Effective Gauge Theories of Spin Systems

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A large variety of microscopic gauge theories can be written for antiferromagnetic spin systems, including $U(1), SU(2)$, and $Z_N$. I consider the question of the appropriate effective gauge theory for such systems. I show that while an $SU(N)$ anti-ferromagnet can be written microscopically as a $Z_N$ gauge theory, for unfrustrated systems, with a two-sublattice structure, there is always an effective $U(1)$ gauge field. The dispersion relation for the gauge field is shown to depend on the presence or absence of charge-conjugation symmetry. Frustrated systems can break the gauge group to a discrete group, but this appears to always involve introducing a gap for the spinons.

I. INTRODUCTION

Spin systems naturally give rise to dynamics on a constrained Hilbert space. The spin-1/2 Heisenberg antiferromagnet arises from a Hubbard model of fermions, constrained to single occupancy per site by strong on-site repulsion, while a dimer model leads to a space with each site participating in one dimer. In consequence, it is not surprising that a gauge theory should be the correct description. A variety of effective gauge theories have been suggested, including the $U(1)$ gauge theory [1], the $SU(2)$ gauge theory [2,3], and the $Z_2$ gauge theory [4].

It has been shown that the $Z_2$ gauge symmetry is correct for short-range RVB states on a triangular lattice [5], and that there may be a deconfined phase. A similar argument has been made by considering dimer states on the square lattice with diagonal bonds [6]. Finally, a $Z_2$ symmetry has been shown for the $SU(2)$ model on the square lattice with diagonal bonds [7]. For unfrustrated systems with holes, a $Z_2$ gauge theory has again been proposed [8], though it has been criticized on the grounds of neglecting a $U(1)$ symmetry [9]. Emphasizing the different possible gauge theories for the same model, while the Rokhsar-Kivelson (RK) model on a square lattice can be described by a $U(1)$ gauge theory [10] and has gapless excitations suggestive of a $U(1)$ theory, there is an exact microscopic mapping to a $Z_2$ gauge theory [11]. The different behavior of the excitations on square and triangular lattices suggests that discrete gauge symmetries require frustration.

To better understand the correct gauge theory description, I consider an $SU(N)$ antiferromagnet. I assume that the ground state is known, and consider possible excitations above the ground state in a single-mode approximation. I show that an unfrustrated system with a two-sublattice structure there is always an effective $U(1)$ symmetry, with the dispersion relation for the excitations depending on the presence or absence of charge-conjugation symmetry. In contrast, I show that microscopically a $Z_N$ gauge theory suffices, but I point out the difficulty in the microscopic derivation of the gauge theory. However, I show that for frustrated systems one may generate an effective $Z_2$ (or other discrete group) gauge theory.

II. TWO-SUBLATTICE $U(1)$ GAUGE SYMMETRY IN DIMER STATES

The Affleck-Marston (AM) large $N$ theory [12] and the RK model both have gapless excitations, with differing dispersion relations. I will show that the AM theory can also be written as a model of dimers, and that an appropriate generalization of the RK single-mode operator acting on the AM ground state produces exactly the AM gauge excitations. The different dispersion relation is related to the presence or absence of charge-conjugation symmetry.

Consider a square-lattice, or other two-sublattice, $SU(N)$ antiferromagnet. Label the sublattices A and B. Use the fermionic representation of spins following Affleck and Marston [1]. Consider a Hamiltonian

$$H = -\sum_{i,j} \left( \frac{J}{N} |\psi^\dagger \psi|^2 - \frac{j}{N^3} |\psi_i^\dagger \psi_j^\dagger|^4 \right),$$

where the Hamiltonian acts on nearest neighbor lattice sites $i, j$. Let $N - m$ fermions be placed on each site in sublattice A and $m$ fermions be placed on each site in sublattice B. Two possibilities are of interest, depending on $m$. If $m = N/2$, charge conjugation symmetry is present and, depending on the ratio $J/J$, the large $N$ mean-field solution of Hamiltonian (1) leads to either a state with gapless spinons in the $\pi$-flux phase [12] or to a state with spin-Peierls order and gapped excitations. If $m \neq N/2$, the spinon spectrum is always gapped; for $m < N$, there is an effective dimer model [13].

For either possibility, we may make a particle-hole transformation on the sites in the A sublattice (after transformation, we have an $Sp(N)$ system [13]). The Hamiltonian becomes

$$H = -\sum_{i,j} \left( \frac{J}{N} |\psi^\dagger \psi|^2 - \frac{j}{N^3} |\psi_i^\dagger \psi_j^\dagger|^4 \right).$$

Consider a dimer state

$$\prod_n (\hat{c}_n, \hat{c}_n^\dagger) \equiv \prod_n (\psi_n^\dagger \psi_n^\dagger) |0\rangle,$$

where $|0\rangle$ is the no-particle vacuum state. All the sites $\vec{i}$ are on the A sublattice and all the sites $\vec{j}$ are on the B...
sublattice. Each site participates in \( m \) dimers. Hamiltonian (4) acts on a dimer state to produce another dimer state. Further, the Hamiltonian preserves the sublattice structure in the (non-orthogonal) dimer basis.

Rokhsar and Kivelson [3] noticed the existence of gapless “resonon” modes in a dimer model at the RK point where the wavefunction is an equal amplitude superposition of dimer states. These modes may be obtained by introducing a gauge field \( \vec{A}(\vec{x}) \), where \( \vec{x} \) is a point on the dual lattice. The ground state wavefunction, \( \Psi_0 \), is a superposition of different dimer states \( \Psi_a \) with amplitudes \( A_a \). Take each dimer state,

\[
\Psi_a = \prod_n (\hat{t}_{n,a}, \hat{\tau}_{n,a}),
\]

and change the phase of every dimer in that state as

\[
(\vec{i}, \vec{j}) \rightarrow e^{i\hat{A}(\vec{i}, \vec{j})/2} (\vec{i}, \vec{j}) (\vec{i}, \vec{j})^\dagger.
\]

In the limit that \( A \) is small, the superposition of these dimer states, orthogonalized with respect to \( \Psi_0 \), is

\[
\sum_a A_a \sum_{n'} A(\vec{i}', \vec{j}') e^{i\hat{A}(\vec{i}', \vec{j}')/2} (\vec{i}', \vec{j}') - (\vec{i}, \vec{j}) \prod_n (\hat{t}_{n,a}, \hat{\tau}_{n,a}).
\]

Choosing \( \hat{A}(\vec{x}) = A e^{i\vec{k} \cdot \vec{x}} \), with \( \vec{k} \perp \vec{A} \), this is an excitation above the ground state with momentum \( k \). Rokhsar and Kivelson showed that this mode has dispersion \( E \propto k^2 \), under the assumptions that the ground state \( \Psi_0 \) was an equal amplitude superposition of dimer states and that the overlap between dimer states can be ignored (valid in the large \( N \) limit).

Similarly, Affleck and Marston found gapless \( U(1) \) gauge fields about the large \( N \) flux-phase solution with \( m = N/2 \). In the flux-phase, the Hamiltonian (4) is decoupled by a field \( t_{ij} \); the mean-field solution is described by the hopping Hamiltonian

\[
H = \sum_{i,j} t_{ij}^{\text{flux}} \psi_i^\dagger \psi_j + \text{h.c.}
\]

with a mean-field solution \( t_{ij}^{\text{flux}} \). After the particle-hole transformation, the ground state of Hamiltonian (4) is described by a collection of Cooper pairs with particles in the pair on opposite sublattices. Therefore, the solution of Hamiltonian (4) can be described as a superposition of dimer states \( \Psi_a \) given by equation (6), with amplitudes \( A_a \); due to the mean-field nature of the solution, the number of dimers that each lattice site participates in is not fixed.

For \( m \neq N/2 \), the mean-field Hamiltonian (4) has an additional chemical potential term, alternating on the two sublattices. After the particle-hole transformation, this state, too, can be described by a projection of dimer states. In the limit of large chemical potential, the AM state evolves into the ground state of the RK point.

The fluctuations in the amplitude of \( t \) are gapped, but there are gapless phase fluctuations, which give rise to a \( U(1) \) gauge field. It may be shown that the solution of the mean-field Hamiltonian in the presence of a transverse gauge field (14) reduces to, for weak fields, a superposition of states given by equation (6). Intuitively this may be understood as follows: a fluctuation in \( t \) which is pure gauge, so that

\[
t_{ij} = t_{ij}^{\text{flux}} e^{i(\theta_i - \theta_j)},
\]

multiplies the Green’s function by \( e^{i(\theta_i - \theta_j)} \) and hence multiplies each dimer by \( e^{i(\theta_i - \theta_j)} \). Taking \( \theta_i = \vec{A} \cdot \vec{x}_i \), and then letting \( A \) become space-dependent yields equation (6). Thus, not only does the AM ground state evolve into the RK ground state as \( m \) varies, but the gauge excitations also are connected.

To elucidate why the AM gauge field has \( E \propto |k| \) while the RK gauge field has \( E \propto k^2 \), consider excitations in single-mode approximation about an arbitrary dimer state, assumed to be the ground state of some unspecified Hamiltonian that preserves the sublattice structure of dimer states. Take a gauge field excitation at wavevector \( k \), while the dimers have range \( l \). The dispersion of the gauge field will depend on the presence or absence of charge conjugation symmetry.

For a given dimer state, \( \Psi_a \), define the “current” \( \hat{J}(\vec{i}) \), for \( \vec{i} \) on sublattice \( A \), to be equal to the sum over sites \( \vec{j} \) on sublattice \( B \), such that \( \vec{i} \) and \( \vec{j} \) are connected by a dimer, of \( \vec{i} - \vec{j} \). Then, the state (6) can be obtained from the ground state by acting with operator

\[
O(\vec{k}, \vec{A}) = \sum_{\vec{i}} \hat{A} \cdot \hat{J}(\vec{i}) e^{i\vec{k} \cdot \vec{x}}.
\]

The energy of the excited state is equal to \( f(O)/s(O) \), where \( f(O) \) is the oscillator strength \( f(O) = \frac{1}{2} \langle \Psi_0 | O \dagger (H, O) | \Psi_0 \rangle \), and the structure factor \( s(O) = \langle \Psi_0 | O \dagger O | \Psi_0 \rangle \).

If we sum the current \( \hat{J}(\vec{i}) \) over an area \( X \) with length scale much longer than \( l \), we obtain

\[
\sum_{\vec{i}} \hat{J}(\vec{i}) = \sum_{\vec{j}} N(\vec{j}) \vec{j} - \sum_{\vec{i}} N(\vec{i}) \vec{i},
\]

where the sums extend over \( \vec{i}, \vec{j} \) in \( X \) and \( N(\vec{i}), N(\vec{j}) \) is defined to be the number of dimers which connect \( \vec{i} \) or \( \vec{j} \) to a site outside of \( X \). Eq. (11) implies a conservation law for the current on length scales greater than \( l \), as nonvanishing contributions to the total current arise only for \( \vec{i}, \vec{j} \) near the boundary of \( X \).

First we consider \( m << N \), so that the overlap between dimer states may be ignored. Overlap produces ferromagnetic correlations between spins on the same sublattice, absent in the infinite \( N \) limit. The overlap between
dimer states leads to a loop gas \([13]\); for large \(l\) and small \(N\) there is a phase with arbitrarily long loops. The most likely result of this phase transition is long range Néel order. Restricting our attention to systems without long-range order permits us to ignore this possibility.

Suppose \(k^{-1} \gg l\). The oscillator strength \(f(O)\) must vary as \((\bar{k} \times \bar{A})^2\): the vanishing of \(f(O)\) for longitudinal excitations is due to current conservation, while the \(k^2\) variation is required by analyticity and \(f(O) = 0\) at \(k = 0\).

The longitudinal structure factor vanishes for small \(k\). The transverse structure factor vanishes exactly for certain translational symmetry-breaking states, such as a columnar state in a dimer model, as well as for C-breaking states \([14]\) in which, after the particle-hole transformation, each lattice site participates in \(m/4\) dimers with each of its four neighbors. Even in the event of weak fluctuations about these states, if the long-range dimer order is preserved the structure factor vanishes as \(k^2\), leading to a gap to gauge modes. Conversely, if the transverse structure factor vanishes at \(k = 0\), on length scales much greater than \(l\) there is no net current. If two holon excitations are created in the system on opposite sublattices (these consist of lattice sites with \(2\) holon excitations are created in the system on opposite sublattices (these consist of lattice sites with \(m - 1\) dimers), current conservation requires that there be a net current of \(1\) along a string connecting the two sites, which implies that along the string the system is in an excited state, so that the energy of the state is proportional to the spinon separation. Thus, we argue that the vanishing of the transverse structure factor implies confinement of holons. However, a nonvanishing structure factor implies gapless gauge excitations with \(E \propto |k|^2\).

Now we consider \(m \approx N/2\); since overlap in the dimer basis becomes important, we use the fermion basis. This gives the dispersion for the AM gauge field, taking the ground state to be fermions with eq. \(\text{(7)}\), and excitation given by eq. \(\text{(10)}\). For \(m = N/2\) we will obtain the same \(E \propto k^2\) dispersion relation as was found previously by integrating over the fermions: the oscillator strength is proportional to \((\bar{k} \times \bar{A})^2\), but the structure factor will be proportional to \(|k|^2\).

The operator that creates state \(\text{(10)}\) from the ground state is

\[
O(k) = \tilde{A}(\bar{i} + \bar{j}) \cdot (\bar{i} - \bar{j}) \times a^\dagger(\bar{i})a^\dagger(\bar{j})a(\bar{i}^{\prime})a(\bar{j}^{\prime})\psi(\bar{i}, \bar{j})\psi(\bar{i}^{\prime}, \bar{j}^{\prime}).
\] (11)

Here \(\psi\) is the Cooper pair wavefunction. Undoing the particle-hole transformation so that there is a Fermi sea, the destruction of the Cooper pair creates a particle-hole excitation at opposite momenta: \(k_1, -k_1\). The creation of the Cooper pair creates a particle-hole excitation at momenta \(k_2 + k/2, -k_2 + k/2\). Both \(k_1, k_2\) lie within the Fermi sea. Therefore, \(k_1 = \pm k_2 + k/2\), with \(-k_1\) within the Fermi sea and \(\mp k_2 + k/2\) outside the Fermi sea, so that \(O(k)\) creates one particle and one hole. The phase space volume for this is proportional to \(|k|\), giving the desired result for the structure factor. To reconcile this result with the nonvanishing structure factor found above in the dimer basis, one must include overlap; the structure factor of \(|k|\) found in the fermion basis for \(m = N/2\) is a result of cancellations due to overlap.

For \(m \neq N/2\), there are two separate bands, and, using the fermion basis, the structure factor is constant for small \(k\), leading to \(E \propto k^2\). Equivalently, the cancellations due to overlap in the dimer basis are no longer exact for \(m \neq N/2\) and the structure factor becomes constant. For \(m \neq N/2\), after integrating over fermions the effective action for the AM gauge field has a Maxwell term, so that the quadratic dispersion relation is surprising; however, the gauge field is coupled to an alternating charge density on each site, producing a term in the action proportional to \((\bar{A}(\bar{i}) - \bar{A}(\bar{j}))\), the difference in the time component of the gauge field on the two sublattices. Such a term would otherwise be forbidden by charge-conjugation symmetry.

As a result, there is a flux in the gauge field from sites on the A sublattice to the B sublattice. The dynamics in this space of states, in which there is a given amount of flux leaving each site on the A sublattice going to a site on the B sublattice, is closely related to the dynamics of a dimer model and gives the \(k^2\) dispersion.

### III. Microscopic Gauge Theories

Despite the \(U(1)\) gauge symmetry of the effective theory of an unfrustrated anti-ferromagnet, the microscopic field theory may be written as a \(Z_N\) gauge theory. Formally, a system with Hamiltonian \(H\) given by equation \(\text{(1)}\) can be written as a functional integral

\[
Z = \int [d\psi^\dagger][d\psi][dt_{ij}]e^{-S[\psi,\psi^\dagger, t_{ij}]},
\] (12)

with action

\[
S = \sum_{ij} \psi_i^\dagger (\partial_t \delta_{ij} + t_{ij}) \psi_j^\dagger + f(|t_{ij}|)dt.
\] (13)

for an appropriate function \(f\). The decoupling field, \(t\), obeys \(t_{ij} = \overline{t_{ji}}\). However, in contrast to the large \(N\) AM gauge theory \(\text{(1)}\), for finite \(N\) the above decomposition is still valid if the variables \(t_{ij}\) take only the values \(t_{ij} = |t_{ij}|e^{\theta_{ij}}\), with \(\theta\) an integer multiple of \(2\pi/N\).

Then, by integrating out the fermions, a plaquette action is induced for \(t\), yielding a \(Z_N\) gauge theory. However, as argued above, there is always a \(U(1)\) symmetry present for such an antiferromagnet on a bipartite lattice. In consequence, the microscopic derivation of the gauge theory should not be trusted.

The field \(t_{ij}\) is conjugate to \(\overline{\psi_i^\dagger \psi_j^\dagger}\). Within an \(SU(N)\) theory, one finds that \((t_{ij})^N\) vanishes within the \(m\)-particle-per-site Hilbert space for any \(m < N\). This
is the formal device that lets one replace $t$ with a $Z_N$ gauge field. However, plaquette operators for $t$ are generated under an RG. There are two possible forms of these. One is $t_{ij}t_{jk}t_{kl}t_{li}$ around a plaquette. For $SU(2)$, this operator is self-cube \([17]\), which suggests a $Z_2$ gauge theory. However, another, more symmetric, plaquette term is $\left( S_{ij}^{\mu} S_{jk}^{\nu} S_{kl}^{\rho} S_{li}^{\sigma} \right)$, where the spin operator $S = \psi_i^{\dagger} \psi_i - \frac{1}{2} \delta^{\mu\nu}$. This operator cyclically permutes the spins on sites $i, j, k, l$. The fourth power of this operator is equal to unity, which suggests that the gauge symmetry is at least $Z_4$; by continuing to larger traces of spin operators, the full $U(1)$ symmetry is restored.

IV. FRUSTRATED ANTIFERROMAGNETS

In contrast to this result for unfrustrated antiferromagnets, a frustrated antiferromagnet may have a discrete gauge symmetry. Due to the lack of a two-sublattice structure, the type of $U(1)$ symmetry considered above is not possible.

On the two-sublattice systems, one may take conjugate representations of $SU(N)$ on alternate sublattices. For a frustrated system, this is not possible, and one must either use the group $Sp(N)$ \([3]\), or use the self-conjugate $SU(N)$ system with $m = N/2$. The two-sublattice $U(1)$ symmetry we consider above is a $U(1)$ symmetry between conjugate representations of $SU(N)$; for a frustrated $SU(2)$ theory, in addition to forming singlet operators $\psi_{\vec{i}}^{\mu} \psi_{\vec{j}}^{\nu}$ between sites $\vec{i}, \vec{j}$ on opposite sublattices, one may also form singlets $\epsilon^{\mu\nu} \psi_{\vec{i}}^{\mu} \psi_{\vec{j}}^{\nu}$ on the same sublattice, destroying the $U(1)$ symmetry. In general, the symmetry is broken to $Z_N$.

Examples of effective $Z_2$ theories include the dimer model on the triangular lattice \([3]\) and the frustrated square lattice $SU(2)$ theory \([2]\). These theories lead to gapped, deconfined spinons. No frustrated system with gapless spinons has been found. From the point of view of the $SU(2)$ mean-field theory, this is to be expected. For the unfrustrated square-lattice, the $\pi$-flux phase enlarges the unit cell of the lattice to two sites, so there is an integer number of each flavor of spinon per unit cell. One would expect such a system to be a band insulator; however, the two-sublattice structure leads to a zero-energy state and keeps the spinons gapless. As soon as this structure is lost, the spinons are expected to become gapped.

V. CONCLUSION

The effective gauge theories that describe spin systems have been considered from excitations above the ground state. I have argued that, despite the possibility of a microscopic $Z_N$ gauge theory, there is always an effective $U(1)$ symmetry for unfrustrated spin systems. While for frustrated spin systems an effective $Z_2$, or other discrete gauge symmetry, is possible, this appears to coincide with a gap to spinon excitations.

The discussion above, however, is specific to spin systems. It is possible for systems with holes to break the symmetry to $Z_2$ as suggested by Senthil and Fisher \([4]\).

The effect of the $U(1)$ gauge excitations is unclear. While Marston \([18]\) has argued that instantons in the $U(1)$ gauge theory of fermionic $SU(N)$ antiferromagnets do not necessarily lead to confinement, another possibility is that gauge theories of two dimensional antiferromagnets with gapless, deconfined spinons occur only as theories of a critical point, such as the RK point. In this case, while some examples may be found of gapless spinons in two-dimensions, there will always be relevant perturbations that will lead to some form of ordering and to spinon confinement.

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