Non-triviality in a totally asymmetric one-dimensional Boolean percolation model on a half-line

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Abstract

It is well known that there are two regimes in a standard one-dimensional Boolean percolation model: either the entire space is covered a.s., or the covered volume fraction is strictly less than one. The aim of this work is to demonstrate that there is a third possibility in a Boolean model with totally asymmetric grains on a half-line: a.s. there is no unbounded component, but the covered volume fraction is one. An explicit condition is given characterizing the existence of an unbounded occupied component.

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1 Introduction

Let $\mathbb{R}_+ = [0, \infty)$ and let $\mathcal{P}$ be a Poisson point process on $\mathbb{R} \times (0, +\infty)$ with intensity measure $\lambda du \times \mu$. Here $\mu$ is a probability distribution on $(0, +\infty)$, and $\lambda > 0$. We consider totally asymmetric Boolean percolation (TABP) on $\mathbb{R}$ and $\mathbb{R}_+$ defined as follows. On $\mathbb{R}$ the occupied part of the space is the union

$$\mathcal{U} = \bigcup_{(u, \rho_u) \in \mathcal{P}} [u, u + \rho_u].$$

On $\mathbb{R}_+$ the occupied part of the space is

$$\mathcal{U}_+ = \bigcup_{(u, \rho_u) \in \mathcal{P}: u \geq 0} [u, u + \rho_u].$$

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The model just described belongs to the class of the germ-grain models [CSKM13, Chapter 6], [SW08, Chapter 4]. The intervals $[u, u + \rho_u], (u, \rho_u) \in \mathcal{P}$, are the grains, and the points $u$ are the germs. A discrete-space equivalent of TABP appears in [Lam70, Section 3], where the grains are interpreted as fountains that wet sites to their right.

The properties of the corresponding symmetric model are well known. Let $\ell$ be one-dimensional Lebesgue measure. For $Q = \mathbb{R}_+$ or $Q = \mathbb{R}$, define the covered volume fraction ([MR96, Chapter 5]) as

$$\lim_{n \to \infty} \frac{\ell([-n, n] \cap Q \cap \mathcal{U})}{\ell([-n, n] \cap Q)} \quad (1)$$

Let $\rho$ be a random variable with distribution $\mu$. In the standard one-dimensional Boolean model the grains are given by $[u - \rho_u, u + \rho_u]$, and only two regimes exist ([MR96, Sections 3.2 and 5.1]):

- If $\mathbb{E}\rho < \infty$ a.s. no unbounded occupied component occurs and the covered volume fraction is $1 - e^{-2\lambda\mathbb{E}\rho} \in (0, 1)$.

- If $\mathbb{E}\rho = \infty$ for any $\lambda > 0$ a.s. the entire space is covered.

These two regimes exhaust all possible scenarios also for TABP on $\mathbb{R}$. Indeed,

$$\mathcal{U} = \bigcup_{(u, \rho_u) \in \mathcal{P}} [u, u + \rho_u] = \bigcup_{(u, \rho_u) \in \mathcal{P}} \left[u + \frac{\rho_u}{2} - \frac{\rho_u}{2}, u + \frac{\rho_u}{2} + \frac{\rho_u}{2}\right] = \bigcup_{(v, \rho_v) \in \mathcal{P}'} [v - \rho_v, v + \rho_v],$$

where $\mathcal{P}' = \{(v, \rho_v) : v = u + \frac{\rho_u}{2}, \rho_v = \frac{\rho_u}{2} \text{ for some } (u, \rho_u) \in \mathcal{P}\}$ is a Poisson point process on $\mathbb{R} \times (0, +\infty)$ with intensity $\lambda du \times \mu_2$, where $\mu_2$ is the distribution of $\rho/2$. Note that $\mathcal{P}'$ is the image of $\mathcal{P}$ under the linear transformation $(a, b) \mapsto (a + \frac{b}{2}, \frac{b}{2})$. Thus we are essentially in the settings of the Poisson Boolean model [MR96]. Hence for TABP on $\mathbb{R}$ given by $\mathcal{U}$

- If $\mathbb{E}\rho < \infty$ a.s. no unbounded occupied component occurs and the covered volume fraction is $1 - e^{-\lambda\mathbb{E}\rho} \in (0, 1)$.

- If $\mathbb{E}\rho = \infty$ for any $\lambda > 0$ a.s. the entire space is covered.

The aim of the this work is to demonstrate that for TABP on $\mathbb{R}_+$ (given by $\mathcal{U}_+$) there is another alternative. It appears in item (iii) of Theorem 1.1, where $\mathbb{E}\rho = \infty$ and the covered volume fraction is 1 and despite that no unbounded component exists. Furthermore, the necessary and sufficient conditions for the existence of an unbounded occupied component for $\mathcal{U}$ and $\mathcal{U}_+$ are different.

**Theorem 1.1.** Consider totally asymmetric Boolean percolation on $\mathbb{R}_+$.  

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(i) Assume that $E\rho < \infty$. Then a.s. no unbounded occupied component exists, and the covered volume fraction is $1 - e^{-\lambda E\rho}$.

(ii) Assume that $\int_0^\infty e^{-\lambda E(\rho \wedge t)} dt < \infty$. Then a.s. there exists an unbounded occupied component.

(iii) Assume that $E\rho = \infty$ and $\int_0^\infty e^{-\lambda E(\rho \wedge t)} dt = \infty$. Then a.s. there is no unbounded occupied component, however the expected size of the occupied component containing 1 is infinite and the covered volume fraction is 1.

**Corollary 1.2.** Consider totally asymmetric Boolean percolation on $\mathbb{R}_+$. A.s. there exists an unbounded occupied component if and only if $\int_0^\infty e^{-\lambda E(\rho \wedge t)} dt < \infty$. A.s. there is no unbounded occupied component if and only if $\int_0^\infty e^{-\lambda E(\rho \wedge t)} dt = \infty$.

Note that the assumption $\int_0^\infty e^{-\lambda E(\rho \wedge t)} dt < \infty$ in (iii) of Theorem 1.1 implies $E\rho = \infty$. All possible distributions $\mu$ are covered by Theorem 1.1 and items (i), (ii), and (iii) constitute distinct cases. It is interesting to note that in the settings of (iii) the expected size of an occupied component is infinite, yet despite that a.s. every occupied component is finite.

In [BK20] similar results are collected for a discrete version of the model. In particular, the discrete space equivalent of Theorem 1.1 can be found in [BK20, Section 3]. As one might expect, in dimension $d \geq 2$ the geometry of the Boolean model and the interplay between occupied and vacant components are more intricate. There is a substantial amount of literature on the subject. We mention only a few recent works on the connectivity properties (such as the decay of the probability of reaching a remote point from the origin in the subcritical regime, or the sharpness of the phase transition) [DCRT20, ATT18] and the capacity functional (that is, the probability that the occupied part of the space intersects a given compact set) [LPZ17].

The proof of Theorem 1.1 is located Section 2. An example of a distribution satisfying conditions of Theorem 1.1, (iii), can be found in Remark 2.1.

## 2 Proof of Theorem 1.1

By [SW08, Theorem 9.3.5], the Boolean model is ergodic, thus the sequence $\{\psi_n\}_{n \in \mathbb{Z}}$ defined by

$$\psi_n = \ell ([n, n+1] \cap \mathcal{U})$$

is ergodic too. Hence

$$\lim_{x \to +\infty} \frac{\ell ([0, x] \cap \mathcal{U})}{x} = E\psi_0 = P \{0 \in \mathcal{U} \} = 1 - e^{-\lambda E\rho}.$$
Proof of Theorem 1.1. We start with (i). Assume \( \mathbb{E}\rho < \infty \). The fact that a.s. no unbounded occupied component exists is the content of [MR96, Theorem 3.1]. We have

\[
\mathbb{E}\# \{(u, \rho_u) \in \mathcal{P} : u < 0, (u + \rho_u) \geq 0\} = \mathbb{E}\# \{\mathcal{P} \cap \{(u, p) : u < 0, p \geq |u|\}\}
\]

\[
= \int \int \lambda du \mu(dp) = \lambda \int \mu(dp) \int du = \lambda \int p\mu(dp) = \lambda \mathbb{E}\rho < \infty.
\]

Hence a.s. \( \# \{(u, \rho_u) \in \mathcal{P} : u < 0, (u + \rho_u) \geq 0\} < \infty \) and

\[
\sup_{(u, \rho_u) \in \mathcal{P} : u < 0} (u + \rho_u) < \infty. \tag{4}
\]

Consequently by (3) a.s.

\[
\lim_{x \to +\infty} \frac{\ell([0, x] \cap \mathcal{U}_+)}{x} = \lim_{x \to +\infty} \frac{\ell([0, x] \cap \mathcal{U})}{x} = 1 - e^{-\lambda \mathbb{E}\rho}. \tag{5}
\]

Now we proceed to (ii) and (iii). For \( t \geq 0 \) the probability that \( t \) is not covered by \( \mathcal{U}_+ \)

\[
\mathbb{P} \{t \notin \mathcal{U}_+\} = \mathbb{P} \{\mathcal{P} \cap \{(u, p) : 0 \leq u \leq t, p \geq t - u\} = \emptyset\} \tag{6}
\]

\[
= \exp \left\{ - \int \int_{(u, p) : 0 \leq u \leq t, p \geq t - u} \lambda du \mu(dp) \right\}
\]

\[
= \exp \left\{ -\lambda \int_0^\infty \mu(dp) \int_{(t-p)v0}^t du \right\} = \exp \left\{ -\lambda \int_0^\infty (t \land p)\mu(dp) \right\}
\]

\[
e e^{-\lambda \mathbb{E}(t \land p)}. \tag{7}
\]

Set \( \tau_1 = 0, \xi_1 = \inf\{t > \tau_1 : t \in \mathcal{U}_+\}, \tau_2 = \inf\{t > \xi_1 : t \notin \mathcal{U}_+\}, \) and so on, so that for \( n \in \mathbb{N}, n \geq 2 \)

\[
\tau_n = \inf\{t > \xi_{n-1} : t \notin \mathcal{U}_+\}, \quad \xi_n = \inf\{t > \tau_n : t \in \mathcal{U}_+\}
\]

A.s. on \( \{\tau_n < \infty\}, [\tau_n, \xi_n] \) is the \( n \)-th connected component of \( \mathbb{R}_+ \setminus \mathcal{U}_+ \), whereas \([\xi_n, \tau_{n+1}]\) is the \( n \)-th connected component of \( \mathcal{U}_+ \). A.s. on \( \{\tau_n < \infty\}, \xi_n = \min\{u > \tau_n : (u, \rho_u) \in \mathcal{P} \text{ for some } \rho_u \in \mathbb{R}_+\}, \)

therefore given \( \{\tau_n < \infty\}, \xi_1 - \tau_1, \xi_2 - \tau_2, \ldots, \xi_n - \tau_n \) is a sequence of i.i.d random variables distributed exponentially with mean \( 1/\lambda \).

Let \( N_v \) be the total number of vacant components, \( N_v = \min\{n \in \mathbb{N} : \tau_{n+1} = \infty\} \) (\( N_v \) may take the value \( +\infty \)). We have

\[
\mathbb{E}[\ell(\mathbb{R}_+ \setminus \mathcal{U}_+)] = \mathbb{E}[\ell(\mathbb{R}_+ \setminus \mathcal{U}_+)|N_v] = \mathbb{E}\left[ \sum_{i=1}^{N_v} (\xi_i - \tau_i) \right] = \frac{N_v}{\lambda} = \frac{\mathbb{E}N_v}{\lambda}. \tag{8}
\]
On the other hand by (6)
\[ \mathbb{E}\ell(\mathbb{R}_+ \setminus \mathcal{U}_+) = \int_0^\infty \mathbb{P}\{t \notin \mathcal{U}_+\} \, dt = \int_0^\infty e^{-\lambda \mathbb{E}(t \land \rho)} \, dt. \]  
(9)

Conditionally on \( \{\tau_n < \infty\} \), the distribution of \( \tau_{n+1} - \xi_n \) is the same as the (unconditional) distribution of \( \tau_2 - \xi_1 \). Therefore \( N_v \) has a geometric distribution with parameter \( p = \mathbb{P}\{\tau_2 = \infty\} \), that is, \( \mathbb{P}\{N_v = m\} = p(1-p)^{m-1}, \ m = 1, 2, ... \) if \( p = 0 \), then a.s. \( N_v = \infty \). In particular, \( \mathbb{E}N_v = 1/p \) if \( p > 0 \). Note that \( \mathbb{E}N_v < \infty \) if and only if \( p > 0 \). By (8) and (9)
\[ \mathbb{E}N_v = \lambda \int_0^\infty e^{-\lambda \mathbb{E}(t \land \rho)} \, dt. \]  
(10)

Assume that \( \int_0^\infty e^{-\lambda \mathbb{E}(\rho \land t)} \, dt < \infty \). Then by (10), \( \mathbb{E}N_v < \infty \) and hence \( N_v \) is a.s. finite. In this case the \( N_v \)-th occupied component is unbounded, and (ii) is proven.

Assume that \( \mathbb{E}\rho = \infty \) and \( \int_0^\infty e^{-\lambda \mathbb{E}(\rho \land t)} \, dt = \infty \). By (10), \( \mathbb{E}N_v = \infty \), hence \( p = 0 \) and \( N_v = \infty \) a.s. Thus, there are a.s. infinitely many vacant components, that is there is no unbounded occupied component. The expected size of an occupied component exceeds \( \mathbb{E}\rho \), hence it is infinite. Let \( l_1, l_2, ... \) be the lengths of the consecutive occupied components (that is, \( l_m = \tau_{m+1} - \xi_m \)). Then \( \mathbb{E}l_i = \infty, i = 1, 2, ..., \) and a.s. \( \frac{1}{n} \sum_{i=1}^n l_i \to \infty, n \to \infty \). Since a.s. the average length of the vacant component \( \frac{1}{n} \sum_{i=1}^n (\xi_i - \tau_i) \to \frac{1}{\lambda} \), the covered volume fraction is 1.

**Remark 2.1.** An example of \( \lambda \) and \( \rho \) satisfying conditions of item (iii) of Theorem 1.1 is \( \lambda = 1 \) and
\[ \mathbb{P}\{\rho > y\} = \frac{1}{y}, \ y \geq 1. \]  
(11)

The density of \( \rho \) is
\[ f_\rho(x) = \frac{1}{x^2}, \ x \geq 1, \]  
(12)
\[ \mathbb{E}\rho = \int_1^\infty \frac{x \, dx}{x^2} = \infty, \]  
(13)
and for \( t \geq 1 \)
\[ \mathbb{E}(t \land \rho) = t \mathbb{P}\{\rho > t\} + \mathbb{E}\{\rho \mathbb{1}\{\rho \leq t\}\} = t \frac{1}{t} + \int_1^t \frac{x \, dx}{x^2} = 1 + \ln t, \]  
(14)

hence
\[ \int_0^\infty e^{-\mathbb{E}(\rho \land t)} \, dt \geq \int_0^\infty e^{-1-\ln t} \, dt = e^{-1} \int_1^\infty \frac{dt}{t} = \infty. \]  
(15)

**Remark 2.2.** An alternative proof of (iii) of Theorem 1.1 based on the recurrence properties of Markov processes [MT93] can be found in the first Arxiv version of this paper.
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