Reduction of order and Fadeev-Jackiw formalism in generalized electrodynamics

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Abstract

The aim of this work is to discuss some aspects of the reduction of order formalism in the context of the Fadeev-Jackiw symplectic formalism, both at the classical and the quantum level. We start by reviewing the symplectic analysis in a regular theory (a higher derivative massless scalar theory), both using the Ostrogradsky prescription and also by reducing the order of the Lagrangian with an auxiliary field, showing the equivalence of these two approaches. The interpretation of the degrees of freedom is discussed in some detail. Finally, we perform the similar analysis in a singular higher derivative gauge theory (the Podolsky electrodynamics), in the reduced order formalism: we claim that this approach have the advantage of clearly separating the symplectic structure of the model into a Maxwell and a Proca (ghost) sector, thus simplifying the understanding of the degrees of freedom of the theory.

Keywords: Constrained Systems; Gauge Theories; Higher derivative theories; Classical Field Theory; Quantum Field Theory
I. INTRODUCTION

Constrained systems are a basic tool for theoretical research in different contexts such as gauge theories and the field theory approach for gravity, for example. The pioneers in this treatment were Dirac and Bergmann (DB) [1–3] whose works established the standard method to study theories with constraints, providing generalized brackets appropriate to quantize these systems. When the dynamics of a singular Lagrangian formulated in configuration space is translated to a Hamiltonian formulation in phase space, the first constraints that appear, from the definition of the canonical momenta, are called the Dirac primary constraints. The condition that these should not change over time (consistency condition) may generate additional constraints, called secondary constraints, for which consistency conditions are again applied, and so forth. When this process ends and we arrive at a complete set of constraints, we may classify them as being of first or second class, according to the vanishing or not of their canonical Poisson brackets. This Dirac-Bergmann algorithm, including its classification of constraints, has a meaning associated with the physical degrees of freedom [4]. This provides a first approach to the connection between classical and quantum dynamics, the classical dynamics described in the phase space by the observables and (Poisson\Dirac) brackets, and the quantum dynamics described in Hilbert space by the operators and commutator\anti-commutators. A second approach begins in a study by Dirac about the connection between a dynamics described in configuration space and its resulting quantum dynamics, where we see the emergence of a very important object called the transition amplitude [5]. Feynman later used Dirac’s idea to describe the quantum Lagrangian mechanics with the path integral formalism [6]; afterwards, an elegant variational principle [7] of the quantum action was developed by Schwinger, utilizing as a guide the Heisenberg description [8].

The need to describe the interactions of nature along the lines of a relativistic dynamics leads us to build a covariant language with gauge symmetry [9], which has more degrees of freedom then the physical ones, hence the necessity of introducing constrains. The connections between classical and quantum physical systems with constraints, in a functional formalism, was first formulated by Faddeev (for first class constraints), and later the ideas where extended by Senjanovic (including second class constraints) [10]. The quantization procedure of a gauge theory is in principle possible for the physical degrees of freedom only,
and thus we loose the explicit covariance of the equations: in order to maintain it at the quantum level, Faddeev, Popov and DeWitt built a method in which additional, non-physical ghost fields, are introduced [11].

The canonical quantization gained new life with the Fadeev-Jackiw (FJ) method, developed in the 1980’s [12]. The (FJ) formalism pursues a classical geometric treatment based on the symplectic structure of the phase space and it is only applied to first order Lagrangians. The 2-form symplectic matrix associated with the reduced Lagrangian allows us to obtain the generalized brackets in the reduced phase space without the need to follow Dirac’s method step by step [13]. The (FJ) method has some very useful properties, such as not needing to distinguish the types of constraints and the Dirac’s conjecture, and therefore evoked much attention. Barcelos and Wotzasek introduced one procedure of dealing with constraints in the (FJ) method [14, 15]; on the other hand, despite the quantization being essentially canonical, the path integral quantization was also constructed in [16, 17]. We can find in the literature many studies of the equivalence between the (DB) and (FJ) formalisms [18–21], which can be proved in many (but not all) cases.

When Ostrogradski constructed Lagrangian theories with higher order derivatives in classical mechanics, a new field of research was opened [22]. The main idea of these theories is simple: we introduce additional higher order terms such as to preserve the original symmetries of the problem [23]. Bopp, Podolsky and Schwed [24] proposed a generalized electrodynamics in an endeavor to get rid of the infinites in quantum electrodynamics (QED), starting from a higher order Lagrangian, corresponding to the usual QED Lagrangian augmented by a quadratic term in the divergence of the field-strength tensor, which by dimensional reasons exhibits a free parameter that can be identified as the inverse of a mass, the Podolsky’s mass $m$. This modification gives the correct (finite) expression for the self-force of charged particles, as shown by Frenkel, and interesting effects produced by the presence of external sources [25, 26]. At the quantum level, higher derivative theories have in general the property of better behaved (or even absent) ultraviolet divergences in a sense closely related to the Pauli-Villars-Rayski regularization scheme [27, 28], but also sometimes exhibit Hamiltonians without a lower limit [29] due to the presence of states with negative norm (ghosts), leading to the breakdown of unitarity [30]. Several procedures to avoid this problem have been already been studied [31], one approach being a careful investigation of the analytic structure of the Green functions as discussed in [33–35]. Another way to
implement terms with higher order derivatives without breaking the stability of the theory has recently been proposed using the concept of Lagrangian anchors [32], an extension of the Noether theorems in the sense that one can define a class of conserved quantities associated with a given symmetry. For instance, the symmetry due to time translations will lead to two conserved Hamiltonians, one of them will have regularizing properties but will break the stability because the energy is not bounded from below, whereas the other recovers the stability but loses the regularizing properties. This leads to a new perspective on the unitarity problem of higher derivative theories, that makes use of the formalisms of reduction of order [36, 37] and the concepts of complexation of the Lagrangian [38].

Given the advantages in using the (FJ) formalism to deal with the constraint structure of gauge theories, it is natural to apply this formalism to the quantization of gauge theories with higher order derivatives. In doing so, one needs to bring the Lagrangian to a first order form, and the two most known ways to do so are either by extending the number of the canonical momenta (the Ortogradsky formalism) or by reducing the order of derivatives using auxiliary fields (which we call the reduced order formalism). The first approach was considered in [39], where the BRST quantization of the higher derivative Podolski electrodynamics was described, using the (FJ) method to deal with the constraints. On the other hand, recently the same model was also considered in the reduced order formalism [36], but using the Dirac procedure for the constraints analysis. In this work, we will show that the still unused path, that is, bringing the Lagrangian to a first order form with the reduced order formalism while working with the constraints using the (FJ) formalism, has a very nice property, which is the clear separation of degrees of freedom during the calculations, separating the non-massive sector from the massive (ghost) one.

This work is organized as follows. In Sec.II we review the main conceptual aspects of Fadeev-Jackiw formalism. In Sec.III we apply the (FJ) symplectic analysis to a simple higher derivative scalar model in both Ostrogradsky and reduced order formalisms, discussing their equivalence at classical and quantum level, as well as the interpretation of their degrees of freedom. In Sec.IV we present the reduced order version of the (FJ) formalism in a singular higher derivative gauge theory (Podolsky electrodynamics). Sec.V contains our conclusions and perspectives.
II. REVIEW OF FADEEV-JACKIW FORMALISM

We start with a brief review of the elementary aspects of the (FJ) formalism. Starting with a Lagrangian $L(q_i, \dot{q}_i)$, by means of a Lagrange transformation we define the canonical momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and Hamiltonian $H = p_i \dot{q}_i - L$. With the aim of writing the symplectic structure, the Lagrangian has to be cast as a first order expression in the velocities $\dot{\xi}_i$, where hereafter $\xi_i$ represents the set of all the canonical variables in the theory (at this point, $\xi_i$ corresponds to the set of the $q_i$ and $p_i$). More explicitly, the Lagrangian has to be brought up to the form

$$L(\xi, \dot{\xi}) = a_i(\xi) \dot{\xi}_i - V(\xi_i),$$

where we identify $a_i(\xi) = p_i$ and $V = H$. The equations of motion are derived as usual from the principle of least action,

$$\delta S = \int dt \left[ \frac{\partial L}{\partial \xi_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\xi}_i} \right) \right] \delta \xi_i = 0,$$

where $S = \int dt L$. Taking into account the explicit form of $L$ as a linear function in $\dot{\xi}_i$ given in (1), we have $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\xi}_i} \right) = \dot{a}_i (\xi) = \frac{\partial a_i}{\partial \xi_j} \dot{\xi}_j$ as well as $\frac{\partial L}{\partial \xi_i} = \frac{\partial a_i}{\partial \xi_i} \dot{\xi}_i - \frac{\partial V}{\partial \xi_i}$. Finally, introducing the symplectic matrix $f_{ij}$,

$$f_{ij} = \frac{\partial a_j}{\partial \xi_i} - \frac{\partial a_i}{\partial \xi_j},$$

we can rewrite the equations of motion as

$$f_{ij} \dot{\xi}_j = \frac{\partial V}{\partial \xi_i}.$$  

(4)

In the regular case, $f_{ij}$ has an inverse $f^{ij}$, and this last equation can immediately be solved for the velocities $\dot{\xi}_i$ as follows,

$$\dot{\xi}_i = f^{ij} \frac{\partial V}{\partial \xi_j} = \{\xi_i, V\}_P = \{\xi_i, \xi_j\}_P \frac{\partial V}{\partial \xi_j},$$

(5)

on the other hand, when $f_{ij}$ is singular, there is no inverse matrix since $\det[f] = 0$, thus establishing the existence of zero modes. This can be seen clearly by considering the problem of eigenvalues and eigenvectors

$$[f] v_a = \omega_a v_a,$$

$$\det[f - \omega_a I] = 0,$$

(6)

(7)
from which it follows that $\det[f] = \prod_a \omega_a$. Hence if $[f]$ is singular, $\det[f] = 0$ and we have null eigenvalues. Let $\{v_n\}$ be the set of linearly independent null eigenvectors: when we multiply Eq. (4) by each of the $v_n$ we obtain

$$v_n[f][\dot{\xi}] = v_n \frac{\partial V}{\partial [\dot{\xi}]} = \Omega_n^{(1)} = 0, \quad (8)$$

which represents an initial set of constraints on the dynamics. They can be enforced by means of Lagrange multipliers $\lambda_n^{(1)}$, augmenting the initial Lagrangian by the term $\sum_n \lambda_n^{(1)} \Omega_n^{(1)}$. Alternatively, taking into account that the constraint does not evolve in time ($\Omega = 0$) and that the Lagrangian is defined up to total time derivatives, it follows that a term such as

$$\frac{d}{dt} \left( \lambda_n^{(1)} \Omega_n^{(1)} \right) = \dot{\lambda}_n^{(1)} \Omega_n^{(1)} + \lambda_n^{(1)} \dot{\Omega}_n^{(1)} \quad (9)$$

does not modify the dynamics, so we can actually write a first iterated Lagrangian as

$$L^{(1)} = a_i^{(1)} \dot{\xi}^i + \sum_n \dot{\lambda}_n^{(1)} \Omega_n^{(1)} - V^{(1)}, \quad (10)$$

where

$$V^{(1)} = V|_{\Omega^{(1)}=0}. \quad (11)$$

At this point, one can enlarge the set of canonical variables $\xi_i$ including the $\lambda_n^{(1)}$. A new iteration can be started, taking $L^{(1)}$ as the initial Lagrangian, and the procedure continues until a non singular symplectic matrix $f_{ij}$ is obtained – a process which, in the case of gauge theories, involves also the inclusion of gauge fixing conditions into the Lagrangian.

After a nonsingular symplectic matrix $f_{ij}$ is obtained at the end of the (FJ) procedure, the transition amplitude is written as

$$Z = \int D\xi \sqrt{\det[f]} \exp[iS]. \quad (12)$$

The crucial point to understand the previous equation is based in the Darboux theorem, which states that by an appropriate change of canonical coordinates ($\xi_i \rightarrow \xi'_i$), we can write the symplectic part of the Lagrangian, in the canonical form, as

$$L(\dot{Q}_i, P_i) = P_i \dot{Q}^i - H(Q_i, P_i), \quad (13)$$

where

$$P_i \dot{Q}^i = \frac{1}{2} \omega^{ij} \dot{\xi}_i \xi'_j, \quad (14)$$
$Q^i$ and $P_i$ being canonical variables obeying the standard Poisson algebra, and $[\omega]$ the antisymmetric block matrix,

$$[\omega] = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \quad (14)$$

In fact, Eq. (13) can be written in the following form,

$$L(\xi') = a^n \dot{\xi}'_i - H(\xi'), \quad (15)$$

where

$$a^n = \frac{1}{2} \xi'^j \omega^{ij}. \quad (16)$$

Here we can identify the arbitrary vector potential (one-form) as

$$a' = a^n d\xi'_i, \quad (16)$$

whose associated field strength (two-form) is given by

$$da' = \frac{\partial a^n}{\partial \xi_j} d\xi'_i d\xi'_j \quad (17)$$

$$= \frac{1}{2} \left( \frac{\partial \xi'_i}{\partial \xi'_a} \omega^{ij} \frac{\partial \xi'_j}{\partial \xi'_b} \right) d\xi'_a d\xi'_b. \quad (18)$$

The fact that the action $S = \int a_i d\xi'^i - \int H dt$ is invariant under canonical transformations leads us to define the symplectic matrix as

$$f_{ij} = \frac{\partial \xi'_i}{\partial \xi_j} \omega^{ij}, \quad (19)$$

while, by the Schwinger variational principle of quantum action, $\delta Z = \langle \delta \hat{S} \rangle = \delta SZ$, we have

$$Z = \int DQDP \exp \left[ i \int dt \left( P_i \dot{Q}^i - H(Q_i, P_i) \right) \right], \quad (20)$$

or, in other words, $Z = \int D\xi' \exp[iS']$, where $S' = a'_i d\xi'^i - \int H' dt$. Therefore, by a canonical transformation ($\xi'_i \to \xi_i$), we write

$$Z = \int D\xi \det \left( \frac{\partial \xi'^n}{\partial \xi_j} \right) \exp[iS], \quad (21)$$

wherein we see that

$$\det \left( \frac{\partial \xi'^n}{\partial \xi_j} \right) = \sqrt{\det \left( \frac{\partial \xi'_i}{\partial \xi_a} \omega^{ij} \frac{\partial \xi'_j}{\partial \xi_b} \right)} = \sqrt{\det[F]}. \quad (22)$$

As stated in [17], it is important that the final result actually does not depend on the explicit form of the transformation ($\xi'_i \to \xi_i$), but only on the symplectic structure of the theory, which is solved by the (FJ) procedure.
III. TOY MODEL AS A PROOF OF CONCEPT

In this section, we consider a rather simple higher derivative theory, based on a massless real scalar field. The aim is to gain insight in the physical interpretations of such theories, and to present in a simpler setting the procedure to be considered in connection to the Podolsky theory in the next section. We will work out both the Ostrogradsky and the reduced order approach, and we will explicitly verify that both routes lead to the same quantum theory. The connections between the Ostrogradsky formalism (starting with a fourth order Lagrangian), the reduction of order formalism with an auxiliary field (starting with a first order Lagrangian, directly suitable to the application of the (FJ) method), and the final first order description (Hamiltonian), should be such that in any description we have the same propagating degrees of freedom, which in the present case are two: one being the original massless and the other one, massive, whose physical interpretation is of a ghost (unphysical) mode. We will briefly comment on some recent ideas on how to interpret the presence of this ghost mode.

A. Ostrogradsky formalism

We begin with the Lagrangian density \( \mathcal{L}_{Ostro} \)

\[
\mathcal{L}_{Ostro} = \frac{1}{2} \partial_\mu \phi \left( 1 + \frac{\Box}{m^2} \right) \partial^\mu \phi ,
\]

(23)

so the corresponding, fourth order equation of motion is given by

\[
\Box \left( \Box + m^2 \right) \phi = 0 .
\]

(24)

According to the Noether theorem, the conserved quantity corresponding to the time translation invariance of the action is the Hamiltonian density

\[
\mathcal{H}_{Ostro} = \pi \partial_0 \phi + P \partial_0 Q - \mathcal{L}_{Ostro} ,
\]

(25)

where the additional canonical coordinate \( Q = \partial_0 \phi \) was introduced to account for the higher order time derivatives. The canonical momenta are given by

\[
\pi = \left( 1 + \frac{\Box}{m^2} \right) \partial_0 \phi , \quad P = -\frac{\Box}{m^2} \phi ,
\]

(26)
and therefore
\[ H_{Ostro}(\phi, Q; \pi, P) = \pi Q - \frac{1}{2} m^2 P^2 - P \partial_k \partial^k - \frac{1}{2} D^2 - \partial_k \phi \partial^k \phi. \] (27)

The first order Lagrangian can be written as
\[ \mathcal{L}_{Ostro} = \pi \partial_0 \phi + P \partial_0 Q - H_{Ostro}, \] (28)
where the canonical one form of the symplectic variables \( \xi = (\phi, \pi, Q, P) \) corresponds to
\[ a_\phi = \pi, \quad a_\pi = 0, \quad a_Q = P, \quad a_P = 0. \] (29)

Therefore, we obtain the symplectic matrix
\[
[f] = \begin{bmatrix}
\phi & \pi & Q & P \\
\phi & 0 & -1 & 0 & 0 \\
\pi & 1 & 0 & 0 & 0 \\
Q & 0 & 0 & 0 & -1 \\
P & 0 & 0 & 1 & 0
\end{bmatrix} \delta^3(\vec{x} - \vec{y}),
\] (30)

and, as \( \det[f] = 1 \), the inverse matrix exists, and can be readily obtained as \( [f]^{-1} = -[f] \).

As a consequence, the fundamental non null Poisson brackets read
\[ \{\phi(x), \pi(y)\}_P = \delta^3(\vec{x} - \vec{y}), \quad \{Q(x), P(y)\}_P = \delta^3(\vec{x} - \vec{y}). \] (31)

Now, going to the quantum language, the transition amplitude is given in view of Eq. (12), as
\[ Z_{Ostro} = \int D\phi D\pi DQ DP \exp \left\{ i \int d^4x \left[ \pi \partial_0 \phi + P \partial_0 Q - H_{Ostro} \right] \right\} \]
\[ = \int D\phi D\pi DQ DP \exp \left\{ i \int d^4x \left[ \pi \partial_0 \phi + P \partial_0 Q - \pi Q + \frac{1}{2} m^2 P^2 + P \partial_k \partial^k \phi + \frac{1}{2} Q^2 \right] \right\}. \] (32)

After integration in \( DQD\pi \), and completing the squares we obtain as our final result the gaussian functional,
\[ Z_{Ostro} = \int D\phi DP \exp \left\{ i \int d^4x \left[ \frac{1}{2} m^2 \left( P + \frac{\partial_\mu \partial^\mu \phi}{m^2} \right)^2 - \left( \frac{\Box \phi}{m^2} \right)^2 - \frac{1}{2} \phi \Box^2 \phi \right] \} \] (33)
\[ = N \int D\phi \exp \left\{ -i \int d^4x \phi \Box \left( 1 + \frac{\Box}{m^2} \right) \phi \right\} \] (34)
\[ = N \det \left[ -\frac{1}{16} (\Box + m^2) \right]. \] (35)
B. Reduced order with an auxiliary field

Instead of dealing with the higher derivatives via the Ostrograsdkym method, one may also introduce an auxiliary field \(Z\), starting with the Lagrangian

\[
\mathcal{L}_{\text{red}} = \frac{1}{2} \phi \Box Z - \frac{1}{8} m^2 \phi \phi + \frac{1}{4} m^2 \phi Z - \frac{1}{8} m^2 ZZ ,
\]

whose corresponding equations of motion are given by

\[
\left(1 + 2 \frac{\Box}{m^2}\right) \phi = Z , \quad \left(1 + 2 \frac{\Box}{m^2}\right) Z = \phi .
\]

These set of coupled equations are equivalent to Eq. (24), as can be seen by direct substitution. The canonical Hamiltonian is given by

\[
\mathcal{H}_{\text{red}} = \pi \partial_0 \phi + \theta \partial_0 Z - \mathcal{L}_{\text{red}} \text{ with the respective canonical momenta defined as}
\]

\[
\pi = \frac{\partial \mathcal{L}_{\text{red}}}{\partial (\partial_0 \phi)} = - \frac{1}{2} \partial_0 Z ,
\]

\[
\theta = \frac{\partial \mathcal{L}_{\text{red}}}{\partial (\partial_0 Z)} = - \frac{1}{2} \partial_0 \phi ,
\]

or, more explicitly,

\[
\mathcal{H}_{\text{red}} = -2 \pi \theta + \frac{1}{2} \partial_i \phi \partial^i Z + \frac{1}{8} m^2 \phi \phi - \frac{1}{4} m^2 \phi Z + \frac{1}{8} m^2 ZZ .
\]

Therefore the canonical one form of the symplectic variables \(\xi = (\phi, \pi, Z, \theta)\) is given by

\[
a_\phi = \pi , \quad a_\pi = 0 , \quad a_Z = \theta , \quad a_\theta = 0 ,
\]

and the corresponding symplectic matrix is

\[
[f] = \begin{bmatrix}
\phi & \pi & Z & \theta \\
\phi & 0 & -1 & 0 \\
\pi & 1 & 0 & 0 \\
Z & 0 & 0 & 0 \\
\theta & 0 & 0 & 1 \\
\end{bmatrix} \delta^3(\vec{x} - \vec{y}) ,
\]

which again is a non-singular, unitary determinant matrix, with inverse \([f]^{-1} = -[f]\). The corresponding fundamental non null Poisson brackets are

\[
\{\phi(x), \pi(y)\}_P = \delta^3(\vec{x} - \vec{y}) , \quad \{Z(x), \theta(y)\}_P = \delta^3(\vec{x} - \vec{y}) .
\]
Quantization is achieved by calculating the transition amplitude which in this case reads

$$Z_{\text{red}} = \int D\phi D\pi DZ D\theta \exp \left\{ i \int d^4 x \left[ \pi \partial_0 \phi + \theta \partial_0 Z - \mathcal{H}_{\text{red}} \right] \right\}$$

$$= \int D\phi D\pi DZ D\theta \exp \left\{ i \int d^4 x \left[ \pi \partial_0 \phi + \theta \partial_0 Z + 2\pi \theta - \frac{1}{2} \partial_i \phi \partial^i Z - \frac{1}{8} m^2 \phi \right. \right.$$  

$$\left. + \frac{1}{4} m^2 \phi Z - \frac{1}{8} m^2 ZZ \right\}.$$ \hfill (44)

Integrating in $D\pi D\theta$, one obtains

$$Z_{\text{red}} = \int D\phi DZ \exp \left\{ i \int d^4 x \left[ -\frac{1}{2} \partial_\mu \phi \partial^\mu Z - \frac{1}{8} m^2 \phi \phi + \frac{1}{4} m^2 \phi Z - \frac{1}{8} m^2 ZZ \right] \right\},$$

and therefore

$$Z_{\text{red}} = \int D\phi DZ \exp \left\{ i \int d^4 x \left[ \phi \ Z \right] \left[ \begin{array}{cc} -\frac{m^2}{8} & \left( \frac{m^2}{4} + m^2 \right) \\ \left( \frac{m^2}{4} + m^2 \right) & -\frac{m^2}{8} \end{array} \right] \left[ \begin{array}{c} \phi \\ Z \end{array} \right] \right\},$$ \hfill (46)

$$= N \int D\phi \exp \left\{ -i \int d^4 x \phi \Box \left( 1 + \frac{\Box}{m^2} \right) \phi \right\},$$ \hfill (47)

which reduces to the determinant of the square matrix appearing in Eq. (46). The determinant of course involves both the discrete matrix indices as well as the continuous spacetime indices (coordinates): calculating explicitly the first part gives

$$\det \left[ \begin{array}{cc} -\frac{m^2}{8} & \left( \frac{m^2}{4} + m^2 \right) \\ \left( \frac{m^2}{4} + m^2 \right) & -\frac{m^2}{8} \end{array} \right] = \det \left[ -\frac{1}{16} \Box (\Box + m^2) \right],$$ \hfill (48)

which agrees with Eq. (35). We therefore verify that the equivalence between the classical equations of motion in the Ostrogradsky and Reduction of order prescriptions, seen in Eqs. (4) and (37), hold also at the quantum level, when we compare the transition amplitude obtained in both prescriptions.

C. Characterization of the degrees of freedom

It is a common feature of higher derivatives theories to present additional, non physical degrees of freedom. This can be clearly seen in the present model, in the reduced order formalism as discussed in the previous subsection. The coupled equations of motion for the $\phi$ and $Z$ field, given in Eq. (37), can be written in matrix notation as

$$M \begin{pmatrix} \phi \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$ \hfill (49)
where

\[
M = \begin{pmatrix}
(1 + \frac{2}{m^2} \Box) & -1 \\
-1 & (1 + \frac{2}{m^2} \Box)
\end{pmatrix}.
\]

(50)

The dynamics can be rewritten in order to make manifest the fact that it involves two independent degrees of freedom. At the matrix level, this amounts to the problem of diagonalization of the matrix \(M\). The eigenvalues of \(M\) are determined by the equation

\[
\det[M - \lambda I] = 0,
\]

whose solutions are

\[
\lambda_{\pm} = \left(1 + \frac{2}{m^2} \Box\right) \mp 1.
\]

(51)

So the matrix \(M\) is unitarily equivalent to a matrix describing two degrees of freedom, one being massless, and the other massive. Indeed, by means of a proper linear transformation,

\[
\phi = \alpha A + \beta B, \quad (52)
\]

\[
Z = \alpha A - \beta B, \quad (53)
\]

the Lagrangian in Eq. (36) can be brought to the following form,

\[
\mathcal{L}'_{\text{red}} = \alpha^2 \left[\frac{1}{2} A \Box A \right] - \beta^2 \left[\frac{1}{2} B \Box B + \frac{m_p^2}{2} B^2 \right].
\]

(54)

This last equation explicitly separates the two degrees of freedom present in the model. For real \(\alpha\) and \(\beta\), clearly the \(B\) mode appears with a “wrong sign” in the Lagrangian, and will in fact violate the stability of the Hamiltonian. Therefore, \(B\) should be interpreted as a nonphysical (ghost) degree of freedom.

The presence of ghosts is a longstanding issue in the quantization of higher derivative models. Recently, it has been pointed out that, at least in the free case, these ghosts could be reinterpreted as physical particles after a proper complexification: this was discussed for the Pais-Uhlenbeck oscillator in [38]. In the present case, one may note that the choice

\[
\phi = A + iB, \quad (55)
\]

\[
Z = A - iB,
\]

recovers the stability of the Hamiltonian, at the price of losing its Hermiticity, since the sets of fields \((\phi, Z)\) and \((A, B)\) could not be both real, and therefore either \(\mathcal{L}_{\text{red}}\) or \(\mathcal{L}'_{\text{red}}\) would also be complex.
If we try to interpret the imaginary part of the field $\phi$ as a massive physical degree of freedom, so that both $A$ and $B$ are real degrees of freedom associated with the real and imaginary parts of the field $\phi$, it may seem that by complexifying the original Lagrangian we are increasing the degrees of freedom to four (complex $\phi$ and $Z$ fields). Actually, the balance in the degrees of freedom can be preserved with the introduction the condition $Z = \bar{\phi}$ by means of a Lagrange multiplier $\lambda$ into the reduced order complex scalar Lagrangian,

$$\mathcal{L} = \frac{1}{2} \phi \Box Z + \frac{1}{8} m^2 \phi \phi - \frac{1}{4} m^2 \phi Z + \frac{1}{8} m^2 Z Z + \lambda (Z - \bar{\phi}),$$  \hspace{1cm} (56)

$\phi$ and $Z$ being now complex fields. The equations of motion are given by

$$\left(1 + \frac{2}{m^2} \Box \right) \phi = Z - \frac{4}{m^2} \lambda, \hspace{1cm} (57)$$

$$\left(1 + \frac{2}{m^2} \Box \right) Z = \phi, \hspace{1cm} (58)$$

$$Z = \bar{\phi}, \hspace{1cm} (59)$$

which can be combined and brought into the form

$$\Box \left(1 + \frac{1}{m^2} \Box \right) \phi = 0, \hspace{1cm} (60)$$

$$\Box (\phi + Z) = 0, \hspace{1cm} (61)$$

$$\Box + m^2) (\phi - Z) = 0, \hspace{1cm} (62)$$

where we conclude that $\lambda = 0$, $\phi = A + iB$ and $Z = A - iB$. Substituting this in (56), we end up with

$$\mathcal{L} = \frac{1}{2} A \Box A + \frac{1}{2} B \Box B + \frac{1}{2} m^2 B^2. \hspace{1cm} (63)$$

In summary: as $\phi$ and $Z$ are complex fields we start with four degrees of freedom described by the complex Lagrangian (56), while the higher derivative real scalar theory has only two degrees of freedom. We match the number of degrees of freedom in both formulation by enforcing the condition $Z = \bar{\phi}$ via a Lagrange multiplier.

A more general prescription to quantize higher derivative theories, circumventing the problem of the stability of the Hamiltonian, have been discussed in [32], using the concept of Lagrangian anchors. Essentially, it involves an extension of the Noether theorems, defining a class of conserved quantities associated with a given symmetry. For time translations, this
procedure can lead to different conserved quantities which could be in principle be identified with a Hamiltonian, some of them would have regularizing properties but will break the stability because the energy is not bounded from below, whereas the other recovers the stability but loses the regularizing properties, seen in the self-energy of a particles and ultraviolet divergences. It would be an interesting endeavor to investigate this approach for more involved models, something that we will not try in this work.

IV. HD PODOLSKY THEORY IN THE (FJ) FORMALISM

There is a simple and intuitive way to understand the connection between the physical degrees of freedom and the appearance of constraints in gauge theories. The equivalence between the dynamics described in the phase space and the one described in the configuration space, through the (DB) constraint analysis, should be such that the number of equations of motion multiplied by their order must be independent of the way in which the dynamics is described. The conditions that must be added in order to maintain this equality are the constraints.

For the Podolsky theory of electrodynamics [40], it is known that the dynamics is described in the configuration space by four equations of fourth order in derivatives, while in the phase space, we have sixteen first order equations. As usual, however, the covariant formulation is highly redundant. When describing only the physical degrees of freedom of the theory, in configuration space we have to consider five second order equations (Maxwell+Proca), so we know the theory must involve six constraints, by consistency. Within the (DB) formalism, half of these constraints are the first-class, arising from the definition of canonical momenta and consistency conditions, while the other half are gauge fixing conditions that transform the first-class constraints into second-class, such that all the Lagrange multipliers can be determined.

Although the (FJ) formalism does not implement major changes in the quantization process of a regular theory, in a singular theory there might be considerable simplifications when adopting the symplectic formalism instead of the usual (DB) algorithm. We apply the (FJ) method to discuss the quantization of the Podolsky electrodynamics but, differently from what was done in [39], we start by writing the theory in the reduction of order formalism, by means of the introduction of an additional auxiliary field $B^\mu$, following [36]. We will show
that this technique allows us to write the sympletic matrix in a block structure, separating the Maxwell and Proca sectors. This makes the treatment of the different degrees of freedom of the model particularly simple and clear.

Concretely, we start with,

\[ L_{\text{red}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{a^2}{2} B_\mu B^\mu + a^2 \partial_\mu B_\nu F^{\mu\nu}, \]

where

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

Up to surface terms, we can also write

\[ L_{\text{red}} = \frac{1}{2} A^\mu (\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu) A^\nu - \frac{a^2}{2} B_\mu B^\mu - a^2 B^\mu (\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu) A^\mu, \]

which leads directly to the coupled equations of motion

\[ (\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu) A^\nu = a^2 (\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu) B^\nu, \]

and

\[ (\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu) A^\nu = -B_\mu. \]

A direct consequence of the last equation is that \( \partial_\mu B^\mu = 0 \). Additionally, one may decouple the previous two equations, obtaining

\[ (1 + a^2 \Box)(\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu) A^\nu = 0; \ (1 + a^2 \Box)B_\mu = 0. \]

Classically the reduced order Lagrangian density \( L_{\text{red}} \) is equivalent to the following Ostrogradsky Lagrangian density, up to surface terms,

\[ L_{\text{Ostro}} = -\frac{1}{4} F^{\mu\nu}(1 + a^2 \Box) F_{\mu\nu} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{a^2}{2} \partial_\nu F^{\mu\nu} \partial^\rho F_{\mu\rho}. \]

Also, the classical coupled equations of motion (67) and (68) can be written as

\[
\begin{bmatrix}
T_{\mu\nu} & a^2 T^\nu_{\mu} \\
T^\mu_{\nu} & -\eta^{\mu\nu}
\end{bmatrix}
\begin{bmatrix}
A^\nu \\
B_\nu
\end{bmatrix}
= 0,
\]

wherein we have the definition \( T_{\mu\nu} \equiv \eta_{\mu\nu} \Box - \partial_\mu \partial_\nu \). Implicitly in the analysis we have a problem of eigenvalues and eigenvectors and the diagonalization of a matrix since

\[
\det
\begin{bmatrix}
T_{\mu\nu} & a^2 T^\nu_{\mu} \\
T^\mu_{\nu} & -\eta^{\mu\nu}
\end{bmatrix}
= \det
\begin{bmatrix}
T_{\mu\nu} & 0 \\
0 & -(T^{\mu\nu} + a^2 \eta^{\mu\nu})
\end{bmatrix}
= -3 \Box (1 + a^2 \Box),
\]

15
making explicit the Maxwell (the $T_{\mu\nu}$ factor) and Proca (the $-(T_{\mu\nu} + a^2 \eta_{\mu\nu})$ factor) physical degrees of freedom (2+3, respectively) of the theory, as well as the problem of instability due to the negative sign of the massive mode.

Due to the fact that $L_{\text{red}}$ is of second order, we can define the usual canonical momenta

$$\pi^i = \frac{\partial L_{\text{red}}}{\partial \dot{A}_i} = F^{0i} + a^2 (\partial^0 B^i - \partial^i B^0), \quad \theta^i = \frac{\partial L_{\text{red}}}{\partial \dot{B}_i} = a^2 F^{0i},$$

leading to

$$L_{\text{red}} = \frac{1}{a^2} \pi^i \dot{\theta}_i - \frac{1}{2a^4} \theta^i \dot{\theta}_i + \frac{1}{4} F^{ij} F_{ij} + a^2 \partial^i B_j F^{ij} + \frac{a^2}{2} B_\mu B^\mu - A^0 \partial^0 \pi_i - B^0 \partial^i \theta_i.$$  \hspace{1cm} (73)

By a Legendre transform, we obtain the canonical Hamiltonian

$$H_{\text{red}} = \pi_i \dot{A}^i + \theta_i \dot{B}^i - L_{\text{red}},$$

which, up to surface terms, leads to

$$H_{\text{red}}(A_i, \pi_i, B_i, \theta_i, A_0, B_0) = \frac{1}{a^2} \pi^i \dot{\theta}_i + \frac{1}{2a^4} \theta^i \dot{\theta}_i + \frac{1}{4} F^{ij} F_{ij} - a^2 \partial^i B_j F^{ij} + \frac{a^2}{2} B_\mu B^\mu - A^0 \partial^0 \pi_i - B^0 \partial^i \theta_i.$$  \hspace{1cm} (74)

We can now construct the symplectic structure in the (FJ) formalism, starting by writing

$$L_{\text{red}} = \frac{1}{a^2} \pi^i \dot{A}^i + \theta_i \dot{B}^i - \mathcal{V}^{(0)},$$

where

$$\mathcal{V}^{(0)} = \pi^i \dot{\theta}_i + \frac{1}{2a^4} \theta^i \dot{\theta}_i + \frac{1}{4} F^{ij} F_{ij} - a^2 \partial^i B_j F^{ij} + \frac{a^2}{2} B_\mu B^\mu - A^0 \partial^0 \pi_i - B^0 \partial^i \theta_i.$$  \hspace{1cm} (75)

The symplectic variables are up to this point $\xi = (A_i, \pi_i, B_i, \theta_i, A_0, B_0)$ and the canonical one-form is given by

$$a_{A_i} = \pi_i, \quad a_{\pi_i} = 0, \quad a_{B_i} = \theta_i, \quad a_{\theta_i} = 0, \quad a_{A_0} = 0, \quad a_{B_0} = 0.$$  \hspace{1cm} (76)

therefore, the symplectic matrix can be written as

$$[f] = \begin{bmatrix}
A_j & \pi_j & B_j & \theta_j & A_0 & B_0 \\
A_i & 0 & -\delta_{ij} & 0 & 0 & 0 \\
\pi_i & \delta_{ij} & 0 & 0 & 0 & 0 \\
B_i & 0 & 0 & 0 & -\delta_{ij} & 0 \\
\theta_i & 0 & 0 & \delta_{ij} & 0 & 0 \\
A_0 & 0 & 0 & 0 & 0 & 0 \\
B_0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\delta^3(\vec{x} - \vec{y}).$$  \hspace{1cm} (77)

16
Clearly, \( \det[f] = 0 \) signaling a singular system, as expected. The following eigenvectors have null eigenvalues,

\[
\begin{align*}
u &= (0, 0, 0, 0, u^{13}, 0), \\
v &= (0, 0, 0, 0, 0, v^{14}),
\end{align*}
\]  

(80) (81)

and the respective constraint equations are

\[
\begin{align*}
\Omega_1 &= \int dx dy \frac{\delta \mathcal{V}^{(0)}(y)}{\delta \pi_i(x)} = \int dx u^{13} \partial^i \pi_i = 0, \\
\Omega_2 &= \int dx dy \frac{\delta \mathcal{V}^{(0)}(y)}{\delta B_0(x)} = \int dx u^{13} (a^2 B_0 - \partial^i \theta_i) = 0.
\end{align*}
\]  

(82) (83)

We enforce the previous constraint equations into \( \mathcal{L}_{\text{red}} \) using Lagrange multipliers,

\[
\mathcal{L}_{\text{red}} = \pi_i \dot{A}^i + \theta_i \dot{B}^i + \lambda^a \Omega_a - \mathcal{V}^{(2)}, \quad a = 1, 2,
\]

(84)

where

\[
\mathcal{V}^{(2)} = \mathcal{V}^{(0)}|_{\Omega_a=0} = \frac{1}{a^2} \partial^i \pi_i + \frac{1}{2a^4} \partial^i \theta_i + \frac{1}{4} F^{ij} F_{ij} - a^2 \partial_i B_j F^{ij} - \frac{a^2}{2} B_0 B^0 + \frac{a^2}{2} B_i B^i.
\]

(85)

So from this augmented symplectic structure, we have the following one form vectors

\[
a^{(2)}_{A_i} = \pi_i, \quad a^{(2)}_{\pi_i} = 0, \quad a^{(2)}_{B_0} = 0, \quad a^{(2)}_{B_1} = \theta_i, \quad a^{(2)}_{\lambda_1} = 0, \quad a^{(2)}_{\lambda_2} = (a^2 B_0 - \partial^i \theta_i).
\]

(86)

At this point, when calculating the symplectic matrix, one realizes the main advantage in the present formalism, since \([f]\) turns out to be a block diagonal matrix

\[
[f] = \begin{bmatrix} [M] & 0 \\ 0 & [P] \end{bmatrix},
\]

(87)

where \([M]\) corresponds to the massless Maxwell sector of the theory,

\[
[M] = \begin{bmatrix} A_i & \pi_j & \lambda_1 \\ A_i & 0 & -\delta_{ij} & \partial_i \\ \pi_i & \delta_{ij} & 0 & 0 \\ \lambda_1 & \partial_j & 0 & 0 \end{bmatrix} \delta^3(x - y),
\]

(88)
and \([P]\) to the massive Proca sector

\[
[P] = \begin{bmatrix}
B_j & \theta_j & B_0 & \lambda_2 \\
B_i & 0 & -\delta_{ij} & 0 & \partial_i \\
\theta_i & \delta_{ij} & 0 & 0 & 0 \\
B_0 & 0 & 0 & -a^2 & 0 \\
\lambda_2 & \partial_j & 0 & a^2 & 0 \\
\end{bmatrix}
\delta^3(\vec{x} - \vec{y}).
\] (89)

Needless to say, the structure of \([f]\) implies that

\[
\det[f] = \det[M] \det[P].
\] (90)

The neat separation between the Maxwell and Proca sectors is a distinctive feature of the (FJ) formalism applied to the reduced order Podolsky electrodynamics, which does not happen within the Ortogradsky formalism [39].

First, let us work with the Maxwell sector. As expected, \(\det[M] = 0\) so \([M]\) is singular, and the null eigenvector is of the form \(v = (0, v^\pi_j, v^{\lambda_1}), j = 1, 2, 3\), corresponding to the constraint equation

\[
\int dxdy v_i^\pi \frac{\delta V^{(2)}(y)}{\delta A_i(x)} = \int dx \partial_i \partial_j v^{\lambda_1} \left[-\partial_i F^{ij} + \frac{a^2}{2} \partial_i \left(\partial^j B^j - \partial^j B^j \right)\right] = 0.
\] (91)

This zero mode does not generate any additional constraints and, consequently, the symplectic matrix remains singular, which is a characteristic of gauge theories: a gauge fixing condition should be introduced in order to obtain a non singular symplectic matrix. Inspired by the form of the fourth-order equations of motion for \(A_\mu\), Eq. (69), as well as the analysis presented in [39], we use generalized Coulomb gauge fixing conditions in the form

\[
A_0 = 0, \quad (1 + a^2 \Box) \vec{\nabla} A = \Omega_3 = 0.
\] (92)

For more details on the gauge fixing of the Podolsky theory we refer the reader to [40]. When this gauge condition is included in \(\mathcal{L}_{\text{red}}\) using a Lagrange multiplier \(\lambda_3 \Omega_3\), we obtain the following \([M]\) matrix for the Maxwell sector

\[
[M] = \begin{bmatrix}
A_j & \pi_j & \lambda_3 & \lambda_2 \\
\pi_i & \delta_{ij} & 0 & \partial_i \\
\lambda_3 & 0 & (1 + a^2 \vec{\nabla}^2) \partial_j & 0 \\
\lambda_2 & \partial_j & 0 & 0 \\
\end{bmatrix}
\delta^3(\vec{x} - \vec{y}).
\] (93)

18
which is a regular matrix, with \(\text{det}[M] = [(1 + a^2 \nabla^2) \nabla^2]^2\), and its inverse can be calculated almost immediately

\[
[M]^{-1} = \frac{1}{(1 + a^2 \nabla^2) \nabla^2} \times \\
\begin{pmatrix}
A_j & \pi_j & \lambda_3 & \lambda_2 \\
A_i & 0 & -(1 + a^2 \nabla^2) \nabla^2 \delta_{ij} + \partial_i \partial_j & 0 & \partial_i \\
\pi_i & (1 + a^2 \nabla^2) \nabla^2 \delta_{ij} - \partial_i \partial_j & 0 & \partial_i & 0 \\
\lambda_3 & 0 & \partial_j & 0 & 1 \\
\lambda_2 & \partial_j & 0 & 1 & 0
\end{pmatrix} \delta^3(\vec{x} - \vec{y}).
\] (94)

From this, one easily identifies the Dirac brackets between the dynamics variables in the generalized Lorenz gauge

\[
\{A_i, \pi_j\}_D = \left[ -\delta_{ij} + \frac{\partial_i \partial_j}{(1 + a^2 \nabla^2) \nabla^2} \right] \delta^3(\vec{x} - \vec{y}).
\] (95)

Now, we consider the Proca sector. One way to calculate the determinant of \([P]\) is to notice that, for any block matrix of the form

\[
P_{n \times n} = \begin{pmatrix}
A_{m \times m} & B_{m \times n-m} \\
C_{n-m \times m} & D_{m \times m}
\end{pmatrix},
\] (96)

if \(D\) has an inverse, the following identity holds

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \times \begin{pmatrix}
I & 0 \\
-D^{-1}C & I
\end{pmatrix} = \begin{pmatrix}
A - BD^{-1}C & B \\
0 & D
\end{pmatrix},
\] (97)

and therefore

\[
\text{det} P = \text{det} (A - BD^{-1}C) \text{det} D.
\] (98)

Applied to Eq. (89), this leads to \(\text{det} [P] = a^4\). The Proca sector is therefore regular, and we obtain the following inverse symplectic matrix

\[
[P]^{-1} = \begin{pmatrix}
B_j & \theta_j & B_0 & \lambda_2 \\
B_i & 0 & -\delta_{ij} - \frac{1}{a^2} \partial_i & 0 \\
\theta_i & \delta_{ij} & 0 & 0 & 0 \\
B_0 & \frac{1}{a^2} \partial_j & 0 & 0 & \frac{1}{a^2} \\
\lambda_2 & 0 & 0 & -\frac{1}{a^2} & 0
\end{pmatrix} \delta^3(\vec{x} - \vec{y}),
\] (99)
corresponding to the following Dirac brackets between the dynamic variables,

\[
\{ B_i, \theta_j \}_D = -\delta_{ij} \delta^3(\vec{x} - \vec{y}). \tag{100}
\]

From now on we are ready to construct the quantum description of this theory. According to Eq. (12), the transition amplitude is given by

\[
Z_{\text{red}} = \int DA_i DB_i D\theta_i D\lambda_\alpha \times \frac{\sqrt{\det[M] \det[P]}}{\det[1 + a^2 \nabla^2] \det[1 + a^2 \Box \nabla A]} \exp \left[ i \int d^4x \left( \pi_i \dot{A}^i + \theta_i \dot{B}^i + \lambda^a \pi_a - \mathcal{V}^{(2)} \right) \right], \quad a = 1, 2, 3. \tag{101}
\]

Identifying \( \lambda_1 = A_0 \), we can write

\[
Z_{\text{red}} = Na^2 \int DA_0 DA_i DB_i D\theta_i \det \left[ \left( 1 + a^2 \nabla^2 \right) \nabla A \right] \delta \left( 1 + a^2 \Box \nabla A \right) \times \delta \left( a^2 B_0 - \partial^\nu \theta_\nu \right) \exp \left[ i \int d^4x \left( \pi_i \dot{A}^i + \theta_i \dot{B}^i + A_0 \left( \partial_i \pi^i \right) - \frac{1}{a^2} \pi^i \theta_i - \frac{1}{2a^4} \theta^2 \theta_i \right. \right.
\]
\[
\left. \left. - \frac{1}{4} F^{ij} F_{ij} + a^2 \partial_i B_j F^{ij} + \frac{a^2}{2} B_0 B^0 - \frac{a^2}{2} B_i B^i \right) \right]. \tag{102}
\]

Integration in \( D\pi_i \) leads to the appearance of a delta function \( \delta \left( F^{0i} - \frac{1}{a^2} \theta^i \right) \), and further integrations in \( D\theta_i \) and \( B_0 \) leads to

\[
Z_{\text{red}} = Na^2 \int DA_0 DA_i DB_i \det \left[ \left( 1 + a^2 \nabla^2 \right) \nabla A \right] \delta \left( 1 + a^2 \Box \nabla A \right) \times \exp \left[ i \int d^4x \left( a^2 F_{0i} \partial^0 B^i - \frac{1}{2} F^{0i} F_{0i} - \frac{1}{4} F^{ij} F_{ij} + \right. \right.
\]
\[
\left. + a^2 \partial_i B_j F^{ij} + \frac{a^2}{2} \partial_i F^{0i} \partial^0 F_{0j} - \frac{a^2}{2} B_i B^i \right) \right]. \tag{103}
\]

Some algebraic manipulations are now in order. Up to a surface term, we have

\[
a^2 F_{0i} \partial^0 B^i + a^2 \partial_i B_j F^{ij} = a^2 \partial^\nu F_{i\nu} B^i, \tag{105}\]

and completing the squares,

\[
a^2 \partial^\nu F_{i\nu} B^i - \frac{a^2}{2} B_i B^i = -\frac{a^2}{2} (B^i + \partial_\nu F^{i\nu})^2 + \frac{a^2}{2} \partial_\nu F^{i\nu} \partial^\nu F_{i\nu}. \tag{106}\]

Thus, by translation invariance of the functional integral, the integration in \( DB_i \) amounts to a \( A^a \) independent Gaussian integral, which can be incorporated in the normalization factor.
As a consequence, the transition amplitude can be cast as

\[
Z_{\text{red}} = N' \int DA_0 DA_i \det \left[ \left( 1 + a^2 \nabla^2 \right) \nabla^2 \right] \delta \left( (1 + a^2 \Box) \nabla A \right) \times 
\times \exp \left[ i \int d^4 x \left( -\frac{1}{2} F^{0i} F_{0i} - \frac{1}{4} F^{ij} F_{ij} + \frac{a^2}{2} \partial_\nu F^{0i} \partial^\nu F_{0i} + \frac{a^2}{2} \partial_\nu F^{\mu \nu} \partial^\rho F_{\mu \rho} \right) \right],
\]

(107)

where

\[
N' = Na^2 \int DB_i \exp \left[ -i \int d^4 x \frac{a^2}{2} (B^i + \partial_\nu F^{i \nu})^2 \right].
\]

(108)

Here, we kept the seemingly dependence of \(N'\) on \(A^\mu\) for clarity purposes. So we rewrite explicitly the following transition amplitude in the generalized Coulomb gauge

\[
Z_{\text{red}} = N' \int DA_\mu \det \left[ \left( 1 + a^2 \nabla^2 \right) \nabla^2 \right] \delta \left( (1 + a^2 \Box) \nabla A \right)
\]

\[
\times \exp \left[ i \int d^4 x \left( -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \frac{a^2}{2} \partial_\nu F^{\mu \nu} \partial^\rho F_{\mu \rho} \right) \right]
\]

\[
= Z_{\text{Ostro}}.
\]

(109)

We therefore verify that the equivalence between the classical equations of motion in the reduction of order and Ostrogradsky prescriptions, seen in Eqs. (64) and (70), hold also at the quantum level, when we compare the transition amplitude obtained in both prescriptions.

V. FINAL REMARKS

Our main objective was to discuss the use of the (FJ) formalism for higher derivatives theories, in particular discussing how, when the order of the equations of motion are reduced by the introduction of auxiliary fields, the dynamics can be put in a more transparent form, with an explicit separation of the relevant degrees of freedom.

These ideas were first presented in a toy model involving a massless scalar field as the physical degree of freedom. We presented both the classical and quantum basic developments of the model, both in the Ortogradsky and the reduction of order approach, showing their equivalence, but also pointed out that, in the latter case, one can neatly disentangle the two degrees of freedom present in the model: one physical massless scalar and a ghost massive one. We also briefly discussed some recent approaches toward a consistent understanding of these ghost fields, which present themselves as a longstanding problem for higher derivative theories.
Afterwards, we discussed the Podolsky electrodynamics. This is a well known higher derivative gauge theory: the (FJ) quantization procedure have already been used for this model in the Ortogradsky formalism \[^{[39]}\], while the reduced order formalism was also considered in \[^{[36]}\] together with the (DB) quantization procedure. We pointed out that the combination of the reduced order with the (FJ) formalism presents itself as a simple way to study the constraint structure and the quantization of this theory, since the relevant degrees of freedom (Maxwell+Proca) are clearly separated. Our results are consistent with the ones obtained in the other formalism, as we have explicitly shown.

It is worth noticing that Podolsky electrodynamics breaks the dual symmetry \[^{[41]}\]

\[
\begin{align*}
\vec{E} &\to \vec{B} \\
\vec{B} &\to -\vec{E}
\end{align*}
\]

(110)

that led Dirac to consider the existence of magnetic monopoles. Hence, a study of Podolsky equations in the vacuum may shed some light on the question of the existence of monopoles as two Dirac strings (solenoids) have an interaction associated with the Podolsky mass \[^{[26]}\]. Besides, the fact that the Podolsky characteristic length is associated with the size of the electron \[^{[42]}\] could lead us to explore, by electron-positron scattering, the existence of magnetic monopoles and some effects of Maxwell $\to$ Podolsky phase transition in view of Landau theory, superconductivity and thermodynamics of magnetic materials. In fact, it is speculated that the Maxwell $\to$ Podolsky transition apparently would be possible from the point of view of a mechanism which breaks the dual symmetry. These speculations derive from our ignorance associated with the mechanisms behind the self-interaction of the particles and their sizes and deserve rigorous scientific analysis.

Finally, we think that the natural next step of this investigation would be the extension of this discussion for the non Abelian and gravity case. These matters will be further elaborated and requires deeper investigations.

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[1] P. A. M. Dirac. Can. J. Math. 2, 129 (1950); P. A. M. Dirac. Phys. Rev. 114, 924 (1959)
[2] J. L. Anderson and P. G. Bergmann. Phys. Rev. 83, 1018 (1951); P. G. Bergmann and R. Schiller. Phys. Rev. 89, 4 (1953); P. G. Bergmann and I. Goldberg. Phys. Rev. 98, 531 (1955).
[3] P. A. M. Dirac, Lectures on Quantum Mechanics, 1st. edn. (Yeshiva University, New York, 1964).
[4] K. Sundermeyer, Constrained Dynamics, Lecture Notes in Physics Vol. 169 (Springer, New York, 1982); H. J. Rothe and K. D. Rothe, Classical and quantum dynamics of constrained Hamiltonian systems, 1st edn. (World Scientific, New Jersey, 2010); M. Henneaux and C. Teitelboim, Quantization of gauge systems, 1st. edn. (Princeton University Press, 1994).
[5] P. A. M. Dirac, Physikalische Zeitschrift der Sowjetunion 3, 64 (1933).
[6] R. P. Feynman, Phys. Rev. 76, 6 (1949); R. Feynman, A New Approach to Quantum Theory, 1st edn. (World Scientific, New Jersey 2005).
[7] E. Noether, Nachr. d. König. Gesellsch.d. Wiss. zu Göttingen, Math-phys. Klasse, 235 (1918); M. A. Tavel’s English translation.
[8] J. S. Schwinger, Phys. Rev. 82, 914 (1951); Symbolism of Atomic Measurements, 1st edn. (Springer, New York 2001).
[9] R. Utiyama, Phys. Rev. 101, 1597 (1956).
[10] L. D. Faddeev, Teor. Mat. Fiz. 1, 3 (1969); Theor. Math. Phys. 1, 1 (1969); P. Senjanovic, Ann. Phys. 100, 227 (1976).
[11] L. D. Faddeev and V. N. Popov, Phys. Lett. B 25, 29 (1967); B. S. DeWitt, Phys. Rev. 160, 1113 (1967).
[12] L. Faddeev and R. Jackiw, Phys. Rev. Lett. 60, 1692 (1988).
[13] R. Jackiw, (Constrained) Quantization Without Tears, arXiv:hep-th/9306075v1 (1993).
[14] J. Barcelos-Neto and C. Wotzasek, Mod. Phys. Lett. A 7, 1737 (1992); J. Barcelos-Neto and C. Wotzasek, Int. J. Mod. Phys. A 7, 4981 (1992).
[15] H. Montani and C. Wotzasek, Mod. Phys. Lett. A 8, 3387 (1993).
[16] L. Liao and Y. C. Huang. Ann. Phys. 322, 2469 (2007).
[17] D. J. Toms. Phys. Rev. D, 92, 105026 (2015).
[18] L. Liao, Y.C. Huang, Ann. Phys. (N.Y.) 322, 2 469 (2007).
[19] L. Liao, Y.C. Huang, Phys. Rev. D 75, 025 025 (2007).
[20] J. Ramos, Canada. Jour. Phys , 95(3), 225 (2017).
[21] B. M. Pimentel and G. E. R. Zambrano, Nuclear and Particle Physics Proceedings 267–269, 183 (2015).
[22] M. Ostrogradski, Mem. Ac. St. Petersburg VI, 4, 385 (1850); J. S. Chang, Proc. Camb. Philos. Soc. 44, 76 (1948).
[23] R. R. Cuzinatto, C. A. M. de Melo and P.J. Pompeia,Ann. Phys. (N.Y.) 322, 1211 (2007).
[24] F. Bopp, Ann. Phys. (Leipzig) 430, 345 (1940); B. Podolsky, Phys. Rev. 62, 68 (1942); B. Podolsky and C. Kikuchy, Phys. Rev. 65, 228 (1944); B. Podolsky and P. Schwed, Rev. Mod. Phys. 20, 4 (1948).
[25] J. Frenkel, Phys. Rev. E 54, 5859 (1996); A. E. Zayats, Ann. Phys. 342, 11 (2014).
[26] F. A. Barone, G. Flores-Hidalgo and A. A. Nogueira, Phys. Rev. D 88, 105031 (2013); F. A. Barone, G. Flores-Hidalgo and A. A. Nogueira, Phys. Rev. D 91, 027701 (2015).
[27] W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949).
[28] J. Rayski, Acta. Phys. Pol. 9, 129 (1948); Phys. Rev. 75, 1961 (1949).
[29] A. Pais and G. E. Uhlenbeck, Phys. Rev. 79, 145 (1950).
[30] W. Heisenberg, Nucl. Phys. 4, 532 (1957).
[31] S. W. Hawking and T. Hertog, Phys. Rev. D 65, 103515 (2002); A. V. Smilga, Nucl. Phys. B 706, 598 (2005); C. M. Bender and P. D. Mannheim, Phys. Rev. Lett. 100, 110402 (2008); A. V. Smilga, SIGMA 5, 017 (2009).
[32] D. S. Kaparulin, S. L. Lyakhovich, and A. A. Sharapov, Eur. Phys. J. C 74, 3072 (2014); D. S. Kaparulin and S. L. Lyakhovich, Russian. Phys. J. 59, 12 (2017).
[33] K. S. Stelle, Phys. Rev. D 16, 953 (1977); Gen. Rel. Grav. 9, 353 (1978). T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 79, 451 (2010).
[34] M. Asorey, J. L. Lopez and I. L. Shapiro, 1DFTUZ 96-15, September, (1996); M. Asorey, L. Rachwal and I. Shapiro, Unitary Issues in Some Higher Derivative Field Theories, Galaxies (2018).
[35] T. D. Lee and G. C. Wick, Nucl. Phys. B 9, 209 (1969); Phys. Rev. D 2, 1033 (1970); A. Accioly, P. Gaete, J. H. Neto, E. Scatena and R. Turcati, Mod. Phys. Lett. A 26, 26 (2011); D. Anselmi and M. Piva, J. High Energy Phys 06, 066 (2017); Phys. Rev. D 96, 045009 (2017);
D. Anselmi, J. High Energy Phys 06, 086 (2017); D. Anselmi, J. High Energ. Phys 02, 141 (2018).

[36] R. Thibes, Braz. Journ. Phys. 47, 72 (2017).

[37] F. J. de Urries and J. Julve, arXiv:hep-th/9812020v1 (1998).

[38] M. Raidal and H. Veermäe, Nucl. Phys. B 916, 607 (2017).

[39] R. Bufalo and B. M. Pimentel, Eur. Phys. Jour. C 74, 2993 (2014).

[40] C. A. P Galvão and B.M. Pimentel, Can. J. Phys. 66, 460 (1988); A. A. Nogueira and B. M. Pimentel, Phys. Rev. D 95, 065034 (2017).

[41] F. T. Brandt, J. Frenkel and D. G. C. McKeon, Mod. Phy. Lett. A 31, 32 (2016).

[42] R. Bufalo, B. M. Pimentel and D. E. Soto, Phys. Rev. D 90, 085012 (2014).