Time-reversed quantum trajectory analysis of micromaser correlation properties and fluctuation relations

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The micromaser is examined with the aim of understanding certain of its properties based on a time-reversed quantum trajectory analysis. The background theory of master equations derived from a repeated interaction model perspective is briefly reviewed and extended by taking into account the more general renewal process description of the sequence of interactions of the system with incoming ancilla, and results compared with other recent (and not so recent) approaches that use this generalisation. The results are then specialised to the micromaser, and a quantum trajectory unravelling of the micromaser dynamics is formulated that enables time-reversed quantum trajectories, defined according to the Crooks approach, to, first, be shown to arise naturally in the analysis of micromaser and atomic beam correlations, and second used in the formulation of a fluctuation relation for the probabilities of trajectories and their time-reversed counterparts.

I. INTRODUCTION

The one-atom maser, or micromaser, has been an important tool in the experimental [1, 2] and theoretical [3, 4] study of the fundamental properties of cavity quantum electrodynamics for over three decades. It is a deceptively simple device: a single quantized mode of a high-\(Q\) (\(\sim 10^{10}\)) cavity driven by excited two-level atoms passing through the cavity at a rate \(R\) sufficiently low that at no time is there more than one atom in the cavity. The two atomic levels are Rydberg states that, in free space, are very long lived, but within the cavity, they couple strongly to the cavity microwave field. This field is, in turn, coupled to an external thermal bath kept at a temperature near absolute zero.

Recently, analogues of the micromaser beyond its original quantum optical setting have been realised e.g., for a nanomechanical resonator coupled to a superconducting single-electron transistors [5, 6]. Also recently, there have been a number of theoretical developments in the theory of open quantum systems which are of immediate relevance to the theory of micromaser. The first such is the growing interest in repeated interaction, or collisional models of open quantum systems, of which it can be argued the micromaser is a very early example. A second development is in the context of quantum thermodynamics where detailed fluctuation relations for heat exchange and entropy production have been formulated which relate the probabilities of a quantum process conditioned on the outcomes of sequences of random quantum jumps (i.e., a quantum trajectory) proceeding forward in time to its time-reversed dual quantum trajectory.

The aim of the work to be presented here is to address these two developments in the context of micromaser theory. The first is motivated by the fact that recently proposed collisional models [7–9], and in particular [10], that make use of renewal theory to describe the random spacing between successive collisions, can lead to somewhat different dynamical equations, i.e., master equations, for a general system, and hence also for the micromaser. The derivation of one such class of equations, an extension of that done in [11, 12] is presented in Section [VI] and the origin of the differences are discussed there. The second is motivated by the fact that certain correlation properties of the micromaser cavity field and its associated post-interaction atomic beam, which have been shown to be expressible in a natural way in terms of time-reversed quantum trajectories, are shown in Section [VII A] to be an example of time-reversal defined in a specific sense due to Crooks [13]. This is then followed, in Section [VII] by an investigation into the micromaser treated as a non-equilibrium thermodynamic system, leading to a discussion of how time-reversed quantum trajectories can be defined in this case, this in turn enabling a derivation of a detailed fluctuation relation [14] for the entropy flow between the cavity field reservoir and the atomic beam incident on the cavity.

Conclusions and acknowledgements then close out the paper.

II. BACKGROUND

A. Relation to collisional models

The micromaser cavity-field is an open quantum system and as such its dynamics are given by tracing over the atomic beam and thermal bath to yield a master equation for the reduced density operator \(\rho\) for the cavity field. The original derivation of the master equation by Filipowicz et al [2] was recast in the operations-effects language of open quantum systems by [15] and in terms of a quantum field description of the atomic beam in [11, 12]. But it is clear from the structure of the model that the micromaser is an early example of a collisional or repeated interaction model that has re-

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ceived significant attention of late [9, 10, 16, 31]. The atomic beam here plays the role, in collisional models, of a stream of elementary ancillas or `units’ that interact with (i.e., collide with) the system $S$, then move on, making way for the next incoming unit. Such models have proved to be a promising tool to analyze the dynamics of quantum Markovian and non-Markovian systems in that the master equation for the reduced system can be obtained in many cases without any of the approximations (such as the Born-Markov-secular approximations) usually needed for typical microscopic derivations. As such, the micromaser expressed in the repeated interaction picture has attracted interest from the point of view of a rigorous mathematical analysis [31].

B. The role of quantum trajectory methods

One feature of the model is that the atoms, after passing through the cavity, will become entangled with the cavity field state. Moreover, these atoms are available for detection – the state of the atoms, i.e., whether excited or not, can be measured by field-ionization techniques, and consequently information on the cavity field can be extracted from the atomic beam measurements. This inspired, very early on, Meystre and Wright [32] to carry out numerical simulations of the micromaser dynamics based on such measurements, which they described as ‘quantum trajectories’, a terminology (along with the notion of ‘unravelling’) later introduced by Alsing and Carmichael [33] in the development of the wave function Monte-Carlo or quantum trajectory method [34, 35], one of the now central theoretical tools used to analyse the properties of open quantum systems.

The distinction between the earlier work of Meystre and Wright and the later developments of quantum trajectory theory is that in the latter, the trajectories are generated by the intrinsic probabilistic dynamics of the system whereas for the micromaser, these probabilities are imposed externally by the arrival statistics of the atoms in the beam. In the standard model, these arrival statistics are taken to be Poissonian, but were later generalised using a renewal process model in [11, 12], this leading to a non-Markovian master equation for the cavity field. A general quantum trajectory analysis taking into account decay of the cavity field can be found in [12, 37, 38].

But the quantum trajectory method applied to the micromaser yields more than simulations of cavity dynamics. The atoms in the emergent atomic beam will also be correlated (and indeed entangled if the mean time between atomic arrivals is much less than the decay time of the cavity field [39]). In the limit of a cavity field reservoir at zero temperature, it has been shown by a quantum trajectory based analysis that measurements of atomic beam correlations lead directly to the cavity field intensity correlation function (i.e., $g(2)$(τ)), [12, 40] (a result also obtained by non-trajectory methods in [41]), and by a suitable quantum interference scheme, leads to the cavity field correlation function, $g(1)$(τ) and hence the cavity field spectrum [42] (also obtainable via measuring the decay of introduced coherence [43, 44]).

These results come about because of a naturally emerging ‘dual’ relationship between the quantum trajectories that contribute to $g(2)$(τ) (and $g(1)$(τ)) and those that contribute to the atomic beam correlation function: they are time-reversed conjugates, and can be understood as an example of conjugate time-reversed trajectories defined in the sense of Crooks [13], a connection that is elaborated on further below.

The natural role of quantum trajectories in the analysis of the micromaser is seen to play a further role when the thermodynamic properties of the micromaser are examined.

C. Thermodynamic properties of the micromaser

In its normal mode of operation [3], the atoms in the incident atomic beam are all wholly in their excited state which can be interpreted as corresponding to an infinite negative temperature. But if the atoms incident on the cavity are drawn from a thermodynamic source at a finite temperature $T_a$, the atoms will be in a mixed state with Boltzmann probabilities for the ground and excited states. The micromaser then assumes the character of a thermodynamic device operating between the two reservoirs at different temperatures, that of the atomic source, $T_a$ and that of the cavity field reservoir, $T_c$. The micromaser interpreted in this fashion has been investigated from a thermodynamic perspective in a recent paper [46] which is based explicitly on a collisional or repeated interaction interpretation of the micromaser. This work, as far as the micromaser is concerned, limits itself to a particular set of thermodynamic issues, but as shown here, the micromaser also provides a natural setting via a quantum trajectory treatment for the analysis of fluctuation theorems of Crooks [14].

III. IMPULSIVE COLLISIONAL MODELS

The collisional model for open quantum systems, of which the micromaser is an early example, involves the system of interest $S$ undergoing interaction with a succession of ancilla, and between such interactions, the system evolves unitarily according to its own intrinsic Hamiltonian, unless the system itself is coupled to an external reservoir in which case a more general non-unitary evolution will occur. In the simplest case, these ancilla are independent, and are all prepared in the same state, and in the extreme instance, the interaction time is sufficiently short on the time scale of evolution of the system that this interaction is effectively impulsive, and can be modelled as an instantaneous change in the state of the system.
The result of the system-ancilla interaction is determined by the detailed nature of the system, the ancilla, and their interaction, but in general can be expressed, for a collision initiated at time \( t \) as

\[
\rho(t + \tau_{int}) = \text{Tr}_a \left[ U(\tau_{int}) \rho(t) \otimes \rho_a U^\dagger(\tau_{int}) \right].
\] (1)

which, on putting \( \rho_a = \sum_n p_n |n\rangle \langle n | \) and with \( \sqrt{\mathcal{F}_n} (m|U(\tau_{int})|n) = L_{mn} (\tau_{int}) \), these being operators on the Hilbert space of the system, Eq. (1) can be written

\[
\rho(t + \tau_{int}) - \rho(t) = \sum_{mn} \left( L_{mn} (\tau_{int}) \rho(t) L_{mn}^\dagger (\tau_{int}) \right)
\]

\[
- \frac{1}{2} \left( L_{mn}^\dagger (\tau_{int}) L_{mn} (\tau_{int}), \rho(t) \right),
\] (2)

which is of Lindblad form. We can write this as

\[
\rho(t + \tau_{int}) = (1 + \mathcal{F}_a(\tau_{int})) \rho(t)
\] (3)

so that the quantum map \( \mathcal{F}_a(\tau_{int}) \) represents the change in the state of \( S \) as a consequence of the interaction of \( S \) with the ancilla over the interval \( \tau_{int} \).

In the limit of zero interaction time, which limit will also require the interaction strength to become infinite, this can be written, with \( \epsilon \) infinitesimal, as

\[
\rho(t + \epsilon) = (1 + \mathcal{F}_a) \rho(t - \epsilon).
\] (4)

Between these collisions, the system will evolve freely. The system could be assumed to be isolated or open but here we are interested in the models in which the system is weakly coupled to a thermal reservoir, in which case the system evolution will be described by a Lindblad evolution

\[
\rho(t + \tau) = e^{\mathcal{L}_S \tau} \rho(t + \epsilon)
\] (5)

where \( \mathcal{L}_S \) is the Lindblad superoperator appropriate for the system reservoir interaction.

Now assume that the system is initially prepared in a state \( \rho(0) \), and that a steady stream of ancilla interact with the system in the manner described by Eq. (5), with the first ancilla interacting at time \( t_1 > 0 \), and subsequent ancilla arriving at times \( t_2, t_3 \ldots \). The state of the system at a time \( t \), \( \rho_c(t) \), with the subscript \( c \) indicating conditioned on collisions occurring at times \( t_1, t_2, \ldots \), will then be

\[
\rho_c(t) = e^{\mathcal{L}_S t} \theta(t_1 - t) + e^{\mathcal{L}_S (t-t_1)} \left( 1 + \mathcal{F}_a \right) e^{\mathcal{L}_S t_1} \rho(0) \theta(t_2 - t) \theta(t - t_1)
\]

\[
+ e^{\mathcal{L}_S (t-t_2)} \left( 1 + \mathcal{F}_a \right) e^{\mathcal{L}_S (t_2-t_1)} \left( 1 + \mathcal{F}_a \right) e^{\mathcal{L}_S t_1} \rho(0) \theta(t_3 - t) \theta(t - t_2) + \ldots
\] (6)

where \( \theta(t) \) is the unit step function. Taking the derivative with respect to time and using \( \theta'(t) = \delta(t) \) we find that \( \rho(t) \) is given by the following differential equation

\[
\dot{\rho}_c(t) = \mathcal{L}_S \rho_c(t) + I(t) \mathcal{F}_a \rho_c(t - \epsilon)
\] (7)

where appears the quantity

\[
I(t) = \sum_n \delta(t - t_n)
\] (8)

sometimes referred to as generalised shot noise [47].

The arrival times \( t_1, t_2, \ldots \) will in general be random, so that \( I(t) \) will itself be a stochastic process. The aim then is to derive the master equation for the density operator for the system averaged over all realizations of \( I(t) \). Thus we are seeking \( \rho(t) = \langle \rho_c(t) \rangle \), for which we need to specify the statistical properties of \( I(t) \).

A. Renewal process master equation

A particularly useful approach to arriving at a model for the stochastic properties of \( I(t) \) is to treat the arrival times as a renewal process [48] in which is specified a waiting time distribution \( w(\tau) \), such that \( w(\tau) d\tau \) is the probability that the next collision will occur in the time interval \( (\tau, \tau + d\tau) \) after the previous. The simplest case of a renewal process is a Poissonian beam with mean arrival rate \( R \), for which the waiting time distribution is exponential:

\[
w(\tau) = R^{-1} e^{-R\tau}.
\] (9)

In this case, averaging over the arrival times can be readily shown to yield the Markovian master equation

\[
\dot{\rho} = \mathcal{L}_S \rho + R \mathcal{F}_a \rho = \mathcal{L} \rho
\] (10)

where \( \mathcal{L} = \mathcal{L}_S + R \mathcal{F}_a \) is a Lindblad operator.

For other choices of \( w(\tau) \), the analysis relies on results of the theory of renewal processes, and is somewhat more involved. It has been applied in the case of the microwave maser in [11, 12], and recently in the general analysis of the class of collisional models formulated in terms of renewal processes in [7, 10, 49]. Amongst other results, the master equation turns out to be non-Markovian. This has been demonstrated in the case of the maser in [11, 12] and again recently, though leading to slightly
different results, in \[\text{[10]}\] for reasons explained below. Here, a variation on these derivations alternative to, and more straightforward than, the derivation presented in \[\text{[12]}\], that makes explicit use of the stationary statistics of the shot noise \(I(t)\) is presented.

The solution to the conditional master equation Eq. (11) can be written as a Dyson series and the average taken over all realisations of \(I(t)\) to yield the corresponding expansion for the system density operator \(\rho(t) = \langle \rho_a(t) \rangle\),

\[
\rho(t) = e^{\mathcal{L}_S t} \rho(0) + \sum_{n=1}^{\infty} \int_0^t \int_0^{\tau_n} \ldots \int_0^{\tau_2} \frac{d\tau_n \ldots d\tau_2 \ldots d\tau_1}{\tau_n! \tau_{n-1}! \ldots \tau_2!} \langle \langle I(\tau_n) I(\tau_{n-1}) \ldots I(\tau_2) I(\tau_1) \rangle \rangle
\]

\[
\times e^{\mathcal{L}_S (\tau_n - \tau_{n-1})} \mathcal{F}_a e^{\mathcal{L}_S (\tau_{n-1} - \tau_{n-2})} \mathcal{F}_a \ldots e^{\mathcal{L}_S (\tau_2 - \tau_1)} \mathcal{F}_a e^{\mathcal{L}_S \tau_1} \rho(0)
\]

(11)

where the initial time \(t = 0\) will, in general, lie between successive arrivals. In this expression there appears the multitime shot noise correlation function \(\langle \langle I(\tau_n) \ldots I(\tau_1) \rangle \rangle\). If the stream of ancilla is assumed to have been initiated in the infinite past, then \(I(t)\) will be a stationary stochastic process and as such will be a function of time differences only. Further, the ranges of integration always involve an infinitesimal offset, so that the correlation function is only required for times satisfying the strict inequality \(t_n > t_{n-1} > \ldots > t_2 > t_1\). In that case, a singular value of the correlation function for equal time arguments will not contribute, and it can be shown [17, 50] that, on averaging over the collision times, the correlation function is given by

\[
\langle \langle I(\tau_n) I(\tau_{n-1}) \ldots I(\tau_2) I(\tau_1) \rangle \rangle
\]

\[
= R^n g(\tau_n - \tau_{n-1}) g(\tau_{n-1} - \tau_{n-2}) \ldots g(\tau_2 - \tau_1)
\]

\[
t_n > t_{n-1} > \ldots > t_1 \quad (12)
\]

where \(g(\tau) = R^{-2} \langle \langle I(t) I(t + \tau) \rangle \rangle\) is a normalized ‘intensity’ correlation function for the incident ancilla, a function of time differences only, as expected for a stationary process. In this expression \(R = \langle \langle I(t) \rangle \rangle\) is the mean collision rate and is given by

\[
R^{-1} = \int_0^\infty \tau w(\tau) d\tau
\]

(13)

while \(g(t)\), the normalised intensity correlation function, also known as the renewal density [47, 48], or sprinkling distribution, satisfies the integral equation

\[
R g(t) = w(t) + R \int_0^t w(\tau) g(t - \tau) d\tau.
\]

(14)

Eq. (12) can be substituted into Eq. (11) from which, as shown in [11, 12], can be derived the master equation for \(\rho(t)\)

\[
\dot{\rho} = \mathcal{L}_S \rho + R \mathcal{F}_a \int_0^t e^{\mathcal{L}_S (t - \tau)} \mathcal{K}(t - \tau) \rho(\tau) d\tau
\]

(15)

which, by virtue of the appearance of a memory kernel \(\mathcal{K}(t)\), describes a generally non-Markovian evolution.

The Laplace transform of the memory kernel \(\mathcal{K}(t)\) is

\[
\mathcal{K}(s) = (1 - (\hat{g}(s) - s^{-1}) R \mathcal{F}_a)^{-1}.
\]

(16)

For Poissonian statistics, \(g(t) = 1\), the memory kernel becomes a delta function, \(\mathcal{K}(t) = \delta(t)\) and the master equation reduces to the standard result Eq. (10).

The master equation derived above can be contrasted with that found in, e.g., [11], (and in particular in the supplement [51]) where an analysis is made on the basis of slightly different assumptions concerning the implementation of the renewal process description of the interaction times. These concern the choice of the exclusive probability densities for jumps corresponding to the action of \((1 + \mathcal{F}_a)\) at times \(t_1, t_2, \ldots\), the difference in outcome lying in the early \(t\) dependence of \(\rho(t)\) on the initial state \(\rho(0)\). A comparison can be made by taking the Laplace transform of Eq. (15) and rearranging terms, so that this equation can be cast in the form

\[
\dot{\rho} = \mathcal{L}_S \rho + R \mathcal{F}_a \int_0^t e^{\mathcal{L}_S (t - \tau)} \hat{\mathcal{K}}(t - \tau) \rho(\tau) d\tau
\]

\[- R \mathcal{F}_a e^{\mathcal{L}_S (g(t) - 1) \rho(0)}
\]

(17)

where \(\hat{\mathcal{K}}(s) = s \hat{g}(s)\). For sufficiently large \(t\), \(g(t) \rightarrow 1\), so the term on the right hand side depending on the initial state \(\rho(0)\) will become negligible and we are left with the result derived in [51] (which was incorrectly stated as being the result found in [12]),

\[
\dot{\rho} = \mathcal{L}_S \rho + R \mathcal{F}_a \int_0^t e^{\mathcal{L}_S (t - \tau)} \hat{\mathcal{K}}(t - \tau) \rho(\tau) d\tau.
\]

(18)

The reason for this is best understood by considering the quantum trajectory unravelling of \(\rho(t)\).

### B. Quantum trajectory unravelling

The appearance of a non-Markovian master equation belies the fact that the dynamics possesses a straightforward quantum trajectory unravelling. This is because the non-Markovianity has its origins in an externally imposed
source of noise, in contrast to non-Markovianity that arises in systems coupled to a quantum reservoir, where a ‘back-flow’ of information produced by the system-reservoir dynamics is the underlying cause of the non-Markov behaviour.

The required unravelling can be obtained by first substituting Eq. (12) into Eq. (11), doing so yielding a series expansion which, in spite of appearances, is in fact not a possible quantum jump unravelling as \( F_a \) is not a jump operator (the trace of its output is zero). But after some reorganisation of terms, as shown in [3] the required expansion expressed in terms of the jump operator \( 1 + F_a \) can be arrived at

\[
\rho(t) = e^{L^t \rho(0)} + \sum_{n=1}^{\infty} \int_0^t dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1 p_f(t - t_n) w(t_n - t_{n-1}) \ldots w(t_2 - t_1) p_1(t_1)
\]

\[
\times e^{L^t (t-t_n)} (1 + F_a) e^{L^t (t_n - t_{n-1})} (1 + F_a) \ldots (1 + F_a) e^{L^t t_1} \rho(0)
\]

where

\[
p_f(t - t_n) = 1 - \int_0^{t-t_n} w(\tau)d\tau
\]

is the probability that there is no collision in the time interval \((t_n, t)\) after the final collision at time \(t_n\),

\[
p_1(t_1) dt = p_f(t_1) Rd t_1
\]

known as the residual time probability distribution, is the probability that no collision occurs in the time interval \((0, t_1)\), and a collision, the first after \(t = 0\), occurs in the interval \((t_1, t_1 + dt_1)\), and

\[
p_0(t) = 1 - \int_0^t p_1(\tau)d\tau
\]

is the probability that there are no collisions in the time interval \((0, t)\). That the probabilities \(p_0(t)\) and \(p_1(t)\) take the form that they do is a consequence of the shot noise \(I(t)\) being stationary. Starting the evolution at an arbitrary initial time \(t = 0\) entails introducing the residual time probability for the arrival of the first ancilla after \(t = 0\), given as above by \(p_1(t)\) [18].

Any unravelling into an ensemble of quantum trajectories can then be carried out by simulating the ancilla interaction times according to the set of probabilities \(p_0(t), p_1(t), w(t)\) and \(p_f(t)\).

The above result was derived starting from the form Eq. (12) for the shot noise correlation. An alternate procedure is to construct the exclusive probability for a sequence of collisions, here given by the product in Eq. (19), \(p_f(t - t_n) w(t_n - t_{n-1}) \ldots w(t_2 - t_1) p_1(t_1)\). The master equation of [15], Eq. (18), can then be shown to arise if we set \(p_1(t) = w(t)\), i.e., the waiting time distribution \(p_1(t)\) for the arrival of the first ancilla after the initial time \(t = 0\) is replaced by the intercollision waiting time distribution \(w(t)\). Thus this master equation corresponds to a different modeling of the statistics of the arrival times – one in which the the initial system state is set at a time immediately after a collision – but which only has an impact for short times, after which Eq. (17) reduces to Eq. (18).

### C. Relaxation to steady state

The steady state solution to the general master equation Eq. (15), achieved when \(\dot{\rho} = 0\), plays an essential role in determining the time-reversal quantum trajectory properties of the system, required in the discussion later in Section V of fluctuation relations. This steady state is most easily determined by working with the Laplace transform of Eq. (15), and using \(\rho(\infty) = \rho_{ss} = \lim_{s \to 0} s \rho(s)\) is given by the solution of

\[
L_S \rho_{ss} + R F_a g(-L_S)(-L_S) \rho_{ss} = 0.
\]

For Poissonian statistics \((g = 1)\) the above reduces to

\[
L_S \rho_{ss} + R F_a \rho_{ss} = 0.
\]

As an example of the steady state solution for non-Poissonian statistics, we can consider the case of

\[
g(t) = A e^{-\Gamma t} + 1
\]

which is the renewal density (intensity correlation) for super-bunched (for \(A > 1\)) ancilla interactions. In this case, in the limit of \(\Gamma \to 0\), the steady state can be readily shown to be

\[
\rho_{ss} = (1 + A)^{-1}(\rho_A + A \rho_{eq})
\]

where \(\rho_{eq}\) is the equilibrium state for the cavity in the absence of any ancilla, while \(\rho\) is the steady state solution to

\[
L_S \rho_A + A R F_a \rho_A = 0
\]

i.e., the steady state for the system with a Poissonian interaction rate \(AR\). This result is easy to understand: excitations by the ancilla will occur in Poissonian bursts at an enhanced rate \(AR\), separated by quiescent intervals in which the cavity will relax to its equilibrium state.
IV. THE MICROMASER

The micromaser is an early example of a collisional or repeated interaction model that has recently been analyzed from the perspective of its thermodynamic properties \[40\]. For the micromaser, the system is a single-mode cavity field of frequency $\omega_S$ which interacts with a succession of qubits (highly excited Rydberg atoms) with a transition frequency $\omega_n$ near or on resonance with the cavity field frequency. These atoms are typically, but not necessarily, prepared in a fully inverted state. Between qubit interactions, the cavity is weakly damped by coupling to its thermal environment of temperature $T_c$. The system Hamiltonian is then $H_S = \omega_S a^\dagger a$ while the effect of the coupling to the external environment is described by the usual Lindblad form

$$L_S\rho = -i\omega_S \{a^\dagger a, \rho\} + (\bar{n} + 1)\gamma \{a, a^\dagger \rho\} + \bar{n}\gamma \{a^\dagger a, \rho\}$$

where $\bar{n} = (e^{\hbar\omega_n/kT_n} - 1)^{-1}$.

The ancilla are qubits with Hamiltonian $H_a = \frac{1}{2}\omega_a\sigma_z$ that interact with the cavity field for a brief period $\tau_{int}$, this interaction being described by the Jaynes-Cummings interaction $V = \Omega (\sigma_+ a^\dagger + \sigma_- a)$. The prepared state of the qubits will be assumed to be diagonal in their energy basis $\rho_a = p|e\rangle\langle e| + (1-p)|g\rangle\langle g|$. In the original model for the micromaser, the qubits were assumed to be fully inverted, $p = 1$, but in general they will be taken to have exited from a thermal reservoir of temperature $T_a$, so that $p = (e^{\hbar\omega_a/kT_a} - 1)^{-1}$.

Assuming exact resonance between the qubit and the cavity field, $\omega_a = \omega_S = \omega$, the interaction of the $n$th qubit with the cavity is described by Eq. (28) where, in the impulsive limit $\tau_{int} \to 0$ and $\Omega \to \infty$ with $\theta = \Omega\tau_{int}$ held fixed, we find that the Lindblad operators, from Eq. (2), $L_{mn} = \sqrt{\rho_m(n)}\exp(-iV\tau)|n\rangle$, $m, n = e, g$ associated with the interaction of the cavity field with an incident atom are

$$L_{ee} = \sqrt{p}\cos(\theta\sqrt{N} + 1) \quad L_{ge} = \sqrt{p}\sin(\theta\sqrt{N})\frac{a^\dagger}{\sqrt{N}}$$

$$L_{gg} = \sqrt{1-p}\cos(\theta\sqrt{N}) \quad L_{eg} = \sqrt{1-p}\sin(\theta\sqrt{N} + 1)\frac{a}{\sqrt{N+1}}$$

In terms of these operators $(1 + F_a)\rho$ is

$$(1 + F_a)\rho = L_{ee}\rho L_{ee}^\dagger + L_{ge}\rho L_{ge}^\dagger + L_{gg}\rho L_{gg}^\dagger + L_{eg}\rho L_{eg}^\dagger$$

The operators $L_{mn}\sqrt{dt}$ are the Kraus operators corresponding to measurements made on the qubits exiting the cavity. Thus, for instance, $L_{ee}\rho L_{ee}^\dagger$ is a mapping of the state of the cavity conditioned on an incident qubit in its excited state being measured to be in its excited state on exiting the cavity, while $L_{eg}\rho L_{eg}^\dagger$ is a mapping of the cavity state conditioned on an incident qubit in its ground state being measured to be in its excited state, with a photon thereby having been absorbed from the cavity field. These ‘quantum jumps’ will occur with a probability $\text{Tr}_a[L_{mn}\rho L_{mn}^\dagger dt]$ in the time interval $(t, t + dt)$.

Any realisation of the measurements implied by these operators will require either a measurement of the qubit state (whether excited or ground) prior to interacting with the cavity field, followed by a measurement after the interaction ceases, or by assuming that there are in fact two distinguishable beams, one of excited atoms, the other of ground state atoms, the first arriving at a rate $pR$, the second at a rate $(1-p)R$, which each beam subject to separate measurements after interaction has ceased.

A. Micromaser master equation

It is typically the case that an exponential waiting time distribution is assumed for the incident atoms, in which case the master equation reduces to the form Eq. (10), and is given by

$$(\dot{\rho} = L_{ee}\rho + L_{ge}\rho L_{ge}^\dagger - \frac{1}{2}\{L_{ee}\rho L_{ee}^\dagger, \rho\} + L_{ge}\rho L_{ge}^\dagger - \frac{1}{2}\{L_{ge}\rho L_{ge}^\dagger, \rho\} + L_{eg}\rho L_{eg}^\dagger - \frac{1}{2}\{L_{eg}\rho L_{eg}^\dagger, \rho\}$$

a result first obtained, for $p = 1$, i.e., where the incident atoms are all fully excited, by $[3, 4]$.

B. Weak coupling limit

In the above master equation, the coupling to the atomic beam reservoir is not assumed weak, in contrast to the usual weak coupling assumption made in deriving Markov master equations. The weak coupling limit of $\theta \ll 1$ is nevertheless revealing. Provided the mean photon numbers in the cavity are not too high, this becomes, with $\bar{n}_a = p\theta^2 = (e^{\hbar\omega_a/kT_a} - 1)^{-1}$

$$\dot{\rho} = -i\omega [a^\dagger a, \rho] + (\bar{n} + 1)\gamma \{a, a^\dagger \rho\} + \bar{n}\gamma \{a^\dagger a, \rho\}$$

$$+ \bar{n}_a R (a^\dagger a - \frac{1}{2}\{a^\dagger a, \rho\}) + (\bar{n} + 1)R (a^\dagger a - \frac{1}{2}\{a^\dagger a, \rho\})$$

indicating the beam acts as a thermal reservoir at the temperature $T_a$ of the beam atoms, and has been commonly used in this fashion, see e.g., $[52, 53]$.

This result holds true even in the case of non-exponential waiting times. In that case, since $F_a \sim \theta^2$, the Laplace transform of the memory kernel $\tilde{K}(s)$, Eq.
can be replaced by unity, so that \( K(t) \sim \delta(t) \) and the non-Markovian master equation reduces to Eq. (10) from which Eq. (32) follows again.

C. Steady State

The steady state of the cavity field \( \rho_{ss} = \rho(\infty) \) will be given by Eq. (24). It is diagonal in the number basis,

\[
\rho_{ss} = \sum_{n=0}^{\infty} P_{ss}(n) |n\rangle \langle n|
\]

where the probability of finding \( n \) photons in the cavity at steady state, \( P_{ss}(n) \), is given by

\[
P_{ss}(n) = P_{ss}(0) \prod_{m=1}^{n} \frac{pR \sin^2(\sqrt{m} \theta)/m + \gamma n}{(1 - p)R \sin^2(\sqrt{m} \theta)/m + \gamma (n + 1)}
\]

and with \( P_{ss}(0) \) determined by the requirement that

\[
\sum_{n=0}^{\infty} P_{ss}(n) = 1.
\]

D. Quantum trajectory analysis

The master equation for the micromaser is of Lindblad form, and so is amenable to standard quantum trajectory analysis [12, 37, 55].

Introducing the operators \( C_{-1} \) and \( D_{-1} \)

\[
C_{-1} = \sqrt{(n + 1) \gamma} a
\]

\[
D_{-1} = \sqrt{(1 - p) R \sin(\theta \sqrt{N + 1})} \sqrt{N + 1} a
\]

which represent a loss of a photon from the cavity to the reservoir, or absorbed by an atom respectively, and

\[
C_1 = \sqrt{pR \sin(\theta \sqrt{N})} a^+ \\
D_1 = \sqrt{(1 - p) R \sin(\theta \sqrt{N})} a
\]

which represent a gain of a photon from the cavity reservoir, or from an atom respectively, and finally, for convenience, a further pair of operators associated with the atom passing through the cavity without giving up or absorbing a photon are defined by

\[
C_0 = \sqrt{pR \cos(\theta \sqrt{N + 1})} \\
D_0 = \sqrt{(1 - p) R \cos(\theta \sqrt{N})},
\]

we can write the master equation as

\[
\dot{\rho} = -i \omega [a^+ a, \rho] + \sum_{i=1}^{1} \left[ C_i \rho C_i^+ - \frac{1}{2} \left\{ C_i^+ C_i, \rho \right\} + D_i \rho D_i^+ - \frac{1}{2} \left\{ D_i^+ D_i, \rho \right\} \right].
\]

The dynamics of the micromaser can then be unravelled in terms of the jump operators \( J_i \) and \( K_i \) defined by

\[
J_i \rho = C_i \rho C_i^+ \quad \text{and} \quad K_i \rho = D_i \rho D_i^+
\]

with the jumps for \( i = \pm 1 \) representing the gain or loss of a single photon from the cavity field, and \( i = 0 \) representing no change in the cavity field photon number. Between jumps the system evolution is determined by the non-Hermitian Hamiltonian

\[
H_c = \omega a^+ a - \frac{1}{2} i \sum_i \left[ C_i^+ C_i + D_i^+ D_i \right]
\]

with the between-jumps evolution given by the superoperator \( \mathcal{L}_c \):

\[
\mathcal{L}_c \rho = -i \left[ H_c, \rho - \rho H_c^\dagger \right].
\]

A Dyson series decomposition of the full dynamics then reads

\[
\rho(t) = \rho_c^{(0)}(t) P^{(0)}(t) + \sum_{n=1}^{\infty} \sum_{L_1} \sum_{L_2} \cdots \sum_{L_n} \int_0^t dt_n \int_0^{t_n} \cdots \int_0^{t_2} dt_1 \times P^{(n)}(t; L_{i_1}, t_1, \ldots, L_{i_n}, t_n) \rho_c^{(n)}(t; L_{i_1}, t_1, \ldots, L_{i_n}, t_n)
\]

where the state of the cavity field at time \( t \) conditioned on the sequence of jumps \( \mathcal{L}_i \in \{ J_i, K_i \} \) occurring at times \( t_1, t_2, \ldots, t_n \) is

\[
\rho_c^{(n)}(t; L_{i_1}, t_1, \ldots, L_{i_n}, t_n) = \frac{e^{\mathcal{L}_c(t-t_n)} L_{i_1} e^{\mathcal{L}_c(t_n-t_{n-1})} L_{i_{n-1}} \cdots e^{\mathcal{L}_c(t-t_1)} L_{i_1} e^{\mathcal{L}_c t_1} \rho(0)}{\text{Tr}[e^{\mathcal{L}_c(t-t_n)} L_{i_1} e^{\mathcal{L}_c(t_n-t_{n-1})} L_{i_{n-1}} \cdots e^{\mathcal{L}_c(t-t_1)} L_{i_1} e^{\mathcal{L}_c t_1} \rho(0)]}.
\]

A quantum trajectory \( \gamma \) can then be defined as a sequence of states

\[
\gamma \equiv \{ \rho_c^{(0)}(t), \rho_c^{(1)}(t; L_{i_1}, t_1), \rho_c^{(2)}(t; L_{i_2}, t_2, L_{i_1}, t_1), \ldots \}.
\]
conditioned on the sequence of measurements implied by the jump operators $\mathcal{L}_i$. Such a trajectory occurs with a probability $P^{(n)}[\gamma] (dt)^n$ and is given by the trace of the final state of the sequence, i.e.,

$$P^{(n)}[\gamma] = P^{(n)}(t; \mathcal{L}_{i_1}, t_{i_1}, \ldots, \mathcal{L}_{i_n}, t_n) = \text{Tr}[e^{\mathcal{L}_c(t-t_n)} \mathcal{L}_{i_n} e^{\mathcal{L}_c(t_{n-1}-t_n-1)} \mathcal{L}_{i_{n-1}} \ldots \mathcal{L}_{i_1} e^{\mathcal{L}_c t_n} \rho(0)]$$

(44)

If the initial state $\rho(0)$ is a pure state, $\rho(0) = |\psi(0)\rangle\langle\psi(0)|$, and since the jump operators map pure states into pure states, the quantum trajectory can be written as a sequence of pure states, $|\psi_c^{(n)}(t)\rangle$, with the probability of a trajectory occurring then given by the norm $\langle\psi_c^{(n)}(t)|\psi_c^{(n)}(t)\rangle$.

V. TIME REVERSED QUANTUM TRAJECTORIES

Time reversed quantum trajectories have gained significant attention in recent times in the context of understanding fluctuation theorems of statistical mechanics in a quantum setting. The original classical fluctuation theorems [14-54] relate the probabilities to observe particular classical microscopic trajectories related by time reversal and typically take the form

$$\frac{p_F(x)}{p_R(-x)} = \exp[a(x - b)]$$

(45)

where $x$ can be, for instance, entropy produced or energy (heat) transported, $p_F(x)$ is the probability of amount $x$ being transported in the ‘forward’ direction, and $p_R(-x)$ is the probability transported in the ‘backward’ direction. These theorems have been extended into the quantum regime, [55-58] with the role of the microscopic classical trajectories played by quantum trajectories.

The essential idea in constructing a time-reversed quantum trajectory lies in associating with any trajectory in the forward time direction, a conjugate trajectory in the time reversed direction. It serves to refine the notion of a quantum trajectory at this point, which is to assume that the initial state of the the forward process is an eigenstate of some observable, which in the case of the micromaser will invariably be an eigenstate of the photon number operator $\hat{N}$, $|n\rangle$ say. A series of $k$ quantum jumps $\mathcal{L}_i$ interleaved with no-jump non-Hermitian evolution generated by $\mathcal{L}_c$ is then projected onto the final state $|\tilde{m}\rangle$ at time $t$. If we adopt the notation $\gamma_{nm} \equiv \{n, 0; \mathcal{L}_{c_1}, t_{1}; \ldots \mathcal{L}_{c_{k-1}}, t_{k-1}; \mathcal{L}_{c_k}, t_k; m, t\}$ (with time increasing from left to right, opposite to how it occurs in the expression Eq. (44)) to represent such a quantum trajectory then the time reversed quantum trajectory $\tilde{\gamma}$ is then taken to start at the time reversed state $|\tilde{n}\rangle = \Theta|m\rangle$ where $\Theta$ is the time reversal operator, and end at time $t$ in the time reversed state $|\tilde{m}\rangle$:

$$\tilde{\gamma}_{m\tilde{n}} = \{\tilde{m}, 0; \mathcal{L}_{c_1}; t - t_k; \ldots; \mathcal{L}_{c_2}; t - t_2; \mathcal{L}_{c_1}; t - t_1; \tilde{n}, t\}$$

(46)

where the $\tilde{x}$ indicate that the time reversed counterparts of $x$ are to be inserted. These conjugate trajectories will then occur with the conditional probability $P[\gamma_{nm}]$, i.e., conditioned on the initial state being $|m\rangle$ for the forward trajectory, and $\tilde{P}[\tilde{n}\tilde{m}]$ the conditional probability for the backward trajectory.

For the micromaser, the cavity field Hamiltonian $H_S = \omega a^\dagger a$ is time reversal invariant, so we can set $\Theta|m\rangle = |\tilde{m}\rangle = |m\rangle$, and since from Eq. (40), $\Theta_i H \Theta_i^{-1} = -iH_i^\dagger$, we have

$$\tilde{\mathcal{L}}_i = i [H_i \rho - \rho H_i^\dagger]$$

(47)

which leaves the task of determining the form for the time reversed jump operators $\mathcal{L}_i$.

The notion of a time-reversed quantum trajectory is not unique, this arising through the different approaches to defining the time reversed jump operators [57]. The various possibilities that have been proposed can be shown to be closely related to one another [60] and in particular to one introduced by Crooks [13], which is based, for systems that have reached steady state, on imposing a time symmetric condition on the probabilities for a forward and its time-reversed trajectory. If a trajectory $\gamma$ specified by a sequence of jumps in the forward direction, starting in the steady state occurs with probability $P[\gamma]$, then a time reversed dual trajectory $\tilde{\gamma}$ is then that trajectory involving a sequence of jumps in the reversed direction for which the steady state probability $\tilde{P}[\tilde{\gamma}]$ of observing $\tilde{\gamma}$ is the same as observing the trajectory $\gamma$ in the original process: $\tilde{P}[\tilde{\gamma}] = P[\gamma]$ [13]. Crooks shows that this leads to the following prescription for constructing the time reversed (dual) jump operators

$$\tilde{\mathcal{L}}_i = \Theta \rho_s^{1/2} \mathcal{L}_i^\dagger \rho_s^{-1/2} \Theta^{-1}$$

(48)

where $\rho_s$ is the steady state density operator, as given by Eq. (34) for the micromaser cavity field. The above expression is that which was originally derived by Crooks, though without the pre and post factors $\Theta \ldots \Theta^{-1}$.

This idea is developed further below for application to the micromaser, but first it is shown how the condition arises in a natural fashion under certain circumstances relating cavity field and atomic beam correlation functions.

VI. DUAL QUANTUM TRAJECTORIES FOR THE MICROMASER

In the case of the micromaser the notion of a dual trajectory has a direct physical interpretation in that instance in which the atoms in the incoming atomic beam are fully excited, $p = 1$, and the cavity field reservoir is at zero temperature. In that case, the master equation Eq. (33) reduces to

$$\dot{\rho} = -i\omega [a^\dagger a, \rho] + \sum_{i=1}^{C_0} \left[ C_i \rho C_i^\dagger - \frac{1}{2} \left( C_i^\dagger C_i, \rho \right) \right]$$

(49)
with the steady state given by \( \rho_{ss} = \sum_n P_{ss}(n) |n\rangle\langle n| \), with \( P_{ss}(n) \) from Eq. (34) with \( \bar{n} = 0 \) and \( p = 1 \):

\[
P_{ss}(n) = P_{ss}(0) \left( \frac{R}{\gamma} \right)^n \prod_{m=1}^{n} \frac{\sin^2(\sqrt{\bar{m}}\theta)}{m} \tag{50}
\]

with

\[
C_{-1} = \sqrt{\bar{a}}, \quad C_1 = \sqrt{R} \sin\left(\frac{\theta \sqrt{\bar{N}}}{\bar{N}}\right) a^1. \tag{51}
\]

The steady state is time reversal invariant, \( \Theta \rho_{ss} \Theta^{-1} = \rho_{ss} \), so that \( P_{ss}(n) = \langle n | \rho_{ss} | n \rangle = \langle \bar{n} | \rho_{ss} | \bar{n} \rangle = P_{ss}(n) \).

For the micromaser, the relevant probability will be the probability of measuring \( m \) photons in the cavity at time \( t = 0 \) (when the cavity field is already at steady state) and \( n \) photons in the cavity at a time \( t \) later for a particular quantum trajectory \( \gamma \). This probability, \( P_{nm}[\gamma] \) is given by

\[
P_{nm}[\gamma] = |\langle m | e^{-iH_c(t-t_k)} C_{ik} e^{-iH_e(t-t_{n+1})} C_{in} \ldots C_{it} e^{-iH_c(t)} |m\rangle|^2 P_{ss}(n) \tag{52}
\]

where the unit operator \( I \) has been introduced between neighbouring operator factors. We now proceed by inserting \( \rho_{ss}^{-1/2} \rho_{ss}^{-1/2} = I \) and making use of \([H_c, \rho_{ss}] = 0 \) to yield

\[
P_{nm}[\gamma] = |\langle n | I e^{-iH_c(t-t_k)} I \rho_{ss}^{-1/2} C_{ik} \rho_{ss}^{-1/2} I \ldots I \rho_{ss}^{-1/2} C_{it} \rho_{ss}^{-1/2} I e^{-iH_c(t)} |m\rangle|^2 P_{ss}(n). \tag{53}
\]

Substituting the decomposition of the unit operator in terms of the time reversal operator \( \Theta \), \( \Theta^{-1} \Theta = I \), using \( \Theta \Theta^\dagger \Theta^{-1} = -iH_c \) and making the substitution from the Crooks definition, Eq. (48), \( \tilde{C}_i^\dagger = \Theta \rho_{ss}^{-1/2} C_i \rho_{ss}^{-1/2} \Theta^{-1} \) then yields

\[
P_{nm}[\gamma] = |\langle \bar{n} | \Theta^{-1} e^{iH_c(t-t_k)} \tilde{C}_{ik}^\dagger \ldots \tilde{C}_{it}^\dagger e^{-iH_c(t)} |\bar{m}\rangle|^2 \tilde{P}_{ss}(n) \tag{54}
\]

In terms of the time reversed states \( |\bar{n}\rangle = \Theta |m\rangle \), and using \( \langle \beta | \Theta^{-1} A \Theta |\alpha\rangle = \langle \tilde{\alpha} | A^\dagger | \tilde{\beta}\rangle \) this then is

\[
P_{nm}[\gamma] = |\langle \bar{n} | e^{-iH_c(t)} \tilde{C}_{i1} \ldots \tilde{C}_{it} e^{-iH_e(t-t_k)} |\bar{n}\rangle|^2 \tilde{P}_{ss}(n). \tag{55}
\]

Since the cavity field Hamiltonian \( H_S = \omega a^\dagger a \) is time reversal invariant we can set for the eigenstates \( |n\rangle \), \( \Theta |n\rangle = |\bar{n}\rangle \). Further by making the substitutions \( t_l \rightarrow t - t_{k-l} \) to reverse the time order, we have the following probability for the time reversed trajectory \( \gamma \):

\[
P_{nm}[\gamma] = |\langle m | e^{-iH_e(t-t_k)} \tilde{C}_{i1} \ldots \tilde{C}_{it} e^{-iH_c(t)} |n\rangle|^2 \tilde{P}_{ss}(n) \tag{56}
\]

which is the same probability as the forward trajectory \( \gamma \), \( P_{nm}[\gamma] \).

Using Eq. (55), it readily follows that \( \tilde{C}_i = C_{-i} \): the jump operator that adds a photon to the cavity field \( C_1 \) through the de-excitation of an atom, is mapped into a jump operator that removes a photon from the field, \( C_{-1} \), through loss to the cavity reservoir, and vice versa, while

\[
C_0 \text{ is left unchanged. Thus we have}
\]

\[
P_{nm}[\gamma] = |\langle m | e^{-iH_e(t-t_k)} \tilde{C}_{i1} \ldots \tilde{C}_{it} e^{-iH_c(t)} |n\rangle|^2 \tilde{P}_{ss}(n) = P_{nm[\gamma]} \tag{57}
\]

a result made use of below.

### A. Field and atomic beam correlation

The significance of this result lies in the fact that the cavity field intensity correlation function, \( g^{(2)}(t) \), given by

\[
g^{(2)}(t) = \frac{\langle a^\dagger(0) a^\dagger(t) a(t) a(0) \rangle}{\langle a^\dagger a \rangle^2} \tag{58}
\]

\[
= \frac{N_{ex}}{\langle a^\dagger a \rangle^2} \sum_{m,n=0}^{\infty} \sin^2(\theta \sqrt{n+1}) m P(m, t; m, 0) \]

and the correlation function for the detection of ground state atoms emerging from the cavity, \( g_1(t) \)

\[
g_1(t) = \frac{\langle C_1^\dagger(0) C_1^\dagger(t) C_1(t) C_1(0) \rangle}{\langle C_1 C_1 \rangle^2} \tag{59}
\]

\[
= \frac{N_{ex}}{\langle a^\dagger a \rangle^2} \sum_{m,n=0}^{\infty} \sin^2(\theta \sqrt{n+1}) m P(m, t; n, 0)
\]

depend on the total probability \( P(m, t; n, 0) \) of observing \( m \) photons in the cavity at time \( t \) given that \( n \) were observed at time 0 as a generalised sum over the probabilities \( P_{nm}[\gamma] \) for all the quantum trajectories connecting the initial state \( |m\rangle \) to the final state \( |n\rangle \):
\[ P(n, t; m, 0) = \sum_{\gamma} P_{nm}[\gamma] \]

\[ = \sum_{k=0}^{\infty} \sum_{i_1 = -1}^{1} \sum_{i_2 = -1}^{1} \cdots \sum_{i_k = -1}^{1} \int_0^t dt_k \int_0^{t_k} dt_{k-1} \cdots \int_0^{t_2} dt_1 \]

\[ \times \left| \langle n|e^{-iH_c(t-t_k)}C_{i_k}e^{-iH_c(t_{k-1})}C_{i_{k-1}} \rangle \right|^2 P_{ss}(m) \]

\[ = \sum_{k=0}^{\infty} \sum_{i_1 = -1}^{1} \sum_{i_2 = -1}^{1} \cdots \sum_{i_k = -1}^{1} \int_0^t dt_k \int_0^{t_k} dt_{k-1} \cdots \int_0^{t_2} dt_1 \]

\[ \times \left| \langle m|e^{-iH_c(t-t_k)}C_{i_k}e^{-iH_c(t_{k-1})}C_{i_{k-1}} \rangle \right|^2 P_{ss}(n) \]

where we have used \( P_{nm}[\gamma] = P_{mn}[\gamma] \), to arrive at the last line, and where the \( k = 0 \) contribution to the sum is to be identified with that due to the no-jump trajectory.

If we now make a change of summation index \(-i_l \rightarrow i_{k-l+1}\) we get

\[ P(n, t; m, 0) = \sum_{k=0}^{\infty} \sum_{i_1 = -1}^{1} \sum_{i_2 = -1}^{1} \cdots \sum_{i_k = -1}^{1} \int_0^t dt_k \int_0^{t_k} dt_{k-1} \cdots \int_0^{t_2} dt_1 \]

\[ \left| \langle m|e^{-iH_c(t-t_k)}C_{i_k}e^{-iH_c(t_{k-1})}C_{i_{k-1}} \rangle \right|^2 P_{ss}(n) \]

and hence the equality of the two correlation functions, \( g^{(2)}(t) = g_1(t) \), this equality coming about since for each trajectory \( \gamma \) contributing to one correlation function, the dual time reversed trajectory \( \tilde{\gamma} \) contributes with equal probability to the other correlation function.

More succinctly, this result amounts to showing that since, by the quantum regression theorem we have

\[ \langle a^\dagger(0)a^\dagger(t)a(t)a(0) \rangle = \text{Tr}_S \left[ \mathcal{J}_{-1}e^{Lt_{-1}}\mathcal{J}_{-1}\rho_{ss} \right] \]

(62)

then, on using the above procedure we get

\[ \langle a^\dagger(0)a^\dagger(t)a(t)a(0) \rangle \propto \text{Tr}_S \left[ e^{Lt_{-1}}\mathcal{J}_{-1}\rho_{ss} \right] \]

(63)

\[ = \text{Tr}_S \left[ e^{Lt_{-1}}\mathcal{J}_{1}\rho_{ss} \right] \]

(64)

the last term here being proportional to \( g_1(t) \).

This essentially means that the intensity correlations of the cavity field are encoded in the correlation properties of the atomic beam emerging from the cavity. This same connection between correlation properties of the emergent atomic beam and the cavity field can also be shown to extend to field correlations and a homodyne-like experiment performed on the emergent atomic beam \[12\], i.e., the cavity field spectrum \[13\] \[14\] \[15\] \[16\] can also be measured by studies of atomic beam correlations.

B. More general cases

If the same approach is adopted in the general case of \( p \neq 1, \bar{n} \neq 0 \), for which the steady state is now given by the more complex expression Eq. (61), we find that the dual of \( \mathcal{J}_{-1} \) is no longer readily identifiable as representing a measurement on the atomic beam, and the above method will fail. Whether or not a relationship can be found by other means, or even exists, remains an open question.

If the incident beam is not Poissonian, then the above relationship also appears not to hold. For instance, if the case of a super-bunched beam, with \( g(t) \) as given by Eq. (25), for which the steady state is given by Eq. (26), the dual relationship between the jump operators \( C_1 \) and \( C_{-1} \), ceases to hold, so the above procedure for constructing time-reversed quantum trajectories will break down.

VII. A FLUCTUATION RELATION FOR THE MICROMASER

A detailed quantum fluctuation theorem involves relating the probability \( P[\gamma] \) to observe a quantum trajectory \( \gamma \) in the forward time direction to the probability \( P[\tilde{\gamma}] \) of its time-reversed conjugate \( \tilde{\gamma} \). Such a theorem provides a measure of the irreversibility of the dynamics for a given trajectory. But in contrast to the previous result, where the notion of a time reversed trajectory emerged as a natural part of the calculation, there arises the matter of defining the time-reversed quantum jump operators in order to derive a detailed fluctuation theorems, in the form of the ratio Eq. (45).

The Crooks approach is not without its difficulties as found for instance in application to driven quantum systems, or systems for which there is no steady state fixed point. Problems of a different nature arise here,
traced to the fact that the cavity field is coupled to two reservoirs. If the dual operators are defined with respect to the steady state $\rho_{ss}$, no meaningful fluctuation relation arises. This difficulty has been discussed in [60], and the argument can be made that the time-reversed jump operators ought to be those that satisfy the Crooks condition (that the forward and reversed trajectories have the same probability at steady state) for each jump operator in the presence of its associated reservoir only. Thus, for the jump operators associated with the coupling of the cavity field to the thermal reservoir, we will require that the duals to the jump operators $C_{-1} = \sqrt{\gamma/(n+1)}a$ and $D_1 = \sqrt{\gamma/na}$ be given by

$$\tilde{C}_{-1} = \Theta \rho_{ss}^{1/2} C_{-1}^{\dagger} \rho_{ss}^{-1/2} \Theta \big|_{R=0} = D_1 = e^{-\hbar \omega/2kT_a} C_{-1}^{\dagger} \big|_{R=0} = e^{-\hbar \omega/2kT_a}$$

and

$$\tilde{D}_1 = \Theta \rho_{ss}^{1/2} D_1^{\dagger} \rho_{ss}^{-1/2} \Theta \big|_{R=0} = C_{-1} = e^{\hbar \omega/2kT_a} D_1^{\dagger} \big|_{R=0} = e^{\hbar \omega/2kT_a}$$

where $\rho_{ss}|R=0 = (1 - e^{\hbar \omega/kT_a}) e^{-N \hbar \omega/kT_a}$ is cavity field steady state in the absence of an atomic beam reservoir. For the jump operators associated with an atom passing through the cavity, $C_1 = \sqrt{\rho_{ss}(\sin(\theta \sqrt{N})/\sqrt{N})^{a}_1}$ and $D_{-1} = \sqrt{1 - p_{ss}(\sin(\theta \sqrt{N + 1})/\sqrt{N + 1})a}$ we find

$$\tilde{C}_1 = \Theta \rho_{ss}^{1/2} C_1^{\dagger} \rho_{ss}^{-1/2} \Theta \big|_{\gamma=0} = D_{-1} = e^{\hbar \omega/2kT_a} C_1^{\dagger} \big|_{\gamma=0}$$

and

$$\tilde{D}_{-1} = \Theta \rho_{ss}^{1/2} D_{-1}^{\dagger} \rho_{ss}^{-1/2} \Theta \big|_{\gamma=0} = C_1 = e^{-\hbar \omega/2kT_a} D_{-1}^{\dagger} \big|_{\gamma=0}$$

where, from Eq. (34), $\rho_{ss}|_{\gamma=0} = (1 - e^{\hbar \omega/kT_a}) e^{-N \hbar \omega/kT_a}$ is the cavity field steady state in the presence of the atomic beam only. The remaining operators $C_0$ and $D_0$ are unaffected.

With these results at hand it is now straightforward to determine the ratio of the forward and backward trajectory probabilities. Thus we wish to compare the two conditional probabilities $P[\gamma_{mn}]$ for the forward trajectory $\gamma_{mn}$ and $P[\tilde{\gamma}_{nm}]$ for the time reversed dual trajectory $\tilde{\gamma}_{nm}$ (i.e., excluding boundary terms [60] depending on the probabilities of the initial states of the forward and reverse processes):

$$\frac{P[\gamma_{mn}]}{P[\tilde{\gamma}_{nm}]} = \frac{|\langle n | e^{-iH_c(t-t_a)} L_{i_1} \ldots L_{i_k} e^{-iH_c t_1} | m \rangle|^2}{|\langle m | e^{-iH_c(t-t_a)} \tilde{L}_{i_k} \ldots \tilde{L}_{i_1} e^{-iH_c t_1} | n \rangle|^2}$$

where $L_{i} \in \{C_1, D_1\}$. Each pairing in the numerator and denominator of the operator $C_1$ and its dual $D_{-1}$ will contribute a factor $e^{-\hbar \omega/kT_a}$, $D_1$ and its dual $C_{-1}$ a factor $e^{\hbar \omega/kT_a}$ and so on. As the number states are eigenstates of $H_c$, and the jump operators map number states into number states, the remaining factors in the numerator and denominator cancel exactly and we are left with

$$P[\gamma_{mn}] = \exp[(\Delta E_a(\gamma)/kT_a + \Delta E_c(\gamma)/kT_c)]P[\gamma_{mn}]$$

where $\Delta E_a(\gamma)$ and $\Delta E_c(\gamma)$ are the total energies gained by the cavity field through cavity reservoir and the atomic beam induced quantum jumps, respectively in the forward process.

This result is independent of the Rabi factors so holds true irrespective of the strength of the coupling of the field to the atoms, i.e., it is not a weak system-reservoir interaction result. It is also independent of the initial and final states, a general result not specific to the micromaser [60], and as such is dependent solely on the history of quantum jumps, so the subscripts $m$ and $n$ can be suppressed. Finally, there is no dependence on the atomic arrival times $t_1, t_2 \ldots$. Thus the above result will remain true even if the arrival times are described by a more general process (e.g., a renewal process) as it is meaningful to unravel the system dynamics as an ensemble of quantum trajectories in spite of the master equation being non-Markovian, as discussed in Section III B.

The ratios $-\Delta E_c/T_c$ and $-\Delta E_a/T_a$ can be recognized as the entropy flows $\Delta S_c$ and $\Delta S_a$ from the cavity reservoir and atomic beam respectively, so that we have

$$P[\gamma] = e^{-(\Delta S_c(\gamma) + \Delta S_a(\gamma))/k} P[\gamma]$$

or equivalently

$$P[\gamma] = e^{-(\Delta S_c(\tilde{\gamma}) + \Delta S_a(\tilde{\gamma}))/k} P[\gamma]$$

as an example of a general set of quantum trajectory derived fluctuation theorems [54, 55, 56].

It should be made clear that the time reversed trajectories are explicitly constructed for comparison with the forward trajectories, but both represent possible physically realisable forward trajectories. Thus the comparison of the probability of the two trajectories embodied in Eq. (71) is a comparison of two forward trajectories, with one having a reversed sequence of quantum jumps. So this final result tells us that the trajectory for which the total entropy change $\Delta S_a + \Delta S_c$ is greater than zero will be exponentially more likely than its reversed counterpart, an outcome consistent with the second law.

VIII. SUMMARY AND CONCLUSIONS

The micromaser, an early example of a collisional or repeated interaction model of an open quantum system, has been investigated here with attention paid to an understanding of its properties based on a time-reversed quantum trajectory analysis. The master equation for a general impulsive repeated interaction model was derived in the general setting of a renewal process describing the interaction time statistics. The approach developed makes it possible to show the impact of the underlying assumption of stationary statistics of these interaction times, edifying the differences between this approach and those of other researchers, most recently [1, 10], that yield somewhat different master equations, and showing that they are asymptotically in agreement.
This work then provided a background in which the notion of time-reversed quantum trajectories (TRQT) could be formulated for the micromaser. The Crooks prescription [13] for constructing TRQTs is then shown to arise in a natural fashion (i.e., no put in ‘by hand’) when analysing the relationship between micromaser field correlation properties and those of the atomic beam, a relationship that breaks down if the incident atomic beam is not Poissonian.

Attention was then given to using the Crooks approach to define a class of TRQTs suitable for investigating thermodynamic quantum trajectory fluctuation relations for the micromaser in the sense of introduced by [54]. However, in this case, it is now necessary to define what is meant by a time-reversed quantum trajectory. In other words, in contrast to the previous instance, a time-reversed quantum trajectory has to be constructed ‘by hand’. The fact that the micromaser is a system interacting with two reservoirs also implies that further care be taken in constructing the TRQTs [60]. The results show that neither the strong coupling of the micromaser field to the atoms, nor non-Poissonian arrival statistics has any impact on the fluctuation relation, which is of the generally expected form.

Further work can focus on the detailed thermodynamic properties of the micromaser, in particular with respect to the action of the atomic beam, the circumstances under which it can be considered a work reservoir or a thermal reservoir [40]. And of course, a natural extension of this work would be to those cases in which the atomic beam possesses coherence, or to the phaseonium model of [62].

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enables Eq. (11) to be written in terms of $\sigma(t)$ as

$$\dot{\rho}(t) = \mathcal{L}_S \rho(t) + R\mathcal{F}_a \sigma$$  \hspace{1cm} (A2)$$

Taking the Laplace transform of Eq. (A2) yields, with $\bar{s} = s - \mathcal{L}_S$$

$$\bar{s}\dot{\rho}(s) - \rho(0) = R\mathcal{F}_a \tilde{\sigma}(s)$$  \hspace{1cm} (A3)$$

while for Eq. (A1) we get

$$\tilde{\sigma}(s) = (1 - R\bar{g}(\bar{s})\mathcal{F}_a)^{-1} \bar{s}^{-1} \rho(0).$$  \hspace{1cm} (A4)$$

Combining Eq. (A3) and (A4) by eliminating $\rho(0)$ then gives $\tilde{\sigma}(s) = \tilde{K}(\bar{s}) \tilde{\rho}(s)$ where

$$\tilde{K}(\bar{s}) = (1 - (\bar{g}(\bar{s}) - s^{-1})R\mathcal{F}_a)^{-1}.$$  \hspace{1cm} (A5)$$

Substituting this into Eq. (A3) and inverting the Laplace transform then gives the required master equation

$$\frac{d\rho}{dt} = \mathcal{L}_S \rho + R\mathcal{F}_a \int_0^t e^{\mathcal{L}_S(t-\tau)}K(t-\tau)\rho(\tau)d\tau.$$  \hspace{1cm} (A6)$$

Appendix B: Expansion of non-Markovian master equation

The Laplace transform density operator $\tilde{\rho}(s)$ can be written

$$\tilde{\rho}(s) = \left(\bar{s} - R\mathcal{F}_a \tilde{K}(\bar{s})\right)^{-1} \rho(0)$$ \hspace{1cm} (B1)$$

which on substituting for $\tilde{K}$, Eq. (A5) leads to

$$\tilde{\rho}(s) = \left(1 + \bar{s}^{-1}R\mathcal{F}_a (1 - \bar{g}(\bar{s})R\mathcal{F}_a)^{-1}\right) \bar{s}^{-1} \rho(0).$$ \hspace{1cm} (B2)$$

From the defining equation for $g(t)$, Eq. (14), we have

$$R\bar{g}(\bar{s}) = \frac{\bar{w}(s)}{1 - \bar{w}(s)}$$ \hspace{1cm} (B3)$$

which can be used to reduce the expression for $\tilde{\rho}(s)$ to

$$\tilde{\rho}(s) = \bar{s}^{-1} \left(1 - R\frac{1 - \bar{w}(\bar{s})}{\bar{s}}\right) \rho(0)$$

$$+ \frac{1 - \bar{w}(\bar{s})}{\bar{s}}(1 + \mathcal{F}_a)(1 - \bar{w}(\bar{s})(1 + \mathcal{F}_a))^{-1} R\frac{1 - \bar{w}(\bar{s})}{\bar{s}} \rho(0).$$ \hspace{1cm} (B4)$$

Inverting the Laplace transform then gives
\[ \rho(t) = e^{L_{\sigma}t} \left( 1 - \int_0^t d\tau R \left( 1 - \int_0^\tau d\tau' w(\tau') \right) \right) \rho(0) \\
+ \sum_{n=1}^\infty \int_0^t dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1 \\
\times \left( 1 - \int_0^{t-t_n} d\tau w(\tau) \right) w(t_n - t_{n-1}) \ldots w(t_2 - t_1) R \left( 1 - \int_0^{t_1} d\tau' w(\tau') \right) \\
\times e^{L_{\sigma}(t-t_n)} (1 + F_a) e^{L_{\sigma}(t_n-t_{n-1})} (1 + F_a) \ldots (1 + F_a) e^{L_{\sigma}t_1} \rho(0). \] (B5)

The interpretation of this expansion is given in Section IIIA.