Amplify–Quantize–Forward Relay Channel With Quadrature Amplitude Modulation

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ABSTRACT This paper considers an amplify–quantize–forward (AQF) relay channel with \( M \)-ary quadrature amplitude modulation, where the relay implements a uniform quantization for the amplified received signal and forwards it to the destination. Two step-size determinant methods for the uniform quantization at the relay are proposed by deriving the power and the mean square error for the quantized signal. Given the step size, the maximum-likelihood detection is presented, and a linear combining detection called equivalent maximum ratio combining is proposed to reduce the detection complexity at the destination in the AQF relay channel. The simulation results confirm the superiority of the proposed quantization methods and detection algorithms.

INDEX TERMS Detection, maximum likelihood, quadrature amplitude modulation, quantize, relay.

I. INTRODUCTION

Wireless communications are assisted by relays to improve communication reliability and data rate [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. Even though the capacity for the relay channel with one source (S), one destination (D), and one relay (R) is unknown, the cutset upper bound and the achievable lower bounds on the capacity by the partial decode–forward (DF) and the compress–forward (CF) were derived in [1]. Moreover, it was proved that both the partial DF and CF achieve within constant bits of the cutset bound [2].

The typical relaying operations for the partial DF and CF are DF relaying and amplify–forward (AF) relaying, respectively [3]. The DF relaying makes a hard decision which eliminates the influence of random noises at the relay [3], [4], [5], [6], [7]. However, the DF relaying requires decoding and re-encoding processes at the relay, which increases the burden on the relay and makes the DF unsuitable for resource-limited relays. The AF relaying does not require these complicated processes, and only requires amplification and retransmission of noisy symbols [3], [8], [9], [10]. Since half-duplex relays could not receive and transmit at the same time, the amplified received signal could not be immediately forwarded to the destination and must be quantized and stored in memory before retransmission [11], [12], [13], [14].

There are two kinds of quantize–forward (QF) relaying: one is phase quantization [11], [12], [13] where the relay uniformly quantizes phases of received signals; and the other one is scale quantization [14] which quantizes scales of received signals. In [11], a uniform phase quantization was introduced and the maximum-likelihood (ML) detection at the destination was presented. With high complexity, the ML detection is not easy to be applied in practice. To overcome this difficulty, a cooperative linear combining that properly linearly combines received signals at the destination was proposed [13]. In [14], the authors introduced two scale-quantization methods: a naive QF and an amplify–QF (AQF). While the naive QF directly quantizes the received signal at the relay, the AQF amplifies the received signal first and then quantizes the amplified version. It was shown that the AQF relaying is optimal by means of throughput on the RD link with a lower complexity than the naive QF. To the best of my knowledge, there has been no much study on the AQF relay channel related to step sizes of the scale quantization at the relay and detection algorithms at the destination so far.
In this paper, the AQF relay channel with the uniform scale quantization [10], [15] at the relay is addressed for the quadrature amplitude modulation (QAM). In the uniform quantization, the step size is usually given by the ratio of the difference between the maximum and minimum values of the input signal and the number of quantization levels [15]. However, the difference between the maximum and minimum values of the amplified received signal is infinitely large due to randomness of the noise. Furthermore, there is a power constraint for the quantized signal at the relay. Under the power constraint, the best step size can be determined by deriving the symbol error probability (SEP). Since the SEP is too difficult to derive, another parameter, a mean square error (MSE) for the quantized signal, is assumed to reduce the detection complexity. The EMRC detection is presented, and a linear combining detection algorithm called equivalent maximum ratio combining (EMRC) is proposed to reduce the detection complexity. The EMRC detection achieves excellent performance with lower complexity than the ML detection.

In summary, the main contributions of this paper are as follows.

- Two suboptimal algorithms for the determinant of the step size are proposed by deriving the power and the MSE for the quantized signal as functions of the step size. (Section II-C)
  - Given the SR channel coefficient, the power and the MSE for the quantized signal are derived in Theorem 1 and 2.
  - The power and the MSE for the quantized signal in the Rayleigh fading channel are derived in closed-forms in Lemma 2, which greatly simplifies the process of the step-size determinant algorithms.

- Given the step size, the ML detection in the AQF relay channel is presented for the first time. (Section III-B)

- The EMRC detection with lower complexity and good performance is proposed by deriving an equivalent signal-to-noise ratio (SNR) and a proper combining weight for the S-R-D link. (Section IV)

The rest of this paper is organized as follows. In Section II, the channel model and the quantization method at the relay in the AQF relay channel are described. Section III presents the ML detection for both the AF relay channel and the AQF relay channel. Section IV proposes the EMRC detection which linearly combines the received signals with a proper ratio at the destination. In Section V, numerical results are provided to validate the proposed methods of the step-size determinant and performance of the ML and the EMRC detections. Finally, the conclusion is given in Section VI.

Throughout the paper, the following notations are used. $\mathbb{C}$ denotes the set of complex numbers; $\text{Re}(x)$ and $\text{Im}(x)$ mean the real and imaginary parts of a complex number $x \in \mathbb{C}$, respectively; $|x|$ and $\angle x$ denote the amplitude and the phase of $x = |x|e^{j\angle x} \in \mathbb{C}$, respectively; $x^*$ denotes the complex conjugate for $x \in \mathbb{C}$; $x \sim \mathcal{CN}(0, \sigma^2)$ denotes that $x$ is a circularly symmetric complex Gaussian random variable with zero mean and variance $\sigma^2$; and $\mathbb{E}[\cdot]$ denotes the expectation with respect to the random variables in the argument.

II. SYSTEM DESCRIPTION

A. CHANNEL MODEL

A half-duplex AQF relay channel with one source, one destination, and one relay shown in Figure 1 is considered. It is assumed that the instantaneous channel state information (CSI) of the SR link is known at both the relay and the destination, and the CSIs of the SD and RD links are known at the destination. In the first phase, the source broadcasts one uniformly distributed $M$-QAM symbol $x$ to the relay and the destination, where $x \in S = S + jS$, $S = \{ \pm \sqrt{\frac{3}{2(M-1)}}, \pm 3 \sqrt{\frac{3}{2(M-1)}}, \ldots, \pm (\sqrt{M} - 1) \sqrt{\frac{3}{2(M-1)}} \}$ and $\mathbb{E}[|\text{Re}(x)|^2] = \mathbb{E}[|\text{Im}(x)|^2] = 1/2$. During the second phase, the relay transmits $\hat{x}_R$ to the destination after some operation to the received signal. Subsequently, the respective received signals at the relay and destination can be written as

$$y_{SR} = h_{SR}x + z_{SR}$$

$$y_{SD} = h_{SD}x + z_{SD}$$

$$y_{RD} = h_{RD}x + z_{RD}$$

where $h_{SR}, h_{SD}, h_{RD} \in \mathbb{C}$ are the channel coefficients of the SR, SD, and RD links, respectively, $z_{SR} \sim \mathcal{CN}(0, \sigma^2)$ is the noise term at the relay, and $z_{SD}, z_{RD} \sim \mathcal{CN}(0, \sigma^2)$ are the noise terms at the destination in the first and second phases, respectively.

For the AF relaying, the signal $x_R$ is the normalized $y_{SR}$, i.e., $x_R = \tilde{y} = \alpha y_{SR}$, where $\alpha = 1/\sqrt{\mathbb{E}[|y_{SR}|^2]}$. The instantaneous SNRs for the SR, SD, and RD links are $\gamma_{SR} = |h_{SR}|^2/\sigma^2$, $\gamma_{SD} = |h_{SD}|^2/\sigma^2$, and $\gamma_{RD} = |h_{RD}|^2/\sigma^2$.

Note that the respective transmit SNRs for the received signals, $\gamma_{SR}, \gamma_{SD}, \gamma_{RD}$, are all linearly proportional to $\rho = 1/\sigma^2$. The instantaneous SNRs for the SR, SD, and RD links are

$$\gamma_{SR} = |h_{SR}|^2/\sigma^2 = |h_{SR}|^2 \rho$$

$$\gamma_{SD} = |h_{SD}|^2/\sigma^2 = |h_{SD}|^2 \rho$$

$$\gamma_{RD} = |h_{RD}|^2/\sigma^2 = |h_{RD}|^2 \rho$$

The quantization method at the relay will be explained in detail in the following sections.

B. QUANTIZATION METHOD AT RELAY

Let $f_i(t) = \text{Re}(t), f_2(t) = \text{Im}(t), \tilde{y}_i = f_i(\tilde{y}), i = 1, 2$. Given $x$ and $h_{SR}$, we have the probability density function (pdf) of $\tilde{y}_i, i = 1, 2$ as

$$p(\tilde{y}_i|x, h_{SR}) = \frac{\exp(-\frac{(\tilde{y}_i-\alpha f_i(h_{SR}, t))^2}{\alpha^2 \sigma^2})}{\sqrt{\pi \alpha^2 \sigma^2}}$$

In practical systems, any estimated CSIs should be quantized and stored in memory to avoid repeated channel estimation and time delay. Since this work focuses on the quantization problem for the received signal at the relay and the detection algorithms at the destination, the exact CSIs are assumed to simplify the analysis.
and the mid-point quanta are chosen as \( q_k = \frac{2k-1}{2} \Delta \) for \( k = 1, \ldots, \frac{L}{2} \), where \( L = 2^b \) is the number of quantization levels per dimension. After quantization, we have \( x_R = Q(\tilde{y}_1) + jQ(\tilde{y}_2) \), where \( Q(\tilde{y}_i) \in Q = \{ \pm q_1, \pm q_2, \ldots, \pm q_L/2 \}, i = 1, 2 \). For example in Figure 2, if \(-k_{i-1} < \tilde{y}_i \leq -k_{i-1} \), then \( Q(\tilde{y}_i) = -q_{k_i} \), and if \( k_{i-1} \leq \tilde{y}_i \leq k_{i-1} \), then \( Q(\tilde{y}_i) = q_{k_i} \).

### C. Determinant of Step Size for Quantization

The step size \( \Delta \) is usually given by the ratio of the difference between the maximum and minimum values of the input signal and the number of quantization levels, \( L \) [15]. In the AQF relay channel, however, the difference between the maximum and minimum values of \( \tilde{y}_i \) is infinite due to the randomness of the noise, and the usual method could not be applied in the quantization at the relay. Moreover, the quantized signals are transmitted in a certain power constraint at the relay, i.e., \( E[|x_R|^2] \leq 1 \). Under the power constraint, the best step size can be determined by deriving the SEP. Since the SEP is too difficult to derive, we can define another parameter, the MSE for the quantized signal, \( x_R \), i.e.,

\[
E[(\tilde{y} - x_R)^2] = E[(\tilde{y}_1 - f_1(x_R))^2] + \sum_{i=1}^{\frac{L}{2}} kQ(k\Delta - \alpha f_i(h_{SR}x)) \frac{\alpha^2}{\sigma^2} / \sqrt{2}.
\]

**Theorem 1:** Given \( h_{SR} \) the power of \( f_1(x_R) \) for the 4-QAM symbol set, \( \tilde{S} \), and the number of quantization levels, \( L \), is derived as

\[
E[f_1^2(x_R)|h_{SR}] = 2 \sum_{k=1}^{L} q_k^2 P(q_k|h_{SR})
\]

\[
= \frac{\Delta^2}{4} + \frac{4\Delta^2}{M} \sum_{x \in \tilde{S}} kQ(k\Delta - \alpha f_1(h_{SR}x)) \frac{\alpha^2}{\sigma^2} / \sqrt{2}.
\]

where

\[
P(q_k|h_{SR}) = \frac{1}{M} \sum_{x \in \tilde{S}} [Q\left(\frac{l_{k-1} - \alpha f_1(h_{SR}x)}{\alpha^2/\sqrt{2}}\right) - Q\left(\frac{l_k - \alpha f_1(h_{SR}x)}{\alpha^2/\sqrt{2}}\right)].
\]

**Proof:** see Appendix A.
Theorem 2: Given $h_{SR}$, the MSE of $f_1(x_R)$ for the M-QAM symbol set, $\tilde{S}$, and the number of quantization levels, $L$, is derived as

$$E[(\tilde{y}_1 - f_1(x_R))^2 | h_{SR}]$$

$$= \frac{\Delta^2}{4} + 1 - \frac{2\Delta}{M} \sum_{x \in \tilde{S}} \alpha f_1(h_{SR}x)Q\left(\frac{-f_1(h_{SR}x)}{\alpha \sqrt{2}}\right)$$

$$- \frac{\Delta \alpha \sigma}{\sqrt{\pi M}} \sum_{x \in \tilde{S}} e^{-\frac{(f_1(h_{SR}x))^2}{2 \sigma^2}} - \frac{2 \Delta \alpha \sigma}{\sqrt{\pi M}} \sum_{x \in \tilde{S}} \frac{\xi-1}{k} e^{-\frac{(\Delta - \alpha f_1(h_{SR}x))^2}{2 \sigma^2}}$$

$$+ \frac{4 \Delta}{M} \sum_{k=1}^{L} \sum_{x \in \tilde{S}} (k \Delta - \alpha f_1(h_{SR}x))Q\left(\frac{k \Delta - \alpha f_1(h_{SR}x)}{\alpha \sigma \sqrt{2}}\right).$$

(9)

Proof: see Appendix B.

Taking expectation for (7) and (9) corresponding to $h_{SR}$, the power and the MSE can be derived for a specific distributed fading channel.

Remark 1: Observing the two theorems, one can find that the expectations related to $h_{SR}$ are really complicated. Moreover, the two formulas are related to $\sigma$, thus the best $\Delta$ for each SNR level should be derived separately. It makes the determinant of $\Delta$ more complicated even with numerical search.

Instead, high-SNR approximations can be considered. When $\sigma^2 \rightarrow 0$, $\alpha f_1(h_{SR}x)$ is approximated as

$$\alpha f_1(h_{SR}x) = \alpha |h_{SR}| |x| \cos(\theta_h + \theta_x) \approx |x| \cos(\theta_h + \theta_x)$$

(10)

where $x = |x| e^{i\theta_x}$, $h_{SR} = |h_{SR}| e^{i\theta_h}$, and $\alpha = \frac{1}{\sqrt{|h_{SR}|^2 + \sigma^2}}$.

Moreover, if $t > 0$, $Q\left(\frac{t}{\alpha \sigma \sqrt{2}}\right) \approx 0$; if $t = 0$, $Q\left(\frac{t}{\alpha \sigma \sqrt{2}}\right) = 1/2$; if $t < 0$, $Q\left(\frac{t}{\alpha \sigma \sqrt{2}}\right) \approx 1$, when $\sigma^2 \rightarrow 0$, i.e.,

$$Q\left(\frac{t}{\alpha \sigma \sqrt{2}}\right) \approx \frac{1}{2} (1 - \text{sign}(t)).$$

(11)

Applying these approximations, the following lemma is derived.

Lemma 1: When $\sigma^2 \rightarrow 0$, the power and the MSE for $f_1(x_R)$ in high SNR regions for the AQF relay channel with the M-QAM symbol set, $\tilde{S}$, and the number of quantization levels, $L$, are derived as

$$E[f_1^2(x_R)]$$

$$\approx \frac{(L-1)^2}{4} \Delta^2 + \frac{2 \Delta}{M} \sum_{x \in \tilde{S}} \frac{\xi-1}{k} \int_0^{2\pi} \text{sign}(k \Delta - |x| \cos(\theta_h + \theta_x)) |p(\theta)| d\theta$$

and

$$E[(\tilde{y}_1 - f_1(x_R))^2]$$

$$\approx \frac{(L-1)^2}{4} \Delta^2 + \frac{2 \Delta}{M} \sum_{x \in \tilde{S}} \frac{\xi-1}{k} \int_0^{2\pi} \text{sign}(k \Delta - |x| \cos(\theta_h + \theta_x)) |p(\theta)| d\theta$$

(12)

respectively, where $|p(\theta)| = \text{sign}(\theta_h + \theta_x)$. The probability of $f_1(x_R) = q_k$ in high SNR regions for the Rayleigh-fading AQF relay channel is derived as

$$P(q_k) \approx \frac{1}{\pi M} \sum_{x \in \tilde{S}} \left( \text{arccos}^*\left(\frac{l_k-1}{|x|}\right) - \text{arccos}^*\left(\frac{l_k}{|x|}\right) \right)$$

(17)

for $k = 1, 2, \ldots, \frac{L}{2}$, where $l_k$ is the one in (6).

Proof: see Appendix D.
Remark 2: The step size with smaller MSE or with larger signal power are expected to obtain better SEP. However, there is no guarantee that the step sizes that minimize the MSE and maximize the power at the same time exist.

Hence, two suboptimal algorithms for the determinant of the step size are proposed by applying Lemma 2.

- MMSE: find the step size that minimizes the MSE with the power constraint $E[f_i^2(x_R)] \leq 0.5$.
- Maximum power with MMSE (MP-MMSE): find step sizes with the maximum power $E[f_i^2(x_R)] = 0.5$ and select the one that minimizes the MSE among the candidate step sizes.

Some examples for the two determinant algorithms of $\Delta$ are given in Section V-A.

Given a step size, the quantization method at the relay is determined, and detection algorithms are addressed in the following sections.

III. ML DETECTION AT DESTINATION

A. AF RELAYING

Substituting $x_R = \hat{y} = \alpha y_{SR}$ into (3), we have the S-R-D link for the AF relaying as

$$y_{RD} = h_{RD}(h_{SR}x + z_{SR}) + z_{RD} = ah_{RD}h_{SR}x + h_{RD}z_{SR} + z_{RD}. \tag{18}$$

Considering (2) and (18), the ML detection for the AF relay channel is written as

$$\hat{x} = \arg \max_{x \in S} e^{\frac{|y_{SD} - h_{SD}x|^2}{\sigma^2}} e^{-\frac{|y_{RD} - ah_{RD}h_{SR}x|^2}{\sigma^2(h_{RD})^2 \alpha^2 + 1}}. \tag{19}$$

B. AQF RELAYING

In the AQF relay channel with $x_R = Q(\hat{y})$ in (3), the ML detection that maximizes the likelihood function, $p(y_{SD}, y_{RD}|x, h_{SR}, h_{SD}, h_{RD})$ is written as

$$\hat{x} = \arg \max_{x \in S} \left[ p(y_{SD}|x, h_{SD}) \right. \\
\left. \times \sum_{v_1 \in Q} \left( p(y_{RD}|v_1, v_2, h_{RD}) P(v_1, v_2|x, h_{SR}) \right) \right] \\
= \arg \max_{x \in S} \left[ e^{-\frac{|y_{SD} - h_{SD}x|^2}{\sigma^2}} \right. \\
\left. \times \sum_{v_1 \in Q} e^{-\frac{|y_{RD} - h_{RD}(v_1 + v_2)|^2}{\sigma^2}} P(v_1, v_2|x, h_{SR}) \right] \tag{20}$$

where $v_1$ and $v_2$ denote the possible quantized symbols for the real and imaginary parts at the relay, respectively.

\[ P(v_1, v_2|x, h_{SR}) = P(v_1|x, h_{SR})P(v_2|x, h_{SR}). \]

\[ P(v_i|x, h_{SR}) = \int_{l_{i-1}}^{l_i} p(\tilde{y}_i|x, h_{SR}) d\tilde{y}_i \]

\[ = \int_{l_{i-1}}^{l_i} \exp\left(-\frac{(\tilde{y}_i - \alpha f_i(h_{SR}x))^2}{\alpha^2 \sigma^2} \right) \frac{1}{\sqrt{\pi \alpha^2 \sigma^2}} d\tilde{y}_i \]

\[ = Q\left(\frac{l_{i-1} - \alpha f_i(h_{SR}x)}{\sqrt{\alpha^2 \sigma^2/2}}\right) - Q\left(\frac{l_i - \alpha f_i(h_{SR}x)}{\sqrt{\alpha^2 \sigma^2/2}}\right) \tag{21} \]

for $v_i > 0,$

\[ P(v_i|x, h_{SR}) = \int_{-l_{i-1}}^{-l_i} p(\tilde{y}_i|x, h_{SR}) d\tilde{y}_i \]

\[ = Q\left(-l_{i-1} - \alpha f_i(h_{SR}x)\right) - Q\left(-l_i - \alpha f_i(h_{SR}x)\right) \tag{22} \]

for $v_i < 0,$ and $l_i = \frac{\|v_i\|}{\Delta} + \frac{1}{2}$ for $i = 1, 2.$

Remark 3: The implementation of the ML detection requires about $O(2^M)$ times computations of exponential functions and $Q$ functions. With a large number of quantization bits, the ML detection becomes more complicated.

To reduce the detection complexity, linear detection methods that do not consider all possible quantized signals ($v_1$ and $v_2$ in (20)) can be applied.

IV. LINEAR DETECTION AT DESTINATION

A linear detection linearly combines received signals in a proper ratio and makes a decision based on the combined equivalent channel model. In the AQF relay channel, there are two received signals at the destination, $y_{SD}$ and $y_{RD},$ via the SD link and the S-R-D link, respectively. While it is easy to find the combining weight for $y_{SD},$ the combining weight for $y_{RD}$ is not trivial due to the nonlinearity of the relay operation similar to the DF relay channel [4] and the QF relay channel [13]. In the following sections, an equivalent SNR for the S-R-D link $\gamma_{SRD}^{eq}$ is derived first, and a new linear detection called EMRC is proposed by finding the combining weight for $y_{RD}$ related to $\gamma_{SRD}^{eq}$ in the AQF relay channel.

A. EQUIVALENT SNR FOR S-R-D LINK

In two-hop relay systems, the error probability for the S-R-D link is upper bounded by the sum of the error probabilities of SR and RD links assuming perfect RD and SR links, respectively. The equivalent SNRs for the S-R-D link in both DF and AF relaying are approximated by $\gamma_{SRD}^{eq} = \min(\gamma_{SR}, \gamma_{RD})$ [3], [4], [8], [9], i.e., $\gamma_{SRD}^{eq} = \gamma_{SR}$ for the perfect SR link and $\gamma_{SRD}^{eq} = \gamma_{RD}$ for the perfect RD link.

In the AQF relay channel, the equivalent SNR for the S-R-D link can also be approximated as the minimum of the equivalent SNRs for the SR and RD links, i.e., $\gamma_{SRD}^{eq} = \min(\gamma_{SR}^{eq}, \gamma_{RD}^{eq})$. It is trivial that the equivalent SNR for the RD link is $\gamma_{RD}^{eq}$, but $\gamma_{SR}^{eq}$ is no longer equal to $\gamma_{SR}$.  

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To achieve $\gamma_{\text{SR}}^{eq}$, the error probability $P(x_R \neq Q(\alpha h_{\text{SR}} x) | h_{\text{SR}})$ is considered by averaging the error probabilities for all possible $x$ as

$$P(x_R \neq Q(\alpha h_{\text{SR}} x) | h_{\text{SR}}) = \frac{1}{M} \sum_{x \in S} P(x_R \neq Q(\alpha h_{\text{SR}} x) | x, h_{\text{SR}})$$

$$= \frac{1}{M} \sum_{x \in S} \left[ 1 - \left( 1 - P(f_1(x_R) \neq Q(\alpha f_1(h_{\text{SR}} x) | x, h_{\text{SR}}) \right) \right]$$

$$= \frac{1}{M} \sum_{x \in S} P(f_1(x_R) \neq Q(\alpha f_1(h_{\text{SR}} x) | x, h_{\text{SR}})$$

$$+ \frac{1}{M} \sum_{x \in S} P(f_2(x_R) \neq Q(\alpha f_2(h_{\text{SR}} x) | x, h_{\text{SR}}).$$

(23)

Let $k_i(x) = \frac{Q(\alpha f_i(h_{\text{SR}} x) | x, h_{\text{SR}})}{P(f_i(x_R) \neq Q(\alpha f_i(h_{\text{SR}} x) | x, h_{\text{SR}})} + \frac{1}{2}$ and $s_i(x) = \text{sign}(\alpha f_i(h_{\text{SR}} x) | x, h_{\text{SR}})$, $i = 1, 2$. Then $k_i(x)$ and $s_i(x)$ are the quantum index and the sign of $Q(\alpha f_i(h_{\text{SR}} x) | x, h_{\text{SR}})$, respectively, and $Q(\alpha f_i(h_{\text{SR}} x) | x, h_{\text{SR}}) = s_i(x) k_i(x)$. The error event $f_i(x_R) = Q(\tilde{y}_i(x) | x, h_{\text{SR}})$ happens if $\tilde{y}_i < k_i(x) - 1$ or $\tilde{y}_i > k_i(x)$ for the case of $f_i(h_{\text{SR}} x) > 0$, and if $\tilde{y}_i > k_i(x) - 1$ or $\tilde{y}_i < k_i(x)$ for the case of $f_i(h_{\text{SR}} x) < 0$ as shown in Figure 2, i.e., the error event happens if $s_i(x) \tilde{y}_i < k_i(x) - 1$ or $s_i(x) \tilde{y}_i > k_i(x)$. Hence, the error probability given an $x$ in (23) is derived as

$$P(f_1(x_R) \neq Q(\alpha f_1(h_{\text{SR}} x) | x, h_{\text{SR}}) = \int_{k_i(x) - 1 < k_i(x)} p(\tilde{y}_i | x, h_{\text{SR}}) d\tilde{y}_i$$

$$+ \int_{s(x) \tilde{y}_i < k_i(x) - 1} p(\tilde{y}_i | x, h_{\text{SR}}) d\tilde{y}_i$$

$$= \frac{Q \left( \frac{2(\alpha f_1(h_{\text{SR}} x) - s_1(x) l_{k_i(x) - 1})^2}{\alpha^2 \sigma^2} \right)}{\alpha^2 \sigma^2}$$

$$+ \frac{Q \left( \frac{2(\alpha f_1(h_{\text{SR}} x) - s_1(x) l_{k_i(x)}^2}{\alpha^2 \sigma^2} \right)}{\alpha^2 \sigma^2}.$$

(24)

The details for the derivation of (24) are given in Appendix E. Due to the symmetry between the real and imaginary parts, it is trivial that

$$\frac{1}{M} \sum_{x \in S} P(f_1(x_R) \neq Q(\alpha f_1(h_{\text{SR}} x) | x, h_{\text{SR}})$$

$$= \frac{1}{M} \sum_{x \in S} P(f_2(x_R) \neq Q(\alpha f_2(h_{\text{SR}} x) | x, h_{\text{SR}})$$

(25)

and then the average error probability in the left-hand side of (23) is rewritten as

$$P(x_R \neq Q(\alpha h_{\text{SR}} x) | h_{\text{SR}})$$

$$\approx \frac{1}{2} \sum_{x \in S} P(f_1(x_R) \neq Q(\alpha f_1(h_{\text{SR}} x) | x, h_{\text{SR}})$$

(26)

where $P(f_1(x_R) \neq Q(\alpha f_1(h_{\text{SR}} x) | x, h_{\text{SR}})$ is the one in (24) for $i = 1$. Observing (24), there are a sum of two $Q$ functions, thus a common $Q$-function approximation, $Q(d_1 \rho) + Q(d_2 \rho) \approx Q(\rho \min(d_1, d_2))$, $d_1 > 0$, $d_2 > 0$ for very large $\rho$ [13] may be applied. However, the values inside the $Q$ function are very small even for some quite large $\rho$ if $b$ is not small, and the approximation does not work anymore. Observing Figure 2, the minimum of $(\alpha f_1(h_{\text{SR}} x) - s_1(x) l_{k_i(x) - 1})^2$ and $(\alpha f_1(h_{\text{SR}} x) - s_1(x) l_{k_i(x)})^2$ are less than $\frac{\Delta^2}{3}$ and $\Delta$ decreases when $b$ increases as shown in Section V-A. Hence, $P(x_R \neq Q(\alpha h_{\text{SR}} x) | h_{\text{SR}})$ will be approximated corresponding to different cases of $b$.

1) THE CASE OF SMALL $b$

For this case, the values inside the $Q$ functions in (24) are large enough in the interested SNR regions. Applying the approximation $Q(d_1 \rho) + Q(d_2 \rho) \approx Q(\rho \min(d_1, d_2))$, $d_1 > 0$, $d_2 > 0$ for large $\rho$, (24) for $i = 1$ is approximated as

$$P(f_1(x_R) \neq Q(\alpha f_1(h_{\text{SR}} x) | x, h_{\text{SR}})$$

$$\approx Q \left( \frac{2d_{\min}^2(x)}{\alpha^2 \sigma^2} \right)$$

(27)

where $d_{\min}^2(x) = \min \left[ (\alpha f_1(h_{\text{SR}} x) - s_1(x) l_{k_i(x) - 1})^2, (\alpha f_1(h_{\text{SR}} x) - s_1(x) l_{k_i(x)})^2 \right]$. Substituting (27) into (26), the error probability at the relay can be upper bounded as

$$P(x_R \neq Q(\alpha h_{\text{SR}} x) | h_{\text{SR}}) \leq 2Q \left( \frac{2d_{\min}^2(x)}{\alpha^2 \sigma^2} \right).$$

(28)

where

$$d_{\min}^2 = \min_{x \in S} d_{\min}^2(x)$$

$$= \min_{x \in S} \left[ \min_{x \in S} (\alpha f_1(h_{\text{SR}} x) - s_1(x) l_{k_i(x) - 1})^2, \right.$$

$$\left. (\alpha f_1(h_{\text{SR}} x) - s_1(x) l_{k_i(x)})^2 \right].$$

(29)

Since the $b$-bit-per-dimension uniformly quantized signal $x_R$ can be seen as a $2^{2b}$-QAM symbol, the upper bound on $P(x_R \neq Q(\alpha h_{\text{SR}} x) | h_{\text{SR}})$ in (28) can be approximated by the average SEP for $2^{2b}$-QAM symbols in an additive Gaussian noise channel such as

$$2Q \left( \frac{2d_{\min}^2(x)}{\alpha^2 \sigma^2} \right) = 2^{2b - 1 - \frac{\Delta^2}{2}} Q \left( \frac{\Delta^2 \gamma_{\text{SR}}^{eq}}{2} \right).$$

(30)

where the coefficient $\frac{\Delta^2}{2} = \frac{2^{2b - 1}}{2b - 1}$ is because there are $2^b$ possible symbols in each dimension, i.e., the SEP contains one $Q$ function when the two outermost symbols are transmitted and two $Q$ functions when the other $2^b - 2$ symbols are transmitted. $2$ in front of the fraction is due to two dimensions corresponding to real and imaginary parts. Then we have the equivalent SNR as

$$\gamma_{\text{SR}}^{eq} = \frac{2}{\Delta^2} \left[ Q^{-1} \left( \frac{2^{2b - 1}}{2b - 1} \right) Q \left( \frac{\Delta^2 d_{\min}^2(x)}{\alpha^2 \sigma^2} \right) \right]^2. \quad (31)$$
Applying the approximation $Q(t) \approx \frac{e^{-t^2/2}}{t\sqrt{2\pi}}$ for large $t$ by the upper and lower bounds on $Q$ function in [17, (5)-(6)], we can drag the multiplier of the $Q$ function, $\frac{2^{b-1}}{2b-1}$, inside the $Q$ function in (31) as

$$y_{SR}^{eq} \approx \frac{2}{\Delta^2} \left[ Q^{-1}\left(Q\left(\frac{2\gamma_{\min}^2 + 2\ln(2b-1)}{\alpha^2\sigma^2}\right)\right)^2 \right]$$

$$= \frac{4}{\Delta^2\alpha^2\sigma^2} \gamma_{\min}^2 + \frac{4}{\Delta^2\ln(2b-1)} 2^{b-1} - 1. \tag{32}$$

2) THE CASE OF LARGE $b$

For large $b$, we have $\alpha f_1(h_{SR}x) \approx Q(\alpha f_1(h_{SR}x)) = s_1(x)q_{l_1(x)}$ and so $\alpha f_1(h_{SR}x) - s_1(x)q_{l_1(x)} \approx 0$ except for the case of $l_1(x) = \infty$. Ignoring the cases of $l_1(x) = \infty$, the error probability at the relay becomes

$$P(x_r \neq Q(\alpha h_{SR}x)|h_{SR}) \approx 2^{2b-1} - 1 \left(\frac{\Delta^2}{2\alpha^2\sigma^2}\right).$$

Comparing to the SER in the equivalent additive Gaussian noise channel with $2^b$-QAM symbols i.e., in the right-hand side of (30), the equivalent SNR is approximated as

$$y_{SR}^{eq} = \frac{1}{\alpha^2\sigma^2} \approx \frac{|h_{SR}|^2}{\sigma^2} = y_{SR}. \tag{33}$$

Summarizing the results for different values of $b$, the equivalent SNR for the S-R-D link is

$$y_{SRD}^{eq} = \min(y_{SR}^{eq}, y_{RD}) \tag{34}$$

where $y_{SR}^{eq}$ is the one in (32) for small $b$ and the one in (33) for large $b$.

B. EQUIVALENT MAXIMUM RATIO COMBINING (EMRC)

To properly combine $y_{SD}$ and $y_{RD}$, two received signals through the SD link and the S-R-D link, respectively, the corresponding combining weights should be determined. The combining weight for the SD link corresponding to (2) is $w_{SD} = h_{SR}^*$ and the SNR is $y_{SD}$.

For the S-R-D link, the combining weight $w_{SRD}$ is not trivial. Since $x_r = Q(\alpha y_{SR}) = Q(\alpha h_{SR}x + \alpha z_{SR})$, $x_r$ may be equal to $Q(\alpha h_{SR}x)$ or not due to the randomness of $z_{SR}$. In the case of perfect quantization at the relay, i.e., $x_r = Q(\alpha h_{SR}x)$, $y_{RD}$ can be expressed as

$$y_{RD} = h_{RD}Q(\alpha h_{SR}x) + z_{RD} = h_{RD}h_{SR}x + z_{RD} \tag{35}$$

where $h_{SR}^* = \frac{Q(\alpha h_{SR}x)}{x}$. Then the combining weight for $y_{RD}$ is derived as

$$w_{SRD} = \frac{y_{RD}^{eq}}{|h_{RD}|^2|h_{SR}|^2/\sigma^2}h_{RD}^*h_{SR}^* \tag{36}$$

where $y_{RD}^{eq}$ is the one in (34).

Linearly combining two received signals $y_{SD}$ and $y_{RD}$ with the corresponding combining weights $w_{SD}$ and $w_{SRD}$, an equivalent channel model is derived as

$$w_{SD}y_{SD} + w_{SRD}y_{RD} = (w_{SD}h_{SR}^* + w_{SRD}h_{RD}^*)x + w_{SD}z_{SD} + w_{SRD}z_{RD} \tag{37}$$

where $|h_{SR}^*|^2 = y_{SR}^{eq}\sigma^2$ is the equivalent squared channel gain for the S-R-D link. Comparing the squared distances between $w_{SD}y_{SD} + w_{SRD}y_{RD}$ and $(|h_{SD}|^2 + |h_{SRD}^*|^2)x$ for all possible $x \in \mathcal{S}$, the EMRC detection is described as

$$\hat{x} = \arg\min_{x \in \mathcal{S}} |w_{SD}y_{SD} + w_{SRD}y_{RD} - (|h_{SD}|^2 + |h_{SRD}^*|^2)x|^2$$

where $w_{SD} = h_{SR}^*$, $w_{SRD}$ is the one in (36), and $|h_{SRD}^*|^2 = y_{SR}^{eq}\sigma^2$. Unlike the ML detection in (20), the EMRC does not require $2^b$ times summations for all possible quantized signals $v_1 \in Q$ and $v_2 \in Q$.

**Remark 4**: Given the channel coefficients $h_{SR}$, $h_{SD}$, $h_{RD}$, the combining weights $w_{SD}$ and $w_{SRD}$, and the channel gains $|h_{SD}|^2$ and $|h_{SRD}^*|^2$ only need to be calculated once. Therefore, the EMRC in (37) requires $O(M)$ times simple computations, unlike the ML detection in (20) that requires $O(2^BM)$ times complicated computations such as exponential functions and $Q$ functions.

For some special cases, the derivations of the combining weights and the equivalent SNRs become much simpler, and the detection complexity of the EMRC can be reduced more.

C. SPECIAL EXAMPLE

**Example 1**: The case of $M = 4$ and $b = 1$

Since $x \in \{\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\}$ and $Q(\alpha h_{SR}x) \in \{\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\}$, $x$ and $Q(\alpha h_{SR}x)$ have the same possible phases and different amplitude. Thus $h_{SR}$ can be simplified as

$$h_{SR} = \frac{Q(\alpha h_{SR}x)}{x} = \frac{\Delta}{\sqrt{2}} e^{i\arg(Q(\alpha h_{SR}x))} = \frac{\Delta}{\sqrt{2}} e^{i\theta_0} \tag{38}$$

| $\Delta$ | $E[|f_1(x)|^2]$ | $E[(g_1 - f_1(x))^2]$ | $P(q_1)$ | $P(q_2)$ |
|---|---|---|---|---|
| 0.5 | 0.5 | 0.5 | 0.02127 | 0.1126 | 0.0997 |
| 0.5 | 0.5 | 0.5 | 0.1997 | 0.4316 | 0.05 |
| 1.4142 | 1.4142 | 1.4142 | 0.3003 | 0.0684 | 0.0 |

**Table 1**: Comparison for various $\Delta$ satisfying $E[f_1^2(x)] = 0.5$ in Lemma 2 for $M = 4, b = 2$. 
where \( \theta_x = \angle x, \theta_h = \angle h_{SR} \), and
\[
g(\theta) = \begin{cases} 
0 & -\frac{\pi}{4} < \theta < \frac{\pi}{4} \\
\frac{\pi}{2} & \frac{\pi}{2} < \theta < \frac{3\pi}{2} \\
\frac{\pi}{2} & \frac{3\pi}{2} < \theta < \frac{7\pi}{2} \\
\frac{\pi}{2} & \frac{7\pi}{2} < \theta < 2\pi .
\end{cases}
\]

The weight \( w_{SRD} \) for the EMRC detection in (37) is
\[
w_{SRD} = \frac{\Delta}{\sqrt{2}} \min(g_{SR}, g_{RD}) h_{RD} e^{-i g(h)} \tag{39}
\]
where \( g(h) = \frac{\Delta h_{RD}^2}{2\sigma^2} \) and
\[
g_{SR} \approx \frac{2}{\Delta^2\sigma^2} \min \left[ (f_1(h_{SR}) - f_2(h_{SR}))^2, (f_1(h_{SR}) + f_2(h_{SR}))^2 \right]. \tag{40}
\]

**Example 2:** The case of large \( b \)

In this case, we have \( h_{SR} = \frac{Q(\alpha h_{SR}))}{\alpha} \approx \frac{oh_{SR}}{\alpha} = \alpha h_{SR} \), \( E[|x_R|^2] \approx 1 \) and \( \gamma_{SR}^q \approx \gamma_{SR} \). Hence, \( w_{SRD} \) for the EMRC detection in (37) becomes
\[
w_{SRD} = \min(\gamma_{SR}, \gamma_{RD}) \frac{h_{RD}^2 h_{SR}^2}{\alpha |h_{SR}|^2 h_{RD}^2} \tag{41}
\]
where \( \gamma_{SR} = \frac{|h_{SR}|^2}{\alpha^2} \) and \( \gamma_{RD} = \frac{|h_{RD}|^2}{\alpha^2} \).

For the above examples, the computation of \( w_{SRD} \) is simple, and the detection complexity of the EMRC is further reduced.

**V. NUMERICAL RESULT**

**A. DETERMINANT OF STEP SIZE**

In this section, the determinant of the step size \( \Delta \) in the Rayleigh-fading channel are evaluated by applying two suboptimal algorithms, the MMSE and the MP-MMSE.

First, to help understand the step-size determinant algorithms, two simple examples are described in detail.

| \( \Delta \) | \( b = \frac{1}{2} \log_2 M \) | \( b = \frac{1}{2} \log_2 M + 1 \) | \( b = \frac{1}{2} \log_2 M + 2 \) |
|---|---|---|
| \( M = 4 \) | 1.273 | 0.569 | 0.268 |
| \( M = 16 \) | 0.638 | 0.354 | 0.174 |

1) THE CASE OF \( M = 4 \) AND \( b = 1 \)

For this case, the power and the MSE in Lemma 2 are simplified as \( E[f_1^2(x_R)] = \frac{\Delta^2}{4} \) and
\[
E[(\hat{y}_1 - f_1(x_R))^2] \approx \frac{\Delta^2}{4} + \frac{2\Delta}{\pi} \left( \frac{\Delta}{4} \right) + \frac{1}{4} - \frac{2\Delta}{\pi^2} \tag{42}
\]
respectively.

- MMSE algorithm: When \( \Delta = \frac{4}{\pi} \approx 1.273 \), the MSE has the minimum value \( \frac{1}{4} - \frac{2\Delta}{\pi^2} \approx 0.0947 \) and the corresponding power is \( E[f_1^2(x_R)] = 0.4051 \).

- MP-MMSE algorithm: When \( \Delta = \sqrt{2} \approx 1.4142 \), the power reaches the maximum value \( 0.5 \), and the corresponding MSE is \( E[(\hat{y}_1 - f_1(x_R))^2] = 0.0997 \) which is slightly larger than 0.0947, the result for the MMSE algorithm.

2) THE CASE OF \( M = 4 \) AND \( b = 2 \)

For this case, the power and the MSE in Lemma 2 are written as
\[
E[f_1^2(x_R)] \approx \frac{\Delta^2}{4} + \frac{4\Delta^2}{\pi} \arccos^*(\Delta) \tag{43}
\]
and
\[
E[(\hat{y}_1 - f_1(x_R))^2] \approx \frac{\Delta^2}{4} + \frac{4\Delta^2}{\pi} \arccos^*(\Delta) + \frac{2\Delta}{\pi^2} \tag{44}
\]
respectively.

- MMSE algorithm: When \( \Delta = 0.569 \), the MSE has the minimum value 0.02095 and the corresponding power is \( E[f_1^2(x_R)] \approx 0.4789 \).

- MP-MMSE algorithm: Applying (43) and (44), \( P(q_1), P(q_2) \), the power, and the MSE for \( f_1(x_R) \) corresponding to various \( \Delta \) are given in Figure 3, and the exact values with the maximum power are listed in Table 1. From Figure 3 and Table 1, one can find that there are three values of \( \Delta \) satisfying \( E[f_1^2(x_R)] = 0.5 \), i.e., \( \Delta = 0.5869, \Delta = 0.9770, \) and \( \Delta = 1.4142 \), and the corresponding MSES are 0.02127, 0.1126, and 0.0997, respectively. Since the case of \( \Delta = 0.5869 \) obtains the minimum MSE 0.02127 among the three values, \( \Delta = 0.5869 \) is chosen as the step size. Note that the quantization with \( \Delta = 1.4142 \) (\( P(q_1) = 0.5 \) and \( P(q_2) = 0 \)) corresponds to the case of \( b = 1 \) with the MSE 0.0997. It confirms that the quantized signal approaches...
TABLE 3. Determinant of the step size by the MP-MMSE algorithm which minimizes the MSE with $E[f_{2}(x_{R})] = 0.5$ applying Lemma 2.

| $\Delta$ | $b = \frac{1}{2} \log_{2} M$ | $b = \frac{1}{2} \log_{2} M + 1$ | $b = \frac{1}{2} \log_{2} M + 2$ |
|-----------|-------------------------------|-------------------------------|-------------------------------|
| $M = 4$   | 1.4142                        | 0.5869                        | 0.271                         |
| $M = 16$  | 0.668                         | 0.3606                        | 0.174                         |

In the similar way, the step sizes for other $M$ and $b$ by the MMSE and MP-MMSE algorithms are derived and summarized in Tables 2 and 3, respectively.

Next, the simulations for the AQF relay channel are presented to compare the performance of the MMSE and MP-MMSE algorithms. The SERs for 4QAM symbols by applying the ML detection at the destination over Rayleigh fading channel, i.e., $h_{SR}, h_{SD}, h_{RD} \sim CN(0, 1)$ are shown in Figures 4 and 5. In Figure 4, the MMSE and MP-MMSE algorithms with Lemma 2 (the high-SNR approximation) are compared to the algorithms by applying Theorem 1 and 2 (without the high-SNR approximations) denoted as MMSE-$\sigma$ and MP-MMSE-$\sigma$. Without the high-SNR approximations, a new step size should be obtained for each SNR level, and each of them should be determined by numerically searching the power in Theorem 1 and the MSE in Theorem 2 for the given $\sigma$ on various Rayleigh fading channels. From the figure, one can find that the proposed MMSE and MP-MMSE algorithms obtain very close SER performance as the MMSE-$\sigma$ and MP-MMSE-$\sigma$, respectively. Figure 5 compares the SER curves for the MMSE and MP-MMSE algorithms by applying Lemma 2 and shows that the MP-MMSE algorithm obtains slightly better performance than the MMSE algorithm when $M = 4$.

In the following section, the simulations are performed with the step sizes by applying only one of the determinant algorithms, the MP-MMSE.

B. DETECTION ALGORITHM

Using the step size of the MP-MMSE algorithm in Table 3, the SERs of the ML detection in Section III and the...
For both algorithms and the detection algorithms. The simulation results confirmed the proposed quantization and the MSE for the quantized signal. Given the step size, the algorithms for the determinant of the step size in the uniform channel with 16QAM.

VI. CONCLUSION

In this paper, the AQF relay channel with \( M \)-QAM was presented. With the power constraint, two suboptimal algorithms for the determinant of the step size in the uniform quantization at the relay were proposed by deriving the power and the MSE for the quantized signal. Given the step size, the ML detection was introduced, and the EMRC was proposed to reduce the detection complexity at the destination. The simulation results confirmed the proposed quantization algorithms and the detection algorithms.

APPENDIX A

PROOF OF THEOREM 1

Given \( h_{\text{SR}} \), the power for \( f_1(\chi_{\text{SR}}) \) is written as

\[
E[f_1^2(\chi_{\text{SR}})|h_{\text{SR}}] = \sum_{k=1}^{\frac{b}{2}} q_k^2 P(q_k|h_{\text{SR}}) + \sum_{k=1}^{\frac{b}{2}} (-q_k)^2 P(-q_k|h_{\text{SR}}) \\
= 2 \sum_{k=1}^{\frac{b}{2}} q_k^2 P(q_k|h_{\text{SR}})
\]

where

\[
P(q_k|h_{\text{SR}}) = \int_{l_{k-1}}^{l_k} p(\tilde{y}_1|h_{\text{SR}}) d\tilde{y}_1
\]

\[
= \int_{l_{k-1}}^{l_k} \frac{1}{M} \sum_{x \in S} \frac{\exp\left(\frac{l_{k-1} - \alpha f_1(h_{\text{SR}})}{\alpha^2 \sigma^2} \right)}{\sqrt{\pi \alpha^2 \sigma^2}} d\tilde{y}_1
\]

\[
= \frac{1}{M} \sum_{x \in S} \left[ Q\left(\frac{l_{k-1} - \alpha f_1(h_{\text{SR}})}{\alpha^2 \sigma^2} \right) - Q\left(\frac{l_{k} - \alpha f_1(h_{\text{SR}})}{\alpha^2 \sigma^2} \right) \right]
\]

(46)

and (45) is due to \( p(\tilde{y}_1|h_{\text{SR}}) = p(-\tilde{y}_1|h_{\text{SR}}) \). Substituting \( q_k = \frac{2k-1}{2} \Delta \) and (46) into (45), Theorem 1 is proved as

\[
E[f_1^2(\chi_{\text{SR}})|h_{\text{SR}}] = 2 \sum_{k=1}^{\frac{b}{2}} \left( \frac{2k-1}{2} \Delta \right)^2 \int_{l_{k-1}}^{l_k} p(\tilde{y}_1|h_{\text{SR}}) d\tilde{y}_1
\]

\[
= 2 \left[ \frac{\Delta^2}{2} \left( \int_0^{\infty} p(\tilde{y}_1|h_{\text{SR}}) d\tilde{y}_1 - \int_{l_1}^{\infty} p(\tilde{y}_1|h_{\text{SR}}) d\tilde{y}_1 \right)
\]

\[
+ \sum_{k=2}^{\frac{b}{2}-1} \left( \frac{2k-1}{2} \Delta \right)^2 \times \left( \int_{l_{k-1}}^{\infty} p(\tilde{y}_1|h_{\text{SR}}) d\tilde{y}_1 - \int_{l_k}^{\infty} p(\tilde{y}_1|h_{\text{SR}}) d\tilde{y}_1 \right)\right]
\]

\[
= \Delta^2 \left[ \frac{1}{2} + 8 \sum_{k=1}^{\frac{b}{2}-1} k \int_{l_{k-1}}^{\infty} p(\tilde{y}_1|h_{\text{SR}}) d\tilde{y}_1 \right]
\]

\[
= \Delta^2 \left[ \frac{1}{2} + \frac{4\Delta^2}{M} \sum_{x \in S} \sum_{k=1}^{\frac{b}{2}-1} k Q\left(\frac{k\Delta - \alpha f_1(h_{\text{SR}})}{\alpha^2 \sigma^2} \right) \right]
\]

APPENDIX B

PROOF OF THEOREM 2

Given \( h_{\text{SR}} \), the MSE for \( f_1(\chi_{\text{SR}}) \) is written as

\[
E[(\tilde{y}_1 - f_1(\chi_{\text{SR}}))^2|h_{\text{SR}}] = E[\tilde{y}_1^2|h_{\text{SR}}] - 2E[\tilde{y}_1 f_1(\chi_{\text{SR}})|h_{\text{SR}}] + E[f_1^2(\chi_{\text{SR}})|h_{\text{SR}}] \]

(47)

where

\[
E[\tilde{y}_1^2|h_{\text{SR}}] = E[f_1(\alpha y_{\text{SR}})^2|h_{\text{SR}}] - \alpha^2 E[(f_1(h_{\text{SR}}) f_1(x) - f_2(h_{\text{SR}}) f_2(x) + f_1(z_{\text{SR}}))^2|h_{\text{SR}}] + \alpha^2 \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)
\]

(48)

and the last term in (47) equals the one in (7). To derive the middle term, the following result is given first:

\[
\int_{-\infty}^{\infty} \tilde{y}_1 p(\tilde{y}_1|h_{\text{SR}}) d\tilde{y}_1
\]
Applying (49), \(E[\tilde{y}f_1(x_R)|h_{SR}]\) is derived as
\[
E[\tilde{y}f_1(x_R)|h_{SR}] = \frac{4\Delta}{M} \sum_{k=1}^{1} \sum_{x \in S} \left( k \Delta - \alpha f_1(h_{SR}) \right)Q\left( \frac{k \Delta - \alpha f_1(h_{SR})}{\alpha / \sqrt{2}} \right) + \frac{\Delta^2}{4} \sum_{k=1}^{1} \sum_{x \in S} \left| x \right| \cos(\theta_h + \theta) + 2 \Delta \sum_{k=1}^{1} \left( k \Delta - \frac{1}{M} \sum_{x \in S} \left| k \Delta - |x| \cos(\theta_h + \theta) \right| \right) \quad (52)
\]
where (52) is due to
\[
\frac{2}{M} \sum_{x \in S} \left( k \Delta - \alpha f_1(h_{SR}) \right)Q\left( \frac{k \Delta - \alpha f_1(h_{SR})}{\alpha / \sqrt{2}} \right)
\]
\[
\approx \frac{1}{M} \sum_{x \in S} \left( k \Delta - \alpha f_1(h_{SR}) \right) \left( 1 - \text{sign}(k \Delta - \alpha f_1(h_{SR})) \right)
\]
\[
\approx k \Delta - \frac{1}{M} \sum_{x \in S} \alpha f_1(h_{SR}) \left( \text{sign}(k \Delta - \alpha f_1(h_{SR})) \right)
\]
\[
= k \Delta - \frac{1}{M} \sum_{x \in S} |k \Delta - \alpha f_1(h_{SR})| \quad (53)
\]
for \(k = 0, \ldots, \frac{L}{2} - 1\). Simplifying (52) and taking expectation related to \(\theta_h\), Lemma 1 is proved.

**APPENDIX D**

**PROOF OF LEMMA 2**

For the Rayleigh fading channel with \(p(\theta) = \frac{1}{2\pi} \), (12), (14), and (13) are rewritten as
\[
E[f_1^2(x_R)] \approx \frac{(L - 1)^2}{4} \Delta^2 - \frac{2\Delta^2}{M} \sum_{k=1}^{1} \sum_{x \in S} k \frac{1}{2\pi} \int_0^{2\pi} \text{sign}(k \Delta - |x| \cos(\theta_h + \theta_x)) d\theta \quad (53)
\]
\[
P(q_k) \approx \frac{1}{2\pi} \sum_{x \in S} \frac{1}{2\pi} \int_0^{2\pi} \left[ \text{sign}(l_k - |x| \cos(\theta_h + \theta_x)) - \text{sign}(l_{k-1} - |x| \cos(\theta_h + \theta_x)) \right] d\theta \quad (54)
\]
and
\[
E[(\tilde{y}_1 - f_1(x_R))^2] = \frac{(L - 1)^2}{4} \Delta^2 + \frac{\Delta}{2} \sum_{x \in S} \frac{1}{2\pi} \int_0^{2\pi} \left| k \Delta - |x| \cos(\theta_h + \theta_x) \right| d\theta
\]
\[
= \frac{(L - 1)^2}{4} \Delta^2 - \frac{2\Delta}{M} \sum_{x \in S} \frac{1}{2\pi} \int_0^{2\pi} \left[ \text{sign}(l_k - |x| \cos(\theta_h + \theta_x)) - \text{sign}(l_{k-1} - |x| \cos(\theta_h + \theta_x)) \right] d\theta \quad (55)
\]
respectively.
To simplify the equations, we first deal with the following integrals for a constant $c \geq 0$:

\[
\int_0^{2\pi} \text{sign}(c - |x| \cos(\theta + \theta_x)) d\theta = 2\pi - 4 \arccos \left( \frac{c}{|x|} \right) \tag{56}
\]

and

\[
\int_0^{2\pi} \left| c - |x| \cos(\theta + \theta_x) \right| d\theta = 2\pi - 4 \arccos \left( \frac{c}{|x|} \right) \tag{57}
\]

Since cos function is a periodic function, we have

\[
\int_0^{2\pi} g(c - |x| \cos(\theta + \theta_x)) d\theta = \int_{\theta_0}^{2\pi+\theta_0} g(c - |x| \cos \phi) d\phi
\]

\[
= \int_0^{2\pi} g(c - |x| \cos(\theta + \theta_x)) d\theta
\]

for both cases $g(t) = \text{sign}(t)$ and $g(t) = |t|$ corresponding to (56) and (57), respectively. Depending on the non-negative constant $c$, the integrals can be computed as follows.

1) WHEN $c \leq |x|$ If $\arccos \left( \frac{c}{|x|} \right) < \phi < 2\pi - \arccos \left( \frac{c}{|x|} \right)$, then $c - |x| \cos \phi > 0$; otherwise, $c - |x| \cos \phi < 0$. The respective integrals in (56) and (57) are derived as

\[
\int_0^{2\pi} \text{sign}(c - |x| \cos \phi) d\phi = 2\pi - 4 \arccos \left( \frac{c}{|x|} \right)
\]

and

\[
\int_0^{2\pi} \left| c - |x| \cos \phi \right| d\phi
\]

\[
= \int_{\arccos \left( \frac{c}{|x|} \right)}^{2\pi - \arccos \left( \frac{c}{|x|} \right)} (c - |x| \cos \phi) d\phi
\]

\[
- \int_0^{\arccos \left( \frac{c}{|x|} \right)} (c - |x| \cos \phi) d\phi
\]

\[
- \int_{2\pi - \arccos \left( \frac{c}{|x|} \right)}^{2\pi} (c - |x| \cos \phi) d\phi
\]

\[
= c(2\pi - 4 \arccos \left( \frac{c}{|x|} \right)) + 4|x| \sin(\arccos \left( \frac{c}{|x|} \right))
\]

\[
= c(2\pi - 4 \arccos \left( \frac{c}{|x|} \right)) + 4\sqrt{|x|^2 - c^2}.
\]

2) WHEN $c > |x|$ In this case, we have

\[
\int_0^{2\pi} \text{sign}(c - |x| \cos \phi) d\phi = 2\pi
\]

and

\[
\int_0^{2\pi} \left| c - |x| \cos \phi \right| d\phi = \int_0^{2\pi} (c - |x| \cos \phi) d\phi = 2\pi c.
\]

Combining the cases of $c \leq |x|$ and $c > |x|$, we have

\[
\int_0^{2\pi} \text{sign}(c - |x| \cos(\theta + \theta_x)) d\theta
\]

\[
= \int_0^{2\pi} \text{sign}(c - |x| \cos \phi) d\phi
\]

\[
= 2\pi - 4 \arccos \left( \frac{c}{|x|} \right)
\]

and

\[
\int_0^{2\pi} \left| c - |x| \cos(\theta + \theta_x) \right| d\theta
\]

\[
= \int_0^{2\pi} \left| c - |x| \cos \phi \right| d\phi
\]

\[
= c(2\pi - 4 \arccos \left( \frac{c}{|x|} \right)) + 4\sqrt{\max \left( 0, |x|^2 - c^2 \right)}
\]

\[
= c(2\pi - 4 \arccos \left( \frac{c}{|x|} \right)) + 4\sqrt{\max \left( 0, |x|^2 - c^2 \right)}.
\]

Applying (59) in (53) and (54) and applying (60) in (55), Lemma 2 is proved.

**APPENDIX E**

**PROOF OF (24)**

If $s_i(x) > 0$, we have

\[
\int_{s_i(x)\tilde{y}_i<l_i(x)-1} p(\tilde{y}_i|x, h_{SR})d\tilde{y}_i + \int_{s_i(x)\tilde{y}_i>l_i(x)} p(\tilde{y}_i|x, h_{SR})d\tilde{y}_i
\]

\[
= \int_{-\infty}^{l_i(x)-1} \frac{\exp\left( -\sqrt{2\alpha f_i(h_{SR})} \tilde{y}_i \left( \frac{s_i(x)\tilde{y}_i-l_i(x)+1}{a^2\sigma^2} \right) \right)}{\sqrt{\pi a^2\sigma^2}} d\tilde{y}_i + \int_{l_i(x)}^{\infty} \frac{\exp\left( -\sqrt{2\alpha f_i(h_{SR})} \tilde{y}_i \left( \frac{s_i(x)\tilde{y}_i-l_i(x)}{a^2\sigma^2} \right) \right)}{\sqrt{\pi a^2\sigma^2}} d\tilde{y}_i
\]

\[
= \left( \frac{2\alpha f_i(h_{SR}) - l_i(x)-1}{a^2\sigma^2} \right)
\]

\[
+ \left( \frac{2\alpha f_i(h_{SR}) - l_i(x)}{a^2\sigma^2} \right)
\]

If $s_i(x) < 0$, we have

\[
\int_{s_i(x)\tilde{y}_i<l_i(x)-1} p(\tilde{y}_i|x, h_{SR})d\tilde{y}_i + \int_{s_i(x)\tilde{y}_i>l_i(x)} p(\tilde{y}_i|x, h_{SR})d\tilde{y}_i
\]

\[
= \int_{-\infty}^{-l_i(x)-1} \frac{\exp\left( -\sqrt{2\alpha f_i(h_{SR})} \tilde{y}_i \left( \frac{s_i(x)\tilde{y}_i-1}{a^2\sigma^2} \right) \right)}{\sqrt{\pi a^2\sigma^2}} d\tilde{y}_i + \int_{-\infty}^{l_i(x)} \frac{\exp\left( -\sqrt{2\alpha f_i(h_{SR})} \tilde{y}_i \left( \frac{s_i(x)\tilde{y}_i+1}{a^2\sigma^2} \right) \right)}{\sqrt{\pi a^2\sigma^2}} d\tilde{y}_i
\]

\[
= \left( \frac{2\alpha f_i(h_{SR}) + l_i(x)-1}{a^2\sigma^2} \right)
\]

\[
+ \left( \frac{2\alpha f_i(h_{SR}) + l_i(x)}{a^2\sigma^2} \right)
\]

Summarizing the two cases, we have (24).

**REFERENCES**

[1] T. M. Cover and A. A. El Gamal, “Capacity theorems for the relay channel,” *IEEE Trans. Inf. Theory*, vol. IT-25, no. 5, pp. 572–584, Sep. 1979.
REFERENCES

[2] X. Jin and Y. Kim, “The approximate capacity of the MIMO relay channel,” IEEE Trans. Inf. Theory, vol. 63, no. 2, pp. 1167–1176, Feb. 2017.

[3] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, “Cooperative diversity in wireless networks: Efficient protocols and outage behavior,” IEEE Trans. Inf. Theory, vol. 50, no. 12, pp. 3062–3080, Dec. 2004.

[4] T. Wang, A. Cano, G. B. Giannakis, and J. N. Laneman, “High-performance cooperative demodulation with decode-and-forward relays,” IEEE Trans. Commun., vol. 55, no. 7, pp. 1427–1438, Jul. 2007.

[5] X. Jin, J.-S. No, and D.-J. Shin, “Relay selection for decode-and-forward cooperative network with multiple antennas,” IEEE Trans. Wireless Commun., vol. 10, no. 12, pp. 4068–4079, Dec. 2011.

[6] X. Jin and H.-N. Kim, “Switched-power two-layer superposition coding in cooperative decode-forward relay systems,” IEEE Trans. Wireless Commun., vol. 15, no. 3, pp. 2193–2204, Mar. 2016.

[7] X. Jin and H.-N. Kim, “A new switching superposition strategy in decode-forward relay system,” IEEE Trans. Veh. Technol., vol. 67, no. 8, pp. 7826–7830, Aug. 2018.

[8] M. O. Hasna and M.-S. Alouini, “Performance analysis of two-hop relayed transmissions over Rayleigh-fading channels,” in Proc. IEEE 56th Veh. Technol. Conf., Birmingham, AL, USA, Dec. 2002, pp. 1992–1996.

[9] A. Ribeiro, X. Cai, and G. B. Giannakis, “Symbol error probabilities for general cooperative links,” IEEE Trans. Wireless Commun., vol. 4, no. 3, pp. 1264–1273, May 2005.

[10] J.-C. Lin, H.-K. Chang, M.-L. Ku, and H. V. Poor, “Impact of imperfect source-to-relay CSI in amplify-and-forward relay networks,” IEEE Trans. Veh. Technol., vol. 66, no. 6, pp. 5056–5069, Jun. 2017.

[11] M. R. Souryal and H. You, “Quantize-and-forward relaying with M-ary phase shift keying,” in Proc. IEEE Wireless Commun. Netw. Conf., Mar. 2008, pp. 42–47.

[12] I. Avram, N. Aerts, H. Bruneel, and M. Moeneclaey, “Quantize and forward cooperative communication: Channel parameter estimation,” IEEE Trans. Wireless Commun., vol. 11, no. 3, pp. 1167–1179, Mar. 2012.

[13] X. Jin, “Cooperative linear combining on quantize-forward relay channel with M-ary phase shift keying,” IEEE Trans. Veh. Technol., vol. 71, no. 5, pp. 5645–5650, May 2022.

[14] A. Steiner and S. S. Shamai (Shitz), “Single-user broadcasting protocols over a two-hop relay fading channel,” IEEE Trans. Inf. Theory, vol. 52, no. 11, pp. 4821–4838, Nov. 2006.

[15] S. Haykin, Communication Systems, 4th ed. New York, NY, USA: Wiley, 2001.

[16] J. G. Proakis, Digital Communications, 4th ed. New York, NY, USA: McGraw-Hill, 2001.

[17] P. O. Börjesson and C. E. Sundberg, “Simple approximations of the error function Q(x) for communications applications,” IEEE Trans. Commun., vol. COM-27, no. 3, pp. 639–643, Mar. 1979.

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