The Ostrowski Expansions Revealed

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Abstract

We provide algorithms for the absolute and alternating Ostrowski Expansions of the continuum and provide proofs for their uniqueness.

1 Introduction

The various algorithms that construe the Ostrowski Expansions rely on the continued fraction expansion of a fixed irrational number \( \alpha \) in the interval in order to represent other real numbers ‘base–\( \alpha \)’ and ‘base–(\( -\alpha \))’. They are utilized in a broad range of applications, ranging from Diophantine Approximation [1, 4] to symbolic dynamics and coding theory [5, 6], for a thorough survey refer to [2]. Given \( r \in \mathbb{R} \), we define the floor \( \lfloor r \rfloor \) of the real number \( r \) to be the largest integer smaller than or equal to \( r \) and expand \( r \) as a continued fraction using the following iteration scheme:

Algorithm 1: continued fraction expansion

\[
\begin{align*}
\text{input} &: r \in \mathbb{R} \\
\text{output} &: \ell \in \mathbb{N}^\infty := \mathbb{Z}_{\geq 1} \cup \{\infty\}, \ a_0 \in \mathbb{Z}, \ \langle a_k \rangle_1^\ell \subset \mathbb{Z}_{\geq 1} \\
1 & \text{set } a_0 := \lfloor r \rfloor, \ \alpha_0 := r - a_0, \ \ell := \infty, \ k := 1; \\
2 & \text{while } \alpha_k > 0 \text{ do} \\
3 & \quad \text{set } a_k := \lfloor 1/\alpha_{k-1} \rfloor; \\
4 & \quad \text{set } \alpha_k := 1/\alpha_{k-1} - a_k \in [0, 1); \\
5 & \quad \text{set } k := k + 1; \\
6 & \text{end} \\
7 & \text{set } \ell := k - 1;
\end{align*}
\]

The proof of the existence and uniqueness for this expansion as well as the assertion of the rest of the claims made in this section can be found in the classical exposition [3]. This iteration process will terminate with a finite value \( \ell \) precisely when \( \alpha \) is rational. The assignment of the digit \( a_k \) in line–3 yields the inequality

\begin{equation}
\begin{aligned}
a_k \alpha_{k-1} &\leq 1 < (a_k + 1)\alpha_{k-1}, \ &1 \leq k < \ell + 1,
\end{aligned}
\end{equation}

(1)
(where $\infty + 1 := \infty = \ell$ when applicable). After we rewrite the assignment in line–4 as $\alpha_{k-1} = (a_k + \alpha_k)^{-1}$, we obtain the expansion

$$r = a_0 + \alpha_0 = a_0 + \frac{1}{a_1 + \alpha_1} = a_0 + \frac{1}{a_1 + \alpha_1} = \ldots = a_0 + \frac{1}{a_1 + \alpha_1},$$

whose truncation at the $k < \ell + 1$ step yields the convergent

$$\frac{p_k}{q_k} := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ldots + \frac{1}{a_k}}}}.$$

We fix $\alpha \in (0, 1) \setminus \mathbb{Q}$ throughout and, after we plug it as input in the algorithm $\square$ we obtain the value $a_0 = 0$ and the infinite digit sequence $\langle a_k \rangle_1^\infty$. We end this section by quoting two well known facts about the resulting sequence of convergents $\langle p_k/q_k \rangle_0^\infty$, namely the recursion equation

$$q_{-1} = p_0 := 0, \quad p_{-1} = q_0 := 1, \quad p_k = a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}, \quad k \geq 1 \quad (2)$$

and the inequality

$$\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}, \quad k \geq 0. \quad (3)$$

## 2 Basic definitions and identities

The base–$\alpha$ and base–$(-\alpha)$ Ostrowski Expansions are dot products of two sequences: a digit sequence and a sequence of certain coefficients depending on $\alpha$, which will we now define and study. After applying algorithm $\square$ and letting $\langle p_k/q_k \rangle_0^\infty$ be as in equation (2), we define the coefficients

$$\theta_k := q_k \alpha - p_k, \quad k \geq -1. \quad (4)$$

Utilizing this definition and the equations (2) in an induction argument, we arrive at the recursion equations

$$\theta_{-1} = -1, \quad \theta_0 = \alpha, \quad \theta_k = \theta_{k-2} + a_k \theta_{k-1}, \quad |\theta_k| = |\theta_{k-2}| - a_k |\theta_{k-1}|, \quad k \geq 1. \quad (5)$$

Multiplying both sides of the inequality (3) by $q_k$ yields the inequality

$$|\theta_k| < q_k^{-1}, \quad k \geq 0. \quad (6)$$
Assuming that $\theta_k = -\theta_{k-1}\alpha_k$ for some $k \in \mathbb{N}$, where $\alpha_k$ is as in line–4 of algorithm 1, we use this recursion relationship to obtain the equality

$$
\theta_{k+1} = \theta_{k-1} + a_{k+1}\theta_{k} = \theta_{k-1}(1 - a_{k+1}\alpha_k) = \theta_{k-1}\alpha_k \left( \frac{1}{\alpha_k} - a_{k+1} \right) = -\theta_k\alpha_{k+1}.
$$

Since we have $\theta_0 = \alpha = -\theta_{-1}\alpha$, that is, the assumption holds for $k = 0$, we have just proved by induction that

$$
\theta_k = -\theta_{k-1}\alpha_k = (-1)^k\alpha_0\alpha_1...\alpha_k, \quad k \geq 0,
$$

(7)

In particular, this shows that the sequence $\langle \theta_k \rangle_{-1}^\infty$ is alternating as in

$$
|\theta_k| = (-1)^k\theta_k, \quad k \geq -1.
$$

(8)

In tandem with the inequality (1), we obtain the inequality

$$
\frac{a_{k+1}|\theta_k|}{|\theta_{k-1}|} = a_{k+1}\alpha_k < 1 < (a_{k+1} + 1)\alpha_k = \frac{(a_{k+1} + 1)|\theta_k|}{|\theta_{k-1}|},
$$

that is,

$$
a_{k+1}|\theta_k| < |\theta_{k-1}| < (a_{k+1} + 1)|\theta_k|, \quad k \geq 0.
$$

(9)

Since by definition (5), we have $|\theta_0| < |\theta_{-1}|$ and since $|\alpha_k| < 1$ by its definition in line–4 of algorithm 1, the formula (7) also asserts that the sequence $\langle |\theta_k| \rangle_{-1}^\infty$ is strictly decreasing to zero, that is,

$$
|\theta_{k+1}| < |\theta_k| \to 0, \quad \text{as} \quad k \to \infty.
$$

(10)

While, by the alternating series test, this is enough to assert the convergence of the series $\sum_{k=1}^{\infty} a_k|\theta_{k-1}|$, this series, in fact, converges absolutely:

**Proposition 2.1.** The infinite series $\sum_{k=1}^{\infty} a_k|\theta_{k-1}|$ converges for all $\alpha$.

**Proof.** By the inequality (9), we see that $\sum_{k=1}^{\infty} a_k|\theta_{k-1}| < \sum_{k=-1}^{\infty} |\theta_k|$ and by the recursion equation (2) and the inequality (6) we have

$$
\frac{|\theta_{k+1}|}{|\theta_k|} < \frac{q_k}{q_{k+1}} = \frac{q_k}{a_{k+1}q_k + q_{k-1}} < \frac{1}{a_{k+1}}, \quad k \geq 0.
$$

Thus, as long as $\limsup_{k \to \infty} \langle a_k \rangle \geq 2$, we conclude convergence from the simple comparison and ratio tests. When $\limsup_{k \to \infty} \langle a_k \rangle = 1$, then $\alpha$ must be a noble number, whose continued fraction expansion ends with a tail of 1’s. By the limit comparison test, we need only establish the convergence for this tail, that is for $\alpha$ where $a_k = 1$ for all $k \geq 1$. After using the assignments of line–3 and line–4 in algorithm 1 we write

$$
\alpha_k = \frac{1}{1 + \alpha_{k+1}} = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \alpha_k}}}, \quad k \geq 0.
$$
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The solution for the resulting quadratic equation is the golden section \( \phi := \alpha_k = .5(-1 + 5^{.5}) \approx .618 \). Then the identity (7) and the geometric sum formula assert that

\[
\sum_{k=1}^{\infty} a_k|\theta_{k-1}| = \sum_{k=0}^{\infty} |\theta_k| = \sum_{k=0}^{\infty} \alpha_0\alpha_1...\alpha_k = \sum_{k=0}^{\infty} \phi^{k+1} = 1 + \phi,
\]

which proves convergence for this case as well.

After fixing a finite index \( n \geq 1 \), we can now use formula (5) to rewrite the tail \( \sum_{k=n}^{\infty} a_k|\theta_{k-1}| \) as the telescoping series

\[
a_n|\theta_{n-1}| + a_{n+1}|\theta_n| + ... = (|\theta_{n-2}| - |\theta_n|) + (|\theta_{n-1}| - |\theta_{n+1}|) + ... = |\theta_{n-2}| + |\theta_{n-1}|.
\]

(11)

After plugging \( n := 1 \) and plugging the values for \( \theta_{-1} \) and \( \theta_0 \) as in the equation (5), we use this identity to explicitly evaluate the sum

\[
\sum_{k=1}^{\infty} a_k|\theta_{k-1}| = 1 + \alpha.
\]

(12)

We use the relationship (5) again, we can write \( \sum_{k=1}^{\infty} a_k \theta_{k-1} \) as the telescopic series

\[
a_1 \theta_0 + a_2 \theta_1 + a_3 \theta_2 + ... = (\theta_1 - \theta_{-1}) + (\theta_2 - \theta_0) + (\theta_3 - \theta_1) + ... = -\theta_{-1} - \theta_0,
\]

and evaluate this sum as

\[
\sum_{k=1}^{\infty} a_k \theta_{k-1} = 1 - \alpha.
\]

(13)

Subtracting the sum (13) from the sum (12) and dividing by two yields the self representations

\[
\alpha = \sum_{k=1}^{\infty} a_{2k}|\theta_{2k-1}| = -\sum_{k=1}^{\infty} a_{2k} \theta_{2k-1}.
\]

(14)

Adding the sum (13) to the sum (12) and then dividing by two yields the expansion of unity

\[
1 = \sum_{k=0}^{\infty} a_{2k+1} \theta_{2k} = \sum_{k=0}^{\infty} a_{2k+1}|\theta_{2k}|.
\]

(15)

3 The Absolute Ostrowski Expansion

The base–\( \alpha \) Absolute Ostrowski Expansion is a sum of the form \( \sum_{k=1}^{\infty} d_k|\theta_{k-1}| \), where \( 0 \leq d_k \leq a_k \) along with all its finite truncations. While a simple comparison to the convergent series in proposition 2.1 proves its existence, it is by no means unique. For instance, using the definition (5) of \( \theta_0 := \alpha \) we see that after setting \( \ell = d_1 := 1 \), we obtain the self-expansion, which is different from formula (14). To achieve uniqueness, we will require the digit sequence to adhere to the so called Markov Conditions. We say that the sequence \( \langle b_k \rangle_1^\infty \) is \( \alpha \)-admissible when:
If $b_k = a_k$, then $b_{k+1} = 0$.

(iii) for infinitely many odd and even indexes $k$ we have $b_k \leq a_k - 1$.

We then expand this definition to the finite digit sequence $\langle b_k \rangle_1^\ell$ with $\ell < \infty$ and $b_\ell \geq 1$ by testing these conditions against the infinite sequence $\langle b_k \rangle_1^\infty$ obtained by letting $b_k := 0$ for all $k \geq \ell + 1$.

**Theorem 3.1.** For all $\ell \in \mathbb{N}^\infty$ and $\alpha$–admissible digit sequences $\langle b_k \rangle_1^\ell$, we have $\sum_{k=1}^\ell b_k |\theta_{k-1}| \in (0, 1)$. Furthermore, for every real number $\beta \in (0, 1)$, there exists a unique limit $\ell \in \mathbb{N}^\infty$ and an $\alpha$–admissible digit sequence $\langle b_k \rangle_1^\ell$ (with $b_\ell \geq 1$ when $\ell$ is finite) such that $\beta = \sum_{k=1}^\ell b_k |\theta_{k-1}|$.

**Proof.** Given a limit $\ell \in \mathbb{N}^\infty$ and an $\alpha$–admissible digit sequence $\langle b_k \rangle_1^\ell$, we first show that $\sum_{k=1}^\ell b_k |\theta_{k-1}| \in (0, 1)$. When $\ell = 0$ we obtain the vacuous expansion of nullity and when $1 \leq \ell < \infty$ we first pad this sequence with a tail of zeros and obtain the $\alpha$–admissible sequence $\langle b_k \rangle_1^\infty$. If $b_1 \leq a_1 - 1$ then we use the identity (12) as well as condition–(i) to obtain the inequality

$$0 < \sum_{k=1}^\ell b_k |\theta_{k-1}| \leq \sum_{k=1}^\infty b_k |\theta_{k-1}| \leq (a_1 - 1)\theta_0 + \sum_{k=2}^\infty b_k |\theta_{k-1}| < (a_1 - 1)\theta_0 + \sum_{k=2}^\infty a_k |\theta_{k-1}|$$

$$= (a_1 - 1)\theta_0 - a_1\theta_0 + \sum_{k=2}^\infty a_k |\theta_{k-1}| = (a_1 - 1)\theta_0 - a_1\theta_0 + (1 + \alpha) = 1.$$
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\[ = (1 - |\theta_{2n-1}|) + (|\theta_{2n-1}| - |\theta_{2n+1}|) - |\theta_{2n}| + (|\theta_{2n}| + |\theta_{2n+1}|) = 1 \]

and conclude that \( \sum_{k=1}^{\ell} b_k |\theta_{k-1}| \in (0, 1). \)

Given \( \beta \in (0, 1) \) we obtain the limit \( \ell \) and the sequence \( \langle b_k \rangle_1 \) using the following iteration scheme:

**Algorithm 2:** Absolute Ostrowski Expansion

- **input:** the base \( \alpha \in (0, 1) \setminus \mathbb{Q} \), the initial seed \( \beta \in (0, 1) \)
- **output:** the limit \( \ell \in \mathbb{Z}_{\geq 0} \), the \( \alpha \)-admissible digit sequence \( \langle b_k \rangle_1 \)

1. Use algorithm 1 and formula (5) to obtain the sequence \( \langle |\theta_k| \rangle_0 \);  
2. Set \( \beta_0 := \beta, \ell := \infty, k = 1; \)
3. While \( \beta_{k-1} > 0 \) do  
   4. Set \( b_k := \lfloor \beta_{k-1}/|\theta_{k-1}| \rfloor; \)
   5. Set \( \beta_k := \beta_{k-1} - b_k|\theta_{k-1}|; \)
   6. Set \( k := k + 1; \)
4. Set \( \ell := k - 1; \)

Since \( \beta_0 > 0 \), we must have \( b_k \geq 1 \) at least once. The assignment of \( b_k \) in line–4 yields the inequality

\[ b_k |\theta_{k-1}| \leq \beta_{k-1} < (b_k + 1)|\theta_{k-1}|, \quad 1 \leq k < \ell + 1. \quad (16) \]

By the assignments of line–4 and line–5, we have

\[ \frac{\beta_{k-1}}{|\theta_{k-1}|} = b_k + \frac{\beta_k}{|\theta_{k-1}|} = \left| \frac{\beta_{k-1}}{|\theta_{k-1}|} \right| + \frac{\beta_k}{|\theta_{k-1}|}, \quad 1 \leq k < \ell + 1, \]

that is, \( b_k \) and \( \beta_k \) are the quotient and remainder of the division of \( \beta_{k-1} \) by \( |\theta_{k-1}| \), hence \( \beta_k < |\theta_{k-1}|. \) This inequality and the inequalities (9) and (16) imply that

\[ b_k |\theta_{k-1}| \leq \beta_{k-1} < |\theta_{k-2}| < (a_k + 1)|\theta_{k-1}|, \quad 1 \leq k < \ell + 1. \]

Then for all \( k \) we have \( 0 \leq b_k \leq a_k \) and, since the sequence \( \langle |\theta_k| \rangle_0 \) is strictly decreasing to zero, we must also have \( b_k \geq 1 \) at least once, thus satisfying condition–(i). A simple comparison of the the sum

\[ \beta = \beta_0 = b_1|\theta_0| + \beta_1 = b_1|\theta_0| + b_2|\theta_1| + \beta_2 = \ldots = \sum_{k=1}^{\ell} b_k |\theta_{k-1}| \]

to the convergent series in proposition 2.1 establishes its convergence and confirms that \( \beta = \sum_{k=1}^{\ell} b_k |\theta_{k-1}|. \) Furthermore, the Archimedean property of the field of real numbers asserts the uniqueness of the quotient \( b_k \) and remainder \( \beta_k \) in each iteration. Since this iteration terminates precisely when \( \ell \) is finite, \( \beta_{\ell-1} > 0 \) and \( \beta_{\ell} = 0, \) the limit \( \ell \) must be unique with \( b_\ell \geq 1 \) whenever it is finite.
To establish condition–(ii), suppose \( b_k = a_k \). Then we use the recursion formula (5) and the iterative definitions of \( b_k \) and \( \beta_k \) in line–4 and line–5 to obtain the inequality

\[
\beta_k = -b_k|\theta_{k-1}| + \beta_{k-1} = -a_k|\theta_{k-1}| + \beta_{k-1} = |\theta_k| - |\theta_{k-2}| + \beta_{k-1}
\]

\[
= |\theta_k| - |\theta_{k-2}| - b_k|\theta_{k-2}| + |\beta_{k-2}| < |\theta_k| - (b_{k-1} + 1)|\theta_{k-2}| + (b_{k-1} + 1)|\theta_{k-2}| = |\theta_k|.
\]

Thus \( \beta_k/|\theta_k| < 1 \) so that in line–4 of the next iteration we must assign \( b_{k+1} := 0 \) as desired.

Finally, to establish condition–(iii), we assume by contradiction that \( b_k \leq a_k - 1 \) for only finitely many odd indexes \( k \). Then we must have \( \ell = \infty \) and there is some index \( n \geq 1 \) for which \( b_{2k+1} = a_{2k+1} \) for all \( k \geq n \). After we apply algorithm (2) to the inputs \( \alpha := \alpha_{n-1} \) and \( \beta := \beta_{n-1} \), we use the unitary representation (15) to arrive at the contradiction

\[
1 = \sum_{k=0}^{\infty} a_{2k+1}|\theta_{2k}| = \sum_{k=0}^{\infty} b_{2k+1}|\theta_{2k}| \leq \sum_{k=1}^{\infty} b_k|\theta_{k-1}| = \beta < 1.
\]

Thus \( b_k \leq a_k - 1 \) for infinitely many odd indexes \( k \). To show this is true for infinitely many even indexes as well, we simply rewrite \( \beta_0 := \beta_1 \) so that all even indexes now become odd and repeat the previous argument.

**Corollary 3.2.** Every real number \( r \) can be uniquely expanded base–\( \alpha \) as

\[
r = \sum_{k=0}^{\ell} b_k|\theta_{k-1}|,
\]

where \( \ell \in \mathbb{Z}_{\geq 0}, b_0 \in \mathbb{Z} \) and \( \langle b_k \rangle \) is an \( \alpha \)-admissible digit sequence.

**Proof.** If \( r \) is an integer we set \( \ell := 0, b_0 := r \) so that, by the definition of \( \theta_{-1} = -1 \) in the recursive formula (5), we obtain the vacuous expansion \( r = b_0|\theta_{-1}| \). Furthermore, since \( \sum_{k=1}^{\ell} b_k|\theta_{k-1}| \in (0, 1) \) for all \( \alpha \)-admissible digit sequences \( \langle b_k \rangle \), this expansion is unique. Otherwise, we set \( b_0 := \lfloor r \rfloor \) and apply the theorem to \( \beta_0 := r - b_0|\theta_{-1}| = r - \lfloor r \rfloor \in (0, 1) \) and obtain the desired expansion. If \( \langle b'_k \rangle \) is another \( \alpha \)-expansion for \( r \), then \( \sum_{k=1}^{\ell} b'_k|\theta_{k-1}| \in (0, 1) \), hence we must have \( b'_0 = b_0 = \lfloor r \rfloor \). The uniqueness of this expansion now guarantees that \( \ell = \ell' \) and \( b_k = b'_k \) for all \( 1 \leq k < \ell \).

### 4 The Alternating Ostrowski Expansion

The base–\((-\alpha)\) Alternating Ostrowski Expansion is a sum of the form \( \sum_{k=1}^{\infty} d_k\theta_{k-1} \), where \( 0 \leq d_k \leq a_k \) along with all its finite truncations. As in the absolute case, uniqueness is not guaranteed. For instance, after setting \( \ell := \infty, c_1 := a_1 - 1, c_{2k+1} := a_{2k+1} \) and \( c_{2k} = 0 \) for all \( k \), we use the definition (4) of \( \theta_0 := \alpha \) and the identity (15) to see that

\[
\sum_{k=1}^{\ell} c_k\theta_{k-1} = \sum_{k=0}^{\infty} c_{2k+1}\theta_{2k} = \sum_{k=0}^{\infty} a_{2k+1}\theta_{2k} - \theta_0 = 1 - \alpha
\]
is a different expansion than the one in the identity (13). To achieve uniqueness, we will require the digit sequence to satisfy a refinement of the Markov Conditions. Given \( \ell \in \mathbb{N}^\infty \) and a sequence \( \langle c_k \rangle_1^\ell \), we say this sequence is \(( -\alpha )\)-admissible when it is \( \alpha \)-admissible and:

(i) if \( c_{k+1} = 0 \) then \( c_k = a_k \) for all \( 1 \leq k < \ell - 1 \).

(ii) if \( \ell = \infty \) then \( c_k \geq 1 \) for infinitely many odd and even indexes \( k \).

**Theorem 4.1.** For all \( \ell \in \mathbb{Z}_{\geq 0}^\infty \) and \(( -\alpha )\)-admissible digit sequences \( \langle c_k \rangle_1^\ell \), we have \( \sum_{k=1}^\infty c_k \theta_{k-1} \in ( -\alpha, 1 ) \). Furthermore, for every real number \( \gamma \in ( -\alpha, 1 ) \), there exists a unique limit \( \ell \in \mathbb{Z}_{\geq 0}^\infty \) and a \(( -\alpha )\)-admissible digit sequence \( \langle c_k \rangle_1^\ell \) (with \( c_\ell \geq 1 \) when \( \ell \) is finite) such that \( \gamma = \sum_{k=1}^\ell c_k \theta_{k-1} \).

**Proof.** Given a limit \( \ell \in \mathbb{N}^\infty \) and a \(( -\alpha )\)-admissible digit sequence \( \langle c_k \rangle_1^\ell \), we use condition–(i), condition–(iii) and the identities (8), (14) and (15) to obtain the inequality

\[
-\alpha = \sum_{k=1}^\infty a_{2k} \theta_{2k-1} < \sum_{k=1}^\ell c_k \theta_{k-1} \leq \sum_{k=0}^\ell c_{2k+1} \theta_{2k} < \sum_{k=0}^\infty a_{2k+1} \theta_{2k} = 1
\]

and assert that \( \sum_{k=1}^\ell c_k \theta_{k-1} \in ( -\alpha, 1 ) \).

Define the ceiling \( \lceil r \rceil \) of the real number \( r \) to be the smallest integer larger than or equal to \( r \). Given \( \gamma \in ( -\alpha, 1 ) \), we obtain the index \( \ell \) and the sequence \( \langle c_k \rangle_1^\ell \) using the following iteration scheme:

**Algorithm 3:** Alternating Ostrowski Expansion

```plaintext
input: the base \( \alpha \in (0, 1) \setminus \mathbb{Q} \), the initial seed \( \gamma \in ( -\alpha, 1 ) \\
output: the limit \( \ell \in \mathbb{Z}_{\geq 0}^\infty \), the \(( -\alpha )\)-admissible digit sequence \( \langle c_k \rangle_1^\ell \\
1 use algorithm [1] and formula (5) to obtain the sequence \( \langle \theta_k \rangle_0^\infty \); \\
2 set \( \gamma_0 := \gamma, \ell := \infty, k := 1; \\
3 \text{while } \gamma_{k-1} \neq 0 \text{ do} \\
4 \quad \text{set } c_k := \min \{ \lceil \gamma_{k-1}/\theta_{k-1} \rceil, a_k \}; \\
5 \quad \text{set } \gamma_k := \gamma_{k-1} - c_k \theta_{k-1}; \\
6 \quad \text{set } k := k+1; \\
7 \text{end} \\
8 \text{set } \ell := k-1;
```

This iteration may terminate with a positive finite value for \( \ell \) or continue indefinitely in which case \( \ell = \infty \). We define the parity \( \rho(k) \) of \( k \) to be one (zero) precisely when \( k \) is odd (even), that is, \( \rho(k) := \lceil k/2 \rceil - \lfloor k/2 \rfloor \). We will first prove by induction that

\[
\gamma_k \in ( -\theta_{k-\rho(k)}, -\theta_{k-1+\rho(k)} ), \quad 0 \leq k < \ell.
\]  

(17)

By the definitions (5) of \( \theta_{-1} = -1 \) and \( \theta_0 = \alpha \), the definition of \( \gamma \) in the hypothesis and the assignment of line–2, we have \( \gamma_0 = \gamma \in ( -\alpha, 1 ) = ( -\theta_0, -\theta_{-1} ) \), hence the base case \( k = 0 \)
holds. After we assume its validity for $k - 1$, we prove it is also true for $k$ by considering the two cases $\rho(k) \in \{0, 1\}$ separately.

- If $\rho(k) = 0$, then, by the induction assumption, we have $-\theta_{k-2} < \gamma_{k-1} < -\theta_{k-1}$. If in line-4, we set $c_k = \lfloor \gamma_{k-1}/\theta_{k-1} \rfloor$, then from from formula (8) and the assignments of line-5 we obtain

$$\frac{\gamma_k}{|\theta_{k-1}|} = -\frac{\gamma_k}{\theta_{k-1}} = c_k - \frac{\gamma_{k-1}}{\theta_{k-1}} = \left\lfloor \frac{\gamma_{k-1}}{\theta_{k-1}} \right\rfloor - \frac{\gamma_{k-1}}{\theta_{k-1}} \geq 0,$$

hence $\gamma_k \geq 0$. We use this inequality to obtain

$$-1 < \frac{\gamma_{k-1}}{\theta_{k-1}} - \left\lfloor \frac{\gamma_{k-1}}{\theta_{k-1}} \right\rfloor = \frac{\gamma_{k-1}}{\theta_{k-1}} - c_k = \gamma_k = -\frac{\gamma_k}{|\theta_{k-1}|}$$

and conclude that $0 \leq \gamma_k < -\theta_{k-1}$. If $c_k = a_k \leq \lfloor \gamma_{k-1}/\theta_{k-1} \rfloor - 1$, then

$$\frac{\gamma_k}{|\theta_{k-1}|} = a_k - \frac{\gamma_{k-1}}{\theta_{k-1}} \leq \left\lfloor \frac{\gamma_{k-1}}{\theta_{k-1}} \right\rfloor - 1 - \frac{\gamma_{k-1}}{\theta_{k-1}} \leq 0,$$

hence $\gamma_k \leq 0$. The recursion formula (5), the assignment of line-5 and the induction assumption will now yield

$$0 \leq -\gamma_k = a_k \theta_{k-1} - \gamma_{k-1} < a_k \theta_{k-1} + \theta_{k-2} = \theta_k.$$

Conclude that $-\theta_k < \gamma_k < -\theta_{k-1}$, which is the desired statement for the even index $k$.

- If $\rho(k) = 1$, then, by the induction assumption, we have $-\theta_{k-1} < \gamma_{k-1} < -\theta_{k-2}$. If in line-4, we set $c_k = \lceil \gamma_{k-1}/\theta_{k-1} \rceil$, then from from formula (8) and the assignments of line-5 we obtain

$$\frac{\gamma_k}{|\theta_{k-1}|} = \frac{\gamma_k}{\theta_{k-1}} = \frac{\gamma_{k-1}}{\theta_{k-1}} - c_k = \gamma_k = -\frac{\gamma_k}{|\theta_{k-1}|}$$

hence $\gamma_k \leq 0$. We use this inequality to obtain

$$-1 < \frac{\gamma_{k-1}}{\theta_{k-1}} - \left\lceil \frac{\gamma_{k-1}}{\theta_{k-1}} \right\rceil = \frac{\gamma_{k-1}}{\theta_{k-1}} - c_k = \gamma_k = -\frac{\gamma_k}{|\theta_{k-1}|}$$

and conclude that $-\theta_{k-1} < \gamma_k \leq 0$. If $c_k = a_k \leq \lceil \gamma_{k-1}/\theta_{k-1} \rceil - 1$, then

$$\frac{\gamma_k}{|\theta_{k-1}|} = \frac{\gamma_{k-1}}{\theta_{k-1}} - a_k \geq \frac{\gamma_{k-1}}{\theta_{k-1}} - \left\lceil \frac{\gamma_{k-1}}{\theta_{k-1}} \right\rceil + 1 \geq 0,$$

hence $\gamma_k \geq 0$. The recursion formula (5), the assignment of line-5 and the induction assumption will now yield

$$0 \leq \gamma_k = \gamma_{k-1} - a_k \theta_{k-1} \leq -\theta_{k-2} - a_k \theta_{k-1} = -\theta_k.$$

Conclude that $-\theta_{k-1} < \gamma_k < -\theta_k$ for this case, which is the desired statement for the odd index $k$. This concludes the proof and asserts the validity of formula (17).
To establish condition–(i), assume that $\ell > 0$. Clearly by its definition in line–4, we have $c_k \leq a_k$. Furthermore, by formula \((\ref{17})\) we either have have $-|\theta_k| = -\theta_k < \gamma_k$ when $\rho(k) = 0$ or $\gamma_k < -\theta_k = -|\theta_k|$ when $\rho(k) = 1$. In either case we see that $\gamma_k/\theta_k > -1$, so by the definition of $c_k$ in line–4 we conclude that $0 \leq c_k \leq a_k$ for all $k$ as desired. A simple comparison of the absolute terms in the the sum

$$
\gamma = \gamma_0 = c_1\theta_0 + \gamma_1 = c_1\theta_0 + c_2\theta_1 + \gamma_2 = \ldots = \sum_{k=1}^{\ell} c_k\theta_{k-1},
$$
to the convergent series in proposition \([2.1]\) establishes its convergence and confirms that $\gamma = \sum_{k=1}^{\ell} c_k\theta_{k-1}$.

To prove uniqueness, we split $\gamma$ into its positive and negative parts and invoke the uniqueness of the Absolute Ostrowski Expansion. More precisely, suppose $\langle c'_k \rangle_1^\ell$ is a $(\alpha)$–admissible sequence such that $\gamma = \sum_{k=1}^{\ell} c'_k\theta_{k-1}$. We first pad this sequence with an infinite tail of zeros whenever $\ell$ is finite and then define the terms

$$
b^0_k := \begin{cases} c_k/2, & \rho(k) = 0 \\ 0, & \rho(k) = 1 \end{cases}, \quad b^1_k := \begin{cases} 0, & \rho(k) = 0 \\ c_k/2, & \rho(k) = 1 \end{cases}
$$

and the factors

$$
\gamma^+ := \sum_{k=0}^{\infty} c_{2k+1}\theta_{2k} = \sum_{k=1}^{\infty} b^1_k|\theta_{k-1}|, \quad \gamma^- := -\sum_{k=1}^{\lceil \ell/2 \rceil} c_{2k}\theta_{2k-1} = \sum_{k=1}^{\infty} b^0_k|\theta_{k-1}|,
$$

so that $\gamma = \gamma^+ - \gamma^-$. If $\langle c'_k \rangle_1^\ell$ is another $(\alpha)$–admissible sequence such that $\gamma = \sum_{k=1}^{\ell} c'_k\theta_{k-1}$, then, we also pad it with a tail of zeros when applicable. Since both the sequences $\langle b^0_k \rangle_1^\infty$ and $\langle b^1_k \rangle_1^\infty$ are $\alpha$–admissible, the uniqueness of the absolute expansion implies that

$$
\gamma^+ = \sum_{k=0}^{\infty} c_{2k+1}|\theta_{2k}| = \sum_{k=1}^{\infty} b^1_k|\theta_{k-1}| = \sum_{k=0}^{\infty} c'_{2k+1}|\theta_{2k}|
$$
and

$$
\gamma^- = \sum_{k=0}^{\infty} c_{2k}|\theta_{2k-1}| = \sum_{k=1}^{\infty} b^0_k|\theta_{k-1}| = \sum_{k=0}^{\infty} c'_{2k}|\theta_{2k-1}|,
$$
hence $\langle c_k \rangle_1^\infty = \langle c'_k \rangle_1^\infty$. Furthermore, we must have $\ell = \ell'$ for otherwise we will obtain two distinct representations for either

$$
\gamma^+ = \sum_{k=0}^{\lceil \ell/2 \rceil} c_{2k+1}\theta_{2k} = \sum_{k=0}^{\lceil \ell'/2 \rceil} c_{2k+1}\theta_{2k} \quad \text{or} \quad \gamma^- = \sum_{k=1}^{\lceil \ell/2 \rceil} c_{2k}\theta_{2k-1} = \sum_{k=1}^{\lceil \ell'/2 \rceil} c_{2k}\theta_{2k-1},
$$
contrary to the uniqueness of the absolute expansion. Conclude that this alternating expansion is also unique.
To establish condition–(ii), suppose \( c_{k+1} = 0 \). Then by its assignment in line–4, we must have
\[
\frac{\gamma_k}{\theta_k} < \left\lfloor \frac{\gamma_k}{\theta_k} \right\rfloor = c_{k+1} = 0
\]
so that, using formulas (8), we obtain that \( \gamma_k > 0 \) precisely when \( k \) is odd. Since \( \theta_{k-1} > 0 \) precisely when \( k \) is even, by the assignment of line–5, we will have
\[
c_k = \frac{c_k \theta_{k-1}}{\theta_{k-1}} < \frac{\gamma_k + c_k \theta_{k-1}}{\theta_{k-1}} = \frac{\gamma_{k-1}}{\theta_{k-1}} \leq \left\lfloor \frac{\gamma_{k-1}}{\theta_{k-1}} \right\rfloor.
\]
Therefore, by its definition in line–4, we conclude that \( c_k = a_k \). Finally, to establish condition–(iii), we assume by contradiction that \( c_k \geq 1 \) for only finitely many odd indexes \( k \). Then we must have \( \ell = \infty \) and there is some index \( n \geq 1 \) for which \( c_{2k+1} = 0 \) for all \( k \geq n \). Then by condition-(ii) we must have \( c_{2k} = a_{2k} \) for all \( k \geq n \). After we apply algorithm 3 to the inputs \( \alpha := a_{n-1} \) and \( \gamma := \gamma_{n-1} \), we use the self representation (14) to arrive at the contradiction
\[
-\alpha < \gamma = \sum_{k=1}^{\infty} c_k \theta_{k-1} = \sum_{k=0}^{\infty} c_{2k+1} \theta_{2k} + \sum_{k=1}^{\infty} c_{2k} \theta_{2k-1} = \sum_{k=1}^{\infty} a_{2k} \theta_{2k-1} = -\alpha.
\]
If \( c_k \geq 1 \) for only finitely many even indexes \( k \), then we must have \( \ell = \infty \) and there is some index \( n \geq 1 \) for which \( c_{2k} = 0 \) for all \( k \geq n \). Then by condition-(ii) we must have \( c_{2k-1} = a_{2k-1} \) for all \( k \geq n \). After we apply algorithm 3 to the inputs \( \alpha := a_{n-1} \) and \( \gamma := \gamma_{n-1} \), we use the unitary representation (15) to arrive at the contradiction
\[
1 = \sum_{k=0}^{\infty} a_{2k+1} \theta_{2k} = \sum_{k=0}^{\infty} c_{2k+1} \theta_{2k} + \sum_{k=1}^{\infty} c_{2k} \theta_{2k-1} = \sum_{k=1}^{\infty} c_k \theta_{k-1} = \gamma < 1.
\]

**Corollary 4.2.** Every real number \( r \) can be uniquely expanded base–\((-\alpha)\) as
\[
r = \sum_{k=0}^{\ell} c_k \theta_{k-1},
\]
where \( \ell \in \mathbb{Z}_{\geq 0} \), \( c_0 \in \mathbb{Z} \) and \( \langle c_k \rangle_{\ell} \) is a \((-\alpha)\)-admissible digit sequence with \( c_1 \geq 1 \).

**Proof.** If \( r \) is an integer we set \( \ell := 0 \), \( c_0 := -r \) and use the definition of \( \theta_{-1} := -1 \) in the recursive formula (5) to obtain the unique vacuous expansion \( r = c_0 \theta_{-1} \). Otherwise, we set \( c_0 := -\lfloor r \rfloor \) and apply the theorem to \( \gamma_0 := r - c_0 \theta_{-1} = r - \lfloor r \rfloor \in (0, 1) \). Since \( \gamma_0 / \theta_0 > 0 \), by its definition in line–4 of algorithm 3 we set \( c_1 \geq 1 \) and derive the desired expansion. If \( \langle c_k' \rangle_{\ell} \) is another \((-\alpha)\)-admissible with \( c_1' \geq 1 \) then so is the sequence obtained from the concatenation of \( \langle c_1' \rangle_{1} \) with \( \langle c_k' \rangle_{\ell} \). If we are further supplied with an integer \( c_0' \) such that \( r = \sum_{k=0}^{\ell} c_k' \theta_{k-1} \), then by the theorem we have
\[
-\alpha = -\theta_0 < (c_1' - 1)\theta_0 + \sum_{k=2}^{\ell} c_k' \theta_{k-1} < 1,
\]
hence \( \sum_{k=0}^{\ell} c_k' \theta_{k-1} \in (0, 1) \). Thus we must have \( c_0' = c_0 = -\lfloor r \rfloor \) and then the uniqueness for this expansion guarantees that \( \ell = \ell' \) and \( c_k = c_k' \) for all \( 1 \leq k < \ell \).
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