Synchronization of the cardiac pacemaker model with delayed pulse-coupling

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Abstract
We consider the integrate-and-fire model of the cardiac pacemaker with delayed pulsatile coupling. Sufficient conditions of synchronization are obtained for identical and non-identical oscillators.

Keywords: Cardiac pacemaker model; Identical oscillators; Non-identical oscillators; Firing in unison; Delayed pulse-coupling; Discontinuous dynamics; Fixed points.

1. Introduction

In paper [1] C. Peskin proposed a model of cardiac pacemaker, where signal of fire arises not from an outside stimuli, but in the population of cells itself. Well known conjectures of self-synchronization were formulated. Solution of these conjectures for identical oscillators [1, 2] stimulated intensive investigations [3]-[11].

Delays arise naturally in many biological models [12]. In particular, they were considered in firefly models [13] as delayed between stimulus and response, and in continuously coupled neuronal oscillators [14]. Authors of [4] considered the phenomenon for the Mirollo and Strogatz analysis, [2]. Identical oscillators were investigated. Two oscillators dynamics is discussed mathematically, and the multi-oscillatory system by computer simulations. It was found that the excitatory model of two units “can get only out-of-phase synchronization since in-phase synchronization proved to be not stable.” In paper [5] a model without leakage was discussed, that is, oscillators increase at a constant rate between moments of firing. It was found that a periodic solution is reached after a finite time. Consequently, research of integrate-and-fire oscillators, which admit discontinuous dynamics; fixed points.

2. The couple of identical oscillators

Let us start to analyze two identical oscillators, which satisfy, if they do not fire, the following differential equations

\[ x'_i = S - \gamma x_i, \quad (2.1) \]

where \( 0 \leq x_i \leq 1, i = 1, 2 \). It is assumed that \( S, \gamma \) are positive numbers and \( \kappa = \frac{S}{\gamma} > 1 \).

When \( x_i(t) = 1, \) then the oscillator fires, \( x_i(t^+) = 0 \). The firing changes value of the another oscillator, \( x_j \), such that

\[ x_i(t^+) = 0, \text{ if } x_i(t) \geq 1 - \epsilon, \quad (2.2) \]

and

\[ x_i(t + \tau^+) = x_i(t + \tau) + \epsilon, \text{ if } x_i(t) < 1 - \epsilon. \quad (2.3) \]

We have that

\[ x_i(s) = x_i(t)e^{-\gamma(s-t)} + \int_t^s e^{-\gamma(u-t)} S \, du \]

near \( t \).

In what follows, assume that

\[ \frac{\kappa - 1}{\kappa - 1 + \epsilon} < e^{-\gamma \tau}. \quad (2.4) \]

Then, from

\[ \|x_i(s)\| \leq \|x_i(t)\|e^{-\gamma(s-t)} + \int_t^s e^{-\gamma(u-t)} S \, du \leq (1 - \epsilon)e^{-\gamma \tau} + \kappa(1 - e^{-\gamma \tau}), \]

and \( x_i(t) < 1 - \epsilon \), we obtain that \( x_i(s) < 1 \), for all \( s \in [t, t + \tau] \). In other words oscillator \( x_i \) does not achieve the threshold within interval \([t, t + \tau]\), if the distance of \( x_i(t) \) to threshold is more than \( \epsilon \). This is important for the construction of the prototype map, and makes a sense of condition (2.3).

One must emphasis that couplings of units are not only delayed in our model. By (2.2) oscillators interact instantaneously, if they are near threshold. This assumption is natural as firing provokes another oscillator, which being close to
threshold "is ready" to react instantaneously. Otherwise, the interaction is retarded.

Next, we shall construct the prototype map. Fix a moment \( t = \xi \), when \( x_1 \) fires, and suppose that oscillators are not synchronized. In interval \([\xi, \xi + \tau]\) oscillator \( x_2 \) moves by law \( x_2(t) = x_2(\xi) e^{-\gamma(t-\xi)} + \int_\xi^t e^{-\gamma(t-u)} S du \), and

\[
x_2(\xi + \tau) = [x_2(\xi) - \kappa] e^{-\gamma \tau} + \kappa.
\]

(2.5)

Denote \( t = \eta \), the firing moment of \( x_2 \), then

\[
x_2(\eta) = [x_2(\xi + \tau) + \epsilon] e^{-\gamma(\tau - \eta)} + \kappa[1 - e^{-\gamma(\tau - \eta)}].
\]

The equation \( x_2(\eta) = 1 \) implies that

\[
e^{-\gamma\eta} = \frac{1 - \kappa}{x_2(\xi) - \kappa + \epsilon_1}.
\]

(2.6)

where \( \epsilon_1 = \epsilon e^{-\gamma \tau} \). Since \( x_1(\eta) = \kappa[1 - e^{-\gamma(\tau - \eta)}] \), we have that

\[
x_1(\eta) = \kappa \frac{1 - (x_2(\xi) + \epsilon)}{x_2(\xi) - \kappa + \epsilon_1}.
\]

(2.7)

Introduce the following map

\[
L_D(v, \epsilon) = \kappa \frac{1 - (v + \epsilon)}{v - (v + \epsilon)},
\]

(2.8)

such that \( x_1(\eta) = L_D(x_2(\xi), \epsilon) \). If \( t = \xi \) is the next to \( \eta \) firing moment of \( x_2 \), then one can similarly find that \( x_2(\xi) = L_D(x_1(\eta), \epsilon) \). One can see that the map \( L_D \) can be useful for our investigation, since it evaluates alternatively the sequence of values \( x_1 \) and \( x_2 \) at firing moments.

Take \( \tau > 0 \) so small that

\[
e^{-\gamma \tau} > \epsilon.
\]

(2.9)

From (2.9) it implies that \( \epsilon_1 < 1 \).

One can evaluate that

\[
L_D(1 - \epsilon_1, \epsilon) = 0.
\]

and the derivatives of the map in \((0, 1 - \epsilon_1)\) satisfy

\[
L_D'(v, \epsilon) = \kappa \frac{1 - \kappa}{(v - (v + \epsilon))^2} < 0,
\]

(2.10)

and

\[
L_D''(v, \epsilon) = 2\kappa \frac{1 - \kappa}{(v - (v + \epsilon))^3} < 0
\]

(2.11)

We can easily find that there is a fixed point of the map,

\[
v^* = \left( \kappa - \frac{\epsilon_1}{2} \right) - \sqrt{\kappa^2 - \kappa + \frac{\epsilon_1^2}{4}}.
\]

(2.12)

and

\[
L_D'(v^*, \epsilon) < -1.
\]

(2.13)

That is, fixed point \( v^* \) is a repeller.

Now, we will define an extension of \( L_D \) on \([0, 1]\) in the following way. Let

\[
\omega = \frac{1 - \epsilon_1}{\kappa - 1 + \epsilon}.
\]

(2.14)

One can see that \( 1 - \epsilon < \omega < 1 \), if

\[
e^{\gamma \tau} < \frac{\kappa}{\kappa - 1 + \epsilon}.
\]

(2.15)

In what follows, we assume that \( \epsilon \) is sufficiently small such that (2.4) implies (2.15). We set \( L_D(0, \epsilon) = \omega \), and define \( L_D(v, \epsilon) = 0 \), if \( 1 - \epsilon_1 \leq v \leq 1 \). Since \( L_D : [0, 1] \rightarrow [0, 1] \) is a monotonic continuous function and \([0, 1]\) is an invariant set of this map, it is convenient for analysis by using iterations. The graph of this map is seen in Figure 1. To emphasize a significance of this map for the present analysis, let us see how iterations of it can help to observe the synchronization. Fix \( t_0 \geq 0 \), a firing moment, \( x_1(t_0) = 1, x_1(t_0+) = 0 \). While the couple \( x_1, x_2 \) does not synchronize, there exists a sequence of moments \( t_0 < t_1 < \ldots \) such that \( x_1 \) fires at \( t_i \) with even \( i \) and \( x_2 \) with odd indices.

Denote \( u_i = x_1(t_i) \), if \( i \) is odd, and \( u_i = x_2(t_i) \), if \( i \) is even. One can easily see that \( u_{i+1} = L_D(u_i, \epsilon), i \geq 0 \). The pair synchronizes if and only if there exists \( j \geq 1 \) such that \( x_1(t) \neq x_2(t) \), if \( t \leq t_j \), and \( x_1(t) = x_2(t) \), for \( t > t_j \). In particular, both oscillators have to fire at \( t_j \). That is, inequalities \( 1 - \epsilon \leq u_{i+1} < 1 \) are valid. It is possible if \( 0 \leq u_{j-1} \leq L_D(1 - \epsilon) \). In particular, we have that \( L_D(0) = \omega \) satisfies this condition. In the same time, if \( 1 - \epsilon_1 \leq u_{j-1} < 1 \), then \( u_{j-1} = 0 = L_D(u_{j-3}) \) and \( 1 - \epsilon < u_{j-1} = \omega < 1 \) again. That is, we have found that if there exists an integer \( k \geq 1 \) such that \( 1 - \epsilon \leq L_D^k(v) \leq 1 \), then the motion \((x_1(t), x_2(t))\) with \( x_1(t_0+) = v, x_2(t_0+) = 0 \), synchronizes at the \( k \)-th firing moment. Conversely, if a motion \((x_1(t), x_2(t))\)
synchronizes, then one can find a firing moment, \( t_0 \), such that 
\[ x_1(t_0 + \epsilon) = 0, x_2(t_0 + \epsilon) = v, v \in [0,1], \]
and a number \( k \) such that 
\[ 1 - \epsilon \leq L_D^k(v) \leq 1. \]

Thus, the last discussion confirms that the analysis of synchronization is consistent fully with the dynamics of the introduced map \( L_D(v, \epsilon) \) on \([0,1]\), and the map \( L_D \) can be applied as the main instrument of the paper. That is why, we use this function as a prototype map in our investigations.

Now, by applying properties of \( L_D \), and analyzing self-compositions of the map, one can easily obtain that for all \( k \geq 0 \) functions \( L_D^k \) have only one fixed point, \( v^* \), and \([L_D^k(v^*, \epsilon)] > 1\). We skip the discussion as it is respectively simple, and request a large place. Since all the maps \( L_D^k \) have one and the same fixed point, \( v^* \), there is not a \( k \)-periodic motion, \( k > 1 \), of the map. Consequently, for arbitrary point \( v \neq v^* \) one has a stabilized trajectory present in Figure 1. The couple synchronizes when \( L_D^k(v, \epsilon) \geq 1 - \epsilon \).

Next, we investigate the rate of synchronization. Set \( a_0 = L_D^{-1}(0) = 0, a_{k+1} = L_D^{-1}(a_k), k = 0, 1, 2, \ldots \) (See Figure 2).

Denote by \( S_k \) the region of \([0,1]\), which points \( v \) are synchronized after exactly \( k \) iterations of the map \( L_D \). One can see that 
\[ S_0 = [1 - \epsilon, 1], S_1 = [a_0, a_2] \text{ and } S_k = [a_{k-1}, a_{k+1}], \]
if \( k \geq 3 \), is an odd positive integer, and \( S_1 = [a_{k-1}, a_{k+1}] \), if \( k \geq 2 \), is an even positive integer. One can see that \( a_k \to v^* \) as \( k \to \infty \). We shall call \( S_k, k \geq 0 \), the rate intervals.

From the discussion has been made above it follows that there is no a finite time such that all points of the unite square synchronize. The closer \( v \) is to the equilibrium \( v^* \) the later is the moment of synchronization.

Set \( T = \frac{1}{2} \ln \frac{1}{1-\epsilon} \) and denote by \( \hat{T} \) the time needed for solution \( u(t, 0, v') \) of the equation \( u' = S - \gamma u \), to achieve threshold. Since all oscillators fire within an interval of length \( T \) and the distance between two firing moments of an oscillator are not less than \( \hat{T} \), we can conclude that the following theorem is correct.

**Theorem 2.1.** Assume that (2.4) and (2.5) are valid. If \( t_0 \geq 0 \) is a firing moment, \( x_i(t_0) = 1, x_i(t_0 + \epsilon) = 0, \) and \( x_2(t_0 + \epsilon) \in S_m \) for some natural number \( m \), then the couple \( x_1, x_2 \) of continuously coupled identical biological oscillators synchronizes within the time interval \([t_0 + \frac{m}{2} \hat{T}, t_0 + T_m]\).

3. **Non-identical oscillators: the general case.**

To make our investigation closer to the real world problems one has to consider an ensemble of non-identical oscillators. We will discuss the following system of equations

\[
x_i' = (S + \mu_i) - (\gamma + \xi_i)x_i,
\]
where \( 0 \leq x_i \leq 1 + \xi_i, i = 1, 2, \ldots, n \). The constants \( S \) and \( \gamma \) are the same as in the last section such that \( \kappa = \frac{S}{\gamma} > 1 \). Moreover, constants \( \mu_i \) and \( \xi_i \) are sufficiently small for \( \kappa_i = \frac{S\mu_i}{\gamma x_i} > 1 \).

When \( x_i(t) = 1 + \xi_i \) then the oscillator fires, \( x_i(t+\epsilon) = 0 \). The firing changes values of other oscillators \( x_i, i \neq j \), such that

\[
x_i(t+\epsilon) = 0, \text{ if } x_i(t) \geq 1 - \epsilon
\]
and, if \( x_i(t) < 1 - \epsilon \), then

\[
x_i(t + \tau + \epsilon) = x_i(t + \tau) + \epsilon + \xi_i.
\]

In what follows, we call real numbers \( \epsilon, \mu_i, \xi_i, \xi_i, \) parameters, assuming the first one is positive. Moreover, constants \( \mu_i, \xi_i, \xi_i, \xi_i \) will be called parameters of perturbation. Assume that they are zeros to obtain the model of identical oscillators.

We have that an exhibitory model is under discussion, that is \( \epsilon + \xi_i > 0 \) for all \( i \). Coupling is all-to-all such that each firing elicits jumps in all non-firing oscillators. If several oscillators fire simultaneously, then other oscillators react as just it one oscillator fires. In other words, any firing acts only as a signal which abruptly provokes a state change, the intensity of the signal is not important, and pulse strengths are not additive. We have that

\[
x_i(s) = x_i(t)e^{-\gamma(\xi_j + \eta)} + \int_{\gamma}^{t\eta} e^{-\gamma(\xi_j + \eta)}(S + \mu_idu),
\]

near \( t \).

If one assume that condition (2.4) is valid, and constants \( \mu_i \) and \( \xi_i \) are sufficiently small such that

\[
\frac{\kappa_i - 1}{\kappa_i - 1 + \epsilon} < e^{-\gamma/\xi_i \tau},
\]
then \( x_i(s) < 1 \) for all \( s \in [t, t + \tau] \), if \( x(t) < 1 - \epsilon \).

In this section we begin with analysis of a couple of oscillators of the ensemble of \( n \) oscillators, and find that the couple synchronizes if parameters close to zero. Then synchronization of the ensemble will be proved.

Consider the model of \( n \) non-identical oscillators given by relations (2.1) and (2.3). Fix two of them, let say, \( x_1, x_r \).
Lemma 3.1. Assume that conditions (2.2) and (2.2) are valid, \( l_0 \geq 0 \) is a firing moment such that \( x_i(t_0) = 1 + \xi_i, x_{i}(l_0) = 0 \). If parameters are sufficiently close to zero, and absolute values of parameters of perturbation are sufficiently small with respect to \( \epsilon \), then the couple \( x_1, x_2 \) synchronizes within the time interval \([a_0, a_1]\) and within the time interval \([l_0 + m + (m + 1)T], l_0 + m + (m + 1)T\), if \( x_i(l_0) \in S_{m}, m \geq 1 \).

Proof. If \( 1 + \xi_i - \epsilon - \epsilon_i \leq x_i(l_0) \leq 1 + \xi_i \), then two oscillators fire simultaneously, and we have only to prove the persistence of the synchrony, that will be discussed later. So, fix another oscillator \( x_j(t) \) such that \( 0 \leq x_j(t_0) < 1 + \xi_i - \epsilon - \epsilon_i \). While the couple is not synchronized, there is a sequence of firing moments, \( t_i \), such that \( 0 \leq t_i < t_2 < \ldots \), and oscillator \( x_j \) fires at \( t_i \), with \( i \) even, and \( x_j \) fires at \( t_i \) with odd \( i \). For the sake of brevity let \( u_i = x_j(t_i), i = 2j + 1, u_i = x_j(t_i), i = 2j, j \geq 0 \). In what follows we shall show how to evaluate \( u_i \), through \( L_{c}(u_i) \). Consider \( i \) even. There are \( k \leq n - 2 \) distinct firing moments of the motion \( x(t) \) in the interval \((t_0, t_{i+1})\). Denote by \( t_j < \theta_1 < \theta_2 < \ldots < \theta_k < t_{i+1} \), the moments of firing, at least one of the coordinates of \( x(t) \). We have that

\[
x_i(t_0 + \tau) = \xi_i - \epsilon_i \leq x_i(t_0) \leq 1 + \xi_i,
\]

and continuously depends on parameters and \( x_i(t_i) \).

We have also that

\[
x_i(t_1 + \tau) = \xi_i(1 - \epsilon - \epsilon_i) + \kappa_i(1 - \epsilon - \epsilon_i),
\]

\[
x_i(t_2 + \tau) = \xi_i(1 - \epsilon - \epsilon_i),
\]

\[
\ldots \ldots \ldots
\]

\[
x_i(t_{i+1}) = \xi_i(1 - \epsilon - \epsilon_i),
\]

and within the time interval \([a_0, a_1]\) we have that \( x_i(t_0 + \tau) = \kappa_i(1 - \epsilon - \epsilon_i) \).

Set \( \delta_i(u_i, \zeta_i) = \kappa_i - \kappa_i \). One can see that \( \delta_i(0, 0) = 0 \). Use (3.21) and (3.23) to obtain

\[
x_i(t_{i+1}) = (x_i(t_1 + \tau) + \epsilon)i(1 - \epsilon - \epsilon_i) + \kappa_i(1 - \epsilon - \epsilon_i),
\]

\[
\kappa_i(1 - \epsilon - \epsilon_i),
\]

\[
\ldots \ldots \ldots
\]

\[
x_i(t_{i+1}) = \kappa_i(1 - \epsilon - \epsilon_i),
\]

and the last expression tends to zero as all of its arguments tend to zero. Next, by utilizing (3.22) and (3.29) we have that \( t_{i+1} = L_{c}(u_i) \in \Phi_2(\epsilon_i, \zeta_2, \delta_2, \tau) \), where

\[
\Phi_2(\epsilon_i, \zeta_2, \delta_2, \tau) = \frac{\xi_i + \epsilon_i + \Phi_i(\epsilon_i, \zeta_i, \delta_i, \tau)}{S - |\mu_i| - \gamma - |\zeta_i|}.
\]

Now, by applying the last inequality, (3.25) and (3.28), one can see that

\[
|L_{c}(u_i) - K_{c}(u_i)| = |x_i(t_{i+1}) - \psi(t_{i+1})| \leq |x_i(t_{i+1}) - x_i(t_{i+1})| + |x_i(t_{i+1}) - \psi(t_{i+1})| \leq \Phi_2(S + |\mu_i| + \gamma + |\zeta_i|) + \Phi_1.
\]

That is, difference \( L_{c}(u_i) - u_i \) can be made arbitrarily small if the parameters are sufficiently close to zero. Moreover, we should assume smallness of absolute values of the parameters.
of perturbation with respect to $\epsilon$, to satisfy (3.22). This convergence is uniform with respect to $u_0$. We can also vary the number of points $\Theta$ and their location in the intervals $(t_j, t_{j+1})$ between 0 and $n - 1$. The convergence is indifferent with respect to these variations, too.

Consider $L^p_D(u_0, \epsilon)$. It is true that $L^p_D(u_0, \epsilon) \in [1 - \epsilon, 1]$. Assume, without loss of generality, that $m$ is an even number. Since $L_D$ is a continuous function, we can find recurrently, by applying the following sequence of inequalities $|u_i - L^p_D(u_0, \epsilon)| \leq |u_i - L_D(u_0, \epsilon)| + |L_D(u_0, \epsilon) - L^p_D(u_0, \epsilon)|$, $i = 1, 2, \ldots$, that either $1 + \xi_l - \epsilon < u_m < 1 + \xi_l$ or $1 + \xi_l - \epsilon < u_{m+1} < 1 + \xi_l$, if the parameters are sufficiently small. From the notation it implies that each of the last two inequalities brings the couple to synchronization.

Since each of the iterations of $L_D$ is done within interval with length not more than $T$, we obtain now that the couple $x_i, x_j$ is synchronized not later than $t = t_0 + (m + 1)T$.

We have found that oscillators $x_i$ and $x_j$ fire in unison at some moment $t = \Theta$. Next, we show that they will save the state, being different. To find conditions for this, let us denote by $\tau > \Theta$ the next moment of firing of the couple. Let say, $x_i$ fires at this moment. Thus, we have that $x_i(\Theta + \tau) = x_j(\Theta + \tau) = 0$. Then $x_i(t) = x_j(t), t \leq \Theta \leq \tau$. It is clear that to satisfy $x_i(\tau + \epsilon) > x_j(\tau + \epsilon) = 0$, we need $1 + \xi_l - \epsilon < \epsilon < 1 + \xi_l$. By applying formula (3.22) again, this time with $t_\Theta = \Theta, t_{\Theta+1} = \tau$, one can easily obtain that the inequality is correct if parameters are close to zero and absolute values of the parameters of perturbation are small with respect to $\epsilon$. Thus, one can conclude that if a couple of oscillators is synchronized at some moment of time than it continues to fire in unison for ever. The lemma is proved.

Let us extend the result of the last Lemma for the whole ensemble.

**Theorem 3.1.** Assume that (2.2) and (2.9) are valid, $t_0 \geq 0$ is a firing moment such that $x_i(t_0) = 1 + \xi_j, x_j(t_0) = 0$. If the parameters are sufficiently close to zero, and absolute values of parameters of perturbation are sufficiently small with respect to $\epsilon$, then the motion $x_i(t)$ of the system synchronizes within the time interval $[t_0, t_0 + T]$, if $x_i(t_0) \neq [a_0, a_1], i \neq j$, and within the time interval $[t_0 + \max_{a_j} k - 1, t_0 + (\max_{a_j} k + 1)T]$, if there exist $x_i(t_0 + k) \in [a_0, a_1]$ for some $s \neq j$ and $x_i(t_0 + k) \in S_j, i \neq j$.

**Proof.** Consider the collection of couples $(x_i, x_j), i \neq j$. Each of these pairs synchronizes by the last Lemma within interval $[t_0 + \max_{a_j} k - 1, t_0 + (\max_{a_j} k + 1)T]$. The theorem is proved.

Let us introduce a more general system of oscillators such that Theorem (3.1) is still true.

Consider a system of $n$ oscillators given such that if $i$-th oscillator does not fire or jump up, it satisfies $i$-th equation of system (2.3). If several oscillators $x_i, x_j, i = 1, 2, \ldots, k$, fire such that $x_i(t) = 1 + \phi(t, x(t), x_i(t - \tau_i), i), \phi(t, x(t), x_i(t - \tau_i)) < \xi_i, i = 1, 2, \ldots, n$, and $x_i(t) = 0$, then all other oscillators $x_{i+n}, p = k + 1, k + 1, \ldots, n$, change their coordinates by law

$$x_i(t) = 0, \quad x_i(t) \geq 1 - \epsilon$$

and, if $x_i(t) < 1 - \epsilon$,

$$x_i(t + \tau) = x_i(t + \tau) + \epsilon + \sum_{i=1}^{k} \xi_i, \quad (3.31)$$

One can easily see that the last theorem is correct for the model just have been described, if $\epsilon + \sum_{i=1}^{k} \xi_i > 0$, for all possible $k, l$ and $i_\bot$, and we assume that $\xi_i$ are also parameters of perturbation. Moreover, one can easily see that initial conditions for thresholds conditions can be chosen arbitrarily with values in the domain of the system.

**Remark 3.1.** Our preliminary analysis shows that the dynamics in a neighborhood of $v^*$ can be very complex. We do not exclude that a chaos appearance can be observed, and trajectories may belong to a fractal, if parameters are not small. It does not contradict the zero Lebesgue measure of non-synchronized points. Possibly, analysis of non-identical oscillators with not small parameters is of significant interests to explore arrhythmias, earthquakes, chaotic flashing of fireflies, etc.

**Remark 3.2.** The time of synchronization for a given initial point does not increase if number of oscillators increases (but the parameters needed to be closer to zero). This property, possibly, can be accepted as a small-world phenomenon.

4. The simulation result

To demonstrate our main result numerically, let us consider a model of 100 oscillators, which initial values are randomly uniform distributed in $[0, 1]$. Their differential equations are of form

$$x_i' = (4.1 + 0.01 \cdot \text{sort}(\text{rand}(1), n)) - (3.2 + 0.01 \cdot \text{sort}(\text{rand}(1), n)) x_i,$$

and thresholds

$$1 + 0.005 \cdot \text{sort}(\text{rand}(1), n), i = 1, 2, \ldots, 100,$$

where deviations of coefficients the threshold are also uniformly random in $[0, 1]$. We place the result of simulation with $\epsilon = 0.06$ and $\tau = 0.002$ in Figure 3, where the state of the system is shown at the initial moment, before the 183-th jump, before the 366-th jump and the last is before the 549-th jump. That is, it is obvious that eventually all oscillators fire in unison.

5. Conclusion

The cardiac pacemaker model of identical and non-identical oscillators with delayed pulse-couplings is investigated in the paper. We apply the method of investigation proposed in [13], which is based on a specially defined map. The map is, in fact, a Poincaré map if one considers the identity of oscillators. Sufficient conditions are found such that delay involvement in the Peskin’s model does not change the synchronization result for identical and non-identical oscillators [1, 2, 13]. The result has
a biological sense, since retardation is often presents in biological processes and if one proves that a phenomenon preserves even with delays, that makes us more confident that the model is adequate to the reality. Moreover, the method of treatment of models with delay can be useful for neural networks and earthquake faults analysis. All proved assertions can be easily specified with ε = 0, to obtain synchronization of the Peskin’s model for identical [1] and nonidentical [15] oscillators. In particular, from the theorem of Section 2 one can obtain the result of Example 2.2 of [15].

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