A LIMIT APPROACH TO GROUP HOMOLOGY

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Abstract. In this paper, we consider for any free presentation $G = F/R$ of a group $G$ the coinvariance $H_0(G, R^{\otimes n}_{ab})$ of the $n$-th tensor power of the relation module $R_{ab}$ and show that the homology group $H_{2n}(G, \mathbb{Z})$ may be identified with the limit of the groups $H_0(G, R^{\otimes n}_{ab})$, where the limit is taken over the category of these presentations of $G$. We also consider the free Lie ring generated by the relation module $R_{ab}$, in order to relate the limit of the groups $\gamma_n R/\gamma_n R, F$ to the $n$-torsion subgroup of $H_{2n}(G, \mathbb{Z})$.

0. Introduction

It is well-known that one may use a presentation of a group $G$ as the quotient $F/R$, where $F$ is a free group, in order to calculate its (co-)homology. Besides Hopf’s formula for the second homology group $H_2(G, \mathbb{Z})$ (cf. [1, Chapter II, Theorem 5.3]), another example supporting that claim is the existence of the Gruenberg resolution [3]. Using Quillen’s description of the cyclic homology of an algebra over a field of characteristic 0 as the limit of a suitable functor over the category of extensions of the algebra (cf. [6]), the homology groups $H_n(G, Q)$ are described in [2] as the limits of certain functors over the category of group extensions $G = K/H$ (here, the group $K$ is not necessarily free).

Working in the same direction, we obtain in this paper a description of the even homology of $G$ with coefficients in an arbitrary $\mathbb{Z}G$-module $M$ as the limit of a functor over the category $\mathfrak{P}$ of all free presentations $G = F/R$. More precisely, we use the associated relation module $R_{ab} = R/[R, R]$ and prove that there is an isomorphism

$$H_{2n}(G, M) \simeq \lim_{\leftarrow} H_0(G, M \otimes R^{\otimes n}_{ab}),$$

where the limit is taken over $\mathfrak{P}$. We note that the technique used in the present paper allows us to interpret only the even homology of $G$ as a limit. Together with the free associative ring $TR_{ab}$ on $R_{ab}$ (which is built up by the tensor powers $R^{\otimes n}_{ab}, n \geq 0$), we may also consider the free Lie ring $\mathfrak{L}R_{ab}$ on $R_{ab}$. The Lie ring $\mathfrak{L}R_{ab}$ is graded and its homogeneous component in degree $n \geq 1$ consists of the abelian group $\gamma_n R/\gamma_{n+1} R$, where $(\gamma_i R)_{i \geq 1}$ is the lower central series of $R$. Then, the inclusion $\mathfrak{L}R_{ab} \subseteq TR_{ab}$ induces a natural map

$$l_n : \gamma_n R/\gamma_n R, F \rightarrow H_0(G, R^{\otimes n}_{ab})$$

for all $n \geq 1$. The group $\gamma_n R/\gamma_n R, F$ is the kernel of the free central extension

$$1 \rightarrow \gamma_n R/\gamma_n R, F \rightarrow F/\gamma_n R, F \rightarrow F/\gamma_n R \rightarrow 1$$

and can be identified, in view of Hopf’s formula, with the homology group $H_2(F/\gamma_n R, \mathbb{Z})$. It has been studied by many authors; a survey of the corresponding results may be found in [9]. As an example, we note that the torsion subgroup of $\gamma_n R/\gamma_n R, F$, which is shown in [loc.cit.]
to be an $n$-torsion group if $n \geq 3$, may be identified with the kernel of the so-called Gupta representation of $F/\langle \gamma_n R, F \rangle$ (cf. [4,8,10]). Confirming the existence of a close relationship between the groups $\gamma_n R/\langle \gamma_n R, F \rangle$ and the torsion in the homology of $G$, we show that the $l_n$’s induce an additive map

$$\ell_n : \lim \gamma_n R/\langle \gamma_n R, F \rangle \to \lim H_0(G, R_{ab}^{\otimes n}) \cong H_{2n}(G, \mathbb{Z}),$$

whose image is contained in the $n$-torsion subgroup of $H_{2n}(G, \mathbb{Z})$.

The contents of the paper are as follows: In Section 1, we explain how one can use dimension shifting by the powers of the relation module $R_{ab}$, which is associated with a presentation $G = F/R$, in order to embed the homology groups $H_{2n}(G, -)$ into $H_0(G, - \otimes R_{ab}^{\otimes n})$ for all $n \geq 1$. In the following Section, we record some generalities about limits and prove a simple criterion for them to vanish. In Section 3, we define the presentation category $\mathcal{P}$ of $G$ and prove the existence of an isomorphism between $H_{2n}(G, -)$ and the limit of the $H_0(G, - \otimes R_{ab}^{\otimes n})$’s. Finally, in the last Section, we consider the free Lie ring on the relation module $R_{ab}$ and relate the limit of the quotients $\gamma_n R/\langle \gamma_n R, F \rangle$ to the $n$-torsion subgroup of $H_{2n}(G, \mathbb{Z})$.

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1. Relation modules and dimension shifting in homology

In this Section, we consider a group $G$ and fix a presentation of it as the quotient of a free group $F = F(S)$ on a set $S$ by a normal subgroup $R$. We note that the conjugation action of $F$ on $R$ induces an action of $F$ on the abelianization $R_{ab} = R/[R,R]$, which is obviously trivial when restricted to $R$. Therefore, the latter action induces an action of $G$ on $R_{ab}$. The abelian group $R_{ab}$, endowed with the $G$-action defined above, is referred to as the relation module of the given presentation.

The augmentation ideal $\mathfrak{f} \subseteq ZF$ of $F$ is well-known to be free as a $ZF$-module; in fact, it is free on the set $\{s-1 : s \in S\}$. In particular, the $ZG$-module $ZG \otimes_{ZF} \mathfrak{f}$ is free on the set $\{1 \otimes (s-1) : s \in S\}$. Moreover, it follows from [1, Chapter II, Proposition 5.4] that there is an exact sequence of $ZG$-modules

$$0 \to R_{ab} \xrightarrow{\mu} ZG \otimes_{ZF} \mathfrak{f} \xrightarrow{\sigma} ZG \xrightarrow{\varepsilon} Z \to 0,$$

where $\mu$ maps $r[R,R]$ onto $1 \otimes (r-1)$ for all $r \in R$, $\sigma$ maps $1 \otimes (s-1)$ onto $sR-1$ for all $s \in S$ and $\varepsilon$ is the augmentation homomorphism. We note that $R$, being a subgroup of the free group $F$, is itself free; therefore, the relation module $R_{ab}$ is $Z$-free. Since this is also the case for the other three terms of the exact sequence (1), we conclude that the latter is $Z$-split. We shall refer to the exact sequence (1) as the relation sequence associated with the given presentation of $G$. The map $\mu$ therein was defined by Magnus in [5]; it will be referred to as the Magnus embedding.

**Lemma 1.1.** Let $M$ be a $ZG$-module. Then, there are natural isomorphisms $H_i(G, M) \cong H_{i-2}(G, M \otimes R_{ab})$ for all $i \geq 3$, where $G$ acts on $M \otimes R_{ab}$ diagonally.

**Proof.** Since the relation sequence (1) is $Z$-split, we may tensor it with $M$ and obtain the exact sequence of $ZG$-modules (with diagonal action)

$$0 \to M \otimes R_{ab} \to M \otimes (ZG \otimes_{ZF} \mathfrak{f}) \to M \otimes ZG \to M \to 0.$$
If $N$ is a free $ZG$-module, then the $ZG$-module $M \otimes N$ (with diagonal action) is known to be isomorphic with an induced module (cf. [1, Chapter III, Corollary 5.7]); in particular, the homology of $G$ with coefficients in $M \otimes N$ vanishes in positive degrees. Since the $ZG$-modules $ZG \otimes_{ZP} P$ and $ZG$ are free, we may use the exact sequence (2) and dimension shifting, in order to obtain the existence of natural isomorphisms, as claimed.

\begin{proof}
The result follows by induction on $n$, using Lemma 1.1.
\end{proof}

\begin{corollary}
Let $M$ be a $ZG$-module. Then, there are natural isomorphisms $H_{2n}(G, M) \simeq H_2(G, M \otimes R_{ab}^{\otimes n+1})$ and $H_{2n+1}(G, M) \simeq H_1(G, M \otimes R_{ab}^{\otimes n})$ for all $n \geq 1$.

\begin{proof}

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There are isomorphisms $H_{2n}(G, \mathbb{Z}) \simeq H_2(G, R_{ab}^{\otimes n-1})$ and $H_{2n+1}(G, \mathbb{Z}) \simeq H_1(G, R_{ab}^{\otimes n})$ for all $n \geq 1$.

\begin{proof}

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\begin{remark}
The dimension shifting in the homology of a group $G$, which is associated with the relation module $R_{ab}$ as above, may be alternatively described by using cap products; see, for example, [11, §2.3]. More precisely, let $\chi \in H^2(G, R_{ab})$ be the cohomology class that classifies the group extension
\[
1 \rightarrow R/[R, R] \rightarrow F/[R, R] \rightarrow G \rightarrow 1,
\]
as in [1, Chapter IV, Theorem 3.12]. Then, the dimension shifting isomorphisms above are induced by the cap product maps with $\chi$ or with suitable powers of it.

We consider a $ZG$-module $M$ and note that the Lyndon-Hochschild-Serre spectral sequence associated with the extension
\[
1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1
\]
induces in low degrees the exact sequence
\[
0 \rightarrow H_2(G, M) \rightarrow H_0(G, H_1(R, M)) \rightarrow H_1(F, M) \rightarrow H_1(G, M) \rightarrow 0.
\]
Since $M$ is trivial as a $ZR$-module, we have $H_1(R, M) = M \otimes R_{ab}$ and hence the latter exact sequence reduces to
\[
0 \rightarrow H_2(G, M) \rightarrow H_0(G, M \otimes R_{ab}) \rightarrow H_1(F, M) \rightarrow H_1(G, M) \rightarrow 0.
\]
We note that the above embedding of $H_2(G, M)$ into $H_0(G, M \otimes R_{ab})$, which is provided by the $d^2$-differential of the spectral sequence, is known to coincide (up to a sign) with the cap product map with the cohomology class $\chi \in H^2(G, R_{ab})$ defined in Remark 1.4. In particular, replacing $M$ by $M \otimes R_{ab}^{\otimes n-1}$, we conclude that there is an exact sequence
\[
0 \rightarrow H_2(G, M \otimes R_{ab}^{\otimes n-1}) \xrightarrow{\partial} H_0(G, M \otimes R_{ab}^{\otimes n}) \rightarrow H_1(F, M \otimes R_{ab}^{\otimes n-1}) \rightarrow H_1(G, M \otimes R_{ab}^{\otimes n-1}) \rightarrow 0
\]
for all $n \geq 1$.

Taking into account Corollary 1.2 and Remark 1.4, we may state the following result.

\begin{proposition}
Let $M$ be a $ZG$-module and consider the cohomology class $\chi \in H^2(G, R_{ab})$ defined in Remark 1.4. Then, there is an exact sequence
\[
0 \rightarrow H_{2n}(G, M) \xrightarrow{\chi} H_0(G, M \otimes R_{ab}^{\otimes n}) \rightarrow H_1(F, M \otimes R_{ab}^{\otimes n-1}) \rightarrow H_1(G, M \otimes R_{ab}^{\otimes n-1}) \rightarrow 0
\]
\end{proposition}
for all \( n \geq 1 \). In particular, there is an exact sequence

\[
0 \longrightarrow H_{2n}(G, \mathbb{Z}) \overset{\chi_{2n}}{\longrightarrow} H_0(G, R_{ab}^{\otimes n}) \longrightarrow H_1(F, R_{ab}^{\otimes n-1}) \longrightarrow H_1(G, R_{ab}^{\otimes n-1}) \longrightarrow 0
\]

for all \( n \geq 1 \). \( \square \)

2. Some generalities on limits

Let \( C \) be a small category, \( Ab \) the category of abelian groups and \( \mathcal{F} : C \to Ab \) a functor. Then, the limit \( \varinjlim \mathcal{F} \) of \( \mathcal{F} \) is the subgroup of the direct product \( \prod_{c \in C} \mathcal{F}(c) \), consisting of those families \( (x_c)_c \) which are compatible in the following sense: For any two objects \( c, c' \in C \) and any morphism \( a \in \text{Hom}_C(c, c') \), we have \( \mathcal{F}(a)(x_c) = x_{c'} \in \mathcal{F}(c') \). We often denote the abelian group \( \varinjlim \mathcal{F} \) by \( \varinjlim \mathcal{F}(c) \).

Let \( \mathcal{F}, \mathcal{G} \) be two functors from \( C \) to \( Ab \). Then, a natural transformation \( \eta : \mathcal{F} \to \mathcal{G} \) induces an additive map

\[
\varinjlim \eta : \varinjlim \mathcal{F} \to \varinjlim \mathcal{G},
\]

by mapping any element \( (x_c)_c \in \varinjlim \mathcal{F} \) onto \( (\eta_c(x_c))_c \in \varinjlim \mathcal{G} \). In this way, \( \varinjlim \) itself becomes a functor from the functor category \( Ab^C \) to \( Ab \).

The proof of the following result is straightforward.

**Lemma 2.1.** The limit functor \( \varinjlim : Ab^C \to Ab \) is left exact. \( \square \)

We recall that the coproduct of two objects \( a, b \) of \( C \) is an object \( a \star b \), which is endowed with two morphisms \( \iota_a : a \to a \star b \) and \( \iota_b : b \to a \star b \), having the following universal property: For any object \( c \) of \( C \) and any pair of morphisms \( f : a \to c \) and \( g : b \to c \), there is a unique morphism \( h : a \star b \to c \), such that \( h \circ \iota_a = f \) and \( h \circ \iota_b = g \). The morphism \( h \) is usually denoted by \( (f, g) \).

As an example, we note that the coproduct of two abelian groups \( M \) and \( N \) in the category \( Ab \) is the direct sum \( M \oplus N \), endowed with the obvious inclusion maps. For any abelian group \( T \) and any pair of additive maps \( f : M \to T \) and \( g : N \to T \), the additive map \( (f, g) : M \oplus N \to T \) is given by \( (m, n) \mapsto f(m) + g(n) \), \( (m, n) \in M \oplus N \).

The following elementary vanishing criterion will be used twice in the sequel.

**Lemma 2.2.** Let \( C \) be a small category and \( \mathcal{F} : C \to Ab \) a functor. We assume that:

(i) Any two objects \( a, b \) of \( C \) have a coproduct \( (a \star b, \iota_a, \iota_b) \) as above.

(ii) For any two objects \( a, b \) of \( C \) the morphisms \( \iota_a : a \to a \star b \) and \( \iota_b : b \to a \star b \) induce a monomorphism

\[
(\mathcal{F}(\iota_a), \mathcal{F}(\iota_b)) : \mathcal{F}(a) \oplus \mathcal{F}(b) \to \mathcal{F}(a \star b)
\]

of abelian groups.

Then, the limit \( \varinjlim \mathcal{F} \) is the zero group.

**Proof.** Let \( (x_c)_c \in \varinjlim \mathcal{F} \) be a compatible family and fix an object \( a \) of \( C \). We consider the coproduct \( a \star a \) of two copies of \( a \) and the morphisms \( \iota_1 : a \to a \star a \) and \( \iota_2 : a \to a \star a \). Then, we have

\[
\mathcal{F}(\iota_1)(x_a) = x_{a \star a} = \mathcal{F}(\iota_2)(x_a)
\]
and hence the element \((x_a, -x_a)\) is contained in the kernel of the additive map
\[
(\mathfrak{F}(t_1), \mathfrak{F}(t_2)) : \mathfrak{F}(a) \oplus \mathfrak{F}(a) \to \mathfrak{F}(a \ast a).
\]
In view of our assumption, this latter map is injective and hence \(x_a = 0\). Since this is the case for any object \(a\) of \(C\), we conclude that the family \((x_c)_c\) is the zero family, as needed. \(\square\)

3. A LIMIT FORMULA FOR \(H_{2n}(G, -)\)

We fix a group \(G\) and define the category of presentations \(\mathfrak{P} = \mathfrak{P}(G)\), as follows: The objects of \(\mathfrak{P}\) are pairs of the form \((F, \pi)\), where \(F\) is a free group and \(\pi : F \to G\) a surjective group homomorphism. Given two objects \((F, \pi)\) and \((F', \pi')\) of \(\mathfrak{P}\), a morphism from \((F, \pi)\) to \((F', \pi')\) is a group homomorphism \(\varphi : F \to F'\) such that \(\pi' \circ \varphi = \pi\). Since the groups that are involved are free, we note that for any two objects \((F, \pi)\) and \((F', \pi')\) of \(\mathfrak{P}\) there is at least one morphism from \((F, \pi)\) to \((F', \pi')\).

Given an object \((F, \pi)\) of \(\mathfrak{P}\), we may consider the group ring \(\mathbb{Z}F\), the augmentation ideal \(\mathfrak{f}\), the kernel \(R = \ker \pi\), the relation module \(R_{ab}\) and the associated Magnus embedding
\[
\mu : R_{ab} \to \mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f}.
\]
It is clear that all these depend naturally on the object \((F, \pi)\) of \(\mathfrak{P}\). Moreover, this is also true for the cohomology class \(\chi \in H^2(G, R_{ab})\) defined in Remark 1.4. Therefore, invoking the naturality of the low degrees exact sequence which is induced by the Lyndon-Hochschild-Serre spectral sequence with respect to the group extension and the coefficient module, we conclude that the dimension shifting isomorphisms as well as the exact sequences of Proposition 1.5 are natural with respect to the morphisms of \(\mathfrak{P}\). In view of the left exactness of the limit functor (cf. Lemma 2.1), we thus obtain an exact sequence
\[
\begin{align*}
0 \to H_{2n}(G, M) \to \lim H_0(G, M \otimes R_{ab}^{\otimes n}) \to \lim H_1(F, M \otimes R_{ab}^{\otimes n-1})
\end{align*}
\]
for all \(n \geq 1\), where the limits are taken over the category \(\mathfrak{P}\).

**Lemma 3.1.** Let \((F, \pi)\) and \((F', \pi')\) be two objects of the presentation category \(\mathfrak{P}\) of \(G\).

(i) The coproduct \((F, \pi) \ast (F', \pi')\) is provided by the object \((F'', \pi'')\) of \(\mathfrak{P}\), where \(F''\) is the free product of \(F\) and \(F'\) and \(\pi'' : F'' \to G\) the homomorphism which extends both \(\pi\) and \(\pi'\).

(ii) Let \(\iota : (F, \pi) \to (F'', \pi'')\) and \(\iota' : (F', \pi') \to (F'', \pi'')\) be the structural morphisms of the coproduct \((F'', \pi'')\). Then, the induced maps \(\iota_* : R_{ab} \to R_{ab}''\) and \(\iota'_* : R_{ab}' \to R_{ab}''\) between the corresponding relation modules are both split monomorphisms of \(\mathbb{Z}G\)-modules.

**Proof.** Assertion (i) is clear and, because of symmetry, we only have to prove assertion (ii) for the structural morphism \(\iota\). We note that the additive map \(\iota_* : R_{ab} \to R_{ab}''\) is obtained by restricting \(\iota\) and then passing to the quotients. We choose a morphism \(\varphi : (F', \pi') \to (F, \pi)\) in \(\mathfrak{P}\) and consider the morphism \(\lambda = (id_F, \varphi) : (F'', \pi'') \to (F, \pi)\), which extends both the identity of \((F, \pi)\) and \(\varphi\). Then, \(\lambda\) restricts to a group homomorphism \(\lambda_0 : R'' \to R\), which is a left inverse of the restriction \(\iota_0 : R \to R''\) of \(\iota\) and satisfies the equality
\[
\lambda_0(\iota(x)r'' \iota(x)^{-1}) = x \lambda_0(r'') x^{-1}
\]
for all \(x \in F\) and \(r'' \in R''\). It follows that the additive map \(\lambda_* : R_{ab}'' \to R_{ab}\), which is induced from \(\lambda_0\) by passage to the quotients, is a \(\mathbb{Z}G\)-linear left inverse of \(\iota_*\). \(\square\)

We can now state and prove our first main result.
Theorem 3.2. Let $M$ be a $\mathbb{Z}G$-module. Then, there is an isomorphism of abelian groups
\[
H_{2n}(G, M) \xrightarrow{\sim} \lim_{\to} H_0(G, M \otimes R_{ab}^{\otimes n}),
\]
where the limit is taken over the category $\mathfrak{P}$ of presentations of $G$ for all $n \geq 1$. In particular, there is an isomorphism
\[
H_{2n}(G, \mathbb{Z}) \xrightarrow{\sim} \lim_{\to} H_0(G, R_{ab}^{\otimes n})
\]
for all $n \geq 1$.

Proof. Let $\mathfrak{F} : \mathfrak{P} \rightarrow Ab$ be the functor which maps an object $(F, \pi)$ of $\mathfrak{P}$ onto the abelian group $H_1(F, M \otimes R_{ab}^{\otimes n-1})$. In view of the exact sequence (3), the result will follow if we show that $\lim \mathfrak{F} = 0$. To that end, we shall apply the criterion established in Lemma 2.2. We have to verify that conditions (i) and (ii) therein are satisfied. To that end, we fix two objects $(F, \pi)$ and $(F', \pi')$ of $\mathfrak{P}$ and denote by $R_{ab}$ and $R_{ab}'$ the corresponding relation modules.

In view of Lemma 3.1(i), the objects $(F, \pi)$ and $(F', \pi')$ have a coproduct in $\mathfrak{P}$, which is provided by $(F'', \pi'')$, where $F''$ is the free product of $F$ and $F'$. Let $R_{ab}''$ denote the relation module that corresponds to the coproduct $(F'', \pi'')$. We have to prove that the map
\[
H_1(F, M \otimes R_{ab}^{\otimes n-1}) \oplus H_1(F', M \otimes R_{ab}'^{\otimes n-1}) \longrightarrow H_1(F'', M \otimes R_{ab}''^{\otimes n-1}),
\]
which is induced by the inclusions of $F$ and $F'$ into $F''$, is injective. To that end, we note that the corresponding Mayer-Vietoris exact sequence shows that the natural map
\[
H_1(F, M \otimes R_{ab}^{\otimes n-1}) \oplus H_1(F', M \otimes R_{ab}'^{\otimes n-1}) \longrightarrow H_1(F'', M \otimes R_{ab}''^{\otimes n-1})
\]
is injective. Therefore, it only remains to prove that the natural maps
\[
H_1(F, M \otimes R_{ab}^{\otimes n-1}) \longrightarrow H_1(F, M \otimes R_{ab}'^{\otimes n-1})
\]
and
\[
H_1(F', M \otimes R_{ab}'^{\otimes n-1}) \longrightarrow H_1(F', M \otimes R_{ab}''^{\otimes n-1})
\]
are injective. We may now complete the proof invoking Lemma 3.1(ii), which itself implies that the natural map $R_{ab}^{\otimes n-1} \longrightarrow R_{ab}'^{\otimes n-1}$ (resp. $R_{ab}'^{\otimes n-1} \longrightarrow R_{ab}''^{\otimes n-1}$) is a split monomorphism of $\mathbb{Z}G$-modules and hence of $\mathbb{Z}F$-modules. \hfill \Box

4. The limit of the $\gamma_n R/\gamma_{n+1} R, F$’s

Let $H$ be a group. We recall that the lower central series $(\gamma_n H)_{n \geq 1}$ of $H$ is given by $\gamma_1 H = H$ and $\gamma_{n+1} H = [\gamma_n H, H]$ for all $n \geq 1$. Then, the graded Lie ring $Gr H = \bigoplus_{n=1}^{\infty} Gr^n H$ of $H$ is defined in degree $n$ to be the (additively written) abelian group $Gr^n H = \gamma_n H/\gamma_{n+1} H$. The Lie bracket on $Gr H$ is defined by letting
\[
(x \gamma_{n+1} H, y \gamma_{m+1} H) = [x, y] \gamma_{n+m+1} H,
\]
where $[x, y] = x^{-1} y^{-1}xy$ for all $x \in \gamma_n H$ and $y \in \gamma_m H$ (cf. [7, Chapter 2]).

On the other hand, if $A$ is an abelian group then we may consider the free associative ring on $A$, i.e. the tensor ring $TA = \bigoplus_{n=0}^{\infty} A^{\otimes n}$. We recall that the multiplication in $TA$ is defined by concatenation of tensors. The associated Lie ring $LTA$ is equal to $TA$ as an abelian group, whereas its Lie bracket is defined by letting $(x, y) = xy - yx$ for all $x, y \in TA$. The free Lie ring on $A$ is the Lie subring $\mathfrak{L}A$ of $LTA$ generated by $A$. In fact, $\mathfrak{L}A$ is a graded subring
of $LTA$, whose homogeneous component $\mathfrak{L}_nA \subseteq A^{\otimes n}$ of degree $n$ is generated as an abelian group by the left normed $n$-fold commutators $(x_1, \ldots, x_n)$, $x_1, \ldots, x_n \in A$.

We now consider a group $H$ and its abelianization $H_{ab} = H/[H, H]$. Then, in view of the universal property of the free Lie ring $\mathfrak{L}H_{ab}$, the identity map of $H_{ab} = \mathfrak{L}_1H_{ab}$ into $H_{ab} = Gr^1H$ extends to a graded Lie ring homomorphism

$$\kappa : \mathfrak{L}H_{ab} \to Gr H.$$ 

It is clear that $\kappa$ depends naturally on $H$. In particular, for all $n \geq 1$ there is an additive map

$$\kappa_n : \mathfrak{L}_nH_{ab} \to \gamma_nH/\gamma_{n+1}H,$$

which is natural in $H$. We note that if the group $H$ is free then the map $\kappa$ (and hence all of the $\kappa_n$'s) is bijective; cf. [7, Chapter 4, Theorem 6.1].

We now fix a group $G$ and consider an object $(F, \pi)$ of the presentation category $\mathfrak{P}$ of $G$ with kernel $R = \ker \pi$. We specialize the discussion above to $R$ and note that the terms of its lower central series are normal subgroups of $F$; in particular, $F$ acts on each quotient $Gr^nR = \gamma_nR/\gamma_{n+1}R$, by letting $x \cdot y\gamma_{n+1}R = xyx^{-1}\gamma_{n+1}R$ for all $x \in F$ and $y \in \gamma_nR$. The latter action being trivial on $R$, it induces an action of $G$ on the $Gr^nR$'s. Endowed with that action, the abelian group $Gr^nR$ is referred to as the $n$-th higher relation module associated with the given presentation. (For $n = 1$, we recover the relation module $Gr^1R = R_{ab}$.) It is clear that the induced action of $G$ on $Gr R$ is compatible with the Lie bracket. On the other hand, the diagonal action of $G$ on the tensor powers $R_{ab}^{\otimes n}$ induces a $G$-action on $TR_{ab}$, which is compatible with multiplication. In particular, $G$ acts on the associated Lie ring $LTR_{ab}$ by Lie ring automorphisms. It is easily seen that the action of any group element on $LTR_{ab}$ restricts to a Lie ring automorphism of the free Lie ring $\mathfrak{L}R_{ab}$. In particular, $\mathfrak{L}R_{ab}$ is a $ZG$-submodule of $LTR_{ab}$ and the homogeneous component $\mathfrak{L}_nR_{ab}$ is a $ZG$-submodule of $R_{ab}^{\otimes n}$ for all $n \geq 1$.

In view of the naturality of the additive map (4) with respect to group homomorphisms, we conclude that the additive map

$$\kappa_n : \mathfrak{L}_nR_{ab} \to \gamma_nR/\gamma_{n+1}R$$

is $ZG$-linear for all $n \geq 1$. Moreover, since the group $R$ is free (being a subgroup of the free group $F$), the latter map is an isomorphism. For all $n \geq 1$ we consider the $ZG$-linear map

$$\lambda_n : \gamma_nR/\gamma_{n+1}R \to R_{ab}^{\otimes n},$$

which is defined as the composition

$$\gamma_nR/\gamma_{n+1}R \xrightarrow{\kappa_n^{-1}} \mathfrak{L}_nR_{ab} \xleftarrow{\mathfrak{L}_n} R_{ab}^{\otimes n}.$$ 

Since the group $H_0(G, \gamma_nR/\gamma_{n+1}R)$ is identified with $\gamma_nR/[\gamma_nR, F]$, the $ZG$-linear map $\lambda_n$ defined above induces an additive map

$$l_n : \gamma_nR/[\gamma_nR, F] \to H_0(G, R_{ab}^{\otimes n})$$

for all $n \geq 1$. The abelian groups $J_n^G(R_{ab}, Z) = \ker l_n$ have been studied in [10] by M.W. Thomson, who proved that they are $n$-torsion for all $n \geq 1$. It is clear that $l_n$ depends naturally on the object $(F, \pi)$ of the presentation category $\mathfrak{P}$ of $G$. Therefore, taking limits over $\mathfrak{P}$, we obtain the additive map

$$\ell_n = \lim l_n : \lim \gamma_nR/[\gamma_nR, F] \to \lim H_0(G, R_{ab}^{\otimes n}).$$
We can now state our second main result.

**Theorem 4.1.** Let $n$ be an integer with $n \geq 2$. Then, under the isomorphism between the homology group $H_{2n}(G, \mathbb{Z})$ of $G$ and the limit $\lim H_0(G, R_{ab}^{\otimes n})$, which is established in Theorem 3.2, the image of the additive map $\ell_n$ defined above is contained in the $n$-torsion subgroup $H_{2n}(G, \mathbb{Z})[n]$ of $H_{2n}(G, \mathbb{Z})$.

The proof of the Theorem will occupy the remaining of the Section. Let $(F, \pi)$ be an object of the presentation category $\mathcal{P}$ of $G$ and consider the associated Magnus embedding

$$\mu : R_{ab} \longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}F} f,$$

and the $n$-th tensor power map

$$\mu^{\otimes n} : R_{ab}^{\otimes n} \longrightarrow (\mathbb{Z}G \otimes_{\mathbb{Z}F} f)^{\otimes n},$$

which is also ZG-linear. The composition

$$\gamma_n R/\gamma_{n+1} R \xrightarrow{\lambda_n} R_{ab}^{\otimes n} \xrightarrow{\mu^{\otimes n}} (\mathbb{Z}G \otimes_{\mathbb{Z}F} f)^{\otimes n}$$

is a ZG-module map, which induces, by applying the functor $H_0(G, \_)$, the composition

$$\gamma_n R/\gamma_n R, F \xrightarrow{\lambda_n} H_0(G, R_{ab}^{\otimes n}) \xrightarrow{\mu^{\otimes n}} H_0(G, (\mathbb{Z}G \otimes_{\mathbb{Z}F} f)^{\otimes n}).$$

We shall denote the latter composition by $\varphi_n$. We deduce the existence of an exact sequence

$$0 \longrightarrow J_n^G(R_{ab}, \mathbb{Z}) \longrightarrow \ker \varphi_n \xrightarrow{l_n} H_0(G, R_{ab}^{\otimes n}),$$

where $l_n|$ denotes the restriction of $l_n$ to the subgroup $\ker \varphi_n \subseteq \gamma_n R/\gamma_n R, F$. The key point is that, as shown in [8], the map $l_n$ maps $\ker \varphi_n$ into the $n$-torsion subgroup $H_0(G, R_{ab}^{\otimes n})[n]$ of $H_0(G, R_{ab}^{\otimes n})$ for all $n \geq 2$. Therefore, we conclude that there is an exact sequence

$$0 \longrightarrow J_n^G(R_{ab}, \mathbb{Z}) \longrightarrow \ker \varphi_n \xrightarrow{l_n} H_0(G, R_{ab}^{\otimes n})[n].$$

We shall now consider the commutative diagram with exact rows

$$\begin{array}{ccc}
0 & \longrightarrow & J_n^G(R_{ab}, \mathbb{Z}) \\
\| & \parallel & \downarrow \\
0 & \longrightarrow & J_n^G(R_{ab}, \mathbb{Z})
\end{array} \xrightarrow{\ker \varphi_n} \downarrow \xrightarrow{l_n} H_0(G, R_{ab}^{\otimes n})[n] \xrightarrow{\downarrow} 0$$

where both unlabelled vertical arrows are the corresponding inclusion maps. Since all maps involved are natural with respect to the given object $(F, \pi)$ of the presentation category $\mathcal{P}$ of $G$, we may invoke Lemma 2.1 in order to obtain a commutative diagram with exact rows

$$\begin{array}{ccc}
0 & \longrightarrow & \varprojlim J_n^G(R_{ab}, \mathbb{Z}) \\
\| & \parallel & \downarrow \\
0 & \longrightarrow & \varprojlim J_n^G(R_{ab}, \mathbb{Z})
\end{array} \xrightarrow{\ell_n} \varprojlim H_0(G, R_{ab}^{\otimes n})[n] \xrightarrow{\downarrow} 0$$

\[\text{As shown in [10, Proposition 1], if } n \geq 2 \text{ then the kernel of } \varphi_n \text{ can be identified with the kernel of a certain matrix representation of the group } F/\gamma_n R, F, \text{ which was defined by C.K. Gupta and N.D. Gupta in [4].}\]
Since the limit \( \lim H_0(G, R_{ab}^\otimes n) [n] \) of the \( n \)-torsion subgroups is identified with the \( n \)-torsion subgroup of the limit \( \lim H_0(G, R_{ab}^\otimes) \), the assertion in the statement of Theorem 4.1 follows from the next result.

**Lemma 4.2.** The additive map \( \lim \ker \varphi_n \longrightarrow \lim \gamma_n R/[\gamma_n R, F] \), which is induced by the inclusions \( \ker \varphi_n \hookrightarrow \gamma_n R/[\gamma_n R, F] \), is an isomorphism for all \( n \geq 1 \).

**Proof.** In view of Lemma 2.1, the exact sequence

\[
0 \longrightarrow \ker \varphi_n \longrightarrow \gamma_n R/[\gamma_n R, F] \xrightarrow{\varphi_n} H_0 \left( G, (ZG \otimesZF)^\otimes n \right),
\]

which is associated with an object \((F, \pi)\) of \( \mathfrak{P} \) as above, induces an exact sequence

\[
0 \longrightarrow \lim \ker \varphi_n \longrightarrow \lim \gamma_n R/[\gamma_n R, F] \xrightarrow{\phi_n} \lim H_0 \left( G, (ZG \otimesZF)^\otimes n \right),
\]

where \( \phi_n = \lim \varphi_n \). Hence, the result will follow if we show that \( \lim H_0(G, (ZG \otimesZF)^\otimes n) = 0 \).

To that end, we shall apply the criterion established in Lemma 2.2. We have to verify that conditions (i) and (ii) therein are satisfied. In view of Lemma 3.1, any two objects \((F, \pi)\) and \((F', \pi')\) of \( \mathfrak{P} \) have a coproduct, which is provided by \((F'', \pi'')\), where \( F'' \) is the free product of \( F \) and \( F' \). Therefore, if \( F \) (resp. \( F' \)) is free on the set \( S \) (resp. \( S' \)), then \( F'' \) is free on the disjoint union \( S'' \) of \( S \) and \( S' \). It follows that the \( ZG \)-modules \( ZG \otimesZF, ZG \otimesZF', ZG \otimesZF'' \) are free on the sets \( \{1 \otimes (s-1) : s \in S\}, \{1 \otimes (s'-1) : s' \in S'\} \) and \( \{1 \otimes (s''-1) : s'' \in S''\} \) respectively. Hence, the inclusions of \( F \) and \( F' \) into \( F'' \) induce an isomorphism of \( ZG \)-modules

\[
(ZG \otimesZF) \oplus (ZG \otimesZF') \xrightarrow{\sim} ZG \otimesZF''.
\]

Therefore, considering \( n \)-th tensor powers, we conclude that the natural map

\[
(ZG \otimesZF)^\otimes n \oplus (ZG \otimesZF')^\otimes n \longrightarrow (ZG \otimesZF'')^\otimes n
\]

is a split monomorphism of \( ZG \)-modules. Therefore, applying the functor \( H_0(G, \underline{\_}) \), we conclude that the natural map

\[
H_0 \left( G, (ZG \otimesZF)^\otimes n \right) \oplus H_0 \left( G, (ZG \otimesZF')^\otimes n \right) \longrightarrow H_0 \left( G, (ZG \otimesZF')^\otimes n \right)
\]

is a (split) monomorphism of abelian groups, as needed. \qed

**Remarks 4.3**

(i) Let \((F, \pi)\) be an object of the presentation category \( \mathfrak{P} \) of \( G \). Then, as shown in [8, Theorem 2], the kernel \( \ker \varphi_n \) of the additive map \( \varphi_n \) constructed above coincides with the torsion subgroup of \( \gamma_n R/[\gamma_n R, F] \) for all \( n \geq 2 \). Therefore, it follows from [9] that \( \ker \varphi_n \) is an \( n \)-torsion group if \( n \geq 3 \) and a 4-torsion group if \( n = 2 \). Since this is also the case for the limit of these groups, we may invoke Lemma 4.2 in order to conclude that \( \lim \gamma_n R/[\gamma_n R, F] \) is an \( n \)-torsion group if \( n \geq 3 \) and a 4-torsion group if \( n = 2 \). The latter assertion provides another proof of Theorem 4.1, in the case where \( n \geq 3 \).

(ii) It follows from the proof of Theorem 4.1 given above that there is an exact sequence of abelian groups

\[
0 \longrightarrow \lim \gamma_n R/[\gamma_n R, F] \longrightarrow H_{2n}(G, Z)[n],
\]

where the limits are taken over the presentation category \( \mathfrak{P} \) of \( G \), for all \( n \geq 2 \). In order to obtain an embedding of the limit \( \lim \gamma_n R/[\gamma_n R, F] \) into the \( n \)-torsion subgroup \( H_{2n}(G, Z)[n] \)
of the homology group $H_{2n}(G, \mathbb{Z})$, at least in the case where $n \geq 3$, one may ask whether the abelian group $\lim J_n^G(R_{ab}, \mathbb{Z})$ is zero. Following M.W. Thomson, who studied the vanishing of the group $J_n^G(R_{ab}, \mathbb{Z})$ in [10], we consider the following special cases:

(i) Assume that $G$ is a finite group of order relatively prime to $n$. Then, the homology group $H_{2n}(G, \mathbb{Z})$ has no non-trivial $n$-torsion elements and the group $J_n^G(R_{ab}, \mathbb{Z})$ vanishes for any presentation $G = F/R$ (cf. [10, Theorem 2(ii)]). Therefore, taking into account the exact sequence above, it follows that $\lim \gamma_n R/\gamma_n R, F = 0$ for all $n \geq 2$.

(ii) Assume that the cohomological dimension of $G$ is $\leq 2$. Then, the group $J_n^G(R_{ab}, \mathbb{Z})$ vanishes for any presentation $G = F/R$ (cf. [10, Theorem 2(iii)]), whereas the homology group $H_{2n}(G, \mathbb{Z})$ vanishes for all $n \geq 2$. Therefore, taking into account the exact sequence above, it follows that $\lim \gamma_n R/\gamma_n R, F = 0$ for all $n \geq 2$.

(iii) Let $(F, \pi)$ be an object of the presentation category $\mathfrak{P}$ of $G$ and consider a $\mathbb{Z}G$-module $M$. We also consider the Magnus embedding

$$\mu : R_{ab} \longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}F} f$$

and the $\mathbb{Z}G$-linear map

$$\mu_{n,M} = id_M \otimes \mu^{\otimes n} : M \otimes R_{ab}^{\otimes n} \longrightarrow M \otimes (\mathbb{Z}G \otimes_{\mathbb{Z}F} f)^{\otimes n}.$$

Then, as shown in [10, Lemma 8], the kernel of the induced additive map

$$\overline{\mu_{n,M}} : H_0(G, M \otimes R_{ab}^{\otimes n}) \longrightarrow H_0(G, M \otimes (\mathbb{Z}G \otimes_{\mathbb{Z}F} f)^{\otimes n})$$

is identified with the homology group $H_{2n}(G, M)$ for all $n \geq 1$. Since the exact sequence

$$0 \longrightarrow H_{2n}(G, M) \longrightarrow H_0(G, M \otimes R_{ab}^{\otimes n}) \overline{\mu_{n,M}} H_0(G, M \otimes (\mathbb{Z}G \otimes_{\mathbb{Z}F} f)^{\otimes n})$$

depends naturally on the object $(F, \pi)$ of $\mathfrak{P}$, we may invoke Lemma 2.1 in order to obtain an exact sequence of abelian groups

$$0 \longrightarrow H_{2n}(G, M) \longrightarrow \lim H_0(G, M \otimes R_{ab}^{\otimes n}) \longrightarrow \lim H_0(G, M \otimes (\mathbb{Z}G \otimes_{\mathbb{Z}F} f)^{\otimes n}),$$

where the limits are taken over the category $\mathfrak{P}$. Using exactly the same argument as in the proof of Lemma 4.2, we can show that $\lim H_0(G, M \otimes (\mathbb{Z}G \otimes_{\mathbb{Z}F} f)^{\otimes n}) = 0$. We conclude that the group $H_{2n}(G, M)$ is isomorphic with the limit $\lim \gamma_n R_{ab}$ for all $n \geq 1$, obtaining thereby an alternative proof of Theorem 3.2.$^2$

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$^2$This argument was communicated to us by R. Stöhr.
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