FINITE TEMPERATURE MATRIX THEORY

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Abstract

We present the way the Lorentz invariant canonical partition function for Matrix Theory as a light-cone formulation of M-theory can be computed. We explicitly show how when the eleventh dimension is decompactified, the $\mathcal{N}=1$ eleven dimensional SUGRA partition function appears. We also provide a high temperature expansion which captures some structure of the canonical partition function when interactions amongst D-particles are on. The connection with the semi-classical computations thermalizing the open superstrings attached to a D-particle is also clarified through a Born-Oppenheimer approximation. Some ideas about how Matrix Theory would describe the complementary degrees of freedom of the massless content of eleven dimensional SUGRA are discussed. Comments about possible connections to black hole physics are also made.

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1 Introduction

It has been intense the attention that the proposal known as Matrix Theory \cite{1,2} has attracted from the community of physicists working on the field String Theory has become after the so called second string revolution. It seems that after a first boost of frenetic activity few physical quantities have been computed so as to test to what extent we can at least recover what we knew before from D-brane dynamics. At present, it could be considered pretentious trying to get exact non perturbative results from Matrix Theory. In Theoretical Physics, the way out of this kind of situations is to wait for a more powerful formulation of what we have at hand. Today’s form of Matrix Theory has some drawbacks. First of all the original formulation was one in the IMF. This means that one should take a subtle $N \rightarrow \infty$ limit to recover Lorentz covariance. A reformulation of the conjecture by using discrete light-cone quantization (DLCQ from now on) seems to provide a meaning for the computations at finite $N$. In DLCQ, it is the light-like coordinate $x^-$ that gets compactified. In this form, the conjecture would establish a correspondence between finite $N$ Matrix Theory and the DLCQ of M-theory. Support for this correspondence has been given in \cite{3}. Another related point is that the present form of Matrix Theory is not a background independent one. A list of Matrix Models for given backgrounds is in need and, up to now, things do not go very fast to this respect. Something as modest as a formulation of Matrix Theory for toroidal backgrounds is still lacking. Consequently, Matrix Theory might be nothing more than another step towards getting a more complete description of what in the past we called String Theory. It is then not clear how far we will be able to go.

It seems however that there are some ingredients that could show up as advantages. After all we have a Hamiltonian and perhaps this could be used to compute physical quantities that from their own definition are adapted to that formulation. This would be the case of the canonical partition function, $Z(\beta)$. This work will be devoted to computing this quantity, although the task will not be exactly accomplished. The interest of the work will be that of getting a better knowledge of what Matrix Theory Thermodynamics might be. From the very beginning it seems clear that with $T = 1/\beta$ as the thermodynamical temperature we use to define the canonical ensemble, the canonical partition function appears to have only sense in the DLCQ picture because the IMF formulation of the conjecture is actually throwing away energy configurations it seems one should take into account at finite temperature. We will focus on the case in which all spatial dimensions are open with the exception of $x^-$. One might argue that owing to the fact, also known in string perturbation theory, that compact spaces increase the number of degrees of freedom even to the extent of making difficult a formulation of a Matrix Model for $T^n$ for $n \geq 6$, one should compute $Z(\beta)$ for a $T^9$ space and get from here $Z(\beta)$ for $\mathbb{R}^9$. This prevention assumes that this tremendous amount of degrees of freedom do not fully disappear after decompactification.
Such an attitude is based on the extended misconception that a microcanonical description of perturbative string theory gives a decompactification limit which is different from that of the microcanonical picture gotten directly from \( Z(\beta) \) for an already open box (cf. [5]). Then there is no a priori reason to think that a formulation of Matrix Theory on a \( T^9 \) space, if there is such a thing, would not give what we have now as its decompactification limit. In fact, by computing the free energy of the D-strings, by the way, we will check that the extra degrees of freedom from the extra compact dimension decouple when its length goes to infinity.

For the time being, in section 2, we will expose the technique to compute \( Z(\beta) \) for a theory which is formulated in the light-cone frame. The main point is that if \( T = 1/\beta \) can be treated as the temperature one could measure with a thermometer, one has to admit that it has to transform as the energy, i.e. \( p^0 \).

In other words \( Z(\beta) = \text{Tr} e^{-\beta p^0} \) is a Lorentz invariant quantity we are going to compute knowing only a description in a light-cone frame. We will present this computation in the DLCQ form of the Matrix Theory conjecture. In section 3 we will study the limiting case in which one has free (classical geometry) configurations and what happens when off-diagonal terms (open stretching strings) are taken into account. In section 4, we will study a classical system of matrices or what more properly could be defined as the classical statistical mechanics associated with Matrix Theory and we will relate it to Matrix Theory at finite temperature by computing its quantum corrections. In section 5 we will perturbatively compute the partition function by a finite temperature extension of the computation in [8]. We will finally get out the conclusions in section 5.

### 2 The canonical free energy from a light-cone frame description

The canonical partition function for the simplest relativistic system of a gas of particles of mass \( m \) is by definition (see [9])

\[
Z(\beta) = \text{Tr} e^{-\beta p^0}.
\]  

(1)

This is an invariant quantity whenever we admit that the temperature \( T \) transforms as the energy, i.e. the zero component of the momentum. This is in physical terms a consequence of the interpretation of the temperature as measuring the average kinetic energy of the system. If the system is known in the light-cone frame, one can use the relation \( p^0 = \frac{1}{2}(p^+ + p^-) \) to write

\[
Z(\beta) = \text{Tr} e^{-\frac{1}{\beta}(p^+ + p^-)} = \text{Tr}_{p^+, p^-} \left[ e^{-\frac{1}{\beta}\left(p^+ + \frac{m^2}{p^+}\right)} e^{-\beta \frac{p^2}{2p^+}} \right].
\]  

(2)
Let us perform the trace by first taking a very illustrative path. One can compute a single particle partition function by performing the trace in (2) using a basis \( |p^+ > \otimes |\vec{p}_r > \) of single particle momentum eigenstates. If the space volume goes to infinity, we have

\[
\frac{L}{2\pi} \int_0^{+\infty} dp^+ e^{-\frac{\beta p^+}{2}} e^{-\frac{m^2}{2p^+}} \frac{L^{d_r}}{(2\pi)^{d_r}} \int d^{d_r} p \ e^{-\frac{\beta p^+}{2p^+}}.
\]  

(3)

The change of variables \( s = \frac{\beta}{p^+} \) and an elementary integration take us to

\[
\frac{L^{d-1}\beta}{(2\pi)^{\frac{d}{2}}} \int_0^{+\infty} ds \ s^{-\frac{d}{2}} e^{-\frac{\beta^2}{2s}} e^{-\frac{m^2}{2}}.
\]  

(4)

Where \( d \) is the space-time dimension, \( d = d_r + 2 \), and \( L \rightarrow +\infty \). The variable \( s = \frac{\beta}{p^+} \) is the Schwinger proper time, and we have the well known proper time representation of the Helmholtz free energy. Eq. (4) can be recognized as \( -\beta F_1(\beta) \equiv Z_1(\beta) \) for a particle of mass \( m \) and also as the same magnitude per degree of freedom for a quantum field with particle excitations of mass \( m \). In both cases one has Maxwell-Boltzmann statistics.

To get \( Z(\beta) \), i.e. the canonical partition function, a further exponentiation and another sum are needed (see \([10]\)). The relation, for particles obeying Bose statistics, is

\[
\ln Z(\beta) = \sum_{r=1}^{+\infty} \frac{Z_1(r\beta)}{r}.
\]  

(5)

The exponentiation gives the multi-particle system and the sum over \( r \) gives bosonic quantum statistics versus Maxwell-Boltzmann which corresponds in fact to the \( r = 1 \) term in the sum. So one has two aspects which, in principle, are different. One can have a multi-particle description but with Maxwell-Boltzmann statistics instead of Bose or Fermi quantum statistics. This subtlety shows up clearly as soon as one deals with the computation of the canonical partition function for string perturbation theory in the light-cone gauge. It is worth noticing here that the single-particle partition function depends linearly on the open (infinite) volume and then the multi-particle partition function depends exponentially on the volume.

Eq. (5) is the result of computing \( Z(\beta) \) as the trace of \( e^{-\beta p^0} \) using a basis, in the Fock space, of multi-particle eigenstates of the energy \( p^0 \) assuming Bose statistics. The rigorous way of getting (5) is assuming that our spectrum is actually discrete. Physically this can be achieved by enclosing the system in a finite volume box. One can then write

\[
Z(\beta) = \prod_{\vec{k}} \left( 1 \pm e^{-\beta \omega_{\vec{k}}} \right)^{\mp 1}.
\]  

(6)
Here $\omega_\vec{k}$ is the energy, that is usually understood as a dispersion relation and the signs distinguish bosons from fermions. Taking the logarithm of both sides, one gets the Helmholtz free energy as an infinite sum. It is from here that one gets the sum over $r$ by expanding $\ln (1 \pm x)$. Physically it is not necessary to assume that the compactified theory is known if one makes the hypothesis that, after taking the infinite volume limit that convert the sum over discrete momenta into an integral, the remnant degrees of freedom are those of the formulation of the system for an infinite box which is supposed to be what is known.

In quantum field theory one can check that $\ln Z (\beta)$ can also be obtained by a Euclidean path integral computation of the covariantly defined theory on $\mathbb{R}^{d-1} \times S^1_\beta$. After clarifying some questions related to modular invariance, the same correspondence will work for perturbative string theory, but a light-cone gauge computation based on the transverse Hamiltonian does not produce a multi-string partition function on its own. Let us explain what we mean. The starting point is again Eq. (1) which can be rewritten as

$$Z (\beta) = \frac{L}{2\pi} \int_{0}^{+\infty} dp^+ e^{-\frac{\beta p^+}{2}} z_T \left( \frac{\beta}{2p^+\alpha'} \right)$$

Here, $z_T$ is defined as a transverse partition function given by

$$z_T (\lambda) = \text{Tr} e^{-\lambda H_T}$$

with $H_T$ the light-cone gauge Hamiltonian associated with the light-cone gauge (transverse) action that for the bosonic part reads

$$S_{\text{bosonic}}^{\text{l.c.}} = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \partial_\alpha X^i \partial^\alpha X^i \quad i = 1, ..., 8$$

$\lambda$ is a sort (it is a dimensionless quantity) of inverse transverse temperature. With this action (and is fermionic part), $z_T$ can be computed by performing a Euclidean path integral with periodic (anti-periodic) boundary conditions in the transverse temperature $\lambda$ for the coordinates (32-spinor $S$) as fields in two dimensions. The result one finally obtains for a string one-loop computation is again a Maxwell-Boltzmann Helmholtz free energy proportional to the infinite volume in nine dimensions; i.e. a single string partition function $Z_{\text{1-string}} (\beta)$.

The supersymmetrical version of Eq.(3) gives the free energy for the multiple-string gas. In short, it can be written

$$\ln Z (\beta) = \sum_{r=\text{odd}>0} \frac{Z_{\text{1-string}} (\beta r)}{r}$$

Where one assumes that the single partition function includes the sum over Maxwell-Boltzmann contributions coming from the bosonic and fermionic degrees of freedom. It is worth noticing again that the single-string partition function depends linearly on the infinite volume while the multiple-string partition function
exponentially increases its free energy with the volume. In terms of the canonical entropy, it is clear that the Helmholtz free energy has to depend linearly on the volume as the entropy does.

Let us finally tackle our goal which is to get the form of the partition function for the Matrix Model. One first notes that in the DLCQ, $p^+$ is discrete because $x^-$ is a circle of radius $R$. From an eleven dimensional point of view, what one would first write down for the partition function of the model would be

$$Z_{\text{matrix}} = \sum_{N=1}^{\infty} e^{-\beta N} \text{Tr} e^{-\beta RH(N)}$$

where $H = RH(N)$ is the Hamiltonian for the SYM with $U(N)$ symmetry given, with the gauge $A_0 = 0$, by

$$H = R \text{Tr} \left( \frac{1}{2} \Pi_i^2 - \frac{1}{16\pi^2\alpha'} [Y^i, Y^j]^2 - \frac{i}{2\pi\alpha'} \frac{1}{r^3/2} \pi \gamma^i [Y_j, \theta] \right)$$

Physically Eq. (11) should be interpreted as the classical partition function for a single eleven dimensional object. The question is that we have to determine what is meant by the trace after the sum. After knowing the features Matrix Theory have revealed since the conjecture was formulated, it appears reasonable to us to study particular configurations in order to try to identify the single object one could use to make statistical mechanics the way it has been presented here for particles, fields and perturbative strings. At first sight one knows that in terms of ten dimensional physics one has systems of $N$ free D0-branes and also bound states at threshold formed by $N$ D0-branes. Let us then study the simplest picture which is that of the free (classical geometry) configurations in the Matrix Model.

3 Matrix Theory free configurations

A simple analysis of the Hamiltonian we have written down shows that in Matrix Theory there is no well defined meaning for the concept of position interpreted as a way of defining a standard geometry for the configuration space of the blocks we play with which are D0-branes. The reason is well known and results from the natural thickness of Dirichlet branes given by the fact that they have open strings on them and actually at ultrashort distances stretching strings can glue together two D0-branes, for example. These stretching strings actually introduce some kind of fuzziness that makes a classical geometrical interpretation for the configuration space of the branes an approximate concept. If the $N \times N$ matrices we use to write $H_{(N)}$ are diagonal, then a great simplification results because commutators get null and the Hamiltonian reduces to the simple form
\[ H_{\text{free}}^{(N)} = \text{Tr} \left( \frac{1}{2} \Pi_i^2 \right). \]  

(13)

with each \( \Pi_i \) being a diagonal \( N \times N \) matrix. It is clear because of the form of the Hamiltonian that the classical (geometry) configurations correspond to a free theory. The states we want to perform the trace over in Eq. (11) are \([\otimes_{i=1}^9 |p^i\rangle \otimes |\tau\rangle]_N\) with the subscript \( N \) indicating that the states are actually arranged in an \( N \) component vector for a given \( N \) and \(|\tau\rangle\) carries the Pauli spin information. With them, one can make the following guess

\[ Z_{1}\text{free} = \sum_{N=1}^{+\infty} e^{-\beta \frac{N}{\pi}} \text{Tr} e^{-\beta R H_{\text{free}}^{(N)}} = \sum_{N=1}^{+\infty} 256^N \frac{V_9^N}{N!} \left( \frac{2\pi}{R\beta} \right)^{\frac{9}{2}N} e^{-\frac{1}{2}R\beta}. \]  

(14)

Here the \( N! \) factor comes from the permutation group which is the discrete symmetry that survives from \( U(N) \) after truncating the matrices to the diagonal ones. \( V_9 \) is the spatial nine-dimensional volume.

Now the next step would be that of substituting this single partition function into Eq. (10) and getting \( Z(\beta) \) after an exponentiation. However there is something remarkable in Eq. (14). It has been constructed as a single particle function an as such should be identified with \(-\beta F_1(\beta)\) and then one would expect it to depend linearly on the volume and this is not the case. In fact one can write

\[ Z_{1}\text{free} = \exp \left[ 256 \frac{V_9}{(2\pi)^9} \left( \frac{2\pi}{R\beta} \right)^{\frac{9}{2}} e^{-\frac{1}{2}R\beta} \right] - 1, \]  

(15)

i.e., this partition function depends exponentially on the nine-dimensional volume. This is simply the result of the fact that the Matrix Model contains second quantization in itself in the sense that the first quantization of an object in eleven dimensions gives the description of ten dimensional multi-objects which are sets of free D0-branes. Looking at it this way, Eq. (14) would define the single object as one with single partition function

\[ \frac{256 V_9}{(2\pi)^9} \left( \frac{2\pi}{R\beta} \right)^{\frac{9}{2}} e^{-\frac{1}{2}R\beta}. \]  

(16)

The second quantization of this single object would produce composite systems of free D0-branes. In other words, a given total momentum would result from the sum of the individual momenta of each free D0-brane. Then the special bound states at threshold, special because they are bound states of \( n \) D0-branes but with zero relative energy, are missing and we are not taking into account that the given total momentum can also be shared by several bound states at threshold. The consequence is that we have to modify our single object to include these states and that is easy. Our single object will be a sum over partons of any
positive RR charge. This is similar to the image of the single fundamental string in the analog model as a collection of an infinite number of fields with masses running from zero to infinity. At last, the single object partition function will be

$$Z^\text{free}_1(\beta) = \sum_{k=1}^{+\infty} \frac{256 V_9}{(2\pi)^9} \left(\frac{2k\pi}{R\beta}\right)^\frac{9}{2} e^{-\beta \frac{k}{2R}}$$

(17)

With this, we can now apply the recipes in Eq. (10) to get

$$Z^\text{free}(\beta) = \exp \left[ \sum_{r=\text{odd}>0} \frac{1}{r} Z^\text{free}_r(\beta r) \right].$$

(18)

The relationship between $Z^\text{free}(\beta)$ and our starting point in Eq. (14) is very clear. Eq. (14) coincides with the $r = k = 1$ term in $Z^\text{free}(\beta)$ after expanding the exponential. In other words it is the multi-object Maxwell-Boltzmann version of our system without counting the D0-brane bound states at threshold.

We see in (17) that our single object is a superposition of bound-states composed by $k$ D-particles. The dynamical degrees of freedom of each parton does not contain the dynamics of its components, i.e. we do not have $V_9^k$ or any permutation property of the $k$ D-particles as in a standard gas with the same number of components; it only contains the dynamics of the center of mass. This feature is interesting because it does not prevent the interpretation of a given $k$-parton as a Schwarzschild black hole (see [6]). We will explain it in more detail in section 5.

It is necessary to check the nature of our single object by identifying into what the single partition function turns after taking the $R \rightarrow +\infty$ limit that would open the eleventh dimension up. It is an easy task to perform such limit because amounts to converting the infinite sum over positive integers $k$ into an integral. One has

$$\sum_{k=1}^{+\infty} \frac{256 V_9}{(2\pi)^9} \left(\frac{2k\pi}{R\beta}\right)^\frac{9}{2} e^{-\beta \frac{k}{2R}} \rightarrow \int_0^{+\infty} dk \frac{(2\pi R)}{2\pi} 256 \frac{V_9}{(2\pi)^9} \left(\frac{2k\pi}{\beta}\right)^\frac{9}{2} e^{-\beta \frac{k}{2}} =$$

$$256 \pi^{-11/2} \Gamma\left(\frac{11}{2}\right) V_9 (\beta)^{-10},$$

(19)

with $V_{10}$ the space volume in eleven dimensions. This, through Eq. (18), gives $-\beta F(\beta)$ for the massless field content of $\mathcal{N} = 1$ SUGRA in eleven dimensions. Our single object becomes an eleven dimensional supergraviton when opening up the light-like dimension as, may be, one could have expected from the very beginning. At finite temperature we have to sum over the complete tower of
longitudinal momentum modes, that is, we have to take into account the contribution of all the finite-N Matrix Models, this allows us to avoid the problem of relating the finite-N model to the eleven dimensional supergravity \cite{12}.

The question now is whether we can use all this analysis to treat interactions among the D0-branes. At first sight one would say that all the treatment for particles, fields, strings and the Matrix Model itself is only accurate for free single objects. We know this is not true for fields and strings. The proper time formalism is also accurate for including interactions perturbatively through corrections $\delta m^2$ to $m^2$. One only has to change the mass squared by the mass squared plus the loop corrections. This can be done for strings in the analog model in which the free energy is the sum over the field content of the string (cf. \cite{21}). We will actually see in section 5, that this can also be done for the Matrix Model in the approach in which a perturbative expansion around a fixed background is done. In some sense, what the light-cone description of M-theory should provide for the massless content of $\mathcal{N} = 1$ SUGRA is analogous to the massive tower of ten dimensional Planck masses that promote the ten dimensional supergravities to the vibrational modes of the full fundamental superstring theories. The Matrix Model interactions through stretching strings are the sources for these massive companions, although we do not see any a priori reason to believe that the relevant degrees of freedom could be described as some kind of massive quantum fields beyond this fixed background perturbation expansion approach.

4 Statistical mechanics of $U(\mathcal{N})$ matrices

We have already stressed as one of the advantages of the present formulation of Matrix Theory that it is based upon a relatively simple Hamiltonian. This provides us with a different way to do our computations without any need to do a hard path integral nor to restrict ourselves to a particular perturbation series that strongly depends on the background -the vacuum- we choose to expand around. At least, as we shall see, it is possible to find global, non perturbative features of the partition function that the field perturbation theory cannot capture.

The Hamiltonian formulation fundamentally differs from the Lagrangian one in the mathematical objects that contain the complexity of the system. In the Lagrangian formulation, the function that defines the system is the action, calculated over the trajectories or paths. This necessarily leads to a path integral if we want to calculate any physical quantity. On the other hand, in the Hamiltonian formulation, the system is defined by a hermitian operator acting over wave functions. Physical magnitudes are calculated through summations or integrals over real or Grassmann variables. The complexity is thus transported into the search for appropriate wave functions as well as their matrix elements with the operators that appear in the Hamiltonian.

Let us rewrite the transverse Matrix Theory Hamiltonian with every $\hbar$ and
\[ \alpha' \text{ in it} \]

\[ H = R \text{Tr} \left( \frac{1}{2h} \Pi^2_i - \frac{1}{16\alpha'\hbar^2} [Y^i, Y^j]^2 - \frac{i}{2\pi(\alpha'\hbar)^{3/2}} \pi \gamma^j [Y_j, \theta] \right), \quad (20) \]

with the constraint

\[ \pi = -i\sqrt{\frac{\hbar}{\alpha'}} \theta^i \quad (21) \]

Again, we have chosen the gauge \( A_0 = 0 \). Besides, one has to impose another condition that arises from the Gauss law. If we take the Arnowitt-Fickler gauge in the 9 + 1 theory, the condition after dimensional reduction to 0 + 1 is \( Y_9^a = 0 \).

The only consequence is that now, the bosonic index \( i \) takes values from 1 to 8. This gauge election does not need ghost fields. This is general for all the 'gluons' except for that related to the identity. This one decouples from all the rest and does not even appear in the Gauss constraints because trivially commutes with any other matrix.

More comments on the particular form of this Hamiltonian are also in need. Four different operators appear: the fermionic and bosonic positions \( \theta \) and \( Y \) and their conjugate momenta \( \Pi \) and \( \pi \). The commutation relations among them are various. Firstly, all of them are \( N \times N \) matrices and therefore they obey the \( U(N) \) algebra. Secondly, they are either bosonic or fermionic creation operators of one-dimensional world-line fields or, equivalently, operators with either real or Grassmann numbers as eigenvalues from the ten dimensional point of view. So they correspondingly commute or anti-commute according to their nature. Finally, each coordinate and its conjugate momentum have canonical Heisenberg commutation rules too.

Other noticeable characteristic of the Hamiltonian is the ubiquitous appearance of \( \hbar \) in every term. The physical origin of this fact is that, from the eleven-dimensional point of view, the BPS masses of the D0-branes are Kaluza-Klein energies and, therefore, purely quantum mechanical. Besides, let us remember that the Planck length in any dimension has also got a purely quantum mechanical nature. As it stands, it is impossible to say what is the limit of the Hamiltonian as \( \hbar \to 0 \), or even if there is any.

The first step we shall take is to expand the matrix operators in terms of the generators of the \( U(N) \) algebra in order to avoid complications related to their matrix nature. This yields

\[ \text{In the special case of a 0 + 1 dimensionally reduced theory in the hamiltonian formalism, the Gauss constraints can be completely written in terms of the structure constants of the gauge group, that is} \]

\[ G_\rho = F_\rho^{\mu} Y^\mu_j \Pi^j - iF_\rho^{\mu} \theta^\mu \theta^\nu. \]

Let us note that in the free case in which the gauge symmetry is broken to \( U(1)^N \) these conditions do not impose any extra constraint to the election \( A_0 = 0 \) and so we keep nine physical scalar fields for each D-particle. This justifies the calculation in section 3.
\begin{equation}
H = \frac{R}{2\hbar} \Pi^k_\lambda \Pi^k_\lambda - \frac{R}{16\pi\alpha}\mathcal{F}^{\mu\rho}F_{\rho}^{\tau\gamma}Y^{i}_\gamma Y^{j}_\gamma Y^{j}_\epsilon - \frac{iR}{2\pi(\hbar\alpha')^{3/2}} \pi^{\mu}Y^{i}_\gamma F_{\mu}^{\sigma\nu}\gamma_i \theta_{\nu}
\end{equation}

where $F_{\mu}^{\sigma\nu}$ are the structure constants of the gauge group. To make the trace over the exponential of this Hamiltonian we have to choose a base of wave functions. The most appropriate is formed with eigenvectors of the $16N^2$ fermionic coordinates and the $8N^2$ bosonic momenta. We could denote them by $|\Pi^k_\lambda \theta^a_\mu >$ where the indices mean that the state is defined with the $(16 + 8)N^2$ eigenvalues. This quantum numbers do not always completely determine the system. At least we know that when two or more of the coordinates coincide there appear bound states at threshold that cannot be described if we do not add another number. We shall ignore this fact since we shall not give precise numerical results in this section and this degeneracy would only alter the value of certain coefficients. The complete calculation we want to perform is

\begin{equation}
Z = \sum_{N=1}^{\infty} e^{-\beta \frac{N\hbar}{2\pi}} \int \left( \prod_{k,\lambda} d\Pi^k_\lambda \right) \left( \prod_{l,\mu} d\theta^l_\mu \right) < \Pi^k_\lambda \theta^a_\mu | e^{-\beta H(\Pi^k_\lambda \theta^a_\mu)} | \Pi^k_\lambda \theta^a_\mu >
\end{equation}

We shall first carry out the integral and then sum over the index $N$. Let us begin with the integral over the fermions

\begin{equation}
I_f = (\hbar\alpha')^{4(N^2-1)} \int d\theta e^{-\theta^a M^a_b \theta_b}
\end{equation}

with

\begin{equation}\{\theta^a, \theta^b\} = \delta^{ab} \sqrt{\hbar\alpha'}\end{equation}

and where the indices shown are labels related to the gauge group as well as to the sixteen different fermionic degrees of freedom. The constant before the integral comes from the relation between the coordinates and their momenta. Its function is to make the integral dimensionless.

\begin{equation}
M = i\beta \hbar^{-1}\alpha'^{-2} R \frac{1}{2\pi} \gamma_i \sigma^{\mu}\nu F_{\mu\nu}
\end{equation}

It is easy to expand the exponential in a Grassmann series. The matrix $M$ is $16(N^2-1) \times 16(N^2-1)$ so that the $8(N^2-1)^{th}$ term is the first to contribute. Here again, the fermions related to the identity are not taken into account because they do not appear in the Hamiltonian. As regards to this first term, we can proceed just ignoring all the anti-commutators because any term coming from them and depending on $\hbar$ cannot have all the variables we need for the integral not to be null. Therefore, it is possible to order the operators separating the coordinates from the momenta, and to convert it into a double integral of Grassmann numbers, not operators. The result is known to be
\[
\int d\theta \frac{1}{n!} \sum_{\text{combinations}} <\theta | \prod (-\theta, m_{ij} \theta_j) | \theta > = \int d\theta \frac{1}{n!} \sum_{\text{combinations}} \pm \left( \prod_{i=1}^{n} \theta_i \right) \left( \prod_{j=1}^{n} m_{ij} \right) \left( \prod_{j=1}^{n} \theta_j \right) = \det^{1/2} M. \tag{27}
\]

The next term does not contribute because there is a repeated coordinate. There are some terms with \(8(N^2 - 1) + 4\) operators that do contribute. Let us show how with an example

\[
\theta_1 m_{12} \theta_2 m_{21} \theta_1 [\cdots] = m_{12}^2 \frac{1}{2} \delta^2 \alpha' \theta_1 \theta_1 [\cdots] = m_{12}^2 \frac{1}{4} \delta \alpha' [\cdots]. \tag{28}
\]

The dots inside the square brackets stand for the terms that we have already calculated. This procedure can be generalized to give a series in \(\hbar\). Sadly, we cannot say much about the exact value of the integrals, so we write it this way

\[
I_f = \sum_{k=0}^{\infty} \hbar^{4(N^2-1)+k} \alpha'^4(N^2-1)+k D^{(8N^2-8+2k)}(y) \tag{29}
\]

where

\[
D^{(8N^2)}(y) = \det^{1/2} \left( i \hbar^{-1} \alpha'^{-2} R \frac{1}{2\pi} Y^i \gamma^\mu F_{\mu}^{\sigma\nu} \gamma_i \right) \tag{30}
\]

and each \(D^{(l)}\) is homogeneous of degree \(l\) in the coordinates and in general, in the elements of the matrix \(M\). With this, we have exactly calculated the classical term and obtained the dependence on the physical magnitudes of the quantum series.

On the other hand, this is only valid for \(N > 1\). If \(N = 1\), no fermionic operator appears in the Hamiltonian and the integral yields null. This is to be interpreted as a consequence of the discrete nature of the spinorial degrees of freedom. In other words, the 'volume' occupied by the Grassmann variables has null measure according to Berezin's rules of integration. The appropriate count of the fermionic degrees of freedom in this case is the sum over spin polarizations that we made for the free configurations (classical geometry). When the Hamiltonian depends on the spin, then the integral does contain all the information. The classical configurations are included although they do not contribute. Both in the fermionic and the bosonic case, they lie on spaces that have zero measure if compared to the global phase space.

In order to perform the bosonic integral, we expand the exponential separating the terms in the Hamiltonian with coordinate or momentum dependence. Symbolically we write
\[ e^{-\hat{H}(p) - \hat{H}(y)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} [\hat{H}(p) + \hat{H}(y)]^n. \]  

(31)

Where \( p \) and \( y \) stand for any momentum or coordinate operator. We shall keep this notation from here on. Once the binomial is expanded, we are left with a series in the coordinates and the momenta with no particular ordering. Taking advantage of the canonical commutation rules we can expand each term as a finite series in \( \hbar \). The coefficients of the series will depend on the particular order of each term and on whether the momentum and coordinate operators do have the same indices or not. An example is

\[ \hat{p}^{2l} \hat{y}^{4m} \bigg|_{\text{disordered}} = \hat{p}^{2l} \hat{y}^{4m} \bigg|_{\text{ordered}} + \hbar \times c_1 \times \hat{p}^{2l-1} \hat{y}^{4m-1} \bigg|_{\text{ordered}} + \hbar^2 \times c_2 \times \hat{p}^{2l-2} \hat{y}^{4m-2} \bigg|_{\text{ordered}} + O(\hbar^3) \]  

(32)

with \( c_1 \) and \( c_2 \) some constants.

The integration is carried out inserting the expansion of the identity between the coordinate and momentum operators so as to transform them into numbers. This insertion need just be done once in each term thanks to the ordering. This way we make only one integration over the phase space and we shall be able to extract common properties of all terms. One of them would be integrated to give

\[ I_b = \int dp \, dy \, (8N^2 + 2s + 8N^2 + 2k^2) \, D(8N^2 + 2k)(y) a^s (ap^2)^{-s} b^{s/2} (by^4)^{m-s/2}. \]  

(33)

Where

\[ a = \frac{\beta R}{2\hbar} \quad \text{and} \quad b = \frac{\beta R}{16\pi^2 \hbar^2 \alpha'^3}. \]  

(34)

The number of commutations that have been needed to order the operators is \( 2s \). We only consider even terms because the others are related to odd integrals that yield null. We shall ignore again the contribution of the identity, as a generator of \( U(N) \). It can be easily calculated later. Now we change variables \( p \to a^{-1/2} p \) and \( y \to b^{-1/4} y \) and take advantage of the homogeneity properties of the \( D \)-functions to simply get

\[ I_b = \left( \frac{\beta R}{\alpha'} \right)^{3s + \frac{3}{2} k} \times \int dp' \, dy' \, D(8N^2 + 2k)(y') \hat{p}'^{2l-2s} \hat{y}'^{4m-2s}. \]  

(35)

It is manifest from here the remarkable scaling property that the dependence on the physical magnitudes \( \beta \) and \( R \) is the same for all terms in the series indexed by \( l \) and \( m \). This is quite fortunate since all the bosonic integrals are, by themselves, polinomially divergent. We had to make this expansion in order to get the
quantum series that disordered is hidden in the expansion of the exponential in Eq. (31). Now we can sum up the series over \( l \) and \( m \) before integrating. These are the integrals that should be finite, at least for physical reasons, because they are the coefficients of the quantum series.

Namely, what we wanted was to expand the original exponential of the Hamiltonian in terms of the \( \bar{h} \) that appears in the commutators. We know that this is feasible because it is physically meaningful. Then we expanded the exponential and the binomials that appeared, and after that we expanded again each term as the series in \( \bar{h} \) that we were looking for. As we have found that the physical behaviour of the terms depend only on the index \( s \) of the quantum series, we are allowed to sum back again every term with common \( s \), that is, to undo the first two expansions. Therefore the partition function for a fixed \( N \) is

\[
Z^{(N>1)} = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} g(N, s, k) \left( \frac{\beta R}{\alpha'} \right)^{\frac{3}{2}+\frac{3k}{2}} e^{-\frac{\beta \bar{h}}{2} R^2} N. \tag{36}
\]

As we mentioned, this is not valid for \( N = 1 \).

The way we have made the calculation seems to appear a little clearer if we separately compute the 'classical' \( s = 0, k = 0 \) term. That amounts to ignoring the commutation rules from the beginning or equivalently truncating the series in Eq. (32) to the first term, i.e. the \( O(1) \) in \( \bar{h} \). That way, we can reconstruct the exponential and get

\[
Z^{\text{classical}}_{\text{transverse}} = \int dp < p | e^{-\beta \hat{H}(p)} \det^{1/2} M(\hat{x}) e^{-\beta \hat{H}(x)} | p > = \int dp \int dx \int dx' < p | e^{-\beta \hat{H}(p)} | p' > < p' | x > < x | \det^{1/2} M(\hat{x}) e^{-\beta \hat{H}(x)} | x' > < x' | p > = \int dp \int dx \det^{1/2} M(\hat{x}) e^{-\beta \hat{H}(x)}. \tag{37}
\]

To finally arrive at

\[
Z^{(N>1)}_{\text{classical}} = \sum_{N=2}^{\infty} \frac{e^{-\beta \frac{N\bar{h}}{2}}}{N!} \int \left( \prod_{k, \lambda} dy^k_{\lambda} \right) \det^{1/2} \left( \gamma_{ij} F_{\mu \nu}^i \right) e^{-\frac{1}{2} [y', y]^2}. \tag{38}
\]

We explicitly show all the indices for this simple case and again, we have ignored the \( N = 1 \) term. The \( N! \) factor comes again as the remnant of the gauge symmetry along the minima of the potential. It is precisely the same result as in Eq. (36) and we can see that the integral that appears as a coefficient is finite and perfectly defined. Apart from the dependence on the temperature, the expression differs from the calculation with the free configurations in the absence of the volume in the final result. This is due to the fact that volume is a physical quantity that only has a complete sense for those configurations. Here, it is taken into account in the flat, divergent directions of the integral. There, we would have different
powers of the volume depending on whether we integrate over the D0-brane wave functions or their bound states. So even when we do not see powers of the nine dimensional volume, our computation is a partition function and as such it is not a linear function of $V_9$. The fact that these integrals include by themselves the effects of bound states can be seen by just inserting a $(-1)^F$ operator. This would turn this function into the Witten index, which is precisely used to count such states. These states also include different dependences on the volume even before performing the sum in $N$, because they act as if certain directions along the classical phase space were frozen.

Let us now complete the calculation by adding the $N = 1$ term, which exactly equals the integral over the fields related to the identity -the center of mass of the system. This is special in several ways. Not only is it independent of the fermionic variables, but neither does it depend on the bosonic coordinates. Since it is just a function of the nine bosonic momenta and the Gauss constraints are trivial, it does not receive any quantum correction. This is clear when we relate this special matrix configuration with a single object in eleven dimensions: one supergraviton on its light-cone. One single object is always free and its statistics is irrelevant. It was already computed in the previous section, so we simply write

$$Z = 256 V_9 (2\pi R \beta \hbar)^{-9/2} \left[ e^{-\frac{\beta \hbar}{2R}} + \sum_{N=2}^{\infty} \sum_{q=0}^{\infty} f(N, q) \left( \frac{\beta R}{\alpha'} \right)^{2q} e^{-\frac{\beta \hbar}{2R} N} \right].$$  \hspace{1cm} (39)$$

We have been able to gather the two quantum series -the one of the fermions and that of the bosons- into a single one because they are expansions in exactly the same parameter. One of the most amazing characteristics of the partition function here calculated is its peculiar dependence on $\hbar$. It only appears in the center of mass contribution and in the exponential coming from the longitudinal sum. For all the $N > 1$ terms we have supposedly made a classical approximation followed by a quantum series that would correct it; nevertheless the $\hbar$'s that we include in the commutators are exactly canceled by those that appear in the Hamiltonian. So we arrive at the bizarre conclusion that the 'quantum' series is, in fact, not quantum at all. Moreover, this series is an expansion in the same parameter as the series in the longitudinal momentum $N$, therefore, we are tempted to conclude that the true natures of both series are not clearly distinguishable. The function is not very different from its $\hbar \to 0$ limit, the only role of this constant seems to be to serve as a kind of chemical potential that measures the cost of adding longitudinal momentum to the system.

This poses the important question of how to relate this result with the approximations we have made in previous sections and eventually, how to take the limits that would lead us to the different string theories. Different classical limits will not appear in this theory just as it is usual in traditional quantum theories. According to them, specific quantum statistical properties were effects related to
the nature of the fundamental fields. However, in Matrix Theory we have seen in a previous section that the only way of including quantum statistics is to assume that they appear as a consequence of the residual gauge interactions. This connects with the surprising behaviour of the quantum series. That is why we shall see that all these effects come out because of the appearance of new degrees of freedom related to the off-diagonal terms. The classical limits are reached when those degrees of freedom decouple getting infinitely massive.

Nevertheless, the decoupling of degrees of freedom is always obscure when we look at the partition function because physical magnitudes are related to its logarithm and derivatives thereof. Different contributions to, for example, the Helmholtz free energy are not summed up but multiplied inside the partition function. Therefore, if some of the effects are taken to be negligible, the whole partition function may tend to zero. The best way to circumvent this difficulty is, then, precisely, to calculate the Helmholtz free energy. It is, basically, the logarithm of $Z$, which carries the advantage of turning products into sums, and so, clarifying the discrimination of the contributions.

Let us, for example, take the classical limit

$$Z = Z_{\text{classical}} Z_{\text{other}}$$

$$\beta F = -\ln Z_{\text{classical}} - \ln Z_{\text{other}}$$

(40)

We know that the classical partition function diverges when $\hbar \to 0$ so that the other term must tend to zero in order for the complete function to remain finite. Looking at the expression for the free energy one can see that the non-classical term acquires an infinite energy while the classical one gets less and less energetic as we get nearer to the limit.

It is clear that the factorization of the partition function is quite arbitrary as we can separate any two parts by putting the appropriate parameter and then decouple one of them by taking a certain limit for the parameter. This is a consequence of the fact that we can organize the degrees of freedom of the theory in many ways -with or without physical meaning- and that, indeed, the information that the partition function holds is global so that the behaviour of particular configurations may be, as we have seen, quite hidden. The only physical way of knowing which is the correct expansion in each case is to go back to the Hamiltonian, then decide which degrees of freedom we are interested in according to the limit we are choosing (classical, large distances, ...), and only after that should we integrate exclusively the appropriate configurations. Some limits will be calculated in next section.

Let us not forget that the series is itself a limit in the sense that it is only a good expansion for $(\beta R) \to 0$. Its validity is basically determined by the coefficients $f(N, q)$. What we know is that the $f(N, 0)$ carry a $1/N!$ factor with more $N$ dependence coming from the Gaussian-like integrals and that the other coefficients come from a quantum series that should be well behaved. If we fix
the radius of the eleventh dimension, and look at the series at high temperature, we can see that the thermodynamical behaviour is completely determined by the coefficients \(f(N,0)\) but it is clear that the quantum corrections are the only ones that are affected by variations of the temperature. The question of what is the mechanism through which the theory remains finite without any ultraviolet cut-off is unresolved yet, because it is related to the growth of the coefficients with \(N\). This would tell us how the theory distributes its degrees of freedom. An analogous high temperature expansion can be obtained for perturbative strings by making a high temperature expansion for the free energy of each field in the string. This always gives a Laurent series for \(\beta \to 0\) with leading term \(\beta^{-d}\) with \(d - 1\) the number of open spatial dimensions for the Helmholtz free energy. The point is that the number of degrees of freedom per mass level grows too fast so as to get the high temperature series a non convergent one. Actually, no high temperature divergence appears because the canonical equilibrium gets broken at the Hagedorn temperature. This is the way modular invariance works as an ultraviolet cut-off at finite temperature in fundamental strings.

On the other hand, for a fixed temperature, the series is not able to describe systems with large light cone radius. It seems to be adapted to be a good description of type IIA String Theory. Indeed, one just has to make the substitution \(R \to g_s\sqrt{\alpha'/\hbar}\) to recognize the series in \(N\) as a String Theory non-perturbative expansion with the exponential of the inverse of the string coupling as expansion parameter. Each term is further corrected by a weak coupling expansion, which represents the quantization around the different solitonic sectors. This quantum series recovers its natural parameter \(\hbar\). So this would be the partition function of type IIA String Theory including the non-perturbative effects. Nevertheless, there is no exclusively perturbative term in this series so that strings themselves do not seem to be accounted for. This is a consequence of the particular choice of reference frame that we have made in the eleven-dimensional theory which we began with. The length \(R\) we have been writing is not exactly the \(R^{11}\) that is what is directly related to the string coupling but the same radius measured by a strongly boosted observer. The same happens to the temperature, it is an energy and so it is affected by Lorentz transformations. Anyway, this is not much worrying since we have calculated a scalar dimensionless quantity; that is, the partition function is the same no matter if it is calculated in the light-cone or in any other reference frame. This way, we can invert the boost and make the substitution

\[
\left.\frac{\beta R}{\alpha'}\right|_{\text{light-cone}} \to \left.\frac{\beta R^{11}}{\alpha'}\right|_{\text{at rest}} \approx \left.\frac{\beta R}{\alpha'}\right|_{\text{light-cone}} \times \sqrt{2\left(\frac{R^{11}}{R}\right)^2}. \tag{41}
\]

with

\[
\frac{R^{11}}{R} \to \infty \tag{42}
\]
It may be useful to remind that when we say 'at rest' we mean that the observer measures the invariant radius of the eleventh dimension, the smallest, so that, somehow, he is at rest with respect to the compact circle. In this reference frame, the parameter is much bigger so that the series is useless except for extremely high temperatures. Therefore, in spite of being Lorentz invariant, the series is only adapted to describe light-cone objects. Now the relation to the type IIA string is more direct but still, we do not seem to have any string! The reason for this is that our light-cone calculation has 'integrated out' the string degrees of freedom and spreaded them over the D0-brane ones. The degeneracies that are counted by the series in the index $N$ do not either come from strings nor D0-branes but rather from what in Solid State Physics would be called *quasi*-D0-*branes*. These are the original branes, but dressed by the closed type IIA strings, other D-branes and anti-D-branes and interacting through them. Those are the solitons to which the series refers and this is the reason why we do not have any zero mode. The conclusion is that our calculation corresponds to the partition function of type IIA strings, but it is adapted to a point of view different from the usual one.

5 The field-theoretical approach

In the previous section we have computed the partition function for the Matrix Model using the commutation properties of the matrices that describe the D0-brane dynamics. This section is devoted to a calculation of $Z(\beta)$ using the perturbative expansion for the $U(N)$ Super Yang-Mills Quantum Mechanics. This type of calculations are closely related to those done in [8, 11], where the one loop effective potential between two D0-branes is obtained. We will also compute the thermal degrees of freedom for the case of the Matrix Model on $T^2$, using the dual description in terms of the D-strings dynamics. Comparing both analysis we will check $T$-Duality.

It is assumed that, when studying the interaction of non-BPS states, the correspondence between Matrix Model and Supergravity calculations may be realized at finite temperature [13, 11, 12]. This picture comes from the idea that the non-extremality of the D-branes would be included into an entropy, and it is closely related to the Supersymmetry breaking produced by the different statistics of fermionic and bosonic fields.

We will work with the Euclidean version of the SYM action, that is (in the temporal gauge, $A_0 = 0$)

$$S^E = \int_0^\beta d\tau \text{Tr} \left[ \frac{\dot{Y}^i \dot{Y}^j}{2R} + \frac{1}{\sqrt{\alpha'}} g^{i\hat{j}} \dot{\theta} + \frac{R}{16\pi^2 \alpha' \beta} \left[ Y^i, Y^j \right]^2 + \frac{R}{2\pi \alpha' \beta} g^{i\hat{j}} \left[ Y^j, \theta \right] \right]$$

(43)

\[3\]In a recent paper [10] closely related to this part of our work, A. Tseytlin has analyzed these ideas.
where the Euclidean time $\tau$ is taken to be a circle of length $\beta$ and we have set $\hbar = 1$ again. The $Y^i$ are nine scalars fields and the $\theta$ are Grassmann fields of the transverse $SO(9)$ rotation group. Both are matrices of the adjoint of $U(N)$.

We will compute the one-loop correction to the free partition function for the fields that appear in the Matrix Lagrangian, and compare to the results obtained in the previous sections. The physical meaning of this type of calculations will be explained in detail below. We will show how this description, like the calculation presented in [17], only takes into account the thermal properties of the fields attached to the D0-branes without thermalizing the D-particle itself.

As usual in quantum field theory, we can obtain the corrections to the partition function by using the Feynman diagram techniques. Expanding the interaction term of the action in the Path Integral functional we have

$$
Z(\beta, N) = Z_0(\beta) \left[ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} Z_n(\beta) \right],
$$

where $Z_n$ stands for the $n$-vertex correlation functions. As an example, in a simple model with two Yukawa-coupled real fields we would have

$$
Z_n = \int \prod_{i,j,k} dx_i dy_j dz_k \int D\phi D\psi (\lambda \phi(x_i) \psi(y_j) \phi(z_k))^n e^{-(S_{\text{free}}[\phi] + S_{\text{free}}[\psi])} \tag{45}
$$

and

$$
Z_0(\beta) = [\det (K(x, \beta, \phi))]^{\frac{(-1)^F}{2}} [\det (K(x, \beta, \psi))]^{\frac{(-1)^F}{2}} \tag{46}
$$

with $F$ the fermionic number. The above expressions may be trivially generalized to the momentum space.

We will start by computing and studying the tree level contribution to the Helmholtz free energy by using the Schwinger representation of the field propagator. Following this method we write

$$
-\beta F_0(\beta) = -\text{Tr} \left[ (-1)^F \int_0^\infty \frac{dt}{t} e^{-(k^2 + m^2) t} \right]. \tag{47}
$$

In our case we have massless fields in $d = 0 + 1$ dimensions, then the trace in the above expression has to be taken over the Matsubara frequencies and over internal degrees of freedom which are the representations of the transverse $SO(9)$ and the $U(N)$ gauge group. As the result of a blind calculation, we would propose

$$
-\beta F_0(\beta) = \frac{N^2}{2} \int_0^\infty dt \frac{1}{t} \left[ 9 \theta_3 \left( 0, \frac{2\pi i t}{\beta^2} \right) - 8 \theta_2 \left( 0, \frac{2\pi i t}{\beta^2} \right) \right]. \tag{48}
$$

Here the $N^2$ factor comes from the $U(N)$ degrees of freedom, and the multiplicative factors of the thermal modular functions reflect the number of bosonic (9) and fermionic (8) fields. Some comments about the previous expression are
needed. If we were studying a second quantized theory of objects living in eight dimensions we would expect the Helmholtz free energy to grow proportionally to the eight-dimensional volume. As we see $F_0(\beta)$ in (48) is volume independent. This fact shows that what we are really doing is to thermalize the internal degrees of freedom of the D0-branes, that is, the strings attached to them.

This type of situation has already been studied from the string point of view. However it seems that there are some subtleties to be taken into account to establish the possible connections. In [17] the thermal free energy of an open superstring gas in presence of a Dp-brane has been computed, obtaining for the special $p=0$ case

$$-\beta F^{\text{String}}(\beta) = -\frac{N^2}{8} \int_0^\infty dt \frac{1}{2} \left[ \theta_3 \left(0, \frac{2\pi it}{\beta^2} \right) - \theta_2 \left(0, \frac{2\pi it}{\beta^2} \right) \right] f(t) \quad (49)$$

where $f(t)$ is the partition function of the Type I string theory. If one takes the zero temperature limit one recovers the vanishing vacuum energy of a supersymmetric theory. This property have been used by Polchinsky [18] to show the BPS nature of the D-branes. If we want to compare this expression with (48) we must consider only the massless field content of the string spectrum giving

$$-\beta F^{m=0}(\beta) = -\frac{N^2}{2} \int_0^\infty dt t^{-1} \left[ 8 \theta_3 \left(0, \frac{2\pi it}{\beta^2} \right) - 8 \theta_2 \left(0, \frac{2\pi it}{\beta^2} \right) \right] =$$

$$= -\frac{8N^2}{2} \int_0^\infty dt t^{-1} \theta_4 \left(0, \frac{2\pi it}{\beta^2} \right). \quad (50)$$

Where the 8 corresponds to the number of degrees of freedom of the massless field. Remember that at this level the string spectrum is composed by a vector and a Majorana-Weyl fermion. Both belong in their respective $SO(8)$ representation. However in [18] we would have obtained a different result that could be written

$$-\beta F_0(\beta) = -\beta F^{m=0}(\beta) - \frac{N^2}{2} \int_0^\infty dt t^{-1} \theta_3 \left(0, \frac{2\pi it}{\beta^2} \right). \quad (51)$$

The last term in the previous equation prevents $F_0(\beta)$ to vanish at $T=0$ because of the $\theta_3(0, 0)$ zero mode. This property of the Helmholtz free energy we have obtained for the Matrix Lagrangian seems to break the supersymmetric nature of the theory in such a way that the D-particles would become non-BPS states, but there are some details related to the gauge we have chosen that will restore our standard knowledge about D-branes.

In the string calculation, the Dp-brane configuration is obtained by compactifying Type I string theory on a torus with radii $R_{p+1}, ..., R_{d-1}$ which then are taken to zero. The free energy is then calculated integrating the momentum of

\footnote{If we neglect the zero mode of the fields, we see that both expressions vanish in the zero temperature limit, but they will do it in a different way.}
the string coordinates with Neumann boundary conditions. When we do it we
fix the D-brane at a given position in the Dirichlet direction; only allowing fluc-
tuations on the D-brane world-volume. In other words this type of computation
only takes into account the string’s knowledge of the full \(d\)-dimensional space-
time, that gives the eight bosonic and fermionic contributions in (49). In the
particular case of a D-particle we assume it to be fixed at a given point in space,
and we compute the string contributions to \(F(\beta)\). From the point of view of the
SYM Quantum Mechanics that describes the D-particle dynamics, and then from
the Matrix Model one, this kind of configuration must correspond to an election
of a fixed background. However this is not enough. If we let all the D-particle
coordinates fluctuate we would once more have nine bosonic contributions with-
out connection to the string calculation. What we are forced to do is to assume
that there is one non-fluctuating direction. To choose the frozen direction we can
take advantage of the worldsheet conformal invariance of string theory that for-
bids the vibrations along the longitudinal direction of the string. In our case this
coordinate coincides with the straight line that connects the D-particles \([11, 13]\).
Consequently, if we want to relate the Matrix Model computation and its string
origin we have to freeze this direction.

We can analyze this problem from a more technical perspective. As we said
the theory that describe the D-particle dynamics is a SYM quantum mechanics
coming from a dimensional reduction of the theory in \(d = 9 + 1\) to \(d = 0 + 1\).
We can start our analysis taking the Arnowitt-Fickler gauge in the initial SYM
quantum field theory, that is

\[ n_\mu A^b_\mu = 0 \tag{52} \]

where \(n_\mu\) are components of a unitary vector. Here the gauge fields are functions
of the full ten-dimensional space, then after the dimensional reduction we have

\[ n_0 A^b_0 + n_j Y^b_j = 0 \quad j = 1, \ldots, 9. \tag{53} \]

Remember that the condition in (52) does not totally reduce the degrees of free-
dom to the physical ones. So we must take another gauge condition to complete
the reduction. Here we can choose the temporal gauge \((A^b_0 = 0)\), that reduces
the relation in (53) to a constraint between the scalars fields \(Y^b_j\). This picture is
what underlies the physical idea of freezing the vibrational modes of the string
along the longitudinal direction, and corresponds to the picture described in the
previous section for the hamiltonian formalism.

We could have done the same analysis by choosing another gauge and intro-
ducing the adequate ghost contribution. As in \([8]\) it is possible to choose the
background field gauge, in the special case of the coinciding D-brane configura-
tion. In this case the spectrum of the theory is composed of eight on-shell bosons,
after considering the ghost contribution, eight fermion fields, and the interactions
between them are the same we have in the temporal gauge.
To summarize, we have then shown that, for the \( d = 0 + 1 \) gauge theory describing the dynamics of our system, the temporal gauge does not completely reduce the degrees of freedom to the physical ones. Another reduction is then needed. This constraint has to be taken from the dimensionally-reduced theory we started from. In the case of the D-particle dynamics it is its string vibration origin what determines the reduction.

Let us come back to the volume dependence of (50). From an exact M-theoretic point of view we should compute the contribution coming from the nine directions. In fact we would expect a linear volume dependence of the Helmholtz free energy. These contributions could be studied by assuming that we are separating the physical degrees of freedom by means of a Born-Oppenheimer approximation \[11\]. We are assuming that the phase space of the theory may be factorized in terms of the dynamics of the D-particles as objects moving in the nine transverse directions and their vibrational degrees of freedom. The eight transverse string modes are decoupled from the motion of the D0-brane which must be studied as we did in the previous section. Finally we may describe the full dynamics in terms of a phase space coming from the motion along the nine dimensions and characterized by the presence of the string vibrational degrees of freedom for each point in that space. This idea have been also proposed in relation with \( p \)-brane and black-hole physics \[6,16,19\].

Let us start a more detailed study assuming that the \( N \) D0-branes are located in a background characterized by

\[
Y_1 = \begin{pmatrix}
  x_1 & 0 & 0 & \cdots \\
  0 & x_1 & 0 & \cdots \\
  \vdots & 0 & x_1 & \cdots \\
  \vdots & \vdots & 0 & \ddots \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]  

This election corresponds to a fluctuation of the D-particles orthogonal to the \( Y_1 = x_1 \) fixed plane. This configuration completely decouples the \( Y_1 \) field from the action. On the other hand we consider the transverse vibration of the massless fields connecting the branes. We should do that taking the following parameterization

\[
(Y_k)^i_j = x_k \delta^i_j + \lambda \phi^i_j
\]

with \( \phi^i_j \in U(N) \). This corresponds to coinciding D0-brane positions, preserving the full \( U(N) \) gauge symmetry. This configuration exactly coincides with the string approach to the thermal strings stretched between \( N \) D-branes in \[17\].

It is possible to generalize our study to the separated branes configurations. As it is well known, separating the D-branes breaks the \( U(N) \) symmetry down to \( U(N_1) \times U(N_2) \). It is easy to see how this breaking occurs. As an example we
can look at the $N = 2$ case. Taking the branes in different positions in the frozen direction, we have

$$Y_1 = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \quad \text{and} \quad Y_i = (\phi_i^o T^a)$$

(56)

where $T^a$ are the group generators. One can easily obtain the mass correction to the scalar fields, coming from the interaction term in the action which is

$$\text{Tr} [Y_1, Y_i]^2 = \phi_i^{1} \phi_i^{1} \frac{(x_1 - x_2)^2}{4} + \phi_i^{2} \phi_i^{2} \frac{(x_1 - x_2)^2}{4}$$

(57)

that is diagonalized in terms of the massive fields $\phi_i^1$ and $\phi_i^2$. This property was firstly pointed out by E. Witten in [20]. Finally the free energy reads

$$- \beta F(\beta) = - \frac{8}{2} \sum_{i=1}^{2} N_i^2 \int_0^{\infty} dt \, t^{-1} e^{-\frac{(x_1 - x_2)^2}{2}} \theta_4 \left( 0, \frac{2 \pi it}{\beta^2} \right).$$

(58)

The same arguments hold for an arbitrary $N$ case. If we separate this system into two objects with $N_1$ and $N_2$, the free energy will have the form

$$- \beta F(\beta) \simeq - \frac{N_1^2}{2} G_1(\beta) - \frac{N_2^2}{2} G_2(\beta).$$

(59)

Where $G_1(\beta)$ and $G_2(\beta)$ are the functions of the temperature corresponding to the thermal degrees of freedom of the fields living attached to $N_i$ block of D-particles. Following this procedure it is possible to change the $N^2$ dependence in (50) into a linear one on $N$, that corresponds to a $U(1)^N$ gauge group. The description we have shown in terms of the string physics included in the D0-brane dynamics may be done without any reference to it. As we have said for the overlapping D-brane configuration, here we may also fix any other gauge condition and obtain the same results as with the temporal (Arnowitt-Fickler) gauge. The spectrum obtained pulling apart the particle positions is, as expected, gauge independent. It is worth noticing that, as in quantum field and string theories [21], the interaction can be partially absorbed as a correction for the masses of the fields.

Now let us carry out the computation of the one-loop correction to the free energy. As we did at tree level we have to decide how many string directions we take into account. Assuming the philosophy of the Born-Oppenheimer approximation we have explained before, we will fix the corresponding longitudinal direction. Following [8, 4] we rewrite the action (43) in the usual SYM form, which reads

$$S^E = \frac{1}{g^2} \int_0^\beta d\tau Tr \left[ \dot{Y}^i \dot{Y}^i + \theta^T \dot{\theta} + \frac{1}{2} [Y^i, Y^j]^2 + \theta^T \gamma^j [Y^i, \theta] \right]$$

(60)

with

$$g = \frac{R^{3/2}}{2 \pi \alpha'^{3/2}}$$

(61)
An adequate rescaling of the bosonic and fermionic fields with the SYM coupling \( g \) allows us to study the Feynman rules of this theory. It is also useful to explicitly write down the matrix fields in terms of the U(N) generators, obtaining:

\[
S^E = \int_0^\beta d\tau \left[ Y^i_\alpha \dot{Y}^i_\alpha + \theta^T_i \dot{\theta}^i + \frac{g^2}{4} Y^i_\alpha Y^j_\beta Y^k_\gamma \gamma^f F^{abc} F^{fge} + g \theta^T_i \gamma^j Y^i_\alpha \theta_j F^{cde} \right].
\] (62)

To be complete we can write a formal expression for the \( n \)-vertex correction to the Helmholtz free energy, which is

\[
F(\beta) = -\frac{1}{\beta} \log Z_0(\beta) - \frac{1}{\beta} \log \left[ 1 + \sum_{n=1}^{\infty} \frac{Z_n(\beta)}{n!} \right].
\] (63)

We show how to compute the functions \( Z_n \) for a very simple case in (45). In the case of the Matrix Model we shall write \( Z_n(\beta) \) as

\[
Z_n = \int DY_\alpha D\theta e^{-\left[ S^E_{\text{free}} + S^E_{\text{free}} \right]} \sum_{k+p=n} \frac{(-1)^k}{k!} \frac{(-1)^{2p}}{p!} \left[ I(Y^i_\alpha) \right]^k \left[ I(\theta, Y^i_\alpha) \right]^{2p} \] (64)

where \( I(Y^i_\alpha) \) and \( I(\theta, Y^i_\alpha) \) are the interaction terms appearing in the Matrix Lagrangian.

Now we can explicitly write the Feynman rules of the Lagrangian in (62)

Scalar propagator = \( \frac{\delta^{ab} \delta_{ij}}{p^2} \) with \( p = p_0 = \frac{2\pi n}{\beta} (n \in Z) \)

Fermion propagator = \( -i\frac{\delta^{ab}}{\gamma^k p_k} \) with \( p = p_0 = \frac{(2n+1)\pi}{\beta} (n \in Z) \)

and the interaction vertices are

\[
i) \quad \frac{g^2}{4} Y^i_\alpha Y^j_\beta Y^k_\gamma \gamma^f F^{abc} F^{fge} \rightarrow -g^2 \left[ F^{abc} F^{fge} (\delta_{ij} \delta_{kl} - \delta_{ij} \delta_{kl}) + \text{permutations} \right]
\]

\[
ii) \quad g \theta^T_i \gamma^j Y^i_\alpha \theta_j F^{cde} \rightarrow -ig F^{cde} \gamma^j\] (65)

in both cases the total momentum of the incoming fields is zero. It is easy to check that the correlation functionals in (45) vanish for an odd number of fermionic fields, leaving the perturbative expansion of \( Z(\beta) \) to be a power series in \( g^2 \).

Finally we must take care of the discrete nature of the momenta when we compute the loop integrals. The one-loop correction we want to obtain corresponds to the diagrams in Fig. 1.

The purely scalar contribution is

\[
Z^{\text{scalars}}_1(\beta) = -336 g^2 F^{ade} F^{ade} \frac{\beta^3}{(2\pi)^3} \sum_{n,m} \frac{1}{n^2 m^2} \] (66)

Here we take the normalization \( \text{Tr} (T^a T^b) = \delta^{ab} \)
Figure 1: The one-loop Feynman diagrams involving scalar fields (a) and the coupling of fermions and scalar fields (b)

The factor 336 appears because of the 8 bosonic oscillating directions, i.e. after freezing the longitudinal one.

The contribution coming from the diagram involving fermionic fields gives

\[ Z_2^{\text{fermi}}(\beta) = -2g^2 \gamma_j \gamma_j F^{\text{fae}} F^{\text{fae}} \frac{\beta^3}{\pi^3} \sum_{n,m} \frac{1}{2n+1} \frac{1}{2m+1} \frac{1}{(n-m)^2} \]

\[ = -256 g^2 \left[ \frac{(N^2 - 1)(N^2 - 2)}{2} \right] \frac{\beta^3}{\pi^3} \sum_{n,m} \frac{1}{2n+1} \frac{1}{2m+1} \frac{1}{(n-m)^2} \]  

(67)

where we have taken the trace over the fermionic degrees of freedom included into the \( \gamma_j \). We are now able to study some properties of the one-loop correction to the partition function, which takes the form

\[ Z_1(\beta) = -Z_1^{\text{scalars}}(\beta) + \frac{1}{2} Z_2^{\text{fermi}}(\beta). \] 

(68)

Then the corresponding free energy looks like

\[ F(\beta) = -\frac{1}{\beta} \log Z_0(\beta) - \frac{1}{\beta} \log [1 + Z_1(\beta)] = \]

\[ = F_0(\beta) - \frac{1}{\beta} \log \left[ 1 + g^2 \left[ \frac{(N^2 - 1)(N^2 - 2)}{2} \right] \beta^3 \frac{G}{\pi^3} \right] \]

(69)
where $G$ is a constant coming from (68). The value of this constant can be obtained by a generalized-$\zeta$ function regularization. In the zero-temperature case this contribution vanishes by dimensional regularization because of the fact that there is no dimensionful parameter involved. This means that we are dealing with a high temperature expansion whose relationship to the $T = 0$ case is subtle.

We can now check what happens if we consider the Matrix Model compactified on a torus. The dynamics of $N$ D0-branes on $T^n$ is described by a system of $N$ D$n$-branes on the dual torus [1, 2, 4, 22]. In our case we start with the Matrix Model on $S^1(R) \times (S^1(L) \times \mathbb{R}^8)$, where the first $S^1$ stands for the compactified eleventh dimension. In this case the dual description is given by a system of D-strings wrapped around a cylinder of length $\Sigma = 1/L$. The theory that describes this system is a $d = 1 + 1$ SYM quantum field theory coming from dimensional reduction from $d = 9+1$. Because of the appearance of the $\Sigma = 1/L$ multiplicative factor in the dual action [22] we can redefine the coupling constant of the theory in such a way we can write

$$g^2_{\text{dual}} = \tilde{g}^2 = g^2 \Sigma$$

(70)

where $\tilde{g}$ is the effective coupling constant in the Kaluza-Klein sense.

In this case the gauge condition in (52) reduces to the following form

$$n_0 A^b_0 + n_1 A^b_1 + n_2 Y^b_j = 0$$

(71)

then the problem of gauge fixing in this case is exactly the same as the one we have analyzed before. After taking the temporal gauge we have to choose the additional condition coming from the residual constraint that relates the eight scalar fields $Y_j$ and the $A_1$ gauge field. Initially we have two possible options, either we can set the gauge field to zero, assuming that there are no propagating modes on the D-string, or we can stop the string vibration in one direction. In the latter case we will have a purely Neumann string mode propagating on the D-string and interacting with seven scalars. In both cases we will have eight bosonic and fermionic degrees of freedom. At the tree level the free energy is the same as in (50), except for the Kaluza-Klein modes contribution

$$- \beta F^\Sigma = - \frac{N^2}{2} \int_0^\infty dt \ t^{-1} \left[ 8 \theta_3 \left( 0, \frac{2\pi i t}{\beta^2} \right) - 8 \theta_2 \left( 0, \frac{2\pi i t}{\beta^2} \right) \right] \theta_3 \left( 0, \frac{2\pi i t}{\Sigma^2} \right) =$$

$$= - \frac{8 N^2}{2} \int_0^\infty dt \ t^{-1} \theta_4 \left( 0, \frac{2\pi i t}{\beta^2} \right) \theta_3 \left( 0, \frac{2\pi i t}{\Sigma^2} \right).$$

(72)

As we can easily see, we recover the free decompactified limit, that is the Matrix Model result, when we take the dual radius $\Sigma$ to zero.

It is now easy to compute the one-loop corrections to the Helmholtz free energy for the compactified case. The interaction terms we have are independent of the gauge reduction and exactly map into the open Matrix-Model ones. We
To check $T$-duality in this expression, we have to rewrite the dimensionless coupling $\tilde{g}$ in terms of the initial Matrix parameter, the coupling constant $g$ and finally take the limit $\Sigma \to 0$, which reduces this to (66). The same arguments hold for the fermionic contribution (67). We can give a general argument for the $n$-vertex function in (64). In this case, taking the limit of the open Matrix Model (that corresponds to the dimensionally reduced D-string system), one gets

$$Z_n \propto \left( \frac{\tilde{g}^2 \beta^3}{\Sigma} \right)^n = (2\pi)^{-2n} \left( \frac{\beta R}{\alpha'} \right)^{3n}. \quad (74)$$

This parameter is explicitly independent of the compactification radius, therefore it is self-dual and then coincident with that of the already open Matrix Model.

The expression in (64) corresponds to the term with $g^{2n}$ (or $\tilde{g}^{2n}$, when we compactify one dimension) in the perturbative expansion of the free energy

$$F^\Sigma(\beta) = F^\Sigma_0(\beta) - \frac{1}{\beta} \log \left[ 1 + \sum_{n=0}^{\infty} \frac{g^{2n} \beta^{3n}}{\Sigma^n} G_n \left( \frac{\beta}{\Sigma} \right) \right] \quad (75)$$

where $G_n$ is the function that comes from the diagram of order $\tilde{g}^{2n}$. The regularization arguments given for (69) hold here too. This function depends on the Kaluza-Klein modes of the compactified dimension but, in the small $\Sigma$ limit, it only takes into account the corresponding zero mode. The behaviour of the free energy of a D-string system (Eq. (75)) allows us to explicitly check $T$-duality order by order in perturbation theory. In fact if we computed a general $n$-loop contribution we would see that its dependence of the coupling constant, the temperature, and the length of the compact dimension would map, after taking the corresponding limit, into the $d = 0 + 1$ SYM computation of the same contribution. This property is not surprising because of the string origin of the $T$-duality of the D-brane dynamics.

We can now go back to the physics involved in the Born-Oppenheimer approximation. The complete partition function of the finite-$N$ Matrix model has to be obtained multiplying the internal (stringy) degrees of freedom and those of the translational D-particle dynamics. In terms of phase spaces we can express the complete phase space as the direct product of the translational and internal ones. This assumption amounts to the following form for the partition function

$$Z_{\text{Born-Oppenheimer}}(\beta) = Z_{\text{Free}}(\beta) \times Z_{\text{Internal}}(\beta). \quad (76)$$
Finally, let us mention that adding the internal partition function to one appropriate term in the sum in (17) we recover the desired $N^2$ dependence coming from the internal unbroken $U(N)$ SYM degrees of freedom which are necessary to fit the entropy of a given $N$-parton in (17) as that of a Schwarzschild black hole of the model in [4], and obtain the Bekenstein-Hawking area law.

6 Conclusions

Matrix theory shows several different aspects when one looks at its thermodynamical behaviour. When the distances between objects are large, they can be described by a semiclassical expansion around what can be later interpreted as the Kaluza-Klein modes of a supergraviton in eleven dimensions. This is, in fact, the low energy limit of M-theory and it is the only case in which we have been able to take the radius of the eleventh dimension to infinity and see what are the predictions of the model for a completely open universe. We have seen that, as maybe one could have expected from the beginning, it precisely corresponds to the results obtained for eleven-dimensional supergravity, defined in an open space.

One interesting point would be the connection between our work and the fact that the semiclassical Born-Oppenheimer approximation for the supergraviton modes justifies their election in [6] as candidates to describe black holes. The contribution of their center of mass does not spoil the arguments exclusively based on the Super Yang Mills entropy. Another point of view is taken in [7], where the model supposes that a black hole consists of a gas of distinguishable D-particles with Boltzmann statistics. In this case, the separation between fast and slow modes of the system helps us justify the dependence of the entropy with the number of D-particles. Indeed, the field theory gauge symmetry is broken to $U(1)^N$ and it seems that $S \propto N$ may not be taken as an assumption. It keeps being unclear how the discernibility is acquired in this model. Maybe one could get some insight about this possible connection from the calculation in section four, where the whole gauge group is included. The question is that we do not really see how the black hole enters into the stage in our picture.

In section four we have tried to calculate some global aspects of the partition function including interactions and quantum effects. We have seen that the consequence of taking canonical Heisenberg commutators between the coordinate fields and their conjugate momenta is the appearance of a series whose natural parameter is not $\hbar$ as usual but the light-cone radius. In fact, the parameter is the same as that of the series coming from the sum over the possible values of $p^+$. We see this as a confirmation of the idea that this theory is quantum in a broader sense that the presently known quantum theories and that String Theories and Supergravities come out as classical limits of a moduli space.

It is known, both in field and string theory, that the corrections to the free
theory coming from interactions can always be seen as the appearance of effective masses that correct the original ones. We wanted to see that in our case, but to do it, we needed to make a Born-Oppenheimer approximation. It consists in separating the 'movements' of the system in fast and slow ones. That is what we made in section 5. We suppose that we can divide the system in clusters of supergravitons with weak interactions among them. Each of these objects has low energy so we can separate the fields that propagate along their world-line (fast movements) and the displacements of their centers of mass through the target space (slow ones). The world-line calculation has been performed and represents a first attempt to include interactions as well as quantum effects that do not have statistical nature. We have obtained a high temperature expansion that corresponds to the series in the Super Yang-Mills coupling constant. This calculation and the equivalent one using T-duality and D-strings can be used to relate this approximation with one loop expansions in string theory like those made in [17]. They can also serve as a finite temperature check of T-duality, that holds to all orders in perturbation theory, as expected.
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References

[1] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, ”M Theory As A Matrix Model: A Conjecture”,Phys. Rev.D55 (1997) 5112-5128, hep-th/9610043.

[2] A. Bilal, ”M(atrix) Theory: a Pedagogical Introduction”, LPTENS-97/43, hep-th/9710130
T Banks,”Matrix Theory” [hep-th/9710231]
D. Bigatti, L. Susskind, ”Review of Matrix Theory” [hep-th/9712072].

[3] L. Susskind, ”Another Conjecture about M(atrix) Theory”, SU-ITP-97-11 and hep-th/9704080
N. Seiberg, ”Why is the Matrix Model Correct?”, Phys. Rev. Lett.79 (1997) 3577-3580, hep-th/9710009
S. Hellerman and J. Polchinski, ”Compactification in the Lightlike Limit”, NSF-ITP-97-139 and hep-th/9711037.
D. Bigatti, L. Susskind,”A note on discrete light cone quantization” hep-th/9711063.

[4] W. Fischler, E. Halyo, A. Rajaraman and L. Susskind, ”The Incredible Shrinking Torus”, Nucl. Phys.B501 (1997) 409-426, hep-th/9703102
A. Sen, ”D0 branes on T^n and Matrix Theory”, MRI-PHY/P9709 and hep-th/9709220.

[5] M. Laucelli Meana, M. A. R. Osorio, and J. Puente Peñalba, ”The String Density of States from the Convolution Theorem”, Phys. Lett.B400 (1997) 275-283, hep-th/9701122.
”Counting Closed String States in a Box”, Phys. Lett.B408 (1997) 183-191, hep-th/9705183.

[6] T. Banks, W. Fischler, I. R. Klebanov and L. Susskind, ”Schwarzschild Black Holes from Matrix Theory”, Phys. Rev. Lett. 80 (1998) 226-229, hep-th/9709091.
I. R. Klebanov and L. Susskind, ”Schwarzschild Black Holes in Various Dimensions from Matrix Theory”, Phys. Lett.B416 (1998) 62-66, hep-th/9709108.
[7] T. Banks, W. Fischler, I. R. Klebanov and L. Susskind, "Schwarzschild Black Holes from Matrix Theory II" hep-th/9711003.
H. Liu and A. A. Tseytlin, "Statistical Mechanics of D0-branes and Black Hole Thermodynamics", JHEP 01 (1998) 010, hep-th/9712063.
N. Ohta and J. Zhou "Euclidean Path Integral, D0-branes and Schwarzschild Black Holes in Matrix Theory", hep-th/9801023.
D. A. Lowe "Statistical origin of Black Hole Entropy", hep-th/9802173.

[8] K. Becker and M. Becker, "A Two-Loop Test of M(atrix) Theory", hep-th/9705091.

[9] E. Álvarez, "Strings at Finite Temperature", Nucl. Phys. B269 (1986) 596.

[10] E. Álvarez and M. A. R. Osorio, "Superstrings at Finite Temperature", Phys. Rev D36 (1987) 1175-1183.

[11] M. Claudson and M. B. Halpern, "Supersymmetric Ground State Wave Functions", Nucl. Phys. B250 (1985) 689.
U. H. Danielsson, G. Ferretti and B. Sundborg, "D-particle Dynamics and Bound States", Int. J. Mod. Phys. A11(1996) 5463-5478, hep-th/9603081.
D. Kabat and P. Pouliot, "A Comment on Zero-brane Quantum Mechanics", Phys. Rev. Lett. 77 (1996) 1004-1007, hep-th/9603127.

[12] D. Kabat and W. Taylor "Linearized supergravity from Matrix theory", hep-th/9712185.
M. Raamsdonk "Conservation of Supergravity Currents from Matrix Theory" hep-th/9803003.

[13] M. R. Douglas, D. Kabat, P. Pouliot, S. H. Shenker, "D-branes and Short Distances in String Theory", Nucl. Phys. B485 (1997) 85-127, hep-th/9608024.

[14] C.G. Callan and J. Maldacena "D-Brane approach to black hole quantum mechanics", Nucl. Phys. B 472 (1996) 591, hep-th/9602043.

[15] J. Maldacena and A. Strominger "Black hole greybody factors and D-brane spectroscopy", Phys. Rev. D 55 (1997) 861, hep-th/9609026.

[16] A. Tseytlin "Open superstring partition function in constant gauge field background at finite temperature", hep-th/9802133.

[17] M. A. Vázquez-Mozo, "Open String Thermodynamics and D-Branes" Phys. Lett. B388 (1996) 494, hep-th/9607052.
M. B. Green, Nucl. Phys. B381 (1992) 201.
[18] J. Polchinski, ”Dirichlet-Branes and Ramond-Ramond Charges”, *Phys. Rev. Lett.* **75** (1995) 4724-4727, [hep-th/9510017](https://arxiv.org/abs/hep-th/9510017). "TASI Lectures on D-Branes", [hep-th/9611050](https://arxiv.org/abs/hep-th/9611050).

[19] I. R. Klebanov and A. A. Tseytlin, ”Entropy of Near-Extremal Black p-branes”, *Nucl. Phys.* **B475** (1996) 164-178, [hep-th/9604089](https://arxiv.org/abs/hep-th/9604089). S. S. Gubser, I. R. Klebanov and A. W. Peet, ”Entropy and Temperature of Black 3-Branes”, *Phys. Rev.* **D54** (1996) 3915-3919, [hep-th/9602135](https://arxiv.org/abs/hep-th/9602135).

[20] E. Witten, ”Bound States of Strings and p-Branes”, *Nucl. Phys.* **B460** (1996) 335-350, [hep-th/9510135](https://arxiv.org/abs/hep-th/9510135).

[21] G. Moore, ”Modular Forms and Two-Loop String Physics”, *Phys. Lett.* **B176** (1986) 369-379.

[22] W. Taylor, ”D-brane field theory on compact spaces”, *Phys. Lett.* **B394** (1997) 283-287, [hep-th/9611042](https://arxiv.org/abs/hep-th/9611042). ”Lectures on D-branes, Gauge Theory and M(atrices)”, PUPT-1762 and [hep-th/9801182](https://arxiv.org/abs/hep-th/9801182).