Analyzing QCD Sum Rules for Heavy Baryons
at Next-to-Leading Order in $\alpha_S$

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Abstract

In this talk I consider QCD sum rules for the ground state heavy baryons to leading order in $1/m_Q$ and at next-to-leading order in $\alpha_S$ within the context of Heavy Quark Symmetry. The analysis is done at a fixed scale $\mu = 1$ GeV. The evolution behaviour of the residues is determined by the two-loop anomalous dimension of the heavy baryon currents calculated earlier. I compare the diagonal, non-diagonal and constituent type QCD sum rules and show that the diagonal sum rules are the most reliable one. As central values for the bound state energies I find $m(\Lambda_Q) - m_Q \simeq 760$ MeV and $m(\Sigma_Q) - m_Q \simeq 940$ MeV. The central values for the residues are given by $F(\Lambda_Q) \simeq 0.030$ GeV$^3$ and $F(\Sigma_Q) \simeq 0.038$ GeV$^3$.

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1 Introduction

The correlator of two baryonic currents is the fundamental building block for stating QCD sum rules for heavy baryons, i.e. baryons which contain one heavy quark. In this talk I present the calculation of first order radiative corrections to QCD sum rules within the Heavy Quark Effective Theory (HQET) in the limit of infinite mass for this quark [1]. The calculation of first order radiative corrections [2, 3] make sense because of the knowledge of the two-loop anomalous dimension [4]. The interpolating currents for the heavy baryons can be written as

\[ J_B = [(q_1)^T C \tau (q_2)^j]) \Gamma' Q^k \varepsilon_{ijk}, \]  

(1)

where the index \( T \) means transposition, \( C \) is the charge conjugation matrix, \( \tau \) is a matrix in flavour space, and \( i, j \) and \( k \) are color indices. \( \Gamma \) and \( \Gamma' \) are the light-side and heavy-side Dirac structures of the current vertices. For each ground-state baryon there are two independent current representations [5] carrying the same quantum numbers, for the \( \Lambda_Q \)-type baryons they are e.g. given by

\[ J_{\Lambda_1} = [(q_1)^T C \tau \gamma_5 (q_2)^j]) Q^k \varepsilon_{ijk} \]  

and  \[ J_{\Lambda_2} = [(q_1)^T C \tau \gamma_5 \bar{v} (q_2)^j]) Q^k \varepsilon_{ijk}. \]  

(2)

2 The two-point correlator

The two-point correlator can be constructed as

\[ \Pi_{ij}(\omega = p \cdot v) = i \int \langle 0 | T \{ J_i(x) \bar{J}_j(0) \} | 0 \rangle e^{ipx} d^4x = \Gamma' \frac{1 + \frac{\bar{v}}{2} \Gamma' \frac{1}{4} \text{tr}(\Gamma \bar{\Gamma}) 2 \text{tr}(\tau \tau^\dagger) P_{ij}(\omega). \]  

(3)

\( i \) and \( j \) stand for the two independent current representations for the ground state baryons. This gives rise to two independent types of correlators, namely the diagonal correlators \( \langle J_1 \bar{J}_1 \rangle \) and \( \langle J_2 \bar{J}_2 \rangle \) and the non-diagonal correlators \( \langle J_1 \bar{J}_2 \rangle \) and \( \langle J_2 \bar{J}_1 \rangle \) as well as linear combinations of these cases. Following the standard QCD sum rule method [6], the correlator is calculated in the Euclidean region \( -\omega \approx 1 \sim 2 \text{ GeV} \) including perturbative and non-perturbative contributions. In the Euclidean region the non-perturbative contributions are expected to form a convergent series. The non-perturbative effects are taken into account by employing an operator product expansion (OPE) for the time-ordered product of the currents. One then has

\[ \langle T \{ J(x), J(0) \} \rangle = \sum_d C_d(x^2) O_d = \]  

\[ = C_0(x^2) O_0 + C_3(x^2) O_3 + C_4(x^2) O_4 + C_5(x^2) O_5 + \ldots \]  

(4)

where the \( O_d \) are vacuum expectation values of local operators whose mass dimensions are labelled by their subscripts \( d \). \( O_0 = 1 \) corresponds to the so-called perturbative term, \( O_3 = \langle \bar{q}q \rangle \) is a quark condensate term, \( O_4 = \alpha_s \langle G^2 \rangle \) is a gluon condensate term, \( O_5 = g_s \langle \bar{q} \sigma_{\mu\nu} G^{\mu\nu} q \rangle \) is a mixed quark-gluon condensate, and so on. The expansion coefficient \( C_d(x^2) \) are the corresponding Wilson coefficients of the operator product expansion. The
different diagrams are shown in Fig. 1.

A straightforward dimensional analysis shows that the operator product expansion of the diagonal correlators contain only even-dimensional terms, while the expansion of the non-diagonal correlators contain only odd-dimensional terms. This classification is preserved when radiative corrections are included, assuming the light quarks to be massless.

3 Construction of QCD sum rules

The QCD sum rules can be constructed by taking care of two possible expressions for the two-point correlator. On the one hand, the scalar correlator function $P(\omega)$ satisfies the dispersion relation

$$P(\omega) = \int_0^\infty \frac{\rho(\omega')d\omega'}{\omega' - \omega - i0} + \text{subtraction},$$

where $\rho(\omega) = \text{Im}(P(\omega))/\pi$ is the spectral density. On the the phenomenological side of view, the two-point correlator is represented by the spectral representation

$$\Pi(\omega) = \frac{1}{2} \sum_X \frac{|\langle 0| J_X |X \rangle|^2}{\omega_X - \omega - i0} + \text{subtraction}. (6)$$

The scalar correlator function can thus be expressed by the residues $F_B$ given by

$$\langle 0| J_\Lambda |\Lambda_Q \rangle = F_\Lambda u, \quad \langle 0| J_\Sigma |\Sigma_Q \rangle = F_\Sigma u \quad \text{and} \quad \langle 0| J_{\Sigma^*}^\nu |\Sigma^*_{Q} \rangle = \frac{1}{\sqrt{3}} F_{\Sigma^*}^\nu u \quad \text{(7)}$$

in the form

$$P(\omega) = \frac{1}{2} |F_B|^2 \frac{1}{\bar{\Lambda} - \omega - i0} + \sum_{X \neq B} \frac{1}{2} |F_X|^2 \frac{1}{\omega_X - \omega - i0} + \text{subtraction}, \quad \text{(8)}$$

where $\bar{\Lambda} = m_B - M_Q$ is the ground state energy of the baryon. The main assumption of the QCD sum rule method is that the remaining sum can be approximated by the integral of the spectral density given by the dispersion relation starting from some threshold energy $E_C$.

The combination of the phenomenological and the theoretical identity for the correlator function leads to

$$\frac{1}{2} |F_B|^2 \frac{1}{\bar{\Lambda} - \omega - i0} = \int_0^{E_C} \frac{\rho(\omega')d\omega'}{\omega' - \omega - i0} + \text{subtraction}. \quad \text{(9)}$$

This formula is not useful anyhow, since the spectral density is reliable only for negative values of $\omega$, while the integral is to be calculated mainly at $\omega = \bar{\Lambda}$. This region of integration can be reached by an extrapolation using higher and higher derivatives when $\omega$ goes to $-\infty$. This extrapolation is expressed by the Borel transformation

$$\hat{B}f(T) = \hat{B}_T^{(\omega)}(f(\omega)) := \lim_{\omega, n \to \infty} \frac{(-\omega)^{n+1}}{n!} \frac{d^n}{d\omega^n} f(\omega), \quad T = \frac{-\omega}{n} \text{ fixed.} \quad \text{(10)}$$
This Borel transformation leads to the final form of the QCD sum rules, namely

\[ \frac{1}{2} |F_B|^2 e^{-\bar{\Lambda}/T} = \int_0^{E_C} \rho(\omega) e^{-\omega/T} d\omega. \]  \hspace{1cm} (11)

The Borel transformation also cancels the subtraction term. The constant ratio \( T \) is called the Borel parameter. It is an unphysical quantity in units of an energy, and the obtained values should be mostly independent on this parameter. This is the main criterion in analyzing the sum rules.

4 Diagonal and non-diagonal correlators

I don’t want to go into details concerning the calculations of first order radiative corrections to the two-point correlators (for details see [2, 3]). We calculate radiative corrections only to the leading terms in the operator product expansion because the non-leading contributions are small. It should be stressed that we could make use of the same MATHEMATICA package which was constructed for the calculation of two-loop corrections of the currents itself. For the case of the diagonal correlator (for an overview over the contributing diagrams see Fig. 2) we ended up with a renormalized spectral density

\[ \rho^\text{ren}_0(\omega) = \rho^\text{Born}_0(\omega) \left[ 1 + \frac{\alpha_s}{4\pi} r(\omega/\mu) \right], \]  \hspace{1cm} (12)

where

\[ \rho^\text{Born}_0(\omega) = \frac{\omega^5}{20\pi^4} \quad \text{and} \quad r(\omega/\mu) = r_1 \ln \left( \frac{\mu}{2\omega} \right) + r_2 \]  \hspace{1cm} (13)

with

\[ r_1 = \frac{8}{3} (n^2_\gamma - 4n_\gamma + 6) \quad \text{and} \quad r_2 = \frac{8}{15} \left( 38n^2_\gamma - 137n_\gamma + 273 + 60\zeta(2) \right), \]  \hspace{1cm} (14)

which results in the sum rule

\[ \frac{1}{2} F^2(\mu) e^{-\bar{\Lambda}/T} = \frac{N_c}{\pi^4} \left[ T^6 \left( f_5(x_C) + \frac{\alpha_s}{4\pi} \left( \ln \frac{\mu}{2T} \right) f_5(x_C) - f_5^l(x_C) \right) r_1 + f_5(x_C) r_2 \right] + \] 
\[ + c E_G^4 T^2 f_1(x_C) + E_Q^6 \exp \left( -\frac{2E_0^2}{T^2} \right) \]  \hspace{1cm} (15)

where \( c \) is a Clebsch-Gordan type factor which takes the values \( c = 1 \) for the \( \Lambda_Q \)-type and \( c = -1/3 \) for the \( \Sigma_Q \)-type doublet \( \{ \Sigma_Q, \Sigma^*_Q \} \) ground state baryons. In order to simplify the notation I have introduced the abbreviations

\[ x_C := \frac{E_C}{T}, \quad E_0 := \frac{m_0}{4}, \quad (E_Q)^3 := -\frac{\pi^2}{2N_c} \langle \bar{q}q \rangle \quad \text{and} \quad (E_G)^4 := \frac{\pi \alpha_s \langle G^2 \rangle}{32N_c(N_c - 1)}. \]  \hspace{1cm} (16)

The sum rule is expressed in terms of the modified Gamma functions

\[ f_n(x) := \int_0^x \frac{x^m}{n!} e^{-x'} dx' = 1 - e^{-x} \sum_{m=0}^{n} \frac{x^m}{m!}, \quad f^l_n(x) := \int_0^x \frac{x^m}{n!} \ln x e^{-x'} dx'. \]  \hspace{1cm} (17)
I have omitted the index for the current. Later on we will see that the sum rule analysis is indeed independent of the chosen current within the assumed error bars. So we can use only one residue also for the non-diagonal case. For the non-diagonal correlator (cf. Fig. 3) we obtain a spectral density part

\[ \rho_{\text{ren}}^3(\omega) = \rho_{\text{Born}}^3(\omega) \left[ 1 + \frac{\alpha_s}{4\pi} r(\omega/\mu) \right], \]  

(18)

where

\[ \rho_{\text{Born}}^3(\omega) = -\langle \bar{q}q \rangle_{\text{ren}} \left( \frac{\pi^2}{6} \omega^2 \right) \quad \text{and} \quad r(\omega/\mu) = r_1 \ln \left( \frac{\mu}{2\omega} \right) + r_2 \]  

(19)

with

\[ r_1 = \frac{4}{3} (2n_\gamma^2 - 8n_\gamma + 7 + 2(n_\gamma - 2)s_\gamma) \quad \text{and} \]

\[ r_2 = \frac{2}{3} (8n_\gamma^2 - 28n_\gamma + 37 + 8n_\gamma s_\gamma - 12s_\gamma + 8\zeta(2)). \]  

(20)

Here the sum rule has the form

\[ \frac{1}{2} F^2(\mu) e^{-\Lambda/T} = \frac{2N_c!}{\pi^4} \left[ E_Q^3 T^3 \left( f_2(x_C) + \frac{\alpha_s}{4\pi} \left( \ln \left( \frac{\mu}{2T} \right) f_2(x_C) - f_1^i(x_C) \right) r_1 + f_2(x_C) r_2 \right) \right] \]

\[ - E_Q^3 E_0^3 T \left( 1 - \frac{c}{2} \right) f_0(x_C) + \frac{2}{3} \left( 1 - \frac{c}{2} \right) \frac{E_Q^3 E_0^4}{T} + \frac{\alpha_s C_F E_Q^3}{36\pi T^3}. \]  

(21)

5 Numerical analysis

Having the necessary formulae at hand we can start the numerical analysis of the sum rules. In doing this we use the following numerical input values for the condensate contributions \[ \langle \bar{q}q \rangle = -0.23 \text{ GeV}^3 \] (quark condensate), \[ \alpha_s \langle G^2 \rangle = 0.04 \text{ GeV}^4 \] (gluon condensate), and \[ g_s \langle \bar{q} \sigma_{\mu\nu} G^{\mu\nu} q \rangle = m_0^2 \langle \bar{q}q \rangle \quad \text{with} \quad m_0^2 = 0.8 \text{ GeV}^2 \] (mixed quark-gluon condensate).

There are in general two strategies for the numerical analysis of the QCD sum rules. The first strategy fixes the bound state energy \( \Lambda \) from the outset by choosing a specific value for the pole mass of the heavy quark and then extracts a value for the residue \( F \). In order to obtain information from the sum rules which is independent of specific input values, we adopt a second strategy, namely to determine both \( \Lambda \) and \( F \) by finding simultaneous stability values for them with respect to the Borel parameter \( T \).

The first step in carrying out the numerical analysis of the sum rules is to find a sum rule “window” for the allowed values of the Borel parameter \( T \). The parameter range of \( T \) is constrained by two different physical requirements. The first is that the convergence of the OPE expansion must be secured. We therefore demand that the subleading term
Table 1: Results of the diagonal sum rule analysis for the continuum threshold parameter $E_C$, the bound state energy $\bar{\Lambda}$, and the residuum $F$ for $\Lambda_Q$-type and $\Sigma_Q$-type currents, analyzed to leading order (L.O.) as well as next-to-leading order (N.L.O.)

| Baryon type state | $E_C$ (GeV) | $\bar{\Lambda}$ (GeV) | $F$ (GeV$^3$) |
|-------------------|------------|------------------------|---------|
| $\Lambda_Q$ (L.O.) | $1.2 \pm 0.1$ | $0.77 \pm 0.05$ | $0.022 \pm 0.001$ |
| $\Lambda_Q$ (N.L.O.) | $1.1 \pm 0.1$ | $0.77 \pm 0.05$ | $0.027 \pm 0.002$ |
| $\Sigma_Q$ (L.O.) | $1.4 \pm 0.1$ | $0.96 \pm 0.05$ | $0.031 \pm 0.002$ |
| $\Sigma_Q$ (N.L.O.) | $1.3 \pm 0.1$ | $0.94 \pm 0.05$ | $0.038 \pm 0.003$ |

In the OPE does not contribute more than 30% of the leading order term. This gives a lower limit for the Borel parameter. The upper limit is determined by the requirement that the contributions from the excited states plus the physical continuum (even after Borel transformation) should not exceed the bound state contribution. This requirement is necessary in order to guarantee that the sum rules are as independent as possible of the model-dependent assumptions concerning the profile of the theoretical spectral density, i.e. the model of the continuum. The lower limit of $E_C$ is given by the requirement that the indicated window should be kept open. For the rest, $E_C$ is a free floating variable which is only limited by the stability requirements on $\bar{\Lambda}$ and $F$.

As an example I want to show the analysis of the diagonal sum rule for the $\Lambda_Q$-type state in Fig. 4. Fig 4(a) shows the dependence of the bound state energy $\bar{\Lambda}$ on the Borel parameter $T$ and Fig. 4(b) shows the dependence of the residue on $T$, both for the leading order sum rule. Fig. 4(c) and Fig. 4(d) show the same dependencies for the radiatively corrected sum rules. The same analysis is repeated for the $\Sigma_Q$-type baryons. The results of the numerical analysis both without and with radiative corrections are given in Table 1.

The numerical results for the non-diagonal sum rules are given in Table 2. Assuming relative errors of 10% for the bound state energy and 20% for the residue, the obtained values are in agreement with the results of the analysis of the diagonal sum rules, where the values for the $\Sigma_Q$-type baryon are the more reliable one.

For the linear combination $J = (J_1 + J_2)/2$ of currents the light-side Dirac structure can be seen to appear in the form $1/2(1 + \gamma^\mu)\Gamma$, i.e. one has the projector factor $P_+ = (1 + \gamma^\mu)/2$ which projects on the large components of the light quark fields. We refer to this particular linear combination of currents as the constituent type current. This linear combination of currents is expected to have minimum overlap with the heavy ground state baryons in the constituent quark model, i.e. where the light diquark state in the heavy baryon is taken to be composed of on-shell light quarks. This model emerges in the large $N_c$-limit.

The use of a constituent type interpolating current $J = (J_1 + J_2)/2$ combines the two sum rule formulas for the diagonal and the non-diagonal case, taking one half of each part. The numerical results of the analysis are given in Table 3. The constituent type sum rules show an improved stability on the Borel parameter $T$ as compared to the non-diagonal
Table 2: Results of the non-diagonal sum rule analysis for the continuum threshold parameter $E_C$, the bound state energy $\bar{\Lambda}$, and the residuum $F$ for $\Lambda_Q$-type and $\Sigma_Q$-type currents, analyzed to leading order (L.O.) as well as next-to-leading order (N.L.O.)

| Baryon type state | $E_C$       | $\bar{\Lambda}$ | $F$              |
|-------------------|-------------|-----------------|-----------------|
| $\Lambda_Q$ (L.O.) | 1.0 ± 0.10 GeV | 0.75 ± 0.10 GeV | 0.024 ± 0.002 GeV³ |
| $\Lambda_Q$ (N.L.O.) | 1.0 ± 0.10 GeV | 0.72 ± 0.10 GeV | 0.032 ± 0.003 GeV³ |
| $\Sigma_Q$ (L.O.) | 1.5 ± 0.10 GeV | 1.16 ± 0.10 GeV | 0.045 ± 0.003 GeV³ |
| $\Sigma_Q$ (N.L.O.) | 1.2 ± 0.10 GeV | 0.94 ± 0.10 GeV | 0.039 ± 0.004 GeV³ |

Table 3: Results of the constituent type mixed sum rule analysis for the continuum threshold parameter $E_C$, the bound state energy $\bar{\Lambda}$, and the residuum $F$ for $\Lambda_Q$-type and $\Sigma_Q$-type currents, analyzed to leading order (L.O.) as well as next-to-leading order (N.L.O.)

| Baryon type state | $E_C$       | $\bar{\Lambda}$ | $F$              |
|-------------------|-------------|-----------------|-----------------|
| $\Lambda_Q$ (L.O.) | 1.1 ± 0.10 GeV | 0.77 ± 0.10 GeV | 0.034 ± 0.004 GeV³ |
| $\Lambda_Q$ (N.L.O.) | 1.1 ± 0.10 GeV | 0.77 ± 0.10 GeV | 0.032 ± 0.004 GeV³ |
| $\Sigma_Q$ (L.O.) | 1.3 ± 0.10 GeV | 1.03 ± 0.10 GeV | 0.045 ± 0.004 GeV³ |
| $\Sigma_Q$ (N.L.O.) | 1.2 ± 0.10 GeV | 0.94 ± 0.10 GeV | 0.036 ± 0.004 GeV³ |

sum rules, but the stability is not as good as in the diagonal case. Within the assumed errors the results are again in agreement with both the diagonal and the non-diagonal sum rule analysis.

6 Comparison with experimental values

Taking the experimental results for charm-quark baryons, namely $m(\Lambda_c) = 2284.9 \pm 0.6$ MeV and $m(\Sigma_c^+) = 2453.5 \pm 0.9$ MeV, our central values $\bar{\Lambda}(\Lambda_Q) = 760$ MeV and $\bar{\Lambda}(\Sigma_Q) = 940$ MeV predict a mean pole mass of $m_c = 1520$ MeV for the charm quark. For the experimental value $m(\Lambda_b) = 5642 \pm 50$ MeV for the mass of the baryon $\Lambda_b$, our central value $\bar{\Lambda}(\Lambda_Q) = 760$ MeV for the bound state energy suggests a pole mass of $m_b = 4880$ MeV for the bottom quark.
7 Conclusion and outlook

- The determination of the two-loop anomalous dimension is essential for the evolution behaviour of the current to first order perturbation theory.

- The presented results as well as the related computer package are applicable and extendable to the calculation of three-point correlators and thus the determination of the Isgur-Wise function. This work will be started in the near future.

- Another extend is the calculation of correlators including massive lines within exact QCD. This will be done to test the convergence of the $1/m_Q$ expansion in HQET.

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Figure Captions

Fig. 1: Diagrams representing the first few vacuum expectation values of local operators due to the operator product expansion of the two-point correlator.

Fig. 2: Radiative corrections to the diagonal correlator.
(0) Lowest order two-loop contribution,
(1)–(4) $O(\alpha_s)$ three-loop contributions.

Fig. 3: Radiative corrections to the non-diagonal correlator given by the dimension three condensate contribution.
(0) Lowest order one-loop contribution,
(1)–(4) $O(\alpha_s)$ two-loop contributions.

Fig. 4: Sum rule results on the non-perturbative parameters of the $\Lambda_Q$ as functions of the Borel parameter $T$. Shown are five curves for five different values of the threshold energy $E_C$ spaced by 100 MeV around the central value $E_C = E_{C}^{\text{best}}$. $E_C$ grows from bottom to top. These are in detail
(a) lowest order sum rule results for the bound state energy $\bar{\Lambda}(\Lambda)$;
(b) lowest order sum rule results for the absolute value of the residue $F_{\Lambda}$;
(c) $O(\alpha_s)$ sum rule results for the bound state energy $\bar{\Lambda}(\Lambda)$;
(d) $O(\alpha_s)$ sum rule results for the absolute value of the residue $F_{\Lambda}$. 
Figure 1
Figure 3
