Generalized Newton-Leibniz Formula and the Embedding of the Sobolev Functions into Hölder Spaces

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Abstract

It is well-known that the embedding of the Sobolev space of weakly differentiable functions into Hölder spaces holds if the integrability exponent is higher than the space dimension of the domain. In this paper, the embedding of the Sobolev functions into the Hölder spaces is expressed in terms of the minimal weak differentiability requirement independent of the integrability exponent. The proof is based on the generalized Newton-Leibniz formula in the rectangular prism and inductive application of the Sobolev trace embedding results. The particular motivation for the new embedding theorem appears in proving compactness of the family of multilinear interpolations of discrete grid functions arising as solutions of the discretized PDE problems.

Key words: generalized Newton-Leibniz formula, Sobolev spaces, weakly differentiable functions, Hölder spaces, embedding theorems

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1 Introduction and Main Result

Let \( Q \subset \mathbb{R}^n, n \geq 2 \) be a bounded domain with Lipschitz boundary, and \( W^1_p(Q), 1 < p \leq \infty \) be a Sobolev space of weakly differentiable functions \( u \in L^p(Q) \) with first order weak derivatives in \( L^p(Q), i = 1, ..., n \). Originally discovered in the celebrated paper [1], the concept of Sobolev spaces became a trailblazing idea in many fields of mathematics. The goal of this paper is to analyze embedding of \( W^1_p(Q) \) into Hölder spaces \( C^{0,p}(Q), 0 < \mu \leq 1 \) [2].

Standard notation will be employed for embedding of Banach spaces:

- \( B_1 \hookrightarrow B_2 \) means bounded embedding of \( B_1 \) into \( B_2 \), i.e. \( B_1 \subset B_2 \), and
  \[ \|u\|_{B_2} \leq C\|u\|_{B_1}, \forall u \in B_1, \text{ for some constant } C. \]
- \( B_1 \Subset B_2 \) denotes compact embedding of \( B_1 \) into \( B_2 \), meaning that \( B_1 \hookrightarrow B_2 \), and every bounded subset of \( B_1 \) is precompact in \( B_2 \).

If \( n = 1 \), the equivalency class of elements of \( W^1_p(Q) \) always contain an absolutely continuous element, which is Hölder continuous with exponent \( 1 - p^{-1} \), i.e. there is a bounded embedding

\[ W^1_p(Q) \hookrightarrow C^{0,1-p}(Q). \tag{1.1} \]

This fails to be true if \( n \geq 2 \) and \( p \leq n \). However, there is a bounded embedding

\[ W^1_p(Q) \hookrightarrow C^{0,1-p}(Q) \text{ if } p > n. \tag{1.2} \]

Hence, stretching the integrability exponent \( p \) beyond space dimension \( n \) implies the Hölder continuity. In particular, elements of the Hilbert space \( H^1(Q) = W^1_2(Q) \), are not Hölder continuous in general, if \( n \geq 2 \). The main goal of this paper is to express the Hölder continuity of elements of \( W^1_p(Q) \) in terms of weak differentiability requirements.

**Problem:** What are the minimal weak differentiability requirements on elements of \( W^1_p(Q) (1 < p \leq n) \) to be Hölder continuous?

We introduce new Sobolev-Banach space \( \tilde{W}^1_p(Q) \) as the following subspace of \( W^1_p(Q) \)

\[ \tilde{W}^1_p(Q) = \left\{ u \in W^1_p(Q) \bigg| \frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} \in L^p(Q), i_1 < \cdots < i_k, k = 2, n \right\}. \]

equipped with the norm

\[ \|u\|_{\tilde{W}^1_p(Q)} := \begin{cases} \left( \|u\|_{L^p(Q)}^p + \sum_{k=1}^n \sum_{i_1 < \cdots < i_k} \left\| \frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} \right\|_{L^p(Q)}^p \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \|u\|_{L^\infty(Q)} + \sum_{k=1}^n \sum_{i_1 < \cdots < i_k} \left\| \frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} \right\|_{L^\infty(Q)}, & \text{if } p = \infty. \end{cases} \]

If \( p = 2 \), \( \tilde{H}^1(Q) := \tilde{W}^1_2(Q) \) is a Hilbert space with inner product

\[ (f,g)_{\tilde{H}^1(Q)} = (f,g)_{L^2(Q)} + \sum_{k=1}^n \sum_{i_1 < \cdots < i_k} \left( \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}, \frac{\partial^k g}{\partial x_{i_1} \cdots \partial x_{i_k}} \right)_{L^2(Q)}. \]
The main result of this paper is to prove embedding of $\tilde{W}_p^1(Q)$ into H"older spaces. Proof is based on generalized Newton-Leibniz formula in rectangular prisms in $\mathbb{R}^n$. Let $x, x' \in \mathbb{R}^n$ with $x_i < x'_i, i = 1, n$ are fixed and $P$ be $n$-dimensional prism

$$P = \{ \eta \in \mathbb{R}^n : x_i \leq \eta_i \leq x'_i, i = 1, n \}$$

with vertex $x$ (or $x'$) called a low (or top) corner of $P$. For any subset $\{i_1, ..., i_k\} \subset \{1, ..., n\}, k = 1, n$, let

$$P_{i_1...i_k} = P \cap \{ \eta \in \mathbb{R}^n : \eta_l = x_l, l \neq i, j = 1, k \}$$

be a $k$-dimensional sub-prism with low corner $x$. Note that $P_{i_1...i_k}$ is invariant with respect to permutation of multiindex $i_1 \cdots i_k$, and it coincides with $P$ if $k = n$.

The main result of the paper reads:

**Theorem 1.** Let $1 < p \leq \infty$. For arbitrary bounded Lipschitz domain $Q \in \mathbb{R}^n, n \geq 2$, the following bounded and compact embeddings hold

$$\tilde{W}_p^1(Q) \hookrightarrow C^{0,1-\frac{1}{p}}(Q), \quad (1.3)$$

$$\tilde{W}_p^1(Q) \subset C^{0,\mu}(Q), 0 < \mu < 1 - \frac{1}{p}. \quad (1.4)$$

The equivalency class of every element of $\tilde{W}_p^1(Q)$ possesses a representative in $C^{0,1-\frac{1}{p}}(Q)$, which satisfies the following **generalized Newton-Leibniz formula**: 

$$u(x') - u(x) = \sum_{k=1}^{n} \sum_{i_1 < ... < i_k}^{n} \int_{P_{i_1...i_k}} \frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}} d\eta_{i_1} \cdots d\eta_{i_k} \quad (1.5)$$

where $x, x' \in Q$ are such that $P \subset Q$, with $P$ being a corresponding $n$-dimensional prism with low and top corner at $x$ and $x'$ respectively.

**Remark 2.** If $n = 1$, (1.5) coincides with the Newton-Leibniz formula. Note that for $\forall k$ there are $\binom{n}{k}$ integrals in (1.5) along all $k$-dimensional sub-prisms $P_{i_1...i_k}$ with low corner $x$. Therefore, altogether there are

$$\sum_{k=1}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} - 1 = (1 + 1)^n - 1 = 2^n - 1$$

integrals in (1.5) along all sub-prisms of $P$ with low corner at $x$. In particular, Theorem 1 implies that

$$\frac{\partial^k u}{\partial x_{i_1} \cdots x_{i_k}} \in L_p(P_{i_1...i_k}), k = 1, n-1, \quad (1.6)$$

in the sense of traces.
1.1 Notations

Let \( Q \) is a domain in \( \mathbb{R}^n \). We use the standard notation for Banach spaces \( C^k(Q) \) of \( k \)-times continuously differentiable functions on \( Q \), and we simply write \( C(Q) \), if \( k = 0 \). \( C_0^\infty(Q) \) is a space of infinitely differentiable and compactly supported functions. The following standard notation will be used for Hölder spaces:

- For \( 0 < \gamma \leq 1 \), Hölder space \( C^{0,\gamma}(Q) \) is the Banach space of elements \( u \in C(Q) \) with finite norm
  \[
  \| u \|_{C^{0,\gamma}(Q)} := \| u \|_{C(Q)} + [u]_{C^{0,\gamma}(Q)}
  \]
  where
  \[
  [v]_{C^{0,\gamma}(Q)} := \sup_{x, x' \in Q, x \neq x'} \frac{|v(x) - v(x')|}{|x - x'|^\gamma}
  \]

Throughout the paper we use standard notations for \( L_p(Q), 1 \leq p \leq \infty \) spaces; the following standard notations are used for Sobolev spaces \( W^s_p(Q) \):

- For \( 1 \leq p \leq \infty \), Sobolev space \( W^1_p(Q) \) is the Banach space of measurable functions on \( Q \) with finite norm
  \[
  \| u \|_{W^1_p(Q)} := \left\{ \begin{array}{ll}
  \left( \| u \|_{L_p(Q)}^p + \| Du \|_{L_p(Q)}^p \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\
  \| u \|_{L_\infty(Q)} + \| Du \|_{L_\infty(Q)}, & \text{if } p = \infty,
  \end{array} \right.
  \]
  where \( Du \) is a gradient of \( u \). In particular, if \( p = 2 \), \( H^1(Q) := W^1_2(Q) \) is a Hilbert space with inner product
  \[
  (f, g)_{H^1(Q)} = (f, g)_{L_2(Q)} + (Df, Dg)_{L_2(Q)}
  \]

- For \( s = (s_1, ..., s_n) \in \mathbb{Z}_+^n, 1 \leq p \leq \infty \), anisotropic Sobolev space \( W^s_p(Q) \) is the Banach space of measurable functions on \( Q \) with finite norm
  \[
  \| u \|_{W^s_p(Q)} := \left\{ \begin{array}{ll}
  \left( \| u \|_{L_p(Q)}^p + \sum_{i=1}^n \sum_{k=1}^{s_i} \left\| \frac{\partial^{s_i} u}{\partial x_i^k} \right\|_{L_p(Q)}^p \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\
  \| u \|_{L_\infty(Q)} + \sum_{i=1}^n \sum_{k=1}^{s_i} \left\| \frac{\partial^{s_i} u}{\partial x_i^k} \right\|_{L_\infty(Q)}, & \text{if } p = \infty
  \end{array} \right.
  \]
  If \( p = 2 \), \( H^s(Q) := W^s_2(Q) \) is a Hilbert space with inner product
  \[
  (f, g)_{H^s(Q)} = (f, g)_{L_2(Q)} + \sum_{i=1}^n \sum_{k=0}^{s_i} \left( \frac{\partial^k f}{\partial x_i^k}, \frac{\partial^k g}{\partial x_i^k} \right)_{L_2(Q)}
  \]
  Note that the size of the vector \( s \) coincides with the dimension of \( Q \).
  In particular, for domains \( Q \subset \mathbb{R}^k, 1 \leq k \leq n \), and fixed \( j \in \{1, ..., k\} \), we consider Sobolev spaces \( W^s_p(Q) \) of the weakly \( x_j \)-differentiable functions, where \( s = (s_i)_{i=1}^k \in \mathbb{Z}_+^k \) and \( s_i = \delta_{ij} \) is a Kronecker symbol.
2 Motivation

The motivation and application for the main result of this paper arises in approximation theory. The new Sobolev space \( \tilde{W}_p^1(Q) \) appears to be a natural space for multilinear interpolation of discrete grid functions arising as solutions of discretized PDE problems, as well as discretized optimal control problems for systems with distributed parameters. Theorem 1 provides a key compactness criterion for the family of multilinear interpolations of solutions of PDE problems discretized via finite differences. [3, 4].

Let \( h > 0 \) and cut \( \mathbb{R}^n \) by the planes
\[
x_i = k_i h, \ i = 1, \ldots, n \ \forall \ k_i \in \mathbb{Z}.
\]
into a collection of elementary cells with length \( h \) in each \( x_i \)-direction. For every \( h > 0 \) and multi-index \( \alpha = (k_1, \ldots, k_n) \) we define a cell \( P_\alpha^h \) as
\[
P_\alpha^h = \{ x \in \mathbb{R}^n | k_i h \leq x_i \leq (k_i + 1)h, \ i = 1, \ldots, n \}, \tag{2.1}
\]
and consider the collection of cells which have non-empty intersection with \( Q \):
\[
\mathcal{S}_h = \{ P_\alpha^h | P_\alpha^h \cap Q \neq \emptyset \} \tag{2.2}
\]
We now introduce exterior approximation of \( Q \) as follows:
\[
Q_h = \bigcup_{P_\alpha^h \in \mathcal{S}_h} P_\alpha^h \tag{2.3}
\]
Obviously, we have \( \overline{Q} \subset Q_h \). Consider a lattice
\[
\mathcal{L} = \left\{ x \in \mathbb{R}^n \mid \exists \alpha \in \mathbb{Z}^n \text{ s.t. } x_i = k_i h, \ i = 1, \ldots, n \right\}.
\]
Denote \( x_\alpha := (k_1 h, \ldots, k_n h) \), and consider a natural bijection \( \mathcal{A} : x_\alpha \mapsto \alpha \). Given a set \( X \subset \mathcal{L} \), we write \( \mathcal{A}(X) \) as the corresponding indexing set. Moreover, if \( X \subset \mathbb{R}^n \), then \( \mathcal{L}(X) := \mathcal{L} \cap X \). When \( X = \mathcal{L}(Y) \subset \mathbb{R}^n \), we’ll agree to write \( \mathcal{A}(Y) \) instead of \( \mathcal{A}(\mathcal{L}(Y)) \).

Let \( \{ u_\alpha : \alpha \in \mathcal{A}(Q_h) \} \) be a discrete grid function. We adopt the notation
\[
\alpha \pm e_i := (k_1, \ldots, k_i \pm 1, \ldots, k_n),
\]
and introduce standard notation for finite differences of grid functions \( u_\alpha \):
\[
u_{\alpha x_i} = \frac{u_{\alpha + e_i} - u_\alpha}{h}, \ u_{\alpha x_i} = \frac{u_\alpha - u_{\alpha - e_i}}{h}, \ u_{\alpha x_i x_j} = (u_{\alpha x_i})_{x_j}, \text{ etc.}
\]
Let \( u_h' \) be a multilinear interpolation of \( \{ u_\alpha : \alpha \in \mathcal{A}(Q_h) \} \), which assigns the value \( u_\alpha \) to each grid point of \( Q_h \), and is piecewise linear with respect to each variable \( x_i \) when the other variables are fixed. Precisely,
\[
u_h'(x) = u_\alpha + \sum_{k=1}^n \sum_{i_1, \ldots, i_k} u_{\alpha x_{i_1} \ldots x_{i_k}} \prod_{1 \leq j \leq k} (x_{i_j} - k_{i_j} h), \ x \in P_\alpha^h \subseteq \overline{Q}.
\]
One can easily check that the multilinear interpolation \( u'_h \) is an element of \( \tilde{W}_p^1(Q) \). Indeed, for \( \forall \ m = 1, ..., n - 1 \) and \( 1 \leq i_1 < \cdots < i_m \leq n \), we have
\[
\frac{\partial^m u'_h(x)}{\partial x_1 \cdots \partial x_m} = u_{a_{x_1 \cdots x_m}} + \sum_{k=1}^{n-m} \sum_{i_1 < \cdots < i_k \neq i_1, \ldots, i_m} u_{a_{x_{i_1} \cdots x_{i_k}}} \prod_{1 \leq j \leq k} (x_{i_j} - k_{i_j} h),
\]
and
\[
\frac{\partial^m u'_h(x)}{\partial x_1 \cdots \partial x_n} = u_{a_{x_1 \cdots x_n}},
\]
for \( x \in \text{int} P^n_h \subset Q \). Note that every indicated partial derivative of \( u \) is piecewise constant with respect to the differentiated variables, and piecewise linear with respect to the non-differentiated variables, when all the other variables are fixed. Therefore, none of the other weak derivatives of \( u \) besides the ones included in \( \tilde{W}_2^1(Q) \) exist. Hence, the multilinear interpolations are typical elements of the Sobolev-Banach space \( \tilde{W}_p^1(Q) \).

3 Proof of Theorem 1

Step 1. Assuming that \( u \in C^n(P) \), we prove (1.5) by induction in terms of the space dimension \( n \). If \( n = 1 \), it coincides with the Newton-Leibniz formula. Assume that (1.5) is true, and demonstrate that it is true if \( n \) is replaced with \( n + 1 \). Let \( x, x' \in \mathbb{R}^{n+1} \) with \( x_i < x'_i, i = 1, n + 1 \), are fixed. We have
\[
u(x') - u(x) = (u(x') - u(\bar{x}, x'_{n+1})) + (u(\bar{x}, x'_{n+1}) - u(x)), \tag{3.1}
\]
where \( \bar{x} = (x_1, ..., x_n) \). Applying (1.5) to the first term and the Newton-Leibniz formula to the second term in (3.1), we derive
\[
u(x') - u(x) = \sum_{k=1}^{n} \sum_{i_1 < \cdots < i_k} \int_{P_{1 \cdots k}} \frac{\partial^k u(\bar{\eta}, x'_{n+1})}{\partial x_{i_1} \cdots \partial x_{i_k}} d\eta_{i_1} \cdots d\eta_{i_k}
+ \int_{x'_{n+1}}^{x_{n+1}} \frac{\partial u(\bar{x}, \eta)}{\partial x_{n+1}} d\eta. \tag{3.2}
\]
Applying Newton-Leibniz to all but the last integrand, we have
\[
u(x') - u(x) = \sum_{k=1}^{n} \sum_{i_1 < \cdots < i_k} \int_{x'_{n+1}}^{x_{n+1}} \int_{P_{1 \cdots k}} \frac{\partial^{k+1} u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k} \partial x_{n+1}} d\eta_{i_1} \cdots d\eta_{i_k} d\eta_{n+1}
+ \int_{x_{n+1}}^{x'_{n+1}} \frac{\partial u(\bar{x}, \eta)}{\partial x_{n+1}} d\eta + \sum_{k=1}^{n} \sum_{i_1 < \cdots < i_k} \int_{x_{n+1}}^{x'_{n+1}} \frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}} d\eta_{i_1} \cdots d\eta_{i_k}. \tag{3.3}
\]
which imply that
\[ u(x') - u(x) = \sum_{k=1}^{n+1} \sum_{i_1, \ldots, i_k=1}^{n+1} \int_{P_{i_1 \cdots i_k}} \frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}} \, d\eta_{i_1} \cdots d\eta_{i_k}, \tag{3.4} \]

where we use the same notation for the prism \( P \), as well as its corresponding sub-prisms in \( \mathbb{R}^{n+1} \). Indeed, divide all \( 2^{n+1} - 1 \) sub-prisms of \( P \) with low corner at \( x \) into two groups depending on whether or not the edge \( p_{n+1} \) joining vertices \( x \) and \( (\tilde{x}, x'_{n+1}) \) is contained in it. The first two terms on the right hand side of (3.3) contain all \( 2^n \) terms of (3.4) with integrals along sub-prisms containing the edge \( p_{n+1} \), and the last term on the right hand side of (3.3) contains all \( 2^n - 1 \) integrals in (3.4) along sub-prisms which do not contain the edge \( p_{n+1} \). This completes the proof by induction.

**Step 2.** Now we prove that for \( u \in \dot{W}^1_p(Q) \), each of the \( 2^n - 1 \) integrals on the right hand side of (1.5) is finite, and in particular, (1.6) is satisfied. Existence of the integral with \( k = n \) on the right hand side of (1.5) follows from definition of \( \dot{W}^1_p(Q) \) and Hölder inequality. Existence of the remaining \( 2^n - 2 \) trace integrals in (1.5) follows from Sobolev trace embedding theorem ([2]) via induction argument. First, we demonstrate that the claim is true if \( k = n - 1 \). Then we show that the claim is true for any \( k < n - 1 \), provided it is true for \( k + 1 \). Indeed, if \( k = n - 1 \), for each of the \( n \) integrals we select a unique integer \( j \) satisfying
\[ j \in \{1, \ldots, n\} \cap \{i_1, \ldots, i_k\}^c \tag{3.5} \]

and define a multi-index \( s = (s_1, \ldots, s_n) \in \mathbb{Z}^n_k \), where \( s_i = \delta_{ij} \) is a Kronecker symbol. We have
\[ \frac{\partial^{n-1} u}{\partial x_{i_1} \cdots \partial x_{i_{n-1}}} \in W^s_p(P). \tag{3.6} \]

\( n - 1 \)-dimensional prism \( P_{i_1 \cdots i_{n-1}} \) is a boundary of \( P \) on the hyperplane \( x_j = \text{const} \). Sobolev trace bounded embedding result implies [2]:
\[ W^s_p(P) \hookrightarrow L_p(P_{i_1 \cdots i_{n-1}}), \tag{3.7} \]

which proves \( n \) relations (1.6) in the case \( k = n - 1 \). It remains to be proven that the claim is true for \( k \), if it is so for \( k + 1 \). For any of the \( \binom{n}{k} \) integrals in (1.5) along the \( k \)-dimensional prism \( P_{i_1 \cdots i_k} \) we select any integer \( j \) satisfying (3.5), and define a multiindex \( s = (s_1, \ldots, s_{k+1}) \in \mathbb{Z}^{k+1}_k \), where \( s_i = \delta_{ij} \) is a Kronecker symbol. Noting that \( P_{i_1 \cdots i_{k+1}} \) is invariant with respect to permutations of the multi-index \( i_1 \cdots i_{k+1} \), and due to the induction assumption we have
\[ \frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} \in W^s_p(P_{i_1 \cdots i_{k+1}}). \tag{3.8} \]

\( k \)-dimensional prism \( P_{i_1 \cdots i_k} \) is a boundary of \( k+1 \)-dimensional prism \( P_{i_1 \cdots i_{k+1}} \) on the hyperplane \( x_j = \text{const} \). Sobolev trace embedding result implies:
\[ W^s_p(P_{i_1 \cdots i_{k+1}}) \hookrightarrow L_p(P_{i_1 \cdots i_k}), \tag{3.9} \]
which proves the relations (1.6) for all \( k \)-dimensional integrals.

Step 3. Next, we are going to derive the interior Hölder seminorm estimate of \( u \in C^n(Q) \) in terms of \( \tilde{W}_p^1(Q) \). Let \( x, x' \in Q \), and \( \delta > 0 \) is chosen such that

\[
P^\delta := \{ \eta \in \mathbb{R}^n : x_i \leq \eta_i \leq x'_i + \delta, \ i = 1, n \} \subset Q.
\]

We prove the following estimate:

\[
|u(x) - u(x')| \leq C(p, \delta, n) \| u \|_{\tilde{W}_p^1(Q)} |x - x'|^{\frac{p-1}{p}}, \tag{3.10}
\]

where the exponent \( \frac{p-1}{p} \) must be replaced with 1, if \( p = \infty \). The proof of (3.10) is based on (1.5). By using Hölder inequality the integral on the right hand side of (1.5) with \( k = n \) is estimated as follows

\[
\left| \int_P \frac{\partial^n u(\eta)}{\partial x_1 \cdots \partial x_n} d\eta \right| \leq |P| \left\| \frac{\partial^n u}{\partial x_1 \cdots \partial x_n} \right\|_{L_p(P)}, \tag{3.11}
\]

where \( |P| \) denotes volume of the prism \( P \). For \( k = 1, \ldots, n-1 \), estimation of any of the \( k \)-dimensional integrals on the right hand side of (1.5) will be pursued in \( n - k \) steps. Consider typical \( k \)-dimensional integral in (1.5) along the prism \( P_{i_1 \cdots i_k} \). The idea is based on successive application of the trace embedding result (3.9) \( n - k \) times. First we select any integer \( j \) from (3.5), and assign it to multiindex component \( i_{k+1} \). By using Hölder inequality we have

\[
\left| \int_{P_{i_1 \cdots i_k}} \frac{\partial^k u(\eta)}{\partial x_{i_1} \cdots \partial x_{i_k}} d\eta_{i_1} \cdots d\eta_{i_k} \right| \leq |P_{i_1 \cdots i_k}| \left\| \frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}} \right\|_{L_p(P_{i_1 \cdots i_k})} \tag{3.12}
\]

Consider a function

\[
\zeta(\eta) = 1 - \frac{\eta_{i_{k+1}} - x'_{i_{k+1}}}{x'_{i_{k+1}} - x_{i_{k+1}} + \delta}, \tag{3.13}
\]

which satisfy

\[
0 \leq \zeta \leq 1, \quad \left| \frac{\partial \zeta}{\partial \eta_{i_{k+1}}} \right| \leq \frac{1}{\delta} \tag{3.14}
\]
We have
\[
\int_{P_{1\ldots i_k}} \left| \frac{\partial^k u(\eta)}{\partial x_{i_1} \ldots \partial x_{i_k}} \right|^p d\eta_1 \ldots d\eta_k = \int_{P_{1\ldots i_k}} \left| \frac{\partial^k u(\eta)}{\partial x_{i_1} \ldots \partial x_{i_k}} \right|^p d\eta_1 \ldots d\eta_k
\]
\[
= - \int_{P_{1\ldots i_k}} \int_{x_{i_k}+\delta} x'_{i_k+\delta} \frac{\partial}{\partial x_{i_k+1}} \left( \left| \frac{\partial^k u(\eta)}{\partial x_{i_1} \ldots \partial x_{i_k}} \right|^p d\eta_{i_k+1} d\eta_1 \ldots d\eta_k 
\]
\[
= - \int_{P_{1\ldots i_k}} \int_{x_{i_k}+\delta} x'_{i_k+\delta} \left[ \frac{\partial \zeta}{\partial x_{i_k+1}} \left| \frac{\partial^k u(\eta)}{\partial x_{i_1} \ldots \partial x_{i_k}} \right|^p + \zeta \frac{\partial}{\partial x_{i_k+1}} \right] \left| \frac{\partial^{k+1} u(\eta)}{\partial x_{i_1} \ldots \partial x_{i_k} \partial x_{i_k+1}} \right| d\eta_{i_k+1} d\eta_1 \ldots d\eta_k
\]
\[
\text{sgn} \left( \frac{\partial^k u(\eta)}{\partial x_{i_1} \ldots \partial x_{i_k}} \right) \left| \frac{\partial^{k+1} u(\eta)}{\partial x_{i_1} \ldots \partial x_{i_k} \partial x_{i_k+1}} \right| \] (3.15)

By using Young’s inequality and (3.14), from (3.15) it follows
\[
\int_{P_{1\ldots i_k}} \left| \frac{\partial^k u(\eta)}{\partial x_{i_1} \ldots \partial x_{i_k}} \right|^p d\eta_1 \ldots d\eta_k \leq \int_{P_{1\ldots i_k}} \int_{x_{i_k}+\delta} x'_{i_k+\delta} \left[ \left| \frac{\partial^{k+1} u(\eta)}{\partial x_{i_1} \ldots \partial x_{i_k} \partial x_{i_k+1}} \right| 
\]
\[
+ \left( p - 1 + \frac{1}{b} \right) \left| \frac{\partial^k u(\eta)}{\partial x_{i_1} \ldots \partial x_{i_k}} \right|^p \right] d\eta_{i_k+1} d\eta_1 \ldots d\eta_k. \] (3.16)

If \( k = n - 1 \), we derive an upper bound of the right hand side of (3.16) by replacing integration domain with \( Q \), and after using this upper bound in (3.12) we have
\[
\left| \int_{P_{1\ldots i_k}} \frac{\partial^k u(\eta)}{\partial x_{i_1} \ldots \partial x_{i_k}} d\eta_1 \ldots d\eta_k \right| \leq \| P_{1\ldots i_k} \|^{\frac{2^k}{p}} C \| u \|_{\tilde{W}^{2^k}_p(\mathbb{R}^n)}, \] (3.17)

where \( C = 2^k \left( p - 1 + \frac{1}{b} \right)^{\frac{1}{p}} \). If \( k < n - 1 \), then we select any integer \( j \) from (3.5) with \( k \) replaced by \( k + 1 \), and assign it to multiindex component \( i_{k+2} \). Then we define \( \zeta \) as in (3.13) with \( k \) replaced by \( k + 1 \), and apply similar estimation like in (3.15), (3.16) to two integrals on the right hand side of (3.16). After repeating this cycle \( n - k \) times, we eventually deduce (3.17), where \( C \) depends on \( n, p, \delta \). Finally, using all estimates (3.17), from (1.5), the desired estimate (3.10) follows.

**Step 4.** Since \( \partial Q \) is Lipschitz, there exists an extension \( \tilde{u} \in \tilde{W}^1_p(\mathbb{R}^n) \) such that [2]
\[
\begin{align*}
\tilde{u} &= u \text{ a.e. in } Q \\
\tilde{u} &\text{ has a compact support} \\
\| \tilde{u} \|_{\tilde{W}^1_p(\mathbb{R}^n)} &\leq C \| u \|_{\tilde{W}^1_p(\mathbb{R}^n)}
\end{align*}
\] (3.18)

According to the approximation theorem ([2]) we can select a sequence \( u_m \in C^\infty_0(\mathbb{R}^n) \) such that
\[
\| u - \tilde{u} \|_{\tilde{W}^1_p(\mathbb{R}^n)} \to 0, \text{ as } m \to \infty. \] (3.19)
Applying estimate (3.10) to $u_m$ with fixed $\delta = 1$, we have

$$\| u_m \|_{C^{0,1-\frac{1}{p}}(\mathbb{R}^n)} \leq C \| u_m \|_{\tilde{W}^{1,p}_p(\mathbb{R}^n)}. \tag{3.20}$$

Now fix $x \in \mathbb{R}^n$. By using (3.20) and Hölder inequality we deduce

$$|u_m(x)| \leq \int_{|y-x| \leq 1} |u_m(x) - u_m(y)| \, dy + \int_{|y-x| \leq 1} |u_m(y)| \, dy \leq [u_m]_{C^{0,1-\frac{1}{p}}(\mathbb{R}^n)} + \Gamma_n^{\frac{1}{p}} \| u_m \|_{L_p(\mathbb{R}^n)} \leq C \| u_m \|_{\tilde{W}^{1,p}_p(\mathbb{R}^n)}, \tag{3.21}$$

where $\Gamma_n$ is a volume of the unit ball in $\mathbb{R}^n$. From (3.20), (3.21) it follows that

$$\| u_m \|_{C^{0,1-\frac{1}{p}}(\mathbb{R}^n)} \leq C \| u_m \|_{\tilde{W}^{1,p}_p(\mathbb{R}^n)}. \tag{3.22}$$

Equivalently, we have

$$\| u_m - u_l \|_{C^{0,1-\frac{1}{p}}(\mathbb{R}^n)} \leq C \| u_m - u_l \|_{\tilde{W}^{1,p}_p(\mathbb{R}^n)}, \tag{3.23}$$

for all $m, l \geq 1$, whence there exists a function $u_* \in C^{0,1-\frac{1}{p}}(\mathbb{R}^n)$ such that

$$\| u_m - u_* \|_{C^{0,1-\frac{1}{p}}(\mathbb{R}^n)} \rightarrow 0, \text{ as } m \rightarrow \infty. \tag{3.24}$$

From (3.18) it follows that $u_* = u$, a.e. on $Q$, so that $u_*$ is in the equivalency class of $u$. Passing to limit as $m \rightarrow \infty$, from (3.22) it also follows that

$$\| u_* \|_{C^{0,1-\frac{1}{p}}(\mathbb{R}^n)} \leq C \| \bar{u} \|_{\tilde{W}^{1,p}_p(\mathbb{R}^n)}. \tag{3.25}$$

From (3.18), (3.25) we deduce

$$\| u_* \|_{C^{0,1-\frac{1}{p}}(Q)} \leq C \| \bar{u} \|_{\tilde{W}^{1,p}_p(Q)}, \tag{3.26}$$

which proves the bounded embedding (1.3). Compact embedding result (1.4) follows from (1.3) and Arzela-Ascoli’s theorem.

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