Spectral theory of dynamical systems as
diffraction theory of sampling functions

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We consider topological dynamical systems over $\mathbb{Z}$ and, more generally, locally compact, $\sigma$-compact abelian groups. We relate spectral theory and diffraction theory. We first use a recently developed general framework of diffraction theory to associate an autocorrelation and a diffraction measure to any $L^2$-function over such a dynamical system. This diffraction measure is shown to be the spectral measure of the function. If the group has a countable basis of the topology one can also exhibit the underlying autocorrelation by sampling along the orbits. Building on these considerations we then show how the spectral theory of dynamical systems can be reformulated via diffraction theory of function dynamical systems. In particular, we show that the diffraction measures of suitable factors provide a complete spectral invariant.

Introduction

During the last three decades, the mathematical theory of aperiodic order has become a field of substantial interest. This is not the least due to the discovery (honored with a Noble Prize in Chemistry) of certain materials - later known as quasicrystals - exhibiting this form of order [26]. The discovery of such substances came as a complete surprise to physicists and materials scientists alike. These materials are characterized by their remarkable diffraction properties: They exhibit pure point diffraction (indicating long range order) and at the same time their diffraction patterns exhibit symmetries which are incompatible with a lattice structure. Hence, their structure exhibits long range aperiodic order.

The mathematical study of diffraction of quasicrystals and, more generally, the mathematical study of aperiodic order has profited tremendously from ideas and methods from dynamical systems and stochastic processes, see e.g. the survey collections [4, 16, 3] and the monograph [2]. On the other hand, recent years have also seen a flow of ideas in the other direction, i.e. from the study of aperiodic order to dynamical systems. In fact, inspired by diffraction theory for quasicrystals, a characterization for pure discrete spectrum is given for general dynamical systems in [18]. Similarly, the proof of recent results of regularity of tame systems was paved by considerations on certain models of aperiodic order [13].

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Another instance of a flow of ideas from the study of aperiodic order to the treatment of general dynamical systems concerns spectral theory of dynamical systems. This is the topic of the present article. Specifically, it is possible to consider the spectral measures arising in the study of dynamical systems as diffraction measures. Certainly, this will not come as a surprise to the experts. In fact, for special systems an indication of this is already given in [7]. However, it does not seem to be discussed explicitly and in any form of detail for general dynamical systems in the existing literature. As this point of view may be useful for further developments, we present a rather complete discussion here. In this discussion the connection between spectral theory and diffraction theory is investigated along two different (but related) lines.

The first line is provided in the first part of the article starting with Section 1. There, we show that the framework for diffraction theory developed in [20] is general enough to not only cover the ‘usual’ situations treated in diffraction theory but also allows one to recover the spectral measures as diffraction measures. As a by-product, we obtain in a rather simple way a structural understanding of (generalizations of) recent results of [17]. These considerations can be cast in the framework of dynamical systems over general locally compact, $\sigma$-compact abelian groups. To ease the presentation and as this is a particularly relevant case, we have decided to first present the case of dynamical systems over the group of integers in Section 1. Subsequently, we consider the case of general locally compact, $\sigma$-compact abelian groups in Section 2. Most of the considerations concerning the integers can also be adapted to treat actions of the semigroup of natural numbers rather than the integers, see also [24] for a treatment of related problems on one-sided subshifts. We leave the details to the interested reader.

Diffraction theory was originally developed for dynamical systems of point sets and generalizations thereof. In this context, the most general framework is provided by the diffraction theory for measure dynamical systems developed in [5, 21]. The considerations discussed above show that on the formal level diffraction theory for measure dynamical systems and spectral theory of general dynamical systems can be treated on the same footing. This also begs the question whether there is an intrinsic connection between diffraction theory for measure dynamical systems and spectral theory for general dynamical systems. An affirmative answer to this question constitutes the second line of connecting diffraction theory and spectral theory. The corresponding discussion is given in the second part of this article. This starts in Section 3 and builds up on the first part of the article. In Section 3 we recall basic the diffraction theory for measure dynamical systems following [5]. Moreover, we introduce a special class of measure dynamical systems, termed function dynamical systems, which are of particular relevance for our subsequent considerations. We then show in Section 4 that any dynamical system has canonical factors which are function dynamical systems and that the diffraction of these function dynamical systems encodes the spectral theory of the original system. On the structural level this is our main result. It may be summarized as follows (compare Theorem 4.3 and Proposition 4.6 for precise statements):

**Result.** The spectral theory of a general dynamical systems is the diffraction theory of those of its factors which are function dynamical systems.

We can use this to show for any dynamical system that the diffraction measures of its function dynamical system factors form a complete spectral invariant (Theorem 4.5) and so do the diffraction measures of its translation bounded measure dynamical system factors (Corollary 4.7). This type of result has so far only been known for a very restricted class of measure dynamical systems, i.e. Delone dynamical systems of finite local complexity [7]. Beyond the structural
insights, our results may also be of interest for specific questions. In fact, they allow us to
study arbitrary dynamical systems via methods developed for the study of measure dynamical
systems. As a concrete application we provide in Section 5 a criterion for discrete spectrum
based on recent investigations of measure dynamical systems carried out in [27]. This can be
seen as a variant of a well-known criterion for \(\mathbb{Z}\).

We also point out that our results give that any spectral measure (of a continuous function)
is a diffraction measure. Hence, they allow one to generate - in a perfectly natural way -
interesting examples of diffraction measures.

The preceding results are set in the topological category in that they deal with continuous
functions and factor maps. It is also possible to deal with their analogues in a measurable
setting. This is discussed in the Section 6.

Finally, for the convenience of the reader we include in an appendix a short review of basics
of the diffraction theory developed in [20].

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1. Dynamical systems over \(\mathbb{Z}\)

In this section, we first introduce our main actors. These are dynamical systems and their
spectral theory as well as diffraction theory. Throughout the article all dynamical systems are
topological dynamical systems.

Diffraction theory has been developed in various contexts in various degrees of generality in
the last decades, see e.g. the survey [3] for a gentle introduction. Here, we follow the most
general framework given in [20] (see Appendix as well). In fact, the framework of [20] can be
slightly simplified for actions of the group of integers and this is the setting we present. One of
these simplifications is that we can deal with functions throughout and do not have to consider
measures. Having conveniently set up the framework, the derivation of the main results is then
straightforward.

Throughout we denote the group of integers by \(\mathbb{Z}\) and the vector space of functions on \(\mathbb{Z}\)
with finite support by \(C_c(\mathbb{Z})\). For any \(n \in \mathbb{Z}\) the characteristic function of \(\{n\} \subset \mathbb{Z}\) is denoted
by \(1_n\). The convolution between \(h : \mathbb{Z} \to \mathbb{C}\) and \(\varphi \in C_c(\mathbb{Z})\) is the function

\[
h \ast \varphi : \mathbb{Z} \to \mathbb{C}, n \mapsto \sum_{k \in \mathbb{Z}} h(n-k)\varphi(k).
\]

For \(\varphi \in C_c(\mathbb{Z})\) we define \(\tilde{\varphi} \in C_c(\mathbb{Z})\) via

\[
\tilde{\varphi}(n) = \overline{\varphi(-n)}
\]
for all \( n \in \mathbb{Z} \). A function \( p \) on \( \mathbb{Z} \) is positive definite if
\[
\sum_{n \in \mathbb{Z}} p(n)(\varphi \ast \tilde{\varphi})(n) \geq 0
\]
for all \( \varphi \in C_c(\mathbb{Z}) \). By a result of Bochner any positive definite function \( p \) is the Fourier transform of a unique measure \( \varrho \) on the unit circle
\[
\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \},
\]
i.e.
\[
p(n) = \int_{\mathbb{T}} z^n d\varrho(z)
\]
for all \( n \in \mathbb{Z} \), see e.g. [9].

By a dynamical system we mean a triple \((X, \alpha, m)\) consisting of a compact space \( X \), a continuous action \( \alpha : \mathbb{Z} \times X \rightarrow X \) of \( \mathbb{Z} \) on \( X \) and an \( \alpha \)-invariant probability measure \( m \) on \( X \). Clearly, this action is completely determined by \( \alpha_1 \) as we have \( \alpha_n = (\alpha_1)^n \) for any \( n \in \mathbb{Z} \). The dynamical system is ergodic if any \( \alpha \)-invariant measurable subset of \( X \) has measure zero or one.

Any dynamical system \((X, \alpha, m)\) comes naturally with the Hilbert space \( L^2(X, m) \) with inner product
\[
\langle f, g \rangle := \int_X f \overline{g} d\mu
\]
as well as a unitary map \( U \) on \( L^2(X, m) \) defined via
\[
Uf = f \circ \alpha_{-1}.
\]

The unitary \( U \) is known as Koopman operator. Such operators play an enormous role in the study of dynamical systems, see e.g. the recent monograph [12] for the (ever increasing) interest in this type of operators.

Consider now an \( f \in L^2(X, m) \). Then, a short computation shows
\[
\sum_{n \in \mathbb{Z}} \langle U^n f, f \rangle (\varphi \ast \tilde{\varphi})(n) = \| \sum_n \varphi(n) U^n f \|^2 \geq 0
\]
for all \( \varphi \in C_c(\mathbb{Z}) \). Hence, the function
\[
\mathbb{Z} \rightarrow \mathbb{C}, n \mapsto \langle U^n f, f \rangle,
\]
is positive definite. By the theorem of Bochner mentioned above, there exists then a unique measure \( \varrho^f \) on the unit circle with
\[
\langle U^n f, f \rangle = \int_{\mathbb{T}} z^n d\varrho^f(z)
\]
for all \( n \in \mathbb{Z} \). This measure is called the spectral measure of \( f \), see e.g. [23]. Spectral theory of dynamical systems is the study of these spectral measures.

We now turn to diffraction theory. It is developed to describe the outcome of diffraction experiments. Here, the piece of matter to be analyzed in the diffraction experiment is modeled by
a function and the diffraction measure of this function describes the outcome of the diffraction experiment. A discussion of the physics behind this can be found in [10]. The mathematical side is developed in the fundamental work [15], see the monograph [2] as well for a general discussion. Specifically, the autocorrelation of a function $h: \mathbb{Z} \to \mathbb{C}$ is defined as the pointwise limit of the functions

$$
\frac{1}{2n} \sum_{-n \leq k,l \leq n} h(k) \overline{h(l)} 1_{k-l}
$$

if this limit exists. This autocorrelation can easily be seen to be positive definite. Hence, by Bochner theorem, it is the Fourier transform of a positive measure on $\mathbb{T}$. This measure is called the diffraction measure of $h$.

**Remark.** Diffraction theory in one dimension is often developed for measures on $\mathbb{R}$. In this section, we rather deal with functions on $\mathbb{Z}$. It is possible to consider functions on $\mathbb{Z}$ as measures on $\mathbb{R}$ by associating to $h: \mathbb{Z} \to \mathbb{R}$ the measure $\delta_h := \sum_{x \in \mathbb{Z}} h(x) \delta_x$ (with $\delta_p$ being the unit point mass at $p$). The autocorrelation and diffraction measure of $h$ and of $\delta_h$ are related. The diffraction measure of $\delta_h$ is a periodic extension (with period 1) of the diffraction measure on $h$, where $\mathbb{T}$ is considered as $[0, 1) \subset \mathbb{R}$, see [5, 7] for details.

In typical diffraction situations, one is not given one function but rather a family of functions arising from a dynamical system. This is captured by the somewhat more involved framework of diffraction arising from dynamical systems. In order to set this up one needs a dynamical system together with one more ingredient. Let $\beta$ be the natural action of $\mathbb{Z}$ on $C_c(\mathbb{Z})$ i.e. $(\beta_n \varphi)(k) = \varphi(k - n)$. The mentioned ingredient is then a map

$$
\mathcal{N}: C_c(\mathbb{Z}) \to L^2(X, m)
$$

satisfying the following two properties (see Appendix as well):

(N1) $\mathcal{N}$ is linear.

(N2) $\mathcal{N}$ is equivariant (i.e. $\mathcal{N}(\beta_n \varphi) = U^n \mathcal{N}(\varphi)$ for all $n \in \mathbb{Z}$ and $\varphi \in C_c(\mathbb{Z})$).

Any such map then comes with a unique function $\gamma = \gamma^{(\mathcal{N})}$ on $\mathbb{Z}$ satisfying

$$
\sum_{n \in \mathbb{Z}} \gamma(n)(\varphi * \tilde{\psi})(n) = \langle \mathcal{N}(\varphi), \mathcal{N}(\psi) \rangle
$$

for all $\varphi, \psi \in C_c(\mathbb{Z})$. Indeed, a short calculation confirms that $\gamma$ is given by $\gamma = \sum_n c_n 1_n$ with

$$
c_n := c_n^{(\mathcal{N})} := \langle \mathcal{N}(1_n), \mathcal{N}(1_0) \rangle.
$$

Note that $c_n$ satisfies

$$
c_n = \langle U^n \mathcal{N}(1_0), \mathcal{N}(1_0) \rangle
$$

by equivariance of $\mathcal{N}$. The function $\gamma$ is called the autocorrelation of $\mathcal{N}$. Again, $\gamma$ is easily be seen to be positive definite. Hence, it has a Fourier transform, which is a measure on $\mathbb{T}$. This measure is called the diffraction measure of $\mathcal{N}$.

This immediately gives the following.

**Lemma 1.1** (Autocorrelation and spectral measure). Let $(X, \alpha, m)$ be a dynamical system and $\mathcal{N}: C_c(\mathbb{Z}) \to L^2(X, m)$ satisfy (N1) and (N2). Then, the spectral measure of $N(1_0)$ is the diffraction measure of $\mathcal{N}$. 

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Proof. By the preceding discussion we have
\[ \gamma^N(n) = c_n = \langle U^n N(1_0), N(1_0) \rangle = \int z^n d\varrho N(1_0) \]
for all \( n \in \mathbb{Z} \). Now, the result follows by taking Fourier transforms on both sides. \( \square \)

Remark. In the general situation treated in [20] existence of \( \gamma \) is not automatically satisfied. For \( \mathbb{Z} \) (and more generally for discrete groups) it is automatically satisfied. \( \diamond \)

From the defining properties of \( N \) it is not hard to see that \( N(1_0) \) is an \( L^2 \)-function and the map \( N \) is determined by \( N(1_0) \). In fact - as will be discussed more thoroughly below in this section - any element from \( L^2(X, m) \) gives rise to such a map \( N \). So, such functions \( N \) are in one-to-one correspondence with functions in \( L^2(X, m) \).

Under a suitable ergodicity assumption, it is possible to understand the autocorrelation of \( N \) as the autocorrelation of the sampling along \( N(1_0) \), i.e. as the autocorrelation of the sampling functions
\[ Z \rightarrow \mathbb{C}, n \mapsto N(1_0)(\alpha_n(x)) \]
for \( x \in X \). This is discussed next.

Lemma 1.2. Let \((X, \alpha, m)\) be ergodic and \( N : C_c(\mathbb{Z}) \rightarrow L^2(X, m) \) as above. Set \( f := N(1_0) \). Then, for \( m \)-almost every \( x \in X \), the autocorrelation \( \gamma^N \) is equal to the autocorrelation of the function \( Z \rightarrow \mathbb{C}, n \mapsto f(\alpha_n(x)) \).

Proof. This follows by a standard computation from Birkhoff’s ergodic theorem. We include it for the convenience of the reader: We have to show that the functions \( \gamma^x_n \) with
\[ \gamma^x_n = \frac{1}{2n} \sum_{-n \leq j, l \leq n} f(\alpha_j(x)) \overline{f(\alpha_l(x))} 1_{j-l} \]
converge pointwise to \( \gamma^N \) for almost every \( x \in X \). Fix an index \( k \). Then, the coefficient of \( 1_k \) in \( \gamma^N \) is given by
\[ \langle N(1_k), N(1_0) \rangle = \langle f \circ \alpha_{-k}, f \rangle. \]
On the other hand, the coefficient of \( 1_k \) in \( \gamma^x_n \) for sufficiently large \( n \) is essentially given by
\[ \frac{1}{2n} \sum_{-n \leq l \leq n} f(\alpha_{-k-l}(x)) \overline{f(\alpha_l(x))}. \]

By Birkhoff’s ergodic theorem, this can easily be seen to converge for \( m \)-almost every \( x \in X \) to
\[ \int_X f(\alpha_{-k}x) \overline{f(x)} dm(x) = \langle f \circ \alpha_{-k}, f \rangle. \]

As there are only countably many \( k \in \mathbb{Z} \), we find pointwise convergence of the functions in question for \( m \)-almost every \( x \in X \). This finishes the proof. \( \square \)

Remark. (a) Note that the previous lemma provides a connection between the two versions of diffraction theory discussed above, viz the version based on individual functions and the version based on dynamical systems.

(b) If the system is uniquely ergodic (i.e. there exists only one \( \alpha \)-invariant probability measure on \( X \)) and \( N(1_0) \) is Riemann integrable, the limit in the preceding lemma can easily be shown to exist for all \( x \in X \) by Oxtoby’s theorem. \( \diamond \)
Remark (Situation for real f). For real valued f the map N is also real (i.e. satisfies $N(\varphi) = N(\varphi)$ for all $\varphi \in C_c(Z)$). This brings an additional symmetry into play. More specifically, it implies that $\gamma$ is real-valued. Now, $\gamma$ is positive definite and, hence, satisfies $\gamma(n) = \gamma(-n)$ for all $n \in \mathbb{Z}$. Taken together this then gives that $\gamma$ is symmetric under reflection $R : \mathbb{Z} \to \mathbb{Z}, n \mapsto -n$. In particular, $\gamma$ is also the autocorrelation of the dynamical system $(X, \alpha^R, m)$ with respect to $N^R$, where $\alpha^R$ is the reflected action defined by $\alpha^R_n = \alpha_{-n}$ and $N^R$ is defined by $N^R(\varphi) := N(\varphi \circ R)$. In particular, $\gamma^N$ also arises by sampling along the function $n \mapsto f(\alpha_n(x))$ for almost every $x \in X$.

Let now $(X, \alpha, m)$ be a dynamical system. Let $f \in L^2(X, m)$ be given and define $N^f : C_c(Z) \to L^2(X, m)$ via $N^f(\varphi) := \sum_{n \in \mathbb{Z}} \varphi(n) U_n f$. It is not hard to see that $N^f$ satisfies the properties (N1) and (N2). Hence, there is an autocorrelation and a diffraction associated to $N^f$. We denote this autocorrelation by $\gamma^f$ and this diffraction by $\omega^f$ and call it the autocorrelation and diffraction measure of $f$, respectively.

Clearly, $N^f(1_0) = f$ holds. Hence, from Lemma 1.1 we immediately derive the main result of this section.

**Theorem 1.3** (Spectral measure as diffraction measure). Let $(X, \alpha, m)$ be a dynamical system and $f \in L^2(X, m)$ be given. Then, $\omega^f = \gamma^f$.

If the system is ergodic, it is possible to compute the autocorrelation almost surely as the autocorrelation of the sequences arising from sampling along $f$. Indeed, we have the following immediate consequence of the previous theorem and Lemma 1.2.

**Corollary 1.4** (Pointwise sampling). Let $(X, \alpha, m)$ be ergodic and $f \in L^2(X, m)$ be real valued. Then, for $m$-almost every $x \in X$, the autocorrelation $\gamma^f$ is equal to the autocorrelation of the function $Z \to \mathbb{R}, n \mapsto f(\alpha_{-n}(x))$.

Remark. If the dynamical system is uniquely ergodic, then the autocorrelation exists for all $x \in X$, whenever $f$ is continuous (or, more generally, Riemann integrable).

Remark (Relationship with [17]). The previous result immediately implies that dynamical systems whose spectral measures have certain properties will give rise to diffraction measures with these properties as well.

In particular, any dynamical system with pure discrete spectrum, i.e. pure point spectral measures, will give rise to pure point diffraction measures $\omega^f$ associated to its real valued functions $f$. Such systems are at the core of the theory of aperiodic order (see e.g. [2, 5, 16] for recent introductory and survey type treatments of this topic).

Likewise, a dynamical system with sufficiently strong mixing properties to ensure absolute continuity of its spectral measures will have diffraction measures $\omega^f$ which are absolutely continuous (up to a possible atom at 0). Such systems include Bernoulli systems and have also attracted attention in the context of diffraction theory, see e.g. last chapter of [2] or the survey article [1].

This provides also a structural understanding of the phenomena arising in the examples given in [17]. Indeed, the examples treated there are exactly dynamical systems with pure point
spectrum and dynamical systems with strong mixing properties respectively. Note, however, that \[17\] deals also with non-invertible systems, whereas we have restricted our attention to invertible systems. Finally, let us point out that existence of sampling along orbits is also already discussed in \[17\].

2. Dynamical systems over general locally compact abelian groups

In the last section, we have considered dynamical systems with actions of the group of integers. Here, we consider dynamical systems over general locally compact, \(\sigma\)-compact, abelian groups and indicate how the theory developed above carries over to this case as well. This requires some further technical elaborations not present in the case of \(\mathbb{Z}\).

We consider a compact space \(X\) equipped with a continuous action
\[G \times X \rightarrow X, (t, x) \mapsto \alpha_t(x),\]
of a locally compact, \(\sigma\)-compact, abelian group \(G\) and a probability measure \(m\), which is invariant under the action of \(G\). We then call \((X, G, m)\) a dynamical system over the space \(X\) and write \(tx\) instead of \(\alpha_t(x)\) for \(t \in G\) and \(x \in X\). Likewise the composition on \(G\) itself is written additively.

The dual group of \(G\) (i.e. the set of all continuous group homomorphisms from \(G\) to the unit circle \(\mathbb{T}\)) is denoted as \(\hat{G}\). Haar measure on \(G\) is denoted as \(m_G\). The space of continuous functions on \(G\) with compact support is denoted by \(C_c(G)\) and the convolution between a measure \(\eta\) on \(G\) and a \(\varphi \in C_c(G)\) is the function \(\eta * \varphi : G \rightarrow \mathbb{C}\) with
\[\eta * \varphi(t) = \int_G \varphi(t - s)d\eta(s)\]
for \(t \in G\). A function \(F\) on \(G\) (whose restrictions to compact sets belong to \(L^1(G, m_G)\)) may be considered as the measure \(Fm_G\) and we can accordingly define \(F * \varphi := (Fm_G) * \varphi\) for \(\varphi \in C_c(G)\). For \(\varphi \in C_c(G)\) we define \(\hat{\varphi} \in C_c(G)\) by \(\hat{\varphi}(t) = \varphi(-t)\) for \(t \in G\).

The action of \(G\) on \(X\) induces unitary operators \(T_t : L^2(X, m) \rightarrow L^2(X, m)\) with
\[T_t f = f((-t) \cdot)\]
for each \(t \in G\). Here, \(L^2(X, m)\) is the Hilbert space of (equivalence classes of) square integrable functions on \(X\). It is equipped with the inner product
\[\langle f, g \rangle = \int_X f\overline{g} \, dm\]
and the associated norm
\[\|f\| := \|f\|_2 := \sqrt{\langle f, f \rangle}\]
for \(f, g \in L^2(X, m)\). Then, for each \(f \in L^2(X, m)\) there exists a unique finite measure \(\rho f\) on \(\hat{G}\) with
\[\langle T_t f, f \rangle = \int_{\hat{G}} \eta(t)d\rho f(\eta)\]
for all \( t \in G \). This measure is called the spectral measure of \( f \).

Let now \((X,G,m)\) be a dynamical system and consider \( f \in L^2(X,m) \). We can then define the map

\[ N^f : C_c(G) \to L^2(X,m) \text{ via } N^f(\phi) := \int \phi(t) T_t f \, dm_G(t). \]

Note that \( N^f \) maps indeed in \( L^2 \) with the bound

\[ \|N^f(\phi)\| \leq \int_G |\phi(t)||T_t f| \, dt = \|f\| \|\phi\|_{L^1(G,m_G)} \]

for all \( \phi \in C_c(G) \). Clearly, this map is linear, and equivariant and thus satisfies the conditions (N1) and (N2) given in the appendix. In fact, it is not hard to see that it also satisfies (N3). This is discussed next.

We have to show that there exists a (unique) measure \( \gamma^f \) on \( G \) with

\[ \gamma^f(\varphi \ast \tilde{\psi}) = \langle N^f(\varphi), N^f(\psi) \rangle \]

for all \( \varphi, \psi \in C_c(G) \). In fact, \( \gamma^f \) is just given by \((t \mapsto \langle T_t f, f \rangle) m_G\), i.e. satisfies

\[ \int \varphi(t) d\gamma^f(t) = \int \varphi(t) \langle T_t f, f \rangle \, dt \]

for all \( \varphi \in C_c(G) \). Indeed, unwinding the definitions and using the invariance of Haar measure we find

\[ \int_G (\varphi \ast \tilde{\psi})(t) \langle T_t f, f \rangle \, dm_G(t) = \int_G \left( \int_G \varphi(t+s) \overline{\psi(s)} \, dm_G(s) \right) \langle T_t f, f \rangle \, dm_G(t) \]

\[ = \int_G \int_G \varphi(t+s) \overline{\psi(s)} \langle T_t f, f \rangle \, dm_G(t) \, dm_G(s) \]

\[ = \int_G \int_G \varphi(t) \overline{\psi(s)} \langle T_{t-s} f, f \rangle \, dm_G(t) \, dm_G(s) \]

\[ = \langle N^f(\varphi), N^f(\psi) \rangle \]

for all \( \varphi, \psi \in C_c(G) \). As this is just the defining property of \( \gamma^f \), we obtain the claimed representation.

From this formula for \( \gamma^f \) we can also directly see that it is the Fourier transform of the spectral measure. More specifically, for any \( \varphi \in C_c(G) \) we obtain

\[ \gamma^f(\varphi) = \int_G \varphi(t) \langle T_t f, f \rangle \, dm_G(t) \]

\[ = \int_G \varphi(t) \left( \int_{\hat{G}} \eta(t) d\varphi^f(\eta) \right) \, dm_G(t) \]

\[ = \int_{\hat{G}} \left( \int_G \varphi(t) \eta(t) \, dm_G(t) \right) d\varphi^f(\eta) \]

\[ = \int_{\hat{G}} \tilde{\varphi} \, d\varphi^f. \]

Here, the inverse Fourier transform \( \tilde{\sigma} \) of \( \sigma : G \to \mathbb{C} \) is defined by

\[ \tilde{\sigma}(\eta) = \int \eta(t) \sigma(t) \, dm_G(t). \]
As a consequence of these considerations we find in particular
\[ \gamma^f(\varphi * \tilde{\psi}) = \int_G \varphi \tilde{\psi} \, dq^f \]
for all \( \varphi, \psi \in C_c(G) \). Taking Fourier transforms on both sides one obtains the following result (compare the argument in the last section as well).

**Theorem 2.1 (Spectral measure as diffraction measure).** Let \((X, G, m)\) be a dynamical system and \( f \in L^2(X, m) \) be given. Then, the Fourier transform of \( \gamma^f \) is the spectral measure \( q^f \) of \( f \).

### 3. Diffraction theory of measure dynamical systems

Diffraction theory is usually developed for point sets or, more generally, measures. The leads then to diffraction theory for measure dynamical systems, as developed in [5]. The first aim of this section is to briefly recall this theory. A notable difference of this theory to the considerations above is that there is no need to fix a function. Instead the dynamical system itself already provides enough information to define autocorrelation and diffraction. This is quite remarkable and the second aim of this section is to show how this can be understood in light of the theory developed above via limiting procedures. Moreover, we present a relevant class of measure dynamical systems for subsequent considerations. This class - called function dynamical systems below - does not seem to have been a systematic focus in the development of diffraction theory so far.

Throughout this section \( G \) is a locally compact, \( \sigma \)-compact abelian group. A measure \( \mu \) on \( G \) is called **translation bounded** if its total variation \( |\mu| \) satisfies
\[ \sup |\mu|(t + U) < \infty \]
for one (all) relatively compact open \( U \) in \( G \). We denote that set of all translation bounded measures by \( M^\infty(G) \) and equip it with the vague topology. Then, \( G \) admits a natural action on \( M^\infty(G) \) by translations. More specifically, for \( t \in G \) and \( \mu \in M^\infty(G) \) the measure \( t\mu \) is defined by
\[ t\mu(\varphi) = \mu(\varphi(\cdot + t)) \]
for all \( \varphi \in C_c(G) \).

Whenever \( X \) is a compact subset of \( M^\infty(G) \), which is invariant under the translation action to and \( m \) is an invariant probability measure on \( X \), we call \((X, G, m)\) a dynamical system of translation bounded measures or just TMDS for short. Such a system comes with a canonical map
\[ \mathcal{N} : C_c(G) \rightarrow C(X), \varphi \mapsto \mathcal{N}_\varphi \text{ with } \mathcal{N}_\varphi(\mu) = \mu(\varphi) := \int \varphi(s) \, d\mu(s). \]

Let us emphasize that the existence of such a map is a distinctive feature of TMDS compared to general dynamical systems.

It is not hard to see that \( \mathcal{N} \) is linear and equivariant i.e. satisfies (N1) and (N2). Moreover, as shown in [5], it also satisfies (N3), viz. there exists then a unique translation bounded measure \( \gamma = \gamma^m \) on \((X, G, m)\) with
\[ \gamma(\varphi * \tilde{\psi}) = \langle \mathcal{N}_\varphi, \mathcal{N}_\psi \rangle \]
for all $\varphi, \psi \in C_c(G)$ and all $t \in G$. In fact, \[5\] even contains an explicit formula for $\gamma^m$, see \[14\] as well). The measure $\gamma$ is called the \textit{autocorrelation} of the TMDS. This measure allows for a Fourier transform $\hat{\gamma}$ which is a (positive) measure on $\hat{G}$. It is known as \textit{diffraction} of the TMDS. By the previous discussion and basic results on Fourier transform the diffraction satisfies

$$|\varphi|^2 \hat{\gamma} = \hat{\gamma}^N_{\varphi}$$

for all $\varphi \in C_c(G)$.

In the last sections, autocorrelation and diffraction arose from a dynamical system together with a chosen function. Thus, it is remarkable that for TMDS there is no need to chose a function. Instead the dynamical system alone suffices as piece of data. Intuitively speaking, the reason is that any approximate unit in $C_c(G)$ can be understood as providing a canonical choice of 'function'. More precisely, it is possible to exhibit the maps $N$ and the Fourier transform $\hat{\gamma}$ of the autocorrelation measure $\gamma$ by limiting procedures which involve chosen functions stemming from an approximate unit. Here are the details: If $(X, G, m)$ is a TMDS, we can consider an approximate unit in $C_c(G)$ i.e. a net $(\varphi_\alpha)$ in $C_c(G)$ with $\varphi_\alpha \ast \psi \to \psi$ for all $\psi \in C_c(G)$. We can additionally assume that for any relatively compact neighborhood $U$ of $e \in G$ the supports of the $\varphi_\alpha$ are contained in $U$ for sufficiently large $\alpha$. Now, define $f_\alpha \in L^2(X, m)$ by

$$f_\alpha(\mu) := N_{\varphi_\alpha}(\mu) = \int_G \varphi_\alpha(s)d\mu(s).$$

Then, convergence

$$N^{f_\alpha}(\varphi) \to N_{\varphi}$$

holds for all $\varphi \in C_c(G)$. Indeed, by construction and the defining properties of an approximate unit, we have pointwise (in $\mu \in X$) convergence

$$N^{f_\alpha}(\varphi)(\mu) = \int_G \varphi(t) \int_G \varphi_\alpha(s-t)d\mu(s)d\mu_G(t)$$

$$\to \int_G \hat{\varphi}(s)d\mu(s)$$

as well as a uniform (in $\mu \in X$) bound

$$|N^{f_\alpha}(\varphi)(\mu)| \leq \|f\|_{\infty} |\mu|(K) \leq C < \infty$$

due to the assumption on $X$. Thus, the maps $N^{f_\alpha}$ converge to the map $N$. Moreover, the spectral measures $\mu_{f_\alpha}$ converge to $\hat{\gamma}$ (see Corollary 1 in \[15\]).

For our subsequent considerations a special class of TMDS will be particularly relevant. These are introduced next. Consider the set $C(G)$ of continuous functions on $G$ equipped with the topology of uniform local convergence. In particular, the map

$$\delta_0 : C(G) \longrightarrow \mathbb{C}, f \mapsto f(e),$$

Footnote 1: Note that \[15\] uses a different sign in the definition of $N$ (called $f$ there) as well as has the inner product linear in the second argument. This results in a different display of the formula for $\gamma$, viz. $\gamma \ast \tilde{\varphi} \ast \psi(0) = \langle f_\varphi, f_\psi \rangle$. 

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is continuous. Clearly, the group $G$ acts continuously on $C(G)$ via translations
\[ G \times C(G) \rightarrow C(G), (s, f) \mapsto f(s \cdot). \]
Whenever $Y$ is a compact and invariant subset of $C(G)$ and $m$ an invariant probability measure on $Y$, we call $(Y, G, m)$ a function dynamical system or just FDS for short. Let now an FDS $(Y, G, m)$ be given. As $\delta_0$ is continuous, its range on such a $Y$ is compact. As $Y$ is invariant, we easily find that any $f \in Y$ must be bounded. Moreover, it is not hard to check that the map
\[ Y \rightarrow M^\infty(G), f \mapsto fm_G, \]
is continuous and injective. In this way, any $(Y, G, m)$ can be considered as a TMDS. Hence, FDS are special instances of TMDS.

**Remark.** (a) For discrete groups $G$, the set $C(G)$ can easily be identified with the set of all measures on $G$. Under this identification any TMDS can then be seen as a FDS (and vice versa). So, in this case there is no difference between FDS and TMDS.

(b) As discussed above any FDS can be seen as a TMDS, whose measures all have continuous densities with respect to Haar measure. The converse is not true. More specifically, it is not hard to construct examples of TMDS, all of whose elements have continuous densities, which are not FDS. Consider e.g. on the real numbers $\mathbb{R}$ the measure $\mu = h\lambda$ with $\lambda$ the Lebesgue measure and the density $h : \mathbb{R} \rightarrow [0, \infty)$ given by
\[ h := \sum_{n \in \mathbb{Z}} |n|1_{|n-2-|n|, n+2-|n|]}, \]
where $1_A$ denotes the characteristic function of $A$. Then, the measure $\mu$ is a translation bounded measure. In fact, for any $\delta > 0$ it even satisfies $\mu([s-\delta, s+\delta]) \rightarrow 0, |s| \rightarrow \infty$. This easily gives that the hull of $\mu$ given by
\[ \Omega(\mu) := \{t\mu : t \in \mathbb{R}\} \]
consists of the translates $t\mu$, $t \in \mathbb{R}$, as well as the zero measure. Then, the hull is a TMDS and all of its elements have densities with respect to Lebesgue measure. Moreover, the measures $n\mu$, $n \in \mathbb{N}$, converge clearly to the zero measure. However, the densities of $n\mu$ do not converge pointwise let alone locally uniformly (but rather explode at 0).

4. Spectral theory as diffraction theory of factors

In this section we will discuss how the spectral theory of arbitrary dynamical systems can be considered as diffraction theory of suitable factors. The basic idea is the following: Whenever $(X, G, m)$ is a dynamical system and $f$ is a continuous function on $X$, any $x \in X$ will encode a bounded continuous function
\[ \Phi^f_x : G \rightarrow \mathbb{C} \text{ via } \Phi^f_x(t) := f(\alpha_\cdot x). \]
and $\Phi^f$ can be seen as a factor map (see below for details and definitions). This effectively allows us to consider the situation of $(X, G, m)$ together with an $f \in C(X)$ as an FDS. Now, as discussed in the previous section, for such systems (and even more general TMDS) diffraction theory has been developed in [5] and the theory developed there can be applied to
• identify the autocorrelation $\gamma^f$ as the autocorrelation of a function dynamical system, which is a factor of the original system;
• exhibit the spectral measure $\rho^f$ as the diffraction of a a factor of the original system;
• show pointwise existence of the autocorrelation along sampling of an orbit.

Consider a dynamical system $(X, G, m)$. A dynamical system $(Y, G, n)$ is called a factor of $(X, G, m)$ if there exists a continuous surjective $G$-invariant map $\Phi : X \rightarrow Y$ with $\Phi(m) = n$. Here, $\Phi(m)$ is the push-forward of the measure $m$ by $\Phi$ defined by

$$\int_Y F(y)d\Phi(m)(y) = \int_X (F \circ \Phi)(x) \, dm(x)$$

for measurable $F$ on $X$ with $F \geq 0$. In this situation the map $\Phi$ is called factor map.

Our first aim is to identify all the FDS factors of $(X, G, m)$. To do so, we first consider a continuous $f$ on $X$. Then, we can indeed define for any $x \in X$ the function $\Phi^f_x$ on $G$ via $\Phi^f_x(t) = f(\alpha_t x)$. With the measure $\mu^f_x := \Phi^f_x dt$ we then obviously have

$$\int \varphi(t) d\mu^f_x(t) = \int \varphi(t)f(\alpha_t x) d\mu_G(t) = N^f(\varphi)(x)$$

for $\varphi \in C_c(G)$. It is then not hard to see that

$$\Phi^f : X \rightarrow C(G), x \mapsto \Phi^f_x,$$

is continuous and equivariant. Denote the image of $X$ under $\Phi^f$ by $X^f$ and let $\Phi^f(m)$ be the push-forward of $m$. Then, $(X^f, \alpha, \Phi^f(m))$ is a FDS. It can easily be seen to be ergodic if $(X, G, m)$ is ergodic. Conversely, whenever $\Phi : X \rightarrow Y$ is the factor map between $(X, G, m)$ and the FDS $(Y, G, n)$, the function $f = \delta_0 \circ \Phi$ is continuous with $\Phi = \Phi^f$.

We summarize the preceding considerations in the following proposition.

**Proposition 4.1** (Correspondence between FDS factors and continuous $f$). Let $(X, G, m)$ be a dynamical system. For any continuous $f : X \rightarrow \mathbb{C}$ the FDS $(X^f, G, \Phi^f(m))$ is a factor of $(X, G, m)$ via the factor map $\Phi^f$ and the map

$$C(X) \rightarrow \text{FDS factors of } (X, G, m), \ f \mapsto (X^f, G, \Phi^f(m)),$$

is a bijection (with inverse given by $\Phi \mapsto \delta_0 \circ \Phi$).

Having identified the FDS factors we now turn to study their autocorrelation and diffraction. As is not surprising, the autocorrelation of $(X^f, G, \Phi^f(m))$ and of $f$ are identical.

**Lemma 4.2** (Autocorrelation of $f$ as autocorrelation of a FDS). Let $(X, G, m)$ be a dynamical system and $f : X \rightarrow \mathbb{C}$ continuous. The autocorrelation $\gamma^{\Phi^f(m)}(\varphi, \psi)$ of $(X^f, \alpha, \Phi^f(m))$ is just $\gamma^f$.

**Proof.** From the definition of $\gamma^f$ and of $\gamma^{\Phi^f(m)}$ we find for any $\varphi, \psi \in C_c(G)$

$$\gamma^{\Phi^f(m)}(\varphi \ast \psi) = \langle N_\varphi, N_\psi \rangle_{L^2(X^f, \Phi^f(m))}$$

$$= \int_{X^f} y(\varphi) y(\psi) \, d\Phi^f(m)(y)$$

$$= \int_X \Phi^f_x(\varphi) \Phi^f_x(\psi) \, dm(x)$$

$$= \langle N^f(\varphi), N^f(\psi) \rangle_{L^2(X, m)}$$

$$= \gamma^f(\varphi \ast \psi).$$

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This finishes the proof.

The lemma effectively allows one to reduce the spectral theory (of continuous functions) of dynamical systems to diffraction theory for TMDS. In particular, it is an main step in the proof of the following result.

**Theorem 4.3** (Spectral theory as diffraction theory of FDS factors and vice versa). Let $(X,G,m)$ be a dynamical system. Then the following holds.

(a) For any FDS factor $(Y,G,n)$ with factor map $\Phi$ the equality $\widehat{\gamma^n} = \varrho^{\delta_0 \circ \Phi}$ holds.

(b) For any continuous $f : X \to \mathbb{C}$ the equality $\widehat{\gamma^{\Phi f}(m)} = \varrho^f$ holds.

In particular, the map

$$
\text{FDS factors of } (X,G,m) \to \text{Spectral measures of continuous functions}
$$

$$(Y,G,n) \mapsto \widehat{\gamma^n},$$

is onto.

**Proof.** The preceding lemma and Theorem 2.1 directly give (b). Now, (a) is a direct consequence of (b) and Proposition 4.1. The last statement is then immediate from (a) and (b). □

**Remark.** The theory of TMDS was originally developed in [5] with a view towards dynamical systems of Delone sets and generalizations thereof (as these are the relevant models for aperiodic order). The previous considerations and, in particular, the previous Theorem show that TMDS (and in fact even FDS) arise very naturally in the treatment of spectral theory of any dynamical system. Somewhat informally, the previous theorem can be summarized as giving that spectral theory of continuous functions is the same as diffraction theory of FDS factors. On the structural level this can be seen as a main result of the article. □

We are now heading towards a version of the previous result taking simultaneously all spectral measures into account. To give this version we need some further concepts.

Consider a dynamical system $(X,G,m)$ with associated unitary representation $T$ of $G$. By Stone’s theorem, compare [22, Sec. 36D], there exists a (unique) projection-valued measure $E_T : \{ \text{Borel sets of } \hat{G} \} \to \{ \text{projections on } L^2(X,m) \}$ with

$$
\langle f | E_T(\cdot) f \rangle = \varrho^f
$$

for any $f \in L^2(X,m)$. Then, the projection-valued measure $E_T$ contains the entire spectral information on the dynamical system. It is desirable to encode this spectral information in terms of measures on $\hat{G}$. One way of doing so is via the family of spectral measures. More generally, we introduce the following definition.

**Definition 4.4** (Complete spectral invariant). Let $T$ be the unitary representation associated to $(X,G,m)$, and let $E_T$ be the corresponding projection-valued measure. A family $\{ \sigma_\iota \}$ of measures on $\hat{G}$ (with $\iota$ in some index set $J$) is called a complete spectral invariant when $E_T(A) = 0$ holds for a Borel set $A \subset \hat{G}$ if and only if $\sigma_\iota(A) = 0$ holds for all $\iota \in J$. □

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Theorem 4.5 (Diffractions of FDS factors as complete spectral invariant). Let \((X, G)\) be an arbitrary dynamical system with invariant probability measure \(m\). Then, the diffractions of its FDS factors form a complete spectral invariant.

Proof. This is a direct consequence of Theorem 4.3 and the denseness of \(C(X)\) in \(L^2(X, m)\).

Remark. The main abstract result of [4] deals with special TMDS viz Delone dynamical systems with finite local complexity. It shows that for such systems the diffraction of factors, which are also Delone dynamical systems of finite local complexity, provide a complete spectral invariant. The methods of [4] heavily rely on the assumption of finite local complexity. It is not even clear how to extend them to general TMDS. The previous result now provides an analogue for arbitrary topological dynamical systems.

So far, we have been concerned with FDS factors. It turns out that it is also possible to work with general TMDS factors. Here, the crucial insight is that the diffraction of TMDS factors is also encoded by spectral theory. Here is the corresponding result.

Proposition 4.6 (Diffraction of TMDS factors via spectral measures). Let \((X, G, m)\) be dynamical system and \((Y, G, \Phi(m))\) be a TMDS factor with factor map \(\Phi : X \rightarrow Y\). Let \(\hat{\gamma}\) be the diffraction of \((Y, G, \Phi(m))\). Then,

\[
|\hat{\phi}|^2 \hat{\gamma} = \hat{\varphi}^{\mathcal{N}_\varphi \Phi}
\]

for all \(\varphi \in C_c(G)\).

Proof. In this proof we will deal with both spectral measures of functions on \(X\) and on \(Y\). In order to avoid confusion we therefore write \(\hat{\varphi}^X\) to denote the spectral measure of a function \(f \in C(X)\) and \(\hat{\varphi}^Y\) to denote the spectral measure of a function \(g \in C(Y)\). Similarly, we will write \(T^X_t\) and \(T^Y_t\) to denote the unitary action of \(G\) on \(L^2(X, m)\) and \(L^2(Y, \Phi(m))\) respectively.

As discussed above we have

\[
|\hat{\varphi}|^2 \hat{\gamma} = \hat{\varphi}^{\mathcal{N}_\varphi}
\]

for all \(\varphi \in C_c(G)\). As \(\Phi\) is a factor map, the definition of \(\Phi(m)\) easily gives that

\[
\langle \mathcal{N}_\varphi \circ \Phi, T^Y_t \mathcal{N}_\varphi \circ \Phi \rangle_{L^2(Y, \Phi(m))} = \langle \mathcal{N}_\varphi, T^X_t \mathcal{N}_\varphi \rangle_{L^2(X, m)}
\]

for all \(t \in G\). By the defining properties of the spectral measures, this yields

\[
\hat{\varphi}^{\mathcal{N}_\varphi \Phi} = \hat{\varphi}^{\mathcal{N}_\varphi}
\]

for all \(\varphi \in C_c(G)\). Putting this together we find

\[
|\hat{\varphi}|^2 \hat{\gamma} = \hat{\varphi}^{\mathcal{N}_\varphi} = \hat{\varphi}^{\mathcal{N}_\varphi \Phi}
\]

and the proof is finished.

From the preceding considerations we now obtain the following corollary.

Corollary 4.7 (Diffraction of TMDS factors as complete spectral invariant). Let \((X, G)\) be an arbitrary dynamical system with invariant probability measure \(m\). Then, the diffractions of its TMDS factors form a complete spectral invariant.
Proof. This is a direct consequence of Theorem 4.5 and Proposition 4.6.

As a consequence of Lemma 4.2 it is also possible to show that the autocorrelation exists as a pointwise limit along the orbits of a sampling function. To do so, we need some further concepts. Firstly, recall from [25] that a sequence \((B_n)\) of compact subsets of \(G\) is called a van Hove sequence if

\[
\lim_{n \to \infty} \frac{|\partial^K B_n|}{|B_n|} = 0
\]

for all compact \(K \subset G\). Here, for compact \(B,K\), the “\(K\)-boundary” \(\partial^K B\) of \(B\) is defined as

\[
\partial^K B := ((B + K) \setminus B) \cup ((G \setminus B - K) \cap B),
\]

where the bar denotes the closure. The existence of van Hove sequences for all \(\sigma\)-compact LCA groups is shown in [25, p. 249], see also Section 3.3 and Theorem (3.L) of [28, Appendix]. Moreover, every van Hove sequence is a Følner sequence, i.e.,

\[
\frac{|B_n \triangle (B_n + K)|}{|B_n|} \to 0
\]

for every compact set \(K \subset G\), where \(A \triangle B := (A \setminus B) \cap (B \setminus A)\).

Theorem 4.8 (Sampling for continuous functions). Let \((X,\alpha,m)\) be ergodic and assume that \(G\) has a countable basis of topology. Let \((B_n)\) be a van Hove sequence along which the Birkhoff ergodic theorem holds. Let \(f : X \to \mathbb{C}\) be continuous. Then, the measures

\[
\gamma^f_{x,n} := \frac{1}{m_G(B_n)} \mu^f_x|B_n \ast \hat{\mu}^f_{\Phi^f(m)}|B_n
\]

converge vaguely to \(\gamma^f\) for almost every \(x \in X\).

Proof. From [5] we obtain that the limit exists almost everywhere and equals the autocorrelation of the TMDS \((X^f,\alpha,\Phi^f(m))\). By the Lemma 4.2 that autocorrelation is just \(\gamma^f\) and the theorem is proven.

5. Application to discrete spectrum

In this section we will use the material of the previous section to study discrete spectrum. In particular, we will combine this material with a recent result of [27] to provide a characterization of discrete spectrum. This characterization is a variant of well-known characterizations for action of \(\mathbb{Z}\) (see end of the section for details).

Recall that the dynamical system \((X,G,m)\) is said to have discrete spectrum if all spectral measures are pure point measures. We start with the following consequence of our discussion.

Lemma 5.1. Let \((X,G,m)\) be a dynamical system. Then the following assertions are equivalent.

(i) The dynamical system \((X,G,m)\) has discrete spectrum.

(ii) For every \(f \in C(X)\) the dynamical system \((Y_f,G,\Phi^f(m))\) has discrete spectrum.

Proof. By denseness of \(C(X)\) in \(L^2(X,m)\) discrete spectrum is equivalent to \(\rho^f\) being a pure point measure for all \(f \in C(X)\). By Theorem 4.3 this is equivalent to pure pointedness of all measures \(\hat{\gamma}^\Phi^f(m)\) for \(f \in C(X)\). Now, for TMDS a main result of [5] gives that pure pointedness of the diffraction spectrum is equivalent to discrete spectrum of the dynamical system. This yields the desired statement.
Remark. Other features of the dynamical system (mean equicontinuity, equicontinuity etc) can be characterized similarly via factors. We refrain from giving details.

We need some further concepts. Consider an arbitrary van-Hove sequence \((B_n)\). Then, we define the associated mean via

\[
M := M_B : \text{Bounded function on } G \rightarrow [0, \infty), M(h) := \lim_{n \to \infty} \frac{1}{m_G(B_n)} \int_{B_n} h(s)dm_G(s).
\]

Clearly, \(M\) is a seminorm on the space of bounded functions on \(G\). A subset \(A\) of \(G\) is called \(relatively dense\) if there exists a compact set \(K \subset G\) with \(G = \bigcup_{a \in A} (a + K)\).

Now, a uniformly continuous bounded \(h : G \rightarrow \mathbb{C}\) is called \(mean almost periodic\) if for every \(\varepsilon > 0\) the set

\[
\{t \in G : M_G(h - h(t)) < \varepsilon\}
\]

is relatively dense in \(G\). A translation bounded measure \(\mu\) on \(G\) is called \(mean almost periodic\) if for any \(\varphi \in C_c(G)\) the function \(\mu \ast \varphi\) is mean almost periodic. Then, it is not hard to see that a uniformly continuous bounded function \(h : G \rightarrow \mathbb{C}\) is mean almost periodic if and only if the associated measure \(hm_G\) is mean almost periodic (compare Proposition 3.9 in [27]).

**Theorem 5.2.** Let \((X, G, m)\) be an ergodic dynamical system. Assume that \(G\) has a countable basis of the topology and let \((B_n)\) be a van Hove sequence along which the Birkhoff ergodic theorem holds. Then, the following assertions are equivalent.

(i) The dynamical system \((X, G, m)\) has discrete spectrum.

(ii) For every \(f \in C(X)\) the function \(\Phi_f^x\) is mean almost periodic for \(m\) almost every \(x \in X\).

**Proof.** By the preceding lemma, (i) is equivalent to \((Y_f, G, \Phi_f^m)\) having discrete spectrum for all \(f \in C(X)\). Now, \((Y_f, G, \Phi_f^m)\) is a TMDS (and even an FDS) over a locally compact abelian group with countable basis of the topology. Thus, by a main result of [27] discrete spectrum of \((Y_f, G, \Phi_f^m)\) is equivalent to mean almost periodicity of \(ym_G\) for \(\Phi_f^m\) almost every \(y\). As discussed above this is equivalent to mean almost periodicity of the function \(y\) for \(\Phi_f^m\) almost every \(y \in Y_f\). Now, the desired statement follows from the definition of \(\Phi_f^m\). \(\square\)

Let us conclude this section with a consequence of the preceding theorem in the case that \(X\) has a countable basis of the topology. In this case the set \(C(X)\) is separable in the uniform topology. Thus, one can then chose in (ii) one set \(X'\) of full measure in \(X\) such that \(\Phi_f^x\) is mean almost periodic for every \(x \in X'\) and all \(f \in C(X)\). If one now calls a point \(x \in X\) \textit{mean almost periodic} if \(\Phi_f^x\) is mean almost periodic for every \(f \in C(X)\), the previous result can then be restated as follows:

**Corollary 5.3.** Let \((X, G, m)\) be an ergodic dynamical system. Assume that both \(X\) and \(G\) have a countable basis of the topology. Then, the dynamical system \((X, G, m)\) has discrete spectrum if and only if almost every \(x \in X\) is mean almost periodic.
Remark. We could not find the result in this form in the literature. However, it will certainly not be a surprise to experts. Indeed, for ergodic systems over $G = \mathbb{Z}$, it is known that discrete spectrum is equivalent to $\mathbb{Z} \ni n \mapsto f(nx)$ belonging to the Besicovich class for almost every $x \in X$ whenever $f$ is a bounded measurable function on $X$, see Theorem 3.22 in [8]. The condition of Besicovich class seems to be somewhat stronger than mean almost periodicity. On the other hand, clearly, the condition that $f$ is continuous is stronger than $f$ being bounded and measurable. We also note that a related result for subshifts over a finite alphabet can be found in Lemma 5 of [29]. There, discrete spectrum is characterized with a mean almost periodicity condition on points rather than functions.

6. The measurable situation

The previous sections has been set in the topological category. Starting with a dynamical system $(X,G,m)$ we considered continuous functions on $X$ as well as factor maps. From the point of view of the unitary representation $T$ it is also desirable to treat functions $f \in L^2(X,m)$. Roughly speaking this case can be treated by similar means as the case of continuous $f$ by using the theory of square integrable measure dynamical systems from [21] instead of the theory of translation bounded dynamical systems from [3]. A sketch is given in this section.

Recall that we denote by $M(G)$ the set of all measures on $G$. This set is equipped with the vague topology and the associated Borel-$\sigma$-algebra. There is a canonical action of $G$ by translation. Any $\varphi \in C_c(G)$ then gives rise to a continuous function $N_{\varphi}$ on $M(G)$ via

$$N_{\varphi}(\mu) := \int_G \varphi \, d\mu.$$ 

An invariant probability measure $n$ on $M(G)$ is called square integrable if

$$\int_{M(G)} |N_{\varphi}(|\mu|)|^2 \, dn(m) < \infty$$

for all $\varphi \in C_c(G)$. As shown in [21] any square integrable measure $n$ then comes with a unique measure $\gamma = \gamma^n$ satisfying

$$\gamma(\varphi \ast \tilde{\psi}) = \langle N_{\varphi}, N_{\psi} \rangle_{L^2(M(G),n)}$$

for all $\varphi, \psi \in C_c(G)$. This measure $\gamma$ is called the autocorrelation of $n$. Its Fourier transform exists and is a positive measure called the diffraction of $n$. Note that this generalizes the framework of TMDS discussed above.

We now want to proceed as above by defining the measure $\mu_f^x$ on $G$ via

$$\mu_f^x(\varphi) = \int_G \varphi(s)f(\alpha^{-1}s)dm_G(s).$$

Here, we have to overcome the obstacle that this will not necessarily make sense (due to possible unboundedness of $f$). However, it turns out that it will make sense for almost every $x \in X$ under suitable additional assumptions.

**Proposition 6.1.** Let $f \in L^2(X,m)$ be given. Then,

$$\int_X \left( \int_X |(\varphi(t)|f(\alpha^{-1}s)|dm_G(t) \right)^2 \, dm(x) < \infty$$

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for any $\varphi \in C_c(G)$ with $\varphi \geq 0$. In particular, for $m$-almost every $x \in X$ there exists a measure $\mu_x^f G$ with
\[
\int \varphi(t) d\mu_x^f(t) = \int \varphi(t) f(\alpha_t x) dm_G(t)
\]
for $\varphi \in C_c(G)$.

**Proof.** The first inequality follows easily from Cauchy–Schwarz inequality and Fubini’s Theorem. As for the last statement we note that by $\sigma$-compactness of $G$ there exists a increasing sequence of compact sets $K_n$ with
\[
\bigcup_n \text{int}(K_n) = G.
\]
Here, int denotes the interior of a set. For each $K_n$ we can now chose a function $\varphi_n \in C_c(G)$ with $0 \leq \varphi_n \leq 1$ and $\varphi_n = 1$ on $K_n$. By the first statement, we can then find a set $N_n \subset X$ with $m(N_n) = 0$ such that
\[
\int \varphi_n(t) |f(\alpha_t x)| dm_G(t) < \infty
\]
for all $x \not\in N_n$. Hence, we infer that $\mu_x^f$ is a measure for all $x \not\in \bigcup N_n$ and this is the desired statement. \hfill \qed

We denote the set of $x \in X$ for which $\mu_x^f$ is a measure by $X_{\text{meas}}$. By the previous proposition this set has full measure. Moreover, it is clearly invariant. Then the map
\[
\mu^f : X_{\text{meas}} \to M(G)
\]
is measurable. Denote by $\mu^f(m)$ the push-forward of the measure $m$ to $M(G)$. This is an invariant probability measure on $M(G)$ and ergodic if $m$ is ergodic. From Proposition 6.1 we easily infer that $\mu^f(m)$ is square integrable. Hence, we can apply the theory of \cite{21} to $\mu^f(m)$ to conclude existence of $\gamma^{\mu^f(m)}$. Then, arguing as above for the case of TMDS we find
\[
\gamma^{\mu^f(m)} = \gamma^f \text{ and } \hat{\gamma}^{\mu^f(m)} = \hat{\gamma}^f.
\]
Moreover, using the results of \cite{21} we can then also conclude (as above) that Theorem 4.8 remains valid for $f \in L^2(X, m)$.

**Remark.** It may be worthwhile to emphasize the following application of the results of this section: Any measure appearing as spectral measure of a dynamical system can be obtained as diffraction measure of a whole family of functions. In principle, this opens up a way to generate rather naturally diffraction measures with interesting properties.

## A. A short review of diffraction theory

In this section we briefly review the framework for diffraction theory developed in \cite{20}.

The basic pieces of data are given by a dynamical system $(X, G, m)$ (with $G$ a locally compact, $\sigma$-compact abelian group) and a map
\[
\mathcal{N} : C_c(G) \to L^2(X, m)
\]
with the following properties:
\footnote{The article \cite{20} introduced one more assumption, viz that $\mathcal{N}$ is real. This, however, is not needed for the results we discuss here.}

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(N1) $\mathcal{N}$ is linear. 

(N2) $\mathcal{N}$ is equivariant i.e. satisfies $\mathcal{N}(\varphi(\cdot - t)) = \mathcal{N}(\varphi)(\alpha_{-t}(\cdot))$ for all $\varphi \in C_c(G)$ and all $t \in G$. 

(N3) There exists a measure $\gamma$ on $G$ with $\gamma(\varphi \ast \tilde{\psi}) = \langle \mathcal{N}(\varphi), \mathcal{N}(\psi) \rangle_{L^2(X,m)}$ for all $\varphi, \psi \in C_c(G)$. 

In this case $\gamma$ is a positive definite measure called the **autocorrelation** of $\mathcal{N}$ and its Fourier transform is a positive measure called the **diffraction** of $\mathcal{N}$. Moreover, there exists a unique isometry, called **diffraction to dynamics map** 

$$\Theta: L^2(\hat{G}, \mu_f) \longrightarrow L^2(X,m)$$ 

with $\Theta^f(\hat{\sigma}) = N^f(\sigma)$ for any $\sigma \in C_c(G)$.

It turns out that in applications one can often conclude validity of (N3) for maps satisfying (N1) and (N2) from a continuity property. Specifically, the following is shown in [20].

**Lemma A.1.** Let $(X, G, m)$ be a dynamical system. Let $\mathcal{N}: C_c(G) \longrightarrow L^2(X, m)$ satisfy (N1), (N2) and $\mathcal{N}(\varphi) = \mathcal{N}(\overline{\varphi})$ for all $\varphi \in C_c(G)$. If for all $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$ be given and any compact $K \subset G$ there exists a $C_K > 0$ with

$$|\langle \mathcal{N}^f(\varphi), \mathcal{N}^f(\psi) \rangle| \leq C_K \|\varphi\|_{L^p(G)} \|\psi\|_{L^q(G)}$$

for all $\varphi, \psi \in C_c(G)$ with support contained in $K$ then $\mathcal{N}$ satisfies (N3) as well.

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