The Orbifolds of Permutation-Type as Physical String Systems
at Multiples of $c = 26$

III. The Spectra of $\hat{c} = 52$ Strings

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February 1, 2008

Abstract

In the second paper of this series, I obtained the twisted BRST systems and extended physical-state conditions of all twisted open and closed $\hat{c} = 52$ strings. In this paper, I supplement the extended physical-state conditions with the explicit form of the extended (twisted) Virasoro generators of all $\hat{c} = 52$ strings, which allows us to discuss the physical spectra of these systems. Surprisingly, all the $\hat{c} = 52$ spectra admit an equivalent description in terms of generically-unconventional Virasoro generators at $c = 26$. This description strongly supports our prior conjecture that the $\hat{c} = 52$ strings are free of negative-norm states, and moreover shows that the spectra of some of the simpler cases are equivalent to those of ordinary untwisted open and closed $c = 26$ strings.

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1 Introduction

Opening another chapter in the orbifold program [1-11,12-15], this is the third in a series of papers which considers the critical orbifolds of permutation-type as candidates for new physical string systems at higher central charge. In the first paper [16] of this series, we found that the twisted sectors of these orbifolds are governed by new, extended (permutation-twisted) world-sheet gravities – which indicate that the free-bosonic orbifold-string systems of permutation-type can be free of negative-norm states at critical central charge $\hat{c} = 26K$. Correspondingly-extended world-sheet permutation supergravities are expected in the twisted sectors of the superstring orbifolds of permutation-type, where superconformal matter lives at higher multiples of critical superstring central charges.

In the second paper [17] of the series, we found the corresponding twisted BRST systems for all sectors of the free-bosonic orbifolds which couple to the simple case of $\mathbb{Z}_2$-twisted permutation gravity, i.e. for all the twisted strings with $\hat{c} = 52$ matter. The new BRST systems also implied the following extended physical-state conditions for the physical states $\{|\chi\rangle\}$ of each of the $\hat{c} = 52$ strings:

$$\left( \hat{L}_u \left( \left( m + \frac{u}{2} \right) \geq 0 \right) - \delta_{m+\frac{u}{2},0} \frac{12}{8} \right) |\chi\rangle = 0, \quad m \in \mathbb{Z}, \quad \bar{u} = 0,1 \quad (1.1a)$$

$$\left[ \hat{L}_u \left( m + \frac{u}{2} \right), \hat{L}_v \left( n + \frac{v}{2} \right) \right] = \left( m - n + \frac{u-v}{2} \right) \hat{L}_{u+v} \left( m + n + \frac{u+v}{2} \right) + \frac{52}{12} \left( \left( m + \frac{u}{2} \right) \left( \left( m + \frac{u}{2} \right)^2 - 1 \right) \right) \delta_{m+n,\frac{u+v}{2},0}. \quad (1.1b)$$

The algebra in Eq. (1.1b) is called an order-two orbifold Virasoro algebra (or extended, twisted Virasoro algebra) and general orbifold Virasoro algebras [1,18,9,12,16,17] are known to govern all the twisted sectors of the orbifolds of permutation-type at higher multiples of $c = 26$.

The set of all $\hat{c} = 52$ orbifold-strings is a very large class of fractional-moded free-bosonic string systems, including e.g. the twisted open-string sectors of the orientation orbifolds, the twisted closed-string sectors of the generalized $\mathbb{Z}_k$-permutation orbifolds and many others (see Refs. [16,17] and Sec. 2). Starting from the extended physical-state conditions (1.1) (and a right-mover copy of (1.1) on the same $\{|\chi\rangle\}$for the twisted closed-string sectors) this paper begins the concrete study of the physical spectrum of each $\hat{c} = 52$ string.
As the prerequisite for this analysis, I first provide in Sec. 2 the explicit form – in terms of twisted matter fields – of the extended Virasoro generators \( \{ \hat{L}_u (m + \frac{u}{2}), \tau = 0, 1 \} \) of all \( \hat{c} = 52 \) strings. This construction allows us to begin the study of the general \( \hat{c} = 52 \) string spectra in Sec. 3. The same subject is further considered in Sec. 4, where I point out that all the \( \hat{c} = 52 \) spectra admit an equivalent description in terms of generically-unconventional Virasoro generators at \( c = 26 \). This description allows us to see clearly a number of spectral regularities which are only glimpsed in Sec. 3, including strong further evidence that the critical orbifolds of permutation-type can be free of negative-norm states. Moreover, although the generic \( \hat{c} = 52 \) spectrum is apparently new, we are able to show that some of the simpler spectra are equivalent to those of ordinary untwisted open and closed critical strings at \( c = 26 \).

Based on these results, the discussion in Sec. 5 raises some interesting questions about these theories at the interacting level, and speculates on the form of the extended physical-state conditions for more general orbifold-strings of permutation-type. I will return to both of these subjects in succeeding papers of the series.

2 The Twisted Virasoro Generators of \( \hat{c} = 52 \) Strings

As emphasized in Ref. [17], the universal form of the twisted BRST systems and the extended physical-state conditions (1.1) are consequences of their origin in \( \mathbb{Z}_2 \)-twisted permutation gravity, which governs all twisted \( \hat{c} = 52 \) matter.

There are however many distinct \( \hat{c} = 52 \) strings, including the twisted open-string sectors of the orientation orbifolds [12,13,15-17]

\[
\frac{U(1)^{26}}{H_-}, \quad H_- = \mathbb{Z}_2(\text{w.s.}) \times H
\]

and the twisted closed-string sectors of the generalized \( \mathbb{Z}_2 \)-permutation orbifolds [15-17]

\[
\frac{U(1)^{26} \times U(1)^{26}}{H_+}, \quad H_+ = \mathbb{Z}_2(\text{perm}) \times H'
\]
as well as the generalized open-string $\mathbb{Z}_2$-permutation orbifolds and their $T$-duals [15-17]. For the orientation orbifolds in Eq. (2.1), I remind that $H_-$ is any automorphism group of the untwisted closed string $U(1)^{26}$ which includes world-sheet orientation-reversing automorphisms. Indeed the twisted open-string orientation-orbifold sectors correspond to the orientation-reversing automorphisms, which have the form $\tau_\times \omega, \omega \in H$ where the basic automorphism $\tau_-$ exchanges the left- and right-movers of the closed string and $\omega$ is an extra automorphism which acts uniformly on the left- and right-movers of the closed string. Similarly, the automorphism group $H_+$ of the generalized $\mathbb{Z}_2$-permutation orbifolds in (2.2) is generated by elements of the form $\tau_+ \times \omega, \omega \in H'$, where the basic automorphism $\tau_+$ exchanges the two copies of the closed string and the extra automorphism $\omega$ again acts uniformly on the left- and right-movers of each closed string. In both cases, the extra automorphisms $\omega$ in $\tau \times \omega$ may or may not form a group (see the examples at the end of this section).

The spectra of different $\hat{c} = 52$ strings are characterized by their extended (twisted) Virasoro generators, all of which can in fact be written in the following unified form:

$$\hat{L}_u \left( m + \frac{u}{2} \right) = \frac{1}{4} \sum_{r, \mu \nu} G^{n(r)_\mu; -n(r)_\nu}(\sigma) \sum_{v=0}^{1} \sum_{\rho \in \mathbb{Z}} \times$$

$$\times : \hat{J}_{n(r)_{\mu \nu}} \left( p + \frac{n(r)}{\rho(\sigma)} + \frac{u}{2} \right) \hat{J}_{-n(r), \nu; u-v} \left( m - p - \frac{n(r)}{\rho(\sigma)} + \frac{u-v}{2} \right) :M +$$

$$+ \delta_{m+u,0} \hat{\Delta}_0(\sigma)$$

(2.3a)

$$\left[ \hat{J}_{n(r)_{\mu \nu}} \left( m + \frac{n(r)}{\rho(\sigma)} + \frac{u}{2} \right), \hat{J}_{n(s)_{\mu \nu}} \left( n + \frac{n(s)}{\rho(\sigma)} + \frac{v}{2} \right) \right] =$$

$$= 2 \left( m + \frac{n(r)}{\rho(\sigma)} + \frac{u}{2} \right) \delta_{n(r)+n(s),0}\text{mod}(\rho) \delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)}+\frac{u+v}{2},0} G_{n(r),\mu; -n(r),\nu}(\sigma)$$

(2.3b)

$$\left[ \hat{L}_u \left( m + \frac{u}{2} \right), \hat{J}_{n(r)_{\mu \nu}} \left( n + \frac{n(r)}{\rho(\sigma)} + \frac{v}{2} \right) \right]$$

$$= - \left( n + \frac{n(r)}{\rho(\sigma)} + \frac{v}{2} \right) \hat{J}_{n(r)_{\mu \nu}} \left( m + n + \frac{n(r)}{\rho(\sigma)} + \frac{u+v}{2} \right)$$

$$\hat{\Delta}_0(\sigma) = \frac{13}{8} + \frac{1}{2} \sum_{r} \dim [\pi(r)] \left( \frac{n(r)}{\rho(\sigma)} - \frac{1}{2} \right) \left( \theta \left( \frac{n(r)}{\rho(\sigma)} \geq \frac{1}{2} \right) - \frac{n(r)}{\rho(\sigma)} \right)$$

(2.3c)

(2.3d)
Each set of extended Virasoro generators in Eq. (2.3a) satisfies the order-two orbifold Virasoro algebra (1.1b) at \( \hat{c} = 52 \), and the current algebras in Eq. (2.3b) are of the type called \textit{doubly-twisted} in the orbifold program.

For those unfamiliar with the program, I first give a short summary of the standard notation in the result (2.3) – followed by the derivation of the result. As in the extended Virasoro generators themselves, the indices \( u, v \) with fundamental range \( \bar{u}, \bar{v} \in \{0, 1\} \) describe the twist of the basic permutations \( \tau^{\pm} \) in each \( H^{\pm} \). For each extra automorphism \( \omega(\sigma) \) in each \( \tau^{\pm} \times \omega(\sigma) \), the spectral indices \( \{n(r)\} \) and the degeneracy indices \( \{\mu \equiv \mu(n(r))\} \) of each twisted sector \( \sigma \) are determined by the so-called \( H \)-eigenvalue problem [3,5,6]

\[
\omega(\sigma)_a^b U^\dagger(\sigma)_b^{n(r)\mu} = U^\dagger(\sigma)_a^{n(r)\mu} e^{-2\pi i \frac{n(r)}{\rho(\sigma)}} \quad \omega(\sigma) \in H \text{ or } H' \quad (2.4a)
\]

\[
\omega(\sigma)_a^c \omega(\sigma)_b^d G_{cd} = G_{ab}, \quad G_{ab} = G^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.4b)
\]

\[
a, b = 0, 1, \ldots, 25, \quad \bar{\pi}(r) \in (0, 1, \ldots, \rho(\sigma) - 1) \quad (2.4c)
\]

where \( G \) is the untwisted target-space metric of \( U(1)^2 \). The quantity \( \rho(\sigma) \) is the order of \( \omega(\sigma) \) and all indices \( \{n(r)\mu\} \) are periodic modulo \( \rho(\sigma) \), with \( \{\bar{\pi}(r)\} \) the pullback to the fundamental region and \( \dim[\bar{\pi}(r)] \) the size of the subspace \( \bar{\pi}(r) \). The index \( r \) is summed once over the fundamental region in Eqs. (2.3a), (2.3d) and (2.3e). The \textit{twisted metric} \( \mathcal{G}_{\mu}(\sigma) \) and its inverse \( \mathcal{G}^{\mu}(\sigma) \) are defined in terms of the unitary eigenvectors \( U(\sigma) \) of the H-eigenvalue problem

\[
\mathcal{G}_{n(r)\mu; n(s)\nu}(\sigma) = \chi_{n(r)\mu} \chi_{n(s)\nu} U(\sigma)_{n(r)\mu}^a U(\sigma)_{n(s)\nu}^b G_{ab} \quad (2.5a)
\]

\[
= \delta_{n(r)+n(s),0 \mod \rho(\sigma)} \mathcal{G}_{n(r)\mu; n(s)\nu}(\sigma) \quad (2.5b)
\]

\[
\mathcal{G}^{n(r)\mu; n(s)\nu}(\sigma) = \chi^{-1}_{n(r)\mu} \chi^{-1}_{n(s)\nu} G^{ab} U^\dagger(\sigma)_a^{n(r)\mu} U^\dagger(\sigma)_b^{n(s)\nu} \quad (2.5c)
\]

\[
= \delta_{n(r)+n(s),0 \mod \rho(\sigma)} \mathcal{G}^{n(r)\mu; n(s)\nu}(\sigma) \quad (2.5d)
\]

where \( G \) is again the untwisted metric and the \( \chi \)'s are essentially-arbitrary normalization constants. Finally, the standard mode normal-ordering in Eq. 6
(2.3a) is:

\[ :\hat{J}_{n(r)\mu u}(m + \frac{n(r)}{\rho(\sigma)} + \frac{u}{2}) : \hat{J}_{n(s)\nu v}(n + \frac{n(s)}{\rho(\sigma)} + \frac{v}{2}) : M \]  

(2.6)

\[ = \theta \left( \left( m + \frac{n(r)}{\rho(\sigma)} + \frac{u}{2} \right) \geq 0 \right) \hat{J}_{n(s)\nu v}(n + \frac{n(s)}{\rho(\sigma)} + \frac{v}{2}) \hat{J}_{n(r)\mu u}(m + \frac{n(r)}{\rho(\sigma)} + \frac{u}{2}) + \theta \left( \left( m + \frac{n(r)}{\rho(\sigma)} + \frac{u}{2} \right) < 0 \right) \hat{J}_{n(s)\nu v}(n + \frac{n(s)}{\rho(\sigma)} + \frac{v}{2}) \hat{J}_{n(r)\mu u}(m + \frac{n(r)}{\rho(\sigma)} + \frac{u}{2}). \]

It follows that the quantity \( \Delta_0(\sigma) \) in Eqs. (2.3a) and (2.3d)

\[ \hat{J}_{n(r)\mu u} \left( \left( m + \frac{n(r)}{\rho(\sigma)} + \frac{u}{2} \right) \geq 0 \right) |0\rangle_\sigma = 0 \]  

(2.7a)

\[ \rightarrow \hat{L}_\mu \left( \left( m + \frac{u}{2} \right) \geq 0 \right) |0\rangle_\sigma = \hat{\Delta}_0(\sigma) \delta_{m+\frac{u}{2},0} |0\rangle_\sigma \]  

(2.7b)

is the conformal weight of the scalar twist-field state \(|0\rangle_\sigma\) of sector \(\sigma\).

I comment briefly on the derivation of the unified form (2.3) of the \(\hat{c} = 52\) extended Virasoro generators. Essentially this result was given for the twisted open-string sectors of the non-abelian orientation orbifolds in Subsecs. 3.4, 3.5 of Ref. [12], and that result is easily reduced for our abelian case \(U(1)^{26}/H_-\) in Eq. (2.1). With a right-mover copy of the extended Virasoro generators (and \(\overline{\mu} \rightarrow \overline{\mathcal{G}} = 0,1\)), the result also hold for the twisted closed-string sectors of the generalized \(\mathbb{Z}_2\)-permutation orbifolds \((U(1)^{26} \times U(1)^{26})/H_+\) in Eq. (2.2). This follows by the substitution

\[ G \rightarrow \mathcal{G}, \quad \frac{u}{2} \rightarrow \frac{n(r)}{\rho(\sigma)} + \frac{u}{2} \]  

(2.8)

into the known results for the ordinary \(\mathbb{Z}_2\)-permutation orbifolds with trivial \(H^\prime\) (see Ref. [perm] and Subsec. 4.2 of Ref. [16]). Finally, a single copy of the unified form (2.3) holds as well for each twisted sector of the generalized open-string \(\mathbb{Z}_2\)-permutation orbifolds \((U(1)^{26} \times U(1)^{26})_{\text{open}}/H_+\) and all possible T-dualizations of each of these sectors. This conclusion follows because the left-mover extended Virasoro generators of the closed-string orbifolds for each \(H_+\) are the input data for the construction of the corresponding open-string orbifolds [14], and the twisted-current form of each set of extended Virasoro generators is independent of T-dualization [15]. The branes, quasi-canonical algebra and non-commutative geometry of the twisted open strings [13-15,16,17] depend of course on the particular T-dualization, but these will not be needed here.

In what follows I will consider each twisted \(\hat{c} = 52\) string separately, but the reader may find it helpful to bear in mind the complete sector structure of
these orbifold-string systems as labelled by the elements of the automorphism groups $H_{\mp}$. Given a particular extra automorphism $\omega_n \in H$ or $H'$ of order $n$, one may list the following low-order examples:

\begin{align*}
(1; \tau_{\mp}) & \quad (2.9a) \\
(1; \tau_{\mp} \times \omega_2) & \quad (2.9b) \\
(1, \omega_3, \omega_3^2; \tau_{\mp}, \tau_{\mp} \times \omega_3, \tau_{\mp} \times \omega_3^2) & \quad (2.9c) \\
(1, \omega_4^2, \tau_{\mp} \times \omega_4, \tau_{\mp} \times \omega_4^2) & \quad (2.9d) \\
(1, \omega_6^2, \omega_6^4, \tau_{\mp} \times \omega_6, \tau_{\mp} \times \omega_6^3, \tau_{\mp} \times \omega_6^5). & \quad (2.9e)
\end{align*}

For the generalized $\mathbb{Z}_2$-permutation orbifolds ($\tau_+$) all of these sectors are twisted closed strings at $\hat{c} = 52$, while all the sectors of the generalized open-string $\mathbb{Z}_2$-permutation orbifolds ($\tau_+$) and their T-dualizations are twisted open strings at $\hat{c} = 52$. For the orientation orbifolds ($\tau_-$) the sectors before the semicolon are twisted closed strings at $c = 26$ (which form an ordinary space-time orbifold) while the sectors after the semicolon are twisted open strings at $\hat{c} = 52$. More generally, orientation orbifolds always contain an equal number of twisted open and closed strings. In all cases, the twisting is of course trivial for sectors corresponding to the unit element.

3 First Discussion of the $\hat{c} = 52$ String Spectra

To frame this discussion, I remind [1] the reader that the Virasoro primary states of our orbifold CFT’s are defined by the integral Virasoro subalgebra (generated by $\{\hat{L}_0(m)\}$) of the extended Virasoro algebra. Then the extended physical-state conditions (1.1a) tell us that all the physical states $\{|\chi\rangle\}$ of each $\hat{c} = 52$ orbifold-string are Virasoro primary

\begin{align*}
\hat{L}_0(m > 0)|\chi\rangle &= 0 \quad (3.1)
\end{align*}

but only a small subset of these primary states are selected by the rest of the physical-state conditions:

\begin{align*}
\left(\hat{L}_0(0) - \frac{17}{8}\right)|\chi\rangle &= \hat{L}_1 \left((m + \frac{1}{2}) > 0\right)|\chi\rangle = 0. \quad (3.2)
\end{align*}
In what follows, I will refer to the $\hat{L}_0(0)$ condition in Eq. (3.2) as the spectral condition, since it will determine the allowed values of momentum-squared for each $\hat{c} = 52$ string.

The space of physical states of each orbifold-string is then much smaller than the space of states of the underlying orbifold conformal field theory. For the experts, I remark in particular that the extended physical-state conditions generically disallow the characteristic sequence [1,9] of Virasoro primary states known as the principle-primary states [1,9]. This follows first by the spectral condition (which fixes the conformal weight), and second because the physical-state condition

$$\hat{L}_u \left( (m + \frac{u}{2}) > 0 \right) \simeq 0$$

which does not extend to $m = 0$.

I turn now to concretize the spectral condition of each twisted $\hat{c} = 52$ string, using the explicit form (2.3) of its extended Virasoro generators. For this, recall [12,15] first that these generators contain in general two kinds of commuting zero modes (dimensionless momenta), namely \{\hat{J}_{0\mu\nu}(0)\} and \{\hat{J}_{\rho(\sigma)/2,\mu,1}(0)\}, where the latter is relevant only when the order $\rho(\sigma)$ of $\omega(\sigma)$ is even. In what follows, I often refer to these zero modes collectively as \{\hat{J}(0)\}. It is then natural to define the “momentum-squared” operator $\hat{P}^2$ as follows:

$$\hat{L}_0(0) = \frac{1}{4} \left( -\hat{P}^2 + \hat{R}(\sigma) \right) + \hat{\Delta}_0(\sigma)$$

$$\hat{P}^2 \equiv - \sum_{\mu,\nu} \left\{ \mathcal{G}_{0\mu;0\nu}(\sigma) \hat{J}_{0\mu,0\nu}(0) \hat{J}_{0\nu,0\mu}(0) + \mathcal{G}_{\frac{\rho(\sigma)/2}{2},\mu;\frac{\rho(\sigma)/2}{2},\nu}(\sigma) \hat{J}_{\rho(\sigma)/2,\mu,1}(0) \hat{J}_{-\rho(\sigma)/2,\nu,-1}(0) \right\}$$

$$\hat{R}(\sigma) \equiv \left( \sum_{r,\mu,\nu} \sum_{u \in \mathbb{Z}} \right) \mathcal{G}^{n(r)\mu; -n(r)\nu}(\sigma) \times$$

$$\times : \hat{J}_{n(r)\mu\nu} \left( p + \frac{n(r)}{\rho(\sigma)} + \frac{u}{2} \right) \hat{J}_{-n(r)\nu,-\mu} \left( -p - \frac{n(r)}{\rho(\sigma)} + \frac{u}{2} \right) : M.$$
With this decomposition, the spectral condition in Eq. (3.2) takes the simple form:

\[ \hat{P}^2 |\chi\rangle = \left( \hat{P}^2_{(0)} + \hat{R}(\sigma) \right) |\chi\rangle \]  

(3.5a)

\[ \hat{P}^2_{(0)} \equiv 2 \left( \hat{\delta}_0(\sigma) - 1 \right) \]  

(3.5b)

\[ \hat{\delta}_0(\sigma) = \sum_r \text{dim}[\mathcal{N}(r)] \left( \frac{\mathcal{m}(r)}{\rho(\sigma)} - \frac{1}{2} \right) \left( \theta \left( \frac{\mathcal{m}(r)}{\rho(\sigma)} > \frac{1}{2} \right) - \frac{\mathcal{m}(r)}{\rho(\sigma)} \right) \geq 0. \]  

(3.5c)

Although I will continue the discussion primarily in this form, in fact Eqs. (3.4a) and (3.5a) hold only for the twisted open-string sectors of the orbifolds. For the twisted closed-string sectors, we also have right-mover copies of the extended Virasoro generators (2.3), and a corresponding right-mover copy of the extended physical-state conditions (1.1) on the same \{|\chi\rangle\}. For simplicity I will limit the discussion of these sectors here to the case of decompactified zero modes, for which it is appropriate to equate the left and right movers

\[ \hat{J}_R(0) = \hat{J}_L(0) = \frac{1}{\sqrt{2}} \hat{J}(0) \rightarrow \hat{R}^R(\sigma) = \hat{R}^L(\sigma) \]  

(3.6)

where the last equality is level-matching in each twisted sector. Keeping the same definition of the operator \( \hat{P}^2 \) in Eq. (3.4b), the correct closed-string \( \hat{c} = 52 \) spectral condition is then obtained by the substitution

\[ \hat{P}^2 \rightarrow \frac{1}{2} \hat{P}^2 \]  

(3.7)

in both Eqs. (3.4a) and (3.5a). These identifications, and hence \( \hat{P}^2_{(0)} \rightarrow 2\hat{P}^2_{(0)} \), can be used at any point in the discussion below to obtain the corresponding closed-string results.

Returning to the open-string case, one simple solution of the extended physical-state conditions is the ground state \(|0, \hat{J}(0)\rangle_\sigma \) of twisted sector \( \sigma \):

\[ \hat{R}(\sigma)|0, \hat{J}(0)\rangle_\sigma = \hat{L}_u \left( (m + \frac{n}{2}) > 0 \right) |0, \hat{J}(\sigma)\rangle_\sigma = 0 \]  

(3.8a)

\[ \hat{P}^2|0, \hat{J}(0)\rangle_\sigma = \hat{P}^2_{(0)}|0, \hat{J}(0)\rangle_\sigma, \quad \hat{P}^2_{(0)} = -2 + 2\hat{\delta}_0(\sigma) \]  

(3.8b)

This is the “momentum-boosted” twist-field state (see Eq. (2.7)) of that sector, with ground-state mass-squared \( \hat{P}^2_{(0)} \). Moreover Eq. (3.4c) and the commutator (2.3c) give the increments

\[ \Delta(\hat{P}^2) = \Delta(\hat{R}(\sigma)) = 4 \left| m + \frac{\mathcal{m}(r)}{\rho(\sigma)} + \frac{u}{2} \right| \]  

(3.9)
obtained by adding the negatively-moded current
\[ \hat{J}_{n(r)\mu u} \left( m + \frac{n(r)}{n(\sigma)} + \frac{u}{2} \right) < 0 \]
to any previous state. The precise content of these excited levels must of course be determined from the remainder of the extended physical-state conditions.

I continue this discussion with some specific examples of \( \hat{c} = 52 \) strings, beginning with the simplest twisted open-string orientation-orbifold sectors [12,13,15,16,17]:

\[ \omega = \mathbb{1} : \quad \rho = 1, \quad \overline{\mu} = 0, \quad U = \mathbb{1}, \quad G = G, \quad \hat{J}_{0a\mu} \left( m + \frac{u}{2} \right) \quad (3.10a) \]
\[ \Delta(\hat{P}^2) = 4 |m + \frac{u}{2}| \quad (\overline{\mu} = 0 \text{ is DD, } \overline{\mu} = 0 \text{ is ND}) \quad (3.10b) \]

\[ \omega = -\mathbb{1} : \quad \rho = 2, \quad \overline{\mu} = 1, \quad U = \mathbb{1}, \quad G = G, \quad \hat{J}_{1a\mu} \left( m + \frac{u+1}{2} \right) \quad (3.11a) \]
\[ \Delta(\hat{P}^2) = 4 |m + \frac{u+1}{2}| \quad (\overline{\mu} = 0 \text{ is DN, } \overline{\mu} = 0 \text{ is NN}) \quad (3.11b) \]

In these cases, the extra automorphisms \( \omega \) act uniformly on the labels \( a = 0, \ldots, 25 \) and \( G \) is the untwisted target space metric in Eq. (2.4b). Although both twisted strings have \( 26 + 26 = 52 \) matter degrees of freedom, note that each example has only one of the two types of zero modes \( \{ \hat{J}(0) \} \): 26 DD zero modes \( \{ \hat{J}_{0a\mu}(0) \} \) for \( \omega = \mathbb{1} \) and 26 NN zero modes \( \{ \hat{J}_{\rho/2,a,1}(0) \} \) for \( \omega = -\mathbb{1} \).

In both cases, the momentum-squared (3.4b) has the schematic form
\[ \hat{P}^2 = \eta^{ab} \hat{J}_a(0) \hat{J}_b(0), \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad (3.12) \]
where \( \eta = -G \) is the standard (west-coast) 26-dimensional target space metric. Then we compute from Eqs. (3.5b) and (3.5c) that both strings share the same tachyonic ground-state mass-squared
\[ \hat{\delta}_0(\sigma) = 0, \quad \hat{\Delta}_0(\sigma) = \frac{13}{8}, \quad \hat{P}^2_{(0)} = -2 \quad (3.13) \]
and the first excited state of each is massless:
\[ \hat{P}^2 \left\{ \begin{array}{l} \hat{J}_{0a1} \left( -\frac{1}{2} \right) \\ \hat{J}_{1a0} \left( -\frac{1}{2} \right) \end{array} \right\} |0, \hat{J}(0)\rangle_\sigma = 0 \quad \text{for} \quad \omega = \begin{cases} \mathbb{1} \\ -\mathbb{1}. \end{cases} \quad (3.14) \]
For this level, I have checked that the $\hat{L}_1 \left( \frac{1}{2} \right) \simeq 0$ gauge eliminates the longitudinal parts of the 26-dimensional “photons”, and moreover the $\hat{L}_1 \left( \frac{1}{2} \right)$ and $\hat{L}_0 \left( 1 \right)$ gauges together eliminate the negative-norm states at the next level:

$$\left( \alpha \left( \hat{J} \left( -\frac{1}{2} \right) \right)^2 + \beta \hat{J}(-1) \right) |0, \hat{J}(0)\rangle_\sigma, \quad \hat{P}^2 = 2. \quad (3.15)$$

Since the increments $\Delta(P^2)$ in Eqs. (3.10b) and (3.11b) are even integers, we are led to suspect that the spectra of these two twisted $\hat{c} = 52$ strings are nothing but the spectrum of an ordinary open $c = 26$ string in disguise \[1\]. I will return to this question in the following section.

A larger subset of twisted $\hat{c} = 52$ strings is the following. For a particular twisted sector $\sigma$, suppose that $\omega = \pm \mathds{1}$ acts uniformly on a set of $d$ labels $a = 0, 1, \ldots, d - 1, d \geq 4$ while a non-trivial element $\omega(\text{perm})$ of some permutation group acts non-trivially on the other $26 - d$ spatial labels. Then Eqs. (2.4),(2.5) and standard results [3,5-7,9] in the orbifold program give the following explicit form of the extended Virasoro generators (2.3) in this sector:

$$\hat{L}_m \left( m + \frac{u}{2} \right) = \delta_{m+\frac{u}{2},0} \Delta_0(\sigma) +$$

$$+ \frac{1}{4} G^{ab}_{(d)} \sum_v \sum_p \hat{J}_{eav} \left( p + \frac{u+v}{2} \right) \hat{J}_{-\epsilon,b,u-v} \left( m - p + \frac{u-v-\epsilon}{2} \right) :M +$$

$$+ \frac{1}{4} \sum_v \sum_j \frac{1}{f_j(\sigma)} \sum_{j=0}^{f_j(\sigma)-1} \sum_p \times$$

$$\times \hat{J}_{jiv} \left( p + \frac{j}{f_j(\sigma)} + \frac{u}{2} \right) \hat{J}_{-\epsilon,j,u-v} \left( m - p - \frac{j}{f_j(\sigma)} + \frac{u-v}{2} \right) :M$$

$$\hat{\delta}_0(\sigma) = \sum_j f_j(\sigma)-1 \sum_{j=0}^{f_j(\sigma)-1} \left( \frac{j}{f_j(\sigma)} - \frac{1}{2} \right) \left( \theta \left( \frac{j}{f_j(\sigma)} > \frac{1}{2} \right) - \frac{j}{f_j(\sigma)} \right) \geq 0 \quad (3.16b)$$

$$\sum_j f_j(\sigma) = 26 - d, \quad 4 \leq d \leq 26. \quad (3.16c)$$

Here $\epsilon = 0$ or 1 for $\omega = \mathds{1}$ or $-\mathds{1}$, $G^{(d)}$ is the restriction of the flat target-space metric (2.4b) to the first $d$ labels, $f_j(\sigma)$ is the size of the $j$th cycle in $\omega(\text{perm}),$

---

1The spectra of these two $\hat{c} = 52$ strings look even more familiar in terms of the dimensionful momenta $k \equiv \hat{J}(0)/\sqrt{2\alpha'_0}$, where $\alpha'_0$ is the conventional open-string Regge slope.
and the previous cases with \( \hat{\delta}_0(\sigma) = 0 \) are included when \( d = 26 \). The half-integer moded currents in the second term of (3.16) satisfy the twisted current algebra (2.3b) with \( G \rightarrow G^{(d)} \). For the permutation-twisted currents in the last term of (3.16), I have used the standard relation 

\[
\frac{n(r)}{\rho(\sigma)} = \frac{j}{f_j(\sigma)}
\]

and (the inverse of) the twisted metric \([3,5-7,9]\)

\[
G_{jj'}^{il}(\sigma) = \delta_{j,l} f_j(\sigma) \delta_{j'+i,0 \text{mod} f_j(\sigma)}
\]

which also determines the twisted current algebra (2.3b) for these currents. Using Eq. (3.16b), we see that the non-trivial element of \( \mathbb{Z}_2 \) on two labels also gives \( \hat{\delta}_0(\sigma) = 0 \) and a \( \hat{P}_2(0) = -2 \) ground state, but a non-trivial element of \( \mathbb{Z}_3 \) on three labels gives a slightly-raised ground state

\[
\hat{\delta}_0(\sigma) = \frac{1}{9}, \quad \hat{\Delta}_0(\sigma) = \frac{121}{72}, \quad \hat{P}_2(0) = -\frac{16}{9}
\]

and no photons.

Given the cycle-structure \( \{f_j(\sigma)\} \) of any extra automorphism \( w(\text{perm}) \) (see e.g. Eq. (3.4) of Ref. [16]), it is straightforward to evaluate the sum in Eq. (3.16b). As an illustration, one finds the simple tachyonic ground-state mass-squares

\[
5 \leq (d = \text{prime}) \leq 23 : \quad \hat{P}_2(0) = -\frac{1}{12}(d - 2 + \frac{1}{26 - d})
\]

in twisted sectors which correspond to the action of any non-trivial element of the cyclic group \( \mathbb{Z}_\lambda \) of prime order on \( 3 \leq (\lambda = 26 - d) \leq 21 \) spatial labels. The result (3.19) includes Eq. (3.18) when \( d = 23 \), but does not extend to the cases \( d = 26, 24 \) with \( \hat{P}_2(0) = -2 \) discussed above. I remind that this result applies only to the open orbifold-strings, while twice these values of \( \hat{P}_2(0) \) are obtained for the closed-string versions.

Further analysis of the \( \hat{c} = 52 \) strings, including the “larger subset” of examples (3.16), is found in the following section.

### 4 Equivalent \( c = 26 \) Description of the \( \hat{c} = 52 \) Spectra

In fact, there exists an entirely equivalent description of all the \( \hat{c} = 52 \) string spectra in terms of generically-unconventional Virasoro generators at \( c = 26 \).
To obtain the $c = 26$ description, I first define the relabelled (unhatted) operators
\[ J_{n(r)\mu} \left( 2m + u + \frac{2n(r)}{\rho(\sigma)} \right) \equiv \hat{J}_{n(r)\mu u} \left( m + \frac{n(r)}{\rho(\sigma)} + \frac{u}{2} \right), \quad \overline{u} = 0, 1 \tag{4.1a} \]
\[ L(2m + u) \equiv 2\hat{L}_u \left( m + \frac{u}{2} \right) - \frac{13}{4} \delta_{m+\frac{u}{2},0} \tag{4.1b} \]
in terms of the hatted operators above. This $1\!-\!1$ map is recognized as a modest generalization of (the inverse of) the order-two orbifold-induction procedure of Borisov, Halpern and Schweigert [1]. Since $M \equiv 2m + u, \overline{u} = 0, 1$ covers the integers once, we then find from (2.3) the explicit form of the $c = 26$ generators:
\[ L(M) = \hat{\delta}_0(\sigma) \delta_{M,0} + \frac{1}{2} \sum_{r,\mu,\nu} G_{n(r)\mu; -n(r),\nu}(\sigma) \sum_{Q \in \mathbb{Z}} J_{n(r)\mu} \left( Q + \frac{2n(r)}{\rho(\sigma)} \right) J_{-n(r),\nu} \left( M - Q - \frac{2n(r)}{\rho(\sigma)} \right) \tag{4.2a} \]
\[ \hat{\delta}_0(\sigma) = \sum_r \text{dim}[\pi(r)] \left( \overline{\pi(r)} \frac{1}{\rho(\sigma)} - \frac{1}{2} \right) \left( \theta \left( \overline{\pi(r)} \frac{1}{\rho(\sigma)} > \frac{1}{2} \right) - \overline{\pi(r)} \frac{1}{\rho(\sigma)} \right) \tag{4.2b} \]
\[ [L(M), L(N)] = (M - N) L(M + N) + \frac{26}{12} M(M^2 - 1) \delta_{M+N,0} \tag{4.2c} \]
\[ \left[ L(M), J_{n(r)\mu} \left( N + \frac{2n(r)}{\rho(\sigma)} \right) \right] = - \left( N + \frac{2n(r)}{\rho(\sigma)} \right) J_{n(r)\mu} \left( M + N + \frac{2n(r)}{\rho(\sigma)} \right) \tag{4.2d} \]
\[ \left[ J_{n(r)\mu} \left( M + \frac{2n(r)}{\rho(\sigma)} \right), J_{n(s)\nu} \left( N + \frac{2n(s)}{\rho(\sigma)} \right) \right] \tag{4.2e} \]
\[ = \delta_{n(r)+n(s),0 \text{mod} \rho(\sigma)} \delta_{M+N,2(\frac{2n(r)+n(s)}{\rho(\sigma)})} G_{n(r)\mu; -n(r),\nu}(\sigma). \]
The expression (4.2b) for $\hat{\delta}_0(\sigma)$ is the same as above, and the mode-normal ordering in Eq. (4.2a)
\[ :J_{n(r)\mu} \left( M + \frac{2n(r)}{\rho(\sigma)} \right) J_{n(s)\nu} \left( N + \frac{2n(s)}{\rho(\sigma)} \right) :M \tag{4.3} \]
\[ = \theta \left( \left( M + \frac{2n(r)}{\rho(\sigma)} \right) \geq 0 \right) J_{n(s)\nu} \left( N + \frac{2n(s)}{\rho(\sigma)} \right) J_{n(r)\mu} \left( M + \frac{2n(r)}{\rho(\sigma)} \right) \]
\[ + \theta \left( \left( M + \frac{2n(r)}{\rho(\sigma)} \right) < 0 \right) J_{n(r)\mu} \left( M + \frac{2n(r)}{\rho(\sigma)} \right) J_{n(s)\nu} \left( N + \frac{2n(s)}{\rho(\sigma)} \right) \]
follows from the $\hat{c} = 52$ ordering (2.6) because the map (4.1) preserves the sign of all arguments.
I emphasize that the $c = 26$ Virasoro generators in Eq. (4.2) are generically-unconventional because the twisted matter is now summed over the fractions \( \{2n/\rho\} \) instead of the conventional orbifold-fractions \( \{n/\rho\} \). This distortion of the “extra twist” is the price we must pay in order to unwind the “basic twist” associated to the basic permutations $\tau_{\pm}$ of $H_{\pm}$.

The map (4.1) also tells us that the $\hat{c} = 52$ momenta \( \{\hat{J}(0)\} \) and the $c = 26$ momenta \( \{J(0)\} \) are identical, and we may record

$$J(0) = \hat{J}(0) : J_{0\mu}(0) = \hat{J}_{0\mu}(0), \quad J_{\rho(\sigma)/2,\mu}(0) = \hat{J}_{\rho(\sigma)/2,\mu,1}(0)$$

(4.4a)

$$P^2 = \hat{P}^2$$

(4.4b)

$$= -\sum_{\mu,\nu} \left\{ G^{0\mu;0\nu}(\sigma)J_{0\mu}(0)J_{0\nu}(0) + G^{\rho(\sigma);-\frac{\mu(\sigma)}{2}J_{\rho(\sigma)/2,\mu}(0)J_{-\rho(\sigma)/2,\nu}(0) \right\}$$

(4.4b)

where the $\hat{c} = 52$ form of $\hat{P}^2$ was given in Eq. (3.4b). Similarly, the “level-number” operator $R(\sigma)$ in the decomposition of $L(0)$ is the same

$$L(0) = -\frac{1}{2} (P^2 + R(\sigma)) + \hat{\delta}_0(\sigma)$$

(4.5a)

$$R(\sigma) = \hat{R}(\sigma)$$

(4.5b)

$$= \left( \sum_{r,\mu,\nu} \sum_{Q \in \mathbb{Z}} G^{n(\mu;0\nu)}(\sigma) \times$$

$$\times J_{n(\mu)} \left( Q + \frac{2n(\mu)}{\rho(\sigma)} \right) J_{-n(\nu)} \left( -Q - \frac{2n(\nu)}{\rho(\sigma)} \right) :M \right)$$

where the $\hat{c} = 52$ form of $\hat{R}(\sigma)$ was given in Eq. (3.4c).

By itself, the inverse orbifold-induction procedure (4.1) is only a relabelling of the operators of the permutation-orbifold CFT’s. The central point of this discussion however is that for the orbifold-string theories – restricted by the extended physical state conditions (1.1) – the map also gives us a completely equivalent $c = 26$ description of the physical spectrum of each $\hat{c} = 52$ orbifold-string. Indeed, it is easily checked that both components $\bar{u} = 0, 1$ of the $\hat{c} = 52$ extended physical-state condition (1.1a) map directly onto the simpler and in fact conventional physical-state condition

$$L(M \geq 0)|\chi\rangle = \delta_{M,0}|\chi\rangle$$

(4.6)
in the 26-dimensional description! A right- mover copy of Eq. (4.6) on the same physical states \{|\chi\rangle\} is similarly obtained in the equivalent \(c = 26\) description of the closed orbifold-strings.

I emphasize that the physical states \{|\chi\rangle\} of the 26-dimensional description (4.6) are exactly the original physical states (1.1a) of the \(\hat{c} = 52\) string. Indeed, each physical state \(|\chi\rangle\) can be regarded as invariant under the map, or each can now be rewritten in 26-dimensional form. In further detail, Eqs. (4.5) and (4.6) give the same spectral condition \(P^2 \simeq P_0^2 + R(\sigma)\), the same physical ground state \(|0, J(0)\rangle\sigma \equiv |0, \hat{J}(0)\rangle_{\sigma}, P_0^2 = \hat{P}_0^2 = -2 + 2\hat{\delta}_0(\sigma)\) (4.7)

and each negatively-moded hatted current in any physical state can be replaced according to Eq. (4.1a) by the corresponding unhatted current mode. Note finally that the commutator (4.2d) and the decomposition (4.5a) give the 26-dimensional increment

\[
\Delta(\hat{P}^2) = \Delta(R(\sigma)) = 2 \left| M + \frac{2n(r)}{\rho(\sigma)} \right|
\]

which results from the addition of \(J_{n(r)}(\mu) \left( M + \frac{2n(r)}{\rho(\sigma)} \right) < 0 \) to any previous state. With \(M = 2m + n\), these are recognized as the same increments (3.9) obtained in the \(\hat{c} = 52\) description.

As simple examples, consider the “larger subset” (3.16) of \(\hat{c} = 52\) strings – whose equivalent \(c = 26\) physical state condition (4.6) now involves the following subset of the \(c = 26\) Virasoro generators (4.2):

\[
L(M) = \delta_{M,0}\hat{\delta}_0(\sigma) + \frac{1}{2}G^a_{(d)} \sum_{Q \in \mathbb{Z}} :J_{ea}(Q + \epsilon)J_{-\epsilon,b}(M - Q - \epsilon);M + \frac{1}{2} \sum_j \frac{1}{f_j(\sigma)} \sum_{j=0}^{f_j(\sigma)-1} \sum_{Q \in \mathbb{Z}} :J_{jj}(Q + \frac{2j}{f_j(\sigma)})J_{-j,j}(M - Q - \frac{2j}{f_j(\sigma)});M \quad (4.9a)
\]

\[
\hat{\delta}_0(\sigma) = \frac{1}{4} \sum_j \sum_{j=0}^{f_j(\sigma)-1} \left( \frac{2j}{f_j(\sigma)} - 1 \right) \left( 2\theta \left( \frac{2j}{f_j(\sigma)} > 1 \right) - \frac{2j}{f_j(\sigma)} \right) \quad (4.9b)
\]

\footnote{Although it is not directly relevant in either description of the \(\hat{c} = 52\) strings, one notes that the conformal weight of the scalar twist-field state \(|0\rangle_{\sigma}\) of sector \(\sigma\) has now shifted from \(\Delta_0(\sigma)\) to \(\delta_0(\sigma)\) in the \(c = 26\) description.}
\[ a, b = 0, \ldots, d - 1, \quad \sum_j f_j(\sigma) = 26 - d, \quad 4 \leq d \leq 26. \] (4.9c)

Recall for the larger subset that \( \epsilon = 0, 1 \) corresponds in the symmetric theory to the action of the extra automorphism \( \omega = \pm 1 \) on the first \( d \geq 4 \) labels \( \{a\} \), while \( f_j(\sigma) \) is the length of the \( j \)-th cycle of the extra permutation \( \omega(\text{perm}) \) which acts on the remaining \( 26 - d \) spatial labels. Shifting the dummy integer \( Q \) by the integer \( \epsilon \), we note that the second term in Eq. (4.9a) is a set of ordinary Virasoro generators for \( d \) untwisted bosons with the ordinary current algebra

\[ [J_a(Q), J_b(P)] = G^{(d)}_{ab} Q \delta_{Q+P,0} \] (4.10)

for both values of \( \epsilon \). The currents in the third term satisfy the twisted current algebra (4.2e) with the permutation-twisted metric (3.17), and the value of \( \hat{\delta}_0(\sigma) \) in Eq. (4.9b) is only a slightly-rewritten form of that given in Eq. (3.16b).

We are now in a position to confirm our suspicions in the previous section about the simplest orbifold-strings, described earlier at \( \hat{c} = 52 \) by the extended Virasoro generators:

\[
\hat{L}_u (m + \frac{u}{2}) = \frac{1}{4} G^{ab} \sum_v \hat{J}_{k v} (p + \frac{u+\epsilon}{2}) \hat{J}_{-\epsilon,b,u-v} (m - p + \frac{u-\epsilon}{2}) :M \\
+ \frac{13}{8} \delta_{m+\frac{u}{2},0}, \quad \epsilon = 0, 1, \quad \epsilon = 0, 1. \] (4.11)

These are now equivalently described by the choice \( d = 26 \) in Eq. (4.9), in which case only the second (ordinary) term of Eq. (4.9a) is non-zero – and then the equivalent physical-state condition (4.6) verifies that the physical spectrum of each of these particular twisted \( \hat{c} = 52 \) strings is indeed equivalent to that of an ordinary untwisted \( c = 26 \) string! These cases include the open-string orientation-orbifold sectors corresponding to \( \tau_- \times (\omega = \pm 1) \) in Eq. (3.10) and their T-duals, as well as the twisted closed-string sectors of the generalized \( \mathbb{Z}_2 \)-permutation orbifolds corresponding to \( \tau_+ \times (\omega = \pm 1) \).

Additionally, consider the following special cases of the extended Virasoro
generators (3.16) at $\hat{c} = 52$

$$\hat{L}_u (m + \frac{u}{2}) = \delta_{m+\frac{u}{2}, \frac{13}{8}} +$$
$$+ \frac{1}{4} G_{(24)}^{ab} \sum_v \sum_p :J_{\epsilon \epsilon \epsilon} (p + \frac{u+\epsilon}{2}) \hat{J}_{-\epsilon, b, u-v} (m - p + \frac{u-v-\epsilon}{2}) :M +$$
$$+ \frac{1}{8} \sum_v \sum_j \sum_p :J_{j \epsilon} (p + \frac{j+u}{2}) \hat{J}_{-j, u-v} (m - p + \frac{u-v+i}{2}) :M$$

which result when the extra automorphism in the symmetric theory acts as $\omega = \pm 1$ on the first $d = 24$ labels and the non-trivial element of a $\mathbb{Z}_2$ on the remaining 2 spatial labels. I have noted in Sec. 3 that $\hat{\delta}_0 (\sigma) = 0$ for these cases as well, and indeed the equivalent $c = 26$ description (4.6) and (4.9) at $d = 24$ now shows that the open and closed orbifold-strings of this type also have the spectrum of ordinary untwisted $c = 26$ strings. The common thread for the orbifold-strings in Eqs. (4.10) and (4.12) is that they are at most half-integer moded, so that the shift $\{n/\rho\} \rightarrow \{2n/\rho\}$ gives integer moding in the $c = 26$ description.

Beyond these simple cases, the $\hat{c} = 52$ strings are apparently new – with $\hat{\delta}_0 (\sigma) \neq 0$, unfamiliar ground-state mass-squares, and fractional modeing (and increments) in either description.

5 Conclusions

We have discussed the physical spectrum of the general $\hat{c} = 52$ orbifold-string, as well as an equivalent but unconventionally-twisted $c = 26$ description of the twisted $\hat{c} = 52$ matter. The equivalent $c = 26$ description holds only for the orbifold-string theories – restricted by the extended physical-state conditions (1.1) – and not in the larger Hilbert space of the underlying orbifold conformal field theories.

In general we have found that the spectra of these orbifold-string systems are unfamiliar. One simple and unexpected conclusion however is that, as string theories restricted by the extended physical-state conditions, the single twisted $\hat{c} = 52$ sector of each of the simplest orbifolds of permutation-type (see Eq. (2.9))

$$(1; \tau_{\pm})$$

$$(1; \tau_{\pm} \times \omega_2), \quad \omega_2^2 = 1$$

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have the same physical spectra as ordinary untwisted $c = 26$ strings. No such equivalence is found of course in the half-integer moded Hilbert space of the full orbifold CFT’s. The list in Eq. (5.1) includes the simplest orientation orbifolds (with $\tau_-$) and their T-duals, as well as the simplest generalized $\mathbb{Z}_2$-permutation orbifolds (with $\tau_+$).

For the simplest orientation orbifolds in particular, the string theories in Eq. (5.1) consist of an ordinary unoriented closed string (the unit element) at $c = 26$ and a $\hat{c} = 52$ twisted open string whose physical spectrum is equivalent to that of an ordinary untwisted $c = 26$ critical open string. Since both the closed- and open-string spectra of these simple orientation orbifolds are equivalent to those of the archtypal orientifold (without Chan-Paton factors), we are led to suspect that orientation orbifolds include orientifolds. I will return in the next paper of this series to consider this question at the interacting level, where we will also be able to ask about the decoupling of null physical states. Following that, I will consider in a succeeding paper the corresponding situation and modular invariance for the simplest permutation orbifold-string systems.

More generally, we have seen that there are many other orientation orbifolds, open-string $\mathbb{Z}_2$-permutation orbifolds and generalized $\mathbb{Z}_2$-permutation orbifolds whose $\hat{c} = 52$ spectra show fractional modeing in both the $\hat{c} = 52$ and the $c = 26$ descriptions. These include in particular the orbifolds in Eq. (2.9) when the order $n$ of the extra automorphism is greater than two.

There is more to say about no-ghost theorems for the general twisted $\hat{c} = 52$ string. The original intuition [16] was that the doubled gauges $\bar{u} = 0, 1$ of the extended physical state condition (1.1) could remove the doubled set of negative-norm states (time-like modes) of the $\hat{c} = 52$ strings – which are also associated with $\bar{u} = 0, 1$. For the simplest $\hat{c} = 52$ strings in Eq. (5.1), this intuition is certainly born out [20]. More generally, the equivalent $c = 26$ description of each spectrum shows that both aspects of the doubling are indeed eliminated at the same time, leaving us with the conventional physical state condition (4.6) and only a single set of time-like modes. This is clearly visible in the set of examples (4.9), where the only time-like modes ($a = 0$) are included in the second term. For the general $\hat{c} = 52$ string, the reader should bear in mind that the twisted metric $\mathcal{G}$ in Eq. (4.2) is only a unitary transformation (2.5) of the untwisted metric $G$ with a single time-like direction. Although not yet a proof, and illustrated here only for $\hat{c} = 52$, I consider this a stronger form of the original arguments [16] that all the critical orbifolds of permutation-type should be free of negative-norm
The next question I wish to address is the following: I have emphasized that the equivalent $c = 26$ Virasoro generators (4.2) are generically-unconventional, being summed over the matter-field fractions $\{2n/\rho\}$ instead of the conventional orbifold fractions $\{n/\rho\}$, but are they actually new Virasoro generators? I do not know the answer to this question in general, but at least some of them can in fact be re-expressed by further mode-relabeling in terms of more familiar Virasoro generators. As examples, consider the special case of the “larger subset” (4.9) when $\omega(\text{perm})$ is one of the elements of order $\lambda$ of each cyclic group $\mathbb{Z}_\lambda$. (These are the particular, single-cycle elements of $\mathbb{Z}_\lambda$ with $f_0(\sigma) = \lambda$.) When $\lambda$ is odd, one finds that the first and third terms of (4.9) can in fact be re-expressed in terms of the conventional Virasoro generators associated to a twisted sector of an ordinary cyclic permutation orbifold $U(1)^\lambda/\mathbb{Z}_\lambda$. [christ]

\[
L_\lambda(M) = \frac{1}{2\lambda} \sum_{j=0}^{\lambda-1} \sum_{Q \in \mathbb{Z}} :J_j(Q + \frac{j}{\lambda}) J_{-j}(M - Q - \frac{j}{\lambda}) :M + \delta_{M,0} \frac{1}{2\lambda} \left( \lambda - \frac{1}{\lambda} \right), \quad c = \lambda = 2l + 1
\]

where I have relabeled the currents $J_j \equiv J_{j\theta}$. To obtain this result from (4.9), one needs the fact that $\{2j/\lambda\} \simeq \{j/\lambda\}$ modulo the integers when $\lambda$ is odd. This observation is consistent with the ground-state mass-squares for prime $\lambda$ in Eq. (3.19). When $\lambda$ is even, I have also checked that the first and third terms of (4.9) can be re-expressed as the sum of two identical commuting Virasoro generators of this type

\[
L_\lambda(M) = L_+^\lambda(M) + \tilde{L}_+^\lambda(M), \quad c = \lambda = 2l
\]

each of which is associated to a twisted sector of $U(1)^{\lambda/2}/\mathbb{Z}_{\lambda/2}$. This result is also obtained by relabeling the modes modulo the integers, and provides us with another way to understand that the ground-state mass-squared is unshifted when $\omega(\text{perm})$ is the non-trivial element of a $\mathbb{Z}_2$.

My final remark is a conjecture, that the extended physical-state conditions for the twisted strings at $\hat{c} = 26\lambda$, $\lambda$ prime will in fact read

\[
\left( \hat{L}_j \left( \left( m + \frac{j}{\lambda} \right) \geq 0 \right) - \delta_{m+\hat{j},0} \hat{\delta}_\lambda \right) \left| \chi \right> = 0, \quad \hat{j} = 0, 1, \ldots, \lambda - 1
\]

(5.4a)
\[ a_\lambda \equiv \frac{13\lambda^2 - 1}{12\lambda} \quad (5.4b) \]

\[ \left[ \hat{L}_j \left( m + \frac{j}{\lambda} \right), \hat{L}_l \left( n + \frac{l}{\lambda} \right) \right] = \left( m - n + \frac{j+l}{\lambda} \right) \hat{L}_j+l \left( m + n + \frac{j+l}{\lambda} \right) + \frac{26\lambda}{12} \left( m + \frac{j}{\lambda} \right) \left( (m + \frac{j}{\lambda})^2 - 1 \right) \delta_{m+n+\frac{j+l}{\lambda},0} \quad (5.4c) \]

where Eq. (5.4c) is an orbifold Virasoro algebra \([1,18,9]\) of order \( \lambda \). This form includes the correct generators \( \{ \hat{L}_j \} \) corresponding to the classical extended Polyakov constraints of Ref. [16], and includes the correct value \( \hat{a}_2 = 17/8 \) studied here for the \( \hat{c} = 52 \) strings. I obtained the system (5.4) by requiring (as we now know for \( \lambda = 2 \)) that it map by the inverse of the order-\( \lambda \) orbifold-induction procedure [1] to the conventional physical-state condition (4.6) with \( \hat{a}_1 = 1 \) at \( c = 26 \). One way to test this conjecture would be the construction of the corresponding twisted BRST systems [17] for these higher values of \( \hat{c} \).

Extensions to include winding number and twisted \( B \) fields at \( \hat{c} = 52 \) are also deferred to another time and place.

**Acknowledgements**

For helpful information, discussions and encouragement, I thank L. Alvarez-Gaumé, K. Bardakci, I. Brunner, J. de Boer, D. Fairlie, O. Ganor, E. Gimon, C. Helfgott, E. Kiritsis, R. Littlejohn, S. Mandelstam, J. McGreevy, N. Obers, A. Petkou, E. Rabinovici, V. Schomerus, K. Schoutens, C. Schweigert and E. Witten. This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC02-O5CH11231 and in part by the National Science Foundation under grant PHY00-98840.

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