Dynamics of relativistic particle with Lagrangian dependent on acceleration

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Abstract

Models of relativistic particle with Lagrangian $L(k_1)$, depending on the curvature of the worldline $k_1$, are considered. By making use of the Frenet basis, the equations of motion are reformulated in terms of the principal curvatures of the worldline. It is shown that for arbitrary Lagrangian function $L(k_1)$ these equations are completely integrable, i.e., the principal curvatures are defined by integrals. The constants of integration are the particle mass and its spin. The developed method is applied to the study of a model of relativistic particle with maximal proper acceleration, whose Lagrangian is uniquely determined by a modified form of the invariant relativistic interval. This model gives us an example of a consistent relativistic dynamics obeying the principle of a superiorly limited value of the acceleration, advanced recently.

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1 Introduction

Lagrangians depending on higher derivatives of the curve coordinates have been recently considered in many problems: the null-dimensional (particle-like) version of the rigid string [1–7], the model of boson-fermion transmutation in external Chern-Simons field [8–11], the polymer theory [12].

Lagrangians of this kind also occur in connection with a conjecture about the existence of the maximal proper acceleration of a massive particle in arbitrary motion introduced by E.R. Caianiello and coll. [13–19]; they proposed a new geometric approach to quantum mechanics in which the commutators between coordinates and momenta are interpreted as the components of the curvature tensor of the eight dimensional space–time tangent bundle TM, that, in this scheme, acquires a metric structure. This fundamental constant of nature has been also introduced starting from the Heisenberg’s uncertainty relations [20–22], putting in evidence, in such a way, the deeply interconnection between the maximal acceleration and the extended nature of particles, whose finite extension cannot be neglected without introducing in quantum field theory troubles and divergencies connected with the point-like approximation. Recently these variational problems also became interesting for mathematicians [23]. The list of references is certainly incomplete; however, it illustrates the continuing interest in the subject.

Investigation of these models, in the framework of the classical variational calculus, gives rise to very complicated nonlinear differential equations of order 2p (p is the highest order of the derivatives in the Lagrangian), which, practically, cannot be subject to analysis.

However, a considerable advance can be achieved by applying the following basic result from the classical differential geometry [24, 25]: any smooth curve $x^\mu(s)$, $\mu = 0, 1, \ldots, D - 1$ in $D$-dimensional flat space-time is completely determined (up to rotations as a whole) by specifying the $D - 1$ principal curvatures $k_i(s)$, $i = 1, 2, \ldots, D - 1$, where $s$ is the worldline length. Therefore, one can try to derive the Euler-Lagrange equations in terms of the principal curvatures $k_i(s)$, $i = 1, 2, \ldots, D - 1$ of the worldline, rather than in terms of the coordinates $x^\mu(s)$. The order of the differential equations for $k_i(s)$ will be necessarily lower than the order for $x^\mu(s)$, because the curvature $k_i(s)$ is a function of the derivatives of $x$ up to order $i + 1$ ($i = 1, 2, \ldots, D - 1$).

For Lagrangians of the form $\mathcal{L}(k_1, k_2, \ldots, k_{D-1})$ the Euler-Lagrange equations can be always reformulated in terms of $k_i(s)$. In the present paper it will be shown that the equations of motion for $k_i(s)$, generated by arbitrary Lagrangian function $\mathcal{L}(k_1(s))$, are always integrable by quadratures. Furthermore, the constants of integration are expressed in terms of the Noether invariants for the given Lagrangian. If one assumes that the Lagrangian $\mathcal{L}(k_1)$ defines a model of relativistic particle, then the integration constants turn out to be the mass and spin of the particle. In this case $k_1(s)$ is the proper acceleration of the particle.

In Section 2, given an arbitrary Lagrangian $\mathcal{L}(k_1)$, we derive the equations of motion in terms of the principal curvatures, $k_i(s)$, of the trajectory, by making use of the Hamilton principle. The derivation proposed by us is very simple and clear; we shall only apply
the standard Frenet equations describing the moving $D$-hedron of the curve; the Griffiths’ approach [23, 26], using the Cartan formalism of exterior differential forms, is similar but more involved.

In Section 3 the equations of motion for principal curvatures $k_i(s)$ are integrated. The constants of integration are expressed in terms of the particle mass and its spin.

In Section 4 the efficiency of our formalism is illustrated by investigating the model of a relativistic particle with maximal proper acceleration.

In Section 5 we draw some conclusion and discuss shortly the results. In Appendix A Lagrangians linear in $k_1$ are considered; in Appendix B the Hamiltonian formalism for the model of a relativistic particle with maximal proper acceleration is presented.

2 Euler-Lagrange equations in terms of the principal curvatures

Let us consider a reparametrization-invariant action

$$S = \int \mathcal{L}(k_1) \, ds$$

(2.1)

with a Lagrangian $\mathcal{L}$ depending only on the first curvature $k_1$ (on the particle acceleration) of the worldline $x^\mu(s)$, $\mu = 0, 1, \ldots, D - 1$ in $D$-dimensional Minkowski space. Here $ds^2 = dx^\mu dx_\mu$, and $s$ is the natural parameter along the worldline $x^\mu(s)$ (its length)

$$\frac{dx^\mu}{ds} \frac{dx_\mu}{ds} = 1.$$  

(2.2)

The Lorentz metric $\eta_{\mu\nu}$ with a signature $(+, -, \ldots, -)$ will be used. For shortening, the differentiation with respect to the natural parameter $s$ will be denoted by a dot. The first curvature (or simply curvature) is defined by

$$k_1^2(s) = -\ddot{x}_\mu \ddot{x}^\mu = -\dot{x}^2.$$  

(2.3)

Following the Hamiltonian principle, we require

$$\delta S = \delta S_1 + \delta S_2 =$$

(2.4)

$$= \int ds \mathcal{L}'(k_1) \delta k_1(s) + \int \mathcal{L}(k_1) \delta ds = 0,$$

where the variations $\delta k_1(s)$ and $\delta ds$ are generated by the variation of the worldline $\delta x^\mu(s)$. The prime on the Lagrangian function $\mathcal{L}(k_1)$ denotes the differentiation with respect to its argument $k_1$.

At every point on the worldline $x^\mu(s)$ we can associate a Frenet basis [24, 25], an orthonormal $D$-hedron $e^\mu_a(s)$, (the Latin index is a $D$-hedron index, the Greek index specifies the space-time component), which is formed by the unit time-like tangent vector

$$e^\mu_0(s) = \frac{dx^\mu}{ds}, \quad e^2_0 = \dot{x}^2 = 1$$

(2.5)
and by a set of \( D - 1 \) unit space-like vectors \( e^\mu_i(s) \):

\[
e^\mu_i e^\mu_j = - \delta_{ij}, \quad e^\mu_0 e^\mu_j = 0, \quad 1 \leq i, j \leq D - 1.
\]

Raising and lowering of indices are made by the corresponding metric tensors

\[
\eta_{ab} = \text{diag} (1, -1, \ldots, -1), \quad \eta_{\mu\nu} = \text{diag} (1, -1, \ldots, -1).
\]

The orthonormality condition for Frenet basis is:

\[
e^\mu_i a e^\mu_j b = \eta_{ab}, \quad 0 \leq a, b \leq D - 1, \quad \mu = 0, 1, \ldots, D - 1.
\]

We shall need the Frenet equations describing the change of the Frenet basis under motion of its origin along the worldline

\[
e^\mu_a = \omega^b_a e^\mu_b, \quad \omega_{ab} + \omega_{ba} = 0.
\]

Nonzero elements of the matrix \( \omega \) are determined by the principal curvatures

\[
\omega_{a,a+1} = - \omega_{a+1,a} = k_{a+1}(s), \quad a = 0, 1, \ldots, D - 2.
\]

Let us express the variation \( \delta x^\mu(s) \) in terms of the Frenet basis

\[
\delta x^\mu(s) = \varepsilon^a(s) e^\mu_a(s),
\]

\( \mu = 0, 1, \ldots, D - 1, \quad a = 0, 1, \ldots, D - 1. \)

The variations \( \delta ds \) and \( \delta k_1(s) \) encountered in (2.4) can be expressed now in terms of the functions \( \varepsilon^a(s) \) from (2.11). For variation \( \delta ds \) one obtains

\[
\delta ds = \delta \sqrt{dx_\mu dx^\mu} = \frac{dx_\mu \delta dx^\mu}{ds} = \dot{x}_\mu d(\delta x^\mu).
\]

Substituting (2.12) into (2.4) and integrating by parts, the second term in (2.4) acquires the form

\[
\delta S_2 = - \int d(\mathcal{L}(k_1) \mathcal{K}_\mu) \delta x^\mu =
\]

\[
= - \int \mathcal{L}'(k_1) k_1 (\dot{x}_\mu \delta x^\mu) \, ds - \int \mathcal{L}(k_1) (\ddot{x}_\mu \delta x^\mu) \, ds.
\]

By making use of eqs. (2.8) – (2.11) we obtain for the variation \( \delta S_2 \)

\[
\delta S_2 = - \int \mathcal{L}'(k_1) k_1 \varepsilon_0(s) ds - \int \mathcal{L}(k_1) k_1(s) \varepsilon^1(s) ds.
\]

Now we proceed to calculate the variation \( \delta k_1 \). From the definition (2.3) we deduce

\[
k_1(s) \delta k_1(s) = -(\ddot{x}_\mu \delta x^\mu) = -(\dot{e}_0 \delta x^\mu).
\]
Here it should be taken into account that the operations $\delta$ and $d/ds$ do not commute. The direct calculation shows that

$$(2.16) \quad \left[ \delta, \frac{d}{ds} \right] = - \left( e^a_{\mu} \frac{d}{ds} \delta x^\mu \right) \frac{d}{ds}.$$ 

Applying this formula one finds

$$(2.17) \quad \delta \ddot{x}^\mu = \frac{d^2}{ds^2} \delta x^\mu - 2 \dot{x}^\mu \left( e^a_{\nu} \frac{d}{ds} \delta x^\nu \right) - e^a_{\nu} \frac{d}{ds} \left( e^a_{\mu} \frac{d}{ds} \delta x^\nu \right).$$

Substituting (2.17) into (2.15) and taking into account the Frenet equations (2.9) we obtain

$$(2.18) \quad \delta k_1(s) = \epsilon^0 \dot{k}_1 - \epsilon^1 + \epsilon^2 \left( k_1^2 + k_2^2 \right) - 2 \epsilon^2 k_2 - \epsilon^3 \dot{k}_2 - \epsilon^3 k_2 k_3. $$

Thus, the variation of the first curvature of the curve, $k_1(s)$ depends on the variations of the worldline coordinates only along the directions $e^0_\mu$, $e^1_\mu$, $e^2_\mu$, and $e^3_\mu$ (on the functions $\epsilon^a(s)$, $a = 0, 1, 2, 3$) and on the first three curvatures $k_1$, $k_2$ and $k_3$.

Substituting (2.18) into (2.4), integrating by parts, and taking into account (2.14) we obtain

$$(2.19) \quad \delta S = \int ds \left\{ \left[ \left( k_1^2 + k_2^2 \right) \mathcal{L}'(k_1) - \frac{d^2}{ds^2} \mathcal{L}'(k_1) \right] - k_1 \mathcal{L}(k_1) \right\} \epsilon^1(s) + 
\quad + \left[ 2 \frac{d}{ds} (\mathcal{L}'(k_1) k_2) - \dot{k}_2 \mathcal{L}'(k_1) \right] \epsilon^2(s) - \mathcal{L}'(k_1) k_2 k_3 \epsilon^3(s) \right\} = 0.$$ 

The functions $\epsilon^i(s)$, $i = 1, 2, 3$ are arbitrary; therefore we deduce from (2.19) three equations

$$(2.20) \quad \frac{d^2}{ds^2} \mathcal{L}'(k_1) = \left( k_1^2 + k_2^2 \right) \mathcal{L}'(k_1) - k_1 \mathcal{L}(k_1),$$

$$(2.21) \quad 2 \frac{d}{ds} (\mathcal{L}'(k_1) k_2) = \dot{k}_2 \mathcal{L}'(k_1),$$

$$(2.22) \quad \mathcal{L}'(k_1) k_2 k_3 = 0.$$ 

A set of equations equivalent to eqs. (2.20) – (2.22) has been derived in the book [23] (see also [26]) by making use of rather complicated formal mathematical methods, based on the exterior Cartan forms. We consider that the use of simple Frenet equations familiar to physicists is enough.

To satisfy eq. (2.22) one has to put $k_3 = 0$. From here it follows that all the higher curvatures $k_4$, $k_5$, . . . , $k_{D-1}$ vanish also [23]. Thus we have for arbitrary $D$

$$(2.23) \quad k_n(s) = 0, \quad n = 3, 4, \ldots, D - 1.$$ 

Equation (2.21) can be explicitly integrated. Namely, rewriting this equation in the form

$$2 \mathcal{L}''(k_1) k_2 dk_1 + \mathcal{L}'(k_1) dk_2 = 0$$

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we obtain
\[ 2 \mathrm{d} \ln \mathcal{L}'(k_1) + \mathrm{d} (\ln k_2) = 0. \]  

Therefore,
\[ (\mathcal{L}'(k_1))^2 k_2 = C, \]  

where \( C \) is an integration constant.

Eq. (2.25) allows us to eliminate \( k_2(s) \) from eq. (2.20). As a result, one non-linear equation of the second order for the worldline curvature, \( k_1(s) \), arises
\[ \frac{d^2}{ds^2} (\mathcal{L}'(k_1)) = \left( k_1^2 + \frac{C^2}{(\mathcal{L}'(k_1))^4} \right) \mathcal{L}'(k_1) - k_1 \mathcal{L}(k_1). \]

Thus, we have derived a complete set of equations (see eqs. (2.23), (2.25), and (2.26)) for principal curvatures \( k_i(s) \), \( i = 1, 2, \ldots, D - 1 \) of the worldline of the particle.

3 Integrability of the Euler-Lagrange equations for principal curvatures

It is remarkable that a first integral for eq. (2.26) can be find in a general form for arbitrary Lagrangian \( \mathcal{L}(k_1) \). This first integral naturally arises analysing the Euler-Lagrange equations written in terms of the worldline coordinates \( x^\mu(s) \). The constants of integration turn out to be the particle mass and its spin.

In this section we shall use an arbitrary parametrization of the worldline \( x^\mu(\tau) \). From now on, a dot over \( x \) will denote the differentiation with respect to evolution parameter \( \tau \); prime on the Lagrangian function \( \mathcal{L} \) will, as before, denote the differentiation with respect to its argument \( k_1 \). The action (2.1) assumes the form
\[ S = \int L(\dot{x}, \ddot{x}) \, d\tau = \int \mathcal{L}(k_1) \sqrt{\dot{x}^2} \, d\tau, \]

where
\[ L(\dot{x}, \ddot{x}) = \sqrt{\dot{x}^2} \mathcal{L}(k_1), \]

\[ k_1^2 = \frac{(\dot{x} \ddot{x})^2 - \dot{x}^2 \ddot{x}^2}{(\dot{x}^2)^3}. \]

Introducing the conserved Lorentz vector of energy-momentum
\[ P^\mu = \frac{d}{d\tau} \left( \frac{\partial L}{\partial \ddot{x}_\mu} \right) - \frac{\partial L}{\partial \dot{x}_\mu}, \quad \mu = 0, 1, \ldots, D - 1, \]

the Euler-Lagrange equations generated by (3.1) are written as
\[ \frac{d}{d\tau} P^\mu = 0, \quad \mu = 0, 1, \ldots, D - 1. \]
Now we pass in eq. (3.4) from an arbitrary evolution variable \( \tau \) to the natural parameter \( s \), using the following relations

\[
\frac{d}{d\tau} = \sqrt{\dot{x}^2} \frac{ds}{d\tau}; \quad \frac{d^2x_\mu}{ds^2} = \frac{x^2 \ddot{x}_\mu - (\dot{x} \ddot{x}) \dot{x}_\mu}{(\dot{x}^2)^2}; \quad k_1 \frac{\partial k_1}{\partial \dot{x}_\mu} = -\frac{1}{\dot{x}^2} \frac{d^2x_\mu}{ds^2};
\]

(3.6)

\[
k_1 \frac{\partial k_1}{\partial \dot{x}_\mu} = \frac{1}{(\dot{x}^2)^4} \left\{ x^2 (\dot{x} \ddot{x}) \ddot{x}^\mu + \left[ 2 \dot{x}^2 \dddot{x}^2 - 3 (\dot{x} \dddot{x})^2 \right] \dot{x}^\mu \right\}
\]

As a result, the energy-momentum vector acquires the form

(3.7)

\[
P^\mu = \frac{dx_\mu}{ds} (2 L' k_1 - L) + \frac{d^2x_\mu}{ds^2} \frac{dk_1}{ds} \left( \frac{L'}{k_1^2} - \frac{L''}{k_1} \right) - \frac{L'}{k_1} \frac{d^3x_\mu}{ds^3}.
\]

Further we shall use the direct consequences of the definitions (2.2) and (2.3)

(3.8)

\[
\frac{dx_\mu}{ds} \frac{d^2x_\mu}{ds^2} = 0; \quad \frac{dx_\mu}{ds} \frac{d^3x_\mu}{ds^3} = k_1^2; \quad \frac{d^2x_\mu}{ds^2} \frac{d^3x_\mu}{ds^3} = -k_1 \frac{dk_1}{ds}.
\]

The second curvature (or torsion), \( k_2 \), is defined in the following way [27]

(3.9)

\[
k_1^2 k_2^2 = \det_G \left( \frac{dx_\mu}{ds}, \frac{d^2x_\mu}{ds^2}, \frac{d^3x_\mu}{ds^3} \right),
\]

where \( \det_G (dx_\mu/ds, d^2x_\mu/ds^2, d^3x_\mu/ds^3) \) is the Gramm determinant for vectors \( dx_\mu/ds, d^2x_\mu/ds^2, d^3x_\mu/ds^3 \) [28]. It enables one to express \( (d^3x_\mu/ds^3)^2 \) in terms of \( k_1 \) and \( k_2 \)

(3.10)

\[
\left( \frac{d^3x_\mu}{ds^3} \right)^2 = k_1^4 - \left( \frac{dk_1}{ds} \right)^2 - k_1^2 k_2^2.
\]

Bearing in mind eq. (2.25), the torsion \( k_2 \) can be eliminated from (3.10)

(3.11)

\[
\left( \frac{d^3x_\mu}{ds^3} \right)^2 = k_1^4 - \left( \frac{dk_1}{ds} \right)^2 - k_1^2 \frac{C^2}{(L'(k_1))^4}.
\]

Now we square the right and the left hand sides of eq. (3.7) and take into account (3.8) and (3.11). As a result, we obtain

(3.12)

\[
M^2 \equiv P^2 = L^2 - \left( \frac{d}{ds} L' \right)^2 - 2 L L' k_1 + (L')^2 k_1^2 - \frac{C^2}{(L')^2}.
\]

It turns out that eq. (3.12) is the first integral for the Euler-Lagrange equation (2.26) derived in the previous Section. If \( L'' \neq 0 \), one can be convinced of this relevant result by direct differentiation of (3.12) with respect to \( s \). For Lagrangians linear in \( k_1 \) equations (2.26) and (3.12) should be treated as independent ones because the differentiation of (3.12) identically gives zero (see Appendix A).
It follows that the order of eq. (2.26) can be reduced by one. From (3.12) we obtain

\[
\frac{dk_1}{ds} = \pm \sqrt{f(k_1)},
\]

where

\[
f(k_1) = \frac{1}{(\mathcal{L}')^2} \left\{ \mathcal{L}^2 - 2 \mathcal{L} \mathcal{L}' k_1 + (\mathcal{L}')^2 k_1^2 - \frac{C^2}{(\mathcal{L}')^2} - M^2 \right\}.
\]

Integration of (3.13) gives

\[
\int_{k_{10}}^{k_1} \frac{dx}{\sqrt{f(x)}} = \pm (s - s_0),
\]

where \(k_{10} = k_1(s_0)\).

Thus, formula (3.15) defines the curvature of the worldline as a function of \(s\). Equations (2.23) and (2.25) determine the remaining curvatures. As a result, for arbitrary Lagrangian \(\mathcal{L}(k_1)\), the problem of finding the principal curvatures of the worldline is reduced to quadratures.

Another interesting result can be obtained from the study of eqs. (3.4), (3.5), that enables us to derive the relation (2.25) between curvature and torsion in a new way. It turns out that the integration constant \(C\) can be expressed in terms of the particle spin and mass.\footnote{In the models defined by action (2.1) spin of the particle proves to be non zero at the classical level already. Its value is ultimately determined by the initial conditions for corresponding equations of motion.} Let us show this explicitly.

The invariance of the action (2.1) under Lorentz transformations entails the conservation of the angular-momentum tensor

\[
M_{\mu\nu} = \sum_{a=1}^{2} (q_{a\mu}p_{a\nu} - q_{a\nu}p_{a\mu}),
\]

where the canonical variables \(q^a_{\mu}\) and \(p^a_{\mu}\) are defined as follows

\[
q_{1\mu} = x_{\mu}, \quad q_{2\mu} = \dot{x}_{\mu},
\]

\[
p_{1\mu} = P_{\mu} = -\frac{\partial L}{\partial \dot{x}_\mu} - \frac{dp_2}{d\tau}, \quad p_{2\mu} = -\frac{\partial L}{\partial \dot{x}_\mu}.
\]

The spin \(S\) of the particle will be also a conserved quantity. In the case of \(D\)-dimensional space-time spin \(S\) is defined by [29]

\[
S^2 = \frac{W}{M^2},
\]

where

\[
W = \frac{1}{2} M_{\mu\nu}M^{\mu\nu} p_1^2 - (M_{\mu\nu}p_1^\mu)^2, \quad M^2 = p_1^2.
\]
For \( D = 4 \) the invariant \( W \) is the squared Pauli-Lubanski vector with sign minus

\[
W = -w_\mu w^\mu, \quad w_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} p_1^\sigma.
\]

From these equations it follows

\[
M^2 S^2 = k_2^2 (\mathcal{L}')^4.
\]

Hence, the integration constant \( C \) in (2.25) is given by

\[
C^2 = M^2 S^2.
\]

After quantization, \( S^2 \) in (3.21) should be replaced in the following way

\[
S^2 \rightarrow S (S + D - 1), \quad S = 0, 1, \ldots,
\]

where \( D \) is the dimension of the space-time.

Therefore, dealing with the Euler-Lagrange equations written in terms of the curve coordinates \( x^\mu(s) \), we have derived the basic equations (3.12) and (3.20) for the principal curvatures \( k_1 \) and \( k_2 \) and have specified the integration constants.\(^3\) Probably, the equations (2.23) can be derived in this way also. However, checking this possibility is beyond the scope of our consideration.

In conclusion of this section a general note concerning the mass spectrum in the models in question should be done. From eqs. (3.12) and (3.21) it follows that the tachyonic states with \( M^2 < 0 \) are unavoidably present in these theories already at the classical level \([30]\). It takes place at any values of the particle spin \( S \).

### 4 Particle with maximal proper acceleration

We apply the approach developed above to the model of a relativistic particle with maximal proper acceleration. The Lagrangian of this model is uniquely determined by a modified form of the infinitesimal space-time interval

\[
d\tilde{s}^2 = \left( 1 - L_0^2 \left( \frac{d^2 x_\mu}{ds^2} \right)^2 \right) ds^2,
\]

where \( L_0 \) is a fundamental constant with dimension of length. The physical motivation for considering such a metric form can be found in papers \([13 – 18]\). Here we make only a short comment: mathematically, the introduction of the line element (4.1) (the metric structure of the second order) means that submanifolds in the Minkowski space-time are now Kawaguchi spaces \([31]\) rather than Riemannian spaces. In particular, the worldline

\(^2\)In paper \([4]\) the Euler-Lagrange equations in terms of \( x^\mu(s) \) have been used for obtaining some auxiliary conditions that should be satisfied by \( k_1(s) \).
should be geometrically treated as a one-dimensional Kawaguchi space. In pure geometry
the Kawaguchi spaces are considered for a long time [32, 33].

Substituting the interval $ds$ in a standard action for spinless relativistic particle

\begin{equation}
S_0 = -m \int ds
\end{equation}

by $d\tilde{\sigma}$ we obtain

\begin{equation}
S = -\mu_0 \int \sqrt{M_0^2 - k_1^2} \, ds,
\end{equation}

where $\mu_0 = m/M_0$, $M_0 = L_0^{-1}$. When $M_0^2 \to \infty$ the action (4.3) reduces to (4.2).

Obviously, the investigation of the Euler-Lagrange equations for action (4.3) in terms
of particle coordinates is practically hopeless task. However, by making use of the method
developed above, we can easily show that the acceleration is actually superiorly limited by
$M_0^2$. We shall make use of eqs. (3.13) – (3.15), substituting there

\begin{equation}
\mathcal{L}(k_1) = -\mu_0 \sqrt{M_0^2 - k_1^2}.
\end{equation}

Integral in eq. (3.15) can be easily expressed in terms of the elementary functions. However, the final formula turn out to be rather complicated. For simplicity, we exactly
integrate eq. (3.13) for particle spin $S$ equal to zero. From eqs. (3.13), (3.14) and (4.4)
we deduce

\begin{equation}
\left( \frac{dk}{d\bar{s}} \right)^2 = (1 - k^2) \left[ 1 - \mu^{-2} (1 - k^2) \right],
\end{equation}

where $k(\bar{s})$ is the dimensionless proper acceleration, $k^2 = k_1^2/M_0^2$, and $\bar{s} = M_0 s$ is the
dimensionless length of worldline. Here a new parameter $\mu$ is introduced, $\mu^2 = (m/M)^2$,
where $M$ is the particle mass $M^2 = P^2$. We confine ourselves to a positive $M^2$ (tachyonic
solutions are not considered).

Obviously, the initial values for $k(\bar{s})$ should belong to the interval

\begin{equation}
0 \leq k^2 < 1.
\end{equation}

Therefore in the solutions of the equation (4.5) we have to take only branches that have
common points with the interval (4.6). Depending on the value of the parameter $\mu^2$ we
have here two possibilities.

If

\begin{equation}
\mu^2 = \frac{m^2}{M^2} \leq 1,
\end{equation}

then $k^2(\bar{s})$ varies in the region

\begin{equation}
1 - \mu^2 \leq k^2 < 1
\end{equation}

and it is defined by

\begin{equation}
k^2(\bar{s}) = 1 - \mu^2 \frac{(\coth \bar{s})^2 - 1}{(\coth \bar{s})^2 - \mu^2}.
\end{equation}
The constant $s_0$ in eq. (3.15) is chosen in such a way that

\begin{equation}
(4.10) \quad k^2(0) = 1 - \mu^2.
\end{equation}

From (4.9) it follows that

\begin{equation}
(4.11) \quad k^2(\bar{s}) \to 1^{-}
\end{equation}

when $\bar{s} \to \pm \infty$. If

\begin{equation}
(4.12) \quad \mu^2 = \frac{m^2}{M^2} > 1,
\end{equation}

then $k^2$ varies in the region

\begin{equation}
(4.13) \quad 0 \leq k^2 < 1
\end{equation}

and it is defined by

\begin{equation}
(4.14) \quad k^2(\bar{s}) = 1 - \mu^2 \frac{(\tanh \bar{s})^2 - 1}{(\tanh \bar{s})^2 - \mu^2}.
\end{equation}

The boundary values of $k$ in this case are

\begin{equation}
(4.15) \quad k^2(0) = 0, \quad k^2(\bar{s}) \to 1^{-} \text{ if } \bar{s} \to \pm \infty.
\end{equation}

The behaviour of $k^2(\bar{s})$ near 1 can be easily investigated in the general case of nonzero spin $S$. To this end an approximate differential equation for $k^2(\bar{s})$ should be used. If $k^2 \to 1^{-}$ then we get from eqs. (3.13), (3.14) and (4.4)

\begin{equation}
(4.16) \quad \left( \frac{dk}{d\bar{s}} \right)^2 \approx 1 - k^2, \quad k^2 \to 1^{-}.
\end{equation}

Integration of this equation in the range of $k^2$ under consideration gives

\begin{equation}
(4.17) \quad k^2(\bar{s}) \approx \tanh^2(\bar{s} - \bar{s}_0), \quad \bar{s} \to \pm \infty.
\end{equation}

Hence, the proper acceleration of the particle obeys the restriction

\begin{equation}
(4.18) \quad k^2(\bar{s}) < 1.
\end{equation}

In conclusion of this section it should be noted the following. In the models with the Lagrangians (see papers [4, 30, 34 – 36])

\begin{equation}
(4.19) \quad \mathcal{L}(k_1) = -m - \alpha k_1(s),
\end{equation}

\begin{equation}
(4.20) \quad \mathcal{L}(k_1) = -\alpha k_1(s)
\end{equation}

the particle spin $S$ and its mass $M$ are not independent; they have to obey a mass-spin relation determining the Regge trajectory (see Appendix A). In the model under consideration with action (4.3) $M$ and $S$ are arbitrary independent constants of motion. This distinction is easily explained by different numbers of constraints in the phase space in these models. The point is that all of these Lagrangians are singular or degenerate ones and as a result the corresponding phase spaces are restricted by constraints. In the model (4.19) there are four Hamiltonian constraints and in the model (4.20) the number of such constraints is five. In the model (4.3) there are only two constraints (see Appendix B where the Hamiltonian formalism for this model is constructed). This means that the models (4.19) and (4.20) possess extra symmetries in addition to the reparametrization invariance and such symmetries are absent in the model (4.3).
5 Conclusion

The formulae (2.20) – (2.26), (3.13) – (3.15), and (3.17) provide a complete solution to the problem of obtaining equations of motion for arbitrary Lagrangians $\mathcal{L}(k_1)$ in terms of the principal curvatures of the worldline, and of integrating these equations by quadratures; moreover, the integration constants are expressed in terms of the particle mass $M$ and its spin $S$. For comparison sake, we note that in the book [23] and in paper [26] the complete integrability of the equations of motion was been proved only for the simplest Lagrangians $\mathcal{L}(k_1)$ linear and quadratic in $k_1$, without elucidating the physical meaning of the integration constants.

In present paper we do not touch upon the problem of recovering the world trajectory by means of its principal curvatures. If all the curvatures $k_i$, $i = 1, 2, \ldots, D - 1$ are constants, then this problem can be solved easily (see, for example, [27]). Obviously, in general case one encounters difficulties. However, it seems to us that the key properties of the dynamics in the models under consideration are determined fully enough. We have relate the physical characteristics of the particle (its mass and spin) with geometrical invariants (principal curvatures) of its world trajectory.

As regards the model (4.3), it gives us nontrivial example of consistent relativistic dynamics obeying the principle of superiorly limited proper acceleration.

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Appendix A

Here Lagrangians linear in $k_1$ (see eqs. (4.19) and (4.20)) will be considered in the framework of the geometrical approach developed in the present paper. For Lagrangian (4.19) equations (2.20) and (3.21) give

\[ k_1^2 (1 - \alpha) + k_1 m + \alpha^{-3} M^2 S^2 = 0. \]

Hence, curvature of the world trajectory should be a constant expressed in terms of the particle mass $M$ and spin $S$ and the parameters of the model $m$ and $\alpha$. In view of this
equation (3.12) reduces to the following mass-spin relation

\[ M^2 = \frac{m^2}{1 + \alpha^{-2}S^2}. \]  

(A.2)

Transition to the quantum theory can be made by the substitution (3.22). Therefore, we obtained the basic results concerning this model without dealing with the constraints that appear in the Lagrangian or Hamiltonian treatments of this model.

In the case of the Lagrangian (4.20) we easily deduce from eq. (2.21) that the torsion \( k_2 \) in this model should be a constant. Then it follows from (3.12) and (3.20) that the mass and spin are equal to zero. Equation (3.12) can be rewritten as

\[ P^2 = -\alpha^{-2}C^2 = \alpha^{-2}W. \]  

(A.3)

It means that Pauli-Lubanski vector \( w^\mu \) is now isotropic as well as \( P^\mu \) and they are proportional each other. Therefore, parameter \( \alpha \) is a helicity of the massless particle. In quantum case it takes only integer and half-odd-integer values.

**Appendix B**

Here we shortly consider the Hamiltonian formalism for the model (4.3). Canonical variables \( q_\alpha^a, p_\alpha^a, a = 1, 2 \) are introduced according to eqs. (3.17). By making use of the definition of \( p_2^\mu \) in (3.17) and the explicit form for \( L \) in (4.3) we find primary constraint

\[ \varphi(q, p) = p_2^\mu q_2^\mu \approx 0, \]  

(B.1)

where \( \approx \) means a weak equality [33]. There are no other primary constraints in the model under consideration. It is an essential distinction of this model as compared with the models (4.19) and (4.20).

Canonical Hamiltonian is introduced in the standard way

\[ H = -p_1^\mu \dot{x}_\mu - p_2^\mu \dot{x}_\mu - L. \]  

(B.2)

In terms of the canonical variables \( H \) assumes the form

\[ H = -p_1 q_2 + M_0 \sqrt{q_2^\mu (\mu_0^2 - q_2^\nu q_2^\nu)}. \]  

(B.3)

The dynamics in the phase space is determined by the total Hamiltonian

\[ H_T = H + \lambda(\tau) \varphi(q, p), \]  

(B.4)

where \( \lambda(\tau) \) is the Lagrange multiplier. The equations of motion in the phase space are written as follows

\[ \frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \{ f, H_T \}, \]  

(B.5)
where $f$ is an arbitrary function of the canonical variables and evolution parameter $\tau$ and \{\ldots, \ldots\} stands for the Poisson brackets

$$\{f, g\} = \sum_{a=1}^{2} \left( \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q_a} - \frac{\partial f}{\partial q_a} \frac{\partial g}{\partial p_a} \right).$$

(B.6)

Requirement of the stationarity of the primary constraint (A.1)

$$\frac{d\varphi}{d\tau} = \{\varphi, H_T\} \approx 0$$

(B.7)

results in secondary constraint

$$\{\varphi, H_T\} = \{\varphi, H\} = H \approx 0.$$  

(B.8)

As one would expect, the canonical Hamiltonian in the model in question vanishes in a weak sense. It is actually a consequence of the reparametrization invariance of the initial action (4.3).

Finally, in the model (4.3) there are only two constraints in the phase space, $\varphi$ and $H$. The Hamiltonian formalism presented here can be used as the basis of the canonical quantization of the model under consideration.
References

[1] A. M. Polyakov: Nucl. Phys., B286, 406 (1986).

[2] H. Kleinert: Phys. Lett., B174, 335 (1986).

[3] M. S. Plyushchay: Mod. Phys. Lett., A3, 1299 (1988); Phys. Lett., B253, 50 (1991).

[4] H. Arod´ z, A. Sitarz and P. Wegrzyn: Acta Phys. Polonica, B20, 921 (1989).

[5] M. Pavsic: Phys. Lett., B205, 231 (1988).

[6] J. Grundberg, J. Isberg, U. Lindström and H. Nordström: Phys. Lett., B231, 61 (1989).

[7] J. Isberg, U. Lindström and H. Nordström: Mod. Phys. Lett., A5, 2491 (1990).

[8] A. M. Polyakov: Mod. Phys. Lett., 3A, 325 (1988).

[9] V. V. Nesterenko: Class. Quantum Grav. 9, 1101 (1992).

[10] M. S. Plyushchay: Phys. Lett., B235, 47 (1990); B262, 71 (1991); it Nucl. Phys., B362, 54 (1991).

[11] S. Iso, C. Itoi and H. Mukaida: Phys. Lett., B236, 287 (1990); Nucl. Phys., B346, 293 (1990).

[12] A. L. Kholodenko: Ann. Phys., 202, 186 (1990).

[13] E. R. Caianiello: II Nuovo Cimento, B59, 350 (1980).

[14] E. R. Caianiello: Lett. Nuovo Cimento, 32, 65 (1981).

[15] E. R. Caianiello: La rivista del Nuovo Cimento, 15 n.4, (1992)

[16] G. Scarpetta: Nuovo Cimento, 41, 51 (1984).

[17] E. R. Caianiello, A. Feoli, M. Gasperini, and G. Scarpetta: Int. J. Theor. Phys., 29, 131 (1990).

[18] E. R. Caianiello, M. Gasperini, and G. Scarpetta: Nuovo Cimento, 105B, 259 (1990).

[19] G. Fiorentini, M. Gasperini, and G. Scarpetta: Mod. Phys. Lett., A6, 2033(1991).

[20] E. R. Caianiello: Lett. Nuovo Cimento, 41, 370, (1984)

[21] W. R. Wood, G. Papini, Y. Q. Cai, Nuovo Cimento, 104 B, 361, (1989)
[22] A. Feoli, G. Scarpetta: *Accelerated Strings with limited proper acceleration*. In “Structure: from Physics to General Systems” Ed.s M. Marinaro and G. Scarpetta, (World Scientific, Singapore, 1992)

[23] P. A. Griffiths: *Exterior differential systems and the calculus of variations* (Birkhäuser, Boston, 1983).

[24] L. P. Eisenhart: *Riemannian Geometry* (Princeton University Press, Princeton, 1964).

[25] M. P. Do Carmo: *Differential Geometry of Curves and Surfaces* (Prentice-Hall, London, 1976).

[26] T. Dereli, D. H. Hartley, M. Önder, and R. W. Tucker: *Phys. Lett.*, B252, 601 (1990).

[27] Yu. A. Aminov: *Differential Geometry and Topology of Curves* (Nauka, Moscow, 1987).

[28] G. A. Korn and T. M. Korn: *Mathematical Handbook* (McGraw-Hill, 1968).

[29] S. S. Schweber: *An Introduction to Relativistic Quantum Field Theory* (Row Peterson, New York, 1961).

[30] V. V. Nesterenko: *J. Phys. A: Math. Gen.*, 22, 1673 (1989).

[31] A. Kawaguchi: *Proc. Imp. Acad. Tokyo*, 13, 237 (1937).

[32] Tonooka Keinosuke: *J. Math. Soc. Jap.*, No. 4, (1952).

[33] S. L. Synge: *Amer. J. Math.*, 57, 679 (1935).

[34] V. V. Nesterenko: *Int. J. Mod. Phys.*, A6, 3989 (1991).

[35] M. S. Plyushchay: *Mod. Phys. Lett.*, A4, 837 (1988); *Int. J. Mod. Phys.*, A4, 3851 (1989).

[36] C. Batlle, J. Gomis, J. M. Pons, and N. Roman-Roy: *J. Phys. A: Math. Gen.*, 21, 2693 (1989).

[37] P. A. M. Dirac: *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, 1964).