Existence and Concentration Results for the General Kirchhoff-Type Equations

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Abstract
We consider the following singularly perturbed Kirchhoff-type equations

\[-\varepsilon^2 M\left(\varepsilon^{2-N} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right) \Delta u + V(x)u = |u|^{p-2}u \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \quad N \geq 1,\]

where \(M \in C([0, \infty))\) and \(V \in C(\mathbb{R}^N)\) are given functions. Under very mild assumptions on \(M\), we prove the existence of single-peak or multi-peak solution \(u_\varepsilon\) for above problem, concentrating around topologically stable critical points of \(V\), by a direct corresponding argument. This gives an affirmative answer to an open problem raised by Figueiredo et al. (Arch Ration Mech Anal 213(3):931–979, 2014)

Keywords Kirchhoff-type equations · Semiclassical solutions · Topologically stable critical points

Mathematics Subject Classification 35B25 · 35A01

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1 Introduction

In the present paper, we study the existence and the concentration behavior of positive solutions to the general Kirchhoff-type equations

\[-\varepsilon^2 M \left( \varepsilon^{2-N} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + V(x) u = f(u) \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N, \quad (1.1)\]

where \( M \in C([0, \infty)) \), \( f \in C(\mathbb{R}) \), and \( V \in C(\mathbb{R}^N) \) are given functions. If \( M(t) \equiv 1 \), it becomes the well-known nonlinear Schrödinger equation (replace \( \varepsilon \) by \( \delta \)):

\[-\delta^2 \Delta w + V(x) w = f(w), \quad w > 0, w \in H^1(\mathbb{R}^N). \quad (1.2)\]

In the past decades, a lot of work have been devoted to the study of semiclassical solutions for (1.2).

Recalling the pioneering work [33], Floer and Weinstein first studied the existence of single-peak solutions for \( N = 1 \), \( V \in C^2(\mathbb{R}^N) \) and \( f(s) = s^3 \). They construct a single-peak solution concentrating around any given nondegenerate critical point of \( V \). And the higher dimension case with \( f(s) = |s|^{p-2}s \), \( p \in (2, 2^*) \) was studied by Oh [55]. In [33, 55], their arguments are based on a Lyapunov–Schmidt reduction, which requires a linearized nondegeneracy of a solution for a limiting problem. That is, if

\[-\Delta \phi + m\phi - f'(W)\phi = 0 \text{ in } \mathbb{R}^N, \quad \phi \in H^1(\mathbb{R}^N), \]

then \( \phi = \sum_{i=1}^N a_i \frac{\partial W}{\partial x_i} \) for some \( a_i \in \mathbb{R} \). Here, \( m > 0 \), \( W \) is a ground state solution of the autonomous problem

\[-\Delta w + mw = f(w), \quad w \in H^1(\mathbb{R}^N). \quad (1.3)\]

Moreover, they also required that \( f(t)/t \) is nondecreasing on \((0, \infty)\) and the uniqueness of ground states of (1.3). After then, many authors have applied Lyapunov–Schmidt reduction approach to further refined results for more general \( f \) and more general types of critical points of \( V \) (see [2, 4, 28, 41, 42, 56] and references therein). It is known that the linearized nondegeneracy condition holds only for a restricted class of \( f \) for \( N \geq 2 \). We remark that one needs at least the monotonicity of \( \frac{mt - f'(t)t}{mt - f(t)} \) for \( t > T \), where \( T \) is the first positive zero of \( mt - f(t) = 0 \) (see [21]). Even though there is such a restriction on the nonlinearity when one applies the reduction method, the Lyapunov–Schmidt reduction method is a very powerful tool when one constructs very subtle (highly unstable) solutions with continuum peaks as we can see in [28].

Dancer developed a refined finite-dimensional reduction to construct peak solutions without the linearized nondegeneracy condition in [22]. However, he still requires some type of nondegeneracy for the limiting problem.
We also remark that the variational approach is also proved very effective to study the problem (1.2), which does not require the nondegeneracy condition for the limiting problem (1.3). This kind of approach was initiated by Rabinowitz [49] and has been developed further by several authors (see [7, 8, 10–13, 24, 25, 27, 35, 39, 52, 57–59] and the references therein). In [49], Rabinowitz proved the existence of positive solutions of (1.2) for small $\delta$ whenever $V \in C(\mathbb{R}^N, \mathbb{R})$ and \( \liminf_{|x| \to \infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x) =: V_0 > 0 \). Wang proved that these solutions (obtained by Rabinowitz [49]) concentrate around the global minimum points of $V$ as $\delta \to 0$ in [52]. In [24], del Pino and Felmer established a localized version of the result in [49, 52]. Precisely, suppose the following

\( (V_1) \) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^N} V(x) = V_0 > 0$;

\( (V_2) \) there is a bounded domain $O$ such that

$$m := \inf_{x \in O} V(x) < \inf_{x \in \partial O} V(x),$$

they obtained a single-peak solution concentrating around the minimum points of $V$ in $O$, provided that $f$ satisfies some conditions, such as the Ambrosetti–Rabinowitz condition and that the function $t \mapsto f(t)/t$ is nondecreasing. After then, a lot of works are devoted to weak the assumptions on $f$ and construct peak solutions concentrating at more general critical points (such as local maximum points and special saddle points).

Byeon et al. [8, 12] developed a new variational method to explore what the essential features that guarantee the existence of localized ground states are. They studied the nonlinearities of Sobolev subcritical case under the well know Berestycki–Lions conditions, which were first proposed in the pioneer work [6] to guarantee the existence of ground states of (1.3) in the subcritical case. So Byeon and Jeanjean [8] believed that Berestycki–Lions conditions are almost optimal for the subcritical case. Byeon and Tanaka [10] improved the result of [8, 12] by proving the existence of positive solutions of (1.2) also under the Berestycki–Lions conditions, which concentrate at more general critical points of $V$, such as saddle points or local maximum points. We also remark that d’Avenia et al. [23] developed a min-max argument to establish the concentration phenomenon around the saddle points of the potential $V$. In [59], Zhang et al. generalized the result of [8] to the nonlinearities involving critical growth. In [58], Zhang and Zou also established the concentration phenomenon around the saddle points of the potential $V$ for nonlinearities involving critical growth, which generalized the result of subcritical case given by d’Avenia et al. [23].

When $M(t) \neq \text{const}$, He and Zou [37] study (1.1) when $N = 3$ and $M(t) = a + bt$, and show the existence of positive solutions of (1.1) concentrating to global minima of $V$, under some suitable assumptions on $f$ which is of subcritical case. Later, Wang et al. [53] generalized the result to the nonlinearities involving Sobolev critical case. The authors in [37, 53] used the Nehari manifold method and thus the positive solution obtained indeed has the least energy among all nontrivial solutions of (1.1). In [36], Yi He studied Problem (1.1) with the nonlinearity involving Sobolev critical case when $N = 3$ and $M(t) = a + bt$. Zhang et al. [60] investigated problem (1.1) with general
nonlinearities involving critical exponent for $N = 2, 3$. Some other related results we refer to [30, 31, 38, 44, 45, 51] and references therein.

Inspired by the work of [8, 9, 12], Figueiredo et al. [32] studied problem (1.1) for general $M$ by purely variational approach. Under suitable conditions on $M$ and Berestycki–Lions conditions on $f$, they constructed a family of positive solutions $u_{\varepsilon}$ (for sufficiently small $\varepsilon$, and may not be least energy solution of (1.1)) concentrating at a local minimum of $V$ up to a subsequence. Precisely, for $M \in C([0, \infty))$, suppose $(M_1)$ when $N = 1, 2$ and $(M_1) - (M_5)$ when $N \geq 3$ below:

$$(M_1) \text{ There exists } m_0 > 0 \text{ such that } M(t) \geq m_0 > 0 \text{ for any } t \geq 0.$$

$$(M_2) \text{ Set } \tilde{M}(t) := \int_0^t M(s)ds. \text{ Then there holds } \lim_{t \to +\infty} \left\{ \tilde{M}(t) - (1 - 2/N)M(t)t \right\} = +\infty.$$

$$(M_3) \text{ } M(t)/t^{2/(N-2)} \to 0 \text{ as } t \to +\infty.$$

$$(M_4) \text{ The function } M \text{ is nondecreasing in } [0, \infty).$$

$$(M_5) \text{ The function } t \mapsto M(t)/t^{2/(N-2)} \text{ is nonincreasing in } (0, \infty).$$

And assume that $f$ satisfies the following (f1)-(f4).

$$(f_1) \ f \in C(\mathbb{R}), \ f(s) = 0 \text{ for } s \leq 0.$$

$$(f_2) \ -\infty < \liminf_{s \to 0^+} \frac{f(s)}{s} \leq \limsup_{s \to 0^+} \frac{f(s)}{s} < V_0 := \inf_{x \in \mathbb{R}^N} V(x).$$

$$(f_3) \text{ When } N \geq 3, f(s)/s^{2^*-1} \to 0 \text{ as } s \to +\infty \text{ and when } N = 2, f(s)/e^{\alpha s^2} \to 0 \text{ as } s \to +\infty, \text{ for any } \alpha > 0, \text{ where } 2^* := 2N/(N-2).$$

$$(f_4) \text{ There exists } T > 0 \text{ such that if } N \geq 2, (m/2)T^2 < F(T) \text{ and if } N = 1, \frac{1}{2}mT^2 > F(t) \text{ for } t \in (0, T), \frac{1}{2}mT^2 = F(T) \text{ and } mT < f(T), \text{ where } F(t) := \int_0^t f(s)ds.$$  

Then Figueiredo et al. established the following result.

**Theorem A** (concentrating to local minimum points, see [32, Theorem 1.1])

Assume $(V_1) - (V_2)$ and $(f_1) - (f_4)$. In addition, suppose $(M_1)$ when $N = 1, 2$ and $(M_1) - (M_5)$ when $N \geq 3$. Then there exists $\bar{\varepsilon} > 0$ and a family $(u_{\varepsilon})_{0 < \varepsilon \leq \bar{\varepsilon}}$ of positive solutions of (1.1) satisfying the following:

(i) $\text{dist}(x_{\varepsilon}, M) \to 0$ as $\varepsilon \to 0$, where $(x_{\varepsilon})$ be a maximum point of $u_{\varepsilon}$ and $M \equiv \{x \in O : V(x) = m\}$.

(ii) After taking a subsequence $(\varepsilon_n)$, $u_{\varepsilon_n}(\varepsilon_n x + x_{\varepsilon_n}) \to U$ strongly in $H^1(\mathbb{R}^N)$, where $U$ is a positive least energy solution of

$$-M \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u + mu = f(u) \text{ in } \mathbb{R}^N, u \in H^1(\mathbb{R}^N). \quad (1.4)$$

(iii) There exists $C_1, C_2 > 0$ such that

$$u_{\varepsilon}(x) \leq C_1 \exp \left( -C_2 \frac{|x - x_{\varepsilon}|}{\varepsilon} \right) \text{ for all } x \in \mathbb{R}^N \text{ and } 0 < \varepsilon < \bar{\varepsilon}. \quad (1.5)$$

$\square$

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Remark 1.1

(i) If we study problem (1.1) by pure variational method, the difficulty is the presence of nonlocal term $M \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)$, which makes (1.1) more delicate than (1.2).

(ii) Also if we study problem (1.1) by Lyapunov–Schmidt reduction directly, one needs the nondegenerate condition that the kernel of the linearized operator $\mathcal{L} : H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \mapsto L^2(\mathbb{R}^N)$ defined by

\[
\mathcal{L}\varphi = -M \left( \int_{\mathbb{R}^N} |\nabla U|^2 \, dx \right) \Delta \varphi - 2M' \left( \int_{\mathbb{R}^N} |\nabla U|^2 \, dx \right) \nabla U \cdot \nabla \varphi \Delta U + m\varphi - f'(U)\varphi
\]

is spanned by the functions $\frac{\partial U}{\partial x_i}$, $i = 1, \cdots, N$, provided $M, f \in C^1$. We remark that if $N = 3, M(t) = a + bt$ with $a > 0, b > 0$ and $f(u) = |u|^{p-1}u, 1 < p < 5$, Li et al. [46] proved that there exists a unique positive radial solution $U \in H^1(\mathbb{R}^3)$ satisfying

\[
- \left( a + b \int_{\mathbb{R}^3} |\nabla U|^2 \, dx \right) \Delta U + U = U^p, U > 0 \text{ in } \mathbb{R}^3.
\]

Moreover, $U$ is nondegenerate in $H^1(\mathbb{R}^3)$ (see [46, Theorem 1.2]). Then the authors in [46] could apply the Lyapunov–Schmidt reduction to construct a family of solutions concentrating at a local minimum (see [46, Theorem 1.3]). Furthermore, they obtain the local uniqueness of single-peak solutions provided some further assumptions on $V(x)$ (see [46, Theorem 1.4]). However, for the general $M(t)$, the nondegenerate condition is very hard to check even for the nonlinearities are of polynomials. Hence, it is also very hard to study problem (1.1) by a direct Lyapunov–Schmidt reduction.

So Figueiredo et al. raised the following open problem (see [32, Remark 1.2-(iii)]):

“It seems interesting to consider whether one can find a family of solutions of (1.1) which has multi-peaks or which concentrates around other type of critical points of $V$ (local maxima, saddle points and so on). These types of results to (1.2) have been obtained.”

There is little progress on this open problem. As far as we know, in [19], Chen and Ding gave a first affirmative answer to this open problem for $N \geq 3$, by constructing positive solutions concentrating around the local maximum points of $V$, basing on the same assumptions on $M$ and $f$ as in [32] and in addition that $M(t) + (1 - N/2)M'(t)t \neq 0$.

In the present paper, we shall give another affirmative answer to this open problem, by constructing positive single-peak solutions concentrating around local minimum, local maxima, or saddle points. Our assumptions on $M$ are mild. Furthermore, we also
obtain some result about the concentrating positive multi-peak solutions, which seem never be obtained in the related literatures.

The paper is organized as follows. In the next section, we give the corresponding relationship between (1.1) and (1.2) for single-peak solutions and multi-peak solutions, respectively. Taking \( f(s) = |s|^{p-2}s \) with \( 2 < p < 2^* \) as applications, we obtain the existence and multiplicity of single-peak solutions (see Theorem 2.7) and the existence of multi-peak solutions (see Theorem 2.9) under different assumptions on \( V(x) \) and mild assumption on \( M(t) \). In Sect. 3, we shall prove these corresponding theorems. In Sect. 4, we will give some sufficient conditions that guarantee the corresponding theorems to be applied. Theorems 2.7 and 2.9 will be proved in Sect. 5.

### 2 Statement of Main Results

We define

\[
H_\varepsilon = \left\{ u(x) \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} \left( \varepsilon^2 |\nabla u(x)|^2 + V(x)u^2(x) \right) dx < \infty \right\},
\]

and for any \( u(x) \in H_\varepsilon \), denote its norm by

\[
\|u\|_\varepsilon := (u(x), u(x))_{\varepsilon}^{\frac{1}{2}} = \left( \int_{\mathbb{R}^N} \left( \varepsilon^2 |\nabla u(x)|^2 + V(x)u^2(x) \right) dx \right)^{\frac{1}{2}}. \tag{2.1}
\]

For \( u \in L^p(\mathbb{R}^N), 1 \leq p < \infty \), we denote the \( L^p \)-norm by \( \| \cdot \|_p \) for simplicity, i.e.,

\[
\|u\|_p := \left( \int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}.
\]

#### 2.1 Some Correspondent Results

As the reasons stated in Remark 1.1, we will not deal with (1.1) directly. We firstly establish the following corresponding results between (1.1) and (1.2) for single-peak solutions and multi-peak solutions, respectively.

**Theorem 2.1** (correspondence for single-peak solution) Suppose that, under some conditions on \( V \) and \( f \), that there exist \( \delta > 0 \) and a family \( (\omega_\delta)_{0 < \delta < \bar{\delta}} \) of positive solutions of (1.2) satisfying the following:

(a-i) \( \text{dist}(x_\delta, \mathcal{M}) \to 0 \) as \( \delta \to 0 \), where \( \mathcal{M} \) is an isolated set of topologically stable critical points of \( V \) and \( (x_\delta) \) is a maximum point of \( \omega_\delta \);

(a-ii) after taking a subsequence \( (\delta_n), \omega_{\delta_n}(\delta_n x + x_{\delta_n}) \to W \) strongly in \( H^1(\mathbb{R}^N) \), where \( W \) is a positive least energy solution of

\[
- \Delta w + mw = f(w) \text{ in } \mathbb{R}^N, w \in H^1(\mathbb{R}^N), \tag{2.2}
\]

\[
m := V(x_0) \text{ with some } x_0 \in \mathcal{M} \text{ and } x_{\delta_n} \to x_0 \text{ as } n \to +\infty;
\]

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(a-iii) there exists \( C_1, C_2 > 0 \) such that
\[
\omega_\delta(x) \leq C_1 \exp \left( -C_2 \frac{|x - x_\delta|}{\delta} \right) \text{ for all } x \in \mathbb{R}^N \text{ and } 0 < \delta < \bar{\delta}. \tag{2.3}
\]

In addition, suppose, under some conditions on \( M \), that

(b-i) there exists \( \bar{\varepsilon} > 0 \) such that for any \( \varepsilon \in (0, \bar{\varepsilon}) \), there exists \( \delta_\varepsilon \in (0, \bar{\delta}) \) and a positive solution \( \omega_{\delta_\varepsilon} \) of (1.2) such that
\[
\varepsilon^2 M \left( \varepsilon^{2-N} \| \nabla \omega_{\delta_\varepsilon} \|_{L^2}^2 \right) = \delta_\varepsilon^2; \tag{2.4}
\]

(b-ii) it holds that
\[
0 < \inf_{\varepsilon \in (0, \bar{\varepsilon})} \frac{\delta_\varepsilon}{\varepsilon} \leq \sup_{\varepsilon \in (0, \bar{\varepsilon})} \frac{\delta_\varepsilon}{\varepsilon} < +\infty. \tag{2.5}
\]

For \( \varepsilon \in (0, \bar{\varepsilon}) \), define \( u_\varepsilon(x) := \omega_{\delta_\varepsilon}(x) \) and denote a maximum point \( x_{\delta_\varepsilon} \) of \( \omega_{\delta_\varepsilon} \) by \( x_\varepsilon \) for simplicity. Then \( (u_\varepsilon)_{0 < \varepsilon < \bar{\varepsilon}} \) is a family of positive solutions to (1.1) and \( x_\varepsilon \) is also a maximum point of \( u_\varepsilon \). Furthermore,

(c-i) \( \text{dist}(x_\varepsilon, \mathcal{M}) \to 0 \) as \( \varepsilon \to 0 \);
(c-ii) after taking a subsequence \( (\varepsilon_n) \), \( u_{\varepsilon_n}(\varepsilon_n x + x_{\varepsilon_n}) \to U \) strongly in \( H^1(\mathbb{R}^N) \), where \( U \) is a positive least energy solution of
\[
- M(\| \nabla w \|_{L^2}^2) \Delta w + mw = f(w) \text{ in } \mathbb{R}^N, \ w \in H^1(\mathbb{R}^N), \tag{2.6}
\]

\[
m := V(x_0) \text{ with some } x_0 \in \mathcal{M} \text{ and } x_{\varepsilon_n} \to x_0 \text{ as } n \to +\infty; \tag{2.9}
\]

(c-iii) there exist \( C_3, C_4 > 0 \) such that
\[
u_\varepsilon(x) \leq C_3 \exp \left( -C_4 \frac{|x - x_\varepsilon|}{\varepsilon} \right) \text{ for all } x \in \mathbb{R}^N \text{ and } 0 < \varepsilon < \bar{\varepsilon}. \tag{2.7}
\]

**Theorem 2.2** (correspondence for multi-peak solution) Suppose that, under some conditions on \( V \) and \( f \), there exist \( \bar{\delta} > 0 \) and a family \( (\omega_\delta)_{0 < \delta < \bar{\delta}} \) of multi-peak positive solutions of (1.2) such that \( \omega_\delta \) is the form of
\[
\omega_\delta(x) = \sum_{j=1}^{k} W_{p_j} \left( \frac{x - y^{(j)}_\delta}{\delta} \right) + \psi_\delta(x) \tag{2.8}
\]

with \( y^{(j)}_\delta, \psi_\delta(x) \) satisfying
\[
y^{(j)}_\delta \to p_j \text{ and } \| \psi_\delta \| = o(\delta^{N/2}) \tag{2.9}
\]
as $\delta \to 0$ for $j = 1, \cdots, k$, where $W_{P_j}$ is a radial positive ground state solution of

$$
- \Delta w + V(P_j)w = f(w) \text{ in } \mathbb{R}^N, \; w \in H^1(\mathbb{R}^N),
$$

(2.10)

and $P_j, \; j = 1, 2, \cdots, k$, are critical points of $V(x)$. In addition, suppose, under some conditions on $M$, that $M$ satisfies the assumptions (b-i) and (b-ii) in Theorem 2.1.

For $\varepsilon \in (0, \bar{\varepsilon})$, define $u_\varepsilon(x) := \omega_{\delta\varepsilon}(x), \phi_\varepsilon(x) := \psi_{\delta\varepsilon}(x)$ and $x_\varepsilon^{(j)} = y_{\delta\varepsilon}^{(j)}$. Then $(u_\varepsilon)_{0<\varepsilon<\bar{\varepsilon}}$ is a family of multi-peak positive solutions to (1.1). In particular, $u_\varepsilon$ is the form of

$$
u_\varepsilon(x) = \sum_{j=1}^k U_{P_j} \left( \frac{x - x_\varepsilon^{(j)}}{\varepsilon} \right) + \phi_\varepsilon(x)
$$

(2.11)

with $x_\varepsilon^{(j)}, \phi_\varepsilon(x)$ satisfying

$$
x_\varepsilon^{(j)} \to P_j \text{ and } \|\phi_\varepsilon\|_\varepsilon = o(\varepsilon^\frac{N}{2})
$$

(2.12)

as $\varepsilon \to 0$ for $j = 1, \cdots, k$. Here, $(U_{P_1}, \cdots, U_{P_k})$ is a positive solution to the following system

$$
- M \left( \sum_{j=1}^k \|\nabla u_j\|_2^2 \right) \Delta u_j + V(P_j)u_j = f(u_j), \; j = 1, \cdots, k.
$$

(2.13)

Remark 2.3

(i) Indeed, up to a subsequence, we may assume that $\delta\varepsilon \to C_\ast$ as $\varepsilon \to 0$. Then $U_{P_j}(x) := W_{P_j} \left( \frac{x}{C_\ast} \right)$. In particular, $C_\ast$ solves $G(t) = 0$ with $G(t)$ defined by

$$
G(t) := M \left( t^{N-2} \left( \sum_{j=1}^k \|\nabla W_{P_j}\|_2^2 \right) \right) - t^2, \; t > 0.
$$

(2.14)

(ii) Under some conditions on $M$, if $G(t) = 0$ has a unique positive root, one can see that $(U_{P_1}, \cdots, U_{P_k})$ is indeed the unique positive radial solution to the system (2.13). For example, if $M(t) = a + bt, \; a, b > 0, \; N = 3$, one can easily verify that

$$
G(t) = a + b \left( \sum_{j=1}^k \|\nabla W_{P_j}\|_2^2 \right) t - t^2.
$$
It is trivial that $G(t) = 0$ has a unique positive root. In such a case, our result coincides [47, Proposition 3], where the authors deal with (1.1) by Lyapunov–Schmidt reduction method directly, thanks to the nondegeneracy of positive solutions to the limit Kirchhoff problem proved in [46].

(iii) If $G(t) = 0$ has multiple positive roots, it will be interesting to consider the multiplicity of multi-peak solutions concentrating to the same given $k$ critical points of $V(x)$.

### 2.2 Single-Peak Solutions: Existence and Multiplicity

As applications, we firstly concerned with the existence and multiplicity of single-peak solutions of (1.1). We adopt some definitions by Grossi [34].

**Definition 2.4** We say that a function $h : \mathbb{R}^N \mapsto \mathbb{R}$ is homogenous of degree $\alpha \in \mathbb{R}^+$ with respect to $P \in \mathbb{R}^N$ if

$$h(tx + P) = t^\alpha h(x + P)$$

for any $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^N$. (2.15)

**Definition 2.5** (Definition of admissible potential) Let us assume that $V \in C^1(\mathbb{R}^N)$ satisfies

$$|\nabla V(x)| \leq C e^{\gamma|x|}$$

at infinity (2.16) and

$$0 < V_0 \leq V(x) \leq V_1$$

for some $\gamma > 0$. We say that $V$ is an admissible potential at $P \in \mathbb{R}^N$ if there exist continuous functions $h_i : \mathbb{R}^N \mapsto \mathbb{R}$, $R_i : B_{P,r} = \{x \in \mathbb{R}^N : |x - P| < r\} \mapsto \mathbb{R}$ and real numbers $\alpha_i \geq 1$, $i = 1, \cdots, N$, such that

(i) $\frac{\partial V}{\partial x_i}(x) = h_i(x) + R_i(x)$ in $B_{P,r}$;

(ii) $R_i(x) \leq C|x - P|^{\beta_i}$ in $B_{P,r}$ with $\beta_i > \alpha_i$ for any $i = 1, \cdots, N$;

(iii) $h_i(x) = 0$ if and only if $x = P$;

(iv) $h_i$ is homogeneous of degree $\alpha_i$ respect to $P$.

**Definition 2.6** Let $G \in C(\mathbb{R}^N, \mathbb{R}^N)$ be a vector field. We say that $y$ is a stable zero for $G$ if

(i) $G(y) = 0$;

(ii) $y$ isolated;

(iii) if $G_n$ is a sequence of vector fields such that $\|G_n - G\|_{C(B_{y,\rho})} \to 0$ as $n \to \infty$ for some $\rho > 0$, then there exists $y_n$ such that $G_n(y_n) = 0$ and $y_n \to y$ as $n \to \infty$. 

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Set the following vector field

\[
\mathcal{L}_P(y) = \left( \int_{\mathbb{R}^N} h_i(x+y+P)W^2_P(x) \right)_{i=1,\ldots,N},
\]

where \( W_P \in H^1(\mathbb{R}^N) \) is the unique positive radial solution to

\[
- \Delta w + V(P)w = w^{p-1} \text{ in } \mathbb{R}^N.
\]

Define

\[
Z = \{ y \in \mathbb{R}^N \text{ such that } y \text{ is a stable zero of } \mathcal{L}_P \},
\]

then we get the following theorem which is concerned with the existence and multiplicity of single-peak solution concentrating at general critical point \( P \).

**Theorem 2.7** Let \( V \) be an admissible potential at \( P \). Suppose that \( \#Z < \infty \) and

\[
det \text{Jac } \mathcal{L}_P(y) \neq 0
\]

for any \( y \in Z \). In addition, suppose \((M_1)\) when \( N = 1, 2 \) and suppose \((M_1)\) and \((M_3)\) when \( N \geq 3 \). Then there exists \( \bar{\varepsilon} > 0 \) such that for any \( 0 < \varepsilon < \bar{\varepsilon} \), the equation

\[
\begin{cases}
-\varepsilon^2 M \left( \varepsilon^{2-N} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + V(x)u = |u|^{p-2}u \text{ in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N), u > 0 \text{ in } \mathbb{R}^N, 2 < p < 2^* := \frac{2N}{(N-2)_+}
\end{cases}
\]

possesses at least \( \#Z \) different single-peak solutions concentrating at \( x = P \). Precisely, we obtain families of positive solutions \((u^{(k)}_{\varepsilon})_{0<\varepsilon<\bar{\varepsilon}, k=1,\ldots,\#Z}\) with the maximum point at \((P + \varepsilon y^{(k)}_{\varepsilon})_{0<\varepsilon<\bar{\varepsilon}, k=1,\ldots,\#Z}\) such that, for any \( k \in \{ 1, \ldots, \#Z \} \), \( y^{(k)}_{\varepsilon} \) is bounded with \( y^{(k)}_{\varepsilon} \rightarrow y_k \) as \( \varepsilon \rightarrow 0 \). Here, \( \{ y_k, k = 1, \ldots, \#Z \} \) are \( \#Z \) distinct stable zeros of \( \mathcal{L}_P(y) \). In particular, \( u^{(k)}_{\varepsilon}(\varepsilon x + P + \varepsilon y^{(k)}_{\varepsilon}) \rightarrow U \) strongly in \( H^1(\mathbb{R}^N) \), where \( U \) is a positive radial ground state solution of

\[
- M \left( \| \nabla u \|_2^2 \right) \Delta u + V(P)u = |u|^{p-2}u \text{ in } \mathbb{R}^N.
\]

When \( P \) is a nondegenerate critical point of \( V \), a direct conclusion of Theorem 2.7 can be stated as follows.

**Corollary 2.8** (single-peak solution concentrating at nondegenerate critical point) Let \( P \) be a nondegenerate critical point of \( V \) and \( 0 < V_0 \leq V(x) \leq V_1 \). Suppose \((M_1)\) when \( N = 1, 2 \) and suppose \((M_1)\) and \((M_3)\) when \( N \geq 3 \). Then problem (2.22) possesses a family of positive solutions \((u_{\varepsilon})_{0<\varepsilon<\bar{\varepsilon}}\) with the maximum point at \((P + \varepsilon y_{\varepsilon})_{0<\varepsilon<\bar{\varepsilon}}\) such that \( y_{\varepsilon} \rightarrow 0 \) and \( u_{\varepsilon}(\varepsilon x + P + \varepsilon y_{\varepsilon}) \rightarrow U \) strongly in \( H^1(\mathbb{R}^N) \) as \( \varepsilon \rightarrow 0^+ \), where \( U \) is a positive radial ground state solution of (2.23).
2.3 Multi-peak Solutions

The existence of multi-peak solutions for nonlinear Schrödinger equations (1.2) (or its variants) concentrating at the critical points of $V(x)$ also has been studied deeply, we refer to [16, 24, 26, 43, 48]. For the case of a critical nonlinearity, the results on the existence of multi-peak solutions can be seen in [5, 50]. For the case of existence with the concentration phenomena, we refer to [1, 3, 15, 20, 25, 29, 35, 41] and the references therein. For the uniqueness of multi-bump solution, we refer to [14, 18]. There are also some results focused on the multi-peak solutions for (1.1) when $M(t) = a + bt$, $a > 0$, $b > 0$, see [47, 51, 54]. For example, assume that there are bounded disjoint open sets $O_i \subset \mathbb{R}^3$ such that $m_i := \inf_{x \in O_i} V(x) < \inf_{x \in \partial O_i} V(x)$, $i = 1, 2, \cdots, k$. Suppose further (V1) and that $f$ satisfies the Berestycki–Lions condition, by pure variational method, Hu and Shuai in [51] managed to construct multi-peak solutions for (1.1) concentrating on the above $k$ different local minimal points of $V(x)$. In present paper, we shall establish the existence of multi-peak solutions concentrating at much more general topologically stable critical points of $V(x)$.

To state our result on the multi-peak solutions, we consider a class of $V(x)$ as follows:

\begin{itemize}
  \item[(V1)'] $V(x) \in C^1(\mathbb{R}^N)$ and $\inf_{x \in \mathbb{R}^N} =: V_0 > 0$.
  \item[(V2)'] $V(x)$ satisfies the following expansions:
\end{itemize}

\begin{equation}
\begin{aligned}
  V(x) &= V(P_j) + \sum_{i=1}^N b_{j,i}|x_i - P_{j,i}|^\alpha + O(|x - P_j|^{|\alpha|+1}), & x \in B_\eta(P_j), \\
  \frac{\partial V(x)}{\partial x_i} &= \alpha b_{j,i} |x_i - P_{j,i}|^{\alpha-2}(x_i - P_{j,i}) + O(|x - P_j|^\alpha), & x \in B_\eta(P_j),
\end{aligned}
\end{equation}

where $\eta > 0$ is a small constant, $\alpha > 1$, $x = (x_1, \cdots, x_N)$, $P_j = (P_{j,1}, \cdots, P_{j,N})$, $b_{j,i} \in \mathbb{R}$ with $b_{j,i} \neq 0$ for each $i = 1, \cdots, N$, $j = 1, \cdots, k$.

Here comes to our main result about multi-peak solutions.

**Theorem 2.9** Assume that $V(x)$ satisfies (V1)’ and (V2)’. Suppose (M1) when $N = 1, 2$ and suppose (M1) and (M3) when $N \geq 3$. Then there exists $\bar{\varepsilon} > 0$ and a family of positive solutions $(u_\varepsilon)_{0 < \varepsilon < \bar{\varepsilon}}$ of (2.22) concentrating at a set of $k$ different points $\{P_1, \cdots, P_k\} \subset \mathbb{R}^N$. Precisely, $u_\varepsilon$ is the form of

\begin{equation}
  u_\varepsilon(x) = \sum_{j=1}^k W_{P_j} \left( \frac{x - x^{(j)}_\varepsilon}{\varepsilon} \frac{1}{C_\varepsilon} \right) + \phi_\varepsilon(x) \tag{2.25}
\end{equation}

with $x^{(j)}_\varepsilon$, $\phi_\varepsilon(x)$ satisfying

\begin{equation}
  |x^{(k)}_\varepsilon - P_j| = o(\varepsilon) \text{ and } \|\phi_\varepsilon\|_\varepsilon = O(\varepsilon \frac{\varepsilon}{N + \alpha}) \tag{2.26}
\end{equation}

as $\varepsilon \to 0$ for $j = 1, \cdots k$. 

Here, $W_{p_j}$ is a radial positive ground state solution of (2.10) with $f(w) = w^{p-1}$ and $C_*$ is the smallest positive number determined by

$$M \left( C_*^{N-2} \left( \sum_{j=1}^{k} \| \nabla W_{p_j} \|_2^2 \right) \right) = C_*^2. \quad (2.27)$$

3 Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1 By the assumption (b-i) and the definition of $u_\varepsilon$, a direct computation shows that

$$-\varepsilon^2 M \left( \varepsilon^{2-N} \| \nabla u_\varepsilon \|_2^2 \right) \Delta u_\varepsilon + V(x) u_\varepsilon - f(u_\varepsilon) = -\varepsilon^2 M \left( \varepsilon^{2-N} \| \nabla \omega_{\delta_\varepsilon} \|_2^2 \right) \Delta \omega_{\delta_\varepsilon} + V(x) \omega_{\delta_\varepsilon} - f(\omega_{\delta_\varepsilon}) = -\delta_\varepsilon^2 \Delta \omega_{\delta_\varepsilon} + V(x) \omega_{\delta_\varepsilon} - f(\omega_{\delta_\varepsilon}) = 0.$$

Hence, $u_\varepsilon$ is a solution to (1.1) and it is clear that $x_\varepsilon$ is a maximum point of $u_\varepsilon$. Thus the conclusion of (c-i) holds by (a-i).

By the assumption (b-ii), we can find some $K_1 > 0$ such that

$$\frac{\delta_\varepsilon}{\varepsilon} \leq K_1, \forall \varepsilon \in (0, \bar{\varepsilon}). \quad (3.1)$$

Noting that $\delta_\varepsilon < \bar{\delta}$, $\varepsilon \in (0, \bar{\varepsilon})$, by the assumption (a-iii) and the definition of $u_\varepsilon$, we have that

$$u_\varepsilon(x) = \omega_{\delta_\varepsilon}(x) \leq C_1 \exp \left( -C_2 \frac{|x - x_{\delta_\varepsilon}|}{\delta_\varepsilon} \right)$$

$$\leq C_1 \exp \left( -C_2 \frac{|x - x_{\delta_\varepsilon}|}{\varepsilon} \right) \frac{\varepsilon}{\delta_\varepsilon}$$

$$\leq C_1 \exp \left( -C_2 \frac{|x - x_{\delta_\varepsilon}|}{\varepsilon} \right) \frac{\varepsilon}{K_1}$$

for all $x \in \mathbb{R}^N$ and $0 < \varepsilon < \bar{\varepsilon}$.

Hence, the conclusion (c-iii) holds.

After taking a subsequence $(\varepsilon_n)$, by the assumption (b-ii), we may assume that

$$\frac{\delta_{\varepsilon_n}}{\varepsilon_n} \rightarrow C_* \in (0, K_1]. \quad (3.2)$$

Noting that $\delta_{\varepsilon_n} \rightarrow 0$, by the assumption of (a-ii), we have that $\varphi_n(x) : = \omega_{\delta_{\varepsilon_n}}(\delta_{\varepsilon_n} x + x_{\delta_{\varepsilon_n}}) \rightarrow W$ strongly in $H^1(\mathbb{R}^N)$, where $W$ is a positive least energy solution of (2.2).
Hence,

$$\|\nabla u_{\varepsilon_n}\|_2^2 = \|\nabla \omega_{\delta \varepsilon_n}\|_2^2 = \delta_{\varepsilon_n}^{N-2} \|\nabla \varphi_n\|_2^2 = \delta_{\varepsilon_n}^{N-2}(\|\nabla W\|_2^2 + o(1)).$$  \hfill (3.3)

We note that (2.4) in the assumption (b-i) implies that

$$M\left(\varepsilon_n^{2-N} \|\nabla u_{\varepsilon_n}\|_2^2\right) = \left(\frac{\delta_{\varepsilon_n}}{\varepsilon_n}\right)^2.$$  \hfill (3.4)

Let $n \to +\infty$, it follows from $M \in C([0, +\infty))$ that

$$M\left(C_*^{N-2} \|\nabla W\|_2^2\right) = C_*^2.$$  \hfill (3.5)

Put $U(x) := W\left(\frac{1}{C_*} x\right)$, a direct computation shows that $U$ is a positive least energy solution to

$$-C_*^2 \Delta u + mu = f(u) \text{ in } \mathbb{R}^N, u \in H^1(\mathbb{R}^N).$$  \hfill (3.6)

Noting that $\|\nabla U\|_2^2 = C_*^{N-2} \|\nabla W\|_2^2$, so by (3.5) and (3.6), we see that $U$ is a positive least energy solution to (2.6). In particular,

$$u_{\varepsilon_n}(\varepsilon_n x + x_{\varepsilon_n}) = \omega_{\delta \varepsilon_n}(\varepsilon_n x + x_{\varepsilon_n})$$

$$= \omega_{\delta \varepsilon_n}\left(\frac{\varepsilon_n}{\delta_{\varepsilon_n}} x + x_{\delta \varepsilon_n}\right)$$

$$= \omega_{\delta \varepsilon_n}\left(\frac{1}{C_*} + o(1)\right) x + x_{\delta \varepsilon_n})$$

$$\to W\left(\frac{1}{C_*} x\right) = U(x) \text{ in } H^1(\mathbb{R}^N).$$

Hence, the conclusion of (c-ii) holds. We complete the proof of Theorem 2.1. \qed

**Proof of Theorem 2.2** Similar to the proof of Theorem 2.1. We only note that in this case,

$$\|\nabla u_{\varepsilon_n}\|_2^2 = \|\nabla \omega_{\delta \varepsilon_n}\|_2^2 = \delta_{\varepsilon_n}^{N-2}\left(\sum_{j=1}^{k} \|\nabla W_{P_j}\|_2^2 + o(1)\right)$$

and thus $C_*$ satisfies

$$M\left(C_*^{N-2}\left(\sum_{j=1}^{k} \|\nabla W_{P_j}\|_2^2\right)\right) = C_*^2,$$

i.e., $G(C_*) = 0$. We also note that $o(\delta_e^N) = o(e^{-N})$ due to (b-ii). \qed
4 Some Sufficient Conditions to Guarantee (b-i) and (b-ii)

Lemma 4.1 Under suitable assumptions on $V$ and $f$, we assume that $\omega_\delta$ depends continuously on $\delta$ and there exists some $A > 0$ such that $\|\nabla \omega_\delta\|_2^2 \to A$ as $\delta \to 0$. Suppose $(M_1)$ when $N = 1, 2$ and suppose $(M_1)$ and $(M_3)$ when $N \geq 3$. Then the conditions (b-i) and (b-ii) in Theorem 2.1 hold.

Proof For $\delta > 0$ small, and $\varepsilon > 0$ small, we define

$$g_\varepsilon(\delta) := \varepsilon^2 M \left( \varepsilon^{2-N} \|\nabla \omega_\delta\|_2^2 \right) - \delta^2. \quad (4.1)$$

Since $\omega_\delta$ depends continuously on $\delta$ and from the assumption $(M_1)$, we see that $g_\varepsilon(\delta)$ is continuous with respect to $\delta \in (0, \bar{\delta})$. Let us consider the equation $g_\varepsilon(\delta) = 0$. By $(M_1)$, there exists some $K_0 > 0$ large enough such that

$$\begin{cases}
\frac{1}{K^2} M \left( \frac{A}{K} \right) < 1, & \forall K \geq K_0, \text{ if } N = 1, \\
\frac{1}{K^2} M (A) < 1, & \forall K \geq K_0, \text{ if } N = 2.
\end{cases} \quad (4.2)$$

Suppose further $(M_3)$ if $N \geq 3$, for $K$ large enough, $(M_3)$ implies that

$$\frac{1}{K^2} M \left( K^{N-2} A \right) = \frac{1}{K^2} o_K \left( (K^{N-2} A) \frac{2}{N-2} \right) = o_K (1).$$

So there also exists $K_0 > 0$ large enough such that

$$\frac{1}{K^2} M \left( K^{N-2} A \right) < 1, \ \forall K \geq K_0. \quad (4.3)$$

Now, let $K = K_0$ be fixed. We claim that there exists $\delta_1 < \bar{\delta}$ small enough such that

$$g_{\frac{\delta_1}{K}} (\delta) < 0, \ \forall \delta \in (0, \delta_1). \quad (4.4)$$

If not, there exists $\delta_n \to 0$ such that

$$g_{\frac{\delta_n}{K}} (\delta_n) \geq 0. \quad (4.5)$$

Under the assumption, we have that

$$\|\nabla \omega_{\delta_n}\|_2^2 = \delta_n^{N-2} (A + o(1)). \quad (4.6)$$

Hence, by (4.3), for $n$ large enough,

$$\left( \frac{\delta_n}{K} \right)^2 M \left( \left( \frac{\delta_n}{K} \right)^{N-2} \delta_n^{N-2} (A + o(1)) \right) - \delta_n^2$$
\[
= \left( \frac{1}{K^2} M \left( K^{N-2} (A + o(1)) \right) - 1 \right) \delta_n^2 < 0, \quad (4.7)
\]
a contraction. Hence, (4.4) holds. In another word, there exists \( \varepsilon_1 := \frac{\delta_1}{K} > 0 \) small enough such that
\[
g_\varepsilon(K\varepsilon) < 0, \quad \forall \varepsilon \in (0, \varepsilon_1). \quad (4.8)
\]
On the other hand, under the assumption \((M_1)\), we have that
\[
g_\varepsilon(\delta) \geq m_0 \varepsilon^2 - \delta^2 > 0 \text{ for all } \delta \in (0, \sqrt{m_0 \varepsilon}). \quad (4.9)
\]
Define
\[
\delta_\varepsilon := \sup \{ s : g_\varepsilon(\delta) > 0, \quad \forall 0 < \delta < s \}, \quad \varepsilon \in (0, \varepsilon_1). \quad (4.10)
\]
By (4.9) and (4.8), one can see that \( \delta_\varepsilon \) is well defined and thus \( g_\varepsilon(\delta_\varepsilon) = 0 \). In particular,
\[
\sqrt{m_0 \varepsilon} \leq \delta_\varepsilon \leq K \varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_1). \quad (4.11)
\]
Hence, \((b-i)\) and \((b-ii)\) hold. \(\square\)

**Remark 4.2**

(i) We remark that the assumption \((M_3)\) plays an crucial role to guarantee the existence of \( \delta_\varepsilon \). This condition has requirements for dimension, and usually the high-dimensional case is not applicable. Indeed, it is a sufficient but not necessary condition. For example, we take \( M(t) = a + bt \) with \( a, b > 0 \). For the case of \( N \geq 4 \), \((M_3)\) fails for any \( a, b > 0 \). However, if \( a \) and \( b \) are small suitable such that
\[
\inf \left\{ M(A t_i N^{-2}) - t_i^2 : t_i > 0 \right\} = \inf \left\{ a + b A t_i N^{-2} - t_i^2 : t_i > 0 \right\} < 0, \quad (4.12)
\]
then one can prove the existence of \( \delta_\varepsilon \), arguing by contradiction like the case \( N \geq 3 \) in the proof of Lemma 4.1. Then we can also establish the similar result for \( M(t) = a + bt \) with \( N \geq 4 \), which is very difficult to obtain by a direct variational method.

(ii) Under some suitable assumptions on the general nonlinearity \( f \), if \( \inf_{x \in \mathbb{R}^N} V(x) =: V_0 > 0 \) large enough, then the uniqueness and nondegeneracy of positive solution to \((2.10)\) hold (see [40, Theorem 1.3] and [21]). In particular, \( \| \nabla W_{P_j} \|_2 \to +\infty \) as \( V(P_j) \to +\infty \). Hence, \( A \to \infty \) as \( V(P_j) \to +\infty \) and thus \((4.12)\) is not expected provided \( \inf_{x \in \mathbb{R}^N} V(x) =: V_0 > 0 \) large enough. Indeed, if these kinds of concentrating results are valid independent of \( V_0 \), the condition \((M_3)\) is almost necessary for \( N \geq 3 \), see the following Proposition 4.3.
Proposition 4.3 Let $N \geq 3$. $M \in C([0, \infty))$ satisfies $(M_1)$ and
\[
\liminf_{t \to +\infty} \frac{M(t)}{t^{2/(N-2)}} > 0. 
\] (4.13)

Assume that $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the following conditions:
(i) $\lim_{s \to 0} \frac{f(s)}{s} = 0$;
(ii) there exists some $\ell \in (2, 2^*)$ such that $\lim_{|s| \to \infty} \frac{|f(s)|}{|s|^{\ell-1}} < \infty$.

Then there exists some $V_0 > 0$ such that if $\inf_{x \in \mathbb{R}^N} V(x) \geq V_0$, the equation (1.1) has no nontrivial solution in $H^1(\mathbb{R}^N)$ for all $\varepsilon > 0$.

Proof Noting that $u(x)$ solves (1.1) if and only if $u(\varepsilon x)$ solves
\[
- \frac{M(\|\nabla \phi \|^2_2)}{\Delta} \phi + V(\varepsilon x) \phi = f(\phi) \text{ in } \mathbb{R}^N, \phi \in H^1(\mathbb{R}^N),
\] (4.14)
we only need to prove the conclusion holds for $\varepsilon = 1$. By $(M_1)$ and (4.13), it is easy to see that
\[
\inf_{t \geq 0} \frac{M(t)}{t^{\frac{2}{N-2}}} =: \sigma > 0.
\] (4.15)

Thus,
\[
M(\|\nabla u \|^2_2) \|\nabla u \|^2_2 \geq \sigma \|\nabla u \|_{2^*}^2, \forall u \in H^1(\mathbb{R}^N).
\] (4.16)

For any $u$ solves
\[
- \frac{M(\|\nabla u \|^2_2)}{\Delta} u + V(x) u = f(u) \text{ in } \mathbb{R}^N, u \in H^1(\mathbb{R}^N),
\] (4.17)
we have that
\[
M(\|\nabla u \|^2_2) \|\nabla u \|^2_2 + \int_{\mathbb{R}^N} V(x) u^2 \, dx \geq \int_{\mathbb{R}^N} f(u) u \, dx.
\] (4.18)

We remark that under the assumptions on $f$, one can see that for any $\eta > 0$, there exists some $C_\eta > 0$ such that
\[
|f(s)s| \leq \eta |s|^2 + C_\eta |s|^\ell, \forall s \in \mathbb{R}.
\] (4.19)

Hence, by (4.16)–(4.19), we obtain that
\[
\sigma \|\nabla u \|_{2^*}^2 + (V_0 - \eta) \|u\|_2^2 \leq C_\eta \|u\|_\ell^\ell.
\] (4.20)

Recalling the Gagliardo-Nirenberg inequality, there exists some $C_\ell > 0$ such that
\[
\|u\|_\ell^\ell \leq C_\ell \|\nabla u\|_2^\frac{N(\ell-2)}{2} \|u\|_2^\frac{N(\ell-2)}{2 \ell}.
\] (4.21)
Thus, by (4.20) and (4.21), there exists some \( C^* = C_{n, \ell} > 0 \) such that
\[
\sigma \| \nabla u \|_2^{2*} + (V_0 - \eta) \| u \|_2^2 \leq C^* \| \nabla u \|_2^{N(\ell - 2)} \| u \|_2^{\ell - N(\ell - 2)}. \tag{4.22}
\]

By Young inequality, take \( p = \frac{4}{(N-2)(\ell-2)} \) and \( p' = \frac{4}{2(N-2)(\ell-2)} \), we have that
\[
\| \nabla u \|_2^{\frac{N(\ell - 2)}{2}} \| u \|_2^{\frac{\ell - N(\ell - 2)}{2}} \leq \kappa \| \nabla u \|_2^{2*} + (p - 1)p \| \frac{p}{p-1} - \frac{\ell}{\ell-1} \| u \|_2^2, \forall \kappa > 0. \tag{4.23}
\]

By taking \( \kappa = \sigma \), it follows (4.22) and (4.23) that
\[
(V_0 - \eta) \| u \|_2^2 \leq C^*(p - 1)p \| \frac{p}{p-1} - \frac{\ell}{\ell-1} \| u \|_2^2, \text{ where } p = \frac{4}{(N-2)(\ell-2)}. \tag{4.24}
\]

Hence, if \( V_0 > \eta + C^*(p - 1)p \| \frac{p}{p-1} - \frac{\ell}{\ell-1} \| u \|_2^2 \) with \( p = \frac{4}{(N-2)(\ell-2)} \), then \( u \equiv 0 \).

\section{5 Proof of Theorems 2.7 and 2.9}

\textbf{Proof of Theorem 2.7} We firstly recall some known results. Since \( V(x) \) is an admissible potential, by [34, Theorem 4.3], there exists \( \delta > 0 \) such that for any stable zero \( y_k \) of \( \mathcal{L}_P(y) \), one can construct single-peak solutions \( (\omega^{(k)}_{\delta})_{0 < \delta < \delta} \) to equation (1.2) such that

(i) let \( x^{(k)}_{\delta} \) be the maximum point of \( \omega^{(k)}_{\delta} \), then \( x^{(k)}_{\delta} \) can be written as
\[
x^{(k)}_{\delta} = P + \delta y^{(k)}_{\delta} \text{ with } y^{(k)}_{\delta} \to y_k \text{ as } \delta \to 0. \tag{5.1}
\]

(ii) \( \omega_{\delta}(\delta x + x^{(k)}_{\delta}) \to W_P \) in \( H^1(\mathbb{R}^N) \).

(iii) For \( j \neq k \), the single-peak solutions generated by \( y_j \) and \( y_k \) are different.

Furthermore, if \( det \text{ Jac } \mathcal{L}_P(y_k) \neq 0 \), then \( y^{(k)}_{\delta} \) is unique determined in a small neighborhood of \( y_k \) and \( y^{(k)}_{\delta} \to y^{(k)}_{\bar{\delta}} \) as \( \delta \to \bar{\delta} \), where \( \bar{\delta} \) is unique determined in a small neighborhood of \( y_k \) and \( y^{(k)}_{\delta} \to y^{(k)}_{\bar{\delta}} \) as \( \delta \to \bar{\delta} \).

So for any fixed \( k \) and \( \delta > 0 \) small, the uniqueness of \( y^{(k)}_{\delta} \) implies that \( (\omega^{(k)}_{\delta})_{0 < \delta < \delta} \) depends on \( \delta \) continuously. Indeed, we can take \( \eta > 0 \) small enough such that \( B_{\eta}(y_j) \) is an admissible potential, by [34, Theorem 1.1].

Thus, we can take \( \eta > 0 \) small enough such that \( y^{(k)}_{\delta} \to y^{(k)}_{\bar{\delta}} \) for all \( \delta \in (0, \bar{\delta}) \) and \( k = 1, \ldots, \#Z \). Let \( k \) be fixed, for any \( \delta^* \in (0, \bar{\delta}) \) and any sequence \( \delta_n \to \delta^* \). Up to a subsequence, we may assume that \( y^{(k)}_{\delta_n} \to y^{(k)}_{\delta^*} \) in \( B_{\eta}(y_k) \). It is trivial that
\[
W_P \left( \frac{x - (P + \delta_n y^{(k)}_{\delta_n})}{\delta_n} \right) \to W_P \left( \frac{x - (P + \delta^* y^{(k)}_{\delta^*})}{\delta^*} \right) \text{ strongly in } H^1(\mathbb{R}^N). \tag{5.2}
\]

Define

\[\exists \text{ Springer}\]
\[ E_{\delta, p, y} = \{ \begin{array}{ll}
 w(x) \in H^1(\mathbb{R}^N) : \\
 \left( \begin{array}{l}
 (w(x), W_p \left( \frac{x-y}{\delta} \right)) = 0, \\
 (w(x), \frac{w(x) - w(x)}{\delta i}) = 0, i = 1, \ldots, N
\end{array} \right) \end{array} \} \] (5.3)

The standard Lyapunov–Schmidt reduction process implies that \( \omega_\delta^{(k)} \) is of form

\[ \omega_\delta^{(k)}(x) = W_p \left( \frac{x - x_\delta^{(k)}}{\delta} \right) + \psi_\delta^{(k)}(x), \] (5.4)

with \( \psi_\delta^{(k)} \in E_{\delta, p, x_\delta^{(k)}}. \) By the uniformly decay of \( \{ \omega_\delta^{(k)} \} \), we may assume that

\[ \omega_\delta^{(k)} \to \omega^* \text{ strongly in } H^1(\mathbb{R}^N). \] (5.5)

So that \( \omega^* \) is a solution to

\[ -\delta^2 \Delta w + V(x)w = |w|^{p-2}w \text{ in } \mathbb{R}^N, 0 < w \in H^1(\mathbb{R}^N) \] (5.6)

Furthermore, it is of form

\[ \omega^* = W_p \left( \frac{x - (P + \delta^* y_\delta^{(k)})}{\delta^*} \right) + \psi^*, \] (5.7)

with \( \psi_\delta^{(k)} \to \psi^* \text{ strongly in } H^1(\mathbb{R}^N). \) Hence, one can see that \( \psi^* \in E_{\delta, p, P + \delta^* y_\delta^{(k)}}. \)

Hence, by the uniqueness of \( y_\delta^{(k)} \), we have that \( y_\delta^{(k)} = y_\delta^{(k)} \) and thus \( \omega^* = \omega_\delta^{(k)} \). Hence, \( \omega_\delta^{(k)} \) is continuous with respect to \( \delta \in (0, \bar{\delta}) \).

On the other hand, by \( \omega_\delta (\delta x + x_\delta^{(k)}) \to W_p \) in \( H^1(\mathbb{R}^N) \), it is trivial that

\[ \frac{\| \nabla \omega_\delta \|_2^2}{\delta^N} \to A := \| \nabla W_p \|_2^2 \text{ as } \delta \to 0^+. \]

So combining with Lemma 4.1, we see that Theorem 2.1 is applicable here and we finish the proof of Theorem 2.7. \( \square \)

**Proof of Corollary 2.8** It is a direct conclusion of Theorem 2.7. We only note that when \( P \) is a nondegenerate critical point of \( V(x) \), one has that \( Z = \{0\} \) and thus \# \( Z \) = 1. \( \square \)

**Proof of Theorem 2.9** Firstly, we remark that under the assumptions, for the problem

\[ -\delta^2 \Delta w + V(x)w = |w|^{p-2}w \text{ in } \mathbb{R}^N, w \in H^1(\mathbb{R}^N), \] (6.8)
the existence of multi-peak solutions concentrating at critical points of $V(x)$ is standard by Lyapunov–Schmidt reduction method, see for example [17]. Secondly, under the assumptions of Theorem 2.9, Cao et al. also proved the local uniqueness result, see [18, Theorem 1.1]. Precisely, for $\delta$ small enough, $\omega_\delta(x)$ is of form

$$\omega_\delta(x) = \sum_{j=1}^{k} W_{P_j} \left( \frac{x - y^{(j)}_\delta}{\delta} \right) + \psi_\delta(x) \quad (6.9)$$

with $y^{(j)}_\delta$, $\psi_\delta(x)$ satisfying, for $j = 1, \cdots, k$, as $\delta \to 0$,

$$|y^{(j)}_\delta - P_j| = o(\delta) \text{ and } \|\psi_\delta\|_\delta = O(\delta^{N/2 + \alpha}) = o(\delta^{N/2}). \quad (6.10)$$

By $\delta^2 \|\nabla \psi_\delta\|_\delta^2 \leq \|\psi_\delta\|_\delta^2$, we see that $\|\nabla \psi_\delta\|_\delta^2 = o(\delta^{N-2})$. Hence, it follows that

$$\frac{\|\nabla \omega_\delta\|_\delta^2}{\delta^{N-2}} \to A := \sum_{j=1}^{k} \|\nabla W_{P_j}\|_\delta^2 \text{ as } \delta \to 0^+.$$ 

We also remark that the local uniqueness can imply that $(\omega_\delta)_{0 < \delta < \bar{\delta}}$ depends continuously on $\delta$, see also the proof of Theorem 2.7. So the conclusions of Theorem 2.9 follow by Theorem 2.2. We only note that by (6.10) and that

$$0 < \liminf_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon} \leq \limsup_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon} < +\infty,$$

we obtain

$$\|\phi_\varepsilon\|_\varepsilon = \|\psi_{\delta_\varepsilon}\|_\varepsilon = O(\|\psi_{\delta_\varepsilon}\|_{\delta_\varepsilon}) = O(\delta_\varepsilon^{N/2 + \alpha}) = O(\varepsilon^{N/2 + \alpha}).$$

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