Fixed-Point Alignment: Incentive Bayesian Persuasion for Pipeline Stochastic Bayesian Game

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Abstract—This letter studies a general-sum pipeline stochastic game where each period is composed of two pipelined stages. The first stage is a cognitive decision-making in which each agent selects one of multiple information sources (ISs) and then acquires information about the unobserved payoff-relevant state from his selected IS. In the second stage, each agent takes an action based on the information provided by his selected IS. There is a rational Bayesian persuader who controls one IS and aims to influence the agents’ actions to induce her desired dynamic equilibria by solely crafting the information structure of her IS. We restrict attention to a direct Bayesian persuasion in which the persuader incentivizes the agents to select her as the IS at the first stage and consider an equilibrium concept known as pipelined perfect Markov Bayesian equilibrium (PPME) which is a nontrivial extension of Nash equilibria in general stochastic games. We propose a novel design principle termed fixed-point alignment that captures the observation that the agents’ strategic interactions in the second stage induce an indirect cognitive competition (i.e., IS selection) in the first stage and formulate the direct Bayesian persuasion in PPME as a constrained non-linear optimization problem. By decomposing the optimization problem into a two-stage process known as pipelined local fixed-point alignment, we then provide a class of verifiable necessary and sufficient conditions for the implementability of the direct Bayesian persuasion in PPME.

I. INTRODUCTION

Tackling uncertainties is a key cognitive components of rationality in decision makings under incomplete information environment. A Bayesian agent usually handles uncertainties by relying on priors and forming posterior beliefs (see, e.g., [1]) over the unobserved but payoff-relevant state of the game or forms belief hierarchies (beliefs about the state as well as others’ beliefs; see, e.g., [2], [3]) in competitive multiagent environments. The priors and the beliefs constitute an essential component of the environment to shape the agents’ decisions of action choices since they model the uncertainty of the game (due to the unobservability of the state). Bayesian persuasion (e.g., [1], [4], [5], [6], [3]) studies how a principal uses her information advantage to strategically reveal noisy information about the state to the agents, thereby inducing and manipulating agents’ beliefs, to influence the agents’ behaviors in her favor.

In many scenarios, agents have multiple information sources (ISs) that they can choose to get informed about the state. For example, smart vehicles (the agents) can get global traffic conditions (the state) from different navigation apps (the ISs) to choose a route based on the recommendations of the apps. Agents can choose to use a combination of information from multiple ISs or they can individually subscribe a particular IS. In our Bayesian persuasion model, we restrict attention to a scenario when each agent chooses an IS from a finite set of options and assume that the principal controls one of the ISs and is the only strategic information designer while other ISs operate in a take-it-or-leave-it manner.

In this letter, we study a dynamic Bayesian persuasion model in which agents play a novel general-sum incomplete-information stochastic game that we refer to as pipeline stochastic Bayesian game (PSBG). The first stage is a cognitive decision-making process where each agent decides how to get informed about the game. In particular, the agent selects one IS from a set of finite options and then acquires information about the unobserved state from the selected IS. In the second stage, agents start the primitive game play in which agent takes an action based on the information from the selected IS. The state then evolves in a Markovian fashion based on current state and actions. As in canonical stochastic games, agents’ joint action is directly payoff-relevant to every agent in each period, which together with the state determines the single-period payoff realized at the end of that period. On the other hand, agents’ selections of ISs in the first stage is not directly coupled through their payoffs. However, since each agent is informed about the state only through the information provided by his selected IS, the coupling of actions induces indirect interdependence of the agents’ cognitive posture in the first stage through each action’s dependence on the information. Built upon Markov perfect equilibrium, we consider a refinement of Nash Equilibrium defined as pipelined perfect Markov Bayesian equilibrium (PPME) as the central concept of stability in PSBG for the consistency of each agent’s individual pipeline decisions as well as interactions with others.

The principal aims to design a Bayesian persuasion mechanism such that the PSBG admits a direct PPME in which (i) the agents are incentivized to select the principal’s IS at the first stage of the pipeline in every period and (ii) the agents’ behaviors (i.e., actions) coincide with the principal’s goal. Such pipeline setting in dynamic environments distinguishes our work from existing Bayesian persuasion in static settings (e.g., [1], [2], [4], [5]) as well as in dynamic models (e.g., [6], [3], [7]).

We propose a design regime referred to as the fixed-point alignment (FPA) which formulate the dynamic Bayesian persuasion as a non-linear optimization problem. The FPA selects a persuasion strategy for the principal that matches the first fixed point from the optimality of the agents’ selection

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of the principal’s IS at the first stage to the second fixed point of optimal action choices at the second stage. We then decompose the optimization problem of FPA into local fixed-point alignment processes and obtain a set of verifiable necessary and sufficient conditions for implementable dynamic Bayesian persuasion in direct PPME.

The remainder of the letter is organized as follows. In Section II, we describe the PSMG model and define the equilibrium concept PPME. Section III formulates the dynamic Bayesian persuasion in PSMG. Then, we propose the fixed-point alignment and characterize the necessary and sufficient conditions in Section IV. Section V provides the conclusion.

II. PIPELINE STOCHASTIC BAYESIAN GAME

We consider a stochastic game in which each agent makes a sequential two-stage decision-making at each period. The agents only has partial observation about the game due to their unobservability of the realized state which is payoff-relevant to the agents. At the first stage, each agent acquires knowledge about the game by selecting one information source (IS), who observes the realizations of the states, from a finite set of available ISs. Then, each agent privately informed about the game by the selected IS. At the second stage, each agent takes an action based on his knowledge the game obtained at the first stage. We refer to such game as the (two-stage) pipeline stochastic Bayesian game (PSBG). Formally, a PSBG is an infinite-horizon game defined by a tuple \( \text{PSBG} \equiv \langle N, G, K, \{\Theta_i\}_{i \in N}, \{A_i\}_{i \in N}, d_g, T_g, \{\tau^k_i\}_{i \in N, k \in K}, \{R_i\}_{i \in N}, \delta \rangle >, \)

- \( N \equiv [n] \) with \( 1 \leq n < \infty \) is a finite set of agents.
- \( G \) is a finite set of states.
- \( K \equiv [m] \) with \( 1 \leq m < \infty \) is a finite set of information sources (ISs).
- \( \Theta_i \) is a finite set of types for agent \( i \).
- \( A_i \) is a finite set of actions available to each agent \( i \) at each period with \( A_i = \{A_i\}_{i \in N} \).
- \( d_g \in \Delta(G) \) is an initial distribution of the state.
- \( T_g : g \times A \rightarrow \Delta(G) \) is the transition function of the state, such that \( T_g(\cdot|g, a_i) \in \Delta(G) \) specifies the probability distribution of the next-period state when the current state is \( g \) and the current joint action of the agents is \( a_i \equiv (a_i)_{i \in N} \).
- \( \tau^k_i : G \rightarrow \Delta(\Theta_i) \) is IS \( k \)'s stationary type generation rule (generation rule) for agent \( i \).
- \( R_i : G \times A \rightarrow \mathbb{R} \) is the reward function of agent \( i \), for \( i \in N \).
- \( \delta \in (0, 1) \) is the common discount factor.

The game model PSBG is fixed and publicly known at the ex-ante state (i.e., before the game starts). When the period- \( t \) − 1 event is \( h_{t-1} = (g_{t-1}, a_{t-1}) \), a state \( g_{t-1} \) is drawn by a fictitious player known as Nature according to \( T_{g_{t-1}}(\cdot|h_{t}) \) (or \( d_g(\cdot) \) at the initial period) which initiates period- \( t \). For ease of exposition, we refer to the previous-period event \( h_t \in H \equiv G \times A \) as the history of period \( t \). We assume that the history is publicly observed at the end of each period. At the first stage of period \( t \), each agent \( i \) selects an IS \( k \) from \( K \), and then acquires information about the states from that IS. Let \( N_i[k] \subset N \) denote the set of agents who selects IS \( k \) in period \( t \) with \( \prod_{i \in K} N_i[k] = N \) for all \( t \geq 0 \). Upon being selected, IS \( k \) privately determines a type (coherent hierarchy of beliefs or Harsanyi’s type; see, e.g., [8], [9]), denoted by \( \Theta^k_i \equiv \Theta_i, \) for each agent \( i \) in \( N_i[k] \) by privately sending a noisy signal about the state to agent \( i \) (see, e.g., [2], [3] for explicit formations of type due to private signals). Note that the state space \( G \) and each type space \( \Theta_i \) need not be of the same size for all \( i \in N \). At each period \( t \), each IS \( k \) uses a stationary type generation policy \( \tau^k_i(\cdot|g) \in \Delta(\Theta_i) \) to randomly draw a type for agent \( i \) given the state \( g \) if agent \( i \) selects IS \( k \).

The solution to the game PSBG is a stationary strategy profile \( \langle \beta, \pi \rangle = \langle \beta_i, \pi_i \rangle \in \mathcal{E} \) where each \( \beta_i : H \rightarrow K \) and \( \pi_i : \Theta_i \rightarrow \Delta(A_i) \) is each agent \( i \)'s IS selection policy (selection policy) and action policy, respectively. We refer to each agent \( i \)'s decision-making of selecting an IS and then taking an action as a pipeline decision in each period. We assume that agents’ choices of ISs (but not types), denoted by \( K^i \), at the current period are publicly released after each agent receives his signal. Thus, agents do not have to form beliefs over others’ selections of ISs when they decide to take actions. Based on the history \( h_t \) and type \( \theta_{t,i} \), each agent \( i \) forms a belief \( \mu^i(\cdot|\theta_{t,i}, h_t) \in \Delta(G \times \Theta_{t-1,i}) \) over the state \( g_t \) and other agents’ types \( \theta_{t-1,i} \) according to the Bayes’ rule. With a slight abuse of notation, let \( \tau^k_i(\cdot|g) \in \Delta(\Theta_i) \) denote the generation policies for other agents’ types given \( K^i \). Then, applying the Bayes’ rule yields

\[
\mu^i(g_t, \theta_{t-1,i}|\theta_{t,i}, h_t) = \frac{\tau^k_i(\cdot|g_t)|g_t| \tau^k_i(\cdot|\theta_{t,i})T_g(g_t|h_t)}{\sum_{g_t \in G, \theta_{t-1,i} \in \Theta_{t-1,i}} \tau^k_i(\cdot|g_t)|g_t| \tau^k_i(\cdot|\theta_{t-1,i})T_g(g_t|h_t)}.
\]

According to the Ionescu Tulcea theorem [10], \( \{d_g, T_g, \tau^k_i, R_i\} \) and the agents’ strategy profile \( \langle \beta, \pi \rangle \) uniquely define a probability measure, \( P^\beta,\pi \), over \( G \times K^n \times \Theta \times A^n \). Let \( E^\beta,\pi[\cdot] \) denote the expectation operator with respect to \( P^\beta,\pi \). Additionally, given history \( h_t \) and history-type pair \( (h_t, \theta_{t,i}) \), we obtain unique probability measures (as perceived by each agent \( i \)) \( P^\beta,\pi(h_t, \theta_{t,i}) \) and \( P^\beta,\pi(h_t, \theta_{t,i}) \), over \( G \times K^n \times \Theta \times A^n \) and \( G \times \Theta_{t-1,i} \times A^n \times (G \times K^n \times \Theta \times A)^\infty \), respectively. In particular, \( P^\beta,\pi(h_t, \theta_{t,i}) \) models the uncertainty at period \( t \) for each agent at the beginning of the first stage while \( P^\beta,\pi(h_t, \theta_{t,i}) \), \( \theta_{t,i} \), models the uncertainty for each agent \( i \) at the beginning of the second stage. Let \( E^\beta,\pi[\cdot|h_t] \) and \( E^\beta,\pi[\cdot|h_t, \theta_{t,i}] \), respectively, denote the expectation operators with respect to \( P^\beta,\pi(h_t) \) and \( P^\beta,\pi(h_t, \theta_{t,i}) \).

Given \( P^\beta,\pi \), the social welfare (for given strategy profile) is defined as the ex-ante expected discounted sum of all agents’ accumulated rewards over time. That is,

\[
W(\beta, \pi, \tau) \equiv E^\beta,\pi \left[ \sum_{t=0}^{\infty} \delta^t R_t(g_t, a_t) \right].
\]
defined as

\[ J^t_i(h_i|\beta, \pi) \equiv E^\pi_{\theta_i} \sum_{s,t} \delta^{t} R_i(g_i, a_i) | h_t \]. (3)

With a slight abuse of notation, we write \( \mu_i(g_i, \theta_i, h_i) \) and \( \mu_i(\theta_i, h_i) \) as the marginals of \( \mu_i(g_i, \theta_i, h_i) \) over \( G \) and \( \Theta_i \), respectively. For any \( h_i \in H \) and \( \theta_i \in \Theta_i \), each agent \( i \)'s expected immediate reward function is defined as:

\[ \bar{R}_i(h_i, \theta_i, a_i) = \sum_{g_i \in G} R_i(g_i, a_i) \mu_i(g_i, \theta_i, h_i). \]

Given \( P^\tau_{\beta, \pi}[h_i|\theta_i, h_i] \), agent \( i \)'s period-\( t \) history-type (HT) value function is defined as

\[ V^\tau_i(h_i, \theta_i|\beta, \pi) \equiv E^\pi_{\theta_i} \left[ \bar{R}_i(h_i, \theta_i, a_i) \right]_{h_t}. \]

Finally, agent \( i \)'s period-\( t \) history-type-action (HTA) value function is defined as

\[ Q^\tau_i(h_i, \theta_i, a_i|\beta, \pi) \equiv \bar{R}_i(h_i, \theta_i, a_i) + E^\pi_{\theta_i} \left[ \sum_{s,t} \delta^{t-s+1} R_i(g_i, a_i) | h_t, \theta_t \right]. \]

Definition 1 (Pipelined-Sequential Rationality). Fix \( \tau \). We say that an agent’s strategy profile \( \beta, \pi > \) is pipelined-sequential rational to \( \beta, \pi \) if for any \( h_i \in H \) and \( \theta_i \in \Theta_i \),

\[ J^\tau_i(h_i|\beta, \pi) \geq J^\tau_i(h_i|\beta^*, \pi, \pi), \]

\[ V^\tau_i(h_i, \theta_i|\beta, \pi) \geq V^\tau_i(h_i, \theta_i|\beta, \pi^*, \pi), \]

(6) (7)

Definition 2 (Pipelined Perfect Markov Bayesian Equilibrium). A strategy profile \( \beta^*, \pi^* \) constitutes a stationary pipelined perfect Markov Bayesian equilibrium (PPME) if each \( \beta_i^*, \pi^*_i \) is pipelined-sequential rational to \( \beta^*_i, \pi^*_i \), for all \( i \in N \) simultaneously.

III. DYNAMIC PERSUASION

We consider that there is a rational Bayesian persuader known as principal (she) who controls one IS of \( K \). Without loss of generality, we index the principal as IS \( K \). We assume that all other ISs are non-strategic in the sense that they commit to their \( \tau^k \)-kth in a take-it-or-leave-it manner. The principal is rational in that she possesses a goal, \( \kappa(\cdot|g) \in \Delta(A) \) for every \( g \in G \), which is described by a stationary probability distribution of the agents’ actions conditioning on the state, and she aims to induce the agents to take actions that coincides with her goal \( \kappa \) in PPME by strategically generating types according to \( \tau^k \). Let \( PSBG[\tau^k] \) denote the mechanism of the game PSBG when the principal uses \( \tau^k \). Since we focus on stationary strategies, we suppress the time indexes in the notations in the rest of the letter; unless otherwise stated.

The timing of \( PSBG[\tau^k] \) is as follows. At the ex-ante stage, the principal chooses \( \tau^k \) and publicly releases it, given others’ \( \tau^k \). At each period \( t \), each agent \( i \) selects an IS from \( K \) according to \( \beta_i \) and requests a type \( \theta_i \) from that IS. By observing the current state \( g \), each IS \( k \) privately sends a type \( \theta_i^k \) according to \( \tau^k \) to each agent \( i \) in \( N[k] \). After receiving \( \theta_i^k \), each agent \( i \) takes an action \( a_i \) based on \( \theta_i^k \) according to \( \pi_i \). At the end of period \( t \), each agent \( i \) receives a reward \( R_i(g, a) \). The state \( g \) and the joint actions \( a \) are publicly disclosed. Then, the state \( g \) is transitioned to the next state \( g' \) according to \( T_g \).

Definition 3 (Direct Information Design). We say that the principal’s information design is direct if each agent \( i \) chooses the principal’s IS \( k \) at every period. That is, each \( \beta_i(h) = k \), for all \( h \in H \). We call such selection policy as obedient selection policy.

When the principal performs direct information design, we refer to the game \( PSBG[\tau^k] \) as direct mechanism.

Definition 4 (Attainable Goal). In a direct mechanism \( PSBG[\tau^k] \), the principal’s goal \( \kappa \) is attainable by \( \pi \) if

\[ \kappa(a|g) = \sum \pi(a|\theta) \tau^k(\theta|g). \]

Since other ISs are non-strategic, we rewrite the expectation operators and value functions by only showing the principal’s generation rule \( \tau^k \); e.g., \( E^\pi_{\theta_i} \left[ \cdot \right], J^\tau_i \).

Definition 5 (Implementability). We say that a direct mechanism \( PSBG[\tau^k] \) is implementable in PPME with \( \kappa \) if \( PSBG[\tau^k] \) induces a profile \( \beta^0, \pi^0 > \) that constitutes a PPME where \( \beta^0 \) is obedient and the principal’s goal \( \kappa \) is attainable by \( \pi^0 \). We refer to such PPME as obedient PPME.

The notion of implementability requires that when the principal uses the generation rule profile \( \tau^k \), each agent \( i \)’s obedient pipeline decision in each period while believing with probability 1 that others make obedient pipeline decisions is a PPME. To achieve an implementable (direct) mechanism \( PSBG[\tau^k] \), the principal needs to (i) incentivize agents to choose her as their IS at every period and (ii) induce actions that coincide with her goal. We assume, as is standard, that when there are multiple equilibria, tie-breaking is in the principal’s favor.

Let

\[ EV_i^k(h, \theta_i|\beta^0, \pi, V_i^k) \equiv E^\theta_i \left[ V_i^k(h, \theta_i, \bar{\theta}_{i-1}|\beta^0, \pi) | h_t, \theta_t \right]. \]

The obedience is incentivized by the obedience compatibility (OC) constraints which enforce the obedient pipeline decision as a PPME. That is, for all \( i \in N, \beta_i^0, \pi^0_i \),

\[ J_i^k(h|\beta^0, \pi) \geq J_i^k(h|\beta_i^0, \pi^0_i), \]

\[ EV_i^k(h, \theta_i|\beta^0, \pi, V_i^k) \geq EV_i^k(h, \theta_i|\beta_i^0, \pi^0_i, \pi_{i-1}, \pi^0_i, \pi^0_i), \]

(OC1) (OC2)

Define the set of attainable goals in direct mechanisms as:

\[ \Gamma^0 \equiv \left\{ \kappa: G \rightarrow \Delta(A) \mid \kappa(\cdot|g) = \sum \pi(\cdot|\theta) \tau^k(\theta|g) \right\} \]

where \( \beta^0, \pi > \) is a PPME of some IC \( PSBG[\tau^k] \).
Hence, for each $\kappa \in \Gamma^O$, there exists a generation rule profile $\tau^k$ such that the mechanism $PSBG[\tau^k]$ is implementable. With a slight abuse of notation, let

$$W^O(\kappa) \equiv \mathbb{E}^\kappa \left[ \sum_{i=1}^{\infty} \delta^i R_i(\bar{g}, \bar{a}) \right],$$

where $\mathbb{E}^\kappa[\cdot]$ is the expectation operator given $\kappa$ and the game model $PSBG$. Hence, $W^O(\kappa)$ is the social welfare when the agents’ behaviors coincide with the principal’s goal $\kappa$. Hence, there exist $< \beta^O, \pi >$ and $\tau^k$ such that (i) $W^O(\kappa) = W(\beta^O, \pi; \tau^k)$ and (ii) the mechanism $PSBG[\tau^k]$ is implementable for every $\kappa \in \Gamma^O$. In this work, we consider that the principal aims to optimize the social welfare of the multiagent system. That is, the principal’s social-welfare optimization problem is:

$$\max_{\kappa \in \Gamma^O} W^O(\kappa).$$

Following standard dynamic programming argument (see, e.g., [11]), we represent (3), (4), and (5) recursively:

$$J^t_i(\tau; |V^t_i) = \sum_{\theta \in H} V_i^t(\theta, |\tau) J^t_i(\theta; |V^t_i),$$

$$V_i^t(\theta, |\tau) = \sum_{a \in A} \pi(a|\theta; \theta_{-i}) Q_i^t(\theta; a, |V^t_i) + R_i(\theta, a|\theta; |V^t_i),$$

$$Q_i^t(\theta; a, |V^t_i) = \gamma_i^t(\theta; a, |V^t_i).$$

Here, we denote $J^t_i(\tau^k; |V^t_i)$ and $Q_i^t(\tau^k; |V^t_i)$ with $V^t_i$ to highlight their dependence on $V^t_i$ from the Bellman recursions (12)-(14). Note that $h = (g', a')$ is the pair of $(g', a')$ of the previous period. Hence, the last term of (13) is the expected history value of the next period where the expectation is taken over the current state $g$.

Consider the following constrained optimization problem $(OPT[\tau^k])$:

$$\min_{\pi, V} \left\{ \sum_{h, g, \theta} \left( V_i(h, \theta) \right) \right\}$$

subject to, for all $i \in N$, $a_i \in A_i$, $h \in H$, $\theta_i \in \Theta_i$ with $
\sum_{g, h, \theta_{-i}} \gamma_i^t(\theta; a_{-i}|V^t_i) > 0,$

$$\pi(a_i|\theta_i) \geq 0,$$

$$\sum_{a_i \in A_i} \pi(a_i|\theta_i) = 1,$$

$$EV_i^t(h, \theta_i|\pi; V_i) \geq \mathbb{E}_{\pi_{-i}} \left[ Q_i^t(h, \theta_i; a_i, a_{-i}|V_i) | h, \theta_i \right].$$

Proposition 1. A strategy profile $< \beta^O, \pi^O >$ constitutes an obedient PPME of a game $PSBG[\tau^k]$ if and only if it is a global minimum of $(OPT[\tau^k])$ with $Z^t(\beta^O, V^O) = 0$ where $V^O$ is the corresponding optimal HT value function.

Proof. See Appendix A.

Proposition 1 extends the fundamental formulation of finding a Nash equilibrium of a stochastic game as a nonlinear programming (Theorem 3.8.2 of [12]; see also, [13]). Here, the constraints (RG) and (FE) ensure that each candidate $\pi$ is a valid conditional probability distribution and rules out the possible trivial solution $\pi = 0$ for all $i \in N$. The conditions (EQ) and (OB) are two necessary conditions for an obedient PPME of the game $PSBG[\tau^k]$ derived from the optimality of PPME and the Bellman recursions (12) and (13). Here, the obedient selection policy profile $\beta^O$ is not a solution of the problem $(OPT[\tau^k])$; instead, the optimality of being obedient is constrained by (OB) in terms of $\pi$ and $V$ given $\tau^k$.

IV. DESIGN PRINCIPLE: FIXED-POINT ALNIGNMENT

In this section, we first propose a design principle referred to as fixed-point alignment (FPA) for the principal to design an implementable mechanism $PSBG[\tau^k]$. Then, we provide a set of verifiable necessary and sufficient conditions for implementable mechanisms. Suppose that the principal’s IS $k$ is the only IS available to the agents. The principal’s feasible information design should first guarantee that her generation rule profile $\pi^*$ induces an equilibrium $< \pi^*, V^* >$. Since agents are obedient by default, Proposition 1 implies that the point $< \pi^*, V^* >$ needs to be a global minimum of $(OPT[\tau^k])$ that satisfies (RG) (FE), and (EQ) with $Z^t(\pi, V) = 0$. In other words, $V$ has to be a fixed point of the following equation, given $\pi_{-i}$, for all $i \in N$, $h \in H$, $\theta_i \in \Theta_i$:

$$EV_i^t(h, \theta_i|V_i) = \max_{a_i \in A_i} \pi(a_i|\theta_i) \sum_{g, h, \theta_{-i}} Q_i^t(h, \theta_i; a_i, a_{-i}|V_i) | h, \theta_i \right].$$

When there are multiple ISs, the notion of obedient PPME requires that agents’ obedient IS selection is optimal at the first stage of the equilibrium pipeline decisions. Given any $J_i$, define, all $i \in N$, $\theta_i \in \Theta_i$, $h \in H$,

$$\mathbb{I}J_i^t(h, \theta_i|J_i) = \sum_{g, h, \theta_{-i}} \left( \gamma_i^t(h, \theta_i; a_i, a_{-i}|V_i) \right) \sum_{g, h, \theta_{-i}} (\theta_i| V_i) T_i(g|h).$$

The optimality of obedience (i.e., constraint (OB)) implies that the optimal history value function $J_i$ for each agent $i$ needs to be a fixed point: for all $i \in N$, $h \in H$,

$$J_i(h) = \max_{\theta_i} \mathbb{I}J_i^t(h, \theta_i|J_i),$$

which is independent of $V$ or $\pi$.

The underlying idea of FPA is to choose a generation rule profile $\tau^k$ such that $V$ is a fixed point of (EQ1) if and only if $J$ is a fixed point of (OB1) when the principal’s goal is attainable by the agents’ action profile policy. One possible objective function is

$$Z^t(\pi^O, V^O) = 0$$

where $V^O$ is the corresponding optimal HT value function.
which is a function of $t^k_i$, $J$, and $V$, given a goal $\kappa$. Consider the constrained optimization problem (FPA$[\kappa]$), for any $\kappa$:

$$\min_{t^k_i,J,V} Z^{\text{FPA}}(t^k_i,J,V;\kappa) \quad \text{(FPA$[\kappa]$)}$$

subject to (EQ1), (OB1), and $\forall i\in N, \ g\in G, \ \theta_i\in \Theta_i$,

$$t^k_i(\theta_i|g) \geq 0, \quad \text{(RGt}_i)$$

$$\sum_{\theta_i\in \Theta_i} t^k_i(\theta_i|g) = 1, \quad \text{(FT}_i)$$

$$\kappa(a|g) = \sum_{\theta_i\in \Theta_i} \pi(a|\theta_i) t^k_i(\theta_i|g). \quad \text{(AT)}$$

**Proposition 2.** For each $\kappa \in \Gamma[\beta^O]$, let $<\hat{t}^k, \hat{\kappa}>$ satisfy the Bellman recursions (12)-(14). Then, $<\hat{t}^k, \hat{V}, \hat{\kappa}>$ is a global minimum of (FPA$[\kappa]$) with $Z^{\text{FPA}}(\hat{t}^k, \hat{J}, \hat{V}; \hat{\kappa}) = 0$ if and only if $<\hat{t}^k, \hat{V}>$ is a global minimum of (OPT$[\hat{t}^k]$) with $Z^{\hat{\kappa}}(\hat{t}^k, \hat{V}) = 0$.

**Proof.** See Appendix B. □

**A. Local FPA**

For each type $\theta_i$ and any generation profile $t^k_i$, for $i\in N$, we decompose it into $\Theta_i = \Theta_i' \cup \{\hat{\theta}_i\}$, such that the constraint (FT$\_i$) can be decomposed into the following two for: for all $i\in N, \:\sum_{\theta_i\in \Theta_i'} t^k_i(\theta_i|g) \leq 1$ and $t^k_i(\hat{\theta}_i|g) + \sum_{\theta_i\in \Theta_i'} t^k_i(\theta_i|g) = 1$.

Let, for all $i\in N$,

$$IV^k_i(\hat{t}, \theta_i|V) \equiv \sum_{\theta_i\in \Theta_i} V_i(\hat{t}_i, \theta_i, \theta_{-i}) t^k_i(\theta_{-i}|g) T^k_i(g|h).$$

In (14), the dependence of $Q^{\text{FPA}}_i(\cdot|V_i)$ on $V_i$ is through $J_i$’s dependence on $V_i$. Hence, given any $\Theta_i$, we can represent (without of notation) each $Q^{\text{FPA}}_i(\cdot|J_i)$ depending on $J_i$. Let $X^k_i \equiv (J_i, V_i, t^k_i(\cdot|g))$. Define:

$$\lambda_i(X^k_i; h, \theta_i) \equiv J_i(h) - IV^k_i(\hat{t}, \theta_i|V_i).$$

Here, $\lambda_i$ is a function of $J_i, V_i, t^k_i$, and is independent of agents’ policy profile $\pi$. For any $g \in G, h \in H, \theta_i \in \Theta_i$, define the local fixed-point misalignment (misalignment) as follows:

$$M_i(X^k_i, \tau^k_i, g, h) \equiv \sum_{\theta_i'\in \Theta_i'} \lambda_i(X^k_i; h, \theta_i') \tau^k_i(\theta_i'|g) + \lambda_i(X^k_i; h, \hat{\theta}_i) \tau^k_i(\hat{\theta}_i|g).$$

Then, we define the local alignment process (for $i\in N, g \in G, h \in H$) as the process to minimize the misalignment:

$$\min_{X^k_i, \tau^k_i} M_i(X^k_i, \tau^k_i, g, h)$$

s.t., $\tau^k_i(\theta_i'|g) \geq 0, \forall \theta_i' \in \Theta_i', \tau^k_i(\hat{\theta}_i|g) \geq 0, \lambda_i(X^k_i; h, \theta''_i) \geq 0, \forall \theta''_i \in \Theta_i$.\quad (LMM$[g,h]$)

Let $b \equiv \{b(\theta_i')\}_{\theta_i'\in \Theta_i'}$, $c, f \equiv \{f(\theta_i')\}_{\theta_i'\in \Theta_i}$, respectively, denote the Lagrange multipliers with respect to the constraints $\tau^k_i(\theta_i'|g) \geq 0, \forall \theta_i' \in \Theta_i', \tau^k_i(\hat{\theta}_i|g) \geq 0$, and $\lambda_i(X^k_i; h, \theta''_i) \geq 0, \forall \theta''_i \in \Theta_i$, of (LMM$[g,h]$); and the corresponding slack variables are denoted by $q \equiv \{q(\theta_i')\}_{\theta_i'\in \Theta_i'}$, $w \equiv \{w(\theta_i')\}_{\theta_i'\in \Theta_i}$, respectively. We construct the Lagrange function of (LMM$[g,h]$) as:

$$L_i(X^k_i, \tau^k_i, b, c, f, q, w; z, g, h) \equiv M_i(X^k_i, \tau^k_i; h) + \sum_{\theta_i'\in \Theta_i'} b(\theta_i') (q(\theta_i') - \tau^k_i(\theta_i'|g))$$

$$+ c(w - \tau^k_i(\hat{\theta}_i|g)) + \sum_{\theta_i\in \Theta_i} f(\theta_i') (z(\theta_i') - \lambda_i(X^k_i; h, \theta_i)).$$

By taking derivative of $L_i$ with respect to $X^k_i$ and $\tau^k_i(\cdot|g)$, respectively, we obtain

$$\Delta_i(X^k_i, \tau^k_i, f; g, h) = \nabla X^k_i M_i(X^k_i, \tau^k_i; h)$$

$$- \sum_{\theta_i\in \Theta_i} f(\theta_i') \nabla X^k_i \lambda_i(X^k_i; h, \theta_i),$$

and, for all $\theta_i \in \Theta_i$,\quad (18)

$$d(X^k_i, \tau^k_i(\theta_i|g), b(\theta_i'|g); c; h, g) = b(\theta_i|g) + c$$

$$\frac{\partial}{\partial \tau^k_i(\theta_i|g)} M_i(X^k_i, \tau^k_i; h).$$

For any $\theta_i \in \Theta_i$, define

$$\gamma^k_i(J_i, V_i, \pi_{-i}; h, \theta_i, a_i) \equiv EV^k_i(h, \theta_i|V_i)$$

$$- \mathbb{E}_{\hat{\kappa}} \left[ \nabla^2 \gamma^k_i(h, \theta_i, a_i, \pi_{-i}|J_i) \right] h, \theta_i, a_i.$$ (19)

**Definition 6 (Local Admissibility).** Given a goal $\kappa$, we say a generation rule profile $\tau^k$ is locally admissible if there exist $J$ and $V$ such that, for every local alignment process (LMM$[g,h]$)

(i) there exist $b$ and $f$, such that, for every $i\in N$,

$$\Delta_i(X^k_i, \tau^k_i, f; g, h) = 0$$

$$d(X^k_i, \tau^k_i(\theta_i|g), b(\theta_i'|g); c; h, g) = 0, \forall \theta_i \in \Theta_i$$

$$f(\theta_i') \lambda_i(X^k_i; h, \theta_i) = 0, \forall \theta_i \in \Theta_i$$

(ii) and there exists $\pi$ such that, for every $i\in N, \theta_i \in \Theta_i$, with $\tau^k_i(\theta_i|g) > 0$,

$$\kappa(a|g) = \sum_{\theta_i\in \Theta_i} \pi(a|\theta_i) \tau^k_i(\theta_i|g)$$

$$\gamma^k_i(J_i, V_i, \pi_{-i}; h, \theta_i, a_i) = 0, \forall a_i \in A_i.$$ (20)

Here, (20) of local admissibility requires the stationarity (the first two lines) and complementary slackness (the last two lines). Note that the conditions in (20) are in general not the Karush–Kuhn–Tucker (KKT) conditions for the local alignment process (LMM$[g,h]$) because the local admissibility does not require the feasibility of $f$ and $b$ as Lagrange multipliers. The conditions in (21) perform local fixed-point alignment. In particular, (21) requires the generation rule profile $\tau^k$ to align the fixed-points $J_i$ and $V_i$ through the policy profile $\pi$ (the second line of (21)) that attains the goal $\kappa$ (the first line of (21)).

Let $\gamma^k_i[\tau^k_i; h] \equiv \nabla X^k_i \lambda_i(X^k_i; h, \theta_i)$ denote a vector of gradients of $\lambda_i$ with respect to $X^k_i$. 

Condition 1. \( \mathcal{V}(x^*_i; h) \) is a set of linearly independent vectors for all \( x^*_i = (J, V_i, \tau^*_i, (|g|)) \) for all \( i \in N \), given any \( g \in G \) and \( h \in H \).

Condition (1) is regularity condition similar to the linear independence constraint qualification (LICQ) in KKT conditions. However, Condition (1) imposes linear independence only for the constraints \( \{ \lambda_i(X_i^*, h, \theta_i^*) \geq 0, \forall \theta_i^* \in \Theta_i \} \) of \((LMH)(g, h))\).

**Theorem 1.** Fix a goal \( \kappa \). Under Condition 1, a direct mechanism \( PSB[G]^{[\pi^*_i]} \) is implementable in obedient PPME with \( \kappa \) if and only if the generation rule profile \( \pi^*_i \) is locally admissible.

**Proof.** See Appendix C.

Theorem 1 provides necessary and sufficient conditions for characterizing the implementability of the principal’s information design with a goal \( \kappa \). In particular, if there is an algorithm converges to a local admissible \( \pi^*_i \) for a goal \( \kappa \), then, under Condition 1, the direct mechanism \( PSB[G]^{[\pi^*_i]} \) admits an obedient PPME in which the principal’s goal \( \kappa \) is attained. In other words, locally admissible \( \pi^*_i \) with the corresponding \( J \) and \( V \) achieves \( ZFP_{\pi^*_i}(\pi^*_i, J, V; \kappa) = 0 \), the global minimum of \((FPA[\kappa])\). Since usually \( |X_i^*| \gg |\Theta_i| \), for all \( i \in N \), it is straightforward to see that satisfying Condition (1) is much less restrictive than satisfying LICQ of all the constraints of \((FPA[\kappa])\).

The following corollary straightforwardly characterizes the attainable goal in Definition 4 in terms of local admissible generation rule profiles.

**Corollary 1.** A goal \( \kappa \) is attainable in direct mechanisms if and only if there exists a local admissible generation rule profile that achieves \( \kappa \).

V. CONCLUSION

In this letter, we have studied a dynamic Bayesian persuasion in a pipelined stochastic Bayesian game in which each agent periodically makes a pipeline decision of selecting an information sources and then taking an action based on the noisy information acquired from the selected information sources. A principal who controls one of the information sources aims to conduct Bayesian persuasion by first incentivizing the agents to select her information source and then take actions that coincide with her goal. We have proposed a design principle known as fixed-point alignment that formulates the principal’s design as a non-linear optimization problem and characterized a set of verifiable necessary and sufficient conditions for the implementability of the design in a refinement of Nash equilibrium referred to as the pipelined perfect Markov Bayesian equilibrium.

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APPENDIX

A. Proof of Proposition 1

Suppose that \( < \beta^O, \pi^O > \) is an obedient PPME. Then, we can construct \( V^O \) given \( \pi^O \). By the definition of PPME, it is straightforward to show that the constraints (FE), (EQ), and (OB) are satisfied. Therefore, \( < \pi^O, V^O > \) is a feasible solution of \((OPT[\pi^i])\). By construction, \( Z^d(\pi^O, V^O) = 0 \). From the feasibility, \( < \beta^O, \pi^O > \) is a global minimum of \((OPT[\pi^i])\).

Conversely, suppose that \( < \pi^0, V^O > \) is feasible for \((OPT[\pi^i])\) with \( Z^d(\pi^O, V^O) = 0 \). Then, the constraints (EQ) and (OB) imply that, for every \( i \in N \), \( h_i \in H_i \), \( \theta_i \in \Theta_i \) with \( \pi^O_i(\theta_i|g_i) > 0 \) where \( g_i \in G \) with \( T^O(g_i|h_i) > 0 \),

\[
V_i(h_i, \theta_i) \geq \sum_{a_i, \theta_{-i}} Q^O_i(h_i, \theta_i, a_i|V_i) \pi^O_i(a_i|\theta_i) \mu_i(\theta_{-i}|\theta_i). \tag{22}
\]

From \( Z^d(\pi^O, V^O) = 0 \), we have for every \( i \in N \), \( h_i \in H_i \), \( \theta_i \in \Theta_i \) with \( \pi^O_i(\theta_i|g_i) > 0 \) where \( g_i \in G \) with \( T^O(g_i|h_i) > 0 \),

\[
\sum_{h_i, \theta_i} \left( V_i(h_i, \theta_i) - \sum_{a_i, \theta_{-i}} Q^O_i(h_i, \theta_i, a_i|V_i) \pi^O_i(a_i|\theta_i) \mu_i(\theta_{-i}|\theta_i) \right) = 0.
\]

Since (22) holds for \( < \pi^O, V^O > \), we have

\[
\tilde{V}_i(h_i, \theta_i) = \sum_{a_i, \theta_{-i}} Q^O_i(h_i, \theta_i, a_i|\tilde{V}_i) \pi^O_i(a_i|\theta_i) \mu_i(\theta_{-i}|\theta_i).
\]

From iteration, we have that \( V^O \) is the unique optimal HT value function associated with \( \pi^O \). In addition, (OB) implies that given \( V^O \), obedient selection \( \beta^O \) is a PPME selection policy. Therefore, \( < \beta^O, \pi^O > \) is an obedient PPME.

\[\square\]
B. Proof of Proposition 2

For given $\kappa \in \Gamma[\beta^O]$, let $<\tau^k, \pi>$ satisfy (AT). The constraint (AT) is satisfied automatically by $<\tau^k, \pi^O>$. Suppose that $<\beta^O, \pi, V>$ is a global minimum of (OPT[\tau^k]) with $Z^{\tau^k}(\pi, V) = 0$. Then, the constraints (RG\tau_i) and (FT_i) are trivially satisfied. Proposition 1 implies that $<\beta^O, \pi>$ is an obedient PPME that attains the goal $\kappa$. From the construction of $Z^{\tau^k}(\pi)$ and the constraint (EQ), we obtain that $<\tau^k, V, \pi>$ satisfies the constraint (EQ1). According to (12), we construct $J$ as

$$J(h) = \sum_{g, \theta} V(h, \theta) \tau^k(\theta|g) T_g(g|h).$$  \hfill (23)

Then, $Z^{FPA}(\tau^k, J, V; \kappa) = 0$. Since $V, \pi >$ satisfies constraint (OB) given $\tau^k$, $J(h) \geq \sum_{g, \theta} V(h, \theta, \theta_{-i}) \tau^k(\theta_{-i}|g) T_g(g|h)$, for all $\theta_i$ and $h$, which implies constraint (OB1). From the constraints (EQ1) and (OB1), we know that for any feasible $<\tau^k, J', V'>$, $Z^{FPA}(\tau^k, J', V'; \kappa) \geq 0$. Therefore, $<\tau^k, J, V>$ is a global minimum of (FPA[\kappa]) with $Z^{FPA}(\tau^k, J, V; \kappa) = 0$.

Conversely, let $<\tau^k, J, V>$ be a global minimum of (FPA[\kappa]) with $Z^{FPA}(\tau^k, J, V; \kappa) = 0$. Then,

$$f(h) = \sum_{g, \theta} \hat{V}_i(h, \theta) \tau^k(\theta|g) T_g(g|h).$$ \hfill (24)

The constraint (EQ1) directly implies the constraint (EQ). The constraint (OB1) implies

$$f(h) \geq \sum_{g, \theta, a} \left(\hat{R}_i(h, \theta, a) + J_i(g, a)\right) \pi(a|\theta) \tau^k(\theta|g) T_g(g|h).$$ \hfill (25)

The right-hand side (RHS) of (25) can be written as:

$$\text{RHS of (25)} = \sum_{g, \theta, a} \left(\hat{R}_i(h, \theta, a) + \sum_{g} f(h, g, a) \mu_i(g|\theta_i)\right) \times \pi(a|\theta) \tau^k(\theta|g) T_g(g|h).$$ \hfill (26)

Construct

$$\hat{Q}_i(h, \theta, a|\hat{V}_i) = \hat{R}_i(h, \theta, a) + \sum_{g} \left(\sum_{\theta, g} \hat{V}_i(g, a, \theta) \tau^k(\theta|g) T_g(g|h) \mu_i(g|\theta_i)\right).$$

Then, RHS of (25) $= \sum_{g, \theta, a} \hat{Q}_i(h, \theta, a|\hat{V}_i) \pi(a|\theta) \tau^k(\theta|g) T_g(g|h)$.

The constraint (EQ1) implies

$$\hat{V}_i(h, \theta) = \sum_a \hat{Q}_i(h, \theta, a|\hat{V}_i) \pi(a|\theta),$$

and thus $Z^{\tau^k}(\pi, \hat{V}) = 0$. Hence, from (24), we have

$$\sum_{g, \theta} \hat{V}_i(h, \theta) \tau^k(\theta|g) T_g(g|h) \geq \sum_{g, \theta} \hat{V}_i(h, \theta) \tau^k(\theta|g) T_g(g|h),$$ \hfill (27)

which implies the constraint (OB). Therefore, $<\pi, \hat{V}>$ is a global minimum of (OPT[\tau^k]) with $Z^{\tau^k}(\pi, \hat{V}) = 0$. \hfill \Box

C. Proof of Theorem 1

To prove Theorem 1, we need to show that $<\tau^k, J, V>$ is a global minimum of (FPA[\kappa]) (implementability) with $Z^{FPA}(\tau^k, J, V; \kappa) = 0$ for a goal $\kappa$ if and only if $\tau^k$ is local admissible with the goal $\kappa$ (local admissibility). Our proof is based on any fixed goal $\kappa$.

C2. Local Admissibility $\Rightarrow$ Implementability

Fix any $g \in G$ and $h \in H$. Suppose that $\tau^k$ is locally admissible with goal $\kappa$. Because $\Delta_i(X_i^g, \tau^k_i, f; g, h) = 0$, we have

$$\sum_{\theta_i \in \Theta^g_i} \nabla_{X_i^g} \lambda_i(X_i; h, \theta_i^k) \pi_i(\theta_i^k|g) + \nabla_{X_i^g} \lambda_i(X_i; h, \hat{\theta}_i) \tau^k_i(\hat{\theta}_i|g)$$

for all $\theta_i$ and $h$, which implies $<\tau^k, J, \pi>$ is a global minimum of (FPA[\kappa]) with $Z^{FPA}(\tau^k, J, \pi; \kappa) = 0$.

Then, Condition 1 implies

$$f(\theta_i) = \tau^k_i(\theta_i|g), \text{ for all } \theta_i \in \Theta_i.$$ \hfill (28)

The decomposition $\Theta_i = \Theta_i^g \cup \{\hat{\theta}_i\}$ implies that $\theta_i$ can be fully characterized by $\Theta_i^g$. That is,

$$\tau^k_i(\theta_i|g) = 1 - \sum_{\theta_i \in \Theta_i^g} \tau^k_i(\theta_i|g).$$

From $d(X_i^g, \tau^k_i(\theta_i|g), b(\theta_i), c; h, g) = 0$, we have

$$b(\theta_i) = -\lambda_i(X_i^g; h, \theta_i) - \lambda_i(X_i^g; h, \hat{\theta}_i) = 0.$$ \hfill (29)

Therefore, the conditions in (20) leads to $M_i(X_i^g, \tau^k_i; h) = 0$ which implies $Z^{FPA}(\tau^k_i, J, V; \kappa) = 0$. Moreover, the condition $\pi(a_i|\theta_i) \tau^k_i(\theta_i|g) = 0$, $\forall \theta_i \in A_i$ of (21) implies $Z^{\tau^k_i}(\pi, V) = 0$. From Proposition 1, $<\pi, V>$ constitutes an obedient PPME of PSBG[\tau^k]$. Then, based on Proposition 2 and $\kappa(a_i|g) = \sum_{\theta_i \in \Theta_i^g} \pi(a_i|\theta_i) \tau^k_i(\theta_i|g)$ of (21), we have that $<\tau^k, J, V>$ is a global minimum of (FPA[\kappa]) with $Z^{FPA}(\tau^k, J, V; \kappa) = 0$.

C1. Implementability $\Rightarrow$ Local Admissibility

Suppose that a direct mechanism PSBG[\tau^k] is implementable in obedient PPME that achieves the goal $\kappa$. Hence,
there exists a \( \tau^k, J, V > \) is a global minimum of (FPA[\kappa]) with \( Z^{FPA}(\tau^k, J, V; \kappa) = 0 \). First, \( \tau^k(\theta_i|g) \geq 0 \)

According to the constraints of (FPA[\kappa]), every
\[
J_i(h) - \sum_{\theta, g} V(h, \theta) \tau^k(\theta|g) T_g(g|h) \geq 0,
\]
which implies that every \( \lambda_i(X^g_i; h, \theta_i) \geq 0 \).

Because \( Z^{FPA}(\tau^k, J, V; \kappa) = 0 \), we have
\[
J_i(h) - \sum_{\theta, g} V(h, \theta) \tau^k(\theta|g) T_g(g|h) = 0.
\]
Then, from the definition of \( M_i \) in (17), \( M_i = 0 \). Since \( \lambda_i(X^g_i; h, \theta_i) \geq 0 \) for \( \theta_i \in \Theta_i \), we have
\[
\tau^k(\theta_i|g) \lambda_i(X^g_i; h, \theta_i) = 0.
\]
By constructing \( f[\theta_i], b[\theta_i], \) and \( c \) according to (28) and (29), respectively, we can show that there exist Lagrange multipliers such that the conditions in (20) are satisfied.

From Proposition 2, \( < \pi, V > \) constitutes an obedient PPME and is a global minimum of (OPT[\tau^k]) with \( Z^{\tau^k}(\pi, V) = 0 \). From the constraints of (OPT[\tau^k]), we have each
\[
EV_{\tau^k}(h, \theta_i|\pi, V_i) \geq E_{\tau^k}[Q_{\tau^k}(h, \theta_i, a_i, \bar{a}_{-i}|V_i) | h, \theta_i].
\]
Then, \( Z^{\tau^k}(\pi, V) = 0 \) implies
\[
EV_{\tau^k}(h, \theta_i|\pi, V_i) - E_{\tau^k}[Q_{\tau^k}(h, \theta_i, a_i, \bar{a}_{-i}|V_i) | h, \theta_i] = 0.
\]
Therefore, we have
\[
\pi(a_i|\theta_i) \gamma_{\tau^k}(J_i, V_i, \pi_{-i}(\cdot|\theta_i); h, \theta_i, a_i) = 0, \forall a_i \in A_i.
\]
Thus, \( \tau^k \) is locally admissible. \( \square \)