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Generalized logarithmic Hardy-Littlewood-Sobolev inequality

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This paper is devoted to logarithmic Hardy-Littlewood-Sobolev inequalities in the two-dimensional Euclidean space, in presence of an external potential with logarithmic growth. The coupling with the potential introduces a new parameter, with two regimes. The attractive regime reflects the standard logarithmic Hardy-Littlewood-Sobolev inequality. The second regime corresponds to a reverse inequality, with the opposite sign in the convolution term, which allows us to bound the free energy of a drift-diffusion-Poisson system from below. Our method is based on an extension of an entropy method proposed by E. Carlen, J. Carrillo and M. Loss, and on a nonlinear diffusion equation.

1 Main result and motivation

On $\mathbb{R}^2$, let us define the density of probability $\mu = e^{-V}$ and the external potential $V$ by

$$\mu(x) := \frac{1}{\pi (1 + |x|^2)^2} \quad \text{and} \quad V(x) := - \log \mu(x) = 2 \log (1 + |x|^2) + \log \pi \quad \forall x \in \mathbb{R}^2.$$ 

We shall denote by $L^1_+(\mathbb{R}^2)$ the set of a.e. nonnegative functions in $L^1(\mathbb{R}^2)$. Our main result is the following generalized logarithmic Hardy-Littlewood-Sobolev inequality.

\textbf{Theorem 1.1.} For any $\alpha \geq 0$, we have that

$$\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f \, dx + M (1 - \alpha) \left( 1 + \log \pi \right) \geq \frac{2}{M} (\alpha - 1) \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x - y| \, dx \, dy \quad (1)$$

for any function $f \in L^1_+(\mathbb{R}^2)$ with $M = \int_{\mathbb{R}^2} f \, dx > 0$. Moreover, the equality case is achieved by $f_* = M \mu$ and $f_*$ is the unique optimal function for any $\alpha > 0$.

With $\alpha = 0$, the inequality is the classical logarithmic Hardy-Littlewood-Sobolev inequality

$$\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \frac{2}{M} \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x - y| \, dx \, dy + M \left( 1 + \log \pi \right) \geq 0. \quad (2)$$

In that case $f_*$ is an optimal function as well as all functions generated by a translation and a scaling of $f_*$. As long as the parameter $\alpha$ is in the range $0 \leq \alpha < 1$, the coefficient of the right-hand side of (1)
is negative and the inequality is essentially of the same nature as the one with $\alpha = 0$. It can indeed be written as

$$\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f \, dx + M (1 - \alpha) \left( 1 + \log \pi \right) + \frac{2}{M} (1 - \alpha) \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x - y| \, dx \, dy \geq 0.$$ 

For reasons that will be made clear below, we shall call this range the attractive range.

If $\alpha = 1$, the inequality is almost trivial since

$$\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \int_{\mathbb{R}^2} V f \, dx = \int_{\mathbb{R}^2} f \log \left( \frac{f}{f^*} \right) \geq 0 \quad (3)$$

is a straightforward consequence of Jensen’s inequality. Now it is clear that by adding (2) multiplied by $(1 - \alpha)$ and (3) multiplied by $\alpha$, we recover (1) for any $\alpha \in [0, 1]$. As a consequence (1) is a straightforward interpolation between (2) and (3) in the attractive range.

Now, let us consider the repulsive range $\alpha > 1$. It is clear that the inequality is no more the consequence of a simple interpolation. We can also observe that the coefficient $(\alpha - 1)$ in the right-hand side of (1) is now positive. Since

$$G(x) = -\frac{1}{2\pi} \log |x|$$

is the Green function associated with $-\Delta$ on $\mathbb{R}^2$, so that we can define

$$(-\Delta)^{-1} f(x) = (G * f)(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y| \, f(y) \, dy,$$

it is interesting to write (1) as

$$\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f \, dx + \frac{4\pi}{M} (\alpha - 1) \int_{\mathbb{R}^2} f (-\Delta)^{-1} f \, dx \geq M (\alpha - 1) \left( 1 + \log \pi \right). \quad (4)$$

If $f$ has a sufficient decay as $|x| \to +\infty$, for instance if $f$ is compactly supported, we know that $(-\Delta)^{-1} f(x) \sim -\frac{M}{2\pi} \log |x|$ for large values of $|x|$ and as a consequence,

$$\alpha \, V + \frac{4\pi}{M} (\alpha - 1) (-\Delta)^{-1} f \sim 2 (\alpha + 1) \log |x| \to +\infty \quad \text{as} \quad |x| \to +\infty.$$

In a minimization scheme, this prevents the runaway of the left-hand side in (4). On the other hand, $\int_{\mathbb{R}^2} f \log f \, dx$ prevents any concentration, and this is why it can be heuristically expected that the left-hand side of (4) indeed admits a minimizer.

Inequality (2) was proved in [8] by E. Carlen and M. Loss (also see [2]). An alternative method based on nonlinear flows was given by E. Carlen, J. Carrillo and M. Loss in [7]: see Section 2 for a sketch of their proof. Our proof of Theorem 1.1 relies on an extension of this approach which takes into account the presence of the external potential $V$. A remarkable feature of this approach is that it is insensitive to the sign of $\alpha - 1$.

One of the key motivations for studying (4) arises from entropy methods applied to drift-diffusion-Poisson models which, after scaling out all physical parameters, are given by

$$\frac{\partial f}{\partial t} = \Delta f + \beta \nabla \cdot (f \nabla V) + \nabla \cdot (f \nabla \phi) \quad (5)$$
with a nonlinear coupling given by the Poisson equation
\[- \varepsilon \Delta \phi = f.\]  

(6)

Here \( V = -\log \mu \) is the external confining potential and we choose it as in the statement of Theorem 1.1, while \( \beta \geq 0 \) is a coupling parameter with \( V \), which measures the strength of the external potential. We shall consider more general potentials at the end of this paper. The coefficient \( \varepsilon \) in (6) is either \( \varepsilon = -1 \), which corresponds to the attractive case, or \( \varepsilon = +1 \), which corresponds to the repulsive case. In terms of applications, when \( \varepsilon = -1 \), (6) is the equation for the mean field potential obtained from Newton’s law of attraction in gravitation, for applications in astrophysics, or for the Keller-Segel concentration of chemo-attractant in chemotaxis. The case \( \varepsilon = +1 \) is used for repulsive electrostatic forces in semiconductor physics, electrolytes, plasmas and charged particle models.

In view of entropy methods applied to PDEs (see for instance [15]), it is natural to consider the free energy functional
\[ \mathcal{F}_\beta[f] := \int_{\mathbb{R}^2} f \log f \, dx + \beta \int_{\mathbb{R}^2} V f \, dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi f \, dx \]  

(7)

because, if \( f > 0 \) solves (5)-(6) and is smooth enough, with sufficient decay properties at infinity, then
\[ \frac{d}{dt} \mathcal{F}_\beta[f(t, \cdot)] = -\int_{\mathbb{R}^2} f \left| \nabla \log f + \beta \nabla V + \nabla \phi \right|^2 \, dx \]  

(8)

so that \( \mathcal{F}_\beta \) is a Lyapunov functional. Of course, a preliminary question is to establish under which conditions \( \mathcal{F}_\beta \) is bounded from below. The answer is given by the following result.

**Corollary 1.2.** Let \( M > 0 \). If \( \varepsilon = +1 \), the functional \( \mathcal{F}_\beta \) is bounded from below and admits a minimizer on the set of the functions \( f \in L^1_1(\mathbb{R}^2) \) such that \( \int_{\mathbb{R}^2} f \, dx = M \) if and only if \( \beta \geq 1 + \frac{M}{8 \pi} \). It is bounded from below if \( \varepsilon = -1 \), \( \beta \geq 1 - \frac{M}{8 \pi} \) and \( M \leq 8 \pi \). If \( \varepsilon = +1 \), the minimizer is unique. \( \Box \)

As we shall see in Section 3.1, Corollary 1.2 is a simple consequence of Theorem 1.1. In the case of the parabolic-elliptic Keller-Segel model, that is, with \( \varepsilon = -1 \) and \( \beta = 0 \), this has been used in [12, 4] to provide a sharp range of existence of the solutions to the evolution problem. In [6], the case \( \varepsilon = -1 \) with a potential \( V \) with quadratic growth at infinity was also considered, in the study of intermediate asymptotics of the parabolic-elliptic Keller-Segel model.

Concerning the drift-diffusion-Poisson model (5)-(6) and considerations on the free energy, in the electrostatic case, we can quote, among many others, [14, 13] and subsequent papers. In the Euclidean space with confining potentials, we shall refer to [10, 11, 3, 1]. However, as far as we know, these papers are primarily devoted to dimensions \( d \geq 3 \) and the sharp growth condition on \( V \) when \( d = 2 \) has not been studied so far. The goal of this paper is to fill this gap. The specific choice of \( V \) has been made to obtain explicit constants and optimal inequalities, but the confining potential plays a role only at infinity if we are interested in the boundedness from below of the free energy. In Section 3.3, we shall give a result for general potentials on \( \mathbb{R}^2 \): see Theorem 3.4 for a statement.

### 2 Proof of the main result

As an introduction to the key method, we briefly sketch the proof of (2) given by E. Carlen, J. Carrillo and M. Loss in [7]. The main idea is to use the nonlinear diffusion equation
\[ \frac{\partial f}{\partial t} = \Delta \sqrt{f} \]
with a nonnegative initial datum $f_0$. The equation preserves the mass $M = \int_{\mathbb{R}^2} f \, dx$ and is such that
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^2} f \log f \, dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f \left( (-\Delta)^{-1} f \right) \, dx \right) = - \frac{8}{M} \left( \int_{\mathbb{R}^2} \left| \nabla f^{1/4} \right|^2 \, dx \int_{\mathbb{R}^2} f \, dx - \pi \int_{\mathbb{R}^2} f^{3/2} \, dx \right).
\]
According to [9], the Gagliardo-Nirenberg inequality
\[
\| \nabla g \|_2^2 \| g \|_4^4 \geq \pi \| g \|_6^6 \tag{9}
\]
applied to $g = f^{1/4}$ guarantees that the right-hand side is nonpositive. By the general theory of fast diffusion equations (we refer for instance to [17]), we know that the solution behaves for large values of $t$ like a self-similar solution, the so-called Barenblatt solution, which is given by $B(t, x) := t^{-2} f_*(x/t)$.

As a consequence, we find that
\[
\int_{\mathbb{R}^2} f_0 \log f_0 \, dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f_0 \left( (-\Delta)^{-1} f_0 \right) \, dx 
\geq \lim_{t \to +\infty} \int_{\mathbb{R}^2} B \log B \, dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} B \left( (-\Delta)^{-1} B \right) \, dx = \int_{\mathbb{R}^2} f_* \log f_* \, dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f_* \left( (-\Delta)^{-1} f_* \right) \, dx
\]
After an elementary computation, we observe that the above inequality is exactly (2) written for $f = f_0$.

The point is now to adapt this strategy to the case with an external potential. This justifies why solutions are regular enough to perform the computations below and in particular if they have a sufficient decay at infinity to allow all kinds of integrations by parts needed by the method. The answer is twofold. First, we can take an initial datum $f_0$ that is as smooth and decaying as $|x| \to +\infty$ as needed, prove the inequality and argue by density. Second, integrations by parts can be justified by an approximation scheme consisting in a truncation of the problem in larger and larger balls. We refer to [17] for regularity issues and to [15] for the truncation method. In the proof, we will therefore leave these issues aside, as they are purely technical.

**Proof of Theorem 1.1.** By homogeneity, we can assume that $M = 1$ without loss of generality and consider the evolution equation
\[
\frac{\partial f}{\partial t} = \Delta \sqrt{f} + 2 \sqrt{\pi} \nabla \cdot (xf).
\]
1) Using simple integrations by parts, we compute
\[
\int_{\mathbb{R}^2} (1 + \log f) \Delta \sqrt{f} \, dx = -8 \int_{\mathbb{R}^2} \left| \nabla f^{1/4} \right|^2 \, dx
\]
and
\[
\int_{\mathbb{R}^2} (1 + \log f) \nabla \cdot (xf) \, dx = - \int_{\mathbb{R}^2} \frac{\nabla f}{f} \cdot (xf) \, dx = - \int_{\mathbb{R}^2} x \cdot \nabla f \, dx = 2 \int_{\mathbb{R}^2} f \, dx = 2.
\]
As a consequence, we obtain that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} f \log f \, dx = -8 \int_{\mathbb{R}^2} \left| \nabla f^{1/4} \right|^2 \, dx + 8\pi \int_{\mathbb{R}^2} f^{3/2} \, dx \tag{10}
\]
using
\[
\int_{\mathbb{R}^2} f^{3/2} \, dx = \frac{1}{2\sqrt{\pi}}.
\]
2) By elementary considerations again, we find that
\[ 4 \pi \int_{\mathbb{R}^2} f (\Delta)^{-1} \left( \Delta \sqrt{f} \right) dx = -4 \pi \int_{\mathbb{R}^2} f^{3/2} dx \]
and
\[ 4 \pi \int_{\mathbb{R}^2} \nabla \cdot (x f) (\Delta)^{-1} f dx = -4 \pi \int_{\mathbb{R}^2} x f \cdot \nabla (\Delta)^{-1} f dx \]
\[ = \int_{\mathbb{R}^2} f(x) f(y) x \cdot \frac{x - y}{|x - y|^2} dx dy \]
\[ = \int_{\mathbb{R}^2} f(x) f(y) (x - y) \cdot \frac{x - y}{|x - y|^2} dx dy = 1 \]
where, in the last line, we exchanged the variables \( x \) and \( y \) and took the half sum of the two expressions.

This proves that
\[ \frac{d}{dt} \left( 4 \pi \int_{\mathbb{R}^2} f (\Delta)^{-1} f dx \right) = -8 \pi \int_{\mathbb{R}^2} \left( f^{3/2} - \mu^{3/2} \right) dx. \] (11)

3) We observe that
\[ \mu(x) = \frac{1}{\pi \left(1 + |x|^2\right)^{3/2}} = e^{-V(x)} \]
solves
\[ \Delta V = -\Delta \log \mu = 8 \pi \mu \] (12)
and, as a consequence,
\[ \int_{\mathbb{R}^2} V \Delta \sqrt{f} dx = \int_{\mathbb{R}^2} \Delta V \sqrt{f} dx = 8 \pi \int_{\mathbb{R}^2} \mu \sqrt{f} dx. \]

Since
\[ 2 \sqrt{\pi} \int_{\mathbb{R}^2} V \nabla \cdot (x f) dx = -2 \sqrt{\pi} \int_{\mathbb{R}^2} f x \cdot \nabla f dx = -8 \sqrt{\pi} \int_{\mathbb{R}^2} \frac{|x|^2}{1 + |x|^2} f dx \]
\[ = -8 \sqrt{\pi} + 8 \sqrt{\pi} \int_{\mathbb{R}^2} \frac{f}{1 + |x|^2} dx = -8 \sqrt{\pi} + 8 \pi \int_{\mathbb{R}^2} \sqrt{\mu} f dx, \]
we conclude that
\[ \frac{d}{dt} \int_{\mathbb{R}^2} V \sqrt{f} dx = 8 \pi \int_{\mathbb{R}^2} \left( \mu \sqrt{f} + \sqrt{\mu} f - 2 \mu^{3/2} \right) dx. \] (13)

Let us define
\[ \mathcal{F} [f] := \int_{\mathbb{R}^2} f \log f dx + \alpha \int_{\mathbb{R}^2} V f dx + (1 - \alpha) \left(1 + \log \pi\right) + 2 \left(1 - \alpha\right) \int_{\mathbb{R}^2} f(x) f(y) \log |x - y| dx dy. \]

Collecting (10), (11) and (13), we find that
\[ \frac{d}{dt} \mathcal{F} [f(t, \cdot)] = -8 \left( \int_{\mathbb{R}^2} \nabla f^{1/4} dx - 8 \pi \int_{\mathbb{R}^2} f^{3/2} dx \right) - 8 \pi a \int_{\mathbb{R}^2} \left( f^{3/2} - \mu \sqrt{f - \sqrt{\mu} f + \mu^{3/2}} \right) dx. \]

Notice that
\[ \int_{\mathbb{R}^2} \left( f^{3/2} - \mu \sqrt{f - \sqrt{\mu} f + \mu^{3/2}} \right) dx = \int_{\mathbb{R}^2} \varphi \left( \frac{f}{\mu} \right) \mu^{3/2} dx \quad \text{with} \quad \varphi(t) := t^{3/2} - t - \sqrt{t} + 1. \]
and that \(\varphi\) is a strictly convex function on \(\mathbb{R}^+\) such that \(\varphi(1) = \varphi'(1) = 0\), so that \(\varphi\) is nonnegative. On the other hand, by (9), we know that
\[
\int_{\mathbb{R}^2} |\nabla f|^{1/4} \, dx - \pi \int_{\mathbb{R}^2} f^{3/2} \, dx \geq 0
\]
as in the proof of [7]. Altogether, this proves that \(t \mapsto \mathcal{F}[f(t, \cdot)]\) is monotone nonincreasing. Hence
\[
\mathcal{F}[f_0] \geq \mathcal{F}[f(t, \cdot)] \geq \lim_{t \to +\infty} \mathcal{F}[f(t, \cdot)] = \mathcal{F}[f_{\star}] = 0.
\]
This completes the proof of (1).

3 Consequences

3.1 Proof of Corollary 1.2

To prove the result of Corollary 1.2, we have to establish first that the free energy functional \(\mathcal{F}_\beta\) is bounded from below. Instead of using standard variational methods to prove that a minimizer is achieved, we can rely on the flow associated with (5)-(6).

- Repulsive case. Let us consider the free energy functional defined in (7) where \(\phi\) is given by (6) with \(\epsilon = +1\), i.e., \(\phi = -\frac{1}{2\pi} \log |x| \ast f\).

Lemma 3.1. Let \(M > 0\) and \(\epsilon = +1\). Then \(\mathcal{F}_\beta\) is bounded from below on the set of the functions \(f \in L^1_+ (\mathbb{R}^2)\) such that \(\int_{\mathbb{R}^2} f \, dx = M\) if and only if \(\beta \geq 1 + \frac{M}{8\pi}\).

Proof. With \(g = \frac{f}{M}\) and \(\alpha = 1 + \frac{M}{8\pi}\), this means that
\[
\frac{1}{M} \mathcal{F}_\beta[f] - \log M = \int_{\mathbb{R}^2} g \log g \, dx + \beta \int_{\mathbb{R}^2} V g \, dx - \frac{M}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log|x - y| \, dx \, dy
\]
\[
= (\beta - \alpha) \int_{\mathbb{R}^2} V g \, dx + \int_{\mathbb{R}^2} g \log g \, dx + \alpha \int_{\mathbb{R}^2} V g \, dx - 2(\alpha - 1) \int_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log|x - y| \, dx \, dy
\]
\[
\geq (\beta - \alpha) \int_{\mathbb{R}^2} V g \, dx - (1 - \alpha) \left(1 + \log \pi\right)
\]
according to Theorem 1.1; the condition \(\beta \geq \alpha\) is enough to prove that \(\mathcal{F}_\beta[f]\) is bounded from below.

Reciprocally, let us assume that \(\beta < 1 + \frac{M}{8\pi}\) and let \(f_\epsilon(x) := \epsilon^2 f_{\epsilon}(\epsilon x)\). It is then straightforward to check that \(\mathcal{F}_\beta\) is not bounded from below because
\[
\mathcal{F}_\beta[f_\epsilon] \sim -2M(\beta - \alpha) \log \epsilon \to -\infty \quad \text{as} \quad \epsilon \to 0_+.
\]

Proof of Corollary 1.2 with \(\epsilon = +1\). Let us consider a smooth solution of (5)-(6). We refer to [16] for details and to [1] for similar arguments in dimension \(d \geq 3\). According to (8), \(f\) converges as \(t \to +\infty\) to a solution of
\[
\nabla \log f + \beta \nabla V + \nabla \phi = 0.
\]
Notice that this already proves the existence of a stationary solution. The equation can be solved as
\[
f = M \frac{e^{-\beta V - \phi}}{\int_{\mathbb{R}^2} e^{-\beta V - \phi} \, dx}
\]
after taking into account the conservation of the mass. With (6), the problem is reduced to solving

\[-\Delta \psi = M \left( e^{-f - \psi} \frac{e^{-V} - \psi}{\int_{\mathbb{R}^2} e^{-f - \psi} dx} - \mu \right), \quad \psi = (\beta - \gamma) V + \phi, \quad \gamma = \beta - \frac{M}{8\pi} \]

using (12). It is a critical point of the functional \( \psi \rightarrow \mathcal{F}_{M,Y}[\psi] := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 dx + M \int_{\mathbb{R}^2} \psi \mu dx + M \log(\int_{\mathbb{R}^2} e^{-f - \psi} dx) \). Such a functional is strictly convex as, for instance, in [10, 11]. We conclude that \( \psi \) is unique up to an additional constant.

\[\bullet\] Attractive case. Let us consider the free energy functional (7) \( \mathcal{F}_\beta \) where \( \phi \) is given by (6) with \( \epsilon = -1 \), i.e., \( \phi = \frac{1}{2\pi} \log |r| \ast f \). Inspired by [12], we have the following estimate.

**Lemma 3.2.** Let \( \epsilon = -1 \). Then \( \mathcal{F}_\beta \) is bounded from below on the set of the functions \( f \in L^1(\mathbb{R}^2) \) such that \( \int_{\mathbb{R}^2} f dx = M \) if \( M \leq 8\pi \) and \( \beta \geq 1 - \frac{M}{8\pi} \). It is not bounded from below if \( M > 8\pi \).

**Proof.** With \( g = \frac{\pi}{M} \) and \( \alpha = 1 - \frac{M}{8\pi} \), Theorem 1.1 applied to

\[
\frac{1}{M} \mathcal{F}_\beta[f] - \log M = (\beta - \alpha) \int_{\mathbb{R}^2} V g dx + \int_{\mathbb{R}^2} g \log g dx + \alpha \int_{\mathbb{R}^2} V g dx + 2(1 - \alpha) \int_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log |x - y| dx dy \\
\geq (\beta - \alpha) \int_{\mathbb{R}^2} V g dx - (1 - \alpha) (1 + \log \pi)
\]

proves that the free energy is bounded from below if \( M \leq 8\pi \) and \( \beta \geq \alpha \). On the other hand, if \( f_\epsilon(x) := \epsilon^{-2} f(\epsilon^{-1} x) \) and \( M > 8\pi \), then

\[\mathcal{F}_\beta[f_\epsilon] \sim 2M \left( \frac{M}{8\pi} - 1 \right) \log \epsilon \rightarrow -\infty \quad \text{as} \quad \epsilon \to 0_+,\]

which proves that \( \mathcal{F}_\beta \) is not bounded from below.

**Proof of Corollary 1.2 with \( \epsilon = -1 \).** The proof goes as in the case \( \beta = 0 \). We refer to [4] and leave details to the reader.

**Remark 3.3.** Let us notice that \( \mathcal{F}_\beta \) is unbounded from below if \( \beta < 0 \). This follows from the observation that \( \lim_{|y| \to \infty} \mathcal{F}_\beta[f_y] = -\infty \) where \( f_y(x) = f(x + y) \) for any admissible \( f \).

3.2 Duality

When \( \alpha > 1 \), we can write a first inequality by considering the repulsive case in the proof of Corollary 1.2 and observing that

\[\mathcal{F}_{M,Y}[\psi] \geq \min_{M,Y} \mathcal{F}_{M,Y}\]

where \( \psi \in W^{2,1}_{\text{loc}}(\mathbb{R}^2) \) is such that \( \int_{\mathbb{R}^2} (\Delta \psi) dx = 0 \) and the minimum is taken on the same set of functions.

When \( \alpha \in [0, 1) \), it is possible to argue by duality as in [5, Section 2]. Since \( f_* \) realizes the equality case in (1), we know that

\[
\int_{\mathbb{R}^2} f_* \log \left( \frac{f_*}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f_* dx + M (1 - \alpha) \left( 1 + \log \pi \right) \geq \frac{2}{M} (\alpha - 1) \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_* (x) f_* (y) \log |x - y| dx dy
\]
and, using the fact that \( f_* \) is a critical point of the difference of the two sides of (1), we also have that
\[
\int_{\mathbb{R}^2} \log \left( \frac{f}{f_*} \right) (f-f_*) \, dx + \alpha \int_{\mathbb{R}^2} V (f-f_*) \, dx = \frac{4}{M} (\alpha - 1) \int_{\mathbb{R}^2} (f(x) - f_*(x)) f_*(y) \log |x-y| \, dx \, dy.
\]
By subtracting the first identity to (1) and adding the second identity, we can rephrase (1) as
\[
\mathcal{F}_{(1)}[f] := \int_{\mathbb{R}^2} f \log \left( \frac{f}{f_*} \right) \, dx \geq \frac{4 \pi}{M} (1 - \alpha) \int_{\mathbb{R}^2} (f-f_*) (-\Delta)^{-1} (f-f_*) \, dx := \mathcal{F}_{(2)}[f].
\]
Let us consider the Legendre transform
\[
\mathcal{F}_{(1)}^*[g] := \sup_f \left( \int_{\mathbb{R}^2} g f \, dx - \mathcal{F}_{(1)}[f] \right)
\]
where the supremum is restricted to the set of the functions \( f \in L^1_+(\mathbb{R}^2) \) such that \( M = \int_{\mathbb{R}^2} f \, dx \). After taking into account the Lagrange multipliers associated with the mass constraint, we obtain that
\[
M \log \left( \int_{\mathbb{R}^2} e^{8-V} \, dx \right) = \mathcal{F}_{(1)}^*[g] \leq \frac{M}{16 \pi (1 - \alpha)} \int_{\mathbb{R}^2} |\nabla g|^2 \, dx + M \int_{\mathbb{R}^2} g e^{-V} \, dx = \mathcal{F}_{(2)}^*[g].
\]
We can get rid of \( M \) by homogeneity and recover the standard Euclidean form of the Onofri inequality in the limit case as \( \alpha \to 0^+ \), which is clearly the sharpest one for all possible \( \alpha \in [0, 1) \).

### 3.3 Extension to general confining potentials with critical asymptotic growth

As a concluding observation, let us consider a general potential \( W \) on \( \mathbb{R}^2 \) such that
\[
W \in C(\mathbb{R}^2) \quad \text{and} \quad \lim_{|x| \to +\infty} \frac{W(x)}{V(x)} = \beta \quad \text{(} \mathcal{H}_W \text{)}
\]
and the associated free energy functional
\[
\mathcal{F}_{\beta,W}[f] := \int_{\mathbb{R}^2} f \log f \, dx + \beta \int_{\mathbb{R}^2} W f \, dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi f \, dx
\]
where \( \phi \) is given in terms of \( f > 0 \) by (6). With previous notations, \( \mathcal{F}_\beta = \mathcal{F}_{\beta,V} \). Our last result is that the asymptotic behaviour obtained from (\( \mathcal{H}_W \)) is enough to decide whether \( \mathcal{F}_{\beta,W} \) is bounded from below or not. The precise result goes as follows.

**Theorem 3.4.** Under Assumption (\( \mathcal{H}_W \)), \( \mathcal{F}_{\beta,W} \) defined as above is bounded from below if either \( \varepsilon = +1 \) and \( \beta > 1 + \frac{M}{8 \pi} \), or \( \varepsilon = -1 \), \( \beta > 1 - \frac{M}{8 \pi} \), and \( M \leq 8 \pi \). The result is also true in the limit case if \( (W - \beta V) \in L^\infty(\mathbb{R}^2) \) and either \( \varepsilon = +1 \) and \( \beta = 1 + \frac{M}{8 \pi} \), or \( \varepsilon = -1 \), \( \beta = 1 - \frac{M}{8 \pi} \) and \( M \leq 8 \pi \).

**Proof.** If \( (W - \beta V) \in L^\infty(\mathbb{R}^2) \), we can write that
\[
\mathcal{F}_{\beta,W}[f] \geq \mathcal{F}_\beta[f] - M \left\| W - \beta V \right\|_{L^\infty(\mathbb{R}^2)}.
\]
This completes the proof in the limit case. Otherwise, we redo the argument using \( \hat{\beta} V - (\hat{\beta} V - W)_+ \) for some \( \hat{\beta} \in (0, \beta) \) if \( \varepsilon = -1 \), and for some \( \hat{\beta} \in (1 + \frac{M}{8 \pi}, \beta) \) if \( \varepsilon = +1 \).
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