Abstract An introduction to generalized thermodynamics of quantum black holes, in the one-loop approximation, is given. The material is aimed at graduate students. The topics include: quantum evaporation of black holes, Euclidean formulation of quantum theory on black hole backgrounds, the Hartle-Hawking-Israel state, generalized entropy of a quantum black hole and its relation to the entropy of entanglement.

Keywords

Black holes, thermodynamics, statistical mechanics, quantum entanglement.

1 Introduction

Although the perturbative quantum gravity approach has a limited range of applicability, its use in the last decades led to some conceptual issues which are to be addressed in the full-fledged quantum gravity theory. The most important issues include understanding evaporation of quantum black holes and resolution of the information loss paradox, as well as finding a microscopic origin of black hole entropy.

Black holes are specific solutions of the Einstein equations,

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \]  

(1)

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which describe regions of a space-time where the gravitational field is so strong that nothing, including light signals, can escape them. The interior of a black hole is hidden from an external observer. The boundary of the unobservable region is called the horizon. In (1) we use standard notations $R_{\mu\nu}, R, T_{\mu\nu}$ for the Ricci tensor, the scalar curvature, and the stress-energy tensor of matter, respectively. $G$ is the Newton constant. The Schwarzschild and Kerr black holes are solutions to the vacuum equations (1) with $T_{\mu\nu} = 0$.

In recent years our understanding of physics near the black hole horizon received important experimental evidences on the base of direct detection of gravitational waves from binary black hole mergers [1] and observations of shadows of the supermassive black holes [2].

The perturbative quantum gravity, in this Chapter, is treated in the one-loop approximation or as a theory of free quantum fields on black hole geometries. By explaining quantum effects near black holes in these rather restricted models we come to important insights which have been a matter of intensive discussions in a large number of publications.

The concrete aim of this Chapter is to give a self-consistent introduction to generalized thermodynamics of quantum black holes, accessible to graduate students. The material is organized as follows. We start in Sec. 2 with a brief description of black hole solutions by focusing mostly on the Killing structure of the black hole horizon and near-horizon features which are needed to define the first law of black hole mechanics. Quantization of free fields on external backgrounds is presented in Sec. 3. The essence of the Hawking effect is discussed in Sec. 4, by using the so-called s-mode approximation. Thermodynamics of classical black holes is discussed in Sec. 5. The basic concept, the Hartle-Hawking-Israel state, which we use to study quantum black holes, is introduced in Sec. 6. We also give here some elements of a spectral theory of second order elliptic operators and define the Euclidean effective action. From a point of view of stationary observers quantum matter near black hole horizon is in a high-temperature regime. Hence some features of high-temperature hydrodynamics in gravitational fields are discussed in Sec. 7. Finally, in Sec. 8 we consider generalized thermodynamics of quantum black holes, and, in particular, generalized black hole entropy. Quantum corrections to the entropy are discussed in detail. We introduce the notion of entanglement entropy and show that the generalized entropy is partly related to entanglement of states across the black hole horizon. Section 9 contains concluding comments.

We include in this Chapter almost all required definitions and try to show how basic relations can be derived. We use the system of units where $\hbar = c = k_B = 1$ ($k_B$ is the Boltzmann constant), the Lorentzian signature is defined as $(-, +, +, +)$, geometrical conventions coincide with [59].
2 Necessary definitions

We start with a brief description of basic properties of black hole geometries in the near-horizon approximation. For a comprehensive introduction to black hole physics see [59],[32]. A metric of a neutral rotating black hole, which is most interesting from the point of view of physical applications, is the Kerr solution to the Einstein equations (1) in vacuum, $T_{\mu\nu} = 0$,

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{\theta\theta}d\theta^2,$$

(2)

$$g_{tt} = -\left(1 - \frac{2MG}{r}\right), \quad g_{\theta\theta} = \frac{\Sigma}{\Delta}, \quad g_{rr} = \frac{\Sigma}{\Delta}, \quad g_{t\phi} = -\frac{2MGa}{\Sigma}\sin^2\theta, \quad g_{\phi\phi} = \left((r^2 + a^2)^2 - a^2\sin^2\theta \Delta\right)\frac{\sin^2\theta}{\Sigma},$$

(3)

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2MG + a^2.$$

Metric (2) is written in the Boyer–Lindquist coordinates. The Kerr solution is asymptotically flat at large $r$. By analyzing its behavior at large $r$ one concludes that $M$ is the mass of the source, $J = Ma$ is its angular momentum. It is supposed that $MG > a$.

We also need the Schwarzschild solution, which follows from (2)-(5) when $a = 0$,

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2(\sin^2\theta d\phi^2 + d\theta^2),$$

(6)

$$-g_{tt} = g_{rr}^{-1} = 1 - \frac{2MG}{r}.$$

(7)

We denote by $\mathcal{H}$ the event horizon of a black hole. $\mathcal{H}$ is a null hypersurface located at a constant radial coordinate $r$. By the definition, the normal vector $l_{\mu}$ to a null hypersurface is null, $l^2 = 0$. For constant $r$ hypersurfaces $l_{\mu} = \delta_{\mu}^t$. Hence a surface $r = r_0$ is null if $g^{tt}(r_0) = 0$, or $\Delta(r_0) = 0$. This equation has two roots and $\mathcal{H}$ corresponds to the largest root, $r_0 = r_0 = MG + \sqrt{(MG)^2 - a^2}$. For eternal black holes (see Fig. (1)) the horizon $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$ has two components, the future, $\mathcal{H}^+$, and the past $\mathcal{H}^-$ event horizons. The future light cone of any point on $\mathcal{H}^+$ is tangent to $\mathcal{H}^+$ and is directed inside the black hole. Correspondingly, past light cones on $\mathcal{H}^-$ are directed inside the white hole. A detailed discussion of this can be found in [59],[32].

One can consider observers which rotate with respect to the Boyer–Lindquist coordinate grid (and therefore with respect to objects at the spatial infinity) with an angular coordinate velocity $d\phi/dt = -g_{t\phi}/g_{\phi\phi}$. An important property of (2) is that the only possible value for the angular velocity, when $r$ approaches $r_H$, is

$$\Omega_H = \frac{a}{a^2 + r_H^2}.$$

(8)

The parameter $\Omega_H$ is called the angular velocity of the horizon.
Fig. 1 Carter-Penrose diagrams for Minkowsky space-time (a) and for eternal Schwarzschild black hole (b). Lines with arrows are integral lines of the Killing-vector field. The Killing field in the Minkowsky space-time corresponds to Lorentz boosts. The Killing horizons $\mathcal{H}^{\pm}$ intersect at bifurcation 2-surfaces. A constant time section $\Sigma_L \cup \Sigma_R$ on the black hole diagram is the Einstein-Rosen bridge.

Although Kerr solution (2)-(5) looks complicated only few features of the near horizon geometry are required for studying quantum effects we are interested in. These features are related to the structure of time-like isometries and properties of the so called Killing observers. A vector field $\zeta^\mu(x)$ on a manifold $\mathcal{M}$ is called a Killing field if it generates isometries of $\mathcal{M}$. The Killing field obeys the Killing equation

$$\zeta^\mu_{\;\nu} + \zeta^\nu_{\;\mu} = 0 \,.$$  

(9)

For the Kerr solution there is a distinguished Killing vector field, $\zeta = \partial_t - \Omega_H \partial_\phi$, which is null on the horizon

$$\zeta^2 |_{\mathcal{H}} = 0 \,.$$  

(10)

Since $\mathcal{H}$ is the null hypersurface Eq. (10) implies that $\zeta$ is a normal vector to $\mathcal{H}$. Properties of null hypersurfaces say that integral lines of $\zeta$ on $\mathcal{H}$ are geodesics:

$$\nabla_\zeta \zeta = k \zeta \,.$$  

(11)

One can show that $\partial_\zeta k = 0$ on $\mathcal{H}$, and, as a consequence, there is a 2D section, $\mathcal{B}$ of $\mathcal{H}$ where the Killing field is zero,

$$\zeta^\mu |_{\mathcal{B} \in \mathcal{H}} = 0 \,.$$  

(12)

The parameter $k$ in (11) is called the surface gravity of the horizon, the section $\mathcal{B}$ is called the bifurcation surface of the Killing horizons. Examples of Killing fields with bifurcating horizons are shown on Fig. (1) for the case of Minkowsky space-time and eternal Schwarzschild black hole geometry.

The Killing field $\zeta$ is time-like in the wedge to the right from $\mathcal{H}^+$ and $\mathcal{H}^-$. In this region one can define a frame of reference of observers whose 4-velocities $u^\mu$ are directed along $\zeta$. 

\begin{align*}
\zeta_{\mu;\nu} + \zeta_{\nu;\mu} &= 0 \,.
\end{align*}  

(9)
\[ u^\mu = \frac{\zeta^\mu}{\sqrt{B}} \quad ; \quad B = -\zeta^2 = -g_{tt} - 2\Omega g_{\phi\phi} - \Omega^2 g_{\phi\phi} \quad . \] (13)

Such observers are called the Killing observers. For a rotating black hole the given frame of reference is defined in a domain close to \( H \), where \( B > 0 \). The congruence of the trajectories is specified [46] by the acceleration \( w^\mu \), the rotation tensor \( A_{\mu\nu} \) and the local angular velocity \( \Omega(x) \)

\[ w^\mu = \nabla u^\mu \quad , \quad A_{\mu\nu} = \frac{1}{2} h^\lambda_{\mu} h^\rho_{\nu} \left( \nabla_\rho u_\lambda - \nabla_\lambda u_\rho \right) \quad , \quad \Omega(x) = \left( \frac{1}{2} A_{\mu\nu} A^{\mu\nu} \right)^{1/2} \quad , \] (14)

where \( h^\lambda_{\mu} = \delta^\lambda_{\mu} + u_\mu u^\lambda \). Quantities (14) appear under study of quantum systems in thermal equilibrium with the black hole, see Sec. 7.2. Local angular velocity determines rotation of the Killing frame with respect to a local inertial frame.

One can use (11) to relate definition of the surface gravity to the strength of gravity near the horizon,

\[ k = \lim_{r\to r_H} \left( \frac{\sqrt{Bw^2}}{r_H - MG} \right) = \frac{r_H - MG}{a^2 + r_H^2} \quad . \] (15)

The right hand side (r.h.s.) of (15) follows from (3)-(5), (13).

It is convenient to change in the Boyer–Lindquist coordinates \( \phi \) to \( \phi' = \phi - \Omega_H t \). In the new coordinates the Killing vector field is \( \zeta = \partial_t \), that is, the Killing observers do not move with respect to the new coordinate grid. Metric (2) can be rewritten as

\[ ds^2 = -B(dt + a_i dx^i)^2 + h_{ij} dx^i dx^j \quad , \] (16)

where \( x^i = (r, \phi', \theta) \), \( a_i dx^i = a_\phi d\phi' \). The non-vanishing components of acceleration and rotation are \( w_i = (\ln B)_i/2, A_{ij} = \sqrt{B}(a_{i,j} - a_{j,i})/2 \).

One can check that at small \( r - r_H \)

\[ B \simeq 4k^2 h r_H(r - r_H) \quad , \quad h_{rr} \simeq \frac{h r_H}{r - r_H} + O((r - r_H)^2) \quad , \quad h(\theta) = \frac{\Sigma(r_H, \theta)}{2r_H \sqrt{(MG)^2 - a^2}} \quad . \] (17)

Here we took into account that \( h_{rr} = g_{rr} \). It is convenient to introduce a new coordinate \( \rho \):

\[ r - r_H \simeq \frac{\rho^2}{4r_H} \quad . \] (18)

connected with the proper distance \( L \) to the horizon

\[ L(r, \theta) = \int_{r_H}^r dr' \sqrt{h_{rr}} \simeq \sqrt{h} \rho \quad . \] (19)

In the leading approximation \( B \simeq h k^2 \rho^2 \). Since the local angular velocity vanishes near the horizon, \( \Omega = O(\rho) \), terms \( a_i dx^i \) in (16) can be neglected near \( H \).

One comes to the following form of near-horizon black hole metric (2):

\[ u^\mu = \frac{\zeta^\mu}{\sqrt{B}} \quad , \quad B = -\zeta^2 = -g_{tt} - 2\Omega g_{\phi\phi} - \Omega^2 g_{\phi\phi} \quad . \] (13)
\[ ds^2 \simeq h(\theta)(-k^2 \rho^2 dt^2 + d\rho^2) + dl^2 , \]  
\[ dl^2 = \gamma_{\theta \theta} d\theta^2 + \gamma_{\varphi \varphi} (d\varphi')^2 , \] 
where \( \gamma_{\alpha \beta} = h_{\alpha \beta}(r = r_H) \) and (21) is the metric on \( \mathcal{B} \). It can be shown that \( \gamma_{\theta \theta} = g_{\theta \theta}(r_H) \), \( \gamma_{\varphi \varphi} = g_{\varphi \varphi}(r_H) \). Therefore the space-time near \( \mathcal{H} \) has the product structure \( \mathbb{R}^2 \times \mathcal{B} \). In case of the Schwarzschild black hole \( \mathcal{B} \) is \( S^2 \), \( \mathcal{B} \) is closed and has the topology of \( S^2 \) for the Kerr solution, and \( \mathcal{B} = \mathbb{R}^2 \) for flat space-time.

The area \( A \) of \( \mathcal{B} \),
\[ A = \int d\theta d\varphi \sqrt{\text{det} \gamma_{\alpha \beta}} = 4\pi(r_H^2 + a^2) , \] 
is called the area of black hole horizon. It plays an important role in the subsequent discussions. To get (22) one should use (3),(4).

### 3 Classical fields, quantization, quasiparticles

Since stationary black holes have a universal structure near the horizon it is worth studying classical fields in this region by using, as an example, a free scalar field \( \phi \).

The field equation is
\[ (\nabla^2 - m^2)\phi = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g}g^{\mu \nu} \partial_{\nu} \phi) - m^2 \phi = 0 , \] 
where \( m \) is the mass of the field. Analysis of wave equations like (23) on the Kerr background can be found in [21].

According to (20), when \( r \) close \( r_H \), Eq. (23) reduces to:
\[ \left[ -\frac{1}{k^2 \rho^2} \partial_{\rho}^2 + \partial_{\varphi}^2 \right] \phi - L\phi = 0 . \] 
Here \( L = h(\Delta + m^2) \) and \( \Delta \) is the Laplace operator on \( \mathcal{B} \). Solutions to (24), which are interpreted by a Killing observer as excitations with energies \( \omega \), are
\[ i\partial_t \phi = \omega \phi . \] 
If \( \psi_\lambda(\theta, \varphi) \) is an eigen-function of \( L \) with an eigen-value \( \lambda \), solution to (24), (25) can be written as
\[ \phi(t, \rho, \theta, \varphi) = e^{-i\omega t} \phi_\omega(x) \psi_\lambda(\theta, \varphi) , \] 
\[ (-\partial_x^2 + V_\lambda(x)) \phi_\omega(x) = \omega^2 \phi_\omega(x) . \] 
Here \( \rho = Ce^{kx} \) and \( V_\lambda(x) = \lambda C^2 k^2 e^{2kx} \), \( C \) is a constant. Note that \( \mathcal{B} \) is compact and \( h \) is finite on \( \mathcal{B} \). One can show that eigenvalues \( \lambda \) of \( L \) are discrete and positive.

The spectrum of \( \omega \), called the single-particle energies, is defined by a Schroedinger-like equation (27) with an effective potential \( V_\lambda \). We assume that \( C > 0 \) and imply that limit \( \rho \to 0 \) corresponds to \( x \to -\infty \). A straightforward conclusion from (27)
is that fields near the horizon are effectively massless and the spectrum of $\omega$ is continuous. Effects related to non-zero mass $m$ or properties of $L$ are exponentially suppressed.

As an illustration, we give an exact form of $V_\lambda$ for the Schwarzschild black hole ($a = 0$)

$$V_\lambda(x) = B(r) \left( m^2 + \frac{\partial_r B(r)}{r} + \frac{l(l+1)}{r^2} \right), \quad x = \int^r \frac{dr'}{B(r')}, \quad \text{(28)}$$

$B(r) = 1 - r_H/r$. Eigen-values $\lambda = l(l+1)$ belong to the spectrum of a Laplacian on unit 2-sphere. At $r \to r_H$ (28) reduces to (27). Near the horizon, $x \to -\infty$, the potential is exponentially small. The mass of the field dominates, $V_\lambda(x) \simeq m^2$, as $x \to +\infty$. Potential $V_\lambda(x)$ reaches a maximum at some point $x$ whose position depends on $l$.

Since we are interested in near-horizon physics the above analysis suggests a serious technical simplification: one can focus only on dynamics in the $(t, r)$ coordinate sector where all fields are effectively massless.

For further purposes we recall basic elements of quantum field theory on classical backgrounds, for details see [41]. Take, as an example, the free scalar field with equation (23). For a pair of solutions, $f_1$, $f_2$, to (23) one introduces a relativistic inner product

$$\langle f_1, f_2 \rangle = \int \Sigma d\Sigma \mu \left( f_1^* \partial_{\mu} f_2 - \partial_{\mu} f_1^* f_2 \right), \quad \text{(29)}$$

which is taken on a Cauchy hypersurface $\Sigma$. The choice of $\Sigma$ is unimportant since (29) conserves under variations of $\Sigma$. Suppose that $f^{(\pm)}_i$ are solutions enumerated by a certain set of indices $i$, discrete or continuous, and normalized as

$$\langle f^{(\pm)}_i, f^{(\pm)}_j \rangle = \pm \delta_{ij}, \quad \langle f^{(+)j}, f^{(-)}_j \rangle = 0. \quad \text{(30)}$$

Suppose also that any solution to (23) can be written as a linear combination

$$\phi(x) = \sum_i a_i f^{(+)i}_i(x) + \sum_j b^*_j f^{(-)}_j(x). \quad \text{(31)}$$

Quantization procedure implies that $\phi$, $a_i$ and $b^*_j$ are replaced with operators $\hat{\phi}$, $\hat{a}_i$ and $\hat{b}^*_j$ with the following commutation relations:

$$[\hat{a}_i, \hat{a}^*_j] = \delta_{ij}, \quad [\hat{b}_i, \hat{b}^*_j] = \delta_{ij}. \quad \text{(32)}$$

Operators $\hat{a}^*_j, \hat{b}^*_j$ create quasiparticles which, in general, may not carry any definite energy. In stationary space-times with the Killing field $\zeta = \partial_t$ there is a special set of single-particle modes such as (compare with (25))

$$i\partial_t f^{(\pm)}_i = \pm \omega_i^{(\pm)} f^{(\pm)}_i. \quad \text{(33)}$$
ω_i are positive numbers we call single-particle energies. Correspondingly, the Killing observers interpret f_i^{(±)} as excitations with certain energies. Spectrum of single-particle energies can be found from equations like Eq. (27).

In General Relativity definition of a particle depends on the frame of reference where observations are done. To see how different choices of creation and annihilation operators are connected suppose that φ is a real field. One has two decompositions

\[ \hat{\phi}(x) = \sum_i (\hat{a}_i f_i^{(+)}(x) + \hat{\bar{a}}_i \bar{f}_i^{(-)}(x)) = \sum_j (\hat{\bar{a}}_j \bar{f}_j^{(+)}(x) + \hat{a}_j f_j^{(-)}(x)) \]

(34)
corresponding to different definitions of quasiparticles, \(\hat{a}_i, \hat{\bar{a}}_j\). Reality condition implies that \(f_i^{(-)} = (f_i^{(+)})^*, \bar{f}_i^{(-)} = (\bar{f}_i^{(+)})^*\). By using normalization conditions (30) one finds from (34) that

\[ \hat{a}_i = \langle f_i^{(+)}|, \hat{\phi} \rangle = \sum_j (c_{ij}\hat{a}_j + d_{ij}\hat{\bar{a}}_j) \]

(35)

\[ c_{ij} = \langle f_i^{(+)}|, \bar{f}_j^{(+)}) \], \[ d_{ij} = \langle f_i^{(+)}|, \bar{f}_j^{(-)} \rangle \].

(36)

Coefficients \(c_{ij}, d_{ij}\) are called the Bogoliubov coefficients. As follows from (35) the average number of quasiparticles created by \(\hat{a}_i\) is non-trivial, in general,

\[ N_i = \langle \bar{0}|\hat{a}_i^+\hat{a}_i|\bar{0}\rangle = \sum_j |d_{ij}|^2 \]

(37)
in the vacuum state \(|\bar{0}\rangle\) which does not contain particles of the other sort, \(\hat{a}_j|\bar{0}\rangle = 0\).

### 4 Quantum evaporation of a black hole

We present now a sketch of arguments demonstrating the Hawking effect of quantum evaporation of a black hole [44]. The effect is related to physics near the black hole horizon where, according to Sec. 2, the geometry has the universal form, see Eq. (20). In the leading approximation the dynamics occurs in \((t, r)\) or \((t, \rho)\) coordinate sector, while dependence on angles \(\theta\) and \(\varphi\) is unimportant.

For simplicity we consider the behavior of quantum scalar field \(\phi\) outside a spherically symmetric star which collapses and creates a Schwarzschild black hole. We assume that \(\phi\) does not depend on the angles, that is, \(\phi\) is the so called \(s\)-mode. Since mass of the field is not important near the horizon we also assume that \(\phi\) is massless.

Wave equation (23) reduces to \(\nabla^2 \phi = 0\), where \(\nabla^2\) is the operator on a 2D spacetime \(\mathcal{M}_2\). Outside the star

\[ ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)}, \quad B(r) = 1 - \frac{r_H}{r}. \]

(38)
A standard approach is to introduce ingoing, \((v, r)\), and outgoing, \((u, r)\), Eddington-Finkelstein coordinates

\[
u = t - x, \quad v = t + x, \quad x = \int dr' B(r'),
\]

\[
ds^2 = -Bdv^2 + 2dvdr = -Bdu^2 - 2dudr = -B(r)du dv.
\]

Null coordinates \(u, v\) are retarded and advanced times, respectively. Lines of constant \(u\) or \(v\) are radial rays. Past-directed null geodesics end up on the past null infinity denoted by \(\mathcal{I}^-\), future-directed null geodesics end up on the future null infinity \(\mathcal{I}^+\). These infinities can be used as parts of the Cauchy surface \(\Sigma\) where relativistic product (29) is defined. In computations \(\mathcal{I}^+\) and \(\mathcal{I}^-\) are replaced, respectively, with null surfaces \(v = C > 0\) or \(u = -C < 0\), with large \(C\). For an eternal black hole shown on Fig. 1 coordinates \((v, r)\) are continued across \(\mathcal{H}^+\) inside the black hole, the outgoing coordinates can be continued across \(\mathcal{H}^-\).

2D massless scalar fields are easy to analyze since \(M^2\) is conformally flat. The metric can be brought to the form

\[
ds^2 = -e^{2\sigma} du dv,
\]

and a general solution to the wave equation

\[
\nabla^2 \phi = -4e^{-2\sigma} \partial_u \partial_v \phi = 0
\]

is a combination of left-moving and right-moving modes, \(\phi(u, v) = \phi_R(u) + \phi_L(v)\).

Before we proceed with solutions on black hole geometries it is instructive to consider fields on 2D Minkowsky space-time with metric (40), where \(B = 1, x = r\) and \(r > 0\). A complete set of modes is

\[
f^{(+)}_{\omega}(u, v) = \frac{1}{\sqrt{4\pi \omega}} (e^{-i\omega v} - e^{-i\omega u}) , \quad \langle f^{(+)}_{\omega}, f^{(+)}_{\sigma} \rangle = \delta(\omega - \sigma).
\]

Since \(r\) is a radial coordinate with the center at \(r = 0\) each mode is a combination of left-moving and right-moving waves with the Dirichlet boundary condition, \(f^{(+)}_{\omega} = 0\), at \(r = 0\). In these coordinates waves coming from \(\mathcal{I}^-\) are reflected from the center and travel, unchanged, to \(\mathcal{I}^+\), see Fig. 2. Due to this reflection condition null infinities \(\mathcal{I}^\pm\) are the Cauchy surfaces where the relativistic inner product \(\langle f^{(+)}_{\omega}, f^{(+)}_{\sigma} \rangle\) can be defined as

\[
\langle f_1, f_2 \rangle = i \int_{\mathcal{I}^-} dv (f_1^* \partial_v f_2 - \partial_v f_1^* f_2) = i \int_{\mathcal{I}^+} du (f_1^* \partial_u f_2 - \partial_u f_1^* f_2).
\]

Formula (43) also holds on space-times with metric \(ds^2 = -e^{2\sigma} du dv\), it does not depend \(\sigma\).

There is a principle distinction between fields near a collapsing star and fields in the flat space-time: for the star there are ingoing waves, \(\phi_H(v)\), which cannot escape to \(\mathcal{I}^+\). Such waves come after the black hole horizon is formed, cross \(\mathcal{H}\) and move inside the black hole. Schematically the 2D part of a collapsing star is shown on Fig. 2. A general solution to (41) in the null coordinates is a combination of left and right-moving modes of the following form:
\[ \phi(u, v) = \phi_{\text{esc}}(u, v) + \phi_H(v) \quad . \] (44)

By \( \phi_{\text{esc}}(u, v) \) we denote waves which can escape to \( \mathcal{I}^+ \). Like waves in the Minkowsky space-time \( \phi_{\text{esc}}(u, v) \) satisfy a sort of reflection condition. If \( v = 0 \) is the trajectory of the last ingoing ray which escapes the black hole, just before the horizon is formed, \( \phi_{\text{esc}}(u, v), \phi_H(v) \) are defined for \( v < 0 \) or \( v > 0 \), respectively. For solutions (44) the equivalent Cauchy surfaces are \( \mathcal{I}^- \), in far past, or \( \mathcal{I}^+ \cup \mathscr{H} \), in future.

To determine \( \phi_{\text{esc}} \) one can trace modes inside the collapsing star and find the “boundary condition”. This option is complicated and requires additional model assumptions. Another strategy is to define \( \phi_{\text{esc}} \) by requiring its regularity across \( \mathcal{H} \). The fact that \( \phi_{\text{esc}} \) cannot be taken as waves (42) in flat spacetime is clear since outgoing coordinate \( u \) is not analytical on \( \mathcal{H} \).

There are alternative coordinates, the Kruskal coordinates,

\[ \begin{align*}
U(u) &= -ae^{-ku}, \\
V(v) &= be^{kv},
\end{align*} \] (45)

which are both analytic on \( \mathcal{H} \). Here \( a, b \) are some positive constants which can be chosen such that \( ds^2 \simeq -dUdV \) near \( \mathcal{H} \). The event horizon \( \mathcal{H} \) is defined by conditions \( U = 0, \ v > 0 \). It is important that \( U \) and \( V \) can be interpreted near \( \mathcal{H} \) as retarded and advanced times for freely falling observers.

In the black hole exterior \( U > 0 \), and \( U \) can be continued inside the black hole, where \( U < 0 \). Therefore, for example, the following modes:

\[ f_{\text{in}, \omega}^{(+)}(u, v) = \frac{1}{\sqrt{4\pi \omega}} \left( e^{-i\omega v} - e^{-i\omega U(u)} \right), \quad -\infty < u, v < \infty \] (46)

behave well at \( \mathcal{H} \). These modes:

i) are ingoing waves \( e^{-i\omega v} \) with frequency \( \omega \) as measured by freely moving observers in the far past, near \( \mathcal{I}^- \);

ii) are outgoing waves \( e^{-i\omega U} \) with frequency \( \omega \) for freely falling observers near \( \mathcal{H} \);

iii) make a complete set on the Cauchy surfaces \( \mathcal{I}^- \) or \( \mathcal{I}^+ \cup \mathcal{H} \).

Therefore for the collapsing star a quantum state which is experienced as a vacuum by freely falling observers, including observers near \( \mathcal{I}^- \), should be modeled by a state \( |\Psi\rangle \) determined by the condition:

\[ \hat{a}_\omega |\Psi\rangle = 0 \quad , \] (47)

where operators \( \hat{a}_\omega \) are introduced by (31) with respect to modes \( f_{\text{in}, \omega}^{(+)} \) defined in (46).

The \( \phi_H \) part of modes (46) is simply proportional to \( e^{-i\omega v} \) for \( v > 0 \). The escape part, \( \phi_{\text{esc}}, \ v < 0 \), looks as a solution to the wave equation in the presence of a perfectly reflecting accelerated mirror which moves along the trajectory

\[ v = U(u) \quad , \] (48)
Fig. 2 Carter-Penrose diagrams which show propagation of waves from $\mathcal{I}^-$ in Minkowski space-time, (a), and in space-time of a collapsing star, (b). In flat space-time all modes from $\mathcal{I}^-$ reach $\mathcal{I}^+$ unchanged. The right figure shows the "escape" modes $\phi_{\text{esc}}(u,v)$ defined at $v < 0$. These modes leave the star just before the horizon is formed and behave as if reflected from an imaginary mirror. Modes $\phi_H(v)$ are shown as waves from $\mathcal{I}^-$ at $v > 0$. They cross $\mathcal{H}$ and end on the singularity $r = 0$.

see Fig. 2. This fictitious mirror plays the role of a strong gravity, it transforms ingoing waves $e^{-i\omega v}$ in the escape part to outgoing waves $e^{-i\omega U}$.

It is well-known that accelerated mirrors create particles. Analogously the collapsing body creates the flux of the Hawking radiation. To see this define modes

$$f_{\text{out},\sigma}^{(+)}(u,v) = \frac{1}{\sqrt{4\pi\sigma}} \left( e^{-i\sigma u} - e^{-i\sigma \bar{v}(v)} \right), \quad -\infty < u < \infty, \quad v < 0,$$

(49)

where $\bar{v}(v) = -\frac{i}{\sigma} \ln(-\frac{v}{\sigma})$. These out-modes:

i) are reduced to outgoing waves $e^{-i\omega u}$ with frequency $\omega$ as measured by freely moving observer in the far future, near $\mathcal{I}^+$;

ii) satisfy the Dirichlet condition on "trajectory" (48);

iii) do not have $\phi_H(v)$ part;

iv) make a complete set with the same normalization on the null surfaces discussed above.

Define creation and annihilation operators $\hat{b}^+$ and $\hat{b}$ for out-modes $f_{\text{out},\sigma}^{(+)}$. The corresponding quasiparticles are interpreted by observers near $\mathcal{I}^+$ as particles with certain energies. One identifies these particles with the Hawking quanta.

The number of the Hawking quanta in given state is, see Eqs. (36),(37),
\[ N_\sigma = \langle \Psi | \hat{b}_\sigma^\dagger \hat{b}_\sigma | \Psi \rangle = \int d\omega |d_\sigma,\omega|^2, \]  
\[ d_{\sigma,\omega} = \langle f_{\text{out},\sigma}^{(+)} | f_{\text{in},\omega}^{(-)} \rangle = -\frac{1}{2\pi} \sqrt{\frac{\sigma}{\omega}} \int_{-\infty}^{\infty} du e^{i\sigma u + i\omega U(u)}. \] 

The product in (51) is defined on \( \mathcal{I}^+ \). Integral in r.h.s. of (51) can be performed after the substitution \( t = e^{-ka} \)

\[ d_{\sigma,\omega} = -\frac{1}{2\pi k} \sqrt{\frac{\sigma}{\omega}} \exp \left( \frac{\pi \sigma}{2k} \right) \Gamma \left( \frac{1}{2} \right) \left( \omega \sigma \right)^{\frac{1}{2}}, \]  

where \( \Gamma(z) \) is the \( \Gamma \)-function

\[ \Gamma(z) = \int_0^\infty dt \ t^{z-1} e^{-t}. \]  

By using the property \( |\Gamma(iz)|^2 = \pi/(z\sinh \pi z) \) one gets the number of the Hawking quanta in the form:

\[ N_\sigma = \frac{\Gamma_\sigma}{e^{\sigma/T_\text{H}} - 1}. \]  

Coefficients \( \Gamma_\sigma \) are called the grey-body factors. In the considered case \( \Gamma_\sigma \) is a constant which includes a regularized integral over \( \omega \).

A remarkable fact is that the Hawking quanta are distributed according to Planck’s law with a temperature

\[ T_\text{H} = \frac{k}{2\pi} \]  

called the Hawking temperature.

The simplified analysis presented here can be extended beyond the \( s \)-mode approximation and near-horizon approximation. Massless modes which depend on angles will experience a partial reflection on the potential \( V_l(r) \), see (28). This effect yields non-trivial factors \( \Gamma_\sigma \).

The Hawking effect can be interpreted as a process of creation of particle-antiparticle pairs in the strong gravitational field near the horizon. Antiparticles created in this process tunnel inside the black hole, while particles make the Hawking flux. Semiclassical estimations of the tunneling probability [61] are in agreement with (54), see [77] for a review.

If, after the collapse of the star, the black hole evaporates completely, it results in violation of the unitarity and information loss since the initial pure state \( |\psi\rangle \) evolves to a mixed thermal state. This paradox, despite several interesting hypothesis [76], has not been resolved so far.
5 Thermodynamic laws of black holes

5.1 Black hole mechanics

If a black hole appears as a result of the gravitational collapse of a star, it quickly reaches a stationary state characterized by a certain mass $M$ and an angular momentum $J$. By using purely classical Einstein equations, or on the base of definitions of (8), (15), (22), one arrives at the following variational formula [4]:

$$\delta M = T_H \delta S_{BH} + \Omega_H \delta J,$$

(56)

where $T_H$ is given by (55) and

$$S_{BH} = \frac{1}{4G} \mathcal{A}.$$

(57)

$\mathcal{A}$ is the surface area of the horizon (see above) and $G$ is the Newton gravitational constant.

The quantity $S_{BH}$ was introduced in [8]-[10],[44] and is called the Bekenstein-Hawking entropy.

Relation (56) has the form of the first law of thermodynamic where $S_{BH}$ has the meaning of an entropy, $T_H$ is a temperature, and $M$ is an internal energy. If the collapsing matter was not electrically neutral a black hole has an additional parameter, an electric charge $Q$. Then the r.h.s. of (56) would acquire additional term $\Phi_H \delta Q$, where $\Phi_H$ is the difference of the electric potential at the horizon and at infinity. $M,J,Q$ are the only parameters a black hole in the Einstein-Maxwell theory can have. Its metric in the most general case is the Kerr-Newmann metric. This statement is known as the "no-hair" theorem, see e.g. [32].

The Bekenstein-Hawking entropy is one of the most mysterious quantities in black hole thermodynamics. For super-massive black holes with masses of the order of $10^9$ solar masses $S_{BH}$ is of the order of $10^{95}$, it is eight orders of magnitude larger than the entropy of the microwave background radiation in the visible part of the Universe. This raises a natural question about microscopic degrees of freedom whose number is consistent with the Bekenstein-Hawking entropy.

The reason why this question is fundamental is because it goes beyond the black hole physics itself. On one hand, its answer may give important insights into the as yet mysterious nature of quantum gravity. On the other hand, since the thermodynamics of black holes is a low-energy phenomenon, understanding of the black hole entropy may be possible without knowing details of quantum gravity, for example in the framework of the perturbative quantum gravity methods.
5.2 Black holes and Euclidean theory

There is another way to see that black holes look as thermodynamic systems. Consider the partition function of a quantum system with a (normally ordered) Hamiltonian \( \hat{H} \): at temperature \( T = \beta^{-1} \),

\[
Z(\beta) = \text{Tr} e^{-\beta \hat{H}} \quad .
\]

As is known, (58) can be interpreted as a trace of the evolution operator \( \hat{U}(t) = e^{-it\hat{H}} \) with imaginary time interval \( t = -i\beta \). This allows one (see discussion in next sections) to represent \( Z(\beta) \) as a path integral in the corresponding Euclidean quantum theory with periodic or antiperiodic boundary conditions in the Euclidean time \( \tau = it \). In application to black holes in vacuum this implies that instead of the Lorentzian Ricci flat solutions, \( R_{\mu\nu} = 0 \), see (1), we should consider analogous solutions on Riemannian, or Euclidean manifolds with the signature \(+,+,+,+,+\). Such solutions are called gravitational instantons.

Consider the Kerr solution (2)-(5). The corresponding instanton can be obtained from (2)-(5) by the Wick rotation of time, \( t = -i\tau \) and by changing \( a \) to \(-ia\) to ensure that non-diagonal term \( 2g_{t\phi}dt d\phi \) in metric (2) remains real. (It should be noted that quantum theory which we discuss in next sections does not require the Wick rotation of \( a \), so one can work in principle with complex metrics.) By using parameters of the Lorentzian solution, \( r_{H}, \Omega_{H}, k \) one defines analogous parameters \( \bar{r}_{H} = r_{H}(ia), \bar{\Omega}_{H} = \Omega_{H}(ia), \bar{k} = k(ia) \) for the instanton. The Euclidean Kerr solution has analogous symmetries. The Killing vector field \( \zeta = \partial_{\tau} + \bar{\Omega}_{H} \partial_{\phi} \) does not have a horizon, as in the Lorentzian theory, but it has fixed points located on a closed 2D surface \( r = \bar{r}_{H} \), the Euclidean horizon, which we denote by \( \mathcal{B} \), in the same way as the bifurcation surface. Near \( \mathcal{B} \) the Euclidean metric looks as follows (compare with (20)):

\[
ds^2 \simeq \bar{h}(\theta)(\bar{k}^2 \rho^2 d\tau^2 + d\rho^2) + dt^2 \quad , \quad 0 < \tau \leq \beta \quad .
\]

At arbitrary periodicity (59) has a conical singularity at \( \rho = 0 \) (\( r = \bar{r}_{H} \)). The singularity disappears if

\[
\beta = \beta_{H} = \frac{2\pi}{\bar{k}} \quad .
\]

Thus, the regularity condition requires that the period \( \beta_{H} \) coincides with the inverse Hawking temperature (55).

The fact that the Euclidean horizon is a fixed point set of the Killing field associated to time translations also implies a “thermodynamic” form of the gravity action on the black hole instanton. It is easy to check that the Euclidean action, \( I_{E}[\phi] \), say, for a scalar field \( \phi \), between a constant time hypersurface \( \Sigma_{\tau} \) and a hypersurface \( \Sigma_{\tau}' \) with \( \tau' = \tau + \beta \) has the form \( I_{E}[\phi] = \beta H \), where \( H \) has a form of the Hamiltonian of the system. On static solutions, \( \partial_{\tau}\phi = 0, H \) coincides with the energy of the system.

Consider now the Einstein-Hilbert action on Riemannian (Euclidean) manifolds \( \mathcal{M} \).
\[ I_E[g] = -\frac{1}{16\pi G} \left[ \int d^4x \sqrt{g} R + 2 \int_{\partial M} d^3\sqrt{h} K \right]. \] (61)

The last term in the r.h.s. of (61) should be added, according to Gibbons and Hawking [42], when \( M \) has a boundary \( \partial M \). This term depends on the trace \( K \) of the extrinsic curvature of \( \partial M \) and it guarantees that variations \( I_E[g] \) do not contain variations of normal derivatives of the metric on \( \partial M \). To avoid infrared divergences in (61) on asymptotically flat space-times, \( I_E[g] \) is defined with a subtraction of the corresponding action on a flat space-time.

If the Killing field \( \partial_\tau \) does not have fixed points the only boundaries of constant \( \tau \) hypersurfaces belong to \( \partial M \). Then action (61) computed between \( \Sigma_\tau \) and \( \Sigma_\tau + \beta \) has the structure, \( I_E = \beta H \), where \( H \) is a Hamiltonian. On asymptotically flat solutions, after the subtraction in \( I_E[g] \), \( H \) coincides with the ADM mass [47].

Situation is different for black hole instantons since \( \partial_\tau \) has fixed points on \( B \). As a result, \( \Sigma_\tau \) end on \( B \), and the Euclidean horizon becomes an internal boundary. To calculate the action on \( M \) one should consider the near-horizon part \( M_\epsilon \) of \( M \) separately. The metric of \( M_\epsilon \) can be approximated by (59) with the outer boundary located at \( \rho = \epsilon \). The gravitational action taken on a black hole instanton then becomes [42][47]

\[ I_E = \beta H (M - \Omega_H I) - \frac{\mathcal{A}}{4G}, \] (62)

where \( \mathcal{A} \) is the area of \( B \). The first term in the r.h.s. of (62) appears from a domain outside of \( M_\epsilon \) where constant time hypersurfaces do not have intersections. The last term, \( -\mathcal{A}/(4G) \) appears from the boundary term,

\[ \lim_{\epsilon \to 0} \frac{1}{8\pi G} \int_{\partial M_\epsilon} d^3\sqrt{h} K = \frac{\mathcal{A}}{4G}, \] (63)

see discussion of this point, based on topological arguments, in [3].

It follows from (62) that \( I_E/\beta H \) can be interpreted as the free energy of a thermodynamic system with energy \( M \), entropy \( \mathcal{A}/(4G) \), and temperature \( 1/\beta H \), in accord with first law (56).

It will be important for the future discussion to point out that thermodynamic form of the action (62) can be extended to the case when the Euclidean time \( \tau \) in the instanton solution has an arbitrary period \( \beta \). Such a geometry is not regular because of conical singularities at \( B \) with the deficit angle \( 2\pi(1 - \beta/\beta_H) \). For this reason it is not a solution of the Einstein equations near \( B \). The black hole thermodynamics at \( \beta \neq \beta_H \) is called an off-shell approach. Components of the Riemann tensor on manifolds with conical singularities behave as distributions at \( B \), see [39]. In particular, the integral curvature is

\[ \int_{\partial M} d^4x \sqrt{g} R = 4\pi \left( 1 - \frac{\beta}{\beta_H} \right) \mathcal{A} + \int_{M/B} d^4x \sqrt{\bar{g}} R, \] (64)

where \( M/B \) is the regular part of \( M \). By using (63), (64) it is not difficult to check that the off-shell action,
\[ I_E = \beta (M - \Omega H J) - \frac{\partial f}{4G} = I_E(\beta, M, J), \quad (65) \]

holds the thermodynamic form. The advantage of the off-shell formulation is that \( \beta \) is a free parameter which is not related to the mass \( M \) and angular momentum \( J = Ma \) of a black hole. If \( I_E/\beta = F \) is interpreted as a free energy, the Bekenstein-Hawking entropy can be derived by using statistical-mechanical formula:

\[ S_{BH} = \beta^2 \partial_\beta F(\beta, M, J)_{\beta=\beta_H} = (\beta \partial_\beta - 1) I_E(\beta=\beta_H). \quad (66) \]

We use definition (66) in what follows.

To avoid unnecessary complications boundary conditions for a black hole have not been specified in the above analysis. The importance of boundary conditions has been pointed out in [83] for the case of a Schwarzschild black hole inside a spherical cavity of a finite radius. It can be shown by using the Gibbons-Hawking action that such a black hole at certain size of the cavity behaves as a stable thermodynamic system.

Asymptotically anti-de Sitter black hole solutions in the Einstein gravity with a negative cosmological constant have a similar property [49]: they can be in stable equilibrium state when their size is greater than the radius of the anti-de Sitter space.

Thermodynamics of black holes in higher dimensional gravity theories with the negative cosmological constant is used [80] to study finite-temperature gauge theories in the framework of the so called AdS/CFT correspondence [81], [62]. Interestingly, charged black holes in anti-de Sitter space-times have a phase structure similar to that of the van der Waals-Maxwell liquid-gas systems in a space-time of one-dimension lower [20].

### 5.3 Black holes in generalized gravity theories

It should be noted that variational formulas analogous to (56) can be found for asymptotically flat black hole solutions in a general classical theory of gravity arising from a diffeomorphism invariant Lagrangian [79]. In these theories one comes to a generalization of the Bekenstein-Hawking formula (57). Moreover, the black hole entropy can be interpreted as a Noether charge associated to the horizon Killing field \( \zeta \). As an example, consider a modification of the Einstein gravity by terms quadratic in curvatures. The bulk part of gravity action (on Euclidean manifolds) is

\[ I[g, (G, \Lambda, c_i)] = - \int d^4x \sqrt{g} \left[ \frac{R}{16\pi G} - 2\Lambda - c_1 R^2 - c_2 R_{\mu\nu} R^{\mu\nu} - c_3 R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right], \quad (67) \]

where \( R_{\alpha\beta\mu\nu}, R_{\mu\nu}, R \) are the Riemann and Ricci tensor, as well as the scalar curvature, respectively. Introduce 2 normal vectors \( n^\mu_i \) at the bifurcation surface \( \mathcal{B}, (n_i \cdot n_j) = \delta_{ij} \), and define at \( \mathcal{B} \) the following invariants:
\[ R_{ij} = R_{\mu \nu} \delta_i^\mu \delta_j^\nu , \quad R_{ij} = R_{\mu \nu \lambda \rho} \delta_i^\mu \delta_j^\nu \delta_\lambda^\rho \delta_\rho^\lambda . \] (68)

The entropy of stationary black holes in such theories is

\[ S_{\text{BH}}(G, c_i) = \frac{1}{4G} e^\mathcal{A} - \int d^2x \sqrt{\gamma} (8\pi c_1 R + 4\pi c_2 R_{ij} + 8\pi c_3 R_{ijkl}) , \] (69)

where \( \gamma = \det \gamma_{\alpha \beta} \), and \( \gamma_{\alpha \beta} \) is defined in (21).

An alternative way to derive (69) is to use the off-shell approach or the conical singularity method. As was shown in [39] integrals of powers of the curvature tensor can be well-defined on off-shell instantons with conical singularities in the linear approximation in the deficit angle. Such integrals are similar to (64) and yield (69) if one applies (66) for definition of the entropy.

### 5.4 Generalized second law

By considering classical processes with black holes one can conclude that the area of the horizon never decreases, the observation which is reminiscent to the second law. Black hole must have an intrinsic entropy proportional to the horizon area, otherwise processes like a gravitational collapse would be at odds with the second law. The second law of thermodynamic in the presence of black holes can be written in the generalized form

\[ \delta S_{\text{BH}} + \delta S_m \geq 0 , \] (70)

which states that the sum of the Bekenstein-Hawking entropy of a black hole and the entropy of a surrounding matter does not decrease in physical processes.

Generalized second law (70) poses a number of serious questions when it is applied to quantum matter around black holes. The very definition of \( S_m \), when the black hole horizon serves as a boundary of the system, requires clarification. Thermal entropy of a relativistic plasma is not well defined near \( \mathcal{H} \) due to an infinite blue-shift of the energies of quanta in this region. The classical part, \( S^{\text{BH}} \), of the generalized entropy depends on the gravitational coupling \( G \) which participates in the renormalization of the ultraviolet divergences. Can \( S_m \) in (70) be considered as a quantum correction to the classical entropy \( S^{\text{BH}} \)?

Although such questions may look technical, their resolution is a necessary step toward understanding profound conceptual issues brought in theoretical physics by black holes. Perturbative quantum gravity is a testbed where these questions can be dealt with by using conventional quantum field theory.
6 Quantum black holes in thermal equilibrium

6.1 The Hartle-Hawking-Israel state

To proceed with the discussion of generalized second law (70) in case of quantum fields around a black hole one needs to specify the quantum state of the system. If a black hole has an astrophysical mass it evaporates due to the Hawking effect very slowly, as if being in a thermal equilibrium with the radiation. A real equilibrium state can be realized for an eternal black hole placed inside a cavity with perfectly reflecting walls. A classical analogue of this system has been studied in [83].

Consider, for simplicity, quantum fields on space-time of an eternal Schwarzschild black hole. This black hole has two space-like singularities, the future (black hole) singularity, the past (white) hole singularities, and two asymptotically flat, left and right regions, separated by the horizons, see Fig. 1. The horizons $H^\pm$ bifurcate at a two sphere $B$. The structure of the Killing field of the Schwarzschild black hole is almost identical to that of Minkowsky space-time. This fact indicates that, for a black hole, there may be defined a quantum state which is a counterpart of the Minkowsky vacuum. Such a state does exist and is called the Hartle-Hawking-Israel state (HHI-state) [43],[52]. Stationary observers which move along integral lines of $\partial_t$ are analogous to the Rindler observers, and they see the HHI state as a thermal bath at the Hawking temperature.

To come to the definition of this equilibrium state we use results of Sec. 5.2. The fact that classical gravity formulated on gravitational instantons has a thermodynamic form indicates also the importance of quantum theory on Riemannian (Euclidean) manifolds. From now on we use notation $\mathcal{M}_E$ for these geometries. Euclidean QFT has mathematical advantages, which can be explained by using example of a free scalar field $\phi$ with equation

$$P_E \phi = (-\nabla^2 + m^2) \phi = 0$$

(71)
on $\mathcal{M}_E$. Operator $P_E$ is of a Laplace type. Since $\mathcal{M}_E$ is a curved manifold plane waves $e^{ikx}$ are not eigen-functions of $P_E$. However one can act by $P_E$ on a plane wave to get

$$P_E e^{ik_\mu x^\mu} = [k_\mu k_\nu g^\mu_\nu(x) + \ldots] e^{ik_\mu x^\mu}.$$  

(72)

In mathematical applications, like the spectral theory, the important property of $P_E$ is that its leading symbol $\sigma_P(x,k) = k_\mu k_\nu g^\mu_\nu(x)$ is not degenerate and is positive-definite on $\mathcal{M}_E$ (so $P_E$ are called elliptic operators). This behavior of the leading symbols is crucially different for operators on Euclidean and Lorentzian manifolds. For elliptic operators spectral functions, which serve to define other ingredients of the quantum theory, such as the effective action, can be introduced with mathematically meaningful prescriptions since large $k^2$ asymptotics are under control. More on this topic can be found in monograph [41].

One of the key quantities which can be rigorously defined is the heat kernel $K(x,y|t)$ of $P_E$ which is the solution to the following problem:
\[(\partial_t + P_E(x))K(x,y|t) = 0, \quad K(x,y|t)_{t\to 0} = \delta(x,y).\]  

Here \(t\) is a positive parameter, \(P_E(x)\) acts on argument \(x\), \(\delta(x,y) = \delta^4(x-y)/\sqrt{g}\), and the symmetry, \(K(x,y|t) = K(y,x|t)\), is implied. Suppose that \(P_E(x)\) does not have zero eigen-values. Then by using the heat kernel one can define the Green function \(G(x,y)\) of the operator \(P_E(x)\) \(\text{ for } G(x,y) = \delta(x,y).\)  

One can check with help of (73) that \(G_E(x,y) = \int_0^\infty dt\, K(x,y|t).\)  

Hartle and Hawking [43] used extension of \(G_E(x,y)\), via the inverse Wick rotation to the corresponding Lorentzian space-time \(\mathcal{M}\), to define the Green’s function \(G(x,y)\) on \(\mathcal{M}\). It is a unique quantum state fixed in this way that is called the HHI-state.

Below we briefly describe statistical-mechanical interpretation of Euclidean QFT’s for quantum fields on stationary space-times. Let \(\mathcal{M}\) be a stationary space-time with the time-like Killing field \(\zeta = \partial_t\). Let \(g_{\mu\nu}\) be components of metric of \(\mathcal{M}\) in coordinates \(x^\mu(t) = (t,x^i)\). First suppose that this coordinate chart covers \(\mathcal{M}\) globally, that is \(\mathcal{M}\) has the structure \(\Sigma \times R^4\) where \(\Sigma\) are constant \(t\) hypersurfaces. Consider decomposition (31) of the field \(\phi(x)\) on modes which are eigen-functions of the operator \(\zeta = \partial_t\), see (33). Since \(\mathcal{M}\) is stationary one can define a finite-temperature state of \(\phi\) at temperature \(T = \beta^{-1}\). The average of an operator \(\hat{O}\) in this state is:

\[\langle \hat{O} \rangle_\beta = Z^{-1}(\beta)\text{Tr}(\hat{O}e^{-\beta\hat{H}}),\]  

where \(Z(\beta)\) is partition function (58) and the Hamiltonian \(\hat{H}\) : generates evolution along \(t\). One is usually interested in a relation between the Wightman functions

\[G_\beta^+(x,x') = \langle \hat{\phi}(x)\hat{\phi}^+(x') \rangle_\beta, \quad G_\beta^-(x,y) = \langle \hat{\phi}^+(x')\hat{\phi}(x) \rangle_\beta\]  

on \(\mathcal{M}\) and the Green’s function \(G_E(x,y)\) on \(\mathcal{M}_E\). It is assumed that \(\mathcal{M}\) and \(\mathcal{M}_E\) are connected via the Wick rotation \(t \to -i\tau\). The inverse components of the Lorentzian and Euclidean metrics are: \(g_{\mu\nu}^E = -g^{\mu\nu}\), \(g_{\mu\nu}^E = ig^{ik}\), \(g_{ik}^E = g^{ik}\). In general, \(\mathcal{M}_E\) is a complex manifold and \(P_E(x)\) is not self-adjoint but it is still elliptic, which is enough to define with its help corresponding spectral functions and the effective action.

Let \(x(t)\) be points on an integral line of \(\partial_t\) on \(\mathcal{M}\) and \(\bar{x}(\tau) = x(-i\tau)\) be points on corresponding integral line of \(\partial_t\) on \(\mathcal{M}_E\). One can define the two-point function

\[\tilde{G}_\beta(x(z),x'(0)) = \theta(-3z)G_\beta^+(x(z),x'(0)) + \theta(3z)G_\beta^-(x(z),x'(0))\]  

where \(z = t + i\tau\). It can be shown, see e.g. [41] that (78) is an analytic function of \(z\) everywhere in the strip \(-\beta < 3z < \beta\) except the domains where the Wightman functions have singularities, that the periodicity property, \(\tilde{G}_\beta(x(z - i\beta),x'(0)) = \tilde{G}_\beta(x(z),x'(0))\), holds, and that there is the fundamental relation,
between Euclidean and thermal Green’s functions.

In case of black holes coordinates \( x^\mu(t) = (t, x^i) \) cover only a part of \( \mathcal{M} \) located to the right from the horizon \( \mathcal{H}^+ \cup \mathcal{H}^- \), see Fig. 1. Constant time hypersurfaces \( \Sigma \) intersect on \( \mathcal{B} \). The analysis [43] shows that if the Euclidean Green’s function in (79) is defined on a black hole instanton, at \( \beta = \beta_H = 1/T_H \), the corresponding finite-temperature Green’s function can be extended to the entire black hole space-time \( \mathcal{M} \), beyond the domain of stationary coordinates. The quantum state defined by such a Green’s function is the Hartle-Hawking-Israel state. (When applying this analysis to Kerr black holes one should note that \( \zeta \) is time-like in a restricted domain and that non-diagonal components of \( \mathcal{M}_E \) can be made real by going to imaginary values of the angular momentum, see Sec. 5.2.) If \( \beta \) is an arbitrary parameter there are some peculiarities in the behavior of the Green’s functions due to conical singularities discussed in next sections.

By the construction, a Killing observer sees the HHI state as a thermal bath at the Hawking temperature \( T_H \). In the near-horizon approximation (39),(40) the single-particle excitations for such observers are defined by the set of left-moving and right-moving modes

\[
\begin{align*}
    f^{(+)}_{L,\omega}(v) &= \frac{1}{\sqrt{4\pi \omega}} e^{-i\omega v}, \\
    f^{(+)}_{R,\omega}(u) &= \frac{1}{\sqrt{4\pi \omega}} e^{-i\omega u},
\end{align*}
\]

which are eigen-functions of \( \zeta = \partial_t \). In case of an eternal Schwarzschild black hole modes \( f^{(+)}_{L,\omega} \) start from \( \mathcal{I}^- \) and end on the black hole singularity, while \( f^{(+)}_{R,\omega} \) start at the white hole singularity and move to \( \mathcal{I}^+ \), see Fig. 1.

One can also consider modes

\[
\begin{align*}
    \tilde{f}^{(+)}_{L,\omega}(v) &= \frac{1}{\sqrt{4\pi \omega}} e^{-i\omega v^V(v)}, \\
    \tilde{f}^{(+)}_{R,\omega}(u) &= \frac{1}{\sqrt{4\pi \omega}} e^{-i\omega U(u)},
\end{align*}
\]

where \( V \) and \( U \) are Kruskal coordinates (45). The HHI state is the vacuum state for quasiparticles associated with modes (81). To check this one can calculate the number of particles related to (80) in a vacuum state for (81) by using (37). The relevant non-trivial Bogoliubov coefficients between \( R \)-modes are

\[
d^R_{\sigma,\omega} = \langle f^{(+)}_{R,\sigma}, \tilde{f}^{(-)}_{R,\omega} \rangle = -\frac{1}{2\pi} \sqrt{\frac{\sigma}{\omega}} \int_{-\infty}^{\infty} du e^{i\sigma u + i\omega U(u)}. \tag{82}
\]

As follows from (51), the integral in the r.h.s. of (82) is given by (52). Thus (82) corresponds to the thermal distribution of quanta at the Hawking temperature, which is the property of the HHI state. The corresponding coefficient between the left modes has analogous expression.

The HHI state can be defined on a global Cauchy surface where it has an important representation. Consider, as an example, the Carter-Penrose diagram of an eternal Schwarzschild black hole. A constant time hypersurface \( \Sigma_R \cup \Sigma_L \) which goes
from the right to the left world is called the Einstein-Rosen bridge, see Fig. 1. The left and right parts of the bridge are identical and coincide with constant time sections. In HHI wave function, in the configuration representation, depends on field variables \( \phi_R, \phi_L \) set on \( \Sigma_R, \Sigma_L \), correspondingly. The wave function can be represented as transition amplitude in the Euclidean time \( \tau \) between the left and right worlds,

\[
\langle \phi_L, \phi_R | \text{HHI} \rangle = N^{-1/2} \langle \phi_L | e^{-\beta_H \hat{H}} | \phi_R \rangle = \int [D\phi] e^{-I_E[\phi_L, \phi_R]} ,
\]

where \( N \) is a normalization factor, and the Euclidean action,

\[
I_E[\phi_L, \phi_R] = -\frac{1}{2} \int_0^{\beta_H/2} d\tau \int d^3x \phi^* P_E \phi ,
\]

is defined with boundary conditions \( \phi(\tau = 0, x^i) = \phi_R(x^i), \phi(\tau = \beta_H/2, x^i) = \phi_L(x^i) \).

A detailed discussion of this representation, which is a natural generalization of a similar formula for the Minkowsky vacuum [52], can be found in [5].

### 6.2 Effective action and renormalized stress-energy tensor

Vacuum polarization in an external gravitational field \( g_{\mu\nu} \) results a non-trivial average of the stress energy tensor of a quantum field, \( \langle \hat{T}_{\mu\nu} \rangle \), which appears in right-hand side of the Einstein equations (1). The advantage of the Euclidean formulation of the theory is that \( \langle \hat{T}_{\mu\nu} \rangle \) in the HHI state can be derived by variation of the effective action.

An introduction to the effective action approach in perturbative quantum gravity can be found in [14]. The effective action of non-interacting quantum fields can be defined as the following functional on \( \mathcal{M}_E \):

\[
\Gamma[g] = I[g, (G^B, \Lambda^B, c_i^B)] + W[g] .
\]

Here \( I[g] \) is classical gravity action (67) and \( G^B, \Lambda^B, c_i^B \) are bare couplings. The quantum part of the action is

\[
W[g] = \frac{\eta}{2} \ln \det P_E ,
\]

where \( \eta = +1 \) or \( -1 \) for fields with Bose or Fermi statistics, respectively. Expression (86) is motivated by a formal path integral for free fields. It needs a further prescription to deal with ultraviolet divergences. Since \( P_E \) is an elliptic operator one can introduce the heat trace (see details in [41])

\[
K(P_E; t) = \sum_{\lambda} e^{-t\lambda} ,
\]
where the sum is taken over all eigenvalues $\lambda$ of $P_E$. If $\lambda$ are positive one can use the definition:

$$\ln \det P_E = - \int_\delta^\infty \frac{dt}{t} K(P_E; t)$$

(88)

where $\delta > 0$ is a proper cutoff parameter. An important property of (87) is a short $t$ expansion (as $t \to +0$)

$$K(P_E; t) \sim \sum_{p=0}^n t^{-\frac{n-p}{2}} a_p(P_E),$$

(89)

where $n$ is the dimensionality of $\mathcal{M}_E$, and $a_p(P_E)$ with odd $p$ appear if $\mathcal{M}_E$ has boundaries. The heat kernel (or DeWitt-Seeley) coefficients $a_p(P_E)$ determine the divergent part the effective action

$$W_{\text{an}}[g, \delta] = \eta \sum_{p=0}^{n-1} a_p(P_E) \delta^{\frac{n-p}{2}} + \eta a_n(P_E) \ln \delta,$$

(90)

where we used (88). A common prescription to eliminate the ultraviolet divergences is to note that $a_p(P_E)$ with $p = 2k$ are integrals of the $k$-th order polynomials in the curvature tensor. Thus, in four dimensions (90) has the same structure as the bare action $I[g, (G^B, \Lambda^B, c^B)]$. The divergences are eliminated by redefinition of $G^B, \Lambda^B, c^B$,

$$I[g, (G, \Lambda, c)] = I[g, (G^B, \Lambda^B, c^B)] + W_{\text{an}}[g, \delta].$$

(91)

This yields effective action (85) in the renormalized form:

$$\Gamma[g] = I[g, (G, \Lambda, c)] + W_{\text{ren}}[g],$$

(92)

where renormalized, or the UV-finite part is

$$W_{\text{ren}}[g] = \lim_{\delta \to 0} (W[g, \delta] - W_{\text{an}}[g, \delta]).$$

(93)

Formula (92) can be used to derive the gravity equations which allow one to take into account the back-reaction to quantum fields,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \ldots = \frac{8\pi G}{c^4} T^E_{\mu\nu},$$

(94)

$$T^E_{\mu\nu}[g] = \frac{2 \delta W_{\text{ren}}[g]}{\sqrt{\det[g]} \delta g_{\mu\nu}}.$$  

(95)

The dots in (94) are quadratic in curvature terms. To calculate first quantum correction to the metric tensor $g_{\mu\nu}$ it is enough to consider $T^E_{\mu\nu}[g]$ on corresponding classical black hole instanton.

Several remarks are in order.

i) One can analytically continue (94) from the Euclidean to the Lorentzian theory. If we are interested in first quantum correction, the Lorentzian theory is just
the gravity theory sourced by the average of the stress-energy tensor in the Hartle-Hawking-Israel state. On black hole instanton

\[ T_{\mu\nu}^{E}[g] = i^q \langle HH | \hat{T}_{\mu\nu}[g] | HH \rangle, \tag{96} \]

where \( q \) is the number of temporal indexes and the factor \( i^q \) is related to the Wick rotation. Given connection (79) between the Euclidean and finite-temperature Green’s functions relation (96) can be proved when the quantum stress-energy tensor is computed with the help of a Green’s function by using well-known point-splitting procedure. (The point-splitting method is discussed in detail in [11].)

ii) By the construction, (96) yields the stress-energy tensor in stationary coordinates outside the horizon. Since the Green’s function can be analytically extended to the entire space-time, so does (96).

iii) In addition to its transparent physical meaning the HHI state is distinguished by mathematical properties when computations of quantum averages are reduced to finding various spectral functions of elliptic operators \( P_E \).

iv) There are different types of UV regularizations in the effective action, for example, the \( \zeta \)-function regularization [45], [26],[27], the dimensional regularization and the Pauli-Villars regularization [11]. Up to several finite counter-terms all of them yield the same \( W_{\text{ren}} \).

7 Fluid dynamics in gravitational fields

7.1 Finite-temperature QFT’s and effective action

The connection between the Euclidean and finite-temperature Green’s functions suggests that there may exist an analogous relation between the Euclidean effective action \( W \) and the free-energy of the quantum field. Let us start with stationary space-times of the structure \( \Sigma \times R^1 \), where \( \Sigma \) are constant time sections, and assume a time-like Killing field acts globally on \( \mathcal{M} \).

For simplicity we proceed with a free scalar field. Generalization to other fields is possible but requires additional irrelevant details. Wave equation (23) on a stationary space-time can be written as

\[ P(\partial_t, \partial_i) \phi = 0. \tag{97} \]

The relation between Lorentzian and Euclidean operators is \( P_E(\partial_t, \partial_i) = P(i\partial_t, \partial_i). \)

To define statistical-mechanical quantities one needs the single-particle energies \( \omega_i \) introduced in (33). Their spectrum is determined by the problem

\[ (P_0 \omega^2 + P_1 \omega + P_2) \phi_{\omega} = 0, \tag{98} \]
which follows after substitution \( \phi(t,x') = e^{-i\omega t} \phi_0(x') \) to (97). Here \( P_k \) are \( k \)-th order partial differential operators. Eq. (98) is called non-linear spectral problem.

With help of the single-particle spectrum the free energy of the considered system can be written as

\[
F(\beta) = -\beta^{-1} \ln Z(\beta) = \beta^{-1} \sum_i \ln \left( 1 - e^{-\beta \omega_i} \right),
\]

(99)

where \( Z(\beta) \) is given by (58). The summation goes over all single-particle energies \( \omega_i^\pm \). For continuous spectra the sum is replaced with corresponding integrals.

One can formally define the vacuum energy \( E_0 = \frac{1}{2} \sum \omega_i \), where the series diverges and should be considered in the framework of some regularization prescription. Let \( E_{0,\text{ren}} \) be a finite (renormalized) part of \( E_0 \) left after subtracting divergent terms. Then there is the relation between the free energy and Euclidean action (86):

\[
W_{\text{ren}}[g] = \beta (F(\beta) + E_{0,\text{ren}}).
\]

(100)

The left and right parts of (100) coincide up to finite counterterms. Derivation of (100) can be found, e.g. in [41].

On the considered class of geometries \( F(\beta) \) is finite at large \( \omega_i \). As follows from (100) the UV divergent part of the effective action appears from the divergences of the vacuum energy. As we see in Sec. 8.3, the situation changes when the Killing field \( \zeta \) has the horizon.

### 7.2 High-temperature asymptotic

A Killing observer with 4-velocity along \( \zeta = \partial_t \) measures the so called local Tolman temperature

\[
T(x) = \frac{T_0}{\sqrt{-\zeta^2}},
\]

(101)

where \( T_0 = \beta^{-1} \). In asymptotically flat space-times \( T_0 \) is a temperature measured by observers at infinity. Factor \( \sqrt{-\zeta^2} \) relates coordinate time \( t \) to the proper time of the observer. The Tolman temperature increases if position of the observer is taken closer to the black hole horizon. One says that temperature is blueshifted near \( \mathcal{H} \).

Therefore in the near-horizon region the system is effectively at a high temperature regime. Interestingly, high-temperature limit allows an analytic form of the free energy which, in general, can be written in terms of characteristics of the Killing frame of reference, discussed in Sec. 2. For example, in the absence of boundaries the leading terms look as

\[
F(\beta) \simeq - \int \sqrt{-g} d^3x \left[ b_1 T^4(x) + T^2(x)(b_2 R + b_3 m^2 + b_4 \Omega^2(x) + b_5 w_\mu w^\mu + b_6 \nabla w) + O(\ln \beta) \right].
\]

(102)
Here $w_\mu$ is 4-acceleration, $\Omega = \frac{1}{2} (A_\mu A^\mu)^{1/2}$ is the absolute value of the local angular velocity, $A_\mu$ is the rotation tensor, see definitions (14). The structure of (102) is determined by using canonical mass dimensions of all quantities, $b_k$ are numerical coefficients which depend on the considered model and require computations. For a real scalar field with equation $(-\nabla^2 + V)\phi = 0$ one finds [37]

$$F(\beta) \simeq - \int d^3x \sqrt{-g} \left[ \frac{\pi^2}{90} T^4 + \frac{1}{24} T^2 \left( \frac{1}{6} R - V - \frac{2}{3} \Omega^2 \right) + O(\ln \beta) \right]. \quad (103)$$

According with (100) one can split the stress-energy tensor as

$$\langle HHI \mid \hat{T}_{\mu\nu} \mid HHI \rangle = \langle \hat{T}_{\mu\nu} \rangle_\beta + \langle \hat{T}_{\mu\nu} \rangle_{\text{vac}}. \quad (104)$$

Thermal part $\langle T_{\mu\nu} \rangle_\beta$ is determined by the free energy $F(\beta)$, while the vacuum part $\langle T_{\mu\nu} \rangle_{\text{vac}}$ follows from the vacuum energy. Variations of (103) over the metric yield the thermal part at high temperatures. For example, its expression in the case of conformal coupling, $V = R/6$, is [37]

$$\langle T_{\mu\nu} \rangle_\beta \sim \langle g_{\mu\nu} + 4 u_\mu u_\nu \rangle \left( \frac{\pi^2}{90} T^4 - \frac{1}{36} T^2 \Omega^2 \right) + \frac{T^2}{36} \left( 2 A_{\nu\rho} A^{\rho \mu} - u_\nu A_{,\mu} w^\lambda - u_\mu A_{,\nu} w^\lambda + u_\nu A_{,\mu} A^{\lambda \lambda} + u_\mu A_{,\nu} A^{\lambda \lambda} \right) + O(\ln \beta). \quad (105)$$

One can check that $\nabla^\mu \langle T_{\mu\nu} \rangle_\beta = 0$, $g^{\mu\nu} \langle T_{\mu\nu} \rangle_\beta = 0$. High-temperature asymptotics in static space-times have been studied in pioneering papers [24],[25]. Extension to stationary geometries, which requires one to deal with non-linear spectral problems (98) has been suggested in [37], see [41] for more details.

Asymptotics (102), (103) are of increasing interest in modern studies of equilibrium distributions in finite-temperature quantum field theories with rotation and acceleration in flat space-times [7], where $A_\mu$ is interpreted as a thermal vorticity. The interest is related to properties of quark-gluon plasma in heavy ion collisions.

8 Generalized thermodynamics of quantum black holes

8.1 Generalized free energy

The HHI state describes an eternal black hole in thermal equilibrium with its radiation. The system, a black hole and quantum fields in the HHI state, can be called a quantum black hole. The split of this system on classical and quantum components is rather conditional. The classical black hole geometry backreacts to quantum matter, while bare classical couplings participate in renormalization of ultraviolet divergences.
Black hole thermodynamics poses a number of fundamental questions to quantum gravity mentioned in previous Sections. As a first step in resolving the existing paradoxes, it is reasonable to understand thermodynamics of quantum black holes in HHI state. We describe here some steps how it can be done by using advantages of the Euclidean field theory. For simplicity, we consider non-rotating black holes. Extension of the arguments below to black holes with angular momentum is possible but it does not carry principle issues.

The key assumption we adopt is that the generalized free energy of a quantum black hole is determined by the effective action $\Gamma[\bar{g}]$, see (92). The background metric, $\bar{g}$, is a stationary point of $\Gamma[g]$ under certain boundary conditions, which can be defined as in classical case [83]. One also requires that $\bar{g}$ is a black-hole-instanton-like solution with a global Euclidean Killing vector field $\partial_\tau$, $\tau$ being a periodic coordinate with a period $\beta$. We interpret $\beta$ as an inverse temperature.

Variations of $\Gamma[\bar{g}]$ lead to the Einstein equations (94) sourced by the stress-energy tensor of the quantum matter. The functional $W[g]$ is a complicated non-local functional which is known in some approximations, in the second order curvature approximation [6], for example. Quantum corrected Schwarzschild solutions in the framework of these approximations have been discussed in [16],[82].

The generalized free-energy of a quantum black hole is defined as follows:

$$F_{\text{gen}}(\beta) = \beta^{-1} \Gamma[\bar{g}(\beta)] .$$

There are some subtle issues related to normalization of $\Gamma[\bar{g}(\beta)]$ in (106) by analogy with the Gibbons-Hawking subtraction procedure [42]. We do not imply any subtraction in the quantum part of the effective action before we find its statistical-mechanical interpretation. Eq. (106) allows one to introduce generalized energy and entropy of the black hole by using standard definitions of statistical physics

$$E_{\text{gen}}(\beta) = \partial_\beta (\beta F_{\text{gen}}(\beta)) , \quad S_{\text{gen}}(\beta) = \beta^2 \partial_\beta (F_{\text{gen}}(\beta)) .$$

By virtue of (107) parameters of two black holes with slightly different temperatures are related by the first law

$$\delta E_{\text{gen}}(\beta) = \beta^{-1} \delta S_{\text{gen}}(\beta) ,$$

which is the generalization of (56) for $\Omega_H = 0$.

The generalized free energy and entropy can be found in some cases, for example for a Schwarzschild black hole with massless quantum fields [35],[68]. This can be done by using scaling properties of $W_{\text{gen}}(\beta)$, and leads to logarithmic corrections to the Bekenstein-Hawking entropy. The result can be generalized to include higher loops [72]. Corrections for other types of black holes are discussed in [64],[69].

To proceed with (107) we note that derivative of the effective action over $\beta$ can be taken in two steps,

$$\partial_\beta \Gamma[\bar{g}] = \partial_\beta \Gamma[\bar{g}] + \int d^4x \sqrt{\bar{g}} \frac{\delta \Gamma[\bar{g}]}{\delta g_{\mu\nu}(x)} \partial_\beta \bar{g}_{\mu\nu}(x) + B .$$
The first term, $\partial_\beta \Gamma[\tilde{g}]_{\tilde{g}}$, is the derivative over $\beta$ when the bulk metric is fixed. This derivative changes the periodicity of the instanton $\tilde{g}$ and results in conical singularities at the Euclidean horizon $\mathcal{H}$. The second and third terms in the r.h.s. of (109) come out when one differentiates the metric but keeps the periodicity fixed. The third term, $B$, appears from the “inner boundary” $\mathcal{H}$. This term is related to singularities on $\mathcal{H}$. The “inner boundary” $\mathcal{H}$ in the off-shell approach should be taken into account when integrating by parts to get rid of derivatives of $\delta \tilde{g}$. Since there are no singularities on the physical boundary $\partial \mathcal{M}$ we assume that corresponding boundary terms in (109) are excluded by appropriate boundary conditions.

Since $\tilde{g}$ is a stationary point of $\Gamma[g]$ the second term in the r.h.s of (109) disappears, and one gets

$$E_{\text{gen}}(\beta) = \partial_\beta \Gamma[\tilde{g}]_{\tilde{g}} + B \ .$$

(110)

$$S_{\text{gen}}(\beta) = (\beta \partial_\beta - 1) \Gamma[\tilde{g}]_{\tilde{g}} + \beta B \ .$$

(111)

It follows from (92), properties of the classical Euclidean action (62) and (107) that

$$E_{\text{gen}}(\beta) = \bar{M}_H(\beta) + E_q(\beta) + B \ , \ S_{\text{gen}}(\beta) = \bar{S}_{BH}(\beta) + S_q(\beta) + \beta B \ .$$

(112)

Here $\bar{M}_H(\beta)$ and $\bar{S}_{BH}(\beta)$ are the mass and Bekenstein-Hawking entropy (for simplicity we assume that renormalized couplings $c_i = 0$ in (92)) computed by applying classical formula to the quantum corrected instanton $\tilde{g}$. We recall that $\beta^{-1}$ is identified with the Hawking temperature. Other quantum corrections in (112) are defined as

$$E_q(\beta) = \partial_\beta W_{\text{ren}}(\beta) \ , \ S_q(\beta) = (\beta \partial_\beta - 1) W_{\text{ren}}(\beta) \ .$$

(113)

Since we are interested in first order quantum corrections, it is enough to take $W_{\text{ren}}(\beta)$ on the classical instanton.

### 8.2 Energy of a quantum black hole

Consider first classical black holes. The total mass $M$ of a black hole, as measured at infinity, is defined by the differential mass formula [4] :

$$M = M_H + E \ ,$$

(114)

$$E = - \int_\Sigma d^3 x \sqrt{-g} \ T^0_0 \ ,$$

(115)

where $M_H$ is the mass measured at the horizon, and $E$ is the energy of matter outside the horizon. The covariant stress-energy tensor of matter $T^\mu_\nu$ for field theories is determined by the variation of the corresponding action over the metric. The integral in (115) is defined on a constant time slice $\Sigma$ outside the black hole.

Equation (114) suggests the following natural form of the generalized energy of a quantum black hole:

$$E_{\text{gen}}(\beta) = \bar{M}_H(\beta) + E(\beta) \ ,$$

(116)
\[ E(\beta) = -\int_{\Sigma} d^3x \sqrt{-g} \langle HHI | \hat{T}_0^0 | HHI \rangle . \] (117)

\( \hat{M}_H \) is identified with the mass at the horizon, while \( E \) is replaced with the expectation value of the corresponding energy operator in the HHI state.

Consider now quantum correction \( E_q \) in (112). This correction is defined in (113). It follows from (100) and thermal properties of the HHI state that

\[ E_q(\beta) = \langle : \hat{H} : \rangle_\beta + E_{0_{\text{ren}}} = \langle HHI | \hat{H} | HHI \rangle , \] (118)

The r.h.s. of (118) is the renormalized expectation value of the canonical energy operator \( \hat{H} \) without any normal ordering.

Definition of energy by (115) in terms of the metric stress-energy tensor and definition of energy as the Hamiltonian may differ by a total derivative which results in a non-vanishing term when \( \Sigma \) has \( \partial \) as an internal boundary, where the Killing field bifurcate. One can show that [28],[36]

\[ E = H - \beta Q . \] (119)

Here \( \beta \) is inverse Hawking temperature determined by the surface gravity of \( \zeta \), and \( Q \) is an integral over \( \partial \). The quantity \( Q \) is always non-trivial when there are non-minimal couplings between dynamical fields and the background curvature. It can be demonstrated for black holes in a general classical theory of gravity arising from a diffeomorphism invariant Lagrangian that [36]:

i) \( E \) appears in the first law of black hole mechanics and plays the same role as the energy in the differential mass formula (114);

ii) \( H \) is the canonical energy which generates evolution along the Killing time;

iii) \( Q \) in (116) is a Noether charge analogous to Wald’s charge;

iv) the entropy of a black hole in the presence of non-minimal couplings is

\[ S = S_{BH} - Q . \] (120)

To give a typical illustration of non-minimal coupling consider the scalar theory

\[ I[\phi, g] = -\frac{1}{2} \int d^4x \sqrt{-g} \left[ (\nabla \phi)^2 + \xi R \phi^2 + m^2 \phi^2 \right] . \] (121)

A direct check yields [28]

\[ Q = 2\pi \xi \int_{\partial B} d^2y \phi^2(y) . \] (122)

Other examples of non-minimal couplings with non-trivial charges are present for fields with integer spins.

The quantum version of (119) should be the following formula for the average energy (117):

\[ E(\beta) = \langle HHI | \hat{H} | HHI \rangle - \beta \langle HHI | \hat{Q} | HHI \rangle . \] (123)
By comparing (116) with (112) one concludes that
\[ B = -\beta \langle \hat{Q} \rangle, \] (124)

where \( \langle \hat{Q} \rangle = \langle HH| \hat{Q} |HH \rangle \). As can be seen from (122) the expectation value \( \langle \hat{Q} \rangle \) is determined by two-point correlators, like \( \langle \hat{\phi}(x)\hat{\phi}(x') \rangle \) in the limit \( x \to x' \). Since such limits results in UV divergent terms, it is assumed that \( \langle \hat{Q} \rangle \) is a renormalized quantity.

The presented arguments which lead to (124) cannot be considered as a rigorous proof but they pass non-trivial checks which we consider in the next Section.

### 8.3 Generalized black hole entropy and its interpretations

As follows from (112), (124) the generalized entropy of a quantum black hole is the sum of three terms,
\[ S_{\text{gen}}(\beta) = S^{BH}(\beta) + S_q(\beta) - \langle \hat{Q} \rangle. \] (125)

The quantum correction to black hole entropy includes contribution from non-minimal couplings \( \langle \hat{Q} \rangle \). This fact has been first pointed out in [70]. The origin of \( \langle \hat{Q} \rangle \) in (125) can be traced to the delta-function-like term in the scalar curvature on \( B \), which appears in the off-shell approach according to Eq. (64). Let us emphasize that \( S^{BH}(\beta) \) in (125) depends on the renormalized gravitational couplings, UV divergences in \( S_q(\beta) \) and \( \langle \hat{Q} \rangle \) are subtracted.

Consistency of (125) implies that divergences in \( S_q(\beta) \) and \( \langle \hat{Q} \rangle \) are absorbed in the course of renormalization of the bare couplings in \( S^{BH}(\beta) \). To demonstrate this in the one-loop approximation we focus on properties and interpretation of \( S_q(\beta) \). To derive \( S_q(\beta) \) by using (113) the one-loop effective action \( W_{\text{ren}}(\beta) \) has to be considered off-shell, at \( \beta \) slightly different from the Hawking temperature. We denote the corresponding background manifold \( M_\beta \). Near the Euclidean horizon \( M_\beta \) has the structure \( \mathcal{C}_\beta \times B \), where \( \mathcal{C}_\beta \) is a two-dimensional cone with metric
\[ dl^2 = \kappa^2 p^2 d\tau^2 + d\rho^2, \quad 0 \leq \tau < \beta. \] (126)

The nice property of elliptic operators that their spectral functions are well-defined on manifolds with conical singularities.

Conical singularities modify results known for regular manifolds. In particular, the heat coefficients of the heat trace (89) on \( M_\beta \) with \( p \) even look as
\[ a_p(P_E) = A_p + B_p, \] (127)

where \( A_p \) are defined by standard expressions on regular domain of \( M_\beta \). Additional terms \( B_p, p = 2k \), are some curvature polynomials on \( B \) which have non-trivial dependence on \( \beta \). For example [34].
\[ B_2(\beta) = f(\beta)A, \quad B_4(\beta) = \int_{\beta}^{2} \sqrt{\gamma} \, d^2 y \, (g_1(\beta)R + g_1(\beta)R_{ii} + g_3(\beta)R_{ijij}) , \] (128)

where \( f(\beta), g_k(\beta) \) are some functions analytic at \( \beta = 2\pi/k \), curvatures \( R_{ii}, R_{ijij} \) are defined by (68). Definition (113) implies that there must exist a divergent part of \( S_q \) determined by the divergent part of the effective action. In four dimensions, \( n = 4 \),

\[ S_{q,\text{div}}(\delta) = (\beta \partial_\beta - 1)W_{\delta} \simeq s_1 \delta + s_2 \ln \delta + O(\delta^{-1}) , \] (129)

\[ s_1 = -\frac{\eta}{2}(\beta \partial_\beta - 1)B_2|_{\beta = \beta_H} , \quad s_2 = \eta(\beta \partial_\beta - 1)B_4|_{\beta = \beta_H} \] (130)

To get (129) we used (90).

Note that the leading divergence of \( S_q \) is connected with \( B_2 \), and it is proportional to the horizon area \( A \). This divergence behaves as the Bekenstein-Hawking entropy and, as has been suggested in [75], [15], it can be removed by the standard renormalization of the Newton constant.

Consider a standard renormalization procedure in the one-loop effective action. To remove the divergences the bare gravity action should be taken in form (67) with bare couplings \( G^B, \Lambda_c^B, c_i^B \). The relation between bare and renormalized couplings \( G, \Lambda, c_i \) is (91). Given (91) one can show that the following formula holds [40], [70], [29], [71] :

\[ S^{BH}(G, c_i) = S^{BH}(G^B, c_i^B) + S_{q,\text{div}}(\delta) - \langle \hat{Q}(\delta) \rangle_{\text{div}} . \] (131)

The result holds for spins 0, 1/2 and 1. Formula (131) is a non-trivial check which supports (125).

One of the interpretations of \( S_q \) is related to quantum entanglement in the HHI state since states on \( \Sigma_L \) part of the Einstein-Rosen bridge are not accessible for observers in the right part of the eternal black hole, see Fig. 1. Consider a quantum system in a state \( |\psi\rangle \). Suppose some part \( A \) of the system cannot be measured. \( A \) may be, for example, a spatial region. Let \( B \) be a supplement of \( A \). Averages of operators located in \( B \) can be written as

\[ \langle \psi| \hat{O} |\psi\rangle = \text{Tr}_B (\hat{\rho} \hat{O}) , \] (132)

\[ \hat{\rho} = \text{Tr}_A |\psi\rangle \langle \psi| . \] (133)

The density matrix \( \hat{\rho} \) allows different measures of the information loss about states located in \( A \). For example, one can define the Renyi entropy

\[ S^{(\alpha)} = \frac{\text{Tr}_B \hat{\rho}^\alpha}{1 - \alpha} , \quad \alpha > 0 , \quad \alpha \neq 1 , \] (134)

and the entropy of entanglement

\[ S_{\text{ent}} = \lim_{\alpha \to 1} S^{(\alpha)} = -\text{Tr}_B \hat{\rho} \ln \hat{\rho} . \] (135)
A review of quantum entanglement and entanglement measures can be found in [51]. Entanglement in quantum field theories is discussed, e.g. in [17].

In the considered case of an eternal black hole $A$ and $B$ are two sides of the Einstein-Rosen bridge separated by the bifurcation surface $B$, say, $B = \Sigma_R$ and $A = \Sigma_L$. To calculate the entanglement entropy we use representation (83) for the HHI state. If one assumes that $\langle HHI|HHI \rangle = 1$, the normalization factor in (83) is $N = \text{Tr} e^{-\beta_H \hat{H}} = Z(\beta_H)$. This yields for natural parameters $n$

$$
\text{Tr}_B \hat{\rho}^n = \frac{Z(n\beta_H)}{Z^n(\beta_H)} = \exp(-W(n\beta_H) + nW(\beta_H)),
$$

(136)

where $W(\beta)$ should be considered as a regularized action. To get (136) we used (100) and noticed mutual cancellation of the vacuum energies in $W(n\beta_H)$ and in $nW(\beta_H)$. Since $W_{\text{ren}}(\beta)$ can be defined at arbitrary $\beta$ one can replace $n$ in the r.h.s. of (136) with a parameter $\alpha > 0$ and use (134), (135) to see that

$$
S_{en} = S_q .
$$

(137)

That is the entanglement entropy in HHI state is given by formula (113) and is a part of the generalized entropy (125).

The fact that $S_{en}$ is proportional to the horizon area and can be related to the Bekenstein-Hawking entropy was first pointed out in [12], [73], [31]. Since computations of the entanglement entropy based on (113) are reduced to finding effective actions on manifolds with conical singularities $S_{en}$ was also called a geometric entropy. Pioneering studies of $S_{en}$ can be found in [56],[55],[54],[57]. A separate interesting topic is the entanglement entropy of gauge fields which is discussed in a number of publications, see, e.g. [19], [23]. A comprehensive review of entanglement entropy of black holes is [71].

One can also come to (137) with the help of (83) by noticing that the reduced density matrix for the HHI state is thermal, $\hat{\rho} = e^{-\beta_H \hat{H}}/Z(\beta_H)$. This means that the entanglement entropy coincides with the entropy of the thermal atmosphere of a black hole. The first attempts to relate the Bekenstein-Hawking entropy to the thermal atmosphere have been made in [84],[50]. The thermal entropy can be derived from high-temperature asymptotic (102) and it diverges due to the infinite blue-shift of the temperature at the horizon, see (101). To avoid divergences a narrow region near the horizon should be excluded when integrating in (102). If the cutoff is made at a proper distance equal the Planck length the thermal entropy is of the order of the Bekenstein-Hawking entropy [50]. The divergences in the thermal entropy can be eliminated by using other regularizations, for instance, the Pauli-Villars (PV) regularization [22]. Equivalence between divergences of thermal entropy and entanglement entropy in different regularizations is discussed in [29].

To simplify the presentation in last sections we restricted the discussion by static black holes. The analysis can be extended to rotating black holes, see [30] and references therein.
9 Concluding remarks

The aim of this Chapter was an introduction to generalized thermodynamics of quantum black holes and related notions which appeared in last decades. The key conclusion is that the Bekenstein-Hawking entropy is a part of the generalized entropy (125) of a black hole and it is deeply related to quantum effects via the renormalization of the ultraviolet divergences.

Renormalization requires the bare entropy which does not have any statistical meaning in the perturbative quantum gravity. The problem of the bare entropy in (131) can be resolved [53] if the Einstein gravity is entirely induced by quantum effects, so that (125) becomes \( S_{\text{ren}} = S_q - \langle \hat{Q} \rangle \). This mechanism can be checked at a one-loop order in QFT models where the leading ultraviolet divergences are canceled out [33], [28]. The idea of induced gravity was formulated long ago [66], [67], see [78], for a review, but it does not contradict to a modern perspective. From the point of view of the open string theory black hole entropy can be considered as a loop effect [48], in full analogy with its origin in induced gravity. There still remains the problem of statistical interpretation of the “contact” term \( \langle \hat{Q} \rangle \) in the entropy, whose presence is unavoidable to compensate divergences in \( S_q \).

We have not discussed alternative interpretations of the black hole entropy which go beyond the perturbative quantum gravity. A number of promising ideas to statistical explanation of the Bekenstein-Hawking formula are reviewed in [18]. Among them is a remarkable calculation of the entropy of extremal [74] and near-extremal black holes in string theory, more on that can be found in [63].

There is a mounting evidence that the Bekenstein-Hawking entropy formula can be applied for systems which are not black holes. For instance, in higher dimensional anti-de Sitter gravities it allows successful holographic description [60], [65] of the entanglement entropy in dual conformal field theories. In a similar way, \( \mathcal{A}/(4G) \) can be interpreted as as an entanglement entropy in low-energy limit of quantum gravity for regions restricted by minimal surfaces of the area \( \mathcal{A} \) [37],[58].

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