SOLVABLE BASE CHANGE

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Abstract. We determine the image and the fibres for solvable base change.

1. Introduction

The reciprocity conjectures formulated by Langlands give a parametrization of cusp forms associated to $GL_n$ over a global field $K$ by $n$-dimensional complex representations of the Langlands group attached to $K$. The Langlands group, whose existence is yet to be shown, is a vast generalization of the absolute Galois group or the Weil group of $K$, and can be considered in analogy with these latter groups. In this analogy, the theory of base change amounts to restriction of parameters on the Galois theoretic side.

For cyclic extensions of number fields of prime degree, the existence and the characterization of the image and fibres of base change for $GL(2)$ was done by Langlands ([L2]) following earlier work of Saito and Shintani. This was used by Langlands to establish Artin’s conjecture for a class of octahedral two dimensional representations of the absolute Galois group of a number field. The work of Saito, Shintani and Langlands was generalized by Arthur and Clozel to $GL(n)$, for all $n$. In [AC], they proved the existence and characterized the image of the base change transfer for cyclic extensions of number fields of prime degree. However, the proof in the general, cyclic case contained a mistake.

The theorem of Langlands, Arthur and Clozel, gives inductively the existence of the base change transfer corresponding to a solvable extension of number fields for $GL_n$. The problem of characterizing the image and fibres of base change for cyclic extensions of non-prime degree was considered by Lapid and Rogawski in [LN]. This led them to conjecture the non-existence of certain types of cusp forms on $GL(n)$, and they proved this conjecture when $n = 2$. (The general cyclic case has since been settled by other means, see Labesse [Lab].) It was shown in [R] that the conjecture of Lapid and Rogawski allows a characterization of the image and fibres of the base change map for solvable extensions of number fields. In this article, our main aim is to prove the conjecture of Lapid and Rogawski for all $n$.

In order to make this paper more self-contained, we have included here complete proofs of the theorem of Lapid and Rogawski characterising cusp forms on $GL(n)$ whose Galois conjugate by a generator of a cyclic Galois group differs from the original

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form by an Abelian twist (Theorem 2), which was in their paper conditional on Theorem 1; and of the theorem of one of us (Rajan) characterising the image and fiber of base change in a solvable extension (Theorem 3.)

1.1. Main theorem. For a number field $F$, let $\mathbb{A}_F$ denote the adele ring of $F$ and $\mathbb{C}_F$ the group of idele classes of $F$. Given a representation $\pi$ of $GL(n, \mathbb{A}_F)$ and $\sigma$ an automorphism of $F$, define $\sigma \pi$ to be the representation $g \mapsto \pi(\sigma^{-1}(g))$ for $g \in GL(n, \mathbb{A}_F)$. Given an extension $E/F$ of number fields, if $\omega$ is an idele class character of $E$, denote by $\omega_F$ its restriction to the idele class group $C_F$ of $F$. If $\omega$ is a character of $C_F$, define $\omega^E = \omega \circ N_{E/F}$, where $N_{E/F} : C_E \to C_F$ is the norm map on the idele classes of $E$.

Let $E/F$ be a cyclic extension of number fields of degree $d$ and $\sigma$ denote a generator for $\text{Gal}(E/F)$. By $\varepsilon_{E/F}$, we mean an idele class character of $C_F$ corresponding to the extension $E/F$, i.e., a character of $C_F$ of order $d$, vanishing on the subgroup of norms $N_{E/F}(C_E)$ coming from $E$.

The primary aim of this paper is to establish the following conjecture of Lapid and Rogawski ([LR]) for $GL(n)$ for all $n$, proved by them for $GL(2)$:

**Theorem 1** (Statement A, p.178, [LR]). Let $E/F$ be a cyclic extension of number fields of degree $d$ and $\sigma$ denote a generator for $\text{Gal}(E/F)$. Let $\omega$ be an idele class character of $E$, such that its restriction to $C_F \subset C_E$ is $\varepsilon_{E/F}$. Then there does not exist any cuspidal automorphic representation $\pi$ of $GL(n, \mathbb{A}_E)$ such that

$$\sigma \Pi \simeq \Pi \otimes \omega. \quad (1.1)$$

We note that the theorem is obvious if $d$ does not divide $n$, as one sees by considering the restriction to $C_F$ of the central character of $\Pi$.

1.2. Galois conjugate cusp forms up to twisting by a character. From Theorem 1 Lapid and Rogawski derive a structure theorem for cusp forms $\pi$ on $GL(n, \mathbb{A}_E)$, whose Galois conjugate $\sigma(\pi)$ differs from $\pi$ up to twisting by a character $\omega$.

Given a cyclic extension $E/F$ of number fields and an automorphic representation $\pi$ of $GL(n, \mathbb{A}_F)$, denote by $BC_E^F(\pi)$ the base change lift of $\pi$ to an automorphic representation of $GL(n, \mathbb{A}_E)$. For a cusp form $\eta$ on $GL(n, \mathbb{A}_E)$, let $AI_E^F(\eta)$ denote the automorphic representation of $GL(nd, \mathbb{A}_F)$, where $d = \deg(E/F)$, the existence of which was proved for $nd = 2$ in [LL], and for general $n, d$ in [AC]. For a field $F$ let $G_F$ denote the absolute Galois group $\text{Gal}(\bar{F}/F)$ where $\bar{F}$ denotes an algebraic closure of $F$. At the level of Galois representations, base change corresponds to the restriction of representations from $G_F$ to $G_E$, and automorphic induction corresponds to the induction of representations of $G_E$ to $G_F$.

**Theorem 2** (Statement B, p. 179, [LR]). Let $E$ be a number field and $\sigma$ an automorphism of $E$ of order $d$. Let $F$ be the field left fixed by the subgroup of automorphisms of $E$ generated by $\sigma$. Let $\omega$ be an idele class character of $E$ and $\pi$ be a cuspidal
representation of $GL(n, \mathbb{A}_E)$ such that $\sigma(\pi) \simeq \pi \otimes \omega$. Let $K/F$ be the extension corresponding to the character $\omega_F$ of $C_F$, and let $L = KE$. Then

(1) $K \cap E = F$ and $[K : F]$ divides $n$. Let $r = n/[K : F]$ and let $\tau$ be the unique extension of $\sigma$ to $L$ trivial on $K$.
(2) There exists a cuspidal representation $\pi_0$ of $GL(r, \mathbb{A}_F)$ such that
$$\pi = AI_L^E(BC_{L/K}(\pi_0) \otimes \psi),$$
where $\psi$ is a Hecke character of $L$ such that $\tau(\psi)\psi^{-1} = \omega^L$.
(3) Conversely, given $\pi_0$ and $\psi$ as in (ii), the representation
$$\pi = AI_L^E(BC_{L/K}(\pi_0) \otimes \psi)$$
satisfies $\sigma(\pi) \simeq \pi \otimes \omega$. However $\pi$ need not be cuspidal.

1.3. Solvable base change. The following theorem characterizing the image and fibres of the base change transfer for solvable extensions of number fields was established in [R]:

**Theorem 3.** Let $E/F$ be a solvable extension of number fields, and let $\Pi$ be a unitary, cuspidal automorphic representation of $GL_n(\mathbb{A}_E)$.

(1) Suppose $\Pi$ is $Gal(E/F)$-invariant. Then there exists a $Gal(E/F)$-invariant Hecke character $\psi$ of $E$, and a cuspidal automorphic representation $\pi$ of $GL_n(\mathbb{A}_E)$ such that
$$BC_{E/F}(\pi) \simeq \Pi \otimes \psi.$$  
Further $\psi$ is unique up to base change to $E$ of a Hecke character of $F$.

(2) Suppose there exist cuspidal automorphic representations $\pi$, $\pi'$ of $GL_n(\mathbb{A}_F)$ such that,
$$BC_{E/F}(\pi) = BC_{E/F}(\pi') = \Pi.$$  
Then there exists a character $\chi$ of $C_F$ corresponding via class field theory to a character of $Gal(E/F)$, such that
$$\pi' \simeq \pi \otimes \chi.$$  
Moreover if $\chi$ is non-trivial, the representations $\pi$ and $\pi \otimes \chi$ are distinct.

Theorem 3 follows by an inductive argument from Theorem 2.

**Remark.** Suppose $E/F$ is a solvable extension with the property that invariant idele class characters of $E$ descend to $F$. Then any invariant, unitary, cuspidal automorphic representation of $GL_n(\mathbb{A}_E)$ lies in the image of the base change map $BC_{E/F}$. In particular, we recover the classical formulation that invariant, unitary, cuspidal automorphic representations descend if $E/F$ is cyclic.

**Remark.** The motivation for this theorem stems from the following analogous Galois theoretic situation: let $E/F$ be a Galois extension of number fields, and $\rho : G_E \to GL_n(\mathbb{C})$ an irreducible representation of $G_E$. Suppose that $\rho$ is invariant under the action of $G_F$ on the collection of representations of $G_E$. By an application of Schur's lemma, it can be seen that $\rho$ extends as a projective representation, say $\tilde{\eta}$ to $G_F$.
By a theorem of Tate on the vanishing of $H^2(G_F, \mathbb{C}^\times)$ ([SI]), this representation can be lifted to a linear representation $\eta$ of $G_F$. This implies that $\rho \otimes \chi$ descends to a representation of $G_F$ for some character $\chi$ of $G_E$.

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2. Proof of Theorem 2

In this section, assuming the validity of Theorem 1, we give a proof of Theorem 2 modifying the arguments given in [LR].

Let $E/F$ be a cyclic extension of degree $d$, and let $\sigma$ denote a generator for $\text{Gal}(E/F)$. We want to classify the idele class characters $\omega$ of $E$ and cusp forms $\pi$ on $GL(n, \mathbb{A}_E)$ satisfying the condition $\sigma(\pi) \simeq \pi \otimes \omega$. At the level of central characters, this implies $\sigma(\chi_\pi) = \chi_\pi \omega^n$, where $\chi_\pi$ denotes the central character of $\pi$. In particular this implies that the restriction $\omega_F$, of $\omega$ to $C_F$, has finite order dividing $n$. Let $K/F$ be the cyclic extension of $F$ corresponding to $\omega_F$.

We first show that $K \cap E = F$. We prove this by induction on the pair $(d, n)$, assuming the validity of the claim for all extensions $E/F$ of degree less than $d$, or for cusp forms on $GL_m$ for $m < n$. When $n = 1$ or $d = 1$, the assertion is clearly true. So, we now assume that $d > 1$ and $n > 1$.

We initially rule out the following case: $K \subset E$ and $K$ is not equal to $F$. The case $E = K$ is ruled out by Theorem 1. Let $F \subset K \subset E' \subset E$ be an extension of fields, such that the degree of $E/E'$ is $p$, for some rational prime $p$. Let $m = [E' : F]$. The group $\text{Gal}(E/E')$ is generated by $\sigma^m$. We observe,

$$\sigma^m(\pi) \simeq \sigma^{m-1}(\pi \otimes \omega) \simeq \cdots \simeq \pi \otimes \psi,$$

where $\psi = \omega \sigma(\omega) \cdots \sigma^{m-1}(\omega)$. The restriction $\psi_{E'}$ of $\psi$ to $E'$ satisfies,

$$\psi_{E'} = \psi_F \circ N_{E'/F}.$$

Since $K \subset E'$, $\psi_{E'}$ is trivial. Consequently, there exists an idele class character $\eta$ of $C_E$ such that $\eta/\sigma^m(\eta) = \psi$. Hence,

$$\sigma^m(\pi \otimes \eta) \simeq \sigma^m(\pi) \otimes \sigma^m(\eta) \simeq \pi \otimes \psi \otimes \sigma^m(\eta) \simeq \pi \otimes \eta.$$

By the descent theorem for cusp forms invariant with respect to a cyclic extension of prime degree ([AC]), there exists a cuspidal automorphic representation $\rho'$ of $GL_m(\mathbb{A}_{E'})$ such that

$$BC_{E'}^{E}(\rho') = \pi \otimes \eta.$$

Let $\sigma'$ denote the restriction of $\sigma$ to $E'$. Then,

$$BC_{E'}^{E}(\sigma' (\rho')) \simeq \sigma(\pi \otimes \eta) \simeq \pi \otimes \omega \otimes \sigma(\eta) \simeq BC_{E'}^{E}(\rho') \otimes \sigma(\eta)\eta^{-1}\omega.$$
We observe now that $\sigma(\eta)\eta^{-1}\omega$ is invariant under $\text{Gal}(E/E') = \langle \sigma^m \rangle$. From the definition of $\psi$, it follows that $\sigma(\psi)\psi^{-1} = \sigma^m(\omega)\omega^{-1}$. Hence, $\sigma^m(\sigma(\eta)\eta^{-1}\omega) = \sigma(\sigma^m\eta)\sigma^m(\eta)^{-1}\sigma^m(\omega) = \sigma(\eta)\sigma(\psi)^{-1}\psi\eta^{-1}\sigma(\psi)\psi^{-1}\omega = \sigma(\eta)\eta^{-1}\omega$.

Hence there exists an idele class character $\theta$ of $E'$ such that $\theta \circ N_{E/E'} = \sigma(\eta)\eta^{-1}\omega$. We have,

$$BC^E_{E'}(\sigma'(\rho')) \simeq BC^E_{E'}(\rho' \otimes \theta).$$

From the characterization of the fibres of base change for cyclic extensions of prime degree, there exists an idele class character $\chi$ of $E'$ such that

$$\sigma'(\rho') \simeq \rho' \otimes \chi.$$ 

The base change of the character $\chi$ to $E$ is $\sigma(\eta)\eta^{-1}\omega$. Hence $\chi_{K'} \simeq \omega_F$, and $\chi_F$ defines an extension $K'$ of $F$ containing $K$. Since the degree $[E' : F]$ is less than that of $[E : F]$, by the inductive hypothesis $K' \cap E' = F$. This implies that $K = K' \cap E = F$. Hence we have ruled out the case that $K \subset E$ and $K \neq F$.

Let $N = K \cap E$. We want to show $N = F$. Assume now that $\omega_F$ is not trivial. The field $K$ associated to $\omega_F$ is a non-trivial cyclic extension of $F$. By what we have shown above, $K \cap E$ is a proper subfield of $K$. Let $K' \subset K$ be a subfield of $K$ containing $N$, such that its degree over $N$ is a rational prime $p$. We have,

$$\pi \simeq \sigma^d(\pi) \simeq \pi \otimes \omega\sigma(\omega) \cdots \sigma^{d-1}(\omega) \simeq \pi \otimes \omega_F \circ N_{E/F}.$$ 

By class field theory, the character $\omega_F \circ N_{E/F}$ corresponds to the cyclic extension $L = KE$ over $E$. The field $E' = EK'$ is an extension of $E$ of degree $p$ contained inside $L$. The isomorphism $\pi \simeq \pi \otimes \omega_F \circ N_{E/F}$ implies an isomorphism $\pi \simeq \pi \otimes (\omega_F \circ N_{E/F})^k$ for any natural number $k$. Taking $k = [L : E']$, we have $\pi \simeq \pi \otimes \varepsilon_{E'/E}$, where $\varepsilon_{E'/E}$ is an idele class character of $E$ corresponding via class field theory to the extension $E'/E$. By the characterization of automorphic induction ([AC]), there exists a cusp form $\pi'$ on $GL(n/p, A_{E'})$ such that

$$\pi \simeq AI_{E'}^E(\pi').$$

Let $\sigma' \in \text{Gal}(E'/F)$ be an extension of $\sigma$ to $E'$. Then

$$AI_{E'}^E(\sigma'(\pi')) \simeq \sigma(\pi) \simeq \pi \otimes \omega \simeq AI_{E'}^E(\pi' \otimes \omega'),$$

where $\omega' = \omega \circ N_{E'/E}$ is the base change of $\omega$ to $E'$. From the characterization of the fibres of automorphic induction with respect to a cyclic extension of prime degree, it follows that there exists an automorphism $\tau \in \text{Gal}(E'/E)$ such that $\tau \sigma'(\pi) \simeq \pi' \otimes \omega'$. The automorphism $\tau \sigma'$ of $E'$ extends $\sigma$. Renaming $\tau \sigma'$ as $\sigma'$, we see that there exists an automorphism $\sigma'$ of $E'$ extending $\sigma$ such that

$$\sigma'(\pi') \simeq \pi' \otimes \omega'.$$

Let $F'$ be the fixed field of $\sigma'$. Since the fixed field of $\sigma$ is $F$, $E \cap F' = F$. Further $d$ divides the degree of $E'$ over $F'$ as $\sigma'|E = \sigma$, and $[E' : F] = dp$. There are two cases: either $F' = F$ or $F' \neq F$.

Suppose $F' = F$. Then $E'$ is a cyclic extension of $F$ of degree $dp$, and $\omega' = \omega_F$. If $p$ divides $[N : F]$, then $p^2$ divides $[K' : F] = p[N : F]$. It follows that the extension of $F$ defined by $\omega_F$ has a non-trivial intersection with $K' \subset E'$. By the inductive
hypothesis, the extension of $F$ defined by $\omega'_F = \omega^p_F$ should be disjoint with $E'$ over $F$.

This contradiction leads to saying that $p$ is coprime to $[N : F]$. But then $N$ is contained in the cyclic extension $K''$ of $F$ cut out by $\omega^p_F$. By the inductive hypothesis, $K'' \cap E' = F$. It follows that $N = K \cap E = F$.

Before taking up the case $F' \neq F$, we now rule out the case $F' = F$ by showing that $\pi$ is not cuspidal in this case. Since $\omega_F \circ N_{E'/F}$ corresponds to the extension $L$ of $E$ and $E' \subset E$, the character $\omega'_F \circ N_{E'/F} = \omega^p_F \circ N_{E'/F}$ corresponds to the extension $L = KE = K'E'$. The above process can be continued, and the representation $\pi'$ (and hence $\pi$) is automorphically induced from a cuspidal representation $\pi_L$ of $GL(n/r, A_L)$ where $r = [L : E']$. The equation $\sigma'(\pi') \simeq \pi' \otimes \omega'$ implies that $\pi_L$ satisfies the condition,

$$\sigma''(\pi_L) \simeq \pi_L \otimes \omega^L,$$

for some automorphism $\sigma''$ of $L$ extending $\sigma'$ on $E'$, and of order equal to $dp$, the order of $\sigma'$. Now $\omega^L_K$ is the trivial character. The automorphism $\sigma''$ is a non-trivial automorphism of $L$ trivial on $E$, as $\sigma''$ extends $\sigma$ on $E$. But

$$\sigma''(\pi_L) \simeq \pi_L \otimes \omega^L \circ N_{L/K} \simeq \pi_L.$$

This implies that $\pi$ is not cuspidal, contrary to our hypothesis. Hence this rules out the case that $F' = F$.

Hence we are in the situation that $F' \neq F$. Since $d$ divides $[E' : F']$ and $[E' : F] = dp$, $[E' : F'] = d$, we see that $[F' : F] = p$; moreover we saw that $F' \cap E = F$. The extension $E'$ over $F$ is a compositum of the linearly disjoint extensions $E$ and $F'$ over $F$. The character $\omega'_F = \omega_F \circ N_{E'/F}$, corresponds to the extension $KF'$ which contains the compositum $NF' \subset E'$. By the induction hypothesis, $KF' \cap E' = F'$. Hence $NF' = F'$, and this implies that $N = K \cap E = F$.

The compositum of the fields $K'$ and $F'$ is contained inside both $K$ and $E'$. Hence it is disjoint from $E$. If $K' \neq F'$, then the degree of the compositum $[K'F' : F] = p^2$. This contradicts the fact that the degree of $E'$ over $F$ is $dp$. Hence $F' = K'$.

We also obtain that the representation $\pi$ is automorphically induced from a cuspidal representation $\pi'$ of $GL(n/p, A_{E'})$, satisfying

$$\sigma'(\pi') \simeq \pi' \otimes \omega'.$$

The field $E'$ is also the compositum of the fields $E$ and $K'$. The extension of $E$ defined by the character $\omega_{F'} \circ N_{E'/F'}$ is $L$. This can be continued, and we obtain a cuspidal representation $\pi_L$ of $GL(n/r, A_L)$ such that

$$\pi \simeq AI_E^F(\pi_L),$$

where $r = [L : E]$. The representation $\pi_L$ satisfies,

$$\tau(\pi_L) \simeq \pi_L \otimes \omega^L,$$

where $\tau$ is the unique automorphism of $L$ extending $\sigma$ such that the fixed field of $L$ by $\tau$ is equal to $K$. 
Since $\omega^L_K$ is trivial, there exists a Hecke character $\psi$ of $L$ such that $\tau(\psi)\psi^{-1} = \omega^L$.

Then,

$$\tau(\pi_L \otimes \psi^{-1}) \simeq \pi_L \otimes \omega^L \otimes \tau(\psi)^{-1} \simeq \pi_L \otimes \psi^{-1}.$$ 

Hence the cuspidal representation $\pi_L \otimes \psi^{-1}$ is invariant with respect to the cyclic automorphism group $\text{Gal}(L/K)$ generated by $\tau$. To complete the proof of Theorem 2, we now have to establish descent for cyclic extensions.

Let $E/F$ be a cyclic extension of degree $d$ and $\pi$ a cuspidal representation of $GL_n(A_E)$ invariant under the action of $\text{Gal}(E/F) = \langle \sigma \rangle$. Choose an extension $F \subset E' \subset E$ such that $[E : E'] = p$ for some rational prime $p$. By the descent for cyclic extensions of prime degree, there exists a cuspidal representation $\pi'$ of $GL_n(A_{E'})$ which base changes to $\pi$. We need to show that $\pi'$ is left invariant by $\text{Gal}(E'/F)$.

Suppose $\sigma(\pi') \simeq \pi' \otimes \varepsilon$, where $\varepsilon$ is an idele class character of $E'$ which corresponds to the extension $E/E'$ via class field theory. Since $E/F$ is cyclic, $\varepsilon = \eta \circ N_{E'/F}$ for some idele class character $\eta$ of $F$ defining the cyclic extension $E/F$ by class field theory. Then $\varepsilon_F = \eta^{d/p}$. Since $\eta$ is of order $d$, $\varepsilon_F$ defines a non-trivial cyclic extension $F''$ of degree $p$ of $F$ contained inside $E$. Since $E/E'$ is a cyclic extension of degree $p$, this implies that $F'' \subset E'$, thus $E' \cap F'' \neq F$.

This contradicts the first part of Theorem 2 proved above. Hence $\pi'$ is left invariant by $\text{Gal}(E'/F)$, and by induction can be descended to a cuspidal representation of $GL_n(A_F)$.

3. Proof of Theorem 3

In this section, we deduce Theorem 3 from Theorem 2 following the arguments given in [R].

**Lemma 1.** Let $E/F$ be a solvable extension of number fields. Suppose $\pi$ is a cuspidal automorphic representation of $GL_n(A_E)$ such that its base change to $E$ remains cuspidal. Let $\chi$ be a non-trivial idele class character on $F$, such that $\chi \circ N_{E/F}$ is trivial, where $N_{E/F} : C_E \rightarrow C_F$ is the norm map on the idele classes. Then $\pi$ and $\pi \otimes \chi$ are not isomorphic.

**Proof.** The hypothesis implies $\pi \simeq \pi \otimes \chi^k$ for any natural number $k$. Hence we can assume that $\chi$ is of prime order. In this case, $\chi$ cuts out a cyclic extension $E'$ of prime degree contained in $F$. By the characterization of automorphically induced representations ([AC]), it follows that the base change of $\pi$ to $E'$ is not cuspidal. This contradicts the assumption that the base change of $\pi$ to $E$ is cuspidal. \qed

We first prove Part (2) of Theorem 3 characterizing the fibres of the base change lift for solvable extensions of number fields. Suppose $\Pi$ is a cuspidal automorphic representation of $GL_n(A_E)$. Let $\pi$, $\pi'$ be cuspidal automorphic representations of $GL_n(A_F)$ which base change to $\Pi$. We need to show that $\pi' \simeq \pi \otimes \chi$ for some Hecke character $\chi$ on $F$ such that $\chi \circ N_{E/F} = 1$. 


By the results of [AC], the theorem is true for cyclic extensions of prime degree. Let $E \supset E_1 \supset F$ be a tower of Galois extensions of $F$, where $E_1/F$ is of prime degree. Let $BC_{E_1/F}(\pi) = \pi_1$ and $BC_{E_1/F}(\pi') = \pi'_1$. By induction, we assume that the theorem is true for the extension $E/E_1$. We have, $\pi'_1 \simeq \pi_1 \otimes \chi_1$ for some Hecke character $\chi_1$ on $E_1$ such that $\chi_1 \circ N_{E/E_1} = 1$. Let $\sigma$ be a generator of $\text{Gal}(E_1/F)$. We have

$$\pi_1 \otimes \chi_1 \simeq \pi'_1 \simeq \sigma \pi'_1 \simeq \sigma \pi_1 \otimes \sigma \chi_1 \simeq \pi_1 \otimes \sigma \chi_1.$$  

Hence $\pi_1 \simeq \pi_1 \otimes 1^{-\sigma} \chi_1$.

If $\chi_1 \neq 1^{-\sigma} \chi_1$, let $f$ denote the order of $1^{-\sigma} \chi_1$, $p$ a prime dividing $f$, and let

$$\nu = (1^{-\sigma})f/p \chi_1.$$  

$\nu$ is a non-trivial character of $\text{Gal}(E/E_1)$ of order $p$ satisfying,

$$\pi_1 \simeq \pi_1 \otimes \nu.$$  

It follows from the characterization of automorphic induction, that $\pi_1$ is automorphically induced from a cuspidal representation $\pi_\nu$ belonging to the class field $E_\nu$ defined by $\nu$. But $E_\nu \subset E$, and it follows that $\Pi$ is not cuspidal, contrary to our assumption on $\Pi$. Hence we have that $\chi_1$ is invariant by $\text{Gal}(E_1/F)$ and descends to an idele class character $\chi_2$ of $C_F$ such that $\chi_1 = \chi_2 \circ N_{K_1/K_2}$. Then

$$BC_{E_1/F}(\pi_2 \otimes \chi_2) \simeq \pi_1 \otimes \chi_1 \simeq \pi'_1 \simeq BC_{E_1/F}(\pi'_2).$$  

Hence we have a Hecke character $\theta$ corresponding to a character of $\text{Gal}(E_1/F)$, such that

$$\pi'_2 \simeq \pi_2 \otimes \chi_2 \theta;$$  

and $\chi_2 \theta$ defines a character of $\text{Gal}(E/F)$. This proves Part (2) of Theorem 3 as the distinction between $\pi$ and $\pi \otimes \chi$ follows from the properties of automorphic induction.

We now move on to proving Part (1) of Theorem 3. We prove a preliminary lemma, which also proves the uniqueness assertion about $\psi$ in Theorem 3.

**Lemma 2.** Let $E/F$ be a solvable extension, and $\Pi$ be a cuspidal automorphic representation of $GL_n(A_K)$. Suppose $\chi$ is a $\text{Gal}(E/F)$ invariant idele class character of $E$, such that both $\Pi$ and $\Pi \otimes \chi$ are in the image of base change from $F$. Then $\chi$ lies in the image of base change.

**Proof.** The proof is by induction, and is true for cyclic extensions of prime degree. Assume we have $E \supset E_1 \supset F$ with $E/E_1$ prime of degree $p$. Since $\chi$ is invariant, $\chi = \chi_1 \circ N_{E/E_1}$ for some idele class character $\chi_1$ of $C_{E_1}$. Suppose $\pi_1$, $\pi'_1$ are cuspidal automorphic representations of $GL_n(A_{E_1})$ which base change respectively to $\Pi$ and $\Pi \otimes \chi$. Since both $\pi'_1$ and $\pi_1 \otimes \chi_1$ base change to $\Pi \otimes \chi$, by the description of the fibres of base change for a cyclic extension of prime degree, we obtain

$$\pi'_1 \simeq \pi_1 \otimes \chi_1 \eta_1,$$

for some Hecke character $\eta_1$ of $C_{E_1}$ vanishing on $N_{E/E_1}C_E$. 


Assume further, as we may from the hypothesis, that both \( \pi_1 \) and \( \pi'_1 \) lie in the image of base change from \( F \) to \( E_1 \). For any \( \sigma \in \text{Gal}(E_1/F) \),
\[
\pi_1 \otimes \chi_1 \eta_1 \simeq \pi'_1 \simeq \sigma(\pi'_1) \simeq \pi_1 \otimes \sigma(\chi_1 \eta_1).
\]
Hence, \( \pi_1 \simeq \pi_1 \otimes \nu \), where \( \nu = \sigma(\chi_1 \eta_1)(\chi_1 \eta_1)^{-1} \). Since \( \chi \) is \( \text{Gal}(E/F) \)-invariant, we have \( \sigma \chi_1 = \chi_1 \varepsilon_1^i \) and \( \sigma \eta_1 = \varepsilon_1^i \) for some integers \( i, j \), \( \varepsilon_1 \) being a character associated to \( E/E_1 \). Hence \( \nu = \varepsilon_1^i \) for some integer \( l \). Since \( \Pi \) is cuspidal, the cuspidality criterion for automorphic induction implies that \( \nu = 1 \). Hence we get that \( \chi_1 \eta_1 \) is invariant by \( \text{Gal}(E_1/F) \). By induction, \( \chi_1 \eta_1 \) lies in the image of base change from \( F \) to \( E_1 \), and it follows that \( \chi \) also lies in the image of base change from \( F \) to \( E \).

With this lemma, we now proceed to the proof of Part (1) of Theorem \ref{3}. The proof is by induction on the degree of the extension \( E \) over \( F \). By the results of [AC], it is true for extensions of prime degree. We now assume there is a sequence of fields
\[
E \supset E' \supset F,
\]
where \( E'/F \) is a cyclic extension of prime degree \( p \). By the inductive hypothesis, there exists a \( \text{Gal}(E/E') \)-invariant idele class character \( \psi_0 \) of \( E \), and a cuspidal automorphic representation \( \pi' \) of \( GL(n, \mathbf{A}_{E'}) \) such that
\[
\Pi \otimes \psi_0 = \text{BC}_{E/E'}(\pi').
\]
Let \( \tau' \) be a generator of \( \text{Gal}(E'/F) \), and let \( \tau \) be an element of \( \text{Gal}(E/F) \) lifting \( \tau' \). Then
\[
\text{BC}_{E/E'}(\tau'(\pi')) \simeq \tau \Pi \otimes \tau(\psi_0) \simeq \Pi \otimes \tau(\psi_0) \simeq (\Pi \otimes \psi') \otimes \tau(\psi_0)\psi_0^{-1}.
\]
Since \( \text{Gal}(E/E') \) is a normal subgroup of \( \text{Gal}(E/F) \), for any \( \sigma \in \text{Gal}(E/E') \),
\[
\sigma(\tau(\psi_0)) = \tau(\sigma^{-1} \sigma \tau)(\psi_0) = \tau(\psi_0).
\]
Hence \( \tau(\psi_0)\psi_0^{-1} \) is \( \text{Gal}(E/E') \)-invariant. Since both \( \Pi \otimes \psi' \) and \( (\Pi \otimes \psi_0) \otimes \psi_0^{-1+\tau} \) lie in the image of base change from \( E' \) to \( E \), by Lemma \ref{2} there exists an idele class character \( \chi' \) of \( E' \), such that \( \tau(\psi_0)\psi_0^{-1} = \chi' \circ N_{E/E'} \). Hence,
\[
\text{BC}_{E/E'}(\tau'(\pi') \otimes \chi'^{-1}) \simeq \Pi \otimes \psi_0.
\]
By Part (2) of Theorem \ref{3} characterizing the fibres of the base change lift, we conclude that there is an idele class character \( \chi'' \) corresponding via class field theory to a character of \( \text{Gal}(E/E') \), such that
\[
\tau'(\pi') \simeq \pi \otimes \chi' \chi'' = \pi' \otimes \eta',
\]
where \( \eta' = \chi' \chi'' \). Further,
\[
\eta' \circ N_{E/E'} = \chi' \circ N_{E/E'} = \tau(\psi_0)\psi_0^{-1}.
\]
Write the elements of
\[
\text{Gal}(E/F) = \{ \tau^{-i} \sigma \ | \ 0 \leq i < p, \ \sigma \in \text{Gal}(E/E') \}.
\]
We have for \( x \in C_E \),
\[
\eta'_E \circ N_{E/F}(x) = \eta' \left( \prod_{i=0}^{p-1} \prod_{\sigma \in \text{Gal}(E/E')} \tau^{-i} \sigma x \right) = \prod_{i=0}^{p-1} \tau^i \eta' (N_{E/E'}(x))
\]
\[
= \prod_{i=0}^{p-1} \tau^i (\psi_0^{-1}) \psi_0^{-1}
\]
\[
= \tau^p (\psi_0) \psi_0^{-1}.
\]

Since \( \psi_0 \) is Gal\((E/E')\)-invariant and \( \tau^p \in \text{Gal}(E/E') \), it follows that \( \eta'_E \circ N_{E/F} \) is trivial. Hence by Part (1) of Theorem 2 and Equation (3.1), \( \eta'_F \) is trivial.

Let \( \alpha \) be an idele class character of \( E' \), satisfying \( \alpha \tau' (+1) = \eta' \). By equation (3.1),
\[
\tau'(\pi' \otimes \alpha) = \tau'(\pi') \otimes \alpha \eta'^{-1} = \pi' \otimes \alpha.
\]
Hence \( \pi' \otimes \alpha \) is Gal\((E'/F)\)-invariant, and descends to \( F \). Hence we obtain that \( \Pi \otimes \psi_0 \otimes (\alpha \circ N_{E/E'}) \) descends.

To finish the proof, we have to check that \( \psi_0 \otimes (\alpha \circ N_{E/E'}) \) is Gal\((E/F)\)-invariant. For this it is enough to check that \( \psi_0 \otimes (\alpha \circ N_{E/E'}) \) is \( \tau \)-invariant:
\[
\tau(\psi_0 \otimes (\alpha \circ N_{E/E'})) = \tau(\psi_0) \otimes \tau(\alpha) \circ N_{E/E'}
\]
\[
= \tau(\psi_0) \otimes (\alpha \circ N_{E/E'})(\eta' \circ N_{E/E'})^{-1}
\]
\[
= \psi_0 \otimes (\alpha \circ N_{E/E'}),
\]
where the last equality follows from equation (3.2).

4. Trace Formula

We want to prove Theorem 1 ruling out the existence of a cuspidal representation \( \Pi \) of \( GL(n, A_E) \) satisfying Equation 1.1
\[
\sigma \Pi \simeq \Pi \otimes \omega,
\]
where \( \omega \) is an idele class character of \( E \) such that its restriction to \( C_F \) corresponds by Artin reciprocity to a primitive character of the cyclic group Gal\((E/F)\).

If \( \Pi \) satisfies Equation 1.1 it will, for a suitable choice of a function \( \phi \in C_c^\infty(GL(n, A_E)) \), contribute a non-zero term to the trace,
\[
\text{Trace}(I_\theta(R_{\text{disc}} \otimes \omega^{-1})(\phi)). \tag{4.1}
\]
Here \( R_{\text{disc}} \) is the discrete part of the representation of \( GL(n, A_E) \) on
\[
\mathcal{A}_2 := L^2(GL(n, E)A \backslash GL(n, A_E)).
\]
Here \( A \) is \( \mathbb{R}_+^\times \) embedded diagonally into \( GL(n, E_w) \) at all archimedean places \( w \) of \( E \). The operator \( I_\theta \) is given by \( \phi(g) \mapsto \phi(\sigma^{-1}(g)) \), where \( \sigma \) is our chosen generator of \( \Sigma = \text{Gal}(E/F) \).
There is a general formula for the trace in Equation (4.1) due to Kottwitz-Shelstad ([KSI]) and Moeglin-Waldspurger ([MWII]). (In fact this trace must be completed by Arthur’s “discrete terms”, which we will describe presently in our case). The formula is,

\[ T_{\text{disc}}(\phi \times \theta; \omega^{-1}) = \sum_{G' \in \mathcal{E}} \iota(G')ST_{\text{disc}}^G(\phi G'). \tag{4.2} \]

Here \( G' \) runs over the elliptic endoscopic data consisting of triples of the form \((G', G', s)\): these will be reviewed in the next paragraph; \( G' \) is a reductive \( F \)-group and \( \phi G' \) is a function on \( G'({\mathbb{A}}_F) \) associated to \( \phi \).

The so-called ’stable discrete trace formula’ \( ST_{\text{disc}}^G \) will be very simple in our case, as \( G' \) will be a group \( GL(m) \): see paragraph 7. The terms in

\[ T_{\text{disc}}(\phi \times \theta; \omega^{-1}) \]

are as follows:

1. The traces \( \text{Trace}(I_{\phi}(\Pi \otimes \omega^{-1})(\phi)) \) for a cuspidal representation \( \Pi \) of \( GL(n, \mathbb{A}_E) \) such that \( \sigma \Pi \simeq \Pi \otimes \omega \). The operator \( I_{\phi} \) sends an automorphic form \( f(g) \mapsto f(\sigma^{-1}(g)) \), \( g \in GL(n, \mathbb{A}_E) \). The cusp forms occur with multiplicity one and \( I_{\phi} \) is an intertwining operator sending \( \Pi \) to \( \sigma \Pi \).

2. Similar traces where \( \Pi \) belongs to the discrete spectrum (and is not cuspidal) (see [MWII]). This means that \( n = ab \), and that there exists \( \pi_a \), a cuspidal representation of \( GL(a, \mathbb{A}_E) \) such that \( \Pi \) is a quotient of the representation

\[ \rho = \pi_a|.| \frac{1}{t} \boxplus \pi_a|.| \frac{1}{s} \boxplus \cdots \boxplus \pi_a|.| \frac{1}{t} \]

where \( |.| \) denotes the idele norm, seen as a character of \( GL(a, \mathbb{A}_E) \) via the determinant, and the notation \( \boxplus \) denotes, as usual, parabolic induction, here from the parabolic subgroup of \( GL(n) \) of type \((a, \cdots, a)\).

Now if \( \sigma \Pi \simeq \Pi \otimes \omega \), the same is true of \( \rho \). Since the representation \( \pi_a \) is almost tempered, this implies that \( \sigma \pi_a \simeq \pi_a \otimes \omega \). By induction (since \( a < n \)), this is impossible.

3. There are now the discrete terms defined by Arthur, which do not come from the discrete spectrum. We first consider the simplest case. Here \( \Pi = \pi_1 \boxplus \cdots \boxplus \pi_t \), where \( \pi_i \) is a cuspidal representation of \( GL(n_i, \mathbb{A}_E) \), and \( \sum_{i=1}^t n_i = n \). We assume then \( \sigma \Pi \simeq \Pi \otimes \omega \); of course

\[ \sigma \Pi = \sigma \pi_1 \boxplus \cdots \boxplus \sigma \pi_t, \]

and this equivalence implies that there is an element \( s \in W_M \), the Weyl group corresponding to the Levi component \( GL(n_1) \times \cdots GL(n_t) \), such that

\[ \sigma \pi_i \simeq \pi_{s(i)} \omega. \]

We must further assume that \( s \) is ‘regular’, i.e., \( a_M^s = a_G \), where \( a_M \) (resp. \( a_G \)) denote the split component of the centers of \( M \) and \( G \) respectively. This implies that \( M \) is homogeneous \((n = ab)\) and that \( \sigma \pi_i \simeq \pi_{s(i)} \omega \), where \( s \) is a cyclic permutation of order \( b \). The corresponding term is the trace of the product of \( \Pi \omega^{-1}(\phi) \), and of an intertwining operator associated to \( s \times \sigma \), defined by Arthur, acting on the space of \( \Pi \). Its precise form will be irrelevant.
Finally, we can build similar terms with \( \Pi_i \) cuspidal replaced by a residual, discrete spectrum representations as in (2) above.

We note that all the representations of type (1,2) occur with multiplicity 1. Furthermore their Hecke eigenvalues are independent from those of the representations of type (3,4).

In the next paragraph, we compute the right-hand side, i.e., the endoscopic terms.

For more information on the endoscopic stabilisation of the trace formula, and in particular the use of formula (4.2), we refer to [MWII]; in particular sections I.6.4 and X.5.9. Suffice it to say here that if \( \Pi \) is a representation of \( GL(n, A_E) \) occurring in the left-hand side of (4.2), i.e., in the discrete part of Arthur’s trace formula as reviewed above, there will be an endoscopic group \( G' \), and a representation \( \pi' \) of \( G'(A_F) \) such that \( \pi' \) and \( \Pi \) are associated, i.e., the Hecke matrices of \( \Pi \) are deduced at almost all primes from those of \( \pi' \) in a prescribed manner, determined by the endoscopic datum, given in [MWI, Section 6.4]. In our case, there will be a unique datum \( G' \) (or none at all) and the relation between \( \pi' \) and \( \Pi \) will be quite explicit.

5. Endoscopic data

We now consider the right-hand side of Equation (4.2). We must first describe the endoscopic data. We use Waldspurger’s formalism for base change ([W], [BC]).

We consider \( GL(n)/E \) as an \( F \)-group by restriction of scalars and denote it by \( G \). We will sometimes denote by \( G_0 \) the group \( GL(n) \) over \( E \). The generator \( \sigma \) of \( \Sigma = \text{Gal}(E/F) \) acts on \( G \) by \( F \)-automorphisms; as such we denote it by \( \theta \). We fix an isomorphism \( \sigma \mapsto \iota(\sigma) \) between \( \Sigma \) and \( \mathbb{Z}/d\mathbb{Z} \). For \( w \in W_F, \iota(w) \) is then defined by composition.

The connected component of identity of the dual group of \( G \) is \( \hat{G} = GL(n, \mathbb{C})^d = \prod_{v \in \mathbb{Z}/d\mathbb{Z}} GL(n, \mathbb{C}) \); the \( F \)-structure on \( G \) gives an action of \( \text{Gal}(\overline{F}/F) \) on \( \hat{G} \) quotienting through \( \Sigma \):

\[
\sigma(g_1, \ldots, g_d) = (g_{1+\iota(\sigma)}, \ldots, g_{d+\iota(\sigma)}).
\]

(5.1)

Then \( ^L G = \hat{G} \rtimes W_F \), the action of \( W_F \) being so obtained. On the other hand, \( \theta \) defines an automorphism \( \hat{\theta} \) of \( \hat{G} \),

\[
\hat{\theta}(g_1, \ldots, g_d) = (g_2, \ldots, g_d, g_1).
\]

We are given a character \( \omega \) of \( A_E^\times \), which defines via the determinant an abelian character of \( G(A_F) = GL(n, A_E) \). By a result of Langlands [L3] we can see \( \omega \) as an element \( a \in H^1(W_F, Z(\hat{G})) \). Note that \( Z(\hat{G}) = (\mathbb{C}^\times)^d \), the action of \( W_F \) being given by equation (5.1). In general, the element \( a \) is only defined modulo the group,

\[
\ker^1(F, Z(\hat{G})) = \ker \left( H^1(W_F, Z(\hat{G})) \to \bigoplus_v H^1(W_{F_v}, Z(\hat{G})) \right),
\]

where \( v \) ranges over the places of \( F \) (see [KS], [W]).
In our case, however, Shapiro’s lemma implies that
\[ H^1(W_F, Z(\hat{G})) = H^1(W_E, \mathbb{C}^\times), \]
with trivial action of \( W_E \). Hence,
\[ H^1(W_F, Z(\hat{G})) = \text{Hom}_\alpha(W_E, \mathbb{C}^\times) = \text{Hom}_\alpha(C_E, \mathbb{C}^\times), \]
where \( C_E \) is the group of idele classes. Similarly, for a place \( v \) of \( F \),
\[ H^1(W_{F_v}, Z(\hat{G})) = \bigoplus_{w|v} H^1(W_{E_w}, \mathbb{C}^\times). \]
Thus, \( \ker^1(F, Z(\hat{G})) \) is the group of idele class characters that are locally trivial, so
\[ \ker^1(F, Z(\hat{G})) = \{1\}. \]

Now an endoscopic datum for \( (G, \theta, a) \) is a triple, \( G' = (G', \theta', \tilde{s} = s\hat{\theta}) \) subject to the following conditions:

\begin{enumerate}[(E1)]
    
    \item \( G' \) is a quasisplit connected reductive group over \( F \).
    
    \item \( \tilde{s} = s\hat{\theta} \) is a semisimple element in \( \hat{G} \times \Theta \), where \( \Theta = \langle \hat{\theta} \rangle \cong \mathbb{Z}/d\mathbb{Z} \).
    
    \item \( G' \subset L_G \) is a closed subgroup.
    
    \item There exists a split exact sequence,
    \[ 1 \to \hat{G}_{\tilde{s}} \to G' \to W_F \to 1, \]
    where \( \hat{G}_{\tilde{s}} \) is the connected component of the centralizer of \( \tilde{s} \) and \( G' \to W_F \) is induced by the map \( L_G \to W_F \). In particular, \( G' \cap \hat{G} = \hat{G}_{\tilde{s}} \).
    
    \item For \( (g, w) \in G' \),
    \[ s\hat{\theta}(g)w(s)^{-1} = a(w)g, \]
    where \( a(w) \) is a 1-cocycle of \( W_F \) with values in \( Z(\hat{G}) \) and defining \( a \).
\end{enumerate}

We note that any semisimple \( \tilde{s} = s\hat{\theta} \) is conjugate to an element \( \tilde{s} \) such that \( s = (s_0, 1, \cdots, 1) \). In this case, \( \hat{H} = \hat{G}_{\tilde{s}} = \hat{G}_{0, s_0} \) is diagonally embedded in \( \hat{G} \). Here \( \hat{G}_{0, s_0} \) is the centralizer of \( s_0 \) in \( \hat{G}_0 \), which is connected. Thus,
\[ \hat{G}_{\tilde{s}} = \{(h, \cdots, h) \mid h \in \hat{H}\} = \{\text{diag}(h) \mid h \in \hat{H}\}, \]
where \( \text{diag} : GL(n, \mathbb{C}) \to GL(n, \mathbb{C})^d \) is the diagonal map. We look for
\[ \xi : G' \to L_G \tag{5.2} \]
where \( G' \) admits an exact sequence \((E4)\). Thus for \( h \in \hat{H} \),
\[ \xi : (h, 1) \mapsto (\text{diag}(h), 1), \]
while for \( w \in W_F \),
\[ \xi : (1, w) \mapsto (n(w), w) = (n(w), 1)(1, w). \]
Here we have chosen a splitting \( n : W_F \to G' \) for \( G' \). Let us denote by \( h \mapsto ^w h \) the action of \( W_F \) on \( \hat{H} \) coming from \((E4)\). Then
\[ (^w h, 1) = (n(w))(h, 1)(n, w)^{-1}, \]
where we are writing for short \( n = n(w) \) and \( (h, 1) = (\text{diag}(h), 1) \). Hence
\[
(wh, 1) = (n, 1)(1, w)(h, 1)(1, w)^{-1}(n, 1)^{-1} = (n, 1)(h, 1)(n, 1)^{-1},
\]
since \( h \), being diagonal in \( \hat{G} \), is invariant by the action \((5.1)\) of \( W_F \). Write \( n = n(w) = (n_1, \cdots, n_d) \), so
\[
n \text{diag}(h)n^{-1} = (n_1h_1^{-1}, \cdots, n_dh_d^{-1}) = \text{diag}(h'),
\]
for some \( h' \in \hat{H} \).

We now assume that \( s_0 = (s_1, \cdots, s_d) \) is given by diagonal scalar matrices \( s_i \) of degree \( b_i \) with distinct eigenvalues \( t_i \). Then
\[
\hat{H} = \prod_{i=1}^a GL(b_i) \subset GL(n).
\]
Write \( a = a_1 + \cdots + a_r \), \( (a_k \geq 1) \), with
\[
b_1 = b_2 = \cdots = b_{a_1} < b_{a_1+1} = \cdots = b_{a_2} < \cdots
\]
Since \( n_i \) normalizes \( \hat{H} \),
\[
n_i \in \prod_{k=1}^r GL(b_k)^{a_k} \ltimes S_{a_k}
\]
with obvious notation. We choose explicitly as representatives of the Weyl group \( S_{a_k} \) the obvious block matrices with blocks of size \( b_k \) equal to either 0 or 1.

Write \( W = \prod_{k=1}^r S_{a_k} \), so that the normaliser of \( \hat{H} \) is \( \hat{H}W \). By Equation \((5.3)\),
\[
\text{Ad}(n_i)h \equiv \text{Ad}(n_j)h \quad \forall i, j.
\]
Hence \( n_i = h_i\tau \), where \( \tau \in W \) is independent of \( i \) and \( h_i \in \hat{H} \); moreover
\[
\text{Ad}(h_i)h \equiv \text{Ad}(h_j)h \quad \forall i, j.
\]
Thus \( h_i = z_{ij}h_j \) with \( z_{ij} \in Z(\hat{H}) = (\mathbb{C}^\times)^r \). Hence we can write
\[
n(w) = (z_i(w)h(w)\tau(w))_i, \quad \text{where} \quad z_i(w) = (z_{i1}(w), \cdots, z_{ir}(w)) \in (\mathbb{C}^\times)^r. \quad (5.4)
\]
In the stabilisation of the trace formula, we are only interested in the elliptic endoscopic data, i.e., those such that the neutral component of \( Z(\hat{H})^{W_F} \) and of \( Z(\hat{H})^{W_F,\theta} \) coincide. The second group is equal to \( \mathbb{C}^\times \) embedded diagonally in \( GL(n, \mathbb{C})^d \). We have \( Z(\hat{H}) = \prod_{k=1}^r (\mathbb{C}^\times)^{a_k} \) and \( n(w) \) acts by \( \tau(w) \in \prod_k S_{a_k} \). Thus \( Z(\hat{H})^{W_F} \) is the set of fixed points of the \( \tau(w), w \in W_F \). In particular, it contains the product \( \prod_k \mathbb{C}^\times \), embedded diagonally in \( \prod_k GL(b_k) \).

If \( H \) is elliptic, we see that \( r = 1 \), so \( \hat{H} = GL(b)^k \) is homogeneous. Furthermore, \( W_F \) acts on \( (\mathbb{C}^\times)^a \) via \( \tau(w) \in S_a \). The image of \( W_F \) by \( w \mapsto \tau(w) \) must therefore be a transitive subgroup of \( S_a \).

So far we have shown that \( \hat{H} = GL(b)^k \), and
\[
n(w) = (n_i(w)), \quad \text{where} \quad n_i(w) = z_i(w)h(w)\tau(w) \quad (5.5)
\]
with \( z_i(w) \in Z(\hat{H}) \simeq (\mathbb{C}^\times)^r \).

The group \( G' = \hat{H} \rtimes W_F \) is defined as a semi-direct product, by the conjugation action of \( n(w) \) on \( \hat{H} \). Dually, \( H \times_F \bar{F} \simeq GL(b)^{\times}/\bar{F} \), where the rational structure will be described presently. In particular, the derived subgroup of \( H \) is simply connected. This implies (see [KS, Section 2.2]), that \( \mathcal{H}' \) is an \( L \)-group, i.e., that for a suitable choice of section the action of \( W_F \) on \( \hat{H} \) preserves a Borel subgroup and a splitting.

We have seen that \( n(w) = z_i(w)h(w)\tau(w) \) acts by conjugation on \( \hat{H} \). If \( h(w) = 1 \), this is easily seen to preserve a splitting. Conversely, if \( n(w) \) preserves a splitting, one checks that the \( h(w) \in \hat{H} \) must act trivially by conjugation, so we may assume \( h(w) \equiv 1 \). With this section (if it is one), \( \mathcal{H}' \cong L H \) is naturally embedded in \( L G \), whence a homomorphism of \( L \)-groups,

\[
\xi_1: L H \rightarrow L G.
\]

The contribution of this endoscopic datum will be deduced from \( \xi_1 \).

Since now \( n(w) = (z_i(w)\tau(w))_i \), we must still check the cocycle relation

\[
n(ww') = n(w).wn(w'),
\]

where the action of \( w \) is given by the structure of \( L G \). If \( w \in W_F \) is sent to \( \sigma^k \in \Sigma \), with \( k = \iota(w) \), this says that

\[
n_i(ww') = n_i(w)n_{i+k}(w').
\]

Write \( z_i(w) = (z_{i,\alpha}(w)) \) according to the decomposition \( Z(\hat{H}) = (\mathbb{C}^\times)^r \) for \( \alpha = 1, \cdots, a \). Thus

\[
\begin{align*}
    z_i(ww')\tau(ww') &= z_i(w)\tau(w)z_{i+k}(w)\tau(w') \\
    \text{i.e., } z_i(ww') &= z_i(w)\tau z_{i+k}(w')\tau^{-1},
\end{align*}
\]

with \( \tau = \tau(w) \in \mathcal{S}_a \). Now \( \tau((z_{\alpha})) = z_{\tau^{-1}\alpha} \), so the cocycle relation reads:

\[
\begin{align*}
    z_{i,\alpha}(ww') &= z_{i,\alpha}(w)z_{i+k,\tau^{-1}\alpha}(ww'),
\end{align*}
\]

where \( k = \iota(w) \).

6. ENDOSCOPY, WITH CHARACTER

We now have to introduce the character \( \omega \) in the endoscopic computations. This intervenes through formula \((E5)\) in the definition of endoscopic datum. We want to make the element \( a \in H^1(W_F, Z(\hat{G})) \), or rather a representative \( a \in Z^1(W_F, Z(\hat{G})) \), explicit. We write for \( w \in W_F \):

\[
a(w) = (a_i(w)), \quad a_i(w) \in \mathbb{C}^\times.
\]

Since \( \omega \) is a character of \( C_E \), it can be identified with an element of \( H^1(W_E, \mathbb{C}^\times) \). We now need Shapiro's lemma. For its explicit description, we follow Langlands [L1] (see also Serre [S]). Recall that

\[
W_E/W_F \simeq \text{Gal}(E/F) \simeq \mathbb{Z}/d\mathbb{Z},
\]
the isomorphism sending the generator \( \sigma \) of \( \Sigma \) to 1. We choose a representative \( \sigma \in W_F \) of this generator, which we also denote by \( \sigma \). Now \( \{ \sigma, \sigma^2, \ldots, \sigma^d \} \) are representatives of \( W_E \setminus W_F \). Note that \( \sigma^d \neq 1 \) as follows from class field theory, cf. (7.5), (7.6) below.

For any \( w \in W_F \),
\[
\sigma^i(w) = \delta_i(w)\sigma^j,
\]
where \( j \equiv i + k(\text{mod}) \) if \( k = \iota(w) \) (see Equation (5.1)) and \( \delta_i(w) \in W_E \). We set \( a_i(w) = \omega(\delta_i(w)) \). For \( w \in W_E \), \( \sigma^i w = \delta_i(w)\sigma^j \), so
\[
a_i(w) = (\omega(\sigma^i w\sigma^{-i})) = \omega(\sigma^i w) := \omega_i(w),
\]
(6.1) since the lifting \( \sigma \) acts by conjugation, on the abelianized Weil group \( C_E \) of \( E \), through its image in \( \text{Gal}(E/F) \).

Consider our chosen lift \( \sigma \in W_F \). Then
\[
\sigma^{i+1} = \delta_{i+1}(\sigma)\sigma^{i+1},
\]
(6.2)
where \( \delta_i(\sigma) = 1 \) for \( (i = 1, \ldots, d-1) \), \( \delta_d(\sigma) = \sigma^d \in W_E \).

This defines completely \( a(w) \). Now,
\[
\phi : W_F \to Z(\hat{G}) \rtimes W_F, \quad w \mapsto (a_i(w), w),
\]
defines an \( L \)-parameter for the \( L \)-group of \( \text{Res}_{E/F}GL(1) \), corresponding to the character \( \omega \).

We now have to introduce the condition
\[
\omega|_{A^\times_F} = \varepsilon_{E/F}.
\]
(6.4)
In cohomological terms, this is given by the corestriction,
\[
\text{Cor} : H^1(W_E, \mathbb{C}^\times) \to H^1(W_F, \mathbb{C}^\times),
\]
dual to the transfer map \( W_F/W_F^{\text{der}} \to W_E/W_E^{\text{der}} \). Explicitly, this is given as in ([S, Chapter VII, Section 8]) by
\[
w \mapsto \prod_i \delta_i(w).
\]
Our condition is therefore, for \( w \in W_F \):
\[
\prod_i a_i(w) = \prod_i \omega(\delta_i(w)) = \varepsilon_{E/F}(w),
\]
(6.5)
where both sides are seen as characters of the Weil groups (recall that \( \varepsilon_{E/F} \) is here seen as a character of the Galois group). For \( w \in W_E \), \( \varepsilon_{E/F}(w) = 1 \) and the left hand side is
\[
\prod_i \sigma^i\omega(w) = \omega(N_{E/F}z),
\]
where \( z \in C_E \) is the image of \( w \). Since \( \omega \) restricts trivially to \( N_{E/F}(C_E) \subset C_F \), the relation is satisfied.
On the other hand, \( \delta_i(\sigma) = (1, \cdots, 1, \sigma^d) \), thus equation (6.5) is equivalent to
\[
a(\sigma) = (1, \cdots, \zeta),
\]
where \( \zeta = \varepsilon_{E/F}(\sigma) \) is a primitive root of unity of order \( d \). (In particular, \( \omega(\sigma^d) = \zeta \) when \( \sigma^d \) is seen as an element of \( W_{E/F} \), hence of \( C_{E} \)).

Now consider the condition \((E5)\) on the endoscopic group:
\[
s\theta(g)w(s)^{-1} = a(w)g, \quad (g, w) \in \mathcal{G}'.
\]
For \( w = 1 \), this is the condition defining \( \hat{H} = \hat{G}' \). Consider the image \((n(w), w)\) of the section \( n \) for \( w \in W_F \). We have
\[
n(w) = (z_i(w)\tau(w))_i.
\]
Write for simplicity \( \tau, z_i, a_i \) for \( \tau(w), z_i(w), a_i(w) \) respectively. If \( k = \iota(w) \),
\[
w(s) = w(s_0, 1, \cdots, 1) = (1, \cdots, s_0, \cdots, 1),
\]
where \( s_0 \) occurs at the place \( l = d + 1 - k \) with the convention that \( l = 1 \) if \( k = 0 \). The equation (6.7) reads,
\[
(s_0, 1, \cdots, 1)(z_2\tau, \cdots, z_1\tau)(1, \cdots, 1, s_0^{-1}, 1) = (a_i)(z_i\tau).
\]
Thus,
\[
s_0z_2\tau = a_1z_1\tau, \quad z_3\tau = a_2z_2\tau, \quad \cdots \quad z_{l+1}\tau s_0^{-1} = a_lz_l\tau, \quad \cdots \quad z_1\tau = a_dz_d\tau.
\]
(If \( k = 0 \), \( s_0z_2\tau s_0^{-1} = a_1z_1\tau \) etc.; if \( k = 1 \), \( l = d \) and the last equation is \( z_1\tau s_0^{-1} = a_dz_d\tau \).) Write \( s_0^{-1} = \tau s_0^{-1} \tau^{-1} \). Then equation (6.9) is equivalent to,
\[
\begin{align*}
s_0z_2 &= a_1z_1 \\
z_3 &= a_2z_2 \\
\vdots \\
z_{l+1}^{-1}s_0^{-1} &= a_lz_l \\
\vdots \\
z_1 &= a_dz_d.
\end{align*}
\]
Note that all these elements are contained in \( Z(\hat{H}) \), hence commute. (For \( k = 0 \), the first line is \( s_0z_0^{-1} = a_1z_1 \); for \( k = 1 \), the last line is \( z_1^{-1}s_0^{-1} = a_dz_d \).) Taking the product, we see that
\[
s_0^{-1} = \prod a_i = \varepsilon_{E/F}(w),
\]
by equation (6.5), so \( s_0 = \varepsilon_{E/F}(w)^{-1}s_0 \).

Now \( s_0 \in Z(\hat{H}) = (C^\times)^a \), and \( \tau \in \mathfrak{S}_a \). Write \( s_0 = (s_{0,\alpha}) \), \( \alpha = 1, \cdots, a \). Thus \( \tau s_0 = (s_{0,\tau^{-1}\alpha}) \), whence
\[
s_{0,\tau\alpha} = \varepsilon_{E/F}(w)s_{0,\alpha}.
\]
Assume \( w \in W_F \) is sent to the chosen generator \( \sigma \in \Sigma \), so \( \varepsilon_{E/F}(w) = \zeta \). Recall that \( s_0 \in Z(\hat{H}) \) is given by block-diagonal matrices \( s_{0,\alpha} \) of the size \( b \) with distinct
eigenvalues. Equation (6.11) now implies that the $s_{0,\alpha}$ can be partitioned into $a' = a/d$ subsets of the form
\[(s_1, \zeta s_1, \cdots, \zeta^{d-1}s_1; s_2, \zeta s_2, \cdots, \zeta^{d-1}s_2; \cdots); \quad (6.12)\]
the entries being block-diagonal, we assimilate them to scalars. In particular $d | a$, so $d | n$. The scalars $s_j$ ($j = 1, \cdots, a'$) verify $s_j \not= \mu s_j'$ for any $\mu \in \mu_d(\mathbb{C})$. Equation (6.11) now uniquely determines $\tau_1 = \tau(\sigma)$: it is a product of $a'$ $d$-cycles.

Consider now an arbitrary element $w \in W_F$. If $w \in W_E$, $\varepsilon_{E/F}(w) = 1$ and equation (6.11) implies that $\tau = 1$, the eigenvalues being distinct. Thus $W_F$ acts via $\Sigma = W_F/W_E$ and $\varepsilon(w) = \tau_1^{(w)}$. The image of $W_F$ is therefore a cyclic subgroup of $\mathfrak{S}_a$, of order $d$, preserving the strings of length $d$ in equation (6.12). The ellipticity of $H$ now implies that this action is transitive, so $a = d$.

Since we have determined $n(w)$ up to central elements in $\hat{H}$, we have now computed the $F$-group $H$. Indeed, $\hat{H} = GL(b) \times \cdots \times GL(b)$ ($a = d$ factors) and $W_F$ acts via $\Sigma$, cyclically permuting the factors. This implies that $H$ is isomorphic to $\text{Res}_{E/F}(GL(b)/E)$. This embedding into $\hat{G}$ is given on $\hat{H}$ by
\[(h_1, \cdots, h_d) \mapsto \text{diag}(h_1 \oplus \cdots \oplus h_d) \in GL(n, \mathbb{C})^d.\]
We can now summarise the main result of this section:

**Proposition 4.**

1. If $d \not| n$, there exists no elliptic endoscopic group for $(\theta, \omega)$.
2. If $n = db$, there exists (at least) one endoscopic datum for $(\theta, \omega)$ given by the foregoing construction.

To complete the proof of the proposition, we still have to show that we can choose the $z_i(w)$ so as to satisfy equations (E5) and (6.8). We will obtain in fact a more precise result.

The permutation $\tau$ associated to $w \in W_F$ is a cyclic permutation on the indices $\alpha$. We have $\tau(z_\alpha)\tau^{-1} = (z_{\tau^{-1}\alpha})$. We now assume that for $i(w) = k$, $\tau^{-1}(\alpha) = \alpha - k$. The relation (6.6) now reads
\[z_{i,\alpha}(ww') = z_{i,\alpha}(w)z_{i+k,\alpha-k}(w'). \quad (6.13)\]
We consider all indices as elements of $\mathbb{Z}/d\mathbb{Z}$. Now fix $\beta \pmod{d}$. Now equation (6.13) yields for $i + \alpha = \beta$:
\[z_{i,\beta-i}(ww') = z_{i,\beta-i}(w)z_{i+k,\beta-i-k}(w').\]
Set $\xi_i^\beta(w) = z_{i,\beta-i}(w)$: we now have
\[\xi_i^\beta(ww') = \xi_i^\beta(w)\xi_{i+k}^\beta(w').\]
This means that $\xi_i^\beta = (\xi_i^\beta)_i$ is a 1-cocycle of $W_F$ for the action of $W_F$ on $(\mathbb{C}^\times)^d$. However $H^1(W_F, (\mathbb{C}^\times)^d)$, for $W_F$ acting by its action on the dual group of $\text{Res}_{E/F}(GL(1))$ is equal to $H^1(W_E, C^\times) = \text{Hom}(C_E, C^\times)$ by Shapiro’s lemma. Thus we see that each character $\eta_\beta$ of $C_E$ defines such a cocycle, by $\xi_i^\beta(w) = \eta_\beta(\delta_i(w))$. We can then set
\[z_{i,\alpha}(w) = \eta_{i+\alpha}(\delta_i(w)), \quad (6.14)\]
and we see that a section (and therefore a subgroup) is defined by the choice of \( \eta_\beta \).

We still have to fulfill the condition given by equation (6.10). Assume first that \( w \in W_E \). Then \( \tau = 1 \), and the condition is simply
\[
\zeta_{i+1}(w) = a_i(w)\zeta_i(w).
\]
Now \( \zeta_i = (\zeta_{i,\alpha}) \) with \((\zeta_{i,\alpha}) = (\eta_{i+\alpha}(w))_i = (\eta_{i+\alpha}(\sigma^jw))_i \). Here \( \sigma^j \) is the automorphism of \( C_E \) obtained by quotienting the automorphism \( w \mapsto \sigma^jw\sigma^{-j} \) of \( W_E \). So the condition is,
\[
\eta_{i+1+\alpha}(\sigma^{i+1}w) = \omega(\sigma^jw)\eta_{i+\alpha}(\sigma^jw).
\]
(\( w \) being seen as an element of \( C_E \cdot \cdots \)), i.e.,
\[
\eta_{\alpha+1} \circ \sigma = \omega\eta_{\alpha}, \quad \alpha = 1, \cdots, d.
\]
We can now write the condition of equation (6.10) for \( w = \sigma \). Recall from equation (6.3) that \((\delta_i(\sigma)) = (1, \cdots, 1, \sigma^d)\) with \( \sigma^d \in W_E \). We still have \( \zeta_{i,\alpha}(w) = \eta_{i+\alpha}(\delta_i(w)) \), whence
\[
\zeta_{i,\alpha}(\sigma) = 1 \quad 1 \leq i \leq d - 1
\]
\[
\zeta_{d,\alpha}(\sigma) = \eta_{\alpha}(\sigma^d).
\]
Moreover, as before
\[
\eta_{\alpha+1} \circ \sigma(\sigma^d) = \omega(\sigma^d)\eta_{\alpha}(\sigma^d),
\]
and \( \sigma^d\sigma^{-1} = \sigma^d, \ \omega(\sigma^d) = \zeta \), whence
\[
\zeta_{d,\alpha}(\sigma) = (\eta, \zeta\eta, \cdots, \zeta^{d-1}\eta)
\]
where \( \eta = \eta_1(\sigma^d) \). Write \( \zeta_i = (\zeta_{i,\alpha}) = (\zeta_{i,\alpha}(\sigma)) \). Then equation (6.10) for \( \sigma \) reads:
\[
s_0 \zeta_2 = \zeta_1
\]
\[
\zeta_3 = \zeta_2
\]
\[
\vdots
\]
\[
\zeta_d = \zeta_{d-1}
\]
\[
\tau s_0^{-1} \zeta_1 = \zeta \zeta_d
\]
and is obviously not satisfied. Recall that for \( k = k(\sigma) = 1 \) we have set \((\tau s_0)_{\alpha} = s_0, \tau^{-1}\alpha = s_0,\alpha-1 \) and, cf equation (6.11), \( s_0,\alpha = s_0,\alpha+1 = \zeta s_0,\alpha \). We can choose
\[
s_0 = (1, \zeta, \cdots, \zeta^{d-1}).
\]
Substituting \( s_0 \tau s_0^{-1} = \zeta \) in equation (6.17), we get
\[
s_0 \zeta_2 = \zeta_1
\]
\[
\zeta_3 = \zeta_2
\]
\[
\vdots
\]
\[
\zeta_d = \zeta_{d-1}
\]
\[
s_0^{-1} \zeta_1 = \zeta_d.
\]
Now we can replace the $z_i$ by cohomologous elements for the action of $W_F$ (via $\Sigma$) giving the cocycle relation (5.6), i.e.,

$$w(z_{i\alpha}) = z_{i+k,\alpha-k}.$$  

A coboundary is given for $w \in W_F$, $k = \iota(w)$ by

$$\zeta_{i\alpha}(w) = v_{i\alpha}v_{i+k,\alpha-k}^{-1}.$$  

In particular for $w = \sigma$:

$$\zeta_{i\alpha}(\sigma) = v_{i\alpha}v_{i+1,\alpha-1}^{-1}. $$

As we did for $w \in W_E$ we can write $\zeta_{i\alpha} = \zeta_i^\beta$ with $\beta = i + \alpha$, $\zeta_i^\beta = v_i^\beta(v_{i+1}^\beta)^{-1} = u_i^\beta$. The $u_i^\beta$ must then satisfy the condition,

$$\prod_{i+\alpha = \beta} u_i^\beta = 1. \quad (6.19)$$

Our equations then become, with $\alpha = 1, \ldots, d$:

$$s_{0,\alpha}u_2^{2+\alpha} = u_1^{1+\alpha}$$
$$u_3^{3+\alpha} = u_2^{2+\alpha}$$

and so on.

$$u_d^{d-1+\alpha} = u_{d-2}^{d-2+\alpha}$$
$$z_{d,\alpha}u_d^{d+\alpha} = u_{d-1}^{d-1+\alpha}$$
$$s_{0,\alpha}u_1^{1+\alpha} = z_{d,\alpha}u_d^{\alpha}.$$

Write $u = u_2 = (u^\alpha)$. Then $d-3$ lines yield:

$$u_{i+1}^{i+\alpha} = u_i^{\alpha}, \quad i = 2, \ldots, d-2,$$

so

$$u_i^{\alpha} = u^{\alpha-2+i}, \quad i = 2, \ldots, d-1. \quad (6.21)$$

Thus

$$s_{0,\alpha}u_2^{2+\alpha} = u_1^{1+\alpha}$$
$$z_{d,\alpha}u_d^{d+\alpha} = u^{\alpha+2}$$
$$s_{0,\alpha}u_1^{1+\alpha} = z_{d,\alpha}u_d^{\alpha},$$

equations obviously compatible. However we must choose the $u_i^\beta$ verifying equation (6.19). We determine $u_1$ and $u_d$ by the first equations. Thus,

$$u_1^{\alpha} = s_{0,\alpha-1}u^{\alpha+1}, \quad u_d^{\alpha} = z_{d,\alpha}^{d-1}u^{\alpha+2}.$$

The product is then

$$\prod_{i=1}^{d-1} u_i^{\beta-i} = s_{0,\beta-2}z_{d,\beta}^{-1}u_1^{\beta} \prod_{i=2}^{d-1} u_i^{\beta+2-2-i}u_i^{\beta+2}$$

$$= s_{0,\beta-2}z_{d,\beta}^{-1} \prod_{j} u^{\beta+j}.$$
where the product is taken over $2\mathbb{Z}/d\mathbb{Z} \subset \mathbb{Z}/d\mathbb{Z}$. However,

$$s_{0,\beta-2}z_{\beta}^{-1} = \zeta^{\beta-3}\zeta^{-1} \eta = \zeta^{-2}\eta$$

is constant, cf. (6.16). It suffices therefore to impose on $u = u_2$ the two conditions (at most)

$$\prod_{j=\beta} u_{\beta+j} = \zeta^2 \eta^{-1}$$

for $\beta \in \mathbb{Z}/d\mathbb{Z}$, the congruence being modulo $2\mathbb{Z}/d\mathbb{Z}$.

We can now define the new cocycle in $Z^1(W_F, Z(\hat{H}))$ by multiplying the previous map by the coboundary just obtained. It defines a new section, which verifies the defining condition (6.10), obviously for $w \in W_E$, for all powers of $\sigma$, and therefore for $w \in W_F$. This proves the second part of Proposition 4, and moreover it exhibits an explicit section. We have moreover

**Lemma 3.** The new section is conjugate in $\hat{G}$ to the section given by the $z_{i,\alpha}$. In particular they define (up to conjugation in $\hat{G}$) the same embedding $^LH \to ^LG$.

This is clear: if we define by $s_1$ the previous section and $s_2$ the new one, we have by construction

$$s_1(w) = z^1(w)\tau, \quad s_2(w) = z^2(w)\tau,$$

with $\tau = \tau(w)$, and, with $k = \iota(w)$:

$$z^2(w) = z^1(w)z(\tau z^{-1})_{i+k} \tau^{-1},$$

with $z = (z_i) \in Z(\hat{H})^d$. Thus,

$$s_2(w) = z^1(w)z\tau(z_{i+k})^{-1}$$

since $\tau$ is diagonal, so

$$s_2(w) = z(z^1(w)\tau)w(z)^{-1}.$$

Recall that $\xi_1$ was defined by

$$\xi_1 : h \mapsto \text{diag}(h) \quad (h \in \hat{H})$$

$$w \mapsto (s_1(w),w) = (z^1(w)\tau,w) \quad (w \in W),$$

so $(h,w) \mapsto (hz^1(w)\tau,w)$. Similarly we define $\xi_2$ by

$$\xi_2 : h \mapsto \text{diag}(h)$$

$$w \mapsto z(z^1(w)\tau,w)z^{-1}.$$

Since $z \in Z(\hat{H})^d$, it centralizes $\text{diag}(h)$, so $\xi_2$ is conjugate to $\xi_1$. In particular, they will have the same effect on functions of Hecke matrices, the data for endoscopy.

Recall that two endoscopic data,

$$G'_1 = (G'_1, G'_1, \tilde{s}_1) \quad \text{and} \quad G'_2 = (G'_2, G'_2, \tilde{s}_2)$$

1 We could avoid this verification by using Lemma 4.5 of Borel[3]. It will be clearer explicitly to exhibit the conjugation.
are equivalent if there exists \( g \in \hat{G} \) such that
\[
gG_i'g^{-1} = G_2', \quad g\tilde{s}_1'g^{-1} = z\tilde{s}_2
\]
for an element \( z \in Z(\hat{G}) \). (Recall that \( \tilde{s}_i = (s_i, \theta) \) with \( s_i \in \hat{G} \).

**Proposition 5** (Waldspurger). Assume that \( d \mid n \). Then there exists only one equivalence class of endoscopic data for \((G, \theta, a)\).

For the proof we can use our previous construction of a section. It follows from our analysis that we must take \( s_0 = (1, \zeta, \ldots, \zeta^{d-1}) \) (in fact block matrices of size \( b \)) up to a scalar. For two choices we therefore have \( g\tilde{s}_1g^{-1} = z\tilde{s}_2 \) (take \( g \in GL(n)^d \) diagonal). We can therefore assume that \( s_0 \) is fixed. Then \( \hat{H} \subset \hat{G} \) is well defined.

If \( G' \subset \hat{G} \rtimes W_F \) is an endoscopic subgroup, its fiber \( G'_w \) over \( w \in W_F \) is equal to \( \hat{G}_s\mathfrak{s}(w) \) for any section \( s \). In particular it is equal to \( \hat{H}\mathfrak{s}(w) \) in our case, with of course \( \hat{H} \subset \hat{G} \) given by the diagonal embedding. Assume \( s_1 \) and \( s'_1 \) are two sections as before. We obtain \( \xi_1, \xi'_1 \) and \( \xi_2 = z\xi_1z^{-1}, \xi'_2 = z\xi'_1z^{-1} \). The choice of \( z \) does not depend on the choice of the characters \( \eta^\beta \) in the construction of \( s_1, s'_1 \). Then, since \( z \) commutes with \( \text{diag}(\hat{H}) \), we see that the two corresponding subgroups (given by \( \xi_2, \xi'_2 \)) will be the same if \( z^{1}(w) \in \text{diag}(\hat{H})z^{2}(w) \) for \( w \in W \).

We have seen that \( z^{1}(w) \) could be obtained from the characters \( \eta_\alpha \), with \( z_{i,\alpha}(w) = \eta_{i,\alpha}(\delta_i) \) and \( \eta_{\alpha+1} \circ \sigma = \omega \eta_\alpha \). If \( (\eta'_\alpha) \) are another choice, \( \eta'_\alpha = \eta_\alpha \nu_\alpha \) with \( \nu_{\alpha+1} \circ \sigma = \nu_\alpha \). This implies that \( \nu_{\alpha+i+1}(\delta_{i+1}) = \nu_{\alpha+i}(\delta_i) \), and the correction belongs to \( \text{diag}(Z(\hat{H})) \).

This concludes the proof.\(^2\)

### 7. Proof of Theorem 1

7.1. We now have to understand the effect of our homomorphism \( \xi_1 \) as in Equation (5.2) of \( L \)-groups on the data pertinent to the stabilization, i.e., on the data composed of Hecke matrices for almost all primes. Note that
\[
\hat{L}H = GL(b, \mathbb{C})^d \rtimes W_F
\]
is not a direct product, so a ”Hecke matrix” at a prime \( v \) is in fact a conjugacy class in \( \hat{H} \rtimes \text{Frob}_v \), under the conjugation action of \( \hat{H} \).

For this we must first consider a simple case. Assume that \( \omega = 1 \), so we are in the case of non-twisted base change, i.e., characterizing the representations \( \Pi_n \) of \( GL(n, A_E) \) such that \( \sigma\Pi_n \simeq \Pi_n \).

Recall ([AC], [C]) the two natural operations associated to (cyclic) base change. The first is automorphic restriction, denoted earlier by \( BC^E_F \), sending representations of \( GL(n, A_F) \) to representations of \( GL(n, A_E) \). It is associated to the diagonal embedding
\[
\hat{L}G_0 \rightarrow \hat{L} G
\]
\(^2\)Waldspurger’s proof, terser, was more elegant.
where \( {}^L G_0 \) is the \( L \)-group of \( GL(n)/F \), \( {}^L G \) the \( L \)-group of \( \text{Res}_{E/F}(GL(n)/E) \), so \( \hat{G} = GL(n, \mathbb{C})^d \):

\[
(g, w) \mapsto (\text{diag}(g), w) \quad (w \in W_F).
\]

Suppose \( n = db \). The second operation is automorphic induction, denoted by \( AI_E^\tau \), sending representations of \( GL(b, \mathbb{A}_E) \) to those of \( GL(n, \mathbb{A}_F) \). The associated embedding of \( L \)-groups is given by

\[
(g_1, \cdots, g_d) \mapsto g_1 \oplus \cdots \oplus g_d \quad (g_i \in GL(b, \mathbb{C}))
\]

and \( (1, w) \mapsto (\tau(w), w) \)

where \( \tau(w) \in \mathbb{G}_d \) (realized as before by block-scalar matrices in \( GL(n, \mathbb{C}) \)), and

\[
\tau(\oplus g_i)\tau^{-1} = (g_{i+k}),
\]

where \( k = i(w) \), so \( \tau^{-i} = i + k \).

We simply write \( \text{Res} \) and \( \text{Ind} \) for these two operations, the fields being here \( F \) and \( E \). This corresponds to our constructions in ([AC]), taking \( \omega = 1 \). The corresponding operations are described in ([AC] Chapter 3), see also [C]. They are well-defined for representations that are 'induced from cuspidal' ([AC Sections 3.1, 3.6]), i.e., induced from unitary cuspidal representations.

Composing these two operations, we get a homomorphism of \( L \)-groups \( \mu_1 : GL(b)^d \rtimes W_F \to GL(n)^d \rtimes W_F \), given by,

\[
(g_1, \cdots, g_d) \mapsto \text{diag}(g_1 \oplus \cdots \oplus g_d)
\]

\[
(1, w) \mapsto (\text{diag} \tau(w), w).
\] \hspace{1cm} (7.1)

Recall also that for \( \pi_i \) \( (i = 1, \cdots, r) \) representations of \( GL(n_i, \mathbb{A}_E) \), there is an associated representation \( \bigoplus \pi_i \) of \( GL(n, \mathbb{A}_E) \) obtained by parabolic induction \( n = \sum n_i \). We recall the following well known result:

**Proposition 6.** For \( \pi_b \) a representation of \( GL(b, \mathbb{A}_E) \) induced from cuspidal,

\[
\text{Res} \circ \text{Ind}(\pi_b) = \pi_b \bigoplus \sigma \pi_b \bigoplus \cdots \bigoplus \sigma^{d-1}\pi_b.
\]

Consider finite primes \( w \mid v \) of \( E \) over \( F \) where all data are unramified. If \( t_{w'} \) is the Hecke matrix of \( \pi_b \) at such a prime \( w' \), the matrix \( T_v \) of \( \text{Ind}(\pi_b) \) at \( v \) is

\[
T_v = \bigoplus_{w' \mid v} (t_{w'})^{1/f} \oplus \zeta^{1/f} t_{w'}^{1/f} \oplus \cdots \oplus \zeta^{f-1} t_{w'}^{1/f},
\]

where \( \zeta \) is a primitive root of unity of order \( f = [E_w : F_v] \). The Hecke matrix \( T_w \) of \( \text{Res}(\Pi) \) for a representation \( \Pi \) of \( GL(n, \mathbb{A}_E) \) is \( T_v^f \). Thus the Hecke matrix of \( \text{Res} \circ \text{Ind}(\pi_b) \) at a prime \( w \) is \( \bigoplus_{w' \mid v} (t_{w'} \oplus \cdots \oplus t_{w'}) \), equal to \( \bigoplus_{\sigma \in \text{Gal}(E/F)} t_{\sigma w} \), the Hecke matrix of the right-hand side. Since the representations on the two sides of the equality are induced from cuspidal, they are equal.

Now there exists an obvious homomorphism of \( L \)-groups realising the operation

\[
\pi_b \mapsto \pi_b \bigoplus \sigma \pi_b \bigoplus \cdots \bigoplus \sigma^{d-1}\pi_b.
\]
First \((\pi_1, \cdots, \pi_d) \mapsto \pi_1 \sqcup \cdots \sqcup \pi_d\) is given by \(GL(b)^d \to GL(n)^d\),
\[
(g_{ki}) \mapsto \left( \bigoplus_k g_{ki} \right)_i.
\] (7.2)

It is obviously compatible with the operation of the Weil group. On the other hand
\(\pi \mapsto \sigma \pi\) is given by
\[
(g_i) \mapsto (g_{i+1}).
\]

So the composite operation is given by
\[
GL(b)^d \to GL(n)^d
\]
\[
(g_i) \mapsto \left( \bigoplus_k g_{i+k} \right)_i.
\]

It is equivariant for the action of \(W_F\) acting (via the restriction of scalars) on both sides. Thus we get
\[
\mu_0 : GL(b)^d \rtimes W_F \to GL(n)^d \rtimes W_F
\]
\[
(g_i, w) \mapsto \left( \left( \bigoplus_k g_{i+k} \right)_i, w \right).
\] (7.3)

Since the two homomorphisms of \(L\)-groups \(\mu_0\) and \(\mu_1\) have the same effect on representations, they should be conjugate by an element in \(\check{G} = GL(n)^d\). We proceed to exhibit this conjugation. We first consider the connected dual groups. We seek \(P = (P_i) \in GL(n)^d\) such that
\[
P\mu_0(g)P^{-1} = \mu_1(g), \; g = (g_i) \in GL(b)^d.
\]

Thus,
\[
P_i \left( \bigoplus_k g_{i+k} \right) P_i^{-1} = g_1 \oplus \cdots \oplus g_d,
\]
for each \(i\). If \(Q\) is (a block-scalar matrix) associated to a permutation \(\tau \in \mathfrak{S}_d\),
\[
Q \left( \bigoplus_k g_k \right) Q^{-1} = (g_{\tau^{-1}(k)}).
\]

Thus \(P_i\) must be the permutation matrix associated to \(\tau\), where \(\tau(k) = i + k\).

To avoid confusion, we now replace our indices \(k\) by \(\alpha\) (in conformity with the previous section) and write \(k = \iota(w), \; (w \in W_F)\). The conjugation of the homomorphisms on \(W_F\) gives,
\[
(P, 1)(1, w)(P, 1)^{-1} = (\text{diag} \tau(w), w) \; \; (w \in W_F),
\]
so \(P_i P_{i+k}^{-1} = \tau(w)\). (7.4)

The left hand side is associated to \(\tau(\alpha) = \alpha - k\). Thus (with the previous choices) we must take
\[
\tau(w) = \tau_1^{\iota(w)}, \; \; \tau_1(\alpha) = \alpha - 1.
\]

(compare with the formula for \(\tau\) preceding (7.1)).

We have therefore proved:
Lemma 4. With the above notation,
\[ P\mu_0P^{-1} = \mu_1, \]
where \( P = (P_i) \in GL(n, \mathbb{C})^d \) and \( P_i \) is the permutation matrix associated to \( \alpha \mapsto i + \alpha \).

7.2. Now return to the homomorphism \( \mu_0 \) (7.1) realising the operation
\[ \pi \mapsto \bigoplus_{\alpha=1}^d \sigma^\alpha \pi. \]
(\( \pi \) being a representation of \( GL(d, A_E) \)). Let \( \eta_1, \cdots, \eta_d \) be characters of \( C_E \), associated to the parameters \( \eta_\alpha(\delta_i(w)) \in (\mathbb{C}^\times)^d \) as discussed before equation (6.14). Now the homomorphisms (7.2) can be multiplied by the homomorphisms associated with the \( \eta_\alpha \), so we see that
\[ (g_\alpha) \mapsto \left( \bigoplus_{\alpha} g_\alpha \eta_\alpha(\delta_i(w)) \right)_i \]
corresponds to \( (\pi_\alpha) \mapsto \bigoplus \pi_\alpha \otimes \eta_\alpha \). In particular, \( \pi \mapsto \bigoplus \sigma^\alpha \pi_\alpha \otimes \eta_\alpha \), is then given by
\[ \xi_0 : GL(b)^d \rtimes W_F \to GL(n)^d \rtimes W_F \]
\[ (g_1, w) \mapsto [(g_{\alpha + 1}, \eta_\alpha(\delta_i(w))_i, w]. \]
Conjugating by \( P \), we obtain a homomorphism \( \xi_1 \). On \( GL(b)^d \), it coincides with \( \eta_1 \). For \( w \in W_F \) we must compute
\[ (P, 1) \left( \bigoplus_{\alpha} \eta_\alpha(\delta_i(w)) \right)_i (P^{-1}, 1). \]
The \( i \)-th component is
\[ P_i \left( \bigoplus_{\alpha} \eta_\alpha(\delta_i(w)) \right) P_i^{-1} = P_i \left( \bigoplus_{\alpha} \eta_\alpha(\delta_i(w)) \right) P_i^{-1} \tau(w) \]
by (7.4). The conjugation by \( P_i \) is the permutation \( \alpha \mapsto i + \alpha \), so this is
\[ \left( \bigoplus_{\alpha} \eta_{i+\alpha}(\delta_i(w)) \right) \tau(w). \]
In conclusion,

Lemma 5. The map \( \pi \mapsto \bigoplus \sigma^\alpha \pi_\alpha \otimes \eta_\alpha \) is realized by the homomorphism,
\[ \xi_1 : GL(b)^d \rtimes W_F \to GL(b)^d \rtimes W_F \]
\[ \xi_1(g_1, \cdots, g_d) = \text{diag}(g_1 \oplus \cdots \oplus g_d) \]
\[ \xi_1(w) = \left[ \left( \bigoplus_{\alpha} \eta_{i+\alpha}(\delta_i(w)) \right) \tau(w) \right]_i, w]. \]

We note that this is the homomorphism \( \xi_1 \) obtained from endoscopy as in Equations (6.22) and (6.14).
7.3. We can now complete the proof of Theorem 1. Assume first \( d \) does not divide \( n \). We then have

\[
T_{\text{disc}}(\phi \times \theta; \omega^{-1}) = 0.
\]

The cuspidal representation occurring in the discrete trace have multiplicity one, and their families of Hecke eigenvalues (away from a finite set \( S \) of primes) are linearly independent, and independent from those of other representations. Of course only the cuspidal representations such that \( \sigma \Pi \simeq \Pi \otimes \omega \) contribute. We conclude that there are no such representations, as was of course clear from the consideration of the central characters (see the remark after Theorem 1.)

Consider now the case when \( d \) divides \( n \). There is only, up to equivalence, one endoscopic datum \( H \); if \( \pi_b \) is a cuspidal representation of \( GL(b, \mathbb{A}_E) \), the associated map on Hecke matrices sends (up to conjugation) \( t_w(\pi_b) \) to

\[
t_w(\pi_b \otimes \eta_1) \oplus t_w(\sigma \pi_b \otimes \eta_2) \oplus \cdots \oplus t_w(\sigma^{d-1} \pi_b \otimes \eta_d)
\]

at primes where all data are unramified, as follows from the conjugation of \( \xi_1 \) and \( \mu_0 \). This then remains true if \( \pi_b \) is any automorphic representation of \( GL(b, \mathbb{A}_E) \), in particular for those appearing in the discrete trace formula for \( H \). Here again, the sum

\[
\sum_{\sigma \Pi \simeq \Pi \otimes \omega} \text{trace}(I_{\theta}(\Pi \otimes \omega^{-1})(\phi))
\]

over the cuspidal representations yields, evaluated against a function \( \phi \), a linear combination of characters of the Hecke algebra linearly independent from all others, associated to induced representations. The same argument then proves the Theorem.

References

[AC] Arthur, J. and Clozel, L. Simple algebras, base change and the advanced theory of the trace formula, Annals of Maths Studies 120 (1989), Princeton Univ. Press.

[B] Borel, A. Automorphic L-functions. Automorphic forms, representations and L-functions, Proc. Sympos. Pure Math., XXXIII, Part 2, pp 27-61, Amer. Math. Soc., Providence, R.I., 1979.

[BC] Bergeron, N. and Clozel, L. Sur la cohomologie des variétés hyperboliques de dimension 7 triéthaires, Israel J. Math. 222 (2017), no. 1, 333–400.

[C] Clozel, L. Base change for GL(n), Proceedings of the International Congress of Mathematicians, Vol. 1 (Berkeley, Calif., 1986), 791–797, Amer. Math. Soc., Providence, RI, 1987.

[JS1] Jacquet, H.; Shalika, J. A. On Euler products and the classification of automorphic representations, I, Amer. J. Math. 103 (1981), no. 3, 499–558.

[JS2] Jacquet, H.; Shalika, J. A. On Euler products and the classification of automorphic representations, II, Amer. J. Math. 103 (1981), no. 4, 777–815.

[KS] Kottwitz, R. and Shelstad, D. Foundations of twisted endoscopy, Astérisque No. 255 (1999), vi+190 pp.

[L1] Langlands, R. On the classification of irreducible representations of real algebraic groups, Representation theory and harmonic analysis on semisimple Lie groups, 101–170, Math. Surveys Monogr., 31, Amer. Math. Soc., Providence, RI, 1989.

[L2] Langlands, R. Base change for GL(2), Annals of Maths Studies 96 (1980), Princeton Univ. Press.

[L3] Langlands, R. Representations of Abelian Algebraic groups, preprint, 1968.

[Lab] Labesse, Jean-Pierre Noninvariant base change identities, Mém. Soc. Math. France (N.S.) No. 61 (1995).
[LL] Labesse, J.-P. and Langlands, R. *L-indistinguishability for SL(2)*, Canad. J. Math. 31 (1979), 726-785.

[LR] Lapid, E. and Rogawski, J. *On twists of cuspidal representations of GL(2)*, Forum Math. 10 (1998), no. 2, 175–197.

[MWI] Moeglin, C.; Waldspurger, J.-L. *Le spectre résiduel de GL(n)*, Ann. Sci. Ecole Norm. Sup. (4) 22 (1989), no. 4, 605–674.

[MWII] Moeglin, C.; Waldspurger, J. L. *Stabilisation de la formule des traces tordue*, vol.1,2, Progress in Mathematics 316, 317 (2016), Birkhäuser.

[R] Rajan, C.S. *On the image and fibres of solvable base change*, Math. Res. Lett. 9 (2002), no. 4, 499-508.

[S] Serre, J.-P. *Local fields*, GTM 67, Springer, New York.

[S1] Serre, J.-P. *Modular forms of weight one and Galois representations*, Algebraic number fields: L-functions and Galois properties (Proc. Sympos. Univ. Durham, Durham, 1975), pp. 193-268. Academic Press, London, 1977.

[W] Waldspurger, J.-L. *Stabilisation de la partie géométrique de la formule des traces tordue*, Proceedings of the International Congress of Mathematicians - Seoul 2014. Vol. II, 487–504, Kyung Moon Sa, Seoul, 2014.

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