Radion Dynamics in BPS Braneworlds

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Abstract: We examine the moduli dynamics of a specific class of supergravity-inspired BPS braneworlds, clarifying the role of bulk scalar fields in brane collisions. The model contains as a special case the Randall-Sundrum model both with and without a free, massless bulk scalar field. Its low-energy effective theory is derived with a moduli space approximation (MSA) and agrees with the corresponding results derived elsewhere. Rather than stabilising the radion, we look at cosmological evolution of the system stimulated by breaking the BPS condition on the branes. We examine in detail the range of validity of the MSA in both the RS and BPS case, paying particular attention to the divergences that can arise during a collision of the branes. In the absence of perturbations such an event is finite in the RS model, and accurately described by the low-energy effective theory. We demonstrate, however, that a collision is divergent in the BPS case even with an exact FRW geometry.

Keywords: supergravity models, cosmology of theories beyond the SM, physics of the early universe, brane collisions.
1. Introduction

Braneworld models for the Universe have recently been the subject of considerable theoretical interest [1]. Much of this research has been inspired by the work of Horava and Witten [2], who showed that the strong coupling limit of $E8 \times E8$ heterotic String Theory could be described by an eleven-dimensional supergravity with the eleventh dimension made up of the orbifold $S^1/Z_2$, i.e. an interval with reflection symmetry about its endpoints. These endpoints define ten-dimensional submanifolds, the branes, which sit at the orbifold fixed points and define the boundaries of the spacetime. The other six dimensions could consistently be compactified on a Calabi-Yau threefold whose characteristic scale is considerably smaller than the interbrane distance. This theory then led to many toy models of spacetime as a five-dimensional manifold (the bulk) bounded by two branes with a $Z_2$ symmetry.
One of the most well-known of these is the Randall-Sundrum I model \[3\]. Whereas the Horava-Witten model allows a large number of fields to propagate in the bulk, this model has only a bulk cosmological constant. The two boundary branes carry equal and opposite tensions whose magnitudes can be fine tuned such that the effective cosmological constants on the branes vanish, provided the bulk cosmological constant is negative. The resulting low-energy theory can be derived in a number of ways \[4, 5, 6\] and can be formulated in terms of a single modulus field, the radion, representing the proper distance between the branes. This four-dimensional low-energy theory describes the physics as viewed by an observer confined to one of the branes and, in the case of exact cosmological symmetry, turns out to be perfectly well defined even when the size of the fifth dimension vanishes, i.e. the branes collide.

The radion field is classically massless, representing the fact that the brane positions are arbitrary when the brane tensions are fine tuned. This is phenomenologically undesirable, since a massless scalar field is not observed in nature. By detuning the brane tensions from their preferred values one can generate a potential for the radion; one could imagine these detuning potentials on the branes coming either from a more consistent quantum mechanical treatment of the scalar field \[7\] or from sort of SUSY-breaking mechanism. In this paper we shall just insert it by hand.

We shall consider a supergravity-motivated generalisation \[8, 4\] of the Randall-Sundrum model where a scalar field \(\Psi\) is allowed to propagate in the bulk. Its potential \(U\) replaces the Randall-Sundrum bulk cosmological constant and the ‘superpotential’ induced on the branes \(\hat{V}\) replaces the tensions. Again the branes can be Minkowski if \(\hat{V}\) and \(U\) are related by a BPS condition, the analogue of the Randall-Sundrum fine tuning. In fact, this model reduces exactly to the Randall-Sundrum model if its free parameter \(\alpha\), defined by \(\hat{V} \propto \exp \alpha \Psi\), vanishes. This time, the four-dimensional effective theory contains two moduli fields and, by detuning the brane potentials, one can generate dynamics and cosmological solutions as before.

The cosmological consequences of this model have been explored in detail in \[4\]. In this work we focus on brane collisions, events which have been used recently in the literature to try to illuminate and resolve the Big Bang singularity \[9, 10\]. It is already known that perturbations diverge during such an event; in this paper, we investigate how the simple generalisation to BPS bulk scalar fields effects both the dynamics of the radion and the regularity of the collision. Such bulk scalar fields arise naturally in supersymmetric theories \[8\].

This paper is organised as follows. In \(\S2\) we describe the model and derive the exact projected Einstein equations on the brane. In the cosmological context we obtain the Friedmann equation for the induced FRW brane geometry. This is local (i.e. defined purely in terms of quantities on the brane) apart from the well-known Weyl tensor representing the gravitational effects of the bulk. Discussing the low-energy effective theory in \(\S3\) connects this quantity to the moduli fields and allows one to write down an approximate, closed set of equations for the motion of the branes. The dynamics in the Randall-Sundrum case are reviewed in \(\S4\), with the identification of the Weyl tensor explaining the finiteness of the system through a brane collision. The more general case \(\alpha \neq 0\) is studied in \(\S5\), showing...
that in all but the most contrived case the scalar field will diverge during a brane collision. The conclusions are summarised in §.

2. The Model

A general action for the two-brane system is

\[
S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[ R - \frac{1}{2} \partial^2 - U(\Phi) \right] + \int_1 d^4x \sqrt{-h^{(1)}} \left[ \frac{2K}{\kappa_5^2} - \hat{V}(\Phi) \right] + \int_2 d^4x \sqrt{-h^{(2)}} \left[ \frac{2K}{\kappa_5^2} + \hat{V}(\Phi) \right]
\]

ignoring for the time being the presence of possible matter actions for the branes. \(h^{(1)}_{ab}\) and \(h^{(2)}_{ab}\) are the induced metrics on the positive- and negative-tension branes respectively, \(U\) and \(\hat{V}\) > 0 are the bulk and brane potentials and \(K\) is the trace of the extrinsic curvature \(K_{ab}\) taking outward-pointing normals. The factor of \(\kappa_5^2\) ensures that \(\Phi\) is dimensionless.

The index convention to be used is that the metric signature is ‘mostly plus’ \(-++++\), \(a, b, c...\) are five-dimensional indices and \(\mu, \nu, \rho...\) are four-dimensional, running from 0 to 3. The unusual factor of 2 in the Gibbons-Hawking sectors is because of the \(\mathbb{Z}_2\) orbifold nature of the bulk, and the five-dimensional integral in the action is taken to mean an integral over two copies of the bulk spacetime Region I between the two branes, see Fig.4. As described in the introduction the bulk geometry is assumed to be \(\mathbb{Z}_2\)-symmetrical about the position of the two branes, which simplifies the junction conditions. The 1 and 2 on the brane integrals refer to positive and negative tension respectively.

Variation of the action with respect to the metric and the field yield the usual bulk equations and junction conditions for each brane [11]. Considering the positive-tension brane (the analysis for the other brane is equivalent with \(\hat{V} \rightarrow -\hat{V}\)), we define \(h_{ab} \equiv h^{(1)}_{ab}\) and find

\[
G_{ab} = \frac{1}{2} \partial_a \Phi \partial_b \Phi - \frac{1}{4} g_{ab} \partial^2 - \frac{1}{2} U(\Phi) g_{ab}
\]

\[
\Box \Phi = \frac{dU}{d\Phi}
\]

\[
[K_{ab}] = -\kappa_5^2 \left( T_{ab} - \frac{1}{3} T h_{ab} \right)
\]

\[
[n \cdot \partial \Phi] = 2\kappa_5^2 \frac{d\hat{V}}{d\Phi}
\]

where

\[
T_{ab} = -\hat{V}(\Phi) h_{ab}
\]

and \([X]\) means the jump in \(X\) in the same direction as the normal (hence both these quantities are invariant under \(n \rightarrow -n\)). Assuming \(\mathbb{Z}_2\) symmetry the junction conditions
become

\[ -K_{ab} = - \frac{1}{6} \kappa^2 \dot{V}(\Phi) h_{ab} \]  
(2.4)

\[ -n \cdot \partial \Phi = \kappa^2 \frac{d\dot{V}}{d\Phi} \]  
(2.5)

where quantities are evaluated at the edge of Region I, which the normal points outward from as depicted in Fig. 1. The 4D Einstein tensor \( \overline{G}_{ab}(\Phi) \) is given by the standard result

\[
\overline{G}_{ab} = \frac{2}{3} \left\{ G_{cd} h^{c \, a}_a h^{d \, b}_b + \left( G_{cd} n^c n^d - \frac{1}{4} G \right) h_{ab} \right\} + KK_{ab} - K_a^c K_{bc} - \frac{1}{2} \left( K^2 - K^{cd} K_{cd} \right) h_{ab} - E_{ab}
\]  
(2.6)

where \( n^a \) is the spacelike unit normal to the brane and \( E_{ab} \) is the (traceless) electric part of the Weyl tensor

\[ E_{ab} = C_{abcd} n^c n^d \]

Figure 1: Bulk and boundary structure: The bulk topology is \( \Sigma \times S_1 / \mathbb{Z}_2 \), with the branes sitting at the orbifold fixed points. There are therefore two identical copies of the bulk, Regions I & II, with four boundary planes in all as shown, giving rise to two copies of the bulk action and of the GH boundary terms. There is only one copy of each brane worldvolume (and matter) action.

From (2.2),

\[
G_{cd} h^{c \, a}_a h^{d \, b}_b + \left( G_{cd} n^c n^d - \frac{1}{4} G \right) h_{ab} = \frac{1}{2} \nabla_a \Phi \nabla_b \Phi - \frac{5}{16} \nabla \Phi^2 h_{ab} + \frac{3}{16} (n \cdot \partial \Phi)^2 h_{ab} - \frac{3}{8} U h_{ab}
\]

where \( \nabla \) is the covariant derivative of the induced metric \( h \), and from (2.4),

\[
KK_{ab} - K_a^c K_{bc} - \frac{1}{2} \left( K^2 - K^{cd} K_{cd} \right) h_{ab} = - \frac{1}{12} \kappa^2 \dot{V}^2 h_{ab}
\]
Substituting all this back into (2.6) with the junction condition (2.5) we obtain the projected Einstein equation on the brane:

$$\mathbf{G}_{ab} = \frac{1}{3} \nabla_a \Phi \nabla_b \Phi - \frac{5}{24} \nabla^2 h_{ab} - E_{ab}$$

$$+ \frac{1}{8} \kappa_5^2 \left( \frac{d \hat{V}}{d \Phi} \right)^2 h_{ab} - \frac{1}{4} U h_{ab} - \frac{1}{12} \kappa_5^4 \hat{V}^2 h_{ab}$$

(2.7)

If matter were included on the brane via

$$T_{ab} \rightarrow - \hat{V} (\Phi) h_{ab} + \tau_{ab}$$

(2.8)

then there would also be the terms

$$... + \frac{\kappa_5^4}{6} \hat{V} \tau_{ab} + \kappa_5^4 \pi_{ab}$$

(2.9)

where $\pi_{ab}$ depends quadratically on $\tau_{ab}$. This has two consequences; in order to recover the Friedmann equation we must live on the positive-tension brane (at least at the classical level), and be at an energy scale much less than the brane tension so that the resulting $\tau_{00}^2$ term can be regarded as a small correction important only in the early Universe. Note that the effective four-dimensional cosmological ‘constant’ is given by

$$\Lambda_4 = \frac{1}{4} U + \kappa_5^4 \left[ \frac{1}{12} \hat{V}^2 - \frac{1}{8} \left( \frac{d \hat{V}}{d \Phi} \right)^2 \right]$$

which will vanish for potentials satisfying

$$U = \kappa_5^4 \left[ \frac{1}{2} \left( \frac{d \hat{V}}{d \Phi} \right)^2 - \frac{1}{3} \hat{V}^2 \right]$$

(2.10)

Such self-tuned potentials often arise in the context of supergravity models where $U$ and $\hat{V}$ would be derived from the same superpotential [8]. We shall for convenience refer to such potentials as being ‘supersymmetric’ and the function $\hat{V}$ as the superpotential. This relation between the bulk and brane potentials is a generalisation of the usual Randall-Sundrum (henceforth RS) fine tuning, which can be reproduced by taking $U = -2 \Lambda$ and $\hat{V} = \sigma$; (2.10) then reduces to the familiar form

$$\Lambda = -\frac{1}{6} \kappa_5^4 \sigma^2$$

3. Low-energy Effective Theory

Whilst (2.7) gives the exact Einstein equations for an observer on the brane it has little predictive power, since it contains the term $E_{ab}$ which is not defined in terms of data on the brane. Such explicit dependence on the bulk geometry is also present in the Klein-Gordon equation (2.3) which, when written out explicitly in terms of four-dimensional covariant
derivatives, will contain a term of the form \((n \cdot \partial)^2 \Phi\), not prescribed by the value of the field on, and its derivatives along, the brane.

For this reason one seeks a four-dimensional effective theory in which these non-local quantities are replaced by scalar ‘moduli fields’. Such low-energy effective theories can be derived in many different, but essentially equivalent ways. Here we shall employ the moduli space approximation; the result (3.14) agrees with that obtained by perturbative expansion [12].

Note that a four-dimensional description, low energy or otherwise, in terms of an effective action cannot hope to reproduce a Friedmann equation with the terms quadratic in the brane energy-momentum tensor given by (2.9). For example [13, 14], in the RS case, these can be viewed as arising approximately from the trace anomaly in the CFT defined on the branes in the context of the AdS-CFT correspondence, and hence cannot be derived from the variation of an action. Higher order derivative terms could be included in the effective action; these will approximate then more and more closely the effects of \(E_{\mu\nu}\) in the projected Einstein equations, but the quadratic stress-energy terms cannot be obtained this way.

### 3.1 BPS backgrounds

The significance of supersymmetric (henceforth SUSY) potentials is that they allow privileged configurations of the system where the brane positions are arbitrary. These configurations are, in a sense, a ground state of the system, with a high degree of symmetry. We therefore look for solutions for the metric and the scalar field which do not depend on the transverse directions (i.e. are static and transversely homogeneous). We take Gaussian Normal coordinates away from the positive-tension brane

\[
ds^2 = a(z)^2 \eta_{\mu\nu} dx^\mu dx^\nu + dz^2
\]

where \(z\) is the proper distance along the normal to the brane (increasing towards the other brane) and \(x^\mu\) parameterise the flat, transverse foliations. Since we are assuming fine-tuning of the brane tensions the transverse foliations are flat, hence the use of \(\eta_{\mu\nu}\) above. Due to the symmetry of this ‘vacuum’ configuration the branes are at constant \(z\). In the next section, when we relax the requirements of staticity and homogeneity, we will consider more general coordinate systems. Taking \(\Phi(t, x, z) = \Phi(z)\), the Einstein and Klein-Gordon equations become:

\[
\begin{align*}
\frac{a'^2}{a^2} + \frac{a''}{a} &= -\frac{1}{12} \Phi'^2 - \frac{1}{6} U \\
\frac{a'^2}{a^2} &= \frac{1}{24} \Phi'^2 - \frac{1}{12} U \\
\Phi'' + 4 \frac{a'}{a} \Phi' &= \frac{dU}{d\Phi}
\end{align*}
\]
where $' = d/dz$. For potentials $U$ satisfying (2.10) this can be written as

$$
\left( \frac{a'}{a} \right)' = -\frac{1}{6} \Phi'^2
$$

(3.2)

$$
\left( \frac{a'}{a} \right)^2 = \frac{1}{24} \Phi'^2 - \frac{1}{24} \kappa_5^2 \left( \frac{d\hat{V}}{d\Phi} \right)^2 + \frac{1}{36} \kappa_5^4 \hat{V}^2
$$

(3.3)

noting that there are only two independent equations in (3.1). An interesting family of solutions follow from the first-order system

$$
\frac{a'}{a} = -\frac{\kappa_5^2 \hat{V}}{6}, \quad \Phi' = \kappa_5^2 \frac{d\hat{V}}{d\Phi}
$$

(3.4)

which can easily be seen to solve the full Einstein-Klein-Gordon equations (3.2,3.3). In the underlying supergravity theory [8] these are BPS configurations; the ‘no force’ condition between BPS branes manifests itself in that (3.4) implies the junction conditions, which are given by

$$
\left. \frac{a'}{a} \right|_1 = -\frac{\kappa_5^2 \hat{V}}{6} \bigg|_1, \quad \left. \Phi' \right|_1 = \kappa_5^2 \frac{d\hat{V}}{d\Phi} \bigg|_1
$$

(3.5)

where the quantities are evaluated at the positive-tension brane and we have used $K_{ab} = g'_{ab}/2$ for surfaces of constant $z$. In other words, there exist static solutions of the system for arbitrary brane positions. A solution of (3.4) will from now on be referred to as a ‘BPS background’. Note that the free parameters in these solutions are the value of the scalefactor $a$ and the scalar field $\Phi$ on the positive-tension brane.

Note that, in the metric junction condition (3.5), $a'/a$ is proportional to the extrinsic curvature on the brane. Since the normal to the negative-tension brane must point in the opposite direction for consistency (i.e. either both outward or both inward), the corresponding junction condition there will pick up a minus sign. Therefore, in order for the junction conditions to be satisfied at both branes simultaneously, the negative-tension brane must have tension $\hat{V}_-(\Phi) = -\hat{V}_+(\Phi)$ as is assumed from the start in the action (2.1).

As we shall see in the next section, it is desirable to detune the brane potentials from their ‘supersymmetric’ values. This will generate a potential for the moduli fields in the four-dimensional effective theory, which is desirable for both non-trivial dynamics and some hope of their stabilisation. We shall insert this by hand,

$$
V_1(\Phi) = \hat{V}(\Phi) + v(\Phi)
$$

(3.6)

$$
V_2(\Phi) = -\hat{V}(\Phi) + w(\Phi)
$$

where $V_{1,2}$ are the tensions on the two branes, $\hat{V}$ is the supersymmetric value given by (2.10) and $v, w$ are small perturbations, $v, w \ll \hat{V}$. For definiteness, we shall consider only exponential superpotentials,

$$
\hat{V} = \frac{6k}{\kappa_5^2} e^{\alpha \Phi}
$$

(3.7)

which reduce to the RS model on taking the limit $\alpha \rightarrow 0$ whereupon the constant $k$, which has dimensions of inverse length, is the curvature scale of the then AdS bulk.
3.2 The Moduli Space Approximation

In the previous subsection we derived a set of equations (3.4) whose solutions we will use as a ground state for our model. The corresponding metric and scalar field profiles were static and homogeneous, depending only on the bulk coordinate. This profile for the scalar field and the metric satisfies the junction conditions (3.5) at the branes provided the branes are parallel and static. In order to examine small perturbations around this vacuum we allow the brane positions to fluctuate as depicted in Fig. 1, and incorporate the graviton zero mode [8] by generalising the metric ansatz to

$$ds^2 = dz^2 + a(z)^2 \tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu$$  \hspace{1cm} (3.8)$$

In this coordinate system the perturbed brane positions are given by $z = z_1(x)$ and $z = z_2(x)$ (from now on, subscripts 1 and 2 will refer to evaluation at the positive- and negative-tension branes respectively). The radion, the proper distance between the two branes, is given by $r(x) = z_2 - z_1$. This is the local size of the fifth-dimension; the only physical spacetime is that between the two branes.

We shall assume that any $x$-dependence is small, i.e. that transverse derivatives are much smaller than normal derivatives. Also, we shall assume that the SUSY-breaking potentials $v$ and $w$ (and matter actions were we to be considering them) are small, to be consistent with the fact that we must necessarily lose the quadratic terms in the Friedmann equation in using an effective action, as discussed above. In this sense then the effective four-dimensional theory we will obtain is only valid at low energies, specifically for $v, w, \tau_{\mu\nu} \ll \hat{V}$. Radion fluctuations can be regarded as a small perturbation around the BPS configuration; by substituting this perturbed brane positions back into the action, keeping the BPS profile for the metric and scalar field, we shall obtain an effective theory governing the motion of the moduli fields $z_1$ and $z_2$.

For the superpotential (3.7) the BPS profile is given by

$$a(z) = \xi(z)^{1/6\alpha^2}$$
$$\Phi(z) = -\frac{1}{\alpha} \log \xi(z)$$
$$\xi(z) = [6k\alpha^2 (z_0 - z)]$$  \hspace{1cm} (3.9)$$

where $z_0$ is a constant of integration and we have assumed $\alpha \neq 0$. As is generally the case with self-tuned potentials there is a singularity in the bulk at $z = z_0$. The theory will break down when this singularity lies in the physical region of spacetime between the two branes [4].
The action is given by

\[
S_{\text{full}} = S_g + S_\Phi + S_B + S_{\text{GH}}
\]

\[
S_g = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g} R(g)
\]

\[
S_\Phi = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g} \left[ -\frac{1}{2} \partial^2 \Phi^2 - U(\Phi) \right]
\]

\[
S_B = \int_1^2 d^4 x \sqrt{-h(1)} (-\hat{V} - v)
+ \int_2^1 d^4 x \sqrt{-h(2)} (\hat{V} - w)
\]

\[
S_{\text{GH}} = \frac{2}{\kappa_5^2} \int_1^2 d^4 x \sqrt{-h(1)} K_1 + \frac{2}{\kappa_5^2} \int_2^1 d^4 x \sqrt{-h(1)} K_2
\]

reminding that

\[
\int d^5 x \sqrt{-g} ... = 2 \int d^4 x \int_{z_1(x)}^{z_2(x)} dz \sqrt{-\hat{g}} a^4 ...
\]

Keeping the BPS profile for the metric and the field in place, we can evaluate this term by term to obtain a four-dimensional effective theory in terms of the \(z_i\). For \(v = w = 0\) all terms not involving \(x\)-derivatives will sum to zero since they would otherwise generate a potential term for the moduli fields or a cosmological constant term, which both must vanish in this fine-tuned case. Hence we can discard any terms not involving transverse derivatives.

The result, expressed in terms of the induced metric on the positive-tension brane \(h_{\mu\nu}^{(1)} \equiv h_{\mu\nu}\), is derived in the Appendix, and is given by

\[
S = \frac{1}{2\kappa_5^2} \int d^4 x \sqrt{-h} \left[ \Omega^2 R(h) - \gamma^{AB} \nabla \xi_A \cdot \nabla \xi_B - V \right]
\]

with

\[
\Omega^2 = \frac{1}{1 + 3\alpha^2} \left[ \xi_1 - \xi_2 \left( \frac{\xi_2}{\xi_1} \right)^{1/3\alpha^2} \right]
\]

\[
\gamma^{11} = \left[ 6\alpha^4 (1 + 3\alpha^2) \right]^{-1} \frac{1}{\xi_1} \left[ 3\alpha^2 + \left( \frac{\xi_2}{\xi_1} \right)^{1+1/3\alpha^2} \right]
\]

\[
\gamma^{12} = - \left[ 6\alpha^4 \right]^{-1} \frac{1}{\xi_1} \left( \frac{\xi_2}{\xi_1} \right)^{1/3\alpha^2}
\]

\[
\gamma^{22} = \left[ 6\alpha^4 \right]^{-1} \frac{1}{\xi_1} \left( \frac{\xi_2}{\xi_1} \right)^{1/3\alpha^2}
\]

\[
V = 2k\kappa_5^2 \left[ v + w \left( \frac{\xi_2}{\xi_1} \right)^{2/3\alpha^2} \right]
\]
in agreement with the result of [12] (using different normalisations for the fields). Here, and elsewhere, covariant derivatives are taken with respect to the metric \( h_{\mu\nu} \) unless otherwise specified. Finally we put the action into a more easily interpretable form, in terms of the radion and value of the scalar field on the positive-tension brane. For convenience we make the connection to the radion via the approximate conformal factor \( \omega^2 \) relating the two induced metrics,

\[
\begin{align*}
  h_{\mu\nu}^{(2)} &\approx \omega^2 h_{\mu\nu}^{(1)} \\
  \omega^2 &= \left( \frac{a_2(x)}{a_1(x)} \right)^2,
\end{align*}
\]

(3.15)

and define

\[
\psi(x) = 1 - \omega^{2(1+3\alpha^2)} = 1 - \left( \frac{\xi_2}{\xi_1} \right)^{1+1/3\alpha^2}
\]

(3.16)

so that the branes coincide for \( \psi = 0 \) whilst \( \psi = 1 \) signifies an infinite redshift between the two branes; this could either mean their proper separation is infinite or that the second brane has hit the singularity. The value of the scalar field on the positive-tension brane is

\[
\eta(x) = -\frac{1}{\alpha} \log \xi_1
\]

(3.17)

giving our final action as

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-h}e^{-\alpha\eta} \left[ \psi R(h) - \frac{3}{2\beta} \frac{1}{(1-\psi)} \nabla \psi^2 - \frac{1}{2} \psi \nabla \eta^2 \right]
\]

\[
- \int d^4x \sqrt{-h} \left\{ u + (1 - \psi)^{2/\beta} w \right\}
\]

(3.18)

where

\[
\kappa^2 = k \kappa_5^2 \beta \quad \beta = 1 + 3\alpha^2
\]

(3.19)

Since the underlying action has five-dimensional coordinate invariance, the choice of coordinate system used to perform the above dimensional reduction cannot affect the final action, which is a scalar quantity. Furthermore, if one changes the coordinate system so that neither, one, or both of the branes are at fixed positions, each component of the action is still separately invariant for the same reason. This will, of course, depend on which four-dimensional metric is being used, but if one is careful to express everything in terms of the same metric (\( h_{\mu\nu} \), for example) then it is easy to check that each contribution to the total effective action is unchanged. For example, one could use a coordinate system in which the branes are fixed, say at \( y = 0 \) and \( y = 1 \), via the transformation \( z = z_1(x) + r(x) y \).

The scalar field modulus then enters as usual, being defined as the value of the field at, for example, \( y = 0 \), but this time the radion enters directly in the metric,

\[
ds^2 = r(x)^2 dy^2 + 2y \partial_\mu r(x) dx^\mu dy
\]

\[
+ \left[ \left( \frac{a(z(x,y))}{a_1(x)} \right)^2 h_{\mu\nu}(x) + y^2 \partial_\mu r(x) \partial_\nu r(x) \right] dx^\mu dx^\nu
\]

(3.20)
The presence of these extra terms in the metric ensures that the boundary terms still give precisely the same contribution.

### 3.3 The RS Limit

From (3.9), we see that
\[
a(z) \sim a_0 e^{-kz} \quad \eta \equiv \Phi(z_1(x)) \to 0
\]  
(3.21)
as \( \alpha \to 0 \), i.e. one recovers a AdS profile in the bulk and the scalar field vanishes. Hence the RS model should be understood as being recoverable at the level of the action by taking \( \alpha = \eta = 0 \):
\[
S_{RS} = \frac{1}{2k\kappa_5^2} \int d^4x \sqrt{-h} \left[ \psi R - \frac{3}{2(1 - \psi)} \nabla \psi^2 - 2k\kappa_5^2 \left( v + (1 - \psi)^2 w \right) \right]
\]  
(3.22)
which is the standard action found in the literature [5]. Note that setting \( \eta = 0 \) is consistent with the equation of motion obtained from the full action in the limit \( \alpha \to 0 \):
\[
\psi \Box \eta + \nabla \psi \cdot \nabla \eta = 0
\]  
(3.23)

If one takes \( \alpha = 0 \) does not set \( \eta \) to zero, the system describes a RS braneworld with a free, massless bulk scalar field added by hand. With a different normalisation for \( \eta \), the resulting action is in agreement with the result of [6].

### 3.4 Scalar Degrees of Freedom as Goldstone Bosons

The moduli fields have an interesting interpretation in terms of symmetry breaking. In the RS case, the radion can be thought of as the Goldstone boson associated with the breaking of translation invariance. For example, if we fix the coordinate gauge invariance (in the bulk direction) by defining the positive-tension brane to be at \( z = 0 \), there is still a continuous family of ‘vacuum states’ parameterised by the arbitrary position of the negative-tension brane at \( z = z_1 \). Choosing a particular value of \( z_1 \) breaks this symmetry, giving rise to a Goldstone mode which can be identified with the radion. In the presence of the scalar field, one is also free to choose the value of the scalar field e.g. \( z = 0, \Phi_1 \); there are then two continuously deformable parameters \( z_1 \) and \( \Phi_1 \) parameterising the ‘vacuum manifold’, giving rise to two massless degrees of freedom when a specific choice of ground state is made. When the tensions are detuned the moduli develop a potential, and hence a mass, since they are no longer Goldstone bosons - there is no longer a continuous family of static solutions due to the violation of the junction conditions.

### 4. Dynamics for RS

In this section we re-derive the well-known result that tension perturbations in RS generate a potential for the radion (see, for example, [10, 19]) since it will be useful for the next Section where we allow \( \alpha \neq 0 \).
4.1 Equations of motion

The equations of motion which follow from the RS effective action for the positive-tension brane (which we shall be working with for now on unless otherwise specified) are

\[
\psi G_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} \psi - h_{\mu\nu} \Box \psi
+ \frac{3}{2(1 - \psi)} \left( \nabla_{\mu} \psi \nabla_{\nu} \psi - \frac{1}{2} h_{\mu\nu} \nabla \psi^2 \right)
- \kappa^2 h_{\mu\nu} \left[ v + (1 - \psi)^2 w \right]
\]

\[
\Box \psi = -\frac{1}{2} \frac{\nabla \psi^2}{1 - \psi} - \frac{4 \kappa^2}{3} (1 - \psi) \left( v + (1 - \psi) w \right)
\]

where now \(\psi\) is given by \(\psi = 1 - \exp(-2kr)\). \(\kappa\), as defined from the action, is given by

\[
\kappa^2 = k \kappa_3^2
\]

which is also the effective gravitational coupling on the brane; this can be identified directly from the projected Einstein equations in the presence of matter \(\Box \psi\), which give

\[
\kappa_1^2 = \frac{\kappa_4^4}{6} \hat{V}
\]

For the RS case, \(\hat{V} = 6k/\kappa_3^2\), this is consistent with \(\Box \psi\), i.e. \(\kappa = \kappa_4\). In the absence of matter, the projected Einstein equations for the positive-tension brane are

\[
G_{\mu\nu} = -E_{\mu\nu} - \kappa^2 v h_{\mu\nu}
\]

allow the identification, to this level of approximation, of

\[
E_{\mu\nu} = h_{\mu\nu} \Box \psi - \frac{1}{\psi} \nabla_{\mu} \nabla_{\nu} \psi
- \frac{3}{2(1 - \psi)} \frac{1}{\psi} \left( \nabla_{\mu} \psi \nabla_{\nu} \psi - \frac{1}{2} h_{\mu\nu} \nabla \psi^2 \right)
+ \kappa^2 h_{\mu\nu} \frac{1 - \psi}{\psi} \left( v + (1 - \psi) w \right)
\]

from \(\Box \psi\). Note that the tracelessness of \(E_{\mu\nu}\) implies the equation of motion \(\Box \psi\) for \(\psi\).

In fact [13], in the absence of matter, the requirement that a single scalar field coupled to gravity yields Einstein equations consistent with the tracelessness of the Weyl tensor determines that, defining the scalar field to be the coefficient of \(R\) as above, the action must be of the form

\[
S \propto \int d^4 x \sqrt{-h} \left[ \psi R - \frac{3}{2(1 - \psi)} \nabla \psi^2 + A + B (1 - \psi)^2 \right]
\]
When the system is fine-tuned \((A = B = 0)\) one can write the resulting Einstein equations as
\[
G_{\mu\nu} = \kappa^2 T_{\mu\nu}^\psi
\]
and comparing with (4.3) one can identify the effective energy-momentum tensor for \(\psi\) with \(-E_{\mu\nu}\). The fact then that \(T_{\mu\nu}^\psi\) must be traceless is sign of the underlying conformal invariance of the RS effective action \([14]\).

### 4.2 Dynamics

The field \(\psi\) has a non-standard kinetic term and, hence, its dynamics are not immediately obvious from (4.2). Defining
\[
\chi = \sqrt{1 - \psi} = e^{-kr} \tag{4.8}
\]
we find
\[
\Box \chi = \frac{4\kappa^2}{3} \chi \left( v + \chi^2 w \right) \tag{4.9}
\]
i.e. the field \(\chi\) moves in the potential
\[
\Phi(\chi) = \frac{4\kappa^2}{3} \left( \frac{1}{2} \chi^2 v + \frac{1}{4} \chi^4 w \right) \tag{4.10}
\]
This has a turning point at
\[
\chi_c^2 = -\frac{v}{w} \leftrightarrow \psi_c = 1 + \frac{v}{w} \tag{4.11}
\]
which is only inside the physical range \(0 \leq \chi \leq 1\) for \(v\) and \(w\) of opposite sign and \(|v| \leq |w|\). This stationary point is unstable for a de-Sitter positive-tension brane (i.e. positive effective cosmological constant, \(v > 0\)), i.e. an inflating brane, and stable only when the brane is collapsing (\(v < 0\)). Depending on the initial conditions the branes will either be driven...
apart to \( \chi = 0 \) \( \Leftrightarrow r \to \infty \) or will collide, \( \chi = 1 \) \( \Leftrightarrow r = 0 \). However, if there is no stationary point in the potential (for example, if \( v > 0 \), when \( v + w > 0 \)) the branes will always be driven apart. The potential (4.10) is plotted in Fig.2 for two different cases, and Figs.3 and 4 show explicitly the above behaviour.

4.3 Brane Collisions

As we have seen it is possible to produce an initial configuration which will lead to a brane collision, which corresponds to \( \psi = 0 \). Whilst this appears in general to be a singular point of the equations of motion, in the special case of an exact FRW induced geometry the collision (in the Brane Frame, as used here) is regular. We take a FRW metric with scalefactor \( a(t) \), curvature \( K = 0, \pm 1 \) and Hubble parameter \( H = \dot{a}/a \). The Bianchi Identity,

\[
\nabla^\mu G_{\mu\nu} = 0 \Rightarrow \nabla^\mu E_{\mu\nu} = 0 \Rightarrow E_{00} \propto a^{-4} \tag{4.12}
\]
gives, from (4.5),

\[
H^2 + K/a^2 = \frac{C}{a^4} + \frac{\kappa^2}{3} v \tag{4.13}
\]

for some constant dark radiation coefficient \( C \). Hence the Hubble parameter does not diverge in the collision; the equations of motion ensure that, despite appearing to be singular at \( \psi = 0 \) from (4.6), \( E_{00} \) actually only behaves as dark radiation. This is why we have chosen to work in the Brane Frame, rather than the Einstein Frame defined by

\[
h_{\mu\nu} = \psi h_{\mu\nu} \Rightarrow ds^2 = \psi \left( dt^2 - a^2 d\mathbf{x}^2 \right) \equiv d\tilde{t}^2 - \tilde{a}^2 d\mathbf{x}^2,
\]

which gives

\[
\dot{H} \equiv \frac{1}{a} \frac{d\dot{a}}{dt} = \frac{H}{\sqrt{\psi}} + \frac{\dot{\psi}}{2\sqrt{\psi}^3} \tag{4.14}
\]

which will, in general, diverge as \( \psi \to 0 \).

It is therefore possible to follow the evolution of the system through a brane collision by using (4.5) (or, rather, its derivative) in place of (4.1). Taking \( K = 0 \) for simplicity, the equations of motion for \( \psi \) and \( H \) are

\[
\dot{H} = -2H^2 + \frac{2\kappa^2}{3} v \tag{4.15}
\]

\[
\ddot{\psi} = -3H\dot{\psi} - \frac{1}{2} \frac{\dot{\psi}^2}{1 - \psi}
+ \frac{4\kappa^2}{3} (1 - \psi) \left( v + (1 - \psi) w \right) \tag{4.16}
\]

which are manifestly regular as \( \psi \to 0 \). Note that, for \( v > 0 \), (4.13) has analytic solution

\[
H = \kappa \sqrt{\frac{v}{3}} \tanh \left( 2\kappa \sqrt{\frac{v}{3}} t + \text{cnst} \right) \tag{4.17}
\]
i.e. the Universe rapidly approaches de Sitter space, $H \to H_s = \kappa \sqrt{v/3}$. The evolution of the Hubble parameter is completely independent of that of the radion and, in particular, is not affected by $\psi = 0$.

The initial value of $H$ can be determined from the 00-component of (4.1), which gives

$$H^2 + \frac{\dot{\psi}}{\psi}H = \frac{\dot{\psi}^2}{4\psi(1 - \psi)} + \frac{\kappa^2}{3\psi} \left(v + (1 - \psi)^2 w\right)$$

(4.18)

which in general has two solutions. Some possible evolutions of $H$ and $\psi$ for different initial conditions are given in Figs.3 and 4 which uses the potential of Fig.2 ($\psi_c = 0.5$, $H_s/\kappa \approx 0.183$). (4.15) can also be obtained as a linear combination of the three equations of motion obtained from (4.1) and (4.2); using these equations, (4.16), (4.18) and

$$2\dot{H}^2 + 3H^2 = H\frac{\dot{\psi}}{\psi} - \frac{\dot{\psi}^2}{4\psi(1 - \psi)}$$

$$- \frac{\kappa^2}{3\psi} \left((1 - 4\psi) v + (1 - \psi)^2 w\right),$$

(4.19)

gives the same results but more care must be taken with the numerical integration to cope with the divergence of some terms as $\psi \to 0$.

![Figure 3: Brane collision in RS: $\psi(t)$ and $H(t)$ are plotted for initial conditions leading to a brane collision ($\psi = 0$). $H(t) \to H_s$ as expected.](image)

4.4 Accuracy of the MSA

As we have seen, the four-dimensional effective theory contains a single modulus, $\psi$, related to the local radius of the orbifold dimension. The effective theory assumed that $\nabla z_i^2/a_i^2 \ll 1$, and we need to check that these conditions are not violated during the evolution of the system. It is actually sufficient to check that $\nabla r^2/\omega^2 \ll 1$. Taking a Gaussian Normal coordinate system where the positive-tension brane lies at $y = 0$, the metric is given by

$$ds^2 = dy^2 + b(y)^2 h_{\mu\nu} dx^\mu dx^\nu$$

(4.20)

where the scalefactor $b(y)$ is normalised to $b(0) = 1$, hence $h_{\mu\nu}$ is indeed the induced metric. The same effective action (3.22) is then obtained provided $\nabla r^2/\bar{b}^2 \ll 1$, where $\bar{b}(x)$ is the
Figure 4: Branes driven apart: \( \psi(t) \to 1 \) asymptotically for suitable choice of initial conditions. Note that \( \epsilon \) ceases to be small as \( \psi \to 1 \)

scalefactor at the second brane, \( y = r(x) \). Clearly we can identify \( \bar{u}(x) = \omega(x) \), giving the above condition on \( \nabla r^2 \). In other words, although only checking \( \nabla r^2 \) to be suitably small ignores the possibility that \( z_1 \) and \( z_2 \) could oscillate wildly but coherently (i.e. that \( \nabla r^2 \) small \( \Rightarrow \nabla z_i^2 \) small), all that is of physical relevance is the separation, \( r(x) \). In terms of the modulus \( \psi \), the condition is

\[
\epsilon \equiv \left| \frac{\nabla r^2}{\omega^2} \right| = \frac{1}{4k^2} \left| \frac{\nabla \psi^2}{(1 - \psi)^3} \right| \ll 1
\]  

(4.21)

As can be seen from Fig. 4, this suggests that the MSA will become inaccurate as the branes move apart, \( \psi \to 1 \); a given fluctuation will cease to be small as lengthscales shrink on the negative-tension brane. As \( \psi \to 0 \) though, we just need that \( \nabla \psi^2 \ll k^2 \); the approximation appears to be valid through the collision. In all the numerical simulations we shall define our units by \( k = \kappa_5^2 = 1 \).

5. Dynamics with Bulk Scalar

Since the scalar field is assumed to take its BPS profile in the background the effect of non-zero \( \alpha \) on the dynamics is just to alter some of the coefficients in the potential. In particular, we are not attempting to produce a potential capable of stabilising the radion as in [18, 19]. The extra degree of freedom, however, gives rise to key differences in the brane collision.

The gravitational coupling of matter on the brane is now given by

\[
\kappa_4^2(\eta) = k\kappa_5^2e^{\alpha \eta} = \frac{k^2}{\beta}e^{\alpha \eta}
\]  

(5.1)

For simplicity we shall consider still the case where the tension perturbations \( v \) and \( w \) are constants; qualitatively identical results are obtained for perturbations with the same functional form as the superpotential, i.e. \( v(\Phi) = \delta v \exp \alpha \Phi \).
5.1 Equations of Motion

The variation of (3.18) with respect to $h_{\mu\nu}$, $\psi$ and $\eta$ gives

$$
\psi G_{\mu\nu} = \nabla_\mu \nabla_\nu \psi - h_{\mu\nu} \Box \psi - \alpha \psi (\nabla_\mu \nabla_\nu \eta - h_{\mu\nu} \Box \eta)
- 2\alpha (\nabla (\mu \psi \nabla_\nu) \eta - h_{\mu\nu} \nabla \psi \cdot \nabla \eta)
+ \alpha^2 \psi (\nabla_\mu \eta \nabla_\nu \eta - h_{\mu\nu} \nabla \eta^2)
+ \frac{3}{2\beta} \left( \frac{1}{1-\psi} \left( \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} h_{\mu\nu} \nabla \psi^2 \right) \right)
+ \frac{\psi}{2} \left( \nabla_\mu \eta \nabla_\nu \eta - \frac{1}{2} h_{\mu\nu} \nabla \eta^2 \right)
- \kappa^2 e^{\alpha \eta} h_{\mu\nu} \left( v + (1 - \psi)^{2/\beta} w \right)
$$

R = - \frac{3}{2\beta} \frac{\nabla \psi^2}{(1 - \psi)^2} + \frac{1}{2} \frac{\nabla \eta^2}{\beta} + \frac{3\alpha}{\beta} \frac{\nabla \eta \cdot \nabla \psi}{(1 - \psi)}
- \frac{3}{2\beta} \frac{\Box \eta}{(1 - \psi)} - \frac{4\kappa^2}{\beta} e^{\alpha \eta} (1 - \psi)^{2/\beta - 1} w

$$
\psi R = \frac{3}{2\beta} \frac{\nabla \psi^2}{(1 - \psi)} - \frac{1}{2} \frac{\psi \nabla \eta^2}{\alpha} + \frac{\nabla \psi \cdot \nabla \eta}{\alpha} + \frac{\psi \Box \eta}{\alpha}
$$

respectively ($\beta = 1 + 3\alpha^2$). Eliminating $R$ from the last two of these equations yields

$$
\Box \psi = \alpha \nabla \eta \cdot \nabla \psi - \frac{\nabla \psi^2}{2(1 - \psi)}
- \frac{4\kappa^2}{3} e^{\alpha \eta} \left[ (1 - \psi) v + (1 - \psi)^{2/\beta} w \right]
$$

$$
\Box \eta = \alpha \nabla \eta^2 - \frac{\nabla \eta \cdot \nabla \psi}{\psi} - \frac{3\alpha}{2\beta} \frac{\nabla \psi^2}{\psi (1 - \psi)}
+ \frac{4\alpha}{\beta} \kappa^2 e^{\alpha \eta} v
$$

consistent with (4.1) and (4.2). The behaviour of $\psi$ is qualitatively similar to the RS case, with the critical value of $\psi$ modified to

$$
\psi_c = 1 - \left( \frac{-v}{w} \right)^{\frac{1+3\alpha^2}{1+3\alpha^2}}
$$

which is independent of $\eta$. Although there is no corresponding stationary value of $\eta$ unless $\alpha = 0$, (5.3) ensures $\psi = \psi_c$, $\nabla \psi = 0 \Rightarrow \Box \psi = 0$. Note that (5.3) is regular as $\psi \to 0$.

The influence of the bulk scalar field is therefore not great on the effective potential in which the radion moves; however, we shall see that it has a more important role to play during a collision.
5.2 Brane Collisions and Cosmological Evolution

Taking again a flat FRW metric, the Einstein equations (5.2) give the Hubble constraint

\[ H^2 = -\frac{\dot{H}}{\psi} + \alpha H\dot{\eta} + \frac{1}{4\beta \psi (1-\psi)} \dot{\psi}^2 + \frac{\eta^2}{12} \]

\[ + \kappa^2 \frac{e^{\alpha \eta}}{3\psi} \left( v + (1-\psi)^{2/\beta} w \right) \]

and the evolution equation

\[ 2\dot{H} = -3H^2 - \frac{\dot{\psi}}{\psi} - 2H \frac{\dot{\psi}}{\psi} + \alpha \dot{\eta} + 2\alpha H\dot{\eta} \]

\[ - \left( \alpha^2 + \frac{1}{4} \right) \ddot{\eta}^2 - \frac{3}{4\beta \psi (1-\psi)} \dot{\psi}^2 + 2\alpha \frac{\dot{\psi}\dot{\eta}}{\psi} \]

\[ + \kappa^2 \frac{e^{\alpha \eta}}{\psi} \left( v + (1-\psi)^{2/\beta} w \right) \]

\[ = -3H^2 + H \frac{\dot{\psi}}{\psi} - \alpha H\dot{\eta} - \frac{\eta^2}{4} - \frac{1}{4\beta \psi (1-\psi)} \dot{\psi}^2 \]

\[ - \frac{\kappa^2}{3\psi} e^{\alpha \eta} \left[ \left( 1 - \frac{4\psi}{\beta} \right) v + (1-\psi)^{2/\beta} w \right] \]

(5.7)

using (5.3). Finally, substituting in from (5.6) gives the analogue of (4.15):

\[ \dot{H} = -2H^2 - \frac{\dot{\eta}^2}{12} + \frac{2\kappa^2}{3\beta} e^{\alpha \eta} v \]

(5.8)

So again we have manifestly finite equations of motion for \( H \) and \( \psi \) as \( \psi \to 0 \). However, the equation of motion (5.4) is not free of divergences; if \( |\eta| \to \infty \) then, from (5.8), there will be a corresponding divergence in \( H \). Such behaviour is shown in Fig.5.

For the \( \alpha = 0 \) case, i.e. RS with a free, massless scalar field, (5.4) gives

\[ \nabla_\mu (\psi \nabla^\mu \eta) = 0 \Rightarrow \dot{\eta} \propto \frac{1}{a^3 \psi} \]

(5.9)

Hence \( \eta \) will tend to either a constant (if \( \dot{\eta} = 0 \), since the field \( \eta \) then has no time dependence and decouples) or to \( \pm \infty \) as \( \psi \to 0 \) depending on the initial sign of \( \dot{\eta} \).

In the general case it is difficult to make much progress analytically. The numerical results for four sets of initial conditions and values of \( v, w \) and \( \alpha \) are shown in Fig.5, appearing to show that \( \eta \) diverges at the collision with a sign opposite to that of \( \alpha \) (a conclusion supported by thorough numerical investigation, of which Fig.5 is only a sample). This implies that the tension on the branes, given by (3.7), vanishes as the branes collide.

The conclusion is then that, although the scalar field does not have much impact on the dynamics of the radion itself, it will, in general, diverge during a collision (one could envisage a situation where it would not, for example, by setting \( \alpha = \dot{\eta} = 0 \), in which case \( \eta \) would just remain constant and the collision would be regular). This is similar to the evolution of perturbations, which are found to diverge logarithmically in the RS case [10], a feature common to scalar-tensor theories when the gravitational constant changes sign.
When the initial conditions are such that the branes do not collide, \( \psi \) is rapidly driven to 1. \( \eta \) is then approximately governed (for \( \alpha \neq 0 \)) by

\[
\ddot{\eta} + 3H \dot{\eta} \sim \frac{4\alpha}{\beta} \kappa^2 e^{\alpha \eta} \psi
\]

(5.10)

The Hubble constant is slowly driven to zero whilst \( |\eta| \) grows approximately logarithmically (a full numerical solution is given in Fig.6 for \( \alpha = -0.2 \)). There is no analytic solution to the system even if one approximates \( \psi = 1 \); one can, however, predict the late time behaviour

\[
\eta - \eta_0 \sim -\frac{2}{\alpha} \log |1 + t - t_0|
\]

(5.11)

with the timescale for the transition to the logarithmic behaviour increasing without limit as \( \alpha \to 0 \). This is clearly demonstrated in Fig. 7.

### 5.3 A Caveat

As before, the self-consistency of the effective theory can be checked by monitoring the size of the quantities

\[
\epsilon_1 \equiv \left| \frac{\nabla z_1^2}{a_1^2} \right| = \frac{e^{(1/3\alpha - 2\alpha)\eta}}{36k^2\alpha^2} \left| \nabla \eta^2 \right|
\]

\[
\epsilon_2 \equiv \left| \frac{\nabla z_2^2}{a_2^2} \right| = \frac{e^{(1/3\alpha - 2\alpha)\eta}}{36k^2\alpha^2} (1 - \psi)^{-3/(1+3\alpha^2)} \times \left| (1 - \eta) \nabla \eta + 3\alpha \nabla \psi \right|^2
\]

\[\sim \epsilon_1 \quad \text{as } \psi \to 0\]

However, due to the divergence of \( \eta \), these are not necessarily small during the collision, as can be seen in Fig.7. The reason for this divergence, and the reason for its absence in the RS case, can be understood as follows. In our coordinate system in which both the bulk and the scalar field are static, one can interpret a divergence in \( \eta \) as a divergence, or a tending to the singularity \( z^* \), depending on the sign, of the coordinate \( z \) of the positive-tension brane. In the RS case, the two coordinates diverge in the collision. However, the effective theory can be expressed solely in terms of the difference between the two values, the radion, which remains finite. In the more general case, the divergence of the second modulus cannot be removed in such a way. The breakdown of the effective theory can also be understood by rederiving the effective action in a Gaussian-Normal coordinate system about the positive-tension brane, analogously to the procedure used in [6]. One performs a perturbative expansion about a static ground state [12], requiring transverse derivatives of the scalar field to be small. However, this procedure will break down when this condition is no longer met, in particular when \( \eta \) diverges during the collision.

Whilst then we cannot trust the MSA right at the moment of collision, we have a firm numerical tool for identifying dynamically the region in which it is, indeed, valid. We can use the MSA to identify the approach to a collision and, although we have not ruled out the possibility of higher-order effects repelling the branes again, it seems likely that a collision will then take place, accompanied by a divergence in the bulk scalar field.
5.4 Einstein vs. Brane Frame

In this paper we have chosen to work in the Brane Frame, defined by the induced metric on the positive-tension brane, since this has the best regularity properties at the brane collision. For late-time evolution the Einstein Frame is most often used in the literature; from the action (3.18), this can be read off as

\[ g_{\mu\nu} = e^{-\alpha \eta \psi} h_{\mu\nu} \]  

(5.12)

As \( \psi \to 0 \) the two metrics will give wildly different physics, but in the RS case, with or without the scalar field, the two metrics become identical as \( \psi \to 1 \), so the distinction between the two frames becomes unimportant. However, for \( \alpha \neq 0 \), we see from (5.11) that

\[ e^{-\alpha \eta \psi} \propto \frac{1}{t^2} \text{ as } \psi \to 1 \]  

(5.13)

so that, even for late times, there is still a distinction between the two frames. The cosmological evolution of the moduli fields in the Einstein Frame are investigated in detail in [4].

6. Conclusions

In this paper, we studied the evolution of the moduli fields of a particular class of SUGRA-inspired braneworld models, containing the RS model as a special case, with a scalar field in the bulk. In contrast to previous work we worked in the frame of the induced metric of one of the branes in order to study collisions, which are always singular in the Einstein Frame. After discussing the significance of ‘supersymmetric’ pairings of brane interaction and bulk scalar field potentials from a fully five-dimensional viewpoint, we derived an effective theory to discuss the low-energy motions of the system about a BPS ground state configuration with small SUSY-breaking potentials. It was found that, for this specific class of braneworlds, the presence of a bulk scalar field has little qualitative effect on the dynamics of the radion itself, i.e. the motion of the branes through the bulk. However, even in the absence of any perturbations around an FRW induced geometry, its contribution to the brane Friedmann equation causes the Hubble parameter to diverge at the collision even in the Brane Frame. By monitoring the size of certain functions of the field variables which are required to be small in the derivation of the effective action, we can identify the regions in which the MSA itself is valid. Whilst collisions are well-described by it in the RS case, the presence of the bulk scalar field can causes it to break down just before the collision. Although the RS model can be used to construct toy models whose regularity at the collision gives a link between pre- and post-Big Bang phases, where the Big Bang is identified with the moment of collision, this work implies that any attempt to generalise the model to include bulk scalar fields encounters singularities. Recent developments on modelling brane collision in the context of M-Theory [15] suggest that, with more care, these divergences might turn out to be removable; purely within the context of general relativity, however, they appear to be unavoidable.
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7. Appendix

Firstly, the Ricci scalar decomposes as

\[ R(g) = -8 \frac{a''}{a} - 12 \frac{a'^2}{a^2} + \frac{\tilde{R}}{a^2} \]

where \( \tilde{R} \) is the Ricci scalar of the metric \( \tilde{g}_{\mu\nu} \). Writing \( S = \int d^4x \sqrt{-g} \mathcal{L} \), we find

\[ \mathcal{L}_g \supset \frac{\tilde{R}}{\kappa_5^2} \int_{z_1(x)}^{z_2(x)} dz \ a^2(z) \]

\[ = \frac{\xi_1^{1+1/3\alpha^2} - \xi_2^{1+1/3\alpha^2}}{2k\kappa_5^2 (1 + 3\alpha^2)} \tilde{R} \]

(7.1)

where \( \xi_i = \xi(z = z_i(x)) \). The scalar field Lagrangian contains no \( x \)-derivatives (as shown in [16] there is no additional perturbation around the BPS field configuration that needs to be taken into account) and hence makes no overall contribution.

The moduli kinetic terms come from the boundary action. For the \( i \)-brane the induced line element and outward-pointing normal are given by

\[ ds_i^2 = (a_i^2 g_{\mu\nu} + \partial_\mu z_i \partial_\nu z_i) \ dx^\mu dx^\nu \]

\[ \equiv h^{(i)}_{\mu\nu} dx^\mu dx^\nu \]

\[ n_a^{(i)} dx^a = (-1)^i \frac{1}{\sqrt{1 + \hat{\nabla} z_i^2 / a_i^2}} (dz - \partial_\mu z_i dx^\mu) \]

(7.2)

giving

\[ \sqrt{-h^{(i)}} \approx a_i^4 \sqrt{-g} \left( 1 + \frac{1}{2a_i^2 \hat{\nabla} z_i^2} \right) \]

(7.3)

where \( \hat{\nabla} \) is the covariant derivative of \( \tilde{g}_{\mu\nu} \). Here, and from now on, we are assuming that \( \nabla z_i^2 \ll a_i^2 \), i.e. that the brane fluctuation lengthscale is much larger than that of the bulk curvature. Then, from (3.12),

\[ \mathcal{L}_B \supset -a_1^4 v - a_2^4 w + \frac{3k}{\kappa_5^2} \left( \frac{1}{\xi_2} a_2^2 \hat{\nabla} z_2^2 - \frac{1}{\xi_1} a_1^2 \hat{\nabla} z_1^2 \right) \]

(7.4)

where we have from (3.7) and (3.9) that

\[ \hat{V}(\Phi) = \frac{6k}{\kappa_5^2} \frac{1}{\xi} \]
and we have assumed that the SUSY-breaking potentials \(v\) and \(w\) are also small. Finally, we need to compute the Gibbons-Hawking boundary term involving the extrinsic curvatures of the two branes. Taking outward-pointing normals again,

\[
K_i = \frac{1}{\sqrt{-g}} \partial_a \left( \sqrt{-g} n^a_{(i)} \right) \bigg|_{z=z_i(x)} \\
\approx (-1)^i \frac{1}{a^4} \left( a^4 \left( 1 - \frac{1}{2} \tilde{\nabla} z^2 \right) \right) \bigg|_{z=z_i(x)} \\
- (-1)^i \frac{1}{\sqrt{-g} a_4^4} \partial_\mu \left( a^2 \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\nu z_i \right) \bigg|_{z=z_i(x)} \\
= - (-1)^i \frac{4k}{\xi_i} \\
+ \frac{(-1)^i}{a^4} \left[ k \frac{\tilde{\nabla} z^2}{\xi_1} - \frac{1}{a^4} \left\{ \tilde{\nabla}_\mu \left( a^2 \tilde{\nabla}^\mu z_i \right) \right\} \bigg|_{z=z_i(x)} \right]
\]

From (7.3) and (3.12) it then follows that

\[
\mathcal{L}_{GH} \supset 2 \frac{\kappa_5^2}{\kappa_5^2} \left[ \frac{a^2}{\xi_1} \frac{k}{\xi_1} \tilde{\nabla} z^2_1 + \left\{ \tilde{\nabla}_\mu \left( a^2 \tilde{\nabla}^\mu z_1 \right) \right\} \bigg|_{z_1(x)} - (1 \leftrightarrow 2) \right] \\
= 2 \frac{\kappa_5^2}{\kappa_5^2} \left[ \frac{a^2}{\xi_1} \frac{k}{\xi_1} \tilde{\nabla} z^2_1 - 2a_1 \tilde{\nabla}^\mu z_1 \tilde{\nabla}_\mu a_1 - (1 \leftrightarrow 2) \right] \\
= 6k \frac{\kappa_5^2}{\kappa_5^2} \left( \frac{a^2}{\xi_1} \frac{\tilde{\nabla} z^2_1}{\xi_1} - \frac{a^2}{\xi_2} \frac{\tilde{\nabla} z^2_2}{\xi_2} \right)
\]

(7.5)

where integration by parts has been used in the second line, and using (3.9). It is straightforward to show that the omitted terms above some to zero as expected, leaving an effective action

\[
S = \frac{1}{2k\kappa_5^2} \int d^4x \sqrt{-g} \tilde{\Omega}^2 \tilde{R} + \frac{1}{6a^4} \left[ \xi_1^{1/3a^2} - \xi_2^{1/3a^2} \right] \\
- \tilde{\Omega}^2 = \frac{1}{1+3a^2} \left[ \xi_1^{1/3a^2} - \xi_2^{1/3a^2} \right] \\
\tilde{V} = \xi_1^{2/3a^2} v + \xi_2^{2/3a^2} w
\]

(7.6)

As expected, the potential for the moduli fields vanishes for \(v = w = 0\); when supersymmetry is unbroken (i.e. (2.10) holds), the moduli fields are massless. Next we write the action in terms of the induced metric on the positive-tension brane, \(h^{(1)}_{\mu\nu} \equiv h_{\mu\nu}\). To this order it is sufficient to use the approximation

\[
h_{\mu\nu} = a^2_1 \tilde{g}_{\mu\nu} + ...
\]

(7.7)

Using

\[
\tilde{R} = a^2_1 \left( R(h) - 6a_1 \Box a_1^{-1} + ... \right)
\]

(7.8)
we finally obtain, after integration by parts, the action as given by (3.14).

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Figure 5: Divergence of $\eta$ as $\psi \to 0$: For four different sets of parameters and initial conditions, $\eta$ can be seen to diverge at the collision with opposite sign to $\alpha$. This divergence causes a corresponding divergence in the parameter $\epsilon_1$, signalling a breakdown of the effective theory description.
Figure 6: Late time cosmological evolution: For suitable initial conditions $\psi$ is rapidly driven to unity and the branes move far apart. The scalar field grows logarithmically, the Hubble parameter is slowly driven to zero, and the Universe expands rapidly. Here $\alpha = -0.2$.

Figure 7: Late-time $\eta(t)$: The evolution of $-\alpha \eta/2$ as $\psi \to 1$ is plotted, with $\alpha = 0.4, -0.2, 0.1, -0.05$ from top to bottom. The dashed fiducial line with unit gradient demonstrates the late-time behaviour $\eta \sim -\frac{2}{\alpha} \log t$. As $\alpha$ decreases it takes longer for the system to settle into this behaviour.