An explicit formula for the A-polynomial of the knot with Conway’s notation \( C(2n, 3) \)

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ABSTRACT

An explicit formula for the A-polynomial of the knot with Conway’s notation \( C(2n, 3) \) is obtained from the explicit Riley–Mednykh polynomial of it.

Keywords: A-polynomial; explicit formula; knot with Conway’s notation \( C(2n, 3) \); Riley–Mednykh polynomial.

Mathematics Subject Classification: 57M27, 57M25

1. Introduction

In 1994, the A-polynomial, \( A(L, M) \), of a compact 3-manifold \( N \) with a single torus boundary was introduced by Cooper, Culler, Gillet, Long and Shalen in [3]. It’s variables are eigenvalues of the meridian and the longitude under the representations from \( \pi_1 N \) into \( \text{SL}(2, \mathbb{C}) \). One of the main results of [3] is “boundary slopes are boundary slopes”. That is the boundary slope of the Newton polygon of \( A(L, M) \) is the boundary slope of an incompressible surface in \( N \). In 2001, it was shown that the Newton polygon of \( A(L, M) \) is dual to the fundamental polygon of the Culler–Shalen seminorm [1]. The Culler–Shalen seminorm [4] can be used to detect and classify the exceptional surgeries, which is a step toward another proof of the Poincare conjecture. A-polynomial also encodes the deformed structure of \( N \). For example, using the longitude \( L \) in [9, 11], the volumes of the deformed cone-manifolds are computed and in [8, 10], the Chern–Simons invariants [2, 12] of the deformed orbifolds are computed. The non-commutative A-polynomial \( A(L, M, q) \) of a knot is introduced and it is conjectured that
A(\(L, M, 1\)) = B(M)A(L, M^{1/2}) for some polynomial B(M) of M and the conjecture is called AJ conjecture [6]. AJ conjecture is proved for some knots. For example, our knot, the knot with Conway’s notation C(2n, 3) satisfies the AJ conjecture [14]. If AJ conjecture is true, then the colored Jones polynomial detects knottedness [5] as A-polynomial.

With today’s technology, A-polynomial is relatively difficult to compute. Recovering representations from a triangulation of \(N\) and compute a factor of the A-polynomial is another try to compute it [20]. By one by one computation, A-polynomials are known up to eight crossings and most nine crossings and many 10 crossings. For infinite families, recursively, A-polynomials are known for twist knots [13], \((-2, 3, 1 + 2n)\) pretzel knots [7, 19], \(J[m, 2n]\) [13] for \(m\) between 2 and 5 [17] and explicitly, for two-bridge torus knots [3, 13], iterated torus knots [16], and for twist knots [8, 15]. We record here that \(J[3, -2n]\) is the mirror image of \(C(2n, 3)\).

The main purpose of the paper is to find the explicit formula for the A-polynomial of the knot with Conway’s notation \(C(2n, 3)\). Let us denote the knot with Conway’s notation \(C(2n, 3)\) by \(T_{2n}\) and the A-polynomial of the knot with Conway’s notation \(C(2n, 3)\) by \(A_{2n}\). The following theorem gives the explicit formula for the A-polynomial of \(T_{2n}\).

**Theorem 1.1.** A-polynomial \(A_{2n}(L, M)\) is given explicitly by

\[
A_{2n} = \begin{cases} 
\displaystyle \sum_{i=0}^{2n} \left( n + \left[ \frac{i}{2} \right] \right) \left( \frac{LM^{4n} - 1}{1 + LM^{2+4n}} \right)^i \left( \frac{1 + LM^{6+4n}}{M^2 + LM^{4+4n}} \right)^{\left\lfloor \frac{i+1}{2} \right\rfloor} M^{-2n} \left( 1 + LM^{2+4n} \right)^{3n} & \text{if } n \geq 0 \\
\displaystyle \sum_{i=0}^{-2n-1} \left( -n + \left[ \frac{i-1}{2} \right] \right) \left( \frac{(1 - M^2)(M^{-4n} - L)}{LM^2 + M^{-4n}} \right)^i \left( \frac{LM^6 + M^{-4n}}{LM^4 + M^{-2-4n}} \right)^{\left\lfloor \frac{i+1}{2} \right\rfloor} M^{8n+6} \left( LM^2 + M^{-4n} \right)^{-3n-1} & \text{if } n < 0.
\end{cases}
\]

One can consult [8] for solving the recurrence formula. Our writing is parallel with that in [15], which is based on [13].

2. **Proof of Theorem 1.1**

A knot \(K\) is a two bridge knot with Conway’s notation \(C(2n, 3)\), if \(K\) has a regular two-dimensional projection of the form in Fig. 1. Recall that, we denote it by \(T_{2n}\). Let us denote the exterior of \(T_{2n}\) by \(X_{2n}\). The following proposition gives the fundamental group of \(X_{2n}\) [9, 10, 13, 18].
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![Figure 1](image)

*Fig. 1. A two bridge knot with Conway's notation $C(2n,3)$ for $n > 0$ (left) and for $n < 0$ (right).*

**Proposition 2.1.**

$$\pi_1(X_{2n}) = \langle s, t \mid swt^{-1}w^{-1} = 1 \rangle,$$

where $w = (ts^{-1}tst^{-1}s)^n$.

Given a set of generators, $\{s, t\}$, of the fundamental group for $\pi_1(X_{2n})$, we define a representation $\rho: \pi_1(X_{2n}) \to \text{SL}(2, \mathbb{C})$ by

$$\rho(s) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix}, \quad \rho(t) = \begin{bmatrix} M & 0 \\ 2 - M^2 - M^{-2} - x & M^{-1} \end{bmatrix}.$$  

Then $\rho$ can be identified with the point $(M, x) \in \mathbb{C}^2$. When $M$ varies, we have an algebraic set, whose defining equation is the following explicit Riley–Mednykh polynomial.

**Lemma 2.2.** $\rho$ is a representation of $\pi_1(X_{2n})$, if and only if $x$ is a root of the following Riley–Mednykh polynomial $P_{2n} = P_{2n}(x, M)$, which is given explicitly by

$$P_{2n} = \begin{cases} 
\sum_{i=0}^{2n} \left(n + \begin{array}{c} i \\ \frac{1}{2} \end{array} \right) M^{4n}(M^2 + M^{-2} + x - 1)^{\lfloor \frac{1+i}{2} \rfloor} & \text{if } n \geq 0, \\
\sum_{i=0}^{-2n-1} \left(-n - \begin{array}{c} i \\ \frac{1}{2} \end{array} \right) M^{-4n-2}(M^2 - M^{-2} - x + 1)^{\lfloor \frac{1-i}{2} \rfloor} & \text{if } n < 0.
\end{cases}$$

**Proof.** In [9], $P_{2n}$ is give by the following recursive formula.

$$P_{2n} = \begin{cases} 
QP_{2(n-1)} - M^8 P_{2(n-2)} & \text{if } n > 1, \\
QP_{2(n+1)} - M^8 P_{2(n+2)} & \text{if } n < -1
\end{cases}$$

with initial conditions

$$P_{-2} = M^2x^2 + (M^4 - M^2 + 1)x + M^2, \\
P_0 = M^{-2} \quad \text{for } n < 0 \quad \text{and} \quad P_0 = 1 \quad \text{for } n > 0, \\
P_2 = -M^4x^3 + (-2M^6 + M^4 - 2M^2)x^2 + (-M^8 + M^6 - 2M^4 + M^2 - 1)x + M^4.$$
Case 2. $n < 0$. When $i > -2n - 1$ or $i < 0$, $\binom{n + \lfloor \frac{i}{2} \rfloor}{i}$ is undefined and can be considered as zero. Hence the finite sum can be regarded as an infinite sum. Direct computation shows that $f_0 = P_0$ and $f_2 = P_2$. Now, we only need to show that $f_{2n}$ satisfies the recursive relation. We know that $Q$ can be written as $M^4(-x(M^2 + M^{-2} + x - 1)^2 + 2)$.

\[
Q f_{2(n+1)} - M^8 f_{2(n+2)} = M^4(-x(M^2 + M^{-2} + x - 1)^2 + 2) = M^4(-x(M^2 + M^{-2} + x - 1)^2 + 2)
\]

three times.

Case 1. $n \geq 0$. When $i > 2n$ or $i < 0$, $\binom{n + \lfloor \frac{i}{2} \rfloor}{i}$ is undefined and can be considered as zero. Hence the finite sum can be regarded as an infinite sum. Direct computation shows that $f_0 = P_0$ and $f_2 = P_2$. Now, we only need to show that $f_{2n}$ satisfies the recursive relation. We know that $Q$ can be written as $M^4(-x(M^2 + M^{-2} + x - 1)^2 + 2)$.

\[
Q f_{2(n-1)} - M^8 f_{2(n-2)} = M^4(-x(M^2 + M^{-2} + x - 1)^2 + 2)
\]

\[
\times \sum_i \left( n - 1 + \frac{i}{2} \right) M^{4n-4}(M^2 + M^{-2} + x - 1)^i(-x)^{\frac{i+1}{2}}
\]

\[
- M^8 \sum_i \left( n - 2 + \frac{i}{2} \right) M^{4n-8}(M^2 + M^{-2} + x - 1)^i(-x)^{\frac{i+1}{2}}
\]

\[
= \sum_i \left[ \left( n - 2 + \frac{i}{2} \right) + 2 \left( n - 1 + \frac{i}{2} \right) \right] M^4(M^2 + M^{-2} + x - 1)^i(-x)^{\frac{i+1}{2}}
\]

\[
= f_{2n}.
\]

In the last equality, we use the binomial relation

\[
\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}
\]

\[
\text{three times.}
\]
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$$
\times \sum_i \left( -n - 1 + \frac{\left( i - 1 \right)}{2} \right) M^{-4n-6}(M^2 + M^{-2} + x - 1)^i (-x)^{\frac{i+4}{2}}
$$

$$
- M^8 \sum_i \left( -n - 2 + \frac{\left( i - 1 \right)}{2} \right) M^{-4n-10}(M^2 + M^{-2} + x - 1)^i (-x)^{\frac{i+4}{2}}
$$

$$
= \sum_i \left( \left( -n - 2 + \frac{\left( i - 1 \right)}{2} \right) + 2 \left( -n - 1 + \frac{\left( i - 1 \right)}{2} \right) \right)
$$

$$
- \left( -n - 2 + \frac{\left( i - 1 \right)}{2} \right) M^{-4n-2}(M^2 + M^{-2} + x - 1)^i (-x)^{\frac{i+4}{2}}
$$

$$
= \sum_i \left( -n + \frac{\left( i - 1 \right)}{2} \right) M^{-4n-2}(M^2 + M^{-2} + x - 1)^i (-x)^{\frac{i+4}{2}}
$$

$$
= f_{2n}.
$$

In the last equality, we use the binomial relation

$$
\binom{a}{b} = \binom{a - 1}{b - 1} + \binom{a - 1}{b}
$$

three times, again.

Let $l = \text{ww}^*s^{-4n}$ [3, 13], where $w^*$ is the word obtained by reversing $w$. Let $L = \rho(l)_{11}$. Then $l$ is the longitude, which is null-homologus in $X_{2n}$ (you can read a twisted longitude $\text{ww}^*$ from the Schubert normal form of the knot $C(2n, 3)$ and multiply it by $s^{-4n}$, so that the exponent sum of $l$ becomes 0). And, we have

Lemma 2.3 ([9, 10]).

$$
L = -M^{-4n-2}M^{-2} + x
$$

$$
x = -\frac{1 + LM^{6+4n}}{M^2(1 + LM^{2+4n})}.
$$

Now substituting $-\frac{1 + LM^{6+4n}}{M^2(1 + LM^{2+4n})}$ for $x$ into $P_{2n}$, for $n \geq 0$, gives

$$
\sum_{i=0}^{2n} \left( n + \frac{i}{2} \right) M^{4n+i} \left( M^2 + M^{-2} - \frac{1 + LM^{6+4n}}{M^2(1 + LM^{2+4n})} - 1 \right)^i
$$

$$
\times \left( \frac{1 + LM^{6+4n}}{M^2(1 + LM^{2+4n})} \right)^{\frac{i+1}{4}}.
$$
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We observe that
\[ M^2 + M^{-2} = \frac{1 + LM^{6+4n}}{M^2(1 + LM^{2+4n})} - 1 = \frac{(LM^{4n} - 1)(1 - M^2)}{(1 + LM^{2+4n})}. \]

The resulting expression,
\[
\sum_{i=0}^{2n} \left( n + \left\lfloor \frac{i}{2} \right\rfloor \right) M^{4n} \left( \frac{(LM^{4n} - 1)(1 - M^2)}{1 + LM^{2+4n}} \right)^i \left( \frac{1 + LM^{6+4n}}{M^2 + LM^{4+4n}} \right)^{\left\lfloor \frac{i+1}{2} \right\rfloor}
\]

once denominators are cleared and some power of \( M \) is factored out to give a polynomial, gives the \( A \)-polynomial \( A_{2n}(L, M) \). We multiply it by \( M^{-2n}(1 + LM^{2+4n})^{4n} \), so that we have the claimed formula in Theorem 1.1. The following equality, which guarantees that the claimed formula has the constant term 1 and is a polynomial,
\[ c_{2n} = \sum_{i=0}^{2n} \left( n + \left\lfloor \frac{i}{2} \right\rfloor \right) (M^2 - 1)^i \left( \frac{1}{M^2} \right)^{\left\lfloor \frac{i+1}{2} \right\rfloor} M^{-2n} = 1 \]
can be proved by induction,
\[ c_{2n} = \frac{(M^2 - 1)^2 c_{2(n-1)}}{M^4} + \frac{2c_{2(n-1)}}{M^2} - \frac{c_{2(n-2)}}{M^4} \]
which is proven in the following. As in the case of the proof of Lemma 2.2, \( c_{2n} \) can be regarded as an infinite sum. Direct computation shows that \( f_2 = c_2 \) and \( f_4 = c_4 \). Let \( f_{2n} \) be the right side of the claimed formula. We will show that \( f_{2n} = c_{2n} \).

Again as in the case of the proof of Lemma 2.2, \( f_{2n} \) can be written as
\[
\sum_i \left[ \left( n - 2 + \left\lfloor \frac{i}{2} \right\rfloor \right) + 2 \left( n - 1 + \left\lfloor \frac{i}{2} \right\rfloor \right) - \left( n - 2 + \left\lfloor \frac{i}{2} \right\rfloor \right) \right]
\]
\[ \times (M^2 - 1)^i \left( \frac{1}{M^2} \right)^{\left\lfloor \frac{i+1}{2} \right\rfloor} M^{-2n}. \]

Hence as in the case of the proof of Lemma 2.2, by using the binomial relations three times, we have \( f_{2n} = c_{2n} \).

Similarly, for \( n < 0 \), substituting
\[ \frac{1 + LM^{6+4n}}{M^2(1 + LM^{2+4n})} = -\frac{M^{-4n} + LM^6}{M^2(M^{-4n} + LM^2)} \]
for \( x \) into \( P_{2n} \) gives
\[
\sum_{i=0}^{-2n-1} \left( -n + \left\lfloor \frac{i - 1}{2} \right\rfloor \right) M^{-4n-2} \left( -M^2 - M^{-2} + \frac{M^{-4n} + LM^6}{M^2(M^{-4n} + LM^2)} + 1 \right)^i
\]
\[ \times \left( \frac{M^{-4n} + LM^6}{M^2(M^{-4n} + LM^2)} \right)^{\left\lfloor \frac{i+1}{2} \right\rfloor}. \]
We observe that
\[-M^2 - M^{-2} + \frac{M^{-4n} + LM^6}{M^2(M^{-4n} + LM^2)} + 1 = \frac{(M^{-4n} - L)(1 - M^2)}{(M^{-4n} + LM^2)}.\]

The resulting expression,
\[-2n-1 \sum_{i=0}^{i} \left(-n + \left\lfloor \frac{(i - 1)}{2} \right\rfloor \right) M^{-4n-2} \left( \frac{(1 - M^2)(M^{-4n} - L)}{LM^2 + M^{-4n}} \right)^i \left( \frac{LM^6 + M^{-4n}}{LM^4 + M^{-4n-2}} \right)^{\lfloor \frac{i+1}{2} \rfloor} \times \frac{2^{n+4}}{LM} \left( L + M^{-4n-2} \right)^{-3n-1},\]

once denominators are cleared and some power of $M$ is factored out to give a polynomial, gives the $A$-polynomial $A_{2n}(L, M)$. We multiply it by $M^{12n+8}(LM^2 + M^{-4n})^{-3n-1}$, so that we have the claimed formula in Theorem 1.1:
\[-2n-1 \sum_{i=0}^{i} \left(-n + \left\lfloor \frac{(i - 1)}{2} \right\rfloor \right) \left( \frac{(1 - M^2)(M^{-4n} - L)}{LM^2(L + M^{-4n-2})} \right)^i \left( \frac{M^2(L + M^{-4n-6})}{L + M^{-4n-2}} \right)^{\lfloor \frac{i+1}{2} \rfloor} \times \frac{2^{n+4}}{LM} \left( L + M^{-4n-2} \right)^{-3n-1}.\]

Now we want to show that the claimed formula does not have fractions. For $n = -1$, by direct computation, one can show that the claimed formula does not have fractions. For each $n < -1$, fractions can only occur in the following sums.
\[-2n-1 \sum_{i=0}^{i} \left(-n + \left\lfloor \frac{(i - 1)}{2} \right\rfloor \right) \left( (1 - M^2)(-L) \right)^i L^{\frac{i+1}{2}} \times M^{2n+4-2i+2\lfloor \frac{i+1}{2} \rfloor} L^{-3n-1-i-\lfloor \frac{i+1}{2} \rfloor} = \sum_{i=0}^{i} \left(-n + \left\lfloor \frac{(i - 1)}{2} \right\rfloor \right) (M^2 - 1)^i M^{2n+4-2i+2\lfloor \frac{i+1}{2} \rfloor} L^{-3n-1}.\]

Let $c_{2n}$ be the coefficient of $L^{-3n-1}$ of the above sum. Then, one can prove that $c_{2n} = M^4$ by the following recurrence relation,
\[c_{2n} = \frac{(M^2 - 1)^2 c_{2(n+1)}}{M^4} + \frac{2c_{2(n+1)}}{M^2} - \frac{c_{2(n+2)}}{M^4},\]

which can be proved as in the case of $n > 0$.

Now, we are going to compute a part of the coefficient of $L^{-3n-2}$. For each $n < 0$, the term $-L^{-3n-2}$ exists:
\[-2n-1 \sum_{i=-2n-2}^{i} \left(-n + \left\lfloor \frac{(i - 1)}{2} \right\rfloor \right) M^{2n+4-2i+2\lfloor \frac{i+1}{2} \rfloor}.\]
$(-L)^i \left( \frac{1+i}{2} \right) L^{\left\lfloor \frac{1}{2}(i-1) \right\rfloor} M^{-4n-6} L^{-3n-1-i-\left\lfloor \frac{1}{2}i \right\rfloor}$

$= \sum_{i=-2n-2}^{-2n-1} \left( \frac{1+i}{2} \right) L^{-3n-2}.$

And now, we are going to compute a part of the coefficient of $L^0$. For each $n < 0$, the term $M^{12n^2+14n+6}$ exists:

$\sum_{i=-2n-2}^{-2n-1} \left( -n + \left\lfloor \frac{i-1}{2} \right\rfloor \right) M^{2n+4-2i+2\left\lfloor \frac{i}{2} \right\rfloor}$

$\times (M^{-4n})(M^{-4n-6})^{\left\lfloor \frac{i}{2} \right\rfloor} (M^{-4n-2})^{-3n-1-i-\left\lfloor \frac{i}{2} \right\rfloor}$

$= \sum_{i=-2n-1}^{-2n-1} M^{12n^2+12n+6-2i+\left\lfloor \frac{i}{2} \right\rfloor}.$

Hence there does not exist redundant $L$ or $M$ factors.

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