Wave-like Solutions for Bianchi type-I cosmologies in 5D

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Abstract

We derive exact solutions to the vacuum Einstein field equations in 5D, under the assumption that (i) the line element in 5D possesses self-similar symmetry, in the classical understanding of Sedov, Taub and Zeldovich, and that (ii) the metric tensor is diagonal and independent of the coordinates for ordinary 3D space. These assumptions lead to three different types of self-similarity in 5D: homothetic, conformal and “wave-like”. In this work we present the most general wave-like solutions to the 5D field equations. Using the standard technique based on Campbell’s theorem, they generate a large number of anisotropic cosmological models of Bianchi type-I, which can be applied to our universe after the big-bang, when anisotropies could have played an important role. We present a complete review of all possible cases of self-similar anisotropic cosmologies in 5D. Our analysis extends a number of previous studies on wave-like solutions in 5D with spatial spherical symmetry.

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1 Introduction

Nowadays, there are a number of theories suggesting that the universe may have more than four dimensions. Extra dimensions arise naturally in supergravity (11D) and superstring theories (10D), which seek the unification of gravity with the interactions of particle physics, and are expected to become important at very high energies, e.g., near the horizon of black holes \[1\] and during the evolution of the early universe \[2\].

Therefore, it is worthwhile to explore cosmological models in presence of extra dimensions. In this regard, a powerful theoretical tool is provided by Campbell’s theorem \[3\], \[4\], which serves as a ladder to go between manifolds whose dimensionality differs by one. This theorem, which is valid in any number of dimensions, implies that every solution of the 4D Einstein equations with arbitrary energy-momentum tensor can be embedded, at least locally, in a solution of the five-dimensional vacuum Einstein field equations. In this work we will derive exact solutions to the 5D field equations which embed a large family of anisotropic cosmological models of Bianchi type-I that may be applicable to the early universe.

In conventional 4D general relativity, in order to solve the field equations one usually assumes a form for the matter content, i.e., the energy-momentum tensor, and imposes certain symmetries on the spacetime. For example, if we assume empty space and spatial spherical symmetry we obtain the Schwarzschild solution; if we assume that the matter satisfies a barotropic equation of state, and that the spacetime is homogeneous and isotropic, then we obtain the standard Friedmann-Robertson-Walker (FLW) cosmological models.

An important consequence of Campbell’s theorem is, in particular, that for the study of cosmological models embedded in 5D we do not need a five-dimensional energy-momentum tensor. Thus, instead of seeking an embedding of a 4D spacetime with a specified physical energy-momentum tensor, the procedure in 5D is as follows: first one has to find a solution to the fifteen Einstein field equations in vacuum, then the properties of the 4D effective matter source for the 5D solutions are deduced after choosing an embedding. The standard technique consists in isolating the 4D part of the relevant 5D quantities, using them to construct the 4D Einstein solution \(G_{\alpha\beta}\) \((\alpha, \beta = 0, 1, 2, 3)\) and utilizing the field equations of general relativity \(G_{\alpha\beta} = 8\pi T_{\alpha\beta}\) (we use \(c = G = 1\)) to identify the effective energy-momentum tensor \(T_{\alpha\beta}\).

At this point, the natural question is how can we construct solutions to the 5D vacuum Einstein equations that lead to models with ‘reasonable’ physical properties in 4D. Clearly, the best approach to accomplish this is to impose spacetime symmetries, on the 5D metric, that are characteristic of the 4D source that we want to embed in 5D. This is illustrated by a number of 5D solutions, e.g., the Kramer-Gross-Perry-Davidson-Owen solutions \[1\], \[8\] which embed the Schwarzschild solution of general relativity; the ‘standard’ 5D cosmologies \[9\] that reduce to the usual FRW cosmologies with flat space sections, on every hypersurface defined by fixing the fifth coordinate (which we denote as \(\psi\)). This approach allows not only to recover known solutions of 4D general relativity, but also generates new ones that may shed some light on the effects of a putative extra dimension on the physics in 4D.

Following this modus operandi, it is important to investigate five-dimensional cosmological models whose metric tensor is diagonal and independent of the coordinates for ordinary 3D space. This is because they reduce to homogeneous cosmological models with flat spatial sections on every hypersurface defined by fixing the fifth coordinate (which we denote as \(\psi\)). This approach allows not only to recover known solutions of 4D general relativity, but also generates new ones that may shed some light on the effects of a putative extra dimension on the physics in 4D.

In a recent work \[10\] we started a systematic investigation of such models under the assumption that the line element in 5D

\[
dS^2 = e^{\nu(t, \psi)} dt^2 - e^{\lambda(t, \psi)} dx^2 - e^{\mu(t, \psi)} dy^2 - e^{\sigma(t, \psi)} dz^2 + e^{\omega(t, \psi)} d\psi^2,
\]

admits self-similar symmetry, in the sense that all the dimensionless quantities are assumed to be functions of a single variable \(\xi \), which is some combination of the coordinates \(x^0 = t\) and \(x^4 = \psi\). In this way the field equations become a system of ordinary, instead of partial, differential equations. From a physical point of view, the assumption of self-similarity is motivated by a number of studies suggesting that many homogeneous and inhomogeneous cosmological

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¹This standard technique is used in braneworld \[5\] as well as in space-time-matter (or induced matter) theory \[6\]. Both theories employ a 5D Kaluza-Klein type of metric, \(dS^2 = \gamma_{AB} dx^A dx^B + g_{\mu
u}(x^\sigma, \psi) dx^\mu dx^\nu + eF(x^\sigma, \psi) d\psi^2\), where the extra dimension \(\psi\) is not assumed to be compactified as in the original account. Consequently, both theories are mathematically equivalent in the sense that they lead to the same effective energy-momentum tensor in 4D, although they have different physical interpretation \[7\].

²Notation: here the coordinates are assigned as usual, \(x^0 = t\) for time; \(x^{1,2,3} = x, y, z\) for space; \(x^4 = \psi\) for the extra coordinate, and \(\epsilon = \pm 1\) depending on whether the extra dimension is spacelike or timelike.

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models can be approximated by self-similar homothetic models in the asymptotic regimes \[12\], i.e., near the initial cosmological singularity and at late times. In \[10\] we showed that there are three possible choices for the similarity variable which lead to solutions with different physical and mathematical properties. These are (i) \(\xi = t/\psi\); (ii) \(\xi = e^{q t}/e^{q \psi}\), and (iii) \(\xi = \mathcal{E} t + k \psi\), where \(q, \mathcal{E}\) and \(k\) are some constants with the appropriate units. We found the most general solutions to the 5D vacuum field equations \(R_{AB} = 0\) \((A, B = 0, 1, 2, 3, 4)\) corresponding to the first two choices of \(\xi\) and showed that they admit homothetic and conformal symmetry in 5D, respectively.

However, in \[10\] we did not discuss the third case where the metric functions have a dependence of time and the extra coordinate like in traveling waves or pulses propagating along the fifth dimension. In this work we conclude our analysis of self-similar anisotropic cosmologies in 5D by giving a detailed discussion of the field equations and their solutions for the case where the similarity variable is \(\xi = \mathcal{E} t + k \psi\). In short, we will refer to them as “wave-like” solutions. This work extends a number of previous studies of wave-like solutions in 5D with spatial spherical symmetry \[13\].

The paper is organized as follows. In section 2 we present the general integration of the field equations. We will see that the solutions are expressed in terms of one arbitrary function and three dimensionless parameters. In sections 3 we study some particular solutions which are generated by geometrical considerations. In section 4 we will see that the solutions are expressed in terms of one arbitrary function and three dimensionless parameters. In section 5 we present a summary and a complete analysis of all possible cases of self-similar anisotropic cosmologies in 5D.

2 Integrating the field equations

Let us consider the five-dimensional, self-similar, line element

\[
dS^2 = e^\nu(\xi) dt^2 - e^\lambda(\xi) dx^2 - e^{\mu(\xi)} dy^2 - e^{\sigma(\xi)} dz^2 + e^{\omega(\xi)} d\psi^2, \tag{2}
\]

with

\[
\xi = \mathcal{E} t + k \psi, \tag{3}
\]

where \(\mathcal{E}\) and \(k\) are some constants. The five metric functions \(\nu(\xi), \lambda(\xi), \mu(\xi), \sigma(\xi), \omega(\xi)\), as well as the signature of the extra coordinate, have to be determined from the field equations \(R_{AB} = 0\). The 5D Ricci tensor has six nonvanishing components, viz., \(R_{00}, R_{11}, R_{22}, R_{33}, R_{44}, R_{04}\). However, not all of them are independent. We will see bellow that they reduce to two independent equations for three unknown functions.

Setting \(R_{11}, R_{22}, R_{33}\) to zero we obtain

\[
\lambda \xi \left[ (e^2 e^\omega + k^2 e^\nu) \left( \lambda + \frac{2\lambda \xi}{\lambda} + \mu + \sigma \right) + (e^2 e^\omega - k^2 e^\nu) (\omega - \nu) \right] = 0. \tag{4}
\]

\[
\mu \xi \left[ (e^2 e^\omega + k^2 e^\nu) \left( \mu + \frac{2\mu \xi}{\mu} + \lambda + \sigma \right) + (e^2 e^\omega - k^2 e^\nu) (\omega - \nu) \right] = 0. \tag{5}
\]

\[
\sigma \xi \left[ (e^2 e^\omega + k^2 e^\nu) \left( \sigma + \frac{2\sigma \xi}{\sigma} + \mu + \lambda \right) + (e^2 e^\omega - k^2 e^\nu) (\omega - \nu) \right] = 0. \tag{6}
\]

These equations require

\[
\frac{\lambda \xi}{\lambda} = \frac{\mu \xi}{\mu} = \frac{\sigma \xi}{\sigma}. \tag{7}
\]

Therefore, without loss of generality one can set

\[
e^\lambda = A f^{2\alpha}(\xi), \quad e^\mu = B f^{2\beta}(\xi), \quad e^\sigma = C f^{2\gamma}(\xi), \tag{8}
\]
where $A, B, C$ are constants; $f$ is some function of the variable $\xi = (Et+k\psi)$; and $\alpha, \beta$ and $\gamma$ are arbitrary parameters. As a consequence, $R_{11}=0$, $R_{22}=0$ and $R_{33}=0$ reduce to

$$
\left(\epsilon E^2 e^\nu + k^2 e^{\nu'}\right) \left[ \frac{f_{\xi}}{f_\xi} + \frac{f_\xi}{f} (\alpha + \beta + \gamma - 1) \right] + \frac{1}{2} \left(\epsilon E^2 e^\nu - k^2 e^{\nu'}\right) (\omega_\xi - \nu_\xi) = 0. \tag{9}
$$

On the other hand $R_{04}=0$ yields

$$
(\alpha + \beta + \gamma) \left( \frac{2f_{\xi}}{f} - \nu_\xi - \omega_\xi \right) + 2(\alpha^2 + \beta^2 + \gamma^2 - \alpha - \beta - \gamma) \left( \frac{f_\xi}{f} \right) = 0, \tag{10}
$$

from which we get

$$
e^{(\nu+\omega)/2} = E f^{(b/a)} f_\xi, \tag{11}
$$

where $E$ is a constant of integration, and

$$
a \equiv (\alpha + \beta + \gamma), \quad b \equiv (\alpha^2 + \beta^2 + \gamma^2 - \alpha - \beta - \gamma). \tag{12}
$$

We note that $a=0$ implies that $(\alpha^2 + \beta^2 + \gamma^2) f_\xi = 0$, which means that either $\alpha = \beta = \gamma = 0$ or $f = \text{constant}$. In both cases, the spatial sections, $t = \text{constant}$, $\psi = \text{constant}$, are static. Since we are looking for cosmological solutions, in what follows we will assume $a \neq 0$.

From (11) we obtain

$$
e^{\nu/2} = E f^{b/a} f_\xi e^{-\omega/2}. \tag{13}
$$

Feeding this expression back into (9) we get

$$
\left[ (b+c) \frac{f_\xi}{f} + a \left( \frac{f_{\xi\xi}}{f_\xi} - \frac{\omega_\xi}{2} \right) \right] f^{2b/a} f_\xi^2 + \epsilon \left[ c \left( \frac{f_\xi}{f} \right) + \frac{\alpha \omega_\xi}{2} \right] \left( \frac{E}{kE} \right)^2 e^{2\omega} = 0, \tag{14}
$$

where

$$
c \equiv \alpha \beta + \alpha \gamma + \beta \gamma. \tag{15}
$$

Now $R_{00}$ and $R_{44}$, depend on the second derivative of $\nu$. Therefore, after we substitute (13) into them, they become functions of $f_{\xi\xi\xi}$ and $f_{\xi\xi}$; the third and second derivative of $f$. If we isolate $f_{\xi\xi}$ from (14); calculate the third derivative and substitute into $R_{00}$ and $R_{44}$ we find that they vanish identically.

Consequently, the field equations $R_{AB}=0$ reduce to two independent equations, namely (13) and (14) for three unknown metric functions: $\nu(\xi)$, $\omega(\xi)$ and $f(\xi)$. Thus, fixing one of them we obtain the other two. The interesting point here is that the spacetime metric is completely determined by the dynamics of the extra dimension. The opposite is also true, namely, that knowing the metric in 4D we can reconstruct the geometry in 5D.

Thus, in order to obtain specific wave-like solutions one has to complement these equations with some additional information. The question is how to do that in a way that is physically “justifiable”. The next two sections are devoted to the discussion of this question.

### 3 Solutions generated from geometrical considerations in 5D

In this section we present a number of solutions to the above equations that follow from the choice of the coordinate/reference system, and the metric, that are frequently encountered in the literature.

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3Our equations (9)–(11) are the counterparts of equations (22)–(24) in [10]. Although they look alike, these are distinct differential equations.
3.1 Gaussian normal coordinate system

A popular choice in the literature is to use the five degrees of coordinate freedom to set \( g_{4\mu} = 0 \) and \( g_{44} = \epsilon \). This is the so-called “Gaussian normal coordinate system” based on \( \Sigma \psi \). In these coordinates we should set \( \omega = 0 \) in (14), after which we obtain a second order differential equation for \( f \),

\[
a f f_{\xi \xi} + (b + c) f_{\xi}^2 + \epsilon c \left( \frac{E}{kE} \right)^2 f^{-2b/a} = 0,
\]

whose first integral is given by

\[
f_{\xi}^2 = C_1 f^{-2(b+c)/a} - \epsilon \left( \frac{E}{kE} \right)^2 f^{-2b/a},
\]

where \( C_1 \) is a constant of integration. Consequently, the metric

\[
dS^2 = E^2 \left[ C_1 f^{-2c/a} - \epsilon \left( \frac{E}{kE} \right)^2 f^{-2b/a} \right] dt^2 - A f^{2a} dx^2 - B f^{2b} dy^2 - C f^{2\gamma} dz^2 + \epsilon d\psi^2
\]

is a solution of the field equations provided the function \( f(\xi) \) satisfies (17).

3.2 Synchronous reference system

The choice \( g_{00} = 1 \) is usual in cosmology; it corresponds to the so-called synchronous reference system where the time coordinate \( t \) is the proper time at each point. In order to generate the appropriate solution, let us note that (11) is invariant under the change \( \nu \leftrightarrow \omega \). Also, the similarity variable \( \xi = E t + k \psi \) is invariant under the simultaneous change \( t \leftrightarrow \psi \) and \( E \leftrightarrow k \). Therefore, the line element

\[
dS^2 = dt^2 - A f^{2a} dx^2 - B f^{2b} dy^2 - C f^{2\gamma} dz^2 + \epsilon d\psi^2
\]

is also a solution of the field equations \( R_{AB} = 0 \), provided \( f(\xi) \) satisfies the equation (17) with \( E \leftrightarrow k \). Namely,

\[
f_{\xi}^2 = C_1 f^{-2(b+c)/a} - \epsilon \left( \frac{k}{E} \right)^2 f^{-2b/a}.
\]

3.3 Power-law type solutions

A simple inspection of the field equations reveals that there are two simple power-law type solutions. They correspond to the cases where \( \nu = \pm \omega \).

3.3.1 Solutions with \( \nu = \omega \)

For \( \nu = \omega \) there two different solutions depending on whether \( (\epsilon E^2 + k^2) \neq 0 \) or \( (\epsilon E^2 + k^2) = 0 \).

1. For \( (\epsilon E^2 + k^2) \neq 0 \), we substitute \( \nu = \omega \) into (19) and obtain a simple equation for \( f \) whose solution is

\[
f(\xi) = (c_1 \xi + c_2)^{1/a},
\]

where \( c_1 \) and \( c_2 \) are constants of integration. Feeding back into (11) we find that \( e^{\nu} \sim e^{\omega} \sim f^{-2c/a} \). In summary, the line element

\[
dS^2 = M f^{-2c/a} dt^2 - A f^{2a} dx^2 - B f^{2b} dy^2 - C f^{2\gamma} dz^2 + \epsilon N f^{-2c/a} d\psi^2,
\]

where \( M \) and \( N \) are constants, and \( f \) satisfies (21), is a solution of the field equations \( R_{AB} = 0 \).
2. For \((\epsilon^2 + k^2) = 0\), which might happen only if the extra dimension is spacelike \((\epsilon = -1)\) and \(E = \pm k\), equation (9) is identically satisfied. Consequently, the line element

\[
dS^2 = E f^b/\alpha f_c dt^2 - A f^{2\alpha} dx^2 - B f^{2\beta} dy^2 - C f^{2\gamma} dz^2 - E f^b/\alpha f_c d\psi^2,
\]

(23)
is an exact solution of the field equations \(R_{AB} = 0\) for an arbitrary function \(f = f(\xi)\), where \(\xi = E(t \pm \psi)\), and a general choice of the parameters \(\alpha, \beta, \gamma\). In order to avoid misunderstanding, let us emphasize that here \(f\) is arbitrary, i.e., (23) is not necessarily a power-law solution (but we include it here because it belongs to the class of solutions with \(\nu = \omega\)).

### 3.3.2 Solution with \(\nu = -\omega\)

If \(\omega = -\nu\), then from (11) we obtain a first order differential equation for \(f\) whose solution is

\[
f(\xi) = (C_2 \xi + C_3)^{\alpha/(\alpha + b)}, \tag{24}
\]

where \(C_2 \equiv [(a + b)/a E] \) and \(C_3\) is a new constant of integration. For this expression, the field equation (9) yields

\[
(k^2 e^\nu - \epsilon E^2 e^{-\nu}) (a + b) (C_2 \xi + C_3) \nu_1 + 2 \epsilon C_2 (k^2 e^\nu + \epsilon E^2 e^{-\nu}) = 0,
\]

(25)

from which we get

\[
k^2 e^\nu + \epsilon E^2 e^{-\nu} = \text{constant} \times (C_2 \xi + C_3)^{-2c/(\alpha + b)}. \tag{26}
\]

Now, it is not difficult to verify that the line element

\[
dS^2 = e^{\nu(\xi)} dt^2 - A f^{2\alpha} dx^2 - B f^{2\beta} dy^2 - C f^{2\gamma} dz^2 + e^{\nu(\xi)} d\psi^2,
\]

(27)

with the metric functions \(f(\xi)\) and \(e^{\nu(\xi)}\), given by (24) and (26), is an exact solution of the 5D field equations \(R_{AB} = 0\). We note that, from (25) it follows that the case where \(\nu = -\omega = \text{constant}\) requires \(C_2 = 0\), i.e., \(f = \text{constant}\), in which case the 5D manifold is a Minkowski space.

### 4 Solutions generated from physical considerations in 4D

In this section we derive a number of solutions to the 5D field equations that follow from the properties of matter in 4D. In order to make the paper self-consistent, let us restate some concepts that are essential in our discussion. Following the discussion in [10], there are different ways of producing, or embedding, a 4D spacetime in a given five-dimensional manifold (see e.g., [14]). However, the most popular approach is based on three different assumptions. First, that we can use the coordinate frame [15]. Second, that our 4D spacetime can be recovered by going onto a hypersurface \(\Sigma_\psi: \psi = \psi_0 = \text{constant}\), which is orthogonal to the 5D unit vector

\[
n_A = \frac{\delta^A_4}{\sqrt{\epsilon g^{44}}}, \quad n_A n^A = \epsilon, \tag{28}
\]

along the extra dimension. Third, that the physical metric of the spacetime can be identified with the one induced on \(\Sigma_\psi\).

For a line element of the form

\[
dS^2 = g_{\mu \nu}(x^\rho, \psi) dx^\mu dx^\nu + \epsilon \Phi^2(x^\rho, \psi) d\psi^2,
\]

(29)

the induced metric on hypersurfaces \(\Sigma_\psi\) is just \(g_{\mu \nu}\), i.e., the 4D part of the metric in 5D. The crucial moment is that, although the energy-momentum tensor (EMT) in 5D is zero, to an observer confined to making physical measurements in our ordinary spacetime, and not aware of the extra dimension, the spacetime is not empty but contains (effective) matter whose EMT, \((^{(4)}T_{\alpha \beta}\)) is determined by the Einstein equations in 4D, namely

\[
(^{(4)}G_{\alpha \beta} = 8\pi (^{(4)}T_{\alpha \beta} = \epsilon (K_{\lambda \beta} - K_{\lambda \beta}^\alpha) + \frac{\epsilon}{2} g_{\alpha \beta} (K_{\lambda, \rho} K_{\lambda \rho} - (K_{\lambda}^\alpha)^2) - \epsilon E_{\alpha \beta}, \tag{30}
\]

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where \( K_{\mu\nu} \) is the extrinsic curvature

\[
K_{\alpha\beta} = \frac{1}{2} \partial_{\alpha} g_{\beta\gamma} = \frac{1}{2\Phi} \frac{\partial g_{\alpha\beta}}{\partial \psi},
\]

(31)

\( E_{\mu\nu} \) is the projection of the 5D Weyl tensor \((5) C_{ABCD} \) orthogonal to \( \hat{n}^A \), i.e., “parallel” to spacetime, viz.,

\[
E_{\alpha\beta} = (5) C_{\alpha A \beta B} \hat{n}^A \hat{n}^B = -\frac{1}{\Phi} \frac{\partial K_{\alpha\beta}}{\partial \psi} + K_{\alpha\rho} K_{\beta}^\rho - \left( \frac{\alpha}{\Phi} \right),
\]

(32)

and \( \Phi_\alpha \equiv \partial \Phi / \partial x^\alpha \). It is worth mentioning that the effective matter content of the spacetime is the same whether we interpret it in space-time-matter theory, or in a \( \mathbb{Z}_2 \) symmetric brane universe \([7]\).

### 4.1 Properties of the effective energy-momentum tensor

For the case under consideration, the spacetime metric induced on \( \Sigma_\psi \) is given by

\[
ds^2 \equiv dS_\Sigma^2 = g_{\mu\nu} dx^\mu dx^\nu = E^2 f^{2b/a} f^2 e^{-\omega(\xi)} dt^2 - A f^{2\alpha} dx^2 - B f^{2\beta} dy^2 - C f^{2\gamma} dz^2.
\]

(33)

The effective EMT is\(^4\)

\[
\begin{align*}
T_0^0 & = \frac{cE^2 e^\omega}{E^2 f^{(a+b)/a}}, & c \neq 0, \\
T_1^1 & = T_0^0 \left[ \frac{\beta + \gamma - \alpha}{a} + \frac{\omega \xi (\beta + \gamma)f}{2cf_\xi} \right], \\
T_2^2 & = T_0^0 \left[ \frac{\alpha + \gamma - \beta}{a} + \frac{\omega \xi (\alpha + \gamma)f}{2cf_\xi} \right], \\
T_3^3 & = T_0^0 \left[ \frac{\alpha + \beta - \gamma}{a} + \frac{\omega \xi (\alpha + \beta)f}{2cf_\xi} \right].
\end{align*}
\]

(34)

Let us notice that, for a general choice of parameters, the components of the EMT satisfy the following algebraic expressions

\[
(\beta - \gamma)(T_0^0 + T_1^1) + (\beta + \gamma)(T_2^2 - T_3^3) = 0,
\]

\[
(\alpha - \gamma)(T_0^0 + T_2^2) + (\alpha + \gamma)(T_1^1 - T_3^3) = 0,
\]

\[
(\alpha - \beta)(T_0^0 + T_3^3) + (\alpha + \beta)(T_1^1 - T_2^2) = 0.
\]

(35)

They can be interpreted as “equations of state” for the effective density \( T_0^0 \) and the stresses \( T_i^i \), with \( i = 1, 2, 3 \). One can use them to obtain a simple relationship between the stresses, viz.,

\[
(\beta - \gamma)T_1^1 + (\gamma - \alpha)T_2^2 + (\alpha - \beta)T_3^3 = 0.
\]

(36)

In the case of axial symmetry, say along the \( x \)-direction \( (\beta = \gamma) \), they reduce to

\[
T_2^2 = T_3^3, \quad \text{and} \quad T_0^0 = -\frac{\alpha + \beta}{\alpha - \beta} T_1^1 + \frac{2\beta}{\alpha - \beta} T_2^2, \quad \alpha \neq \beta = \gamma.
\]

(37)

In the case of isotropic expansion \( (\alpha = \beta = \gamma) \) the effective EMT behaves like a perfect fluid

\[
T_1^1 = T_2^2 = T_3^3 = nT_0^0, \quad \text{with} \quad n \equiv \frac{1}{3} \left( 1 + \frac{\omega_\xi f}{\alpha f_\xi} \right),
\]

(38)

\(^4\)To simplify the notation, in what follows we suppress the index \( \text{(4)} \) in \( \text{(4)} T_{\alpha\beta} \)
which for \( n = \text{constant} \) is nothing but the barotropic equation of state commonly used in cosmological problems.

In general, we can write

\[
T_1^1 = n_x T_0^0, \quad T_2^2 = n_y T_0^0, \quad T_3^3 = n_z T_0^0,
\]

where

1. For \( \beta \neq -\gamma \)

\[
n_x = \frac{n_x(\alpha + \gamma) + (\alpha - \beta)}{\beta + \gamma}, \quad n_y = \frac{n_x(\alpha + \beta) + (\alpha - \gamma)}{\beta + \gamma}, \quad n_z = \left[ \frac{\beta + \gamma - \alpha}{a} + \frac{\omega \xi (\beta + \gamma) f}{2 e f \xi} \right].
\] (40)

2. For \( \beta = -\gamma \) and \( \alpha \neq -\beta \)

\[
n_x = -1, \quad n_y = \frac{n_x(\alpha - \beta) - 2\beta}{\alpha + \beta}, \quad n_z = \frac{\alpha + 2\beta}{\alpha} \frac{(\alpha + \beta) \omega \xi f}{2 \beta^2 f}.
\] (41)

3. For \( \beta = -\gamma \) and \( \alpha = -\beta \) (\( \alpha = \gamma = -\beta \))

\[
n_x = -1, \quad n_z = -1, \quad n_y = 3 - \frac{\omega \xi f}{\alpha f \xi}.
\] (42)

Clearly, one can use the above expressions to obtain the parameters \((\alpha, \beta, \gamma)\) in terms of \((n_x, n_y, n_z)\). It should be emphasized that, in the wave-like cosmologies under study, the effective matter in 4D behaves like a perfect fluid \((n_x = n_y = n_z)\) only for isotropic expansion. This is different from other models which do allow perfect fluid and anisotropic expansion (see, e.g., [16] and references therein, also the solution with stiff equation of state given in section 5.1.2 of our previous work [10]).

### 4.2 Solutions for constant \( n_x, n_y \) and \( n_z \)

Let us assume that the ratio \( n_i = T_i^i / T_0^0 \) is constant in every direction, which constitutes an extension to anisotropic models of the assumption of barotropic expansion used in FRW cosmologies. Without loss of generality, for the sake of argument let us assume that \( \beta \neq -\gamma \). In this case from (40) it follows that

\[
\omega \xi = D f^{2\eta/a}, \quad \text{where} \quad \eta = \frac{c [(n_x - 1)(\beta + \gamma) + (n_x + 1)\alpha]}{(\beta + \gamma)}, \quad \beta \neq -\gamma
\] (43)

and \( D \) is a constant of integration. Substituting this into (44) we get an equation for \( f \), viz.,

\[
a f \xi \xi + (b + c - \eta) f^2 \xi + \epsilon (c + \eta) \left( \frac{\mathcal{E} D}{kE} \right)^2 f^{2(2\eta - b)/a} = 0,
\] (44)

whose first integral is given by

\[
f^2 \xi = C_1 f^{-2(b + c - \eta)/a} - \epsilon \left( \frac{\mathcal{E} D}{kE} \right)^2 f^{2(2\eta - b)/a},
\] (45)

where \( C_1 \) is a constant of integration. Consequently, the metric

\[
dS^2 = \left[ \tilde{C} f^{-2c/a} - cD \left( \frac{\xi}{k} \right)^2 f^{2\eta/a} \right] dt^2 - Af^{2\alpha} dx^2 - B f^{2\beta} dy^2 - C f^{2\gamma} dz^2 + eD f^{2\eta/a} d\psi^2,
\] (46)

with \( \tilde{C} = E^2 C_1 / D \), is a solution of the 5D field equations \( R_{AB} = 0 \) provided the function \( f(\xi) \) satisfies (45), regardless the signature of the extra dimension and arbitrary parameters \( \alpha, \beta \) and \( \gamma \). It is clear that solutions with \( \beta = -\gamma \), as well as the axially symmetric ones with \( \beta = \gamma \), can be obtained in a similar way.
It should be noted that the solutions in Gaussian normal coordinates (18) are particular cases of (46) with $\eta = 0$. Also, as mentioned above, the parameters can be expressed in terms of the barotropic coefficients. In the present case they are

$$
\beta = \alpha \frac{(1 + n_x - n_y + n_z)}{(1 - n_x + n_y + n_z)}, \quad \gamma = \alpha \frac{(1 + n_x + n_y - n_z)}{(1 - n_x + n_y + n_z)}.
$$

(47)

Certainly, without loss of generality one can add any additional condition on these parameters.

4.3 Radiation-like solutions

It is well known that in the case of radiation (i.e., photons with zero rest mass) as well as for ultra-relativistic matter (i.e., particles with finite rest masses moving close to the speed of light) the trace of the EMT, say $T$, vanishes identically. A simple calculation from (34) gives

$$
T = T_0^0 + T_1^1 + T_2^2 + T_3^3 = T_0^0 \left(2 + \frac{a \omega \xi f}{c f \xi}\right).
$$

(48)

Thus, for $T = 0$ (and $T_0^0 \neq 0$) the second term in (14) vanishes. Consequently, radiation-like solutions have $e^\omega \sim f^{-2c/a}$ and are given by the power-law line element (22).

4.4 Vacuum solutions in 4D

From (34) it follows that $T_{\mu\nu} = 0$ requires $c = 0$ and $\omega \xi = 0$. Thus, vacuum solutions are particular cases (with $c = 0$) of those in Gaussian coordinates (18). Then, from (17) it follows that $f(\xi) \sim \xi^{a/(a+b)}$. Therefore, the 5D line element

$$
dS^2 = dt^2 - \Delta_\xi dx^2 - B\xi^2 dy^2 - C\xi^2 dz^2 + \epsilon d\psi^2,
$$

(49)

with

$$
p_1 = \frac{a}{a+b}, \quad p_2 = \frac{\beta a}{a+b}, \quad p_3 = \frac{\gamma a}{a+b},
$$

(50)

which satisfy $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$, may be interpreted as an embedding for a 4D Kasner spacetime. Clearly, vacuum solutions can also be obtained from those in the synchronous reference system (19) by setting $c = 0$.

5 Other interpretations in 4D

In the case where the extra dimension is spacelike, the solutions to the 5D field equations are invariant under the transformation $(x, y, z) \leftrightarrow \psi$. However, the physics in 4D crucially depends on how the coordinates of our ordinary 3D space are identified. In fact, after such a transformation (i) the spacetime slices $\Sigma_\psi$ are non-flat, and (ii) the effective four-dimensional EMT is traceless.

As an illustrative simple example, let us consider the 5D metric

$$
dS^2 = MF^{-2c/a} dt^2 - A\xi^2 dx^2 - B\xi^2 dy^2 - NF^{-2c/a} dz^2 + \epsilon CF^2 \xi d\psi^2,
$$

(51)

with $F = (c_1 \eta + c_2)^{1/a}$ and $\eta = \mathcal{E} t + k z$, which is a solution of the 5D field equations $R_{AB} = 0$ obtained from the power-law solution (22) after a $z \leftrightarrow \psi$ transformation. Here the metric functions are independent of the extra coordinate $\psi$ and, consequently, the effective EMT is traceless. Therefore, the 4D metric induced on $\Sigma_\psi$ can be interpreted either as anisotropic vacuum solutions in the Randall-Sundrum (RS2) braneworld scenario (10), or as inhomogeneous radiation-like solutions in conventional 4D general relativity. Certainly, the same is true for all the solutions discussed here.

The above discussion highlights the fact that much work is still needed in order to have a clear understanding of 4D physical models as Lorentzian hypersurfaces in pseudo-Riemannian 5D spaces.
6 Summary

According to Campbell’s theorem any solution of the Einstein equations in 4D, with an arbitrary energy-momentum tensor, can be locally embedded in a solution of the vacuum Einstein field equations in 5D. In this paper we have considered a five-dimensional Riemannian manifold whose line element is diagonal and independent of coordinates for ordinary 3D space \((x, y, z)\), which is given by \((\text{I})\). This line element is quite general in the sense that, on every hypersurface \(\Sigma_\psi : \psi = \psi_0\), it can be used or interpreted as a 5D embedding for spatially flat FRW models, as well as for Bianchi type-I cosmologies.

In order to solve the field equations \(R_{AB} = 0\) we have assumed that the 5D manifold presents self-similar symmetry. In the traditional interpretation of Sedov, Taub and Zeldovich \((\text{I1})\), this means that all dimensionless quantities in the theory can be expressed as functions only of a single similarity variable. This assumption, and the field equations, determine the possible shape of the similarity variable. These are (i) \(\xi = t/\psi\); (ii) \(\xi = e^{qt}/e^{q\psi}\), and (iii) \(\xi = \mathcal{E}t + k\psi\), where \(q, \mathcal{E}\) and \(k\) are some constants with the appropriate units. They correspond to three distinct families of self-similar solutions, each of them being parameterized by an arbitrary function of the similarity variable and three arbitrary parameters \(\alpha, \beta, \gamma\).

1. For \(\xi = t/\psi\) the 5D manifold possesses homothetic symmetry and the general solution of the field equations \(R_{AB} = 0\) is given by \((\text{10})\)

\[
dS^2 = E f^{(b/a)} f_\xi dt^2 - A f^{2\alpha} dx^2 - B f^{2\beta} dy^2 - C f^{2\gamma} dz^2 - \Xi f^{(b/a)} f_\xi d\psi^2. \tag{52}
\]

2. For \(\xi = (e^{t/\psi})^q\) the 5D manifold possesses conformal symmetry and the general solution of the field equations \(R_{AB} = 0\) is \((\text{10})\)

\[
dS^2 = E \Xi f^{(b/a)} f_\xi dt^2 - A f^{2\alpha} dx^2 - B f^{2\beta} dy^2 - C f^{2\gamma} dz^2 - E \Xi f^{(b/a)} f_\xi d\psi^2. \tag{53}
\]

3. For \(\xi = \mathcal{E}t + k\psi\), in this work we have seen that the general solution can be written as

\[
dS^2 = E^2 f^{(2b/a)} f_\xi^2 e^{-\omega} dt^2 - A f^{2\alpha} dx^2 - B f^{2\beta} dy^2 - C f^{2\gamma} dz^2 + e^{\omega} d\psi^2, \tag{54}
\]

where the functions \(f\) and \(\omega\) are related by \((\text{14})\). This case is mathematically more complicated than the other two because, after choosing some specific function, say \(f\) for the sake of argument, one still has to integrate \((\text{14})\) in order to concretize the solution.

For the wave-like models discussed here, as well as for the ones discussed in our recent work \((\text{10})\), the arbitrary function needed to specify the solution can be determined either by the choice of the reference/coordinate system, as it is illustrated in section 3, or by imposing certain conditions on the effective EMT in 4D, as we did in section 4. In all cases the parameters \(\alpha, \beta\) and \(\gamma\) are related to the properties of the effective 4D matter. They are not independent and, therefore, without loss of generality, one can impose any algebraic condition on them. This is illustrated by \((\text{17})\), as well as by the solutions discussed in section 5.1 of \((\text{10})\).

According to Campbell’s theorem the connection to 4D is deduced after choosing an embedding \((\text{I})\). In order to keep the spacetime signature \((+, - , - , - )\) in \((\text{52})-\text{(54)}\) the constants \(A - E\) ought to be positive, otherwise they are arbitrary. Thus, the extra dimension must be spacelike for the homothetic and conformal solutions and, in general, it is undefined (\(\epsilon = \pm 1\)) for wave-like solutions. Although an exception is provided by solution \((\text{23})\) which requires \(\epsilon = -1\).

For wave-like solutions the effective matter in 4D cannot be interpreted as perfect fluid, except in the isotropic limit corresponding to \(\alpha = \beta = \gamma\). This is quite different from the homothetic solutions \((\text{52})\) which do allow such interpretation (section 5.1 in \((\text{10})\)). Despite these differences, the anisotropic 5D cosmologies \((\text{52})-\text{(54)}\) share in common the property that, although they are Ricci-flat \((R_{AB} = 0)\), they are not Riemann-flat \((R_{ABCD} = 0)\), except

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From a general-relativistic viewpoint one could argue that, instead of deriving the properties of the EMT from an embedding in 5D, a more ‘physical’ approach would be to find an embedding for a 4D spacetime with a specified physical EMT. However, this seems to face fundamental problems \((\text{10})\).
in the trivial case where \( a = 0 \). The distinction is important because certain solutions of 5D relativity with high degrees of symmetry may have \( R_{ABCD} = 0 \) and be flat in 5D, while possessing curved subspaces in 4D. This is the case of 5D cosmologies with spherical symmetry in ordinary 3D space \[17\], which include the standard 5D cosmologies \[9\].

Our solutions may be applied to the era after the big-bang, where the anisotropy could have played a significant role, and the universe could not have been well described by FRW isotropic models. Besides, the solutions allow different equations of state, among them a radiation-like equation of state, with \( T = 0 \), which is typical of radiation and/or ultra-relativistic matter.

References

[1] A. Davidson and D. Owen, Phys. Lett. B 155, 247(1985).
[2] T. Appelquist, A. Chodos and G.P.O. Freund, Modern Kaluza-Klein Theories. New York: Addison-Wesley (1987); E.W. Kolb and M.S. Turner, The Early Universe. New York: Adisson-Wesley (1990).
[3] J.E. Campbell, A Course of Differential Geometry (Clarendon, Oxford, 1926)
[4] S. Rippl, C. Romero, R. Tavakol, Class.Quant.Grav. 12, (1995)2411; J.E. Lidsey, C. Romero, R. Tavakol, S. Rippl, Class.Quant.Grav. 14, (1997)865; S.S. Seahra, P.S. Wesson, Class. Quant. Grav. 20 (2003)1321; F. Dahia, C. Romero, Class.Quant.Grav. 22 (2005)5005.
[5] L. Randall, R. Sundrum, Mod. Phys. Lett. A13 (1998)2807; L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999)4680; N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B429 (1998)263; N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Rev. D59 (1999)086004; T. Shiromizu, Kei-ichi Maeda and Misao Sasaki, Phys. Rev. D62 (2000)02412.
[6] P.S. Wesson, Phys. Lett. B276 (1992)299; P.S. Wesson, Astrophys. J. 394 (1992)19; P.S. Wesson, J. Ponce de Leon, J. Math. Phys. 33 (1992)3883; P.S. Wesson, Space-time-Matter, World Scientific, Singapore, 1999.
[7] J. Ponce de Leon, Mod. Phys. Lett A16 (2001)2291.
[8] D. Kramer, Acta Phys. Polon. B2 (1970)807; D.J. Gross and M.J. Perry, Nucl. Phys. B226 (1983)29.
[9] J. Ponce de Leon, Gen. Rel. Grav. 20 (1988)539.
[10] J. Ponce de Leon, Self-similar cosmologies in 5D: spatially flat anisotropic models, arXiv:0805.1108
[11] L.I. Sedov, Similarity and Dimensional Methods in Mechanics (Academic Press, New York, 1959); M.E. Cahill and A.H. Tuub, Commun. Math. Phys. 21 (1971)1; G. E. Barenblat and Ya. B. Zel’ dovich, Ann. Rev. Fluid. Mech. 4 (1972)285; R. N. Henriksen, A.G. Emslie and P.S. Wesson, Phys. Rev. D 27 (1983)1219; P.S. Wesson, Phys. Rev. D 34 (1986)3925; B.J. Carr and A. Yahil, Astrophys. J. 360 (1990)330; J. Ponce de Leon, Mon. Not. R. astr. Soc. 250 (1991)69; J. Ponce de Leon, Gen. Rel. Grav. 25 (1993)865.
[12] B.J. Carr and A.A. Coley, Gen. Rel. Grav 37 (2005)2165; J. Wainwright, M.J. Hancock and C. Uggl, C, Class. Quantum Grav. 16 (1999)2577; P.S. Apostolopoulos and M. Tsamparlis, Gen. Rel. Grav. 35 (2003)2051; P.S. Apostolopoulos, Class.Quant.Grav. 20 (2003)3371, Class.Quant.Grav. 22 (2005)323.
[13] H. Liu and P.S. Wesson, Int. J. Mod. Phys. D3 (1994)627; P.S. Wesson, H. Liu and S.S. Seahra, Astron. Astrophys. 358 (2000)425; G.T. Horowitz, I. Low and A. Zee, Phys. Rev. D62 (2000)0860005; J. Ponce de Leon, Int. J. Mod. Phys. D12 (2003)1053, Gen. Rel. Grav. 36 (2004)923.
[14] J. Ponce de Leon, Mod. Phys. Lett. A21 (2006)947; Int. J. Mod. Phys. D15 (2006)1237; Class. Quant. Grav. 23 (2006)3043.
[15] J. Ponce de Leon, Int. J. Mod. Phys. D 11 (2002)1355.
[16] S. Kumar and C.P. Singh, *Astrophys. Space Sci.* **312** (2007) 57.

[17] J. Ponce de Leon, *JCAP* **03** (2008) 021.