Second Moment Polytopic Systems: Generalization of Uncertain Stochastic Linear Dynamics

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Abstract—This article presents a new paradigm to stabilize uncertain stochastic linear systems. Herein, second moment polytopic (SMP) systems are proposed that generalize systems with both uncertainty and randomness. The SMP systems are characterized by second moments of the stochastic system matrices and the uncertain parameters. Further, a fundamental theory for guaranteeing stability of the SMP systems is established. It is challenging to analyze the SMP systems owing to both the uncertainty and randomness. An idea to overcome this difficulty is to expand the SMP systems and exclude the randomness. Because the expanded systems contain only the uncertainty, their stability can be analyzed via robust stability theory. The stability of the expanded systems is equivalent to statistical stability of the SMP systems. These facts provide sufficient conditions for the stability of the SMP systems as linear matrix inequalities (LMIs). In controller design for the SMP systems, the LMIs reduce to cubic matrix inequalities (CMIs) whose solutions correspond to feedback gains. The CMIs are transformed into simpler quadratic matrix inequalities (QMs) that can be solved using optimization techniques. Moreover, solving such nonconvex QMs is relaxed into the iteration of a convex optimization. Solutions to the iterative optimization provide feedback gains that stabilize the SMP systems. As demonstrated here, the SMP systems represent linear dynamics with uncertain distributions and other existing systems such as independently identically distributed dynamics and random polytopes. Finally, a numerical simulation shows the effectiveness of the proposed method.

Index Terms—Linear matrix inequalities (LMIs), robust control, stability of linear systems, stochastic systems, uncertain systems.

I. INTRODUCTION

Uncertainties and randomness occur in various dynamical systems such as semi-autonomous vehicles with human drivers [3] and nanoscale receivers with manufacturing variations [4]. Stabilization of such systems is a crucial task for implementation of control systems in the real world. Thus, this study focuses on stability analysis and controller design for uncertain stochastic linear systems.

Previously, various robust control approaches have been proposed to treat uncertainties in system dynamics. Stability and control of uncertain systems have been discussed using matrix inequalities [5], [6], [7], [8], [9], boundary mapping [10], [11], and variational methods [12]. Probabilistic methods have guaranteed stability of uncertain systems in a probabilistic sense [13], [14], [15]. If the uncertainties are probabilistic, they are treated as time-invariant (TI) stochastic parameters [16], [17]. Such a stochastic representation is efficient for the improvement of statistical control performance. Controller design associated with stochastic parameters reduces to stochastic optimal control problems [16], [17], [18], [19], [20], [21], for which efficient tools such as polynomial chaos expansions [22] have been established.

Further, stability and control of various time-varying (TV) stochastic systems have been discussed. TV stochastic parameters can express multiplicative noises contained in system dynamics [23] rather than external additive noises [24]. Independently identically distributed (i.i.d.) stochastic parameters have been widely analyzed because their Markov properties are tractable in control problems. Stability of systems with i.i.d. parameters has been guaranteed via several approaches such as using stochastic Riccati equations [23], [25] and Kronecker products [26], [27]. Continuous-time cases of i.i.d. parameters are expressed by Wiener processes that have been analyzed [28], [29], [30]. Control problems such as optimal control [23], variance suppression [31], [32], and risk-sensitive control [25], [33] have been addressed for systems with i.i.d. parameters.

A crucial challenge is to treat combination of uncertainties and stochastic parameters, which increases the expressiveness of the systems. The combination has the possibility to represent stochastic parameters obeying uncertain distributions. Such uncertain distributions practically occur and have attracted attention in the field of distributionally robust control [34], [35] in which system stability is not analyzed. Systems containing both TI uncertainty and TV stochastic noise have been considered in [36], [37], [38], and [39]. Unfortunately, their stability is not analyzed in [36] and [37] or is discussed in special cases ignoring the TV noise [38], [39]. Stability and control of random polytopes have been analyzed based on sampling-based matrix inequalities [40] and S-variable approaches [41]. Other systems involving such combinations have also been analyzed [42], [43]. Some variations of stochastic systems are summarized...
in [44]. Although these promising results have combined the uncertainties with the stochastic parameters, their system representations are still limited to specific types.

To generalize systems involving both uncertainties and randomness, this study establishes a fundamental theory to stabilize various types of uncertain stochastic linear systems in a unified manner. The systems are generalized as second moment polytopic (SMP) systems. Herein, stability conditions and controller design methods for the SMP systems are derived. The main contributions of this study are summarized as follows.

1) We present a novel class of uncertain stochastic linear systems, called SMP systems, which can represent various types of systems with both uncertainties and randomness. Although the existing results have treated limited classes such as i.i.d. stochastic systems [23], [25], [26], [27], [31], [32], [33], deterministic polytopic systems [5], [6], [7], [8], and random polytopic systems [1], [2], [40], [41], the SMP systems contain these classes. Furthermore, the SMP systems treat uncertain probability distributions of stochastic system parameters that attract attention in the field of distributionally robust control [34], [35].

2) We show that statistical stability of the SMP systems is equivalent to stability of their expanded systems. The proposed expanded systems are included in deterministic polytopes, which are compatible with existing methods such as [6]. This simplifies the stability analysis of the SMP systems. Our expansion focuses on SMP systems containing both uncertainty and randomness whereas existing expansion techniques have considered specific stochastic systems without uncertainty [26], [27], [28], [29], [30].

3) We derive sufficient conditions for the statistical stability of SMP systems whereas the existing results [1], [6], [7], [8], [23], [26], [27], [28], [29], [30], [40], [41] analyze stability of limited stochastic systems. The conditions are expressed by linear matrix inequalities (LMIs) if the systems are autonomous or controllers are given.

4) We propose a method to design linear feedback controllers that stabilize the SMP systems whereas the existing results [2], [6], [23], [40], [41] design controllers of only limited systems. It is first shown that solutions to quadratic matrix inequalities (QMI) provide the feedback gains. Next, we relax solving the QMIs as an iteration of solving a semidefinite program (SDP) that is convex and easy to solve.

This article is a substantially extended version of our conference papers [1], [2]. This study treats the SMP systems that generalize uncertain stochastic systems although the conference papers consider only random polytopic systems, that is, a special class of the SMP systems. The proposed expanded system for SMP systems is a generalized version of that of the conference papers. Moreover, whereas the paper [2] tackles solving a nonconvex program, this study presents a method to relax such a nonconvex program as an iteration of a convex one. A novel analysis and numerical examples are presented to show the contributions of this study. Moreover, important theoretical points have been reviewed to improve the technical soundness and readability. The rest of this article is organized as follows. Section II provides notations used. Section III outlines SMP systems with problem settings. Section IV discusses the proposed method. Section V presents a demonstrative analysis and numerical examples. Finally, Section VI concludes this article.

II. Notation

This article uses the following notation.

1) \( \mathbb{R}^{n \times m} \): Set of \( n \times m \) real-valued matrices;

2) \( \mathbb{R}^{n} \): Set of \( n \times n \) real-valued symmetric matrices;

3) \( I_n \): \( n \times n \) identity matrix;

4) \([y]_i\): \( i \)th component of a vector \( y \in \mathbb{R}^n \);

5) \([X]_{i,j}\): Component in the \( i \)th row and \( j \)th column of a matrix \( X \in \mathbb{R}^{n \times m} \);

6) \([X]_{i,:}\): \( i \)th column vector of a matrix \( X \in \mathbb{R}^{n \times m} \);

7) \( \text{vec}(X) := \left[ [X]_{1,1}, [X]_{1,2}, \ldots, [X]_{1,m} \right] \): Vectorization of the components of a matrix \( X \in \mathbb{R}^{n \times m} \);

8) \( \text{vech}(X) := \left[ [X]_{1,1}, [X]_{2,2}, \ldots, [X]_{n,n} \right] \): Half vectorization of the lower triangular components of a square matrix \( X \in \mathbb{R}^{n \times n} \);

9) \( X \otimes X_b \in \mathbb{R}^{n_a n_b} \): Kronecker product of matrices \( X_a \in \mathbb{R}^{n_a \times m_a} \) and \( X_b \in \mathbb{R}^{n_b \times m_b} \), given as follows:

\[
X_a \otimes X_b = \begin{bmatrix}
[X_a]_{1,1} X_b & \cdots & [X_a]_{1,m_a} X_b \\
\vdots & \ddots & \vdots \\
[X_a]_{n_a,1} X_b & \cdots & [X_a]_{n_a,m_a} X_b
\end{bmatrix}
\]

10) \( \text{rank}(X) \): Rank of a matrix \( X \in \mathbb{R}^{n \times m} \);

11) \( Y > 0 \) (respectively, \( < 0 \)): the positive (respectively, negative) definiteness of a symmetric matrix \( Y \in \mathbb{R}^{n \times n} \);

12) \( Y \geq 0 \) (respectively, \( \leq 0 \)): the positive (respectively, negative) semidefiniteness of a symmetric matrix \( Y \in \mathbb{R}^{n \times n} \);

13) \( \lambda_i(Y) \): \( i \)th eigenvalue of a symmetric matrix \( Y \in \mathbb{R}^{n \times n} \) such that \( \lambda_1(Y) \geq \lambda_2(Y) \geq \cdots \geq \lambda_n(Y) \);

14) \( \nu_i(Y) \): \( i \)th unit eigenvector corresponding to \( \lambda_i(Y) \) of a symmetric matrix \( Y \in \mathbb{R}_{\text{sym}}^{n} \), i.e., \( Y \nu_i(Y) = \lambda_i(Y) \nu_i(Y) \) and \( \nu_i(Y)^\top \nu_i(Y) = 1 \);

15) \( E[y(v(\theta))] \): Conditional expectation with respect to \( v(\theta) \) obeying a conditional probability density function (PDF) \( p(v(\theta)) \) given \( \theta \) := \( \theta_1, \theta_2, \ldots \). If \( \theta \) is TI, \( E[y(v(\theta))] \) denotes the conditional expectation with \( p(v(\theta)) \). If \( v \) is independent of \( \theta \), \( E[y(v)] \) denotes the expectation with \( p(v) \);

16) \( \text{Cov}[y(v(\theta))]|\theta=\theta_0] := E[y(v(\theta))|\theta=\theta_0]E[y(v(\theta))|\theta=\theta_0]^\top \): Conditional covariance given \( \theta \).

III. SMP Systems With Problem Settings

A. Target Systems Described by Second Moment Polytopes

Consider the following uncertain stochastic linear system:

\[
x_{t+1} = A_t(\theta_t)x_t + B_t(\theta_t)u_t
\]

\[
v_t(\theta_t) := \text{vec} \left( [A_t(\theta_t), B_t(\theta_t)] \right) \sim p(v_t|\theta_t)
\]
where \( x_t \in \mathbb{R}^n, u_t \in \mathbb{R}^m, \) and \( \theta_t \in S_\theta \subset \mathbb{R}^{d_\theta} \) denote the state, control input, and TV uncertain parameter for the discrete time \( t \in \{0, 1, 2, \ldots \} \), respectively, for a given subset \( S_\theta \) and natural number \( d_\theta \). The initial state \( x_0 \) is deterministic, and the uncertain parameter \( \theta_t \) can be stochastic or deterministic. Let \( v_I(\theta_t) \in \mathbb{R}^{n+n+m} \) be the vectorization of the stochastic system matrices \( A_I(\theta_t) \in \mathbb{R}^{n \times n} \) and \( B_I(\theta_t) \in \mathbb{R}^{n \times m} \). The stochastic parameter \( v_I(\theta_t) \) obeys a PDF \( p(v_I(\theta_t)) \) independently with respect to \( t \). The PDF is uncertain because it depends on the uncertain parameter \( \theta_t \).

To characterize this general uncertain stochastic system (1), we propose the notion of a second moment polytope. The notion focuses on the second moment of the stochastic parameter \( v_I(\theta_t) \) that is included in a polytope.

**Definition 1 (SMP systems):** The system (1) is said to be SMP if there exist a positive integer \( N \), vertices \( M^{(k)} \in \mathbb{R}^{n+n+m} \) for \( k \in \{1, 2, \ldots, N\} \), and a function \( \phi : S_\theta \rightarrow \mathbb{P}_N \) that satisfy the following:

\[
\forall t \quad \forall \theta_0, \theta_1, \ldots, \in S_\theta \quad \mathbb{E} \left[ v_I(\theta_t)v_I(\theta_t)^\top \right] = \sum_{k=1}^N [\phi(\theta_t)]_k M^{(k)} \tag{2}
\]

where the codomain \( \mathbb{P}_N \) is the \( N \)-dimensional set as follows:

\[
\mathbb{P}_N := \left\{ \phi \in \mathbb{R}^N \mid \forall k, [\phi]_k \geq 0, \sum_{k=1}^N [\phi]_k = 1 \right\}. \tag{3}
\]

**Definition 2 (TI/TV SMP systems):** An SMP system is said to be TI SMP if \( \theta_t \) is TI, that is, \( \theta_t = \theta \) holds for all \( t \) with a constant \( \theta \). Otherwise, the system is said to be TV SMP.

We introduce one simple example of SMP systems below.

**Example 1 (Simple example of SMP systems):** Suppose that \( v_I(\theta_t), S_\theta, \) and \( N \) satisfy \( \mathbb{E}[v_I(\theta_t)]_{\theta_t} = \mu, \) \( \text{Cov}[v_I(\theta_t)]_{\theta_t} = \sum_{k=1}^{d_{\theta}} [\theta_t]_k \Sigma^{(k)}, S_\theta \subseteq \mathbb{P}_N \), and \( N=d_\theta \), where \( \mu \in \mathbb{R}^{n+n+m} \) and \( \Sigma^{(k)} \in \mathbb{R}^{n+n+m} \) are constants. Then, the system (1) is SMP with \( \phi(\theta_t) = \theta_t \) and \( M^{(k)} = \mu \Sigma^{(k)} + \Sigma^{(k)} \).

To discuss stability and controller design for the system (1), the following assumptions are used throughout this article.

**Assumption 1 (Second moment polytope):**

i) The system (1) is SMP.

ii) Vertices \( M^{(k)} \) and \( N \) satisfying (2) are known although it is not required that the PDF \( p(v_I(\theta_t)) \) is known.

iii) For any \( t \), the values of \( \theta_t \) and \( v_I(\theta_t) \) are unknown.

iv) For any \( t \), the PDF \( p(v_I(\theta_t)) \) is Lebesgue measurable on \( \mathbb{R}^{n+n+m} \) and the following independence holds:

\[
\forall s \geq 1, p(v_0, v_1, \ldots, v_s|\theta_0, \theta_1, \ldots, \theta_s) = \prod_{t=0}^s p(v_I(\theta_t)).
\]

To justify Assumption 1 (i) and (ii), Section V-A presents how to transform various systems into SMP forms. For demonstrated polytopic and/or stochastic systems in Section V-A, the vertices \( M^{(k)} \) can be obtained from the information of system matrices. In general, it is practical to obtain the vertices because the corresponding polytopic form can accept the estimation error of the second moment. In other words, the second moment polytope does not need the exactness of the moment estimation whereas the exact moments are needed in existing analyses of well-known i.i.d. stochastic systems [23, 26, 27].

Assumption 1 (iii) and (iv) are formal descriptions of the problem setting. Assumption 1 (iv) is derived from the properties that \( v_I \) obeys the PDF \( p(v_I(\theta_t)) \) independently with respect to \( t \) and the PDF \( p(v_I(\theta_t)) \) is independent of \( \theta_s \) for \( s \neq t \).

**Remark 1 (Generality of SMP systems):** While \( p(v_I(\theta_t)) \) has the independence described in Assumption 1 (iv), \( p(v_I(\theta_t)) \) is not needed to be identical with respect to \( t \). The sequence of \( \theta_0, \theta_1, \theta_2, \ldots \) can obey various deterministic and stochastic processes. The proposed SMP systems reduce to Markov jump linear systems (with multiplicative noises instead of additional noises) [45, 46] if the sequence obeys a Markov process. Although random polytopic systems are considered in [1] and [2], multiple types of systems can be represented by various PDFs with a bounded second moment of \( v_I(\theta_t) \) as shown in Section V-A. i.i.d. stochastic systems are characterized by PDFs that are invariant regardless of \( t \) and \( \theta_t \). Deterministic polytopic systems are obtained if PDFs are delta functions depending on \( \theta_t \), that is, \( v_I(\theta_t) \) is a deterministic function of \( \theta_t \). Using general PDFs varying according to \( \theta_t \) provides distributionally uncertain systems, which reduce to random polytopic systems in the special case that \( v_I(\theta_t) \) is a linear combination of multiple stochastic parameters.

**Remark 2 (SMP systems with time delay):** SMP systems can treat time delay by extending results in [47]. For example, one-step time delay in (1) yields \( x_{t+1} = A_1(\theta_t)x_t + B_1(\theta_t)u_{t-1} \). This is represented by another system without delay: \( x_{t+1} = A_0(\theta_t)x_t + B_0(\theta_t)u_t \), where \( x_{t+d} := [x_t \quad u_{t-1}]^\top \) and \( A_{d,t}(\theta_t) \) and \( B_{d,t}(\theta_t) \) can be determined from \( A_0(\theta_t) \) and \( B_0(\theta_t) \). Therefore, this system is SMP if \( v_{d,t}(\theta_t) := \text{vec} ([A_{d,t}(\theta_t) \quad B_{d,t}(\theta_t)]) \) obeys Definition 1. This extension may invoke several challenges: reducing computational costs and clarifying the influence of the delay length. The computational costs for the analysis and design will be increased because the dimension of the state \( x_{t+d} \) is greater than that of the original state. A long delay makes stabilization of systems hard essentially.

**B. Problem Statements**

A linear feedback controller is applied to the system (1) as follows:

\[
u_t = -Kx_t
\]

\[
x_{t+1} = (A_1(\theta_t) - B_1(\theta_t)K)x_t
\]

where \( K \in \mathbb{R}^{m \times n} \) is a feedback gain. The feedback system (4) is SMP because of Assumption 1. This study focuses on the following two types of statistical stability for the SMP system (4) by modifying stability notions in [40, Sec. II.C].

**Definition 3 (Robust mean-square stability):** The SMP system (4) is said to be robustly mean-square (MS) stable if \(^2\)

\[
\forall \alpha \in \mathbb{R}^n, \quad \forall \theta_0, \theta_1, \ldots, \in S_\theta, \quad \lim_{t \to \infty} \mathbb{E} \left[ \|x_t\|^2 \right]_{\theta_t} = 0.
\]

**Definition 4 (Exponential robust MS stability):** The SMP system (4) is said to be exponentially robustly MS stable if there exist \( \alpha \in (0, \infty) \) and \( \beta \in (0, 1) \) such that

\[
\forall x_0 \in \mathbb{R}^n, \quad \forall \theta_0, \theta_1, \ldots, \in S_\theta, \quad \forall t \quad \mathbb{E} \left[ \|x_t\|^2 \right]_{\theta_t} \leq \alpha \|x_0\|^2 \beta^t.
\]

**Remark 3:** These stability notions are not equivalent at least for TI SMP systems. While the exponential robust
MS stability implies the robust MS stability, the reverse does not hold in general. For example, consider the 1-D TI SMP system \( x_{t+1} = A_t(\theta_t) x_t \) with \( S_\theta = (0, 1) \), \( N = 2 \), \( \phi(\theta) = [\theta_1 - \theta_2] \), \( M_1 = 1/2 \), and \( M_2 = 1 \). Then, we have \( \mathbb{E}[A_t(\theta_t)] = 1 - \theta_2 / 2 \in (1/2, 1) \). Because of
\[
\sqrt{\mathbb{E}[|x_t|^2]} = (1 - \theta_2 / 2)^{1/2} \mathbb{E}[x_0],
\]
the robust MS stability holds. Meanwhile, for every \( \beta \in (0, 1) \), the case of \( \theta = 1 - \beta^2 \in S_\theta \) yields \( (1 - \theta / 2)^{1/2} = \sqrt{(1 + \beta^2)/2} > \beta \). For any \( \alpha \in (0, \infty) \), the exponential robust MS stability does not hold for large \( t \).

The robust MS stability is advantageous in the sense that the transition probability of \((\theta_t, \theta_1, \theta_2, \ldots)\) is not needed while the transition probability of a Markov parameter is used in [45]. We state the following two main problems.

**Problem 1 (Stability Analysis):** Find necessary and/or sufficient conditions that the SMP system (4) is (exponentially) robustly MS stable for a given feedback gain \( K \).

**Problem 2 (Controller Design):** Design a feedback gain \( K \) such that the SMP system (4) is (exponentially) robustly MS stable.

## IV. PROPOSED METHOD

### A. Overview

We solve Problems 1 and 2 associated with the SMP system (4). An analysis of the SMP system suffers from two factors: the uncertainty of \( \theta_t \) and the randomness of \( v_t(\theta_t) \) given \( \theta_t \). Our key idea to overcome this difficulty is to expand the SMP system so that the randomness is excluded. The expanded system with only the uncertain \( \theta_t \) is included in a deterministic polytopic system. Such an exclusion simplifies the stability analysis and controller design for the SMP system. These details are described in Section IV-B.

Section IV-C presents solutions to Problem 1. We show that the (exponential) robust MS stability of the SMP system reduces to stability of the expanded system. Stability conditions of the expanded system can be derived based on existing results for deterministic polytopic systems. We obtain LMI-based sufficient conditions that the SMP system is stable for a given feedback gain \( K \).

Sections IV-D and IV-E provide solutions to Problem 2, starting from the solutions to Problem 1. Unfortunately, the derived LMI-based conditions become cubic matrix inequalities (CMIs) if the feedback gain is not given but is designed. In Section IV-D, we transform the CMIs into simpler QMIs. The QMIs can be solved via some optimization techniques. Moreover, in Section IV-E, the QMIs are relaxed as an iterative convex program because they are still nonconvex problems. We show that the QMIs are equivalent to LMIs with a rank-one constraint. Solutions to the constrained LMIs are approximately obtained via the iteration of a convex SDP. Finally, the solutions provide stabilizing feedback gains \( K \).

### B. Key Idea: Development of Expanded Systems

In this section, we propose the expanded system to the SMP system (4). First, let us introduce an operator for developing the expanded system.

**Definition 5 (Compression Operator \( C \)):** For any \( Y \in \mathbb{R}^{n \times n} \), the compression operator \( C : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} \) is defined as follows:
\[
C(Y) := C_v Y C_d
\]
\[
\tilde{n} := n(n + 1)/2
\]
where the elimination matrix \( C_v \in \mathbb{R}^{n \times n} \) and duplication matrix \( C_d \in \mathbb{R}^{n \times n} \) are defined such that \( C_v \text{vec} (X) = \text{vec} (X) \) for any \( X \in \mathbb{R}^{n \times n} \) and \( C_d \text{vec} (X_s) = \text{vec} (X_s) \) for any \( X_s \in \mathbb{R}^{n \times n} \).

**Remark 4 (Details of \( C \)):** The details of their definitions are described in [48, Def. 3.1a, 3.1b, 3.2a, and 3.2b]. As an example, \( C_v \) and \( C_d \) for \( n = 2 \) are given as follows:
\[
C_v = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]
\[
C_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

We define the expanded system, using the vertices \( M \) and \( N \) of the SMP system and the compression operator \( C \). The following expanded system for SMP systems is a generalized version of that for random polytopic systems introduced in our conference papers [1], [2].

**Definition 6 (Expanded System):** The \( \tilde{n} \)-dimensional expanded system to the SMP system (4) is defined as follows:
\[
\tilde{x}_{t+1} = C(F(\tilde{\theta}_t, K)) \tilde{x}_t
\]
\[
F(\tilde{\theta}_t, K) := \sum_{k=1}^{N} [\tilde{\theta}_t]_k F^{(k)}(K).
\]

The symbols \( \tilde{x}_t \in \mathbb{R}^{\tilde{n}} \) and \( \tilde{\theta}_t \in \mathbb{R}^{\tilde{n}} \) denote the expanded state and expanded uncertain parameter, respectively, at the time \( t \), where \( \mathbb{S}_\theta \) is the image of \( \phi \) in Definition 1 as follows:
\[
\mathbb{S}_\theta := \{ \phi(\theta) \mid \theta \in S_\theta \} \subseteq P_N \subset \mathbb{R}^N.
\]

For each \( k \in \{1, \ldots, N\} \), \( F^{(k)}(K) \in \mathbb{R}^{n \times n} \) are defined as follows:
\[
F^{(k)}(K) := F^{(k)}_{aa} - F^{(k)}_{ab}(I_n \otimes K) - F^{(k)}_{ba}(K \otimes I_n) + F^{(k)}_{bb}(K \otimes K).
\]

The matrices \( F^{(k)}_{aa} \in \mathbb{R}^{n \times n} \), \( F^{(k)}_{ab} \in \mathbb{R}^{n \times n} \), \( F^{(k)}_{ba} \in \mathbb{R}^{n \times n} \), and \( F^{(k)}_{bb} \in \mathbb{R}^{n \times n} \) are given as follows:
\[
[F^{(k)}_{aa}]_{i,n(j-1)+i} := \text{vec} (M^{(k)}_{i,j})
\]
\[
[F^{(k)}_{ab}]_{i,m(j-1)+i} := \text{vec} (M^{(k)}_{i,n+j})
\]
\[
[F^{(k)}_{ba}]_{i,n(j-1)+i} := \text{vec} (M^{(k)}_{i,n+j})
\]
\[
[F^{(k)}_{bb}]_{i,m(j-1)+i} := \text{vec} (M^{(k)}_{n+i,n+j})
\]
for $i,j \in \{1, \ldots, n\}$ and $i',j' \in \{1, \ldots, m\}$, using the following block matrix form of $M^{(k)}$:

$$M^{(k)} := \begin{bmatrix} M_{1,1}^{(k)} & \cdots & M_{1,m+n}^{(k)} \\ \vdots & \ddots & \vdots \\ M_{n,m,1}^{(k)} & \cdots & M_{n+m,n+m}^{(k)} \end{bmatrix},$$

where $M_{i,j}^{(k)}$ for $i,j \in \{1, \ldots, n+m\}$ are $n \times n$ matrices.

**Definition 7 (TI/TV Expanded Systems):** The expanded system (9) is said to be TI (respectively, TV) if the corresponding SMP system (4) is TI (respectively, TV).

**Example 2 (Demonstration of an expanded system):** For the SMP system in Example 1 with $n = 2$ and $m = 1$, we develop the expanded system, where $\bar{n} = 3$ is given by (6). Because of $M_{i,j}^{(k)} = \mu_{ij}^T + \Sigma_{ij}^{(k)} \in \mathbb{R}^{\bar{n}}_{sym}$ for any $i,j \in \{1, 2, 3\}$, we obtain the following:

$$M_{i,j}^{(k)} = \begin{bmatrix} M_{21}^{(k)} & \cdots & M_{2,m+n+2}^{(k)} \\ \vdots & \ddots & \vdots \\ M_{m,n+1}^{(k)} & \cdots & M_{m+n,m+n+2}^{(k)} \end{bmatrix}.$$ 

Because of (12), $F_{aa}^{(k)}$ is given as follows:

$$F_{aa}^{(k)} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} M_{1,1}^{(k)} & \cdots & M_{1,2}^{(k)} & \cdots & M_{1,2m}^{(k)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ M_{2m,1}^{(k)} & \cdots & M_{2m,2}^{(k)} & \cdots & M_{2m,2m}^{(k)} \end{bmatrix} \end{bmatrix} \end{bmatrix}.$$ 

In a similar manner, we derive the following:

$$F_{ab}^{(k)} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} M_{1,1}^{(k)} & \cdots & M_{1,2}^{(k)} & \cdots & M_{1,2m}^{(k)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ M_{2m,1}^{(k)} & \cdots & M_{2m,2}^{(k)} & \cdots & M_{2m,2m}^{(k)} \end{bmatrix} \end{bmatrix} \end{bmatrix}.$$ 

C. Solutions to Problem 1: Stability Analysis

In this section, we solve Problem 1 by using the expanded system (9) with Theorem 1. We show that (exponential) robust stability of the expanded system is equivalent to the (exponential) robust MS stability of the SMP system (4). Given a feedback gain $K$, stability conditions for the expanded system are derived as LMIs. Therefore, the LMIs are solutions to Problem 1.

First, the two stability notions of the expanded system (9) are defined below.

**Definition 8 (Robust Stability):** The expanded system (9) is said to be robustly stable if

$$\exists \bar{x}_0 \in \mathbb{R}^\bar{n}, \quad \forall \bar{\theta}_0, \bar{\theta}_1, \ldots \in \mathbb{S}_\theta, \lim_{t \to \infty} ||\bar{x}_t|| = 0.$$ 

**Definition 9 (Exponential Robust Stability):** The expanded system (9) is said to be exponentially robustly stable if there exist $\bar{\alpha} \in (0, \infty)$ and $\bar{\beta} \in (0, 1)$ such that

$$\exists \bar{x}_0 \in \mathbb{R}^\bar{n}, \quad \forall \bar{\theta}_0, \bar{\theta}_1, \ldots \in \mathbb{S}_\theta, \quad \forall t, \quad ||\bar{x}_t|| \leq \bar{\alpha}||\bar{x}_0||^{\bar{\beta}}.$$ 

We derive equivalence of the stability between the expanded system and the SMP system as follows.

**Theorem 2 (Equivalence of the stability):** The SMP system (4) is robustly MS stable if and only if the expanded system (9) is robustly stable.

**Proof:** The proof is described in Appendix B.

**Remark 7 (Contributions of Theorems 2 and 3):** The stability analysis of the SMP system (4) reduces to that of the expanded system (9). By regarding the expanded system as a deterministic polytopic system, we can employ various stability analyses for polytopic linear systems, e.g., [6].
In the following, we derive sufficient conditions for the stability of the expanded system (9) based on existing results in [6]. Let us define the following matrix-valued function that is cubic in $P \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times n}$, and $K \in \mathbb{R}^{m \times n}$ with fixed $\beta$:

$$S_{cmi}^{(k)}(P, G, K, \beta) := \begin{bmatrix} \beta^2 P + C(F^{(k)}(K))^\top G \\ G^\top C(F^{(k)}(K)) + G^\top + G - P \end{bmatrix}.$$ 

We derive Theorem 4 below to show that the following conditions are sufficient stability conditions of the expanded system and reduce to those of the SMP system because of Theorems 2 and 3.

(C1.1) There exist $P^{(k)} > 0 \in \mathbb{R}_{sym}^n$ for $k \in \{1, \ldots, N\}$, $G$, and $K$ such that

$$\forall k \in \{1, \ldots, N\}, \quad S_{cmi}^{(k)}(P^{(k)}, G, K, \beta) \geq 0.$$ 

(C1.2) There exist $P^{(k)} > 0 \in \mathbb{R}_{sym}^n$ for $k \in \{1, \ldots, N\}$, $G$, and $K$ such that

$$\forall k \in \{1, \ldots, N\}, \quad S_{cmi}^{(k)}(P^{(k)}, G, K, 1) > 0.$$ 

**Theorem 4 (Solutions to Problem 1):** For a given $\beta \in (0, 1)$, the SMP system (4) is exponentially robustly stable (respectively, robustly stable) if either of the following (i) or (ii) is satisfied:

i) (C1.1) [respectively, (C1.2)] holds and the system is TI.

ii) (C1.1) [respectively, (C1.2)] holds under the constraint that matrices $P^{(k)}$ for all $k$ are identical, that is, $P^{(1)} = P^{(2)} = \cdots = P^{(N)} = P$ for some $P > 0 \in \mathbb{R}_{sym}^n$.

**Proof:** The proof is described in Appendix D.

**Remark 8 (Contributions of Theorem 4):** Theorem 4 provides sufficient stability conditions of both expanded and original SMP systems for a given $\beta$. The conditions reduce to LMIIs of $P^{(k)}$ and $G$ if $K$ is given or if $B_1(\theta_1)K = 0$ holds (autonomous cases). In comparison with the existing results [6] for TI polytopic systems, Theorem 4 overcomes an issue that the expanded systems (9) are not directly regarded as polytopic systems because the operator $C$ and the nonpolytopic set $S_B$ in (11) are involved. Combining the following two facts make the expanded systems polytopic to apply the existing results. First, using the linearity of $C$ yields the polytopic form regardless of $C$ in Appendix D. Second, the polytopic set $P_N$ in (3) contains the set $S_B$. In addition, Theorem 4 covers both the exponential robust stability and robust stability although only the robust stability has been considered in [6]. Even if the SMP system is TV, we can apply these stability results based on the quadratic stability.

**Remark 9:** The value of $\beta \in (0, 1)$ indicates the rate of the exponential robust stability and a smaller value of $\beta$ is better for fast convergence. There are some methods to choose $\beta$. A simplest method is a grid search by which a value close to the minimum is found depending on a grid width. A bisection method is more efficient in a manner similar to [40]. We can update $\beta$ and other decision variables iteratively.

**D. Solutions to Problem 2: QMI-Based Controller Design**

In this section, we solve Problem 2, starting from the stability conditions (C1.1) and (C1.2). Unfortunately, these conditions are CMIIs with respect to $(P, G, K)$ if the feedback gain $K$ is not given but regarded as a decision variable. To avoid solving the CMIIs directly, we transform them into simpler QMIIs in the following.

Let us define the following matrix-valued function that is quadratic in $Q \in \mathbb{R}_{sym}^n$, $H \in \mathbb{R}^{n \times n}$, and $L \in \mathbb{R}^{m \times n}$ with fixed $\beta$:

$$S_{qmi}^{(k)}(Q, H, L, \beta) := \begin{bmatrix} \beta^2 Q + F^{(k)}_a(H \otimes H) - F^{(k)}_a(H \otimes L) \\ F^{(k)}_b(H \otimes H) - F^{(k)}_b(H \otimes L) - \frac{1}{2}F^{(k)}_c(H \otimes H) \end{bmatrix}.$$ 

We derive the following results to simplify stability conditions.

**Theorem 5 (Controller Design via QMIIs):** For a given $\beta \in (0, 1)$, the following (C2.1) and (C2.2) imply (C1.1) and (C1.2), respectively, even under the following conditions (23)–(25):

$$P^{(k)} = G^\top Q^{(k)}G$$

$$G = C(H \otimes H)^{-1}$$

$$K = LH^{-1}.$$ 

(C2.1) There exist $Q^{(k)} > 0 \in \mathbb{R}_{sym}^n$ for $k \in \{1, \ldots, N\}$, $H$, and $L$ such that

$$\forall k \in \{1, \ldots, N\}, \quad S_{qmi}^{(k)}(Q^{(k)}, H, L, \beta) \geq 0.$$ 

(C2.2) There exist $Q^{(k)} > 0 \in \mathbb{R}_{sym}^n$ for $k \in \{1, \ldots, N\}$, $H$, and $L$ such that

$$\forall k \in \{1, \ldots, N\}, \quad S_{qmi}^{(k)}(Q^{(k)}, H, L, 1) > 0.$$ 

Furthermore, if $Q^{(k)}$ for $k \in \{1, \ldots, N\}$ are identical, then $P^{(k)}$ given in (23) are also identical.

**Proof:** The proof is described in Appendix E.

**Remark 10 (Contribution of Theorem 5):** The CMIs are transformed into the QMIIs with respect to $(Q^{(k)}, H, L)$, which are expected to be easier to solve. Theorem 5 guarantees that $K$ in (25) is a stabilizing feedback under some conditions. If (C2.1) [respectively, (C2.2)] holds, we have (C1.1) [respectively, (C1.2)] even under (23)–(25). Satisfying (C1.1) or (C1.2) indicates the stability of the SMP system, provided that additional conditions in Theorem 4 hold. Furthermore, in Section IV-E, the QMIIs are relaxed as an iterative convex program whereas the CMIs and QMIIs are nonconvex.

**Remark 11 (Intuition of Deriving Theorem 5):** Multiplying the QMIIs $S_{qmi}^{(k)}(Q^{(k)}, H, L, \beta) \geq 0$ by a block matrix using $G$ in (24) yields the CMIs $S_{cmi}^{(k)}(P^{(k)}, G, K, \beta) \geq 0$. While such a multiplication is based on robust control design [6], [40], we show that this technique can be applied to matrix inequalities involving the compression operator $C$ and Kronecker product $\otimes$ (the details are found in the proof in Appendix E).

**E. Solutions to Problem 2: Convex Controller Design**

Solving the QMIIs in (C2.1) or (C2.2) is still a nonconvex program. Our conference paper [2] has encountered the same nonconvexity and [1] has not addressed controller design. To overcome this difficulty, we relax the QMIIs as an iterative convex
program that provides two advantages. First, the convex property is efficient for obtaining an optimal solution to the program at each iteration. Second, the complexity of the program can be analyzed, which is to be described in Remark 14. The quadratic terms of $L$ and $H$ in the QMIs are replaced with linear terms of a rank-one matrix $Z$. For any $Z \geq 0 \in \mathbb{R}^{n(m+n)}_{\text{sym}}$, if rank($Z$) = 1 holds, there exist $H \in \mathbb{R}^{n \times n}$ and $L \in \mathbb{R}^{m \times n}$ that satisfy the following:

$$Z = \begin{bmatrix} \text{vec}(H) \\ \text{vec}(L) \end{bmatrix} \begin{bmatrix} \text{vec}(H) \end{bmatrix}^\top.$$  \hfill (27)

Let us define the following matrix-valued function that is linear in $Q' \in \mathbb{R}^{n \times n}_{\text{sym}}$ and $Z \in \mathbb{R}^{n(m+n)}_{\text{sym}}$ with fixed $\beta$:

$$S_{\text{lin}}^{(k)}(Q, Z, \beta) := \left[ \begin{array}{c} \beta^2 Q \\ F_{\text{lin}}(k)(Z) \end{array} \right] + \left[ \begin{array}{c} C(F_{\text{hh}}(Z)) + C(F_{\text{lh}}(Z)) \end{array} \right]^\top - Q.$$  \hfill (28)

The function $F_{\text{lin}}^{(k)}(Z)$ is defined as follows:

$$F_{\text{lin}}^{(k)}(Z) := C(F_{\text{aa}}) F_{\text{hh}}(Z) + F_{\text{ab}} F_{\text{lh}}(Z) - F_{\text{ba}} F_{\text{hh}}(Z) + F_{\text{bb}} F_{\text{hh}}(Z).$$

where $F_{\text{hh}}(Z) \in \mathbb{R}^{n \times n}$, $F_{\text{hl}}(Z) \in \mathbb{R}^{m \times n}$, and $F_{\text{ll}}(Z) \in \mathbb{R}^{m \times m}$ are the following linear functions of $Z \in \mathbb{R}^{n(m+n)}_{\text{sym}}$, using its block matrix form:

$$[F_{\text{hh}}(Z)]_{n(n-1)+i} := \text{vec}(Z_{i,j}),$$
$$[F_{\text{hh}}(Z)]_{n(n-1)+i} := \text{vec}(Z_{i,n+i}),$$
$$[F_{\text{lh}}(Z)]_{n(n-1)+i} := \text{vec}(Z_{i,n+j}),$$
$$[F_{\text{ll}}(Z)]_{n(n-1)+i} := \text{vec}(Z_{n+i,n+j}).$$

$Z_{i,j} \in \mathbb{R}^{m \times n}$, $Z_{n+i,j} \in \mathbb{R}^{m \times n}$, $Z_{i,n+j} \in \mathbb{R}^{n \times m}$, and $Z_{n+i,n+j} \in \mathbb{R}^{n \times m}$ for any $i, j \in \{1, \ldots, n\}$. These functions are defined to satisfy the following relations.

Proposition 1 (Properties of the linear functions): The condition (27) implies the following relations:

$$F_{\text{hh}}(Z) = H \otimes H,$$  \hfill (34)
$$F_{\text{hl}}(Z) = H \otimes L,$$  \hfill (35)
$$F_{\text{lh}}(Z) = L \otimes H,$$  \hfill (36)
$$F_{\text{ll}}(Z) = L \otimes L.$$  \hfill (37)

Proof: The proof is described in Appendix 1 F.

Based on these definitions and Proposition 1, we transform the QMIs into constrained LMIIs as follows.

Theorem 6 (Controller Design via Constrained LMIIs): For a given $\beta \in (0, 1)$, the following (C3.1) and (C3.2) are equivalent to (C2.1) and (C2.2), respectively, under the setting (27):

(C3.1) There exist $Q^{(k)} > 0 \in \mathbb{R}^{n \times n}$ for $k \in \{1, \ldots, N\}$ and $Z \geq 0 \in \mathbb{R}^{n(m+n)}_{\text{sym}}$ such that the following LMIIs and rank($Z$) = 1 hold:

$$\forall k \in \{1, \ldots, N\}, \quad S_{\text{lin}}^{(k)}(Q^{(k)}, Z, \beta) \geq 0.$$  \hfill (38)

(C3.2) There exist $Q^{(k)} > 0 \in \mathbb{R}^{n \times n}$ for $k \in \{1, \ldots, N\}$ and $Z \geq 0 \in \mathbb{R}^{n(m+n)}_{\text{sym}}$ such that the following LMIIs and rank($Z$) = 1 hold:

$$\forall k \in \{1, \ldots, N\}, \quad S_{\text{lin}}^{(k)}(Q^{(k)}, Z, 1) > 0.$$  \hfill (39)

Proof: The proof is described in Appendix G.

Note that the LMIIs in (C3.1) and (C3.2) are just linear in $Q^{(k)}$ and $Z$. A remaining challenge is to exclude the rank constraint rank($Z$) = 1 that invokes the nonconvexity in solving the LMIIs. In the following, we relax the rank-constrained LMIIs as an iterative convex SDP based on the ideas of capped trace norm minimization [49, 50]. Let us define the absolute sum of the eigenvalues of symmetric $Z \geq 0 \in \mathbb{R}^{n(m+n)}_{\text{sym}}$, except the maximum eigenvalue $\lambda_1(Z)$, as follows:

$$\varepsilon(Z) := \sum_{i=2}^{n(m+n)} |\lambda_i(Z)| \geq 0.$$  \hfill (39)

Solving the LMIIs in (C3.1) and (C3.2) with the rank constraint reduces to minimize $\varepsilon(Z)$ under the LMIIs because rank($Z$) = 1 is equivalent to $\varepsilon(Z) = 0$ with $\lambda_1(Z) > 0$. Whereas $\varepsilon(Z)$ is nonlinear in $Z$, we focus on the fact that the minimization of $\varepsilon(Z)$ has a form similar to the capped trace norm minimization [50]. This motivates us to employ the following function $\varepsilon(Z, Z')$ of $Z \in \mathbb{R}^{n(m+n)}_{\text{sym}}$, $Z' \in \mathbb{R}^{n(m+n)}_{\text{sym}}$ for approximating $\varepsilon(Z)$:

$$\varepsilon(Z, Z') := \text{tr}(Z) - \nu_1(Z')^\top Z \nu_1(Z')$$

where $\nu_1(Z')$ is the unit eigenvector corresponding to $\lambda_1(Z')$ defined in Section 2. We show that $\varepsilon(Z, Z')$ is an upper bound of $\varepsilon(Z)$, as follows.

Lemma 1 (Approximate Linear Functions): Suppose that $Z \neq 0$ holds. For any $Z \geq 0$ and $Z' \geq 0$, the following relations hold:

$$\varepsilon(Z, Z') \geq \varepsilon(Z).$$

Proof: The proof is described in Appendix H.

In addition to the above relations, an efficient property is that $\varepsilon(Z, Z')$ is linear in $Z$ with a fixed $Z'$, which is convex. Using these properties, we solve the following convex SDP iteratively instead of (C3.1) or (C3.2). For each $\ell$th iteration, let $Z^{(\ell)}$ be a numerical/approximate solution to the SDP as follows:

$$\min_{Z, Q^{(1)}, \ldots, Q^{(N)}} \varepsilon(Z, Z^{(\ell-1)})$$

subject to

$$\forall k \in \{1, \ldots, N\}, \quad S_{\text{lin}}^{(k)}(Q^{(k)} , Z, \beta) \geq \eta I_{2n},$$

$$\forall k \in \{1, \ldots, N\}, \quad Q^{(k)} \geq \eta I_{n},$$

$$Z \geq 0,$$

$$\text{tr}(Z) \leq Z_{ub}.$$  \hfill (40)

where $\eta \geq 0$ and $Z_{ub} > 0$ are free parameters. The first and second inequalities describe (C3.1) or (C3.2), where the positive value of $\eta$ helps satisfy the positive (semi)definiteness even if numerical errors occur. The inequality $\text{tr}(Z) \leq Z_{ub}$ is employed to avoid divergence of $Z$. Let $Z^{(1)}$ be a solution to (40) when replacing $\varepsilon(Z, Z^{(\ell)})$ with $\text{tr}(Z)$, which is similar to trace norm minimization [51]. After solving the SDP (40) iteratively, we obtain its solution $Z_* = Z^{(\ell)}$. We approximate $Z_*$ by the
Algorithm 1: Iterative SDP to Solve (C3.1) or (C3.2).

Input: Vertices $M^{(k)}$ for $k \in \{1, \ldots, N\}$ and $N$ of the SMP system, $\beta$, the free parameters $\eta$ and $Z_{ab}$ used in the SDP (40), and $\delta$

Output: Rank-one matrix $\tilde{Z}$, and $Q(k)$

1: Calculate $F_{aa}, F_{ab}, F_{ba},$ and $F_{bb}$ according to (12)–(15) using $M^{(k)}$ and $N$

2: Develop the function $S_{k}^{(k)}(Q, Z, \tilde{Z})$ with arguments $(Q, Z)$ in (28) using $F_{aa}, F_{ab}, F_{ba}, F_{bb},$ and $\tilde{Z}$

3: Obtain $Z^{(1)}$ by solving the SDP (40) when replacing $\tilde{Z}$ with $\text{tr}(Z)$,

4: Set $\ell \leftarrow 1$

5: repeat

6: Set $\ell \leftarrow \ell + 1$

7: Obtain $Z^{(\ell)}$ and $Q^{(\ell)}$ by solving the SDP (40) with substituting $Z^{(\ell-1)}$.

8: until $\varepsilon(Z^{(\ell)}) \geq \varepsilon(Z^{(\ell-1)}) - \delta$

9: Obtain a solution $Z_{\ell} \leftarrow Z^{(\ell)}$ to the iterative SDP

10: Obtain the rank-one matrix $\tilde{Z}$, by (41)

following rank-one matrix $\tilde{Z}$ associated with the maximum eigenvalue $\lambda_1(Z_s)$:

$$\tilde{Z} := \lambda_1(Z_s) \nu_1(Z_s) \nu_1(Z_s)^\top.$$  \hspace{1cm} (41)

Note that the approximation residual $Z_s - \tilde{Z} = \sum_{s=1}^{n(n+m)} \lambda_s(Z_s) \nu_s(Z_s) \nu_s(Z_s)^\top$ can be negligible if $\varepsilon(Z_s)$ in (39) is sufficiently decreased.

Algorithm 1 summarizes the proposed iterative SDP. The iteration is terminated when $\varepsilon(Z^{(\ell)}) \geq \varepsilon(Z^{(\ell-1)}) - \delta$ holds, where $\delta \geq 0$ is a small threshold. If (C3.1) or (C3.2) is satisfied by $Z = \tilde{Z}$, in (41) and a solution $Q^{(k)}$ (in the $\ell$th iteration) to the SDP, a stabilizing feedback gain $K$ is obtained. Consequently, solutions to Problem 2 are summarized, using Theorems 4–6.

Corollary 1 (Solutions to Problem 2): The SMP system (4) is exponentially robustly MS stable (respectively, robustly MS stable) if at least one of the following (i)–(iv) is satisfied and if $\tilde{K}$ is given by (25) and (27):

i) (C2.1) [respectively, (C2.2)] is solved and the system is TI.

ii) (C2.1) [respectively, (C2.2)] is solved under the constraint that $Q^{(1)} = Q^{(2)} = \cdots = Q^{(N)}$.

iii) Solutions $Z = \tilde{Z}$, and $Q(k)$ to the iterative SDP (40) satisfy (C3.1) [respectively, (C3.2)] and the system is TI.

iv) Solutions $Z = \tilde{Z}$, and $Q(k)$ to the iterative SDP (40) satisfy (C3.1) [respectively, (C3.2)] under the constraint that $Q^{(1)} = Q^{(2)} = \cdots = Q^{(N)}$.

Remark 12: Algorithm 1 is justified in the sense that $\varepsilon(Z^{(\ell)})$ is successfully decreased via the iteration, that is, $\varepsilon(Z^{(\ell)}) \leq \varepsilon(Z^{(\ell-1)})$. If the following relation holds:

$$\varepsilon(Z^{(\ell)}, Z^{(\ell-1)}) \leq \varepsilon(Z^{(\ell-1)}, Z^{(\ell-1)})$$  \hspace{1cm} (42)

using Lemma 1 with the constraint $Z^{(\ell)} \geq 0$ yields $\varepsilon(Z^{(\ell)}) \leq \varepsilon(Z^{(\ell-1)}, Z^{(\ell-1)}) \leq \varepsilon(Z^{(\ell-1)}, Z^{(\ell-1)}) = \varepsilon(Z^{(\ell-1)})$, where its strict inequality holds if that of (42) holds. For any $\beta \in (0, 1)$ (respectively, $\beta = 1$) and any $Z_{ab} > 0$, if (C3.1) [respectively, (C3.2)] holds, the SDP (40) with $\eta = 0$ has a feasible solution. Then, (42) holds if $Z^{(\ell)}$ is an optimal solution. Even if $\eta$ is set to a small positive value and $Z^{(\ell)}$ contains a small numerical error, (42) is expected to hold, where a numerical suboptimal solution $Z^{(\ell)}$ can be obtained by using solvers such as interior-point methods [52, Ch. 8], [5, Sec. 2.4 and 2.5].

Remark 13: While the proposed iterative SDP is based on [49] and [50], such extension is challenging because the problem solved above is essentially different from problems considered in [49] and [50]. Our problem is to find feasible solutions satisfying LMIs and a rank-one constraint and this problem is relaxed into the convex SDP while [49], [50] consider regularized minimization problems relaxed into convex regularization.

Remark 14: The complexity of solving the SDP (40) is polynomial in two metrics: the total dimension $\eta_1$ of the LMIs and the number $\eta_2$ of decision variables, for fixed precision parameters of ellipsoid and interior point methods under technical assumptions based on [53, Sec. 4.1]. The complexity can be successfully decreased by using the compression operator $C$ because $\eta_1$ and $\eta_2$ with the operator are almost half of those without the operator. Namely, for $m := \kappa n$ with $\kappa \in (0, 1]$, using $C$ yields $\eta_1 = ((3N/2) + 1 + \kappa n^2 + O(n)$ and $\eta_2 = ((N/4) + 1 + (1 + \kappa^2)n^4/2 + O(n^3)$ whereas $\eta_1 = (3N + 1 + \kappa)n^2 + O(n)$ and $\eta_2 = (N + 1 + (1 + \kappa^2)n^4/2 + O(n^3)$ are obtained without $C$.

V. DEMONSTRATION WITH A NUMERICAL SIMULATION

Section V-A demonstrates the applicability of the proposed SMP systems. The proposed method to design stabilizing controllers is evaluated through a numerical example. Sections V-B and V-C describe the simulation settings and results, respectively. The following results are novel and not presented in our conference papers [1], [2].

A. Variety of Second Moment Polytopes

The SMP systems in Definition 1 can describe various classes of uncertain stochastic linear systems, which contain familiar existing systems. We derive certain examples of such classes as follows.

Theorem 7 (Examples of TI SMP systems): Suppose that $\theta_i$ is TI (i.e., $\theta_i = \theta$) and that $S_{\theta}$ is given as follows:

$$S_{\theta} = \left\{ \theta \in \mathbb{R}^{d_\theta} \mid \forall k, [\theta]_k \geq 0, \sum_{k=1}^{d_\theta} [\theta]_k = 1 \right\}. $$  \hspace{1cm} (43)

The following classes are represented as SMP systems:

i) i.i.d. stochastic systems [33]: Suppose that $v_i(\theta) = v_i$ is i.i.d. with respect to $t$ and independent of $\theta$. The system (1) is TI SMP with

$$N = 1$$  \hspace{1cm} (44)

$$\phi(\theta) = 1$$  \hspace{1cm} (45)

$$M^{(1)} = E[v_i v_i^\top].$$  \hspace{1cm} (46)

ii) Deterministic polytopic systems [6]: Suppose that $v_i(\theta)$ is given as follows:

$$v_i(\theta) = \sum_{k=1}^{d_\theta} [\theta]_k v^{(k)}$$
where \(v^{(k)}\) for \(k \in \{1, \ldots, d_\theta\}\) are deterministic vertices. Supposing that the expectation \(E[\tau_t(\theta_t)\tau_t(\theta_t)^\top]|\theta_t]\) can be replaced with the deterministic value \(v_t(\theta_t)\tau_t(\theta_t)^\top\) the system (1) is TI SMP with

\[
N = a_\theta^2
\]

(47)

\[
\phi(\theta) = \text{vec} (\theta \theta^\top)
\]

(48)

\[
M^d(a(k-1)+k) = \frac{v^{(k)}_t v^{(k)}_t^\top + v^{(k)}_t v^{(k)}_t^\top}{2}.
\]

iii) Random polytopic systems [40]: Suppose that \(v_t(\theta)\) is given as follows:

\[
v_t(\theta) = \sum_{k=1}^{d_\theta} [\theta]_k v^{(k)}_t
\]

where the stochastic vertices \(v^{(k)}_t\) for \(k \in \{1, \ldots, d_\theta\}\) are i.i.d. with respect to \(t\) and independent of \(\theta\). The system (1) is TI SMP with (47), (48), and

\[
M^d(a(k-1)+k) = \frac{E[v^{(k)}_t v^{(k)}_t^\top + v^{(k)}_t v^{(k)}_t^\top]}{2}.
\]

(49)

iv) Distributionally uncertain systems: Suppose that \(v_t(\theta)\) is given as follows:

\[
E[\tau_t(\theta)|\theta] = \sum_{k=1}^{d_\theta} [\theta]_k \mu^{(k)}
\]

(50)

\[
\text{Cov} [\tau_t(\theta)|\theta] = \sum_{k=1}^{d_\theta} [\theta]_k\Sigma^{(k)}
\]

(51)

where \(\mu^{(k)}\) and \(\Sigma^{(k)}\) are deterministic. The system (1) is TI SMP with (47), (48), and

\[
M^d(a(k-1)+k) = \frac{\mu^{(k)}\mu^{(k)}^\top + \mu^{(k)}\mu^{(k)}^\top}{2} + \Sigma^{(k)}.
\]

Proof: The proof is described in Appendix I.

Corollary 2 (Examples of TV SMP systems): Suppose that \(\theta_t\) is TV and that \(S_\theta\) is given by (43). Then, the statements (ii)–(iv) in Theorem 7 hold if \(\theta_t, v_t(\theta_t)\), and the TV property are replaced with \(\theta_t, v_t(\theta_t)\), and the TV property, respectively.

Proof: By replacing \(\theta_t\) with \(\theta_t\), we can prove the statements in a manner similar to Theorem 7.

While SMP systems represent random polytopes in Theorem 7 (iii), the following statement shows a difference between the SMP systems and random polytopes.

Proposition 2 (Comparison with random polytopes): There exists a TI SMP system with \(\overline{\tau}(\theta)\) that is not equivalent to every random polytope with \(v_t(\theta_t)\) given in Theorem 7 (iii), in the sense that there is no setting of \(v^{(k)}_t\) for \(k \in \{1, \ldots, d_\theta\}\) satisfying the following conditions:

\[
\forall \theta \in S_\theta, \exists \theta \in S_\theta, E[\overline{\tau}(\theta)\overline{\tau}(\theta)^\top]|\theta] = E [v_t(\theta)\tau_t(\theta)^\top]|\theta,
\]

(52)

\[
\forall \theta \in S_\theta, \exists \theta \in S_\theta, E[\overline{\tau}(\theta)\overline{\tau}(\theta)^\top]|\theta] = E [v_t(\theta)\tau_t(\theta)^\top]|\theta.
\]

(53)

where the notation \(\overline{\bullet}\) denotes variables different from \((\bullet)\).

Proof: The proof is described in Appendix J.
results with the different parameters are plotted in Fig. 2. It can be observed that the systems without control did not converge to the origin in Fig. 2(a). In contrast, the designed controller stabilized the systems as shown in Fig. 2(b). These results indicate that the proposed controller design method can be successfully applied to SMP systems.

D. Estimation and Control Design for SMP Systems

We demonstrate the controller design with estimating an SMP system when the system (1) contains unknown $\omega_{t} \in \mathbb{R}^{3}$ that is i.i.d. from the normal distribution $p(\omega|\theta)$:

$$v_{t}(\theta) = [1.1, 0, 0.3 + [\omega_{t}]_{1}, 0.9, 0, 1 + [\omega_{t}]_{2}]^{T}$$

$$\text{Cov}[\omega|\theta] = \begin{bmatrix} a(\theta) & b(\theta) \\ b(\theta) & c(\theta) \end{bmatrix}$$

$$\theta = \begin{bmatrix} 0.05 & 0.05 \\ 0.05 & 1 \end{bmatrix} + (1 - \theta) \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

where $E[\omega|\theta] = 0$ and $\theta \in [0, 1]$ denotes uncertainty. This implies $E[v_{t}(\theta) v_{t}(\theta)^{T}]|_{d} = V(a(\theta), b(\theta), c(\theta))$, where

$$V(a, b, c) := \begin{bmatrix} 1.1 & 1.1 \\ 0 & 0 \\ 0.3 & 0.3 \\ 0.9 & 0.9 \\ 1.0 & 1.0 \end{bmatrix}^{T} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & c \end{bmatrix}.$$  

Suppose that $\omega_{t}, \theta$, and the functions $(a(\cdot), b(\cdot), c(\cdot))$ are unknown. To design a stabilizing controller, we express this uncertain system as an SMP system, provided that 20 datasets are given. Each $j$th data set consists of $D$ random samples $\tilde{\omega}_{i,j}$ ($i = 1, 2, \ldots, D$) of $\omega$ obeying the distribution $p(\omega|\theta_{j})$, where each $\theta_{j}$ is uniformly randomly sampled on $[0, 1]$.

We obtain an SMP system by estimating $(\tilde{a}, \tilde{\pi}, \tilde{b}, \tilde{c}, \tilde{\tau})$ using the datasets, where $\tilde{a} := \min_{\theta} a(\theta), \tilde{\pi} := \max_{\theta} a(\theta)$, and the others are defined in the same manner. If these values are precisely estimated, the SMP system satisfying the conditions in Definition 1 can be obtained by the following eight vertices:

$$M^{(1)} = V(\tilde{a}, \tilde{\pi}, \tilde{b}, \tilde{c}), \quad M^{(2)} = V(\tilde{a}, \tilde{\pi}, \tilde{b}, \tilde{c}), \quad M^{(3)} = V(\tilde{a}, \tilde{\pi}, \tilde{b}, \tilde{c}),$$

$$M^{(4)} = V(\tilde{a}, \tilde{\pi}, \tilde{b}, \tilde{c}), \quad M^{(5)} = V(\tilde{a}, \tilde{\pi}, \tilde{b}, \tilde{c}), \quad M^{(6)} = V(\tilde{a}, \tilde{\pi}, \tilde{b}, \tilde{c}),$$

$$M^{(7)} = V(\tilde{a}, \tilde{\pi}, \tilde{b}, \tilde{c}), \quad M^{(8)} = V(\tilde{a}, \tilde{\pi}, \tilde{b}, \tilde{c}).$$

We estimate $a \approx \min_{j} \sum_{i=1}^{D} (|\tilde{\omega}_{i,j}|_{1})^{2}/D$, $\pi \approx \max_{j} \sum_{i=1}^{D} (|\tilde{\omega}_{i,j}|_{1})^{2}/D$, and $b \approx \min_{j} \sum_{i=1}^{D} (|\tilde{\omega}_{i,j}|_{1})^{2}/D$, where the others are estimated in the same manner.

Table I shows success rates of the controller design for different data sizes $D$. The success rate is defined as the number of success trials divided by total 1000 trials with different random seeds, where the success trial indicates that Algorithm 1 using the estimated SMP vertices works successfully and the resultant controller stabilizes the systems with 5000 random values of $\theta$. The settings of Algorithm 1 are described in Section V-B. Table I shows that the success rate increases with the data size $D$.

VI. Conclusion

This article presented the concept of general uncertain stochastic systems called SMP systems. The goal of this study was to establish fundamental theory to guarantee stability of the SMP systems. A central idea to address this goal is to develop expanded systems with key properties as shown in Section IV-B. We derived conditions for which the SMP systems are (exponentially) robustly MS stable in Section IV-C. Based on the derived conditions, a method to design stabilizing feedback gains of linear controllers was proposed. Section IV-D showed that feedback gains are solutions to QMIs (into which CMIIs are transformed). In Section IV-E, we relaxed the nonconvex QMIs as an iterative convex SDP that is easy to solve.

SMP systems can widely represent various uncertain stochastic systems such as i.i.d. systems, random polytopic systems, and distributionally uncertain systems as shown in Theorem 7. The effectiveness of the proposed design method was confirmed via a numerical simulation. Future work includes pursuing the possibility that combining this study with H-representation techniques [28], [29], [30], [45] yields novel results for robust higher order moment stability of SMP systems. Various control problems such as optimal control and output feedback control will be considered for SMP systems.

APPENDIX A

Proof of Theorem 1

For brevity of notation, $A_{d,t} := A_{d}(\theta_{t}) - B_{d}(\theta_{t}) K$ denotes the closed-loop matrix in (4). We show (18) using mathematical induction after proving the following relation:

$$\forall t, \quad F(\hat{\theta}_{d}, K) = E[A_{d,t} \otimes A_{d,t}|\theta_{t}]. \quad (54)$$

First, we show (54). For any $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, let us define $v := \text{vec} ([A, B])$ and $M_{i,j} \in \mathbb{R}^{n \times n}$ as follows:

$$vv^{T} := \begin{bmatrix} M_{1,1} & \cdots & \cdots & \cdots \\ \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & M_{n,m,1} & \cdots \\ \cdots & \cdots & \cdots & M_{n,m,n+m+n} \end{bmatrix}.$$  

For any $i' \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, we obtain

$$[A \otimes B]_{i',m(j-1)+j} = \text{vec} ([B]_{i',j}[A]_{i,j}) = \text{vec} (M_{n+v,j}). \quad (55)$$
In a manner similar to this relation, the following results are obtained for \( i,j \in \{1, \ldots, n\} \) and \( i',j' \in \{1, \ldots, m\} \):

\[
A \otimes A = \text{vec} \begin{pmatrix} \mu_{i,j} \end{pmatrix} \quad (56)
\]

\[
B \otimes A = \text{vec} \begin{pmatrix} \mu_{i,n+j} \end{pmatrix} \quad (57)
\]

\[
B \otimes B = \text{vec} \begin{pmatrix} \mu_{i,m+j'} \end{pmatrix} \quad (58).
\]

These properties (55)–(58) hold even if we replace \( A, B, v, \) and \( \mu_{i,j} \) with \( A', B', v', \) and \( \mu_{i,j}' \), respectively. Since (18) was assumed for \( t = 0 \), (18) holds for all \( t \in \{0,1,2,\ldots\} \) by mathematical induction. This completes the proof.

**APPENDIX B**

**PROOF OF THEOREM 2**

We first show the sufficiency. For any \( x_0 \in \mathbb{R}^n \) and any \( \theta_0, \theta_1, \ldots \in S_{\Theta} \), we choose \( x_0, \theta_0, \theta_1, \ldots \) that satisfy (16) and (17). Theorem 1 implies that, for \( i \in \{1, \ldots, n\} \), there exists \( j \) satisfying \( E[|x_i|^2_\theta | \theta] = |x_i| \leq \|x_i\| \). If the expanded system (9) is robustly stable, we obtain the following asymptotic convergence for all \( i \in \{1, \ldots, n\} \):

\[
\lim_{t \to \infty} E[|x_i|^2_\theta | \theta] \leq \lim_{t \to \infty} \|x_i\| = 0.
\]

Therefore, the SMP system (4) is robustly MS stable.

Next, we show the necessity. For any \( \theta_0, \tilde{\theta}_1, \ldots \in \tilde{S}_{\Theta} \), there exist \( \theta_0, \theta_1, \ldots \in S_{\Theta} \) satisfying (17) because of the definition (11) of \( S_{\Theta} \). We consider \( \tilde{x}_0 \in \mathbb{R}^{\tilde{n}} \) such that (16) holds for some \( x_0 \). Then, Theorem 1 gives (18) for all \( t \). For any \( i \) and \( j \), applying the Cauchy–Schwarz inequality [55, Sec. 5.5] to \( x_i, x_j \) yields

\[
E[|x_i||x_j| | \theta] \leq E[|x_i|^2_\theta | \theta] E[|x_j|^2_\theta | \theta].
\]

If the SMP system (4) is robustly MS stable, this property indicates \( \lim_{t \to \infty} E[|x_i||x_j| | \theta] = 0 \). Combining this convergence with (18) gives as follows:

\[
\forall \theta_0, \tilde{\theta}_1, \ldots \in \tilde{S}_{\Theta}, \lim_{t \to \infty} \tilde{x}_t = 0
\]

for any \( \tilde{x}_0 \) that satisfies \( \tilde{x}_0 = \text{vec}(x_0 x_0^T) \) for some \( x_0 \). Therefore, we finally show that (61) holds for all \( \tilde{x}_0 \in \mathbb{R}^{\tilde{n}} \). Let us define \( x_0^{(i\cdot)} \in \mathbb{R}^{n_{ij}} \) such that \( x_0^{(i\cdot)}_j = |x_0^{(i\cdot)}|_j = 1 \) and the other components are zero. Let us define \( x_0^{(i\cdot)} := \text{vec}(x_0^{(i\cdot)} x_0^{(i\cdot)T}) \). The standard basis on \( \mathbb{R}^{\tilde{n}} \) consists of the unit vectors \( \tilde{x}_0^{(i\cdot)} \) and \( \tilde{x}_0^{(j\cdot)} \) for all \( i \in \{1, \ldots, n\} \) and \( j \in \{i+1, \ldots, n\} \). Then, for any \( \tilde{x}_0 \in \mathbb{R}^{\tilde{n}} \), there exist constants \( w_{i,j} \in \mathbb{R} \) satisfying

\[
\tilde{x}_0 = \sum_{i=1}^{n} w_{i,i} \tilde{x}_0^{(i\cdot)} + \sum_{j=i+1}^{n} w_{i,j} (\tilde{x}_0^{(i\cdot)} - \tilde{x}_0^{(i\cdot)})
\]

(62)

Then, for any \( \tilde{x}_0 \in \mathbb{R}^{\tilde{n}} \), we obtain

\[
\tilde{x}_t = \left( \prod_{s=0}^{t-1} C(F(\tilde{\theta}_t, K)) \right) \tilde{x}_0
\]
where \( \bar{x}_t^{(i,j)} := (\prod_{i=0}^{t-1} C(F(\bar{\theta}_i, K)))\bar{x}_0^{(i,j)} \) denotes \( \bar{x}_t \) starting from \( \bar{x}_0 = \bar{x}_0^{(i,j)} \). This implies (61) for all \( \bar{x}_0 \in \mathbb{R}^n \) because (61) with \( \bar{x}_0 = \bar{x}_0^{(i,j)} = \text{vech}(x_0^{(i,j)} - x_0^{(j,i)}) \) holds for any \( i \) and \( j \). This completes the proof.

APPENDIX C
PROOF OF THEOREM 3

To employ (18) for all \( t \in \{0, 1, 2, \ldots \} \), we consider the case that (16) and (17) hold, which are derived later. Because of the definitions of \( \bar{x}_t \) in (18) and \( \text{vech}(\cdot) \), we have \( \sum_{i=1}^{n} \|\bar{x}_t^{(i,j)}\|^2 = \sum_{i=1}^{n} \|\bar{x}_t^{(i,j)}\|^2 \geq \sum_{i=1}^{n} E[\|x_t^{(i,j)}\|^2] \). By using this inequality, the relation \( n = \sum_{i=1}^{n} 1 \), and the following Cauchy–Schwarz inequality [56, Lecture 1]:

\[
\left( \sum_{i=1}^{n} y_{a_i} \right)^2 \leq \left( \sum_{i=1}^{n} y_{a_i}^2 \right) \left( \sum_{i=1}^{n} y_{b_i}^2 \right)
\]

with the settings \( |y_{a_i}| = 1 \) and \( |y_{b_i}| = E[|x_t^{(i,j)}|] \), we obtain

\[
n \|\bar{x}_t\|^2 = \sum_{i=1}^{n} \|\bar{x}_t^{(i,j)}\|^2 \geq \sum_{i=1}^{n} \sum_{j=1}^{n} E[\|x_t^{(i,j)}\|^2] \geq \sum_{i=1}^{n} E \left[ \|x_t^{(i,j)}\|^2 \right] = E \left[ \|x_t\|^2 \right].
\]

Meanwhile, using (60) yields

\[
\|\bar{x}_t\|^2 = \sum_{i=1}^{n} \|\bar{x}_t^{(i,j)}\|^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} E \left[ \|x_t^{(i,j)}\|^2 \right] \geq E \left[ \|x_t\|^2 \right] = E \left[ \|\bar{x}_t\|^2 \right].
\]

Thus, using (60) holds for any \( \bar{x}_0 \in \mathbb{R}^n \) and any \( \bar{\theta}_0, \bar{\theta}_1, \ldots \in \mathbb{S}_\theta \) because there exist \( \bar{x}_0 \) and \( \bar{\theta}_0, \bar{\theta}_1, \ldots \) satisfying (16) and (17). If the expanded system (9) is exponentially robustly stable, substituting (65) with \( t = 0 \) and (64) into (19) yields

\[
n^{-1/2} E \left[ \|x_t\|^2 \right] \leq \|\bar{x}_t\| \leq \tilde{\alpha} \|\bar{x}_0\|^{\tilde{\beta}_t} \leq \tilde{\alpha} \|\bar{x}_0\|^{\tilde{\beta}_t}.
\]

Taking the root of this inequality gives as follows:

\[
\sqrt{E[\|x_t\|^2]} \leq n^{1/4} \tilde{\alpha}^{1/2} \|\bar{x}_0\|^{\tilde{\beta}_t/2}.
\]

Therefore, substituting (20) and \( \tilde{\alpha} \geq n^{1/4} \tilde{\alpha}^{1/2} \) yields the exponential robust MS stability in (5).

Next, we show the necessity. For any \( \bar{\theta}_0, \bar{\theta}_1, \ldots \in \mathbb{S}_\theta \), there exist \( \theta_0, \theta_1, \ldots \in \mathbb{S}_\theta \) satisfying (17) because of the definition (11) of \( \mathbb{S}_\theta \). We consider \( \bar{x}_0 \in \mathbb{R}^n \) such that (16) holds for some \( x_0 \). Then, (64) and (65) can be used. If the SMP system (4) is exponentially robustly MS stable, substituting (64) with \( t = 0 \) and (65) into the squared version of (5) yields as follows:

\[
\|\bar{x}_0\| \leq E \left[ \|x_t\|^2 \right] \leq \alpha^2 \|x_0\|^{2\beta_t} \leq \alpha^2 n^{1/2} \|\bar{x}_0\|^{2\beta_t}.
\]

This completes the proof.

APPENDIX D
PROOF OF THEOREM 4

The proof is complete by proving the stability of expanded systems because Theorems 2 and 3 provide the equivalence of the stability between the expanded systems and corresponding SMP systems. First, we prove the exponential robust stability of the TI expanded systems based on the result in [6, Th. 1 and 2] and in a manner similar to [40, Th. 1 and 2]. As explained in Remark 8, using the linearity of \( C \) yields the polytopic form as follows:

\[
C(F(\bar{\theta}_1, K)) = C(\sum_{k=1}^{N} \bar{\theta}_k, F(\theta_k(K))) = \sum_{k=1}^{N} \bar{\theta}_k C(F(\theta_k(K))).
\]

Thus, using \( P(k) \geq 0, G, \) and \( K \) satisfying (21), we obtain the following result [6, Th. 1 and 2]:

\[
\tilde{\beta}^2 P_s(\bar{\theta}_1) \geq \left( \sum_{k=1}^{N} \bar{\theta}_k C(F(\theta_k(K))) \right)^T \times P_s(\bar{\theta}_1) \left( \sum_{k=1}^{N} \bar{\theta}_k C(F(\theta_k(K))) \right)
\]

(67)
where $P_{s}(\tilde{\theta}) := \sum_{k=1}^{N} \tilde{\theta}_{k} P^{(k)} \succ 0$. Since the expanded system is supposed to be TI, $P_{s}(\tilde{\theta}) \succ 0$ is constant. Iterating (67) with multiplying by $\tilde{x}_{0}$ from the left and right sides yields as follows:

$$\tilde{\beta}^{2} \tilde{x}_{0}^{T} P_{s}(\tilde{\theta}) \tilde{x}_{0} \geq \tilde{x}_{0}^{T} P_{s}(\tilde{\theta}) \tilde{x}_{t}. \quad (68)$$

This implies the exponential stability for all $\tilde{\theta} \in \mathbb{P}_{N}$ [40, Th. 1 and 2]. Whereas the set $\mathbb{S}_{0}$ is not polytopic in general, it is included in the polytopic set $\mathbb{P}_{N}$, that is, $\mathbb{S}_{0} \subseteq \mathbb{P}_{N}$ holds. Thus, the TI expanded system (9) is exponentially robustly stable.

Next, we prove the robust stability of the TI expanded systems. Because of the finiteness of $k \in \{1, \ldots, N\}$, for any $P^{(k)} \succ 0$, $G$, and $K$ that satisfy (22), there exists $\gamma > 0$ such that

$$\forall k, \quad S^{(k)}_{\text{eqm}}(P^{(k)}, G, K, 1) \geq \gamma \begin{bmatrix} I_{n} & 0 \\ 0 & 0 \end{bmatrix} \succeq 0.$$ 

The following result is obtained in a manner similar to (67):

$$P_{s}(\tilde{\theta}) \geq C(\tilde{\theta}, K)^{T} P_{s}(\tilde{\theta}) C(\tilde{\theta}, K) + \gamma I_{n}. \quad (69)$$

This condition with $\tilde{\theta} = \tilde{\theta}$ indicates that, for any $\tilde{\theta} \in \mathbb{S}_{0} \subseteq \mathbb{P}_{N}$, the absolute values of all the eigenvalues of $C(\tilde{\theta}, K)$ is less than one [6, Lemma 1]. This implies the robust stability of the TI expanded system (9).

Finally, we prove the stability of TV expanded systems based on that of the TI systems derived above, considering the quadratic stability. For any sequence $\tilde{\theta}_{0}, \tilde{\theta}_{1}, \ldots \in \mathbb{S}_{0}$, we obtain (67), (68), and (69) with $P_{s}(\tilde{\theta}) = P$. Therefore, the exponential robust stability of the TV expanded system holds. Next, let us denote $C(\tilde{\theta}, K)$ by $F_{t}$ for brevity of notation. Iterating (69) in a manner similar to [23, Lemma 2.2] yields as follows:

$$P \succeq F_{0}^{\top} F_{0} + \gamma I_{n}$$

$$\geq F_{0}^{\top} F_{1}^{\top} F_{1} F_{0} + \gamma (I_{n} + F_{0}^{\top} F_{0})$$

$$\geq F_{0:t+1}^{\top} F_{0:t+1} + \gamma \left( I_{n} + \sum_{s=0}^{t} F_{0:s}^{\top} F_{0:s} \right) \quad (70)$$

where $F_{0:t} := F_{t} F_{t-1} \ldots F_{0}$. The positive definiteness of $P$ in (70) leads to the boundedness of $\sum_{s=0}^{t} F_{0:s}^{\top} F_{0:s} \geq 0$ that implies $\lim_{t \to \infty} F_{0:t}^{\top} F_{0:t} = 0$. Because of $\tilde{x}_{t+1} = F_{0:t} \tilde{x}_{0}$, the robust stability of the TV expanded system holds. This completes the proof.

**APPENDIX E**

**PROOF OF THEOREM 5**

We first show that (C2.1) implies (C1.1), that is, satisfying the QMI s (26) implies that the CMI s (21) hold. Let us suppose that the QMIs (26) are satisfied.

First, we prove the following relation [6] for any $X \in \mathbb{R}^{n \times n}$ and any $Y \succ 0 \in \mathbb{R}^{n_{\text{sym}}}$:

$$X + X^{\top} - Y \succ 0 \implies \exists X^{-1}. \quad (71)$$

The condition $X + X^{\top} - Y \succeq 0$ with any $y \neq 0 \in \mathbb{R}^{n}$ implies

$$y^{\top} (X + X^{\top}) y = 2y^{\top} X y \geq y^{\top} Y y > 0.$$ 

Therefore, any $y \neq 0$ satisfies $X y \neq 0$, implying that $X$ is nonsingular, that is, (71) holds.

Next, any $X \in \mathbb{R}^{n \times n}$ satisfies the following relations [48, Lemma 4.4]:

$$C_{d} C_{e} (X \otimes X) C_{d} = (X \otimes X) C_{d} \quad (72)$$

$$\det(C_{e} (X \otimes X) C_{d}) = \det(X)^{n+1} \quad (73)$$

$$\exists X^{-1} \Rightarrow (C_{e} (X \otimes X) C_{d})^{-1} = C_{e} (X^{-1} \otimes X^{-1}) C_{d}. \quad (74)$$

Because (26), (71), and the positive definiteness of $Q^{(k)}$ hold, $C(H \otimes H)$ is nonsingular. Substituting $C(H \otimes H)$ into (73) gives $\det(H) \neq 0$, that is, $H$ is nonsingular.

Next, we use the settings (23), (24), and (25). By using the relation $(X_{1} \otimes X_{2})(X_{3} \otimes X_{4}) = X_{1} X_{3} \otimes X_{2} X_{4}$ [54, Sec. 3.2.9], (72), and (74), we transform $F_{\text{qmi}}^{(k)}(L, H)$ as follows:

$$G^{\top} F_{\text{qmi}}^{(k)}(L, H) G$$

$$= G^{\top} F_{\text{qmi}}^{(k)}(L, H) C(H \otimes H)^{-1} \quad (75)$$

$$= G^{\top} C_{e} \left( F_{ab}^{(k)}(H \otimes H) - F_{ab}^{(k)}(H \otimes L) \right) \quad (76)$$

$$- F_{ba}^{(k)}(L \otimes H) + F_{bb}^{(k)}(L \otimes L) \right) \left( C_{d} C_{e}(H^{-1} \otimes H^{-1}) C_{d} \right.$$

$$= G^{\top} C_{e} \left( F_{ab}^{(k)}(H \otimes H) - F_{ab}^{(k)}(H \otimes L) \right)$$

$$- F_{ba}^{(k)}(L \otimes H) + F_{bb}^{(k)}(L \otimes L) \right) \left( H^{-1} \otimes H^{-1} \right) \quad (77)$$

$$= G^{\top} C_{e} \left( F_{ab}^{(k)}(I_{n} \otimes I_{n}) - F_{ab}^{(k)}(I_{n} \otimes K) \right)$$

$$- F_{ba}^{(k)}(K \otimes I_{n}) + F_{bb}^{(k)}(K \otimes K) \right) C_{d} \quad (78)$$

$$= G^{\top} C(F^{(k)}(K)).$$

Therefore, if the QMIs (26) hold, the CMI s (21) are satisfied as follows:

$$S_{\text{eqm}}^{(k)}(P^{(k)}, G, K, \tilde{\beta})$$

$$= \begin{bmatrix} \tilde{\beta}^{2} G^{\top} Q^{(k)} G & C(F^{(k)}(K))^{\top} G \\ G^{\top} C(F^{(k)}(K)) & G^{\top} + G^{\top} Q^{(k)} G \end{bmatrix}$$

$$= \begin{bmatrix} G^{\top} & 0 \\ 0 & G^{\top} \end{bmatrix} S_{\text{qmi}}^{(k)}(Q^{(k)}, H, L, \tilde{\beta}) \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}$$

$$\succeq 0 \quad (75)$$

where the above block diagonal matrix using $G$ is nonsingular. We can prove that (C2.2) implies (C1.2) in a manner similar to this proof, by taking the strict inequality in (26) and (75) and substituting $\tilde{\beta} = 1$ into them. In addition, (23) indicates that $P^{(k)}$ are identical if $Q^{(k)}$ are identical. This complete the proof.

**APPENDIX F**

**PROOF OF PROPOSITION 1**

Combining (27) with (33) yields the following:

$$Z = \begin{bmatrix} \text{vec} \ (H) \\ \text{vec} \ (L) \end{bmatrix}^{\top} = \begin{bmatrix} Z_{1,1} & \cdots & Z_{1,2n} \\ \vdots & \ddots & \vdots \\ Z_{2n,1} & \cdots & Z_{2n,2n} \end{bmatrix}.$$
In a manner similar to (55), for any \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, n\} \), we obtain
\[
[\mathbf{H} \otimes \mathbf{L}]_{i, n(j-1)+i} = \begin{bmatrix} [H]_{i,j} & [L]_{i,i} \\ \vdots & \vdots \\ [H]_{n,j} & [L]_{i,i} \end{bmatrix} = \text{vec} \left( [L]_{i,i} [H]_{i,j}^\top \right)
\]
\[
= \text{vec}(Z_{n+i,j}).
\]
In the same manner, we have \([\mathbf{H} \otimes \mathbf{H}]_{i, n(j-1)+i} = \text{vec}(Z_{i,j})\), \([\mathbf{L} \otimes \mathbf{H}]_{i, n(j-1)+i} = \text{vec}(Z_{n+i,j})\), and \([\mathbf{L} \otimes \mathbf{L}]_{i, n(j-1)+i} = \text{vec}(Z_{n+i,n+j})\). Substituting these relations into (29)–(32) yields (34)–(37). This completes the proof.

**APPENDIX G**

**PROOF OF THEOREM 6**

If (C2.1) holds, we use \( Z \) given in (27) of which the rank is zero or one. Because \( H \) is nonsingular, \( Z \neq 0 \) and thus \( \text{rank}(Z) \neq 0 \). Thus, we have \( \text{rank}(Z) = 1 \). Substituting (34)–(37) yields \( S_{k}(k)(Q(k), Z, \theta) = S_{k}(k)(Q(k), H, L, \theta) \geq 0 \) which implies (C3.1). Next, if (C3.1) holds, using (27) gives \( H \) and \( L \) because of \( \text{rank}(Z) = 1 \). Substituting these matrices yields \( S_{k}(k)(Q(k), H, L, \theta) = S_{k}(k)(Q(k), Z, \theta) \geq 0 \) that implies (C2.1). In the same way, we can prove the equivalence between (C2.2) and (C3.2). This completes the proof.

**APPENDIX H**

**PROOF OF LEMMA 1**

For any \( Z \succeq 0, \lambda_i(Z) \geq 0 \) holds for \( i \in \{1, \ldots, n(n+m)\} \). Using \( \nu_i(Z) = \nu_i(Z) - \lambda_i(Z) \nu_i(Z) = \lambda_i(Z) \) and \( \text{tr}(Z) = \sum_{i=1}^{n(n+m)} \lambda_i(Z) \) yields the following:
\[
\tilde{e}(Z, Z^\top) = \left( \sum_{i=1}^{n(n+m)} |\lambda_i(Z)| \right) - |\lambda_1(Z)| = e(Z).
\]
Next, we choose unit eigenvectors \( \nu_i(Z) \) such that \( [\nu_1(Z), \ldots, \nu_{n(n+m)}(Z)] \) is orthogonal because \( Z \neq 0 \) is symmetric [54, Sec. 3.8.7]. This choice enables the use of the spectral decomposition \( I_{n(n+m)} = \sum_{i=1}^{n(n+m)} \nu_i(Z)^\top \nu_i(Z) \) [54, Sec. 3.8.7] and the results in [50, Lemma 4.2] with the replacement of singular values with eigenvalues. Thus, we obtain
\[
\tilde{e}(Z, Z^\top) = \text{tr}(Z) - \text{tr}(\nu_1(Z^\top) \nu_1(Z^\top) Z)
\]
\[
= \text{tr} \left( I_{n(n+m)} - \nu_1(Z^\top) \nu_1(Z^\top) Z \right)
\]
\[
= \text{tr} \left( \sum_{j=2}^{n(n+m)} \nu_j(Z^\top) \nu_j(Z^\top) Z \right)
\]
\[
\geq \sum_{i=1}^{n(n+m)} \nu_i(Z^\top) \nu_i(Z^\top) \lambda_{n(n+m)-i+1}(Z)
\]
\[
= \sum_{i=1}^{n(n+m)} \lambda_{n(n+m)-i+1}(Z)
\]
\[
= \sum_{i=2}^{n(n+m)} |\lambda_i(Z)|.
\]
Note that the above inequality is derived from the property \( \sum_{i=1}^{n} \lambda_i(Y_1 \lambda_{n-i+1}(Y_2) \leq \text{tr}(Y_1 Y_2) \) for any \( Y_1 \geq 0 \in \mathbb{R}^{n \times n} \) and \( Y_2 \geq 0 \in \mathbb{R}^{n \times n} \) [50, Lemma 4.1][57, Th. 2.1]. This completes the proof.

**APPENDIX I**

**PROOF OF THEOREM 7**

First, we prove the statement (i). Using the i.i.d. property and (44)–(46) gives \( E[\nu_i(\theta) \nu_i(\theta)^\top] = E[\nu_i(\theta) \nu_i(\theta)^\top] = M^{(1)} \), which corresponds to the condition (2). Because of (45), \( \phi(\theta) \in \mathbb{P}_N \) holds clearly. All the conditions for the SMP systems in Definition 1 are satisfied, implying the statement (i).

Next, we prove the statement (iii). Using (48) yields \( \theta|\kappa \theta| = \text{vec}(\theta \theta^\top)_{d_\theta(k-1)+k} = [\phi(\theta)]_{d_\theta(k-1)+k} \). Using this relation and (49) yields the following:
\[
E[\nu_i(\theta) \nu_i(\theta)^\top] = \text{vec}(\theta \theta^\top)_{d_\theta(k-1)+k}
\]
\[
= \text{vec}(\theta \theta^\top)_{d_\theta(k-1)+k} + \phi(\theta)_{d_\theta(k-1)+k}
\]
\[
= \sum_{k=1}^{N} \phi(\theta)_{k} M(k).
\]
In addition, for any \( \theta \in \mathbb{S}_\theta \), \( \sum_{k=1}^{N} \phi(\theta)_{k} = \sum_{k=1}^{d_\theta} |\theta_k| \kappa | \theta_k | = 1 \) and \( \phi(\theta) \in \mathbb{P}_N \) holds for any \( \theta \in \mathbb{S}_\theta \). This indicates \( \phi(\theta) \in \mathbb{P}_N \) holds for any \( \theta \in \mathbb{S}_\theta \). This result and (76) satisfy all the conditions for the SMP systems in Definition 1, implying the statement (iii).

Next, we prove the statement (ii), where \( \phi(\theta) \in \mathbb{P}_N \) was already proved. We can derive the condition (2) in a manner similar to (76) by excluding the expectation. Thus, the statement (ii) holds.

Finally, we prove the statement (iv), where \( \phi(\theta) \in \mathbb{P}_N \) was already proved. In a manner similar to (76), we obtain
\[
E[\nu_i(\theta) \nu_i(\theta)^\top] = \text{vec}(\theta \theta^\top)_{d_\theta(k-1)+k}
\]
\[
= \sum_{k=1}^{N} \phi(\theta)_{k} M(k)_{d_\theta(k-1)+k}
\]
Therefore, the statement (iv) holds. This completes the proof.
APPENDIX J

PROOF OF PROPOSITION 2

Let us consider the TI version of the SMP system involving $\bar{\nu}_k(\bar{\theta})$ in Example 1 with the following settings: $n = m = 1$, $d_0 = 2$, $\Sigma = \mathbb{P}^N$. $N = 2$, $\mu = 0$, $\{\Sigma(k)\}_{1,1} = \{\Sigma(k)\}_{2,2} = \sigma_0$, and $\{\Sigma(k)\}_{1,2} = \{\Sigma(k)\}_{2,1} = \sigma_2$, where $\sigma_2$ is a constant satisfying $\sigma_2 \neq \sigma_2$. Let us suppose that this SMP system is represented by a random polytope with $v_k(\theta)$ given in Theorem 7 (iii), that is, (52) and (53) are assumed to hold. Then, we obtain

$$
\sum_{k=1}^{d_0} \sum_{k'=1}^{d_0} [\theta_k]_k E \left[ \left[ v_k(\theta) v_k(\theta)^\top \right]_{k'2} \right] = E \left[ v_1(\theta) v_1(\theta)^\top \right]_{\theta_1} = E \left[ \sigma_0 \bar{\theta}_1 \sigma_1 + \bar{\theta}_2 \sigma_2 \right].
$$

(77)

For each $k$, substituting $[\theta_k]_k = 1$ into (77) yields as follows:

$$
E[v_k(\theta)_1] = \sigma_0.
$$

For each $(k, k') \in \{1, \ldots, d_0\}^2$, substituting $[\theta_k]_k = [\theta_{k'}]_{k'} = 1/2$ into (77) gives the following:

$$
E[v_k(\theta)_1^2] + E[v_k(\theta)_1 v_k(\theta)_{k'}_1] = \frac{1}{4} \sigma_0.
$$

This implies $E[v_k(\theta)_1 v_k(\theta)_{k'}_1] = \sigma_0$. Then, applying the Cauchy–Schwarz inequality [55, Sec. 5.5] to $v_k(\theta)_1 v_k(\theta)_{k'}_1$ gives

$$
E\left[ \left( v_k(\theta)_1^2 \right) \left( v_k(\theta)_1 - v_k(\theta)_{k'}_1 \right)^2 \right] = E[v_k(\theta)_1^2] E[v_k(\theta)_1 - v_k(\theta)_{k'}_1]^2 \\ \leq E[v_k(\theta)_1^2] E[v_k(\theta)_1 - v_k(\theta)_{k'}_1]^2 \\
\leq E[v_k(\theta)_1^2] \left( E[v_k(\theta)_1^2] - 2 E[v_k(\theta)_1 v_k(\theta)_{k'}_1] \right) = 0.
$$

In a similar manner, $E[v_k(\theta)_1 v_k(\theta)_{k'}_1] = 0$ is derived. Therefore, substituting these results into (77) yields the following:

$$
\sum_{k=1}^{d_0} \sum_{k'=1}^{d_0} [\theta_k]_k E \left[ \left[ v_k(\theta)_1 v_k(\theta)_{k'}_1 \right]^2 \right] = 0.
$$

The result contradicts the condition $\sigma \neq \sigma_2$ because $[\theta_1]_1$ is not constant. Therefore, whatever value is set for $v_k(\theta)_1$, there exists $[\theta_1]_1$ such that (77) is never satisfied regardless of $\theta$. This completes the proof.

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