Partitions and functional Santaló inequalities

Joseph Lehec ∗

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Abstract

We give a direct proof of a functional Santaló inequality due to Fradelizi and Meyer. This provides a new proof of the Blaschke-Santaló inequality. The argument combines a logarithmic form of the Prékopa-Leindler inequality and a partition theorem of Yao and Yao.

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Introduction

If $A$ is a subset of $\mathbb{R}^n$ we let $A^\circ$ be the polar of $A$:

$$A^\circ = \{ x \in \mathbb{R}^n \mid \forall y \in A, x \cdot y \leq 1 \},$$

where $x \cdot y$ denotes the scalar product of $x$ and $y$. We denote the Euclidean norm of $x$ by $|x| = \sqrt{x \cdot x}$. Let $K$ be a subset of $\mathbb{R}^n$ with finite measure. The Blaschke-Santaló inequality states that there exists a point $z$ in $\mathbb{R}^n$ such that

$$\text{vol}_n(K) \text{vol}_n(K - z)^\circ \leq \text{vol}_n(B^n_2) \text{vol}_n(B^n_2)^\circ = v_n^2,$$

where $\text{vol}_n$ stands for the Lebesgue measure on $\mathbb{R}^n$, $B^n_2$ for the Euclidean ball and $v_n$ for its volume. It was first proved by Blaschke in dimension 2 and 3 and Santaló [7] extended the result to any dimension. We say that an element $z$ of $\mathbb{R}^n$ satisfying (1) is a Santaló point for $K$.

Throughout the paper a weight is an measurable function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $n$, the function $x \in \mathbb{R}^n \mapsto \rho(|x|)$ is integrable.

Definition 1. Let $f$ be a non-negative integrable function on $\mathbb{R}^n$, and $\rho$ be a weight. We say that $c \in \mathbb{R}^n$ is a Santaló point for $f$ with respect to $\rho$ if the following holds: for all non-negative Borel function $g$ on $\mathbb{R}^n$, if

$$\forall x, y \in \mathbb{R}^n, \quad x \cdot y \geq 0 \Rightarrow f(c + x)g(y) \leq \rho(\sqrt{x \cdot y})^2,$$

then

$$\int_{\mathbb{R}^n} f(x) \, dx \int_{\mathbb{R}^n} g(y) \, dy \leq (\int_{\mathbb{R}^n} \rho(|x|) \, dx)^2.$$

∗LAMA (UMR CNRS 8050) Université Paris-Est
Heuristically, the choice of the weight $\rho$ gives a notion of duality (or polarity) for non-negative functions. Our purpose is give a new proof of the following theorem, due to Fradelizi and Meyer [4].

**Theorem 2.** Let $f$ be non-negative and integrable. There exists $c \in \mathbb{R}^n$ such that $c$ is a Santaló point for $f$ with respect to any $\rho$. Moreover, if $f$ is even then $0$ is a Santaló point for $f$ with respect to any weight.

The even case goes back to Keith Ball in [2], this was the first example of a functional version of (1). Later on, Artstein, Klartag and Milman [1] proved that any integrable $f$ admits a Santaló point with respect to the weight $t \mapsto e^{-t^2/2}$. Moreover in this case the barycenter of $f$ suits (see [5]). Unfortunately this is not true in general; indeed, taking

$$f = 1_{(-2,0)} + 41_{(0,1)}$$

it is easy to check that $f$ has its barycenter at $0$, and that $f(s)g(t) \leq \rho(\sqrt{st})^2$ as soon as $st \geq 0$. However

$$\int_{\mathbb{R}} f(s) \, ds \int_{\mathbb{R}} g(t) \, dt = \frac{9}{4} > 4 = \left( \int_{\mathbb{R}} \rho(|r|) \, dr \right)^2.$$

To prove the existence of a Santaló point, the authors of [4] use a fixed point theorem and the usual Santaló inequality (for convex bodies). Our proof is direct, in the sense that it does not use the Blaschke Santaló inequality; it is based on a special form of the Prékopa-Leindler inequality and on a partition theorem due to Yao and Yao [8].

Lastly, the Blaschke-Santaló inequality follows very easily from Theorem 2: we let the reader check that if $c$ is a Santaló point for $1_K$ with respect to the weight $1_{[0,1]}$ then $c$ is a Santaló point for $K$.

1 **Yao-Yao partitions**

In the sequel we consider real affine spaces of finite dimension. If $E$ is such a space we denote by $\tilde{E}$ the associated vector space. We say that $\mathcal{P}$ is a partition of $E$ if $\bigcup\mathcal{P} = E$ and if the interiors of two distinct elements of $\mathcal{P}$ do not intersect. For instance, with this definition, the set $\{(-\infty,a], [a,+\infty)\}$ is a partition of $\mathbb{R}$. We define by induction on the dimension a class of partitions of an $n$-dimensional affine space.

**Definition 3.** If $E = \{c\}$ is an affine space of dimension 0, the only possible partition $\mathcal{P} = \{c\}$ is a Yao-Yao partition of $E$, and its center is defined to be $c$.

Let $E$ be an affine space of dimension $n \geq 1$. A set $\mathcal{P}$ is said to be a Yao-Yao partition of $E$ if there exists an affine hyperplane $F$ of $E$, a vector $v \in \tilde{E} \setminus \tilde{F}$ and two Yao-Yao partitions $\mathcal{P}_+$ and $\mathcal{P}_-$ of $F$ having the same center $c$ such that

$$\mathcal{P} = \{ A + \mathbb{R}_-v \mid A \in \mathcal{P}_- \} \cup \{ A + \mathbb{R}_+v \mid A \in \mathcal{P}_+ \},$$

The center of $\mathcal{P}$ is then $x$. 
If $A$ is a subset of $E$ we denote by $\text{pos}(A)$ the positive hull of $A$, that is to say the smallest convex cone containing $A$.

A Yao-Yao partition $\mathcal{P}$ of an $n$-dimensional space $E$ has $2^n$ elements and for each $A$ in $\mathcal{P}$ there exists a basis $v_1, \ldots, v_n$ of $E$ such that

$$A = c + \text{pos}(v_1, \ldots, v_n),$$

where $c$ is the center of $\mathcal{P}$. Indeed, assume that $\mathcal{P}$ is defined by $F, v, \mathcal{P}_+, \mathcal{P}_-$ (see Definition 3). Let $A \in \mathcal{P}_+$ and assume inductively that there is a basis $v_1, \ldots, v_{n-1}$ of $E$ such that $A = c + \text{pos}(v_1, \ldots, v_{n-1})$. Then $A + \mathbb{R}_+ v = c + \text{pos}(v, v_1, \ldots, v_{n-1})$.

A fundamental property of this class of partitions is the following

**Proposition 4.** Let $\mathcal{P}$ be a Yao-Yao partition of $E$ and $c$ its center. Let $\ell$ be an affine form on $E$ such that $\ell(c) = 0$. Then there exists $A \in \mathcal{P}$ such that $\ell(x) \geq 0$ for all $x \in A$. Moreover there is at most one element $A$ of $\mathcal{P}$ such that $\ell(x) > 0$ for all $x \in A \setminus \{c\}$.

**Proof.** By induction on the dimension $n$ of $E$. When $n = 0$ it is obvious, we assume that $n \geq 1$ and that the result holds for all affine spaces of dimension $n - 1$. Let $\ell$ be an affine form on $E$ such that $\ell(c) = 0$. We introduce $F, v, \mathcal{P}_+, \mathcal{P}_-$ given by Definition 3. By the induction assumption, there exists $A_+ \in \mathcal{P}_+$ and $A_- \in \mathcal{P}_-$ such that

$$\forall y \in A_+ \cup A_- \quad \ell(y) \geq 0.$$ 

If $\ell(c + v) \geq 0$ then $\ell(x + tv) \geq 0$ for all $x \in A_+$ and $t \in \mathbb{R}_+$, thus $\ell(x) \geq 0$ for all $x \in A_+ + \mathbb{R}_+ v$.

If on the contrary $\ell(c + v) \leq 0$ then $\ell(x) \geq 0$ for all $x \in A_- + \mathbb{R}_- v$, which proves the first part of the proposition. The proof of the ‘moreover’ part is similar.

The latter proposition yields the following corollary, which deals with dual cones: if $C$ is cone of $\mathbb{R}^n$ the dual cone of $C$ is by definition

$$C^* = \{y \in \mathbb{R}^n \mid \forall x \in C, \ x \cdot y \geq 0\}.$$ 

**Corollary 5.** Let $\mathcal{P}$ be a Yao-Yao partition of $\mathbb{R}^n$ centered at $0$. Then

$$\mathcal{P}^* := \{A^* \mid A \in \mathcal{P}\}$$

is also a partition of $\mathbb{R}^n$.

Actually the dual partition is also a Yao-Yao partition centered at $0$ but we will not use this fact.

**Proof.** Let $x \in \mathbb{R}^n$ and $\ell : y \in \mathbb{R}^n \mapsto x \cdot y$. By the previous proposition there exists $A \in \mathcal{P}$ such that $\ell(y) \geq 0$ for all $y \in A$. Then $x \in A^*$. Thus $\cup \mathcal{P}^* = \mathbb{R}^n$. Moreover if $x$ belongs to the interior of $A^*$, then for all $y \in A \setminus \{0\}$ we have $\ell(y) > 0$. Again by the proposition above there is at most one such $A$. Thus the interiors of two distinct elements of $\mathcal{P}^*$ do not intersect.

We now let $\mathcal{M}(E)$ be the set of Borel measure $\mu$ on $E$ which are finite and which satisfy $\mu(F) = 0$ for any affine hyperplane $F$.

**Definition 6.** Let $\mu \in \mathcal{M}(E)$, a Yao-Yao equipartition $\mathcal{P}$ for $\mu$ is a Yao-Yao partition of $E$ satisfying

$$\forall A \in \mathcal{P}, \quad \mu(A) = 2^{-n} \mu(E).$$

We say that $c \in E$ is a Yao-Yao center of $\mu$ if $c$ is the center of a Yao-Yao equipartition for $\mu$. 

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Here is the main result concerning those partitions.

**Theorem 7.** Let \( \mu \in \mathcal{M}(\mathbb{R}^n) \), there exists a Yao-Yao equipartition for \( \mu \). Moreover, if \( \mu \) is even then \( 0 \) is a Yao-Yao center for \( \mu \).

It is due to Yao and Yao [8]. They have some extra hypothesis on the measure and their paper is very sketchy, so we refer to [6] for a proof of this very statement.

2 Proof of the Fradelizi-Meyer inequality

In this section, all integrals are taken with respect to the Lebesgue measure. Let us recall the Prékopa-Leindler inequality, which is a functional form of the famous Brunn-Minkowski inequality, see for instance [3] for a proof and selected applications. If \( \varphi_1, \varphi_2, \varphi_3 \) are non-negative and integrable functions on \( \mathbb{R}^n \) satisfying

\[
\varphi_1(x) \leq \varphi_3(\lambda x + (1 - \lambda) y)
\]

for all \( x, y \) in \( \mathbb{R}^n \) and for some fixed \( \lambda \in (0, 1) \), then

\[
\left( \int_{\mathbb{R}^n} \varphi_1 \right)^\lambda \left( \int_{\mathbb{R}^n} \varphi_2 \right)^{1-\lambda} \leq \int_{\mathbb{R}^n} \varphi_3.
\]

The following lemma is a useful (see [4, 2]) logarithmic version of Prékopa-Leindler. We recall the proof for completeness.

**Lemma 8.** Let \( f_1, f_2, f_3 \) be non-negative Borel functions on \( \mathbb{R}_+^n \) satisfying

\[
f_1(x)f_2(y) \leq \left( f(\sqrt{x_1y_1}, \ldots, \sqrt{x_ny_n}) \right)^2,
\]

for all \( x, y \) in \( \mathbb{R}_+^n \). Then

\[
\int_{\mathbb{R}_+^n} f_1 \int_{\mathbb{R}_+^n} f_2 \leq \left( \int_{\mathbb{R}_+^n} f_3 \right)^2.
\]  \( \tag{6} \)

**Proof.** For \( i = 1, 2, 3 \) we let

\[
g_i(x) = f_i(e^{x_1}, \ldots, e^{x_n})e^{x_1+\cdots+x_n}.
\]

Then by change of variable we have

\[
\int_{\mathbb{R}^n} g_i = \int_{\mathbb{R}_+^n} f_i.
\]

On the other hand the hypothesis on \( f_1, f_2, f_3 \) yields

\[
g_1(x)g_2(y) \leq g_3(\frac{x+y}{2}),
\]

for all \( x, y \) in \( \mathbb{R}^n \). Then by Prékopa-Leindler

\[
\int_{\mathbb{R}^n} g_1 \int_{\mathbb{R}^n} g_2 \leq \left( \int_{\mathbb{R}^n} g_3 \right)^2.
\]

**Theorem 9.** Let \( f \) be a non-negative Borel integrable function on \( \mathbb{R}^n \), and let \( c \) be a Yao-Yao center for the measure with density \( f \). Then \( c \) is a Santaló point for \( f \) with respect to any weight.
Combining this result with Theorem 7 we obtain a complete proof of the Fradelizi-Meyer inequality.

**Proof.** It is enough to prove that if 0 is a Yao-Yao center for \( f \) then 0 is a Santaló point. Indeed, if \( c \) is a center for \( f \) then 0 is a center for
\[
f_c : x \mapsto f(c + x).
\]
And if 0 is a Santaló point for \( f_c \) then \( c \) is a Santaló point for \( f \).

Let \( \mathcal{P} \) be a Yao-Yao equipartition for \( f \) with center 0. Let \( g \) and \( \rho \) be such that (2) holds (with \( c = 0 \)). Let \( A \in \mathcal{P} \), by (1), there exists an operator \( T \) on \( \mathbb{R}^n \) with determinant 1 such that \( A = T(\mathbb{R}^n_+) \). Let \( S = (T^{-1})^* \), then \( S(\mathbb{R}^n_+) = A^* \). Let \( f_1 = f \circ T, f_2 = g \circ S \) and \( f_3(x) = \rho(|x|) \). Since for all \( x, y \) we have \( T(x) \cdot S(y) = x \cdot y \), we get from (2)
\[
f_1(x)f_2(y) \leq \rho(\sqrt{x \cdot y})^2 = f_3(\sqrt{x_1y_1}, \ldots, \sqrt{x_ny_n})^2,
\]
for all \( x, y \) in \( \mathbb{R}^n_+ \). Applying the previous lemma we get (6). By change of variable it yields
\[
\int_A f \int_{A^*} g \leq \left( \int_{\mathbb{R}^n_+} \rho(|x|) \, dx \right)^2.
\]
Therefore
\[
\sum_{A \in \mathcal{P}} \int_A f \int_{A^*} g \leq 2^n \left( \int_{\mathbb{R}^n_+} \rho(|x|) \, dx \right)^2. \tag{7}
\]
Since \( \mathcal{P} \) is an equipartition for \( f \) we have for all \( A \in \mathcal{P} \)
\[
\int_A f = 2^{-n} \int_{\mathbb{R}^n} f.
\]
By Corollary 5, the set \( \{ A^*, A \in \mathcal{P} \} \) is a partition of \( \mathbb{R}^n \), thus
\[
\sum_{A \in \mathcal{P}} \int_A g = \int_{\mathbb{R}^n} g.
\]
Inequality (7) becomes
\[
\int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq 4^n \left( \int_{\mathbb{R}^n_+} \rho(|x|) \, dx \right)^2,
\]
and of course the latter is equal to \( (\int_{\mathbb{R}^n} \rho(|x|) \, dx)^2 \).

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