Lattice QCD at Finite Temperature and Density

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1. Outline

This is a general introduction into the subject aimed at a general theoretical physics audience. Several recent results on the behavior of the cold and hot phases, as well as of the critical line have been included, however no attempt was made at being exhaustive: further material is to be found in the contributions by A. Nakamura and Ph. De Forcrand, as well as in introductory 1 and topical reviews 2.

This writeup is organized as follows: we give a first introduction and physical motivation. Next, we briefly review field theory thermodynamics, and the lattice approach. Technical difficulties (the sign problem) are introduced there, while the three main methods to circumvent them are discussed in the following Section. The following section discusses in some detail the interrelation between the Taylor expansion and the imaginary chemical potential approach. Section VI, VII, VIII give an overview of the results on the critical line, hot and cold phases respectively. A short summary concludes the paper.

2. Introduction

The historical developments of the phase diagram of CD is characterized by an increasing complication: early views were based on asymptotic freedom, and divided
sharply the phase diagram into an hadronic phase and a quark gluon plasma phase. In the late 90’s it has been appreciated that the high density region is much more complicated than previously thought. In the last couple of years, it was the turn of the region above $T_c$ to become more rich: the survival of bound states above the phase transition brought up the idea of a more complicated phase, the strongly interactive quark gluon plasma, characterized by a rich particle spectrum of colored particle (free quarks, diquarks, etc.).

What has the lattice to say in this scenario? Its main task is to provide ab initio calculation of the properties of the phase diagram, including, of course, the precise location of the critical line, and the equilibrium properties of the different phases. One main challenge, of course, is to frame these results into the more general context of real time evolution. In this respect, it is worthwhile to remind ourselves that equilibrium solution are steady state solution of the dynamical Fokker Planck operator; and that lattice calculations will help validating simple models which can be studied out of equilibrium: equilibrium studies are a mandatory first step toward a full real time understanding of the collisions.

3. A brief review of field theory thermodynamics

3.1. Formulation

Let us remind ourselves how to introduce a chemical potential $\mu$ for a conserved charge $\hat{N}$ in the density matrix $\hat{\rho}$ in the Grand Canonical formalism, which is the one appropriate for a relativistic field theory:

$$\hat{\rho} = e^{-\hat{H} - \mu\hat{N}}/T \qquad (1)$$

$$Z(T, \mu) = Tr\hat{\rho} = \int d\phi d\psi e^{-S(\phi, \psi)} \qquad (2)$$

3.2. The Hamiltonian Formalism

My task in this lecture is to discuss numerical results for the Lagrangian formalism. Still, it is very important to mention the Hamiltonian approach: early studies did show a great promise, and the recent efforts might well indicate that this is correct avenue to treat the phase diagram at finite temperature and density. I refer the reader to recent literature on the subject, which contains a full set of references to early work.

3.3. The Lagrangian Formalism

The path integral representation of the grand partition function $Z$ in the Euclidean space gives the temperature as the reciprocal of the imaginary time:

$$S(\phi, \psi) = \int_0^{1/T} dt \int d^d x L(\phi, \psi)$$

(3)
with periodic boundary conditions in time for bosons \( \phi(t = 0, \vec{x}) = \phi(t = 1/T, \vec{x}) \)
and antiperiodic for fermions \( \psi(t = 0, \vec{x}) = -\psi(t = 1/T, \vec{x}) \).

All in all, \( Z \) at finite temperature \( T \) and density \( \mu \) is the partition function of a
statistical system in \( d+1 \) dimension, where \( T \) is the reciprocal of the imaginary time,
and \( \mu \) couples to any conserved charge. This representation, which is the starting
point for a lattice calculation, allows us to deal with thermodynamics and spectrum
exactly on the same footing.

The theory is regularized on a space time lattice: a regular four dimensional grid
with \( N_s \) points in each space directions, \( N_t \) points in the imaginary time direction,
and spacing \( a \). We refer to the very many excellent reviews and textbooks for back-
ground material on lattice field theory, and we briefly summarize here the specific
aspects of lattice QCD thermodynamics which will be useful in the following.

The temperature \( T \) on a lattice is the same as in the continuum: \( T = 1/N_t a \),
\( N_t a \) being the lattice extent in the imaginary time direction (while, ideally, the
lattice spatial size should be infinite). A lattice realization of a finite density of
baryons, instead, poses specific problems: the naive discretization of the continuum
expression \( \mu \bar{\psi} \gamma_0 \psi \) would give an energy \( \epsilon \propto \mu^2 a^2 \)
diverging in the continuum \( (a \to 0) \) limit \(^6\).

The problem could be cured by introducing appropriate counterterms, however
the analogy between \( \mu \) and an external field in the \( 0 \) (temporal) direction offers
a nicer solution by considering the appropriate lattice conserved current \(^6\). This
amounts to the following modification of the fermionic part of the Lagrangian for
the \( 0 \) direction \( L^F_0 \):

\[
L^F_0(\mu) = \bar{\psi}_x \gamma_0 e^{\mu a} \psi_{x+\hat{0}} - \bar{\psi}_{x+\hat{0}} \gamma_0 e^{-\mu a} \psi_x
\]

while the remaining part of the Lagrangian is unchanged. This yields the current:

\[
J_0 = -\partial_\mu L = -\partial_\mu \mathcal{L}^F_0(\mu) = \bar{\psi}_x \gamma_0 e^{\mu a} \psi_{x+\hat{0}} + \bar{\psi}_{x+\hat{0}} \gamma_0 e^{-\mu a} \psi_x
\]

This representation of \( J_0 \) is amenable to a simple interpretation: the time forward
propagation is enhanced by \( e^{\mu a} \), while the time backward propagation is discouraged
by \( e^{-\mu a} \); hence, the link formulation generates a particles–antiparticles asymmetry.
In addition, note that \( \int J_0 = N - \bar{N} \) as it should. An alternative way to look at
the link formulation introduces an explicit dependence on the fugacity \( e^{\mu/T} \) via
an unitary transformation for the fields \(^7\). In this way \( L(\mu) = L(0) \), and the \( \mu 
dependence is on the boundaries, via the fugacity \( e^{\mu/T}; \psi(x+N_T) = -e^{\mu a N_T} \psi(x)
= -e^{\mu/T} \psi(x) \). This is analogous to the continuum case \(^8\).

3.4. Calculational Schemes

Having set up the formalism, the task is to compute

\[
Z = \int dU d\psi e^{-S(U,\psi)}
\]
where from now on the Lagrangian defining the Action will be that of lattice QCD, containing gluon fields $U$ and quark fields $\psi$.

We have two options. We might integrate out gluons first:

$$\int dU d\bar{\psi} d\psi Z(T, \mu, \bar{\psi}, \psi, U) \simeq \int d\bar{\psi} d\psi Z(T, \mu, \bar{\psi}, \psi)$$ (7)

This produce an effective approximate fermion model: the procedure is physically appealing, but not systematically improvable, but for one special (lattice) case (see below). Alternatively, we might integrate out fermions exactly, by taking advantage of the bilinearity of the fermionic part of the Lagrangian

$$L = L_{YM} + L_F = L_{YM} + \bar{\psi}M(U)\psi$$

$$\int dU d\psi d\bar{\psi} Z(T, \mu, \bar{\psi}, \psi, U) = \int dU e^{-\left(S_{YM}(U) - \log(\det M)\right)}$$ (8)

The “effective” model we build this way is exact: the price to pay being that its physical interpretation is not as clear as for effective fermion models. Anyway, this expression is the starting point for numerical calculations: the fact that in many cases they are highly successful tell us that the configuration space is well behaved enough that only a minor subset of configurations, although carefully chosen via importance sampling, suffice to produce reasonable results.

### 3.5. Effective Fermionic Models: analytical approaches

Let us start by following the first idea, namely integrating out the gluon fields so to define an effective fermionic Action. This is a time honored approach, leading, for instance, to the instanton model Hamiltonian, hence to the exciting discoveries on the QCD phase diagram of the last five years [9].

On the lattice, one very interesting approach leading to a fermionic model is provided by the strong coupling expansion: in the infinite gauge coupling limit the Yang Mills term decouples from the Action, and the integral over the gauge fields can be carried out exactly.

The starting point is the QCD lattice Lagrangian:

$$S = -1/2 \sum_x \sum_{j=1}^3 \eta_j(x)[\bar{\chi}(x)U_j(x)\chi(x+j) - \bar{\chi}(x+j)U_j^\dagger(x)\chi(x)]$$ (9)

$$-1/2 \sum_x \eta_0(x)[\bar{\chi}(x)U_0(x)\chi(x+0) - \bar{\chi}(x+0)U_0^\dagger(x)\chi(x)]$$

$$-1/3 \sum_x 6/g^2 \sum_{\mu,\nu=1}^4 \left[1 - reTr U_{\mu\nu}(x)\right]$$

$$+ \sum_x m\bar{\chi}\chi$$

The $\chi, \bar{\chi}$ are the staggered fermion fields living on the lattice sites, the $U$’s are the $SU(N_c)$ gauge connections on the links, the $\eta$’s are the lattice Kogut–Susskind
counterparts of the Dirac matrices, and the chemical potential is introduced via the time link terms $e^\mu$, $e^{-\mu}$ as discussed above. This time we have written down explicitly the lattice Action to show that the pure gauge term

$$S_G = -\frac{1}{3} \sum x 6/g^2 \sum_{\mu,\nu=1}^4 [1 - reTrU_{\mu\nu}(x)]$$

contains the gauge coupling in the denominator, hence it disappears in the infinite coupling limit. Consequently, one can perform independent spatial link integrations, leading to

$$Z = \int \prod \text{timelinks} dU_t d\bar{\chi} d\chi e^{-S_t}$$

where $\sum_{<x,y>}$ means sum over nearest neighboring links, terms of higher order have been dropped, and we recognize a four fermion interaction. Further manipulations yield the mean field effective potential:

$$V_{eff}(\langle \bar{\psi}\psi \rangle, \mu) = 2 \cosh(rN_t N_c \mu) + \frac{\sinh((N_t+1)N_c \langle \bar{\psi}\psi \rangle)}{\sinh(N_t \langle \bar{\psi}\psi \rangle)}$$

which we quote for further reference. A standard analysis of $V_{eff}$ finally gives the condensate as a function of temperature and density, and allows the reconstruction of the phase diagram.

More recently this approach has been furthered both in two and three colors, and new developments on cluster algorithms have appeared as well.

In order to describe in detail the rich physics of the finite density phase, one needs both to include higher order terms into the strong coupling expansion, as well as to go beyond a simple mean field analysis, which assumes an homogeneous background. The question is as to whether such improved strong coupling approaches would be able to generate a four fermion term with the correct flavor structure as well as order of magnitude, thus opening the possibility of a systematically improvable approach to finite density QCD, including the study of the superconducting phase.

3.6. Effective Gluonic Models: Importance Sampling and the positivity issue

Let us write again

$$Z(T, \mu) = \int dU e^{-(S_{YM}(U) - \log(\det M))}$$

When $\det M > 0$ the functional integral can be evaluated with statistical methods, sampling the configurations according to their importance $(S_{YM}(U) - \log(\det M))$. For this to be possible the would-be-measure $(\det M)$ has to be positive.

Let me mention at this point that the factorization method might alleviate the problems of complex measures by guiding the simulations along a sensible path in the phase space. I will not dwell on this interesting development which is not really in the scope of an introductory review, but I wish to call on it the attention of the interested reader, as it really seems to offer some promise, and has been already tested in random matrix models.
In QCD with an even number of flavors, and zero chemical potential, standard importance sampling simulations are possible if $\det M$ is real, which is true if $M^\dagger = -PMP^{-1}$ where $P$ is any non singular matrix. In the most popular lattice fermion formulation this holds: for Wilson fermions $P = \gamma_5$ and for staggered fermions $P = I$ (note that this basically expresses a particle–antiparticle symmetry). We will consider staggered fermions from now on.

Consider now the relationship $M^\dagger(\mu_B) = -M(-\mu_B)$ implying that reality is lost when $\Re \mu \neq 0$: the reality of the determinant is lost, and with it the possibility of doing simulations with non zero chemical potential, when we want to create a particle antiparticle asymmetry. On the other hand a purely imaginary chemical potential does not spoil the reality of the determinant: indeed, even if an imaginary chemical potential can be used to extract information at real chemical potential, it does not create any real particle–antiparticle asymmetry and it is natural that the fermion determinant remains real.

Note that in QCD with two color the determinant remains positive with nonzero real chemical potential: indeed, in that case quarks and antiquarks transform under equivalent representation of the color group and are, essentially, the same particle. Other important models with a real determinant include finite density of isospin $^{16}$ and four fermion models $^{15}$. All in all, if we want to extract information useful for QCD at nonzero baryon density by use of standard MonteCarlo sampling we will have to use information from the accessible region:

$$\Re \mu = 0, \Im \mu \leq 0$$

4. Overview of the methods

To begin with, it is useful to think of the theory in the $T, \mu^2$ plane. Let us then discuss the phase diagram from the perspective of analyticity and positivity of the partition function and of the determinant. One important consideration to keep in mind: the Gran Canonical partition function has to be positive. It is only the determinant which can change sign, or even be complex, on single configurations.

Let us consider a mapping from complex $\mu$ to complex $\mu^2$. Because of the symmetry properties of the theory, this mapping can be done without loss of generality. Let us note then that $Z(\mu^2)$ is real valued for real $\mu^2$: this is a situation familiar from condensed matter: the partition function is real where the external parameter is real, complex otherwise.

The reality region for the partition function represents states which are physically accessible. The reality region for the determinant represents the region which is amenable to an importance sampling calculation: $\Re \mu^2 \leq 0$. The methods which have been applied so far are

- $\mu = 0$ Derivatives, Reweighting, Expanded reweighting
- $\mu^2 \leq 0$ Imaginary chemical potential
Fig. 1. Sketch of the phase diagram in the $\mu^2, T$ plane: the solid line is the chiral transition, the dashed line is the Roberge Weiss transition. Simulations can be carried out at $\mu^2 \leq 0$ and results continued to the physical domain $\mu^2 \geq 0$. The derivative and reweighting methods have been used so far to extract informations from simulations performed at $\mu = 0$. The imaginary chemical potential approaches uses results on the left hand half plane. Different methods could be combined to improve the overall performance.

4.1. Derivatives at $\mu = 0.0$

This is one early attempt at exploring the physics of nonzero quark density: the derivatives can be formally computed at $\mu = 0$.\textsuperscript{17} The obvious limitation is that we do not really know how far from the $\mu = 0$ axis can we get. Nonetheless, such derivatives are interesting per se, and the region where derivatives are clearly different from zero is the natural candidate for the application of other methods.

We would also like to quote new results from the Derivatives on the mass spectrum\textsuperscript{18}, where the first order response of the nucleon (neutron) to baryochemical potential was computed in correspondence of several quark masses.

4.2. Reweighting from $\mu = 0$

Back in the 80’s Ian Barbour and collaborators proposed to calculate $Z(\mu)$ from simulations at $\mu = 0$:

$$Z = \left\langle \frac{|M(\mu)|}{|M(\mu = 0)|} \right\rangle_{\mu = 0}$$

(13)

In other words, the chemical potential $\mu$ of the target ensemble at that of the simulation ensemble – $\mu = 0$ – are different: the properties of the target ensemble
can be inferred from those of the simulation ensemble, provided that there is a sizable overlap between the two.  

At $T = 0$ the Glasgow procedure fails because of a poor overlap (aside, the strong coupling calculations were quite useful to asses these problems), and it is instructive to study the overlap problem as seen in the Gross Neveu model, where there is no sign problem, and the results obtained with reweighting methods can be compared with those of exact simulations.

The distribution of the order parameter (the $\sigma$ particle) helps visualizing the problem; the order parameter distributions in the two phases do not overlap.

The conclusion from these early studies was that reweighting fails in QCD at zero temperature because of a poor overlap, and that the reason behind the failure is practical rather than conceptual: the situation can be ameliorated if a better starting point were used.

### 4.3. Fodor and Katz’s multiparameter reweighting

The prescription for ameliorating the overlap is due to Fodor and Katz, whose Multiparameter reweighting use fluctuations around $T_c$ at $\mu = 0$ to explore the critical region. Making reference to Fig. 2, and oversimplifying: instead of trying to reweight the distribution at zero temperature in the broken phase, which is obviously hopeless, one might hope that a distribution generated at zero density, and close to the critical temperature, bears more resemblance with the target distribution along the critical line, and is thus amenable to a successful reweighting.

The strategy was applied to QCD. The improvement obtained is impressive and produced the first quantitative results for the critical line at nonzero chemical potential in QCD: we will come back to this in the section on results. A multistep reweighting proposed by Crompton might well produce a further improvement.

### 4.4. Taylor Expanded Reweighting

The Bielefeld-Swansea collaboration suggested a Taylor expansion of the reweighting factor as a power series in $\lambda = \mu / T$, and similarly for any operator.

This strategy is computationally very convenient as it greatly simplifies the calculation of the determinant. Expectation values are then given by

$$
\langle O \rangle_{(\beta, \mu)} = \frac{\langle (O_0 + O_1 \lambda + O_2 \lambda^2 + \ldots) \exp(\mathcal{R}_1 \lambda + \mathcal{R}_2 \lambda^2 + \ldots - \Delta S_g) \rangle_{\lambda=0, \beta_0}}{\langle \exp(\mathcal{R}_1 \lambda + \mathcal{R}_2 \lambda^2 + \ldots - \Delta S_g) \rangle_{\lambda=0, \beta_0}}. \quad (14)
$$

Results have been obtained both for the critical line and thermodynamics.

### 4.5. Imaginary Baryon Chemical Potential

This method uses information from all of the negative $\mu^2$ half plane (Fig. 1) to explore the positive, physical relevant region. An imaginary chemical potential $\nu$ in
a sense bridges Canonical and Grand Canonical ensemble

\[ Z_C(N) = \frac{\beta}{2\pi} \int_0^{2\pi/\beta} d\nu Z_{GC}(i\nu)e^{-i\beta\nu N} \]  

The main physical idea behind any practical application is that at \( \mu = 0 \) fluctuations allow the exploration of \( N_b \neq 0 \) hence tell us about \( \mu \neq 0 \). Mutatis mutandis, this is the same condition for the reweighting methods to be effective: the physics of the simulation ensemble has to overlap with that of the target ensemble.

A practical way to use the results obtained at negative \( \mu^2 \) relies on their analytical continuation in the real plane. For this to be effective \( 27 \) \( Z(\mu, T) \) must be analytical, nontrivial, and fulfilling this rule of thumb:

\[ \chi(T, \mu) = \partial \rho(\mu, T)/\partial \mu = \partial^2 \log Z(\mu, T)/\partial \mu^2 > 0 \]  

This approach has been tested in the strong coupling limit \( 27 \) of QCD, in the dimensionally reduced model of high temperature QCD \( 29 \) and, more recently, in the two color model \( 30 \).

5. Taylor expansion and analytic continuation

We discuss here in more detail the interrelation between the Taylor expansion and the imaginary chemical potential approach.

Results obtained at imaginary \( \mu_B \) can be analytically continued to real \( \mu_B \) \( 27, 31, 32 \). In principle, rigorous arguments guarantee that the analytic continuation of a function can be done within the entire analytic domain. In practice, the exact analytic form is not known, and a systematic procedure relying on the Taylor expansion is only valid within the circle of convergence of the series itself. Here, we discuss how to implement the analytic continuation of the critical line and of thermodynamics observables beyond the circle of convergence of the Taylor series in a controlled way.

Let us remind ourselves that an analytic function is locally representable as a Taylor series. The convergence disks can be chosen in such a way that they overlap two by two, and cover the analytic domain. Thus, one way to build the analytic continuation is by connecting all of these convergence disks. The arcs of the convergence circles which are within the region where \( f \) is analytic have a pure geometric meaning, and by no means are an obstacle to the analytic continuation. Assume now that the circle of convergence about \( z = (0, 0) \) has radius unit, i.e. is tangent to the lines which limit the analytic domain; take now a \( z \) value, say \( z_1 = (0, a), 1/2 < a < 1 \) inside the convergence disk as the origin of a new series expansion, which is explicitly defined by the rearrangement \( (z - z_0)^n = (z - z_1 + z_1 - z_0)^n \) As the radius of convergence of the new series will be again one, this procedure will extend the domain of definition of our original function (the two series define restrictions of the same function to the intersection between the two disks), and by 'sliding' the convergence disk we can cover all the analytic strip.
We have sketched above the standard theoretical argument to demonstrate the feasibility of analytic continuation beyond the radius of convergence, and we will show that the Pade’ series is one practical way to accomplish it.

To complete this discussion, let me mention that the radius of convergence of a Taylor expansion about the origin might well be larger than the distance of the origin itself from the nearest singularity. While complex analysis textbooks offer a full discussion of this point, I would like to single out here three cases which might be encountered in usual critical behavior: $f_1(z) = A_1(z)(1 - z/z_c)^{-\lambda}; f_2(z) = A_2(z)\theta(z - z_c); f_3(z) = A_3(z)\theta(z - z_c)(1 - z/z^*)^{-\lambda}$, where $A_n(z)$ is an analytic function.

Case 1 corresponds to an usual critical behavior (second order or larger). Case 2 represents a strong first order phase transition. Case 3 is intermediate between the two, a weak first order transition at $z_c$, and a spinodal point at $z^*$. Correspondingly, we have different radius of convergence of the Taylor series: $r_1 = |z_c|, r_2 = \infty, r_3 = z^*$: in conclusion the radius of convergence of the Taylor expansion for the critical line and thermodynamics observables might be infinite as well a finite, depending on the nature of the Roberge Weiss transition. Conversely, if the nature of the phase transition is known, one can infer from it the radius of convergence of the Taylor series, as done by Gavai and Gupta [46], which in turn locates the critical point.

6. The Critical Line : Taylor vs Pade’

A full discussion of the current status of the critical line we refer to recent reviews [2]. Here, we would like to emphasize more general aspects on the possibility of continuing the results beyond the radius of convergence of the Taylor expansion, by taking advantage of the discussions presented in the Section above.

The radius of convergence of the Taylor representation of the critical line might well be limited by the Roberge Weiss singularities (see again Fig.1). However, as explained before, the Pade’ approximation is not.

We [49] have performed present the Pade’ analysis where we have used data for four [52] and two flavor [31]. The results seem stable beyond $\mu_B = 500MeV (\mu_B/T \simeq 1)$, with the Pade’ analysis in good agreement with Taylor expansion for smaller $\mu$ values. At larger $\mu$ the Taylor expansion seems less stable, while the Pade’ still converges, giving a slope of the critical line larger than the naive continuation of the second order Taylor approximations. The same behavior is suggested by recent results within the canonical approach [50] and the DOS method [51].

We underscore that the possibility of analytically continue the results beyond the radius of convergence of the Taylor series by no means imply that one can blindly extrapolate a lower order approximation! Even when it is possible to achieve

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*The analytic continuation is insensitive to a discontinuous phase transition since it lives on the metastable branch : it follows the secondary minimum and determines the spinodal point $<\bar{\psi}\psi> = A(\mu - \mu^*)^3$. The discontinuity can be related to $\mu - \mu^*$. Both shrinks to zero at the endpoint of a first order transition.*
convergence – via Pade’ approximants, or within the convergence radius of the Taylor series – one has always to cross check different orders of approximation to make sure that convergence has indeed been achieved. For instance, it would then be interesting to repeat the comparisons between the second order results shown in by extending the Taylor series to fourth order and/or by use of Pade’ approximants.

6.1. New Results for Wilson Fermions

It is very important to double check the results coming from different methods and different approaches. The results described so far have been obtained with staggered fermions. First calculations with Wilson fermions at finite density, and imaginary chemical potential, have appeared recently-Chen:2004tb,Luo:2004se,Luo:2004mc. The results are so far in nice agreement with those obtained with the staggered formulation.

7. The Hot Phase and the approach to a Free Gas

The behaviour of the number density at high temperature approaches the lattice Stephan-Boltzmann prediction, with some residual deviation. The deviation from a free field behavior can be parametrised as

$$\Delta P(T, \mu) = f(T, \mu) P_{free}^L(T, \mu)$$

where $P_{free}^L(T, \mu)$ is the lattice free result for the pressure. For instance, in the discussion of Ref. $f(T, \mu) = 2(1 - 2\alpha_s/\pi)$

and the crucial point was that $\alpha_s$ is $\mu$ dependent.

We can search for such a non-trivial prefactor $f(T, \mu)$ by taking the ratio between the numerical data and the lattice free field result $n_{free}^L(\mu I)$ at imaginary chemical potential:

$$R(T, \mu I) = \frac{n(T, \mu I)}{n_{free}^L(\mu I)}$$

A non-trivial (i.e. not a constant) $R(T, \mu I)$ would indicate a non-trivial $f(T, \mu)$. In Ref. we calculated $R(T, \mu I)$ versus $\mu I/T$: the results for $T \geq 1.5T_c$ seem consistent with a free lattice gas, with an fixed effective number of flavors $N_{eff}^f(T)/4 = R(T)$: $N_{eff}^f = 0.92 \times 4$ for $T = 3.5T_c$, and $N_{eff}^f = 0.89 \times 4$ for $T = 1.5T_c$.

7.1. Beyond $\mu/T \simeq 1$ in the Hot Phase

The Pade’ approximants to the results for the chiral condensate in the hot phase have been calculated as well using four flavor data. Again we found that the Pade’ analysis seems capable to produce stable results. We should also note that the Taylor expansion seems stable as well, which might indicate a large (infinite?)
radius of convergence in this range of temperature. Indeed, as noted in\cite{44} the radius of convergence should tend to infinite in the infinite temperature limit, and indeed it has been estimated to be large by the Bielefeld–Swansea collaboration\cite{47}. A detailed investigation at imaginary $\mu$ of the region closer to the critical temperature is in progress\cite{48}.

We also note a possible interplay of thermodynamics and critical behaviour for $T_C < T < T_E \simeq 1.1 T_c$ : the critical line at negative $\mu^2$ would imply, at least for second order and weak first order transitions, $\log P(\mu, T) \propto (\mu^2 - \mu_c^2)\eta$ which is incompatible with a free field behaviour.

8. The Hadronic Phase the Hadron Resonance Gas Model

The Hadron Resonance Gas model might provide as description of QCD thermodynamics in the confined, hadronic phase of QCD: the grand canonical partition function of the Hadron Resonance Gas model\cite{37} has a simple hyperbolic cosine behaviour.

This behaviour could be assessed via a computation of the Taylor coefficients\cite{37}. As a perhaps simpler alternative, the hadron gas model can be framed in the discussion of the phase diagram in the temperature-imaginary chemical potential plane which suggests to use Fourier analysis in this region, as observables are periodic and continuous there\cite{32}.

For observables which are even $(O_e)$ or odd $(O_o)$ under $\mu \to -\mu$ the analytic continuation to real chemical potential of the Fourier series read $O_e[\phi](\mu_T, N_t) = \sum_n a_F^{(n)} \cosh[\sinh(n N_t N_c \mu_I)]$. In a Fourier analysis of the chiral condensate\cite{44} - even and odd observables, respectively - we limited ourselves to $n = 0, 1, 2$ and we assessed the validity of the fits via both the value of the $\chi^2$/d.o.f. and the stability of $a_F^{(0)}$ and $a_F^{(1)}$ given by one and two cosine [sine] fits: when HRG holds true, one term in the Fourier series should suffice. $(\sinh(x) \to \sin(x))$ $n(\mu) = \frac{\partial P(\mu)}{\partial \mu} = K \sin(N_c N_t \mu)$.

We found that one cosine [sine] fit describes reasonably well the data up to $T \simeq 0.985 T_c$; further terms in the expansion did not modify much the value of the first coefficients and does not particularly improve the $\chi^2$/d.o.f.. This means that our data are well approximated by the hadron resonance gas prediction $\Delta P \propto (\cosh(\mu_B/T) - 1)$ in the broken phase up to $T \simeq 0.985 T_c$.

In the same region, we can also computer the mismatch with respect to the HRG in an 'effective mass analysis' style: this analysis\cite{18} confirms that the HRG is consistent with our data within errors up to $T = .985 T_c$.

$$\text{Mismatch} = n(\mu)/\sin(N_c N_t \mu) - k$$

8.1. The critical line from the Hadron Gas

An alternative way to analytically continue the results relies on phenomenological modeling. The Hadron Resonance Gas model might provide a description of QCD
thermodynamics in the confined, hadronic phase of QCD\cite{37,47,44} and can be used to determine the critical line as well. The critical temperature as a function of $\mu_B$ is determined by lines of constant energy density: $\epsilon \simeq 0.5 - 1.0$ GeV/fm\cite{52}. A continuation of the critical line using the HRG ansatz plus a fixed energy (or any other quantity determined at $\mu = 0$) criterion suggests the implicit form for the critical line $T = f(T)\cosh(\mu_B/T)$ with $\lim_{\mu_B/T \to 0} f(T)\cosh(\mu_B/T) = 1 - k\mu^2$. We have naively approximated $f(T) = 1 - k\mu^2$, and used the resulting form to fit the data in the $\mu/B < 1$ range. According to the above discussion, this again can be continued beyond this limit, and also in this case we get a critical line whose slope increases with increasing $\mu_B$\cite{49}.

### 8.2. How to calculate the critical values

These calculations described above require the critical density as an input. Let us then consider:

$$n(i\mu) = a_1\sin(i\mu N_c N_T) + a_2\sin(i2\mu N_c N_T)$$ \hspace{1cm} (21)

Analytic continuation up to $\mu = \mu_c(T)$ gives:

$$n(\mu) = a_1\sinh(\mu N_c N_t) + a_2\sinh(i2\mu N_c N_T)$$ \hspace{1cm} (22)

This gives the critical density at $T = .985 T_c$ and mass = .05 in lattice units: $n_c(\mu_c)/T^3 \simeq 0.44$\cite{44}, once the value of $\mu_c$ has been taken into account. In addition, the mass dependence of the critical density has been estimated\cite{44} from the Maxwell Relations:

$$\partial <\bar{\psi}\psi> /\partial \mu = \partial n(\mu)/\partial m$$ \hspace{1cm} (23)

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### 9. Parting Comments

We have described three different, independent methods which afford a quantitative study of the phases, and phase transitions in QCD. All of the methods exploit physical fluctuations either at zero and purely imaginary chemical potential to explore real baryon chemical potential. The physical idea in a sense is similar, but the systematics is very different. Cross check are then most useful and informative, and in many cases, which we have reviewed, have been performed satisfactorily.

I should reiterate a caveat: both the critical line, and thermodynamics observables in the hadronic phase are very sensitive to the quark masses. I preferred to
concentrate on methods and general idea, rather than on rapidly evolving results. I apologize for possible incomplete presentation, and refer once more to the reviews I cited. The reader who chances into this note at some later stage, might find useful to read the plenary reviews on Thermodynamics at the Latticexx Conferences and that on Lattice at the QuarkMatterxx Conferences.

Finally, again and again, the methods described here are dodges for the sign problem reviewed above. A more complete solution might well be afforded by the Hamiltonian approach reviewed, the density of states formalism, the canonical method, or the strong coupling expansion.

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