Integrable Systems and Harmonic Maps into Lie Groups

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Abstract

We study harmonic maps into Lie groups as a generalisation of the study of other well-known integrable systems, particularly the Toda and self-dual Chern Simons theories.

I. INTRODUCTION

The concept of integrability in infinite dimensions is not clear cut. In classical mechanics, a Hamiltonian system with a $2n$-dimensional phase space is said to be integrable if it has $n$ constants of the motion in involution (with vanishing Poisson brackets). In systems of partial differential equations, there are infinitely many degrees of freedom, and there is no straightforward corresponding definition. Since the equations we are working with are static equations, not flow equations, integrals are not even defined.

There are other features of the systems that are associated with being ‘integrable’. First, the equations are to some degree soluble, meaning that explicit solutions can be found and there exist general methods for constructing solutions, which may be superimposable in some extended sense. Alternatively, it may be possible to find a large number of constants of the motion, or the system may have the Painleve property.

The prototype of the integrable system is the Korteweg-De Vries (KdV) equation. A solitary wave is modelled by a soliton solution to the KdV equation:

$$4u_t - u_{xxx} - 6uu_x = 0$$

where $\kappa$ and $x_0$ are constants. The integrability of the KdV equation can be traced to the existence of a Lax pair, or obtaining the equation is the condition that the two differential operators commute. For KdV, the equation is the condition that the two differential operators

$$L = \partial_x^2 + u \quad \text{and} \quad M = \partial_t - \partial_x^3 - \frac{3}{2}u\partial_x - \frac{3}{4}u_x$$

commute.

In this paper we look at the nature of relationships between some of the better-known integrable systems. We focus on one of the most widely studied integrable system, the Toda model, and find a direct connection to the theory of harmonic maps into Lie groups.
II. TODA THEORY

The Toda field is a multicomponent field in two dimensions satisfying

$$\partial^2 \phi_i + \sum k_{ij} e^{\phi_j} = 0$$  \hspace{1cm} (3)

where $k_{ij}$ is the Cartan matrix of a semisimple Lie algebra. The Liouville equation is obtained from (3) by taking $k$ as the Cartan matrix of the Lie algebra $\text{sl}(2, \mathbb{C})$. The Toda field equations have a formulation in terms of a Lax pair. First, we need some results from the theory of semisimple Lie algebra to describe this. If a Lie algebra $\mathcal{L}$ has a basis $L^i$ and structure constants $f_{ij}^k$ then

$$[L^i, L^j] = \sum_k f_{ij}^k L^k$$  \hspace{1cm} (4)

Associated with $\mathcal{L}$ is its set of roots $\Phi$. These roots are $r$-dimensional vectors, where the rank $r$ of $\mathcal{L}$ is the maximal number of linearly independent commuting generators. A set of simple roots $\Delta = \alpha_i; \ i = 1...r$ is a subset of $\Phi$ such that the difference of any two of its elements is not in $\Phi$. Any root may be expressed

$$\alpha = \sum_i n_i \alpha_i$$  \hspace{1cm} (5)

with the $n_i$ either all non-negative or all non-positive integers. Hence $\Phi$ is divided into two sets $\Phi^+$ and $\Phi^-$ containing positive and negative roots respectively.

The Cartan subalgebra is the maximal abelian subalgebra of $\mathcal{L}$ is called the Cartan subalgebra. For $\text{SU}(n)$, these are just the diagonal matrices. A Chevalley basis of the Lie algebra is a choice of an element $H_i$ of the Cartan for each simple root. The remaining generators $E_{\alpha}$ are labelled by roots. Those labelled by simple roots are for convenience denoted as $E_{\alpha}^\pm := E_{\pm \alpha}$, which for our purposes are just the generators for the off-diagonal parts of the algebra.

In this basis the algebra takes the form

$$[H_i, H_j] = 0$$
$$[H_i, E_j^\pm] = \pm k_{ji} E_j^\pm$$
$$[E_i^+, E_j^-] = \delta_{ij} H_j$$  \hspace{1cm} (6)

The upper and lower triangular matrices are given by

$$\mathcal{L}_{\pm} = \{H_i, E_{\alpha} : \ i = 1...r, \ \alpha \in \Phi^\pm\}$$  \hspace{1cm} (7)

We are now ready to write down the Lax pair and the associated linear problem which codify the Toda theory.

We follow Leznov and Saveliev [13] and consider a connection whose components have values in different subalgebras

$$A_{z, \bar{z}} \in \mathcal{L}_{+, -}$$  \hspace{1cm} (8)

If the connection is trivialisable, that is, it satisfies the linear equations
\begin{equation}
\partial_z g = A_z g, \quad \bar{\partial}_z g = \bar{A}_z g \tag{9}
\end{equation}

then it automatically satisfies the flatness or integrability condition

\begin{equation}
[\partial_z + A_z, \partial_{\bar{z}} + A_{\bar{z}}] = 0 \tag{10}
\end{equation}

An appropriate choice for \( A \) is:

\begin{align*}
A_z &= \sum_i \left( \partial_z \psi_i H_i + \alpha E_i^+ \right) \\
A_{\bar{z}} &= \sum_i \alpha e^{\beta \phi_i} E_i^-
\end{align*} \tag{11}

where \( \alpha, \beta \) are constants and \( \psi, \phi \) are two \( r \)-component classical fields. The algebra is straightforward, and in characteristic fashion we get an equation for each element of the Chevalley basis that appears. Write out (10) explicitly, to get

\begin{equation}
\partial_z A_z - \bar{\partial}_{\bar{z}} A_{\bar{z}} + \sum_{ij} \left( \partial_+ \psi_i \alpha e^{\beta \phi_j} \left[ H_i, E_j^- \right] + \alpha^2 e^{\beta \phi_j} \left[ E_i^+, E_j^- \right] \right) \\
= \sum_i \left( \alpha \beta \partial_+ \phi_i e^{\beta \phi_i} E_i^- - \partial_z \partial_{\bar{z}} \psi_i H_i \right) - \sum_{ij} \alpha \partial_+ \psi_i e^{\beta \phi_j} k_{ji} E_j^- + \sum_i \alpha^2 e^{\beta \phi_i} H_i \\
= 0
\end{equation} \tag{12}

which is satisfied by

\begin{align*}
\psi_i &= \beta \sum_j k_{ij}^{-1} \phi_j \\
\partial_z \partial_{\bar{z}} \psi_i &= \alpha^2 e^{\beta \phi_i} 
\end{align*} \tag{13}

or written in the form of a single equation, the Toda equation itself

\begin{equation}
\partial_z \partial_{\bar{z}} \phi_i = \frac{\alpha^2}{\beta} \sum_j k_{ij} e^{\beta \phi_j} \tag{14}
\end{equation}

### III. CONFORMAL AFFINE TODA THEORY

Because of the appearance of the spectral parameter \( \lambda \) in our Lax pair for the harmonic map, we will find it is useful to study Lie algebras containing an affine parameter, i.e., affine Lie algebras. The corresponding Toda theories are the Affine Toda theories. We will be able to describe a relationship between harmonic maps into Lie groups and all of these models, most interestingly with the so-called Conformal Affine Toda theory. This conformally invariant field theory is based on the affine Lie algebra \( \hat{sl}_2 \) which reduces under certain circumstances to the Liouville theory and the non-conformal sinh-Gordon theory.

We first describe the conformally invariant CAT theory [5]. These models are obtained from the usual Toda field theory by adding two fields which transform in the correct way under conformal transformations. The addition of these fields is facilitated by constructing the affine Lie algebra \( \hat{sl}_2 \). This is the Lie algebra of traceless \( 2 \times 2 \) matrices with entries which are Laurent polynomials in \( \lambda \) (the loop algebra \( \hat{s}l_2 \)). This algebra is centrally extended as follows:

\begin{equation}
\hat{sl}_2' \cong \hat{s}l_2 \oplus \mathbb{C}c . \tag{15}
\end{equation}
The affine Lie algebra $\hat{sl}_2$ is obtained by adding the derivation $d = \lambda \frac{d}{d\lambda}$. The algebra can be decomposed as

$$\hat{sl}_2' = \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$$

where $\mathcal{N}_-$, $\mathcal{N}_+$ are lower and upper triangular matrices respectively, and $\mathcal{H}$, the Cartan sub-algebra, is spanned by the elements $H$, $c$ and $d$. We write in co-ordinates $\Phi : \mathbb{C} \to \mathcal{H}$

$$\Phi = \frac{1}{2} \phi H + \eta d + \frac{1}{2} \xi c$$

where $H, d, c$ generate $\mathcal{H}$.

We follow the construction of Toda field theory shown above, but using complex co-ordinates and a slightly different connection. The Lax pair $(\partial_z + A_z, \partial_{\bar{z}} + A_{\bar{z}})$ is written as

$$A_z = \partial_z \Phi + e^{\Phi} (E_+ + \lambda E_-) e^{-\Phi}$$
$$A_{\bar{z}} = -\partial_{\bar{z}} \Phi + e^{-\Phi} (E_- + \lambda^{-1} E_+) e^{\Phi}.$$ 

The zero curvature condition (10) now gives us, after a little effort, the following set of equations

$$\partial_z \partial_{\bar{z}} \phi = e^{2\phi} - e^{2\eta-2\phi}$$
$$\partial_z \partial_{\bar{z}} \eta = 0$$
$$\partial_z \partial_{\bar{z}} \xi = e^{2\eta-2\phi}.$$ 

This system of three equations is the Conformal Affine Toda theory. The reduction to the sinh-Gordon and Liouville theories are realised by the limits $\eta \to 0$ and $\eta \to -\infty$ respectively.

The set of equations (90)-(92) are conformally invariant, as is the reduction to the Liouville equation.

**IV. WESS-ZUMINO-WITTEN MODELS AND LIOUVILLE THEORY**

O’Raifeartaigh et al [9] have shown that Liouville theory can be regarded as a reduced SL(2,R) Wess-Zumino-Witten theory. Their reduction uses a decomposition into local fields. The energy or action for a group-valued field $g$ is

$$S(g) = -\frac{k}{8\pi} \int d^2 z Tr \left[ (g^{-1} \partial_z g)(g^{-1} \partial_{\bar{z}} g) \right] + \frac{k}{12\pi} \int B^3 Tr \left[ (g^{-1} dg)^3 \right]$$

Any connected semi-simple real Lie group $G$ admits a Gauss decomposition.
where \( Y \) is the direct product \( Y = A \otimes K \) of a simply-connected abelian group \( A \) and a connected semisimple compact group \( K \), and the groups \( X \) and \( Z \) are simply connected and nilpotent; two arbitrary decompositions are connected by an automorphism of \( G \). \[?\]

In the case of \( SL(2, R) \), there exists such a decomposition for regular \( g \) which valid in a neighbourhood of the identity, as follows:

\[
g = ABC
\]  

where

\[
A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \exp(xE_+)
\]

\[
C = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \exp(yE_-)
\]

\[
B = \begin{pmatrix} \exp(\frac{1}{2}\phi) & 0 \\ 0 & \exp(-\frac{1}{2}\phi) \end{pmatrix} = \exp(\frac{1}{2}\phi H)
\]

The Wess-Zumino energy for the product of three matrices \( A, B, C \) can be written as the sum of the energies for the actions for \( A, B \) and \( C \) plus another term:

\[
S(ABC) = S(A) + S(B) + S(C) - \frac{k}{4\Pi} \int d^2\xi \text{ Tr } \left[(A^{-1}\partial_z A)(\partial_z B)B^{-1} + (B^{-1}\partial_z B)(\partial_z C)C^{-1} + (A^{-1}\partial_z A)B(\partial_z C)C^{-1}B^{-1}\right] (30)
\]

Using our parametrization, the energy for the Wess-Zumino model takes the following local form:

\[
\partial_x \partial_y + 2e^{-\phi}\partial_x y \partial_z x = 0
\]

\[
\partial_z(\partial_x e^{-\phi}) = \partial_z(\partial_y e^{-\phi}) = 0
\]

The equations of motion for the Wess-Zumino model can be derived in these co-ordinates as

\[
\partial_x \partial_z \phi + 2e^{-\phi}\partial_x y \partial_z x = 0
\]

\[
\partial_z(\partial_x e^{-\phi}) = \partial_z(\partial_y e^{-\phi}) = 0
\]

Consider special solutions

\[
\partial_x x = \nu e^\phi \quad \partial_x y = \mu e^\phi
\]

where \( \mu, \nu \) are arbitrary constants. Then the system reduces to the Liouville system

\[
\partial_x \partial_z \phi + 2\mu\nu e^\phi = 0
\]
V. SELF-DUAL CHERN SIMONS AND TODA THEORY

In [11], Gerald Dunne defined the self-dual Chern-Simons equations over $SU(N)$ as:

\begin{align}
\partial_z A - \partial_{\bar{z}} A + [A, A] &= \frac{2}{\kappa} [\Psi^\dagger, \Psi] \quad (38) \\
\partial_{\bar{z}} \Psi + [A, \Psi] &= 0 . \quad (39)
\end{align}

He showed that, with for a diagonal $A$ and upper triangular $\Psi$,

\begin{align}
A &= \sum A^\alpha_i H_{\alpha} \quad (40) \\
\Psi &= \sum \psi^\alpha E_{\alpha} \quad (41)
\end{align}

these equations combine to become those of the Toda model:

\begin{equation}
\partial^2 \phi_{\alpha} = -\frac{2}{\kappa} K_{\alpha\beta} \phi_{\beta} . \quad (42)
\end{equation}

where $\ln \phi_{\alpha} \equiv |\psi^\alpha|^2$. He also showed it is possible to make a gauge transformation $u^{-1}$ which combines the self-dual Chern-Simons equations into a single equation:

\begin{equation}
\partial_z \chi = [\chi^\dagger, \chi] \quad (43)
\end{equation}

where

\begin{equation}
\chi = \sqrt{\frac{2}{\kappa}} u \Psi u^{-1} \quad (44)
\end{equation}

Define

\begin{align}
\tilde{A}_z &\equiv A_z - \sqrt{\frac{2}{\kappa}} \Psi \quad (45) \\
\tilde{A}_{\bar{z}} &\equiv A_{\bar{z}} + \sqrt{\frac{2}{\kappa}} \Psi^\dagger \quad (46)
\end{align}

Then the self-dual Chern Simons equations imply that $\tilde{A}$ is flat. Trivialising $\tilde{A}$ as $\tilde{A} = u^{-1} du$, and using the $\chi$ defined above, we find that

\begin{equation}
\partial_z A_z - \partial_{\bar{z}} A_z + [A_z, A_z] - \frac{2}{\kappa} [\Psi^\dagger, \Psi] = g^{-1} \left( \partial_z \chi + \partial_{\bar{z}} \chi^\dagger - 2 [\chi^\dagger, \chi] \right) g . \quad (47)
\end{equation}

This shows that the equations (38) and (39) are equivalent to the single equation (43).

This equation may now be written as the harmonic map equation

\begin{equation}
\partial_z \left( h^{-1} \partial_z h \right) + \partial_{\bar{z}} \left( h^{-1} \partial_{\bar{z}} h \right) = 0 , \quad (48)
\end{equation}

where $h \in SU(N)$ is related to $\chi$ as

\begin{equation}
h^{-1} \partial_z h = 2\chi . \quad (49)
\end{equation}
All SU(2) finite action harmonic maps have the form

\[ h = -h_0 (2p - 1) \]  \hspace{1cm} (50)

where \( p \) is a holomorphic projection valued map

\[ (1 - p) \partial_z p = 0 . \]  \hspace{1cm} (51)

With this condition, we can write a general \( p \) in the defining representation for SU(2) as

\[ p = \frac{M M^\dagger}{M^\dagger M} \]  \hspace{1cm} (52)

where

\[ M = \begin{pmatrix} 1 \\ f(z) \end{pmatrix} \]  \hspace{1cm} (53)

for arbitrary \( f(z) \). It can easily be checked that \( p \) satisfies the correct projectivity, hermiticity and holomorphicity conditions.

Hence, we find that

\[ p = \frac{1}{1 + f\bar{f}} \begin{pmatrix} 1 & \bar{f} \\ f & \bar{f} \end{pmatrix} \]  \hspace{1cm} (54)

Explicitly, the \( \chi \) become

\[ \frac{1}{2} h^{-1} \partial_z h = \frac{1}{2} (2p - 1) \cdot 2 \partial_z p = \partial_z p \]

\[ = \frac{f \partial_z \bar{f}}{(1 + f\bar{f})^2} \begin{pmatrix} -1 & \frac{1}{f} \\ -f & 1 \end{pmatrix} . \]  \hspace{1cm} (55)

The corresponding commutator is

\[ [\chi, \chi^\dagger] = \frac{\partial_z f \partial_z \bar{f}}{(1 + f\bar{f})^2} \begin{pmatrix} 1 - f\bar{f} & 2\bar{f} \\ 2f & -1 + f\bar{f} \end{pmatrix} \]  \hspace{1cm} (57)

Dunne showed that this can be diagonalised by \( u \in SU(2) \)

\[ u = \frac{1}{\sqrt{1 + f\bar{f}}} \begin{pmatrix} 1 & -\bar{f} \\ f & 1 \end{pmatrix} , \]  \hspace{1cm} (58)

so that

\[ u^{-1} [\chi, \chi^\dagger] u = \frac{-\partial_z f \partial_z \bar{f}}{(1 + f\bar{f})^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \]  \hspace{1cm} (59)

Using (44), we see that \( \ln \phi_\alpha \equiv |\psi_\alpha|^2 \) is the quantity

\[ \frac{-\partial_z f \partial_z \bar{f}}{(1 + f\bar{f})^2} \]

where \( \phi \) is the Toda field. It above can also be written as

\[ \partial_z \partial_{\bar{z}} \ln (1 + f\bar{f}) = \partial_z \partial_{\bar{z}} \ln \det(M^\dagger M) \]  \hspace{1cm} (60)

This is the form of the general solution of the classical Liouville equation [12].

7
VI. HARMONIC MAPS AND TODA SYSTEMS

We seek a broader understanding of the relationship of harmonic maps to other integrable systems. In the process we construct a general scheme for reducing certain Harmonic maps into Lie groups to the Toda system and examine the relationships with other models discussed in the previous chapter.

We look mainly at harmonic maps into SL(2,R) or equivalently, SU(1,1), a choice which allows us to compare our constructions to both the work of Dunne [11] and O’Raifertaigh et. al. [9]. Using analysis described by Uhlenbeck in [1] for harmonic maps into unitary groups, we find that the work of Dunne can be used to find solutions for the Liouville system. The second section of this chapter employs the field-theoretic approach of O’Raifertaigh et. al. using a decomposition of the group SU(1,1) to demonstrate an alternative reduction to the Toda model.

We then construct a new framework which puts these reductions in a single context. A requirement that the fields in the Lax pair satisfy certain requirements in their Lie algebraic structure is found to be behind the conditions and \textit{Ansätze} used in [9], [11] and [15], and others.

There exists a relatively straightforward way to associate a harmonic map and the corresponding harmonic map (chiral model) equations with a CAT theory. This method also holds for reducing the non-affine algebra (and the corresponding system of equations) to the simple Toda theory.

The correspondence relies on the following proposition, first proved in [?]:

\textbf{Proposition 1} If \( A = A_z dz + A_{\bar{z}} d\bar{z} \) and \( B = B_z dz + B_{\bar{z}} d\bar{z} \) satisfy

\[
\left[ \partial_{\bar{z}} + A_{\bar{z}} + \lambda B_{\bar{z}}, \partial_z + A_z + \lambda^{-1} B_z \right] = 0 ,
\]

then there exists \( u \) with

\[
\tilde{A} = u^{-1} (Au + du), \quad \tilde{B} = u^{-1} Bu
\]

and

\[
\left[ \partial_{\bar{z}} + (1 - \lambda) \tilde{A}_{\bar{z}}, \partial_z + (1 - \lambda^{-1}) \tilde{A}_z \right] = 0 .
\]

Furthermore, \( 2\tilde{A} = s^{-1}ds \) where \( s \) is harmonic.

\textit{Proof:} The necessary transformations correspond to trivialising (61) at \( \lambda = 1 \), with the gauge transformation \( u \). From [1], since \( C \) is simply connected, we find that \( s \) is a harmonic map where \( \tilde{A} = s^{-1}ds/2 \).

Notice that (20) can be written in the correct format to allow use of Proposition 1, where

\[
A_z = \partial_z \Phi + e^\Phi E_+ e^{-\Phi}
\]

\[
A_{\bar{z}} = -\partial_{\bar{z}} \Phi + e^\Phi E_- e^{-\Phi}
\]

\[
B_z = e^\Phi E_+ e^{-\Phi}
\]

\[
B_{\bar{z}} = e^\Phi E_- e^{-\Phi}.
\]

We write the Lax pair of our harmonic map as

\[
\left( \partial_{\bar{z}} - (1 - \lambda) u^{-1}e^\Phi E_+ e^{-\Phi} u, \partial_z - (1 - \lambda^{-1}) u^{-1}e^\Phi E_- e^{-\Phi} u \right) ,
\]

an equation which holds for all order of \( \lambda \).
VII. A CONDITION FOR THE EQUIVALENCE OF THE HARMONIC MAP AND CONFORMAL AFFINE TODA EQUATIONS

Not all harmonic maps are Toda maps. We now show that Toda maps, and more generally, CAT maps, are a subset of harmonic maps into the appropriate Lie group.

As a brief introduction on the way to a more general result for all semisimple Lie groups, we now describe a condition under which a harmonic map into $SL(2,\mathbb{C})'$ (the subgroup of constant loops in the centrally extended loop group) gives rise to the CAT system of equations.

**Theorem 2** If $s$ is a harmonic map into $SL(2,\mathbb{C})$, then the harmonic map equations for $s$ are gauge equivalent to the Conformal Affine Toda equations (92) for $\phi, \eta, \xi$ if the fields $s^{-1}\partial_z s, s^{-1}\partial_\bar{z} s$ can be simultaneously diagonalised into lower and upper triangular matrices respectively.

**Proof:** Let $\tilde{A}_z = s^{-1}\partial_z s/2, \tilde{A}_{\bar{z}} = s^{-1}\partial_{\bar{z}} s/2$. Thus we can write, for some $\theta \in \hat{sl}_2'$,

$$\tilde{A}_z = e^{-\theta}fE_+ e^\theta, \quad \tilde{A}_{\bar{z}} = e^{-\theta}f'_{-} E_- e^\theta$$  \hspace{1cm} (69)

By proposition 1, the harmonic map equations for $s$ are described by the vanishing of the Lax pair in (63) for all $\lambda \in \mathbb{C}^*$. In terms of the connection given above, this provides the following identities as coefficients for $\lambda^{-1}, \lambda$ respectively:

$$-\partial_z f E_+ - f [\partial_z \theta, E_+] - 2f'f H = 0$$ \hspace{1cm} (70)
$$\partial_z f' E_+ + f' [\partial_z \theta, E_-] - 2f'f H = 0$$ \hspace{1cm} (71)

Writing $\partial_z \theta, \partial_{\bar{z}} \theta$ in terms of its algebraic decomposition

$$\partial_z \theta = \alpha E_+ + \beta H + \gamma E_- + \delta c$$ \hspace{1cm} (72)
$$\partial_{\bar{z}} \theta = \alpha' E_+ + \beta' H + \gamma' E_- + \delta' c$$ \hspace{1cm} (73)

we see that the following relationships are specified:

$$\partial_z f' + \beta f' = 0 \quad \alpha f - f'f = 0$$ \hspace{1cm} (74)
$$\partial_{\bar{z}} f - \beta' f = 0 \quad \gamma' f' - f'f = 0$$ \hspace{1cm} (75)

Using this information, we write

$$\partial_z \theta = f' E_+ - \partial_z (\ln f') H + \gamma E_- + \delta c$$ \hspace{1cm} (76)
$$\partial_{\bar{z}} \theta = f E_- + \partial_{\bar{z}} (\ln f) H + \alpha' E_+ + \delta' c$$ \hspace{1cm} (77)

Apply Proposition 1 in exactly the reverse manner using the inverse of the gauge transformation given by the trivialisation at the identity. In other words, transform to

$$A = u\tilde{A}u^{-1} - du \cdot u^{-1}$$ \hspace{1cm} (78)
$$B = u\tilde{B}u = -u\tilde{A}u^{-1}$$ \hspace{1cm} (79)

where here we have $u = e^\theta$. Define the quantities
\[
\phi_1 = \ln f; \quad \eta_1 = \ln \gamma + \phi_1
\]
\[
\phi_2 = \ln f'; \quad \eta_2 = \ln \alpha' + \phi_2 .
\]

We find the connection becomes
\[
(\partial \bar{z} + A \bar{z} + \lambda B \bar{z}, \partial z + A z + \lambda^{-1} B z)
\]
\[
= \left( \partial \bar{z} - \partial \bar{z} \phi_1 H - e^{\eta - \phi_1} E_+ - \delta c - \lambda e^{\phi_1} E_-; \right. \partial z + \partial \bar{z} \phi_2 H - e^{\eta_2 - \phi_2} E_+ - \delta c - \lambda^{-1} e^{\phi_2} E_+ \left. \right) .
\]

The curvature is given by
\[
\left( -\partial \bar{z} \partial_z (\phi_1 + \phi_2) + 2 e^{\phi_1 + \phi_2} - 2 e^{\eta + \eta_2 - \phi_1 - \phi_2} \right) H
\]
\[
+ \left( -\partial \bar{z} e^{\eta - \phi_1} + \lambda^{-1} \partial_z \phi_1 e^{\phi_2} - \lambda^{-1} \partial_z \phi_2 e^{\phi_2} - \partial_z \phi_2 e^{\eta - \phi_1} \right) E_+
\]
\[
+ \left( \partial_z e^{\eta_2 - \phi_2} + \lambda \partial_z \phi_2 e^{\phi_1} - \lambda^{-1} \partial_z \phi_1 e^{\phi_1} + \partial_z \phi_1 e^{\eta_2 - \phi_2} \right) E_-
\]
\[
+ \left( -\partial \bar{z} \delta' + \partial \bar{z} \delta + e^{\phi_1 + \phi_2} \right) c .
\]

We now introduce the variables \( \xi_1, \xi_2 \) where \( \delta' = \partial \bar{z} \xi_1, \delta_2 = \partial \bar{z} \xi_2 \). We find that, by writing
\[
\eta = \eta_1 + \eta_2; \quad \phi = \phi_1 + \phi_2;
\]
\[
\xi = \xi_1 + \xi_2 + \phi
\]

this system is equivalent to the CAT set:
\[
\partial \bar{z} \partial_z \phi = e^{2\phi} - e^{2\eta - 2\phi}
\]
\[
\partial \bar{z} \partial_z \eta = 0
\]
\[
\partial \bar{z} \partial_z \xi = e^{2\eta - 2\phi} .
\]

Note that a reduction of our result to the unextended \( \text{sl}(2) \) algebra will lead to a subset of the CAT equations, namely equations (90) and (91). Although equation (92) seems somewhat superfluous to requirements, in that it can be safely omitted while preserving the conformal invariance, the central extension plays an important role by including the spectral parameter into the algebra. Also, the conformal invariance of the CAT theory can be described by the invariance of the harmonic map equations for the system described above under the transformation \( f \rightarrow \tilde{f} = g(z) f \).

**VIII. ALGEBRAIC REDUCTION OF HARMONIC MAPS TO TODA SYSTEMS**

In the last section we reduced the harmonic map equations to the Conformal Affine Toda systems for \( \text{SL}(2) \). In this section, we will investigate harmonic maps into \( \text{SU}(1,1) \) and see how we can identify these with solutions of the Toda equations. Later we will generalise this to include all semi-simple Lie groups.

Let us first construct the analog of Uhlenbeck’s uniton for the case of a harmonic map into \( \text{SU}(1,1) \). We will begin by generalising the uniton construction of [1] to the indefinite \( U(q,N-q) \) form. Again, we consider maps into the Grassmannian of \( k \) planes which is a geodesic submanifold.
**Definition:** An \( n,q \)-uniton is a harmonic map \( s : \Omega \to U(q,N - q) \) which has an extended solution

\[
E_\lambda : \mathbb{C}^* \times \Omega \to G = GL(N,C)
\] (93)

with

\[
\begin{align*}
(a) \ & E_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha \text{ for } T_\alpha : \Omega \to gl(N,C) \quad (94) \\
(b) \ & E_1 = I \quad (95) \\
(c) \ & E_{-1} = Q s^{-1} \text{ for } Q \in SU(q,N - q) \text{ constant} \quad (96) \\
(d) \ & (E_\lambda)^* J = J(E_{\lambda^{-1}})^{-1} \quad (97)
\end{align*}
\]

and \( J \) is the automorphism which defines the real form for \( U(q,N - q) \)

**Proposition 3:** \( s : \Omega \to SU(q,N - q) \) is a 1-q-uniton (a holomorphic map) if \( s = Q(2p - 1) \) for constant \( Q \in SU(q,N - q) \), where \( p \) satisfies \( p^* J = Jp, \ p^2 = p, \text{ and } (1 - p) \partial_\bar{z} p = 0 \)

**Proof:** With \( E_\lambda = T_0 + \lambda T_1 \) and \( E_1 = T_0 + T_1 = I \), we have \( E_\lambda = T_0 + \lambda(I - T_0) \).

The reality condition (d) above tells us that

\[
\begin{align*}
(I - T_0^*) J T_0 &= 0 \quad (98) \\
T_0^* J T_0 + (I - T_0^*) J (I - T_0) &= J \quad (99)
\end{align*}
\]

Combining these equations gives \( T_0^* J = J T_0 \) and \( T_0^2 = T_0 \). We can identify \( T_0 \) with \( p \).

As in the previous chapter the necessary condition for \( E_\lambda \) to be an extended solution for a harmonic map as is that \( (1 - p) \partial_\bar{z} p = 0 \)

With this condition, we can write a general \( p \) in the defining representation for \( SU(1,1) \) as

\[
p = M(M^* J M)^{-1} M^* J
\] (100)

where, as before,

\[
M = \begin{pmatrix} 1 \\ f(x_-) \end{pmatrix}
\] (101)

for arbitrary \( f(x_-) \).

Hence, we find that, in this case

\[
p = \frac{1}{1 - f \bar{f}} \begin{pmatrix} 1 & -\bar{f} \\ f & -f \bar{f} \end{pmatrix}
\] (102)

The simplest solutions of the harmonic map equation are now given by the holomorphic maps:

\[
\frac{1}{2} g^{-1} \partial_\bar{z} g = \frac{1}{2}(2p - 1) \cdot 2 \partial_\bar{z} p = \partial_\bar{z} p
\] (103)

\[
= -\frac{f \partial_\bar{z} \bar{f}}{(1 - f \bar{f})^2} \begin{pmatrix} -1 & \frac{1}{f} \\ 1 & -f \end{pmatrix}
\] (104)

We can now find a \( u \in SU(1,1) \)
\[ u = \frac{1}{\sqrt{1-ff}} \begin{pmatrix} 1 & \bar{f} \\ f & 1 \end{pmatrix} \] (105)

such that

\[ u^{-1}[A_+, A_-]u = \frac{\partial_z \bar{f} \partial_{\bar{z}} f}{(1-f\bar{f})} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (106)

This diagonalisation is a critical step, and as was shown in the previous chapter for SU(2), the magnitude of this quantity should be a solution of the appropriate Toda equation. It can be rewritten

\[ u = \frac{1}{\sqrt{1-ff}} \begin{pmatrix} 1 & \bar{f} \\ f & 1 \end{pmatrix} \] (107)

and we find that the magnitude of this expression for the diagonalised commutator above is just

\[ \ln \phi = \partial_z \partial_{\bar{z}} \ln (1-ff) = \partial_z \partial_{\bar{z}} \ln \det(M^*JM) . \] (108)

IX. HARMONIC MAPS AND LIOUVILLE THEORY

Let us now examine the reduction of the SU(1,1) harmonic map to the Toda system using a decomposition into local fields.

Recall that the energy or action for \( g : \Omega \to SU(1,1) \) is

\[ S(g) = -\frac{k}{8\pi} \int d^2 z Tr [(g^{-1} \partial_z g)(g^{-1} \partial_{\bar{z}} g)] \] (109)

As in the previous chapter, where we saw a similar reduction to analyse Wess-Zumino-Witten theory, we use a Gauss decomposition of \( SU(1,1) \). For regular \( g \) in a neighbourhood of the identity, we write:

\[ g = ABC \] (110)

where

\[ A = \begin{pmatrix} 1 & ix \\ 0 & 1 \end{pmatrix} = \exp(ixE_+) \] (111)

\[ C = \begin{pmatrix} 1 & 0 \\ iy & 1 \end{pmatrix} = \exp(iyE_-) \] (112)

\[ B = \begin{pmatrix} \exp\left(\frac{1}{2}\phi\right) & 0 \\ 0 & \exp\left(-\frac{1}{2}\phi\right) \end{pmatrix} = \exp\left(\frac{1}{2}\phi H\right) \] (113)

With this parametrization, the energy for the harmonic map can be written in terms of three real fields as

\[ S(g) = S(x, y, \phi) = -\frac{k}{8\pi} \int d^2 z \left[ \frac{1}{2} \partial_z \phi \partial_{\bar{z}} \phi - e^{-\phi}(\partial_z x \partial_{\bar{z}} y + \partial_y \partial_{\bar{z}} x) \right] \] (114)

A local form of the equations of motion for the harmonic map can now be derived from this:
\[ \partial_z \partial_{\bar{z}} \phi - e^{-\phi}(\partial_z x \partial_{\bar{z}} y + \partial_{\bar{z}} x \partial_z y) = 0 \]  
\[ \partial_z (\partial_{\bar{z}} x e^{-\phi}) + \partial_{\bar{z}} (\partial_z x e^{-\phi}) = 0 \]  
\[ \partial_z (\partial_{\bar{z}} y e^{-\phi}) + \partial_{\bar{z}} (\partial_z y e^{-\phi}) = 0 . \]  

(115)  
\[ \partial_z x = f(z) e^\phi, \quad \partial_{\bar{z}} y = g(z) e^\phi . \]  

(118)

We consider the solution for the above:

\[ \partial_z x = f(z) e^\phi, \quad \partial_{\bar{z}} y = g(z) e^\phi . \]  

Then the system reduces to the following form of the Liouville system:

\[ \partial_z \partial_{\bar{z}} \phi + M(z, \bar{z}) e^\phi = 0 \]  
\[ f(z) \bar{g}(\bar{z}) - \bar{f}(\bar{z}) g(z) - M = 0 . \]  

(119)  
(120)

X. A USEFUL RESULT

We have seen the reduction of the harmonic maps into Lie groups to the Toda model carried out in a number of different ways, and there are others, such as [15], which we have not alluded to. We now ask ourselves if there is any underlying link in the manner in which these reductions are carried out.

We find that these reductions can be observed to be different strategies for taking advantage of the following general result:

**Result 4** Any pencil of connections \((A_z + \lambda A'_z, A_{\bar{z}} + \lambda^{-1} A'_{\bar{z}})\) gives rise to the Toda system of equations for the corresponding semi-simple Lie group if the \(A_z, A_{\bar{z}}\) are upper or lower triangular respectively and the \(A'_z, A'_{\bar{z}}\) are off-diagonal.

**Proof:**

We will begin with the most general case and reduce using our stated condition as needed. We see that most of the work is accomplished by the algebraic structure and the parameter \(\lambda\).

We begin with the stated condition that

\[ [\partial_z + A_z + \lambda A'_z, \partial_{\bar{z}} + A_{\bar{z}} + \lambda^{-1} A'_{\bar{z}}] = 0 \]  

(121)

where \(A\) and \(A'\) take values in the Chevalley basis of the Lie algebra, i.e.

\[ A_z = \sum_{\alpha \in \Phi^+} g_\alpha E_\alpha + \sum_{\alpha \in \Phi^+} f_\alpha E_{-\alpha} + \sum_{\alpha \in \Phi^+} h_\alpha H_\alpha \]  
\[ A_{\bar{z}} = \sum_{\alpha \in \Phi^+} g_\alpha E_\alpha + \sum_{\alpha \in \Phi^+} f_\alpha E_{-\alpha} + \sum_{\alpha \in \Phi^+} h_\alpha H_\alpha \]  

(122)

and

\[ A'_z = \sum_{\alpha \in \Phi^+} g'_\alpha E_\alpha + \sum_{\alpha \in \Phi^+} f'_\alpha E_{-\alpha} + \sum_{\alpha \in \Phi^+} h'_\alpha H_\alpha \]  
\[ A'_{\bar{z}} = \sum_{\alpha \in \Phi^+} g'_\alpha E_\alpha + \sum_{\alpha \in \Phi^+} f'_\alpha E_{-\alpha} + \sum_{\alpha \in \Phi^+} h'_\alpha H_\alpha \]  

(123)

Here \(\Phi^+\) denotes the set of positive roots, and \(H_\alpha, E_{\pm \alpha}\) are the Cartan subalgebra and step generators of the Chevalley basis respectively.
In terms of the first power of \( \lambda \) and the basis of the Lie algebra, the set of equations reduces to:

\[
-\partial_z g_+^\alpha - \sum_\beta K_{\beta \alpha} h_-^\beta g_+^\alpha + \sum_\beta K_{\beta \alpha} h_+^\beta g_-^\alpha = 0 \tag{124}
\]

\[
-\partial_z f_+^\alpha + \sum_\beta K_{\beta \alpha} h_-^\beta f_+^\alpha - \sum_\beta K_{\beta \alpha} h_+^\beta f_-^\alpha = 0 \tag{125}
\]

\[
\partial_- h_+^\alpha + g_+^\alpha f_+^\alpha - f_-^\alpha g_-^\alpha = 0 \tag{126}
\]

and their dual, where \( K \) is the classical Cartan matrix for the Lie algebra. In the case where \( A' \) is off-diagonal, all the \( h' \) vanish, so we find:

\[
\partial_z h_-^\alpha - \partial_z h_+^\alpha + f_-^\alpha g_+^\alpha - f_+^\alpha g_-^\alpha = 0 \tag{127}
\]

The part of (121) independent of \( \lambda \) contributes the following (and its dual):

\[
\partial_z g_-^\alpha - \partial_z g_+^\alpha - \sum_\beta K_{\beta \alpha} h_-^\beta g_+^\alpha + \sum_\beta K_{\beta \alpha} h_+^\beta g_-^\alpha - \sum_\beta K_{\beta \alpha} h_-^\beta g_+^\alpha + \sum_\beta K_{\beta \alpha} h_+^\beta g_-^\alpha = 0 \tag{128}
\]

as well as

\[
\partial_z h_-^\alpha - \partial_z h_+^\alpha + f_-^\alpha g_+^\alpha - f_+^\alpha g_-^\alpha = 0 \tag{129}
\]

When \( A \) is triangular, \( g_+^\alpha = 0 \) and when \( A' \) is off-diagonal, \( h' = 0 \) and we find \( g_-^\alpha = 0 \). Equation (126) and its dual now give us:

\[
\partial_z \left( \ln g_+^\alpha \right) = -\sum_\beta K_{\beta \alpha} h_+^\beta \tag{130}
\]

\[
\partial_z \left( \ln f_+^\alpha \right) = \sum_\beta K_{\beta \alpha} h_-^\beta \tag{131}
\]

Combining these equations we find

\[
\partial_z \partial_z \left[ \ln(g_+^\alpha f_+^\alpha) - \ln(g_-^\alpha f_-^\alpha) \right] = -2 \sum_\beta K_{\beta \alpha} \left( g_+^\alpha f_-^\alpha - g_-^\alpha f_+^\alpha \right) . \tag{132}
\]

With the further result that

\[
\partial_z \ln \left( g_+^\alpha f_+^\alpha \right) = \partial_z \ln \left( f_-^\alpha g_-^\alpha \right) = 0 , \tag{133}
\]

we recognise (132) as the affine toda equations:

\[
\partial_z \partial_z \phi_\alpha + \sum_\beta K_{\beta \alpha} \left( \eta_+ e^{\phi_\alpha} - \eta_- e^{-\phi_\alpha} \right) . \tag{134}
\]

Here

\[
\phi_\alpha = \ln \left( \frac{f_+^\alpha}{f_+^\alpha} \right) \tag{135}
\]

and \( \eta_+^\alpha, \eta_-^\alpha \) are arbitrary holomorphic and anti-holomorphic functions respectively.
XI. SOME CONCLUSIONS

All of the reductions to Toda systems we have seen use the above Result 4, although this is far from clear from a first examination. An understanding of why this follows from the fact that for a large class of harmonic maps, the paramatrised Lax pair can be gauge transformed into our required form. We need the following result.

**Result 5:** If the commutator \([A_z, A_\bar{z}]\) is diagonal, then either \(A_z, A_\bar{z}\) are off-diagonal or the commutator is zero.

**Proof:** Using the expansion of \(A\) as in (122), we find that since the off-diagonal elements of the commutator are given for all \(\alpha, \beta\) by

\[
\begin{align*}
  h^\alpha_+ g^-_-^\alpha - h^\beta_- g^-_+^\alpha &= 0 \\
  h^\beta_+ f^-_-^\alpha - h^\alpha_- f^-_+^\alpha &= 0
\end{align*}
\]

(136)  
(137)

We find that the only case when all \(h^\beta\) do not vanish is the trivial case.

Now when we gauge transform by a group element \(u\) which diagonalises the hermitian quantity \([A_z, A_\bar{z}]\) with \(u^{-1}du\) triangular, our connection is in the appropriate form to apply the Result 4. In the work on self-dual Chern-Simons theory by Dunne [11], it can be seen that for the SU(N) uniton solutions, the necessary gauge transformation is in the required form. These uniton solutions are therefore Toda systems. Moreover, Guest [15], selected *a priori* a connection equivalent to our form for his reduction of the chiral model to the Toda lattice.

O’Raifeartaigh et al [9] carry out essentially this same diagonalisation process but in a different setting. To see this, use the decomposition in (110) and write the commutator of the connection of Section V of this chapter as

\[
C^{-1}B^{-1} \left[ \tilde{\hat{A}}_+, \tilde{\hat{A}}_- \right] BC
\]

(138)

where

\[
\tilde{\hat{A}} = A^{-1}dA + (dB)B^{-1} + B(dC)C^{-1}B^{-1}
\]

(139)

Now we see that the quantity \(BC\) corresponds to the \(u\) in (107) which diagonalises \([\tilde{\hat{A}}_+, \tilde{\hat{A}}_-]\). In light of the conditions (118), the diagonal component

\[
(\partial_z x \partial_{\bar{z}} y - \partial_{\bar{z}} x \partial_z y) e^{-\phi}
\]

(140)

is just the Liouville field \(e^\phi\) up to a holomorphic function. This corresponds to the SU(1,1) Toda solution result in (108) and confirms the solution found by Fujii [10].
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