Surface Operators in $\mathcal{N} = 2$ 4d Gauge Theories

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ABSTRACT: $\mathcal{N} = 2$ four dimensional gauge theories admit interesting half BPS surface operators preserving a $(2, 2)$ two dimensional SUSY algebra. Typical examples are $(2, 2)$ 2d sigma models with a flavor symmetry which is coupled to the 4d gauge fields. Interesting features of such 2d sigma models, such as (twisted) chiral rings, and the $tt^*$ geometry, can be carried over to the surface operators, and are affected in surprising ways by the coupling to 4d degrees of freedom. We describe in detail a relation between the parameter space of twisted couplings of the surface operator and the Seiberg-Witten geometry of the bulk theory. We discuss a similar result about the $tt^*$ geometry of the surface operator. We predict the existence and general features of a wall-crossing formula for BPS particles bound to the surface operator.
1. Introduction and outline

$\mathcal{N} = 2$ gauge theories in four dimensions are a rich theoretical playground. A large class of them can be conveniently engineered as the compactification of a $(2,0)$ six-dimensional SCFT on a punctured Riemann surface $C$. This is a powerful tool to describe and compute protected quantities. Examples include the gauge coupling parameter space and S-duality group, the massless effective Lagrangian [1] [2], the spectrum of BPS particles and the effective Lagrangian for the theory on $\mathbb{R}^3 \times S^1$ [3], the $S^4$ and the instanton partition functions [4], the S-duality action on BPS line operators [5], the expectation value of BPS line operators on $S^4$ [6] [7].
The range of applicability of the 6d engineering approach is not fully explored. One may wonder if it could provide a sort of classification of $\mathcal{N} = 2$ gauge theories in four dimensions. There are two obvious obstructions: the 6d construction might not be surjective, and is surely not injective in the space of 4d theories. Some known gauge theories, notably the superconformal quivers in the shape of an exceptional Dynkin diagram solved in [8], have no known 6d engineering construction. It is possible that such a construction might yet be found, maybe involving a Riemann surface $C$ with the orbifold points. This was the case for superconformal quivers in the shape of a D-type Dynkin diagram [9]. Furthermore, the same four dimensional theory often admits several distinct six-dimensional realizations.

It would be interesting to understand how to define an “inverse map”, an algorithm to identify a six dimensional ancestor for a given 4d theory. It would be even better to be able derive the above mentioned results directly in four dimensions, without invoking an higher dimensional construction. We believe that the the recent work [7] offers a crucial clue, in the form of a certain “minimal” half BPS surface operator. The minimal surface operators descend from a natural surface operator in the $(2,0)$ six-dimensional SCFT. The 6d surface operator sits at a point in the internal Riemann surface $C$. As a consequence, $C$ coincides with the parameter space of the minimal surface operator. To be precise, the surface operator preserves $(2,2)$ SUSY in two dimensions, and $C$ coincides with the parameter space of couplings in the 2d twisted superpotential. The minimal surface operators have a set of massive vacua which, fibered over the parameter space $C$, produce a second curve $\Sigma$ which coincides with the Seiberg-Witten curve of the 4d theory.

These two facts are rather natural from a six dimensional point of view, but become rather striking as soon as one identifies the minimal surface operator as a specific defect in the 4d gauge theory, and forgets about the 6d engineering construction. Different 6d realizations of the same 4d theory correspond to different choices of defects in the 4d theory.

In section 3.1 we aim to show that similar facts are universally true for any $\mathcal{N} = 2$ 4d theories and any possible choice of surface operators: the twisted parameter space of the surface operator and the space of discrete vacua fibered over it encode the Seiberg-Witten geometry of the bulk theory.

One interesting property of massive $(2,2)$ two dimensional theories are the $tt^*$ equations (topological-antitopological fusion). The extension to surface operators in 4d theories turns out to be quite interesting, and is detailed in section 3.2. We mentioned that the six dimensional engineering of 4d theories was used in [3] as a tool to understand the spectrum of massive BPS particles. A crucial role was played by a system of Hitchin equations on $C$, whose spectral curve coincides with the Seiberg-Witten curve. We will show how such Hitchin systems arise generically in any 4d theory, given a choice of a non-trivial surface operator, from the 2d $tt^*$ equations.

We observe that these equations control both the 4d BPS spectrum in the bulk and the 2d BPS spectrum of particles bound to the surface operator, generalizing the results of [10]. We claim in section 3.3 that this implies the existence of a “2d-4d wall-crossing formula” which combines the known 2d and 4d formulae, and will be presented in detail in a separate publication.

Section 4 presents a few examples. Unfortunately, the six dimensional engineering
construction is the only systematic method we have available to solve for the properties of surface operators, and covers most natural choices of 4d theories and surface operators. This makes it rather hard to find examples illustrating the full generality of our conclusions. Rather, we will illustrate how distinct surface operators in the same theory manage to encode the same Seiberg Witten geometry, even though their parameter space and number of vacua differ.

We conclude with some final remarks in section 5

2. SUSY review

We will work both with conformal and with asymptotically free theories, but it is useful to start from the $SU(2,2\vert 2)$ 4d $\mathcal{N} = 2$ superconformal group, and identify the subgroup preserved by a half BPS surface operator. Indeed, all the surface operators which we will discuss are classically conformal invariant. The bosonic subgroup of $SU(2,2\vert 2)$ is $SU(2,2) \times U(1)_R \times SU(2)_R$. $SU(2,2) \sim SO(4,2)$ is the 4d conformal group. A conform invariant surface operator wrapping $R^1, R^1 \in \mathbb{R}^3$ will preserve 2d conformal transformations in $R^{1,1}$, and rotations in the plane perpendicular to the operator. That’s $SO(2,2) \times SO(2)_s \in SO(4,2)$, which is the block-diagonal $SU(1,1) \times SU(1,1) \times U(1)_s$ in $SU(2,2)$. We can complete this to a 2d superconformal subgroup $SU(1,1) \times SU(1,1) \times U(1)_d$ embedded in the obvious block-diagonal way in $SU(2,2)$. This preserves half of the bulk supercharges, and corresponds to a defect with $(2,2)$ 2d SUSY.

The $U(1)_R$ symmetry of the four dimensional theory becomes an R-charge in the 2d SUSY algebra, which we will conventionally denote as the axial $U(1)_A$. The 2d vector R-charge $U(1)_V$ is a linear combination of a $U(1)$ subgroup of $SU(2)_R$ and of the rotation generator $U(1)_s$ in the plane orthogonal to the surface operator. $U(1)_d$ is a second linear combination of these two. The subset of the super(conformal) charges preserved by the line operator is the set commuting with the action of $U(1)_d$: the charge under rotations around the surface operator should be equal to the weight under the $SU(2)_R$ action. Conformal symmetry, and $U(1)_R$, can be broken by 4d Coulomb branch expectation values, mass parameters or gauge coupling scales. The $U(1)_A$ symmetry group (and 2d conformal symmetry) of the surface operator will be then broken as well. Even if the bulk theory is conformal, $U(1)_A$ can still be broken by 2d mass parameters or strong coupling scales.

Let us denote the two $\mathcal{N} = 1$ sets of 4d supercharges of the $\mathcal{N} = 2$ theory as $Q^+_a$ and $Q^-_a$. The sign $\pm$ denotes the $SU(2)_R$ weight. The 4d chirality operator is the product of the chirality operator on the plane of the surface operator and the chirality operator orthogonal to the plane, i.e. 2d chirality and charge under $SO(2)_s$. The surface operator will preserve the component of the 4d chiral spinor $Q^+_a$ which has positive 2d chirality and positive charge under $SO(2)_s$. This 2d supercharge has positive $U(1)_V$ charge and will be denoted as $Q_L$. This supercharge and the conjugate $\bar{Q}_L$ are the left-moving supercharges in the $(2,2)$ 2d supersymmetry algebra. The surface operator will also preserve the component of the 4d

\[ A \text{ second class of half BPS surface operators may exist, preserving a } SU(1,1) \times SU(1,1|2) \times U(1)_d, \text{ i.e. } (0,4) \text{ SUSY in 2d. They will play no role in this paper. One might also consider quarter-BPS surface operators, preserving only } (0,2) \text{ SUSY in 2d.} \]
chiral spinor $Q^\alpha_-$ which has negative 2d chirality and negative charge under $U(1)_s$. This 2d supercharge has negative $U(1)_V$ charge and will be denoted as $Q_R$. This supercharge and the conjugate $\bar{Q}_R$ are the right-moving supercharges in the $(2, 2)$ 2d supersymmetry algebra.

There are two related ways to construct supersymmetric defects in gauge theory. A simple approach is to add to the Lagrangian terms which are integrated on the defect only. They will include kinetic and potential terms for the degrees of freedom on the defect, and couplings to the bulk degrees of freedom. If the bulk Lagrangian has a superspace formulation, the defect will break translations along half of the superspace directions as well, and the defect Lagrangian can be written as an integral over the unbroken superspace directions. Bulk superfields will decompose into a tower of superfields for the restricted defect superspace.

There is an elegant perspective which simplifies the derivation of defect Lagrangian, and furthermore allows one to incorporate naturally a breaking of the bulk gauge group at the defect. To describe a dimension $d$ defect, one simply rewrites the bulk theory as a theory in $d$ dimensions, whose fields are valued in the space of functions of the coordinates transverse to the defect, and whose gauge groups are maps from the transverse space to the original gauge groups. The bulk + defect Lagrangian is just the most general supersymmetric $d$ dimensional Lagrangian coupling this peculiar version of the bulk fields with the defect degrees of freedom.

A crucial role in this paper is played by protected terms in the 2d Lagrangian, i.e. superpotential or twisted superpotential terms, and by their dependence on the bulk fields. It turns out that bulk vector multiplets enter twisted superpotential terms, while bulk hypers enter superpotential terms. Indeed the scalar component of a 4d vector multiplet is annihilated by both sets of antichiral supercharges. As they have opposite $U(1)_V$ charge, the restriction of the scalar to the surface operator is the lowest component of a twisted chiral multiplet. In all examples we will consider, the 2d degrees of freedom are massive in the IR, and the the protected couplings of vector multiplets to a surface operator are encoded in a 2d twisted effective superpotential $W(a, z)$ (See [11] for a beautiful supergravity example). The twisted superpotential depends on the (twisted)couplings $z_a$ of the surface operator and on the Coulomb branch parameters of the 4d theory. It plays a role akin to the effective prepotential for the bulk 4d theory.

The 4d hypermultiplet scalars sit in a doublet of $SU(2)_R$. Consider the complex scalar with positive $SU(2)_R$ weight $q^+$. It is annihilated by $Q^+_\alpha$ and by the conjugate of $Q^-_{\bar{\alpha}}$. Restricted to the surface operator it plays the role of a chiral multiplet, annihilated by the supercharges with positive $U(1)_V$. Hence it can enter the 2d superpotential. These couplings are mostly relevant for the behavior of surface operators in the Higgs branch of the bulk theory, though they can play a role in the Coulomb branch of the bulk theory as well [7]

2.1 4d gauge theory in 2d language

In order to study the surface operators in a 4d gauge theory, one simply recasts the 4d theory in a 2d language, as a 2d gauge theory whose gauge group is the group $G$ of maps
from the transverse plane (parameterized by $x^2, x^3$) to the original 4d gauge group $G$. The trace for the gauge group $\mathcal{G}$ includes the integral over the $x^2, x^3$ directions. The 0, 1 components of the 4d connection take the form of a $\mathcal{G}$ valued connection, sitting in a $(2, 2)$ vector multiplet together with half of the 4d gauginos and the 4d complex scalar. The 4d gauge field in the 2, 3 directions, or better the covariant derivative $D_2 + i D_3$ sits in a $(2, 2)$ chiral multiplet transforming under $\mathcal{G}$. The moment map for the $\mathcal{G}$ action coincides with the transverse field strength $F_{23}$. 4d hypermultiplets also give rise to pairs of 2d chiral multiplets in conjugate representations of $\mathcal{G}$.

The 4d gauge coupling $\tau$ plays a double role: 2d gauge coupling and Kahler parameter for the $D_2 + i D_3$ chiral field. The various pieces of the 4d gauge kinetic term arise from the 2d kinetic terms. In particular, the $F_{23}^2$ term arises from integrating away the 2d auxiliary field D, which couples to the moment map $F_{23}$. The 4d kinetic energy for 4d hypermultiplets is a combination of the 2d kinetic energy, a superpotential term involving the $D_2 + i D_3$ derivative of the hypermultiplets, and the D term potential.

The advantage of this construction is that it makes rather simple to add a surface operator along the 0, 1 directions. For example we can define a surface operator by adding extra 2d chiral multiplets, and possibly 2d gauge fields, to the setup. The overall moment map for the $\mathcal{G}$ gauge action has an extra contribution from the moment map of the 2d matter $\mu^{2d}$, and takes the form $F_{23} + \delta(x^2) \delta(x^3) \mu^{2d}$. We see how SUSY, or more precisely the D-term constraints, force the connection to have a monodromy around the surface operator (see [12] for the corresponding statement in $\mathcal{N} = 4$ SYM). Notice also that the 2d F-term and D-term equations in presence of hypermultiplets coincide with the equations for BPS vortices localized in the 2, 3 directions. This is no coincidence. Surface operators and normalizable vortex solutions are clearly related. A vortex in a Higgs branch or a mixed Higgs-Coulomb branch of a $\mathcal{N} = 2$ theory will flow in the IR to a surface operator in the IR theory. Many beautiful results about vortex operators (see for example the [13] review) can be recast in the language of surface operators.

In two dimensions the effective twisted superpotential receives contributions from 2d instanton (“vortex” in a different sense) configurations, where the chiral fields are covariantly holomorphic, and the magnetic flux is set to be equal to the moment map. In the 4d setup, 4d instantons are a neat example of a 2d instanton for the $\mathcal{G}$ 2d gauge theory (see also [14]): $D_2 + i D_3$ should be covariantly holomorphic and $iF_{01} = F_{34}$. The 4d gauge coupling enters the instanton action as a 2d Kahler parameter for the $D_2 + i D_3$ chiral field. Coupling to 2d matter allows one to study combinations of 2d vortices and 4d instantons, where the moment map for the 2d matter acts as a source for the $iF_{01} = F_{34}$ self-duality condition. We will see from explicit example that these mixed 4d-2d instantons indeed appear correct the effective twisted superpotential and twisted chiral ring relations.

3. Surface operators in the Abelian IR theory

3.1 Seiberg-Witten geometry from surface operators

We will make the following assumptions about the IR behavior of the theory:
The massless degrees of freedom of the 4d theory consist of an abelian gauge theory of rank $r$.

There are no 2d massless degrees of freedom: the 2d theory is massive. (Notice that the Coulomb branch parameters of the 4d theory enter the 2d Lagrangian as twisted masses, so this assumption is not very restrictive.)

The 2d surface operator has a finite number of vacua, parameterized by the expectation values of the operators in the 2d twisted chiral ring, subject to the twisted chiral ring relations. The structure constants of the ring may depend on the parameters $z_a$, and 2d twisted masses, on the 4d Coulomb branch parameters, gauge couplings and mass parameters.

The parameters $z_a$ of the surface operator can be varied by adding a term $\delta z_a \hat{x}_a$ to the twisted superpotential. $\hat{x}_a$ are appropriate elements of the twisted chiral ring. There might be a space of marginal superpotential deformations as well, but it will play no role in the following. We will denote the space of 2d parameters as $\mathcal{P}$.

As long as the surface operator has a good UV definition, $\mathcal{P}$ can depend on the UV gauge couplings of the 4d theory, but not on the 4d Coulomb branch parameters or masses. The 2d IR vacua of the surface operator can be fibered over $\mathcal{P}$ to give a new manifold $\hat{\mathcal{P}}$. $\hat{\mathcal{P}}$ will in general depend on the 4d Coulomb branch parameters and masses. The expectation values $x_a$ of the $\hat{x}_a$ operators will give a useful one form $\lambda = x_a dz_a$ on $\hat{\mathcal{P}}$. The six-dimensional construction gives a canonical example of this setup: $\mathcal{P}$ coincides with the curve $C$, $\hat{\mathcal{P}}$ with the Seiberg-Witten curve and $\lambda$ with the Seiberg-Witten differential.

As the 2d theory is, by assumption, massive, in the IR the surface operator takes the form of a defect in the 4d IR abelian gauge theory. The defect is characterized by the parameters $\alpha^i$ and $\eta_i$, where $\alpha^i$ are the monodromies of the abelian gauge fields around the operator, and $\eta_i$ the 2d theta angles, couplings to the magnetic fluxes on the surface. The two sets of parameters are both angular variables, and are exchanged by the abelian electric-magnetic duality group [12]. The angles are naturally combined in coordinates $t_i = \eta_i + \tau_{ij} \alpha^j$ on an Abelian variety. This Abelian variety with complex structure given by the IR bulk gauge couplings is a familiar object in Seiberg-Witten theory.

What is the relation between the IR parameters $t_i$ and the original parameters $z_a$ of the surface operator? The IR couplings are encoded in an IR effective twisted superpotential, as

$$t_i = \frac{\partial W_{eff}}{\partial a^i}$$ (3.1)

Indeed the superspace integral of $W_{eff}[a^i]$ gives the 2d Lagrangian coupling

$$\frac{\partial W_{eff}}{\partial a^i} F^i_+ + c.c$$ (3.2)
Here $F^i_+$ is the restriction to the surface operator of the self-dual field strength.

The effective twisted superpotential should also control the expectation value $x_a$.

$$x_a = \frac{\partial W_{\text{eff}}}{\partial z_a}$$  \hspace{1cm} (3.3)

We can write

$$\frac{\partial t_i}{\partial z_a} = \frac{\partial x_a}{\partial a^i}$$  \hspace{1cm} (3.4)

Consider the periods of the differentials $\frac{\partial x_a}{\partial a^i}dz_a = \frac{\partial \lambda}{\partial a^i}$ over a one-cycle $\gamma$ in $\hat{P}$. As the cycles starts and ends at the same point in $\hat{P}$, the variation of $t_i$ along the cycle must be a period of the Abelian variety, of the form $n_i + \tau_{ij}m^j$ for some integers $n, m$. Alternatively, in terms of more general coordinates $u_s$ on the 4d Coulomb branch moduli space,

$$\oint_{\gamma \subset \hat{P}} \frac{\partial \lambda}{\partial u_s} = n_i[\gamma] \frac{\partial a^i}{\partial u_s} + m^i[\gamma] \frac{\partial a^P_i}{\partial u_s}$$  \hspace{1cm} (3.5)

This statement strongly resembles the basic structure of the Seiberg-Witten geometry, with $\hat{P}$ playing the role of the SW curve and $x^adz_a$ the role of the SW differential. In particular, this defines a map $\gamma \rightarrow (n_i, m^i)$ from the homology of one-cycles in $\hat{P}$ to the charge lattice of the 4d theory. Up to possible integration constants, one is tempted to guess that under this map the central charges of the 4d theory are reproduced

$$Z_\gamma = \oint_\gamma \lambda$$  \hspace{1cm} (3.6)

Notice that the central charge function is a linear map on an extended lattice $\hat{\Gamma}$, which include the gauge charges, but also the flavor charges of the theory, which multiply the mass parameters. These include the mass parameters of the 4d theory, but also possibly some extra “twisted masses” of the 2d theory. The above, tentative relation would require an extended map from $H_1(\hat{P}, \mathbb{Z})$ to $\hat{\Gamma}$.

It should be possible to derive such relation directly, and not just in a differentiated form. One possibility is to look at dynamical BPS domain walls on the surface operator. On general grounds, the tension of such domain walls is given in terms of the variation of the superpotential across the wall, i.e. $|\Delta W|$. If we use the expression for the effective IR superpotential, this gives

$$\Delta W = \int_{\tilde{z}'}^{\tilde{z}''} \lambda$$  \hspace{1cm} (3.7)

Here $\tilde{z}', \tilde{z}''$ are two points in $\hat{P}$ in the fibre of the same point $z$ in $P$. This can be interpreted as a central charge, which receives contributions from the topological charge of the soliton (the choice of vacua $\tilde{z}', \tilde{z}''$ above $z$) and possibly the gauge and flavor charges of the soliton. More precisely, different solitons may be associated to different choices of paths between $\tilde{z}'$ and $\tilde{z}''$. The difference in their central charge is the integral of $\lambda$ on the difference $\gamma$ between the two paths. $\gamma$ is a closed path, and we ascribe the difference in the central charges

$$Z_\gamma = \oint_\gamma \lambda$$  \hspace{1cm} (3.8)
to the difference in the flavor and gauge charges carried by the two solitons.

A few observations are in order. Both the homology lattice $H_1(\hat{P}, \mathbb{Z})$ and the 4d charge lattice $\hat{\Gamma}$ are local systems over the Coulomb branch of the 4d theory. $\hat{\Gamma}$ has monodromies around loci where massless 4d BPS particles appear. The map from $H_1(\hat{P}, \mathbb{Z})$ to $\hat{\Gamma}$ should intertwine the monodromies of the two local systems. This fact almost answers an important question regarding the image of the map $H_1(\hat{P}, \mathbb{Z})$ to $\hat{\Gamma}$, which in principle may or not be surjective. Indeed, as the image includes at least a non-zero charge vector, it will have to include the linear span of all the images of that charge under monodromy transformations. The result is strengthened by some results we will accumulate in the next sections: bulk 4d BPS particles can form bound states with 2d BPS particles, much like 4d particles can form bound states among themselves. The appearance and disappearance of such bound states is determined by the IR gauge, flavor and topological charges of the particles only. Each time a 4d particle of charge $\gamma$ binds to a 2d domain wall, it implies that $\gamma$ sits in the image of the map from $H_1(\hat{P}, \mathbb{Z})$ to $\hat{\Gamma}$. Monodromies and wall-crossing should be sufficient guarantee surjectivity in non-degenerate cases.

In the case of the minimal surface operator in theories built from six dimensions the map is indeed surjective. All the periods of the 4d theory are reproduced as periods of $\lambda$ over the Seiberg-Witten curve. An even stronger condition is true: the 4d Coulomb branch coincides with the space of possible normalizable deformations of the Seiberg-Witten curve $\hat{P}$. It is unclear from the derivation in this section if a similar statement would hold in a more general setup. The next sections will put further constraints, but will not provide a definitive answer.

3.2 Hitchin systems from surface operators

The $tt^*$ equations are a beautiful property of $(2, 2)$ theories [15], which extends well to surface operators in $\mathcal{N} = 2$ 4d theories. Consider the compactification of the 4d theory on a circle of radius $R$, with the surface operator wrapping $\mathbb{R} \times S^1 \in \mathbb{R}^3 \times S^1$ (and Ramond boundary conditions). Following the 2d story, instead of the discrete fibration $\hat{P}$, it is possible to consider now a vector bundle $\mathcal{V}$ of Ramond vacua over the parameter space $P$. This bundle is endowed with two natural structures: an Hermitian connection $D_a, \bar{D}_a$ and a holomorphic one form $c_a dz_a$ (and $\bar{c}_ad\bar{z}_a$) valued in endomorphisms of $\mathcal{V}$, given by the action of the twisted chiral operator $x_a$ over the Ramond vacua of the theory.

The two objects satisfy a generalization of Hitchin’s equations:

$$[D_a, D_b] = [\bar{D}_a, \bar{D}_b] = [c_a, c_b] = [\bar{c}_a, \bar{c}_b] = 0$$

$$D_a c_b = D_b c_a \quad D_a \bar{c}_b = \bar{D}_b \bar{c}_a \quad D_a \bar{c}_b = 0 \quad \bar{D}_a c_b = 0$$

$$[D_a, \bar{D}_b] + [c_a, \bar{c}_b] = 0 \quad (3.9)$$

These are equivalent to the flatness of the spectral connection

$$\nabla_a = D_a + \frac{R}{\zeta} c_a \quad \bar{\nabla}_a = \bar{D}_a + R\zeta \bar{c}_a \quad (3.10)$$

for all values of $\zeta$. These results admit a very simple and intuitive derivation in terms of supersymmetric Janus configurations, which is included in appendix A.
The relation between $\mathcal{V}$ and $\hat{\mathcal{P}}$ is straightforward: the $c$ endomorphisms which describe the action of twisted chiral ring operators commute, and their eigenvalues are the expectation values of the corresponding operators on $\hat{\mathcal{P}}$. The main difference between the pure 2d case and the case of surface operators is the fact that $c, D$ will also depend on the 4d Coulomb branch parameters. The effect of the 4d Coulomb branch parameters is similar to the effect of 2d twisted masses. The effect of twisted masses on the $tt^*$ equations has been studied before, but the result are unpublished, and possibly lost. [16]. From the very beginning, we will see a strong resemblance with ideas developed in the context of the 4d $tt^*$ equations [17]. Indeed, our final result will be a neat merger of the 2d and 4d $tt^*$ perspectives.

Upon compactification on a circle, the Coulomb branch moduli space of the theory doubles in dimension, as an Abelian variety of electric and magnetic Wilson lines is fibered over the 4d moduli space. The 3d Coulomb branch moduli space $\mathcal{M}$ is an hyperkähler manifold. It has a has a $\mathbb{CP}^1$ worth of complex structures. We will always label the choices of complex structures by an inhomogeneous parameter $\zeta$. The complex structures $\zeta = 0$ or $\zeta = \infty$ are special: the 4d Coulomb branch parameters are holomorphic in these complex structures, and the torus of Wilson lines is an Abelian variety (dual to the one we met in the previous section). The holomorphic functions in other complex structures $\zeta \in \mathbb{C}^*$ are rather more interesting, and are the main subject of the analysis of [17].

A canonical example of such function is the expectation value of a half BPS line operator in the 4d theory, wrapped along the $S^1$ [18]. A BPS line operator stretched along the $x^1$ direction preserves a linear combination $Q^\pm + \zeta \gamma_1 Q^\pm$ of the chiral and anti-chiral supercharges (it preserves $SU(2)_R$). $\zeta$ is a pure phase for a physical operator, but it can be analytically continued to $\mathbb{C}^*$. In terms of a low energy sigma model on the 3d moduli space of vacua $\mathcal{M}$, the linear combination of supercharges $Q^\pm + \zeta \gamma_1 Q^\pm$ kills a certain ring of protected operators, which are holomorphic functions of the scalar fields in complex structure $\zeta$.

A free Abelian example of a BPS Maldacena-Wilson line operator and its expectation value would have the schematic form

$$\langle P \exp \oint \frac{1}{2} \zeta^{-1} a + iA + \frac{1}{2} \bar{\zeta} \bar{a} \rangle = \exp \left[ \pi R \zeta^{-1} a + i \theta + \pi R \zeta \bar{a} \right]$$

(3.11)

Here $a$ is the vector multiplet scalar field and $\theta$ the Wilson line order parameter. Furthermore, every mass parameter of the 4d theory (and twisted mass parameter of the 2d theory) is associated to a flavor symmetry, and we can include an extra flavor Wilson line $\theta_f$ for each mass parameter $m$. It is natural to restrict BPS operators in complex structure $\zeta$ to be functions of the natural combination $\pi R \zeta^{-1} m + i \theta_f + \pi R \zeta \bar{m}$. From now on, every time we mention holomorphic functions on $\mathcal{M}$, we implicitly assume such a dependence on the masses and flavor Wilson lines as well. (Including both 4d masses and 2d twisted masses. Notice that both are expectation values of a background vector multiplet gauging the flavor symmetry.)

The spectral connection $\nabla$ will depend on the choice of 3d vacuum in $\mathcal{M}$ and on the mass parameters and flavor Wilson lines. We would like to identify a sense in which the
spectral connection with given parameter $\zeta$ depends on $M$ holomorphically in complex structure $\zeta$ (and $R$ identified with the radius of the compactification circle).

Indeed, consider the monodromy data of the spectral connection. In simple cases, it will consist of traces of monodromy matrices around cycles of $\mathcal{P}$. In more complicated cases, it will include Stokes data at irregular singularities. In any case, the result is a set of functions on $\mathcal{M}$. We want to argue that such functions will be holomorphic in complex structure $\zeta \in \mathbb{CP}^1$.

The BPS projection for a surface operator and the BPS projection for a line operator in the 4d theory parallel to it are compatible, and intersect on a set of two supercharges. The same amount of supersymmetry is in general preserved by a line operator (a non-dynamical domain wall, or a boundary) inside a surface operator. Such line operators are analogous to supersymmetric boundary conditions in a 2d theory, relating the left and right moving supercharges as $Q_L + \zeta \bar{Q}_R = 0$ and $Q_R + \zeta \bar{Q}_L = 0$.

As proven in [19] and reviewed in the appendix A the correlation function of the 2d theory on a half-cylinder of radius $R$, with such a boundary at one end and a choice of Ramond vacuum at the other end, is a flat section of the spectral connection $\nabla$ with the same $R, \zeta$! This statement can be immediately extended to supersymmetric line operators between two different theories, or the same theory at different values of the parameters $z$ and $z'$, by the reflection trick: an interface between two 2d theories is the same as a boundary condition for the product theory.

The expectation value of a line operator interpolating between given Ramond vacua of the theory at different values $z$ on the left and $z'$ on the right of the parameters will be a matrix flat section $M(z,z')$. (This is a flat section for both a left action of the spectral connection in $z$ and for a right action of the spectral connection in $z'$.) A particularly interesting line operator is the Janus domain wall defined in detail in appendix A. It is defined starting from a trivial line operator at $z = z'$ and continuously deforming the coupling $z$ to a given final value while preserving the same SUSY. The expectation value of the operator only depends on the homotopy class of the path in $\mathcal{P}$ between $z$ and $z'$. For the trivial line operator $M(z',z')$ is the identity matrix, and we can use the flat spectral connection to transport $z$ along the path as we define the Janus line operator. Hence the expectation value of the Janus line operator literally computes the transport matrix for the flat connection. In particular the transport matrix is the expectation of a line operator annihilated by the two supercharges $Q_L + \zeta \bar{Q}_R = 0$ and $Q_R + \zeta \bar{Q}_L = 0$.

The same result will apply for the case of surface operators. The spectral data for the flat connection, which is computed from the transport matrix, gives functions over $\mathcal{M}$ annihilated by the two supersymmetries preserved by the line operator. These two supercharges are two out of the four $Q + \zeta \gamma_i \bar{Q}$ which annihilate holomorphic functions in complex structure $\zeta$. Notice that the kernel of the two supercharges coincides with the kernel of the full set of four supercharges: the four supercharges in the 3d low energy sigma model on $\mathcal{M}$ all have the general form $\psi^i \partial_i^\zeta$ and only differ in the $SU(2)_R$ and spacetime indices of the fermion $\psi$.

In the six dimensional setup in [3], a very specific Hitchin system on $C$ produces holomorphic functions in complex structure $\zeta$ as the monodromy data of the spectral
connection. We see that the \( tt^* \) equations for a generic surface operator can play a similar role. In the six-dimensional setup, the choice of Hitchin system is determined by the requirement that the spectral curve \( \det(x - c(z)) = 0 \) should coincide with the Seiberg-Witten curve and \( xdz \) with the Seiberg-Witten differential. (Remember that \( c(z) \) plays the role of the Higgs field in the Hitchin system.) This is just the expected relation between \( \mathcal{V} \) and \( \mathcal{P} \).

In general the gauge invariant information about \( \mathcal{P} \) encoded in the \( c_a(z) \) can be packaged into a ring of symmetric differentials of various degree \( k \), \( \text{Tr}(c_a dz_a)^k \) on \( \mathcal{P} \). We are now given several, possibly related complex manifolds: the 4d Coulomb branch, the moduli space of possible \( \mathcal{P} \), the moduli space of normalizable deformations of the degree \( k \) differentials. They all coincide in the six dimensional example, but we do not know if that will be true in general. As a step in that direction, it is useful to consider the full map from the 3d Coulomb branch of vacua to the moduli space of solutions of the \( tt^* \) Hitchin-like system. This map extends the map between the 4d Coulomb branch and the moduli space of possible \( \mathcal{P} \).

It is not clear to us if the moduli space of solutions of the multidimensional \( tt^* \) Hitchin-like system would admit a hyperkähler metric, as the equations do not have the form of a hyperkähler quotient in general. We do not actually know how to even define the moduli space manifold precisely, though it should bear some relation to the some space of Higgs bundles on \( \mathcal{P} \). In any case, given any one-dimensional submanifold of \( \mathcal{P} \), we have a well-defined hyperkähler moduli space of solutions of Hitchin equations over it. Hence we have a map from the 3d Coulomb branch to the moduli space of this one dimensional Hitchin system. It is a map between hyperkähler manifolds which commutes with the structure of fibrations by Abelian varieties, and is holomorphic in all complex structures \( \zeta \in \mathbb{CP}^1 \). Such maps between different hyperkähler manifolds are rather uncommon.

It would be interesting to explore if the moduli space of solutions of the multidimensional \( tt^* \) Hitchin system could exactly coincide in general with the 3d Coulomb branch of the theory. For this to be true, it would have to be the case that the \( tt^* \) equations for a genuinely 2d theory had no moduli space.

### 3.3 Wall-crossing and surface operators

In two dimensions, the \( tt^* \) equations are part of an interesting structure [10]: the spectral connection commutes with a pair of connections \( \nabla_\zeta = \zeta \partial_\zeta + A_{2d}, \ \nabla_R = R \partial_R + A_{2d}^R \). All connections have simple poles at \( \zeta = 0, \infty \). In particular the \( \nabla_\zeta \) connection has irregular singularities at \( \zeta = 0, \infty \), which lead to Stokes phenomena. The Stokes factors for the \( \zeta \) connection can be computed in a large radius limit (as \( \nabla_R \) commutes with \( \nabla_\zeta \) and the location of Stokes rays is \( R \) independent, the Stokes factors are also \( R \) independent) and turn out to be in one-to-one correspondence with the 2d BPS particles in the theory.

The \( \nabla_a \) connection commutes with \( \nabla_\zeta \) as well, but the location of the individual Stokes rays is a function of the \( z_a \) (it coincides with the phase of the central charge of the corresponding BPS state). The product of all Stokes factors in a wedge in the \( \zeta \) plane is still invariant, as long as no rays enters or exits the wedge. This leads to a simple wall-crossing formula for the BPS particles of the 2d theory.
The holomorphic functions on $\mathcal{M}$ are governed by a formally similar set of equations [17], which we denote as 4d $tt^*$ equations. In the 4d $tt^*$ setup, one has a compatible set of connections $\mathcal{A}_\zeta, \mathcal{A}_R, \mathcal{A}_u$, along $\zeta, R$ and along the 4d Coulomb branch moduli, for the bundle of functions of the electric and magnetic Wilson line parameters. More concretely, $\mathcal{A}_\zeta, \mathcal{A}_R, \mathcal{A}_u$ are differential operators in the Wilson line parameters. In particular, the connection $\partial_u + \mathcal{A}_u$ has the interpretation of Cauchy-Riemann equations for a holomorphic functions on $\mathcal{M}$. The Stokes data of the connection on the $\zeta$ plane captures the BPS spectrum and wall-crossing of the 4d theory. Rather than finite matrices, the Stokes factors take the form of KS transformations, which are symplectomorphisms of a certain complex torus.

In the context of surface operators we have the 2d $tt^*$ connection $\nabla_a$, and one might wonder if a connection $\nabla_u$ along the 4d Coulomb branch might also exist on the bundle of Ramond vacua of the surface operator, compatible with the connection $\nabla_a$. This is cannot be the case, as the spectral data of $\nabla_a$ depends on the 4d Coulomb branch parameters! As the spectral data defines holomorphic functions on $\mathcal{M}$, we can instead consider a combined 2d-4d connection $\nabla_u + \mathcal{A}_u$. This should be seen as a connection on the bundle of functions of the Wilson line parameters of the 3d theory, valued in $\mathcal{V}$. Here and below $\nabla$ denotes a connection valued in endomorphisms of the finite dimensional bundle $\mathcal{V}$, and $\mathcal{A}$ is the standard 4d connection valued in differential operators.

Similarly, we expect some $\nabla_\zeta + \mathcal{A}_\zeta$ and $\nabla_R + \mathcal{A}_R$. The existence of such connections is a consequence of the (possibly anomalous) scale invariance and $U(1)_A$ symmetries of the combined 2d-4d system. (This was true both in the 2d $tt^*$ and in the 4d $tt^*$ separately.) It is natural to expect that the Stokes data for the combined connection $\nabla_\zeta + \mathcal{A}_\zeta$ will describe the spectrum and wall-crossing of BPS particles bound to the surface operator and their interactions with the 4d particles in the bulk.

It is possible for 4d BPS particles to bind to 2d BPS solitons, giving rise to mixed 2d-4d wall-crossing formulae. Indeed the 2d BPS particles carry 4d gauge charges, and, say, a 4d electron should be able to form bound states to a 2d monopole. The 2d wall-crossing formula expresses the invariance of a product of Stokes factors across the walls of marginal stability. These 2d Stokes factors are finite matrices. The 4d wall-crossing formula involves Stokes factors valued in a group of symplectomorphisms of certain formal variables $x_\gamma$ labeled by the elements $\gamma$ of the charge lattice of the 4d gauge theory. The structure group of the 4d-2d connection appears to be a semi-direct product of the group of symplectomorphisms of the formal variables $x_\gamma$ and of a group of finite matrices valued in the $x_\gamma$.

The detailed formulation and concrete examples of such 2d-4d wall-crossing formula is left to a separate publication.

4. Examples

Several of our examples are based on a simple Type IIA brane construction introduced in [1] to engineer specific $\mathcal{N} = 2$ gauge theories, and extended by [20] to engineer two dimensional (2, 2) sigma models. The construction involves an array of NS5 branes (along
Figure 1: Different brane realizations of a simple quiver gauge theory: $SU(3) \times SU(2)$ with a bifundamental and two $SU(2)$ fundamentals. Vertical lines represent NS5 branes, horizontal D4 branes, circles are D6 branes. (a) Simplest realization. Flavors from semi-infinite D4 branes. (b) Two D6 can also produce the flavors (c) D4 segments are created when moving the D6 branes. (d) An extra D6 has been added to the right, brought to the left.

The construction is extended in [20]: one can add a D2 branes (along the 017 directions) attached to one NS5 brane in order to produce interesting 2d sigma models. As remarked in [7], the construction is actually producing a surface operator. If the 4d theory engineered by the brane setup is trivial, the degrees of freedom living on the surface operator describe a purely 2d theory.
4.1 The $\mathbb{CP}^1$ sigma model

The $\mathbb{CP}^1$ sigma model is the canonical example of a massive $(2,2)$ theory in 2d. It can be described neatly by a linear sigma model, with a $U(1)$ gauge field in two dimensions coupled to two chiral fields $q^i$ of charge $+1$. The only protected coupling is the complexified FI parameters $t$ for the $U(1)$ gauge field. It determines the size of the $\mathbb{CP}^1$ target space by the SUSY constraints $\sum_i |q^i|^2 = t$. It is renormalized at one loop, so that $\exp 2\pi it$ has the dimension of a strong coupling scale squared.

The $U(1)$ gauge symmetry is Higgsed, and eats the overall phase of the $q^i$, leading to the $\mathbb{CP}^1$ sigma model. Notice that the mass of the gauge boson is of order $g_{YM}|t|$, so the linear sigma model is an arbitrarily good description of the $\mathbb{CP}^1$ sigma model as $g_{YM}$ is made very large. $g_{YM}$ is not a protected coupling, and does not affect the protected quantities we are interested in.

There is also a $SU(2)$ flavor symmetry acting on the two chiral fields. In general, we can turn on a twisted mass parameter $m$ in the Cartan of the $SU(2)$. This can be defined as an expectation value for the scalar component of a background vector multiplet, gauging the $SU(2)$ flavor symmetry. If the mass parameter is sufficiently large, the theory is weakly coupled: the massive chiral fields can be integrated out, and one is left with an effective twisted superpotential for the $U(1)$ gauge field scalar partner $x$:

$$-2\pi itx + (x - m) [\log(x - m) - 1] + (x + m) [\log(x + m) - 1]$$ (4.1)

The twisted chiral ring equation is then $2\pi it = \log(x - m) + \log(x + m)$ or $x^2 = m^2 + e^{2\pi it}$. This result is actually valid for all values of $m, t$. The parameter space $\mathcal{P}$ is a cylinder parameterized by $t$. The space $\hat{\mathcal{P}}$ is the curve defined by the equation $x^2 = m^2 + e^{2\pi it}$, and the canonical differential is $\lambda = x dt$. The $e^{2\pi it}$ correction to the $x^2 = m^2$ classical twisted chiral ring can be seen as a 2d 1-instanton effect.

This is an example of a model with a six dimensional construction. The authors of [19] engineered the model with a IIA brane construction (see fig. fig:cp1): two semi-infinite D4 branes ending on the same side of an NS5 brane, and a D2 brane ending on the system. The brane configuration can be lifted to M-theory and reduced to a simple six-dimensional engineering construction, based on the $A_1$ 6d SCFT. [3], [2]. The theory is compactified on a cylinder (or, equivalently, the two punctured sphere), with boundary conditions encoded by the quadratic differential

$$\phi_2 = (m^2 + e^{2\pi it}) dt^2 = (\frac{m^2}{z^2} + \frac{\Lambda^2}{z}) dz^2$$ (4.2)

The second expression is suitable for the two punctured sphere, we defined $e^{2\pi it} = \Lambda^2 z$ in terms of a scale $\Lambda$ and a dimensionless parameter $z$. The D2 brane goes to a minimal surface operator in the setup. In a sense, this construction gives the 2d sigma model as a surface operator in a trivial 4d theory.

The $tt^*$ equations for the model correspond to a $SU(2)$ Hitchin system with a regular singularity at $z = 0$ and an irregular singularity at $z = \infty$. The quadratic differential $\phi_2$ has no normalizable deformation, and the moduli space of solutions to the Hitchin system is zero-dimensional.
Figure 2: A brane realization of the $\mathbb{C}P^1$ sigma model. The dashed line represents a D2 brane ending on the system. If the brane ends on the NS5 brane, the linear sigma model can be recovered. Turning on an FI term moves the D2 along the D4 branes.

4.2 The minimal surface operator in $SU(2)$ Seiberg-Witten theory

The $SU(2)$ Seiberg-Witten theory can be engineered in Type IIA string theory by two D4 brane segments suspended between a pair of NS5 branes. The basic six dimensional engineering involve the compactification of the $A_1$ 6d SCFT on a cylinder, with boundary conditions at the ends corresponding to the quadratic differential [3]

$$\phi_2 = (\frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z})dz^2$$  \hspace{1cm} (4.3)

Here $u$ is the Coulomb branch parameter, expectation value of $\text{Tr}\Phi^2$, where $\Phi$ is the adjoint scalar superpartner of the $SU(2)$ gauge field. The scale $\Lambda$ is related to the UV renormalized gauge coupling $\Lambda^4 = \exp 2\pi i \tau$. The coordinate $z$ is also the parameter of a minimal surface operator. At weak 4d coupling, $|u| \geq |\Lambda^2|$, there are three interesting ranges of values for $z$, depending on which of the three terms in $\phi_2$ dominates.

If $z$ is of order one, $\lambda \sim \frac{\lambda^2}{z}$ with small corrections of order $\frac{\Lambda^2}{z}$. (At weak coupling, $2u \sim \lambda^2$). The IR effective superpotential has the form $a \log z$. The IR couplings take the form of $2\pi it^{IR} \sim \log z = 2\pi it$, roughly $a$ independent up to the order of $\frac{\Lambda^2}{a}$ corrections. The surface operator in this intermediate regime for $z$ is well described by the definition as a gauge theory defect. The parameter $t$ lives on a cylinder, rather than the expected torus, because $t = \eta + \tau \alpha$, but $\tau$ diverges at short distances, where the defect is defined.

As $z$ becomes sufficiently large or sufficiently small, the first or the last terms of the quadratic differential dominate. In terms of the IIA brane picture, the minimal surface operator is exploring the region near either NS5 brane, where the system resembles the one used to engineer an $\mathbb{C}P^1$ sigma model. (See fig. 3) For large $z$, it is instructive to use a coordinate $\Lambda^2 z = e^{2\pi it}$, to get

$$\phi_2 = (\Lambda^4 e^{-2\pi it} + 2u + e^{2\pi it})dt^2$$  \hspace{1cm} (4.4)
This indeed resembles the one for an $\mathbb{CP}^1$ sigma model, coupled to the 4d $SU(2)$ gauge group. The extra correction $\Lambda^4e^{-2\pi it}$ seems to have a simple physical interpretation: a combination of a 2d-4d instanton with 4d instanton number 1, and 2d instanton number $-1$. It would be interesting to compute this term directly in field theory, and understand in detail how the presence of the 4d instanton allows for a negative 2d instanton number.

**4.3 Non-minimal surface operators in $SU(2)$ Seiberg-Witten theory: Coupling to an $\mathbb{CP}^2$ sigma model.**

Now, consider again the brane configuration engineering pure the $SU(2)$ theory and the following manipulation: add a D6 brane to the left of all NS5, D4 branes, and then move it to the right, taking care to follow the appropriate rules of D4 brane creation due to the Hanani-Witten effect. Each time the D6 crosses an NS5 a new D4 brane segment appears. As a result, we are left with three D4 brane segments between the two NS5 branes, and two between the rightmost NS5 and the D6 brane. The SW curve takes a form

$$\Lambda^3z^2 + (x - m)(x^2 - 2u)z + \Lambda(x - m)^2 = 0 \quad (4.5)$$

$m$ is the transverse position of the D6 brane. The $(x - m)$ factors represent the new D4 brane segments attached to the D6 brane. The SW differential is $\lambda = x\frac{dz}{z}$. In practice, this is the usual $SU(2)$ curve, subject to a coordinate redefinition $z \rightarrow z\Lambda(x - m)^{-1}$, which does not change the form of the SW differential.

As we have now three D4 segments ending on the leftmost NS5 brane (see fig. 4), by [20] construction we expect that a D2 brane ending on that NS5 brane will give rise to a $\mathbb{CP}^2$ 2d sigma model. Indeed at weak 4d gauge coupling $2u \sim a^2 \geq \Lambda^2$ and large $z$ the curve approaches

$$\Lambda^3z + (x - m)(x^2 - a^2) = 0 \quad (4.6)$$

which is the correct curve for a $\mathbb{CP}^2$ 2d sigma model with twisted mass parameters $(m, a, -a)$. We see that the 4d $SU(2)$ gauge group is embedded in the $SU(3)$ flavor symmetry by decomposing the fundamental $3 \rightarrow 2 + 1$. This embedding commutes with a residual $U(1)$ flavor symmetry, associated to the $m$ mass parameter.
Figure 4: A brane realization of a non-minimal surface operator in the pure $SU(2)$ theory.

4.4 Non-minimal surface operators in $SU(2)$ Seiberg-Witten theory: Coupling to an $\mathbb{CP}^n$ sigma model.

The above example can be easily generalized to surface operators defined by a coupling of pure $SU(2)$ gauge theory to $\mathbb{CP}^{n-1}$ sigma models, as long as the $SU(2)$ gauge group is embedded as a $2 \times 2$ block in the $SU(n)$ flavor group.

If we simply carry $n - 2$ D6 branes from left to right, the SW curve takes a form

$$\Lambda^n z^2 + (x^2 - 2u)z \prod_{i=1}^{n-2} (x - m_i) + \Lambda^{4-n} \prod_{i=1}^{n-2} (x - m_i)^2 = 0 \quad (4.7)$$

This is the spectral curve for an $SU(2n - 4)$ Hitchin system. It is curious that the surface operator appears to have $n - 4$ extra vacua besides the ones of the $\mathbb{CP}^{n-1}$ sigma model. It would be interesting to explore the dynamics of this system.

4.5 Non-minimal surface operators in $SU(2)$ Seiberg-Witten theory: Triplet coupling to an $\mathbb{CP}^2$ sigma model.

A second natural way to embed $SU(2)$ in $SU(3)$ is to embed it as $SO(3)$, i.e. to use the triplet representation of $SU(2)$. Hence it should be possible to consider a pure $SU(2)$ gauge theory coupled to a $\mathbb{CP}^2$ sigma model this way. It is not quite obvious how to realize this in a brane setup. As we want $SU(2)$ to act as a triplet, we are tempted to take the Seiberg-Witten curve, written as a spectral curve for an $SU(2)$ Hitchin system, i.e. as a
determinant in the fundamental representation \( \det_{(2)}(x - \phi(z)) = 0 \), and rewrite it as a determinant in the triplet representation

\[
det_{(3)}(x - \phi(z)) = x(x^2 + 4z - 8u + 4z^{-1}) = 4z x + x(x^2 - 8u) + 4z^{-1} x = 0 \tag{4.8}
\]

The Seiberg-Witten differential is the usual \( x \frac{dz}{z} \). As this resembles the curve derived from the set of branes in figure ??, albeit with restrictions on the allowed values of the parameters, let us assume we are allowed to mimick the D6 brane manipulations in the figure, to arrive to a simple proposal

\[
4z + x(x^2 - 8u) + 4z^{-1} x^2 = x^3 + 4z^{-1} x^2 - 8ux + 4z = 0 \tag{4.9}
\]

The large \( z \) behavior of this curve reproduces the expected behavior of the \( \mathbb{CP}^2 \) sigma model. This is the spectral curve for a rather reasonable \( U(3) \) Hitchin system. The constraint on the coefficient is actually rather simple: the curve is symmetric under \( x \rightarrow -x \) and \( z \rightarrow -z \). At this point, we recognize an actual six-dimensional configuration: the \( A_2 \) six dimensional SCFT has a \( \mathbb{Z}_2 \) outer automorphism under which the protected operators of odd degree are odd. This is a compactification of the \( A_2 \) theory on a cylinder of coordinate \( z = z^2 \), with a twist line for the outer automorphism along the cylinder and appropriate boundary conditions at the two ends. Going from M-theory to Type IIA by reduction on the circle in the cylinder, it is known that such a twist line will give rise to an \( SO(3) \) gauge theory. This brane construction could be a way to justify our proposed curve.

### 4.6 Surface operators and flavors: \( N_f = 1 \) \( SU(2) \) Seiberg-Witten theory

The \( N_f = 1 \) \( SU(2) \) Seiberg-Witten theory can be engineered in Type IIA string theory by two D4 brane segments suspended between a pair of NS5 branes, together with an extra semi-infinite D4 brane at either end. The six dimensional engineering involve the compactification of the \( A_1 \) 6d SCFT on a cylinder, with boundary conditions at the ends corresponding to the quadratic differential

\[
\phi_2 = \left( \frac{\Lambda^2}{z^4} + \frac{2m\Lambda}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z} \right) dz^2 \tag{4.10}
\]

Here \( u \) is the Coulomb branch parameter, the scale \( \Lambda \) is related to the UV renormalized gauge coupling \( \Lambda^2 = \exp 2\pi i \tau \), \( m \) is the mass parameter for the single flavor of the theory. The coordinate \( z \) is also the parameter of a minimal surface operator. Again, there are are various interesting ranges of values for \( z \).

The most notable point is the asymmetry between the two ends of the cylinder parameterized by \( z \). At weak coupling, and intermediate values of \( z \), we can again describe the surface operator rather well as a defect. The endpoints of the cylinder corresponds to the values of the monodromy parameter \( \alpha \) at which the monodromy vanishes in the adjoint representation. In the presence of matter in a fundamental representation, the two ends will correspond either to a trivial monodromy or to antiperiodic boundary conditions for the matter hypermultiplet.
When the surface operator moves to either end of the cylinder, in the type IIA description we can describe it as a D2 brane ending on either NS5 branes. At large $z$ the D2 brane only communicates with the two finite D4 brane segments, and the theory at the defect is a $U(1)$ linear sigma model with two chiral multiplets of charge 1, i.e. a $\mathbb{CP}^1$ sigma model. Indeed at large $\Lambda^2 z = e^{2\pi i t}$ we see the $\mathbb{CP}^1$ chiral ring relation with interesting 4d instanton corrections.

$$x^2 = e^{2\pi i t} + 2u + 2m\Lambda^3 e^{-2\pi i t} + \Lambda^6 e^{-4\pi i t} \quad (4.11)$$

On the other hand, at small $z$, the D2 brane interacts with the semi-infinite D4 brane as well, giving rise to a $U(1)$ linear sigma model with two chiral multiplets $q^i$ of charge $-1$ and a single one $\tilde{q}$ of charge 1. This is a sigma model with a non-compact target space $O(-1) \to \mathbb{CP}^1$. One might be troubled by the non-compactness of the target space, but there is an allowed superpotential coupling between the matter hypermultiplet and the 2d sigma model, of the form

$$\tilde{q}Q_i q^i \quad (4.12)$$

This coupling is marginal, but probably not exactly marginal. This superpotential forces the identification between the flavor symmetry of the 4d hypermultiplet and the flavor symmetry acting on $\tilde{q}$ (and of the corresponding mass parameters). Any expectation value for $\tilde{q}$ would source the 4d hypermultiplets. It would be interesting to understand in better detail the effect of this term.

### 4.7 Surface operators and flavors: $N_f = 2$ $SU(2)$ Seiberg-Witten theory

The $N_f = 2$ $SU(2)$ Seiberg-Witten theory can be engineered in Type IIA string theory by two D4 brane segments suspended between a pair of NS5 branes, together with two extra semi-infinite D4 brane at either end. There are two distinct choices: two semi-infinite D4 branes at the same end, or at different ends.

The first choice leads to the $A_1$ 6d SCFT compactified on a cylinder, with boundary conditions at both ends resembling the $N_f = 1$ case:

$$\phi_2 = \left( \frac{\Lambda^2}{z^4} + \frac{2m\Lambda}{z^3} + \frac{2u}{z^2} + \frac{2\Lambda m'}{z} + \Lambda^2 \right) dz^2 \quad (4.13)$$

Here $u$ is the Coulomb branch parameter, the scale $\Lambda$ is related to the UV renormalized gauge coupling $\Lambda^2 = \exp 2\pi i T$, $m$, $m'$ are the mass parameters for the two flavors of the theory. It should be rather clear that the minimal surface operator in this setup must be treating the two flavor hypermultiplets in an asymmetric way. In the defect description, the simplest possibility is that the monodromy of the two fundamental hypers differs by a factor of $-1$. This would break the $SO(4)$ flavor symmetry to $SO(2) \times SO(2)$, which is the flavor symmetry manifest in the six-dimensional construction. At both ends, we see a description in terms of the $O(-1) \to \mathbb{CP}^1$ sigma model.

The second choice leads to a compactification with three punctures, two regular ones and an irregular one.

$$\phi_2 = \left( \frac{m_1^2}{z^2} + \frac{m_2^2}{z(z-1)^2} + \frac{2u}{z(z-1)} + \frac{\Lambda^2}{z} \right) dz^2 \quad (4.14)$$
Each regular puncture carries an $SU(2)$ flavor group, realizing the full $SO(4)$ flavor group of the theory. This indicates that the minimal surface operator in this construction treats the two 4d hypermultiplets symmetrically, and that the defect description involves the same monodromy parameter for the two. Near the irregular puncture the behavior of the quadratic differential is the same as near the ends of the $N_f = 0$ cylinder. The brane construction confirms the description as a simple $\mathbb{C}P^1$ sigma model when the D2 brane is near the NS5 brane without semi-infinite D4 branes.

On the other end of the brane system, as observed in [7], we get a $O(-2) \rightarrow \mathbb{C}P^1$ sigma model, i.e., a conifold sigma model. As discussed in [7], this description of the surface operator worldvolume theory is a bit confusing, as the superpotential coupling only preserves an $U(2)$ subgroup of the $SO(4)$ flavor symmetry. In a sense, the 2d sigma model is a description which is valid near one of the regular punctures, where the surface operator takes the form of a defect for one of the flavor symmetry groups.

### 4.8 Surface operators and flavors: $N_f = 3 \ SU(2)$ Seiberg-Witten theory

If we consider a surface operator which treats the three flavors in a symmetric fashion, we encounter an interesting phenomenon. The IIA brane setup leads to a six dimensional construction involving three M5 branes. In the intermediate region of parameters where the defect description should be appropriate, two of the M5 branes represent the D4 brane segments over which the $SU(2)$ gauge theory lives. The third M5 brane represents one of the NS5 branes, which is bent inwards by the pull of the three semi-infinite D4 branes. In particular, the defect theory must three vacua! Two are the usual perturbative ones, but the third vacuum must be non-perturbative. In the third vacuum, the $SU(2)$ gauge symmetry is somehow restored at the defect. It would be interesting to explore the physical meaning of this fact.

### 4.9 Surface operators and product theories: $SU(2) \times SU(2)$ Seiberg-Witten theory

Finally, we would like to find an example of a surface operator coupling two otherwise decoupled bulk theories. We aim to describe the coupling of an $\mathbb{C}P^3$ sigma model to a pair of pure $SU(2)$ gauge theories in the bulk, embedded in a block-diagonal fashion inside the $SU(4)$ flavor symmetry of the sigma model. The 4d gauge group commutes with a diagonal $U(1)$ flavor symmetry. We will borrow a construction from [9], where orbifold planes were added to the standard D4-NS5 construction in order to engineer quivers of unitary groups in the shape of D-type Dynkin diagrams. The M-theory lift of the orbifold plane leads to M5 branes wrapping a orbifolded Riemann surface C, either a cylinder or a torus. There is a $Z_2$ action $t \rightarrow -t$ (or $z \rightarrow 1/z$), with two fixed points.

In order to engineer a pair of $SU(2)$ groups, we want to use an $SO(4)$ Dynkin diagram, which is reproduced by a single orbifold plane, in the presence of two NS5 branes (and their mirror images), and four D4 brane segments stretched between the furthest NS5 brane and its mirror image. The orbifold plane restricts the D4 branes to break according to a certain pattern (see fig. 6 in the reference) so that one really has two sets of two D4 brane
segments, leading to two decoupled $SU(2)$ theories. The counting of degrees of freedom is quite manifest in the SW curve. One starts with a

$$z^2 + P_4(x)z + Q_4(x) + P_4(x)/z + 1/z^2 = 0 \quad (4.15)$$

but has to impose the extra constraint on the degree 4 polynomials $P_4, Q_4$ that at the fixed points $z = \pm 1$ the roots of $x$ should all be double. Hence we can write $P_4(x) = p_2(x)q_2(x), Q_4(x) = p_2(x)^2 + q_2(x)^2 - 2$, so that at $z = \pm 1$ the equation becomes $(p_2(x) \pm q_2(x))^2 = 0$. Without loss of generality, we can set $\Lambda_1p_2(x) = (x + m)^2 - 2u_1, \Lambda_2p_2(x) = (x - m)^2 - 2u_2$.

We claim that $\Lambda_i, u_i$ are the parameters of the two $SU(2)$ theories, and $m$ is the 2d flavor symmetry mass parameter. We do not know of a simple proof that the periods of this SW curve coincide with linear combinations of the periods of the two $SU(2)$ theories and $m$. We checked this result at the first few orders of a weak coupling power expansion. It would be nice to execute a full check through the appropriate Picard-Lefschetz equations. At large $z$, we see the curve for the $\mathbb{CP}^3$ sigma model.

5. Final remarks

Consider the following setup: pick any choice of 4d theory and any one-parameter surface operator with, say, an $SU(2)$ flavor symmetry. If we couple that $SU(2)$ flavor symmetry to an extra $SU(2)$ Seiberg-Witten theory, something remarkable will happen. The curves $\mathcal{P}, \tilde{\mathcal{P}}$ will be modified in a rather interesting way, to accommodate the periods of the pure $SU(2)$ theory, and the new Hitchin system must have a moduli space which is the fibered product of the old moduli space and the moduli space of the pure $SU(2)$ theory (the product is fibered because the $u$ parameter of the pure $SU(2)$ theory determines the mass parameter of the old theory).

It is quite astonishing that such an universal transformation should always be possible. This hints to the existence of some interesting underlying mathematical structure. One could explore, for example, if the Picard-Lefschetz equations satisfied by the periods are modified in some standard fashion.

In this paper we have not addressed the duality between the instanton partition function of 4d theories and 2d conformal blocks. While we expect the duality to also admit a formulation in terms of surface operators, we do not know how could it be extended to situations where the surface operator parameter space has dimension higher than one. For that to be possible, one would need to formulate some sort of higher dimensional generalization of, say, Liouville theory. It would have to be partly topological, as it should only depend on the complex structure of $\mathcal{P}$.

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A. A review of \(tt^*\)

In this appendix we will review the derivation of the \(tt^*\) equations of [15], and combine it in a useful way with some important results derived in [19]. A slight difference from the standard derivation is that we will make less the use of topological twists. We plan to use \(tt^*\) equations for surface operators in \(\mathcal{N} = 2\) four dimensional theories. It is unclear if a useful topological twist of the combined 4d-2d system exists which reduces to the standard twist of (2, 2) theories in 2d. Donaldson-Witten twist probably does the job, but as the surface operator breaks \(SU(2)_R\) of the bulk theory to an \(U(1)\) subgroup, the 4d manifold has to be Kahler. Instead of checking at each step if the properties of the 2d topological twist can hold for surface operators, we prefer to work directly with the physical theory. In particular, we only consider the physical parameter space of the theory, rather than the topological extended parameter space: we only consider deformations by exactly marginal operators. We also minimize the use of the “topological basis” of Ramond vacua.

The \(tt^*\) equations govern the behavior of the Ramond vacua of a massive 2d theory with (2, 2) SUSY compactified on a circle. As in the main text, we will denote the left moving supercharges as \(Q_L\) and \(\overline{Q}_L\), the right moving supercharges as \(Q_R\) and \(\overline{Q}_R\). \(Q_L\) and \(Q_R\) have opposite \(U(1)\) charge.

We assume the theory has an exactly marginal parameter space \(P\). The theory defined on a line has a finite set of Lorentz invariant vacua, which can be fibered over \(P\) to give a ramified cover \(\hat{P}\). The vacua are parameterized by the expectation values of operators in the twisted chiral ring. A particularly interesting set of twisted chiral operators \(\hat{x}_a\) generates deformations of the parameters \(z^a \rightarrow z^a + \delta z^a\) along \(P\), by adding a the twisted 2d superpotential \(x_a \delta z^a\) to the theory. The deformation parameter \(\delta z^a\) can be taken to be infinitesimal or finite. In the latter case the expression \(z^a + \delta z^a\) should be intended as the change in the parameters due to the finite change in the twisted superpotential. The \(x_a\) may or may not generate the whole twisted chiral ring, so their expectation values may or may not be sufficient to separate the vacua of the theory. In any case, they give a projection of \(\hat{P}\) to a ramified cover of \(P\) inside \(T^*P\).

The Ramond vacua of the theory compactified on a circle \(S^1_\beta\) define a hermitean vector bundle \(V\) on \(P\). Ramond vacua are killed by all supercharges. The \(tt^*\) equations involve certain natural connections on \(V\). In order to define a connection on \(V\), we need to be able to compare vacua of the theory at close but distinct values of the parameters. An example of such a comparison is a path integral on a cylinder geometry, with asymptotic value of the parameters \(z^a\) on one side, \(z^a + \delta z^a\) on the other side, and at the two ends appropriate choices of vacua \(\langle v, z| v', z^a + \delta z^a\rangle\). If we can make sense of the change of \(z\) between the two ends, this defines an inner product

\[
\langle v, z| v', z^a + \delta z^a\rangle \quad \text{(A.1)}
\]

and hence a connection.

To this end we need to understand how to define the theory on a line, with different asymptotic values of the parameters \(z^a\), \(z^a + \delta z^a\). This is a 2d example of a “Janus domain wall”. A naive possibility is to simply add a position dependent twisted superpotential.
This term does the job and is hermitean, but breaks all supersymmetries. At the first order of perturbation theory in \( f, \bar{f} \) the cylinder calculation in this background is sensible, and gives a hermitean connection \( D_a, \bar{D}_a \). Indeed, for a change of the profile \( \delta f \) which goes to zero sufficiently fast the change in the twisted superpotential

\[
\int d\sigma d\tau \delta f(\sigma)\{Q_R, \{Q_L, x\}\} + \delta f(\sigma)\{\bar{Q}_R, \{\bar{Q}_L, \bar{x}\}\}
\]

is a sum of Q-exact terms, which annihilate the Ramond vacua at either ends of the cylinder, without any chance of interesting contact terms. Hence the answer only depends on the value of the parameters at the two ends of the cylinder.

On the other hand, for finite \( f, \bar{f} \), the result depends on the specific profile: if we vary the profile, we add Q-exact terms in a SUSY breaking background. In particular, the hermitean connection \( D_a, \bar{D}_a \) has no reason to be flat. If we turn off the anti-holomorphis \( \bar{f} \) deformation parameter though, the superpotential term preserves both \( Q_R \) and \( Q_L \), and the variations in the profile \( \delta f \) are \( Q_R \) and \( Q_L \) exact, hence the “holomorphic” Janus domain wall gives a well defined transport in holomorphic directions: the \((2,0)\) part of the connection \([D_a, D_b]\) vanishes. The same is true for \([\bar{D}_a, \bar{D}_b]\)

To compute the \((1,1)\) part of the connection, we need to be able to deal with both holomorphic and antiholomorphic deformations. To archive that, we will now define a supersymmetric Janus wall, preserving the interesting combinations \( \zeta Q_L + \bar{Q}_R, \zeta Q_R + \bar{Q}_L \).

\[
\int d\sigma d\tau [f(\sigma)\{Q_R, \{Q_L, x\}\} + \zeta^{-1} \partial_\sigma f x] + [\bar{f}(\sigma)\{\bar{Q}_R, \{\bar{Q}_L, \bar{x}\}\} + \zeta \partial_\sigma \bar{f} \bar{x}]
\]

Indeed if we act with \( Q_L + \zeta \bar{Q}_R \) we get

\[
\int d\sigma d\tau [f(\sigma) \partial_R \{Q_L, x\} + \partial_\sigma f \{Q_L, x\}] + [\bar{f}(\sigma) \zeta \partial_L \{\bar{Q}_R, \bar{x}\} + \zeta \partial_\sigma \bar{f} \{\bar{Q}_R, \bar{x}\}]
\]

which integrates by parts to zero. Furthermore, variations of the profile are \( \zeta Q_L + \bar{Q}_R \) or \( \zeta Q_R + \bar{Q}_L \) exact.

This supersymmetric Janus wrapped on a cylinder defines a flat connection \( \nabla_a, \bar{\nabla}_a \) which depends on the spectral parameter \( \zeta \). This connection can be written in terms of \( D_a, \bar{D}_a \) and of the matrices \( c_a \) which give the action of \( x_a \) on the Ramond vacua of the theory

\[
\nabla_a = D_a + \frac{\beta}{\zeta} c_a \quad \bar{\nabla}_a = \bar{D}_a + \beta \zeta \bar{c}_a
\]

Flatness of this spectral connection implies all the \( tt^* \) equations, which are a sort of multidimensional generalization of the Hitchin system on Riemann surfaces.

\[
[D_a, D_b] = [\bar{D}_a, \bar{D}_b] = [c_a, c_b] = [\bar{c}_a, \bar{c}_b] = 0
\]
\[
D_a c_b = D_b c_a \quad \bar{D}_a c_b = \bar{D}_b \bar{c}_a \quad D_a \bar{c}_b = 0 \quad \bar{D}_a c_b = 0
\]
\[
[D_a, \bar{D}_b] + \beta^2 [c_a, \bar{c}_b] = 0
\]
We can now make contact with [19], where a simple result is proven: the “boundary entropy”, i.e. the pairing between a Ramond vacuum and a boundary state, for some boundary condition preserving $\zeta Q_L + \bar{Q}_R$ and $\zeta Q_R + \bar{Q}_L$, is a flat section for the spectral connection $\nabla_a, \bar{\nabla}_a$. Indeed, the fact that the correction terms in the supersymmetric Janus configuration are proportional to the derivative $\partial_x f$, implies that the Janus configuration has a well defined limit as the profile $f$ becomes a step function: the correction terms take the form of a boundary Lagrangian, which cancels the SUSY variation of the bulk superpotential term. In this limit, the Janus configuration becomes a sharp domain wall, which is the same as a boundary condition for a doubled-up theory, the product of the theory at $z$ and the theory at $z + \delta z$.

The matrix element $\langle v, z|v', z^a + \delta z^a \rangle$ is the boundary entropy for this boundary condition, and indeed is killed both by the left action and by the right action of $\nabla_a, \bar{\nabla}_a$. More generally, given a boundary condition for the theory at some $z$, we can define a boundary condition at $z + \delta z$ by adding the integral of the twisted superpotential in the bulk, and the extra $\int \zeta^{-1} \bar{L} + \zeta \bar{L}$ along the boundary to fix the SUSY transformations. This shows directly that the boundary entropy is a flat section for the spectral connection.

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