Asymptotics for the maximum regression depth estimator

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September 27, 2018
Abstract

Notions of depth in regression have been introduced and studied in the literature. Regression depth (RD) of Rousseeuw and Hubert (1999) (RH99), the most famous one, is a direct extension of Tukey location depth (Tukey (1975)) to regression. RD of RH99 satisfies all the desirable axiomatic properties in Zuo (2018) and therefore could serve as a real depth notion in regression.

Like its location counterpart, the most remarkable advantage of notion of the depth in regression is to introduce directly the maximum (or deepest) regression depth estimator for regression parameters in a multi-dimensional setting. Classical questions for the maximum regression depth estimator include (i) is it a consistent estimator (or rather under what sufficient conditions, it is consistent) and (ii) is there any limiting distribution?

Bai and He (1999) (BH99) intended to answer these questions. Unfortunately, they treated a closely related but different depth notion than the intended one. So the above questions remain open. Answering these questions is the main goal of this article.

AMS 2000 Classification: Primary 62G09; Secondary 62G05, 62G15 62G20.

Key words and phrase: regression depth, maximum depth estimator, consistency, limiting distribution, asymptotics.

Running title: Asymptotics for maximum regression depth estimator.
1 Introduction

Depth notions in location have had much attention in the literature. In fact data depth and its applications remain one of the most active research topics in statistics in the last two decades. Most favored notions of depth in location include Tukey (1975) halfspace depth (HD) (popularized by Donoho and Gasko (1992)), Liu (1990) simplicial depth, and projection depth (PD) (Liu (1992) and Zuo and Serfling (2000), promoted by Zuo (2003)), among others.

Depth notions in regression have also been sporadically proposed. Regression depth of Rousseeuw and Hubert (1999) (RH99) (RD), the most famous one, is a direct extension of Tukey HD to regression. Others include Carrizosa depth (Carrizosa (1996)) and the projection regression depth (PRD) induced from Marrona and Yohai (1993) (MY93) and proposed in Zuo (2018). The latter turns out to be the extension of PD to regression.

One of the prominent advantages of depth notions is that they can be directly employed to introduce median-type deepest estimating functionals (or estimators in the empirical case) for the location or regression parameter in a multi-dimensional setting based on a general min-max stratagem. The maximum (deepest) regression depth estimator serves as a robust alternative to the classical least squares or least absolute deviations estimator of the unknown parameter in a general linear regression model:

\[ y = x'\beta + e, \]  

where \( ' \) denotes the transpose of a vector, and random vector \( x = (x_1, \ldots, x_p)' \) and parameter vector \( \beta \) are in \( \mathbb{R}^p \) \( (p \geq 2) \) and random variable \( y \) and \( e \) are in \( \mathbb{R}^1 \). If \( \beta = (\beta_0, \beta_1)' \) and \( x_1 = 1 \), then one has \( y = \beta_0 + x_1'\beta_1 + e \), where \( x_1 = (x_2, \ldots, x_p)' \in \mathbb{R}^{p-1} \). Let \( w = (1, x_1')' \). Then \( y = w'\beta + e \). We use this model or (1) interchangeably depending on the context.

The maximum regression depth estimator possesses the outstanding robustness feature as the univariate location counterpart does. Indeed, the maximum depth estimator induced from \( RD_{RH} \), could, asymptotically, resist up to 33% contamination without breakdown, in contrast to 0% of the classical estimators (see Van Aelst and Rousseeuw (2000) (VAR00)).

It has been quoted and believed in the literature (e.g. Nolan (1999) and Mizera (2002)) that the asymptotics of the maximum depth estimator induced from \( RD_{RH} \) (denoted by \( T_{RD_{RH}}^* \)) have been considered and obtained by Bai and He (1999) (BH99). A closer examination reveals, nevertheless, that BH99 actually defined and treated a notion that is different from the intended \( RD_{RH} \) (see Section 2 for more details). Therefore, the asymptotics of the \( T_{RD_{RH}}^* \) remain open. Providing an answer to the open issue is the major objective of this article.

The rest of article is organized as follows. Section 2 introduces the regression depth, i.e. \( RD_{RH} \) of RH99. Examples of the computation of the \( RD_{RH} \) for population distributions are also given. Section 3 summarizes the important results on the regression depth which are utilized in later sections. In Section 4, sufficient conditions are proposed for the strong and root-n consistency of the maximum regression depth estimator. Section 5 is devoted to the establishment of limiting distribution of the \( T_{RD_{RH}}^* \), where the main tool is the Argmax continuous mapping theorem. Sufficient conditions of the theorem are verified via empirical process theory and especially stochastic equi-continuity and VC-classes of functions. The
limiting distribution is characterized through an Argmax operation over the infimum of a function involving a Gaussian process. The article ends in Section 6 with some brief concluding remarks.

2 Regression depth of Rousseeuw and Hubert (1999)

Definition 2.1 For any $\beta$ and joint distribution $P$ of $(y, x)$ in $\mathbb{R}^p$, RH99 defined the regression depth of $\beta$, denoted by $RD_{HR}(\beta; P)$, to be the minimum probability mass that needs to be passed when tilting (the hyperplane induced from) $\beta$ in any way until it is vertical. The maximum regression depth estimating functional $T_{RD_{RH}}^*$ is defined as

$$T_{RD_{RH}}^*(P) = \arg\max_{\beta \in \mathbb{R}^p} RD_{RH}(\beta; P)$$ (2)

Some characterizations of $RD_{HR}(\beta; P)$, or equivalent definitions are summarized in the following. In the empirical case, depth in RH99 divided by $n$ is identical to the following.

Lemma 2.1. The following statements for $RD_{RH}$ are equivalent.

(i) [Zuo 2018]

$$RD_{HR}(\beta; P) = D_C(\beta; P), \forall \beta \text{ with } \|\beta\| < \infty,$$ (3)

where $D_C(\beta; P) := \sup_{\alpha \in \mathbb{R}^p} P(|r(\beta)| \leq |r(\alpha)|)$ is Carrizosa’s depth (Carrizosa (1996)) and $r(\beta) := y - (1, x')\beta := y - w'\beta$.

(ii) [Zuo 2018]

$$RD_{RH}(\beta; P) = \inf_{\|v_2\|=1, v_1 \in \mathbb{R}} \{ E(I(r(\beta) * (v_1, v_2)' w \geq 0)) = \inf_{v \in \mathbb{S}^{p-1}} E(I(r(\beta) * v' w \geq 0)) \}. \tag{4}$$

(iii) [Van Aelst and Rousseeuw (2000) (VAR00)]

$$RD_{RH}(\beta; P) = \inf_{u \in \mathbb{R}^{p-1}, v \in \mathbb{R}} \{ P(\{\{r(\beta) > 0 \cap x'u < v\} + P(\{r(\beta) < 0 \cap x'u > v\}) \} \}, \tag{5}$$

where it is implicitly assumed that $P(x'u = v) = 0$, and $P(r(\beta) = 0) = 0$.

(iv) [Rousseeuw and Struyf (2004) (RS04)]

$$RD_{RH}(\beta; P) = \inf_{D \in \mathcal{D}} \{ P(\{(\beta) \geq 0 \cap D) + P((\beta) \leq 0 \cap D^c) \}, \tag{6}$$

where $\mathcal{D}$ is the set of all vertical closed halfspaces $D$ (i.e. the boundary of $D$ is parallel to the vertical direction).

Other characterizations are also given in the literature, e.g., in Adrover, Maronna, and Yohai (2002), in Mizera (2002) (page 1689-1690) and in BH99. The latter is specifically defined by

$$rdepth(\beta, Z_n) = \inf_{\|u\|=1, v \in \mathbb{R}} \{ \sum_{i=1}^n I(\{r_i(\beta)(u'x_i - v) > 0\}, \sum_{i=1}^n I(\{r_i(\beta)(u'x_i - v) < 0\} \}, \tag{7}$$
where \( y_i = \beta_0 + \mathbf{x}_i' \beta_1 + e_i, \beta' = (\beta_0, \beta_1) \) \( \in \mathbb{R}^p, \mathbf{x}_i \in \mathbb{R}^{p-1}, r_i(\beta) = y_i - (1, \mathbf{x}_i') \beta, \) and \( Z_n = \{(\mathbf{x}_i, y_i), i = 1, \ldots, n\}. \) That is, the empirical depth of \( \beta \) when \( P = P_n. \)

Furthermore, BH99 actually depended solely on the following alternative definition

\[
\text{rdepth}(\beta, Z_n) = \frac{n}{2} + \frac{1}{2} \inf_{\gamma \in S^{p-1}} \sum_{i=1}^{n} \text{sgn}(y_i - w_i(\beta)) \text{sgn}(w_i(\gamma)),
\]

(8)

where \( S^{p-1} := \{\mathbf{u} \in \mathbb{R}^p : \|\mathbf{u}\| = 1\}, w_i' = (1, \mathbf{x}_i').\)

Remarks 2.1

(I) The definition (7) of BH99 above is not identical to the original definition of RH99. For example, when all sample points lie in the hyperplane \( H_\beta \) determined by \( y = \mathbf{w}' \beta, \) (7) of BH99 determines depth 0 for \( \beta \) while \( RD_{RH}(\beta; P_n) = n. \) Of course, when assume that \( P(\mathbf{x}' \mathbf{u} = v) = 0, \) and \( P(r(\beta) = 0) = 0 \) for any \( \mathbf{u}, v, \) and \( \beta, \) then the two are (a.s.) identical.

(ii) Definition (8) of BH99 is also not identical to the \( RD_{RH} \) of RH99, neither to (7). Let’s consider a non-trivial example, assume we have four sample points in \( \mathbb{R}^2, Z_1 = (1/8, 1), Z_2 = (4/8, 0); Z_3 = (6/8, -1); Z_4 = (7/8, 2), \) then it is not difficult to see that for \( \beta = (0, 0)' \) \( RD_{RH}(\beta, Z_n) = 2; \) (7) gives 1 whereas (8) yields 1.5.

For empirical distributions \( (P = P_n), \) computing \( RD_{RH}(\beta, P) \) is quite straightforward and examples have been given in RH99. For a general distribution (probability measure) \( P, \) it is not easy to determine what is \( RD_{RH}(\beta, P). \) For special classes of distributions, however, one could derive the explicit expression for \( RD_{RH}(\beta, P). \) In the examples below, for simplicity, we again confine our attention to the case \( p = 2. \) That is, we have a simple linear regression model \( y = \beta_0 + \beta_1 x + e. \)

Example 2.1 A random vector \( \mathbf{X} \in \mathbb{R}^p \) is said to be elliptically distributed, denoted by \( \mathbf{X} \sim E(h; \mu, \Sigma), \) if its density is of the form

\[
f(\mathbf{x}) = c |\Sigma|^{-1/2} h \left( (\mathbf{x} - \mu)' \Sigma^{-1} (\mathbf{x} - \mu) \right), \quad \mathbf{x} \in \mathbb{R}^p,
\]

(9)

where \( c \) is some constant so that the \( f(\mathbf{x}) \) is a density function, \( h \) is generally a known function. A straightforward transformation such as \( \mathbf{Z} = \Sigma^{-1/2} (\mathbf{X} - \mu) \) leads to \( \mathbf{Z} \sim E(h; \mathbf{0}, I_p). \)

To determine the expression for \( RD_{RH}(\beta, P) \), we restrict to the case \( h(x) = \exp(-x^2/2), \) i.e., the bivariate normal class. That is, we have \( (y, x) \sim N_2(\mu, \Sigma). \) After applying the transformation above, we can assume w.l.o.g. that \( (y, x) \sim N_2(\mathbf{0}, I_2) \), where \( I_2 \) is a 2 by 2 identity matrix. For any \( \beta = (\beta_0, \beta_1)' \), by the invariance of regression depth (see Zuo (2018) and Section 3), we can just consider, w.l.o.g., the depth of \( \beta \) w.r.t. the \( P \) that corresponds to the bivariate standard normal distribution.

(i) \( \beta = (0, 0)', \) then the regression line is \( y = 0, \) and \( RD_{RH}(\beta; P) = 1/2. \)

(ii) \( \beta_0 = 0 \) and \( \beta_1 > 0 \) (\( \beta_1 < 0 \) can be discussed similarly). Denote the region bounded by the regression line \( y = \beta_1 x \) and the positive \( y \) axis as \( I \) and the region by the line
and horizontal positive \( x \) axis as \( \Pi \), then it is readily seen that
\[
\text{RD}_{RH}(\beta; N(0; I_2)) = 2 \cdot P((y, x) \in I) = 2(1/4 - P((y, x) \in \Pi))
\]
\[
= 1 - 2 \cdot \int_{0}^{\infty} (\Phi(\beta_1 x) d\Phi(x).
\]
where \( \Phi(x) \) is the standard normal cumulative distribution function.

(iii) \( \beta_0 > 0 \) and \( \beta_1 > 0 \) (the case \( \beta_0 > 0 \) and \( \beta_1 < 0 \) and the cases \( \beta_0 < 0 \) can be treated similarly). Denote the region formed by the line with part of the positive \( y \)-axis \( \{x = 0, y \geq \beta_0\} \) as \( I \) and with negative \( y \)-axis \( \{x = 0, y \leq \beta_0\} \) as \( \Pi \), then it is readily seen that
\[
\text{RD}_{RH}(\beta; N(0; I_2)) = P((y, x) \in I) + P((y, x) \in \Pi)
\]
\[
= \int_{-\infty}^{\infty} \Phi((y - \beta_0)/\beta_1) d\Phi(y) - 1/2.
\]

(iv) \( \beta_0 > 0 \) and \( \beta_1 = 0 \) (the case \( \beta_0 < 0 \) and \( \beta_1 = 0 \) can be handled similarly). Denote the region formed by the positive \( x \) part of the line with part of the positive \( y \)-axis \( \{x = 0, y \geq \beta_0\} \) as \( I \) and the other part of the line with negative \( y \)-axis \( \{x = 0, y \leq \beta_0\} \) as \( \Pi \), then it is readily seen that
\[
\text{RD}_{RH}(\beta; N(0; I_2)) = P((y, x) \in I) + P((y, x) \in \Pi) = 1/2.
\]

**Example 2.2** Assume that \((y, x)\) is uniformly distributed over a unit circle centered at \((0,0)\). By invariance of depth, this covers a class of distributions of \( A(y, x) + b \) for any nonsingular \( A \in \mathbb{R}^{2 \times 2} \) and \( b \in \mathbb{R}^2 \).

(i) \( \beta = (0, 0)' \), then the regression line is \( y = 0 \), and \( \text{RD}_{RH}(\beta; P) = 1/2 \).

(ii) \( \beta_0 = 0 \) and \( \beta_1 > 0 \) (\( \beta_1 < 0 \) can be treated similarly). Denote the region bounded by the regression line \( y = \beta_1 x \) and the positive \( y \) axis as \( I \), then it is readily seen that
\[
\text{RD}_{RH}(\beta; P) = 2 \cdot P((y, x) \in I) = 1/2 - |\arctan(\beta_1)|/\pi.
\]

(iii) \( \beta_0 > 0 \) and \( \beta_1 \geq 0 \) (the cases that \((\beta_0 > 0, \beta_1 < 0)\) or \((\beta_0 < 0, \beta_1 \geq (or <) 0)\) can be dealt with similarly) and \( \Delta = 1 + \beta_1^2 - \beta_0^2 > 0 \). That is, the regression line intercepts the unit circle at two points \( x_\pm \), where \( x_\pm = \frac{-\beta_0 \pm \sqrt{1 + \beta_1^2 - \beta_0^2}}{1 + \beta_1^2} \).

(a) Assume that both interception points were with positive \( y \) coordinate. Denote the region formed by the regression line and the circle between the vertical lines \( x = x_- \) and \( x = x_+ \) as \( I \). Then it is readily seen that
\[
\text{RD}_{RH}(\beta; P) = P((y, x) \in I)
\]
\[
= \int_{x_-}^{x_+} \left( \sqrt{1 - x^2} - (\beta_0 + \beta_1 x) \right) dx
\]
\[
= g(x_+) - g(x_-),
\]
where \( g(x) = \left( x\sqrt{1-x^2}/2 + \arctan\left( x/\sqrt{1-x^2}\right)/2 \right) - (\beta_0 x + \beta_1 x^2/2) := g_1(x) - g_2(x) \).

(b) Assume that the y coordinates of the two interception points have different sign. The latter implies that \( \beta_1 \neq 0 \). Denote the region formed by the regression line and the circle and the positive (negative) y-part of vertical line \( x = -\beta_0/\beta_1 \) as I (II) Then it is readily seen that

\[
\text{RD}_{RH}(\beta; P) = \int_{x_+}^{x_+} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydx + \int_{x_-}^{x_-} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydx
\]

\[
= g(x_+) - g(x_-) + 2g_2(-\beta_0/\beta_1) - 2g_2(x_-).
\]

(iv) Other cases, \( \text{RD}_{RH}(\beta; P) = 0 \).

3 Preliminary results

A regression depth functional \( G \) is said to be regression, scale and affine invariant w.r.t. a given \( F_{(y, w)} \) iff, respectively,

\[
G(\beta + b; F_{(y + w'b, w)}) = G(\beta; F_{(y, w)}), \forall b \in \mathbb{R}^p;
\]

\[
G(s\beta; F_{(sy, w)}) = G(\beta; F_{(y, w)}), \forall s (\neq 0) \in \mathbb{R},
\]

\[
G(A^{-1}\beta; F_{(y, A'w)}) = G(\beta; F_{(y, w)}), \forall \text{ nonsingular } p \text{ by } p \text{ matrix } A.
\]

A regression estimating functional \( T(\cdot) \) is said to be regression, scale, and affine equivariant iff, respectively

\[
T(F_{(y + w'b, w)}) = T(F_{(y, w)}) + b, \forall b \in \mathbb{R}^p,
\]

\[
T(F_{(sy, w)}) = sT(F_{(y, w)}), \forall s \in \mathbb{R},
\]

\[
T(F_{(y, A'w)}) = A^{-1}T(F_{(y, w)}), \forall \text{ nonsingular } A \in \mathbb{R}^{p \times p}.
\]

We now summarize some preliminary results on the regression depth and its induced maximum depth estimating functional. Denote them by a generic notation \( D(\cdot; F_Z) \) and \( T^*_D(F_Z) \), respectively (\( F_Z \) and \( P \) are used interchangeably, where \( Z := (y, x) \) or := \( (y, w) \)).

**Lemma 3.1** [Zuo (2018)]

(i) \( D(\beta; F_Z) \) is regression, scale and affine invariant and hence \( T^*_D(F_Z) \) is regression, scale and affine equivariant. Furthermore, \( D(\beta; F_Z) \to 0 \) as \( \|\beta\| \to \infty \).

(ii) \( D(\beta; P) \) is upper-semicontinuous (in \( \beta \)) and continuous if \( P \) has a density.
\[(iii) \sup_{\beta \in \mathbb{R}^p} |D(\beta; F^n_Z) - D(\beta; F_Z)| \to 0 \text{ a.s. as } n \to \infty, \text{ where } F^n_Z \text{ is the empirical version of the distribution } F_Z.\]

In the sequel, we will assume that there exists a unique point $T^*_D(F_Z)$ (or $\beta^*$) that maximizes the underlying regression depth. Equivariance of $T^*_D(F_Z)$ implies that one can assume w.l.o.g. that $T^*_D(F_Z) = 0$.

Uniqueness is guaranteed if $F_Z$ has a strictly positive density and is regression symmetric about a point $\beta$ ($F_Z$ is regression symmetric about $\theta$ if $P(x \in B, r(\theta) > 0) = P(x \in B, r(\theta) < 0)$ for any Borel set $B \in \mathbb{R}^{p-1}$, see RS04).

4 Consistency

For a regression depth functional $D(\beta; F_Z)$, Let $\beta^*(F_Z) = \arg \max_{\beta \in \mathbb{R}^p} D(\beta; F_Z)$, then $\beta_n^* := \beta^*(F^n_Z)$ is a natural maximum regression depth estimator of $\beta^*$.

Is $\beta_n^*$ a consistent estimator? This becomes a very typical question asked in statistics and the argument (or answer) for it is also very standard, almost to the point of cliché as Kim and Pollard (1990) (KP90) commented.

Let’s first deal with the problem in a more general setting. Let $M_n$ be stochastic processes indexed by a metric space $\Theta$ of $\theta$, and $M: \Theta \to \mathbb{R}$ be a deterministic function of $\theta$. The sufficient conditions for the consistency of this type of problem were given in Van Der Vaart (1998) (VDV98) and Van Der Vaart and Wellner (1996) (VW96) and are listed below:

\[C1: \sup_{\theta \in \Theta} |M_n(\theta) - M(\theta)| = o_p(1);\]
\[C2: \sup_{\{\theta: d(\theta, \theta_0) \geq \delta\}} M(\theta) < M(\theta_0), \text{ for any } \delta > 0 \text{ and the metric } d \text{ on } \Theta;\]

Then any sequence $\theta_n$ is consistent for $\theta_0$ providing that it satisfies
\[C3: M_n(\theta_n) \geq M_n(\theta_0) - o_p(1).\]

Lemma 4.1 [Th. 5.7, VDV98] If $C1$ and $C2$ hold, then any $\theta_n$ that satisfying $C3$ is consistent for $\theta_0$.

Remarks 4.1

(I) $C1$ basically requires that the $M_n(\theta)$ converges uniformly in probability to $M(\theta)$. For the depth process $D(\beta; F^n_Z)$ and $D(\beta; F_Z)$, it holds true for $RD_{RH}$ (see RH99) (the convergence is actually almost surely (a.s.) and uniformly).

(II) $C2$ essentially requests that the unique maximizer $\theta_0$ is well separated. This holds true for $RD_{RH}$ as long as $D(\beta; F_Z)$ is upper semi-continuous and $\theta_0$ is unique (see, Lemma 4.1 below).
(III) $C3$ asks that $\theta_n$ is very close to $\theta_0$ in the sense that the difference of images of the two at $M_n$ is within $o_p(1)$.

In KP90 and VW96 a stronger version of $C3$ is required:

$C3'$: $M_n(\theta_n) \geq \sup_{\theta \in \Theta} M_n(\theta) - o_p(1)$,

which implies $C3$. This strong version demands that $\theta_n$ nearly maximizes $M_n(\theta)$. The maximum regression depth estimator $\beta_n^*(:= \theta_n)$ is defined to be the maximizer of $M_n(\theta) := D(\beta; F^n_x)$, hence $C3'$ (and thus $C3$) holds automatically. □

**Theorem 4.1** The maximum regression depth estimator $\beta_n^*$ induced from $RD_{RH}$ is strongly consistent for $\beta^*$ (i.e. $\beta_n^* - \beta^* = o(1)$ a.s.) if $\beta^*$ is unique.

**Proof:** The proof for the consistency of Lemma 4.1 could be easily extended to the strong consistency with a strengthened version of $C1$

$C1'$: $\sup_{\theta \in \Theta} |M_n(\theta) - M(\theta)| = o(1)$ a.s.

In the light of the proof of Lemma 4.1, we need only verify the sufficient conditions $C1'$ and $C2$-$C3$. By (III) of Remark 4.1, $C3$ holds automatically, so we need to verify $C1'$ and $C2$. $C1'$ has been given in Lemma 3.1 for $RD$. So the only thing left is to verify $C2$ for $RD$ which is guaranteed by Lemma 4.2 below. Notice that $RD_{RH}$ meets the sufficient conditions required (see Lemma 3.1). □

**Lemma 4.2** Assume that the regression depth $D(\beta; F_Z)$ is upper semi-continuous in $\beta$ and vanishing when $\|\beta\| \to \infty$. Let $\eta \in \mathbb{R}^p$ be a unique point with $\eta = \arg \sup_{\beta \in \mathbb{R}^d} D(\beta; F_Z)$ and $D(\eta; F_Z) > 0$. Then for any $\varepsilon > 0$, $\sup_{\beta \in N^c_\varepsilon} D(\beta; F_Z) < D(\eta; F_Z)$, where $N^c_\varepsilon = \{ \beta \in \mathbb{R}^P : \|\beta - \eta\| \geq \varepsilon \}$.

**Proof:** Assume conversely that $\sup_{\beta \in N^c_\varepsilon} D(\beta; F_Z) = D(\eta; F_Z)$. Then by the given conditions, there is a sequence of bounded $\beta_j (j = 0, 1, \ldots)$ in $N^c_\varepsilon$ such that $\beta_j \to \beta_0 \in N^c_\varepsilon$ and $D(\beta_j; F_Z) \to D(\eta; F_Z)$ as $j \to \infty$. Note that $D(\eta; F_Z) > D(\beta_0; F_Z)$. The upper-semicontinuity of $D(\cdot; F_Z)$ now leads to a contradiction: for sufficiently large $j$, $D(\beta_j; F_Z) \leq (D(\eta; F_Z) + D(\beta_0; F_Z))/2 < D(\eta; F_Z)$. This completes the proof. □

**Remarks 4.2**

(I) Note that under the assumption that there exists a unique $\eta = \beta^*$ for $RD_{RH}$ (which holds true for special $F_Z$). See the end of section 3. The sufficient conditions in the Lemma are all met in virtue of Lemma 3.1 so that the Lemma holds true for $RD_{RH}$.

(II) The strong consistency in Theorem 4.1 for the maximum regression depth $RD_{RH}$ seems to have also been addressed in Section 2 of BH99 with a very different approach and much more stringent assumptions on the underlying distribution of $y$ and $x$ (see their D1-D4).
However, as was pointed out in Remarks 2.1, BH99 actually did not treat the maximum depth estimator induced from the original $RD_{RH}$. They instead defined a different regression depth notion (see their (1.1)-(1.2)). Their (1.1) does not count the sample points on the regression hyperplane determined by $y = w'\beta$. Their (1.2) is not equivalent to $RD_{RH}$ or to their own (1.1). 

With the establishment of the strong consistency, one naturally wonders about the rate of convergence of the maximum regression depth estimator. Does it possess root-n consistency? To answer the question, we need a stronger version of $C2$ for a general depth notion $D$.

$C2'$: For any $\delta > 0$, $\sup_{\|\beta-\eta\| \geq \delta} D(\beta; P) < \alpha^* - \kappa \delta$, for some positive constant $\kappa$, where $\alpha^* := D(\eta; P) = \sup_{\beta \in \mathbb{R}^p} D(\beta; P)$.

**Remark 4.3:**

Lemma 4.2 clearly provides a choice for the individual $\kappa$ for every $\delta$, but $C2'$ requires a uniform $\kappa$ for any small $\delta > 0$.

Let $D(\beta; P)$ be any regression (or even location) depth functional for $\beta \in \mathbb{R}^p$, We have the following general result for $\beta^*_n = \arg \max_{\beta \in \mathbb{R}^p} D(\beta; P_n)$ and $\beta^* = \arg \max_{\beta \in \mathbb{R}^p} D(\beta; P)$.

**Lemma 4.3** Let $D(\beta; P)$ be a general depth notion that is continuous in $\beta \in \mathbb{R}^p$. If (i) $\sup_{\beta \in \mathbb{R}^p} |D(\beta; P_n) - D(\beta; P)| = O_P(n^{-1/2})$ and (ii) $C2'$ holds, then $\beta^*_n - \beta^* = O_P(n^{-1/2})$.

**Proof:** In virtue of (i) and (ii) and Lemma 4.1, we see that $\beta^*_n - \beta^* = o_P(1)$. Denote by $\Delta_n \sup_{\beta \in \mathbb{R}^p} |D(\beta; P_n) - D(\beta; P)|$. Let $\delta = \Delta_n/\kappa$. In light of $C2'$, we have for every $n$

$$
\sup_{\beta: \|\beta - \beta^*\| \geq \delta} D(\beta; P_n) \leq \sup_{\beta: \|\beta - \beta^*\| \geq \delta} |D(\beta; P_n) - D(\beta; P)| + \sup_{\beta: \|\beta - \beta^*\| \geq \delta} D(\beta; P) < \Delta_n + \alpha^* - \kappa \delta = \alpha^*
$$

which, in conjunction with the consistency of $\beta^*_n$ and the continuity of $D(\beta; P)$ in $\beta$, implies that $\|\beta^*_n - \beta^*\| < \delta$ in probability, which leads to the desired result. 

**Theorem 4.2** $\sup_{\beta \in \mathbb{R}^p} |RD_{RH}(\beta; P_n) - RD_{RH}(\beta; P)| = O_P(n^{-1/2})$, and furthermore $\beta^*_n - \beta^* = O_P(n^{-1/2})$ if $C2'$ holds and $P$ has a density.

**Proof:** In the light of (ii) of Lemma 2.1, write $f(\cdot, \cdot, \beta, \nu) := (\cdot - \nu')\beta (\nu' \cdot) \forall \beta \in \mathbb{R}^p, \nu \in \mathbb{S}^{p-1}$. Then we see that $RD_{RH}(\beta; P) = \inf_{\nu \in \mathbb{S}^{p-1}} P(f(y, w, \beta, \nu) \geq 0)$. Define a class of functions

$$
\mathcal{F} = \{I(f(\cdot, \cdot, \beta, \nu) \geq 0), \ \forall \beta \in \mathbb{R}^p, \text{ and } \nu \in \mathbb{S}^{p-1}\},
$$

where $I(A)$ is the indicator function of a set $A$. We identify the sets with their indicator functions and follow the convention in KP90 that $P(I(A)) = \int_A dP(x)$. Then by Lemma II. 18 of Pollard (1984), $\mathcal{F}$ is a permissible class of functions with polynomial discrimination (see Pollard (1984) Chapter II for the related concepts). Hence $\mathcal{F}$ has VC subgraphs with
constant envelop 1 (see 2.5 and 2.6 of VW96). In the light of Corollary 3.2 of KP90, we have that
\[ \sup_{\beta \in \mathbb{R}^p, \nu \in S^{p-1}} \left| P_n(f(y, w, \beta, \nu) \geq 0) - P(f(y, w, \beta, \nu) \geq 0) \right| = O_P(n^{-1/2}). \]
Thus we have
\[ \sup_{\beta \in \mathbb{R}^p, \nu \in S^{p-1}} \left| \inf_{\nu \in S^{p-1}} P_n(f(y, w, \beta, \nu) \geq 0) - \inf_{\nu \in S^{p-1}} P(f(y, w, \beta, \nu) \geq 0) \right| \leq \sup_{\beta \in \mathbb{R}^p, \nu \in S^{p-1}} \left| P_n(f(y, w, \beta, \nu) \geq 0) - P(f(y, w, \beta, \nu) \geq 0) \right| = O_P(n^{-1/2}), \]
where the inequality follows from the fact that \( |\inf_A f - \inf_A g| \leq \sup_A |f - g| \). It follows that \( \sup_{\beta \in \mathbb{R}^p} |RD_{R_H}(\beta; P_n) - RD_{R_H}(\beta; P)| = O_P(n^{-1/2}) \) so that the first part of the theorem is established.

This first part, in conjunction with \( C2' \) and (ii) of Lemma 3.1, and Lemma 4.3, yields the desired second part of the theorem, that is \( \beta_n^* - \beta^* = O_P(n^{-1/2}) \).

**Remarks 4.4:**

(I) The approach of the first part of the proof could be extended for any depth notions that are defined based on sets that form a VC class such as the location counterpart, Tukey halfspace depth (HD), where one has a class of halfspaces, a VC class of sets.

That is, utilizing the approach one can prove that the maximum Tukey location depth estimator (aka Tukey median) is root-n consistent (uniformly tight) if \( C2' \) hold for the HD and \( P \) has a density. Lemma 6.1 of Donoho and Gasko (1992) guarantees the continuity of HD. For \( C2' \) to hold, a sufficient condition was given in Nolan (1999) ((ii) of Lemma 2), BH99 ((N2) in Theorem 4.1), and Massé (2002) ((b) of Proposition 3.2 and Theorem 3.5). That is, the approach here covers the uniform tightness result in those papers.

(II) BH99 intended to discuss the root-n consistency for the maximum regression depth estimator induced from \( RD_{R_H} \). Unfortunately, they addressed a different process from the intended one (see Remarks 4.2) so that their result does not cover the result here.

**5 Limiting distribution**

With the root-n consistency of the maximum regression depth estimator established, we are now in a position to address the natural question: Does it have a limiting distribution.

Since the tool for establishing the limiting distributions is the Argmax theorem, we first cite it below from VW96 (Theorem of 2.7 of KP90 is an earlier version).
**Theorem 5.1**  
If symmetric about a point, then for that \( \hat{x} \) possess a unique maximum at a random point \( \hat{s} \), which, as a random map into S, is tight. If the sequence \( \hat{s}_n \) is uniformly tight and satisfies \( M_n(\hat{s}_n) \geq \sup_s M_n(s) - o_p(1) \), then \( \hat{s}_n \xrightarrow{d} \hat{s} \), where \( d \) stands for convergence in distribution.

In the light of the Lemma, to establish the limiting distribution for \( \hat{s}_n := \sqrt{n} \beta_n^* \), we need (A) to identify the processes \( M_n \) and \( M \) and show that \( M_n \xrightarrow{d} M \) in \( l^\infty(K) \) for every compact \( K \subset S \). Suppose that almost all sample paths \( s \mapsto M(s) \) are upper semicontinuous and possess a unique maximum at a random point \( \hat{s} \), which is tight, and (C) to show that \( \hat{s}_n \) is uniformly tight and \( M_n(\hat{s}_n) \geq \sup_s M_n(s) - o_p(1) \).

In virtue of Theorem 4.2, part of (C) already holds under certain conditions for \( \hat{s}_n = \sqrt{n} \beta_n^* \). So we need to verify the (A) and (B) and the second part of (C).

By (ii) of Lemma 2.1 and [2], we have that  
\[
\beta^* = \arg\max_{\beta \in \mathbb{R}^p} RD_{RH}(\beta; P) = \arg\max_{\beta \in \mathbb{R}^p} \inf_{v \in \mathbb{S}^{p-1}} E(I_{f(y, w, \beta, v) \geq 0}),
\]
where, \( f(y, w, \beta, v) = (y - w^\prime \beta)v^\prime w \). For a given \( \beta \) define  
\[
V(\beta) = \{ v \in \mathbb{S}^{p-1} : RD_{RH}(\beta; P) = P(f(y, w, \beta, v) \geq 0) = \inf_{v \in \mathbb{S}^{p-1}} P(f(y, w, \beta, v) \geq 0) \},
\]
i.e., the collection of \( v \) at which \( P(f(y, w, \beta, v) \geq 0) \) attains the infimum over \( v \in \mathbb{S}^{p-1} \).

Recall that \( \beta^* \) is assumed (w.l.o.g.) to be \( 0 \). Hereafter \( \beta \) is assumed to be in a small bounded neighborhood \( \Theta \) of \( 0 \) in virtue of Theorem 4.1. Assume for a \( v \in \mathbb{S}^{p-1} \) and a \( \beta \in \Theta \) that  
\[
D1: \quad P(f(y, w, \beta, v) \geq 0) = P(f(y, w, 0, v) \geq 0) - cg(\beta) v^\prime + o(v^\prime),
\]
where \( c > 0 \) and \( g(\beta) \) could be interpreted as the density of \( f(y, w, \beta, v)(x) \) evaluated at \( x = 0 \) when \( \beta = 0 \). That is, the LHS permits a Taylor expansion at \( \beta^* = 0 \). Furthermore,  
\[
D2: \quad V(0) = \mathbb{S}^{p-1}
\]
That is, along any direction \( v \in \mathbb{S}^{p-1} \), \( P(f(y, w, 0, v) \geq 0) = \alpha^* := RD_{RH}(\beta^*, P) \). And  
\[
D3: \quad \inf_{v \in V(0)} g(v) = c_2 > 0, \quad \sup_{v \in V(0)} g(v) < \infty
\]
That is, \( g(v) \) is uniformly positive and bounded over \( V(0) \).

**Theorem 5.1** If \( C2' \) and D1-D3 hold and \( F_Z \) has a strictly positive density and is regression symmetric about a point, then for \( \beta_n^* \) induced from \( RD_{RH} \), as \( n \to \infty \),  
\[
\sqrt{n}(\beta_n^* - \beta^*) \xrightarrow{d} \arg\max_{v \in V(0)} \{ E_P(f(y, w, 0, v) \geq 0) - cg(v)v^\prime \},
\]
where \(E_P\) is the limit of the empirical process \(E_n = \sqrt{n}(P_n - P)\) in \(l^\infty(F)\), a P-Brownian bridge (see Def. VII. 14 of Polard(1984)), and \(F = \{1(f, \cdot, 0, \mathbf{v}) \geq 0, \mathbf{v} \in V(0)\}\).

**Proof:** Assume that \(F_Z\) is regression symmetric about \(\theta \in \mathbb{R}^p\), then by RS04, \(\beta^* = \theta\) is unique. By the equivariance of \(T_{RD_{RH}}(F_Z)\), assume (w.l.o.g.) that \(\beta^* = 0\).

Invoke (ii) of Lemma 2.1, we have that
\[
RD_{RH}(\beta; P) = \inf_{\mathbf{v} \in S^{p-1}} E(\mathcal{I}((y - \mathbf{w}^t \beta) \mathbf{v}^t \mathbf{w} \geq 0)) = \inf_{\mathbf{v} \in S^{p-1}} P(f(y, \mathbf{w}, \beta, \mathbf{v}) \geq 0).
\]

Note that
\[
n^{1/2} \beta_n^* = n^{1/2} \arg \max_{\beta \in \mathbb{R}^p} \inf_{\mathbf{v} \in S^{p-1}} P_n(f(y, \mathbf{w}, \beta, \mathbf{v}) \geq 0).
\]

Hence for any compact \(K\) and \(s \in K \subset \mathbb{R}^p\) and sufficiently large \(n\)
\[
n^{1/2} P_n(f(y, \mathbf{w}, s/n^{1/2}, \mathbf{v}) \geq 0) = n^{1/2} P(f(y, \mathbf{w}, s/n^{1/2}, \mathbf{v}) \geq 0) + E_n(f(y, \mathbf{w}, s/n^{1/2}, \mathbf{v}) \geq 0)
\]
\[= n^{1/2} P(f(y, \mathbf{w}, s/n^{1/2}, \mathbf{v}) \geq 0) + E_n(f(y, \mathbf{w}, 0, \mathbf{v}) \geq 0) + o_P(1)
\]
\[= n^{1/2} P(f(y, \mathbf{w}, 0, \mathbf{v}) \geq 0) - cg(\mathbf{v}) \mathbf{v}' s + o(\mathbf{v}' s)
\]
\[+ E_n(f(y, \mathbf{w}, 0, \mathbf{v}) \geq 0) + o_P(1),
\]

where the second equality follows from the stochastic equi-continuity Lemma VII. 15 of Pollard (1984) (a permissible class of functions with polynomial discrimination and a square-integrable envelope, see the proof of (ii) of Theorem 4.2), the last one follows from the **D1**. Then we can define that
\[
M_n(s) = n^{1/2} \inf_{\mathbf{v} \in S^{p-1}} P_n(f(y, \mathbf{w}, s * n^{-1/2}, \mathbf{v})) - n^{1/2} \alpha^*,
\]

where \(\alpha^* = RD_{RH}(\beta^*; P)\). Note that by (10), it is readily seen that \(\mathbf{s}_n := n^{1/2} \beta_n^*\) maximizes \(M_n(s)\) and is uniformly tight in virtue of Theorem 4.2, therefore (C) is completely verified.

Now we need to verify (A) and (B) for
\[
M(s) := \inf_{\mathbf{v} \in V(0)} \{E_P(f(y, \mathbf{w}, 0, \mathbf{v}) \geq 0) - cg(\mathbf{v}) \mathbf{v}' s\}.
\]

We first establish some lemmas to fulfil the task above.

**Lemma 5.2** In the light of **D2** and **D3**,

**R1:** The sample path of \(M(s)\) is obviously continuous in \(s\) almost surely (a.s.), and furthermore \(M(s) \to -\infty\) as \(\|s\| \to \infty\) a.s.;

**R2:** \(M(s)\) is concave in \(s\) a.s.

**Proof:** Write \(M(s, v) = E_P(f(y, \mathbf{w}, 0, \mathbf{v}) \geq 0) - cg(\mathbf{v}) \mathbf{v}' s\). The continuity and concavity of \(M(s, v)\) in \(s\) is obvious. The assertion on \(M(s)\) follows since the infimum preserves these properties. We need to show the second part of **R1**.
Assume the assertion is false, then there is a sequence \( s_n \) with \( \|s_n\| \to \infty \) and a number \( M \) such that for all \( n \) large enough, \( M \leq M(s_n) \) hold with a positive probability. In light of \( \textbf{D2} \), choose a sequence \( v_n \in V(0) \) such that \( v'_n s_n/\|s_n\| = c_1 = 1 \). Now we have by \( \textbf{D3} \) for all large \( n \) with positive probability

\[
M \leq M(s_n) \leq E_P(f(y, w, 0, v_n) \geq 0) - cg(v_n)v'_n * s_n
\]
\[
\leq E_P(f(y, w, 0, v_n) \geq 0) - cc_1 c_2 \|s_n\|,
\]

which is impossible since \( cc_1 c_2 \|s_n\| \to \infty \) as \( n \to \infty \).

Let \( \widehat{s} \) be a maximizer of \( M(s) \). The existence of a \( \widehat{s} \) is guaranteed by \( \textbf{R1} \) and \( \textbf{R2} \). The tightness of \( \widehat{s} \) is equivalent to its measurability, which is straightforward (see Pollard (1984), for example). Now we have to show that \( \widehat{s} \) is unique. Recall \( M(s, v) = E_P(f(y, w, 0, v) \geq 0) - cg(v)v' * s \). Define

\[
\mathcal{V}(\widehat{s}) := \{ v \in V(0), M(\widehat{s}) = M(\widehat{s}, v) \},
\]

which is clearly non-empty. Suppose that \( \widehat{t} \) is another maximizer of \( M(s) \), then by \( \textbf{R2} \), \( \alpha \widehat{s} + (1 - \alpha) \widehat{t} \) is also a maximum point for every \( \alpha \in [0, 1] \). Following Nolan (1999), one can show that

**Lemma 5.3** If \( \textbf{D2} \) and \( \textbf{D3} \) hold, then

- \( \textbf{R3} \): \( \inf_{v \in \mathcal{V}(\widehat{s})} v' x \leq 0 \); for any \( x \in \mathbb{R}^p \)

- \( \textbf{R4} \): \( \mathcal{V}(\alpha \widehat{s} + (1 - \alpha) \widehat{t}) = \mathcal{V}(\widehat{s}) \cap \mathcal{V}(\widehat{t}) \), for every \( \alpha \in (0, 1) \).

Equipped with the results above, we now are in the position to show that

**Lemma 5.4** If \( \textbf{D2} \) and \( \textbf{D3} \) hold, then \( \widehat{s} \) is unique.

**Proof:** Denote the dimension of the linear space spanned by \( \mathcal{V}(\alpha \widehat{s} + (1 - \alpha) \widehat{t}) \) by \( r \). If \( r = 1 \), then by \( \textbf{R3} \), \( \mathcal{V}(\alpha \widehat{s} + (1 - \alpha) \widehat{t}) = \{ v, -v \} \subset V(0) \) for some \( v \in S^1 \). Note that \( E_P(f(y, w, 0, v) \geq 0) = -E_P(f(y, w, 0, -v) \geq 0) \) and \( g(v) = g(-v) \), therefore for any \( s \)

\[
M(s) = \min \{ E_P(f(y, w, 0, v) \geq 0) - cg(v)v's, E_P(f(y, w, 0, -v) \geq 0) + cg(-v)v's \} \leq 0,
\]

which implies that \( M(\widehat{s}) = 0 \). The uniqueness of \( \widehat{s} \) for \( r = 1 \) follows in a straightforward fashion.

We now assume that \( 2 \leq r \leq p - 1 \). Assume that \( v_1, \cdots, v_r \) are linearly independent and belong to \( \mathcal{V}(\alpha \widehat{s} + (1 - \alpha) \widehat{t}) \) for an \( \alpha \in (0, 1) \). Let \( S \) be any \( r \)-dimensional space that contains both \( \widehat{s} \) and \( \widehat{t} \), then both \( \widehat{s} \) and \( \widehat{t} \) satisfy the linear equations system:

\[
 cg(v_i)v'_i s = E_P(f(y, w, 0, v_i) \geq 0) - M(\widehat{s}), \; i = 1, \cdots, r, \; s \in S
\]

which immediately implies that \( \widehat{s} - \widehat{t} = 0 \) is the only solution of the linear system \( cg(v_i)v_i(\widehat{s} - \widehat{t}) = 0, \; i = 1, \cdots, r \). That is, \( \widehat{s} \) is unique.

\[
\boxed{\text{\textbf{D2}}}
\]
We have verified (B) completely. As we have noticed above \( \tilde{s}_n := n^{1/2} \beta^*_n \) maximizes \( M_n(s) \) defined in (12). To verify (A) and then complete the proof of the theorem, we need only show that \( M_n(s) \rightarrow M(s) \) uniformly in \( s \in K \), where \( K \subseteq \mathbb{R}^p \) is a compact set. Note that by (11)

\[
M_n(s) = \inf_{v \in S^{p-1}} n^{1/2} (P(f(y, w, 0, v) \geq 0) - \alpha^*) - cg(v) v' s + n^{1/2} o(v' s / n^{1/2}) + E_n(f(y, w, 0, v) \geq 0) + o_P(1),
\]

where it is obvious that \( \sup_{s \in K} \sup_{v \in S^{p-1}} |n^{1/2} o(v' s / n^{1/2})| = o(1) \), \( o_P(1) \) is also uniformly in \( v \) and \( s \). Write

\[
\lambda_n(v, s) := n^{1/2} (P(f(y, w, 0, v) \geq 0) - \alpha^*) - cg(v) v' s + E_n(f(y, w, 0, v) \geq 0),
\]

and

\[
M_n^1(s) := \inf_{v \in S^{p-1}} \lambda_n(v, s).
\]

Then it is readily seen that in terms of asymptotic weak convergence in \( l^\infty (K) \), \( M_n(s) \) is equivalent to \( M_n^1(s) \) (that is \( \sup_{s \in K} |M_n(s) - M_n^1(s)| = o_P(1) \)).

Define \( V_n := \{ v \in S^{p-1} : P(f(y, w, 0, v) \geq 0) - \alpha^* > n^{-1/4} \} \); then it is obvious that \( V_n^c \) (the complement of \( V_n \)) decreases to \( V(0) = \cap_n V_n^c \). Write

\[
M_n^2(s) := \inf_{v \in V_n^c} \lambda_n(v, s)
\]

and

\[
M_n^3(s) := \inf_{v \in V(0)} \{ E_P(f(y, w, 0, v) \geq 0) - cg(v) v' s \} = M(s)
\]

We now establish

**Lemma 5.5** If \( D1 - D3 \) hold, then \( \sup_{s \in K} |M_n(s) - M_n^i(s)| = o_P(1) \) for \( i = 1, 2 \), and \( M_n(s) \rightarrow M(s) \) uniformly over \( s \in K \).

**Proof:** We employ three steps to prove the Lemma.

(i) First we show \( \sup_{s \in K} |M_n(s) - M_n^1(s)| = o_P(1) \). In light of (13) and (14), we have

\[
\sup_{s \in K} |M_n(s) - M_n^1(s)| = \sup_{s \in K} \left| \inf_{v \in S^{p-1}} \left( \lambda_n(v, s) + n^{1/2} o(v' s / n^{1/2}) + o_P(1) \right) - \inf_{v \in S^{p-1}} \lambda_n(v, s) \right|
\]

\[
\leq \sup_{s \in K} \sup_{v \in S^{p-1}} |n^{1/2} o(v' s / n^{1/2}) + o_P(1)|
\]

\[
= o_P(1),
\]

where the last equality follows from two facts: (1) the term \( n^{1/2} o(v' s / n^{1/2}) \) in (13) is \( o(1) \) uniformly in \( s \) over \( K \), and (2) the term \( o_P(1) \) in (13) is also uniformly in \( s \) over \( K \) for large enough \( n \), because it is obtained from application of stochastic equi-continuity over a class of functions whose members are close enough within a distance \( \delta > 0 \) w.r.t. seminorm \( \rho_P \) (see Lemma VII. 15 of Pollard (1984)). That is, (i) thus follows.
(ii) Secondly, we show that $\sup_{s \in K} |M^1_n(s) - M^2_n(s)| = o_P(1)$. That is, we need to show for any $\epsilon > 0$
$$P \left( \sup_{s \in K} \left| \inf_{v \in V^c_n} \lambda_n(v, s) - \inf_{u \in S^{p-1}} \lambda_n(u, s) \right| < \epsilon \right) > 1 - \epsilon,$$
for large enough $n$. Or equivalent to show that
$$P \left( \sup_{s \in K} \left( \inf_{v \in V^c_n} \lambda_n(v, s) - \inf_{u \in V_n} \lambda_n(u, s) \right) < \epsilon \right) > 1 - \epsilon,$$
for large enough $n$. It is readily seen that it suffices to show that for large enough $n$
$$P \left( \sup_{s \in K} \left( \inf_{v \in V^c_n} \lambda_n(v, s) - \inf_{u \in V_n} \lambda_n(u, s) \right) < \epsilon \right) > 1 - \epsilon,$$  \hfill (18)
Note that for any $v_0 \in V(0)$,
$$\inf_{v \in V^c_n} \lambda_n(v, s) - \inf_{u \in V_n} \lambda_n(u, s) \leq \lambda_n(v_0, s) - \inf_{u \in V_n} \lambda_n(u, s)$$
So, instead of establish the inequality (19), it suffices to show that for large enough $n$
$$P \left( \sup_{s \in K} \left( \lambda_n(v_0, s) - \inf_{u \in V_n} \lambda_n(u, s) \right) < \epsilon \right) > 1 - \epsilon,$$  \hfill (20)
Note that
$$- \sup_{s \in K} \left( \lambda_n(v_0, s) - \inf_{u \in V_n} \lambda_n(u, s) \right) = \inf_{s \in K} \left( \inf_{u \in V_n} \lambda_n(u, s) - \lambda_n(v_0, s) \right)$$
$$\leq \inf_{s \in K} \inf_{u \in V_n} n^{1/2} \left( P(f(y, w, 0, u) \geq 0) - \alpha^* \right) +$$
$$\inf_{s \in K} \inf_{u \in V_n} \{ c(g(v_0)v_0' - g(u)u')s + E_n((f(y, w, 0, u) \geq 0) - (f(y, w, 0, v_0) \geq 0)) \}$$
$$\geq n^{1/4} - 2c \sup_{v \in S^{p-1}} g(v) \sup_{s \in K} ||s|| - C$$
$$> - \epsilon,$$
for $n$ large enough, which implies that (20) holds automatically. (ii) follows.

(iii) Thirdly, we show that $M_n(s) \overset{d}{\to} M^3_n(s)$ uniformly over $s \in K$. In virtue of (i) and (ii) above, it suffices to show that $M^2_n(s) \overset{d}{\to} M^3_n(s)$ uniformly over $s \in K$. Notice that by D2, $V_n = V(0) = S^{p-1}$ and $P(f(y, w, 0, v)) - \alpha^* = 0$ for any $v \in V(0)$. Therefore,
$$M^2_n(s) = \inf_{v \in V(0)} \left( E_n(f(y, w, 0, v) \geq 0) - cg(v)v's + n^{1/2}(P(f(y, w, 0, v) - \alpha^*)) \right)$$
$$= \inf_{v \in V(0)} \left( E_n(f(y, w, 0, v) \geq 0) - cg(v)v's \right)$$
$$\overset{d}{\to} \inf_{v \in V(0)} \left( E_P(f(y, w, 0, v) \geq 0) - cg(v)v's \right)$$
$$= M^3_n(s),$$
where the second to last step follows from the central limit theorem for empirical process (Theorem VII. 21 of Pollard (1984)) and the continuous mapping theorem. The steps above hold uniformly for $s \in K$. (A) has been verified completely.

So far we have verified (A), (B) and (C). This completes the proof of Theorem 5.1 in light of Lemma 5.1.

Remarks 5.1

(I) $D_2$ holds true for symmetric distributions such as regression symmetric about $\theta$ (in this case, $RD_{RH}(\theta; P) = \alpha^*$ and $V(\theta) = S^{p-1}$, see Lemma 4 of RS04), which implies that the assumption $D_2$ in the theorem could be dropped for such $F_Z$. $D_2$ also holds for the two examples 2.1 and 2.2, where $\beta^* = (0,0)'$ and $\alpha^* = 1/2$.

In the study of the asymptotics of the Tukey Median, Massé (2002) tried to relax $D_2$ to: There exists a $c > 0$ such that
\[
\min_{u \in S^{p-1}} \max_{v \in V(0)} u'v \geq c,
\]
to cover the non-symmetric distribution cases. With this, our proofs above hold until Lemma 5.5, where we have to use the fact that $P(f(y,w,0,v)) = \alpha^*$ over $v \in V(0)$, otherwise the proof will not go through. The latter happens at Massé 2002 (the second line on page 298).

(II) The theorem could be adapted to cover the location counterpart (maximum halfspace depth estimator, Tukey median), The assumptions $D_1$-$D_3$ hold under the conditions given in Nolan (1999) and BH99.

(III) BH99 treated the limit distribution of a maximum depth induced estimator. Unfortunately, their result does not cover the result here.

6 Concluding remarks

The asymptotics of the maximum regression depth estimator $\beta^*_n$ induced from $RD_{RH}$ (RH99) have been investigated and established.

The asymptotics of $\beta^*_n$ were believed to have been already obtained by BH99 and thus were cited in the literature (e.g. Mizera (2002), Nolan (1999), and Massé (2002)) in the last twenty years. BH99 actually defined a very close but different depth notion than the original RH99, notwithstanding. Hence their induced maximum depth estimator is not identical to the intended one. The asymptotics of $\beta^*_n$ were not addressed until this article.

The approaches for root-n consistency and limiting distribution here are quite general and can be adapted to cover other min-max (or max-min) induced estimators, such as the deepest location estimator (multi-dimensional Tukey median).

Sufficient conditions for root-n consistency and limiting distribution in this article might not be optimal ones. Seeking the weakest sufficient conditions for the asymptotics of the maximum regression depth estimator, however, is not the principal goal of this article.

The main technical tools used in this article are empirical theory and the Argmax theorem.
The latter was employed in the ground-breaking article of Kim and Pollard (1990) for the cube root asymptotics. These powerful tools are anticipated to be very useful for the asymptotics of maximum depth estimators induced from the min-max stratagem.

Acknowledgment

The author thanks Dr. Stapleton for his careful English proofreading of the manuscript which has led to improvement.

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