Bi-partite and global entanglement in a many-particle system with collective spin coupling

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Bipartite and global entanglement are analyzed for the ground state of a system of $N$ spin 1/2 particles interacting via a collective spin-spin coupling described by the Lipkin-Meshkov-Glick (LMG) Hamiltonian. Under certain conditions which includes the special case of a super-symmetry, the ground state can be constructed analytically. In the case of an anti-ferromagnetic coupling and for an even number of particles this state undergoes a smooth crossover as a function of the continuous anisotropy parameter $\gamma$ from a separable ($\gamma = \infty$) to a maximally entangled many-particle state ($\gamma = 0$). From the analytic expression for the ground state, bipartite and global entanglement are calculated. In the thermodynamic limit a discontinuous change of the scaling behavior of the bipartite entanglement is found at the isotropy point $\gamma = 0$. For $\gamma = 0$ the entanglement grows logarithmically with the system size with no upper bound, for $\gamma \neq 0$ it saturates at a level only depending on $\gamma$. For finite systems with total spin $J = N/2$ the scaling behavior changes at $\gamma = \gamma_{\text{crit}} = 1/J$.

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I. INTRODUCTION

Since the early days of quantum theory it was realized that quantum systems can possess correlations that do not have a classical counterpart 1, 2, 3. For a long time this phenomenon, called entanglement, has been of interest mostly in the context of foundations of quantum mechanics. With the advent of quantum information science 4 it has been realized that entanglement is an essential resource for efficient computation and communication. This initiated a more systematic study of its properties. While entanglement in small systems is by now well understood, many particle entanglement is still a widely open field. It is well known that quantum correlations and entanglement naturally occur in interacting many-particle systems but we only begin to understand their role in the systems 5.

Recently the entanglement properties of quantum systems near the critical points of quantum phase transitions 6 have attracted much attention. Two-particle entanglement has been studied in terms of the concurrence 7, e.g. in one dimensional spin chains 8, 9, 10, 11, 12. The concurrence contains however only limited information about the global distribution of entanglement, and other measures such as the bi-partite entanglement between blocks of spins may be of larger interest. Bi-partite entanglement has been analyzed e.g. in one dimensional quantum spin chains in 12, where it was shown that it scales logarithmically with the system size at the critical point with a prefactor determined by the universality class and saturates in the noncritical regime.

We here study the bi-partite entanglement in a system of spins with a collective coupling described by a generalization of the Lipkin-Meshkov-Glick (LMG) Hamiltonian 13. Since this Hamiltonian is symmetric under the exchange of particles the Hilbert space separates into subspaces whose dimensions grow only linearly with the number of particles, which makes them numerically accessible. In particular we consider the case of an even number of spins and an anti-ferromagnetic coupling. Furthermore we concentrate on the case where the Hamiltonian can be factorized in a product of two terms each being linear in the total spin operators. Under these conditions, which includes the special case of a super-symmetry (SUSY) 14, 15, the ground state can be constructed explicitly 16. This ground state undergoes a smooth transition from a separable to a maximally entangled state as a function of a parameter $\gamma$, which characterizes the asymmetry between the collective spin coupling in $x$ and $y$ directions. The two-particle concurrence in this system, which due to symmetry is the same for all pair of spins, has been analyzed in 17. Very recently also bi-partite entanglement has been studied in 18 and 19, however for the ferromagnetic version of the LMG model, where there is a quantum phase transition. Although in the anti-ferromagnetic case, considered here, there is no quantum phase transition, we find a discontinuous behavior of the entanglement when $\gamma$ is changed.

After discussing the LMG model and its ground state under the condition of super-symmetry in Sec.II, we analyze the bi-partite entanglement of this state in terms of the von Neumann entropy 20 in Sec.III. We show that for an isotropic interaction ($\gamma = 0$) and in the thermodynamic limit the entropy grows logarithmically with the spin of the subsystem with no upper bound. On the other hand for any non vanishing $\gamma$ the entropy has an upper limit determined solely by $\gamma$. Furthermore it becomes a function of the ratio of the subsystem spin to the total spin rather than a function of the subsystem spin alone. In a finite system the transition between isotropic and anisotropic behavior occurs at $\gamma_{\text{crit}} = J^{-1}$. To understand the saturation of the entanglement quantitatively,
we give in Sec. IV an analytic estimate for the global entanglement by determining the geometric measure of entanglement.

II. COLLECTIVE SPIN COUPLING AND SUPER-SYMMETRY

Let us consider an even number $N$ of spin $1/2$ particles interacting through a nonlinear coupling of the collective spin $J_\mu = \sum_{j=1}^{N} \hat{\sigma}_\mu$, where $\hat{\sigma}_\mu$ denotes the $\mu$’s component of the single-particle spin. The interaction is assumed to be of second order in the total spin and is thus a generalization of the Lipkin-Meshkov-Glick (LMG) Hamiltonian [13]

$$H = \alpha J_z + \beta J_x^2 + J_y^2 - 2\mu J_y.$$  \hspace{1cm} (1)

$\alpha$ and $\beta$ are positive real numbers and thus the coupling is of the anti-ferromagnetic type. $H$ commutes with the total spin $\mathbf{J}^2$ and thus the total Hilbert space separates in sub spaces determined by the spin quantum number $J$. We here restrict ourselves to the case of maximum spin, i.e. $J = N/2$. As has been shown in [10], (1) can be written as a product of two terms linear in the collective spin operators if $\beta = \alpha^2$:

$$H = (\alpha J_z + i\beta J_y - i\mu) (\alpha J_z - i\beta J_y + i\mu) - \mu^2.$$  \hspace{1cm} (2)

$H + \mu^2$ is positive definite, and if $\mu = m$, with $m \in \{-J, -(J-1), \ldots, (J-1), J\}$, $J = N/2$ being the total angular momentum, it possesses a non degenerate ground state with $E = 0$ obeying

$$(\alpha J_z - i\beta J_y + im) |\Psi\rangle = 0.$$  

Since this equation is linear the ground state can be easily constructed, which yields

$$|\Psi\rangle = \mathcal{N}(\gamma, m) \exp[-\gamma J_z] |m_y = m\rangle$$  \hspace{1cm} (3)

where we have introduced the real anisotropy parameter $\gamma$ through $\tanh(\gamma) = \alpha \geq 0$. It is interesting to note that an anisotropy in the spin coupling is reflected here in the non-unitary term $\exp[-\gamma J_z]$. In the fully isotropic limit $\gamma = 0$, the ground state is the state $|m_y = m\rangle$ which is entangled for all $|m| < J$. In the maximally anisotropic case $\gamma = \infty$, the ground state is $|m_z = -J\rangle$, which is a product state. The loss of entanglement in this case is due to the non-unitary term $\exp[-\gamma J_z]$. Changing $\gamma$, e.g. as function of time, from $\infty$ to 0 causes a smooth transition from a factorized to an entangled many-body state.

Due to the symmetry of the coupling all matrix elements of the Hamiltonian between states corresponding to different total spin $J$ vanish exactly even for time-dependent parameter. Thus even though the ground state of (1) becomes degenerate for $\gamma = 0$ with respect to the total spin $J$, the system cannot undergo a first order quantum phase transition upon changing $\gamma$. In fact the degeneracy in $J$ at $\gamma = 0$ can easily be lifted by adding a term $-\lambda J_z$ to (1), which has no effect on $|\Psi\rangle$.

In the following we will restrict ourselves to the most interesting special case $m = 0$. As has been shown in [21] and [22] the collective state $|m = 0\rangle$ has the largest global entanglement and should thus be considered as the state with maximum $N$-particle entanglement. A generalization to arbitrary $m$ values is rather straightforward but less instructive. An additional feature of the $m = 0$ case is the presence of a super-symmetry of the LMG Hamiltonian [10]. As a consequence in every spin sector $J$ the spectrum of (1) has for all values of $\gamma$ a nondegenerate ground state and all excited states are pairwise degenerate [13]. As shown in [10] the energy gap between the ground state and the pair of first excited states does not close.

For $m = 0$ the ground state reads explicitly

$$|\Psi\rangle = \frac{e^{-\gamma J_z}}{\sqrt{P_J(\cosh 2\gamma)}} |m_y = 0\rangle$$  \hspace{1cm} (4)

with $P_J$ being Legendre polynomials.

III. BI-PARTITE ENTANGLEMENT

In the following section we discuss the entanglement between two arbitrary partitions of the $N$ particle system in the SUSY ground state of the LMG model. As mentioned in the introduction it is not important here how the partitioning is done. Due to the symmetry of the Hamiltonian only the number of particles in each partition is of relevance.

A. Entropy of entanglement and distribution of Schmidt coefficients

A generally accepted quantitative measure for the entanglement between two subsystems 1 and 2 if the total system is in a pure state $|\Psi\rangle$ is the von Neumann entropy of either of the two subsystems (entropy of entanglement)

$$S(\Psi) = -\text{tr}_1 \left\{ \rho_1 \ln \rho_1 \right\} = -\text{tr}_2 \left\{ \rho_2 \ln \rho_2 \right\},$$

where

$$\rho_{1,2} = \text{tr}_{2,1} \left\{ |\Psi\rangle \langle \Psi| \right\},$$

are the reduced density matrices. $S(\Psi)$ is essentially a measure for the information loss due to division of the system and ignoring one of the subsystems. If there is entanglement between 1 and 2 in the original pure state $|\Psi\rangle$ of the total system, the entropy is nonzero. On the other hand if $|\Psi\rangle$ factorizes there is no information loss if we ignore one subsystem and the entropy vanishes.
The von Neumann entropy for pure states $S(\Psi)$ is identical to the minimum relative entropy of entanglement $E_2(\Psi)$ \cite{23} with respect to all bi-partite separable states $\sigma \in \mathcal{S}_2$

$$\sigma = \sum_i p_i \sigma^1_i \otimes \sigma^2_i \quad p_i \geq 0, \quad \sum_i p_i = 1.$$  

$$E_2(\Psi) = \min_{\sigma \in \mathcal{S}_2} S(\Psi|\sigma),$$  

$$S(\Psi|\sigma) = \text{tr} (\rho \log_2 \rho - \rho \log_2 \sigma)$$

and $\rho = |\Psi\rangle\langle \Psi|$.  

Calculating the von Neumann entropy of a many-particle system is in general a very nontrivial task due to the exponential growth of the relevant Hilbert space. We will show now that the von Neumann entropy can be related to the variance of the distribution of Schmidt coefficients, arranged in an appropriate order, in the limit of a large number of particles. For the symmetric spin states considered here this variance can easily be calculated, which will be done in the following subsection.

Let $|\Psi\rangle$ denote a pure state of a quantum system consisting of two parts labeled 1 and 2. In the case of finite dimensional spaces, Schmidt’s theorem \cite{24} asserts that any state $|\Psi\rangle$ in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be written in the form:

$$|\Psi\rangle = \sum_{m=1}^\chi \lambda_m \left| \Phi^{(1)}_m \right\rangle \otimes \left| \Phi^{(2)}_m \right\rangle$$  

where $\chi \leq \min\{d_1, d_2\}$, $d_1$ and $d_2$ being the dimensions of the corresponding Hilbert spaces. $\left\{ \left| \Phi^{(1)}_m \right\rangle \right\}$ and $\left\{ \left| \Phi^{(2)}_m \right\rangle \right\}$ are sets of orthonormal states for the subsystems 1 and 2, respectively, and $\lambda_m$ are the positive Schmidt coefficients obeying the sum rule

$$\sum_{m=1}^\chi \lambda_m^2 = 1.$$  

It is easy to see that the entropy of entanglement is related to the Schmidt coefficients via

$$S(\Psi) = -\sum_{m=1}^\chi \lambda_m^2 \log_2 \lambda_m^2.$$

The Schmidt rank, i.e. the number of Schmidt coefficients $\chi$ provides a simple upper bound for the entropy of entanglement $S(\Psi) \leq \log_2 \chi$. If $\chi \sim \min\{d_1, d_2\}$, i.e. if it scales exponentially with the number of particles, $\log_2 \chi$ is a polynomial function of the number of particles in the smaller of the two subsystems. For the symmetric coupling considered here, the dimension of the relevant Hilbert space $d$ increases only linear in the number of particles implying a logarithmic scaling of $\log_2 \chi$ with the system size. Thus also the von Neumann entropy $S$ is expected to scale logarithmically, however with a yet unknown coefficient. In order to calculate this coefficient it is obviously not sufficient to use $\log_2 \chi$ as an estimate for $S(\Psi)$.

It is possible, however, to find a better estimate for $S(\Psi)$ in terms of the variance of the distribution of appropriately ordered Schmidt coefficients. To show this we first note that, as shown in \cite{25}, the following inequality holds

$$-\int |f(x)|^2 \log_2 |f(x)|^2 \, dx \leq \frac{1}{2} \left( 1 + \log_2 \pi e \right) + \log_2 \Delta x$$

(10)

where $\Delta x^2 = \int (x - \bar{x})^2 |f(x)|^2 \, dx$ is the variance of a probability distribution $f(x)$. This inequality becomes an equality if $f(x)$ is a Gaussian function.

For large values of $\chi$ the sum in eq. \cite{26} can be written as an integral with $\lambda_m^2 \rightarrow \lambda^2(m)$, representing a continuous, smooth probability distribution. The functional form of this distribution depends on the ordering of the Schmidt coefficients $\lambda_m$. For symmetric states this ordering can be chosen in such a way, that $\lambda^2(m)$ becomes to a good approximation a Gaussian function \cite{20}. In this case the entropy of entanglement is given by

$$S(\Psi) = \frac{1}{2} \left( 1 + \log_2 \pi e \right) + \log_2 \Delta \lambda,$$

(11)

where $\Delta \lambda$ is the variance of the Schmidt coefficients.

B. Clebsch-Gordan decomposition and bi-partite entanglement

We will now calculate the entropy of entanglement using eq. \cite{11} by finding a suitable bipartite decomposition of the ground state \cite{11}. The scaling of $S(\Psi)$ with the system size will be studied in detail and in particular the prefactor of the logarithm determined. In the limit of a totally isotropic spin coupling $\gamma = 0$ an explicit analytic expression for the variance of the Schmidt coefficients and the entanglement can be given. For nonzero values of $\gamma$ numerical results will be presented.

We start from equation \cite{11} and make use of the Clebsch-Gordan decomposition of total angular momentum $J = N/2$ into $J_1$ and $J_2$ with $J_1 + J_2 = J$:

$$|\Psi\rangle = N(\gamma) e^{-\gamma J_1} |m_y = 0\rangle$$  

$$= N(\gamma) \sum_{m_1} \sum_{m_2} C^{J_1 J_2}_{m_1 m_2} \hat{d}_m^{\dagger}(-i\gamma) |J_1 m_1\rangle \otimes |J_2 m_2\rangle$$

Here $C^{J_1 J_2}_{m_1 m_2}$ are the Clebsch-Gordan (Wigner) coefficients and $m = m_1 + m_2$. $\hat{d}_m^{\dagger}(\beta)$ are the rotation matrices defined as \cite{27}

$$\hat{d}_m^{\dagger}(\beta) = \langle Jm' | e^{-i\beta J_y} | Jm \rangle$$  

(13)

$$= \sqrt{(J + m)! (J - m)!} \left( \frac{\sin \beta}{2} \right)^{m - m'} \left( \frac{\cos \beta}{2} \right)^{m + m'}$$

$$\times P^{(m - m', m + m')}_{J - m} (\cos \beta),$$
where $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials. Since we are considering the special decomposition for $J = J_1 + J_2$, the Wigner coefficients have the binomial distribution:

$$\left(C_{m_1,m_2}^{J_1,J_2}\right)^2 = \frac{2J_1}{(J_1 + m_1)} \frac{2J_2}{(J_2 + m_2)} \frac{2J}{J + m}.$$  \hfill (14)

This relation combined with (13) gives the following decomposition for the wave function $|\Psi\rangle$:

$$|\Psi\rangle = \sum_{m_1 = -J_1}^{J_1} \sum_{m_2 = -J_2}^{J_2} A_{m_1,m_2}(\gamma) |J_1 m_1\rangle \otimes |J_2 m_2\rangle$$ \hfill (15)

with

$$A_{m_1,m_2}(\gamma) = N(\gamma) (i)^{m_1 + m_2} \times$$

$$\times \sqrt{\frac{(J)!^2 (2J_1)! (2J_2)!}{(2J)! (J_1 + m_1)! (J_1 - m_1)! (J_2 + m_2)! (J_2 - m_2)!}} \times \left(\coth \frac{\gamma}{2}\right)^{m_1 + m_2} P^{(\gamma)}_{J_1 m_1 - m_2, m_1 + m_2} (\cosh \gamma).$$ \hfill (16)

(16) is separable if all coefficients $A_{m_1,m_2}$ factorize. This is the case in the limit $\gamma \to \infty$, where the coth term in (16) approaches unity and the Jacobi polynomials factorize in $m_1$ and $m_2$.

1. **Isotropic spin coupling $\gamma = 0$**

Making use of the results of subsection A the summation in (12) can be carried out explicitly for the limit of isotropic spin coupling. In this case one sees from (16) that only coefficients $A_{m_1,m_2}$ with $m_1 + m_2 = 0$ survive. Thus one has the following decomposition

$$|\Psi\rangle = \sum_{m = -J_1}^{J_1} C_{m,m}^{J_1,J_2} |J_1 m\rangle \otimes |J_2 - m\rangle$$

where $J_2 \geq J_1$ was assumed without loss of generality. The Schmidt coefficients are, therefore, the Clebsch-Gordan coefficients. In the limit $J_2 \gg J_1$, they have a Gaussian form [26]

$$\left(C_{m,m}^{J_1,J_2}\right)^2 \approx \exp\left[-\frac{m^2}{J_1 \pi}\right].$$

where $|m| \leq J_1 \ll J_2$. The Gaussian form of the coefficients allows to make use of relation (11) to calculate the von Neumann entropy for $J_1 \ll J$:

$$S \approx \frac{1}{2} \log_2 J_1 + \frac{1}{2}(1 + \log_2 \pi e), \quad \gamma = 0$$ \hfill (17)

In fig[1] we have plotted the von Neumann entropy for $\gamma = 0$ as a function of the subsystem spin $J_1 = N_1/2$ for different values of the total spin $J = N/2$. For $J_1 \ll J$ a logarithmic scaling with prefactor 1/2 is evident. When $J_1$ approaches $J/2$ the entropy saturates since $S$ is symmetric with respect to the replacement $J_1 \leftrightarrow J - J_1$. It is important to note that for $J_1 \ll J$ the von Neumann entropy $S$ does not depend on $J$.

2. **Anisotropic spin coupling $\gamma \neq 0$**

If the spin coupling is anisotropic, i.e. if $\gamma \neq 0$, the double sum in eqs. (12) or (15) remains. Thus in order to discuss the influence of a finite $\gamma$ it is necessary to explicitly evaluate the sum in (9). We have done this numerically for a total particle number up to 200 and subsystems up to 100 particles. The results are shown in figs[2] and [3]. As can be seen from fig[2] in contrast to the isotropic case $\gamma = 0$, the entropy is no longer independent on the total spin if

$$\gamma \geq \gamma_{\text{crit}} \equiv \frac{1}{J}.$$  \hfill (18)

In the thermodynamic limit the critical point is $\gamma = 0$. As can be seen from fig[3] for any $\gamma \geq J^{-1}$ the entropy becomes a function of the logarithm of the fraction of particles $J_1/J = N_1/N$.

Our numerical calculations suggest for $J_1 \ll J$ and $\gamma \gg J^{-1}$ a functional dependence of the form

$$S \sim f(\gamma) \log_2 \left(\frac{J_1}{J}\right), \quad \gamma > \gamma_{\text{crit}}$$

The reduction of entanglement with increasing $\gamma$ is expected. The state $N(\gamma) e^{-\gamma J_1} |m_y = 0\rangle$ is maximally entangled for $\gamma = 0$ and the prefactor $e^{-\gamma J_2}$ corresponds to a local non-unitary operation which always decreases the...
FIG. 2: Entropy of entanglement for $J = 100$ (full line) and $J = 200$ (dashed line) as function of logarithm of subsystem spin $J_1$ for different values of $\gamma$. One recognizes that in contrast to the isotropic case the entropy now depends on the total spin $J$.

FIG. 3: Entropy of entanglement for different values of $\gamma$ as function of $\log_2(J_1/J)$ for $J = 100$ (solid line) and $J = 200$ (dashed line). For $\gamma J \geq 1$ the curves become virtually indistinguishable.

amount of entanglement. For large values of $\gamma$ the state becomes eventually separable.

The most peculiar feature of the von Neumann entropy is the change of the scaling behavior with $J_1$ from $S \sim \log_2 J_1$ for $\gamma \geq \gamma_{\text{crit}}$ to $S \sim \log_2 J_1/J$ for $\gamma \leq \gamma_{\text{crit}}$. The role of $\gamma_{\text{crit}} = J^{-1}$ and the change of the scaling behavior is also reflected in the distribution of ordered Schmidt numbers. As can be seen from fig.4 the fall-off of the Schmidt numbers $\lambda_m$ becomes exponential when $\gamma J$ exceeds unity.

FIG. 4: Ordered distribution of normalized Schmidt numbers for different values of $\gamma J$.

There is no obvious distinction of the point $\gamma = J^{-1}$ in the properties of the system. The system does not undergo a phase transition at this point. Due to the SUSY the qualitative structure of the spectrum is the same for all values of $\gamma$ and there is always an energy gap between the ground and first excited state. Thus the question remains whether there are any physical signatures for the change of the scaling behavior of the entanglement at $\gamma = \gamma_{\text{crit}}$.

IV. GEOMETRIC ESTIMATE FOR GLOBAL ENTANGLEMENT

In the previous section we have discussed the bi-partite entanglement of two partitions of the $N$ spin 1/2 system. We have seen (cf. fig.4) that for $\gamma \geq \gamma_{\text{crit}}$ the von Neumann entropy has a maximum value independent on $J$.

In this section we will quantitatively analyze this maximum by determining the $N$-partite or global entanglement $E_N$ of the SUSY ground state which is an upper bound to the bipartite entanglement $E_2$. Although it is not possible to obtain an analytic expression for $E_N$, we can determine a very good estimate for it given by the geometric measure of entanglement.

A. Relative entropy and geometric measure of entanglement

A many-particle state is called $N$-partite separable, if it can be written as a product of states of all $N$ subsystems. Obviously a bi-partite entangled state is always $N$-partite entangled but not vice versa. A quantitative measure of many-particle or global entanglement of a state $\rho$ is the minimum relative entropy which determines the minimum distance between $\rho$ and the set $S_N$.
of $N$-partite product states $\sigma$:

$$E_N = \min_{\sigma \in \mathcal{S}_N} S(\rho||\sigma)$$  \hspace{1cm} (18)$$

where

$$S(\rho||\sigma) \equiv \text{tr}(\rho \log_2 \rho - \rho \log_2 \sigma),$$  \hspace{1cm} (19)$$

$\sigma \in \mathcal{S}_N$ being an $N$-partite separable state

$$\sigma = \sum_{i=1}^N p_i \rho_i^1 \otimes \rho_i^2 \otimes ... \otimes \rho_i^N$$  \hspace{1cm} (20)$$

with $p_i > 0$, and $\sum_i p_i = 1$. For the bi-partite case $E_N$ is equivalent to the entanglement of formation [23], which in the case of pure states is identical to the von Neumann entropy.

Since the set $\mathcal{S}_N$ is smaller than $\mathcal{S}_2$ for any partitioning, $\mathcal{S}_N \subset \mathcal{S}_2$, it follows immediately

$$E_N(\Psi) \geq E_2(\Psi),$$

i.e. the global entanglement represents an upper bound to the bi-partite entanglement.

In order to compute $E_N$ for any state $\rho$, one has to find its closest product state $\sigma$. This is in general a quite difficult task and can be done only in very special cases. There is however a lower bound to $E_N$ which gives a good estimate for the behavior of the global entanglement. This lower bound is the geometric entanglement $E_G(\Psi)$ [21, 22],

$$E_G(\Psi) \equiv -2 \log_2 \Lambda_{\max}(\Psi),$$  \hspace{1cm} (21)$$

where

$$\Lambda_{\max}(\Psi) = \max_\phi |\langle \phi|\Psi \rangle|$$  \hspace{1cm} (22)$$

is the maximum overlap of $|\Psi\rangle$ with an $N$-partite separable state $|\phi\rangle$. $E_G(\Psi)$ is not an entanglement monotone and thus in the strict sense not a valid measure of entanglement. It does give however a close lower bound to $E_N$ which for some states such as the Dicke states is a tight lower bound, i.e. $E_G = E_N$ [21]. The geometric entanglement can easily be calculated for states which are permutation invariant, which is the case for the SUSY ground state [4].

**B. Geometric measure of entanglement for the SUSY state**

To calculate the geometric entanglement $E_G$ or equivalently the maximum overlap $\Lambda_{\max}$ of the SUSY ground state [4] with $N$-partite separable states it is sufficient to construct the most general $N$-partite separable state which is invariant under permutation of spins [21]. This state is given by rotations of the state $|m_z = -J\rangle$:

$$|\phi\rangle = e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\xi J_z}|m_z = -J\rangle.$$  

Calculating the overlap of $|\phi\rangle$ with $\Psi$ and maximizing it with respect to the real parameters $\alpha, \beta$ and $\xi$ leads to $|\phi\rangle = |m_z = -J\rangle$.

The corresponding entanglement eigenvalue reads

$$\Lambda_{\max}(\gamma) = \frac{\sqrt{(2J)!}}{2^JJ!} \frac{e^{\gamma J}}{\sqrt{P_J(\cosh 2\gamma)}}$$  \hspace{1cm} (23)$$

1. **isotropic spin coupling $\gamma = 0$**

In the isotropic case [28] reduces to [21]

$$\Lambda_{\max}(\gamma = 0) = \frac{\sqrt{(2J)!}}{2^JJ!}$$  \hspace{1cm} (24)$$

and thus the geometric entanglement is given by $E_G(\Psi) = \frac{1}{2} \log_2 J$. Since the SUSY state for $\gamma = 0$ is the Dicke state $|J, m_y = 0\rangle$ the geometric entanglement is a tight lower bound to the relative entropy and thus

$$E_N(\Psi) = E_G(\Psi) = \frac{1}{2} \log_2 J.$$  \hspace{1cm} (25)$$

2. **anisotropic spin coupling $\gamma \neq 0$**

It is obvious that the largest entanglement is obtained for $\gamma = 0$ where the maximum overlap $\Lambda_{\max}$ with separable states is smallest. On the other hand for $\gamma \rightarrow \infty$ the state becomes identical to the separable state $|m_z = -J\rangle$. The same conclusion can of course be obtained from eq. [23] employing the asymptotic expansion of the Legendre polynomials.

![FIG. 5: Geometric measure of entanglement as function of $\log_2 J$ for different values of the anisotropy parameter $\gamma$. One recognizes a saturation at $J > \gamma^{-1}$.](image)

In fig[4] we have plotted the geometric entanglement as a function of $J$ for different values of $\gamma$. For sufficiently small values of $J$ one recognizes a logarithmic growth which saturates when $J$ exceeds the value $\gamma^{-1}$. One can
easily obtain an analytic expression for the saturation value of $E_G$. Making use of the asymptotics of the Legendre polynomials for large $J$ and $\gamma \neq 0$

$$P_J(\cosh 2\gamma) \underset{\text{large } J}{\longrightarrow} \frac{1}{\sqrt{1-e^{-2\gamma}}} e^{2\gamma J} \frac{(2J-1)!!}{2^J J!}$$

one arrives at the simple expression

$$\Lambda_{\text{max}}(\gamma) = (1-e^{-4\gamma})^{1/4}$$

leading to

$$E_G(\gamma, J) \underset{\text{large } J}{\longrightarrow} -\frac{1}{2} \log_2(1-e^{-4\gamma}).$$

Comparing the numerical values for $E_2$ obtained in the previous section one finds that $E_G < E_2$. This shows that in the case of non-isotropic coupling $\gamma \neq 0$ the geometric entanglement is not a tight lower bound to the global entanglement, i.e. here $E_G < E_2$. Thus $E_G$ can only be used as qualitative measure for the global entanglement.

V. CONCLUSIONS

In the present paper we have studied the bi-partite entanglement between blocks of spins in the antiferromagnetic Lipkin-Meshkov-Glick model, which describes a nonlinear coupling of collective spins, under conditions of super-symmetry. The super-symmetry of the model allows for an explicit construction of the ground state which undergoes a smooth transition from a separable to a maximally entangled state when changing the anisotropy of the collective spin coupling. Making use of the Clebsch-Gordan decomposition of angular momenta, the von Neumann entropy which quantifies the bi-partite entanglement can be calculated analytically in the isotropic case or numerically in the case of anisotropic coupling.

Although the structure of the spectrum stays always the same with one nondegenerate ground state and pairwise degenerate excited states, and no level crossing or merging occurs, the entanglement shows a discontinuous behavior at the isotropy point. When the anisotropy parameter $\gamma$ vanishes exactly, the von Neumann entropy grows logarithmically with the number of particles in the subsystem. For any nonvanishing value of $\gamma$ (in the thermodynamic limit) the entropy saturates at a finite value determined by $\gamma$. The maximum bi-partite entanglement can be estimated by the geometric measure of global entanglement, which has been determined analytically. Furthermore in this case the entropy becomes a function of the ratio of particle number in the subsystem to the total particle number rather than a function of the subsystem size alone. For finite systems the transition between the two cases happens at a small but finite value of $\gamma$ corresponding to the inverse of the total number of spins. A discontinuous scaling behavior of entanglement is usually attributed to level crossings and quantum phase transitions. This is not the case in the present system and the question remains whether there are any physical signatures of the discontinuous transition.

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