The Sarason sub-symbol and the Recovery of the Symbol of Densely Defined Toeplitz Operators over the Hardy Space

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Abstract

While the symbol map for the collection of bounded Toeplitz operators is well studied, there has been little work on a symbol map for densely defined Toeplitz operators. In this work a family of candidate symbols, the Sarason sub-symbols, is introduced as a means of reproducing the symbol of a densely defined Toeplitz operator. This leads to a partial answer to a question posed by Donald Sarason in 2008. In the bounded case the Toeplitzness of an operator can be classified in terms of its Sarason sub-symbols. This justifies the investigation into the application of the Sarason sub-symbols on densely defined operators. It is shown that analytic closed densely defined Toeplitz operators are completely determined by their Sarason sub-symbols, and it is shown for a broader class of operators that they extend closed densely defined Toeplitz operators (of multiplication type).

1. Introduction

The study of bounded Toeplitz operators over the Hardy space $H^2(T)$ is a well developed subject where there are several equivalent definitions of a Toeplitz operator. The simplest definition of a bounded Toeplitz operator is an extension of the definition of a Toeplitz matrix. In this case an operator, $T$, is called a Toeplitz operator if the matrix representation of the operator, with respect to the orthonormal basis $\{e^{in\theta}\}_{n=0}^{\infty}$ is constant along the diagonals. Algebraically, this can be represented as $S^*TS = T$. Here $S = M_z$ is the shift operator for the Hardy space. If the coefficients corresponding to each diagonal of the matrix are the Fourier coefficients of

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a function \( \phi \in L^\infty(\mathbb{T}) \), then \( T = P_{H^2(\mathbb{T})}M_\phi \). Here \( P_{H^2(\mathbb{T})} \) is the projection from \( L^2(\mathbb{T}) \to H^2(\mathbb{T}) \), and \( M_\phi \) is the bounded multiplication operator from \( H^2(\mathbb{T}) \to L^2(\mathbb{T}) \) given by \( M_\phi f = \phi f \). Finally the converse is true, the bounded operator given by \( T_\phi = P_{H^2(\mathbb{T})}M_\phi \) with \( \phi \in L^\infty(\mathbb{T}) \) satisfies \( S^*TS = T \).

When the bounded condition is relaxed to closed and densely defined, the corresponding definitions of Toeplitz operators are no longer equivalent. For instance, if the coefficients of an upper triangular matrix,

\[
\begin{pmatrix}
a_1 & a_2 & a_3 \\
0 & a_1 & a_2 \\
0 & 0 & a_1 \\
\vdots & \ddots & \ddots
\end{pmatrix},
\]

are the coefficients of a Smirnov class function, \( \phi \in N^+ \), then the operator defined by the closure of this matrix, call it \( T \), is densely defined, and the operator is the adjoint of a densely defined multiplication operator (an analytic Toeplitz operator) \( M_\phi \). Morally \( M_\phi \) is given by a lower triangular matrix of the form

\[
\begin{pmatrix}
\bar{a}_1 & 0 & 0 \\
\bar{a}_2 & \bar{a}_1 & 0 \\
\bar{a}_3 & \bar{a}_2 & \bar{a}_1 \\
\vdots & \ddots & \ddots
\end{pmatrix}
\]

though \( \langle M_\phi z^n, z^m \rangle \) may not be well-defined, since \( z^n \) isn’t necessarily in the domain of \( M_\phi \). Unlike its bounded counterpart, \( T \) cannot be represented by a multiplication operator as \( PM_\phi \) since its domain is strictly larger than the domain of \( M_\phi \). The operator \( T \) does satisfy the following algebraic equations:

1. \( D(T) \) is \( S \)-invariant,
2. \( S^*TS = T \),
3. If \( f \in D(T) \) and \( f(0) = 0 \), then \( S^*f \in D(T) \).

These can be seen as the densely defined analogue of the algebraic condition for bounded Toeplitz operators. Therefore \( T \) satisfies the algebraic conditions for being a Toeplitz operator, but is not a Toeplitz operator in the multiplication sense. However, \( T \) is a closed extension of a multiplication type Toeplitz operator. At the close of [1] the following problem was posed:
Question 1. Is it possible to characterize those closed densely defined operators $T$ on $H^2(\mathbb{T})$ with the above three properties? Moreover, is every closed densely defined operator on $H^2(\mathbb{T})$ that satisfies these conditions determined in some sense by a symbol?

This paper aims to address the second half of this question. If a closed densely defined operator, $T$, satisfies the three algebraic conditions above, henceforth a Sarason-Toeplitz operator, then is $T$ the extension of an operator of the form $PM_\phi$ where $M_\phi$ is a densely defined multiplication operator from $H^2$ to $L^2$?

Various investigations into unbounded operators have been performed as far back as the 1950s. Hartman and Wintner [2] investigated the self-adjointness of unbounded Toeplitz matrices, and Hartman continued the work in [3]. Rosenblum in [4] and Rovnyak in [5] investigated the resolvent of unbounded Toeplitz operators. Further results for unbounded analytic and co-analytic Toeplitz operators can be found in [6, 7, 1, 8].

For bounded Toeplitz operators the recovery of the symbol of a Toeplitz operator can be achieved through the symbol map on $\mathcal{T}$, the algebra generated by the collection of Toeplitz operators in $L(H^2)$. Douglas demonstrated that there is a unique multiplicative mapping, $\phi$ from $\mathcal{T}$ to $L^\infty$ such that $\phi(T_fT_g) = \phi(T_f)\phi(T_g) = fg$ [9, 10]. This fact was proven again in [11] by Halmos and Barria using the limits along the diagonals of a Toeplitz matrix in order to find the symbol in $L^\infty$.

The Hardy space can be identified with analytic functions of the disc $\mathbb{D}$ such that the Taylor coefficients of these functions are square summable. By this viewpoint, $H^2$ is a reproducing kernel Hilbert space (RKHS) over $\mathbb{D}$ with the kernel functions $k_w(z) = (1 - \bar{w}z)^{-1}$ for $|w| < 1$.

In the case of bounded Toeplitz operators, the Berezin transform, a tool particular to the study of RKHSs, is sufficient for the recovery of the of $L^\infty$ functions via radial limits of the Berezin transform of a bounded Toeplitz operator [12]. The Berezin transform creates a function based on an operator as follows: $\hat{V}(w) = \|k_w\|^{-2}(Vk_w, k_w)$. For several different function spaces, such as the Bargman space, the Fock space, and the Hardy space, certain properties of the Berezin transform correspond to properties of the operator. In certain cases, if $V$ is a bounded Toeplitz operator, then if $\hat{V}(w)$ vanishes as $w$ approaches the boundary of the domain of the corresponding RKHS, $V$ is then compact. This was shown to hold for the Bergman space [13], and the Fock space [14, 15, 16]. This property also holds when $V$ is a finite sum of finite products of Toeplitz operators on the Bergman space [13], and further investigations of the Berezin transform on the Toeplitz algebra can
be found in [17]. For the Hardy space, a bounded Toeplitz operator, \( T_\phi \), is compact iff \( \phi \equiv 0 \) [9]. Similar investigations have been carried out for other settings [16, 18, 19, 20].

In more general cases, the well definedness of the Berezin transform and the recovery of the symbol of a Sarason-Toeplitz operator is no longer clear. In the special case of a densely-defined analytic (or co-analytic) Toeplitz operator on the Hardy space with symbol \( \phi \), the recovery of the symbol can be accomplished by the use of the Berezin transform. In this case, the adjoint of an analytic Toeplitz operator has the reproducing kernels as eigenvectors, \( k_z \), with eigenvalues \( \phi(z) \) [1]. Thus
\[
\tilde{T}(z) = (1 - |z|^2)(k_z, T^*k_z) = (1 - |z|^2)(k_z, \phi(z)k_z) = \phi(z).
\]

The application of the Berezin transform requires the kernel functions \( k_w(z) = (1 - \bar{w}z)^{-1} \) to be in the domain of a operator or in the domain of its adjoint. Thus, the investigation of a new method is justified for the recovery of the symbol of a densely defined Sarason-Toeplitz operator.

We introduce the Sarason sub-symbol, which depends on a choice of a function in \( D(T) \), as a family of symbol maps for Sarason-Toeplitz operators. In the development, it will be demonstrated that for the bounded case the Sarason sub-symbol is unique iff the operator is Toeplitz. Thus the uniqueness of the Sarason sub-symbol provides another equivalent definition for a bounded Toeplitz operator. Subsequently it is demonstrated that the Sarason sub-symbol for an analytic Toeplitz operator is unique and determines the operator. The rest of the paper is concerned with classes of Toeplitz operators for which the existence of the Sarason sub-symbol can be established, and it demonstrates sufficient conditions to show that \( T \) is a closed extension of a multiplication type Toeplitz operator.

2. Adjoints of Sarason-Toeplitz Operators

For bounded Toeplitz operators, the Sarason Problem has been long settled [9, 21]. Indeed, if a Toeplitz operator is bounded, then it can be represented by an \( L^\infty \) function. Suarez characterized all closed densely defined operators on \( H^2(T) \) that commute with the adjoint of the shift operator [8], and Sarason gives a different treatment of operators that commute with the shift operator, the so called analytic Toeplitz operators. Both of these collections of operators satisfy the Sarason-Toeplitz condition. In addition, the analytic Toeplitz operators are precisely the operators of multiplication by an function in the Smirnov class, \( N^+ \) [1]. Suarez’s operators are
the adjoints of these analytic Toeplitz operators and are called co-analytic
Toeplitz operators \[8\]. Thus the above classes of Sarason-Toeplitz operators
are completely characterized by a symbol.

Analytic and co-analytic Toeplitz operators both satisfy the Sarason
conditions. The following property generalizes this relationship.

**Proposition 2.1.** If \( T \) is a Sarason-Toeplitz operator then so is \( T^* \).

**Proof.** \( T \) is a closed densely defined operator, which means that \( T^* \) is closed
and densely defined as well. Thus \( D(T^*) \) is nonempty.

To demonstrate that \( T^* \) has a shift invariant domain, take \( g \in D(T^*) \).
By definition this means that \( \tilde{L}(f) = \langle Tf, g \rangle \) is a continuous functional. In
order to show that \( zg \in D(T^*) \) it must be established that \( L(f) = \langle Tf, zg \rangle \)
is continuous. Note that \( zD(T) \subset D(T) \), and \( zD(T) \) has co-dimension 1 in
\( D(T) \). Thus there exists an \( f_0 \in D(T) \) such that
\[
D(T) = \mathbb{C}\{f_0\} \oplus zD(T).
\]

The functional \( L \) is continuous on \( \mathbb{C}\{f_0\} \), since it is finite dimensional.
Therefore it suffices to show that \( L \) is continuous on \( zD(T) \). If \( f = zh \)
for some \( h \in D(T) \), then
\[
L(f) = L(zh) = \langle Tzh, zg \rangle = \langle Th, g \rangle = \tilde{L}(h).
\]
Thus \( L \) is continuous on \( zD(T) \), since \( \tilde{L} \) is continuous on \( D(T) \).

Now suppose that \( g \in D(T^*) \) and \( g(0) = 0 \), and consider the functional
\( L_2(f) = \langle Tf, S^*g \rangle \) defined for \( f \in D(T) \). This functional can be rewritten
as
\[
L_2(f) = \langle S^*TSf, S^*g \rangle = \langle TSf, g \rangle := \tilde{L}_2(Sf).
\]
It follows that \( L_2(f) \) is continuous, since \( \tilde{L}_2(Sf) \) is continuous with respect
to \( f \).

Finally for all \( f \in D(T^*) \) and \( g \in D(T) \) we have,
\[
\langle T^*f, g \rangle = \langle f, Tg \rangle = \langle f, S^*TSg \rangle = \langle S^*T^*Sf, g \rangle,
\]
which yields the second condition. \( \square \)

3. The Sarason sub-symbol

While the Berezin transform can be applied to recover the symbol of
densely defined analytic and co-analytic Toeplitz operators, it is not clear if
it can be used to recover the symbol of more general densely defined Toeplitz
operators.
operators. This is because the functions $k_z$ are required to be in the domain of either the operator or the adjoint of the operator for the Berezin transform to be well defined. Instead, in this section the Sarason sub-symbol will be introduced as a candidate for the recovery of the symbol of densely defined Sarason-Toeplitz operators.

As a motivating example for the definition of the Sarason sub-symbol, first suppose that $T$ is a bounded Toeplitz operator with symbol $\phi \in L^\infty$. In this case

$$a_n = \begin{cases} \langle T1, z^n \rangle & n \geq 0 \\ \langle Tz^n, 1 \rangle & n < 0 \end{cases}$$

are the Fourier coefficients of $\phi$. Thus $\phi$ can be reconstructed as follows

$$\phi(e^{i\theta}) = \sum_{n=1}^{\infty} \langle Tz^n, 1 \rangle e^{-in\theta} + \sum_{n=0}^{\infty} \langle T1, z^n \rangle e^{in\theta}.$$

While it is not expected that $1 \in D(T)$ in general, given any function $f \in D(T)$ the domain of the densely defined operator $TMf$ contains the polynomials, since $D(T)$ is shift invariant. The Sarason sub-symbol is defined as follows:

**Definition** Let $T$ be an operator with a shift invariant domain $D(T)$. For $f \in D(T) \setminus \{0\}$ the Sarason sub-symbol corresponding to $f$ is given by $R_f = h_f/f$ where

$$h_f = \sum_{n=1}^{\infty} \langle Tf z^n, 1 \rangle e^{-in\theta} + \sum_{n=0}^{\infty} \langle T1, z^n \rangle e^{in\theta}$$

where this series is convergent in some sense. The partial Sarason sub-symbol corresponding to $f$ is given by $R_{f,N} = h_{f,N}/f$ where

$$h_{f,N} = \sum_{n=1}^{N} \langle Tf z^n, 1 \rangle e^{-in\theta} + \sum_{n=0}^{\infty} \langle T1, z^n \rangle e^{in\theta}.$$

Heuristically, if $T$ is a Toeplitz operator associated with multiplication by the symbol $\phi$, then $h_f = \phi \cdot f$. The question of well definedness of the Sarason sub-symbol depends on the convergence of the series contained in the definition of $h_f$. When $\phi \in L^\infty$, $h_f = \phi \cdot f$, and is a well defined function in $L^2$. More specifically we can characterize all bounded Toeplitz operators by means of the Sarason sub-symbol.
Proposition 3.1. Let $V$ be a bounded operator on $H^2$. The operator $V$ is a Toeplitz operator iff the Sarason sub-symbol is independent of the choice of $f \in H^2$.

Proof. Suppose that $V = T_\phi$ is a Toeplitz operator with symbol $\phi$.

$$h_f = \sum_{n=1}^{\infty} \langle T_\phi(fz^n), 1 \rangle_{H^2} e^{-in\theta} + \sum_{n=0}^{\infty} \langle T_\phi f, z^n \rangle_{H^2} e^{in\theta}$$

$$= \sum_{n=1}^{\infty} \langle \phi f z^n, 1 \rangle_{L^2} e^{-in\theta} + \sum_{n=0}^{\infty} \langle \phi f, z^n \rangle_{L^2} e^{in\theta} = \phi \cdot f.$$

Thus $R_f = h_f/f = \phi$ is independent of the choice of $f$.

Now suppose that $V$ is not a Toeplitz operator. This means there is a pair of integers $n, m \in \mathbb{N}$ such that $n < m$ (without loss of generality) and $\langle V z^n, z^m \rangle \neq \langle V 1, z^{m-n} \rangle$. In this case consider the two Sarason sub-symbols

$$R_1 = \sum_{k=1}^{\infty} \langle V z^k, 1 \rangle e^{-ik\theta} + \sum_{k=0}^{\infty} \langle V 1, z^k \rangle e^{ik\theta} = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} \text{ and}$$

$$R_{z^n} = e^{-in\theta} \left( \sum_{k=0}^{\infty} \langle V z^{n+k}, 1 \rangle e^{-ik\theta} + \sum_{k=0}^{\infty} \langle V z^n, z^k \rangle e^{ik\theta} \right) = e^{-in\theta} \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}.$$

The difference of the two sub-symbols yields

$$R_1 - R_{z^n} = e^{-in\theta} \sum_{k=-\infty}^{\infty} (a_{k-n} - b_k) e^{ik\theta}.$$

The coefficient $(a_m - b_m) \neq 0$ by construction. Therefore $R_1 \neq R_{z^n}$. □

Thus every bounded Toeplitz operator is characterized by the uniqueness of its Sarason sub-symbols. This motivates the investigation into densely defined operators. The following sections investigate the interplay between the Sarason sub-symbols and densely defined Sarason-Toeplitz operators.

4. Analytic Densely Defined Toeplitz Operators

Just as in Proposition 3.1, an analytic densely defined Toeplitz operator is completely characterized by a symbol. As shown in [1], these operators are precisely the multiplication operators with symbols, $\phi$, in the Smirnov class of functions. That is, each $\phi$ can be written as a ratio of $H^\infty$ functions $b/a$ where $|a(e^{i\theta})|^2 + |b(e^{i\theta})|^2 = 1$ for all $\theta$ and $a$ an outer function. In this setting the Sarason sub-symbol is unique.
**Theorem 4.1.** Given a Sarason-Toeplitz operator \( T \), there exists a symbol \( \phi \in N^+ \) for which \( T = M_\phi \) iff \( \langle Tzf, 1 \rangle = 0 \) for all \( f \in D(T) \). Moreover, the Sarason sub-symbol is unique.

**Proof.** The forward direction follows since \( T_\phi = M_\phi \) for \( \phi \in N^+ \). This means \( TS = ST \), and \( \langle Tzf, 1 \rangle = \langle zTf, 1 \rangle = 0 \) since \( 1 \in (zH^2)^\perp \).

In order to establish sufficiency, let \( f_1 = \sum_{n=0}^{\infty} a_n z^n, f_2 = \sum_{n=0}^{\infty} b_n z^n \in D(T) \setminus \{0\} \). By hypothesis, \( h_{f_i} = \sum_{n=0}^{\infty} \langle Tf_i, z^n \rangle z^n = Tf_i \in H^2 \) for \( i = 1, 2 \).

In order to establish uniqueness of the symbol, \( R_{f_1} = R_{f_2} \), consider the function \( h_{1}f_2 - h_{2}f_1 \in L^1(T) \). The Fourier series of \( h_{1}f_2 \) and \( h_{2}f_1 \) can be computed through convolution. Hence,

\[
\begin{align*}
h_{1}f_2 &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \langle T f_1, z^{n-k} \rangle b_k \right) z^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \langle T(z^k f_1), z^n \rangle b_k \right) z^n, \text{ and} \\
h_{2}f_1 &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \langle T(z^k f_2), z^n \rangle a_k \right) z^n.
\end{align*}
\]

The second equality follows since \( S^*TSf = Tf \), and \( \langle TSf, 1 \rangle = 0 \) implies that \( TSf(0) = 0 \) and \( SS^*TSf = TSf \). This leads to

\[
H := h_{1}f_2 - h_{2}f_1 = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \langle T(z^k f_1), z^n \rangle b_k - \sum_{k=0}^{n} \langle T(z^k f_2), z^n \rangle a_k \right) z^n.
\]

In order to establish that each coefficient is in fact zero, consider, for arbitrary \( n \), the coefficient of \( z^n \):

\[
\begin{align*}
\hat{H}(n) &= \sum_{k=0}^{n} \langle T(z^k f_1), z^n \rangle b_k - \sum_{k=0}^{n} \langle T(z^k f_2), z^n \rangle a_k \\
&= \left( \langle f_1 \left[ \sum_{k=0}^{n} b_k z^k - f_2 \right], z^n \rangle - f_2 \left[ \sum_{k=0}^{n} a_k z^k - f_1 \right] \right), z^n \rangle.
\end{align*}
\]

The \( H^2 \) function inside of \( T \) is in fact in the domain of \( T \) by the properties of Sarason-Toeplitz operators, and this function has a zero of order greater than \( n \) at zero. Denote by \( z^{n+1}F_n \), the function in the argument of \( T \). By our hypothesis,

\[
\hat{H}(n) = \langle T(z^{n+1}F_n), z^n \rangle = \langle T(zF_n), 1 \rangle = 0.
\]

Therefore \( R_{f_1} = R_{f_2} \) for any choice of \( f_1, f_2 \in D(T) \setminus \{0\} \), so let \( \phi = R_{f_1} \) be the proposed symbol for the Sarason-Toeplitz operator \( T \). \( h_f = Tf \in H^2 \).
for each \( f \in D(T) \). Further, given any \( z \in \mathbb{D} \) there exists \( f_z \in D(T) \) such that \( f_z(z) \neq 0 \) (this follows from the density of \( D(T) \) in \( H^2 \)). Thus \( \phi = T f_z / f_z \) is analytic at \( z \) for every point \( z \in \mathbb{D} \). Finally note that for each \( f \in D(T) \), \( M_\phi f = \phi f = (T f / f) f = T f \). Thus \( T = M_\phi \) is a densely defined multiplication operator with an analytic symbol. By [1], \( \phi \in \mathbb{N}^+ \).

\[ \text{Corollary 4.2. A Sarason-Toeplitz operator } T \text{ on } H^2 \text{ is analytic (} ST = TS \text{) iff } (Tzf, 1) = 0 \text{ for all } f \in D(T). \]

5. Symbols that are ratios of \( L^2 \) functions and \( H^2 \) functions

In the case of an analytic densely defined Toeplitz operator with symbol \( \phi \) (expressed as \( \phi = b/a \) in canonical form), the domain is given by \( D(T) = aH^2 \) (c.f. [1] Proposition 5.3). This means that there is an outer function, in particular \( a \), in the domain of \( T \). Moreover, since \( T = M_\phi \), it is clear that \( h_a = \phi a \in H^2 \). Therefore, the existence of an outer function \( f \in D(T) \) for which \( h_f \) is well defined is straightforward in the case of analytic Toeplitz operators.

When we consider a co-analytic Toeplitz operator of the form \( M_\phi^* \), its domain is given by \( \mathcal{H}(b) \), the de Branges-Rovnyak space corresponding to \( b \). Since their introduction in [22], de Branges-Rovnyak spaces have seen extensive study and applications [23, 24, 25, 26, 27, 28, 29]. Recent surveys and texts, such as [30, 31], give an extensive introduction to such spaces. The space \( \mathcal{H}(b) \) contains \( aH^2 \) as a subspace [1]. Hence, it also has an outer function in its domain. In particular, if \( f = a \cdot p \), where \( p \) is a polynomial, then \( h_f \) (corresponding to \( M_\phi^* \)) is in \( L^2 \). Since \( a \) is an outer function, the collection of all such \( f \) is dense in \( H^2 \). Therefore, the set
\[
D_2(T) = \{ f \in D(T) : h_f \in L^2 \}
\]
is dense in \( D(T) = D(M_\phi^*) = \mathcal{H}(b) \).

The answer to the question of the nonemptiness (as well as density) of the space \( D_2(T) \) is unknown for general Sarason-Toeplitz operators. In this section, the applicability of the Sarason sub-symbol is extended to include functions of the form \( B/A \) where \( B \in L^2 \) and \( A \) is an \( H^2 \) outer function.

\[ \text{Lemma 5.1. Let } \phi \text{ be a measurable function on the unit circle that can be written as the ratio of an } L^2 \text{ function and an } H^2 \text{ outer function. Let } \]
\[
D(M_\phi) = \{ f \in H^2 : \phi \cdot f \in L^2 \}.
\]
The operator \( M_\phi : D(M_\phi) \to L^2 \) is a closed densely defined operator on \( H^2 \).
Proof. Write \( \phi = B/A \) where \( B \in L^2 \) and \( A \in H^2 \) is an outer function. Since \( B \cdot p \in L^2 \) for every polynomial \( p(z) \), we see that \( A \cdot p \in D(M_{\phi}) \) for every polynomial \( p \). Therefore, \( D(M_{\phi}) \) is dense in \( H^2 \) by the outer property of \( A \).

Now suppose that \( \{f_n\} \subset D(M_{\phi}) \) and \( f_n \to f \in H^2 \). Suppose further that \( M_{\phi}f_n \to F \in L^2 \). Since \( f_n \to f \) in the \( L^2 \) norm, there exists a subsequence, \( \{f_{n_j}\} \), such that \( f_{n_j} \to f \) almost everywhere. Since \( A \) is an outer function, \( A(e^{i\theta}) \neq 0 \) for almost every \( \theta \). Thus \( \phi f_{n_j} \to \phi f \) almost everywhere.

The subsequence \( \phi f_{n_j} \to F \) in \( L^2 \) and so there is a subsequence \( \phi f_{n_{jk}} \to F \) almost everywhere. However, this subsequence also converges almost everywhere to \( \phi f \). Thus we may conclude that \( \phi f = F \) almost everywhere, which completes the proof. \( \square \)

**Theorem 5.2.** Let \( T \) be a Sarason-Toeplitz operator. If there is an \( H^2 \) outer function \( f \in D(T) \) such that \( \sum_{n=1}^{\infty} \langle T(z^n f), 1 \rangle z^n \in L^2 \), then \( T \) extends a closed densely defined operator of the form \( T_{\phi} = PM_{\phi} \) where \( \phi = R_f \) is the ratio of an \( L^2 \) function and an \( H^2 \) outer function. Moreover, \( D_2(T) \) is a dense subset of \( D(T) \).

**Proof.** Let \( f \) be an \( H^2 \) outer function in \( D(T) \), and let \( h_f \) be the corresponding numerator of the Sarason Sub-symbol corresponding to \( f \). Express \( h_f = \sum_{n=-\infty}^{\infty} b_n z^n \). By the properties of Sarason-Toeplitz operators, \( b_{n-m} = \langle T(z^m f), z^n \rangle \). Now consider the operator \( T_{R_f} = PM_{R_f} \), which is closed and densely defined by Lemma 5.1.

Since the domain of \( T \) is shift invariant, \( f \cdot p \in D(T) \) for every polynomial \( p \). Moreover, \( h_f \cdot p \in L^2 \) for every polynomial \( p \). It follows that \( D_2(T) \subset D(T) \) is dense in \( H^2 \) since \( f \) is an outer function. Define the set \( F := \{ f \cdot p : p \) is a polynomial \} \( \subset D_2(T) \).

Let \( p(z) = a_k z^k + \cdots + a_1 z + a_0 \) be a polynomial of degree \( k \in \mathbb{N} \). The product of \( h(z) \) and \( p(z) \) can be calculated as follows:

\[
h(z) \cdot p(z) = \sum_{n=-\infty}^{\infty} \left( \sum_{m=0}^{k} b_{n-m} a_m \right) z^n = \sum_{n=-\infty}^{\infty} \left( \sum_{m=0}^{k} \langle T(a_m z^m f), z^n \rangle \right) z^n = \sum_{n=-\infty}^{\infty} \langle T(fp), z^n \rangle z^n = w(z) + T(fp)(z).
\]

Where \( w(z) \in \overline{H^2} \) since \( h_f \in L^2 \). In particular, this means \( T_{R_f}(fp) = P(hp) = T(fp) \) for all polynomials \( p \). Hence, \( T \) agrees with \( T_{R_f} \) on a dense
domain, and $T$ extends $T_{R_f} |_F$. Finally, by Lemma 5.1, $T_{R_f} |_F$ is closable, and $T_{R_f} |_F \subset T$ implies that $T_{R_f}^{**} \subset T^{**} = T$. 

The above theorem relies on the ability to find an outer function in $D_2(T)$. Once such a function is found, $T$ is shown to be a closed extension of the corresponding operator $PM_{R_f}$. When such a function does not exist, it can be shown that $T$ is the limit of multiplication type Toeplitz operator on a restricted domain.

**Proposition 5.3.** Suppose $T$ is a Sarason-Toeplitz operator, let $f \in D(T)$, and define $F = \{ f \cdot p : p$ is a polynomial $\}$. There exists a sequence of multiplication type Toeplitz operators, $T_{\phi_M} = PM_{\phi_M}$ such that $T_{\phi_M}$ converges to $T$ strongly on all of $F$. Moreover, these operators have a common dense domain.

**Proof.** Let $f \in D(T)$ and let $p(z) = a_k z^k + \cdots + a_1 z + a_0$ be a polynomial of degree $k$. Now, as in Theorem 5.2, consider the product $h_{f,N}(z)p(z)$:

$$h_{f,N}(z) \cdot p(z) = \sum_{n=-N}^{\infty} \left( \sum_{m=0}^{\min(k,n+N)} b_{n-m} a_m \right) z^n$$

$$= \sum_{n=-N}^{k-N-1} \langle T(f(a_0 + \cdots + a_{n+N} z^{n+N})), z^n \rangle z^n + \sum_{n=k-N}^{\infty} \langle T(fp), z^n \rangle z^n.$$

Therefore,

$$T_{R_{f,N}}(fp)(z) = P(h_Np)(z)$$

$$= \sum_{n=0}^{k-N-1} \langle T(f(a_0 + \cdots + a_{n+N} z^{n+N})), z^n \rangle z^n + \sum_{n=\min(k-N,0)}^{\infty} \langle T(fp), z^n \rangle z^n.$$

The left sum is empty for large enough $N$, therefore $\{ T_{R_{f,N}}(fp) \}$ is constant for large enough $N$. This means $T_{R_{f,N}}(fp) \to T(fp)$ as $N \to \infty$.

In order to find a common domain for each of these Toeplitz operators, consider the inner-outer factorization $f = f_i f_o$. The functions, $\tilde{h}_N = h_{f,N}/f_i \in L^2$ since $f_i$ has modulus 1 on the circle. Thus $\tilde{h}_{N} f_o p \in L^2$ for all polynomials $p$. This implies that

$$F_0 = \{ f_o p : p$ is a polynomial $\} \subset D(T_{R_{f,N}})$$

for all $N \in \mathbb{N}$. 

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6. An Example of a Non-Sarason Toeplitz Operator

This section is concerned with demonstrating that a densely defined Toeplitz matrix does not necessarily define a Sarason-Toeplitz operator. In particular, this section will extend an upper triangular Toeplitz matrix and demonstrate that the domain of the extension is not shift invariant. An upper triangular Toeplitz matrix is a matrix of the form

\[
\begin{pmatrix}
\gamma_0 & \gamma_1 & \gamma_2 \\
0 & \gamma_0 & \gamma_1 \\
0 & 0 & \gamma_0 \\
\vdots & \ddots & \ddots
\end{pmatrix}.
\]

As an operator over \(H^2\), this matrix has a natural dense domain, namely the polynomials. The density of the domain does not depend on the sequence \(\{\gamma_n\}_{n \in \mathbb{N}}\). Following Sarason [1], this operator may be extended as

\[
Tf = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} n! \hat{f}(n + m) \right) z^m
\]

where the domain of \(T\) is the collection of functions in \(H^2\) for which \(Tf \in H^2\).

**Theorem 6.1.** Let \(T\) be the extension of an upper triangular matrix given by

\[
Tf = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} n! \hat{f}(n + m) \right) z^m.
\]

The domain of \(T\) is defined to be \(D(T) = \{f \in H^2 : Tf \in H^2\}\). Every function \(f \in D(T)\) is an entire function and can be written as \(f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}\) where \(\sum_{n=0}^{\infty} a_n\) converges.

**Lemma 6.2.** The sequence \(\{c_m = \sum_{n=1}^{\infty} (n+1)^{-m}\}_{m=2}\) is an \(l^2\) sequence.

**Proof of Lemma.** Each term of the sequence can be bounded by

\[
\int_0^{\infty} \frac{1}{(1+x)^m} dx = \frac{1}{m-1}.
\]

Thus, \(c_m\) is bounded by an \(l^2\) sequence, and so it is also \(l^2\).

**Proof of Theorem 6.1.** First suppose that \(f \in D(T)\). By definition, the zeroth coefficient of \(Tf\) is given by \(\sum_{n=0}^{\infty} n! \hat{f}(n)\), which must be a convergent...
series. Declaring \( a_n = n! \hat{f}(n) \), it can be seen that \( \sum a_n \) converges. Moreover, since \( f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \), the function \( f \) must be an entire function.

For the other direction, suppose that \( f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \) where \( \sum a_n \) converges. Define \( d_0 = \sum_{n=0}^{\infty} n! \hat{f}(n) = \sum_{n=0}^{\infty} a_n \). Note that since \( a_n \) converges so does \( \sum_{n=0}^{\infty} a_n b_n \) for any positive monotonically decreasing sequence \( \{b_n\} \). Thus for each \( m = 1, 2, \ldots \) the series

\[
d_m = \sum_{n=0}^{\infty} n! \hat{f}(n + m) = \sum_{n=0}^{\infty} a_{n+m} \frac{1}{(n+1)(n+2) \cdots (n+m)}
\]

converges. This enables us to define \( T f \) formally as \( \sum_{m=0}^{\infty} d_m z^m \).

In order to demonstrate that \( d_m \) is in \( l^2 \), write \( d_m \) as follows:

\[
d_m = \frac{a_m}{m!} + \sum_{n=1}^{\infty} \frac{a_{n+m}}{(n+1)(n+2) \cdots (n+m)} := s_m + t_m.
\]

The sequence \( \{s_m\} \in l^2 \) since \( a_m \to 0 \). The sequence \( \{t_m\} \) is bounded by \( \sum_{n=1}^{\infty} (n+1)^{-m} = c_m \) for sufficiently large \( m \), since \( |a_m| < 1 \) for \( m \) sufficiently large. By Lemma 6.2, \( t_m \) is in \( l^2 \). This completes the proof of the theorem. \( \square \)

**Corollary 6.3.** The domain of the operator given in Theorem 6.1 is not shift invariant.

**Proof.** The function \( f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!} \in D(T) \), since \( \sum_{n=1}^{\infty} (-1)^n/n \) converges. Now consider the function

\[
z f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^{n+1}}{n!} = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n-1} \frac{z^n}{(n-1)!} = \sum_{n=2}^{\infty} \frac{(-1)^{n-1} z^n}{n-1}.
\]

The series \( \sum_{n=2}^{\infty} (-1)^{n-1} \frac{z^n}{n-1} \) does not converge, which means \( z f(z) \not\in D(T) \). \( \square \)

By applying the same techniques used in proving Theorem 6.1, a slightly weaker result can be found when \( n! \) is replaced by a sequence of complex numbers \( \{\gamma_n\} \) with the growth condition \( |\gamma_{n+1}| > (n + 1)|\gamma_n| \).

**Theorem 6.4.** Let \( \{\gamma_n\} \) be a sequence of complex numbers as described above, and define the operator \( T f = \sum_{m=0}^{\infty} \bigg( \sum_{n=0}^{\infty} \gamma_n \hat{f}(n + m) \bigg) z^m \) with the domain \( D(T) = \{ f \in H^2 : T f \in H^2 \} \). The operator \( T \) is densely defined and functions of the form \( f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{\gamma_0} \) where \( \sum |a_n| < \infty \) are in its domain.
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