The relative Gel’fand-Kalinin-Fuks cohomology groups of the formal Hamiltonian vector fields on 6-dimensional plane

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1 Introduction

At the beginning of research of the (relative) Gel’fand-Fuks cohomology group, Gel’fand-Kalinin-Fuks ([2]) got that $H^w_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R})) = 0$ for $w = 2, 4, 6$ and the $H^7_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R})) \cong \mathbb{R}$, whose generator is called the Gel’fand-Kalinin-Fuks class. The next non-trivial result in this context is $H^9_{GF}(\mathfrak{ham}_2, \mathfrak{sp}(2, \mathbb{R})) \cong \mathbb{R}$, which is discovered by S. Metoki ([9]) in 1999. In short, $\mathfrak{ham}_2$ is the Lie algebra of the Hamiltonian vector fields of the formal polynomials on $\mathbb{R}^2$.

D. Kotschick and S. Morita ([7]) research $H^w_{GF}(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))$ and determined the whole spaces while $w \leq 10$, where $\mathfrak{ham}^0_2$ is the Lie subalgebra of the Hamiltonian vector fields of the formal polynomials which vanish at the origin of $\mathbb{R}^2$.

Inspired by [7], we are interested in studying higher weight or higher dimensions. When $n = 1$, we have got some results for higher weight cases in [11], and when $n = 2$ for lower weight cases in [10]. A big difference of methodology in the two cases happens when getting the irreducible decomposition of the tensor product of irreducible representations of $Sp(2n, \mathbb{R})$. When $n = 1$, it is lucky we can use the Clebsch-Gordan rule. When $n = 2$, in [10] we have used the Littlewood-Richardson rule.

In this paper, we deal with the case of $n = 3$. The main target is again the space of homogeneous polynomials. If we put $S_k(\mathbb{R}^{2n})$ be the vector space of order $k$ homogeneous polynomials of $x_1, x_2, \ldots, x_{2n}$ of $\mathbb{R}^{2n}$, then it is known that $\dim S_k(\mathbb{R}^{2n}) = \frac{(2n - 1 + k)!}{(2n - 1)!k!}$. The table below shows how fast the dimension of $S_k(\mathbb{R}^{2n})$ increases in accordance with $k$ or $n$:

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By the methodology we have used so far, we encountered some difficulty in computations when \( n = 3 \).
So, this time, instead of the Littlewood-Richardson rule, we would like to use of the crystal base theory by M. Kashiwara ([4], [5], [15]).

Abbreviating the relative cochain space \( C^j_{GF}(\mathfrak{ham}^0_6, sp(6, \mathbb{R}))_w \) by \( C^j_w \), we introduce our first result and the second result about Betti numbers for weight 2, 4 and 6.

**Theorem 1:** Dimensions of each relative cochain complexes of weight 2 or 4 are as below:

| \( n \) | \( k \) | 1  | 2  | 3  | 4  | 5  | 6  | \( \cdots \) |
|--------|--------|----|----|----|----|----|----|-------|
| 1      | \( \text{dim} \) | 0  | 1  | 2  | 3  | 4  | 5  | \( \cdots \) |
| 2      | \( \text{dim} \) | 0  | 0  | 1  | 3  | 4  | 7  | \( \cdots \) |
| 3      | \( \text{dim} \) | 0  | 1  | 1  | 0  | 4  | 7  | \( \cdots \) |

The Euler characteristic number of \( H^\bullet_{GF}(\mathfrak{ham}^0_6, Sp(6, \mathbb{R}))_2 = 1 \) and the Euler characteristic number of \( H^\bullet_{GF}(\mathfrak{ham}^0_6, Sp(6, \mathbb{R}))_4 = 2 \).

The table of dimensions of each relative cochain complexes with weight 6 is

| \( \text{dim} \) | \( C^1_6 \) | \( C^2_6 \) | \( C^3_6 \) | \( C^4_6 \) | \( C^5_6 \) | \( C^6_6 \) |
|-----------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0               | 1           | 1           | 0           | 4           | 7           |             |

and the Euler characteristic number of \( H^\bullet_{GF}(\mathfrak{ham}^0_6, Sp(6, \mathbb{R}))_6 = 3 \).

The Euler characteristic number means in this paper the alternating sum of Betti numbers except 0-dimensional.

**Theorem 2:** The Betti numbers of weight 2 relative Gel’fand-Kalinin-Fuks cohomology groups are \( b^0_2 = 1, b^1_2 = 0, b^2_2 = 1 \).

About the Betti numbers of weight 4 relative Gel’fand-Kalinin-Fuks cohomology groups, \( b^4_4 = 2, b^0_4 = 1 \) and the others are 0.

In the case of weight 6, we have \( b^0_6 = 1, b^1_6 = b^2_6 = b^3_6 = b^4_6 = 0, b^5_6 = 2, \) and \( b^6_6 = 5 \).

## 2 Preliminary

We are interested in the relative Gel’fand-Kalinin-Fuks cohomology groups of \( \mathfrak{ham}^0_{2n} \) when \( n = 3 \). We review the notions we deal with in this paper, but we refer the precise definitions to [11].
We can split the polynomial functions by their homogeneity, and we regard the Lie algebras as follows.

\[ \mathfrak{ham}_{2n} = \left( \bigoplus_{p=1}^{\infty} \mathcal{S}_p(\mathbb{R}^{2n}) \right)^\wedge \] is a Lie algebra

\[ \mathfrak{ham}^0_{2n} = \left( \bigoplus_{p=2}^{\infty} \mathcal{S}_p(\mathbb{R}^{2n}) \right)^\wedge \] is a subalgebra of \( \mathfrak{ham}_{2n} \)

and

\[ \mathfrak{ham}^1_{2n} = \left( \bigoplus_{p=3}^{\infty} \mathcal{S}_p(\mathbb{R}^{2n}) \right)^\wedge \] is a subalgebra of \( \mathfrak{ham}_{2n} \)

where \((\quad)^\wedge\) means the completion with the Krull topology.

The cochain complex is the exterior algebra of the dual spaces of \( \mathcal{S}_p(\mathbb{R}^{2n}) \)'s, and we have the notion of “weight” on the cochain complex.

**Definition 2.1** Let \( \mathcal{S}_\ell \) be the dual space of \( \ell \)-homogeneous polynomial functions \( \mathcal{S}_\ell(\mathbb{R}^{2n}) \), and define the weight of each non-zero element of \( \mathcal{S}_\ell \) to be \( \ell - 2 \). For each non-zero element of \( \mathcal{S}_{\ell_1} \wedge \mathcal{S}_{\ell_2} \wedge \cdots \wedge \mathcal{S}_{\ell_s} \) \((\ell_1 \leq \ell_2 \leq \cdots \leq \ell_s)\), we define its weight to be \( \sum_{i=1}^{\ell} (\ell_i - 2) \).

**Proposition 2.1 (cf.[7],[11])** The coboundary operator \( d \) of the Gel’fand-Kalinin-Fuks cochain complex preserves the weight, namely, if a cochain \( \sigma \) is of weight \( w \), then \( d \sigma \) is also of weight \( w \).

Hence we can decompose the total space of cochain complex by degree and weight: namely,

\[ C^m_{GF}(\mathfrak{ham}_{2n})_w = \text{Linear Span of } \{ \sigma \in \Lambda^{k_1} \mathcal{S}_1 \wedge \Lambda^{k_2} \mathcal{S}_2 \wedge \cdots \wedge \Lambda^{k_s} \mathcal{S}_s \mid \sum_{j=1}^{s} k_j = m, \sum_{j=1}^{s} k_j(j - 2) = w, s = 1, 2, \ldots \} \]

and we can define the cohomology group \( H^m_{GF}(\mathfrak{ham}_{2n})_w \).

\( C^\bullet_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w \) is the subspace of \( C^\bullet_{GF}(\mathfrak{ham}_{2n})_w \) characterized by \( k_1 = 0 \). If we restrict our attention to the cochain complex relative to \( \mathfrak{sp}(2n, \mathbb{R}) \), then it turns out \( k_2 = 0 \) and

\[ C^\bullet_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w = \left( C^\bullet_{GF}(\mathfrak{ham}^1_{2n})_w \right)^\text{triv} \]

(1)

(cf.[11]).

**Remark 2.1 (cf.[7])** Since we have the negative Identity matrix in \( \mathfrak{sp}(2n, \mathbb{R}) \), we see that the relative cochain complex of odd weight must be the zero space, and hence we only deal with the complexes of even weights.
From (1), when we study \( C^*_G(\mathfrak{ham}_{2n}^0, \mathfrak{sp}(2n, \mathbb{R}))_w \), we first look at the subcomplex \( C^*_G(\mathfrak{ham}_{2n}^1)_w \), namely, \( k_1 = k_2 = 0 \). So, for \( w = 2, 4, 6 \), we consider finite sequences of non-negative integers \((k_3, k_4, \ldots, k_s)\) satisfying

\[
\sum_{j=3}^s k_j = m \quad \text{and} \quad \sum_{j=3}^s k_j(j - 2) = w.
\]

(2)

**Proposition 2.2** (cf. [10]) When \( n \geq 2 \) and weight = 2, 4 or 6, the non-trivial sub-cochain complexes are as follows:

\[
C^1_G(\mathfrak{ham}_{2n}^1)_2 = \mathfrak{S}_4, \quad C^2_G(\mathfrak{ham}_{2n}^1)_2 = \Lambda^2 \mathfrak{S}_3
\]

\[
C^1_G(\mathfrak{ham}_{2n}^1)_4 = \mathfrak{S}_6, \quad C^2_G(\mathfrak{ham}_{2n}^1)_4 = (\mathfrak{S}_3 \wedge \mathfrak{S}_5) \oplus \Lambda^2 \mathfrak{S}_4 \cong (\mathfrak{S}_3 \otimes \mathfrak{S}_5) \otimes \Lambda^2 \mathfrak{S}_4,
\]

\[
C^3_G(\mathfrak{ham}_{2n}^1)_4 = \Lambda^2 \mathfrak{S}_3 \wedge \mathfrak{S}_4 \cong \Lambda^2 \mathfrak{S}_3 \otimes \mathfrak{S}_4, \quad C^4_G(\mathfrak{ham}_{2n}^1)_4 = \Lambda^4 \mathfrak{S}_3
\]

In the above, we identify the exterior product \( \mathfrak{S}_3 \wedge \mathfrak{S}_5 \) with the tensor product \( \mathfrak{S}_3 \otimes \mathfrak{S}_5 \) as vector spaces, and we often use this identification without comments.

\[
C^1_G(\mathfrak{ham}_{2n}^1)_6 = \mathfrak{S}_8, \quad C^2_G(\mathfrak{ham}_{2n}^1)_6 = (\mathfrak{S}_3 \otimes \mathfrak{S}_7) \oplus (\mathfrak{S}_4 \otimes \mathfrak{S}_6) \oplus \Lambda^2 \mathfrak{S}_5
\]

\[
C^3_G(\mathfrak{ham}_{2n}^1)_6 = (\Lambda^2 \mathfrak{S}_3 \otimes \mathfrak{S}_6) \oplus (\mathfrak{S}_3 \otimes \mathfrak{S}_4 \otimes \mathfrak{S}_5) \oplus \Lambda^3 \mathfrak{S}_4
\]

\[
C^4_G(\mathfrak{ham}_{2n}^1)_6 = (\Lambda^3 \mathfrak{S}_3 \otimes \mathfrak{S}_5) \oplus (\Lambda^2 \mathfrak{S}_3 \wedge \Lambda^2 \mathfrak{S}_4)
\]

\[
C^5_G(\mathfrak{ham}_{2n}^1)_6 = \Lambda^4 \mathfrak{S}_3 \otimes \mathfrak{S}_4, \quad C^6_G(\mathfrak{ham}_{2n}^1)_6 = \Lambda^6 \mathfrak{S}_3
\]

The next job we have to do is to pick up all the trivial representations in \( C^m_G(\mathfrak{ham}_{2n}^1)_w \).

### 3 Crystal Base Theory

Our group is \( Sp(2n, \mathbb{R}) \) and each irreducible representation space of \( Sp(2n, \mathbb{R}) \) is parameterized by a partition \( \mu \) of length less than \( n \). We denote its space by \( V_\mu \). When \( W \) is a representation space of \( Sp(2n, \mathbb{R}) \) and let us assume \( W \) is decomposed into irreducible subspaces as \( W = W_1 \oplus W_2 \oplus W_3 \). If \( W_1 \) and \( W_3 \) are isomorphic with \( V_\lambda \), and \( W_2 \) is isomorphic with \( V_\mu \), then we may denote \( W \cong 2V_\lambda \oplus V_\mu = 2V_\lambda + V_\mu \) (often), and \( W^\lambda = W_1 \otimes W_3 \cong 2V_\lambda \) and \( W^\mu = W_2 \cong V_\mu \).

#### 3.1 Crystal base

According to [4] and [15], we have the crystal basis for \( V_\mu \), say \( \text{base}_\mu \) consists of the semistandard \( C \)-tableaux on the shape \( \mu \), where the numbers from \( 1, \ldots, n, \pi, \ldots, \overline{\pi} \) are printed, where \( \overline{k} := 2n + 1 - k \).
for $1 \leq k \leq n$.
For a given SS tableau, we pick a column, say $J$. Then $J$ is a sequence of increasing integers of $1, 2, \ldots, n, n+1 = \overline{n}, \ldots, 2n = \overline{2n}$. By $\text{pos}_J(a)$, we mean the position of the element $a$ in $J$. For example, if $J = \begin{array}{c}
2 \\
5 \\
6 
\end{array}$, then $\text{pos}_J(2) = 1$, $\text{pos}_J(5) = 2$ and $\text{pos}_J(6) = 3$.

SS $C$-tableau is a SS tableau with 2 more conditions:

C-1 is a requirement for each column $J$ of $\mu$ such that if $k, \overline{k}$ belong to $J$, then

$$\text{pos}_J(k) + (|J| + 1 - \text{pos}_J(\overline{k})) \leq k.$$ 

C-2 is a requirement for each successive two columns $L, R$ of $\mu$ (so that $|L| \geq |R|$) satisfy $L(i) \leq R(i)$ for $i = 1..|R|$ (which is one of conditions of SS Tableau) and for each $1 \leq a \leq b \leq n$, if $(a, b)$-configuration then $a, b$ must satisfy $\text{pos}(b) - \text{pos}(a) + \text{pos}(\overline{a}) - \text{pos}(\overline{b}) < b - a$.

We assume $1 \leq a \leq b \leq n$ (and so $1 \leq a \leq b < \overline{b} \leq \overline{a} \leq 2n$) and assume that $a \in L, \overline{a} \in R$ and $b, \overline{b}$ belongs to $L$ or $R$.

$(a, b)$-configuration is when

$$\text{pos}_L(a) \leq \text{pos}_R(b) < \text{pos}_R(\overline{b}) \leq \text{pos}_R(\overline{a}) \quad \text{or} \quad \text{pos}_L(a) \leq \text{pos}_L(b) < \text{pos}_L(\overline{b}) \leq \text{pos}_L(\overline{a})$$

is satisfied.

**Example 3.1** When $n = 2$ and $\mu = (1, 1)$, the $\text{cbase}_\mu$ consists of

\[
\begin{array}{cccccc}
1 & 1 & 2 & 2 & 3 & 4 \\
2 & 3 & 3 & 4 & 4 & 4 \\
\end{array}
\]

We remark that $\begin{array}{c}
1 \\
4 = \overline{1} 
\end{array}$ is a SSTab, but not a member of $\text{cbase}_\mu$ because of the condition C-1.

The number of $\text{cbase}_\mu$ is equal to $\dim V_\mu$ gotten by Dimension formula coming from Weyl’s character formula.

**Example 3.2** When $n = 2$ and $\nu = (2, 2)$, the $\text{cbase}_\nu$ consists of

\[
\begin{array}{cccccccccccc}
1 & 1 & 1 & 2 & 1 & 2 & 1 & 3 & 1 & 1 & 1 & 2 \\
2 & 2 & 2 & 3 & 2 & 3 & 2 & 4 & 2 & 3 & 3 & 3 \\
3 & 4 & 3 & 4 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]
From the condition (C-1), each column must be one of Example 3.1. Accounting SS Tab property, then candidates should be \( \binom{5 - 1 + 2}{2} = 15 \), but \( \begin{array}{cc}
2 & 2 \\
3 & 3 
\end{array} \) has \((2, 2)\)-configuration property but does not satisfy the condition (C-2), and is not a member of \( \text{cbase}_\mu \).

### 3.2 Decomposition of tensor product by crystal base

Take another irreducible representation \( V_\lambda \). Then the irreducible decomposition of the tensor product \( V_\lambda \otimes V_\mu \) is given by combinatorially from \( \lambda \) and the crystal base of \( \mu \) as follows:

- For each tableau \( T \) of \( \text{cbase}_\mu \),
  1. Stand at the top of the right most column of \( T \).

     **action:** from the top to downward, let the number on the cell apply to the Young diagram by the next rule:

     - If the number \( k \) on the cell of \( T \) satisfies \( k \leq n \), then the new diagram is made by adding one cell to the \( k \)-th row of the old Young diagram. If the new diagram is not Young diagram, stop this process.
     - If the number \( k \) satisfies \( k > n \), then the new diagram is made by deleting one cell from the \((2n + 1 - k)\)-th row of the old Young diagram. If the new diagram is not Young diagram, stop this process.

  2. If we still have a Young diagram, then move the one left column and do **the action** above, and so on.

  3. At the left most column, if we could get finally a Young diagram, say \( \nu \), after the all actions, then \( V_\nu \) is one factor of \( V_\lambda \otimes V_\mu \).

#### 3.2.1 A concrete example of the action of crystal base to a Young diagram

When \( n = 3 \), the crystal base of \( \mu = (2, 1) = \begin{array}{c}
1 \\
1 \\
2 \\
\end{array} 
\) is known as

\[
\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 3 & 3 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

Apply \( \begin{array}{c}
1 \\
2 
\end{array} \) to \( \lambda = (1, 1) = \begin{array}{c}
1 \\
1 
\end{array} \) then we have
Apply \( \begin{bmatrix} 1 & 1 \\ 3 \end{bmatrix} \) to \( \lambda = (1, 1) = \begin{array}{c} \text{•} \\ \text{•} \\ \end{array} \) then by the notation of partitions, we have

Not Young diagram \((3, 1, -1) \overset{2}{\leftarrow} (3, 1) \overset{1}{\leftarrow} (2, 1) \overset{1}{\leftarrow} (1, 1)\)

Apply \( \begin{bmatrix} 1 & 3 \\ 3 \end{bmatrix} \) to \( \lambda = (1, 1) = \begin{array}{c} \text{•} \\ \text{•} \\ \end{array} \) then by the notation of partitions, we have

\((2, 1, 0) \overset{2}{\leftarrow} (2, 1, 1) \overset{1}{\leftarrow} (1, 1, 1) \overset{1}{\leftarrow} (1, 1, 0)\)

Apply \( \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix} \) to \( \lambda = (1, 1) = \begin{array}{c} \text{•} \\ \text{•} \\ \end{array} \) then by the notation of partitions, we have

\((2, 1, 0) \overset{2}{\leftarrow} (2, 0, 0) \overset{1}{\leftarrow} (1, 0, 0) \overset{1}{\leftarrow} (1, 1, 0)\)

Apply \( \begin{bmatrix} 2 & 1 \\ 1 \end{bmatrix} \) to \( \mu = (2, 1) = \begin{array}{c} \text{•} \\ \text{•} \\ \end{array} \) then by the notation of partitions, we have

trivial \((0, 0) \overset{1}{\leftarrow} (1, 0) \overset{1}{\leftarrow} (1, 1) \overset{1}{\leftarrow} (2, 1).\)

### 3.3 Pick up trivial representations from \( V_\lambda \otimes V_\mu \)

It is well-known that \( \text{triv} \otimes V_\mu \cong V_\mu \) for each \( \mu \). If we look at this result by the crystal base theory, we see that there is the unique element \( J \) of the crystal base of \( V_\mu \) and satisfies \( J \cdot \text{triv} = \mu \), where \( \cdot \) means the action of \( J \) to each partition.

**Proposition 3.1** The SS C-tableau \( J \) of the shape \( \mu \), satisfying \( J \cdot \text{triv} = \mu \) is defined by printing number \( k \) for the cells of the \( k \)-row of the shape of \( \mu \). We denote this SS C-tableau by \( T_\mu \).

If a SS C-tableau \( J \) of the shape \( \mu \) satisfies \( J \cdot \text{triv} = \lambda \), then \( \lambda = \mu \) and \( J = T_\mu \). If \( \mu \neq \text{triv} \), then at each step of \( J \cdot \text{triv} \), the Young diagram is not trivial.

**Proof:** The first operation acting to the trivial Young diagram is 1. Thus, the right most and the top entry of \( J \) is 1. By SS Tab properties, the first row of \( J \) is printed by only 1. Now by (C1) condition, \( J \) does have no \( \overline{T} \). Let \( x \) be the entry of left most of the second row of \( J \). Then the first part of the sequence of operations is \( x, 1, 1, \ldots, 1 \) with \( x \neq \overline{T} \). Keeping the Young diagram property implies \( x = 2 \). We claim that \( J \) does have no \( \overline{2} \). The reason is: if \( \overline{2} \in J \), take a column \( L \) of \( J \) with \( \overline{2} \in L \). Then \( \text{pos}_L(2) = 2 \).
and \( \text{pos}_L(\mathcal{L}) = |L| \) and conflicts with (C1) condition. Repeating the same discussion, we concludes our proof.

We denote the diagram of height \( h \) of one cell by \( s_h \). Any Young diagram of length less than \( n \) is expressed as

\[
\mu = s_n p_n s_{n-1} p_{n-1} \cdots s_2 p_2 s_1 p_1
\]

where \( p_i \geq 0 \) \((i = 1..n)\).

\[
\begin{align*}
\mu_1 &= p_n + p_{n-1} + \cdots + p_2 + p_1 & p_n = \mu_n \\
\mu_2 &= p_n + p_{n-1} + \cdots + p_2 & p_{n-1} = \mu_{n-1} - \mu_n \\
\vdots & & \vdots \\
\mu_n &= p_n & p_1 = \mu_1 - \mu_2
\end{align*}
\]

holds good. Conversely, for a given partition \( \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0) \), we define the sequence of \( p_i \) \((i = 1..n)\) the above relations. Thus, sometimes we regard of Young diagram as the (horizontal!?) collection of columns.

Let \( t_j = \) be a SS C-tableau on \( s_j \). Then \( T_\mu = t_n p_n t_{n-1} p_{n-1} \cdots t_2 p_2 t_1 p_1 \).

**Definition 3.1** For a partition or Young diagram \( \mu \) of length less than \( n \), define the SS C-tableau \( T_\mu \) by printing number \( k \) for the cells of the \( k \)-row of \( T_\mu \). In other words, \( T_\mu = t_n p_n t_{n-1} p_{n-1} \cdots t_2 p_2 t_1 p_1 \), where \( p_j \) \((j = 1..n)\) are defined as (3).

**Example 3.3** When \( n = 3 \) and \( \mu = (4, 2, 1) \), \( T = \)

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & & \\
\vdots & & & \\
3 & & &
\end{array}
\]

then \( T \cdot \text{triv} \) becomes \( \mu \) as

\[
(4, 2, 1) \rightarrow (4, 2, 0) \rightarrow (4, 1, 0) \rightarrow (3, 1, 0) \rightarrow (3, 0, 0) \rightarrow (2, 0, 0) \rightarrow (1, 0, 0) \rightarrow (0, 0, 0)
\]

We look the above flow reversely, i.e.,

\[
(4, 2, 1) \rightarrow (4, 2, 0) \rightarrow (4, 1, 0) \rightarrow (3, 1, 0) \rightarrow (3, 0, 0) \rightarrow (2, 0, 0) \rightarrow (1, 0, 0) \rightarrow (0, 0, 0)
\]

the action does not come from a SS C-tableau. Instead of this, we apply bar-operation for each element.
of $T$ and get a new tableau $T' = \begin{array}{cccc}
1 & 2 & 3 \\
2 & 2 & \\
3 & & \\
\end{array}$. Although $T'$ is not SS Tab, but reversing the elements of each column of $T'$, we get a SS C-tableau $\hat{T} = \begin{array}{ccc}
3 & 2 & \\
2 & & \\
1 & & \\
\end{array}$. Acting $\hat{T}$ on the partition $\mu = (4,2,1)$, we get the trivial representation as below:

$$(0,0,0) \overset{\hat{T}}{\longleftarrow} (1,0,0) \overset{\hat{T}}{\longleftarrow} (1,1,0) \overset{\hat{T}}{\longleftarrow} (2,1,1) \overset{\hat{T}}{\longleftarrow} (2,2,1) \overset{\hat{T}}{\longleftarrow} (3,2,1) \overset{\hat{T}}{\longleftarrow} (4,2,1)$$

**Definition 3.2** We put $\hat{t}_j = j$ for $j = 1..n$. Let $\mu$ be a partition of length less than or equal to $n$. Define a tableau by $\hat{T}_\mu = \hat{t}_n^{p_n}\hat{t}_{n-1}^{p_{n-1}}\ldots\hat{t}_2^{p_2}\hat{t}_1^{p_1}$, where $p_j$ ($j = 1..n$) are defined as in (3). It is easy to see that $\hat{T}_\mu$ is a SS C-tableau on the shape $\mu$ and satisfies $\hat{T}_\mu \cdot \mu = \text{triv}$ because of

$$\hat{T}_\mu \cdot \mu = (\hat{t}_n^{p_n}\hat{t}_{n-1}^{p_{n-1}}\ldots\hat{t}_2^{p_2}\hat{t}_1^{p_1}) \cdot (s_n^{p_n}s_{n-1}^{p_{n-1}}\ldots s_2^{p_2}s_1^{p_1})$$

$$= (\hat{t}_n^{p_n}\hat{t}_{n-1}^{p_{n-1}}\ldots\hat{t}_2^{p_2}) \cdot (s_n^{p_n}s_{n-1}^{p_{n-1}}\ldots s_2^{p_2}s_1^{p_1})$$

$$= (\hat{t}_n^{p_n}\hat{t}_{n-1}^{p_{n-1}}\ldots\hat{t}_2^{p_2}) \cdot (s_n^{p_n}s_{n-1}^{p_{n-1}}\ldots s_2^{p_2}s_1^{p_1})$$

$$\vdots$$

$$= \text{triv}$$

As reverse situations of Proposition 3.1, we have

**Lemma 3.1** Let $J$ be a member of the crystal base of the irreducible representation $V_\mu$ of $Sp(2n, \mathbb{R})$, i.e. $J$ is a SS C-tableau on the shape $\mu$. If $J \cdot \lambda = \text{triv}$, then $\lambda = \mu$ and $J = \hat{T}_\mu$.

**Proof:** Suppose $J \cdot \lambda = \text{triv}$. Then the last operation of $J$ is $\overline{1}$. This means $J$ has $\overline{1}$ on the bottom of the left most column, say $L$. Then it turns out $J$ does not have the number 1. The reason is that if $J$ has 1, then $L$ can be characterized as the top entry is 1 and the bottom entry is $\overline{1}$. This conflicts with the condition (C-1).

$J$ having no 1 means that on the way of successive actions $J \cdot \lambda$, the trivial Young diagram does not appear until the final stage.
Let us imagine the one more past shape of the Young diagram. The possibilities are two, \( \text{triv} \leftarrow \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \downarrow \end{array} \) or \( \text{triv} \leftarrow \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \downarrow \end{array} \). In the first case, \( J \) becomes \( \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \downarrow \end{array} \cdots \begin{array}{c} \uparrow \\ \downarrow \end{array} \). In the second case, \( L \) is in form of \( \begin{array}{c} \uparrow \\ \downarrow \\ \vdots \end{array} \), and we claim that the number 2 does not live in \( J \) because of (C-1) condition.

Repeating the same arguments, we may assume that there is a maximal number \( k \) satisfying \( \ell \in L \) but \( j \notin J \) for \( j = 1..k \).

Again, imagine the Young diagram one step before. The three possibilities: \( [1^k] \leftarrow [1^{k-1}], [1^k] \leftarrow [2, 1^{k-1}] \), or \( [1^k] \leftarrow [1^{k+1}] \). The first case is \( y = k \) (impossible), the second case is \( y = \ell \) and the third case is \( y = k+1 \) (impossible). From those, we conclude \( L = \ell_k \) and if \( J \) has the second column from the left, say \( M \), the bottom entry of \( M \) is \( \ell \) and is just the successive series of \( \ell, \ell \ldots \) upward. We continue the discussion step by step and reach to the right most column. Thus, it turns out \( J = \ell_{\mu} \).

We write down \( \lambda = s_{nq_n} \cdots s_2 q_2 s_1 q_1 \), and consider the contributions of \( \ell_{\mu} \) to \( \lambda \), then

\[
\ell_{\mu} \cdot \lambda = (\ell_{p_n} \hat{\ell}_{p_{n-1}} \cdots \hat{\ell}_2 \hat{p}_1) \cdot (s_{nq_n} s_{n-1q_{n-1}} \cdots s_2 q_2 s_1 q_1) \\
= (\ell_{p_n} \hat{\ell}_{p_{n-1}} \cdots \hat{\ell}_2 \hat{p}_1) \cdot (s_{nq_n} s_{n-1q_{n-1}} \cdots s_2 q_2 s_1 q_1) \text{ if } q_1 - p_1 \geq 0 \\
= (\ell_{p_n} \hat{\ell}_{p_{n-1}} \cdots \hat{\ell}_3 \hat{p}_3) \cdot (s_{nq_n} s_{n-1q_{n-1}} \cdots s_2 q_2 s_1 q_1) \text{ if } q_1 - p_1 \geq 0, q_2 - p_2 \geq 0 \\
\vdots \\
= (s_{nq_n}^{-p_n} s_{n-1q_{n-1}}^{-p_{n-1}} s_3 q_3 - p_3 s_2 q_2 s_1 q_1 - p_1) \text{ if } q_j - p_j \geq 0 \text{ for } j = 1..n
\]

and we conclude \( \mu_j = \lambda_j \) ( \( j = 1..n \)), namely \( \lambda = \mu \).

**Proposition 3.2** Let \( W \) and \( Z \) be \( Sp(2n, \mathbb{R}) \)-representation space and assume that they have irreducible decompositions like \( W = \bigoplus_{\lambda \in I} \xi_{W}^{\lambda} V_{\lambda} \) \( (\xi_{W}^{\lambda} \in \mathbb{N}) \) and \( Z = \bigoplus_{\mu \in J} \xi_{Z}^{\mu} V_{\mu} \) \( (\xi_{Z}^{\mu} \in \mathbb{N}) \). Then the tensor product \( W \otimes Z \) of \( W \) and \( Z \) has the trivial representation of the multiplicity \( \sum_{\lambda \in I \cap J} \xi_{W}^{\lambda} \xi_{Z}^{\lambda} \), i.e.,

\[
(W \otimes Z)^{\text{triv}} \cong \left( \sum_{\lambda \in I \cap J} \xi_{W}^{\lambda} \xi_{Z}^{\lambda} \right)^{\text{triv}}
\]

**Corollary 3.3** Let \( W \) be a \( Sp(2n, \mathbb{R}) \) representation and assume that \( W \) has an irreducible decomposition
as $W = \oplus_{\lambda \in I} \xi^\Lambda_W V_{\lambda} \ (\xi^\Lambda_W \in \mathbb{N})$. Then we have

$$\dim (W \otimes \mathcal{S}_h)^{triv} = \xi^{[h,0^{n-1}]}_W$$

4 The relative cochain complexes

When $n \geq 2$ and weight=2, 4 or 6, we have the non-trivial sub-cochain complexes $C_\text{GF}(\text{ham}^1_{2n})_w$ in Proposition 2.2. We apply elementary irreducible decomposition rules to those spaces $C_\text{GF}(\text{ham}^3_{2n})_w$ and have the $C_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_w$ as follows.

$$C^1_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_2 = 0, \quad C^2_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_2 = (\Lambda^2 \mathcal{S}_3)^{triv}$$

$$C^1_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_4 = 0, \quad C^2_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_4 \cong (\Lambda^2 \mathcal{S}_4)^{triv}$$

$$C^3_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_4 = ((\Lambda^2 \mathcal{S}_3) \otimes \mathcal{S}_4)^{triv}, \quad C^4_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_4 = (\Lambda^4 \mathcal{S}_3)^{triv}$$

$$C^1_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_6 = 0, \quad C^2_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_6 = (\Lambda^2 \mathcal{S}_3)^{triv}$$

$$C^3_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_6 = ((\Lambda^3 \mathcal{S}_3) \otimes \mathcal{S}_6)^{triv} \oplus ((\Lambda^2 \mathcal{S}_3 \otimes \mathcal{S}_4) \otimes \mathcal{S}_3)^{triv} \oplus (\Lambda^3 \mathcal{S}_4)^{triv}$$

$$C^4_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_6 = ((\Lambda^3 \mathcal{S}_3) \otimes \mathcal{S}_5)^{triv} \oplus ((\Lambda^2 \mathcal{S}_3 \otimes \Lambda^2 \mathcal{S}_4)^{triv}$$

$$C^5_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_6 = ((\Lambda^4 \mathcal{S}_3) \otimes \mathcal{S}_4)^{triv}, \quad C^6_\text{GF}(\text{ham}^0_{2n}, \text{sp}(2n, \mathbb{R}))_6 = (\Lambda^6 \mathcal{S}_3)^{triv}$$

4.1 Splitting $\Lambda^p \mathcal{S}_q$ when $n=3$

As we see above lists, we need some irreducible factors or complete irreducible decomposition of $\Lambda^p \mathcal{S}_q$ case by case, which is an $Sp(6, \mathbb{R})$-invariant subspace of $\otimes \mathcal{S}_q$.

We recall here the fundamental fact which our discussion is completely based on. We state the fact for $n=3$ even though it is valid for a general $n$.

Let $T$ be the subgroup of $Sp(6, \mathbb{R})$ consisting of diagonal matrices, $U$ be the subgroup of $Sp(6, \mathbb{R})$ consisting of upper triangle matrices whose diagonal entries are 1, and $U^-$ be the subgroup of $Sp(6, \mathbb{R})$ consisting of lower triangle matrices whose diagonal entries are 1. $U$ is called the maximal unipotent subgroup of $Sp(6, \mathbb{R})$.

Let $W$ be a representation of $Sp(6, \mathbb{R})$. $x \in W$ is called a maximal vector if it satisfies

$$x \neq 0, \quad g \cdot x = x \quad (\forall g \in U), \quad t \cdot x = \lambda(t)x \quad (\forall t \in T, \exists \text{ weight } \lambda) .$$

(4)
The maximal vector is unique up to scalar multiple.

**Fact:** Let \( W \) be a representation of \( Sp(6, \mathbb{R}) \). Then the next hold.

1. Let \( x \) a maximal vector. Then the \( Sp(6, \mathbb{R}) \)-invariant subspace of \( W \) generated by \( x \) is an irreducible subspace. The irreducible space is also generated by \( \{(U^-)^k \cdot x \mid k = 0, 1, 2, \ldots\} \).

2. \( W \) can be decomposed into irreducible subspaces which are one-to-one corresponding to the set of maximal vectors.

To find maximal vectors, we consider the infinitesimal version of the second condition of (4), namely, \( \xi \cdot x = 0 \) for \( (\forall \xi \in u) \), where \( u \) is the Lie algebra of \( U \) and is 9-dimensional.

The infinitesimal action \( \xi \in u \) behaves as an ordinary derivation of degree 0 and for each 1-cochain \( \sigma \) and for each polynomial \( f \), we have

\[
\langle \xi \cdot \sigma, f \rangle = -\langle \sigma, \xi \cdot f \rangle = -\langle \sigma, \{\hat{J}(\xi), f\} \rangle
\]

where \( J \) is the momentum mapping of the natural symplectic action and \( \{,\} \) is the standard Poisson bracket on \( \mathbb{R}^6 \). In general, we have to deal with a big simultaneous linear homogeneous equations of \( 9 \times \dim W \)-equations with \( \dim W \)-variables.

Since our spaces are concrete and special, namely they are (sum of wedge products of) dual of homogeneous polynomials, we devise a simple but effective treatment for the third condition of (4). By the Crystal Base theory we know the decomposition of \( \bigotimes^p \mathfrak{S}_q \) and this decomposition suggests all the possibilities of maximal weights which appear in \( \Lambda^p \mathfrak{S}_q \).

We explain this trick in the case of \( \Lambda^2 \mathfrak{S}_5 \). If we write each basic element of \( \mathfrak{S}_5 \) by \( z_A \) where \( A = (a_1, a_2, \ldots, a_6) \) with \( a_j \in \mathbb{N}^+ \) and \( \sum_{j=1}^6 a_j = 5 \). A basis of \( \Lambda^2 \mathfrak{S}_5 \) consists of \( z_A \wedge z_B \) with \( A < B \) in the lexicographic order. We see that \( \dim \mathfrak{S}_5 = 252 \) when \( n = 3 \) and so \( \dim \Lambda^2 \mathfrak{S}_5 = 31626 \). Observing the torus action of \( Sp(6, \mathbb{R}) \) on this space, we see that

\[
[(a_1 - a_6) + (b_1 - b_6), (a_2 - a_5) + (b_2 - b_5), (a_3 - a_4) + (b_3 - b_4)]
\]

means the weight of \( z_A \wedge z_B \). If we try to find some basis of the trivial space, then the candidates are \( z_A \wedge z_B \) with \( [(a_1 - a_6) + (b_1 - b_6), (a_2 - a_5) + (b_2 - b_5), (a_3 - a_4) + (b_3 - b_4)] = [0, 0, 0] \) and we see the dimension is 330. We stress that the number is drastically reduced. We prepare a general vector of the form \( v = \sum_{j=1}^{330} c_j w_j \), where \( w_j \) is one of \( z_A \wedge z_B \) with the requirement above. By invariance by the maximal unipotent subgroup \( U \), we have 2159 linear equations of \( c_j \). Solving the linear equations, we see that
the freedom is 1, namely the solution space is 1-dimensional, namely $\Lambda^2 \mathfrak{S}_5$ has the trivial representation space with multiplicity 1. If we restart from another weight $\lambda = [\lambda_1, \lambda_2, \lambda_3]$, we may continue the same discussion and get the multiplicity of $V_{\lambda}$ in $\Lambda^2 \mathfrak{S}_5$. This discussion works well for general $\Lambda^p \mathfrak{S}_q$.

We show the irreducible decompositions of some $\Lambda^p \mathfrak{S}_q$ without proof.

**Proposition 4.1** When $n = 3$, we have

\begin{align*}
\Lambda^2 \mathfrak{S}_3 &\cong V_{[0,0,0]} + V_{[1,1,0]} + V_{[2,2,0]} + V_{[3,3,0]} + V_{[4,0,0]} + V_{[5,1,0]} \\
\Lambda^2 \mathfrak{S}_4 &\cong V_{[2,0,0]} + V_{[3,1,0]} + V_{[4,2,0]} + V_{[5,3,0]} + V_{[6,0,0]} + V_{[7,1,0]} \\
\Lambda^2 \mathfrak{S}_5 &\cong V_{[0,0,0]} + V_{[1,1,0]} + V_{[2,2,0]} + V_{[3,3,0]} + V_{[4,0,0]} + V_{[4,4,0]} \\
&\quad + V_{[5,1,0]} + V_{[5,5,0]} + V_{[6,2,0]} + V_{[7,3,0]} + V_{[8,0,0]} + V_{[9,1,0]} \\
\Lambda^3 \mathfrak{S}_3 &\cong V_{[2,1,0]} + 3V_{[3,0,0]} + 2V_{[3,1,1]} + V_{[3,2,0]} + V_{[3,3,3]} + 2V_{[4,1,0]} \\
&\quad + V_{[4,2,1]} + V_{[4,3,0]} + 2V_{[5,2,0]} + V_{[5,3,1]} + V_{[6,1,0]} + V_{[6,3,0]} + V_{[7,0,0]} + V_{[7,1,1]} \\
\Lambda^3 \mathfrak{S}_4 &\cong 2V_{[2,0,0]} + V_{[2,1,1]} + V_{[2,2,2]} + 3V_{[3,1,0]} + V_{[3,2,1]} + V_{[3,3,0]} + V_{[3,3,2]} + V_{[4,0,0]} \\
&\quad + V_{[4,1,1]} + 4V_{[4,2,0]} + 2V_{[4,3,1]} + V_{[4,4,2]} + 3V_{[5,1,0]} + 3V_{[5,0,0]} + 2V_{[5,1,1]} + 3V_{[5,3,0]} + V_{[5,3,2]} \\
&\quad + V_{[5,4,1]} + V_{[5,5,2]} + 3V_{[6,0,0]} + 2V_{[6,1,1]} + 2V_{[6,2,0]} + V_{[6,3,1]} + V_{[6,3,2]} \\
&\quad + 2V_{[6,4,0]} + 2V_{[6,1,0]} + V_{[7,2,1]} + 2V_{[7,3,0]} + V_{[7,4,1]} + V_{[7,5,0]} + 2V_{[8,2,0]} + V_{[8,3,1]} \\
&\quad + V_{[9,1,0]} + V_{[9,3,0]} + V_{[10,0,0]} + V_{[10,1,1]} \\
\Lambda^4 \mathfrak{S}_3 &\cong 3V_{[0,0,0]} + 4V_{[1,1,0]} + V_{[2,1,1]} + 6V_{[2,2,0]} + 3V_{[3,1,0]} + 4V_{[3,2,1]} + 7V_{[3,3,0]} + V_{[3,3,2]} \\
&\quad + 4V_{[4,0,0]} + 5V_{[4,1,1]} + 4V_{[4,2,0]} + 3V_{[4,2,2]} + 4V_{[4,3,1]} + V_{[4,4,0]} + 4V_{[4,4,0]} + 7V_{[5,1,0]} \\
&\quad + 5V_{[5,2,1]} + 2V_{[5,5,0]} + 3V_{[5,3,2]} + 2V_{[5,4,1]} + 2V_{[5,5,0]} + 2V_{[6,0,0]} + 3V_{[6,1,1]} + 5V_{[6,2,0]} \\
&\quad + V_{[6,2,2]} + 3V_{[6,3,1]} + V_{[6,3,3]} + V_{[6,4,0]} + V_{[6,4,2]} + V_{[6,6,0]} + V_{[7,1,0]} + 2V_{[7,2,1]} \\
&\quad + 3V_{[7,3,0]} + V_{[7,4,1]} + V_{[8,0,0]} + V_{[8,1,1]} + V_{[8,2,0]} + V_{[8,3,1]} + V_{[9,1,0]} \\
\end{align*}

**Remark 4.1** As we mentioned in [10] when $n = 2$, we know the dimensions of each irreducible components above by Weyl's dimension formula and we compare the both sides of the equation above in dimension, and we can check validity of our decomposition by comparing the dimensions.

### 4.2 The dimensions of relative cochain complexes

We abbreviate the space $C_{\text{GF}}^m(\mathfrak{ham}_w^0, \mathfrak{sp}(6, \mathbb{R}))_w$ by $C^m_w$. Applying Proposition 4.1, we see that

$$\dim C^m_w = \dim (\Lambda^2 \mathfrak{S}_3)^{\text{triv}} \overset{(5)}{=} 1$$
\[
\dim \mathcal{C}_1^2 = \dim (\Lambda^2 \mathcal{S}_4)^{\text{triv}} (6) = 0
\]
\[
\dim \mathcal{C}_2^3 = \dim ((\Lambda^2 \mathcal{S}_3) \otimes \mathcal{S}_4)^{\text{triv}} = \dim (\Lambda^2 \mathcal{S}_3)^{[4,0,0]} (5) = 1
\]
\[
\dim \mathcal{C}_3^4 = \dim (\Lambda^4 \mathcal{S}_3)^{\text{triv}} (10) = 3
\]

Thus, for weight = 2 and 4 we get the dimensions of each cochain complexes.

| dim | $\mathcal{C}_1^2$ | $\mathcal{C}_2^3$ | $\mathcal{C}_3^4$ | $\mathcal{C}_4^5$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| dim | 0               | 0               | 0               | 0               |

The Euler characteristic number of $H^*_{GF}(\mathfrak{ham}_6^{0}, Sp(6, \mathbb{R}))_2 = 1$ and the Euler characteristic number of $H^*_{GF}(\mathfrak{ham}_6^{0}, Sp(6, \mathbb{R}))_2 = 2$. Those results are the first part of Theorem 1.

When weight = 6, we again apply Proposition 4.1 to our decompositions, we see that

\[
\dim \mathcal{C}_6^5 = \dim (\Lambda^2 \mathcal{S}_3)^{\text{triv}} (7) = 1
\]
\[
\dim \mathcal{C}_6^5 = \dim ((\Lambda^2 \mathcal{S}_3) \otimes \mathcal{S}_6)^{\text{triv}} + \dim (\mathcal{S}_3 \otimes \mathcal{S}_4 \otimes \mathcal{S}_5)^{\text{triv}} + \dim (\Lambda^2 \mathcal{S}_4)^{\text{triv}}
\]
\[
= \dim ((\Lambda^2 \mathcal{S}_3)^{[6,0,0]} + \dim (\mathcal{S}_3 \otimes \mathcal{S}_4 \otimes \mathcal{S}_5)^{\text{triv}} + 0 = \dim (\mathcal{S}_3 \otimes \mathcal{S}_4 \otimes \mathcal{S}_5)^{\text{triv}}
\]
\[
\dim \mathcal{C}_6^5 = \dim ((\Lambda^2 \mathcal{S}_3) \otimes (\Lambda^2 \mathcal{S}_4))^{\text{triv}}
\]
\[
= \dim (\Lambda^2 \mathcal{S}_3)^{[5,0,0]} + \dim ((\Lambda^2 \mathcal{S}_3)^{[2,0,0]} + \dim (\Lambda^2 \mathcal{S}_3)^{[3,1,0]} + \dim (\Lambda^2 \mathcal{S}_3)^{[4,2,0]})
\]
\[
+ \dim (\Lambda^2 \mathcal{S}_3)^{[5,3,0]} + \dim (\Lambda^2 \mathcal{S}_3)^{[6,0,0]} + \dim (\Lambda^2 \mathcal{S}_3)^{[7,1,0]}
\]
\[
= 0
\]

When $n = 3$ we see the crystal base of $[3,0,0]$ consists of the next 56 tableaux.

| 1 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 | 1 | 3 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 3 | 3 | 3 | 3 | 1 | 1 | 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 3 | 3 |
| 3 | 3 | 3 | 3 | 3 | 1 | 1 | 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 3 | 3 |
| 2 | 3 | 2 | 3 | 2 | 1 | 3 | 2 | 2 | 3 | 3 | 2 | 3 | 3 | 2 | 1 | 2 | 2 |
| 3 | 2 | 2 | 3 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 3 | 1 | 1 | 3 | 1 | 2 | 3 | 1 | 3 | 3 | 1 | 3 | 3 | 1 | 3 | 3 | 1 |
| 3 | 2 | 1 | 2 | 2 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 3 | 1 | 1 | 3 | 1 |
| 3 | 2 | 1 | 2 | 2 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 3 | 1 | 1 | 3 | 1 |
By the Crystal Base theory, we get the irreducible decomposition of $\mathcal{S}_3 \otimes \mathcal{S}_4 \cong \mathcal{S}_4 \otimes \mathcal{S}_3$ by the action of each tableau to the Young diagram \[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
& & & & & & & & & & & & \\
\end{array}\]. The action of crystal basis of type \[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
& & & & & & & & & & & & \\
\end{array}\]. Avoiding those actions, we have

**Proposition 4.2**

$\mathcal{S}_3 \otimes \mathcal{S}_4 \cong \{4, 0, 0\} \cdot \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
1 & 1 & 1 \\
\end{array} + \{4, 0, 0\} \cdot \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
1 & 1 & 2 \\
\end{array} + \{4, 0, 0\} \cdot \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
1 & 2 & 2 \\
\end{array}$

$\cong V_{[7,0,0]} + V_{[6,1,0]} + V_{[5,2,0]} + V_{[4,3,0]} + V_{[5,0,0]} + V_{[4,1,0]} + V_{[3,2,0]} + V_{[3,0,0]} + V_{[2,1,0]} + V_{[1,0,0]}$

And so

$$\dim \mathcal{C}^3_6 = \dim (\mathcal{S}_3 \otimes \mathcal{S}_4 \otimes \mathcal{S}_5)^{triv} = \dim (\mathcal{S}_3 \otimes \mathcal{S}_4)^{[5,0,0]} = 1.$$ 

So far, concerning to $\dim \mathcal{C}^6_6$, we have got the dimensions for degree less than 6 as follows:

| $\dim$ | $\mathcal{C}^1_6$ | $\mathcal{C}^2_6$ | $\mathcal{C}^3_6$ | $\mathcal{C}^4_6$ | $\mathcal{C}^5_6$ | $\mathcal{C}^6_6$ |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0     | 0               | 1               | 1               | 0               | 4               |

By Proposition 4.1, $\mathcal{C}^6_6 = (\Lambda^6 \mathcal{S}_3)^{triv}$ and $\dim (\Lambda^6 \mathcal{S}_3) = 32468436$. The number of possibilities of the maximal weight vectors is 146. The subspace of $\Lambda^6 \mathcal{S}_3$ generated by $z_A_1 \wedge z_A_2 \wedge z_A_3 \wedge z_A_4 \wedge z_A_5 \wedge z_A_6$ with $A_1 < A_2 < A_3 < A_4 < A_5 < A_6$, where $A_j$ are 6-dim vector of non-negative integer with the sum of them is 3, and $\sum_{j=1}^{6} (A_j(1) - A_j(6)) = \sum_{j=1}^{6} (A_j(2) - A_j(5)) = \sum_{j=1}^{6} (A_j(3) - A_j(4)) = 0$ has the dimension 204894. By the condition that a general element is invariant under $U$, we have 1553660 linear equations with 204894 unknown variables. Solve this huge linear equations with help of symbol calculus, we see the dimension of the null space (the kernel space) is 7. Thus, we have the table of dimensions of each cochain complexes with weight 6 as

| $\dim$ | $\mathcal{C}^1_6$ | $\mathcal{C}^2_6$ | $\mathcal{C}^3_6$ | $\mathcal{C}^4_6$ | $\mathcal{C}^5_6$ | $\mathcal{C}^6_6$ |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0     | 0               | 1               | 1               | 0               | 4               | 7               |

and the Euler characteristic number of $H^*_G(\mathfrak{g}_6^0, Sp(6, \mathbb{R}))_6 = 3$. Those are the second part of Theorem 1. In fact, we have the complete irreducible decomposition of $\Lambda^6 \mathcal{S}_3$ as next.
Proposition 4.3

\[ \Lambda^2 \mathfrak{g}_3 \cong V_{[12,2,0]} + V_{[12,0,0]} + V_{[11,4,1]} + V_{[11,3,2]} + 3V_{[11,3,0]} + 2V_{[11,2,1]} + V_{[11,1,0]} + V_{[10,6,0]} + 2V_{[10,5,1]} + V_{[10,4,4]} + 3V_{[10,4,2]} + 4V_{[10,4,0]} + 2V_{[10,3,3]} + 7V_{[10,3,1]} + 4V_{[10,2,2]} + 7V_{[10,2,0]} + 5V_{[10,1,1]} + 3V_{[10,0,0]} + 4V_{[9,6,3]} + 3V_{[9,6,1]} + 4V_{[9,5,2]} + 8V_{[9,5,0]} + 5V_{[9,4,3]} + 14V_{[9,4,1]} + 14V_{[9,3,2]} + 13V_{[9,3,0]} + 18V_{[9,2,1]} + 14V_{[9,1,0]} + V_{[8,8,0]} + 3V_{[8,7,1]} + V_{[8,6,4]} + 6V_{[8,6,2]} + 4V_{[8,6,0]} + 6V_{[8,5,3]} + 16V_{[8,5,1]} + 3V_{[8,4,4]} + 22V_{[8,4,2]} + 24V_{[8,4,0]} + 10V_{[8,3,3]} + 39V_{[8,3,1]} + 20V_{[8,2,2]} + 30V_{[8,2,0]} + 20V_{[8,1,1]} + 9V_{[8,0,0]} + V_{[7,7,4]} + V_{[7,7,2]} + 5V_{[7,7,0]} + 5V_{[7,6,3]} + 12V_{[7,6,1]} + 3V_{[7,5,4]} + 24V_{[7,5,2]} + 15V_{[7,5,0]} + 19V_{[7,4,3]} + 49V_{[7,4,1]} + 41V_{[7,3,2]} + 55V_{[7,3,0]} + 57V_{[7,2,1]} + 29V_{[7,1,0]} + 3V_{[6,6,4]} + 7V_{[6,6,2]} + 15V_{[6,6,0]} + V_{[6,5,5]} + 14V_{[6,5,3]} + 33V_{[6,5,1]} + 7V_{[6,4,4]} + 53V_{[6,4,2]} + 41V_{[6,4,0]} + 26V_{[6,3,3]} + 82V_{[6,3,1]} + 41V_{[6,2,2]} + 68V_{[6,2,0]} + 41V_{[6,1,1]} + 15V_{[6,0,0]} + 5V_{[5,5,4]} + 16V_{[5,5,2]} + 28V_{[5,5,0]} + 24V_{[5,4,3]} + 62V_{[5,4,1]} + 70V_{[5,3,2]} + 56V_{[5,3,0]} + 84V_{[5,2,1]} + 57V_{[5,1,0]} + 7V_{[4,4,4]} + 25V_{[4,4,2]} + 40V_{[4,4,0]} + 22V_{[4,3,3]} + 73V_{[4,3,1]} + 48V_{[4,2,2]} + 54V_{[4,2,0]} + 46V_{[4,1,1]} + 25V_{[4,0,0]} + 21V_{[3,3,2]} + 46V_{[3,3,0]} + 51V_{[3,2,1]} + 27V_{[3,1,0]} + 6V_{[2,2,2]} + 32V_{[2,2,0]} + 15V_{[2,1,1]} + 3V_{[2,0,0]} + 14V_{[1,1,0]} + 7V_{[0,0,0]}

5 Betti numbers

The definition of Betti number is \( b^\bullet_w := \dim H^\bullet_w(\mathfrak{h}, \mathfrak{g}, Sp(6, \mathbb{R})) \) and so

\[
\begin{align*}
    b^\bullet_w &= \dim (\ker(d_j : \mathfrak{e}_w^j \to \mathfrak{e}_w^{j+1}))) - \dim (d_{j-1}(\mathfrak{e}_w^{j-1})) \\
    &= \dim (\mathfrak{e}_w^j) - \dim (d_j(\mathfrak{e}_w^j)) - \dim (d_{j-1}(\mathfrak{e}_w^{j-1})) \\
    &= \dim (\mathfrak{e}_w^j) - \text{rank}(d_j) - \text{rank}(d_{j-1})
\end{align*}
\]

From the definition of the coboundary operator, the operator has skew-derivation of degree 1 and for each 1-cochain \( \sigma, d(\sigma)(f, g) := -\langle \sigma, \{f, g\} \rangle \) where \( \{f, g\} \) is the Poisson bracket of the standard symplectic structure of \( \mathbb{R}^6 \).

When the weight is 2, we see \( b^1_2 = \dim \mathfrak{e}_2^j \) directly.

In general, to investigate the Betti numbers, we shall know the rank of the consecutive coboundary operators. For that purpose, we have to prepare concrete bases of the cochain complexes.

As commented briefly just before Proposition 4.3, each cochain is generated by \( z_A \) where \( A \) is 6-dimensional multi-index with non-negative integers. We may regard \( z_A \) as the dual of \( \frac{x_1^{a_1} x_2^{a_2} \cdots x_6^{a_6}}{a_1! a_2! \cdots a_6!} \)
where \( A = (a_1, a_2, a_3, a_4, a_5, a_6) \) with \( a_j \in \mathbb{N}^+ \) (\( j = 1 \ldots 6 \)). \( z_A \in \mathfrak{S}_h \) is equivalent to \( \sum_{j=1}^{6} a_j = h \).

In this paper, we use the abbreviation \( Z_{a_1a_2a_3}^{a_4a_5a_6} \) instead of \( z[a_1, a_2, a_3, a_4, a_5, a_6] \).

When the weight is 4, as a basis of \( \mathfrak{C}_4^2 \) we have

\[
- Z_{101}^{100} \wedge Z_{001}^{200} \wedge Z_{000}^{202} - \frac{1}{2} Z_{011}^{001} \wedge Z_{102}^{000} \wedge Z_{100}^{100} + \frac{1}{2} Z_{001}^{200} \wedge Z_{003}^{000} \wedge Z_{010}^{300} + \frac{1}{2} Z_{011}^{001} \wedge Z_{003}^{000} \wedge Z_{010}^{300} + \frac{1}{2} Z_{011}^{110} \wedge Z_{010}^{200} \wedge Z_{000}^{202} + \frac{1}{2} Z_{011}^{110} \wedge Z_{003}^{000} \wedge Z_{010}^{300}
\]

\[+ (1228 \text{ terms}) \]

\[
- \frac{1}{2} Z_{100}^{200} \wedge Z_{101}^{100} \wedge Z_{010}^{201} + \frac{1}{2} Z_{200}^{000} \wedge Z_{001}^{100} \wedge Z_{010}^{201} - Z_{102}^{000} \wedge Z_{001}^{110} \wedge Z_{101}^{110} - \frac{1}{2} Z_{102}^{000} \wedge Z_{001}^{110} \wedge Z_{100}^{100} - \frac{1}{2} Z_{102}^{000} \wedge Z_{011}^{110} \wedge Z_{100}^{100}
\]

\[\text{(You may see the full form of the above vector at } \text{http://www.math.akita-u.ac.jp/~mikami/GKF_R6/c3w4d6.pdf}, \text{ where we denote } Z_{101}^{100} \wedge Z_{102}^{100} \wedge Z_{000}^{202} \text{ by } w(100101,000102,202000) \text{ for example, and the } d\text{-image is the next form, which is not 0.} \]

\[
- 6 Z_{120}^{000} \wedge Z_{001}^{200} \wedge Z_{010}^{110} - Z_{110}^{100} \wedge Z_{000}^{120} \wedge Z_{000}^{120} + 8 Z_{111}^{000} \wedge Z_{100}^{100} \wedge Z_{011}^{100} \wedge Z_{000}^{120} + 8 Z_{111}^{000} \wedge Z_{120}^{100} \wedge Z_{011}^{100} \wedge Z_{000}^{120} + 7 Z_{111}^{000} \wedge Z_{120}^{100} \wedge Z_{020}^{010} \wedge Z_{000}^{120} + (2645 \text{ terms})
\]

\[\text{(cf. } \text{http://www.math.akita-u.ac.jp/~mikami/GKF_R6/d-image-c3w4d6.pdf (21 pages))}. \text{ Thus, } \text{rank}(d_3) = 1 \text{ and this implies } b_1^3 = 0 \text{ and } b_2^3 = 2 \text{ and the others are 0. This completes a proof to the first half of Theorem 2.} \]

When the weight is 6, as a basis of \( \mathfrak{C}_6^2 \) we have

\[
- \frac{1}{2} Z_{010}^{211} \wedge Z_{010}^{211} - \frac{1}{2} Z_{010}^{211} \wedge Z_{012}^{100} + \frac{1}{4} Z_{200}^{120} \wedge Z_{200}^{120} + \frac{1}{2} Z_{110}^{110} \wedge Z_{021}^{001} + \frac{1}{2} Z_{120}^{120} \wedge Z_{001}^{110} + \frac{1}{2} Z_{120}^{120} \wedge Z_{010}^{201} + \frac{1}{4} Z_{120}^{120} \wedge Z_{001}^{110} + Z_{120}^{120} \wedge Z_{010}^{201}
\]

\[+ (111 \text{ terms}) \]

\[
- \frac{1}{2} Z_{130}^{130} \wedge Z_{031}^{001} - \frac{1}{6} Z_{130}^{130} \wedge Z_{031}^{001} - \frac{1}{6} Z_{130}^{130} \wedge Z_{031}^{001} + \frac{1}{4} Z_{122}^{122} \wedge Z_{031}^{001} + \frac{1}{4} Z_{122}^{122} \wedge Z_{031}^{001} - \frac{1}{2} Z_{110}^{110} \wedge Z_{021}^{211} - \frac{1}{2} Z_{121}^{121} \wedge Z_{010}^{201}
\]

\[\text{(cf. } \text{http://www.math.akita-u.ac.jp/~mikami/GKF_R6/c3w6d6.pdf}) \text{ and the } d\text{-image is the next form, which is not 0.} \]

\[
- \frac{1}{4} Z_{030}^{200} \wedge Z_{030}^{200} \wedge Z_{012}^{010} + \frac{1}{2} Z_{030}^{200} \wedge Z_{012}^{010} + \frac{1}{12} Z_{030}^{200} \wedge Z_{030}^{200} \wedge Z_{010}^{201} + \frac{1}{12} Z_{030}^{200} \wedge Z_{030}^{200} \wedge Z_{010}^{201} - \frac{1}{12} Z_{030}^{200} \wedge Z_{030}^{200} \wedge Z_{010}^{201}
\]

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The number of summands of $r_1$ is 150340. Here we may write $\#r_1 = 150340$. Then we have $\#r_2 = 21612, \#r_3 = 153466, \#r_4 = 148660, \#r_5 = 155512, \#r_6 = 3276$, and $\#r_7 = 148600$.

You will understand it is hard to show them in this paper. You will see them at http://www.math.akita-u.ac.jp/~mikami/GKF6/c6w6-jd6.pdf (j = 1, ..., 7). The first vector spends 1567 pages to show, the second vector 226 pages, the third vector 1599 pages, the fourth vector 1549 pages, the fifth vector 1620 pages, the sixth vector 35 pages, and the last seventh vector spends 1548 pages.

If we try to find a concrete basis of $C_6 = (\Lambda^4 \mathfrak{S}_3) \otimes \mathfrak{S}_4$, we can use our strategy of splitting $\Lambda^r \mathfrak{S}_q$ and we have got them, say $r_1, r_2, r_3, r_4, r_5, r_6, r_7$.

**Proposition 5.1**

The next are small part of them. In order to know the rank of $d : C_5^j \rightarrow C_6^0$, we have to fix some bases of both spaces, whose dimensions are 4 and 7 respectively. Concerning to finding a basis of $C_6 = (\Lambda^6 \mathfrak{S}_3)^{triv}$, we can use our strategy of splitting $\Lambda^r \mathfrak{S}_q$ and we have got them, say $r_1, r_2, r_3, r_4, r_5, r_6, r_7$.

You will understand it is hard to show them in this paper. You will see them at http://www.math.akita-u.ac.jp/~mikami/GKF6/c6w6-jd6.pdf (j = 1, ..., 7). The first vector spends 1567 pages to show, the second vector 226 pages, the third vector 1599 pages, the fourth vector 1549 pages, the fifth vector 1620 pages, the sixth vector 35 pages, and the last seventh vector spends 1548 pages.

If we try to find a concrete basis of $C_6 = (\Lambda^4 \mathfrak{S}_3) \otimes \mathfrak{S}_4$ directly, then from $\dim ((\Lambda^4 \mathfrak{S}_3) \otimes \mathfrak{S}_4) = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \cdot \dim \mathfrak{S}_4$, we have to handle 46278540 variables. But, as we discussed in the section of Crystal base, trivial representation spaces live in $(\Lambda^4 \mathfrak{S}_3)^{triv} \otimes \mathfrak{S}_4$. Then we only deal with $126^2 = 15876$ variables ($\dim \mathfrak{S}_4 = 126$). Proposition 4.1 (10) tells us that the multiplicity is 4. Thus, we first fix the 4 linearly independent maximal vectors, say $w_1, w_2, w_3, w_4$ of weight $[4, 0, 0]$ of $\Lambda^4 \mathfrak{S}_3$. Let us say $W_j$ be the irreducible subspace of the maximal vector $w_j$ ($j = 1, 2, 3, 4$). Using the Fact (2), We get a basis of the space $W_j$ of 126-dimensional. On each the tensor product space $W_j \otimes \mathfrak{S}_4$, after a long calculation, we get the maximal vector of weight $[0, 0, 0]$. Those are member of a concrete basis of $C_6 = ((\Lambda^4 \mathfrak{S}_3) \wedge \mathfrak{S}_4)^{triv}$. The next are small part of them.

**Proposition 5.2**

\[
\begin{align*}
&+ \frac{1}{12} Z_{002}^{10} \wedge Z_{002}^{01} \wedge Z_{120}^{00} \wedge Z_{120}^{01} \wedge Z_{120}^{02} + \frac{1}{12} Z_{002}^{10} \wedge Z_{002}^{01} \wedge Z_{120}^{00} \wedge Z_{120}^{01} \wedge Z_{120}^{02} + \frac{1}{4} Z_{002}^{10} \wedge Z_{002}^{01} \wedge Z_{120}^{00} \wedge Z_{120}^{01} \wedge Z_{120}^{02} \\
&+ (6510 \text{ terms}) \\
&- \frac{1}{4} Z_{020}^{110} \wedge Z_{110}^{010} \wedge Z_{002}^{010} + \frac{1}{4} Z_{020}^{110} \wedge Z_{110}^{010} \wedge Z_{002}^{010} + \frac{1}{2} Z_{020}^{110} \wedge Z_{110}^{010} \wedge Z_{002}^{010} \\
&+ \frac{1}{4} Z_{020}^{110} \wedge Z_{110}^{010} \wedge Z_{002}^{010} + \frac{1}{4} Z_{020}^{110} \wedge Z_{110}^{010} \wedge Z_{002}^{010} + \frac{1}{4} Z_{020}^{110} \wedge Z_{110}^{010} \wedge Z_{002}^{010}.
\end{align*}
\]
Theorem 5.1  Let $w_j$ ($j = 1, \ldots, 4$) be the basis described in Proposition 5.2 and $r_k$ ($k = 1, \ldots, 7$) be the basis in Proposition 5.1. Then the coboundary operator $d$ has the representation below:

$$
[d(w_1), d(w_2), d(w_3), d(w_4)] = [r_1, r_2, r_3, r_4, r_5, r_6, r_7]
$$

And so we have $\text{rank}(d) = 2.$

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