The monad of saturated prefilters

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Abstract
This paper addresses the problem whether the functor of (quantale-valued) saturated prefilters is a monad on the category of sets. It is shown that when the quantale is the unit interval equipped with a continuous t-norm &, the following conditions are equivalent:

1. The functor of saturated prefilters is a monad.
2. The functor of ⊤-filters is a monad.
3. The functor of bounded saturated prefilters is a monad.
4. The implication operator of & is continuous at each point off the diagonal.

Keywords Quantale, Continuous t-norm, Saturated prefilter, Bounded saturated pre-filter, Conical Q-semifilter, Bounded Q-semifilter, Monad

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1 Introduction

That the functor of filters is a monad on the category of sets is of crucial importance in topology and order theory. The multiplication of the filter monad helps us to express iterative limits in topology; the Eilenberg-Moore algebras of the monad of (not necessarily proper) filters are precisely the continuous lattices and hence the injective $T_0$ spaces; the Eilenberg-Moore algebras of the monad of ultrafilters are precisely the compact and Hausdorff spaces; and etc. There exists a large number of works that are related to applications of the filter monad in topology and order theory, for example, the monographs [10, 14, 32] and the articles [1, 6, 8, 31, 34, 38].

The notion of filter has been extended to the enriched (=quantale-valued in this paper) setting in different ways, resulting in prefilters [25, 26], ⊤-filters [15, 16, 17], functional ideals [28, 29, 30] (for Lawvere’s quantale), and (various kinds of) $Q$-filters (or, fuzzy filters) [7, 9, 16, 17]. The functor of $Q$-filters in the sense of [7, 9] and the functor of $Q$-filters in the sense of [16, 17] are both a monad on the category of sets, see [9, 17]. It is shown in [5] that the functor of functional ideals is also a monad on the category of sets. But, it is still unknown whether the functor of prefilters is a monad.

In this paper, we address the problem whether saturated prefilters [15, 26], a special kind of prefilters, form a monad. Since a monad consists of a functor and two natural transformations, the problem will be precisely formulated in Section 3. The answer depends on the structure of the quantale. A complete solution is presented in Section 5 in the case

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that the quantale $Q$ is the interval $[0, 1]$ equipped with a continuous t-norm. The key idea is to identify the functor of saturated prefilters with a subfunctor of the monad of $Q$-semifilters. In Section 6 and Section 7, the same technique is applied to investigate whether the functor of $\top$-filters and that of bounded saturated prefilters are monad, respectively.

## 2 Preliminaries

A complete lattice $L$ is **meet continuous** if for all $p \in L$, the map $p \land - : L \rightarrow L$ preserves directed joins, that is,

$$p \land \bigvee D = \bigvee_{d \in D} (p \land d)$$

for each directed set $D \subseteq L$.

In a complete lattice $L$, $x$ is way below $y$, in symbols $x \ll y$, if for every directed set $D \subseteq L$, $y \leq \bigvee D$ always implies that $x \leq d$ for some $d \in D$. It is clear that for all $p \in L$, \[ \downarrow p = \{x \in L \mid x \ll p\} \] is a directed subset of $L$. A complete lattice $L$ is **continuous** if $p = \bigvee \downarrow p$ for all $p \in L$. It is known that each continuous lattice is meet continuous \[10\].

In this paper, by a quantale we mean a commutative unital quantale in the sense of \[36\]. In the language of category theory, a quantale is a small, complete and symmetric monoidal closed category \[3, 19\]. Explicitly, a quantale

$$Q = (Q, \&, k)$$

is a commutative monoid with $k$ being the unit, such that the underlying set $Q$ is a complete lattice with a bottom element 0 and a top element 1, and that the multiplication $\&$ distributes over arbitrary joins. The multiplication $\&$ determines a binary operator $\rightarrow$, sometimes called the implication operator of $\&$, via the adjoint property:

$$p \& q \leq r \iff q \leq p \rightarrow r.$$

Given a quantale $(Q, \&, k)$, we say that

- **Q is integral**, if the unit $k$ coincides with the top element of the complete lattice $Q$;

- **Q is meet continuous**, if the complete lattice $Q$ is meet continuous; and

- **Q is continuous**, if the complete lattice $Q$ is continuous.

Lawvere’s quantale $(\lfloor 0, \infty \rfloor^{\text{op}}, +, 0)$ is clearly integral and continuous. Quantales obtained by endowing the unit interval $[0, 1]$ with a continuous t-norm are of particular interest in this paper. A continuous t-norm \[20\] is, actually, a continuous map $\& : [0, 1]^2 \rightarrow [0, 1]$ that makes $([0, 1], \&, 1)$ into a quantale. Given a continuous t-norm $\&$, the quantale $Q = ([0, 1], \&, 1)$ is clearly integral and continuous. The way below relation in $[0, 1]$ is as follows: $x \ll y$ if either $x = 0$ or $x < y$. Such quantales play a decisive role in the BL-logic of Hájek \[13\].

**Example 2.1.** Some basic continuous t-norms and their implication operators:

1. The Gödel t-norm:

$$x \& y = \min\{x, y\}, \quad x \rightarrow y = \begin{cases} 1, & x \leq y, \\ y, & x > y. \end{cases}$$

The implication operator $\rightarrow$ of the Gödel t-norm is continuous except at $(x, x), x < 1$. 

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(2) The product t-norm:

\[ x \&_P y = xy, \quad x \rightarrow y = \begin{cases} 1, & x \leq y, \\ y/x, & x > y. \end{cases} \]

The implication operator \( \rightarrow \) of the product t-norm is continuous except at \((0, 0)\). The quantale \(([0, 1], \&_P, 1)\) is isomorphic to Lawvere’s quantale \(([0, \infty]^\text{op}, +, 0)\) \[23\].

(3) The Lukasiewicz t-norm:

\[ x \&_L y = \max\{0, x + y - 1\}, \quad x \rightarrow y = \min\{1 - x + y, 1\}. \]

The implication operator \( \rightarrow \) of the Lukasiewicz t-norm is continuous on \([0, 1]^2\).

Let \& be a continuous t-norm. An element \( p \in [0, 1] \) is idempotent if \( p \& p = p \).

**Proposition 2.2.** ([20, Proposition 2.3]) Let \& be a continuous t-norm on \([0, 1]\) and \( p \) be an idempotent element of \&. Then \( x \& y = \min\{x, y\} \) whenever \( x \leq p \leq y \).

It follows immediately that \( y \rightarrow x = x \) whenever \( x < p \leq y \) for some idempotent \( p \).

Another consequence of Proposition 2.2 is that for any idempotent elements \( p, q \) with \( p < q \), the restriction of \& to \([p, q]\), which is also denoted by \&', makes \([p, q]\) into a commutative quantale with \( q \) being the unit element. The following theorem, known as the ordinal sum decomposition theorem, plays a prominent role in the theory of continuous t-norms.

**Theorem 2.3.** ([20, 35]) Let \& be a continuous t-norm. If \( a \in [0, 1] \) is non-idempotent, then there exist idempotent elements \( a^-, a^+ \in [0, 1] \) such that \( a^- < a < a^+ \) and that the quantale \(([a^-, a^+], \&)\) is either isomorphic to \(([0, 1], \&_P, 1)\) or to \(([0, 1], \&_L, 1)\). Conversely, for each set of disjoint open intervals \( \{(a_n, b_n)\}_n \) of \([0, 1]\), the binary operator

\[ a_n + (b_n - a_n)T_n\left(\frac{x - a_n}{b_n - a_n}, \frac{y - a_n}{b_n - a_n}\right), \quad (x, y) \in [a_n, b_n]^2, \]

\[ \min\{x, y\}, \quad \text{otherwise} \]

is a continuous t-norm, where each \( T_n \) is a continuous t-norm on \([0, 1]\).

Let \( Q = (Q, \&_Q, k) \) be a quantale. A \( Q \)-category \([11, 23, 37]\) (a.k.a. \( Q \)-order, see e.g. [2, 22]) consists of a set \( A \) and a map \( a : A \times A \rightarrow Q \) such that

\[ k \leq a(x, x) \quad \text{and} \quad a(y, z) \& a(x, y) \leq a(x, z) \]

for all \( x, y, z \in A \). It is customary to write \( A \) for the pair \((A, a)\) and \( A(x, y) \) for \( a(x, y) \) if no confusion would arise.

A \( Q \)-functor \( f : A \rightarrow B \) between \( Q \)-categories is a map \( f : A \rightarrow B \) such that

\[ A(x, y) \leq B(f(x), f(y)) \]

for all \( x, y \in A \).
Example 2.4. \((Q, d_L)\) is a \(Q\)-category, where
\[
d_L(p, q) = p \rightarrow q
\]
for all \(p, q \in Q\). More generally, for each set \(X\), \((X, \text{sub}_X)\) is a \(Q\)-category, where for all \(\lambda, \mu \in Q^X\),
\[
\text{sub}_X(\lambda, \mu) = \bigwedge_{x \in X} \lambda(x) \rightarrow \mu(x).
\]
For each map \(f : X \longrightarrow Y\) and each \(\lambda \in Q^X\), define \(f(\lambda) \in Q^Y\) by letting
\[
f(\lambda)(y) = \bigvee_{f(x) = y} \lambda(x).
\]
Then, for all \(\lambda \in Q^X\) and \(\mu \in Q^Y\),
\[
\text{sub}_Y(f(\lambda), \mu) = \text{sub}_X(\lambda, \mu \circ f).
\]
This equation is an instance of Kan extensions in the theory of enriched categories \([3, 19, 23]\).

3 The problem

A filter on a set \(X\) is an upper set of \((P(X), \subseteq)\) that is closed under finite meets and does not contain the empty set \(\emptyset\). The notion of filter has been extended to the quantale-valued setting in different ways: prefilters and \(Q\)-semifilters.

Definition 3.1. (Lowen, \([25]\)) A prefilter \(F\) on a set \(X\) is a subset of \(Q^X\) and such that

1. \(k_X \in F\), where \(k_X\) is the constant map \(X \longrightarrow \mathbb{Q}\) with value \(k\);
2. if \(\lambda, \mu \in F\) then \(\lambda \land \mu \in F\);
3. if \(\lambda \in F\) and \(\lambda \leq \mu\) then \(\mu \in F\).

It should be warned that in the above definition, a prefilter \(F\) is allowed to contain the constant map \(0_X\), in which case \(F = Q^X\); but, in \([25]\) a prefilter is required not to contain \(0_X\).

Definition 3.2. (Höhle, \([15\text{, Definition 1.5}]\)) Let \(F\) be a prefilter on a set \(X\). Then we say that

1. \(F\) is saturated if \(\lambda \in F\) whenever \(\bigvee_{\mu \in F} \text{sub}_X(\mu, \lambda) \geq k\).
2. \(F\) is a \(\top\)-prefilter if it is saturated and \(\bigvee_{x \in X} \lambda(x) \geq k\) whenever \(\lambda \in F\).

It is clear that saturated prefilters on \(X\) are closed with respect to intersection. For each prefilter \(F\), the smallest saturated prefilter containing \(F\) is called the saturation of \(F\).

For each set \(X\), let \(\text{SPF}(X)\) denote the set of all saturated prefilters on \(X\). For each map \(f : X \longrightarrow Y\) and each saturated prefilter \(F\) on \(X\), let
\[
f(F) := \{\lambda \in Q^Y \mid \lambda \circ f \in F\}.
\]
Then \( f(F) \) is a saturated prefilter on \( Y \). That \( f(F) \) is a prefilter is clear, to see that it is saturated, suppose that

\[
\bigvee_{\lambda \circ f \in F} \text{sub}_Y(\lambda, \mu) \geq k.
\]

Then

\[
\bigvee_{\lambda \circ f \in F} \text{sub}_X(\lambda \circ f, \mu \circ f) \geq \bigvee_{\lambda \circ f \in F} \text{sub}_Y(\lambda, \mu) \geq k,
\]

which implies that \( \mu \circ f \in F \), hence \( \mu \in f(F) \). In this way, we obtain a functor

\[
\text{SPF} : \text{Set} \rightarrow \text{Set}.
\]

For each map \( f : X \rightarrow Y \) and each \( \top \)-filter \( F \) on \( X \), \( f(F) \) is clearly a \( \top \)-filter on \( Y \). So, assigning to each set \( X \) the set \( \top\text{-Fil}(X) \) of all \( \top \)-filters on \( X \) defines a functor

\[
\top\text{-Fil} : \text{Set} \rightarrow \text{Set},
\]

which is a subfunctor of \( \text{SPF} \).

Let \( X \) be a set. For each \( x \in X \), let

\[
\varnothing_X(x) := \{ \lambda \in Q^X \mid \lambda(x) \geq k \};
\]

(3.1)

for each saturated prefilter \( \mathcal{F} \) on \( \text{SPF}(X) \), let

\[
n_X(\mathcal{F}) := \{ \lambda \in Q^X \mid \lambda \in \mathcal{F} \},
\]

where for each saturated prefilter \( F \) on \( X \),

\[
\tilde{\lambda}(F) = \bigvee_{\mu \in F} \text{sub}_X(\mu, \lambda).
\]

As we shall see in Proposition 5.3, \( \varnothing = \{ \varnothing_X \}_X \) is a natural transformation \( \text{id} \rightarrow \text{SPF} \) and \( n = \{ n_X \}_X \) is a natural transformation \( \text{SPF}^2 \rightarrow \text{SPF} \). The formulas (3.1) and (3.2) appears in Yue and Fang [39] for \( \top \)-filters (also see Remark 6.4 below). So, the proof of the naturality of \( \eta \) and \( \mu \) in [39, Section 3] also verifies that \( \varnothing \) and \( n \) are natural transformations. Any way, Proposition 5.3 shows that these formulas are derived in a natural way.

**Problem 3.3.** When is the triple \((\text{SPF}, n, \varnothing)\) a monad?

### 4 The monad of Q-semifilters

The following definition is a slight modification of that of Q-filters in [9, 16, 15].

**Definition 4.1.** A Q-semifilter on a set \( X \) is a map \( \mathcal{F} : Q^X \rightarrow Q \) subject to the following conditions: for all \( \lambda, \mu \in Q^X \),

(F1) \( \mathcal{F}(k_X) \geq k \);

(F2) \( \mathcal{F}(\lambda) \wedge \mathcal{F}(\mu) \leq \mathcal{F}(\lambda \wedge \mu) \);

(F3) \( \text{sub}_X(\lambda, \mu) \leq \mathcal{F}(\lambda) \rightarrow \mathcal{F}(\mu) \); in other words, \( \mathcal{F} : (Q^X, \text{sub}_X) \rightarrow (Q, d_L) \) is a Q-functor.
A $Q$-semifilter $\mathcal{F}$ is called a $Q$-filter if it satisfies moreover

(F4) $\mathcal{F}(p_X) \leq p$ for all $p \in Q$, where $p_X$ is the constant map $X \to Q$ with value $p$.

The condition (F3) is equivalent to

(F3') $p \& \mathcal{F}(\lambda) \leq \mathcal{F}(p \& \lambda)$ for all $p \in Q$ and $\lambda \in Q^X$.

And, (F3) implies that the inequalities in (F2) and (F4) are actually equalities.

For each set $X$, let $Q\text{-SemFil}(X)$ be the set of all $Q$-semifilters on $X$. For each map $f : X \to Y$ and each $\mathcal{F} \in Q\text{-SemFil}(X)$, let

$$f(\mathcal{F}) : Q^Y \to Q$$

be given by

$$f(\mathcal{F})(\mu) = \mathcal{F}(\mu \circ f).$$

Then $f(\mathcal{F})$ is a $Q$-semifilter on $Y$. Therefore, we obtain a functor

$$Q\text{-SemFil} : \text{Set} \to \text{Set}.$$ 

Likewise, assigning to each set $X$ the set $Q\text{-Fil}(X)$ of all $Q$-filters on $X$ defines a functor

$$Q\text{-Fil} : \text{Set} \to \text{Set},$$

which is a subfunctor of $Q\text{-SemFil}$.

For each set $X$ and each $x$ of $X$, the map

$$e_X(x) : Q^X \to Q, \quad e_X(x)(\lambda) = \lambda(x)$$

is a $Q$-filter. It is clear that $e = \{e_X\}_X$ is a natural transformation $\text{id} \to Q\text{-SemFil}$.

For each $Q$-semifilter $\mathcal{F}$ on $Q\text{-SemFil}(X)$, define

$$m_X(\mathcal{F}) : Q^X \to Q$$

by

$$m_X(\mathcal{F})(\lambda) = \mathcal{F}(\hat{\lambda}),$$

where $\hat{\lambda} : Q\text{-SemFil}(X) \to Q$ is given by

$$\hat{\lambda}(\mathcal{G}) = \mathcal{G}(\lambda).$$

Then $m_X(\mathcal{F})$ is a $Q$-semifilter on $X$. That $m_X(\mathcal{F})$ satisfies (F1) and (F2) is obvious. To see that it satisfies (F3), let $\lambda, \mu \in Q^X$. Then

$$\text{sub}_X(\lambda, \mu) \leq \bigwedge_{\mathcal{G} \in Q\text{-SemFil}(X)} (\mathcal{G}(\lambda) \to \mathcal{G}(\mu))$$

$$= \text{sub}_{Q\text{-SemFil}(X)}(\hat{\lambda}, \hat{\mu})$$

$$\leq \mathcal{F}(\hat{\lambda}) \to \mathcal{F}(\hat{\mu})$$

$$= m_X(\mathcal{F})(\lambda) \to m_X(\mathcal{F})(\mu).$$

The $Q$-semifilter $m_X(\mathcal{F})$ is called the diagonal ($Q$-semifilter) of $\mathcal{F}$.

It is routine to check that $m = \{m_X\}_X$ is a natural transformation $Q\text{-SemFil}^2 \to Q\text{-SemFil}$. Moreover, we have:
Proposition 4.2. (C.f. [17, Theorem 2.4.2.2]) The triple \((Q\text{-SemFil}, m, e)\) is a monad on the category of sets.

Proof. Routine verification, in fact, \((Q\text{-SemFil}, m, e)\) is a submonad of the double \(Q\)-powerset monad as we see below.

Let \((S, \mu, \eta)\) be a monad on the category of sets; and let \(T\) be a subfunctor of \(S\). Suppose that \(T\) satisfies the following conditions:

- for each set \(X\) and each \(x \in X\), \(\eta_X(x) \in T(X)\). So \(\eta\) is also a natural transformation from the identity functor to \(T\).
- \(T\) is closed under multiplication in the sense that for each set \(X\) and each \(H \in T T(X)\), \(\mu_X \circ (i * i)_X(\mathbb{H}) \in T(X)\), where \(i\) is the inclusion transformation of \(T\) to \(S\) and \(i * i\) stands for the horizontal composite of \(i\) with itself. So \(\mu\) determines a natural transformation \(T^2 \to T\), which is also denoted by \(\mu\).

Then, \((T, \mu, \eta)\) is a monad and the inclusion transformation \(i : T \to S\) is a monad morphism. In this case, \(T\) is called a submonad of \((S, \mu, \eta)\) [33, 34].

Given a set \(B\), the contravariant functor

\[ P_B : \text{Set}^{\text{op}} \to \text{Set}, \]

which sends each set \(X\) to (the \(B\)-powerset) \(B^X\), is right adjoint to its opposite

\[ P_B^{\text{op}} : \text{Set} \to \text{Set}^{\text{op}}. \]

The resulting monad \((P, m, e)\) is called the \(B\)-double powerset monad [17, Remark 1.2.7], where,

\[ P(X) = P_B \circ P_B^{\text{op}}(X) = B^{B^X}; \]

the unit \(e_X : X \to P(X)\) is given by

\[ e_X(x)(\lambda) = \lambda(x) \]

for all \(x \in X\) and \(\lambda \in B^X\); and the multiplication \(m_X : P P(X) \to P(X)\) is given by

\[ m_X(H)(\lambda) = H(\lambda) \]

for all \(H : B^{P(X)} \to B\) and \(\lambda \in B^X\) with \(\lambda : P(X) \to B\) given by

\[ \lambda(\mathfrak{A}) = \mathfrak{A}(\lambda) \]

for all \(\mathfrak{A} \in P(X)\).

The monad \((Q\text{-SemFil}, m, e)\) is clearly a submonad of the double \(B\)-powerset monad; \(Q\)-Fil is a submonad of \((Q\text{-SemFil}, m, e)\), hence a submonad of the double \(B\)-powerset monad.
Remark 4.3. The construction of the submonads \((\text{Q-SemFil}, m, e)\) and \((\text{Q-Fil}, m, e)\) is typical: when the set \(B\) comes with some structures, we may be able to formulate some submonads of the double \(B\)-powerset monad by aid of the structures on \(B\). The filter monad and the ultrafilter monad are important examples of this construction. To see this, let \(B = \{0, 1\}\), viewed as a lattice with \(0 < 1\). Then, assigning to each set \(X\) the set
\[
\{ \lambda : 2^X \to 2 \mid \lambda(X) = 1, \lambda(\emptyset) = 0, \lambda(A \cap B) = \lambda(A) \land \lambda(B) \}
\]
defines a submonad of \((P, m, e)\) — the monad of proper filters; assigning to each set \(X\) the set
\[
\{ \lambda : 2^X \to 2 \mid \lambda \text{ is a lattice homomorphism} \}
\]
defines a submonad of \((P, m, e)\) — the monad of ultrafilters. Furthermore, assigning to each set \(X\) the set
\[
P^+(X) := \{ \lambda : 2^X \to 2 \mid \lambda \text{ is a right adjoint} \}
\]
defines a submonad of \((P, m, e)\), which is essentially the covariant powerset monad.

5 The monad of saturated prefilters

For a \(\text{Q}\)-semifilter \(\mathfrak{F}\) on \(X\), the set
\[
\Gamma_X(\mathfrak{F}) = \{ \lambda \in Q^X \mid \mathfrak{F}(\lambda) \geq k \}
\]
is a saturated prefilter on \(X\). To see that \(\Gamma_X(\mathfrak{F})\) is saturated, suppose that
\[
\bigvee_{\lambda \in \Gamma_X(\mathfrak{F})} \text{sub}_X(\lambda, \mu) \geq k.
\]
Then,
\[
\mathfrak{F}(\mu) \geq \bigvee_{\lambda \in \Gamma_X(\mathfrak{F})} \text{sub}_X(\lambda, \mu) \& \mathfrak{F}(\lambda) = \bigvee_{\lambda \in \Gamma_X(\mathfrak{F})} \text{sub}_X(\lambda, \mu) \geq k;
\]
so \(\mu \in \Gamma_X(\mathfrak{F})\).

Conversely, if \(Q\) is a meet continuous quantale, then for each prefilter \(F\) on \(X\), the map
\[
\Lambda_X(F) : Q^X \to Q, \quad \Lambda_X(F)(\lambda) = \bigvee_{\mu \in F} \text{sub}_X(\mu, \lambda)
\]
is easily verified to be a \(\text{Q}\)-semifilter.

Convention. From now on, all quantales are assumed to be meet continuous.

The assignment \(X \mapsto \Gamma_X\) is a natural transformation
\[
\Gamma : \text{Q-SemFil} \to \text{SPF};
\]
the assignment \(X \mapsto \Lambda_X\) is a natural transformation
\[
\Lambda : \text{SPF} \to \text{Q-SemFil}.
\]
Furthermore, since for each prefilter $F$ and each $Q$-semifilter $\mathfrak{F}$ on a set $X$,

$$F \subseteq \Gamma_X(\mathfrak{F}) \iff \forall \lambda \in F, \mathfrak{F}(\lambda) \geq k$$
$$\iff \forall \lambda \in F, \forall \mu \in Q^X, \text{sub}_X(\lambda, \mu) \leq \mathfrak{F}(\mu)$$
$$\iff \Lambda_X(F) \leq \mathfrak{F},$$

it follows that $\Lambda_X$ and $\Gamma_X$ form a Galois connection [10] between the partially ordered sets of prefilters and $Q$-semifilters on $X$ with $\Lambda_X$ being the left adjoint.

**Proposition 5.1.** For each prefilter $F$ on a set $X$, the saturation of $F$ is given by

$$\Gamma_X \circ \Lambda_X(F) = \left\{ \lambda \in Q^X \mid \bigvee_{\mu \in F} \text{sub}_X(\mu, \lambda) \geq k \right\}.$$

In particular, $F$ is saturated if and only if $F = \Gamma_X(\mathfrak{F})$ for some $Q$-semifilter $\mathfrak{F}$, if and only if $F = \Gamma_X \circ \Lambda_X(F)$.

**Proof.** This follows immediately from the adjunction $\Lambda_X \dashv \Gamma_X$ and the fact that

$$\lambda \in \Gamma_X \circ \Lambda_X(F) \iff \Lambda_X(F)(\lambda) = \bigvee_{\mu \in F} \text{sub}_X(\mu, \lambda) \geq k$$

for all $\lambda \in Q^X$.

The adjunction $\Lambda_X \dashv \Gamma_X$ leads to the following:

**Definition 5.2.** A $Q$-semifilter $\mathfrak{F}$ is conical if $\mathfrak{F} = \Lambda_X(F)$ for some prefilter $F$.

Let $f : X \to Y$ be a map and $\mathfrak{F}$ be a conical $Q$-semifilter on $X$. Since

$$f(\mathfrak{F})(\mu) = \mathfrak{F}(\mu \circ f)$$
$$= \bigvee_{\lambda \in \Gamma_X(\mathfrak{F})} \text{sub}_X(\lambda, \mu \circ f)$$
$$= \bigvee_{\lambda \in \Gamma_X(\mathfrak{F})} \text{sub}_Y(f(\lambda), \mu)$$
$$= \bigvee_{\gamma \in f(\Gamma_X(\mathfrak{F}))} \text{sub}_Y(\gamma, \mu),$$

it follows that $f(\mathfrak{F})$ is conical. Therefore, we obtain a functor

$$\text{ConSF} : \text{Set} \to \text{Set}.$$ 

It is clear that a $Q$-semifilter $\mathfrak{F}$ is conical if and only if $\Lambda_X \circ \Gamma_X(\mathfrak{F}) = \mathfrak{F}$, so, $F \mapsto \Lambda_X(F)$ establishes a bijection between saturated prefilters and conical $Q$-semifilters on $X$. Since for each map $f : X \to Y$ and each prefilter $F$ on $X$,

$$f(\Lambda_X(F)) = \Lambda_X(f(F)),$$

it follows that the functor of saturated prefilters is naturally isomorphic to the functor of conical $Q$-semifilters:

$$\text{SPF} \xrightarrow{\Lambda} \text{ConSF} \xleftarrow{\Gamma}$$ (5.1)
Thus, in order to make the functor of saturated prefilters into a monad, it suffices to do so for the functor of conical $Q$-semifilters.

The functor $\text{ConSF}$ is a obviously subfunctor of $\text{Q-SemFil}$. Furthermore, it is a retract of $\text{Q-SemFil}$, as we see now. For each set $X$, define

$$c_X : \text{Q-SemFil}(X) \longrightarrow \text{ConSF}(X)$$

by letting

$$c_X(\mathcal{F}) := \Lambda_X \circ \Gamma_X(\mathcal{F})$$

for each $Q$-semifilter $\mathcal{F}$. Then $c_X$ is right adjoint and left inverse to the inclusion

$$i_X : \text{ConSF}(X) \longrightarrow \text{Q-SemFil}(X).$$

This implies that $c_X(\mathcal{F})$ is the largest conical $Q$-semifilter that is smaller than or equal to $\mathcal{F}$, so we call it the conical coreflection of $\mathcal{F}$. We note that for each $\lambda \in Q^X$, 

$$\mathcal{F}(\lambda) \geq k \iff c_X(\mathcal{F})(\lambda) \geq k.$$

Since $c = \{c_X\}_X$ and $i = \{i_X\}_X$ are natural transformations, we get natural transformations

$$d := c \circ e \quad \text{and} \quad n := c \circ m \circ (i \ast i),$$

where $e$ and $m$ refer to the unit and the multiplication of the monad $(\text{Q-SemFil}, m, e)$, respectively.

![Diagram](image)

We spell out the details of $d$ and $n$ for later use. For each set $X$ and each $x$ of $X$, since the $Q$-semifilter $e_X(x)$ is conical, it follows that

$$d_X(x) = e_X(x).$$

Since for each conical $Q$-semifilter $\mathcal{F}$ on $\text{ConSF}(X)$ and each $\xi : \text{Q-SemFil}(X) \longrightarrow Q$,

$$(i \ast i)_X(\mathcal{F})(\xi) = \mathcal{F}(\xi \circ i_X),$$

it follows that $n_X(\mathcal{F})$, the conical coreflection of $m_X \circ (i \ast i)_X(\mathcal{F})$, is given by

$$n_X(\mathcal{F})(\lambda) = \bigvee_{m_X \circ (i \ast i)_X(\mathcal{F})(\mu) \geq k} \text{sub}_X(\mu, \lambda) = \bigvee_{\mathcal{F}(\mu) \circ i_X \geq k} \text{sub}_X(\mu, \lambda)$$

for all $\lambda \in Q^X$.

Let $\Gamma$ and $\Lambda$ be the natural isomorphisms in Equation (5.1). Then,

$$\Gamma \circ d$$

is a natural transformation $\text{id} \longrightarrow \text{SPF}$ and

$$\Gamma \circ n \circ (\Lambda \ast \Lambda)$$

is a natural transformation $\text{SPF}^2 \longrightarrow \text{SPF}$. The following proposition says that $\Gamma \circ d$ and $\Gamma \circ n \circ (\Lambda \ast \Lambda)$ are, respectively, the natural transformations $d$ and $n$ given via the formulas (3.1) and (3.2).
Proposition 5.3. \( d = \Gamma \circ d \) and \( n = \Gamma \circ n \circ (\Lambda \ast \Lambda) \).

Proof. (1) is obvious. As for (2), let \( \lambda \in Q^X \). Then

\[
\lambda \in \Gamma_X \circ n_X \circ (\Lambda \ast \Lambda)_X(F) \iff m_X \circ (i \ast i)_X \circ (\Lambda \ast \Lambda)_X(F)(\lambda) \geq k 
\iff (\Lambda \ast \Lambda)_X(F) (\lambda \circ i_X) \geq k 
\iff \Lambda_{\text{SPF}} (X)(\lambda \circ i_X \circ \Lambda_X) \geq k 
\iff \lambda_X(F) \subseteq m_X \circ (i \ast i)_X \circ (\Lambda \ast \Lambda)_X(F) \subseteq \text{satur}(F).
\]

Since for each saturated prefilter \( F \) on \( X \),

\[
\hat{\lambda} \circ i_X \circ \Lambda_X(F) = \Lambda_X(F)(\lambda) = \bigvee_{\mu \in F} \text{sub}_X(\mu, \lambda) = \text{\textlambda}(F),
\]

it follows that \( \lambda \in \Gamma_X \circ n_X \circ (\Lambda \ast \Lambda)_X(F) \) if and only if \( \text{\textlambda} \in \text{\textmathbb{F}} \), hence \( n = \Gamma \circ n \circ (\Lambda \ast \Lambda) \). \( \square \)

We say that conical \( Q \)-semifilters are closed under multiplication if for each set \( X \) and each conical \( Q \)-semifilter \( F \) on \( \text{\text{ConSF}}(X) \), the \( Q \)-semifilter (on \( X \))

\[
m_X \circ (i \ast i)_X(F)
\]

is conical, where \( i : \text{\text{ConSF}} \rightarrow \text{\text{Q-SemFil}} \) denotes the inclusion transformation. By definition, \( m_X \circ (i \ast i)_X(F) \) is the diagonal \( Q \)-semifilter

\[
m_X(i_X(F)) : Q^X \rightarrow Q, \quad \lambda \mapsto \text{\textnormal{\textmathbb{F}}} (\lambda \circ i_X).
\]

The following conclusion is obvious.

Proposition 5.4. If the conical \( Q \)-semifilters are closed under multiplication, then \( (\text{\text{ConSF}}, n, d) \) is a submonad of \( (\text{\text{Q-SemFil}}, m, e) \).

There is an easy-to-check condition for \( \text{\text{ConSF}} \) to be a submonad of the \( Q \)-semifilter monad:

Proposition 5.5. Let \( Q \) be a quantale such that for each \( p \in Q \), the map

\[
p \rightarrow - : Q \rightarrow Q
\]

preserves directed joins. Then, \( \text{\text{ConSF}} \) is a submonad of the \( Q \)-semifilter monad.

We prove two lemmas first. The first is a slight extension of \cite[Proposition 9]{11}; the second is a slight extension of \cite[Corollary 3.13]{21}). Proofs are included here for convenience of the reader.

Lemma 5.6. A \( Q \)-semifilter \( \mathcal{F} \) on a set \( X \) is conical if and only if

\[
\mathcal{F}(\lambda) = \bigvee \{ p \in Q \mid \mathcal{F}(p \rightarrow \lambda) \geq k \}
\]

for all \( \lambda \in Q^X \).
Proof. Necessity. Suppose that $\mathcal{F}$ is conical. If $\mathcal{F}(p \to \lambda) \geq k$, then

$$p \leq \text{sub}_X(p \to \lambda, \lambda) \leq \mathcal{F}(p \to \lambda) \to \mathcal{F}(\lambda) \leq \mathcal{F}(\lambda),$$

hence

$$\mathcal{F}(\lambda) \geq \bigvee \{p \in Q \mid \mathcal{F}(p \to \lambda) \geq k\}.$$

As for the converse inequality, for each $\nu \in \Gamma_X(\mathcal{F})$, let $p_\nu = \text{sub}_X(\nu, \lambda)$. Since $\nu \leq p_\nu \to \lambda$, then $\mathcal{F}(p_\nu \to \lambda) \geq k$, and consequently,

$$\mathcal{F}(\lambda) = \bigvee_{\nu \in \Gamma_X(\mathcal{F})} \text{sub}_X(\nu, \lambda) \leq \bigvee \{p \in Q \mid \mathcal{F}(p \to \lambda) \geq k\}.$$

Sufficiency. It suffices to check that $\mathcal{F} \leq \Lambda_X \circ \Gamma_X(\mathcal{F})$ since $\Lambda_X$ is left adjoint to $\Gamma_X$. For all $p \in Q$ with $\mathcal{F}(p \to \lambda) \geq k$, one has $p \to \lambda \in \Gamma_X(\mathcal{F})$, so,

$$p \leq \text{sub}_X(p \to \lambda, \lambda) \leq \Lambda_X \circ \Gamma_X(\mathcal{F})(\lambda).$$

Therefore, $\mathcal{F}(\lambda) \leq \Lambda_X \circ \Gamma_X(\mathcal{F})(\lambda)$ since $\mathcal{F}(\lambda) = \bigvee \{p \in Q \mid \mathcal{F}(p \to \lambda) \geq k\}. \quad \square$

Lemma 5.7. Let $Q$ be a quantale such that for each $p \in Q$, the map $p \to - : Q \to Q$ preserves directed joins. Then, a $Q$-semifilter $\mathcal{F}$ on a set $X$ is conical if and only if

$$\mathcal{F}(p \to \lambda) = p \to \mathcal{F}(\lambda)$$

for all $p \in Q$ and $\lambda \in Q^X$.

Proof. If $\mathcal{F}$ is conical, then for each $p \in Q$,

$$\mathcal{F}(p \to \lambda) = \bigvee_{\nu \in \Gamma_X(\mathcal{F})} \text{sub}_X(\nu, p \to \lambda)$$

$$= \bigvee_{\nu \in \Gamma_X(\mathcal{F})} (p \to \text{sub}_X(\nu, \lambda))$$

$$= p \to \bigvee_{\nu \in \Gamma_X(\mathcal{F})} \text{sub}_X(\nu, \lambda)$$

$$= p \to \mathcal{F}(\lambda).$$

Conversely, assume that $\mathcal{F}(p \to \lambda) = p \to \mathcal{F}(\lambda)$ for all $p \in Q$ and $\lambda \in Q^X$. Then

$$p \leq \mathcal{F}(\lambda) \iff \mathcal{F}(p \to \lambda) \geq k,$$

hence

$$\mathcal{F}(\lambda) = \bigvee \{p \in Q \mid \mathcal{F}(p \to \lambda) \geq k\},$$

and consequently, $\mathcal{F}$ is conical by Lemma 5.6. \quad \square

Proof of Proposition 5.7. It suffices to check that conical $Q$-semifilters are closed under multiplication. Let $\mathcal{F}$ be a conical $Q$-semifilter on $\text{ConSF}(X)$. We need to check that the diagonal $Q$-semifilter $m_X(i_X(\mathcal{F}))$ is conical. For each $\lambda \in Q^X$ and $p \in Q$, since

$$m_X(i_X(\mathcal{F}))(p \to \lambda) = i_X(\mathcal{F})(p \to \lambda)$$
\[
\begin{align*}
&= \mathbb{F}(p \to (\hat{\lambda} \circ i_X)) \\
&= p \to \mathbb{F}(\hat{\lambda} \circ i_X) \\
&= p \to m_X(i_X(F))(\lambda),
\end{align*}
\]

then the conclusion follows immediately from Lemma 5.7. \hfill \Box

**Corollary 5.8.** Let Q be a quantale such that for each \( p \in Q \), the map \( p \to - : Q \to Q \) preserves directed joins. Then, \((\text{SPF}, n, d)\) is a monad on the category of sets.

The condition in Proposition 5.5 is a strong one. For example, for a continuous t-norm \&, the quantale \( Q = ([0,1], \& , 1) \) satisfies the condition in Proposition 5.5 (i.e., \( p \to - : [0,1] \to [0,1] \) preserves directed joins for all \( p \)) if and only if \& is Archimedean. So, there are essentially only two continuous t-norms — the product t-norm and the Lukasiewicz t-norm — that satisfy the condition.

It seems hard to find a sufficient and necessary condition for a general quantale \( Q \) so that \((\text{ConSF}, n, d)\) is monad. However, in the case that \( Q \) is the interval \([0,1]\) equipped with a continuous t-norm, there is one.

**Proposition 5.9.** (22) For each continuous t-norm \& on \([0,1]\), the following conditions are equivalent:

(S1) For each non-idempotent element \( a \in [0,1] \), the quantale \(([a^-, a^+], \& , a^+)\) is isomorphic to \(([0,1], \& , p, 1)\) whenever \( a^- > 0 \).

(S2) The implication \( \to : [0,1]^2 \to [0,1] \) is continuous at every point off the diagonal \( \{(x, x) \mid x \in [0,1]\} \).

(S3) For each \( p \in (0,1] \), the map \( p \to - : [0,p) \to [0,1] \) preserves directed joins.

We say that a continuous t-norm satisfies the condition \((S)\) if it satisfies one, hence all, of the equivalent conditions (S1)–(S3). Every continuous Archimedean t-norm satisfies the condition \((S)\); the ordinal sum decomposition theorem guarantees that there exist many continuous t-norms that satisfy the condition \((S)\) but are not Archimedean, with the Gödel t-norm being an example.

**Theorem 5.10.** Let \( Q = ([0,1], \& , 1) \) with \& being a continuous t-norm. Then the following conditions are equivalent:

(1) The t-norm \& satisfies the condition \((S)\).

(2) \( \text{ConSF} \) is a submonad of \((\text{Q-SemFil}, m, e)\).

(3) The triple \((\text{ConSF}, n, d)\) is a monad.

(4) The triple \((\text{SPF}, n, d)\) is a monad.

We make some preparations first.

**Lemma 5.11.** If \( \{\mathcal{F}_i\}_i \) is a directed family of conical \( Q \)-semifilters on \( X \), then so is \( \bigvee_i \mathcal{F}_i \).

**Proof.** Let \( F = \bigcup_i \Gamma_X(\mathcal{F}_i) \). Then, \( F \) is a prefilter and \( \Lambda_X(F) = \bigvee_i \mathcal{F}_i \). \hfill \Box
**Proposition 5.12.** Conical Q-semifilters are closed under multiplication if and only if the following conditions are satisfied:

1. If \( \{ \mathcal{F}_i \}_i \) is a family of conical Q-semifilters on \( X \), then so is \( \bigwedge_i \mathcal{F}_i \).
2. If \( \mathcal{F} \) is a conical Q-semifilter on \( X \), then so is \( p \mapsto \mathcal{F} \) for each \( p \in Q \).

**Proof.**

**Sufficiency.** If we can show that for each conical Q-semifilter \( F \) on the set \( \text{ConSF}(X) \),

\[
m_X \circ (i \ast i)_X(F) = \bigvee_{F(\xi) \geq k} \bigwedge_{\mathcal{G} \in \text{ConSF}(X)} (\xi(\mathcal{G}) \to \mathcal{G}),
\]

then the conclusion follows immediately since each directed join of conical Q-semifilters is conical.

In fact, for each \( \lambda \in Q^X \),

\[
m_X \circ (i \ast i)_X(F)(\lambda) = (\hat{\lambda} \circ i_X)
= \bigvee_{F(\xi) \geq k} \bigwedge_{\mathcal{G} \in \text{ConSF}(X)} (\xi, \hat{\lambda} \circ i_X)
= \bigvee_{F(\xi) \geq k} \bigwedge_{\mathcal{G} \in \text{ConSF}(X)} (\xi(\mathcal{G}) \to \mathcal{G})(\lambda).
\]

**Necessity.** Let \( F \) be the prefilter on \( \text{ConSF}(X) \) consisting of maps \( \xi : \text{ConSF}(X) \to Q \) with \( \xi(\mathcal{F}_i) = 1 \) for all \( i \) and let \( F = \Lambda_X(F) \). Then, for each \( \lambda \in Q^X \),

\[
m_X \circ (i \ast i)_X(F)(\lambda) = (\hat{\lambda} \circ i_X)
= \bigvee_{\xi \in F} \bigwedge_{\mathcal{G} \in \text{ConSF}(X)} (\xi, \hat{\lambda} \circ i_X)
= \bigvee_{\xi \in F} \bigwedge_{\mathcal{G} \in \text{ConSF}(X)} (\xi(\mathcal{G}) \to \mathcal{G})(\lambda),
\]

which shows that \( \bigwedge_i \mathcal{F}_i = m_X \circ (i \ast i)_X(F) \), so \( \bigwedge_i \mathcal{F}_i \) is conical. This proves (1).

As for (2), assume that \( \mathcal{F} \) is a conical Q-semifilter on \( X \) and \( p \in Q \). Let \( F \) be the prefilter on \( \text{ConSF}(X) \) consisting of maps \( \xi : \text{ConSF}(X) \to Q \) with \( \xi(\mathcal{F}) \geq p \) and let \( F = \Lambda_X(F) \). Then, for each \( \lambda \in Q^X \),

\[
m_X \circ (i \ast i)_X(F)(\lambda) = (\hat{\lambda} \circ i_X)
= \bigvee_{\xi \in F} \bigwedge_{\mathcal{G} \in \text{ConSF}(X)} (\xi, \hat{\lambda} \circ i_X)
= p \mapsto \mathcal{F}(\lambda).
\]

Therefore, \( p \mapsto \mathcal{F} = m_X \circ (i \ast i)_X(F) \) and is, consequently, conical. \( \square \)

**Lemma 5.13.** Let \( Q \) be a continuous quantale. Then, a Q-semifilter \( \mathcal{F} \) is conical if and only if \( \mathcal{F}(p \mapsto \lambda) \geq k \) whenever \( p \ll \mathcal{F}(\lambda) \).

**Proof.** It suffices to check that \( \mathcal{F} \leq \Lambda \circ \Gamma(\mathcal{F}) \). If \( p \ll \mathcal{F}(\lambda) \), then \( \mathcal{F}(p \mapsto \lambda) \geq k \) by assumption. Hence, \( p \mapsto \lambda \in \Gamma(\mathcal{F}) \), so, \( p \leq \text{sub}_X(p \mapsto \lambda, \lambda) \leq \Lambda \circ \Gamma(\mathcal{F})(\lambda) \), and consequently, \( \mathcal{F}(\lambda) \leq \Lambda \circ \Gamma(\mathcal{F})(\lambda) \). \( \square \)

**Proposition 5.14.** Let \( Q \) be a continuous quantale. If each member of the family \( \{ \mathcal{F}_i \}_i \) is a conical Q-semifilter on a set \( X \), then so is the meet \( \bigwedge_i \mathcal{F}_i \).

**Proof.** If \( p \ll \bigwedge_i \mathcal{F}_i(\lambda) \), then for each \( i \), \( p \ll \mathcal{F}_i(\lambda) \), hence \( \mathcal{F}_i(p \mapsto \lambda) \geq k \), and consequently, \( \bigwedge_i \mathcal{F}_i(p \mapsto \lambda) \geq k \). Therefore, the conclusion holds by Lemma 5.13. \( \square \)
Proof of Theorem 5.10. That (3) ⇔ (4) is clear. We prove that (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (2) Since \([0,1],\&\), 1 is a continuous quantale, by propositions 5.12 and 5.14, it suffices to prove that for each conical Q-semifilter \(\mathfrak{F} : [0,1]^X \rightarrow [0,1]\) and each \(p \in [0,1]\), \(p \rightarrow \mathfrak{F}\) is conical. Let \(G = \{\lambda \in [0,1]^X \mid p \leq \mathfrak{F}(\lambda)\}\).

Then \(G\) is a prefilter on \(X\). We show in two steps that

\[ p \rightarrow \mathfrak{F} = \Lambda_X(G), \]

which implies that \(p \rightarrow \mathfrak{F}\) is conical.

**Step 1.** \(\Lambda_X(G) \leq p \rightarrow \mathfrak{F}\).

For each \(\lambda \in G\), since \(\mathfrak{F}(\lambda) \geq p\), it follows that for all \(\mu \in [0,1]^X\),

\[ \text{sub}_X(\lambda, \mu) \leq \mathfrak{F}(\lambda) \rightarrow \mathfrak{F}(\mu) \leq p \rightarrow \mathfrak{F}(\mu), \]

hence \(\text{sub}_X(\lambda, -) \leq p \rightarrow \mathfrak{F}\) and consequently,

\[ \Lambda_X(G) = \bigvee_{\lambda \in G} \text{sub}_X(\lambda, -) \leq p \rightarrow \mathfrak{F}. \]

**Step 2.** \(p \rightarrow \mathfrak{F} \leq \Lambda_X(G)\).

First, we claim that if \(\lambda \in \Gamma_X(\mathfrak{F})\) then \(p \& \lambda \in G\). In fact, if \(\mathfrak{F}(\lambda) = 1\), then

\[ p \leq \text{sub}_X(\lambda, p \& \lambda) \leq \mathfrak{F}(\lambda) \rightarrow \mathfrak{F}(p \& \lambda) = \mathfrak{F}(p \& \lambda), \]

hence \(p \& \lambda \in G\).

Now we check that \(p \rightarrow \mathfrak{F}(\lambda) \leq \Lambda_X(G)(\lambda)\) for all \(\lambda \in [0,1]^X\).

If \(p \leq \mathfrak{F}(\lambda)\), then \(\lambda \in G\) and consequently,

\[ \Lambda_X(G)(\lambda) = \bigvee_{\mu \in G} \text{sub}_X(\mu, \lambda) \geq \text{sub}_X(\lambda, \lambda) = 1 \geq p \rightarrow \mathfrak{F}(\lambda). \]

If \(p > \mathfrak{F}(\lambda)\), then

\[ p \rightarrow \mathfrak{F}(\lambda) = p \rightarrow \bigvee_{\mu \in \Gamma_X(\mathfrak{F})} \text{sub}_X(\mu, \lambda) \]

\[ = \bigvee_{\mu \in \Gamma_X(\mathfrak{F})} (p \rightarrow \text{sub}_X(\mu, \lambda)) \]

\[ = \bigvee_{\mu \in \Gamma_X(\mathfrak{F})} \text{sub}_X(p \& \mu, \lambda) \]

\[ \leq \bigvee_{\gamma \in G} \text{sub}_X(\gamma, \lambda) \]

\[ = \Lambda_X(G)(\lambda). \]

(2) ⇒ (3) Clear by definition.

(3) ⇒ (1) For each map \(h : X \rightarrow \text{ConSF}(Y)\), let \(h^\sharp\) be the composite

\[ \text{ConSF}(X) \xrightarrow{\text{ConSF}(h)} \text{ConSF}^2(Y) \xrightarrow{n_Y} \text{ConSF}(Y). \]
Since $(\text{ConSF}, n, d)$ is a monad, it follows that
\[ g^\sharp \circ f^\sharp = (g^\sharp \circ f)^\sharp \]
for any $f : X \to \text{ConSF}(Y)$ and $g : Y \to \text{ConSF}(Z)$, see e.g. \[32\, 33\].

In the following we derive a contradiction if $\&$ does not satisfy the condition (S). Suppose that $p, q \in [0, 1]$ are idempotent elements of $\&$ such that $0 < p < q$ and that the restriction of $\&$ on $[p, q]$ is isomorphic to the Łukasiewicz t-norm. Pick $t, s \in (p, q)$ with $t \& s = p$.

Let $X = [0, 1]$. Consider the constant maps $f : X \to \text{ConSF}(X)$ and $g : X \to \text{ConSF}(X)$ given as follows: $f$ sends every $x$ to the conical $Q$-semifilter $\mathcal{F}$ generated by the prefilter
\[ \{ \nu : X \to [0, 1] | \nu \geq t_X \}; \]
$g$ sends every $x$ to the conical $Q$-semifilter $\mathcal{G}$ on $X$ generated by the prefilter
\[ \{ \nu : X \to [0, 1] | \exists n \geq 1, \nu \geq 1_{A_n} \}, \]
where $A_n = \{ 1/m | m \geq n \}$. By definition, for all $\mu \in [0, 1]^X$,
\[ f(x)(\mu) = \mathcal{F}(\mu) = t \to \bigwedge_{y \in X} \mu(y) \]
and
\[ g(x)(\mu) = \mathcal{G}(\mu) = \bigvee_{n \geq 1} \bigwedge_{m \geq n} \mu(1/m). \]

We claim that $g^\sharp \circ f^\sharp \neq (g^\sharp \circ f)^\sharp$, contradicting that $(\text{ConSF}, n, d)$ is a monad. To see this, let $\gamma(x) = p(1 - x)$ and let $\mathcal{H}$ be the conical $Q$-semifilter generated by the prefilter
\[ \{ \nu : X \to [0, 1] | \nu \geq s_X \}; \]
that is,
\[ \mathcal{H}(\mu) = s \to \bigwedge_{y \in X} \mu(y). \]

In the following we show in two steps that $g^\sharp \circ f^\sharp(\mathcal{H})(\gamma)$ is not equal to $(g^\sharp \circ f)^\sharp(\mathcal{H})(\gamma)$.

**Step 1.** $g^\sharp \circ f^\sharp(\mathcal{H})(\gamma) = 1$.

Since
\[ \mathbf{m}_X (f(\mathcal{H}))(\lambda) = f(\mathcal{H})(\lambda) \]
\[ = \mathcal{H}(\lambda \circ f) \]
\[ = \mathcal{H}(x \mapsto \mathcal{F}(\lambda)) \]
\[ = s \to \mathcal{F}(\lambda), \]

it follows that $f^\sharp(\mathcal{H})$ is the conical coreflection of $s \to \mathcal{F}$. Since
\[ s \to \mathcal{F}(\mu) = s \to \left( t \to \bigwedge_{y \in X} \mu(y) \right) = p \to \bigwedge_{y \in X} \mu(y), \]
it follows that \( s \rightarrow \mathfrak{F} \) is generated by \( \{ \nu \mid \nu \geq p_X \} \), so \( s \rightarrow \mathfrak{F} \) is conical and \[
f^\sharp(\mathfrak{F})(\mu) = p \rightarrow \bigwedge_{y \in X} \mu(y).
\]

Since
\[
m_X(g(f^\sharp(\mathfrak{F}))) (\lambda) = g(f^\sharp(\mathfrak{F}))(\lambda) = f^\sharp(\mathfrak{F})(\lambda) = f^\sharp(\mathfrak{F})(x \mapsto \mathfrak{G}(\lambda)) = p \rightarrow \mathfrak{G}(\lambda),
\]

it follows that \( g^\sharp \circ f^\sharp(\mathfrak{F}) \) is the conical coreflection of \( p \rightarrow \mathfrak{G} \).

Because
\[
p \rightarrow \mathfrak{G}(\mu) = 1 \iff p \leq \mathfrak{G}(\mu) = \bigvee_n \bigwedge_{m \geq n} \mu(1/m),
\]

therefore
\[
g^\sharp \circ f^\sharp(\mathfrak{F})(\gamma) = \bigvee \left\{ \text{sub}_X(\mu, \gamma) \mid p \leq \bigwedge_{n \geq n} \mu(1/m) \right\} = 1.
\]

**Step 2.** \( (g^\sharp \circ f)^\sharp(\mathfrak{F})(\gamma) \leq p \).

Since
\[
m_X(g^\sharp(\mathfrak{F})) (\lambda) = g^\sharp(\mathfrak{F})(\lambda) = \mathfrak{F}(\lambda) = \mathfrak{F}(x \mapsto g^\sharp(\mathfrak{F})(\mu)) = \mathfrak{F}(\mu),
\]

it follows that \( (g^\sharp \circ f)^\sharp(\mathfrak{F}) \) is the conical coreflection of \( s \rightarrow g^\sharp(\mathfrak{F}) \).

Since
\[
m_X(g(\mathfrak{F})) (\lambda) = g(\mathfrak{F})(\lambda) = \mathfrak{F}(\lambda) = \mathfrak{F}(x \mapsto \mathfrak{G}(\lambda)) = t \rightarrow \mathfrak{G}(\lambda),
\]

it follows that \( g^\sharp(\mathfrak{F}) \) is the conical coreflection of \( t \rightarrow \mathfrak{G} \). Thus,
\[
g^\sharp(\mathfrak{F})(\mu) = \bigvee \left\{ \text{sub}_X(\lambda, \mu) \mid t \leq \mathfrak{G}(\lambda) \right\}
\]
\[
= \bigvee \left\{ \text{sub}_X(\lambda, \mu) \mid t \leq \bigvee_n \bigwedge_{m \geq n} \lambda(1/m) \right\}.
\]

If \( s \leq g^\sharp(\mathfrak{F})(\mu) \), then there exist \( r > p \) and \( \lambda \in [0, 1]^X \) such that \( t \leq \bigvee_n \bigwedge_{m \geq n} \lambda(1/m) \) and that \( r < \text{sub}_X(\lambda, \mu) \). Since \( t, r > p \), then for \( m \) large enough, \( \mu(1/m) \geq p \) and consequently, 
\[
\mu(1/m) \rightarrow \gamma(1/m) = \mu(1/m) \rightarrow (p(1 - 1/m)) = p(1 - 1/m) \leq p.
\]

Therefore,
\[
(g^\sharp \circ f)^\sharp(\mathfrak{F})(\gamma) = \bigvee \left\{ \text{sub}_X(\mu, \gamma) \mid s \leq g^\sharp(\mathfrak{F})(\mu) \right\} \leq p.
\]

\( \square \)
6 The monad of \( \top \)-filters

This section concerns whether the functor of \( \top \)-filters can be made into a monad. The idea is to relate \( \top \)-filters to conical \( Q \)-filters. By a conical \( Q \)-filter we mean, of course, a \( Q \)-semifilter that is conical and is a \( Q \)-filter.

Assigning to each set \( X \) the set \( \text{ConFil}(X) \) of all conical \( Q \)-filters on \( X \) defines a subfunctor of \( Q \)-Fil:

\[
\text{ConFil} : \text{Set} \to \text{Set}.
\]

For a meet continuous and integral quantale \( Q \), the functor of \( \top \)-filters is naturally isomorphic to the functor of conical \( Q \)-filters, as we see now.

**Proposition 6.1.** Let \( Q \) be a meet continuous and integral quantale. Then a saturated prefilter \( F \) on a set \( X \) is a \( \top \)-filter if and only if \( \Lambda_X(F) \) is a \( Q \)-filter.

**Proof.** If \( F \) is a \( \top \)-filter then for each \( p \in Q \),

\[
\Lambda_X(F)(p_X) = \bigvee_{\mu \in F} \bigwedge_{x \in X} (\mu(x) \to p) = \bigvee_{\mu \in F} \left( \bigvee_{x \in X} \mu(x) \right) \to p = p,
\]

showing that \( \Lambda_X(F) \) is a \( Q \)-filter. If \( F \) is not a \( \top \)-filter, there is some \( \mu \in F \) such that \( \bigvee_{x \in X} \mu(x) = q < 1 \). Then,

\[
\Lambda_X(F)(q_X) \geq \bigwedge_{x \in X} (\mu(x) \to q) = 1 > q,
\]

which implies that \( \Lambda_X(F) \) is not a \( Q \)-filter. \( \square \)

Therefore, the correspondence \( F \mapsto \Lambda_X(F) \) establishes a bijection between \( \top \)-filters and conical \( Q \)-filters. Thus, the functor \( \top \)-Fil is naturally isomorphic to \( \text{ConFil} \):

\[
\text{ConFil} \cong \text{ConFil} : \text{Set} \to \text{Set}.
\]

**Proposition 6.2.** Let \( Q \) be a meet continuous and integral quantale. Then the conical coreflection of each \( Q \)-filter is a \( Q \)-filter.

**Proof.** It suffices to check that for each \( Q \)-filter \( \mathfrak{F} \) on a set \( X \), \( \Gamma_X(\mathfrak{F}) \) is a \( \top \)-filter. If \( \Gamma_X(\mathfrak{F}) \) is not a \( \top \)-filter, then, by the argument of the above proposition, there is some \( q < 1 \) such that \( \Lambda_X(\mathfrak{F})(q_X) = 1 \), contradicting that \( \Lambda_X(\mathfrak{F})(q_X) \leq \mathfrak{F}(q_X) = q \). \( \square \)

Hence the natural transformation \( c : \text{Q-SemFil} \to \text{ConSF} \) in the previous section restricts to a natural transformation \( c : \text{Q-Fil} \to \text{ConFil} \). In particular, \( \text{ConFil} \) is a retract of \( \text{Q-Fil} \).

Since \( \text{Q-Fil} \) is a submonad of \( (\text{Q-SemFil}, m, e) \), it follows that

\[
n := c \circ m \circ (i \ast i)
\]

is a natural transformation \( \text{ConFil}^2 \to \text{ConFil} \).

\[
\begin{array}{ccc}
\text{ConFil}^2 & \xrightarrow{\cong} & \text{Q-SemFil}^2 \\
\downarrow n & & \downarrow m \\
\text{ConFil} & \xleftarrow{c} & \text{Q-SemFil}
\end{array}
\]

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Consequently, 

\[ n := \Gamma \circ n \circ (\Lambda \ast \Lambda) \quad (6.2) \]

is a natural transformation \( \mathcal{T} \rightarrow \mathcal{T} \).

It is clear that \( d = \{d_X\}_X \) is a natural transformation \( \text{id} \rightarrow \text{ConFil} \) and \( \mathcal{D} = \{\mathcal{D}_X\}_X \) is a natural transformation \( \text{id} \rightarrow \mathcal{T} \rightarrow \mathcal{T} \). Now, we state the problem of this section.

**Problem 6.3.** Let \( Q \) be a meet continuous and integral quantale; and let \( n : \text{ConFil}^2 \rightarrow \text{ConFil} \) and \( n : \mathcal{T} \rightarrow \mathcal{T} \) be the natural transformations given in Equations (6.1) and (6.2), respectively. When is the triple

\[ (\mathcal{T}, n, d) \]

a monad? Or equivalently, when is the triple

\[ (\text{ConFil}, n, d) \]

a monad?

**Remark 6.4.** It follows from Proposition 5.3 that the natural transformations \( d : \text{id} \rightarrow \mathcal{T} \rightarrow \mathcal{T} \) and \( n : \mathcal{T} \rightarrow \mathcal{T} \) coincide, respectively, with the natural transformations \( \eta \) and \( \mu \) in [39, Section 3] defined via the formulas (3.1) and (3.2).

Similar to Proposition 5.5 and Corollary 5.8, it can be shown that if \( Q \) is a meet continuous and integral quantale such that the map \( p \rightarrow - : Q \rightarrow Q \) preserves directed joins for each \( p \in Q \), then, \( \text{ConFil} \) is a submonad of the \( Q \)-semifilter monad, hence \( (\mathcal{T}, n, d) \) is a monad [39, Lemma 3.1].

The main result of this section presents a complete solution to Problem 6.3 in the case that \( Q \) is the interval \([0, 1]\) equipped with a continuous t-norm.

**Theorem 6.5.** Let \( Q = ([0, 1], \& \& 1) \) with \& being a continuous t-norm. Then the following conditions are equivalent:

1. The t-norm \& satisfies the condition (S).
2. \( \text{ConFil} \) is a submonad of \( (Q \text{-SemFil}, m, e) \).
3. The triple \( (\text{ConFil}, n, d) \) is a monad.
4. The triple \( (\mathcal{T}, n, d) \) is a monad.

**Proof.**

(1) \( \Rightarrow \) (2) By Theorem 5.10, we know that conical \( Q \)-semifilters are closed under multiplication in this case. Since \( Q \)-filters are closed under multiplication, it follows that conical \( Q \)-filters are closed under multiplication. Therefore, \( \text{ConFil} \) is a submonad of \( (Q \text{-SemFil}, m, e) \).

(2) \( \Rightarrow \) (3) By definition of \( n \) and \( d \).

(3) \( \Rightarrow \) (1) A slight improvement of the proof of (3) \( \Rightarrow \) (1) in Theorem 5.10 will suffice. Let \( \mathfrak{F} \) be the conical \( Q \)-filter generated by the prefilter

\[ \{\nu : X \rightarrow [0, 1] \mid \nu(1) = 1, \nu \geq t_X\}; \]

let \( \mathfrak{G} \) be the conical \( Q \)-filter on \( X \) generated by the prefilter

\[ \{\nu : X \rightarrow [0, 1] \mid \nu \geq 1_{A_n} \text{ for some } n \geq 1\}, \]
Consider the maps
\[ f, g : X \rightarrow \text{ConFil}(X) \]
given by
\[ f(x) = \begin{cases} \mathfrak{f}, & x < 1, \\ d_X(1), & x = 1 \end{cases} \]
and
\[ g(x) = \begin{cases} \mathfrak{g}, & x < 1, \\ d_X(1), & x = 1. \end{cases} \]
Then, via similar calculations, one sees that \( g^\sharp \circ f^\#(\gamma) \) is not equal to \((g^\sharp \circ f)^\#(\gamma)\), where
\[ \gamma(x) = \begin{cases} p(1 - x), & x < 1, \\ 1, & x = 1 \end{cases} \]
and \( \mathfrak{f} \) is the \( Q \)-filter generated by the prefilter \( \{ \nu : X \rightarrow [0, 1] | \nu(1) = 1, \nu \geq s_X \} \).

(3) \( \iff \) (4) By definition.

7 The monad of bounded saturated prefilters

Functional ideals \([28, 29, 30]\), postulated for Lawvere’s quantale \(([0, \infty]^\text{op}, +, 0)\), are a special kind of saturated prefilters. Bounded saturated prefilters are an extension of functional ideals to the \( Q \)-valued setting with \( Q \) being the interval \([0, 1]\) equipped with a continuous t-norm \( \& \). This section investigates whether the functor of bounded saturated prefilters is a monad.\(^1\)

The main result says that this happens exactly when the continuous t-norm \( \& \) satisfies the condition (S). So, in this section, \( Q \) is always assumed to be \(([0, 1], \& , 1)\) with \( \& \) being a continuous t-norm.

A map \( \lambda : X \rightarrow [0, 1] \) is said to be bounded (precisely, bounded below), if \( \lambda \geq \epsilon_X \) for some \( \epsilon > 0 \).

Let \( F \) be a saturated prefilter on a nonempty set \( X \). Since \([0, 1]\) is linearly ordered,
\[ B_F := \{ \mu \in F | \mu \text{ is bounded} \} \]
is clearly a prefilter on \( X \). We say that \( F \) is a \textit{bounded saturated prefilter} if it is the saturation of \( B_F \); that is to say,
\[ \lambda \in F \iff \bigvee_{\mu \in B_F} \text{sub}_X(\mu, \lambda) = 1. \]

Let \( \text{BSP}(X) \) denote the set of bounded saturated prefilters on \( X \).

**Proposition 7.1.** For a continuous t-norm \( \& \), the following conditions are equivalent:

1. Every saturated prefilter is bounded.
2. \( \& \) is isomorphic to the Lukasiewicz t-norm.

**Lemma 7.2.** A continuous t-norm \( \& \) is isomorphic to the Lukasiewicz t-norm if and only if
\[ \bigvee_{p \geq 0} (p \rightarrow 0) = 1. \]

\(^1\)We thank gratefully Dirk Hofmann for bringing this problem to our attention.
Proof. Necessity is clear. As for sufficiency, first we show that \& has no idempotent element in \((0, 1)\), hence \& is either isomorphic to the Łukasiewicz t-norm or to the product t-norm. If, on the contrary, \(b \in (0, 1)\) is an idempotent element, then for each \(p > 0\), \(p \to 0 \leq b < 1\), a contradiction. If \& is isomorphic to the product t-norm, then for each \(p > 0\), \(p \to 0 = 0\), a contradiction. Therefore, \& is isomorphic to the Łukasiewicz t-norm.

Proof of Proposition 7.1 (1) \(\Rightarrow\) (2) Consider the largest prefilter \(F\) on a singleton set; that is, \(F\) is the unit interval \([0, 1]\). By assumption, \(F\) is bounded. Then
\[
\bigvee_{p > 0} (p \to 0) = 1,
\]
which, by Lemma 7.2 implies that \& is isomorphic to the Łukasiewicz t-norm.

(2) \(\Rightarrow\) (1) Without loss of generality, we may assume that \& is, not only isomorphic to, the Łukasiewicz t-norm. Let \(F\) be a saturated prefilter on a nonempty set \(X\). Since for all \(p > 0\) and \(\mu \in F\), \(\lambda X \lor \mu\) is bounded and
\[
\text{sub}_X (\lambda X \lor \mu, \mu) = \bigwedge_{x \in X} (p \to \mu(x)) \geq 1 - p,
\]
it follows that
\[
1 = \bigvee_{p > 0} \text{sub}_X (\lambda X \lor \mu, \mu).
\]
Therefore, \(F\) is bounded.

Convection. Because of Proposition 7.1 in the remainder of this section, we agree that \& is a continuous t-norm that is not isomorphic to the Łukasiewicz t-norm, unless otherwise specified.

Proposition 7.3. Every element of a bounded saturated prefilter is bounded.

Proof. Let \(F\) be a bounded saturated prefilter and \(B_F\) be the set of all bounded elements of \(F\). By definition, for each \(\mu \in F\), we have
\[
\bigvee_{\lambda \in B_F} \text{sub}_X (\lambda, \mu) = 1.
\]
Let \(b\) be an idempotent element of \& in \((0, 1)\) if \& has one, otherwise, take an arbitrary element in \((0, 1)\) for \(b\). Then there exist some \(\lambda \in F\) and \(a > 0\) such that \(\lambda \geq a X\) and \(b \leq \text{sub}_X (\lambda, \mu)\). Then for each \(x\), we have \(0 < a \& b \leq \mu(x)\), showing that \(\mu\) is bounded.

Example 7.4 (Functional ideals, I). Functional ideals play an important role in the theory of approach spaces, see, e.g. [4, 5, 27, 28, 29, 30]. This example shows that functional ideals are essentially bounded saturated prefilters for the product t-norm. For each \(X\), let \(B X\) denote the set of all bounded functions \(X \to [0, \infty]\). Then, a functional ideal on \(X\) in the sense of [28, 29, 30] is a subset \(\mathcal{J}\) of \(B X\) subject to the following conditions:

(i) If \(\lambda \in \mathcal{J}\) and \(\mu \leq \lambda\) (pointwise) then \(\mu \in \mathcal{J}\).

(ii) If \(\lambda, \mu \in \mathcal{J}\) then there is some \(\gamma \in \mathcal{J}\) such that \(\gamma(x) \geq \max\{\lambda(x), \mu(x)\}\) for all \(x \in X\).
(iii) $\mathcal{I}$ is saturated in the sense that for each $\lambda : X \rightarrow [0, \infty]$:

\[(\forall \epsilon > 0, \exists \mu \in \mathcal{I}, \lambda \leq \mu + \epsilon) \Rightarrow \lambda \in \mathcal{I}.
\]

Let $\&$ be the product t-norm. Since the correspondence $x \mapsto e^{-x}$ is an isomorphism between Lawvere’s quantale $([0, \infty]^\text{op}, +, 0)$ and the quantale $Q = ([0, 1], \& , 1)$, it follows that a functional ideal on a set $X$ is essentially a bounded saturated prefilter on $X$ (with respect to the product t-norm).

For each saturated prefilter $F$ on a set $X$, let $g_F(F)$ be the set of bounded elements in $F$. Then $g_F(F)$ is a bounded saturated prefilter on $X$, and it is the largest bounded saturated prefilter contained in $F$, called the bounded coreflection of $F$. Moreover,

\[g_F(F) = \{\lambda \vee \epsilon_F | \lambda \in F, \epsilon > 0\}.
\]

Given a map $f : X \rightarrow Y$ and a bounded saturated prefilter $F$ on $X$, let

\[f_B(F) = g_Y(f(F)) = \{\mu \in [0, 1]^Y | \mu \text{ is bounded and } \mu \circ f \in F\}.
\]

Then the assignment

\[X \xrightarrow{f} Y \mapsto \text{BSP}(X) \xrightarrow{f_B} \text{BSP}(Y)
\]

defines a functor

\[
\text{BSP} : \text{Set} \rightarrow \text{Set}.
\]

Moreover, $g = \{g_F\}_X$ is a natural transformation SPF $\rightarrow$ BSP, which is an epimorphism in the category of endofunctors on Set.

We would like to warn the reader that though BSP is a subset of SPF for every set $X$, the functor BSP is, in general, not a subfunctor of SPF.

For each $x \in X$, let $\tilde{\delta}_F(x)$ be the bounded coreflection of $\delta_X(x)$, i.e.,

\[\tilde{\delta}_F(x) = \{\lambda \in [0, 1]^X | \lambda(x) = 1, \lambda \text{ is bounded}\}.
\]

Then, $\tilde{\delta} = \{\tilde{\delta}_F\}_X$ is a natural transformation id $\rightarrow$ BSP.

For each bounded saturated prefilter $F$ on $\text{BSP}(X)$, let

\[\tilde{n}_F(X) = \{\lambda \in [0, 1]^X | \lambda \text{ is bounded}, \tilde{\lambda} \in F\},
\]

where

\[\tilde{\lambda}(F) = \bigvee_{\mu \in F} \text{sub}_{X}(\mu, \lambda)
\]

for each bounded saturated prefilter $F$ on $X$. Put differently, $\tilde{n}_F(X)$ is the bounded coreflection of $n_X(k_X(F))$, where $k_X$ refers to the inclusion of $\text{BSP}(X)$ in SPF. Then,

\[\tilde{n}_F(X) = \{\tilde{n}_F\}_X \quad (7.1)
\]

is a natural transformation $\text{BSP}^2 \rightarrow \text{BSP}$ (this fact is contained in Proposition 7.11 below).

Now we are able to state the problem of this section.

**Problem 7.5.** When is the triple $(\text{BSP}, \tilde{n}, \tilde{\delta})$ a monad?
As in previous sections, the key idea to solve this problem is to relate bounded saturated prefilters to certain kind of \(Q\)-semifilters, conical bounded \(Q\)-semifilters in this case.

**Example 7.6** (Functional ideals, II). Let \(\&\) be the product t-norm. Then, the natural transformation \(\tilde{n}\) in Equation (7.1) is essentially the multiplication of the monad of functional ideals in \([5, \text{Subsection 2.3}]\). To see this, first we show that for each bounded saturated prefilter \(F\) on \(X\) and each \(\alpha > 0\),

\[
\alpha \otimes F = \{ \lambda \in [0,1]^X, \exists \mu \in F, \alpha \& \mu \leq \lambda \}
\]

is a bounded saturated prefilter. It suffices to check that \(\alpha \otimes F\) is saturated. Actually,

\[
\bigvee_{\gamma \in \alpha \otimes F} \text{sub}_X(\gamma, \lambda) = 1 \implies \bigvee_{\mu \in F} \text{sub}_X(\alpha \& \mu, \lambda) = 1
\]

\[
\implies \bigvee_{\mu \in F} \text{sub}_X(\mu, \alpha \rightarrow \lambda) = 1
\]

\[
\implies \alpha \rightarrow \lambda \in F
\]

\[
\implies \lambda \in \alpha \otimes F. \quad (\alpha \& (\alpha \rightarrow \lambda) \leq \lambda)
\]

Next, let \(\mu : X \rightarrow [0,1]\) be a bounded map, \(p \in [0,1]\), and let \(F\) be a bounded saturated prefilter on \(\text{BSP}(X)\). Then, for each \(F \in \text{BSP}(X)\),

\[
p \leq \tilde{\mu}(F) \iff p \leq \bigvee_{\lambda \in F} \text{sub}_X(\lambda, \mu)
\]

\[
\iff p \leq \bigvee \{ \alpha \in (0,1] \mid \exists \lambda \in F, \alpha \& \lambda \leq \mu \}
\]

\[
\iff p \leq \bigvee \{ \alpha \in (0,1] \mid \mu \in \alpha \otimes F \}.
\]

Therefore, \(\tilde{\mu}\) is essentially the map \(l_\mu\) in \([5, \text{Subsection 2.2}]\), and consequently, the natural transformation \(\tilde{n}\) is the multiplication of the monad of functional ideals in \([5, \text{Subsection 2.3}]\).

A \(Q\)-semifilter \(\mathcal{F}\) on a nonempty set \(X\) is said to be bounded if \(\mathcal{F}(\mu) < 1\) whenever \(\mu : X \rightarrow [0,1]\) is unbounded.

**Lemma 7.7.** Let \(\mathcal{F}\) be a \(Q\)-semifilter on a set \(X\). If \(\mathcal{F}\) is bounded, then the saturated prefilter \(\Gamma_X(\mathcal{F})\) is bounded. The converse implication also holds when \(\mathcal{F}\) is conical. Therefore, the conical coreflection of a bounded \(Q\)-semifilter is bounded.

*Proof.* If \(\mathcal{F}\) is bounded and \(\mu \in \Gamma_X(\mathcal{F})\), then \(\mathcal{F}(\mu) = 1\), hence \(\mu\) is bounded. As for the converse implication, assume that \(\mathcal{F}\) is conical. Since \(\Gamma_X(\mathcal{F})\) is bounded and

\[
\mathcal{F}(\mu) = \bigvee_{\lambda \in \Gamma_X(\mathcal{F})} \text{sub}_X(\lambda, \mu),
\]

it suffices to check that if \(\mu\) is unbounded, then there is some \(b < 1\) such that \(\text{sub}_X(p_X, \mu) \leq b\) for all \(p > 0\). Take for \(b\) an idempotent element of \(\&\) in \((0,1)\) if \(\&\) has one, otherwise, take an arbitrary element in \((0,1)\) for \(b\). Since \(\mu\) is unbounded, for each \(p > 0\) there is some \(z \in X\) such that \(\mu(z) < b \land p\). Then

\[
\text{sub}_X(p_X, \mu) \leq p \rightarrow \mu(z) \leq b,
\]

as desired. \(\square\)
For each set $X$, let $\text{ConBSF}(X)$ be the set of conical bounded $Q$-semifilters on $X$. For each $Q$-semifilter $\mathfrak{F}$ on $X$, 

$$\vartheta_X(\mathfrak{F}) := \Lambda_X \circ \varrho_X \circ \Gamma_X(\mathfrak{F})$$

is clearly the largest conical bounded $Q$-semifilter that is smaller than or equal to $\mathfrak{F}$, so, we call it the conical bounded coreflection of $\mathfrak{F}$.

For each map $f : X \rightarrow Y$ and each $Q$-semifilter $F$ on $X$, let $f_B(F)$ be the conical bounded coreflection of $f(\mathfrak{F})$, i.e.,

$$f_B(\mathfrak{F}) = \vartheta_Y(f(\mathfrak{F})).$$

Then we obtain a functor

$$\text{ConBSF} : \text{Set} \rightarrow \text{Set}$$

that maps $f$ to $f_B$. We hasten to note that though $\text{ConBSF}(X)$ is a subset of $Q$-$\text{SemFil}(X)$ for each set $X$, the functor $\text{ConBSF}$ is, in general, not a subfunctor of $Q$-$\text{SemFil}$.

**Lemma 7.8.** Let $f : X \rightarrow Y$ be a map. Then, for each conical $Q$-semifilter $\mathfrak{F}$ and each bounded $\lambda \in [0,1]^Y$, $f_B(\mathfrak{F})(\lambda) = f(\mathfrak{F})(\lambda)$.

**Proof.** Since $\lambda$ is bounded, there is some $\epsilon > 0$ such that $\lambda \geq \epsilon_Y$. Since $f(\mathfrak{F})$ is conical, then

$$f(\mathfrak{F})(\lambda) = \bigvee \{ \text{sub}_Y(\mu, \lambda) \mid f(\mathfrak{F})(\mu) = 1 \}$$

$$= \bigvee \{ \text{sub}_Y(\mu \vee \epsilon_Y, \lambda) \mid f(\mathfrak{F})(\mu) = 1 \}$$

$$= \bigvee \{ \text{sub}_Y(\mu, \lambda) \mid f(\mathfrak{F})(\mu) = 1, \mu \text{ is bounded} \}$$

$$= f_B(\mathfrak{F})(\lambda).$$

The correspondence $F \mapsto \Lambda_X(F)$ restricts to a bijection between bounded saturated prefilters and conical bounded $Q$-semifilters. So, $\text{ConBSF}$ is naturally isomorphic to $\text{BSP}$:

$$\text{BSP} \cong \text{ConBSF}.$$ 

Moreover, the assignment to each set $X$ the map

$$\vartheta_X : Q\text{-SemFil}(X) \rightarrow \text{ConBSF}(X)$$

defines a natural transformation

$$\vartheta : Q\text{-SemFil} \rightarrow \text{ConBSF}.$$

For each $x \in X$, let $\tilde{d}_X(x)$ be the conical bounded coreflection of $e_X(x)$, i.e.,

$$\tilde{d}_X(x) = \vartheta_X(e_X(x)).$$

Then $\tilde{d} = \{ \tilde{d}_X \}_X$ is a natural transformation $\text{id} \rightarrow \text{ConBSF}$.

**Lemma 7.9.** If $F$ is a bounded $Q$-semifilter on $\text{ConBSF}(X)$, then the diagonal $Q$-semifilter $m_X(j_X(F))$ is also bounded, where $j_X$ denotes the inclusion $\text{ConBSF}(X) \rightarrow Q\text{-SemFil}(X)$. 

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Proof. We check that if $\mu \in [0, 1]^X$ is unbounded, then $m_X(j_X(F))(\mu) < 1$. Since $F$ is bounded and $m_X(j_X(F))(\mu) = F(\mu \circ j_X)$, it suffices to check that $\mu \circ j_X$ is unbounded. This follows from that
\[
\bigwedge_{\mathcal{G} \in \text{ConBSF}(X)} \mu \circ j_X(\mathcal{G}) = \bigwedge_{\mathcal{G} \in \text{ConBSF}(X)} \mathcal{G}(\mu) \leq \bigwedge_{x \in X} d_X(x)(\mu) \leq \bigwedge_{x \in X} \mu(x) = 0.
\]

Proposition 7.10. For each set $X$, define
\[
\tilde{n}_X : \text{ConBSF}^2(X) \to \text{ConBSF}(X)
\]
by letting $\tilde{n}_X(F)$ be the conical coreflection of the diagonal $Q$-semifilter $m_X(j_X(F))$. Then $\tilde{n} = \{\tilde{n}_X\}_X$ is a natural transformation $\text{ConBSF}^2 \to \text{ConBSF}$.

Proof. We show that for each map $f : X \to Y$, the following diagram is commutative:
\[
\begin{array}{ccc}
\text{ConBSF}^2(X) & \xrightarrow{\tilde{n}_X} & \text{ConBSF}(X) \\
(f_B)_B & \downarrow & \downarrow f_B \\
\text{ConBSF}^2(Y) & \xrightarrow{\tilde{n}_Y} & \text{ConBSF}(Y)
\end{array}
\]

Let $F$ be a conical bounded $Q$-semifilter on $\text{ConBSF}(X)$. Since both $f_B \circ \tilde{n}_X(F)$ and $\tilde{n}_Y \circ (f_B)_B(F)$ are conical, it suffices to check that for every $\lambda \in [0, 1]^Y$,
\[
f_B \circ \tilde{n}_X(F)(\lambda) = 1 \iff \tilde{n}_Y \circ (f_B)_B(F)(\lambda) = 1.
\]

On one hand,
\[
f_B \circ \tilde{n}_X(F)(\lambda) = 1 \iff \lambda \in \Gamma_Y(f_B \circ \tilde{n}_X(F)) \\
\iff \lambda \text{ is bounded and } f(\tilde{n}_X(F))(\lambda) = 1 \\
\iff \lambda \text{ is bounded and } \tilde{n}_X(F)(\lambda \circ f) = 1 \\
\iff \lambda \text{ is bounded and } m_X(j_X(F))(\lambda \circ f) = 1 \\
\iff \lambda \text{ is bounded and } F(\tilde{\lambda} \circ \tilde{j}_X) = 1 \\
\iff \lambda \text{ is bounded and } F(\tilde{\lambda} \circ \tilde{j}_X \circ f) = 1.
\]

On the other hand, since $m_Y(j_Y((f_B)_B(F)))$ is bounded (Lemma 7.9) and
\[
f_B(\tilde{\mathcal{F}})(\lambda) = f(\tilde{\mathcal{F}})(\lambda)
\]
for each $\tilde{\mathcal{F}} \in \text{ConBSF}(X)$ and each bounded $\lambda \in [0, 1]^Y$ (Lemma 7.8), we have
\[
\tilde{n}_Y \circ (f_B)_B(F)(\lambda) = 1 \iff m_Y(j_Y((f_B)_B(F)))(\lambda) = 1 \\
\iff j_Y((f_B)_B(F))(\tilde{\lambda}) = 1 \\
\iff (f_B)_B(F)(\tilde{\lambda} \circ j_Y) = 1 \\
\iff \tilde{\lambda} \circ j_Y \text{ is bounded and } F(\tilde{\lambda} \circ j_Y \circ f_B) = 1 \\
\iff \lambda \text{ is bounded and } F(\tilde{\mathcal{F}} \mapsto f_B(\tilde{\mathcal{F}})(\lambda)) = 1 \\
\iff \lambda \text{ is bounded and } F(\tilde{\mathcal{F}} \mapsto \tilde{\mathcal{F}}(\lambda \circ f)) = 1.
\]
Proposition 7.11. \( \tilde{d} = \Gamma \circ \tilde{d} \) and \( \tilde{n} = \Gamma \circ \tilde{n} \circ (\Lambda \ast \Lambda) \).

Proof. The proof is similar to that of Proposition 5.3 and is included here for convenience of the reader. The first equality is obvious. As for the second, let \( \mathcal{F} \) be a bounded saturated prefilter on \( \text{BSP}(X) \). Let \( j_X \) be the inclusion

\[
\text{ConBSF}(X) \rightarrow \text{Q-SemFil}(X).
\]

Then, for each \( \lambda \in Q^X \),

\[
\lambda \in \Gamma_X \circ \tilde{n}_X \circ (\Lambda \ast \Lambda)_X(\mathcal{F}) \iff m_X(j_X((\Lambda \ast \Lambda)_X(\mathcal{F}))(\lambda)) = 1 \iff (\Lambda \ast \Lambda)_X(\mathcal{F})(\lambda \circ j_X) = 1 \iff \Lambda_{\text{BSP}}(\mathcal{F})(\lambda \circ j_X \circ \Lambda_X) = 1 \iff \lambda \circ j_X \circ \Lambda_X \in \mathcal{F}.
\]

Since for each bounded saturated prefilter \( F \) on \( X \), we have

\[
\tilde{\lambda} \circ j_X \circ \Lambda_X(F) = \Lambda_X(F)(\lambda) = \bigvee_{\mu \in F} \text{sub}_X(\mu, \lambda) = \tilde{\lambda}(F),
\]

it follows that \( \tilde{n} = \Gamma \circ \tilde{n} \circ (\Lambda \ast \Lambda) \).

Theorem 7.12. Let \& be a continuous t-norm that is not isomorphic to the Lukasiewicz t-norm. Then the following conditions are equivalent:

1. The t-norm \& satisfies the condition (S).
2. The triple \((\text{ConBSF}, \tilde{n}, \tilde{d})\) is a monad.
3. The triple \((\text{BSP}, \tilde{n}, \tilde{d})\) is a monad.

Before proving this theorem, we would like to point out that Proposition 2.9 in \cite{5} together with Example 7.6 already imply that \((\text{BSP}, \tilde{n}, \tilde{d})\) is a monad when \& is isomorphic to the product t-norm.

Proof. (1) \(\rightarrow\) (2) Since \& satisfies the condition (S), then for each conical bounded \(Q\)-semifilter \(F\) on \(\text{ConBSF}(X)\), the diagonal \(Q\)-semifilter \(m_X(j_X(F))\) is conical, hence

\[
\tilde{n}_X(F) = m_X(j_X(F)).
\]

In other words, conical bounded \(Q\)-semifilters are “closed under multiplication”. With help of this fact and that two bounded conical \(Q\)-semifilters are equal if and only if they are equal on bounded elements, it is routine to check that \((\text{ConBSF}, \tilde{n}, \tilde{d})\) is a monad. (2) \(\rightarrow\) (1) In the proof of (3) \(\rightarrow\) (1) in Theorem 5.10, replace \(\gamma\) by

\[
\epsilon_X \vee p(1 - x)
\]

for some \(0 < \epsilon < p\) and replace \(\mathcal{G}\) by the conical bounded \(Q\)-filter on \(X\) generated by

\[
\{1_{A_n} \vee \delta_X \mid n \geq 1, \delta > 0\},
\]

where \(A_n = \{1/m \mid m \geq n\}\).

(2) \(\iff\) (3) Proposition 7.11.
8 Summary

Let $Q$ be the quantale $([0, 1], \&, 1)$ with $\&$ being a continuous t-norm. It is proved that the following conditions are equivalent:

1. The implication operator of $\&$ is continuous at each point off the diagonal.
2. The functor $\text{ConSF}$ of conical $Q$-semifilters is a submonad of $(Q\text{-SemFil}, m, e)$.
3. The functor $\text{ConFil}$ of conical $Q$-filters is a submonad of $(Q\text{-SemFil}, m, e)$.
4. The triple $(\text{SPF}, n, d)$ is a monad.
5. The triple $(\top\text{-Fil}, n, d)$ is a monad.
6. The triple $(BSP, \bar{n}, \bar{d})$ is a monad.

When $\&$ is isomorphic to the Lukasiewicz t-norm, the triple $(BSP, \bar{n}, \bar{d})$ coincides with $(\text{SPF}, n, d)$.

The relations among these monads are summarized in the following diagram:

- each vertex is a monad on the category of sets;
- each arrow is a monad morphism;
- each monad in the diagram is power-enriched (see [14] for definition);
- each $i$ is a monomorphism;
- both $\vartheta$ and $\iota$ are an epimorphism;
- each $\Lambda$ is an isomorphism with inverse given by $\Gamma$;
- $\iota \circ i$ is the identity.

These monads are useful in monoidal topology [8, 14] and in the theory of quantale-enriched orders. As an example, let $\&$ be the product t-norm. Since the quantale $([0, 1], \&, 1)$ is isomorphic to Lawvere’s quantale $([0, \infty]^{op}, +, 0)$, it follows that Kleisli monoids and Eilenberg-Moore algebras of the monad $(BSP, \bar{n}, \bar{d})$ are essentially approach spaces [5] and injective approach spaces [12], respectively. As another example, let $\&$ be a continuous t-norm that satisfies the condition (S). Then, Kleisli monoids and Eilenberg-Moore algebras of the monad $(\text{SPF}, n, d)$ are CNS spaces [21] and complete and continuous $Q$-categories [22, 24], respectively.
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