Geometric picture of quantum discord for two-qubit quantum states

Mingjun Shi\textsuperscript{1,3}, Fengjian Jiang\textsuperscript{1}, Chunxiao Sun\textsuperscript{1} and Jiangfeng Du\textsuperscript{1,2,3}

\textsuperscript{1} Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, People’s Republic of China
\textsuperscript{2} Hefei National Laboratory for Physical Sciences at Microscale, Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, People’s Republic of China
E-mail: shmj@ustc.edu.cn and djf@ustc.edu.cn

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Abstract. Among various definitions of quantum correlations, quantum discord has attracted considerable attention. To find an analytical expression for quantum discord is an intractable task. Exact results are known only for very special states, namely two-qubit X-shaped states. We present in this paper a geometric viewpoint, from which two-qubit quantum discord can be described clearly. The known results on X state discord are restated in the directly perceivable geometric language. As a consequence, the dynamics of classical correlations and quantum discord for an X state in the presence of decoherence is endowed with geometric interpretation. More importantly, we extend the geometric method to the case of more general states, for which numerical as well as analytical results on quantum discord have not yet been obtained. Based on the support of numerical computations, some conjectures are proposed to help us establish the geometric picture. We find that the geometric picture for these states has an intimate relationship with that for X states. Thereby, in some cases, analytical expressions for classical correlations and quantum discord can be obtained.

\textsuperscript{3} Authors to whom any correspondence should be addressed.
Correlation is the relationship between different things, and is a pervasive phenomenon in nature. It is the means by which we learn about the external world, and is the bridge with which we communicate with each other and transmit information from one end to the other.

In the classical world, correlations have been well studied from the viewpoint of information theory (see, for example, [1]). However, ‘quantizing’ classical information or correlation is definitely not an effortless task. Difficulties crop up because quantum information, unlike its classical counterpart, is encoded in quantum states that may not be orthogonal and thus may not be distinguished unambiguously; moreover, quantum systems can be correlated in ways inaccessible to classical objects.

One of the prominent features of quantum correlation is entanglement. Entangled states cannot be prepared with the help of local operations and classical communication (LOCC) and thus are nonclassical. Entanglement is indeed an important aspect of quantum correlation and is a prerequisite for many tasks in quantum information processing [2]. Nevertheless, entanglement is not the only aspect of quantum correlation, and the notion of quantum correlation is more general than entanglement. For example, there exists quantum nonlocality without entanglement [3–5].

Various approaches, other than through entanglement, have been proposed for studying the correlations in a composite quantum system. The first attempt to quantify quantum contents of correlations was that by Zurek. The concept of quantum discord was proposed and developed into a measure of the extent to which the underlying correlation of two quantum systems is non-classical [6–8]. The important issue is the existence of quantum correlations beyond entanglement in separable states. Quantification of classical correlations in a bipartite quantum state and splitting the total correlation into a classical and a quantum part were presented by Henderson and Vedral [9]. Later, the Henderson–Vedral (H–V) classical correlation was shown to have an operational meaning: the regularization of H–V classical correlation is just the maximal number of common random bits obtained by one-way LOCC operations in...
excess of communication invested [10, 11]. In [12], Oppenheim et al presented an operational proposal, which comes from thermodynamical consideration, to quantify quantum correlations (see also [13]). By considering the amount of noise required to erase the correlation, Groisman et al [14] gave an operational definition of the quantum, classical and total amounts of correlations in a bipartite quantum state. Recently, Modi et al proposed a unified view of quantum and classical correlations by using the concept of relative entropy as a distance measure [15]. Based on the Hilbert–Schmidt distance, a geometric measure of quantum discord is presented by Dakić et al [16] (see also [17]).

The above considerations shed new light on the properties of the correlation incorporated into a composite quantum system. After the division of total correlation into classical and quantum parts, many works have been devoted to the study of the roles played by different types of correlation in quantum processes, and reveal the relationship between them. These studies involve fuzzy measurement [18], mixed-state quantum computation speed-ups [19, 20], broadcasting of quantum state [21], complete positivity of dynamics [22, 23], complementarity and monogamy relationship between classical and quantum correlations [24–26], dynamics of discord [27, 28], operational interpretations of quantum discord in terms of state merging [29, 30] and the relation between quantum discord and entanglement [31, 32].

However, there is no effective method for calculating the exact results on quantum discord and other measures of quantumness analytically. Unlike the measure of entanglement, the new paradigms of quantumness of correlations are measurement oriented. What should be done in these paradigms is to extract information about system A by measuring another system B. Given a bipartite quantum system in the state \( \rho^{AB} \), when measuring system B gives the outcome \( k \) with probability \( p_k \), system A would be in some post-measurement state \( \rho_k^A \). For a complete measurement on system B, the \( p_k \) and \( p_k \) are the members and probabilities of an ensemble of the local state of system A; that is, \( \rho^A = \sum_k p_k \rho_k^A \). The accessible information about system A with respect to the particular measurement is given by \( S(\rho^A) - \sum_k p_k S(\rho_k^A) \), where \( S(\rho) \) is the von Neumann entropy of a state \( \rho \). The major obstacle is to maximize the accessible information, or equivalently, to minimize the average entropy \( S^A = \sum_k p_k S(\rho_k^A) \), over all possible complete measurements carried out on system B. The explicit analytical results on quantum discord are known only for very special cases: Bell-diagonal states [33], X-shaped states [34] of the two-qubit system and Gaussian states of continuous variable systems [35]. For the Bell-diagonal states and X-shaped states, the surfaces of constant discord are plotted and analyzed in [36, 37].

Considering this problem, we propose in this paper a geometric method to describe the quantum discord of two-qubit quantum states. The geometric method is based on the idea of a quantum steering ellipsoid, which is defined in [38]. A quantum steering ellipsoid is an ellipsoid in three-dimensional real space \( \mathbb{R}^3 \) such that each point in the interior or on the surface represents a post-measurement state of one qubit when a particular measurement has been carried out on the other qubit. We denote the quantum steering ellipsoid by \( \mathcal{E} \). The available post-measurement states are constrained by the \( \mathcal{E} \). In other words, decomposition of one local state, say \( \rho^A \), can only be performed in the \( \mathcal{E} \) (including the surface), namely, \( \rho^A = \sum_k p_k \rho_k^A \) for each \( \rho_k^A \in \mathcal{E} \). We call a post-measurement ensemble optimal if this ensemble minimizes the average entropy. We also call the optimal ensemble the optimal decomposition of the local state. It can be shown that the optimal ensemble can only be found on the surface of \( \mathcal{E} \). This situation can be compared with the optimal signal ensembles studied in [39], where the output states of a noisy quantum channel are restricted to a convex set \( \mathcal{A} \), and the optimal signal ensemble \( \{ p_k^{opt}, \rho_k^{opt} \} \)
is such that the Holevo quality, $\chi = S(\rho) - \sum_k p_k S(\rho_k)$ with each $\rho_k \in \mathcal{A}$ and $\rho = \sum_k p_k \rho_k$, reaches the maximum on $\{p_k^{\text{opt}}, \rho_k^{\text{opt}}\}$.

For two-qubit X states, there are only two candidates for the optimal ensemble. We call them equi-entropy decomposition and quasi-eigendecomposition, respectively. The geometric picture of these two forms of decomposition is clear: equi-entropy decomposition corresponds to a horizontal line segment, while quasi-eigendecomposition corresponds to a vertical one. Then the known results for Bell-diagonal states and X states can be ‘seen’ in this picture. Subsequently, we study the dynamics of classical correlations and quantum discord in the presence of decoherence. In [28], it has been shown that there is a sudden transition from the classical to quantum decoherence regime for some Bell-diagonal states undergoing nondissipative decoherence. We generalize this result to the case of general X states. The sudden transition can even be ‘seen’ in the geometric picture.

Not only can the geometric method be used to recover the known results, but also it should help us find the possible analytical expressions of quantum discord for more general states. Following this line of thought, we consider a class of two-qubit states that have more complicated forms than X states. For these states, we cannot give a thoroughly analytical procedure to derive the classical correlations or quantum discord. However, numerical computations give us interesting results. We find that, just like the case of X states, there are only two possibilities as to the optimal post-measurement ensemble. One is equi-entropy decomposition, and the other, although not the quasi-eigendecomposition, has an intimate relation with the quasi-eigendecomposition. For the former, we can write out analytical expressions for classical correlations and quantum discord, whereas for the latter, further research is needed to characterize its property. We think that these phenomena revealed by numerical work should not be accidental coincidences. If these phenomena can be verified analytically, they will give us a geometric insight into the quantum discord.

In section 2, we give a brief overview of the concepts of classical correlations and quantum discord. In section 3, we introduce a very useful tool the quantum steering ellipsoid for evaluating the quantum discord of two-qubit states. We present in section 4 the geometric picture to evaluate and describe the quantum discord of X states. For X states undergoing decoherence, the dynamics of classical correlations and quantum discord is studied and depicted geometrically in section 5. More general states are considered in sections 6 and 7. Several conjectures and numerical tests are presented therein. Section 8 concludes the paper.

Throughout this paper, the logarithm has base 2. Numerical computations have been performed by using Mathematica 8.0.

2. Classical correlations and quantum discord

Consider a bipartite quantum system composed of particle A and particle B, which are possessed by Alice and Bob, respectively. The state of the whole system is described by a density matrix $\rho_{AB}$. Total correlation between particle A and particle B is usually measured by the mutual information, that is,

$$ I(\rho_{AB}) = S(\rho^A) + S(\rho^B) - S(\rho_{AB}), $$

(1)

where $\rho^A$ and $\rho^B$ are local states of A and B, respectively, $\rho^{A(B)} = \text{Tr}_{B(A)}(\rho_{AB})$. Mutual information quantifies the strength of the correlation. For product state $\rho_{AB} = \rho^A \otimes \rho^B$, the entropy $S(\rho_{AB})$ is additive, namely, $S(\rho_{AB}) = S(\rho^A) + S(\rho^B)$, and it follows that the
mutual information for any product state is zero. For a maximally entangled state, such as \( \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A \otimes |i\rangle_B \), with \( d \) being the dimension of the Hilbert space of subsystem \( A \) or \( B \), the mutual information reaches its maximal value, \( 2 \log_2 d \). It is shown that quantum mutual information is just the minimal rate of randomness that is required to completely erase all the correlations in \( \rho_{AB} \) [14].

The total correlation \( I \) can be split into a quantum part \( Q \) and a classical part \( C \); namely, \( I = Q + C \). There are several ways to define the measure of classical correlations. Here we adopt the definition given by Henderson–Vedral [9], which quantifies the information gained about one subsystem from the measurement on the other.

Suppose Bob carries out POVM measurements on his particle \( B \). The set of positive operator valued measure (POVM) elements is denoted by \( \mathcal{M} = \{M_k\} \) with \( M_k \geq 0 \) and \( \sum_k M_k = 1 \). The probability that outcome \( k \) is obtained is given by \( p_k = \text{Tr}[\rho_{AB} (1 \otimes M_k)] \).

The post-measurement state of particle \( A \) that corresponds to the outcome \( k \) is

\[
\rho_A^k = \frac{1}{p_k} \text{Tr}_B[\rho_{AB} (1 \otimes M_k)].
\]

Considering all POVM elements \( M_k \)'s, the post-measurement states of particle \( A \) are characterized by the ensemble \( \{p_k, \rho_A^k\} \). Note that Alice’s local state \( \rho^A \) remains unchanged; namely, \( \rho^A = \sum_k p_k \rho_A^k \) for any post-measurement ensemble. In other words, Bob’s POVM measurements induce a decomposition of Alice’s local state \( \rho^A \) into the ensemble \( \{p_k, \rho_A^k\} \).

The information about particle \( A \) that is acquired by Bob’s specific POVM \( \mathcal{M} \) is given by

\[
S(\rho^A) - \sum_k p_k S(\rho_A^k).
\]

The dependence on the measurement procedure can be removed by maximization over all possible POVMs. The classical correlation is then defined as

\[
C^- = \max_{\mathcal{M}} \left[ S(\rho^A) - \sum_k p_k S(\rho_A^k) \right] = S(\rho^A) - \min_{\mathcal{M}} \sum_k p_k S(\rho_A^k),
\]

where the maximization and minimization are taken over all of Bob’s POVM measurements. The left arrow over \( C \) indicates the situation that Bob makes a measurement to acquire information about Alice’s system. Similarly, if Alice performs the POVM, \( \mathcal{N} = \{N_j\} \), we can define the information gained about particle \( B \) by measuring particle \( A \) as

\[
C^- = \max_{\mathcal{N}} \left[ S(\rho^B) - \sum_j p_j S(\rho_B^j) \right] = S(\rho^B) - \min_{\mathcal{N}} \sum_j p_j S(\rho_B^j).
\]

Generally, \( C^- \neq C^- \), meaning that the classical information gain given in the above sense is asymmetric.

It is natural to define quantum correlation as the difference between total correlation and classical correlation, namely,

\[
Q^- = I - C^- = \min_{\mathcal{M}} \sum_k p_k S(\rho_A^k) + S(\rho^B) - S(\rho_{AB}).
\]

Similarly for \( Q^- = I - C^- \).
Quantum correlation $Q$ is also called quantum discord. Quantum discord, which is originally defined as the difference between two classically identical (but quantumly distinct) formulae that measure the amount of mutual information of a pair of quantum systems \cite{6,7}, aims to capture all the quantum correlations, not limited to entanglement. There is a fundamental difference between entanglement and discord for mixed states, although they are equivalent for pure states. A typical example of this is the class of separable states with nonvanishing discord \cite{8}. Other ways of defining of quantum discord can be found in \cite{40}.

To obtain quantum discord or classical correlation, one has to make considerable effort to minimize the average entropy $\overline{S}^A = \sum_k p_k S(\rho_k^A)$ over all possible measurements on particle B. In the next section, we introduce a useful tool, the quantum steering ellipsoid, which will help us to establish a geometric picture of these concepts.

3. The quantum steering ellipsoid

First we express the states and POVM elements in the Hilbert–Schmidt space. Let $\rho^{AB}$ be a two-qubit state shared by Alice and Bob. It can be written as $\rho^{AB} = \frac{1}{2} \sum_{\alpha,\beta=0}^{3} R_{\alpha\beta} \sigma_\alpha \otimes \sigma_\beta$, where $\sigma_0$ is the $2 \times 2$ identity matrix, $\sigma_i$ ($i = 1, 2, 3$) are Pauli matrices and $R_{\alpha\beta} = \text{Tr}[\rho^{AB} (\sigma_\alpha \otimes \sigma_\beta)]$ are all real numbers. We arrange the 16 coefficients $R_{\alpha\beta}$ into a $4 \times 4$ matrix $R = (R_{\alpha\beta})$. Note that $R_{00}$ is just the trace of $\rho^{AB}$ and is equal to one. We write $M_k$, one element of Bob’s POVM, as $M_k = \sum_{\alpha=0}^{3} x_k,\alpha \sigma_\alpha$. Similarly, the state $\rho_k^A$ (see (2)) can be expressed as $\rho_k^A = \frac{1}{4} \sum_{\alpha=0}^{3} (y)_k,\alpha \sigma_\alpha$.

Let us define two four-component vectors in row form, $y_k = (y_{k,0}, \bar{y}_k)$ and $x_k = (x_{k,0}, \bar{x}_k)$, where $\bar{y}_k = (y_{k,1}, y_{k,2}, y_{k,3})$ and $\bar{x}_k = (x_{k,1}, x_{k,2}, x_{k,3})$. Note that $y_{k,0} = 1$ for all $k$ and $\bar{y}_k$ is the Bloch vector of $\rho_k^A$. Direct calculation shows the following equation,

$$p_k \ y_k = x_k \ R^T, \quad p_k = \sum_{\alpha=0}^{3} R_{\alpha\alpha} x_k,\alpha,$$

where the superscript $T$ means matrix transpose. Equation (5) provides the relationship between Bob’s measurement and the corresponding components in Alice’s ensemble. For entangled states, the matrix $R$ is of full rank, and the vector $y_k$ is in one-to-one correspondence to the vector $x_k$. In the following, we will treat $R$ as a full rank matrix. Some states with singular $R$ will be discussed in section 6.

Also note that (5) imposes constraints on the vector $y_k$: although $x_k$ can represent any projective measurement, the vector $y_k$ cannot be arbitrary. For example, $y_k$ cannot represent a pure state unless $\rho^{AB}$ is a pure state. It is pointed out in \cite{38} that the allowed $y_k$ must satisfy

$$y_k \left[ R^{-T} \eta \ R^{-1} \right] (y_k)^T \geq 0,$$

where $R^{-T} = (R^{-1})^T$ and $\eta = \text{diag}(1, -1, -1, -1)$. In fact, (6) comes from the requirement that each $M_k$ is nonnegative. Noting that the three-component vector $\bar{y}_k = (y_{k,1}, y_{k,2}, y_{k,3})$ is the Bloch vector of $\rho_k^A$, we can see that (6) describes an ellipsoidal region in three-dimensional (3D) real space. This means that, for each $\rho_k^A$ allowed to appear in Alice’s ensemble $\{p_k, \rho_k^A\}$, the corresponding Bloch vector $\bar{y}_k$ is constrained to be within an ellipsoid (including the surface). The ellipsoid given by (6) is called ‘steering ellipsoid’ in \cite{38}. We denote it by $e$.

The steering ellipsoid renders a concrete geometric picture when we perform the minimization of the average entropy $\overline{S}^A$. Each point belonging to $e$ corresponds to some $\rho_k^A$. By noting that the set $e$ is convex and the entropy function is concave, we see that the minimal
value of $S^A$ must be attained on the surface of $\mathcal{E}$. For any point on the surface of $\mathcal{E}$, i.e. for any vector $\mathbf{y}$ such that the equality in (6) holds, the corresponding vector $\mathbf{x}$ (see (5))—must satisfy $x_0^2 = x_1^2 + x_2^2 + x_3^2$. Such a vector $\mathbf{x}$ represents the rank-one element of Bob’s POVM that can be taken to be proportional to the 1D projector $\Pi$, namely, $M = m\Pi$. With the factor $m$ absorbed into the probability $p$, we say that any point on the surface of $\mathcal{E}$ is induced by Bob’s projective measurement. Therefore in order to obtain an optimal ensemble of Alice’s state, Bob need only perform projective measurements. This fact has been pointed out by Hamieh et al in [41]. We give a geometric description here.

4. Quantum discord of two-qubit X states

For a general two-qubit state, $\mathcal{E}$ is too complicated to be dealt with. However, for a specific class of states, called X states, we will show that the geometric picture is very clear. In various situations, X states have been used to demonstrate significant quantum phenomena; for example, entanglement sudden death or birth [42, 43], dynamics of quantum and classical correlations [27, 44] and sudden transition between classical and quantum decoherence [28]. The density matrix of a general two-qubit X state is

$$\rho = \begin{pmatrix} a & 0 & 0 & u e^{i\mu} \\ 0 & b & v e^{i\nu} & 0 \\ 0 & v e^{-i\nu} & c & 0 \\ u e^{-i\nu} & 0 & 0 & d \end{pmatrix},$$

(7)

where $a + b + c + d = 1$ and $u, v \geq 0$. It is required that $u^2 \leq ad$ and $v^2 \leq bc$ to ensure the positivity of the density matrix. By local unitary operations, the off-diagonal entries can be transformed to real ones. Since all correlations are invariant under local unitary operations, it suffices to consider X states with all the entries of density matrix being real. However, we will retain the form given by (7) for later references.

The matrix $R$ is given by

$$R = \begin{pmatrix} 1 & 0 & 0 & a - b \\ 0 & 2u \cos \mu & -2u \sin \mu & 0 \\ 0 & +2v \cos \nu & +2v \sin \nu & 0 \\ a + b & 0 & 0 & a - b \\ c - d & -c - d \end{pmatrix},$$

(8)

When $\det(R) = 16(bc - ad)(u^2 - v^2) \neq 0$, the inverse $R^{-1}$ exists. From (6), we can write the equation of the ellipsoid $\mathcal{E}$, i.e.

$$\frac{y_1^2}{l_1^2} + \frac{y_2^2}{l_2^2} + \frac{y_3^2}{l_3^2} = 1,$$

(9)
Figure 1. Schematic plot of quantum steering ellipsoid $\varepsilon$ of a two-qubit X state. In the $(O y_1 y_2 y_3)$ frame, the center of $\varepsilon$, denoted by $O'$, lies on the $y_3$-axis. Point A represents the state of qubit A.

where $y'_1 = y_1 \cos \phi - y_2 \sin \phi$, $y'_2 = y_1 \sin \phi + y_2 \cos \phi$, $y'_3 = y_3 - Y_3$ with $\phi = (\mu + \nu)/2$, and the three major axes are given by

$$
\begin{align*}
    l_1 &= \frac{u + v}{\sqrt{(a + c)(b + d)}}, \\
    l_2 &= \frac{|u - v|}{\sqrt{(a + c)(b + d)}}, \\
    l_3 &= \frac{|ad - bc|}{(a + c)(b + d)}, \\
    Y_3 &= \frac{ab - cd}{(a + c)(b + d)}.
\end{align*}
$$

The $y_3$-axis intersects the LE at two points, G and H. Consider a line parallel to the $y'_1$-axis and passing through point A. This line will intersect the LE at two points, E and F (see figure 2).

We now proceed to find Alice’s optimal ensemble $\{p_k, \rho_k^A\}^{opt}$. When Bob performs complete projective measurements, Alice’s ensemble has two components, i.e. $k = 1, 2$, and both $\rho_1^A$ and $\rho_2^A$ are on the surface of $\varepsilon$. In the following, we demonstrate a geometric picture to describe how to obtain Alice’s optimal ensemble and thereby the value of $S_{\text{min}}^A$.
Schematic plot of the largest ellipse given by $(y_1'/l_1)^2 + (y_3'/l_3)^2 = 1$. Two pairs of points, (E,F) and (G,H), are the only candidates for Alice’s optimal ensemble, which will give the minimal value of average entropy $S^A$.

Figure 2. Schematic plot of the largest ellipse given by $(y_1'/l_1)^2 + (y_3'/l_3)^2 = 1$. Two pairs of points, (E,F) and (G,H), are the only candidates for Alice’s optimal ensemble, which will give the minimal value of average entropy $S^A$.

Remember that point A stands for Alice’s local state $\rho^A$. Then $\rho^A$ can be expressed as either of the two forms of the convex sum: $\rho^A = p_G \rho_G + p_H \rho_H$ or $\rho^A = p_E \rho_E + p_F \rho_F$, where $\rho_G$ is the state corresponding to point G and the probability $p_G = AH/\phi$, and similarly for other states and probabilities. These two forms of the convex sum lead us to the average entropies,

$$S_{GH} = p_G S(\rho_G) + p_H S(\rho_H),$$

$$S_{EF} = p_E S(\rho_E) + p_F S(\rho_F) = S(\rho_E) = S(\rho_F).$$

We call (12) quasi-eigendecomposition, meaning that $\rho^A$, $\rho_G$ and $\rho_H$ have the same eigenstate. We call (13) equi-entropy decomposition, meaning that $\rho^A = p_E \rho_E + p_F \rho_F$ with $S(\rho_E) = S(\rho_F)$.

Now we state our main result. With Bob carrying out POVM measurement on his particle, the optimal post-measurement ensemble of Alice’s state is given by $\{p_G, \rho_G, p_H, \rho_H\}$ or $\{p_E = 1/2, \rho_E; p_F = 1/2, \rho_F\}$, and the minimal value of $\overline{S}^A$ is

$$\overline{S}^A_{\min} = \min\{S_{GH}, S_{EF}\}.$$  

It follows that the classical correlation and quantum discord are given, respectively, by

$$C^-(\rho) = S(\rho^A) - \min\{S_{GH}, S_{EF}\},$$

$$Q^-(\rho) = \min\{S_{GH}, S_{EF}\} + S(\rho^B) - S(\rho^{AB}).$$

The result (14) can be derived by using the conclusion in [34]. As pointed out in [34], there are two candidates for Bob’s measurements that will induce Alice’s optimal ensemble. In our notations, these candidates are denoted by four-component vectors $\mathbf{x}_\pm$: (i) $\mathbf{x}_\pm = (\frac{1}{2}, 0, 0, \pm \frac{1}{2})$;
(ii) $x_\pm = \left(\frac{1}{2}, \pm \frac{1}{2} \sin \theta, \pm \frac{1}{2} \cos \theta, 0\right)$, where the measurement parameter $\theta$ will be determined later.

For case (i), it follows from (5) that

\[ p_+ = a + c, \quad y_+ = \left(1, 0, 0, \frac{a - c}{a + c}\right), \]

\[ p_- = b + d, \quad y_- = \left(1, 0, 0, \frac{b - d}{b + d}\right). \]

It is easy to see that this case results in the two points G and H in figure 2.

For case (ii), we have $p_+ = p_- = 1/2$ and $y_\pm = (1, \vec{y}_\pm)$ with Bloch vectors $\vec{y}_\pm$ given by

\[ (\vec{y}_\pm)^T = \begin{pmatrix} \pm 2u \sin(\theta - \mu) & \pm 2v \sin(\theta + \mu) \\ \mp 2u \cos(\theta - \mu) & \pm 2v \cos(\theta + \mu) \\ a + b - c - d \end{pmatrix}. \]

Because $|\vec{y}_+| = |\vec{y}_-|$, the entropy of the corresponding state is equal to each other, namely, $S(\rho_+) = S(\rho_-)$. It follows that the average entropy is given by $p_+ S(\rho_+) + p_- S(\rho_-) = S(\rho_+) = S(\rho_-)$. To obtain classical correlation, we will maximize $|\vec{y}_\pm|$ over the parameter $\theta$. In fact, the maximal value of $|\vec{y}_+|$ is attained when $\theta = (\pi + \mu - \nu)/2$. In this situation, the Bloch vectors $\vec{y}_\pm$ are given by

\[ \vec{y}_\pm = (\pm 2(u + v) \cos \phi, \mp 2(u + v) \sin \phi, a + b - c - d), \]

where $\phi = (\mu + \nu)/2$. It is straightforward to check that $\vec{y}_+$ and $\vec{y}_-$ just correspond to the vectors $\vec{O} \vec{F}$ and $\vec{O} \vec{E}$ in figure 2, respectively. Thus, we have proved that the two pairs of points, (G,H) and (E,F), stand for the only two candidates for Alice’s optimal ensemble. Now the problem of finding quantum discord for two-qubit X states is reduced to a simple geometrical one. The only thing we have to take into account is the steering ellipsoid $\mathcal{E}$ and the largest ellipsoidal section.

4.1. Bell-diagonal states

To appreciate the geometric picture, let us consider a specific class of X states, i.e. Bell-diagonal states. Although the quantum discord of Bell-diagonal states has been calculated explicitly in [33], we would like to provide a more concrete interpretation.

For a Bell-diagonal state $\rho_{\text{BD}}$ given by

\[ \rho_{\text{BD}} = \frac{1}{4} \sum_{\mu=0}^{4} t_\mu \sigma_\mu \otimes \sigma_\mu, \]

where $t_0 = 1$, $1 \pm t_3 \geq |t_1 \mp t_2|$ and it is assumed that $t_1 t_2 t_3 \neq 0$, the steering ellipsoid $\mathcal{E}$ has the standard form, i.e.

\[ \frac{y_1^2}{t_1^2} + \frac{y_2^2}{t_2^2} + \frac{y_3^2}{t_3^2} = 1. \]

Moreover, Alice’s local state, represented by point A, coincides with the origin point O. Without loss of generality, we assume that $|t_1| \geq |t_2| \geq |t_3|$. It is not difficult to see that the LE is given by

\[ \frac{y_1^2}{t_1^2} + \frac{y_2^2}{t_2^2} = 1. \]
According to the previous analysis, if Bob carries out a two-element POVM measurement, the minimal value of $\overline{S}_A$ is attained at the pair of points (E,F) or the pair (G,H). Note that $\text{OE} = \text{OF} = |t_1|$, $\text{OG} = \text{OH} = |t_2|$ and $|t_1| \geq |t_2|$. It follows that $\overline{S}_{\text{min}}^A = S(\rho_{E})$. Or more generally, $\overline{S}_{\text{min}}^A = \min\{h(|t_1|), h(|t_2|), h(|t_3|)\}$, with the function $h(x)$ defined by

$$h(x) = -\frac{1+x}{2} \log \frac{1+x}{2} - \frac{1-x}{2} \log \frac{1-x}{2}$$

for $x \in [0, 1]$.

5. Geometric picture of the dynamics of quantum discord

As an application of our result, let us consider the dynamics of quantum discord or classical correlation. Recently, this problem received considerable attention [27, 28, 44]. It has been shown that for some Bell-diagonal states passing through a phase damping channel, the classical correlation can be unaffected by decoherence. And more interestingly, the dynamics exhibit a sudden transition from the classical to the quantum decoherence regime [28]. We will show that in the geometric picture these phenomena can be ‘seen’ clearly even for a general X state.

When each particle of a two-qubit quantum system undergoes the phase damping process, the evolution of the state is expressed as

$$\rho_{AD} = \sum_{i,j=1}^{2} (K_i \otimes K_j) \rho (K_i \otimes K_j)^\dagger,$$

where $K_1 = \text{diag}(\gamma, 1)$ and $K_2 = \text{diag}(\sqrt{1-\gamma^2}, 0)$ are the Kraus operators representing the phase damping channel and $\gamma = e^{-\Gamma t}$ with $\Gamma$ being the phase damping rate. Here we assume that qubit A and B endure the same noisy environment. At the initial time $t = 0$, the steering ellipsoid $\mathcal{E}(0)$ is given by (9). At time $t > 0$, the ellipsoid is transformed to $\mathcal{E}(t)$, which is expressed by

$$\frac{y_1'^2}{(\gamma^2 l_1)^2} + \frac{y_2'^2}{(\gamma^2 l_2)^2} + \frac{y_3'^2}{l_3^2} = 1.$$  

That is, with $\gamma$ decreasing from 1 to 0, the radius of the ellipsoid along the $y_1'$-axis and that along the $y_2'$-axis decrease continuously from $l_1$ and $l_2$, respectively, to zero, whereas the radius along the $y_3'$-axis (i.e. $y_3$-axis) remains the same. Since $l_1 \geq l_2$, the LE is given by $[y_1'(\gamma^2 l_1)]^2 + [y_3'/l_3]^2 = 1$ in the time evolution. The LE will shrink to the $y_3'$-axis, namely, points G and H remain fixed and points E and F gradually approach the $y_3'$-axis. Also note that Alice’s local state $\rho^A$ is not affected by phase damping and thus point A is fixed.

We now show that the dynamics of quantum discord $Q^-$ and classical correlation $C^-$ can be demonstrated clearly in the geometric picture. To this end, it suffices to consider the minimal average entropy $\overline{S}_{\text{min}}^A$. See figure 3. There are only two possibilities with respect to the initial value of $\overline{S}_{\text{min}}^A$. One is that at $t = 0$, Alice’s optimal ensemble is determined by points G and H, and then $\overline{S}_{\text{min}}^A(t = 0) = S_{GH}$. At time $t$, points E and F move to $E_t$ and $F_t$, respectively, while points G and H remain unchanged. It follows from $\text{OE} < \text{OE}$ that $S_{E_tF_t} > S_{EF} > S_{GH}$. So in this
Visual interpretation of the dynamics of classical correlation and quantum discord. When an X state passes through a phase damping channel, the steering ellipsoid shrinks to the GH line with G and H fixed. If at \( t = 0 \) the optimal ensemble is determined by G and H, then this ensemble remains unchanged and is always optimal afterwards. In this case, the classical correlation is constant. If the initial optimal ensemble is determined by E and F, then the classical correlation will decrease with time: the points \( E_t \) and \( F_t \) determine the optimal ensemble at time \( t \). There must be a critical time \( \bar{t} \) after which the optimal ensemble is given by \( \{ \rho_G, \rho_H \} \) and then the classical correlation remains invariant.

In case, \( S_{\text{min}}^A \) is always given by \( S_{\text{GH}} \) and remains invariant. As a consequence, classical correlation \( C \leftarrow \) does not change during the time evolution.

The other is that initially Alice’s optimal ensemble is described by two points, E and F; that is, \( S_{\text{min}}^A (t = 0) = S_{EF} < S_{\text{GH}} \). Time evolution will make \( S_{E,F_t} \) larger continuously, until the evolution reaches the critical time, denoted by \( \bar{t} \), such that \( S_{E,F_t} = S_{\text{GH}} \). For \( t > \bar{t} \), we have \( S_{E,F_t} > S_{\text{GH}} \) and then \( S_{\text{min}}^A (t > \bar{t}) = S_{\text{GH}} \). This means that after the critical time \( \bar{t} \), the classical correlation \( C \leftarrow \) does not change any longer. In other words, and the phenomena presented in [27, 28] also arise in general X states.

It is noted that the discussion presented above is not limited to a phase-damping channel. In fact, the geometric picture applies to any quantum channel that preserves the X form of the state, such as all unital channels in the canonical form (e.g. the Pauli channel), and the amplitude damping channel. Classical correlations may not remain constant in these more general cases.

### 6. States with singular \( R \)

The geometric method presented in sections 3 and 4 is based on the 3D quantum steering ellipsoid, which requires a nonsingular coefficient matrix \( R \). To extend this idea to the case of singular \( R \), we consider in this section two classes of states. One is the class of X states with \( \det(R) = 0 \), which is in fact a supplement to the content of section 4. The other is such a class of states coming from mixing two pure product states. For these states, we put forward a conjecture about the geometric description of the quantum discord.
6.1. X states with \( \det(R) = 0 \)

It follows from the coefficient matrix \( R \) given by (8) that when \( ad - bc = 0 \) or \( u = v \), the determinant of \( R \) vanishes. Recalling the ellipsoid given by (9) and the parameters given by (10) and (11), we have the following cases.

When \( ad - bc = 0 \) and \( u \neq v \), we see that \( l_3 = 0 \) and \( r^A_3 \) (the \( y_3 \)-component of \( \vec{r}^A \)) is equal to \( Y_3 \). Then the ellipsoid degenerates to the ellipse. In the \((O'\ y'_1', y'_2', y'_3')\) frame, the equation of the ellipse is

\[
\frac{y'_1'^2}{l'_1^2} + \frac{y'_2'^2}{l'_2^2} = 1.
\]

And the position of Alice’s local state happens to be on the origin point \( O' \). Since \( l_1 > l_2 \), the minimal value of \( \bar{S}^A \) is given by

\[
\bar{S}^A_{\min} = h(l_1) = - \frac{1 + l_1}{2} \log \frac{1 + l_1}{2} - \frac{1 - l_1}{2} \log \frac{1 - l_1}{2}.
\]

When \( ad - bc = 0 \) and \( u = v \neq 0 \), it follows that \( l_2 = l_3 = 0 \) and \( r^A_3 = Y_3 \). We have two points given by \( y'_1 = l_1 \) and \( y'_1 = -l_1 \), which will determine the optimal ensemble of Alice’s state.

When \( ad - bc \neq 0 \) and \( u = v \neq 0 \), we have \( l_2 = 0 \), and the ellipsoid degenerates to an ellipse in the \( y'_1', y'_3' \) plane; that is,

\[
\frac{y'_1'^2}{l'_1^2} + \frac{y'_3'^2}{l'_3^2} = 1.
\]

In this case, \( \bar{S}^A_{\min} \) can be easily obtained with reference to figure 2.

The last case is that \( u = v = 0 \). In this case, the density matrix \( \rho \) takes the diagonal form. If \( ad - bc \neq 0 \), the state is classically correlated and the ellipsoid reduces to the two points G and H in figure 2, which determine the quantity \( \bar{S}^A_{\min} \). If \( ad - bc = 0 \), the state is a trivial product state.

6.2. A mixture of two pure product states

In this subsection, we will find the quantum and classical correlations in such states that can be written as

\[
\rho = \lambda \left| \psi_1 \right\rangle \left\langle \psi_1 \right| \otimes \left| \psi_2 \right\rangle \left\langle \psi_2 \right| + (1 - \lambda) \left| \phi_1 \right\rangle \left\langle \phi_1 \right| \otimes \left| \phi_2 \right\rangle \left\langle \phi_2 \right|,
\]

where \( \lambda \in [0, 1] \), and \( \left| \psi_i \right\rangle \) and \( \left| \phi_i \right\rangle \) (\( i = 1, 2 \)) are the states of particle A and B, respectively. Since the correlations remain invariant under local unitary transformations, it suffices to consider the states with the following form,

\[
\rho = \lambda \left| 0 \right\rangle \left\langle 0 \right| \otimes \left| 0 \right\rangle \left\langle 0 \right| + (1 - \lambda) \left| \psi \right\rangle \left\langle \psi \right| \otimes \left| \varphi \right\rangle \left\langle \varphi \right|,
\]

where \( \left| \psi \right\rangle = \cos \alpha |0\rangle + \sin \alpha |1\rangle \), and \( \left| \varphi \right\rangle = \cos \beta |0\rangle + \sin \beta |1\rangle \) with \( \alpha, \beta \in [0, \frac{\pi}{2}] \).

A special case of (16), where both \( \left| \psi \right\rangle \) and \( \left| \varphi \right\rangle \) are set to be \( \left| + \right\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \), is discussed in [9] (see also [41]), and numerical evaluation is performed to enquire about the classical
correlation therein. Here, we consider a more general case, and will put forward a conjecture about the exact value of quantum discord or classical correlation from a geometric viewpoint.

Suppose that Bob carries out POVM measurement on his qubit to acquire information about Alice’s qubit. As stated earlier, we need only consider projective measurements. Let Bob’s measurement operators be $M_+$ and $M_-; \text{ that is,}$

$$M_{\pm} = \frac{1}{2} \pm x_1 \sigma_x \pm x_2 \sigma_y \pm x_3 \sigma_z,$$

with $x_1^2 + x_2^2 + x_3^2 = 1/4$. Bob’s measurement will give the result ‘+’ with probability $p_+$ and the result ‘−’ with probability $p_-$. The corresponding post-measurement states of qubit A are $\rho^A_+$ and $\rho^A_−$, respectively.

We define two four-component vectors $\mathbf{x}_+$ and $\mathbf{x}_-$ as

$$\mathbf{x}_\pm = (\frac{1}{2}, \vec{x}) = (\frac{1}{2}, \pm x_1, \pm x_2, \pm x_3).$$

We denote by $\vec{y}_\pm = (y_{\pm, 1}, y_{\pm, 2}, y_{\pm, 3})$ the Bloch vectors of $\rho^A_{\pm}$, respectively. Then from (5) we have

$$p_\pm = \frac{1}{2} \pm x_1 (1 - \lambda) \sin 2\beta \pm x_3 [\lambda + (1 - \lambda) \cos 2\beta],$$

$$y_{\pm, 1} = \frac{1}{p_\pm} \frac{1}{2} (1 - \lambda) \sin 2\alpha \pm x_1 (1 - \lambda) \sin 2\alpha \sin 2\beta \pm x_3 (1 - \lambda) \sin 2\alpha \cos 2\beta,$$

$$y_{\pm, 2} = 0,$$

$$y_{\pm, 3} = \frac{1}{p_\pm} \frac{1}{2} [\lambda + (1 - \lambda) \cos 2\alpha] \pm x_1 (1 - \lambda) \cos 2\alpha \sin 2\beta \pm x_3 [\lambda + (1 - \lambda) \cos 2\alpha \cos 2\beta].$$

(21)

We can see from these expressions that

$$y_{\pm, 3} + y_{\pm, 1} \tan \alpha = 1. \quad (22)$$

Equation (22) means that the two points corresponding to the Bloch vectors $\vec{y}_+$ and $\vec{y}_-$ are located on the line $L$ that lies in the $y_1 y_3$ plane and passes through the point $(0, 1)$ with the slope-$\tan \alpha$. Note that the Bloch vector of $\rho^A$ is given by

$$\vec{r}^A = ((1 - \lambda) \sin 2\alpha, \ 0, \ \lambda + (1 - \lambda) \cos 2\alpha).$$

Then the point A, denoting the state $\rho^A$, is also on the line $L$. See figure 4.

Before using this picture to find the quantum discord of the state given by (16), let us make some remarks.

Roughly speaking, the set of post-measurement states of qubit A (i.e. $\rho^A_+$ and $\rho^A_-$) are restricted to the line $L$. For convenience, we denote the two states by points E and F, respectively, in figure 4. Assume that E is on the left side of points A and F on the right side of A. Then it should be noted that E cannot be located on the left side of point $(0, 1)$, because the length of OE cannot be larger than one. For the same reason, F cannot be on the right side of point $(\sin 2\alpha, \cos 2\alpha)$. Then the line segment EF, which represents the set of all available post-measurement states of qubit A, slides along line $L$ between point $(0, 1)$ and point $(\sin 2\alpha, \cos 2\alpha)$. This picture is somewhat different from that presented in section 4, where the steering ellipsoid takes up a fixed region for a given state and does not depend on the choice of...
The measurements made by Bob on qubit B. But here the line segment EF is ‘moving’, in the sense that both the length and the position of EF depend on Bob’s measurements.

Now we propose a conjecture about minimal average entropy $S_A^{\min}$. Given a two-qubit state (16), Bob carries out two-element POVM measurement on qubit B. If Alice’s post-measurement ensemble $\{p_k, \rho^A_k\}_{k=+, -}$ minimizes the average entropy $S_A^A$, then $S(\rho^A_+) = S(\rho^A_-)$.

In other words, OE = OF is the necessary condition that must be satisfied in order that the average entropy $S_A^A$ takes the minimal value. To test this conjecture, we select randomly $1.5 \times 10^5$ two-qubit states with the form given by (16), and for each state we calculate numerically the value of $S_A^{\min}$ and the corresponding measurement parameters, namely, $x_i$ with $i = 1, 2, 3$. From (19)–(21), we obtain the coordinates of points E and F and also the length of OE and OF. In figure 5, we plot the value of OE – OF for the $1.5 \times 10^5$ states. We see that $|OE – OF| \sim 10^{-6}$. Numerical results confirm our conjecture.

If the conjecture is indeed true, we need only consider the situation that $S(\rho^A_+) = S(\rho^A_-)$ or OE = OF. It follows that

$$S_A^A = p_+ S(\rho^A_+) + p_- S(\rho^A_-) = S(\rho^A_+) = S(\rho^A_-).$$

To obtain $S_A^{\min}$, we need only to maximize OE or OF under the condition that OE = OF. This is not a difficult task. Following this line of thought, we obtain the classical correlation and quantum discord of states (16) and plot the results in figures.

In figures 6 and 7, we plot the classical correlation $C^\leftarrow$ and quantum discord $Q^\leftarrow$ for the states given by (16) with $\lambda = 0.5$ and $\lambda = 0.7$, respectively. They are very similar to each other, but it should be noted that the plot of $Q^\leftarrow$ in figure 6 is symmetric with respect to the parameter $\beta$, while it is not the case in 7. To see this, refer to figure 8.
Figure 5. Numerical test of the conjecture that $OE = OF$ is the necessary condition for $\{\rho_E, \rho_F\}$ to be the optimal ensemble of $\rho^A$. $1.5 \times 10^5$ random states are tested and the values of $|OE - OF|$ are calculated. The results show that almost $|OE - OF| \sim 10^{-6}$.

Figure 6. Classical correlation $\mathcal{C}^-$ and quantum discord $\mathcal{Q}^-$ of the states given by (16) with $\lambda = \frac{1}{2}$.

It can be seen that among all states given by (16) with fixed $\lambda$, the one with maximal classical correlation is of the form
$$\lambda |0\rangle \langle 0| \otimes |0\rangle \langle 0| + (1 - \lambda) |1\rangle \langle 1| \otimes |1\rangle \langle 1|,$$
and the one with maximal quantum discord is given by
$$\lambda |0\rangle \langle 0| \otimes |0\rangle \langle 0| + (1 - \lambda) |1\rangle \langle 1| \otimes |+\rangle \langle +|.$$
Figure 7. Classical correlation $\mathcal{C}^{-}$ and quantum discord $Q^{-}$ of the states given by (16) with $\lambda = 0.7$.

Figure 8. Quantum discord $Q^{-}$ of the states given by (16) with $\alpha = \pi/6$ and $\lambda = 0.7$. It is not symmetric with respect to $\beta \in [0, \pi/2]$.

It is not difficult to see that the choice of $\lambda = 1/2$ renders the largest classical correlation and quantum discord.

7. More general states

It is desirable to apply the geometric picture for a wider class of states. This section is devoted to extending the discussion in section 4 about X states to a more general case. Recall that in the geometric picture of X states the Alice local state $\rho^A$ is located on the $y_3$ (or $y'_3$)-axis, which is one of the symmetric axes of the ellipsoid $\mathcal{E}$ (see figure 1). How about the case that point A deviates from the $y_3$-axis? We will discuss in this section this type of state. To begin with, let us consider an example.
The example we will consider comes from [45]. Suppose a two-qubit pure state is given by
\[ |\psi^{AB}\rangle = \frac{1}{\sqrt{2}} (|\psi_0\rangle_A |0\rangle_B + |\psi_1\rangle_A |1\rangle_B), \]
where \(|\psi_0\rangle_A = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \) and \(|\psi_1\rangle_A = \frac{4}{3} |0\rangle + \frac{3}{3} |1\rangle\). Let qubit A pass through a quantum channel, the Kraus operators of which are given by
\[ A_1 = |0\rangle \langle 0| + \frac{1}{\sqrt{2}} |1\rangle \langle 1|, \quad A_2 = \frac{1}{\sqrt{2}} |0\rangle \langle 1|. \]
Then the output state of the channel is
\[ \rho^{AB} = \frac{1}{2} \sum_{i=1}^{2} (A_i \otimes 1) |\psi^{AB}\rangle \langle \psi^{AB}| (A_i \otimes 1)^\dagger. \]
Note that \(\rho^{AB}\) here is not an X state. With Bob carrying out the measurement, the steering ellipsoid is given by
\[ y_1^2 + y_2^2 + 2 \left( y_3 - \frac{1}{2} \right)^2 = \frac{1}{2}. \]
The Bloch vector of \(\rho^A\) is
\[ \vec{r}^A = \left( \frac{49}{50\sqrt{2}}, 0, \frac{57}{100} \right). \]

The ellipsoid \((23)\) is symmetric under rotation about the \(y_3\)-axis. The form is similar to that for X states. However, the vector \(\vec{r}^A\) does not lie on the \(y_3\)-axis (see figure 9). We cannot obtain the \(S^A_{\text{min}}\) analytically, but numerical evaluation reveals an interesting result: considering any line passing through point A and intersecting the ellipsoid at the two points E and F, the minimal value of \(S^A\) is reached at such E and F that OE = OF, or
\[ S^A_{\text{min}} = p_E S_E + p_F S_E = S_E = S_F, \]
where \(S_E\) or \(S_F\) have analytical expressions. We can see that it is the equi-entropy decomposition. Geometric description is clearly demonstrated in figure 9.

The numerical value of \(S^A_{\text{min}}\) is equal to 0.2804, and classical correlation and quantum discord is given by
\[ C^- = 0.0118, \quad Q^- = 0.0338. \]

The above example motivates us to consider more general two-qubit states. That is, the quantum steering ellipsoid can be translated along the \(y_3\)-axis and no rotation is allowed. The position of the local state \(\rho^A\) is restricted to the \(y_1\ y_3\) plane.

We take into consider action the states \(\rho^{AB}\) with the following \(R\) matrix,
\[ R = \begin{pmatrix} 1 & s_1 & 0 & s_3 \\ r_1 & t_{11} & 0 & t_{13} \\ 0 & 0 & t_{22} & 0 \\ r_3 & t_{31} & 0 & t_{33} \end{pmatrix}. \]

\[ \text{In [45] it is } C^- \text{ that is taken into account. Here we have interchanged particles A and B in order to calculate } C^- . \]
Figure 9. The ellipsoid $\mathcal{E}$ given by (23) is cut by a plane that is parallel to the $y_1y_2$ plane and contains point $A$. The intersection is a circle. Passing through point $A$, any chord of the circle leads to the optimal ensemble of $\rho^A$.

It is assumed that $R$ is nonsingular, namely, $\det(R) \neq 0$. Obviously, the Bloch vector of $\rho^A$ is given by $\vec{r}^A = (r_1, 0, r_3)$. Assuming that $s_1, t_{13} \neq 0$, we choose the parameters $t_{11}$ and $t_{33}$ as

$$t_{11} = \frac{r_1 - s_3t_{13}}{s_1}, \quad t_{33} = \frac{r_1r_3s_1 - r_1t_{31} + s_3t_{13}t_{31}}{s_1t_{13}}.$$

Under these conditions, we construct the states $\rho^{AB}$ randomly. The quantum steering ellipsoid $\mathcal{E}$ is given by

$$\frac{y_1^2}{\ell_1^2} + \frac{y_2^2}{\ell_2^2} + \frac{(y_3 - Y_3)^2}{\ell_3^2} = 1,$$

where

$$\ell_1^2 = \frac{r_1^2(1 - s_1^2) - 2r_1s_3t_{13} + (s_1^2 + s_3^2)t_{13}^2}{s_1^2(1 - s_1^2 - s_3^2)},$$

$$\ell_2^2 = \frac{t_{22}^2}{1 - s_1^2 - s_3^2},$$

$$\ell_3^2 = \frac{[r_1^2(1 - s_1^2) - 2r_1s_3t_{13} + t_{13}^2(s_1^2 + s_3^2)](r_3s_1 - t_{31})^2}{s_1^2t_{13}(1 - s_1^2 - s_3^2)^2},$$

$$Y_3 = \frac{r_3s_1t_{13} - r_1s_3(r_3s_1 - t_{31}) - t_{13}t_{31}(s_1^2 + s_3^2)}{s_1t_{13}(1 - s_1^2 - s_3^2)}.$$

Consider a class of lines passing through point $A$ and intersecting the $\mathcal{E}$ at points $M$ and $N$. It follows that $\rho^A = p_M\rho_M + p_N\rho_N$. Corresponding to this decomposition of $\rho^A$, the average
Entropy is given by $S_{MN} = p_M S_M + p_N S_N$. We will find the minimal $S^A$ among all these lines. Numerical results can be classified into the following two categories.

**Class I**: The optimal line is parallel to the $y_1$-axis or the $y_2$-axis. We denote by E and F the intersection points with $e$. The minimal $S^A$ is given by $S_{\min}^A = S_E = S_F$, meaning that it is an equi-entropy decomposition. Moreover, let $A'$ be the point that is on the line segment EF and symmetric to point A about the $y_3$-axis. Then any point $A''$ between the points A and $A'$ has the same minimal value of average entropy, namely

$$S_{\min}^{A''} = S_E = S_F$$

for $A'' \in AA'$.

**Class II**: The optimal line is not parallel to the $y_1 y_2$ plane. In this case, let us consider the point $\tilde{A}$ with the coordinate $(0, 0, r_3)$. Point $\tilde{A}$ in fact corresponds to the projection of the vector $\vec{r}^A$ onto the $y_3$-axis. We find that the minimal value of the average entropy $\tilde{S}^A$ is given by the two points G and H, which are the upper and the lower apex of the ellipsoid $e$, respectively; that is,

$$\tilde{S}_{\min}^A = p_G S_G + p_H S_H.$$

It is a quasi-eigendecomposition.

In the case of Class I, $S_{\min}^A$ have analytical expressions. For Class II, we only see that it has a relationship with quasi-eigendecomposition. More effort is necessary in order to gain further insights.

8. Conclusion

We present a geometric method as to how to describe and evaluate the minimal average entropy, which is the major obstacle in the computation of classical correlations and quantum discord. For two-qubit states, the available ensemble of post-measurement states of qubit A, which comes from the measurements carried out on qubit B, is restricted to the quantum steering ellipsoid. The optimal ensemble can only be found on the surface of the ellipsoid.

For two-qubit X states, the geometric method provides a clear picture as well as exact results. We show that for X states the optimal decomposition is either equi-entropy decomposition or quasi-eigendecomposition. In the geometric picture, equi-entropy decomposition corresponds to a horizontal line segment, while the quasi-eigendecomposition corresponds to a vertical one. When an X state passes through some quantum channels, the dynamics of classical correlation and quantum discord can be easily analyzed in the geometric picture.

We extend the discussion of X states to a more general case by relaxing the requirement that the reduced density matrices are of diagonal form. Little is known about the classical correlations or quantum discord of these states. We perform numerical computations. A consequence of the numerical results is the following interesting alternative: the optimal decomposition is either equi-entropy decomposition, or intimately related to the quasi-eigendecomposition. Combining these numerical results with the exact results for X states, we think that the geometric method may be generalized analytically rather than numerically. Further study in this direction might be worthwhile.

The geometric viewpoint presented in this paper offers an alternative way to interpret and compute classical correlations and quantum discord. It is also useful in elucidating issues related
to decoherence. The problem remaining is to verify the conclusions we have drawn by the numerical method. If these conclusions are indeed true, they will be helpful in working out the exact results on quantum discord analytically.

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References

[1] Cover T M and Thomas J A 1991 *Elements of Information Theory* (New York: Wiley)
[2] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 *Rev. Mod. Phys.* 81 865
[3] Bennett C H, DiVincenzo D P, Fuchs C A, Mor T, Rains E, Shor P W, Smolin J A and Wootters W K 1999 *Phys. Rev. A* 59 1070
[4] Bennett C H, DiVincenzo D P, Mor T, Shor P W, Smolin J A and Terhal B M 1999 *Phys. Rev. Lett.* 82 5385
[5] Horodecki M, Sen A, Sen U and Horodecki K 2003 *Phys. Rev. Lett.* 90 047902
[6] Zurek W H 2003 *Rev. Mod. Phys.* 75 715
[7] Zurek W H 2000 *Ann. Phys.* 9 855
[8] Ollivier H and Zurek W H 2001 *Phys. Rev. Lett.* 88 017901
[9] Henderson L and Vedral V 2001 *J. Phys. A: Math. Gen.* 34 6899
[10] Devetak I and Winter A 2004 *IEEE Trans. Inf. Theory* 50 3183
[11] Devetak I 2005 *Phys Rev. A* 71 062303
[12] Oppenheim J, Horodecki M, Horodecki P and Horodecki R 2002 *Phys. Rev. Lett.* 89 180402
[13] Horodecki M, Horodecki P, Horodecki R, Oppenheim J, Sen A, Sen U and Synak-Radtke B 2005 *Phys. Rev. A* 71 062307
[14] Groisman B, Popescu S and Winter A 2005 *Phys. Rev. A* 72 032317
[15] Modi K, Paterek T, Son W, Vedral V and Williamson M 2010 *Phys. Rev. Lett.* 104 080501
[16] Dakić B, Vedral V and Brukner Č 2010 *Phys. Rev. Lett.* 105 190502
[17] Luo S and Fu S 2010 *Phys. Rev. A* 82 034302
[18] Vedral V 2003 *Phys. Rev. Lett.* 90 050401
[19] Datta A and Vidal G 2007 *Phys. Rev. A* 75 042310
[20] Datta A, Shaji A and Caves C M 2008 *Phys. Rev. Lett.* 100 050502
[21] Piani M, Horodecki P and Horodecki R 2008 *Phys. Rev. Lett.* 100 090502
[22] Rodriguez-Rosario C A, Modi K, Kuah A-m, Shaji A and Sudarshan E C G 2008 *J. Phys. A: Math. Theor.* 41 205301
[23] Shabani A and Lidar D A 2009 *Phys. Rev. Lett.* 102 100402
[24] Oppenheim J, Horodecki K, Horodecki M, Horodecki P and Horodecki R 2003 *Phys. Rev. A* 68 022307
[25] Badziag P, Horodecki M, Sen A and Sen U 2003 *Phys. Rev. Lett.* 91 117901
[26] Koashi M and Winter A 2004 *Phys. Rev. A* 69 022309
[27] Maziero J, Céleri L C, Serra R M and Vedral V 2009 *Phys. Rev. A* 80 044102
[28] Mazzola L, Piilo J and Maniscalco S 2010 *Phys. Rev. Lett.* 104 200401
[29] Madhok V and Datta A 2011 *Phys. Rev. A* 83 032323
[30] Cavalcanti D, Aolita L, Boixo S, Modi K, Piani M and Winter A 2011 *Phys. Rev. A* 83 032324
[31] Streltsov A, Kampermann H and Bruß D 2011 *Phys. Rev. Lett.* 106 160401
[32] Piani M, Gharibian S, Adesso G, Calsamiglia J, Horodecki P and Winter A 2011 *Phys. Rev. Lett.* 106 220403
[33] Luo S 2008 *Phys. Rev. A* 77 042303
[34] Ali M, Rau A R P and Alber G 2010 *Phys. Rev. A* 81 042105

*New Journal of Physics* 13 (2011) 073016 (http://www.njp.org/)
[35] Giorda P and Paris M G A 2010 Phys. Rev. Lett. 105 020503
[36] Lang M D and Caves C M 2010 Phys. Rev. Lett. 105 150501
[37] Li B, Wang Z-X and Fei S-M 2011 Phys. Rev. A 83 022321
[38] Verstraete F 2002 A study of entanglement in quantum information theory PhD Thesis Katholieke Universiteit Leuven
[39] Schumacher B and Westmoreland M D 2001 Phys. Rev. A 63 022308
[40] Brodutch A and Terno D R 2010 Phys. Rev. A 81 062103
[41] Hamieh S, Kobes R and Zaraket H 2004 Phys. Rev. A 70 052325
[42] Yu T and Eberly J H 2006 Phys. Rev. Lett. 97 140403
[43] López C E, Romero G, Lastra F, Solano E and Retamal J C 2008 Phys. Rev. Lett. 101 080503
[44] Maziero J, Werlang T, Fanchini F F, Céleri L C and Serra R M 2010 Phys. Rev. A 81 022116
[45] Synak-Radtke B and Horodecki M 2004 J. Phys. A: Math. Gen. 37 11465