A Survey on Automorphism Groups of Finite $p$-Groups

Geir T. Helleloid
Department of Mathematics, Bldg. 380
Stanford University
Stanford, CA 94305-2125
guir@math.stanford.edu

March 30, 2022

Abstract

This survey on the automorphism groups of finite $p$-groups focuses on three major topics: explicit computations for familiar finite $p$-groups, such as the extraspecial $p$-groups and Sylow $p$-subgroups of Chevalley groups; constructing $p$-groups with specified automorphism groups; and the discovery of finite $p$-groups whose automorphism groups are or are not $p$-groups themselves. The material is presented with varying levels of detail, with some of the examples given in complete detail.

1 Introduction

The goal of this survey is to communicate some of what is known about the automorphism groups of finite $p$-groups. The focus is on three topics: explicit computations for familiar finite $p$-groups; constructing $p$-groups with specified automorphism groups; and the discovery of finite $p$-groups whose automorphism groups are or are not $p$-groups themselves. Section 2 begins with some general theorems on automorphisms of finite $p$-groups. Section 3 continues with explicit examples of automorphism groups of finite $p$-groups found in the literature. This includes the computations on the automorphism groups of the extraspecial $p$-groups (by Winter [65]), the Sylow $p$-subgroups of the Chevalley groups (by Gibbs [22] and others), the Sylow $p$-subgroups of the symmetric group (by Bondarchuk [4] and Lentoudis [40]), and some $p$-groups of maximal class and related $p$-groups. Section 4 presents several theorems showing how to prescribe various quotients of the automorphism group of a finite $p$-group. Section 5 focuses on the order of the automorphism groups, concluding with many examples of finite $p$-groups whose automorphism group is a $p$-group. Finally, Section 6 contains some other miscellaneous results on the topic. The material within is presented with varying levels of detail; in particular, some of the more explicit examples
have been given in full detail. Most of the necessary terminology is defined; some of the background material can be found by Huppert [32].

There are aspects of the research on the automorphism groups of finite $p$-groups that are largely omitted in this survey. We mention three here. The first is the conjecture that $|G| \leq |\text{Aut}(G)|$ for all non-cyclic finite $p$-groups $G$ of order at least $p^3$. This has been verified for many families of $p$-groups, and no counter-examples are known; there is an old survey by Davitt [17]. The second is the (large) body of work on finer structural questions, like how the automorphism group of an abelian $p$-group splits or examples of finite $p$-groups whose automorphism group fixes all normal subgroups. The third is the computational aspect of determining the automorphism group of a finite $p$-group. Eick, Leedham-Green, and O’Brien [19] describe an algorithm for constructing the automorphism group of a finite $p$-group. This algorithm has been implemented by Eick and O’Brien in the GAP package AutPGroup [20]. There are references in [19] to other related research as well.

There are a few other survey papers that overlap with this one. Corsi Tani [12] has a survey of examples of finite $p$-groups whose automorphism group is a $p$-group; all these examples are included in the survey along with some others. Starostin [56] and Mann [48] have surveys of questions on finite $p$-groups, each of which includes a section on automorphism groups; Starostin focuses on specific examples related to the $|G| \leq |\text{Aut}(G)|$ conjecture and finer structural questions.

All $p$-groups mentioned in this survey will be finite and $p$ will always denote a prime.

2 General Theorems

Here we summarize some basic theorems about the automorphism group of a finite $p$-group $G$, for the most part following the survey of Mann [48]. First, we can identify two subgroups of $\text{Aut}(G)$ which are themselves $p$-groups. Let $\text{Aut}_c(G)$ be the automorphisms of $G$ which induce the identity automorphism on $G/Z(G)$ (these are called the central automorphisms of $G$), and let $\text{Aut}_f(G)$ be the automorphisms of $G$ which induces the identity automorphism on $G/\Phi(G)$. Then $\text{Aut}_c(G)$ and $\text{Aut}_f(G)$ are $p$-groups. More results on $\text{Aut}_c(G)$ are given by Curran and McCaughan [15].

The next result is a theorem of Gaschütz [21], which states that all finite $p$-groups have outer automorphisms. Furthermore, unless $G \cong C_p$, there is an outer automorphism whose order is a power of $p$. It is an open question of Berkovich as to whether this outer automorphism can be chosen to have order $p$. Schmid [55] extended Gaschütz’ theorem to show that if $G$ is a finite nonabelian $p$-group, then the outer automorphism can be chosen to act trivially on the center. Furthermore, if $G$ is neither elementary abelian nor extra-special, then $\text{Out}(G)$ has a non-trivial normal $p$-subgroup. Webb [61] proved Gaschütz’s theorem and Schmid’s first generalization in a simpler way and without group cohomology. If $G$ is not elementary abelian or extra-special, then Müller [52]
shows that $\text{Aut}_f(G) > \text{Inn}(G)$.

As mentioned in Section 1, one prominent open question is whether or not $|G| \leq |\text{Aut}(G)|$ for all non-cyclic $p$-groups $G$ of order at least $p^3$. A related question concerns the automorphism tower of $G$, namely

$$P_0 = G \to G_1 = \text{Aut}(G) \to G_2 = \text{Aut}(G_1) \to \cdots,$$

where the maps are the natural maps from $G_i$ to $\text{Inn}(G_i)$. For general groups $G$, a theorem of Wielandt shows that if $G$ is centerless, then the automorphism tower of $G$ becomes stationary in a finite number of steps. Little is known about the automorphism tower of finite $p$-groups. In particular, it is not known whether there exist finite $p$-groups $G$ other than $D_8$ with $\text{Aut}(G) \cong G$.

3 The Automorphisms of Familiar $p$-Groups

There are several familiar families of finite $p$-groups whose automorphisms have been described in a reasonably complete manner. The goal of this section to present these results as concretely as possible. We begin with a nearly exact determination of the automorphism groups of the extraspecial $p$-groups. The next subsection discusses the maximal unipotent subgroups of Chevalley groups and Steinberg groups, for which Gibbs [22] describes six types of automorphisms that generate the automorphism group. For type $A_l$, Pavlov [54] and Weir [64] have (essentially) computed the exact structure of the automorphism group. The last three subsections summarize what is known about the automorphism groups of the Sylow $p$-subgroups of the symmetric group, $p$-groups of maximal class, and certain stem covers. We note that Barghi and Ahmedy [6] claim to determine the automorphism group of a class of special $p$-groups constructed by Verardi [59]; unfortunately, as is pointed out in the MathSciNet review of [6], the proofs in this paper are incorrect.

3.1 The Extraspecial $p$-Groups

Winter [65] gives a nearly complete description of the automorphism group of an extraspecial $p$-group. (Griess [25] states many of these results without proof.) Following Winter’s exposition, we will present some basic facts about extraspecial $p$-groups and then describe their automorphisms.

Recall that a finite $p$-group $G$ is special if either $G$ is elementary abelian or $Z(G) = G' = \Phi(G)$. Furthermore, a non-abelian special $p$-group $G$ is extraspecial if $Z(G) = G' = \Phi(G) \cong C_p$. The order of an extraspecial $p$-group is always an odd power of $p$, and there are two isomorphism classes of extraspecial $p$-groups of order $p^{2n+1}$ for each prime $p$ and positive integer $n$, as proved in Gorenstein [24]. When $p = 2$, both isomorphism classes have exponent 4. When $p$ is odd, one of these isomorphism classes has exponent $p$ and the other has exponent $p^2$. 


Any extraspecial $p$-group $G$ of order $p^{2n+1}$ has generators $x_1, x_2, \ldots, x_{2n}$ satisfying the following relations, where $z$ is a fixed generator of $Z(G)$:

$$[x_{2i-1}, x_{2i}] = z \quad \text{for } 1 \leq i \leq n,$$

$$[x_i, x_j] = 1 \quad \text{for } 1 \leq i, j \leq n \text{ and } |i - j| > 1, \text{ and}$$

$$x_i^p \in Z(G) \quad \text{for } 1 \leq i \leq 2n.$$

When $p$ is odd, either $x_i^p = 1$ for $1 \leq i \leq 2n$, in which case $G$ has exponent $p$, or $x_1^p = z$ and $x_i^p = 1$ for $2 \leq i \leq 2n$, in which case $G$ has exponent $p^2$. When $p = 2$, either $x_i^2 = x_i^2 = 1$ for $1 \leq i \leq 2n$, or $x_1^2 = x_2^2 = z$ and $x_i^2 = 1$ for $3 \leq i \leq 2n$.

Recall that if two groups $A$ and $B$ have isomorphic centers $Z(A) \cong Z(B)$, then the central product of $A$ and $B$ is the group

$$(A \times B)/\{(z, z^{-1} \phi) : z \in Z(A)\}.$$

All extraspecial $p$-groups can be written as iterated central products as follows. If $p$ is odd, let $M$ be the extraspecial $p$-group of order $p^3$ and exponent $p$, and let $N$ be the extraspecial $p$-group of order $p^3$ and exponent $p^2$. The extraspecial $p$-group of order $p^{2n+1}$ and exponent $p$ is the central product of $n$ copies of $M$, while the extraspecial $p$-group of order $p^{2n+1}$ and exponent $p^2$ is the central product of $n - 1$ copies of $M$ and one copy of $N$. If $p = 2$, the extraspecial 2-group of order $2^{2n+1}$ and $x_1^2 = x_2^2 = 1$ is isomorphic to the central product of $n$ copies of the dihedral group $D_8$, while the extraspecial 2-group of order $2^{2n+1}$ and $x_1^2 = x_2^2 = z$ is isomorphic to the central product of $n - 1$ copies of $D_8$ and one copy of the quaternion group $Q_8$.

When $G$ has exponent $p$, we can view the group more concretely. The extraspecial $p$-group of order $p^{2n+1}$ and exponent $p$ is isomorphic to the group of $(n + 1) \times (n + 1)$ matrices over $\mathbb{F}_p$ with ones along the diagonal, arbitrary entries in the rest of the first row and the last column, and zeroes elsewhere.

In [65], Winter states the following theorem on the automorphism groups of the extraspecial $p$-groups for all primes $p$.

**Theorem 3.1 (Winter [65]).** Let $G$ be an extraspecial $p$-group of order $p^{2n+1}$. Let $I = \text{Inn}(G)$ and let $H$ be the normal subgroup of $\text{Aut}(G)$ which acts trivially on $Z(G)$. Then

1. $I \cong (C_p)^{2n}$.

2. $\text{Aut}(G) \cong H \rtimes \langle \theta \rangle$, where $\theta$ has order $p - 1$.

3. If $p$ is odd and $G$ has exponent $p$, then $H/I \cong \text{Sp}(2n, \mathbb{F}_p)$, and the order of $H/I$ is $p^{n^2} \prod_{i=1}^{n} (p^{2i} - 1)$.

4. If $p$ is odd and $G$ has exponent $p^2$, then $H/I \cong Q \rtimes \text{Sp}(2n - 2, \mathbb{F}_p)$, where $Q$ is a normal extraspecial $p$-group of order $p^{2n-1}$, and the order of $H/I$ is $p^{n^2} \prod_{i=1}^{n-1} (p^{2i} - 1)$. The group $Q \rtimes \text{Sp}(2n - 2, \mathbb{F}_p)$ is isomorphic to the subgroup of $\text{Sp}(2n, \mathbb{F}_p)$ consisting of elements whose matrix $(a_{ij})$ with respect to a fixed basis satisfies $a_{11} = 1$ and $a_{ii} = 0$ for $i > 1$. 


5. If \( p = 2 \) and \( G \) is isomorphic to the central product of \( n \) copies of \( D_8 \), then \( H/I \) is isomorphic to the orthogonal group of order \( 2^{n(n-1)+1}(2^n-1) \prod_{i=1}^{n-1} (2^{2i} - 1) \) that preserves the quadratic form \( \xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{2n-1}\xi_{2n} \) over \( \mathbb{F}_2 \).

6. If \( p = 2 \) and \( G \) is isomorphic to the central product of \( n-1 \) copies of \( D_8 \) and one copy of \( Q_8 \), then \( H/I \) is isomorphic to the orthogonal group of order \( 2^{n(n-1)+1}(2^n+1) \prod_{i=1}^{n-1} (2^{2i} - 1) \) that preserves the quadratic form \( \xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{2n-1}\xi_{2n} + \xi_{2n}^2 \) over \( \mathbb{F}_2 \).

The automorphisms in \( \text{Aut}(G) \) can be described more explicitly. First, to define the automorphism \( \theta \) in Theorem 3.1, let \( m \) be a primitive root modulo \( p \) with \( 0 < m < p \). Define \( \theta \) by \( \theta(x_{2i-1}) = x_{2i-1}^m \) and \( \theta(x_{2i}) = x_{2i} \) for \( 1 \leq i \leq n \) and by \( \theta(z) = z^m \).

As for \( \text{Inn}(G) \), it is clear that \( \text{Inn}(G) \cong G/Z(G) \) acts trivially on \( Z(G) \) and \( G/Z(G) \) and that \( |\text{Inn}(G)| = p^{2n} \). The elements of \( \text{Inn}(G) \) are given explicitly by the \( p^{2n} \) automorphisms \( \sigma \) where \( \sigma(z) = z^a \) and \( \sigma(x_i) = x_i z^{d_i} \) for each \( i \) and some integers \( 0 \leq d_i < p \).

It remains to describe \( H \). For \( x \in G \), let \( \mathcal{Z} \) denote the coset \( Z(G)x \). Now \( G/Z(G) \) becomes a non-degenerate symplectic space over \( \mathbb{F}_p \) with the symplectic form \( (\mathbf{x}, \mathbf{y}) = a \), where \( [x, y] = a \) and \( 0 \leq a < p \). The symplectic group \( \text{Sp}(2n, \mathbb{F}_p) \) acts on \( G/Z(G) \), preserving the given symplectic form. Let \( T \in \text{Sp}(2n, \mathbb{F}_p) \) and let \( A = (a_{ij}) \) be the matrix of \( T \) relative to the basis \( \{\mathbf{x}_i\} \) (with \( 0 \leq a_{ij} < p \)). Each element \( x \in G \) can be uniquely expressed as \( x = \prod_{i=1}^{2n} x_i^{a_i} \) with \( 0 \leq a_i, c < p \). Define \( \phi : G \rightarrow G \) by

\[
\phi(x) = \left[ \prod_{i=1}^{2n} \left( \prod_{j=1}^{a_i} x_i^{d_{ij}} \right)^{a_i} \right] z^c.
\]

Then \( \phi \) is an automorphism of \( G \) if and only if \( T \) is in the subgroup of \( \text{Sp}(2n, \mathbb{F}_p) \) given in Theorem 3.1.

While Winter's results do give a complete description of the automorphisms in \( H \), we can say a bit more about the structure of \( H \); namely, whether or not \( H \) splits over \( I \). As Griess proves in [26], when \( p = 2 \), \( H \) splits if \( n \leq 2 \) and does not split if \( n \geq 3 \). Griess also states, but does not prove, that when \( p \) is odd, \( H \) always splits over \( I \). This observation is also made in, and can be deduced from, Isaacs [43] and [44] and Glasby and Howlett [24]. A short exposition of this proof when \( p \) is odd and \( G \) has exponent \( p \) was communicated via the grouppub-forum mailing list by Martin Isaacs [35]. Let \( J/I \) be the central involution of the symplectic group \( H/I \), and let \( T \) be a Sylow 2-subgroup of \( J \). Then \( J \) is normal in \( H \), \( |T| = 2 \), and the non-identity element of \( T \) acts on \( I \) by sending each element to its inverse. Then \( I = C_I(T) = I \cap N_H(I) \). On the other hand, by the Frattini argument, \( H = JN_H(T) \), and since \( T \leq J \cap N_H(T) \), it follows that \( H = IN_H(T) \). But this means that \( N_H(T) \) is a complement of \( I \) in \( H \), and so \( H \) splits over \( I \).

According to Griess, the proof when \( G \) has exponent \( p^2 \) is more technical.
| Type | Chevalley Group |
|------|-----------------|
| $A_l$ | $\text{PSL}_{l+1}(\mathbb{F}_q)$ |
| $B_l$ | $\text{P}(\text{O}_{2l+1}^+(\mathbb{F}_q))$ |
| $C_l$ | $\text{PSp}_{2l}(\mathbb{F}_q)$ |
| $D_l$ | $\text{P}(\text{O}_{2l}^-(\mathbb{F}_q))$ |

Table 1: The Chevalley groups of types $A_l$, $B_l$, $C_l$ and $D_l$.

3.2 The Maximal Unipotent Subgroups of a Chevalley Group

Associated to any simple Lie algebra $\mathcal{L}$ over $\mathbb{C}$ and any field $K$ is the Chevalley group $G$ of type $\mathcal{L}$ over $K$. Table 1 lists the Chevalley groups of types $A_l$, $B_l$, $C_l$, and $D_l$ over the finite field $\mathbb{F}_q$, as given in Carter [11]. A few clarifications are necessary: the entry for type $B_l$ requires that $\mathbb{F}_q$ have odd characteristic; $\text{O}_{2l+1}(\mathbb{F}_q)$ is the orthogonal group which leaves the quadratic form $\xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{2l+1}^2$ invariant over $\mathbb{F}_q$; and $\text{O}_{2l}(\mathbb{F}_q)$ is the orthogonal group which leaves the quadratic form $\xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{2l-1}\xi_{2l}$ invariant over $\mathbb{F}_q$.

Gibbs [22] examines the automorphisms of a maximal unipotent subgroup of a Chevalley group over a field of characteristic not two or three. We are only interested in finite groups, so from now on we will let $K = \mathbb{F}_q$, where $\mathbb{F}_q$ has characteristic $p > 3$ and $q = p^n$. After some preliminaries on maximal unipotent subgroups, we will present his results.

Let $\Sigma$, $\Sigma^+$, and $\pi$ denote the sets of roots, positive roots, and fundamental roots, respectively, of $\mathcal{L}$ relative to some Cartan subalgebra. Then the Chevalley group $G$ is generated by $\{x_r(t) : r \in \Sigma, t \in \mathbb{F}_q\}$. One maximal unipotent subgroup $U$ of $G$ is constructed as follows. As a set,

$$U = \{x_r(t) : r \in \Sigma^+, t \in \mathbb{F}_q\}.$$ 

For any $r, s \in \Sigma^+$ and $t, u \in \mathbb{F}_q$, the multiplication in $U$ is given by

$$x_r(t)x_r(u) = x_r(t + u)$$

$$[x_s(u), x_r(t)] = \left\{ \prod_{ir+js \in \Sigma} x_{ir+js}(C_{ij,rs}(-t)^iu^j) : \begin{array}{ll} 1 & : r + s \text{ is not a root} \\ \prod x_{ir+js}(C_{ij,rs}(-t)^iu^j) & : r + s \text{ is a root.} \end{array} \right.$$ 

Here $i$ and $j$ are positive integers and $C_{ij,rs}$ are certain integers which depend on $\mathcal{L}$. The order of $U$ is $q^N$, where $N = |\Sigma^+|$, and $U$ is a Sylow $p$-subgroup of $G$.

Gibbs [22] shows that $\text{Aut}(G)$ is generated by six types of automorphism, namely graph automorphisms, diagonal automorphisms, field automorphisms, central automorphisms, extremal automorphisms, and inner automorphisms. Let the subgroup of $\text{Aut}(G)$ generated by each type of automorphism be denoted by $P$, $D$, $F$, $C$, $E$, and $I$ respectively. Let $P_r$ be the additive group generated
by the roots of \( \mathcal{L} \) and let \( r_N \) be the highest root. Label the fundamental roots \( r_1, r_2, \ldots, r_l \).

1. **Graph Automorphisms:** An automorphism \( \sigma \) of \( P_r \) that permutes both \( \pi \) and \( \Sigma \) induces a graph automorphism of \( U \) by sending \( x_r(t) \) to \( x_r(t) \) for all \( r \in \Sigma^+ \) and \( t \in \mathbb{F}_q \). Graph automorphisms correspond to automorphisms of the Dynkin diagram, and so types \( A_l \) (\( l > 1 \)), \( D_l \) (\( l > 4 \)), and \( E_6 \) have a graph automorphism of order 2, while the graph automorphisms in type \( D_4 \) form a group isomorphic to \( S_3 \).

2. **Diagonal Automorphisms:** Every character \( \chi \) of \( P_r \) with values in \( \mathbb{F}_q^* \) induces a diagonal automorphism which maps \( x_r(t) \) to \( x_r(\chi(r)t) \) for all \( r \in \Sigma^+ \) and \( t \in \mathbb{F}_q \).

3. **Field Automorphisms:** Every automorphism \( \tau \) of \( \mathbb{F}_q^* \) induces a field automorphism of \( U \) which maps \( x_r(t) \) to \( x_r(\sigma(t)) \) for all \( r \in \Sigma^+ \) and \( t \in \mathbb{F}_q \).

4. **Central Automorphisms:** Let \( \tau_i \) be endomorphisms of \( \mathbb{F}_q^* \). These induce a central automorphism that maps \( x_r(t) \) to \( x_r(t)x_{r_N}(\sigma_i(t)) \) for \( i = 1, \ldots, l \) and all \( t \in \mathbb{F}_q \).

5. **Extremal Automorphisms:** Suppose \( r_j \) is a fundamental root such that \( r_N - r_j \) is also a root. Let \( u \in \mathbb{F}_q^* \). This determines an extremal automorphism which acts trivially on \( x_r(t) \) for \( i \neq j \) and sends \( x_{r_j}(t) \) to

\[
x_{r_j}(t)x_{r_N-r_j}(ut)x_{r_N}((1/2)N_{r_N-r_j}r_ju^2).
\]

Here, \( N_{r_N-r_j} \) is a certain constant that depends on the type. In type \( C_l \), \( r_N - 2r_j \) is also a root, and the map that acts trivially on \( x_i(t) \) for \( i \neq j \) and sends \( x_{r_j}(t) \) to

\[
x_{r_j}(t)x_{r_N-2r_j}(ut)x_{r_N-r_j}((1/2)N_{r_N-2r_j}r_ju^2)x_{r_N}((1/3)C_{12,r_N-r_j}r_ju^3)
\]

is also an automorphism of \( U \).

Steinberg [57] showed that the automorphism group of a Chevalley group of a finite field is generated by graph, diagonal, field, and inner automorphisms, which shows that \( P \), \( D \), and \( F \) are, in fact, subgroups of Aut(\( U \)). It is easy to see that the central automorphisms are automorphisms, and a quick computation verifies this for the extremal automorphisms as well. Note that multiplying an extremal automorphism by a judicious choice of central automorphism, the \( x_{r_N}(\cdot) \) term in the description of the extremal automorphisms disappears. Therefore, a functionally equivalent definition of extremal automorphisms omits the \( x_{r_N}(\cdot) \) term, and this is what we will use for what follows.

Gibbs does not compute the precise structure of Aut(\( U \)). This has been done in type \( A_l \) however; Pavlov [54] computes Aut(\( U \)) over \( \mathbb{F}_p \), while Weir [64] computes it over \( \mathbb{F}_q \) (although his computations contain a mistake which we will address in a moment). We will present the result for type \( A_l \) as explicitly as
possible, pausing to note that it does seem feasible to compute the structure of \( \text{Aut}(U) \) for other types in a similar manner.

As mentioned before, in type \( A_l \), we can view \( U \) as the set of \( (l+1) \times (l+1) \) upper triangular matrices with ones on the diagonal and arbitrary entries from \( \mathbb{F}_q \) above the diagonal. There are \( \binom{l+1}{2} \) positive roots in type \( A_l \), given by \( r_i + r_{i+1} + \cdots + r_j \) for \( 1 \leq i \leq j \leq l \). Let \( E_{i,j} \) be the \( (l+1) \times (l+1) \) matrix with a 1 in the \((i,j)\)-entry and zeroes elsewhere. Then \( x_r(t) = I + tE_{i,j} \), where \( r = r_i + r_{i+1} + \cdots + r_j \). In particular, \( x_r(t) = I + tE_{i,i+1} \). In type \( A_l \), some of the given types of automorphisms admit simpler descriptions; we will be content to describe their action on the elements \( x_r(t) \).

As mentioned, in type \( A_l \) there is one nontrivial graph automorphism of order 2, and it acts by reflecting all matrices in \( U \) across the anti-diagonal. The diagonal automorphisms correspond to selecting \( \chi_1, \ldots, \chi_n \in \mathbb{F}_q^* \) and mapping \( x_r(t) \) to \( x_r(\chi_i t) \). This is equivalent to conjugation by a diagonal matrix of determinant 1. The diagonal automorphisms form an elementary abelian subgroup of order \( (p-1)^n \). The field automorphisms of \( \mathbb{F}_q \) are generated by the Frobenius automorphism and form a cyclic subgroup of order \( n \).

Let \( a_1, \ldots, a_n \) generate the additive group of \( \mathbb{F}_q \). Then \( x_r(a_j) \) generate \( U \). The central automorphisms are generated by the automorphisms \( x_r(a_m) \) which send \( x_r(a_m) \) to \( x_r(a_m)x_{r_N}(1) \) and fix \( x_r(a_k) \) for \( i \neq j \) and \( k \neq m \), where \( j = 2, \ldots, l-1 \) and \( m = 1, \ldots, n \). (When \( j = 1 \) or \( j = l \), this automorphism is inner.) The extremal automorphisms are generated by the automorphism that sends \( x_r(t) \) to \( x_r(t)x_{r_N-r_i}(1) \) and fixes \( x_r(t) \) for \( 2 \leq i \leq l \) and the automorphism that sends \( x_r(t) \) to \( x_r(t)x_{r_N-r_i}(t) \) and fixes \( x_r(t) \) for \( 1 \leq i \leq l-1 \). Finally the inner automorphism group is, of course, isomorphic to \( U/Z(U) \) (the center of \( U \) is generated by \( x_{r_N}(t) \)).

It is not hard to use these descriptions to deduce that

\[
\text{Aut}(U) \cong ((I \rtimes (E \times C)) \rtimes (D \rtimes F)) \rtimes P.
\]

Furthermore \( E \times C \) is elementary abelian of order \( q^{n(n-2)/2} \), \( D \) is elementary abelian of order \( (q-1)^l \), \( F \) is cyclic of order \( n \), and \( I \) has order \( q^{(l^2+1)/2} \). It follows that the order of \( \text{Aut}(U) \) is

\[
2n(q-1)^l q^{(l^2+1)/2 + 2n/2}.
\]

The error in Weir’s paper stems from his claim that any \( g \in \text{GL}_n(\mathbb{F}_q) \) acting on \( \mathbb{F}_q \) induces an automorphism of \( U \) that maps \( x_r(a_k) \) to \( x_r(g(a_k)) \), generalizing the field automorphisms. However, it is clear that \( g \) must, in fact, be a field automorphism, as for any \( t, u \in \mathbb{F}_q \),

\[
[x_r(t), x_r(u)] = x_{r_1+r_2(tu)} = [x_r(tu), x_r(1)].
\]

Applying \( g \) to all terms shows that \( g(tu) = g(t)g(u) \).

3.3 Sylow \( p \)-Subgroups of the Symmetric Group

The automorphism groups of Sylow \( p \)-subgroups of the symmetric group for \( p > 2 \) were examined independently by Bondarchuk and Lentoudis.
Their results are reasonably technical. They do show that the order of the automorphism group of the Sylow $p$-subgroup of $S_{p^m}$, which is isomorphic to the $m$-fold iterated wreath product of $C_p$, is

$$(p - 1)^m p^{n(m)},$$

where

$$n(m) = p^{m - 1} + p^{m - 2} + \cdots + p^2 + \frac{1}{2}(m^2 - m + 2)p - 1.$$  

### 3.4 $p$-Groups of Maximal Class

A $p$-group of order $p^n$ is of maximal class if it has nilpotence class $n - 1$. It is not too hard to prove some basic results about the automorphism group of an arbitrary $p$-group of maximal class. Our presentation follows Baartmans and Woeppe J Section 1].

**Theorem 3.2.** Let $G$ be a $p$-group of maximal class of order $p^n$, where $n \geq 4$ and $p$ is odd. Then $\text{Aut}(G)$ has a normal Sylow $p$-subgroup $P$ and $P$ has a $p'$-complement $H$, so that $\text{Aut}(G) \cong H \rtimes P$. Furthermore, $H$ is isomorphic to a subgroup of $C_{p - 1} \times C_{p - 1}$.

The proof of this theorem begins by observing that $G$ has a characteristic cyclic series $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = 1$; that is, each $G_i$ is characteristic and $G_i / G_{i+1}$ is cyclic (see Huppert [32] Lemmas 14.2 and 14.4]). By a result of Durbin and McDonald [18], $\text{Aut}(G)$ is supersolvable and so has a normal Sylow $p$-subgroup $P$ with $p'$-complement $H$, and the exponent of $\text{Aut}(G)$ divides $p^t(p - 1)$ for some $t > 0$. The additional result about the structure of $H$ comes from examining the actions of $H$ on the characteristic cyclic series and on $G / \Phi(G)$. Baartmans and Woeppe J remark that the above theorem holds for any finite $p$-group $G$ with a characteristic cyclic series.

Baartmans and Woeppe J follow up these general results by focusing on automorphisms of $p$-groups of maximal class of exponent $p$ with a maximal subgroup which is abelian. More specifically, the characteristic cyclic series can be taken to be a composition series, in which case $G_i = \gamma_i(G)$ for $i \geq 2$ and $G_1 = C_G(G_2 / G_1)$, where $\gamma_i(G)$ is the $i$-th term in the lower central series of $G$. Baartmans and Woeppe J assume that $G_2$ is abelian.

In this case, they show by construction that $H \cong C_{p - 1} \times C_{p - 1}$. Furthermore, $P$ is metabelian of nilpotence class $n - 2$ and order $p^{2n - 3}$. (Recall that a *metabelian group* is a group whose commutator subgroup is abelian.) The subgroup $\text{Inn}(G) \cong G / Z(G)$ has order $p^{n-1}$ and maximal class $n - 2$. The commutator subgroup $P'$ is the subgroup of $\text{Inn}(G)$ induced by $G_2$. Baartmans and Woeppe J do explicitly describe the automorphisms of $G$, but the descriptions are too complicated to include here.

Other authors who investigate automorphisms of certain finite $p$-groups of maximal class include: Abbasi [1]; Miech [40], who focuses on metabelian groups of maximal class; and Wolf [66], who looks at the centralizer of $Z(G)$ in certain subgroups of $\text{Aut}(G)$. 


Finally, in [35], Juhász considers more general $p$-groups than $p$-groups of maximal class. Specifically, he looks at $p$-groups $G$ of nilpotence class $n - 1$ in which $\gamma_1(G)/\gamma_2(G) \cong C_{p^m} \times C_{p^m}$ and $\gamma_i(G)/\gamma_{i+1}(G) \cong C_{p^n}$ for $2 \leq i \leq n - 1$. He refers to such groups as being of type $(n, m)$. Groups of type $(n, 1)$ are the $p$-groups of maximal class of order $p^n$.

Assume that $n \geq 4$ and $p > 2$. As with groups of maximal class, the automorphism group of a group $G$ of type $(n, m)$ is a semi-direct product of a normal Sylow $p$-subgroup $P$ and its $p'$-complement $H$, and $H$ is isomorphic to a subgroup of $C_{p^{m}}$ for $2 \leq i \leq n - 1$. Juhász' results are largely technical, dealing with the structure of $P$, especially when $G$ is metabelian.

3.5 Stem Covers of an Elementary Abelian $p$-Group

In [63], Webb looks at the automorphism groups of stem covers of elementary abelian $p$-groups. We start with some preliminaries on stem covers. A group $G$ is a central extension of $Q$ by $N$ if $N$ is a normal subgroup of $G$ lying in $Z(G)$ and $G/N \cong Q$. If $N$ lies in $[G, G]$ as well, then $G$ is a stem extension of $Q$. The Schur multiplier $M(Q)$ of $Q$ is defined as the second cohomology group $H^2(Q, C^*)$, and it turns out that $N$ is isomorphic to a subgroup of $M(Q)$. Alternatively, $M(Q)$ can be defined as the maximum group $N$ so that there exists a stem extension of $Q$ by $N$. Such a stem extension is called a stem cover.

Webb takes $Q$ to be elementary abelian of order $p^n$ with $p$ odd and $n \geq 2$. Let $G$ be a stem cover of $Q$. Then $N = Z(G) = [G, G] \cong M(Q) \cong Q \wedge Q$ and has order $p_{(2)}$. Therefore $\text{Aut}_c(G)$ are the automorphisms of $G$ which act trivially on $G/N \cong Q$. Each automorphism $\alpha \in \text{Aut}_c(G)$ corresponds uniquely to a homomorphism $\overline{\alpha} \in \text{Hom}(Q, N)$ via the relationship $gN\overline{\alpha} = g^{-1} \cdot g\alpha$ for all $g \in G$. Of course, $\text{Hom}(Q, N)$ is an elementary abelian $p$-group of order $n\begin{pmatrix} n \\ 2 \end{pmatrix}$, and so $\text{Aut}(G)$ is an extension of a subgroup of $\text{Aut}(Q) \cong \text{GL}(n, F_p)$ by an elementary abelian $p$-group of order $n\begin{pmatrix} n \\ 2 \end{pmatrix}$. Webb proves that the subgroup of $\text{Aut}(Q)$ in question is usually trivial, leading to her main theorem.

**Theorem 3.3 (Webb [63]).** Let $G$ be elementary abelian of order $p^n$ with $p$ odd. As $n \to \infty$, the proportion of stem covers of $G$ with elementary abelian automorphism group of order $p_{(2)}$ tends to 1.

4 Quotients of Automorphism Groups

Not every finite $p$-group is the automorphism group of a finite $p$-group. A recent paper in this vein is by Cutolo, Smith, and Wiegold [16], who show that the only $p$-group of maximal class which is the automorphism group of a finite $p$-group is $D_8$. But there are several extant results which show that certain quotients of the automorphism group can be arbitrary.
4.1 The Central Quotient of the Automorphism Group

Theorem 4.1 (Heineken and Liebeck [27]). Let $K$ be a finite group and let $p$ be an odd prime. There exists a finite $p$-group $G$ of class 2 and exponent $p^2$ such that $\text{Aut}(G)/\text{Aut}_c(G) \cong K$.

The construction given by Heineken and Liebeck can be described rather easily. Let $K$ be a group on $d$ generators $x_1, x_2, \ldots, x_d$. Let $D^t(K)$ be the directed Cayley graph of $K$ relative to the given generators. Form a new digraph $D(K)$ by replacing every arc in $D^t(K)$ by a directed path of length $i$ if the original arc corresponded to the generator $x_i$. Then $\text{Aut}(D(K)) = K$.

Let $v_1, v_2, \ldots, v_m$ be the vertices of $D(K)$. Let $G$ be the $p$-group generated by elements $v_1, v_2, \ldots, v_m$ where

1. $G'$ is the elementary abelian $p$-group freely generated by
   \[ \{ [v_i, v_j] : 1 \leq i < j \leq m \}. \]

2. For each vertex $v_i$, if $v_i$ has outgoing arcs to $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$, then
   \[ v_i^G = [v_i, v_{i_1} \cdots v_{i_k}] \]

Heineken and Liebeck show that $\text{Aut}(G)/\text{Aut}_c(G) \cong K$ when $|K| \geq 5$. (They give a special construction for $|K| < 5$.) As Webb [63] notes, $G$ is a special $p$-group.

They are actually able to determine the automorphism group of $G$ much more precisely, at least when $|K| \geq 5$. Let $S$ be the set of vertices of $D(K)$ consisting of the group-point $e$ corresponding to the identity of $K$ and all vertices that can be reached along a directed path from $e$ that does not pass through any other group-points. Assume that the vertices of $D(K)$ are labeled so that $v_1, \ldots, v_h$ are the elements of $S$. The central automorphisms which fix $v_{s+1}, v_{s+2}, \ldots, v_m$ generate an elementary abelian $p$-group $U$ of rank $s|G'| = (1/2)ks^2(ks - 1)$. Every central automorphism of $G$ is of the form $\prod_{v \in K} v^{-1}a_v$, where the elements $a_v \in U$ and $v \in K$ are uniquely determined. Thus $\text{Aut}_c(G)$ is the direct product of the conjugates of $U$ in $\text{Aut}(G)$ and $\text{Aut}(G) = U \wr K$. It follows that $\text{Aut}(G)$ has order $kp^l$, where $l = (1/2)k^2s^2(ks - 1)$ and $s = (1/2)d(d + 1) + 1$ when $d \geq 2$ and $s = 1$ when $d = 1$.

Lawton [69] modified Heineken and Liebeck’s techniques to construct smaller groups $G$ with $\text{Aut}(G)/\text{Aut}_c(G) \cong K$. He uses undirected graphs which are much smaller, and the $p$-group $G$ which he defines is significantly simpler.

Webb [62] uses similar, though more complicated techniques, to obtain further results. She defines a class of graphs called $Z$-graphs; it turns out that almost all finite graphs are $Z$-graphs (that is, the proportion of graphs on $n$ vertices which are $Z$-graphs goes to 1 as $n$ goes to infinity). To each $Z$-graph $\Lambda$, Webb associates a special $p$-group $G$ for which $\text{Aut}(G)/\text{Aut}_c(G) \cong \text{Aut}(\Lambda)$. The set of all special $p$-groups that arise from $Z$-graphs on $n$ vertices is denoted by $G(p, n)$. 

11
Theorem 4.2 (Webb). Let \( p \) be any prime. Then almost all of the groups in \( \mathcal{G}(p, n) \) (as \( n \to \infty \)) have automorphism group \((C_p)^r\), where \( r = n^2(n-1)/2 \).

The reason the group \((C_p)^r\) arises as the automorphism group is that for \( G \in \mathcal{G}(p, n) \), \( \text{Aut}_c(G) \) is isomorphic to \( \text{Hom}(G/Z(G), Z(G)) \), and hence to \((C_p)^r\) (for essentially the same reason as in Subsection 3.3). Webb then shows that \( \text{Aut}(G)/\text{Aut}_c(G) \) is usually trivial.

Theorem 4.3 (Webb). Let \( K \) be a finite group which is not cyclic of order at most five. Then for any prime \( p \), there is a special \( p \)-group \( G \in \mathcal{G}(p, 2|K|) \) with \( \text{Aut}(G)/\text{Aut}_c(G) \sim K \).

In particular, Theorem 4.3 extends Heineken and Liebeck’s result to the case \( p = 2 \). Note that in Theorems 4.1 and 4.3 the constructed groups are special and \( \text{Aut}_c(G) = \text{Aut}_f(G) \), so that these theorems also prescribe \( \text{Aut}(G)/\text{Aut}_f(G) \).

The \( p = 2 \) analogue of Heineken and Liebeck’s result was discussed by Hughes [31].

4.2 The Quotient \( \text{Aut}(G)/\text{Aut}_f(G) \)

Bryant and Kovács [9] look at prescribing the quotient \( \text{Aut}(G)/\text{Aut}_f(G) \), taking a different approach from that of Heineken and Liebeck in that they assign \( \text{Aut}(G)/\text{Aut}_f(G) \) as a linear group (and they do not bound the class of \( G \)).

Theorem 4.4 (Bryant and Kovács [9]). Let \( p \) be any prime. Let \( K \) be a finite group with dimension \( d \geq 2 \) as a linear group over \( \mathbb{F}_p \). Then there exists a finite \((d\text{-generator})\ p\text{-group} \ G \) such that \( \text{Aut}(G)/\text{Aut}_f(G) \sim K \).

This theorem is (essentially) non-constructive, in contrast to the results of Heineken and Liebeck. To understand the main idea, let \( F \) be the free group on \( d \) generators and let \( F_n \) be the \( n \)-th term in the Frattini series of \( F \). There is an action of \( \text{GL}(d, \mathbb{F}_p) \) on \( F_n/F_{n+1} \). If \( U \) is a normal subgroup of \( F \) with \( F_{n+1} \leq U \leq F_n \), then \( G = F/U \) is a finite \( p \)-group and \( \text{Aut}(G)/\text{Aut}_f(G) \) is isomorphic to the normalizer of \( U \) in \( \text{GL}(d, \mathbb{F}_p) \) (see [28] Theorems 2.7 and 2.8 for more details). Bryant and Kovács show that if \( n \) is large enough, then \( F_n/F_{n+1} \) contains a regular \( \mathbb{F}_p\text{GL}(d, \mathbb{F}_p) \)-module, which shows that any subgroup \( K \) of \( \text{GL}(d, \mathbb{F}_p) \) occurs as the normalizer of some normal subgroup \( U \) of \( F \) with \( F_n \leq U \leq F_{n+1} \).

5 Orders of Automorphism Groups

The first two subsections describe some general theorems about the orders of automorphism groups of finite \( p \)-groups. The third subsection gives the order of the automorphism group of an abelian \( p \)-group, and the last subsection offers many explicit examples of \( p \)-groups whose automorphism group is a \( p \)-group. Helleloid and Martin [28] have proved that, in several asymptotic senses, the automorphism group of a finite \( p \)-group is almost always a \( p \)-group. However, a question raised in Mann [13] remains unanswered:
Question. Fix a prime $p$. Let $v_n$ be the proportion of $p$-groups with order at most $p^n$ whose automorphism group is a $p$-group. Is it true that $\lim_{n \to \infty} v_n = 1$?

5.1 Nilpotent Automorphism Groups

In [67], Ying states two results about the occurrence of automorphism groups of $p$-groups which are $p$-groups, the second being a generalization of a result of Heineken and Liebeck [26].

**Theorem 5.1.** If $G$ is a finite $p$-group and $\text{Aut}(G)$ is nilpotent, then either $G$ is cyclic or $\text{Aut}(G)$ is a $p$-group.

**Theorem 5.2.** Let $p$ be an odd prime and let $G$ be a finite two-generator $p$-group with cyclic commutator subgroup. Then $\text{Aut}(G)$ is not a $p$-group if and only if $G$ is the semi-direct product of an abelian subgroup by a cyclic subgroup.

Heineken and Liebeck [26] also have a criterion which determines whether or not a two-generator $p$-group of class 2 has an automorphism of order 2 or if the automorphism group is a $p$-group. If $p$ is an odd prime and $G$ is a $p$-group that admits an automorphism which inverts some non-trivial element of $G$, then $G$ is an s.i. group (a some-inversion group). Clearly if $G$ is an s.i. group, it has an automorphism of order 2. If $G$ is not an s.i. group, it is called an n.i. group (a no-inversion group).

**Theorem 5.3.** Let $p$ be an odd prime and let $G$ be a two-generator $p$-group of class 2. Choose generators $x$ and $y$ such that

$$\langle x, G' \rangle \cap \langle y, G' \rangle = G',$$

and suppose that

$$\langle x \rangle \cap G' = \langle x^{p^m} \rangle, \quad \langle y \rangle \cap G' = \langle y^{p^n} \rangle.$$

1. If either $x^{p^m} = 1$ or $y^{p^n} = 1$, then $G$ is an s.i. group.

2. If $x^{p^m} = [x, y]^{p^k} \neq 1$ and $y^{p^n} = [x, y]^{p^l} \neq 1$ with $(r, p) = (s, p) = 1$, and $(n - l + k - m)(k - l)$ is non-negative, then $G$ is an s.i. group.

3. If $m, n, k,$ and $l$ are defined as in (2) and $(n - l + k - m)(k - l)$ is negative, then $G$ is an n.i. group and its automorphism group is a $p$-group.

5.2 Wreath Products

In [30], Horoševskii gives the following two theorems on the order of the automorphism group of a wreath product.

**Theorem 5.4.** Let $A$ and $B$ be non-identity finite groups, and let $A_1$ be a maximal abelian subgroup of $A$ which can be distinguished as a direct factor of $A$. Then

$$\pi(\text{Aut}(A \wr B)) = \pi(A) \cup \pi(B) \cup \pi(\text{Aut}(A)) \cup \pi(\text{Aut}(B)) \cup \pi(\text{Aut}(A_1 \wr B)).$$
Theorem 5.5. Let $P_1, P_2, \ldots, P_m$ be non-identity finite $p$-groups. Then
\[ \pi(\text{Aut}(P_1 \wr P_2 \wr \cdots \wr P_m)) = \bigcup_{i=1}^{m} \pi(\text{Aut}(P_i)) \cup \{p\}. \]

Thus given any finite $p$-groups whose automorphism group is a $p$-group, we can construct infinitely many more by taking iterated wreath products.

5.3 The Automorphism Group of an Abelian $p$-Group

Macdonald [45, Chapter II, Theorem 1.6] calculates the order of the automorphism group of an abelian $p$-group. The literature does contain some more technical results on the structure of such an automorphism group.

Theorem 5.6. Let $G$ be an abelian $p$-group of type $\lambda$. Then
\[ |\text{Aut}(G)| = p^{|\lambda|+2n(\lambda)} \prod_{i \geq 1} \phi_{m_i(\lambda)}(p^{-1}), \]
where $m_i(\lambda)$ is the number of parts of $\lambda$ equal to $i$, $n(\lambda) = \sum_{i \geq 1} \binom{\lambda}{2}$, and $\phi_m(t) = (1-t)(1-t^2) \cdots (1-t^m)$.

5.4 Miscellaneous $p$-Groups Whose Automorphism Group is a $p$-Group

In this subsection, we collect constructions of finite $p$-groups whose automorphism groups are $p$-groups.

The first example of a finite $p$-group whose automorphism group is a $p$-group was given by Miller [51], who constructed a non-abelian group of order 64 with an abelian automorphism group of order 128. Generalized Miller’s construction, Struik [58] gave the following infinite family of 2-groups whose automorphism groups are abelian 2-groups:

\[
G = \langle a, b, c, d : a^{2^n} = b^2 = c^2 = d^2 = 1, [a, c] = [a, d] = [b, c] = [c, d] = 1, bab = a^{2^{n-1}}, bdb = cd \rangle,
\]
where $n \geq 3$. ($G$ can be expressed as a semi-direct product as well.) Struik shows that $\text{Aut}(G) \cong (C_2)^6 \times C_{2^{n-2}}$. (As noted in [58], it turns out that revision problem #46 on p. 237 of Macdonald [46] asks the reader to show that $\text{Aut}(G)$ is an abelian 2-group.) Also, Jamali [36] has constructed, for $m \geq 2$ and $n \geq 3$, a non-abelian $n$-generator group of order $2^{2m+n-2}$ with exponent $2^m$ and abelian automorphism group $(C_2)^n \times C_{2^{m-2}}$.

More examples of 2-groups whose automorphism groups are 2-groups are given by Newman and O’Brien [53]. As an outgrowth of their computations on 2-groups of order dividing 128, they present (without proof) three infinite families of 2-groups for which $|G| = |\text{Aut}(G)|$. They are, for $n \geq 3$,
1. $C_{2^{n-1}} \times C_2$,

2. $\langle a, b : a^{2^{n-1}} = b^2 = 1, a^b = a^{1+2^{n-2}} \rangle$, and

3. $\langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [b, a] = 1, a^c = a^{1+2^{n-4}}, b^c = ba^{2^{n-3}} \rangle$.

Moving on to finite $p$-groups where $p$ is odd, for each $n \geq 2$ Horoševski [29] constructs a $p$-group with nilpotence class $n$ whose automorphism group is a $p$-group, and for each $d \geq 3$ he constructs a $p$-group on $d$ generators for each $d \geq 3$ whose automorphism group is a $p$-group. (He gives explicit presentations for these groups.)

Curran [13] shows that if $(p - 1, 3) = 1$, then there is exactly one group of order $p^5$ whose automorphism group is a $p$-group (and it has order $p^5$). It has the following presentation:

$$G = \langle a, b : b^p = [a, b]^p = [a, b, b]^p = [a, b, b, b]^p = [a, b, b, b, b]^p = 1, a^p = [a, b, b, b, b] = [b, a, b]^{-1} \rangle.$$  

When $(p - 1, 3) = 3$, there are no groups of order $p^5$ whose automorphism group is a $p$-group. However, in this case, there are three groups of order $p^5$ which have no automorphisms of order 2. Curran also shows that $p^6$ is the smallest order of a $p$-group which can occur as an automorphism group (when $p$ is odd).

Then, in [14], Curran constructs 3-groups $G$ of order $3^n$ with $n \geq 6$ where $|\text{Aut}(G)| = 3^{n+3}$ and $p$-groups $G$ for certain primes $p > 3$ with $|\text{Aut}(G)| = p|G|$. The MathSciNet review of [14] remarks that F. Menegazzo notes that for odd $p$ and $n \geq 3$, the automorphism group of

$$G = \langle a, b : a^{p^n} = 1, b^{p^n} = a^{p^{n-1}}, a^b = a^{1+p} \rangle$$

has order $p|G|$.

Ban and Yu [5] prove the existence of a group $G$ of order $p^n$ with $|\text{Aut}(G)| = p^{n+1}$, for $p > 2$ and $n \geq 6$. In [26], Heineken and Liebeck construct a $p$-group of order $p^6$ and exponent $p^2$ for each odd prime $p$ which has an automorphism group of order $p^{10}$.

Jonah and Konvisser [37] exhibit $p + 1$ nonisomorphic groups of order $p^8$ with elementary abelian automorphism group of order $p^{10}$ for each prime $p$. All of these groups have elementary abelian and isomorphic commutator subgroups and commutator quotient groups, and they are nilpotent of class two. All their automorphisms are central.

Malone [47] gives more examples of $p$-groups in which all automorphisms are central: for each odd prime $p$, he constructs a nonabelian finite $p$-group $G$ with a nonabelian automorphism group which comprises only central automorphisms. Moreover, his proof shows that if $F$ is any nonabelian finite $p$-group with $F' = Z(F)$ and $\text{Aut}_c(F) = \text{Aut}(F)$, then the direct product of $F$ with a cyclic group of order $p$ has the required property for $G$.

Caranti and Scoppola [10] show that for every prime $p > 3$, if $n \geq 6$, there is a metabelian $p$-group of maximal class of order $p^n$ which has automorphism
group of order $p^{2(n-2)}$, and if $n \geq 7$, there is a metabelian $p$-group of maximal class of order $p^n$ with an automorphism group of order $p^{2(n-2)+1}$. They also show the existence of non-metabelian $p$-groups ($p > 3$) of maximal class whose automorphism groups have orders $p^7$ and $p^9$.

6 Miscellaneous Results

This section contains a brief mention of several results which seem to be worth including.

In [2], Adney and Yen examine the automorphism group of a finite $p$-group $G$ of class 2 with no abelian direct factor, where $p$ is odd. Under certain conditions on $G$, they show that $|G|$ divides $|\text{Aut}(G)|$; the MathSciNet review of this article states that M. Newman can prove this with no extra conditions on $G$. Ban and Yu have several papers on which groups can be the automorphism group of a $p$-group, focusing on groups of small order. As an example, see [4].

Beisiegel [7] shows that if $G$ is a $p$-group and not elementary abelian, then $\text{Aut}(G)$ is $p$-constrained. Furthermore, if $G$ has a cyclic commutator subgroup and $p > 2$, then $\text{Aut}(G)$ is not a $p$-group if and only if $\text{Aut}(G)$ contains an involution. Menegazzo [19] studies the automorphism groups of finite non-abelian 2-generated $p$-groups with cyclic commutator subgroup for odd primes $p$. He exhibits presentations for the relevant groups and computes the orders of $\text{Aut}(G)$, $O_p(\text{Aut}(G))$, and $\text{Aut}_f(G)$.

Liebeck [44] obtains an upper bound for the class of $\text{Aut}_f(G)$. Finally, Wang and Zhang [60] discuss the automorphism groups of some $p$-groups.

References

[1] G. Q. Abbasi, Automorphism groups of certain metabelian $p$-groups of maximal class, Punjab Univ. J. Math. (Lahore) 17/18 (1984/85), 55–62.

[2] J. E. Adney and T. Yen, Automorphisms of a $p$-group, Illinois J. Math. 9 (1965), 137–143.

[3] A. H. Baartmans and J. J. Woeppe, The automorphism group of a $p$-group of maximal class with an abelian maximal subgroup, Fund. Math. 93 (1976), no. 1, 41–46.

[4] G. N. Ban and S. X. Yu, The orders of the automorphism groups of a class of $p$-groups, Acta Math. Sinica 35 (1992), no. 4, 570–574.

[5] , A counterexample to Curran’s third conjecture, Adv. in Math. (China) 23 (1994), no. 3, 272–274.

[6] A. R. Barghi and M. M. Ahmedy, On automorphisms of a class of special $p$-groups, Arch. Math. (Basel) 77 (2001), no. 4, 289–293.
[7] B. Beisiegel, *Finite p-groups with nontrivial p'-automorphisms*, Arch. Math. (Basel) **31** (1978/79), no. 3, 209–216.

[8] Yu. V. Bondarchuk, *Structure of automorphism groups of the sylow p-subgroup of the symmetrical group sp, (p = 2)*, Ukr. Mat. Zh. **36** (1984), no. 6, 688–694.

[9] R. M. Bryant and L. G. Kov´acs, *Lie representations and groups of prime power order*, J. London Math. Soc. (2) **17** (1978), 415–421.

[10] A. Caranti and C. M. Scoppola, *A remark on the orders of p-groups that are automorphism groups*, Boll. Un. Mat. Ital. A (7) **4** (1990), no. 2, 201–207.

[11] R. W. Carter, *Simple groups and simple Lie algebras*, J. London Math. Soc. **40** (1965), 193–240.

[12] G. Corsi Tani, *Finite p-groups with nilpotent automorphism group*, Rend. Sem. Mat. Fis. Milano **58** (1988), 55–66 (1990).

[13] M. J. Curran, *Automorphisms of certain p-groups (p odd)*, Bull. Austral. Math. Soc. **38** (1988), no. 2, 299–305.

[14] ______, *A note on p-groups that are automorphism groups*, Rend. Circ. Mat. Palermo (2) Suppl. (1990), no. 23, 57–61.

[15] M. J. Curran and D. J. McCaughan, *Central automorphisms of finite groups*, Bull. Austral. Math. Soc. **34** (1986), no. 2, 191–198.

[16] G. Cutolo, H. Smith, and J. Wiegold, *p-groups of maximal class as automorphism groups*, Illinois J. Math. **47** (2003), no. 1-2, 141–156, Special issue in honor of Reinhold Baer (1902–1979).

[17] R. M. Davitt, *On the automorphism group of a finite p-group with a small central quotient*, Canad. J. Math. **32** (1980), no. 5, 1168–1176.

[18] J. R. Durbin and M. McDonald, *Groups with a characteristic cyclic series*, J. Algebra **18** (1971), 453–460.

[19] B. Eick, C. R. Leedham-Green, and E. A. O’Brien, *Constructing automorphism groups of p-groups*, Comm. Algebra **30** (2002), no. 5, 2271–2295.

[20] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2005, packages AutPGrp and SmallGroups (http://www.gap-system.org).

[21] Wolfgang Gaschütz, *Nichtabelsche p-Gruppen besitzen äussere p-Automorphismen*, J. Algebra **4** (1966), 1–2.

[22] J. A. Gibbs, *Automorphisms of certain unipotent groups*, J. Algebra **14** (1970), 203–228.
[23] S. P. Glasby and R. B. Howlett, *Extraspecial towers and Weil representations*, J. Algebra **151** (1992), no. 1, 236–260.

[24] D. Gorenstein, *Finite groups*, Harper & Row Publishers, New York, 1968.

[25] Robert L. Griess, Jr., *Automorphisms of extra special groups and nonvanishing degree 2 cohomology*, Pacific J. Math. **48** (1973), 403–422.

[26] H. Heineken and H. Liebeck, *On p-groups with odd order automorphism groups*, Arch. Math. (Basel) **24** (1973), 464–471.

[27] ———, *The occurrence of finite groups in the automorphism group of nilpotent groups of class 2*, Arch. Math. (Basel) **25** (1974), 8–16.

[28] G. T. Helleloid and U. Martin, *The automorphism group of a finite p-group is almost always a p-group*, J. Algebra, to appear, available at arXiv:math.GR/0602039.

[29] M. V. Horoševskiĭ, *The automorphism groups of finite p-groups*, Algebra i Logika **10** (1971), 81–86, English translation in Algebra and Logic **10** (1971), 54–57.

[30] ———, *The automorphism group of wreath products of finite groups*, Sibirsk. Mat. Ñ. **14** (1973), 651–659, 695, English translation in Siberian Math. J. **14** (1973), 453–458.

[31] A. Hughes, *Automorphisms of nilpotent groups and supersolvable orders*, The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), Proc. Sympos. Pure Math., vol. 37, Amer. Math. Soc., Providence, R.I., 1980, pp. 205–207.

[32] B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967.

[33] I. M. Isaacs, *Extensions of group representations over nonalgebraically closed fields*, Trans. Amer. Math. Soc. **141** (1969), 211–228.

[34] ———, *Symplectic action and the Schur index*, Representation theory of finite groups and related topics (Proc. Sympos. Pure Math., Vol. XXI, Univ. Wisconsin, Madison, Wis., 1970), Amer. Math. Soc., Providence, R.I., 1971, pp. 73–75.

[35] ———, *Re: [group-pub-forum] the automorphism group of the extraspecial p-groups*, September 27, 2006, sent to the group-pub-forum mailing list at group-pub-forum@lists.maths.bath.ac.uk.

[36] A.-R. Jamali, *Some new non-abelian 2-groups with abelian automorphism groups*, J. Group Theory **5** (2002), no. 1, 53–57.

[37] D. Jonah and M. Konvisser, *Some non-abelian p-groups with abelian automorphism groups*, Arch. Math. (Basel) **26** (1975), 131–133.
[38] A. Juhász, The group of automorphisms of a class of finite $p$-groups, Trans. Amer. Math. Soc. 270 (1982), no. 2, 469–481.

[39] R. Lawton, A note on a theorem of Heineken and Liebeck, Arch. Math. (Basel) 31 (1978/79), no. 5, 520–523.

[40] P. Lentoudis, Détermination du groupe des automorphismes du $p$-groupe de Sylow du groupe symétrique de degré $p^m$: l'idée de la méthode, C. R. Math. Rep. Acad. Sci. Canada 7 (1985), no. 1, 67–71.

[41] ———, Erratum: “Determining the automorphism group of the Sylow $p$-group of the symmetric group of degree $p^m$: the idea of the method”, C. R. Math. Rep. Acad. Sci. Canada 7 (1985), no. 5, 325.

[42] ———, Le groupe des automorphismes du $p$-groupe de Sylow du groupe symétrique de degré $p^m$: résultats, C. R. Math. Rep. Acad. Sci. Canada 7 (1985), no. 2, 133–136.

[43] P. Lentoudis and J. Tits, Sur le groupe des automorphismes de certains produits en couronne, C. R. Acad. Sci. Paris Sér. I Math. 305 (1987), no. 20, 847–852.

[44] H. Liebeck, The automorphism group of finite $p$-groups, J. Algebra 4 (1966), 426–432.

[45] I. G. Macdonald, Symmetric functions and Hall polynomials, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995.

[46] Ian D. Macdonald, The theory of groups, Clarendon Press, Oxford, 1968.

[47] J. J. Malone, $p$-groups with nonabelian automorphism groups and all automorphisms central, Bull. Austral. Math. Soc. 29 (1984), no. 1, 35–37.

[48] A. Mann, Some questions about $p$-groups, J. Austral. Math. Soc. Ser. A 67 (1999), no. 3, 356–379.

[49] F. Menegazzo, Automorphisms of $p$-groups with cyclic commutator subgroup, Rend. Sem. Mat. Univ. Padova 90 (1993), 81–101.

[50] R. J. Miech, The metabelian $p$-groups of maximal class, Trans. Amer. Math. Soc. 236 (1978), 93–119.

[51] G. A. Miller, A non-abelian group whose group of isomorphisms is abelian, Messenger Math. 43 (1913), 124–125, (or G.A. Miller, Collected works, vol. 5, 415–417).

[52] O. Müller, On $p$-automorphisms of finite $p$-groups, Arch. Math. (Basel) 32 (1979), no. 6, 533–538.
[53] M. F. Newman and E. A. O’Brien, A CAYLEY library for the groups of order dividing 128, Group Theory (Singapore, 1987), de Gruyter, Berlin, 1989, pp. 437–442.

[54] P. P. Pavlov, Sylow p-subgroups of the full linear group over a simple field of characteristic p, Izvestiya Akad. Nauk SSSR. Ser. Mat. 16 (1952), 437–458.

[55] P. Schmid, Normal p-subgroups in the group of outer automorphisms of a finite p-group, Math. Z. 147 (1976), no. 3, 271–277.

[56] A. I. Starostin, Finite p-groups, J. Math. Sci. (New York) 88 (1998), no. 4, 559–585, Algebra, 5.

[57] Robert Steinberg, Automorphisms of finite linear groups, Canad. J. Math. 12 (1960), 606–615.

[58] R. R. Struik, Some nonabelian 2-groups with abelian automorphism groups, Arch. Math. (Basel) 39 (1982), no. 4, 299–302.

[59] L. Verardi, A class of special p-groups, Arch. Math. (Basel) 68 (1997), no. 1, 7–16.

[60] Y. Wang and X. Zhang, The orders of automorphism groups of some families p-groups, J. Guangxi Univ. Nat. Sci. Ed. 29 (2004), no. 1, 50–53.

[61] U. H. M. Webb, An elementary proof of Gaschütz’ theorem, Arch. Math. (Basel) 35 (1980), no. 1-2, 23–26.

[62] ______, The occurrence of groups as automorphisms of nilpotent p-groups, Arch. Math. (Basel) 37 (1981), no. 6, 481–498.

[63] U. M. Webb, The number of stem covers of an elementary abelian p-group, Math. Z. 182 (1983), no. 3, 327–337.

[64] A. J. Weir, Sylow p-subgroups of the general linear group over finite fields of characteristic p, Proc. Amer. Math. Soc. 6 (1955), 454–464.

[65] D. L. Winter, The automorphism group of an extraspecial p-group, Rocky Mountain J. Math. 2 (1972), no. 2, 159–168.

[66] B. Wolf, A note on p′-automorphism of p-groups P of maximal class centralizing the center of P, J. Algebra 190 (1997), no. 1, 163–171.

[67] J. H. Ying, On finite groups whose automorphism groups are nilpotent, Arch. Math. (Basel) 29 (1977), no. 1, 41–44.