EXTENDED ORBIT PROPERTIES ON SURFACES

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ABSTRACT. In this paper, we study “demi-caractéristique” and (Poisson) stability in the sense of Poincaré. Using the definitions à la Poincaré for \(\mathbb{R}\)-actions \(v\) on compact connected surfaces, we show that \("R\)-closed" \(\Rightarrow\) “pointwise almost periodicity (p.a.p.)” \(\Rightarrow\) “recurrence” \(\Rightarrow\) non-wandering. Moreover, we show that the action \(v\) is “recurrence” with \(|\text{Sing}(v)| < \infty\) iff \(v\) is regular non-wandering. If there are no locally dense orbits, then \(v\) is “p.a.p.” iff \(v\) is “recurrence” without “orbits” containing infinitely singularities. If \(|\text{Sing}(v)| < \infty\), then \(v\) is “\(R\)-closed” iff \(v\) is “p.a.p.”.

1. INTRODUCTION AND PRELIMINARIES

In the Poincaré celebrated paper [P] which is an origin of dynamical systems, he studied surface flows. In the series of the relative works, he used the slightly different definitions from the presence notations (e.g. semi-characteristics, limit cycles). On the other hand, the following fact for topological dynamics on compact metrizable spaces is known: \(R\)-closed \(\subset\) p.a.p. \(\subset\) recurrent \(\subset\) non-wandering. In this paper, we study a surface flow using the notations of Poincaré. In particular, we study “demi-caractéristique” and (Poisson) stability in the sense of Poincaré (We call these extended positive orbits and extended recurrence). Precisely, we show the following relation for \(\mathbb{R}\)-actions on compact surfaces:

\[
\text{extended } R\text{-closed } \subset\subset\text{ extended p.a.p. } \subset\subset\text{ extended recurrent } \subset\subset\text{ non-wandering.}
\]

Moreover, we show that the \(\mathbb{R}\)-action \(v\) on a compact surface \(S\) is extended recurrence with at most finitely many singularities if and only if \(v\) is regular non-wandering. If \(v\) has no locally dense orbits, then \(v\) is extended recurrence with \(|\text{Sing}(v) \cap O_{\text{ex}}(x)| < \infty\) for each point \(x \in S\) if and only if \(v\) is extended p.a.p.. If \(|\text{Sing}(v)| < \infty\), then \(v\) is extended \(R\)-closed if and only if \(v\) is extended p.a.p..

Recall “demi-caractéristique” in the sense of Poincaré. Let \(v\) be an \(\mathbb{R}\)-action on a surface \(S\). For a singular point \(x\) of \(S\), we call that \(x\) is a (topological) saddle for a \(C^1\) \(\mathbb{R}\)-action if there is a neighborhood of \(x\) which is locally homeomorphic to a neighborhood of a saddle for a \(C^1\) \(\mathbb{R}\)-action. For a point \(x\) of \(S\), define \(O_i^+(x)\) as follows:

\[
O_{i+1}^+(x) := O_i^+(x) \cup \bigcup_{x' \in O_i^+(x)} \{W^u(\omega(x')) \mid \omega(x') : \text{saddle}\}
\]

for any successor ordinal \(i\), and \(O_\nu^+ := \bigcup_{\nu > \nu} O_\nu^+\) for any limit ordinal \(\nu\). Here \(W^u(y) := \{z \in S \mid \alpha(z) = \{y\}\}\). Put \(O_{\text{ex}}^+(x) := \bigcup\{O_\nu^+(x) \mid \nu : \text{ordinal}\}\) is called the extended positive orbit of \(x\). Note Poincaré called this demi-caractéristique. Similarly, we defined the extended negative orbit \(O_{\text{ex}}^-(x)\) and so define the extended...
Recall the following fundamental fact. If there are extended limit cycles, there is a wandering point $x$ in $P$ such that $O(x) = O_{ex}(x)$.
Proof. Suppose that there is an extended limit cycle $C$. We may assume that there is a point whose omega limit set is $C$. Then there are uncountably many proper orbits each of whose omega limit set is $C$. Since the set of saddles are countable, there is a proper orbit $O$ whose extend orbit is coincident with itself such that $\omega(O) = C$. This implies that each point of $O$ is wandering.

**Lemma 2.4.** If $P$ consists of at most finitely many orbits, then $v$ is non-wandering.

**Proof.** It suffices to show that each point $x \in P$ is non-wandering. Indeed, By the flow box theorem, we have $x \in E \cup LD \cup \text{Per}(v)$. Since each point of $E \cup LD \cup \text{Per}(v)$ is either positive or negative recurrent, we have that $x$ is non-wandering and so $v$ is non-wandering.

**Lemma 2.5.** Suppose that $v$ is extended recurrent. For any point $x$ which is regular or is a saddle, there is a neighborhood $U$ such that $U - O_{ex}(x)$ contains no singularities.

**Proof.** If $O_{ex}(x)$ contains no saddles, then it contains no singularities and so the flow box theorem implies the assertion. Thus we may assume that $O_{ex}(x)$ contains saddles points. By the definition of extended orbits, we obtain that $O_{ex}(x) \cap \text{Sing}(v)$ consists of saddles points. Since each saddle $p$ has a neighborhood $U_p$ such that $U_p - \{p\}$ consists of regular points. By the flow box theorem, there is a neighborhood $U$ of $O_{ex}(x)$ such that $U - O_{ex}(x)$ contains no singularities.

The extended recurrence implies the (usual) non-wandering property.

**Lemma 2.6.** If $v$ is extended recurrent, then $v$ is non-wandering.

**Proof.** Note $S - P = E \cup LD \cup \text{Per}(v) \cup \text{Sing}(v)$. If $\text{int}P = \emptyset$, then the closedness of $\text{Sing}(v)$ implies that $E \cup LD \cup \text{Per}(v) \supset P$ and so $v$ is non-wandering. Thus it suffices to show $\text{int}P = \emptyset$. Indeed, recall that the set of saddles are countable. For any $x \in P$, the extended recurrence implies that the omega (resp. alpha) limit set of $x$ is a saddle. Therefore $P$ consists of countable orbits. Since $S$ is a Baire space, we have that $\text{int}P = \emptyset$.

Recall that a continuous $\mathbb{R}$-action $v$ is regular if each singularity of $v$ is locally homeomorphic to a non-degenerated singularity of a $C^1$ vector field. Note the non-wandering flow $v$ has no no exceptional orbits such that $LD \cup \text{Per}(v) \supset S - \text{Sing}(v)$, by Lemma 2.1 [Y2].

**Proposition 2.7.** Suppose that $v$ is non-wandering. Then $v$ is regular if and only if $v$ is extended recurrent and has finitely many singularities. Moreover, if $v$ is regular, then either $O_{ex}(x)$ is closed or both $O^+_{ex}(x)$ and $O^-_{ex}(x)$ are locally dense for any $x \in S$.

**Proof.** Suppose that $v$ is regular. The regularity implies that each singularity is either a center, a saddle, a sink, or a source. By the non-wandering property, we have that there are no limit cycles and that each singularity is either a center or a saddle. By the regularity, the set of saddles is finite. Since the omega (resp. alpha) limit set of each non-closed proper orbit is a saddle, we have that the set of non-closed proper orbits are finite. It suffices to show that each point $x \in S$ whose extended orbit is not closed but proper is extended recurrent. Indeed, we may assume that there are no $y \in O_{ex}^+(x) - O^+(x)$ such that $x \notin O_{ex}^+(y)$. Since each
saddle has two local (un)stable manifolds, both $O^+_{\text{ex}}(x)$ and $O^-_{\text{ex}}(x)$ are not closed. Since the union of non-closed proper orbits is finite and since each non-closed proper orbit is a saddle connection, we have that $O^+_{\text{ex}}(x)$ (resp. $O^-_{\text{ex}}(x)$) contains a locally dense orbit. Let $U$ be a neighborhood of $O_{\text{ex}}(x)$ such that $U - O_{\text{ex}}(x)$ contains no singularities. By the finiteness of saddles, there is a arbitrary thin connected open subset $U_x \subseteq U$ which is disjoint from the union of heteroclinic connections in $O^+_{\text{ex}}(x)$ and whose closure contains a curve $C^+$ in $O^+_{\text{ex}}(x)$ from $x$ to a point in a locally dense orbit $O^+$ such that the orientations of $C^+$ and $O^+_{\text{ex}}(x)$ are same. By locally density, we have $C^+ \subset U_x \cap O^+$ and so that $x \in O^+ \subseteq O^+_{\text{ex}}(x) - O^+(x)$. By the symmetry, this implies that $x$ is extended recurrent and so $v$ is extended recurrent.

Conversely, suppose that $v$ is extended recurrent and has finitely many singularities. By the finiteness of singularities, we have $\overline{\text{Per}(v)} \cap \text{LD} = S$. Since each connected component $C$ of the boundary of $\overline{\text{Per}(v)}$ consists of proper orbits and finitely many singularities, by the extended recurrence, we have that $C$ is either a center or a closed extended orbit and so each singularity contained in $C$ is a center or a saddle. On the other hand, the boundary of LD consists of proper orbits and finitely many singularities. The extended recurrence implies that each singularity in the boundary is a saddle. Thus $v$ is regular. \qed

Now we describe an $\mathbb{R}$-action which has non-closed extended orbits and which is not recurrent but extended recurrent. Consider an irrational rotation on $\mathbb{T}^2$ and a rational rotation on $\mathbb{T}^2$. Removing a point from each torus, paste the metric completions of them such that the intersection is a circle which consists of two saddles and two heteroclinic connections. Then we obtain an extended recurrent $\mathbb{R}$-action on a closed oriented surface with genus 2 which is not recurrent and has non-closed extended orbits. Notice that this example shows also that the extended orbits are different from chain recurrent components. We obtain the following dichotomy for extended recurrent $\mathbb{R}$-actions.

**Lemma 2.8.** Suppose that $v$ is extended recurrent. For any point $x$ whose extended orbit is not closed, either there is a singular point in $O_{\text{ex}}(x)$ which is not a saddle or there is a locally dense orbit $O$ such that $O_{\text{ex}}(x) \cap \overline{O} \neq \emptyset$.

**Proof.** Since extended recurrence implies non-wandering property, there are no exceptional orbits and $\text{LD} \sqcup \text{Per}(v) \supseteq S - \text{Sing}(v)$. Suppose that $O_{\text{ex}}(x) \cap \text{LD} = \emptyset$. Then $O_{\text{ex}}(x)$ consists of proper orbits and saddles. The extended recurrence implies that the omega (resp. alpha) limit set of each proper orbit in $O_{\text{ex}}(x)$ is a saddle. The non-closedness of $O_{\text{ex}}(x)$ implies that $O_{\text{ex}}(x)$ contains infinitely many saddles. Since saddles are isolated, a convergence point of saddles is a singular point which is not a saddle. This singularity is desired. Suppose that $O_{\text{ex}}(x) \cap \text{LD} \neq \emptyset$. By the Maier Theorem [M], the set of closures of locally dense orbits is finite and so there is a locally dense orbit $O$ such that $O_{\text{ex}}(x) \cap \overline{O} \neq \emptyset$. \qed

Note that there is an extended recurrent $\mathbb{R}$-action with a non-closed proper extended orbit with infinitely many saddles on a disk. Indeed, let $S := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2\}$. Consider circles $S_n := \{(x, y) \in \mathbb{R}^2 \mid (x - 3/2^n)^2 + y^2 = 2^{-n}\}$ for each $n \geq 2 \in \mathbb{Z}$. Let $O := \bigcup_{n \geq 2} S_n$. Define an $\mathbb{R}$-action $v$ with an extended orbit $O$ such that the origin is a fixed point which is not a saddle, the outside of $\overline{O}$ consists of periodic orbits, and each open disk bounded by $S_n$ is a center disk. Then $v$ is
3. Pointwise almost periodicity

We define extended versions of pointwise almost periodicity. An \( \mathbb{R} \)-action \( v \) on a topological space \( X \) is said to be extended pointwise almost periodic (extended p.a.p.) if the set \( \{ O_{\text{ex}}(x) \mid x \in X \} \) of closures of extended orbits is a decomposition of \( X \).

Lemma 3.1. If \( v \) is an extended p.a.p. \( \mathbb{R} \)-action on a compact surface, then \( v \) is extended recurrent and \( |\text{Sing}(v) \cap O_{\text{ex}}(x)| < \infty \) for each \( x \in S \).

Proof. Fix each regular point \( y \in S \) such that \( O_{\text{ex}}(y) \) contains singularities. By definition of extended orbits, the singularities in \( O_{\text{ex}}(y) \) are saddles. If \( O_{\text{ex}}(y) \) contains a singular point \( p \) which is not a saddle, then \( O_{\text{ex}}(p) = \{ p \} \not\subseteq O_{\text{ex}}(y) \) which contradicts to the extended p.a.p.. Thus \( O_{\text{ex}}(y) \cap \text{Sing}(v) \) consists of finitely many saddles. Since the set of saddles is countable, the extended p.a.p. property implies that \( P \) consists of countably many orbits. The flow box theorem implies that \( P \subseteq \mathbb{R} \cap \text{LD} \cup \text{Per}(v) \) and so that \( v \) is non-wandering. Fix any point \( x \in P \) whose extended orbit is not closed. We show that either omega or alpha limit set \( L \) of \( x \) is a saddle. Otherwise there is a point \( z \in O_{\text{ex}}(x) \) whose omega (resp. alpha) limit set is not a saddle. Then \( O(z) = O_{\text{ex}}(z) \) and \( O(z) \cap \omega(z) = \emptyset \). On the other hand, \( O_{\text{ex}}(x) \subseteq \omega(z) \) and so \( O_{\text{ex}}(x) \subseteq \omega(z) \). Since \( z \notin \omega(z) \), we have \( z \notin O_{\text{ex}}(x) \), which contradicts to the choice of \( z \). Since \( |\text{Sing}(v) \cap O_{\text{ex}}(x)| < \infty \), each of \( O_{\text{ex}}^+(x) \) and \( O_{\text{ex}}^-(x) \) contains locally dense orbits. By symmetry, it suffices to show that \( x \in O_{\text{ex}}^+(x) - O^+(x) \). Indeed, we may assume that there is no point \( y \in O_{\text{ex}}^+(x) - O^+(x) \) with \( x \in O_{\text{ex}}^+(y) \). Since \( \text{Sing}(v) \cap O_{\text{ex}}(x) \) consists of finitely many saddles, there is a thin connected open subset \( U_x \) without singularities whose closure contains a curve in \( O_{\text{ex}}^+(x) \) from \( x \) to a point \( w \in \text{LD} \) such that the orientations of the curve and \( O_{\text{ex}}^+(x) \) are compatible. Then \( x \in O_{\text{ex}}^+(w) \subseteq O_{\text{ex}}(x) - O^+(x) \).

Proposition 3.2. Suppose that \( v \) is a non-identical \( \mathbb{R} \)-action without locally dense orbits on a compact surface \( S \). The following are equivalent:

1) \( v \) is extended p.a.p..
2) \( v \) is extended recurrent and \( |\text{Sing}(v) \cap O_{\text{ex}}(x)| < \infty \) for each \( x \in S \).
3) \( v \) consists of closed extended orbits.

Proof. Obviously 3) \( \Rightarrow \) 1). By Lemma 3.1 we have that 1) \( \Rightarrow \) 2). Suppose that 2) holds. Moreover suppose that there is a non-closed extended orbit \( O_{\text{ex}}(x) \). By Lemma 2.8 there is a singularity \( z \) in \( O_{\text{ex}}(x) \) which is not a saddle. The extended recurrence implies that \( \{ z \} \neq \alpha(y) \) and \( \{ z \} \neq \omega(y) \) for any \( y \neq z \in S \). This contradicts to the Ura-Kimura-Bhatia theorem (cf. Theorem 1.6 [3]). Thus \( v \) consists of closed extended orbits.

Note that there is an \( \mathbb{R} \)-action on a connected closed surface which is not extended p.a.p. but extended recurrent and whose singularities consists of two saddles. Indeed, consider two irrational rotations on \( \mathbb{T}^2 \). Let \( T_1, T_2 \) be the metric completions of the resulting surfaces by removing one point from each torus. Then \( T_i \) is...
homeomorphic to a torus minus an open disk. Paste them such that the resulting surface $S = T_1 \cup T_2$ is a closed orientable surface with genus 2 and that the intersection $T_1 \cap T_2$ is a circle which consists of two saddles and two heteroclinic connections. Let $v$ be the resulting $\mathbb{R}$-action on $S$. The extended orbit closure of each point of $(\text{int} T_i) \setminus \mathcal{O}_{\text{ex}}(x)$ for a point $x \in T_1 \cap T_2$ is $T_i$, and the extended orbit closure of each point $x \in \mathcal{O}_{\text{ex}}(x_1) \cup \mathcal{O}_{\text{ex}}(x_2)$ is $S$, where any $x_i \in T_i$. Then $v$ is not extended p.a.p.. The extended recurrence is obviously.

4. Extended $R$-closedness

Define extended versions of $R$-closedness. An $\mathbb{R}$-action $v$ on a compact surface $S$ is said to be extended $R$-closed if $R_{\text{ex}} := \{(x, y) \mid y \in \overline{\mathcal{O}_{\text{ex}}(x)}\}$ is closed.

**Lemma 4.1.** If $v$ is extended $R$-closed, then $v$ is extended p.a.p..

**Proof.** First we show that $R_{\text{ex}}$ is symmetric. Indeed, the definition of extended orbits implies that $\{(x, y) \mid y \in \mathcal{O}_{\text{ex}}(x)\}$ is symmetric. For any $y \in \overline{\mathcal{O}_{\text{ex}}(x)}$, let $(y_n)$ be a sequence of points in $\mathcal{O}_{\text{ex}}(x)$ converging to $y$. Since $x \in \mathcal{O}_{\text{ex}}(y_n)$, we have $(y_n, x) \in R_{\text{ex}}$. The extended $R$-closedness implies $(y, x) \in R_{\text{ex}}$ and so $x \in \overline{\mathcal{O}_{\text{ex}}(y)}$.

The closure of each extended orbit contains at most finitely many singularities and either $\omega(x)$ or $\alpha(x)$ is a saddle for any $x \in P$. Hence $P$ consists of at most countably many orbits. By the flow box theorem, we obtain that $P \subseteq \overline{\text{LD}} \cup \overline{\text{Per}(v)} \cup \overline{E}$. This implies that $v$ is non-wandering. Fix any point $x \in S$. By symmetry, it suffices to show that $\overline{\mathcal{O}_{\text{ex}}(y)} \subseteq \overline{\mathcal{O}_{\text{ex}}(x)}$ for any $y \in \mathcal{O}_{\text{ex}}^+(x)$. We may assume that $\mathcal{O}_{\text{ex}}(x)$ is not closed. Then there is a point $z \in \mathcal{O}_{\text{ex}}^+(y)$ whose orbit is locally dense. Since the set of recurrent points is dense, there is a recurrent point $w \in \text{int} \overline{O^+(z)}$ whose orbit is locally dense. For any $z' \in \overline{\mathcal{O}_{\text{ex}}^-(z)}$, we have $w \in \overline{\mathcal{O}_{\text{ex}}(z')}$ and so $z' \in \overline{\mathcal{O}_{\text{ex}}(w)} = \overline{O^+(w)} \subseteq \overline{O^+(z)} \subseteq \overline{\mathcal{O}_{\text{ex}}(x)}$. Then $\mathcal{O}_{\text{ex}}(y) \subseteq \mathcal{O}_{\text{ex}}(z) \subseteq \mathcal{O}_{\text{ex}}(x)$ and so $\overline{\mathcal{O}_{\text{ex}}(y)} \subseteq \overline{\mathcal{O}_{\text{ex}}(x)}$. \qed

For a singular point $x$, we call that $x$ is an extended center if there is a neighborhood $U$ of $x$ such that $U \setminus \{x\}$ consists of extended periodic orbits and centers.

**Lemma 4.2.** Suppose that $v$ is non-identical extended $R$-closed and $S$ is connected. Then $\overline{\text{LD}} \cap \overline{\text{Sing}(v)}$ is finite and all singularities are saddles and extended centers.

**Proof.** Since $v$ is non-wandering, there are no exceptional orbits and $\overline{\text{Per}(v)} \cup \overline{\text{LD}} \supseteq S \setminus \overline{\text{Sing}(v)}$. By the extended $R$-closedness, we have that each connected component of the boundary of $\overline{\text{Per}(v)}$ (resp. $\overline{\text{LD}}$) is contained in one extended orbit and so that $\overline{\text{Per}(v)} \cap \overline{\text{Sing}(v)}$ consists of saddles and extended centers. The extended recurrence also implies that $\overline{\text{LD}} \cap \overline{\text{Sing}(v)}$ consists of saddles. Since each saddle is isolated, we have that $\overline{\text{LD}} \cap \overline{\text{Sing}(v)}$ is finite. By Lemma 2.34, $\overline{\text{Per}(v)} \cup \overline{\text{LD}}$ is clopen and so $S = \overline{\text{Per}(v)} \cup \overline{\text{LD}}$. Thus each singularity is either a saddle or an extended center. \qed

There is an extended $R$-closed flow with infinitely many saddles. Indeed, consider a center disk and a converging sequence of periodic orbits to the center. Replacing the periodic orbits by homoclinic saddle connections with center disks, we obtain an extended center disk with infinitely many saddles. By doubling this disk, we obtain an extended $R$-closed flow on $S^2$ with two extended centers and with infinitely many saddles. Consider the case with finitely many singularities.
Proposition 4.3. Suppose $|\text{Sing}(v)| < \infty$. Then $v$ is extended $R$-closed if and only if $v$ is extended p.a.p..

Proof. It suffices to show the “if” part. Suppose that $v$ is extended p.a.p.. By Proposition 2.7, we have that each singularity is regular and so is a center or a saddle. By the Mayer Theorem [M], the set of closures of locally dense orbits is finite. Then $S - \overline{LD} \subseteq \text{int} \Per(v)$ consists of periodic orbits, finitely many centers, and finitely many closed extended orbits. By the extended p.a.p. property, we obtain that $\overline{LD}$ consists of finitely many minimal sets with respect to extended orbits. For any connected component $C$ of $\overline{LD}$, there is a neighborhood $U$ of $C$ with $U - C \subseteq \Per(v)$. Consider the quotient map $\pi : S \to S/\O_{\text{ex}}$ of closures of extended orbits. Then $\overline{LD}$ is the inverse image of a finite subset of $S/\O_{\text{ex}}$ and $\pi(S - \overline{LD})$ is a forest (i.e. a disjoint union of trees). Then $S/\O_{\text{ex}}$ is Hausdorff. By Lemma 2.3 [Y], we have that $v$ is extended $R$-closed. □

The finiteness and the non-existence of locally dense orbits imply the following corollary.

Corollary 4.4. Suppose that $v$ is a non-identical $R$-action with finitely many singularities on a compact surface with genus 0. The following are equivalent: 1) $v$ is extended $R$-closed. 2) $v$ is extended p.a.p.. 3) $v$ is extended recurrent. 4) $v$ is regular non-wandering. 5) $v$ consists of closed extended orbits.

5. An extended non-wandering

Naturally, we can define extended non-wandering property as others. It’s easy to see that extended non-wandering property and (usual) non-wandering property are equivalent if the set of singularities are finite. The author don’t know whether these notions are same or not in general.

6. A note for a more generalization of orbits

Let $F$ be a compact invariant set of $v$. Then $F$ is said to be isolated (from minimal sets) if there exists a neighborhood $U$ of $F$ such that any minimal set contained in $U$ is a subset of $F$. $F$ is called a saddle set if there exists a neighborhood $U$ of $F$ such that $G_U \cap F \neq \phi$, where $G_U := \{x \in U - F \mid O^+(x) \not\subseteq U, O^-(x) \not\subseteq U\}$. In the definition of extended orbits of $x$, if we replace saddles with isolated saddle sets, then we call that the resulting extended orbits are generalized extended orbits, denoted by $O^-_{ge}(x), O^+_{ge}(x), O_{ge}(x)$. Also we define some “generalized” notation by replacing saddles with isolated saddle sets. By the definitions, we notice that extended recurrence implies generalized recurrence. Then one can show the generalized version of Lemma 2.5 in the similar fashion if one replaces saddles (resp. ex) with isolated saddle sets (resp. ge). However, this generalization does not imply the generalized version of Lemma 2.6, 2.8. Moreover non-wandering property and generalized recurrence are independent. In fact, the following example is a vector field on $S^2$ which is not non-wandering but generalized recurrent. Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}, D_+ = \{(x, y) \in D \mid x > 0\}, D_- = \{(x, y) \in D \mid x < 0\}, p_+ := (0, 1), p_- := (0, -1)$, and let $v$ be a flow on $D$ such that
Fix(\(v\)) = \{(0, y) \in D\} and that \(\alpha(p) = p_+\) and \(\omega(p) = p_-\) for each point \(p \in D_\pm\). Pasting an open center disk, we obtain a flow \(v'\) on \(S^2\) whose fixed point set consists of Fix(\(v\)) and the center. Then \(p_-, p_+\) are isolated saddle sets and so \(D - Fix(v) \subset O^-_{ge}(y) = O^+_{ge}(y)\) for any \(y \in D - Fix(v)\). This implies that \(v'\) is generalized recurrent. On the other hand, the following example is a vector field on \(S^2\) which is not generalized recurrent but non-wandering. Define a vector field whose orbits consists of \{\((1/n, 0)\), \(\{1/n\} \times (T^1 - \{0\})\), and \(\{x\} \times T^1\) for \(n \in \mathbb{Z}_{>0}\) and for \(x \in T^1 - \{1/m \mid m \in \mathbb{Z}_{>0}\}\). Because \(\alpha(p) = \omega(p) = \{(0, 0)\}\) is a saddle set but not isolated for any \(p \in \{0\} \times (T^1 - \{0\})\) and so \(O_{ex}(p) = \{0\} \times (T^1 - \{0\})\) is not generalised recurrent.

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