Let $\Phi \in \mathfrak{F}$ be a unit vector which is cyclic and separating for $\mathfrak{A}$. This is the case, if $\mathfrak{A}$ is a $\sigma$-finite von Neumann algebra, and by applying Proposition 1.22.

2. The mapping $A \in \mathfrak{A} \mapsto A\Omega \in \mathfrak{F}$ then establishes a one-to-one linear correspondence between $\mathfrak{A}$ and a dense subspace $\mathfrak{A}\Omega$ of $\mathfrak{F}$. Let $S_0$ and $F_0$ two antilinear operators on $\mathfrak{A}\Omega$ and $\mathfrak{A}'\Omega$, respectively. By
Proposition 1.22 \( \Omega \) is cyclic and separating for \( \mathfrak{A} \) and \( \mathfrak{A}' \). Therefore the two antilinear operators \( S_0 \) and \( F_0 \), given by

\[
S_0 A \Omega = A^* \Omega, \quad \text{for} \quad A \in \mathfrak{A} \\
F_0 A' \Omega = A'^* \Omega, \quad \text{for} \quad A' \in \mathfrak{A}'
\]

are both well defined on the dense domains on \( D(S_0) = \mathfrak{A} \Omega \) and \( D(F_0) = \mathfrak{A}' \Omega \). Then follows the definition:

**Definition 1.41**

Define \( S \) and \( F \) as the closures of \( S_0 \) and \( F_0 \), respectively, i.e.,

\[
S = \overline{S_0}, \quad F = \overline{F_0}
\]

where the bar denotes the closure. Let \( \Delta \) be the unique, positive, selfadjoint operator and \( J \) the unique antiunitary operator occurring in the polar decomposition

\[
S = J \Delta^{1/2}
\]

of \( S \). \( \Delta \) is called the *modular operator* associated with the pair \( \{ \mathfrak{A}, \Omega \} \) and \( J \) the *modular conjugation*.

The following proposition provides connections between \( S, F, \Delta \) and \( J \):

**Proposition 1.42** (\[8\] p.89)

The following relations are valid:

\[
\begin{align*}
\Delta &= FS, \quad \Delta^{-1} = SF \\
S &= J \Delta^{1/2}, \quad F = J \Delta^{-1/2} \\
J &= J^*, \quad J^2 = I_\mathcal{H} \\
\Delta^{-1/2} &= J \Delta^{1/2} J
\end{align*}
\]

**Proof.** (see \[8\] pp.89-90)

\( \Delta = S^* S = F S \), and \( S = J \Delta^{1/2} \) by Definition 1.41. Using the fact that for any \( \psi \in D(\overline{S_0}) \) there exists a closed operator \( Q \), on \( \mathcal{H} \), with \( S_0^* = \overline{F_0}, F_0^* = \overline{S_0} \), such that

\[
Q \Omega = \psi, \quad Q^* \Omega = \overline{S_0} \psi
\]

where \( \mathfrak{A}'D(Q) \subseteq D(Q), QQ' \supseteq Q'Q \) for all \( Q' \in \mathfrak{A}' \), with \( S_0 = S_0^{-1} \), it follows by closure that \( S = S^{-1} \), and hence

\[
J \Delta^{1/2} = S = S^{-1} = \Delta^{1/2} J^*,
\]

so that \( J^2 \Delta^{1/2} = J \Delta^{-1/2} J^* \). Since \( J \Delta^{-1/2} J^* \) is a positive operator, and by the uniqueness of the polar decomposition one deduces that

\[
J^2 = I_\mathcal{H}
\]

and then

\[
J^* = J, \quad \Delta^{-1/2} = J \Delta^{1/2} J
\]
But this implies that

\[ F = S^* = (\Delta^{-1/2} J)^* = J \Delta^{-1/2} \]  

(1.86)

and

\[ SF = \Delta^{-1/2} J J \Delta^{-1/2} = \Delta^{-1}. \]  

(1.87)

3. The principal result of the Tomita-Takesaki theory [8, 25] is that the following relations

\[ J^\mathfrak{A} J = \mathfrak{A}', \quad \Delta^i \mathfrak{A} \Delta^{-i} = \mathfrak{A} \]  

(1.88)

hold for all \( t \in \mathbb{R} \).

4. Definition 1.43 Modular automorphism group

Let \( \mathfrak{A} \) be a von Neumann algebra, \( \omega \) a faithful, normal state on \( \mathfrak{A} \), \( (\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega) \) the corresponding cyclic representation, and \( \Delta \) the modular operator associated with the pair \( (\omega(\mathfrak{A}), \Omega_\omega) \). The Tomita-Takesaki theorem establishes the existence of a \( \sigma \)-weakly continuous one-parameter group \( t \mapsto \sigma_\omega^t \) of \( * \)-automorphisms of \( \mathfrak{A} \) through the definition

\[ \sigma_\omega^t(A) = \pi^{-1}_\omega((\Delta^i \pi_\omega(A) \Delta^{-i})). \]  

(1.89)

The group \( t \mapsto \sigma_\omega^t \) is called the modular automorphism group associated with the pair \( (\mathfrak{A}, \omega) \).

5. Definition 1.44 \( C^* \)-dynamical system

A \( C^* \)-dynamical system \( (\mathfrak{G}, \alpha) \) is a \( C^* \)-algebra \( \mathfrak{G} \) equipped with a group homomorphism \( \alpha : G \to \text{Aut}(\mathfrak{G}) \) that is strongly continuous i.e., \( g \mapsto ||\alpha_g(x)|| \) is a continuous map for all \( x \in \mathfrak{G} \).

A von Neumann dynamical system \( (\mathfrak{A}, \alpha) \) is a von Neumann algebra acting on the Hilbert space \( \mathfrak{H} \) equipped with a group homomorphism \( \alpha : G \to \text{Aut}(\mathfrak{A}) \) that is weakly continuous i.e., \( g \mapsto \langle \xi | \alpha_g(x) \eta \rangle \) is continuous for all \( x \in \mathfrak{A} \) and all \( \xi, \eta \in \mathfrak{H} \).

Definition 1.45

A state \( \omega \) on a one-parameter \( C^* \)-dynamical system \( (\mathfrak{G}, \alpha) \) is a \( (\alpha, \beta) \)-KMS state, for \( \beta \in \mathbb{R} \), if for all pairs of elements \( x, y \) in a norm dense \( \alpha \)-invariant \( * \)-subalgebra of \( \alpha \)-analytic elements of \( \mathfrak{G} \), then \( \omega(x \alpha_{\alpha \beta}(y)) = \omega(yx) \).

Remark 1.46 In the case of a von Neumann dynamical system \( (\mathfrak{A}, \alpha) \), a \( (\alpha, \beta) \)-KMS state must be normal (i.e., for every increasing bounded net of positive elements \( x_\lambda \to x \), we have \( \omega(x_\lambda) \to \omega(x) \)). Besides, given \( \alpha : \mathbb{R} \to \text{Aut}(\mathfrak{A}) \), an element \( x \in \mathfrak{G} \) is \( \alpha \)-analytic if there exists a holomorphic extension of the map \( t \mapsto \alpha_t(x) \) to an open horizontal strip \( \{ z \in \mathbb{C} ||\text{Im } z| < r \} \), with \( r > 0 \), in the complex plane. The set of \( \alpha \)-analytic elements is always \( \alpha \)-invariant (i.e., for all \( x \) is analytic, \( \alpha(x) \) is analytic) \( * \)-subalgebra of \( \mathfrak{G} \) that is norm dense in the \( C^* \) case and weakly dense in the von Neumann case.

6. The modular automorphism group associated with \( \omega \) is only the one parameter automorphism group that satisfies the Kubo-Martin-Schwinger KMS-condition with respect to the state \( \omega \), at inverse temperature \( \beta \), i.e.,

\[ \omega(\sigma_\omega^t(x)) = \omega(x), \quad \forall x \in \mathfrak{A} \]  

(1.90)
and for all \(x, y \in \mathfrak{A}\), there exists a function \(F_{x,y} : \mathbb{R} \times [0, \beta] \to \mathbb{C}\) such that:

\[
F_{x,y} \quad \text{is holomorphic on} \quad \mathbb{R} \times [0, \beta],
\]
\[
F_{x,y} \quad \text{is bounded continuous on} \quad \mathbb{R} \times [0, \beta],
\]
\[
F_{x,y}(t) = \omega(\sigma_t^y(x)), \quad t \in \mathbb{R},
\]
\[
F_{x,y}(i\beta + t) = \omega(x\sigma_t^y(y)), \quad t \in \mathbb{R}.
\]

KMS state

Let \(\alpha_i, i = 1, 2, \ldots, N\) be a sequence of non-zero, positive numbers, satisfying:

\[
\sum_{i=1}^N \alpha_i = 1.
\]

Let

\[
\Phi = \sum_{i=1}^N \alpha_i \mathcal{P}_i = \sum_{i=1}^N \alpha_i X_{\mathcal{P}_i} \in B_2(\mathfrak{H}) \quad \text{with} \quad X_{\mathcal{P}_i} = |\zeta_i\rangle \langle \zeta_i|.
\]

Then, we have the following properties:

(i) **Proposition 1.47** \(\Phi\) defines a vector state \(\varphi\) on the von Neumann algebra \(\mathfrak{A}_l\), corresponding to the operators given with \(A\) in the left of the identity operator \(I_\mathfrak{H}\), i.e., \(\mathfrak{A}_l = \{A_\mathfrak{H} = A \vee I_\mathfrak{H} | A \in \mathcal{L}(\mathfrak{H})\}\).

**Proof.**

Indeed, for any \(A \vee I \in \mathfrak{A}_l\), since \(B_2(\mathfrak{H}) \subset \mathcal{L}(\mathfrak{H})\) and \(\mathfrak{A} \subset \mathcal{L}(\mathfrak{H})\), from the Remark 1.26 and the equality (1.34) together, the state \(\varphi\) on \(\mathfrak{A}_l\) may be defined by

\[
\langle \varphi; A \vee I \rangle = \langle \Phi | (A \vee I)(\Phi) \rangle = \text{Tr}[\Phi^* A \Phi] = \text{Tr}[\rho_\varphi A], \quad \text{with} \quad \rho_\varphi = \sum_{i=1}^N \alpha_i \mathcal{P}_i.
\]  

(ii) **Proposition 1.48** The state \(\varphi\) is faithful and normal.

**Proof.**

The state \(\varphi\) is normal by Theorem 1.29 using the fact that \(\rho_\varphi\) is a density matrix and since we have

\[
\langle \varphi; A \vee I \rangle = \text{Tr}[\rho_\varphi A].
\]

Its faithfulness comes from Proposition 1.30 by use of the equivalence (2) \(\Leftrightarrow\) (3), since \(\mathfrak{A}_l \subset \mathcal{L}(\mathfrak{H})\) using the Eq. (1.34) in the Remark 1.26 we have, with \(\mathcal{P} = |\zeta_i\rangle \langle \zeta_i|\),

\[
\langle \varphi; (A \vee I)^*(A \vee I) \rangle := \varphi\left(\{\{A \vee I\}^*(A \vee I)\}\right) = \text{Tr}[\rho_\varphi A^* A] \quad \text{[by 1.34]}
\]
\[
= \sum_{k=1}^N \langle \zeta_k | \rho_\varphi A^* A | \zeta_k \rangle \quad \text{[by 1.67]}
\]
\[
= \sum_{k=1}^N \langle \zeta_k | \left( \sum_{i=1}^N \alpha_i |\zeta_i\rangle \langle \zeta_i| \right) A^* A | \zeta_k \rangle \quad \text{[by 1.93]}
\]
\[
= \sum_{k=1}^N \sum_{i=1}^N \alpha_i \langle \zeta_i | A^* A | \zeta_k \rangle \langle \zeta_k | \zeta_i \rangle
\]
\[
= \sum_{i=1}^N \alpha_i \|A |\zeta_i\|\|^2, \quad \alpha_i > 0, \quad A \in \mathcal{L}(\mathfrak{H})
\]

where the \(\{\zeta_i\}_{i=1}^N\) form an orthonormal basis set of \(\mathfrak{H}\). \(\Phi\) is separating for \(\mathfrak{A}_l\), by use of Theorem 1.29 and the relation

\[
\langle \varphi; (A \vee I)^*(A \vee I) \rangle = 0 \iff \sum_{i=1}^N \alpha_i \|A |\zeta_i\|\|^2 = 0, \quad \forall i = 1, 2, \ldots, N
\]
\[
\iffalse
A \lor I = 0 \iff A = 0.
\fi
\]
(1.96)

Thereby, \( \langle \varphi; (A \lor I)^\ast (A \lor I) \rangle = 0 \) if and only if \( A \lor I = 0 \).

\[\Box\]

(iii) **Proposition 1.49** The vector \( \Phi \) is cyclic and separating for \( \mathfrak{A}_I \).

**Proof.**

If \( X \in \mathcal{B}_2(\mathcal{H}) \) is orthogonal to all \( (A \lor I)\Phi, A \in \mathcal{L}(\mathcal{H}) \), then

\[
\text{Tr}[X^\ast A\Phi] = \sum_{i=1}^{N} \alpha_i^T \langle \zeta_i | X^\ast A \zeta_i \rangle = 0, \quad \forall A \in \mathcal{L}(\mathcal{H}).
\]
(1.97)

Taking \( A = X_{kl} \), it follows from the above equality, \( \langle \zeta_i | X^\ast \zeta_k \rangle = 0 \) and since this holds for all \( k, l \), we get \( X = 0 \). Indeed, let \( \Phi = \sum_{i=1}^{N} \alpha_i^T \varphi_i = \sum_{i=1}^{N} \alpha_i^T X_{ii} = \sum_{i=1}^{N} \alpha_i^T |\zeta_i\rangle \langle \zeta_i| \), then by definition, see Eq.(1.67),

\[
\langle X | (A \lor I) \Phi \rangle = \text{Tr}[X^\ast A \Phi]
\]
(1.99)

such that the orthogonality implies

\[
\langle X | (A \lor I) \Phi \rangle^2 = 0 \iff \sum_{i=1}^{N} \alpha_i^T \langle \zeta_i | X^\ast A \zeta_i \rangle = 0.
\]
(1.100)

Now taking \( A = X_{kl} = |\zeta_k\rangle \langle \zeta_l| \), it follows that

\[
\sum_{i=1}^{N} \alpha_i^T \langle \zeta_i | X^\ast A \zeta_i \rangle = \sum_{i=1}^{N} \alpha_i^T \langle \zeta_i | X^\ast \{ |\zeta_k\rangle \langle \zeta_l| \} | \zeta_i \rangle = \sum_{i=1}^{N} \alpha_i^T \langle \zeta_i | X^\ast \zeta_k \rangle \delta_{il} = \alpha_i^T \langle \zeta_i | X^\ast \zeta_k \rangle.
\]
(1.101)

From (1.99) and (1.100) together, it follows

\[
\sum_{i=1}^{N} \alpha_i^T \langle \zeta_i | X^\ast A \zeta_i \rangle = 0 \iff \alpha_i^T \langle \zeta_i | X^\ast \zeta_k \rangle = 0, \quad \forall \alpha_i > 0.
\]
(1.102)

Thereby,

\[
\langle \zeta_i | X^\ast \zeta_k \rangle = 0, \quad \forall k, l \iff X = 0.
\]
(1.102)
Therefore, we have

\[ \langle X | (A \lor I) \Phi \rangle_2 = 0 \implies X = 0 \]  \hspace{1cm} (1.103)

implying that the set \( \{(A \lor I) \Phi, A \in \mathfrak{A}_l\} \) is dense in \( B_2(\mathfrak{S}) \), proving from the Definition 1.19 that \( \Phi \) is cyclic for \( \mathfrak{A}_l \).

The fact that \( \Phi \) is separating for \( \mathfrak{A}_l \) is obtained through the relation

\[ (A \lor I) \Phi = (B \lor I) \Phi \iff A \lor I = B \lor I, \quad \forall A, B \in \mathfrak{A}_l. \]  \hspace{1cm} (1.104)

**Proof.**

Let \( A, B \in \mathfrak{A}_l \), such that \( (A \lor I) \Phi = (B \lor I) \Phi \), and take \( X \neq 0, X \in B_2(\mathfrak{S}) \). We have

\[
\begin{align*}
\langle X | (A \lor I) - (B \lor I) \Phi \rangle_2 & = \text{Tr} \left[ X^* \{(A \lor I) - (B \lor I) \Phi\} \right]_N \\
& = \text{Tr} \left[ X^*(A - B) \Phi I^* \right]_N \quad \text{[by (1.70)]} \\
& = \sum_{k=1}^N \langle \zeta_k | X^*(A - B) \rangle \left\{ \sum_{i=1}^N \alpha_i^\pm \langle \zeta_i | \xi_k \rangle \right\} \langle \zeta_k | \xi_k \rangle \quad \text{[by (1.92)]} \\
& = \sum_{i=1}^N \sum_{k=1}^N \alpha_i^\pm \langle \zeta_k | X^*(A - B) \rangle \langle \zeta_i | \zeta_k \rangle \\
& = \sum_{i=1}^N \alpha_i^\pm \langle \zeta_i | X^*(A - B) \rangle \langle \zeta_i | \zeta_i \rangle \quad \text{[by (1.98) and (1.100)]} \\
& = \sum_{i=1}^N \alpha_i^\pm \langle \zeta_i | X^*(A - B) \rangle \langle \zeta_i | \zeta_i \rangle
\end{align*}
\]

Taking \( (A \lor I) \Phi = (B \lor I) \Phi \), the equality \( \langle X | (A \lor I) - (B \lor I) \Phi \rangle_2 = 0 \) leads to

\[
\begin{align*}
\langle X | (A \lor I) - (B \lor I) \Phi \rangle_2 = 0 & \iff \sum_{i=1}^N \alpha_i^\pm \langle \zeta_i | X^*(A - B) \rangle \langle \zeta_i | \zeta_i \rangle = 0, \quad \alpha_i > 0 \\
& \iff A \lor I = B \lor I
\end{align*}
\]

which completes the proof.

\[
\square
\]

In the same way, \( \Phi \) is also cyclic for \( \mathfrak{A}_r = \{ A_r = I \lor A | A \in \mathcal{L}(\mathfrak{S})\} \), which corresponds to the operators given with \( A \) in the right of the identity operator \( I_\mathfrak{S} \) on \( \mathfrak{S} \), hence separating for \( \mathfrak{A}_r \), i.e. \( (I \lor A) \Phi = (I \lor B) \Phi \iff I \lor A = I \lor B \).

Then, starting to the above setup, to the pair \( \{ \mathfrak{A}, \varphi \} \) is associated:

- a **one parameter unitary group** \( t \mapsto \Delta^\varphi_t \in \mathcal{L}(\mathfrak{S}) \)
- and a **conjugate-linear isometry** \( J_\varphi : \mathfrak{S} \rightarrow \mathfrak{S} \) that:

\[
\Delta^\varphi_t \mathfrak{A} \Delta^\varphi_{-t} = \mathfrak{A}, \quad t \in \mathbb{R},
\]

\[ J_\varphi \mathfrak{A} J_\varphi = \mathfrak{A}', \]

\[ J_\varphi \circ J_\varphi = I_\mathfrak{S}, \quad J_\varphi \circ \Delta^\varphi_t = \Delta^\varphi_{-t} \circ J_\varphi. \]  \hspace{1cm} (1.109)
Denote the automorphisms by $\alpha_\varphi(t)$, and deal with operators $A \in \mathfrak{A}$ with $\mathfrak{A} \subset \mathcal{L}(\mathfrak{H})$. Then, taking into account the Definition 1.25 and the Remark 1.26 from the expression (1.89), the automorphisms, in this case, satisfy the following relation:

$$\alpha_\varphi(t)[A] = \Delta^{+\beta}_t A \Delta^{-\beta}_t, \quad \forall A \in \mathfrak{A}. \quad (1.110)$$

The KMS condition with respect to the automorphism group $\alpha_\varphi(t), t \in \mathbb{R}$, is obtained for any two $A, B \in \mathfrak{A}$, such that the function

$$F_{A,B}(t) = \langle \varphi; A \alpha_\varphi(t)[B] \rangle$$

has an extension to the strip $\{ z = x + iy | t \in \mathbb{R}, y \in [0, \beta] \} \subset \mathbb{C}$ such that $F_{A,B}(z)$ is analytic in the strip $(0, \beta)$ and continuous on its boundaries. In addition, it also satisfies the boundary condition, at an inverse temperature $\beta$

$$\langle \varphi; A \alpha_\varphi(t+i\beta)[B] \rangle = \langle \varphi; \alpha_\varphi(t)[B] A \rangle, \quad t \in \mathbb{R}. \quad (1.112)$$

Setting the generator of the one-parameter group by $H_\varphi$, the operators $\Delta_{\varphi}^{-+\beta}$ verify the relation

$$\Delta_{\varphi}^{-+\beta} = e^{itH_\varphi} \quad \text{and} \quad \Delta_{\varphi} = e^{-\beta H_\varphi}. \quad (1.113)$$

On $\tilde{\mathfrak{H}} = L^2(\mathbb{R}^2, dxdy)$, $\forall (x, y) \in \mathbb{R}^2$, consider the operators

$$
\begin{align*}
U_1(x, y) &= W[U(x, y) \lor I_\beta] W^{-1}, \\
U_2(x, y) &= W[I_\beta \lor U(x, y)^*] W^{-1}
\end{align*}
$$

and let $\mathfrak{A}_i, i = 1, 2$, be the von Neumann algebra generated by the unitary operators $\{U_i(x, y)/(x, y) \in \mathbb{R}^2\}$. Then, it follows that:

**Proposition 1.50** (1)

(i) The algebra $\mathfrak{A}_1$ is the commutant of the algebra $\mathfrak{A}_2$ (i.e. each element of $\mathfrak{A}_1$ commutes with every element of $\mathfrak{A}_2$) and vice versa with a factor, i.e.

$$\mathfrak{A}_1 \cap \mathfrak{A}_2 = \mathbb{C} I_{\tilde{\mathfrak{H}}}. \quad (1.115)$$

Considering the antiunitary operator $J_\beta$ (i.e. $\langle J_\beta \phi | J_\beta \psi \rangle = \langle \psi | \phi \rangle$, $\forall \phi, \psi \in \tilde{\mathfrak{H}} = L^2(\mathbb{R})$) such that:

$$J_\beta \Psi_{nl} = \Psi_{ln}, \quad J_\beta^2 = I_\beta, \quad J_\beta \Phi_\beta = \Phi_\beta,$$

it comes

$$J_\beta \mathfrak{A}_1 J_\beta = \mathfrak{A}_2. \quad (1.116)$$

The relation (1.116) and the property (i) provide the modular structure of the triplet $\{\mathfrak{A}_1, \mathfrak{A}_2, J_\beta\}$.

(ii) The map $S_\beta : \tilde{\mathfrak{H}} \rightarrow \tilde{\mathfrak{H}}$, $S_\beta[U_1(x, y) \Phi_\beta] = U_1(x, y)^* \Phi_\beta$,

is closable and has the polar decomposition

$$S_\beta = J_\beta \Delta^\frac{1}{2}_\beta, \quad (1.117)$$

where $J_\beta$ is the antiunitary operator, with

$$J_\beta \Psi_{nl} = \Psi_{ln}, \quad J_\beta^2 = I_\beta, \quad J_\beta \Phi_\beta = \Phi_\beta, \quad (1.118)$$

$$J_\beta \mathfrak{A}_1 J_\beta = \mathfrak{A}_2. \quad (1.119)$$
Indeed, $J_{\beta}$ is by definition an antiunitary operator, then, it’s self-adjoint and symmetric and consequently closable. Also, $\Delta_{\beta}^{\frac{1}{2}}$ being self-adjoint by definition, it’s then closable. From (1.117), the map $S_{\beta}$ given as the product of two closable operators, is then closable.

**Proof of (1.117).** The proof is achieved as follows:

The vectors $\Psi_{jk}$, $j, k = 0, 1, 2, \ldots, \infty$, form an orthonormal basis of $\tilde{\mathcal{H}} = L^2(\mathbb{R}^2, dx dy)$. We have $\Phi_{\beta} = \sum_{i=0}^{\infty} \lambda_{ii}^\beta \Psi_{ii}$. Applying $U_1(x, y)$ to both sides leads to

$$U_1(x, y)\Phi_{\beta} = \sum_{i=0}^{\infty} \lambda_{ii}^\beta U_1(x, y) \Psi_{ii}.$$ 

Since $\sum_{j,k=0}^{\infty} |\Psi_{jk}\rangle \langle \Psi_{jk}| = I_{\tilde{\mathcal{H}}}$, we get

$$U_1(x, y)\Phi_{\beta} = \sum_{i=0}^{\infty} \lambda_{ii}^\beta U_1(x, y) \Psi_{ii} = \sum_{i,j,k=0}^{\infty} \lambda_{ii}^\beta \langle \Psi_{jk}|U_1(x, y)\Psi_{ii}\rangle_{\tilde{\mathcal{H}}} \Psi_{jk}.$$ 

From the relations

$$\phi_{nl} = |\phi_n\rangle \langle \phi_l| \quad \text{and} \quad \mathcal{W}\phi_{nl} = \Psi_{nl}, \quad n, l = 0, 1, 2, \ldots, \infty$$

we get

$$\mathcal{W}\phi_{jk} = \mathcal{W}(|\phi_j\rangle \langle \phi_k|) = \Psi_{jk}, \quad \forall j, k.$$ 

Using the fact that $\phi_i$, $i = 0, 1, 2, \ldots, \infty$, form a basis of $\tilde{\mathcal{H}} = L^2(\mathbb{R})$, we have

$$\langle \Psi_{jk}|U_1(x, y)\Psi_{ii}\rangle_{\tilde{\mathcal{H}}} = Tr \left[ |\phi_k\rangle \langle \phi_j|U(x, y)\phi_i\rangle \langle \phi_i| \right]$$

$$= \sum_{l=0}^{\infty} \langle \phi_l|\phi_k\rangle \langle \phi_j|U(x, y)\phi_i\rangle \delta_{il}$$

$$= \delta_{ik} \langle \phi_j|U(x, y)\phi_i\rangle_{\tilde{\mathcal{H}}}$$

$$= \delta_{ik} \langle \phi_j|U(x, y)\phi_i\rangle_{\tilde{\mathcal{H}}} = (2\pi)^{\frac{1}{2}} \delta_{ik} \mathcal{W}(\phi_i)\phi_j(x, y).$$

Thus $\langle \Psi_{jk}|U_1(x, y)\Psi_{ii}\rangle_{\tilde{\mathcal{H}}} = (2\pi)^{\frac{1}{2}} \delta_{ik} \Psi_{ji}(x, y)$. From (1.120), it comes

$$U_1(x, y)\Phi_{\beta} = (2\pi)^{\frac{1}{2}} \sum_{i,j,k=0}^{\infty} \lambda_{ii}^\beta \Psi_{ji}(x, y) \Psi_{jk} \delta_{ki} \quad \text{i.e.}$$

$$U_1(x, y)\Phi_{\beta} = (2\pi)^{\frac{1}{2}} \sum_{i,j=0}^{\infty} \lambda_{ii}^\beta \Psi_{ji}(x, y) \Psi_{ji}.$$ 

Let us calculate $U_1(x, y)^*\Phi_{\beta}$. We have

$$U_1(x, y)^*\Phi_{\beta} = \sum_{j=0}^{\infty} \lambda_{jj}^\beta U_1(x, y)^* \Psi_{jj} = \sum_{i,j,k=0}^{\infty} \lambda_{jj}^\beta \langle \Psi_{ik}|U_1(x, y)^*\Psi_{jj}\rangle_{\tilde{\mathcal{H}}} \Psi_{ik}, \quad (1.120)$$

where

$$\langle \Psi_{ik}|U_1(x, y)^*\Psi_{jj}\rangle_{\tilde{\mathcal{H}}} = Tr \left[ |\phi_k\rangle \langle \phi_i|U(x, y)^*\phi_j\rangle \langle \phi_j| \right]$$

$$= \sum_{l=0}^{\infty} \langle \phi_l|\phi_k\rangle \langle \phi_i|U(x, y)^*\phi_j\rangle \delta_{jl}$$

$$= \langle \phi_j|\phi_k\rangle \langle \phi_i|U(x, y)^*\phi_j\rangle = \delta_{jk} \langle \phi_j|U(x, y)\phi_i\rangle_{\tilde{\mathcal{H}}}^*$$

$$= (2\pi)^{\frac{1}{2}} \delta_{kj} \mathcal{W}(\phi_i)\phi_j(x, y).$$
Putting (1.121) in (1.120) leads to

\[ U_1(x, y)^* \Phi_\beta = (2\pi)^{\frac{1}{2}} \sum_{i, j = 0}^{\infty} \lambda_i^\frac{1}{2} \Psi_{ji}(x, y) \Psi_{kj}. \]  

(1.122)

From (1.120), we have

\[ U_1(x, y) \Phi_\beta = (2\pi)^{\frac{1}{2}} \sum_{i, j = 0}^{\infty} \lambda_i^\frac{1}{2} \Psi_{ji}(x, y) \Psi_{ji}. \]

Applying \( S_\beta \) to both sides of the equality gives

\[ S_\beta U_1(x, y) \Phi_\beta = (2\pi)^{\frac{1}{2}} \sum_{i, j = 0}^{\infty} \lambda_i^\frac{1}{2} S_\beta \Psi_{ji}(x, y) S_\beta \Psi_{ji}. \]

Since \( S_\beta [U_1(x, y) \Phi_\beta] = U_1(x, y)^* \Phi_\beta \), then \( S_\beta \Psi_{ji}(x, y) = \Psi_{ji}(x, y) \).

Thus,

\[ S_\beta U_1(x, y) \Phi_\beta = (2\pi)^{\frac{1}{2}} \sum_{i, j = 0}^{\infty} \lambda_i^\frac{1}{2} S_\beta \Psi_{ji}(x, y) S_\beta \Psi_{ji}, \]

which rewrites

\[ S_\beta U_1(x, y) \Phi_\beta = U_1(x, y)^* \Phi_\beta = (2\pi)^{\frac{1}{2}} \sum_{i, j = 0}^{\infty} \lambda_i^\frac{1}{2} \Psi_{ji}(x, y) S_\beta \Psi_{ji}. \]  

(1.123)

From the relations (1.122) and (1.123) together, it follows that

\[ \lambda_i^\frac{1}{2} S_\beta \Psi_{ji} = \lambda_j^\frac{1}{2} \Psi_{ij} \quad \text{i.e.} \quad S_\beta \Psi_{ji} = \left[ \frac{\lambda_j}{\lambda_i} \right]^\frac{1}{2} \Psi_{ij}, \]  

(1.124)

for all \( \Psi_{ij} \in \tilde{H} \), \( i, j = 0, 1, 2, \ldots, \infty \).

\[ \square \]

**Proof of (1.118).**

Consider the operator \( J_\beta \) with \( J_\beta \Psi_{ji} = \Psi_{ij} \). We have

\[ J_\beta^2 \Psi_{ji} = J_\beta \Psi_{ij} = \Psi_{ji}, \forall i, j \quad \text{i.e.} \quad J_\beta^2 = I_\tilde{\beta}. \]

Besides,

\[ \Phi_\beta = \sum_{i = 0}^{\infty} \lambda_i^\frac{1}{2} \Psi_{ii} \quad \text{i.e.} \quad J_\beta \Phi_\beta = \sum_{i = 0}^{\infty} \lambda_i^\frac{1}{2} (J_\beta \Psi_{ii}) = \sum_{i = 0}^{\infty} \lambda_i^\frac{1}{2} \Psi_{ii} = \Phi_\beta. \]

Thus \( J_\beta \Phi_\beta = \Phi_\beta \). Therefore, \( J_\beta \) is such that

\[ J_\beta \Psi_{nl} = \Psi_{nl}, \quad J_\beta^2 = I_\tilde{\beta}, \quad J_\beta \Phi_\beta = \Phi_\beta. \]

From (1.117) and

\[ \Delta_\beta = \sum_{n, l = 0}^{\infty} \frac{\lambda_n}{\lambda_l} |\Psi_{nl}|^2 \Psi_{nl} = e^{-\beta H}, \quad H = H_1 - H_2, \]
we get
\[ S_\beta = J_\beta \Delta_\beta^{\frac{1}{2}} \quad \text{and} \quad \Delta_\beta = \sum_{n,l=0}^{\infty} \frac{\lambda_n}{\lambda_l} |\Psi_{nl}\rangle \langle \Psi_{nl}|. \]

Using this above relations yields
\[
\forall i, j, \quad (J_\beta \Delta_\beta^{\frac{1}{2}} |\Psi_{ji}\rangle = J_\beta \left[ \sum_{n,l=0}^{\infty} \left( \frac{\lambda_n}{\lambda_l} \right)^{\frac{1}{2}} |\Psi_{nl}\rangle \delta_{nl} \delta_{il} \right] = J_\beta \left( \frac{\lambda_j}{\lambda_i} \right)^{\frac{1}{2}} |\Psi_{ji}\rangle.
\]

From (1.124),
\[
S_\beta |\Psi_{ji}\rangle = \left( \frac{\lambda_j}{\lambda_i} \right)^{1/2} |\Psi_{ij}\rangle = (J_\beta \Delta_\beta^{\frac{1}{2}}) |\Psi_{ji}\rangle, \quad \forall i, j \quad (1.125)
\]
i.e.
\[
S_\beta = J_\beta \Delta_\beta^{\frac{1}{2}}. \quad (1.126)
\]

**Proposition 1.51** (14)

If \( \{ \lambda_n \}_{n=0}^{\infty} \) is a sequence of nonzero positive numbers such that \( \sum_{n=0}^{\infty} \lambda_n = 1 \), then the vector
\[
\Phi = \sum_{n=0}^{\infty} \lambda_n^n |\Psi_{nn}\rangle \quad (1.127)
\]
is cyclic (that is the set of vectors \( \{ A\Phi / A \in \mathfrak{A}_1 \} \) is dense in \( \tilde{\mathfrak{H}} \)) and separating (i.e. if \( A\Phi = 0 \), for all \( A \in \mathfrak{A}_1 \) then \( A = 0 \)) for \( \mathfrak{A}_1 \).

**Proof.**

Let \( X \in B_2(\tilde{\mathfrak{H}}) \) and consider the operator \( \mathcal{W}[U(x,y) \vee I_{\tilde{\mathfrak{H}}}] \mathcal{W}^{-1} \in \mathfrak{A}_1 \). Taking \( U(x,y) \vee I_{\tilde{\mathfrak{H}}} \), we have, since \( \mathcal{W} \) is unitary,
\[
(X|\mathcal{W}[U(x,y) \vee I_{\tilde{\mathfrak{H}}}] |\mathcal{W}^{-1}\Phi_{\tilde{\mathfrak{H}}} \rangle = (X|(U(x,y) \vee I_{\tilde{\mathfrak{H}}}) \Phi_{B_2(\tilde{\mathfrak{H}})} \rangle
\]
and the complex conjugate of (1.128) given by
\[
(X|(U(x,y) \vee I_{\tilde{\mathfrak{H}}}) \Phi_{B_2(\tilde{\mathfrak{H}})} \rangle = \text{Tr}[((X^* U(x,y)) \Phi^*)^*(2\pi)^{\frac{1}{2}}(\mathcal{W} \Phi)(x,y)] \quad \text{[by (1.76)]} \quad (1.129)
\]
and the modulus squared
\[
(X|(U(x,y) \vee I_{\tilde{\mathfrak{H}}}) \Phi_{B_2(\tilde{\mathfrak{H}})} \rangle^* (X|(U(x,y) \vee I_{\tilde{\mathfrak{H}}}) \Phi_{B_2(\tilde{\mathfrak{H}})} \rangle = |(X|(U(x,y) \vee I_{\tilde{\mathfrak{H}}}) \Phi_{B_2(\tilde{\mathfrak{H}})} \rangle|^2 \quad (1.130)
\]
Then, by integrating over \( \mathbb{C} \) the modulus squared
\[
(X|(U(x,y) \vee I_{\tilde{\mathfrak{H}}}) \Phi_{B_2(\tilde{\mathfrak{H}})} \rangle^* (X|(U(x,y) \vee I_{\tilde{\mathfrak{H}}}) \Phi_{B_2(\tilde{\mathfrak{H}})} \rangle = |(X|(U(x,y) \vee I_{\tilde{\mathfrak{H}}}) \Phi_{B_2(\tilde{\mathfrak{H}})} \rangle|^2 \quad (1.130)
\]
Since $\mathcal{W}$ is unitary, we may write

$$\left< \mathcal{W}\Phi(x,y)|\mathcal{W}\Phi(x,y) \right> = \int_C \sum_{i,j=0}^{\infty} \sum_{l,k=0}^{\infty} (\Psi_{jk}\lambda_i^l \lambda_k^j) \mathcal{W} \phi_j(x,y) \mathcal{W} \phi_i(x,y) dx dy |\Psi_i \rangle \langle \Psi_i |$$

$$= \sum_{i,j=0}^{\infty} \sum_{l,k=0}^{\infty} \lambda_i^l \lambda_k^j$$

with respect to $x, y$, we get

$$\int_C \left< (\mathcal{W}\Phi)(x,y)|(\mathcal{W}\Phi)(x,y) \right> dx dy = \sum_{i=0}^{\infty} \lambda_i = 1.$$

(1.131)

Since $\mathcal{W}$ is unitary, we may write

$$\int_C \left< (\mathcal{W}\Phi)(x,y)|(\mathcal{W}\Phi)(x,y) \right> dx dy = 0 \implies |\left< X|\mathcal{W}(U(x,y) \lor I_{\tilde{B}})|\mathcal{W}^{-1}\Phi \right>\rangle |^2 = 0$$

$$\implies |\left< X|(U(x,y) \lor I_{\tilde{B}})|\Phi \right>_{\mathcal{B}_2(\tilde{B})}|^2 = 0$$

$$\implies X = 0.$$

(1.132)

This implies that the set $\{\mathcal{W}(U(x,y) \lor I_{\tilde{B}})|\mathcal{W}^{-1}\Phi, \mathcal{W}(U(x,y) \lor I_{\tilde{B}})|\mathcal{W}^{-1} \in \mathfrak{A}_1\}$ is dense in $\tilde{B}$, proving from the Definition 1.19 that $\Phi$ is cyclic for $\mathfrak{A}_1$.

The fact that $\Phi$ is separating for $\mathfrak{A}_1$ is obtained through the relation

$$\mathcal{W}(U(x,y) \lor I_{\tilde{B}})|\mathcal{W}^{-1}\Phi = \mathcal{W}(U'(x,y) \lor I_{\tilde{B}})|\mathcal{W}^{-1}\Phi \iff \mathcal{W}(U(x,y) \lor I_{\tilde{B}})|\mathcal{W}^{-1} = \mathcal{W}(U'(x,y) \lor I_{\tilde{B}})|\mathcal{W}^{-1}.$$  

(1.133)

Proof.

Let $U(x,y), U'(x,y)$ such that $\mathcal{W}(U(x,y) \lor I_{\tilde{B}})|\mathcal{W}^{-1}\Phi = \mathcal{W}(U'(x,y) \lor I_{\tilde{B}})|\mathcal{W}^{-1}\Phi$. Take $X \neq 0, X \in \mathcal{B}_2(\tilde{B})$ and set $\Phi = \sum_{i=1}^{N} \lambda_i^l |\zeta_i \rangle \langle \zeta_i |$. We have

$$\left< X|\mathcal{W}\left\{ (U(x,y) \lor I_{\tilde{B}}) - (U'(x,y) \lor I_{\tilde{B}}) \right\}|\mathcal{W}^{-1}\Phi \rangle_{\tilde{B}} = \left< X|\left\{ (U(x,y) \lor I_{\tilde{B}}) - (U'(x,y) \lor I_{\tilde{B}}) \right\}|\Phi \right>_{\mathcal{B}_2(\tilde{B})} = \text{Tr} \left[ X^* \left\{ (U(x,y) \lor I_{\tilde{B}}) - (U'(x,y) \lor I_{\tilde{B}}) \right\} \Phi \right]$$

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Then, we have

\[
\sum_{k=1}^{N} \langle \zeta_k | X^* \{ U(x, y) - U'(x, y) \} | \zeta_k \rangle = \sum_{k=1}^{N} \sum_{i=1}^{N} \lambda_i^k \langle \zeta_k | X^* \{ U(x, y) - U'(x, y) \} \zeta_i \zeta_k \rangle
\]

which completes the proof.

1.4 Modular theory-Thermal state

Here, we deal with some notions on thermal state as developed in the literature. For more details, see for example [124, 112, 124, 26] for the thermal equilibrium state at inverse temperature \( \beta \),

\[
\text{let the two von Neumann algebras be given by}
\]

1. Let \( \alpha_i, i = 1, 2, \ldots, N \) be a sequence of non-zero, positive numbers, satisfying \( \sum_{i=1}^{N} \alpha_i = 1 \). Let

\[
\Phi = \sum_{i=1}^{N} \alpha_i \rho_i = \sum_{i=1}^{N} \alpha_i X_{ii} \in B_2(H)
\]

where \( \rho_i = X_{ii} = |\zeta_i\rangle \langle \zeta_i| \).

2. Consider the thermal equilibrium state at inverse temperature \( \beta \), corresponding to the harmonic oscillator Hamiltonian \( H_{OSC} = \frac{1}{2}(P^2 + Q^2) \), with \( H_{OSC} \phi_n = \omega(n + \frac{1}{2}) \phi_n, n = 0, 1, 2, \ldots \), the density matrix

\[
\rho_\varphi = \frac{e^{-\beta H_{OSC}}}{\text{Tr}[e^{-\beta H_{OSC}}]} = (1 - e^{-\omega \beta}) \sum_{n=0}^{\infty} e^{-n\omega \beta} \langle \phi_n | \langle \phi_n |\n\]

with

\[
\Phi = \left[ 1 - e^{-\omega \beta} \right] \sum_{n=0}^{\infty} e^{-\frac{\omega \beta}{2} n} \langle \phi_n | \langle \phi_n | .
\]

3. Let the two von Neumann algebras be given by

\[
\mathfrak{A}_l = \{ A_l = A \lor I | A \in L(\mathfrak{H}) \}, \quad \mathfrak{A}_r = \{ A_r = I \lor A | A \in L(\mathfrak{H}) \}
\]

where \( \mathfrak{A}_l \) corresponds to the operators given with \( A \) in the left, and \( \mathfrak{A}_r \) corresponds to the operators given with \( A \) in the right of the identity operator \( I \) on \( \mathfrak{H} \), respectively.

\( \Phi \) defines a vector state \( \varphi \), called KMS state, on the von Neumann algebra \( \mathfrak{A}_l \), with for any \( A \lor I \in \mathfrak{A}_l \), one has the state \( \varphi \) on \( \mathfrak{A}_l \) by

\[
\langle \varphi; A \lor I \rangle = \langle \Phi | (A \lor I) \Phi \rangle_2 = Tr[\Phi^* A \Phi] = Tr[\rho_\varphi A], \quad \text{with} \quad \rho_\varphi = \sum_{n=1}^{N} \alpha_n \rho_n
\]
with $P_n = |\phi_n\rangle \langle \phi_n|$, where

$$
\rho_\beta = \frac{e^{-\beta H_{OSC}}}{\text{Tr}[e^{-\beta H_{OSC}}]} = (1 - e^{-\omega^2}) \sum_{n=0}^{\infty} e^{-n\omega^2} |\phi_n\rangle \langle \phi_n|
$$

(1.141)

and

$$
H_\phi = -\frac{1}{\beta} \sum_{n=0}^{\infty} (\ln \alpha_n) P_n, \quad \alpha_n = (1 - e^{-\omega^2}) e^{-n\omega^2}.
$$

(1.142)

### 1.5 Coherent states built from the harmonic oscillator thermal state

Consider the density matrix corresponding to the Hamiltonian $H_{OSC}$ such that

$$
\rho_\beta = \frac{e^{-\beta H_{OSC}}}{\text{Tr}[e^{-\beta H_{OSC}}]} = (1 - e^{-\omega^2}) \sum_{n=0}^{\infty} e^{-n\omega^2} |\phi_n\rangle \langle \phi_n|, \quad \text{Tr}[e^{-\beta H_{OSC}}] = \frac{e^{-\beta \omega}}{1 - e^{-\beta \omega}}.
$$

(1.143)

Take the cyclic vector $\Phi$ of the von Neumann algebra $A_1$ generated by the unitary operator

$$
U_1(x,y) = W[U(z,y) \lor I] W^{-1}, \quad \text{with } W \text{ given in (1.70)}
$$

(1.144)

such that $\Phi = \Phi_{\beta}$ with the $\lambda_n$ corresponding to the thermal state $\Phi_{\beta}$

$$
\Phi_{\beta} = [1 - e^{-\omega^2}] \frac{1}{2} \sum_{n=0}^{\infty} e^{-n\omega^2} |\Psi_n\rangle, \quad \text{i.e., } \lambda_n = (1 - e^{-\omega^2}) e^{-n\omega^2}.
$$

(1.145)

The CS, denoted $|z, \bar{z}, \beta\rangle_{\text{KMS}}$, built from the thermal state $\Phi_{\beta}$, are given by

$$
|z, \bar{z}, \beta\rangle_{\text{KMS}} = U_1(z) |\Phi_{\beta}\rangle = e^{z A_1^1 - \bar{z} A_1^1} |\Phi_{\beta}\rangle.
$$

(1.146)

with $U_1(z) := U_1(x,y) = e^{z A_1^1 - \bar{z} A_1^1}$, where the actions of the annihilation and creation operators, $A_1$ and $A_1^1$ are given by

$$
A_1^1 |\Psi_{nl}\rangle = \sqrt{n+1} |\Psi_{n+1,l}\rangle, \quad A_1 |\Psi_{nl}\rangle = \sqrt{n} |\Psi_{n-1,l}\rangle.
$$

(1.147)

**Proposition 1.52 (17)**

*Using the fact that for any normalized vector $\phi \in \mathcal{F} = L^2(\mathbb{R})$, the vectors $U(z)\phi, z \in \mathbb{C}$, where $U(z) := U(x,y)$ satisfy

$$
\frac{1}{2\pi} \int_{\mathbb{C}} |U(z)\phi\rangle \langle U(z)\phi| \, dxdy = I_{\mathcal{F}}
$$

(1.148)

from the isometry $W$, the CS $|z, \bar{z}, \beta\rangle_{\text{KMS}}$ satisfy the resolution of the identity condition

$$
\frac{1}{2\pi} \int_{\mathbb{C}} |z, \bar{z}, \beta\rangle_{\text{KMS}} |z, \bar{z}, \beta\rangle_{\text{KMS}} \, dxdy = I_{\mathcal{F}}, \quad \mathcal{F} = L^2(\mathbb{R},dxdy).
$$

(1.149)

*Proof of (1.148).*

Consider the unitary operator $\forall (x,y) \in \mathbb{R}^2, \quad U(x,y) = e^{-i(xQ+yP)}$. Let

$$
\phi_\alpha(x,y) = e^{-i(xQ+yP)} \phi \equiv U(z)\phi, \quad \forall \phi \in \mathcal{F}.
$$

Show that $\frac{1}{2\pi} \int_{\mathbb{R}^2} |\phi_\alpha(x,y)\rangle \langle \phi_\alpha(x,y)| \, dxdy = I_{\mathcal{F}}$.

\[
\forall q \in \mathbb{R}, \text{ set } \phi_\alpha(x,y)(q) = (\pi)^{-\frac{1}{4}} e^{-i(x^2+\bar{z}q)} e^{-i(x+y)q}.
\]
such that \( \forall \psi, \xi \in \mathcal{H} \), we have

\[
\langle \xi | \phi_{\alpha(x,y)} \rangle_{\mathcal{H}} = (\pi)^{-\frac{3}{2}} e^{-\frac{ix^2}{2}} \int_{\mathbb{R}} \overline{\xi(q)} e^{iyq} e^{-\frac{(q-x)^2}{2}} dq,
\]

\[
\langle \phi_{\alpha(x,y)} | \psi \rangle_{\mathcal{H}} = (\pi)^{-\frac{3}{2}} e^{iyq} e^{-\frac{(q'-x)^2}{2}} \psi(q')dq'.
\]

Then,

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2} \langle \xi | \phi_{\alpha(x,y)} \rangle \langle \phi_{\alpha(x,y)} | \psi \rangle dx dy = \frac{1}{2\pi} \int_{\mathbb{R}^2} dx dy \int_{\mathbb{R}^2} dq dq' \overline{\xi(q)} e^{iy(q-q')} e \left[ \frac{(x-g)^2}{2} - \frac{(q'-x)^2}{2} \right] \psi(q')
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} dx \int_{\mathbb{R}^2} dq dq' \overline{\xi(q)} \left[ \frac{1}{2\pi} \int_{\mathbb{R}} e^{iy(q-q')} dq \right] e \left[ \frac{(x-g)^2}{2} - \frac{(q'-x)^2}{2} \right] \psi(q')
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} dq dq' \overline{\xi(q)} \psi(q') \delta(q - q') e \left[ \frac{(x-g)^2}{2} - \frac{(q'-x)^2}{2} \right]
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^2} e^{-(x-g)^2/\pi} \overline{\xi(q)} \psi(q) dx dq = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(x-g)^2} \int_{\mathbb{R}} \overline{\xi(q)} \psi(q) dq
\]

Thus

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2} |\phi_{\alpha(x,y)}\rangle \langle \phi_{\alpha(x,y)} | dx dy = I_{\mathcal{H}} \quad \text{i.e.} \quad \frac{1}{2\pi} \int_{\mathcal{L}} \langle U(z)\phi | U(z)\phi \rangle dx dy = I_{\mathcal{H}}.
\]

Using the property of the isometry \( \mathcal{W} \), it comes that the states \( |z, \bar{z}, \beta\rangle^{\text{KMS}} \) satisfy the following resolution of the identity

\[
\frac{1}{2\pi} \int_{\mathcal{C}} |z, \bar{z}, \beta\rangle^{\text{KMS}} \langle z, \bar{z}, \beta| dx dy = I_{\mathcal{H}}.
\] (1.150)

**Proof.**

By definition of the states \( |z, \bar{z}, \beta\rangle^{\text{KMS}} \), we have:

\[
|z, \bar{z}, \beta\rangle^{\text{KMS}} = U_1(z) \Phi_\beta = (2\pi)^{\frac{3}{2}} \sum_{i,j=0}^{\infty} \lambda_i^\dagger \Psi_{ji}(x,y) |\Psi_{ji}\rangle,
\]

\[
^{\text{KMS}} \langle z, \bar{z}, \beta| = U_1(z)^\dagger \Phi_\beta = (2\pi)^{\frac{3}{2}} \sum_{i,k=0}^{\infty} \langle \Psi_{ik}| \lambda_k^\dagger \Psi_{ik}(x,y).
\] (1.151)

Thereby

\[
|z, \bar{z}, \beta\rangle^{\text{KMS}} \langle z, \bar{z}, \beta| = 2\pi \sum_{i,j=0}^{\infty} \sum_{l,k=0}^{\infty} \lambda_i^\dagger \lambda_k^\dagger \mathcal{W} \phi_{ji}(x,y) \mathcal{W} \phi_{lk}(x,y) |\Psi_{ji}\rangle \langle \Psi_{lk}|.
\] (1.152)
By integrating the two members of the Eq. (1.152) over C, we get by using the Wigner map $\mathcal{W}$,

$$
\frac{1}{2\pi} \int_C |z, z, \beta\rangle_{KMS-KMS} \langle z, z, \beta| dxdy = \int_C \sum_{i,j=0}^{\infty} \sum_{l,k=0}^{\infty} \lambda^2_{l,j} \lambda^2_{k,i} \langle \psi_{ji} | \langle \psi_{lk} | \mathcal{W} \phi_j(x,y) \mathcal{W} \phi_k(x,y) | dxdy =
\sum_{i,j=0}^{\infty} \sum_{l,k=0}^{\infty} \lambda^2_{l,j} \lambda^2_{k,i} \langle \psi_{ji} | \int_C \mathcal{W} \phi_j(x,y) \mathcal{W} \phi_k(x,y) | dxdy
= \sum_{i,j=0}^{\infty} \sum_{l,k=0}^{\infty} \lambda^2_{l,j} \lambda^2_{k,i} \langle \psi_{ji} | \delta_{ij} \delta_{kl}
= \sum_{i,j=0}^{\infty} | \psi_{ji} \rangle \langle \psi_{ji} |
= I_{\delta},
$$

(1.153)

From $(A_1^x | \psi_{0n}) = \sqrt{n!} | \psi_{nn} \rangle$ and $U_1(z) | \psi_{0n} \rangle = e^{-\frac{iz^2}{2}} \sum_{k=0}^{\infty} \frac{(z A_1^x)^k}{k!} | \psi_{0n} \rangle$, it comes that:

$$
U_1(z) | \psi_{nn} \rangle = \frac{1}{\sqrt{n!}} \left( A_1^x - \bar{z} I_{\delta} \right)^n U_1(z) | \psi_{0n} \rangle = \frac{1}{\sqrt{n!}} \left( -\frac{\partial}{\partial z} - \frac{\bar{z}}{2} \right)^n U_1(z) | \psi_{0n} \rangle.
$$

(1.154)

**Proof of the first equality of (1.154).** The operators $P$ and $Q$ verify the following relations:

$$
[Q, P^n] = n P^{n-1} [Q, P] = i\hbar n P^{n-1}, \quad [Q^n, P] = n Q^{n-1} [Q, P] = i\hbar n Q^{n-1}.
$$

(1.155)

We establish that

$$
e^{-iP_u}Qe^{iP_u} = Q - u, \quad e^{-iQ_u}Pe^{iQ_u} = P + u, \quad \forall u \in \mathbb{R}.
$$

(1.156)

Multiplying the first and second equalities of (1.156), by $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$, respectively, provides:

$$
e^{-iP_u} Q \frac{Q}{\sqrt{2}} e^{iP_u} = Q - \frac{u}{\sqrt{2}}, \quad e^{-iQ_u} \left( \frac{iP}{\sqrt{2}} \right) e^{iQ_u} = -\frac{iP}{\sqrt{2}} - \frac{iu}{\sqrt{2}}.
$$

(1.157)

Setting $u = -x$ and $u = -y$, in the first and second relations of (1.157), respectively, gives with replacing $P$ by $P_1$, and $Q$ by $Q_1$, respectively:

$$
e^{iP_1 x} \frac{Q_1}{\sqrt{2}} e^{-iP_1 x} = \frac{Q_1}{\sqrt{2}} + \frac{x}{\sqrt{2}}, \quad e^{iQ_1 y} \left( -\frac{iP_1}{\sqrt{2}} \right) e^{-iQ_1 y} = -\frac{iP_1}{\sqrt{2}} + \frac{y}{\sqrt{2}}.
$$

(1.158)

From $A_1^x = \frac{Q_1 - iP_1}{\sqrt{2}}$, set $z = \frac{-x + iy}{\sqrt{2}}$. Summing both equalities of (1.158) gives:

$$
e^{iP_1 x} \frac{Q_1}{\sqrt{2}} e^{-iP_1 x} + e^{iQ_1 y} \left( -\frac{iP_1}{\sqrt{2}} \right) e^{-iQ_1 y} = A_1^x - \bar{z} I_{\delta},
$$

(1.159)

Since $U_1(z) = e^{z A_1^x - \bar{z} A_1} = \langle (P_1 x + Q_1 y) \rangle$, it follows: $U_1(z) = e^{(P_1 x + Q_1 y)}$. This latter equality with (1.160) together lead to:

$$
(A_1^x - \bar{z} I_{\delta}) U_1(z) = e^{iP_1 x} \frac{Q_1}{\sqrt{2}} e^{-iP_1 x} e^{(P_1 x + Q_1 y)} + e^{iQ_1 y} \left( -\frac{iP_1}{\sqrt{2}} \right) e^{-iQ_1 y} e^{(P_1 x + Q_1 y)} = e^{(P_1 x + Q_1 y)} e^{-iP_1 x} e^{iQ_1 y} \left( -\frac{iP_1}{\sqrt{2}} \right) = e^{z A_1^x - \bar{z} A_1} \left( \frac{Q_1}{\sqrt{2}} - \frac{iP_1}{\sqrt{2}} \right) = U_1(z) A_1^x.
$$

(1.160)

From (1.160), we have

$$
U_1(z) | \psi_{1n} \rangle = (A_1^x - \bar{z} I_{\delta}) U_1(z) | \psi_{0n} \rangle
$$

(1.161)
such that, by recursion, we get

\[ U_1(z)|\Psi_{nn}\rangle = \frac{1}{\sqrt{n!}} \left( A_1^\dagger - \bar{z} I_\beta \right)^n U_1(z)|\Psi_{0n}\rangle. \tag{1.162} \]

**Proof of the second equality of (1.154).** Considering \( A_1^\dagger = \frac{1}{\sqrt{2}} (Q_1 - iP_1) \), where \( z = \frac{1}{\sqrt{2}} (y - ix) \), with

\[ \frac{\partial}{\partial x} = -i \frac{\partial}{\partial z} + i \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial y} = \frac{1}{\sqrt{2}} \frac{\partial}{\partial z} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial \bar{z}} \tag{1.163} \]

provides

\[ A_1^\dagger = -\frac{\partial}{\partial z} - \frac{i}{2} I_\beta, \quad \text{i.e.,} \quad A_1^\dagger - \bar{z} I_\beta = -\frac{\partial}{\partial z} - \frac{i}{2} I_\beta \tag{1.164} \]

which completes the proof. \( \square \)

Since

\[ |z, \bar{z}, \beta\rangle_{\text{KMS}} = U_1(z)|\Phi_\beta\rangle = (1 - e^{-\omega \beta})^\frac{1}{2} \sum_{n=0}^{\infty} e^{-n \frac{z \bar{z}}{2}} U_1(z)|\Psi_{nn}\rangle, \tag{1.165} \]

setting \(|z; n\rangle = U_1(z)|\Psi_{0n}\rangle\) and from (1.154), it comes

\[ |z, \bar{z}, \beta\rangle_{\text{KMS}} = (1 - e^{-\omega \beta})^\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} e^{-n \frac{z \bar{z}}{2}} \left( -\frac{\partial}{\partial z} - \frac{i}{2} I_\beta \right)^n |z; n\rangle. \tag{1.166} \]

Set \(|z; n\rangle = U_2(z)|\Psi_{n0}\rangle\) with \((A_2)^n|\Psi_{n0}\rangle = \sqrt{n!}|\Psi_{nn}\rangle\) and

\[ U_2(z)|\Psi_{n0}\rangle = e^{-\frac{iy}{2}} \sum_{k=0}^{\infty} \frac{(z A_2^\dagger)^k}{k!} |\Psi_{n0}\rangle. \tag{1.167} \]

Taking \(|z; 0\rangle = U_2(z)|\Psi_{00}\rangle\) leads to \((A_2)^n|z; 0\rangle = \sqrt{n!}|z; n\rangle\) with

\[ U_2(z) \left[ (A_2)^n |\Psi_{00}\rangle \right] = \sqrt{n!} U_2(z)|\Psi_{n0}\rangle. \tag{1.168} \]

Then, \(|z; n\rangle = \frac{1}{\sqrt{n!}} (A_2)^n|z; 0\rangle.\) In (1.168), we get

\[ |z, \bar{z}, \beta\rangle_{\text{KMS}} = (1 - e^{-\omega \beta})^\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} e^{-n \frac{z \bar{z}}{2}} \left( -\frac{\partial}{\partial z} - \frac{i}{2} \right)^n (A_2)^n |z; 0\rangle. \tag{1.169} \]

Using the relation \(U_1(x, y)^* = U_1(-x, -y)\), the CS \(|-z, -\bar{z}, \beta\rangle_{\text{KMS}}\) are obtained as follows:

\[ |-z, -\bar{z}, \beta\rangle_{\text{KMS}} = S_{\beta}|z, \bar{z}, \beta\rangle_{\text{KMS}}. \tag{1.169} \]

Indeed, by definition \(|z, \bar{z}, \beta\rangle_{\text{KMS}} = U_1(z)|\Phi_\beta\rangle := U_1(x, y)|\Phi_\beta\rangle\) such that

\[ S_{\beta} [U_1(x, y)|\Phi_\beta\rangle] = U_1(x, y)^*|\Phi_\beta\rangle = U_1(-x, -y)|\Phi_\beta\rangle, \quad \text{i.e.,} \quad U_1(-z)|\Phi_\beta\rangle = e^{-\frac{\partial}{\partial z}} (A_2)^n |\Phi_\beta\rangle \tag{1.170} \]

where \(e^{-\frac{\partial}{\partial z}} (A_2)^n |\Phi_\beta\rangle = |-z, -\bar{z}, \beta\rangle_{\text{KMS}}\), leading to (1.169).

The CS (1.169) satisfy a resolution of the identity analogue to (1.150), i.e.,

\[ \frac{1}{2\pi} \int_C | -z, -\bar{z}, \beta\rangle_{\text{KMS}} KMS (-z, -\bar{z}, \beta) dx dy = I_\beta. \tag{1.171} \]

**Proof.** Similar to the proof of (1.150). \( \square \)
Proposition 2.1

Given the Hilbert space $\mathcal{H}$ set of vectors $\{\phi \}$ in "bra-ket" notation given by $|\cdot\rangle\langle\cdot|$ $B$ where equivalent to the Hilbert space of square integrable function, with operators $a, a^\dagger$ obey the Fock space $\mathcal{H}_c = \text{span} \left\{ |n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle \right\}_{n=0}^{\infty}$. This space is isomorphic to the boson Fock space $\mathcal{F} = \{ |n\rangle \}_{n=0}^{\infty}$, where the annihilation and creation operators $a, a^\dagger$ obey the Fock algebra $[a, a^\dagger] = 1$.

The physical states of the system represented on $\mathcal{H}_q$ known as the set of Hilbert-Schmidt operators, is equivalent to the Hilbert space of square integrable function, with

$$\mathcal{H}_q = \{ \psi(\hat{x}_1, \hat{x}_2) : \psi(\hat{x}_1, \hat{x}_2) \in B(\mathcal{H}_c), \text{tr}_c(\psi(\hat{x}_1, \hat{x}_2)^\dagger, \psi(\hat{x}_1, \hat{x}_2)) < \infty \}$$

where $B(\mathcal{H}_c)$ is the set of bounded operators on $\mathcal{H}_c$. $\mathcal{H}_q$ is defined as the set of bounded operators, with the form $|\cdot\rangle\langle\cdot|$, acting on the classical configuration space $\mathcal{H}_c$, with a general element of the quantum Hilbert space, in "bra-ket" notation given by

$$|\psi\rangle = \sum_{n,m=0}^{\infty} c_{n,m} |n,m\rangle,$$

with $\{n, m\} := |n\rangle\langle m| \}_{n,m=0}^\infty$ a basis of $\mathcal{H}_q$. By considering the unitary Wigner map $\mathcal{W} : \mathcal{B}_2(\mathcal{N}) \rightarrow L^2(\mathbb{R}^2, dx\,dy)$ let us discuss a correspondence between $L^2(\mathbb{R}^2, dx\,dy)$ and $\mathcal{B}_2(\mathcal{N})$.

Proposition 2.1 Given the Hilbert space $\mathcal{N} = L^2(\mathbb{R})$, the inverse of the map $\mathcal{W}$ is defined on the dense set of vectors $f \in L^2(\mathbb{R}^2, dx\,dy)$, comprising the image of $\mathcal{N} \otimes \mathcal{N}$ under $\mathcal{W}$, as follows:

$$\mathcal{W}^{-1} : L^2(\mathbb{R}^2, dx\,dy) \rightarrow \mathcal{N} \otimes \mathcal{N}$$

the integral is defined weakly, $|\phi\rangle\langle\psi|$ is an element in $\mathcal{B}_2(\mathcal{N}) \simeq \mathcal{N} \otimes \mathcal{N}$ and $f = \mathcal{W}(|\phi\rangle\langle\psi|)$.

Proof.

Let us derive the inverse of the map $\mathcal{W}$ on $L^2(\mathbb{R}^2, dx\,dy)$ where the group $G$ and the Duffo-Moore operator $C$ with domain $\mathcal{D}(C^{-1})$ given in $\mathcal{N}$ are identified here to $\mathbb{R}^2$, $I_\mathcal{N}$, the identity operator on $\mathcal{N} = L^2(\mathbb{R})$, respectively, with $\mathcal{D}(C^{-1}) = \mathcal{N}$ and $\mathcal{D}(C^{-1})^\dagger = \mathcal{N}$. Consider an element in $\mathcal{B}_2(\mathcal{N}) \simeq \mathcal{N} \otimes \mathcal{N}$ of the type $|\phi\rangle\langle\psi|$, with $\phi, \psi \in \mathcal{N}$ and let $f = \mathcal{W}(|\phi\rangle\langle\psi|)$.

For $\phi', \psi' \in \mathcal{N}$, we have from the definition of $\mathcal{W}$ in (2.7)

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \langle \phi'| U(x, y) \psi' \rangle \mathcal{W}(|\phi\rangle\langle\psi|)(x, y) dx\,dy = \int_{\mathbb{R}} \int_{\mathbb{R}} \langle \phi'| U(x, y) \psi' \rangle \mathcal{W}(\langle\phi| \langle\psi\rangle)(x, y) dx\,dy\,\mathcal{W}(\langle\phi| \langle\psi\rangle)(x, y) dx\,dy$$

(2.5)

By the orthogonality relations, we get

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \langle \phi'| U(x, y) \psi' \rangle \mathcal{W}(|\phi\rangle\langle\psi|)(x, y) dx\,dy = \langle \phi'| \phi \rangle \langle \psi| \psi' \rangle.$$

(2.6)

The relation $|\langle \phi'| \phi \rangle \langle \psi| \psi' \rangle| \leq ||\phi'||||\psi'||||\phi||||\psi||$, implies that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \langle \phi'| U(x, y) \psi' \rangle \mathcal{W}(|\phi\rangle\langle\psi|)(x, y) dx\,dy \leq ||\phi'||||\psi'||||\phi||||\psi||$$

(2.7)

Then, (2.6) holds for all $\phi', \psi' \in \mathcal{N}$. Thus, as a weak integral, we obtain

$$|\phi\rangle\langle\psi| = \int_{\mathbb{R}} \int_{\mathbb{R}} U(x, y) \mathcal{W}(|\phi\rangle\langle\psi|)(x, y) dx\,dy$$

(2.8)

which completes the proof.
3 Application

Consider the motion of an electron placed in the xy-plane and subjected to a constant magnetic field, pointing along the positive z-direction with the Hamiltonian given, in some convenient units, by\( [2] \)

\[
H_{\text{elec}} = \frac{1}{2}(p - A)^2 = \frac{1}{2}(p_x + \frac{y}{2})^2 + \frac{1}{2}(p_y - \frac{x}{2})^2, \quad p_x = -i\frac{\partial}{\partial x}, \quad p_y = -i\frac{\partial}{\partial y}.
\]  

(3.1)

For \( A^\dagger = (\frac{B}{2}y, -\frac{B}{2}x) \), \( ||B|| = 1 \), one defines on \( \mathcal{H} = L^2(\mathbb{R}^2, dxdy) \) the operators

\[
Q_- = p_x + \frac{y}{2}, \quad P_- = p_y - \frac{x}{2} \quad \text{such that} \quad [Q_-, P_-] = iI_{\mathcal{H}}.
\]  

(3.2)

Next, taking \( A^\dagger = (\frac{B}{2}y, -\frac{B}{2}x) \) one defines \( \tilde{\mathcal{H}} = L^2(\mathbb{R}^2, dxdy) \) the operators

\[
Q_+ = -p_y + \frac{x}{2}, \quad P_+ = -p_x - \frac{y}{2} \quad \text{such that} \quad [Q_+, P_+] = iI_{\tilde{\mathcal{H}}}.
\]  

(3.3)

The two sets of operators \( \{Q_\pm, P_\pm\} \) mutually commute as follows

\[
[Q_+, Q_-] = [P_+, Q_-] = [Q_+, P_-] = [P_+, P_-] = 0
\]  

(3.4)

such that the corresponding Hamiltonians given by

\[
H^\dagger := H_- = \frac{1}{2}(P_- + Q_-)^2, \quad H^\dagger := H_+ = \frac{1}{2}(P_+ + Q_+)^2,
\]  

(3.5)

satisfy the commutation relation \([H_+, H_-] = 0\), with

\[
Q_+ = \frac{1}{\sqrt{2}}(A^*_+ + A_+), \quad P_+ = \frac{i}{\sqrt{2}}(A^*_+ - A_+),
\]  

(3.6)

\[
Q_- = \frac{i}{\sqrt{2}}(A^*_+ - A_-), \quad P_- = -\frac{1}{\sqrt{2}}(A^*_+ + A_-).
\]  

(3.7)

The system \( \{A_\pm, A^\dagger_\pm\} \) forming an irreducible set of operators on the chiral boson Fock space \( \mathcal{F} = \{n_\pm\}_{n_\pm = 0}^\infty \), has the following realization on the states \( |n_+, n_-\rangle = |n_\rangle\langle n_-|, \quad n_\pm = 0, 1, 2, \ldots \) of the Hilbert space \( \mathcal{H}_q \) as follows:

\[
A_+|n_+, n_-\rangle := \sqrt{n_+}|n_+ - 1, n_-\rangle \quad A^\dagger_+|n_+, n_-\rangle := \sqrt{n_+ + 1}|n_+ + 1, n_-\rangle,
\]  

(3.8)

\[
A_-|n_+, n_-\rangle := \sqrt{n_-}|n_+, n_- - 1\rangle \quad A^\dagger_-|n_+, n_-\rangle := \sqrt{n_- + 1}|n_+, n_- + 1\rangle.
\]  

(3.9)

We have

\[
|n_+, n_-\rangle = \frac{1}{\sqrt{n_+!n_-!}}(A^\dagger_+)^{n_+}(A^\dagger_-)^{n_-}|0\rangle\langle 0|
\]  

(3.10)

where \( A^\dagger_- \) may have an action on the right by \( A_- \) on \( |0\rangle\langle 0| \). \(||n_+, n_-\rangle|| = 1\) and \( |0\rangle\langle 0| \) stands for the vacuum state on \( \mathcal{H}_q \).

Consider the map

\[
\mathcal{V} : L^2(\mathbb{R}^2, dxdy) \rightarrow \mathcal{H} \otimes \tilde{\mathcal{H}} = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}),
\]  

(3.11)

with \( \mathcal{V} = \mathcal{I} \circ \mathcal{W}^{-1} \), where \( \mathcal{I} : \mathcal{H} \otimes \tilde{\mathcal{H}} \rightarrow \mathcal{H} \otimes \tilde{\mathcal{H}} \)
\[ \forall |\phi\rangle\langle\psi| \in \mathcal{H} \otimes \overline{\mathcal{H}}, \mathcal{I}(\phi(x)\overline{\psi(y)}) = \phi(x)\psi(y), x, y \in \mathbb{R}, \phi, \psi \in \mathcal{H}. \] (3.12)

For e.g., taking \( X = |\phi\rangle\langle\psi| \in \mathcal{H} \otimes \overline{\mathcal{H}} \),

\[ \mathcal{V}(\mathcal{W}(Q \otimes I_{\bar{\mathcal{H}}}(X))(x, y) = \mathcal{V}(\mathcal{W}(Q \otimes I_{\bar{\mathcal{H}}}(|\phi\rangle\langle\psi|))(x, y) = (Q \otimes I_{\bar{\mathcal{H}}})\phi(x)\psi(y). \] (3.13)

From the following relations [1]

\[ \mathcal{W}\left(\frac{I_{\bar{\mathcal{H}}} \lor Q}{I_{\bar{\mathcal{H}}} \lor P}\right)\mathcal{W}^{-1} = \left(\begin{array}{c} Q_{-} \\ P_{-}\end{array}\right), \quad \mathcal{W}\left(\frac{Q \lor I_{\bar{\mathcal{H}}}}{P \lor I_{\bar{\mathcal{H}}}}\right)\mathcal{W}^{-1} = \left(\begin{array}{c} Q_{+} \\ P_{+}\end{array}\right), \] (3.14)

using (3.13), we get on \( \mathcal{H} \otimes \overline{\mathcal{H}} \)

\[ Q \otimes I_{\bar{\mathcal{H}}} = \mathcal{V}Q_{+}\mathcal{V}^{-1}, \quad P \otimes I_{\bar{\mathcal{H}}} = \mathcal{V}P_{+}\mathcal{V}^{-1}, \] (3.15)

\[ I_{\bar{\mathcal{H}}} \otimes Q = \mathcal{V}Q_{-}\mathcal{V}^{-1}, \quad I_{\bar{\mathcal{H}}} \otimes P = \mathcal{V}P_{-}\mathcal{V}^{-1}. \] (3.16)

which lead to the annihilation and creation operators denoted by \( a_L, a_L^\dagger \) and \( b_R, b_R^\dagger \), acting on the left and on the right on \( \mathcal{H} \otimes \overline{\mathcal{H}} \), respectively, provided as follows:

\[ \mathcal{V}A_{+}\mathcal{V}^{-1} = \frac{1}{\sqrt{2}} [(Q + iP) \otimes I_{\bar{\mathcal{H}}}] = a \otimes I_{\bar{\mathcal{H}}} := a_L \]

\[ \mathcal{V}A_{+}^\dagger\mathcal{V}^{-1} = -\frac{1}{\sqrt{2}} [(Q - iP) \otimes I_{\bar{\mathcal{H}}}] = a^\dagger \otimes I_{\bar{\mathcal{H}}} := a_L^\dagger, \]

\[ \mathcal{V}A_{-}\mathcal{V}^{-1} = \frac{1}{\sqrt{2}} [I_{\bar{\mathcal{H}}} \otimes (iQ - P)] = I_{\bar{\mathcal{H}}} \otimes b := b_R \]

\[ \mathcal{V}A_{-}^\dagger\mathcal{V}^{-1} = \frac{1}{\sqrt{2}} [I_{\bar{\mathcal{H}}} \otimes (-iQ - P)] = I_{\bar{\mathcal{H}}} \otimes b^\dagger := b_R^\dagger. \] (3.17)

Given a state \(|m \rangle \langle n| \) on \( \mathcal{H}_q \simeq \mathcal{H} \otimes \overline{\mathcal{H}} \), we have the left and right annihilation operators:

\[ a_L|m \rangle \langle n| = (a \otimes I_{\bar{\mathcal{H}}})|m \rangle \langle n| = a|m \rangle \langle n|I_{\bar{\mathcal{H}}} = \sqrt{m}|m - 1 \rangle \langle n| \]

\[ b_R|m \rangle \langle n| = (I_{\bar{\mathcal{H}}} \otimes b)|m \rangle \langle n| = I_{\bar{\mathcal{H}}}|m \rangle \langle n|b = \sqrt{n + 1}|m \rangle \langle n + 1|. \] (3.18)

Taking \( \{|\phi_m \rangle \langle \psi_n|\}_{m,n=0}^\infty \) as an orthonormal basis of \( \mathcal{H}_q \simeq \mathcal{H} \otimes \overline{\mathcal{H}} \), we get

\[ \mathcal{I}(a_L|\phi_m \rangle \langle \psi_n| |x, y\rangle) = \mathcal{I}(\sqrt{m}|\phi_{m-1} \rangle \langle \psi_{n-1}| |x, y\rangle) = \sqrt{m}\phi_{m-1}(x)\psi_{n-1}(y) \]

\[ \mathcal{I}(b_R|\phi_m \rangle \langle \psi_n| |x, y\rangle) = \mathcal{I}(I_{\bar{\mathcal{H}}}|\phi_m \rangle \langle \psi_n|b |x, y\rangle = \sqrt{n + 1}|\phi_m \rangle \langle \psi_{n+1}|b = \sqrt{n + 1}\phi_m(x)\psi_{n+1}(y). \] (3.19)

### 3.1 Coherent states construction

Using the operators \( \{A_{\pm}, A_{\pm}^\dagger\} \), since there exists a set of eigenstates \(|z_\pm \rangle \) satisfying

\[ A_{\pm}|z_\pm \rangle = z_\pm|z_\pm \rangle, \quad \langle z_\pm|A_{\pm}^\dagger = \langle z_\pm| \overline{z_\pm} \] (3.20)
having complex eigenvalues $z_\pm$ with

$$|z_\pm\rangle = e^{-\frac{|z_\pm|^2}{2}} e^{(z_\pm A_\pm)} |0\rangle$$

(3.21)
given in terms of the chiral Fock basis and provided by the Baker-Campbell-Hausdorff identity

$$e^{(z_\pm A_\pm - \bar{z}_\pm A_\pm)} = e^{-\frac{|z_\pm|^2}{2}} e^{(z_\pm A_\pm)} e^{-\bar{z}_\pm A_\pm},$$

(3.22)
the CS of the noncommutative plane denoted by $|z_\pm\rangle$ related to the Hamiltonian $H_+ + H_-$ are defined as

$$|z_\pm\rangle = |z_+\rangle \langle z_-| = e^{-\frac{1}{2}(z_+^2 + z_-^2)} \sum_{n_+,n_-=0}^{\infty} \frac{z_+^{n_+} z_-^{n_-}}{\sqrt{n_+! n_-!}} |n_+\rangle \langle n_-|. $$

(3.23)
They satisfy the resolution of the identity

$$\frac{1}{\pi^2} \int_{\mathbb{C}^2} |z_\pm\rangle \langle z_\pm| d^2 z_+ d^2 z_- \equiv \mathbb{I}_q$$

(3.24)
where

$$\mathbb{I}_q = \sum_{n_+,n_-=0}^{\infty} |n_+,n_-\rangle \langle n_+,n_-|$$

(3.25)
is the identity operator on $\mathcal{H}_q$ given by

$$\mathbb{I}_q = \frac{1}{\pi} \int_C dz d\bar{z} |z\rangle e^{\bar{z} \partial_z} |z\rangle.$$ 

(3.26)

**Proof.**

We have from the definition the following relation

$$\frac{1}{\pi^2} \int_{\mathbb{C}^2} d^2 z_+ d^2 z_- |z_+\rangle \langle z_+||z_-\rangle \langle z_-| \equiv \frac{1}{\pi^2} \int_{\mathbb{C}^2} d^2 z_+ d^2 z_- |z_\pm\rangle (z_\pm)$$

(3.27)
where

$$|z_\pm\rangle = |z_+\rangle \langle z_-|$$

$$= e^{-\frac{1}{2}(z_+^2 + z_-^2)} \sum_{\tilde{n}_+,\tilde{n}_-=0}^{\infty} \frac{\tilde{z}_+^{\tilde{n}_+} \tilde{z}_-^{\tilde{n}_-}}{\sqrt{\tilde{n}_+! \tilde{n}_-!}} |\tilde{n}_+\rangle \langle \tilde{n}_-|.$$ 

(3.28)

By taking a state $|\psi\rangle$ on $\mathcal{H}_q$, we obtain

$$\frac{1}{\pi^2} \int_{\mathbb{C}^2} d^2 z_+ d^2 z_- |z_\pm\rangle (z_\pm) |\psi\rangle$$

$$= \frac{1}{\pi^2} \int_{\mathbb{C}^2} d^2 z_+ d^2 z_- |z_\pm\rangle \langle z_+| \sum_{\tilde{n}_+=0}^{\infty} |\tilde{n}_-\rangle \langle \tilde{n}_+||z_\pm\rangle \langle z_-| \psi \langle \tilde{n}_+|$$

$$= \frac{1}{\pi^2} \int_{\mathbb{C}^2} d^2 z_+ d^2 z_- |z_\pm\rangle \langle z_-| \psi \langle z_+|$$

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(3.29)

such that

\[
\frac{1}{\pi^2} \int_{C^2} d^2z_+ d^2z_- |z_\pm| (z_\pm) = |\Psi\rangle
\]  

(3.30)

In order to provide an equivalence between (3.24) and (3.26), let us consider the following relations

\[
\mathbb{I}_q |\Psi\rangle = \frac{1}{\pi^2} \int_{C^2} dzd\zbar dzd\zbar |z\rangle |z| \langle z| |\psi\rangle = \frac{1}{\pi^2} \int_{C^2} dzd\zbar dzd\zbar |z + u| |z\rangle |z + u| \langle z + u| |\psi\rangle
\]

(3.31)

where \( w = z + u \) with \( d^2w = d^2u \), and \( e^{u\partial_z} f(z) = f(z + u) \). Then, set

\[
\frac{1}{\pi} \int_{C} d^2ue^{-|u|^2} |z\rangle |e^{u\partial_z} e^{u\partial_{\zbar}} |z\rangle \langle z| |\psi\rangle = \frac{1}{\pi} \int_{C} d^2ue^{-|u|^2} |z\rangle |e^{u\partial_z} e^{u\partial_{\zbar}} |z\rangle \langle z| |\psi\rangle
\]

(3.32)

and

\[
I = |z\rangle |e^{u\partial_z} e^{u\partial_{\zbar}} |z\rangle |\psi\rangle.
\]

(3.33)

We have

\[
I = \sum_{n,m=0}^{\infty} |n\rangle |m\rangle e^{-\bar{z}z} \frac{z^n m^m}{\sqrt{n}! \sqrt{m}!} \left[ \sum_{m,m'=0}^{\infty} (m|\psi|n) e^{-\bar{z}z} \frac{z^n m^m}{\sqrt{n}! \sqrt{m}!} \right] e^{u\partial_z} e^{u\partial_{\zbar}}
\]

(3.34)

Let

\[
K(z) = \left( e^{-\bar{z}z} \frac{z^n m^m}{\sqrt{n}! \sqrt{m}!} \right) e^{u\partial_z} e^{u\partial_{\zbar}}
\]

(3.35)

We obtain

\[
K(z) = \frac{1}{\sqrt{m}!} \frac{1}{\sqrt{m}!} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!} \left( \bar{u}^k \partial_z^k [z^m e^{-\bar{z}z}] \right) \frac{1}{l!} \left( u^l \partial_{\zbar}^l [z^m e^{-\bar{z}z}] \right)
\]

(3.36)

which supplies, by performing a radial parametrization,

\[
\frac{1}{\pi} \int_{C} d^2ue^{-|u|^2} K(z)
\]
allowing to obtain (3.31) under the form:

\[ Z = \prod_{k} \langle z^{m_{k}} e^{-\bar{z}z} \rangle \]

Besides,

\[ (\langle e^{-\bar{z}z} \frac{z^{m'}}{\sqrt{m!}} \rangle e^{-\bar{z}z} \frac{z^{m}}{\sqrt{m!}}) = \frac{1}{\sqrt{m'! \sqrt{m!}}} \sum_{k=0}^{\infty} \frac{1}{k!} \partial_{k}^{k} \langle z^{m'} e^{-\bar{z}z} \rangle \partial_{k}^{k} \langle z^{m} e^{-\bar{z}z} \rangle \]  

which implies that

\[ \frac{1}{\pi} \int_{C} d^{2} u e^{-|u|^{2}} K(z) = \langle e^{-\bar{z}z} \frac{z^{m'}}{\sqrt{m'!}} \rangle e^{\bar{z} \partial_{z}} \langle e^{-\bar{z}z} \frac{z^{m}}{\sqrt{m!}} \rangle. \]  

Then,

\[ \frac{1}{\pi} \int_{C} d^{2} u e^{-|u|^{2}} |z \rangle \langle z| e^{\bar{z} \partial_{z} + u \partial_{z}} |z \rangle \langle z| = \sum_{n,m=0}^{\infty} |n' \rangle \langle m'| e^{-\bar{z}z} \frac{z^{m'}}{\sqrt{m'!}} e^{\bar{z} \partial_{z}} \langle e^{-\bar{z}z} \frac{z^{m}}{\sqrt{m!}} \rangle \sum_{n,m=0}^{\infty} |n \rangle \langle m| e^{-\bar{z}z} \frac{z^{m}}{\sqrt{m!}} \]  

allowing to obtain (3.31) under the form:

\[ \mathbb{I}_{q} |\psi\rangle = \frac{1}{\pi} \int_{C} dz d\bar{z} \frac{1}{\pi} \int d^{2} u e^{-|u|^{2}} |z \rangle \langle z| e^{\bar{z} \partial_{z} + u \partial_{z}} |z \rangle \langle z| \]  

which completes the proof.

\[ \square \]

### 3.2 Density matrix and diagonal elements

Considering that the quantum system obeys the canonical distribution \[19][12], let us take the partition function \(Z\) as that of a composite system made of two independent systems such that is the product of the partition functions of the components, i.e. \(Z = Z_{+} Z_{-}\). The diagonal elements of the density operator \(\hat{\rho} = \frac{1}{Z} e^{-\beta H}\) in the CS \(|z_{+}, z_{-}\) representation, given by
\[ |z_+, z_-\rangle = |z_+\rangle_\downarrow \otimes |z_-\rangle_\downarrow \]
\[ = e^{-|z_+|^2 - |z_-|^2} \sum_{n,n'=0}^\infty \sum_{m,m'=0}^\infty \frac{z_n^m \bar{z}_{n'}^m}{\sqrt{n!} \sqrt{n'!} \sqrt{m!} \sqrt{m'!}} |n, n', m, m'\rangle \]  

(3.42)

is then derived, with \( \hbar = 1 \), as

\[
(z_+, z_- | \hat{\rho} | z_+, z_-) = \frac{1}{\mathcal{Z}} \sum_{n_+, n_-=0}^\infty e^{-\beta H} |(n_+, n_- | z_+, z_-)|^2
\]
\[
= \left\{ \sum_{m_+, m_-=0}^\infty \left( \frac{m_+ + m_-}{(m_+ + m_-)!} \right)^2 \right\} \left\{ \sum_{n_+, n_-=0}^\infty \frac{z_{n_+}^{m_+} \bar{z}_{n_-}^{m_-}}{n_+! n_-!} \right\}
\]
\[
= \frac{1}{\mathcal{Z}} \sum_{n_+, n_-=0}^\infty e^{-\beta H} e^{-|z_+|^2} \frac{|z_+|^{2n_+}}{n_+!} \sum_{n_-=0}^\infty e^{-\beta H} e^{-|z_-|^2} \frac{|z_-|^{2n_-}}{n_-!}
\]
\[
(3.43)
\]

Performing the variables changes \( r_+ = [1 - e^{-\beta}]^{1/2} |z_+| \) and \( r_- = [1 - e^{-\beta}]^{1/2} |z_-| \) with \( \frac{d^2z}{\pi} = rdrd\varphi, \ r \in [0, \infty), \varphi \in (0, 2\pi) \), we obtain

\[
Tr \hat{\rho} = \frac{1}{\pi^2} \int_{C^2} dz_+ dz_- (z_+, z_- | \hat{\rho} | z_+, z_-)
\]
\[
= \frac{1}{\pi^2} \int_{C^2} d^2 z_+ d^2 z_- [1 - e^{-\beta}] \ e^{-|z_+|^2} [1 - e^{-\beta}] \ e^{-|z_-|^2}
\]
\[
= \frac{1}{\pi^2} \int_0^\infty r_+ dr_+ [e^{-r_+^2}] \int_0^{2\pi} d\theta_+ \times \int_0^\infty r_- dr_- [e^{-r_-^2}] \int_0^{2\pi} d\theta_-
\]
\[
= 1,
\]

(4.44)

with here \( n_\pm = 0 \), where we have used the following integral

\[
\int_0^\infty \frac{1}{n_+!} 2r^{2n_+ + 1} e^{-r^2} dr = 1,
\]

(4.45)

ensuring that the normalization condition of the density matrix is accomplished. The right-hand side of (3.42) corresponds to the product of two harmonic oscillators Husimi distributions [10].

### 3.3 Lowest Landau levels and reproducing kernel

Let us make a relationship between the quantum numbers \( n_\pm \in \mathbb{N} \) which label the energy levels per sector and the quantum numbers \( n, m \) where \( n \) labels the levels and \( m \) describes the degeneracy [5]. Fixing \( n_- = 0 \) (resp. \( n = 0 \)), one obtains a state corresponding to the quantum number \( m \) in the lowest Landau level (LLL) given by

\[
\phi_{n=0, m}(z_+, \bar{z}_+) = \frac{1}{\sqrt{2\pi l_0^2 m!}} \left( \frac{z_+}{\sqrt{2l_0}} \right)^{m} e^{-|z_+|^2/4l_0^2}
\]

(3.46)

where \( l_0 = \sqrt{\frac{\hbar}{2e}} \equiv 1 \) (with \( \hbar = 1, e = 1 \)) is the scale of lengths associated with the Landau problem.
Equivalently, fixing \( n_+ = 0 \) (resp. \( m = 0 \)), one gets a state centered at the origin \((m = 0)\) in the Landau level \( n \) given by

\[
\phi_{n,m=0}(\bar{z}_-,\bar{z}_-) = \frac{1}{\sqrt{2\pi \ell_0^2 m!}} \left( \frac{\bar{z}_-}{\sqrt{2\ell_0}} \right)^n e^{-|z_-|^2/4\ell_0^2}.
\] (3.47)

Consider the projector onto the LLL given by

\[
\mathbb{P}_0 = \sum_{m=0}^{\infty} |0, m)(0, m|.
\] (3.48)

In the LLL \( |0, \bar{z}_+\rangle \), the state \(|0, \bar{z}_+\rangle \), where \( \bar{z}_+ = x_+^1 - i x_+^2 \), is such that

\[
(0, \bar{z}_+|0, m, m) = e^{-\frac{1}{2} |\bar{z}_+|^2} \frac{\bar{z}_m}{\sqrt{m!}}, \quad \bar{z}_+ = \frac{z_+ - \bar{z}_+}{\ell_0 \sqrt{2}}.
\] (3.49)

and also

\[
(0, \bar{z}_+|0, m, \bar{z}_+) = e^{-\frac{1}{2} |\bar{z}_+|^2} \frac{\bar{z}_m}{\sqrt{m!}}
\] (3.50)

The matrix elements of the projector \( \mathbb{P}_0 \) are obtained as

\[
(0, \bar{z}_+|0, \bar{z}_+) = e^{-\frac{1}{2} |\bar{z}_+|^2 + |\bar{z}_+|^2 - 2 \bar{z}_+ |\bar{z}_+|}.
\] (3.51)

Let \( |\psi\rangle \in \mathcal{H}_q \), a state given on the LLL by

\[
|\psi\rangle = \sum_{m=0}^{\infty} a_m |0, m\rangle, \quad a_m \in \mathbb{C}.
\] (3.52)

We obtain that \( |\psi\rangle \) is analytic up to the Landau gaussian factor \( e^{-\frac{1}{2} |\bar{z}_+|^2} \) as follows:

\[
(0, \bar{z}_+|\psi\rangle = e^{-\frac{1}{2} |\bar{z}_+|^2} f(\bar{z}_+), \quad f(\bar{z}_+) = \sum_{m=0}^{\infty} a_m \bar{z}_m \frac{1}{\sqrt{m!}} \in L^2_{\text{hol}}(\mathbb{C}, dv(z, \bar{z}))
\] (3.53)

with \( dv(z, \bar{z}) = \frac{e^{-|\bar{z}_+|^2}}{2\pi} dz d\bar{z} \).

Next, let the projection operator

\[
\mathbb{P}_{\text{hol}} : L^2(\mathbb{C}, dv(z, \bar{z})) \longrightarrow L^2_{\text{hol}}(\mathbb{C}, dv(z, \bar{z}))
\] (3.54)

which is an integral operator with

\[
K(\bar{z}_+, \bar{z}_+') = e^{\frac{1}{2} \left| \bar{z}_+ \right|^2} |(0, \bar{z}_+|0, \bar{z}_+')\rangle = e^{\bar{z}_+ \bar{z}_+'}
\] (3.55)

being a reproducing kernel for \( L^2_{\text{hol}}(\mathbb{C}, dv(z, \bar{z})) \) \[\text{[2]}\]. \( L^2_{\text{hol}}(\mathbb{C}, dv(z, \bar{z})) \) being the subspace of the Hilbert space \( L^2(\mathbb{C}, dv(z, \bar{z})) \), consisting of holomorphic functions in the variable \( z \) which are \( dv \)-square integrable on \( \mathbb{C} \). Thereby,

\[
\frac{1}{\pi} \int_{\mathbb{C}} e^{\bar{z}_+ \bar{z}_+'} (\mathcal{O}f)(\bar{z}_+) e^{-|z_+'|^2} d^2z_+ = \left( \mathbb{P}_{\text{hol}} \mathcal{O} \mathbb{P}_{\text{hol}} f \right)(\bar{z}_+)
\] (3.56)
3.4 Statistical properties

Let us consider the operators given on $\mathcal{H}_q \otimes \mathcal{H}_q$ by

$$\hat{P}_X = \frac{-i\hbar}{\sqrt{2\theta}} [a_R - a_R^\dagger, \cdot] \quad \hat{P}_Y = \frac{-\hbar}{\sqrt{2\theta}} [a_R + a_R^\dagger, \cdot]$$

$$\hat{X} = \sqrt{\theta} [a_R + a_R^\dagger] \quad \hat{Y} = i \sqrt{\theta} [a_R^\dagger - a_R].$$

From (3.18), we obtain

$$[a_R - a_R^\dagger, |\tilde{n}\rangle \otimes |m\rangle] = \sqrt{n + 1} |\tilde{n}\rangle \otimes |m\rangle - \sqrt{n} |\tilde{n}\rangle \otimes |m\rangle \langle n|. \quad (3.59)$$

We get the following expressions:

$$\langle \Delta \hat{X} \rangle^2 = \frac{\theta^2}{2}, \quad \langle \Delta \hat{Y} \rangle^2 = \frac{\theta^2}{2} \quad (3.60)$$

$$\langle \Delta \hat{P}_X \rangle^2 = \frac{\hbar^2}{\theta}, \quad \langle \Delta \hat{P}_Y \rangle^2 = \frac{\hbar^2}{\theta} \quad (3.61)$$

leading to the following uncertainties:

$$[\Delta \hat{X} \Delta \hat{Y}]^2 = \frac{\theta^2}{4} \geq \frac{1}{4} |\langle [\hat{X}, \hat{Y}] \rangle|^2,$$

$$[\Delta \hat{X} \Delta \hat{P}_X]^2 = \frac{\hbar^2}{2} \geq \frac{1}{4} |\langle [\hat{X}, \hat{P}_X] \rangle|^2,$$

$$[\Delta \hat{Y} \Delta \hat{P}_Y]^2 = \frac{\hbar^2}{2} \geq \frac{1}{4} |\langle [\hat{Y}, \hat{P}_Y] \rangle|^2,$$

$$[\Delta \hat{P}_X \Delta \hat{P}_Y]^2 = \frac{\hbar^4}{40\theta^2} \geq \frac{1}{4} |\langle [\hat{P}_X, \hat{P}_Y] \rangle|^2 = 0. \quad (3.62)$$

**Concluding remarks**

We have first dealt with some preliminaries about definitions, and remarkable properties on Hilbert-Schmidt operators and the Tomita-Takesaki modular theory. Then, the construction of CS built from the thermal state has been achieved with the resolution of the identity discussed. Besides, some detailed proofs have been provided in the study of the modular theory and Hilbert-Schmidt operators. The relation between the noncommutative quantum mechanics formalism and the modular theory, both using Hilbert-Schmidt operators, has been analyzed by use of the Wigner map as an interplay between them. After, the physical model of the motion of a charged particle on the flat plane xy in the presence of a constant magnetic field along the z-axis has been taken as illustrative example. CS have been constructed in the physical model Hilbert space. Then, the density matrix, the projection onto the lowest Landau level (LLL) and some statistical properties, in the CS basis, have been discussed.

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