FOLDING PHENOMENON OF MAJOR-BALANCE IDENTITIES ON
RESTRICTED INVOLUTIONS

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Abstract. In this paper we prove a refined major-balance identity on the 321-avoiding
involutions of length \( n \), respecting the leading element of permutations. The proof is based
on a sign-reversing involution on the lattice paths within a \( \lfloor \frac{n}{2} \rfloor \times \lceil \frac{n}{2} \rceil \) rectangle. Moreover, we prove affirmatively a question about refined major-balance identity on the 123-avoiding
involutions, respecting the number of descents.

1. Introduction

Let \( \mathfrak{S}_n \) be the set of permutation of \( \{1, 2, \ldots, n\} \). A permutation \( \sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n \)
is called 321-avoiding (123-avoiding, respectively) if it has no decreasing (increasing, re-
spectively) subsequence of length three. Let \( \mathfrak{S}_n(321) \) (\( \mathfrak{S}_n(123) \), respectively) be the set of
321-avoiding (123-avoiding, respectively) permutations in \( \mathfrak{S}_n \). It is known that \(|\mathfrak{S}_n(321)| = C_n = \frac{1}{n+1} \binom{2n}{n} \), the \( n \)th Catalan number.

1.1. Sign-balance for restricted permutations. The sign-balance of restricted permuta-
tions is an interesting theme in enumerative combinatorics. Simion and Schmidt \[8\] proved the following sign-balance property of \( \mathfrak{S}_n(321) \).

\[
\sum_{\sigma \in \mathfrak{S}_n(321)} (-1)^{\text{inv}(\sigma)} q^{\text{ldes}(\sigma)} = \begin{cases} C_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}, \end{cases}
\]

where \( \text{inv}(\sigma) = \#\{(\sigma_i, \sigma_j) : \sigma_i > \sigma_j \text{ and } i < j\} \) is the inversion number of \( \sigma \). Adin and Roichman \[1\] proved a refinement of this result, respecting the position of the last descent (ldes) of \( \sigma \), i.e., \( \text{ldes}(\sigma) = \max\{i : \sigma_i > \sigma_{i+1} \text{ and } 1 \leq i \leq n-1\} \).

**Theorem 1.1 (Adin-Roichman).** For all \( n \geq 1 \), the following identities hold.

\[
(i) \quad \sum_{\sigma \in \mathfrak{S}_{2n+1}(321)} (-1)^{\text{inv}(\sigma)} q^{\text{ldes}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n(321)} q^{2 \cdot \text{ldes}(\sigma)},
\]

\[
(ii) \quad \sum_{\sigma \in \mathfrak{S}_{2n}(321)} (-1)^{\text{inv}(\sigma)} q^{\text{ldes}(\sigma)} = (1-q) \sum_{\sigma \in \mathfrak{S}_n(321)} q^{2 \cdot \text{ldes}(\sigma)}.
\]

Later, Reifegerste \[7\] proved an analogous refinement for another permutation statistic,
the length of the longest increasing subsequence of the permutations. Eu et al. \[4\] turned
to other families of restricted permutations and obtained refined sign-balance result for
321-avoiding alternating permutations, respecting the leading element and the last element
of permutations, respectively.

These results share a folding phenomenon that with respect to a certain statistic the
sign-balance generating function for restricted permutations of length \( 2n \) essentially equals
the ordinary generating function for the permutations of length \( n \). Eu et al. \cite{Eu} described the folding phenomenon in the framework

\[
\sum_{\pi \in \mathcal{S}_n} (-1)^{\text{maj}(\pi)} q^{\text{des}(\pi)} = f(q) \sum_{\pi \in \mathcal{S}_n} q^{2 \cdot \text{maj}(\pi)},
\]

where \( \mathcal{S}_n \) is a family of combinatorial objects of size \( n \) with statistics \( \text{stat}_1 \) and \( \text{stat}_2 \), and \( f(q) \) is a rational function. In this paper, we present an instance of such a phenomenon on 321-avoiding involutions.

1.2. Major-balance for 321-avoiding involutions. The descent set of \( \sigma \) is defined as \( \text{Des}(\sigma) = \{ i : \sigma_i > \sigma_{i+1}, 1 \leq i \leq n-1 \} \), and the descent number (\( \text{des} \)) and major index (\( \text{maj} \)) of \( \sigma \) are defined by

\[
\text{des}(\sigma) = \sum_{i \in \text{des}(\sigma)} 1 \quad \text{and} \quad \text{maj}(\sigma) = \sum_{i \in \text{des}(\sigma)} i.
\]

Recall that a permutation \( \sigma \) is called an involution if and only if \( \sigma^{-1} = \sigma \). Let \( \mathcal{I}_n(321) \) (\( \mathcal{I}_n(123) \), respectively) be the set of involutions in \( \mathfrak{S}_n(321) \) (\( \mathfrak{S}_n(123) \), respectively). Simion and Schmidt \cite{Simion} proved that \( |\mathcal{I}_n(321)| = |\mathcal{I}_n(123)| = \binom{n}{n/2} \). Recently, Eu et al. \cite{Eu} proved the following refined major-balance result on 321-avoiding involutions.

**Theorem 1.2 (Eu-Fu-Pan-Ting).** For all \( n \geq 1 \), the following identities hold.

(i) \[
\sum_{\sigma \in \mathcal{I}_n(321)} (-1)^{\text{maj}(\sigma)} q^{\text{des}(\sigma)} = \sum_{\sigma \in \mathcal{I}_n(321)} q^{2 \cdot \text{des}(\sigma)},
\]

(ii) \[
\sum_{\sigma \in \mathcal{I}_{n+2}(321)} (-1)^{\text{maj}(\sigma)} q^{\text{des}(\sigma)} = (1 - q) \sum_{\sigma \in \mathcal{I}_n(321)} q^{2 \cdot \text{des}(\sigma)},
\]

(iii) \[
\sum_{\sigma \in \mathcal{I}_{n+1}(321)} (-1)^{\text{maj}(\sigma)} q^{\text{des}(\sigma)} = \sum_{\sigma \in \mathcal{I}_n(321)} q^{2 \cdot \text{des}(\sigma)}.
\]

Meanwhile, they asked a question about an analogous result for 123-avoiding involutions.

**Conjecture 1.3 (Eu-Fu-Pan-Ting).** For all \( n \geq 1 \), the following identities hold.

(i) \[
\sum_{\sigma \in \mathfrak{S}_n(123)} (-1)^{\text{maj}(\sigma)} q^{\text{des}(\sigma)} = q \sum_{\sigma \in \mathfrak{S}_n(123)} q^{2 \cdot \text{des}(\sigma)},
\]

(ii) \[
\sum_{\sigma \in \mathfrak{S}_{n+2}(123)} (-1)^{\text{maj}(\sigma)} q^{\text{des}(\sigma)} = (1 - q) q^2 \sum_{\sigma \in \mathfrak{S}_{n+2}(123)} q^{2 \cdot \text{des}(\sigma)},
\]

(iii) \[
\sum_{\sigma \in \mathfrak{S}_{n+1}(123)} (-1)^{\text{maj}(\sigma)} q^{\text{des}(\sigma)} = (-1)^n q^2 \sum_{\sigma \in \mathfrak{S}_n(123)} q^{2 \cdot \text{des}(\sigma)}.
\]

1.3. Our work. For a permutation \( \sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n \), let \( \text{lead}(\sigma) \) denote the first element of \( \sigma \), i.e., \( \text{lead}(\sigma) = \sigma_1 \). Recall that the \( q \)-binomial coefficients are polynomials defined as

\[
\binom{n}{k}_q := \frac{[n]_q}{[k]_q [n-k]_q},
\]

where \([n]_q = [1]_q [2]_q \cdots [n]_q \) and \([i]_q = 1 + q + \cdots + q^{i-1} \).

In addition to answering the above question, one of the main results in this paper is the following enumeration of joint distributions for two statistics of 321-avoiding involutions.

**Theorem 1.4.** We have
Theorem 1.5. For all $n \geq 1$, we have

\begin{align*}
(i) \quad & \sum_{\sigma \in \mathcal{I}_n(321)} (-1)^{maj(\sigma)} q^{lead(\sigma)} = \frac{1}{q} \sum_{\sigma \in \mathcal{I}_{2n}(321)} q^{2 \cdot lead(\sigma)}, \\
(ii) \quad & \sum_{\sigma \in \mathcal{I}_{n+2}(321)} (-1)^{maj(\sigma)} q^{lead(\sigma)} = \left( \frac{1}{q} - 1 \right) \sum_{\sigma \in \mathcal{I}_{2n+1}(321)} q^{2 \cdot lead(\sigma)}, \\
(iii) \quad & \sum_{\sigma \in \mathcal{I}_{n+3}(321)} (-1)^{maj(\sigma)} q^{lead(\sigma)} = \left( \frac{2}{q} - 1 \right) \sum_{\sigma \in \mathcal{I}_{2n+1}(321)} q^{2 \cdot lead(\sigma)}, \\
(iv) \quad & \sum_{\sigma \in \mathcal{I}_{n+1}(321)} (-1)^{maj(\sigma)} q^{lead(\sigma)} = \left( \frac{1}{q} - 1 \right) \sum_{\sigma \in \mathcal{I}_{2n+1}(321)} q^{2 \cdot lead(\sigma)} + \sum_{\sigma \in \mathcal{I}_{2n}(321)} q^{2 \cdot lead(\sigma)}.
\end{align*}

2. A bijection between 321-avoiding permutations and grand Dyck paths

Let $m, n$ be positive integers. A Dyck path of length $2n$ is a lattice path from $(0,0)$ to $(n,n)$, using north step $(0,1)$ and east step $(1,0)$, that stays weakly above the line $y = x$. A partial Dyck path of length $n$ is a lattice path from $(0,0)$ to the line $x = n$ staying weakly above the line $y = x$. Let $\mathcal{P}_n$ be the set of partial Dyck paths of length $n$. A grand Dyck path of length $m+n$ is a lattice path from $(0,0)$ to $(m,n)$ without restriction. Let $\mathcal{B}(n,m)$ denote the set of grand Dyck paths from $(0,0)$ to $(m,n)$. Let $N$ and $E$ denote a north step and an east step, respectively.

We shall give combinatorial proofs of Theorem 1.4 and Theorem 1.5 on the basis of a bijection between $\mathcal{I}_n(321)$ and $\mathcal{B}(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$ given by Barnabei et al. [2]. With the partial Dyck paths $\mathcal{P}_n$ being the intermediate stage, the bijection is the composition of two maps $\delta : \mathcal{I}_n(321) \rightarrow \mathcal{P}_n$ and $\xi : \mathcal{P}_n \rightarrow \mathcal{B}(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$. First, we describe the map $\delta : \mathcal{I}_n(321) \rightarrow \mathcal{P}_n$ given by Deutsch et al. [3].

2.1. The bijection $\delta : \mathcal{I}_n(321) \rightarrow \mathcal{P}_n$. Given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{I}_n(321)$, we associate $\sigma$ with a path $\tau = \delta(\sigma) = z_1 \ldots z_n \in \mathcal{P}_n$, where $z_i = N$ if $\sigma_i \geq i$ and $z_i = E$ if $\sigma_i < i$.

To find $\delta^{-1}$, we label the steps of $\tau$ from left to right by $1, 2, \ldots, n$. Proceeding from right to left across $\tau$, couple each $E$ step with the nearest uncoupled $N$ step to its left. Then the cycle structure of the involution $\delta^{-1}(\tau)$ can be determined by taking the labels of a coupled
pair as a transposition and an uncoupled \(N\) step as a fixed point. Recall that a permutation is an involution if its cycle structure contains no cycle of length greater than two.

**Example 2.1.** Consider \(\sigma = 2136745810911 \in \mathcal{A}_{11}(321)\). The partial Dyck path \(\tau = \delta(\sigma)\) is shown on the left hand side of Figure 1. For the inverse map, if we label the steps from left to right by 1, 2, \ldots, 11 and traverse \(\tau\) backward, then we obtain the cycle structure of \(\sigma = \delta^{-1}(\tau)\), i.e., \((12)(3)(46)(57)(8)(910)(11)\).

![Figure 1](image)

**Figure 1.** An example for the two-stage bijection \(\xi \circ \delta\) between \(\mathcal{P}_{11}\) and \(\mathcal{D}(5, 6)\).

A peak at position \(i\) of \(\tau\) is an occurrence \(z_i z_{i+1} = \text{NE}\), which is sometimes identified with the point \(p\) between \(z_i\) and \(z_{i+1}\). Note that the coordinate \((x, y)\) of \(p\) satisfies \(x + y = i\) and that every descent \(\sigma_i > \sigma_{i+1}\) of \(\sigma\) is carried to a peak at position \(i\) of \(\tau\). We observe that if all fixed points in \(\sigma\) are ignored, each east step of \(\tau\) must be coupled with a remaining north step to its left. A valley of \(\tau\) is an occurrence of \(\text{EN}\). A lattice point with coordinate \((x, y)\) is said to be even (odd, respectively) if \(x + y\) is even (odd, respectively). For convenience, we say that a peak or valley \(p\) is odd (even, respectively) if \(p\) is an odd (even, respectively) lattice point. Let \(\text{sump}(\tau)\) be the sum of the \(x\)-coordinates and \(y\)-coordinates of all peaks in \(\tau\).

2.2. **The bijection** \(\xi : \mathcal{P}_n \rightarrow \mathcal{D}([\frac{k}{3}], [\frac{n}{2}])\). Let \(\tau \in \mathcal{P}_n\) be a partial Dyck path with \(k\) more \(N\) steps than \(E\) steps \((0 \leq k \leq n)\). Match the \(N\) steps and \(E\) steps that face each other, in the sense that the line segment from the midpoint of \(N\) to the midpoint of \(E\) has slope 1 and stays below the path \(\tau\). Then we construct the path \(\xi(\tau) \in \mathcal{D}([\frac{k}{3}], [\frac{n}{2}])\) by changing the first \([\frac{k}{3}]\) unmatched \(N\) steps into \(E\) steps. Note that a peak \((x, y)\) in \(\tau\) is carried to a peak \((x', y')\) in \(\xi(\tau)\) with \(x + y = x' + y'\).

**Example 2.2.** Following Example 2.1, consider the path \(\tau = z_1 \cdots z_{11} \in \mathcal{P}_{11}\) shown on the left hand side of Figure 1. Note that \(\text{sump}(\tau) = 15\). The unmatched \(N\) steps are \(z_3, z_8\) and \(z_{11}\). Then the corresponding path \(\xi(\tau) \in \mathcal{D}(5, 6)\) is obtained from \(\tau\) by changing \(z_3\) and \(z_8\) into \(E\) steps, shown on the right hand side of Figure 1. Note that the peaks \((1, 4)\) and \((3, 6)\) of \(\tau\) are carried to the peaks \((2, 3)\) and \((5, 4)\) of \(\xi(\tau)\), respectively and \(\text{sump}(\xi(\tau)) = 15\).

To construct \(\xi^{-1}\), given a grand Dyck path \(\pi \in \mathcal{D}([\frac{p}{3}], [\frac{q}{2}])\), match the \(N\) steps and \(E\) steps that face each other in \(\pi\). Then the path \(\xi^{-1}(\pi)\) is recovered from \(\pi\) by changing the remaining unmatched \(E\) steps into \(N\) steps.

The following properties about the statistics \(\text{maj}(\sigma)\) and \(\text{lead}(\sigma)\) hold.
Lemma 2.3. Given a permutation $\sigma \in \mathcal{S}_n(321)$ with $\text{lead}(\sigma) = \ell$, let $\tau = \delta(\sigma) \in \mathcal{P}_n$ and let $\pi = \xi(\tau) \in \mathcal{B}(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$. Then the following results hold.

(i) $\text{maj}(\sigma) = \text{sump}(\tau) = \text{sump}(\pi)$.

(ii) $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor + 1$.

(iii) The path $\pi$ passes through the points $(0, \ell - 1)$ and $(1, \ell - 1)$.

(iv) The number of permutations $\sigma \in \mathcal{S}_n(321)$ with $\text{lead}(\sigma) = \ell$ is $\binom{n-\ell}{n/2-1}$.

Proof. Note that every $i \in \text{Des}(\sigma)$ corresponds to a peak $(x, y)$ of $\tau$ and a peak $(x', y')$ of $\pi$ with $x + y = x' + y' = i$. The assertion (i) follows.

Since $\text{lead}(\sigma) = \ell$, $\sigma_\ell = 1$. We observe that either $\ell = 1$ or $\sigma_1 < \cdots < \sigma_{\ell-1}$ if $\ell > 1$ since $\sigma$ is 321-avoiding. If $\ell > 1$ then the entries $2, \ldots, \ell - 1$ appear to the right of $\sigma_\ell$ and hence $2\ell - 2 \leq n$. Moreover, by the construction of the maps $\delta$ and $\xi$, we observe that the grand Dyck path $\pi$ has the prefix $N^{\ell-1}E$. The assertions (ii) and (iii) follow.

Note that the number of permutation $\sigma \in \mathcal{S}_n(321)$ with $\text{lead}(\sigma) = \ell$ coincides with the number of lattice paths from the point $(1, \ell - 1)$ to the point $(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor)$. The assertion (iv) follows. \hfill \square

3. A combinatorial proof of Theorem 1.4

Define the $k$-th elementary symmetric functions in $n$ variables

$$e_k = e_k(x_1, x_2, \ldots, x_n) := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1}x_{i_2}\cdots x_{i_k}.$$  

Note that $e_0(x_1, \ldots, x_n) = 1$ and $e_k(x_1, \ldots, x_n) = 0$ for $k > n$. Recall the principle specialization of $e_k(x_1, \ldots, x_n)$ (see e.g. [9, Prop. 7.8.3]):

$$e_k(1, 1, \ldots, 1) = \binom{n}{k} \quad \text{and} \quad e_k(1, q, \ldots, q^{n-1}) = q^{k \binom{n}{k}}.$$  

Proof of Theorem 1.4. (i) By the bijection $\xi \circ \delta$ between $\mathcal{S}_n(321)$ and $\mathcal{B}(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$, a permutation $\sigma \in \mathcal{S}_n(321)$ with $\text{des}(\sigma) = k$ is mapped to a grand Dyck path $\pi = \xi(\delta(\sigma))$ with $k$ peaks, say $(x_1, y_1), \ldots, (x_k, y_k)$ with $0 \leq x_1 < \cdots < x_k \leq \lfloor \frac{n}{2} \rfloor - 1$ and $1 \leq y_1 < \cdots < y_k \leq \lfloor \frac{n}{2} \rfloor$. Moreover, $\text{maj}(\sigma) = \text{sump}(\pi) = x_1 + \cdots + x_k + y_1 + \cdots + y_k$. Hence

$$\sum_{\sigma \in \mathcal{S}_n(321) \atop \text{des}(\sigma) = k} q^{\text{maj}(\sigma)} = \sum_{0 \leq x_1 < \cdots < x_k \leq \lfloor \frac{n}{2} \rfloor - 1 \atop 1 \leq y_1 < \cdots < y_k \leq \lfloor \frac{n}{2} \rfloor} q^{x_1 + \cdots + x_k} \cdot q^{y_1 + \cdots + y_k} = e_k(1, q, \ldots, q^{\lfloor n/2 \rfloor - 1}) \cdot e_k(q, q^2, \ldots, q^{\lfloor n/2 \rfloor}) = q^{k^2 \binom{\lceil n/2 \rceil}{k} \binom{\lfloor n/2 \rfloor}{k}}.$$  

(ii) Let $\sigma \in \mathcal{S}_n(321)$ be a permutation with $\text{lead}(\sigma) = \ell$ and $\text{des}(\sigma) = k$. Then the corresponding grand Dyck path $\pi = \xi(\delta(\sigma))$ can be factorized as $\pi = N^{\ell-1}E\mu$. If $\ell = 1$ then the segment $\mu$ contains $k$ peaks, say $(x_1, y_1), \ldots, (x_k, y_k)$ with $1 \leq x_1 < \cdots < x_k \leq \lfloor \frac{n}{2} \rfloor - 1$.
and $1 \leq y_1 < \cdots < y_k \leq \left\lfloor \frac{n}{q} \right\rfloor$. Hence
\[ \sum_{\sigma \in \mathcal{S}_n(321) \atop \text{des}(\sigma) = k} q^{\text{maj}(\sigma)} = e_k(q, q^2, \ldots, q^{\left\lfloor \frac{n}{q} \right\rfloor - 1}) \cdot e_k(q, q^2, \ldots, q^{\frac{n}{q}}) \]
\[ = q^{k^2 + k} \left[ \frac{n}{q} \right] - 1 \choose k \right] \left[ \frac{n}{q} \right] \right]_q. \]
Otherwise $\ell > 1$, and the segment $\mu$ contains another $k-1$ peaks, say $(x_1, y_1), \ldots, (x_{k-1}, y_{k-1})$ with $1 \leq x_1 < \cdots < x_{k-1} \leq \left\lfloor \frac{n}{q} \right\rfloor - 1$ and $\ell \leq y_1 < \cdots < y_{k-1} \leq \left\lfloor \frac{n}{q} \right\rfloor$. Hence
\[ \sum_{\sigma \in \mathcal{S}_n(321) \atop \text{des}(\sigma) = k} q^{\text{maj}(\sigma)} = q^{\ell - 1}e_{k-1}(q, q^2, \ldots, q^{\frac{n}{q} - 1})e_{k-1}(q^\ell, q^{\ell+1}, \ldots, q^{\frac{n}{q}}) \]
\[ = q^{(k-1)^2 + (k-1)\ell + \ell - 1} \left[ \frac{n}{q} \right] - 1 \choose k - 1 \right] \left[ \frac{n}{q} \right] - \ell + 1 \right] \right]_q. \]
The assertion follows. \hfill \square

We remark that Barnabei et al. \cite{Barnabei} proved (i) of Theorem 1.4 by establishing a bijection between the paths in $\mathcal{B}(\left\lfloor \frac{n}{q} \right\rfloor, \left\lfloor \frac{n}{q} \right\rfloor)$ and the partitions whose Young diagrams fit inside the $\left\lfloor \frac{n}{q} \right\rfloor \times \left\lfloor \frac{n}{q} \right\rfloor$-rectangle so that the descent set of $\sigma \in \mathcal{S}_n(321)$ is carried to the hook-decomposition of the mapped partition.

With the result in (ii) of Theorem 1.4, we give an arithmetic verification of Theorem 1.3 as follows. For positive integers $m, n$, we have the following facts (i) $[m]_{q=-1} = 0$ if and only if $m$ is even, and (ii) if $m, n$ have the same parity, then
\[ \lim_{q \to -1} \frac{[n]_q}{[m]_q} = \begin{cases} \frac{n}{m} & \text{if } m, n \text{ are even} \\ 1 & \text{if } m, n \text{ are odd} \end{cases} \]
Making use of the above facts, we observe that
\[ \left[ \frac{n}{k} \right]_{q=-1} = \lim_{q \to -1} \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[1]_q[2]_q \cdots [k]_q} = \begin{cases} 0 & \text{if } n \text{ is even and } k \text{ is odd} \\ \left[ \frac{n}{k} \right] \left[ \frac{k}{2} \right] & \text{otherwise}. \end{cases} \tag{1} \]
Now, we verify the identity in (i) of Theorem 1.5. By (ii) of Theorem 1.4, we have the left hand side
\[ \sum_{\sigma \in \mathcal{S}_n(321) \atop \text{des}(\sigma) = k} (-1)^{\text{maj}(\sigma)} = \lim_{q \to -1} \sum_{k \geq 0} q^{k^2 + k\ell + \ell - 1} \left[ \frac{2n-1}{k} \right]_q \left[ \frac{2n-\ell+1}{k} \right]_q. \]
If $\ell$ is odd, say $\ell = 2\ell' - 1$ then by Eq. (1) we have
\[ \sum_{\sigma \in \mathcal{S}_n(321) \atop \text{lead}(\sigma) = 2\ell' - 1} (-1)^{\text{maj}(\sigma)} = \sum_{k' \geq 0} \binom{n-1}{k'} \binom{n-\ell'+1}{k'} \]
\[ = \binom{2n-\ell'}{n-1} = \{ \sigma \in \mathcal{S}_{2n}(321) : \text{lead}(\sigma) = \ell' \}. \]
If \( \ell \) is even, say \( \ell = 2\ell' \), then
\[
\sum_{\sigma \in \mathcal{A}_4(n,321) \atop \text{lead}(\sigma) = 2\ell'} (-1)^{\text{maj}(\sigma)} = \sum_{k \geq 0} \binom{n-1}{\left\lfloor \frac{k}{2} \right\rfloor} \binom{n-\ell'}{\left\lfloor \frac{k}{2} \right\rfloor} = 0.
\]
This agrees with the right hand side of (i) of Theorem 1.5. The other identities (ii), (iii) and (iv) of Theorem 1.5 can be verified in a similar manner.

4. A combinatorial proof of Theorem 1.5

For any grand Dyck path \( \pi \in \mathcal{B}(n,m) \), we factorize \( \pi \) as \( \pi = \mu_0 \mu_1 \cdots \mu_d \), where each segment \( \mu_{2i}, \mu_{2i+1} \) is a maximal sequence of consecutive N steps (E steps, respectively). This is called the primal factorization of \( \pi \). Note that \( \mu_0 \) is empty if \( \pi \) starts with an east step.

According to the length of \( \mu_0 \), we partition the set \( \mathcal{B}(n,m) \) into subsets \( \mathcal{B}_j(n,m) \) for \( 0 \leq j \leq n \), where \( \mathcal{B}_j(n,m) \) consists of the paths pass the points \((0,j)\) and \((1,j)\). By (iii) of Lemma 2.3, we have the following result.

Lemma 4.1. For \( 0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \), the paths in \( \mathcal{B}_j(\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil) \) are in one-to-one correspondence with the permutations \( \sigma \in \mathcal{I}_n(321) \) with \( \text{lead}(\sigma) = j + 1 \).

4.1. The case \( \Phi_1 : \mathcal{B}(2n,2n) \to \mathcal{B}(2n,2n) \). To prove (i) of Theorem 1.5 we shall establish a sump-parity-reversing involution \( \Phi_1 : \mathcal{B}(2n,2n) \to \mathcal{B}(2n,2n) \) while preserving the initial segment from the beginning to the first east step. Let \( \mathcal{F}(2n,2n) \subseteq \mathcal{B}(2n,2n) \) be the set of fixed points of the map \( \Phi_1 \). The set \( \mathcal{F}(2n,2n) \) can be constructed from \( \mathcal{B}(n,n) \) as follows.

For each path \( \omega \in \mathcal{B}(n,n) \), we form a path \( \gamma(\omega) \) by duplicating every step of \( \omega \). Then \( \gamma(\omega) \in \mathcal{B}(2n,2n) \). Note that the peaks and valleys of \( \gamma(\omega) \) are all even lattice points. Moreover, every path without odd peaks and odd valleys in \( \mathcal{B}(2n,2n) \) can be reduced to a path in \( \mathcal{B}(n,n) \) by a reverse operation. For example, for \( \omega = \text{NEENEN} \in \mathcal{B}(3,3) \), the path \( \gamma(\omega) \) is shown as Figure 2.

![Figure 2](image.png)

**Figure 2.** Construction of the path \( \gamma(\omega) \) for \( \omega = \text{NEENEN} \in \mathcal{B}(3,3) \).

The set \( \mathcal{F}(2n,2n) \) is defined by
\[
\mathcal{F}(2n,2n) = \{ \gamma(\omega) : \omega \in \mathcal{B}(n,n) \},
\]
and for \( 0 \leq i \leq 2n \), the subset \( \mathcal{F}_i(2n,2n) \) is defined by
\[
\mathcal{F}_i(2n,2n) = \mathcal{F}(2n,2n) \cap \mathcal{B}_i(2n,2n).
\]
Note that \( \mathcal{F}_i(2n,2n) \) is empty if \( i \) is odd. We have following immediate observation.
Lemma 4.2. For $0 \leq j \leq n$ and any path $\omega \in B_j(n,n)$, the following properties hold.

(i) $\gamma(\omega) \in F_{2j}(2n,2n)$ and $\text{sump}(\gamma(\omega)) = 2 \cdot \text{sump}(\omega)$.
(ii) $|F_{2j}(2n,2n)| = |B_j(n,n)|$ and $|F_{2j+1}(2n,2n)| = 0$.
(iii) The set $F(2n,2n)$ consists of all the paths without odd peaks and odd valleys in $B(2n,2n)$.

Now, we construct the involution $\Phi_1$ on $B(2n,2n)$.

Algorithm A

Given a path $\pi \in B(2n,2n)$, let $\pi = \mu_0 \mu_1 \cdots \mu_d$ be the primal factorization of $\pi$. If every segment $\mu_i$ contains an even number of steps then $\Phi_1(\pi) = \pi$. Otherwise, find the greatest integer $k$ such that $\mu_k$ contains an odd number of steps. The path $\Phi_1(\pi)$ is obtained from $\pi$ by interchanging the first step of $\mu_k$ and the last step of $\mu_{k-1}$.

Example 4.3. Let $\pi$ be the path shown on the left hand side of Figure 3 with the primal factorization $\pi = \mu_0 \mu_1 \cdots \mu_5$. Then $\mu_4 = \text{NNN}$ is the last segment of odd length. Hence $\Phi_1(\pi)$ is obtained from $\pi$ by interchanging the first step of $\mu_4$ and the last step of $\mu_3$, shown on the right hand side of Figure 3.

![Figure 3](image)

**Figure 3.** An example for sump-parity-reversing involution on grand Dyck paths.

Lemma 4.4. For the primal factorization $\pi = \mu_0 \mu_1 \cdots \mu_d$ of a path $\pi \in B(2n,2n)$, if $k$ is the greatest integer such that $\mu_k$ contains an odd number of steps then $k \neq 0$ and $k \neq 1$, i.e., $\Phi_1(\pi)$ preserves the first segment $\mu_0$ of $\pi$.

Proof. The assertion follows from the fact that $\pi$ has $2n$ east steps and $2n$ north steps. □

Proposition 4.5. For $0 \leq i \leq 2n$, the map $\Phi_1$ establishes a refined involution on $B_i(2n,2n) - F_i(2n,2n)$. Moreover, a path $\pi$ is carried to a path $\Phi_1(\pi)$ such that $\text{sump}(\Phi_1(\pi))$ has the opposite parity of $\text{sump}(\pi)$.

Proof. Given a path $\pi \in B(2n,2n)$, suppose $\Phi_1(\pi) \neq \pi$. By Lemma 4.2, $\pi$ contains a peak (or valley) which is an odd lattice point. By algorithm A, we find the last odd peak (or valley) $p$. The path $\Phi(\pi)$ is obtained by interchanging the N and E steps adjacent at $p$. This changes the last odd peak (or valley) of $\pi$ into the last odd valley (or peak) of $\Phi(\pi)$. Moreover, by Lemma 4.4 $\Phi_1$ is an involution, restricted to each subset $B_i(2n,2n) - F_i(2n,2n)$. We observe that there is exactly one odd lattice point affected. Hence $\text{sump}(\Phi_1(\pi))$ has the opposite parity of $\text{sump}(\pi)$. □
Proof of (i) of Theorem 1.5.

\[
\sum_{\sigma \in \mathcal{F}_n(321)} (-1)^{\text{maj}(\sigma)} q^{\text{lead}(\sigma)} = 2^n \left( \sum_{i=0} \sum_{\pi \in \mathcal{B}_i(2n,2n)} (-1)^{\text{sum}(\pi)} \right) q^{i+1}
\]
\[
= \sum_{j=0}^{n} \left( \sum_{\pi \in \mathcal{F}_{2j}(2n,2n)} q^{2j+1} \right) 
\]
\[
= \sum_{j=0}^{n} |\mathcal{B}_{j}(n,n)| q^{2j+1} 
\]
\[
= \frac{1}{q} \sum_{\sigma \in \mathcal{F}_n(321)} q^{2\text{lead}(\sigma)}. 
\]

4.2. The case \( \Phi_2 : \mathcal{B}(2n+1,2n+1) \to \mathcal{B}(2n+1,2n+1) \). To prove (ii) of Theorem 1.5 we shall establish a sump-parity-reversing involution \( \Phi_2 : \mathcal{B}_i(2n+1,2n+1) \to \mathcal{B}_i(2n+1,2n+1) \) for \( 0 \leq i \leq 2n+1 \). Let \( \mathcal{F}(2n+1,2n+1) \subseteq \mathcal{B}(2n+1,2n+1) \) be the set of fixed points of the map \( \Phi_2 \). For \( 0 \leq i \leq 2n+1 \), we define

\[
\mathcal{F}_i(2n+1,2n+1) = \mathcal{F}(2n+1,2n+1) \cap \mathcal{B}_i(2n+1,2n+1).
\]

The set \( \mathcal{F}(2n+1,2n+1) \) can be constructed from \( \mathcal{B}(n,n+1) \) as follows. For each path \( \omega \in \mathcal{B}_j(n,n+1) \) \( (0 \leq j \leq n) \), we form a path \( \gamma(\omega) \) by duplicating every step of \( \omega \). Note that \( \gamma(\omega) \) is from \( (0,0) \) to \( (2n+2,2n) \) with the prefix \( N^{2j}EE \). Factorize \( \gamma(\omega) \) as \( \gamma(\omega) = N^{2j}EE\beta \).

Then we create two paths \( \phi_1(\omega), \phi_2(\omega) \in \mathcal{B}(2n+1,2n+1) \) from \( \gamma(\omega) \) by

\[
\phi_1(\omega) = N^{2j}EN\beta, \quad \phi_2(\omega) = N^{2j+1}E\beta,
\]

i.e., \( \phi_1(\omega) \) (respectively, \( \phi_2(\omega) \)) is obtained from \( \gamma(\omega) \) by changing the second (respectively, first) east step into a north step. For example, let \( \omega = \text{NEENE} \in \mathcal{B}_1(2,3) \). Then the path \( \gamma(\omega) \) is shown on the left hand side of Figure 4 and the paths \( \phi_1(\omega), \phi_2(\omega) \) are shown on the right hand side of Figure 4.

![Figure 4](image_url)

**Figure 4.** Construction of the paths \( \phi_1(\omega) \) and \( \phi_2(\omega) \) for \( \omega = \text{NEENE} \).

For \( 0 \leq j \leq n \), define

\[
\mathcal{F}_{2j}(2n+1,2n+1) = \{ \phi_1(\omega) : \omega \in \mathcal{B}_j(n,n+1) \},
\]
\[
\mathcal{F}_{2j+1}(2n+1,2n+1) = \{ \phi_2(\omega) : \omega \in \mathcal{B}_j(n,n+1) \}.
\]

Note that the segment \( \beta \) of \( \phi_1(\omega), \phi_2(\omega) \) goes from \( (1,2\ell+1) \) to \( (2n+1,2n+1) \) and that every segment in the primal factorization of \( \beta \) contains an even number of steps. We have following immediate observation.
Lemma 4.6. For $0 \leq j \leq n$ and any path $\omega \in \mathcal{B}_j(n, n+1)$, the following properties hold.

(i) The path $\phi_1(\omega) \in \mathcal{B}_j(2n+1, 2n+1)$ contains a unique odd valley $(1, 2j)$ and no odd peaks.

(ii) The path $\phi_2(\omega) \in \mathcal{B}_{j+1}(2n+1, 2n+1)$ contains a unique odd peak $(0, 2j+1)$ and no odd valleys.

(iii) $|\mathcal{F}_j(2n+1, 2n+1)| = |\mathcal{F}_{j+1}(2n+1, 2n+1)| = |\mathcal{B}_j(n, n+1)|$.

It follows that $\text{sump}(\phi_1(\omega))$ is even and $\text{sump}(\phi_2(\omega))$ is odd. Note that every path $\pi \in \mathcal{B}(2n+1, 2n+1)$ contains at least one odd valley or odd peak since $\pi$ has $2n+1$ east steps and $2n+1$ north steps. In fact, the set $\mathcal{F}(2n+1, 2n+1)$ consists of all the paths in $\mathcal{B}(2n+1, 2n+1)$ either containing a unique odd valley in the line $x=1$ and no odd peaks, or containing a unique odd peak in the line $x=0$ and no odd valleys.

Now, we construct the involution $\Phi_2$ on $\mathcal{B}(2n+1, 2n+1)$.

Algorithm B

Given a path $\pi \in \mathcal{B}(2n+1, 2n+1)$, let $\mu_0\mu_1 \cdots \mu_d$ be the primal factorization of $\pi$. According to the parity of the length of $\mu_0$, there are two cases.

Case 1. $\mu_0$ is odd, say $\mu_0 = N^{2j+1}$. Find the greatest integer $k \geq 1$ such that $\mu_k$ contains an odd number of steps. If $k = 1$ then let $\Phi_2(\pi) = \pi$. Otherwise, the path $\Phi_1(\pi)$ is obtained from $\pi$ by interchanging the first step of $\mu_k$ and the last step of $\mu_{k-1}$.

Case 2. $\mu_0$ is even, say $\mu_0 = N^{2j}$. If $\mu_1 = E$ and $\mu_2$ is the only other segment containing an odd number of steps (i.e., $\mu_t$ is of even length for all $t \geq 3$) then let $\Phi_2(\pi) = \pi$. Otherwise, find the greatest integer $k \geq 3$ such that $\mu_k$ contains an odd number of steps. Then the path $\Phi_2(\pi)$ is obtained from $\pi$ by interchanging the first step of $\mu_k$ and the last step of $\mu_{k-1}$.

It is obvious that the map $\Phi_2$ preserves the segment $\mu_0$, i.e., $\Phi_2$ is a map restricted to each subset $\mathcal{B}_i(2n+1, 2n+1)$ for $0 \leq i \leq 2n+1$.

Example 4.7. Let $\pi$ be the path shown on the left hand side of Figure 5 with the primal factorization $\pi = \mu_0\mu_1 \cdots \mu_7$. Then $\mu_5 = EEE$ is the last segment of odd length. Hence $\Phi_2(\pi)$ is obtained from $\pi$ by interchanging the first step of $\mu_5$ and the last step of $\mu_4$, shown on the right hand side of Figure 5.

![Figure 5](image)

By Lemma the same argument as in the proof of Proposition 4.5, the assertion is proved.

Proposition 4.8. For $0 \leq i \leq 2n+1$, the map $\Phi_2$ establish a refined involution on $\mathcal{B}_i(2n+1, 2n+1) - \mathcal{F}_i(2n+1, 2n+1)$. Moreover, a path $\pi$ is carried to a path $\Phi_2(\pi)$ such that $\text{sump}(\Phi_2(\pi))$ has the opposite parity of $\text{sump}(\pi)$. 
Proof. Given a path $\pi \in \mathcal{B}(2n + 1, 2n + 1)$, suppose $\Phi_2(\pi) \neq \pi$. By Lemma 4.5, $\pi$ either contains an odd peak $(x, y)$ with $x > 0$ or contains an odd valley $(x', y')$ with $x' > 1$. By algorithm B, we find the last odd peak (or valley) $p$ and construct the path $\Phi_2(\pi)$ by interchanging the $N$ and $E$ steps adjacent at $p$. By the same argument as in the proof of Proposition 4.3, the assertion is proved. □

Proof of (ii) of Theorem 1.5

$$
\sum_{\sigma \in \mathcal{F}_{2n+2}(321)} (-1)^{\text{maj}(\sigma)} q^{\text{lead}(\sigma)} = \sum_{i=0}^{2n+1} \left( \sum_{\pi \in \mathcal{B}_i(2n+1, 2n+1)} (-1)^{\text{sum}(\pi)} \right) q^{i+1}
$$

$$
= \sum_{j=0}^{n} \left( \sum_{\pi \in \mathcal{F}_j(2n+1, 2n+1)} q^{2j+1} - \sum_{\pi \in \mathcal{F}_{j+1}(2n+1, 2n+1)} q^{2j+2} \right)
$$

$$
= \sum_{j=0}^{n} |\mathcal{B}_j(n, n+1)|q^{2j+1} - \sum_{j=0}^{n} |\mathcal{B}_j(n, n+1)|q^{2j+2}
$$

$$
= \left( \frac{1}{q} - 1 \right) \sum_{\pi \in \mathcal{F}_{2n+2}(321)} q^{2\text{lead}(\sigma)}.
$$

4.3. The case $\Phi_3 : \mathcal{B}(2n+1, 2n+2) \rightarrow \mathcal{B}(2n+1, 2n+2)$. To prove (iii) of Theorem 1.5, we shall establish a sum-parity-reversing involution $\Phi_3 : \mathcal{B}_i(2n+1, 2n+2) \rightarrow \mathcal{B}_i(2n+1, 2n+2)$ for $0 \leq i \leq 2n+1$.

The set $\mathcal{F}(2n+1, 2n+2)$ can be constructed from $\mathcal{B}(n, n+1)$ as follows. For each path $\omega \in \mathcal{B}_j(n, n+1)$ ($0 \leq j \leq n$), we form a path $\gamma(\omega)$ by duplicating every step of $\omega$. Note that $\gamma(\omega)$ is from $(0, 0)$ to $(2n+2, 2n)$ with the prefix N2EE. Factorize $\gamma(\omega)$ as $\gamma(\omega) = N^2EE\beta$. Then we create three paths $\psi_0(\omega), \psi_1(\omega), \psi_2(\omega) \in \mathcal{B}(2n+1, 2n+2)$ from $\gamma(\omega)$ by

$$\psi_0(\omega) = \gamma(\omega)N,$$

$$\psi_1(\omega) = N^2EN\beta E,$$

$$\psi_2(\omega) = N^2E\beta E.$$

Note that $\psi_0(\omega)$ is obtained from $\gamma(\omega)$ by appending a north step in the end and that $\psi_1(\omega)$ (respectively, $\psi_2(\omega)$) is obtained from $\gamma(\omega)$ inserting a north step after (respectively, before) the first east step and moving the second east step to the end. For example, for $\omega = \text{NEENE} \in \mathcal{B}_1(2, 3)$, the paths $\psi_0(\omega), \psi_1(\omega)$ and $\psi_2(\omega)$ are shown in Figure 6.

We define the refinement of the set $\mathcal{F}(2n+1, 2n+2)$. For $0 \leq j \leq n$ let

$$\mathcal{F}_{2j}(2n+1, 2n+2) = \{ \psi_0(\omega), \psi_1(\omega) : \omega \in \mathcal{B}_j(n, n+1) \},$$

$$\mathcal{F}_{2j+1}(2n+1, 2n+2) = \{ \psi_2(\omega) : \omega \in \mathcal{B}_j(n, n+1) \}.$$

We have the following properties of the fixed points $\mathcal{F}(2n+1, 2n+2)$.

Lemma 4.9. For $0 \leq j \leq n$ and any path $\omega \in \mathcal{B}_j(n, n+1)$, the following properties hold.

(i) The path $\psi_0(\omega)$ contains no odd peaks and odd valleys.

(ii) The path $\psi_1(\omega)$ contains a unique odd valley $(1, 2j)$ and no odd peaks.
(iii) The path $\psi_2(\omega)$ contains a unique odd peak $(0, 2j + 1)$ and no odd valleys.
(iv) $|\mathcal{F}_{2j}(2n + 1, 2n + 2)| = 2|\mathcal{B}_j(n, n + 1)|$ and $|\mathcal{F}_{2j+1}(2n + 1, 2n + 2)| = |\mathcal{B}_j(n, n + 1)|$.

It follows that $\text{sump}(\psi_0(\omega)), \text{sump}(\psi_1(\omega))$ are even and $\text{sump}(\psi_2(\omega))$ is odd. In fact, $\mathcal{F}(2n + 1, 2n + 2)$ consists of all the paths in $\mathcal{B}(2n + 1, 2n + 2)$ containing on odd valley $(x, y)$ with $x \geq 2$ and no odd peak $(x', y')$ with $x' \geq 1$.

Now, we construct the involution $\Phi_3$ on $\mathcal{B}(2n + 1, 2n + 2)$.

**Algorithm C**

Given a path $\pi \in \mathcal{B}(2n + 1, 2n + 2)$, let $z$ denote the last step of $\pi$. Let $\pi'$ be the path obtained from $\pi$ by removing $z$. We consider the following two cases according to the step $z$.

Case 1. $z = N$. Then $\pi'$ goes from $(0, 0)$ to $(2n + 2, 2n)$. Applying algorithm A to the primal factorization of $\pi'$, we determine the path $\Phi_1(\pi')$ associated with $\pi'$ under the map $\Phi_1$. Then the corresponding path $\Phi_3(\pi)$ is obtained from $\Phi_1(\pi')$ by appending a north step, i.e., $\Phi_3(\pi) = \Phi_1(\pi')N \in \mathcal{B}(2n + 1, 2n + 2)$.

Case 2. $z = E$. Then $\pi'$ goes from $(0, 0)$ to $(2n + 1, 2n + 1)$. Applying algorithm B to the primal factorization of $\pi'$, we determine the path $\Phi_2(\pi')$ associated with $\pi'$ under the map $\Phi_2$. Then the corresponding path $\Phi_3(\pi) \in \mathcal{B}(2n + 1, 2n + 2)$ is obtained from $\Phi_2(\pi')$ by appending an east step, i.e., $\Phi_3(\pi) = \Phi_2(\pi')E \in \mathcal{B}(2n + 1, 2n + 2)$.

Note that the construction of the map $\Phi_3$ in Case 1 (respectively, Case 2) of algorithm C is similar to the construction of $\Phi_1$ by algorithm A (respectively, $\Phi_2$ by algorithm B). The following property of the map $\Phi_3$ can be proved by the same argument as in the proofs of Propositions 4.5 and 4.8.

**Proposition 4.10.** For $0 \leq i \leq 2n + 1$, the map $\Phi_3$ establishes a refined involution on $\mathcal{B}_i(2n + 1, 2n + 2) - \mathcal{F}_i(2n + 1, 2n + 2)$. Moreover, a path $\pi$ is carried to a path $\Phi_3(\pi)$ such that $\text{sump}(\Phi_3(\pi))$ has the opposite parity of $\text{sump}(\pi)$. 
Proof of (iii) of Theorem 1.5.

\[ \sum_{\sigma \in \mathcal{F}_{4n+3}(321)} (-1)^{\text{maj}(\sigma)} q^{\text{lead}(\sigma)} = \sum_{i=0}^{2n+1} \left( \sum_{\pi \in \mathcal{F}_i(2n+1,2n+2)} (-1)^{\text{sump}(\pi)} \right) q^{i+1} \]

\[ = \sum_{j=0}^{n} \left( \sum_{\pi \in \mathcal{F}_{2j}(2n+1,2n+2)} q^{2j+1} - \sum_{\pi \in \mathcal{F}_{2j+1}(2n+1,2n+2)} q^{2j+2} \right) \]

\[ = \sum_{j=0}^{n} 2|\mathcal{B}_j(n,n+1)| q^{2j+1} - \sum_{j=0}^{n} |\mathcal{B}_j(n,n+1)| q^{2j+2} \]

\[ = \left( \frac{2}{q} - 1 \right) \sum_{\sigma \in \mathcal{F}_{2n+1}(321)} q^{2\text{lead}(\sigma)}. \]

4.4. The case \( \Phi_4 : \mathcal{B}(2n,2n+1) \rightarrow \mathcal{B}(2n,2n+1) \). To prove (iv) of Theorem 1.5, we shall establish a sump-parity-reversing involution \( \Phi_4 : \mathcal{F}_i(2n,2n+1) \rightarrow \mathcal{F}_i(2n,2n+1) \) for \( 0 \leq i \leq 2n \). Let \( \mathcal{F}(2n,2n+1) \subseteq \mathcal{B}(2n,2n+1) \) be the set of fixed points of the map \( \Phi_4 \). For \( 0 \leq i \leq 2n \), we define

\[ \mathcal{F}_i(2n,2n+1) = \mathcal{F}(2n,2n+1) \cap \mathcal{B}_i(2n,2n+1). \]

The set \( \mathcal{F}(2n,2n+1) \) can be constructed from \( \mathcal{B}(n,n+1) \) as follows. For each path \( \omega \in \mathcal{B}_j(n,n+1) \) (\( 0 \leq j \leq n \)), we form a path \( \gamma(\omega) \) by duplicating every step of \( \omega \). We consider the following two cases according to the last step \( z \) of \( \omega \):

- \( z = \text{E} \). Then the last two steps of \( \gamma(\omega) \) are east steps. Let \( \varphi_0(\omega) \) be the path obtained from \( \gamma(\omega) \) by removing the last step.
- \( z = \text{N} \). Then the last two steps of \( \gamma(\omega) \) are north steps. Factorize \( \gamma(\omega) \) as \( \text{N}^{2j}\text{E}\beta\text{N} \) and let \( \varphi_1(\omega) \) (respectively, \( \varphi_2(\omega) \)) be the path obtained from \( \gamma(\omega) \) by inserting a north step after (respectively, before) the first east step and then removing the second east step and the last step, i.e.,

\[ \varphi_1(\omega) = \text{N}^{2j}\text{EN}\beta\text{N} \]

\[ \varphi_2(\omega) = \text{N}^{2j+1}\beta\text{N} \]

For example, for \( \omega_1 = \text{NEENENE} \in \mathcal{B}_1(3,4) \), the path \( \varphi_0(\omega_1) \) is shown as the left hand side of Figure 7. For \( \omega_2 = \text{EENENEN} \in \mathcal{B}_0(3,4) \), the paths \( \varphi_1(\omega_2), \varphi_2(\omega_2) \) are shown as the right hand side of Figure 7.

We define the refinement of the set \( \mathcal{F}(2n,2n+1) \). For \( 0 \leq j \leq n \), let \( \mathcal{F}_{2j}(2n,2n+1) = \mathcal{F}^{E}_{2j}(2n,2n+1) \cup \mathcal{F}^{N}_{2j}(2n,2n+1) \), where

\[ \mathcal{F}^{E}_{2j}(2n,2n+1) = \{ \varphi_0(\omega) : \omega \in \mathcal{B}_j(n,n+1) \text{ ends with an east step} \}, \]

\[ \mathcal{F}^{N}_{2j}(2n,2n+1) = \{ \varphi_1(\omega) : \omega \in \mathcal{B}_j(n,n+1) \text{ ends with a north step} \}, \]

\[ \mathcal{F}_{2j+1}(2n,2n+1) = \{ \varphi_2(\omega) : \omega \in \mathcal{B}_j(n,n+1) \text{ ends with a north step} \}. \]

We have the following properties of the fixed points \( \mathcal{F}(2n+1,2n+2) \).

**Lemma 4.11.** For \( 0 \leq j \leq n \) and any path \( \omega \in \mathcal{B}_j(n,n+1) \), the following properties hold.

(i) The path \( \varphi_0(\omega) \) contains no odd peaks and odd valleys.
Figure 7. Examples of the maps $\varphi_0$, $\varphi_1$, and $\varphi_2$.

(ii) The path $\varphi_1(\omega)$ contains a unique odd valley $(1, 2j)$ and no odd peaks.
(iii) The path $\varphi_2(\omega)$ contains a unique odd peak $(0, 2j + 1)$ and no odd valleys.
(iv) $|F_{2j}(2n + 1, 2n + 2)| = |B_j(n, n + 1)|$.
(v) $|F_{2j+1}(2n + 1, 2n + 2)| = |B_j(n, n + 1)| - |B_j(n, n)|$.

It follows that $\sum^p(\varphi_0(\omega)), \sum^p(\varphi_1(\omega))$ are even and $\sum^p(\varphi_2(\omega))$ is odd. Now, we construct the involution $\Phi_4$ on $B(2n, 2n + 1)$.

Algorithm D

Given a path $\pi \in B(2n, 2n + 1)$, let $z$ denote the last step of $\pi$. Let $\pi'$ be the path obtained from $\pi$ by removing $z$. We consider the following two cases according to the step $z$.

Case 1. $z = E$. Then $\pi'$ goes from $(0, 0)$ to $(2n, 2n)$. By algorithm A, we determine the path $\Phi_1(\pi') \in B(2n, 2n)$ associated with $\pi'$ under the map $\Phi_1$. Then the corresponding path $\Phi_4(\pi)$ is obtained from $\Phi_1(\pi')$ by appending an east step in the end, i.e., $\Phi_4(\pi) = \Phi_1(\pi')E$.

Case 2. $z = N$. Then $\pi'$ goes from $(0, 0)$ to $(2n + 1, 2n - 1)$. By the same method as in algorithm B, we determine the path $\Phi_2(\pi') \in B(2n - 1, 2n + 1)$ associated with $\pi'$ under the map $\Phi_2$. Then the corresponding path $\Phi_4(\pi) \in B(2n, 2n + 1)$ is obtained from $\Phi_2(\pi')$ by appending a north step in the end, i.e., $\Phi_4(\pi) = \Phi_2(\pi')N$.

The following property of the map $\Phi_4$ can be proved by the same argument as in the proofs of Propositions 4.5 and 4.8 since the construction of $\Phi_4$ in Case 1 (respectively, Case 2) of algorithm D is similar to the construction of $\Phi_1$ (respectively, $\Phi_2$).

Proposition 4.12. For $0 \leq i \leq 2n$, the map $\Phi_4$ establishes a refined involution on $B_i(2n, 2n + 1) - F_i(2n, 2n + 1)$. Moreover, a path $\pi$ is carried to a path $\Phi_4(\pi)$ such that $\sum^p(\Phi_4(\pi))$ has the opposite parity of $\sum^p(\pi)$. 

Proof of (iv) of Theorem 1.5

\[
\sum_{\sigma \in \mathcal{I}_{n+1}(321)} (-1)^{\text{maj}(\sigma)} q^{\text{lead}(\sigma)} = \sum_{i=0}^{2n} \left( \sum_{\pi \in \mathcal{R}_n(2n, 2n+1)} (-1)^{\text{sum}_{\pi}(\sigma)} \right) q^{i+1} \\
= \sum_{j=0}^{n} \left( \sum_{\pi \in \mathcal{R}_j(2n, 2n+1)} q^{2j+1} - \sum_{\pi \in \mathcal{R}_{j+1}(2n, 2n+1)} q^{2j+2} \right) \\
= \sum_{j=0}^{n} \left( |\mathcal{R}_j(n, n+1)| - |\mathcal{R}_j(n, n+1)|q + |\mathcal{R}_j(n, n)|q \right) q^{2j+1} \\
= \left( \frac{1}{q} - 1 \right) \sum_{\sigma \in \mathcal{I}_{2n+1}(321)} q^{2\text{lead}(\sigma)} + \sum_{\sigma \in \mathcal{I}_{2n}(321)} q^{2\text{lead}(\sigma)}. 
\]

5. Analogous results for 123-avoiding involutions

It is known that by the Robinson-Schensted-Knuth (RSK) algorithm an involution \( \sigma \in \mathcal{I}_n(321) \) (respectively, \( \sigma \in \mathcal{I}_n(123) \)) is associated with a pair \((Q, Q)\) of identical \(n\)-cell standard Young tableaux with at most two rows (respectively, columns). Let \( Q^T \) be the transpose of \( Q \) and let \( \sigma^T \) be the preimage of the pair \((Q^T, Q^T)\) under the RSK correspondence. Then \( \sigma \leftrightarrow \sigma^T \) is a bijection between \( \mathcal{I}_n(321) \) and \( \mathcal{I}_n(123) \).

Lemma 5.1. We have

\[
\text{Des}(\sigma^T) = \{i : i \notin \text{Des}(\sigma), 1 \leq i \leq n - 1\}. 
\]

Proof. It is known that a descent \( \sigma_i > \sigma_{i+1} \) in \( \sigma \) is translated to the ‘descent’ of the recording tableau \( Q \) that the entry \( i + 1 \) is in a row lower than the row of \( i \). For \( 1 \leq i \leq n - 1 \), if \( i \in \text{Des}(\sigma) \) then \( i \) (respectively, \( i + 1 \)) is in the first (respectively, second) row in \( Q \). Then \( i + 1 \) is either in the same column as \( i \) or in a column to the left of the column of \( i \) in \( Q \). Then \( i + 1 \) is not in a lower row than the row of \( i \) in \( Q^T \). Hence \( i \notin \text{Des}(\sigma^T) \).

Otherwise, \( i \notin \text{Des}(\sigma) \). Then \( i + 1 \) is not in a row lower than the row of \( i \) in \( Q \). The element \( i \) is in the first row or the second row. In either case, the element \( i + 1 \) is in a column to the right of the column of \( i \). Then \( i + 1 \) is in row lower than the row of \( i \) in \( Q^T \). Hence \( i \in \text{Des}(\sigma^T) \).

We obtain the joint distribution of major index and descent number for 123-avoiding involutions.

Corollary 5.2. We have

\[
\sum_{\sigma \in \mathcal{I}_n(123) \atop \text{des}(\sigma) = n - 1 - k} q^{\text{maj}(\sigma)} = q^{\binom{n}{2} + k^2 - nk} \left[ \frac{\binom{n}{2}}{k} \right]_q \left[ \frac{\binom{n}{2}}{k} \right]_q.
\]

Proof. Substituting \( q^{-1} \) for \( q \) in (i) of Theorem 1.4 we have

\[
\sum_{\sigma \in \mathcal{I}_n(321) \atop \text{des}(\sigma) = k} q^{-\text{maj}(\sigma)} = q^{-k^2} \left[ \frac{\binom{n}{2}}{k} \right] q^{-1} \left[ \frac{\binom{n}{2}}{k} \right] = q^{k^2 - nk} \left[ \frac{\binom{n}{2}}{k} \right] \left[ \frac{\binom{n}{2}}{k} \right]_q.
\]
By Lemma 5.1, \( \text{des}(\sigma) = n - 1 - \text{des}(\sigma^T) \) and \( \text{maj}(\sigma) = \binom{n}{2} - \text{maj}(\sigma^T) \) for any \( \sigma \in \mathcal{I}_n(123) \), we have

\[
\sum_{\sigma \in \mathcal{I}_n(123)} q^{\text{maj}(\sigma)} = \sum_{\sigma^T \in \mathcal{I}_n(321)} q^{\binom{n}{2} - \text{maj}(\sigma^T)}
\]

\[
= q^{\binom{n}{2} + k^2 - nk} \left\lfloor \frac{n}{2} \right\rfloor q \left\lfloor \frac{n}{2} \right\rfloor ,
\]

as required. \( \square \)

With the bijection \( \sigma \leftrightarrow \sigma^T \) between \( \mathcal{I}_n(321) \) and \( \mathcal{I}_n(123) \), we can prove affirmatively Conjecture 1.3. In fact, this result is essentially equivalent to Theorem 1.2.

Theorem 5.3. For all \( n \geq 1 \), we have

\[
\begin{align*}
(\text{i}) & \quad \sum_{\sigma \in \mathcal{I}_n(123)} (-1)^{\text{maj}(\sigma)} q^{\text{des}(\sigma)} = q \sum_{\sigma \in \mathcal{I}_{2n}(123)} q^{2 \text{des}(\sigma)}, \\
(\text{ii}) & \quad \sum_{\sigma \in \mathcal{I}_{2n+2}(123)} (-1)^{\text{maj}(\sigma)} q^{\text{des}(\sigma)} = (1 - q)q^2 \sum_{\sigma \in \mathcal{I}_{2n}(123)} q^{2 \text{des}(\sigma)}, \\
(\text{iii}) & \quad \sum_{\sigma \in \mathcal{I}_{2n+1}(123)} (-1)^{\text{maj}(\sigma)} q^{\text{des}(\sigma)} = (-1)^n q^2 \sum_{\sigma \in \mathcal{I}_{2n}(123)} q^{2 \text{des}(\sigma)}.
\end{align*}
\]

Proof. Note that \( \text{des}(\sigma) = n - 1 - \text{des}(\sigma^T) \) and \( \text{maj}(\sigma) = \binom{n}{2} - \text{maj}(\sigma^T) \) for any \( \sigma \in \mathcal{I}_n(123) \). Making use of the identity in (i) of Theorem 1.2, we have

\[
\sum_{\sigma \in \mathcal{I}_{4n}(123)} (-1)^{\text{maj}(\sigma)} q^{\text{des}(\sigma)} = \sum_{\sigma^T \in \mathcal{I}_{4n}(321)} (-1)^{\binom{4n}{2} - \text{maj}(\sigma^T)} q^{4n - 1 - \text{des}(\sigma^T)}
\]

\[
= q \sum_{\sigma^T \in \mathcal{I}_{2n}(321)} q^{2(2n - 1 - \text{des}(\sigma^T))} \quad \text{(by (i) of Theorem 1.2)}
\]

\[
= q \sum_{\sigma \in \mathcal{I}_{2n}(123)} q^{2 \text{des}(\sigma)}.
\]

The assertion (i) follows. Making use of the the identities in (ii) and (iii) of Theorem 1.2, the assertions (ii) and (iii) can be proved straightforward in the same manner. \( \square \)

References

[1] R.M. Adin, Y. Roichman, Equidistribution and sign-balance on 321-avoiding permutations, Sémin. Loth. Combin. 51 (2004) B51d.
[2] M. Barnabei, F. Bonetti, S. Elizalde, M. Silimbani, Descent sets on 321-avoiding involutions and hook decompositions of partitions, J. Combin. Theory Ser. A 128 (2014) 132–148.
[3] E. Deutsch, A. Robertson, D. Saracino, Refined restricted involutions, European J. Combin. 6 (1985) 383–406.
[4] S.-P. Eu, T.-S. Fu, Y.-J. Pan, C.-T. Ting, Sign-balance identities of Adin-Roichman type on 321-avoiding alternating permutations, Discrete Math. 312 (2012) 2228–2237.
[5] S.-P. Eu, T.-S. Fu, Y.-J. Pan, C.-T. Ting, Baxter Permutations, Maj-balances, and Positive Braids, Electronic J. Combin. 19 Issue 3 (2012) P26.
[6] A. Reifegerste, Refined sign-balance on 321-avoiding involutions, Electronic J. Combin. 49 (2015) 250–264.
[7] R. Simion, F.W. Schmidt, Restricted permutations, European J. Combin. 6 (1985) 383–406.
[9] R.P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge Univ. Press, New York/Cambridge, UK, 1999.

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