Linear in Temperature Resistivity and Associated Mysteries

Chandra M. Varma*

Physics Department, University of California, Berkeley, CA. 94704

(Dated: August 19, 2019)

Abstract

Recent experimental results: (i) the measurement of the singular specific heat in cuprates and the earlier such results in some heavy fermion compounds, (ii) the measurement of the single-particle scattering rates, (iii) the density fluctuation spectrum in cuprates and (iv) the long standing results on the linear temperature dependence of the resistivity, are used to show that a theory of the quantum-criticality in these compounds based on the solution of the dissipative 2D - XY model quantitatively gives the temperature and frequency dependence of such properties with one parameter - the dimensionless coupling constant of the fermions to the quantum-critical fluctuations. The asymptotic low frequency or temperature dependences persist to an upper cut-off scale which is given by another parameter in the theory - the upper cut-off of the derived fluctuations. This can be read from the measured singularity in the specific heat or the saturation of the single-particle relaxation rate. The two parameter had earlier been estimated from microscopic parameters to within a factor of about 2. The cross-over exponent from quantum-criticality to Fermi-liquid is estimated from specific heat and resistivity data to be about 1/2, also in agreement with the results on the model. The simplicity of the results depends on the discovery that orthogonal topological excitations in space and in time determine the fluctuations near criticality such that the space and time metrics are free of each other.

arXiv:1908.05686v1 [cond-mat.str-el] 15 Aug 2019
I. INTRODUCTION

Innumerable papers in the last two decades point out that the linear in $T$ resistivity extending to asymptotic low temperatures in cuprates, some of the Fe based compounds and some heavy fermion compounds is a great mystery and the most important unsolved problem in condensed matter physics. It was realized very early that fundamental new principles must be involved in understanding these and various other associated observed properties related to the breakdown of the quasi-particle concept, and tentative directions of future research laid out [1], [2]. Not surprisingly, given the enormous attention devoted to this problem by physicists from a variety of different backgrounds, other novel points of views have also developed. These include a branch of string theory, physics of black holes and applications of theory of quantum chaos.

Finally, a solution of these remarkable problems can be said to have been achieved, based on comparing detailed quantitative predictions of a theory with a variety of different experiments, some only recently available even on this prolifically worked at problem. The solution is quite subtle and relies on quantum-criticality governed by topological excitations. The answers are unusual, but typical of most subtle problems, extra-ordinarily simple. For a brief review, see Ref. [3]. The model solved is the 2D-dissipative quantum XY model which can be solved by renormalization group methods [4] as well as Kosterlitz’s solution of the classical XY model [5], and checked in detail by quantum Monte-carlo calculations with some additional results [6, 7]. The applicability of the model to the quantum-criticality of the cuprates is straight-forward and to anisotropic antiferromagnets has also been discussed [8] [9].

Some essential aspects of the fluctuation spectra, based on a close reading of a variety of experiments were suggested much earlier [2], before the microscopic basis was understood and an appropriate theoretical framework for deriving the results formulated. Now that the foundations of the unusual criticality have been found, some important aspects are changed. But one of the central results, that the fermions form a marginal Fermi-liquid, which followed from the assumed quantum-critical spectra [2] and which was the basis for several predictions remains unchanged.
Related to the physics of the normal state anomalies is also the aspect of the theory giving quantitatively a theory of the d-wave superconductivity in these compounds [10] with the same two parameters. This have also been compared in detail with experiments [11] but are not our purview here.

For the Fe-based compounds and the heavy-fermions, as wide a variety of relevant experimental results as in the cuprates are not yet available. But one can argue from what are available that the same principles are at work. This raises an unresolved issue which will be briefly described.

This brief paper is organized as follows: First, the four classes of experiments mentioned are summarized and the two important parameters giving their frequency or temperature dependence extracted from each of them. These parameters are shown to be directly related in different experiments. Next, the results of the theory are briefly summarized to show how the correct frequency and temperature dependence in each of the experiments is obtained and why two parameters describe all of them quantitatively.

II. EXPERIMENTAL RESULTS

A. Specific heat near quantum-criticality

First the evidence that there is indeed a quantum-critical point in the cuprates relevant to the linear in T resistivity and other related properties is shown. This is unexceptional for the heavy-fermions and the Fe-based compounds where the anomalous properties occur in the region of the easily evident anti-ferromagnetic quantum-critical point. In the hole-doped cuprates, the antiferromagnetic correlation length in the region of the linear in T resistivity and other anomalous properties is only of order a lattice constant. When a phase diagram with a quantum critical point in this region was first proposed for the cuprates [13], it was a matter of contentious debate. It is more or less commonly accepted now. However the clear thermodynamic evidence for quantum criticality was discovered only last year through the measurement of the the singularity in the measured specific heat [12], which was predicted in 1989 [2]. The symmetries broken along the line $T^*(p)$ ending at the quantum-critical point are best observed by polarized neutron scattering [14] although at least five different techniques have observed some or other of the broken symmetries predicted at the transition
Figure 1. The electronic Specific heat on $La_{2-p}A_pCuO_4$, taken from Fig. S10 of Ref. [12]. The left panel shows the data at 0.5 K, the lowest temperature measured (in a magnetic field to remove superconductivity). The right panel shows the temperature dependence of the specific heat nearest to the critical composition $p \approx p_c$. The extraction of the electronic component from the total specific heat is fully discussed in [12].

[15], [16], [17], [18], [19], [20], [21], [22], [23], [24]. Fig. (1) from Ref. [12] presents both the singularity in the specific heat very close to the critical point in a measurement down to 0.5 K together with the crossover in the singularity on either side of the quantum-critical point.

The relevant specific heat data from Ref. [12] is shown in Fig. (1). Very close to quantum-criticality, the electronic specific heat fits

$$\frac{C_{el}}{k_B T}(p_c) = \gamma \left(1 + g \ln \frac{T_x}{T}\right).$$

The logarithmic enhancement of the specific heat is of-course equivalent to the basic postulates of a marginal Fermi-liquid - that the quasi-particle residue go to zero at the critical point as

$$z_p(\omega, T) = \frac{1}{1 + g_p \ln \frac{\pi T x}{x}}, \quad x = max(\pi T, \omega).$$

Following the summary of the theory in Sec. III below, I have assumed that both the coupling constant $g$ and the cut-off $T_x$ may have weak dependence on direction of the momentum $p$. 
at the Fermi-surface. The experimental $\bar{g}$ and $T_x$ in the specific heat may be taken as the averages of the parameters in $z_p$.

What is plotted in Fig. (1) is not the total specific heat divided by $T$, but $C_{el}/T$ obtained by subtracting from the total specific heat at a given $p$, all but an observed constant (electronic or Fermi-liquid) contribution to the total specific heat $C_v/T$ at $p = 0.16$. Both are measured at a magnetic field of 8 Tesla to eliminate superconductivity. This serves to eliminate the nuclear Schottky contribution and the phonon contribution. Using $\gamma \approx 5 \text{mJoules/mole}K^2$ at $p = 0.24$, we may read $\bar{g}$ and the cut-off $T_x$, from the slope and the intercept by extending the dashed red-line to 0 in the right side of the figure to be:

$$\bar{g} \approx 0.4 \pm 0.1, \quad T_x \approx 1,200 \pm 300 K.$$  (3)

The error bars come from the large region over which an extrapolation is necessary to deduce $T_x$ and the (smaller) uncertainty in $\gamma$.

From Fig. (1), one can also deduce the crossover temperature $\xi_T^{-1}(p-p_c)$ to a Fermi-liquid

$$\frac{C_{el}(p)}{T} = \gamma (1 + \bar{g} \ln(T_x/\sqrt{T^2 + \xi_T^{-2}(p)}),$$  (4)

$$\frac{(\xi_T)^{-1}}{T_x} \propto \left(\frac{p-p_c}{p_c}\right)^{-\zeta}$$  (5)

Given the error bars, $\zeta$ cannot be determined too accurately. Assuming that the background specific heat coefficient $\gamma$ is independent of $T$ for range of $p$ between 0.24 and 0.35, $\zeta \approx 0.5$ is estimated. The calculations summarized in Sec. III do give this value.

**B. Single-particle Relaxation rate**

Inelastic single-particle relaxation rate began to be reliably measured in the year 2,000 [26] and showed a relaxation rate proportional to $T$ for $T$ much larger than $\omega$ and proportional to $\omega$ in the opposite limit and with evidence that it is nearly independent of momentum both perpendicular to the Fermi-surface and along the Fermi-surface. The most complete such measurements arrived in 2,005 [25]. I show the relaxation rate in different directions on the Fermi-surface for $\omega >> T$ in Fig. (2) in that paper. Later measurements have shown a relaxation rate about 20% smaller [27]. The deduction of the single-particle relaxation rate as a function of frequency, or what is the same thing, the imaginary part of the single particle self-energy $Im\Sigma(p,\omega)$ reproduced in Fig. (2) is from the energy dependence of
Figure 2. Single-particle scattering rate measured by ARPES taken from [25]. The top part shows the momentum distribution curves at various energies with a fit with which the parameters of the self-energy are extracted. The bottom panel on the left shows the points on the Fermi-surface and the directions in which the data was taken. The bottom panel on the right shows the extracted parameters for the self-energy.

the momentum distribution curves, which are also shown in the figure for various energies. Fits to the energy distribution curves for fixed momenta, also done in the same paper [25] gives results consistent with the parameters deduced. The momentum distribution function, fits very well a Lorentzian. The Lorentzian form is evidence that the relaxation rate is also independent of momentum perpendicular to the Fermi-surface [28]. To convert to the scattering rate as a function of energy, one must multiply by the band-structure velocity at the measured $\omega$. The low frequency departure from linearity as a function of $\omega$ is due to impurity scattering and finite temperature. The data also deduces a frequency (and
Figure 3. The full width of the momentum distribution curves, which all fit a Lorentzian form very well, as a function of energy up to high energy for the three compounds which are measurable by ARPES. The open black circles are data for optimally doped Bi2201 (nodal cut, from [29]) The red-crosses are for “optimally doped” Bi2212 (nodal cut, from [30]), the full green circles are for La$_{2-p}$Sr$_p$CuO$_4$ for $p=0.17$ at about 20 degrees from the nodal direction and the blue squares are for the same compound at $p=0.145$ from the nodal direction. The last two set of points are from [31]

temperature) independent relaxation rate which is quite angle dependent. It is not possible in this experiment to disentangle the small angle impurity scattering contribution [28] and the angle-dependent width due to bi-layer splitting in this quantity.

The parameter $b = 0.7 \pm 0.1$ shown in fig. (2) is independent within this error bar of the momentum along the Fermi-surface. It is defined through $\text{Im} \Sigma = a + b \omega$ in the legend in the figure. Given the definition of the parameter $g_{\vec{p}}$ in Eq. (2), $b = \frac{\pi}{2} g_{\vec{p}}$. This experiment
therefore determines \( \gamma = 0.4 \pm 0.1 \). This should be compared with \( \gamma \) deduced from the specific heat in Eq. (3). It must be remembered that the specific heat is measured in a different compound than the scattering rates. However, when a variety of compounds are measured, as in the resistivity results [32] quoted below, the variation of this parameter appears to be no more than 50%. The measurements of the single-particle line shapes in \( La_{2-x}Sr_xCuO_4 \) [31] [33] show angle dependence in the scattering rate increasing by about 50% from the \((\pi, \pi)\) to the \((\pi, 0)\) directions with a value in the former about the same as that shown in Fig. (2).

Fig. (3) shows a compilation of data of the line-width in the momentum distribution for a fixed frequency up to frequencies of about 0.5 eV in three different compounds that are measurable by ARPES. The low frequency departure from linearity as a function of \( \omega \) is due to impurity scattering and finite temperature. The upper cut-off in the relaxation rate given by the theory, when it must change to frequency independent, is within 50% of the \( \pi T_x \) in the specific heat data shown in Fig. (1). This is no more than a requirement of causality. Later measurements in Bi2212 show [27] that this upper cut-off is angle dependent - about 0.5 eV near the diagonal and decreasing to about 0.2 eV towards the \((\pi, 0)\) -directions. Below about 0.4 eV, the cut-off is the same as the bottom of the band measured from the chemical potential [25].

C. Resistivity

Fig. (4) presents the region of temperature as a function of doping \( p \) in \( La_{2-x}Sr_xCuO_4 \) in which the resistivity is linear in \( T \) [34]. The dashed lines give the temperature below which resistivity begins to deviate from linear in \( T \). So the dashed line in the right marks the temperature cut-off \( \xi_T^{-1}(p - p_c) \). This is the right way to plot a quantum-critical property which has crossovers at finite temperatures for \( p \neq p_c \). (Plotting the temperature dependence from low temperatures to high temperatures across the cross-over by a power law expansion in temperature is the wrong thing to do but has been indulged in by many authors including those of [34] in other papers.) The data is consistent with linear in \( T \) resistivity to arbitrary low temperatures at near the critical doping and in many compounds remains the same up to temperatures at which they begin to melt or decompose, about 1000 K. Recently data from several compounds has been collected [32] and summarized after a careful estimate of
parameters such as electron density and velocity in terms of a transport relaxation rate

\[ \tau_{tr}^{-1} = \alpha k_B T / h, \quad \text{for} \quad p \approx p_c \]  

(6)

I will identify

\[ \alpha \equiv \frac{\pi}{2} g_{tr}. \]  

(7)

where \( g_{tr} \propto g \); their relation is discussed in Sec. III. There is one fault in the deduction of the dimensionless parameter \( \alpha \) in Ref. [32] which is otherwise a very useful and careful paper.

The effective mass in the formula for conductivity should be the band-structure mass and not the renormalized many body mass, which occurs for example in the specific heat. This follows from a Ward identity [35], [36], [37], [38] which is a consequence of the continuity equation. This is a subtle point which is dealt with in Sec.III. Here I simply note that if a renormalized mass were to be used, which is logarithmically infinite at criticality at low temperatures, the scattering rate would tend logarithmically \( \to 0 \) as \( T \to 0 \) which would be quite unphysical. Moreover even if only a constant mass enhancement factor is put in as done in Ref. [32], the momentum transport scattering rate would be significantly larger than the single-particle scattering rate. The general theorem is that the transport scattering rates must always be smaller or equal to the single-particle scattering rates, as discussed further below in Section III. The correct estimate of \( \alpha \) is therefore about a factor of 3 (the effective many body enhancement estimated in [32]) lower than that given. Ref. ([32]) finds \( \alpha \) varying in different cuprate compounds to be 0.7 \( \pm \) 0.2 to 1.2 \( \pm \) 0.3. The corrected value then varies from about 0.25 \( \pm \) 0.1 to about 0.4 \( \pm \) 0.1. (Would Planck prefer his limit to have a coefficient \( 1/\pi \) and not 1?!) \( g_{tr} \) would then be about 2/3 of these numbers. \( g_{tr} \) is in fact calculated to be about \( (2/3) \tilde{g} \) in Sec III, quite consistent with the single-particle scattering rates and the specific heat.

The resistivity phase diagram Fig. (4) should be compared with Fig. (1) for specific heat near criticality. More data near the critical point would be nice to have. Given what we have, one may deduce similar value of the cross-over exponent \( \zeta \approx 0.5 \) from this plot as well.

1. Minimum scattering length

As already mentioned, the single-particle scattering rate gives an upper limit to the transport scattering rate or the inverse width of the momentum distribution function gives
a minimum limit to the transport scattering length $\ell_{tr}$. The maximum in the width of the momentum distribution may be read from Fig. (3). It is about 0.4 ($\pi/a$) at an energy of about 0.4 eV (corresponding to a temperature $\omega_{cx}/\pi$, of about 1600 K). The mean-free path is the half-width and $k_F$ is about $0.8\pi/a$ near critical doping. The fears that $k_F\ell_{tr}$ or $\ell_{tr}/a$ is smaller than 1, the so-called ”Mott-Ioffe-Regel limit” are therefore unfounded. We appear to be a factor of about 5 on the safe side of it. The basis of this limit and its precise statement at high temperatures, where the use of the uncertainty principle is not evidently valid, itself needs further study.
resembling that of early Raman studies (16, 17). This continuum plasmon fades into a featureless, energy-independent continuum suggesting it is electronic in origin. The continuum was found to be essentially isotropic in the cuprates. (Fig. 1.

To evaluate this possibility, we determined the polarizability of the sample in the presence of strong screening (24, 25). The long-ranged part of the Coulomb interaction, revealing the dielectric function of the system. The difference between the dielectric function of the sample and that of the vacuum is highly anomalous in the region $v_F q / \omega \lesssim 1$, showing the fit to the square of this quantity whose coefficient is related to the compressibility and the scattering rate using the Einstein relation.

D. Density correlations

Great technical developments have led to a laboratory instrument [39] to measure the density correlations accurately over a wide range of frequencies and over the entire Brillouin zone. These are shown in Fig. (5). Quite generally, the Einstein relation gives the conductivity

$$\sigma(\omega) = e^2 \kappa D(\omega).$$

$\kappa$ is the compressibility which is equal to the density of states at the Fermi-surface for non-interacting fermions. $D(\omega)$ is the diffusion function. Continuity equation gives that the
imaginary part of the (screened) density correlation function $\Pi''(q, \omega)$ for $v_F q << \omega$,

$$\Pi''(q, \omega) = \frac{\kappa q^2 D(\omega)}{\omega}. \quad (9)$$

The possible quantum-critical aspects arise in the possible renormalization of $\kappa$ and the frequency dependence of the diffusion function $D(\omega)$. One can write

$$D(\omega) = \frac{v_F^2}{2} \tau_{tr}(\omega), \quad (10)$$

again with no renormalization in $v_F$ from its band-structure value and $\tau_{tr}$ is the transport relaxation rate. (9) is appropriate when the velocity is isotropic. When it is anisotropic an appropriate average is called for which takes into account the direction of measurement of the density correlations.

Given $\tau_{tr}(\omega) \propto \omega^{-1}$, $\Pi''(q, \omega) \propto \frac{q^2}{\omega^2}$ follows. This is consistent with the optical conductivity if the conductivity is $\propto 1/\omega$ in the range of the data. (Actually both the logarithmic dependence of the mass and the upper cut-off begin to play a visible role in the optical conductivity above a frequency of about 0.1 eV, but within the accuracy of the density correlation function data, they are unimportant.)

To compare quantitatively, the experimental results shown in Fig. (5) are fitted to find,

$$\Pi''(q, \omega) = (3 \pm 0.5) \times 10^{-3} eV^{-1} \AA^{-3} \left(\frac{v_F q}{\omega}\right)^2. \quad (11)$$

A bare fermi-velocity of about 2eV $\AA$ obtained from ARPES measurements is used to get this result. The numerical coefficient then is equal to $\kappa(\pi/2)g_{tr}/2$. The theory of the density correlations in the limit $v_F q/\omega << 1$ [40] summarized below allows no singular corrections to the compressibility but Fermi-liquid corrections are allowed. With the dimensionless $g_{tr}$ from the resistivity measurements of about 0.3, we get $\kappa \approx 10^{-2}/(eV \AA^3)$. An unrenormalized $\kappa$ is the density of states near the chemical potential. Such a density of states is about 2 states /[2eV(16 x 12)\AA^3] $\approx (1/2)10^{-2}/(eV \AA^3)$.

Actually (9) is obeyed up to nearly $v_F q = \omega$ below which it is nearly a constant. The constant part is indeed even more remarkable than the part discussed above. A theory for that [41] has also been provided.

E. Resistivity and specific heat in Heavy Fermions and Fe based compounds

Temperature dependent resistivity proportional to T is measured from 30 mK to about 0.6 K in the AFM quantum-critical region of the compound CeCu$_{6-2x}$Au$_x$ at $x = 0.1$ with
cross-over on both sides [42]. Correspondingly, the specific heat follows Eq. (1) with cross-over in either side. See Fig. (6). From the specific heat, one deduces \( g \approx 0.8 \), and \( T_x \approx 10K \).

Using an effective Fermi-energy of about 20 K, corresponding to the background specific heat in the nearby Fermi-liquid compositions, a slope in resistivity of about 0.5 is obtained. The cut-off \( T_x \) is similar to what is directly deduced from the measurements of the fluctuation spectra [43], [9]. The energy scales in this compound are too small to be measurable in single-particle spectra by ARPES.

In the Fe based compounds, evidence for the existence of a quantum critical point [47] has been noted through linear in \( T \) dependence of the resistivity. Resistivity for one of the compounds showing one of the clearest linear in \( T \) dependence is shown in (7) together with the thermopower of another which also shows such a resistivity. There is no single-particle spectra to compare with and neither are the basic band-parameters, an average fermi-velocity etc. known, or a background Fermi-liquid specific heat estimated to get enough parameters to determine \( g \) from a scattering rate. Thermopower, which is the entropy per thermally excited particle does vary as \( T \log T \), consistent with the theoretical point of view in this paper. But a quantitative estimate of the parameter \( g \) from its magnitude is not possible since there is no estimate of \( \gamma \). These and the single-particle scattering rates will hopefully be available in the future.

Figure 6. Resistivity of \( Cu_{5.9}Au_{0.1}Cu_6 \) and specific heat at various pressures and dopings of \( CeCu_6 \) across the antiferromagnetic quantum critical point. The figures are taken from Ref. [44].
Figure 7. Left: Phase diagram of \(BaFe_2(As_{1-x}P_x)_2\) and Resistivity at criticality \((x \approx 0.3)\). Taken from \cite{45}. Thermopower \(S\) divided by \(T\) across AFM quantum criticality in \(K_xSr_{1-x}Fe_2As_2\), effectively showing the \(T \ln T\) dependence of the specific heat at quantum-criticality. From Ref. \cite{46}. The same paper also gives resistivity showing \(\propto T\) behaviour in a similar range in temperature and composition.

III. FOUNDATION OF THE RESULTS IN THE THEORY

The theory has been briefly summarized \cite{3} and detailed references given. Only the conclusions will be summarized here. The model solved is the 2D dissipative quantum XY model in which one calculates the frequency and momentum-dependence of the correlations of the angle of the quantum rotor \(e^{i\theta(r,\tau)}\). The coupling to Fermions leads to a dissipation of the Caldeira-Leggett form, which is an essential part of the model. The applicability of the model to the problems has been discussed. The most important result is that the correlation functions are products of a function of space and of imaginary time \(\tau\) and that the spatial correlation length is proportional to logarithm of the temporal correlation. This leads effectively to relative freedom of the temporal and spatial metric near criticality and
to very unusual but simple results for physical properties in terms of just two parameters.

\[
G(\mathbf{r}, \mathbf{r}', \tau, \tau') = \left\langle e^{i \theta(\mathbf{r}, \tau)} e^{-i \theta(\mathbf{r}', \tau')} \right\rangle
\]  
(12)

\[
= G_0 \left( \tau_c - \tau' \right) \left( \ln \left| (\mathbf{r} - \mathbf{r}') / a \right| \right) e^{-|\tau - \tau'|/\xi_r} e^{-|\mathbf{r} - \mathbf{r}'|/\xi_r},
\]  
(13)

\[
\frac{\xi_r}{a} = \ln \frac{\xi_r}{\tau_c},
\]  
(14)

\(\tau_c^{-1}\) is the high energy cut-off in the theory (\(\omega_{cx} = \pi T_{cx}\) in the analysis of the experiments.) The thermal Fourier transform of \(\frac{1}{\tau - \tau'}\) gives the function \(\tanh(\omega/2T)\). The asymptotic low energy and high energy forms of this were the phenomenological assumption made in 1989 [2].

The variation of the correlation length \(\xi_r\) as a function of \((p - p_c)\) or equivalently to the parameters of the xy model has been obtained by quantum Monte-carlo calculations [7]. When the variation is due to the variation in the ratio of the kinetic energy parameter to the interaction energy parameter for a fixed dissipation, the exponent \(\zeta\) defined through the experiments on specific heat and resistivity above, is approximately 1/2, consistent with the data.

A. Some measurable properties

The fermions scatter off the fluctuations with the same coupling function as the one which leads to the dissipation of the fluctuations. The fluctuations serve as the irreducible vertex [35] in a calculation of the properties of the fermions. Because of the product form of (12), the calculations for the properties of the fermions can be done easily and precisely [3]. Of relevance to this paper is the retarded self-energy of the fermions, which is calculated to be

\[
\Sigma(\mathbf{p}, \omega) = g_p \left( i \frac{\pi}{2} \max(|\omega|, \pi T) + \omega \ln(\frac{\omega_{cx}}{\omega}) \right), \text{ for } \max(\omega, \pi T) \lesssim \omega_{cx},
\]  
(15)

\[
= i \frac{\pi}{2} g_p \omega_{cx}, \text{ for } \max(\omega, T) \gg \omega_{cx}.
\]

\(g_p\) includes in it the amplitude of the fluctuations and the density of states of the fermions. It also depends on the anisotropy of the band-structure. It is independent of \(\mathbf{p}\) for a circular Fermi-surface. For the band-structure of Bi2212 near optimum doping, it is estimated to increase by less than about 2 in going from the \((\pi, \pi)\) directions to the \((\pi, 0)\) directions. For a square lattice, the correction to isotropy varies as \(\cos 4\theta(\mathbf{p})\).
The electronic specific heat can be written directly in terms of integrals over the imaginary part of the self-energy [48]. The result is particularly simple when the self-energy is nearly momentum independent. Then the specific heat singularity is simply given by the inverse of the renormalization given in Eq. (2) which follows from (15). There is no renormalization of the compressibility if the self-energy is momentum independent as follows from a Ward identity.

The conductivity may be calculated in three different ways: From the Kubo formula, which simplifies enormously when the self-energy is independent of momentum [2]. Then the momentum transport scattering rate is equal to the single-particle scattering rate. One may calculate including the vertex correction due to the small angular dependence in the self-energy using the Boltzmann equation, for example as in Refs. [49][50]. The transport scattering rate is then necessarily smaller than the single-particle scattering rate. A straightforward calculation shows that with a factor of 2 variation in the self-energy, increasing in the direction where the velocity is least, the coefficient of the linear in T resistivity is about 2/3 of that of the maximum self-energy.

The density-density correlation has also been calculated directly [40], [41] giving the form shown in Fig. (5) with the coefficient related to the resistivity as noted already.

For all the properties considered in this paper, there are only two parameters $g$ and $\omega_c$ in terms of which every property considered is given. These two parameters were roughly estimated in Ref. [10] from the kinetic and interaction energies of a copper-oxide three orbital model. This gave $g \approx 1$ and $\omega_c \approx 0.5eV$.

1. Effective mass in resistivity

I now briefly consider the point about the effective mass in the expression for the resistivity or other transport properties, using the Kubo formula. This issue arises even in Fermi-liquids in which the self-energy is very weakly momentum-dependent compared to its frequency dependence [36]. An application in that case to the resistivity in heavy fermions is given in Eq. (3) of Ref. [38], where the Kadowaki-Woods [51] relation was derived. For applications to the problems of interest in this paper, the $T^2$ on the left side of that equation should be replaced with $T$ and the $Im\Sigma$ on the right side by the relevant part of Eq. (15) here.

The Kubo formula for conductivity is equivalent to the evaluation of the diagram Fig.
Figure 8. The Kubo formula for the current-current correlation in terms of the bare current operator, the renormalized current operator and the exact single-particle Green’s functions. The dc conductivity is \((1/\epsilon)\) times the imaginary part of the current-current correlation at \(q \to 0\).

(8) for the current-current correlations. The vertex on the left is the bare (band-structure) velocity operator \(v\), the one on the right is the renormalized velocity operator \(v_{renorm}\), and the lines are the exact Green’s functions. In the appropriate limit for calculation of dc conductivity, the matrix elements of the operator \(v_{renorm}\) are given by a Ward identity [35] which is a reflection of the equation for conductivity:

\[
v_{renorm} = \Lambda \, v
\]

(16)

\[
\text{Lim}_{\epsilon \to 0} \text{Lim}_{q \to 0} \Lambda(p, \omega; q, \epsilon) = 1 - \frac{1}{v} \frac{\partial \Sigma(p, \omega, T)}{\partial p}.
\]

(17)

If \(\Sigma(p, \epsilon, T)\) is independent of \(p\), the relevant limit of \(v_{renorm} = v\). The sum over frequencies and integration over momentum over the Green’s functions in (8), for the same conditions can be easily carried out. The conductivity in the \(\alpha\) direction is then,

\[
\sigma(T) = \frac{e^2\langle v_{\alpha}^2 \rangle N(0)}{2\text{Im} \Sigma(p_F, 0, T)}.
\]

(18)

\(N(0)\) is the bare (band-structure) density of states. One can show that the lack of any
renormalization of the effective mass in the conductivity is true only for frequency much smaller than the temperature.

B. Related Matters

Equally interesting as the experiments discussed in this paper is the fact that although the normal self-energy is nearly momentum independent, pairing is in the d-wave channel. This is the fundamental paradox of superconductivity in the cuprates. A full explication of this through the momentum dependence of the vertex coupling the fermions to the critical fluctuations [10] and verification through the measurement of the spectrum of the pairing self-energy in ARPES experiments [11] has been provided.

It should be obvious that hardly anything about superconductivity in the cuprates or the heavy-fermion compounds or the Fe based compounds can be understood without understanding the normal state properties. The application of the dissipative quantum xy model to the cuprates is rather clear given the symmetries changed at $T^*(x)$. The application of the same model is not a surprise close to the AFM critical point in the heavy fermions and the Fe compounds. The anisotropic AFM maps to such a model [52] (see also erratum) and the point of view is supported by the fit to the measured fluctuation spectra by the calculated fluctuations [9]. The unanswered question is why the theory works over such a wide temperature range when the xy anisotropy is so small that the classical transition would crossover to that of the xy model only very close to the transition. The answer has probably to do with the possibility that with criticality of the kind discovered in the model, which for some purposes may be regarded as having a dynamical critical exponent $\rightarrow \infty$, the cross-over to the anisotropic model occurs over essentially the entire range below the ultraviolet cut-off.

No other theoretical ideas and calculations on a physical model have explained the temperature and frequency dependence of the properties discussed here in cuprates or heavy fermions or the Fe based compounds, leave alone get the parameters. The models which are extensions of the dynamical classical critical phenomena to quantum-criticality [53], [54], and extensively worked on since are known not to give any of the experimental properties noted. The closest that comes is a local model of $SU(N)$-spins in the limit $N \rightarrow \infty$ [55] which leads to the fluctuation spectra in frequency which was suggested in [2]. Besides the
difficulty of how such a model can be an effective physical model, it has the problem that an extensive ground state entropy is inevitably tied to its results. Modification of the model to get rid of this entropy [56] also alter its properties to that of a Fermi-liquid.

Recently, experiments in twisted bi-layer graphene reveal a linear in $T$ resistivity [57], [58], [59] in the phase diagram in a region which has the quantum-critical shape in the phase diagram at the boundary between the insulator and the superconductor, extending to asymptotic low temperatures when magnetic field is applied to suppress superconductivity. One may speculate that the relevant model for critical fluctuations is again of the xy variety, with the $U(1)$ symmetry being that in valley space, and which is broken in the correlated insulator. This point of view needs further work.

I speculate that quantum-critical fluctuations of a variety of (soft) vertex models coupled to fermions, (many of which for 2D classical problems are treated in Baxter’s book [60]) which classically are not in the Ginzburg- Landau, Wilson-Fisher class are generically related to the 2D- dissipative XY model in quantum version of the problems. They may all be governed by topological excitations with relative freedom of temporal and spatial fluctuations. The scaling of the metric of space and time different from the flat world is the fundamental aspect of any quantum-critical problem. That the spatial and temporal metric become free relative to each other is a unique property which has led to the extra-ordinarily simple results which explain the observed quantum-criticality in the problems discussed. The model itself is richer than the application noted here. For example, there is a critical region to a phase in it [4, 6, 7] in which the correlation functions are of product form in space and time but with the temporal correlation length proportional logarithmically to the spatial correlation length. One may speculate, of course at great peril, that this is the appropriate description of another quantum-critical problem - inflation in the early universe.

Acknowledgement: I wish to thank the many many experimentalists who have taken the time to explain their experimental results to me. The data quoted here is of-course a very tiny fraction of the mutually consistent data that exists in the literature. I especially wish to thank Bastien Michon and Louis Taillefer for providing me versions of figures from their paper suitable for me and for answering questions about the data. This paper was written
while at Aspen Center for Physics.

[1] P. W. ANDERSON, Science 235, 1196 (1987), https://science.sciencemag.org/content/235/4793/1196.full.pdf.
[2] C. M. Varma, P. B. Littlewood, S. Schmitt-Rink, E. Abrahams, and A. E. Ruckenstein, Phys. Rev. Lett. 63, 1996 (1989).
[3] C. M. Varma, Reports on Progress in Physics 79, 082501 (2016).
[4] C. Hou and C. M. Varma, Phys. Rev. B 94, 201101 (2016).
[5] J. Kosterlitz, J. Phys. C 7 (1974).
[6] L. Zhu, Y. Chen, and C. M. Varma, Phys. Rev. B 91, 205129 (2015).
[7] L. Zhu, C. Hou, and C. M. Varma, Phys. Rev. B 94, 235156 (2016).
[8] C. M. Varma, Phys. Rev. Lett. (2015).
[9] C. M. Varma, L. Zhu, and A. Schröder, Phys. Rev. B 92, 155150 (2015).
[10] V. Aji, A. Shekhter, and C. M. Varma, Phys. Rev. B 81, 064515 (2010).
[11] J. M. Bok, J. J. Bae, H.-Y. Choi, C. M. Varma, W. Zhang, J. He, Y. Zhang, L. Yu, and X. J. Zhou, Science Advances 2 (2016), 10.1126/sciadv.1501329, https://advances.sciencemag.org/content/2/3/e1501329.full.pdf.
[12] B. Michon, C. Girod, S. Badoux, J. Kamark, Q. Ma, M. Dragomir, H. A. Dabkowska, B. D. Gaulin, J. S. Zhou, S. Pyon, T. Takayama, H. Takagi, S. Verret, N. Doiron-Leyraud, C. Marce- nat, L. Taillefer, and T. Klein, Nature 567, 218 (2019).
[13] C. M. Varma, Phys. Rev. B 55, 14554 (1997).
[14] P. Bourges and Y. Sidis, Comptes Rendus Physique 12, 461 (2011).
[15] A. Kaminski, S. Rosenkranz, H. M. Fretwell, J. C. Campuzano, Z. Li, H. Raffy, W. G. Cullen, H. You, C. G. Olson, C. M. Varma, and H. Höchst, Nature 416, 610 (2002).
[16] B. Leridon, P. Monod, D.Colson, and A. Forget, EPL (Europhysics Letters) 87, 17011 (2009).
[17] L. Zhao, C. A. Belvin, R. Liang, D. A. Bonn, W. N. Hardy, N. P. Armitage, and D. Hsieh, Nature Physics 13, 250 EP (2016).
[18] J. Xia, E. Schemm, G. Deutscher, S. A. Kivelson, D. A. Bonn, W. N. Hardy, R. Liang, W. Siemens, G. Koster, M. M. Fejer, and A. Kapitulnik, Phys. Rev. Lett. 100, 127002 (2008).
[19] Y. Lubashevsky, L. Pan, T. Kirzhner, G. Koren, and N. P. Armitage, Phys. Rev. Lett. 112,
147001 (2014).

[20] J. Zhang, Z. Ding, C. Tan, K. Huang, O. O. Bernal, P.-C. Ho, G. D. Morris, A. D. Hillier, P. K. Biswas, S. P. Cottrell, H. Xiang, X. Yao, D. E. MacLaughlin, and L. Shu, Science Advances 4 (2018), 10.1126/sciadv.aao5235, https://advances.sciencemag.org/content/4/1/eaao5235.full.pdf.

[21] A. Shekhter, B. J. Ramshaw, R. Liang, W. N. Hardy, D. A. Bonn, F. F. Balakirev, R. D. McDonald, J. B. Betts, S. C. Riggs, and A. Migliori, Nature 498, 75 (2013).

[22] H. Murayama, Y. Sato, R. Kurihara, S. Kasahara, Y. Mizukami, Y. Kasahara, H. Uchiyama, A. Yamamoto, E. G. Moon, and J. Cai, arXiv e-prints, arXiv:1805.00276 (2018), arXiv:1805.00276 [cond-mat.supr-con].

[23] Y. Sato, S. Kasahara, H. Murayama, Y. Kasahara, E.-G. Moon, T. Nishizaki, T. Loew, J. Porras, B. Keimer, T. Shibauchi, and Y. Matsuda, Nature Physics 13, 1074 (2017), arXiv:1706.05214 [cond-mat.supr-con].

[24] A. Mukherjee and et al., (2019).

[25] A. Kaminski, H. M. Fretwell, M. R. Norman, M. Randeria, S. Rosenkranz, U. Chatterjee, J. C. Campuzano, J. Mesot, T. Sato, T. Takahashi, T. Terashima, M. Takano, K. Kadowaki, Z. Z. Li, and H. Raffy, Phys. Rev. B 71, 014517 (2005).

[26] T. Valla, A. Fedorov, P. D. Johnson, Q. Li, G. D. Gu, and N. Koshizuka, Phys. Rev. Lett. 85, 828 (2000).

[27] J. M. Bok, J. H. Yun, H.-Y. Choi, W. Zhang, X. J. Zhou, and C. M. Varma, Phys. Rev. B 81, 174516 (2010).

[28] E. Abrahams and C. Varma, Proc. Nat. Ac. Sciences 97, 5714 (2000).

[29] W. Meevasana, F. Baumberger, K. Tanaka, F. Schmitt, W. R. Dunkel, D. H. Lu, S.-K. Mo, H. Eisaki, and Z.-X. Shen, Phys. Rev. B 77, 104506 (2008).

[30] J. Graf, G.-H. Gweon, K. McElroy, S. Y. Zhou, C. Jozwiak, E. Rotenberg, A. Bill, T. Sasagawa, H. Eisaki, S. Uchida, H. Takagi, D.-H. Lee, and A. Lanzara, Phys. Rev. Lett. 98, 067004 (2007).

[31] J. Chang, S. Pailhés, M. Shi, M. Månsson, T. Claesson, O. Tjernberg, J. Voigt, V. Perez, L. Patthey, N. Momono, M. Oda, M. Ido, A. Schnyder, C. Mudry, and J. Mesot, Phys. Rev. B 75, 224508 (2007).

[32] A. Legros, S. Benhabib, W. Tabis, F. Laliberte, M. Dion, M. Lizaire, B. Vignolle, D. Vignolles,
H. Raffy, Z. Z. Li, P. Auban-Senzier, N. Doiron-Leyraud, P. Fournier, D. Colson, L. Taillefer, and C. Proust, Nature Physics 15, 142 (2019).

[33] L. Zhu, V. Aji, A. Shekhter, and C. Varma, Phys. Rev. Lett. 100, 057001 (2008).

[34] N. Hussey, R. Cooper, X. Xiaofong, Y. Wang, I. Mouzopoulou, B. Vignolle, and C. Proust, Phil. Trans. R. Soc. 369, 1626 (2011).

[35] P. Nozières, Theory of Interacting Fermi Systems (Benjamin, New York, 1960).

[36] C. M. Varma, Phys. Rev. Lett. 55, 2723 (1985).

[37] R. E. Prange and L. P. Kadanoff, Phys. Rev. 134, A566 (1964).

[38] K. Miyake, T. Matsuura, and C. Varma, Solid State Communications 71, 1149 (1989).

[39] M. Mitrano, A. A. Husain, S. Vig, A. Kogar, M. S. Rak, S. I. Rubeck, J. Schmalian, B. Uchoa, J. Schneeloch, R. Zhong, G. D. Gu, and P. Abbamonte, Proceedings of the National Academy of Sciences 115, 5392 (2018), https://www.pnas.org/content/115/21/5392.full.pdf.

[40] A. Shekhter and C. M. Varma, Phys. Rev. B 79 79 (2009).

[41] C. M. Varma, Phys. Rev. B 96, 075122 (2017).

[42] H. Löhneysen, A. Rosch, M. Vojta, and P. Wölfle, Rev. Mod. Phys. 79, 1015 (2007).

[43] A. Schröder and et al., Phys. Rev. Lett. 80, 5623 (1998).

[44] H. von Löhneysen, J. Phys.: Condens. Matter 8, 9689 (1996).

[45] I. M. Hayes, Z. Hao, N. Maksimovic, S. K. Lewin, M. K. Chan, R. D. McDonald, B. J. Ramshaw, J. E. Moore, and J. G. Analytis, Phys. Rev. Lett. 121, 197002 (2018).

[46] B. Lv, M. Gooch, B. Lorenz, F. Chen, A. Guloy, and C. Chu, New J. Phys. 11, 025013 (2009).

[47] T. Shibauchi, A. Carrington, and Y. Matsuda, Annual Review of Condensed Matter Physics 5, 113 (2014), https://doi.org/10.1146/annurev-conmatphys-031113-133921.

[48] A. A. Abrikosov, L. Gorkov, and I. Dzyaloshinski, Methods of Quantum Field Theory in Statistical Physics (Prentice Hall, NJ, 1963).

[49] C. M. Varma and E. Abrahams, Phys. Rev. Lett. 86, 4652 (2001).

[50] E. Abrahams and C. M. Varma, Phys. Rev. B 68, 094502 (2003).

[51] K. Kadowaki and S. Woods, Solid State Communications 58, 507 (1986).

[52] C. M. Varma, Phys. Rev. Lett. 115, 186405 (2015).

[53] J. A. Hertz, Phys. Rev. B 14, 1165 (1976).

[54] T. Moriya, Spin Fluctuations in Itinerant Electron Magnetism (Springer-Verlag, Berlin, 1985).

[55] S. Sachdev and J. Ye, Phys. Rev. Lett. 70, 3339 (1993).
[56] D. Chowdhury, Y. Werman, E. Berg, and T. Senthil, Phys. Rev. X 8, 031024 (2018).

[57] Y. Cao, D. Chowdhury, D. Rodan-Legrain, O. Rubies-Bigorda, K. Watanabe, T. Taniguchi, T. Senthil, and P. Jarillo-Herrero, arXiv:1901.03710 (2019).

[58] X. Lu, P. Stepanov, W. Yang, M. Xie, M. A. Aamir, I. Das, C. Urgell, T. Watanabe, Kenji and-Taniguchi, G. Zhang, A. Bachtold, A. H. MacDonald, and D. K. Efetov, arXiv:1903.06513 (2019).

[59] H. Polshyn, M. Yankowitz, S. Chen, Y. Zhang, K. Watanabe, T. Taniguchi, C. R. Dean, and A. F. Young, arXiv:1902.00763 (2019).

[60] R. J. Baxter, Exactly Solved Models in Statistical Mechanics (Academic, London, 1982).