ON A THEOREM BY BOJANOV AND NAIDENOV APPLIED TO FAMILIES OF GEGENBAUER-SOBOLEV POLYNOMIALS

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Abstract. Let \( \{Q^{(\alpha)}_{n,\lambda}\}_{n \geq 0} \) be the sequence of monic orthogonal polynomials with respect the Gegenbauer-Sobolev inner product

\[
\langle f, g \rangle_S := \int_{-1}^{1} f(x)g(x)(1 - x^2)^{\alpha - \frac{1}{2}}dx + \lambda \int_{-1}^{1} f'(x)g'(x)(1 - x^2)^{\alpha - \frac{1}{2}}dx,
\]

where \( \alpha > -\frac{1}{2} \) and \( \lambda \geq 0 \). In this paper we use a recent result due to B.D. Bojanov and N. Naidenov \( [3] \), in order to study the maximization of a local extremum of the \( k \)th derivative \( \frac{d^k}{dx^k}Q^{(\alpha)}_{n,\lambda} \) in \([-M_{n,\lambda}, M_{n,\lambda}]\), where \( M_{n,\lambda} \) is a suitable value such that all zeros of the polynomial \( Q^{(\alpha)}_{n,\lambda} \) are contained in \([-M_{n,\lambda}, M_{n,\lambda}]\) and the function \( |Q^{(\alpha)}_{n,\lambda}| \) attains its maximal value at the end-points of such interval. Also, some illustrative numerical examples are presented.

Key words and phrases: Sobolev orthogonal polynomials; oscillating polynomials.

1. Introduction

Extremal properties for general orthogonal polynomials is an interesting subject in approximation theory and their applications permeate many fields in science and engineering \([5, 18, 21, 28, 29]\). Although it may seem an old subject from the view point of the standard orthogonality \([5, 18, 29]\), this is not the case neither in the general setting (cf. \([11-14, 20]\)) nor from the view point of Sobolev orthogonality, where it remains like a partially explored subject \([1]\). In fact, new results continue to appear in some recent publications \([10, 12, 24, 26, 27]\).

Let \( d\mu(x) = (1 - x^2)^{\alpha - \frac{1}{2}}dx \) with \( \alpha > -\frac{1}{2} \), be the Gegenbauer measure supported on the interval \([-1, 1]\). We consider the following Sobolev inner product on the linear space \( \mathbb{P} \) of polynomials with real coefficients.

\[
\langle f, g \rangle_S := \int_{-1}^{1} f(x)g(x)d\mu(x) + \lambda \int_{-1}^{1} f'(x)g'(x)d\mu(x),
\]

where \( \lambda \geq 0 \). Let \( \{Q^{(\alpha)}_{n,\lambda}\}_{n \geq 0} \) denote the sequence of monic orthogonal polynomials with respect to \( \langle \cdot, \cdot \rangle_S \). These polynomials are usually called monic Gegenbauer-Sobolev polynomials \([7, 8, 15-17, 25]\), and it is known that the zeros of these polynomials are in the real line \([15, 16]\), and
therefore they belong to other important class of algebraic polynomials, namely the oscillating polynomials [3, 19].

The main result of [3] allows to guarantee the maximal absolute value of higher derivatives of a symmetric oscillating polynomial on a finite interval are attained in the end-points of such interval, whenever the maximal absolute value of the polynomial is attained in the end-points of that interval. Then, [3] Section 4] contains a brief explanation about applications of previous result to orthogonal polynomials on the real line associated to symmetric weights. We focus our attention on that last part of [3, Section 4] in order to enlarge the range of application of [3, Theorem 1] to the class of Gegenbauer-Sobolev polynomials corresponding to the inner product (1.1).

The paper is structured as follows. Section 2 provides some background about structural properties of the Gegenbauer and Gegenbauer-Sobolev polynomials corresponding to the inner product (1.1), respectively. Section 3 contains some well-known characteristics of the class of oscillating polynomials on a finite interval. We also state there our main result (see Theorem 3.2) and provide some illustrative numerical examples. Throughout this paper, the notation \( u_n \sim v_n \) means that the sequence \( \{u_n, v_n\} \) converges to 1 as \( n \to \infty \). We will denote by \( \mathbb{P}_n \) and \( \|f\|_\infty \), the space of polynomials of degree at most \( n \) and the uniform norm of \( f \) on the interval in consideration, respectively. Any other standard notation will be properly introduced whenever needed.

2. Basic facts: Gegenbauer and Gegenbauer-Sobolev orthogonal polynomials

For \( \alpha > -\frac{1}{2} \) we denote by \( \{\hat{C}_n^{(\alpha)}\}_{n \geq 0} \) the sequence of Gegenbauer polynomials, orthogonal on \([-1, 1]\) with respect to the measure \( d\mu(x) \) (cf. [29, Chapter IV]), normalized by

\[
\hat{C}_n^{(\alpha)}(1) = \frac{\Gamma(n+2\alpha)}{n\Gamma(2\alpha)}.
\]

It is clear that this normalization does not allow \( \alpha \) to be zero or a negative integer. Nevertheless, the following limits exist for every \( x \in [-1, 1] \) (see [29, formula (4.7.8)].)

\[
\lim_{\alpha \to 0} \hat{C}_n^{(\alpha)}(x) = T_0(x), \quad \lim_{\alpha \to 0} \frac{\hat{C}_n^{(\alpha)}(x)}{\alpha} = \frac{2}{n}T_n(x),
\]

where \( T_n \) is the \( n \)th Chebyshev polynomial of the first kind. In order to avoid confusing notation, we define the sequence \( \{\hat{C}^{(0)}_n\}_{n \geq 0} \) as follows.

\[
\hat{C}^{(0)}_0(1) = 1, \quad \hat{C}^{(0)}_n(1) = \frac{2}{n}, \quad \hat{C}^{(0)}_n(x) = \frac{2}{n}T_n(x), \quad n \geq 1.
\]

We denote the \( n \)th monic Gegenbauer orthogonal polynomial by

\[
C_n^{(\alpha)}(x) = (h_n^{\alpha})^{-1}\hat{C}^{(\alpha)}_n(x),
\]

where the constant \( h_n^{\alpha} \) (cf. [29, formula (4.7.31)]) is given by

\[
h_n^{\alpha} = \frac{2^{n-1}n^{n+\alpha}}{\Gamma(n+\alpha)}, \quad \alpha \neq 0,
\]

(2.3)

\[
h_n^0 = \lim_{\alpha \to 0} \frac{2^{n-1}n^{n+\alpha}}{\Gamma(n+\alpha)} = \frac{2^n}{n}, \quad n \geq 1.
\]

(2.4)
Then for \( n \geq 1 \), we have \( C_n^{(0)}(x) = \lim_{\alpha \to 0} (h_n^{(\alpha)})^{-1} C_n^{(\alpha)}(x) = \frac{1}{2^n} T_n(x) \).

**Proposition 2.1.** Let \( \{ C_n^{(\alpha)} \}_{n \geq 0} \) be the sequence of monic Gegenbauer orthogonal polynomials. Then the following statements hold.

1. **Three-term recurrence relation.** For every \( n \in \mathbb{N} \),
   \[
   x C_n^{(\alpha)}(x) = C_{n+1}^{(\alpha)}(x) + \gamma_n^{(\alpha)} C_{n-1}^{(\alpha)}(x), \quad \alpha > -\frac{1}{2}, \quad \alpha \neq 0,
   \]
   with initial conditions \( C_0^{(\alpha)}(x) = 1 \), \( C_1^{(\alpha)}(x) = x \), and recurrence coefficient \( \gamma_n^{(\alpha)} = \frac{n(n+2\alpha-1)}{4(n+\alpha)(n+\alpha-1)} \).
2. For every \( n \in \mathbb{N} \) (see [29, formula (4.7.14)]),
   \[
   \| C_n^{(\alpha)} \|^2 = \int_{-1}^{1} |C_n^{(\alpha)}(x)|^2 d\mu(x) = \pi 2^{1-2\alpha-2n} \frac{n! \Gamma(n+2\alpha)}{\Gamma(n+\alpha+1) \Gamma(n+\alpha)}.
   \]
3. **Structure relation** (see [29, formula (4.7.15)]). For every \( n \geq 2 \),
   \[
   C_n^{(\alpha-1)}(x) = C_n^{(\alpha)}(x) + \xi_n \alpha C_{n-2}^{(\alpha)}(x),
   \]
   where
   \[
   \xi_n = \frac{(n+2)(n+1)}{4(n+\alpha+1)(n+\alpha)}, \quad n \geq 0.
   \]
4. For every \( n \in \mathbb{N} \) (see [29, formula (4.7.14)]),
   \[
   \frac{d}{dx} C_n^{(\alpha)}(x) = n C_{n-1}^{(\alpha+1)}(x).
   \]

Some well-known properties of the monic Gegenbauer-Sobolev orthogonal polynomials corresponding to the inner product (1.1) are the following.

**Proposition 2.2.** Let \( \{ Q_{n,\alpha}^{(\alpha)} \}_{n \geq 0} \) be the sequence of monic orthogonal polynomials with respect to (1.1). Then the following statements hold.

1. The polynomials \( Q_{n,\alpha}^{(\alpha)} \) are symmetric, i.e.,
   \[
   Q_{n,\alpha}^{(\alpha)}(-x) = (-1)^n Q_{n,\alpha}^{(\alpha)}(x).
   \]
2. The zeros of \( Q_{n,\alpha}^{(\alpha)} \) are real and simple, and they interlace with the zeros of the monic Gegenbauer orthogonal polynomials \( C_n^{(\alpha)} \). Furthermore, for \( \alpha \geq \frac{1}{2} \) they are all contained in the interval \([-1, 1]\) and for \(-\frac{1}{2} < \alpha < \frac{1}{2}\) there is at most a pair of zeros symmetric with respect to the origin outside the interval \([-1, 1]\), (cf. [13, 16]).
3. **[13, Lemma 5.1].** For \( \alpha \geq \frac{1}{2} \), we have \( Q_{n,\alpha}^{(\alpha)}(1) > 0 \).

It is worthwhile to point out that in the case \(-\frac{1}{2} < \alpha < \frac{1}{2}\), no global properties about the sign \( Q_{n,\alpha}^{(\alpha)}(1) \) can be deduced (cf. [15]).

However, the location of zeros of Sobolev orthogonal polynomials is not a trivial problem. For instance, if we consider \((\mu_0, \mu_1)\) a vector of compactly supported positive measures on the real line with finite total mass and the following Sobolev inner product on the linear space \( P \) of polynomials with real coefficients.
\begin{equation}
\langle f, g \rangle_{(\mu_0, \mu_1)} := \int f(x)g(x)d\mu_0(x) + \int f'(x)g'(x)d\mu_1(x),
\end{equation}

then, simple examples show that the zeros of these Sobolev orthogonal polynomials do not necessarily remain in the convex hull of the union of the supports of the measures \( \mu_k, k = 0, 1 \), and they can be complex. In this regard some numerical experiments may be found in [9]. In particular, the boundedness of the zeros of Sobolev orthogonal polynomials is an open problem [1, 10], but as was stated in [10], it could be obtained as a consequence of the boundedness of the multiplication operator \( Mf(z) = zf(z) \): If \( M \) is bounded and \( \|M\| \) is its operator norm (induced by (2.11)), then all the zeros of the Sobolev orthogonal polynomials \( Q_n \) are contained in the disc \( \{z \in \mathbb{C} : |z| \leq \|M\| \} \).

Indeed, if \( x_0 \) is a zero of \( Q_n \) then \( xp(x) = x_0 p(x) + Q_n(x) \) for a polynomial \( p \in \mathbb{P}_{n-1} \). Since \( p \) and \( Q_n \) are orthogonal, we get
\[ |x_0|^2 \|p\|_{(\mu_0, \mu_1)}^2 = \|xp\|_{(\mu_0, \mu_1)}^2 - \|Q_n\|_{(\mu_0, \mu_1)}^2 \leq \|xp\|_{(\mu_0, \mu_1)}^2 = \|Mp\|_{(\mu_0, \mu_1)}^2 \leq \|M\|^2 \|p\|_{(\mu_0, \mu_1)}^2, \]
which yields the above result.

Thus, in the last decades the question whether or not the multiplication operator \( M \) is bounded has been a topic of interest to investigators in the field, since it turns out to be a key for the location of zeros and for establishing the asymptotic behavior of orthogonal polynomials with respect to diagonal (or non-diagonal) Sobolev inner products (cf. [10, 26, 27] and the references therein).

From the structure relation (2.7) and [17, formula (3)] (see also [7, Proposition 1]) the following connection formula can be obtained.

**Proposition 2.3.** For \( \alpha > -\frac{1}{2} \),
\begin{equation}
C_n^{(\alpha-1)}(x) = Q_n^{(\alpha)}(x) - d_{n-2}(\alpha)Q_{n-2,\lambda}^{(\alpha)}(x), \quad n \geq 2,
\end{equation}
where
\begin{equation}
d_n(\alpha) = \xi_n^{(\alpha)} \frac{\|C_n^{(\alpha)}\|_{\mu_0}^2}{\|Q_n^{(\alpha)}\|_{S}^2}.
\end{equation}
Moreover,
\begin{equation}
d_n(\alpha) \sim \frac{1}{16\lambda n^2}.
\end{equation}

### 3. Maximization of Local Extremum of the Derivatives for Families of Gegenbauer-Sobolev Polynomials

A polynomial \( P \in \mathbb{P} \) is said oscillating (see [2, 19, 22, 23]) if it has all its zeros on the real line \( \mathbb{R} \). For example, the classical orthogonal polynomials on subsets of \( \mathbb{R} \) (Hermite, Laguerre and Jacobi polynomials [6, 20, 29]), orthogonal polynomials for weights in the Nevai class \( M(0,1) \) [21], including whose orthogonal with respect to weights belonging to Levin-Lubinsky class \( \mathcal{W} \) [13], and a broad class of Sobolev orthogonal polynomials [7, 9, 15, 17, 25] constitute an important family of oscillating polynomials. Usually, when all zeros of a polynomial \( P \in \mathbb{P}_n \) with \( \deg(P) = n \), are contained in a given finite interval \( [a,b] \), it is called oscillating polynomial on \( [a,b] \), (see [3, 19].)
We denote by $\text{Osc}(\mathbb{R})$ and $\text{Osc}[a, b]$ the classes of oscillating polynomials on $\mathbb{R}$ and $[a, b]$, respectively. For any $P \in \text{Osc}[a, b]$ with $\deg(P) = n$, we consider the vector $h(P) = (h_0(P), \ldots, h_n(P))$, where $h_j(P) = |P(t_j)|$, $0 \leq j \leq n$, $t_0 = a$, $t_n = b$, and $t_1 \leq t_2 \leq \cdots \leq t_{n-1}$ are the zeros of $P'$.

Amongst the main characteristics of the class $\text{Osc}[a, b]$ we list the following.

i) $P' \in \text{Osc}[a, b]$, for all $P \in \text{Osc}[a, b]$.

ii) Any $P \in \text{Osc}[a, b]$ is completely determined by its local extrema and the values at the end-points of the interval $[a, b]$ (cf. [2, Theorem 2], [4, Remark 1].)

iii) For $P \in \text{Osc}[a, b]$ with $\deg(P) = n$, there exists a monotone dependence of the parameters $h_j(P')$ on the parameters $h_0(P), \ldots, h_n(P)$ of $P$ (cf. [4, Lemma 3].)

iv) If $P \in \text{Osc}[a, b]$ with $\deg(P) \geq 3$ and $\|P\| = |P(a)|$, then each local extremum of $P'$ from the first half (i.e., with an index less than or equal to $\left\lfloor \frac{n-1}{2} \right\rfloor$, and $[\cdot]$ denoting the integer part of $\cdot$) is smaller in absolute value than $|P'(a)|$.

More precisely, the property iv) was stated in the following theorem.

**Theorem 3.1.** ([3, Theorem 1]) Let $P \in \text{Osc}[a, b]$ with $\deg(P) \geq 3$. Assume that $\|P\|_\infty = |P(a)| = 1$. Then

\[
(3.15) \quad |P'(\tau_j)| < |P'(a)|, \quad \text{for } j = 0, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor,
\]

where $\tau_1 \leq \cdots \leq \tau_{n-2}$ are the zeros of $P''$.

**Corollary 3.1.** ([3, Corollary 1]) Let $P \in \text{Osc}[-1, 1]$ be a symmetric polynomial, with $\deg(P) = n$. Assume that $\|P\|_\infty = |P(1)| = 1$. Then

\[
(3.16) \quad \|P^{(k)}\|_\infty = |P^{(k)}(1)|, \quad \text{for } k = 1, \ldots, n.
\]

As a consequence of the combination of Theorem 3.1 (or Corollary 3.1) and the structural properties of the sequence $\{Q_{n, \lambda}^{(a)}\}_{n \geq 0}$ given in the previous section, we can obtain the maximization of local extremum of the derivatives for the sequence $\{Q_{n, \lambda}^{(a)}\}_{n \geq 0}$ as follows.

Let $\{Q_{n, \lambda}^{(a)}\}_{n \geq 0}$ be the sequence of monic orthogonal polynomials with respect to $\{1, 1\}$. Let us consider $x_{n, \lambda}^{(a)[1]} < x_{n, \lambda}^{(a)[2]} < \cdots < x_{n, \lambda}^{(a)[n]}$ the zeros of the Gegenbauer-Sobolev polynomial $Q_{n, \lambda}^{(a)}$ and $N$ the maximum value attained by $|Q_{n, \lambda}^{(a)}(x)|$ on the interval $[x_{n, \lambda}^{(a)[1]}, x_{n, \lambda}^{(a)[n]}]$. Then $M_{n, \lambda}$ can be defined to be the minimal real point such that $x_{n, \lambda}^{(a)[n]} < M_{n, \lambda}$ and $|Q_{n, \lambda}^{(a)}(M_{n, \lambda})| = N$, i.e., $M_{n, \lambda}$ is the point closest to $x_{n, \lambda}^{(a)[n]}$ where the maximal absolute value of the polynomial $Q_{n, \lambda}^{(a)}$ is attained. Notice that $M_{n, \lambda}$ also depends on the parameter $\alpha$ and $Q_{n, \lambda}^{(a)} \in \text{Osc}[-M_{n, \lambda}, M_{n, \lambda}]$. Thus, we can consider the following normalized polynomials

\[
(3.17) \quad Q_{n, \lambda}^{(a)}(x) := \frac{Q_{n, \lambda}^{(a)}(x)}{Q_{n, \lambda}^{(a)}(M_{n, \lambda})}, \quad x \in [-M_{n, \lambda}, M_{n, \lambda}], \quad n \geq 0.
\]

**Theorem 3.2.** Let $\{Q_{n, \lambda}^{(a)}\}_{n \geq 0}$ be the sequence of orthogonal polynomials given in (3.17). Then

\[
\frac{d^k}{dx^k} Q_{n, \lambda}^{(a)}(x) \quad \text{attains its maximal value on the interval } [-M_{n, \lambda}, M_{n, \lambda}] \text{ at the end-points, for } \alpha > -\frac{1}{2}
\]

and $1 \leq k \leq n$. 


Proof. It suffices to follow the proof of Theorem 3.1 (or Corollary 3.1) given in [3, Theorem 1 (or Corollary 1)] by making the corresponding modifications. □

Notice that from a numerical point of view the value $M_{n,\lambda}$ can be difficult to obtain for $n$ large enough. However, for any value $K > 0$ such that $N < |Q_{\alpha_{n,\lambda}}(x)|$ for $x < -K$ and $x > K$, the result of Theorem 3.2 remains true on the interval $[-K, K]$.

We finish this section providing some illustrative numerical examples (with the help of MAPLE) about the above result for different values of $n$, $\alpha$ and $\lambda$ (see Figure 1 and Figure 2 below).

![Graphs](image)

**Figure 1.** Graphics of $|\frac{d^k}{dx^k} q_{n,\lambda}^{(\alpha)}|$ for $n = 4$, $\alpha = \lambda = 1$, $M_{n,\lambda} = 0.9926198253$ and $k = 0, 1, 2$, respectively.
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Figure 2. Graphics of $|\frac{d^k}{dx^k} q_{n,\lambda}^{(\alpha)}|$ for $n = 7$, $\alpha = -\frac{1}{4}$, $\lambda = \frac{1}{2}$, $M_{n,\lambda} = 1.091516326$ and $k = 0, 2, 3$, respectively.

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