Asymptotic analysis of multi-lumps solutions in the Kadomtsev-Petviashvili-(I) equation

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Abstract

Inspired by the works of Y. Ohta and J. Yang, one constructs the lumps solutions in the Kadomtsev-Petviashvili-(I) equation using the Grammian determinants. It is shown that the locations of peaks will depend on the real roots of Wronskian of the orthogonal polynomials for the asymptotic behaviors in some particular cases. Also, one can prove that all the locations of peaks are on a vertical line when time approaches $-\infty$, and then they will be on a horizontal line when time approaches $\infty$, i.e., there is a rotation $\frac{\pi}{2}$ after interaction.

Keywords: Grammian Determinant, Lumps Solutions, Orthogonal Polynomials, Wronskian

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1 Introduction

The lumps solutions of Kadomtsev-Petviashvili(KP)-(I) equation correspond to the rationally decaying solutions (like $O \left( \frac{1}{x^2 + y^2} \right)$), and they were first found in [15] and constructed by direct algebraic methods [2, 3, 14]. It is known that the Schrödinger equation can be used to linearize the KP-(I) equation via the inverse scattering transformation (see (2) below). The lumps solutions of reflectionless potentials have discrete spectrum for the Schrödinger equation, and the associated wave functions are meromorphic functions with respect to the spectrum; moreover, in general, the associated wave functions can be solved by the Fredholm-like integral equation [3]. Also, the lumps solutions of DS equation were investigated in [16]. On the other hand, the asymptotic analysis of these lumps solutions are also noteworthy. These lumps can be thought as a collection of individual humps that generically evolve with non-uniform dynamics, and their asymptotic behaviors, that is, $|t| \to \infty$, can be described by the locations and poles structure of the associated meromorphic wave functions in a complicated way [2, 3]. It is not clear how to describe the locations of the peaks of these lumps solutions as $|t| \to \infty$ for KP-(I) equation.

The basic shallow water waves equation of 2+1 integrable model is the KP equation describing small amplitude with slow variations in the direction transverse to the wave propagation [4, 11, 12, 13]:

$$u_t + u_{xxx} + 6uu_x \pm \partial_x^{-1}3u_{yy} = 0,$$  \hspace{1cm} (1)

where $u$ is the height of water wave and $x, y$ are the space coordinates, $t$ being the time. Here the subscripts denote partial derivatives and

$$\partial_x^{-1}u(x, y, t) = \int_{-\infty}^{x} u(x', y, t) dx'.$$

The ”+” is the KP-(II) equation and the ”-” is the KP-(I) equation. It depends on the surface tension of water waves. The integrability structure of the KP equation can be found in [8] and its physical origin, the inverse scattering transformation can be found in [1].

To construct the lumps solutions of KP-(I) equation, one introduces the Grammian solution structure [2, 8]. The KP-(I) equation can be written as the compatibility of the linear equations:

$$i\psi_y + \psi_{xx} + w\psi = 0$$
$$\psi_t + 4\psi_{xxx} + 6w\psi_x + w\psi = 0, \quad w_x = u.$$  \hspace{1cm} (2)

The adjoint of (2) is

$$-i\psi_y^* + \psi_{xx}^* + w\psi^* = 0$$
$$\psi_t^* + 4\psi_{xxx}^* + 6(w\psi^*)_x - w\psi^* = 0, \quad w_x = u.$$  \hspace{1cm} (3)
Let \( \{\psi_j, \psi^*_j\}, j = 1, 2, 3, \ldots, m \), be distinct solutions of (2) and (3) respectively, corresponding to a given solution \( u(x, y, t) \) of KP-(I) equation. Then a new solution \( \hat{u} \) is given by

\[
\hat{u} = u + 2\partial_x^2 \ln(\text{det}M(x, y, t)),
\]

where \( \text{det}M(x, y, t) \) is the Grammian determinant of the \( m \times m \) matrix with elements

\[
M_{lj} = \int_{-\infty}^{x} \psi_l(x', y, t)\psi^*_j(x', y, t)dx', l, j = 1, 2, 3, \ldots, m.
\]

The paper is organized as follows. In section 2, we obtain the lumps solutions using the Grammian determinant structure and the elementary Schur functions. Section 3 is used to analyze the asymptotic behaviors of lumps solutions. It is shown that the locations of peaks will depend on the real roots of Wronskian of the orthogonal polynomials for some special cases. The section 4 is devoted to the concluding remarks.

2 Grammian Determinant Solutions

In this section, we construct the multi-lumps solutions (4) of the KP-(I) equation.

Without loss of generality, we choose the seed solution \( u = w = 0 \) for the asymptotic analysis. Inspired by the work in [17, 18], then one considers the rogue waves solutions in KP-(I) equation. Let

\[
\begin{align*}
\psi_1 &= A_1 e^{\xi_1}, & \xi_1 &= p_1 x + i q_1^2 y - 4 p_1^3 t, & p_1, p_2, \ldots, p_m \in C \\
\psi_j &= B_j e^{\eta_j}, & \eta_j &= q_j x - i q_j^2 y - 4 q_j^3 t, & q_1, q_2, \ldots, q_m \in C
\end{align*}
\]

where \( A_l \) and \( B_j \) are differential operators defined by

\[
A_l = \sum_{k=0}^{n_l} c_{lk}(p_l \partial_{p_l})^{n_l-k}, \quad B_j = \sum_{s=0}^{n_j} d_{js}(q_j \partial_{q_j})^{n_j-s},
\]

where the coefficients \( c_{lk} \) and \( d_{js} \) are complex numbers. One supposes that

\[
A_l e^{\xi_l} = P_l e^{\xi_l}, \quad B_j e^{\eta_j} = Q_j e^{\eta_j},
\]

where \( P_l(x, y, t) \) and \( Q_j(x, y, t) \) are functions of \( x, y, t \). Then, using integration by parts, we can get

\[
M_{lj} = e^{\xi_l+\eta_j} \sum_{\nu=0}^{n_l+n_j} \frac{(-1)^{\nu}}{(p_l+q_j)^{\nu+1}} \partial_x^\nu P_l Q_j.
\]

One can take the following constraints to get real solutions of \( u \)

\[
q_j = \bar{p}_j, \quad d_{lj} = \bar{c}_{lj}
\]
such that the matrix $M$ is Hermitian $M_{ij} = \bar{M}_{ji}$ and $Q_j = \bar{P}_j$.

To find the functions $P_l$, we define $S_r(x, y, t, p)$ as follows: $(r \geq 0)$

$$(p \partial_p)^r e^{\xi(p)} = S_r(x, y, t, p)e^{\xi(p)}, \quad \xi(p) = xp + ip^2y - 4p^3t.$$  \hspace{1cm} (10)

Then one has

$$(p \partial_p)^{r+1} e^{\xi(p)} = (p \partial_p)[(p \partial_p)^r e^{\xi(p)}] = p\partial_p(S_r) + f(x, y, t, p)S_r)e^{\xi(p)} = S_{r+1}e^{\xi(p)},$$

where

$$\frac{\partial \xi(p)}{\partial p} = f(x, y, t, p) = x + 2ipy - 12p^3t.$$  \hspace{1cm} (11)

Hence we get

$$S_{r+1} = p\partial_p(S_r) + pfS_r.$$  \hspace{1cm} (12)

For example,

$S_0 = 1,$

$S_1 = pf,$

$S_2 = pf + p^2f_p + p^2f^2,$

$S_3 = pf + 3p^2f^2 + p^3f^3 + 3p^2f_p + 3pf_p^2f_p + p^3f_{pp},$

$S_4 = pf + 6p^3f_{pp} + p^4f_{ppp} + 18p^3f_p + 3p^4f^2 + 4p^4f_{pp} + 7p^2f_p + 7p^2f^2 + 6p^3f^3$

$+ 6p^4f^2f_p + p^4f^4,$

Now, we have

$$P_l = \sum_{k=0}^{n_l} c_{lk}S_{n_l-k}.$$  

On the other hand, we can express $S_r$ as elementary Schur functions in the following way. The elementary Schur functions $H_r$ are defined by

$$e^{\sum_{k=1}^{\infty} x_k \lambda^k} = \sum_{r=0}^{\infty} H_r(\bar{x})\lambda^r,$$

where $\bar{x} = (x_1, x_2, x_3, \cdots)$. In general, we have

$$H_r(\bar{x}) = \sum_{n_1 + 2n_2 + 3n_3 + \cdots + rn_r = r} \frac{x_1^{n_1}x_2^{n_2}x_3^{n_3} \cdots x_r^{n_r}}{n_1!n_2! \cdots n_r!}.$$  \hspace{1cm} (13)

For example,

$$H_0 = 1, \quad H_1 = x_1, \quad H_2 = \frac{1}{2}x_1^2 + x_2, \quad H_3 = \frac{1}{6}x_1^3 + x_1x_2 + x_3,$$

$$H_4 = x_4 + x_1x_3 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3x_1 + \frac{1}{24}x_4,$$

$$H_5 = x_5 + x_1x_4 + x_3x_2 + \frac{1}{2}x_3x_1^2 + \frac{1}{2}x_2^2x_1 + \frac{1}{6}x_2x_3 + \frac{1}{120}x_5^2, \cdots.$$
We remark that $H_r$ has the basic property
\[
\frac{\partial H_r}{\partial x_s} = H_{r-s}.
\] (14)

Using the functional identity,
\[
e^{\lambda p\partial_p}F(p) = F(e^\lambda p),
\]
one has, defining that $\xi = xp + ip^2 y - 4p^3 t = px_1 + p^2 x_2 + p^3 x_3 = \sum_{\mu=1}^3 p^\mu x_\mu$,
\[
e^{-\xi}e^{\lambda p\partial_p}e^\xi = \exp(\sum_{\mu=1}^3 (e^{\mu\lambda} - 1)p^\mu x_\mu) = \exp\left(\sum_{r=1}^3 \frac{\lambda^r}{r!} \sum_{\mu=1}^3 \mu^r p^\mu x_\mu\right)
\] (15)
\[= \sum_{r=0}^\infty \frac{\lambda^r}{r!} H_r(\tilde{\chi}_r(p)),
\]
where $\tilde{\chi}_r(p) = (\chi_1(p), \chi_2(p), \chi_3(p), \ldots, \chi_r(p)) = (x_1, x_2, x_3, \ldots, x_r)$, i.e.,
\[
x_n = \chi_n(p) = \sum_{\mu=1}^3 \frac{\mu^nx_\mu}{n!} = \frac{xp + ip^22^ny - 4p^33^nt}{n!}, \quad n = 1, 2, 3, \ldots, r.
\] (16)

Comparing the coefficient of $\lambda^r$ of (15), one gets
\[
e^{-\xi}(p\partial_p)^r e^\xi = H_r(\tilde{\chi}_r(p)).
\]
Hence
\[
S_r = r!H_r(\tilde{\chi}_r(p)).
\] (17)

For the case $m = 1$ in (4), we obtain $u = 2\partial^2_{xx} \ln F_n$, where
\[
F_n = M_{11} = \sum_{\nu=0}^n \frac{(-1)^\nu}{(p_1 + \bar{p}_1)^{\nu+1}} \partial_{\nu}^2(|P_1|^2),
\]
and
\[
P_1 = c_{10}S_n + c_{11}S_{n-1} + c_{12}S_{n-2} + \cdots + c_{1n}S_0.
\]

Under the condition (9), the matrix (5) is Hermitian and positive. It can be seen as follows. For any non-zero column vector $v = (v_1, \ldots, v_m)^T$ and $\tilde{v}$ being its complex transpose, we have
\[
\tilde{v} M v = \sum_{l,j=1}^m \tilde{v}_l M_{lj} v_j = \sum_{l,j=1}^m \tilde{v}_l v_j A_l B_j \int_{-\infty}^x e^{\xi_l + \eta_j} dx
\]
\[= \int_{-\infty}^x \left( \sum_{l,j=1}^m \tilde{v}_l v_j A_l B_j e^{\xi_l + \eta_j} \right) dx = \int_{-\infty}^x \left| \sum_{l=1}^m \tilde{v}_l A_l e^{\xi_l} \right|^2 dx.
\] (19)
From \((5)\) it follows that

\[
M_{lj} = A_l B_j \frac{1}{p_i + q_j} e^{\xi_l + \eta_j}.
\]

By using the operator relation

\[
(p_l \partial_{p_l}) e^{\xi_l} = e^{\xi_l} (p_l \partial_{p_l} + \hat{\xi}_l), \quad (q_j \partial_{q_j}) e^{\eta_j} = e^{\eta_j} (q_j \partial_{q_j} + \hat{\eta}_j),
\]

where

\[
\hat{\xi}_l = p_l \partial_{p_l} \xi_l = xp_l + 2i p_l^2 y - 12 p_l^3 t, \quad \hat{\eta}_j = q_j \partial_{q_j} \eta_j = xq_j - 2i q_j^2 y - 12 q_j^3 t,
\]

we also have

\[
M_{lj} = e^{\xi_l + \eta_j} \sum_{k=0}^{n_l} c_{lk} (p_l \partial_{p_l} + \hat{\xi}_l)^{n_l-k} \sum_{s=0}^{n_j} d_{js} (q_j \partial_{q_j} + \hat{\eta}_j)^{n_j-s} \frac{1}{p_l + q_j}.
\]

### 3 Asymptotic Analysis

In this section, we can investigate the asymptotic behaviors of the multi-lumps solutions defined by \((5)\), \((8)\) and \((9)\) or \((21)\). From this asymptotic analysis, one can get the orthogonal polynomials, of which the real roots are used to determine the locations of peaks as \(|t| \to \infty\). Without loss of generality, we assume that \(c_{10} = c_{20} = \cdots = c_{n0} = 1\) by \((4)\). Let’s define

\[
\hat{A}_j = \sum_{k=0}^{n_j} c_{jk} (p_j \partial_{p_j} + \hat{\xi}_j)^{n_j-k},
\]

and it’s not difficult to see that

\[
[\hat{A}_l, \hat{A}_j] = 0, \quad [\hat{A}_l, \bar{\hat{A}}_j] = 0, \quad [\bar{\hat{A}}_l, \hat{A}_j] = 0, \quad l \neq j.
\]

Notice that the differential operators in \((17)\) also have the properties \((22)\). By \((22)\) and the reality condition \((38)\), one has the expression

\[
\tau(x, y, t) := e^{-(\sum_{j=1}^{m} p_j + \bar{p}_j)} \det M(x, y, t) \\
= e^{-(\sum_{j=1}^{m} p_j + \bar{p}_j)} A_1 A_2 \cdots A_m \bar{A}_1 \bar{A}_2 \cdots \bar{A}_m \\
\frac{\prod_{1 \leq i < j \leq m} (p_i - p_j)(\bar{p}_i - \bar{p}_j) e^{\sum_{j=1}^{m} p_j + \bar{p}_j}}{\prod_{j=1}^{m} (p_i + \bar{p}_j)} \\
= \hat{A}_1 \hat{A}_2 \cdots \hat{A}_m \bar{A}_1 \bar{A}_2 \cdots \bar{A}_m \frac{\prod_{1 \leq i < j \leq m} (p_i - p_j)(\bar{p}_i - \bar{p}_j)}{\prod_{j=1}^{m} (p_i + \bar{p}_j)},
\]

(24)
where one uses the Cauchy determinant formula
\[
\det \left( \frac{1}{x_i + y_j} \right)_{1 \leq i, j \leq m} = \prod_{1 \leq i < j \leq m} (x_i - x_j)(y_i - y_j) / \prod_{i,j=1}^m (x_i + y_j).
\]

For example, when \( n = 3 \), we have
\[
\tau(x, y, t) = \det \left( \begin{array}{ccc}
\hat{A}_1 \hat{A}_1 & 1/p_1 + p_1 & 1/p_1 + p_2 & 1/p_1 + p_3 \\
\hat{A}_2 \hat{A}_1 & 1/p_2 + p_1 & 1/p_2 + p_2 & 1/p_2 + p_3 \\
\hat{A}_3 \hat{A}_1 & 1/p_3 + p_1 & 1/p_3 + p_2 & 1/p_3 + p_3
\end{array} \right)
\]
\[
= \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_1 \hat{A}_2 \hat{A}_3 \det \left( \begin{array}{ccc}
1/p_1 + p_1 & 1/p_2 + p_1 & 1/p_3 + p_1 \\
1/p_2 + p_2 & 1/p_2 + p_2 & 1/p_3 + p_2 \\
1/p_3 + p_3 & 1/p_3 + p_3 & 1/p_3 + p_3
\end{array} \right)
\]
\[
= \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_1 \hat{A}_2 \hat{A}_3 (p_1 - p_2)(p_2 - p_3)(p_3 - p_1)(\hat{p}_1 - \hat{p}_2)(\hat{p}_1 - \hat{p}_3)(\hat{p}_2 - \hat{p}_3)
\]
\[
\prod_{i,j=1}^3 (p_i + \hat{p}_j).
\]

From (13), (8) and (17), we see that the solution \( u(x, y, t) = 2\partial_x^2 \ln \tau(x, y, t) \) can also be expressed in terms of the elementary Schur functions (13). As \( \sqrt{x^2 + y^2} \to \infty \), the leading order behavior of \( \tau(x, y, t) \) consists of terms arising from where all the highest-order differential operators, \((p_1 \partial p_1)^{n_1}, (p_2 \partial p_2)^{n_2}, \ldots, (p_m \partial p_m)^{n_m}\) and their complex conjugates, act on the factor \( \prod_{1 \leq i < j \leq m} (p_i - p_j)(\hat{p}_i - \hat{p}_j) e^{\sum_{j=1}^m p_j + \hat{p}_j} \). It is similar to the case in [2]. This yields
\[
\tau \approx W \hat{W},
\]
where
\[
W = e^{-\sum_{j=1}^m p_j (p_1 \partial p_1)^{n_1} (p_2 \partial p_2)^{n_2} \cdots (p_m \partial p_m)^{n_m}} \prod_{1 \leq i < j \leq m} (p_i - p_j)e^{\sum_{j=1}^m p_j}
\]
\[
= W \mathcal{R}(S_{n_1}(p_1), S_{n_2}(p_2), S_{n_3}(p_3), \ldots, S_{n_m}(p_m)) = \det \partial_{x_i}^{n_i - 1}(S_{n_i}(p_j))
\]
(25)
is the Wronskian of the elementary Schur polynomials of degrees \( n_1, n_2, n_3, \ldots, n_m \). Here we have used (10) and the Vandermonde determinant formula. Furthermore, using (13), (16) and (17), we can get
\[
\frac{\partial S_{n_j}}{\partial x} = (n_j)! \left( \frac{\partial H_{n_j}}{\partial x_1} \right) + \frac{\partial H_{n_j}}{\partial x_2} + \cdots + \frac{\partial H_{n_j}}{\partial x_{n_j}}
\]
\[
= (n_j)! p(H_{n_j-1} + \frac{1}{2!} H_{n_j-2} + \frac{1}{3!} H_{n_j-3} + \cdots + \frac{1}{(n_j)!} H_0).
\]

Finally, one yields, up to an overall constant factor \( p^{m(m-1)/2}(n_1)!(n_2)!(n_3)! \cdots (n_m)! \),
\[
\tau \approx W \hat{W} \approx \Omega \hat{\Omega}, \quad \text{as} \quad \sqrt{x^2 + y^2} \to \infty
\]
(26)
where
\[
\Omega(x, y, t) = \det \begin{pmatrix}
H_{n_1}(p_1) & H_{n_2}(p_2) & H_{n_3}(p_3) & \cdots & H_{n_m}(p_m) \\
H_{n_1}(p_1) & H_{n_2}(p_2) & H_{n_3}(p_3) & \cdots & H_{n_m}(p_m) \\
H_{n_1}(p_1) & H_{n_2}(p_2) & H_{n_3}(p_3) & \cdots & H_{n_m}(p_m) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
H_{n_1}(p_1) & H_{n_2}(p_2) & H_{n_3}(p_3) & \cdots & H_{n_m}(p_m)
\end{pmatrix}.
\]

We see that \( \tau(x, y, t) = O(|f|^{2\rho}), \rho = \sum_{j=1}^{m} n_j - \frac{m(m-1)}{2}. \)

Next, we assume \( p_1 = p_2 = \cdots = p_m = a + bi. \) In the limit \( t \to \pm \infty, u(x, y, t) \)
genERICALLY has \( \rho \) distinct peaks whose locations are asymptotically given by \( [2] \)

\[
x^{(j)}(t) = 12(a^2 + b^2)t + c_j |t|^q + o(|t|^q) \\
y^{(j)}(t) = 12at + w_j |t|^q + o(|t|^q), \quad j = 1, 2, 3, \cdots, \rho, \quad 1/3 \leq q \leq 1/2, \quad (27)
\]

where \( q \) depends on \( m \) and \( n_1, n_2, n_3, \cdots, n_m. \) We remark here why \( 1/3 \leq q \leq 1/2 \)
is not explained in \( [2]. \) The real constants \( c_j \) and \( w_j \) can be found as follows. We
plug the forms \( (27) \) into \( (26) \) and then we have

\[
\Omega(x, y, t) = (M(c, w; a, b) + R(c, w; a, b))|t|^q + o(|t|^q), \quad (28)
\]

where \( M(c, w) \) and \( R(c, w) \) are real polynomials of \( c \) and \( w. \) Letting \( M(c, w; a, b) = R(c, w; a, b) = 0, \) we can get exactly \( \rho \) (topological charge) real roots \((c_j, w_j), \) counting multiplicity, that is,

\[
M(c_j, w_j; a, b) = R(c_j, w_j; a, b) = 0. \quad (29)
\]

Also, from \( (11) \) and \( (27), \) one has

\[
f = (c-2bw+2iaw)|t|^q + o(|t|^q), \quad f_p = -24at + 2iw|t|^q, \quad f_{pp} = -24t, \quad f_{ppp} = \cdots = 0. \quad (30)
\]

To find \( q, \) we define the weights for \( S_n \) in \( (13) \) by

\[
weight(f) = 1, \quad weight(f_p) = 2, \quad weight(f_{pp}) = 3, \quad weight(f_{ppp}) = 4, \cdots. \quad (31)
\]

In the expansion \( (26), \) to obtain the highest powers in \( |t| \) by \( (17) \) and \( (30), \) we consider only the terms \( f^\alpha f^\beta f^\gamma_p (\alpha, \beta, \gamma \geq 0) \) of the highest weight \( \rho. \) Then we have

\[
\alpha + 2\beta + 3\gamma = \rho, \quad \alpha q + \beta + \gamma = pq.
\]

The right-hand equation means we assume these terms have the highest power in
\(|t|. \) So

\[
q = \frac{\beta + \gamma}{\rho - \alpha} = \frac{\beta + \gamma}{2\beta + 3\gamma}. \quad (32)
\]
It is not difficult to see that $1/3 \leq q \leq 1/2$. Also, let’s find the condition that the highest power in $|t|$ is the same. Suppose $f^{n'} f_p f^{\gamma'}_{pp}$ is another term having the highest power in $|t|$, i.e., from (32),

$$\frac{\beta + \gamma}{2\beta + 3\gamma} = \frac{\beta' + \gamma'}{2\beta' + 3\gamma'}.$$  

Then a simple calculation gets

$$\beta' \gamma' = \gamma' \beta', \quad \text{or} \quad \frac{\beta}{\beta'} = \frac{\gamma}{\gamma'}.$$  

The equation (33) will imply the condition of those terms having the highest power in $|t|$.

Now, we observe that each elementary Schur polynomial $H_r$ in (13) has the highest degree $r$ in $x_1 = xp + 2ip^2 y - 12p^2 t$. One utilizes the Maple software to do the following computations. For simplicity, we set $a = 1, b = 0$, i.e., $p = 1$. By (16) and (27) one has

$$\begin{align*}
x_1 &= x + 2iy - 12t = (c + 2iw)|t|^q + o(|t|^q), \\
x_2 &= \frac{x}{2} + 2yi - 18t = -12t + (\frac{c}{2} + 2iw)|t|^q + o(|t|^q).
\end{align*}$$  

(34)

3.1 $m = 1$, $n_1 = n = \rho$

Among the terms of highest weight in $S_n$, from (30) the terms $f^n$ and $f^{n-2}f_p$ of weight $n$ has the highest powers in $t$, that is, $\alpha = n - 2, \beta = 1, \gamma = 0$ in (32). Then $nq = (n-2)q + 1$. Hence $q = 1/2$. For example, in $S_4$, the terms $f_{ppp}, f_p^2, f_{pp}, f^2 f_p, f^4$ have the highest weight 4. We see that $f^4$ and $f^2 f_p$ has the highest powers $t^4q$ and $t^{2q+1}$ from (30). Then we get $q = 1/2$.

- For $t \to -\infty$, we consider $c = 0$ in (34), and as $t \to -\infty$, one gets $\sqrt{x^2 + y^2} \to \infty$ and consequently, the asymptotic expansion (26) is used. Plugging (34) into $H_r$, we assume

$$H_r(t) = \frac{\Gamma_r(w)}{r!} (2i)^r (-t)^{r/2} + O(|t|^{r+1}),$$  

(35)

where $\Gamma_r(w)$ is a polynomial in $w$. To find the polynomials $\Gamma_r(w)$, we see that from (35)

$$\begin{align*}
&\frac{\partial H_r}{\partial t} = \frac{\Gamma_r(w)}{r!} (2i)^r \frac{r}{2} (-t)^{(r-2)/2} (-1) + O(|t|^{r+2}) \\
&\quad = \left( \frac{\partial H_r}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial H_r}{\partial x_2} \frac{\partial x_2}{\partial t} + \cdots + \frac{\partial H_r}{\partial x_r} \frac{\partial x_r}{\partial t} \right) \\
&\quad = \frac{\Gamma_{r-1}(w)}{(r-1)!} (2i)^{r-1} 2iw(t)^{(r-1)/2} \frac{1}{2\sqrt{-t}} + (-12) \frac{\Gamma_{r-2}(w)}{(r-2)!} (2i)^{r-2} (-t)^{(r-2)/2} \\
&\quad + O(|t|^{r+2}).
\end{align*}$$  

(36)
Then one has the recursive relation after a simple calculation,
\[ w \Gamma_r(w) = \Gamma_{r+1}(w) + 6r \Gamma_{r-1}(w), \quad \Gamma_0 = 1, \quad \Gamma_1 = w. \] (37)

Also, it follows
\[ \frac{d\Gamma_r(w)}{dw} = r \Gamma_{r-1}(w) \] (38)
from
\[
\frac{\partial H_r}{\partial w} = \left( \frac{\partial H_r}{\partial x_1} \frac{\partial x_1}{\partial w} + \frac{\partial H_r}{\partial x_2} \frac{\partial x_2}{\partial w} + \cdots + \frac{\partial H_r}{\partial x_r} \frac{\partial x_r}{\partial w} \right) = (2i)^{r-1}(-t)^{(r-1)/2} \frac{\Gamma_{r-1}(w)}{(r-1)!} \sqrt{2i} + O(|t|^{r-2}). \] (39)

By (48) and (38), it can be seen that \( \Gamma_r(w) \) satisfies the linear second-order differential equation \(( r \geq 0)\)
\[ \frac{d^2 \Gamma_r(w)}{dw^2} - \frac{w}{6} \frac{d \Gamma_r(w)}{dw} + \frac{r}{6} \Gamma_r(w) = 0. \] (40)

The orthogonal polynomial solutions (48) of (40) have \( r \) real roots for \( \Gamma_r(w) \), as is proved in page 22 [9] using the Christoffel-Darboux identity. We list a few polynomials for reference. For example,

\[
\begin{align*}
\Gamma_0(w) &= 1, \quad \Gamma_1(w) = w, \quad \Gamma_2(w) = (w^2 - 6), \\
\Gamma_3(w) &= w^3 - 18w, \quad \Gamma_4(w) = w^4 - 36w^2 + 108, \quad \Gamma_5(w) = w^5 - 60w^3 + 540w, \\
\Gamma_6(w) &= w^6 - 90w^4 + 1620w^2 - 3240, \quad \Gamma_7(w) = w^7 - 126w^5 + 3780w^3 - 22680w, \\
\Gamma_8(w) &= w^8 - 168w^6 + 7560w^4 - 90720w^2 + 136080.
\end{align*}
\]

We remark that the terms in \( \Gamma_r(w) \) come from the condition (33).

- For \( t \to \infty \), the calculation is similar. But we consider \( w = 0 \) in (34). Plugging (34) into \( H_r \), we assume
\[ H_r(t) = \frac{G_r(c)}{r!} (t)^{r/2} + O(|t|^{(r+1)/2}), \] (41)
where \( G_r(c) \) is a polynomial in \( c \). To find the polynomials \( G_r(c) \), in a similar way as (36) and (39), we have
\[
\begin{align*}
c G_r(c) &= G_{r+1}(c) + 24r G_{r-1}(c), \quad G_0 = 1, \quad G_1 = c, \\
\frac{dG_r(c)}{dc} &= r G_{r-1}(c).
\end{align*}
\] (42) (43)

By (42) and (43), it can be also seen that \( G_r(c) \) satisfies the linear second-order differential equation \(( r \geq 0)\)
\[ \frac{d^2 G_r(c)}{dc^2} - \frac{c}{24} \frac{d G_r(c)}{dc} + \frac{r}{24} G_r(c) = 0. \] (44)
Also, the orthogonal polynomial solutions \((42)\) of \((44)\) have \(r\) real roots for \(G_r(c)\). We list a few polynomials for reference. For example,

\[
\begin{align*}
G_0(c) &= 1, \\
G_1(c) &= c, \\
G_2(c) &= (c^2 - 24), \\
G_3(c) &= c^3 - 72c, \\
G_4(c) &= c^4 - 144c^2 + 1728, \\
G_5(c) &= c^5 - 240c^3 + 8640c, \\
G_6(c) &= c^6 - 360c^4 + 25920c^2 - 207360, \\
G_7(c) &= c^7 - 504c^5 + 60480c^3 - 1451520c, \\
G_8(c) &= c^8 - 672c^6 + 120960c^4 - 5806080c^2 + 34836480.
\end{align*}
\]

Therefore, from \((27)\) one knows that all the peaks are on one vertical line when time approaches \(-\infty\), and then they will be on a horizontal line when time approaches \(\infty\). Consequently, there is a rotation \(\pi/2\) after interaction of lumps.

### 3.2 \(m = 2, n_1 = 1, n_2 = n, \rho = n\)

In this case, by \((26)\), we have

\[
\Omega_{(1,n)}(x, y, t) = \det \left( \begin{array}{cc} H_1(1) & H_n(1) \\ H_{n-1}(1) & \end{array} \right) = H_1(1)H_{n-1}(1) - H_n(1) 
\]

(45)

To find \(q\), we consider it in the following way. One focus on the terms \(f^n\) and \(f^{n-2}f_p\) in \(S_n\), that is, \(x_1^n\) and \(x_1^{n-2}x_2\) in \(H_n\). By \((13)\), the coefficient of \(x_1^{n-2}x_2\) in \(H_n\) is \(\frac{1}{(n-2)!}\); consequently, the coefficient of this term in \((13)\) is \(\frac{1}{(n-2)!} - \frac{1}{(n-2)!}\). Hence when \(n = 3\), we get \(q = 1/3\); in other cases, we get \(nq = (n-2)q + 1\), i.e., \(q = 1/2\).

When \(n = 3\) and \(t \to -\infty\), one has in \((28)\)

\[
M(c, w) = 1/3 c^3 - 4cw^2 - 16, \quad R(c, w) = 2c^2w - 8/3 w^3;
\]

and \(t \to \infty\), one has

\[
M(c, w) = 1/3 c^3 - 4cw^2 + 16, \quad R(c, w) = 2c^2w - 8/3 w^3.
\]

A little algebra shows both of them have three real roots. So there are three peaks when \(|t| \to \infty\). Also, after interaction, the peak in the \(x\)-axis, that is \(w = 0\), will slow down.

On the other hand, from \((34)\), the real roots in \((29)\) for \(q = 1/2\) come from either \(c = 0\) or \(w = 0\) as \(t \to \pm\infty\) (see below). Therefore, there are two more cases to be discussed.

- As \(t \to -\infty\), we consider \(w = 0\) in \((34)\). Then \(x_1 = c\sqrt{-t}\) and \(x_2 = -12t + c\sqrt{-t}\). Plugging \((34)\) into \(H_r\), we assume

\[
H_r(t) = \frac{\hat{\Gamma}_r(c)}{r!}(-t)^{r/2} + O(|t|^{\frac{r+1}{2}}),
\]

(46)
where \( \hat{\Gamma}_r(w) \) is also a polynomial in \( c \). To find the polynomials \( \hat{\Gamma}_r(c) \), we see that from (46)

\[
\frac{\partial H_r}{\partial t} = \hat{\Gamma}_r(c) \frac{r}{r!} (-t)^{(r-2)/2} (-1) + O(|t|^{\frac{r-1}{2}})
\]

\[
= \left( \frac{\partial H_r}{\partial x_1 \partial t} + \frac{\partial H_r}{\partial x_2 \partial t} + \cdots + \frac{\partial H_r}{\partial x_r \partial t} \right)
\]

\[
= - \frac{\hat{\Gamma}_{r-1}(c)}{(r-1)!} (-t)^{(r-1)/2} \frac{c}{2\sqrt{-t}} + (-12) \frac{\hat{\Gamma}_{r-2}(c)}{(r-2)!} (-t)^{(r-2)/2}
\]

\[
+ O(|t|^{\frac{r-1}{2}}). \quad (47)
\]

Then one has the recursive relation after a simple calculation,

\[
c \hat{\Gamma}_r(c) = \hat{\Gamma}_{r+1}(c) - 24r \hat{\Gamma}_{r-1}(c), \quad \Gamma_0 = 1, \quad \Gamma_1 = c. \quad (48)
\]

When compared with the recursive relation (42), the equation (48) has a minus sign before 24. Also, it follows

\[
\frac{d \hat{\Gamma}_r(c)}{dc} = r \hat{\Gamma}_{r-1}(c) \quad (49)
\]

from

\[
\frac{\partial H_r}{\partial c} = \left( \frac{\partial H_r}{\partial x_1 \partial w} + \frac{\partial H_r}{\partial x_2 \partial w} + \cdots + \frac{\partial H_r}{\partial x_r \partial w} \right)
\]

\[
= (-t)^{(r-1)/2} \frac{\hat{\Gamma}_{r-1}(c)}{(r-1)!} \sqrt{-t} + O(|t|^{\frac{r-1}{2}}). \quad (50)
\]

By (48) and (49), it can be seen that \( \hat{\Gamma}_r(c) \) satisfies the linear second-order differential equation ( \( r \geq 0 \))

\[
\frac{d^2 \hat{\Gamma}_r(c)}{dc^2} + \frac{c}{24} \frac{d \hat{\Gamma}_r(c)}{dc} - \frac{r}{24} \hat{\Gamma}_r(c) = 0. \quad (51)
\]

The orthogonal polynomial solutions (48) of (40) have no real root for even degree and zero is the only real root for odd degree. We list a few polynomials for reference. For example,

\[
\hat{\Gamma}_0(c) = 1, \quad \hat{\Gamma}_1(c) = c, \quad \hat{\Gamma}_2(c) = (c^2 + 24),
\]

\[
\hat{\Gamma}_3(c) = c^3 + 72c, \quad \hat{\Gamma}_4(c) = c^4 + 144c^2 + 1728, \quad \hat{\Gamma}_5(c) = c^5 + 240c^3 + 8640c,
\]

\[
\hat{\Gamma}_6(c) = c^6 + 360c^4 + 25920c^2 + 207360, \quad \hat{\Gamma}_7(c) = c^7 + 504c^5 + 60480c^3 + 1451520c,
\]

\[
\hat{\Gamma}_8(c) = c^8 + 672c^6 + 120960c^4 + 5806080c^2 + 34836480.
\]
• For \( t \to \infty \), we consider \( c = 0 \) in (34). Then \( x_1 = 2wi\sqrt{t} \) and \( x_2 = -12t + 2wi\sqrt{t} \). Plugging (34) into \( H_r \), we assume

\[
H_r(t) = \frac{\hat{G}_r(w)}{r^n}(2it)^{r/2} + O(|t|^{(r-1)/2}),
\]

where \( \hat{G}_r(w) \) is a polynomial in \( w \). To find the polynomials \( \hat{G}_r(w) \), in a similar way as (47) and (50), we have

\[
w\hat{G}_r(w) = \hat{G}_{r+1}(w) - 6\hat{G}_{r-1}(w), \quad \hat{G}_0 = 1, \quad \hat{G}_1 = w, \tag{53}
\]

\[
\frac{d\hat{G}_r(w)}{dw} = r\hat{G}_{r-1}(w). \tag{54}
\]

(53) and (54) can be also seen that \( \hat{G}_r(w) \) satisfies the linear second-order differential equation ( \( r \geq 0 \))

\[
\frac{d^2\hat{G}_r(c)}{dc^2} + \frac{r}{6} \frac{d\hat{G}_r(c)}{dc} - \frac{r}{6} \hat{G}_r(c) = 0. \tag{55}
\]

The orthogonal polynomial solutions (53) of (55) also have no real root for even degree and zero is the only real root for odd degree. We list a few polynomials for reference. For example,

\[
\hat{G}_0(w) = 1, \quad \hat{G}_1(w) = w, \quad \hat{G}_2(w) = (w^2 + 6), \quad \hat{G}_3(w) = w^3 + 18w, \quad \hat{G}_4(w) = w^4 + 36w^2 + 108, \quad \hat{G}_5(w) = w^5 + 60w^3 + 540w, \quad \hat{G}_6(w) = w^6 + 90w^4 + 1620w^2 + 3240, \quad \hat{G}_7(w) = w^7 + 126w^5 + 3780w^3 + 22680w, \quad \hat{G}_8(w) = w^8 + 168w^6 + 7560w^4 + 90720w^2 + 136080.
\]

Also, the number of real roots of the Wronskian of the orthogonal polynomials has been investigated [7, 10]. Let the multi-index \((n_1, n_2, n_3, \cdots, n_m)\) be related to the partition \( \zeta = (\zeta_1, \zeta_2, \cdots, \zeta_m) \) by

\[
n_j = \zeta_j + j - 1, \quad j = 1, 2, 3, \cdots, m, \quad d_\zeta = p - q,
\]

where \( p, q \) are the numbers of odd and even elements in \( n_j \), respectively. One defines the length of the partition \( \zeta \) is \(|\zeta| = \sum_{j=1}^{m} \zeta_j = \zeta_1 + \zeta_2 + \cdots + \zeta_m \). For a symmetric case, the Wronskian \( \phi_\zeta \) has the well-defined parity

\[
\phi_\zeta(-x) = (-1)^{|\zeta|}\phi_\zeta(x), \tag{56}
\]

where \( x = c \) or \( x = w \). For the symmetric property (56), we have the

**Theorem [7]:**

1. The number of simple real roots of the Wronskian is \( \sum_{j=1}^{m} (-1)^{m-j} \zeta_j - \frac{|d_\zeta + (m-2[m/2])|}{2} \).
(2) The multiplicity of 0 is \( \frac{d(d+1)}{2} \).

For \( m = 1 \) and \( n_1 = n \), one has the cases \( \Gamma_r(w) \) and \( G_r(c) \). It is not difficult to see that (45) also satisfies the symmetric property (56) as \( t \to \pm \infty \) for \( \Gamma_r(w) \) and \( G_r(c) \).

- \( n = 2k+1(k \geq 2) \): We have \( d_z = 2 \) and \( (\zeta_1, \zeta_2) = (1, 2k) \). From the Theorem, one knows that there are \( 2k - 1 - 1 = 2k - 2 \) simple real roots, and \( (0, 0) \) is the triple roots. So we obtain \( 2k - 2 + 3 = 2k + 1 \) real roots as \( t \to -\infty \) for \( \Gamma_r(w) \) and \( t \to \infty \) for \( G_r(c) \). For example, taking \( n = 5 \), we have as \( t \to \infty \)

\[
\Omega_{(1,5)}(c, t) = \det \left( \begin{array}{c} ct + o(t) \frac{G_5(c)t^{5/2}}{G_5(c)t^2 + o(t^2)} \end{array} \right) = \left( \frac{1}{30} c^5 - 4c^3 \right)t^{5/2} + o(t^{5/2}).
\]

The equation \( \frac{1}{30} c^5 - 4c^3 = 0 \) has five real root, and \( c = 0 \) is a triple root.

- \( n = 2k(k \geq 1) \): We have \( d_z = 0 \) and \( (\zeta_1, \zeta_2) = (1, 2k-1) \). From the Theorem, one knows that there are \( 2k - 1 - 1 - 0 = 2k - 2 \) simple real roots, and \( (0, 0) \) is not a root. In fact, another two real roots can be obtained from \( \hat{\Gamma}_r(c) \) and \( \hat{G}_r(w) \) as \( t \to \pm \infty \). It can be seen as follows. For example, one considers \( t \to \infty \). Using (45), (54) and (52), we know that

\[
\Omega_{(1,2k)}(x, y, t) = 2wit^{1/2} \frac{\hat{G}_{2k-1}(w)}{(2k-1)!} \left( 2i \right)^{2k-1} \frac{1}{t^{2k-1}} - \hat{G}_{2k}(w) + O(|t|^{(2k-1)/2})
\]

\[
= t^k (2i)^{2k} \frac{1}{(2k)!} \left( 2k w \hat{G}_{2k-1}(w) - \hat{G}_{2k}(w) \right) + O(|t|^{(2k-1)/2})
\]

From (53), it is known that

\[
\hat{G}_{2k}(w) = \alpha_{2k} w^{2k} + \alpha_{2(k-1)} w^{2(k-1)} + \alpha_{2(k-2)} w^{2(k-2)} + \cdots + \alpha_2 w^2 + \alpha_0,
\]

where \( \alpha_{2k}, \alpha_{2(k-1)}, \cdots, \alpha_2, \alpha_0 \) are positive integers. Therefore,

\[
w \frac{d\hat{G}_{2k}(w)}{dw} = \hat{G}_{2k}(w) = L(w) - \alpha_0,
\]

where

\[
L(w) = (2k-1)\alpha_{2k} w^{2k} + (2k-3)\alpha_{2(k-1)} w^{2(k-1)} + (2k-5)\alpha_{2(k-2)} w^{2(k-2)} + \alpha_2 w^2.
\]

Since \( (2k-1)\alpha_{2k}, (2k-3)\alpha_{2(k-1)}, (2k-5)\alpha_{2(k-2)}, \cdots, \alpha_2 \) are positive integers, we obtain \( L(w) \geq 0 \), and the only minima is at \( w = 0 \). By \( \alpha_0 > 0 \), it is known that the equation (57) only has two real roots. As \( t \to -\infty \), one can consider similarly for \( \hat{\Gamma}_r(c) \).
One remarks here that if $n_1 \neq 1$, then some real roots of (29) come from neither $c = 0$ nor $w = 0$. For example, we take $n_1 = 2, n_2 = 4$ as an example. Then $\Omega_{(2, 4)}(x, y, t) = H_2 H_3 - H_1 H_4$ and $q = 1/2$. When one considers $t \to \infty$ and uses (34), one has

$$M(c, w) = \frac{1}{24} c^5 - 2 c^3 - \frac{5}{3} c^3 w^2 + 72 c + \frac{10}{3} c w^4 + 24 c w^2 = 0,$$

$$R(c, w) = \frac{5}{12} c^4 w - \frac{10}{3} c^3 w^2 - 12 c^2 w + 144 w + 16 w^3 + 4/3 w^5 = 0.$$

They have $\rho = 2 + 4 - 1 = 5$ real roots, and some real roots $(c, w) \neq (0, 0)$. For $t \to -\infty$, it can be considered similarly, and also some real roots $(c, w) \neq (0, 0)$.

### 4 Concluding Remarks

We study the asymptotic behaviors of lump solutions of KP-(I) equation using the Grammian determinant structure. In particular, for $m = 1$ and $p = 1$, it is found that the positions $\zeta$ of peaks can be described by the real roots of orthogonal polynomials; moreover, the peaks are on one vertical line when time approaches $-\infty$, and then they will be on a horizontal line when time approaches $\infty$, i.e., there is a rotation $\pi/2$ after interaction. Also, for $m = 2$ and $p = 1$, in some special case, as $|t| \to \infty$, the locations of peaks will depend on the real roots of Wronskian of the orthogonal polynomials. The asymptotic structures are different, depending on the partition structure (even or odd); however, there is also a rotation $\pi/2$ after interaction.

In [2], the lump solutions (rationally decaying potentials) of KP-(I) equation are constructed in a similar way. These solutions are classified by the pole structure of the corresponding meromorphic eigenfunctions and a set of integers of quantity called the charge. When comparing with their lump solutions, one has different function (11) and different recursive relation (12). Especially, the relation (17) is important to get the orthogonal polynomials for the asymptotic analysis. There is no such relation in [2]. To the best of my knowledge, this kind of orthogonal polynomials have never been used in the asymptotic analysis for any rationally decaying solutions in KP-(I) equation.

In [5, 6, 19, 21], the multi-rogue wave solutions of KP-(I) equation are constructed from the NLS hierarchy via different approach. It is the so-called NLS-KP correspondence. The authors choose appropriate parameters such that at some intermediate time all the peaks will collide together to form the only extreme rogue wave, i.e., $P_m$ breather of the KP-(I) equation. On the other hand, by a similar consideration in [17], we also could choose $p_\alpha = q_\alpha = 1, \alpha = 1, 2, 3, \ldots, m$ and
\((n_1, n_2, n_3, \ldots, n_m) = (1, 3, 5, 7, \ldots, 2m - 1)\) for some particular parameters \(c_{lk}\) to obtain the \(P_m\) breather of the KP-(I) equation. As mentioned above, the asymptotic structures will be related to the partition structures. In \([20]\), there are concentric rings with in the center Peregrine breathers for the NLS equation using different partitions and parameters. From this point of view, our asymptotic structures only depend on the partitions \(\zeta\), that is, the exponent \(q\) in \((32)\) and the real roots of the polynomials \(M(c, w)\) and \(R(c, w)\) in \((28)\). It could be interesting to investigate such structures in KP-(I) equation when compared with the NLS equation. The research in this direction will be published elsewhere.

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