Conformal Lagrangians from the (formal) near boundary analysis of AdS gauge fields

Alexander Chekmenev\textsuperscript{a,b} and Maxim Grigoriev\textsuperscript{a,c}

\textsuperscript{a}Lebedev Institute of Physics, Leninsky ave. 53, 119991 Moscow, Russia

\textsuperscript{b}Moscow Institute of Physics and Technology, Institutskiy per. 7, Dolgoprudny, 141700 Moscow region, Russia

\textsuperscript{c}Institute for Theoretical and Mathematical Physics, Lomonosov Moscow State University, 119991 Moscow, Russia

Abstract

A simple generating procedure for Lagrangians of conformal gauge fields of mixed-symmetry type is presented. The construction originates from the analysis of the near-boundary behaviour of the associated AdS gauge fields using the ambient space approach to leading boundary values. As an illustration we apply the procedure to the simplest mixed-symmetry conformal gauge field, described by the two-row Young diagram, and derive the explicit component form of the respective Lagrangian.
1 Introduction

Conformal higher spin gauge theories attract considerable attention because they give tractable examples of interacting Lagrangian theories that extend conformal gravity and are holographically related to higher spin gauge theories in AdS space of one dimension higher. The simplest example of conformal higher spin fields are totally symmetric gauge fields in even dimensional Minkowski space, which are known as Fradkin-Tseytlin fields. They provide a field content of the conformal higher spin gravity [1, 2].

Conformal fields in $d$-dimensions are intimately related to their associated AdS fields living in $d + 1$-dimensional AdS. More precisely, conformal fields can be identified as boundary values of the respective AdS fields. Furthermore, the Lagrangian for conformal fields can be derived as a logarithmically-divergent part of the effective action for the respective AdS fields [3-5]. In the extension [6] of this approach to higher spin theories one starts with the Lagrangian of higher spin gauge fields in the bulk.
It turns out that equations of motion of conformal fields can be inferred from the bulk dynamics without resorting to the Lagrangian formulation in AdS. More precisely, the conformally invariant equations on boundary values arise as conditions ensuring that the unconstrained boundary value can be lifted to an on-shell bulk field. This interpretation dates back to the celebrated Fefferman-Graham [7] construction. The extension of this approach to HS theories was proposed in [8–10] (see also [11]) and gave rise to a rather concise formulation of Fradkin-Tseytlin fields and their higher-depth generalizations. As far as conformal (gauge) fields of general symmetry type are concerned the generalization of the approach becomes somewhat inevitable because it leads to a concise and handful formulation of mixed symmetry conformal gauge fields [12] at the level of equations of motion.

In contrast to the equations of motion, Lagrangians of generic conformal mixed-symmetry (gauge) fields can not be obtained from the Lagrangians of the respective AdS fields simply because the later are not yet known in the general case. However, in particular cases, where Lagrangian description in the bulk is available, the standard strategy works [13]. Lagrangians for a rather general conformal (gauge) fields have been proposed in [14] from a different perspective and their interpretation in terms of the bulk dynamics remains somewhat unclear.

In this work we derive Lagrangian description of a wide class of conformal mixed-symmetry fields directly from their bulk dynamics. Although the Lagrangian description in the bulk is not available the bulk equations of motion (more precisely, their ambient space version) naturally provide us with the gauge-invariant kinetic operator defined on the leading boundary values of the bulk fields. Moreover we succeeded to build an inner product with respect to which the kinetic operator is formally self-adjoint and hence immediately gives us a local and gauge invariant Lagrangian. We show that at least in the simplest cases our Lagrangian is equivalent to the one of [14]. The advantage of our approach is that it makes the relation to bulk dynamics manifest and is conformally invariant by construction.

Additional motivation of this work has to do with already mentioned long-standing problem of Lagrangian formulation for general mixed-symmetry gauge fields in AdS. It is tempting to expect that this may help to lift the construction to the bulk, leading to the Lagrangian description of AdS fields.

The paper is organized as follows: In Section 2 we review the ambient description of AdS fields and fix a class of fields we work with. In Section 3 we recall how this description leads to a concise formulation of the conformal equations of motion satisfied by the leading boundary values. Section 4 we prose the inner product that makes kinetic operator formally self-adjoint, giving a gauge invariant Lagrangian. Appendix A contains some technical details of the construction.
2 Ambient description of AdS fields

Our approach to conformal Lagrangians originates from the ambient description of AdS gauge fields proposed in [15–17]. Here we follow [17]. This approach is based on the extensive use of the ambient space.

More specifically, tensor fields on $AdS_{d+1}$ are described in terms of tensor fields defined on the ambient space $\mathbb{R}^{d+2}/\{0\}$, which is a pseudo-Euclidean space of signature $d+2$ with the origin excluded. We use Cartesian coordinates $X^A, A = 0, \ldots, d+1$ on the ambient space, where components of the metric are $\eta_{AB} = \eta((\partial/\partial X^A, \partial/\partial X^B))$. $AdS_{d+1}$ can be understood as hyperboloid $X^2 = -1$ embedded in the ambient space. The restriction of $o(d, 2)$ transformations to the hyperboloid gives the algebra of infinitesimal AdS isometries. Although in this way one can not describe the most general negative constant curvature spaces it turns out that the resulting description is covariant and hence applicable in the general case. This is because the ambient space construction is eventually implemented in the fiber of a suitable fiber bundle rather than in the space-time. This is achieved through the appropriate version of the parent formulation [15, 16].

To work with tensor fields on the ambient space we employ the language of generating functions. To this end we consider the algebra of polynomials in the auxiliary coordinates $P^A_i, i = 1, \ldots, n-1, A = 0, \ldots, d+1$. This algebra contains ambient tensors with $n-1$ groups of totally symmetric indexes and hence is wide enough to contain all the relevant representations of $o(d-1,2)$. We then introduce the ambient space function $\Phi$ with values in the algebra which we treat as a generating function for AdS fields. More precisely, the coefficients entering in the expansion of $\Phi(X, P_i)$ in $P^A_i$ are precisely tensor fields on the ambient space.

The ambient space tensor fields form a natural representation space of the algebra $o(d, 2)$ of isometries. In terms of generating function $o(d, 2)$ are represented as differential operators:

$$J_{AB} = X_A \frac{\partial}{\partial X^B} + P^A_i \frac{\partial}{\partial P^B_i} - (A \leftrightarrow B).$$

The space of ambient tensors is also a module over $sp(2n)$, which together with $o(d-1,2)$ module structure gives a standard setting of Howe duality. More precisely, respective groups centralise each other in this bimodule.

In what follows we only need to introduce notation for the following $sp(2n)$ generators:

$$T^{ij} = \frac{\partial}{\partial P^A_i} \frac{\partial}{\partial P^A_j}, \quad N^j_i = P^A_i \frac{\partial}{\partial P^A_j}, \quad N^i = N^i_i, \quad N_X = X^A \frac{\partial}{\partial X^A},$$

$$\Box = \frac{\partial}{\partial X^A} \frac{\partial}{\partial X^A}, \quad S^i = \frac{\partial}{\partial P^A_i} \frac{\partial}{\partial X^A}, \quad S^i_i = P^A_i \frac{\partial}{\partial X^A}, \quad S^i = X^A \frac{\partial}{\partial P^A_i}. \quad (2.2)$$

They form a subalgebra of $sp(2n)$. 

4
A generic mixed symmetry field of spin \( \{s_1, s_2, \ldots, s_{n-1}\} \) (it is assumed that \( s_1 \geq s_2 \geq \ldots \geq s_{n-1} \) and \( n - 1 \leq \left[ \frac{d}{2} \right] \)) on \( d + 1 \)-dimensional AdS space can be described by the following constraints, which depend on extra real parameter \( \Delta \) and positive integer parameter \( p \leq n - 1 \):

**Purely algebraic constraints** These are generalized tracelessness, Young-symmetry and spin-weight conditions:

\[
T^{ij} \Phi = 0, \quad N_i^j \Phi = 0, \quad i < j, \quad N_i \Phi = s_i \Phi. \tag{2.3}
\]

**Tangent constraints**

\[
\bar{\Sigma}^{i\hat{\alpha}} \Phi = 0, \quad \hat{\alpha} = p + 1, \ldots, n - 1 \tag{2.4}
\]

their role is to reduce tensor in \( d + 2 \) dimensions to (a collections of) tensors in \( d + 1 \).

**Radial weight constraint**

\[(N_X + \Delta) \Phi = 0, \tag{2.5}\]

where \( \Delta \) is a parameter of the theory. Roughly speaking, this constraint fixes the radial dependence of \( \Phi \).

**Equations of motion (and partial gauges)**

\[\Box \Phi = 0, \quad S^i \Phi = 0. \tag{2.6}\]

In contrast to the above ones these are essentially differential constraints because they do involve \( X^A \) derivatives along the hyperboloid and, being rewritten in terms of tensor fields on the hyperboloid, are precisely the equations of motion together with partial gauge conditions.

**Gauge invariance** The above constraints in general describe a reducible system. Indeed, the space of fields satisfying the constraints has an invariant submodule, which gives rise to the following linear gauge transformation:

\[
\delta \chi^\alpha \Phi = S^\dagger_\alpha \chi^\alpha, \quad \alpha = 1, \ldots, p, \tag{2.7}
\]

where gauge parameters \( \chi^\alpha \) satisfy the same constraints as \( \Phi \) except those involving \( N_X, N_i, N_i^j \) which are replaced by

\[
(N_X + \Delta - 1) \chi^\alpha = 0, \tag{2.8}
\]

\[
N_i \chi^\alpha = s_i \chi^\alpha - \delta_i^\alpha \chi^\alpha, \tag{2.9}
\]

\[
N_i^j \chi^\alpha = -\delta_i^\alpha \delta_j^\beta \chi^\beta \quad i < j. \tag{2.10}
\]
**Extra tangent constraint.** For $\Delta$ generic the above system is irreducible. But for special $\Delta$, namely such that $\Delta = t + p - s_p$, where $t \in \{1, 2, s_p - s_{p+1}\}$ the extra condition needs to be imposed for the system to be irreducible:

$$\mathcal{S}^{\dagger t} \Phi = 0.$$  \hfill (2.11)

### 2.1 Different types of AdS (gauge) fields

**Massive fields** For $\Delta$ generic the gauge invariance is purely algebraic and can be completely removed by a proper gauge condition $\mathcal{S}^{\dagger \alpha} \Phi = 0$. Such fields are called massive and the full set of constraints/equations of motion describing irreducible field reads as:

$$T^{ij} \Phi = 0, \quad N_i^j \Phi = 0, \quad N_i \Phi = s_i \Phi, \quad (2.12)$$

$$\mathcal{S}^{\dagger i} \Phi = 0, \quad (2.13)$$

$$\left( N_X + \Delta \right) \Phi = 0, \quad (2.14)$$

$$\Box \Phi = 0, \quad \mathcal{S}^i \Phi = 0. \quad (2.15)$$

**(Partially) massless fields** If $\Delta, p, s_i$ are such that there exists $t \in \{1, 2, \ldots, s_p - s_{p+1}\}$ satisfying $\Delta = t + p - s_p$ the gauge transformation is not completely algebraic and the field is a genuine gauge field called (partially) massless. In particular for $t = 1$ it is called massless.

Note that for $t = 1$ (i.e. massless field) and $s_1 = s_2 = \ldots = s_p$ the field is associated to a unitary $o(d - 1, 2)$-module [18]. The important technical point is that thanks to constraint algebra for a unitary massless field “all tangent constraints” hold:

$$\mathcal{S}^{\dagger i} \Phi = 0.$$  \hfill (2.16)

**Critical fields** Among the fields we consider there are so called critical. They correspond to cases where $\Delta = \frac{d}{2} - \ell$, $\ell = 1, 2, 3, \ldots$. These are fields whose space of solutions contains a submodule of the form $(X^2)^\ell \Phi^+$. Such fields are of particular interest because they lead to a nontrivial conformal equations on their leading boundary values while noncritical fields correspond to off-shell boundary values.

In particular, all (partially) massless fields in odd-dimensional AdS space are critical. Among massive fields only those with $\Delta = \frac{d}{2} - \ell$, $\ell = 1, 2, 3, \ldots$ are critical.

Boundary values of unitary massless fields had been described at the level of equations of motion in [12]. In this work we expand the description to massive fields and propose a systematic construction of the respective Lagrangians. In what follows we restrict ourselves to massive or unitary massless mixed-symmetry fields or generic totally symmetric fields.
3 Conformal fields as boundary values

A useful way to describe a conformal boundary of AdS space is to identify it with rays of the null-cone. This can be equipped with the metric by identifying conformal boundary with a section of the null-cone and pulling back the ambient metric to the section. In what follows we chose to work with Minkowski metric on the boundary but the formalism can be naturally generalised [11] to generic conformally flat boundary metrics.

To study boundary values the strategy is to consider the ambient system in the vicinity of the section of a null-cone and to identify values of the ambient field on the section as its boundary value. It turns out that constraints/EOMs on the ambient field give rise to constraints on boundary values. These can also be seen as obstructions to lift an unconstrained boundary field to the field subject to the constraints of the previous section. This approach was put forward in [8, 9, 11, 12], where it was shown that in this way one indeed arrives at a concise formulation (at the level of equations of motion) of the (generalized) Fradkin-Tseytlin fields on the boundary.

A crucial technical tool employed in [8, 9, 11, 12] is the parent formulation approach which allows to perform the ambient construction in the fiber of a suitable fiber bundle rather than in the space-time. This is achieved by replacing $X^A$ coordinates with formal variables $Y^A$ and bringing the entire system to geometrical 1st order (in space time) form. After this one can safely consider the system to be defined in generic coordinates on either AdS or conformal boundary. Such description gives both manifestly local and manifestly conformal description. Moreover, it originates from BV-BRST framework and hence properly takes into account gauge systems. Here we closely follow the exposition of [12] to which we refer for further details.

Using the parent technique and identifying the conformal boundary with the section $X^+ = 1$ of the hypercone $X^2 = 0$ in the standard light-cone coordinates $X^+, X^-, x^a$, $a = 0, \ldots, d-1$, system (2.12)-(2.15) can be equivalently rewritten in terms of function $\phi$ of the boundary coordinates $x^a$ and commuting variables $p_i^a, w_i, u$, where $p_i^a \equiv P_i^a$, $w_i = P^-\text{ while } u$ is a certain redefinition of the coordinate $X^-$. The resulting system reads as:

\begin{equation}
\tilde{\Box} \phi + \frac{\partial}{\partial u} \left( d - 2(\Delta + u \frac{\partial}{\partial u}) \right) \phi = 0, \quad (3.1)
\end{equation}

\begin{equation}
(\partial_{p_i} \cdot \partial) \phi + \frac{\partial}{\partial w_i} \left( d + n_i - \Delta - 1 - 2u \frac{\partial}{\partial u} \right) \phi + \sum_{j \neq i} \frac{\partial}{\partial w_j} (p_j \cdot \partial_{p_i}) \phi = 0, \quad (3.2)
\end{equation}

\begin{equation}
(n_i + n_{w_i} - s_i) \phi = 0, \quad (3.3)
\end{equation}

\begin{equation}
(p_i \cdot \partial p_j) \phi + w_j \frac{\partial}{\partial w_j} \phi = 0, \quad i < j, \quad (3.4)
\end{equation}

\begin{equation}
(\partial_{p_i} \cdot \partial_{p_j}) \phi - 2u \frac{\partial}{\partial w_i} \frac{\partial}{\partial w_j} \phi = 0, \quad (3.5)
\end{equation}
where \( n_{w_i} = w_i \frac{\partial}{\partial w_i} \) and \( \tilde{\Box} = \partial^a \tilde{\partial}_a, \tilde{\partial}^a = \partial^a + \sum_i p_i a \frac{\partial}{\partial w_i} \).

Equation (3.1) determines \( u \)-dependence of \( \phi \) and imposes on \( \phi_0 = \phi \vert_{u=0} \) the equation \( \tilde{\Box} \phi_0 = 0 \). Recall that for critical fields \( \ell = \frac{d}{2} - \Delta \), where in the (partially-) massless case \( \Delta = t + p - s_p \). Note that equation (3.1) doesn’t determine coefficient before \( u^\ell \) in terms of \( \phi_0 \). That corresponds to a subleading solution, describing a conserving current. But we are more interested on the leading solution and the equations it satisfies.

At \( u = 0 \) equations (3.2)-(3.5) uniquely determine \( \phi_0 \) for a given initial data \( \phi_0(x, p_i) = \phi_0 \vert_{w_i=0} \) satisfying

\[
(n_i - s_i) \phi_0 = 0, \quad (\partial_{p_i} \cdot \partial_{p_j}) \phi_0 = 0, \quad (p_i \cdot \partial_{p_j}) \phi_0 = 0 \quad i < j,
\]

(3.6)

which are precisely the conditions that \( \phi_0 \), as a function in \( x^a \), takes values in the irreducible module with weights \( s_1, \ldots, s_{n-1} \) of the Lorentz \( o(d-1, 1) \) subalgebra of \( o(d, 2) \). In other words, there is a map \( \pi : \phi_0 \mapsto \phi_0 \vert_{w_i=0} \) which sends solutions of (3.2)-(3.5) to the space of unconstrained fields with values in irreducible Lorentz tensors. In appendix [A] we show that this map is bijective, i.e. given \( \phi_0(x, p) \) satisfying (3.6) there exists a unique \( \phi_0(x, p, w) \) satisfying (3.2)-(3.5).

Given that \( \pi \) is bijective the equations induced on \( \phi_0 \) can be written as

\[
(\tilde{\Box} \phi_0) \vert_{w_i=0} = 0, \quad \phi_0 \vert_{w_i=0} = \phi_0,
\]

\[
(\partial_{p_i} \cdot \partial) \phi_0 + \frac{\partial}{\partial w_i} \left( d + s_i - \Delta - i - \sum_{j \leq i} n_{w_j} \right) \phi_0 + \sum_{i < j} (p_j \cdot \partial_{p_i}) \frac{\partial}{\partial w_j} \phi_0 = 0,
\]

(3.7)

where the equations in the second line are interpreted as the constraints determining the \( w_j \)-dependence in a unique way (see [12] and Appendix [A] for details). These equations are by construction conformally invariant though the invariance is not manifest in this form.

In the case of massless fields, i.e. \( t = 1 \) and \( \Delta = 1 + p - s_p \), the equations on \( \phi_0 \) encoded in (3.7) are invariant under the following gauge transformations:

\[
\delta \phi_0 = \left( \sum_\alpha (p_\alpha \cdot \tilde{\partial}) \lambda_\alpha \right) \bigg|_{w_i=0},
\]

(3.8)

where \( \lambda_\alpha(x, p, w) \) is itself determined in terms of terms of the gauge parameter \( \lambda_0^\alpha \) via \( \lambda_0^\alpha = \lambda_\alpha \vert_{w_i=0} \) and the following equations

\[
(\partial_{p_i} \cdot \partial) \lambda_\alpha + \frac{\partial}{\partial w_i} \left( d + \tilde{s}_i - \tilde{\Delta} - i - \sum_{j \leq i} n_{w_j} \right) \lambda_\alpha + \sum_{i < j} (p_j \cdot \partial_{p_i}) \frac{\partial}{\partial w_j} \lambda_\alpha = 0,
\]

(3.9)

where \( \tilde{\Delta} = \Delta - 1, \tilde{s}_\alpha = s_\alpha - 1. \)
In the case of totally symmetric (partially)-massless field in the bulk (i.e. \( n = 2, \Delta = t + 1 - s \)), the gauge transformation takes the form
\[
\delta \phi_{00} = (\Pi (p \cdot \hat{\partial})^t \lambda)|_{w=0},
\]
(3.10)
where \( \Pi \) denotes projection to the traceless component. In this case (3.9) takes the following simple form
\[
(\partial p \cdot \partial) \lambda + \frac{\partial}{\partial w} (d + s - \Delta - 1 - n_w) \lambda = 0.
\]
(3.11)

4 Conformal Lagrangians

It turns out the equations on \( \phi_{00} \) encoded in (3.7) has the same tensor structure as \( \phi_{00} \) itself and hence have a chance to be Euler-Lagrange for some Lagrangians. As we are going to see in this section this is indeed the case.

To see this let us consider the operator \( \mathcal{A} = \pi \circ \tilde{\Box}^\ell \circ \pi^{-1} : \phi_{00} \mapsto (\tilde{\Box}^\ell \pi^{-1} \phi_{00})|_{w_i=0}. \) In terms of \( \mathcal{A} \) the first equation in (3.7) take the following form \( \mathcal{A} \phi_{00} = 0 \). Next we introduce the formal inner product
\[
\langle \phi, \chi \rangle = \int d^d x \langle \phi, \chi \rangle_0,
\]
(4.1)
where \( \langle \cdot, \cdot \rangle_0 \) is the standard inner product on polynomials in \( p_a^i \) determined by the Minkowski metric \( \eta_{ab} \). For instance, the corresponding formal conjugation rules read as:
\[
x^\dagger = x, \quad \partial_a^\dagger = -\partial_a, \quad p_i^{a\dagger} = \eta^{ab} \partial_{p_b^i}.
\]
(4.2)
Note that the inner product restricts to the subspace (3.6) of Lorentz irreducible tensor fields.

We claim that \( \mathcal{A} \) preserves the space (3.6) of irreducible Lorentz tensors and is formally symmetric with respect to the above inner product. This guaranties that equations \( \mathcal{A} \phi_{00} = 0 \) follow from the following Lagrangian:
\[
L = \langle \phi_{00}, \mathcal{A} \phi_{00} \rangle = \langle \phi_{00}, (\tilde{\Box}^\ell \phi_{00})|_{w_i=0} \rangle = \langle \phi_{00}, \tilde{\Box}^\ell \phi_{00} \rangle|_{w_i=0},
\]
(4.3)
where \( \phi_{00} \) is an irreducible Lorentz tensor, \( \phi_0 = \pi^{-1} \phi_{00} \) is its unique lift via the last equation in (3.7).

Moreover, the Lagrangian is gauge invariant. Indeed, equation on \( \phi_{00} \) is gauge invariant: \( \mathcal{A} \delta \phi_{00} = 0 \) so that
\[
\delta \langle \phi_{00}, \mathcal{A} \phi_{00} \rangle = \langle \delta \phi_{00}, \mathcal{A} \phi_{00} \rangle + \langle \phi_{00}, \mathcal{A} \delta \phi_{00} \rangle = 0
\]
(4.4)
because \( \mathcal{A} \) is symmetric.

In the rest of this section we demonstrate that \( \mathcal{A} \) indeed preserves (3.6) and is formally symmetric there.
4.1 Invariance of the Lorentz irreducible subspace

First of all, we demonstrate that the space of irreducible Lorentz tensors is invariant under \( \mathcal{A} \). In other words, if \( \phi_{00} \) satisfies (3.6) then \( (\tilde{\Box}^\ell \phi_0)|_{w=0} \) also does so provided \( \phi_0 \) is constructed as above. For this it is sufficient to show that if \( \phi_0 \) is a solution of (3.2)-(3.5) at \( u = 0 \), then \( \psi_0 = \tilde{\Box}^\ell \phi_0 \) satisfies (3.3)-(3.5) at \( u = 0 \). Let us write down explicitly the system of equations on \( \phi_0 \).

\[(d - \Delta - 1) \frac{\partial}{\partial w_i} \phi_0 + D^i \phi_0 = t^{ij} \phi_0 = (n_{p_i} + n_{w_i} - s_i) \phi_0 = Y^{ij} \phi_0 = 0 \quad i < j, \quad (4.5)\]

where

\[D^i := (\partial_{p_i} \cdot \tilde{\partial}) \equiv (\partial_{p_i} \cdot \partial) + \sum_j \frac{\partial}{\partial w_j} (p_j \cdot \partial_{p_i}),\]

\[Y^{ij} := (p_i \cdot \partial_{p_j}) + w_i \frac{\partial}{\partial w_j}, \quad t^{ij} := (\partial_{p_i} \cdot \partial_{p_j}).\]

We need to show that

\[t^{ij} \psi_0 = (n_{p_i} + n_{w_i} - s_i) \psi_0 = Y^{ij} \psi_0 = 0 \quad i < j. \quad (4.7)\]

It follows from simple algebra that

\[[n_{p_i} + n_{w_i} - s_i, \tilde{\partial}] = [Y^{ij}, \tilde{\partial}] = 0, \quad (4.8)\]

\[[t^{ij}, \tilde{\partial}^{\ell}] = 2\ell \tilde{\partial}^{\ell-1} \frac{\partial}{\partial w_i} \left( D^j + \left( \frac{d}{2} + \ell - 1 \right) \frac{\partial}{\partial w_j} \right) + (i \leftrightarrow j). \quad (4.9)\]

The expression in parenthesis is exactly the operator in the first equation in (4.5) because for critical fields \( \ell = \frac{d}{2} - \Delta \).

4.2 Formal symmetry

There remains to show that \( \mathcal{A} = \mathcal{A}^\dagger \). To this end let us observe first that elements of the form \( (p_i \cdot p_j) \chi \) are orthogonal to \( \phi_{00} \) because \( (p_i \cdot p_j)^\dagger = (\partial_{p_i} \cdot \partial_{p_j}) \). So for any \( \chi_{00}, \phi_{00} \) satisfying (3.0) one has

\[\langle \chi_{00}, (\tilde{\Box}^\ell \phi_0)|_{w_i=0} \rangle = \langle \chi_{00}, \left( \Box + 2 \sum_i (p_i \cdot \partial) \frac{\partial}{\partial w_i} \right)^\ell \phi_0 \bigg|_{w_i=0} \rangle, \quad (4.10)\]

where \( \phi_0 = \pi^{-1} \phi_{00} \).

Then we recall the fact proved in [12] that equations (4.5) can be solved order by order in \( \mathbb{Z}_{\geq 0} \)-grading of weighted powers of \( w_i : \deg w_{n-1} = 1, \deg w_{n-2} = s_{n-1} + 1, \deg w_{n-3} = s_{n-2} \deg w_{n-2} + 1 \) and so on: \( \deg w_{n-1} = s_i \deg w_i + 1 \) (see Appendix [A] for more details). Furthermore, the coefficient in front of \( (w_i)^{k_1} \ldots (w_{n-1})^{k_{n-1}} \) in \( \phi_0 \) has the
form $O_{k_1\ldots k_{n-1}}\phi_{00}$, where $O_{k_1\ldots k_{n-1}}$ is a linear differential operator of order $k_1+\ldots+k_{n-1}$ and has homogeneity $-k_i$ in $p_i$.

It follows from (3.4) that it is a homogeneous differential operator of order $2\ell$ in $x^a$ that preserves homogeneity in $p_k$. But any such operator is symmetric on the subspace of solutions of (3.6) with respect to the inner product $\langle \cdot, \cdot \rangle$ described above. Indeed, the linear span of $\square$, $(p_k \cdot \partial)$, $(\partial_{p_k} \cdot \partial)$, $(p_j \cdot \partial_{p_i})$, $j > i$ is closed under commutator. Let us introduce ordering

$$(p_j \cdot \partial_{p_i}) < \square < (p_k \cdot \partial) < (\partial_{p_k} \cdot \partial) < (p_{k+1} \cdot \partial) < (\partial_{p_{k+1}} \cdot \partial) \quad j > i. \quad (4.11)$$

After reordering $\mathcal{P}$ would become a sum of terms with $(p_i \cdot \partial_{p_j})$ at the left and terms of the form $\square^{k_0}(p_1 \cdot \partial)^{k_1}(\partial_{p_1} \cdot \partial)^{k_1} \ldots (p_{n-1} \cdot \partial)^{k_{n-1}}(\partial_{p_{n-1}} \cdot \partial)^{k_{n-1}}$ with $k_0+k_1+\ldots+k_{n-1} = \ell$. The former ones produce terms orthogonal to $\phi_{00}$ because $(p_j \cdot \partial_{p_i})$ is symmetric under (4.2) while the later ones are obviously symmetric under (4.2).

### 4.3 Examples

#### 4.3.1 "Hook"-type field

As an illustration of the construction let us consider the simplest mixed-symmetry field, the so-called "hook"-type field which correspond to $d=4, s_1 = 2, s_2 = 1, p = 1$. This field has a symmetry type described by Young diagram (2, 1), with the gauge parameter described by Young diagram (1, 1). In this case $\Delta = 1 + p - s_p = 0, \ell = \frac{4}{2} - \Delta = 2$.

Let us introduce notations for the coefficients of $\phi_0$ as follows:

$$\phi_0 = \phi_{00} + w_1\phi_{10} + w_2\phi_{01} + \frac{1}{2}(w_1^2\phi_{20} + w_1w_2\phi_{11}) + \frac{1}{2}(w_1^2)w_2\phi_{21}. \quad (4.12)$$

The subsystem (A.5) determining the $w_i$-dependence of $\phi_0$ read as

| degree | $w$ | equation |
|--------|-----|-----------|
| 0      | 1   | $\phi_{00}$ |
| 1      | $w_2$ | $(\Delta + 2 - d)\phi_{01} = (\partial_{p_2} \cdot \partial)\phi_{00}$ |
| 2      | $w_1$ | $(\Delta - d)\phi_{10} = (\partial_{p_1} \cdot \partial)\phi_{00} + (p_2 \cdot \partial_{p_1})\phi_{01}$ |
| 3      | $w_1w_2$ | $(\Delta - d)\phi_{11} = (\partial_{p_1} \cdot \partial)\phi_{01}$ |
| 4      | $(w_1)^2$ | $(\Delta + 1 - d)\phi_{20} = (\partial_{p_1} \cdot \partial)\phi_{10} + (p_2 \cdot \partial_{p_1})\phi_{11}$ |
| 5      | $(w_1)^2w_2$ | $(\Delta + 1 - d)\phi_{21} = (\partial_{p_1} \cdot \partial)\phi_{11}$ |

It follows from (3.4) that $\phi_{21} = 0$ so that

$$(\square^2 \phi_0)|_{w=0} = \square^2 \phi_{00} + 4\square(p_1 \cdot \partial)\phi_{10} + 4\square(p_2 \cdot \partial)\phi_{01} + 4(p_1 \cdot \partial)^2\phi_{20} + 8(p_1 \cdot \partial)(p_2 \cdot \partial)\phi_{11} + \ldots. \quad (4.14)$$
where ellipses denote terms that are in the images of \((p_j \cdot p_j)\) and \((p_j \cdot \partial p_i)\) \(i < j\). These terms ensure that the expression in the RHS is Lorentz irreducible.

Using (4.13) we express \(\phi_{ij}\) in terms of \(\phi_{00}\) and substitute it in the last equation. So according to (4.3) the Lagrangian is

\[
L = \left\{ \phi_{00}, \left( \Box^2 - \Box (p_1 \cdot \partial) (\partial p_1 \cdot \partial) - \frac{5}{2} \Box (p_2 \cdot \partial) (\partial p_2 \cdot \partial) + \frac{5}{3} (p_1 \cdot \partial) (\partial p_1 \cdot \partial) (p_2 \cdot \partial) (\partial p_2 \cdot \partial) + \frac{1}{3} (p_1 \cdot \partial)^2 (\partial p_1 \cdot \partial)^2 \right) \phi_{00} \right\}
\]  

(4.15)

In components (\(\phi_{00} = P^a_1 P^b_1 P^c_2 \phi_{abc}\), where \(\phi_{abc} = \phi_{bac}, \phi_{abc} + \phi_{acb} + \phi_{bca} = 0\)):

\[
\frac{1}{2} L = \phi_{abc} \Box^2 \phi_{abc} + 2 \partial_e \phi^{ebe} \Box \partial^f \phi_{feb} + \frac{5}{2} \partial_e \phi^{abe} \Box \partial^f \phi_{abf} + \frac{3}{2} \partial a \partial b \phi^{abe} \partial^c \partial^f \phi_{ecf},
\]

(4.16)

which reproduces the special case of the general Lagrangian proposed by Vasiliev [14].

The explicit form of this Lagrangian of the “hook” field was obtained in [13] starting from the Lagrangian [19] of the respective “hook”-type field on AdS.

### 4.3.2 Totally-symmetric fields

In order to make connections to the literature let us consider the case of symmetric fields.

Equations on leading boundary value emerges for critical \(\Delta = \frac{d}{2} - \ell\), \(\ell \in \mathbb{Z}^\geq 0\). They are gauge invariant for even \(d\) and \(\Delta = 1, \ldots, 2 - s\), which corresponds to the case of (partially) massless fields. For other critical values of \(\Delta\) and/or odd \(d\) it describes massive fields. Thus critical values of \(\Delta\) for even \(d\) can be grouped as follows:

\[
\Delta = \frac{d}{2} - 1, \ldots, 2 - s, \ 2 - s, \ 1 - s, \ldots, -\infty.
\]

(4.17)

They are called **special**, **part-short**, **short** and **long** in [20].

The solution to the second equation of (3.7) is

\[
\phi_0 = \sum_{k=0}^{s} \varphi_k w^k, \quad \varphi_0 = \phi_{00}, \quad \left( \frac{d}{2} - 1 + s - k + \ell \right) \varphi_k = -\left( \partial_p \cdot \partial \right) \varphi_{k-1}.
\]

(4.18)

\[
\mathcal{A} \phi_{00} = (\Box^\ell \phi_0) |_{w=0} = \sum_{k=0}^{\ell} \binom{\ell}{k} \Box^{\ell-k} 2^k (p \cdot \partial)^k \varphi_k + \ldots,
\]

(4.19)

where ellipses denote traceful terms that cancel in the Lagrangian. Now (4.3) takes the following form:

\[
L = \sum_{k=0}^{\min(s,\ell)} \binom{\ell}{k} 2^k \langle \phi_{00}, \Box^{\ell-k} (p \cdot \partial)^k \varphi_k \rangle = \sum_{k=0}^{\min(s,\ell)} \binom{\ell}{k} (-2)^k \langle (\partial_p \cdot \partial)^k \phi_{00}, \Box^{\ell-k} \varphi_k \rangle.
\]

(4.20)
Up to an overall number this is exactly $\mathcal{L}$ from eq. (3.2) of [20]. It’s gauge invariant for even $d$ and $1 \leq \Delta \leq 2 - s$.

5 Conclusions

In this work we have proposed a simple generating procedure for Lagrangians of a wide class of mixed-symmetry type conformal fields. The class involves total symmetric fields, conformal fields associated to unitary mixed-symmetry fields in AdS as well as generic massive fields.

It seems that the construction is also applicable to conformal fields associated to nonunitary mixed-symmetry fields on AdS but proving this requires extra technical steps which we leave for a future work. Mention also that the proposed Lagrangians have something in common with ordinary derivative Lagrangians proposed in [21, 22] for totally symmetric fields.

An important and more conceptual point is to understand what the formal symmetry of the kinetic operator means in terms of the bulk dynamics. This may also shed some light on the long-standing problem of constructing a proper Lagrangian description of generic mixed-symmetry fields on AdS. Note that Lagrangians for some special classes of mixed-symmetry fields are available in the literature [19, 23, 24].

Acknowledgments

We are grateful to R. Metsaev for useful discussions. The work of A.C. was supported by the Russian Science Foundation grant 18-72-10123 in association with the Lebedev Physical Institute. The work of M.G. was supported by the RFBR grant 18-02-01024.

A Existence of the lift

We want to show that the system (4.5) is consistent and has a unique solution $\phi_0$ such that $\phi_0|_{w_i=0} = \phi_{00}$ for any $\phi_{00}$ satisfying (3.6).

From spin constraints it follows that $\phi_0$ is polynomial in $w$ with $w_i$ degree no more than $s_i$. So essentially it’s the finite dimensional linear algebra problem about nonhomogeneous system of linear equations. We follow steps similar to Gaussian elimination: transform the system into row echelon form, indicate a subsystem providing a unique solution and then show that other equations are consistent with that solution.
**Row echelon form**  Taking into account $w$-extended Young and spin constraints we get the equivalent system

\[ D_i \phi_0 = t^{ij} \phi_0 = (n_{p_i} + n_{w_i} - s_i)\phi_0 = Y_i \phi_0 = 0 \quad i < j, \quad (A.1) \]

where

\[ A_i := (\partial_{p_i} \cdot \partial), \quad B_i := \frac{\partial}{\partial w_i} \left( d + s_i - \Delta - i - \sum_{j \leq i} n_{w_j} \right), \quad y_j^i := p_j \cdot \partial_{p_i}; \quad (A.2) \]

\[ D_i := A_i + B_i + \sum_{j > i} y_j^i \frac{\partial}{\partial w_j}. \quad (A.3) \]

**Subsystem**  Let us denote by $\phi_0 |_{m_1 \ldots m_{n-1}}$ coefficient before $w_{m_1} \ldots w_{m_{n-1}}$. Equation $(D_i \phi_0) |_{m_1 \ldots m_{n-1}} = 0$ in components reads

\[ A_i \phi_{m_1 \ldots m_{n-1}} + \left( d + s_i - \Delta - i - \sum_{k \leq i} m_k - 1 \right) \phi_{m_i+1 \ldots} + \sum_{k > i} y_k^i \phi_{m_k+1 \ldots} = 0. \quad (A.4) \]

Consider the subsystem

\[ (D_i \phi_0) |_{0 \ldots 0 m_i \ldots m_{n-1}} = 0 \quad i = 1, \ldots, n - 1, \quad m_j = 0, \ldots, s_j - \delta_j^i \quad (A.5) \]

(equations with $m_i = s_i$ are trivial due to spin constrain). In this case

\[ d + s_i - \Delta - i - \sum_{k \leq i} m_k - 1 = \left( \frac{d}{2} - i \right) + (s_i - 1 - m_i) + \ell > 0 \quad (A.6) \]

so (A.5) can be used to solve order by order in $w_i$ in the order defined by $\mathbb{Z}_{\geq 0}$-grading of weighted powers of $w_i : \deg w_{n-1} = 1, \deg w_{i-1} = s_i \deg w_i + 1$.

More concretely, in this way one first solves the equation (A.5) with $i = n - 1$ in the subspace of $w_j$-independent elements with $j < i$. Then one uses the solution as the initial data for the equation with $i = n - 2$ and solves it in the subspace of $w_j$-independent elements with $j < i$. And so on: solution of (A.5) with $i > j$ in the subspace of $w_k$-independent elements with $k \leq i$ us used as the initial data for (A.5) with $i = j$ to get the solution in the subspace of $w_k$-independent elements with $k < i$.

**Consistency**  By construction $\phi_0 |_{m_1 \ldots m_{n-1}} = \mathcal{P} \phi_{00}$ where $\mathcal{P}$ is a polynomial in $A_i^i, y_j^i \ i < j$ such that $\deg_{p_i} \mathcal{P} = -m_i$. So spin constraints are satisfied. Trace constraints follow from the algebra:

\[ [A^k, t^{ij}] = 0, \quad [t^{ij}, y_j^m] = \delta_j^i \delta^m_j + (i \leftrightarrow j). \quad (A.7) \]

For divergency-like constraint we use double induction by Young row number $i = n - 1, \ldots, 1$ and cell number $m_i = 0, \ldots, s_i$. 

14
Assume that equations
\[ (\mathcal{D}^j \phi_0)|_{0...0m_i,m_{i+1}...m_{n-1}} = 0 \quad j > i \] (A.8)
hold for some \( i \) and \( m_i \) and for arbitrary \( m_{i+1}, \ldots, m_{n-1} \). Then acting with \( A^j, j > i \) on \( (\mathcal{D}^i \phi_0)|_{0...0m_i,m_{i+1}...m_{n-1}} = 0 \) (which is given), rearranging terms and using
\[ (\mathcal{D}^j \phi_0)|_{0...0m_i,m_{k+1}...} = 0 \quad k \geq j, \]
\[ (\mathcal{D}^j \phi_0)|_{0...0m_i,m_{n-1}} = 0, \]
\[ (\mathcal{D}^j \phi_0)|_{0...0m_i,m_{k+1}...} = 0 \quad k > i. \] (A.9)
we arrive at
\[ (d + s_i - \Delta - i - m_i - 1)(\mathcal{D}^j \phi_0)|_{0...0m_i+1m_{i+1}...m_{n-1}} = 0. \] (A.10)

Analogously for \( w \)-extended Young constraints. Assume that equations
\[ (Y^k_j \phi_0)|_{0...0m_i,m_{i+1}...m_{n-1}} = 0 \quad i \leq j < k \] (A.11)
hold for some \( i \) and \( m_i \) and for arbitrary \( m_{i+1}, \ldots, m_{m-1} \). Acting with \( y^k_j, k > j \) on \( (\mathcal{D}^j \phi_0)|_{0...0m_i,m_{n-1}} = 0 \) and using
\[ (\mathcal{D}^j \phi_0)|_{0...0m_i+1...m_{k+1}...} = 0, \]
\[ (\mathcal{D}^k \phi_0)|_{0...0m_i,m_{n-1}} = 0, \]
\[ (Y^k_l \phi_0)|_{0...0m_i,m_{l+1}...} = 0 \quad j < l < k, \]
\[ (Y^k_j \phi_0)|_{0...0m_i,m_{...}} = 0, \]
\[ (Y^k_j \phi_0)|_{0...0m_i,m_{j+1}...} = 0 \quad l > j. \] (A.12)
we arrive at
\[ (d + s_i - \Delta - i - m_i - 1)(Y^k_j \phi_0)|_{0...0m_i+1m_{i+1}...m_{n-1}} = 0. \] (A.13)
The base case \( (Y^2_{n-2} \phi_0)|_{0...0m_{n-1}} = y_{n-2}^2 \phi_0|_{0...0m_{n-1}} \) is true because \( \phi_0|_{0...0m_{n-1}} \sim (A^{n-1})^m \phi_0 \).

This ends the proof of consistency.

Note that from \( Y^i_j \phi_0 = 0 \) it follows that elements \( \phi_0|_{s_i...m_{...}} \) with maximum \( m_i = s_i \) and \( m_j \neq 0, i < j \) vanish. Also, if \( \phi_0|_{m_i...m_{i+1}...} = 0 \) then \( \phi_0|_{m_i...m_{i+1}...+1} = 0 \). This implies that elements with \( \sum_{k \geq i} m_k > s_i \) vanish.

**Bibliography**

[1] A. Y. Segal, *Conformal higher spin theory*, *Nucl. Phys. B664* (2003) 59–130, [hep-th/0207212].
[2] A. A. Tseytlin, *Semiclassical quantization of superstrings: AdS(5) x S(5) and beyond*, Int. J. Mod. Phys. A18 (2003) 981–1006 [hep-th/0209116].

[3] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, Phys. Rept. 323 (2000) 183–386 [hep-th/9905111].

[4] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, JHEP 07 (1998) 023, [hep-th/9806087].

[5] K. Skenderis, *Lecture notes on holographic renormalization*, Class. Quant. Grav. 19 (2002) 5849–5876 [hep-th/0206087].

[6] R. R. Metsaev, *Gauge invariant two-point vertices of shadow fields, AdS/CFT, and conformal fields*, Phys. Rev. D81 (2010) 106002 [0907.4678].

[7] C. Fefferman and C. Graham, *Conformal Invariants*, Astérisque, Numero Hors Serie (1985) 95–116.

[8] X. Bekaert and M. Grigoriev, *Notes on the ambient approach to boundary values of AdS gauge fields*, J. Phys. A46 (2013) 214008 [1207.3439].

[9] X. Bekaert and M. Grigoriev, *Higher order singletons, partially massless fields and their boundary values in the ambient approach*, Nucl. Phys. B876 (2013) 667–714 [1305.0162].

[10] X. Bekaert, M. Grigoriev and E. D. Skvortsov, *Higher Spin Extension of Fefferman-Graham Construction*, Universe 4 (2018) 17 [1710.11463].

[11] M. Grigoriev and A. Hancharuk, *On the structure of the conformal higher-spin wave operators*, JHEP 12 (2018) 033 [1808.04320].

[12] A. Chekhmenev and M. Grigoriev, *Boundary values of mixed-symmetry massless fields in AdS space*, Nucl. Phys. B913 (2016) 769–791 [1512.06443].

[13] K. Alkalaev, *Massless hook field in AdS(d+1) from the holographic perspective*, JHEP 01 (2013) 018 [1210.0217].

[14] M. Vasiliev, *Bosonic conformal higher-spin fields of any symmetry*, Nucl. Phys. B829 (2010) 176–224 [0909.5226].

[15] G. Barnich and M. Grigoriev, *Parent form for higher spin fields on anti-de Sitter space*, JHEP 08 (2006) 013 [hep-th/0602166].

[16] K. B. Alkalaev and M. Grigoriev, *Unified BRST description of AdS gauge fields*, Nucl. Phys. B835 (2010) 197–220 [0910.2699].

[17] K. Alkalaev and M. Grigoriev, *Unified BRST approach to (partially) massless and massive AdS fields of arbitrary symmetry type*, Nucl. Phys. B853 (2011) 663–687 [1105.6111].

[18] R. Metsaev, *All conformal invariant representations of d-dimensional anti-de Sitter group*, Mod. Phys. Lett. A10 (1995) 1719–1731.

[19] L. Brink, R. R. Metsaev and M. A. Vasiliev, *How massless are massless fields in AdS(d)*, Nucl. Phys. B586 (2000) 183–205 [hep-th/0005136].

[20] R. R. Metsaev, *Long, partial-short, and special conformal fields*, JHEP 05 (2016) 096 [1604.02091].

[21] R. Metsaev, *Ordinary-derivative formulation of conformal low spin fields*, JHEP 1201 (2012) 064 [0707.4437].

[22] R. R. Metsaev, *Ordinary-derivative formulation of conformal totally symmetric arbitrary spin bosonic fields*, JHEP 06 (2012) 062 [0709.4392].

[23] Y. M. Zinoviev, *First order formalism for mixed symmetry tensor fields* [hep-th/0304067].

[24] K. Alkalaev, O. Shaynkman and M. Vasiliev, *Lagrangian formulation for free mixed-symmetry bosonic gauge fields in (A)dS(d)*, JHEP 0508 (2005) 069 [hep-th/0501108].