Twisted Klein curves modulo 2

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Abstract

We give an explicit description of all 168 quartic curves over the field of two elements that are isomorphic to the Klein curve over an algebraic extension. Some of the curves have been known for their small class number [12], [11], others for attaining the maximal number of rational points [15], [13].

The Klein curve is the famous curve

\[ K : X^3Y + Y^3Z + Z^3X = 0 \]

The equation goes back to F. Klein who derived it as a model for the modular curve \( X(7) \) [9]. In characteristic zero, the curve is the unique curve of genus three, up to isomorphism, with the maximal number of 168 automorphisms. Various aspects of the curve are described in the recent papers [8], [14], [6], in [7], and in other contributions to [10]. For further and older references dealing with the many fascinating combinatorial and arithmetic properties of the curve, we refer to these publications.

In this paper, we investigate the different models of the Klein curve that exist in characteristic two. First, we present six different inequivalent models that we found at different places in the literature, followed by a brief discussion of different models in characteristic zero. Then we recall the special behaviour of the bitangents in even characteristic and we present the main theorem that classifies all 168 twisted curves in terms of their generic tangent. Finally, we show that this description immediately reveals the zeta functions for each of the 168 twisted curves.
The results of the paper were obtained while visiting the University of Puerto Rico at Rio Piedras, September 1995 - May 1996. A first preprint was distributed at the conference “Geometry, Algorithms and Arithmetic in the Theory of Error Correcting Codes,” Guadeloupe, April 1-5, 1996. The results were presented under the title “Twisted Klein curves” at the AMS-Benelux meeting in Antwerpen, May 22-24, 1996. This version is unchanged from the January 30, 2001 version. We added two references of related interest: [2] (Example 7.1), kindly pointed out to us by David Joyner, and [16].

1 Six inequivalent models modulo 2

The behaviour of the Klein curve is special in all the characteristics that divide 168. In this paper, we are particularly interested in the different models that exist in characteristic two. Theorem 1 shows that each model is characterised by a transformation $P \in PSL(3, 2)$, such that the generic tangent of the curve at the point $(x, y, z)$ passes through $(x^2, y^2, z^2)P^t$ and $(x^8, y^8, z^8)(P^t)^3$. Two elements of $PSL(3, 2)$ define equivalent curves if and only if they belong to the same conjugacy class. The conjugacy classes of $PSL(3, 2)$ are of size 1 (identity), 24 (elements of order 7 with even trace), 24 (order 7, odd trace), 56 (order 3), 42 (order 4), and 21 (order 2). Up to equivalence we find six different models. For each of the six conjugacy classes, the corresponding curve has 0 (identity), 7 (order 7, even trace), 0 (order 7, odd trace), 3 (order 3), 2 (order 4), or 4 (order 2) rational points over the binary field.

The Klein curve modulo 2 represents a curve with 3 rational points. In [9], Klein gives a different model

$$K' \equiv \frac{1}{21\sqrt{7}}(X^4 + 21X^2YZ - 147Y^2Z^2 + 49XY^3 + 49XZ^3) = 0$$

which modulo 2 represents a curve with 4 rational points (all points not on the line $Y = Z$). L.E. Dickson, in his paper “Quartic curves modulo 2” [5], classifies up to equivalence all the quartics that have 0, 7, 6, or 5 rational points (for the last case, he finds twenty five equivalence classes and “Quartics with 1, 2, 3, or 4 real points have not been treated since they would probably not present sufficient novelty to compensate for the increased length of the...
investigation”). The paper provides us with three more inequivalent models,

\[ \alpha : X^4 + Y^4 + Z^4 + X^2Y^2 + Y^2Z^2 + Z^2X^2 + X^2YZ + XY^2Z + XY^2Z^2 = 0 \]

\[ A : X^3Y + X^2Y^2 + XZ^3 + X^2Z^2 + Y^3Z + YZ^3 = 0 \]

\[ \gamma_{1,0} : X^4 + Y^4 + Z^4 + X^3Y + XZ^3 + Y^3Z + XY^2Z = 0 \]

The model \( \alpha \) is the unique model invariant under the full group \( PSL(3, 2) \). The model \( A \) has the maximum number of rational points for a genus 3 curve. An equivalent model appears in \cite{15}, \cite{13}. We will denote it \( X_{N=7} \).

\[ X_{N=7} : XY(X + Y)(X + Z) + YZ(Z + Y)Y + ZX(X + Z)Z = 0 \]

The model \( \gamma_{1,0} \), with no rational points, has trivial Jacobian. An equivalent model appears in the list by Madan and Queen \cite{12} of the seven congruence function fields which have class number one and genus different from zero. We will denote it \( X_{h=1} \).

\[ X_{h=1} : Y^4 + XY^3 + (X^2 + XZ)Y^2 + (X^3 + Z^3)Y + (X^4 + XZ^3 + Z^4) = 0 \]

The classes for \( A \) and \( \gamma_{1,0} \) are of same size and differ by a translation by \( \alpha \). It happens that \( X_{N=7} = \gamma_{1,0} + \alpha \), and \( X_{N=7} \) and \( \gamma_{1,0} \) have the same seven rational automorphisms. For the class with 2 rational points, we give the model

\[ X_{N=2} : X^4 + Y^4 + Z^4 + X^3Y + XZ^3 + Y^3Z + XY^2Z^2 + X^2Y^2 + X^2Z^2 + Y^2Z^2 + YZ^3 = 0 \]

Comparison with \( \gamma_{1,0} \) shows that the rational points are those not on \( X(Y + Z) = 0 \). Properties of the different models are summarized in the Appendix.

\section{Models in characteristic zero}

The Klein curve arises in characteristic zero as a model for the elliptic modular curve \( X(7) \). In particular this yields that the curve has the simple group \( PSL(2, 7) \) of order 168 as automorphism group. Many of the interesting properties of the Klein curve are directly related to this group. F. Klein derived the model

\[ X^3Y + Y^3Z + Z^3X = 0 \]
as follows. A plane curve of degree four contains 24 flexpoints, that is points where the tangent intersects with multiplicity at least three. The group acts faithfully on the set of flexpoints and each flexpoint has stabilizer of order seven. There are eight subgroups of order seven, each fixing at most three points. It follows that each subgroup of order seven fixes three points, all of them flexpoints. The subgroup fixing a flexpoint also fixes the fourth point of intersection on its tangent, which must therefore be a flexpoint with the same stabilizer. Three flexpoints connected by their tangents form an oriented triangle, that Klein called a \textit{Wendedreiecke}. The Klein quartic is now the curve with \textit{triangle of inflection tangents} $XYZ = 0$.

It is convenient to view the action of the automorphism group on the curve through its action on the eight \textit{triangles of inflection tangents}. Each triangle has stabilizer of order 21. Let $\zeta$ be a primitive seventh root of unity. The stabilizer of the triangle $XYZ = 0$ is generated by

$$
\tau = \begin{pmatrix}
\zeta & 0 & 0 \\
0 & \zeta^4 & 0 \\
0 & 0 & \zeta^2
\end{pmatrix}
$$

and the cyclic permutation $\rho$ of the coordinates $X, Y, Z$. It has 14 elements of order three. The full group has 56 elements of order three and each automorphism of order three appears in the stabilizer of two triangles. Say the other triangle stabilized by the cyclic permutation $\rho$ has a vertex $(a : b : c)$. Klein showed that

$$
\sigma = \begin{pmatrix}
a & b & c \\
b & c & a \\
c & a & b
\end{pmatrix}
$$

acts on the curve, for

$$
a = (\zeta^5 - \zeta^2)/\sqrt{-7}, \quad b = (\zeta^3 - \zeta^4)/\sqrt{-7}, \quad c = (\zeta^6 - \zeta)/\sqrt{-7},
$$

with $\sqrt{-7} = \zeta + \zeta^2 - \zeta^3 + \zeta^4 - \zeta^5 - \zeta^6$. The automorphism $\sigma$ is of order two and interchanges the two triangles. Indeed, $\sigma$ is of order two only if $f = (x-a)(x-b)(x-c)$ is of the form $x^3 \pm x^2 - r$, and it acts on the Klein curve only if $r = 0, \pm 1/7$.

The full automorphism group $PSL(2, 7)$ of the Klein curve is visible from its action on the eight \textit{triangles of inflection tangents}. Let $\Delta_\infty$ be the triangle
$XYZ = 0$, let $\Delta_0 = \sigma \Delta_\infty$ be the other real triangle, and let $\Delta_i = \tau^i \Delta_0$, for $i = 0, 1, \ldots, 6$. The action of $\sigma$ and $\tau$ on the triangles is described by the permutations $(1 6)(2 3)(4 5)(0 \infty)$ and $(1 2 3 4 5 6 0)$ respectively. And an isomorphism to $PSL(2, 7)$ is given by

$$
\tau \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \sigma \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
$$

Burnside pointed out that $\tau$ and $\sigma$ generate the automorphism group. Klein had originally included the cyclic permutation $\rho$ in the generating set.

The two triangles of inflection tangents that are left invariant by the cyclic permutation $\rho$ are the only two of the eight triangles with real vertices. If we change the orientation in the triangle $XYZ = 0$, we obtain the different model

$$X^3Z + Y^3X + Z^3Y = 0.$$ 

If we change the orientation in the other real triangle, we obtain the different model

$$O_4 : \frac{1}{7}((X^4 + Y^4 + Z^4) - 3(X^2Y^2 + Y^2Z^2 + Z^2X^2) + 3(X^2YZ + XY^2Z + XY^2Z^2) + 6(X^3Z + Y^3X + Z^3Y)) = 0.$$ 

This model was derived independently as Equation (1.22) in [7]. After reduction modulo $p = 2$, the model agrees with the invariant model $\alpha$ and has all automorphisms defined over the base field. Ciani [3] gives a model over $Q(\sqrt{-7})$

$$C_4 : X^4 + Y^4 + Z^4 + 3\omega(Y^2Z^2 + Z^2X^2 + X^2Y^2) = 0, \quad \omega^2 + \omega + 2 = 0.$$ 

It coincides with Equation (1.11) in [7]. The following model (6) has the $S_3$ symmetry of $C_4$ and the same binary reduction as $O_4$,

$$A_4 : 7s_1(s_1^3 + s_3) - (2s_1^2 + s_2)^2 = 0.$$

All of the last three curves have their automorphisms defined over $Q(\sqrt{-7})$. The six real flexpoints of the last curve have $s_1 = -1, s_2 = -2, s_3 = +1$, i.e. have coordinates $a, b, c$ such that, for $\theta = 2\pi/7$,

$$(x - a)(x - b)(x - c) = x^3 + x^2 - 2x - 1 = (x - 2 \cos(\theta))(x - 2 \cos(2\theta))(x - 2 \cos(4\theta))$$
3 Bitangents in even characteristic

In characteristic zero, a non-singular quartic curve admits 28 bitangents. In even characteristic, the number of bitangents is at most 7. In that case, the bitangents define a Fano plane. All non-singular quartic curves with seven bitangents are in the same isomorphism class. The unique curve with bitangents \( X, Y, Z, X + Y, Y + Z, X + Y + Z, X + Z \) is the invariant curve

\[
\alpha : X^4 + Y^4 + Z^4 + Y^2Z^2 + Z^2X^2 + X^2Y^2 + X^2YZ + Y^2ZX + Z^2XY = 0
\]

Uniqueness follows for example by considering that the bitangents must touch the curve in the fourteen points of \( P^2(F_4) \) not in \( P^2(F_2) \). Other curves with seven bitangents will have their bitangents defined over some extension of \( F_2 \). In those cases, the equation of the curve can be recovered from the bitangents by substitution in \( \alpha \) of three non-concurrent bitangents.

The Klein curve has bitangents

\[
K : a^2X + aY + a^4Z = 0, \quad a^8 + a = 0, \quad a \neq 0.
\]

It happens that the models chosen by L.E. Dickson have a similar description of the bitangents. The model \( A \) has bitangents

\[
A : b^2X + bY + b^4Z = 0, \quad b^8 + b^2 + b = 0, \quad b \neq 0.
\]

The model \( \gamma_{1,0} \) has bitangents

\[
\gamma_{1,0} : c^2X + cY + c^4Z = 0, \quad c^8 + c^4 + c = 0, \quad c \neq 0.
\]

If we choose different but equivalent models, the bitangents change but are still parametrized by the same values. The curve \( X_{N=7} \), which is equivalent to \( A \), has bitangents

\[
X_{N=7} : bX + b^{16}Y + b^8Z = 0, \quad b^8 + b^2 + b = 0, \quad b \neq 0.
\]

The curve \( X_{h=1} \), which is equivalent to \( \gamma_{1,0} \), has bitangents

\[
X_{h=1} : cX + c^8Y + c^2Z = 0, \quad c^8 + c^4 + c = 0, \quad c \neq 0.
\]

We chose the model \( X_{N=2} \) such that it has bitangents

\[
X_{N=2} : d^2X + dY + d^4Z = 0, \quad d^8 + d^4 + d^2 + d = 0, \quad d \neq 0.
\]
For the invariant curve, the bitangents have coefficients, in matrix form,
\[ \alpha : \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \]
So that the coefficients appearing in a given row are the zeros of \((x^8 + x^4)/x\). After reduction, Klein’s model \(K’\) has bitangents, with \(F_4 = \{0, 1, \omega, \bar{\omega}\}, \]
\[ K’ : \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & \omega & \bar{\omega} & \omega & 1 & 1 & \bar{\omega} \\ 0 & \bar{\omega} & \omega & \bar{\omega} & 1 & 1 & \omega \end{bmatrix} \]
Substitution of \(Z = Z + X\) in \(K’\) gives the equivalent reduced model
\[ X_{N=4} : X^2YZ + X^2Y^2 + X^2Z^2 + Y^2Z^2 + XY^3 + XZ^3 + X^3Y + X^3Z = 0 \]
with bitangents,
\[ X_{N=4} : \begin{bmatrix} 1 & \omega & \bar{\omega} & 0 & 1 & \bar{\omega} \\ 0 & \omega & \bar{\omega} & \omega & 1 & \bar{\omega} \\ 0 & \bar{\omega} & \omega & \bar{\omega} & 1 & \omega \end{bmatrix} \]
The coefficients appearing in a given row are the zeros of \((x^8 + x^2)/x\). The only non-trivial additive polynomial that can not be used to define coefficients of a Fano plane is \(x^8 + x^4 + x^2\).

In all cases, the set of bitangents is defined over \(F_2\). The Frobenius automorphism acts on the set of bitangents. For a careful choice of the coefficients of the bitangents (such that the linear expressions for the bitangents are closed under addition), the action is linear and can be represented by an element of \(PSL(3, 2)\).

**Definition 1** To a quartic curve over \(F_2\) with seven bitangents, we associate the unique element \(R \in PSL(3, 2)\) such that the action of \(R\) on the bitangents agrees with the action of the Frobenius automorphism: \((a^2, b^3, c^2)^T = R(a, b, c)^T\), for each bitangent \(ax + by + cz = 0\).

In the next section, we give the inverse map, which gives for each \(R \in PSL(3, 2)\) the unique quartic curve such that the Frobenius action on the bitangents corresponds with the action of \(R\).
4 Description of the 168 twisted curves

The group $\text{PSL}(3, 2)$ is the full automorphism group of the curve $\alpha$. In [4], L.E. Dickson gives the modular invariants of the general linear group over a finite field. In particular, he shows that $\alpha$ is the smallest projective invariant of $\text{PSL}(3, 2)$ acting on $F_2[X, Y, Z]$. The other generating invariants are of degree six and seven, and the three invariants are algebraically independent [4], [1]. Consider the additive polynomial in the variable $T$ with coefficients in $F_2[X, Y, Z]$ that cancels at $X, Y, Z$,

$$\prod_{v \in \langle X, Y, Z \rangle} (T + v) = T^8 + I_4(X, Y, Z)T^4 + I_6(X, Y, Z)T^2 + I_7(X, Y, Z)T.$$

The algebra $F_2[I_4, I_6, I_7]$ is the ring of $\text{PSL}(3, 2)$ invariants in $F_2[X, Y, Z]$. The coefficients follow by solving

$$\begin{pmatrix} X & X^2 & X^4 \\ Y & Y^2 & Y^4 \\ Z & Z^2 & Z^4 \end{pmatrix} \begin{pmatrix} I_7 \\ I_6 \\ I_4 \end{pmatrix} = \begin{pmatrix} X^8 \\ Y^8 \\ Z^8 \end{pmatrix}.$$

For $I_4$, we have

$$I_4(X, Y, Z) = \det \begin{pmatrix} X & X^2 & X^4 \\ Y & Y^2 & Y^8 \\ Z & Z^2 & Z^4 \end{pmatrix} / \det \begin{pmatrix} X & X^2 & X^4 \\ Y & Y^2 & Y^4 \\ Z & Z^2 & Z^4 \end{pmatrix}.$$

Thus, $I_4$ is the unique non-trivial plane curve (i.e. different from a line), for which the line through a point $(x : y : z)$ and its Frobenius $F(x : y : z) = (x^2 : y^2 : z^2)$ also contains $F^3(x : y : z) = (x^8 : y^8 : z^8)$.

Let $C$ be a curve isomorphic to $I_4$. Then $C$ can be represented as a plane quartic. The canonical divisor of a quartic curve is given by the intersection of a line with the curve. Therefore the isomorphism preserves collinear points and can be represented by a linear transformation. Let $(x, y, z) \mapsto A(x, y, z) = (x', y', z')$ be an isomorphism from $C$ to $I_4$. We have that $(x, y, z)$ is a point on $C$ if and only if $(x', y', z')$ is a point on $I_4$ if and only if

$$(x', y', z'), F(x', y', z'), F^3(x', y', z')$$

are collinear if and only if

$$A(x, y, z), F(A(x, y, z)), F^3(A(x, y, z))$$
are collinear if and only if

$$(x, y, z), (A^{-1} FA)(x, y, z), (A^{-1} FA)^3(x, y, z)$$

are collinear. We claim that $(A^{-1} FA)(x, y, z) = (PF)(x, y, z)$, for some $P \in PSL(3, 2)$. Let $A^{(2)}$ be the isomorphism $A$ with coefficients squared. We have

$$(A^{-1} FA)(x, y, z) = (A^{-1} A^{(2)} F)(x, y, z)$$

and $A^{(2)} A^{-1} \in \text{Aut}(I_4) = PSL(3, 2)$, say $A^{(2)} A^{-1} = Q$. The element $Q$ is determined by the curve $C$ only up to conjugacy, for if we replace $A$ with $SA$, for $S \in \text{Aut}(I_4) = PSL(3, 2)$, then $(SA)^{(2)}(SA)^{-1} = SQS^{-1}$. On the other hand, $A^{-1} QA = A^{-1} A^{(2)}$ does not change when $A$ is replaced with $SA$. The latter matrix has coefficients in $F_2$, for $Q^2 = A^{(2)} A^{-1} Q = QA^{(2)} A^{-1}$, or

$$A^{-1} QA = (A^{(2)})^{-1} QA^{(2)} = (A^{-1} QA)^{(2)}$$

and $A^{-1} A^{(2)} = A^{-1} QA$ belongs to $PSL(3, 2)$.

**Theorem 1** A curve is isomorphic to the invariant curve $I_4$ if and only if it is of the form

$$\det\left(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}
P \left(\begin{pmatrix} X^2 \\ Y^2 \\ Z^2 \end{pmatrix}
P^3 \left(\begin{pmatrix} X^8 \\ Y^8 \\ Z^8 \end{pmatrix}\right)\right)/\det\left(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}
P \left(\begin{pmatrix} X^2 \\ Y^2 \\ Z^2 \end{pmatrix}
P^2 \left(\begin{pmatrix} X^4 \\ Y^4 \\ Z^4 \end{pmatrix}\right)\right)\right)$$

for some $P \in PSL(3, 2)$. The rational automorphisms of such a curve are given by the elements of $PSL(3, 2)$ that commute with $P$. The action of the Frobenius on the bitangents, as in Definition 1, is given by the matrix $R = P^t$. The tangent at the point $(x, y, z)$ has intersection divisor

$$2(x, y, z) + (x^2, y^2, z^2) P^t + (x^8, y^8, z^8) (P^t)^3$$

In particular, $P \in PSL(3, 2)$ is determined by the generic tangent of the curve.

**Proof.** The curve is uniquely determined by the matrix $P = A^{-1} A^{(2)}$, for any isomorphism $A$ from $C$ to $I_4$. Now $S \in PSL(3, 2)$ defines an automorphism of $C$ if and only if $AS$ is an isomorphism from $C$ to $I_4$ if and only if $(AS)^{-1}(AS)^{(2)} = S^{-1} PS = P$. Let the columns of the matrix $B$ contain the coefficients of three non concurrent bitangents. Then $B^{(2)} = RB$. On the other hand, $B^t$ defines an isomorphism from $C$ to $I_4$. And $P = (B^t)^{-1}(B^t)^{(2)} = R^t$. qed
The general construction of twisted models for a given object with non-trivial automorphisms is given by elementary galois cohomology. The case that we consider here seems to be particularly favourable for an explicit treatment. The invariant curve $I_4$ and all its automorphisms are defined over the base field, and the equation of the curve $I_4$ is a simple relation among Frobenius images of the generic point on the curve.

As illustration of the theorem, we may write for the Klein curve,

$$X^3Y + Y^3Z + Z^3X = \det \begin{pmatrix} X & Y^2 & X^8 \\ Y & Z^2 & Y^8 \\ Z & X^2 & Z^8 \end{pmatrix} / \det \begin{pmatrix} X & Y^2 & Z^4 \\ Y & Z^2 & X^4 \\ Z & X^2 & Y^4 \end{pmatrix}.$$ 

5 Zeta functions for the twisted curves

The forms $I_4$ and $I_6$ define a pair of dual curves (up to an inseparable morphism of degree two), and are birationally equivalent. For both curves, all points on the tangent at $P$ are images of $P$ under the Frobenius automorphism $F$. And in both cases, the zeta function is determined by the intersection divisor of a generic tangent. For the tangent $L_P$ of $I_4$ at $P$,

$$L_P \cap I_4 = P + P + F(P) + F^3(P),$$

and the Frobenius eigenvalues are zeros of

$$(2 + t + t^3) = (1 + t)(2 - t + t^2).$$

Similarly, for the tangent $L_P$ of $I_6$ at $P$,

$$L_P \cap I_6 = P + P + P + P + F^2(P) + F^3(P),$$

and the Frobenius eigenvalues are zeros of

$$(4 + t^2 + t^3) = (2 + t)(2 - t + t^2).$$

Thus, both $I_4$ and the normalization of $I_6$ have zeta polynomial

$$L(t) = (1 - t + 2t^2)^3.$$
For the twisted models we see, either by elimination of $P^t$ in the generic tangent (given in Theorem 1) or by using properties of the zeta function, that the Frobenius eigenvalues are reciprocal zeros of

$$\prod_{\zeta^m=1} (1 - \zeta t + 2\zeta^2 t^2)$$

where $m$ is the order of the matrix $P$. For the relevant values $m = 2, 3, 4, 7$, we find the zeta polynomial immediately from the factorization of the above product into irreducibles and the known number of rational points. Let

$$z^+ = (1 - t + 2t^2)$$
and
$$z^- = (1 + t + 2t^2).$$

The two irreducible factors $z^+_7, z^-_7$ of degree six represent the two different conjugacy classes for $P$ of order 7.

- even trace: $L(t) = z^+_7 = 1 + 4t + 9t^2 + 15t^3 + 18t^4 + 16t^5 + 8t^6$
- odd trace: $L(t) = z^-_7 = 1 - 3t + 2t^2 + t^3 + 4t^4 - 12t^5 + 8t^6$

The factor $z^+$ corresponds, as is well-known [4], to the elliptic curve with complex multiplication by $-7$. In the cases $m = 2, 3, 4$, it appears as the quotient of the twist by a rational automorphism. In the case $m = 7$, the quotient gives the projective line. We give the elliptic curve as the quotient, in characteristic zero, of the Klein curve by the cyclic permutation of the coordinates. Let $B$ be the divisor sum of the two points in the intersection of the Klein curve and the invariant bitangent $X + Y + Z = 0$. Recall that the Klein curve has two triangles of inflection tangents that are invariant under the cyclic permutation. Let $\Delta_\infty$ and $\Delta_0$, respectively, be the divisor sums of their vertices, respectively. Let $x, y, z \in k[X, Y, Z]$ intersect the Klein curve as follows

$$(x) = 2\Delta_\infty + 2\Delta_0, \quad (y) = \Delta_\infty + \Delta_0 + 3B, \quad (z) = 4\Delta_\infty.$$

We obtain

$$x = X^2Y + Y^2Z + Z^2X + XYZ = ((X^3Y + Y^3Z + Z^3X) + s^2_\nu)/s_1,$$
$$y = (X + Y + Z)(XY + YZ + ZX), \quad z = XYZ.$$
The morphism \((-x : y : z)\) maps \(K\) onto the elliptic curve
\[ E : y^2 + 5xy = x^3 - x^2 + 7x. \]

In fact,
\[
y^2z - 5xyz + x^3 + x^2z + 7xz^2 = (X^3Y + Y^3Z + Z^3X)(X^3Y^2 + Y^3Z^2 + Z^3X^2).
\]

Replacing \(x = ((X^3Y + Y^3Z + Z^3X) + s_1^2)/s_1\) by \(x = s_2^2/s_1\) in the morphism, we find, as in [7],
\[
y^2z - 5xyz + x^3 + x^2z + 7xz^2 = (X^3Y + Y^3Z + Z^3X)(X^3Z + Y^3X + Z^3Y)
\]
The triangles \(\Delta_\infty\) and \(\Delta_0\) are mapped to the point at infinity and the origin respectively. The only ramification is at \(B\).

A Summary of curves

The curves appearing in this paper are summarized in the following table. It gives the number of equivalent curves in a class, the structure of the subgroup of rational automorphisms, and properties of the matrix \(P\). The matrix \(P\) for each of the curves follows (as determined by Theorem 1).

| Curve | # | Rational automorphisms | \(P\) order | trace |
|-------|---|------------------------|-------------|-------|
| \(\alpha = I_4\) | 1 | \(PSL(3,2)\) | 1 | 1 |
| \(A, X_{N=7}\) | 24 | \(Z/7Z\) | 7 | 0 |
| \(\gamma_{1,0}, X_{h=1}\) | 24 | \(Z/7Z\) | 7 | 1 |
| \(K\) | 56 | \(Z/3Z\) | 3 | 0 |
| \(X_{N=2}\) | 42 | \(Z/4Z\) | 4 | 1 |
| \(K', X_{N=4}\) | 21 | \(D_4\) | 2 | 1 |
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
A
\begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\gamma_{1,0} \\
0 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
X_{\gamma=1} \\
0 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
K' \\
1 & 0 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
X_{\gamma=4} \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\]

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