Cosmic censorship and stationary states of half-spin particles in the field of Reissner-Nordström naked singularity

M V Gorbatenko¹, V P Neznamov¹,²*, E Yu Popov¹ and I I Safronov¹

¹Russian Federal Nuclear Center – All-Russian Research Institute of Experimental Physics, Sarov, Mira 37, Nizhni Novgorod region, Russia, 607188
²National Research Nuclear University MEPhI, Moscow, Russia
E-mail: neznamov@vniief.ru

Abstract. The paper explores quantum mechanics of half-spin particle motion in the field of Reissner-Nordström (RN) naked singularity. It is shown that for any quantum mechanical Dirac particle, irrespective of availability and sign of its electrical charge, the RN naked singularity is separated by an infinitely high positive potential barrier. With like charges of a particle and the source of the RN naked singularity, near the origin there exists the second completely impenetrable potential barrier. It has been proved that in the field of the RN naked singularity, bound states of half-spin particles can exist. The conditions for appearance of such states were revealed and computations were performed to find energy eigenvalues and eigenfunctions.

1. The Reissner-Nordström metric

The line element is

\[ ds^2 = f_{RN} dt^2 - \frac{dr^2}{f_{RN}} - r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \]

where \( f_{RN} = \left( 1 - \frac{r_0}{r} + \frac{r_Q^2}{r^2} \right) \), \( r_0 = \frac{2GM}{c^2} \) is the gravitational radius of the Schwarzschild field, and \( r_Q = \sqrt{\frac{GQ}{c^2}} \) is the "charge" radius, \( G \) is the gravitational constant, \( c \) is the velocity of light.

The case \( r_0 < 2r_Q \) corresponds to the naked singularity. In this case, there is always \( f_{RN} > 0 \) and the domain of wave functions is the entire region \( r \in [0, \infty) \).

Below, we will analyze the behavior of effective potentials of the Dirac equation in the field of RN naked singularity.

For obtaining the Schrödinger-type equation with the self-conjugate Hamiltonian the initial Dirac Hamiltonian must be also self-conjugate.

Earlier, a self-conjugate Hamiltonian of a half-spin particle of mass \( m \) and charge \( e \) for the Reissner-Nordström metric was derived in [1]

\[ H_N = \sqrt{f_{RN}} \beta m - i \alpha^1 \left( f_{RN} \frac{\partial}{\partial r} + \frac{1}{r} - \frac{r_0}{2r} \right) \]

\[ -i \sqrt{f_{RN}} \alpha^1 \left[ \alpha^2 \left( \frac{\partial}{\partial \varphi} + \frac{1}{2} \cot \theta \right) + \alpha^3 \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right] + \frac{eQ}{r}. \]

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

Published under licence by IOP Publishing Ltd
In (2), $\alpha^k, \beta$ are Dirac matrices.

After separation of variables (see, for instance, [1]), the system of equations for radial functions $F_{R-N}(\rho), G_{R-N}(\rho)$ takes the form of

\[
\begin{align*}
\frac{dF_{R-N}(\rho)}{d\rho} &+ \left( \frac{1+\sqrt{1-\rho^2}}{\rho} - \frac{\alpha}{\rho^2} \right) F_{R-N}(\rho) - \left( \varepsilon - \frac{\alpha_{em}}{\rho} + \sqrt{R_{R-N}} \right) G_{R-N}(\rho) = 0, \\
\frac{dG_{R-N}(\rho)}{d\rho} &+ \left( \frac{1-\sqrt{1-\rho^2}}{\rho} - \frac{\alpha}{\rho^2} \right) G_{R-N}(\rho) + \left( \varepsilon - \frac{\alpha_{em}}{\rho} - \sqrt{R_{R-N}} \right) F_{R-N}(\rho) = 0.
\end{align*}
\] (3)

In (3), dimensionless variables have been introduced $\rho = \frac{r}{c}; \varepsilon = \frac{E}{m}; \alpha = \frac{r_a}{2l_c} = \frac{G M m}{\hbar c} = \frac{M m}{M_p^2}, \alpha_Q = \frac{r_Q}{l_c} = \frac{\sqrt{G m}}{\hbar c} = \frac{\sqrt{1}}{M_p} m Q |c|; \alpha_{em} = \frac{e Q}{\hbar c} = \alpha_{f s} Q, l_c = \frac{\hbar}{mc}$ is the Compton wave-length of a Dirac particle; $E$ is the energy of a Dirac particle; $M_P = \frac{\hbar c}{G}$ is the Planck mass; $\alpha_{f s}$ is the electromagnetic fine structure constant; $\alpha, \alpha_{em}$ are gravitational and electromagnetic coupling constants; $\alpha_Q$ is the dimensionless constant characterizing the source of the electromagnetic field in the Reissner-Nordström metric; $\kappa = \pm \left( j + \frac{1}{2} \right); M_P = 2.2 \cdot 10^{-5} g \left( 1.2 \cdot 10^{19} GeV \right) ; \quad \alpha_{f s} \approx \frac{1}{137}$.  

2. Effective potentials for the field of Reissner-Nordström naked singularity

From the system of equations (3), we will derive the second-order equation for the function $\psi(\rho)$ proportional to $F(\rho)$.  

\[
\frac{d^2 \psi(\rho)}{d\rho^2} + 2 \left( E_{grav} - U_{eff}(\rho) \right) \psi(\rho) = 0.
\] (4)

2.1 Irrespective of signs of charges $Q, e$, the leading term of the effective potential near the origin is

\[
U_{eff} = \frac{3}{8\rho^2} + O \left( \frac{1}{\rho} \right) \text{at } \rho \to 0.
\] (5)

2.2 With like signs of charges $Q, e$ near the origin there is the second potential barrier of the form

\[
U_{eff}|_{\rho \to \rho_{cl}} \approx \frac{1}{4} \left( \frac{1}{\varepsilon - \frac{\alpha_{em}}{\rho} + \sqrt{1 - \frac{2\alpha}{\rho} + \frac{\alpha_Q^2}{\rho^4}}} \right)^2 \left( \frac{\alpha_{em}}{\rho^2} + \frac{\alpha}{\rho^2} - \frac{\alpha_Q^2}{\rho^4} \right)^2 \left( 1 - \frac{2\alpha}{\rho} + \frac{\alpha_Q^2}{\rho^2} \right)^2.
\] (6)

For all the cases considered by the authors, the coefficient at the leading singularity in (6)

\[
A = \frac{1}{4} \left( \frac{\alpha_{em}}{\rho_{cl}^2} + \frac{\alpha}{\rho_{cl}^2} - \frac{\alpha_Q^2}{\rho_{cl}^4} \right)^2 \left( 1 - \frac{2\alpha}{\rho_{cl}} + \frac{\alpha_Q^2}{\rho_{cl}^2} \right)^2 > \frac{3}{8}.
\] (7)

It is well-known that such potential barriers are impenetrable to quantum-mechanical particles [3].

Figure 1 presents the typical form of $U_{eff}(\rho)$ for the case of like signs of charges $Q, e$ and $\varepsilon > 1$.

One can see two regions $\rho < \rho_{cl}$ and $\rho > \rho_{cl}$ separated by a potential barrier impenetrable to particles. In the outer region, there are no stationary states of Dirac particles. In the inner region $0 < \rho < \rho_{cl}$, the existence of stationary bound states of half-spin particles is possible.
2.3 For opposite signs of charges $Q, e$ at certain values of initial parameters the typical form of the potential $U_{\text{eff}}(\rho)$ is given in figure 2. The form of the potential does not change for the case of an uncharged Dirac particle either. Such $U_{\text{eff}}(\rho)$ relations testify to a possibility of existence of stationary bound states of half-spin particles in both cases.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure1.png}
\caption{Behavior of effective potentials of the Dirac equation in the field of the RN naked singularity at $\alpha = 0.25$, $\alpha_Q = 0.5$, $\alpha_{em} = 1$, $\kappa = -1$, $\varepsilon = 1.5$, $\rho_0 = 0.3675$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure2.png}
\caption{Behavior of effective potentials of the Dirac equation in the field of the RN naked singularity at $\alpha = 0.25$, $\alpha_Q = 0.5$, $\alpha_{em} = -1$, $\kappa = -1$, $\varepsilon = 0.6$.}
\end{figure}

3. Determination of stationary bound states of half-spin particles in the field of Reissner-Nordström naked singularity

To solve the problem, we will use the system of equations (3).

Let us determine that
\[
\tan \Phi (\rho) = \frac{F(\rho)}{G(\rho)}, \quad A^2(\rho) = F^2(\rho) + G^2(\rho).
\] (8)

Then,
\[
F(\rho) = A(\rho) \sin \Phi(\rho), \quad G(\rho) = A(\rho) \cos \Phi(\rho).
\] (9)

From the equations (3), one can derive an equation for the phase $\Phi(\rho)$ in Vronsky’s form [4] and an equation for $A(\rho)$
\[
\frac{d\Phi}{d\rho} = \frac{1}{f_{RN}} \left( \varepsilon - \frac{\alpha_{em}}{\rho} \right) + \frac{\cos(2\Phi)}{\sqrt{f_{RN}}} - \frac{\kappa}{\rho \sqrt{f_{RN}}} \sin(2\Phi),
\] (10)

\[
\frac{d(\ln A)}{d\rho} = -\frac{1}{\rho f_{RN}} + \frac{\alpha}{\rho^2 f_{RN}} + \frac{\kappa}{\rho \sqrt{f_{RN}}} \cos(2\Phi) + \frac{1}{\sqrt{f_{RN}}} \sin(2\Phi).
\] (11)

3.1 Boundary condition

For $\rho \to \infty$, the leading terms of the asymptotics are
\[
F = Ce^{-\rho \sqrt{1-\varepsilon^2}}, \quad G = \frac{\sqrt{1-\varepsilon}}{1+\varepsilon} F, \quad \tan \Phi = -\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}.
\] (12)

At $\rho \to 0$, the asymptotic behavior of wave functions was identified in [5]. In our symbols
\[ F(\rho) = f_0 \rho^\frac{3}{2} + f_1 \rho^\frac{5}{2} + f_2 \rho^\frac{7}{2} + ... \]
\[ G(\rho) = g_0 \frac{1}{\rho^\frac{1}{2}} + g_1 \rho^\frac{1}{2} + g_2 \rho^\frac{3}{2} + ... \]  
(13)

It follows from (13) that
\[ \tan(\Phi) = \left. \frac{F(\rho)}{G(\rho)} \right|_{\rho \to 0} = \frac{|\alpha_Q| - \alpha_{em}}{2\alpha_Q^2} \rho^2 = 0, \quad \Phi(\rho = 0) = k\pi, \quad k = 0, 1, 2... \]  
(14)

### 3.2 Determination of the discrete energy spectrum

For three versions of \( \alpha, \alpha_Q, \alpha_{em} \) with boundary conditions (12) - (14) discrete spectra of states of Dirac particles in the field of RN naked singularity were determined due to the solution to the equation (10).

### 4. Discussions

The consideration of the quantum mechanics of the half-spin particle motion in the field of Reissner-Nordström (RN) naked singularity resulted in drawing the following conclusions:

- for any quantum-mechanical half-spin particle, irrespective of availability and sign of the electrical charge, the RN naked singularity is separated by the infinitely high positive potential barrier \( \sim \frac{3}{8}r_\text{cl}^{-2} \). This agrees with the conclusions [2] concerning the motion of spinless particles in the field of certain singular metrics. As it is noted in [2] the availability of a repulsive barrier shielding the singularity poses no threat to the cosmic censorship.

- with like charges of a Dirac particle and the source of RN naked singularity near the origin there exists the second completely impenetrable positive potential barrier \( \sim \frac{A}{(r-r_\text{cl})^2} \). For elementary charges of a particle and the source of RN naked singularity at \( \alpha_{em} \gg \alpha, \alpha_{em} \gg \alpha_Q \) and at particle energies \( E \sim mc^2 \), the value \( r_\text{cl} \) is close to half a classical radius of a charged particle \( r_\text{e} = \frac{e^2}{me^2} \). At the value of the particle energy \( E \gg mc^2 \), the radius \( r_\text{cl} \) decreases inversely to \( E \left( r_\text{cl} = \frac{r_\text{e}}{E} \right) \).

- the analysis of the effective potentials and the direct numerical solutions to the Dirac equation have shown that the stationary bound states of half-spin particles in the field of RN naked singularity can exist in case of opposite charges of particles and the RN field source. In case of like charges, the bound states of a Dirac particle can exist under the potential barrier in the region \( 0 \leq r \leq r_\text{cl} \). In the region \( r > r_\text{cl} \), there is no bound state of half-spin particles at the explored values of initial parameters.

- the stationary bound states of uncharged Dirac particles implemented due to the forces of gravitational interaction alone were obtained by direct computations of the Dirac equation in the field of RN naked singularity.

Let us pay attention to the difference in the dynamics of classical and quantum half-spin particles. It is well-known that in the classical case there appears the geodesic incompleteness of the RN naked singularity. The geodesics cannot pass across the surface whose radius is equal to half a classical radius of the source of RN naked singularity: \( r_\text{cl} = \frac{Q^2}{2Me^2} \). However, in quantum mechanics, depending on sings of charges, a Dirac particle can either be reflected from the repulsive barrier (6) or be in one of stationary bound states.

### References

[1] Gorbatenko M V, Neznamov V P, Preprint gr-qc/1302.2557
[2] Horowitz G T and Marolf D 1995 Phys. Rev. D10 5670
[3] Dittrich J, Exner P 1985 J. Math. Phys 26 (8) 2000-08
[4] Vronsky M A, Gorbatenko M V, Kolesnikov N S, Neznamov V P, Popov E Yu, Safronov I I, Preprint gr-qc/1301.7595.
[5] Pekeris C L and Frankowski K 1986 Proc. Natl. Acad. Sci. USA 83 1978-82