Abstract
Let $\Sigma_g$ be a closed surface of genus $g \geq 2$ and $\Gamma_g$ denote the fundamental group of $\Sigma_g$. We establish a generalization of Voiculescu’s theorem on the asymptotic $*$-freeness of Haar unitary matrices from free groups to $\Gamma_g$. We prove that for a random representation of $\Gamma_g$ into $\text{SU}(n)$, with law given by the volume form arising from the Atiyah-Bott-Goldman symplectic form on moduli space, the expected value of the trace of a fixed non-identity element of $\Gamma_g$ is bounded as $n \to \infty$. The proof involves an interplay between Dehn’s work on the word problem in $\Gamma_g$ and classical invariant theory.
1 Introduction

In a foundational series of papers [Voi85, Voi86, Voi87, Voi90, Voi91], Voiculescu developed a robust theory of non-commuting random variables that became known as non-commutative probability theory. One of the initial landmarks of this theory is the following result. Let $F_r$ denote the non-commutative free group of rank $r$. Let $U(n)$ denote the group of $n \times n$ complex unitary matrices. For any $w \in F_r$ we obtain a word map $w : U(n)^r \to U(n)$ by substituting matrices for generators of $F_r$. Let $\mu_{\text{Haar}}(U(n)^r)$ denote the probability Haar measure on $U(n)^r$ and $\text{tr} : U(n) \to \mathbb{C}$ the standard trace.

A simplified version of Voiculescu’s result [Voi91, Thm. 3.8] can be formulated as follows:

**Theorem 1.1** (Voiculescu). For any non-identity $w \in F_r$, as $n \to \infty$

$$\lim_{n \to \infty} \int \text{tr}(w(x))d\mu_{\text{Haar}}^r(x) = o_w(n).$$

(1.1)

We describe the interpretation of Theorem 1.1 as convergence of non-commutative random variables momentarily. Before this, we explain the main result of the current paper.

Another way to think about the integral (1.1), that invites generalization, is to identify $U(n)^r$ with $\text{Hom}(F_r, U(n))$ and $\mu_{\text{Haar}}^r$ as a natural probability measure on this representation variety. Now it is natural to ask whether there are other infinite discrete groups rather than $F_r$ such that $\text{Hom}(F_r, U(n))$ has a natural measure, and whether similar phenomena as in Theorem 1.1 may hold. The main point of this paper is to establish the analog of Theorem 1.1 when $F_r$ is replaced by the fundamental group of a compact surface of genus at least 2.

We now explain this generalization of Theorem 1.1; for technical reasons it superficially looks slightly different. For $g \geq 2$ let $\Sigma_g$ denote a closed topological surface of genus $g$. We let $\Gamma_g$ denote the fundamental group of $\Sigma_g$ with explicit presentation

$$\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

1. The integral (1.1) is equal to 0 if $w \notin [F_r, F_r]$, the commutator subgroup of $F_r$ [MP15, Claim 3.1], and if $w \in [F_r, F_r]$, the value of (1.1) is for $n \geq n_0(w)$ the same as the corresponding integral over $SU(n)^r \leq U(n)^r$, where $SU(n)$ is the subgroup of determinant one matrices [Mag21, Prop. 3.1]. So in all cases of interest we can replace $U(n)$ by $SU(n)$ in (1.1).

2. Since $\text{tr} \circ w$ is invariant under the diagonal conjugation action of $SU(n)$ on $\text{Hom}(F_r, SU(n)) \cong SU(n)^r$, the integral $\int \text{tr}(w(x))d\mu_{\text{Haar}}^{SU(n)^r}(x)$ can be written as an integral over $\text{Hom}(F_r, SU(n))/\text{PSU}(n)$.

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1Voiculescu’s result in [Voi91, Thm. 3.8] is more general than what we state here, also involving a deterministic sequence of unitary matrices.
3. Now turning to $\Gamma_g$, the most natural measure on $\text{Hom}(\Gamma_g, \text{SU}(n))/\text{PSU}(n)$ to replace
the measure induced by Haar measure on $\text{Hom}(F_r, \text{SU}(n))/\text{PSU}(n)$ is called the Atiyah-
Bott-Goldman measure. The definition of this measure involves removing singular parts
of $\text{Hom}(\Gamma_g, \text{SU}(n))/\text{PSU}(n)$. Indeed, let $\text{Hom}(\Gamma_g, \text{SU}(n))^\text{irr}$ denote the collection of ho-
omorphisms that are irreducible as linear representations. Then

$$\mathcal{M}_{g,n} \overset{\text{def}}{=} \text{Hom}(\Gamma_g, \text{SU}(n))^\text{irr}/\text{PSU}(n)$$

is a smooth manifold [Gol84]. Moreover there is a symplectic form $\omega_{g,n}$ on $\mathcal{M}_{g,n}$ called
the Atiyah-Bott-Goldman form after [AB83, Gol84]. This symplectic form gives, in the
usual way, a volume form on $\mathcal{M}_{g,n}$ denoted by $\text{Vol}_{\mathcal{M}_{g,n}}$. For many more details see
Goldman [Gol84] or the prequel paper [Mag21, §§2.7].

For any $\gamma \in \Gamma$, we obtain a function $\text{tr}_\gamma : \text{Hom}(\Gamma_g, \text{SU}(n)) \to \mathbb{C}$ defined by

$$\text{tr}_\gamma(\phi) \overset{\text{def}}{=} \text{tr}(\phi(\gamma)).$$

This function descends to a function $\text{tr}_\gamma : \mathcal{M}_{g,n} \to \mathbb{C}$. We are interested in the expected value

$$\mathbb{E}_{g,n}[\text{tr}_\gamma] \overset{\text{def}}{=} \frac{\int_{\mathcal{M}_{g,n}} \text{tr}_\gamma \ d\text{Vol}_{\mathcal{M}_{g,n}}}{\int_{\mathcal{M}_{g,n}} d\text{Vol}_{\mathcal{M}_{g,n}}}.$$

The main theorem of this paper is the following.

**Theorem 1.2.** Let $g \geq 2$. If $\gamma \in \Gamma_g$ is not the identity, then $\mathbb{E}_{g,n}[\text{tr}_\gamma] = O_\gamma(1)$ as $n \to \infty$.

The non-commutative probabilistic consequences of Theorem 1.2 will be discussed in the
next section.

### 1.1 Non-commutative probability

We follow the book [VDN92]. A non-commutative probability space is a pair $(\mathcal{B}, \tau)$ where $\mathcal{B}$
is a unital algebra and $\tau$ is a linear functional on $\mathcal{B}$ such that $\tau(1) = 1$. A random variable
in $(\mathcal{B}, \tau)$ is an element of $\mathcal{B}$. If $(\{(\mathcal{B}, \tau_n)\}_{n=1}^\infty)$ are a family of non-commutative probability
spaces and $(X_1^{(n)}, \ldots, X_r^{(n)}) \in \mathcal{B}^r$ is a family of random variables in $(\mathcal{B}, \tau_n)$ for each $n$, we
define the notion of a joint limiting distribution as follows. Let $\mathbb{C}\langle x_1, \ldots, x_r \rangle$ denote the
free non-commutative unital algebra in indeterminates $x_1, \ldots, x_r$. For a linear functional
$\tau_\infty : \mathbb{C}\langle x_1, \ldots, x_r \rangle \to \mathbb{C}$ with $\tau(1) = 1$, we say that $(X_1^{(n)}, \ldots, X_r^{(n)})$ converge as $n \to \infty$ to
the limit distribution $\tau_\infty$ if for all $p \in \mathbb{N}$, and $i_1, \ldots, i_p \in [r]$,

$$\lim_{n \to \infty} \tau_n(X_{i_1}^{(n)}X_{i_2}^{(n)} \ldots X_{i_p}^{(n)}) = \tau_\infty(x_{i_1}x_{i_2} \ldots x_{i_p}).$$

A very concrete example of this phenomenon is as follows. The function

$$\tau_n : F_r \to \mathbb{C}, \quad \tau_n(w) \overset{\text{def}}{=} \frac{1}{n} \int \text{tr}(w(x))d\mu_{\text{U}(n)} \text{Haar}(x)$$

extends to a linear functional $\tau_n$ on the algebra $\mathbb{C}[F_r]$ with $\tau_n(\text{id}) = 1$. From this point of
view, Theorem 1.1 implies the following statement:
**Theorem 1.3** (Voiculescu). Let \( r \geq 0 \) and \( X_1, \ldots, X_r \) denote the generators of \( F_r \), and \( \bar{X}_1, \ldots, \bar{X}_r \) denote their inverses, i.e. \( \bar{X}_i = X_i^{-1} \). The random variables \( X_1, \ldots, X_r, \bar{X}_1, \ldots, \bar{X}_r \) in the non-commutative probability spaces \( (C[F_r], \tau_n) \) converge as \( n \to \infty \) to a limiting distribution \( \tau_\infty : C(x_1, \ldots, x_r, \bar{x}_1, \ldots, \bar{x}_r) \to C \) that is completely determined by \((1.1)\). Indeed, if \( w \) is any monomial in \( x_1, \ldots, x_r, \bar{x}_1, \ldots, \bar{x}_r \), then \( \tau_\infty(w) = 1 \) if and only if after identifying \( \bar{x}_i \) with \( x_i^{-1} \), \( w \) reduces to the identity in \( F_r = \langle x_1, \ldots, x_r \rangle \), and \( \tau_\infty(w) = 0 \) otherwise.

In the language of [Voi91], this implies that in the limiting non-commutative probability space \((C(x_1, \ldots, x_r, \bar{x}_1, \ldots, \bar{x}_r), \tau_\infty)\), the subalgebras
\[
A_i \overset{\text{def}}{=} C(x_1, \bar{x}_1), \ldots, A_r \overset{\text{def}}{=} C(x_r, \bar{x}_r)
\]
are a free family of subalgebras: if \( a_j \in A_{ij} \) for \( j \in [q] \), \( i_1 \neq i_2 \neq \cdots \neq i_q \), and \( \tau_\infty(a_j) = 0 \) for \( j \in [q] \) then
\[
\tau_\infty(a_1 a_2 \cdots a_q) = 0.
\]
Accordingly, it is said [Voi91, Thm. 3.8] that if \( \{u_j(n) : j \in [r]\} \) are independent Haar-random elements of \( U(n) \), the family \( \{\{u_j(n), u_j^*(n)\} : j \in [r]\} \) of sets of random variables are asymptotically free.

Asymptotic freeness does not correctly capture the asymptotic behavior of the expected values \( E_{g,n}[\text{tr}_\gamma] \), however, an analog of Theorem 1.3 is implied by Theorem 1.2. For \( \gamma \in \Gamma_g \) let
\[
\tilde{\tau}_n(\gamma) \overset{\text{def}}{=} \frac{1}{n} E_{g,n}[\text{tr}_\gamma].
\]

**Corollary 1.4.** Let \( g \geq 2 \), \( a_1, b_1, \ldots, a_g, b_g \) denote the generators of \( \Gamma_g \), and \( \bar{a}_1, \bar{b}_1, \ldots, \bar{a}_g, \bar{b}_g \) denote their inverses. The random variables \( a_1, b_1, \ldots, a_g, b_g, \bar{a}_1, \bar{b}_1, \ldots, \bar{a}_g, \bar{b}_g \) in the non-commutative probability spaces \( (C[\Gamma_g], \tilde{\tau}_n) \) converge as \( n \to \infty \) to a limiting distribution
\[
\tilde{\tau}_\infty : C(x_1, \ldots, x_g, y_1, \ldots, y_g, \bar{x}_1, \ldots, \bar{x}_g, \bar{y}_1, \ldots, \bar{y}_g) \to C,
\]
where \( x_i \) (resp. \( y_i, \bar{x}_i, \bar{y}_i \)) corresponds to \( a_i \) (resp. \( b_i, \bar{a}_i, \bar{b}_i \)). This can be described explicitly as follows. If \( w \) is any monomial in \( x_1, \ldots, x_g, y_1, \ldots, y_g, \bar{x}_1, \ldots, \bar{x}_g, \bar{y}_1, \ldots, \bar{y}_g \), then \( \tilde{\tau}_\infty(w) = 1 \) if and only if \( w \) maps to the identity under the map
\[
C(x_1, \ldots, x_g, y_1, \ldots, y_g, \bar{x}_1, \ldots, \bar{x}_g, \bar{y}_1, \ldots, \bar{y}_g) \to C[\Gamma_g]
\]
obtained by identifying \( x_i, y_i, \bar{x}_i, \bar{y}_i \) with the corresponding elements of \( \Gamma_g \). If \( w \) does not map to the identity under this map, then \( \tilde{\tau}_\infty(w) = 0 \).

Notice that the estimate given in Theorem 1.2 is stronger than needed to establish Corollary 1.4.

### 1.2 Related works and further questions

The most closely related existing result to Theorem 1.2 is a theorem of the author and Puder [MP20, Thm. 1.2] that establishes Theorem 1.2 when the family of groups SU(n) is replaced by...
the family of symmetric groups \( S_n \), and \( \text{tr} \) is replaced by the character fix given by the number of fixed points of a permutation. In this case, the result is phrased in terms of integrating over \( \text{Hom}(\Gamma_g, S_n) \) with respect to the uniform probability measure. The corresponding result for \( \text{Hom}(F_r, S_n) \) was proved much longer ago by Nica in [Nic94].

The problem of integrating geometric functions like \( \text{tr}_\gamma \) over \( M_{g,n} \) is also connected to the work of Mirzakhani since, as Goldman explains in [Gol84, §2], the Atiyah-Bott-Goldman symplectic form generalizes the Weil-Petersson symplectic form on the Teichmüller space of genus \( g \) Riemann surfaces. In [Mir07], Mirzakhani developed a method for integrating geometric functions on moduli spaces of Riemann surfaces with respect to the Weil-Petersson volume form. Although there is certainly a similarity between (ibid.) and the current work, here the emphasis is on \( n \to \infty \), whereas (ibid.) caters to the regime \( g \to \infty \); the target group playing the role of \( SU(n) \) is always \( PSL(2, \mathbb{R}) \).

We now take the opportunity to mention some questions that Theorem 1.2 leads to. In the paper [Voi91], Voiculescu is able to boost Theorem 1.1 from a convergence in distribution result to a result on convergence in probability, that is, for any \( \varepsilon > 0 \), and fixed \( w \in F_r \), the Haar measure of the set

\[
\{ \phi \in \text{Hom}(F_r, U(n)) : |\text{tr}(\phi(w))| \leq \varepsilon n \}
\]

tends to one as \( n \to \infty \) [Voi91, Thm. 3.9]. To do this, Voiculescu uses that the family of measure spaces \( (\text{Hom}(F_r, U(n)), \mu_{\text{Haar}}^{U(n)}) \) form a Levy family in the sense of Gromov and Milman [GM83]. This latter fact relies on an estimate for the first non-zero eigenvalue of the Laplacian on \( \text{Hom}(F_r, U(n)) \). It is interesting to ask whether a similar phenomenon holds for the family of measure spaces \( (M_{g,n}, \mu_{\text{ABG}}^{M_{g,n}}) \) where \( \mu_{\text{ABG}}^{M_{g,n}} \) is the probability measure corresponding to \( \text{Vol}_{M_{g,n}} \). The fact that \( M_{g,n} \) is non-compact seems to be a significant complication in answering this question using isoperimetric inequalities.

In the prequel to this paper [Mag21] it was proved that for any fixed \( \gamma \in \Gamma_g \), there is an infinite sequence of rational numbers \( a_{-1}(\gamma), a_0(\gamma), a_1(\gamma), \ldots \in \mathbb{Q} \) such that for any \( M \in \mathbb{N} \),

\[
E_{g,n}[\text{tr}_\gamma] = a_{-1}(\gamma)n + a_0(\gamma) + \frac{a_1(\gamma)}{n} + \cdots + \frac{a_{M-1}(\gamma)}{n^{M-1}} + O_{g,M} \left( \frac{1}{n^M} \right)
\]

as \( n \to \infty \). Theorem 1.2 implies that \( a_{-1}(\gamma) = 0 \) if \( \gamma \neq \text{id} \). It is also interesting to understand the other coefficients of this series. This has been accomplished when \( \Gamma_g \) is replaced by \( F_r \) by the author and Puder in [MP19] where in fact it is proved that

\[
E_{F_r,n}[\text{tr}_w] \overset{\text{def}}{=} \int \text{tr}(w(x))d\mu_{\text{Haar}}^{U(n)}(x)
\]

is given by a rational function of \( n \) and in particular can be expanded as in (1.2). The corresponding coefficients of the Laurent series of \( E_{F_r,n}[\text{tr}_w] \) are explained in terms of Euler characteristics of subgroups of mapping class groups. One corollary is that as \( n \to \infty \)

\[
E_{F_r,n}[\text{tr}_w] = O \left( \frac{1}{n^{2\text{cl}(w)-1}} \right)
\]

where \( \text{cl}(w) \) is the the commutator length of \( w \): the minimal number of commutators that \( w \) can be written as a product of, or \( \infty \) if \( w \notin [F_r, F_r] \). We guess that an estimate like (1.3)
should hold for $\mathbb{E}_{g,n}[\text{tr}_g]$ where commutator length in $F_r$ is replaced by commutator length in $\Gamma_g$.

Another strengthening of Theorem 1.1 is the \textit{strong asymptotic freeness} of Haar unitaries. This states that for any complex linear combination

$$\sum_w a_w w \in \mathbb{C}[F_r],$$

almost surely w.r.t. Haar random $\phi \in \text{Hom}(F_r, U(n))$ as $n \to \infty$, we have

$$\|\sum_w a_w \phi(w)\| \to \|\sum_w a_w w\|_{\text{Op}(\mathbb{C}[F_r])}$$

where the left hand side is the operator norm on $\mathbb{C}^n$ with standard Hermitian inner product and the norm on the right hand side is the operator norm in the regular representation of $F_r$. This result was proved by Collins and Male in [CM14]. It is probably very hard to extend this result to $\Gamma_g$; the proof of Collins and Male relies on seminal work of Haagerup and Thorbjørnsen [HT05] in a way that does not obviously extend to $\Gamma_g$.

We finally mention that the expected values $\mathbb{E}_{g,n}[\text{tr}_g]$ arise as a limiting form of expected values of Wilson loops in 2D Yang-Mills theory, when the coupling constant is set to zero. This will not be discussed in detail here, we refer the reader instead to the introduction of [Mag21].

\section*{1.2.1 Notation}

We write $\mathbb{N}$ for the natural numbers and $\mathbb{N}_0 \overset{\text{def}}{=} \mathbb{N} \cup \{0\}$. We write $[n] \overset{\text{def}}{=} \{1, \ldots, n\}$ for $n \in \mathbb{N}$ and $[k, \ell] \overset{\text{def}}{=} \{k, k+1, \ldots, \ell\}$ for $k, \ell \in \mathbb{N}$. If $A$ and $B$ are two sets we write $A \setminus B$ for the elements of $A$ not in $B$. If $H$ is a group and $h_1, h_2 \in H$ we write $[h_1, h_2] \overset{\text{def}}{=} h_1 h_2 h_1^{-1} h_2^{-1}$. We let $\text{id}$ denote the identity element of a group. We let $[H, H]$ be the subgroup of $H$ generated by elements of the form $[h_1, h_2]$; this is called the commutator subgroup of $H$. If $V$ is a complex vector space, for $q \in \mathbb{N}_0$ we let

$$V \otimes^q \overset{\text{def}}{=} \underbrace{V \otimes V \otimes \cdots \otimes V}_{q}.$$ 

We use Vinogradov notation as follows. If $f$ and $h$ are functions of $n \in \mathbb{N}$, we write $f \ll h$ to mean that there are constants $n_0 \geq 0$ and $C_0 \geq 0$ such that for $n \geq n_0$, $f(n) \leq C_0 h(n)$. We write $f = O(h)$ to mean $f \ll |h|$. We write $f \asymp h$ to mean both $f \ll h$ and $h \ll f$. If in any of these statements the implied constants depend on additional parameters we add these parameters as subscript to $\ll, O$, or $\asymp$. Throughout the paper we view the genus $g$ as fixed and so any implied constant may depend on $g$.

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2 Background

2.1 Representation theory of symmetric groups

Let $S_k$ denote the symmetric group of permutations of $[k] \defeq \{1, \ldots, k\}$, and $\mathbb{C}[S_k]$ denote the group algebra. The group $S_0$ is by definition the group with one element.

If we refer to $S_\ell \leq S_k$ with $\ell \leq k$ we always view $S_\ell$ as the subgroup of permutations that fix every element of $[\ell + 1, k] \defeq \{\ell + 1, \ldots, k\}$. We write $S'_\ell \leq S_k$ for the subgroup of permutations that fix every element of $[k - r]$. As a consequence we obtain fixed inclusions $\mathbb{C}[S_\ell] \subset \mathbb{C}[S_k]$ for $\ell$ and $k$ as above. When we write $S_\ell \times S_{k-\ell} \leq S_k$, the first factor is $S_\ell$ and the second factor is $S'_{k-\ell}$.

A Young diagram $\lambda$ is a left-aligned contiguous collection of identical square boxes in the plane, such that the number of boxes in each row is non-increasing from top to bottom. We write $\lambda_i$ for the number of boxes in the $i$th row of $\lambda$ and say $\lambda \vdash k$ if $\lambda$ has $k$ boxes. We write $\ell(\lambda)$ for the number of rows of $\lambda$. For each $\lambda \vdash k$ there is a Young subgroup

$$S_\lambda \defeq S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_{\ell(\lambda)}} \leq S_k$$

where the factors are subgroups in the obvious way, according to the increasing order of $[k]$.

The equivalence classes of irreducible representations of $S_k$ are in one-to-one correspondence with Young diagrams $\lambda \vdash k$. Given $\lambda$, the construction of the corresponding irreducible representation $V^\lambda$ can be done for example using Young symmetrizers as in [FH91, Lec. 4]. We write $\chi_\lambda$ for the character of $S_k$ associated to $V^\lambda$ and $d_\lambda \defeq \chi_\lambda(\text{id}) = \dim V^\lambda$. Given $\lambda \vdash k$, the element

$$p_\lambda \defeq \frac{d_\lambda}{k!} \sum_{\sigma \in S_\ell} \chi_\lambda(\sigma)\sigma \in \mathbb{C}[S_k]$$

is a central idempotent in $\mathbb{C}[S_k]$.

If $G$ is a compact group, $(\rho, W)$ is an irreducible representation of $G$, and $(\pi, V)$ is any finite-dimensional representation of $G$, the $(\rho, W)$-isotypic subspace of $(\pi, V)$ is the invariant subspace of $V$ spanned by all irreducible direct summands of $(\pi, V)$ that are isomorphic to $(\rho, W)$. When $\rho$ and $\pi$ can be inferred from $W$ and $V$ we call this simply the $W$-isotypic subspace of $V$. If $H \leq G$ is a subgroup, and $(\rho, W)$ is an irreducible representation of $H$, then the $W$-isotypic subspace of $V$ for $H$ is the $W$-isotypic subspace of the restriction of $(\pi, V)$ to $H$.

If $(\pi, V)$ is any finite-dimensional unitary representation of $S_k$, and $\lambda \vdash k$, then $V$ is also a module for $\mathbb{C}[S_k]$ by linear extension of $\pi$ and $\pi(p_\lambda)$ is the orthogonal projection onto the $V^\lambda$-isotypic subspace of $V$.

For any compact group $G$ we write $(\text{triv}_G, \mathbb{C})$ for the trivial representation of $G$. The following lemma can be deduced for example by combining Young’s rule [FH91, Cor. 4.39] with Frobenius reciprocity.

**Lemma 2.1.** Let $k \in \mathbb{N}_0$, and $\lambda \vdash k$. The $(\text{triv}_{S_\lambda}, \mathbb{C})$-isotypic subspace of $V^\lambda$ for the group $S_\lambda$ is one-dimensional, i.e. $(\text{triv}_{S_\lambda}, \mathbb{C})$ occurs with multiplicity one in the restriction of $V^\lambda$ to $S_\lambda$. 


2.2 Representation theory of $U(n)$ and $SU(n)$

Every irreducible representation of $U(n)$ restricts to an irreducible representation of $SU(n)$, and all equivalence classes of irreducible representations of $SU(n)$ arise in this way. The equivalence classes of irreducible representations of $U(n)$ are parameterized by dominant weights, that can be thought of as non-increasing sequences

$$\Lambda = (\Lambda_1, \ldots, \Lambda_n) \in \mathbb{Z}^n,$$

also known as the signature. We write $W^\Lambda$ for the irreducible representation of $U(n)$ corresponding to the signature $\Lambda$. Let $T(n)$ denote the maximal torus of $U(n)$ consisting of diagonal matrices diag(exp($i\theta_1$), ..., exp($i\theta_n$)) where all $\theta_j \in \mathbb{R}$. Associated to the signature $\Lambda$ is the character $\xi_\Lambda$ of $T(n)$ given by

$$\xi_\Lambda (\text{diag}(\exp(i\theta_1), \ldots, \exp(i\theta_n))) \overset{\text{def}}{=} \exp \left( i \left( \sum_{j=1}^n \Lambda_j \theta_j \right) \right).$$

The highest weight theory says that, among other things, the $\xi_\Lambda$-isotypic subspace of $W^\Lambda$ for $T(n)$ is one-dimensional. Any vector in this subspace is called a highest weight vector of $W^\Lambda$.

Given $k, \ell \in \mathbb{N}_0$ and fixed Young diagrams $\mu \vdash k$, $\nu \vdash \ell$, we define a family of representations of $U(n)$ as follows. For $n \geq \ell(\mu) + \ell(\nu)$ define

$$\Lambda_{\mu,\nu}(n) \overset{\text{def}}{=} (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}; 0, \ldots, 0; -\nu_{\ell(\nu)}, -\nu_{\ell(\nu)-1}, \ldots, -\nu_1).$$

We let $(\rho_n^{[\mu,\nu]}, W_n^{[\mu,\nu]})$ denote the irreducible representation of $U(n)$ corresponding to $\Lambda_{\mu,\nu}(n)$ when $n \geq \ell(\mu) + \ell(\nu)$. We let $D_{[\mu,\nu]}(n) \overset{\text{def}}{=} \dim W_n^{[\mu,\nu]}$ and $s_{[\mu,\nu]}(g) \overset{\text{def}}{=} \text{tr}(\rho_n^{[\mu,\nu]}(g))$ for $g \in U(n)$. If $\mu \vdash k$ and $\nu \vdash \ell$ then as $n \to \infty$

$$D_{[\mu,\nu]}(n) \asymp n^{k+\ell} \quad (2.1)$$

by [Mag21, Cor. 2.3] (alternatively [EI16, Lem. 3.5]).

We now present a version of Schur-Weyl duality for mixed tensors due to Koike [Koi89]. The very definition of $U(n)$ makes $\mathbb{C}^n$ into a unitary representation of $U(n)$ for the standard Hermitian inner product. We let $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{C}^n$. If $(\rho, W)$ is any finite dimensional representation of $U(n)$ we write $(\bar{\rho}, \bar{W})$ for the dual representation where $\bar{W}$ is the space of complex linear functionals on $W$. The vector space $\mathbb{C}^n$ has a dual basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$ given by $\bar{e}_j(v) \overset{\text{def}}{=} \langle v, e_j \rangle$. Throughout the paper we frequently use certain canonical isomorphisms e.g.

$$(\mathbb{C}^n)^\otimes p \cong (\bar{\mathbb{C}}^n)^\otimes p, \quad \text{End}(W) \cong W \otimes \bar{W}$$

to change points of view on representations; if we use non-canonical isomorphisms we point them out.

Let $T^{k,\ell}_n \overset{\text{def}}{=} (\mathbb{C}^n)^\otimes k \otimes (\bar{\mathbb{C}}^n)^\otimes \ell$. With the natural inner product induced by that on $\mathbb{C}^n$, this is a unitary representation of $U(n)$ under the diagonal action and also a unitary representation of $S_k \times S_\ell$ where $S_k$ acts by permuting the indices of $(\mathbb{C}^n)^\otimes k$ and $S_\ell$ acts by permuting the
indices of \((\mathbf{C}^n)^\otimes \ell\). We write \(\pi_n^{k,\ell} : U(n) \rightarrow \text{End}(\mathcal{T}_n^{k,\ell})\) and \(\rho_n^{k,\ell} : C[S_k \times S_\ell] \rightarrow \text{End}(\mathcal{T}_n^{k,\ell})\) for these representations. By convention \((\mathbf{C}^n)^{\otimes 0} = \mathbf{C}\). Recall that \(S'_\ell\) is our notation for the second factor of \(S_k \times S_\ell\). The actions of \(U(n)\) and \(S_k \times S_\ell\) on \(\mathcal{T}_n^{k,\ell}\) commute. We use the notation, for \(I = (i_1, \ldots, i_k)\) and \(J = (j_1, \ldots, j_\ell)\)

\[
e_i \overset{\text{def}}{=} e_{i_1} \otimes \cdots \otimes e_{i_k} \in (\mathbf{C}^n)^\otimes k, \quad \hat{e}_j \overset{\text{def}}{=} \hat{e}_{j_1} \otimes \cdots \otimes \hat{e}_{j_\ell} \in (\mathbf{C}^n)^\otimes \ell,
\]

where \(\hat{e}_j\) is the dual element to \(e_j\). We write \(I \sqcup J\) for the concatenation \((i_1, \ldots, i_k, j_1, \ldots, j_\ell)\).

For \(k, \ell \geq 1\) let \(\mathcal{T}_n^{k,\ell}\) denote the intersection of the kernels of the mixed contractions \(c_{pq} : \mathcal{T}_n^{k,\ell} \rightarrow \mathcal{T}_n^{k-1,\ell-1}, p \in [k], q \in [\ell]\) given by

\[
c_{pq}(e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes \hat{e}_{j_1} \otimes \cdots \otimes \hat{e}_{j_\ell}) = \delta_{ipq} e_{i_1} \otimes \cdots \otimes e_{i_{p-1}} \otimes e_{i_{p+1}} \otimes \cdots \otimes e_{i_k} \otimes \hat{e}_{j_1} \otimes \cdots \otimes \hat{e}_{j_{q-1}} \otimes \hat{e}_{j_{q+1}} \otimes \cdots \otimes \hat{e}_{j_\ell}, \quad (2.2)
\]

where \(\delta_{ipjq}\) is the Kronecker delta. If \(k = 1\) or \(\ell = 1\) then the definition is extended in the natural way, interpreting an empty tensor of \(e_i\) or \(\hat{e}_i\) as 1. If either \(k = 0\) or \(\ell = 0\) then \(\mathcal{T}_n^{k,\ell} = \mathcal{T}_n^{k,\ell}\) by convention. The space \(\mathcal{T}_n^{k,\ell}\) is an invariant subspace under \(U(n) \times S_k \times S_\ell\) and hence a unitary subrepresentation of \(\mathcal{T}_n^{k,\ell}\). On \(\mathcal{T}_n^{k,\ell}\) there is an analog of Schur-Weyl duality due to Koike.

**Theorem 2.2.** [Koi89, Thm. 1.1] There is an isomorphism of unitary representations of \(U(n) \times S_k \times S_\ell\)

\[
\mathcal{T}_n^{k,\ell} \cong \bigoplus_{\mu \vdash k, \nu \vdash \ell, \ell(\mu) + \ell(\nu) \leq n} W_n^{[\mu,\nu]} \otimes V^\mu \otimes V^\nu. \quad (2.3)
\]

Next we explain how to construct \(U(n)\)-subrepresentations of \(\mathcal{T}_n^{k,\ell}\) isomorphic to \(W_n^{[\mu,\nu]}\). Suppose that \(\xi \in \mathcal{T}_n^{k,\ell}\) is a non-zero vector such that under the isomorphism (2.3),

\[
\xi \cong w \otimes v \quad (2.4)
\]

for \(w \in W_n^{[\mu,\nu]}\) and \(v \in V^\mu \otimes V^\nu\). Then \(U(n)\), \(\xi\) linearly spans a \(U(n)\)-subrepresentation of \(\mathcal{T}_n^{k,\ell}\) isomorphic to \(W_n^{[\mu,\nu]}\). The following argument to construct such a vector \(\xi\), given \(\mu \vdash k, \nu \vdash \ell\), appears implicitly in [Koi89] and is elaborated in [BCH94]. For \(n \geq \ell(\mu) + \ell(\nu)\) let

\[
\tilde{\theta}_n^{[\mu,\nu]} = e_1 \otimes \cdots \otimes e_1 \otimes e_2 \otimes \cdots \otimes e_2 \otimes \cdots \otimes e_{\ell(\mu)} \otimes \cdots \otimes e_{\ell(\mu)} \otimes \hat{e}_{\ell(\nu)+1} \otimes \cdots \otimes \hat{e}_{\ell(\nu)+1} \quad (2.5)
\]

This vector is in the \(\xi^{[\mu,\nu]}\)-isotypic subspace of \(\mathcal{T}_n^{k,\ell}\) for the maximal torus \(T(n)\) of \(U(n)\) where \(\xi^{[\mu,\nu]}\) is the character of \(T(n)\) corresponding to the highest weight in \(W_n^{[\mu,\nu]}\).

Let \(p_\mu \in C[S_k]\), \(p_\nu \in C[S'_\ell]\) be the projections defined in \S 2.1. Let \(\rho_n^{k} : S_k \rightarrow \text{End}(\mathcal{T}_n^{k,\ell})\) denote the representation of \(S_k\) described above and \(\hat{\rho}_n^{\ell} : S'_\ell \rightarrow \text{End}(\mathcal{T}_n^{k,\ell})\) that of \(S'_\ell\). Clearly
these two representations commute. Now let
\[ \theta_{[\mu,\nu]}^n \overset{\text{def}}{=} \rho_k^\mu(p_\mu)\rho_\ell^\nu(p_\nu) \tilde{\theta}_{[\mu,\nu]}^n \in T_n^{k,\ell}. \] (2.6)

Now this is in the same isotypic subspace for \( T(n) \) as before since \( S_k \times S_\ell \) commutes with \( U(n) \). Moreover it is in the subspace of \( T_n^{k,\ell} \) corresponding to \( W_n^{[\mu,\nu]} \otimes V^\mu \otimes V^\nu \) under the isomorphism (2.3). The intersection of the two subspaces of \( T_n^{k,\ell} \) just discussed corresponds via (2.3) to \( Cw \otimes V^\mu \otimes V^\nu \) where \( w \) is a highest-weight vector in \( W_n^{[\mu,\nu]} \) and hence \( \theta_{[\mu,\nu]}^n \) takes the form of (2.4) as desired.

Of course we also want to know \( \theta_{[\mu,\nu]}^n \neq 0 \).

**Lemma 2.3.** Suppose that \( k, \ell \in \mathbb{N}_0, \mu \vdash k, \nu \vdash \ell, \) and \( \theta_{[\mu,\nu]}^n \) is as in (2.6) for \( n \geq \ell(\mu) + \ell(\nu) \). We have \( \| \theta_{[\mu,\nu]}^n \| > 0 \) and \( \| \theta_{[\mu,\nu]}^n \| \) does not depend on \( n \), only on \( \mu, \nu \).

**Proof.** Recall the definition of Young subgroups \( S_\mu, S_\nu \) from \S 2.1. Letting \( \tilde{\theta} = \tilde{\theta}_{[\mu,\nu]}^n \) (as in (2.5)) and \( \theta = \theta_{[\mu,\nu]}^n \) we have

\[
\theta = \frac{d_\mu d_\nu}{k! \ell!} \sum_{[\sigma_1] \in S_\mu \atop [\sigma_2] \in S_\ell} \chi_\mu(\sigma_1) \chi_\nu(\sigma_2) \rho_n^k(\sigma_1) \rho_n^\ell(\sigma_2) \tilde{\theta},
\]

\[
= \frac{d_\mu d_\nu}{k! \ell!} \sum_{[\sigma_1] \in S_\mu \atop [\sigma_2] \in S_\ell} \left( \sum_{\tau_1 \in S_\mu} \chi_\mu(\sigma_1 \tau_1) \chi_\nu(\sigma_2 \tau_2) \right) \rho_n^k(\sigma_1) \rho_n^\ell(\sigma_2) \tilde{\theta},
\]

\[
= \frac{d_\mu d_\nu}{k! \ell!} \sum_{[\sigma_1] \in S_\mu \atop [\sigma_2] \in S_\ell} \chi_\mu(\sigma_1 \tau_1) \left( \sum_{\tau_2 \in S_\nu} \chi_\nu(\sigma_2 \tau_2) \right) \rho_n^k(\sigma_1) \rho_n^\ell(\sigma_2) \tilde{\theta}.
\]

The second equality used that \( \tilde{\theta} \) is invariant under \( S_\mu \times S_\nu \).

By Lemma 2.1, there is a one dimensional subspace of invariant vectors for \( S_\mu \) in \( V^\mu \). If \( v_\mu \in V^\mu \) is a unit vector in this space then

\[
\sum_{\tau_1 \in S_\mu} \chi_\mu(\sigma_1 \tau_1) = |S_\mu| \langle \sigma_1 v_\mu, v_\mu \rangle. \] (2.7)

Since the vectors \( \rho_n^k(\sigma_1) \rho_n^\ell(\sigma_2) \tilde{\theta} \) for \( [\sigma_1] \in S_k / S_\mu \) and \( [\sigma_2] \in S_\ell / S_\nu \) are orthogonal unit vectors, this gives

\[
\| \theta \|^2 = \left( \frac{d_\mu d_\nu}{k! \ell!} \right)^2 \sum_{[\sigma_1] \in S_\mu \atop [\sigma_2] \in S_\ell} \left( \sum_{\tau_1 \in S_\mu} \chi_\mu(\sigma_1 \tau_1) \right)^2 \left( \sum_{\tau_2 \in S_\nu} \chi_\nu(\sigma_2 \tau_2) \right)^2.
\]

This clearly does not depend on \( n \). To see that \( \| \theta \| \neq 0 \), the contribution from any \( [\sigma_1], [\sigma_2] \) is non-negative and the contribution from \( \sigma_1 = \text{id}, \sigma_2 = \text{id} \) is by (2.7) equal to \( \left( \frac{d_\mu d_\nu}{k! \ell!} \right)^2 |S_\mu|^2 |S_\nu|^2 \neq 0 \).

Recall that we write \( \pi_n^{k,\ell} : U(n) \rightarrow \text{End}(T_n^{k,\ell}) \) for the diagonal representation of \( U(n) \) on \( T_n^{k,\ell} \). By the remarks surrounding (2.4), Lemma 2.3 implies the following corollary.
Corollary 2.4. Suppose \( n \geq \ell(\mu) + \ell(\nu) \). The subspace
\[
W_n(\theta_{[\mu,\nu]}^n) \overset{\text{def}}{=} \text{span}\{\pi_n^{k,\ell}(u)\theta_{[\mu,\nu]}^n : u \in U(n)\} \subset \hat{T}_n^{k,\ell}
\]
is, under \( \pi_n^{k,\ell} \), a \( U(n) \)-subrepresentation of \( \hat{T}_n^{k,\ell} \) isomorphic to \( W_n^{[\mu,\nu]} \).

2.3 The Weingarten calculus

The Weingarten calculus is a method based on Schur-Weyl duality that allows one to calculate integrals of products of matrix coefficients in the defining representation of \( U(n) \) in terms of sums over permutations. It was developed initially by physicists in [Wei78, Xu97] and then more precisely by Collins and Śniady [CS06, Cor. 2.4].

We present two formulations of the Weingarten calculus. Given \( k \in \mathbb{N} \), \( n \in \mathbb{N} \), the Weingarten function with parameters \( n, k \) is the following element of \( C[S_k] \) [CS06, eq. (9)]
\[
W_{g_{n,k}} \overset{\text{def}}{=} \frac{1}{(k!)^2} \sum_{\lambda \vdash n} \frac{d_\lambda^2}{D_\lambda(n)} \sum_{\sigma \in S_k} \chi_\lambda(\sigma)\sigma. \tag{2.8}
\]

We write \( W_{g_{n,k}}(\sigma) \) for the coefficient of \( \sigma \) in (2.8). The following theorem was proved by Collins and Śniady [CS06, Cor. 2.4].

Theorem 2.5. For \( k \in \mathbb{N} \) and for \( i_1, i'_1, j_1, j'_1, \ldots, i_k, i'_k, j_k, j'_k \in [n] \)
\[
\int_{u \in U(n)} u_{i_1 j_1} \cdots u_{i_k j_k} \bar{u}_{i'_1 j'_1} \cdots \bar{u}_{i'_k j'_k} \mu_{U(n)}(u) \tag{2.9}
\]
\[
= \sum_{\sigma, \tau \in S_k} \delta_{i_1' j_1(1)} \cdots \delta_{i_k' j_k(1)} \delta_{j_1' j_1(1)} \cdots \delta_{j_k' j_k(1)} W_{g_{n,k}}(\tau \sigma^{-1}),
\]
where \( \delta_{pq} \) is the Kronecker delta function.

It is sometimes more flexible to reformulate Theorem 2.5 in terms of projections. Here \( u \in U(n) \) acts on \( A \in \text{End}((\mathbb{C}^n)^\otimes k) \) by \( A \mapsto \pi_n^k(u) A \pi_n^k(u^{-1}) \), \( \pi_n^k : U(n) \to \text{End}((\mathbb{C}^n)^\otimes k) \) the diagonal action. Let \( \rho_n^k : C[S_k] \to \text{End}((\mathbb{C}^n)^\otimes k) \) be the map induced by \( S_k \) permuting coordinates of \((\mathbb{C}^n)^\otimes k\). Write \( P_{n,k} \) for the orthogonal projection in \( \text{End}((\mathbb{C}^n)^\otimes k) \) onto the \( U(n) \)-invariant vectors. The following proposition is due to Collins and Śniady [CS06, Prop. 2.3].

Proposition 2.6 (Collins-Śniady). Let \( n, k \in \mathbb{N} \). Suppose \( A \in \text{End}((\mathbb{C}^n)^\otimes k) \). Then
\[
P_{n,k}[A] = \rho_n^k(\Phi[A] \cdot W_{g_{n,k}})
\]
where
\[
\Phi[A] \overset{\text{def}}{=} \sum_{\sigma \in S_k} \text{tr}(A \rho_n^k(\sigma^{-1}))\sigma.
\]

Later we will need the following bound for the Weingarten function due to Collins and Śniady [CS06, Prop. 2.6]

Proposition 2.7. For any fixed \( \sigma \in S_k \), \( W_{g_{n,k}}(\sigma) = O_k(n^{-k-|\sigma|}) \) as \( n \to \infty \), where \( |\sigma| \) is the minimum number of transpositions that \( \sigma \) can be written as a product of.
2.4 Free groups and surface groups

Let $F_{2g} \overset{\text{def}}{=} \langle a_1, b_1, \ldots, a_g, b_g \rangle$ be the free group on $2g$ generators $a_1, b_1, \ldots, a_g, b_g$ and $R_g \overset{\text{def}}{=} [a_1, b_1] \cdots [a_g, b_g] \in F_{2g}$. Therefore we have a quotient map $F_{2g} \to \Gamma_g$ given by reduction modulo $R_g$. We say that $w \in F_{2g}$ represents the conjugacy class of $\gamma \in \Gamma_g$ if the projection of $w$ to $\Gamma_g$ is in the conjugacy class of $\gamma$ in $\Gamma_g$.

Given $w \in F_{2g}$ we view $w$ as a combinatorial word in $a_1, a_1^{-1}, b_1, b_1^{-1}, \ldots, a_g, a_g^{-1}, b_g, b_g^{-1}$ by writing it in reduced (shortest) form; i.e., $a_1$ does not follow $a_1^{-1}$ etc. We say that $w$ is cyclically reduced if the first letter of its reduced word is not the inverse of the last letter. The length $|w|$ of $w \in F_{2g}$ is the length of its reduced form word. We say $w \in F_{2g}$ is a shortest element representing the conjugacy class of $\gamma \in \Gamma_g$ if it has minimal length among all elements representing the conjugacy class of $\gamma$. If $w$ is a shortest element representing some conjugacy class in $\Gamma_g$ then $w$ is cyclically reduced.

For any group $H$, the commutator subgroup $[H, H] \leq H$ is the subgroup generated by all elements of the form $[h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1}$ with $h_1, h_2 \in H$. It is not hard to see that if $\gamma \in [\Gamma_g, \Gamma_g]$, and $w$ represents the conjugacy class of $\gamma$, then $w \in [F_{2g}, F_{2g}]$.

2.5 Witten zeta functions

Witten zeta functions appeared first in Witten’s work [Wit91] and were named by Zagier in [Zag94]. The Witten zeta function of $SU(n)$ is defined, for $s$ in a half-plane of convergence, by

$$
\zeta(s; n) \overset{\text{def}}{=} \sum_{\mu, \nu \in SU(n)} \frac{1}{(\dim W)^s} \quad (2.10)
$$

where $SU(n)$ denotes the equivalence classes of irreducible representations of $SU(n)$. Indeed, the series $(2.10)$ converges for $\Re(s) > \frac{2}{n}$ by a result of Larsen and Lubotzky [LL08, Thm. 5.1] (see also [HS19, §2]). Also relevant to this work is a result of Guralnick, Larsen, and Manack [GLM12, Thm 2., also eq. (7)] that states for fixed $s > 0$

$$
\lim_{n \to \infty} \zeta(s; n) = 1. \quad (2.11)
$$

2.6 Results of the prequel paper

By [Mag21, Prop. 1.5], if $\gamma \notin [\Gamma_g, \Gamma_g]$, $E_{g,n}[\text{tr}_\gamma] = 0$ for $n \geq n_0(\gamma)$. This proves Theorem 1.2 in this case. Hence in the rest of the paper we need only consider $\gamma \in [\Gamma_g, \Gamma_g]$ and hence $w \in [F_{2g}, F_{2g}]$ if $w \in F_{2g}$ represents the conjugacy class of $\gamma$.

For each $w \in F_{2g}$, we have a word map $w : U(n)^{2g} \to U(n)$ obtained by substituting matrices for the generators of $F_{2g}$. For example, if $u_1, v_1, \ldots, u_g, v_g \in U(n)$ then $R_g(u_1, v_1, \ldots, u_g, v_g) = [u_1, v_1] \cdots [u_g, v_g]$. We begin with the following result from the prequel paper [Mag21, Cor. 1.8].

**Proposition 2.8.** Suppose that $g \geq 2$, $\gamma \in \Gamma_g$, and $w \in F_{2g}$ represents the conjugacy class of $\gamma$. For any $B \in \mathbb{N}$ we have as $n \to \infty$

$$
E_{g,n}[\text{tr}_\gamma] = \zeta(2g - 2; n)^{-1} \sum_{\mu, \nu \text{ Young diagrams}} D_{\mu, \nu}(n) I_n(w, [\mu, \nu]) + O_{B,w,g} \left( n |w|^{-2 \log B} \right). \quad (2.12)
$$
where
\[ I_n(w, [\mu, \nu]) \overset{\text{def}}{=} \int \text{tr}(w(x)) s_{[\mu, \nu]}(R_g(x)) d\mu^{\text{Haar}}_{\text{SU}(n)_{2g}}(x) \] (2.13)

Notice that for \( n \geq 2B \) the right hand side of (2.12) makes sense, i.e. \( D_{[\mu, \nu]}, s_{[\mu, \nu]} \) are well-defined. We also have by [Mag21, Prop. 3.1] and \( s_{[\mu, \nu]} = s_{\nu, \mu} \) the following proposition.

**Proposition 2.9.** Let \( w \in [F_{2g}, F_{2g}] \). Then for any fixed \( \mu, \nu \) and \( n \geq \ell(\mu) + \ell(\nu) \)
\[ I_n(w, [\mu, \nu]) = J_n(w, [\nu, \mu]) \overset{\text{def}}{=} \int \text{tr}(w(x)) s_{[\nu, \mu]}(R_g(x)) d\mu^{\text{Haar}}_{U(n)_{2g}}(x). \]

This is convenient as it will allow us to use the Weingarten calculus directly as it is presented in §§2.3 for \( U(n) \) rather than \( SU(n) \). By using Proposition 2.9, taking a representative \( w \in F_{2g} \) of the conjugacy class of \( \gamma \) and taking \( B \) such that \( |w| - 2\log B \leq -1 \) in Proposition 2.8 we obtain the following result from which we begin the new arguments of this paper.

**Corollary 2.10.** Let \( \gamma \in [\Gamma_g, \Gamma_g] \) and \( w \in [F_{2g}, F_{2g}] \) be a representative of the conjugacy class of \( \gamma \in \Gamma \). Then there exists a finite set \( \tilde{\Omega} \) of pairs \( (\mu, \nu) \) of Young diagrams that
\[ E_{g,n}[\text{tr} \gamma] = \zeta(2g - 2; n)^{-1} \sum_{(\mu, \nu) \in \tilde{\Omega}} D_{[\mu, \nu]}(n) J_n(w, [\mu, \nu]) + O_{w,g} \left( \frac{1}{n} \right). \]

As we know \( \lim_{n \to \infty} \zeta(2g - 2, n) = 1 \) by (2.11), we have now reduced the proof of Theorem 1.2 to establishing suitable bounds for the integrals \( J_n(w, [\mu, \nu]) \) where we can view \( \mu, \nu \) as fixed Young diagrams since \( \tilde{\Omega} \) is finite.

## 3 Combinatorial Integration

### 3.1 Setup and motivation

The main result of this §3 is the following.

**Theorem 3.1.** For \( \gamma \in \Gamma_g, \gamma \neq \text{id} \), if \( w \in F_{2g} \) is a cyclically shortest word representing the conjugacy class of \( \gamma \), we have for \( \mu \vdash k, \nu \vdash \ell \) fixed,
\[ D_{[\mu, \nu]}(n) J_n(w, [\mu, \nu]) = O_{w,k,\ell}(1) \]
as \( n \to \infty \).

Accordingly, since we know the large \( n \) behaviour of \( D_{[\mu, \nu]}(n) \) from (2.1), in this section we wish to estimate
\[ J_n(w, [\mu, \nu]) = \int \text{tr}(w(x)) s_{[\mu, \nu]}(R_g(x)) d\mu^{\text{Haar}}_{U(n)_{2g}}(x) \]
for fixed \( \mu \vdash k, \nu \vdash \ell \). We begin by discussing why the most straightforward approach to this problem leads to serious complications. It is possible to approach the problem by writing \( s_{[\mu, \nu]}(h) \) as a fixed finite linear combinations of functions
\[ p_{\mu'}(h)p_{\nu'}(h^{-1}) \]
where \( p_{\mu'}(h) \) (resp. \( p_{\mu'}(h^{-1}) \)) is a power sum symmetric polynomial of the eigenvalues of \( h \) (resp. \( h^{-1} \) or \( \tilde{h} \)). See for example [Mag21, §§3.3] for one way to do this. The coefficients of this expansion are fixed, but not transparent, since they involve Littlewood-Richardson coefficients. In any case, this approach leads to writing \( J_n(w, [\mu, \nu]) \) as a finite linear combination of integrals of the form

\[
\int \text{tr}(w(x))\text{tr} \left( R_g(x)^{k_1} \right) \cdots \text{tr} \left( R_g(x)^{k_p} \right) \text{tr} \left( R_g(x)^{-\ell_1} \right) \cdots \text{tr} \left( R_g(x)^{-\ell_q} \right) d\mu_{U(n)2g}^{\text{Haar}}(x) \tag{3.1}
\]

where \( \sum k_j = |\mu| \) and \( \sum \ell_j = |\nu| \).

The work of the author and Puder in [MP19] gives a full asymptotic expansion for (3.1) as \( n \to \infty \). However these estimates are not sufficient for the current paper and to motivate the rest of this §3 we explain briefly the issues involved. However, this discussion is not needed to understand the arguments that we will make to prove Theorem 3.1.

The main result of [MP19] gives a full ‘genus’ expansion of (3.1) in terms of surfaces and maps on surfaces dictated by \( w \in \mathcal{F}_{2g} \). Roughly speaking, every term in this expansion comes from a homotopy class of map \( f \) from an orientable surface \( \Sigma_f \) to \( \bigvee_{i=1}^r S^1 \); to contribute to (3.1) the surface \( \Sigma_f \) has one boundary component that maps to \( w \) at the level of the fundamental groups, \( p \) boundary components that map respectively to \( R_g^{k_1}, \ldots, R_g^{k_p} \) at the level of fundamental groups, and \( q \) boundary components that map respectively to \( R_g^{-\ell_1}, \ldots, R_g^{-\ell_q} \) at the level of fundamental groups. The contribution of the pair \((f, \Sigma_f)\) to (3.1) is of the form \( c(f, \Sigma_f) n^{\chi(\Sigma_f)} \); the coefficient \( c(f, \Sigma_f) \) is an Euler characteristic of a symmetry group of \((f, \Sigma_f)\) and is not easy to calculate in general. However, one could still hope to get decay of (3.1) by controlling the possible \( \chi(\Sigma_f) \) that could appear.

There are two issues with this. The first one is that if \( w \) is not the shortest element representing the conjugacy class of \( \gamma \) then we get bounds that are not helpful. For a very simple example, let \( w = R_g^\ell, \gamma = \text{id}_g \), and consider the potential contribution from \( p = 0, q = 1, \ell_1 = \ell \). Then for any \( \nu \) with \(|\nu| = \ell \) there is contribution to \( J_n(w, [\varnothing, \nu]) \) that is a multiple of

\[
\int \text{tr} \left( R_g(x)^{\ell} \right) \text{tr} \left( R_g(x)^{-\ell} \right) d\mu_{U(n)2g}^{\text{Haar}}(x).
\]

Here, in the theory of [MP19] there is a \((\Sigma_f, f)\) that is an annulus, one boundary component corresponding to \( w = R_g^\ell \) and one corresponding to \( R_g^{-\ell} \), so we can only bound the corresponding contribution to \( D_{[\varnothing, \nu]}(n)J_n(w, [\varnothing, \nu]) \) by using [MP19] on the order of \( D_{[\varnothing, \nu]}(n) \propto n^\ell \). On the other hand, any approach that works to establish Theorem 3.1 (for \( \gamma \neq \text{id} \)) should extend to show when \( \gamma = \text{id}_g D_{[\varnothing, \nu]}(n)J_n(w, [\varnothing, \nu]) \ll n \) as \( E_{g,n}[\text{tr}_{\text{id}}] = n \).

Indeed, this phenomenon extends to words of the form \( w_0 R_g^\ell \) and more generally to words that are not shortest representatives of some conjugacy class in \( \Gamma_g \). It means that even if we use something similar in spirit to [MP19], to prove Theorem 3.1 we must incorporate the theory of shortest representative words. This indeed takes place in §§4.3–§§4.5; the topological result proved there hinges on this theory.

The second issue is a little more subtle and only appears for ‘mixed’ representations, i.e., both \( \mu, \nu \neq \varnothing \). In this case, suppose \( w \) is a shortest element representing some conjugacy class in \( \Gamma_g \) and \( w \in [\mathcal{F}_{2g}, \mathcal{F}_{2g}] \). This means that there is a pair \((f_0, \Sigma f_0)\) where \( \Sigma f_0 \) has one boundary component that maps to \( w \) at the level of the fundamental groups. Let us take \([\mu, \nu] = [(k), (k)] \), i.e each Young diagram has one row of \( k \) boxes. This means we get a
potential contribution to $D_{[\mu,\nu]}(n)J_n(w, [\mu, \nu])$ that is a constant multiple of

$$D_{[(k), (k)]}(n) \int_{x \in U(n)g} \text{tr}(w(x)) \text{tr}(R_g(x)^k) \text{tr}(R_g(x)^{-k}) \, d\mu_{\text{Haar}}^{U(n)g}(x) \tag{3.2}$$

Now, for every $k \in \mathbb{N}$, there is $(f, \Sigma_f)$ contributing to (3.2) with one component that is $(f_0, \Sigma_{f_0})$ and the other is an annulus with boundary components corresponding to $R_g^k, R_g^{-k}$. Since the annulus has Euler characteristic 0, and $D_{[(k), (k)]} \asymp n^{2k}$, the order of this contribution to $D_{[(k), (k)]}(n)J_n(w, [(k), (k)])$ is potentially $\gg n^{2k} n^{\chi(\Sigma_{f_0})}$. For large enough $k$ the exponent here is arbitrarily large, which is clearly catastrophic. In reality, this contribution must cancel with some other contribution but we do not know how to see these cancellations.

To bypass this we produce a refined version of the Weingarten calculus that leads to a restricted set of surfaces, for instance, not including the ones causing the problem above as well as all generalizations of this issue. The restriction we obtain is summarized in the forbidden matching property below (§§3.4) and property P4 (§§4.3).

### 3.2 Proof of Theorem 3.1 when $k = \ell = 0$

Here we give a proof of Theorem 3.1 when $k = \ell = 0$. This will allow us to bypass the slightly confusing issue of using the Weingarten function $W_{\Sigma, n, k+\ell}$ when $k + \ell = 0$ in §§3.3.

If $k = \ell = 0$ then the only possible $\mu \vdash k, \nu \vdash \ell$ are empty Young diagrams $\mu = \nu = \emptyset$, and $W^{[0,0]}_n$ is the trivial representation of $U(n)$, so $D_{[0,0]}(n) = 1$ for all $n \geq 1$ and $s_{[0,0]}(h) = 1$ for all $h \in U(n)$. We then have

$$D_{[0,0]}(n)J_n(w, [\emptyset, \emptyset]) = J_n(w, [\emptyset, \emptyset]) = \int \text{tr}(w(x))d\mu_{\text{Haar}}^{U(n)g}(x). \tag{3.3}$$

If $w \in F_{2g}$ is a cyclically shortest word representing the conjugacy class of $\gamma \in \Gamma_g$ with $\gamma \neq \text{id}$, then $w \neq \text{id}$. It then follows from (1.1) that $D_{[0,0]}(n)J_n(w, [\emptyset, \emptyset]) = o_w(n)$ as $n \to \infty$, but in fact, (3.3) is given by a rational function of $n$ for $n \geq n_0(w)$ by a straightforward application of the Weingarten calculus [MP19]. This implies $D_{[0,0]}(n)J_n(w, [\emptyset, \emptyset]) = O_w(1)$ as $n \to \infty$, as required.

This proves Theorem 3.1 when $k = \ell = 0$. Hence in the rest of this §3 we can assume $k + \ell > 0$.

### 3.3 A projection formula

Here we develop an integral calculus that is more powerful than the usual Weingarten calculus and allows us to directly tackle $J_n(w, [\mu, \nu])$ without writing it in terms of integrals as in (3.1). The key point is that our method leads to the forbidden matchings property of §§3.4 and property P4 of §§4.3.

We now view $k, \ell, \mu \vdash k, \nu \vdash \ell$ as fixed, assume $k + \ell > 0$, $n \geq \ell(\mu) + \ell(\nu)$, and write $\theta = \theta_{[\mu, \nu]}^n$ as in (2.6), suppressing the dependence on $n$. Let $W_n(\theta)$ be defined as in Corollary 2.4. Thus $W_n(\theta)$ is an irreducible summand of $\hat{T}_n^{k,\ell}$ isomorphic to $W_n^{[\mu, \nu]}$ for the group $U(n)$.

Our first task is to compute the orthogonal projection $\mathfrak{q}_n^\theta$ onto $W_n(\theta)$. Let $P_\theta$ denote the orthogonal projection in $\hat{T}_n^{k,\ell}$ onto $\theta$. We also view $P_\theta$ as an element of $\text{End}(\hat{T}_n^{k,\ell})$ by restriction.
Under the canonical isomorphism $\text{End}(\mathcal{T}^k_\ell) \cong \mathcal{T}^k_\ell \otimes \mathcal{T}^k_\ell$ we have $P_\theta \cong \frac{\theta \otimes \bar{\theta}}{||\theta||^2}$ and also from (2.6)

$$P_\theta = \frac{1}{||\theta||^2}P^k_\ell(p_{\mu})\rho^n_\ell(p_{\nu})[\bar{\theta}_{\mu,\nu}] \otimes \bar{\theta}_{\mu,\nu}\rho^n_\ell(p_{\mu})\rho^n_\ell(p_{\nu});$$

here the inner square bracket is interpreted as an element of $\text{End}(\mathcal{T}^k_\ell)$. By Schur’s lemma we have

$$q_\theta^\rho = D_{[\mu,\nu]}(n) \int_{h \in U(n)} \pi_n(h)P_\theta P_n(h^{-1})d\mu_{U(n)}(h)$$

since the right hand side is an element of $\text{End}(W_n(\theta)) \subset \text{End}(\mathcal{T}^k_\ell)$ that commutes with $\pi_n(\mathbf{U}(n))$, so it is a multiple of $q_\theta^\rho$, and it has the correct trace.

On the other hand, we can view $\mathcal{T}^{k,\ell}_n \otimes \mathcal{T}^{k,\ell}_n \cong \mathcal{T}^{k,\ell+k,\ell}_n$ by the canonical isomorphism

$$\mathcal{T}^{k,\ell}_n \otimes \mathcal{T}^{k,\ell}_n \cong (C^n)^{\otimes k} \otimes (C^n)^{\otimes \ell} \otimes (C^n)^{\otimes k} \otimes (C^n)^{\otimes \ell}$$

followed by the following fixed isomorphism

$$\varphi : e_j^I \otimes e_j^J \mapsto e_{I \cup J} \otimes e_{I \cup J}.$$ 

Finally, there is a canonical isomorphism $\mathcal{T}^{k,\ell+k,\ell}_n \cong \text{End}((C^n)^{\otimes k+\ell})$. So combining these we fix isomorphisms

$$\text{End}(\mathcal{T}^{k,\ell}_n) \cong \mathcal{T}^{k,\ell}_n \otimes \mathcal{T}^{k,\ell}_n \cong (C^n)^{\otimes k+\ell} \cong \text{End}((C^n)^{\otimes k+\ell}).$$  

We view the outer two isomorphisms as fixed identifications. These isomorphisms are of unitary representations of $U(n)$ when everything is given its natural inner product. Moreover for $\sigma = (\sigma_1, \sigma_2) \in S_k \times S_\ell$ and $\tau = (\tau_1, \tau_2) \in S_k \times S_\ell$ we have for $A \in \text{End}(\mathcal{T}^{k,\ell}_n)$

$$\varphi[\rho^n_\ell(\sigma_1)\rho^n_\ell(\sigma_2)A\rho^n_\ell(\tau_1)\rho^n_\ell(\tau_2)] = \rho^{k+\ell}_n(\sigma_1, \tau_1^{-1})\varphi[A]\rho^{k+\ell}_n(\tau_1, \sigma_2^{-1}),$$

recalling that $\rho^{k+\ell}_n : C[S_{k+\ell}] \to \text{End}((C^n)^{\otimes k+\ell})$ is the representation by permuting coordinates.

We now return to the calculation of $q_\theta$ in (3.5). We have

$$q_\theta = D_{[\mu,\nu]}(n)\varphi^{-1}[P_{n,k+\ell}[\varphi(P_\theta)]]$$

where $P_{n,k+\ell}$ is projection onto the $U(n)$-invariant vectors (by conjugation) in $\text{End}((C^n)^{\otimes k+\ell})$. This can now be done using the classical Weingarten calculus. By Proposition 2.6 we have

$$P_{n,k+\ell}[\varphi(P_\theta)] = \rho^{k+\ell}_n(\Phi[\varphi(P_\theta)] \cdot Wg_{n,k+\ell})$$

where

$$\Phi[\varphi(P_\theta)] = \sum_{\sigma \in S_{k+\ell}} \text{tr}(\varphi(P_\theta)\rho^{k+\ell}_n(\sigma^{-1})\sigma).$$
By (3.8) and (3.4), and since e.g. $\chi_\mu(g) = \chi_\mu(g^{-1})$ we obtain
\[
\varphi(P_\theta) = \frac{1}{\|\theta\|_2^2} \varphi \left( \rho_n^k(p_\mu) \hat{\rho}_n^\ell(p_\nu) [\hat{\theta}_{\mu,\nu}^n] \otimes \tilde{\theta}_{\mu,\nu}^n \right) \\
= \frac{1}{\|\theta\|_2^2} \rho_n^{k+\ell}(p_\mu \otimes p_\nu) \varphi \left( \hat{\theta}_{\mu,\nu}^n \otimes \tilde{\theta}_{\mu,\nu}^n \right) \rho_n^{k+\ell}(p_\mu \otimes p_\nu)
\]
where
\[
p_\mu \otimes p_\nu \overset{\text{def}}{=} \frac{d_\mu d_\nu}{k! \ell!} \sum_{\sigma = (\sigma_1, \sigma_2) \in S_k \times S_\ell} \chi_\mu(\sigma_1) \chi_\nu(\sigma_2) \sigma \in C[S_{k+\ell}].
\]
Now using that $\Phi$ is a $C[S_{k+\ell}]$-bimodule morphism [CŚ06, Prop. 2.3 (1)] we obtain
\[
\Phi[\varphi(P_\theta)] = \frac{1}{\|\theta\|_2^2} p_\mu \otimes p_\nu \Phi \left[ \varphi \left( \hat{\theta}_{\mu,\nu}^n \otimes \tilde{\theta}_{\mu,\nu}^n \right) \right] p_\mu \otimes p_\nu \\
= \frac{1}{\|\theta\|_2^2} p_\mu \otimes p_\nu \left( \sum_{\sigma \in S_{k+\ell}} \text{tr} \left( \varphi \left( \hat{\theta}_{\mu,\nu}^n \otimes \tilde{\theta}_{\mu,\nu}^n \right) \rho_n^{k+\ell}(\sigma^{-1}) \right) \right) p_\mu \otimes p_\nu.
\]
Now, $\text{tr}(\varphi \left( \hat{\theta}_{\mu,\nu}^n \otimes \tilde{\theta}_{\mu,\nu}^n \right) \rho_n^{k+\ell}(\sigma^{-1}))$ is equal to 1 if and only if $\sigma$ is in $S_\mu \times S_\nu \leq S_k \times S_\ell$, and is 0 otherwise. So we obtain
\[
\Phi[\varphi(P_\theta)] = \frac{1}{\|\theta\|_2^2} p_\mu \otimes p_\nu \left( \sum_{\sigma \in S_\mu \times S_\nu} \sigma \right) p_\mu \otimes p_\nu,
\]
hence from (3.10)
\[
P_{n,k+\ell}[\varphi(P_\theta)] = \rho_n^{k+\ell}(z_\theta)
\]
where
\[
z_\theta \overset{\text{def}}{=} \sum_{\tau \in S_{k+\ell}} z_\theta(\tau) \tau \overset{\text{def}}{=} \frac{1}{\|\theta\|_2^2} p_\mu \otimes p_\nu \left( \sum_{\sigma \in S_\mu \times S_\nu} \sigma \right) p_\mu \otimes p_\nu W_{n,k+\ell} \in C[S_{k+\ell}]. \tag{3.11}
\]
Therefore we obtain the following proposition.

**Proposition 3.2.** We have
\[
q_\theta = \varphi^{-1}[\rho_n^{k+\ell}(z_\theta)].
\]

We can use the bound for the coefficients of $W_{n,k+\ell}$ from Proposition 2.7 to infer a bound on the coefficients $z_\theta(\tau)$. For $\sigma \in S_{k+\ell}$, let $\|\sigma\|_{k,\ell}$ denote the minimum $m$ for which
\[
\sigma = \sigma_0 t_1 t_2 \cdots t_m
\]
where $\sigma_0 \in S_k \times S_\ell$ and $t_1, \ldots, t_m$ are transpositions in $S_{k+\ell}$.

**Lemma 3.3.** For all $\tau \in S_{k+\ell}$ and $\theta = \theta_{\mu,\nu}^n$ as above, $z_\theta(\tau) = O_{k,\ell}(n^{-k-\ell-\|\tau\|_{k,\ell}})$ as $n \to \infty$.

**Proof.** Referring to (3.11), as $n \to \infty$, $\|\theta\|^{-2} = O_{k,\ell}(1)$ by Lemma 2.3 and the coefficients of
\[ p_\mu \otimes \nu \left( \sum_{\sigma \in S_k \times S_\ell} \sigma \right) \] are clearly \( O_{k, \ell}(1) \), so \( z_\theta \) has the form
\[
\left( \sum_{\sigma \in S_k \times S_\ell} A(\sigma) \sigma \right) W_{g_{n,k+\ell}}
\]
where each \( A(\sigma) \) is \( O_{k, \ell}(1) \). This means
\[
z_\theta(\tau) = \sum_{\sigma \in S_k \times S_\ell, \sigma' \in S_{k+\ell} : \sigma \sigma' = \tau} A(\sigma) W_{g_{n,k+\ell}}(\sigma').
\]
The order of any of the finitely many summands above is \( n^{-k-\ell-|\sigma'|} \) by Proposition 2.7, and the minimum possible value of \( |\sigma'| \) is \( \|\tau\|_{k, \ell} \).

Before moving on, it is useful to explain the operator \( \varphi^{-1}[\rho_{n}^{k+\ell}(\pi)] \) for \( \pi \in S_{k+\ell} \). For \( I = (i_1, \ldots, i_{k+\ell}) \) let \( I'(I; \pi) \) be such that \( I'(I; \pi) = \{i_{\pi(1)}, \ldots, i_{\pi(k)}\} \), and \( J'(I; \pi) = \{i_{\pi(k+1)}, \ldots, i_{\pi(k+\ell)}\} \). As an element of \( (C^n)^{\otimes k+\ell} \), \( \rho_{n}^{k+\ell}(\pi) \) is given by
\[
\sum_{I = (i_1, \ldots, i_k), J = (j_{k+1}, \ldots, j_{k+\ell})} e^{I'(I; \pi)} e^{J'(\pi; J)} \otimes e^{\ell_{I \cup J}},
\]
so from (3.6)
\[
\varphi^{-1}[\rho_{n}^{k+\ell}(\pi)] = \sum_{I = (i_1, \ldots, i_k), J = (j_{k+1}, \ldots, j_{k+\ell})} e^{I'(I; \pi)} e^{J'(J; \pi)} \otimes e^{\ell_{I \cup J}}.
\] (3.12)

### 3.4 A combinatorial integration formula

In this rest of this §3 we assume \( g = 2 \). All proofs extend to \( g \geq 3 \). We write \( \{a, b, c, d\} \) for the generators of \( F_4 \) and \( R \) be defined as \( \{a, b\}[c, d] \). Assume both \( \gamma \) and \( w \) are not the identity and \( w \in [F_4, F_4] \) according to the remarks at the beginning of §§2.6. We write \( w \) in reduced form:
\[
w = f_1^{\epsilon_1} f_2^{\epsilon_2} \cdots f_{|w|}^{\epsilon_{|w|}}, \quad \epsilon_u \in \{\pm 1\}, \quad f_u \in \{a, b, c, d\},
\] (3.13)
where if \( f_u = f_{u+1} \), then \( \epsilon_u = \epsilon_{u+1} \). For \( f \in \{a, b, c, d\} \) let \( p_f \) denote the number of occurrences of \( f \) in (3.13). The expression (3.13) implies that for \( h \) be defined as \( (h_a, h_b, h_c, h_d) \in U(n)^4 \),
\[
\text{tr}(w(h)) = \sum_{i,j \in [n]} (h_{f_1}^{\epsilon_1})_{i_1i_2} (h_{f_2}^{\epsilon_2})_{i_2i_3} \cdots (h_{f_{|w|}}^{\epsilon_{|w|}})_{i_{|w|}i_1}.
\] (3.14)

Working with this expression will be cumbersome so we explain a diagrammatic way to think about (3.14). This will be the starting point for how we eventually understand \( J_n(w, [\mu, \nu]) \) in terms of decorated surfaces. We begin with a collection of intervals as follows.

**w-intervals and the w-loop**

Firstly, for every \( j \in [w] \), with \( f_j = f \) as in (3.13) and \( \epsilon_j = 1 \) we take a copy of \([0, 1]\) and direct it from 0 to 1.

In our constructions, every interval will have two directions: the *intrinsic direction* (which is the direction from 0 to 1) and the *assigned direction*. In the case just discussed, these agree, but in general they will not.
Figure 3.1: Illustration of the $w$-loop for $w = a^2ba^{-2}b^{-1}$. The solid intervals are $w$-intervals and the dashed intervals are $w$-intermediate-intervals. We also label each interval by the set e.g. $I_{a,w}$ to which they belong.

We write $[0,1]_{f,j,w}$ for such an interval and $I^+_{f,w}$ for the collection of these intervals.

For every $j \in [w]$, with $f_j = f$ as in (3.13) and $\epsilon_j = -1$ we take a copy of $[0,1]$ and direct this interval from 1 to 0. We write $[0,1]_{f^{-1},j,w}$ for such an interval and $I^-_{f,w}$ for the collection of these intervals.

All the intervals described above are called $w$-intervals. There are $|w|$ of these intervals in total.

Between each $[0,1]_{f_j,j,w}$ and $[0,1]_{f_{j+1}^{-1},j+1,w}$ we add a new interval connecting $1_{f_j,j,w}$ to $0_{f_{j+1}^{-1},j+1,w}$, where the indices $j$ run mod $|w|$. These intervals added are called $w$-intermediate-intervals. Note that these intervals together with the $w$-intervals now form a closed cycle that is paved by $2|w|$ intervals alternating between $w$-intervals and $w$-intermediate-intervals. Starting at $[0,1]_{f_{j^*},1,w}$, reading the directions and $f$-labels of the $w$-intervals so that every $w$-interval is traversed from 0 to 1 spells out the word $w$. The resulting circle is called the $w$-loop and the previously defined orientation of this loop is now fixed. See Figure 3.1 for an illustration of the $w$-loop in a particular example.

We now view the indices $i_j$ as an assignment

$$a : \{\text{end-points of } w\text{-intervals}\} \rightarrow [n],$$

$$a(0_{f,j,w}) \overset{\text{def}}{=} i_j, \ a(1_{f,j,w}) = i_{j+1}, \ a(0_{f^{-1},j,w}) = i_j, \ a(1_{f^{-1},j,w}) = i_{j+1}.$$

The condition that $a$ comes from a single collection of $i_j$ is precisely that if two end points of $w$-intervals are connected by a $w$-intermediate-interval, they are assigned the same value by $a$. Let $A(w)$ denote the collection of such $a$. If $I$ is any copy of $[0,1]$ we write $0_I$ for the copy
of 0 and 1 for the copy of 1 in $I$. We can now write

$$
\text{tr}(w(h)) = \sum_{a \in A(w)} \prod_{f \in \{a, b, c, d\}} \left( \prod_{i \in I_{f, w}} h_{a(i_0)a(i_1)} \right) \left( \prod_{i \in I_{f, w}} h_{a(1)a(0)} \right).
$$

Now let $v_p$ be an orthonormal basis for $W_n(\theta)$. We have

$$
S_{[\mu, \nu]}(R(g(h_a, h_b, h_c, h_d))) = \sum_{p_i} \langle h_a v_{p_2}, v_{p_1} \rangle \langle h_b v_{p_3}, v_{p_2} \rangle \langle h_a^{-1} v_{p_4}, v_{p_3} \rangle \langle h_b^{-1} v_{p_5}, v_{p_4} \rangle
$$

$$
\langle h_c v_{p_6}, v_{p_5} \rangle \langle h_d v_{p_7}, v_{p_6} \rangle \langle h_c^{-1} v_{p_8}, v_{p_7} \rangle \langle h_d^{-1} v_{p_9}, v_{p_8} \rangle.
$$

Here we have written e.g. $h_a v_{p_2}$ for $\pi_n^{k_f}(h_a)v_{p_2}$ to make things easier to read. Next we write each $v_p = \sum_{I,J} \beta^J_{p_I} e^I_f$, where $\beta^J_{p_I} \overset{\text{def}}{=} \langle v_p, e^I_f \rangle$. We then have

$$
\langle h_a v_{p_2}, v_{p_1} \rangle \langle h_b v_{p_3}, v_{p_2} \rangle \langle h_a^{-1} v_{p_4}, v_{p_3} \rangle \langle h_b^{-1} v_{p_5}, v_{p_4} \rangle
$$

$$
\times \langle h_c v_{p_6}, v_{p_5} \rangle \langle h_d v_{p_7}, v_{p_6} \rangle \langle h_c^{-1} v_{p_8}, v_{p_7} \rangle \langle h_d^{-1} v_{p_9}, v_{p_8} \rangle
$$

$$
= \sum_{r_f, R_f, V_f, f, V_f, U_f, e_f, S_f} \beta^V_{p_2 a} \beta^V_{p_1 a} \beta^{U_a}_p \beta^{U_b}_p \beta^{\bar{U}_b}_p \beta^{\bar{U}_a}_p \beta^{v_a}_p \beta^{U_c}_p \beta^{U_d}_p \beta^{\bar{U}_d}_p \beta^{\bar{V}_b}_p \beta^{v_b}_p \beta^{v_a}_p \beta^{v_b}_p \beta^{v_c}_p \beta^{v_d}_p \beta^{u_a}_p \beta^{u_b}_p \beta^{u_c}_p \beta^{u_d}_p \beta^{u_e}_p
$$

$$
\langle h_a e_{s_a}, e_{r_f} \rangle \langle h_b e_{s_b}, e_{r_f} \rangle \langle h_a^{-1} e_{s_a}, e_{r_f} \rangle \langle h_b^{-1} e_{s_b}, e_{r_f} \rangle \langle h_c e_{s_c}, e_{r_f} \rangle \langle h_d e_{s_d}, e_{r_f} \rangle \langle h_c^{-1} e_{s_c}, e_{r_f} \rangle \langle h_d^{-1} e_{s_d}, e_{r_f} \rangle.
$$

We calculate

$$
\langle h_f e_{s_f}, e_{r_f} \rangle \langle h_f^{-1} e_{r_f}, e_{s_f} \rangle
$$

$$
= \langle h_f e_{s_f}, e_{r_f} \rangle \langle h_f e_{s_f}^{-1}, e_{r_f} \rangle \langle h_f e_{s_f}, e_{r_f} \rangle \langle h_f e_{s_f}^{-1}, e_{r_f} \rangle
$$

$$
= \langle h_f e_{s_f \cup V_f}, e_{r_f \cup U_f} \rangle \langle h_f e_{S_f \cup V_f}, e_{R_f \cup U_f} \rangle.
$$

We now want a diagrammatic interpretation of (3.15) similarly to before. We make the following constructions.

**R-intervals.**

For each $j \in [k]$ and $f \in \{a, b, c, d\}$, we make a copy of $[0, 1]$, direct it from 0 to 1, label it by $f$, and also number it by $j$. We write $I_{f, R}$ for the collection of these intervals. These correspond to occurrences of $f$ in $R$.

For each $j \in [k]$ and $f \in \{a, b, c, d\}$, we make a copy of $[0, 1]$, direct it from 1 to 0, label it by $f$, and also number it by $j$. We write $I_{f, R}$ for the collection of these intervals. These correspond to occurrences of $f^{-1}$ in $R$.

(These two constructions of $k$ intervals correspond to the presence of $f$ and $f^{-1}$ each exactly once in $R$.)

These intervals are called **R-intervals**. There are 8k R-intervals in total (for general $g$, there are $4gk$ of these intervals).

**R^{-1}-intervals.**

For each $j \in [k + 1, k + \ell]$ and $f \in \{a, b, c, d\}$, we make a copy of $[0, 1]$, direct it from 0 to 1, label it by $f$, and also number it by $j$. We write $I_{f, R^{-1}}$ for the collection of these intervals.
Figure 3.2: Here is shown the R-intervals (left) and the $R^{-1}$-intervals (right). We have indicated their assigned direction and label (which $f$ they correspond to). We have also, for each endpoint of an interval, indicated which index function, e.g. $r_a$, has this endpoint in its domain.

These correspond to occurrences of $f$ in $R^{-1}$.

For each $j \in [k+1, k+\ell]$ and $f \in \{a, b, c, d\}$, we make a copy of $[0, 1]$, direct it from 1 to 0, label it by $f$, and also number it by $j$. We write $\mathcal{I}_{f,R}^-$ for the collection of these intervals. These correspond to occurrences of $f^{-1}$ in $R^{-1}$.

These intervals are called $R^{-1}$-intervals. There are $8\ell R^{-1}$ intervals in total (for general $g$, there are $4g\ell$ of these intervals). See Figure 3.2 for an illustration of the $R$ and $R^{-1}$ intervals.

We now view (by identifying endpoints of intervals with the given numbers of intervals in $[k+\ell]$)

\[
\begin{align*}
    r_f : & \{0_i : i \in \mathcal{I}_{f,R}^+\} \to [n], & R_f : & \{1_i : i \in \mathcal{I}_{f,R}^-\} \to [n], \\
n_f : & \{1_i : i \in \mathcal{I}_{f,R}^+\} \to [n], & S_f : & \{0_i : i \in \mathcal{I}_{f,R}^-\} \to [n], \\
U_f : & \{1_i : i \in \mathcal{I}_{f,R-1}^-\} \to [n], & u_f : & \{0_i : i \in \mathcal{I}_{f,R-1}^+\} \to [n], \\
V_f : & \{0_i : i \in \mathcal{I}_{f,R-1}^-\} \to [n], & v_f : & \{1_i : i \in \mathcal{I}_{f,R-1}^+\} \to [n].
\end{align*}
\]

We obtain from (3.16)

\[
\begin{align*}
& \langle h_a e_{s_a}, e_{r_a} \rangle \langle h_b e_{s_b}, e_{r_b} \rangle \langle h_a^{-1} e_{R_a}, e_{S_a} \rangle \langle h_b^{-1} e_{U_a}, e_{V_a} \rangle \\
& \langle h_c e_{s_c}, e_{r_c} \rangle \langle h_d e_{s_d}, e_{r_d} \rangle \langle h_c^{-1} e_{U_d}, e_{V_d} \rangle \langle h_d^{-1} e_{R_d}, e_{S_d} \rangle \\
= & \Pi_{f} \Pi_{i^+ \in \mathcal{I}_{f,R}^+} \Pi_{i^- \in \mathcal{I}_{f,R}^-} \Pi_{j^+ \in \mathcal{I}_{f,R-1}^+} \Pi_{j^- \in \mathcal{I}_{f,R-1}^-} \\
& h_{r_f(0_{i+})} s_f(1_{i+}) h_{u_f(0_{i+})} v_f(1_{i+}) h_{R_f(1_{i-})} S_f(0_{i-}) h_{U_f(1_{i-})} V_f(0_{i-}).
\end{align*}
\]
With this formalism we obtain

\[
\mathcal{J}_n(w, [\mu, \nu]) = \sum_{p_1} r_f, R_f, V_f, \nu_f, U_f, u_f, s_f, S_f \in \mathcal{A}(w) \beta^{s_1}_{p_{s_1}^{u_1}} \beta^{s_2}_{p_{s_2}^{u_2}} \beta^{s_3}_{p_{s_3}^{u_3}} \beta^{s_4}_{p_{s_4}^{u_4}} \beta^{t_1}_{p_{t_1}^{u_1}} \beta^{t_2}_{p_{t_2}^{u_2}} \beta^{t_3}_{p_{t_3}^{u_3}} \beta^{t_4}_{p_{t_4}^{u_4}} \beta^{u_1}_{p_{u_1}^{t_1}} \beta^{u_2}_{p_{u_2}^{t_2}} \beta^{u_3}_{p_{u_3}^{t_3}} \beta^{u_4}_{p_{u_4}^{t_4}} \prod_{f \in \{a, b, c, d\}} \int_{h \in U(n)} \prod_{i \in J_u} \prod_{j \in J_v} \prod_{k \in J_a} \prod_{l \in J_b} \prod_{m \in J_c} \prod_{n \in J_d} \prod_{r \in J_f} \prod_{s \in J_g} \prod_{t \in J_h} \prod_{u \in J_i} \prod_{v \in J_j} \prod_{w \in J_k} \prod_{x \in J_l} \prod_{y \in J_m} \prod_{z \in J_n} dh.
\]

(3.17)

For each \( f \), the integral in (3.17) can be done using the Weingarten calculus (Theorem 2.5). To do this, fix bijections for each \( f \in \{a, b, c, d\} \)

\[
\mathcal{J}_f^{+} \overset{\text{def}}{=} J_{f,R}^{+} \cup J_{f,R-1}^{+} \cup J_{f,w}^{+} \cong [k + \ell + pf]
\]

\[
\mathcal{J}_f^{-} \overset{\text{def}}{=} J_{f,R}^{-} \cup J_{f,R-1}^{-} \cup J_{f,w}^{-} \cong [k + \ell + pf]
\]

such that

\[
J_{f,w}^{+} \cong [k + \ell + 1, k + \ell + pf], \ J_{f,w}^{-} \cong [k + \ell + 1, k + \ell + pf]
\]

and

\[
J_{f,R}^{+} \cong [k], \ J_{f,R}^{-} \cong [k], \ J_{f,R-1}^{+} \cong [k + 1, \ell], \ J_{f,R-1}^{-} \cong [k + 1, k + \ell]
\]

(3.18)
correspond to the original numberings of \( J_{f,R}^{+}, J_{f,R}^{-}, J_{f,R-1}^{+}, J_{f,R-1}^{-} \).

Hence if \( \sigma_f, \tau_f \in S_{k+\ell+pf} \) we view \( \sigma_f, \tau_f : \mathcal{J}_f^{+} \to \mathcal{J}_f^{-} \) by the above fixed bijections. For each \( f \in \{a, b, c, d\} \) we say \( (a, r_f, u_f, R_f, U_f) \to \sigma_f \) if for all \( i \in J_f^{+}, i' \in J_f^{-} \) with \( \sigma_f(i) = i' \), we have

\[
[r_f \sqcup u_f \sqcup a](0_1) = [R_f \sqcup U_f \sqcup a](1_1);
\]

here we wrote e.g. \([r_f \sqcup u_f \sqcup a]\) for the function that \( a, r_f, u_f \) induce on \( \{0_1 : i \in J_f^{+}\} \). Similarly we say \( (a, s_f, v_f, S_f, V_f) \to \tau_f \) if for all \( i \in J_f^{+}, i' \in J_f^{-} \) with \( \tau_f(i) = i' \) we have

\[
[s_f \sqcup v_f \sqcup a](1_1) = [S_f \sqcup V_f \sqcup a](0_1).
\]

Theorem 2.5 translates to

\[
\int_{h \in U(n)} \prod_{i \in J_u} \prod_{j \in J_v} \prod_{k \in J_a} \prod_{l \in J_b} \prod_{m \in J_c} \prod_{n \in J_d} \prod_{r \in J_f} \prod_{s \in J_g} \prod_{t \in J_h} \prod_{u \in J_i} \prod_{v \in J_j} \prod_{w \in J_k} \prod_{x \in J_l} \prod_{y \in J_m} \prod_{z \in J_n} dh
\]

\[
= \sum_{\sigma_f, \tau_f \in S_{k+\ell+pf}} \text{Wg}_n, k+\ell+pf \left( \sigma_f \tau_f^{-1} \right) 1\{ (a, r_f, u_f, R_f, U_f) \to \sigma_f, (a, s_f, v_f, S_f, V_f) \to \tau_f \},
\]
so putting this into (3.17) gives

\[ \mathcal{J}_n(w, [\mu, \nu]) = \sum_{\sigma_f, \tau_f \in S_{k+\ell+p_f}} \left( \prod_{f \in \{a,b,c,d\}} W_{g_{n,k+\ell+p_f}}(\sigma_f \tau_f^{-1}) \right) \sum_{p_i} \sum_{(a,r_f, R_f, V_f, U_f, u_f, s_f, S_f) \rightarrow \sigma_f} \sum_{(a,s_f, V_f, S_f, \tau_f) \rightarrow \tau_f} \beta_{V_a} \bar{\sigma}_{U_a} \beta_{V_b} \bar{\sigma}_{U_b} \beta_{V_c} \bar{\sigma}_{U_c} \beta_{V_d} \bar{\sigma}_{U_d} \beta_{p_a S_a} \beta_{p_b S_b} \beta_{p_c S_c} \beta_{p_d S_d} \beta_{p_a R_a} \beta_{p_b R_b} \beta_{p_c R_c} \beta_{p_d R_d} \beta_{p_a V_a} \beta_{p_b V_b} \beta_{p_c V_c} \beta_{p_d V_d} \beta_{p_a U_a} \beta_{p_b U_b} \beta_{p_c U_c} \beta_{p_d U_d} \bar{\sigma}_{V_a} \bar{\sigma}_{V_b} \bar{\sigma}_{V_c} \bar{\sigma}_{V_d} \bar{\sigma}_{U_a} \bar{\sigma}_{U_b} \bar{\sigma}_{U_c} \bar{\sigma}_{U_d} \bar{\sigma}_{V_a} \bar{\sigma}_{V_b} \bar{\sigma}_{V_c} \bar{\sigma}_{V_d}. \]

Here we make our main improvement over the classical Weingarten calculus. We introduce the following beneficial property that the \(\sigma_f, \tau_f\) possibly have.

**Forbidden matchings property:** For every \(f \in \{a, b, c, d\}\) the following hold: neither \(\sigma_f\) nor \(\tau_f\) map any element of \(\mathcal{J}_{f,R}^+\) to an element of \(\mathcal{J}_{f,R-1}^-\), or map an element of \(\mathcal{J}_{f,R-1}^+\) to an element of \(\mathcal{J}_{f,R}^-\).

We have the following key lemma.

**Lemma 3.4.** If for some \(f \in \{a, b, c, d\}\), \(\sigma_f\) and \(\tau_f\) do not have the *forbidden matchings* property, then for any choice of \(p_1, \ldots, p_8\)

\[ \sum_{a \in A(w), r_f, R_f, V_f, U_f, u_f, s_f, S_f} \sum_{(a,r_f, u_f, R_f, U_f) \rightarrow \sigma_f} \sum_{(a,s_f, V_f, S_f, \tau_f) \rightarrow \tau_f} \beta_{V_a} \bar{\sigma}_{U_a} \beta_{V_b} \bar{\sigma}_{U_b} \beta_{V_c} \bar{\sigma}_{U_c} \beta_{V_d} \bar{\sigma}_{U_d} \beta_{p_a S_a} \beta_{p_b S_b} \beta_{p_c S_c} \beta_{p_d S_d} \beta_{p_a R_a} \beta_{p_b R_b} \beta_{p_c R_c} \beta_{p_d R_d} \beta_{p_a V_a} \beta_{p_b V_b} \beta_{p_c V_c} \beta_{p_d V_d} \beta_{p_a U_a} \beta_{p_b U_b} \beta_{p_c U_c} \beta_{p_d U_d} \bar{\sigma}_{V_a} \bar{\sigma}_{V_b} \bar{\sigma}_{V_c} \bar{\sigma}_{V_d} \bar{\sigma}_{U_a} \bar{\sigma}_{U_b} \bar{\sigma}_{U_c} \bar{\sigma}_{U_d} \bar{\sigma}_{V_a} \bar{\sigma}_{V_b} \bar{\sigma}_{V_c} \bar{\sigma}_{V_d} = 0. \]

**Proof.** Indeed suppose \(\sigma_a\) matches an element \(i \in \mathcal{J}_{a,R}^+\) with \(j \in \mathcal{J}_{a,R-1}^-\); \(\sigma_a(i) = j\). With our given fixed bijections \((3.18)\), \(i\) corresponds to an element of \([k]\) and \(j\) corresponds to an element of \([k+1, k+\ell]\). Without loss of generality in the argument suppose that \(i\) corresponds to 1 and \(j\) corresponds to \(k+1\). The condition \(\sigma_a(i) = j\) and \((a, r_a, u_a, R_a, U_a) \rightarrow \sigma_f\) means that as functions on \([k]\) and \([k+1, k+\ell]\), \(r_a(1) = U_a(k+1)\). There are no other constraints on these values.

Then for all variables in \((3.19)\) fixed apart from \(r_a\) and \(U_a\), and all values of \(r_a, U_a\) fixed other than \(r_a(1)\) and \(U_a(k+1)\) the ensuing sum over \(r_a, U_a\) is

\[ \sum_{r_a(1) = U_a(k+1)} \beta_{p_a U_a}. \]

But recalling the contraction operators from \((2.2)\), this sum is the coordinate of \(e_{r_a(2)} \otimes \cdots \cdots \otimes e_{r_a(k+2)} \otimes \cdots \otimes e_{r_a(k+\ell)} \in c_{1,1}(v_{p_2})\). But \(c_{1,1}(v_{p_2}) = 0\) because \(v_{p_2} \in \mathcal{T}_n^{k,\ell} \). \(\square\)

We henceforth write \(\sum_{\sigma_f, \tau_f} \) to mean the sum is restricted to \(\sigma_f, \tau_f\) satisfying the *forbid-
den matchings property. Lemma 3.4 now implies

\[ \mathcal{J}_n(w, [\mu, \nu]) \]

\[ = \sum_{\sigma_f, \tau_f \in S_{f+k+\ell}} \left( \prod_{f \in \{a,b,c,d\}} W_{g_n,k+\ell+p_f}^g(\sigma_f \tau_f^{-1}) \right) \sum_{p_1 \in A(w), r_f, R_f, V_f, \sigma_f, U_f, u_f, s_f, S_f} \sum_{(a, s_f, V_f, \tau_f) \mapsto \sigma_f} \beta_{p_2 s_a \beta_{p_1 R_a} \beta_{p_3 s_b} \beta_{p_2 V_b} \beta_{p_4 R_a} \beta_{p_3 s_s} \beta_{p_4 S_b} \beta_{p_6 s_e} \beta_{p_5 r_c} \beta_{p_7 s_d} \beta_{p_6 R_c} \beta_{p_7 s_e} \beta_{p_5 R_d} \beta_{p_8 S_d}} \]

Moreover, we can significantly tidy up (3.20). For everything in (3.20) fixed except for e.g. \( p_2 \), the ensuing sum over \( p_2 \) is

\[ \sum_{p_2} \beta_{p_2 s_a} \beta_{p_2 r_b} = \sum_{p_2} \langle \mathbb{U}_b^u, v_{p_2} \rangle = \langle \mathbb{q}_\theta \mathbb{U}_b^u, \mathbb{e}_s^a \rangle. \]

Therefore executing the sums over \( p_i \) in (3.20) we replace the sum over \( p_i \) and the product over \( \beta \)-terms by

\[ \langle \mathbb{q}_\theta \mathbb{U}_b^u, \mathbb{c}_s^a \rangle = \sum_{\pi \in S_{k+\ell}} z_{\theta}(\pi) \langle \varphi^{-1}[\rho_n^{k+\ell}(\pi)] \mathbb{e}_s^a \rangle \]

By Proposition 3.2 we have e.g.

\[ \langle \mathbb{q}_\theta \mathbb{U}_b^u, \mathbb{c}_s^a \rangle = \sum_{\pi \in S_{k+\ell}} z_{\theta}(\pi) \langle \varphi^{-1}[\rho_n^{k+\ell}(\pi)] \mathbb{e}_s^a \rangle \]

Now recall from (3.12) that

\[ \varphi^{-1}[\rho_n^{k+\ell}(\pi)] = \sum_{I = (i_1, \ldots, i_k), J = (j_{k+1}, \ldots, j_{k+\ell})} e_{I \cup J}(I; \pi) \otimes e_{J}(I; \pi). \]

This means that \( \langle \varphi^{-1}[\rho_n^{k+\ell}(\pi)] \mathbb{e}_s^a \rangle \) is either equal to 0 or 1 and \( \langle \varphi^{-1}[\rho_n^{k+\ell}(\pi)] \mathbb{e}_s^a \rangle = 1 \) if and only if, letting (3.18) induce identifications

\[ \{ l_i : i \in \mathcal{J}_{a,R} \} \supseteq [k], \{ l_i : i \in \mathcal{J}_{b,R} \} \supseteq [k+1, k+\ell], \]

\[ \{ 0_i : i \in \mathcal{J}_{a,R} \} \supseteq [k], \{ 0_i : i \in \mathcal{J}_{b,R} \} \supseteq [k+1, k+\ell], \]

via their given indexing of intervals, we have \([s_a \sqcup U_b] \circ \pi = [r_b \sqcup V_a]\), where e.g. \( s_a \sqcup U_b \) is the function either on endpoints of intervals or on \([k+\ell]\) induced by the union of \( s_a \) and \( U_b \).
Hence, repeating this argument,

\[
(3.21) = \sum_{\pi_1, \ldots, \pi_8 \in S_{k+\ell}} \left( \prod_{i=1}^{8} z_\theta(\pi_i) \right)
\]

Putting all these arguments together gives

\[
J_n(w, [\mu, \nu]) = \sum_{\sigma_f, \tau_f \in S_{b+\ell}} \sum_{\pi_1, \ldots, \pi_8 \in S_{k+\ell}} \left( \prod_{f \in \{a, b, c, d\}} W_{g_{n, k+\ell+p_f}}(\sigma_f^{-1}) \right) \left( \prod_{i=1}^{8} z_\theta(\pi_i) \right)
\]

This formula says that we can calculate \( J_n(w, [\mu, \nu]) \) by summing over some combinatorial data of matchings (the \( \sigma_f, \tau_f, \pi_i \)) a quantity that we can understand well times a count of the number of indices that satisfy the prescribed matchings. To formalize this point of view we make the following definition.

**Definition 3.5.** A matching datum of the triple \((w, k, \ell)\) is a pair \((\sigma_f, \tau_f, \pi_i)\) in \(S_{k+\ell+p_f} \times S_{k+\ell+p_f}\) as above, satisfying the forbidden matchings property for each \(f \in \{a, b, c, d\}\), together with \((\pi_1, \ldots, \pi_8) \in (S_{k+\ell})^8\). We write

\[
\text{MATCH}(w, k, \ell)
\]

for the finite collection of all matching data for \((w, k, \ell)\).

Given a matching datum \{\(\sigma_f, \tau_f, \pi_i\}\), we write \(\mathcal{N}(\{\sigma_f, \tau_f, \pi_i\})\) for the number of choices of \(a \in A(w), r_f, R_f, V_f, v_f, U_f, u_f, s_f, S_f\) such that

\[
(a, r_f, u_f, R_f, U_f) \rightarrow \sigma_f, (a, s_f, v_f, S_f, V_f) \rightarrow \tau_f,
\]

\[
[a_s \cup U_b] \circ \pi_1 = [r_b \cup V_a], [s_b \cup v_a] \circ \pi_2 = [a_s \cup V_b],
\]

\[
[R_a \cup v_b] \circ \pi_3 = [s_b \cup u_a], [R_b \cup U_c] \circ \pi_4 = [r_c \cup u_b],
\]

\[
[s_c \cup U_d] \circ \pi_5 = [r_d \cup V_c], [s_d \cup v_c] \circ \pi_6 = [s_c \cup V_d],
\]

\[
[R_c \cup v_d] \circ \pi_7 = [s_d \cup u_c], [R_d \cup U_a] \circ \pi_8 = [r_a \cup u_d].
\]

With this notation, we have proved the following theorem.
Theorem 3.6. For \( k + \ell > 0, \mu \vdash k \) and \( \nu \vdash \ell, \) \( w \in [F_4, F_4], \) we have

\[
\mathcal{J}_n(w, [\mu, \nu]) = \sum_{\{\sigma_f, \tau_f, \pi_i\} \in \text{MATCH}(w, k, \ell)} \left( \prod_{i=1}^{8} z_\theta(\pi_i) \right) \prod_{f \in \{a, b, c, d\}} W_{\sigma_f, n+\ell+pf}(\sigma_f \tau_f^{-1}) N(\{\sigma_f, \tau_f, \pi_i\}). \tag{3.23}
\]

We conclude this section by bounding the terms \( z_\theta(\pi_i) \) and \( W_{\sigma_f, n+\ell+pf}(\sigma_f \tau_f^{-1}) \) using Proposition 2.7 and Lemma 3.3. Note that \( \sum_{f \in \{a, b, c, d\}} p_f = \frac{|w|}{2} \). This yields

Corollary 3.7. For \( k + \ell > 0, \mu \vdash k \) and \( \nu \vdash \ell, \) \( w \in [F_4, F_4], \) we have

\[
\mathcal{J}_n(w, [\mu, \nu]) \ll_{k, \ell, w} n^{-4k-4\ell-\frac{|w|}{2}} \sum_{\{\sigma_f, \tau_f, \pi_i\} \in \text{MATCH}(w, k, \ell)} n^{-\sum_f |\sigma_f \tau_f^{-1}| - \sum_{i=1}^{8} k_i \ell_i} N(\{\sigma_f, \tau_f, \pi_i\}). \tag{3.24}
\]

We will proceed in the next section to understand all the quantities in (3.24) in topological terms by constructing a surface from each \( \{\sigma_f, \tau_f, \pi_i\}. \)

4 Topology

4.1 Construction of surfaces from matching data

We now show how a datum in \( \text{MATCH}(w, k, \ell) \) can be used to construct a surface such that the terms appearing in (3.23) can be bounded by topological features of the surface. This construction is similar to the constructions of [MP19, MP15], but with the presence of additional \( \pi_i \) adding a new aspect. We continue to assume \( g = 2 \) for simplicity. We can still assume that \( \gamma \in [\Gamma_2, \Gamma_2] \) and hence \( w \in [F_4, F_4]. \)

Construction of the 1-skeleton

\( \pi \)-intervals. The identifications of the previous section mean that we view

\[
\begin{align*}
\pi_1 : & \{0_i : i \in \mathcal{J}^+_{b,R} \cup \mathcal{J}^-_{a,R-1} \} \rightarrow \{1_i : i' \in \mathcal{J}^+_{a,R} \cup \mathcal{J}^-_{b,R-1} \}, \\
\pi_2 : & \{0_i : i \in \mathcal{J}^-_{a,R} \cup \mathcal{J}^+_{b,R-1} \} \rightarrow \{1_i : i' \in \mathcal{J}^+_{b,R} \cup \mathcal{J}^-_{a,R-1} \}, \\
\pi_3 : & \{0_i : i \in \mathcal{J}^-_{b,R} \cup \mathcal{J}^+_{a,R-1} \} \rightarrow \{1_i : i' \in \mathcal{J}^+_{a,R} \cup \mathcal{J}^-_{b,R-1} \}, \\
\pi_4 : & \{0_i : i \in \mathcal{J}^+_{c,R} \cup \mathcal{J}^-_{b,R-1} \} \rightarrow \{1_i : i' \in \mathcal{J}^+_{b,R} \cup \mathcal{J}^-_{c,R-1} \}, \\
\pi_5 : & \{0_i : i \in \mathcal{J}^-_{c,R} \cup \mathcal{J}^+_{c,R-1} \} \rightarrow \{1_i : i' \in \mathcal{J}^+_{c,R} \cup \mathcal{J}^-_{d,R-1} \}, \\
\pi_6 : & \{0_i : i \in \mathcal{J}^-_{c,R} \cup \mathcal{J}^+_{d,R-1} \} \rightarrow \{1_i : i' \in \mathcal{J}^+_{d,R} \cup \mathcal{J}^-_{c,R-1} \}, \\
\pi_7 : & \{0_i : i \in \mathcal{J}^-_{d,R} \cup \mathcal{J}^+_{c,R-1} \} \rightarrow \{1_i : i' \in \mathcal{J}^+_{c,R} \cup \mathcal{J}^-_{d,R-1} \}, \\
\pi_8 : & \{0_i : i \in \mathcal{J}^+_{a,R} \cup \mathcal{J}^+_{d,R-1} \} \rightarrow \{1_i : i' \in \mathcal{J}^+_{d,R} \cup \mathcal{J}^-_{a,R-1} \}.
\end{align*}
\tag{4.1}
\]

We add an arc between any two interval endpoints that are mapped to one another by some \( \pi_i \). All the intervals added here are called \( \pi \)-intervals. The purpose of this construction is that
the conditions concerning $\pi_i$ in (3.22) correspond to the fact that two end-points of intervals connected by a $\pi$-interval are assigned the same value in $[n]$ by the relevant functions out of $r_f, R_f, V_f, v_f, U_f, u_f, s_f, S_f$ (at most one of these functions has any given interval endpoint in its domain).

The $\pi$-intervals together with the $R$-intervals and $R^{-1}$ intervals form a collection of loops that we call $R^{\pm}-\pi$-loops.

$\sigma$-arcs and $\tau$-arcs. Recall from the previous sections that we view $\sigma_f, \tau_f : I_f^+ \to I_f^-$. We add an arc between each $0_i$ and $1_i'$ with $\sigma_f(i) = i'$ and between each $1_i$ and $0_i'$ with $\tau_f(i) = i'$. These arcs are called $\sigma_f$-arcs and $\tau_f$-arcs respectively. Any $\sigma_f$-arc (resp. $\tau_f$-arc) is also called a $\sigma$-arc (resp. $\tau$-arc). Notice even though an arc is formally the same as an interval, we distinguish these types of objects. The only arcs that exist are $\sigma$-arcs and $\tau$-arcs. The purpose of this construction is that the conditions pertaining to $\sigma_f, \tau_f$ in (3.22) are equivalent to the fact that two end-points of intervals connected by a $\sigma$-arc or $\tau$-arc are assigned the same value in $[n]$ by the relevant functions out of $a, r_f, R_f, V_f, v_f, U_f, u_f, s_f, S_f$.

After adding these arcs, every endpoint of an interval has exactly one arc emanating from it. We have therefore now constructed a trivalent graph $G(\{\sigma_f, \tau_f, \pi_i\})$. The number of vertices of this graph is the twice the total number of $w$-intervals, $R$-intervals, and $R^{-1}$-intervals which is $2(|w| + 8(k + \ell))$. Therefore we have

$$\chi(G(\{\sigma_f, \tau_f, \pi_i\})) = -(|w| + 8(k + \ell)). \tag{4.2}$$

(For general $g$, we have $\chi(G(\{\sigma_f, \tau_f, \pi_i\})) = -(|w| + 4g(k + \ell))$.) Moreover, the conditions in (3.22) are now interpreted purely in terms of the combinatorics of this graph.

Gluing in discs

There are two types of cycles in $G(\{\sigma_f, \tau_f, \pi_i\})$ that we wish to consider:

- Cycles that alternate between following either a $w$-intermediate interval or a $\pi$-interval and then either a $\sigma$-arc or a $\tau$-arc. These cycles are disjoint from one another, and every $\sigma$ or $\tau$-arc is contained in exactly one such cycle. We call these cycles type-I cycles. For every type-I cycle, we glue a disc to $G(\{\sigma_f, \tau_f, \pi_i\})$ along its boundary, following the cycle. These discs will be called type-I discs. (These are analogous to the $o$-discs of [MP19].)

- Cycles that alternate between following either a $w$-interval, an $R$-interval, or an $R^{-1}$-interval and then either a $\sigma$-arc or a $\tau$-arc. Again, these cycles are disjoint, and every $\sigma$ or $\tau$-arc is contained in exactly one such cycle. We call these cycles type-II cycles. For every type-II cycle, we glue a disc to $G(\{\sigma_f, \tau_f, \pi_i\})$ identifying the boundary of the disc with the cycle. These discs will be called type-II discs. (These are similar to the $z$-discs of [MP19].)
Because every interior of an interval meets exactly one of the glued-in discs, and every arc has two boundary segments of discs glued to it, the object resulting from gluing in these discs is a decorated topological surface that we denote by

\[ \Sigma(\{\sigma_f, \tau_f, \pi_i\}). \]

The boundary components of \( \Sigma(\{\sigma_f, \tau_f, \pi_i\}) \) consist of the \( w \)-loop and the \( R^\pm \)-\( \pi \)-loops. It is not hard to check that \( \Sigma(\{\sigma_f, \tau_f, \pi_i\}) \) is orientable with an orientation compatible with the fixed orientations of the boundary loops corresponding to traversing every \( w \)-interval or \( \pm 1 \)-interval from 0 to 1.

We view the given CW-complex structure, and the assigned labelings and directions of the intervals that now pave \( \partial \Sigma \) as part of the data of \( \Sigma(\{\sigma_f, \tau_f, \pi_i\}). \) The number of discs of \( \Sigma(\{\sigma_f, \tau_f, \pi_i\}) \) is connected to the quantities appearing in Proposition 3.6 as follows.

**Lemma 4.1.** \( N(\{\sigma_f, \tau_f, \pi_i\}) = n \# \{\text{type-I discs of } \Sigma(\{\sigma_f, \tau_f, \pi_i\})\}. \)

**Proof.** The constraints on the functions \( a, r_f, R_f, V_f, V_f, U_f, u_f, s_f, S_f \) in (3.22) now correspond to the fact that altogether, they assign the same value in \( [n] \) to every interval end-point in the same type-I-cycle, and there are no other constraints between them. \( \square \)

The quantities \( |\sigma_f \tau_f^{-1}| \) in (3.24) can also be related to \( \Sigma(\{\sigma_f, \tau_f, \pi_i\}) \) as follows.

**Lemma 4.2.** We have

\[ \prod_{f \in \{a,b,c,d\}} n^{-|\sigma_f \tau_f^{-1}|} = n^{-d(k+\ell)} \frac{|w|}{\pi} n^{\# \{\text{type-II discs of } \Sigma(\{\sigma_f, \tau_f, \pi_i\})\}}. \]

**Proof.** Recalling the definition of \( |\sigma_f \tau_f^{-1}| \) from Proposition 2.7, we can also write

\[ |\sigma_f \tau_f^{-1}| = k + \ell + p_f - \# \{\text{cycles of } \sigma_f \tau_f^{-1}\}. \]

The cycles of \( \{\sigma_f \tau_f^{-1} : f \in \{a,b,c,d\}\} \) are in 1:1 correspondence with the type-II cycles of \( \Sigma(\{\sigma_f, \tau_f, \pi_i\}) \) and hence also the type-II discs. Therefore

\[ \prod_{f \in \{a,b,c,d\}} n^{-|\sigma_f \tau_f^{-1}|} = n^{-d(k+\ell)} n^{\sum_{f \in \{a,b,c,d\}} (-p_f + \# \{\text{cycles of } \sigma_f \tau_f^{-1}\})} \]

\[ = n^{-d(k+\ell)} \frac{|w|}{\pi} n^{\# \{\text{type-II discs of } \Sigma(\{\sigma_f, \tau_f, \pi_i\})\}}. \]

\( \square \)

We are now able to prove the following.

**Theorem 4.3.** For \( k + \ell > 0 \), \( \mu \vdash k \) and \( \nu \vdash \ell \), \( w \in [F_4, F_4] \), we have

\[ J_n(w, [\mu, \nu]) \llw_{k,\ell} \sum_{\{\sigma_f, \tau_f, \pi_i\} \in MATCH(w,k,\ell)} n^{-\sum_{i=1}^{n} \|\pi_i\|_{k,\ell} n^{\chi(\Sigma(\{\sigma_f, \tau_f, \pi_i\}))}}. \]

**Proof.** Combining Lemmas 4.1 and 4.2 with Corollary 3.7 gives

\[ J_n(w, [\mu, \nu]) \llw_{k,\ell} n^{-8k-8\ell-|w|} \sum_{\{\sigma_f, \tau_f, \pi_i\} \in MATCH(w,k,\ell)} n^{-\sum_{i=1}^{n} \|\pi_i\|_{k,\ell} n^{\# \{\text{discs of } \Sigma(\{\sigma_f, \tau_f, \pi_i\})\}}}. \]

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Then from (4.2) we obtain
\[
\mathcal{J}_n(w,[\mu,\nu]) \leq \sum_{\{\sigma_f,\tau_f,\pi_i\} \in \text{MATCH}(w,k,\ell)} n - \sum_{i=1}^{8} \|\pi_i\|_{k,\ell} \chi(G(\{\sigma_f,\tau_f,\pi_i\})) + \# \text{discs of } \Sigma(\{\sigma_f,\tau_f,\pi_i\})
\]
\[
= \sum_{\{\sigma_f,\tau_f,\pi_i\} \in \text{MATCH}(w,k,\ell)} n - \sum_{i=1}^{8} \|\pi_i\|_{k,\ell} \chi(\Sigma(\{\sigma_f,\tau_f,\pi_i\})).
\]

\[\Box\]

### 4.2 Two simplifying surgeries

Theorem 4.3 suggests that we now bound

\[
\chi(\Sigma(\{\sigma_f,\tau_f,\pi_i\})) - \sum_{i=1}^{8} \|\pi_i\|_{k,\ell}
\]

for all \(\{\sigma_f,\tau_f,\pi_i\} \in \text{MATCH}(w,k,\ell)\). To do this, we make some observations that simplify the task. If \(C\) is a simple closed curve in a surface \(S\), then compressing \(S\) along \(C\) means that we cut \(S\) along \(C\) and then glue discs to cap off any new boundary components created by the cut.

Suppose that we are given \(\{\sigma_f,\tau_f,\pi_i\} \in \text{MATCH}(w,k,\ell)\). Then \(\{\sigma_f,\tau_f,\pi_i\}\) is also in \(\text{MATCH}(w,k,\ell)\) (the forbidden matching property continues to hold). It is not hard to see that

\[
\chi(\Sigma(\{\sigma_f,\sigma_f,\pi_i\})) \geq \chi(\Sigma(\{\sigma_f,\tau_f,\pi_i\})).
\]

Indeed, the \(\tau_f\) arcs can be replaced by \(\sigma_f\)-parallel arcs inside the type-II discs of \(\Sigma(\{\sigma_f,\tau_f,\pi_i\})\). The resulting surface’s arcs may not cut the surface into discs, but this can be fixed by (possibly repeatedly) compressing the surface along simple closed curves disjoint from the arcs, leaving the combinatorial data of the arcs unchanged but only potentially increasing the Euler characteristic.

It remains to deal with the sum \(\sum_{i=1}^{8} \|\pi_i\|_{k,\ell}\).

Suppose again that an arbitrary \(\{\sigma_f,\tau_f,\pi_i\} \in \text{MATCH}(w,k,\ell)\) is given. For each \(i \in [8]\) write

\[
\pi_i = \pi_i^* \sigma_i
\]

where \(\pi_i^* \in S_k \times S_\ell\), \(\sigma_i = (\pi_i^*)^{-1} \pi_i \in S_{k+\ell}\), and \(|\sigma_i| = \|\pi_i\|_{k,\ell}\). Let \(X_0 \overset{\text{def}}{=} \Sigma(\{\sigma_f,\tau_f,\pi_i\})\).

Take \(\Sigma(\{\sigma_f,\tau_f,\pi_i\})\) and add to it all the \(\pi_i^*\)-intervals that would have been added if \(\pi_i\) was replaced by \(\pi_i^*\) for each \(i \in [8]\) in its construction. The resulting object \(X_1\) is the decorated surface \(X_0\) together with a collection of \(\pi_i^*\)-intervals with endpoints in the boundary of \(X_0\), and interiors disjoint from \(X_0\). This adds \(8(k+\ell)\) edges to \(X_0\) and hence

\[
\chi(X_1) = \chi(\Sigma(\{\sigma_f,\tau_f,\pi_i\})) - 8(k+\ell).
\]

Now we consider all cycles that for any fixed \(i \in [8]\), alternate between \(\pi_i\)-intervals and \(\pi_i^*\)-intervals. The number of these cycles is the total number of cycles of the permutations \(\{ (\pi_i^*)^{-1} \pi_i : i \in [8] \}\). On the other hand, the number of cycles of \((\pi_i^*)^{-1} \pi_i\) is

\[
k + \ell - \|(\pi_i^*)^{-1} \pi_i\| = k + \ell - |\sigma_i| = k + \ell - \|\pi_i\|_{k,\ell}
\]

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So in total there are $8(k + \ell) - \sum_i \|\pi_i\|_{k,\ell}$ of these cycles. For every such cycle, we glue a disc along its boundary to the cycle. The resulting object is denoted $X_2$. Now, $X_2$ is a topological surface, and we added $8(k + \ell) - \sum_i \|\pi_i\|_{k,\ell}$ discs to $X_1$ to form $X_2$, so

$$\chi(X_2) = \chi(X_1) + 8(k + \ell) - \sum_i \|\pi_i\|_{k,\ell} = \chi(\Sigma(\{\sigma_f, \tau_f, \pi_i\})) - \sum_i \|\pi_i\|_{k,\ell}.$$ 

Now ‘forget’ all the original $\pi_i$-intervals from $X_2$ to form $X_3$. The surface $X_3$ is a decorated surface in the same sense as $X_0$, except the connected components of $X_3 - \{\text{arcs}\}$ may not be discs. Similarly to before, by sequentially compressing $X_3$ along non-nullhomotopic simple closed curves disjoint from arcs, if they exist, we obtain a new decorated surface $X_4$. Moreover, and this is the main point, $X_4$ is the same as $\Sigma(\{\sigma_f, \tau_f, \pi_i^*\})$ in the sense that they are related by a decoration-respecting cellular homeomorphism. Compression can only increase the Euler characteristic, so we obtain

$$\chi(\Sigma(\{\sigma_f, \tau_f, \pi_i^*\})) \geq \chi(X_3) = \chi(X_2) = \chi(\Sigma(\{\sigma_f, \tau_f, \pi_i\})) - \sum_i \|\pi_i\|_{k,\ell}.$$ 

Combining these two arguments proves the following proposition.

**Proposition 4.4.** For any given $\{\sigma_f, \tau_f, \pi_i\}$, there exist $\pi_i^* \in S_k \times S_\ell$ for $i \in [8]$ such that

$$\chi(\Sigma(\{\sigma_f, \sigma_f, \pi_i^*\})) - \sum_{i=1}^{8} \|\pi_i^*\|_{k,\ell} = \chi(\Sigma(\{\sigma_f, \sigma_f, \pi_i^*\})) \geq \chi(\Sigma(\{\sigma_f, \tau_f, \pi_i\})) - \sum_{i=1}^{8} \|\pi_i\|_{k,\ell}.$$ 

This has the following immediate corollary when combined with Theorem 4.3. Let

$$\text{MATCH}^s(w, k, \ell)$$

denote the subset of MATCH($w, k, \ell$) consisting of $\{\sigma_f, \sigma_f, \pi_i\}$ (i.e. $\sigma_f = \tau_f$ for each $f \in \{a, b, c, d\}$) with $\pi_i \in S_k \times S_\ell$ for each $i \in [8]$.

**Corollary 4.5.** For $k + \ell > 0$, $\mu \vdash k$ and $\nu \vdash \ell$, $w \in [F_4, F_4]$, we have

$$\mathcal{J}_n(w, [\mu, \nu]) \ll_{w, k, \ell} \max_{\{\sigma_f, \sigma_f, \pi_i\} \in \text{MATCH}^s(w, k, \ell)} \chi(\Sigma(\{\sigma_f, \sigma_f, \pi_i\})).$$

The benefit to having $\pi_i \in S_k \times S_\ell$ for $i \in [8]$ is the following. Suppose now that $\{\sigma_f, \sigma_f, \pi_i\} \in \text{MATCH}^s(w, k, \ell)$. Recall that the boundary loops of $\Sigma(\{\sigma_f, \sigma_f, \pi_i\})$ consist of one $w$-loop and some number of $R^\pm$-loops. The condition that each $\pi_i \in S_k \times S_\ell$ means that no $\pi$-interval ever connects an endpoint of a $R$-interval with an endpoint of an $R^{-1}$-interval. So every boundary component of $\Sigma(\{\sigma_f, \sigma_f, \pi_i\})$ that is not the $w$-loop contains either only $R$-intervals or only $R^{-1}$-intervals, and in fact, when following the boundary component and reading the directions and labels of the intervals according to traversing each from 0 to 1, reads out a positive power of $R$ (in the former case of only $R$-intervals) or a negative power of $R^{-1}$ (in the latter case of only $R^{-1}$-intervals). The sum of the positive powers of $R$ in boundary loops is $k$, and the sum of the negative powers of $R$ is $-\ell$. Knowing this boundary structure is extremely important for the arguments in the next sections.
4.3 A topological result that proves Theorem 3.1

Here, in the spirit of Culler [Cul81], we explain another way to think about the surfaces $\Sigma(\{\sigma_f, \sigma_f, \pi_i\})$ for $\{\sigma_f, \sigma_f, \pi_i\} \in \text{MATCH}^*(w, k, \ell)$ that is easier to work with than the construction we gave. At this point we also show how things work for general $g \geq 2$. An arc in a surface $\Sigma$ is a properly embedded interval in $\Sigma$ with endpoints in the boundary $\partial \Sigma$.

**Definition 4.6.** For $w \in F_{2g}$, we define $\text{surfaces}(w, k, \ell)$ to be the set of all decorated surfaces $\Sigma^*$ as follows. A decorated surface $\Sigma^* \in \text{surfaces}(w, k, \ell)$ is an oriented surface with boundary, with compatibly oriented boundary components, together with a collection of disjoint embedded arcs that cut $\Sigma^*$ into topological discs. One boundary component is assigned to be a $w$-loop, and every other boundary component is assigned to be either a $R$-loop or an $R^{-1}$-loop. Each arc is assigned a transverse direction and a label in $\{a_1, b_1, \ldots, a_g, b_g\}$. Every arc-endpoint in $\partial \Sigma^*$ inherits a transverse direction and label from the assigned direction and label of its arc. We require that $\Sigma^*$ satisfy the following properties.

**P1** When one follows the $w$-loop according to its assigned orientation, and reads $f$ when an $f$-labeled arc-endpoint is traversed in its given direction, and $f^{-1}$ when an $f$-labeled arc-endpoint is traversed counter to its given direction, one reads a cyclic rotation of $w$ in reduced form, depending on where one begins to read.

**P2** When one follows any $R$-loop according to its assigned orientation in the same way as before, one reads (a cyclic rotation) of some positive power of $R_g$ in reduced form. The sum of these positive powers over all $R$-loops is $k$.

**P3** When one follows any $R^{-1}$-loop according to its assigned orientation in the same way as before, one reads (a cyclic rotation) of some negative power of $R_g$ in reduced form. The sum of these negative powers over all $R^{-1}$-loops is $-\ell$.

**P4** No arc connects an $R$-loop to an $R^{-1}$-loop.

Given a surface $\Sigma(\{\sigma_f, \sigma_f, \pi_i\})$ with $\{\sigma_f, \sigma_f, \pi_i\} \in \text{MATCH}^*(w, k, \ell)$, all the type-II discs of the surface are rectangles. Hence, by collapsing each $w$-interval, $R$-interval, and $R^{-1}$-interval to a point, and collapsing every type-II rectangle to an arc, we obtain a CW-complex that is a surface with boundary, cut into discs by arcs. Every arc inherits a transverse direction and label from the compatible assigned directions and labels of the intervals in the boundary of its originating type-II rectangle. We call this modified surface $\Sigma^* = \Sigma^*(\{\sigma_f, \pi_i\})$. It clearly satisfies **P1-P3** and **P4** follows from the forbidden matchings property. (Of course, when $g = 2$, we identify $\{a, b, c, d\}$ with $\{a_1, b_1, a_2, b_2\}$.) We also have $\chi(\Sigma(\{\sigma_f, \sigma_f, \pi_i\})) = \chi(\Sigma^*(\{\sigma_f, \pi_i\}))$. With Definition 4.6 and the remarks proceeding it, we can now state a further consequence of Corollary 4.5 as it extends to general $g \geq 2$.

**Corollary 4.7.** For $k + \ell > 0$, $\mu \vdash k$, $\nu \vdash \ell$, $w \in [F_{2g}, F_{2g}]$, as $n \to \infty$

$$J_n(w, [\mu, \nu]) \ll_{w, k, \ell} n^{\max\{\chi(\Sigma^*): \Sigma^* \in \text{surfaces}(w, k, \ell)\}}.$$  

In order for Corollary 4.7 to give us strong enough results it needs to be combined with the following non-trivial topological bound.

**Proposition 4.8.** If $w \in [F_{2g}, F_{2g}]$ is a shortest element representing the conjugacy class of $\gamma \in \Gamma_g$, $w \neq \text{id}$, and $\Sigma^* \in \text{surfaces}(w, k, \ell)$ then $\chi(\Sigma^*) \leq -(k + \ell)$.
Remark 4.9. Proposition 4.8 is by no means a trivial statement and one has to use that \( w \) is a shortest element representing the conjugacy class of some element of \( \Gamma_g \). For example, if \( w = \gamma^g \), then \( w \) represents the conjugacy class of \( \text{id} \), but for \( k = 0 \) and \( \ell = 1 \) there is an ‘obvious’ annulus in surfaces\((w,0,1)\). This has \( \chi = 0 > -(k + \ell) = -1 \). Proposition 4.8 also requires \( w \neq \text{id} \); if \( w = \text{id} \) then for \( k = 0 \) and \( \ell = 1 \) one can take a disc with no arcs as a valid element of surfaces\((\text{id},0,0)\). This has \( \chi = 1 > -(k + \ell) = 0 \). In fact this disc is ultimately responsible for \( \mathbb{E}_{g,n}[\text{id}] = n \).

The proof of Proposition 4.8 is self-contained and given in §§4.5. Before doing this, we prove Theorem 3.1.

Proof of Theorem 3.1 given Proposition 4.8. Since Theorem 3.1 was proved when \( k = \ell = 0 \) in §§3.2, we can assume \( k + \ell > 0 \). Then combining Corollary 4.7 and Proposition 4.8 gives

\[
\mathcal{J}_n(w, [\mu, \nu]) \ll_{w,k,\ell} n^{-(k+\ell)}.
\]

On the other hand, \( D_{[\mu,\nu]}(n) = O(n^{k+\ell}) \) from (2.1). Therefore \( D_{[\mu,\nu]}(n)\mathcal{J}_n(w, [\mu, \nu]) \ll_{w,k,\ell} 1 \).

4.4 Work of Dehn and Birman-Series

As we mentioned in §§3.1, to prove Proposition 4.8 we have to use the fact that \( w \in [F_{2g}, F_{2g}] \) is a shortest element representing the conjugacy class of \( \gamma \in \Gamma_g \). We use a combinatorial characterization of such words that stems from Dehn’s algorithm [Deh12] for solving the problem of whether a given word represents the identity in \( \Gamma_g \). The ideas of Dehn’s algorithm were refined by Birman and Series in [BS87]. In [MP20], the author and Puder used Birman and Series’ results (alongside other methods) to obtain the analog of Theorem 1.2 when the family of groups \( SU(n) \) is replaced by the family of symmetric groups \( S_n \). Similar consequences of the work of Dehn, Birman, and Series that we used in (ibid.) will be used here.

We now follow the language of [MP20] to state the results we need in this paper. We stress one more time that these results are simple and direct consequences of the work of Birman and Series.

We view the universal cover of \( \Sigma_g \) as a disc tiled by \( 4g \)-gons that we call \( U \). We assume every edge of this tiling is directed and labeled by some element of \( \{a_1, b_1, \ldots, a_g, b_g\} \) such that when we read counter-clockwise along the boundary of any octagon we read the reduced cyclic word \([a_1, b_1] \cdots [a_g, b_g]\). By fixing a basepoint \( u \in U \) we obtain a free cellular action of \( \Gamma_g \) on \( U \) that respects the labels and directions of edges and identifies the quotient \( \Gamma_g \backslash U \) with \( \Sigma_g \); this gives a description of \( \Sigma_g \) as a \( 4g \)-gon with glued sides as is typical.

Now suppose that \( \gamma \in \Gamma \) is not the identity. The quotient \( A_\gamma = \langle \gamma \rangle \backslash U \) of \( U \) by the cyclic group generated by \( \gamma \) is an open annulus tiled by infinitely many \( 4g \)-gons. The edges of \( A_\gamma \) inherit directions and labels from those of the edges of \( U \). The point \( u \in U \) maps to some point denoted by \( x_0 \in A_\gamma \).

Now let \( w \in F_{2g} \) be an element that represents \( \gamma \), and identify \( w \) with a combinatorial word by writing \( w \) in reduced form. Beginning at \( x_0 \), and following the path spelled out by \( w \) beginning at \( x_0 \), we obtain an oriented closed loop \( L_w \) in the one-skeleton of \( A_\gamma \). If \( w \) is a shortest element representing the conjugacy class of \( \gamma \), then this loop \( L_w \) must not have self-intersections. In this case, that we from now assume, \( L_w \) is therefore a topologically
Figure 4.1: Illustration of a piece $P$ of $\hat{L}_w$ in the case when the reduced form of $w$ contains $a_ga_1^{-1}b_2^{-1}$ as a subword. The edges of $L_w$ are in bold. The piece is indicated by the dotted lines. This piece $P$ has $e(P) = 2$, $he(P) = 7$, and $\chi(P) = 1$. Note that a piece may also run along the other side of $L_w$.

embedded circle in the annulus $A_\gamma$ that is non-nullhomotopic and cuts $A_\gamma$ into two annuli $A_\gamma^\pm$.

Every vertex of $A_\gamma$ has $4g$ incident half-edges each of which has an orientation and direction given by the edge they are in. Going clockwise, the cyclic order of the half-edges incident at any vertex is

'\text{$a_1$-outgoing, $b_1$-incoming, $a_1$-incoming, $b_1$-outgoing, ... , $a_g$-outgoing, $b_g$-incoming, $a_g$-incoming, $b_g$-outgoing'}'.

We define $\hat{L}_w$ to be the loop $L_w$ with all incident half edges in $A_\gamma$ attached. We call the new half-edges added hanging half-edges.

Moreover, we thicken up $\hat{L}_w$ by viewing each edge of $L_w$ as a rectangle, each hanging half-edge as a half-rectangle, and each vertex replaced by a disc. In other words, we take a small neighborhood of $\hat{L}_w$ in $A_\gamma$. We now think of $\hat{L}_w$ as the thickened version. This is a topological annulus, where the hanging half-edges have become stubs hanging off. A piece of $\hat{L}_w$ is a contiguous collection of hanging half-rectangles and rectangle sides following edges of $L_w$ in the boundary of $\hat{L}_w$. Such a piece is in either $A_\gamma^+$ or $A_\gamma^-$. Given a piece $P$ of $\hat{L}_w$ we write $e(P)$ for the number of rectangle sides following edges of $L_w$, and $he(P)$ for the number of hanging-half edges in $P$. We say that a piece $P$ has Euler characteristic $\chi(P) = 0$ if it follows an entire boundary component of $\hat{L}_w$, and $\chi(P) = 1$ otherwise as we view it as an interval running along the rectangle sides and around the sides of the hanging half-rectangles. See Figure 4.1 for an illustration of a piece of $\hat{L}_w$.

Birman and Series prove in [BS87, Thm. 2.12(a)] that if $w$ is a shortest element representing the conjugacy class of $\gamma \in \Gamma_g$ then there are strong restrictions on the pieces of $\hat{L}_w$ that can appear. This has the following consequence which is given by [MP20, Proof of Lem. 4.20].

**Lemma 4.10.** If $w$ is a shortest element representing the conjugacy class of $\gamma \in \Gamma_g$, and both
\[ \gamma \text{ and hence } w \text{ are non-identity, then for any piece } P \text{ of } \hat{L}_w, \text{ we have} \]
\[ e(P) \leq (2g - 1)h(P) + 2g\chi(P). \]

**Proof.** Since \( w \) is a shortest element representing some non-identity conjugacy class in \( \Gamma_g \), in the language of [MP20], \( L_w \) is a boundary reduced tiled surface. Then the proof of [MP20, Lem. 4.20] contains the result stated in the lemma. \( \square \)

This inequality plays a crucial role in the next section.

### 4.5 Proof of Proposition 4.8

Suppose that \( g \geq 2 \) and \( w \in [F_{2g}, F_{2g}] \) is a non-identity shortest element representing the conjugacy class of \( \gamma \in \Gamma_g \). In particular, \( w \) is cyclically reduced. We let \( R = R_g \). Now fix \( k, \ell \in \mathbb{N}_0 \) and suppose \( \Sigma^* \in \text{surfaces}(w, k, \ell) \). The arcs of \( \Sigma^* \) are of three different types:

- **WR** An arc with one endpoint in the \( w \)-loop and one endpoint in an \( R \) or \( R^{-1} \)-loop.
- **RR** An arc with both endpoints in \( R \) or \( R^{-1} \) loops. By property \( P4 \), the endpoints of such an arc are both in \( R \)-loops or both in \( R^{-1} \)-loops.
- **WW** An arc with both endpoints in the \( w \)-loop.

The boundary of any disc of \( \Sigma^* \) alternates between segments of \( \partial \Sigma^* \) and arcs. A disc is a **pre-piece disc** if its boundary contains exactly one segment of the \( w \)-loop. A disc is called a **junction disc** if it is not a pre-piece disc. We say that a junction disc is **piece-adjacent** if it meets a WR-arc-side.

To be precise, we view all discs as open discs, and hence not containing any arcs. A disc meets certain arc-sides along its boundary; it is possible for a disc to meet both sides of the same arc and we view this scenario as the disc meeting two separate arc-sides. We say an arc-side has the same type WR/RR/WW as its corresponding arc.

Note that any pre-piece disc cannot meet any WW-arc-side: if it did, the disc could only meet this one arc-side together with one segment of the \( w \)-loop and this would contradict the fact that \( w \) is cyclically reduced since the arc matches a letter \( f \) with a cyclically adjacent letter \( f^{-1} \) of \( w \). It is also clear that any pre-piece disc meets exactly 2 WR-arc-sides: the ones that emanate from the sole segment of the \( w \)-loop. So in light of \( P4 \) a pre-piece disc takes one of the forms shown in Figure 4.2.

We define a **piece of** \( \Sigma^* \) to be a connected component of

\[ \{\text{pre-piece discs}\} \cup \{\text{WR-arcs}\}. \]

A piece of \( \Sigma^* \) is therefore either a contiguous collection of pre-piece discs that meet only along WR-arcs, or a single WR-arc. If \( P \) is a piece of \( \Sigma^* \), either \( \chi(P) = 1 \), or \( \chi(P) = 0 \), in which case \( P \) meets the entire \( w \)-loop and is the unique piece.

We now have **two** definitions of pieces; pieces of \( \hat{L}_w \) and pieces of \( \Sigma^* \). These are, as the names suggest, closely related, and this is the key observation in the proof of Proposition 4.8. Indeed, the reader should carefully consider Figure 4.3 that leads to the following lemma. In analogy to pieces of \( \hat{L}_w \), if \( P \) is any piece of \( \Sigma^* \), we write \( e(P) \) for the number of WR-arcs.
Figure 4.2: Possible forms of pre-piece discs. The number of $R$-loop segments or $R^{-1}$-loop segments is at least 1 and bounded given $k$ and $\ell$. The arrows denote the orientations of the boundary loops.

Figure 4.3: Given a segment of the $w$-loop corresponding to a juncture between letters $a_1^{-1}b_2^{-1}$ in $w$, if this segment is part of a pre-piece disc then some possible forms of that disc are shown above. This juncture between letters of $w$ corresponds to a vertex in $L_w$. The right hand illustration shows the neighborhood of this vertex in the annulus $A_\gamma$, where the bold lines correspond to half-edges of $L_w$. The right hand picture actually almost determines the left hand pictures. Indeed, given the $a_1$ arc on the top-left, the next arc has to be a $b_2$ arc with the given direction, since only $b_2^{-1}$ cyclically precedes $a_1$ in $R_g$ or any power of $R_g$. Then the next arc $a_2$ with its direction is determined since only $a_2$ cyclically precedes $b_2$ in $R_g$. This continues until an arc labeled by $b_2$ and with an incoming direction is reached, as in the right arc of the top-left picture. At this point, the boundary of the disc may close up. (This is analogous to what happens in the bottom picture, where an analogous pattern occurs.) The only indeterminacy is that after reaching a $b_2$ arc with an incoming direction for the first time, the entire pattern shown in the right hand picture may repeat any number of times, as long as $k$ and $\ell$ allow it. The upshot of this is that any pre-piece disc has at least as many incident RR-arc-sides as there are hanging half-edges on the corresponding side of $L_w$, at the corresponding vertex.
Lemma 4.11. If \( w \) is a shortest element representing the conjugacy class of \( \gamma \in \Gamma_g, k, \ell \in \mathbb{N}_0 \), and \( \Sigma^* \in \text{surfaces}(w, k, \ell) \) then for any piece \( P \) of \( \Sigma^* \), we have
\[
e(P) \leq (2g - 1)h(P) + 2g\chi(P).
\]

Proof. Given any piece \( P \) of \( \Sigma^* \), it contains a consecutive (possibly cyclic) series of WR-arcs that correspond to a contiguous collection of edges in the loop \( L_w \). The discs of \( P \) correspond to certain vertices of \( L_w \); each of these vertices has two emanating half-edges belonging to the edges defined by WR-arcs of \( P \). The piece \( P \) can either meet only \( R \)-loops or meet only \( R^{-1} \)-loops.

We define a piece \( P' \) of \( \hat{L}_w \) corresponding to \( P \) as follows. If \( P \) meets \( R \)-loops, then \( P' \) consists of rectangle sides along the edges of \( L_w \) corresponding to the WR-arcs of \( P \) together with all hanging half-edges at vertices corresponding to discs of \( P \) that are on the left of \( L_w \) as it is traversed in its assigned orientation (corresponding to reading \( w \) along \( L_w \)). If \( P' \) meets \( R^{-1} \)-loops, then \( P' \) is defined similarly with the modification that we include instead hanging half-edges on the right of \( L_w \). Figure 4.3 together with its captioned discussion now shows that
\[
h(P) \leq h(P'),
\]
and \( e(P) = e(P') \) by construction. We also have \( \chi(P') = \chi(P) \). Therefore Lemma 4.10 applied to \( P' \) implies
\[
e(P) = e(P') \leq (2g - 1)h(P') + 2g\chi(P') \leq (2g - 1)h(P) + 2g\chi(P).
\]

Let \( N_{RR} \) be the number of RR-arcs, \( N_{WR} \) the number of WR-arcs, and \( N_{WW} \) the number of WW-arcs in \( \Sigma^* \). In the following we refer to discs of \( \Sigma^* \) simply as discs. Since there are \( 4g(k + \ell) \) incidences between arcs and \( R \)-loops or \( R^{-1} \) loops we have
\[
2N_{RR} + N_{WR} = 4g(k + \ell).
\] (4.3)

Let \( \Sigma_1 \) be the surface formed by cutting \( \Sigma^* \) along all RR-arcs. We have
\[
\chi(\Sigma_1) = \sum_{\text{discs } D} \left( 1 - \frac{d'(D)}{2} \right)
\]
where \( d'(D) \) is the number of arc-sides meeting \( D \) that are not of type RR. (In other words,
$d'(D)$ is the degree of $D$ in the dual graph of $\Sigma_1$.) We partition this sum according to

\[ \chi(\Sigma_1) = S_0 + S_1 + S_2, \]

\[ S_0 \overset{\text{def}}{=} \sum_{\text{pre-piece discs } D} \left( 1 - \frac{d'(D)}{2} \right), \]

\[ S_1 \overset{\text{def}}{=} \sum_{\text{piece-adjacent junction discs } D} \left( 1 - \frac{d'(D)}{2} \right), \]

\[ S_2 \overset{\text{def}}{=} \sum_{\text{not piece-adjacent junction discs } D} \left( 1 - \frac{d'(D)}{2} \right). \]

Note first that a pre-piece disc has $d'(D) = 2$ (cf. Fig 4.2). Hence $S_0 = 0$. We deal with $S_1$ next. For a disc $D$ of $\Sigma^*$, let $d_{WR}(D)$ denote the number of WR-arc-sides meeting $D$. Note that a piece-adjacent junction disc $D$ has $d_{WR}(D) > 0$ by definition. We rewrite $S_1$ as

\[ S_1 = \sum_{\text{piece-adjacent junction discs } D} \left( 1 - \frac{d'(D)}{2} \right) \frac{1}{d_{WR}(D)} \sum_ {\text{incidences between } D \text{ and WR-arc-sides}} 1 \]

\[ = \sum_{\text{pieces } P} \sum_{\text{incidences between } P \text{ and some junction disc } D \text{ along WR-arc}} \frac{1}{d_{WR}(D)} \left( 1 - \frac{d'(D)}{2} \right) \]

\[ = \sum_{\text{pieces } P} \sum_{\text{incidences between } P \text{ and some junction disc } D \text{ along WR-arc}} Q(D) \quad (4.4) \]

where for a piece-adjacent junction disc $D$

\[ Q(D) \overset{\text{def}}{=} \frac{1}{d_{WR}(D)} \left( 1 - \frac{d'(D)}{2} \right). \]

Suppose that $D$ is a piece-adjacent junction disc. By parity considerations, $d_{WR}(D)$ is even. We estimate $Q(D)$ by splitting into two cases. If $d_{WR}(D) = 2$ then $d'(D) \geq 3$ since otherwise, $D$ would meet only 2 WR arc-sides and other RR arc-sides, hence be a pre-piece disc and not be a junction disc. In this case

\[ Q(D) = \frac{1}{2} \left( 1 - \frac{d'(D)}{2} \right) \leq \frac{1}{2} \left( 1 - \frac{3}{2} \right) = -\frac{1}{4}. \]

Otherwise, $d_{WR}(D) \geq 4$ and since $d'(D) \geq d_{WR}(D)$, we have

\[ Q(D) \leq \frac{1}{d_{WR}(D)} \left( 1 - \frac{d_{WR}(D)}{2} \right) = \frac{1}{d_{WR}(D)} - \frac{1}{2} \leq \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}. \]

So we have proved that for all piece-adjacent junction discs $D$, $Q(D) \leq -\frac{1}{4}$. Putting this into (4.4) gives

\[ S_1 \leq -\frac{1}{4} \sum_{\text{pieces } P} \sum_{\text{incidences between } P \text{ and some junction disc } D \text{ along WR-arc}} 1 \]

\[ = -\frac{1}{4} \sum_{\text{pieces } P} 2\chi(P) = -\frac{1}{2} \sum_{\text{pieces } P} \chi(P). \quad (4.5) \]
We now turn to $S_2$. Here is the key moment where $w \neq \text{id}$ is used\textsuperscript{2}. Since $w \neq \text{id}$, any disc must meet an arc. Indeed, the only other possibility is that the boundary of the disc is an entire boundary loop that has no emanating arcs. This hypothetical boundary loop cannot be an $R$ or $R^{-1}$-loop, so it has to be the $w$-loop. But this would entail $w = \text{id}$.

Hence any disc contributing to $S_2$ meets no WR-arc-side, but meets some arc-side. Therefore it meets only WW-arcs or only RR-arcs. Every disc $D$ contributing to $S_2$ meeting only WW-arcs gives a non-positive contribution since $w$ is cyclically reduced hence $d'(D) \geq 2$. Every disc $D$ contributing to $S_2$ meeting only RR-arcs, which we will call an \textit{RR-disc}, has $d'(D) = 0$ and hence contributes 1 to $S_2$.

This shows

$$S_2 \leq \#\{\text{RR-discs}\}. \tag{4.6}$$

In total combining $S_0 = 0$ with (4.5) and (4.6) we get

$$\chi(\Sigma_1) \leq \#\{\text{RR-discs}\} - \frac{1}{2} \sum_{\text{pieces } P \text{ of } \Sigma^*} \chi(P).$$

To obtain $\Sigma^*$ from $\Sigma_1$ we have to glue all cut RR-arcs, of which there are $N_{RR}$. Each gluing decreases $\chi$ by 1 so

$$\chi(\Sigma^*) \leq \#\{\text{RR-discs}\} - N_{RR} - \frac{1}{2} \sum_{\text{pieces } P \text{ of } \Sigma^*} \chi(P).$$

Using Lemma 4.11 with the above gives

$$\chi(\Sigma^*) \leq \#\{\text{RR-discs}\} - N_{RR} - \frac{1}{2} \sum_{\text{pieces } P \text{ of } \Sigma^*} \chi(P)$$

$$\leq \#\{\text{RR-discs}\} - N_{RR} - \frac{1}{4g} \sum_{\text{pieces } P \text{ of } \Sigma^*} \varepsilon(P) + \frac{(2g - 1)}{4g} \sum_{\text{pieces } P \text{ of } \Sigma^*} \text{hc}(P)$$

$$= \#\{\text{RR-discs}\} - N_{RR} - \frac{N_{WR}}{4g} + \frac{(2g - 1)}{4g} \sum_{\text{pieces } P \text{ of } \Sigma^*} \text{hc}(P). \tag{4.7}$$

Let $\text{hc}'(\Sigma^*)$ denote the total number of RR-arc-sides meeting RR-discs. Every RR-disc has to meet at least $4g$ arc-sides; this observation is similar to the reasoning in Figure 4.3. Therefore

$$\text{hc}'(\Sigma^*) \geq 4g \#\{\text{RR-discs}\}. \tag{4.8}$$

Every RR-arc-side either meets a piece $P$ and contributes to $\text{hc}(P)$ or a disc meeting only RR-arc-sides and contributes to $\text{hc}'(\Sigma^*)$. Hence

$$\text{hc}'(\Sigma^*) + \sum_{\text{pieces } P \text{ of } \Sigma^*} \text{hc}(P) = 2N_{RR}. \tag{4.9}$$

\textsuperscript{2}Although technically, $w \neq \text{id}$ was used to define $L_w$ and pieces etc, if $w$ is the identity the proof of Proposition 4.8 could, a priori, circumvent these definitions.
Combining (4.3), (4.8), and (4.9) with (4.7) gives

\[
\chi(\Sigma^*) \leq \frac{h_\gamma'(\Sigma^*)}{4g} - \frac{N_{RR}}{4g} - \frac{N_{WR}}{4g} + \frac{(2g - 1)}{4g} \sum_{\text{pieces } P \text{ of } \Sigma^*} h_\gamma(P)
\]

(4.8)

\[
\chi(\Sigma^*) \leq \frac{h_\gamma'(\Sigma^*)}{4g} - \frac{N_{RR}}{4g} - \frac{N_{WR}}{4g} + \frac{(2g - 1)N_{RR}}{2g} - \frac{(2g - 1)}{4g} h_\gamma'(\Sigma^*)
\]

(4.9)

\[
= -\frac{1}{4g} (2N_{RR} + N_{WR}) - \frac{2g - 2}{4g} h_\gamma'(\Sigma^*)
\]

\[
\leq -\frac{1}{4g} (2N_{RR} + N_{WR}) \overset{(4.3)}{=} -\frac{4g(k + \ell)}{4g} = -(k + \ell).
\]

This completes the proof of Proposition 4.8. □

5 Proof of main theorem

5.1 Proof of Theorem 1.2

Proof of Theorem 1.2. Assume \(\gamma \in [\Gamma_g, \Gamma_g]\) is not the identity and that \(w \in [F_{2g}, F_{2g}]\) is a shortest element representing the conjugacy class of \(\gamma\), hence also not the identity. By Corollary 2.10 we have

\[
E_{g,n}[\text{tr}_\gamma] = \zeta(2g - 2; n)^{-1} \sum_{(\mu, \nu) \in \tilde{\Omega}} D_{[\mu, \nu]}(n)J_n(w, [\mu, \nu]) + O_{w,g}(\frac{1}{n}),
\]

where \(\tilde{\Omega}\) is a finite collection of pairs of Young diagrams. We know \(\lim_{n \to \infty} \zeta(2g - 2; n) = 1\) from (2.11) and for each fixed \((\mu, \nu), D_{[\mu, \nu]}(n)J_n(w, [\mu, \nu]) = O_{w,\mu,\nu}(1)\) by Theorem 3.1. Hence \(E_{g,n}[\text{tr}_\gamma] = O_\gamma(1)\) as \(n \to \infty\) as required. □

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