Motivated by our recent work, presented in the gravitation and cosmology conference [1], in this paper, we present two new equations of hydrostatic equilibrium related to two kinds of gravity models. First, we consider a spherical symmetric metric to obtain the hydrostatic equilibrium equation of stars in 4-dimensional Einstein-Λ gravity, and then extend this equation to the arbitrary dimensions. Second, considering Gauss-Bonnet (GB) gravity as a generalization of the Einstein gravity, we extract the hydrostatic equilibrium equation in 5-dimensions and show that in special case, this equation turns into the hydrostatic equilibrium equation (HEE) of Einstein-Λ gravity in 5-dimension. Then we generalize GB results to the higher dimensions.

I. INTRODUCTION

Einstein gravity can explain the solar system phenomena successfully, but when we intend to study beyond the solar system or when the gravity is so strong, this theory encounters with some problems. On the other hand, the observations show that the expansion of our universe is currently undergoing a period of acceleration [2]. In order to interpret this expansion, some various candidates have been proposed by many authors. One of these interesting theories is cosmological constant [3].

In agreement to string theory, the gravity theories predict the existence of more than four dimensions such as string theory. In addition, the string theory predicts the existence of higher order space-time curvature terms in the gravitational Lagrangian in combinations known as the Lovelock polynomials [4]. Lovelock gravity is a natural generalization of Einstein gravity, so it gives equations of motion with no more than the second derivatives of the metric.

On the other hand, stars reach the equilibrium state due to the balance between gravitational force and internal pressure. So, in order to study a star, we should obtain the hydrostatic equilibrium equation (HEE) and calculate other physical properties of the stars. It is notable that the neutron and quark stars are in the category of high mass, high density and small radius stars, and therefore they are called compact stars. Due to this fact, we need to take into account effects of general relativity such as the curvature of spacetime in the studying the compact stars. The first HEE for stars in the Einstein gravity and for 4-dimensions of spacetime have been studied by Tolman, Oppenheimer and Volkov (TOV) [5]. Also, the physical characteristics of stars using TOV equation have been investigated by many authors [6]. If one is interested to study the structure and evolution of stars in different gravities, the HEE must be obtained. Therefore, in recent years, the generalizations and modifications of this equation were of special interests for many authors (for more details see [7]). On the other hand, recently, Alberto A. Garcia-Diaz obtained the (2 + 1)-dimensional HEE for a static star in the presence of cosmological constant [8].

It is notable that, the generalization and extension of TOV equation has not been studied in Einstein-Λ gravity to higher dimensions of spacetime, therefore, one of our aim is to construct the HEE of Einstein-Λ gravity in 4-dimensions, and then extend it to the higher dimensions ($d > 4$).

In this paper, we intend to restrict ourself to the first three terms of Lovelock gravity. The first two terms are the Einstein-Hilbert term with cosmological constant and the third term is known as the Gauss-Bonnet (GB) term [9]. Another our aim in this paper is to construct the HEE in GB gravity in 5-dimensions and extend it to $d$-dimensions ($d \geq 5$).

The outline of our paper is as follows. In next section, we consider a spherical symmetric metric and obtain the HEE in Einstein-Λ gravity for arbitrary dimensions. In section III, we consider the GB gravity and extract a global equation of hydrostatic equilibrium for compact stars in the higher dimensions. Finally, we finish our paper with some closing remarks.

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II. EQUATION OF HYDROSTATIC EQUILIBRIUM IN EINSTEIN GRAVITY WITH COSMOLOGICAL CONSTANT

Here we present the Einstein gravity with cosmological constant. The action of this gravity is given by

$$I_G = -\frac{1}{16\pi} \int_M d^4x\sqrt{-g}\{R - 2\Lambda\} + I_{M\text{att}},$$  \hspace{1cm} (1)$$

where $R$ is the Ricci scalar, $\Lambda$ is the cosmological constant and $I_{M\text{att}}$ is the action of matter field. Varying the action (1) with respect to the metric tensor $g_{\mu\nu}$, the equation of motion for this gravity can be written as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu},$$  \hspace{1cm} (2)$$

where $G_{\mu\nu}$, $T_{\mu\nu}$ are the Einstein and energy-momentum tensors, respectively.

Here, we want to obtain the static solutions of Eq. (2). For this purpose, we assume a spherical symmetric spacetime in the following form,

$$ds^2 = f(r)dt^2 - \frac{dr^2}{g(r)} - r^2d\Omega_k^2,$$  \hspace{1cm} (3)$$

where $f(r)$ and $g(r)$ are the unknown metric functions of this metric and

$$d\Omega_k^2 = d\theta_1^2 + \prod_{i=2}^{d-2} \sum_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2.$$  \hspace{1cm} (4)$$

On the other hand, the energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = (\rho + P) U_\mu U_\nu + Pg_{\mu\nu},$$  \hspace{1cm} (5)$$

where $\rho$ and $P$ are density and pressure of the fluid which are measured by local observer, respectively, and $U_\mu$ is the fluid four-velocity. Using Eqs. (2) and (5) and the metric introduced in Eq. (3), we can obtain the components of energy-momentum for $(3 + 1)$-dimensions as follows

$$T^0_0 = \rho \quad \& \quad T^1_1 = T^2_2 = T^3_3 = -P.$$  \hspace{1cm} (6)$$

Now, we consider the metric (3) and the Eq. (6) for perfect fluid and obtain the components of Eq. (2) in the following form,

$$rg' + g + \Lambda r^2 + r^2 \rho - 1 = 0,$$  \hspace{1cm} (7)$$

$$rgf' - f(g + \Lambda r^2 + r^2 P - 1) = 0,$$  \hspace{1cm} (8)$$

$$f [g'(rf' + 2f) + 4rf (\Lambda - P) + 2g (rf'' + f')] - gf' = 0,$$  \hspace{1cm} (9)$$

where $f$, $g$, $\rho$ and $P$ are functions of $r$. we note that the prime and double prime denote the first and second derivatives with respect to $r$, respectively.

Using Eqs. (7-9) and after some calculation, we have

$$\frac{dP}{dr} + \frac{f'(\rho + P)}{2f} = 0.$$  \hspace{1cm} (10)$$

Now, we obtain $f'$ from the Eq. (8) as follows,

$$f' = \frac{f(1 - g - \Lambda r^2 + r^2 P)}{rg}.$$  \hspace{1cm} (11)$$

Then, to obtain $g(r)$ using Eq. (7), we have

$$g(r) = 1 - \Lambda \frac{r^2}{3} - \frac{M}{r},$$  \hspace{1cm} (12)$$
where $M$ is a function of $r$ ($M(r)$). We insert $M$ with $\int 4\pi r^2 \rho(r)dr$ in the above equation which represents the inner mass of the star. By considering Eqs. (11) and (12), and inserting them in Eq. (10), we can extract the HEE in Einstein-Λ gravity for $(3 + 1)$-dimension as the following form

$$\frac{dP}{dr} = \left(\frac{\pi M - r^3 (\Lambda - \frac{\pi}{2} P)}{r (\Lambda r^3 + 3 \left(\frac{M}{4\pi} - r\right))}\right) (\rho + P),$$ (13)

We see that this equation is different from TOV equation, and it leads to TOV equation when $\Lambda$ is zero [5].

Now, we are going to obtain the HEE in Einstein-Λ gravity for higher dimensions ($d \geq 5$ where $d$ represents the dimensionally of spacetime). For this purpose we must obtain Eq. (6) for arbitrarily dimensions, therefore we have

$$T^0_0 = \rho \quad \& \quad T^1_1 = T^2_2 = T^3_3 = \ldots = T^{d-1}_{d-1} = -P.$$ (14)

We use the above equation to obtain a global relation for the HEE in higher dimensions for Einstein-Λ gravity. Using Eqs. (2) and (14), the components of the equation of motion Einstein-Λ gravity can be written as

$$\frac{(d-2)}{2} r g' + \frac{(d-2)(d-3)}{2} (g-1) + \Lambda r^2 + r^2 \rho = 0,$$ (15)

$$\frac{(d-2)}{2} r g f' + \frac{(d-2)(d-3)}{2} (g-1) + r^2 f(\Lambda - P) = 0,$$ (16)

$$2(d-3) r g' f^2 + 2(d-3) r g f f' + r^2 g' f f' + 2r^2 g f f'' - r^2 g f'^2$$

$$+ 2(d-4)(d-3) g f^2 + 4\Lambda r^2 f^2 - 4r^2 f^2 P - 2(d-4)(d-3) f^2 = 0.$$ (17)

We obtain Eq. (10) in the same manner as described for $(3 + 1)$-dimension, then using Eq. (16), we get $f'$ as follows

$$f' = \frac{2 f \left(\frac{(d-2)(d-3)}{2} (1 - g) - r^2 (\Lambda - P)\right)}{rg(d-2)}.$$ (18)

We calculate $g(r)$ of Eq. (15) as follows

$$g(r) = 1 - \frac{2\Lambda}{(d-1)(d-2)} r^2 - \frac{M \Gamma(d-1)}{2\pi \frac{\pi}{2} r^{(d-3)}}.$$ (19)

It should be noted that we have used $M = \int 2\pi^{(d-1)/2} \rho(r)dr$ in the above equation and also $\Gamma$ is the gamma function, which satisfies some conditions as $\Gamma(1/2) = \sqrt{\pi}$; $\Gamma(1) = 1$; $\Gamma(x + 1) = x\Gamma(x)$.

Using Eqs. (18) and (19) and inserting them in Eq. (10), we can extract the HEE in the Einstein-Λ gravity for $d$-dimensions as the following form

$$\frac{dP}{dr} = \left(\frac{(d-1)(d-3)\Gamma(d-1)}{4\pi \frac{\pi}{2} r^{(d-1)}} M - r^{d-1} \left(\Lambda - d\frac{1}{2} P\right)\right) (\rho + P),$$ (20)

where for 4-dimensional limit, Eq. (20) is reduced to Eq. (13) as expected. It is seen that when we insert $d = 3$, the Eq. (20) reduces to the HEE obtained by Albert A. Garcia-Diaz [8], therefore we could obtain a global form of HEE for arbitrary dimensions ($d \geq 3$).

We are going to continue our paper with considering GB gravity as a generalization of Einstein gravity and extract HEE.
III. EQUATION OF HYDROSTATIC EQUILIBRIUM IN GAUSS-BONNET (GB) GRAVITY

Here, we consider the GB gravity in the presence of energy-momentum tensor. The action of this gravity is

$$I_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \left( R - 2\Lambda + \alpha L_{GB} \right) + I_{Matt}, \quad (21)$$

where $\alpha$ is the Gauss-Bonnet coefficient and $L_{GB}$ is the Lagrangian of GB gravity,

$$L_{GB} = R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4R_{\mu\nu} R^{\mu\nu} + R^2. \quad (22)$$

In above equation, $R_{\mu\nu}$ and $R_{\mu\nu\gamma\delta}$ are the Ricci and Riemann tensors of the manifold $\mathcal{M}$, respectively. Varying the action $(21)$ with respect to the metric tensor $g^\mu_\nu$, the equation of motion is obtained as the following form

$$G^\nu_\mu + \Lambda g^\nu_\mu + \alpha H^\nu_\mu = T^\nu_\mu, \quad (23)$$

where $H^\nu_\mu$ is the divergence-free symmetric tensor,

$$H^\nu_\mu = 4R^{\rho\sigma} R_{\rho\sigma} - 2R_{\mu}^{\rho\sigma\lambda} R_{\nu\rho\sigma\lambda} - 2RR_{\mu\nu}^{\nu} + 4R_{\mu\lambda} R^{\lambda\nu} + \frac{1}{2} g_{\mu\nu} L_{GB}. \quad (24)$$

We note that GB term is a topological term in 4-dimension and does not contribute.

One of our purposes in this section is to extract the equation of hydrostatic equilibrium for GB gravity in $(4 + 1)$-dimensions. Using Eqs. $(23), (14)$ and metric $(3)$, the components of the equation of motion in GB gravity can be written as

$$g' \left( 3r^2 + 12\alpha \left( g - 1 \right) \right) + 6r \left( g - 1 \right) + 2r^3 \left( \Lambda + \rho \right) = 0, \quad (25)$$

$$gf' \left( 3r^2 + 12\alpha g \left( g - 1 \right) \right) + 6rf \left( g - 1 \right) + 2r^3 f \left( \Lambda - P \right) = 0, \quad (26)$$

$$gf f'' \left( 2r^2 + 8\alpha \left( g - 1 \right) \right) + 4f^2 \left( g - 1 \right) + 4r^2 f^2 \left( \Lambda - P \right)$$

$$+ f' \left( 4rgf + gf' \left( r^2 + 4\alpha \left( 3g - 1 \right) \right) - gf' \left( r^2 + 4\alpha \left( g - 1 \right) \right) \right) = 0. \quad (27)$$

We combine the above equations and achieve a new equation similar to Eq. $(10)$. Using Eq. $(26)$, we have

$$f' = \frac{2rf \left( 3 - 3g - \Lambda r^2 + r^2 P \right)}{3g \left( r^2 + 4\alpha \left( g - 1 \right) \right)}. \quad (28)$$

Also, we can obtain the metric function $g(r)$ in Eq. $(25)$ as follows,

$$g(r) = \frac{12\alpha - 3r^2 + \Upsilon}{12\alpha}, \quad (29)$$

where $\Upsilon = \sqrt{9r^4 - 12\alpha (\Lambda r^4 + 2M/r^2)}$. Insert $g(r)$ in Eq. $(28)$ and using Eq. $(10)$, we get the HEE for GB gravity in $(4 + 1)$-dimensions $[1]$

$$\frac{dP}{dr} = \frac{3r \left( 3r^2 - \Upsilon - 4\alpha r^2 (\Lambda - P) \right)}{\left( 3r^2 - 12\alpha - \Upsilon \right) \Upsilon} \left( \rho + P \right). \quad (30)$$

One can use the expansion of Eq. $(30)$ for small values of $\alpha \ (\alpha \rightarrow 0)$ to find that it reduces to the Eq. $(20)$ for 5-dimension.

Now, we want to obtain the HEE for GB gravity in $d$-dimensions. We consider the metric $(3)$ in $d$-dimensions and use Eqs. $(23)$ and $(14)$ for this purpose. The nonzero component of Eq. $(23)$ for $d$-dimensions can be written as $[1]$

$$(d - 2)r^3 g' + (d - 2)(d - 3)r \left[ r + 2(d - 4)\alpha g' \right] \left( g - 1 \right)$$

$$+ \alpha (d - 2)(d - 3)(d - 4)(d - 5)(g - 1)^2 + 2r^4 (\Lambda + \rho) = 0, \quad (31)$$

where
\[(d - 2)r^3 g f' + (d - 2)(d - 3)r [rf + 2(d - 4)\alpha g f'] (g - 1) + \alpha(d - 2)(d - 3)(d - 4)(d - 5)f(g - 1)^2 + 2r^4f(\Lambda + \rho) = 0,\]  
\[(32)\]

\[r^{d-3} \left[ 1 + \frac{2\alpha(d - 3)(d - 4)(g - 1)}{r^2} \right] g \left( 2f f'' - f^2 \right) - 4r^{d-3} \left[ (P - \Lambda) + \frac{(d - 3)(d - 4)}{2r^2} \right] f^2 + 2r^{d-4} \left[ (d - 3)(1 + 2g') + \frac{4\alpha(d - 4)(d - 5)}{r^2} \right] g' f^2 + 2r^{d-3} \left[ 2\alpha(d - 3)(d - 4) \left( \frac{(d - 5)(g - 1)}{r^3} f + \frac{3g - 1}{r^2} f' \right) + f' \right] g' f + 2(d - 1)r^{d-5} \left[ (d - 4) + \frac{6\alpha(d - 6)^2(g - 1)^2}{r^2} \right] g f^2 = 0.\]  
\[(33)\]

We combine the above equations and obtain a new equation which is similar to Eq. (10). We use Eqs. (31) and (32) to obtain two functions \(f'\) and \(g(r)\),

\[f' = f \left[ (d - 3)(g - 1) + \frac{2\alpha(d - 3)(d - 4)(g - 1)^2}{rg(r^2 + 2\alpha(d - 3)(d - 4)(g - 1))} \right],\]  
\[(34)\]

\[g(r) = \frac{2\alpha(d - 2)(d - 3)(d - 4)r^{d-6} - (d - 2)r^{d-1} + \Xi}{2\alpha(d - 2)(d - 3)(d - 4)r^{d-2}},\]  
\[(35)\]

where \(\Xi\) is

\[\Xi = \sqrt{(d - 2)r^{d-1} \left[ (d - 2) - 8\alpha(d - 3)(d - 4) \left( \frac{\Lambda}{d - 1} + \frac{M\Gamma \left( \frac{d - 1}{2} \right)}{2\pi \Gamma \left( \frac{d - 1}{2} \right)} \right) \right]}\]  
\[(36)\]

Now, we can obtain the HEE for GB gravity in \(d\)-dimensions by using Eqs. (10), (34) and (35). Therefore we can write

\[\frac{dP}{dr} = \frac{(2\alpha(d - 3)(d - 4)K - (d - 2))r^{d-1} + \Xi r^{d-1}}{(2\alpha(d - 3)(d - 4) - r^2)r^{d-1} + \frac{\Xi}{d-2}}\]  
\[(37)\]

where

\[K = \frac{4\Lambda}{d - 1} - \frac{(d - 5)M\Gamma \left( \frac{d - 1}{2} \right)}{2\pi \Gamma \left( \frac{d - 1}{2} \right)r^{d-1}} - P.\]  
\[(38)\]

The expansion of Eq. (37) for small values of \(\alpha\) leads to Eq. (20).

**IV. CLOSING REMARKS**

In the first part of this paper, we considered a spherical symmetric metric and extracted the hydrostatic equilibrium equation of stars in 4-dimensional Einstein-\(\Lambda\) gravity and extracted this equation for \(d\)-dimensions \((d \geq 3)\). Then, we considered the Gauss-Bonnet gravity as a generalization of Einstein gravity and extracted the HEE in 5-dimensions. Then, we generalized the HEE to the higher dimensions of spacetime and obtained a general relationship for this equation. We found that for \(\alpha \to 0\) limit, the HEE for GB gravity reduces to Einstein one as one expected. Considering the HEE obtained in this paper, we can investigate the structure and evolution of compact stars such as the neutron stars and quark stars in arbitrary dimensions.
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