EXACT ASYMPTOTIC VOLUME AND VOLUME RATIO OF SCHATTEN UNIT BALLS

ZAKHAR KABLUCHKO, JOSCHA PROCHNO, AND CHRISTOPH THÄLE

Abstract. The unit ball $B^n_p(ℝ)$ of the finite-dimensional Schatten trace class $S^n_p$ consists of all real $n \times n$ matrices $A$ whose singular values $s_1(A), \ldots, s_n(A)$ satisfy $s_1^p(A) + \ldots + s_n^p(A) \leq 1$, where $p > 0$. Saint Raymond [Studia Math. 80, 63–75, 1984] showed that the limit

$$\lim_{n \to \infty} n^{1/2 + 1/p} (\text{Vol} B^n_p(ℝ))^{1/n^2}$$

exists in $(0, \infty)$ and provided both lower and upper bounds. In this paper we determine the precise limiting constant based on ideas from the theory of logarithmic potentials with external fields. A similar result is obtained for complex Schatten balls. As an application we compute the precise asymptotic volume ratio of the Schatten $p$-balls, as $n \to \infty$, thereby extending Saint Raymond’s estimate in the case of the nuclear norm ($p = 1$) to the full regime $1 \leq p \leq \infty$ with exact limiting behavior.

1. Introduction and main results

1.1. Introduction. The Schatten trace classes $S^n_p$ ($0 < p \leq \infty$), consisting of all compact linear operators on a Hilbert space for which the sequence of their singular values belongs to the sequence space $ℓ_p$, are one of the most important classes of unitary operator ideals. Their analysis, particularly in the finite-dimensional setting, has a long tradition in asymptotic geometric analysis and the local theory of Banach spaces. For example, Gordon and Lewis [8] obtained that the space $S_1$ does not have local unconditional structure, Tomczak-Jaegermann [20] demonstrated that this space (which is naturally identified with the projective tensor product $ℓ_2 \otimes_π ℓ_2$) has Rademacher cotype 2, and König, Meyer and Pajor [12] proved the boundedness of the isotropic constants of $S^n_p$ ($1 \leq p \leq \infty$). More recently, Guédon and Paouris [9] have established concentration of mass properties for the unit balls of Schatten $p$-classes $S^n_p$, Barthe and Cordero-Erausquin [3] studied variance estimates, Radke and Vritsiou [14] proved the thin-shell conjecture, and Hinrichs, Prochno and Vy bíral [10] computed the entropy numbers for their natural embeddings.

1.2. Asymptotic volume of Schatten balls. In [17], Saint Raymond studied the volumetric properties of unit balls in finite-dimensional real and complex Schatten $p$-classes. For $0 < p \leq \infty$, let $S^n_p(ℂ)$ denote the space of all $n \times n$ matrices $A$ with entries from

\begin{itemize}
\item 2010 Mathematics Subject Classification. Primary: 52A23. Secondary: 46B06, 60B20, 46B07, 47B10, 52A21, 31A15.
\item Key words and phrases. Asymptotic geometric analysis, convex bodies in high dimensions, logarithmic potential theory with external field, Schatten classes, Schatten balls, Ullman distribution, volume, volume ratio.
\end{itemize}
$\mathbb{F} \in \{ \mathbb{R}, \mathbb{C} \}$ equipped with the Schatten $p$- (quasi)norm

$$\|A\|_{S^p} = \begin{cases} \left( \sum_{j=1}^{n} s_j(A)^p \right)^{1/p} & : p < \infty \\ \max \{ s_1(A), \ldots, s_n(A) \} & : p = \infty, \end{cases}$$

where $s_1(A), \ldots, s_n(A)$ are the singular values of $A$. If we denote by

$$B^n_p(\mathbb{F}) = \{ A \in S^n_p : \|A\|_{S^p} \leq 1 \}$$

the corresponding Schatten unit ball, Saint Raymond proved asymptotic formulas for their volume, showing that, as $n \to \infty$,

$$\left( \text{Vol}_n^2 B^n_p(\mathbb{R}) \right)^{1/n^2} \sim n^{-\frac{1}{2} - \frac{1}{p}} \sqrt{2\pi e^{3/2} \Delta(p/2)}$$

and

$$\left( \text{Vol}_{2n^2} B^n_p(\mathbb{C}) \right)^{(1/(2n^2)}} \sim n^{-\frac{1}{2} - \frac{1}{p}} \sqrt{\pi e^{3/2} \Delta(p/2)},$$

where $\Delta(p) \in (0, \infty)$ is certain constant and $\text{Vol}_N$ stands for the Lebesgue measure of dimension $N \in \mathbb{N}$. Here and below we shall write $a_n \sim b_n$ for two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ whenever $a_n/b_n \to 1$, as $n \to \infty$. For the parameter $\Delta(p)$ that appears in (1) and (2) he provided both lower and upper bounds. However, with the exception of $\Delta(1) = e^{-1/2}$ and $\Delta(\infty) = 1/4$ no explicit values of $\Delta(p)$ seem to be known. With this paper we want to shed light on the precise value of $\Delta(p)$ and the asymptotic volume of the unit balls in finite-dimensional Schatten $p$-classes for all $0 < p \leq \infty$. Concomitantly, we shall study another important quantity related to the geometry of Banach spaces, the (asymptotic) volume ratio of $S^n_p(\mathbb{F})$; see below. Our main result is the explicit computation of $\Delta(p)$.

**Theorem 1.** For all $0 < p \leq \infty$ we have

$$\Delta(p) = \frac{1}{4} \left( \frac{2\sqrt{\pi} \Gamma(p + 1)}{e^{1/2} \Gamma(p + \frac{1}{2})} \right)^{1/p}.$$
where the infimum is taken over all ellipsoids $\mathcal{E}$ which are contained in $K$. One can in fact show that there is a unique ellipsoid $\mathcal{E}$, referred to as the John ellipsoid, of maximal volume which is contained in $K$. The volume ratio is a very powerful concept in asymptotic geometric analysis that has its origin in the groundbreaking works of Szarek [18], and Szarek and Tomczak-Jaegermann [19] who extracted this core notion behind a famous result of Kašin on nearly Euclidean decompositions of $\ell^n_1$ and successfully applied their ideas to study several classes of finite-dimensional normed spaces using this affine invariant. Since then the volume ratio appeared in many places, for example, in an estimate of the volume ratio in terms of the Rademacher cotype-2 constant by Bourgain and Milman [5], in Ball’s volume ratio inequality [2], or in Bourgain, Klartag and Milman’s reduction of the hyperplane conjecture [4]. It has also significant applications in approximation theory. We refer the reader to the monographs [1, 6, 20] for further background material.

Before we state our next result let us recall that Szarek and Tomczak-Jaegermann [19, Proposition 3.1] proved that, for all $n \in \mathbb{N}$,

$$
\text{vr}(B^n_1(\mathbb{F})) \leq 32000,
$$

at the same time obtaining bounds for general unitary operator ideals of operators on Hilbert spaces [19, Proposition 3.2]. Later, Saint Raymond [17, Théorème 9] obtained asymptotically, as $n \to \infty$,

$$
\text{vr}(B^n_1(\mathbb{R})) \sim \sqrt{\frac{\Delta(1/2)}{\Delta(1)}} \leq \frac{2}{e^{1/4}}.
$$

Our main result, Theorem 1, can be used to determine asymptotically, as $n \to \infty$, the precise volume ratio of the Schatten $p$-balls $B^n_p(\mathbb{F})$ in the full regime $1 \leq p \leq \infty$.

**Theorem 2.** Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. For $1 \leq p < 2$ we have that, as $n \to \infty$,

$$
\text{vr}(B^n_p(\mathbb{F})) \sim \sqrt{\frac{\Delta(p/2)}{\Delta(1)}} = \frac{1}{2} \left( \frac{\Gamma\left(\frac{p}{2} + 1\right)}{\Gamma\left(\frac{p}{2} + \frac{1}{2}\right)} \right)^{\frac{1}{p}} \sqrt{e^{\frac{1}{p} - \frac{1}{2}} (4\pi)^{\frac{1}{p}}},
$$

while if $2 \leq p \leq \infty$ we have, as $n \to \infty$,

$$
\text{vr}(B^n_p(\mathbb{F})) \sim n \frac{1}{2} \left( \frac{\Gamma\left(\frac{p}{2} + 1\right)}{\Gamma\left(\frac{p}{2} + \frac{1}{2}\right)} \right)^{\frac{1}{p}} \sqrt{e^{\frac{1}{p} - \frac{1}{2}} (4\pi)^{\frac{1}{p}}}.\]

**1.4. Organization of the paper.** In Section 2.1 we present the reduction trick of Saint Raymond’s discrete variational problem to a corresponding continuous problem that allows us to apply methods and ideas from the theory of logarithmic potentials with external fields. Section 2.2 is devoted to the Ullman distribution, which is the unique maximizer in this variational problem, while in Section 2.3 we present the proof of Theorem 1. Finally, in Section 3 we prove Theorem 2.

**2. Proof of Theorem 1**

We start with the following observation. The equality $\Delta(\infty) = 1/4$ was established by Saint Raymond [17, Corollaire 4 on p. 69]. For $0 < p < \infty$, which is always assumed in
the following, Saint Raymond [17, p. 70] characterized the constant \( \Delta(p) \) as the limit of \( \Delta_n(p) \), as \( n \to \infty \), where

\[
\log \Delta_n(p) = \sup_{0 \leq t_1 \leq \ldots \leq t_n} \left( \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \log |t_i - t_j| - \frac{1}{p} \log \left( \frac{1}{n} \sum_{i=1}^{n} t_i^p \right) \right).
\]

He then showed that the positive sequence \( \Delta_n(p) \) is decreasing, which implies that it converges to a limit, as \( n \to \infty \). By providing bounds on this limit, he showed that it is non-zero, but, as (1) and (2) show, these bounds were not sharp. We shall compute the limiting constant precisely.

2.1. Reduction to the continuous problem. The first step in the computation of \( \Delta(p) \) is to replace the supremum over the points \( 0 \leq t_1 \leq \ldots \leq t_n \) in (3) by its continuous version, namely the supremum over probability measures on the positive half-line. For this purpose, let \( \mathcal{M}_1^p(\mathbb{R}) \) be the set of all probability measures on \( \mathbb{R} \) with \( \int_{\mathbb{R}} |x|^p \mu(dx) < \infty \). Similarly, denote by \( \mathcal{M}_1^p(\mathbb{R}+) \) the set of all probability measures on \( \mathbb{R}+ = [0, \infty) \) that satisfy \( \int_{\mathbb{R}+} x^p \mu(dx) < \infty \). Let us write \( \delta_0 \) for the Dirac measure at 0.

On the set \( \mathcal{M}_1^p(\mathbb{R}) \setminus \{ \delta_0 \} \) we consider the functional

\[
J_p(\mu) := \int_{\mathbb{R}} \int_{\mathbb{R}} \log |x - y| \mu(dx) \mu(dy) - \frac{1}{p} \log \int_{\mathbb{R}} |x|^p \mu(dx),
\]

which takes values in \( \mathbb{R} \cup \{-\infty\} \). We shall now demonstrate that the limit of \( (\log \Delta_n(p))_{n \in \mathbb{N}} \) coincides with the supremum that \( J_p(\cdot) \) takes on the set \( \mathcal{M}_1^p(\mathbb{R}+) \setminus \{ \delta_0 \} \).

Proposition 3. For all \( 0 < p < \infty \) we have

\[
\lim_{n \to \infty} \log \Delta_n(p) = \sup_{\mu \in \mathcal{M}_1^p(\mathbb{R}+) \setminus \{ \delta_0 \}} J_p(\mu).
\]

Proof. We split the proof into a lower and an upper bound.

Lower bound. Let us prove that

\[
\lim_{n \to \infty} \log \Delta_n(p) \geq J_p(\mu)
\]

for an arbitrary probability measure \( \mu \in \mathcal{M}_1^p(\mathbb{R}+) \setminus \{ \delta_0 \} \). We assume that

\[
\int_{\mathbb{R}+} \int_{\mathbb{R}+} \log |x - y| \mu(dx) \mu(dy) \neq -\infty,
\]

because otherwise the statement is evident. Since by the definition of \( \mathcal{M}_1^p(\mathbb{R}+) \setminus \{ \delta_0 \} \) the \( p \)-th moment of \( \mu \) is finite, the above double integral cannot take the value \( +\infty \), so it must be finite. Let \( V_1, V_2, \ldots \) be i.i.d. non-negative random variables with probability distribution \( \mu \). The strong law of large numbers for \( U \)-statistics, see [13, Theorem 3.1.1], yields the almost sure convergence

\[
\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \log |V_i - V_j| \overset{a.s.}{\rightarrow} \mathbb{E} \log |V_1 - V_2|.
\]
On the other hand, the strong law of large numbers for sums of i.i.d. random variables yields

\[
\frac{1}{p} \log \left( \frac{1}{n} \sum_{i=1}^{n} V_i^p \right) \xrightarrow{a.s.} \frac{1}{p} \log \mathbb{E} V_1^p, \quad n \to \infty.
\]

By (3), we have that for each realization

\[
\log \Delta_n(p) \geq \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \log |V_i - V_j| - \frac{1}{p} \log \left( \frac{1}{n} \sum_{i=1}^{n} V_i^p \right).
\]

Therefore, using (6) and (7), we get

\[
\log \Delta(p) = \lim_{n \to \infty} \log \Delta_n(p) \geq \mathbb{E} \log |V_1 - V_2| - \frac{1}{p} \log \mathbb{E} V_1^p = \mathcal{I}_p(\mu),
\]

thus proving the lower bound (5).

**Upper bound.** Our next aim is to prove that there is a sequence of probability measures \(\nu_1, \nu_2, \ldots \in \mathcal{M}_p^r(\mathbb{R}_+ \setminus \{0\})\) such that

\[
\liminf_{n \to \infty} \mathcal{I}_p(\nu_n) \geq \log \Delta(p).
\]

Saint Raymond [17, Lemme 6 on p. 71] showed that there is a maximizer of the right-hand side of (3), which we denote by \((t_{1,n}^*, \ldots, t_{n,n}^*)\), and which has the following properties:

\[
0 = t_{1,n}^* < \ldots < t_{n,n}^*, \quad \frac{1}{n} \sum_{i=1}^{n} (t_{i,n}^*)^p = 1, \quad \text{and} \quad t_{i,n}^* - t_{i-1,n}^* \geq n^{-C}
\]

for all \(n \geq 2, i \in \{2, \ldots, n\}\) and some constant \(C = C(p) > 0\). In fact, Saint Raymond normalized the \(\ell_p\)-norm of the maximizer to be 1, but for us it is more convenient to set it to be \(n^{1/p}\), as above. This is possible since the expression on the right-hand side of (3) remains unchanged if we replace \((t_1, \ldots, t_n)\) by \((at_1, \ldots, at_n)\) for \(a > 0\). For \(n \geq 2\), we put \(\varepsilon_n := n^{-2C}\) and consider an absolutely continuous probability measure \(\nu_n\) on \(\mathbb{R}_+\) with Lebesgue density

\[
f_n(t) = \frac{1}{n\varepsilon_n} \mathbb{I}_{[0,\varepsilon_n]}(t) + \frac{1}{n\varepsilon_n} \sum_{i=2}^{n} \mathbb{I}_{[t_{i,n}^*-\varepsilon_n, t_{i,n}^*]}(t), \quad t \in \mathbb{R}_+.
\]

Note that \(\nu_n\) is the uniform distribution on the union of the intervals \(B_{i,n} := [0, \varepsilon_n]\) and \(B_{i,n} := [t_{i,n}^*-\varepsilon_n, t_{i,n}^*]\), for \(i = 2, \ldots, n\). For sufficiently large \(n\), the intervals \(B_{1,n}, \ldots, B_{n,n}\) are disjoint by (9) and we have

\[
\int_{\mathbb{R}_+} t^p f_n(t) \, dt = \frac{1}{n\varepsilon_n} \sum_{i=1}^{n} \int_{B_{i,n}} t^p \, dt \leq \frac{\varepsilon_n^p}{n} + \frac{1}{n} \sum_{i=2}^{n} (t_{i,n}^*)^p = \frac{\varepsilon_n^p}{n} + \frac{1}{n} \sum_{i=1}^{n} (t_{i,n}^*)^p.
\]

We claim that

\[
\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f_n(x) f_n(y) \log |x - y| \, dx \, dy = \frac{2}{n^2} \sum_{1 \leq i < j \leq n} \log |t_{i,n}^* - t_{j,n}^*| - o(1),
\]

where \(o(1)\) denotes a term that tends to zero as \(n \to \infty\).
thus proving the upper bound (8). Here, we used that

\[ \exists Z. KABLUCHKO, J. PROCHNO, AND C. THÄLE \]

\[ \text{analogous estimate also holds for } \varepsilon. \]

\[ \text{Proof of (11).} \]

Observe that each summand on the right-hand side represents the “interaction” between two independent random variables with uniform distribution on the interval \([0,1]\), then \(t_{i,n}^* - \varepsilon_n X\) and \(t_{i,n}^* - \varepsilon_n Y\) are uniformly distributed on the interval \(B_{i,n}\) and we can write

\[ \frac{1}{n^2 \varepsilon_n^2} \int_{B_{i,n}} \int_{B_{i,n}} \log |x - y| \, dx \, dy = \frac{\mathbb{E} \log |(t_{i,n}^* - \varepsilon_n X) - (t_{i,n}^* - \varepsilon_n Y)|}{n^2} \]

\[ = \log \varepsilon_n + \frac{\mathbb{E} \log |X - Y|}{n^2} = O \left( \frac{\log n}{n^2} \right) \]

by the choice of \(\varepsilon_n\). Here we write \(a_n = O(b_n)\) for two sequences \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) if there exists a constant \(M \in (0,\infty)\) such that \(|a_n| \leq Mb_n\) for all sufficiently large \(n\). An analogous estimate also holds for \(i = 1\). For the sum of self-interaction terms we thus obtain the upper bound

\[ \sum_{i=1}^n \frac{1}{n^2 \varepsilon_n^2} \int_{B_{i,n}} \int_{B_{i,n}} \log |x - y| \, dx \, dy = O \left( \frac{\log n}{n} \right) = o(1). \]
Case 2: Interactions between different intervals. Take some $i, j \in \{2, \ldots, n\}$ with $i \neq j$. If $X$ and $Y$ are, as above, independent random variables with uniform distribution on the interval $[0, 1]$, then the random variables $t_{i,n}^* - \varepsilon_n X$ and $t_{j,n}^* - \varepsilon_n Y$ are uniformly distributed on the intervals $B_{i,n} = [t_{i,n}^* - \varepsilon_n, t_{i,n}^*]$ and $B_{j,n} = [t_{j,n}^* - \varepsilon_n, t_{j,n}^*]$, respectively. Thus,

$$
\frac{1}{n^2 \varepsilon_n^2} \int_{B_{i,n}} \int_{B_{j,n}} \log |x - y| \, dx \, dy = \frac{\mathbb{E} \log |(t_{i,n}^* - \varepsilon_n X) - (t_{j,n}^* - \varepsilon_n Y)|}{n^2} 
$$

$$
= \frac{1}{n^2} \mathbb{E} \log |t_{i,n}^* - t_{j,n}^*| + \frac{1}{n^2} \mathbb{E} \log \left| 1 + \varepsilon_n \frac{Y - X}{t_{i,n}^* - t_{j,n}^*} \right|.
$$

Recalling that $|t_{i,n}^* - t_{j,n}^*| > n^{-C}$ and $\varepsilon_n = n^{-2C}$, we arrive at

$$
\frac{1}{n^2} \mathbb{E} \log \left| 1 + \varepsilon_n \frac{Y - X}{t_{i,n}^* - t_{j,n}^*} \right| = \frac{1}{n^2} O \left( \frac{\varepsilon_n}{|t_{i,n}^* - t_{j,n}^*|} \right) = O \left( \frac{1}{n^{2+C}} \right).
$$

The same estimate applies if $i = 1$ and $j \in \{2, \ldots, n\}$, but this time the uniform distribution on the intervals $B_{1,n} = [0, \varepsilon_n]$ and $B_{j,n} = [t_{j,n}^* - \varepsilon_n, t_{j,n}^*]$ is represented by the random variables $\varepsilon_n X$ and $t_{j,n}^* - \varepsilon_n Y$.

Taking together the estimates of Case 1 and Case 2, we arrive at

$$
\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} f_n(x) f_n(y) \log |x - y| \, dx \, dy = \frac{2}{n^2} \sum_{1 \leq i < j \leq n} \log |t_{i,n}^* - t_{j,n}^*| - o(1),
$$

which completes the proof of (11). □

2.2. The Ullman distribution: Maximizer of the functional $\mathcal{J}_p$. In view of Proposition 3 it remains to compute the supremum of the functional $\mathcal{J}_p(\mu)$ over $\mu \in \mathcal{M}_1^p(\mathbb{R}_+) \setminus \{\delta_0\}$. In fact, the maximizer of the same functional over the larger space $\mathcal{M}_1^p(\mathbb{R}) \setminus \{\delta_0\}$ is known to be the so-called Ullman distribution.

Let $0 < p < \infty$. We say that a random variable $U$ which takes values in the interval $[-1, 1]$ has Ullman distribution with parameter $p$, and write $U \sim \mathcal{U}(p)$, if its Lebesgue density is given by

$$
h_p(x) := \frac{p}{\pi} \int_{|x|}^{1} \frac{t^{p-1}}{\sqrt{t^2 - x^2}} \, dt, \quad x \in [-1, 1].
$$

The Ullman distribution appears as the equilibrium distribution for electric charges on the real line in the external field of the form of a constant multiple of $|x|^p$. Namely, for a probability measure $\mu \in \mathcal{M}_1^p(\mathbb{R}) \setminus \{\delta_0\}$ consider the energy functional

$$
\mathcal{E}_p(\mu) := \int_{\mathbb{R}} \int_{\mathbb{R}} \log \frac{1}{|x - y|} \mu(dx) \mu(dy) + 2 \int_{\mathbb{R}} Q_p(x) \mu(dx),
$$

with the external field

$$
Q_p(x) = \frac{\sqrt{\pi} \Gamma \left( \frac{p}{2} \right)}{2 \Gamma \left( \frac{p+1}{2} \right)} |x|^p, \quad x \in \mathbb{R}.
$$
Then the unique minimizer of $\mathcal{E}_p$ is the Ullman distribution on the interval $[-1, 1]$ with density $h_p$ (see, e.g., [16, Theorem 5.1 on p. 240]). As an easy consequence, one can derive the following proposition (see [11, Lemma 3.6]).

**Proposition 4.** Let $p > 0$. The only maximizers of the functional

$$
\mathcal{J}_p(\mu) = \int_{\mathbb{R}} \int_{\mathbb{R}} \log |x - y| \mu(dx) \mu(dy) - \frac{1}{p} \log \int_{\mathbb{R}} |x|^p \mu(dx)
$$

over $\mathcal{M}_1^p(\mathbb{R}) \backslash \{\delta_0\}$ are probability measures with densities $\frac{1}{b} h_p\left(\frac{x}{b}\right)$, $b > 0$, where $h_p$ is the Ullman density (12).

In the following we shall also need two more properties of the Ullman distribution, which can be verified by direct computation; see, e.g., [11, Section 2.5].

**Lemma 5.** Let $p > 0$ and let $U \sim \mathcal{U}(p)$ and $V \sim \mathcal{U}(p)$ be two independent Ullman random variables. Then

$$
\mathbb{E}|U|^p = \int_{-1}^{1} h_p(x)|x|^p \, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)}{2\sqrt{\pi} \Gamma\left(\frac{p+2}{2}\right)}
$$

and

$$
\mathbb{E} \log |U - V| = \int_{-1}^{1} \int_{-1}^{1} h_p(x) h_p(y) \log |x - y| \, dx \, dy = -\log 2 - \frac{1}{2p}.
$$

Finally, we are able to maximize $\mathcal{J}_p(\mu)$ over $\mathcal{M}_1^p(\mathbb{R}_+) \backslash \{\delta_0\}$.

**Proposition 6.** For each $p > 0$ we have

$$
\sup_{\mu \in \mathcal{M}_1^p(\mathbb{R}_+) \backslash \{\delta_0\}} \mathcal{J}_p(\mu) = -2 \log 2 + \frac{1}{p} \log \left(\frac{2\sqrt{\pi} \Gamma(p+1)}{\sqrt{\pi} \Gamma(p+\frac{1}{2})}\right).
$$

**Proof.** We reduce the problem on the half-line to the problem on the whole line by a trick known in the theory of orthogonal polynomials [15, §6]. Let $V$ be a random variable with distribution $\mu \in \mathcal{M}_1^p(\mathbb{R}_+) \backslash \{\delta_0\}$ and denote by $\tilde{V}$ an independent copy of $V$. Then we are interested in maximizing the expression

$$
\mathcal{J}_p(\mu) = \mathbb{E} \log |V - \tilde{V}| - \frac{1}{p} \log \mathbb{E} V^p
$$

over all possible choices for $V \geq 0$ with $\mathbb{E} V^p < \infty$ and $\mathbb{P}[V = 0] < 1$. Independently of $V$, consider a symmetric Rademacher random variable $\varepsilon$ with $\mathbb{P}[\varepsilon = 1] = \mathbb{P}[\varepsilon = -1] = 1/2$ and define $U := \varepsilon \sqrt{V}$. Then $U$ has the same distribution as $-U$, and $U^2 = V$. Let also
\( \tilde{U} \) be an independent copy of \( U \). With this notation, we can write
\[
J_p(\mu) = \mathbb{E} \log |V - \tilde{V}| - \frac{1}{p} \log \mathbb{E} V^p
\]
\[
= \mathbb{E} \log |U^2 - \tilde{U}^2| - \frac{1}{p} \log \mathbb{E} |U|^{2p}
\]
\[
= \mathbb{E} \log |U - \tilde{U}| + \mathbb{E} \log |U + \tilde{U}| - \frac{1}{p} \log \mathbb{E} |U|^{2p}
\]
\[
= 2 \left( \mathbb{E} \log |U - \tilde{U}| - \frac{1}{2p} \log \mathbb{E} |U|^{2p} \right)
\]
\[
= 2 J_{2p}(\mathcal{L}_U),
\]
where in the penultimate line we used that \( U + \tilde{U} \) has the same distribution as \( U - \tilde{U} \), and where \( \mathcal{L}_U \) is the probability distribution of \( U \). Since \( \mathcal{L}_U \in \mathcal{M}_{2p}(\mathbb{R}) \setminus \{\delta_0\} \), Proposition 4 with \( p \) replaced by \( 2p \) yields that
\[
J_p(\mu) = 2 J_{2p}(\mathcal{L}_U) = 2 \int_{\mathbb{R}} \int_{\mathbb{R}} \log |x - y| h_{2p}(x) h_{2p}(y) \, dx \, dy - \frac{1}{p} \log \int_{\mathbb{R}} |x|^{2p} h_{2p}(x) \, dx.
\]
Moreover, if \( U \) would have Lebesgue density \( h_{2p} \), then the previous inequality would turn into an equality. The right-hand side above can be computed explicitly using Lemma 5. In fact,
\[
J_p(\mu) \leq -2 \log 2 - \frac{1}{2p} - \frac{1}{p} \log \left( \frac{\Gamma(p + \frac{1}{2})}{\sqrt{\pi} \Gamma(p + 1)} \right) - 2 \log 2 + \frac{1}{p} \log \left( \frac{2\sqrt{\pi} \Gamma(p + 1)}{\sqrt{\pi} \Gamma(p + \frac{1}{2})} \right)
\]
with equality if \( \mu \) is the distribution of \( \sqrt{\|U\|} \), where \( U \sim \mathcal{U}(2p) \) is Ullman distributed with parameter \( 2p \).

2.3. Proof of Theorem 1. By combining Proposition 3 with Proposition 6 we obtain
\[
\log \Delta(p) = \lim_{n \to \infty} \log \Delta_n(p) = \sup_{\mu \in \mathcal{M}_{2p}(\mathbb{R}) \setminus \{\delta_0\}} J_p(\mu) = -2 \log 2 + \frac{1}{p} \log \left( \frac{2\sqrt{\pi} \Gamma(p + 1)}{\sqrt{\pi} \Gamma(p + \frac{1}{2})} \right).
\]
By exponentiating, we arrive at the required formula for \( \Delta(p) \). \qed

3. Proof of Theorem 2

Let us recall some definitions and provide some additional preliminaries. Let \( X \) be a real \( N \)-dimensional Banach space with unit ball \( B_X \). If we are given a complex Banach space, we ignore the complex structure and consider the space as a real one, so that \( N \) is the dimension over \( \mathbb{R} \). We denote by \( \mathcal{E}_X \) the (unique) maximal volume ellipsoid that is contained in \( B_X \). The volume ratio of \( X \) is then defined as
\[
vr(X) = \left( \frac{\text{Vol}_N(B_X)}{\text{Vol}_N(\mathcal{E}_X)} \right)^{1/N},
\]
where \( \text{Vol}_N(\cdot) \) stands for the usual \( N \)-dimensional Lebesgue measure. Note that if \( K \) is an \( N \)-dimensional symmetric convex body, \( K \) is the unit ball of an \( N \)-dimensional Banach space \( X_K \) and \( vr(X_K) \) coincides with the definition of the volume ratio presented in the
introduction. Let us recall from [21, Section 16] that a Banach space $X$ is said to have enough symmetries if the only operators that commute with every isometry of $X$ are multiples of the identity. If $X$ is $N$-dimensional and has enough symmetries, it is known that $E_X$ is a suitable multiple of the Euclidean unit ball of the same dimension. More precisely,

\[
E_X = \| \text{id} : \ell_2^N \to X \|^{-1} \mathbb{B}_2^N,
\]

where $\ell_2^N$ is the $N$-dimensional Euclidean space with the Euclidean unit ball $\mathbb{B}_2^N$ and $\text{id} : \ell_2^N \to X$ stands for the identity operator from $\ell_2^N$ to $X$ with the standard operator norm $\| \text{id} : \ell_2^N \to X \|$. We also recall from [7] that the Schatten classes $S_p^n(F)$, where $F \in \{\mathbb{R}, \mathbb{C}\}$, is in fact a Banach space with enough symmetries. In what follows, for $F \in \{\mathbb{R}, \mathbb{C}\}$ we denote by $\text{Mat}_n(F)$ the set of all $n \times n$ matrices with entries from $F$.

**Proof of Theorem 2.** According to what has been said above, we need to compute the operator norm

\[
\| \text{id} : S_2^n(F) \to S_p^n(F) \|,
\]

where we used the fact that the Schatten 2-ball $B_2^n(F)$ is just the Euclidean unit ball of the appropriate dimension (namely $n^2$ if $F = \mathbb{R}$ and $2n^2$ if $F = \mathbb{C}$). We first observe that

\[
\| \text{id} : S_2^n(F) \to S_p^n(F) \| = \sup_{\|A\|_{S_2^n} \leq 1} \|A\|_{S_p^n} \leq \begin{cases} 
\frac{1}{p-\frac{1}{2}} : & 1 \leq p < 2 \\
1 : & 2 \leq p \leq \infty,
\end{cases}
\]

since for $1 \leq p < 2$,

\[
\|A\|_{S_p^n} \geq \left( \sum_{i=1}^{n} s_i(A)^{p} \right)^{1/p} \leq n^{\frac{1}{p}-\frac{1}{2}} \left( \sum_{i=1}^{n} s_i(A)^{2} \right)^{1/2} = n^{\frac{1}{p}-\frac{1}{2}} \|A\|_{S_2^n},
\]

by the inequality between the generalized means. On the other hand, let $A_1 \in \text{Mat}_n(F)$ be $n^{-1/2}$ times the $n \times n$ identity matrix, which has singular values $s_1(A_1), \ldots, s_n(A_1)$ all equal to $n^{-1/2}$. This shows that, if $1 \leq p < 2$,

\[
\| \text{id} : S_2^n(F) \to S_p^n(F) \| \geq \left( \sum_{j=1}^{n} n^{-p/2} \right)^{1/p} = n^{\frac{1}{p}-\frac{1}{2}}.
\]

Also, if $2 \leq p \leq \infty$ we take $A_2 = (a_{ij}) \in \text{Mat}_n(F)$ to be the $n \times n$ with all entries equal to 0, except for setting $a_{11} = 1$. In this case $s_1(A_2) = 1$ and $s_2(A_2) = \ldots = s_n(A_2) = 0$ and so

\[
\| \text{id} : S_2^n(F) \to S_p^n(F) \| \geq 1.
\]
Let $\beta = 1$ if $F = \mathbb{R}$ and $\beta = 2$ if $F = \mathbb{C}$. Then, taking together the upper and lower bound and plugging this into (13) and (14), we conclude from (1) and (2) that

$$\nu r(B^n_p(F)) = \begin{cases} \frac{n^{\frac{1}{p} - \frac{1}{2}} \left( \frac{\text{vol}(B^n_p(F))}{\text{vol}(B^n_2(F))} \right)}{n^{\frac{1}{p}}} : 2 \leq p \leq \infty & \quad \text{if } n^{\frac{1}{p} - \frac{1}{2}} \sqrt{\Delta(p/2)} \approx \frac{1}{n^{\frac{1}{p} - \frac{1}{2}} \sqrt{\Delta(1)}} : 1 \leq p < 2 \\
\frac{\text{vol}(B^n_p(F))}{n^{\frac{1}{p}}} : 2 \leq p \leq \infty & \quad \text{if } n^{\frac{1}{p} - \frac{1}{2}} \sqrt{\Delta(p/2)} \approx \frac{1}{n^{\frac{1}{p} - \frac{1}{2}} \sqrt{\Delta(1)}} : 1 \leq p < 2 \\
\frac{n^{\frac{1}{p} - \frac{1}{2}} \sqrt{\Delta(p/2)}}{n^{\frac{1}{p}}} : 2 \leq p \leq \infty & \quad \text{if } n^{\frac{1}{p} - \frac{1}{2}} \sqrt{\Delta(p/2)} \approx \frac{1}{n^{\frac{1}{p} - \frac{1}{2}} \sqrt{\Delta(1)}} : 1 \leq p < 2 \\
\frac{n^{\frac{1}{p} - \frac{1}{2}}}{n^{\frac{1}{p}}} : 1 \leq p < 2 & \quad \text{if } n^{\frac{1}{p} - \frac{1}{2}} \sqrt{\Delta(p/2)} \approx \frac{1}{n^{\frac{1}{p} - \frac{1}{2}} \sqrt{\Delta(1)}} : 1 \leq p < 2 \\
\frac{n^{\frac{1}{p} - \frac{1}{2}} \sqrt{\Delta(p/2)}}{n^{\frac{1}{p}}} : 2 \leq p \leq \infty & \quad \text{if } n^{\frac{1}{p} - \frac{1}{2}} \sqrt{\Delta(p/2)} \approx \frac{1}{n^{\frac{1}{p} - \frac{1}{2}} \sqrt{\Delta(1)}} : 1 \leq p < 2 \\
\end{cases}$$

Applying now Theorem 1 and simplifying the resulting expression completes the proof. \hfill \Box

Remark 7. Theorem 2 implies that $\sup_{n \in \mathbb{N}} \nu r(B^n_p(F))$ is finite for $1 \leq p \leq 2$. In fact, it is possible to give an explicit upper bound on this quantity. Indeed, since the John ellipsoid is just a rescaled Euclidean ball, its volume can be computed exactly. It remains to provide an explicit upper bound on the volume of $B^n_p(F)$. Using the estimate in [17, p. 73], it suffices to provide an explicit upper bound on $\Delta_n(p/2)$. To this end, one can estimate the error terms in the proof of the upper bound of Proposition 3. We refrain from providing the details.

Acknowledgement. JP has been supported by a Visiting International Professor (VIP) Fellowship from the Ruhr University Bochum. ZK and CT were supported by the DFG Scientific Network Cumulants, Concentration and Superconcentration.

References

[1] S. Artstein-Avidan, A. Giannopoulos, and V.D. Milman. Asymptotic Geometric Analysis. Part I, volume 202 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2015.
[2] K. Ball. Volume ratios and a reverse isoperimetric inequality. J. London Math. Soc. (2), 44(2):351–359, 1991.
[3] F. Barthe and D. Cordero-Erausquin. Invariances in variance estimates. Proc. Lond. Math. Soc. (3), 106(1):33–64, 2013.
[4] J. Bourgain, B. Klartag, and V.D. Milman. Symmetrization and isotropic constants of convex bodies. In Geometric aspects of functional analysis, volume 1850 of Lecture Notes in Math., pages 101–115. Springer, Berlin, 2004.
[5] J. Bourgain and V.D. Milman. New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$. Invent. Math., 88:319–340, 1987.
[6] S. Brazitikos, A. Giannopoulos, P. Valettas, and B.-H. Vritsiou. Geometry of Isotropic Convex Bodies, volume 196 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2014.
[7] A. Defant and C. Michels. Norms of tensor product identities. Note Mat., 25(1):129–166, 2005/06.
[8] Y. Gordon and D.R. Lewis. Absolutely summing operators and local unconditional structures. Acta Math., 133:27–48, 1974.
[9] O. Guédon and G. Paouris. Concentration of mass on the Schatten classes. Ann. Inst. H. Poincaré Probab. Statist., 43(1):87–99, 2007.
[10] A. Hinrichs, J. Prochno, and J. Vybíral. Entropy numbers of embeddings of Schatten classes. J. Funct. Anal., 273(10):3241 – 3261, 2017.
[11] Z. Kabluchko, J. Prochno, and C. Thäle. Intersection of unit balls in classical matrix ensembles. Preprint, 2018.
[12] H. König, M. Meyer, and A. Pajor. The isotropy constants of the Schatten classes are bounded. *Math. Ann.*, 312(4):773–783, 1998.

[13] V.S. Koroljuk and Y.V. Borovskich. *Theory of U-Statistics*, volume 273 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994.

[14] J. Radke and B.-H. Vritsiou. On the thin-shell conjecture for the Schatten classes. *ArXiv e-prints*, February 2016.

[15] E.A. Rakhmanov. Asymptotic properties of orthogonal polynomials on the real axis. *Mat. Sb. (N.S.)*, 119(161)(2):163–203, 303, 1982.

[16] E.B. Saff and V. Totik. *Logarithmic potentials with external fields*, volume 316 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1997. Appendix B by Thomas Bloom.

[17] J. Saint Raymond. Le volume des idéaux d’opérateurs classiques. *Studia Math.*, 80(1):63–75, 1984.

[18] S. Szarek. On Kashin’s almost Euclidean orthogonal decomposition of $l_1^n$. *Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys.*, 26(8):691–694, 1978.

[19] S. Szarek and N. Tomczak-Jaegermann. On nearly Euclidean decomposition for some classes of Banach spaces. *Compositio Math.*, 40(3):367–385, 1980.

[20] N. Tomczak-Jaegermann. The moduli of smoothness and convexity and the Rademacher averages of trace classes $S_p$ (1 ≤ p < ∞). *Studia Math.*, 50:163–182, 1974.

[21] N. Tomczak-Jaegermann. *Banach-Mazur Distances and Finite-Dimensional Operator Ideals*, volume 38 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman, Harlow; Wiley, New York, 1989.

Zakhar Kabluchko: Institut für Mathematische Stochastik, Westfälische Wilhelms-Universität Münster, Germany

E-mail address: zakhar.kabluchko@uni-muenster.de

Joscha Prochno: School of Mathematics & Physical Sciences, University of Hull, United Kingdom

E-mail address: j.prochno@hull.ac.uk

Christoph Thäle: Fakultät für Mathematik, Ruhr-Universität Bochum, Germany

E-mail address: christoph.thaele@rub.de