SINGULAR ARMA SYSTEMS: A STRUCTURE THEORY

MANFRED DEISTLER
Institute for Statistics and Mathematical Methods in Economics
Vienna University of Technology
Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria

Abstract. Singular vector ARMA systems are vector ARMA (VARMA) systems with singular innovation variance or equivalently with singular spectral density of the corresponding VARMA process. Such systems occur in linear dynamic factor models, e.g. if the dimension of the static factors is strictly larger than the dimension of the dynamic factors or in linear dynamic stochastic equilibrium models, if the number of outputs is strictly larger than the number of shocks. We describe the relation of factor models and singular ARMA systems and a realization procedure for singular ARMA systems. Finally we discuss kernel systems.

1. Introduction. Singular ARMA systems are multivariate ARMA systems with singular innovation variance: Let the ARMA system be of the form

\[ a(z)y_t = b(z)\varepsilon_t \]  

where \( z \) is a complex variable as well as the backward shift on the integers \( \mathbb{Z} \),

\[ a(z) = \sum_{j=0}^{p} a_j z^j \quad a_j \in \mathbb{R}^{n \times n} \]

\[ b(z) = \sum_{j=0}^{p} b_j z^j \quad b_j \in \mathbb{R}^{n \times n} \]

and where \( (\varepsilon_t) \) is white noise, i.e. \( \mathbb{E}\varepsilon_t = 0 \) and \( \mathbb{E}\varepsilon_t\varepsilon_t' = \delta_{st} \Sigma \). We will assume throughout stability, i.e.

\[ \det a(z) \neq 0 \quad |z| \leq 1 \]  

and we impose the miniphase assumption, i.e.

\[ \det b(z) \neq 0 \quad |z| \leq 1. \]  

For singular ARMA systems \( \Sigma \) is singular of rank \( q < n \), say. Write \( \Sigma \) as

\[ \Sigma = bb', \quad b \in \mathbb{R}^{n \times q} \]

where \( b \) is unique up to orthogonal postmultiplication. Then (1) can be written as

\[ a(z)y_t = b(z)b\nu_t = b(z)\nu_t \]

where \((\nu_t)\) is white noise with \( \mathbb{E}\nu_t\nu_t' = I_q \).

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Under the stability condition (2), there exists a steady state solution of the form
\[ y_t = a^{-1}(z)b(z)e_t = a^{-1}(z)b(z)e_t \] (5)
Then the singular ARMA process \((y_t)\) is the only linearly regular stationary solution of (1). The solution of (1) may have a linearly singular component, which however we do not consider here.

The miniphase condition for \(\bar{b}(z)\) is
\[ \bar{b}(z) \text{ has full rank } q, |z| \leq 1. \] (6)
Let
\[ k(z) = a^{-1}(z)\bar{b}(z) = \sum_{j=0}^{\infty} k_j z^j, k_j \in \mathbb{R}^{n \times q}, \] (7)
denote the transfer function from \((\nu_t)\) to \((y_t)\).

The spectral density (written as a function of \(z \in \mathbb{C}\)) of the singular ARMA process \((y_t)\) then is of the form
\[ f(z) = (2\pi)^{-1} k(z)k^*(z) \] (8)
where \(k^*(z) = k(z^{-1})'\).

Clearly \(f(z)\) has normal rank equal to \(q\) (i.e. it has rank \(q\), except for a finite number of points). Let \(\tilde{f}(\lambda) = f(e^{-i\lambda}), \lambda \in [-\pi, \pi]\) and \(\gamma(j) = E y_{t+j} y_t'\). Then, as is well known,
\[ \tilde{f}(\lambda) = (2\pi)^{-1} \sum_{j=-\infty}^{\infty} e^{-i\lambda j}\gamma(j), \] (9)
and
\[ \gamma(j) = \int_{-\pi}^{\pi} e^{i\lambda j} f(\lambda) d\lambda. \] (10)
As is also well known, every rational \(n \times q \ (q \leq n)\) matrix \(k(z)\) of rank \(q\) has a Smith-McMillan form (see e.g. [7] Hannan and Deistler (2012))
\[ k(z) = u(z)d(z)v(z), \] (11)
where \(u(z)\) and \(v(z)\) are \(n \times n\) and \(q \times q\) respectively unimodular polynomial matrices (i.e. \(\det u(z) = \text{constant} \neq 0\), \(\det v(z) = \text{constant} \neq 0\) and
\[ d(z) = \begin{pmatrix} \varepsilon_1(z) \\ \psi_1(z) \\ \vdots \\ \varepsilon_q(z) \\ \psi_q(z) \end{pmatrix}, \] (12)
where \(\varepsilon_i, \psi_i\) for \(i = 1, ..., q\) are relatively prime monic polynomials, \(\varepsilon_i\) divides \(\varepsilon_{i+1}\) and \(\psi_{i+1}\) divides \(\psi_i\), \(i = 1, ..., q - 1\); \(d(z)\) is unique for given \(k(z)\), whereas \(u(z)\) and \(v(z)\) are not unique in general. The zeros of \(\varepsilon_i\) are the zeros of \(k\) and the zeros of \(\psi_i\) are the poles of \(k\).

Now assume that \(k(z)\) satisfies the stability assumption, i.e. that \(k(z)\) has no poles for \(|z| \leq 1\) and the miniphase assumption that \(k(z)\) has full rank \(q\) for \(|z| \leq 1\).
1. In addition we throughout assume that \((a(z), b(z))\) is left coprime, which is equivalent to
\[
\text{rk}(a(z), b(z)) = n \quad \forall z \in \mathbb{C},
\]
where \(\text{rk}\) denotes the rank, see e.g. [7] Hannan and Deistler (2012). Under this assumption \(k(z)\) is stable if and only if (2) holds and miniphase if and only if (6) holds.

From the spectral factorization theorem (see e.g. [10] Rozanov (1967), [3] Deistler Scherrer (2018)) we know that for a given \(f(z)\), a stable and miniphase transferfunction is unique up to right postmultiplication by orthogonal matrices. The transfer function is unique for given \(f(z)\) if we e.g. postulate that the submatrix of \(k_0\) consisting of the first \(q\) rows is nonsingular, lower triangular and has ones on its main diagonal.

Due to the stability and the miniphase assumption, (5) is already the Wold decomposition, i.e. \((\varepsilon_t)\) and \((\nu_t)\) are innovations for \((y_t)\) and, in particular, \((\nu_t)\) is unique for given \((y_t)\) up to orthogonal premultiplication \(O\nu_t\). As \(n > q\) holds, the left inverse of \(k(z)\) is not unique. A special causal left inverse of \(k(z)\) is given by (compare (11))
\[
k^{-}(z) = v^{-1}(z)(d'(z)d(z))^{-1}d'(z)u^{-1}(z).
\]
As is easily seen, \(k^{-}(z)\) has no poles and zeros for \(|z| \leq 1\) and we have
\[
\nu_t = k^{-}(z)y_t.
\]

The remaining parts of the paper are organized as follows: In section 2 we deal with the relation of singular ARMA systems and factor models, in order to take into account, for instance, observational noise. Section 3 presents a state space realization algorithm for singular ARMA systems, section 4 deals with the corresponding ARMA realizations. The singularity of the spectral density of singular ARMA processes implies an exact (i.e. noiseless) linear dynamic relation between the components of the process. This is discussed in section 5.

2. Factor Models and Singular ARMA Systems. As is easy to see, due to e.g. observational noise, singular ARMA processes, i.e. ARMA processes with a singular spectral density, are unlikely to occur “in practice”. For this reason a noise model has to be added. This leads to linear dynamic factor models or linear dynamic errors in variables models (see e.g. [11] Scherrer and Deistler (1998) or [5] Forni, Hallin, Lippi and Reichlin (2000)) where the observations have to be denoised, in order to obtain a latent process with a singular spectral density. Singular ARMA processes also occur in DSGE (dynamic stochastic general equilibrium) models, see e.g. [9] Komunjer and Ng (2011), when the number of shocks is smaller than \(n\).

A convenient presentation of the latent process then is of the form
\[
y_t = Lf_t, \quad L \in \mathbb{R}^{n \times r}
\]
where \(f_t\) are the so called minimal static factors, which are obtained as follows: Write
\[
\gamma(0) = Ey_ty'_t = RR', \quad R \in \mathbb{R}^{n \times r}.
\]
If we assume that in addition $\gamma(0)$ is of rank $r < n$, then a minimal static factor is given by

$$f_t = (R'R)^{-1}R'y_t.$$  \hspace{1cm} (18)

As is easy to see, a minimal static factor $f_t$ is unique up to premultiplication by a constant, nonsingular matrix $T$ and the factor loading matrix $L$ is unique up to postmultiplication by $T^{-1}$.

The advantage of extracting a static factor process ($f_t$) for ($y_t$) is that in modeling the dynamics of ($y_t$), the dimension of the parameter space can be reduced. As can be shown, even for $q < n$, we may have $r = n$, see [6] Forni et al. (2015). According to this reference, the latent process in a linear dynamic factor model is only required to have a Wold decomposition (5) with $q < n$, and we allow for either $r = n$ or $q = r$. The $\nu_t$’s are (minimal) dynamic factors.

Note that, because (16) and (18) are static linear transformations, the innovation spaces for ($y_t$) and ($f_t$) are the same. Clearly ($f_t$) is a process with rational spectral density and therefore has an ARMA representation

$$c(z)f_t = d(z)\nu_t = d(z)\varepsilon_t$$ \hspace{1cm} (19)

where $c(z), d(z)$ are $r \times r$ polynomial matrices satisfying (2) and (3) respectively, $d(z) = d(z)b$ and ($c(z), d(z)$) is assumed to be relatively left prime. The corresponding transfer function

$$w(z) = c^{-1}(z)d(z)b$$ \hspace{1cm} (20)

then is stable and miniphase.

A representation of $k(z), n > q$ of the form

$$k(z) = L \left[ \begin{array}{c} w(z) \\ \vdots \end{array} \right]_{n \times q}$$ \hspace{1cm} (21)

$r \leq n, q \leq r$ will be called an **FM-type representation**.

### 3. State Space Realizations for FM-type Transfer-Function Representations

Here we are concerned with construction of $L$ and $w(z)$ from $k(z)$ and with the corresponding realizations of $w(z)$. For the realization problem we refer to [8] Ho and Kalman (1966) and [2] Akaike (1974).

We commence from the (block) Hankel matrix

$$H_k = \begin{pmatrix} k_0 & k_1 & k_2 & \ldots \\ k_1 & k_2 & k_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$ \hspace{1cm} (22)

of the transfer function $k(z)$. Then, from the Wold representation (5) we have:

$$\begin{pmatrix} y_t \\ \hat{y}_{t+1|t} \\ \hat{y}_{t+2|t} \\ \vdots \\ \hat{y}_{t+s|t} \end{pmatrix} = \begin{pmatrix} k_0 & k_1 & k_2 & \ldots \\ k_1 & k_2 & k_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \nu_t \\ \nu_{t-1} \\ \nu_{t-2} \\ \vdots \end{pmatrix}$$ \hspace{1cm} (23)

where $\hat{y}_{t+s|t}$ is the (best linear) least squares predictor of $y_{t+s}$ from the infinite past $y_{t-s}, s \geq 0$. 
We now e.g. select the first basis rows of $H_k$ which span the rowspace of $H_k$. Let $S \in \mathbb{R}^{s \times \infty}$ be the corresponding selector matrix, where $s$ denotes the rank of $H_k$. Then a minimal state space system
\begin{align}
x_{t+1} &= Ax_t + Bu_t \tag{24} \\
y_t &= Cx_t \tag{25}
\end{align}
is defined by (see e.g. [2] Akaike (1974))
\begin{align}
S \begin{pmatrix} k_1, & k_2, & k_3, & \ldots \\
k_2, & k_3, & k_4, & \ldots \\
& \cdots
\end{pmatrix} &= ASH_k \tag{26} \\
B &= S \begin{pmatrix} k_0 \\
k_1 \\
& \vdots
\end{pmatrix} \tag{27}
\end{align}
and
\begin{align}
(k_0, k_1, \ldots) &= CSH_k. \tag{28}
\end{align}
A minimal state is given by
\begin{align}
x_t &= SY_t^- \tag{29}
\end{align}
Equations (26), (27) and (28) give unique $(A, B, C)$, since the rows of $SH_k$ are linearly independent by definition. This $(A, B, C)$ is called the echelon canonical form (see e.g. [7] Hannan and Deistler (2012)). Alternative choices of $S$ lead to alternative canonical forms. We have:
\begin{align}
k(z) &= C(z^{-1}I - A)^{-1}B = \sum_{j=0}^{\infty} CA^j Bz^j. \tag{30}
\end{align}

We now decompose $k(z)$ as follows: In a first step we have:

**Theorem 3.1.** Let $(y_t)$ be a singular ARMA process with Wold representation (5). Then the following statements are equivalent:
(i) $r < n$, i.e. $r \gamma(0) = r < n$
(ii) $(k_0, k_1, \ldots)$ has rank $r$
(iii) The left kernel of $\overline{f}(\lambda)$ contains exactly $(n-r) > 0$ linearly independent constant (i.e. not dependent on $\lambda$) vectors.

**Proof.** From
\begin{align}
\gamma(0) = \int_{-\pi}^{\pi} \overline{f}(\lambda)d\lambda = (2\pi)^{-1} \int_{-\pi}^{\pi} k(e^{-i\lambda})k(e^{-i\lambda})^*d\lambda = 2\pi \sum_{j=0}^{\infty} k_jk_j' \tag{31}
\end{align}
(i) $\Leftrightarrow$ (ii) follows immediately.
If the left kernel of $\overline{f}(\lambda)$, for all $\lambda \in [-\pi, \pi]$, contains $(n-r)$ linearly independent constant vectors, then by (31) these vectors are also contained in the left kernel of $\gamma(0)$.
Thus $r \gamma(0) \leq r$. Now assume $r \gamma(0) < r$, then there exists a vector $g \in \mathbb{R}^n$ say with $g\gamma(0)g' = 0$, where $g$ is not an element of the left kernel of $\overline{f}(\lambda)$, $\forall \lambda \in [-\pi, \pi]$ and thus,
\begin{align}
g \left( \int_{-\pi}^{\pi} \overline{f}(\lambda)d\lambda \right)g' > 0
\end{align}
which gives a contradiction. 
\hfill $\square$
Now, let \( S_1 \in \mathbb{R}^{r \times n} \) denote the matrix selecting the first basis rows from \((k_0, k_1, ... )\) spanning the row space of \((k_0, k_1, ... )\). Then we may define a matrix \( L \) by

\[
LS_1(k_0, k_1, ...) = (k_0, k_1, ...)
\]

and

\[
f_t = S_1 y_t = S_1(k_0, k_1, ...) \nu_t
\]

is a minimal static factor. For simplicity of presentation, we will assume that \( f_t \) consists of the first \( r \) elements of \( y_t \), and then \( S_1 = (I_r \mid 0) \). This can always be achieved by reordering the entries in \( y_t \). We have

\[
L = \begin{pmatrix}
I_r \\
L_2
\end{pmatrix}
\]

for a suitably chosen \( L_2 \).

Then we have

\[
f_t = S_1 k(z) \nu_t = w(z) \nu_t
\]

where

\[
w(z) = \sum_{j=0}^{\infty} S_1 k_j z^j = \sum_{j=0}^{\infty} w_j z^j
\]

is the transfer function from \((\nu_t)\) to \((f_t)\).

Note that from

\[
S_1(k_j, k_{j+1}, \ldots) = (w_j, w_{j+1}, \ldots)
\]

we see that the first basis rows of

\[
\tilde{H}_k = \begin{pmatrix}
k_1, & k_2, & k_3, & \ldots \\
k_2, & k_3, & k_4, & \ldots \\
& \cdots & & \cdots
\end{pmatrix}
\]

and of

\[
\tilde{H}_w = \begin{pmatrix}
w_1, & w_2, & w_3, & \ldots \\
w_2, & w_3, & w_4, & \ldots \\
& \cdots & & \cdots
\end{pmatrix}
\]

are the same. Let \( S_2 \in \mathbb{R}^{(s-r) \times \infty} \) denote the selection matrix selecting the first basis rows for the row space of \( \tilde{H}_k \). Then clearly

\[
S = \begin{pmatrix}
S_1, & 0, & \ldots \\
0, & S_2
\end{pmatrix}.
\]

A minimal state space system \((A_w, B_w, C_w, w_0)\), for \((f_t)\), where

\[
\begin{align*}
\dot{x}_{t+1} &= A_w \dot{x}_t + B_w \nu_t \\
f_t &= C_w \dot{x}_t + w_0 \nu_t
\end{align*}
\]
then is given by:

\[ S_2 \begin{pmatrix} k_2, k_3, k_4, \ldots \\ k_3, k_4, k_5, \ldots \\ \vdots \end{pmatrix} = A_w S_2 \tilde{H}_k \quad (38) \]

\[ B_w = S_2 \begin{pmatrix} k_1 \\ k_2 \\ \vdots \end{pmatrix} \quad (39) \]

and

\[ S_1(k_1, k_3, \ldots) = C_w S_2 \tilde{H}_k \quad (40) \]

and the state is given by:

\[ \tilde{x}_t = (0_{1 \times n}, S_2) Y_t^- \quad (41) \]

Let us summarize:

**Theorem 3.2.** Consider an n-dimensional singular ARMA process \( (y_t) \) with spectral density \( f \) of normal rank \( q < n \). Then:

1. Under (2) and (6) the \( n \times q \) transfer function \( k(z) \) in (7) is unique up to orthogonal postmultiplication.
2. For given \( k(z) \) an FM-type representation \( k(z) = Lw(z), L \in \mathbb{R}^{n \times r} \) is obtained from (32) and (35). A minimal static factor is given by (33) and the dynamics of the static factor is described by (34) and (35).
3. A minimal state space representation for the static factor is given by (38), (39) and (40).
4. The minimal static factors are unique up to postmultiplication by a constant nonsingular matrix.

4. **VARMA and VAR Representations.** As described e.g. in [7] Hannan and Deistler (2012), Chapter 2, an echelon form VARMA realization \( (a_w, b_w) \) for \( w(z) \) satisfying (2), (6) and (13) can be obtained commencing from the Hankel matrix \( \tilde{H}_w \).

Let

\[ w(z) = u_w(z) d_w(z) \psi_w \quad (42) \]

denote the Smith-McMillan form of \( w(z) \) and let \( L^\perp \in \mathbb{R}^{n \times (n-r)} \) be the completion of \( L \) to a nonsingular matrix \( (L, L^\perp) \in \mathbb{R}^{n \times n} \). Then

\[ k(z) = Lw(z) = (L, L^\perp) \begin{pmatrix} u_w & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} d_w \\ 0 \end{pmatrix} \psi_w. \quad (43) \]

As

\[ (L, L^\perp) \begin{pmatrix} u_w & 0 \\ 0 & I \end{pmatrix} \]

is unimodular (43) is a Smith-McMillan form of \( k(z) \). This implies that \( k(z) \) and \( w(z) \) have the same pole-zero structure in terms of \( \varepsilon_1/\psi_1, \ldots, \varepsilon_q/\psi_q \).

The tall matrix \( b_w(z) \), for prescribed order \( p \), is generically of full rank \( q \) for all \( z \in \mathbb{C} \). The intuition behind this is, that, generically, there are no zeros common to
all $q \times q$ minors of $\bar{b}(z)$. For an exact proof see [1] Anderson and Deistler (2008).

Now, as easily seen from the Smith form (i.e. the Smith McMillan form for the special case of a polynomial matrix) of $\bar{b}(z)$, if $\bar{b}(z)$ is zeroless, there exists a matrix $\bar{b}_c(z)$ such that $(\bar{b}(z), \bar{b}_c(z))$ is unimodular. Therefore:

$$a_w(z)f_t = (\bar{b}(z), \bar{b}_c(z)) \begin{pmatrix} \nu_t \\ 0 \end{pmatrix}$$

leads to the singular VAR model

$$(\bar{b}(z), \bar{b}_c(z))^{-1}a_w(z)f_t = \begin{pmatrix} \nu_t \\ 0 \end{pmatrix}$$

since $(\bar{b}(z), \bar{b}_c(z))^{-1}$ is polynomial.

5. The Kernel System. Let $g$ denote a rational $(n - q) \times n$ matrix of rank $(n - q)$, whose rows form a basis for the left kernel of $k(z)$. Thus, in particular,

$$g(z)k(z) = 0 \quad (44)$$

and $g(z)$ is unique up to premultiplication by nonsingular rational $(n - q) \times (n - q)$ matrices. Therefore, by extracting the least common denominator polynomial of the elements of $g(z)$, the basis can always be chosen as polynomial, and by extracting all non-unimodular common left factors of $g(z)$, as left coprime.

Clearly (44) implies

$$g(z)k(z)\nu_t = g(z)y_t = 0 \quad (45)$$

which gives an exact linear dynamic relation ("comovement") between the component process in $(y_t)$. Adding noise to (44) leads to an errors-in-variables representation or to a factor model.

As is easy to see, the left kernels of $k(z)$ and of the spectral density $f(z)$ are the same, thus, by Theorem 3.1, the first $(n - r)$ rows of $g(z)$ may be chosen to be constant and we may write

$$g(z) = \begin{pmatrix} G \\ \tilde{g}(z) \end{pmatrix} \quad (46)$$

where $G \in \mathbb{R}^{(n-r)\times n}$ is a basis for the left kernel of the factor loading matrix $L$ and $\tilde{g}(z)$ is polynomial and relatively left coprime.

Now, write

$$G = (G_1 \mid G_2)$$

and

$$\tilde{g}(z) = (\tilde{g}_1(z) \mid \tilde{g}_2(z))$$

where $G_1$ is $(n-r) \times (n-q)$ and $\tilde{g}_1(z)$ is $(r-q) \times (n-q)$, then we can write in an obvious notation:

$$g(z) = (g_1(z) \mid g_2(z)) = \begin{pmatrix} G_1 \\ \tilde{g}_1(z) \\ \tilde{g}_2(z) \end{pmatrix} \quad (47)$$

where $g_1(z)$ is $(n-q) \times (n-q)$. We assume that $g_1(z)$ is nonsingular, which can always be assumed by reordering the components in $(y_t)$ as

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} \begin{pmatrix} (n-q) \\ q \end{pmatrix}$$
Then (45) gives

\[ y_{1,t} = -g_1(z)^{-1} g_2(z)y_{2,t}. \]  

(48)

Using the Smith form of \( g_1(z) \) we may extract those zeros in \( g_1(z) \) where \( |z| \leq 1 \), such that (48) is a causal linear dynamic transformation.

Thus we have

**Theorem 5.1.** Let \((y_t)\) be a \(n\)-dimensional ARMA process with spectral density of rank \(q < n\), then

1. there exist \((n-q)\) exact linear, in general, dynamic relations (48) between the component processes of \((y_t)\).
2. There exist exactly \((n-r)\) linear static transformations between the component processes of \((y_t)\).

6. **Conclusion.** Singular VARMA processes occur e.g. in linear dynamic factor models or in DSGE models (the latter are used in macroeconomics) after ”denoising” of the observations (i.e. after estimating the latent variables). We describe the structure of singular ARMA systems and their realization by state space and ARMA systems. Finally we describe the exact linear dynamic relations between the components of singular ARMA processes.

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**REFERENCES**

[1] B. D. O. Anderson and M. Deistler, Properties of zero-free transfer function matrices, *SICE Journal of Control, Measurement and System Integration*, 1 (2008), 284–292.

[2] H. Akaike, Stochastic theory of minimal realization, *IEEE-TAC*, 19 (1974), 667–674.

[3] M. Deistler and W. Scherrer, *Modelle der Zeitreihenanalyse*, Birkhäuser-Springer, Cham, 2018.

[4] M. Deistler, B. D. O. Anderson, A. Filler, C. Zinner and W. Chen, Generalized dynamic factor models - An approach via singular autoregressions, *European Journal of Control*, 16 (2010), 1–14.

[5] M. Forni, M. Hallin, M. Lippi and L. Reichlin, The generalized dynamic factor model: identification and estimation, *The Review of Economics and Statistics*, 82 (2000), 540–554.

[6] M. Forni, M. Hallin, M. Lippi and P. Zaffaroni, Dynamic factor models with infinite dimensional factor spaces: one-sided representations, *Journal of Econometrics*, 185 (2015), 359–371.

[7] E. J. Hannan, M. Deistler, *The Statistical Theory of Linear Systems*, SIAM Classics in Applied Mathematics, Philadelphia, 2012.

[8] B. L. Ho and R. E. Kalman, Effective construction of linear state-variable models from input/output functions, *Regelungstechnik*, 14 (1966), 545–592.

[9] I. Komunjer and S. Ng, Dynamic identification of dynamic stochastic general equilibrium models, *Econometrica*, 79 (2011), 1995–2032.

[10] Y. A. Rozanov, *Stationary Random Processes*, Holden Day, San Francisco, 1967.

[11] W. Scherrer and M. Deistler, A structure theory for linear dynamic errors-in-variables models, *SIAM J. Cont. Opt.*, 36 (1998), 2148–2175.

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E-mail address: manfred.deistler@tuwien.ac.at