ON THE GEOMETRY OF KÄHLER–POISSON STRUCTURES

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Abstract. We prove that the Riemannian geometry of almost Kähler manifolds can be expressed in terms of the Poisson algebra of smooth functions on the manifold. Subsequently, Kähler–Poisson algebras are introduced, and it is shown that a corresponding purely algebraic theory of geometry and curvature can be developed. As an illustration of the new concepts we give an algebraic proof of the statement that a bound on the (algebraic) Ricci curvature induces a bound on the eigenvalues of the (algebraic) Laplace operator, in analogy with the well-known theorem in Riemannian geometry.

As the correspondence between Poisson brackets of smooth functions and commutators of operators lies at the heart of quantization, a purely Poisson algebraic proof of, for instance, such a “Gap Theorem”, might lead to an understanding of spectral properties in a corresponding quantum mechanical system.

1. Introduction

In a series of papers, the possibility of expressing differential geometry of Riemannian submanifolds as Nambu-algebraic expressions in the function algebra has been investigated [AHH10a, AHH10b, AHH10c, AHH12]. More precisely, it was shown that on an $n$-dimensional submanifold $\Sigma$, geometric objects can be written in terms of a $n$-ary alternating multi-linear map acting on the embedding functions. One of the original motivations for studying the problem came from matrix regularizations of surfaces in the context of “Membrane Theory” (cp. [Hop82]), where smooth functions are mapped to hermitian matrices such that the Poisson bracket of functions correspond to the commutator of matrices (as the matrix dimension becomes large). In this context, matrices corresponding to the embedding coordinates of a surface arise as solutions to equations, which contain matrices associated to surfaces of arbitrary genus. In order to identify the topology of a solution, it is desirable to be able to compute geometric invariants in terms of the embedding matrices and their commutators. This was illustrated in [AHH10a] where formulas for the discrete scalar curvature and the discrete genus were presented.

A natural generalization to higher dimensional manifolds is to require that the $n$-ary multi linear bracket corresponds to a $n$-ary “commutator” of matrices. However, there is no natural candidate for such a $n$-ary map, and it is hard to construct explicit realizations. One may then ask the following question: Is there a particular class of manifolds (of dimension greater than two) for which one can use a Poisson bracket on the space of smooth functions to express geometric quantities? In the following we shall demonstrate that almost Kähler manifolds provide a context where an affirmative answer can be given.

Once a theory of Riemannian differential geometry in terms of Poisson brackets has been developed, one wonders whether the obtained formulas make sense in an arbitrary Poisson algebra? That is, can one use the results to develop a theory
of Poisson algebraic geometry? As expected, this is not possible in general, and one might search for an intrinsic definition of algebras for which a theory of differential geometry can be implemented. It turns out that starting from the simple assumption that the square of the Poisson bivector is (proportional to) a projection operator, one can in fact reproduce several standard results with purely algebraic methods. For instance, if the sectional curvature is independent of the choice of tangent plane, then it follows that the sectional curvature is in the center of the Poisson algebra (i.e. a “Poisson-constant”). Moreover, one can prove that a bound on the Ricci curvature induces a bound on the eigenvalues of the Laplace operator. This framework opens up for an algebraic treatment of Riemannian geometry, which have many potential applications. In particular, as the correspondence between Poisson brackets and operator commutators lies at the heart of quantization, the presented results should have an impact on the quantization of geometrical systems. Moreover, concerning the original motivation, our result fits nicely with the fact that for any (quantizable) compact Kähler manifold there exists a matrix regularization [BMS94], and it suggests a way to define matrix regularizations without any reference to a manifold. Finally, we hope that it is possible to extend the results to non-commutative Poisson algebras, providing an interesting approach to non-commutative geometry.

The structure of the paper is as follows. In Section 2 we recall some concepts for Riemannian submanifolds, and in Section 3 Kähler–Poisson structures are introduced together with some basic results. Section 4 is devoted to the reformulation of Riemannian geometry of Kähler submanifolds in terms of Poisson brackets of the embedding coordinates. In Section 5 we formulate a theory of algebraic Riemannian geometry for Kähler–Poisson algebras and show that analogues of differential geometric theorems can be proven in the purely algebraic setting.

2. Preliminaries

Let \((M, \eta)\) be a Riemannian manifold of dimension \(m\), and let \((\Sigma, g)\) be a \(n\)-dimensional submanifold of \(M\) with induced metric \(g\). Given local coordinates \(x^1, \ldots, x^m\) on \(M\), we consider \(\Sigma\) as embedded in \(M\) via \(x^i(u^1, \ldots, u^n)\) where \(u^1, \ldots, u^n\) are local coordinates on \(\Sigma\). Indices \(i, j, k, \ldots\) run from 1 to \(m\) and indices \(a, b, c, \ldots\) run from 1 to \(n\). The covariant derivative on \(M\) is denoted by \(\bar{\nabla}\) (with Christoffel symbols \(\bar{\Gamma}^i_{jk}\)) and the covariant derivative on \(\Sigma\) by \(\nabla\) (with Christoffel symbols \(\Gamma^a_{bc}\)). The tangent space \(T\Sigma\) is regarded as a subspace of the tangent space \(TM\) and at each point of \(\Sigma\) one can choose \(e_a = (\partial_a x^i)\partial_i\) as basis vectors in \(T\Sigma\), and in this basis we define \(g_{ab} = \eta(e_a, e_b)\). Moreover, we introduce an orthonormal basis of \(T\Sigma^\perp\), given by the vectors \(N_A\) for \(A = 1, \ldots, p\).

The formulas of Gauss and Weingarten split the covariant derivative in \(M\) into tangential and normal components as

\[
\bar{\nabla}_X Y = \nabla_X Y + \alpha(X, Y) \tag{2.1}
\]

\[
\bar{\nabla}_X N_A = -W_A(X) + D_X N_A \tag{2.2}
\]

where \(X, Y \in T\Sigma\) and \(\bar{\nabla}_X Y, W_A(X) \in T\Sigma\) and \(\alpha(X, Y), D_X N_A \in T\Sigma^\perp\). By expanding \(\alpha(X, Y)\) in the basis \(\{N_1, \ldots, N_p\}\) one can write (2.1) as

\[
\bar{\nabla}_X Y = \nabla_X Y + \sum_{A=1}^{p} h_A(X, Y)N_A, \tag{2.3}
\]
and we set \( h_{A,ab} = h_A(e_a, e_b) \). From the above equations one derives the relation
\[
(2.4) \quad h_{A,ab} = -\eta(e_a, \nabla_b N_A),
\]
as well as Weingarten’s equation
\[
(2.5) \quad h_A(X, Y) = \eta(W_A(X), Y),
\]
which implies that \((W_A)^a_b = g^{ac} h_{A,cb}\).

From formulas (2.1) and (2.2) one also obtains Gauss’ equation, i.e. an expression for the curvature \( R \) of \( \Sigma \) in terms of the curvature \( \bar{R} \) of \( M \), as
\[
(2.6) \quad R(X, Y, Z, V) = \bar{R}(X, Y, Z, V) + \eta(\alpha(X, Z), \alpha(Y, V)) - \eta(\alpha(X, V), \alpha(Y, Z))
\]
where \( R(X, Y, Z, V) \equiv \eta(R(Z, V)Y, X) \) and \( X, Y, Z, V \in T\Sigma \).

3. Kähler Poisson structures

In case \( \Sigma \) is a surface, a generic Poisson structure may be written as
\[
\{f_1, f_2\} = \frac{1}{\rho} \varepsilon^{ab}(\partial_a f_1)(\partial_b f_2)
\]
for some density \( \rho \), where \( \varepsilon^{ab} \) is the totally anti-symmetric Levi-Civita symbol. Setting \( g^{ab} = \varepsilon^{ab}/\rho \) and \( g = \det(g_{ab}) \), one notes that
\[
(3.1) \quad \frac{g}{\rho^2} g^{ab} = g^{ap} g^{bq} g_{pq} = \frac{1}{\rho} \varepsilon^{ap} \varepsilon^{bq} g_{pq}
\]
since the right hand side is simply the cofactor expansion of the inverse metric. If \( \Sigma \) is embedded in \( M \) via the embedding coordinates \( x^i \), the above relation allows one to write many of the differential geometric objects of \( \Sigma \) in terms of \( \{x^i, x^j\} \) \cite{AHH10c}. It was also shown that for higher dimensional submanifolds one obtains a similar description using \( n \)-ary Nambu brackets \cite{AHH12, AHH10b}. One may ask if there is a class of \( n \)-dimensional submanifolds (with \( n > 2 \)) that allows for a description in terms of Poisson brackets? It turns out that relation (3.1) plays a key role in the answer to this question. Therefore, we make the following definition.

**Definition 3.1.** Let \((\Sigma, g)\) be a Riemannian manifold and let \( \theta \) be a Poisson bivector on \( \Sigma \). If there exists a strictly positive \( \gamma \in C^\infty(\Sigma) \) such that
\[
(3.2) \quad \gamma^2 g^{-1}(\sigma, \tau) = g(\theta(\sigma), \theta(\tau))
\]
for all 1-forms \( \sigma, \tau \), then \( \theta \) is called an almost Kähler–Poisson structure on \((\Sigma, g)\). In local coordinates, the above relation is equivalent to
\[
(3.3) \quad \gamma^2 g^{ab} = g^{ap} g^{bq} g_{pq}.
\]
Moreover, we call the function \( \gamma^2 \) the characteristic function of the almost Kähler–Poisson structure.

Note that such “self-dual” Poisson structures, and corresponding geometric formulas, have also been studied in the context of matrix models for gravity \cite{BS10a, BS10b}.

The reason for calling it an almost Kähler–Poisson structure is that the Poisson structure induced from the Kähler form always fulfills (3.2), and that the existence of an almost Kähler–Poisson structure implies that \( \Sigma \) is an almost Kähler manifold.
Proposition 3.2. Let \((\Sigma, g)\) be an almost Kähler manifold with Kähler form \(\omega\). The Poisson bracket on \(C^\infty(\Sigma)\), given as

\[
\{f_1, f_2\} = \omega(X_{f_1}, X_{f_2})
\]

(3.4)

where \(\omega(X_f, Y) = df(Y)\) for all \(Y \in T\Sigma\), defines an almost Kähler–Poisson structure on \(\Sigma\) with characteristic function \(\gamma^2\).

Proof. It is a standard fact that formula (3.4) defines a Poisson bracket for any symplectic form \(\omega\), and the Poisson bivector is the inverse of \(\omega\). Let us now show that formula (3.3) is fulfilled with \(\gamma^2 = 1\).

Let \(J^a_b\) be the almost complex structure of \(\Sigma\). The Kähler form is then written as \(\omega_{ab} = g_{ac}J^c_b\). The inverse of \(\omega\) is then computed to be \(\theta_{ab} = -\omega_{ab} = \omega_{pq}g^{pa}g^{qb} = -J^a_cg_{cb}\).

Using the fact that \(g\) is an hermitian metric one obtains

\[
\theta_{ap} \theta_{bq} g^{pq} = J^a_r g^{rp}J^b_s g^{sq}g_{pq} = g^{rp}J^a_r J^b_s = g^{ab},
\]

which proves that \(\theta\) is an almost Kähler–Poisson structure with \(\gamma^2 = 1\). \(\square\)

On the other hand, it immediately follows from (3.3) that \(J^a_b \equiv \gamma^{-1}\theta^a_b\) is an almost complex structure, i.e. \(J^2(X) = -X\) for all \(X \in T\Sigma\). However, it is not necessarily integrable. Given a Riemannian manifold \((\Sigma, g)\) with an almost Kähler–Poisson structure \(\theta\) one can define \(\tilde{g} = \gamma^{-1}g\), and since \(\theta\) is Kähler–Poisson structure it is easy to see that \(\tilde{g}\) is an hermitian metric with respect to the complex structure defined above. Furthermore, one defines the Kähler form \(\Omega\) as

\[
\Omega(X, Y) = \tilde{g}(J(X), Y),
\]

which implies that \(\Omega_{ab} = -\frac{1}{\gamma^2}\theta_{ab}\). Since \(\theta\) is an almost Kähler–Poisson structure, it follows that \(\Omega\) is the inverse of \(\theta\), i.e. \(\Omega_{ab}\theta^{bc} = \delta^c_a\), which implies that \(d\Omega = 0\) since \(\theta\) satisfies the Jacobi identity. Hence, \((\Sigma, \tilde{g}, J)\) is an almost Kähler manifold. Thus, we have proved the following statement:

Proposition 3.3. Let \((\Sigma, g)\) be a Riemannian manifold such that there exists a Kähler–Poisson structure \(\theta\) with characteristic function \(\gamma^2\). Then \((\Sigma, \gamma^{-1}g, J)\) is an almost Kähler manifold, with \(J^a_b = \gamma^{-1}\theta^a_b\).

4. Differential geometry of embedded Kähler–Poisson structures

In the following we shall assume that \(\Sigma\) is a submanifold of \(M\) and that there exists an almost Kähler–Poisson structure \(\theta\) on \(\Sigma\) with characteristic function \(\gamma^2\). The submanifold \(\Sigma\) is embedded in \(M\) via the embedding coordinates \(x^1, \ldots, x^m\). Our main goal is to express geometric properties of \(\Sigma\) in terms of Poisson brackets of embedding coordinates (and components of normal vectors) and the geometric objects of the ambient manifold \(M\), such as the metric \(\eta_{ij}\) and the Christoffel symbols \(\Gamma^i_{jk}\). In particular, derivatives should only appear as Poisson brackets.

Let us define

\[
P^{ij} = \{x^i, x^j\},
\]

(4.1)
which will also be considered as a map $\mathcal{P} : TM \to TM$ through $\mathcal{P}(X) = \mathcal{P}^{ij} \eta_{jk} X^k \partial_l$.

Furthermore, for all $u \in C^\infty(\Sigma)$ we introduce

\begin{equation}
D^i(u) = \frac{1}{\gamma} \{u, x^k\} \{x^i, x^l\} \eta_{kl}
\end{equation}

as well as

\begin{equation}
D^{ik} = D^i(x^k) \quad \hat{\nabla}^i = D^{ik} \nabla_k \quad \nabla_X Y = X^i \hat{\nabla}^i Y,
\end{equation}

where $\nabla$ is the covariant derivative on $M$. As opposed to $\nabla$, the derivative $\hat{\nabla}$ can be expressed in terms of Poisson brackets; i.e.

\begin{equation}
\hat{\nabla}^i X^k = D^i(X^k) + D^i(x^l) \Gamma_{lm}^k X^m.
\end{equation}

For upcoming calculations, it is convenient to note that

\begin{equation}
D^i(u) = \frac{1}{\gamma^2} g^{ab} \eta_{jk} \eta_i \theta^{pq} (\partial_p x^i) (\partial_q x^j) \eta_{kl} (\partial_k u) = \frac{1}{\gamma^2} g^{ab} \theta^{pq} \eta_{jk} \eta_i (\partial_k u)
\end{equation}

which is independent of $\theta$.

Let us gather some properties of $\mathcal{P}$ in the next proposition.

**Proposition 4.1.** Let $J = \gamma^{-1} \theta$ denote the associated almost complex structure on $\Sigma$. For all $X \in TM$ and $Y \in T\Sigma$ it holds that

\begin{align*}
(4.6) & \quad \mathcal{P}(Y) = \gamma J(Y) \\
(4.7) & \quad \mathcal{P}^2(X) = -\gamma^2 \eta(X, e_a) g^{ab} e_b \\
(4.8) & \quad \text{Tr} \mathcal{P}^2 \equiv (\mathcal{P}^2)_i^i = -n \gamma^2.
\end{align*}

In particular, it follows that $-\gamma^{-2} \mathcal{P}^2$ is the orthogonal projection onto $T\Sigma$.

**Proof.** Let us start by proving (4.6). By definition we obtain

\[
\mathcal{P}(Y) = \theta^{ab} (\partial_a x^i) (\partial_b x^j) Y_j \partial_i = \theta^{ab} (\partial_a x^i) (\partial_b x^j) Y^c (\partial_c x^k) \eta_{jk} \partial_i
\]

\[
= \theta^{ab} (\partial_a x^i) g_{bc} Y^c \partial_i = \theta^a_c (\partial_a x^i) Y^c \partial_i = \theta^a_c Y^c \partial_a = \gamma J(Y).
\]

Let us also prove formula (4.7). One obtains

\[
\mathcal{P}^2(X) = \theta^{ab} (\partial_a x^i) (\partial_b x^j) \eta_{jk} \theta^{pq} (\partial_p x^k) (\partial_q x^l) X_l \partial_i
\]

\[
= \theta^{ab} \theta^{pq} g_{bp} (\partial_a x^i) (\partial_q x^l) X_l \partial_i,
\]

and by using the fact that $\theta$ is an almost Kähler–Poisson structure one gets

\[
\mathcal{P}^2(X) = -\gamma^2 g^{aq} (\partial_a x^i) (\partial_q x^l) X_l \partial_i
\]

\[
= -\gamma^2 g^{aq} g(e_q, X) e_a.
\]

Formula (4.8) can be proven in a similar way. \hfill \Box

The above result shows that $\gamma^{-1} \mathcal{P}$ is an extension of the complex structure of $T\Sigma$ to $TM$ such that $-\gamma^{-2} \mathcal{P}^2$ is the orthogonal projection onto $T\Sigma$. Note that

\begin{equation}
-\frac{1}{\gamma^2} (\mathcal{P}^2)^{ij} = D^{ij}
\end{equation}
and that the projection $\Pi$, onto $T\Sigma^\perp$, can then be written as

$$\Pi^{ij} = \sum_{A=1}^{P} N^i_A N^j_A = \eta^{ij} - \mathcal{D}^{ij}. \tag{4.10}$$

For convenience, we shall also consider the map $\mathcal{D} : TM \to TM$, defined as $\mathcal{D}(X) = D^i(x^k)X_k \partial_i$. It follows from Proposition 4.1 that $\nabla_X$ coincides with $\nabla_X$ for all $X \in T\Sigma$. Namely

$$\nabla_X Y = \mathcal{D}(\nabla_X Y), \tag{4.11}$$

since $X_i D^i(x^k) = X^k$ for all $X \in T\Sigma$. Having at hand the covariant derivative in the direction of a vector in $T\Sigma$, together with the projection operator, enables us to obtain the covariant derivative on $\Sigma$ as

$$\nabla_X Y = \mathcal{D}(\nabla_X Y), \tag{4.12}$$

which can again be written in terms of Poisson brackets. In the same way, this gives us the second fundamental form as

$$\alpha(X, Y) = \Pi(\nabla_X Y). \tag{4.13}$$

Apart from $P$, there is another fundamental object given as

$$B_A^{ij} = -\gamma^2 \hat{\nabla}^i N_A^j \tag{4.14}$$

and one notes that $B_A$ can be written in terms of Poisson brackets as

$$B_A^{ij} = \{x^i, x^k\} \eta_{kl}\{x^l, N_A^j\} + \{x^i, x^k\} \eta_{kl}\{x^l, x^{m_1}\} \hat{\Gamma}^j_{m_1 m_2} N_A^{m_2}. \tag{4.15}$$

Just as $\gamma^{-1}P$ is an extension of the almost complex structure on $T\Sigma$ to $TM$, the map $\gamma^{-2}B_A$ is an extension of the Weingarten map.

**Proposition 4.2.** For $X \in TM$ it holds that

$$B_A(X) = -\gamma^2 \eta(X, \nabla_{e_a} N_A) g^{ab} e_b, \tag{4.16}$$

and for $Y \in T\Sigma$ one obtains

$$B_A(Y) = \gamma^2 W_A(Y), \tag{4.17}$$

where $W_A$ is the Weingarten map associated to the normal vector $N_A$.

**Proof.** By using (4.15) one obtains

$$B_A(X) = -\gamma^2 \left( \hat{\nabla}^i N_A^j \right) X_j \partial_i = -\gamma^2 g^{ab} (\partial_a x^i) (\hat{\nabla}_k N_A^j) X_j \partial_i$$

$$= -\gamma^2 \eta(X, \nabla_{e_a} N_A) g^{ab} e_a,$$

and for $Y \in T\Sigma$ one gets

$$B_A(Y) = -\gamma^2 Y^c \eta(e_c, \nabla_{e_a} N_A) g^{ab} e_a = \gamma^2 Y^c h_A e_b g^{ab} e_a$$

$$= \gamma^2 (W_A)^c e_a = \gamma^2 W_A(Y),$$

from the definition of the Weingarten map. \[\square\]

It turns out that the extension of the Weingarten map is such that the action on normal vectors gives the covariant derivative in $T\Sigma^\perp$.

**Proposition 4.3.** For $X \in T\Sigma$ it holds that

$$\eta(B_A(N_B), X) = -\gamma^2 (D_X)_{AB}, \tag{4.17}$$

where $(D_X)_{AB}$ is defined through $D_X N_A = (D_X)_{AB} N_B$. 


Proof. It holds that \((D_X)_{AB} = \eta(D_X N_A, N_B)\), and from Weingarten’s formula one obtains
\[
(D_X)_{AB} = \eta(\bar{\nabla}_X N_A + W_A(X), N_B) = \eta(\bar{\nabla}_X N_A, N_B).
\]
On the other hand, one gets
\[
\eta(B_A(N_B), X) = -\gamma^2 \eta(N_B, \bar{\nabla}_e N_A)g^{ab}\eta(e_b, X) = -\gamma^2 X^c\eta(N_B, \bar{\nabla}_e N_A)\delta^a_c
\]
which proves the statement. □

Recall Weingarten’s formula
\[
\bar{\nabla}_X N_A = -W_A(X) + D_X N_A
\]
for all \(X \in T\Sigma\). Having both \(W_A\) and \(D_X\) expressed in terms of \(B_A\) implies that Weingarten’s formula gives a nontrivial relation involving derivatives of normal vectors.

**Proposition 4.4.** For all \(X \in T\Sigma\) is holds that
\[
(\bar{\nabla}_i N^k_A) X_k = N^i_A (\bar{\nabla}^k \Pi_{il}) X_k,
\]
from which it follows that
\[
\bar{\nabla}_k N^k_A = N^i_A (\bar{\nabla}^i \Pi_{il}).
\]

Proof. For \(X \in T\Sigma\) we have previously shown that the following holds
\[
\bar{\nabla}_X N^i_A = \bar{\nabla}_X N^i_A = (\bar{\nabla}^k N^i_A) X_k
\]
\[
W_A(X)^i = - (\bar{\nabla}^i N^k_A) X_k
\]
\[
D_X N^i_A = X_k (\bar{\nabla}^k N^i_A)_{(N_B)} N^j_B = \Pi^i_l (\bar{\nabla}^k N^i_A) X_k.
\]
Therefore, Weingarten’s formula can be rewritten as
\[
(\bar{\nabla}^k N^i_A) X_k = (\bar{\nabla}^i N^k_A) X_k + \Pi^i_l (\bar{\nabla}^k N^i_A) X_k
\]
\[
= (\bar{\nabla}^i N^k_A) X_k + (\bar{\nabla}^k \Pi^i_l N^j_A) X_k - N^i_A (\bar{\nabla}^k \Pi^j_l) X_k
\]
\[
= (\bar{\nabla}^i N^k_A) X_k + (\bar{\nabla}^k N^i_A) X_k - N^i_A (\bar{\nabla}^k \Pi^j_l) X_k,
\]
from which it follows that
\[
(\bar{\nabla}_i N^k_A) X_k = N^i_A (\bar{\nabla}^k \Pi_{il}) X_k.
\]
In particular, one may replace \(X_k\) by \(D^i_k\), giving
\[
\bar{\nabla}_k N^k_A = N^i_A (\bar{\nabla}^i \Pi_{il}),
\]
since \(D^i_k \bar{\nabla}^k = \bar{\nabla}^i\). □

Let us now turn to Gauss’ equation and the curvature of \(\Sigma\). By using Weingarten’s equation and Proposition 4.4 one finds the following formulas.

**Proposition 4.5.** For \(X, Y, Z, V \in T\Sigma\) it holds that
\[
\eta(\alpha(X, Y), \alpha(Z, V)) = \frac{1}{2} \gamma^i \sum_{A=1}^p \eta(B_A(X), Y) \eta(B_A(Z), V)
\]
\[
= X_i Y_j Z_k V_l (\bar{\nabla}^j \Pi_{il}^m) (\bar{\nabla}^i \Pi_{jl}^m).
\]
In the same way, the scalar curvature is computed to be
\[
R(\mathbf{4.20}) = \mathbf{R}_{TTM} \text{ projection operators before tracing over } T \in T \Sigma, \text{ one immediately obtains}
\]
\[
\eta(\alpha(X,Y), \alpha(Z,V)) = X_i Y_j Z_k V_l (\hat{\nabla}^i N_A^j) (\hat{\nabla}^k N_A^l),
\]
and by applying Proposition 4.4 twice one arrives at the desired result. \qed

From Proposition 4.5 and Gauss’ equation the following result is immediate.

**Proposition 4.6.** Let \( \bar{R} \) and \( R \) be the curvature tensors of \( M \) and \( \Sigma \) respectively. For \( X, Y, Z, V \in T \Sigma \) it holds that
\[
R(X, Y, Z, V) = X^i Y^j Z^k V^l \left[ \bar{R}_{ijkl} + (\hat{\nabla}_k \Pi_{im}) (\bar{\nabla}_i \Pi^m_j) - (\bar{\nabla}_i \Pi_{im}) (\hat{\nabla}_k \Pi^m_j) \right]
\]
\[
= \bar{R}(X, Y, Z, V) + \frac{1}{\gamma^2} \sum_{A=1}^{p} \left[ \eta(\mathcal{B}_A(X), Z) \eta(\mathcal{B}_A(Y), V) - \eta(\mathcal{B}_A(X), V) \eta(\mathcal{B}_A(Y), Z) \right].
\]

To compute the Ricci curvature and the scalar curvature, one needs to take the trace over \( T \Sigma \) of the tensor in Proposition 4.6. This can be done by applying projection operators before tracing over \( TM \). That is, the Ricci tensor of \( \Sigma \) can be computed as \( R_{ik} = \bar{R}_{ijkl} \mathcal{D}^l_m \mathcal{D}^l = \bar{R}_{ijkl} \mathcal{D}^{ijkl} \), which implies that
\[
R_{ik} = \mathcal{D}^{ij} \bar{R}_{ijkl} + (\hat{\nabla}_k \Pi_{im}) (\hat{\nabla}_i \Pi^m) - (\hat{\nabla}_i \Pi_{im}) (\hat{\nabla}_k \Pi^m). \tag{4.20}
\]

In the same way, the scalar curvature is computed to be
\[
R = \mathcal{D}^{ij} \bar{R}_{ijkl} + (\hat{\nabla}_k \Pi_{im}) (\hat{\nabla}^i \Pi^m) - (\hat{\nabla}^i \Pi_{im}) (\hat{\nabla}_k \Pi^m), \tag{4.21}
\]
which (by using Proposition 4.3) is equal to
\[
R = \mathcal{D}^{ij} \bar{R}_{ijkl} + (\hat{\nabla}_k \Pi_{im}) (\hat{\nabla}^i \Pi^m) - \frac{1}{2} (\hat{\nabla}^i \Pi_{km}) (\hat{\nabla}_l \Pi_{km}). \tag{4.22}
\]

### 4.1. The Codazzi-Mainardi equations.
For submanifolds there are two fundamental sets of equations: Gauss’ equations and the Codazzi-Mainardi equations. Having considered Gauss’ equations in the previous section, let us now turn to the Codazzi-Mainardi equations. These equations express the normal component of the curvature in terms of covariant derivatives of the second fundamental form and covariant derivatives of normal vectors. That is, for all \( X, Y, Z \in T \Sigma \) it holds that
\[
\Pi(R(X, Y) Z) = \sum_{A=1}^{p} \left[ (\nabla_X h_A)(Y, Z) - (\nabla_Y h_A)(X, Z) \right] N_A
\]
\[
+ \sum_{A=1}^{p} \left[ h_A(Y, Z) D_X N_A - h_A(X, Z) D_Y N_A \right].
\]

Let us now try to rewrite these equations in terms of Poisson brackets.

**Proposition 4.7.** For \( X, Y, Z \in T \Sigma \) it holds that
\[
(\nabla_X h_A)(Y, Z) - (\nabla_Y h_A)(X, Z) = \eta \left( (\hat{\nabla}_X \gamma^{-2} \mathcal{B}_A)(Y) - (\hat{\nabla}_Y \gamma^{-2} \mathcal{B}_A)(X), Z \right)
\]
\[
= X^i Y^j Z^k \left( \hat{\nabla}_j \hat{\nabla}_k N_A - \hat{\nabla}_k \hat{\nabla}_j N_A \right).
\]
Proof. Let us start by using Weingarten’s equation to rewrite
\[
(\nabla_X h_A)(Y, Z) = X \cdot h_A(Y, Z) - h_A(\nabla_X Y, Z) - h_A(Y, \nabla_X Z) \\
= X \cdot g(W_A(Y), Z) - g(W_A(\nabla_X Y), Z) - g(W_A(Y), \nabla_X Z) \\
= g((\nabla_X W_A)(Y), Z).
\]
On the other hand, one gets
\[
\eta((\tilde{\nabla}_X \gamma^{-2} B_A)(Y), Z) = \eta((\nabla_X W_A)(Y), Z) - \frac{1}{\gamma^2} \eta(B_A(\alpha(X, Y)), Z),
\]
which implies that
\[
(\nabla_X h_A)(Y, Z) - (\nabla_Y h_A)(X, Z) = \eta\left((\tilde{\nabla}_X \gamma^{-2} B_A)(Y) - (\tilde{\nabla}_Y \gamma^{-2} B_A)(X)\right),
\]
since \(\tilde{\nabla}_X = \tilde{\nabla}_Y\) for all \(X \in T\Sigma\). The second formula is obtained by simply inserting
the definition of \(B_A\) in the above expression. \(\square\)

**Proposition 4.8.** For \(X, Y, Z \in T\Sigma\) it holds that
\[
\sum_{B=1}^{p} \left[ h_B(Y, Z)(D_X)_{BA} - h_B(X, Z)(D_Y)_{BA} \right] = X_i Y_j Z_k \left[ \tilde{\nabla}^i \nabla^j N_A^i - \tilde{\nabla}^k \nabla^j N_A^i \right].
\]

**Proof.** Since \(h_A(Y, Z) = \eta(\alpha(Y, Z), N_A)\) it follows from (4.13) that
\[
h_B(Y, Z) = \Pi^k B \left( \tilde{\nabla}^j Y_k \right) N_A^j,
\]
and from Proposition 4.8 that
\[
(D_X)_{BA} = (\tilde{\nabla}^i N_A^j) X_i (N_A)_k = -(\tilde{\nabla}^i N_A^j) X_i (N_B)_k.
\]
Thus, one obtains
\[
\sum_{B=1}^{p} h_B(Y, Z)(D_X)_{BA} = -\Pi^k \Pi^m X_j Z_l \left( \tilde{\nabla}^i Y_k \right) \left( \tilde{\nabla}^j N_A^m \right)
\]
and using the definition of \(\Pi^{ij}\) one gets
\[
X_j Z_l Y_k \left( \tilde{\nabla}^i \Pi^m \right) \left( \tilde{\nabla}^j N_A^m \right) = X_j Z_l Y_k \left( \tilde{\nabla}^i \Pi^m \tilde{\nabla}^j N_A^m \right) = X_j Z_l Y_k \left( \tilde{\nabla}^i \nabla^j N_A^k \right)
\]
Now, let us rewrite the second term using Proposition 4.8
\[
X_j Z_l Y_k \tilde{\nabla}^i (D_m N_A^j) = X_j Z_l Y_k \tilde{\nabla}^i (D_m N_A^j \tilde{\nabla}^m \Pi^i_n) = X_j Z_l Y_k \tilde{\nabla}^i (N_A^i \tilde{\nabla}^k \Pi^i_n)
\]
Comparing this with (4.23) one obtains
\[
\sum_{B=1}^{p} \left[ h_B(Y, Z)(D_X)_{BA} - h_B(X, Z)(D_Y)_{BA} \right] = X_j Z_l Y_k \left( \tilde{\nabla}^i N_A^i \right)
\]
which is equal to the stated formula. \(\square\)

Since \(\eta(N_A, \Pi(R(X, Y) Z)) = -\tilde{R}_{jki} X^i Y^j Z^k N_A^i\) one can now formulate the Codazzi-
Mainardi equations in the following way.
Proposition 4.9 (The Codazzi-Mainardi equations). For all $X, Y, Z \in T\Sigma$ it holds that

\[ X^j Y^i Z^k N_A^i R_{ijkl} = X_i Y_j Z_k \left[ \nabla^i \nabla^k - \nabla^i \nabla^k + \nabla^k \nabla^i - \nabla^i \nabla^k \right] . \]

4.2. Covariant derivatives and curvature. As we have seen, the operator $\hat{\nabla}_X$ coincides with the covariant derivative in the ambient space $M$, in the direction of a vector $X \in T\Sigma$. Thus, expressions of the type $g(\nabla_X Y, Z)$, where $X, Y, Z \in T\Sigma$, can be computed as $\eta(\nabla_X Y, Z)$. In particular, one obtains the following formulas.

Proposition 4.10. Let $\nabla$ denote the covariant derivative on $\Sigma$. For $u, v \in C^\infty(\Sigma)$ and $X \in T\Sigma$ it holds that

\begin{align}
(4.24) & \quad \nabla u = \nabla^i(u) \partial_i \\
(4.25) & \quad \text{div}(X) = \hat{\nabla}_i X^i \\
(4.26) & \quad \Delta(u) = \hat{\nabla}_i \hat{\nabla}^i(u) \\
(4.27) & \quad |\nabla^2 u|^2 = \hat{\nabla}_i \hat{\nabla}^j(u) \hat{\nabla}_j \hat{\nabla}^i(u),
\end{align}

where $\text{div}(X)$ is the divergence of $X$ on $\Sigma$, and $\Delta$ is the Laplace operator on $\Sigma$.

Proof. Let us prove equation (4.25) and equation (4.26). The remaining formulas can be proven in an analogous way. Since $D^{ik} = g^{ab}(\partial_a x^i)(\partial_b x^k)$, Gauss’ formula gives

\[ \hat{\nabla}_i X^i = D^{ik} \hat{\nabla}_i X_k = g^{ab}(\partial_a x^i) \hat{\nabla}_e x^k = g^{ab}(\partial_a x^i) \left( (\nabla_b X^c)(e_c) + \alpha(e_b, X)_i \right) \]

\[ = g^{ab} g_{ac}(\nabla_b X^c) = \nabla_b X^b = \text{div}(X). \]

We prove equation (4.26) by making use of normal coordinates on $\Sigma$. Thus, we assume that $u^1, \ldots, u^n$ is a set of normal coordinates on $\Sigma$. In particular, this implies that $\nabla_a = \partial_a$ and $\partial_a g_{bc} = 0$. With the help of Gauss formula, one computes

\[ \hat{\nabla}_i \hat{\nabla}^i(u) = \eta_{ij} g^{ab}(\partial_a x^i) \hat{\nabla}_e x^k \left( g^{pq}(\partial_p x^j)(\partial_q u) \right) \]

\[ = \eta_{ij} g^{ab}(\partial_a x^i) \hat{\nabla}_b \left( g^{pq}(\partial_q u) \right)(\partial_p x^j) \]

\[ = g^{ab} g_{ap} g^{pq}(\partial^2_{bq} u) = g^{ba}(\partial^2_{bq} u), \]

which is equal to $\Delta(u)$ in normal coordinates.

Let us investigate how the operator $\hat{\nabla}$ is related to curvature. By the very definition of curvature, it arises as the commutator of two covariant derivatives; is there a similar relation for $\hat{\nabla}$? In fact, when contracted with vectors in $T\Sigma$, $\hat{\nabla}$ fulfills a curvature equation analogous to the one of $\nabla$.

Proposition 4.11. For any $u \in C^\infty(\Sigma)$, $Z \in TM$ and $X, Y \in T\Sigma$ it holds that

\begin{align}
(4.28) & \quad X^i Y^j [\hat{\nabla}_i, \hat{\nabla}_j] Z^k = R_{ijkl} X^i Y^j Z^l \\
(4.29) & \quad X^i Y^j [\hat{\nabla}_i, \hat{\nabla}_j](u) = 0.
\end{align}
Proof. One computes
\[ X_i Y_j \hat{\nabla}^i \hat{\nabla}^j Z^k = X_i Y_j D^{il} \hat{\nabla}_i \left( D^{jm} \hat{\nabla}_m Z^k \right) \]
\[ = X_i Y_j D^{il} D^{jm} \hat{\nabla}_i \hat{\nabla}_m Z^k + X_i Y_j D^{il} (\hat{\nabla}_l D^{jm}) \hat{\nabla}_m Z^k \]
\[ = X_i Y_j D^{il} D^{jm} \hat{\nabla}_i \hat{\nabla}_m Z^k + X_i Y_j D^{il} D^{jm} \bar{R}_{nmlm} Z^n + X_i Y_j D^{il} (\hat{\nabla}_l D^{jm}) \hat{\nabla}_m Z^k. \]

Let us now prove that the last term is symmetric in \( X \) and \( Y \). From Gauss’ formula it follows that, for \( X, Y \in T \Sigma \) and \( V = U + N \), with \( U \in T \Sigma \) and \( N \in T \Sigma^\perp \), for any tensor of the form \( T_{ik} = T^{ab}(e_a)_i(e_b)_k \),
\[ (\hat{\nabla}_X \hat{T})(Y, V) = (\nabla_X T)(Y, U) - \hat{T}(Y, W_N(X)). \]

Applying this to the expression above, with \( T^{ab} = g^{ab} \) and \( U_m = \nabla_m Z^k \), gives
\[ X_i Y_j D^{il} (\hat{\nabla}_l D^{jm}) \hat{\nabla}_m Z^k = (\hat{\nabla}_X \hat{T})(Y, (\nabla^m Z^k)\partial_m) = -\hat{T}(Y, W_N(X)) \]
\[ = g^{ab}(e_a)_i(e_b)_k Y^i W_N(X)^k = g(Y, W_N(X)) \]
\[ = h_N(X, Y), \]
where the last equality is Weingarten’s equation. Thus, the expression is symmetric since the second fundamental form is symmetric. Hence, one obtains
\[ X_i Y_j \hat{\nabla}^i \hat{\nabla}^j Z^k = X_i Y_j \hat{\nabla}^i \hat{\nabla}^j Z^k + X_i Y_j D^{il} D^{jm} \bar{R}_{nmlm} Z^n \]
\[ + X_i Y_j D^{il} (\hat{\nabla}_l D^{jm}) \hat{\nabla}_m Z^k - X_i Y_j D^{jm} (\hat{\nabla}_m D^{il}) \hat{\nabla}_l Z^k \]
\[ = X_i Y_j \hat{\nabla}^i \hat{\nabla}^j Z^k + X^i Y^m \bar{R}_{nmlm} Z^n. \]

Equation (4.29) is proven in an analogous way. \( \Box \)

Let us illustrate that the operator \( \hat{\nabla} \) is also related to the curvature on \( \Sigma \). Namely, consider the following equation
\[ (\nabla^a u) \nabla_a \nabla_b \nabla^b u = (\nabla^a u) \nabla_b \nabla_a \nabla^b u - \mathcal{R}(\nabla u, \nabla u), \]
where \( \mathcal{R} \) is the Ricci curvature of \( \Sigma \), which is a particular instance of the relation between curvature and covariant derivatives on \( \Sigma \). Let us rewrite this equation in terms of \( \hat{\nabla}^i \). From Proposition 4.10 it immediately follows that
\[ (\nabla^a u) \nabla_a \nabla_b \nabla^b u = \hat{\nabla}^i(u) \hat{\nabla}_i \hat{\nabla}^j(u), \]
and from
\[ (\nabla^a u) \nabla_a \nabla^b u = 2(\nabla^a u) \nabla_b \nabla_a \nabla^b u + 2 |\nabla^2 u|^2, \]
on one obtains
\[ (\nabla^a u) \nabla_a \nabla^b u = \frac{1}{2} \hat{\nabla}^i \hat{\nabla}^j (\hat{\nabla}^i \hat{\nabla}^j(u)) - \hat{\nabla}^i \hat{\nabla}^j (u) \hat{\nabla}^i \hat{\nabla}^j(u) \]
\[ = \nabla^i \hat{\nabla}^j (u) \hat{\nabla}^j (u) + [\hat{\nabla}^i, \hat{\nabla}^j](u) \hat{\nabla}^i \hat{\nabla}^j(u), \]
by again using Proposition 4.10. Thus, we can write eq. (4.30) as
\[ \hat{\nabla}^i(u) \nabla_i \nabla^i \nabla^j(u) = \nabla^i \hat{\nabla}^j(u) \hat{\nabla}^j(u) + [\nabla^i, \hat{\nabla}^j](u) \nabla^i \hat{\nabla}^j(u) \]
\[ - \mathcal{R}(\nabla u, \nabla u). \]

Turning this equation around, it gives an expression for the Ricci curvature evaluated at \( \nabla u \). Can the same expression be directly derived from equation (4.29)? Let us now prove that it is indeed possible.
Proposition 4.12. For all $u \in C^\infty(\Sigma)$ it holds that
\[
(\tilde{\nabla}_k \Pi_{lm}) (\tilde{\nabla}_i \Pi^{lm}) \tilde{\nabla}^i (u) \tilde{\nabla}^k (u) = \tilde{\nabla}_l \left( \tilde{\nabla}^k (u) \tilde{\nabla}^i \tilde{\nabla}_k (u) \right) - \tilde{\nabla}^i \left( \tilde{\nabla}^k (u) \tilde{\nabla}_k \tilde{\nabla}_i (u) \right)
- (\tilde{\nabla}_i \Pi_{lm})(\tilde{\nabla}_k \Pi^{lm}) \tilde{\nabla}^i (u) \tilde{\nabla}^k (u) = -\tilde{\nabla}^k (u) \tilde{\nabla}_k \tilde{\nabla}^i (u) + \tilde{\nabla}^k (u) \tilde{\nabla}^i \tilde{\nabla}_k (u) - D^j \tilde{R}_{ijk} \tilde{\nabla}^k (u) \tilde{\nabla}^i (u).
\]

Proof. Let us prove the second formula. One computes
\[
- (\tilde{\nabla}_i \Pi_{lm})(\tilde{\nabla}_k \Pi^{lm}) \tilde{\nabla}^i (u) \tilde{\nabla}^k (u) = \Pi_{im} \tilde{\nabla}_l \tilde{\nabla}^i (u) (\tilde{\nabla}_k \Pi^{lm}) \tilde{\nabla}^k (u)
= \tilde{\nabla}_l \tilde{\nabla}_i (u) \tilde{\nabla}^k (u) (\tilde{\nabla}_k \Pi^{lm}) = -\tilde{\nabla}_l \tilde{\nabla}_i (u) \tilde{\nabla}^k (u) (\tilde{\nabla}_k D^j)
= -\tilde{\nabla}^k (u) \tilde{\nabla}_l \left( D^{ji} \tilde{\nabla}_i (u) \right) + \tilde{\nabla}^k (u) D^{jk} \tilde{\nabla}_i \tilde{\nabla}_j (u),
\]
and by applying Proposition 4.11 to $[\tilde{\nabla}_k, \tilde{\nabla}_i] \tilde{\nabla}_j (u)$ a curvature term appears and one obtains the stated formula. □

Applying the above result to $\mathcal{R}(\nabla u, \nabla u)$, by using formula (4.20) for the Ricci curvature, one reproduces equation (4.32).

4.3. Integrable structures and Kähler manifolds. Let us investigate the important case when $\Sigma$ is a Kähler manifold with respect to $g$ and $\mathcal{J} = \gamma^{-1} \theta$. In particular, this implies that the complex structure, which is now integrable, is parallel with respect to the Levi-Civita connection, which allows for a simplification of several formulas in Section 4.2. The reason for never having to consider the derivative of the Poisson bivector so far, is that everything was expressed in terms of $D^i (u)$, which in local coordinates becomes
\[
D^i (u) = g^{ab} (\partial_a x^i) (\partial_b u),
\]
due to the fact that $\theta$ is an almost Kähler–Poisson structure. Thus, there are no explicit dependencies on $\theta$ left, and any derivative acting on $D^i (u)$ will only produce derivatives of the metric.

For Kähler manifolds, one need not worry about derivatives of $\gamma^{-1} \theta$, since the complex structure is covariantly constant, and instead of $\nabla$, one may consider $\tilde{\nabla}$, defined by
\[
\tilde{\nabla}^i = \tilde{D}^{ijk} \tilde{\nabla}_k \equiv \frac{1}{\gamma} \{ x^i, x^k \} \nabla_k.
\]
Recall from Proposition 4.1 that $\tilde{D}$ is in fact the associated complex structure. Therefore, the equation $\nabla \mathcal{J} = 0$ can be formulated as follows.

Proposition 4.13. Assume that $(\Sigma, g, \mathcal{J})$ is a Kähler manifold. For any $X, Y \in T\Sigma$ it holds that
\[
X^i Y^k (\tilde{D}^{ijk}) = 0.
\]

Proof. Let us assume that $u^a$ is a set of normal coordinates on $\Sigma$. One obtains
\[
X^j Y^k (\tilde{D}^{ijk}) = \frac{1}{\gamma} X^j Y^k \{ x^i, x^j \} \tilde{\nabla}_l \frac{1}{\gamma} \{ x^l, x^k \}
= \frac{1}{\gamma} X^j Y^k \theta^{ab} (\partial_a x^i) \nabla_{c} \frac{1}{\gamma} \theta^{pq} (\partial_p x^j) (\partial_q x^k)
= \frac{1}{\gamma} X^j Y^k \theta^{ab} (\partial_a x^i) \nabla_{c} \frac{1}{\gamma} \theta^{pq} (\partial_p x^j) (\partial_q x^k)
\]
since $\nabla_{e_i}$ coincides with $\nabla_{e_k}$ when contracted with vectors in $T\Sigma$. Now, we use the fact that $\nabla^{-1} \theta = 0$ and normal coordinates to obtain
\[ X_j Y_k (\tilde{\nabla}^i \tilde{D}^i) = \frac{1}{\gamma^2} X_j Y_k \theta^{pq} (\partial_a x^i) \partial_b ((\partial_p x^j)(\partial_q x^k)) = 0, \]
since $X_j \partial^2_{ab} x^j = X^c (\partial_c x^i) \eta_{ij} \partial^2_{ab} x^j = 0$ in normal coordinates. \qed

Since $\tilde{\nabla}^i = -\tilde{D}^{ik} \tilde{\nabla}_{k}$, it is easy to see why one is allowed to replace $\tilde{\nabla}$ by $\tilde{\nabla}$ in many of the formulas in Section 4.2. For instance
\[ \tilde{\nabla}^i \tilde{\nabla}_i (u) = \tilde{D}^{ik} \tilde{\nabla}_{k} \tilde{D}^i (u) = D^{ik} \tilde{\nabla}_{k} \tilde{\nabla}_i (u) + \tilde{D}^{ik} \tilde{\nabla}_{k} \tilde{D}^l (u) = \tilde{\nabla}^i \tilde{\nabla}_i (u). \]

Let us end this section with a couple of words about integration. Assume that $\Sigma$ is a closed manifold, in which case Stoke’s theorem tells us that
\[ \int_\Sigma \text{div}(X) = 0 \]
for all $X \in T\Sigma$. If $\Sigma$ carries an almost Kähler–Poisson structure, it follows from Proposition 4.10 that
\[ \int_\Sigma \tilde{\nabla}_i X^i = 0, \]
which implies that the standard rule for partial integration also holds for $\tilde{\nabla}$. In case $\Sigma$ is a Kähler manifold, a similar formula holds for $\nabla$, since one can show that $\tilde{\nabla}_i X^i = -\nabla^i (\tilde{D}^{ik} X^k)$.

5. Kähler–Poisson algebras

In this section we shall consider the algebraic version of the previous results, and find an intrinsic definition of a Poisson algebra that corresponds to a (complex) function algebra on a submanifold of $\mathbb{R}^m$. Thus, we consider Poisson algebras $\mathcal{A}$ (with a unit) over $\mathbb{C}$ generated (as algebras) by $m$ elements $x^1, \ldots, x^m$, for which we denote
\[ \mathcal{P}^{ij} = \{x^i, x^j\}. \]

It is interesting to carry through the constructions that will follow below, when $\mathcal{A}$ is an arbitrary algebra generated by $x^1, \ldots, x^m$; however, as the goal of the current exposition is rather to explore what kind of results that can be obtained in the algebraic setting, we shall at this stage avoid unnecessary complications by considering $\mathcal{A}$ to be the field of fractions of the polynomial ring $\mathbb{C}[x^1, \ldots, x^m]$. This is analogous to considering Poisson brackets on $\mathbb{R}^m$ that restrict to functions on a subspace (which is identified with the submanifold $\Sigma$). For instance (see also Section 4), for an arbitrary polynomial $C(x^1, x^2, x^3)$ one may define a Poisson bracket of functions on $\mathbb{R}^3$ by setting $\{f, g\} = \varepsilon^{ijk} (\partial_i f)(\partial_j g)(\partial_k C)$. Since $\{f, C\} = 0$ for all $f$, the Poisson bracket restricts to the quotient algebra $\mathbb{C}[x^1, x^2, x^3]/(C)$, which may be identified with the polynomial functions on the level set $\Sigma = \{(x^1, x^2, x^3) : C(x^1, x^2, x^3) = 0\}$.

We let $\mathcal{A}$ have the structure of a $*$-algebra by setting $(x^i)^* = x^i$, and we assume that the Poisson structure is such that $\{u, v\}^* = \{u^*, v^*\}$ for all $u, v \in \mathcal{A}$. Although the position of indices (upper or lower) will not matter in what follows (as the ambient manifold is thought of as $\mathbb{R}^m$), we shall keep the notation from differential geometry and also assume that all repeated indices are summed over from 1 to $m$. 
Let $\text{Der}(A)$ denote the module of derivations generated by $\partial_i \equiv \partial_{x^i}$ for $i = 1, \ldots, m$. We equip this module with a bilinear form $(\cdot, \cdot)$ defined through
\begin{equation}
(X, Y) \equiv (X^i\partial_i, Y^j\partial_j) = X^iY_i,
\end{equation}
and we extend the involution to $\text{Der}(A)$ by setting $X^* = (X^i)^*\partial_i$. Furthermore, we define $\mathcal{P} : \text{Der}(A) \to \text{Der}(A)$ through
\begin{equation}
\mathcal{P}(X) = \mathcal{P}^i_jX^j\partial_i.
\end{equation}
In terms of the Poisson tensor $\mathcal{P}^{ij}$, the defining relation for an almost Kähler–Poisson structure can be formulated as $\mathcal{P}^{ij}(X) = -\gamma^2\mathcal{P}(X)$. We shall take this as a definition for almost Kähler–Poisson algebras.

**Definition 5.1.** Let $\mathcal{A}$ be the field of fractions of $\mathbb{C}[x^1, \ldots, x^m]$, and let $(\cdot, \cdot)$ be a Poisson structure on $\mathcal{A}$, for which we set $\mathcal{P}^{ij} = \{x^i, x^j\}$. The Poisson algebra $(\mathcal{A}, (\cdot, \cdot))$ is called an almost Kähler–Poisson algebra if there exists an invertible hermitian $\gamma^2 \in \mathcal{A}$ such that
\begin{equation}
\mathcal{P}^{i_kk} = -\gamma^2\mathcal{P}^{ij}.
\end{equation}
We shall introduce the notation of Section 4 and define for $u \in \mathcal{A}$ and $X \in \text{Der}(\mathcal{A})$
\begin{align*}
\mathcal{D}^i(u) &= \frac{1}{\gamma^2}\{u, x^k\}\mathcal{P}^{i_k} \\
\Pi^i &= \delta^{ik} - \mathcal{D}^k \\
\mathcal{D}(X) &= \mathcal{D}_k^iX^k\partial_i, \quad \mathcal{D}(X, Y) = (\mathcal{D}(X), Y) \quad \text{and} \quad \Pi(X) = \Pi^kX^k\partial_i.
\end{align*}
Note that the defining relation (5.4) implies that $\mathcal{D}$ is a projector on $\text{Der}(\mathcal{A})$, i.e. $\mathcal{D}^2(X) = \mathcal{D}(X)$ for all $X \in \text{Der}(\mathcal{A})$. Therefore, the concepts of tangent space and normal space arise naturally as projections of $\text{Der}(\mathcal{A})$.

**Definition 5.2.** Let $\mathcal{A}$ be an almost Kähler–Poisson algebra. The tangent space $\mathcal{X}(\mathcal{A})$ is defined as
\begin{equation}
\mathcal{X}(\mathcal{A}) = \{\mathcal{D}(X) : X \in \text{Der}(\mathcal{A})\},
\end{equation}
and the normal space $\mathcal{N}(\mathcal{A})$ is defined as
\begin{equation}
\mathcal{N}(\mathcal{A}) = \{\Pi(X) : X \in \text{Der}(\mathcal{A})\}.
\end{equation}
Clearly, it holds that $\text{Der}(\mathcal{A}) = \mathcal{X}(\mathcal{A}) \oplus \mathcal{N}(\mathcal{A})$, $\mathcal{D}(X) = X$ for all $X \in \mathcal{X}(\mathcal{A})$, and $(X, N) = 0$ for all $X \in \mathcal{X}(\mathcal{A})$ and $N \in \mathcal{N}(\mathcal{A})$. In the current situation, all modules are vector spaces, since $\mathcal{A}$ is a field. Note however that the above setup allows for an extension to more general rings.

It follows from (5.4) that
\begin{equation}
\gamma^2 = -(\mathcal{P}^{i_jk}/(\mathcal{P}^2)^k),
\end{equation}
and in case there exists a basis $X_1, \ldots, X_n, Y_1, \ldots, Y_p$ of $\text{Der}(\mathcal{A})$ such that $X_1, \ldots, X_n$ is a basis for $\mathcal{X}(\mathcal{A})$ and $Y_1, \ldots, Y_p$ is a basis of $\mathcal{N}(\mathcal{A})$, it is easy to see that
\begin{equation}
\gamma^2 = -\frac{1}{n}\text{Tr}\mathcal{P}^2
\end{equation}
\footnote{Let us leave the square of $\gamma^2$ in order to keep the analogy with differential geometry; however, we shall in general not assume that there exists a square root of $\gamma^2$.}
\footnote{Strictly speaking, $\mathcal{X}(\mathcal{A})$ should rather be called the tangent bundle, since we are always considering analogues of global objects on a manifold.}
where \(\operatorname{Tr} \mathcal{P}^2 = (\mathcal{P}^2)i\). The dimension of \(\mathcal{X}(\mathcal{A})\), i.e. \(n\), is called the geometric dimension of \(\mathcal{A}\).

In the differential geometric setting, the special case of a Kähler manifold was studied. How may one proceed to define a Kähler–Poisson algebra? Let us start from Proposition 4.13, which states that the following relation holds

\[X_j Y_k (\tilde{\nabla}^i \tilde{D}^{jk}) = 0,\]

where \(\tilde{D}^{jk} = \frac{1}{\gamma} \{ x^j, x^k \}\) and \(\tilde{\nabla}^i = \tilde{D}^{ik} \nabla_k\). In the case of \(\mathbb{R}^m\), this statement can be written as

\[\frac{1}{\gamma} \{ x^i, \frac{1}{\gamma} \{ x^j, x^k \} \} \mathcal{P}_{jl} \mathcal{P}_{km} = 0.\]

However, this expression depends on the existence of a square root of \(\gamma^2\), which in general does not exist. Let us expand the above expression as

\[\frac{1}{\gamma} \{ x^i, \frac{1}{\gamma} \{ x^j, x^k \} \} \mathcal{P}_{jl} \mathcal{P}_{km} = \frac{1}{2 \gamma^2} \{ x^i, \gamma \} \{ x^j, x^k \} \mathcal{P}_{jl} \mathcal{P}_{km} - \frac{1}{2 \gamma^2} \{ x^i, \gamma \} \mathcal{P}_{lm}.\]

Thus, one is lead to the following additional requirement on an almost Kähler–Poisson algebra

\[\{ x^i, \{ x^j, x^k \} \} \mathcal{P}_{jl} \mathcal{P}_{km} = \frac{1}{2} \{ x^i, \gamma \} \mathcal{P}_{lm}.\]

However, in the following we shall focus on the general case of almost Kähler–Poisson algebras.

5.1. Curvature. Since the formulas for curvature in Section 4 are expressed in terms of Poisson brackets, it is natural to introduce curvature in almost Kähler–Poisson algebras. It was shown (see eq. (4.12)) that the covariant derivative on \(T\Sigma\) can be computed as

\[\nabla_X Y = \mathcal{D}(\tilde{\nabla} X Y)\]

for all \(X, Y \in T\Sigma\). Let us make this into a definition in the algebraic setting.

**Definition 5.3.** Let \(\mathcal{A}\) be an almost Kähler–Poisson algebra. For any \(X \in \mathcal{X}(\mathcal{A})\), the **covariant derivative** \(\nabla_X : \mathcal{X}(\mathcal{A}) \to \mathcal{X}(\mathcal{A})\) is defined as

\[\nabla_X Y = \mathcal{D}^{ik} X^l (Y_k) \partial_i,\]

and in components we shall also write \(\nabla_k Y^i = \mathcal{D}^{ij} Y_k \partial_j\). Furthermore, for \(u \in \mathcal{A}\) we set \(\nabla_X (u) = X^i \partial_i (u)\).

**Proposition 5.4.** Let \(\mathcal{A}\) be an almost Kähler–Poisson algebra. For all \(X, Y, Z \in \mathcal{X}(\mathcal{A})\) and \(u \in \mathcal{A}\), the covariant derivative has the following properties

1. \(\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z\),
2. \(\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z\),
3. \(\nabla_{(uX)} Y = u \nabla_X Y\),
4. \(\nabla_X (uY) = \nabla_X (u) Y + u \nabla_X Y\).
Proof. The first three properties are immediate from the definition. Let us prove the last one. One computes
\[
\nabla_X (uY) = D^i_k X^l D_l (u Y_k) = u D^i_k X^l D_l (Y_k) + D^i_k Y_k X^l D_l (u)
\]
\[
= u \nabla_X Y + Y^i \nabla_X (u),
\]
since \( Y \in \mathcal{X}(A) \). \( \square \)

The above result shows that \( \nabla \) has all the properties one expects from an affine connection. We shall also extend the action of the covariant derivative to tensors in a standard manner; for instance
\[
(\nabla_X T)(Y, Z) = \nabla_X T(Y, Z) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z).
\]

The following lemma is important proving further properties of the covariant derivative and its associated curvature. It is an algebraic analogue of the fact that in a Riemannian manifold \( M \), it holds that \( \nabla_i \nabla_j (u) = \nabla_j \nabla_i (u) \) for \( u \in C^\infty(M) \).

**Lemma 5.5.** Let \( A \) be an almost Kähler–Poisson algebra. For all \( u \in A \) it holds that
\[
[D^i, D^j](u) P_{ik} P_{jl} = 0.
\]

**Proof.** One computes
\[
D^i D^j (u) P_{ik} P_{jl} = \frac{1}{2} \{ D^j (u), x_m \} P^m P_{ik} P_{jl} = \{ D^j (u), x^m \} D_{mk} P_{jl}
\]
\[
= \{ D^j (u), x_k \} P_{jl} = \{ D^j (u) P_{jl}, x_k \} - \{ P_{jl}, x_k \} D^j (u)
\]
\[
= \{ \{ u, x_l \}, x_k \} - \{ P_{jl}, x_k \} D^j (u).
\]

Using the Jacobi identity in both terms gives
\[
D^i D^j (u) P_{ik} P_{jl} = - \{ \{ x_l, x_k \}, u \} - \{ x_k, u \}, x_l \}
+ \{ P_{lk}, x_j \} D^j (u) + \{ P_{kj}, x_l \} D^j (u)
\]
\[
= \{ \{ u, x_k \}, x_l \} - \{ P_{jk}, x_l \} D^j (u)
- \{ \{ x_l, x_k \}, u \} + \{ P_{lk}, u \}
\]
\[
= \{ \{ u, x_k \}, x_l \} - \{ P_{jk}, x_l \} D^j (u) = D^j D^j (u) P_{ik} P_{jl},
\]
by comparing with the result of the previous computation. \( \square \)

The action of on \( A \) of an element \( X \in \mathcal{X}(A) \), has previously been defined as \( X(u) = X^i D_i (u) \). By setting \([X, Y]^t = X(Y^t) - Y(X^t)\) one can show that \([X, Y]\) is again an element of \( \mathcal{X}(A) \) and that the connection is torsion free with respect to this commutator.

**Proposition 5.6.** If \( X, Y \in \mathcal{X}(A) \) then it follows that \([X, Y] \in \mathcal{X}(A)\). Moreover, the covariant derivative in an almost Kähler–Poisson algebra has no torsion, i.e.
\[
\nabla_X Y - \nabla_Y X - [X, Y] = 0
\]
for all \( X, Y \in \mathcal{X}(A) \).
Proof. One computes

\[ D([X, Y])^i = D^{ik}[X, Y]_k = D^{ik} X^i D_l(Y_k) - D^{ik} Y^i D_l(X_k) \]

\[ = X^i D_l(Y^i) - Y_k X^i D_l(D^{ik}) - Y^i D_l(X_k) + X_l Y^i D_l(D^{ik}) \]

\[ = [X, Y]^i - Y_k X^i D_l(D^{ik}) + X_k Y^i D_l(D^{ik}). \]

The last two terms cancel by Lemma 5.5, which shows that \( D([X, Y]) = [X, Y] \).

Let us now show that the connection is torsion free. Writing \( \nabla_X Y^i - \nabla_Y X^i = D^{ik} X^i D_l(Y_k) - D^{ik} Y^i D_l(X_k) = D([X, Y])^i \),

it follows from the previous calculation that this equals \([X, Y]^i\). □

Furthermore, one can show that the connection \( \nabla \) is a metric connection.

**Proposition 5.7.** In an almost Kähler–Poisson algebra with covariant derivative \( \nabla \) it holds that

\[ \nabla_X (Y, Z) - (\nabla_X Y, Z) - (Y, \nabla_X Z) = 0 \]

for all \( X, Y, Z \in \mathcal{X}(\mathcal{A}) \).

**Proof.** One computes

\[ \nabla_X (Y, Z) - (\nabla_X Y, Z) - (Y, \nabla_X Z) \]

\[ = X^m D_m(Y^i Z^j) - D^{ik} X^m D_m(Y_k) Z_i - Y_l D^{ik} X^m D_m(Z_k) \]

\[ = X^m D_m(Y^i Z^j) - X^m D_m(Y_k) Z^i - Y^k X^m D_m(Z_k) = 0, \]

which shows that the connection is metric. □

Let us now proceed and define curvature in the usual manner.

**Definition 5.8.** Let \( \mathcal{A} \) be an almost Kähler–Poisson algebra and let \( \nabla \) be the covariant derivative of \( \mathcal{A} \). For \( X, Y, Z \in \mathcal{X}(\mathcal{A}) \) we define the curvature tensor \( R \) via

\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \]

and we write \( R(X, Y, Z, V) = (R(Z, V)Y, X) \) as well as \( R(X, Y, Z) = R(X, Y)Z \) for \( X, Y, Z, V \in \mathcal{X}(\mathcal{A}) \).

We continue by proving the Bianchi identities.

**Proposition 5.9.** Let \( \mathcal{A} \) be an almost Kähler–Poisson algebra and let \( R \) be the curvature tensor of \( \mathcal{A} \). For all \( X, Y, Z, V \in \mathcal{X}(\mathcal{A}) \) it holds that

\[ R(X, Y, Z) + R(Z, X, Y) + R(Y, Z, X) = 0 \]

\[ (\nabla_X R)(Y, Z, V) + (\nabla_Y R)(Z, X, V) + (\nabla_Z R)(X, Y, V) = 0. \]

**Proof.** The first Bianchi identity \( \text{(5.12)} \) is proven by acting with \( \nabla_Z \) on the torsion free condition \( \nabla_X Y - \nabla_Y X - [X, Y] = 0 \), and then summing over cyclic permutations of \( X, Y, Z \). Since \([[[X, Y], Z], X] + [[Y, Z], X] + [[Z, X], Y] = 0\), the desired result follows. The second identity is obtained by a cyclic permutation (in \( X, Y, Z \)) of \( R(\nabla_X Y - \nabla_Y X - [X, Y], Z, V) = 0 \). One has

\[ 0 = R(\nabla_X Y - \nabla_Y X - [X, Y], Z, V) + \text{cycl}. \]

\[ = R(\nabla_Z X, Y, V) + R(X, \nabla_Z Y, V) - R([X, Y], Z, V) + \text{cycl}. \]
On the other hand, one has
\[ (\nabla_Z R)(X, Y, V) = \nabla_Z R(X, Y, V) - R(\nabla_Z X, Y, V) - R(X, \nabla_Z Y, V) - R(X, Y, \nabla_Z V), \]
and substituting this into the previous equation yields
\[ 0 = \nabla_Z R(X, Y, V) - (\nabla_Z R)(X, Y, V) - R(X, Y, \nabla_Z V) - R([X, Y], Z, V) + \text{cycl.} \]
After inserting the definition of \( R \), and using that \([ [X, Y], Z] + \text{cycl.} = 0\), the second Bianchi identity follows.

The following proposition shows that the usual symmetries of the curvature tensor also hold in the algebraic setting.

**Proposition 5.10.** The curvature \( R \) of an almost Kähler–Poisson algebra has the following properties
\[
\begin{align*}
(5.14) & \quad R(X, Y, Z, V) = -R(X, Y, V, Z) = -R(Y, X, Z, V) \\
(5.15) & \quad R(X, Y, Z, V) = R(Z, V, X, Y)
\end{align*}
\]
for all \( X, Y, Z, V \in \mathcal{X}(\mathcal{A}) \).

**Proof.** The first identity, \( R(X, Y, Z, V) = -R(X, Y, V, Z) \), follows directly from the definition of \( R \). Let us now prove that \( R(X, Y, Z, V) = -R(Y, X, Z, V) \). It is easy to show that for any \( u \in \mathcal{A} \) it holds that \( \nabla_X \nabla_Y (u) - \nabla_Y \nabla_X (u) - \nabla_{[X,Y]}(u) = 0 \).

By setting \( u = (X, Y) \) one obtains
\[ \nabla_Z \nabla_V (X, Y) - \nabla_Z \nabla_V (X, Y) - \nabla_{[Z,V]} (X, Y) = 0, \]
and since \( \nabla \) is a metric connection, it follows that
\[ \nabla_Z \left[ (\nabla_V X, Y) + (X, \nabla_V Y) \right] - \nabla_V \left[ (\nabla_Z X, Y) + (X, \nabla_Z Y) \right] - (\nabla_{[Z,V]} X, Y) - (X, \nabla_{[Z,V]} Y) = 0. \]
A further expansion of the derivatives yields
\[ (\nabla_Z \nabla_V X, Y) + (X, \nabla_Z \nabla_V Y) - (\nabla_V \nabla_Z X, Y) - (X, \nabla_V \nabla_Z Y) - (\nabla_{[Z,V]} X, Y) - (X, \nabla_{[Z,V]} Y) = 0, \]
which is equivalent to
\[ (R(Z, V)X, Y) = -(R(Z, V)Y, X). \]

It is a standard algebraic result that any quadrilinear map satisfying \((5.14)\) and \((5.15)\) also satisfies \((5.15)\) \cite{KN96}.

One can now derive Gauss’ formula, in the form of Proposition 4.6

**Proposition 5.11.** Let \( \mathcal{A} \) be an almost Kähler–Poisson algebra with curvature tensor \( R \). For \( X, Y, Z \in \mathcal{X}(\mathcal{A}) \) it holds that
\[ R(X, Y, Z, V) = X^k Y^l Z^m \partial^i \left[ (\partial_k \Pi^m_{\Pi^i_j}) (\partial_l \Pi_{jm}) - (\partial_l \Pi^m_{\Pi^i_j}) (\partial_k \Pi_{jm}) \right]. \]

**Proof.** One computes that
\[ (\nabla_X \nabla_Y Z)^i = \partial^{jk} X^i \partial_l (\partial_{km} Y^n \partial_n (Z^m)) = \partial^{jk} X^i \partial_l (\partial_{km} Y^n \partial_n (Z^m)) + \partial^{mk} X^i \partial_l (Y^n \partial_n (Z^m)) + \partial^{mk} X^i Y^n \partial_l \partial_n (Z^m). \]
Now, one uses Lemma [5.5] in the last term to obtain
\[
\mathcal{D}^i_m X^i Y^m \mathcal{D}_n (Z^m) = \mathcal{D}^i_m X^i Y^m \mathcal{D}_n (\mathcal{D}(Z^m)) = \mathcal{D}^{ik}_m \mathcal{D}_n (\mathcal{D}^{ik}(X^i Y^m \mathcal{D}_n (\mathcal{D}(Z^m))))
\]
\[
= \mathcal{D}^{ik} Y^m \mathcal{D}_n (\mathcal{D}(\mathcal{D}^{ik}(X^i Y^m \mathcal{D}_n (\mathcal{D}(Z^m)))) - \mathcal{D}^{ik} Y^m \mathcal{D}_n (\mathcal{D}(\mathcal{D}(Z^m))))
\]
\[
= (\nabla_Y \nabla_X Z)^i - \mathcal{D}^{ik} Y^m \mathcal{D}_n (\mathcal{D}(\mathcal{D}(Z^m))))
\]
which implies that
\[
[\nabla_X, \nabla_Y] Z^i = \mathcal{D}^{ik} X^i \mathcal{D}_l (\mathcal{D}(\mathcal{D}(Z^m)))) - \mathcal{D}^{ik} Y^m \mathcal{D}_n (\mathcal{D}(\mathcal{D}(Z^m))))
\]
\[
+ \mathcal{D}^{ik} X^i \mathcal{D}_n (\mathcal{D}(\mathcal{D}(Z^m)))) - \mathcal{D}^{ik} Y^m \mathcal{D}_n (\mathcal{D}(\mathcal{D}(Z^m))))
\]
One easily checks that the two last terms equal \( \nabla_{[X,Y]} Z^i \), and therefore it holds that
\[
R(X,Y) Z^i = \mathcal{D}^{ik} X^i \mathcal{D}_l (\mathcal{D}(\mathcal{D}(Z^m)))) - \mathcal{D}^{ik} Y^m \mathcal{D}_n (\mathcal{D}(\mathcal{D}(Z^m))))
\]
\[
\text{Let us consider the first of these two terms (as the other one is obtained by interchanging } X \text{ and } Y \text{)}
\]
\[
\mathcal{D}^{ik} X^i \mathcal{D}_l (\mathcal{D}(\mathcal{D}(Z^m)))) = - \mathcal{D}^{ik} X^i \mathcal{D}_l (\mathcal{D}(\mathcal{D}(Z^m))))
\]
\[
\text{Just as Leibnitz rule holds only for vectors in } \mathcal{X}(A), \text{ as was shown in Proposition 5.4, the corresponding rule for tensors will hold as long as they are invariant under projections, i.e.}
\]
\[
(5.16) \quad \nabla_i T_{k_1 \cdots k_n}^{i_1 \cdots i_m} = \mathcal{D}^{i_1}_{k_1} \cdots \mathcal{D}^{i_m}_{k_m} \mathcal{D}^{i_1}_{i_1} \cdots \mathcal{D}^{i_m}_{i_m} \mathcal{D}_i (T_{k_1 \cdots k_n}^{i_1 \cdots i_m}).
\]
\[
\text{for } m = 1, \ldots, N. \text{ Such tensors will be called } \text{tangential}, \text{ and it is clear from the definition that the covariant derivative of a tangential tensor is again a tangential tensor. For instance, one computes that for tangential tensors } T_{kl} \text{ and } X^i
\]
\[
\nabla_i (T_{kl} X^i) = \mathcal{D}^{i}_{k} \mathcal{D}_i (T_{ml} X^i) = \mathcal{D}^{m}_{k} \mathcal{D}_i (X^i) + \mathcal{D}^{m}_{k} \mathcal{D}_i (T_{ml})
\]
\[
= T_{kl} \mathcal{D}_i^i X^i + \mathcal{D}^{m}_{k} \mathcal{D}_i (T_{ml})
\]
\[
\text{Note that the above definition in 5.10 coincides, for tangential tensors, with the previous index-free definition. For instance, one easily computes that}
\]
\[
(\nabla_X T)(Y,Z) = \nabla_X (T(Y,Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z) = X^i Y^j Z^k \nabla_i T_{jk}
\]
\[
\text{for } X, Y, Z \in \mathcal{X}(A) \text{ and } T \text{ a tangential tensor. With this notation, one computes that the following relation holds}
\]
\[
(5.18) \quad (R(X,Y) Z)^i = R_{jkl}^i Z^j X^k Y^l = X^k Y^l (\nabla_k \nabla_l Z^i - \nabla_l \nabla_k Z^i).
\]
Let us also note that the Codazzi-Mainardi equations, in the form of Proposition 4.9 (with $M = \mathbb{R}^m$)

$$0 = X_i Y_j Z_k \left[ D^i D^k (N^j_A) - D^i D^j (N^k_A) + D^k D^j (N^i_A) - D^k D^i (N^j_A) \right],$$

are satisfied due to Lemma 5.5.

It is a standard theorem in differential geometry that if the sectional curvature only depends on the point (and not on the choice of tangent plane) then the sectional curvature is constant (if the dimension is greater than or equal to three). In the following, we shall derive an analogous theorem for almost Kähler–Poisson algebras. Let us first define the sectional curvature.

**Definition 5.12.** Let $\mathcal{A}$ be an almost Kähler–Poisson algebra with curvature tensor $R$. For any $X, Y \in \mathcal{X}(\mathcal{A})$, the sectional curvature is defined as

$$K(X, Y) = \frac{R(X, Y, X, Y)}{(X, X)(Y, Y) - (X, Y)^2}.$$  

For almost Kähler–Poisson algebras, if $K(X, Y)$ is independent of $X$ and $Y$, then the sectional curvature is in the center of the Poisson algebra.

**Proposition 5.13.** Let $\mathcal{A}$ be an almost Kähler–Poisson algebra with curvature tensor $R$ and geometric dimension $n \geq 3$. If $K(X, Y) = k \in \mathcal{A}$ for all $X, Y \in \mathcal{X}(\mathcal{A})$ then $\{k, u\} = 0$ for all $u \in \mathcal{A}$.

**Proof.** It is a standard algebraic result that if $R$ and $R'$ are two quadri-linear maps, satisfying (5.12) and (5.14), and $R(X, Y, X, Y) = R'(X, Y, X, Y)$ for all $X, Y \in \mathcal{X}(\mathcal{A})$ then $R(X, Y, Z, V) = R'(X, Y, Z, V)$ for all $X, Y, Z, V \in \mathcal{X}(\mathcal{A})$ [KN96]. Hence, if we define

$$R'(X, Y, Z, V) = (X, Z)(Y, V) - (X, V)(Y, Z),$$

for which $K(X, Y) = 1$ for all $X, Y \in \mathcal{X}(\mathcal{A})$, it follows that $R(X, Y, Z, V) = k R'(X, Y, Z, V)$. Since $\nabla$ is a metric connection one has $(\nabla_U R')(X, Y, Z, V) = 0$ and

$$(\nabla_U R)(X, Y, Z, V) = (\nabla_U k R')(X, Y, Z, V) = \nabla_U (k R')(X, Y, Z, V).$$

If we sum this identity over cyclic permutations of $U, X, Y$ the left hand side will vanish due to the second Bianchi identity and one is left with

$$0 = \nabla_U (k \left((X, Z)(Y, V) - (X, V)(Y, Z)\right)$$

$$+ \nabla_X (k \left((Y, Z)(U, V) - (Y, V)(U, Z)\right)$$

$$+ \nabla_Y (k \left((U, Z)(X, V) - (U, V)(X, Z)\right).$$

Given an arbitrary $X \in \mathcal{X}(\mathcal{A})$ one can always find $Y, Z \in \mathcal{X}(\mathcal{A})$ such that $(X, Y) = (X, Z) = (Y, Z) = 0$, since the geometric dimension of $\mathcal{A}$ is at least 3. For such vectors, the above relation becomes

$$-\nabla_X (k)(Y, V)(U, Z) + \nabla_Y (k)(U, Z)(X, V) = 0,$$

and for $U = Z$ and $V = Y$ one obtains

$$\nabla_X (k)(Y, Y)(Z, Z) = 0,$$
which implies that $\nabla X(k) = 0$. Thus, for all $X \in \mathcal{X}(A)$ it holds that $X^i \mathcal{D}_i(k) = 0$, which is equivalent to $\mathcal{D}^i \mathcal{D}_i(k) = 0$ for $l = 1, \ldots, m$. Writing out this equation yields

$$0 = \mathcal{D}^i \mathcal{D}_i(k) = \mathcal{D}^i(k) = \frac{1}{\gamma^2} \mathcal{P}^{lm} \mathcal{P}^{im} (\partial_i k),$$

and multiplying by $\mathcal{P}_{ij}$ gives

$$0 = \frac{1}{\gamma^2} \mathcal{P}_{ij} \mathcal{P}^{lm} \mathcal{P}^{im} (\partial_i k) = \mathcal{P}^{ij} \partial_i k = \{k, x_j\}$$

which implies that $\{k, u\} = 0$ for all $u \in A$. □

5.2. Tracial states, Stoke’s theorem and orderings. As an important illustration of how the developed algebraic techniques can be used, we aim to prove that a bound on the Ricci curvature induces a bound on the eigenvalues of the Laplace operator, which is a standard result for compact Riemannian manifolds. To achieve this goal we shall, in this section, introduce concepts of integration and ordering on $A$.

Definition 5.14. Let $A$ be an almost Kähler–Poisson algebra. A state on $A$ is a $C^*$-linear map $\int_A : A \rightarrow \mathbb{C}$ such that

$$(5.19) \quad \int_A a^* = \overline{\int_A a} \quad \text{and} \quad \int_A a^* a \geq 0$$

for all $a \in A$.

In a noncommutative $*$-algebra, a state that fulfills $\int_A XY = \int_A YX$, or $\int_A [X, Y] = 0$, is called a tracial state. An analogous extension to commutative Poisson algebras would be to require $\int_A \{a, b\} = 0$. However, what one needs for calculations is an equation in correspondence with the fact that the integral (over a closed manifold) of the divergence of a vector field is zero, which allows one to perform “partial integration”. In Section 4.3 it was shown that $\int_\Sigma \nabla_i X^i = 0$, which motivates the following definition.

Definition 5.15. Let $\int_A$ be a state on an almost Kähler–Poisson algebra. The state is called tracial if

$$(5.20) \quad \int_A \nabla_i X^i = 0$$

for all $X \in \mathcal{X}(A)$.

Note that in the case when a square root of $\gamma^2$ exists, and the algebra fulfills the additional condition $(5.9)$, a tracial state in a Kähler–Poisson algebra fulfills

$$\int_A \frac{1}{\gamma} \{x^i, X_i\} = 0,$$

which is in analogy with a tracial state in a noncommutative algebra.

Definition 5.16. An almost Kähler–Poisson algebra with a tracial state is called a geometric almost Kähler–Poisson algebra.

\[3\] A slightly more appropriate $*$-algebraic term is positive linear functional, but for simplicity we have chosen a less cumbersome terminology.
As usual, any state induces a $\mathbb{C}$-valued sesquilinear form on $A$ via
\begin{equation}
\langle u, v \rangle = \int_A u^* v, \tag{5.21}\end{equation}
which may be extended to $T\Sigma$ by setting
\begin{equation}
\langle X, Y \rangle = \int_A (X^*, Y). \tag{5.22}\end{equation}

Let us now introduce a preorder on $A$.

**Definition 5.17.** Let $A$ be an almost Kähler–Poisson algebra. We say that $a$ is positive, and write $a \geq 0$ if and only if
\begin{equation}
a = \frac{\sum_{i=1}^{N} u_i^* u_i}{\sum_{k=1}^{N'} v_k^* v_k}, \tag{5.23}\end{equation}
for some elements $u_i, v_k \in A$. Moreover, we write $a \geq b$ whenever $a - b \geq 0$.

Let us state some of the properties of this ordering.

**Proposition 5.18.** The relation $\geq$ has the following properties

1. $a \geq a$,
2. if $a \geq b$ and $b \geq c$ then $a \geq c$,
3. if $a \geq b$ then $a + c \geq b + c$ for all $c \in A$,
4. if $a \geq 0$ and $b \geq 0$ then $ab \geq 0$,

i.e. $\geq$ is a ring preorder.

**Proof.** It is immediate from (5.22) that sums and products of positive elements are again positive elements. **Reflexivity**: $a \leq a$. This is equivalent to $a - a \geq 0$, which is true since $0 = 0^* 0$. **Transitivity.** Assume that $a \geq b$ and $b \geq c$. By definition, this means that $a - b$ and $b - c$ are positive, which implies that their sum, i.e. $a - c$, is positive. Thus, $a \geq c$. \hfill $\square$

In particular, one notes that $(X^*, X) \geq 0$ for all $X \in \text{Der}(A)$, which enables us to prove the Cauchy–Schwarz inequality for tensors.

**Proposition 5.19.** Let $T^{ij}$ be a tangential tensor. Then it holds that
\begin{equation}
(\text{Tr} D)^2 (T^{ij})^* T_{ij} \geq (\text{Tr} D)(\text{Tr} T)^*(\text{Tr} T). \tag{5.24}\end{equation}

**Proof.** First, we note that for a tangential tensor it holds that
\begin{equation}
\text{Tr} T = T_i^i = D^{ik} T_{ik}, \end{equation}
and writing
\begin{equation}
(\text{Tr} D)T^{ij} = (\text{Tr} D)T^{ij} - (\text{Tr} T)D^{ij} + (\text{Tr} T)D^{ij}, \end{equation}
it follows that (recall that $(D^{ij})^* = D^{ij}$)
\begin{equation}
(\text{Tr} D)^2 (T^{ij})^* T_{ij} = \left( (\text{Tr} D)T^{ij} - (\text{Tr} T)D^{ij} \right)^* \left( (\text{Tr} D)T_{ij} - (\text{Tr} T)D_{ij} \right) + (\text{Tr} T)^*(\text{Tr} T)(\text{Tr} D) \tag{5.25}\end{equation}
since
\begin{equation}
(\text{Tr} D)T^{ij} - (\text{Tr} T)D^{ij} \end{equation}
is zero.
Using that the first term in (5.23) is positive gives
\[(\text{Tr} D)^2 (T^{ij})^* T_{ij} \geq (\text{Tr} D)(\text{Tr} T)^*(\text{Tr} T),\]
which completes the proof. □

If the geometric dimension of the algebra is \( n \), the inequality can be written as
\[(T^{ij})^* T_{ij} \geq \frac{1}{n} (\text{Tr} T)^*(\text{Tr} T).\]

In the following the above inequality will be applied to the tensor \( \nabla_i \nabla_j(u) \), where \( u \) is a hermitian element of \( \mathcal{A} \), which gives
\[(\nabla_i \nabla_j(u)) \nabla_j(u) \geq \frac{1}{n} (\nabla_i \nabla_i(u))^2.\]

(5.24)

5.3. Eigenvalues of the Laplace operator. Let us now proceed to define the Laplace operator, and to show that its eigenvalues are bounded by the Ricci curvature. We start by introducing the Laplace operator, together with some of its properties.

**Definition 5.20.** The operator \( \Delta : \mathcal{A} \to \mathcal{A} \), defined as
\[(5.25) \Delta(u) = \nabla^i \nabla_i(u),\]
is called the Laplace operator on \( \mathcal{A} \). An eigenvector of \( \Delta \) is an element \( u \in \mathcal{A} \) such that \( \Delta(u) = \lambda u \) for some \( \lambda \in \mathbb{C} \). The complex number \( \lambda \) is then called an eigenvalue of \( \Delta \).

**Proposition 5.21.** In a geometric almost Kähler–Poisson algebra, the Laplace operator is a self-adjoint operator with respect to the sesquilinear form \( \langle \cdot, \cdot \rangle \). Hence, for any eigenvector \( u \) with \( \langle u, u \rangle > 0 \), the corresponding eigenvalue is real.

**Proof.** Since \( \nabla^i(u)^* = \nabla^i(u^*) \) it follows that \( \Delta(u)^* = \Delta(u^*) \). As the state is tracial, it follows that
\[\langle \Delta(u), v \rangle = \int_{\mathcal{A}} \nabla^i \nabla_i(u^*) v = -\int_{\mathcal{A}} \nabla^i(u^*) \nabla_i(v) = \int_{\mathcal{A}} u^* \Delta(v) = \langle u, \Delta(v) \rangle.\]

Let \( u \) be an eigenvector of \( \Delta \) with eigenvalue \( \lambda \). Then it holds that
\[\langle \lambda, u, u \rangle = \langle \Delta(u), u \rangle = \langle u, \Delta(u) \rangle = \lambda \langle u, u \rangle,\]
from which it follows that \( \lambda = \bar{\lambda} \) since \( \langle u, u \rangle > 0 \). □

Without any further assumptions on the algebra, eigenvectors of the Laplace operator may in general have degenerate features. Let us therefore restrict to a particular class of eigenvectors.

**Definition 5.22.** Let \( u \) be an eigenvector of the Laplace operator with eigenvalue \( \lambda \). The eigenvector is called non-degenerate if \( \langle u, u \rangle > 0 \) and \( \langle \nabla(u), \nabla(u) \rangle > 0 \).

**Proposition 5.23.** Let \( u \) be a non-degenerate eigenvector of the Laplace operator with eigenvalue \( -\lambda \). Then it follows that \( \lambda > 0 \).

**Proof.** One computes that
\[\lambda \langle u, u \rangle = -\langle u, \Delta(u) \rangle = -\int_{\mathcal{A}} u^* \Delta(u) = \int_{\mathcal{A}} \nabla_i(u)^* \nabla_i(u) = \langle \nabla(u), \nabla(u) \rangle.\]

Since \( u \) is assumed to be non-degenerate, it holds that both \( \langle u, u \rangle \) and \( \langle \nabla(u), \nabla(u) \rangle \) are strictly positive, which implies that \( \lambda > 0 \). □
We shall now prove, in the purely algebraic setting, a classical theorem of differential geometry saying that a bound on the Ricci curvature induces a bound on the eigenvalues of the Laplace operator (corresponding to non-degenerate eigenvectors) on a compact manifold.

**Theorem 5.24.** Let \( A \) be a geometric almost Kähler–Poisson algebra with geometric dimension \( n \geq 2 \), and let \( -\lambda \neq 0 \) be an eigenvalue of the Laplace operator corresponding to a non-degenerate eigenvector \( u \). If there exists a real number \( \kappa > 0 \) such that \( R(X^*,X) \geq \kappa(X^*,X) \) for all \( X \in \mathcal{X}(A) \) then \( \lambda \geq n\kappa/(n-1) \).

**Proof.** First we note that one can always choose the eigenvector \( u \) to be hermitian, since \( u^* \) is also an eigenvector of \( \Delta \) with eigenvalue \( \lambda \). Let us start by writing

\[
(5.26) \quad \int_A (\Delta(u))^2 = -\lambda \int_A u\Delta(u) = -\lambda \int_A u\nabla_iu = \lambda \int_A \nabla_i(u)\nabla_i(u),
\]

since the state is assumed to be tracial. One the other hand one gets

\[
\int_A (\Delta(u))^2 = \int_A \nabla_i\nabla_i(u)\nabla_i\nabla_i(u) = -\int_A \nabla_k(u)\nabla_k\nabla_i(u),
\]

and using equation (5.18) gives

\[
\int_A (\Delta(u))^2 = -\int_A \left[ \nabla_k(u)\nabla_i\nabla_i(u) - R(\nabla u, \nabla u) \right].
\]

After partial integration one obtains

\[
\int_A (\Delta(u))^2 = \int_A \left[ \nabla_i\nabla_k(u)\nabla_i\nabla_k(u) + R(\nabla u, \nabla u) \right].
\]

Now, using the inequality (5.24) together with the assumption that \( R(X^*,X) \geq \kappa(X^*,X) \) for all \( X \in \mathcal{X}(A) \) gives

\[
\int_A (\Delta(u))^2 \geq \frac{1}{n} \int_A (\Delta(u))^2 + \kappa \int_A \nabla_i(u)\nabla_i(u)
\]

\[
= \left( \frac{\lambda}{n} + \kappa \right) \int_A \nabla_i(u)\nabla_i(u),
\]

Now, we compare this expression with (5.26) and conclude that

\[
\frac{1}{n} (\lambda(n-1) - n\kappa) \int_A \nabla_i(u)\nabla_i(u) = \frac{1}{n} (\lambda(n-1) - n\kappa) \int_A \nabla_i(u)\nabla_i(u) \geq 0,
\]

which implies \( \lambda \geq n\kappa/(n-1) \) since \( u \) is a non-degenerate eigenvector. \( \square \)

6. Examples

Let us consider two examples of Kähler–Poisson algebras that are constructed in an algebraic way, although they have clear geometrical interpretations.

6.1. A simple flat example. Let \( A \) be generated by

\[
\{x^i\} = \{p^1, \ldots, p^n, q^1, \ldots, q^n, n^1, \ldots, n^p\}
\]

and we shall let indices \( a, b, c, \ldots \) run from \( 1 \) to \( n \) and indices \( A, B, C, \ldots \) from \( 1 \) to \( p \). We introduce a Poisson structure defined by

\[
\{p^a, p^b\} = \{q^a, q^b\} = \{n^A, x^i\} = 0
\]

\[
\{p^a, q^b\} = \delta^{ab}\cdot 1
\]
with $\gamma \in \mathbb{R}$. It is easy to check that it is a Kähler–Poisson algebra with characteristic function $\gamma^2 \cdot 1$. The projection operators $D, \Pi$ become

$$
(D_{ik}) = \text{diag}(1, \ldots, 1, 0, \ldots, 0) \quad \text{and} \quad (\Pi_{ik}) = \text{diag}(0, \ldots, 0, 1, \ldots, 1),
$$

from which it follows that $\mathcal{A}(\mathcal{A})$ is $2n$-dimensional and a basis is given by $\partial_1, \ldots, \partial_{2n}$. Hence, $D = \partial_i$ and $R(X, Y, Z, V) = 0$ for all $X, Y, Z, V \in \mathcal{A}(\mathcal{A})$.

### 6.2. Algebras defined by a polynomial.

Let us introduce a Poisson algebra which has been used to construct matrix regularizations of surfaces \cite{ABH+09a,ABH+09b,Arn08b,AS09,Arn08a}. Let $\mathcal{A} = \mathbb{C}[x^1, x^2, x^3]$ be the polynomial algebra in three variables together with the Poisson structure

$$
\{x^i, x^j\} = \varepsilon^{ijk} \partial_k C
$$

where $C$ is an arbitrary (hermitian) element of $\mathcal{A}$, and $\varepsilon^{ijk}$ is the totally antisymmetric Levi-Civita symbol. It is easy to check that $\mathcal{A}$ is an almost Kähler–Poisson algebra with

$$
\gamma^2 = (\partial_1 C)^2 + (\partial_2 C)^2 + (\partial_3 C)^2.
$$

Moreover, one also can check that $\mathcal{A}$ is a Kähler–Poisson algebra. The projection operator $D_{ik}$ is computed to be

$$
D_{ik} = \delta_{ik} - \frac{1}{\gamma^2} (\partial_i C)(\partial_k C),
$$

which gives $\Pi_{ik} = (\partial_i C)(\partial_k C)/\gamma^2$. Hence, the geometric dimension of $\mathcal{A}$ is 2, and a basis for $\mathcal{A}(\mathcal{A})$ (which is then one-dimensional) is given by $\sum_{i=1}^3 (\partial_i C)\partial_i$. By using Gauss formula (in Proposition 5.11), one computes the curvature to be

$$
R(X, Y, Z, V) = \frac{1}{\gamma^2} \left( (\partial_{ik}^2 C)(\partial_{jl} C) - (\partial_{il}^2 C)(\partial_{jk} C) \right) X^i Y^j Z^k V^l.
$$

For instance, choosing $C = x^2 + y^2 + z^2 - r^2 1$, with $r \in \mathbb{R}$, one computes that $\mathcal{A}$ has constant curvature, i.e. $K(X, Y) = 1/\gamma^2$ for all $X, Y \in \mathcal{A}(\mathcal{A})$. Note that if one considers the quotient algebra $\mathcal{A}/\langle C \rangle$ (to which the Poisson structure restricts), the sectional curvature will be a constant, i.e. $K(X, Y) = 1/r^2$.

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