A NOTE ON HOMOLOGICAL PROPERTIES OF NAKAYAMA ALGEBRAS

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Abstract. Using the resolution quiver for a connected Nakayama algebra, a fast algorithm is given to decide whether its global dimension is finite or not and whether it is Gorenstein or not. The latter strengthens a result of Ringel.

1. Introduction

Let $A$ be a connected Nakayama algebra. Following [6, 3, 7], its resolution quiver is defined as follows: the vertex set is the set of non-isomorphic simple $A$-modules; there is a unique arrow from each simple $A$-module $S$ to $\gamma(S) = \tilde{\tau} \text{soc} P(S)$. Here, $P(S)$ is the projective cover of $S$ and ‘soc’ is the socle of a module. If $A$ has a simple projective module, denote by $S_{\text{inj}}$ the unique simple injective $A$-module up to isomorphism. Then

$$\tilde{\tau}(S) = \begin{cases} \tau(S), & \text{if } S \text{ is not projective} \\ S_{\text{inj}}, & \text{otherwise} \end{cases}$$

for each simple $A$-module $S$, where $\tau$ is the Auslander-Reiten translation [1].

Let $A$ be a connected Nakayama algebra. Denote by $R(A)$ its resolution quiver. It is known that each connected component of $R(A)$ has a unique cycle. For a cycle $C$ in $R(A)$ with vertices $S_1, S_2, \cdots, S_m$, the weight $w(C)$ of $C$ is $\sum_{k=1}^{m} c_k$. Here, $n$ is the number of non-isomorphic simple $A$-modules and $c_k$ is the composition length of $P(S_k)$. It turns out that $w(C)$ is an integer and all cycles in $R(A)$ have the same weight [7]. The weight $w(C)$ is called the weight of the algebra $A$.

The resolution quiver is a very efficient tool for investigating the homological properties of Nakayama algebras. The Gorenstein projective modules for Nakayama algebras are described by resolution quivers [6]. Resolution quivers are also used to study the singularity categories of Nakayama algebras [8].

The resolution quiver of a connected Nakayama algebra gives a fast algorithm to decide whether it is Gorenstein or not and whether it is CM-free or not [6]. In this paper, we show that the resolution quiver of a connected Nakayama algebra also gives a fast algorithm to decide whether its global dimension is finite or not.

More precisely, we have the following.

Proposition 1.1. Let $A$ be a connected Nakayama algebra. Then $A$ has finite global dimension if and only if its resolution quiver is connected and its weight is 1.

As a consequence of Proposition 1.1, the resolution quiver is connected for a connected Nakayama algebra with finite global dimension.

For a connected Nakayama algebra, recall from [6] that a cycle in its resolution quiver is called black provided that the projective dimension of each simple module on this cycle is not equal to 1.

The following result strengthens [6, Proposition 5(a)].
Proposition 1.2. Let $A$ be a connected Nakayama algebra with infinite global dimension. Then $A$ is a Gorenstein algebra if and only if all cycles in its resolution quiver are black.

Let $A$ be a connected Nakayama algebra. Take a complete set $\{S_1, S_2, \cdots, S_n\}$ of pairwise non-isomorphic simple $A$-modules. The Cartan matrix $C_A$ of $A$ is an $n \times n$ matrix $(c_{ij})$, where $c_{ij}$ is the number of copies of $S_i$ appearing in a composition series for the projective cover of $S_j$. Denote by $C_A^T$ the transpose of $C_A$.

Denote by $c$ the number of cycles and by $b$ the number of black cycles in the resolution quiver of $A$.

The following result gives a connection between Cartan matrices and resolution quivers.

Proposition 1.3. Let $A$ be a connected Nakayama algebra.

1. The rank of $C_A$ is $n + 1 - c$.
2. If $b$ is nonzero, then the rank of $(C_A, C_A^T)$ is $n + 1 - b$.

The paper is organised as follows. The proofs of Proposition 1.1 and Proposition 1.2 are given in Section 2 and Section 3, respectively. In Section 4, we study the connection between Cartan matrices and resolution quivers for Nakayama algebras and prove Proposition 1.3.

2. Retractions and resolution quivers

Let $A$ be a connected Nakayama algebra. Denote by $n = n(A)$ the number of non-isomorphic simple $A$-modules. Take a sequence $(S_1, S_2, \cdots, S_n)$ of pairwise non-isomorphic simple $A$-modules such that the radical of $P_i$ is a factor module of $P_{i+1}$ for $1 \leq i \leq n - 1$ and the radical of $P_n$ is a factor module of $P_1$. Here, $P_i$ is the projective cover of $S_i$. Denote by $c_i$ the composition length of $P_i$. The admissible sequence for $A$ is given by $c(A) = (c_1, c_2, \cdots, c_n)$; see [1, Chapter IV. 2].

Following [1], there exists a map $f_A : \{1, 2, \cdots, n\} \rightarrow \{1, 2, \cdots, n\}$ such that $n$ divides $f_A(i) - (c_i + i)$. For $1 \leq i \leq n$, we have $\gamma(S_i) = S_{f_A(i)}$. Then for $1 \leq i, j \leq n$, there is an arrow $S_i \rightarrow S_j$ in the resolution quiver of $A$ if and only if $f_A(i) = j$.

Suppose now that $A$ is not selfinjective. If $A$ has no simple projective modules, after possible cyclic permutations, we may assume that its admissible sequence is normalized [3], that is, $p(A) = c_1 = c_n - 1$. Here, $p(A)$ is the minimal integer among $c_i$. For convenience, if $A$ has a simple projective module, its admissible sequence is always normalized.

Following [3], there exists a left retraction $\eta : A \rightarrow L(A)$, where $L(A)$ is a connected Nakayama algebra with admissible sequence $c(L(A)) = (c'_1, c'_2, \cdots, c'_{n-1})$ such that $c'_i = c_i - \left\lceil \frac{c_i + i - 1}{n} \right\rceil$ for $1 \leq i \leq n - 1$. In particular, $n(L(A)) = n(A) - 1$. Here, $\lfloor x \rfloor$ is the largest integer not greater than a real number $x$. The corresponding sequence of simple $L(A)$-modules is denoted by $(S'_1, S'_2, \cdots, S'_{n-1})$.

We need the map $\pi : \{1, 2, \cdots, n\} \rightarrow \{1, 2, \cdots, n - 1\}$ such that $\pi(i) = i$ for $1 \leq i \leq n - 1$ and $\pi(n) = 1$.

Recall from [7, Lemma 2.1] the following result. For the convenience of the reader, we give a proof here.

Lemma 2.1. Let $A$ be a connected Nakayama algebra which is not selfinjective. Then $\pi f_A(i) = f_{L(A)}(i)$ for $1 \leq i \leq n - 1$.

Proof. Let $f_A(i) = j$ and $c_i + i = kn + j$ with $k \in \mathbb{N}$. For $1 \leq i \leq n - 1$, we have

$$c'_i + i = c_i + i - \left\lceil \frac{c_i + i - 1}{n} \right\rceil = kn + j - \left\lceil \frac{kn + j - 1}{n} \right\rceil = k(n - 1) + j. \quad (2.1)$$

It follows that $\pi f_A(i) = \pi(j) = f_{L(A)}(i)$. \hfill $\square$
Denote by $\gamma'$ the map $\tilde{\tau} \text{soc} P(-)$ for $L(A)$. It follows from Lemma 2.1 that $\gamma'(S'_i) = S'_{i''(f)}$ for $1 \leq i \leq n - 1$. Hence the resolution quiver of $L(A)$ can be obtained from the resolution quiver of $A$ just by “merging” the vertices $S_1$ and $S_n$.

Observe that $\gamma(S_n) = \gamma(S_1)$ if $A$ has no simple projective modules and $\gamma(S_n) = S_1$ if $A$ has a simple projective module. In particular, the vertices $S_1$ and $S_n$ lie on the same connected component in the resolution quiver $R(A)$ of $A$. Then $R(A)$ and $R(L(A))$ have the same number of connected components. It follows that they have the same number of cycles since each connected component of a resolution quiver has a unique cycle.

Let $C$ be a cycle with vertices $S_{x_1}, S_{x_2}, \cdots, S_{x_s}$ in $R(A)$. The weight $w(C)$ of $C$ is $\sum_{k=1}^{s} \frac{c_{x_k}}{w(x_k)}$, and the size of $C$ is the number of vertices on $C$.

The following result strengthens [7, Lemma 2.2], where the connected Nakayama algebra is required to have no simple projective modules.

**Lemma 2.2.** Let $A$ be a connected Nakayama algebra which is not selfinjective. Then there exists a weight preserving bijection between the set of cycles in $R(A)$ and the set of cycles in $R(L(A))$. Moreover, if $A$ has no simple projective modules or the simple projective $A$-module does not lie on a cycle in $R(A)$, then the bijection also preserves the size.

**Proof.** If $A$ has no simple projective modules, then the bijection follows from [7, Lemma 2.2]. We may assume that $A$ has a simple projective module $S_n$.

Let $C$ be a cycle with vertices $S_{x_1}, S_{x_2}, \cdots, S_{x_s}$ in $R(A)$. Assume $x_{i+1} = f_A(x_i)$.

Here, we identify $x_{s+1}$ with $x_1$. Let $c_{x_i} + x_i = k_i n + x_{i+1}$ with $k_i \in \mathbb{N}$. Then

$$w(C) = \sum_{i=1}^{s} \frac{c_{x_i}}{n} = \sum_{i=1}^{s} k_i.$$ 

It follows from (2.1) that for $x_i < n$, we have $c'_{x_i} + x_i = k_i (n - 1) + x_{i+1}$.

There exist two cases:

**Case 1:** $S_n$ does not lie on $C$. Then $S'_{x_1}, S'_{x_2}, \cdots, S'_{x_s}$ form a cycle $C'$ in $R(L(A))$.

Observe that $c'_{x_i} + x_i = k_i (n - 1) + x_{i+1}$ for $1 \leq i \leq s$. Then

$$\sum_{i=1}^{s} c'_{x_i} = (n - 1) \sum_{i=1}^{s} k_i.$$ 

Therefore, $w(C) = w(C')$.

**Case 2:** $S_n$ lies on $C$. Since the admissible sequence of $A$ is normalized, we have $x_n = n$ and $x_1 = 1$. It follows that $S'_{x_{n-1}}, S'_{x_{n-2}}, \cdots, S'_{x_1}$ form a cycle $C''$ in $R(L(A))$.

Observe that $k_{n-1} = 1$ and $c'_{x_i} + x_i = k_i (n - 1) + x_{i+1}$ for $1 \leq i \leq s - 1$. Then

$$\sum_{i=1}^{s-1} c'_{x_i} = (n - 1) \sum_{i=1}^{s-1} k_i + n - 1 = (n - 1) \sum_{i=1}^{s} k_i.$$ 

Therefore, $w(C) = w(C'')$.

We have shown that there exists an injective weight preserving map from the set of cycles in $R(A)$ to the set of cycles in $R(L(A))$. Since $R(L(A))$ and $R(A)$ have the same number of cycles, this map is also surjective. This finishes our proof. □

Recall from [3, Theorem 3.8] that there exists a sequence of left rejections

$$A = A_0 \xrightarrow{\eta_0} A_1 \xrightarrow{\eta_1} A_2 \cdots \rightarrow A_{r-1} \xrightarrow{\eta_{r-1}} A_r$$ (2.2)

such that each $A_i$ is a connected Nakayama algebra, each $\eta_i : A_i \rightarrow A_{i+1}$ is a left retraction and $A_r$ is selfinjective; the global dimension of $A$ is finite if and only if $A_r$ is simple.
For a connected Nakayama $A$, denote by $c(A)$ the number of cycles and by $w(A)$ the weight of a cycle in $R(A)$. We mention that $w(A)$ is an integer and all cycles in $R(A)$ have the same weight; see [7].

We now prove Proposition 1.1.

**Proof of Proposition 1.1.** Applying Lemma 2.2 to (2.2) repeatedly, we obtain $c(A) = c(A_r)$ and $w(A) = w(A_r)$. Observe that $A_r$ is simple if and only if $c(A_r) = 1$ and $w(A_r) = 1$. Then the global dimension of $A$ is finite if and only if $c(A) = 1$ and $w(A) = 1$. Each connected component of a resolution quiver has a unique cycle. Then $c(A) = 1$ if and only if the resolution quiver of $A$ is connected. □

3. Two Maps on Simple Modules

Let $A$ be a connected Nakayama algebra. For an $A$-module $M$, denote by $\text{pd } M$ its projective dimension and by $\text{id } M$ its injective dimension. Recall that the syzygy $\Omega(M)$ of $M$ is the kernel of its projective cover $\gamma: P(M) \longrightarrow M$. Dually, the cosyzygy $\Omega^{-1}(M)$ of $M$ is the cokernel of its injective envelope $\delta: M \longrightarrow I(M)$.

It is known that $A$ has a simple projective module if and only if it has a simple injective module. In this case, denote by $S_{\text{inj}}$ the unique simple injective $A$-module up to isomorphism. Then there exists map $\tilde{\tau}$ defined by

$$\tilde{\tau}(S) = \begin{cases} \tau(S), & \text{if } S \text{ is not projective} \\ S_{\text{inj}}, & \text{otherwise} \end{cases}$$

for each simple $A$-module $S$, where $\tau$ the Auslander-Reiten translation [1].

Take a complete set $\mathcal{S}$ of pairwise non-isomorphic simple $A$-modules. Recall from [4, 6] that there exist two maps $\gamma, \psi: \mathcal{S} \rightarrow \mathcal{S}$ given by

$$\gamma(S) = \tilde{\tau} \text{soc } P(S) \text{ and } \psi(S) = \tilde{\tau}^{-1} \text{top } I(S).$$

for each $S$ in $\mathcal{S}$. Here, ‘soc’ is the socle and ‘top’ is the top of a module.

The map $\gamma$ determines the resolution quiver for $A$. Denote by $A^{\text{opp}}$ the opposite algebra of $A$ and by $\gamma^{\text{opp}}$ the map $\tilde{\tau} \text{soc } P(-)$ for $A^{\text{opp}}$. Then $D\psi(S) = \gamma^{\text{opp}}(DS)$ for each $S$ in $\mathcal{S}$, where $D$ is the usual dual for finitely generated $A$-modules. Hence the resolution quiver of $A^{\text{opp}}$ is isomorphic to the quiver determined by the map $\psi$.

The following terminology is taken from [6]. A simple $A$-module $S$ is called $\gamma$-black provided that $\text{pd } S$ is not equal to 1; it is called $\gamma$-cyclic provided that $\gamma^n(S) = S$ for some integer $m > 0$. Dually, one can define $\psi$-black and $\psi$-cyclic simple $A$-modules.

We need the following lemma.

**Lemma 3.1.** Let $S$ and $T$ be simple $A$-modules.

1. If $S$ is not projective, then $\psi(T) = S$ if and only if $T$ is a composition factor in $\Omega^2(S)$.
2. If $\text{pd } \psi(S)$ is odd, then $\text{pd } S$ is odd and $\text{pd } S \leq \text{pd } \psi(S) - 2$.
3. $S$ is $\psi$-cyclic if and only if $\text{pd } S$ is not odd.
4. $A$ has infinite global dimension if and only if the set of $\psi$-cyclic simple $A$-modules is exactly the set of simple $A$-modules of infinite projective dimension.

**Proof.** If $A$ has no simple projective modules, then the arguments follow from [5, Section 3]. We mention that they are valid for any connected Nakayama algebra. □

We need the following.

**Lemma 3.2.** Let $M$ be an indecomposable $A$-module. Then either $\text{id } M \leq 1$ or $\text{soc } \Omega^{-2}(M) = \psi(\text{soc } M)$.

**Proof.** This is dual to [6, Lemma 2]; see also [4]. □
The following lemma provides a connection between the maps $\gamma$ and $\psi$.

**Lemma 3.3.** A simple $A$-module $S$ is $\gamma$-black if and only if $\psi \gamma(S) = S$.

**Proof.** Suppose that $S$ is a $\gamma$-black simple $A$-module. If $S$ is projective, then by definition $\psi \gamma(S) = S$. If $\text{pd} S \geq 2$, then by Lemma 3.2 we have $\gamma(S) = \text{soc} \Omega^1(S)$.

It follows from Lemma 3.1(1) that $\psi \gamma(S) = S$.

Suppose $\psi \gamma(S) = S$. If $S$ is projective, then $S$ is $\gamma$-black. If $S$ is not projective, then it follows from Lemma 3.1(1) that $\gamma(S)$ is a composition factor in $\Omega^2(S)$. In particular, $\Omega^2(S)$ is nonzero and thus $\text{pd} S \geq 2$.

Recall that a cycle in the resolution quiver of $A$ is called black provided that each vertex on this cycle is $\gamma$-black.

We have the following observation.

**Proposition 3.4.** Let $C$ be a cycle in the resolution quiver of $A$. Then the following statements are equivalent.

1. The vertices of $C$ form a $\psi$-cycle.
2. Each vertex on $C$ is $\psi$-cyclic.
3. $C$ is a black cycle.

**Proof.** (1) $\implies$ (2) This is obvious.

(2) $\implies$ (3) By Lemma 3.1(3) each $\psi$-cyclic simple $A$-module is $\gamma$-black.

(3) $\implies$ (1) Assume that the vertices of $C$ are $S_1, S_2, \cdots, S_m$ with $S_{i+1} = \gamma(S_i)$ for $i \geq 1$. Here, we identify $S_{m+1}$ with $S_1$.

Since $C$ is a black cycle, each $S_i$ is $\gamma$-black and each id $S_i$ is not odd for $i \geq 1$.

It follows from Lemma 3.5 that $\psi \gamma(S_i) = \psi(S_{i+1}) = S_i$.

We claim that each $\text{pd} S_i$ is not odd and thus each $S_i$ is $\psi$-cyclic by Lemma 3.1(3).

Since $\psi(S_{i+1}) = S_i$ for $i \geq 1$, it follows that $S_m, S_{m-1}, \cdots, S_1$ form a $\psi$-cycle.

For the claim, suppose to the contrary that the projective dimension of some $S_i$ is odd. Then by Lemma 3.2(2) $S_{i+1}$ is odd and $\text{pd} S_{i+1} \leq \text{pd} S_i - 2$. By induction, we infer that $\text{pd} S_{i+k} \leq \text{pd} S_i - 2k$ for $k \geq 1$. This is a contradiction.

Suppose that global dimension of $A$ is infinite. In particular, $A$ has no simple projective modules. The following lemma describes projective $A$-modules of finite injective dimension and injective $A$-modules of finite projective dimension.

**Lemma 3.5.** Let $A$ be a connected Nakayama algebra with infinite global dimension, and let $P$ be an indecomposable projective $A$-module and $I$ be an indecomposable injective $A$-module.

1. The injective dimension of $P$ is infinite if and only if $P$ is a nontrivial submodule of $P(S)$ with $S$ a $\gamma$-cyclic simple $A$-module.
2. The projective dimension of $I$ is infinite if and only if $I$ is a nontrivial factor module of $I(S)$ with $S$ a $\psi$-cyclic simple $A$-module.

**Proof.** (1) “$\Leftarrow$” If the injective dimension of $P$ is infinite, then the injective dimension of its cosyzygy $\Omega^{-1}(P)$ is also finite. It follows that $\Omega^{-1}(P)$ contains at least one composition factor with infinite injective dimension.

We claim that $\Omega^{-1}(P)$ contains at most one composition factor which is $\gamma$-cyclic. It follows from Lemma 3.1(4) that $\Omega^{-1}(P)$ contains precisely one composition factor which is $\gamma$-cyclic. Since $P$ is a nontrivial submodule of $P(T)$ for each composition factor $T$ in $\Omega^{-1}(P)$, we infer that $P$ is a nontrivial submodule of $P(S)$ with $S$ a $\gamma$-cyclic simple $A$-module.

For the claim, observe that $\text{soc} P(T) = \text{soc} P$ and $\gamma(T) = \gamma(\text{top} P)$ for each composition factor $T$ in $\Omega^{-1}(P)$. Since the composition factors in $\Omega^{-1}(P)$ have the same image under the map $\gamma$, at most one of them is $\gamma$-cyclic.
Suppose that \( P \) is a nontrivial submodule of \( P(S) \) with \( S \) a \( \gamma \)-cyclic simple \( A \)-module. By the previous claim, one can show that there exists only one composition factor in \( \Omega^{-1}(P) \) which is \( \gamma \)-cyclic, namely \( S \). Since the injective dimension of \( S \) is infinite by Lemma 5.1(4), the injective dimension of \( \Omega^{-1}(P) \) is infinite. Then the injective dimension of \( P \) is infinite.

(2) This is dual to (1).

Recall that an Artin algebra \( A \) is called a Gorenstein algebra if both \( \text{id}A \) and \( \text{pd}DA \) are finite. Here, \( D \) is the usual dual for finitely generated \( A \)-modules.

We are now ready to prove Proposition 1.2. Indeed, using the maps \( \gamma \) and \( \psi \) as above, several characterizations are given to decide whether a connected Nakayama algebra with infinite global dimension is Gorenstein or not. It strengthens [6, Proposition 5(a)].

**Proposition 3.6.** Let \( A \) be a connected Nakayama algebra with infinite global dimension. Then the following statements are equivalent.

1. \( A \) is a Gorenstein algebra.
2. Each \( \gamma \)-cyclic simple \( A \)-module is \( \gamma \)-black.
3. Each \( \psi \)-cyclic simple \( A \)-module is \( \psi \)-black.
4. The set of \( \gamma \)-cyclic simple \( A \)-modules is exactly the set of \( \psi \)-cyclic simple \( A \)-modules.

**Proof.** (1) \( \iff \) (2). Let \( S \) be a \( \gamma \)-cyclic simple \( A \)-module. By Lemma 3.5(1) the projective cover of \( S \) has no nontrivial projective submodules. Then the projective dimension of \( S \) greater than 1 and thus \( S \) is \( \gamma \)-black.

Similarly, we have (1) \( \iff \) (3).

(2) \( \implies \) (4). Following Proposition 3.4 the set of \( \gamma \)-cyclic simple \( A \)-modules is contained in the set of \( \psi \)-cyclic simple \( A \)-modules. By [7, Corollary 2.3] the two sets have the same finite number of modules. Then they must coincide.

(4) \( \implies \) (2). Since each \( \psi \)-cyclic simple \( A \)-module is \( \gamma \)-black, it follows that each \( \gamma \)-cyclic simple \( A \)-module is \( \gamma \)-black.

Similarly, one can prove that (3) and (4) are equivalent.

(2) + (3) \( \implies \) (1). By Lemma 3.5 all projective \( A \)-modules have finite injective dimension and all injective \( A \)-modules have finite projective dimension. Then \( A \) is a Gorenstein algebra.

We mention that the global dimension condition in Proposition 3.6 cannot be omitted; see [6, Example 2].

4. Cartan matrices and resolution quivers

In this section, we study the connection between the Cartan matrix and the resolution quiver for a fixed connected Nakayama algebra \( A \).

Denote by \( n = n(A) \) the number of non-isomorphic simple \( A \)-modules. Take a complete set \( \{S_1, S_2, \ldots, S_n\} \) of pairwise non-isomorphic simple \( A \)-modules. The Cartan matrix \( C_A = (c_{ij}) \) of \( A \) is an \( n \times n \) matrix, where \( c_{ij} \) is the number of copies of \( S_i \) appearing in a composition series for the projective cover of \( S_j \).

Recall that two \( n \times n \) integer matrix \( X \) and \( Y \) are \( \mathbb{Z} \)-equivalent provided that there exist invertible integer matrices \( P \) and \( Q \) such that \( PXQ = Y \). For an \( n \times n \) integer matrix, its Smith normal form is the \( n \times n \) diagonal integer matrix

\[
\text{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0)
\]

where \( d_1, d_2, \ldots, d_r \in \mathbb{N}^+ \) and \( d_i \) divides \( d_{i+1} \) for \( 1 \leq i \leq r - 1 \). The Smith normal form always exists and is unique; it is \( \mathbb{Z} \)-equivalent to the original matrix.
Proposition 4.1. Let $A$ be a connected Nakayama algebra. Then the Smith normal form of its Cartan matrix $C_A$ is the diagonal matrix
\[
\text{diag}(1, \cdots, 1, w(A), 0, \cdots, 0)
\]
with $c(A) - 1$ zeros on the diagonal. In particular, the rank of $C_A$ is $n(A) + 1 - c(A)$.

Proof. Recall [2, Section 2] in Section 2. For $1 \leq i \leq r - 1$, it is easy to show that the Cartan matrix $C_{A_i}$ of $A_i$ and the block diagonal matrix $\text{diag}(1, C_{A_{i+1}})$ are Z-equivalent.

By Lemma 2.2 we have $c(A_i) = c(A_{i+1})$ and $w(A_i) = w(A_{i+1})$. Then it suffices to prove the assertion for the selfinjective algebra $A_r$.

Assume that $A$ is a connected selfinjective Nakayama algebra. Denote by $m$ its radical length. Then the admissible sequence of $A$ is $(m, m, \cdots, m)$. It is routine to show that the resolution quiver of $A$ consists of $\gcd(m, n)$ cycles of the same weight $w = \frac{m}{\gcd(m, n)}$. Here, ‘gcd’ is the greatest common divisor.

Let $m = kn + r$ with $k \in \mathbb{N}$ and $1 \leq r \leq n$. After possible permutations on simple $A$-modules, the Cartan matrix $C_A$ is a circulant matrix given by
\[
c_{ij} = \begin{cases} k + 1, & \text{if } 0 \leq i - j < r \text{ or } j - i > n - r \\ k, & \text{otherwise}. \end{cases}
\]

It follows from [3] that the Smith normal form of the Cartan matrix $C_A$ is the diagonal matrix $\text{diag}(1, \cdots, 1, w, 0, \cdots, 0)$ with $\gcd(m, n) - 1$ zeros on the diagonal. Then the rank of $C_A$ is $n + 1 - \gcd(m, n)$. This finishes our proof. □

Remark 4.2. Use the notation in the Proposition 4.1.

(1) Following [2, Theorem 6], the global dimension of $A$ is finite if and only if the determinant of the Cartan matrix $C_A$ is 1. In fact, by [3, Proposition 2.2(5)] left retractions also preserve the determinants of Cartan matrices. Then the determinant of the Cartan matrix $C_A$ is $w(A)$ if the resolution quiver of $A$ is connected; see also [2, Lemma 2]. This provides another proof of Proposition 1.1.

(2) Denote by $A^\text{op}$ the opposite algebra of $A$. Then the Cartan matrix of $A^\text{op}$ is the transpose of $C_A$. Since $C_A$ and its transpose $C_A^T$ have the same Smith normal form, it follows that $c(A) = c(A^\text{op})$ and $w(A) = w(A^\text{op})$; see also [8, Proposition 5.2].

Let $X$ be a subset of $\{S_1, S_2, \cdots, S_n\}$. There exists a $n \times 1$ vector $\xi_X$ associated with $X$, where the $i$-th entry of $\xi_X$ is 1 if $S_i$ is in $X$ and the $i$-th entry of $\xi_X$ is 0 if $S_i$ is not in $X$ for $1 \leq i \leq n$. Denote by $1$ the $n \times 1$ vector $(1, \cdots, 1)^T$.

We have the following observation.

Proposition 4.3. Let $A$ be a connected Nakayama algebra. Denote by $\Gamma$ the set of cycles and by $B^r$ the set of black cycles in the resolution quiver of $A$.

(1) The vectors $\{\xi_C\}_{C \in \Gamma}$ are maximal linearly independent solutions to the linear system $C_A \xi = w(A)1$.

(2) The vectors $\{\xi_C\}_{C \in \Gamma^r}$ are the entire nonnegative integer solutions to the linear system $C_A \xi = w(A)1$.

(3) The vectors $\{\xi_E\}_{E \in B^r}$ are maximal linearly independent solutions to the linear system $C_A^T \xi = C_A \xi = w(A)1$. 

Denote by $c(A)$ the number of cycles and by $w(A)$ the weight of a cycle in the resolution quiver of $A$.

The following result provides a connection between the Cartan matrix and the resolution quiver for $A$.

Proposition 4.4. Let $A$ be a connected Nakayama algebra. Then the Smith normal form of its Cartan matrix $C_A$ is the diagonal matrix
\[
\text{diag}(1, \cdots, 1, w(A), 0, \cdots, 0)
\]
with $c(A) - 1$ zeros on the diagonal. In particular, the rank of $C_A$ is $n(A) + 1 - c(A)$.

Proof. Recall [2, Theorem 6], the global dimension of $A$ is finite if and only if the determinant of the Cartan matrix $C_A$ is 1. In fact, by [3, Proposition 2.2(5)] left retractions also preserve the determinants of Cartan matrices. Then the determinant of the Cartan matrix $C_A$ is $w(A)$ if the resolution quiver of $A$ is connected; see also [2, Lemma 2]. This provides another proof of Proposition 1.1.

(2) Denote by $A^\text{op}$ the opposite algebra of $A$. Then the Cartan matrix of $A^\text{op}$ is the transpose of $C_A$. Since $C_A$ and its transpose $C_A^T$ have the same Smith normal form, it follows that $c(A) = c(A^\text{op})$ and $w(A) = w(A^\text{op})$; see also [8, Proposition 5.2].

Let $X$ be a subset of $\{S_1, S_2, \cdots, S_n\}$. There exists a $n \times 1$ vector $\xi_X$ associated with $X$, where the $i$-th entry of $\xi_X$ is 1 if $S_i$ is in $X$ and the $i$-th entry of $\xi_X$ is 0 if $S_i$ is not in $X$ for $1 \leq i \leq n$. Denote by $1$ the $n \times 1$ vector $(1, \cdots, 1)^T$.

We have the following observation.

Proposition 4.3. Let $A$ be a connected Nakayama algebra. Denote by $\Gamma$ the set of cycles and by $B^r$ the set of black cycles in the resolution quiver of $A$.

(1) The vectors $\{\xi_C\}_{C \in \Gamma}$ are maximal linearly independent solutions to the linear system $C_A \xi = w(A)1$.

(2) The vectors $\{\xi_C\}_{C \in \Gamma^r}$ are the entire nonnegative integer solutions to the linear system $C_A \xi = w(A)1$.

(3) The vectors $\{\xi_E\}_{E \in B^r}$ are maximal linearly independent solutions to the linear system $C_A^T \xi = C_A \xi = w(A)1$. 

Denote by $c(A)$ the number of cycles and by $w(A)$ the weight of a cycle in the resolution quiver of $A$. 

Proof. (1) Since $C$ is a cycle in the resolution quiver, for $1 \leq i \leq n$ the simple $A$-module $S_i$ appears exactly $w_i(A)$ times in the direct sum $\oplus_{i \in C} P_i(S)$. It follows that $C_A S_i = w_i(A)1$. Since the vectors $\{\xi_C\}_{C \in \Gamma}$ have disjoint support, they are linearly independent. Then we obtain $\{\xi_C\}_{C \in \Gamma}$ linearly independent solutions to the linear system $C_A \xi = w(A)1$. By Proposition 3.4, the number of these solutions is $n + 1 - \text{rank} C_A$. Therefore, the solutions $\{\xi_C\}_{C \in \Gamma}$ are maximal.

(2) This follows from (1).

(3) Let $E$ be a cycle in the resolution quiver of $A$. By Proposition 3.4, the cycle $E$ is black if and only if the vertices of $E$ form a $\psi$-cycle. By (2), the vertices of $E$ form a $\psi$-cycle if and only if $C_A^* \xi_E = w(A)1$. Then the vectors $\{\xi_E\}_{E \in B \Gamma}$ are linearly independent solutions to the linear system $C_A^* \xi = C_A \xi = w(A)1$.

Denote by $\Psi$ the set of $\psi$-cycles for $A$. We claim that the vectors $\{\xi_C\}_{C \in \Gamma \cup \Psi}$ are linearly independent. For a solution $\xi$ to the desired linear system, by (1) we have $\xi = \sum_{C \in \Gamma} a_C \xi_C = \sum_{D \in \Psi} b_D \xi_D$. Observe that the intersection $\Gamma \cap \Psi$ is the set of black cycles. Since the vectors $\{\xi_C\}_{C \in \Gamma}$ are linearly independent, we have $\xi = \sum_{C \in B \Gamma} a_C \xi_C$. This proves (3).

For the claim, let $\sum_{C \in \Gamma \cup \Psi} a_C \xi_C = 0$. For a non-black cycle $C \in \Gamma$, it follows from Proposition 3.4 that there exists some $S_i$ on $C$ which is not $\psi$-cyclic. Observe that the $i$-th entry of $\sum_{C \in \Gamma \cup \Psi} a_C \xi_C$ is $a_C$. It follows that $a_C = 0$ for each non-black cycle $C \in \Gamma$. Since the vectors $\{\xi_C\}_{C \in \Psi}$ have disjoint support, $a_C = 0$ for each $\psi$-cycle $C$. Therefore, the vectors $\{\xi_C\}_{C \in \Gamma \cup \Psi}$ are linearly independent. $\square$

Denote by $b(A)$ the number of black cycles in the resolution quiver of $A$.

We have the following.

Corollary 4.4. Let $A$ be a connected Nakayama algebra.

(1) $b(A)$ is nonzero if and only if $\text{rank} \begin{pmatrix} C_A^* \\ C_A \end{pmatrix} = \text{rank} \begin{pmatrix} C_A^T & 1 \\ C_A & 1 \end{pmatrix}$.

(2) If $b(A)$ is nonzero, then $b(A) = n(A) + 1 - \text{rank} (C_A, C_A^*)$.

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