VORTICES ON SURFACES WITH CYLINDRICAL ENDS

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Abstract. We consider Riemann surfaces obtained from nodal curves with infinite cylinders in the place of nodal and marked points, and study the space of finite energy vortices defined on these surfaces. To compactify the space of vortices, we need to consider stable vortices - these incorporate breaking of cylinders and sphere bubbling in the fibers. The space of stable vortices modulo gauge transformations is compact and Hausdorff under the Gromov topology. Moreover, it is homeomorphic to the moduli space of quasimaps defined by Ciocan-Fontanine, Kim and Maulik in [CKM11].

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The moduli space of stable quasimaps is described by Ciocan-Fontanine, Kim and Maulik in [CKM11]. This is a different way of compactifying maps from genus g n-pointed curves C to a GIT quotient $X//G$, compared to the stable map compactification $\overline{M}_{g,n}(X//G)$. Here $X//G$ is a GIT quotient obtained by the linearized action of a complex reductive group G on an affine variety X, we assume $X//G$ is non-singular. The curve C is allowed to vary and degenerate to a nodal curve, and in addition the map $C \to X/G$ is allowed to acquire isolated base points (i.e. map to $X\backslash X^{ss}$). For separatedness of the moduli space, it is required that base points don’t coincide with marked points or nodal points of C. This paper provides a symplectic version of stable quasimaps.

Suppose $K \subset G$ is a maximal compact subgroup. In symplectic geometry, the maps $C \to X/G$ are analogous to $K$-vortices from C to X. When X is as above, the action of K on X is Hamiltonian and has a moment map $\Phi : X \to \mathfrak{k}^* \simeq \mathfrak{k}$. A vortex
\((A, u)\) consists of a connection \(A\) on a principal \(K\)-bundle \(P \to C\) and a holomorphic section \(u : C \to P \times_K X\) with respect to \(\overline{\Omega}_A\) that satisfies the equation

\[ F_A + \Phi(u) \text{vol}_C = 0. \]

This equation requires a choice of area form on \(C\). To make sure that base points are away from special points on \(C\), we ‘blow up’ the area form at the special points. Punctured neighborhoods of these points will be isometric to cylinders. This ensures that the limit of \(u\), as we approach the special points, lies in \(\Phi^{-1}(0)\) and hence is semistable. We define a smooth family of metrics, called the \textit{neck-stretching metrics} on stable nodal genus \(g\), \(n\)-pointed curves such that the metric blows up at special points in the above-mentioned way, so any such curve now corresponds to a Riemann surface with cylindrical ends. The space of vortices representing a given equivariant class in \(H^K_2(X)\) defined on stable nodal curves with neck-stretching metric is not compact. To compactify it, we allow breaking of cylinders (as in Floer theory) and sphere bubbles in fibers as in the work of Ott ([Ott09]), and the new objects are called stable vortices. The space of stable vortices modulo \(K\)-gauge transformations is compact and Hausdorff under the Gromov topology.

Suppose \(\overline{M}_{g,n}\) is the coarse moduli space of stable nodal curves of genus \(g\) with \(n\) marked points. We assume \(n \geq 1\), and for stability \(n + 2g - 3 \geq 0\). If a vortex \((A, u)\) has finite energy, a removal of singularity result applies at the cylindrical ends. This ensures that \(u\) represents a class in \(H^K_2(X)\). Let \(MV^K_{g,n}(X, \beta)\) be the space vortices \((A, u)\) on stable genus \(g\), \(n\)-pointed curves equipped with the neck-stretching metric, such that \([u] = \beta \in H^K_2(X)\) modulo the group of (unitary) gauge transformations. Removal of singularity at the cylindrical ends ensures that the evaluation maps

\[ ev_i : MV^K_{g,n}(X) \to X/\!/G \quad (A, u) \mapsto \lim_{z \to z_i} Ku(z_i) \]

are well-defined for marked points \(z_1, \ldots, z_n\). By the definition of \(MV^K_{g,n}(X)\), there is a forgetful map

\[ st : MV^K_{g,n}(X) \to \overline{M}_{g,n}. \]

In the compactification of \(MV^K_{g,n}(X)\), the domain may not be stable, \(st\) is defined as the stabilization of the domain, achieved by contracting unstable components. Our first result is

**Theorem 0.1.** Suppose \((X, \omega, K, \Phi)\) is a Kähler Hamiltonian \(K\)-manifold that is either compact or equivariantly convex at \(\infty\). The compactification of \(MV^K_{g,n}(X, \beta)\), called \(\overline{MV}^K_{g,n}(X, \beta)\) is a compact Hausdorff space under the Gromov topology. The forgetful map \(st\) and the evaluation maps \(ev_i\) are well-defined and continuous on \(MV^K_{g,n}(X, \beta)\).

Compactification of the space of symplectic vortices has been studied by many authors - [MiRT09], [Zil12] and [Ott09]. Ziltener ([Zil12]) looks at compactifying vortices on the complex plane \(\mathbb{C}\). Besides sphere bubbling in the fibers, there is bubbling at infinity that produces sphere bubbles in \(X/\!/G\) and vortices on \(\mathbb{C}\) attached to these bubbles. The situation for vortices on cylindrical ends is simpler in comparison. Mundet-Tian ([MiRT09]) have constructed a compactification for
vortices with varying domain curve, equipped with a finite volume metric. In that case, when the domain curve degenerates to a nodal curve, the map $u$ can degenerate to a chain of gradient flow lines of the moment map $\Phi$. When the domain curve degenerates to a nodal curve, the map $u$ can degenerate to a chain of gradient flow lines of the moment map $\Phi$ ($K = S^1$, so $i\Phi$ maps to $\mathbb{R}$).

Allowing infinite volume at nodal points helps us avoid these structures.

Let $Q\text{map}_{g,n}(X//G, \beta)$ be the space of stable quasimaps whose domains are stable $n$-pointed curves of genus $g$ and which represent the homology class $\beta \in H^2_{K}(X)$. In [CKM11], $X$ is required to be an affine variety for the proof of compactness of $Q\text{map}_{g,n}(X//G, \beta)$. For any polarization $L \to X$, $X$ can be realized as a Kähler Hamiltonian $K$-manifold that is equivariantly convex at infinity.

**Theorem 0.2.** Let $g$, $n$, $K$, $G$, $X$ and $\beta \in H^2_{K}(X)$. There is a homeomorphism

$$\Psi: Q\text{map}_{g,n}(X//G, \beta) \to \overline{MV}_{g,n}(X, \beta)$$

that commutes with the evaluation maps $ev_i$ and the forgetful map to $\overline{M}_{g,n}(X)$.

The proof of this theorem doesn’t require that $X$ be affine. It only requires that $Q\text{map}_{g,n}(X//G, \beta)$ is compact. Therefore, we expect $Q\text{map}_{g,n}(X//G, \beta)$ to be compact in more general situations, provided $X$ is aspherical. This is to rule out sphere bubbling in the fibers. In order to remove the asphericity assumption, the definition of quasimaps will have to be broadened to include such bubbles.

In the proof of theorem 0.2, the bijection $\Psi$ is constructed using techniques in [VW13]. This is essentially a Hitchin-Kobayashi correspondence. The notion of stability required by this correspondence - of the point at infinity (marked points and nodal points in the case of $Q\text{map}$) mapping to $X^{ss}$ - is part of the definition of quasimaps. The proof of continuity of $\Psi$ relies on the convexity of the moment map - but there are analytical difficulties arising because of the non-compact domains. To overcome these, we crucially rely on a stronger version of removal of singularity at infinity for vortices. The original such result proved by Zil tener in his thesis ([Zil06]) for the affine case gives only $L^p$ control on the decay of the connection. But in the cylindrical case, we are able to show a similar result (proposition 2.8) giving $W^{1,p}$ control.

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1. **Description of neck stretching metrics**

1.1. **Stable curves, gluing.** A compact complex nodal curve $C$ is obtained from a collection of smooth compact curves $(C_1, \ldots, C_k)$ by identifying a collection of distinct nodal points

$$w_i^+ \sim w_i^-, \quad w_i^\pm \in C_{\alpha^\pm(i)}, \quad i = 1, \ldots, m.$$  

Points on the curves $C_i$ that are not nodal points are called smooth points. A nodal curve with marked points comes with a collection of $n$ distinct smooth points $\{z_1, \ldots, z_n\}$. A marked nodal curve is stable if it has finite automorphism group, i.e. every genus 0 component has at least 3 special points (marked or nodal point).
and a genus 1 component has at least 1 special point. The \emph{genus} of a nodal curve is the genus of a “smoothing” of the curve. For example, the genus of the curve in the figure is 1.

A \emph{family of nodal curves} over a scheme $S$ is a proper flat morphism $\pi : C \to S$ such that each fiber $C_s$, $s \in S$ is a nodal curve. One can ask if there is a space $M$ and a family $U \to M$ such that for any family $C \to S$, there is a unique map $\phi : S \to M$ such that $C$ is isomorphic to the pullback $\phi^*U$. For nodal curves (of genus $g$ and $n$ marked points), such a family doesn’t exist. This is because there are curves with a non-trivial automorphism group. For, if such a family existed, the fiber over $[C] \in M$ would be $C/\text{Aut}(C)$. For stable curves, the automorphism group is finite. In that case, there is a \emph{coarse moduli space} $\overline{M}_{g,n}$ - this means for any $C \to S$, there is a unique map $\phi : S \to \overline{M}_{g,n}$ such that $C$ is isomorphic to the pullback $\phi^*U$. $\overline{M}_{g,n}$ has the structure of a complex orbifold, i.e. locally it is homeomorphic to a neighborhood of the quotient of $\mathbb{C}^n$ by a finite group.

The combinatorial type of a marked nodal curve $(C, z)$ is a \emph{modular graph} $\Gamma = (\text{Vert}(\Gamma), \text{Edge}(\Gamma), \text{Edge}_\infty(\Gamma))$ and a genus function $g : \text{Vert}(\Gamma) \to \mathbb{Z}_{\geq 0}$. The vertices are the components of $C$, the edges in $\text{Edge}(\Gamma)$ are nodes in $C$ and the infinite edges $\text{Edge}_\infty(\Gamma)$ correspond to markings. An edge $w \in \text{Edge}(\Gamma)$ has two end points $w^\pm$, so it is incident on two vertices $\iota(w^\pm)$. It is possible that these two vertices are the same - see for example figure 1. For the markings, there is an ordering of the set $\text{Edge}_\infty(\Gamma)$ by a bijection $\{1, \ldots, n\} \to \text{Edge}(\Gamma)$ and each edge $z \in \text{Edge}_\infty(\Gamma)$ is incident on only one vertex $\iota(z)$.

A modular graph is \emph{stable} if any curve corresponding to it is stable. Also, for a graph $\Gamma$, there is a \emph{stabilization} $\text{st}(\Gamma)$ which is the combinatorial type of the stabilization of the curve $C_\Gamma$. A \emph{morphism} $f : \Gamma_1 \to \Gamma_2$ of modular graphs corresponds to a sequence of moves - each either contracts an edge or removes an edge that is a loop. In terms of the related nodal curves, each move smooths out a nodal singularity. Based on the combinatorial type $\Gamma(C, z)$, we can define a stratification of $\overline{M}_{g,n}$. If there is a morphism $\Gamma(C) \to \Gamma(C')$, then $[C'] \preceq [C]$. The lowest stratum consists of points representing smooth curves, and it is an open set in $\overline{M}_{g,n}$. Denote by $M^\Gamma_{g,n} \subset \overline{M}_{g,n}$ the subspace parametrizing curves of combinatorial type $\Gamma$. Then,

$$\overline{M}_{g,n} \setminus M^\Gamma_{g,n} \subseteq \bigcup_{\Gamma \prec \Gamma'} M^{\Gamma'}_{g,n}.$$

Let $C$ be a nodal curve of combinatorial type $\Gamma$. We next describe how neighborhoods of $[C]$ in $M^\Gamma_{g,n}$ and $\overline{M}_{g,n}$ are related to each other - this is done through deformation theory. Suppose $C$ is a compact curve. Then a \emph{deformation of $C$ by a}
Figure 1. A nodal curve and its combinatorial type. Both vertices have genus 0.

A pointed scheme \((S, 0), 0 \in S\), is a proper flat morphism \(\phi : C_\Sigma \to S\) plus an isomorphism between C and the central fiber \(\phi^{-1}(0)\). A deformation \(C_\Sigma\) of \(C\) is universal if given any other deformation \(C_{\Sigma'} \to (S', 0)\), for any sufficiently small neighborhood \(U\) of 0, there is a unique morphism \(\iota : U \to S\) such that \(C_\Sigma\) is isomorphic to the fibered product \(C_\Sigma \times_S U\). Stable curves possess universal deformations. Suppose \(\pi : C_\Sigma \to (S, 0)\) is a universal deformation of the curve \(C\). If \(C\) has a non-trivial stabilizer \(G\), then for a sufficiently small neighborhood \(V\) of 0, the action of \(G\) extends to compatible actions on \(V\) and \(C_V\) (theorem 6.5, chapter 11, [ACG11]). If \(V\) is small enough and \(G\)-invariant, then there is an injection \(V/G \hookrightarrow M_{g,n}\), and thus deformations of curves provide holomorphic orbifold charts for \(M_{g,n}\). The space of infinitesimal universal deformations of \(C\), denoted by \(\text{Def}(C)\), is the tangent space \(T_0 S\). If \(C\) is of combinatorial type \(\Gamma\), \(\text{Def}_{\Gamma}(C)\) is the tangent space \(T_0 S_{\Gamma}\). For a nodal curve of type \(\Gamma\), given a universal deformation \(S_{\Gamma}\) of type \(\Gamma\), we can construct a universal deformation \(S_{\Gamma}\) using a gluing procedure. By Proposition 3.32 in [HM98],

\[
\text{Def}(C)/\text{Def}_{\Gamma}(C) \simeq \oplus_{w \in \text{Edge}(\Sigma)} T_{w^+} \Sigma_{\iota(w^+)} \oplus T_{w^-} \Sigma_{\iota(w^-)}.
\]

Lemma 1.1. Let \(\Sigma\) be as above and let \(g\) be a metric on \(\Sigma\) that is flat in the neighborhoods of nodal points \(w^\pm, w \in \text{Edge}(\Sigma)\). To every small \(\delta \in \oplus_{w \in \text{Edge}(\Sigma)} T_{w^+} \Sigma_{\iota(w^+)} \oplus T_{w^-} \Sigma_{\iota(w^-)}\), we can associate a curve \(\Sigma^\delta\).

Proof. Let \(\delta = (\delta_w)_{w \in \text{Edge}(\Sigma)}\), \(\delta_w \in T_{w^+} \Sigma_{\iota(w^+)} \oplus T_{w^-} \Sigma_{\iota(w^-)}\) gives a map

\[
\Phi_{\delta_w} : T_{w^+} \Sigma_{\iota(w^+)} \setminus \{0\} \to T_{w^-} \Sigma_{\iota(w^-)} \setminus \{0\}, \quad x \otimes \Phi_{\delta_w}(x) = \delta_w.
\]

Define the glued surface

\[
\Sigma^\delta := \Sigma - \{w^\pm : w \in \text{Edge}(\Sigma)\}/(\exp(z_w) \sim \exp(\Phi_{\delta_w}(z_w))),
\]

where \(z_w \in T_{w^+} \Sigma_{\iota(w^+)}\) and the exponential map is with respect to the metric \(g\). If \(\delta_w = 0\), then the node \(w\) stays in place in \(\Sigma^\delta\). \(\square\)

This gluing process can be done in families also. Suppose, we have a family of metrics on curves parametrized by \(S_{\Gamma}\) that are flat in the neighborhood of nodes. Let \(I_{\Gamma} \to S_{\Gamma}\) be a vector bundle whose fiber over \(s \in S_{\Gamma}\) is

\[
I_{\Gamma,s} := \oplus_{w \in \text{Edge}(\Gamma)} T_{w^+} \Sigma_{\iota(w^+)} \oplus T_{w^-} \Sigma_{\iota(w^-)}.
\]
In a neighborhood of the zero section of $I_\Gamma$, we can associate a glued curve to every point. Theorem 3.17, chapter 11 in [ACG11] says that this is a universal deformation of $\Sigma$.

1.2. Riemann surfaces with cylindrical ends.

**Definition 1.2.** $\Sigma$ is a Riemann surface with cylindrical ends if $C := \Sigma$ is a nodal curve with marked points $z_1, \ldots, z_n$, nodal points $w_1, \ldots, w_k$ and $\Sigma = C \setminus \{z_1, \ldots, z_n, w_1, \ldots, w_k\}$ with the following property: for any $z = z_i, w_i^\pm$ there is a neighborhood of $z$ in the normalization of $C$, $N(z) \subset \tilde{C}$ and an isometry

$$\rho_z : N(z) \setminus \{z\} \rightarrow \{r + i\theta : r \geq 0, \theta \in \mathbb{R}/\mathbb{Z}\}.$$ 

The metric on the right hand side $dr^2 + d\theta^2$. We further require that the volume of $\Sigma \setminus (\bigcup_{z=z_i,w^\pm} N(z))$ is finite and any $f \in \text{Aut}(C)$ preserves the metric on $\Sigma$.

Given $g, n$ such that $n + 2g - 3 \geq 0$, the goal of this section is to show that to any stable $n$-pointed curve of genus $g$, we can associate a Riemann surface with cylindrical ends, and that the metric varies smoothly as we vary $C$. In other words, if $\mathcal{C} \to S$ is a family of such curves, we need to put a metric on the fibers of $\mathcal{C} \setminus (z_i(S) \cup w_i(S)) \to S$ that varies smoothly with $s \in S$. We call such a family of metrics neck-stretching metrics, because they stretch the ‘neck region’ in the glued surfaces described in lemma 1.1. That, such a family exists, is the content of the next proposition.

**Proposition 1.3.** There exists a family of neck-stretching metrics on stable curves parametrized by $\overline{\mathcal{M}}_{g,n}$.

**Proof.** We inductively construct a family of neck-stretching metrics, starting our description from the maximal strata. For a curve $C$, $[C]$ is in a maximal stratum iff all the components of $C$ have genus 0 and three special points. Consider the normalization of $C$ (i.e. Remove the identifications $w_i^+ \sim w_i^-$). Around each special point $p$, take a small neighborhood $N(p)$. Embed $N(p) \setminus \{p\}$ holomorphically into the cylinder $\{z = r + i\theta : r > 0, \theta \in \mathbb{R}/\mathbb{Z}\}$. The standard Euclidean metric on the cylinder is pulled back to give a metric on $N(p) \setminus \{p\}$. The metrics on these neighborhoods can be extended smoothly to the rest of the sphere. $\text{Aut}(C)$ is trivial for the maximal stratum.

Next, we aim to define a family of neck-stretching metrics on curves of type $\Gamma$ that extends a family of metrics $\mathcal{G}$ defined on curves of type $\Gamma'$ for all $\Gamma' \succ \Gamma$. We first extend $g$ to curves parametrized by a neighborhood of $\cup_{\Gamma' \succ \Gamma} \overline{\mathcal{M}}^\Gamma_{g,n}$ in $\overline{\mathcal{M}}_{g,n}$. Suppose $\Sigma$ is of type $\Gamma'$ and $w$ is a node in $\Sigma$, then under the neck-stretching metric, there is an isometry from a neighborhood of $w^\pm$ in $\Sigma_{\ell(w^\pm)}$ to the semi-infinite cylinder

$$\rho_{w^\pm} : N(w^\pm) \setminus \{w^\pm\} \rightarrow \{z = r + i\theta : \pm s > 0, t \in \mathbb{R}/\mathbb{Z}\}.$$ 

Let $f^+, f^-$ denote the maps $z \mapsto e^{-2\pi z}, z \mapsto e^{2\pi z}$ respectively. Define another metric $g'$ in the neighborhoods $N(w^\pm)$ that is the pullback of the Euclidean metric by $f^\pm \circ \rho_{w^\pm}$. $g'$ extends smoothly over the lifts $w^+, w^-$. $g'$ varies smoothly as we vary
$\Sigma_2$ \[ w^+ - w^- + \Sigma_1 \]

Figure 2. Nodal curve in figure 1 with neck-stretching metric

$[\Sigma]$ in $M^\Gamma_{g,n}$. ($g'$ is not defined on all of $\Sigma$, but this is enough for our purpose.) Using lemma 1.1, we can associate a glued surface to every point in the neighborhood of the zero section of $I_{S^\Gamma'} \to S^\Gamma'$. For any $\Sigma$ of type $\Gamma'$, let $\Sigma^\delta$, where $\delta = (\delta_w)_{w \in \text{Edge}(\Gamma)}$. $f^\pm \circ \rho_{w^\pm}$ gives holomorphic coordinates in a neighborhood of lifts of nodal points $w^\pm$ under these $\delta_w \in \mathbb{C}$.

We now describe the metric on $\Sigma^\delta$ using the one on $\Sigma$. Let $l_w = L_w + it_w := -\ln \delta_w \in \mathbb{R}_{\geq 0} \times S^1$. The identification $\exp(z) \sim \exp(\Phi_{w} z)$ used in the construction of $C^\delta$ can be re-written as

$$(\rho_{w^+})^{-1}(r + i\theta) \sim (\rho_{w^-})^{-1}(l_w + r + i\theta).$$

$\Sigma^\delta$ is obtained from $\Sigma$ by replacing the semi-infinite cylinders corresponding to $w^\pm$ with a finite cylinder: for $w^+$, we discard the part of the cylinder with coordinates $s > L_w$ and and for $w^-$ the part with coordinates $s < -L_w$ is discarded, the reminder of the cylinders is identified using (2). The two regions being identified are isometric, so we automatically get the neck stretching metric on all of $\Sigma^\delta$. If we assume that the metrics on curves of type $\succ \Gamma$ are invariant under the action of the automorphism group of the curve, the condition would be satisfied for the metrics we have constructed also. $g$ can now be extended smoothly to all of $M^\Gamma_{g,n}$ (no canonical choice here).

The choice of a family of neck-stretching metrics is not unique, but the space of all such metrics is contractible. So, we make a choice and fix it for the rest of the paper. Now, we can talk about a ‘Riemann surface with cylindrical ends corresponding to a stable nodal curve’.

2. Preliminaries: vortices on surfaces with cylindrical ends

2.1. Definitions. Let $G$ be a complex reductive Lie group, so it is the complexification of a maximal compact subgroup $K$. Let $\Sigma$ be a Riemann surface and $P \to \Sigma$ a principal $K$-bundle. Let $(X,\omega)$ be a Kähler manifold on which $G$ acts holomorphically.
Definition 2.1. 1. (Hamiltonian actions) A moment map is a $K$-equivariant map $\Phi$ such that $\iota(\xi_X)\omega = d\langle \Phi, \xi \rangle$, $\forall \xi \in \mathfrak{k}$, where $\xi_X \in \text{Vect}(X)$ given by the infinitesimal action of $\xi$ on $X$. The action of $K$ is Hamiltonian if there exists a moment map $\Phi : X \to \mathfrak{k}^\vee$. Since $K$ is compact, $\mathfrak{k}$ has an Ad-invariant metric. We fix such a metric and $\mathfrak{k}^\vee$ and so the moment map becomes a map $\Phi : X \to \mathfrak{k}$. We assume $X$ is equipped with a Hamiltonian action and fix the moment map.

2. (Affine space of connections) Let $(\Sigma, j)$ be a Riemann surface and $P \to \Sigma$ a principal $K$-bundle over it. In this paper, we mostly don’t encounter closed Riemann surfaces, so we may assume that $P = \Sigma \times K$. The space of connections is then the affine space $\mathcal{A}(P) := d + \Omega^1(\Sigma, \mathfrak{k})$.

On any bundle over $\Sigma$, where the fiber has a $K$-action, a connection $A = d + a$ defines a covariant derivative $d_A$. For example on the bundle $\Sigma \times X$,

$$d_A := u \mapsto du + a_u \in \Omega^1(\Sigma, u^*TX).$$

At a point $x \in \Sigma$, $a_u(x)$ is the infinitesimal action of $a(x)$ at $u(x)$. The curvature of a connection $A = d + a$ is

$$F_A := da + [a \wedge a]/2 \in \Omega^2(\Sigma, \mathfrak{k}).$$

The curvature varies with the connection as

$$F_{A_{ta}} = F_A + td_A a + t^2[a \wedge a].$$

3. (Gauge transformations) A gauge transformation is an automorphism of $P$ - it is an equivariant bundle map $P \to P$. The group of gauge transformations on $P$ is denoted $\mathcal{K}(P)$. On the trivial bundle $\Sigma \times K$, $k \in \mathcal{K}(P)$ is a map $k : \Sigma \to K$. It acts on a connection $A = d + a$ as

$$k(A) = d + (dkk^{-1} + \text{Ad}_k a).$$

A connection $A$ determines a holomorphic structure on any associated bundle of $P$, whose fiber is a complex manifold. One such example is $P(X) = P \times_K X$. Let us call this complex structure $\overline{\mathcal{A}}_A$.

Definition 2.2. 1. (Gauged holomorphic maps) A gauged holomorphic map $(A, u)$ from $P$ to $X$ consists of a connection $A = d + a$ and a section $u$ of $P(X)$ that is holomorphic with respect to $\overline{\mathcal{A}}_A$. The space of gauged holomorphic maps from $P$ to $X$ is called $\mathcal{H}(P, X)$.

2. (Symplectic vortices) A symplectic vortex is a gauged holomorphic map that satisfies

$$F_A + \Phi(u)\omega = 0,$$

where $\omega \in \Omega^2(\Sigma)$ is an area form on $\Sigma$.

3. (Energy) The energy of a gauged holomorphic map $(A, u)$ is

$$E(A, u) := \int_{\Sigma} (|F_A|^2 + |d_A u|^2 + |\Phi \circ u|^2)\omega.$$
We assume the following in the rest of this article.

**Assumption 2.3.** $K$ acts freely on $\Phi^{-1}(0)$.

This assumption can be relaxed to ‘$K$ acts on $\Phi^{-1}(0)$ with finite stabilizers’, which would mean $X/G$ may have orbifold singularities. All our techniques would apply in that case, but we keep the stronger assumption to make the exposition simpler.

**Assumption 2.4 (Equivariant convexity at infinity).** The moment map $\Phi : X \to \mathfrak{k}$ is proper. There is a proper $K$-invariant function $f : X \to \mathbb{R}_{\geq 0}$, and a constant $c > 0$ such that

$$f(x) \geq c \implies \langle \nabla_\xi \nabla f(x), \xi \rangle + \langle \nabla J_\xi \nabla f(x), J_\xi \rangle \geq 0$$

for all non-zero $\xi \in T_x X$ and

$$f(x) \geq c \implies df(x)(J\Phi(x)) \geq 0.$$

The above assumption is an equivariant version of the idea of convexity in symplectic geometry introduced by Eliashberg and Gromov [EG91]. It implies that the image of a finite energy vortex is contained in the compact set $\{ f \leq c \}$ (see [CGMiRS02]). This is required for the Gromov compactness result (theorem 3.6) and the Hitchin Kobayashi correspondence (theorem 7.3).

2.2. **Asymptotic behavior on cylindrical ends.** On a non-compact base space, vortices with finite energy have good asymptotic properties - see [Zil09]. In this section we show that if the base space has cylindrical ends, we can get an improved result - proposition 2.8.

**Proposition 2.5 (Decay for vortices on the half cylinder).** Let $\Sigma$ be a half cylinder

$$\Sigma := \{ r + i\theta : r \geq 0, \theta \in \mathbb{R}/2\pi \mathbb{Z} \}$$

with the standard metric $\omega_\Sigma = dr \wedge d\theta$. Suppose $K$ acts freely on $\Phi^{-1}(0)$ and $(A,u)$ is a finite energy vortex from the trivial bundle $\Sigma \times K$ to $X$. Then, there exist constants $\gamma, C > 0$ so that

$$|F_A(z)|^2 + |d_A u(z)|^2 + |\Phi(u(z))|^2 \leq Ce^{-\gamma r}.$$

where $z = r + i\theta$.

**Remark 2.6.** This result is similar to Theorem 1.3 in [Zil09] and the proof carries over. It is an equivariant version of the isoperimetric inequality. But this result is much weaker, because in [Zil09], $\gamma$ is arbitrarily close to 1. The hypothesis in [Zil09] place some condition on $X$ in order to make the metric on $\Sigma$ ‘admissible’ - this can be dropped if we don’t require an optimal value of $\gamma$.

The following lemma is an easy consequence of proposition 2.5

**Proposition 2.7 (Removal of singularity for vortices at infinity).** Suppose $(A, u)$ is a finite energy vortex on the half cylinder $\{ r + i\theta : r \geq 0, \theta \in \mathbb{R}/\mathbb{Z} \}$. Then, there exist $x_0 \in \Phi^{-1}(0)$ and $k_0 \in C^1(\mathbb{R}/\mathbb{Z}, K)$ such that

$$\lim_{r \to \infty} \max_{\theta} d(x_0, k_0(\theta)u(r + i\theta)) = 0.$$
Suppose, the restriction of $A$ in s-gauge to the circle $\{r = r_0\} \simeq S^1$ is $d + \text{ad}\theta$, there exist constants $c, \gamma > 0$ such that for all $r_0 \geq 0$,

$$|k_0^{-1} \partial_t k_0 + a(r_0, \cdot)| < ce^{-r_0\gamma}. \quad (3)$$

This ensures that the $K$-orbit $Ku(\pm \infty)$ is well-defined.

**Proposition 2.8.** Let $p > 1$. Suppose $(A, u)$ is a finite energy vortex on a trivial K-bundle on the half cylinder $\Sigma = \{r + i\theta : r \geq 0, \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$. There is a gauge transformation $k \in W^{2,p}_{\text{loc}}(\Sigma, K)$, such that if $kA = d + a$, then,

$$\|a\|_{W^{1,p}(n \leq r \leq n + 1)} \leq ce^{-n\gamma}$$

for some positive constants $c, \delta > 0$. Hence $\|a\|_{W^{1,p}(\Sigma)} < \infty$.

**Proof.** We define the following subsets of $\Sigma$. Fix any $\epsilon < \frac{1}{2}$. Let $U$ be a closed set with smooth boundary containing $U' = \{r + i\theta : 1 - \epsilon < r < 2 + \epsilon, -\pi < \theta < \pi + \epsilon\}$, and only very slightly bigger than $U$. Let for any integer $n \geq 0$,

$$U_{0,n} := U + n/m, \quad U_{1,n} := U + n/m + i\pi,$$

$\hat{U}_n = \{r + i\theta : n/m + 1 - \epsilon \leq r \leq n/m + 2 + \epsilon, -\pi \leq \theta \leq 2\pi + \epsilon\}$,

$$U_n = \hat{U}_n/\{\theta \sim \theta + 2\pi : -\epsilon \leq \theta \leq \epsilon\} \subset \Sigma.$$

$m$ is an integer chosen so that $\|F_A\|_{L^p(U_{1,n})} < \kappa$ for all $i = 0, 1, n \geq 0$, where $\kappa$ is the constant in theorem 2.1 in [Uhl82]. This bound ensures that the connection can be put in Coulomb gauge in each of these sets.

By Uhlenbeck’s local theorem (theorem 2.1 in [Uhl82]), there is a gauge transformation $g_{i,n}$ on $U_{i,n}$ for all $n \geq 0, i = 0, 1$ such that $g_{i,n}A$ is in Coulomb gauge, i.e. denoting $a_{i,n} = g_{i,n}A - d$,

$$\|a_{i,n}\|_{W^{1,p}(U_{i,n})} \leq c\|F_A\|_{L^p(U_{i,n})}, \quad d^*a_{i,n} = 0, \quad \text{where } c \text{ is independent of } (i, n). \quad (4)$$

First, we patch all these gauge transformations to get one on $\cup_n \hat{U}_n$. Next, we take the components of $U_{0,n} \cap \hat{U}_{1,n}$ given by $\pi + \epsilon < \theta < \pi + \epsilon$, we call this $V_n$. On $V_n$, let $g_{1,n} = e^{\xi_n^2}g_{0,n}$. Let $\psi$ be a cut-off function on $[\pi - \epsilon, \pi + \epsilon]$ that is 1 in the neighbourhood of $\pi - \epsilon$ and supported away from $\pi + \epsilon$. Let $h_n = e^{\psi(\xi_n)}g_{0,n}$. Define a gauge transformation $g_n$ on $\hat{U}_n$ as being equal to $h_n$ on $V_n$ and equal to $g_{0,n} or$ $g_{1,n}$ outside $V_n$. Define $a_n := g_nA - d$. We have to show that

$$\|a_n\|_{W^{1,p}(\hat{U}_n)} \leq c\|F_A\|_{L^p(\hat{U}_n)}$$

For this it is enough to show that $\|h_n\|_{W^{2,p}(V_n)} \leq c\|F_A\|_{L^p(\hat{U}_n)}$, which in turn can be shown by a similar bound on $\|\xi_n\|_{W^{2,p}(V_n)}$. The condition in (4) is unchanged if $g_{1,n}$ is multiplied by a constant factor - so we can assume there is a point $p \in V_n$ on which $g_{0,n}$ and $g_{1,n}$ agree. We know $a_{1,n} = d\xi_n + Ad_{c,n}a_{0,n}$. By a standard argument (see for example proof of lemma 2.4 in [Uhl82]), we get

$$\|\xi_n\|_{W^{2,p}(V_n)} \leq c(\|a_{0,n}\|_{W^{1,p}(U_{0,n})} + \|a_{1,n}\|_{W^{1,p}(U_{1,n})}) \leq c\|F_A\|_{L^p(U_{1,n})}.$$
By a similar process, we can patch $g_n$ and $g_{n+1}$ for each $n$ and obtain a gauge transformation on all of $\tilde{g}$ on $\tilde{U} = \cup_n U_n$. Denote $\tilde{a} := \tilde{g}A - d$. We have bounds
\[
\|\tilde{a}\|_{W^{1,p}((\tilde{U}_n) \setminus \{z\})} \leq c \|F_A\|_{L^p(U_n)} \leq ce^{-\gamma n}.
\]

We can’t use a similar patching technique for the other component of $U_{0,n} \cap U_{1,n} -$ given by $[-\epsilon, \epsilon]$ and $[2\pi - \epsilon, 2\pi + \epsilon]$. This is because we can’t make the gauge transformations agree at any point as we no longer have the flexibility of modifying anything by a constant. But, we can get a similar bound if we can show that
\[
d(\tilde{g}(r + i\theta))^{-1} \tilde{g}(r + (\theta + 2\pi)), \text{Id}) \leq ce^{-\gamma r}.
\]

This is proved as follows. Let $\gamma_r$ denote the loop $[0, 2\pi] \ni \theta \mapsto r + i\theta \in \Sigma$. Denote by $\text{Hol}_r$ the holonomy of $A$ around $\gamma_r$. By proposition 2.7,
\[
d(\text{Hol}_r, \text{Id}) \leq ce^{-\gamma r}.
\]

Now, denote by $\text{Hol}_r^{\text{new}}$ the holonomy of $\tilde{g}A$ along the path $[0, 2\pi] \ni \theta \mapsto r + i\theta \in \tilde{U}$ - note that the end points are not identified in $\tilde{U}$, but since we are on a trivial bundle there is a canonical identification between the fibers at the end points. So, we have
\[
\text{Hol}_r^{\text{new}} = \tilde{g}(r + 2\pi i) \text{Hol}_r \tilde{g}(r)^{-1}.
\]

Recall $\text{Hol}_r^{\text{new}}$ is defined as follows: denote $\tilde{a} = \tilde{a}_r dr + \tilde{a}_\theta d\theta$, let $h_r : [0, 2\pi] \to K$ be given by the ODE
\[
h_r^{-1} \frac{dh_r}{d\theta} = \tilde{a}_\theta, \quad h_r(0) = \text{Id},
\]

then $\text{Hol}_r^{\text{new}} := h_r(2\pi)$. Since $\|\tilde{a}_\theta\|_{W^{1,p}((\tilde{U}_n) \setminus \{z\})} \leq ce^{-\gamma n}$, we have $\sup_{\theta} |a_\theta(r + i\theta)| \leq ce^{-\gamma r}$, and this implies $d(\text{Hol}_r^{\text{new}}, \text{Id}) \leq ce^{-\gamma r}$. Together with (6) and (7), (5) follows.

**Corollary 2.9.** Let $p > 2$. Suppose $(A, u) \in W^{1,p}_\text{loc} \times W^{2,p}_\text{loc}$ is a finite energy vortex on a Riemann surface $\Sigma$ with cylindrical ends. We assume $\Sigma$ has one component and marked points $z_1, \ldots, z_n$. Then, there exists a gauge transformation $k \in W^{2,p}_\text{loc}(\Sigma, K)$ and $\lambda_1, \ldots, \lambda_n \in \mathfrak{k}$ so that if $kA = d + \lambda_i d\theta + a$,
\[
\|a\|_{W^{1,p}(\rho^{-1}(n \leq r \leq n+1))} \leq ce^{-\gamma n}
\]

and hence, $\|a\|_{W^{1,p}(N(z_i) \setminus \{z_i\})} < \infty$.

**Proof.** It is enough to assume that there is only one cylindrical end. Apply proposition 2.8 to the restriction of $(A, u)$ to the half cylinder $N(z) \setminus \{z\}$, and call the resultant gauge transformation $k_1$. The homotopy equivalence class of the map $k_1|_{\{r=0\}} : \mathbb{R}/2\pi \mathbb{Z} \to K$ will contain a geodesic loop $\theta \mapsto e^{-\lambda \theta}$, where $\lambda \in \mathfrak{k}$ and $e^{2\pi \lambda} = \text{Id}$. Now, $e^{i\beta} k_1 A = d + \lambda d\theta + a$ and $a$ satisfies (8). The gauge transformation $k := e^{i\beta} k_1 : [0, \infty) \times \mathbb{R}/2\pi \mathbb{Z} \to K$ is homotopic to the constant identity map, so using a cut-off function $k$ can be defined on all of $\Sigma$ in such a way that it is identity away from a small neighborhood of $N(z)$.

□
Corollary 2.10. Suppose $0 < 1 - \frac{2}{p} < \gamma$ and $(A,u)$ be a finite energy vortex as in corollary 2.9. Then, there is a principal $K$-bundle over $\Sigma$ so that $(A,u)$ extends to a $L^p_{\text{loc}} \times W^{1,p}_{\text{loc}}$ gauged holomorphic map on $P$.

Proof. The bundle $P$ can be defined as follows. The trivial bundle $N(z_i) \times K$ is glued to $\Sigma \times K$ using transition function

$$k : N(z_i) \setminus \{z_i\} \to K, \quad \theta \mapsto e^{-\lambda_i \theta}.$$ 

On the bundle $\Sigma \times K$, $A|_{N(z_i) \setminus \{z_i\}} = d + \lambda_i d\theta + a_i$ and $a_i$ satisfies an exponential bound as in (8). Then, on the bundle $N(z_i) \times K$, $A|_{N(z_i) \setminus \{z_i\}} = \text{Ad}_e^{-\lambda_i \theta} a_i$. Using (8), $a_i$ and hence, $\text{Ad}_e^{-\lambda_i \theta} a_i$ is in $L^p(N(z_i))$. This is because: the coordinates in $N(z_i)$ are obtained by $e^{-\rho_i}$ to the $\rho_i$ coordinates. This stretches unit vectors by a factor of $e^{-r}$ and the 1-form $a_i \in L^p(N(z_i))$ if $-\gamma + 1 - \frac{2}{p} < 0$.

Using proposition 2.7, we can see that $ku$ extends continuously over $z_i$, by elliptic regularity $u \in W^{1,p}(N(z_i))$. □

Remark 2.11 (Equivariant homology class). Suppose $(A,u)$ is a finite energy vortex on a surface $\Sigma$ with cylindrical ends, then by the removal of singularity proposition 2.7 and corollary 2.9, there is a $K$-bundle $P$ on $\Sigma$ such that $(A,u)$ extends to a gauged holomorphic map over $\Sigma$. So, $[u]$ is an equivariant homology class $H_K^2(X)$. Theorem 3.1 in [CGS00] is also applicable, which says that vortices are energy minimizers in the equivariant homology class and

$$E(A,u) = \langle [\omega - \Phi], [u] \rangle,$$

where $\omega - \Phi \in H_K^2(X)$ and $\langle \cdot, \cdot \rangle$ is the pairing between equivariant homology and cohomology.

3. Gromov convergence for vortices

Suppose $C$ is an $n$-pointed stable nodal curve of genus $g$ and that $\Sigma$ is a Riemann surface obtained by the neck stretching procedure applied on $C$. The components of $\Sigma$ are $\Sigma_1, \ldots, \Sigma_k$, obtained by puncturing the respective components in $C$. Stability of $C$ implies that each genus 0 component has at least 3 cylindrical ends and each genus 1 component has at least 1 cylindrical end.

A vortex on $\Sigma$ is a tuple $(A_i, u_i)_{1 \leq i \leq k}$, where $(A_i, u_i)$ are finite energy vortices on $\Sigma_i$ that satisfy the connectedness condition:

$$Gu_i(w^+) = Gu_i(w^-)$$

for every nodal point $w$ in $\Sigma$. By the finite energy condition, the limits $Ku(z_i)$, $Ku(w_i^\pm)$ exist - see proposition 2.7. Recall that $\mathcal{M}_{g,n}(X)$ denote the space of $K$-vortices from this family of neck-stretched Riemann surfaces to $X$ modulo the action of the gauge group.

To compactify this space, we need stable vortices. A stable vortex on $\Sigma$ would consists of a finite energy vortex on each of its components. In addition,
• there may be a path of cylindrical vortices (vortices on $S^1 \times \mathbb{R}$) at each marked point $z_i$.
• Any edge $w \in \text{Edge}(\Sigma)$ may be replaced by a path of cylindrical vortices joining $w^+$ to $w^-$.
• There may be trees of spherical bubbles on fibers $P(X)_x$ where $x \in \Sigma$ or it is in the domain $(S^1 \times \mathbb{R})$ of one of the cylindrical vortices.

We give a rigorous definition below.

**Definition 3.1** (Stable vortex on a Riemann surface with cylindrical ends). Suppose $C$ is an $n$-pointed stable nodal curve of genus $g$ and $\Sigma$ is the corresponding Riemann surface with cylindrical ends. A stable vortex on $\Sigma$ is modelled on a graph $\Gamma$ such that $st(\Gamma) = \Gamma(\Sigma)$. $\text{vert}(\Gamma)$ can be partitioned as

$$\text{vert}(\Gamma) = \text{vert}(st(\Gamma)) \cup \Gamma_C \cup \Gamma_S.$$  

The vertices $\Gamma_C$ and $\Gamma_S$ are *cylindrical* and *spherical* respectively. The set of cylindrical vertices decomposes into (possibly empty) subsets

$$\Gamma_C = (\bigcup_{z \in \text{Edge}_\infty(\Gamma)} \Gamma_C(z)) \cup (\bigcup_{w \in \text{Edge}(\Gamma)} \Gamma_C(w)).$$

and each set is equipped with an ordering. For $x = z$ or $w$ as above,

$$\Gamma_C(z) = \{\alpha_{x,1}, \ldots, \alpha_{x,|C(z)|}\}.$$

A *stable vortex* on $\Sigma$ modelled on the graph $\Gamma$ is a tuple $((w_\alpha)_{\alpha \in \text{st}(\Gamma) \cup \Gamma_C}, (u_\alpha)_{\alpha \in \Gamma_S})$ that satisfies:

**Maps:** For $\alpha \in \text{st}(\Gamma)$, $\Sigma_\alpha$ is the corresponding component of $\Sigma$ and for $\alpha \in \text{vert}(\Gamma_C)$, $\Sigma_\alpha$ is the cylinder $\{r + i\theta : r \in \mathbb{R}, \theta \in \mathbb{R}/\mathbb{Z}\}$ with Euclidean metric. In either case, $u_\alpha := (A_\alpha, u_\alpha)$ is a finite energy vortex from the trivial bundle $\Sigma_\alpha \times G$ to $X$. For $\alpha \in \Gamma_S$, $u_\alpha$ is a spherical fiber bubble i.e. it is a holomorphic map from $(\mathbb{P}^1)_x$ to $P(X)_x$ for some $x \in \Sigma \cup \bigcup_{\beta \in \Gamma_C} \Sigma_\beta$.

**Nodal points:** Suppose $w \in \text{Edge}(\Gamma)$. Let $\alpha = \iota(w^+)$ and $\beta = \iota(w^-)$.

- If $\alpha, \beta \in \Gamma_S$, then $u_\alpha$ and $u_\beta$ map to the same fiber $P(X)_x$. $w^+ \in (\mathbb{P}^1)_\alpha$, $w^- \in (\mathbb{P}^1)_\beta$ and $u_\alpha(w^+) = u_\beta(w^-)$.
- If $\alpha \in \Gamma_S$ and $\beta \in \text{st}(\Gamma) \cup \Gamma_C$, then $w^+ \in (\mathbb{P}^1)_\alpha$ and $w^- \in (S^1 \times \mathbb{R})_\beta$ (or $\Sigma_\beta$ if $\beta \in \text{st}(\Gamma)$) and $u_\alpha(w^+) = u_\beta(w^-)$.
- If $\alpha, \beta \in \Gamma_C$, then $\alpha$ and $\beta$ are consecutive elements in a set $C_z$ for $z \in \text{Edge}(\Gamma) \cup \text{Edge}_\infty(\Gamma)$. We assume $\alpha = \alpha_{z,j}$ and $\beta = \alpha_{z,j+1}$. Then, $w^+$ and $w^-$ are the north ($+\infty$) and south ($-\infty$) poles of $S^1 \times \mathbb{R}$ respectively, and $Ku_\alpha(w^+) = Ku_\beta(w^-)$. (For finite energy vortices $Ku(\pm \infty)$ is well-defined by proposition 2.7.)
- If $\alpha \in \text{st}(\Gamma)$ and $\beta \in \Gamma_C$, there are 3 possibilities
  - $\beta = \alpha_{z,i,1}$ for some marked point $z_i$ of $\Sigma$. $w^+ = z_i \in \overline{\Sigma} \setminus \Sigma$ and $w^-$ is the south pole.
  - $\beta = \alpha_{w_0,1}$ for some $w_0 \in \text{Edge}(\Sigma)$. $w^+ = w_0^+ \in \overline{\Sigma} \setminus \Sigma$ and $w^-$ is the south pole.
  - $\beta = \alpha_{w_0,|C(w_0)|}$ for some $w_0 \in \text{Edge}(\Sigma)$. $w^- = w_0^- \in \overline{\Sigma} \setminus \Sigma$ and $w^+$ is the north pole.
In all cases, \( K_{u_\alpha}(w^+) = K_{u_\beta}(w^-) \).

- If \( \alpha, \beta \in st(\Gamma) \), then this edge corresponds \( w_0 \in \text{Edge}(\Sigma) \), so \( w^+ = w_0^+ \) and \( w^- = w_0^- \). \( w^\pm \in \Sigma \setminus \Sigma \). In this case, \( |\text{C}(w_i)| = 0 \). \( K_{u_\alpha}(w^+) = K_{u_\beta}(w^-) \).

**Marked points:** Corresponding to each the marked points \( z_i \) on \( \Sigma \) \((1 \leq i \leq n)\), there is a marked point \( z_i^\Gamma \) on \( \Sigma^\Gamma \). If \( |\text{C}(z_i)| = 0 \), then \( z_i^\Gamma = z_i \). If not, \( z_i^\Gamma \) is the north pole of the cylinder corresponding to the vertex \( \alpha_{z_i}|\text{C}(z_i)| \in \Gamma|\text{C}(z_i)| \).

**Stability:** For \( \alpha \in \Gamma_C \cup \Gamma_S \), if \( u_\alpha \) is the constant map then the number of marked points plus nodal points on \( \alpha \) is at least 3.

For \( \alpha \in \text{vert}(st(\Gamma)) \cup \Gamma_C \), we denote by \( Z_\alpha \) the set of points \( x \) in \( \Sigma_\alpha \) where there are sphere bubbles in the fiber \( P(X)_x \).

**Remark 3.2.** To define convergence on a sequence of Riemann surfaces, these surfaces must be identified to each other. First, we consider curves of the same combinatorial type. Suppose \( C \) is a nodal curve of type \( \Gamma \) and \( \text{Def}_\Gamma(C) \) be a universal deformation of type \( \Gamma \) parametrized by \((S, 0)\). Then, there is a smooth family of diffeomorphisms

\[
\tag{10} h_s : C_s \to C_0, \quad s \in S, \quad h_0 = \text{Id}
\]

satisfying the condition that \( h_s \) preserves special points and in a neighborhood \( N(z) \) of a special point \( z = z_i, w_i^\pm \), \( h_s \) is a bi-holomorphism. We can demand the further condition that under the neck-stretching metric \( h_s \) be an isometry in the neighborhoods \( N(z) \). To understand the behavior of curves of different combinatorial type, we consider \( C_\nu \) a sequence of smooth curves and \( C \) a nodal curve such that \([C_\nu]\) converges to \([C]\) in \( \overline{M}_{g,n} \) as \( \nu \to \infty \). \( \Sigma_\nu \) and \( \Sigma \) are Riemann surfaces with cylindrical ends corresponding to \( C_\nu \) and \( C \) respectively. For each \( \nu \), we can find a nodal curve \( C_\nu^\nu \in \text{Def}_\Gamma(C) \) of the same combinatorial type as \( C \) so that \( C_\nu \) can be obtained from \( C_\nu^\nu \) by gluing. Denote by \((\Sigma_\nu^\nu, g_\nu)\) the corresponding Riemann surface with cylindrical ends. We call the gluing parameter \( \delta(\nu) \) and write \( \delta(\nu) = (\delta_w)_w \in \text{Edge}(C) \), where \( \delta_w \in T_{w^+} \Sigma_{\nu}(w^+) \otimes T_{w^-} \Sigma_{\nu}(w^-) \). As in (10), we have diffeomorphisms

\[
h_\nu : \Sigma_\nu^\nu \to \Sigma.
\]

Denote \((h_\nu^{-1})^*g_\nu\) also by \( g_\nu \in \otimes^2(T^*\Sigma) \), and we have \( g_\nu \to g \) in \( C^\infty(\Sigma) \). We break up \( \Sigma_{\delta(\nu)} \) into the following Riemann surfaces. Let

\[
\Sigma_{\nu,\alpha} = \Sigma_{\alpha} \cup \bigcup_{w^\pm ; i(w^\pm) = \alpha} (\rho_{w^\pm})^{-1}\{r + i\theta : \pm r > \text{Re}(l_w(\nu)), \theta \in \mathbb{R}/\mathbb{Z}\},
\]

where \( l_w(\nu) = -\ln \delta_w(\nu) \). Then, \( \Sigma_\nu = \Sigma_{\nu,1} \cup \cdots \cup \Sigma_{\nu,k} \sim, \) where \( \sim \) is an equivalence relation like in (2) corresponding to every edge in \( \Gamma(C) \). Note that \( l_w(\nu) \to \infty \) as \( \nu \to \infty \). The benefit of these constructions is that, now \( \Sigma_{\nu,\alpha} \subset (\Sigma, g_\nu) \), and so we can talk about convergence on \( \Sigma \). Denote by \( j_\nu \) the complex structure associated to \( g_\nu \). A gauged holomorphic map \( w \) defined on \( \Sigma_\nu \) can be pulled back to one on \( \Sigma_{\nu,1} \cup \cdots \cup \Sigma_{\nu,k} \) and restricted to \( \Sigma_{\nu,\alpha} \) - this object is also denoted by \( w \). We define Gromov convergence for a sequence of finite energy vortices \((A_\nu, u_\nu)\) defined on \( \Sigma_\nu \). The limit will be a stable vortex on \( \Sigma \). □

**Notation 3.3.** For \( \Sigma \), a Riemann surface with cylindrical ends, we denote by \( \text{Cyl}(\Sigma) \subset \Sigma \), the parts of \( \Sigma \) which are isometric to a cylinder - these are neighborhoods of
edges, and if there is a nodal curve $\Sigma'$ from which $\Sigma$ is obtained by gluing, the resulting finite cylinders are also in $\text{Cyl}(\Sigma)$. For example, in (11), the finite cylinders $\rho_{w+}^{-1}\{r+i\theta: 0 \leq r \leq \text{Re}(l_w(\nu)), \theta \in \mathbb{R}/\mathbb{Z}\} \subset \Sigma_{\nu,\alpha}$ are included in $\text{Cyl}(\Sigma_{\nu})$. We use the the maps $\rho_{w\pm}$ to describe the coordinates in these finite cylinders also.

**Definition 3.4** (Gromov Convergence). Suppose $\{C_\nu\}_{\nu \in \mathbb{N}}$ be a sequence of $n$-pointed curves of genus $g$ and $C$ is a stable nodal curve such that $[C_\nu] \to [C]$ in $\bar{M}_{g,n}$. Let $\Sigma_\nu$, $\Sigma$ be Riemann surfaces with cylindrical ends corresponding to $C_\nu$, $C$. Let $w_\nu := (A_\nu, u_\nu)$ be a sequence of vortices on the trivial bundle $\Sigma_\nu \times G$. We say the sequence *Gromov converges* to a stable vortex $((w_\alpha)_\alpha \in \Gamma_1(\Sigma))$, $((u^a)_\alpha \in \Gamma_{S^1}((z_i)_{1 \leq i \leq k})$ if the following conditions are satisfied.

**Map:**
(a) For each $\alpha \in \text{vert}(st(\Gamma))$, there exist a sequence of gauge transformations $k_{\nu,\alpha}$ on $\Sigma_{\nu,\alpha}$ so that $k_{\nu,\alpha}(A_{\nu,\alpha}, u_{\nu,\alpha})$ converges in $C^\infty$ to $(A_\alpha, u_\alpha)$ on compact subsets of $\Sigma_\alpha \setminus Z_\alpha$.

(b) For every $\alpha \in \Gamma_{C(\nu)}$ for some $z \in \text{Edge}_C(\Sigma)$, there exist
- a sequence of numbers $s_{\nu,\alpha} \to \infty$ which define a sequence of translations
  \[ \phi_{\nu,\alpha} : ((-s_{\nu,\alpha}, \infty) \times S^1)_\alpha \to (S^1 \times \mathbb{R}_{\geq 0})_{z_i}, \]
  \[ z \mapsto z + s_{\nu,\alpha} \]
  mapping a part of $\Sigma_\alpha$ to the cylindrical end corresponding to $z_i$.
- a sequence of gauge transformations $k_{\nu,\alpha}$ so that $k_{\nu,\alpha}(\phi_{\nu,\alpha}^* (A_\nu, u_\nu))$
  converges to $(A_\alpha, u_\alpha)$ on compact subsets of $(S^1 \times \mathbb{R})_\alpha \setminus Z_\alpha$.

(c) For every $\alpha \in \Gamma_{C(w)}$ for some $w \in \text{Edge}(\Sigma)$, there exist
- a sequence of numbers $0 < s_{\nu,\alpha} < l_i(\nu)$ such that $s_{\nu,\alpha}, l_i(\nu) - s_{\nu,\alpha} \to \infty$ as $\nu \to \infty$, which define a sequence of translations
  \[ \phi_{\nu,\alpha} : ((-s_{\nu,\alpha}, l_i(\nu) - s_{\nu,\alpha}) \times S^1)_\alpha \to ((0, l_i(\nu)) \times S^1)_{w_i^+} \]
  \[ z \mapsto z + s_{\nu,\alpha} \]
  mapping a part of $\Sigma_\alpha$ to the cylindrical end corresponding to $w_i^+$.
- a sequence of gauge transformations $k_{\nu,\alpha}$ so that $k_{\nu,\alpha}(\phi_{\nu,\alpha}^* (A_\nu, u_\nu))$
  converges to $(A_\alpha, u_\alpha)$ on compact subsets of $(S^1 \times \mathbb{R})_\alpha \setminus Z_\alpha$.

(d) $\forall \alpha \in \Gamma_S$, there is a sequence $\Omega_{\nu,\alpha} \subset C \subset (\mathbb{P}^1)_\alpha$ that exhaust $C$, and a sequence of rational maps $\phi_{\nu,\alpha}: \Omega_{\nu,\alpha} \to \Sigma_\nu$ so that $u_\alpha \circ \phi_{\nu,\alpha}$ converges to $u_\alpha$ on compact subsets of $(\mathbb{P}^1)_\alpha \setminus Z_\alpha$.

**Rescaling:**
- For $\alpha, \beta \in \Gamma_{C(\nu)}$ (where $\gamma = w_i$ or $z_i$) with $\alpha < \beta$, $s_{\nu,\beta} - s_{\nu,\alpha} \to \infty$ as $\nu \to \infty$.
- Suppose $\alpha \in \Gamma_S$ is such that $u_\alpha$ maps to $P(X)_x$ and $x \in \Sigma_{\beta}$, $\beta \in \text{st}(\Gamma) \cup \Gamma_C$. Then $\phi_{\nu,\alpha}^{-1} \circ \phi_{\nu,\alpha}: (\mathbb{P}^1)_\alpha \to \Sigma_\beta$ converges to $z_{\beta,\alpha}$ on compact subsets of $(\mathbb{P}^1)_\alpha \setminus \{z_{\beta,\alpha}\}$.
- Suppose $\alpha, \beta \in \Gamma_S$ are fiber bubbles in the same fiber $P(X)_x$. Then, $(\phi_{\nu,\beta})^{-1} \circ \phi_{\nu,\alpha}: (\mathbb{P}^1)_\beta \to (\mathbb{P}^1)_\beta$ converges to $z_{\beta,\alpha}$ in $C^\infty$ on compact subsets of $(\mathbb{P}^1)_\beta \setminus \{z_{\beta,\alpha}\}$.

**Energy:** $\lim_{\nu \to \infty} E(w_\nu) = \sum_{\alpha \in \text{st}(\Gamma) \cup \Gamma_C} E((A_\alpha, u_\alpha)) + \sum_{\alpha \in \Gamma_S} E(u_\alpha)$. 

Remark 3.5. The definition of Gromov convergence is independent of the choice of $h$ in (10).

The main theorem is

**Theorem 3.6 (Gromov compactness).** Suppose $\{C_\nu\}_{\nu \in \mathbb{N}}$ are $n$-pointed genus $g$ curves and $C$ is a stable nodal curve such that $[C_\nu] \to [C]$ in $\overline{M}_{g,n}$. Given $w_\nu = (A_\nu, u_\nu)$ a sequence of vortices defined on $\Sigma_\nu$, the Riemann surface with cylindrical ends corresponding to $C_\nu$, that satisfy

$$\sup_\nu E(w_\nu) < \infty.$$ 

Then, after passing to a subsequence, the sequence converges to a stable vortex $w = (A, u)$ on $\Sigma$.

The evaluation maps are continuous under the Gromov topology, and the equivariant homology class $[u_\nu]$ is preserved in the limit. The following proposition is proved in section 4.

**Proposition 3.7 (Continuity of $ev_i$).** Assume the setting of theorem 3.6.

(a) Suppose the limit stable vortex $(A, u)$ is modelled on a graph $\Gamma$. For each of the marked points $z_1, \ldots, z_n$,

$$\lim_{\nu \to \infty} Ku_\nu(z_i) = Ku(z_i^\Gamma).$$

($z_i^\Gamma$ is defined as part of the definition of a stable vortex.)

(b) If $[u_\nu] = \beta \in H^K_2(X)$ for all $\nu$, then $[u] = \beta$.

**Remark 3.8.** We have defined Gromov convergence and stated the Gromov compactness theorem in the case when the sequence of vortices $(A_\nu, u_\nu)$ is defined on smooth curves and the limit stable vortex is defined on a possibly nodal curve. This can easily be generalized to the case when $(A_\nu, u_\nu)$ are stable vortices on possibly nodal curves. After proving theorems 3.6 and 3.7, we’d have a compact space $\overline{MV}^K_{g,n}(X, \beta)$ of stable vortices modulo gauge transformations.

4. **Proof of Gromov compactness**

4.1. **Convergence modulo breaking of cylinders.**
Proposition 4.1 (Sphere bubbling). Let $\Sigma = B_1(z) \subset \mathbb{R}^2$ be a neighborhood of a point on a surface, and suppose there is a sequence of metrics $g_\nu$ on $\Sigma$ converging to $g$. Denote $\Sigma_\nu := (\Sigma, g_\nu)$. Let $(A_\nu, u_\nu)$ be a sequence of vortices on the trivial bundle $\Sigma_\nu \times K$ to $X$ such that

$$\sup_{\nu} E(A_\nu, u_\nu) < \infty, \quad \sup_{\nu} |dA_{\nu} u_\nu|_{L^\infty(K)} < \infty$$

for all compact subsets of $K \subset \Sigma \setminus \{z\}$. Then, there is a stable vortex $(A, u)$ modelled on a tree $T = \{\Sigma\} \cup V_S$, where $V_S$ is a sphere bubble tree in the fiber $P(X)_z$, such that $(A_\nu, u_\nu)$ Gromov converges to $(A, u)$.

This is proved by Ott [Ott09] using the following technique: a vortex $(A_\nu, u_\nu)$ can be viewed as a $J_{A_\nu}$-holomorphic curve from $C$ to $P(X)$. Here $J_{A_\nu}$ is the complex structure on $P(X)$ corresponding to the Dolbeault operator $\overline{\partial}_{A_\nu}$ and the sequence $J_{A_\nu}$ converges to $J_A$. Now, one can apply the result of Gromov convergence of $J$-holomorphic curves. Knowing this result, we can carry out the rest of the proof assuming the target $X$ is aspherical - so there is no sphere bubbling.

Proposition 4.2 (Convergence of vortices modulo breaking of cylinders). Let $\Sigma_\nu$, $\Sigma$ be as defined in theorem 3.6, and let $w_\nu := (A_\nu, u_\nu)$ be a sequence of vortices on the trivial bundle $\Sigma_\nu \times G$ that satisfy

$$\sup_{\nu} E(w_\nu) < \infty.$$ 

Assume $X$ is aspherical.

After passing to a subsequence, there exists a vortex $(A, u)$ on $\Sigma$ (possibly not satisfying the connectedness condition (9)) and a sequence of gauge transformations $k_\nu \in K(P)$ such that $k_\nu(A_\nu, u_\nu)$ converges to $(A, u)$ in $C^\infty$ on compact subsets of $\Sigma$. Further,

$$\lim_{\nu \to \infty} E(w_\nu) = E(w_0) + \sum_{z \in \text{Edge}_\infty(\Sigma)} E_z + \sum_{w \in \text{Edge}(\Sigma)} E_w,$$

where $E_z$ and $E_w$ are defined as

$$E_z := \lim_{R \to \infty} \lim_{\nu \to \infty} E((A_\nu, u_\nu), \rho_z^{-1}\{s > R\}),$$

$$E_w := \lim_{R \to \infty} \lim_{\nu \to \infty} E((A_\nu, u_\nu), \rho_w^{-1}\{R < s < l_w(\nu)\}).$$

The proof of this theorem is analogous to proof of proposition 37 in [Zil12]. It uses a combination of Uhlenbeck compactness on non-compact domains (Theorem A’ in [Weh04]) and elliptic regularity on $u_\nu$. The only difference here is that the metric on $\Sigma$ is not fixed - but this doesn’t create any changes because Uhlenbeck compactness ([Weh04]) and elliptic regularity for pseudoholomorphic curves (proposition B.4.2 in [MS04]) are valid when we have a sequence of converging metrics.
4.2. **Breaking of cylinders.** In proposition 4.2, $E_z$ and $E_w$ represent energy that escapes to infinity on cylindrical ends. This leads to breaking of cylinders. To analyze that we need quantization of energy for vortices on a cylinder and that the ends of the cylinders connect in the image.

**Proposition 4.3.** Suppose $(A, u)$ is a finite energy vortex on $\mathbb{R} \times S^1$.

(a) (Annulus Lemma) There exist constants $C$, $\delta$, $E_C > 0$ that satisfy the following: for any $s_0, s_1 \in \mathbb{R}$, if $E := E((A, u), [s_0, s_1] \times \mathbb{R}) < E_C$, then for any $T > 0$

$$E((A, u), [s_0 + T, s_1 - T] \times S^1) \leq Ce^{-\delta T}E$$

\[\sup_{z, z' \in [s_0 + T, s_1 - T] \times S^1} d(Ku(z), Ku(z')) \leq Ce^{-\delta T}\sqrt{E}\]

(b) (Quantization of energy for vortices) If $E((A, u), \mathbb{R} \times S^1) < E_C$. ($E_C$ from (a)). Then, $E(A, u) = 0$.

**Proof.** The proof of (a) is identical to the proof of the corresponding result for vortices in $\mathbb{C}$ in [Zil12]. To prove (b) consider a vortex $(A, u)$ on $\mathbb{R} \times S^1$ satisfying $E(A, u) \leq E_C$. For any $s_0 < s_1$ and $T > 0$, by part (a),

$$E((A, u), [s_0, s_1] \times S^1) \leq Ce^{-\delta T}E((A, u), [s_0 - T, s_1 + T] \times S^1) \leq Ce^{-\delta T}E_C.$$

It follows that $E(A, u, [s_0, s_1]) = 0$. Since the choice of $s_0$, $s_1$ is arbitrary, $E(A, u) = 0$. \hfill $\square$

**Lemma 4.4** (Breaking of cylinders). Suppose $w_\nu := (A_\nu, u_\nu)$ is a sequence of vortices on $\Sigma_0 = [0, \infty) \times S^1$ and that $w_\nu$ $C^\infty$-converges to $w := (A, u)$ in compact subsets of $[0, \infty) \times S^1$. Also, suppose

$$m_0 := \lim_{R \to \infty} \limsup_{\nu \to \infty} E((A_\nu, u_\nu), [R, \infty) \times S^1) > 0.$$

Then, there exists a finite energy non-constant vortex $(A_1, u_1)$ on a cylinder $\Sigma_1 = (\mathbb{R} \times S^1)$, a sequence of translations $\phi_\nu : [0, \infty) \times S^1 \to \mathbb{R} \times S^1$ given by $r + i\theta \mapsto r + r_\nu + i\theta$ with $\lim_\nu r_\nu = \infty$ such that after passing to a sub-sequence, the following are satisfied:

(a) Let $(A_\nu^1, u_\nu^1) := \phi_\nu^* (A_\nu, u_\nu)$. There is a sequence of gauge transformations $k_\nu$ so that $k_\nu(A_\nu^1, u_\nu^1)$ converges to $(A_1, u_1)$ in $C^\infty$ on compact subsets of $\mathbb{R} \times S^1$.

(b) Let $m_1 := \lim_{R \to \infty} \limsup_{\nu \to \infty} E((A_\nu^1, u_\nu^1), [R, \infty) \times S^1)$. Then,

$$m_0 = E(A_1, u_1) + m_1.$$

(c) $Ku(\infty) = Ku_1(-\infty)$.

**Proof.** The proof runs parallel to the proof of proposition 4.7.1 in [MS04]. Pick $0 < \delta < \min\{E_C, m_0\}$ (from proposition 4.3). Define the sequence $s_\nu$ as

$$E(w_\nu, [s_\nu, \infty) \times S^1) = m_0 - \frac{\delta}{2}. $$
Step 1: \( \lim_{R \to \infty} \lim_{\nu \to \infty} E(w_\nu, [s_\nu - R, \infty) \times S^1) = m_0 \).

We observe that (12) implies that for any sequence \( R_\nu \to \infty \),
\[
\lim_{\nu \to \infty} E(w_\nu, [R_\nu, \infty) \times S^1) \leq m_0
\]
and that there exists a sequence \( \sigma_\nu \to \infty \) for which equality is attained, i.e.
\[
\lim_{\nu \to \infty} E(w_\nu, [\sigma_\nu, \infty) \times S^1) = m_0.
\]

This implies for every \( T \geq 0 \),
\[
\lim_{\nu \to \infty} E(w_\nu, [\sigma_\nu - T, \infty) \times S^1) = m_0.
\]

The result of step 1 would be true if \( s_\nu - \sigma_\nu \) is bounded above. So, we assume \( \lim (s_\nu - \sigma_\nu) = \infty \). From (13) and (14),
\[
\lim_{\nu \to \infty} E(w_\nu, [\sigma_\nu, s_\nu) \times S^1) = \frac{\delta}{2}
\]
We split up the cylinders \([\sigma_\nu, s_\nu) \times S^1\) into 3 parts and show that in the limit all the energy is focused in the part at the \( s_\nu \) end. First, we handle the middle part. By the annulus lemma (proposition 4.3 (a)), there exist constants \( c, \delta > 0 \) so that for any \( T > 0 \),
\[
\lim_{\nu \to \infty} E(w_\nu, [\sigma_\nu + T, s_\nu - T)] \times S^1 \leq \frac{\delta}{2}ce^{-\delta T}
\]
For the part at the \( \sigma_\nu \)-end, we prove the following.

Claim.
\[
\lim_{\nu \to \infty} E(w_\nu, [\sigma_\nu, \sigma_\nu + T] \times S^1) = 0
\]
for all \( T \geq 0 \).

Proof. Consider the sequence of rescaled vortices \( \tilde{w}_\nu(z) = w_\nu(z + \sigma_\nu) \). Then, for all \( T \geq 0 \),
\[
\lim_{\nu \to \infty} E(\tilde{w}_\nu, [-T, T) \times S^1) = \lim_{\nu \to \infty} E(w_\nu, [\sigma_\nu - T, \sigma_\nu + T] \times S^1)
\leq \lim_{\nu \to \infty} E(w_\nu, [\sigma_\nu - T, s_\nu])
\leq \frac{\delta}{2},
\]
where the last inequality follows from (13) and (15). By quantization of energy (proposition 4.3), the above limit is 0. So, the claim follows. \( \square \)

Together with (16) and (17), this implies, for all \( T \geq 0 \)
\[
(18) \quad \lim_{\nu \to \infty} E(w_\nu, [s_\nu - T, s_\nu]) \geq \frac{\delta}{2}(1 - ce^{-\delta T}).
\]
Step 1 is proved because, for all \( T \geq 0 \)
\[
m_0 \geq \lim_{\nu \to \infty} E(w_\nu, [s_\nu - T, \infty) \times S^1) \geq m_0 - \frac{\delta ce^{-\delta T}}{2}.
\]
Step 2: Finishing the proof.
Let \( w^1_\nu(z) := w_\nu(z + s_\nu) \), so Step 1 implies that
\[
\lim_{R \to \infty} \lim_{\nu \to \infty} E(w^1_\nu, (-R, \infty) \times S^1) = m_0.
\]

Repeating the steps in the proof of proposition 4.2, parts (a) and (b) of the proposition follow. \( w^1 \) is non-constant, because \( E(w^1, (-\infty, 0]) = \lim_\nu E(w^1_\nu, (-\infty, 0] \times S^1) = \frac{\delta}{2} > 0 \). Now we come to (c). Define
\[
E(R) = \lim_{\nu \to \infty} E(u_\nu, [R, s_\nu - R] \times S^1).
\]

If \( R \) is large enough that there is no sphere bubbling for \( u \) on \( s > R \) and no bubbling for \( v \) on \( s < -R \), then
\[
E(R) = E(u, (R, \infty) \times S^1) + E(v(-\infty, -R)).
\]

So, \( \lim_{R \to \infty} E(R) = 0 \). For large enough \( R \), apply the annulus lemma on \( u_\nu \) on the cylinder \([R-1, s_\nu - R + 1]\) with \( T = 1 \). Then,
\[
\sup_{z, z' \in [R, s_\nu - R] \times S^1} d(Ku_\nu(z), Ku_\nu(z')) \leq Ce^{-\delta} \sqrt{E_\nu(R-1)}
\]

Taking limit \( \nu \to \infty \),
\[
\sup_{Re(z) \geq R, Re(z') \leq -R} d(Ku(z), Ku^1(z')) \leq Ce^{-\delta} \sqrt{E(R-1)}.
\]

Letting \( R \to \infty \), we get \( Ku(\infty) = Ku^1(-\infty) \). □

Lemma 4.4 implies the following proposition.

**Proposition 4.5** (Breaking of cylinders at marked points). Suppose \( w_\nu = (A_\nu, u_\nu) \) is a sequence of vortices on the trivial \( K \) bundle on the cylinder \( \Sigma := [0, \infty) \times S^1 \). We assume the sequence converges to a finite energy vortex \( w = (A, u) \) smoothly on compact subsets of \([0, \infty) \times S^1 \) and the following limit exists.
\[
m_0 := \lim_{R \to \infty} \limsup_{\nu \to \infty} E((A_\nu, u_\nu), [R, \infty) \times S^1).
\]

Then, there is a stable vortex \((A, u)\) modelled on the tree \( \{\Sigma\} \cup V_C \) so that a subsequence of \((A_\nu, u_\nu)\) Gromov-converges to \((A, u)\), except there is no stability requirement on \( \Sigma \). Here \( V_C \) consists of a path of \( m \) cylindrical vertices \( \{\alpha_1, \ldots, \alpha_m\} \) and \( \alpha_1 \) is connected to \( \Sigma \).

For the set-up of the next proposition, suppose \( l_\nu = L_\nu + it_\nu \in \mathbb{R}_{\geq 0} \times S^1 \) be a sequence such that \( \lim_{\nu \to \infty} L_\nu = \infty \). Let \( \Sigma^+_\nu = [0, L_\nu] \times S^1 \) and \( \Sigma^-_\nu = [-L_\nu, 0] \times S^1 \) identified to each other by the isomorphism
\[
r_\nu : \Sigma^-_\nu \to \Sigma^+_\nu, \quad z \mapsto z + l_\nu.
\]
\( \Sigma^+ \) and \( \Sigma^- \) denote \( \mathbb{R}_{\geq 0} \times S^1 \) and \( \mathbb{R}_{\leq 0} \times S^1 \) respectively.
Proposition 4.6 (Breaking of cylinders at nodal points). Suppose $w_\nu := (A_\nu, u_\nu)$ be a sequence of vortices on the trivial bundle $\Sigma^+ \times K$. We assume there are sequences of gauge transformations $k_\nu^\pm$ such that $k_\nu^+ w_\nu$ (resp. $k_\nu^- r_\nu^+ w_\nu$) converges to a finite energy vortex $w^+$ (resp. $w^-$) smoothly on compact subsets of $\Sigma^+$ (resp. $\Sigma^-$). Also, we assume the following limit exists.

$$m_0 := \lim_{R \to \infty} \limsup_{\nu \to \infty} E((A_\nu, u_\nu), [R, L_\nu) \times S^1).$$

Then, there is a stable vortex $(A, u)$ modelled on the tree $\{\Sigma^+, \Sigma^-\} \cup V_C$ so that a subsequence of $(A_\nu, u_\nu)$ Gromov-converges to $(A, u)$, except there is no stability requirement on $\Sigma^+$ and $\Sigma^-$. Here, $V_C$ consists of a path of $m$ nodes $\{\alpha_1, \ldots, \alpha_m\}$ connecting $\Sigma^+$ and $\Sigma^-$. 

Proof. We observe

$$\lim_{\nu \to \infty} E(w_\nu, [0, L_\nu]) = E(w_{\Sigma^+}) + m_0.$$ 

The proof is by induction on the integer $\left\lfloor \frac{m_0}{E_C} \right\rfloor$, where $E_C$ is from the proposition 4.3. If $m_0 > 0$, as in the proof of lemma 4.4, pick $0 < \delta < \min\{E_C, m_0\}$ and define a sequence $0 < s_\nu < L_\nu$ that satisfies

$$E((A_\nu, u_\nu), [s_\nu, \infty) \times S^1) = m_0 - \frac{\delta}{2}.$$ 

We divide our analysis into three cases

Case 1: $m_0 > 0$ and $\lim_{\nu \to \infty} (L_\nu - s_\nu) = \infty$.

In this case, the conclusions carry over: Modulo gauge, the sequence $w_\nu^1 := w_\nu (\cdot + s_\nu)$ converges to a vortex $(A^1, u^1)$ defined on $S^1 \times \mathbb{R}$.

$$m_0 = E(w^1) + m_1$$

where

$$m_1 := \lim_{R \to \infty} \limsup_{\nu \to \infty} E(w_\nu^1, [R, L_\nu - s_\nu) \times S^1).$$

Also, $w^1$ is non-constant and $K u^+(\infty) = u^1(-\infty)$. $w^1$ is the first cylindrical vortex $\alpha_1$ in the path connecting $\Sigma^+$ to $\Sigma^-$. We set $s_{\nu, \alpha_1} := s_\nu$. Pick $R'_0$ so that $E(w^1, [s < R'_0]) > E_C$ and such that $Z_0 \cap ([R'_0, \infty) \times S^1) = \phi$. By the induction hypothesis, the proposition is true for the sequence of vortices $w^1_\nu$ defined on $[R'_0, L_\nu - s_\nu]$. This would give us a path of cylindrical vertices connecting $\Sigma_{\alpha_1}$ to $\Sigma^-$. We label the vertices $\{\alpha_2, \ldots, \alpha_m\}$. We have a sequence of translation maps $\Sigma_{\alpha_i} \to \Sigma_{\alpha_1}$ for $i > 1$. This can be composed with the sequence

$$\phi_{\nu, \alpha_1} : ((-s_{\nu, \alpha_1}, L_\nu - s_{\nu, \alpha_1}) \times S^1)_{\alpha_1} \to ((0, L_\nu) \times S^1)_{\Sigma^+}, \quad z \mapsto z + s_{\nu, \alpha_1}$$

to yield $\phi_{\nu, \alpha_1}$ for $i = 2, \ldots, m$.

From the induction hypothesis, we obtain

$$m_1 = \sum_{i=2}^m E(w_i) + E(w_{\Sigma^-}).$$
Combined with (20) and (19), this establishes the energy equality for Gromov convergence.

Case 2: $m_0 > 0$ and $\lim_{\nu \to \infty} (L_{\nu} - s_{\nu}) = L < \infty$.

In this case $m = 0$. The sequences $w_{\nu}(\cdot + l_{\nu})$ and $w_{\nu}(\cdot + l_{\nu})$ (recall $l_{\nu} = L_{\nu} + it_{\nu}$) will have the same limit up to reparametrization. But $w_{\nu}(\cdot + l_{\nu}) = r_{\nu}^* w_{\nu}$. Imitating the proof of lemma 4.4, we get $m_0 = E(w^-)$ and $Ku^+(\infty) = Ku^-(\infty)$.

Case 3: $m_0 = 0$

This implies

$$\lim_{\nu \to \infty} E(r_{\nu}^* w_{\nu}, [0, 1] \times S^1) = \lim_{\nu \to \infty} E(w_{\nu}, [L_{\nu} - 1, L_{\nu}] \times S^1) = 0.$$ 

So, $E(w^-) = 0$, and $u^-$ maps to $Ku^+(\infty)$.

We next prove proposition 3.7: this is about the continuity of the evaluation map and preservation of homology class under Gromov convergence.

Proof of proposition 3.7 (a). We focus on a marked point $z_0$ and a sequence of vortices $w_{\nu} = (A_{\nu}, u_{\nu})$ defined on trivial $K$-bundles on cylinders $[0, \infty) \times S^1$. The proposition is equivalent to showing - 'Given $w_{\nu}$ converges to $w = (A, u)$ in $C^\infty$ on compact subsets of $[0, \infty) \times S^1$ and

$$\lim_{K \to \infty} \lim_{\nu \to \infty} E(w_{\nu}, [R, \infty) \times S^1) = 0,$$

then $\lim_{\nu \to \infty} Ku_{\nu}(\infty) = Ku(\infty)$.' First, we observe that the limits $Ku_{\nu}(\infty)$, $Ku(\infty)$ exist using the removal of singularity theorem (proposition 2.7). Next, using (21) and the annulus lemma (proposition 4.3 (a), the convergence $Ku_{\nu}(re^{i\theta}) \to Ku_{\nu}(\infty)$ as $r \to \infty$ is uniform for all $\nu$. This proves the result.

Proof of proposition 3.7 (b). The conservation of equivariant homology class can be proved in a way similar to chapter 5 in Ziltener’s thesis [Zil06]. But, the proof is quite transparent if the sequence $(A_{\nu}, u_{\nu})$ is part of a continuous family as in the quasimap case. We use that to give an indirect proof in the cases when $X$ is affine and hence $Qmap(X//G, \beta)$ is compact. From the proof of theorem 0.2 in section 7, we know

$$\Psi : \bigsqcup_{\beta \in H_2^G(X)} Qmap(X//G, \beta) \to \bigsqcup_{\beta \in H_2^G(X)} MV^K(X, \beta)$$

is continuous, bijective and preserves $\beta$. We remark that $H^K_*(X)$ and $H_*^G(X)$ are isomorphic because $G/K$ is contractible. In [CKM11], it is shown that a family in $Qmap(X//G, \beta)$ parametrized by $S\backslash \{0\}$ extends to a family over $S$. The map $u$ restricted to the central fiber represents the same equivariant homology class as $[u_s]$ for $s \in S \backslash \{0\}$ because there is a cobordism between $u_0$ and $u_s$. So, $Qmap(X//G, \beta)$ is compact. So, after passing to a sub-sequence, $\Psi^{-1}(A_{\nu}, u_{\nu})$ converges to $w_{\infty} \in Qmap(X//G, \beta)$. So, $[\Psi(w_{\infty})] = [(A, u)] = \beta$.

We finally show that the Gromov limit is unique.
Lemma 4.7 (Uniqueness of Gromov limit). Assume the setting in theorem 3.6. Suppose the sequence of vortices \((A_\nu, u_\nu)\) converges to a stable vortex \((A, u)\). Then this limit is unique up to 1) gauge transformations, 2) translation on cylindrical components i.e. \(\alpha \in \Gamma_C\), 3) reparametrization of spherical components i.e. \(\alpha \in \Gamma_S\), 4) action of the finite automorphism group on \(C\).

Proof. In proposition 4.2, suppose \((A_\nu, u_\nu)\) is a converging sub-sequence. Then the bubbling set is unique because these are the points where the energy density blows up. The limit \((A, u)\) is unique up to gauge transformation because: \(A\) is unique up to gauge because the space \(A/K\) is Hausdorff (see lemma 4.2.4 in \([DK90]\)). Once we know gauge transformations \(k_\nu\) such that \(k_\nu A_\nu\) converges to \(A\), then the limit \(k_\nu u_\nu\), if it exists, is obviously unique. The uniqueness of the sphere bubble trees in the fibers \(P(X)_z\) can be proved in the same way as the proof of uniqueness of limit of \(J\)-holomorphic curves (section 5.4 in \([MS04]\)). The uniqueness is up to reparametrization of components in \(\Gamma_S\).

Next, consider the formation of cylindrical bubbles. Step 1 of the proof of lemma 4.4 shows that the choice of the sequence \(\{ s_\nu \}_{\nu}\) is unique in the following sense. If there is another sequence \(s'_\nu\) for which lemma 4.4 is satisfied, then after passing to a subsequence \(\lim_{\nu} s'_\nu - s_\nu = L < \infty\). In that case, the limit \(w'_\nu\) will just be a re-parametrization of \(w : w_1(\cdot) = w(L + \cdot)\). Suppose not. If \(L = \infty\), then \(m_0 - m_1\) amount of energy would be lost, and by part (d) of the lemma, we know \(m_0 - m_1\) is positive. So, lets assume \(s'_\nu < s_\nu\) and \(L = -\infty\). In Step 1, we showed \(\lim_{\nu}(s_\nu - \sigma_\nu)\) is finite, so \(s'_\nu < \sigma_\nu\). Therefore,
\[
\lim_{\nu \to \infty} E(w_\nu, [s'_\nu, \infty) \times S^1) = m_0.
\]

By the same arguments as were used for \(\sigma_\nu\), we can say \(\lim_{\nu}(s_\nu - s'_\nu)\) is finite. The rest of cylinder-breaking proof is an application of this lemma, and hence the limit stable vortex is unique.

The fact that the limit is unique only up to the action of the finite group \(\text{Aut}(C)\), is hidden because we have fixed a choice of diffeomorphisms \(h_s : C_s \to C_0\). Modifying the family \(h_s\) can have the effect of pulling back \((A, u)\) by an element in \(\text{Aut}(C)\).

5. \(\overline{MV}^K_{g,n}(X)\) is Hausdorff

On the space \(\overline{MV}^K_{g,n}(X)\), Gromov topology is defined as: a set \(S \subset \overline{MV}^K_{g,n}(X)\) is closed if for any Gromov convergent sequence in \(S\), the limit also lies in \(S\). In this section we show

Proposition 5.1. On \(\overline{MV}^K_{g,n}(X)\), convergence in the Gromov topology coincides with Gromov convergence. \(\overline{MV}^K_{g,n}(X)\) is Hausdorff under this topology.

At a first glance, the first statement may appear obvious. But it is true only if we know that \(\overline{MV}^K_{g,n}(X)\) is Hausdorff and first countable - see discussion in section 5.6 of \([MS04]\). We follow the approach in \([MS04]\), borrowing some ideas from \([IP13]\). For
any stable vortex \( w \) on a Riemann surface with cylindrical ends \( \Sigma \) (corresponding to stable nodal curve \( C \)) modelled on a graph \( \Gamma \), we define a metric-like function

\[
d_w : \overline{\mathcal{M}_{g,n}^K}(X) \times \overline{\mathcal{M}_{g,n}^K}(X) \to [0, \infty)
\]

that satisfies the following:

**D1:** \( d_w(w, w) = 0 \).

**D2:** A sequence \( w_\nu \) in \( \overline{\mathcal{M}_{g,n}^K}(X) \) Gromov converges to \( w \) if and only if

\[
\lim_{\nu \to \infty} d_w(w, w_\nu) = 0.
\]

**D3:** If a sequence \( w_\nu \) in \( \overline{\mathcal{M}_{g,n}^K}(X) \) Gromov converges to \( w' \), then

\[
\lim \sup_{\nu \to \infty} d_w(w, w_\nu) \leq d_w(w, w').
\]

In addition, if we know that Gromov limits are unique, then proposition 5.1 follows using proposition 5.6.5 in [MS04].

The forgetful map \( \pi : \overline{\mathcal{M}_{g,n}^K}(X) \to \mathcal{M}_{g,n} \) is continuous. Since the properties D1-D3 are regarding a small neighborhood of \( w \), \( d_w \) can be defined so that \( d_w(w_0, w_1) \) is finite only if both \( \pi(w_0) \) and \( \pi(w_1) \) are in a neighborhood \( [C] \in S_C \subset \overline{\mathcal{M}_{g,n}} \). We assume \( S_C \) is small enough that for any \( [C'] \in S \), the discussion in remark 3.2 applies, i.e. if \( \Sigma' \) is the Riemann surface with cylindrical ends corresponding to \( C' \), we can write \( \Sigma' = \bigsqcup_{\alpha \in \Gamma(\Sigma)} \Sigma'_\alpha / \sim \), so that \( \Sigma'_\alpha \subset (\Sigma, g_{\Sigma'}) \). The metrics \( g_{\Sigma'} \) and \( g_{\Sigma} \) agree on cylindrical ends. While defining \( d_w \), we don’t keep track of metrics and just use \( g_{\Sigma} \) everywhere. The cylinders \( \Sigma_\alpha = (S^1 \times \mathbb{R})_\alpha, \alpha \in \Gamma_C \) are equipped with the standard metric. For a Riemann surface \( \Sigma \) and two points \( p_0, p_1 \) lying on the same component of \( \text{Cyl}(\Sigma) \) (see remark 3.2 for definition), the quantity \( p_1 - p_0 \in \mathbb{R} \times \mathbb{R} / 2\pi \mathbb{Z} \) makes sense.

We don’t consider sphere bubbles in the fibers \( P(X)_x \), i.e. \( \Gamma_S = \emptyset \). So, the domain of \( w \) is \( \Sigma \cup \sqcup_{\alpha \in \Gamma_C} \Sigma_\alpha \). Sphere bubbles can be incorporated into the definition of \( d_w \) in a manner similar to that of [MS04] and is left to the reader. \( C_\Gamma \), the curve corresponding to the domain of \( w \), can be stabilized by adding \( k \) marked points \( \overline{\pi} = (x_\alpha)_{\alpha \in \Gamma_C} \) - one on each cylindrical bubble \( \Sigma_\alpha \). Without loss of generality, we can assume \( x_\alpha \) is the origin in the coordinate system \( \mathbb{R} \times \mathbb{R} / \mathbb{Z} \). Now, \( [(C_\Gamma, \overline{\pi})] \in \overline{M}_{g,n+k} \). For stable vortices \( w_0, w_1 \), we define \( d_w(w_0, w_1) \) when there are morphisms \( \Gamma \xrightarrow{f} \Gamma(w_0) \xrightarrow{g} \Gamma(w_1) \). If not, \( d_w(w_0, w_1) := \infty \). To calculate \( d(w_0, w_1) \), the domain of \( w_0, w_1 \) also have to be realized as stable \( n + k \)-pointed curves. For this we choose marked points \( \overline{x}_0, \overline{x}_1 \), calculate \( d(w_0, w_1; \overline{x}_0, \overline{x}_1) \) and then take infimum over all choices. \( \overline{x}_i \in C_{\Gamma(w_i)} \) are chosen in a way that

- \( f \) and \( g \) can be extended to \( \Gamma' \xrightarrow{f'} \Gamma'(w_0) \xrightarrow{g'} \Gamma'(w_1) \), where \( \Gamma'(w_i) \) is \( \Gamma(w_i) \) with the additional marked points.
- \( \overline{x}_i \in \text{Cyl}(\Sigma_i) \cup (\sqcup_{\alpha \in \Gamma(w_i)} \Sigma_\alpha) \). Recall that \( \Sigma_i \simeq C_{st}(\Gamma(w_i)) \), so we have to include the cylindrical \( \alpha \in \Gamma(w_i) \) which are unstable in the domain.
- \( \overline{x}_0 \) and \( \overline{x}_1 \) are compatible with the metric. If the marked points \( x_{0,i}, x_{0,j} \) lie in the same component \( \alpha \in \text{Vert}(\Gamma'_0) \), then \( x_{0,i} - x_{0,j} = x_{1,i} - x_{1,j} \) in \( \mathbb{R} \times \mathbb{R} / 2\pi \mathbb{Z} \).
Given $L > 0$ and a choice of marked points $\overline{x}_0, \overline{x}_1$ on the domains of $w_0$, $w_1$ respectively, we can construct a Riemann surface $\tilde{\Sigma}_{i,L,\overline{x},\alpha}$ which is the disjoint union of the following:

\[ \tilde{\Sigma}_{i,L,\overline{x},\alpha} = \begin{cases} \tilde{\Sigma}_{i,\alpha} \setminus \bigcup_{j \in \text{Edge}(\gamma_i)} \{ z_j \in \Sigma : \rho^{(\gamma_i)}_{z_j}^{-1}\{ r > L \} \} \\ \bigcup_{j \in \text{Edge}(\gamma_i)} \{ z_j \in \Sigma : \rho^{(\gamma_i)}_{z_j}^{-1}\{ \pm r > L \} \} \\ \{ x_i,\alpha + z : |\text{Re}(z)| \leq L \} \cap \text{Cyl}(\Sigma_i, f_i(\alpha)) \end{cases} \quad \alpha \in \text{vert}(\text{st}(\Gamma)) \]

In the last line, $f_0 := f$, $f_1 = g \circ f$. For $\alpha \in \text{Vert}(\text{st}(\Gamma))$, $\tilde{\Sigma}_{i,L,\overline{x},\alpha}$ is defined by pull-back on $\tilde{\Sigma}_i$. On both $\tilde{\Sigma}_i$ and the cylindrical vertices of $\Gamma(w_i)$, the principal $K$-bundle is trivializable. We fix a trivialization, and by restriction define $w$ on the trivial $K$-bundle on $\tilde{\Sigma}_{i,L,\overline{x}}$. Now, both $w_0$, $w_1$ are defined on the trivial bundle on $\Sigma_0 := \tilde{\Sigma}_{0,L,\overline{x}_0} \cap \tilde{\Sigma}_{1,L,\overline{x}_1}$. Define $\rho(w_0, w_1; \overline{x}_0, \overline{x}_1, L) := \| A_0 - A_1 \|_{C^1(\tilde{\Sigma}_{0,L,\overline{x}_0} \cap \tilde{\Sigma}_{1,L,\overline{x}_1})} + \| d_x(u_0, u_1) \|_{C^0(\tilde{\Sigma}_{0,L,\overline{x}_0} \cap \tilde{\Sigma}_{1,L,\overline{x}_1})}$.

Given $(w_0, w_1; \overline{x}_0, \overline{x}_1)$, define

\[ \rho(w_0, w_1; \overline{x}_0, \overline{x}_1) := \inf_{k \in K(\Sigma_0)} \sum_{L \in \mathbb{Z}_{\geq 0}} 2^{-L} \frac{\rho(kw_0, w_1; \overline{x}_0, \overline{x}_1, L)}{1 + \rho(kw_0, w_1; \overline{x}_0, \overline{x}_1, L)} \]

\[ \rho(w_0, w_1) := \inf_{f_0, f_1} \left( \rho(w, w'; s) + d_{\mathcal{M}_{\theta,n+k}^{\theta}}((C_0, \overline{x}_0), [(C_1, \overline{x}_1)]) \right) \]

**Proof of proposition 5.1.** : By the definition of $d_w$, it is easy to see that D1-D3 are satisfied. By lemma 4.7, Gromov limits are unique up to gauge transformation. The proof follows using proposition 5.6.5 in [MS04].

6. Quasimaps to GIT Quotients

In this section, we define quasimaps and recall why the moduli space of stable quasimaps is proper. For details, refer to the paper by Ciocan-Fontanine, Kim and Maulik [CKM11]. Suppose $X$ is an affine variety and $G$ a reductive complex algebraic group acting on $X$, the action is linearized by a character $\theta \in \xi(G)$. Any character $\theta$ determines a one-dimensional representation $\mathbb{C}_\theta$ of $G$, and hence a $G$-equivariant line bundle $L_\theta = X \times \mathbb{C}_\theta$ - it is ample since $X$ is affine. We fix a $\theta$ and under this linearization, let $X^s$, $X^{ss}$ denote the stable and semistable locus of $X$ respectively and that $X^s/G$ is the GIT quotient. We assume $X^s = X^{ss}$ and that $G$ acts freely on $X^s$.

**Definition 6.1.** Given integers $g$, $n \geq 0$ and a class $\beta \in H^2_G(X)$, an $n$-pointed genus $g$ quasimap of class $\beta$ to $X^s/G$ consists of the data $(C, p_1, \ldots, p_n, P, u)$, where

- $(C, p_1, \ldots, p_n)$ is a nodal curve of genus $g$ with $n$ marked points,
- $P$ is a principal $G$-bundle on $C$,
- $u \in \Gamma(C, P \times_G X)$ such that $(P, u)$ is of class $\beta$,
satisfying: there is a finite set $B$ so that $u(C \backslash B) \subset X^{ss}$. The points in $B$ are called the base points of the quasimap. For a component $C' \subset C$, $u$ is constant on $C'$ if $u(C') \subset X^{ss}$ and $u_{f|C'} : C' \to X//G$ is a constant. $(C, p_1, \ldots, p_n, P, u)$ is stable if the base points are disjoint from the nodes and markings on $C$ and

- every genus 0 component $C'$ of $C$ has at least 2 marked or nodal points. If it has exactly 2 marked points, then $u$ is non-constant on $C'$,
- if for a genus 1 component $C'$ of $C$, $u$ is constant on $C'$, then $C$ has at least one marked or nodal point.

An isomorphism between two quasimaps $(C, p, P, u)$ and $(C', p', P', u')$ consist of isomorphisms $f : (C, p) \to (C', p')$ and $\sigma : P \to f^*P'$ such that under the induced isomorphism $\sigma_X : P \times_G X \to f^*(P' \times_G X)$, $u$ maps to $u'$.

**Definition 6.2.** A family of quasimaps over a base scheme $S$ consists of the data $(\pi : C \to S, \{p_i : S \to C\}_{i=1,\ldots,k}, \mathcal{P}, u)$ where $C \to S$ is a proper flat morphism such that each geometric fiber $C_s$, $s \in S$ is a connected curve, $p_i$ are sections of $\pi$, $\mathcal{P} \to C$ is a principal $G$-bundle and $u : C \to \mathcal{P} \times_G X$ is a section. For each $s \in S$, $(C_s, u(s), \mathcal{P}_s, u|_{C_s})$ is a quasimap.

We describe the compactification of $Qmap_{g,n}(X//G, \beta)$ from [CKM11]. Let $(S, 0)$ be a pointed curve and let $S^0 = S \backslash \{0\}$. Let $((C, p_i), P, u)$ be a $S^0$-family of stable quasimaps. After possibly shrinking $S$ and an etale base change, the base points can be regarded as additional sections $y_i : S^0 \to C$. The family gives a rational map $[u] : C \to X//G$ which is regular on $C \backslash B$. Since $X//G$ is projective, after shrinking $S$ (removing closed sets where the extension requires blowing up), $[u]$ extends to a map on all of $C$, denoted by $[u_{reg}]$. Stability of the quasimaps implies that the family $(C, (p, y), [u_{reg}])$ is a family of stable vortices to $X//G$. By compactness of the moduli space of stable vortices, after an etale base change possibly ramified at 0, there is a family of stable vortices

$$(\hat{C}, (\underline{p}, \underline{y}), [\hat{u}]) \to S, \quad [\hat{u}] : \hat{C} \to X//G$$

extending $(C, [u_{reg}])$. The surface $\hat{C}$ has at most nodal singularities in the central fiber $\hat{C}_0$. Pull back the principal $G$-bundle $X^s \to X//G$ via $[\hat{u}]$ to obtain a principal $G$-bundle $\hat{P}$ and an induced section $\hat{u} : \hat{C} \to \hat{P} \times_G X$.

Next, consider all maximal sub-trees of sphere bubbles (rational tails) $\Gamma_1, \ldots, \Gamma_N$ in $\hat{C}_0$ that have none of the marked points $p_i$ and meets the rest of the curve $(\hat{C}_0 \backslash \Gamma_i)$ in a single point $z_i$. Contract these subtrees in $\hat{C}$ and call the resulting surface $\hat{C}$ and let $(\overline{P}, \overline{u}) := (\hat{P}, \hat{u})|_{\overline{C}}$. $(\overline{P}, \overline{u})$ is well-defined on $C \backslash \{z_1, \ldots, z_N\}$. Base points $y_j$ may come together at the points $z_1, \ldots, z_N$, but these points are away from the markings $p_i$. By lemma 4.3.2 in [CKM11], $\overline{P}$ extends to a principle $G$-bundle on $C$ (i.e. there exists a suitable trivialization of $P$ in a neighborhood of 0). By Hartog’s theorem, since $X$ is affine, $\overline{u}$ extends to all of $C$. 
7. A Homeomorphism between Quasimaps and Vortices

The proof of 0.2 is carried out in this section.

7.1. From a $G$-variety to a Hamiltonian $K$-manifold. As in [CKM11], we assume the target $X$ is affine. By proposition 2.5.2 in [CKM11], there is a vector space $V$ with a linear $G$-action and a $G$-equivariant closed embedding $X \hookrightarrow V$. The $G$-action on $V \cong \mathbb{C}^n$ is given by a homomorphism $\phi : G \rightarrow GL(n)$. Let $K \subset G$ be a maximal compact subgroup of $G$. Choose coordinates on $V$ such that $\rho(K) \subset U(n)$. The action of $U(n)$ on $V$ with the standard symplectic form $\sum_i(dx_i \wedge dy_i)$ is Hamiltonian with moment map $\Phi_{U(n)} : x \mapsto \frac{i}{2}zz^* \in u(n)$.

Here, $u(n)^*$ is identified with $u(n)$ via the inner product $(A, B) = \text{trace}(A^* B)$. The moment map for the $K$-action is $\Phi' = \phi^* \circ \Phi_{U(n)} : X \rightarrow \mathfrak{k}^*$ where $\phi^* : u(n)^* \rightarrow \mathfrak{k}^*$.

But, recall that the GIT quotient of $X$ is defined in terms of a polarization - which is a line bundle $X \times \mathbb{C}_g$. And, a choice of polarization is equivalent to a choice of moment map. The moment map is unique up to an additive term in $\mathfrak{z}(\mathfrak{k^*})$. So, the moment map corresponding to the given polarization is $\Phi = \Phi' + c$, where $c \in \mathfrak{z}(\mathfrak{k^*})$.

By the Kempf-Ness theorem ([KN79]), a point $x$ is semi-stable iff its orbit-closure intersects $\Phi^{-1}(0)$. Together with the stable=semistable assumption, we get $X^{ss} = G\Phi^{-1}(0)$.

The $G$-action on $X^{ss}$ being free is same as the $K$-action on $\Phi^{-1}(0)$ being free. It is easy to see that assumption 2.4 is satisfied for the moment map $\Phi + c$ for any $c \in \mathfrak{z}(\mathfrak{k^*})$.

In general, any quasiprojective polarized $G$-variety can be embedded equivariantly into projective space $\mathbb{P}(V)$ in a way that the $G$-action is given by a homomorphism $G \rightarrow GL(V)$. Under the Fubini-Studi form, the $K$-action is Hamiltonian and produces a moment map on $X$. But this won’t satisfy the convexity assumption 2.4 unless $X$ is projective.

7.2. Correspondence between vortices and stable quasimaps. A prestable quasimap on a curve $C$ to a GIT quotient $X//G$ is a pair $(P_C, u)$ where $P_C \rightarrow C$ is a principal $G$-bundle and $u : C \rightarrow P_C \times_G X$ is holomorphic such that $u$ maps special points on $C$ to $X^{ss}$. By choosing a section $\sigma : C \rightarrow P_C/K$, we get a principal $K$-bundle $P \rightarrow C$ and a gauged holomorphic map $(A, u)$ on $C$. Suppose $[C] \in \overline{M}_{g,n}$ and $\Sigma$ is a Riemann surface with cylindrical ends corresponding to $C$. Theorem 7.3 says that a gauged holomorphic map can be ‘weakly complex gauge transformed’ to a finite energy vortex on $\Sigma$ and this vortex is unique up to unitary gauge transformations (i.e. $k : \Sigma \rightarrow K$). This shows that $\Psi$ in theorem 0.2 is a bijection.
Definition 7.1 (Complex gauge transformations). The complexified gauge group $\mathcal{G}(P)$ consists of sections $g : \Sigma \to P \times_K G$. Recall that

\begin{equation}
K \times \mathfrak{k} \to G
(k, s) \mapsto ke^{is}
\end{equation}

is an isomorphism. So, a complex gauge transformation $g$ can be written as $g = ke^{i\xi}$, where $k \in K(P)$ and $\xi \in \text{Lie}(K(P)) = \Gamma(P(\mathfrak{k}))$.

Definition 7.2. (Weak gauge transformations and gauged holomorphic maps) Let $p > 2$. We call a gauged holomorphic map $(A, u)$ on $P \to C$ weak if it is smooth on $C \setminus \{z_1, \ldots, z_n\}$ and for $1 \leq i \leq n$, on a neighbourhood of $z_i$ - $N(z_i) \subset C$, $(A, u)|_{N(z_i)} \in L^p \times W^{1,p}$. A (complex) gauge transformation on $P$ is weakly extendable if it is smooth on $C \setminus \{z_1, \ldots, z_n\}$ and in $W^{1,p}$ in a neighbourhood of $z_i$. Denote by $G(\text{we})$ the group of weakly extendable gauge transformations.

In the following theorem, $X$ is allowed to be a Hamiltonian Kähler manifold which may not have the structure of an algebraic variety, so we overload notation and define $X^{ss} := G\Phi^{-1}(0)$.

Theorem 7.3. ([VW13] theorem 2.10, remark 2.11) Suppose $X$ is Kähler manifold with Hamiltonian action of a compact Lie group $K$, which is either compact or equivariantly convex at infinity with a proper moment map (i.e. satisfies assumption 2.4). Let $G$ be the complexification of $K$, and suppose $G$ acts freely on $X^{ss}$.

Let $C, \Sigma$ be as above. Let $(A, u)$ be a gauged holomorphic map from $\Sigma$ to $X$ that extends to a map over some principal bundle $P \to C$, and suppose $u(z_i) \in X^{ss}$ for all the marked point $z_1, \ldots, z_n$. For any $p > 2$, there is a weakly extendable complex gauge transformation $g \in G(\text{we})$ such that $g(A, u)$ is a smooth finite energy symplectic vortex, which is unique up to left multiplication by a unitary gauge transformation.

Conversely, given any finite energy symplectic vortex on $\Sigma$, there is a $K$-bundle $P \to C$ so that $(A, u)$ extends to a weak gauged holomorphic map on $P$. There is a weakly extendable complex gauge transformation $g \in G(\text{we})$ so that $g(A, u)$ is smooth over $C$. The gauged holomorphic map $g(A, u)$ is unique up to complex gauge transformations in $G(\text{we})$.

Outline of proof. We first prove the first statement - going from a gauged holomorphic map on $C$ to a vortex on $\Sigma$.

Step 1: We can pick a reduction of the bundle $\sigma : C \to P_C/K$ so that

(a) there is a trivialization of the bundle $P|_{\Sigma}$ so that on any cylindrical end $N(z_i) \setminus \{z_i\}$, $A = d + \lambda d\theta$.
(b) $u(z_i) \in \Phi^{-1}(0)$.

We start out with an arbitrary choice of reduction. The first condition is achieved by working on a compact neighborhood of $z_i$ in $C$ and applying a complex gauge
transformation that makes the connection flat in this neighborhood. For the second condition, we can use a constant gauge transformation - since \( u(z_i) \in X^{ss} \) and any semistable \( G \)-orbit in \( X \) contains a unique \( K \)-orbit in \( \Phi^{-1}(0) \).

**Step 2:** \((A,u)\) can be complex gauge transformed so that it is a finite energy vortex on the cylindrical ends - i.e. outside of a compact subset of \( \Sigma \).

In a chart around \( z_i - N(z_i) \subset \Sigma \), there is a trivialization of \( P \) so that the connection is trivial, so \( u : N(z_i) \to X \) is holomorphic. Since \( \Phi \) is analytic on \( X \), \( \Phi \circ u \) is analytic on \( N(z_i) \). So, close to \( z_i \), \( |\Phi \circ u| \leq c|z - z_i| \). Switching to working with \( \Sigma \) with cylindrical coordinates on \( N(z_i) \setminus \{z_i\} \), we get \( |\Phi \circ u| \leq ce^{-r} \). This implies that \( \|\Phi \circ u\|_{L^2(\Sigma)} < \infty \) and since \( A \) is flat on the cylindrical ends, the \( L^2 \) norm of the curvature \( F_A \) is finite on \( \Sigma \) which gives us that \( \|\ast F_A + \Phi(u)\|_{L^2(\Sigma)} < \infty \). Now, by an implicit function theorem argument (proposition 5.3 in [VW13]) on each of the cylindrical ends, for large enough \( r_i \), one can find a complex gauge transformation \( e^{i\xi_i} \) that makes \((A,u)\) a finite energy vortex on \( p_i^{-1}(\{ r \geq r_i \}) \).

**Step 3:** \((A,u)\) can be complex gauge transformed to a finite energy vortex on all of \( \Sigma \).

We are given that \((A,u)\) satisfies the vortex condition on \( \Sigma \setminus \Sigma_0 \), where \( \Sigma_0 \subset \Sigma \) is compact. This step uses a Hitchin-Kobayashi correspondence for symplectic vortices on compact surfaces with boundary proved by the author in [Ven12]. It says: ‘Suppose \( \Omega \) is a compact Riemann surface with boundary and \((A,u)\) is a gauged holomorphic map from \( \Omega \times K \) to \( X \) such that \( \ast F_A + \Phi(u)|_{\partial \Omega} = 0 \). Then, there exists a complex gauge transformation \( e^{i\xi} \) such that \( e^{i\xi}(A,u) \) is a vortex on \( \Omega \). \( \xi \in W^{2,p}(\Omega,G) \) and \( \xi|_{\partial \Omega} = 0 \).’ We can potentially apply this result to \( \Sigma \), but since the \( \xi \) produced by it is not in \( W_0^{2,p} \), it can’t be extended to \( W_{loc}^{2,p} \) in a way that preserves the vortex condition on \( \Sigma \setminus \Sigma_0 \). To get around this issue, we exhaust \( \Sigma \) by a sequence of compact subsets \( \Sigma_0 \subset \Sigma_1 \subset \cdots \subset \Sigma \). The above-mentioned Hitchin-Kobayashi result is applied in each of these subsets, it can be shown that the limit is indeed a finite energy vortex on \( \Sigma \) that is related to the initial gauged holomorphic map by a weak complex gauge transformation.

The second part is proved by corollary 2.10. Refer to [VW13] for proofs of uniqueness and other details.

**Remark 7.4** (Bijection between \( Qmap_{g,n}(X//G, \beta) \) and \( \overline{MV}^K_{g,n}(X, \beta) \)). Theorem 7.3 produces a bijection
\[
\Psi : Qmap_{g,n}(X//G, \beta) \to \overline{MV}^K_{g,n}(X, \beta).
\]
This is because - given a stable quasimap \((C,\bar{z}, P, u)\), we can apply the above result on each component of \( C \) to produce an element of \( \overline{MV}^K_{g,n}(X) \) - it will be a stable vortex on \( C = st(C) \in \overline{M}_{g,n} \) with neck-stretching metric. The components in \( C \) that are not present in \( C \) are all rational and will have at least two special points (by the stability of quasimap) and these would correspond to the cylindrical vertices on the vortex side, with the two special points mapped to the two ends of the cylinder.

For the reverse step, we need to choose the complex gauge transformation in each component in way that the nodal points \( z_i^+ \) and \( z_i^- \) map to the same point. Note that
since $X$ is affine, in particular it is aspherical, the components of the stable vortex would consist of components of $C$ and some cylindrical vertices. The cylindrical vertices are non-constant and have 2 special points - so a stable vortex would indeed produce a stable quasimap. Finally it is easy to see that $\Psi$ preserves the equivariant homology class $\beta \in H^G_2(X)$. □

7.3. **Continuity for smooth curves.** In this and the following section, we prove the continuity of the map $\Psi$ in theorem 0.2. Together with the fact that $\Psi$ is a bijection and the spaces $\overline{MV}_{g,n}^K(X,\beta)$ and $Qmap(X//G,\beta)$ are compact and Hausdorff, we can conclude that $\Psi$ is a homeomorphism.

To prove continuity of $\Psi$, the setting is : let $S \subset \mathbb{C}$ be a neighborhood of 0. Given a family of quasimaps $Q_S$ parametrized by $S$, $Q_S = (C \to S, \{z_i : S \to C\}_{i=1,...,k}, P_C, u)$, our first step is to make a preliminary choice of reduction $\sigma : C \to P_C/K$ (lemma 7.6) so that we have a family of gauged holomorphic maps $w_s = (A_s, u_s)$ parametrized by $S$. Each member of the family can be complex gauge transformed by $g_s$ to a vortex with cylindrical ends. We show that for any sequence $s_i$ in $S$ converging to 0, $g_{s_i} w_{s_i} \text{ Gromov converges to } g_0 w_0$. The strategy is roughly the following: $g_0 w_{s_i}$ converges to $g_0 w_0$. We show that for gauged holomorphic maps close to vortices, there exist small complex gauge transformations that make it a vortex - we can find a sequence $\xi_i \to 0$ such that $e^{i\xi_i} g_0 w_{s_i}$ is a vortex. We first prove the result in the case that for all $s \in S$, the curve $C_s$ is smooth i.e. it has only one component.

**Proposition 7.5.** Given a family of quasimaps $Q_S = (C \to S, \{z_i : S \to C\}_{i=1,...,k}, P_C, u)$ parametrized by $S$, where $S \subset \mathbb{C}$ is a neighborhood of 0. Suppose $C_0$ is a smooth curve, then for any sequence $s_\nu \to 0$ as $\nu \to \infty$, $\Psi(w_{s_\nu})$ Gromov converges to $\Psi(w_0)$. ($\Psi$ is as described in remark 7.4.)

**Proof.** Since $C_0$ is smooth, we can assume by shrinking $S$ that all $C_s$ are smooth (i.e. no nodal singularities). Also, there is a family of diffeomorphisms $h_s : C_s \to C_0$ that are biholomorphisms in the neighborhoods of marked points $z_i$ and $h_0 = \text{Id}$. Under this diffeomorphism, the corresponding Riemann surfaces with cylindrical ends can be written as $\Sigma_s = (\Sigma, g_s)$. $g_s = g_0$ on $N(z_i) \setminus \{z_i\}$ and $g_s \to g_0$ in $C^\infty(\Sigma \setminus (\cup_i N(z_i) \setminus \{z_i\}))$. In this proof, we ignore this variation in metric i.e. $g_s = g_0$, all the ideas carry over easily to the general case. We have a sequence $s_\nu \to 0$ in $S$.

Using lemma 7.6, choose a reduction $\sigma : C \to P_C/K$ so that the gauged holomorphic maps $w_s := (A_s, u_s)$ are in standard form close to marked points - i.e. they satisfy conditions (a) and (b) in lemma 7.6. We fix a trivialization $P_s|_{\Sigma_s}$ so that $A_s = d + \lambda_i d\theta$ on $N(z_i)s$ for all $s$. We observe that under this trivialization,

\begin{equation}
\|A_s - A_0\|_{W^{1,p}(\Sigma)}, \|d(u_s, u_0)\|_{L^\infty(\Sigma)}, \|\Phi(u_s) - \Phi(u_0)\|_{L^p(\Sigma)} \to 0 \text{ as } s \to 0.
\end{equation}

By theorem 7.3, there is a weak complex gauge transformation $g_0$, on $P_0$ that takes $w_0$ to a finite energy vortex on $\Sigma_0$. Under the trivialization $P_s|_{\Sigma_s}$ in Step 1, by lemma 7.9 we get

$$\|g_0\|_{W^{2,p}(\rho_s, \{n \le r \le n+1\})} \le c e^{-\gamma s}.$$
Then, the family of gauged holomorphic maps \( g_0w_s \) continues to satisfy (23). To the converging sequence \( g_0w_{s_n} \to g_0w_0 \), apply lemma 7.10. This will give a sequence \( \xi_{s_n} : \Sigma \to \mathfrak{k} \) converging to 0 in \( W^{2,p}(\Sigma) \) such that \( e^{\xi_{s_n}}g_0w_{s_n} \) is a vortex on \( \Sigma_\nu \). So, the vortices \( e^{\xi_{s_n}}g_0w_{s_n} \) converge to \( g_0w_0 \) in \( W^{1,p}(\Sigma) \times W^{2,p}_{loc}(\Sigma) \). Also \( e^{\xi_{s_n}}g_0w_{s_n}, g_0w_0 \) are in the gauge orbits \( \Psi(w_{s_n}), \Psi(w_0) \) respectively.

Lemma 7.6 (Standard form around marked points). Given a family of quasimaps \( Q_S \) such that all the fibers in \( C \to S \) are smooth (i.e. none of the fibers \( C_s \) has nodal singularities), there is a reduction \( \sigma : C \to P_C/K \) and \( \lambda_1, \ldots, \lambda_n \in \mathfrak{k} \) satisfying \( e^{2\pi\lambda_i} = \text{Id} \), such that

(a) for every \( s \in S \), there is a trivialization \( \tau \) of the restriction \( P_s \to \Sigma_s \) so that the cylindrical end \( N(z_i) \setminus \{z_i\}, A_s = d + \lambda_i d\theta \) and

(b) \( u(z_i(s)) \in \Phi^{-1}(0) \) for all \( s \in S \) and \( 1 \leq i \leq n \).

\( \sigma, \tau \) are smooth in every fiber, as we vary \( s, \sigma_s, \tau_s \) vary continuously in the \( W^{2,p}_{loc} \) topology.

Remark 7.7. If \( \Sigma \) is a Riemann surface with cylindrical ends, a gauged holomorphic map \((A,u)\) on \( \Sigma \) is said to be in standard form close to marked points if it satisfies conditions (a) and (b) in the lemma above.

Proof of lemma 7.6. On \( P_C \to C \), we start out with any choice of reduction. That we can find a family of complex gauge transformations \( e^{\xi_s} \in G(P_s) \) satisfying conditions (a) and (b) is the content of Step 1 in the proof of 7.3. The new task is to prove continuity of \( e^{\xi_s} \) as \( s \) varies. For this, we first give more details about how \( \xi_s \) is constructed.

We use the fact that on a surface \( \Omega \) with boundary, a connection can be complex gauge transformed to a flat connection by \( e^{\xi_0} \), where \( \xi \in \Gamma(\Omega, P(\mathfrak{k})) \) and \( \xi|_{\partial \Omega} = 0 \) (see for example theorem 1 in [Don92]). We apply this result on \( N(z_i)_s \) to obtain \( \xi_{i,s} \) for every \( s \in S \). Trivialize \( P_s|_{N(z_i)_s} \) so that \( e^{\xi_{i,s}}A_s \) is the trivial connection. Let \( g_{i,s} \in G \) be such that \( \Phi(g_{i,s}e^{\xi_{i,s}}u(z_i(s))) = 0, g_{i,s}e^{\xi_{i,s}}A_s \) is still the trivial connection on \( N(z_i)_s \). By using a cut-off function (that varies smoothly with \( s \) and vanishes away from the cylindrical ends, \( g_{i,s} \xi_{i,s}, 1 \leq i \leq n \) can be patched together to obtain \( \xi_s \).

In order to show that \( \xi_s \) varies continuously with \( s \), it is enough to show that \( \xi_{i,s} \) varies continuously with \( s \) for each \( i \). It can easily be seen that the other steps preserve continuity in \( s \). The argument used for this is the prototype of the argument for continuity used in this section. We assume there is only one cylindrical end so that we can drop \( i \) from the notation. Pick a sequence \( s_0 \to 0 \) in \( S \). \( e^{\xi_{0}}A_0 \) is flat on \( N(z)_0 \), denoting \( A'_0 := e^{\xi_0}A_{s_0} \) we see that \( \|F_{A'_0}\|_{L^p(N(z(s_0)))} \to 0 \). By an implicit function theorem argument for large enough \( i \), we can find \( \xi_i \in W^{2,p} \) such that \( \|\xi_i\|_{W^{2,p}} \leq c\|F_{A'_i}\|_{L^p} \). The argument is similar to the one in proof of lemma 7.10 and so is omitted here.

Finally, we show that the condition (a) is satisfied fiberwise. Again, we can assume that there is only one marked point. Pick trivializations of \( P_s \) over \( \Sigma \) and \( N(z)_s \).
that vary continuously with $s$ and such that $A_s$ is trivial on $N(z)_s$. Suppose the transition functions are $k_s : N(z)_s \setminus \{z(s)\} \to K$. The loops $\theta \mapsto k_s(r + i \theta)$ are in the same homotopy class for all $r \geq 0$ and $s \in S$. Pick a geodesic loop $[0,2\pi] \ni \theta \mapsto e^{\lambda \theta}$ that is in this homotopy class - as in the proof of corollary 2.9, so there is a gauge transformation on $\Sigma_s$ that transforms $A_s|_{N(z)_s}$ to $d + \lambda d\theta$.

Remark 7.8 (Sobolev space of sections on non-compact base space). Suppose $P \to \Sigma$ is a principal $K$-bundle and $E = P \times_K V$ is an associated vector bundle. If $\Sigma$ is compact $W^{k,p}(\Sigma,E)$-topology on the space of sections of $E$ can be defined independent of the trivialization of the bundle $E \to \Sigma$. But this is not so if $\Sigma$ is non-compact. So one has to specify which trivialization is being used.

If $(A,u)$ is a gauged holomorphic map on $\Sigma$ that is in standard form, we can have a stronger bound on the complex gauge transformation $g$ that makes $(A,u)$ a vortex on $\Sigma$.

Lemma 7.9. Let $p > 2$. Suppose $C$ is a smooth curve with $n$ marked points and $\Sigma$ is the corresponding Riemann surface with cylindrical ends. Suppose $(A,u)$ is a smooth gauged holomorphic map on a $K$-bundle $P \to C$ that is in standard form close to marked points, and the weak complex gauge transformation $g \in G(P)_{\text{weak}}$ transforms $(A,u)$ to a finite energy vortex on $\Sigma$. There are constants $c$ and $0 < \gamma < 1$ such that in the trivialization of $P|_{\Sigma}$ found in lemma 7.6,

$$(24) \quad \|g\|_{W^{2,p}(\{n \leq r \leq n+1\})} \leq ce^{-\gamma n}.$$ 

Hence, $g \in W^{2,p}(\Sigma)$.

Here to take the $W^{2,p}$ norm of $G$, we use a trivialization of $G$ in a neighbourhood of $\text{Id}$ using the exponential map $\exp : g \to G$.

Proof of lemma 7.9. We assume that $\Sigma$ has only one cylindrical end $N(z)$. $(A,u)$ being in standard form means $\Phi(u(z)) = 0$ and there is a trivialization of the bundle $P|_{\Sigma}$ so that $A|_{N(z) \setminus \{z\}} = d + \lambda d\theta$. The transition function between $P|_{\Sigma}$ and $P|_{N(z)} \simeq N(z) \times K$ is $\theta \mapsto e^{-\lambda \theta}$. We use these trivializations to describe gauge transformations, for example $k \in K(P)$ is given by functions $k : \Sigma \to K$ and $\hat{k} : N(z) \to K$ such that $\hat{k} = e^{-\lambda \theta} k e^{\lambda \theta}$ on $N(z) \setminus \{z\}$. We’re given $(A_1,u_1) := g(A,u)$ is a vortex.

Step 1: There is a trivialization of $P|_{\Sigma}$ so that $A|_{\Sigma} = d + \lambda d\theta$ and $A_1|_{\Sigma} = d + \lambda d\theta + a$ where $a$ satisfies the exponential bound (8).

By corollary 2.9 we can assume, possibly after modifying $g$ by a unitary gauge transformation, that $A_1|_{N(z) \setminus \{z\}} = d + \lambda_1 d\theta + a$ for some $\lambda_1 \in \mathfrak{k}$ and $a$ satisfies (8).

Since $A_1$ extends to an $L^p$ connection over $P$, we can say $\lambda = \lambda_1$.

Step 2: For any positive constant $\gamma_1 < \gamma$ there exists $c > 0$, so that on $N(z) \setminus \{z\}$,

$$(25) \quad d(\text{Id}, g(r,\theta)) \leq ce^{-\gamma_1 n}.$$ 

Since $g \in G(P)_{\text{weak}}$, $\hat{g} \in W^{1,p}(N(z)) \hookrightarrow C^0$. So, $\hat{g}(z)$ is well-defined and since $\Phi(u(z)) = 0$, $\hat{g}(z) \in K$. We claim, we can assume $\hat{g}(z) = \text{Id}$. If not, define a gauge transformation $k_\infty \in K(P)$ as $\hat{k}_\infty : N(z) \to K$ is the constant $\hat{g}(z)$ and
$k_\infty = e^{\lambda_0}g(z)e^{-\lambda_0}$ on $N(z)\setminus\{z\}$. Using a cut-off function $k_\infty$ is extended to all of $\Sigma$ so that it is identity away from a neighborhood of $N(z)$. It can be verified that $k_\infty^{-1}A_1$ satisfies (8). So, $g$ can be replaced by $k_\infty^{-1}g$.

Given a $\gamma_1 < \gamma$, pick $p$ such that $\gamma_1 < 1 - \frac{2}{p} < \gamma$. We know $\hat{g} \in W^{1,p}(N(z))$. For any compact domain $U \subset N(z)$ containing $z$, we have the Sobolev embedding (see theorem B.1.11 in [MS04])

$$W^{1,p}(U) \rightarrow C^{0,\gamma_1}(U),$$

where $C^{0,\gamma_1}(U)$ is the space of $C^0(U)$ functions with finite Hölder norm

$$\|f\|_{C^{0,\gamma_1}} := \sup_{x \in U} |f(x)| + \sup_{x, y \in U} \frac{|f(x) - f(y)|}{d(x, y)^{\gamma_1}}.$$ 

Then,

$$d(\text{Id}, \hat{g}(r, \theta)) \leq ce^{-\gamma_1 r}.$$ 

The same holds for $g(r, \theta) = e^{\lambda_0} \hat{g}(r, \theta)e^{-\lambda_0}$ on $N(z)\setminus\{z\}$.

**Step 3: Finishing the proof.**

A connection $A$ on $P$ defines an operator $\overline{\partial}_A$ on the $G$-bundle $P_\Sigma$. A complex gauge transformation $g$ acts on $\overline{\partial}_A$ as: $\overline{\partial}_gA = g \circ \overline{\partial}_A \circ g^{-1}$. In our context $a := gA - A$ and

$$\overline{\partial}_gA - \overline{\partial}_A = a^{0,1} = g\overline{\partial}_A(g^{-1}) = -(\overline{\partial}_Ag)g^{-1} \implies a^{0,1}g = \overline{\partial}_Ag.$$

We obtain the required $W^{2,p}$ bound on $g$ (24) by elliptic bootstrapping - it relies on the similar asymptotic $W^{1,p}$ bound on $a$ (8) and $L^p$ bound on $g$ coming from (25). We apply elliptic regularity to domains of the form $\rho^{-1}(\{n \leq r \leq n + 1\})$ so that the constants are same for all $n$.

**Lemma 7.10.** Let $p > 2$. Suppose $\Sigma$ is a Riemann surface with cylindrical ends and there is a sequence of gauged holomorphic maps $w_\nu = (A_\nu, u_\nu)$ on $\Sigma$ that converges to a finite energy vortex $w_\infty = (A_\infty, u_\infty)$ in the following sense:

$$\|A_\nu - A_\infty\|_{W^{1,p}(\Sigma)}, \|d(u_\nu, u_\infty)\|_{L^\infty(\Sigma)}, \|\Phi(u_\nu) - \Phi(u_\infty)\|_{L^p(\Sigma)} \rightarrow 0 \quad \text{as } s \rightarrow 0.$$

The $W^{1,p}$ norm is taken in a trivialization $\Sigma \times K$ in which $A_\infty = d + \lambda_0 \theta + a_\infty$ on the cylindrical ends $N(z_i)\setminus\{z_i\}$ and $a_\infty$ satisfies the exponential $W^{1,p}$-bound (8). Then we can find complex gauge transformations $e^{i\xi_i}$ so that $\xi_i \rightarrow 0$ in $W^{2,p}(\Sigma)$ and $e^{i\xi_i}w_i$ is a vortex.

**Proof.** For a gauged holomorphic map $w$, let

$$\mathcal{F}_w : \Gamma(\Sigma, \xi) \rightarrow \Gamma(\Sigma, \xi) \quad \xi \mapsto \ast F_{e^{i\xi}w}.$$ 

It linearization at $\xi = 0$ is

$$\mathcal{D}\mathcal{F}_w(0) : \xi \mapsto d_A^*d_A\xi + u^*d\Phi(J\xi).$$ 

Since $A_\infty = d + \lambda_0 \theta + a_\infty$ on the cylindrical ends $N(z_i)\setminus\{z_i\}$, the operator $d_A^* := \xi \mapsto d\xi + [\lambda, \xi]d\theta + [a_\infty, \xi]$ extends to a continuous operator between

$$W^{k+1,p}(\Sigma, \xi) \rightarrow W^{k,p}(\Sigma, \xi).$$
for \( k = 0, 1 \). And since \( \| A_{k} - A_{v} \|_{W^{1,p}(\Sigma)} \) is bounded, the same is true for the operators \( d_{A_{v}} \), also. Writing \( d_{A_{v}}^{*} = \ast d_{A_{v}} \), we see that \( d_{A_{v}}^{*} : W^{k+1,p}(\Sigma, \xi) \to W^{k,p}(\Sigma, \xi) \) is also a bounded operator for \( k = 0, 1 \) and for all \( v \) including \( v = \infty \). For \( x \in X \), let

\[
L_{x} \in \text{End}(\xi) := \xi \mapsto d\Phi(J_{x}).
\]

The image of each \( u_{v} \) is a bounded operator. The operator \( d_{A_{v}}^{*} \) is invertible for all \( i \), let the operator \( u_{v}^{*} d\Phi(J_{x}) : W^{2,p} \to L^{p} \) be invertible because it is multiplication by a \( C^{0} \) function. Therefore, for all \( i \),

\[
DF_{w_{\nu}} : W^{2,p}(\Sigma, \xi) \to L^{p}(\Sigma, \xi)
\]

is a bounded operator.

**Step 1:** For all \( \nu \), including \( \nu = \infty \), \( DF_{w_{\nu}} : W^{2,p} \to L^{p} \) is invertible and the norm of the inverses are uniformly bounded i.e. \( \| DF_{w_{\nu}}^{-1} \| \leq c \) for all \( \nu \). \( DF_{w_{\nu}} : W^{2,p} \to L^{p} \) is 1-1 because for \( \xi \in W^{2,p}(\Sigma, \xi) \),

\[
\int_{\Sigma} (d_{A_{v}}^{*} d_{A_{v}} \xi + u_{v}^{*} d\Phi(J_{\xi}(u_{v})), \xi)_{\xi} = \| d_{A_{v}} \xi \|_{L^{2}}^{2} + \int_{\Sigma} \omega_{u_{v}}((\xi)(u_{v}), J_{\xi}(u_{v})).
\]

The right hand side is positive if \( \xi \neq 0 \).

We now focus on the case that \( w = w_{\infty} \) is a vortex on \( \Sigma \). As earlier, we assume \( A_{\infty} = d + \lambda_{0} d\theta + a_{i} \) on \( N(z_{i}) \). Next, we observe that for \( x \in X \), if the infinitesimal action of \( \xi \) is free, the map \( L_{x} : \xi \to \xi \) is positive because \( \langle L_{x} \xi, \xi \rangle = \omega_{x}(a_{x}, J_{s}) \).

Since \( K \) acts freely on \( \Phi^{-1}(0) \), for a small \( e > 0 \), there is a constant \( c \) such that

\[
(27) \quad c^{-1}|\xi| \leq |L_{x}(\xi)| \leq c|\xi|
\]

for all \( x \in \{ |\phi(x)| < \epsilon \} \). The set \( \{ |\phi \circ u| \geq e \} \subset \Sigma \) is compact. \( L_{u} = \xi \mapsto u^{*} d\Phi(J_{\xi}(u_{\nu})) \) is a section of the bundle \( P(\text{End}(\xi)) \to C \). The bundle \( P(\xi) \to C \) is trivial. So, one can define a section \( \mathcal{L} \in W^{2,p}(\Sigma, P(\text{End}(\xi))) \) that is same as \( L_{u} \) on \( \{ |\phi \circ u|^{2} < e \} \subset \Sigma \) and \( \text{Im}(\mathcal{L}) \subset P(\text{Aut}(\xi)) \). Now, there is a constant \( c \) (possibly different from the one in (27)) so that for any \( x \in \Sigma \) and \( \xi \in P(\xi)(x) \),

\[
c^{-1}|\xi| \leq |L_{x}(\xi)| \leq c|\xi|.
\]

The operator \( \text{Id} + d_{A_{\infty}}^{*} d_{A_{\infty}} : W^{2,p}(\Sigma, \xi) \to L^{p}(\Sigma, \xi) \) is invertible, and for all \( \xi \in W^{2,p}(\Sigma, \xi) \),

\[
c^{-1} \| (\mathcal{L} + d_{A_{\infty}}^{*} d_{A_{\infty}}) \xi \|_{L^{p}} \leq \| (\text{Id} + d_{A_{\infty}}^{*} d_{A_{\infty}}) \xi \|_{L^{p}} \leq c \| (\mathcal{L} + d_{A_{\infty}}^{*} d_{A_{\infty}}) \xi \|_{L^{p}}.
\]

So, \( \mathcal{L} + d_{A_{\infty}}^{*} d_{A_{\infty}} : W^{2,p} \to L^{p} \) is also invertible. \( \mathcal{L} - L_{u} : W^{2,p} \to L^{p} \) is a compact operator, since it is multiplied by a compactly supported section. Adding \( \mathcal{L} - L_{u} \) preserves the Fredholm index of the operator, so \( DF_{w_{\infty}}(0) = d_{A_{\infty}}^{*} d_{A_{\infty}} + L_{u} \) has Fredholm index 0, it is invertible because it is 1-1.

Next, we consider \( w_{\nu} \) in the sequence.

\[
\| d_{A_{\infty}} - d_{A_{v}}^{*} d_{A_{v}} \| \leq c \| A_{\infty} - A_{v} \|_{W^{1,p}(\Sigma)},
\]

\[
\| L_{u_{\nu}} - L_{u_{\nu}} \| \leq c \sup_{p \in \Sigma} d_{X}(u_{\infty}(p), u_{\nu}(p)).
\]
This follows from inequalities as in (Notation 7.12) and the fact that there exists constants $c$, $\epsilon > 0$ such that if $A$ satisfies $\|A - A_\infty\|_{W^{1,p}(\Sigma)} \leq \epsilon$, then $\|e^{i\xi}A - A\|_{W^{1,p}} \leq c\|\xi\|_{W^{2,p}(\Sigma)}$ (see lemma A.3).

**Step 3:** Completing the proof By the implicit function theorem - proposition A.2, if $\|F_{w_i}\|_{L^p} < \frac{\delta}{2}$, then there exists $\xi_{w_i}$ such that $e^{i\xi_{w_i}}w_i$ is a vortex and $\|\xi\|_{W^{2,p}} \leq C\|F_{w_i}\|_{L^p}$. Since $F_{w_i} \to 0$ in $L^p$, the lemma is proved. $\square$

### 7.4. Continuity: Nodal degeneration

In this section we finish the proof of continuity of the map $\Psi : Qmap_{g,n}(X//G)$ and $\Mc V^K_{g,n}(X)$. In particular, we prove:

**Proposition 7.11.** Given a family of quasimaps $Q_S = (\C \to S, \{z_i : S \to C\})_{i=1,...,k}$, $\P_C, u)$ parametrized by $S$, where $S \subset \C$ is a neighborhood of 0. We assume that $C_0$ is a curve with a single node $w$ and $C_s$ are smooth curves obtained from $C_0$ by the gluing procedure. Then for any sequence $s_n \to 0$ as $n \to \infty$, $\Psi(w_{s_n})$ Gromov converges to $\Psi(w_0)$. ($\Psi$ is as described in remark 7.4.)

We justify why this indeed proves continuity of $\Psi$. Firstly note that it is enough to consider a family $\C \to S$ where $\C_s$ is smooth for all $s \neq 0$. This is because if $\C_s$ is nodal for $s \neq 0$, we can assume that the combinatorial type $\Gamma(\C_s)$ is constant over $S\setminus\{0\}$ and then we can work with the normalization of $\C_s$ and show continuity component-wise. In the proposition we further assume that $C_0$ has a single node and the family $\C \to S$ is obtained by applying the gluing procedure on $C_0$. We recall the description of $\C \to S$ and the family of Riemann surfaces $\Sigma_s$ with the neck-stretching metric. Let $\tilde{C}_0$ be the normalization of $C_0$ with marked points $w^+$ and $w^-$ corresponding to the node. Let $\Sigma_0$ be the Riemann surface with cylindrical ends corresponding to $C_0$. For $s \in S\setminus\{0\}$, let $l_s = L_s + it_s := -\ln s$. The Riemann surfaces $\Sigma_s$ are obtained by gluing:

$$\Sigma_s = (\Sigma_0 \setminus (\rho_{w^+}^{-1}\{r > L_s\} \cup \rho_{w^-}^{-1}\{r < -L_s\})) / \sim, \quad \rho_{w^+}^{-1}(z) \sim \rho_{w^-}^{-1}(z + l_s).$$

We denote

$$\tilde{\Sigma}_s = \Sigma_0 \setminus (\rho_{w^+}^{-1}\{r > L_s\} \cup \rho_{w^-}^{-1}\{r < -L_s\})..$$

To prove continuity of $\Psi$, it is enough to consider the family $\C \to S$ of the above form because

(a) if $C_0$ has more than one node, the ideas in the proof can be applied at every node.

(b) In the $\Sigma_s$ described above, let $g_s$ denote the metric by $g_s$. We know $g_s|_{\Omega} = g_0|_{\Omega}$. Given $C$ has one node, for the most general setting one must consider $(\Sigma_s, g_s)$ such that $g_s' \to g_\infty$ in $C^\infty(\Omega)$ and $g_s = g'_s$ on $N_s$. Our proofs easily carry over to this case with variation in metric.

**Notation 7.12.** We need some notation for proof of proposition 7.11.
1. \((N_s, \rho_s^\pm, \Omega)\) Let \(N_s\) be the cylindrical part in \(\Sigma_s\). We can put co-ordinates in two ways on \(N_s\) namely

\[
\rho_s^+: N_s \to \{r + i\theta : 0 \leq r \leq L_s\} \quad \text{that coincides with } \rho_{w^+}
\]

\[
\rho_s^-: N_s \to \{r + i\theta : -L_s \leq r \leq 0\} \quad \text{that coincides with } \rho_{w^-}.
\]

They’ll be related as: \(\rho_s^+ - l_s = \rho_s^-\). Let \(\Omega := \Sigma \setminus \text{int}(N(w^+)) \cup N(w^-))\). Then, we can write \(\Sigma_s = \Omega \cup N_s\).

2. (Identification of bundles \(P_s\)) To talk about convergence of gauged holomorphic maps, the bundles \(P_s\) have to be identified to each other. We have an inclusion \(\hat{\Sigma}_s \subset \Sigma_0\) (see (30)) and a quotient map \(\pi_s: \hat{\Sigma}_s \to \Sigma_s\). Assume \(\hat{\Sigma}_0 = \Sigma_0\), so that \(\Sigma \to S\) is a smooth family. We can pick a smooth family of bundle isomorphisms \(h_s: \pi_s^*P_s \to P_0|_{\hat{\Sigma}_s}\) with \(h_0 = \text{Id}\). The choice of \(h\) is unique up to homotopy, so we fix \(h\) and talk as if the bundle \(P_s\) is defined on a part of \(\Sigma_0\).

3. \((\pi_s^{-1}, \pi_s^{1, \pm}, g_s: N_s \to K)\) Recall \(\hat{\Sigma}_s \subset \Sigma\) are such that the quotient map \(\pi_s: \hat{\Sigma}_s \to \Sigma_s\) is one-one everywhere except on a cylinder \(N_s\) where it is two-one. So, an inverse \(\pi_s^{-1}\) is well-defined on \(\Omega\). On \(N_s\), there are two right inverses denoted by \(\pi_s^{1, \pm}\). The bundle \(\tilde{P}_s := \pi_s^*P_s = P_0|_{\hat{\Sigma}_s}\) is trivializable. The bundle \(P_s \to \Sigma_s\) is obtained from \(\tilde{P}_s \to \hat{\Sigma}_s\) by gluing together \((\pi_s^{-1})^*\tilde{P}_s\) and \((\pi_s^{-1})^*\tilde{P}_s\) by a transition function \(g_s: N_s \to K\).

4. \((\Omega_s, \Delta_s^\pm, r: \Delta_s^+ \to \Delta_s^-)\) \(\Sigma_s\) can be realized in the following way: Define the following subsets of \(\Sigma\).

\[
\Omega_s := \Sigma \setminus (\rho_{w^+}^{-1}\{r > (L_s - \Delta)/2\} \cup \rho_{w^+}^{-1}\{r < (-L_s + \Delta)/2\})
\]

\[
\Delta_s^+ := \rho_{w^+}^{-1}\{(L_s - \Delta)/2 \leq r \leq (L_s + \Delta)/2\},
\]

\[
\Delta_s^- := \rho_{w^+}^{-1}\{(-L_s - \Delta)/2 \leq r \leq (-L_s + \Delta)/2\},
\]

and a diffeomorphism \(r: \Delta_s^+ \to \Delta_s^- := \rho_{w^+}^{-1} \circ (-l_s) \circ \rho_{w^+}\). Then, \(\Sigma_s = (\Omega_s \cup \Delta_s^+ \cup \Delta_s^-)/r\).

5. (Cut-off function \(\phi_s\)) Suppose \(\phi: \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z} \to [0, 1]\) is a cut-off function invariant in the second coordinate, 0 on \((-\infty, -\Delta/2]\) and 1 on \([\Delta/2, \infty)\). Here \(\Delta\) is a small positive constant that will be determined later in the proof. Let \(\phi_z(x) := \phi(x - z)\) for any \(z \in \mathbb{R}\). Define a cut-off function on \(\hat{\Sigma}_s\) in the following way:

\[
\phi_s := \begin{cases} 
1 & \text{on } \Omega \\
(1 - \phi_{L_s/2}) \circ \rho_{w^+} & \text{on } \pi_{s^+}(N_s) \\
\phi_{L_s/2} \circ \rho_{w^-} & \text{on } \pi_{s^-}(N_s)
\end{cases}
\]

6. \((\Omega'_s, W^{k,p}(\Sigma_s, P_s(\mathfrak{t})))\) There are many equivalent ways of defining a norm on \(W^{2,p}(\mathfrak{t})\). We fix the following (maybe state this before the lemma, because the statement depends on it). Let \(\Omega'_s := \Omega_s \cup \Delta_s^+ \cup \Delta_s^- \subset \hat{\Sigma}_s\)

\[
\|\sigma\|_{W^{2,p}(\Sigma_s, P_s(\mathfrak{t}))) := \|\pi_s^*\sigma\|_{W^{2,p}(\Omega'_s, \mathfrak{t})}.
\]

\(\Box\)
Remark 7.13 (Model for $C \to S$ near the node $w$). A neighborhood of $w$ (the nodal point in $C_0$) in $C$ can be mapped to a neighborhood of the origin in $\mathbb{C}^2$ such that $w$ is mapped to the origin. This is done in a way that $C_s$ is mapped to $\{xy = s\}$ for all $s \in S$ (see section 3, chapter 11 in [ACG11]). We spell out this map explicitly to make it clear how it fits in with the cylindrical ends construction. For smooth curves, i.e. for $s \neq 0$,

$$\Phi : N_s \to \{xy = s\} \subset \mathbb{C}^2 \quad z \mapsto (e^{\rho^+_s(z)} - l_s, e^{-\rho^+_s(z)}).$$

For $s = 0$,

$$\Phi : \begin{cases} N(w^+) \to \{x = 0\} \subset \mathbb{C}^2 & z \mapsto (0, e^{-\rho^+_w(z)}) \\ N(w^-) \to \{y = 0\} \subset \mathbb{C}^2 & z \mapsto (e^{\rho^-_w(z)}, 0) \end{cases}$$

Lemma 7.14. Given a family of quasimaps $Q_S = (C \to S, \{z_i : S \to C\}_{i=1,...,k}, P_C, u)$ where $C \to S$ satisfies the assumptions mentioned above, there is a reduction $\sigma : S \to P_C/K$ and a trivialization of the restriction $P_0 \to \Sigma_0$ such that

1. $\Phi : N_s \to \{xy = s\} \subset \mathbb{C}^2 \quad z \mapsto (e^{\rho^+_s(z)} - l_s, e^{-\rho^+_s(z)}).$

2. Under this trivialization, $\|A_s - A_0\|_{W^{1,p}(\Omega'_s)} \to 0$ as $s \to 0$.
3. $\|d(u_s, u_0)\|_{L^\infty(\Omega'_s \cap N(w^\pm))} \to 0$ and $\|\Phi(u_s) - \Phi(u_0)\|_{L^p(\Omega'_s \cap N(w^\pm))} \to 0$ as $s \to 0$.
4. The functions $g_s : N_s \to K$ are uniformly bounded in $C^2$.
5. $F_{A_0, u_0} \equiv 0$, i.e. $(A_0, u_0)$ is a vortex on $\Sigma$.

(b), (c) and (e) imply $\|F_{w_s}\|_{L^p(\Sigma_s)} \to 0$ as $s \to 0$.

Proof. Step 1: There is a reduction $\sigma : C \to P_C/K$ such that

1. $F_{A_s} = 0$ in neighborhoods of marked points $z_i$, $\Phi(u_s(z_i)) = 0$ and
2. there is a trivialization $\tau_1$ of $P \to C$ in a neighborhood of $W$ in which the quantities

$$\|A_s - A_0\|_{W^{1,p}(\Omega'_s \cap N(w^\pm))}, \|d(u_s, u_0)\|_{L^\infty(\Omega'_s \cap N(w^\pm))}, \|\Phi(u_s) - \Phi(u_0)\|_{L^p(\Omega'_s \cap N(w^\pm))} \to 0 \quad \text{as } s \to 0.$$  

We'll prove the statement for the positive ends $N(w^+)$, that for the negative end will follow analogously. We focus on a neighborhood of $w$ in $C$ and work with the local model. So, we denote $\Phi^*(A_s, u_s) : (A_s, \tilde{u}_s)$. Pick a reduction $\sigma : C \to P_C/K$ such that $u(w) \in \Phi^{-1}(0)$ and in a neighborhood of $(0,0)$, $F_{\tilde{A}_0 \mid x=0, F_{\tilde{A}_0 \mid y=0} = 0$. We deduce the convergence properties of $A_s, u_s$ working in the local model - these will continue to hold on $\Sigma$ because for a 1-form $a$, the norms $\|a\|_{L^p}$ and $\|a\|_{L^p}$ decrease when there is a conformal change of coordinates that increases the volume, and for $u$ the $L^\infty$ norm is unchanged by change of coordinates.

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Footnote:

1. The space $W^{1,p}(\Sigma, P(t))$ is well-defined only if $\Sigma$ is compact, so we instead work with $W^{1,p}(\Sigma, t)$ with a fixed trivialization.
Figure 4. The region $\Omega_s'$ on the curves $C_s$ and $C_0$

By choosing a reduction $\sigma$, we get a smooth connection $A$ on the bundle $\mathcal{P}$ on $B_1 \subseteq \mathbb{C}^2$, $A_s$ is the restriction of $A$ to \{xy = s\}. Pick a trivialization $\pi_1$ of the bundle $\mathcal{P}$ on $B_1 \subset \mathbb{C}^2$ so that $\hat{A}_0|_{x=0}$, $\hat{A}_0|_{y=0}$ are the trivial connections. We can assume $|A|$, $|\nabla A|$, $|\nabla^2 A|$, $|\nabla u| < c_1$. In order to compare $\hat{A}_s$ and $\hat{A}_0$, recall points in \{xy = s\} are identified to those in \{x = 0\} as $(x, y) \mapsto (0, y)$. $\hat{A}_s$ is pulled back to a connection on \{x = 0\} and this is also called $\tilde{A}_s$. Then,

$$\|A_s - A_0\|_{W^{1,p}(\Omega_s^e \cap N(w^+))} \leq \|\hat{A}_s - \hat{A}_0\|_{W^{1,p}(\{x=0, e^{(-L_s-\Delta)/2} \leq |y| \leq 1\})} \leq c_1 e^{2\Delta/e^{(-L_s+\Delta)/2}}.$$ (See figure 4.) This shows $\|A_s - A_0\|_{W^{1,p}(\Omega_s^e \cap N(w^+))} \to 0$. Similarly for $u_s$, we have

$$\|d(u_s, u_0)\|_{L^\infty(\Omega_s^e \cap N(w^+))} = \|d(\hat{u}_s, \hat{u}_0)\|_{L^\infty(\{x=0, e^{(-L_s-\Delta)/2} \leq |y| \leq 1\})} \leq c_1 e^{(-L_s+\Delta)/2}.$$ Next, note that $u(B_1)$ is compact in $X$, so $|d\Phi|$ is bounded by a constant $c_2$ in this set. Then,

$$|\Phi(u_s) - \Phi(u_0)| \leq c_2 c_1 e^{(-L_s+\Delta)/2}$$

$$\implies \|\Phi(u_s) - \Phi(u_0)\|_{L^p(\Omega_s^e \cap N(w^+))} \leq c_1 c_2 L_{s}^{1/p} e^{(-L_s+\Delta)/2}.$$ As $s \to 0$, $L_s \to \infty$ and so, $\|\Phi(u_s) - \Phi(u_0)\|_{L^p(\Omega_s^e \cap N(w^+))} \to 0$.

The above discussion only fixes the reduction $\sigma$ in a neighborhood of $w$. Around marked points, using lemma 7.6, $\sigma$ can be chosen so that the first condition is satisfied. These can be patched together without any problems because the fibers of the bundle $\mathcal{P}_C/K \to C$ are contractible.

**Step 2:** There is a trivialization $\tau_2$ of the bundles $\pi_s^* P_s \to \tilde{\Sigma}_s$ that varies smoothly with $s$ so that (a)-(d) are satisfied.

Because of the first condition in Step 1, we can produce a trivialization $\tau_2$ of the bundles $\pi_s^* P_s \to \tilde{\Sigma}_s$ so that (a) is satisfied. Suppose $\tau_1$ and $\tau_2$ are related by the gauge transformation $k: \cup_{s \in S \setminus \{0\}} N_s \cup N_w \to K$. We focus on the + side.
i.e. $\pi_{s,+}^{-1}(N_s)$ and $N_{w,+}$. For all $s \in S$ and any $R \geq 0$ let $\gamma_{s,R}$ denote the loop $\pi_{s,+}^{-1} \circ (\rho_T)^{-1}\{r = R, 0 \leq \theta \leq 2\pi\}$. The homotopy class of $k|_{\gamma_{s,R}}$ is constant for all $s \in S$ and $R \geq 0$. Suppose this class contains the geodesic loop $\theta \mapsto e^{\lambda^+ \theta}$ for some $\lambda^+ \in \mathfrak{k}$. Then, as in the proof of corollary 2.9, $\tau_2$ can be altered so that $k|_{\gamma_{s,R}} = e^{\lambda^+ \theta}$.

Then, we see that the convergence in (31) holds in the trivialization $\tau_2$ also. Because of smooth variation of $\tau_2$, it is easy (use the fact that $\tau_2$ satisfies (a)) to see that $A_s|_{\Omega}$, $u_s|_{\Omega}$ and $\Phi(u_s)|_{\Omega}$ converge in the respective Sobolev spaces, so (a) and (b) hold in this trivialization. $\lambda^-$ can be chosen so that $e^{\lambda^+ \theta}$ and $e^{\lambda^- \theta}$ commute. Then, under the trivialization $\tau_2$, $g_s = e^{(\lambda^+ - \lambda^-) (\theta + t_s)}$.

**Step 3:** *Finishing the proof.* Using theorem 7.3, there is a complex gauge transformation $e^{i\xi_0}$ on $P_0 \to \Sigma_0$ that makes $(A_0, u_0)$ a vortex. By lemma 7.9, $\xi_0$ decays exponentially in the sense of (24) at cylindrical ends corresponding to $z_1, \ldots, z_n$ and $w^\pm$. $\xi_s \in \Gamma(\Sigma_s, P_s(\mathfrak{k}))$ is defined as follows: let $\xi_s = \phi_s \xi_0 : \Sigma_s \to \mathfrak{k}$

$$
\xi_s = \begin{cases} 
(\pi_{s,-1})^* \xi_s & \text{on } \Omega \\
(\pi_{s,+1})^* \xi_s & \text{on } N_s
\end{cases}
$$

The reduction $\sigma$ picked in Step 1 is altered by the family of complex gauge transformations $e^{i\xi_s}$. It remains to show (b) and (c) are still satisfied. For this we work on $\pi_s^* P_s \to \Omega_s,$

$$
\pi_s^* \xi_s = \begin{cases} 
\xi_s & \text{on } \Omega_s, \\
\xi_s + \text{Ad}_g(r^* \xi_s) & \text{on } \Delta_s^+ \\
\xi_s + \text{Ad}_g^{-1}((r^{-1})^* \xi_s) & \text{on } \Delta_s^-
\end{cases}
$$

We know $g$ explicitly, we can say $\|g_s\|_{W^{2,p} (\Delta_s^\pm)}$ is bounded for all $s$. This, together with the exponential decay (24) of $\xi_0$ show that (b) and (c) also hold for the new reduction and the trivialization $\tau_2$. 

**Lemma 7.15.** Let $p > 2$. Let $\Sigma$, $\Sigma$ be Riemann surfaces with cylindrical ends as in notation 7.12 above. Given a sequence of gauged holomorphic maps $w_\nu = (A_\nu, u_\nu)$ on the bundles $P_\nu \to \Sigma_\nu$ that converges to a finite energy vortex $w_\infty = (A_\infty, u_\infty)$ on $\Sigma$ in the following sense:

$$
\|A_\nu - A_\infty\|_{W^{1,p}(\Sigma_\nu)} \to 0 \quad \|d(u_\nu, u_0)\|_{L^2(\Sigma_\nu)} \to 0
$$

as $\nu \to \infty$. Then we can find complex gauge transformations $e^{i\xi_\nu}$ on the bundle $P_\nu \to \Sigma_\nu$ so that $\|\xi_\nu\|_{W^{2,p}(\Sigma_\nu, P_\nu(\mathfrak{k}))} \to 0$ and $e^{i\xi_\nu} w_\nu$ is a vortex on $\Sigma_\nu$.

**Proof.** The proof is similar to the proof of lemma 7.10. The proof of invertibility of $DF_{w_\nu}(0)$ and a uniform bound on the inverse is more complicated and is carried out in steps 1A and 1B.

**Step 1A:** An approximate inverse $\hat{Q}_\nu$ for $DF_{\nu}(0)^{-1}$ satisfying $\|\hat{Q}_\nu\| \leq C$. 

By the arguments in the proof of lemma 7.10, the operator

$$
DF_{w_\nu}(0) : W^{2,p}(\Sigma, \mathfrak{k}) \to L^p(\Sigma, \mathfrak{k})
$$
is invertible. The inverse $Q_\infty := D\mathcal{F}_{\omega^\pm}(0)^{-1}$ is used to construct an approximate inverse $\hat{Q}_\nu$ for

$$DF_{\omega^\pm}(0) : W^{2,p}(\Sigma_\nu, P_\nu(\nu)) \to L^p(\Sigma_\nu, P_\nu(\nu)).$$

Given $\eta \in L^p(\Sigma_\nu, P_\nu(\nu))$, we first produce $\eta_\infty \in L^p(\Sigma, \nu)$. For this take the lift $\tilde{\eta} := \pi_\nu^\ast \eta \in L^p(\Sigma_\nu, \nu)$, and extend it by 0 to get $\eta_\infty : \Sigma \to \nu$. $\hat{Q}_\nu \eta$ is constructed from $\zeta_\infty := Q_\infty \eta_\infty$ as follows.

Let $\tilde{\zeta} \in W^{2,p}(\Sigma_\nu, \nu) := \phi_\nu \zeta_\infty |_{\Sigma_\nu}$. Finally define

$$\zeta = \hat{Q}_\nu \eta := \begin{cases} (\pi_\nu^{-1})^{*} \tilde{\zeta} & \text{on } \Omega, \\ (\pi_\nu^{-1})^{*} \tilde{\zeta} + (\pi_{\nu^{-1}})^{*} \tilde{\zeta} & \text{on } N_\nu. \end{cases}$$

Next we show a bound on $\|\hat{Q}_\nu\|$ independent of $\nu$. It is easy to see that the first 3 steps - $\eta \mapsto \tilde{\eta}$, $\tilde{\eta} \mapsto \zeta_\infty$ and $\zeta_\infty \mapsto \tilde{\zeta}$ have norm bounds independent of $\nu$. For the last step $\tilde{\zeta} \mapsto \zeta : W^{2,p}(\Sigma, \nu) \to W^{2,p}(\Sigma_\nu, P_\nu(\nu))$, we need to bound $\|\pi^\ast_\nu \zeta\|_{W^{2,p}(\Sigma_\nu, \nu)}$ in terms of $\tilde{\zeta}$. Now,

$$\pi^\ast_\nu \zeta = \begin{cases} \tilde{\zeta} & \text{on } \Omega_\nu, \\ \tilde{\zeta} + g_\nu(r^+ \tilde{\zeta}) & \text{on } \Delta^+_\nu, \\ \tilde{\zeta} + g_\nu^{-1}(r^{-1} \tilde{\zeta}) & \text{on } \Delta^-_\nu. \end{cases}$$

Since we have uniform $C^2$ bounds on $g_\nu$, the last step $\tilde{\zeta} \mapsto \zeta$ also has a uniform bound.

**Step 1B:** $\|DF_{\nu}(0)\hat{Q}_\nu - \text{Id}\| \leq \frac{1}{2}$, for large enough $\nu$. For any $\eta \in L^p(\Sigma_\nu, P_\nu(\nu))$, we have

$$\|DF_{\nu}(0)\hat{Q}_\nu \eta - \eta\|_{L^p(\Sigma_\nu, P_\nu(\nu))} \leq \|DF_{\nu}(0)\hat{Q}_\nu \eta - \eta\|_{L^p(\Omega_\nu, \nu)}$$

and

$$\|DF_{\nu}(0)\tilde{\zeta} - \phi_\nu \eta\|_{L^p(\Delta^+_\nu, \nu)} + \|DF_{\nu}(0)\zeta - \phi_\nu \eta\|_{L^p(\Delta^-_\nu, \nu)} = T1 + T2 + T3$$

To bound $T1$, on $\Omega_\nu$, we can write $\eta = D\mathcal{F}_s \hat{Q}_\nu \eta$. Then,

$$T1 \leq \|(DF_{\nu}(0) - D\mathcal{F}_s(0))\|_{L^p(\Sigma_\nu, P_\nu(\nu))} \leq \|\hat{Q}_\nu\| \cdot \|\eta\|_{L^p(\Sigma_\nu)} \leq c_\nu \|\hat{Q}_\nu\| \cdot \|\eta\|_{L^p(\Sigma_\nu)},$$

where $c_\nu \to 0$ as $\nu \to \infty$. This uses (28) and the convergence (32). $T2$ and $T3$ are bounded in a similar way to each other. We show the case of $T2$.

$$\|DF_{\nu}(0)\tilde{\zeta} - \phi_\nu \eta\|_{L^p(\Delta^+_\nu, \nu)} \leq \|DF_{\nu}(0)\tilde{\zeta} - D\mathcal{F}_s(0)\tilde{\zeta}\|_{L^p(\Delta^+_\nu, \nu)}$$

and

$$\|DF_{\nu}(0)\zeta - \phi_\nu \eta\|_{L^p(\Delta^-_\nu, \nu)} = T2A + T2B$$

$T2A$ is bounded in a similar way to $T1$ and gives $T2A \leq c_\nu \|\hat{Q}_\nu\| \cdot \|\eta\|_{L^p(\Sigma_\nu)}$, where $\lim_{\nu \to \infty} c_\nu = 0$. To work on $T2B$, we observe that on $\Delta^+_\nu$, $\eta = D\mathcal{F}_s(0)\zeta_\infty$ and so,

$$T2B = \|DF_{\nu}(0)\phi_\nu \zeta_\infty - \phi_\nu D\mathcal{F}_s(0)\zeta_\infty\|_{L^p} = \|\Delta_{A\infty} \phi_\nu \zeta_\infty - \phi_\nu \phi_\nu \Delta_{A\infty} \zeta_\infty\|_{L^p}$$

Recall $A_\infty = d + \lambda^+ d\theta + a^+$, where $\lambda^+ \in \nu$ and $\|a^+\|_{W^{1,p}(N(\nu^+))} < \infty$. Then,

$$\Delta_{A\infty} \phi_\nu \zeta_\infty - \phi_\nu \phi_\nu \Delta_{A\infty} \zeta_\infty = (\Delta \phi_\nu) \zeta_\infty + 2(\nabla \phi_\nu) \cdot (\nabla \zeta_\infty) + \star((\lambda^+ d\theta + a^+) \wedge (d \phi_\nu \otimes \zeta_\infty))$$
The bounds on \( a^+ , \lambda^+ \) are independent of \( \eta , \nu \). So,
\[
T2B \leq c\| \zeta_{\infty} \|_{W^{2,p}(\Delta^+_T,\nu)} (\| \nabla \phi \|_{Lp} + \| \Delta \phi \|_{Lp}) \leq c_{2} \| \eta \|_{Lp(\Sigma_{\nu})} (\| \nabla \phi \|_{Lp} + \| \Delta \phi \|_{Lp}).
\]
where the \( L^p \) norms of \( \nabla \phi \) and \( \Delta \phi \) are taken on \([-\Delta/2, \Delta/2] \times S^1 \). These norms can be made smaller by enlarging \( \Delta \) and stretching out \( \phi \). We fix \( \Delta \) such that
\[
\| \nabla \phi \|_{Lp([-\Delta/2, \Delta/2] \times S^1)} + \| \Delta \phi \|_{Lp([-\Delta/2, \Delta/2] \times S^1)} \leq \frac{1}{4c_{2}}.
\]
Putting things together in the above discussion, we get
\[
\| D_{\nu} \hat{Q}_{\nu} \eta - \eta \|_{Lp(\Sigma_{\nu})} \leq (\frac{1}{4} + c_{\nu}) \| \eta \|_{Lp(\Sigma_{\nu})},
\]
where the constants \( c_{\nu} \) are such that \( \lim_{\nu \to \infty} c_{\nu} = 0 \). So, by taking \( \nu \) large enough, Step 1B is proved.

Step 1B shows that \( D_{\nu}(0) \hat{Q}_{\nu} - \text{Id} \) is invertible, so \( D_{\nu}(0)^{-1} = \hat{Q}_{\nu}(D_{\nu}(0)\hat{Q}_{\nu})^{-1} \) and hence \( \| D_{\nu}(0)^{-1} \| \leq 2 \| \hat{Q}_{\nu} \| \). The rest of the proof - steps 2 and 3 - are identical as the proof of lemma 7.10.

**Proof of proposition 7.11.** The proof follows by applying lemma 7.14 followed by lemma 7.15 to the family \( Q_{\delta} \).

\[ \Box \]

**Appendix A. Some analytic results**

The first two results are standard.

**Proposition A.1** (Sobolev multiplication). *(Theorem 4.39 in [Ada75]*) Let \( \Omega \subseteq \mathbb{R}^n \), not necessarily compact and the boundaries of \( \Omega \) are smooth. Suppose \( p > 1 \) and \( k \geq 0 \) is an integer such that \( kp \geq n \). Then there is a constant \( c(k,p,n) \) such that
\[
\| uv \| \leq c \| u \| \cdot \| v \|,
\]
where \( \| \cdot \| = \| \cdot \|_{W^{k,p}(\Omega)} \). So, \( W^{m,p}(\Omega) \) is a Banach algebra.

**Proposition A.2** (Implicit function theorem). *(Proposition A.3.4 in [MS04]*) Let \( F : X \to Y \) be a differentiable map between Banach spaces. \( DF(0) \) has an inverse \( Q \), with \( \| Q \| \leq c \). For all \( x \in B_{\delta} \), \( \| DF(x) - DF(0) \| < \frac{1}{2c} \). If \( \| F(0) \| < \frac{1}{2c} \), then \( F(x) = 0 \) has a solution in \( B_{\delta} \). \( x \) is the unique solution in \( B_{\delta} \).

**Lemma A.3** (Norm bound for \( G_{C} \) action on \( A \)). Let \( \Omega \) be a Riemann surface (possibly non-compact). Let \( A_{0} \) be a connection on the bundle \( \Omega \times K \) for which the operator \( d_{A_{0}} : W^{2,p} \to W^{1,p} \) is bounded. For any \( \epsilon > 0 \) and a connection \( A \) satisfying \( \| A - A_{0} \|_{W^{1,1}} < \epsilon \) and \( \xi \in W^{2,p}(\Omega, \mathbb{F}) \) satisfying \( \| \xi \|_{W^{2,p}} < 1 \), \( (\exp \epsilon i)A - A \in W^{1,p} \) and there is a constant \( C \) so that
\[
\| (\exp \epsilon i)A - A \|_{W^{1,1}(\Omega)} \leq C \| \xi \|_{W^{2,p}(\Omega)}.
\]
\( C \) depends on \( \epsilon \) and the norm of \( d_{A_{0}} : W^{2,p} \to W^{1,p} \).

Suppose \( \Omega \) is a Riemann surface with cylindrical ends and \( \Omega_{1} \subset \Omega_{2} \subset \cdots \subset \Omega \) a sequence of subsets exhausting \( \Omega \), whose boundaries \( \partial \Omega_{i} \) are smooth and are
translates of each other. Then the constant \(C\) can be picked so that it satisfies (35) for all \(\Omega_i\).

Proof. The infinitesimal action of \(i\xi\) on a connection \(A\) is \(*d_A\xi\). Suppose \(a_t : [0, 1] \to W^{1,p}(\Omega, \xi)\) is the solution of the ODE

\[
\frac{da_t}{dt} = *d_{A_0 + a_t} \xi = *d_{A_0} \xi + *[a_t \wedge \xi], \quad a_0 = A - A_0.
\]

Then, \((\exp i\xi) A = A + a_1\).

\[
\frac{d}{dt} \|a_t\|_{W^{1,p}} \leq \|\frac{da_t}{dt}\|_{W^{1,p}} \leq c_1(\|\xi\|_{W^{2,p}} + \|a_t\|_{W^{1,p}} \|\xi\|_{W^{2,p}}) \leq c_1(1 + \|a_t\|_{W^{1,p}}),
\]

since \(\|\xi\|_{W^{2,p}} < 1\). The constant \(c_1\) depends on the norm of \(d_{A_0} : W^{2,p} \to W^{1,p}\). Now, for any \(t\), \(\|a_t\| < c_2 := e^{c_1}(1 + \epsilon) - 1\), which implies \(\frac{d}{dt} \|a_t\|_{W^{1,p}} \leq c_1(1 + c_2)\|\xi\|_{W^{2,p}}\). This proves the first part of the result.

The constant \(c_1\) above depends only on the constants occurring in Sobolev multiplication on the given domain. On Euclidean domains, Sobolev multiplication constants are independent of the domain. Since \(\Omega\) can be constructed by using a finite number of Euclidean charts, the constants can be picked uniformly for all \(\Omega_i\). \(\square\)

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