Research Article

Unifications of Continuous and Discrete Fractional Inequalities of the Hermite–Hadamard–Jensen–Mercer Type via Majorization

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The main objective of the paper is to develop an innovative idea of bringing continuous and discrete inequalities into a unified form. The desired objective is thus obtained by embedding majorization theory with the existing notion of continuous inequalities. These notions are applied to the latest generalized form of the inequalities, popularly known as the Hermite–Hadamard–Jensen–Mercer inequalities. Moreover, the frequently-used Caputo fractional operators are employed, which are rightly considered critical, especially for applied problems. Both weighted and unweighted forms of the developed results are discussed. In addition to this, some bounds are also provided for the absolute difference between the left- and right-sides of the main results.

1. Introduction

The field of mathematical inequalities contributes to a wide area of research in mathematics. With the passage of time, this field has emerged as a separate discipline, despite the fact that it was being used as a tool earlier. The addition of the notion of convexity enriched its literature and stimulated a new trend among researchers. As a result, many new inequalities came to the surface. These inequalities are (but not limited to) Ostrowski inequalities [1], Jensen’s inequalities [2], the Jensen–Mercer inequalities [3], Fejér inequalities [4], Hermite–Hadamard inequalities [5], and their various variants. The Hermite–Hadamard inequality is believed to be the most widespread inequality in the literature and has received much attention in the last few years. This inequality is defined as follows:

If $\phi : I \longrightarrow \mathbb{R}$ is a convex function with $\vartheta, \theta \in I$ such that $\vartheta < \theta$ then

$$\phi \left( \frac{\vartheta + \theta}{2} \right) \leq \frac{1}{\theta - \vartheta} \int_{\vartheta}^{\theta} \phi(u) du \leq \frac{\phi(\vartheta) + \phi(\theta)}{2}. \quad (1)$$

The direction of the inequality given in (1) reverses whenever the function $\phi$ is concave. This inequality has been established for different generalized convex functions, for example, $s$–convex [6], $\eta$–convex [7], strongly convex [8], and coordinate convex function [9]. Research works in this field have also been extended to the theory of fractional calculus. As there are multiple numbers of fractional operators but due to our interest, we limit ourselves to the well-known Caputo fractional operators. Their definition is given as follows:

Definition 1 (Caputo fractional derivative operators). Consider a function $\phi \in C^n[\vartheta, \theta]$ (the space of functions whose $n^{th}$-derivative exist and continuous on
Let us consider a convex function $\phi$ defined on the interval $[\vartheta, \theta]$. Theorem 2 extended the Jensen–Mercer inequality given as follows:

$$\int_\vartheta^\theta cD^n_\sigma \phi(z) = \int_\vartheta^\theta \sum_{\tau=0}^{n} (-1)^\tau \binom{n}{\tau} \phi(\theta z) \bigg|_{z=\bar{\tau}} \bigg|_{z=1} \bigg|_{z=\vartheta} \bigg|_{z=\theta} \bigg|_{z=\bar{\tau}} \bigg|_{z=1} \bigg|_{z=\vartheta} \bigg|_{z=\theta}$$

$$\int_\vartheta^\theta cD^n_\vartheta \phi(z) = \int_\vartheta^\theta \sum_{\tau=0}^{n} (-1)^\tau \binom{n}{\tau} \phi(\theta z) \bigg|_{z=\bar{\tau}} \bigg|_{z=1} \bigg|_{z=\vartheta} \bigg|_{z=\theta} \bigg|_{z=\bar{\tau}} \bigg|_{z=1} \bigg|_{z=\vartheta} \bigg|_{z=\theta}$$

Where $cD^n_\sigma \phi(z)$ and $cD^n_\vartheta \phi(z)$ stand for the left- and right-sided Caputo fractional derivative operators, respectively.

The inequality (1) can be obtained from (4) when $n = 1$, $\vartheta = 0$, $\sigma = \bar{\vartheta}$ and $\theta = 1$. Some more work related to Hermite–Jensen–Mercer inequalities via fractional operators can be found in [11–13] and the references therein. The associated Hermite–Jensen–Mercer inequality in terms of Caputo fractional operators is defined as follows [14]:

**Theorem 1** (Hermite–Jensen–Mercer inequality). Consider a function $\phi$ defined on the interval $[\vartheta, \theta]$, such that $\phi \in C^n[\vartheta, \theta]$ and $\phi(n)$ is convex on $[\vartheta, \theta]$ with $[x_1, y_1] \subset [\vartheta, \theta]$, $\alpha > 0$, then we have

$$\phi(n)(\theta + \vartheta - \frac{x_1 + y_1}{2}) \leq \frac{2^{n-1} \Gamma(n + 1)}{(y_1 - x_1)^{n+1}} \left( cD^n_\theta \phi(\vartheta, \theta) + cD^n_\vartheta \phi(\theta, \vartheta) \right) $$

$$\frac{n}{2} \left( \phi(n)(\theta) + \phi(n)(\vartheta) - \phi(n)(x_1) + \phi(n)(y_1) \right)$$

(4)

The following lemmas will help us to prove our main results [23].

**Lemma 1.** Let us consider a convex function $\phi$ defined on the interval $I$, $r \times I$ real matrix $(x_{is})$, and two $l$– tuples $\delta = (\delta_1, \ldots, \delta_l), \ p = (p_1, \ldots, p_l)$, such that $\delta_i, x_{is} \in I$, $\sigma_i \geq 0$, $\Sigma_s \sigma_i = 1$, $p_i \geq 0$, with $p_i \neq 0$, $\eta = 1/p_i$, for all $i = 1, 2, \ldots, r, s \in \{1, \ldots, l\}$. If for each $i = 1, 2, \ldots, r$, $x_{is}$ is a decreasing $l$–tuples and satisfying

$$\sum_{s=1}^{k} p_{is} x_{is} \leq \sum_{s=1}^{k} p_{is} \delta_i, \quad \text{for} \quad k = 1, 2, \ldots, l - 1, \quad \sum_{s=1}^{l} x_{is} = \sum_{s=1}^{l} \delta_i$$

Then

$$\phi\left( \sum_{s=1}^{k} \eta p_{is} \delta_i - \sum_{s=1}^{l} \eta p_{is} x_{is} \right) \leq \sum_{s=1}^{l} \eta p_{is} \phi(\delta_i) - \sum_{s=1}^{l} \eta p_{is} x_{is} \phi(\delta_i)$$

(8)

**Lemma 2.** Let us consider a convex function $\phi$ defined on the interval $I$, $r \times I$ real matrix $(x_{is})$, and two $l$– tuples $\delta = (\delta_1, \ldots, \delta_l), \ p = (p_1, \ldots, p_l)$, such that $\delta_i, x_{is} \in I$, $\sigma_i \geq 0$, $\Sigma_s \sigma_i = 1$, $p_i \geq 0$, with $p_i \neq 0$, $\eta = 1/p_i$, for all $i = 1, 2, \ldots, r, s \in \{1, \ldots, l\}$. If for each $i = 1, 2, \ldots, r$, $x_{is}$ is monotonically in the same sense and
The present paper is summarized as follows: first of all, Theorem 3 is devoted to the establishment of a new unified form of Hermite–Hadamard–Jensen–Mercer inequality. This objective is achieved by utilizing the majorized \( l \)-tuples in the context of Caputo fractional operators. A slightly different variant of Theorem 3 is presented in the form of Theorem 4. In order to verify and provide proof of the fact that the newly-obtained results are the unifications and generalizations of those already existing results, Remark 1 and Remark 2 are presented. In addition to this, weighted versions of the obtained results are also provided, taking the weighted generalized Mercer’s inequality into account. These weighted results can be traced to Theorem 5 and Theorem 6. Moreover, two new identities, connected with the right- and left-most terms in the main results are obtained. These results are discussed in Theorem 7, Theorem 8, Theorem 9, Theorem 10, and Theorem 11. Remark 5, Remark 6, and Remark 7 show that the newly-derived identities also generalize those previously-defined identities, while Remark 8 discusses the previous version of Theorem 10. Corollary 1 gives details about a previous bound while Corollary 2, and Corollary 3 provide information about the classical integral versions of Theorem 9 and Theorem 11. At the end, conclusion of the overall attempt is presented.

2. Main Results

The following theorem presents the Hermite–Hadamard–Jensen–Mercer fractional inequality for the Caputo fractional operators.

**Theorem 3.** Let us consider a function \( \phi \in C^n(I) \) and \( \delta = (\delta_1, \ldots, \delta_l) \), \( x = (x_1, \ldots, x_l) \), \( y = (y_1, \ldots, y_l) \) are three \( l \)-tuples, such that \( \delta_s, x_s, y_s \in I \), for all \( s \in \{1, \ldots, l\} \), \( x_s > y_s \), \( a > 0 \). If \( \phi^{(\alpha)} \) is a convex function on \( I \), \( x \prec \delta \), and \( y \prec \delta \), then

\[
\phi^{(\alpha)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left( \frac{x_s + y_s}{2} \right) \right) \leq \frac{\Gamma(n - a + 1)}{2^{\alpha} (\sum_{s=1}^{l-1} (y_s - x_s))^n} 
\cdot \left\{ \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s \right) \phi^{(\alpha)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s \right) + (-1)^a \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s \right) \phi^{(\alpha)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s \right) \right\}
\leq \phi^{(\alpha)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s \right) + \phi^{(\alpha)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s \right)
\leq \sum_{s=1}^{l} \phi^{(\alpha)}(\delta_s) - \frac{\sum_{s=1}^{l-1} \phi^{(\alpha)}(x_s) + \sum_{s=1}^{l-1} \phi^{(\alpha)}(y_s)}{2}.
\]
Proof. It can be written as

\[
\phi^{(n)}\left(\sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} \left(\frac{x_{i} + y_{i}}{2}\right)\right) = \phi^{(n)}\left\{ \frac{1}{2} \left\{ \sum_{i=1}^{l} \left( \frac{x_{i} + y_{i}}{2} \right) \right\} \right\},
\]

\[
= \phi^{(n)}\left\{ \frac{1}{2} \left\{ t \left( \sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} x_{i} \right) + (1-t) \left( \sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} y_{i} \right) \right\} \right\}.
\]

(12)

Since \( \phi^{(n)} \) is a convex function, therefore (12) gives the following inequality:

\[
\phi^{(n)}\left(\sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} \left(\frac{x_{i} + y_{i}}{2}\right)\right) \leq \frac{1}{2} \left\{ \phi^{(n)}\left\{ t \left( \sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} x_{i} \right) + (1-t) \left( \sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} y_{i} \right) \right\} \right\}
\]

\[
+ \phi^{(n)}\left\{ t \left( \sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} y_{i} \right) + (1-t) \left( \sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} x_{i} \right) \right\} \right\}.
\]

(13)

By multiplying both sides of (13) by \( t^{n-\alpha-1} \) and then integrating over \( t \in [0,1] \), we get

\[
\frac{1}{n-\alpha} \phi^{(n)}\left(\sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} \left(\frac{x_{i} + y_{i}}{2}\right)\right)
\]

\[
\leq \frac{1}{2} \left\{ \int_{0}^{1} t^{n-\alpha-1} \phi^{(n)}\left\{ t \left( \sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} x_{i} \right) + (1-t) \left( \sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} y_{i} \right) \right\} \right\} dt
\]

\[
+ \int_{0}^{1} t^{n-\alpha-1} \phi^{(n)}\left\{ t \left( \sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} y_{i} \right) + (1-t) \left( \sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} x_{i} \right) \right\} \right\} dt \right\}.
\]

(14)

In order to apply the definition of the Caputo fractional operators in (14), first, we show that

\[
\sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} y_{i} < \sum_{i=1}^{l} \delta_{i} - \sum_{i=1}^{l-1} x_{i} \tag{15}
\]

By the hypotheses, we have \( x < \delta \) and \( y < \delta \), therefore

\[
\sum_{i=1}^{l} y_{i} - \sum_{i=1}^{l-1} x_{i} = x_{l} - y_{l}. \tag{16}
\]

Also,
By substituting (17) in (16), and adding $\sum_{i=1}^{l} \delta_i$ to both sides, we get

$$
\frac{1}{n-\alpha} \phi^{(n)} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} \frac{x_i + y_i}{2} \right) \leq \frac{\Gamma(n-\alpha)}{2(\sum_{i=1}^{l} (y_i - x_i))^{n-\alpha}} \left\{ \left( c \sum_{i=1}^{l} \delta_i \cdot \sum_{i=1}^{l} y_i \right)^{\frac{t}{n}} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} x_i \right) \right. \\
+ (-1)^n \left( c \sum_{i=1}^{l} \delta_i \cdot \sum_{i=1}^{l} y_i \right)^{\frac{1-t}{n}} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} y_i \right) \right\}, 
$$

and so

$$
\phi^{(n)} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} \frac{x_i + y_i}{2} \right) \leq \frac{\Gamma(n-\alpha + 1)}{2(\sum_{i=1}^{l} (y_i - x_i))^{n-\alpha}} \left\{ \left( c \sum_{i=1}^{l} \delta_i \cdot \sum_{i=1}^{l} y_i \right)^{\frac{t}{n}} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} x_i \right) \right. \\
+ (-1)^n \left( c \sum_{i=1}^{l} \delta_i \cdot \sum_{i=1}^{l} y_i \right)^{\frac{1-t}{n}} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} y_i \right) \right\}. 
$$

Thus, the first inequality of (11) is completed. Now, using the convexity of $\phi^{(n)}$, we obtain the second inequality in the following manner:

$$
\phi^{(n)} \left( t \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} x_i \right) \right) + \left( 1 - t \right) \phi^{(n)} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} y_i \right) \leq t \phi^{(n)} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} x_i \right) + \left( 1 - t \right) \phi^{(n)} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} y_i \right), 
$$

$$
\phi^{(n)} \left( t \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} y_i \right) \right) + \left( 1 - t \right) \phi^{(n)} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} x_i \right) \leq t \phi^{(n)} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} y_i \right) + \left( 1 - t \right) \phi^{(n)} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} x_i \right). 
$$

Adding (21) and (22), and then applying Theorem 2 for $r = 1$ and $\sigma_1 = 1$, we obtain

$$
\phi^{(n)} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} x_i \right) + \phi^{(n)} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} y_i \right) \leq \phi^{(n)} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} x_i \right) + \phi^{(n)} \left( \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} y_i \right) \\
\leq 2 \sum_{i=1}^{l} \delta_i - \sum_{i=1}^{l-1} \left( \sum_{i=1}^{l} \phi^{(n)} (x_i) + \sum_{i=1}^{l} \phi^{(n)} (y_i) \right) . 
$$

By multiplying both sides of (23) by $t^{n-\alpha-1}$ and then integrating over $t \in [0, 1]$, we get the second and third inequality in (11). 

**Remark 1.** For the hypothesis of Theorem 3, if $l = 2$, then we get the following inequality:
\[
\phi^{(n)}\left(\delta_1 + \delta_2 - \frac{x_1 + y_1}{2}\right) \leq \frac{\Gamma(n - \alpha + 1)}{2(y_1 - x_1)^{n-a}} \left\{ \left(\frac{cD^a_{(\delta_1 + \delta_2 - x_1)} \phi}{(\delta_1 + \delta_2 - x_1)} \right) \left(\delta_1 + \delta_2 - x_1\right) + (-1)^n \left(\frac{cD^a_{(\delta_1 + \delta_2 - y_1)} \phi}{(\delta_1 + \delta_2 - y_1)} \right) \left(\delta_1 + \delta_2 - y_1\right) \right\} ,
\]
\[
\leq \frac{\phi^{(n)}(\delta_1 + \delta_2 - x_1) + \phi^{(n)}(\delta_1 + \delta_2 - y_1)}{2} \leq \phi^{(n)}(\delta_1) + \phi^{(n)}(\delta_2) - \frac{\phi^{(n)}(x_1) + \phi^{(n)}(y_1)}{2}.
\]

Moreover, for \( n = 1 \) and \( \alpha = 0 \), we obtain the result of Kian and Moslehian [35].

**Remark 2.** If we take \( x_1 = \delta_1, \ y_1 = \delta_2 \), then inequality (24) reduces to inequality (2.2) in [15].

Another result for the Hermite–Hadamard–Jensen–Mercer fractional inequality is given as follows:

**Theorem 4.** Let all the conditions in the hypothesis of Theorem 3 hold. Then,

\[
\phi^{(n)}\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left(\frac{x_s + y_s}{2}\right)\right) \leq 2^{n-\alpha-1} \frac{\Gamma(n - \alpha + 1)}{\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \gamma_s} \left\{ (-1)^n \left(\frac{cD^a_{(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} (x_s + y_s/2))} \phi}{\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s} \right) \left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s\right) \right\} \leq \sum_{s=1}^{l} \phi^{(n)}(\delta_s) - \frac{\sum_{s=1}^{l-1} \phi^{(n)}(x_s) + \sum_{s=1}^{l-1} \phi^{(n)}(y_s)}{2}.
\]

**Proof.** Let us consider \( t \in [0, 1] \). To prove the required result, we proceed as follows:

\[
\phi^{(n)}\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left(\frac{x_s + y_s}{2}\right)\right) = \phi^{(n)}\left\{ \frac{1}{2} \left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s + \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s\right) \right\} ,
\]
\[
= \phi^{(n)}\left\{ \frac{1}{2} \left(\sum_{s=1}^{l} \delta_s - \frac{t}{2} \sum_{s=1}^{l-1} x_s + \frac{2 - t}{2} \sum_{s=1}^{l-1} y_s + \sum_{s=1}^{l} \delta_s - \frac{t}{2} \sum_{s=1}^{l-1} y_s + \frac{2 - t}{2} \sum_{s=1}^{l-1} x_s\right) \right\} .
\]

Since \( \phi^{(n)} \) is a convex function, therefore (26) gives the following inequality:

\[
\phi^{(n)}\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left(\frac{x_s + y_s}{2}\right)\right) \leq \frac{1}{2} \left\{ \phi^{(n)}\left(\sum_{s=1}^{l} \delta_s - \frac{t}{2} \sum_{s=1}^{l-1} x_s + \frac{2 - t}{2} \sum_{s=1}^{l-1} y_s\right) + \phi^{(n)}\left(\sum_{s=1}^{l} \delta_s - \frac{t}{2} \sum_{s=1}^{l-1} y_s + \frac{2 - t}{2} \sum_{s=1}^{l-1} x_s\right) \right\} .
\]

By multiplying both sides of (27) by \( t^{n-\alpha-1} \) and then integrating over \( t \in [0, 1] \), we obtain
\[
\frac{1}{n - \alpha} \phi^{(n)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left( \frac{x_s + y_s}{2} \right) \right) \leq \frac{1}{2} \left\{ \int_{0}^{1} t^{n-\alpha-1} \phi^{(n)} \left( \sum_{s=1}^{l} \delta_s - \left( \frac{t}{2} \sum_{s=1}^{l-1} x_s + \frac{2 - t}{2} \sum_{s=1}^{l-1} y_s \right) \right) \, dt \right. \\
+ \left. \int_{0}^{1} t^{n-\alpha-1} \phi^{(n)} \left( \sum_{s=1}^{l} \delta_s - \left( \frac{t}{2} \sum_{s=1}^{l-1} y_s + \frac{2 - t}{2} \sum_{s=1}^{l-1} x_s \right) \right) \, dt \right\} \\
= \frac{1}{2 \left( \sum_{s=1}^{l-1} y_s - x_s \right)^{n-\alpha}} \left\{ \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left( x_s + y_s \right) \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s \right)^{n-\alpha+1} \, du \right. \\
+ \left. \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left( x_s + y_s \right) \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s \right)^{n-\alpha+1} \, du \right\}.
\]

Following the same procedure, as given in the proof of Theorem 3, we can show that

\[
\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left( x_s + y_s \right) < \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s, \quad \text{and} \quad \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left( x_s + y_s \right) > \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s.
\]

Now, from (28), we deduce

\[
\frac{1}{n - \alpha} \phi^{(n)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left( x_s + y_s \right) \right) = \frac{2^{n-\alpha-1} \Gamma \left( n - \alpha \right)}{\left( \sum_{s=1}^{l-1} \left( y_s - x_s \right) \right)^{n-\alpha}} \\
\times \left\{ \left( -1 \right)^{l} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left( x_s + y_s \right) \right)^{n-\alpha+1} \phi^{(n)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s \right) \right. \\
+ \left. \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s \right)^{n-\alpha+1} \phi^{(n)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s \right) \right\}.
\]

So, we have

\[
\phi^{(n)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left( x_s + y_s \right) \right) \leq \frac{2^{n-\alpha-1} \Gamma \left( n - \alpha + 1 \right)}{\left( \sum_{s=1}^{l-1} \left( y_s - x_s \right) \right)^{n-\alpha}} \\
\times \left\{ \left( -1 \right)^{l} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \left( x_s + y_s \right) \right)^{n-\alpha+1} \phi^{(n)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s \right) \right. \\
+ \left. \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s \right)^{n-\alpha+1} \phi^{(n)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s \right) \right\}.
\]
This proves the first inequality in (25).

In order to prove the second inequality of (25), we use Theorem 2 for \( r = 2, a_1 = t/2, \) and \( a_2 = 2 - t/2 \) as follows:

\[
\phi^{(n)}\left( \sum_{j=1}^{l} \delta_j \left( \frac{t}{2} \sum_{j=1}^{l-1} x_j + \frac{2 - t}{2} \sum_{j=1}^{l-1} y_j \right) \right) \leq \sum_{j=1}^{l} \phi^{(n)}(\delta_j) - \left( \frac{t}{2} \sum_{j=1}^{l-1} \phi^{(n)}(x_j) + \frac{2 - t}{2} \sum_{j=1}^{l-1} \phi^{(n)}(y_j) \right),
\]

Adding (32) and (33), we get

\[
\phi^{(n)}\left( \sum_{j=1}^{l} \delta_j \left( \frac{t}{2} \sum_{j=1}^{l-1} x_j + \frac{2 - t}{2} \sum_{j=1}^{l-1} y_j \right) \right) + \phi^{(n)}\left( \sum_{j=1}^{l} \delta_j \left( \frac{t}{2} \sum_{j=1}^{l-1} y_j + \frac{2 - t}{2} \sum_{j=1}^{l-1} x_j \right) \right) \leq 2 \sum_{j=1}^{l} \phi^{(n)}(\delta_j) - \left( \sum_{j=1}^{l} \phi^{(n)}(x_j) + \sum_{j=1}^{l} \phi^{(n)}(y_j) \right).
\]

By multiplying both sides of (34) by \( t^{m-a-1} \) and then integrating over \( t \in [0, 1] \), we obtain the second inequality of (25).

We establish the following result for the Caputo fractional operators on the basis of Lemma 1.

\[
\sum_{j=1}^{k} p_j x_j \leq \sum_{j=1}^{k} p_j \delta_j, \quad \sum_{j=1}^{k} p_j y_j \leq \sum_{j=1}^{k} p_j \delta_j, \quad \text{for } k = 1, \ldots, l - 1,
\]

\[
\sum_{j=1}^{l} p_j \delta_j = \sum_{j=1}^{l} p_j x_j, \quad \sum_{j=1}^{l} p_j \delta_j = \sum_{j=1}^{l} p_j y_j,
\]

then

\[
\phi^{(n)}\left( \sum_{j=1}^{l} \eta p_j \delta_j - \eta \sum_{j=1}^{l-1} \frac{p_j x_j + p_j y_j}{2} \right) \leq \frac{\Gamma(n - \alpha + 1)}{2(\sum_{j=1}^{l} \eta p_j y_j - \eta p_j x_j)^{n-a}} \left\{ \left( \frac{cD^\alpha}{\sum_{j=1}^{l} \eta p_j x_j - \sum_{j=1}^{l} \eta p_j y_j} \right) \left( \sum_{j=1}^{l} \eta p_j \delta_j - \sum_{j=1}^{l} \eta p_j x_j \right) \right\} \\
+ \frac{(-1)^{n}}{2}\left( \frac{cD^\alpha}{\sum_{j=1}^{l} \eta p_j x_j - \sum_{j=1}^{l} \eta p_j y_j} \right) \left( \sum_{j=1}^{l} \eta p_j \delta_j - \sum_{j=1}^{l} \eta p_j y_j \right) \left( \sum_{j=1}^{l} \eta p_j \delta_j - \sum_{j=1}^{l} \eta p_j x_j \right)
\]

\[
\leq \phi^{(n)}\left( \sum_{j=1}^{l} \eta p_j \delta_j - \sum_{j=1}^{l} \eta p_j x_j \right) + \phi^{(n)}\left( \sum_{j=1}^{l} \eta p_j \delta_j - \sum_{j=1}^{l} \eta p_j y_j \right) \\
\leq \frac{1}{2} \sum_{j=1}^{l} \eta p_j \phi^{(n)}(\delta_j) - \frac{1}{2} \sum_{j=1}^{l} \eta p_j \phi^{(n)}(x_j) + \frac{1}{2} \sum_{j=1}^{l-1} \eta p_j \phi^{(n)}(y_j).
\]
Proof. It can be written as

\[
\phi^{(n)}\left(\sum_{s=1}^{l} \eta_s \delta_s - \eta \sum_{s=1}^{l-1} p_s \left(\frac{X_s + Y_s}{2}\right)\right)
= \frac{1}{2} \left\{ t \left( \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s x_s \right) + (1-t) \left( \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s y_s \right) \right\} 
+ t \left( \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s y_s \right) + (1-t) \left( \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s x_s \right) 
\]

(37)

Since \(\phi^{(n)}\) is a convex function, therefore (37) gives the following inequality:

\[
\phi^{(n)}\left(\sum_{s=1}^{l} \eta_s \delta_s - \eta \sum_{s=1}^{l-1} p_s \left(\frac{X_s + Y_s}{2}\right)\right) \leq \frac{1}{2} \left\{ t \left( \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s x_s \right) + (1-t) \left( \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s y_s \right) \right\} 
+ \phi^{(n)}\left( \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s y_s \right) + (1-t) \left( \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s x_s \right) 
\]

(38)

By multiplying both sides of (38) by \(t^{n-a-1}\) and then integrating over \(t \in [0,1]\), we get

\[
\frac{1}{n-a} \phi^{(n)}\left(\sum_{s=1}^{l} \eta_s \delta_s - \eta \sum_{s=1}^{l-1} p_s \left(\frac{X_s + Y_s}{2}\right)\right) \leq \frac{1}{2} \left\{ \int_{0}^{1} t^{n-a-1} \phi^{(n)}\left( t \left( \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s x_s \right) + (1-t) \left( \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s y_s \right) \right) \ dt \right\} 
+ \int_{0}^{1} t^{n-a-1} \phi^{(n)}\left( \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s y_s \right) + (1-t) \left( \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s x_s \right) \ dt \right\} 
\]

(39)

In order to apply the definition of the Caputo fractional operators in (39), first, we show that

\[
\sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s p_s x_s < \sum_{s=1}^{l} \eta_s \delta_s - \sum_{s=1}^{l-1} \eta_s p_s x_s \]

(40)

Given that \(\sum_{s=1}^{l} \eta_s \delta_s = \sum_{s=1}^{l} \eta_s p_s x_s\), we have

Also, \(x_i > y_i \Rightarrow x_i > y_i \Rightarrow x_i - y_i > 0\).

By substituting (42) in (41), and adding \(\sum_{s=1}^{l} \eta_s \delta_s\) to both sides, we get

\[
\sum_{s=1}^{l} p_s y_s - \sum_{s=1}^{l-1} p_s X_s = p_1 x_1 - p_1 y_1 
\]

(41)
\[
\sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s < \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s,
\]
(43)

Now (39) implies

\[
\frac{1}{n-\alpha} \phi^{(n)} \left( \sum_{s=1}^{l} \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right) \leq \frac{1}{2} \left( \sum_{s=1}^{l} \left( \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right) \right)
\]

\[
\times \left\{ \left( \sum_{s=1}^{l} \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right) \right\} \times \left( \sum_{s=1}^{l} \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right)
\]

\[
+ (-1)^{\alpha} \left\{ \left( \sum_{s=1}^{l} \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right) \right\} \times \left( \sum_{s=1}^{l} \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right)
\]

(44)

and so

\[
\phi^{(n)} \left( \sum_{s=1}^{l} \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right) \leq \frac{1}{2} \left( \sum_{s=1}^{l} \left( \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right) \right)
\]

\[
\times \left\{ \left( \sum_{s=1}^{l} \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right) \right\} \times \left( \sum_{s=1}^{l} \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right)
\]

\[
+ (-1)^{\alpha} \left\{ \left( \sum_{s=1}^{l} \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right) \right\} \times \left( \sum_{s=1}^{l} \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right)
\]

(45)

Thus, we achieved the first inequality of (36).

To prove the second inequality, from the convexity of \( \phi^{(n)} \) we may write that

\[
\phi^{(n)} \left( t \left( \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s \right) + (1-t) \left( \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s \right) \right) \leq t \phi^{(n)} \left( \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s \right) + (1-t) \phi^{(n)} \left( \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s \right),
\]

(46)

\[
\phi^{(n)} \left( t \left( \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s \right) + (1-t) \left( \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s \right) \right) \leq t \phi^{(n)} \left( \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s \right) + (1-t) \phi^{(n)} \left( \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s \right).
\]

(47)

Adding (46) and (47) and then using Lemma 1 for \( r = 2 \), \( \sigma_1 = t \), and \( \sigma_2 = 1-t \), we obtain

\[
\phi^{(n)} \left( \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s \right) + (1-t) \phi^{(n)} \left( \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s \right)
\]

\[
+ \frac{1}{n-\alpha} \left( \sum_{s=1}^{l} \eta_p \psi_s - \eta \sum_{s=1}^{l-1} \eta_p \psi_s \right)
\]

\[
\leq \phi^{(n)} \left( \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s \right) + \phi^{(n)} \left( \sum_{s=1}^{l} \eta_p \psi_s - \sum_{s=1}^{l-1} \eta_p \psi_s \right)
\]

\[
\leq 2 \left( \sum_{s=1}^{l} \eta_p \phi^{(n)} \delta_s \right) - \left( \sum_{s=1}^{l} \eta_p \phi^{(n)} (x_s) + \sum_{s=1}^{l-1} \eta_p \phi^{(n)} (y_s) \right).
\]

(48)
Let us consider a function \( \phi \in C^n(I) \), such that \( \phi^{(n)} \) is a convex function on \( I \) and \( \delta = (\delta_1, \ldots, \delta_l) \),

\[
\phi^{(n)} \left( \sum_{s=1}^l \eta_p \delta_s - \sum_{s=1}^{l-1} \eta_p x_s \right) \leq \frac{\Gamma(n - \alpha + 1)}{2 \left( \sum_{s=1}^l (\eta_p y_s - \eta_p x_s) \right)^{\alpha - 1}} \\
\cdot \left\{ \left( \mathcal{D}^\alpha \left( \sum_{s=1}^l \eta_p \delta_s - \sum_{s=1}^{l-1} \eta_p x_s \right) \right)^{\phi} \left( \sum_{s=1}^l \eta_p \delta_s - \sum_{s=1}^{l-1} \eta_p y_s \right) \right\} \\
+ (-1)^n \left( \mathcal{D}^\alpha \left( \sum_{s=1}^l \eta_p \delta_s - \sum_{s=1}^{l-1} \eta_p x_s \right) \right)^{\phi} \left( \sum_{s=1}^l \eta_p \delta_s - \sum_{s=1}^{l-1} \eta_p y_s \right) \\
\leq \frac{\phi^{(n)} \left( \sum_{s=1}^l \eta_p \delta_s - \sum_{s=1}^{l-1} \eta_p x_s \right) + \phi^{(n)} \left( \sum_{s=1}^l \eta_p \delta_s - \sum_{s=1}^{l-1} \eta_p y_s \right)}{2} \\
\leq \sum_{s=1}^l \eta_p \phi^{(n)} (\delta_s) + \sum_{s=1}^l \eta_p \phi^{(n)} (y_s).
\]

**Proof.** By using Lemma 2 and following the procedure given in the proof of Theorem 5, we can obtain (50). □

**Remark 3.** Theorem 5 and Theorem 6 provide weighted forms of Theorem 3.

**Remark 4.** The weighted versions of Theorem 4 can be obtained in a similar fashion.

### 3. Bounds Associated with the Main Results

In this section, we discover two new identities associated with the right- and left-sides of the main results. Then

\[
\phi^{(n)} \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} y_s \right) + \phi^{(n)} \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} x_s \right) - \frac{\Gamma(n - \alpha + 1)}{2 \left( \sum_{s=1}^l (y_s - x_s) \right)^{\alpha - 1}} \\
\cdot \left\{ \left( \mathcal{D}^\alpha \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} y_s \right) \right)^{\phi} \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} x_s \right) \right\} \\
+ (-1)^n \left( \mathcal{D}^\alpha \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} y_s \right) \right)^{\phi} \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} y_s \right) \\
= \frac{\sum_{s=1}^l (y_s - x_s)}{2} \int_0^1 \left( t^{n-\alpha} - (1-t)^{n-\alpha} \right) \phi^{(n)} \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} (tx_s + (1-t)y_s) \right) dt.
\]
Proof. To prove our required result, we consider that

\begin{equation}
I = \int_0^1 \left( t^{n-\alpha} - (1 - t)^{n-\alpha} \right) \phi^{(n+1)} \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} (tx_s + (1-t)y_s) \right) dt,
\end{equation}

\begin{equation}
= \int_0^1 t^{n-\alpha} \phi^{(n+1)} \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} (tx_s + (1-t)y_s) \right) dt - \int_0^1 (1 - t)^{n-\alpha} \phi^{(n+1)} \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} (tx_s + (1-t)y_s) \right) dt
\end{equation}

\begin{equation}
= I_1 - I_2.
\end{equation}

Assuming that \( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} y_s < \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} x_s \) and using integration by parts formula, we obtain

\begin{equation}
I_1 = \int_0^1 t^{n-\alpha} \phi^{(n+1)} \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} (tx_s + (1-t)y_s) \right) dt = \frac{\Gamma(n - \alpha + 1)}{\left( \sum_{s=1}^{l-1} (y_s - x_s) \right)^{n-\alpha+1}} \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} x_s \right). 
\end{equation}

Similarly,

\begin{equation}
I_2 = \int_0^1 (1 - t)^{n-\alpha} \phi^{(n+1)} \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} (tx_s + (1-t)y_s) \right) dt = \frac{\Gamma(n - \alpha + 1)}{\left( \sum_{s=1}^{l-1} (y_s - x_s) \right)^{n-\alpha+1}} \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} x_s \right). 
\end{equation}
Now, we have

\[
I = \frac{\phi^{(n)}(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s) + \phi^{(n)}(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s)}{\sum_{s=1}^{l-1} (y_s - x_s)} \frac{\Gamma(n - \alpha + 1)}{\left(\sum_{s=1}^{l-1} (y_s - x_s)\right)^{n-\alpha+1}} \times \left\{ \left(\sum_{s=1}^{l} \sum_{r=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} y_s\right) \phi\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s\right) + (-1)^n \left(\sum_{s=1}^{l} \sum_{r=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} x_s\right) \phi\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s\right) \right\}.
\]

(55)

Multiplying both sides by \(\sum_{s=1}^{l-1} (y_s - x_s)/2\), we get (51).

\[\phi^{(n)}(\delta_1) + \phi^{(n)}(\delta_2) - \frac{\Gamma(n - \alpha + 1)}{2(\delta_2 - \delta_1)^{n-\alpha}} \left\{ \left(\sum_{s=1}^{l} \sum_{r=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} y_s\right) \phi\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s\right) + (-1)^n \left(\sum_{s=1}^{l} \sum_{r=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} x_s\right) \phi\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s\right) \right\} \]

(56)

Remark 6. If we take \(\alpha = 0\) and \(n = 1\) in Remark 5, then the equality (56) gives

\[
\frac{\phi(\delta_1) + \phi(\delta_2)}{2} - \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \phi(u) du = \frac{\delta_2 - \delta_1}{2} \int_0^1 (t^n - (1 - t)^n) \phi^{(n+1)}(t \delta_2 + (1 - t) \delta_1) dt.
\]

(57)

The equality (57) has been proved by Dragomir and Agarwal [5].

The following results have been established on the basis of Lemma 3:

\[
\left| \frac{\phi^{(n)}(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s) + \phi^{(n)}(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s)}{2} \frac{\Gamma(n - \alpha + 1)}{\left(\sum_{s=1}^{l-1} (y_s - x_s)\right)^{n-\alpha}} \times \left\{ \left(\sum_{s=1}^{l} \sum_{r=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} y_s\right) \phi\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s\right) + (-1)^n \left(\sum_{s=1}^{l} \sum_{r=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} x_s\right) \phi\left(\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s\right) \right\} \right| \leq \frac{\sum_{s=1}^{l-1} |y_s - x_s|}{n - \alpha + 1} \left\{ \frac{1}{2} \sum_{s=1}^{l-1} \phi^{(n+1)}(\delta_s) \right\}.
\]

(58)

\[\text{Theorem 7. Let us consider a differentiable function } \phi \text{ defined on } I, \text{ such that } \phi \in C^{n+1}(I) \text{ and } \delta = (\delta_1, \ldots, \delta_l), \ x = (x_1, \ldots, x_l), \ y = (y_1, \ldots, y_l) \text{ are three } l \text{-tuples, such that } \delta_s, x_s, y_s \in I, \text{ for all } s \in \{1, \ldots, l\}, x_s > y_s, a > 0. \text{ If } \delta \text{ majorizes } x, y, \text{ and } |\phi^{(n+1)}| \text{ is convex on } I, \text{ then}
\]
Proof. From Lemma 3, it follows that

\[
\left| \frac{\phi^{(n)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} y_s \right) + \phi^{(n)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \chi_s \right)}{2} - \frac{\Gamma(n, \alpha+1)}{2 \left( \sum_{s=1}^{l-1} (y_s - \chi_s) \right)^{n-\alpha}} \right|
\]

\times \left\{ \left( \frac{C^\delta}{\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \chi_s} \right) \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} \chi_s \right) \right\}^{n-\alpha}

= \left| \frac{\sum_{s=1}^{l-1} (y_s - \chi_s)}{2} \int_0^1 (t^{n-a} - (1-t)^{n-a}) \phi^{(n+1)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} (tx_s + (1-t)y_s) \right) dt \right|

\leq \left| \frac{\sum_{s=1}^{l-1} (y_s - \chi_s)}{2} \int_0^1 (t^{n-a} - (1-t)^{n-a}) \left| \phi^{(n+1)} \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} (tx_s + (1-t)y_s) \right) \right| dt \right|.

(59)

Using Theorem 2 for \( r = 2, \sigma_1 = t, \) and \( \sigma_2 = 1-t \) in (59) as a consequence of the convexity of \( |\phi^{(n+1)}| \), we obtain

\[
\leq \frac{\sum_{s=1}^{l-1} |y_s - \chi_s|}{2} \int_0^1 (t^{n-a} - (1-t)^{n-a}) \left\{ \sum_{s=1}^{l} |\phi^{(n+1)}(\delta_s)| - \left( t \sum_{s=1}^{l} \phi^{(n+1)}(x_s) + (1-t) \sum_{s=1}^{l-1} \phi^{(n+1)}(y_s) \right) \right\} dt.

(60)

Now finding \( C_1 \) and \( C_2 \), we have

\[
C_1 = \int_0^{1/2} (1-t)^{n-a} - t^{n-a} \left\{ \sum_{s=1}^{l} |\phi^{(n+1)}(\delta_s)| - \left( t \sum_{s=1}^{l} \phi^{(n+1)}(x_s) + (1-t) \sum_{s=1}^{l-1} \phi^{(n+1)}(y_s) \right) \right\} dt
\]

\[
= \left( \frac{1}{2} t \int_0^{1/2} (1-t)^{n-a} - t^{n-a} dt \right) \left\{ \sum_{s=1}^{l} \phi^{(n+1)}(x_s) \right\}

\times \left( \frac{1}{2} t \int_0^{1/2} (1-t)^{n-a} - t^{n-a} dt + \sum_{s=1}^{l-1} \phi^{(n+1)}(y_s) \right) \int_0^{1/2} (1-t)^{n-a} - t^{n-a} (1-t) dt

= \frac{1}{2} t \int_0^{1/2} (1-t)^{n-a} - t^{n-a} dt \left( \sum_{s=1}^{l} \phi^{(n+1)}(x_s) \right)

\times \left( \sum_{s=1}^{l-1} \phi^{(n+1)}(y_s) \right) \left( \int_0^{1/2} (1-t)^{n-a} - t^{n-a} dt + \int_0^{1/2} t^{n-a} dt \right)

\quad + \sum_{s=1}^{l-1} \phi^{(n+1)}(y_s) \left( \int_0^{1/2} (1-t)^{n-a} dt - \int_0^{1/2} (1-t)^{n-a} dt \right).
\]
\[
\frac{1}{n-\alpha + 1} \left( \frac{1}{n-\alpha + 1} - \frac{2^{\alpha-1}}{n-\alpha + 1} \right) \sum_{s=1}^{l-1} \phi^{(n)}(\delta_s) \left( \frac{1}{n-\alpha + 1} - \frac{2^{\alpha-1}}{n-\alpha + 1} \right)
\]

\[= C_1 + C_2 \]

Inserting (62) in (60), we achieve (58).

\[\text{Corollary 1. If we take } l = 2, x_1 = \delta_1, \text{ and } y_1 = \delta_2 \text{ in Theorem 7, then inequality (58) reduces to}\]

\[
\left| \phi^{(n)}(\delta_1) + \phi^{(n)}(\delta_2) - \Gamma(n-\alpha + 1) \frac{c}{2(\delta_2 - \delta_1)^{\alpha+1}} \left( cD_{\delta_1}^\alpha \phi(\delta_2) + (-1)^c D_{\delta_2}^\alpha \phi(\delta_1) \right) \right| \leq \frac{1}{n-\alpha + 1} \left( \frac{1}{2^{\alpha-1}} \right) \left( \phi^{(n)}(\delta_1) + \phi^{(n)}(\delta_2) \right) \]

or

\[\phi^{(n)}(\delta_1 + \delta_2) - \Gamma(n-\alpha + 1) \left( \frac{1}{2(\delta_2 - \delta_1)^{\alpha+1}} \left( cD_{\delta_1}^\alpha \phi(\delta_2) + (-1)^c D_{\delta_2}^\alpha \phi(\delta_1) \right) \right) \leq \frac{1}{n-\alpha + 1} \left( \frac{1}{2^{\alpha-1}} \right) \left( \phi^{(n)}(\delta_1) + \phi^{(n)}(\delta_2) \right) \]

which is proved in [15].

\[\text{Theorem 8. Let us consider a differentiable function } \phi \text{ defined on } I, \text{ such that } \phi \in C^{n+1} (I) \text{ and } \delta = (\delta_1, \ldots, \delta_l),\]

\[x = (x_1, \ldots, x_l), \ y = (y_1, \ldots, y_l) \text{ are three } l-\text{tuples, such that } \delta_1, x_1, y_1 \in I, \text{ for all } s \in [1, \ldots, l], \ x_s \geq y_s, \ a > 0. \text{ If } q > 1, \delta \text{ majorizes } x, y, \text{ and } |\phi^{(n)}(\delta)| \text{ is convex on } I, \text{ then}\]
By applying power mean inequality to the above integral,
\[
\left| \frac{\phi^{(n)}(\sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} y_s) + \phi^{(n)}(\sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} x_s)}{2} - \frac{\Gamma(n - \alpha + 1)}{2^{n-\alpha}} (\sum_{s=1}^{l-1} (y_s - x_s))^{n-\alpha} \right|
\]
\[
\times \left\{ \left( c D^\beta \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} y_s \right)^{\gamma} \right) \left( \sum_{s=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} x_s \right) + (-1)^q \left( c D^\beta \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} y_s \right)^{\gamma} \right) \left( \sum_{s=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} y_s \right) \right\}
\]
\[
\leq \frac{\sum_{s=1}^{l-1} |y_s - x_s|}{n - \alpha + 1} \left( \sum_{s=1}^{l-1} |\phi^{(n+1)}(\delta_s)|^q - \frac{\sum_{s=1}^{l-1} |\phi^{(n-1)}(y_s)|^q + \sum_{s=1}^{l-1} |\phi^{(n-1)}(y_s)|^q}{2} \right)^{1/q}.
\]

**Proof.** From Lemma 3, it follows that
\[
\left| \frac{\phi^{(n)}(\sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} y_s) + \phi^{(n)}(\sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} x_s)}{2} - \frac{\Gamma(n - \alpha + 1)}{2^{n-\alpha}} (\sum_{s=1}^{l-1} (y_s - x_s))^{n-\alpha} \right|
\]
\[
\times \left\{ \left( c D^\beta \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} y_s \right)^{\gamma} \right) \left( \sum_{s=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} x_s \right) + (-1)^q \left( c D^\beta \left( \sum_{s=1}^l \delta_s - \sum_{s=1}^{l-1} y_s \right)^{\gamma} \right) \left( \sum_{s=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} y_s \right) \right\}
\]
\[
= \frac{\sum_{s=1}^{l-1} (y_s - x_s)}{2} \int_0^1 (t^{n-\alpha} - (1 - t)^{n-\alpha}) \phi^{(n+1)} \left( \sum_{s=1}^{l-1} (t \delta_s - \sum_{s=1}^{l-1} (t \delta_s + (1 - t) y_s)) \right) dt
\]
\[
\leq \frac{\sum_{s=1}^{l-1} |y_s - x_s|}{2} \int_0^1 |t^{n-\alpha} - (1 - t)^{n-\alpha}| \phi^{(n+1)} \left( \sum_{s=1}^{l-1} (t \delta_s - \sum_{s=1}^{l-1} (t \delta_s + (1 - t) y_s)) \right) dt.
\]

By applying power mean inequality to the above integral, we obtain
\[
\leq \frac{\sum_{s=1}^{l-1} |y_s - x_s|}{2} \left( \int_0^1 |t^{n-\alpha} - (1 - t)^{n-\alpha}| dt \right)^{1-1/q}
\]
\[
\times \left( \int_0^1 |t^{n-\alpha} - (1 - t)^{n-\alpha}| \phi^{(n+1)} \left( \sum_{s=1}^{l-1} (t \delta_s - \sum_{s=1}^{l-1} (t \delta_s + (1 - t) y_s)) \right) dt \right)^q
\]
\[
= \frac{\sum_{s=1}^{l-1} |y_s - x_s|}{2} \left( \int_0^{1/2} (1 - t)^{n-\alpha} - t^{n-\alpha} dt + \int_{1/2}^1 (t^{n-\alpha} - (1 - t)^{n-\alpha}) dt \right)^{1-1/q}
\]
\[
\times \left( \int_0^1 |t^{n-\alpha} - (1 - t)^{n-\alpha}| \phi^{(n+1)} \left( \sum_{s=1}^{l-1} (t \delta_s - \sum_{s=1}^{l-1} (t \delta_s + (1 - t) y_s)) \right) dt \right)^q
\]

Since \(|\phi^{(n+1)}|^q\) is convex, therefore using Theorem 2 for \(r = 2, \sigma_1 = t, \) and \(\sigma_2 = 1 - t\) in (66), we obtain
\[ \sum_{s=1}^{l} \left( \int_{0}^{1/2} (1-t)^{n-a} - t^{n-a} \right) dt + \int_{1/2}^{1} \left( t^{n-a} - (1-t)^{n-a} \right) dt \]

\[ = \sum_{s=1}^{l-1} \frac{1}{y_s - x_s} \left( \int_{0}^{1/2} (1-t)^{n-a} - t^{n-a} \right) dt + \int_{1/2}^{1} \left( t^{n-a} - (1-t)^{n-a} \right) dt \]

\[ \times \sum_{s=1}^{l-1} \left( \int_{0}^{1/2} (1-t)^{n-a} - t^{n-a} \right) dt + \int_{1/2}^{1} \left( t^{n-a} - (1-t)^{n-a} \right) dt \]

\[ = \sum_{s=1}^{l-1} \frac{1}{y_s - x_s} \left( \int_{0}^{1/2} (1-t)^{n-a} - t^{n-a} \right) dt + \int_{1/2}^{1} \left( t^{n-a} - (1-t)^{n-a} \right) dt \]

By calculating these simple integrals, we get (64). □

**Lemma 4.** Let all the conditions in the hypothesis of Lemma 3 hold. Then,

Another lemma is established as follows:

\[ \frac{2^{n-a-1} \Gamma(n-a+1)}{\sum_{s=1}^{l-1} (y_s - x_s)} \left( \sum_{s=1}^{l-1} \sum_{t=0}^{l-1} \left( x_s + \frac{y_s}{2} \right) \right) \left( \sum_{s=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} x_s \right) \]

\[ + \left( -1 \right)^n \left( \sum_{s=1}^{l-1} \sum_{t=0}^{l-1} \left( x_s + \frac{y_s}{2} \right) \right) \left( \sum_{s=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} x_s \right) \]

\[ = \sum_{s=1}^{l-1} \frac{1}{y_s - x_s} \left( \int_{0}^{1/2} t^{n-a} \phi^{(n+1)} \left( \sum_{s=1}^{l-1} \delta_s - \frac{2 - t}{2} \sum_{s=1}^{l-1} x_s + \frac{t}{2} \sum_{s=1}^{l-1} y_s \right) dt \]

\[ - \int_{1/2}^{1} t^{n-a} \phi^{(n+1)} \left( \sum_{s=1}^{l-1} \delta_s - \frac{2 - t}{2} \sum_{s=1}^{l-1} x_s + \frac{t}{2} \sum_{s=1}^{l-1} y_s \right) dt \].

**Proof.** It can be easily proved by following the procedure given in the proof of Lemma 3. □

**Theorem 9.** Let all the conditions in the hypothesis of Theorem 7 hold. Then,

**Remark 7.** When we take \( l = 2, x_1 = \delta_1, \) and \( y_1 = \delta_2 \) in Lemma 4, then it reduces to the equality (3.1) in [16].
Proof. From Lemma 4, it follows that

\[
\frac{2^{n-a-1} \Gamma(n - \alpha + 1)}{\left(\sum_{s=1}^{t-1} (y_s - x_s)\right)^{\alpha - a}} \left(\sum_{i=1}^{t-1} \frac{\delta_s}{\left(\sum_{s=1}^{t-1} \left(y_s + x_s/2\right)\right)^{\alpha - a}}\right) + (-1)^n \left(\sum_{i=1}^{t-1} \frac{\delta_s}{\left(\sum_{s=1}^{t-1} \left(y_s + x_s/2\right)\right)^{\alpha - a}}\right) - \phi^{(n)} \left(\sum_{s=1}^{t-1} \frac{\delta_s}{\left(\sum_{s=1}^{t-1} \left(y_s + x_s/2\right)\right)^{\alpha - a}}\right)
\]

(69)

By utilizing Theorem 2 for \( r = 2, \sigma_1 = 2 - t, \) and \( \sigma_2 = t/2 \) in (70), we obtain

\[
\frac{\sum_{s=1}^{t-1} (y_s - x_s)}{4} \left\{ \int_0^1 t^{n-a} \frac{\phi^{(n+1)}(\delta_s)}{\left(\sum_{s=1}^{t-1} \left(y_s + x_s/2\right)\right)^{\alpha - a}} \right\} \left(\sum_{i=1}^{t-1} \frac{\delta_s}{\left(\sum_{s=1}^{t-1} \left(y_s + x_s/2\right)\right)^{\alpha - a}}\right) \mathrm{d}r
\]

(71)
This finishes the proof.

\[
\left| \frac{1}{y_1 - x_1} \int_{\delta_1, \delta_2, \ldots, \delta_l} \phi(u) du - \phi \left( \delta_1 + \delta_2 - \frac{x_1 + y_1}{2} \right) \right| \leq \frac{|y_1 - x_1|}{4} \left| \left| \phi' (\delta_1) + \phi' (\delta_2) - \frac{\phi' (x_1) + \phi' (y_1)}{2} \right| \right|.
\]

(72)

**Theorem 10.** Let us consider a differentiable function \( \phi \) defined on \( I \), such that \( \phi \in C^{n+1} (I) \) and \( \delta = (\delta_1, \ldots, \delta_l) \), \( x = (x_1, \ldots, x_l) \), \( y = (y_1, \ldots, y_l) \) are three \( l \)- tuples, such that \( \delta_i, x_i, y_i \in I \), for all \( i \in \{1, \ldots, l\} \), \( x_i > y_i, \alpha > 0 \). If \( q > 1 \) such that \( 1/p + 1/q = 1 \), \( \delta \) majorizes \( x, y \), and \( |\phi^{(n+1)}|^q \) is convex on \( I \), then

\[
\frac{2^{n-a-1} \Gamma (n - \alpha + 1)}{(\sum_{s=1}^{l} (y_s - x_s))^{\alpha-a}} \left| \left( D^a (\sum_{r=1}^{l} \delta_r - \sum_{s=1}^{l} (x_r + y_r/2))^a \phi \left( \sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} x_s \right) \right. \right. \nonumber \\
+ (-1)^r \left( D^a (\sum_{s=1}^{l} \delta_s - \sum_{s=1}^{l-1} (x_s + y_s/2))^a \phi \left( \sum_{s=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} x_s \right) \right. \nonumber \\
- \phi^{(n)} \left( \sum_{s=1}^{l-1} \delta_s - \sum_{s=1}^{l-1} (x_s + y_s/2) \right) \right| \nonumber \\
\left. \left. \left( \sum_{s=1}^{l} y_s - x_s \right)^{1/p} \right) \right| \right| \left( \sum_{s=1}^{l} \phi^{(n+1)} (\delta_s) \left( 1 \right) + \sum_{s=1}^{l} \phi^{(n+1)} (y_s) \right) \right|.
\]

(73)

**Proof.** From Lemma 4, it follows that

\[
= \frac{\sum_{s=1}^{l} (y_s - x_s)}{4} \left\{ \int_0^1 t^{n-a} \phi^{(n+1)} \left( \sum_{s=1}^{l} \delta_s - \left( \frac{2 - t}{2} \sum_{s=1}^{l} x_s + \frac{t - 1}{2} \sum_{s=1}^{l} y_s \right) \right) dt \right\},
\]

(74)

By applying Hölder’s inequality to the above integral, we have
\[
\leq \frac{\sum_{s=1}^{l-1} |y_s - x_s|}{4} \left\{ \left( \int_0^1 t^{(n-a)p} \, dt \right)^{1/p} \left( \int_0^1 |\phi^{(n+1)}(\sum_{s=1}^{l-1} \delta_s - \left(\frac{2-t}{2} \sum_{s=1}^{l-1} x_s + \frac{t}{2} \sum_{s=1}^{l-1} x_s\right)||^q \, dt \right)^{1/q} \\
+ \left( \int_0^1 t^{(n-a)p} \, dt \right)^{1/p} \left( \int_0^1 |\phi^{(n+1)}\left(\sum_{s=1}^{l-1} \delta_s - \left(\frac{2-t}{2} \sum_{s=1}^{l-1} y_s + \frac{t}{2} \sum_{s=1}^{l-1} x_s\right)||^q \, dt \right)^{1/q} \right) \right\}
\]

(75)

Since \(|\phi^{(n+1)}|^q\) is convex, therefore using Theorem 2 for \(r = 2, \sigma_1 = 2 - t/2,\) and \(\sigma_2 = t/2\) in (75), we obtain

\[
\leq \frac{\sum_{s=1}^{l-1} |y_s - x_s|}{4} \left( \frac{1}{np - ap + 1} \right)^{1/p} \left( \int_0^1 \left( \sum_{s=1}^{l-1} |\phi^{(n+1)}(\delta_s)|^q - \left(\frac{2-t}{2} \sum_{s=1}^{l-1} |\phi^{(n+1)}(x_s)|^q + \frac{t}{2} \sum_{s=1}^{l-1} |\phi^{(n+1)}(y_s)|^q\right) \right) \, dt \right)^{1/q} \\
+ \left( \int_0^1 \sum_{s=1}^{l-1} |\phi^{(n+1)}(\delta_s)|^q - \left(\frac{2-t}{2} \sum_{s=1}^{l-1} |\phi^{(n+1)}(x_s)|^q + \frac{t}{2} \sum_{s=1}^{l-1} |\phi^{(n+1)}(y_s)|^q\right) \, dt \right)^{1/q} \right) \right\}
\]

(76)

By using Minkowski’s inequality, we get

\[
\leq \frac{\sum_{s=1}^{l-1} |y_s - x_s|}{16} \left( \frac{4}{np - ap + 1} \right)^{1/p} \left\{ 4^{1/q} \cdot 2 \sum_{s=1}^{l-1} |\phi^{(n+1)}(\delta_s)| - (3^{1/q} + 1) \left( \sum_{s=1}^{l-1} |\phi^{(n+1)}(x_s)| + \sum_{s=1}^{l-1} |\phi^{(n+1)}(y_s)| \right) \right\}
\]

(77)

This completes the proof. \qed
Remark 8. If we choose \( l = 2 \) in Theorem 10, then we get inequality (36) in [14].

\[
\begin{align*}
2^{n-\alpha-1} \Gamma(n-\alpha+1) & \left( \frac{1}{\left( \sum_{s=1}^{l-1} (y_s - x_s) \right)^{\alpha}} \right)^{n/\alpha} \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1}
\end{align*}
\]

Theorem 11. Let all the conditions in the hypothesis of Theorem 8 hold. Then,

\[
\begin{align*}
2^{n-\alpha-1} \Gamma(n-\alpha+1) & \left( \frac{1}{\left( \sum_{s=1}^{l-1} (y_s - x_s) \right)^{\alpha}} \right)^{n/\alpha} \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1}
\end{align*}
\]

Proof. From Lemma 4, it follows that

\[
\begin{align*}
2^{n-\alpha-1} \Gamma(n-\alpha+1) & \left( \frac{1}{\left( \sum_{s=1}^{l-1} (y_s - x_s) \right)^{\alpha}} \right)^{n/\alpha} \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1}
\end{align*}
\]

By applying power mean inequality to the above integral, we get

\[
\begin{align*}
2^{n-\alpha-1} \Gamma(n-\alpha+1) & \left( \frac{1}{\left( \sum_{s=1}^{l-1} (y_s - x_s) \right)^{\alpha}} \right)^{n/\alpha} \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1} \\
& \left( \frac{1}{\left( \sum_{s=1}^{l-1} (x_s + y_{s/2}) \right)^{\alpha}} \right)^{l-1}
\end{align*}
\]
\begin{align*}
&\leq \frac{\sum_{i=1}^{l-1} |y_i - x_i|}{4} \left\{ \left( \int_0^1 t^{n-a} \mathrm{d}t \right)^{1-1/q} \left( \int_0^1 t^{n-a} \left( \sum_{i=1}^l \delta_i \left( \frac{2-t}{2} \sum_{j=1}^{l-1} y_j + \frac{t}{2} \sum_{j=1}^{l-1} x_j \right) \right)^q \mathrm{d}t \right)^{1/q} \\
&\quad + \left( \int_0^1 t^{n-a} \left( \sum_{i=1}^l \delta_i \left( \frac{2-t}{2} \sum_{j=1}^{l-1} y_j + \frac{t}{2} \sum_{j=1}^{l-1} x_j \right) \right)^q \mathrm{d}t \right)^{1/q} \right\} \\
&= \frac{\sum_{i=1}^{l-1} |y_i - x_i|}{4} \left\{ \left( \int_0^1 t^{n-a} \left( \frac{1}{n-\alpha+1} \left( \sum_{i=1}^l |\phi^{(n+1)} (\delta_i)|^q - \left( \frac{2-t}{2} \sum_{j=1}^{l-1} |\phi^{(n+1)} (x_j)|^q + \frac{t}{2} \sum_{j=1}^{l-1} |\phi^{(n+1)} (x_j)|^q \right) \right)^q \mathrm{d}t \right)^{1/q} \\
&\quad + \left( \frac{1}{n-\alpha+1} \sum_{i=1}^l |\phi^{(n+1)} (\delta_i)|^q - \left( \frac{n-\alpha + 3}{2(n-\alpha + 1)(n-\alpha + 2)} \sum_{i=1}^l |\phi^{(n+1)} (x_i)|^q + \frac{1}{2(n-\alpha + 2)} \sum_{i=1}^l |\phi^{(n+1)} (x_i)|^q \right) \right)^{1/q} \right\} \\
&= \frac{\sum_{i=1}^{l-1} |y_i - x_i|}{4} \left\{ \left( \frac{1}{n-\alpha+1} \sum_{i=1}^l |\phi^{(n+1)} (\delta_i)|^q - \left( \frac{n-\alpha + 3}{2(n-\alpha + 1)(n-\alpha + 2)} \sum_{i=1}^l |\phi^{(n+1)} (y_i)|^q + \frac{1}{2(n-\alpha + 2)} \sum_{i=1}^l |\phi^{(n+1)} (x_i)|^q \right) \right)^{1/q} \right\}.
\end{align*}

Due to the convexity of $|\phi^{(n+1)}|^q$, using Theorem 2 for $r = 2$, $\sigma_1 = 2 - t/2$, and $\sigma_2 = t/2$ in (80), we have

\begin{align*}
&= \frac{\sum_{i=1}^{l-1} |y_i - x_i|}{4} \left\{ \left( \int_0^1 t^{n-a} \left( \frac{1}{n-\alpha+1} \sum_{i=1}^l |\phi^{(n+1)} (\delta_i)|^q - \left( \frac{2-t}{2} \sum_{j=1}^{l-1} |\phi^{(n+1)} (y_j)|^q + \frac{t}{2} \sum_{j=1}^{l-1} |\phi^{(n+1)} (x_j)|^q \right) \right)^q \mathrm{d}t \right)^{1/q} \\
&\quad + \left( \frac{1}{n-\alpha+1} \sum_{i=1}^l |\phi^{(n+1)} (\delta_i)|^q - \left( \frac{n-\alpha + 3}{2(n-\alpha + 1)(n-\alpha + 2)} \sum_{i=1}^l |\phi^{(n+1)} (y_i)|^q + \frac{1}{2(n-\alpha + 2)} \sum_{i=1}^l |\phi^{(n+1)} (x_i)|^q \right) \right)^{1/q} \right\} \\
&\quad \leq \frac{\sum_{i=1}^{l-1} |y_i - x_i|}{4} \left\{ \left( \int_0^1 t^{n-a} \left( \sum_{i=1}^l \delta_i \left( \frac{2-t}{2} \sum_{j=1}^{l-1} y_j + \frac{t}{2} \sum_{j=1}^{l-1} x_j \right) \right)^q \mathrm{d}t \right)^{1/q} \\
&\quad + \left( \frac{1}{n-\alpha+1} \sum_{i=1}^l |\phi^{(n+1)} (\delta_i)|^q - \left( \frac{n-\alpha + 3}{2(n-\alpha + 1)(n-\alpha + 2)} \sum_{i=1}^l |\phi^{(n+1)} (y_i)|^q + \frac{1}{2(n-\alpha + 2)} \sum_{i=1}^l |\phi^{(n+1)} (x_i)|^q \right) \right)^{1/q} \right\}.
\end{align*}

Hence, the proof is completed.

\textbf{Corollary 3.} For $l = 2$, $n = 1$, and $\alpha = 0$, Theorem 11 gives the following inequality:

\begin{align*}
\frac{1}{y_1 - x_1} \int_{\delta_1 + \delta_2 - x_1 + y_1}^{\delta_1 + \delta_2 - x_1} \phi(u) \mathrm{d}u - \phi\left( \delta_1 + \delta_2 - \frac{x_1 + y_1}{2} \right) \leq \frac{|y_1 - x_1|}{2(2\beta^{\beta} - 1)} \left\{ \left( \frac{\phi\left( \delta_1 \right) + \phi\left( \delta_2 \right)}{2} - \frac{2|\phi\left( x_1 \right)|^q}{2} \right)^{1/q} \\
&\quad + \left( \frac{\phi\left( \delta_1 \right) + \phi\left( \delta_2 \right)}{2} - \frac{2|\phi\left( y_1 \right)|^q + |\phi\left( x_1 \right)|^q}{2} \right)^{1/q} \right\}.
\end{align*}

\textbf{Remark 9.} We can also obtain weighted versions for all the results derived in this section.
4. Conclusion

A new idea in the form of unified inequalities has been put forward. Tools that helped during the development of the main results are the notions of some existing inequalities, majorization theory, and various forms of convex functions. The results have been put up in the context of Hermite–Hadamard–Jensen–Mercer inequalities. The selection of the present areas of inequalities has been made on the basis of their consistent attraction for researchers and their vast applicability in enormous fields. Both the weighted and unweighted versions of the obtained results have been presented. Moreover, some new identities for differentiable functions have been derived. Using these identities and considering the convexity of $|\phi^{(n+1)}|$ and $|\phi^{(n+1)}|^q (q > 1)$, bounds for the absolute difference of the right- and left-sides of the main results have been provided.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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