Perturbations and absorption cross-section of infinite-radius black rings

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We study scalar field perturbations on the background of non-supersymmetric black rings and of supersymmetric black rings. In the infinite-radius limit of these geometries, we are able to separate the wave equation, and to study wave phenomena in its vicinities. In this limit, we show that (i) the non-supersymmetric case is stable against scalar field perturbations, (ii) the low energy absorption cross-section for scalar fields is equal to the area of the event horizon in the supersymmetric case, and proportional to it in the non-supersymmetric situation.

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I. INTRODUCTION

In four dimensions, asymptotically flat black hole spacetimes have two universal features: their horizons have spherical topology and their properties are uniquely fixed by their conserved charges. This is no longer true for black holes in (asymptotically flat) higher dimensions. For instance, in five dimensions, the Einstein theory allows not only the existence of Kerr-like black holes with topology $S^3$, the Myers-Perry black hole [1], but also black holes with topology $S^1 \times S^2$, found by Emparan and Reall and dubbed as rotating black rings [2]. In five dimensions, a black hole can rotate along two distinct planes. The rotating black ring of [2] has angular momentum only along $S^1$. Setting one of the angular momenta equal to zero in the Myers-Perry black hole, there is an upper bound for the ratio between the angular momentum $J$ and the mass $M$ of the black hole, $J^2/M^3 \leq 32/(27\pi)$, while for the black ring there is a lower bound in the above ratio, $J^2/M^3 \geq 1/\pi$, i.e., there is a minimum rotation that is required in order to prevent the black ring from collapsing. Now, what is quite remarkable is that for $1/\pi \leq J^2/M^3 \leq 32/(27\pi)$ there are spherical black holes and black rings with the same values of $M$ and $J$. This clearly shows that the uniqueness theorems of four dimensions cannot be extended to non-static black holes in five dimensions. One of the parameters that characterizes the black ring is its radius, $R$. When this radius goes to zero the black ring reduces to the Myers-Perry black hole, while in the infinite-radius limit it yields a boosted black string. The black ring solution can be extended in order to include electric charge $\mathcal{Q}$ [3], as well as magnetic dipole charge $\mathcal{P}$ (the existence of this last solution was suggested in [4]). The most general known seven-parameter family of black ring solutions, that includes the above solutions as special cases, was presented in [5]. It is characterized by three conserved charges, three dipole charges, two unequal angular momenta, and a parameter that measures the deviation from the supersymmetric configuration. It is generally believed that these non-supersymmetric solutions are classically unstable. The process of Penrose extraction was analyzed in the black ring background [6], and a ultrarelativistic boost of the black ring was considered in [7]. Recently, a different black ring solution with angular momentum along one of the axis of $S^2$ was found in [8], but it has conical pathologies.

Bena and Kraus conjectured that supersymmetric black rings should also exist [9, 10]. Indeed, the first example of a supersymmetric black ring that is a solution of five-dimensional minimal supergravity was found by Elvang, Emparan, Mateos and Reall [11], using the method developed in [12], and soon after, Gauntlett and Gutowski found a solution that describes a set of concentric supersymmetric black rings [13]. These discoveries triggered the research of more general supersymmetric black ring solutions [14-27]. These studies, together with [14], progressively confirmed that upon oxidation to higher dimensions supersymmetric black rings
become equivalent to another class of solutions known as supertubes [28-34], as first suggested in [35]. It is now well established that the more general supersymmetric black ring can be obtained by dimensional reduction of an eleven dimensional supertube solution of M-theory with three independent charges and three independent dipole charges, that consists of three orthogonal M2-branes that carry the conserved charges and three stacks of M5-branes whose number gives the three dipole charges. This solution can be Kaluza-Klein reduced to a solution of type IIA supergravity and then dualized to a type IIB supergravity solution. The solution then describes a three charge D1-D5-P system with D1, D5 and Kaluza-Klein monopole dipoles. So, the most general supertube is specified by three charges, three dipole moments and by the radius of the black ring. It reduces to the simplest supersymmetric black ring of [13] in the case of three equal charges and three equal dipoles; in the zero-radius limit it reduces to the BMPV supersymmetric black hole solution with spherical topology [33]; and in the infinite-radius limit it yields the supersymmetric black string solution found earlier in [12]. The supersymmetric black rings are expected to be stable. It has been also conjectured that horizonless three-charge supertubes might account for the microstates of supersymmetric five dimensional black holes [36-40]. A statistical counting of the Bekenstein-Hawking entropy of super-symmetric rings was provided in [23, 41], and recently further progress on the understand of entropy properties of the black ring system has been achieved [42, 43, 44]. For recent detailed reviews of the black ring system see, e.g., [19, 45].

In this paper we will consider scalar wave perturbations both in non-supersymmetric (Sec. II) and supersymmetric black ring (Sec. III) backgrounds. It seems impossible to separate the scalar wave equation in the black ring (supersymmetric or not) background. Nevertheless, in the infinite-radius limit of the non-supersymmetric black ring, which yields a boosted black string, the wave equation does separate (Sec. III D). We will then compute the absorption cross-section of a scalar wave that impinges on a boosted black string (Sec. III E). Finally, we will also show that in the infinite-radius limit of the supersymmetric black ring with three equal charges and three equal dipole moments, which yields a supersymmetric black string, the wave equation can be separated (Sec. III D). We will then compute the absorption cross-section of a scalar wave that impinges on a supersymmetric black string (Sec. III E).

II. SCALAR PERTURBATIONS IN THE BLACK RING AND IN THE BOOSTED BLACK STRING

A. The black ring

The five-dimensional rotating black ring found by Emparan and Reall [2] is (here we write the metric in the form displayed in [4] after using the results of [46])

\[ ds^2 = -\frac{F(x)}{F(y)}\left(dt + R\sqrt{\lambda}\nu(1 + y)d\psi\right)^2 + \frac{R^2}{(x-y)^2}\left[-F(x)\left(G(y)d\psi^2 + \frac{F(y)}{G(y)}dy^2\right)\right. \]

\[ + F(y)^2\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right), \]

(1)

where

\[ F(\xi) = 1 - \lambda \xi, \quad G(\xi) = (1 - \xi^2)(1 - \nu \xi). \]

(2)

The coordinate \( x \) varies between \(-1 \leq x \leq 1\). In order to avoid conical singularities the period of the angular coordinates \( \phi \) and \( \psi \) must be given by

\[ \Delta \phi = \Delta \psi = \frac{2\pi \sqrt{1+\lambda}}{1+\nu}, \]

(3)

and the parameters \( \lambda \) and \( \nu \) must satisfy the relation

\[ \lambda = \frac{2\nu}{1+\nu^2}, \quad 0 < \nu < 1. \]

(4)

This condition guarantees that the rotation of the ring balances the gravitational self-attraction of the ring. The black ring has a curvature singularity at \( y = 1/\lambda \). The regular event horizon is at \( y = 1/\nu \). The ergosphere is located at \( y = \pm \infty \) (these two points are identified). The solution is asymptotically flat with the spatial infinity being located at \( x = -1 \) and \( y = -1 \).

B. The wave equation of the black ring

The evolution of a scalar field \( \Phi \) is governed by the curved space Klein-Gordon equation

\[ \frac{\partial}{\partial \nu^\mu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial \nu^\nu} \Phi\right) = 0, \]

(5)

where \( g \) is the determinant of the metric [40]. Following the proposal of [47-48] for the C-metric, let us try the following wavefunction ansatz

\[ \Phi = (x-y)\Psi(x,y)e^{-i(\omega t - m_\phi \phi - m_\psi \psi)}. \]

(6)

1 The motivation to try the ansatz used by [47, 48] in the C-metric solution comes from the fact that the black ring can be constructed by Wick-rotating the electrically charged Kaluza-Klein C-metric [2]. For a recent discussion on the C-metric see, e.g., [51].
For an equilibrium black ring the period of \( \phi \) and \( \psi \) must be given by (3). Consequently, \( m_\phi \) and \( m_\psi \) must have the general form

\[
m_\phi, m_\psi = \frac{1 + \nu}{\sqrt{1 + \lambda}} n, \tag{7}
\]

with \( n \) an integer number. Using the ansatz (3) we get

\[
0 = \frac{\partial}{\partial x} \left( F(x)G(x) \frac{\partial}{\partial x} \Psi \right) - \frac{\partial}{\partial y} \left( F(y)G(y) \frac{\partial}{\partial y} \Psi \right) + \Psi \left[ \frac{F(y)^2}{G(y)} \left( m_\phi + R \sqrt{\lambda} \nu \omega (1 + y) \right)^2 + \frac{R^2 F(y)^2 \omega^2}{(x - y)^2} - \frac{m_\phi^2 F(x)^2}{G(x)} \right] + \Psi \left[ \frac{(F(x)G(x))^\prime + (F(y)G(y))^\prime}{x - y} + 2 \frac{F(y)G(y) - F(x)G(x)}{(x - y)^2} \right]. \tag{8}
\]

Using the relation

\[
\frac{(F(x)G(x))^\prime + (F(y)G(y))^\prime}{x - y} + 2 \frac{F(y)G(y) - F(x)G(x)}{(x - y)^2} = \nu(x - y) - 2 \lambda \nu (x^2 - y^2)
\]

in (8) we have

\[
0 = \frac{\partial}{\partial x} \left( F(x)G(x) \frac{\partial}{\partial x} \Psi \right) - \frac{\partial}{\partial y} \left( F(y)G(y) \frac{\partial}{\partial y} \Psi \right) + \Psi \left[ \frac{F(y)^2}{G(y)} \left( m_\phi + R \sqrt{\lambda} \nu \omega (1 + y) \right)^2 + \frac{R^2 F(y)^2 \omega^2}{(x - y)^2} \right] + \Psi \left[ - \frac{m_\phi^2 F(x)^2}{G(x)} + \nu(x - y) - 2 \lambda \nu (x^2 - y^2) \right]. \tag{10}
\]

Unfortunately, the above simplification is not enough to allow the separability of this wave equation: the \( R^2 F(y)^2 \omega^2/(x - y)^2 \) term seems to prevent separation. Separability is possible only for two special situations: the first is the time-independent or static perturbation case \( (\omega = 0) \). This case will be discussed in appendix A. The other case is the infinite-radius limit, \( R \to \infty \), of the black ring. This limit yields a boosted black string and in the next subsections we will study wave perturbations in this background.

**C. The boosted black string**

There is a special limit of the black ring, namely the infinite-radius limit, for which it is possible to separate the wave equation. We will do this separation in Sec. 11 and then, in Secs. 11E and 11F, we shall study wave propagation phenomena in this limit of the black ring, with the hope that the main qualitative features of propagation in this background limit are also valid for the general black ring.

The infinite-radius limit is defined by taking

\[
R \to \infty, \quad \lambda \to 0, \text{ and } \nu \to 0, \tag{11}
\]

in the black ring solution (11), and keeping \( R \lambda \) and \( R \nu \) constants. In particular, take

\[
R \lambda = r_H \cosh^2 \sigma, \quad R \nu = r_H \sinh^2 \sigma, \tag{12}
\]

and introduce the following coordinates

\[
r = -\frac{R F(y)}{y}, \quad \cos \theta = x, \quad \varpi = R \psi. \tag{13}
\]

Then the black ring solution (11) goes over to the so-called boosted black string solution

\[
ds^2 = -\bar{f} \left( dt - \frac{r_H \sinh 2 \sigma}{2 \bar{f}} d\varpi \right)^2 + \frac{\bar{f}}{f} d\varpi^2 + \frac{1}{f} dr^2 + r^2 d\Omega_2^2, \tag{14}
\]

where

\[
d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2, \quad f = 1 - \frac{r_H}{r}, \quad \bar{f} = 1 - \frac{r_H \cosh^2 \sigma}{r}. \tag{15}
\]

The name boosted black string comes from the fact that this solution can also be constructed by applying a Lorentz boost with boost angle \( \sigma \) to the geometry: 4-dimensional Schwarzschild \( \times \mathbb{R} \) (3). For an equilibrium black ring that satisfies (3), in the infinite-radius limit one has \( \lambda = 2 \nu \) and the boost angle is given by \( \tanh \sigma = 1/\sqrt{2} \). Our results will however apply to a general \( \sigma \).

In the above limit the points \( y = 1/\lambda, y = 1/\nu, y = \pm \infty \) and \( y = -1 \) of the black ring have, respectively, a direct correspondence to the points \( r = 0, r = r_H, r = r_H \cosh^2 \sigma \), and \( r = \infty \) of the boosted black string. These points represent the curvature singularity \( (r = 0) \), the regular event horizon \( (r = r_H) \), the static limit of the ergosphere \( (r = r_H \cosh^2 \sigma) \), and the asymptotic infinity \( (r = \infty) \) of the boosted black string.

Three important parameters of the boosted black string are the temperature of its horizon \( T_H \), the linear velocity of the horizon \( V_H = -\frac{\partial \bar{f}}{\partial r} \bigg|_{r=0} \), and the area per unit length of the horizon \( A_H \), given by

\[
T_H = (4 \pi r_H \cosh \sigma)^{-1}, \quad V_H = - \tanh \sigma, \quad A_H = 4 \pi r_H^2 \cosh \sigma. \tag{16}
\]

Note also that \( A_H T_H = r_H \).

**D. The wave equation of the boosted black string**

The evolution of a scalar field \( \Phi \) in the background of (13) is governed by the curved space Klein-Gordon equation (5). Using the ansatz

\[
\Phi = \Psi(r)e^{-i(\omega t + k \varpi)} Y_{lm}(\theta, \phi), \tag{17}
\]
where $Y_{lm}$ are the usual spherical harmonics, we get the following equation for the radial wavefunction $\Psi$ (see also \ref{radial_eq} for a similar wave equation)

$$
\left[ \frac{r^2 \Delta}{f} \omega^2 - \Delta(l+1) - \frac{r^2}{4f} (\omega r_H \sinh 2\sigma + 2krf)^2 \right] \Psi \\
+ \Delta \partial_r (\Delta \partial_r \Psi) = 0,
$$

where $\Delta \equiv r^2 f = r^2 - r_H r$. At this point it is worth noting that this radial wave equation for the boosted black string can also be obtained directly from the wave equation of the black ring \cite{boosted_black_string}, after applying to it the coordinate change \cite{coordinate_transform} plus the identification $\Psi \rightarrow -y^2 \Psi$. In fact, the coordinate substitution \cite{coordinate_transform} implies that

$$
y = -\frac{R}{r} \tilde{f}^{-1}, \quad F(y) = \tilde{f}^{-1}, \quad G(y) = -\frac{R^2 f}{r^2} \tilde{f}^{-3}.
$$

Then, equation \cite{radial_eq} separates and one recovers \cite{coordinate_transform}: the term preventing separation in \cite{radial_eq}, $R^2 F(y)^2 \omega^2/(x - y)^2$, goes over to $\omega^2 f/\tilde{f}$.

At first glance it seems like there is a singularity at the zero of $\tilde{f}$, in the wave equation \cite{coordinate_transform}. This is not however the case since the potentially dangerous terms cancel out. Indeed it is easy to show that the sum of the two terms proportional to $\tilde{f}^{-1}$ in \cite{coordinate_transform} yields

$$
\frac{r^2 \Delta}{f} \omega^2 - \frac{r^2 \Delta}{4f} \omega^2 \sinh^2 2\sigma = \omega^2 r^3 (r + r_H \sinh^2 \sigma).
$$

We will now show that in the limit of small $\omega$'s it is possible to find a valid solution to equation \cite{coordinate_transform}. We will first compute the scalar absorption cross-section of this geometry (Sec. \ref{absorption_cross_section}), and we will then show that the metric \cite{coordinate_transform} is stable to scalar field perturbations that might develop in the vicinity of the horizon (Sec. \ref{perturbations}).

The method we shall use here, known as matched asymptotic expansions, has been widely used with success for the computation of scattering cross-section of black holes \cite{scattering_cross_section}, and also for computing instabilities of massive fields in the Kerr background \cite{instabilities_kerr}. We will assume that $1/\omega \gg r_H$, i.e., that the Compton wavelength of the scalar particle is much larger than the typical size of the black hole; we divide the space outside the event horizon in two regions, namely, the near-region, $r - r_H \ll 1/\omega$, and the far-region, $r - r_H \gg r_H$. These two regions have a non-zero intersection, namely the region $r_H \ll r - r_H \ll 1/\omega$. Thus, in this overlapping region we can match the near-region and the far-region solutions to get a solution to the problem. This allows us to study reflection coefficients and absorption cross-sections (Sec. \ref{reflection_coefficients}). When the correct boundary conditions are imposed upon the solutions, we shall get a defining equation for $\omega$, and the stability or instability of the spacetime depends basically on the sign of the imaginary component of $\omega$. We will show that the spacetime is stable against scalar perturbations (Sec. \ref{stability}).

E. Absorption cross-section of the boosted black string

In this section we will compute the absorption cross-section and decay rates for a massless scalar field that impinges on a boosted black string. More specifically, we will consider an ingoing wave that is sent from asymptotic infinity towards the black string. During the scattering process, part of the incident flux will be absorbed into the event horizon and the rest will be sent back to infinity. The problem is well defined only for $\omega > k$. Indeed, as we shall see, the system has a potential barrier of height $k$ at the asymptotic infinity. Thus, if we want to send a wave from infinity towards the event horizon, it will have to carry a frequency higher than the barrier.

1. The near region solution

First, let us focus on the near-region in the vicinity of the horizon, $r - r_H \ll 1/\omega$. We assume $\omega r_H \ll 1$ and define the new variable

$$
z = 1 - \frac{r_H}{r}.
$$

We have $\Delta = r_H^2 z/(1 - z)^2$, $\Delta \partial_r = r_H z \partial_z$, and the horizon is at $z = 0$. Therefore, neglecting the term $\omega^2 r^2 \Delta / f$ (this is a good approximation as long as $l \neq 0$) and remembering that $\sinh 2\sigma = 2 \sin \sigma \cosh \sigma$, the radial wave equation \cite{coordinate_transform} is written as

$$
z(1 - z) \partial_z^2 \Psi + (1 - z) \partial_z \Psi \\
+ \left[ \frac{l(l+1)}{1 - z} + \frac{1 - z}{z} \mathcal{Y}^2 \right] \Psi = 0,
$$

where we have defined

$$
\mathcal{Y} \equiv r_H \cosh \sigma (\omega - k \tanh \sigma) \\
= \frac{\omega - k |V_H|}{4\pi T_H},
$$

with $T_H$ and $V_H$ being, respectively, the temperature and the velocity of the horizon of the boosted black string defined in \cite{boosted_black_string}. Through the definition

$$
\Psi = z^l \mathcal{Y} (1 - z)^{|l+1|} F,
$$

the near-region radial wave equation becomes

$$
z(1 - z) \partial_z^2 F + \left[ (1 + i 2 \mathcal{Y}) - \left[ 1 + 2(l+1) + i 2 \mathcal{Y} \right] z \right] \partial_z F \\
- \left[ (l+1)^2 + i 2 \mathcal{Y} (l+1) \right] F = 0.
$$

This wave equation is a standard hypergeometric equation \cite{hypergeometric_eq} of the form

$$
z(1 - z) \partial_z^2 F + [c - (a + b + 1)z] \partial_z F - ab F = 0,
$$
with
\[ a = l + 1 + i 2 \Upsilon, \quad b = l + 1, \quad c = 1 + i 2 \Upsilon. \] (27)

The general solution of this equation in the neighborhood of \( z = 0 \) is \( \lambda z^l e^{\eta r} F\left(a - c + 1, b - c + 1, 2 - c, z\right) + B F\left(a, b, c, z\right) \). Using (24), one finds that the most general solution of the near-region equation is
\[ \Psi = A z^{-i \Upsilon} (1 - z)^{l+1} F\left(a - c + 1, b - c + 1, 2 - c, z\right) + B z^{i \Upsilon} (1 - z)^{l+1} F\left(a, b, c, z\right). \] (28)

The first term represents an ingoing wave at the horizon \( z = 0 \), while the second term represents an outgoing wave at the horizon. We are working at the classical level, so there can be no outgoing flux across the horizon, and thus one sets \( B = 0 \) in (28). One is now interested in the large \( r, z \to 1 \), behavior of the ingoing near-region solution. To achieve this aim one uses the \( z \to 1 - z \) transformation law for the hypergeometric function [55], and thus in the large \( \chi \) domain of the far-region. Now, when \( \chi \to +\infty \), one has \( U(\tilde{a}, \tilde{b}, \chi) \sim \chi^{-\tilde{b}} \), and thus the far-region behavior of the near-region solution is then given by
\[ \Psi \sim A l \Gamma(1 - i 2 \Upsilon) \left[ \frac{r^l}{\Gamma(l + 1) \Gamma(l + 1 - i 2 \Upsilon)} + \frac{r^{l+1}}{\Gamma(l) \Gamma(-l - i 2 \Upsilon)} \right]. \] (30)

2. The far region solution

In the far-region, \( r - r_H \gg r_H \), the wave equation (18) reduces to
\[ \partial^2_r (r \Psi) + \left[ \omega^2 - k^2 + \frac{r_H}{r} (\omega \sin \sigma - k \cosh \sigma)^2 - \frac{l(l + 1)}{r^2} \right] (r \Psi) = 0. \] (31)

Defining
\[ \eta^2 \equiv k^2 - \omega^2, \quad \rho \equiv \frac{r_H (\omega \sin \sigma - k \cosh \sigma)^2}{2 \eta}, \quad \chi = 2 \eta r, \] (32)
then equation (31) becomes
\[ \partial^2_{\chi} (\chi \Psi) + \left[ -\frac{1}{4} + \rho \frac{1}{\chi} - \frac{l(l + 1)}{\chi^2} \right] (\chi \Psi) = 0. \] (33)

This is a standard Whittaker equation [55], \[ \partial^2_{\chi} W + \left[ -\frac{1}{4} + \frac{\rho}{\chi} - \frac{l(l + 1)}{\chi^2} \right] W = 0, \] with
\[ W = \chi R, \quad \mu = l + 1/2. \] (34)

The most general solution of this equation is \( W = \chi^{\mu+1/2} e^{-\chi/2} [\alpha M(\tilde{a}, \tilde{b}, \chi) + \beta U(\tilde{a}, \tilde{b}, \chi)] \), where \( M \) and \( U \) are Whittaker's functions with \( \tilde{a} = 1/2 + \mu - \rho \) and \( \tilde{b} = 1 + 2 \mu \). In terms of the parameters that appear in (33) one has
\[ \tilde{a} = l + 1 - \rho, \quad \tilde{b} = 2 l + 2. \] (35)

The far-region solution of (33) is then given by
\[ \Psi = \alpha \chi^{l} e^{-\chi/2} M(\tilde{a}, \tilde{b}, \chi) + \beta \chi^{l} e^{-\chi/2} U(\tilde{a}, \tilde{b}, \chi). \] (36)

In the absorption cross-section problem that we are dealing with in this section, an incident wave coming from infinity is scattered by the black string and part of it is reflected back. The boundary condition at infinity includes both ingoing and outgoing waves and \( \alpha, \beta \) has these two contributions. We will need to identify the contribution coming from the ingoing wave and the one due to the outgoing wave at spatial infinity, in the large \( \chi \) domain of the far-region. Now, when \( \chi \to +\infty \), one has \( U(\tilde{a}, \tilde{b}, \chi) \sim \chi^{-\tilde{b}} \) and \( M(\tilde{a}, \tilde{b}, \chi) \sim \chi^{\tilde{b} - 1} e^{\chi \Gamma(\tilde{b} - 1)}/\Gamma(\tilde{b} - 1) \) [55], and thus in the large \( \chi = 2 \eta r \) regime, the far-region solution behaves as
\[ \Psi \sim \alpha \frac{(2 l + 1) \chi^{-l - 1}}{\Gamma(l + 1 - \rho)} \chi^{-1 - l + \rho} e^{\chi/2} + \beta \chi^{-1 + l + \rho} e^{-\chi/2}. \] (37)

To compute the fluxes, it will be important to note that the first term proportional to \( e^{\chi/2} \) represents an ingoing wave, while the second term proportional to \( e^{-\chi/2} \) represents an outgoing wave\(^2\).

To do the matching of the far and near regions, we will also need to know the small \( \chi \) behavior, \( \chi \to 0 \), of the far-region solution. In this regime one has \( U(\tilde{a}, \tilde{b}, \chi) \sim \chi^{\tilde{b} - 1} \Gamma(\tilde{b} - 1)/\Gamma(\tilde{b}) \) and \( M(\tilde{a}, \tilde{b}, \chi) \sim 1 \) [57]. The small \( \eta r \) behavior of the far-region solution is then given by
\[ \Psi \sim \alpha (2 \eta r)^l + \beta \frac{(2 l + 1) \chi^{l-1}}{\Gamma(l + 1 - \rho)} (2 \eta r)^{-l - 1}. \] (38)

\(^2\) To justify this statement first note that in the absorption cross-section problem the frequency of the incident wave must be greater than the potential barrier term, i.e., in (33) one has \( \eta^2 = k^2 - \omega^2 < 0 \). We choose \( \eta = -i \sqrt{\omega^2 - k^2} \) and thus \( \chi = -i \sqrt{\omega^2 - k^2} \). Now, from (31) one has \( \Phi \propto \Psi e^{i \omega t} \), and thus \( e^{\chi/2 - i \omega t} = e^{-i (\eta r + \omega t)} \) represents an ingoing wave while the term \( e^{-\chi/2 - i \omega t} = e^{i (\eta r - \omega t)} \) describes an outgoing wave.
3. Matching conditions. Absorption cross-section

When \( r_H \ll r - r_H \ll 1/\omega \), the near-region solution and the far-region solution overlap, and thus one can match the large \( r \) near-region solution \( 37 \) with the small \( \eta r \) far-region solution \( 35 \). This matching yields

\[
\alpha = \frac{\Gamma(2l + 1) \Gamma(1 - i2\Upsilon)}{\Gamma(l + 1) \Gamma(l + 1 - i2\Upsilon)} \frac{(2\eta)^{l - 1}}{r_H} A, \\
\beta = \frac{\Gamma(l + 1 - \rho) \Gamma(-2l - 1) \Gamma(1 - i2\Upsilon)}{\Gamma(2l + 1) \Gamma(-l) \Gamma(-l - i2\Upsilon)} \frac{(2\eta)^{l + 1}}{r_H} A, 
\]

where one has \( \beta \ll \alpha \), since \( \eta \ll 1 \).

We are now in position to compute the relevant fluxes for the absorption cross-section. The conserved flux associated to the radial wave equation \( 18 \) is

\[
F = \frac{2\pi}{l} (\Psi^* \Delta \partial_r \Psi - \Psi \Delta \partial_r \Psi^*). 
\]

The incident flux from infinity is computed using the incoming contribution of the large \( \eta r \) far-region solution \( 37 \), i.e., \( \Psi_{\text{inc}} = \alpha \chi^{-1-\rho} e^{i\chi/2} \Gamma(2l + 2) / \Gamma(l + 1 - \rho) \). The complex conjugate \( \Psi_{\text{inc}}^* \) is found noting that \( \eta = -i|\eta| \) implies, from \( 39 \), that \( \rho = i|\rho| \). The derivatives of the wave solution are calculated using \( \chi = 2\eta r \), and thus \( \partial_r = 2\eta \partial_\chi \). After taking the limit \( r \to \infty \), the incident flux from infinity is given by

\[
F_{\text{inc}} = \frac{\pi}{(\omega^2 - k^2)^{1/2}} \frac{[\Gamma(2l + 2)]^2}{\Gamma(l + 1 - \rho) \Gamma(l + 1 + \rho)} |\alpha|^2. 
\]

The flux absorbed by the boosted black string, i.e., the ingoing flux across the horizon is computed using \( 28 \) with \( B = 0 \) (recall that \( r \to r_H \) corresponds to \( z \to 0 \)). The derivatives of the wave solution are taken using \( \Delta \partial_r = r_H \partial_z \partial_z \). After taking the limit \( z \to 0 \) and noting that in this limit \( F(a, b, c, z) = [F(a, b, c, z)]^* \to 1 \), the flux across the horizon is given by

\[
F_{\text{abs}} = A_H (\omega - k |V_H|) |A|^2. 
\]

where the area per unit length \( A_H \) of the horizon of the boosted black string is given in \( 19 \).

The absorption cross-section is given by \( \sigma_{\text{abs}} = |\pi/((\omega^2 - k^2))| F_{\text{abs}} / F_{\text{inc}} \). The factor \( \pi/((\omega^2 - k^2)) \) converts the partial wave cross-sections into plane wave cross-sections. Explicitly, the absorption cross-section is then given by

\[
\sigma_{\text{abs}} = A_H (\omega - k |V_H|) N, 
\]

where \( N \) is

\[
N = (\omega^2 - k^2)^{-1/2} \frac{(2A_H T_H)^2}{\Gamma(2l + 2)} \frac{\Gamma(l + 1 - \rho) \Gamma(l + 1 + \rho)}{[\Gamma(2l + 2)]^2} \times \frac{[\Gamma(l + 1)]^2}{\Gamma(2l + 1)} \frac{\Gamma(l + 1 - i2\Upsilon)}{\Gamma(1 - i2\Upsilon)}^2. 
\]

Using the properties of the gamma function, \( 131 \) and \( 132 \) one finds that \( N \) is given by

\[
N = (\omega^2 - k^2)^{-1/2} \frac{(2A_H T_H)^2}{[(2l)! (2l + 1)!]^2} \frac{\pi |\rho|}{\sinh(\pi |\rho|)} \times \prod_{j=1}^l (j^2 + |\rho|^2) (j^2 + 4\Upsilon^2). 
\]

The absorption cross-section \( 133 \) of a boosted black string has some features that deserve a closer look:

(i) The factor \( N \) is a well-behaved real positive factor as long as \( \omega > k \). For \( \omega < k \) and \( \sigma_{\text{abs}} \) are pure imaginary numbers. Physically, we can understand this behavior by noting that the quantum number that specifies the momentum of wave along the boosted direction provides a natural potential height \( k \) at infinity. Therefore, if we want to send a wave from infinity towards the boosted black hole, its frequency must be higher than the height of the potential.

So the absorption cross-section problem is only well posed when \( \omega > k \), and \( 133 \) is always positive. The above requirement together with the fact that the boost velocity satisfies \( |V_H| = \tanh \sigma \leq 1 \), implies that the factor \( (\omega - k |V_H|) \) in \( 133 \) is always positive. This means that the potential barrier at infinity prevents the existence of superradiant scattering in the boosted black string background. This peculiar behavior was first studied \( 8 \), and sets a significant difference between the wave scattering on a rotating Myers-Perry black hole and on a boosted black string. Indeed, in a Myers-Perry black hole the absorption cross-section is proportional to a power of \( t \) times \( (\omega - k \Omega_H) \), where \( \Omega_H \) is the angular velocity at the horizon. In this case there is no potential barrier at infinity, and thus there is no lower bound for the frequency of the wave that we can send from infinity towards the black hole. This allows the existence of a superradiant regime: for \( \omega < k \Omega_H \), the absorption cross-section of the Myers-Perry black hole is negative, i.e., during the scattering process in the ergosphere, the scalar wave extracts rotational energy from the black hole and the amplitude of the scattered wave is bigger than the one of the incident wave.

(ii) For black holes that have horizons with spherical topology it is a known universal result that the low energy absorption cross-section of a massless scalar wave is given by the area of the black hole horizon \( 56 \). We may ask if this is still the case for black objects with topology \( 3 \times S^2 \), as is the case of the boosted black string. One first notes that the quantum numbers \( l \) and \( k \) are independent, and thus choosing \( l = 0 \) imposes no restrictions on the value of \( k \). So, inserting \( l = 0 \) in \( 133 \) one concludes that the low absorption cross-section is not given.
simply by the area of the horizon, it also depends on $\omega$ and $k$.

(iii) As some trivial checks to our expression \( \sigma_{\text{abs}} \), we note that when we set the boost parameter $\sigma$ equal to zero one gets the cross-section of a black string (i.e., Schwarzschild times $R$). Moreover, if we also set $k = 0$, we recover the absorption cross-section of the Schwarzschild black hole.

Given the absorption cross-section we can also compute the decay rate,

$$
\Gamma_{\text{decay}} = \frac{\sigma_{\text{abs}}}{e^{\frac{\omega}{2r_m}} - 1} = e^{-\frac{\omega}{2r_m}} \left( \omega^2 - k^2 \right)^{1/2} 2^{2l} |\Gamma(l + 1 - i2\Upsilon)|^2 \\
\times (A_H T_H)^{2l+1} \frac{|\Gamma(l + 1 + \rho)|^2 |\Gamma(l + 1)|^2}{|\Gamma(2l + 2)|^2 |\Gamma(2l + 1)|^2}.
$$

(F) Stability study of the boosted black string

In this section we will show that a boosted black string is stable against massless field perturbations. We will first do an analytical study of the wave equation, in a certain regime, and then we will solve it numerically. Both results agree, in the regime where the analytical calculation holds. One might conjecture that this metric is unstable to superradiant effects (the mechanism at play in this instability was recently described in \[5, 7, 8, 9\]), since it is an extended rotating black object. The massless scalar field acquires an effective mass (due to the extra dimension) and this could lead to superradiant amplification in the ergosphere and consequent instability \[7, 8\]. Nevertheless, we will show that boosted black strings are stable, to massless scalar field perturbations. We still expect the Gregory-Laflamme instability \[8\] to be at work here, and so the geometry should be unstable to gravitational perturbations.

1. Analytical study of the stability

The near region solution:

The results found in Sec. \[11\] are also valid in the present problem. So, the near-region wave equation is given by \[22\]. Requiring only ingoing flux at the horizon $z = 0$, the near-region wave solution that satisfies the physical boundary condition is then given by \[25\] with $B = 0$. For large $r$ the near-region solution behaves as \[30\], that we reproduce here again for the sake of clarity:

$$
\Psi \sim A \Gamma(1 - i2\Upsilon) \left[ \frac{r_H^{-l} \Gamma(2l + 1)}{\Gamma(l + 1) \Gamma(l + 1 - i2\Upsilon)} \right] r^l + \frac{i l + 1}{\Gamma(l) \Gamma(-l - i2\Upsilon)} r^{-l-1}.
$$

The far region solution:

The far-region solution is given by \[38\], subjected to \[39\]. The most general wave solution of this far-region is described by \[30\] and \[35\]. The behavior of the far-region solution in the large $\chi = 2\eta r_H$ regime is given by \[37\], where the first term proportional to $e^{r/x^2/2}$ represents an ingoing wave, while the second term proportional to $e^{-r/x^2}$ represents an outgoing wave.

The stability problem differs significantly from the absorption cross-section problem (that we dealt with in Sec. \[11\]) mainly due to the nature of the boundary conditions that are imposed in the far-region. In the stability problem one perturbs the boosted black string outside its horizon, which generates a wave that propagates both into the horizon and out to infinity. Therefore, at $r \rightarrow +\infty$, our physical system has only an outgoing wave (while the absorption cross-section problem of Sec. \[11\] also had an ingoing wave), and thus one sets $\alpha = 0$ in \[38\]:

$$
\Psi = \beta \chi e^{-r/x^2} U(a, \beta, \chi).
$$

Now, we turn our attention to the small $\chi$ behavior, $\chi \rightarrow 2\eta r_H$, of the far-region solution, which is described by \[38\], with $\alpha = 0$ as justified in the last paragraph. Whittaker’s equation also describes the hydrogen atom system, even though one must be cautious since the boundaries in our case are $\chi \sim 2\eta r_H$ and $\chi = \infty$, while in the hydrogen atom the boundaries are $\chi = 0$ and $\chi = \infty$. The wave equation for the electron wavefunction $\Psi$ in the hydrogen atom is of the type \[38\], and at spatial infinity it is given by \[45\]. The inner boundary condition is that $\Psi$ must be regular at the origin, $\chi \rightarrow 0$. For small values of $\chi$, the solution is described by $\Psi \sim \beta \left[ \Gamma(2l + 1)/\Gamma(l + 1 - \rho) \right] (2\pi)^{l-1/2}$. So, when $\chi \rightarrow 0$, the wavefunction $\Psi$ diverges, $r^{-l-1} \rightarrow \infty$. In order to have a regular solution there we must then demand that $\Gamma(l + 1 - \rho) \rightarrow \infty$. This occurs when the argument of the gamma function is a non-positive integer, $\Gamma(-n) = \infty$ with $n = 0, 1, 2, \cdots$. Therefore, the requirement of regularity imposes the condition $l + 1 - \rho = -n$, in the hydrogen atom. Since $\rho$ is related to $\omega$, the demanding regularity at the inner boundary amounts to a natural selection of the allowed frequencies in the hydrogen atom. Now, let us come back to the boosted black string background. In the spirit of \[5, 9\], we expect that the presence of an horizon induces a small complex imaginary part in the allowed frequencies, $\omega_i = \text{Im}[\omega]$, that describes the slow decay of the amplitude of the wave if $\omega_i < 0$, or the slowly growing instability of the mode if $\omega_i > 0$. Now, from \[52\], one sees that a frequency $\omega$ with a small imaginary part corresponds to a complex $\rho$ with a small imaginary part that we will denote by $\delta \rho \equiv \text{Im}[\rho]$. Therefore, guided by the hydrogen atom, we set that in the boosted black string case one has

$$
\rho = (l + 1 + n) + i\delta \rho,
$$

ically symmetric black holes that have been done previously in the literature. In this case it was also assumed that the general expression for the absorption cross-section could be extended to the $l = 0$ case.
with \( n \) being a non-negative integer, and \( \delta \rho \) being a small quantity. In particular, this means that onwards, the arguments of the Whittaker’s function \( U(\tilde{a}, \tilde{b}, \chi) \) previously defined in (34) are to be replaced by
\[
\tilde{a} = -n - i\delta \rho , \quad \tilde{b} = 2l + 2 .
\]

What we have done so far was to use the hydrogen atom paradigm to guess the form of the eigenfrequencies in our case. At the end, we will verify consistency of all our assumptions. In order to match the far-region with the near-region, we will need to find the small \( \chi \) behavior of the far-region solution (48). The Whittaker’s function \( M(\tilde{a}, \tilde{b}, \chi) \) can be expressed in terms of the Whittaker’s function \( U(\tilde{a}, \tilde{b}, \chi) \). Inserting this relation on (48), the far-region solution can be written as
\[
\Psi = \beta \chi^l e^{-\chi/2} \frac{\pi}{\sin(\pi b)} \left[ \frac{M(\tilde{a}, \tilde{b}, \chi)}{\Gamma(1 + \tilde{a} - \tilde{b}) \Gamma(b)} \right] - \chi^{1-\tilde{b}} \frac{M(1 + \tilde{a} - \tilde{b}, 2 - \tilde{b}, \chi)}{\Gamma(\tilde{a}) \Gamma(2 - \tilde{b})} ,
\]
with \( \tilde{a} \) and \( \tilde{b} \) defined in (50). Now, we want to find the small \( \chi \) behavior of (51), and to extract \( \delta \rho \) from the gamma function. This is done in (53), yielding for small \( \delta \rho \) and for small \( \chi \) the result
\[
\Psi \sim \beta (-1)^n \frac{(2l + 1 + n)!}{(2l + 1)!} \chi^{n+1} (2\eta)^l + \beta (-1)^{n+1} (2l) n! (i\delta \rho) (2\eta)^{-l-1} .
\]

Matching condition:

The quantity \( \delta \rho \) cannot take any value. Its allowed values are selected by requiring a match between the near-region solution (47) and the far-region solution (52). So, the allowed values of \( \delta \rho \) are those that satisfy the matching condition
\[
-\frac{(2l)!}{(2l + 1)!} \chi^{n+1} (2\eta)^l = \frac{\Gamma(l + 1)}{\Gamma(2l + 1)} \frac{\Gamma(-2l + 1) \Gamma(l + 1 - 2\Sigma)}{\Gamma(-l + 2\Sigma)} .
\]

Using (28) and the gamma function relations (30) we get
\[
\delta \rho = -2\Sigma (2\eta H)^{(2l + 1)} \frac{(2l + 1 + n)!}{n!} \left[ \frac{l}{(2l)!} \right] \prod_{j=1}^{l} (j^2 + 4\Sigma^2) .
\]

For \( l = 0 \) this would be
\[
\delta \rho = -4\eta H (n + 1) \Sigma .
\]

Condition (49) together with (32) leads to
\[
\frac{r_H (\omega \sinh \sigma - k \cosh \sigma)^2}{2\sqrt{k^2 - \omega^2}} = l + n + 1 + i\delta \rho ,
\]
with \( \delta \rho \) given by (55). This is therefore an algebraic equation for the characteristic values of the frequency \( \omega \). All these values must be consistent with the assumptions made: \( k \ll 1 \) and \( \omega \ll 1 \). If \( \omega \) has a positive imaginary part, then the mode is unstable: because the field has the time dependence \( e^{-i\omega t} \), a positive imaginary part for \( \omega \) means the amplitude grows exponentially as time goes by. We have performed an extensive search for unstable modes, and didn’t find any. Condition (56) is solved by many values of \( \omega, k, \sigma \), but the only ones satisfying the assumptions seem to be marginally stable modes with \( \omega \approx k \), i.e., purely real frequencies. As we shall see in the next section, these purely real modes are also found numerically.

2. Numerical analysis

The appropriate boundary conditions for a (regular) scalar field evolving in this geometry were described in the previous section: ingoing waves at the horizon, and outgoing waves near infinity. These boundary conditions together with the wave equation (18) form an eigenvalue problem for \( \omega \): only for certain values of \( \omega \) one can satisfy both conditions simultaneously. The characteristic frequencies that do so are called quasinormal frequencies (61). The numerical method used here to compute the quasinormal frequencies is described in more detail in Appendix C.

Using the numerical code, we have looked for stable and unstable modes of the boosted black string. We found only stable modes, which indicates that this boosted black string is stable to scalar perturbations. The numerical results for the stable modes are shown in Figs. 11-13. For each \( k, l, , \sigma \), there is a large number (possibly an infinity) of frequencies that satisfy the boundary conditions. We order them by magnitude of imaginary part. Thus, the mode with lowest imaginary part (in magnitude) is called the fundamental mode and labeled with an integer \( n = 0 \), the mode with the second lowest imaginary part is called the first overtone \( n = 1 \), etc.

For very small values of \( \sigma \) we expect to recover the well known Schwarzschild results (62, 63), since in this limit the boosted black string is just a Schwarzschild black hole, with an extra dimension. Thus, the characteristic frequencies should be equal to the characteristic frequencies of a massive field (with mass \( k \)) in the Schwarzschild spacetime. Now, the values of these frequencies are listed in (62) for massless fields and are computed in (63) for general massive fields. The fundamental frequency of massless scalar fields in the Schwarzschild spacetime is \( \omega r_+ \sim 0.221 - i 0.2098 \) for \( l = 0 \) and \( \omega r_+ \sim 0.5858 - i 0.1954 \) for \( l = 1 \). It is apparent from Figs. 11-13 that for small values of \( k \) and \( \sigma \), the real part and imaginary part do go to the Schwarzschild results. For large values of the boost parameter \( \sigma \) the numerical results show that the real part of the frequency tends
FIG. 1: The real part of the fundamental mode \((n = 0)\) of \(l = 0\) perturbations, as a function of the boost parameter \(\sigma\). Here we show the characteristic frequencies for several values of \(k\). For small values of \(\sigma\) and \(k\), the real part of the frequency yields \(\text{Re}(\omega) \sim 0.22\), which is the Schwarzschild value.

FIG. 2: The imaginary part of the fundamental mode \((n = 0)\) of \(l = 0\) perturbations, as a function of the boost parameter \(\sigma\). For small values of \(\sigma\) and \(k\), the imaginary part of the frequency yields \(\text{Im}(\omega) \sim 0.21\), which is the Schwarzschild value.

FIG. 3: The same as Figs 1 and 2 but for \(l = 1\).

to \(k\), \(\text{Re}(\omega) \sim k\), while the imaginary part goes to zero in a \(k\)-independent manner. This is basically the regime found through the analytical approach in the previous section.

The results for boosted black strings are in a way the same as the “boosted results” of unboosted black strings. We know that unboosted black strings are stable to scalar perturbations \([64]\), and thus one could expect boosted black strings to be trivially stable. However, the actual situation is not as simple, because, for example, when one compactifies after a boost (so the solution changes globally) one gets ergosurfaces that were not there before the boost. This gives rise to new phenomena, such as the Penrose process, etc, that were not present in the unboosted solution. Thus, stability of boosted black strings does not seem to follow trivially from stability of unboosted black strings.

III. SCALAR PERTURBATIONS OF THE SUPERSYMMETRIC BLACK RING AND BLACK STRING

A. The supersymmetric black ring

The supersymmetric black ring with three equal charges and three equal dipole moments has a gravitational field given by \([13]\)

\[
 ds^2 = -f^2 (dt + \omega_\psi d\psi + \omega_\phi d\phi)^2 + f^{-1} ds^2(\mathbb{R}^4), 
\]

where the range of coordinates is \(-1 \leq x \leq 1\), \(-\infty \leq y \leq -1\). The infinity is at \(x = y = -1\), and the event horizon is at \(y = -\infty\). The period of the angular coordinates are \(\Delta \phi = 2\pi\) and \(\Delta \psi = 2\pi\). The constant \(R\) sets the radius of the black ring. The functions appearing in \((57)\) are

\[
 f^{-1} = 1 + \frac{Q - q^2}{2R^2} (x - y) - \frac{q^2}{4R^2} (x^2 - y^2), 
\]

\[
 \omega_\psi = \frac{3}{2} q(1 + y) + \frac{q}{8R^2} (1 - y^2) \left[ 3Q - q^2 (3 + x + y) \right], 
\]

\[
 \omega_\phi = -\frac{q}{8R^2} (1 - x^2) \left[ 3Q - q^2 (3 + x + y) \right]. 
\]

The electromagnetic potential of the supersymmetric black ring is given by

\[
 A = \frac{\sqrt{3}}{2} \left[ f(dt + \omega_\psi d\psi + \omega_\phi d\phi) - \frac{q}{2} [(1 + x) d\phi + (1 + y) d\psi] \right]. 
\]
The constants $Q$ and $q$ are proportional to the total electric charge and to the dipole charge, respectively. One has $-1 \leq x \leq 1$, $-\infty \leq y \leq -1$, so we demand that $Q \geq q^2$ in order to guarantee that $f^{-1} \geq 0$ and that the metric has the correct signature. It is also possible to show that in order to avoid naked closed timelike curves we must demand that $R < (Q - q^2)/(2q)$. If this condition is satisfied the solution can also be extended through $y = -\infty$, and thus this hypersurface is a regular event horizon. Note also that $\omega_\psi(x = \pm 1) = 0$ and $\omega_\psi(y = -1) = 0$ which insures that there are no Dirac-Misner string pathologies. For a detailed account of the properties of this solution see [13, 17].

B. The wave equation of the supersymmetric black ring

We now consider the evolution of a minimal scalar in the geometry (67). The BPS ring solution of minimal 5D SUGRA has no scalars built in, but we can consider generic embeddings into 10D string theory or 11D M theory, thereby getting minimal scalars in the reduced general 5D theory (corresponding for example to a graviton polarized along the internal dimensions [65]). The evolution of a minimal scalar field $\Phi$ is governed by the curved space Klein-Gordon equation (5). We have tried several ansatz, but the one that yields the simplest expression is again the ansatz (6), this time with both $m_\phi$ and $m_\psi$ given simply by an integer number since the periods of the angular coordinates of the supersymmetric black ring are $\Delta \phi = 2\pi$ and $\Delta \psi = 2\pi$. We get the following wave equation

\[ 0 = (x^2 - 1)(x - y)^2(y^2 - 1) f^3 \left[ \frac{\partial}{\partial x} \left( x^2 - 1 \frac{\partial \Psi}{\partial x} \right) \right. \]
\[ \left. - \frac{\partial}{\partial y} \left( y^2 - 1 \frac{\partial \Psi}{\partial y} \right) \right] + \Psi \left[ \omega_\phi^2 R^2 (x^2 - 1)(y^2 - 1) \right. \]
\[ + (x - y)^2 f^3 \left[ - m_\phi^2 (x^2 - 1) + m_\psi^2 (y^2 - 1) \right. \]
\[ + (y^2 - 1) \omega_\phi (2m_\phi \omega + \omega^2 \omega_\phi) \]
\[ + (x^2 - 1) \omega_\psi (2m_\psi - \omega \omega_\psi) \] \hspace{1cm} (68)

This wave equation cannot be separated, at least in these coordinates. However, as occurred with the non-supersymmetric case, in the infinite-radius limit it is possible to do the separation of the wave equation. We turn our attention to this case in the next subsections.

C. The supersymmetric black string

The infinite-radius limit of the five-dimensional supersymmetric black ring yields the supersymmetric black string first found in [12]. To obtain this limit [13], one defines a charge density

\[ \bar{Q} = \frac{Q}{2R}, \] \hspace{1cm} (61)

and the new coordinates

\[ r = \frac{R}{y}, \quad x = \cos \theta, \quad \eta = R \psi. \] \hspace{1cm} (62)

Taking the $R \to \infty$ limit, while keeping $\bar{Q}$, $q$, $r$ and $\eta$ fixed, one gets the gravitational field of the supersymmetric black string. In this infinite radius limit of a (supersymmetric) black ring, the charge of the ring becomes a charge density of the black string, and the dipole charge of the ring becomes a conserved charged of the string. The black string geometry is described by

\[ ds^2 = -f^2 dt^2 + \omega_\phi d\phi^2 + d\eta^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \] \hspace{1cm} (63)

where

\[ f^{-1} = 1 + \frac{\bar{Q}}{r} + \frac{q^2}{4r^2}, \]
\[ \omega_\phi = \frac{3q}{2r} \left( \frac{3q \bar{Q}}{4r^2} + \frac{q^2}{8r^3} \right), \] \hspace{1cm} (64)

and the electromagnetic potential is given by

\[ A = \frac{\sqrt{3}}{2} \left( f dt + \omega_\phi d\phi \right) - \frac{q}{2} \left( 1 + \cos \theta \right) d\phi - \frac{d\eta}{r}. \] \hspace{1cm} (65)

From the discussion of last subsection and using (62), one knows that the event horizon of the solution is at $r = 0$. One has $g_{\eta \eta} = (3q^2)/(\bar{Q}^2 - q^2) + O(r)$, thus in order to avoid closed timelike curves near the horizon (this condition also guarantees absence of closed timelike curves in the full solution) we must demand that $\bar{Q}^2 > q^2$. Moreover, $g_{\eta \eta} = 2r/q + O(r^2)$ and therefore the angular velocity of the event horizon is

\[ \Omega_H = -\frac{g_{\eta x}}{g_{\eta \eta}} \bigg|_{r=0} = 0. \] \hspace{1cm} (67)

A supersymmetric asymptotically flat black object, must have a non-rotating event horizon (this feature was first identified in the case of the BMPV black hole [63]). The supersymmetric black string has a translational invariance along the string direction and is flat in the transverse directions, and its horizon is also not rotating. In particular, this means that there is no ergosphere, and thus we cannot remove kinetic energy from the supersymmetric black string using the Penrose process or a superradiant extraction phenomena.

The area per unit length of the supersymmetric black string horizon is

\[ A_H = \pi q \sqrt{3(\bar{Q}^2 - q^2)}. \] \hspace{1cm} (68)
D. The wave equation of the supersymmetric black string

The evolution of a scalar field $\Phi$ is again dictated by the Klein-Gordon equation, and this time we try the ansatz:

$$\Phi = \frac{\Psi(r)e^{-i(\omega t - mn)}}{r},$$

(69)

which yields the following radial wave equation

$$\frac{\partial^2 \Psi}{\partial r^2} + \left(\frac{\omega^2}{r^3} - (m + \omega \eta)^2\right) \Psi = 0.$$  

(70)

In the next subsection we will find the reflection coefficients and the absorption cross-section of a scalar wave impinging upon a supersymmetric black string from infinity. We will use again the method of matched asymptotic expansions, in which we divide the space outside the event horizon in two regions, namely, the near-region, and the far-region, and then we will match the two solutions in the intersection region.

E. The absorption cross-section of the supersymmetric black string

For simplicity we will consider only the case $m = 0$. In the near region, i.e., in the vicinity of the horizon $r = 0$, the radial wave equation (70) can be written as

$$\frac{\partial^2 \Psi}{\partial r^2} + \left(3q^2(Q^2 - q^2) / 16r^4\right) \Psi = 0.$$  

(71)

The general solution of this equation is

$$\Psi = A r e^{-i\varphi \omega} + B r e^{i\varphi \omega},$$  

(72)

where

$$a = \frac{3q^2(Q^2 - q^2)}{16},$$  

(73)

is a positive real number as a consequence of condition (66). We want only ingoing waves at the horizon so we impose $A = 0$. We also set (this is just a normalization) $B = 1$. For small $a$, the large-$r$ behavior of this solution is

$$\Psi \sim r + i\sqrt{a} \omega.$$  

(74)

On the other hand, in the far region, i.e., for $r \to \infty$, the radial wave equation (70) reduces to the wave equation in the flat background:

$$\frac{\partial^2 \Psi}{\partial r^2} + \omega^2 \Psi = 0,$$  

(75)

which has the plane wave solutions,

$$\Psi = Ce^{i\omega r} + De^{-i\omega r}.$$  

(76)

For small $r$ this solution behaves as

$$\Psi \sim C(1 + i\omega r) + D(1 - i\omega r).$$  

(77)

We are now able to perform a match between (74) and (77), yielding

$$C = i - 1 + \sqrt{a} \omega^2 / 2\omega, \quad D = i 1 + \sqrt{a} \omega^2 / 2\omega.$$  

(78)

For small $\sqrt{a} \omega^2$, we have

$$|C|^2 / |D|^2 \sim 1 - 4\sqrt{a} \omega^2.$$  

(79)

Thus, the reflection coefficient $R \equiv 1 - |C|^2/|D|^2$ is given by

$$R = q \sqrt{3(Q^2 - q^2) \omega^2}.$$  

(80)

We have also computed numerically this reflection coefficient, and the results agreed with (80) to typically within better than $10^{-2}\%$ for small values of $\omega$, $q$, $Q$ (small here means that $\omega Q^2$ and $\omega^2$ are less than $10^{-3}$).

We can also compute the relevant fluxes for the absorption cross-section. The conserved flux associated to the radial wave equation (70) is

$$F = \frac{2\pi}{i} (\Psi^* \partial_r \Psi - \Psi \partial_r \Psi^*).$$  

(81)

The incident wave from infinity is $\Psi_{\text{inc}} = De^{-i\omega r}$, and thus the incident flux from infinity is given by

$$F_{\text{inc}} = 4\pi |D|^2 \omega.$$  

(82)

On the other hand the ingoing wave across the horizon is $\Psi_{\text{abs}} = r e^{i\sqrt{a} \omega}$, and therefore the flux absorbed by the black string horizon is given by

$$F_{\text{abs}} = 4\pi \sqrt{a} \omega.$$  

(83)

The absorption cross-section is given by $\sigma_{\text{abs}} = (\pi / \omega^2) F_{\text{abs}}/F_{\text{inc}}$, where the factor $\pi / \omega^2$ converts the partial wave cross-sections into plane wave cross-sections. Explicitly, the absorption cross-section is then given by

$$\sigma_{\text{abs}} = \frac{4\pi \sqrt{a}}{1 + 2\sqrt{a} \omega^2} \sim A_H.$$  

(84)

where in the last approximation we have assumed small values of $\sqrt{a} \omega^2$, and the fact that $A_H = 4\pi \sqrt{a}$. Thus the absorption cross-section is equal to the area, as occurs generally with spherical black holes (64).

As a check to (66) and (71), we note that the reflection coefficient $R$ can be written as $R = (F_{\text{inc}} - F_{\text{out}}) / F_{\text{inc}} = F_{\text{abs}} / F_{\text{inc}}$. So one has $\sigma_{\text{abs}} = (\pi / \omega^2) R$, as it should be.
IV. CONCLUSIONS

In [47, 48], the authors have shown how to separate the wave equation for massless fields in the C-metric background. Motivated by the fact that the non-supersymmetric black ring solution [2] can be constructed by Wick-rotating the electric charged Kaluza-Klein C-metric, we have tried to separate the wave equation in the black ring background using the same ansatz. Even though the wave equation simplifies considerably, it is still not separable. A similar problem exists in the Hamilton-Jacobi equation for geodesics in this spacetime [67]. While we cannot rule out separability, it is likely that an investigation of waves in this geometry will have to be done using full 2-D numerical simulations.

The only instance where we were able to separate the wave equation in the full geometry was for time-independent perturbations. A study of the static equation indicates that one can anchor a scalar field to a rotating black ring, and this seems to indicate that a black ring can have scalar hair, at least in this perturbative approach.

On the other hand, the problem of wave propagation simplifies considerably in the infinite-radius limit of the non-supersymmetric black ring (which yields a boosted black string), where the equation separates. We computed the absorption cross section of the infinite-radius non-supersymmetric black ring, and we show it is not simply given by the area of the horizon; it also depends on the frequency of the wave and on the height $k$ of the potential barrier at infinity. We have shown that this geometry is stable against scalar perturbations, using both an analytical and a numerical approach. In principle, one might still expect the Gregory-Laflamme instability [60] to be at work here, and so the geometry might be unstable to gravitational perturbations.

We have also studied propagation of scalar waves in the background of a supersymmetric black ring [13]. Again, in the infinite radius limit that yields the supersymmetric black string of [12], we have been able to separate the wave equation. The scalar low energy absorption cross-section is given by the area per unit length of the black string horizon. This value should be reproduced directly by use of correlation functions, once the correct microscopic description of the five-dimensional BPS black rings is understood.

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APPENDIX A: TIME-INDEPENDENT PERTURBATIONS IN THE NON-SUPERSYMMETRIC BLACK RING

An interesting particular situation where the wave equation in the full geometry separates is the static case. For time-independent perturbations, i.e., $\omega = 0$, the wave equation (3) for the non-supersymmetric black ring can be separable. Indeed, in this case (10) yields

$$0 = \frac{\partial}{\partial x} \left( F(x) G(x) \frac{\partial \Psi}{\partial x} \right) - \frac{\partial}{\partial y} \left( F(y) G(y) \frac{\partial \Psi}{\partial y} \right) +$$

$$\Psi \left[ \frac{F(y)^2}{G(y)} m_\psi^2 - \frac{F(x)^2}{G(x)} m_\phi^2 + \nu (x - y - 2\lambda(x^2 - y^2)) \right].$$

(A1)

which is a separable equation. Using the ansatz $\Psi = X(x) Y(y)$ we get

$$\frac{\partial}{\partial x} \left( F(x) G(x) \frac{\partial X}{\partial x} \right) +$$

$$X \left[ - \frac{F(x)^2}{G(x)} m_\phi^2 + \nu x - 2\lambda x^2 \right] = LX \quad (A2)$$

and

$$\frac{\partial}{\partial y} \left( F(y) G(y) \frac{\partial Y}{\partial y} \right) +$$

$$Y \left[ - \frac{F(y)^2}{G(y)} m_\phi^2 + \nu y - 2\lambda y^2 \right] = -LY \quad (A3)$$

where $L$ is a separation constant.

1. The $x$ equation

Near $x = 1$, and for $m_\phi \neq 0$ equation (A2) behaves as

$$X \sim (x - 1) \alpha, \quad (A4)$$

where

$$\alpha = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{m_\phi^2(1 - \lambda)}{(1 - \nu^2)}} \right). \quad (A5)$$

For a black ring in equilibrium satisfying (4) this yields

$$\alpha = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{m_\phi^2}{1 + \nu^2}} \right). \quad (A6)$$
Near $x = -1$ and $m_\phi \neq 0$ we have
\[ X \sim (x + 1)^\beta, \quad (A7) \]
where
\[ \beta = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{m_\phi^2 (1 + \lambda)}{(1 + \nu)^2}} \right). \quad (A8) \]

For $m_\phi = 0$, we have
\[ X \sim A_1 + A_2 \log (x \pm 1), \quad x \to \pm 1. \quad (A9) \]

Since the field must be regular everywhere outside the horizon, one must choose the plus sign equations [A8-A9], and put $A_2 = 0$ in (A9). Therefore, the equation for $X$ is a standard eigenvalue problem, which determines the value of the separation constant $L$.

2. The $y$ equation

Near $y = -1$, the wavefunction $Y$ behaves as equations [A8-A9], with the replacement $m_\phi \to m_\phi$.

Near the horizon $y = 1/\nu$, equation [A9] behaves as
\[ Y \sim (y - 1/\nu)\alpha, \quad (A10) \]
where
\[ \alpha = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{4m_\phi^2 (1 - \lambda/\nu)}{(\nu - 1/\nu)^2}} \right). \quad (A11) \]

For equilibrium black rings this last expression is equal to
\[ \alpha = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{4m_\phi^2 \nu^2}{\nu^4 - 1}} \right). \quad (A12) \]

Using (A10) this yields
\[ \alpha = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{4\nu^2}{\nu^4 - 1}} \right), \quad (A13) \]
where $n$ is an integer. Now, for $1 + \frac{4\nu^2}{\nu^4 - 1} < 0$, which always happens for $\nu$ sufficiently close to 1, the square root is a pure complex number, and any behavior is acceptable. If a scalar field is well behaved at spatial infinity, it will be well behaved at the horizon. Thus, it looks like one can anchor a scalar field to a rotating black ring, and this seems to indicate that a black ring can have scalar hair, at least in this perturbative approach. Notice that this is only possible for $\nu$ close to 1, and thus rotating Myers-Perry black holes are not included in this case.

**APPENDIX B: GAMMA FUNCTION RELATIONS**

In this appendix we derive some gamma functions relations that are used in the main body of the text.

i) We start with the relations needed to make the transition from (41) into (45). Using only the gamma function property, $\Gamma(1 + x) = x\Gamma(x)$, we can show that
\[ \frac{\Gamma(l + 1 - i2\gamma)}{\Gamma(1 - i2\gamma)} = \prod_{j=1}^{l} (j - i2\gamma). \quad (B1) \]

Moreover, using the properties $\Gamma(1 + x) = x\Gamma(x)$ and $\Gamma(1 - ix) = -i\pi / \sinh(\pi x)$, and noting that $\rho = i|\rho|$ yields
\[ \Gamma(l + 1 - \rho)\Gamma(l + 1 + \rho) \]
\[ = \Gamma(i|\rho|) \Gamma(1 - i|\rho|) i|\rho| \prod_{j=1}^{l} (j^2 + \rho^2) \]
\[ = \frac{\pi|\rho|}{\sinh(\pi|\rho|)} \prod_{j=1}^{l} (j^2 + |\rho|^2). \quad (B2) \]

ii) Now, we extract $\delta\rho$ from the gamma functions that appear in (51) in order to get (52).

The properties $M(\tilde{a}, \tilde{b}, \chi) = 0$ and $\Gamma(x)\Gamma(1-x) = \pi / \sin(\pi x)$ with $x = \tilde{b} - \tilde{a}$ and then with $x = \tilde{b} - 1$ allow us to write (51) as
\[ \Psi \sim \beta \chi^l e^{-\chi/2} \left[ \sin[\pi(\tilde{b} - \tilde{a})] \Gamma(\tilde{b} - \tilde{a}) \sin(\pi \tilde{b}) \right] \Gamma(\tilde{b}) + \chi^{1-5} \Gamma(\tilde{b} - 1) \right]. \quad (B3) \]

Use of (50) with $\delta\rho \sim 0$ yields
\[ \frac{\sin[\pi(\tilde{b} - \tilde{a})]}{\sin(\pi \tilde{b})} \frac{\Gamma(\tilde{b} - \tilde{a})}{\Gamma(\tilde{b})} \sim (-1)^n \frac{(2l + 1 + n)!}{(2l + 1)!}. \quad (B4) \]

To simplify the second term in between brackets in (B3), we use $\Gamma(\tilde{a})\Gamma(1-\tilde{a}) = \pi / \sin(\pi \tilde{a})$ with $\tilde{a}$ defined in (51) to get
\[ \frac{1}{\Gamma(-n - i\delta\rho)} \sim \frac{n!}{\pi \sin[\pi(n + i\delta\rho)]} \sim (-1)^{n+1} n! i\delta\rho, \quad (B5) \]

where in the first approximation we used $\delta\rho \sim 0$ to obtain $\Gamma(1+n+i\delta\rho) \sim \Gamma(1+n)$, and in the second approximation we used $\sin(x+y) = \sin x \cos y + \cos x \sin y$ together with $\sin(\pi\delta\rho) \sim i\pi\delta\rho$ (valid for small $\delta\rho$).

The relations (B3-B5) allow us to go from (51) into (52).

iii) Finally, we derive the relations needed to make the transition from (53) into (54). Use of $\Gamma(1 + x) = x\Gamma(x)$ yields
\[ \frac{\Gamma(l + 1 - i2\gamma)}{\Gamma(-l - i2\gamma)} = i (-1)^{l+1} 2\gamma \prod_{j=1}^{l} (j^2 + 4\gamma^2), \]
\[
\frac{\Gamma(-2l - 1)}{\Gamma(-l)} = (-1)^{l+1} \frac{l!}{(2l + 1)!}.
\] (B6)

APPENDIX C: NUMERICAL APPROACH TO THE COMPUTATION OF THE CHARACTERISTIC FREQUENCIES

In this appendix we describe the Frobenius expansion method that is used to obtain the numerical results on the stability presented in Sec. II F. The problem is reduced to the computation of a continued fraction, which is rather easy to implement.\[65\]

The radial equation, \[18\] can be cast in the same form as that presented in \[5\], if we define \( \Psi = \frac{1}{r} \Phi \). In this case \( \Phi \) satisfies

\[
\frac{d^2}{dr^2} \Phi + V \Phi = 0,
\] (C1)

with \( dr/dr_+ = 1 - r_H/r \), and

\[
V = f \left( \frac{\omega^2}{f} - \frac{f'}{r} - \frac{l(l+1)}{r^2} \right) - \frac{r_H}{2r} \left( k + \frac{r_H \sinh \sigma}{2r f} \right)^2.
\] (C2)

We can set \( r_H = 1 \) and measure everything in units of \( r_H \). The wave equation \( (C2) \) has the asymptotic behavior (ingoing waves at the horizon)

\[
\Phi \sim (r - 1)^{-i \omega \cosh \sigma - k \sinh \sigma}, \quad r \to 1.
\] (C3)

Near infinity we have

\[
\Phi \sim e^{i \sqrt{\omega^2 - k^2} r}, \quad r \to \infty.
\] (C4)

The perturbation function \( \Phi \) can be expanded around the horizon as

\[
\Phi = e^{i\omega_H r_H} \{a_\infty + \omega_H (2\omega_\infty)^{-1} (k \cosh \sigma - \omega \sinh \sigma)^2 \}
\times (r - 1)^{-i \omega_H} \sum_{k=0}^{\infty} a_k \left( \frac{r - 1}{r} \right)^k,
\] (C5)

where \( a_0 \) is taken to be \( a_0 = 1 \). Here, \( \omega_H \) and \( \omega_\infty \) have been defined as

\[
\omega_H = \omega \cosh \sigma - k \sinh \sigma, \quad \omega_\infty = \sqrt{\omega^2 - k^2},
\] (C6)

where the branch of \( \sqrt{\omega^2 - k^2} \) has been chosen such that \(-\pi/2 < \arg(\sqrt{\omega^2 - k^2}) \leq \pi/2 \). The expansion coefficients \( a_k \) in equation \( (C5) \) are determined from the three-term recurrence relation, given by

\[
\alpha_0 a_1 + \beta_0 a_0 = 0, \quad \alpha_k a_{n+1} + \beta_k a_n + \gamma_k a_{n-1} = 0 \quad \text{(for } n = 1, 2, 3, \cdots \text{),}
\]

where

\[
\alpha_n = 32(1 + n)(1 + n + 2i k \sinh \sigma - 2i \omega \cosh \sigma) \omega_\infty^2, \quad \beta_n =
\]

\[
-16(k^2 - \omega^2)[-2 + (5 - 3 \cosh 2k) k^2 - 2l(l + 1) - 4n - 4n^2 + 4i \omega \cosh \sigma + 8i \omega \cosh \sigma + 5\omega^2
\]

\[
+ 3\omega^2 \cosh 2 \sigma - 2ik(2 + 4n) \sinh \sigma - 3i \omega \sinh 2 \sigma]
\]

\[
+ 8l((7 + \cosh 2 \sigma + 14n + 2n \cosh 2 \sigma - i \omega \cosh 3 \sigma - 15i \omega \cosh \sigma \omega^2 + i (3 \sinh 3 \sigma - 15 \sinh \sigma) k^2 + \omega^2)
\]

\[
- 14n + 2 \cosh 2 \sigma + 2n \cosh 2 \sigma - 3i \omega \cosh 3 \sigma + 15i \omega \cosh \sigma k^2 + \omega^2) \sinh 2 \sigma + 3i(3 \sinh 3 \sigma + 5 \sinh \sigma \omega^2) \omega_\infty,
\]

\[
\gamma_n =
\]

\[
(-35 + 28 \cosh 2 \sigma - \cosh 4 \sigma) k^4 - 4(16 \sinh \sigma + (14 \sinh 2 \sigma - \sinh 4 \sigma) \omega_\infty^2) k^3 + 4k \omega_\infty^2 (16 \sinh \sigma + (14 \sinh 2 \sigma - \sinh 4 \sigma) \omega_\infty) + (-32 n^2 + 64 \i \omega \cosh \sigma + 2(35 - 3 \cosh 4 \sigma) \omega_\infty^2) k^2 - (32 n^2 + 64 \i \omega \cosh \sigma + (35 + 28 \cosh 2 \sigma + \cosh 4 \sigma) \omega_\infty^2) \omega^2 + 8 \omega_\infty^2 \left( (3 \sinh 3 \sigma - 7 \sinh \sigma) k^3 + \omega^2 \right)
\]

\[
+ \omega^2 (3 \cosh 3 \sigma + 7 \omega \cosh \sigma) k^2 - 2i(3 + \cosh 2 \sigma) n^2 + \omega^2 (3 \cosh 3 \sigma + 7 \cosh \sigma + (4 \i \omega \sinh 2 \sigma + 3 \omega \sinh 3 \sigma + 7 \omega \sinh \sigma) k)\omega_\infty^2
\]

One can see that the expanded eigenfunction \( \Phi \) satisfies the quasinormal boundary conditions at the horizon and at infinity if the series expansion in equation \( (C5) \) converges at spatial infinity. This convergence condition can be converted into an algebraic equation for the frequency \( \omega \) given by the continued fraction equation \[65\].

\[
\beta_0 - \alpha_0 \gamma_1 \left( \frac{\beta_1 - \alpha_1 \gamma_2}{\beta_2 - \alpha_2 \gamma_3} \right)^{-1} = 0.
\] (C7)

This continued fraction equation can be solved numerically \[70\].
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