Variation entropy: a continuous local generalization of the TVD property using entropy principles.

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Abstract
This paper presents the notion of a variation entropy. This concept is an entropy framework for the system of equations that results when taking the gradient of a nonlinear conservation law. It appears that all semi-norms are admissible variation entropies. This provides insight into the total variation diminishing property and justifies it from entropy principles. The framework allows to derive new local variation diminishing properties in the continuous form. This can facilitate the design of new numerical methods for problems that contain discontinuities.

1. Introduction

Violent disturbances emerging from sudden changes in velocity, pressure and temperature, known as shock waves appear everywhere in nature, science and industrial applications. Examples are water-air flows, supersonic flights, the water hammer phenomena, shock-bubble interaction, material impact and sudden changes in crow dynamics. The behavior of these phenomena is usually governed by nonlinear conservation laws. The development of numerical techniques for the solution procedure of conservation laws is challenging because higher-order methods produce oscillations near shocks. There exists a large class of numerical methods which aim to tackle these oscillations via reducing to first-order spatial accuracy at the shock wave. All of these techniques augment the numerical method in one way or another with artificial diffusion or viscosity in the shock wave region.

Most numerical methods developed for problems involving shock waves has been performed with the finite-difference and finite-volume approaches. These methods are often well-established and show (very) good performance in numerical computations. They can often be linked to one of the following. The concept of flux limiters (MUSCL), see e.g. [1–5], reduces the scheme at the shock to first-order by adding diffusion. The monotonicity property introduced by Harten in 1983 [6] precludes the creation of local extrema and ensures that local minimaxa (maxima) are non-decreasing (non-increasing). Perhaps the most relevant in numerical simulations is the maximum principle, see [6] or the more recent work of [7, 8]. This principle states that the solution values remain between the minimum and maximum of the initial condition. This is in particular important in simulations of physical quantities that should remain positive, e.g. densities and also, in the case of two-fluid problems, volume-fractions. A negative density or a volume-fraction exceeding the zero-one range can directly lead to a blow-up of the simulations. Therefore numerical methods that preclude this by design are often sought after, see e.g. for compressible two-fluid flow simulations [9–11].

Possibly the most famous property is the total variation diminishing (TVD) property [6, 12, 13]. The total variation diminishing schemes preclude the growth of the total variation of the solution. These methods ensure that the numerical solution φ of a PDE satisfies

\[ TV(\phi^{n+1}) \leq TV(\phi^n), \] (1)

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where \( n \) denotes the time-level. Here the total variation \( TV \) is defined in one and two dimensions respectively as:

\[
TV(\phi) = \sum_j |\phi_{j+1} - \phi_j|, \\
TV(\phi) = \sum_{j,k} (|\Delta y|\phi_{j+1,k} - \phi_{j,k}| + \Delta x|\phi_{j,k+1} - \phi_{j,k}|),
\]

with the subscript the spatial index and with \(| \cdot |\) the absolute value. This definition of the total variation is based on the discrete approximation of the \( L_1 \)-norm of the gradient, and are on a global level. It is important to emphasize that this definition assumes an underlying Cartesian grid that is possibly non-uniform and does not have a straightforward equivalent for unstructured grids. Furthermore, the TVD property does not readily extend to multiple dimensions. This has been the motivation for the development of local extremum diminishing (LED) schemes [14, 15]. In the two-dimensional case, the definition of the total variation depends on the orientation of the grid, i.e. it would change if one would rotate the grid. This is undesirable since (i) this total variation does not have a continuous counterpart and (ii) generally one does not want the numerical method to depend on the coordinate system. The latter is closely linked to the concept of objectivity. Note that all of the above mentioned features and properties are satisfied by the exact solution and are in the discrete setting closely linked. In particular cases (e.g. one-dimensional, scalar) one implies another [6].

In the framework of finite element methods several stabilized methods have been proposed to deal with spurious wiggles in the solution profiles of convection-dominated problems. The well-known methods are the Streamline upwind-Petrov Galerkin (SUPG) [16], the Galerkin/least-squares method [17] and the variational multiscale method [18–20]. The latter methods offer rich possibility for design new stabilized methods and have gained a lot of attention recently [21–25]. In the direction of TVD schemes and maximum principles, several VMS methods have been proposed. For example a total variation bounding constraint [26] and the maximum principle [27] has been enforced in the VMS context. When shocks waves form, plain stabilized methods are not sufficient and additional dissipation mechanisms are necessary. These mechanisms are called discontinuity capturing (DC) operators [28, 29], and are sometimes residual-based [30–32] or entropy-residual based [33]. We refer for more details on DC to the review paper of Hughes et al. [34].

To single out the physically relevant solutions the concept of entropy solutions has been proposed by Kruvzkov in his seminal 1970 paper [35]. The entropy solution is a limiting case of a generalized solution which perturbs the conservation law with a diffusion term and can be used to prove existence, uniqueness and stability theorems. In the case of systems of conservation laws, Friedrichs, Kurt and Lax show in 1971 that if an additional conserved quantity is a convex function of the solution then the system of equations can be symmetrized and provides a corresponding entropy inequality [36]. Harten continues the research on symmetrizability of systems of conservation laws which possess entropy functions [37]. Additionally, he provides symmetric formulations in conservative variables for the Euler equations of gas dynamics. Tadmor shows a year later, in 1984, that the concepts of symmetrizability, having an entropy function and having a so-called skew-selfadjoint form are equivalent [38]. Furthermore Tadmor identifies in [39] that any symmetric system of conservation laws is equipped with a one-parameter family of entropy functions. The work of Harten and Tadmor has been generalized by Hughes et al. [40] to the compressible Navier-Stokes equations with heat conduction effects. The corresponding finite element schemes satisfy by design the second law of thermodynamics, see also [41].

Although total variation diminishing schemes have proven their power and relevance, its use seem to be restricted to finite-difference and finite-volume discretizations. Moreover, the different concepts of total variation diminishing schemes and entropy functions/entropy variables both target to improve the solution quality at shock waves. Despite that they serve the same goal, a clear connection (on the continuous level) is missing.

The current paper aims to bridge this gap. To that purpose, we introduce the notion of a variation entropy. This tool allows us to derive the total variation concept, in fact, we show that a class of total variation solutions equals that of the variation entropy solutions. The total variation solutions are presented...
in the continuous setting and are as such not restricted to a particular discretization.

The remainder of the paper can be summarized as follows. Section 2 provides a brief summary of the classical entropy solutions and introduces some of the required notation. In Section 3 the concept of the variation entropy is presented. This section identifies all the possible variation entropies. Section 4 takes a closer look into the various options of variation entropies. In particular, the well-known TVD property is here presented in an entropy context. Finally, Section 5 draws the conclusions and outline avenues for future research.

2. Classical entropy solutions

Let \( \Omega \subset \mathbb{R}^d \) be an open connected domain. Let us consider the scalar conservation problem: find \( \phi : \Omega \times I \to \mathbb{R} \) such that

\[
\partial_t \phi + \nabla \cdot f = 0, \quad (x, t) \in \Omega \times I,
\]

subject to the initial condition \( \phi(x, 0) = \phi_0(x) \in L^\infty(\Omega) \). Here \( f = f(\phi) \in C(\Omega, \mathbb{R}) \) is the (nonlinear) flux, the spatial coordinate denotes \( x \in \Omega \), the time is \( t \in I = (0, T) \) with \( T > 0 \). Solutions of (3) can contain discontinuities (shocks, rarefaction waves) which motivates the search for weak solutions. A weak solution \( \phi \in L^\infty(\Omega, \mathbb{R}^+) \) of (3) satisfies

\[
\int_{\mathbb{R}^+} \int_{\Omega} \partial_t \phi \psi + f(\phi) \nabla \psi \, d\Omega \, dt + \int_{\Omega} \phi_0 \psi_0 \, d\Omega = 0
\]

for all test functions \( \psi \in C^1_c(\Omega, \mathbb{R}^+) \) with \( \psi_0(x) = \psi(x, 0) \). Weak solutions are generally not unique. One option of establishing uniqueness is to consider vanishing viscosity limit solutions \( \phi^\epsilon : \Omega \times I \to \mathbb{R} \) satisfying

\[
\partial_t \phi^\epsilon + \nabla \cdot f^\epsilon = \epsilon \Delta \phi^\epsilon, \quad (x, t) \in \Omega \times I,
\]

with \( f^\epsilon = f(\phi^\epsilon) \). This concept is closely linked to entropy conditions which provide a single physically admissible solution. Let \( \eta = \eta(\phi^\epsilon) \in C^1(\mathbb{R}) \) be a convex function. Multiplying (5) by \( \partial \eta / \partial \phi^\epsilon \) yields\(^1\):

\[
\partial_t \eta + \nabla \cdot q^\epsilon \leq \epsilon \Delta \eta,
\]

where the flux \( q^\epsilon \) is given by

\[
\frac{\partial q^\epsilon}{\partial \phi^\epsilon} = \frac{\partial \eta}{\partial \phi^\epsilon} \frac{\partial f}{\partial \phi^\epsilon}.
\]

Employing the inequality that holds for convex \( \eta \):

\[
\frac{\partial \eta}{\partial \phi^\epsilon} \Delta \phi^\epsilon = \Delta \eta - \frac{\partial^2 \eta}{\partial (\phi^\epsilon)^2} \nabla \phi^\epsilon \cdot \nabla \phi^\epsilon \leq \Delta \eta,
\]

we get

\[
\partial_t \eta + \nabla \cdot q \leq \epsilon \Delta \eta.
\]

The distinguished limit of a vanishing viscosity \( \epsilon \) yields the entropy condition

\[
\partial_t \eta + \nabla \cdot q = 0,
\]

\(^1\)Note that this can be interpreted as a change of variables, see also [40].
which holds for all convex \( \eta \) where the flux \( q \) is given by
\[
\frac{\partial q}{\partial \phi} = \frac{\partial \eta}{\partial \phi} \frac{\partial f}{\partial \phi}.
\] (11)

The entropy condition tells us that the entropy \( \eta \) dissipates at shock waves. Solutions that satisfy this condition are called entropic or entropy solutions. The function \( \eta \) is referred to as the entropy function and \( q \) as the entropy flux.

In case \( \Omega \) is a periodic domain or has no-inflow boundary conditions, integration of (10) over \( \Omega \) leads to a decay of the overall entropy:
\[
\frac{d}{dt} \int_{\Omega} \eta(\phi(x, t)) \, d\Omega \leq 0, \quad \text{for all } t \geq 0,
\] (12)
or equivalently:
\[
\int_{\Omega} \eta(\phi(x, t)) \, d\Omega \leq \int_{\Omega} \eta(\phi_0(x)) \, d\Omega, \quad \text{for all } t \geq 0.
\] (13)

Note that this is the usual \( L^2 \)-stability from linear theory for hyperbolic equations. We refer to [42] for more details.

**Remark 2.1.** Note that we can write \( q = \eta \frac{\partial f}{\partial \phi} \) for divergence-free flux derivatives: \( \nabla \cdot (\frac{\partial f}{\partial \phi}) = 0 \).

**Remark 2.2.** We can also consider flux functions of the form \( f = f(\phi, \nabla \phi) \). For the sake of simplicity we restrict ourselves to the case where the matrix \( \frac{\partial f}{\partial \nabla \phi} \) is of the form \(-kI\) with the scalar \( k \geq 0 \) and the identity matrix \( I \). The relation (6) now takes the form
\[
\partial_t \eta + \nabla \cdot q' = (k + \epsilon) \frac{\partial \eta}{\partial \phi^\epsilon} \Delta \phi^\epsilon.
\] (14)

Again using (8) provides:
\[
\partial_t \eta + \nabla \cdot q^\epsilon - k \Delta \eta \leq \epsilon \Delta \eta.
\] (15)

### 3. The variation entropy

In this section we introduce the notion of a variation entropy. First we present the main concept and subsequently we provide the corresponding analysis.

**3.1. The concept**

We present the variation entropy for an augmented scalar conservation law\(^2\) with a convective, diffusive and source component, i.e. we consider:
\[
\partial_t \phi + \nabla \cdot f = s, \quad (x, t) \in \Omega \times I,
\] (16)
with flux \( f = f(\phi, \nabla \phi) \) and source term \( s = s(\phi) \). Assume that the matrix \( \frac{\partial f}{\partial \nabla \phi} \) has only negative eigenvalues. The main idea is to look at the associated entropy relation of the spatial gradient (or variation) of the conservation law instead of that of the plain conservation law, i.e. consider the system of equations:
\[
\nabla (\partial_t \phi) + \nabla (\nabla \cdot f) = \nabla s.
\] (17)

\(^2\)We use the term ‘augmented’ to indicate that it concerns a conservation law with a source term. Strictly speaking a conservation law does not include a source term.

\(^3\)The extension to systems of conservation laws is straightforward. We discuss the scalar case for the sake of simplicity.
This allows to identify shockwaves which are characterized by large gradients. In Figure 1 we sketch the concept.

![Figure 1: The concept of the variation entropy. In the classical approach one considers the entropy of the conservation law. The idea of the variation entropy approach is to first take the gradient of the conservation law and subsequently introducing the entropy concept.](image)

In the remainder of this subsection we present the evolution equation of the variation entropy and provide the entropy condition linked to the variation entropy.

**Lemma 3.1.** (Evolution equation) Let $\eta : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable convex function of the gradient of $\phi$, i.e. $\eta = \eta(\nabla \phi)$. Its temporal evolution reads:

$$\partial_t \eta + \nabla \cdot q = \mathcal{A} + \mathcal{D} + \mathcal{S},$$

where the flux $q$, the non-conservative terms $\mathcal{A}$ and $\mathcal{D}$ and the source term $\mathcal{S}$ are respectively given by:

\begin{align}
q &= \left( \frac{\partial \eta}{\partial \nabla \phi} \cdot \nabla \phi \right) \frac{\partial f}{\partial \phi} + \frac{\partial f}{\partial \nabla \phi} \nabla \eta, \\
\mathcal{A} &= (H_\eta \nabla \phi) \cdot \left( H_\phi \frac{\partial f}{\partial \phi} \right), \\
\mathcal{D} &= (H_\phi H_\eta) : \left( \frac{\partial f}{\partial \nabla \phi} H_\phi \right), \\
\mathcal{S} &= \frac{\partial s}{\partial \phi} \nabla \phi \cdot \frac{\partial \eta}{\partial \nabla \phi}.
\end{align}

with the contraction defined as $A : B = \text{Tr}(AB^T)$. Here $(H_\phi)_{mn} = \partial^2 \phi / \partial x_m \partial x_n$ the (symmetric) Hessian of $\phi$ and $(H_\eta)_{mn} = \partial^2 \eta / \partial \nabla_{m \phi} \partial \nabla_{n \phi}$ the (symmetric) Hessian of $\eta$.

**Proof.** (Lemma 3.1) Taking the gradient of the conservation law (16) delivers:

$$\partial_t (\nabla \phi) + \nabla (\nabla \cdot f) = \nabla s.$$

We proceed with following the steps in the previous section with now the entropy depending on the gradient of $\phi$: $\eta = \eta(\nabla \phi)$. Hence, we take the inner product of (20) with the variation entropy variables $\partial \eta / \partial \nabla \phi$.

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\[ ^4 \text{Also this can be understood as a change of variables.} \]
(vector-valued) and find:

\[ \partial_t \eta + \frac{\partial \eta}{\partial \nabla \phi} \cdot \nabla (\nabla \cdot f) = \frac{\partial s}{\partial \phi} \nabla \phi \cdot \frac{\partial \eta}{\partial \nabla \phi} . \tag{21} \]

The temporal term and source term are in their final form. Considering the second term of (21) in isolation we interchange the gradient and divergence operators and use the product rule:

\[
\frac{\partial \eta}{\partial \nabla \phi} \cdot (\nabla (\nabla \cdot f)) = \frac{\partial \eta}{\partial \nabla \phi} \cdot (\nabla \cdot (\nabla f)^T) = \nabla \cdot \left( \nabla f \frac{\partial \eta}{\partial \nabla \phi} \right) - \left( \nabla \left( \frac{\partial \eta}{\partial \nabla \phi} \right) \right)^T \cdot \nabla f. \tag{22} \]

Here we use the notation \( \nabla \cdot \mathbf{T} = \partial_{x_j} T_{ij} \) for the divergence of a tensor. The first term of (22) yields the flux term on the left-hand side of (18). Using the chain rule we get straightforwardly

\[
\nabla f \frac{\partial \eta}{\partial \nabla \phi} = \left( \frac{\partial \eta}{\partial \nabla \phi} \cdot \nabla \phi \right) \frac{\partial f}{\partial \phi} + \left( \frac{\partial f}{\partial \nabla \phi} H_{\phi} \right) \frac{\partial \eta}{\partial \nabla \phi} . \tag{23} \]

Next, by using the identity

\[
\left( \nabla \left( \frac{\partial \eta}{\partial \nabla \phi} \right) \right)^T = H_{\phi} \eta, \tag{24a} \]

the second term of (22) can be written as

\[
\left( \nabla \left( \frac{\partial \eta}{\partial \nabla \phi} \right) \right)^T : \nabla f = (H_{\eta} \nabla \phi) \cdot \left( H_{\phi} \frac{\partial f}{\partial \phi} \right) + (H_{\phi} H_{\eta}) : \left( \frac{\partial f}{\partial \nabla \phi} H_{\phi} \right). \tag{25} \]

Combining (21), (22), (23) and (25) leads to the claim.

The flux term on the left-hand side of (18), composed of a convective and a diffusive component, redistributes \( \eta \) over the domain. We postpone the discussion of the role of the remaining terms, which are closely linked to the variation entropy condition (given below), to the next subsection.

**Definition 3.2.** (Variation entropy condition) The variation entropy condition, i.e. the entropy condition of the variation entropy \( \eta = \eta(\nabla \phi) \) reads:

\[
\partial_t \eta + \nabla \cdot \mathbf{q} - \mathcal{D} \leq 0. \tag{26} \]

**Definition 3.3.** (Variation entropy) A function \( \eta = \eta(\nabla \phi) \) that satisfies the variation entropy condition (26) is termed a variation entropy function or variation entropy and the corresponding flux \( \mathbf{q} \) is called the variation entropy flux. The non-conservative scalars \( \mathcal{A}, \mathcal{D} \) are in this case referred to as variation entropy advection and variation entropy diffusion respectively. Lastly, \( \mathcal{F} \) is called the variation entropy source.

Note that definition 3.2 does not involve any dependence on the variation entropy advection and variation entropy source. The reason for that will be clarified in the following subsection.

The variable \( \eta \) plays the role of an entropy. At this moment it is not clear whether \( \eta \) is an actual variation entropy. In the next subsection we check, or under what conditions, the variation entropy condition is fulfilled.

**Remark 3.4.** By adding a constant to a variation entropy one again arrives at a variation entropy. For the sake of convenience we take \( \eta(0) = 0 \) for all variation entropies.
3.2. Analysis of variation entropies

The main objective of this subsection is to identify conditions for which the variation entropy condition is satisfied. We analyze the non-conservative terms $D$ and $A$ and the source $S$ separately. After that we close with merging them in the main theorem, Theorem 3.9. We start-off by showing that the dissipation term $D$ on the right-hand side of (18) destroys entropy.

**Lemma 3.5. (Negativity of $D$)** The term $D$ on the right-hand side of (18) contributes to dissipation of the variation, i.e. it takes negative values only.

*Proof.* We make use of the convexity of $\eta$ and the negative eigenvalues of the diffusivity matrix. The convexity of the function $\eta = \eta(\nabla \phi)$ implies that the Hessian $H_\eta$ is positive semidefinite:

$$y^T H_\eta y \geq 0 \quad \text{for all } y \in \mathbb{R}^d.$$

This implies

$$H_\phi H_\eta : H_\phi \geq 0,$$

and via negative eigenvalues of $\partial f / \partial \nabla \phi$ it directly leads to the claim:

$$D = \left( H_\phi H_\eta \right) : \left( \frac{\partial f}{\partial \nabla \phi} H_\phi \right) \leq 0. \quad (29)$$

Note that negative eigenvalues of the diffusion matrix are an essential requirement for well-posedness. Positive eigenvalues create variation entropy which leads to a blow-up of the solutions.

The advection term $A$ on the right-hand side of (18) might create entropy. When one seeks for variation entropy solutions the influence of this term is unwanted. The following lemma provides the conditions for which $A$ vanishes.

**Lemma 3.6. (Conditions for vanishing of $A$)** The variation entropy advection term $A = \left( H_\eta \nabla \phi \right) \cdot \left( H_\phi \frac{\partial f}{\partial \phi} \right)$ vanishes for a convex $\eta$ if the variation entropy is given by

$$\eta = \eta(\nabla \phi) = \hat{\eta}(r, \theta) = F(\theta) r, \quad (30)$$

where the radius $r$ and the angles $\theta$ are the spherical coordinates corresponding to the coordinates $\nabla \phi$. In the 2-dimensional case the scalar-valued function $F = F(\theta)$ should satisfy

$$F(\theta) + F''(\theta) \geq 0. \quad (31)$$

The restrictions on the function $F$ in the 3-dimensional case are more involved. We refer the interested reader to Appendix A for the details.

**Remark 3.7.** Note that $\hat{\eta} = r F$ is not differentiable in the origin. However the origin is an important part of the domain since this is where $\phi$ attains a (local) extremum. A possible way to bypass non-differentiability is to regularize $\eta$. In order to not distract the reader with technicalities required for regularization, we do not regularize in this section. We apply regularization in a particular case in subsection 4.4.

*Proof.* (Lemma 3.6) The advection term $A$ vanishes if and only if the vectors $H_\eta \nabla \phi$ and $H_\phi \partial f / \partial \phi$ are orthogonal. Since the sign of each of the components of the advective speed $\partial f / \partial \phi$ is undetermined, the
entries of $H_\phi \delta f / \delta \phi$ can be positive or negative. Thus the only vector independent of $\partial f / \partial \phi$ orthogonal to $H_\phi \delta f / \delta \phi$ is the zero vector. Hence the problem translates into:

Find $\eta = \eta(v)$ such that:

- $H_\eta v = 0$,
- $\eta$ is convex.

First we focus on the differential equation. Observe that the system of equation can be cast into:

$$ \frac{\partial}{\partial v} \left( v \frac{\partial \eta}{\partial v} \right) = \frac{\partial \eta}{\partial v}. \quad (32) $$

Integration of the $i$-th equation with respect to $v_i$ provides

$$ v \cdot \frac{\partial \eta}{\partial v} = \eta + c, \quad (33) $$

where the $i$-th component of $c$ depends on all components of $v$ except for the $i$-th one. The left-hand side is identical for all equations, thus we find that $c_i = c$ is a constant independent of $v$. For the sake of simplicity we take $c = 0$:

$$ v \cdot \frac{\partial \eta}{\partial v} = \eta. \quad (34) $$

We now switch to spherical coordinates, i.e. the coordinates consist of a radial coordinate $r$ and $d - 1$ angular coordinates $\theta \in [0, 2\pi)$, $\varphi_m \in [0, \pi]$, $m = 1, \ldots, d - 2$. The transformation is given by:

$$ v_1 = r \cos \theta \prod_{l=1}^{d-2} \sin \varphi_l, $$

$$ v_2 = r \sin \theta \prod_{l=1}^{d-2} \sin \varphi_l, $$

$$ v_m = r \cos \varphi_{m-2} \prod_{l=m-1}^{d-2} \sin \varphi_l \quad \text{for } m = 3, \ldots, d - 1, $$

$$ v_d = r \cos \varphi_{d-2}. \quad (35) $$

For $d = 3$ the third line drops out, and for $d = 2$ both the third and last line vanish. Both cases reduce to the well-known transformations. Consult [43] for a derivation of a similar form. The direct consequence $r \partial v / \partial r = v$ provides

$$ r \frac{\partial \eta}{\partial r} = r \frac{\partial \eta}{\partial v} \cdot \frac{\partial v}{\partial r} = v \cdot \frac{\partial \eta}{\partial v}. \quad (36) $$

This allows us to cast (33) into the differential equation

$$ r \frac{\partial \eta}{\partial r} = \eta. \quad (37) $$

The corresponding solution follows straightforwardly

$$ \tilde{\eta} = \tilde{\eta}(r, \varphi_1, \ldots, \varphi_{d-2}, \theta) = \tilde{\eta}(r, \theta) = F(\theta)r, \quad (38) $$

with $F$ a scalar-valued function.
We characterize the convexity of $\eta$ by the positivity of the eigenvalues of the Hessian. We restrict ourselves here to the 2-dimensional case in which $F = F(\theta)$. The Hessian in polar coordinates takes the form:

$$H_\eta = \frac{F(\theta) + F''(\theta)}{r} \begin{pmatrix} \sin^2 \theta & -\cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos^2 \theta \end{pmatrix}. \quad (39)$$

Note that this is in line with the first demand. The eigenvalues $\iota_1, \iota_2$ of $H_\eta$ are

$$\iota_1 = 0, \quad (40)$$

$$\iota_2 = \frac{F(\theta) + F''(\theta)}{r}, \quad (41)$$

and convexity of $\eta$ follows when $F(\theta) + F''(\theta) \geq 0$.

We focus on the source term $\mathcal{S}$ before moving to the main theorem.

**Lemma 3.8.** (Condition for vanishing of $\mathcal{S}$) The variation entropy source term $\mathcal{S} = \partial s/\partial \phi \nabla \phi \cdot \partial \eta/\partial \nabla \phi$ vanishes for a convex $\eta$ if and only if the variation entropy is of the form (in polar coordinates) $\hat{\eta} = F(\theta)$.

**Proof.** The derivative of the source term, $\partial s/\partial \phi$, can take both positive and negative values. Thus the variation entropy source $\mathcal{S}$ vanishes for a general $s = s(\phi)$ if and only if

$$\nabla \phi \cdot \frac{\partial \eta}{\partial \nabla \phi} = 0. \quad (42)$$

Switching to spherical coordinates and following the notation in the proof of 3.6 yields:

$$\frac{\partial \hat{\eta}}{\partial r} = 0. \quad (43)$$

The only solution is $\hat{\eta} = F(\theta)$.

Thus only when the variation entropy is independent of $r$, i.e. $\hat{\eta} = F(\theta)$, the source term contribution in the entropy evolution vanishes. This does not comply with the form in Lemma 3.6. We do not proceed with this form. Subsequently, in the non-trivial case the source term will have a contribution to the evolution of the variation entropy.

We are now ready to present the main theorem.

**Theorem 3.9.** (Characterization of a variation entropy) Let the source be zero. The function $\eta$ is a variation entropy if and only if it is given by

$$\eta = \eta(\nabla \phi) = \hat{\eta}(r, \theta) = F(\theta)r, \quad (44)$$

where $r$ and $\theta$ are the polar coordinates corresponding to the coordinates $\nabla \phi$. In the 2-dimensional case the scalar-valued function $F = F(\theta)$ should satisfy

$$F(\theta) + F''(\theta) \geq 0. \quad (45)$$

**Proof.** The non-conservative advective contribution $\mathcal{A}$ vanishes if and only if $\eta = \eta(\nabla \phi) = \hat{\eta}(r, \theta)$ is of the form given in Theorem 3.6. The non-conservative diffusive term $\mathcal{D}$ takes negative values only (Proposition 3.5). Thus we are left with

$$\partial_t \eta + \nabla \cdot \mathbf{q} - \mathcal{D} = 0. \quad (46)$$
We apply the vanishing viscosity technique. To that purpose we first evaluate the expression:

$$\Delta \eta = \nabla \cdot (\nabla \eta) = \nabla \cdot \left( \frac{\partial \eta}{\partial \nabla \phi} H_\phi \right) = (H_\phi H_\eta) : H_\phi + \frac{\partial \eta}{\partial \nabla \phi} \cdot \Delta(\nabla \phi).$$

(47)

The first term\(^7\) is positive, see the proof of Lemma 3.5, thus we conclude

$$\frac{\partial \eta}{\partial \nabla \phi} \cdot \Delta(\nabla \phi) \leq \Delta \eta.$$  (48)

Consider vanishing viscosity limit solutions \(\nabla \phi^\epsilon : \Omega \times I \to \mathbb{R}\) satisfying the strong problem:

$$\partial_t (\nabla \phi^\epsilon) + \nabla \nabla \cdot f^\epsilon = \epsilon \Delta (\nabla \phi^\epsilon),$$  (49)

where the superscript \(\epsilon\) indicates dependence on \(\nabla \phi^\epsilon\). Taking the inner product with \(\partial \eta / \partial \nabla \phi^\epsilon\) provides

$$\partial_t \eta + \nabla \cdot q^\epsilon - \mathcal{D}^\epsilon = \epsilon \frac{\partial \eta}{\partial \nabla \phi^\epsilon} \cdot \Delta(\nabla \phi^\epsilon).$$  (50)

Applying (48) we arrive at:

$$\partial_t \eta + \nabla \cdot q^\epsilon - \mathcal{D}^\epsilon \leq \epsilon \Delta \eta.$$  (51)

In the limit of vanishing viscosity \(\epsilon\) we get the variation entropy condition:

$$\partial_t \eta + \nabla \cdot q - \mathcal{D} \leq 0.$$  (52)

**Remark 3.10.** The viscosity term \(\epsilon \Delta(\nabla \phi^\epsilon)\) in (49) appears in the evolution of the variation entropy as:

$$\partial_t \eta + \nabla \cdot q^\epsilon - (H_\phi^\epsilon H_\eta^\epsilon) \left( \left( \frac{\partial f}{\partial \nabla \phi^\epsilon} + \epsilon I \right) H_\phi^\epsilon \right) = \epsilon \Delta \eta.$$  (53)

Hence, the viscosity term indeed acts as a diffusion contribution by effectively adding \(\epsilon I\) to the diffusion matrix of the conservation law.

**Lemma 3.11.** (Form of the variation entropy source) The source contribution \(\mathcal{I}\) takes the form \(\mathcal{I} = \eta \partial s / \partial \phi\) for \(\eta = \hat{\eta}(r, \theta) = r F(\theta)\), where \(r\) and \(\theta\) are the spherical coordinates corresponding to the coordinates \(\nabla \phi\).

**Proof.** This directly follows from the proof of Lemma 3.6. \(\square\)

**Corollary 3.12.** (Evolution equation of the variation entropy) The evolution equation of a variation entropy \(\eta = \eta(\nabla \phi)\) reads:

$$\partial_t \eta + \nabla \cdot q = \mathcal{D} + \mathcal{I},$$  (54)

\(^7\)Note that its classical entropy counterpart is \(\nabla \phi \cdot \nabla \phi \partial^2 \eta / \partial \phi^2\).
where the flux \( q \), the non-conservative term \( \mathcal{D} \) and the source term \( \mathcal{S} \) are respectively given by:

\[
q = \frac{\partial f}{\partial \phi} \eta + \frac{\partial f}{\partial \nabla \phi} \nabla \eta, \\
\mathcal{D} = (H_\phi H_\eta) : \left( \frac{\partial f}{\partial \nabla \phi} H_\phi \right), \\
\mathcal{S} = \frac{\partial s}{\partial \phi} \eta.
\]

We emphasize that this form closely resembles a conservation law with convection, diffusion and reaction components.

**Corollary 3.13.** (Decay of variation entropy) Let \( \Omega \) be a domain without boundaries. Assume that the matrix \( \partial f/\partial \nabla \phi \) has only negative eigenvalues and that the source is zero. The variation entropy decays in time:

\[
\int_{\Omega} \eta(\nabla \phi(x, t)) \, d\Omega \leq \int_{\Omega} \eta(\nabla \phi_0(x)) \, d\Omega, \quad \text{for all } t > 0.
\]

**Proof.** We start from the variation entropy condition:

\[
\partial_t \eta + \nabla \cdot q \leq \mathcal{D} \leq 0.
\]

Integration of (57) yields, due to the fact that there are no contributions on the boundary:

\[
\frac{d}{dt} \int_{\Omega} \eta(\nabla \phi(x, t)) \, d\Omega \leq 0,
\]

for all \( t > 0 \).

**Remark 3.14.** The reaction term is the only term that can create variation entropy.

**4. A closer look into variation entropies**

Here we take a closer look into variation entropies. We provide a characterization of variation entropies, then present some simple examples of variation entropies, discuss objective variation entropies and explicitly discuss the standard 2-norm variation entropy. We close this section with some comments on the TVD property, the maximum principle and the monotonicity property.

**4.1. Characterization of variation entropies**

Theorem 3.9 says that variation entropies are of the form \( \tilde{\eta} = rF(\theta) \) satisfying a convexity condition. This form is required for the variation entropy advection \( \mathcal{A} \) to vanish. This term in its turn vanishes when a homogeneity property is fulfilled, when is stated in the following lemma.

**Lemma 4.1.** (Homogeneous function) A variation entropy is a positive homogeneous function of degree 1.

**Proof.** In the proof of Lemma 3.6 we see that the variation entropy is of the form \( \tilde{\eta} = rF(\theta) \) if the homogeneity property is fulfilled:

\[
\mathbf{v} \cdot \frac{\partial \eta}{\partial \mathbf{v}} = \eta.
\]

By using Euler’s homogeneous function theorem we see that (34) is equivalent to the positive homogeneous property of degree 1:

\[
\eta(\alpha \mathbf{v}) = \alpha \eta(\mathbf{v}), \quad \text{for all } \alpha \geq 0.
\]
Thus the claim follows.

Lemma 4.2. (Sub-additivity) The convexity demand of $\eta$ may be slightly simplified by noting that for positive homogeneous function of degree 1 convexity is equivalent to sub-additivity, i.e

$$\eta(v_1 + v_2) \leq \eta(v_1) + \eta(v_2), \quad \text{for all } v_1, v_2 \in \mathbb{R}^d.$$  \hspace{1cm} (61)

Theorem 4.3. (Semi-norm) All semi-norms are variation entropies.

Proof. The axioms of a seminorm are absolutely homogeneous and sub-additive. The absolute homogeneous demand is a specific case of the homogeneous property.

Needless to say, all norms are also variation entropies.

We remark that a linear combinations of variation entropies form again a variation entropy.

Corollary 4.4. (Linear combination) Let $\eta_k$ be variation entropies given by $\eta_k = r F_k(\theta)$, for $k = 1, \ldots, n$ for some integer $n$. The linear combination $\eta := \sum_k \alpha_k \eta_k$ with $\alpha_k \in \mathbb{R}^+$ is a variation entropy.

Proof. This is a direct consequence of Theorem 3.9.

Remark 4.5. One can relax the choice of $\alpha_k$ to $\alpha_k \in \mathbb{R}$ as long as $\sum_k \alpha_k (F_k'' + F_k) \geq 0$.

4.2. Some examples of variation entropies

In the following we discuss some options for the variation entropy.

4.2.1. The standard 2-norm

The simplest choice of a diminishing variation entropy is that of a constant $F$. This leads to the well-known variation measured in the 2-norm. We postpone the corresponding discussion to the next subsection.

4.2.2. A linear variation entropy

The second obvious choice is to take functions $F$ that fulfill the convexity condition with equality. In two dimensions these functions are a linear combination of $\cos \theta$ and $\sin \theta$. We present this in the following proposition.

Proposition 4.6. (Linear variation entropy) The function $\eta = \eta(\nabla \phi) = a \cdot \nabla \phi$, with $a \in \mathbb{R}^d$ is a variation entropy.

Proof. Trivially $\eta = \eta(\nabla \phi) = a \cdot \nabla \phi$ is a semi-norm and thus a variation entropy. For the sake for simplicity consider $d = 2$. Indeed we can write $\eta$ in the form $r F(\theta)$:

$$a \cdot \nabla \phi = \|\nabla \phi\|_2 \left( a_1 \frac{\partial \phi}{\|\nabla \phi\|_2} + a_2 \frac{\partial \phi}{\|\nabla \phi\|_2} \right)$$

$$= r \left( a_1 \cos \theta + a_2 \sin \theta \right)$$

$$= r F(\theta),$$  \hspace{1cm} (62)

with $a^T = (a_1, a_2)$.

In this example there is no space for the variation entropy to decrease. Employing this in a numerical computation might lead to issues.

4.2.3. The standard 1-norm

The previous example naturally leads to the 1-norm variation entropy, i.e. $\eta = \|\nabla \phi\|_1$. In a similar fashion as in Proposition 4.6 we find in 2-dimensions $F + F'' = 0$. Thus also for the 1-norm there is also no space for the variation entropy to decrease. We remark that the 1-norm is not differentiable along the axis.
4.2.4. A quadratic form

We now turn our focus to a variant of the 2-norm.

**Proposition 4.7.** (Variation entropy based on a quadratic form) The function $\eta = \eta(\nabla \phi) = \|\nabla \phi\|_A$ defined as $\|\nabla \phi\|_A := \nabla \phi^T A \nabla \phi$, with $A \in \mathbb{R}^{d \times d}$ a symmetric positive definite matrix, is a variation entropy.

**Proof.** One can easily verify that $\|\nabla \phi\|_A$ is a norm. For $d = 2$ we additionally present a proof is similar as the proof of Proposition 4.6. Let the components of $A$ be given by

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$  

(63)

A direct calculation provides

$$\nabla \phi^T A \nabla \phi = a \partial_x \phi^2 + 2b \partial_x \phi \partial_y \phi + c \partial_y \phi^2.$$  

(64)

Next, we can trivially write

$$\|\nabla \phi\|_A = \|\nabla \phi\|_2 \left( a \|\nabla \phi\|_2^2 + 2b \partial_x \phi \partial_y \phi + c \|\nabla \phi\|_2^2 \right)^{1/2}.$$  

(65)

Switching to polar coordinates with radial distance $r$ and angle $\theta$, the right-hand side of (65) takes the form

$$r F(\theta) = r \left( a \cos^2 \theta + 2b \cos \theta \sin \theta + c \sin^2 \theta \right)^{1/2}.$$  

(66)

We proceed with the convexity condition. A straightforward evaluation provides:

$$F(\theta) + F''(\theta) = \frac{\det A}{F(\theta)^3},$$  

(67)

where $\det A$ is the determinant of $A$. From (67) we see that the convexity condition

$$F(\theta) + F''(\theta) \geq 0$$  

(68)

is fulfilled for a positive definite the matrix $A$. Applying Theorem 3.9 completes the proof.

**Remark 4.8.** The symmetric positive definite matrix $A \in \mathbb{R}^{d \times d}$ may depend on $\theta$ (the spherical coordinate angles).

We explicitly state the evolution equation of $\eta = \|\nabla \phi\|_A$:

$$\partial_t \eta + \nabla \cdot \mathbf{q} = \mathcal{D} + \mathcal{J},$$  

(69)

where the flux $\mathbf{q}$, the non-conservative diffusive term $\mathcal{D}$ and the source term $\mathcal{J}$ are respectively given by:

$$\mathbf{q} = \eta \frac{\partial f}{\partial \phi} + \left( \frac{\partial f}{\partial \nabla \phi} H_\phi \right) \nabla \phi^T \mathbf{A},$$  

(70a)

$$\mathcal{D} = H_\phi \left( \frac{A}{\eta} - \frac{1}{\eta^2} \mathbf{A} \nabla \phi : \mathbf{A} \nabla \phi \right) \cdot \left( \frac{\partial f}{\partial \nabla \phi} H_\phi \right),$$  

(70b)

$$\mathcal{J} = \frac{\partial s}{\partial \phi} \eta.$$  

(70c)

\footnote{Note that these requirements are necessary in order for $\|\nabla \phi\|_A$ to be well-defined.}
It follows from (3.12) by noting that the derivatives take the form

\[
\frac{\partial \eta}{\partial \nabla \phi} = \frac{1}{\eta} A \nabla \phi,
\]

\[
(H_\eta) = \frac{1}{\eta} A - \frac{1}{\eta^3} A \nabla \phi (A \nabla \phi)^T.
\]

Remark that the Hessian is indeed positive definite and that \(A = 0\) since

\[
(H_\eta) \nabla \phi = \frac{1}{\eta} A \nabla \phi - \frac{1}{\eta^3} A \nabla \phi (\nabla \phi)^T A \nabla \phi = 0.
\]

Remark 4.9. By corollary 4.4, the superposition of \(\eta = a \cdot \nabla \phi\) and \(\eta = \| \nabla \phi \|_A\) is also a variation entropy.

4.2.5. \(p\)-norm

The function given by \(\eta(v) = \|v\|_p\) is a variation entropy. Indeed, it can be written in the form \(rF(\theta)\). To see this, note the identity:

\[
\|v\|_p = \|v\|_2 \left( \sum_{k=1}^d \left( \frac{|v_k|}{\|v\|_2} \right)^p \right)^{1/p}.
\]

Next, we switch to spherical coordinates via the transformation given in (35). Each of the terms in the sum can be written as a function of the angular coordinates:

\[
\left( \frac{|v_k|}{\|v\|_2} \right)^p = F_k(\theta).
\]

Thus the \(p\)-norm is of the form \(\hat{\eta} = rF(\theta)\). Also note that the matrix-vector product \(H_\eta v\) vanishes.

4.3. Objectivity

A reasonable demand on the continuous level is to ask for objectivity (frame-invariance) of the variation entropy. Here we focus on rotation invariance, i.e.

\[
\eta(R \nabla \phi) = \eta(\nabla \phi)
\]

for all rotation matrices \(R\), i.e. \(R^T = R^{-1}\). We start off with a theorem.

**Theorem 4.10.** (Objectivity) The only objective variation entropy is the total variation measured in the 2-norm, \(\eta = \| \nabla \phi \|_2\) (up to multiplication by a constant).

**Proof.** Let the angle of the rotation matrix \(R\) be \(\varphi\) and let \(\nabla \phi = (r \cos \theta, r \sin \theta)\). Then a direct computation results in

\[
\eta(R \nabla \phi) = rF(\theta + \varphi) = \hat{\eta}(r, \theta + \varphi).
\]

Demanding \(\eta(R \nabla \phi) = \eta(\nabla \phi)\) provides that \(F(\theta + \varphi) = F(\theta)\), i.e. \(F\) is a constant.

Thus the only objective \(p\)-norm is the norm with \(p = 2\). In particular, we want to emphasize that the 1-norm is not objective. This makes it unsuitable for usage in non-Cartesian grid computations.
The variation entropy \( \eta = a \cdot \nabla \phi \) is only objective in the trivial case of \( a = 0 \). However, it can be made objective by rotating the vector \( a \), i.e. by taking \( \tilde{a} = R a \) for a rotated \( \nabla \phi \). Consider for example the case where \( a \) represents a convection velocity. A rotation of \( \nabla \phi \) naturally implies the same rotation of the convective velocity.

Consider now the variation entropy \( \eta = \| \nabla \phi \|_A \). This variation entropy is only objective for \( A = I \). Similarly as in the previous example, rotation of \( \nabla \phi \) via pre-multiplication by the rotation matrix \( R \) implies that \( A \) should be replaced by \( \tilde{A} = R A R^T \) to make this variation entropy objective. Thus clearly the total variation based on the 2-norm, \( \| \nabla \phi \|_2 \), is objective.

We end this subsection with an overview of the variation entropy results presented in Figure 2.

---

**Figure 2:** Overview of objective variation entropy results. The variation entropy \( \| \nabla \phi \|_p \) is not objective unless \( p = 2 \). We classify \( \eta = \| \nabla \phi \|_A \) and \( \eta = a \cdot \nabla \phi \) as objective since these can be made objective by changing \( a \) and \( A \).

---

### 4.4. Variation measured in the regularized 2-norm

Due to its great importance, we discuss the variation measured in the 2-norm here separately. In particular we consider a regularized version in order to allow evaluation everywhere. Thus we consider the case where the variation entropy function is the regularized absolute value operator \( \| \cdot \|_{\varepsilon,2} : \mathbb{R}^d \to \mathbb{R}_+ \) which is defined for \( b \in \mathbb{R}^d, \varepsilon > 0 \) as:

\[
\|b\|_{\varepsilon,2}^2 := b \cdot b + \varepsilon^2. 
\]  

(77)

Notice that

\[
\|b\|_2 \leq \|b\|_{\varepsilon,2} \leq \|b\|_2 + \varepsilon,
\]

(78)

see also Figure 3.
Figure 3: Plot of the regularized norm in (a) 1-dimension and (b) 2-dimensions. Here $\varepsilon = 0.25$.

The regularized absolute value has the derivatives

$$
\partial_b \|b\|_{\varepsilon,2} = \frac{b}{\|b\|_{\varepsilon,2}} \quad (79a)
$$

$$
\partial_b^2 \|b\|_{\varepsilon,2} = \left( I - \frac{bb^T}{\|b\|_{\varepsilon,2}^2} \right) \frac{1}{\|b\|_{\varepsilon,2}}, \quad (79b)
$$

which exist everywhere. The homogeneity constraint is violated by a term that scales with $\varepsilon^2$:

$$
b \cdot \partial_b \|b\|_{\varepsilon,2} - \|b\|_{\varepsilon,2} = \frac{\varepsilon^2}{\|b\|_{\varepsilon,2}}. \quad (80)
$$

Also the term that appears in $\mathcal{A}$ scales with $\varepsilon^2$:

$$
\partial_b^2 \|b\|_{\varepsilon,2} b = \varepsilon^2 \frac{b}{\|b\|_{\varepsilon,2}^2}. \quad (81)
$$

**Corollary 4.11.** (Evolution equation of a regularized variation entropy) The regularized variation $\eta = \eta_\varepsilon = \|\nabla \phi\|_{\varepsilon,2}$ satisfies the evolution equation:

$$
\partial_t \eta_\varepsilon + \nabla \cdot q_\varepsilon = \mathcal{D}_\varepsilon + \mathcal{F}_\varepsilon + \mathcal{R}_\varepsilon, \quad (82)
$$

where the flux $q_\varepsilon$, the non-conservative terms $\mathcal{A}_\varepsilon$ and $\mathcal{D}_\varepsilon$ and the source term $\mathcal{F}_\varepsilon$ are respectively defined as:

$$
q_\varepsilon = \frac{\partial f}{\partial \phi} \eta_\varepsilon + \frac{\partial f}{\partial \nabla \phi} \nabla \eta_\varepsilon, \quad (83a)
$$

$$
\mathcal{D}_\varepsilon = \frac{1}{\eta_\varepsilon^2} \left( \|\nabla \phi\|_2^2 I - \nabla \phi \nabla \phi^T \right) \mathbf{H}_\phi : \left( \frac{\partial f}{\partial \nabla \phi} H_\phi \right), \quad (83b)
$$

$$
\mathcal{F}_\varepsilon = \frac{\partial s}{\partial \phi} \eta_\varepsilon, \quad (83c)
$$

$$
\mathcal{R}_\varepsilon = \frac{\varepsilon^2}{\eta_\varepsilon} \left( \nabla \cdot \left( \frac{\partial f}{\partial \phi} \right) + \frac{1}{\eta_\varepsilon^2} H_\phi : \left( \frac{\partial f}{\partial \nabla \phi} H_\phi \right) - \frac{\partial s}{\partial \phi} \right). \quad (83d)
$$
Proof. A direct substitution of $\eta = \eta_\varepsilon$ into (18)-(19) using (78)-(81) yields

$$\partial_t \eta_\varepsilon + \nabla \cdot \mathbf{q} = \mathcal{A} + \mathcal{D} + \mathcal{S},$$

where the flux $\mathbf{q}$, the non-conservative terms $\mathcal{A}$ and $\mathcal{D}$ and the source term $\mathcal{S}$ are respectively given by:

$$\mathbf{q} = \frac{\partial f}{\partial \phi} \left( \eta_\varepsilon - \frac{\varepsilon^2}{\eta_\varepsilon} \right) + \frac{\partial f}{\partial \nabla \phi} \nabla \eta,$$  \hspace{1cm} (85a)

$$\mathcal{A} = \frac{\varepsilon^2}{\eta_\varepsilon^2} \eta_\varepsilon \nabla \phi \cdot \mathbf{H}_\phi : \left( \frac{\partial f}{\partial \phi} \right),$$  \hspace{1cm} (85b)

$$\mathcal{D} = \frac{1}{\eta_\varepsilon} \left( I - \nabla \phi \nabla \phi^T \right) \mathbf{H}_\phi : \left( \frac{\partial f}{\partial \phi} \right),$$  \hspace{1cm} (85c)

$$\mathcal{S} = \frac{\partial s}{\partial \phi} \left( \eta_\varepsilon - \frac{\varepsilon^2}{\eta_\varepsilon} \right).$$  \hspace{1cm} (85d)

The divergence of the flux writes as

$$\nabla \cdot \mathbf{q} = \nabla \cdot \left( \frac{\partial f}{\partial \phi} \left( \eta_\varepsilon - \frac{\varepsilon^2}{\eta_\varepsilon} \right) + \frac{\partial f}{\partial \nabla \phi} \nabla \eta \right),$$

$$= \nabla \cdot \left( \frac{\partial f}{\partial \phi} \eta_\varepsilon + \frac{\partial f}{\partial \nabla \phi} \nabla \eta \right) + \frac{\varepsilon^2}{\eta_\varepsilon^2} \nabla \phi \cdot \left( \mathbf{H}_\phi \frac{\partial f}{\partial \phi} \right) - \frac{\varepsilon^2}{\eta_\varepsilon} \nabla \cdot \left( \frac{\partial f}{\partial \phi} \right).$$  \hspace{1cm} (86)

The diffusion term $\mathcal{D}$ can be written as

$$\mathcal{D} = \left( \frac{1}{\eta_\varepsilon} \left( I - \nabla \phi \nabla \phi^T \right) \mathbf{H}_\phi \right) : \left( \frac{\partial f}{\partial \phi} \right),$$

$$= \frac{1}{\eta_\varepsilon^2} \left( I - \nabla \phi \nabla \phi^T \right) \mathbf{H}_\phi : \left( \frac{\partial f}{\partial \phi} \right) + \frac{\varepsilon^2}{\eta_\varepsilon^2} \mathbf{H}_\phi : \left( \frac{\partial f}{\partial \phi} \right)$$

$$= \mathcal{D}_\varepsilon + \frac{\varepsilon^2}{\eta_\varepsilon^2} \mathbf{H}_\phi : \left( \frac{\partial f}{\partial \phi} \right).$$  \hspace{1cm} (87)

Substitution of (86)-(87) into (84)-(85) proves the claim. \hfill \Box

Consider the one dimensional case of (82)-(83), i.e.

$$\frac{\partial}{\partial t} \| \partial_x \phi \|_{\varepsilon,2} + \partial_x \left( \frac{\partial f}{\partial \phi} \right) \| \partial_x \phi \|_{\varepsilon,2} + \frac{\partial f}{\partial (\partial_x \phi)} \partial_x \left( \| \partial_x \phi \|_{\varepsilon,2} \right) = \frac{\partial s}{\partial \phi} \| \partial_x \phi \|_{\varepsilon,2} + \mathcal{R}_\varepsilon.$$  \hspace{1cm} (88)

We focus on the last term on the right-hand side. It can be written as

$$\mathcal{R}_\varepsilon = \left( \frac{\partial f}{\partial \phi} - \frac{\partial s}{\partial \phi} \right) g_1^\varepsilon (\partial_x \phi) + \left( (\partial_x \phi)^2 \frac{\partial f}{\partial \partial_x \phi} \right) g_2^\varepsilon (\partial_x \phi),$$  \hspace{1cm} (89)

where the functions $g_1^\varepsilon$ and $g_2^\varepsilon$ are defined as

$$g_1^\varepsilon (\partial_x \phi) = \frac{\varepsilon^2}{\| \partial_x \phi \|_{\varepsilon,2}},$$  \hspace{1cm} (90a)

$$g_2^\varepsilon (\partial_x \phi) = \frac{\varepsilon^2}{\| \partial_x \phi \|_{\varepsilon,2}^3}.$$  \hspace{1cm} (90b)
In Figure 4 we plot \( g_1^\varepsilon \) and \( g_2^\varepsilon \) for several values of \( \varepsilon \).

Figure 4: Plot of the functions \( g_1^\varepsilon \) (a) and \( g_2^\varepsilon \) (b). On the horizontal axis \( \partial_x \phi \) varies.

The function \( g_1^\varepsilon \) vanishes in the limit \( \varepsilon \downarrow 0 \). It value at the origin is \( g_1^\varepsilon(0) = \varepsilon \). On the other hand, \( g_2^\varepsilon \) behaves as a (scaled) delta distribution centered at the origin. The value at the origin is \( g_2^\varepsilon(0) = \varepsilon^{-1} \) and the area under the profile \( g_2^\varepsilon \) is 2 (which is independent of \( \varepsilon \)). Thus the regularization focuses the diffusion contribution at points where \( \partial_x \phi \) approaches zero, i.e. the extrema of \( \phi \). For \( \varepsilon \downarrow 0 \) we conclude that in one dimension variation entropy can either be produced by diffusion at local extrema or by the source term.

In the multi-dimensional case \( (d > 1) \) the variation entropy diffusion \( \mathcal{D} \) does not vanish. This is a clear separation of the 1-dimensional case and the multi-dimensional case. We explicitly state the temporal evolution of the non-regularized variation. The limit of \( \varepsilon \downarrow 0 \) in (82)-(83) yields:

\[
\frac{\partial}{\partial t} \| \nabla \phi \|_2^2 + \nabla \cdot \mathbf{q} = \mathcal{D} + \mathcal{S},
\]

where the flux \( \mathbf{q} \), the non-conservative terms \( \mathcal{A} \) and \( \mathcal{D} \) and the source term \( \mathcal{S} \) are respectively defined as:

\[
\mathbf{q} = \frac{\partial f}{\partial \phi} \| \nabla \phi \|_2 + \frac{\partial f}{\partial \nabla \phi} \nabla \| \nabla \phi \|_2,
\]

\[
\mathcal{D} = \frac{1}{\| \nabla \phi \|_2} \left( \left( I - \frac{\nabla \phi \nabla \phi^T}{\| \nabla \phi \|_2^2} \right) \mathbf{H}_\phi \right) : \left( \frac{\partial f}{\partial \nabla \phi} \mathbf{H}_\phi \right),
\]

\[
\mathcal{S} = \frac{\partial s}{\partial \phi} \| \nabla \phi \|_2,
\]

which is not defined for \( \nabla \phi = 0 \).

**Remark 4.12.** The well-known total variation diminishing (TVD) constraint:

\[
\frac{d}{dt} \int_\Omega \| \nabla \phi(x, t) \|_2 \, d\Omega \leq 0
\]

or

\[
\int_\Omega \| \nabla \phi(x, t) \|_2 \, d\Omega \leq \int_\Omega \| \nabla \phi_0(x) \|_2 \, d\Omega,
\]

is a variation entropy \( L^2 \)-stability result.
4.5. A note on the TVD property, the maximum principle and the monotonicity property

We conclude this section with a note on the link between TVD, the maximum principle and the monotonicity property. It is well-known that in one-dimension these are all very similar. For example consider the one-dimensional case in Figure 5.

• The total variation is

\[
TV(\phi) = (\phi_A - 0) + (\phi_B - \phi_A) + (\phi_C - \phi_B) + (\phi_C - \phi_D) + (0 - \phi_D)
\]

\[
= 2 \sum_k (\phi_{k}^{\text{max}} - \phi_{k}^{\text{min}}),
\]

(95)

where \(\phi_{k}^{\text{max}}\) and \(\phi_{k}^{\text{min}}\) refer to local maxima and minima respectively. This obviously decreases if the three jumps shrink, but also decreases, for instance, when \(B\) increases as long as \(C\) shrinks enough.

• The monotonicity property states that (i) no new local extrema may be created and (ii) that the local extrema \(B\) and \(D\) may not decrease and \(A\) and \(C\) may not increase.

• The governing conservation law satisfies the maximum principle if the profile does not exceed the local extrema \(C\) and \(D\).

In a two-dimensional case the link is less clear. An increase of the total variation in one direction can be compensated with a larger decrease in the perpendicular direction. Additionally, the length of a jump is also relevant. For instance, a jump of double height and half length yields the same total variation. This is also the case when the grid-based definition (2b) is used. Consider a pipe flow with variable cross section governed by the Euler equations of gas dynamics. A decrease of the pipe diameter leads to an increase in velocity. The total variation of the velocity can diminish. Despite the standard maximum principle and the monotonicity property do not hold.

5. Conclusions

In this paper we have developed the new concept variation entropy for nonlinear conservation laws. This studies the use of entropy variables for the system that results when taking the gradient of the conservation law. It appears that all semi-norms are suitable entropy variables for this system. A particular choice of a
semi-norm leads to the evolution of the variation measured in the 2-norm. This establishes a link between on the one hand entropy solutions and on the other the total variation diminishing property. All the evolution equations are (also) presented in a local continuous form. The framework allows for the design of new variation entropies and thus new local variation diminishing properties in the continuous form.

The variation entropy framework considers the gradient of the solution instead of the solution itself. This is a more natural and suitable approach when dealing with shock waves, which are characterized by their large gradients. We hope that this sheds some light on both entropy solutions and TVD property.

This paper opens several doors for future research, of which we mention a few.

- A relevant topic that deserves more attention is the link between TVD, the maximum principle and the monotonicity property in multiple dimensions. We feel that a firm connection (on the continuous level) is still missing.

- An attractive path is the construction of discontinuities capturing mechanisms in finite element methods based on the variation entropy. This would narrow the gap between on the one hand entropy and TVD solutions and on the other discontinuities capturing mechanisms. We have work on this in progress and aim to report on it in the near future.

- In view of the continuous interest in the development of TVD schemes over the past decades, we believe that these studies are very valuable and hope that the framework presented in this paper appears to be useful.

Appendix A. The 3-dimensional version of the convexity condition in spherical coordinates

Lemma Appendix A.1. (Convexity condition in spherical coordinates) Let the dimension \( d = 3 \). Assume a zero source term, i.e. \( s = 0 \). The variation entropy \( \eta = \eta(\nabla \phi) \) diminishes in time, i.e.

\[
\int_\Omega \eta(\nabla \phi(x,t)) \, d\Omega \leq \int_\Omega \eta(\nabla \phi_0(x)) \, d\Omega, \tag{A.1}
\]

for all \( t > 0 \), if and only if the variation entropy is given by

\[
\eta = \eta(\nabla \phi) = \eta(r, \varphi) = F(\theta, \varphi)r, \tag{A.2}
\]

where \( r \) and the \( \varphi \) are the polar coordinates corresponding to \( \nabla \phi \). The scalar-valued function \( F = F(\varphi, \theta) \) satisfies

\[
A \geq B \geq 0, \tag{A.3}
\]

with

\[
A = 4 \left( 2F + \frac{\partial F}{\partial \varphi} \cot \varphi + \frac{\partial^2 F}{\partial \varphi^2} \csc^2 \varphi \right), \\
B = \sqrt{2} \csc^2 \varphi \left( \frac{\partial F}{\partial \varphi} \right)^2 (1 - \cos 4\varphi) + 32 \left( \frac{\partial F}{\partial \theta} \right)^2 + 8 \left( \frac{\partial^2 F}{\partial \theta^2} \right)^2 + 32 \left( \frac{\partial^2 F}{\partial \varphi \partial \theta} \right)^2 - 4 \left( \frac{\partial F}{\partial \varphi \partial \theta} \right)^2 \sin^2 \varphi \\
+ 8 \left( \frac{\partial^2 F}{\partial \varphi \partial \theta} \right)^2 \sin(2\varphi) \right)^{1/2}.
\]

Proof. We provide the details of the restriction (A.3) on \( F \). We follow the same procedure as in the 2-dimensional case and thus we show that the eigenvalues of the Hessian are positive. Therefore we employ
spherical coordinates:

\[ v_1 = r \cos \theta \sin \varphi \]  
\[ v_2 = r \sin \theta \sin \varphi \]  
\[ v_3 = r \cos \varphi \]  

The first derivatives can be written in spherical coordinates as:

\[ \frac{\partial}{\partial v_1} = \cos \theta \sin \varphi \frac{\partial}{\partial r} - \sin \theta \frac{\partial}{\partial \theta} + \frac{\cos \theta \cos \varphi}{r} \frac{\partial}{\partial \varphi} \]  
\[ \frac{\partial}{\partial v_2} = \sin \theta \sin \varphi \frac{\partial}{\partial r} + \frac{\cos \theta}{r \sin \varphi} \frac{\partial}{\partial \theta} + \frac{\sin \theta \cos \varphi}{r} \frac{\partial}{\partial \varphi} \]  
\[ \frac{\partial}{\partial v_3} = \cos \varphi \frac{\partial}{\partial r} - \frac{\sin \varphi}{r} \frac{\partial}{\partial \varphi} \]  

The computation of the second derivatives is straightforward but at the same time quite involved. Here we only provide the resulting components of the Hessian, which are

\[ \frac{\partial^2 \eta}{\partial v_1^2} = \frac{\sin^2 \varphi}{r} F - \frac{\sin \theta \cos \theta}{r} \frac{\partial F}{\partial \theta} + \frac{\sin \theta \cos \theta}{r \sin \varphi} \frac{\partial F}{\partial \varphi} + \frac{\sin^2 \theta}{r \sin^2 \varphi} \frac{\partial^2 F}{\partial \theta^2} + \frac{\sin^2 \theta \cos \varphi}{r \sin \varphi} \frac{\partial^2 F}{\partial \varphi^2} \]  
\[ \frac{\partial^2 \eta}{\partial v_2^2} = \frac{\cos^2 \theta}{r} F + \frac{\sin \theta \cos \theta}{r \sin \varphi} \frac{\partial F}{\partial \theta} - \frac{\sin \theta \cos \theta}{r \sin^2 \varphi} \frac{\partial F}{\partial \varphi} + \frac{\cos^2 \theta}{r \sin^2 \varphi} \frac{\partial^2 F}{\partial \theta^2} + \frac{\cos^2 \theta \cos \varphi}{r \sin \varphi} \frac{\partial^2 F}{\partial \varphi^2} \]  
\[ \frac{\partial^2 \eta}{\partial v_3^2} = \left( F + \frac{\partial^2 F}{\partial \varphi^2} \right) \frac{\sin^2 \varphi}{r} \]  
\[ \frac{\partial^2 \eta}{\partial v_1 \partial v_2} = \frac{\partial^2 \eta}{\partial v_2 \partial v_1} = \frac{\partial^2 \eta}{\partial v_1 \partial v_2} = \frac{\frac{\partial^2 \eta}{\partial v_1 \partial v_2}}{r} = \cos^2 \theta \frac{\sin^2 \theta}{r \sin^2 \varphi} \frac{\partial F}{\partial \theta} - \frac{\sin \theta \cos \theta \cos \varphi}{r \sin \varphi} \frac{\partial F}{\partial \varphi} + \frac{\sin \theta \cos \theta}{r \sin^2 \varphi} \frac{\partial^2 F}{\partial \theta^2} + \frac{\sin \theta \cos \theta \cos \varphi}{r \sin \varphi} \frac{\partial^2 F}{\partial \varphi^2} \]  
\[ \frac{\partial^2 \eta}{\partial v_1 \partial v_3} = \frac{\partial^2 \eta}{\partial v_3 \partial v_1} = -\frac{\frac{\partial^2 \eta}{\partial v_1 \partial v_3}}{r} = -\frac{\sin \varphi \cos \theta \cos \varphi}{r} \left( F + \frac{\partial^2 F}{\partial \varphi^2} \right) - \frac{\sin \theta \cos \varphi}{r \sin \varphi} \frac{\partial F}{\partial \theta} + \frac{\sin \theta}{r} \frac{\partial^2 F}{\partial \varphi \partial \theta} \]  
\[ \frac{\partial^2 \eta}{\partial v_2 \partial v_3} = \frac{\partial^2 \eta}{\partial v_3 \partial v_2} = -\frac{\frac{\partial^2 \eta}{\partial v_2 \partial v_3}}{r} = -\frac{\sin \varphi \sin \theta \cos \varphi}{r} \left( F + \frac{\partial^2 F}{\partial \varphi^2} \right) + \frac{\cos \theta \cos \varphi}{r \sin \varphi} \frac{\partial F}{\partial \theta} - \frac{\cos \theta}{r} \frac{\partial^2 F}{\partial \varphi \partial \theta} \]
The eigenvalues of the Hessian can be computed to be:

\[
\lambda_1 = 0,
\]

\[
\lambda_2 = \frac{1}{8r} \left[ 8F + 4 \frac{\partial F}{\partial \varphi} \cot \varphi + 4 \frac{\partial^2 F}{\partial \theta^2} \csc^2 \varphi 
+ \sqrt{2} \csc^2 \varphi \left( \frac{\partial F^2}{\partial \varphi} (1 - \cos 4\varphi) + 32 \frac{\partial F^2}{\partial \theta} + 8 \frac{\partial^2 F}{\partial \theta^2} + 32 \left( \frac{\partial^2 F}{\partial \varphi \partial \theta} \right)^2 - \left( \frac{\partial F}{\partial \theta} \right)^2 \right) \sin^2 \varphi 
+ 8 \left( \frac{\partial F}{\partial \varphi} \frac{\partial^2 F}{\partial \theta^2} - 4 \frac{\partial^2 F}{\partial \varphi \partial \theta} \frac{\partial F}{\partial \theta} \right) \sin(2\varphi) \right) \right]^{1/2},
\]

(A.16)

\[
\lambda_3 = \frac{1}{8r} \left[ 8F + 4 \frac{\partial F}{\partial \varphi} \cot \varphi + 4 \frac{\partial^2 F}{\partial \theta^2} \csc^2 \varphi
- \sqrt{2} \csc^2 \varphi \left( \frac{\partial F^2}{\partial \varphi} (1 - \cos 4\varphi) + 32 \frac{\partial F^2}{\partial \theta} + 8 \frac{\partial^2 F}{\partial \theta^2} + 32 \left( \frac{\partial^2 F}{\partial \varphi \partial \theta} \right)^2 - \left( \frac{\partial F}{\partial \theta} \right)^2 \right) \sin^2 \varphi 
+ 8 \left( \frac{\partial F}{\partial \varphi} \frac{\partial^2 F}{\partial \theta^2} - 4 \frac{\partial^2 F}{\partial \varphi \partial \theta} \frac{\partial F}{\partial \theta} \right) \sin(2\varphi) \right) \right]^{1/2}.
\]

(A.17)

Positivity of the eigenvalues leads to the restrictions on $F$. □

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