A PROOF OF THE 4-VARIABLE CATALAN POLYNOMIAL OF THE DELTA CONJECTURE

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Abstract. In The Delta Conjecture [HRW15], Haglund, Remmel and Wilson introduced a four variable \(q, t, z, w\) Catalan polynomial, so named because the specialization of this polynomial at the values \((q, t, z, w) = (1, 1, 0, 0)\) is equal to the Catalan number \(\frac{1}{n+1}\binom{2n}{n}\). We prove the compositional version of this conjecture (which implies the non-compositional version) that states that the coefficient of \(s_{r,1^{n-r}}\) in the expression \(\Delta h_r \nabla C_\alpha\) is equal to a weighted sum over decorated Dyck paths.

1. Introduction

In the search for a representation theoretical interpretation for Macdonald symmetric functions, Haiman defined the module of diagonal harmonics [Hai94] as a quotient of the polynomial ring in two sets of \(n\) variables. For a given integer \(n\), the diagonal harmonics are a bi-graded \(S_n\)-module and they have dimension \((n+1)^n - 1\). Garsia and Haiman [GarHai96] took a (at the time conjectured) formula for the bi-graded Frobenius characteristic of the diagonal harmonics and defined for each \(n\) a rational function in two parameters \(q\) and \(t\) which is equal to the bi-graded multiplicity of the alternating representation in the module. This rational function is now known to have a simpler form as a polynomial in \(q\) and \(t\) with non-negative integer coefficients. This expression is known as the \(q,t\)-Catalan polynomial since at \(q = t = 1\) it specializes to the Catalan number \(\frac{1}{n+1}\binom{2n}{n}\).

Important progress was made in this area with the introduction of the notation of two linear symmetric operators \(\nabla\) and \(\Delta f\) which have Macdonald symmetric functions as eigenfunctions [BG99, BGHT99]. The expression \(\nabla(e_n)\) was conjectured to be equal to the Frobenius image of the character of the module of diagonal harmonics and the \(q,t\)-Catalan polynomial is the coefficient of \(e_n\) in this expression. The operators \(\nabla\) and \(\Delta f\) gave notation to extend the types of symmetric function expressions which were conjectured to be Schur positive for representation theoretic reasons to expressions which are conjectured to be Schur positive because of computer experimentation with linear algebra calculations.

Haglund conjectured [Hag03] and shortly after Garsia and Haglund [GarHag02] proved a combinatorial interpretation for the \(q,t\)-Catalan polynomial. They showed that there were two statistics on Dyck paths (called area and bounce) such that the rational expression for the \(q,t\)-Catalan is equal to the sum over all Dyck paths \(D\) with weight \(q^{\text{area}(D)}t^{\text{bounce}(D)}\).

Around this same period, Haiman [Hai02] proved the conjecture that \(\nabla(e_n)\) was equal to the representation theoretic interpretation as the Frobenius image of the diagonal harmonics. Haiman also guessed at a second statistic (dinv, short for diagonal inversions) such that...
the $q,t$-Catalan is equal to to the sum over all Dyck paths with weight $q^{\text{dinv}(D)}t^{\text{area}(D)}$ and Haglund later showed with a bijection why the two combinatorial expressions are equivalent.

With the conjectures on the $q,t$-Catalan polynomial resolved, Haglund, Haiman, Loehr, Remmel and Ulyanov [HHLRU05] extended the combinatorial interpretations for the coefficient of $e_n$ in $\nabla(e_n)$ to other coefficients. They conjectured the coefficient of any monomial symmetric function in terms of labelled Dyck paths (also known as parking functions) and this became known as the Shuffle Conjecture. The Shuffle Conjecture takes its name because the coefficient of a monomial is equal to the number of labelled Dyck paths whose reading word is a shuffle of segments of length the parts of the partition.

Researchers also considered coefficients of $\nabla$ and $\Delta f$ acting on other symmetric functions and extended the combinatorial interpretations to coefficients in these expressions (e.g. [EHKK03, Hag04, CL06, LW07, LW08] and for a survey of results in this area up to 2008 see [Hag08]).

In particular, a refinement of the Shuffle Conjecture was proposed by Haglund, Morse and the author [HMZ12] that gave a symmetric function expression for the labelled Dyck paths which touch the diagonal at a given composition. Some progress on this Compositional Shuffle Conjecture was made [GXZ10, Hic10, DGZ13, Hic14, GXZ14a, GXZ14b] before it was finally proven by Carlsson and Mellit [CM15]. By the time that Carlsson and Mellit had announced their proof, there was already a rational slope version of the compositional shuffle conjecture [BGLX16]. The arms race of conjecture vs. proof in this area did not stay out of balance for long and a proof of this result was announced earlier this year by Mellit [Mel16].

Haglund, Remmel and Wilson [HRW15] recently announced a conjecture for some combinatorial expressions involving $\Delta f$ and $\nabla$ in a sequence of conjectures that generalize the Shuffle Conjecture from labelled Dyck paths to decorated labelled Dyck paths and called this the Delta Conjecture. There does not currently exist a compositional version of this conjecture which might be helpful if progress is to be made on proving it, but they did define $q,t,w,z$-Catalan polynomial for which there exists a compositional version. The $q,t,w,z$-Catalan polynomial specializes to the $q,t$-Catalan polynomial in the case that $w = z = 0$ and to $\frac{(2n)!}{n!}\binom{2n}{n}$ in the case that $q = t = w = z = 1$. It is this conjecture that we will prove here.

Namely we will show the following theorem (this is Theorem 10; note: we leave the definitions of the symmetric function used in the statement of this theorem to Section 2 and the combinatorial objects to Section 3):

**Theorem 1.** For non-negative integers $k, \ell$ and a composition $\alpha$ of size $n - \ell$, 

\[
\left\langle \Delta h_{\ell} \nabla C_{\alpha}, s_{k+1,1}^{\alpha-1-k} \right\rangle = \sum_{D \in D_{n-\ell}^\alpha \atop \text{peak}_{\alpha}(D) = k \atop \text{rise}_{\alpha}(D) = \alpha} q^{\text{dinv}(D)} t^{\text{area}_{\alpha}(D)}
\]

where the sum is over all $\circ$-decorated Dyck paths with $k$ $\circ$-decorated peaks and $\ell$ $\circ$-decorated double rises and rise-touch composition equal to $\alpha$. 
Note that the resolution of this conjecture does not prove all of the conjectures made in Section 7 of [HRW15] because there was a second combinatorial interpretation stated for the coefficient \( \langle \Delta h_\ell \nabla e_{n-k-1} s_{k+1, 1^{n-k-1}} \rangle \) that does not seem to be compatible with the compositional version and our proof relies on this connection.

2. Symmetric functions

The symmetric function results on Macdonald symmetric functions that we will use here almost all come from a series of early papers on the subject [Mac88, G92, GarHai95, GarHai96, BG99, BGHT99, GHT99, GarHag02]. These results have proven to be very prescient in the utility of the identities, notation and techniques developed. We will be able to prove our symmetric function recurrence by using the groundwork paved in these references. The book by J. Haglund [Hag08] collects many of the identities that we will use in a review of the literature and hence will provide a useful reference for their use. The only additional ingredient that we will use are the creation operators and symmetric functions introduced in [HMZ12] which play an important role in developing recurrences for the coefficients in which we are interested.

2.1. Symmetric function notation. The main reference we will use for symmetric functions is [Mac95]. The standard bases of the symmetric functions that will appear in our calculations the complete \( \{ h_\lambda \}_{\lambda} \), elementary \( \{ e_\lambda \}_{\lambda} \), power \( \{ p_\lambda \}_{\lambda} \) and Schur \( \{ s_\lambda \}_{\lambda} \) bases.

The ring of symmetric functions can be thought of as the polynomial ring in the power sum generators \( p_1, p_2, p_3, \ldots \). As we are working with Macdonald symmetric functions involving two parameters \( q \) and \( t \) we will consider this polynomial ring over the field \( \mathbb{Q}(q,t) \).

We will make extensive use of plethystic notation in our calculations and arbitrary alphabets. This is a notational addition that introduces union and difference of alphabets. Alphabets will be represented as sums of monomials \( X = x_1 + x_2 + x_3 \ldots \) and then the expression \( f[X] \) represents the symmetric function \( f \) as an element of \( \Lambda \) with \( p_k \) replaced by \( x_1^k + x_2^k + x_3^k + \cdots \). We have the identities that \( p_k[X + Y] = p_k[X] + p_k[Y] \), \( p_k[X - Y] = p_k[X] - p_k[Y] \), and on the elementary and homogeneous bases we also have the alphabet addition formulae which say

\[
(2) \quad e_n[X + Y] = \sum_{k=0}^{n} e_k[X] e_{n-k}[Y] \quad \text{and} \quad h_n[X + Y] = \sum_{k=0}^{n} h_k[X] h_{n-k}[Y].
\]

The notation \( \epsilon \) is a common tool to express a second sort of negative sign when working with symmetric functions with alphabets where \( p_k[\epsilon X] = (-1)^k p_k[X] \). This is different from the negative of the alphabet expressed as \( p_k[-X] = -p_k[X] \). In general \( f[-\epsilon X] = (\omega f)[X] \) where \( \omega \) is the fundamental algebraic involution which sends \( e_k \) to \( h_k \), \( s_\lambda \) to \( s_{\lambda'} \) and \( p_k \) to \( (-1)^{k-1} p_k \).

There is a special element \( \Omega \) in the completion of the symmetric functions that we will be using. It is defined as \( \Omega = \sum_{n \geq 0} h_n \). It has the property for arbitrary alphabets \( X \) and \( Y \), \( \Omega[X + Y] = \Omega[X] \Omega[Y] \). In addition, it has the property that for any two dual basis
\( \{a_\lambda\}_\lambda \) and \( \{b_\lambda\}_\lambda \) with respect to the standard scalar product \( \langle s_\lambda, s_\mu \rangle = \chi(\lambda = \mu) \), we have

\[
\Omega[XY] = \sum_\lambda a_\lambda[X]b_\lambda[Y].
\]

2.2. Macdonald symmetric function toolkit and \( q, t \) notation. Macdonald symmetric functions that are used here are a transformation of the bases presented in \textbf{Mac95}. They are the symmetric functions that are the Frobenius image of the Garsia-Haiman modules \textbf{GarHai93} indexed by a partition. The symmetric functions

\[
\tilde{H}_\mu[X; q, t] = \sum_{\lambda | \mu} K_{\lambda\mu}(q, 1/t) t^{n(\mu)} s_\lambda[X]
\]

where \( K_{\lambda\mu}(q, t) \) are the Macdonald \( q, t \)-Kostka coefficients and \( n(\mu) = \sum_{i \geq 1} (i - 1) \mu_i \). The basis elements \( \{\tilde{H}_\mu\}_\mu \) are orthogonal with respect to the scalar product

\[
\langle p_\lambda, p_\mu \rangle_* = \chi(\lambda = \mu) (-1)^{|\lambda|+\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} (1 - q^{\lambda_i})(1 - t^{\lambda_i})
\]

and are sometimes defined by this property.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{arm, leg, co-arm and co-leg of a cell of the diagram}
\end{figure}

If we identify the partition \( \mu \) with the collection of cells \( \{(i, j) : 1 \leq i \leq \mu_i, 1 \leq j \leq \ell(\mu)\} \), then for each cell \( c \in \mu \) we refer to the the arm, leg, co-arm and co-leg (denoted respectively as \( a_\mu(c), \ell_\mu(c), a'_\mu(c), \ell'_\mu(c) \)) as the number of cells in the segments labeled in Figure 1. The typical shorthand for the polynomial expressions in \( q \) and \( t \) are

\[
B_\mu = \sum_{c \in \mu} q^{a'_\mu(c)} t^{\ell'_\mu(c)}, T_\mu = \prod_{c \in \mu} q^{a_\mu(c)} t^{\ell_\mu(c)} \quad \text{and} \quad w_\mu = \prod_{c \in \mu} (q^{a_\mu(c)} - t^{\ell_\mu(c)+1})(t^{\ell'_\mu(c)} - q^{a'_\mu(c)+1})
\]

Also set \( M = (1 - q)(1 - t) \) and \( D_\mu = MB_\mu - 1 \).

The following linear operators were introduced in \textbf{BG99} \textbf{BGHT99} which are at the basis of the conjectures relating symmetric function coefficients and \( q, t \)-combinatorics in this area. Define

\[
\nabla(\tilde{H}_\mu) = T_\mu \tilde{H}_\mu \quad \text{and} \quad \Delta_f(\tilde{H}_\mu) = f[B_\mu] \tilde{H}_\mu.
\]
Note that if $n = |\mu|$, then $e_n[B_\mu] = T_\mu$, hence for a symmetric function $f$ of homogeneous degree $n$, $\Delta_{e_n}(f) = \nabla(f)$, so the operators $\Delta_f$ are seen as a more general operator than $\nabla$.

Following other references and introduce the shorthand notation $f^*[X] = f[\frac{X}{M}]$. This notation can then be used to relate the $*$-scalar product with the usual scalar product $(f, g)$ where the Schur functions are orthonormal since $(f, g) = (f, \omega g^*)_\ast$. It is known that $\langle \tilde{H}_\lambda, \tilde{H}_\mu \rangle_\ast = \chi(\lambda = \mu)w_\lambda$, then follows that

$$(7) \quad \Omega \left[ -\epsilon_X Y \right] = \sum_{n \geq 0} e_n^*[XY] = \sum_{\mu+n} \frac{\tilde{H}_\mu[X]\tilde{H}_\mu[Y]}{w_\mu}.$$ 

We will use one of the forms of Macdonald-Koornwinder reciprocity in our calculations (see [Mac95] p. 332 or [GHT99]),

$$(8) \quad \tilde{H}_\mu[1 + uD_\lambda] \prod_{c \in \mu} (1 - ut^\ell(c)q^\ell(c)) = \tilde{H}_\lambda[1 + uD_\mu] \prod_{c \in \lambda} (1 - ut^\ell(c)q^\ell(c)).$$

The form of this identity that we are most interested here is found by setting $u = 1/u$, clearing the denominators, and letting $u \to 0$ to obtain

$$(9) \quad \tilde{H}_\mu[D_\nu] = (-1)^{|\mu|+|\nu|} \tilde{H}_\nu[D_\mu] \frac{T_\mu}{T_\nu}.$$ 

2.3. Pieri rules and summation formulae. Define coefficients $h^+_1 \tilde{H}_\mu = \sum_{\nu \rightarrow \mu} c_{\mu\nu} \tilde{H}_\nu$ and $h_1 \tilde{H}_\nu = \sum_{\mu \leftarrow \nu} d_{\mu\nu} \tilde{H}_\mu$. It was proven in [GarHai95] (Corollary 1.1) that they are related by the identity,

$$(10) \quad d_{\gamma\tau} = Mc_{\gamma\tau} \frac{w_\tau}{w_\gamma}.$$ 

The following identity has been used frequently in work on the Shuffle Conjecture but a full proof did not appear until recently in [GHXZ16]. For $s \geq 0$,

$$(11) \quad e_{s-1}[D_\gamma] = (-1)^{s-1} \sum_{\nu \leftarrow \gamma} d_{\nu\gamma} \left( \frac{T_\nu}{T_\gamma} \right)^s + \chi(s = 0)$$

where we have denoted $\chi(true) = 1$ and $\chi(false) = 0$ so that the term $\chi(s = 0)$ only appears in the case that $s = 0$. The other sum of Pieri coefficients for Macdonald polynomials was proven in a $q, t$ hook walk by Garsia and Haiman [GarHai95] for $s \geq 0$,

$$(12) \quad h_{s+1}[D_\gamma] = Mt^s q^s \sum_{\tau \rightarrow \gamma} c_{\gamma\tau} \left( \frac{T_\gamma}{T_\tau} \right)^s - \chi(s = 0).$$

In order to prove the combinatorial formula for the $q, t$-Catalan polynomial, Garsia and Haglund introduced a generalization of the Pieri coefficients and proved a summation
formula which we will use here. They defined coefficients \(d_{\mu \nu}^f\) and \(c_{\mu \nu}^\perp\) where \(\nu \subseteq \mu\) as
\[
(13) \quad f \tilde{H}_\nu = \sum_{\mu} d_{\mu \nu}^f \tilde{H}_\mu \quad \text{and} \quad f^\perp \tilde{H}_\mu = \sum_{\nu} c_{\mu \nu}^\perp \tilde{H}_\nu .
\]
These coefficients are related by
\[
(14) \quad c_{\mu \nu}^\perp w_\nu = d_{\mu \nu}^f w_\mu .
\]
The summation formula from [GarHag02] (see pp. 698-701) we will use here is
\[
(15) \quad \sum_{\nu \subseteq \mu, m - d \leq |\nu| \leq m} c_{\mu \nu}^{\omega \gamma} = \nabla^{-1} \left( (\omega g) \left[ \frac{X - \epsilon}{M} \right] \right) |_{X \rightarrow D_\mu} .
\]

2.4. Symmetric functions indexed by compositions and creation operators. The work of Haglund, Morse and the author [HMZ12] extended the Shuffle Conjecture to a compositional refinement. The Compositional Shuffle Conjecture implies the original Shuffle Conjecture, and it was this version of the conjecture that was proven in [CM15].

The compositional refinement came by defining for each composition \(\alpha\) symmetric functions \(B_\alpha[X; q]\) and \(C_\alpha[X; q]\). These symmetric functions have the property that the combinatorial expression in terms of labeled Dyck paths for \(\nabla B_\alpha[X; q]\) is in terms of paths which touch in \textit{at least} in the positions specified by the composition \(\alpha\) and \(\nabla C_\alpha[X; q]\) which touches the diagonal in \textit{exactly} the positions specified by the composition \(\alpha\).

Both of these symmetric functions are defined in terms of creation operators. For any symmetric function \(P[X]\) define
\[
(16) \quad \mathbb{B}_m P[X] = P \left[ X + \epsilon \left( \frac{1 - q}{u} \right) \right] \Omega [-\epsilon u X] |_{u=m}
\]
and
\[
(17) \quad \mathbb{C}_m P[X] = (-q) P \left[ X + \epsilon \left( \frac{1 - q}{u} \right) \right] \Omega \left[ \frac{\epsilon u X}{q} \right] |_{u=m} .
\]
Then for any composition \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})\), set \(C_\alpha = C_\alpha[X; q] = \mathbb{C}_{\alpha_1} \mathbb{C}_{\alpha_2} \ldots \mathbb{C}_{\alpha_{\ell(\alpha)}}\).

We can define the symmetric functions \(B_\alpha\) in a similar manner, but for our purposes, we only need ([HMZ12] equation (5.11) and (5.12)) the \(C_\alpha\) symmetric functions and the fact that for \(m \geq 0\),
\[
(18) \quad \mathbb{B}_m (C_\alpha) = q^{\ell(\alpha)} \sum_{\beta \models m} C_{\alpha, \beta} .
\]

We will denote \(\mathbb{B}_m^*\) and \(\mathbb{C}_m^*\) operators which are dual to \(\mathbb{B}_m\) and \(\mathbb{C}_m\) with respect to the \(*\)-scalar product (that is \(\langle \mathbb{B}_m f, g \rangle_* = \langle f, \mathbb{B}_m^* g \rangle_*\)). Since \(\langle e_n^*[XY], f[X] \rangle_* = f[Y]\), the expressions can be verified by calculating that \(\mathbb{B}_m^* e_n^*[XY] = \mathbb{B}_m^* Y e_n^*[XY]\) since \(\langle \mathbb{B}_m^* e_n^*[XY], f[Y] \rangle_* = \mathbb{B}_m f[X] = \langle e_n^*[XY], \mathbb{B}_m^* f[Y] \rangle_* \langle \mathbb{B}_m^* Y e_n^*[XY], f[Y] \rangle_*\) for all symmetric functions \(f[X]\) where
we have denoted $B_m^X$ to be the $B_m$ operator acting on the $X$ variables and $B_m^Y$ to be the $*$-dual operator acting on the $Y$ variables.

\begin{align*}
B_m^* P[X] &= P[X + \frac{M}{u}] \Omega \left[ \frac{-uX}{1-t} \right] \bigg|_{u^{-m}} \\
C_m^* P[X] &= (-q) P[X - \frac{M}{qu}] \Omega \left[ \frac{-uX}{1-t} \right] \bigg|_{u^{-m}}.
\end{align*}

3. The combinatorial recurrence

In [HRW15] there are four combinatorial interpretation stated in Conjecture 7.2 for the symmetric function expression $\langle \Delta_{h_i} \nabla C_{\alpha}, s_{k+1,1}^{\alpha_{-k-1}} \rangle$ (where $\alpha$ is a composition and $k, \ell \geq 0$) in terms of decorated Dyck paths. It seems that only one of these is correct (after a slight modification of the definitions as stated).

What we will do in the beginning of this section is introduce the definitions necessary to state the combinatorial interpretation. In Section 3.1 we state and prove a recurrence on the generating function for the combinatorial objects. Then in Section 4 we will show a symmetric function identity that demonstrates the coefficients also satisfy the same recurrence. This will imply by an inductive argument (because for small values of the indices we can verify that the combinatorial values agree with the symmetric function coefficients) that the symmetric function coefficients agree with the combinatorial generating function.

The basic element of the recursive construction given to us by the symmetric function recurrence is a rotation of the first part around to the end of the Dyck path while deleting the first up step and the first right step that touches the diagonal. This combinatorial recurrence and the effect on the area and dinv statistics first appears in [Hic10]. Assuming that we know the size of the piece that is being rotated around, the process is reversible. This is exactly the same sort of recursive construction that appeared in the proofs of certain coefficients of the Shuffle Conjecture [GXZ10, GXZ14a, GXZ14b]. In this recurrence we identify how it interacts with a decoration on a Dyck path.

We will use the notation $D_n$ to denote the set of Dyck paths with $n$ vertical steps. Dyck paths may be encoded by the area sequence $(a_1(D), a_2(D), \ldots, a_n(D))$ where $a_i(D)$ is the number of full cells in the the $i$th row which are above the diagonal but below the Dyck path. The area sequence of the Dyck paths are characterized the property that $a_1(D) = 0$ and $0 \leq a_{i+1}(D) \leq a_i(D) + 1$ for $1 \leq i < n$. The area statistic on Dyck paths is $\text{area}(D) = \sum_{i=1}^{n} a_i(D)$.

The diagonal inversion statistic on Dyck paths is the number of pairs $(i, j)$ with $i < j$ such that either $a_i(D) = a_j(D)$ or $a_i(D) = a_j(D) + 1$ (the diagonal inversions of the Dyck path). Order the rows from largest area value to smallest and from right to left. The $i$th row in this order has area $a_k(D)$ for some $k$ and let $b_i(D)$ be the number of diagonal inversions of the form $(k, j)$ with $a_k(D) = a_j(D)$ or $(j, k)$ with $a_j(D) = a_k(D) + 1$. The $b_i(D)$ represents all of the diagonal inversions between the $i$th vertical step in this order and all those that come before. Set $\text{dinv}(D) = \sum_{i=1}^{n} b_i(D)$. 
The Dyck path on the left first touches the diagonal after 7 vertical/horizontal steps and has area$(D) = 17$. The Dyck path on the right has the red part of the path moved to the end with the orange segments removed. The resulting path has area $11 = 17 - (7 - 1)$.

**Example 2.** To ensure that the definitions are clear to this point we list the sequences for the Dyck paths pictured in Figure 2. The $a$-sequence is

$$(0, 1, 2, 2, 1, 1, 2, 0, 1, 1, 2, 0, 1, 2)$$

and the $b$-sequence is

$$(0, 1, 2, 3, 4, 5, 5, 6, 6, 7, 6, 6, 4)$$

The $a$ and $b$ sequences of the right Dyck path have a clear relationship to the left Dyck path. The statistics for the right path have the $a$-sequence given by

$$(0, 1, 1, 2, 1, 0, 1, 2, 0, 1, 1, 0, 1)$$

and the $b$-sequence is

$$(0, 1, 2, 3, 4, 5, 5, 6, 6, 7, 6, 6, 4)$$

The combinatorial interpretation of the symmetric function coefficients are in terms of decorated Dyck paths. The set of $\circ$-decorated Dyck paths are Dyck paths where where each vertical segment except for the rightmost one in the highest diagonal can either be labeled with a $\circ$ or not. Denote the set of $\circ$-decorated Dyck paths by $D_n^\circ$. Clearly there are $\frac{2^n - 1}{n+1} \binom{2n}{n}$ such $\circ$-decorated Dyck paths.

Every vertical edge of a Dyck path is either followed by a second vertical edge and is called a double rise, or it is followed by a horizontal edge and it is called a peak. A $\circ$-decorated double rise on a $\circ$-decorated Dyck path is a row where the first vertical segment of two consecutive vertical segments has a $\circ$-decoration (i.e. a $\circ$-decorated row with $a_i(D) = a_{i+1}(D) - 1$). Let $\text{Rise}_\circ(D)$ be the set of indices of the rows of the $\circ$-decorated
double rises and \( \text{rise}_\circ(D) = |\text{rise}_\circ(D)| \) be the number of \( \circ \)-decorated double rises. We will also set for \( \circ \)-decorated Dyck paths, \( \text{area}_\circ(D) = \text{area}(D) - \sum_{i \in \text{rise}_\circ(D)} a_{i+1}(D) = \text{area}(D) - \text{rise}_\circ(D) - \sum_{i \in \text{rise}_\circ(D)} a_i(D) \).

A \( \circ \)-decorated peak on an \( \circ \)-decorated Dyck path is a row where the vertical segment is followed by a horizontal segment and has a \( \circ \)-decoration. There is a reading order of the vertical segments which are ordered by reading them from the highest diagonal from right to left. We will denote the set of indices of the \( \circ \)-decorated peaks (following the reading order) by \( \text{Peak}_\circ(D) \) and the number of \( \circ \)-decorated peaks by \( \text{peak}_\circ(D) = |\text{Peak}_\circ(D)| \).

We note that by the definition of the \( \circ \)-decorated Dyck paths, \( 1 \notin \text{Peak}_\circ(D) \) since the first vertical segment in this order cannot be decorated. Also remark that rows that form a peak will have \( b_i(D) > b_{i-1}(D) \) (except the first one where \( b_1(D) = 0 \)). This is because a peak will have a diagonal inversion with all the same positions that the preceding vertical segment had, plus one with that previous vertical segment.

Also denote a restricted diagonal inversion statistic \( \text{dinv}_\circ(D) = \text{dinv}(D) - \sum_{i \in \text{Peak}_\circ(D)} b_i(D) \).

Define the following multivariate analogues of \( \frac{2n-1}{n+1} \binom{2n}{n} \),

\[
\overline{\text{Cat}}_n(q, t, z, w) = \sum_{D \in D_n} q^{\text{dinv}(D)} t^{\text{area}(D)} \prod_{b_i(D) > b_{i-1}(D)} \left( 1 + \frac{z}{q^{b_i(D)}} \right) \\
\times \prod_{a_i(D) > a_{i-1}(D)} \left( 1 + \frac{w}{t^{a_i(D)}} \right) \\
= \sum_{D \in D^n_\circ} q^{\text{dinv}_\circ(D)} t^{\text{area}_\circ(D)} z^{\text{peak}_\circ(D)} w^{\text{rise}_\circ(D)} .
\]

There is a compositional refinement that we are focusing on in this paper. For a \( \circ \)-decorated Dyck path \( D \) with \( \text{rise}_\circ(D) = \ell \), define the rise-touch composition of \( D \) to be the sequence of numbers of vertical steps which are not \( \circ \)-decorated double rises between the places the path touches the diagonal. This is more simply determined by looking at the usual touch composition for the Dyck path without the decorations and then subtracting from each part of the composition the number of \( \circ \)-decorated double rises in each segment. Where it is necessary to abbreviate, the rise-touch composition will be denoted \( \alpha^{\text{Rise}}(D) \).

It is the case that \( \alpha^{\text{Rise}}(D) \) is a composition of \( n - \ell \).

**Example 3.** To ensure that the combinatorial object and the definitions that we are considering are clear, we provide an example of a \( \circ \)-decorated Dyck path here.
In the $\circ$-decorated Dyck path above, $\text{rise}_\circ(D) = 5$ and $\text{peak}_\circ(D) = 3$. The rise-touch composition is equal to $(2, 3, 1, 5)$. The $b$-sequence for this path is $(0, 0, 1, 2, 1, 2, 3, 1, 2, 3, 3, 4, 4, 5, 4, 3)$.

We next define a generating function for the $\circ$-decorated Dyck paths such that the rise-touch composition is equal to $\alpha$. Fix non-negative integers $k$ and $\ell$ and a composition $\alpha$. Then set

$$
\overline{\text{Cat}}^{\text{Rise}}_{\circ, k, \ell}(q, t) = \sum_{D \in \mathcal{D}_{\circ^{\ell}} \atop \text{peak}_\circ(D) = k \atop \text{area}_\circ(D) = \alpha} q^{\text{dinv}_\circ(D)} t^{\text{area}_\circ(D)}
$$

By definition,

$$
\overline{\text{Cat}}_{n}(q, t, z, w) = \sum_{k, \ell \leq n-1} \sum_{\alpha} \overline{\text{Cat}}^{\text{Rise}}_{\circ, k, \ell}(q, t) z^k w^\ell.
$$

this is because if there are $\ell$ double rises which are $\circ$-decorated, then the rise-touch composition of the $\circ$-decorated Dyck path will be some composition $\alpha$ of $n - \ell$ and so both the left and right hand side of the equation is equal to a weighted sum over all $\circ$-decorated Dyck paths of size $n$.

3.1. Combinatorial recurrence on $\overline{\text{Cat}}^{\text{Rise}}_{\circ, k, \ell}(q, t)$. We will show in this subsection that for for non-negative integers $k, \ell$ such that $k < |\alpha|$ that $\overline{\text{Cat}}_{\circ, k, \ell}(q, t)$ satisfies a recurrence involving the first part of the partition. We consider two cases, one where the first part of the composition $\alpha$ is greater than 1 and a second where $\alpha_1 = 1$.

**Proposition 4.** For $a \geq 1$, and $k, \ell \geq 0$, and a fixed composition $\beta$ we have the combinatorial recurrence,

$$
\overline{\text{Cat}}^{\text{Rise}}_{\circ, (a+1, \beta), k, \ell}(q, t) = t^a q^{\ell(\beta)} \sum_{\gamma = a} \overline{\text{Cat}}^{\text{Rise}}_{\circ, (\beta, \gamma), k, \ell}(q, t) + t^a q^{\ell(\beta)} \sum_{\gamma = a+1} \overline{\text{Cat}}^{\text{Rise}}_{\circ, (\beta, \gamma), k, \ell-1}(q, t).
$$
Proof. To prove equation (36), we will divide the set of $\circ$-decorated Dyck paths into two subsets: either the first vertical step of the $\circ$-decorated Dyck path is decorated or it is not decorated. On both sets we will perform a cyclic rotation as described at the beginning of this section. We will refer to the piece of the Dyck path up to the first point that it touches the diagonal as first (the part pictured in red in Figure 2) and the piece of the Dyck path after the first part as rest (the part pictured in blue in Figure 2).

The circle decorations travel with the vertical edges of the path and the vertical edge which is deleted may either be decorated or not. The sequence of $b$-values for the cyclically rotated path is exactly the same as the $b$-sequence for the original path except the last entry (which corresponds to the deleted edge) is deleted. This is because diagonal inversions within the first piece or within the rest piece are still diagonal inversions after the cyclic rotation and diagonal inversions between the first piece and the rest piece will switch from being on the same diagonal to being on the diagonal below (and vice versa).

**Case 1**: Cyclic rotation gives a bijection between $\circ$-decorated Dyck paths $D$ with rise-touch composition $(a+1, \beta)$ and peak$_{\circ}(D) = k$ and rise$_{\circ}(D) = \ell$ where the first vertical step is not decorated and $\circ$-decorated Dyck paths $D'$ with rise-touch composition $(\beta, \gamma)$ with $\gamma \models a$ and peak$_{\circ}(D') = k$ and rise$_{\circ}(D') = \ell$.

Say that there are $r$ $\circ$-decorated double rises in the first piece. Since the first vertical step is not decorated, deleting the first edge from a piece of a Dyck path which touches the diagonal after $a + r + 1$ steps, the rise-touch composition will be a composition $\gamma$ of size $a$ and deleting the first vertical step has the effect of removing $a$ cells contributing to the area (one for each peak or non-decorated double rise row in the first piece of the Dyck path), so area$_{\circ}(D) = $ area$_{\circ}(D') + a$. Moreover since there is one diagonal inversion of the form $(1, j)$ for each $j > 1$ with $a_j(D) = 0$, then $b_n(D) = \ell(\beta)$ (where $n$ is equal to the length of $D$) and dinv$_{\circ}(D) = $ dinv$_{\circ}(D') + \ell(\beta)$.

**Example 5.** Consider the first part of the Dyck path being the following piece of length 8. The first part of the rise-touch composition of this Dyck path will be 8 minus the number of double rises which are $\circ$-decorated in this piece (in this case there are 2).

![Diagram](image-url)

We consider this example to see how the rise-touch composition and $\circ$-area is affected by deleting the first vertical step and last horizontal step and ensure that the definitions are clear to this point. Since the $\circ$-decorations are in rows 2 and 6 are $\circ$-decorated then this implies that the $\circ$-area contribution from this first piece is 9 and length of the first step of
the rise-touch composition is 6. Deleting the first vertical and last horizontal step then the rise-touch composition will have contribution \((3, 2)\) and the \(\circ\)-area statistic will contribute \(4 = 9 - (6 - 1)\) from this piece.

**Case 2:** Cyclic rotation also gives a bijection between \(\circ\)-decorated Dyck paths \(D\) with rise-touch composition \((a + 1, \beta)\) and peak_{\(\circ\)}(\(D\)) = \(k\) and rise_{\(\circ\)}(\(D\)) = \(\ell\) where the first vertical step is decorated and \(\circ\)-decorated Dyck paths \(D'\) with rise-touch composition \((\beta, \gamma)\) with \(\gamma \models a + 1\) and peak_{\(\circ\)}(\(D'\)) = \(k\) and rise_{\(\circ\)}(\(D'\)) = \(\ell - 1\). Deleting the \(\circ\)-decorated vertical step will reduce the number of \(\circ\)-decorations by one.

Say that there are \(r\) \(\circ\)-decorated double rises in the first piece and the Dyck path first touches the diagonal after \(a + 1 + r\) steps. Notice that the rise-touch composition of the first piece of the path after deleting the first step will be a composition \(\gamma\) of size \(a + 1\). Deleting the first vertical step has the effect of removing \(a\) cells contributing to the area (one for each peak or non-decorated double rise row in the first piece of the Dyck path), so area_{\(\circ\)}(\(D\)) = area_{\(\circ\)}(\(D'\)) + \(a\). Moreover since there is one diagonal inversion of the form \((1, j)\) for each \(j > 1\) with \(a_j(\(D\)) = 0\), then \(b_n(\(D\)) = \ell(\beta)\) and dinv_{\(\circ\)}(\(D\)) = dinv_{\(\circ\)}(\(D'\)) + \(\ell(\beta)\).

**Example 6.** Consider the same first piece of the Dyck path as in Example 5 but assume now that rows 1, 2 and 6 are labelled. The length of the first step of the rise-touch composition will be 5 which is the same size as the resulting rise-touch composition of \((3, 2)\) when we delete the first vertical and last horizontal step.

\[
\begin{array}{c}
\text{The } \circ\text{-area of the contribution from the first piece is 8 before deleting the first vertical and last horizontal step and it is 4 after.}
\end{array}
\]

The two cases are disjoint and together cover all \(\circ\)-decorated Dyck paths and the weights agree between those on the left and right hand side, so equation (36) holds. \(\square\)

Next we consider the case where \(\alpha = (1, \beta)\) (that is, the \(a = 0\) case) and we see that the recurrence is slightly different.

**Proposition 7.** For \(k, \ell \geq 0\), and a composition \(\beta\) we have the combinatorial recurrence,

\[
\text{Cat}_{\text{Rise}}^{(1, \beta), k, \ell}(q, t) = q^\ell(\beta) \text{Cat}_{\text{Rise}}^{\beta, k, \ell}(q, t) + \text{Cat}_{\text{Rise}}^{\beta, k - 1, \ell}(q, t) + q^\ell(\beta) \text{Cat}_{\text{Rise}}^{\beta, 1, k, \ell - 1}(q, t).
\]

**Proof.** In the case that the first part of the rise-touch composition is 1, there are three types of \(\circ\)-decorated Dyck paths which contribute to this expression: the Dyck path starts with a non-decorated vertical step followed by a horizontal step, it begins with a \(\circ\)-decorated
vertical step followed by a horizontal step or it begins with some number of ◦-decorated double rises followed by a peak followed by the same number of horizontal steps (in the picture below, the example representing this case has two ◦-decorated double rises, but in general there are potentially between 1 and ℓ ◦-decorated double rises).

Case 1: The Dyck paths \( D \) which begin with a non-decorated vertical step followed by a horizontal step with rise-touch composition of the form \((1, \beta)\) and \( \text{peak}_o(D) = k \) and \( \text{rise}_o(D) = \ell \) are in bijection with the Dyck paths \( D' \) with rise-touch composition \( \beta \) and \( \text{peak}_o(D) = k \) and \( \text{rise}_o(D) = \ell \) by removing the first vertical and horizontal steps. Since there are \( \ell(\beta) \) diagonal inversions of the form \((1, j)\) for each \( j > 1 \) with \( a_j(D) = 0 \), then \( \text{dinv}_o(D) = \text{dinv}_o(D') + \ell(\beta) \). The area doesn’t change by deleting the first vertical and horizontal step so \( \text{area}_o(D) = \text{area}_o(D') \).

Case 2: The Dyck paths \( D \) with rise-touch composition of the form \((1, \beta)\) which begin with a ◦-decorated vertical step followed by a horizontal step and peak \( \text{peak}_o(D) = k - 1 \) and \( \text{rise}_o(D) = \ell \). The bijection simply to remove the first vertical and horizontal steps (and hence one of the decorations on the peaks). Since that first peak of \( D \) is ◦-decorated, it does not contribute to the ◦-dinv statistic and \( \text{dinv}_o(D) = \text{dinv}_o(D') \). Moreover the ◦-area does not change by removing the first vertical and horizontal step so \( \text{area}_o(D) = \text{area}_o(D') \).

Case 3: The Dyck paths \( D \) with rise-touch composition equal to \((1, \beta)\) which begin with a ◦-decorated vertical step followed by a horizontal step and peak \( \text{peak}_o(D) = k \) and \( \text{rise}_o(D) = \ell \) are in bijection with the Dyck paths \( D' \) with rise-touch composition \( \beta, 1 \), peak \( \text{peak}_o(D) = k \) and \( \text{rise}_o(D) = \ell - 1 \) by a cyclic rotation described at the beginning of this section. We note that the ◦-area does not change with the cyclic rotation because the first row is a ◦-decorated double rise so \( \text{area}_o(D) = \text{area}_o(D') \). Since there is one diagonal inversion of the form \((1, j)\) for each \( j > 1 \) with \( a_j(D) = 0 \), then \( \text{dinv}_o(D) = \text{dinv}_o(D') + \ell(\beta) \).

The three cases are disjoint, they cover all possible ◦-decorated Dyck paths with rise-touch composition beginning with a 1 and the weights on the left hand side of the equation agree with those on the right and side, hence equation \((27)\) holds.

4. The symmetric function recurrence

In this section we will provide a proof of the following symmetric function identity that agrees with the combinatorial recurrence on the generating function for ◦-decorated Dyck paths.
Theorem 8. For $k \geq 0$, and for integers $d,r,m$,
\begin{align}
(28) \quad C_{k+1}^* \Delta_h \nabla (e_r^* h_m^*) &= t^k B_k^* \Delta_h \nabla (e_r^* h_{m-1}) + t^k B_k^* \Delta_h \nabla (e_r^* h_m^*) \\
(29) \quad + \chi(k = 0) \Delta_h \nabla (e_{r-1}^* h_m^*)
\end{align}

Notice that at $d = 0$, one of the terms is equal to 0. This case of this identity is equivalent to the recurrence used in [GXZ10] to prove the Schröder case of the compositional shuffle conjecture.

We have as a consequence the following expression of coefficients which have combinatorial meaning.

Corollary 9. For non-negative integers $k, \ell$ and $a$ and for a composition $\beta$,
\begin{align}
(30) \quad & \left\langle \Delta_h \nabla C_{a+1}(C\beta), s_{k+1,1|\beta|+a-k} \right\rangle = t^a q^{\ell(\beta)} \sum_{\gamma = a} \left\langle \Delta_h \nabla C_{\gamma}, s_{k+1,1|\beta|+a-k-1} \right\rangle \\
(31) \quad & + t^a q^{\ell(\beta)} \sum_{\gamma = a+1} \left\langle \Delta_h \nabla C_{\gamma}, s_{k+1,1|\beta|+a-k} \right\rangle + \chi(a = 0) \left\langle \Delta_h \nabla (C\beta), s_{k,1|\beta|-k} \right\rangle
\end{align}

Proof. This identity is derived by taking the *-scalar product with $C\beta$ on both sides of the equation (28)–(29). We begin by taking the *-scalar product of $C\beta$ and the left hand side of (28). We note that since $\{H_{\mu}\}_{\mu}$ is an orthogonal basis with respect to the *-scalar product and are eigenvectors of the operators $\nabla$ and $\Delta_h$, these operators are self dual with respect to the *-scalar product and commute with each other.

(32) \quad $\left\langle C\beta, C_{a+1}^* \Delta_h \nabla (e_r^* h_n^*) \right\rangle_s = \left\langle \Delta_h \nabla C_{a+1}(C\beta), e_r^* h_n^* \right\rangle_s = \left\langle \Delta_h \nabla C_{a+1}(C\beta), h_r e_n \right\rangle$.

Similarly, the scalar product with the other three terms of the right hand side of (28)–(29) have expressions involving only $C_\alpha$ once equation (18) is applied to the expressions of the form $B_m(C\beta)$. We conclude that
\begin{align}
(33) \quad & \left\langle \Delta_h \nabla C_{a+1}(C\beta), h_{k+1} e_{|\beta|+a-k} \right\rangle = t^a q^{\ell(\beta)} \sum_{\gamma = a} \left\langle \Delta_h \nabla C_{\gamma}, h_{k+1} e_{|\beta|+a-k-1} \right\rangle \\
(34) \quad & + t^a q^{\ell(\beta)} \sum_{\gamma = a+1} \left\langle \Delta_h \nabla C_{\gamma}, h_{k+1} e_{|\beta|+a-k} \right\rangle + \chi(a = 0) \left\langle \Delta_h \nabla (C\beta), h_k e_{|\beta|-k} \right\rangle
\end{align}

We derive equations (30)–(31) because $s_{\ell+1,1|\beta|+a-\ell} = \sum_{r=0}^{|\beta|+a-\ell} (-1)^r h_{\ell+1+r|\beta|+a-\ell-r}$.

Note that in the case that $k = 0$ and $r = 1$ that $s_{r-1,1n+1-r}$ is 0 if $n > 0$ and $s_{r-1,1n+1-r} = 1$ if $n = 0$.

Theorem 10. For non-negative integers $k$ and $\ell$ and a composition $\alpha$,
\begin{align}
(35) \quad \text{Cat}_{\alpha,k,\ell}(q,t) &= \left\langle \Delta_h \nabla (C_{\alpha}), s_{k+1,1|\alpha|-k-1} \right\rangle.
\end{align}
Proof. We have just established in the previous section (combining Propositions 4 and 7) that
\[
\text{Cat}^{\text{Rise}}_{(a+1, \beta), k, \ell}(q, t) = t^a q^\ell \sum_{\gamma | a} \text{Cat}^{\text{Rise}}_{(\beta, \gamma), k, \ell}(q, t) + t^a q^\ell \sum_{\gamma | a+1} \text{Cat}^{\text{Rise}}_{(\beta, \gamma), k, \ell-1}(q, t)
\]
(36)
\[
+ \chi(a = 0) \text{Cat}^{\text{Rise}}_{\beta, k-1, \ell}(q, t).
\]
This combinatorial recurrence agrees with Corollary 9 in the sense that if
\[
\text{Cat}^{\text{Rise}}_{(\beta, \gamma), k, \ell}(q, t) = \left\langle \Delta h_\ell \nabla C_{\beta, \gamma}, s_{k+1,1|\beta+a-k-1} \right\rangle
\]
and
\[
\text{Cat}^{\text{Rise}}_{(\beta, \gamma), k, \ell-1}(q, t) = \left\langle \Delta h_{\ell-1} \nabla C_{\beta, \gamma}, s_{k+1,1|\beta+a-k} \right\rangle
\]
and
\[
\text{Cat}^{\text{Rise}}_{\beta, k-1, \ell}(q, t) = \left\langle \Delta h_k \nabla (C_\beta), s_{k,1|\beta-k} \right\rangle,
\]
then
\[
\text{Cat}^{\text{Rise}}_{(a+1, \beta), k, \ell}(q, t) = \left\langle \Delta h_\ell \nabla C_{a+1}(C_\beta), s_{k+1,1|\beta+a-k} \right\rangle.
\]
(37)
The indices of the right hand side of this recurrence have the property that either the value of \(\ell\) is lower or the size of the composition \(\alpha\) is smaller. Therefore we proceed by induction by assuming that equation (35) holds true for compositions which are of smaller size and smaller values of \(\ell\). Then it remains to show that it is true for a base case.

We note that if \(\alpha = (1)\), then
\[
\left\langle \Delta h_\ell \nabla (C_1), s_{k+1,1-k} \right\rangle = 1
\]
if and only if \(k = 0\) and it is equal to 0 otherwise. Similarly, \(\text{Cat}_{(1), k, \ell}(q, t) = 1\) if and only if \(k = 0\) (and 0 otherwise) because the generating function for \(\circ\)-decorated Dyck paths has one term for the Dyck path consisting of \(\ell \circ\)-decorated double rises, a peak, followed by horizontal steps back to the diagonal. \(\square\)

In order to prove our symmetric function identity from Theorem 10 we break the calculation into lemmas that will hopefully make a long calculation a little easier to follow.

Lemma 11.
\[
\frac{T_\nu}{w_\nu} h_d[B_\mu] \tilde{H}_\mu[D_\nu] = \sum_{a \geq 0} (-1)^a h_{d-a} \left[\frac{1}{M}\right] \sum_{\gamma \geq \nu \atop |\gamma| = |\nu| + a} \tilde{H}_\mu[D_\gamma] \frac{T_\gamma}{w_\gamma} C_{\gamma\nu}^{\circ a}
\]
(38)
Proof. First we apply equation 2 to \(h_d[B_\mu]\) (the alphabet addition formula) and show that
\[
h_d[B_\mu] = h_d \left[ \frac{MB_\mu - 1}{M} + \frac{1}{M} \right] = \sum_{a \geq 0} h_d \left[ \frac{D_\mu}{M} \right] h_{d-a} \left[ \frac{1}{M} \right].
\]
(39)
To the left hand side of equation (38) we apply the reciprocity formula (9), and then use the generalized Pieri coefficients that were introduced in [GarHag02] from equation (13), and then reapply the reciprocity formula to derive

\[
\frac{T_\mu}{w_\nu} h_d[B_\mu] H_\mu[D_\nu] = (-1)^{|\mu|+|\nu|} \frac{T_\mu}{w_\nu} \sum_{a \geq 0} h_a \left[ \frac{D_\mu}{M} \right] H_\nu[D_\mu] h_{d-a} \left[ \frac{1}{M} \right]
\]

\[
= (-1)^{|\mu|+|\nu|} \frac{T_\mu}{w_\nu} \sum_{a \geq 0} \sum_{\gamma \geq \nu} \hat{h}_\gamma[D_\mu] h_{d-a} \left[ \frac{1}{M} \right]
\]

\[
= \frac{T_\mu}{w_\nu} \sum_{a \geq 0} \sum_{\gamma \geq \nu} (-1)^a \frac{T_\gamma}{T_\mu} \hat{h}_\gamma[D_\gamma] h_{d-a} \left[ \frac{1}{M} \right] \tilde{c}_{\gamma \nu}^e.
\]

Now recall that we can convert the \( d_{f \mu} \) coefficients to \( c_{e \mu}^{\perp} \) coefficients using equation (14), the resulting equation is equal to the right hand side of the equation stated in (38). □

Now the coefficients \( c_{e \mu}^{\perp} \) have the sum over \( \nu \) that was also calculated by Garsia and Haglund [GarHag02] and we need this expression that we state in the following lemma.

**Lemma 12.** For a a non-negative integer and for a fixed partition \( \gamma \),

\[
\sum_{\nu \subseteq \gamma} c_{e \nu}^{\perp} = e_a[B_\gamma]
\]

**Proof.** Of course if \( |\gamma| < a \) then both the left and right hand sides of this expression are equal to 0. From the identity in equation (15) in the special case of \( g = e_a \) we have

\[
\sum_{\nu \subseteq \gamma, |\nu| = |\gamma| - a} c_{e \nu}^{\perp} = \nabla^{-1} h_a \left[ \frac{X - \epsilon}{M} \right] \bigg|_{X \to D_\gamma}.
\]

We know that \( \nabla^{-1} h_a \left[ \frac{X}{M} \right] = e_a \left[ \frac{X}{M} \right] \) and hence we apply the alphabet addition formulas to simplify the right hand side of the equation (14) as

\[
\nabla^{-1} h_a \left[ \frac{X - \epsilon}{M} \right] \bigg|_{X \to D_\gamma} = \sum_{b \geq 0} \nabla^{-1} h_{a-b} \left[ \frac{X}{M} \right] h_{b-1} \left[ \frac{-\epsilon}{M} \right] \bigg|_{X \to D_\gamma} = \sum_{b \geq 0} e_{a-b} \left[ \frac{D_\gamma}{M} \right] e_b \left[ \frac{1}{M} \right] = e_a \left[ \frac{MB_\gamma - 1}{M} + \frac{1}{M} \right] = e_a[B_\gamma].
\]

The following result gives us an expression for a kernel which we can use to apply the \( \mathbb{B} \) and \( \mathbb{C} \) operators.

**Proposition 13.** For non-negative integers \( d, r \) and \( n \)

\[
\Delta h_d \nabla (e_r h_n^*) = \sum_{a \geq 0} \sum_{\gamma} (-1)^{r+a} e_{n+r} \left[ \frac{XD_\gamma}{M} \right] h_{d-a} \left[ \frac{1}{M} \right] h_{n+a-|\gamma|} \left[ \frac{-1}{M} \right] \frac{T_\gamma}{w_\gamma} e_a[B_\gamma]
\]
where the sum over $\gamma$ is over all partitions of size smaller than or equal to $n + a$.

Proof. We begin by introducing a false set of variables and using the fact that $e^*_r[X]h^*_n[X] = \langle e^*_n[rXW], h_r[W]e_n[W] \rangle$. Then equation (7) implies

\begin{equation}
\Delta_h \nabla(e^*_r h^*_n) = \langle \Delta_h \nabla e^*_n[rXW], h_r[W]e_n[W] \rangle
\end{equation}

\begin{equation}
= \sum_{\mu \vdash n + r} \frac{h_d[B_\mu]T_\mu \tilde{H}_\mu[X]}{w_\mu} \langle \tilde{H}_\mu[W], h_r[W]e_n[W] \rangle.
\end{equation}

Now a special case of the Macdonald coefficients that are known (see [Mac95] Exercise 2 p. 362) is the scalar product \( \langle \tilde{H}_\mu[W], h_r[W]e_n[W] \rangle = e_n[B_\mu] \). To apply our Lemma \ref{lem:1} we need an expression with \( D_\mu = MB_\mu - 1 \), hence by the alphabet addition formulae, we have

\begin{equation}
= \sum_{\mu \vdash n + r} \frac{h_d[B_\mu]T_\mu \tilde{H}_\mu[X]}{w_\mu} e_n[B_\mu]
\end{equation}

\begin{equation}
= \sum_{\mu \vdash n + r} \frac{h_d[B_\mu]T_\mu \tilde{H}_\mu[X]}{w_\mu} e^*_n[(MB_\mu - 1) + 1]
\end{equation}

\begin{equation}
= \sum_{\mu \vdash n + r} \sum_{k \geq 0} \frac{h_d[B_\mu]T_\mu \tilde{H}_\mu[X]}{w_\mu} e^*_k[D_\mu] e^*_n[k].
\end{equation}

We can then expand the expression $e^*_k[D_\mu] = \sum_{\nu \vdash k} \frac{\tilde{H}_\nu[D_\mu]}{w_\nu}$ so that we can apply the reciprocity formula from equation (9).

\begin{equation}
= \sum_{\mu \vdash n + r} \sum_{k \geq 0} \sum_{\nu \vdash k} \frac{h_d[B_\mu]T_\mu \tilde{H}_\mu[X]}{w_\mu} \frac{\tilde{H}_\nu[D_\mu]}{w_\nu} e^*_n[k]
\end{equation}

\begin{equation}
= \sum_{\mu \vdash n + r} \sum_{k \geq 0} \sum_{\nu \vdash k} (-1)^{n + r + k} \frac{h_d[B_\mu]\tilde{H}_\mu[X]}{w_\mu} \frac{T_\nu \tilde{H}_\mu[D_\nu]}{w_\nu} e^*_n[k].
\end{equation}

At this point we can apply equation \ref{eq:38} and simplify the power of $-1$ by the expression $(-1)^{n+k} e^*_n[k] = h^*_n[k][1]$. We also combine the sum over $k \geq 0$ and $\nu \vdash k$ to just be a sum over all partitions $\nu$. The sum is actually finite because the expression $h^*_n[k][1]$ is equal to 0 if $|\nu| > n$. We then interchange the sum over $\nu$ and $\gamma$ then we have the following
Proof.

Lemma 14. For a non-negative integer $c$ and integer $m$,

\[
(56) \quad \mathbb{C}_m e_c \left[ \frac{X D_\gamma}{M} \right] = \sum_{b \geq 0} q^{-b+1} (-1)^{m+1} e_{c-b} \left[ \frac{X D_\gamma}{M} \right] h_b [D_\gamma] e_{b-m} \left[ \frac{X}{1-t} \right]
\]

Proof.

(57) \quad \mathbb{C}_m e_c \left[ \frac{X D_\gamma}{M} \right] = (-q) e_c \left[ \frac{X - M}{qu} \right] D_\gamma \left[ \frac{-uX}{1-t} \right] u^{-m}

(58) \quad = \sum_{a \geq 0} \sum_{b \geq 0} u^{a-b} q^{-b+1} (-1)^{a+b+1} e_{c-b} \left[ \frac{X D_\gamma}{M} \right] h_b [D_\gamma] e_a \left[ \frac{X}{1-t} \right] |_{u^{-m}}

so when we take the coefficient of $u^{-m}$ in $u^{a-b}$, then $-m = a - b$ or $a = b - m$ and we obtain the expression stated in equation (56).

We also develop a full expression for $\mathbb{B}_m^*$ on the kernel from Proposition 13 because to prove the theorem we will expand the left hand side of (28) and need to recognize when we have the right hand side of that equation.

Lemma 15. For all integers $m, r$ and $a$ and for non negative integer $d$,

\[
(59) \quad \mathbb{B}_m^* \Delta_h \nabla (e_r^* h_n^*) = \sum_{a \geq 0} \sum_{b \geq 0} \sum_{\tau} (-1)^{r+a+b+m} e_b [D_\tau] e_{n+r-b} \left[ \frac{X D_\tau}{M} \right] \times
\]

\[
(60) \quad \times e_{b-m} \left[ \frac{X}{1-t} \right] h_{d-a}^* [1] h_{n+a-\tau}^* \left[ -1 \right] \frac{T_\tau}{w_\tau} e_a [B_\tau]
\]
Proof. As we did in the previous lemma, we will first develop an expression for the action of $B^*_m$ on $e^*_{n+r}[XD_\gamma]$.

\begin{equation}
B^*_m e^*_{n+r}[XD_\gamma] = e^*_{n+r}\left((X + \frac{M}{u}) D_\tau\right) \Omega \left[\frac{-uX}{1-t}\right]_{u^{-m}}(61)
\end{equation}

\begin{equation}
= \sum_{a \geq 0} \sum_{b \geq 0} u^{a-b} e_b[D_\tau] e_{n+r-b} \left[\frac{XD_\tau}{M}\right] h_a \left[\frac{-X}{1-t}\right]_{u^{-m}}(62)
\end{equation}

Now when we take the coefficient of $u^{-m}$ in this expression $-m = a - b$, hence $a = b - m$ and our expression becomes

\begin{equation}
B^*_m e^*_{n+r}[XD_\gamma] = \sum_{b \geq 0} (-1)^{b+m} e_b[D_\tau] e_{c-b} \left[\frac{XD_\tau}{M}\right] e_{b-m} \left[\frac{X}{1-t}\right](64)
\end{equation}

Now we apply that expression to the kernel that we derived in Proposition 13.

\begin{equation}
B^*_m \Delta_{h_d} \nabla(e^*_r h^*_a)
\end{equation}

\begin{equation}
= \sum_{a \geq 0} \sum_{b \geq 0} \sum_{\gamma} (-1)^{r+a+b} e^*_{a+b}[XD_\gamma] h^*_{n+a-|\gamma|-1} \left[\frac{T_\tau}{w_\gamma} e_a[B_\gamma]\right](65)
\end{equation}

\begin{equation}
= \sum_{a \geq 0} \sum_{b \geq 0} \sum_{\gamma} (-1)^{r+a+b+m} e^*_{n+r-b}[XD_\gamma] h_b[D_\gamma] \times
\end{equation}

\begin{equation}
\times e_{b-m} \left[\frac{X}{1-t}\right] h^*_{n+a-|\gamma|-1} \left[\frac{T_\tau}{w_\gamma} e_a[B_\gamma]\right].(69)
\end{equation}

Now we know that since $k \geq 0$, then $b \geq 1$ since $e_{b-k-1} = 0$ for $b = 0$. In fact, it may be helpful to make the replacement $b \to b + 1$ and (69) and we can apply equation (12) to...
\[ h_{b+1}[D_\gamma]. \]

\[ (70) \quad = \sum_{a \geq 0} \sum_{\gamma} \sum_{b \geq 0} q^{-b} (-1)^{r+a+k} e_{n+r-b-1}^*[XD_\gamma] h_{b+1}[D_\gamma] \times \]

\[ \times e_{b-k} \left[ \frac{X}{1-t} \right] h_{d-a}^*[1] h_{n+a-\mid \gamma\mid}^*[1] \frac{T_{\gamma}}{\nu_{\gamma}} e_a[B_\gamma] \]

\[ (71) \quad = \sum_{a \geq 0} \sum_{\gamma} \sum_{b \geq 0} \sum_{\tau \rightarrow \gamma} (-1)^{r+a+k} M^{b} c_{\tau}^\gamma \left( \frac{T_{\gamma}}{T_{\tau}} \right)^{b} e_{n+r-b-1}^*[XD_\gamma] \times \]

\[ \times e_{b-k} \left[ \frac{X}{1-t} \right] h_{d-a}^*[1] h_{n+a-\mid \gamma\mid}^*[1] \frac{T_{\gamma}}{\nu_{\gamma}} e_a[B_\gamma] \]

\[ (72) \quad = \sum_{a \geq 0} \sum_{\gamma} \sum_{b \geq 0} \sum_{\gamma \rightarrow \tau} (-1)^{r+a+k} \quad \]

\[ (73) \quad = \chi(k = 0) \sum_{a \geq 0} \sum_{\gamma} (-1)^{r+a} e_{n+r-1}^*[XD_\gamma] h_{d-a}^*[1] h_{n+a-\mid \gamma\mid}^*[1] \left[ -1 \right] \frac{T_{\gamma}}{\nu_{\gamma}} e_a[B_\gamma] \]

Notice already that equation (74) is equal to \( +\chi(k = 0) \Delta_{k}^{\nu}(e_{r-1}^*[h_{n}^*]) \). Then it remains to expand equations (72)–(73). For this we use \( B_\gamma = B_\tau + \frac{T_{\gamma}}{T_{\tau}} \) and \( D_\gamma = D_\tau + M^{\frac{T_{\gamma}}{T_{\tau}}} \). In this case \( e_a[B_\gamma] = e_a[B_\tau] + e_{a-1}[B_\tau] \frac{T_{\gamma}}{T_{\tau}} \) and \( e_{n}[XD_\gamma] = \sum_{c \geq 0} e_{n-c}[XD_\tau] e_c[X] \frac{T_{\gamma}}{T_{\tau}} \). We will also use the identity \( t^b e_{b-k} \left[ \frac{X}{1-t} \right] = t^b e_{b-k} \left[ \frac{X}{1-t} \right] \) to show that (72)–(73) is equivalent to the following expression:

\[ (75) \quad = t^k \sum_{a \geq 0} \sum_{\gamma} \sum_{b \geq 0} \sum_{\tau \rightarrow \gamma} \sum_{c \geq 0} (-1)^{r+a+k} M c_{\tau}^\gamma \left( \frac{T_{\gamma}}{T_{\tau}} \right)^{b} e_{n+r-b-1-c}[XD_\tau] e_c[X] \frac{T_{\gamma}}{T_{\tau}} \]

\[ \times e_{b-k} \left[ \frac{tX}{1-t} \right] h_{d-a}^*[1] h_{n+a-\mid \gamma\mid}^*[1] \frac{T_{\gamma}}{\nu_{\gamma}} \left( e_a[B_\tau] + e_{a-1}[B_\tau] \frac{T_{\gamma}}{T_{\tau}} \right). \]

We will regroup the \( T_{\gamma}/T_{\tau} \) terms and break the expression into two separate sums. We will also replace \( c_{\tau}^\gamma \) with \( d_{\gamma}^\tau \) using equation (10). Then equations (75)–(76) are equivalent to

\[ (77) \quad = t^k \sum_{a \geq 0} \sum_{\gamma} \sum_{b \geq 0} \sum_{\tau \rightarrow \gamma} \sum_{c \geq 0} (-1)^{r+a+k} d_{\gamma}^\tau \left( \frac{T_{\gamma}}{T_{\tau}} \right)^{b+c+1} e_{n+r-b-1-c}[XD_\tau] e_c[X] \times \]

\[ \times e_{b-k} \left[ \frac{tX}{1-t} \right] h_{d-a}^*[1] h_{n+a-\mid \gamma\mid}^*[1] \frac{T_{\gamma}}{\nu_{\gamma}} e_a[B_\gamma] \right). \]

\[ (78) \quad + t^k \sum_{a \geq 0} \sum_{\gamma} \sum_{b \geq 0} \sum_{\tau \rightarrow \gamma} \sum_{c \geq 0} (-1)^{r+a+k} d_{\gamma}^\tau \left( \frac{T_{\gamma}}{T_{\tau}} \right)^{b+c+2} e_{n+r-b-1-c}[XD_\tau] e_c[X] \times \]

\[ \times e_{b-k} \left[ \frac{tX}{1-t} \right] h_{d-a}^*[1] h_{n+a-\mid \gamma\mid}^*[1] \frac{T_{\gamma}}{\nu_{\gamma}} e_{a-1}[B_\tau] \right). \]

In both of these sums, we can interchange the sum over partitions \( \gamma \) and then over \( \tau \rightarrow \gamma \) to a sum over partitions \( \tau \) and then over \( \gamma \leftarrow \tau \). In this case the sums of the form \( \sum_{\gamma \leftarrow \tau} d_{\gamma}^\tau \left( \frac{T_{\gamma}}{T_{\tau}} \right)^{h} = (-1)^{n-1} e_{n-1}[D_\tau] \) (where \( n \in \{b+c+1, b+c+2\} \)) by equation (11).
since in both of the expressions we have \( n > 1 \) so there is no \( \chi(n = 0) \) term. Equations (77)–(80) are equivalent to

\[
(81) \quad t^k \sum_{a \geq 0} \sum_{b \geq 0} \sum_{c \geq 0} \sum_{\tau} (-1)^{r+a+k+b+c} e_{b+c}[D_r] e_{n+r-b-1-c}[XD_r] e_c[X] \times \\
\times e_{b-k} \left[ \frac{tX}{1-t} \right] h_{d-a}[1] h_{n+a-|\tau|-1}[t-X] T_w e_a[B_r]. 
\]

(82) \quad + t^k \sum_{a \geq 0} \sum_{b \geq 0} \sum_{c \geq 0} \sum_{\tau} (-1)^{r+a+k+b+c+1} e_{b+c+1}[D_r] e_{n+r-b-1-c}[XD_r] e_c[X] \times \\
\times e_{b-k} \left[ \frac{tX}{1-t} \right] h_{d-a}[1] h_{n+a-|\tau|-1}[t-X] T_w e_a[B_r]. 

To make the next step of our calculation clearer, we will interchange the sum over \( b \) and \( c \), change the sum over \( b \geq 0 \) to one of \( b \geq c \) and replace \( b \rightarrow b - c \). Equations (81)–(84) are equivalent to

\[
(85) \quad = t^k \sum_{a \geq 0} \sum_{b \geq 0} \sum_{c \geq 0} \sum_{\tau} (-1)^{r+a+k+b} e_{b}[D_r] e_{n+r-b-1}[XD_r] e_c[X] \times \\
\times e_{b-c-k} \left[ \frac{tX}{1-t} \right] h_{d-a}[1] h_{n+a-|\tau|-1}[t-X] T_w e_a[B_r]. 
\]

(86) \quad + t^k \sum_{a \geq 0} \sum_{b \geq 0} \sum_{c \geq 0} \sum_{\tau} (-1)^{r+a+k+b+1} e_{b+1}[D_r] e_{n+r-b-1}[XD_r] e_c[X] \times \\
\times e_{b-c-k} \left[ \frac{tX}{1-t} \right] h_{d-a}[1] h_{n+a-|\tau|-1}[t-X] T_w e_a[B_r]. 

Now that we can interchange the summations a second time and \( c \geq 0, b \geq c \) is equivalent to \( b \geq 0, 0 \leq c \leq b \). Notice now that

\[
(89) \quad \sum_{0 \leq c \leq b} e_c[X] e_{b-c-k} \left[ \frac{tX}{1-t} \right] = e_{b-k} \left[ \frac{tX}{1-t} \right] = e_{b-k} \left[ \frac{X}{1-t} \right]. 
\]

We now have that equations (85)–(88) are equivalent to

\[
(90) \quad = t^k \sum_{a \geq 0} \sum_{b \geq 0} \sum_{c \geq 0} \sum_{\tau} (-1)^{r+a+k+b} e_{b}[D_r] e_{n+r-b-1}[XD_r] \times \\
\times e_{b-k} \left[ \frac{X}{1-t} \right] h_{d-a}[1] h_{n+a-|\tau|-1}[t-X] T_w e_a[B_r]. 
\]

(91) \quad + t^k \sum_{a \geq 0} \sum_{b \geq 0} \sum_{c \geq 0} \sum_{\tau} (-1)^{r+a+k+b+1} e_{b+1}[D_r] e_{n+r-b-1}[XD_r] \times \\
\times e_{b-k} \left[ \frac{X}{1-t} \right] h_{d-a}[1] h_{n+a-|\tau|-1}[t-X] T_w e_a[B_r]. 

We then note that equation (90)–(91) is the right hand side of the expression derived in Lemma 15 with \( m \to k \) and \( n \to n-1 \) and hence is equal to \( t^k \sum \Delta_h \nabla (e^* h^* \nabla \Delta h) \). As well
we have that (92)–(93) with $b \rightarrow b - 1$ and $a \rightarrow a + 1$ is equivalent to the expression
\[
\sum_{a \geq -1} \sum_{b \geq 1} (-1)^{r+a+k+b+1} e_b [D_{r}] e_{a-r-b} [X D_{r}] \times
\]
\[
\times e_{b-k-1} \left[ \frac{X}{1-t} \right] h^*_{d-a-1} [1] h^*_{n+a-\tau} \left[ -1 \right] T_{\mu} e_{a} [B_{r}] \]
\]
and with a few exceptions of terms which are equal to 0 (i.e. $a = -1$ and $b = 0$) this is precisely the right hand side of the expression in Lemma 15 with $m \rightarrow k + 1$, $d \rightarrow d - 1$ and hence is equal to $t^k B^*_k \Delta h^*_{d-1} \nabla (e^*_r h^*_n)$. \hfill \Box

5. Remarks

Section 7 of [HRW15] has two conjectures that we do not resolve here. These techniques may potentially be adapted to prove them as well, but some additional work remains.

The first conjecture which does not follow directly from Theorem 10 is a symmetry property that is part of Conjecture 7.1 in [HRW15].

**Conjecture 16.** For non-negative integers $k, \ell$ and $n > k + \ell$,
\[
\langle \Delta h_{k} \nabla e_{n-\ell} , s_{k+1,1^n-k-\ell-1} \rangle = \langle \Delta h_{\ell} \nabla e_{n-k} , s_{\ell+1,1^n-k-\ell-1} \rangle .
\]

We hope that an algebraic proof of this result will follow from the calculations in the previous section and are continuing to investigate this possibility.

The second conjecture is a combinatorial interpretation in terms of a second type of decorated Dyck paths, $\ast$-decorated Dyck paths, where vertical edges except for the bottom most left one can be decorated with an $\ast$. This other combinatorial interpretation does not seem to be compatible with the compositional version. However it is conjectured that it is compatible with coefficients of the form $\langle \Delta h_{k} \nabla E_{n-k,p} , s_{\ell+1,1^n-k-\ell-1} \rangle$ where
\[
E_{n-k,p} = \sum_{\alpha \vdash n-k \atop \ell(\alpha) = p} C_{\alpha} .
\]

Potentially recurrences on these coefficients that are compatible with the interpretation in terms of $\ast$-decorated Dyck paths can be derived from Corollary 9.

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