Ramsey numbers of Boolean lattices

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Abstract

The poset Ramsey number $R(Q_m, Q_n)$ is the smallest integer $N$ such that any blue-red coloring of the elements of the Boolean lattice $Q_N$ has a blue induced copy of $Q_m$ or a red induced copy of $Q_n$. The weak poset Ramsey number $R_w(Q_m, Q_n)$ is defined analogously, with weak copies instead of induced copies. It is easy to see that $R(Q_m, Q_n) \geq R_w(Q_m, Q_n)$.

Axenovich and Walzer \cite{1} showed that $n + 2 \leq R(Q_2, Q_n) \leq 2n + 2$. Recently, Lu and Thompson \cite{7} improved the upper bound to $5\frac{3}{2}n + 2$. In this paper, we solve this problem asymptotically by showing that $R(Q_2, Q_n) = n + O(n/\log n)$.

In the diagonal case, Cox and Stolee \cite{6} proved $R_w(Q_n, Q_n) \geq 2n + 1$ using a probabilistic construction. In the induced case, Bohman and Peng \cite{2} showed $R(Q_n, Q_n) \geq 2n + 1$ using an explicit construction. Improving these results, we show that $R_w(Q_m, Q_n) \geq n + m + 1$ for all $m \geq 2$ and large $n$ by giving an explicit construction; in particular, we prove that $R_w(Q_2, Q_n) = n + 3$.

1 Introduction

Background and definitions. The classical Ramsey theorem asserts that for any $m$ and $n$, there is an integer $N$ such that every blue-red edge coloring of the complete graph on $N$ vertices contains a blue clique on $m$ vertices or a red clique on $n$ vertices. Determining the smallest such integer $N$, known as the Ramsey number is a central problem in combinatorics. More generally, for any two graphs $G$ and $H$, the Ramsey number is the smallest integer $N$ such that every blue-red edge coloring of the complete graph on $N$ vertices contains a red copy of $G$ or a blue copy of $H$. Several natural variations of these problems such as multicolor Ramsey numbers, and hypergraph Ramsey numbers are major subjects of ongoing research. For further examples, we refer the reader to the surveys \cite{5, 8}.

In this paper, we will study poset Ramsey numbers. A partially ordered set (or a poset for short) is a set with an accompanying relation $\leq$ which is transitive, reflexive, and antisymmetric. A Boolean lattice of dimension $n$, denoted by $Q_n$, is the power set of $\{1, 2, \ldots, n\}$ equipped with the inclusion relation. If $(P, \leq)$ and $(Q, \leq')$ are posets, then an injection $f : P \to Q$ is order-preserving if $f(x) \leq' f(y)$ whenever $x \leq y$; we say that $f(P)$ is a weak copy of $P$ in $Q$ and that $P$ is a weak subposet of $Q$. An injection $f : P \to Q$ is an order-embedding if $f(x) \leq' f(y)$ if and only if $x \leq y$; we say that $f(P)$ is an induced copy of $P$ in $Q$ and that $P$ is an induced subposet of $Q$. 
For posets $P_1$ and $P_2$, the (induced) poset Ramsey number $R(P_1, P_2)$ is defined to be the smallest integer $N$ such that every blue-red coloring of the elements of the Boolean lattice $Q_N$ contains an induced copy of $P_1$ whose elements are blue or an induced copy of $P_2$ whose elements are red. Similarly, the weak poset Ramsey number $R_w(P_1, P_2)$ is defined to be the smallest integer $N$ such that every blue-red coloring of the elements of the Boolean lattice $Q_N$ contains a weak copy of $P_1$ whose elements are blue or a weak copy of $P_2$ whose elements are red. (For convenience, we will call a copy of poset $P$ all of whose elements are blue is called a blue copy of $P$, and a copy of poset $P$ all of whose elements are red is called a red copy of $P$.) It is easy to see that $R(P_1, P_2) \geq R_w(P_1, P_2)$. The focus of this paper is the natural problem when $P_1$ and $P_2$ are Boolean lattices $Q_m$ and $Q_n$ for $m, n \in \mathbb{N}$. Recently, variants of this problem, such as rainbow poset Ramsey numbers have been studied in [3, 4, 6].

**Induced poset Ramsey numbers.** For the diagonal poset Ramsey number $R(Q_n, Q_n)$, Axenovich and Walzer [1] showed that $2n \leq R(Q_n, Q_n) \leq n^2 + 2n$. Walzer [10] improved the upper bound to $R(Q_n, Q_n) \leq n^2 + 1$. Recently, Lu and Thompson [7] further improved it to $R(Q_n, Q_n) \leq n^2 - n + 2$. On the other hand, Cox and Stolee [6] showed that for $n \geq 13$, $R_w(Q_n, Q_n) \geq 2n + 1$, which implies that $R(Q_n, Q_n) \geq 2n + 1$.

More generally, Axenovich and Walzer [1] showed that $n + m \leq R(Q_m, Q_n) \leq mn + n + m$ for any integers $n, m \geq 1$. Lu and Thompson [7] improved this bound by showing that $R(Q_m, Q_n) \leq (m - 2 + \frac{9m - 9}{(2m - 3)(m + 1)})n + m + 3$ for all $n \geq m \geq 4$. See [1, 6, 7, 10] for several other interesting results.

For the off-diagonal poset Ramsey number $R(Q_2, Q_n)$, Axenovich and Walzer [1] showed that $n + 2 \leq R(Q_2, Q_n) \leq 2n + 2$. Recently, Lu and Thompson [7] improved the upper bound by proving that $R(Q_2, Q_n) \leq \frac{5}{2}n + 2$. In this paper, we determine $R(Q_2, Q_n)$ asymptotically by proving the following theorem.

**Theorem 1.** For every $c > 2$, there exists an integer $n_0$ such that for all $n \geq n_0$, we have

$$R(Q_2, Q_n) \leq n + cn \frac{n}{\log_2 n}.$$

Combining Theorem 1 with the lower bound $R(Q_2, Q_n) \geq n + 2$, we obtain that $R(Q_2, Q_n)$ is asymptotically equal to $n$. We prove Theorem 1 in Section 2. In fact, it follows from our proof of Theorem 1 that for all $n \geq 2$, we have $R(Q_2, Q_n) \leq n + 6.14 \frac{n}{\log_2 n}$.

**Weak poset Ramsey numbers.** A chain of length $k$ is a poset of $k$ distinct, pairwise comparable elements and is denoted by $C_k$. Cox and Stolee [6] showed that $R_w(C_k, Q_n) = n + k - 1$; since $Q_n$ is a weak subposet of $C_{2^n}$, this implies that $R_w(Q_m, Q_n) \leq n + 2^m - 1$. The lower bound $R_w(Q_m, Q_n) \geq m + n$ is obtained by a simple “layered” coloring of $Q_{m+n-1}$ considered by Axenovich and Walzer [1], which is described as follows. The collection of all subsets of $[N]$ of a given size $k$ is called a layer. A coloring of $Q_N$ is layered if for every layer, all sets on that layer have the same color. A layered coloring of $Q_{m+n-1}$ with $m$ blue layers and $n$ red layers does not contain a (weak) blue copy of $Q_m$ or a (weak) red copy of $Q_n$. Therefore, $R_w(Q_m, Q_n) \geq m + n$ (which implies $R(Q_m, Q_n) \geq m + n$). Despite the work of several researchers, so far this lower bound on $R_w(Q_m, Q_n)$ has not been improved except in the diagonal case: Cox and Stolee [6] showed that $R_w(Q_n, Q_n) \geq 2n + 1$ for $n \geq 13$ using a probabilistic construction. Recently, in the induced case, Bolman and Peng [2] gave an explicit construction showing the bound $R(Q_n, Q_n) \geq 2n + 1$. Note that these constructions showing $R(Q_n, Q_n) \geq 2n + 1$ cannot be layered.
We give an explicit construction which yields a lower bound on \( R_w(\mathcal{Q}_m, \mathcal{Q}_n) \) for all \( m \) and \( n \geq 68 \), thereby generalizing the results of Bohman and Peng to the weak poset case, and additionally extending their results and those of Cox and Stolee to the off-diagonal case.

**Theorem 2.** For any \( m \geq 2 \) and \( n \geq 68 \), we have

\[
R_w(\mathcal{Q}_m, \mathcal{Q}_n) \geq m + n + 1.
\]

Note that Theorem 2 shows that \( R_w(\mathcal{Q}_2, \mathcal{Q}_n) = n + 3 \) since \( R_w(\mathcal{Q}_2, \mathcal{Q}_n) \leq n + 2^2 - 1 = n + 3 \) by the upper bound mentioned earlier.

We prove Theorem 2 in Section 3.2. The construction and the proof of Theorem 2 are simpler if we restrict ourselves to the case of \( m = 2 \) and consider induced subposets rather than weak subposets. Therefore, in order to illustrate the main ideas of our construction, we present a short proof showing the special case \( R(\mathcal{Q}_2, \mathcal{Q}_n) \geq n + 3 \) (for \( n \geq 18 \)) in Section 3.1. We also give a probabilistic construction for \( m \geq 3 \) and \( n \) sufficiently large in Section 3.3 by generalizing a construction of Cox and Stolee.

# 2 Upper bound: Proof of Theorem 1

Let \( k = \left\lfloor \frac{n}{\log_2 n} \right\rfloor \). Assume that \( \mathcal{B}, \mathcal{R} \subset \mathcal{Q}_{n+k} \) such that \( \mathcal{B} \sqcup \mathcal{R} = \mathcal{Q}_{n+k} \), and further assume that \( \mathcal{Q}_2 \) is not an induced subposet of \( \mathcal{B} \), and \( \mathcal{Q}_n \) is not an induced subposet of \( \mathcal{R} \).

Before continuing with the proof of Theorem 1, let us provide an outline of the proof.

**Outline of the proof.** We attempt to define an order-embedding \( \varphi \) from \( \mathcal{Q}_n \) into \( \mathcal{R} \) recursively, starting with \( \emptyset \), in such a way that the image of each set only depends on the images of its proper subsets. For every \( A \subseteq [n] \), \( \varphi(A) \) will be a superset of \( A \), possibly containing some additional elements from \( [n+k] \setminus [n] \).

If \( \emptyset \in \mathcal{R} \), then we set \( \varphi(\emptyset) = \emptyset \). More generally, in order for \( \varphi \) to be order-preserving, for any set \( A \in \mathcal{Q}_n \), \( \varphi(A) \) must be a superset of the images of all proper subsets of \( A \); as long as the minimal set that is a superset of \( A \) and also has this property is in \( \mathcal{R} \), we set it as \( \varphi(A) \). If instead this minimal set is in \( \mathcal{B} \), then we proceed to add elements of \( [n+k] \setminus [n] \) to it, in an order determined by some arbitrary permutation \( \pi \) of \( [n+k] \setminus [n] \), until we obtain a set that is in \( \mathcal{R} \). Throughout this recursive procedure, in addition to the injection \( \varphi \), we construct a function \( \alpha \) where \( \alpha(A) \) records the number of elements of \( [n+k] \setminus [n] \) we need to include in \( \varphi(A) \) as a result of hitting sets in \( \mathcal{B} \) while attempting to embed \( A \) (and its subsets, during previous steps of the recursion); and another function \( f \), where \( f(A) \) records an actual chain of length \( \alpha(A) \), consisting of sets in \( \mathcal{B} \) that we have encountered while trying to embed \( A \) and its subsets.

For any fixed permutation \( \pi \) of \( [n+k] \setminus [n] \), the above embedding procedure can only fail if, at some point, as we try to define \( \varphi(A) \) for some \( A \in \mathcal{Q}_n \), we hit a set in \( \mathcal{B} \), but we have already “used up” all \( k \) elements of \( [n+k] \setminus [n] \), so there are no elements left to add. In this event, we obtain a chain of length \( k + 1 \), contained in \( \mathcal{B} \). As \( \mathcal{Q}_n \) is not an induced subposet of \( \mathcal{R} \), the procedure must fail for all \( k! \) permutations \( \pi \) of \( [n+k] \setminus [n] \). This way, we can obtain a chain of length \( k + 1 \) inside \( \mathcal{B} \), corresponding to each of these permutations. We show that these \( k! \) chains must all be distinct. We then show that the existence of \( k! \) distinct chains of length \( k + 1 \) inside \( \mathcal{B} \) implies that \( \mathcal{Q}_2 \) is an induced subposet of \( \mathcal{B} \), a contradiction.
Now we continue with the proof of Theorem 1.

At the core of the proof is Claim 3. We will use the following notation: for a chain of sets $C$ in $Q_{n+k}$ of length $l$, we denote its sets by $(q_0, q_1, \ldots, q_{l-1})$ where $q_0 \subseteq q_1 \subseteq \ldots \subseteq q_{l-1}$.

Claim 3. Let $\pi : [n+k] \setminus [n] \to [n+k] \setminus [n]$ be a permutation. There exist $\varphi : Q_n \to \mathcal{R} \cup \{\emptyset\}$ (where $\emptyset$ is an arbitrary element, distinct from the members of $\mathcal{R}$, and used solely to indicate failure to produce an induced map into $\mathcal{R}$), $\alpha : Q_n \to \{0, 1, \ldots, k, k + 1\}$ and $f : Q_n \to C^{\leq k+1}(B)$, where $C^{\leq k+1}(B)$ is the family of all chains of length at most $k + 1$ in $B$, with the following properties:

P1. If $B, A \in Q_n$ and $\varphi(B), \varphi(A) \in \mathcal{R}$, then $B \subseteq A \iff \varphi(B) \subseteq \varphi(A)$. (This implies that if $\emptyset \notin \text{Im} \varphi$, then $Q_n$ is an induced subposet of $\mathcal{R}$.)

P2. If $B \subseteq A \in Q_n$, then $\alpha(B) \leq \alpha(A)$.

P3. If $\alpha(A) = k + 1$, then $\varphi(A) = \emptyset$. Otherwise $\varphi(A) \cap [n] = A$, and $\varphi(A) = A \cup \{\pi(n + 1), \pi(n + 2), \ldots, \pi(n + \alpha(A))\}$.

P4. For every $A \in Q_n$, $f(A) = (f(A)_0, f(A)_1, \ldots, f(A)_{\alpha(A)-1})$ is a chain in $\mathcal{B}$ of length $\alpha(A)$ with the property that $f(A)_0 \subseteq [n] = \{\pi(n + 1), \pi(n + 2), \ldots, \pi(n + i)\}$.

P5. If $A \in Q_n$ such that $1 \leq \alpha(A) \leq k$, then $f(A)_{\alpha(A)-1} \subseteq \varphi(A)$. (In fact this implies that $f(A)_{\alpha(A)-1} \subseteq \varphi(A)$, since the elements of $f(A)$ are in $\mathcal{B}$, while $\varphi(A)$ is in $\mathcal{R}$. We do not use this observation.)

Proof. We construct the functions $\varphi$, $\alpha$ and $f$ recursively, and simultaneously prove the above properties by induction: we set the values of these functions on a set $A \in Q_n$ in such a way that they only depend on the values of the functions on proper subsets of $A$. (This includes the case of $A = \emptyset$ where no proper subsets exist, which we do not treat in a special way for most of the proof. One can also consider the proof as a recursion and induction on the size of the set $A$.) Let us fix an $A \in Q_n$. Now we will define the values $\varphi(A), \alpha(A)$ and $f(A)$, and then prove that P1 to P5 hold for this set $A$ under the assumption that they hold for every proper subset of $A$.

If there exists a $B \not\subseteq A$ such that $\varphi(B) = \emptyset$, then we pick such a set $B$ arbitrarily, and set $\varphi(A) = \emptyset$, $\alpha(A) = k + 1$ and $f(A) = f(B)$. Otherwise let

$\beta = \min\{i \in \{0, 1, \ldots, k\} : (\forall B \not\subseteq A : \alpha(B) \leq i)\}$

$= \begin{cases} \max_{B \subseteq A} \alpha(B) & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset, \end{cases}$

and let

$$C = A \cup \{\pi(n + 1), \pi(n + 2), \ldots, \pi(n + \beta)\} = A \cup \bigcup_{B \subseteq A} \varphi(B)$$

(note that $\{\pi(n + 1), \pi(n + 2), \ldots, \pi(n + \beta)\} = \emptyset$ if $\beta = 0$). We get the last equality by applying P3 to the proper subsets of $A$. We want $\varphi(A)$ to be a superset of $C$. If $C \in \mathcal{R}$, we set $\varphi(A) = C$. If $C \in \mathcal{B}$, we keep adding $\pi(n + \beta + 1), \pi(n + \beta + 2), \ldots$ to it, until the set is not in $\mathcal{B}$, if possible. That is, let

$$\alpha(A) = \begin{cases} \min\{i \in \{\beta, \beta + 1, \ldots, \beta + \min\left\{C \cup \{\pi(n + \beta + 1), \pi(n + \beta + 2), \ldots, \pi(n + i)\} \in \mathcal{R}\right\} \} & \text{if such } i \text{ exists}, \\ k + 1 & \text{otherwise}. \end{cases}$$
Then let
\[
\varphi(A) = \begin{cases} 
C \cup \{\pi(n + \beta + 1), \pi(n + \beta + 2), \ldots, \pi(n + \alpha(A))\} & \text{if } \alpha(A) \leq k, \\
\emptyset & \text{if } \alpha(A) = k + 1.
\end{cases}
\]

Note that in the first case,
\[
C \cup \{\pi(n + \beta + 1), \pi(n + \beta + 2), \ldots, \pi(n + \alpha(A))\} = A \cup \{\pi(n + 1), \pi(n + 2), \ldots, \pi(n + \alpha(A))\}.
\]

Furthermore if \(A = \emptyset\), set \(f(A) = ()\), an empty chain. Otherwise pick a set \(B \subseteq A\) such that \(\alpha(B) = \beta\). We set \(f(A)\) to be a chain of length \(\alpha(A)\) in \(B\):
\[
f(A)_i = \begin{cases} 
f(B)_i & \text{if } 0 \leq i < \beta, \\
A \cup \{\pi(n + 1), \pi(n + 2), \ldots, \pi(n + i)\} & \text{if } \beta \leq i < \alpha(A).
\end{cases}
\]

Note that these definitions of \(\varphi(A), \alpha(A)\) and \(f(A)\) only depend on the values of these functions for proper subsets of \(A\), so our recursive definitions make sense. It is easy to check that the definitions of \(\varphi\) and \(\alpha\) satisfy \([P4]\) and \([P5]\) which together imply \([P1]\), \([P2]\) and \([P3]\) are also trivially satisfied when \(\alpha(A) = 0\). If \(\alpha(A) = \beta = k + 1\), we have defined \(f(A) = f(B)\) for some \(B \subseteq A\) such that \(\varphi(B) = \emptyset\); then \([P4]\) follows because it holds for \(B\) by induction, and \([P5]\) is trivial.

Now we prove \([P4]\) and \([P5]\) when \(\alpha(A) > 0\) and \(\beta \leq k\). In the case where \(\alpha(A) = \beta\) (equivalently if \(C \in \mathcal{R}\), \(\varphi(A) = C\) and \(\alpha(A) = \alpha(B)\)), then \(f(A) = f(B)\) is a chain satisfying \([P4]\) by induction. Since \([P5]\) holds for \(B\) by induction, we get that \(f(A)_{\alpha(A)-1} = f(B)_{\alpha(B)-1} \subseteq \varphi(B) \subseteq \varphi(A)\), so \([P5]\) is satisfied for \(A\) as well. If \(\alpha(A) > \beta\), then \([P5]\) follows from the definitions of \(f(A)\) and \(\varphi(A)\). Furthermore, \(A \cup \{\pi(n + 1), \pi(n + 2), \ldots, \pi(n + i)\} \in B\) for \(\beta \leq i < \alpha(A)\) because \(\alpha(A)\) was chosen as the smallest \(i \geq \beta\) such that \(A \cup \{\pi(n + 1), \pi(n + 2), \ldots, \pi(n + i)\} \in \mathcal{R}\); this is enough to show \([P4]\) if \(\beta = 0\). Finally, if \(\alpha(A) > \beta > 0\), then \(f(A)\) is obtained by concatenating the chains \(f(B)\) and \((A \cup \{\pi(n + 1), \pi(n + 2), \ldots, \pi(n + i)\})\) \(\beta \leq i < \alpha(A)\). By induction \(f(B)\) is a chain satisfying the conditions of \([P4]\), and \(B\) satisfies \([P5]\) so \(f(B)_{\beta-1} \subseteq \varphi(B)\). Using that \([P3]\) holds for \(B\) by induction, we also have \(\varphi(B) = B \cup \{\pi(n + 1), \pi(n + 2), \ldots, \pi(n + \alpha(B))\} \subseteq B \cup \{\pi(n + 1), \pi(n + 2), \ldots, \pi(n + \beta)\}\) (recall that \(B \not\subseteq A\) and \(\beta = \alpha(B)\)). Thus \(f(A)\) is indeed a chain satisfying \([P4]\). The proof of the properties \(P1\) to \(P5\) of \(\varphi, \alpha\) and \(f\) is now complete.

For an arbitrary permutation \(\pi: [n + k] \setminus [n] \to [n + k] \setminus [n]\), let \(\varphi^\pi, \alpha^\pi\) and \(f^\pi\) be the maps given by \(\text{Claim 3}\). If \(\text{Im} \, \varphi^\pi \subseteq \mathcal{R}\), then \(\varphi^\pi\) shows that \(Q_n\) is an induced subposet of \(\mathcal{R}\) by \([P1]\). Assume that this is not the case. Then, for some \(A \in Q_n\), \(\varphi^\pi(A) = \emptyset\), \(\alpha^\pi(A) = k + 1\) by \([P3]\) and \(f^\pi(A)\) is a chain of length \(k + 1\) in \(B\) by \([P4]\). By \([P4]\) we have \(\pi(n + i) = (f^\pi(A)_i \setminus f^\pi(A)_{i-1}) \setminus [n]\) when \(1 \leq i \leq \alpha^\pi(A) - 1\), so if \(\alpha^\pi(A) = k + 1\), then one can recover the permutation \(\pi\) from the chain \(f^\pi(A)\).

Under our assumption that \(Q_n\) is not an induced subposet of \(\mathcal{R}\), we get a distinct chain \(f^\pi\) of length \(k + 1\) in \(B\) for each of the \(k!\) permutations \(\pi\) of \([n + k] \setminus [n]\), with the property that
\[\forall 0 \leq i \leq k: f^\pi_i \setminus [n] = \{\pi(n + 1), \pi(n + 2), \ldots, \pi(n + i)\}.
\]

We claim that the map \(\pi \mapsto (f^\pi_0, f^\pi_k)\) is injective. Let \(\pi_1\) and \(\pi_2\) be two different permutations of \([n + k] \setminus [n]\). Let \(i = \min_{j \in \{0, \ldots, k\}} \pi_1(n + j) \neq \pi_2(n + j)\). Then \(\pi_1(i) \in f^\pi_i\),
\[ \pi_1(i) \notin \pi_2, \pi_2(i) \in i \] and \( \pi_1(i) \notin \pi_1 \) and \( i \) are unrelated. So if \( f_i^\pi_1 = f_i^\pi_2 \) and \( f_k^\pi_1 = f_k^\pi_2 \), then \( B \) would contain an induced copy of \( Q_2 \), a contradiction.

Since the map \( \pi \mapsto (f_0^\pi, f_k^\pi) \) is injective,
\[
k! \leq \left(2^{n+k}\right)^2 = 2^{2(n+k)}.
\]
Approximating the left-hand side:
\[
k! > \left(\frac{k}{e}\right)^k = 2^{k \log_2 k - \log_2 e}, \quad \text{so}
\]
\[
k(\log_2 k - \log_2 e) < 2(n + k).
\]
Since \( k = \left\lfloor \frac{n}{\log_2 n} \right\rfloor \),
\[
k \log_2 k > \left(1 - \frac{1}{\log_2 n} \right) \log_2 c + \log_2 n - \log_2 \log_2 n - 1 = cn(1 - o(1)).
\]
Since \( c > 2 \), (2) contradicts (1) for sufficiently large \( n \). This completes the proof of Theorem 1.

**Remark.** It follows the above proof that for all \( n \geq 2 \), we have \( R(Q_2, Q_n) \leq n + 6.14 \frac{n}{\log_2 n} \). Here we give a sketch of the calculations.

For \( c = 6.14 \), we have \( k = \left\lfloor \frac{n}{\log_2 n} \right\rfloor > 5.611 \frac{n}{\log_2 n} \) for every integer \( n \geq 2 \), and therefore we have \( k \log_2 k > 5.611 \frac{n}{\log_2 n} \log_2 n(1 - \frac{1}{\log_2 n}) + \log_2 5.611 \geq 2.977n + 13.96 \frac{n}{\log_2 n} \).

Using (1), it can be shown that \( 0.8797k \log_2 k \leq k(\log_2 k - \log_2 e) \leq 2n + 12.28 \frac{n}{\log_2 n} \) for every \( n \geq 2 \), contradicting the lower bound on \( k \log_2 k \) shown earlier.

### 3 Lower bounds

#### 3.1 An explicit construction showing \( R(Q_2, Q_n) \geq n + 3 \)

In this subsection, we prove a special case of Theorem 2 to illustrate the basic ideas of the construction. The fully general proof of Theorem 2 presented in Section 3.2 is significantly more involved (primarily due to the fact that it is more difficult to deduce properties of a weak map \( Q_n \to Q_{n+m} \).

**Theorem 4.** For \( n \geq 18 \), there exist \( B, R \subset Q_{n+2} \) such that \( B \cup R = Q_{n+2} \), \( Q_2 \) is not an induced subposet of \( B \), and \( Q_n \) is not an induced subposet of \( R \).

Let \( k = \left\lfloor \frac{n}{2} \right\rfloor \). Let \( B = \left(\frac{n+2}{k}\right) \cup \left(\frac{n+2}{k+3}\right) \), with some sets of size \( k + 1 \) which we will add later. Assume for a contradiction that \( Q_n \) is an induced subposet of \( R \). Let \( \varphi : Q_n \to R \) be an injection such that \( \varphi(A) \subseteq \varphi(B) \) if and only if \( A \subseteq B \).

For any maximal chain \( \emptyset \subseteq A_1 \subseteq \ldots \subseteq A_{n-1} \subseteq [n] \), the sets in its image satisfy \( \varphi(\emptyset) \subseteq \varphi(A_1) \subseteq \ldots \subseteq \varphi(A_{n-1}) \subseteq \varphi([n]) \), and none of the sets in the image are of size \( k \) or \( k + 3 \).

So for every \( A \subseteq [n] \),
\[
|\varphi(A)| = \begin{cases} |A| & \text{if } |A| \leq k - 1, \\ |A| + 1 & \text{if } k \leq |A| \leq k + 1, \\ |A| + 2 & \text{if } k + 2 \leq |A|, \end{cases}
\]
For $a \in [n]$, let $\tilde{\varphi}(a)$ denote the unique element of $\varphi(\{a\})$. The map $\tilde{\varphi} : [n] \to [n + 2]$ is an injection. Note that, for a set $A \subseteq [n]$, $\tilde{\varphi}[A]$ denotes the image of $A$ under $\tilde{\varphi}$, and for a set $B \subseteq [n + m]$, $\tilde{\varphi}^{-1}[B]$ denotes the preimage of $B$ under $\tilde{\varphi}$.

Let $X = \{\tilde{\varphi}(a) : a \in [n]\}$ and $Y = [n + 2] \setminus X = \{y, z\}$. We have $|X| = n$ and $|Y| = 2$. We claim that for every $A \subseteq [n]$, $\varphi(A) \cap X = \tilde{\varphi}[A]$. Indeed, for every $b \in X$, there is an $a \in [n]$ such that $\tilde{\varphi}(a) = b$, and we have $b = \tilde{\varphi}(a) \in \tilde{\varphi}[A] \iff a \in A \iff \{a\} \subseteq A \iff \{\tilde{\varphi}(a)\} = \varphi(\{a\}) \iff b = \tilde{\varphi}(a) \in \varphi(A)$.

From (3), $\varphi(A)$ contains neither $y$ nor $z$ if $|A| \leq k - 1$, exactly one of them if $k \leq |A| \leq k + 1$, and both if $k + 2 \leq |A|$.

**Claim 5.** For every set $A \in \binom{[n]}{k}$, $\varphi(A)$ contains the same element of $Y$.

*Proof.* Assume for a contradiction that some sets of the form $\varphi(A)$ contain $y$, and others contain $z$. Since the Johnson graph – whose vertices are $\binom{[n]}{k}$, and whose edges connect sets with symmetric difference 2 – is connected, there would be two sets $A, B \in \binom{[n]}{k}$ with a symmetric difference of size 2 such that $y \in \varphi(A)$ and $z \in \varphi(B)$. Then $|A \cup B| = k + 1$, and $\varphi(A \cup B)$ would contain both $y$ and $z$ as it would have to be a superset of both $\varphi(A)$ and $\varphi(B)$, contradicting that it contains exactly one of $y$ and $z$. □

Now we specify which sets of $\binom{[n+2]}{k+1}$ are added to $B$ in addition to $\binom{[n+2]}{k} \cup \binom{[n+2]}{k+1}$. Our goal is to add these sets in such a way that for every map $\varphi : Q_n \to Q_{n+2}$, assuming that $\text{Im} \varphi \subseteq \mathcal{R}$, and $\varphi$ is an order-embedding, the above observations lead to a contradiction. (The map $\varphi$, and the variables dependent on it such as $X, Y$ and $z$, are not fixed now; we have to set $B$ in such a way that the existence of any order-embedding $\varphi : Q_n \to \mathcal{R}$ leads to a contradiction.) For every distinct $y, z \in [n + 2]$, pick a set $C_{y, z} \in \binom{[n+2]}{k+1}$ such that $y \in C_{y, z}$ but $z \notin C_{y, z}$, and $C_{y, z}$ is at a symmetric difference of size at least 4 from every previously chosen set $C_{y', z'}$, and add $C_{y, z}$ to $B$. We can do this by greedily picking sets $C_{y, z}$ one-by-one: At each step, we have picked at most $(n + 2)(n + 1) - 1$ sets so far, and each previously picked set $C_{y', z'}$ blocks at most $1 + k(n - k)$ choices (because there are at most that many sets containing $y$ but not $z$ with a symmetric difference of size at most 2 from $C_{y', z'}$). In total, there are $\binom{n+2}{k+1}$ sets $C \in \binom{[n+2]}{k+1}$ that satisfy $y \in C$ and $z \notin C$. For $n \geq 18$ and $k = \left\lceil \frac{n}{2} \right\rceil$, we have

$$\binom{n}{k} > \left((n + 2)(n + 1) - 1\right)(1 + k(n - k)),$$

so we can always choose a set $C_{y, z}$ which satisfies the required conditions.

After adding such sets $C_{y, z}$ for every distinct $y, z \in [n + 2]$, the resulting family $B$ will not contain a copy of $Q_2$. Indeed, a copy of $Q_2$ in $B$ would have to consist of a set of size $k$, a set of size $k + 3$, and two sets of size $k + 1$; but the latter two sets would need to have a symmetric difference of size 2.

Now assume for a contradiction that $\mathcal{R} = 2^{[n+2]} \setminus B$ contains an induced copy of $Q_n$. Consider an arbitrary injection $\varphi : Q_n \to \mathcal{R}$, and define $\tilde{\varphi}$, $X$ and $Y = \{y, z\}$ as before, and apply Claim 5. We can assume without loss of generality that for every $A \in \binom{[n]}{k}$, we
have \( y \in \varphi(A) \), and thus \( \varphi(A) = \tilde{\varphi}(A) \cup \{ y \} \). There is a set \( C_{y,z} \in \binom{(n+2)}{k+1} \cap B \) such that \( y \in C_{y,z} \), but \( z \notin C_{y,z} \). We have \( \tilde{\varphi}^{-1}[C_{y,z}] \in \binom{n}{k} \), and
\[
\varphi(\tilde{\varphi}^{-1}[C_{y,z}]) = \{ \tilde{\varphi}(a) : a \in [n], \tilde{\varphi}(a) \in C_{y,z} \} \cup \{ y \} = C_{y,z} \in B,
\]
contradicting that the image of \( \varphi \) is in \( \mathcal{R} \).

3.2 An explicit construction showing \( R_w(\mathcal{Q}_m, \mathcal{Q}_n) \geq m + n + 1 \)

Theorem 2 will be an immediate consequence of the following theorem.

**Theorem 6.** Let \( n, m \in \mathbb{N} \) such that \( m \geq 2 \) and \( n \geq \sqrt{32m + 200} + 18 \). There exist \( B, \mathcal{R} \subset \mathcal{Q}_{n+m} \) such that \( B \cup \mathcal{R} = \mathcal{Q}_{n+m} \), \( \mathcal{Q}_m \) is not a weak subposet of \( B \), and \( \mathcal{Q}_n \) is not a weak subposet of \( \mathcal{R} \).

To see that Theorem 2 follows from Theorem 6, notice that we can assume without loss of generality that \( n \geq m \), so for \( n \geq 68 \), we have \( n \geq \sqrt{32n + 260} + 18 \).

For values of \( m \) less than 67, the threshold for \( n \) in the hypothesis of Theorem 6 is smaller than 68: for instance, it holds for \( m = 2 \) and \( n \geq 36 \). Also note that in the proof of Lemma 8, we use Bertrand’s postulate to obtain a prime \( N \leq p < 2(N - 1) \). By finding the smallest prime greater than or equal to \( N \), one may be able to relax the requirement on \( k \) in Lemma 8 and thereby extend Theorem 6 to somewhat smaller values of \( n \), for a given \( m \).

In the proof of Theorem 6, we will use Lemma 8 which will follow from Lemma 7.

**Lemma 7** (Olson [9]). Let \( p \) be a prime, and let \( A \subseteq [p] \) such that \( |A| \geq \sqrt{4p - 3} \). Then for every \( a \in \mathbb{Z} \), there is a subset \( B \subseteq A \) such that
\[
\sum B \equiv a \pmod{p}.
\]

**Lemma 8.** Let \( N, k \in \mathbb{N} \) such that \( N > 3 \) and \( k \geq \sqrt{8N - 15} \). Then there is a constant-weight code \( C \subset \binom{[N]}{k+1} \) such that the symmetric difference between any two sets is of size at least 4, and the following holds:

Let \( n, m \in \mathbb{N} \) such that \( n + m = N \) and \( k \leq n - \sqrt{8N - 15} \). For every \( Y \in \binom{[N]}{m} \) and \( y \in Y \), there is a set \( C \in \binom{[N \setminus Y]}{k} \) such that \( C \cup \{ y \} \in C \).

**Proof.** By Bertrand’s postulate, there is a prime \( p \) such that \( N \leq p < 2(N - 1) \). Let \( d \in [p] \) be a fixed constant, and let
\[
C = \left\{ S \in \binom{[N]}{k+1} : \sum S \equiv d \pmod{p} \right\}.
\]

Let \( Y \in \binom{[N]}{m} \) and \( y \in Y \). We have to find a set \( C \) that satisfies the condition in the statement. Let \( l = \sqrt{8N - 15} \). It follows from the conditions of the lemma that \( n \geq 2l \). Let \( [N \setminus Y] = \{ x_1, \ldots, x_n \} \) such that \( x_1 < x_2 < \ldots < x_n \). For \( i = 1, \ldots, l \), let \( a_i = x_i \) and \( b_i = x_{n-i+1} \). The numbers \( b_i - a_i \) are in \([p]\), and they are different because \( b_1 - a_1 > b_2 - a_2 > \ldots > b_l - a_l \). Let \( a = \sum_{i=1}^l a_i \). Let \( E \) be a subset of \( \{ x_{i+1}, \ldots, x_{n-l} \} \) with \( k - l \) elements, and let \( e = \sum E \). (It follows from the conditions that \( 0 \leq k - l \leq n - 2l \)).
Since $l \geq \sqrt{8N - 15} \geq \sqrt{4p - 3}$, by Lemma 7, there is a subset of \( \{b_i - a_i : 1 \leq i \leq l \} \) such that its sum is congruent with \( d - y - e - a \). That is, there is a set \( I \subseteq [l] \) such that

\[
\sum_{i \in I} (b_i - a_i) \equiv d - y - e - a \pmod{p}.
\]

Let

\[
C = E \cup \left\{ a_i : i \notin I \right\} \cup \left\{ b_i : i \in I \right\}.
\]

We have

\[
\sum (C \cup \{y\}) = \sum E + \sum_{i=1}^{l} a_i + \sum_{i \in I} (b_i - a_i) + y \equiv d \pmod{p},
\]

so \( C \in \mathcal{C} \).

Let \( k \) be an integer between \( \sqrt{8(n + m) - 15} \) and \( n - 1 - \sqrt{8(n + m) - 15} \) inclusive. (The conditions of the proposition imply that \( \sqrt{8(n + m) - 15} + 1 \leq n - 1 - \sqrt{8(n + m) - 15} \), therefore such an integer \( k \) exists.) Let \( B \supseteq \left( \bigcup_{m}^{n+m} \right) \cup \left( \bigcup_{k+3}^{n+m} \right) \cup \ldots \cup \left( \bigcup_{k+m+1}^{n+m} \right) \cup \mathcal{C} \), where \( \mathcal{C} \) is given by Lemma 8 using \( n + m \) in the place of \( N \). First we show that the family \( B \) does not contain a \( \mathcal{Q}_{m} \). Indeed, any two sets in \( B \) of size \( k + 1 \) have a symmetric difference of size at least 4. A copy of \( \mathcal{Q}_{m} \) in \( B \) would consist of a set of size \( k \), some sets of size \( k + 3, k + 4, \ldots, k + m + 1 \) corresponding to the sets of size \( 2 \) to \( m \) of the \( \mathcal{Q}_{m} \), and \( \mathcal{Q}_{m} \) of size \( k + 1 \) corresponding to the singletons of \( \mathcal{Q}_{m} \). The latter \( m \) sets would need to have a symmetric difference of size 2.

Assume for a contradiction that \( \mathcal{Q}_{n} \) is a subposet of \( \mathcal{R} \). Let \( \varphi : \mathcal{Q}_{n} \to \mathcal{R} \) be an injection that preserves relations.

For any maximal chain \( \emptyset \subseteq A_1 \subseteq \ldots \subseteq A_m \subseteq \mathcal{Q}_{n} \), we have \( \varphi(\emptyset) \subseteq \varphi(A_1) \subseteq \ldots \subseteq \varphi(A_m) \subseteq \varphi(\mathcal{Q}_{n}) \), and none of the sets in the image are of size \( k \) or \( k + 3, k + 4, \ldots, k + m + 1 \). So for every \( A \subseteq [n] \),

\[
|\varphi(A)| = \begin{cases} |A| & \text{if } |A| \leq k - 1, \\ |A| + 1 & \text{if } k \leq |A| \leq k + 1, \\ |A| + m & \text{if } k + 2 \leq |A|, \end{cases}
\]

(5)

thus the image of every singleton is a singleton, and the image of the complement of every singleton is the complement of a singleton).

For \( a \in [n] \), let \( \varphi_1(a) \) denote the unique element of \( \varphi(\{a\}) \), and let \( \varphi_2(a) \) denote the unique element of \( [n + m] \setminus \varphi([n] \setminus \{a\}) \). Note that, for a set \( A \subseteq [n] \), \( \varphi_1(A) \) denotes the image of \( A \) under \( \varphi_1 \), and for a set \( B \subseteq [n + m] \), \( \varphi_1^{-1}[B] \) denotes the preimage of \( B \) under \( \varphi_i \).

The maps \( \varphi_1 \) and \( \varphi_2 \) are injections. Furthermore, for any distinct \( a, b \in [n] \), it holds that \( \{a\} \subseteq [n] \setminus \{b\} \), so \( \{\varphi_1(a)\} = \varphi(\{a\}) \subseteq \varphi([n] \setminus \{a\}) = [n + m] \setminus \varphi_2(b) \), so \( \varphi_1(a) \neq \varphi_2(b) \). Now take the sets \( \{\varphi_1(a), \varphi_2(a)\} \subseteq [n + m] \) for each \( a \in [n] \); these sets have 1 or 2 elements, depending on whether \( \varphi_1(a) = \varphi_2(a) \). Based on the observations in this paragraph, if \( a \neq b \), we have

\[
\{\varphi_1(a), \varphi_2(a)\} \cap \{\varphi_1(b), \varphi_2(b)\} = \emptyset.
\]
Since \( \bigcup_{a \in [n]} \{ \varphi_1(a), \varphi_2(a) \} \subseteq [n + m] \), we have \( \sum_{a \in [n]} |\{ \varphi_1(a), \varphi_2(a) \}| \leq n + m \), so the number of these sets which have 2 elements is at most \( m \); in other words,

\[
|\{a \in [n] : \varphi_1(a) \neq \varphi_2(a)\}| \leq m.
\]

Let

\[
D = \{ a \in [n] : \varphi_1(a) = \varphi_2(a) \},
\]

\[
E = [n] \setminus D,
\]

\[
X_{12} = \varphi_1[D] = \varphi_2[D],
\]

\[
X_1 = \varphi_1[E],
\]

\[
X_2 = \varphi_2[E],
\]

\[
X_\emptyset = [n + m] \setminus (X_{12} \cup X_1 \cup X_2).
\]

Then

\[
[n + m] = X_{12} \cup X_1 \cup X_2 \cup X_\emptyset,
\]

\[
\text{Im} \; \varphi_1 = X_{12} \cup X_1,
\]

\[
\text{Im} \; \varphi_2 = X_{12} \cup X_2,
\]

\[
|X_{12}| = |D| \geq n - m,
\]

\[
|X_1| = |X_2| = |E| \leq m,
\]

\[
|X_{12}| + |X_1| = |D| + |E| = n,
\]

\[
|E| + |X_\emptyset| = (n + m) - (|X_{12}| + |X_1|) = m.
\]

We have that, for every \( A \subseteq [n] \),

\[
\forall a \in A : \varphi_1(a) \in \varphi(A)
\]

(10)

because \( a \in A \Rightarrow \{ a \} \subseteq A \Rightarrow \{ \varphi_1(a) \} = \varphi(\{ a \}) \subseteq \varphi(A) \Rightarrow \varphi_1(a) \in \varphi(A) \). (Equivalently, \( \varphi_1[A] \subseteq \varphi(A) \).) Symmetrically,

\[
\forall a \in [n] \setminus A : \varphi_2(a) \notin \varphi(A)
\]

(11)

because \( a \in [n] \setminus A \Rightarrow A \subseteq [n] \setminus \{ a \} \Rightarrow \varphi(A) \subseteq \varphi([n] \setminus \{ a \}) = [n + m] \setminus \{ \varphi_2(a) \} \Rightarrow \varphi_2(a) \notin \varphi(A) \).

For an \( A \subseteq [n] \), let

\[
F(A) = \begin{cases} 
\varphi_2(a) & \text{if } a \in A, \\
\varphi_1(a) & \text{if } a \notin A : a \in E
\end{cases}
\cup X_\emptyset.
\]

By (10), (11) and (11), we have

\[
\varphi(A) \cap ([n + m] \setminus F(A)) = \varphi(A) \cap \left( X_{12} \cup \begin{cases} 
\varphi_1(a) & \text{if } a \in A, \\
\varphi_2(a) & \text{if } a \notin A : a \in E
\end{cases} \right)
\]

(12)

\[
= \varphi(A) \cap \begin{cases} 
\varphi_1(a) & \text{if } a \in A, \\
\varphi_2(a) & \text{if } a \notin A : a \in [n]
\end{cases} = \varphi_1[A],
\]

10
and therefore
\[ |φ(A) \cap ([n + m] \setminus F(A))| = |φ_1[A]| = |A|. \tag{13} \]

Note that \(|F(A)| \stackrel{0}{=} m\). The elements of \(F(A)\) are the only elements of \([n + m]\) such that \(m\) does not determine whether they are elements of \(φ(A)\). In particular,
\[ F(A) \subseteq [n + m] \setminus φ_1[A]. \tag{14} \]

From (3) and (13), \(φ(A)\) contains no element of \(F(A)\) if \(|A| \leq k - 1\), exactly one if \(k \leq |A| \leq k + 1\), and all elements of \(F(A)\) if \(k + 2 \leq |A|\). For \(A \in \binom{[n]}{k} \cup \binom{[n]}{k+1}\), let \(f(A)\) be the single element of \(φ(A) \cap F(A)\).

**Claim 9.** One of the following holds:

**A1.** There is a \(y \in X_0 \cup X_2\) such that, for every \(A \in \binom{[n]}{k}\), we have \(f(A) = y\). (In fact in this case \(y \in X_0\), since for a \(y \in X_2 \) and \(A \in \binom{[n]}{k}\) we would have \(y \notin F(A)\). We do not use this.)

**A2.** There is a \(y \in X_1\) such that, for every \(A \in \binom{[n]}{k}\) \(\setminus \{φ_1^{-1}(y)\}\), we have \(f(A) = y\). (Note that when \(φ_1^{-1}(y) \notin A\), \(y \in F(A)\) holds by the definition of \(F(A)\).)

First we show that Theorem 6 follows from this claim. We use the constant-weight code \(C \subseteq B\) given by Lemma 8. If \(\text{[A1]}\) holds in Claim 9 then we use the statement (4) in Lemma 8 with the same \(n\) and \(m\) as in (4), \(X_2 \cup X_0\) in the place of \(Y\), and \(y\) as given by Claim 9. There is a set \(C \in (X_2 \cup X_1)\) such that \(C \cup \{y\} \in C \subseteq B\). Then \(φ_1^{-1}[C] \in Q_n\), and \(φ^{-1}(φ_1^{-1}[C]) = C \cup \{y\} \in B\), contradicting that the image of \(φ\) is in \(R\). If \(\text{[A2]}\) holds in Claim 9 then we use the statement (4) with \(n - 1\) in the place of \(n\), \(m + 1\) in the place of \(m\), \(X_2 \cup X_0 \cup \{y\}\) in the place of \(Y\), and \(y\) as given by Claim 9. There is a set \(C \in \binom{(X_2 \cup X_1) \setminus \{φ_1^{-1}(y)\}}{k}\) such that \(C \cup \{y\} \in C \subseteq B\). Then \(φ_1^{-1}[C] \in \binom{[n]}{k} \setminus \{φ_1^{-1}(y)\}\) \(\subseteq Q_n\), and \(φ^{-1}(φ_1^{-1}[C]) = C \cup \{y\} \in B\), contradicting that the image of \(φ\) is in \(R\).

To prove Claim 9 we need the following.

**Claim 10.** If \(A, B \in \binom{[n]}{k}\) with a symmetric difference of size 2, and \(f(A) \neq f(B)\), then at least one of the following holds:

- \(f(A) = φ_1(b)\) where \(\{b\} = B \setminus A\). (This implies \(b \in E\) and \(f(A) \in X_1\).)
- \(f(B) = φ_1(a)\) where \(\{a\} = A \setminus B\). (This implies \(a \in E\) and \(f(B) \in X_1\).)

**Proof.** Indeed, \(|A \cup B| = k + 1\), so
\[ φ(A \cup B) = φ_1[A \cup B] \cup \{f(A \cup B)\} = φ_1[A \cap B] \cup \{φ_1(a), φ_1(b), f(A \cup B)\}. \tag{15} \]

Furthermore,
\[ φ(A \cup B) \supset φ(A) = φ_1[A] \cup \{f(A)\} = φ_1[A \cap B] \cup \{φ_1(a), f(A)\} \text{ and} \tag{16} \]
\[ φ(A \cup B) \supset φ(B) = φ_1[B] \cup \{f(B)\} = φ_1[A \cap B] \cup \{φ_1(b), f(B)\}. \tag{17} \]

By (15), (16) and (17), we have \(|\{φ_1(a), f(A), φ_1(b), f(B)\}| \leq |\{φ_1(a), φ_1(b), f(A \cup B)\}| = 3\), therefore the elements on the left-hand side of the inequality are not distinct. We know that \(φ_1\) is an injection, and by (14), \(f(S) \notin φ_1[S]\) for any \(S\) of size \(k\) or \(k + 1\). We have assumed \(f(A) \neq f(B)\). It follows that \(f(A) = φ_1(b)\) or \(f(B) = φ_1(a)\). This completes the proof of Claim 10. \qed
We first prove Claim 9 under the condition \( m \leq n - \sqrt{8(n + m) - 15} - 1 \), as the proof is simpler than the proof for arbitrary \( m \). (For large \( n \), this condition holds whenever the ratio of \( m \) and \( n \) is not very close to 1.)

**Proof of Claim 9 when \( m \leq n - \sqrt{8(n + m) - 15} - 1 \).** At the beginning of the proof of Theorem 6, we chose an arbitrary \( k \) between \( \sqrt{8(n + m) - 15} \) and \( n - 1 - \sqrt{8(n + m) - 15} \). Now we will assume that \( k \leq n - m \); this is satisfied by choosing e.g. \( k = \sqrt{8(n + m) - 15} \).

Then, by (8), there exists an \( B \in \binom{[n]}{k} \) such that \( B \subseteq D \). We show that Claim 9 holds with \( y = f(B) \). In fact, since \( B \cap E = \emptyset \), we have \( f(B) \in F(B) = X_\emptyset \), and we show that \( f(A) = f(B) \) for every \( A \in \binom{[n]}{l} \).

Take an \( A \in \binom{[n]}{l} \). We can get from \( B \) to \( A \) by replacing one element at a time, in such a way that we never add an element that is not an element of \( A \), and we never remove an element of \( A \) (whether it is also an element of \( B \), or we have added it). In particular, we never remove an element of \( E \). That is, there is a sequence \( B = B_0, B_1, \ldots, B_l = A \) such that \( |B_i \triangle B_{i+1}| = 2 \) and \( b_i^- \in D \) where \( \{b_i^-\} = B_i \setminus B_{i+1} \). Let \( b_{i+1}^+ \in [n] \) such that \( \{b_{i+1}^+\} = B_{i+1} \setminus B_i \).

We show by induction that \( f(B_i) = f(B) \) for every \( i = 0, \ldots, l \). Assume that \( f(B) = f(B_i) \neq f(B_{i+1}) \). By Claim 10 either \( f(B_i) = \varphi_1(b_{i+1}^+) \) or \( f(B_{i+1}) = \varphi_1(b_i^+) \). The former implies \( f(B_i) = f(B) \in X_1 \), contradicting that it is in \( X_\emptyset \). The latter implies \( b_i^+ \in E \), contradicting that it is in \( D \).

**Proof of Claim 9.** The general form of Claim 9 will be a consequence of the following lemma.

**Lemma 11.** Let \( n, l \in \mathbb{N} \) such that \( n \geq 5 \) and \( 1 \leq l \leq n - 3 \), and let \( X \) and \( Y \) be disjoint sets such that \( |X| = n \). Let \( g : \binom{X}{l} \to X \cup Y \) be a function such that for every \( A \in \binom{X}{l} \), \( g(A) \notin A \); and for every \( A, B \in \binom{X}{l} \) with a symmetric difference of size 2, where \( g(A) \neq g(B) \), at least one of \( \{g(A)\} = B \setminus A \) and \( \{g(B)\} = A \setminus B \) holds. Then there is a \( y \in X \cup Y \) such that \( g(A) = y \) for every \( A \in \binom{X}{l} \).\( y \).

**Proof.** We prove Lemma 11 by induction on \( l \).

If \( l = 1 \), the sets are singletons, and every symmetric difference is of size 2. We define a graph on \( X \): we connect two elements \( a \) and \( b \) if \( g(\{a\}) \neq g(\{b\}) \). This graph is the complement of a graph whose components are complete graphs (with the components defined by the values of \( a \mapsto g(\{a\}) \)). For every \( a, b \in X \) such that \( ab \) is an edge, we have \( g(\{a\}) = b \) or \( g(\{b\}) = a \). Direct the graph such that we have the directed edge \( (a, b) \) when \( g(\{a\}) = b \) (we may direct some edges in both directions).

The out-degree of every vertex is at most 1. Thus the number of edges is at most \( n \). By our assumptions \( n \geq 4 \); the only graphs with these properties on at least 5 vertices (ignoring the directions of the edges) are the empty graph and a star on \( n \) vertices. If it is an empty graph, then \( g(\{a\}) \) is the same for every \( a \in X \) (and it is necessarily in \( Y \)); the statement of Lemma 11 holds with \( y = g(\{a\}) \). If the graph is a star on \( n \) vertices, let \( a \) be the center. Since at most one edge is directed outward from \( a \), all but at most one edge is directed towards \( a \). That is, \( g(\{b\}) = a \) for all but at most one \( b \in X \setminus \{a\} \). Since the leaves of the star are not connected, \( g \) has the same values on them as singletons, so in fact \( g(\{b\}) = a \) for every \( b \in X \setminus \{a\} \), and the lemma holds with \( y = a \).
We use the induction hypothesis with $\tilde{t} = l - 1$, $\tilde{n} = n - 1$, $\tilde{X} = X \setminus \{a\}$, $\tilde{g}(\tilde{A}) = g(\{a\} \cup \tilde{A})$ for $\tilde{A} \in (\tilde{X})^{\tilde{l}}$, and $Y$ unchanged. Note that since $g(\{a\} \cup \tilde{A}) \notin \{a\} \cup \tilde{A}$, in fact $\tilde{g}(\tilde{A}) \in \tilde{X} \cup Y$ and $\tilde{g}(\tilde{A}) \notin \tilde{A}$, so the conditions of the induction hypothesis hold. So there is a $b \in \tilde{X} \cup Y$ such that $g(\tilde{A}) = b$ for every $\tilde{A} \in (\tilde{X})^{\tilde{l}}$; equivalently, $b \in (X \cup Y) \setminus \{a\}$ such that $g(A) = b$ for every $A \in (X)^l$ that contains $a$ but not $b$. If $b$ were in $Y$, then $g(A)$ would be the same for every $A \in (X)^l$ that contains $a$, contradicting our assumption. So $b \in X$. (See Figure 1)

Take a $C \in (X \setminus \{a, b\})$. We show that $g(C) \in \{a, b\}$. Take an arbitrary $c \in C$. $(C \setminus \{c\}) \cup \{a\}$ contains $a$ but not $b$, so $g((C \setminus \{c\}) \cup \{a\}) = b$. Since $|C \triangle ((C \setminus \{c\}) \cup \{a\})| = 2$, either $g(C) = g((C \setminus \{c\}) \cup \{a\}) = b$, or $g(C) = a$, or $g((C \setminus \{c\}) \cup \{a\}) = c$ — but the last option is false.

Now we show that $g(C)$ is the same for every $C \in (X \setminus \{a, b\})$. (See Figure 2) If this is not the case, there are $C, D \in (X \setminus \{a, b\})$ such that $|C \triangle D| = 2$, and $g(C) = a$ but $g(D) = b$. But this implies that $C \setminus D = \{b\}$ or $D \setminus C = \{a\}$, which is impossible because $a, b \notin C, D$.

We already know that $g(A) = b$ for every $A \in (X)^l$ that contains $a$ but not $b$. If $g(C) = b$ for every $C \in (X \setminus \{a, b\})$, then the lemma holds with $y = b$ (see Figure 3). So assume instead that $g(C) = a$ for every $C \in (X \setminus \{a, b\})$ (Figure 4).

Let $B \in (X)^l$ such that it contains $b$ but not $a$. We show that $g(B) = a$. Take two different, arbitrary elements $c, d \in X \setminus (B \cup \{a\})$. (There are at least two such elements because $l \leq n - 3$. See Figure 5) Since $|B \triangle ((B \setminus \{b\}) \cup \{c\})| = 2$, either $g(B) = g((B \setminus \{b\}) \cup \{c\}) = a$, or $g(B) = c$, or $g((B \setminus \{b\}) \cup \{c\}) = b$ — but the last option is false. So if $g(B) \neq a$, then $g(B) = c$. By the same reasoning applied with $d$ in the place of $c$, if $g(B) \neq a$, then $g(B) = d$, a contradiction. So $g(B) = a$ for every $B \in (X)^l$ that contains $b$ but not $a$. Since we already know that $g(C) = a$ for every $C \in (X \setminus \{a, b\})$, this implies that the lemma holds with $y = a$ (see Figure 6). This completes the proof of Lemma 1.

Using Lemma 1, we show Claim 9. Let $l = k$, $X = X_{12} \cup X_1$, $Y = X_\emptyset \cup X_2$, and for a $B \in (X_{12} \cup X_1)^l$, let $g(B) = f(\varphi^{-1}_1[B])$. (Since $\varphi_1$ is an injection and its image
Figure 3: In this case, Lemma 11 holds with $y = b$.

Figure 4: We assume that the pairs marked in Figure 2 are assigned $a$ instead.

Figure 5: We show that $g(B) = a$ for $B = \{b, e\}$, for arbitrary choice of $c \neq a, b$.

Figure 6: Lemma 11 holds with $y = a$.

is $X_{12} \cup X_1$, we have $\varphi_1^{-1}[B] \in \left(\begin{array}{c} n \\ k \end{array}\right) = \text{Dom } f$.) The conditions of Lemma 11 hold by Claim 10. By Lemma 11 there is a $y \in [n + m]$ such that $f(\varphi_1^{-1}[B]) = g(B) = y$ for every $B \in \left(\begin{array}{c} X_{12} \cup X_1 \end{array}\right) \setminus \{y\}$ (where $(X_{12} \cup X_1) \setminus \{y\}$ may coincide with $X_{12} \cup X_1$).

If $y \in X_\emptyset \cup X_2$, then for every $A \in \left(\begin{array}{c} n \\ k \end{array}\right)$, we have $\varphi_1[A] \in \left(\begin{array}{c} (X_{12} \cup X_1) \setminus \{y\} \end{array}\right) = \left(\begin{array}{c} X_{12} \cup X_1 \end{array}\right) \setminus \{y\}$, and $f(A) = f(\varphi_1^{-1}[\varphi_1[A]]) = y$, so $A1$ holds in Claim 9. If $y \in X_{12} \cup X_1$, then for every $A \in \left(\begin{array}{c} n \\ (\varphi_1^{-1}[\varphi_1[A]]) \end{array}\right)$, we have $\varphi_1[A] \in \left(\begin{array}{c} (X_{12} \cup X_1) \setminus \{y\} \end{array}\right)$, and $f(A) = f(\varphi_1^{-1}[\varphi_1[A]]) = y$. Since $f(A) \in F(A) \subseteq X_1 \cup X_2 \cup X_\emptyset$, we also have $y \in X_1$, so $A2$ holds in Claim 9.

3.3 A probabilistic construction showing $R_w(Q_m, Q_n) \geq m + n + 1$ when $m \geq 3$

Theorem 12. If $n, m \in \mathbb{N}$, $n$ is sufficiently large, and $m \geq 3$, then there exist $\mathcal{B}, \mathcal{R} \subset Q_{n+m}$ such that $\mathcal{B} \cup \mathcal{R} = Q_{n+m}$, $Q_m$ is not a weak subposet of $\mathcal{B}$, and $Q_n$ is not a weak subposet of $\mathcal{R}$.

In most of this subsection, we prove Theorem 12. The core of the random construction will be in Claim 13. In the proof of Claim 13 we will use the asymmetric version of the Lovász Local Lemma.

Lemma 13 (Asymmetric Lovász Local Lemma). Let $A$ be a collection of events. For $A \in \mathcal{A}$, let $\Gamma(A)$ be the set of those events in $\mathcal{A}$, other than $A$ itself, that are not independent of $A$. If there is a function $x : \mathcal{A} \rightarrow [0, 1)$ such that for every $A \in \mathcal{A}$, we have

$$P(A) \leq x(A) \prod_{B \in \Gamma(A)} (1 - x(B)),$$

(18)
then there is a non-zero probability that none of the events occur.

Claim 14. If \( n, m \in \mathbb{N} \), \( n \) is sufficiently large, and \( 3 \leq m \leq n \), then there is a family of sets \( \mathcal{F} \subset \binom{[n+m]}{m} \) such that

(i) for each \( S \in \binom{[n+m]}{m-1} \), \( \mathcal{F} \) contains at least 2 supersets of \( S \), and

(ii) for each \( T \in \binom{[n+m]}{m+1} \), \( \mathcal{F} \) contains at most \( m - 1 \) subsets of \( T \).

Proof. Let \( p = (4(m + 1)(n^2 - 1)e)^{-1/m} \). Let \( \mathcal{F} \) be a collection of sets given by taking each set \( F \in \binom{[n+m]}{m} \) independently at random with probability \( p \).

For any \( S \in \binom{[n+m]}{m-1} \), let \( A_S \) be the event in which \( \mathcal{F} \) contains at most 1 superset of \( S \), and for any \( T \in \binom{[n+m]}{m+1} \), let \( B_T \) be the event in which \( \mathcal{F} \) contains at least \( m \) subsets of \( T \). We have

\[
P(A_S) = (n + 1)(1 - p)^n p + (1 - p)^{n+1} \quad \text{and} \quad P(B_T) = (m + 1)p^m(1 - p) + p^{m+1}.
\]

A given event \( A_S \) is independent of an event of the form \( B_T \) unless there is a set \( F \in \binom{[n+m]}{m} \) such that \( S \subset F \subset T \), i.e., if \( S \subset T \). There are \( \frac{(n+1)m}{2} \) such events \( B_T \). \( A_S \) is not independent of another event \( A_{S'} \) if there is a \( F \in \binom{[n+m]}{m} \) such that \( S, S' \subset F \), i.e., if the symmetric difference of \( S \) and \( S' \) is of size 2. There are \( (m - 1)(n + 1) \) such events \( A_{S'} \). By symmetry, a given event \( B_T \) is independent of all but \( \frac{(m+1)m}{2} \) events \( A_S \), and is independent of all but \( (n - 1)(m + 1) \) other events of the form \( B_{T'} \).

We want to use Lemma 13 to prove that there is a non-zero probability that none of the events \( A_S \) and \( B_T \) occur, and thus \( \mathcal{F} \) fulfills the conditions of Claim 14. Define a function

\[
x : \left\{ A_S : S \in \binom{[n+m]}{m-1} \right\} \cup \left\{ B_T : T \in \binom{[n+m]}{m+1} \right\} \to [0, 1) \quad \text{by}
\]

\[
x(E) = \begin{cases} y := \frac{1}{4(n-1)(n+1)} & \text{if } E = A_S \text{ for some } S \in \binom{[n+m]}{m-1}, \\ z := \frac{1}{4(n-1)(n+1)} & \text{if } E = B_T \text{ for some } T \in \binom{[n+m]}{m+1}. \end{cases}
\]

For an event \( E \) in the domain of \( x \), let \( \Gamma(E) \) be the set of other events that are not independent of \( E \). We will use the bounds

\[
e^{-x} \geq 1 - x \geq e^{-2x}, \tag{19}
\]

which hold when \( 0 \leq x \leq \frac{1}{2} \). For any set \( T \in \binom{[n+m]}{m+1} \), we have

\[
x(B_T) \prod_{E' \in \Gamma(B_T)} (1 - x(E')) = \frac{E' = A_S}{z(1 - y)^{(m+1)m/2} (1 - z)^{(n-1)(m+1)}} \geq ze^{-2(y(m+1)m/2 + z(n-1)(m+1))} > \frac{1}{4(n-1)(n+1)e}.
\]

\[
= (m + 1)p^m > (m + 1)p^m(1 - p) + p^{m+1} = P(B_T).
\]

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For any set $S \in \binom{[n+m]}{m-1}$, we have
\[
x(A_S) \prod_{E' \in \Gamma(A_S)} (1 - x(E')) = y \left(1 - z\right)^{(n+1)n/2} \left(1 - y\right)^{(m-1)(n+1)}
\]
\[
\geq ye^{-2(z(n+1)n/2+y(m-1)(n+1))} > \frac{1}{4(m-1)(n+1)e} \geq \frac{1}{4(n-1)(n+1)e},
\]
and
\[
P(A_S) = (n+1)(1-p)^n + (1-p)^{n+1} < ((n+1)p + 1)(1-p)^n
\]
\[
\leq \left((n+1)p + 1\right) \cdot e^{-pm}
\]
\[
< \left((n+1)(4(m+1)(n^2 - 1)e)^{1/m} + 1\right) \cdot e^{-\left(4(m+1)e\right)^{-1/m} \cdot n^{1-2/m}}.
\]
On the right-hand side of (20), $\left((n+1)(4(m+1)(n^2 - 1)e)^{1/m} + 1\right)$ is increasing in $m$ and $e^{-\left(4(m+1)e\right)^{-1/m} \cdot n^{1-2/m}}$ is decreasing for $m \geq 3$. So, by replacing $m$ with $n$ in the first factor, and $m$ with $3$ in the second factor, we have
\[
P(A_S) \leq \left((n+1)(4(n+1)(n^2 - 1)e)^{1/n} + 1\right) \cdot e^{-\left(16e\right)^{-1/3} \cdot n^{1/3}}
\]
\[
\leq \frac{1}{4(n-1)(n+1)e} < x(A_S) \prod_{E' \in \Gamma(A_S)} (1 - x(E'))
\]
when $n$ is sufficiently large. Therefore the function $x$ satisfies the inequality (18) required by the asymmetric Lovász Local Lemma, so $\mathcal{F}$ has the desired properties.

Now we are ready to prove Theorem 12 using the family of sets constructed in Claim 14. We may assume without loss of generality that $m \leq n$. Let $\mathcal{F} \subset \binom{[n+m]}{m}$ be the family of sets given by Claim 14. Let $\mathcal{B} = \binom{[n+m]}{0} \cup \binom{[n+m]}{1} \cup \ldots \cup \binom{[n+m]}{m-2} \cup \mathcal{F} \cup \binom{[n+m]}{m+1}$, and let $\mathcal{R} = Q_{n+m} \setminus \mathcal{B}$.

Assume that $\mathcal{B}$ contains a weak copy of $Q_m$ provided by the injection $\varphi : Q_m \rightarrow \mathcal{B}$. Note that $\mathcal{B}$ has height $m + 1$ as a poset. Therefore, for $A \in Q_m$, $|\varphi(A)| = m$ if $|A| = m - 1$, and $|\varphi(A)| = m + 1$ if $A = [m]$. The $m$ sets of size $m - 1$ in $Q_m$ are mapped to subsets of $\varphi([m])$ in $\mathcal{F} = \binom{[n+m]}{m} \setminus \mathcal{B}$. But, by (ii) in Claim 14, only at most $m - 1$ subsets of $\varphi([m])$ are in $\mathcal{F}$, a contradiction.

Similarly, assume that $\mathcal{R}$ contains a weak copy of $Q_n$ provided by the injection $\varphi : Q_n \rightarrow \mathcal{R}$. Note that $\mathcal{R}$ has height $n + 1$. Therefore, for $A \in Q_n$, $|\varphi(A)| = m - 1$ if $A = \emptyset$, $|\varphi(A)| = m$ if $|A| = 1$, and $|\varphi(A)| = |A| + m$ if $|A| \in \{2, 3, \ldots, n\}$. The $n$ singletons of $Q_n$ are mapped to supersets of $\varphi(\emptyset)$ in $\binom{[n+m]}{m} \cap \mathcal{R} = \binom{[n+m]}{m} \setminus \mathcal{F}$. But, by (i) in Claim 14, at least 2 supersets of $\varphi(\emptyset)$ are in $\mathcal{F}$, so at most $n - 1$ are in $\binom{[n+m]}{m} \setminus \mathcal{F}$, a contradiction.

**Remark.** The above proof of Theorem 12 cannot be easily made to work for $m = 2$. More precisely, the following claim holds.

**Claim 15.** The conclusion of Claim 14 does not hold for $m = 2$, and in fact there is no $\mathcal{F} \subset \binom{[n+2]}{2}$ such that, for $\mathcal{B} = \emptyset \cup \mathcal{F} \cup \binom{[n+2]}{3}$ and $\mathcal{R} = Q_{n+2} \setminus \mathcal{B}$, $Q_2$ is not a subposet of $\mathcal{B}$, and $Q_n$ is not a subposet of $\mathcal{R}$.
Proof. A family of sets $F \subset \binom{[n+2]}{2}$ that satisfies the condition (i) in Claim 14 contains, for any $S \in \binom{[n+2]}{1}$, a pair of sets $A, B$ such that $S \subset A, B$. Note that $A$ and $B$ have a symmetric difference of size $2$. Then $A, B \subset A \cup B \in \binom{[n+2]}{3}$, which contradicts the condition (ii) in Claim 14. So the two conditions of Claim 14 cannot be satisfied at the same time by a family of sets $F \subset \binom{[n+2]}{2}$.

It is easy to check that the conditions (i) and (ii) on $F$ in Claim 14 are not only sufficient, but also necessary for the above coloring to satisfy the conditions of Theorem 12, that is, to have no $Q_2$ as a subposet of $B = \{\emptyset\} \cup F \cup \binom{[n+2]}{3}$, and no $Q_n$ as a subposet of $R = Q_{n+2} \setminus B$. \hfill \Box

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