Optimization of Harmonious coloring problem in Uniform theta graph and torus Network

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Abstract- Coloring of graphs has been extraordinarily widened areas of investigation. A coloring of a graph can be depicted by a capacity that maps pieces of a graph into some plan of numbers commonly called labels or hues, all together that some property is satisfied. In this article we deal with the harmonious chromatic number of central graph of generalized uniform theta graph and torus network. A coloring $c: V \to \mathbb{N}$ of the nodes is a harmonious coloring if and only if the coloring is proper and for each and every line $(e_1, e_2) \in E$ the line color $(c(e_1), c(e_2))$ is distinctive that is it emerge solely once.

Keywords: Harmonious coloring, Central graph, Uniform theta graph, Torus Network.

1. Introduction
Graph coloring rise ordinarily in various convenient conditions where it is needed to fragment a great deal of things into social occasion with the goal that the people of each get-together are normally amazing as shown by some measure. An imaginative method of coloring a graph richly is harmonious coloring. A coloring $c: V \to \mathbb{N}$ of the nodes is a harmonious coloring if and only if the coloring is proper and for each and every line $(e_1, e_2) \in E$ the line color $(c(e_1), c(e_2))$ is distinctive that is it emerge solely once. In the year 1982 Frank Harray and Plantholt first posted the above way of coloring. Nevertheless the finest part is delivered by Lee Hopcroft and Krishnamoorty[5]. Lu has acquired evaluations for harmonious chromatic number of a meager families of graphs[6,8]. He has additionally found the bounds for binary and trinary tree[7]. The lower bound of harmonious chromatic number might be considered as an edge accepting an interesting shading pair[1,4]. The greatest part of this coloring is concern with estimation results[2]. So instigator get down to shot at central, line and total of some known graphs[3,4]. In this article we are indeed interested in determining the minimum number of colors used in coloring the central graph of uniform theta graph and torus network harmoniously. Thus a central graph is the subdivision of each edge by a node exactly once and joining all the non adjacent nodes.

2. Main Results
In this section we discuss the minimum number of colors used in central graph of Uniform theta graph and Torus network which is colored harmoniously.
2.1. Uniform Theta Graph

A generalised theta graph \( \theta(s_1, s_2, \ldots, s_q) \) consist of a pair of end nodes joined by \( \alpha \) internally disjoint paths of lengths \( s_1, s_2, \ldots, s_q \geq 1 \). These end nodes are called north pole (N) and south pole (S) and the paths are called longitudes. A generalized theta graph in which all the paths are of same length is called a uniform theta graph denoted by \( \theta(n, \alpha)[1] \). Hence \( n \) denotes the number of internal vertices in each longitude.

Let \( l_1, l_2, \ldots, l_\alpha \) denote the longitudes of the uniform theta graph \( \theta(n, \alpha) \). The number of vertices and edges in \( \theta(n, \alpha) \) are \( \alpha n + 2 \) and \( \alpha(n + 1) \) respectively. Its diameter is \( n + 1 \).

We utilize a specific naming of nodes of the Uniform theta graph in request to define an appropriate coloring. First let \( n \) be odd. We partition the vertex set into two sections called the upper and lower parts. The nodes \( v_i \)'s represent the upper half nodes and the nodes \( w_i \)'s for the lower half nodes. The nodes in the upper half are named from left to right beginning from the centre of the longitudes and listed towards the North Pole. Likewise in the lower half are named from left to right beginning from the centre and listed towards the South Pole. Let \( V_1 = \{v_1, v_2, \ldots, v_{\lceil n/2 \rceil} \} \) and \( V_2 = \{w_1, w_2, \ldots, w_{\lfloor n/2 \rfloor} \} \) then the node set is \( V = V_1 \cup V_2 \cup \{N, S\} \). See Figure 1, a uniform theta graph with \( n = 3 \) and \( \alpha = 6 \).

![Figure 1: A uniform theta graph G with n = 3 and \( \alpha = 6 \)](image)

2.2. Theorem

Let \( G = \theta(n, \alpha) \) then \( \chi_h \{C[\theta(n, \alpha)]\} = \Delta[C[\theta(n, \alpha)]] + \Delta[\theta(n, \alpha)] + 1 \).

Proof

Let the node set of \( G \) be \( V = V_1 \cup V_2 \cup \{N, S\} \) where \( V_1 = \{v_1, v_2, \ldots, v_{\lceil n/2 \rceil} \} \) and \( V_2 = \{w_1, w_2, \ldots, w_{\lfloor n/2 \rfloor} \} \) by the annotation of central graph let \( u_i \) be the nodes of subdivision of the lines of \( \theta(n, \alpha) \). Fix colors to the nodes \( V = V_1 \cup V_2 \cup \{N, S\} \) as \( c_i : 1 \leq i \leq \alpha n + 2 \) which is in turn is the maximum degree of central graph of uniform theta graph with the addition of one, that is \( \Delta[C[\theta(n, \alpha)]] + 1 \). The balance \( \alpha(n + 1) \) nodes are colored with \( c_{\alpha n + 3}, c_{\alpha n + 4}, c_{\alpha n + 5}, \ldots, c_{\alpha n + \alpha + 2} \) in anticlockwise direction which is in turn is the maximum degree of G that is \( \Delta[\theta(n, \alpha)] = \alpha \). Thus the above defined coloring is harmoniously colored with least number of colors. Hence \( \chi_h \{C[M(s, t)]\} = \alpha n + \alpha + 2 \), that is \( \chi_h \{C[M(s, t)]\} = \Delta[C[\theta(n, \alpha)]] + \Delta[\theta(n, \alpha)] + 1 \).
2.3. Torus Network
The torus is one of the well known geographies for interconnecting processors to manufacture elite multi computers. In mathematics, a torus is a donut moulded surface of insurgency created by rotating a hover about a hub coplanar with the circle. It is a work with fold over connections. Torus are bipartite if and just if all side lengths are even. They are Hamiltonian, regular and vertex symmetric.
A one-dimensional torus is just a cycle or a ring. A two dimensional torus network contains $st$ nodes, organized in two dimensions with $s$, $t$ nodes per dimension. A $s \times t$ torus is denoted by $TR(s,t)$.

An $s \times t$ torus $TR(s,t)$ is defined as a graph with vertex set $V = \{(j,k): 1 \leq j \leq s, 1 \leq k \leq t\}$ and edge set $E = \{(j_1,k_1),(j_2,k_2): (j_2 = (j_1 + 1)(mod s) \land k_2 = k_1) \lor (j_1 = j_2 \land k_1 = (k_1 + 1)(mod t))\}$. The number of nodes in $TR(s,t)$ is $st$ and the number of edges is $2st$. See Figure 2 a torus network with $s = 4$ and $t = 5$.

![Figure 2: Torus TR(4,5)](image)

2.4. Theorem
Let $G = TR(s,t)$ then $\chi_h[C[TR(s,t)]] = \Delta[C[TR(s,t)]] + \Delta[TR(s,t)] + 1$.

Proof
Let the node set of $G$ be $V = \{v_{n(j-1)+k}: j = 1,2,3, \ldots, s$ and $k = 1,2,3, \ldots, t\}$ and by the annotation of central graph let $w_i$ be the nodes of subdivision of the line of $TR(s,t)$. Fix colors to the nodes $V = \{v_{n(j-1)+k}: j = 1,2,3, \ldots, s$ and $k = 1,2,3, \ldots, t\}$ as $c_i: 1 \leq i \leq st$ which is in turn is the maximum degree of central graph of extended mesh with the addition of one that is $\Delta[C[TR(s,t)]] + 1$. The balance $2st$ nodes are colored with $c_{st+1}$, $c_{st+2}$, $c_{st+3}$, $c_{st+4}$ colors in anticlockwise direction which is in turn is the maximum degree of $G$ that is $\Delta[TR(s,t)] = 4$. Obviously the above defined coloring is harmoniously colored with least number of colors. Hence $\chi_h[C[TR(s,t)]] = st + 4$, that is $\chi_h[C[TR(s,t)]] = \Delta[C[TR(s,t)]] + \Delta[TR(s,t)] + 1$.

3. Conclusion
In most of the coloring problems optimization is a challenge on certain classes of graphs. In this work, we have optimized the harmonious coloring number of central graph of uniform theta graph and Torus Network.

Our future work is to investigate the performance of the other classes of graphs such as line, total middle of some interconnection networks.

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