MULTI-OBJECTIVE ROBUST CROSS-MARKET MIXED PORTFOLIO OPTIMIZATION UNDER HIERARCHICAL RISK INTEGRATION

HAN YANG
Department of Mathematics
Sichuan University
Chengdu, Sichuan, 610064, China

JIA YUE
Department of Economic Mathematics
Southwestern University of Finance and Economics
Chengdu, Sichuan, 610074, China

NAN-JING HUANG*
Department of Mathematics
Sichuan University
Chengdu, Sichuan, 610064, China

(Communicated by Ken Siu)

Abstract. In this paper, we consider a multi-objective robust cross-market mixed portfolio optimization model under hierarchical risk integration in the international financial market consisting of finite sub-markets. It is difficult to describe the dependent structure accurately by the traditional copula theory because of the dependent structures of the risk assets in finite sub-markets are different usually. By employing the hierarchical risk integration method, we establish the multi-objective robust cross-market mixed portfolio model in which the worst-case value at risk is used as the risk measurement and the transaction costs, skewness and investment proportion limitation are all considered. We provide a new algorithm to calculate the worst-case value at risk of the cross-market mixed portfolio and give a numerical experiment to show the superiority of the model considered in this paper.

1. Introduction. The famous Markowitz mean-variance model, proposed by Markowitz [27] in 1952, is a pioneer of portfolio theory. Markowitz portfolio theory opens the door of modern portfolio theory, which lays the foundation for the portfolio decision. Various theoretical results, numerical algorithms and applications have been studied extensively for the mean-variance models in the literature. By using the value at risk (VaR) to replace the variance as a risk measurement, Chang et al. [6] studied the portfolio credit risk model with random loss given default and Su [34] investigated the stock indices in developed and emerging markets. In connection

2010 Mathematics Subject Classification. 91G50, 91G60, 91B30, 62H20, 62E17, 62P99, 65C20.
Key words and phrases. Cross-market mixed portfolio, hierarchical risk integration, multi-objective robust optimization, worst-case value at risk.

This work was supported by the National Natural Science Foundation of China (11471230, 11671282, 11801462).

* Corresponding author: Nan-jing Huang.
with the studies of the mean-variance model with the investment proportion limitation constraint can be found in [4, 17, 19, 41]. Concerned with the studies for the mean-variance model with the transaction costs constraints, we refer the reader to [18, 29, 40, 43].

It is well known that the robust optimization theory has been widely used to study the portfolio selection and others by many authors (see, for example, [9, 12, 13, 16, 26, 37, 45] and the references therein). We note that Ghaoui et al. [13] put forward the concept of the worst-case value at risk (WCVaR) to measure the risk when the distribution of portfolio’s return is partially known and gave an application to the robust portfolio selection. Recently, Zymler et al. [45] used WCVaR in the study of European option portfolio and gave two improved forms of WCVaR. On the other hand, the multi-objective optimization theory has been gradually applied to the study of the portfolio optimization models considering the higher order moment information; for instance, we refer the reader to [2, 7, 21, 24]. As pointed out by Arditti and Levy [2], the higher order moments of the portfolio returns can not to be ignored, unless there is good reason to prove that the return rate of risk assets follow the normal distribution or the higher order moment information is not related to the investor’s decision. However, two conditions mentioned above are almost untenable in practice. Therefore, it is necessary to consider the portfolio optimization models with the higher order moments of the portfolio returns.

The skewness of a random variable is the third standardized moment which can be used to describe the asymmetry of the probability distribution of a real-valued random variable about its mean. Konno and Suzuki [21] proposed a multi-objective portfolio optimization model based on mean-variance-skewness and three computational schemes for solving an associated non-concave maximization problem. Thereafter, Lai et al. [22] introduced the skewness and kurtosis into portfolio model to set up a multi-objective portfolio optimization model and gave the solving process.

It is worth mentioning that the ability to accurately measure the risk of portfolio determines the quality of the optimal portfolio. In order to obtain accurate portfolio risk measurement, we need to quantify the relevance of risk assets at first. We know that the covariance matrix of risk assets’ returns is used to describe the correlation of the risk asset in the traditional Markowitz model. The covariance can only reflect the linear correlation between two risk assets, but the dependent structure of risk assets is much more complicated than the linear structure in practice. Therefore, it is necessary to find a new technique to accurately depict and quantify the correlation of risk assets. We find that copula [31] is a good tool to describe the correlation of random variables from the research of correlation analysis. Copula, joined the multivariate distribution functions to their one-dimensional marginal distribution functions, is a way of studying scale-free measures of dependence [28].

The copulas have been used extensively in financial analysis and risk management such as the measurement of credit risk, the correlation measurement of capital market and the measurement of market risk. Sak et al. [30] use t-copula to construct the joint distribution function of stocks in New York Stock Exchange (NYSE). They used the improved Monte Carlo method to estimate the parameters of t-copula and gave an empirical study of NYSE. It can be seen from the results of the study that the t-copula estimate by the improved Monte Carlo method can accurately measure the correlation of stocks in NYSE. Ghorbel and Trabelsi [14] use extreme value copula methods to study the portfolio risk management of energy commodities (WTI oil, natural gas, heating oil). They compared the accuracy of different copula
and found that EVT-copula can exactly measure of the correlation among the three energy products. Similarly, Zhan and Zhang [42] considered the Chinese stock market and chose the SV model and normal copula to describe the dependent structure of stocks. Recently, Zhou et al. [44] studied the grouped risk aggregation problem under mixed operation by using copulas. They divided the basic risk into several groups according to the actual situation and describe the dependency structure among the basic risks in each group by different copulas. Then they calculated the integrated risk of mixed operation in two steps. The first step is to describe the dependence of the risks in the same group with a copula and calculate the integrated risk in group. The second step is to describe the dependence of the group’s integrated risk with a copula and calculate the risk of mixed operation.

We note that all of the researches in portfolio management with copula we mentioned above only focus on the portfolio of financial assets in a single market. However, with the rapid development of global economic integration, the investors’ investment target is no longer restricted by region, no matter where they can invest in the financial markets in different countries at the same time. We call the behavior of investing in different financial markets at the same time that the cross-market mixed portfolio. Moreover, we find that the copula used to characterize the dependent structure is from different type for different markets which means the dependence structure in different financial markets is not the same. Different dependency structure means that in different financial markets we need to use different types of copula or the same type of copula with different parameters to describe the relevance of the risk assets in the markets. The traditional copula method is not applicable in the cross-market mixed investment, because it can not characterize the difference of different financial markets’ dependence structure by only one copula. Therefore, how to apply copula to describe the correlation of the risk assets in the cross-market mixed portfolio becomes and obtain the optimal investment strategy is a very valuable question. The main purpose of this paper is to solve the correlation measurement problem by using the method in [44] and establish the multi-objective robust cross-market mixed portfolio model under hierarchical risk integration.

This paper is organized as follows. Section 2 presents some preliminaries of the knowledge which will be used in this paper. In Section 3, by employing the hierarchical risk integration method, we establish the multi-objective robust cross-market mixed portfolio model involving WCVaR, transaction costs, skewness and investment proportion limitation. Moreover, we provide a numerical algorithm to estimate WCVaR of the cross-market mixed portfolio. A numerical example for the cross-market mixed portfolio management is given in Section 4 and some conclusions are presented in Section 5.

2. Preliminaries. In this section, we mainly introduce some preliminary knowledge needed in the following discussion.

**Definition 2.1.** ([35]) Let $X$ be the rate of return of the portfolio, $F_X$ be the distribution of $X$ and $\beta$ be the confidence level. Then the $\beta$-VaR of $X$ is defined as

$$\text{VaR}_\beta(X) = -F_X^{-1}(1 - \beta).$$

where, $F_X^{-1}$ is the inverse of the distribution function $F_X$.

**Definition 2.2.** ([13]) Let $X$ be the rate of return of the portfolio, $\beta$ be the confidence level, and $F_X$ be the distribution of $X$ selected from $\mathcal{P}$, where $\mathcal{P}$ is the
set of all known distributions which can be used as the distribution of $X$. The WCVaR of $X$ is defined as

$$ WCVaR_\beta(X) = \min_{\gamma} \gamma \quad s.t. \quad \gamma = \sup_{P_X \in \Psi} \text{Prob}\{\gamma \leq -X\} \leq 1 - \beta. $$

Based on Definitions 2.1 and 2.2, we can give a more concise definition of WCVaR as follows.

**Definition 2.3.** Let $X$ be the rate of return of the portfolio, $\beta$ be the confidence level, $F_X$ be the distribution and $\Psi$ be the collection of all possible distributions. We can define the WCVaR of $X$ as

$$ WCVaR_\beta(X) = \max_{F_X \in \Psi} VaR_\beta(X). $$

**Definition 2.4.** ([22]) Assume that there are $N$ risk assets $X_1, X_2, \ldots, X_N$. Let $w = (w_1, w_2, \ldots, w_N)$ be the portfolio, $R = (R_1, R_2, \ldots, R_N)^\top$ be the rate of return of $N$ risk assets, and $\overline{R}$ be the mean of $R$. Then the skewness of portfolio $w$’s return can be defined as

$$ Skew = E\left[\left(w(R - \overline{R})\right)^3\right] = E(R_w - \overline{R_w})^3, $$

where $R_w = w \cdot R$ is the rate of return of portfolio, $\overline{R_w} = w \cdot \overline{R}$ is the mean of $R_w$. Assume that $r_1, r_2, \ldots, r_T$ are the samples of $R_w$, then (2) can be rewritten as

$$ Skew = \frac{1}{T} \sum_{t=1}^{T} (r_t - \overline{r})^3, $$

where $\overline{r} = \frac{1}{T} \sum_{t=1}^{T} r_t$ is the mean of the $T$ samples.

**Definition 2.5.** ([20]) Let $E_1, E_2, \ldots, E_n$ be nonempty subsets of $\mathbb{R}$, $F$ be an $n$-dimensional real function with domain $\text{Dom}(F) = E_1 \times \cdots \times E_n$, and $H = [a_1, b_1] \times \cdots \times [a_n, b_n]$, where $[a_i, b_i] \subseteq E_i$ for $i = 1, \ldots, n$. Then the $F$-volume of $H$ can be given by

$$ V_F(H) = \sum \text{sgn}(c) F(c), $$

where

$$ \text{sgn}(c) = \begin{cases} 1, & \text{if } c_k = a_k \text{ for an even number of } k, \\ -1, & \text{if } c_k = a_k \text{ for an odd number of } k. \end{cases} $$

Equivalently, the $F$-volume of $H = [a, b]$ can be written as

$$ V_F(H) = \Delta^b_a F(t) = \Delta^b_{a_n} \Delta^b_{a_{n-1}} \cdots \Delta^b_{a_1} F(t), $$

where the $n$-first order differences of $F$ is defined as

$$ \Delta^b_{a_k} F(t) = F(t_1, \ldots, t_{k-1}, b_k, t_{k+1}, \ldots, t_n) - F(t_1, \ldots, t_{k-1}, a_k, t_{k+1}, \ldots, t_n). $$

**Definition 2.6.** ([20]) An $n$-dimensional real function $F$ is called $n$-increasing if $V_F(H) \geq 0$ for all closed interval $H \subseteq \text{Dom}(F)$, and $F$ is called grounded if $F(t) = 0$ for all $t \in \text{Dom}(F)$ with $t_k = a_k$ for at least one $k \in \{1, 2, \ldots, n\}$.

**Definition 2.7.** ([28]) An $n$-dimensional copula is a real function $C$ with the following properties:

1. $C : I^n \to I$, where $I = [0, 1]$;
2. $C$ is grounded and $n$-increasing;
(3) $C$ has (1-dimensional) margins $C_k(x_k) = C(1, \cdots, 1, x_k, 1, \cdots, 1)$ for $k = 1, 2, \cdots, n$, satisfying $C_k(u) = u$ for all $u \in I$.

**Lemma 2.8.** ([21]) Let $F$ be an $n$-dimensional distribution function with the 1-dimensional marginal distributions $F_1, F_2, \cdots, F_N$. Then there exists an $n$-copula $C$ such that, for all $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$,
$$F(x_1, x_2, \cdots, x_n) = C(F_1(x_1), F_2(x_2), \cdots, F_n(x_n)). \tag{3}$$
If $F_1, F_2, \cdots, F_n$ are all continuous, then copula $C$ is unique; otherwise, copula $C$ is non-unique on $I^n$ but determined solely on $\text{Ran}(F_1) \times \text{Ran}(F_2) \times \cdots \times \text{Ran}(F_n)$. On the other hand, if $C$ is an $n$-copula and $F_1, F_2, \cdots, F_n$ are marginal distributions, then the function $F$ defined by (3) is an $n$-dimensional distribution with margins $F_1, F_2, \cdots, F_n$.

**Definition 2.9.** ([38]) A function $g(t)$ is completely monotonic on an interval $E$ if it is continuous and has derivatives of all orders which satisfies
$$(−1)^k \frac{d^k}{dt^k} g(t) \geq 0 \tag{4}$$
for all $t$ in the interior of $E$ and $k = 0, 1, 2, \cdots$.

**Definition 2.10.** ([28]) Let $\varphi$ be a continuous strictly decreasing function from $I$ to $[0, \infty]$ such that $\varphi(0) = \infty$ and $\varphi(1) = 0$. Let $\varphi^{-1}$ denote the inverse of $\varphi$ and $C_\varphi$ be the function from $I^n$ to $I$ defined as
$$C_\varphi(u) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + \cdots + \varphi(u_n))$$
where $u = (u_1, u_2, \cdots, u_n)$. When $\varphi^{-1}$ is completely monotonic on $[0, \infty)$, $C_\varphi$ is an $n$-copula for all $n \geq 2$ and it is called an Archimedean copula.

3. Multi-objective robust cross-market mixed portfolio optimization model. In this section, in order to solve optimal portfolio selection problems in international financial markets, we introduce the robust cross-market mixed portfolio optimization model under the hierarchical risk integration, in which the skewness, transaction cost and investment proportion limitation are all considered.

3.1. Cross-market mixed portfolio. In this subsection, we consider the international financial market composed of $M$ different sub-markets and assume that there are $n_i$ risk assets in the $i$-th sub-market for $i = 1, 2, \cdots, M$. Figure 1 shows the mathematical structure of this financial market.

![Figure 1. The Structure of Financial Market.](image)
When we want to make a cross-market mixed portfolio decision in such an international financial market, it is inevitable to consider the dependence structure of risk assets to measure the risk of portfolio. The cross-market mixed portfolio consists of the portfolio on the risky assets in each sub-market and the portfolio on the sub-markets. Figure 2 shows the structure of the cross-market mixed portfolio which not only exhibits the difference between the cross-market mixed portfolio and the traditional portfolio, but also has obvious advantage considering the hierarchical risk integration of the portfolio. It is worth noting that when all of the risk assets come from the same market, this portfolio problem will be degenerated into the classic one.

Let the investment proportion on the $j$-th risk asset in the $i$-th sub-market be $w_{i,j}$ and the proportion on the $i$-th sub-market be $w_i$ for $j = 1, 2, \cdots, n_i$ with $i = 1, 2, \cdots, M$. Assume that the number of the risk assets in all sub-markets is $N$, that is, $N = n_1 + n_2 + \cdots + n_M$. In order to make the sum of the investment proportion on all the assets in the cross-market mixed portfolio is one, we ask $\sum_{j=1}^{n_i} w_{i,j} = 1$ for $i = 1, 2, \cdots, M$ and $\sum_{i=1}^{M} w_i = 1$. Therefore, we can get the cross-market mixed portfolio $w \in [0, 1]^N$ in the international financial market as follows:

$$w = (w_1 w_{1,1}, \cdots, w_1 w_{1,n_1}, w_2 w_{2,1}, \cdots, w_2 w_{2,n_2}, \cdots, w_M w_{M,1}, \cdots, w_M w_{M,n_M}).$$ \hspace{1cm} (5)

3.2. Copula based hierarchical risk integration of cross-market mixed portfolio. Arbenz et al. [1] first proposed the concept of copula based hierarchical risk integration by employing the rooted tree theory [8]. Compared with the traditional copula theory, Arbenz et al [1] described the dependence structure of high dimensional random vector by the aggregation of low dimensional copula, which overcomes the difficulty causing by the increase of dimensions. Thereafter, Zhou et al. [44] used the copula based hierarchical risk integration theory to measure the risk (VaR) of mixed operation and gave an algorithm to calculate the VaR.

From Figure 1, we can assume that the dependence structure of the risk assets in each sub-market are different and there are correlations among sub-markets. Thus, we use the hierarchical risk integration theory [1] to study the dependence structure of the risk assets and give an algorithm to calculate the risk of the cross-market mixed portfolio.
Assume that all the \( N \) risk assets come from \( M \) different financial sub-markets and the \( i \)-th sub-market has \( n_i \) risky assets with \( N = \sum_{i=1}^{M} n_i \). Let the cross-market mixed portfolio be defined as (5), \( X_{i,j} \) be the rate of return of the \( j \)-th risk asset from the \( i \)-th sub-market, \( X_i = w_{i,1}X_{i,1} + w_{i,2}X_{i,2} + \cdots + w_{i,n_i}X_{i,n_i} \) be the rate of return of the portfolio in \( i \)-th sub-market and \( X = w_1X_1 + \cdots + w_MX_M \) be the rate of return of the cross-market mixed portfolio for \( j = 1, 2, \ldots, n_i \) and \( i = 1, 2, \ldots, M \). From the above assumptions, the rate of return of the cross-market mixed portfolio can be expressed as follows:

\[
X = w_1w_{1,1}X_{1,1} + \cdots + w_1w_{1,n_1}X_{1,n_1} + \cdots + w_Mw_{M,1}X_{M,1} + \cdots + w_Mw_{M,n_M}X_{M,n_M}.
\]  

(6)

In order to obtain the hierarchical risk integration of the cross-market mixed portfolio based on copula, we use WCVaR as the risk measurement and make the following assumptions. Because of the different correlation in each sub-market, we use different copulas to characterize these dependence structures. Let the dependence structure of risk assets in the \( i \)-th sub-market can be described by an \( n_i \)-dimensional copula \( C_i \) for \( i = 1, 2, \ldots, M \) and use an \( M \)-dimensional copula \( C_0 \) to describe the dependence structure of all sub-markets. In addition, we use \( \mathcal{C} \) to represent the set of all copulas. Then we can define a set of copulas

\[
\mathcal{C} = \{ C_i | C_i \in \mathcal{C}_i, i = 0, 1, 2, \ldots, M \},
\]

where \( \mathcal{C}_i \) is a subset of \( \mathcal{C} \) with \( m_i \) different copulas.

**Theorem 3.1.** Let \( F_{i,j} \) be the distribution of the return on risk asset \( X_{i,j} \) for \( j = 1, 2, \ldots, n_i \) with \( i = 1, 2, \ldots, M \). Let the return \( X \) of the cross-market mixed portfolio be defined as (6). Assume that \( C_i \) is the copula of \( (X_{i,1}, \ldots, X_{i,n_i}) \) with \( i = 1, 2, \ldots, M \) and \( C_0 \) is the copula of \( X_1, \ldots, X_M \). Then the distribution \( F(x) \) of \( X \) is uniquely determined by

\[
C_i(i = 0, 1, \ldots, M), \quad F_{i,j}(i = 1, \ldots, M; j = 1, \ldots, n_i)
\]

and

\[
w = (w_1w_{1,1}, \ldots, w_1w_{1,n_1}, \ldots, w_Mw_{M,1}, \ldots, w_Mw_{M,n_M}).
\]

**Proof.** For \( i = 1, \ldots, M \), let \( F_i(x_i) \) denote the distribution of \( i \)-th sub-market’s portfolio return, the copula \( C_i \) be given, and \( F_{i,1}, \ldots, F_{i,n_i} \) be the distributions of \( X_{i,1}, \ldots, X_{i,n_i} \), respectively. Then, by Lemma 2.8, we know that the joint distribution of \( (X_{i,1}, \ldots, X_{i,n_i}) \) can be uniquely expressed as follows for \( i = 1, 2, \ldots, M \).

\[
P[X_{i,1} \leq x_{i,1}, \ldots, X_{i,n_i} \leq x_{i,n_i}] = C_i[F_{i,1}(x_{i,1}), \ldots, F_{i,n_i}(x_{i,n_i})].
\]  

(7)

This implies that the distribution of \( X_i \) can be uniquely given as follows:

\[
F_i(x_i) = P[X_i \leq x_i] = P[w_{i,1}X_{i,1} + \cdots + w_{i,n_i}X_{i,n_i} \leq x_i] = \int_{\mathbb{R}^{n_i}} 1\{w_{i,1}x_{i,1} + \cdots + w_{i,n_i}x_{i,n_i} \leq x_i\}dC_i[F_{i,1}(x_{i,1}), \ldots, F_{i,n_i}(x_{i,n_i})]
\]

for \( i = 1, 2, \ldots, M \), where \( 1(\cdot) \) is the indicator function. Clearly, \( F_i(x_i) \) only depends on \( F_{i,1}, \ldots, F_{i,n_i} \), \( (w_{i,1}, \ldots, w_{i,n_i}) \) and \( C_i \) for \( i = 1, \ldots, M \).
Algorithm 3.1. Set sample size \( K \in \mathbb{N} \) and cross-market mixed portfolio \( w \) as \((9)\).

**Step 1.** Let \( C_{i,j} \) be the copula in \( C_i \) and use historical data to estimate the parameters in \( C_{i,j} \) for \( j = 1, 2, \ldots, m_i \). Then, we select a copula \( C_i \) from \( \{ C_{i,j} \mid j = 1, 2, \ldots, m_i \} \) to use in the following steps for \( i = 0, 1, 2, \ldots, M \). Set \( t = 1 \).

**Step 2.** For each given \( i = 1, 2, \ldots, M \), assume that random vector \( (U_{i,1}, \ldots, U_{i,n_i}) \) has the same copula with the vector \( (X_{i,1}, \ldots, X_{i,n_i}) \), that is, the copula of \( (U_{i,1}, \ldots, U_{i,n_i}) \) is \( C_i \). Moreover, assume that \( U_{i,j} \sim U[0, 1] \) is the uniform distribution on \( [0, 1] \) for \( j = 1, 2, \ldots, n_i \). Generate \( K \) independent samples \( \{ (u_{i,1}, \ldots, u_{i,n_i}) \mid k = 1, 2, \ldots, K \} \) from \( (U_{i,1}, \ldots, U_{i,n_i}) \).

**Step 3.** For \( i = 1, 2, \ldots, M \), let \( x_{i,j} = F_{i,j}^{-1}(u_{i,j}), j = 1, 2, \ldots, n_i \). Then we can get \( K \) samples, denoted by \( \{ (x_{i,1}, \ldots, x_{i,n_i}) \mid k = 1, 2, \ldots, K \} \), from random vector \( (X_{i,1}, \ldots, X_{i,n_i}) \) with copula \( C_i \).

**Step 4.** For \( i = 1, 2, \ldots, M \), let \( x_i = \sum_{j=1}^{n_i} w_{i,j} x_{i,j} \), where \( w_{i,j} \) is the portfolio ratio on \( j \)-th risk asset in \( i \)-th sub-market and \( x_{i,j} \) is the sample of \( X_{i,j} \). Then we can get \( K \) samples of \( X_i \), denoted by \( \{ (x_i)_{k} \mid k = 1, 2, \ldots, K \} \).

**Step 5.** Assume that random vector \( (U_1, \ldots, U_M) \) has the same copula with the vector \( (X_1, \ldots, X_M) \), that is, the copula of \( (U_1, \ldots, U_M) \) is \( C_0 \). In addition, \( U_m \sim U[0, 1] \) is the uniform distribution on \( [0, 1] \) for \( m = 1, 2, \ldots, M \). Generate \( K \) independent samples \( (u_1, \ldots, u_M) \) from \( (U_1, \ldots, U_M) \).

**Step 6.** For \( i = 1, 2, \ldots, M \), sort \( \{ (x_{i,1}, \ldots, x_{i,K}) \mid k = 1, 2, \ldots, K \} \) with the ascending order.

Then we get a new sample sequence \( \{ (x_{i,1})_{k} \mid k = 1, 2, \ldots, K \} \), where \( (x_{i})_{k} \) is the \( k \)-th order statistics of \( \{ (x_{i,1}, \ldots, x_{i,K}) \} \).

**Step 7.** For \( i = 1, 2, \ldots, M \), let \( (x'_{i})_{k} = (x_{i})_{k}^{*} \) for \( k = 1, 2, \ldots, K \). After reordering, the sequence \( \{ (x_{1})_{k}^{*} \}, \{ (x_{2})_{k}^{*} \}, \ldots, \{ (x_{M})_{k}^{*} \} \) are independent now. Let \( (x'_{1})_{k}, (x'_{2})_{k}, \ldots, (x'_{M})_{k} = (x'_{1})_{k}, (x'_{2})_{k}, \ldots, (x'_{M})_{k} \). Then the sequence \( \{ (x'_{1})_{k}, (x'_{2})_{k}, \ldots, (x'_{M})_{k} \mid k = 1, 2, \ldots, K \} \) is the sample of the vector \( (X_1, X_2, \ldots, X_M) \).

**Step 8.** Let \( (x)_{k} = \sum_{i=1}^{M} w_{i,x_{i}} \), where \( k = 1, 2, \ldots, K \). Then it is easy to see that \( \{(x)_{k} \mid k = 1, 2, \ldots, K\} \) is the sample sequences of \( X = \sum_{i=1}^{M} \sum_{j=1}^{n_i} w_{i,j} X_{i,j} \).

**Step 9.** Sort \( \{ (x_{1})_{k}^{*}, \ldots, (x_{K})_{k}^{*} \} \) with the ascending order and get the reordering sample \( \{ (x_{1})_{k}, \ldots, (x_{K})_{k} \} \).
Step 10. For a given $\beta$, let $K\beta = [K(1 - \beta)]$. Then the estimation of VaR of portfolio $w$ is $-(x)_K^{\beta}$ for chosen $\{C_i|i = 0, 1, 2, \cdots, M\}$ and let $VaR^{(t)}_\beta = -(x)_{K(t)}^{\beta}$. Set $t = t + 1$.

Step 11. Repeat steps 1 to 10 for all the elements in $C_i$ with $i = 0, 1, 2, \cdots, M$. Then we can get a collection of all possible VaR as follows:

$$V = \{VaR^{(t)}_\beta | t = 1, 2, \cdots, T\},$$

where $T$ is the number of all possible combinations of $\{C_i|i = 0, 1, 2, \cdots, M\}$.

Step 12. Let

$$WCVaR_\beta = \max_{t=1,2,\cdots,T} VaR^{(t)}_\beta.$$

Then $WCVaR_\beta$ is WCVaR of the cross-market mixed portfolio under the confidence level $\beta$.

We note that in Algorithm 3.1, since the real distributions of $X_{i,j}$ are unknown for $j = 1, 2, \cdots, n_i$ and $i = 1, 2, \cdots, M$, we use the empirical distributions of $X_{i,j}$ instead of the real ones. Therefore, it is necessary to prove the convergence of Algorithm 3.1.

**Theorem 3.2.** For given copulas $C_i$, $i = 0, 1, 2, \cdots, M$, the estimation of $VaR_\beta$ for $X$ in Algorithm 3.1 converges to the real $VaR_\beta$ for $X$. Moreover, $WCVaR_\beta$ obtained by Algorithm 3.1 is also convergent.

**Proof.** Similar to the proofs of Lemmas 4.1 and 4.2, and Theorem 4.3 in [44], the conclusions of Theorem 3.2 can be easily proved and so we omit it here. □

### 3.3. Multi-objective robust cross-market mixed portfolio optimization model

To obtain the portfolio optimization model in the international financial market, we improve the traditional Markowitz model. First of all, we use WC-VaR as the risk measurement to replace the variance in the traditional Markowitz model and consider the impact of the skewness on the portfolio optimization problem. Secondly, we add constraints of the transaction cost and investment proportion limitation in the traditional Markowitz model. Combining all of the above improvements, we can get the following multi-objective robust cross-market mixed portfolio optimization model.

$$\min_w WCVaR_\beta = \sup_{C_i \in \mathcal{C}_i, i = 0, 1, \cdots, M} VaR_\beta$$

$$\max_w Skew = \frac{1}{T} \sum_{i=1}^{T} (r_i - \bar{r})^3$$

$$\text{s.t. } \begin{cases} w^\top I = 1, \\ \frac{1-c}{1+c} w^\top e - \frac{2c}{1+c} \geq \mu, \\ w_{\text{min}} \leq w \leq w_{\text{max}}, \end{cases}$$

where $w$ is the cross-market mixed portfolio, $C_i$ is the $i$-th copula for $i = 0, 1, \cdots, M$, $\mathcal{C}_i$ is the subset of $\mathcal{C}$, $Skew$ is the skewness of the rate of return on the cross-market mixed portfolio, $T$ is the sample size, $r_i$ is the $i$-th sample of cross-market mixed portfolio’s return, $I$ is the vector with all of its elements are 1, $c$ is the transaction cost ratio, $e$ is the expected return vector for all risk assets, $\mu$ is the given minimum expected rate of return for the portfolio, $w_{\text{min}}$ and $w_{\text{max}}$ are the lower and upper bound of the portfolio proportion, respectively.
Clearly, model (10) is a multi-objective optimization problem with constraints, which is a complex non-linear and non-convex optimization problem. In general, it is difficult to solve model (10) directly through the known optimization methods. We note that some results for solving multi-objective optimization model with the same form as model (10) were given in [5, 22, 23]. Following the work [5, 22, 23], model (10) can be decomposed into the multiple single-objective optimization model as follows:

\[
\min_w Z = \frac{\xi_1}{V^*} + \frac{\xi_2}{S^*} \quad \quad (11)
\]

\[
\begin{align*}
\text{s.t.} & \quad \sup_{i=0,1, \ldots, M} VaR_{\beta} - \xi_1 = V^*, \\
& \quad \frac{1}{T} \sum_{i=1}^{T} (r_i - \bar{r})^3 + \xi_2 = S^*, \\
& \quad w^T I = 1, \\
& \quad 1\frac{1-w}{1+c} w^T e - \frac{2c}{1+c} \geq \mu, \\
& \quad w_{\min} \leq w \leq w_{\max},
\end{align*}
\]

where \(\xi_1\) and \(\xi_2\) are global variables, \(\lambda_1\) and \(\lambda_2\) are two given parameters that represent, respectively, investors’ risk preference for the worst-case VaR and skewness, \(V^*\) is the optimal value of the following optimization problem:

\[
\min_w WCVaR_{\beta} = \sup_{i=0,1, \ldots, M} VaR_{\beta} \quad \quad (12)
\]

\[
\begin{align*}
\text{s.t.} & \quad w^T I = 1, \\
& \quad 1\frac{1-w}{1+c} w^T e - \frac{2c}{1+c} \geq \mu, \\
& \quad w_{\min} \leq w \leq w_{\max},
\end{align*}
\]

and \(S^*\) is the optimal value of the following optimization problem:

\[
\max_w Skew = \frac{1}{T} \sum_{i=1}^{T} (r_i - \bar{r})^3 \quad \quad (13)
\]

\[
\begin{align*}
\text{s.t.} & \quad w^T I = 1, \\
& \quad 1\frac{1-w}{1+c} w^T e - \frac{2c}{1+c} \geq \mu, \\
& \quad w_{\min} \leq w \leq w_{\max}.
\end{align*}
\]

It is well known that the genetic algorithm is commonly used to generate high-quality solutions to optimization and search problems. We note that Li and Xu [23] solved the multi-objective portfolio selection problem by using the genetic algorithm. Thus, we can employ the genetic algorithm to solve problem (11).

4. A numerical example. In this section, we give a numerical example to illustrate the effectiveness and robustness of the model presented in this paper. In the following example, we consider four important stock indices from two different regions, that is, Shanghai Stock Exchange Composite Index (SZSZS) and Hang Seng Index (HSI) from the Asian stock market, Deutscher Aktienindex (DAX) and Amsterdam Stock Exchange Index (AEX) from the European stock market. Thus, the sub-markets are Asian and European stock markets, the risk assets are SZSZS, HSI, DAX and AEX. We select the data from January 2010 to December 2015 as the sample set and the data from January 2016 to March 2017 as the test set. Moreover,
we only consider three most commonly used Archimedean copula, that is, Gumbel copula, Clayton copula and Frank copula.

- **Gumbel copula:**
  \[
  \exp\left\{-\left[(-\ln(u))^{1/\theta} + (-\ln(v))^{1/\theta}\right]^{\theta}\right\}, \quad \theta \in (0, 1].
  \]

- **Clayton copula:**
  \[
  (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \quad \theta \in (0, +\infty).
  \]

- **Frank copula:**
  \[
  -\frac{1}{\theta} \ln\left[1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right], \quad \theta \in (-\infty, +\infty)\setminus\{0\}.
  \]

We would like to mention that Gumbel copula, Clayton copula and Frank copula reflect the upper tail correlation, the lower tail correlation and the symmetric correlation, respectively.

---

**Figure 3. Index Trend from January 2010 to March 2017.**

Figure 3 shows the trend of the four stock indices. We use the index of the first trading day of 2016 as the standard to standardize all the indices in the time period, the standardized index is shown in Figure 4. In addition, we calculate the daily rate of return of the four stock indices in the sample period as the data foundation (see Figure 5).

By computing, we can obtain the moment information of daily return rate from January 2010 to December 2016 as Table 1, which tell us that the expected returns of all indices are positive and DAX has the highest expected rate of return 0.0483%. In addition, Table 1 shows that SZSS and HSI have negative skewness, while DAX and AEX have the positive skewness, which means DAX and AEX are more likely to generate positive returns. The fact that all the kurtosis values of four indices
are greater than 3 shows that the distributions of the rate of return for four indices have a thicker tail than the normal distribution.

We can calculate the correlation coefficient matrix and rank correlation coefficient Kendall’s $\tau$ [21] of four indices SZZS, HSI, DAX and AEX as Tables 2 and 3, respectively.
Table 1. Moment information of daily return rate from January 2010 to December 2016.

| Name | Mean(%) | Variance | Skewness | Kurtosis |
|------|---------|----------|----------|----------|
| SZZS | 0.0170  | 2.1816   | -0.3702  | 7.6127   |
| HSI  | 0.0072  | 1.3876   | -0.0621  | 6.0437   |
| DAX  | 0.0483  | 1.7579   | -0.1611  | 5.2904   |
| AEX  | 0.0243  | 1.3845   | -0.1214  | 5.8596   |

Table 2. Correlation Coefficient Matrix.

|       | SZZS  | HSI    | DAX    | AEX    |
|-------|-------|--------|--------|--------|
| SZZS  | 1.0000| 0.5265 | 0.1463 | 0.1700 |
| HSI   | 1.0000| 0.3596 | 0.3900 |        |
| DAX   | 1.0000| 0.9146 |        |        |
| AEX   | 1.0000|        |        |        |

Table 3. Rank Correlation Coefficient Kendall’s τ.

|       | DJIA  | DAX    | SZZS   | N225   |
|-------|-------|--------|--------|--------|
| DJIA  | 1.0000| 0.3575 | 0.0865 | 0.1060 |
| DAX   | 1.0000| 0.2321 | 0.2458 |        |
| SZZS  | 1.0000| 0.7144 |        |        |
| N225  | 1.0000|        |        |        |

We can see from Tables 2 and 3 that there is a strong correlation between DAX and AEX and so is SZZS and HSI. Thus, it is reasonable to divide the financial market into Asia market and Europe market. In order to illustrate the effectiveness and robustness of our model in the cross-market mixed portfolio optimization problem, we will compare our model with others. The descriptions of Models 1-4 are as follow and the model settings of Models 1-4 are given in Table 4.

- Model 1 is the classical Markowitz mean-variance model with the transaction cost and investment proportion limitation.
- Model 2 is the robust portfolio model with the transaction cost and investment proportion limitation which regard the financial market as a whole without considering the different dependent structure of different sub-markets.
- Model 3 is the robust cross-market mixed portfolio model with the transaction cost and investment proportion limitation under hierarchical risk integration, but it is a single-objective optimization model without skewness.
- Model 4 is the multi-objective robust cross-market mixed portfolio model (11).

We set $\lambda_1 = 3, \lambda_2 = 1, \beta = 0.95, c = 0.01\%, \mu = 0.01\%, w_{\min} = (0, 0, 0, 0), w_{\max} = (0.4, 0.4, 0.4, 0.4)$. Similar to the work due to Li and Xu [23], by employing the genetic algorithm, we can obtain the optimal portfolio of Model 4. Some numerical results of the portfolios for Models 1-4 are shown in Table 5. We can see from Table 5 that the optimal portfolio obtained by Model 4 has the smallest WCVar
Table 4. Model Settings.

| Item                        | Model 1 | Model 2 | Model 3 | Model 4 |
|-----------------------------|---------|---------|---------|---------|
| Risk Measurement            | Variance| WC VaR  | WC VaR  | WC VaR  |
| Hierarchical Risk Integration| NO      | NO      | YES     | YES     |
| Skewness                    | NO      | NO      | NO      | YES     |
| Transaction Cost            | YES     | YES     | YES     | YES     |
| Investment Proportion Limitation | YES     | YES     | YES     | YES     |

compared with Models 2 and 3. This means that we can get a robust optimal portfolio by considering the different dependent structure of different sub-markets and skewness. In addition, the daily maximum loss of the optimal portfolio obtained by Model 4 is the lowest during the test period, which further verifies its robustness. It is worth to mention that the optimal portfolio obtained by Model 4 also gains the best rate of return. Thus, the robust optimal portfolio obtained by Model 4 is effective. Furthermore, we show the return curve of Model 4 in Figure 6.

Table 5. Statistic Comparison.

| Item                        | Model 1 | Model 2 | Model 3 | Model 4 |
|-----------------------------|---------|---------|---------|---------|
| WC VaR of optimal portfolio | -       | 1.7816  | 1.7254  | 1.6914  |
| Total Rate of Return(%)     | 15.4084 | 16.1731 | 17.6015 | 18.3447 |
| Daily Rate of Return(%)     | 0.0518  | 0.0541  | 0.0586  | 0.0609  |
| Variance                    | 0.8661  | 0.8876  | 0.9833  | 1.0135  |
| Daily Maximum Loss(%)       | -5.6033 | -5.3498 | -5.1381 | -4.9078 |

Figure 6. Return Curve of Model 4.
5. Conclusions. In this paper, we describe the dependent structure of risk assets in different financial sub-markets and give the mathematical expression of cross-market mixed portfolio by using the hierarchical risk integration method based on copulas. In order to apply our method to the cross-market mixed portfolio optimization problem, we select WCVaR as the risk measurement and give a numerical algorithm to estimate WCVaR of the cross-market mixed portfolio. Moreover, we put forward the multi-objective robust cross-market mixed portfolio optimization model to solve the cross-market mixed portfolio optimization problem. Through numerical results, we find that, compared with some known results, the robust optimal portfolio obtained in this paper has the smallest WCVaR and gains the best rate of return.

We would like to point out that the model of this paper only considers the fact that the different sources of financial sub-markets affect the dependence structure of risk assets. However, the dependence structure among the same risk assets may be affected by significant event and time. Thus, it would be valuable to study the time-varying risk measurement of the cross-market mixed portfolio model. On the other hand, the portfolio models with cardinality constraints have been studied extensively in recent years (see, for example, [33, 36] and the references therein). Therefore, it would be an interesting and important to study the cross-market mixed portfolio model with cardinality constraints in the future work.

Acknowledgments. The authors are grateful to the editor and the referees for their valuable comments and suggestions.

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Received August 2017; revised August 2018.

E-mail address: fishersponge@163.com
E-mail address: yuejia@swufe.edu.cn
E-mail address: nanjinghuang@hotmail.com