LIMIT LAWS OF TRANSIENT EXCITED RANDOM WALKS ON INTEGERS

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ABSTRACT. We consider excited random walks (ERWs) on \( \mathbb{Z} \) with a bounded number of i.i.d. cookies per site without the non-negativity assumption on the drifts induced by the cookies. Kosygina and Zerner \cite{KosyginaZerner2006} have shown that when the total expected drift per site, \( \delta \), is larger than 1 then ERW is transient to the right and, moreover, for \( \delta > 4 \) under the averaged measure it obeys the Central Limit Theorem. We show that when \( \delta \in (2,4] \) the limiting behavior of an appropriately centered and scaled excited random walk under the averaged measure is described by a strictly stable law with parameter \( \delta/2 \). Our method also extends the results obtained by Basdevant and Singh \cite{BasdevantSingh2008} for \( \delta \in (1,2] \) under the non-negativity assumption to the setting which allows both positive and negative cookies.

Résumé. On considère des marches aléatoires excitées sur \( \mathbb{Z} \) avec un nombre borné de cookies i.i.d. à chaque site, ceci sans l'hypothèse de positivité. Auparavant, Kosygina et Zerner \cite{KosyginaZerner2006} ont établi que si le drift totale moyenne par site, \( \delta \), est strictement supérieur à 1, alors la marche est transiente (vers la droite) et, de plus, pour \( \delta > 4 \) il y a un théorème central limite pour la position de la marche. Ici, on démontre que pour \( \delta \in (2,4] \) cette position, convenablement centrée et réduite, converge vers une loi stable de paramètre \( \delta/2 \). L’approche permet également d’étendre les résultats de Basdevant et Singh \cite{BasdevantSingh2008} pour \( \delta \in (1,2] \) à notre cadre plus général.

1. Introduction and main results

Excited random walk (ERW) on \( \mathbb{Z}^d \) was introduced by Benjamini and Wilson in \cite{BenjaminiWilson2004}. They proposed to modify the nearest neighbor simple symmetric random walk by giving it a positive drift (“excitation”) in the first coordinate direction upon reaching a previously unvisited site. If the site had been visited before, then the walk made unbiased jumps to one of its nearest neighbor sites. See \cite{KosyginaMountford2008}, \cite{Bandyopadhyay2006}, \cite{Mountford2007} and references therein for further results about this particular model.

Zerner (\cite{Zerner2008}, \cite{Zerner2009}) generalized excited random walks by allowing to modify the transition probabilities at each site not just once but any number of times and, moreover, choosing them according to some probability distribution. He obtained the criteria for recurrence and transience and the law of large numbers for i.i.d. environments on \( \mathbb{Z}^d \) and strips and also for general stationary ergodic environments on \( \mathbb{Z} \). It turned out that this generalized model had interesting behavior even for \( d = 1 \), and this case was further studied in \cite{KosyginaZerner2006}, \cite{BasdevantSingh2008}, \cite{BasdevantSingh2009}.

Results obtained in all these works rely on the assumption that projections of all possible drifts on some fixed direction are non-negative. In fact, the branching processes framework introduced in \cite{KosyginaMountford2008} for random walks in random environment \((d = 1)\) and employed in \cite{BasdevantSingh2008}, \cite{BasdevantSingh2009} for excited random walks, does not depend on the positivity assumption, and it seems natural to use this approach for extending the analysis to environments which allow both positive and negative drifts. This was done in \cite{KosyginaZerner2006}, where the authors discussed recurrence and transience, laws of large numbers, positive speed, and the averaged central limit theorem for multi-excited

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random walks on $\mathbb{Z}$ in i.i.d. environments with bounded number of “excitations” per site. We postpone further discussion of known results for $d = 1$ and turn to a precise description of the model considered in this paper.

Given an arbitrary positive integer $M$ let

$$\Omega_M := \{((\omega_z(i))_{i \in \mathbb{N}})_{z \in \mathbb{Z}} \mid \omega_z(i) \in [0, 1], \text{ for } i \in \{1, 2, \ldots, M\}$$

$$\text{and } \omega_z(i) = 1/2, \text{ for } i > M, \ z \in \mathbb{Z}.\$$

An element of $\Omega_M$ is called a cookie environment. For each $z \in \mathbb{Z}$, the sequence $\{\omega_z(i)\}_{i \in \mathbb{N}}$ can be thought of as a pile of cookies at site $z$, and $\omega_z(i)$ is referred to as “the $i$-th cookie at $z$.” The number $\omega_z(i)$ is equal to the transition probability from $z$ to $z + 1$ of a nearest-neighbor random walk upon the $i$-th visit to $z$. If $\omega_z(i) > 1/2$ (resp. $\omega_z(i) < 1/2$) the corresponding cookie will be called positive (resp. negative), $\omega_z(i) = 1/2$ will correspond to a “placebo” cookie or, equivalently, the absence of an effective $i$-th cookie at site $z$.

Let $\mathbb{P}$ be a probability measure on $\Omega_M$, which satisfies the following two conditions:

(A1) Independence: the sequence $(\omega_z(\cdot))_{z \in \mathbb{Z}}$ is i.i.d. under $\mathbb{P}$.

(A2) Non-degeneracy:

$$\mathbb{E} \left[ \prod_{i=1}^M \omega_0(i) \right] > 0 \text{ and } \mathbb{E} \left[ \prod_{i=1}^M (1 - \omega_0(i)) \right] > 0.$$

Notice that we do not make any independence assumptions on the cookies at the same site.

It will be convenient to define our ERW model using a coin-toss construction. Let $(\Sigma, \mathcal{F})$ be some measurable space equipped with a family of probability measures $P_{x, \omega}$, $x \in \mathbb{Z}$, $\omega \in \Omega_M$, such that for each choice of $x \in \mathbb{Z}$ and $\omega \in \Omega_M$ we have ±1-valued random variables $B_i(z)$, $z \in \mathbb{Z}$, $i \geq 1$, which are independent under $P_{x, \omega}$ with distribution given by

$$P_{x, \omega}(B_i(z) = 1) = \omega_z(i) \quad \text{and} \quad P_{x, \omega}(B_i(z) = -1) = 1 - \omega_z(i). \quad (1.1)$$

Let $X_0$ be a random variable on $(\Sigma, \mathcal{F}, P_{x, \omega})$ such that $P_{x, \omega}(X_0 = x) = 1$. Then an ERW starting at $x \in \mathbb{Z}$ in the environment $\omega$, $X := \{X_n\}_{n \geq 0}$, can be defined on the probability space $(\Sigma, \mathcal{F}, P_{x, \omega})$ by the relation

$$X_{n+1} := X_n + B_i(X_n), \quad n \geq 0. \quad (1.2)$$

Informally speaking, upon each visit to a site the walker eats a cookie and makes one step to the right or to the left with probabilities prescribed by this cookie. Since $\omega_z(i) = 1/2$ for all $i > M$, the walker will make unbiased steps from $z$ starting from the $(M + 1)$-th visit to $z$.

Events $\{B_i(z) = 1\}$, $i \in \mathbb{N}$, $z \in \mathbb{Z}$, will be referred to as “successes” and events $\{B_i(z) = -1\}$ will be called “failures”.

The consumption of a cookie $\omega_z(i)$ induces a drift of size $2\omega_z(i) - 1$ with respect to $P_{x, \omega}$. Summing up over all cookies at one site and taking the expectation with respect to $\mathbb{P}$ gives the parameter

$$\delta := \mathbb{E} \left[ \sum_{i \geq 1} (2\omega_0(i) - 1) \right] = \mathbb{E} \left[ \sum_{i=1}^M (2\omega_0(i) - 1) \right], \quad (1.3)$$

which we call the average total drift per site. It plays a key role in the classification of the asymptotic behavior of the walk.

We notice that there is an obvious symmetry between positive and negative cookies: if the environment $(\omega_z)_{z \in \mathbb{Z}}$ is replaced by $(\omega_z')_{z \in \mathbb{Z}}$ where $\omega_z'(i) = 1 - \omega_z(i)$, for all $i \in \mathbb{N}$, $z \in \mathbb{Z}$, then
\( X' := \{X'_n\}_{n \geq 0} \), the ERW corresponding to the new environment, satisfies
\[
(1.4) \quad X' \overset{D}{=} -X,
\]
where \( \overset{D}{=} \) denotes the equality in distribution. Thus, it is sufficient to consider, say, only non-negative \( \delta \) (this, of course, allows both negative and positive cookies), and we shall always assume this to be the case.

Define the averaged measure \( P_x \) by setting
\[
P_x(\cdot) = \mathbb{E}(P_{x,\omega}(\cdot)).
\]
Below we summarize known results about this model.

**Theorem 1.1** ([15]). Assume (A1) and (A2).

(i) If \( \delta \in [0, 1] \) then \( X \) is recurrent, i.e. for \( \mathbb{P}\)-a.a. \( \omega \) it returns \( P_{0,\omega}\)-a.s. infinitely many times to its starting point. If \( \delta > 1 \) then \( X \) is transient to the right, i.e. for \( \mathbb{P}\)-a.a. \( \omega \), \( X_n \to \infty \) as \( n \to \infty \) \( P_{0,\omega}\)-a.s..

(ii) There is a deterministic \( v \in [0, 1] \) such that \( X \) satisfies for \( \mathbb{P}\)-a.a. \( \omega \) the strong law of large numbers,
\[
(1.5) \quad \lim_{n \to \infty} \frac{X_n}{n} = v \quad P_{0,\omega}\text{-a.s.}.
\]
Moreover, \( v = 0 \) for \( \delta \in [0, 2] \) and \( v > 0 \) for \( \delta > 2 \).

(iii) If \( \delta > 4 \) then the sequence
\[
B_t^n := \frac{X_{\lceil tn \rceil} - \lfloor tn \rfloor v}{\sqrt{n}}, \quad t \geq 0
\]
converges weakly under \( P_0 \) to a non-degenerate Brownian motion with respect to the Skorohod topology on the space of càdlàg functions.

This theorem does not discuss the rate of growth of the ERW when it is transient but has zero linear speed \((1 < \delta \leq 2)\). It also leaves open the question about fluctuations when \( \delta \leq 4 \).

The rate of growth of the transient cookie walk with zero linear speed was studied in [2] for the case of deterministic spatially homogeneous non-negative cookie environments. For further discussion we need some notation for the limiting stable distributions that appear below. Given \( \alpha \in (0, 2] \) and \( b > 0 \), denote by \( Z_{\alpha,b} \) a random variable (on some probability space) whose characteristic function is determined by the relation
\[
(1.6) \quad \log \mathbb{E} e^{iuZ_{\alpha,b}} = \begin{cases} -b|u|^\alpha \left(1 - i \frac{u}{|u|} \tan \left( \frac{\pi \alpha}{2} \right) \right), & \text{if } \alpha \neq 1; \\ -b|u| \left(1 + \frac{2i}{\pi} \frac{u}{|u|} \log |u| \right), & \text{if } \alpha = 1. \end{cases}
\]
Observe that \( Z_{2,b} \) is a centered normal random variable with variance \( 2b \). The weak convergence with respect to \( P_0 \) will be denoted by \( \Rightarrow \).

**Theorem 1.2** ([2]). Let \( \omega_z(i) = p_i \in [1/2, 1), i \in \mathbb{N} \) for all \( z \in \mathbb{Z} \), where \( p_i = 1/2 \) for \( i > M \), and \( \delta \) be as in (1.5), that is \( \delta = \sum_{i=1}^{M} (2p_i - 1) \).

(i) If \( \delta \in (1, 2) \) then there is a positive constant \( b \) such that as \( n \to \infty \)
\[
X_n/\sqrt{n} \Rightarrow (Z_{\delta/2,b})^{-\delta/2}.
\]

(ii) If \( \delta = 2 \) then \( (X_n \log n)/n \) converges in probability to some constant \( c > 0 \).

The above results also hold if \( X_n \) is replaced by \( \sup_{i \leq n} X_i \) or \( \inf_{i \geq n} X_i \).

The proof of Theorem 1.2 used the non-negativity of cookies, though this assumption does not seem to be essential for most parts of the proof. It is certainly possible that the approach presented in [2] could yield the same results without the non-negativity assumption.
The functional central limit theorem for ERWs with $\delta \in (0,1)$ in stationary ergodic non-negative cookie environments was obtained in [7]. The limiting process is shown to be Brownian motion perturbed at extrema (see, for example, [6], [5]).

The main results of this paper deal with the case when $\delta \in (2,4]$, though they apply also to $\delta \in (1,2]$. Moreover, our approach provides an alternative proof of Theorem 1.2 for general cookie environments that satisfy conditions (A1) and (A2).

We establish the following theorem.

**Theorem 1.3.** Let $T_n = \inf\{j \geq 0 \mid X_j = n\}$ and $v$ be the speed of the ERW (see (1.5)). The following statements hold under the averaged measure $P_0$.

(i) If $\delta \in (2,4)$ then there is a constant $b > 0$ such that as $n \to \infty$

$$\frac{T_n - v^{-1}n}{n^{2/\delta}} \Rightarrow Z_{\delta/2,b}, \text{ and}$$

$$\frac{X_n - vn}{n^{2/\delta}} \Rightarrow -v^{1+2/\delta}Z_{\delta/2,b}. \text{ (1.7)}$$

(ii) If $\delta = 4$ then there is a constant $b > 0$ such that as $n \to \infty$

$$\frac{T_n - v^{-1}n}{\sqrt{n \log n}} \Rightarrow Z_{2,b}, \text{ and}$$

$$\frac{X_n - vn}{\sqrt{n \log n}} \Rightarrow -v^{3/2}Z_{2,b}. \text{ (1.8)}$$

Moreover, (1.8) and (1.10) hold if $X_n$ is replaced by $\sup_{i \leq n} X_i$ or $\inf_{i \geq n} X_i$.

The paper is organized as follows. In Section 2 we recall the branching processes framework and formulate two statements (Theorem 2.1 and Theorem 2.2), from which we later infer Theorem 1.3. Section 8 explains the idea of the proof of Theorem 2.2 and studies properties of the approximating diffusion process. In Section 9 we determine sufficient conditions for the validity of Theorem 2.2. Section 5 contains the main technical lemma (Lemma 5.3). It is followed by three sections, where we use the results of Section 5 to verify the sufficient conditions of Section 4 and prove Theorem 2.1. The proof of Theorem 1.3 is given in Section 9. The Appendix contains proofs of several technical results.

### 2. Reduction to branching processes

From now on we assume that $\delta > 1$, i.e. $X$ is transient to the right. Suppose that the random walk $\{X_n\}_{n \geq 0}$ starts at 0. At first, we recall the framework used in [1], [2], and [15]. The main ideas go back at least to [16] and [14].

For $n \in \mathbb{N}$ and $k \leq n$ define

$$D_{n,k} = \sum_{j=0}^{T_n-1} \mathbb{1}_{\{X_j = k, X_{j+1} = k-1\}},$$

the number of jumps from $k$ to $k-1$ before time $T_n$. Then

$$T_n = n + 2 \sum_{k \leq n} D_{n,k} = n + 2 \sum_{0 \leq k \leq n} D_{n,k} + 2 \sum_{k < 0} D_{n,k}. \text{ (2.1)}$$

The last sum is bounded above by the total time spent by $X_n$ below 0. Since $X_n$ is transient to the right, this sum is $P_0$-a.s. finite, and, therefore, for any $\alpha > 0$

$$\lim_{n \to \infty} \frac{1}{n^\alpha} \sum_{k < 0} D_{n,k} = 0 \quad P_0\text{-a.s..} \text{ (2.2)}$$
This will allow us to conclude that fluctuations of $T_n$ are determined by those of $\sum_{0 \leq k \leq n} D_{n,k}$, once we have shown that the latter are of order $n^{2/3}$.

We now consider the “reversed” process $(D_{n,n}, D_{n,n-1}, \ldots, D_{n,0})$. Obviously, $D_{n,n} = 0$ for every $n \in \mathbb{N}$. Moreover, given $D_{n,n}, D_{n,n-1}, \ldots, D_{n,k+1}$, we can write

$$D_{n,k} = \sum_{j=1}^{D_{n,k+1}+1} \text{(\# of jumps from } k \text{ to } k-1 \text{ between the } (j-1)\text{-th and } j\text{-th jump from } k \text{ to } k+1 \text{ before time } T_n), \; k = 0, 1, \ldots, n-1.$$ 

Here we used the observation that the number of jumps from $k$ to $k+1$ before time $T_n$ is equal to $D_{n,k+1} + 1$ for all $k \leq n-1$. The expression “between the 0-th and the 1-st jump” above should be understood as “prior to the 1-st jump”.

Fix an $\omega \in \Omega_M$ and denote by $F^{(k)}_m$ the number of “failures” in the sequence $B^{(k)}$ (see (1.1) with $z$ replaced by $k$) before the $m$-th “success”. Then, given $D_{n,k+1}$,

$$D_{n,k} = F^{(k)}_{D_{n,k+1}+1}.$$

Since the sequences $B^{(k)}$, $k \in \mathbb{Z}$, are i.i.d. under $P_0$, we have that $F^{(k)}_m \overset{D}{=} F^{(n-k-1)}_m$ and can conclude that the distribution of $(D_{n,n}, D_{n,n-1}, \ldots, D_{n,0})$ coincides with that of $(V_0, V_1, \ldots, V_n)$, where $V = \{V_k\}_{k \geq 0}$ is a Markov chain defined by

$$V_0 = 0, \quad V_{k+1} = F^{(k)}_{V_k+1}, \; k \geq 0.$$ 

For $x \geq 0$ we shall denote by $[x]$ the integer part of $x$ and by $P^V_x$ the measure associated to the process $V$, which starts with $[x]$ individuals in the 0-th generation. Observe that $V$ is a branching process with the following properties:

(i) $V$ has exactly 1 immigrant in each generation (the immigration occurs before the reproduction) and, therefore, does not get absorbed at 0.

(ii) The number of offspring of the $m$-th individual in generation $k$ is given by the number of failures between the $(m-1)$-th and $m$-th success in the sequence $B^{(k)}$. In particular, if $V_k \geq M$ then the offspring distribution of each individual after the $M$-th one is $\text{Geom}(1/2)$ (i.e. geometric on $\{0\} \cup \mathbb{N}$ with parameter $1/2$).

Therefore (here and throughout taking any sum from $k$ to $\ell$ for $k > \ell$ to be zero) we can write

$$(2.3) \quad V_{k+1} = \sum_{m=1}^{M/(V_k+1)} \zeta^{(k)}_m + \sum_{m=1}^{V_k-M+1} \xi^{(k)}_m, \; k \geq 0,$$

where $\{\zeta^{(k)}_m; k \geq 0, m \geq 1\}$ are i.i.d. $\text{Geom}(1/2)$ random variables, vectors $(\zeta^{(k)}_1, \zeta^{(k)}_2, \ldots, \zeta^{(k)}_M)$, $k \geq 0$, are i.i.d. under $P^V_x$ and independent of $\{\zeta^{(k)}_m; k \geq 0, m \geq 1\}$. For each $k \geq 0$ the random variables $\{\zeta^{(k)}_m\}_{m=1}^M$ are neither independent nor identically distributed, but, given that for some $j < M$

$$\sum_{m=1}^j \zeta^{(k)}_m \geq M,$$

that is all cookies at site $k$ have been eaten before the $j$-th jump from $k$ to $(k+1)$, we are left with $\{\zeta^{(k)}_m\}_{m=j+1}^M$ that are independent $\text{Geom}(1/2)$ random variables. Define

$$(2.4) \quad \sigma^V_0 = \inf\{j > 0 \mid V_j = 0\}, \quad S^V = \sum_{j=0}^{\sigma^V_0-1} V_j.$$
Detailed information about the tail behavior of $\sigma_0^V$ and $S^V$ will enable us to use the renewal structure and characterize the tails of $\sum_{0 \leq k \leq n} D_{n,k}$, and, therefore, of $T_n$ as $n \to \infty$. We shall show in Section 9 that the following two statements imply Theorem 1.3.

**Theorem 2.1.** Let $\delta > 1$. Then

\[
\lim_{n \to \infty} n^{\delta} P^V_0 (\sigma_0^V > n) = C_1 \in (0, \infty).
\]

**Theorem 2.2.** Let $\delta > 1$. Then

\[
\lim_{n \to \infty} n^{\delta/2} P^V_0 (S^V > n) = C_2 \in (0, \infty).
\]

**Remark 2.3.** In fact, a weaker result than (2.5) is sufficient for our purpose: there is a constant $B$ such that $n^{\delta} P^V_0 (\sigma_0^V > n) \leq B$ for all $n \in \mathbb{N}$ (see condition (A) in Lemma 4.1). We also would like to point out that the limits in (2.5) and (2.6) exist for every starting point $x \in \mathbb{N} \cup \{0\}$ with $C_1$ and $C_2$ depending on $x$. The proofs simply repeat those for $x = 0$.

For the model described in Theorem 1.2, the convergence (2.5) starting from $x \in \mathbb{N}$ is shown in [2, Proposition 3.1] (for $\delta \in (1, 2)$), and (2.6) for $\delta \in (1, 2]$ is the content of [2, Proposition 4.1]. Theorem 2.1 can also be derived from the construction in [15] (see Lemma 17) and [12] (case $\theta < 0$). We use a different approach and obtain both results directly without using the Laplace transform and Tauberian theorems.

We close this section by introducing some additional notation. For $x \geq 0$ we set

\[
\tau_x^V = \inf\{j > 0 | V_j \geq x\};
\]

\[
\sigma_x^V = \inf\{j > 0 | V_j \leq x\}.
\]

We shall drop the superscript whenever there is no possibility of confusion.

Since the random walk $X$ is transient to the right, $P^V_0 (\sigma_0^V < \infty) = 1$ for every $y \geq 0$. This implies that $P^V_y (\sigma_x^V < \infty) = 1$ for every $x \in [0, y)$.

Let us remark that when we later deal with a continuous process on $[0, \infty)$ we shall simply use the first hitting time of $x$ to record the entrance time in $[x, \infty)$ (or $[0, x]$), given that the process starts outside of the mentioned interval. We hope that denoting the hitting time of $x$ for such processes also by $\tau_x$ will not result in ambiguity.

### 3. The approximating diffusion process and its properties

The bottom-line of our approach is that the main features of branching process $V$ killed upon reaching 0 are reasonably well described by a simple diffusion process.

The parameters of such diffusion processes can be easily computed at the heuristic level. For $V_k \geq M$, (2.3) implies that

\[
V_{k+1} - V_k = \sum_{m=1}^{M} \zeta_m^{(k)} - M + 1 + \sum_{m=1}^{V_k - M + 1} (\xi_m^{(k)} - 1).
\]

By conditioning on the number of successes in the first $M$ tosses it is easy to compute (see Lemma 3.3 in [1] or Lemma 17 in [15] for details) that for all $x \geq 0$

\[
E_x^V \left( \sum_{m=1}^{M} \zeta_m^{(k)} - M + 1 \right) = 1 - \delta.
\]

The term $\sum_{m=1}^{M} \zeta_m^{(k)} - M + 1$ is independent of $\sum_{m=1}^{V_k - M + 1} (\xi_m^{(k)} - 1)$. When $V_k$ is large, the latter is approximately normal with mean 0 and variance essentially equal to $2V_k$. 
Therefore, the relevant diffusion should be given by the following stochastic differential equation:

\[ dY_t = (1 - \delta) \, dt + \sqrt{2Y_t} \, dB_t, \quad Y_0 = y > 0, \quad t \in [0, \tau^Y_0], \]

where for \( x \geq 0 \) we set

\[ \tau^Y_x = \inf\{t \geq 0 \mid Y_t = x\}. \]

Observe that \( \tau^Y_0 < \infty \) a.s., since \( 2Y_t \) is a squared Bessel process of negative dimension \( 2(1 - \delta) \) (for a proof, set \( a = 0 \) and let \( b \to \infty \) in part (ii) of Lemma 3.2 below).

The above heuristics are justified by the next lemma.

**Lemma 3.1.** Let \( Y = \{Y_t\}_{t \geq 0} \) be the solution of (3.3). Fix an arbitrary \( \varepsilon > 0 \). For \( y \in (\varepsilon, \infty) \) let \( V_0 = [ny] \), and define

\[ V^{\varepsilon,n}_t = \frac{V_{[nt] \wedge \sigma^V_t}}{n}, \quad t \in [0, \infty), \]

where \( \sigma^V_t \) is given by (2.8). Then the sequence of processes \( Y^{\varepsilon,n} = \{Y^{\varepsilon,n}_t\}_{t \geq 0} \) converges in distribution as \( n \to \infty \) with respect to the Skorokhod topology on the space of càdlàg functions to the stopped diffusion \( Y^\varepsilon = \{Y_t \wedge \tau^{Y^\varepsilon}_t\}_{t \geq 0}, \quad Y_0 = y \).

**Proof.** We simply apply the (much more general) results of [10]. We first note that our convergence result considers the processes up to the first entry into \(( -\infty, \varepsilon] \) for \( \varepsilon > 0 \) fixed. So we can choose to modify the rules of evolution for \( V \) when \( V_k \leq \varepsilon n \): we consider the process \( (V_n, \varepsilon)_k \geq 0 \) where, with the existing notation,

\[ (3.5) \quad V^{\varepsilon,n}_0 = [ny], \quad V^{\varepsilon,n}_k + 1 = \sum_{m=1}^{M} \zeta_m(k) \quad V^{\varepsilon,n}_k + \varepsilon - M + 1 = \sum_{m=1}^{N} \xi_m(k), \quad k \geq 0. \]

Then (given the regularity of points for the limit process) it will suffice to show the convergence of processes

\[ \tilde{Y}_{\tilde{t}}^{\varepsilon,n} = \frac{V^{\varepsilon,n}_{[n\tilde{t}]}}{n}, \quad \tilde{t} \in [0, \infty), \]

to the solution of the stochastic integral equation

\[ dY_t = (1 - \delta) \, dt + \sqrt{2(Y_t \wedge \varepsilon)} \, dB_t, \quad Y_0 = y > 0, \quad t \in [0, \infty). \]

We can now apply Theorem 4.1 of Chapter 7 of [10] with \( X_n = \tilde{Y}^{\varepsilon,n}_t \). The needed uniqueness of the martingale problem corresponding to operator

\[ Gf = (x \vee \varepsilon)f'' + (1 - \delta)f' \]

follows from [10], Chapter 5, Section 3 (Theorems 3.6 and 3.7 imply the distributional uniqueness for solutions of the corresponding stochastic integral equation, and Proposition 3.1 shows that this implies the uniqueness for the martingale problem).

We shall see in a moment that this diffusion has the desired behavior of the extinction time and of the total area under the path before the extinction (see Lemma 3.3 and Lemma 3.5 below). Unfortunately, these properties in conjunction with Lemma 3.1 do not automatically imply Theorem 2.1 and Theorem 2.2 and work needs to be done to “transfer” these results to the corresponding quantities of the process \( V \). Nevertheless, Lemma 3.1 is very helpful when \( V \) stays large as we shall see later.

In the rest of this section we state and prove several facts about \( Y \). When we need to specify that the process \( Y \) starts at \( y \) at time \( 0 \) we shall write \( Y^y \). Again, whenever there is no ambiguity about which process is being considered we shall drop the superscript in \( \tau^Y_x \) defined in (3.4).
Lemma 3.2. Fix \( y > 0 \).

(i) (Scaling) Let \( \tilde{Y} = \{ \tilde{Y}_t \}_{t \geq 0} \), where \( \tilde{Y}_t = \frac{Y^y_t}{y} \). Then \( \tilde{Y} \overset{D}{=} Y^1 \).

(ii) (Hitting probabilities) Let \( 0 \leq a < x < b \). Then

\[
P^y_y(\tau_x < \tau_y) = \frac{b^y - y^x}{b^x - a^x}.
\]

Proof. Part (i) can be easily checked by Itô’s formula applied to \( \tilde{Y}_t \) or seen from scaling properties of the generator. The proof of part (ii) is standard once we notice that the process \( (Y^y_t)^\delta \) stopped upon reaching the boundary of \([a, b]\) is a martingale. We omit the details. \( \square \)

Lemma 3.3. Let \( Y \) be the diffusion process defined by (3.3). Then

\[
limit_{x \to \infty} x^\delta P^y_1(\tau_0 > x) = C_3 \in (0, \infty). \]

Proof. For every \( \varepsilon > 0 \) and for all \( x > 1/\varepsilon \) we have by Lemma 3.2

\[
x^\delta P^y_1(\tau_0 > x) \geq x^\delta P^y_1(\tau_0 > x | \tau_{\varepsilon x} < \tau_0) P^y_1(\tau_{\varepsilon x} < \tau_0)
\]

\[
\geq x^\delta P^y_{\varepsilon x}(\tau_0 > x)(\varepsilon x)^{-\delta} = \varepsilon^{-\delta} P^y_1(\tau_0 > \varepsilon^{-1}) > 0.
\]

This implies that for each \( \varepsilon > 0 \)

\[
\lim_{x \to \infty} \inf x^\delta P^y_1(\tau_0 > x) \geq \varepsilon^{-\delta} P^y_1(\tau_0 > \varepsilon^{-1}) > 0.
\]

Taking the \( \limsup_{x \to 0} \) in the right-hand side we get

\[
\lim_{x \to \infty} \inf x^\delta P^y_1(\tau_0 > x) \geq \lim_{\varepsilon \to 0} \sup x^\delta P^y_1(\tau_0 > \varepsilon^{-1}) = \lim_{x \to \infty} \sup x^\delta P^y_1(\tau_0 > x).
\]

This would immediately imply the existence of a finite non-zero limit if we could show that

\[
\lim_{x \to \infty} \sup x^\delta P^y_1(\tau_0 > x) < \infty.
\]

This is the content of the next lemma.

Lemma 3.4. \( \limsup_{x \to \infty} x^\delta P^y_1(\tau_0 > x) < \infty. \)

The proof is very similar to the proof of the discrete version (see (A) in Lemma 4.1 and its proof in Section 6) and, thus, is omitted. \( \square \)

The final result of this section can be viewed as the “continuous counterpart” of Theorem 2.2. It concerns the area under the path of \( Y \).

Lemma 3.5. Let \( Y \) be the diffusion process defined by (3.3). Then

\[
\lim_{y \to \infty} y^\delta P^y_1 \left( \int_0^{\tau_0} Y_t \, dt > y^2 \right) = C_4 \in (0, \infty). \]

Proof. The proof uses scaling and follows the same steps as the proof of Lemma 3.3. For every \( \varepsilon > 0 \) and \( y > 1/\varepsilon \) we have

\[
y^\delta P^y_1 \left( \int_0^{\tau_0} Y_t \, dt > y^2 \right) \geq y^\delta P^y_1 \left( \int_0^{\tau_0} Y_t \, dt > y^2 \left| \tau_{\varepsilon y} < \tau_0 \right. \right) P^y_1(\tau_{\varepsilon y} < \tau_0)
\]

\[
\geq y^\delta P^y_{\varepsilon y} \left( \int_0^{\tau_0} Y_t \, dt > y^2 \right)(\varepsilon y)^{-\delta} = \varepsilon^{-\delta} P^y_{\varepsilon y} \left( \int_0^{\tau_0/(\varepsilon y)} Y_{\varepsilon y s} \, ds > \frac{y}{\varepsilon} \right)
\]

\[
= \varepsilon^{-\delta} P^y_{\varepsilon y} \left( \int_0^{\tau_0/(\varepsilon y)} Y_{\varepsilon y s} \, ds > \varepsilon^{-2} \right) = \varepsilon^{-\delta} P^y_1 \left( \int_0^{\tau_0} Y_s \, ds > \varepsilon^{-2} \right) > 0.
\]
This calculation, in fact, just shows that
\[ y^\delta P^Y_1 \left( \int_0^{\tau_0} Y_t \, dt > y^2 \right) \]
is a non-decreasing positive function of \( y \). Therefore, we only need to prove that it is bounded as \( y \to \infty \). But for \( y > 1 \)
\begin{align*}
y^\delta P^Y_1 \left( \int_0^{\tau_0} Y_t \, dt > y^2 \right) &= P^Y_1 \left( \int_0^{\tau_0} Y_t \, dt > y^2 \mid \tau_y < \tau_0 \right) + y^\delta P^Y_1 \left( \int_0^{\tau_0} Y_t \, dt > y^2, \tau_y > \tau_0 \right) \\
&\leq 1 + y^\delta P^V_1 (\tau_0 > y, \tau_y > \tau_0) \leq 1 + y^\delta P^V_1 (\tau_0 > y) .
\end{align*}
An application of Lemma 3.4 finishes the proof. \( \square \)

4. Conditions which imply Theorem 2.2

We have shown that the diffusion process \( Y \) has the desired asymptotic behavior of the area under the path up to the exit time \( \tau_0 \). In this section we give sufficient conditions under which we can “transfer” this result to the process \( V \) and obtain Theorem 2.2.

Lemma 4.1. Suppose that
\begin{enumerate}
\item[(A)] There is a constant \( B \) such that \( n^\delta P^V_0 (\sigma_0 > n) \leq B \) for all \( n \in \mathbb{N} \);
\item[(B)] For every \( \varepsilon > 0 \)
\[ \lim_{n \to \infty} P^V_{\varepsilon n} \left( \sum_{i=0}^{\sigma_0 - 1} V_i > n^2 \right) = P^V_1 \left( \int_0^{\tau_0} Y_t \, dt > \varepsilon^2 \right) ; \]
\item[(C)] \( \lim_{n \to \infty} n^\delta P^V_0 (\tau_n < \sigma_0) = C_5 \).
\end{enumerate}
Then
\[ \lim_{n \to \infty} n^\delta P^V_0 \left( \sum_{i=0}^{\sigma_0 - 1} V_i > n^2 \right) = C_4 C_5, \]
where \( C_4 \) is the constant from Lemma 3.5.

Proof. Fix an \( \varepsilon \in (0, 1) \) and split the path-space of \( V \) into two parts, the event \( H_{n, \varepsilon} := \{ \tau^V_{n \varepsilon} < \sigma_0^V \} \) and its complement, \( H_{n, \varepsilon}^c \).

First, consider the behavior of the total progeny on the event \( H_{n, \varepsilon}^c \). On \( H_{n, \varepsilon}^c \), the process \( V \) stays below \( \varepsilon n \) until the time \( \sigma_0 \). Estimating each \( V_i \) from above by \( \varepsilon n \) and using (A) we get for all \( n \in \mathbb{N} \)
\[ n^\delta P^V_0 \left( \sum_{i=0}^{\sigma_0 - 1} V_i > n^2, \ H_{n, \varepsilon}^c \right) \leq n^\delta P^V_0 (\sigma_0 > n/\varepsilon) \leq (2\varepsilon)^\delta B. \]
Therefore, for all \( n \in \mathbb{N} \)
\[ 0 \leq n^\delta P^V_0 \left( \sum_{i=0}^{\sigma_0 - 1} V_i > n^2 \right) - n^\delta P^V_0 \left( \sum_{i=0}^{\sigma_0 - 1} V_i > n^2, \ H_{n, \varepsilon} \right) \leq (2\varepsilon)^\delta B. \]
Hence, we only need to deal with the total progeny on the event \( H_{n, \varepsilon} \). The rough idea is that, on \( H_{n, \varepsilon} \), it is not unnatural for the total progeny to be of order \( n^2 \). This means that the decay of the probability that the total progeny is over \( n^2 \) comes from the decay of the probability of \( H_{n, \varepsilon} \), which is essentially given by condition (C). This would suffice if we could let \( \varepsilon = 1 \) but
we need $\varepsilon$ to be small, thus, some scaling is necessary to proceed with the argument, and this brings into play condition (B) and the result of Lemma 3.5.

To get a lower bound on $F_n := n^\delta P_0^V \left( \sum_{i=0}^{\sigma_0-1} V_i > n^2, H_{n,\varepsilon} \right)$ we use monotonicity of $V$ with respect to the initial number of particles, conditions (B) and (C), and Lemma 3.5:

$$
\lim_{\varepsilon \to 0} \lim_{n \to \infty} F_n \geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} n^\delta P_0^V (H_{n,\varepsilon}) P_{\varepsilon n}^V \left( \sum_{i=0}^{\sigma_0-1} V_i > n^2 \right)
= C_5 \lim_{\varepsilon \to 0} \varepsilon^{-\delta} P_1^Y \left( \int_0^{\tau} Y_t \, dt > \varepsilon^{-2} \right) = C_4 C_5.
$$

For an upper bound on $F_n$ we shall need two more parameters, $K \in (1, 1/\varepsilon)$ and $R > 1$. At the end, after taking the limits as $n \to \infty$ and then $\varepsilon \to 0$ we shall let $K \to \infty$ and $R \to 1$.

$$
n^{-\delta} F_n = P_0^V \left( \sum_{i=0}^{\tau_{\varepsilon n}-1} V_i + \sum_{i=\tau_{\varepsilon n}}^{\sigma_0-1} V_i > n^2, H_{n,\varepsilon} \right)
\leq P_0^V \left( \sum_{i=\tau_{\varepsilon n}}^{\sigma_0-1} V_i > n^2 (1 - K \varepsilon), \sum_{i=0}^{\tau_{\varepsilon n}-1} V_i \leq K \varepsilon n^2, H_{n,\varepsilon} \right)
+ P_0^V \left( \sum_{i=0}^{\tau_{\varepsilon n}-1} V_i > K \varepsilon n^2, H_{n,\varepsilon} \right)
$$

We bound the first term on the right-hand side by the following sum:

$$
P_0^V \left( \sum_{i=\tau_{\varepsilon n}}^{\sigma_0-1} V_i > n^2 (1 - K \varepsilon), V_{\tau_{\varepsilon n}} \leq R \varepsilon n, H_{n,\varepsilon} \right)
+ P_0^V \left( \sum_{i=\tau_{\varepsilon n}}^{\sigma_0-1} V_i > n^2 (1 - K \varepsilon), V_{\tau_{\varepsilon n}} > R \varepsilon n, H_{n,\varepsilon} \right).
$$

Estimating these terms in an obvious way and putting everything back together we get

$$
n^{-\delta} F_n \leq P_{\text{ren}}^V \left( \sum_{i=0}^{\tau_{\varepsilon n}-1} V_i > n^2 (1 - K \varepsilon) \right) P_0^V (H_{n,\varepsilon})
+ P_0^V (V_{\tau_{\varepsilon n}} > R \varepsilon n, H_{n,\varepsilon}) + P_0^V \left( \sum_{i=0}^{\tau_{\varepsilon n}-1} V_i > K \varepsilon n^2, H_{n,\varepsilon} \right)
= (I) + (II) + (III).
$$

It only remains to multiply everything by $n^\delta$ and consider the upper limits.

Term $n^\delta (I)$ gives the upper bound $C_4 C_5$, in the same way as we got a lower bound by sending $n \to \infty$, $\varepsilon \to 0$, and then $R \to 1$ and using easily verified continuity properties of the relevant distributions. Parameter $K$ disappears when we let $\varepsilon \to 0$.

Term $(II)$ is exponentially small in $n$ for fixed $\varepsilon$ and $R$ (see Lemma 5.1 below), thus $n^\delta (II)$ goes to zero as $n \to \infty$. 
Finally, since \( V_i \leq \varepsilon n \) for all \( i < \tau_{\varepsilon n} \), we get
\[
\begin{align*}
n^\delta P_0^V \left( \sum_{i=0}^{\tau_{\varepsilon n} - 1} V_i > Kn^2, \ H_{n,\varepsilon} \right) & \leq n^\delta P_0^V (\tau_{\varepsilon n} > Kn, H_{n,\varepsilon}) \\
& \leq n^\delta P_0^V (\sigma_0 > Kn, H_{n,\varepsilon}) \leq n^\delta P_0^V (\sigma_0 > Kn) \leq \frac{2^\delta B}{K^\delta}.
\end{align*}
\]
\[\square\]

5. Main tools

The main result of this section is Lemma 5.3 which is a discrete analog of Lemma 3.2 (ii).

We start with two technical lemmas. The first one will be used many times throughout the paper.

**Lemma 5.1.** There are constants \( c_1, c_2 > 0 \) and \( N \in \mathbb{N} \) such that for every \( x \geq N \) and \( y \geq 0 \),
\[
\sup_{0 \leq z < x} P_z^V (V_{\tau_x} > x + y | \tau_x < \sigma_0) \leq c_1 \left( e^{-c_2 y^2/x} + e^{-c_2 y} \right);
\]

\[
\sup_{x < z < 4x} P_z^V (V_{\sigma_z \wedge \tau_x} < x - y) \leq c_1 e^{-c_2 y^2/x}.
\]

This statement is a consequence of the fact that the offspring distribution of \( V \) is essentially geometric. The proof is given in the Appendix.

**Lemma 5.2.** Fix \( a \in (1, 2] \). Consider the process \( V \) with \( |V_0 - a^n| \leq a^{2n/3} \) and let \( \gamma = \inf \{ k \geq 0 | V_k \notin (a^{n-1}, a^{n+1}) \} \). Then for all sufficiently large \( n \)
\[
\begin{align*}
(i) & \ |P^V (\text{dist}(V, (a^{n-1}, a^{n+1})) \geq a^{2(n-1)/3})| \leq \exp(-a^n/4); \\
(ii) & \ |P^V (V_\gamma \leq a^{n-1}) - \frac{a^\delta}{a^\delta + 1}| \leq a^{-n/4}.
\end{align*}
\]

Part (i) is an immediate consequence of Lemma 5.1. The proof of part (ii) is basic but technical and is given in the Appendix.

**Lemma 5.3** (Main lemma). For each \( a \in (1, 2] \) there is an \( \ell_0 \in \mathbb{N} \) such that if \( \ell, m, u, x \in \mathbb{N} \) satisfy \( \ell_0 \leq \ell < m < u \) and \( |x - a^m| \leq a^{2m/3} \) then
\[
\frac{h^+_x (m) - 1}{h^+_a (u) - 1} \leq P_x^V (\sigma_x > \tau_a) \leq \frac{h^+_x (m) - 1}{h^+_a (u) - 1},
\]

where
\[
h^+_a (i) = \prod_{r=\ell+1}^i \left( a^\delta \mp a^{-\lambda r} \right), \quad i > \ell,
\]

and \( \lambda \) is some small positive number not depending on \( \ell \).

**Remark 5.4.** It is to be noted that for fixed \( \ell \) there are \( K_1(\ell) \) and \( K_2(\ell) \) such that
\[
K_1(\ell) \leq \frac{h^+_a (i)}{a^{(i-\ell)\delta}} \leq K_2(\ell) \quad \text{for all } i > \ell
\]

and \( K_i(\ell) \to 1 \) as \( \ell \to \infty, \ i = 1, 2. \)

**Proof of Lemma 5.3.** We will show the upper bound by comparing the process \( V \) with another process \( \tilde{V} \), whose exit probabilities can be estimated by further reduction to an exit problem for a birth-and-death-like Markov chain.
For $i \in \mathbb{N}$ set $x_i = [a^i + a^{2i}/3]$. By monotonicity, it is enough to prove the upper bound when the starting point $x$ is equal to $x_m$. Thus, we set $V_0 = x_m$. The comparison will be done in two steps.

Step 1. We shall construct a sequence of stopping times $\gamma_i$, $i \geq 0$, and a comparison process $\bar{V} = (\bar{V}_k)_{k \geq 0}$ with $x_\ell$ as an absorbing point so that $\bar{V}_k \geq V_k$ for all $k$ before the absorption. Let $\gamma_0 = 0$.

$$
\gamma_1 = \inf\{k > 0 \mid V_k \notin (a^{m-1}, a^{m+1})\}, \quad \bar{V}_k = V_k \quad \text{for} \quad k = 0, 1, \ldots, \gamma_1 - 1,
$$

and at time $\gamma_1$ add to $V_{\gamma_1}$ the necessary number of particles to get

$$
\bar{V}_{\gamma_1} = \begin{cases} x_{m-1}, & \text{if } V_{\gamma_1} \leq a^{m-1}; \\ x_{m+j}, & \text{if } x_{m+j-1} < V_{\gamma_1} \leq x_{m+j}, \quad j \in \mathbb{N}. \end{cases}
$$

Clearly, $\bar{V}_{\gamma_1} \geq V_{\gamma_1}$. By construction, $\bar{V}_{\gamma_1} = x_n$ for some $n \geq m - 1$, $n \neq m$. If $\bar{V}_{\gamma_1} = x_\ell$, then we stop the process.

Assume that we have already defined stopping times $\gamma_i$, $r = 0, 1, \ldots, i$, and the process $\bar{V}_k$ for all $k \leq \gamma_i$ so that $\bar{V}_n = x_n$ for some $n > \gamma_i$. We define $\bar{V}_k$ for $k > \gamma_i$ by applying to it the same branching mechanism as for $V$, namely, (2.3) with $V$ replaced by $\bar{V}$, $k \geq \gamma_i$. Denote by $\gamma_i+1$ the first time after $\gamma_i$ when $\bar{V}$ exits the interval $(a^{n-1}, a^{n+1})$. At time $\gamma_i+1$, if the process exited through the lower end of the interval then we set $\bar{V}_{\gamma_i+1} = x_{n-1}$, if the process exited through the upper end we add to $\bar{V}$ the minimal number of particles needed to get $\bar{V}_{\gamma_i+1} = x_s$ for some $s > n$. If $\bar{V}_{\gamma_i+1} = x_\ell$, then we stop the process. Thus, we obtain a sequence of stopping times $\gamma_i$, $i \geq 0$, and the desired dominating process $\bar{V}$ absorbed at $x_\ell$ such that $\bar{V}_n \in \{x_\ell, x_{i+1}, \ldots\}$, $i \geq 0$.

Step 2. Define a Markov chain $R = (R_j)_{j \geq 0}$ on $\{\ell, \ell + 1, \ldots\}$ by setting

$$
R_j = n \quad \text{if} \quad \bar{V}_{\gamma_j} = x_n, \quad j \geq 0.
$$

The state $\ell$ is absorbing. Let $\sigma^R_\ell = \inf\{j \geq 0 \mid R_j = \ell\}$ and $\tau^R_u = \inf\{j \geq 0 \mid R_j \geq u\}$. By construction,

$$
P^V_{x_m}(\sigma^V_\ell > \tau^V_u) \leq P^\bar{V}_{x_m}(\sigma^\bar{V}_\ell > \tau^\bar{V}_u) = P^R_{x_m}(\sigma^R_\ell > \tau^R_u).
$$

We shall show that $(h^+_a(R_j))_{j \geq 0}$ is a supermartingale with respect to the natural filtration. (We set $h^+_a(\ell) = 1$.) The optional stopping theorem and monotonicity of function $h^+_a$ will immediately imply the upper bound in the statement of the lemma.

For $i > \ell$ we have

$$
E^R_i(h^+_a(R_1)) = h^+_a(i-1)P^R_i(R_1 = i-1)
$$

$$
+ h^+_a(i+1)P^R_i(R_1 = i+1) + \sum_{n = i+2}^{\infty} h^+_a(n)P^R_i(R_1 = n)
$$

By the definition of $h^+_a$ this is less or equal than

$$
(5.3) \quad h^+_a(i) \left( (a^\delta - a^{-\lambda})^{-1}P^R_i(R_1 = i-1) \right)
$$

$$
+ (a^\delta - a^{-\lambda(i+1)})P^R_i(R_1 = i+1) + \sum_{n = i+2}^{\infty} a^{\delta(n-i)}P^R_i(R_1 = n)
$$
By Lemma 5.2 and Lemma 5.1 we have that for all $i > \ell$, where $\ell$ is chosen sufficiently large,

\[
P_i^R(R_1 = i - 1) = \frac{a^{i}}{a^i + 1} + O(a^{-i/4}),
\]

\[
P_i^R(R_1 = i + 1) = \frac{1}{a^i + 1} + O(a^{-i/4}), \text{ and}
\]

\[
P_i^R(R_1 \geq n) \leq P_{n-2}^R(R_1 \geq n) = O(\exp(-a^{n/4})) \text{ for all } n \geq i + 2.
\]

Substituting this into (5.3) and performing elementary computations we obtain

\[
E^R(h_a^+(R_1)) \leq h_a^+(i) \left[ 1 - \frac{a^{\lambda i}}{a^i + 1} \left( a^{-\lambda} - a^{-\delta} \right) + O(a^{-2\lambda i}) \right] \leq h_a^+(i),
\]

provided that $\lambda \leq 1/8$ and $\ell$ (therefore $i$) is sufficiently large.

For the lower bound we argue in a similar manner, except that now we choose $x_m = [a^m - a^{2m/3}] + 1$, assume that $V_0 = x_m$, and construct a comparison process $\tilde{V} = (\tilde{V}_k)_{k \geq 0}$ absorbed at $x_\ell$ so that $\tilde{V}_0 = V_0$ and $\tilde{V}_k \leq V_k$ for all $k$ before the absorption.

More precisely, we let $\gamma_0 = 0$ and assume that we have already defined stopping times $\gamma_r$, $r = 0, 1, \ldots, i$, and the process $V_k$ for all $k \leq \gamma_i$ so that $V_{\gamma_i} = x_n$ for some $n \neq \ell$. We define $V_k$ for $k > \gamma_i$ by (5.3) with $V$ replaced by $\tilde{V}$, $k \geq \gamma_i$. Denote by $\gamma_{i+1}$ the first time after $\gamma_i$ when $\tilde{V}$ exits the interval $(a^{n-1}, a^{n+1})$. At time $\gamma_{i+1}$, if the process exited through the upper end of the interval we set $\tilde{V}_{\gamma_{i+1}} = x_{n+1}$, if the process exited through the lower end we reduce the number of particles by removing the minimal number of particles to ensure that $\tilde{V}_{\gamma_{i+1}} = x_s$ for some $s < n$. If $\tilde{V}_{\gamma_{i+1}} \leq x_\ell$, then we stop the process and redefine $\tilde{V}_{\gamma_{i+1}}$ to be $x_\ell$. This procedure allows us to obtain a sequence of stopping times $\gamma_i$, $i \geq 0$, and the desired comparison process $\tilde{V}$ absorbed at $x_\ell$ such that $V_{\gamma_i} \in \{x_\ell, x_{\ell + 1}, \ldots\}$, $i \geq 0$.

Next, just as in the proof of the upper bound, we construct a Markov chain $R = (R_j)_{j \geq 0}$ and show that $(h_a^-(R_j))_{j \geq 0}$ is a submartingale (with $h_a^-(\ell)$ defined to be 1). The optional stopping theorem and monotonicity of function $h_a^-$ imply the lower bound. \hfill \square

Corollary 5.5. For each non-negative integer $x$ there exists a constant $C_6 = C_6(x)$ such that for every $n \in \mathbb{N}

\[
(5.4) \quad n^\delta P_x^V(\tau_n < \sigma_0) \leq C_6.
\]

Moreover, for each $\varepsilon > 0$ there is a constant $c_3 = c_3(\varepsilon)$ such that for all $n \in \mathbb{N}

\[
(5.5) \quad P_n^V(\sigma_0 > \tau_{c_3 n}) < \varepsilon.
\]

Remark 5.6. In fact, (5.4) will be substantially improved by Lemma 8.1.

Proof. We choose arbitrarily $a \in (1, 2]$ and an $\ell \geq \ell_0$ as in Lemma 5.3 but also such that $a^\ell > x$. We note that it is sufficient to prove the statement for $n$ of the form $[a^n]$. We define stopping times $\beta_i$, $i \in \mathbb{N}$, by

\[
\beta_1 = \inf \{ k > 0 : V_k \geq a^\ell \};
\]

\[
\beta_{i+1} = \inf \{ k > \beta_i : V_k \geq a^{\ell+1} \text{ and } \exists s \in (\beta_i, k) : V_s \leq a^\ell \}.
\]
Lemma 5.1 and the monotonicity of \( V \) with respect to its starting point imply that

\[
P^V_x(\exists r \in [\beta_i, \beta_{i+1}) \mid V_r \geq a^\sigma | \beta_i < \sigma_0)
\leq \frac{h^+_a(\ell + 1) - 1}{h^+_a(u) - 1} + \sum_{k=\ell+1}^{\infty} P^V_x(V_{\beta_i} \geq a^k + a^{2k/3} | \beta_i < \sigma_0) \frac{h^+_a(k + 1) - 1}{h^+_a(u) - 1}
\]

\[
= \frac{h^+_a(\ell + 1) - 1}{h^+_a(u) - 1} \left( 1 + \sum_{k=\ell+1}^{\infty} P^V_x(V_{\beta_i} \geq a^k + a^{2k/3} | \beta_i < \sigma_0) \frac{h^+_a(k + 1) - 1}{h^+_a(u) - 1} \right)
\leq \frac{2(h^+_a(\ell + 1) - 1)}{h^+_a(u) - 1}
\]
supposing, as we may, that \( \ell \) was fixed sufficiently large. Thus,

\[
a^{u_\delta} P^V_x(\sigma_0 > \tau_{c^3}) \leq 2a^{u_\delta} \frac{(h^+_a(\ell + 1) - 1)}{h^+_a(u) - 1} \sum_{i=1}^{\infty} P^V_x(\beta_i < \sigma_0).
\]

The bound (5.3) now follows from noting that \( P^V_x(\beta_i < \sigma_0) \) decays geometrically fast to zero (with a rate which may depend on \( \ell \) but does not depend on \( u \)) and that \( a^{u_\delta}(h^+_a(u) - 1)^{-1} \) is bounded in \( u \) (see Remark 5.4).

To prove (5.5) we notice that by Lemma 5.3 and (5.4) for all \( n > a^\ell \)

\[
P_n(\sigma_0 > \tau_{c^3}) \leq P_n(\sigma_0 > \tau_{c^3}, \sigma_{a^\ell} > \tau_{c^3}) + P_n(\sigma_0 > \tau_{c^3}, \sigma_{a^\ell} < \tau_{c^3})
\]

\[
\leq P_n(\sigma_{a^\ell} > \tau_{c^3}) + \frac{C_6(a^\ell)}{(c_{c^3})^3} = O(c_{c^3}^{-4}).
\]

The constant \( c_3 \) can be chosen large enough to get (5.5) for all \( n \in \mathbb{N} \). \( \square \)

6. PROOF OF (A)

**Proposition 6.1.** There is a constant \( c_4 > 0 \) such that for all \( k, x \in \mathbb{N} \) and \( y \geq 0 \)

\[
P^V_y \left( \sum_{r=1}^{\sigma_0} \mathbb{I} \{ V_r \in [x, 2x) \} > 2xk \right) \leq P^V_y(\rho_0 < \sigma_0)(1 - c_4)^k.
\]

where \( \rho_0 = \inf \{ j \geq 0 \mid V_j \in [x, 2x) \} \).

**Proof.** First, observe that there is a constant \( c > 0 \) such that for all \( x \in \mathbb{N} \)

\[(i) \quad P^V_{2^x}(\sigma_{x/2} < x) > c; \quad (ii) \quad P^V_{x/2}(\sigma_0 < \tau_x) > c.\]

The inequality \((i)\) is an immediate consequence of Lemma 3.1. To prove the second inequality, we fix \( x_0 \in \mathbb{N} \) and let \( x > 2x_0 + 1 \). Then by Corollary 5.5

\[
P^V_{x/2}(\sigma_0 < \tau_x) = P^V_{x/2}(\sigma_0 < \tau_x | \sigma_{x_0} < \tau_2) P^V_{x/2}(\sigma_{x_0} < \tau_x)
\]

\[
\geq (1 - C_6(x_0)x^{-\delta}) P^V_{x/2}(\sigma_{x_0} < \tau_x).
\]

Choosing \( x_0 \) large enough and applying Lemma 5.3 to the last term in the right-hand side we obtain \((ii)\) for all sufficiently large \( x \). Adjusting the constant \( c \) if necessary we can extend \((ii)\) to all \( x \in \mathbb{N} \).

Next, we show that \((i)\) and \((ii)\) imply (6.1) with \( c_4 = c^2 \). Denote by \( \rho_0 \geq 0 \) the first entrance time of \( V \) in \([x, 2x)\) and set

\[
\rho_j = \inf \{ r \geq \rho_{j-1} + 2x \mid V_r \in [x, 2x) \}, \quad j \geq 1.
\]
Notice that for each \( j \geq 1 \), the time spent by \( V \) in \([x, 2x] \) during the time interval \([\rho_j-1, \rho_j] \) is at most \( 2x \). If \( V \) spends more than \( 2xk \) units of time in \([x, 2x] \) before time \( \sigma_0 \) then \( \rho_k < \sigma_0^V \).

Thus,

\[
\tag{6.2}
P_y^V \left( \sum_{r=1}^{\sigma_0} \mathbb{I}_{(V_r \in [x, 2x])} > 2xk \right) \leq P_y^V (\rho_k < \sigma_0) = P_y^V (\rho_k < \sigma_0 | \rho_{k-1} < \sigma_0) P_y^V (\rho_{k-1} < \sigma_0).
\]

Using the strong Markov property, monotonicity with respect to the starting point, and inequalities (i) and (ii) we get

\[
P_y^V (\rho_k < \sigma_0 | \rho_{k-1} < \sigma_0) \leq \max_{x \leq x < 2x} P_z^V (\rho_1 < \sigma_0)
\]

\[
\leq \max_{x \leq x < 2x} \left( P_z^V (\rho_1 < \sigma_0, \sigma_{x/2} < x) + P_z^V (\rho_1 < \sigma_0, \sigma_{x/2} \geq x) \right)
\]

\[
\leq \max_{x \leq x < 2x} \left( P_z^V (\rho_1 < \sigma_0, \sigma_{x/2} < x) P_z^V (\sigma_{x/2} < x) + 1 - P_z^V (\sigma_{x/2} < x) \right)
\]

\[
\leq \max_{x \leq x < 2x} \left( 1 - P_z^V (\sigma_{x/2} < x) (1 - P_z^V (\rho_1 < \sigma_0 | \sigma_{x/2} < x)) \right)
\]

\[
\leq 1 - P_{2x}^V (\sigma_{x/2} < x) (1 - P_{x/2}^V (\rho_0 < \sigma_0))
\]

\[
\leq 1 - P_{2x}^V (\sigma_{x/2} < x) P_{x/2}^V (\tau_x > \sigma_0) \leq 1 - c^2.
\]

Substituting this in (6.2) and iterating in \( k \) gives (6.1). \( \square \)

**Proposition 6.2.** For every \( h > 0 \)

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} n^\delta P_0^V \left( \sum_{i=1}^{\sigma_0} \mathbb{I}_{(V_i < \varepsilon n)} > n h \right) = 0.
\]

**Proof.** Fix \( \varepsilon \in (0, 1) \) and let \( k = k(n, \varepsilon) \) be the smallest integer such that \( 2^k \geq \varepsilon n \). Define intervals \( I_i = [2^{k-i}, 2^{k-i+1}) \), \( i \in \mathbb{N} \), and events

\[
A_i = \left\{ \sum_{r=1}^{\sigma_0} \mathbb{I}_{(V_r \in I_i)} > \frac{h2^{i-1}}{\varepsilon(1+i+1)} |I_i| \right\}.
\]

Intervals \( I_i \) and events \( A_i \) depend on \( n \) but this is not reflected in our notation. Since \( 2^{k-1} < \varepsilon n \) and \( \sum_{i=1}^{k} (i+i+1)^{-1} < 1 \), we have

\[
\left\{ \sum_{i=1}^{\sigma_0} \mathbb{I}_{(V_i < \varepsilon n)} > n h \right\} \subset \bigcup_{i=1}^{k} A_i,
\]

and, therefore,

\[
P_0^V \left( \sum_{i=1}^{\sigma_0} \mathbb{I}_{(V_i < \varepsilon n)} > n h \right) \leq \sum_{i=1}^{k} P(A_i).
\]

Using (6.1) and (6.1) we get

\[
n^\delta P_0^V \left( \sum_{i=1}^{\sigma_0} \mathbb{I}_{(V_i < \varepsilon n)} > n h \right) \leq n^\delta \sum_{i=1}^{k} \left( 1 - c_4 \right) \left[ \frac{2^{1+i-2}}{\varepsilon(1+i+1)} \right] \left[ \frac{1}{2^{i+1}} \right] 2^i \delta,
\]

and this quantity vanishes as \( \varepsilon \to 0 \). \( \square \)
Proof of (A). To obtain (A) we apply (6.4) and Proposition 6.2 to the right-hand side of the following inequality:

\[
n^δP_0^V (σ_0 > n) \leq n^δP_0^V (τ_{εn} < σ_0) + n^δP_0^V (σ_0 > n, τ_{εn} > σ_0) \leq ε^{-δ(εn)}δP_0^V (τ_{εn} < σ_0) + n^δP_0^V \left( \sum_{i=1}^{σ_0} 1\{V_i < εn\} > n \right).
\]

□

7. PROOF OF (B)

We shall need the following fact.

**Proposition 7.1.** For each \( ε > 0 \), there is a constant \( C_7 = C_7(ε) > 0 \) such that

\[
P_n^V (σ_0 > C_7n) < ε \quad \text{for all} \quad n ∈ ℤ.
\]

**Proof.** We have

\[
P_n^V (σ_0 > C_7n) \leq P_n^V (σ_0 > τ_{c_3n}) + P_n^V (σ_0 > C_7n, σ_0 < τ_{c_3n}).
\]

Using (5.5) we can choose \( c_3 > 1 \) so that \( P_n^V (σ_0 > τ_{c_3n}) < ε/2 \) for all \( n ∈ ℤ \). Thus, we only need to estimate the last term. Notice that it is bounded above by the probability that the occupation time of the interval \( (0, c_3n) \) up to the moment \( σ_0 \) exceeds \( C_7n \). The latter can be estimated by Markov inequality:

\[
P_n^V \left( \sum_{r=1}^{σ_0} 1\{V_r < c_3n\} > C_7n \right) \leq (C_7n)^{-1}E_n^V \left( \sum_{r=1}^{σ_0} 1\{V_r < c_3n\} \right).
\]

We claim that the last expectation does not exceed \( 4nc_3/c_4 \) and so we can take \( C_7 > 8c_3/(εc_4) \). Indeed, let \( m \) be the smallest positive integer such that \( 2^m ≥ c_3n \). Then writing the expectation of our non-negative integer-valued random variable as the sum of the probabilities of its tails and using (6.1) to estimate the tails we get

\[
E_n^V \left( \sum_{r=1}^{σ_0} 1\{V_r < c_3n\} \right) \leq \sum_{j=1}^{m} E_n^V \left( \sum_{r=1}^{σ_0} 1\{V_r < 2^{j-1}(2^j)\} \right) \leq \sum_{j=1}^{m} 2j/c_4 \leq \frac{2m+1}{c_4} ≤ \frac{4nc_3}{c_4}.
\]

□

Proof of (B). For every \( α ∈ (0, ε) \) and \( β ∈ (0, 1) \) we have

\[
(7.1) \quad P_{εn}^V \left( \sum_{j=0}^{σ_{αn}-1} V_j > n^2 \right) \leq P_{εn}^V \left( \sum_{j=0}^{σ_{αn}-1} V_j > n^2 \right) \leq P_{εn}^V \left( \sum_{j=0}^{σ_{αn}-1} V_j > (1 - β)n^2 \right) + P_{εn}^V \left( \sum_{j=σ_{αn}}^{σ_{αn}-1} V_j > βn^2 \right).
\]

By Lemma 5.1 for every \( R > 0 \)

\[
(7.2) \quad \lim_{n→∞} P_{εn}^V \left( \sum_{j=0}^{σ_{αn}-1} V_j > Rn^2 \right) = P_{ε}^V \left( \int_0^{σ_α} Y_s \, ds > R \right),
\]

\[
E_n^V \left( \sum_{r=1}^{σ_0} 1\{V_r < c_3n\} \right) \leq \sum_{j=1}^{m} E_n^V \left( \sum_{r=1}^{σ_0} 1\{V_r < 2^{j-1}(2^j)\} \right) \leq \sum_{j=1}^{m} 2j/c_4 \leq \frac{2m+1}{c_4} ≤ \frac{4nc_3}{c_4}.
\]

□

Proof of (B). For every \( α ∈ (0, ε) \) and \( β ∈ (0, 1) \) we have

\[
(7.1) \quad P_{εn}^V \left( \sum_{j=0}^{σ_{αn}-1} V_j > n^2 \right) \leq P_{εn}^V \left( \sum_{j=0}^{σ_{αn}-1} V_j > n^2 \right) \leq P_{εn}^V \left( \sum_{j=0}^{σ_{αn}-1} V_j > (1 - β)n^2 \right) + P_{εn}^V \left( \sum_{j=σ_{αn}}^{σ_{αn}-1} V_j > βn^2 \right).
\]

By Lemma 5.1 for every \( R > 0 \)

\[
(7.2) \quad \lim_{n→∞} P_{εn}^V \left( \sum_{j=0}^{σ_{αn}-1} V_j > Rn^2 \right) = P_{ε}^V \left( \int_0^{σ_α} Y_s \, ds > R \right),
\]

\[
E_n^V \left( \sum_{r=1}^{σ_0} 1\{V_r < c_3n\} \right) \leq \sum_{j=1}^{m} E_n^V \left( \sum_{r=1}^{σ_0} 1\{V_r < 2^{j-1}(2^j)\} \right) \leq \sum_{j=1}^{m} 2j/c_4 \leq \frac{2m+1}{c_4} ≤ \frac{4nc_3}{c_4}.
\]

□
since, as is easily verified, under law \( P_\varepsilon^Y \) the law of \( \int_0^{\sigma_0} Y_s \, ds \) has no atoms. Next, we notice that for all \( x, \beta > 0 \)

\[
(7.3) \quad P_\varepsilon^Y \left( \int_0^{\sigma_0} Y_s \, ds > (1 + \beta)x \right) - P_\varepsilon^Y \left( \int_0^{\sigma_0} Y_s \, ds > \beta x \right) 
\leq P_\varepsilon^Y \left( \int_0^{\sigma_0} Y_s \, ds > x \right) - P_\varepsilon^Y \left( \int_0^{\sigma_0} Y_s \, ds > x \right).
\]

By the strong Markov property and scaling,

\[
P_\varepsilon^Y \left( \int_0^{\sigma_0} Y_s \, ds > \beta x \right) = P_{\alpha}^Y \left( \int_0^{\sigma_0} Y_s \, ds > \beta x \right) = P_1^Y \left( \int_0^{\sigma_0} Y_s \, ds > \beta x \alpha^{-2} \right) \to 0 \quad \text{as } \alpha \to 0.
\]

Letting \( n \to \infty \), then \( \alpha \to 0 \), and finally \( \beta \to 0 \) we obtain from (7.1)–(7.4) that

\[
\lim inf_{n \to \infty} P_{\varepsilon n}^V \left( \sum_{j=0}^{\sigma_0-1} V_j > n^2 \right) \geq P_1^Y \left( \int_0^{\sigma_0} Y_s \, ds > \varepsilon^{-2} \right).
\]

To get the matching upper bound it is enough to show that for every \( \nu > 0 \) and \( \beta \in (0, 1) \) there is an \( \alpha \in (0, \varepsilon) \) such that for all sufficiently large \( n \)

\[
(7.5) \quad P_{\varepsilon n}^V \left( \sum_{j=0}^{\sigma_0-1} V_j > \beta n^2 \right) < 2\nu.
\]

The left-hand side of (7.5) does not exceed

\[
P_{\alpha n}^V (\tau_{c_3 n \alpha} < \sigma_0) + P_{\alpha n}^V \left( \sum_{j=0}^{\sigma_0-1} V_j > \beta n^2, \tau_{c_3 n \alpha} > \sigma_0 \right).
\]

Given \( \nu \), define \( c_3(\nu) \) and \( C_7(\nu) \) as in (5.5) and Proposition 7.1 respectively. Let \( \alpha < \sqrt{\beta/(c_3 C_7)} \).

By (5.5) the first term above is less than \( \nu \) for all \( n > 1/\alpha \). On the set \( \{ \tau_{c_3 n \alpha} > \sigma_0 \} \) the process \( V \) is below \( c_3 n \alpha \) and, thus, the second term is bounded above by \( P_{\alpha n}^V (\sigma_0 > (\beta n)/(c_3 \alpha)) \). By Proposition 7.1 and our choice of \( \alpha \) the latter probability does not exceed \( \nu \). Using relations (7.1)–(7.4) and, again, the absence of atoms for the distribution of \( \int_0^{\sigma_0} Y_s \, ds \) under \( P_1^Y \) for each \( x \geq 0 \), we get the desired upper bound. \( \square \)

8. PROOFS OF (C) AND THEOREM 2.1

First we prove (C) of Lemma 4.1. Then using the approach of Lemma 4.1 we show the convergence claimed in Theorem 2.1.

The next lemma includes (C) as a special case \((k = 0, C_5 = f(0))\).

**Lemma 8.1.** There is a function \( f : \mathbb{N} \cup \{0\} \to (0, \infty) \), such that

\[
\lim_{n \to \infty} n^{\delta} P_k^V (\sigma_0 > \tau_n) = f(k) \quad \text{for each integer } k \geq 0.
\]

We shall need the following proposition.

**Proposition 8.2.** For each \( a \in (1, 2] \) and \( k \geq 0 \)

\[
(8.1) \quad \sum_{j=1}^{\infty} a^{\delta} P_k^V (\sigma_0 > \tau_{a^j}, |\sigma_0 > \tau_{a^{j-1}}| - 1) < \infty.
\]
Proof. By the monotonicity in the initial number of particles we get a lower bound: for all sufficiently large $j$

$$P_k^V(\sigma_0 > \tau_{a^j} \mid \sigma_0 > \tau_{a^{j-1}}) \geq P_{a^{j-1}}^V(\sigma_0 > \tau_{a^j}).$$

For an upper bound we need to take into account the possibility of a large “overshoot”. Let $x = a^{j-1} + a^{\frac{2j-1}{3}}$, then

$$P_k^V(\sigma_0 > \tau_{a^j} \mid \sigma_0 > \tau_{a^{j-1}}) \leq P_k^V(\sigma_0 > \tau_{a^j}) + P_k^V(\sigma_0 > \tau_{a^{j-1}}, V_{\tau_{a^{j-1}}} > x)P_k^V(V_{\tau_{a^{j-1}}} > x \mid \sigma_0 > \tau_{a^{j-1}}) \leq P_k^V(\sigma_0 > \tau_{a^j}) + P_k^V(V_{\tau_{a^{j-1}}} > x \mid \sigma_0 > \tau_{a^{j-1}}).$$

The last probability decays faster than any power of $a^{-j}$ as $j \to \infty$ by Lemma 5.1. Therefore, it is enough to show the convergence of the series

$$\sum_{j=1}^{\infty} |a^{\delta} P_{x_j-1}^V(\sigma_0 > \tau_{a^j}) - 1|,$$

where $x_j = a^j + \epsilon_j$ and $0 \leq \epsilon_j \leq a^\frac{1}{3}$. We have for all sufficiently large $j$ (with $\ell$ chosen appropriately for $a$ as in Lemma 5.3)

$$|P_{x_j-1}^V(\sigma_0 > \tau_{a^j}) - a^{-\delta}| \leq |P_{x_j-1}^V(\sigma_{a^j} > \tau_{a^j}) - a^{-\delta}| + P_{x_j-1}^V(\sigma_0 > \tau_{a^j} > \sigma_{a^j}) \leq |P_{x_j-1}^V(\sigma_{a^j} > \tau_{a^j}) - a^{-\delta}| + P_{x_j-1}^V(\sigma_0 > \tau_{a^j} \mid \tau_{a^j} > \sigma_{a^j}) \leq |P_{x_j-1}^V(\sigma_{a^j} > \tau_{a^j}) - a^{-\delta}| + P_{a^j}^V(\sigma_0 > \tau_{a^j}).$$

By Lemma 5.3

$$\sum_{j=1}^{\infty} |P_{x_{j-1}}^V(\sigma_{a^j} > \tau_{a^j}) - a^{-\delta}| < \infty,$$

and to complete the proof of (8.1) we invoke the bound provided by (5.4) for $x = [a^j].$}

Proof of Lemma 5.1. Fix an arbitrary non-negative integer $k$ and $a \in (1, 2)$. For each $n > a$ there is an $m \in \mathbb{N}$ such that $a^m \leq n < a^{m+1}$. We have

$$a^{m \delta} P_k^V(\sigma_0 > \tau_{a^{m+1}}) \leq n^\delta P_k^V(\sigma_0 > \tau_n) \leq a^{(m+1) \delta} P_k^V(\sigma_0 > \tau_{a^m}).$$

If we can show that

$$(8.2) \quad \lim_{m \to \infty} a^{m \delta} P_k^V(\sigma_0 > \tau_{a^m}) = g(a, k) > 0,$$

for some $g(a, k)$, then

$$0 < a^{-\delta} g(a, k) \leq \liminf_{n \to \infty} n^\delta P_k^V(\sigma_0 > \tau_n) \leq \limsup_{n \to \infty} n^\delta P_k^V(\sigma_0 > \tau_n) \leq a^\delta g(a, k).$$

This implies

$$1 \leq \frac{\limsup_{n \to \infty} n^\delta P_k^V(\sigma_0 > \tau_n)}{\liminf_{n \to \infty} n^\delta P_k^V(\sigma_0 > \tau_n)} \leq a^{2\delta},$$

and we obtain the claimed result by letting $a$ go to 1.
To show (8.2) we set $\ell = \min\{j \in \mathbb{N} \mid a^j > k\}$ and notice that for $m > \ell$

$$a^{\ell \delta} P^V_k (\sigma_0 > \tau_{a^m}) = a^\ell P^V_k (\sigma_0 > \tau_{a^m} \mid \sigma_0 > \tau_{a^{m-1}}) \times a^{(m-\ell)\delta} P^V_k (\sigma_0 > \tau_{a^{m-1}})$$

$$= \ldots = a^{\ell \delta} P^V_k (\sigma_0 > \tau_{a^\ell}) \times \prod_{j=\ell+1}^{m} a^\delta P^V_k (\sigma_0 > \tau_{a^j} \mid \sigma_0 > \tau_{a^{j-1}}).$$

Since all terms in the last product are strictly positive and $a^{\ell \delta} P^V_k (\sigma_0 > \tau_{a^\ell})$ does not depend on $m$, the convergence (8.2) follows from (8.1).

**Proof of Theorem 2.1.** We will show that $\lim_{n \to \infty} n^\delta P^V_0 (\sigma_0 > n) = C_3 C_5$, where $C_3$ and $C_5$ are the same as in Lemma 5.1 and condition (C).

We begin with a lower bound. Fix positive $\varepsilon$ and $\beta \ll \varepsilon$. We have

$$P^V_0 (\sigma_0 > n) \geq P^V_0 (\tau_{e^\varepsilon} < \sigma_0) P^V_{e^\varepsilon} (\sigma_0 > n) \geq P^V_0 (\tau_{e^\varepsilon} < \sigma_0) P^V_{e^\varepsilon} (\sigma_{\beta n} > n).$$

By (C), Lemma 3.1 and scaling (Lemma 3.2 (i))

$$\liminf_{n \to \infty} n^\delta P^V_0 (\sigma_0 > n) \geq C_5 \varepsilon^{-\delta} P^Y_1 (\tau_{\beta/e} > \varepsilon^{-1}).$$

Letting $\beta \to 0$ and then $\varepsilon \to 0$ we obtain via Lemma 3.3 that

$$\liminf_{n \to \infty} n^\delta P^V_0 (\sigma_0 > n) \geq C_3 C_5.$$

The upper bound is slightly more complicated. First, notice that

$$n^\delta P^V_0 (\sigma_0 > n, \tau_{e^\varepsilon} > \sigma_0) \leq n^\delta P^V_0 \left( \sum_{i=1}^{\sigma_0} \mathbb{1}_{\{V_i < e^\varepsilon\}} > n \right).$$

By Proposition 6.2 the right-hand side becomes negligible as $n \to \infty$ and then $\varepsilon \to 0$. Thus, it is enough to estimate $n^\delta P^V_0 (\sigma_0 > n, \tau_{e^\varepsilon} < \sigma_0)$. Let $R \in (1,3/2)$. Then

$$n^\delta P^V_0 (\sigma_0 > n, \tau_{e^\varepsilon} < \sigma_0) \leq n^\delta P^V_0 (\sigma_0 > \tau_{e^\varepsilon} > (R-1)n)$$

$$+ n^\delta P^V_0 (V_{\tau_{e^\varepsilon}} > R e^n, \tau_{e^\varepsilon} < \sigma_0)$$

$$+ n^\delta P^V_0 (\sigma_0 - \tau_{e^\varepsilon} > (2 - R)n, V_{\tau_{e^\varepsilon}} \leq R e^n).$$

By Proposition 6.2 the first term on the right-hand side vanishes for every fixed $R > 1$ when we let $n \to \infty$ and then $\varepsilon \to 0$. By Lemma 5.1 the lim sup $n \to \infty$ of the second term is zero. Thus it will be sufficient to bound the last term. For $\beta \ll \varepsilon$ let us define $\sigma_{\beta n}^\varepsilon$ to be the the first time after $\tau_{e^\varepsilon}$ that $V$ falls below $\beta n$. Then the last term is bounded above by

$$n^\delta \left( P^V_0 (\sigma_0 - \sigma_{\beta n}^\varepsilon > (R-1)n) + P^V_0 (\sigma_{\beta n}^\varepsilon - \tau_{e^\varepsilon} > (3-2R)n, V_{\tau_{e^\varepsilon}} \leq R e^n) \right)$$

$$\leq \left( P_{\beta n} (\sigma_0 > (R-1)n) + P_{Re^n}^V (\sigma_{\beta n}^\varepsilon > (3-2R)n) \right) n^\delta P^V_0 (\tau_{e^\varepsilon} < \sigma_0).$$

Taking lim sup $n \to \infty$ and then letting $\beta \to 0$ we obtain (by Proposition 7.1, Lemma 3.1 and (C)) the following upper bound for lim sup $n \to \infty$ $n^\delta P^V_0 (\sigma_0 - \tau_{e^\varepsilon} > (2 - R)n, V_{\tau_{e^\varepsilon}} \leq R e^n)$

$$C_5 \varepsilon^{-\delta} P^Y_{Re^n} (\tau_0 > (3 - 2R)) = C_5 \varepsilon^{-\delta} P^Y_1 (\tau_0 > \varepsilon^{-1}(3 - 2R)/R).$$

As $\varepsilon \to 0$ and then $R \to 1$, the latter expression converges by Lemma 3.3 to $C_5 C_3$. This completes the proof.
9. Proof of Theorem 1.3

Let $\delta > 2$. By (2.1) and (2.2), it is enough to show that as $n \to \infty$
\begin{equation}
\frac{2 \sum_{k=0}^{n} D_{n,k} - (v^{-1} - 1)n}{n^{2/\delta}} \overset{d}{=} \frac{2 \sum_{j=0}^{n} V_j - (v^{-1} - 1)n}{n^{2/\delta}}
\end{equation}
converges in distribution to $Z_{\delta/2,b}$ for some $b > 0$. Define the consecutive times when $V_j = 0$,
$$
\sigma_{0,0} = 0, \quad \sigma_{0,i} = \inf\{j > \sigma_{0,i-1} | V_j = 0\}, \quad i \in \mathbb{N},
$$
the total progeny of $V$ over each lifetime, $S_i = \sum_{j=0}^{\sigma_{0,i-1}} V_j$, $i \in \mathbb{N}$, and the number of renewals
up to time $n$, $N_n = \max\{i \geq 0 | \sigma_{0,i} \leq n\}$. Then $(\sigma_{0,i} - \sigma_{0,i-1}, S_i)_{i \geq 1}$ are i.i.d. under $P_0^V$.
Moreover, $\sigma_{0,i} - \sigma_{0,i-1} \overset{d}{=} \sigma_0^V$ and $S_i \overset{d}{=} S^V$, $i \in \mathbb{N}$. By Theorem 2.2 the distribution of $S^V$ is in
the domain of attraction of the law of $Z_{\delta/2,b}$ for some $b > 0$ (see, for example [9], Chapter 2, Theorem 7.7.) Since by Theorem 2.1 (in fact, the upper bound (A) is sufficient) the second moment
of $\sigma_0^V$ is finite, it follows from standard renewal theory (see, for example, [9], Theorems II.5.1 and II.5.2) that
$$
\frac{N_n}{n} \overset{a.s.}{\to} \lambda := (E_0^V \sigma_0)^{-1},
$$
and for each $\varepsilon > 0$ there is $c_5 > 0$ such that $P_0^V(|N_n - \lambda n| > c_5 \sqrt{n}) < \varepsilon$ for all sufficiently large $n$.
Using the fact that $T_n/n \to v^{-1}$ a.s. as $n \to \infty$ and relations (2.1) and (9.1) we get that $E_n^V S_i = (v^{-1} - 1)/(2\lambda)$.

Proof of part (i). Let $\delta \in (2,4)$. We have
$$
\frac{\sum_{j=0}^{n} V_j - (v^{-1} - 1)n/2}{n^{2/\delta}} = \frac{\sum_{i=1}^{N_n} (S_i - E_0^V S_i)}{n^{2/\delta}} + E_0^V S_1 \frac{N_n - \lambda n}{n^{2/\delta}} + \sum_{j=\sigma_{0,N_n}}^{n} \frac{V_j}{n^{2/\delta}}.
$$
By Theorem I.3.2, [9], the first term converges in distribution to $Z_{\delta/2,5}$. The second term
converges to zero in probability by the above mentioned facts from renewal theory. The last term is bounded above by $S_{N_n+1}/n^{2/\delta}$, which converges to zero in probability. This finishes the proof of (1.7), which immediately gives (1.8) with $X_n$ replaced by $\inf_{1 \leq n} X_i$, since \{$\inf_{1 \leq n} X_i < m$\} = \{$T_m > n$\}.

Next we show (1.8) with $X_n$ replaced by $\inf_{1 \leq n} X_i$. The proof is the same as in, for example, [2], p. 849. We observe that for all $m,n,p \in \mathbb{N}$
$$
\{\sup_{i \leq n} X_i < m\} \subset \{\inf_{i \geq n} X_i < m\} \subset \{\sup_{i \leq n} X_i < m + p\} \cup \{\inf_{i \geq T_m + p} X_i < m\}.
$$

The following lemma completes the proof of part (i).

Lemma 9.1.
$$
\lim_{k \to \infty} \sup_{n \geq 1} P_0 \left( \inf_{i \geq T_n} X_i < n - k \right) = 0.
$$

We postpone the proof of this lemma until the end of the section.

Proof of part (ii). Let $\delta = 4$. Theorem 2.2 implies that the distribution of $S^V$ is in the domain
of attraction of the normal distribution (III.1, XVII.5). Norming constants are easily computed to be (see [11], XVII.5, (5.23) with $C = 1$) $\sqrt{C_2 n \log n}$. The constant $b$, which appears in
the statement is equal to $C_2/2$. Relations (1.9) and (1.10) follow in the same way as for part (i).

Proof of Lemma 9.1. Let $P_{n,k} := P_0 (\inf_{i \geq T_n} X_i < n - k)$.

Step 1. The supremum over $n \geq 1$ can be reduced to the maximum over $n \in \{1,2,\ldots,k\}$:
$$
\sup_{n \geq 1} P_{n,k} = \max_{1 \leq n \leq k} P_{n,k}.
$$
Indeed, consider \( P_{k,k} \) and \( P_{m+k,k} \) for \( m \geq 1 \). The corresponding events \( \{ \inf_{t \geq T_k} X_i < 0 \} \) and \( \{ \inf_{t \geq T_{m+k}} X_i < m \} \) depend on the behavior of the process only at times when \( X_i \) is in \([0, \infty)\) and \([m, \infty)\) respectively. But at times \( T_0 \) and \( T_m \) the walk is at 0 and \( m \) respectively, and the distributions of the environments starting from the current point to the right of it are the same under \( P_0 \). We conclude that \( P_{k,k} = P_{m+k,k} \). This is essentially the content of Lemma 10 from [19]. The proof does not use the positivity of cookies so it can be applied here.

Step 2. We list four elementary properties of \( \{ P_{n,k} \} \), \( n, k \geq 1 \).

(a) \( P_{n,k} \geq P_{n,k+m} \) for all \( 1 \leq n \leq k \) and \( m \geq 0 \);
(b) \( P_{n,k+m} \) converges to 0 as \( m \to \infty \) for each \( k \geq n \geq 1 \);
(c) \( P_{n,k} \geq P_{n+m,k+m} \) for all \( n \leq k \) and \( m \geq 0 \);
(d) \( P_{n+m,k+m} \) converges to 0 as \( m \to \infty \) for each \( k \geq n \geq 1 \).

Inequality (a) is obvious. Part (b) follows from the transience of \( X \). Namely, \( \inf_{t \geq T_n} X_i > -\infty \) a.s. but \( n - (k + m) \to -\infty \) as \( n \to \infty \). Inequality (c) is also obvious: since \( T_n < T_{n+m} \) we have
\[
\{ \inf_{t \geq T_n} X_i < n - k \} \subseteq \{ \inf_{t \geq T_{n+m}} X_i < n - k \} = \{ \inf_{t \geq T_{n+m}} X_i < (n + m) - (k + m) \}.
\]
The convergence in (d) again follows from the transience: \( X_i \to \infty \) as \( i \to \infty \) a.s. implies that \( \inf_{t \geq T_{n+m}} X_i \to \infty \) as \( m \to \infty \) a.s. but \( n - (k + m) \to -\infty \) as \( m \to \infty \) stays constant.

Step 3. Take any \( \varepsilon > 0 \) and using (d) choose an \( m \) so that \( P_{m,m} \leq \varepsilon \). Properties (a) and (c) imply that \( P_{n,n+1} \leq \varepsilon \) for all \( i \geq 0 \) and \( n \geq m \). Using (b), for \( n = 1, 2, \ldots, m - 1 \) choose \( k_n \) so that \( P_{n,k_n} \leq \varepsilon \). Let \( K = \max_{1 \leq n \leq m} k_n \) (naturally, we set \( k_m = m \)). Then \( P_{n,k} \leq \varepsilon \) for all \( n \leq k \) and \( k \geq K \) that is \( \max_{1 \leq n \leq k} P_{n,k} \leq \varepsilon \) for all \( k \geq K \). \( \square \)

Remark 9.2. Theorem 2.1 and Theorem 2.2 imply Theorem 1.4 for general cookie environments satisfying conditions (A1) and (A2). The proof is the same as in Section 6 of [2] and uses Lemma 9.1.

Appendix A. Proofs of technical results

We shall need the following simple lemma.

Lemma A.1. Let \( (\xi_i)_{i \in \mathbb{N}} \) be i.i.d. \( \text{Geom}(1/2) \) random variables. There exists a constant \( c_0 > 0 \) such that for all \( x, y \in \mathbb{N} \)
\[
P \left( \sum_{i=1}^{x} (\xi_i - 1) \geq y \right) \leq e^{-c_0 y^2/x} \vee e^{-c_0 y}.
\]
Proof. Let \( \varphi(t) = \log E e^{t(\xi - 1)} = -t - \log(2 - e^t), \quad t \in [0, \log 2) \). Then \( \varphi'(0) = 0 \) and there is a constant \( C > 0 \) such that \( \varphi(t) \leq Ct^2 \) for all \( t \in [0, (\log 2)/2] \). By Chebyshev’s inequality, for each \( t \in [0, (\log 2)/2] \)
\[
\log P \left( \sum_{i=1}^{x} (\xi_i - 1) \geq y \right) \leq x \varphi(t) - yt \leq Cxt^2 - yt,
\]
and, therefore,
\[
\log P \left( \sum_{i=1}^{x} (\xi_i - 1) \geq y \right) \leq - \max_{t \in [0, (\log 2)/2]} (yt - Cxt^2)
\]
\[
\leq - \frac{1}{4} \min \left\{ \frac{y^2}{Cx}, y \log 2 \right\}.
\]
This gives the desired inequality with \( c_0 = \frac{1}{4} \min \{ C^{-1}, \log 2 \} \). \( \square \)
Proof of Lemma [5.1] We shall prove part (i). The proof of part (ii) is very similar and is omitted.

To prove part (i) it is enough to show the existence of \( c_1, c_2 > 0 \) such that \([5.1]\) holds for all \( x \geq 2M + 1 \) and \( y \geq 6M \). The extension to all \( y \geq 0 \) is done simply by replacing the constant \( c_1 \) with

\[
\max \left\{ c_1, \left( e^{-c_2(6M)^2/(2M+1)} + e^{-6Mc_2} \right)^{-1} \right\},
\]

since the left hand side of \([5.1]\) is at most 1 and the right hand side of \([5.1]\) is increasing in \( x \) and decreasing in \( y \). We have

\[
P_z(V_{\tau_x} > x + y, \tau_x < \sigma_0) = \sum_{n=1}^{\infty} P_z(V_{\tau_x} > x + y, \tau_x = n, \tau_x < \sigma_0)
\]

\[
= \sum_{n=1}^{\infty} P_z(V_n > x + y, V_n \geq x, \tau_x = n, \tau_x < \sigma_0)
\]

\[
= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} P_z(V_n > x + y, V_n \geq x, V_{n-1} = r, 0 < V_j < x, j \in \{1, \ldots, n - 2\})
\]

\[
= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} P_z(V_n > x + y, V_n \geq x, V_{n-1} = r)
\]

\[
\times P_z(V_n \geq x, V_{n-1} = r, 0 < V_j < x, j \in \{1, \ldots, n - 2\})
\]

\[
= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} P_z(V_n > x + y | V_{n-1} = r) \frac{P_z(\tau_x < \sigma_0, \tau_x = n, V_{n-1} = r)}{P_z(V_n \geq x | V_{n-1} = r)}
\]

\[
= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{P_z(V_1 > x + y)}{P_z(V_1 \geq x)} P_z(\tau_x < \sigma_0, \tau_x = n, V_{n-1} = r)
\]

\[
\leq \max_{0 \leq r < x} \frac{P_z(V_1 > x + y)}{P_z(V_1 \geq x)} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} P_z(\tau_x < \sigma_0, \tau_x = n, V_{n-1} = r)
\]

\[
= \max_{0 \leq r < x} \frac{P_z(V_1 > x + y)}{P_z(V_1 \geq x)} P_z(\tau_x < \sigma_0).
\]

Thus,

\[
\max_{0 \leq z < x} P_z(V_{\tau_x} > x + y | \tau_x < \sigma_0) \leq \max_{0 \leq z < x} \frac{P_z(V_1 > x + y)}{P_z(V_1 \geq x)}.
\]

To estimate the last ratio we recall that \( V_1 \) is the sum of the number of offspring produced by each of \( z \) particles and by the immigrant particle. The offspring distribution of at most \( M \) particles can be affected by the cookies. For notational convenience we shall use \( \xi_j, j = 1, 2, \ldots, z + 1 \), to denote the number of offspring of the \( j \)-th particle. All that was said in Section 2 about \( \xi_m^{(k)} \) applies now to to \( \xi_m, m = 1, 2, \ldots, M \), and \( \xi_m, m > M \), are just i.i.d. Geom(1/2) random variables. Abbreviate \( \xi_j - 1 \) by \( \xi'_j, j \in \mathbb{N} \). With this notation, \( V_1 = z + 1 + \sum_{m=1}^{z+1} \xi'_m \). Let \( \mathcal{G}_0 = \{\emptyset, \Omega\} \), \( \mathcal{G}_n, n \in \mathbb{N} \), be the \( \sigma \)-algebra generated by \( (\xi_m)_{m \leq n} \), and

\[
N_x = \inf \left\{ n \geq 1 \left| z + 1 + \sum_{m=1}^{n} \xi'_m \geq x \right. \right\}.
\]
Then \( \{ V_1 > x + y \} \subset \{ V_1 \geq x \} \subset \{ N_x \leq z + 1 \} \) and

\[
\frac{P_z(V_1 > x + y)}{P_z(V_1 \geq x)} = \frac{P_z(V_1 > x + y)}{P_z(V_1 \geq x)} = \frac{P_z(V_1 > x + y)}{E_z(1_{\{ N_x \leq z + 1 \}} P_z(V_1 \geq x \mid G_{N_x})))}
\]

The proof of part (i) will be complete as soon as we show that

(a) the is a constant \( c_7 > 0 \) such that \( P_z(V_1 \geq x \mid G_{N_x}) \geq c_7 \) for all \( 0 \leq z < x \) and \( x \geq 2M + 1 \);

(b) there is a constant \( c_8 > 0 \) such that for all \( x \geq 2M + 1 \) and \( y \geq 6M \)

\[
(A.1) \quad P_z(V_1 > x + y \mid N_x \leq z + 1) \leq \left( e^{-cy} \vee e^{-cy^2/x} \right).
\]

Proof of (a). Consider two cases: (i) \( z \geq M \) and (ii) \( x - z \geq M + 1 \). In case (i) at least one of \( \xi_m, 1 \leq m \leq z + 1 \), has a Geom(1/2) distribution. The distribution of each \( \xi_m \) is supported on the non-negative integers, and, trivially, \( \sum_{m=1}^{M} \xi_m \geq -M \). Therefore, on the event \( \{ N_x \leq z + 1 \} \)

\[
(A.2) \quad P_z(V_1 \geq x \mid G_{N_x}) \geq P_z\left( \sum_{m=N_x+1}^{z+1} \xi_m \geq 0 \mid G_{N_x} \right) \geq \min_{n \geq M+1} P\left( \sum_{m=M+1}^{n} \xi_m \geq M \right) \geq c_7 > 0,
\]

since for all \( n \geq M + 1 \) the probabilities under the minimum sign are strictly positive and by the Central Limit Theorem

\[
\lim_{n \to \infty} P\left( \sum_{m=M+1}^{n} \xi_m \geq M \right) = \frac{1}{2}.
\]

We recall that when \( N_x = z + 1 \) the first sum in \( (A.2) \) is empty and the probability above is equal to 1, so the lower bond holds for this case as well.

Case (ii) is even simpler. We have \( x - z \geq M + 1 \). This implies that all cookies at site 0 have already been used for offspring of the first \( N_x \) particles. Therefore, the sum \( \sum_{m=N_x+1}^{z+1} \xi_m \) (when it is not zero) is equal to the non-trivial sum of centered Geom(1/2) random variables. Therefore, in case (ii) we again get \( (A.2) \) where the rightmost \( M \) is replaced by 0.

Proof of (b). Observe that

\[
\{ V_1 > x + y \} \subset \left\{ \sum_{m=N_x+1}^{z+1} \xi_m \geq y/2, N_x \leq z + 1 \right\} \cup \left\{ z + 1 + \sum_{m=1}^{N_x} \xi_m - x \geq y/2 \right\}.
\]

Using the assumption \( y \geq 6M \) and Lemma A.1 we get

\[
P_z\left( \sum_{m=N_x+1}^{z+1} \xi_m \geq y/2, N_x \leq z + 1 \right) = E_z\left( 1_{\{ N_x \leq z + 1 \}} P_z\left( \sum_{m=N_x+1}^{z+1} \xi_m \geq y/2 \mid G_{N_x} \right) \right)
\]

\[
\leq \max_{1 \leq n \leq x} P_z\left( \sum_{m=M+1}^{M+n} \xi_m \geq y/2 - M \right) P_z(N_x \leq z + 1)
\]

\[
\leq \max_{1 \leq n \leq x} P_z\left( \sum_{m=M+1}^{M+n} \xi_m \geq y/3 \right) P_z(N_x \leq z + 1)
\]

\[
\leq \left( e^{-cy^2/(9x)} \vee e^{-cy/3} \right) P_z(N_x \leq z + 1).
\]
Finally, we estimate the probability of the second set:

$$P_z \left( z + 1 + \sum_{m=1}^{N_x} \xi'_m - x \geq y/2, N_x \leq z + 1 \right)$$

$$= \sum_{n=1}^{z+1} \sum_{m=1}^{N_x} P_z \left( \sum_{m=1}^{n} \xi'_m \geq y/2 + x - z - 1, N_x = n \right)$$

$$= \sum_{n=1}^{z+1} \sum_{\ell=1-n}^{x-z-2} P_z \left( \sum_{m=1}^{n} \xi'_m \geq y/2 + x - z - 1, N_x = n, \sum_{m=1}^{n-1} \xi'_m = \ell \right)$$

$$= \sum_{n=1}^{z+1} \sum_{\ell=1-n}^{x-z-2} P_z \left( \sum_{m=1}^{n} \xi'_m \geq y/2 + x - z - 1, N_x = n, \sum_{m=1}^{n-1} \xi'_m = \ell \right) \times P_z \left( N_x = n, \sum_{m=1}^{n-1} \xi'_m = \ell \right)$$

$$= \sum_{n=1}^{z+1} \sum_{\ell=1-n}^{x-z-2} P_z \left( \xi_n \geq y/2 + x - z - \ell, N_x = n, \sum_{m=1}^{n} \xi'_m = \ell \right) \times P_z \left( N_x = n, \sum_{m=1}^{n-1} \xi'_m = \ell \right)$$

$$\leq 2^{-y/3} P_z (N_x \leq z + 1).$$

This finishes the proof. □

Proof of part (ii) of Lemma 5.2. Let $s \in C_0^\infty ([0, \infty))$ be a non-negative function such that $s(x) = x^s$ on $(2/(3a), 3a/2)$. Fix an $n$ such that $a^{n-1} > M$ and define the process $U^n := (U^n_k)_{k \geq 0}$ by

$$U^n_k = s \left( \frac{V_k \wedge n}{a^n} \right).$$

We shall show that when $n$ is large $U^n$ is close to being a martingale (with respect to its natural filtration $(\mathcal{F}_k)_{k \geq 0}$). $U^n$ is just a discrete version of the martingale used in the proof of Lemma 3.2.

On the event $\{ \gamma > k \}$ we have

$$E(U^n_{k+1} \mid \mathcal{F}_k) = E \left( s \left( \frac{V_{k+1}}{a^n} \right) \mid \mathcal{F}_k \right) = E \left( s \left( \frac{V_k + V_{k+1} - V_k}{a^n} \right) \mid \mathcal{F}_k \right),$$

and

$$E(U^n_{k+1} \mid \mathcal{F}_k) - U^n_k =

E \left[ s' \left( \frac{V_k}{a^n} \right) \frac{V_{k+1} - V_k}{a^n} \mid \mathcal{F}_k \right] + \frac{1}{2} E \left[ s'' \left( \frac{V_k}{a^n} \right) \frac{(V_{k+1} - V_k)^2}{a^{2n}} \mid \mathcal{F}_k \right] + r^n_k =

- \frac{\delta - 1}{a^n} s' \left( \frac{V_k}{a^n} \right) + \frac{1}{2} E \left[ s'' \left( \frac{V_k}{a^n} \right) \frac{(V_{k+1} - V_k)^2}{a^{2n}} \mid \mathcal{F}_k \right] + r^n_k.$$
By part (i), we obtain that
\[ \zeta^{(k)} = \zeta^{(k)} - 1 \] and \[ \zeta^{(k)} = \zeta^{(k)} - 1. \] This allows us to pass to the limit as \( k \to \infty \). \( \zeta \) is the error, which we shall estimate later. By (3.1), the second term on the right hand side of the above equality is equal to
\[ \frac{1}{2a^{2n}} s'' \left( \frac{V_k}{a^n} \right) E \left[ \left( 1 + \sum_{m=1}^{M} \xi_m^{(k)} + \sum_{m=1}^{M} \xi_m^{(k)'r} \right)^2 \left| \mathcal{F}_k \right. \right], \]
where \( \xi_m^{(k)} = \xi_m^{(k)} - 1 \) and \( \xi_m^{(k)} = \xi_m - 1. \) Since \( \sum_{m=1}^{M} \xi_m^{(k)}r \) is independent from all \( \xi_m^{(k)}r \), \( m \geq 1 \), and \( V_k \), the last formula reduces to
\[ \frac{1}{2a^{2n}} s'' \left( \frac{V_k}{a^n} \right) \left( 2(V_k - M + 1) + E \left[ \left( 1 + \sum_{m=1}^{M} \xi_m^{(k)} \right)^2 \right] \right). \]
Using the fact that \( x s''(x) + (1 - \delta)s'(x) = 0 \) for \( x \in (2/(3a), 3a/2) \), we get that on the event \( \{ \gamma > k \} \)
\[ E(U^n_{k+1} \mid \mathcal{F}_k) - U^n_k = \frac{1}{a^{2n}} s'' \left( \frac{V_k}{a^n} \right) \left( 1 - M + \frac{1}{2} E \left[ \left( 1 + \sum_{m=1}^{M} \xi_m^{(k)} \right)^2 \right] \right) + r^n_k. \]
The first term on the right hand side is bounded in absolute value by \( K_1/a^{3n} \) for some constant \( K_1 \). Thus it remains to estimate \( r^n_k \). By Taylor’s expansion \( r^n_k \) is bounded by
\[ \frac{1}{6} ||s'''||_\infty E \left[ \left( \frac{V_k}{a^n} \right)^3 \left| \mathcal{F}_k \right. \right] \leq \frac{1}{6} ||s'''||_\infty \left( E \left[ \left( \frac{V_k}{a^n} \right)^4 \left| \mathcal{F}_k \right. \right] \right)^{1/4}. \]
Writing again the difference \( V_{k+1} - V_k \) in terms of geometric random variables, using independence of \( \sum_{m=1}^{M} \xi_m^{(k)}r \) from all \( \xi_m^{(k)}r \), \( m \geq 1 \), and the fact that \( V_k < a^{n+1} \) on \( \{ \gamma > k \} \) we find that
\[ E \left[ \left( \frac{V_k}{a^n} \right)^4 \left| \mathcal{F}_k \right. \right] \leq \frac{K_2}{a^{2n}}, \]
and, therefore, \( |r^n_k| \leq K_3/a^{3n/2}. \) Let \( R^n_0 = 0 \) and for \( k \geq 1 \) set
\[ R^n_k = \sum_{j=1}^{k+\gamma} \frac{1}{a^{2n}} s'' \left( \frac{V_j}{a^n} \right) \left( 1 - M + \frac{1}{2} E \left[ \left( 1 + \sum_{m=1}^{M} \xi_m^{(j)} \right)^2 \right] \right) + r^n_j. \]
Then \( U^n_k - R^n_k \) is a martingale with the initial value \( U^n_0 \). Our bounds on the increments of the process \( (R^n_k)_{k \geq 0} \) and Proposition A.2 below imply that
\[ E|R^n_\gamma| \leq \frac{K_4}{a^{3n/2}} E\gamma \leq \frac{K_5}{a^{n/2}}. \]
This allows us to pass to the limit as \( k \to \infty \) and conclude that \( U^n_0 = E U^n_\gamma - E R^n_\gamma \). Thus,
\[ U^n_0 - \frac{K_5}{a^{n/2}} \leq E U^n_\gamma \leq P \left( V_\gamma \in \left[ a^{n+1}, a^{n+1} + a^{2(n-1)/3} \right] \right) s(a + a^{-(n+2)/3}) \]
\[ + P \left( V_\gamma \in \left( a^{n-1} - a^{2(n-1)/3}, a^{n-1} \right] \right) s(a^{1}) \]
\[ + E \left( U^n_\gamma \mathbb{1}_{\left( d(V_\gamma, (a^{n-1}, a^{n+1})) \geq a^{2(n-1)/3} \right)} \right) . \]
By part (i), we obtain that
\[ P(V_\gamma \geq a^{n+1}) a^{\delta} + P(V_\gamma \leq a^{n-1}) a^{-\delta} \geq U^n_0 - K_6/a^{n/3}. \]
Similarly we get
\[ P(V_\gamma \geq a^{n+1})a^{\delta} + P(V_\gamma \leq a^{n-1})a^{-\delta} \leq U_0^n + K_7/a^{n/3}. \]

This completes the proof. \(\square\)

**Proposition A.2.** There exists \(C_8 \in (0, \infty)\) so that for all \(x > 0\),
\[
\sup_{x \leq y \leq 2x} P^V_y \left( \sigma_{x/2}^V > n x \right) < C_8 x.
\]

**Proof.** By the usual compactness considerations and Lemma 3.1 there exists \(c > 0\) such that \(P^V_y(\sigma_{x/2}^V < x) > c\) for all \(x > 0\) and \(y \in [x, 2x]\). From this and the Markov property applied to successive re-entries to the interval \([x, 2x]\) (see the proof of Proposition 6.1 for details), we obtain
\[
P^V_y \left( \sigma_{x/2}^V \sum_{r=0}^{\sigma_{x/2}^V} \mathbb{1}_{V_r \in [x, 2x]} > nx \right) \leq (1 - c)^n,
\]
and the result follows. \(\square\)

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