Numerical-based theoretical analysis on the decay of homogeneous turbulence affected by small strain based on constant and linear strain variations

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Abstract. This study clarifies the effects of small strain on the decay of homogeneous turbulence by focusing on the temporal profile of the small strain. Small strain is defined in such a way as to not affect the anisotropy of the homogeneous turbulence. We apply the framework of the standard $k$–$\varepsilon$ model to examine the effects of small strain. Constant and linearly varying small strains are studied. The effects of linearly varying small strain are found to be greater than those of constant strain. To discuss the results, we derive an analytical solution that describes the effects of the two types of small strain. Although the form of the analytical solutions is the same for constant and linearly varying strains, the coefficients used in the analytical solutions, for which an equation is also obtained, differ. The difference observed in the effects of the two types of small strain is thus caused by the difference in coefficient.

1. Background and purpose
Axisymmetric strain is the purest form of strain. Knowledge of the effects of axisymmetric strain is important in engineering applications, where it can facilitate the development and improvement of turbulence models. In experiments, axisymmetric strain takes the form of contraction or expansion. Following the work of Prandtl [1], velocity fluctuations in homogeneous turbulence that passes through a contraction are characterized by each simple relation. The effects of an axisymmetric strain with a relatively higher contraction ratio than that examined by this study [1] on turbulence have been studied in previous theoretical and experimental works (e.g., [2, 3]). Contractions with low contraction ratios are used to improve the isotropy of grid-generated turbulence (e.g., [4-6]). Although grid-generated turbulence involves weak axisymmetric anisotropy (e.g., [7, 8]), a weak contraction can reduce this anisotropy.

The effects small strain on the decay of homogeneous turbulence were studied in a recent experiment in which the strain rate of the small strain was set to be smaller than that of the above-mentioned strain and in which the effects of the small strain on anisotropy were small [9]. This experiment showed that small strain reduces the turbulent kinetic energy (TKE) in decaying homogeneous turbulence. Suzuki et al. [10] studied the effects of weak fluid acceleration due to wind tunnel blockage on the decay of grid-generated turbulence; this weak fluid acceleration is equivalent to the small strain of the mean flow due to the continuity equation. By focusing on the insensitivity of the
anisotropy to small strain, they derived a scheme that validates the effects of the small strain on decaying grid-generated turbulence using the framework of the standard $k-\varepsilon$ model. In a previous experiment [9], the temporal variation of the small strain had a Gaussian-like shape, whereas a constant small strain was used in Suzuki et al.’s study [10].

The purpose of the present study is to clarify the influence of the shape of the temporal variation of small strain on the effects of small strain on the decay of homogeneous turbulence. Constant and linearly varying small strains are applied. A governing equation that describes the effects of small strain on the decay of homogeneous turbulence is derived and used to solve the present research problem. As shown in this study, the effects of small strain depend on its temporal variation, with those of linearly varying strain being greater than those of constant strain. A solution approximating the numerical results of the governing equation is derived to discuss the cause of the observed effects of small strain. Although the form of the derived solutions is the same for both constant and linearly varying small strains, the derived formulas of the coefficients used in the solutions are different for these two types of small strain.

2. Governing equations and numerical simulation

TKE and its dissipation in decaying homogeneous turbulence each follow a power law when the production term due to small strain is absent [4-12]. Using nondimensional time $t$, the power laws for $k$ and $\varepsilon$ are given respectively as:

$$k(t) = k_o \ t^n \text{ and } \varepsilon(t) = \varepsilon_o \ t^{(n+1)}, \text{ where } t = (t' - t'_o)/t'_c$$

(1)

Here, $n$ is the decay exponent of the homogeneous turbulence (e.g., [4, 5]), $k_o = k(1)$, and $\varepsilon_o$ is defined as $\varepsilon_o = (nU_o/M) \ k_o$, where $U_o$ is the inflow velocity and $M$ is the mesh size of a turbulence-generating grid; these are also defined as the characteristic velocity and the length of grid-generated turbulence, respectively. $t'$ and $t'_o$ are the actual time and the virtual origin of time, respectively, and $t'_c = M/U_o$. Two nondimensional functions, $f(t)$ and $g(t)$, which characterize the effects of small strain, are then introduced. These two nondimensional functions are defined to satisfy the following relations:

$$k(t) = f(t) \ k_o \ t^n, \ \varepsilon(t) = g(t) \ \varepsilon_o \ t^{(n+1)}$$

(2)

Decaying homogeneous turbulence with a production term arising from small strain, which has axisymmetric anisotropy, is analyzed in this study. The standard $k-\varepsilon$ model was applied to analyze the effects of small strain on decaying homogeneous turbulence. In decaying homogeneous turbulence, the
governing equations of the \( k-\varepsilon \) model yield a simplified set of equations [13]. From this simplified set of equations, the governing equations of \( f(t) \) and \( g(t) \) are derived as follows [10]:

\[
\frac{df}{dt} = P_s S(t) f(t) + n (t) - g(t))/t \tag{3}
\]

\[
\frac{dg}{dt} = C_{d1} P_o S(t) g(t) + (n+1) (t) f(t) - g(t))/t, \text{ where} \tag{4}
\]

\[S(t) = (\frac{dU/dx}{(U_o/M)} \text{ and } P_o = 2(a - 1)/ (1 + 2a). \]

Here, \( a = \left< \frac{u^2}{(w^2)} \right> / \left< \frac{x^2}{(u^2)} \right> \) is a parameter that characterizes the anisotropy of the homogeneous turbulence (e.g., [8]), and \( U \) and \( x \) are the streamwise mean velocity and streamwise direction, respectively.

In a previous study [10] that simulated the effects of a constant small strain on the decay of homogeneous turbulence, \( S(0) = S_o \) was used. In the present study, in addition to a constant small strain, a linearly varying small strain, \( S(t) = (dS) \ t \), was used, and their effects were compared. Figure 1 shows these two types of small strain, along with the corresponding profiles of TKE, which are not affected by the small strain. The values of \( dS \) and \( S_o \) for the numerical simulation were determined as follows. The small strain is present up to \( t = t_i \) in the present simulation. The integrals of \( S(t) \) for constant and linearly varying strains over the interval \( t = 0 \) to \( t_i \) are respectively \( I_{S0} = S_o t_i \) and \( I_{dS0} = (1/2) dS \ t_i. \)

From \( I_{S0} \) and \( I_{dS0} \), the value of \( dS \) is calculated using the following relation: \( dS = 2 S_o / t_i. \)

The governing equations, equation (3) and (4), were numerically solved. The value of \( S_o \) was set to 0.03, 0.01, 0.003, 0.001, and 0.0003 based on the previous experiments [9, 10] on grid-generated turbulence. The simulation was run up to \( t = 100 \) using the standard fourth-order Runge–Kutta scheme. The limit of \( t_i \), was set to 200, where \( t \) is comparable to the streamwise direction normalized by the mesh size in grid-generated turbulence. Using \( dS = 2 S_o / t_i \), the values of \( dS \) can be calculated from the values of \( S_o \). The magnitude of anisotropy was \( a = 0.5 \) based on previous experiments of grid-generated turbulence (e.g. [8]). This value of anisotropy yields \( P_o = -0.5. \) The decay exponent \( n \) was set to two theoretical values, namely 6/5 and 10/7. The model coefficient \( C_{d1} \) was set to 1.44, which is one of the most standard values.

3. Derived solution by using numerical results

3.1. Modelling of the governing equation by numerical results

First, the effects of small strain on TKE and its dissipation are investigated. Figure 2 shows these effects. The small strain, constant or linearly varying, reduces TKE and its dissipation. The temporal evolution of the effect of constant small strain differs from that of linearly varying small strain. Total strain \( \sigma(t) \) is introduced to more clearly describe the effects on TKE and its dissipation: \( \sigma(t) = S_o t \) or \( \sigma(t) = (1/2) dS t_i \), where the integrand is the local strain rate at the time of interest. Figure 2(b) shows the effects of small strain on TKE and its dissipation with respect to total strain. As shown, the relative effects of constant and linearly varying small strains are different. The relative effects \( f(t) \) and \( g(t) \) depend on the temporal profile of small strain; they are larger for linearly varying small strain.

The numerical results of this study demonstrate that the effects of linearly varying small strain are larger than those of constant small strain. To discuss the reason for this, this study attempted to solve the governing equations. The governing equations of the relative effects \( f(t) \) and \( g(t) \) include the term \( h(t) = (\sigma(t) - g(t))/t \) and thus can be analytically solved by deriving an equation for this term. As shown in the present results, the effects of small strain on the decay of homogeneous turbulence are qualitatively the same for different strain magnitudes. The formula for the approximation of the term \( h(t) \) can be simplified when the strain magnitude is small. The term \( h(t) = (\sigma(t) - g(t))/t \) is calculated for linearly varying strain based on the previous study [10]. Analytical solutions of the governing equations at small magnitudes of small strain were then derived to discuss the dependence of the effects of small strain on the decay of homogeneous turbulence on the temporal profile of small strain.

Figure 3(a) shows temporal profiles of the term \( h(t) = (\sigma(t) - g(t))/t \) at various magnitudes of linearly varying small strain. As shown, the profiles of \( h(t) \) depend on the strain magnitude \( S_o \). The
value of $h(t)$ increased with increasing strain magnitude $S_o$. The profiles of $h(t)$ are initially approximately linear. This linear evolution was found at all strain magnitudes. The linear evolution of $h(t)$ was the focus of this part of the study. Figure 3(b) shows the temporal profiles of $h(t)/t$ at various strain magnitudes $S_o$. As shown, $h(t)/t$ is initially constant. The width of the region in which $h(t)/t$ is constant depends on the strain magnitude $S_o$. The value of $h(t)/t$, which is equal to the linear gradient of $h(t)$, depends directly on magnitude $S_o$. The gradient is small when strain magnitude $S_o$ is small.

The analytical solution of the relative effects is obtained by substituting the derived equation for the term $(f(t) - g(t))/t$ into the governing equations. As shown in figure 3(b), $h(t)/t$ is constant when the strain magnitude is small. The following simple approximation is introduced: $h(t)/t = C$, where $C$ is a constant whose value is set to the initial value of $h(t)/t$. The above function approximates $h(t)/t$ with sufficient accuracy, as shown in figure 3(b). The effects of linearly varying small strain can be characterized by total strain $\sigma(t) = (1/2) dS t^2$ (see figure 2(b)). The above approximation is rewritten in the following form based on the numerical results: $h(t)/t = (1/2) C_o dS$, where the coefficient $C_o$ is a constant.

3.2. Analytical solutions of the governing equations and discussion

Using the approximation of $h(t)$, the governing equation of $f(t)$ yields a linear ordinary differential equation. The analytical solution of $g(t)$ was derived using the derived analytical solutions of $f(t)$ and $h(t)$, which are:

$$f(t) = \exp((1/2) P_o (dS) \hat{r}^2) - (nC_o/(2P_o)) [1 - \exp((1/2) P_o dS \hat{r}^2)]$$

$$g(t) = f(t) - (1/2) C_o dS \hat{r}^2.$$
Figure 3. Effects of linearly varying small strain on linear difference $h(t) = (f(t) - g(t))/t$. (a) Temporal variations of $h(t)$. Variation of $h(t)$ depends on magnitude of small strain. $h(t)$ initially varies linearly. Initial linear gradients of $h(t)$ are negligibly affected by decay exponent $n$. (b) Temporal variations of $h(t)/t$. This function reveals linearity of gradient of $h(t)$. $h(t)/t$ can be approximated as $h(t)/t = C$ with sufficient accuracy, where $C$ is a constant. Initial linear gradient of $h(t)$ depends directly on magnitude of small strain. Therefore, approximation can be rewritten as $h(t)/t = (1/2) C_o dS$, where $C_o$ is a constant (Color version available online).

Both equations include the constant $C_o$, for which an equation was derived. Using the two analytical solutions, the two sides of the governing equation of $g(t)$ can be approximated as:

\[(\text{LHS}) = (1/2) [2P_o + C_o(n - 2)] dS t + \ldots \quad \text{and} \quad (7)\]

\[(\text{RHS}) = (1/2) [2 C_{o1}P_o + C_o(n + 1)] dS t + \ldots \quad \text{(8)}\]

where LHS and RHS denote the left- and right-hand sides of the governing equation of $g(t)$, respectively. These approximations are derived using a small value of $dS$. Equating LHS to RHS, a formula for the constant $C_o$ is derived as:

\[C_o = - (2/3) (C_{o1} - 1) P_o, \text{therefore, } h(t) = - (1/3) (C_{o1} - 1) P_o dS t. \quad \text{(9)}\]

The equation for $C_o$ is obtained using $P_o$ and $C_{o1}$ and does not include the decay exponent $n$.

Figure 4(a) validates the derived equation for the constant $C_o$ in equation (9). The values of $C_o$ obtained using the derived equation agree with the numerical results, validating the derived equation for $C_o$. Using the derived equation, the analytical solutions for the functions of the relative effects for linearly varying strain are obtained as:

\[f(t) = \exp(P_o \sigma(t)) - (2/3) [n(C_{o1} - 1)/2] [1 - \exp(P_o \sigma(t))] \quad \text{and} \quad \text{(10)}\]

\[g(t) = \exp(P_o \sigma(t)) - (2/3) [n(C_{o1} - 1)/2] [1 - \exp(P_o \sigma(t))] + (2/3) (C_{o1} - 1) P_o \sigma(t), \quad \text{(11)}\]

where $\sigma(t) = (1/2) dS \vec{r}$. As shown in figure 4(b), the analytical solutions agree with the numerical results in the range of $t = 0 - 0.2$. This agreement validates the derived analytical solutions.

This study compares the analytical solutions for linearly varying strain with those for constant strain using the results of a previous study [10]. The approximate solutions that describe the effects of constant small strain can be derived as:

\[f(t) = \exp(P_o \sigma(t)) - [n(C_{o1} - 1)/2] [1 - \exp(P_o \sigma(t))] \quad \text{and} \quad \text{(12)}\]

\[g(t) = \exp(P_o \sigma(t)) - [n(C_{o1} - 1)/2] [1 - \exp(P_o \sigma(t))] + (1/2) (C_{o1} - 1) P_o \sigma(t), \quad \text{(13)}\]
where $\sigma(t) = S_0 t$. The derived solution for the effects of linearly varying small strain is different from that for those of constant small strain. Although the forms of the derived solutions are the same for the two types of strain, the constants in the solutions are different, as shown in figure 4(c). This difference in constants is the cause of the difference between the effects of the two types of small strain.

The mathematical forms derived in this study satisfy the governing equations. Therefore, the mathematical forms could be considered to be analytical solutions of the governing equations. These analytical solutions may not be exact general solutions. Note that profiles of the exact general solutions ($f(t)$ and $g(t)$) similar remarkably with those of the numerical results. The present analytical solutions could be derived by assuming small magnitude of $S_0$. When the magnitude of $S_0$ is larger, mathematical form of the derived solutions could be more complicated.

Also, this study discusses the present analytical solutions further. The discussion of this study focuses on one/two equation modeling, which is used in the application. One equation model could be derived for $f(t)$ with the neglected second term of the governing equation of $f(t)$. When this one equation model is used, the analytical solution of $f(t)$ could be derived as $f(t) = \exp(P_0 \sigma(t))$ as shown in a recent study [12]. Therefore, results of the one equation model are insensitive to the difference between constant and linear strain variations. This derived solution of the one equation model is compared with equation (10) and (12). The derived solution of the one equation model is equal to the first term of the solutions of $f(t)$ in equation (10) and (12). Therefore, the dependency of effects of the small strain on the strain variations could not be described by the one equation model and could
certainly be described by the two-equation model. This statement, which would be significant in the application, could be strictly found by using the analytical solutions.

4. Summary and future works
The effects of small strain on the decay of homogeneous turbulence were investigated using the framework of the standard \( k-\varepsilon \) model. In decaying homogeneous turbulence without a small strain, the TKE and its dissipation each follow a power law; however, with a small strain, they deviate from their respective power laws.

Linearly varying small strain was found to have a greater effect than that of constant strain on the decay of homogeneous turbulence. To determine the cause of this difference, analytical solutions for the relative effects that approximated the numerical results were derived by obtaining an equation that describes the difference between the linear gradients of the relative effects included in the governing equations. Although the analytical solutions for constant and linearly varying small strains were similar, the coefficients in the analytical solutions differed. This difference is the cause of the observed difference in effects.

In future works, the cause of the greater effects observed with linearly varying strain should be further examined through numerical simulations. The sensitivity of TKE dissipation to the small strain is higher than that of TKE itself [10-12]. The approximate solutions of the effects of small strain derived here can be used to further investigate the effects of small strain on the decay of homogeneous turbulence.

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