DERIVING THE POMERON FROM A EUCLIDEAN-MINKOWSKIAN DUALITY

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After a brief review, in the first part, of some relevant analyticity properties of the loop–loop scattering amplitudes in gauge theories, when going from Minkowskian to Euclidean theory, in the second part we shall see how they can be related to the still unsolved problem of the s–dependence of the hadron–hadron total cross–sections.

1 Loop–loop scattering amplitudes

Differently from the parton–parton scattering amplitudes, which are known to be affected by infrared (IR) divergences, the elastic scattering amplitude of two colourless states in gauge theories, e.g., two \( q \bar{q} \) meson states, is expected to be an IR–finite physical quantity. It was shown in Refs.\cite{1,2,3} that the high–energy meson–meson elastic scattering amplitude can be approximately reconstructed by first evaluating, in the eikonal approximation, the elastic scattering amplitude of two \( q \bar{q} \) pairs (usually called “dipoles”), of given transverse sizes \( \vec{R}_{1\perp} \) and \( \vec{R}_{2\perp} \) respectively, and then averaging this amplitude over all possible values of \( \vec{R}_{1\perp} \) and \( \vec{R}_{2\perp} \) with two proper squared wave functions \( |\psi_1(\vec{R}_{1\perp})|^2 \) and \( |\psi_2(\vec{R}_{2\perp})|^2 \), describing the two interacting mesons. The high–energy elastic scattering amplitude of two dipoles is governed by the (properly normalized) correlation function of two Wilson loops \( W_1 \) and \( W_2 \), which follow the classical straight lines for quark \((X(+)) \) and antiquark \((X(-)) \) trajectories:

\[
\mathcal{M}(|\mu|)(s, t; \vec{R}_{1\perp}, \vec{R}_{2\perp}) = -i \left| \vec{q}_\perp \right|^2 \int d^2 \vec{z}_1 e^{i \vec{q}_\perp \cdot \vec{z}_1} \left[ \frac{\langle W_1 W_2 \rangle}{\langle W_1 \rangle \langle W_2 \rangle} - 1 \right] , \tag{1}
\]

where \( s \) and \( t = -\left| \vec{q}_\perp \right|^2 \) are the usual Mandelstam variables. More explicitly the Wilson loops \( W_1 \) and \( W_2 \) are so defined:

\[
W_1^{(T)} \equiv \frac{1}{N_c} \text{Tr} \left\{ P \exp \left[ -ig \oint_{C_1} A_\mu(x) dx^\mu \right] \right\} ,
\]

\[
W_2^{(T)} \equiv \frac{1}{N_c} \text{Tr} \left\{ P \exp \left[ -ig \oint_{C_2} A_\mu(x) dx^\mu \right] \right\} , \tag{2}
\]

where \( P \) denotes the “path ordering” along the given path \( C \) and \( A_\mu = A_\mu^a T^a \); \( C_1 \) and \( C_2 \) are two rectangular paths which follow the classical straight lines for quark \((X(+)) \) (forward in proper time \( \tau \)) and antiquark \((X(-)) \) (backward in \( \tau \)) trajectories, i.e.,

\[
C_1 \to X_\mu^{(\pm 1)}(\tau) = \vec{z}_\mu + \frac{p_1^\mu}{m} \tau \pm \frac{R_{1\mu}}{2},
\]

\[
C_2 \to X_\mu^{(\pm 2)}(\tau) = \frac{p_2^\mu}{m} \tau \pm \frac{R_{2\mu}}{2}, \tag{3}
\]
and are closed by straight–line paths at proper times $\tau = \pm T$, where $T$ plays the role of an IR cutoff, which must be removed at the end ($T \to \infty$). Here $p_1$ and $p_2$ are the four–momenta of the two quarks and of the two antiquarks with mass $m$, moving with speed $\beta$ and $-\beta$ along, for example, the $x^1$–direction:

$$p_1 = m(\cosh \frac{x_1}{2}, \sinh \frac{x_1}{2}, 0, 0),$$
$$p_2 = m(\cosh \frac{x_2}{2}, -\sinh \frac{x_2}{2}, 0, 0),$$

(4)

where $\chi = 2 \text{arctanh}\beta > 0$ is the hyperbolic angle between the two trajectories $(+1)$ and $(+2)$. Moreover, $R_1 = (0, 0, \vec{R}_1 \perp)$, $R_2 = (0, 0, \vec{R}_2 \perp)$ and $z = (0, 0, \vec{z}_\perp)$, where $\vec{z}_\perp = (z^2, \vec{z})$ is the impact–parameter distance between the two loops in the transverse plane.

The expectation values $\langle \tilde{W}_1 \tilde{W}_2 \rangle$, $\langle \tilde{W}_1 \rangle$, $\langle \tilde{W}_2 \rangle$ are averages in the sense of the QCD functional integrals:

$$\langle O[A] \rangle = \frac{1}{Z} \int [dA] \det(Q[A]) e^{iS_A} O[A],$$

(5)

where $Z = \int [dA] \det(Q[A]) e^{iS_A}$, $S_A$ is the pure–gauge (Yang–Mills) action and $Q[A]$ is the quark matrix.

It is convenient to consider also the correlation function of two Euclidean Wilson loops $\tilde{W}_1$ and $\tilde{W}_2$ running along two rectangular paths $\tilde{C}_1$ and $\tilde{C}_2$ which follow the following straight–line trajectories:

$$\tilde{C}_1 \to X^{(\pm 1)}_{E\mu}(\tau) = z_{E\mu} + \frac{p_{1E\mu}}{m} \tau \pm \frac{R_{1E\mu}}{2},$$
$$\tilde{C}_2 \to X^{(\pm 2)}_{E\mu}(\tau) = \frac{p_{2E\mu}}{m} \tau \pm \frac{R_{2E\mu}}{2},$$

(6)

and are closed by straight–line paths at proper times $\tau = \pm T$. Here $R_{1E} = (0, \vec{R}_1 \perp, 0)$, $R_{2E} = (0, \vec{R}_2 \perp, 0)$ and $z_E = (0, \vec{z}_\perp, 0)$. Moreover, in the Euclidean theory we choose the four–vectors $p_{1E}$ and $p_{2E}$ to be:

$$p_{1E} = m(\sin \frac{\theta}{2}, 0, 0, \cos \frac{\theta}{2}),$$
$$p_{2E} = m(-\sin \frac{\theta}{2}, 0, 0, \cos \frac{\theta}{2}),$$

(7)

$\theta \in (0, \pi)$ being the angle formed by the two trajectories $(+1)$ and $(+2)$ in Euclidean four–space.

Let us introduce the following notations for the normalized loop–loop correlators in the Minkowskian and in the Euclidean theory, in the presence of a finite IR cutoff $T$:

$$G_M(\chi; T; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \equiv \frac{\langle \tilde{W}_1^{(T)} \tilde{W}_2^{(T)} \rangle}{\langle \tilde{W}_1^{(T)} \rangle \langle \tilde{W}_2^{(T)} \rangle},$$
$$G_E(\theta; T; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \equiv \frac{\langle \tilde{W}_1^{(T)} \tilde{W}_2^{(T)} \rangle_E}{\langle \tilde{W}_1^{(T)} \rangle_E \langle \tilde{W}_2^{(T)} \rangle_E},$$

(8)
where the expectation values $\langle \ldots \rangle_E$ are averages in the sense of the Euclidean functional integrals:

$$
\langle O[A^{(E)}] \rangle_E = \frac{1}{Z^{(E)}} \int [dA^{(E)}] \det (Q^{(E)}[A^{(E)}]) e^{-S^{(E)}_A} O[A^{(E)}],
$$

$$
Z^{(E)} = \int [dA^{(E)}] \det (Q^{(E)}[A^{(E)}]) e^{-S^{(E)}_A}.
$$

(9)

As already stated in Ref.\textsuperscript{5}, the two quantities in Eq. (8) (with $\chi > 0$ and $0 < \theta < \pi$) are expected to be connected by the same analytic continuation in the angular variables and in the IR cutoff which was already derived in the case of Wilson lines\textsuperscript{5,6,7}, i.e.:

$$
G_E(\theta; T; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) = G_M(\chi \to i\theta; T \to -i\chi; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}),
$$

$$
G_M(\chi; T; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) = G_E(\theta \to -i\chi; T \to i\theta; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}).
$$

(10)

Indeed it can be proved\textsuperscript{8}, simply by adapting step by step the proof derived in Ref.\textsuperscript{5} from the case of Wilson lines to the case of Wilson loops, that the analytic continuation (10) is an exact result, i.e., not restricted to some order in perturbation theory or to some other approximation, and is valid both for the Abelian and the non–Abelian case. This result is derived under the assumption that the function $G_M$, as a function of the complex variable $\chi$, is analytic in a domain $\mathcal{D}_M$ which includes the positive real axis ($\text{Re}\chi > 0, \text{Im}\chi = 0$) and the imaginary segment ($\text{Re}\chi = 0, 0 < \text{Im}\chi < \pi$); and, therefore, the function $G_E$, as a function of the complex variable $\theta$, is analytic in a domain $\mathcal{D}_E = \{\theta \in \mathbb{C} \mid i\theta \in \mathcal{D}_M\}$, which includes the real segment ($0 < \text{Re}\theta < \pi, \text{Im}\theta = 0$) and the negative imaginary axis ($\text{Re}\theta = 0, \text{Im}\theta < 0$). The validity of this assumption is confirmed by explicit calculations in perturbation theory\textsuperscript{6,8,9}. Eq. (10) is then intended to be valid for every $\chi \in \mathcal{D}_M$ (i.e., for every $\theta \in \mathcal{D}_E$).

As we have said above, the loop–loop correlation functions (8), both in the Minkowskian and in the Euclidean theory, are expected to be IR–finite quantities, i.e., to have finite limits when $T \to \infty$, differently from what happens in the case of Wilson lines. One can then define the following loop–loop correlation functions with the IR cutoff removed:

$$
C_M(\chi; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \equiv \lim_{T \to \infty} \left[ G_M(\chi; T; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) - 1 \right],
$$

$$
C_E(\theta; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) \equiv \lim_{T \to \infty} \left[ G_E(\theta; T; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) - 1 \right].
$$

(11)

It has been proved in Ref.\textsuperscript{8} that, under certain analyticity hypotheses in the complex variable $T$ [hypotheses which are also sufficient to make the relations (10) meaningful], the two quantities (11), obtained after the removal of the IR cutoff ($T \to \infty$), are still connected by the usual analytic continuation in the angular variables only:

$$
C_E(\theta; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) = C_M(\chi \to i\theta; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}),
$$

$$
C_M(\chi; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}) = C_E(\theta \to -i\chi; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}).
$$

(12)

This is a highly non–trivial result, whose general validity is discussed in Ref.\textsuperscript{8} where it was also explicitly verified in the simple case of quenched QED, where vacuum polarization effects, arising from the presence of loops of dynamical fermions, are neglected,
and in the case of a non–Abelian gauge theory with \( N_c \) colours, up to the order \( \mathcal{O}(g^4) \) in perturbation theory. Indeed, the validity of the relation (12) has been also recently verified in Ref.\(^8\) by an explicit calculation up to the order \( \mathcal{O}(g^6) \) in perturbation theory.

As said in Ref.\(^8\), if \( G_M \) and \( G_E \), considered as functions of the complex variable \( T \), have in \( T = \infty \) an “eliminable isolated singular point” [i.e., they are analytic functions of \( T \) in the complex region \( |T| > R \), for some \( R \in \mathbb{R}^+ \), and the finite limits (11) exist when letting the complex variable \( T \rightarrow \infty \)], then, of course, the analytic continuation (12) immediately derives from Eq. (10) (with \( |T| > R \), when letting \( T \rightarrow +\infty \). (For example, if \( G_M \) and \( G_E \) are analytic functions of \( T \) in the complex region \( |T| > R \), for some \( R \in \mathbb{R}^+ \), and they are bounded at large \( T \), i.e., \( \exists B_{M,E} \in \mathbb{R}^+ \) such that \( |G_{M,E}(T)| < B_{M,E} \) for \( |T| > R \), then \( T = \infty \) is an “eliminable singular point” for both of them.) But the same result (12) can also be derived under different conditions. For example, let us assume that \( G_E \) is a bounded analytic function of \( T \) in the sector \( 0 \leq \arg T \leq \frac{
abla}{2} \), with finite limits along the two straight lines on the border of the sector: \( G_E \rightarrow G_{E1} \), for (\( \Re T \rightarrow +\infty \), \( \Im T = 0 \)), and \( G_E \rightarrow G_{E2} \), for (\( \Re T = 0 \), \( \Im T \rightarrow +\infty \)). And, similarly, let us assume that \( G_M \) is a bounded analytic function of \( T \) in the sector \( -\frac{
abla}{2} \leq \arg T \leq 0 \), with finite limits along the two straight lines on the border of the sector: \( G_M \rightarrow G_{M1} \), for (\( \Re T \rightarrow +\infty \), \( \Im T = 0 \)), and \( G_M \rightarrow G_{M2} \), for (\( \Re T = 0 \), \( \Im T \rightarrow -\infty \)). We can then apply the “Phragmén–Lindelöf theorem” (see, e.g., Theorem 5.64 in Ref.\(^10\)) to state that \( G_{E2} = G_{E1} \) and \( G_{M2} = G_{M1} \). Therefore, also in this case, the analytic continuation (12) immediately derives from Eq. (10) when \( T \rightarrow \infty \).

2 How a pomeron–like behaviour can be derived

The relation (12) allows the derivation of the loop–loop scattering amplitude (11), which we rewrite as

\[
\mathcal{M}_{(ll)}(s,t; \vec{R}_{1\perp}, \vec{R}_{2\perp}) = -i \, 2s \, \tilde{C}_M(\chi \rightarrow \infty, t; \vec{R}_{1\perp}, \vec{R}_{2\perp}),
\]

(13)

\( \tilde{C}_M \) being the two–dimensional Fourier transform of \( C_M \), with respect to the impact parameter \( \vec{z}_\perp \), at transferred momentum \( \vec{q}_\perp \) (with \( t = -q^2_\perp \)), i.e.,

\[
\tilde{C}_M(\chi, t; \vec{R}_{1\perp}, \vec{R}_{2\perp}) \equiv \int d^2 \vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} C_M(\chi; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}),
\]

(14)

from the analytic continuation \( \theta \rightarrow -i \chi \) of the corresponding Euclidean quantity:

\[
\tilde{C}_E(\theta, t; \vec{R}_{1\perp}, \vec{R}_{2\perp}) \equiv \int d^2 \vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} C_E(\theta; \vec{z}_\perp, \vec{R}_{1\perp}, \vec{R}_{2\perp}),
\]

(15)

which can be evaluated non–perturbatively by well–known and well–established techniques available in the Euclidean theory.

We remind that the hadron–hadron elastic scattering amplitude \( \mathcal{M}_{(hh)} \) can be obtained by averaging the loop–loop scattering amplitude (13) over all possible dipole transverse separations \( \vec{R}_{1\perp} \) and \( \vec{R}_{2\perp} \) with two proper squared hadron wave functions:

\[
\mathcal{M}_{(hh)}(s, t) = \int d^2 \vec{R}_{1\perp} |\psi_1(\vec{R}_{1\perp})|^2 \int d^2 \vec{R}_{2\perp} |\psi_2(\vec{R}_{2\perp})|^2 \mathcal{M}_{(ll)}(s, t; \vec{R}_{1\perp}, \vec{R}_{2\perp}).
\]

(16)
For a detailed description of the procedure leading from the loop–loop scattering amplitude \( M_{(ll)} \) to the hadron–hadron elastic scattering amplitude \( M_{(hh)} \), we refer the reader to Refs. 1, 2, 3, 11. See also Ref. 12 and references therein.

Denoting with \( C_{M}^{(hh)} \) and \( C_{E}^{(hh)} \) the quantities obtained by averaging the corresponding loop–loop correlation functions \( C_{M} \) and \( C_{E} \) over all possible dipole transverse separations \( \vec{R}_{1\perp} \) and \( \vec{R}_{2\perp} \), in the same sense as in Eq. (16), i.e.,

\[
C_{M}^{(hh)}(\chi; \vec{z}_{\perp}) \equiv \int d^{2}\vec{R}_{1\perp}\left|\psi_{1}(\vec{R}_{1\perp})\right|^{2} \int d^{2}\vec{R}_{2\perp}\left|\psi_{2}(\vec{R}_{2\perp})\right|^{2} \times C_{M}(\chi; \vec{z}_{\perp}, \vec{R}_{1\perp}, \vec{R}_{2\perp}),
\]

\[
C_{E}^{(hh)}(\theta; \vec{z}_{\perp}) \equiv \int d^{2}\vec{R}_{1\perp}\left|\psi_{1}(\vec{R}_{1\perp})\right|^{2} \int d^{2}\vec{R}_{2\perp}\left|\psi_{2}(\vec{R}_{2\perp})\right|^{2} \times C_{E}(\theta; \vec{z}_{\perp}, \vec{R}_{1\perp}, \vec{R}_{2\perp}),
\]

(17)

we can write:

\[
M_{(hh)}(s,t) = -i \ 2s \ C_{M}^{(hh)}(\chi \to \infty, t),
\]

(18)

where, as usual:

\[
\tilde{C}_{M}^{(hh)}(\chi, t) \equiv \int d^{2}\vec{z}_{\perp} e^{i\vec{q}_{\perp} \cdot \vec{z}_{\perp}} C_{M}^{(hh)}(\chi; \vec{z}_{\perp}),
\]

\[
\tilde{C}_{E}^{(hh)}(\theta, t) \equiv \int d^{2}\vec{z}_{\perp} e^{i\vec{q}_{\perp} \cdot \vec{z}_{\perp}} C_{E}^{(hh)}(\theta; \vec{z}_{\perp}).
\]

(19)

Clearly, by virtue of the relation (12), we also have that:

\[
\dot{C}_{M}^{(hh)}(\chi, t) = \tilde{C}_{E}^{(hh)}(\theta \to -i\chi, t).
\]

(20)

We also remind that, in order to obtain the correct \( s \)–dependence of the scattering amplitude (18), one must express the hyperbolic angle \( \chi \) between the two loops in terms of \( s \), in the high–energy limit \( s \to \infty \) (i.e., \( \chi \to \infty \)):

\[
cosh \chi = \frac{s - m_1^2 - m_2^2}{2m_1 m_2}, \quad \text{i.e.:} \quad \chi \sim \log \left( \frac{s}{m_1 m_2} \right)
\]

(21)

where \( m_1 \) and \( m_2 \) are the masses of the two hadrons considered.

This approach has been extensively used in the literature in order to address, from a theoretical point of view, the still unsolved problem of the asymptotic \( s \)–dependence of hadron–hadron elastic scattering amplitudes and total cross sections.

For example, in Ref. 13 the loop–loop Euclidean correlation functions have been evaluated in the context of the so–called “loop–loop correlation model” in which the QCD vacuum is described by perturbative gluon exchange and the non–perturbative “Stochastic Vacuum Model” (SVM), and then they have been continued to the corresponding Minkowskian correlation functions using the above–mentioned analytic continuation in the angular variables: the result is an \( s \)–independent correlation function
\( \hat{C}_M(\chi \to \infty, t; \vec{R}_{1\perp}, \vec{R}_{2\perp}) \) and, therefore, a loop–loop scattering amplitude \( a \) linearly rising with \( s \). By virtue of the “optical theorem”,

\[
\sigma_{\text{tot}}^{(hh)}(s) \sim \frac{1}{s} \text{Im} \mathcal{M}_{(hh)}(s, t = 0),
\]

(22)

this should imply (apart from possible \( s \)-dependences in the hadron wave functions!) \( s \)-independent hadron–hadron total cross sections in the asymptotic high–energy limit, in apparent contradiction to the experimental observations, which seem to be well described by a “pomeron–like” high–energy behaviour (see, for example, Ref. \ref{12} and references therein):

\[
\sigma_{\text{tot}}^{(hh)}(s) \sim \sigma_0^{(hh)} \left( \frac{s}{s_0} \right)^{\epsilon_P}, \quad \text{with : } \epsilon_P \simeq 0.08.
\]

(23)

In Refs. \ref{2}, \ref{3} a possible \( s \)-dependence in the hadron wave functions was advocated in order to reproduce the phenomenological “pomeron–like” high–energy behaviour of the total cross sections. However, it would be surely preferable to ascribe the universal high–energy behaviour of hadron–hadron total cross sections (the only dependence on the initial–state hadrons is in the multiplicative constant \( \sigma_0^{(hh)} \) in Eq. (23)) to the same fundamental quantity, i.e., the loop–loop scattering amplitude. (For a different, but still phenomenological, approach in this direction, using the SVM, see Ref. \ref{11}.)

The same approach, based on the analytic continuation from Euclidean to Minkowskian correlation functions, has been also adopted in Ref. \ref{14} in order to study the one–instanton contribution to both the line–line (see also Ref. \ref{15}) and the loop–loop scattering amplitudes: one finds that, after the analytic continuation, the colour–elastic line–line and loop–loop correlation functions decays as \( 1/s \) with the energy. (Instead, the colour–changing inelastic line–line correlation function is of order \( s^0 \) and dominates at high energy. In a further paper \ref{16}, instanton–induced inelastic collisions have been investigated in more detail and shown to produce total cross sections rising with \( s \).)

A behaviour like the one of Eq. (23) seems to emerge directly (apart from possible undetermined \( \log s \) prefactors) when applying the Euclidean–to–Minkowskian analytic–continuation approach to the study of the line–line/loop–loop scattering amplitudes in strongly coupled (confining) gauge theories using the AdS/CFT correspondence \ref{17}, \ref{18}.

(In a previous paper \ref{19} the same approach was also used to study the loop–loop scattering amplitudes in the \( N = 4 \) SYM theory in the limit of large number of colours, \( N_c \to \infty \), and strong coupling.)

It has been also recently proved in Ref. \ref{9} by an explicit perturbative calculation, that the loop–loop scattering amplitude approaches, at sufficiently high energy, the BFKL–pomeron behaviour \ref{20}.

The way in which a pomeron–like behaviour can emerge, using the Euclidean–to–Minkowskian analytic continuation, was first shown in Ref. \ref{6} in the case of the line–line (i.e., parton–parton) scattering amplitudes and can be easily readapted to the case of the loop–loop scattering amplitudes.

One simply starts by writing the Euclidean hadronic correlation function in a partial–wave expansion:

\[
\hat{C}_{E}^{(hh)}(\theta, t) = \sum_{l=0}^{\infty} A_l(t) P_l(\cos \theta),
\]

(24)
which, by making a Sommerfeld–Watson transformation, can be rewritten in the following way:

\[
\tilde{C}_E^{(hh)}(\theta, t) = \frac{1}{2i} \int_C A_l(t) P_l(-\cos \theta) \frac{d\theta}{\sin(\pi l)},
\]

(25)

where “C” is a contour in the complex \(l\)-plane, running anticlockwise around the real positive \(l\)-axis and enclosing all non–negative integers, while excluding all the singularities of \(A_l\). Eq. (25) can be verified after recognizing that \(P_l(-\cos \theta)\) is an integer function of \(l\) and that the singularities enclosed by the contour \(C\) of the expression under integration in the Eq. (25) are simple poles at the non–negative integer values of \(l\). So the right–hand side of (25) is equal to the sum of the residues of the integrand in these poles and this gives exactly the right–hand side of (24). The “minus” sign in the argument of the Legendre function \(P_l\) into Eq. (25) is due to the following relation, valid for integer values of \(l\):

\[
P_l(-\cos \theta) = (-1)^l P_l(\cos \theta).
\]

(26)

Then, we can reshape the contour \(C\) into the straight line \(Re(l) = -\frac{1}{2}\). Eq. (25) then becomes

\[
\tilde{C}_E^{(hh)}(\theta, t) = -\sum_{Re(\alpha_n) > -\frac{1}{2}} \frac{\pi r_n(t) P_{\alpha_n(t)}(-\cos \theta)}{\sin(\pi \alpha_n(t))} - \frac{1}{2i} \int_{-\frac{1}{2} + i\infty}^{\frac{1}{2} - i\infty} A_l(t) P_l(-\cos \theta) \frac{d\theta}{\sin(\pi l)} dl,
\]

(27)

where \(\alpha_n(t)\) is a pole of \(A_l(t)\) in the complex \(l\)-plane and \(r_n(t)\) is the corresponding residue. We have assumed that \(A_l\) vanishes enough rapidly as \(|l| \to \infty\) in the right half–plane, so that the contribution from the infinite contour is zero. Eq. (27) immediately leads to the asymptotic behavior of the scattering amplitude in the limit \(s \to \infty\), with a fixed \(t (|t| \ll s)\). In fact, making use of the analytic extension (20) when continuing the angular variable, \(\theta \to -i\chi\), we derive that

\[
\tilde{C}_M^{(hh)}(\chi, t) = \tilde{C}_E^{(hh)}(-i\chi, t) = -\sum_{Re(\alpha_n) > -\frac{1}{2}} \frac{\pi r_n(t) P_{\alpha_n(t)}(-\cosh \chi)}{\sin(\pi \alpha_n(t))} - \frac{1}{2i} \int_{-\frac{1}{2} + i\infty}^{\frac{1}{2} - i\infty} A_l(t) P_l(-\cosh \chi) \frac{d\theta}{\sin(\pi l)} dl.
\]

(28)

The hyperbolic angle \(\chi\) is linked to \(s\) by the relation (21). The asymptotic form of \(P_\alpha(z)\) when \(|z| \to \infty\) is well known. It is a linear combination of \(z^\alpha\) and of \(z^{-\alpha-1}\). When \(Re(\alpha) > -1/2\), this last term can be neglected. Therefore, in the limit \(s \to \infty\), with a fixed \(t (|t| \ll s)\), we are left with the following expression:

\[
\tilde{C}_M^{(hh)}(\chi \to \infty, t) \sim \sum_{Re(\alpha_n) > -\frac{1}{2}} \beta_n(t)s^{\alpha_n(t)}.
\]

(29)
The integral in Eq. (28), usually called the *background term*, vanishes at least as $1/\sqrt{s}$. Eq. (29) allows to immediately extract the scattering amplitude according to Eq. (18):

$$M_{(hh)}(s, t) = -i \, 2s \, \tilde{C}_M^{(hh)}(\chi \to \infty, t) \sim -2i \sum_{\text{Re}(\alpha_n) > -1/2} \beta_n(t)s^{1+\alpha_n(t)}.$$  

(30)

This equation gives the explicit $s$–dependence of the scattering amplitude at very high energy ($s \to \infty$) and small transferred momentum ($|t| \ll s$). As we can see, this amplitude comes out to be a sum of powers of $s$. This sort of behavior for the scattering amplitude was first proposed by Regge and $1 + \alpha_n(t)$ is often called a “Regge pole”.

In the original derivation, the asymptotic behavior was recovered by analytically continuing to very large imaginary values the angle between the trajectories of the two exiting particles in the $t$–channel process. Instead, in our derivation, we have used the Euclidean–to–Minkowskian analytic continuation and we have analytically continued the Euclidean correlator to very large (negative) imaginary values of the angle $\theta$ between the two Euclidean Wilson loops. As in the original derivation, we have assumed that the singularities of $A_l(t)$ in the complex $l$–plane (at a given $t$) are only simple poles in $l_n = \alpha_n(t)$. If there are other kinds of singularities, different from simple poles, their contribution will be of a different type and, in general, also logarithmic terms (of $s$) may appear in the amplitude.

Denoting with $\bar{\alpha}(t)$ the pole with the largest real part (at that given $t$), we thus find that:

$$\tilde{C}_M^{(hh)}(\chi, s \to \infty) \log \left( s \over m_1 m_2 \right), t) \sim \beta(t) s^{\beta(t)}.$$  

(31)

This implies, for the hadron–hadron elastic scattering amplitude (30), the following high–energy behaviour:

$$M_{(hh)}(s, t) = -i \, 2s \, \tilde{C}_M^{(hh)}(\chi \to \infty, t) \sim -i \, 2\beta(t)s^{1+\beta(t)},$$  

(32)

and, therefore, by virtue of the optical theorem:

$$\sigma_{\text{tot}}^{(hh)}(s) \sim \frac{1}{s} \text{Im} M_{(hh)}(s, t = 0) \sim \sigma_0^{(hh)} \left( \frac{s}{s_0} \right)^{\epsilon_P}, \quad \text{with : } \epsilon_P = \text{Re}[[\pi(0)].$$  

(33)

We want to stress two important issues which clarify under which conditions we have been able to derive this *pomeron*–like behaviour for the elastic amplitudes and the total cross sections.

i) We have ignored a possible energy dependence of hadron wave functions and we have thus ascribed the high–energy behaviour of the Minkowskian hadronic correlation function exclusively to the *fundamental* loop–loop correlation function. With this hypothesis, the coefficients $A_l$ in the partial–wave expansion and, as a consequence, the coefficients $\beta_n$ and $\alpha_n$ in the Regge expansion do not depend on $s$, but they only depend on the Mandelstam variable $t$.

ii) However, this is not enough to guarantee the experimentally–observed universality (i.e., independence on the specific type of hadrons involved in the reaction) of the *pomeron* trajectory $\bar{\alpha}(t)$ in Eq. (32) and, therefore, of the *pomeron* intercept $\epsilon_P$ in Eq. (33). In fact, the partial–wave expansion of the hadronic correlation function can be
considered, by virtue of Eqs. (17) and (19), as a result of a partial–wave expansion of the fundamental loop–loop Euclidean correlation function (15), i.e.,

$$\tilde{C}_E(\theta, t; \vec{R}_{1\perp}, \vec{R}_{2\perp}) = \sum_{l=0}^{\infty} A_l(t; \vec{R}_{1\perp}, \vec{R}_{2\perp}) P_l(\cos \theta), \quad (34)$$

which is then averaged with two proper squared hadron wave functions:

$$\tilde{C}_E^{\text{(hh)}}(\theta, t) = \int d^2 \vec{R}_{1\perp} |\psi_1(\vec{R}_{1\perp})|^2 \int d^2 \vec{R}_{2\perp} |\psi_2(\vec{R}_{2\perp})|^2 \tilde{C}_E(\theta, t; \vec{R}_{1\perp}, \vec{R}_{2\perp}). \quad (35)$$

If we now repeat for the partial–wave expansion (34) the same manipulations that have led us from Eq. (24) to Eq. (29), we arrive at the following Regge expansion for the loop–loop Minkowskian correlator:

$$\tilde{C}_M(\chi \rightarrow \infty, t; \vec{R}_{1\perp}, \vec{R}_{2\perp}) \sim \sum_{\text{Re}(a_n) > -\frac{1}{2}} b_n(t; \vec{R}_{1\perp}, \vec{R}_{2\perp}) s^{a_n(t; \vec{R}_{1\perp}, \vec{R}_{2\perp})}, \quad (36)$$

where $a_n(t; \vec{R}_{1\perp}, \vec{R}_{2\perp})$ is a pole of $A_l(t; \vec{R}_{1\perp}, \vec{R}_{2\perp})$ in the complex $l$–plane. After inserting the expansion (36) into the expression for the Minkowskian hadronic correlation function, i.e.,

$$\tilde{C}_M^{\text{(hh)}}(\chi, t) = \int d^2 \vec{R}_{1\perp} |\psi_1(\vec{R}_{1\perp})|^2 \int d^2 \vec{R}_{2\perp} |\psi_2(\vec{R}_{2\perp})|^2 \tilde{C}_M(\chi, t; \vec{R}_{1\perp}, \vec{R}_{2\perp}), \quad (37)$$

one in general finds a high–energy behaviour which hardly fits with that reported in Eqs. (31) and (32) with a universal pomeron trajectory $\overline{\alpha}(t)$, unless one assumes that, for each given loop–loop correlation function with transverse separations $\vec{R}_{1\perp}$ and $\vec{R}_{2\perp}$, (at least) the position of the pole $a_n(t; \vec{R}_{1\perp}, \vec{R}_{2\perp})$ with the largest real part does not depend on $\vec{R}_{1\perp}$ and $\vec{R}_{2\perp}$, but only depends on $t$. (Maybe this is a rather natural assumption if one believes that the pomeron trajectory is, after all, determined by an even more fundamental quantity, that is the line–line, i.e., parton–parton, correlation function.) If we denote this “common” pole with $\overline{\alpha}(t)$, we then immediately recover the high–energy behaviour (31), where the coefficient in front is given by:

$$\overline{\beta}(t) = \int d^2 \vec{R}_{1\perp} |\psi_1(\vec{R}_{1\perp})|^2 \int d^2 \vec{R}_{2\perp} |\psi_2(\vec{R}_{2\perp})|^2 b_n(t; \vec{R}_{1\perp}, \vec{R}_{2\perp}), \quad (38)$$

and therefore, differently from the universal function $\overline{\alpha}(t)$, explicitly depends on the specific type of hadrons involved in the process.

In conclusion, we have shown that the Euclidean–to–Minkowskian analytic–continuation approach can, with the inclusion of some extra (more or less plausible) assumptions, easily reproduce a pomeron–like behaviour for the high–energy total cross sections. However, we should also keep in mind that the pomeron–like behaviour (23) is, strictly speaking, forbidden (at least if considered as a true asymptotic behaviour) by the well–known Froissart–Lukaszuk–Martin (FLM) theorem (see also (22), (23)), according to which, for $s \rightarrow \infty$:

$$\sigma_{\text{tot}}(s) \leq \frac{\pi}{m_{\pi}^2} \log^2 \left( \frac{s}{s_0} \right), \quad (39)$$
where $m_\pi$ is the pion mass and $s_0$ is an unspecified squared mass scale.

In this respect, the Pomeron–like behaviour $23$ can at most be regarded as a sort of pre–asymptotic (but not really asymptotic!) behaviour of the high–energy total cross sections (see, e.g., Refs. 9,24 and references therein). Immediately the following question arises: why our approach, which was formulated so to give the really asymptotic large–s behaviour of scattering amplitudes and total cross sections, is also able to reproduce pre–asymptotic behaviours [violating the FLM bound (39)] like the one in (23)? The answer is clearly that the extra assumptions, i.e., the models, which one implicitly or explicitly assumes in the calculation of the Euclidean correlation functions $\tilde{C}_E$ play a fundamental role in this respect. This is surely a crucial point which, in our opinion, should be further investigated (and maybe also better formulated) in the future. A great help could be provided by a direct lattice calculation of the loop–loop Euclidean correlation functions. Clearly a lattice approach can at most give (after having overcome a lot of technical difficulties) only a discrete set of $\theta$–values for the above–mentioned functions, from which it is clearly impossible (without some extra assumption on the interpolating continuous functions) to get, by the analytic continuation $\theta \to -i\chi$, the corresponding Minkowskian correlation functions (and, from this, the elastic scattering amplitudes and the total cross sections). However, the lattice approach could provide a criterion to investigate the goodness of a given existing analytic model (such as: Instantons, SVM, AdS/CFT, BFKL and so on ...) or even to open the way to some new model, simply by trying to fit the lattice data with the considered model. This would surely result in a considerable progress along this line of research.

References

1. O. Nachtmann, in Perturbative and Nonperturbative aspects of Quantum Field Theory, edited by H. Latal and W. Schweiger (Springer–Verlag, Berlin, Heidelberg, 1997).
2. H.G. Dosch, E. Ferreira and A. Krämer, Phys. Rev. D 50 (1994) 1992.
3. E.R. Berger and O. Nachtmann, Eur. Phys. J. C 7 (1999) 459.
4. H. Verlinde and E. Verlinde, hep–th/9302104.
5. E. Meggiolaro, Nucl. Phys. B 625 (2002) 312.
6. E. Meggiolaro, Z. Phys. C 76 (1997) 523.
7. E. Meggiolaro, Eur. Phys. J. C 4 (1998) 101.
8. E. Meggiolaro, Nucl. Phys. B 707 (2005) 199.
9. A. Babansky and I. Balitsky, Phys. Rev. D 67 (2003) 054026.
10. E.C. Titchmarsh, The theory of functions, 2nd ed. (Cambridge University Press, London, 1939).
11. A.I. Shoshi, F.D. Steffen and H.J. Pirner, Nucl. Phys. A 709 (2002) 131.
12. S. Donnachie, G. Dosch, P. Landshoff and O. Nachtmann, Pomeron Physics and QCD (Cambridge University Press, Cambridge, 2002).
13. A.I. Shoshi, F.D. Steffen, H.G. Dosch and H.J. Pirner, Phys. Rev. D 68 (2003) 074004.
14. E. Shuryak and I. Zahed, Phys. Rev. D 62 (2000) 085014.
15. A.E. Dorokhov and I.O. Cherednikov, Ann. Phys. 314 (2004) 321.
16. M.A. Nowak, E.V. Shuryak and I. Zahed, Phys. Rev. D 64 (2001) 034008.
17. R.A. Janik and R. Peschanski, Nucl. Phys. B 586 (2000) 163.
18. R.A. Janik, Phys. Lett. B 500 (2001) 118.
19. R.A. Janik and R. Peschanski, Nucl. Phys. B 565 (2000) 193.
20. V.S. Fadin, E.A. Kuraev and L.N. Lipatov, Phys. Lett. B 60 (1975) 50;
   I.I. Balitsky and L.N. Lipatov, Sov. Journ. Nucl. Phys. 28 (1978) 822.
21. T. Regge, Nuovo Cimento 14, 951 (1959); 18, 947 (1960).
22. M. Froissart, Phys. Rev. 123 (1961) 1053;
   A. Martin, Il Nuovo Cimento 42A (1966) 930;
   L. Lukaszuk and A. Martin, Il Nuovo Cimento 47A (1967) 265.
23. W. Heisenberg, Zeitschrift für Physik 133 (1952) 65.
24. A.B. Kaidalov, in \textit{At the frontier of particle physics}, edited by M. Shifman (World
   Scientific, Singapore, 2001), vol. 1, 603–636; \texttt{hep-ph/0103011}. 