Are gauge shocks really shocks?

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Abstract
The existence of gauge pathologies associated with the Bona–Masso family of generalized harmonic slicing conditions is proven for the case of simple 1+1 relativity. It is shown that these gauge pathologies are true shocks in the sense that the characteristic lines associated with the propagation of the gauge cross, which implies that the name ‘gauge shock’ usually given to such pathologies is indeed correct. These gauge shocks are associated with places where the spatial hypersurfaces that determine the foliation of spacetime become non-smooth.

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1. Introduction
When studying the dynamical evolution of spacetime, it is important to choose coordinates that allow one to cover as large a region of spacetime as possible without becoming pathological. In the 3+1 approach, this choice of coordinates naturally separates in two different aspects: the choice of a time coordinate, that is, of a foliation of spacetime into spatial hypersurfaces (also known as the ‘slicing’), associated with the lapse function \( \alpha \), and the choice of the way in which the spatial coordinates (the ‘time lines’) propagate from one hypersurface to the next, associated with the shift vector \( \beta^i \).

With respect to the choice of slicing, many different ways to choose the lapse function are possible. One can, for example, specify the lapse directly as a function of the geometric variables. Alternatively, one can obtain the lapse as a solution of some specific differential equation. Elliptic slicing conditions are typically obtained when one enforces some geometric condition on the spatial hypersurfaces. An example of an elliptic slicing condition is ‘maximal slicing’ \([1]\), which requires that the spatial volume elements remain constant during the evolution and has strong singularity avoiding properties, making it particularly well suited for studies of black hole spacetimes.

Another possibility is to specify an evolution equation for the lapse function and evolve this function in time as just another dynamical quantity. This last approach has the advantage that it is much easier computationally to evolve the lapse than to solve an elliptic equation.
for it. One particular family of evolution-type slicing conditions was introduced by Bona and Masso as a generalization of the harmonic time coordinate condition [2]. This family has the important property of allowing one to construct strongly hyperbolic formulations of the Einstein evolution equations that include the slicing condition. Also, some members of the Bona–Masso family have been shown to mimic the singularity avoiding properties of maximal slicing.

In 1997, I studied different members of the Bona–Masso family and found that, unless a specific condition was imposed, gauge pathologies could easily develop [3]. I called those pathologies ‘gauge shocks’ as they appeared as discontinuities in the lapse and metric functions that developed from smooth initial data. In [4], I strengthened the case for the existence of these gauge shocks by providing a purely kinematic argument independent of the Einstein equations, and more recently in [5] gauge shocks have again been predicted together with blow-ups associated with constraint violation. Still, there have been some doubts in the numerical relativity community about both the relevance and the reality of gauge shocks. Recently, it has even been claimed by Bona et al [6] that genuine gauge shocks are completely discarded.

The purpose of this paper is therefore to provide a rigorous proof in the case of simple (1+1)-dimensional relativity that gauge pathologies associated with the Bona–Masso family of slicing conditions do develop, and that these pathologies are genuine shocks in the following sense: they correspond to discontinuities in the solutions of the underlying hyperbolic differential equations that develop from smooth initial data and are associated with the crossing of the characteristic lines.

It is important to stress the fact that the term ‘shock’ will be used here strictly in this restricted sense of the crossing of the characteristic lines. In hydrodynamics, the word shock is associated also with the subsequent propagation of these discontinuities (‘shock waves’). However, since at a discontinuity the differential equations break down, one can only talk about so-called ‘weak’ solutions. Such weak solutions are not unique, and one needs to specify an extra physical principle known as an ‘entropy condition’ to identify the correct solution. In the case of gauge shocks, however, once the discontinuity develops our gauge has in fact broken down. It is unlikely that one can find an entropy condition in this case as the gauge can be chosen arbitrarily, so any weak solution should be acceptable in principle (though they would all be singular). Note also that we are really interested in finding how such gauge shocks can best be avoided since discontinuities in the gauge variables are clearly undesirable. We are not interested in propagating these discontinuities1.

This paper is organized as follows. In section 2, I review the basic equations of 1+1 relativity, and show that they can in fact be written as a system of conservation laws, both in terms of the geometric variables and in terms of the eigenfields. My main argument for the existence of gauge shocks is presented in section 3, first showing that blow-ups of the eigenfields can develop in finite time, and then showing that the characteristic lines associated with the gauge propagation do cross. I also show a numerical example showing how a gauge shock develops. I conclude in section 4. Appendix A describes the initial data used for the numerical simulations, and finally appendix B counters the argument given by Bona et al in [6] to claim that gauge shocks do not occur.

1 It has been suggested that because of the fact that in this case there is no entropy condition, the name ‘gauge shock’ is not very appropriate and one should just call these singularities ‘gauge pathologies’. However, this name is simply too general as gauge pathologies can develop in many different ways. The pathologies discussed here are of a very specific type, and they develop precisely as shocks do, i.e. because the characteristic lines associated with the propagation of the gauge cross.
2. Einstein equations in 1+1 dimensions

Let us consider vacuum general relativity in one spatial dimension. It is well known that in such a case the gravitational field is trivial and there are no true dynamics. However, one can still have non-trivial gauge dynamics that can be used as a simple example of the type of behaviour one can expect in the higher dimensional case.

As slicing condition we will use the Bona–Masso family of generalized harmonic slicing conditions \[2\]
\[
\frac{\partial}{\partial t} \alpha = -\frac{\alpha}{2} f(\alpha) K,
\]
with \(K = K^x_x\) the trace of the extrinsic curvature.

The ADM [7, 8] evolution equations in the 1+1 case can be written in first-order form as
\[
\frac{\partial}{\partial t} \alpha = -\alpha^2 f K,
\]
\[
\frac{\partial}{\partial t} g = -2\alpha g K,
\]
and
\[
\frac{\partial}{\partial t} D\alpha + \frac{\partial}{\partial x}(\alpha f K) = 0,
\]
\[
\frac{\partial}{\partial t} Dg + \frac{\partial}{\partial x}(2\alpha K) = 0,
\]
\[
\frac{\partial}{\partial t} K + \frac{\partial}{\partial x}(\alpha D\alpha/g) = \alpha(K^2 - D\alpha Dg/2g),
\]
where we have defined \(g := g_{xx}, D\alpha := \partial_x \ln \alpha\) and \(Dg := \partial_x \ln g\).

Before doing an analysis of the characteristic structure of this system of equations, it is important to note that the evolution equation for \(K\) can in fact be rewritten as a conservation law in the following way:
\[
\frac{\partial}{\partial t}(g^{1/2} K) + \frac{\partial}{\partial x}(\alpha D\alpha/g^{1/2}) = 0.
\]
If we now define the vector \(\vec{v} := (D\alpha, Dg, \tilde{K})\), with \(\tilde{K} := g^{1/2} K\), then the evolution equations for the first-order variables can be written as a conservative system of the form
\[
\frac{\partial}{\partial t} \vec{v} + \frac{\partial}{\partial x}(M \vec{v}) = 0,
\]
with the characteristic matrix \(M\) given by
\[
M = 
\begin{pmatrix}
0 & 0 & \alpha f/g^{1/2} \\
0 & 0 & 2\alpha g^{1/2} \\
\alpha g^{1/2} & 0 & 0
\end{pmatrix}.
\]
The characteristic matrix has the following eigenvalues:
\[
\lambda_0 = 0, \quad \lambda_{\pm} = \pm\alpha(f/g)^{1/2},
\]
with corresponding eigenvectors
\[
\vec{e}_0 = (0, 1, 0), \quad \vec{e}_{\pm} = (f, 2, \pm f^{1/2}).
\]
Since the eigenvalues are real for \(f > 0\) and the eigenvectors are linearly independent, the system (2.8) is strongly hyperbolic. The eigenfunctions are given by
\[
\vec{\omega} = R^{-1} \vec{v},
\]
with \(R\) the matrix of column eigenvectors. We find (using an adequate choice of normalization, see below)
\[
\omega_0 = D\alpha/f - Dg/2, \quad \omega_{\pm} = \tilde{K} \pm D\alpha/f^{1/2},
\]
which can be easily inverted to give
\[ \tilde{K} = \frac{(\omega_+ + \omega_-)}{2}, \]  
\[ D_\alpha = \frac{f^{1/2} (\omega_+ - \omega_-)}{2}, \]  
\[ D_g = \frac{\omega_+ - \omega_-}{f^{1/2}} - 2\omega_0. \]

It is important to note that with the eigenfunctions scaled as above, their evolution equations also turn out to be conservative and have the simple form
\[ \partial_t \tilde{\omega} + \partial_x (\Lambda \tilde{\omega}) = 0, \]
with \( \Lambda = \text{diag}\{\lambda_i\} \). If, however, the eigenfunctions are rescaled in the way \( \omega'_i = F_i(\alpha, g) \omega_i \), then the evolution equations for the \( \omega'_i \) will in general no longer be conservative and non-trivial sources will be present. The important point is that there is in fact one normalization in which the equations are conservative, namely the one given in (2.13).

3. Gauge shocks in 1+1 relativity

There are two different ways in which one can see that the evolution equations derived in the previous section develop singular solutions. Let us start by looking at the evolution equations for the travelling eigenfunctions \( \omega_{\pm} \)
\[ \partial_t \omega_\pm + \partial_x (\lambda_\pm \omega_\pm) = 0. \]  
We now rewrite these equations as
\[ \partial_t \omega_\pm + \lambda_\pm \partial_x \omega_\pm = -\omega_\pm \partial_x \lambda_\pm. \]

Using the expressions for \( \lambda_\pm \), and denoting \( f' \equiv df/d\alpha \), one finds
\[ \partial_t \lambda_\pm = \frac{\alpha f^{1/2}}{2g^{3/2}} \partial_\alpha g \pm \frac{f^{1/2}}{g^{1/2}} \left( 1 + \frac{af'}{2f} \right) \partial_\alpha \omega_\pm \]
\[ = \lambda_\pm \left[ \left( f + \frac{af'}{2} \right) \frac{D_\alpha}{f} - \frac{D_f}{2} \right] \]
\[ = \lambda_\pm \left[ \left( f - 1 + \frac{af'}{2} \right) \frac{\omega_+ - \omega_-}{2f^{1/2}} + \omega_0 \right], \]
and finally
\[ \partial_t \omega_\pm + \lambda_\pm \partial_x \omega_\pm = \lambda_\pm \omega_\pm \left[ \left( 1 - f - \frac{af'}{2} \right) \frac{\omega_+ - \omega_-}{2f^{1/2}} - \omega_0 \right]. \]

Assume now that we are in a region such that \( \omega_0 = \omega_- = 0 \). It is clear that \( \omega_0 \) will not be excited, since it does not evolve, nor will \( \omega_- \) be excited, since from the equation above we see that all its sources vanish. The evolution equation for \( \omega_+ \) then simplifies to
\[ \partial_t \omega_+ + \lambda_+ \partial_x \omega_+ = \frac{\lambda_+}{2f^{1/2}} \left( 1 - f - \frac{af'}{2} \right) \omega_+^2. \]
This equation shows that \( \omega_+ \) will blow up along its characteristics unless the term in parenthesis vanishes:
\[ 1 - f - \frac{af'}{2} = 0. \]
The last condition has been derived several times before [3–5], and can be easily integrated to give

\[ f(\alpha) = 1 + k/\alpha^2, \quad (3.7) \]

with \( k \) an arbitrary constant. Note that harmonic slicing given by \( f = 1 \) is of this form, but \( f = \text{constant} \neq 1 \) is not. Note also that ‘1+log’ slicing for which \( f = 2/\alpha \), though not of the form (3.7), nevertheless satisfies condition (3.6) at places where \( \alpha = 1 \).

In some cases, it is even possible to predict exactly when a blow-up will occur. In order to see this we first define the rescaled eigenfunctions \( \Omega_{\pm} := \alpha \omega_{\pm}/g^{1/2} \). For their evolution equations we now find

\[ \partial_t \Omega_{\pm} + \lambda_{\pm} \partial_x \Omega_{\pm} = \left( 1 - f - \frac{\alpha f'}{2} \right) \frac{\Omega_{\pm}^2}{2} + \left( 1 - f + \frac{\alpha f'}{2} \right) \frac{\Omega_{\pm} \Omega_{\mp}}{2}. \quad (3.8) \]

Note that with this new scaling, all contributions from \( \omega_0 \) to the sources have disappeared. If we now assume that we have initial data such that \( \Omega_- = 0 \), then the evolution equation for \( \Omega_+ \) simplifies to

\[ \partial_t \Omega_+ + \lambda_+ \partial_x \Omega_+ = \left( 1 - f - \frac{\alpha f'}{2} \right) \frac{\Omega_+^2}{2}. \quad (3.9) \]

which can be rewritten as

\[ \frac{d\Omega_-}{dt} = \left( 1 - f - \frac{\alpha f'}{2} \right) \frac{\Omega_+^2}{2}, \quad (3.10) \]

with \( d/dt \) the derivative along the characteristic. It is clear that we have the same condition for avoiding blow-ups as before. But the last equation has a very important property: for constant \( f \) the coefficient of the quadratic source term is itself also constant. In that case the equation can be easily integrated to find (assuming \( f \neq 1 \))

\[ \Omega_+ = \frac{\Omega_+^0}{1 - (1 - f) \Omega_+^0 t/2}, \quad (3.11) \]

where \( \Omega_+^0 = \Omega_+(t = 0) \). The solution will then blow-up at a finite time given by \( t^* = 2/[(1 - f) \Omega_+^0] \). Clearly, this time will be in the future if \( (1 - f) \Omega_+^0 > 0 \), otherwise it will be in the past. Since in general \( \Omega_+^0 \) will not be constant in space, the first blow-up will occur at the time

\[ T^* = 2/[(1 - f) \max(\Omega_+^0(x))]. \quad (3.12) \]

Figures 1 and 2 show the numerical evolution of the eigenfield \( \omega_\pm \) and the lapse function \( \alpha \), in a case when \( f = 1/2 \), for initial data such that \( \omega_- = 0 \) and using a resolution of \( \Delta x = 0.003 \ 125 \) (see appendix A for details on how to construct such initial data). For the initial data used here, according to (3.12) a blow-up is expected at time \( T^* = 9.98 \). The plots show both the initial data (dotted lines) and the numerical solution at time \( t = 10 \) (solid lines), just after the expected blow-up. We clearly see how the eigenfield \( \omega_\pm \) has developed a large spike, while the lapse has developed a sharp gradient (the solution does not become infinite because the numerical method used has some inherent dissipation\(^2\)).

If one repeats the simulation at different resolutions one finds that the numerical solution converges up to a time \( t \sim 10 \), and after that convergence fails, indicating that even though the

\(^2\) For those readers interested in the numerical details, the code uses a method of line integration in time, with either 3-step iterative Crank–Nicholson or fourth-order Runge–Kutta. For the spatial differentiation, the code uses a slope limiter method of van Leer’s type, see [9]. No artificial dissipation is introduced other than the inherent dissipation of the slope limiter itself.
Figure 1. Evolution of the eigenfunction $\omega_+$ for initial data representing a pulse travelling to the right. The dotted line shows the initial data and the solid line the numerical solution at $t = 10$. Note how $\omega_+$ has developed a large spike.

Figure 2. Evolution of the lapse function $\alpha$ for initial data representing a pulse travelling to the right. The dotted line shows the initial data and the solid line the numerical solution at $t = 10$. Note how $\alpha$ has developed a sharp gradient.

numerical solution continues past this time, we are no longer solving the original differential equations. To see this we can consider the convergence of the constraint $C_\alpha := D_\alpha - \partial_x \ln \alpha$. Numerically this constraint will not vanish, but it should converge to zero as the resolution is increased. Define now the convergence factor as the ratio of the rms norm of $C_\alpha$ for a run at a given resolution and another run at twice the resolution. Figure 3 shows a plot of the convergence factors as a function of time for runs done at five different resolutions: $\Delta x = 0.05, 0.025, 0.0125, 0.006 25, 0.003 125$. Since the numerical code is second order, the convergence factors should be close to 4. The figure shows that as the resolution is increased the convergence factors approach the expected value of 4 for $t < 10$ (the lines move up), but after that time they drop below 1 indicating loss of convergence.

We have then found that blow-ups do develop for the Bona–Masso family of slicing conditions whenever (3.6) does not hold, and we can even predict the precise time of blow-up formation when $f(\alpha)$ is a constant. The question now is whether these blow-ups are genuine shocks or not. To answer this question let us now consider the evolution of the eigenspeeds themselves along their corresponding characteristic lines. From (2.10) we find that
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Convergence factor

Figure 3. Convergence factors for the constraint $C_\alpha$ for runs done at five different resolutions: $\Delta x = 0.05, 0.025, 0.0125, 0.00625, 0.003125$. As resolution increases, the convergence factors approach the expected value of 4 for $t < 10$ (the curves move up), but after that time they drop sharply indicating loss of convergence.

\[ \partial_t \lambda_{\pm} = \pm \partial_t (\alpha f^{1/2} / g^{1/2}) \] (3.13)

and using now the evolution equations for $\alpha$ and $g$ we obtain

\[ \partial_t \lambda_{\pm} = \alpha \frac{\lambda_{\pm}}{g^{1/2}} \left( 1 - f - \frac{\alpha f'}{2} \right) \tilde{K}. \] (3.14)

In a similar way, we find for the spatial derivative

\[ \partial_x \lambda_{\pm} = \lambda_{\pm} \left[ \frac{D_\alpha f}{f} \left( f + \frac{\alpha f'}{2} \right) - \frac{D_\alpha}{2} \right]. \] (3.15)

The last two equations together imply that

\[ \partial_t \lambda_{\pm} + \lambda_{\pm} \partial_x \lambda_{\pm} = \frac{\alpha}{g^{1/2}} \left[ \left( 1 - f - \frac{\alpha f'}{2} \right) \left( \tilde{K} \mp \frac{D_\alpha}{f^{1/2}} \right) \pm f^{1/2} \left( \frac{D_\alpha}{f} - \frac{D_\alpha}{2} \right) \right], \] (3.16)

or in terms of the eigenfields

\[ \partial_t \lambda_{\pm} + \lambda_{\pm} \partial_x \lambda_{\pm} = \frac{\alpha}{g^{1/2}} \left[ \left( 1 - f - \frac{\alpha f'}{2} \right) \omega_{\pm} \pm f^{1/2} \omega_0 \right]. \] (3.17)

If we now consider a region where $\omega_- = \omega_0 = 0$, then the equation for $\lambda_+$ reduces to

\[ \partial_t \lambda_+ + \lambda_+ \partial_x \lambda_+ = 0. \] (3.18)

This is nothing more than Burgers’ equation, the prototype for studying shock formation. It is easy to understand how this equation implies genuine shocks: the equation shows that characteristic speeds are constant along their corresponding characteristic lines, so if these speeds were not uniform to begin with, and particularly if the derivative of $\lambda_+$ was initially negative at any point, the characteristic lines will inevitably cross.

When the characteristics cross the spatial derivative of $\lambda_+$ will become infinite, and as this derivative is given in terms of eigenfields, the eigenfields will blow up. This shows that the blow-ups we studied above correspond to places where the characteristic lines cross, i.e. they are genuine shocks. The use of the term ‘gauge shocks’ to describe these pathologies is therefore entirely justified.

Figure 4 shows the positions of the characteristics with respect to their initial positions for the simulation discussed above, at a series of different times: $t = 0, 2.5, 5, 7.5, 10$. Note
that for the simulation considered here $f = 1/2$ so the characteristic speed should be close to $\sqrt{f} \sim 0.707$, we would then expect the lines on the plot to move up by approximately $0.707 \times 10 = 7.07$. From the figure we see that even though the characteristics were equally spaced at $t = 0$ (corresponding to a line at $45^\circ$ on the plot), at $t = 10$ this is no longer the case and a plateau has formed. This plateau indicates that those characteristics initially in the region around the origin are now all essentially at the same position $x \sim 7$, or in other words, they are about to cross.

One could wonder what the gauge shocks imply about the geometry of the spacetime being evolved. Since in this case we are simply evolving a foliation in Minkowski spacetime it is clear that the background geometry remains perfectly regular, the only thing that can become pathological are the spatial hypersurfaces that determine the foliation. Figure 5 shows a comparison of the initial hypersurface and the final hypersurface at $t = 10$ as seen in Minkowski spacetime, using the data from the same numerical simulation discussed above.
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(for an easier comparison the final slice has been moved back in time so that it lies on top of the initial slice at the boundaries). The hypersurfaces are reconstructed by explicitly keeping track of the position of the normal observers in the original Minkowski coordinates during the evolution. Note how the initial slice is very smooth (it has in fact a Gaussian profile, see appendix A), while the final slice has developed a sharp kink. This shows that the formation of a gauge shock indicates that the hypersurface, though still spacelike everywhere, is no longer smooth (its derivative is now discontinuous).

4. Conclusions

I have studied the formation of gauge pathologies associated with the Bona–Masso family of slicing conditions in simple 1+1 relativity. The analysis shown here not only recovers previously known results about the existence of such pathologies and the conditions necessary for avoiding them, but also proves that such pathologies arise from the crossing of the characteristic lines associated with the propagation of the gauge. This implies that such pathologies are genuine shocks and hence the name ‘gauge shocks’ used to describe them is entirely justified. The gauge shocks appear as discontinuities in the lapse and spatial metric arising from smooth initial data, and represent places where the spatial hypersurfaces that determine the foliation of spacetime become non-smooth.

It is also important to point out that [5] introduces two different shock avoiding criteria called ‘indirect linear degeneracy’ and the ‘source criteria’. Neither criteria is fully understood at this point, and they are offered in [5] more as indicators that seem to work well in practice than as rigorous conditions necessary for avoiding shocks. In this paper, I decided against using either of those criteria precisely because they are not rigorous. The purpose here has been to show that for a simple case (1+1) one does not need to introduce any ad hoc criteria, and one can instead show that gauge shocks develop by directly analysing the evolution equations and showing that: (1) blow-ups in the eigenfields do happen (and can even be predicted in advance) and (2) the characteristic lines associated with the gauge propagation do cross.

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Appendix A

In this appendix I will discuss how to construct initial data that contain only waves propagating in one direction. Note that the simplest way to construct initial data would be to take a trivial initial slice in Minkowski coordinates, and use a non-trivial initial lapse function to introduce a later distortion. However, since in this case the initial extrinsic curvature vanishes, one sees from (2.13) that inevitably modes travelling in both directions will be excited. We then conclude that in order to have waves propagating in only one direction one has to consider a non-trivial initial slice.

The initial data considered here have already been discussed in [3], and here I just present them for completeness. We start by considering an initial slice given in Minkowski coordinates \((t_M, x_M)\) as

\[ t_M = h(x_M). \] (A.1)
with \( h \) a profile function that decays rapidly, like a Gaussian function. It is then not difficult to show that if we use \( x = x_M \) as our spatial coordinate, the spatial metric and extrinsic curvature turn out to be

\[
g = 1 - h'^2 \quad \Rightarrow \quad D_g = -2h'h''/g, \tag{A.2}
\]

\[
K_{xx} = -h''/\sqrt{g} \quad \Rightarrow \quad \tilde{K} = -h''/g. \tag{A.3}
\]

Assume that we want to have waves travelling to the right (the opposite situation can be done in an analogous way). This means that we want \( \omega_- = 0 \), which implies

\[
D_\alpha = \sqrt{f} \tilde{K} = -\sqrt{f} h''/g. \tag{A.4}
\]

This gives us the following differential equation \( \alpha \):

\[
\partial_\alpha \alpha = -\alpha \sqrt{f} h''/(1 - h'^2). \tag{A.5}
\]

In the particular case when \( f \) is a constant the above equation can be easily integrated to find

\[
\alpha = \left( \frac{1 - h'}{1 + h'} \right)^{\sqrt{f}/2}. \tag{A.6}
\]

From this one can reconstruct the rescaled eigenfunction \( \Omega_1 \) and use equation (3.12) to predict the time of shock formation given a specific form of \( h(x) \).

In the simulation shown in section 3, the profile function \( h \) was taken to be a simple Gaussian of the form

\[
h(x) = e^{-x^2}/10. \tag{A.7}
\]

**Appendix B**

Here I will briefly counter the arguments given by Bona et al [6] to claim that gauge shocks are, in their words, completely discarded. Their main argument is based on looking at the so-called ‘foliation equation’ that looks at the foliation as the evolution of a scalar function \( T \) whose level sets correspond to the 3+1 slices. In [4], I showed that the Bona–Masso slicing condition can be written in covariant form as

\[
\left[ g^{\mu\nu} + \left( 1 - \frac{1}{f(\alpha)} \right) n^\mu n^\nu \right] \nabla_\mu \nabla_\nu T = 0, \tag{B.1}
\]

where \( n^\mu \) is the unit normal vector to the hypersurfaces

\[
n_\mu = -\alpha \nabla_\mu T, \tag{B.2}
\]

and with the lapse function \( \alpha \) given in terms of \( T \) as

\[
\alpha = (\nabla_\mu T \nabla^\mu T)^{-1/2}. \tag{B.3}
\]

Assume for a moment that \( f = 1 \). In that case equation (B.1) reduces to the standard wave equation. Consider now a specific point in spacetime and use locally flat coordinates. It is clear that in such coordinates the foliation equation has constant characteristic speeds equal to 1. This implies that the characteristic lines do not cross, and if the lines do not cross in some set of regular coordinates they will not cross in any regular coordinates. We then conclude that in this case there are no shocks.

But what happens when \( f \neq 1 \)? Bona et al argue that in that case one can always start with an arbitrary regular slice such that \( T = \) constant on that slice. From that one can easily see that, as long as the slice is spacelike, one can always recover all second partial derivatives of \( T \) from equation (B.1), which allows one to construct the next hypersurface. This is certainly
true, but it does not follow from here that one can continue this procedure for any finite time, since all we have done is show that given regular initial data there is locally a regular solution of the foliation equation, but shocks are precisely solutions that fail after a finite time.

In [4], I showed that if one takes equation (B.1) with $f \neq 1$ and considers locally flat coordinates, the characteristic speeds are not constant anymore. There is then no guarantee that they will not cross after a finite time. In fact, in [4] it is shown that if one applies the standard analysis coming from the theory of PDEs to equation (B.1) one finds that shocks will form unless condition (3.6) holds. The problem with the argument of Bona et al is therefore that they failed to consider that when one deals with shocks, smooth initial data always guarantee smooth solutions locally, but not after a finite time.

Bona et al also present other arguments. For example, starting from their equation (6) for the speed of light

$$c = \frac{dl}{dt} = \pm \sqrt{\gamma_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} = \pm \alpha,$$

(B.4)

they obtain the shock avoiding condition $f = k/\alpha^2$ with $k$ constant, which is clearly different from (3.7). But in fact the last equality in the equation above is wrong, the speed of light is not equal to the lapse in a general coordinate system, one is missing a factor of $\sqrt{\gamma_{ii}}$ (with $x^i$ the direction of propagation). Inserting this factor takes us back to the shock avoiding condition (3.7).

Bona et al also consider cosmological-type scenarios (homogeneous and isotropic), and derive different conditions for avoiding blow-ups in the lapse (see equation (18) of [6]). But since in this case there is no propagation and just growth in place, the whole concept of shocks and characteristic speeds is simply non-applicable.

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