A Comparison of $p$-$g$ Tidal Coupling Analyses

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Two recent studies have attempted to constrain the proposed $p$-$g$ tidal instability with gravitational-wave data from GW170817. The studies use Bayesian methods to compare a model that includes $p$-$g$ tidal effects with one that does not. Using the same data, they arrive at very different conclusions. Reyes & Brown find that the observations of GW170817 strongly disfavor the existence of $p$-$g$ mode coupling. However, the LIGO and Virgo Collaborations find that neither model is strongly favored. We investigate the origin of this discrepancy by analyzing Reyes & Brown’s publicly available posterior samples. Contrary to their claims, we find that their samples do not disfavor $p$-$g$ mode coupling.

I. INTRODUCTION

The gravitational wave (GW) observation of a coalescing binary neutron star (NS) system (GW170817 [1]) provides new insights into NS physics, including constraints on the high-density equation of state [2, 3] and tidal deformability [2, 4, 5]. Recently, two papers have attempted to constrain the $p$-$g$ tidal instability with GW170817 ([6, 7]; hereafter RB and LVC, respectively). The instability involves a non-resonant coupling of the linear tidal bulge to high-frequency, pressure-supported modes ($p$-modes) and low-frequency, gravity-supported modes ($g$-modes) within the NS [8–11]. Once unstable, the excited modes continuously drain energy from the orbit and accelerate the rate of GW-driven inspiral. The precise impact on the phasing of the GW signal is, however, unknown due to theoretical uncertainties in how the instability grows and saturates, although studies suggest that its impact might be observable with the current LIGO [12] and Virgo [13] interferometers [10, 14].

RB and LVC attempt to constrain $p$-$g$ effects in GW170817 using the phenomenological model developed by [14]. Both studies employ a modification of the TaylorF2 frequency-domain waveform (see, e.g., [15]) that includes an additional phase correction induced by $p$-$g$ effects. Using Bayesian methods, they compare models with $p$-$g$ effects ($\mathcal{H}_{pg}$) to models without $p$-$g$ effects ($\mathcal{H}_{pg}$) and compute Bayes Factors $B_{pg}^{pg} \equiv \frac{P(D|\mathcal{H}_{pg})}{P(D|\mathcal{H}_{pg})}$, where $D$ refers to the data from GW170817.

While RB and LVC analyze the same data and use the same phenomenological $p$-$g$ waveform, there are differences in their models and priors, which we describe in Section II. Most notably, RB constructs an $\mathcal{H}_{pg}$ model that only includes “detectable $p$-$g$ effects” whereas LVC uses a wider $\mathcal{H}_{pg}$ model. RB finds that their models yield $B_{pg}^{pg} < 10^{-4}$ and LVC finds that their models yield $B_{pg}^{pg} \approx 1$. Thus, RB concludes that the observations strongly disfavor their $\mathcal{H}_{pg}$ model and LVC concludes that the observations do not favor either of their models.

A priori, the disparate $B_{pg}^{pg}$ could be due to differences in the studies’ models and priors. However, we show in Section III that this cannot be the explanation. We use the posterior samples from RB to compute $B_{pg}^{pg}$ using LVC’s method for calculating Bayes Factors [7]. We find that LVC’s method applied to RB’s posterior samples, and thus their models and priors, yield $B_{pg}^{pg} \approx 1$ and not $B_{pg}^{pg} < 10^{-4}$. This indicates that there is an error in how RB calculates $B_{pg}^{pg}$. Our estimate implies that their $\mathcal{H}_{pg}$ model is not disfavored by the data.

II. COMPARISON OF MODELS AND PRIORS

The phenomenological model presented in [14] introduces three $p$-$g$ parameters per NS (indexed by $i \in \{1, 2\}$): an overall amplitude ($A_i$) related to how many modes become unstable, how quickly they grow, and the energy at which they saturate; a turn-on/saturation frequency ($f_i$) that is related to when the modes first become unstable; and a spectral index ($n_i$) that describes how the rate of energy dissipation evolves with the orbital frequency (see [14], RB, and LVC).

The frequency-domain phase shift $\Delta \Psi(f)$ induced by $p$-$g$ effects is given by Equation (3) in RB and Equation (1) in LVC. To account for a possible dependence on the component masses ($m_i$), [14] introduces a Taylor expansion of the $p$-$g$ parameters around $m_i = 1.4M_{\odot}$. LVC keeps the zeroth- and first-order coefficients of the expansion (their Equation (2)). RB keeps only the zeroth order coefficients. However, this should not introduce large discrepancies since LVC and [14] find that the first order terms are not measurable.

In their Equation (3), RB neglects a dependence on the component masses that exists independent of the Taylor expansion. Specifically, in the expression for $\Delta \Psi(f)$, RB have kindly made their posterior samples available at https://github.com/sugwg/gw170817-pg-modes.
there is a factor \( C_i = [2m_i/(m_1 + m_2)]^{2/3} A_i \) and RB assumes \( C_1/A_1 = C_2/A_2 \) even when \( m_1 \neq m_2 \). However, since the difference is small for reasonable ranges of \( m_i \), this should not introduce a large discrepancy.

RB’s priors on the non-p-g parameters are somewhat different from LVC’s. RB considers both a uniform and a Gaussian prior on \( m_i \), whereas LVC considers only a uniform prior on \( m_i \). While RB’s estimates of \( B_{pg}^m \) vary by as much as a factor of \( 10^5 \) for different mass priors, they note that their posterior distributions are qualitatively very similar. In addition, unlike LVC, RB assumes a fixed source location and distance based on the electromagnetic counterparts to GW170817 [3, 16]. However, [14] found that extrinsic parameters do not strongly impact the inference of p-g effects.

RB also assumes the NSs have equal radii and their priors on component spins differ from LVC’s. However, we do not believe this could introduce large discrepancies.

The differences between RB’s and LVC’s priors on the p-g parameters are more substantial. The most significant difference is that LVC assumes uniform priors on the zeroth order coefficients \( \log_{10} A_0 \), \( f_0 \), and \( n_0 \) (following [14]), whereas RB constrains the p-g parameters to values that produce total time-domain phase shifts \( \delta \phi \geq 0.1 \text{rad} \). Thus, RB’s priors on \( \log_{10} A_0 \), \( f_0 \), and \( n_0 \) are not uniform, but favor combinations that produce relatively large p-g tidal effects. RB explains that for \( m_1 = m_2 = 1.4 M_\odot \), there is a 99.98\% overlap between the waveforms from \( H_{pg} \) and \( H_{1pg} \) for values of \((A_0, f_0, n_0)\) that yield \( \delta \phi \approx 0.1 \text{rad} \). As a result, like LVC, their prior still allows certain limits of \( H_{pg} \) to reproduce \( H_{1pg} \).

Finally, LVC focuses on a somewhat narrower bandwidth than RB (minimum frequencies of 30 Hz vs. 20 Hz). LVC does explore minimum frequencies above 30 Hz and find that \( B_{pg}^m \) only varies by factors of order unity (see their Figure 1). Since the gain in signal-to-noise ratio from 30 Hz to 20 Hz is relatively modest (\( \lesssim \) few percent), we do not believe this difference introduces large discrepancies.

III. COMPARISON OF RESULTS

In order to compute Bayes Factors, RB uses thermodynamic integration [17–22] while LVC uses the Savage-Dickey Density Ratio (SDDR, see [23–25] and Appendix A). In principle, both should yield consistent results. However, when we apply the SDDR to RB’s (uniform-mass, small \( f_0 \)) posterior samples, we find \( B_{pg}^m \approx +0.7 \) whereas RB claims \( B_{pg}^m = -21 \) (\( \log_{10} B_{pg}^m = -9.2 \)) for the same posterior samples.3

The SDDR provides a convenient way to estimate Bayes Factors between nested models when the posterior is available for the larger model. In the limit of small \( A_0 \) (see LVC’s Equation (3) and Appendix A),

\[
\lim_{A_0 \to 0} \frac{p(A_0 | \text{data}, H_{pg})}{p(A_0 | H_{pg})} \approx \frac{p(\text{data} | H_{pg})}{p(\text{data} | H_{pg})} = \frac{1}{B_{pg}^m}. \tag{1}
\]

This says that \( B_{pg}^m \) equals the ratio of the marginal distribution of \( A_0 \) a priori to the marginal distribution of

3 RB considers several different mass and \( f_0 \) priors. For the narrow (broad) \( f_0 \) prior, they find \( \log_{10} B_{pg}^m = -9.2, -6.0 \) (\( \log_{10} B_{pg}^m = -6.3, -4.7 \)) for the uniform and Gaussian mass priors, respectively. They summarize their results by stating that they find \( \log_{10} B_{pg}^m < -4 \).
Although the posteriors and priors are not constant as other priors. Since the convenient identity for the evidence $Z$ is small a priori, the evidence should favor small values of small $A_0$. Thus, the average of $\ln p(d|\theta)$ with respect to $p_\beta(\theta|d;H) \propto p(d|\theta)^\beta p(\theta|H)$. With enough temperatures, an estimate for the evidence is obtained as

$$
\ln Z_H = \int_0^1 d\beta \frac{d}{d\beta} \ln Z_H = \int_0^1 d\beta \langle \ln p(d|\theta) \rangle_{p_\beta(\theta|d;H)}
$$

Typically, the set of temperatures is chosen to optimize sampling through parallel tempering (see, e.g., [19, 21]) and the resulting integral is estimated via a trapezoidal approximation. This procedure is repeated for each model separately, and then differences in the evidences yield Bayes Factors.

RB attempt to implement this approach. However, as shown by their public data, they only use 3 temperatures, and the smallest inverse temperature used is $\beta \simeq 0.25$. When checking our implementation of the Savage-Dickey Density Ratio against thermodynamic integration on our own data, we only found convergence with at least 12 temperatures. This means that not only do they poorly resolve the numeric integral, they severely truncate the estimate. Therefore, the RB result contains large systematic errors.

RB rely upon evidence estimates from previous work [4] when estimating Bayes Factors. Examination of the public data from [4] shows that they also used only 3 temperatures and incorrectly truncated the integral. The Bayes factors originally quoted in [4] are therefore also in error (as the erratum in [4] acknowledges).

**V. CONCLUSIONS**

We analyzed the publicly available samples from Reyes & Brown [6]. Using our own method for calculating Bayes Factors [7], we find that their samples yield $B^\text{pg}_{\text{log}} \simeq +2.0$ and not $B^\text{pg}_{\text{log}} < 10^{-4}$. The source of their errors stems from a flawed implementation of thermodynamic integration, which also affected some of the authors’ other work (see erratum in [4]). We therefore conclude that, contrary to Reyes & Brown’s claim, their posterior data do not disfavor $p-g$ mode coupling.

**ACKNOWLEDGMENTS**

The authors thank Steven Reyes and Duncan Brown for making their samples publicly available, and Katherine Chatziioannou, Anuradha Samajdar, Aaron Zimmerman, and the other LVC reviewers for their useful feedback while preparing this note. R. Essick is supported at the University of Chicago by the Kavli Institute for Cosmological Physics through an endowment from the Kavli Foundation and its founder Fred Kavli.

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4 To render our kernel density estimation computationally tractable, we select a random subsample of $\approx 5000$ of their posterior samples. Including more samples would decrease the variance of each distribution shown in Figure 1, but we already rule out RB’s $B^\text{pg}_{\text{log}}$ with only 5000 samples.
Appendix A: Derivation of the Savage-Dickey Density Ratio

According to Bayes theorem

\[ p(A_0, f_0, n_0, \theta | D; \mathcal{H}_{pg}) = \frac{1}{p(D|\mathcal{H}_{pg})} p(D|A_0, f_0, n_0, \theta; \mathcal{H}_{pg}) p(A_0, f_0, n_0, \theta | \mathcal{H}_{pg}), \]  

(A1)

where \( \theta \) refers to all parameters besides the \( p-g \) parameters and \( D \) to the data from GW170817. We drop the first-order terms in the \( p-g \) parameter Taylor expansions for clarity, but they could be included in a straightforward way. The marginal posterior distribution for \( A_0 \) is therefore

\[ p(A_0|D; \mathcal{H}_{pg}) = \frac{1}{p(D|\mathcal{H}_{pg})} \int d\theta df_0 dn_0 p(D|A_0, f_0, n_0, \theta; \mathcal{H}_{pg}) p(A_0, f_0, n_0, \theta | \mathcal{H}_{pg}) \]

\[ = \frac{1}{p(D|\mathcal{H}_{pg})} \int d\theta df_0 dn_0 p(D|A_0, f_0, n_0, \theta; \mathcal{H}_{pg}) p(\theta|A_0; \mathcal{H}_{pg}) p(A_0 | \mathcal{H}_{pg})). \]  

(A2)

Although \( \mathcal{H}_{pg} \) is not formally contained in \( \mathcal{H}_{pg} \) for uniform-in-log\(_10\) \( A_0 \) priors, the lower limit of \( A_0 = 10^{-10} \) (in both studies) is sufficiently small that \( \mathcal{H}_{pg} \) is, to a very good approximation, nested in \( \mathcal{H}_{pg} \). In particular, at \( A_0 = 10^{-10} \), the waveforms of \( \mathcal{H}_{pg} \) and \( \mathcal{H}_{pg} \) match to > 99.999%. Therefore, in the limit \( A_0 \to 10^{-10} \), the likelihood
\( p(D|A_0, f_0, n_0, \theta; \mathcal{H}_{pg}) = p(D|\theta; \mathcal{H}_{pg}) \) and the integral factors. We then have

\[
\lim_{A_0 \rightarrow 10^{-10}} \frac{p(A_0|D; \mathcal{H}_{pg})}{p(A_0|\mathcal{H}_{pg})} = \frac{1}{p(D|\mathcal{H}_{pg})} \lim_{A_0 \rightarrow 10^{-10}} \int d\theta d\varphi_0 d\varphi_0 p(D|A_0, f_0, n_0, \theta; \mathcal{H}_{pg}) p(f_0, n_0, A_0, \theta, \mathcal{H}_{pg}) p(A_0, \mathcal{H}_{pg})
\]

\[
= \frac{1}{p(D|\mathcal{H}_{pg})} \lim_{A_0 \rightarrow 10^{-10}} \left( \int d\theta p(D|\theta; \mathcal{H}_{pg}) p(\theta|A_0, \mathcal{H}_{pg}) \right) \times \left[ \int d\varphi_0 d\varphi_0 p(f_0, n_0|\theta, A_0; \mathcal{H}_{pg}) \right].
\] (A3)

This allows us to integrate away the conditional prior for \( f_0 \) and \( n_0 \) and obtain

\[
\lim_{A_0 \rightarrow 10^{-10}} \frac{p(A_0|D; \mathcal{H}_{pg})}{p(A_0|\mathcal{H}_{pg})} = \frac{1}{p(D|\mathcal{H}_{pg})} \lim_{A_0 \rightarrow 10^{-10}} \int d\theta p(D|\theta; \mathcal{H}_{pg}) p(\theta|\mathcal{H}_{pg}) \left[ \frac{p(\theta|A_0; \mathcal{H}_{pg})}{p(\theta|\mathcal{H}_{pg})} \right]
\]

\[
= \frac{p(D|\mathcal{H}_{pg})}{p(D|\mathcal{H}_{pg})} \lim_{A_0 \rightarrow 10^{-10}} \left( \int d\theta p(D|\theta; \mathcal{H}_{pg}) p(\theta|A_0; \mathcal{H}_{pg}) \frac{p(A_0|\mathcal{H}_{pg})}{p(\theta|\mathcal{H}_{pg})} \right).
\] (A4)

Thus

\[
\lim_{A_0 \rightarrow 10^{-10}} \frac{p(A_0|D; \mathcal{H}_{pg})}{p(A_0|\mathcal{H}_{pg})} = \frac{1}{B_{\mathcal{H}_{pg}}} \lim_{A_0 \rightarrow 10^{-10}} \left( \frac{p(\theta|A_0; \mathcal{H}_{pg})}{p(\theta|\mathcal{H}_{pg})} \right) \frac{p(A_0|\mathcal{H}_{pg})}{p(\theta|D; \mathcal{H}_{pg})},
\] (A5)

where \( \langle x \rangle_p \) denotes the average of \( x \) with respect to the measure defined by \( p \). As we demonstrate in Figure 2, \( \lim_{A_0 \rightarrow 10^{-10}} p(\theta|A_0; \mathcal{H}_{pg}) \approx p(\theta|\mathcal{H}_{pg}) \). The red curves show the conditional prior distributions for component masses \( (m_1, m_2) \), chirp mass \( (\mathcal{M}) \), and mass ratio \( (q = m_2/m_1 \leq 1) \) under RB’s \( \mathcal{H}_{pg} \) priors for \( \log_{10} A_0 \in [-10, -9.9] \). The blue curves show the distributions for an analogous prior with uniform distributions for \( m_1 \) and \( m_2 \) and the same \( \mathcal{M} \) cuts but without any requirement on \( \delta \phi \), which is RB’s corresponding \( \mathcal{H}_{pg} \) prior. While we see some small differences in the marginal distributions, these are all \( \mathcal{O}(1) \). We therefore expect \( \langle p(\theta|A_0; \mathcal{H}_{pg})/p(\theta|\mathcal{H}_{pg}) \rangle _{p(\theta|D; \mathcal{H}_{pg})} \) to be a negligible correction and omit it from the main body of this note (Equation (1)). Therefore, the ratio of the marginal posterior to the marginal prior, when evaluated at sufficiently small \( A_0 \), yields an accurate an estimate of \( B_{\mathcal{H}_{pg}} \).
FIG. 2. Mass priors in RB’s analysis, which enter as a correction factor in Equation (A5). (red) $p(\theta|A_0; \mathcal{H}_{pg})$ with $\log_{10} A_0 \in [-10, -9.9]$. (blue) $p(\theta|\mathcal{H}_{pg})$ assuming uniform priors on component masses and the same $\mathcal{M}$ cuts RB uses for $\mathcal{H}_{pg}$. 