Abstract

For a planar bipartite graph $G$ equipped with a $\text{SL}_n$-local system, we show that the determinant of the associated Kasteleyn matrix counts “$n$-multiwebs” (generalizations of $n$-webs) in $G$, weighted by their web-traces. We use this fact to study random $n$-multiwebs in graphs on some simple surfaces.

1 Introduction

1.1 Multiwebs

Let $G = (V, E)$ be a bipartite graph (which is not necessarily simple, that is, we allow multiple edges between two vertices). We always assume there are the same number $N$ of white vertices as black vertices, $N = |B| = |W|$. An $n$-multiweb $m$ of $G$ is a multiset of edges with degree $n$ at each vertex, that is, a mapping $m : E \to \{0, 1, 2, \ldots\}$ such that for each vertex $v \in G$ we have $\sum_{u \sim v} m_{uv} = n$; that is, each vertex is an endpoint of exactly $n$ edges of $m$, counted with multiplicity. See Figure 1 for a 3-multiweb in a grid. The notion of $n$-multiweb is a generalization of the notion of $n$-web, which is, by definition, an $n$-valent bipartite graph. We let $\Omega_n(G)$ be the set of $n$-multiwebs of $G$. We often call an $n$-multiweb simply a web when the context is clear.

When $n = 1$, an $n$-multiweb of $G$ is a dimer cover of $G$, also known as a perfect matching of $G$. When $n = 2$ an $n$-multiweb of $G$ is a double dimer cover. Dimer covers and double dimer covers are classical combinatorial objects studied starting in the 1960’s by Kasteleyn [Kas61], Temperley/Fisher [TF61], and many others, see e.g. [Ken09] for a survey. Our goal here is to study $n$-fold dimer covers, or equivalently, $n$-multiwebs, for $n \geq 3$.

In [Ken14], $\text{SL}_2$-local systems were used to study topological properties of double dimer covers on planar graphs. We extend this here to $\text{SL}_n$-local...
systems for $n \geq 2$. On a bipartite graph $\mathcal{G}$ on a surface with an $\text{SL}_n$-local system, we define the *trace* of an $n$-multiweb, a small generalization of the trace of an $n$-web. Traces of $n$-webs are used in the study of tensor networks, representation theory, cluster algebras, and knot theory [Jae92, Kup96, Sik01, MP10, FP16, FLL19]. In this paper we study these traces from a probabilistic and combinatorial point of view. To distinguish our trace from the trace of a matrix we should in principle refer to it as a *web trace*. However we say “trace” when there is no risk of confusion (we need to be careful precisely when the web is a loop, because the web trace is not generally equal to the trace of the associated monodromy around the loop; see Section 3.3.2).

Our main result computes the determinant of a certain operator $K(\Phi)$, the *Kasteleyn matrix* for the planar bipartite graph $\mathcal{G}$ in the presence of an $\text{SL}_n$-local system $\Phi$, as a sum of traces of webs:

**Theorem.** Up to a global sign, $\det \tilde{K}(\Phi) = \sum_{n\text{-multiwebs } m \in \Omega_n(\mathcal{G})} \text{Tr}(m)$.

Here $\tilde{K} \in M_{nN}(\mathbb{R})$ is obtained from $K \in M_N(M_n(\mathbb{R}))$ in the obvious way.

For the definitions and precise statement, see below and Theorem 4.1. If we ignore gauge invariance, which is not necessary for the statement, this theorem holds more generally for $M_n$-connections (connections with parallel transports in $M_n(\mathbb{R}) = \text{End}(\mathbb{R}^n)$).

In the case $n = 1$, we have $\text{Tr}(m) = 1$ for any 1-multiweb for an $\text{SL}_1$-local system, or simply the product of edge weights for an $M_1$-connection; in this case $K$ is the usual Kasteleyn matrix. In this sense our result generalizes Kasteleyn’s theorem from [Kas61].

In the case $n = 2$, we give a new proof of (a slightly more general version of) a theorem of [Ken14] enumerating double-dimer covers, see Section 2.3.
As another application of the theorem, in the case $n = 3$ we show how to enumerate isotopy classes of “reduced” 3-multiwebs (see below), on either the annulus or the pair of pants (see Sections 6.1 and 6.2).

1.2 Colorings

For the identity connection, the trace of an $n$-multiweb has a simple combinatorial interpretation. The trace for the identity connection is a signed count of the number of edge-$n$-colorings (see Proposition 3.3 below), and in fact for planar webs, it is the number of edge-$n$-colorings (see Proposition 3.4 below).

Here, an edge-$n$-coloring of an $n$-multiweb $m$ is a coloring of the edges of $m$ with colors from $\mathcal{C} = \{1, 2, \ldots, n\}$ so that at each vertex all $n$ colors are present. More precisely, an edge-$n$-coloring is a map from the edges of $m$ into $2^{\mathcal{C}}$, the set of subsets of $\mathcal{C}$, with the property that, first, an edge of multiplicity $k$ maps to a subset of $\mathcal{C}$ of size $k$, and secondly, the union of the color sets over all edges at a vertex is $\mathcal{C}$. For example, the trace of the multiweb appearing in Figure 2 is $\text{Tr}(m) = 48 \times 24 \times 1 \times 1 \times 1$, thus $m$ has 1152 edge-3-colorings.

For a planar graph $\mathcal{G}$, we define the partition function of $n$-multiwebs to be

$$Z_{nd}(I) := \sum_{m \in \Omega_n(\mathcal{G})} \text{Tr}_+(m),$$

where we use the identity connection $I$. (The + subscript corresponds to a choice of positive cilia, see Section 3.5 below and comments after Corollary 3.5). We also define $Z_d$ to be the number of single dimer covers of $\mathcal{G}$.

**Proposition 1.1.** $Z_{nd}(I) = (Z_d)^n$. 

![Figure 2: An edge-3-coloring of the multiweb of Figure 1.](image)
Proof. There is a natural map from ordered $n$-tuples of single dimer covers to $n$-multiwebs, obtained by taking the union and recording multiplicity over each edge. The fiber over a fixed $n$-fold dimer cover is the number of its edge-$n$-colorings, which is $\text{Tr}_+ (m)$. \hfill \Box

Associated to $Z_{nd}(I)$ is therefore a natural probability measure $\mu_n$ on $n$-multiwebs of $\mathcal{G}$, where a multiweb $m$ has probability $\Pr(m) = \frac{\text{Tr}_+ (m)}{Z_{nd}(I)}$. One of our motivations is to analyze this measure; as a tool to probe this measure we use $\text{SL}_n$-local systems and our main theorem, Theorem 4.1.

1.3 The case $n = 3$

In the case $n = 3$, on a graph on a surface with a flat $\text{SL}_3$-connection $\Phi$, one can use skein relations (see Section 5.1) to write any 3-multiweb as a linear combination of reduced (or non-elliptic) multiwebs. These are multiwebs where each contractible face has 6 or more sides. We can thus rewrite the right-hand side of Theorem 4.1 as a sum over isotopy classes $\lambda$ of reduced multiwebs:

$$Z_{3d}(\Phi) = \sum_{\lambda \in \Lambda_3} C_\lambda \text{Tr}(\lambda).$$

Here $\Lambda_3$ is the set of isotopy classes of reduced 3-multiwebs. By a theorem of Sikora and Westbury [SW07] the coefficients $C_\lambda$ can be extracted from $Z_{3d}(\Phi)$ as $\Phi$ varies over flat connections. See Sections 6.1 and 6.2 for applications.

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2 Background

2.1 $\text{SL}_n$-local systems

Let $\Phi = \{\phi_e\}_{e \in E}$ be an $\text{SL}_n(\mathbb{R})$-local system on $\mathcal{G}$. This is the data of, for each edge $e = bw$ of $\mathcal{G}$, a matrix $\phi_{bw} \in \text{SL}_n(\mathbb{R})$, with $\phi_{wb} = (\phi_{bw})^{-1}$.

A coordinate-free definition can be given as follows. Let $V$ be a $n$-dimensional vector space, fixed once and for all, and associate to each vertex $v$ of $\mathcal{G}$ a copy $V_v = V$; we call $\mathcal{V} := \bigoplus_v V_v$ a $V$-bundle on $\mathcal{G}$. A $\text{GL}(V)$-connection on $\mathcal{V}$ is
a choice of isomorphisms \( \phi_{bw} : V_b \to V_w \) along edges \( e = bw \), with \( \phi_{bw} = \phi_{bw}^{-1} \).

Similarly we can talk about a \( \text{SL}(V) \)-connection. Choosing a basis \( \beta = \{v_i\} \) of \( V \) allows us to talk about a \( \text{SL}_n(\mathbb{R}) \)-connection, also called a \( \text{SL}_n(\mathbb{R}) \)-local system. (In practice, one simply takes \( V = \mathbb{R}^n \) at each vertex.)

More generally, we define an \( \text{End}(V) \)-connection to be the assignment of a linear map \( \phi_{bw} : V_b \to V_w \) to each edge \( e = bw \), not requiring invertibility (and we do not define any linear map from \( w \) to \( b \)). Choosing a basis \( \beta \) of \( V \) allows us to talk about a \( \text{M}_n(\mathbb{R}) \)-connection.

Two \( \text{End}(V) \)-connections \( \Phi = \{\phi_e\}, \Phi' = \{\phi'_e\} \) on the same graph are \( \text{GL}(V) \)-gauge equivalent (resp. \( \text{SL}(V) \)-gauge equivalent) if there are \( A_v \in \text{GL}(V) \) (resp. \( A_v \in \text{SL}(V) \)) such that for all edges \( bw \), we have \( \phi'_{bw} = A_w^{-1} \phi_{bw} A_b \). Similarly, we can talk about \( \text{M}_n(\mathbb{R}) \)-connections being \( \text{GL}_n(\mathbb{R}) \)- or \( \text{SL}_n(\mathbb{R}) \)-gauge equivalent.

Given a local system \( \Phi \) and a closed oriented loop \( \gamma \) on \( G \) with a base vertex \( v \in \gamma \), the monodromy of \( \Phi \) around \( \gamma \) is the composition of the isomorphisms around \( \gamma \) starting at \( v \). The conjugacy class of the monodromy is well-defined independently of gauge and is independent of the base point \( v \).

If \( G \) is embedded on a surface \( \Sigma \), a flat connection is a connection for which the monodromy around any null-homotopic loop is the identity.

### 2.2 Dimer model

For background on the dimer model see [Ken09].

Let \( G = (W \cup B, E) \) be a planar bipartite graph with the same number of white vertices as black vertices: \( N = |W| = |B| \). Let \( \nu : E \to \mathbb{R}_{>0} \) be a positive weight function on its edges. A dimer cover of \( G \) is a perfect matching: a collection of edges covering each vertex exactly once. A dimer cover \( m \) has weight \( \nu_m \) given by the product of its edge weights: \( \nu_m = \prod_{e \in m} \nu_e \). Let \( \Omega_1 \) be the set of dimer covers and let \( Z = \sum_{m \in \Omega_1} \nu_m \) be the sum of weights of all dimer covers. Kasteleyn showed [Kas61] that \( Z = |\det K| \), where the matrix \( K = (K_{wb})_{w \in W, b \in B} \), the (small) Kasteleyn matrix, satisfies

\[
K_{wb} = \begin{cases}
\varepsilon_{wb} \nu_{wb} & w \sim b \\
0 & \text{else}.
\end{cases}
\]

Here the \( \varepsilon_{wb} \in \{\pm 1\} \) are signs chosen using the “Kasteleyn rule”: faces of length \( l \) have \((l/2 + 1) \mod 2\) minus signs. This condition determines \( K \) uniquely up to gauge, that is, up to left- and right-multiplication by a diagonal matrix of \( \pm 1 \)'s (see [Ken09]).

Note that in our definition, \( K \) has rows indexing white vertices and columns indexing black vertices. Some references define the (big) Kasteleyn matrix \( K \)
indexed by all vertices, in which case \( Z = | \det K |^{1/2} \).

## 2.3 Double dimer model

A double-dimer configuration on \( G \) is another name for a 2-multiweb on \( G \). This is a decomposition of the graph into a collection of disjoint (unoriented) loops and doubled edges. If \( \Phi = \{ \phi_{bw} \}_{bw \in E} \) is an \( \text{SL}_2 \)-connection on \( G \), and \( m \) is a 2-multiweb, we can compute the web trace by

\[
\text{Tr}(m) = \prod_{\text{loops } \gamma} \text{Tr}(\phi_{\gamma})
\]

where \( \text{Tr} \) is the matrix trace, and where the product is over loops \( \gamma \) of \( m \) (each with some chosen orientation). Note that \( \text{SL}_2 \) has the special property that \( \text{Tr}(A) = \text{Tr}(A^{-1}) \), so \( \text{Tr}(\phi_{\gamma}) \) is independent of the choice of orientation of \( \gamma \).

In this setting we can construct an associated Kasteleyn matrix \( K = (K_{wb})_{w \in W, b \in B} \) as for single dimer covers, but with entries \( K_{wb} = \varepsilon_{wb} \phi_{bw} \) in \( \text{M}_N(\text{M}_2(\mathbb{R})) \), if \( wb \) are adjacent, where \( \varepsilon_{wb} \) are Kasteleyn signs as before. Note that we use \( \varepsilon_{wb} \phi_{bw} \) for the \( K_{wb} \) entry, and not \( \varepsilon_{wb} \phi_{bw} \). This is because, having chosen \( K \) to have white rows and black columns (rather than the reverse), now both \( \phi_{bw} \) and \( K \) are maps from functions on black vertices to functions on white vertices. Let \( \tilde{K} \) be the \( 2|W| \times 2|B| \) matrix obtained by replacing each entry with its \( 2 \times 2 \) block of numbers. In [Ken14] it was shown that

\[
\det \tilde{K} = \sum_{m \in \Omega_2} \text{Tr}(m).
\] (2)

The main result of the present paper is to generalize this to \( n \geq 3 \). However even in the case \( n = 2 \) our main theorem generalizes (2) to \( \text{M}_2 \)-connections, where the trace of a loop is no longer the trace of its monodromy.

## 3 Web traces

### 3.1 Tensor network definition

A \( n \)-\textit{web} is a bipartite graph, regular of degree \( n \) (we allow parallel edges but each edge has multiplicity 1, unlike a multiweb), embedded on an oriented surface. We consider its edges to be oriented from black to white. We let \( \Phi = \{ \phi_{bw} \} \) be an \( \text{M}_n \)-connection on \( G \). This is just the assignment of an \( n \times n \) matrix \( \phi_{bw} \) to each edge of \( G \), not requiring invertibility.

The trace of a web is a scalar function of \( \Phi \) associated to the web. We define the trace of a web as follows (the definition of trace for a multiweb is
given in Section 3.2 below). This is a standard tensor network definition; see for example [Sik01]. We fix, for each vertex, a linear order of the edges, compatible with the counterclockwise cyclic order at black vertices, and compatible with the clockwise cyclic order at white vertices. We can record this choice of linear order by placing a mark in the wedge at $v$ between the first and last edge. This mark is called the cilium at $v$, see [FR99]. For $n$ odd the trace will be well-defined independently of the cilia, but for $n$ even the trace is only defined up to a sign which depends on the locations of the cilia. To record this dependence we write $\text{Tr}_L$ where $L$ is the choice of cilia.

Let $m$ be a web. Let $V = \mathbb{R}^n$ with fixed basis $e_1, \ldots, e_n$, and $V^*$ its dual with dual basis $f_1, \ldots, f_n$. We associate to each edge $e = bw$ of $m$ a copy $V_e, V_e^*$ of $V$ and $V^*$. Here $V_e$ is associated to the end of $e$ near $b$, and $V_e^*$ to the end near $w$. At each black vertex of $m$ with adjacent vector spaces $V_1, \ldots, V_n$ in the given linear order we associate a canonical vector $v_b \in V_1 \otimes \cdots \otimes V_n$:

$$v_b = \sum_{\sigma \in S_n} (-1)^\sigma e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)},$$

called the codeterminant. Likewise at each white vertex with adjacent vector spaces $V_1^*, \ldots, V_n^*$ in order we associate a canonical codeterminant $v_w^* \in V_1^* \otimes \cdots \otimes V_n^*$:

$$v_w^* = \sum_{\sigma \in S_n} (-1)^\sigma f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}.$$

Note that changing the cilium at a black (resp. white) vertex by one “notch” will change the sign of $v_b$ (resp. $v_w^*$) if $n$ is even, but not if $n$ is odd.

Now along each edge $e = bw$ we have a contraction of tensors using $\phi_{bw}$. That is, we take the tensor product

$$X = \bigotimes_b v_b \in \bigotimes_e V_e$$

of $v_b$ over all black vertices, and the tensor product

$$Y = \bigotimes_w v_w^* \in \bigotimes_e V_e^*$$

of $v_w^*$ over all white vertices. Then we contract component by component along edges: a simple tensor $\bigotimes_{e=bw} v_b$ and a simple tensor $\bigotimes_{e=bw} v_w^*$ contract to give $\prod_e v_w(\phi_{bw}v_b)$, or, in a more symmetric notation $\prod_e \langle v_w | \phi_{bw} | v_b \rangle$. Summing we have

$$\text{Tr}_L(m) := \left\langle \bigotimes_w v_w^* \bigotimes_{e=bw} \phi_{bw} \bigotimes_b v_b \right\rangle.$$

The above definition of trace \textit{a priori} depends on the choice of basis for the $V_e, V_e^*$, but in fact does not depend on these choices, in the following sense:
**Proposition 3.1.** The above definition of the trace of a web (with multiplicity 1 on each edge) is independent of $\text{SL}_n$-change of basis at any vertex. That is, for any $\mathbb{M}_n$-connection the trace of a web is $\text{SL}_n$-gauge invariant (Section 2.1).

**Proof.** An $\text{SL}_n$-change of basis at a black vertex $b$ preserves $v_b$. Likewise at a white vertex. \qed

### 3.1.1 Coordinate-free description

There is an equivalent, coordinate-free definition of the trace $\text{Tr}_L(m)$ of a web, as follows. Recall from Section 2.1 that the vector spaces $V_v = V$ assigned to the vertices $v$ are copies of a fixed vector space $V$.

Let $\Phi$ be a $\text{End}(V)$-connection on $G$. Then the trace of $m$ is the composition

$$\mathbb{C} \xrightarrow{\text{codet}} \bigotimes \text{half edges at black vertices} V \xrightarrow{\Phi} \bigotimes \text{half edges at white vertices} V \xrightarrow{\text{det}} \mathbb{C},$$

evaluated at $1 \in \mathbb{C}$, where the first map $\text{codet}$ is the “codeterminant”, the second $\Phi$ is $\bigotimes_e \phi_e$, and the third $\text{det}$ is the “determinant” map taking a tensor product of $n$ vectors in $V$ to their wedge product in $\wedge^n V \cong \mathbb{C}$; see below for explicit formulas. Both $\text{codet}$ and $\text{det}$ depend on a choice $\beta$ of an isomorphism between $\mathbb{C}$ and $\wedge^n V$, that is, on a volume form on $V$. Note that, although $\text{codet}$ and $\text{det}$ depend on this choice of a basis $\beta$ of $V$, the composition $\text{det} \circ \Phi \circ \text{codet}$ does not; thus, the trace is well-defined.

Explicitly, for $\dim(V) = n$, if $\beta = \{v_i\}$ is the chosen basis for $V$, then $\text{codet} : \mathbb{C} \to V^\otimes n$, at each black vertex, is the linear map defined by the property $1 \mapsto \sum_{\sigma \in S_n} (-1)^{\sigma} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$, and $\text{det} : V^\otimes n \to \mathbb{C}$, at each white vertex, is the map defined by $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \mapsto (-1)^{(i_1, i_2, \ldots, i_n)}$ where the latter sign is zero if $(i_1, i_2, \ldots, i_n)$ does not define a permutation.

The web trace is analogous to the definition of the usual trace $\text{Tr}(\varphi)$ of an endomorphism $\varphi \in \text{End}(V)$ as the composition of “coevaluation” and “evaluation” maps, respectively, $\text{coeval} : \mathbb{C} \to V \otimes V$ and $\text{eval} : V \otimes V \to \mathbb{C}$, depending on the choice of basis $\beta$ but whose composition does not. For the usual trace, the intermediate map is $\varphi \otimes \text{id}_V : V \otimes V \to V \otimes V$.

### 3.2 Multiwebs and their traces

As discussed above, a multiweb is a web allowing edge multiplicities (including zero). More precisely, an $n$-multiweb $m$ is an assignment of edge multiplicities $m_e \geq 0$ to $G$, which is regular of degree $n$: $\sum_{u \sim v} m_{uv} = n$. We define the trace of a multiweb $G$ as follows. Split each edge $e$ of multiplicity $m_e$ into
\( m_e \) parallel edges, removing edges of multiplicity 0; let \( \mathcal{G}' \) be the resulting \( n \)-regular graph. If the original edge has parallel transport \( \phi_{bw} \), put \( \phi_{bw} \) on each of the new edges. The cilia for \( \mathcal{G}' \) are in the same location as they were in \( \mathcal{G} \): the cilia remain outside of the multiple edges. We now define

\[
\text{Tr}_L(\mathcal{G}) := \frac{\text{Tr}_L(\mathcal{G}')}{\prod_{e}(m_e!)}. \tag{3}
\]

That is, we divide the trace of the web \( \mathcal{G}' \) by the factorials of the multiplicities.

Proposition 3.2 below justifies this definition.

### 3.3 Trace examples

#### 3.3.1 Theta graph

Consider the following example for SL\(_3\)-connections (as opposed to M\(_3\)-connections).

Let \( m \) be a “theta” graph consisting of two vertices \( w, b \) with three edges between them \( e^1, e^2, e^3 \), each of multiplicity 1, in counterclockwise order at \( b \) and clockwise order at \( w \) (with respect to a planar embedding of the graph). Let \( A, B, C \) be the parallel transports from \( b \) to \( w \) along edges \( e^1, e^2, e^3 \) respectively (recall that we always orient edges from black to white). Then

\[
v_b = e_r \otimes e_g \otimes e_b - e_r \otimes e_b \otimes e_g + \cdots - e_b \otimes e_g \otimes e_r
\]

and

\[
v_w = f_r \otimes f_g \otimes f_b - f_r \otimes f_b \otimes f_g + \cdots - f_b \otimes f_g \otimes f_r.
\]

The contraction contains 36 terms; for example, contracting the first terms of \( v_b \) and \( v_w \) gives \( A_{rr}B_{gg}C_{bb} \), and contracting the first term of \( v_b \) and the second term of \( v_w \) gives \( -A_{rr}B_{bg}C_{gb} \). Summing, the trace can be written compactly as

\[
\text{Tr}(m) = \text{Tr}(AB^{-1})\text{Tr}(CB^{-1}) - \text{Tr}(AB^{-1}CB^{-1})
\]

(see Section 5 below) or, more symmetrically, as

\[
[xyz] \det(xA + yB + zC), \tag{4}
\]

that is, the coefficient of \( xyz \) in the expansion of the determinant of \( xA + yB + zC \) as a polynomial in \( x, y, z \) (see (11) below); note (4) is valid for any M\(_3\)-connection.

Also notice that when \( A = B = C \) then the trace \( \text{Tr}(m) = +6 \); in this case the SL\(_3\)-local system is trivializable, and the trace is thus the number of edge-3-colorings (see Proposition 3.4 below).
More generally, for the $M_n$-connection on the $n$-theta graph $m$ (two vertices with $n$ edges) consisting of the same matrix $A \in M_n$ attached to each edge, by Proposition 3.2 below the trace $\text{Tr}(m) = n! \det(A)$.

As one last variation, again in the case $n = 3$, note that if the cyclic ordering of the vertices is that induced by the nonplanar embedding of the theta graph in the torus, then its trace $\text{Tr}(m)$ with respect to the identity connection is equal to $-6$ (since we just reverse the cyclic order at one of the two vertices); contrast this with the calculation above and Proposition 3.4.

### 3.3.2 Loop

As another example, let $m$ consist of a cycle $b_1w_1b_2w_2$ with edges $b_iw_i$ of multiplicity 1, and edges $b_iw_{i-1}$ of multiplicity 2, as in Figure 3. At the black vertex $b_1$, say,

$$v_{b_1} = \sum_{\sigma} (-1)^{\sigma_1} e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)} = (e_r \wedge e_g) \otimes e_b + (e_g \wedge e_b) \otimes e_r + (e_b \wedge e_r) \otimes e_g.$$

Here we have used the wedge notation $e \wedge e' := e \otimes e' - e' \otimes e \in \mathbb{R}^3 \wedge \mathbb{R}^3$.

Choosing in addition a $M_3$-connection as in the figure, starting at $b_1$ this leads to

$$\text{Tr}(m) = \text{Tr}((D \wedge D) \cdot C \cdot (B \wedge B) \cdot A).$$

For an $\text{SL}_3$-connection, this becomes $\text{Tr}(m) = \text{Tr}(D^{-1}CB^{-1}A)$, namely the trace of the monodromy when the cycle is oriented clockwise in the figure. In particular, note that even though the cycle is not naturally oriented, the 3-web-trace nevertheless picks out an orientation: the one determined by following from black to white along the non-doubled edges.

Equation (5) needs to be properly interpreted, where $B \wedge B \in M_3(\mathbb{R})$ is defined as follows: Let $\tilde{B} \wedge B \in M_3(\mathbb{R})$ be the matrix of the linear map $\mathbb{R}^3 \wedge \mathbb{R}^3 \rightarrow \mathbb{R}^3 \wedge \mathbb{R}^3$ induced by $B$ written in the basis $\{e_r \wedge e_g, e_r \wedge e_b, e_g \wedge e_r\}$, and let $\varphi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then $B \wedge B := (\varphi \circ \tilde{B} \circ B \circ \varphi^{-1})^T$. Conceptually
speaking, $\varphi$ induces an isomorphism (of $\text{SL}_3$-representations) $\mathbb{R}^3 \wedge \mathbb{R}^3 \cong (\mathbb{R}^3)^*$, and the transpose in the formula for $B \wedge B$ allows one to pass from the dual $(\mathbb{R}^3)^*$ to $\mathbb{R}^3$. It also corresponds to the transpose in the formula for the inverse of $B$ in terms of its cofactor matrix, so that $B \wedge B = B^{-1}$ when $B \in \text{SL}_3(\mathbb{R})$; hence, we have the formula above for the trace $\text{Tr}(m)$ for $\text{SL}_3$-connections.

For general $n$, consider a cycle $b_1w_1b_2w_2\ldots b_\ell w_\ell$ with edges $b_iw_i$ of multiplicity $k$, and edges $b_iw_{i-1}$ of multiplicity $n-k$, and an $M_n$-connection with parallel transport $A_i$ from $b_i$ to $w_i$, and $B_i^{-1}$ from $b_i$ to $w_{i-1}$. Then, we have $\text{Tr}_L(m) = \pm \text{Tr}(C)$, where $C$ is the product

$$C = (\wedge^{n-k} B_\ell)(\wedge^k A_\ell) \cdots (\wedge^{n-k} B_2)(\wedge^k A_2)(\wedge^{n-k} B_1)(\wedge^k A_1)$$

appropriately interpreted, similar to the case $n = 3$.

For an $\text{SL}_n$-connection with total monodromy $M = B_\ell^{-1}A_\ell \cdots B_1^{-1}A_1^{-1}$ clockwise around the loop, we thus have

$$\text{Tr}_L(m) = \pm \text{Tr}(\wedge^k M).$$

Indeed, by first taking advantage of the $\text{SL}_n$-gauge invariance of the web trace (Proposition 3.1) to concentrate the connection $M$ entirely on the edge $b_1w_1$, this is then an immediate consequence of the above equation. We remind the reader that, for $n$ odd the sign is $+$ but for $n$ even the sign depends on the choice of cilia $L$. Note that when $k = n$, corresponding to when $m$ is the $n$-multiedge, this last calculation is consistent with the calculation for the $n$-theta graph of Section 3.3.1: in this case, $\text{Tr}_L(m) = +\det(M) = 1$.

### 3.3.3 Nonplanar example

Consider the complete bipartite graph on three vertices $m = K_{3,3}$. This is nonplanar. For the cyclic ordering on the vertices induced by the embedding of $m$ in the torus, one computes $\text{Tr}(m) = 0$; for example, one can use the skein relations of Section 5 together with the calculation at the very end of Section 3.3.1. On the other hand, $m$ has 12 edge-colorings. This demonstrates that Proposition 3.3 is special to planar graphs.

### 3.4 Traces in terms of edge colorings

The trace of a web or multiweb $m$ can be given a more combinatorial definition as follows. At each vertex $v$ of $m$, choose a coloring of the half-edges at $v$ with $n$ colors $C = \{1, \ldots, n\}$, using each color once, and so that multiple edges get a subset of colors. In other words at a vertex $v$ with edge multiplicities $M_1, \ldots, M_k$, we partition $C$ into $k$ disjoint subsets of sizes $M_1, \ldots, M_k$, one for
each half-edge at \( v \). Note that there are \( \binom{n}{M_1, \ldots, M_k} \) possible such colorings at \( v \). To such a coloring at \( v \) we associate a sign \( c_v \), defined as follows. List the colors according to the linear order on edges at \( v \) and for multiple edges, list the colors in their natural order. Then this list is a permutation of \( \mathcal{C} \) and \( c_v \) is its signature.

Now on each edge \( e \) of multiplicity \( m_e \) there are two sets of colors, each of size \( m_e \), one associated with the black vertex and one at the white vertex. We associate a corresponding matrix element, using the bijection of colors with indices: if edge \( e \) has parallel transport \( A \in \text{SL}_n \) from black to white, and is colored with set \( S \) at the white vertex and \( T \) at the black vertex, then the corresponding matrix element is \( \det(A_{S,T}) \), that is, the determinant of the \( S,T \)-minor of \( A \) (the submatrix of \( A \) with rows \( S \) and columns \( T \)).

To each coloring of all the half-edges we take the product of the associated matrix elements, and multiply this by the sign \( \prod_v c_v \). Then this quantity is summed over all possible colorings to define the trace:

**Proposition 3.2.** The above procedure computes the trace of the \( n \)-multiweb \( m \), that is,

\[
\text{Tr}_L(m) = \sum_{\text{colorings } c} (-1)^{\sum_v c(v)} \prod_{e=bw} \det(\phi_{bw})_{S_e,T_e},
\]

(6)

Proof. If \( \mathcal{G} \) is a web, that is, if all edges have multiplicity 1, then a coloring of the half-edges at \( b \) corresponds to a single term in \( v_b \), and its sign is the coefficient in front of this term. So a coloring of all the half-edges at all vertices corresponds to a single term in the expansion of

\[
\text{Tr}_L(\mathcal{G}) = (\bigotimes_w v_w | \bigotimes_{e=bw} \phi_{bw} | \bigotimes_b v_b)
\]

(7)

when we expand \( \otimes_b v_b \) and \( \otimes_w v_w \) over all permutations. Thus the two formulas (7) and (6) agree for webs.

Now suppose \( \mathcal{G} \) is a multiweb, obtained from a web \( \mathcal{G}' \) by collapsing \( k \) parallel edges, each with parallel transport \( \phi_{bw} \), into a single edge \( e = bw \) of multiplicity \( k \), with parallel transport \( \phi_{bw} \). In \( \mathcal{G} \) a coloring assigns subsets \( S_e, T_e \subset \mathcal{C} \) of size \( k \) to \( e \), and the contribution from this edge is \( \det(\phi_{bw})_{S_e,T_e} \) for the multiweb \( \mathcal{G} \). The corresponding contribution in \( \text{Tr}_L(\mathcal{G}') \) from this set of colorings involves all possible ways of distributing the colors \( S_e \) to the \( k \) half edges at \( w \), and likewise for \( T_e \). There are \( k! \) bijections of \( S_e \) with \( e_1, \ldots, e_k \), and \( k! \) bijections of \( T_e \) with \( e_1, \ldots, e_k \). Each such choice is a term in \( \det(\phi_{bw})_{S_e,T_e} \), and there are \( k! \) choices corresponding to each term; moreover the signs agree. Thus the Proposition holds for \( \mathcal{G} \), using the definition \( \text{Tr}_L(\mathcal{G}') = k! \text{Tr}_L(\mathcal{G}) \) from (3). Splitting any other multiple edges, we argue analogously. This completes the proof. \( \square \)
Figure 4: Suppose every other edge of a face is occupied by a dimer; one can shift all dimers around that face to get a new dimer cover. This is the *face move*. Such moves connect the set of all dimer covers for any planar bipartite graph.

If the connection is *trivial*, we have a nonzero contribution to the trace only when for each edge $e$, $S_e = T_e$, that is, the subsets of colors for the two half-edges are equal. In this case the matrix element is exactly 1 for each edge. The trace is thus the signed number of edge-$n$-colorings, where the sign is $\prod_v c_v$.

**Proposition 3.3.** The trace of an $n$-multiweb for the identity connection is the signed number of its edge-$n$-colorings.

A similar proposition holds for diagonal connections, where now the trace is a weighted, signed number of edge-$n$-colorings.

### 3.5 Planar webs

Proposition 3.3 simplifies for planar webs:

**Proposition 3.4.** The trace of a planar $n$-multiweb for the identity connection is equal to $\pm 1$ times the number of edge-$n$-colorings (with sign $+$ if $n$ is odd).

*Proof.* First note that given an edge-$n$-coloring $C$ of a multiweb $m$, the set of edges containing color $i$ is a dimer cover of $m$: each vertex has exactly one adjacent edge containing color $i$. We will use the fact (originally due to Thurston [Thu90]) that two dimer covers on a planar bipartite graph can be connected by a sequence of “face moves” as in Figure 4.

We now show that two edge-$n$-colorings $C, C'$ have the same sign. The idea is to move the color-1 edges of $C$ to those of $C'$ using face moves. Then move the color-2 edges using face moves, and so on, until $C = C'$.

The proof is different when $n$ is odd and $n$ is even. Suppose first that $n$ is odd.
When we rotate color-1 edges around a face $f$, keeping the other colors fixed, we create a new multiweb with different edge multiplicities. Let us compute how the sign changes. Suppose $f$ has vertices $b_1, w_1, \ldots, b_k, w_k$ in counterclockwise order, and edges $b_iw_i$ have multiplicities $M_i$, and edges $w_ib_{i+1}$ have multiplicities $M'_i$. Since $n$ is odd, we can rotate the cilia at vertices of $f$ so that at each vertex of $f$ the cilia lies in $f$, that is, the linear order starts inside the face $f$. Then the first edge out of $b_i$ is $b_iw_{i-1}$ and the first edge out of $w_{i-1}$ is $w_{i-1}b_i$. Suppose in $C$ edge $b_iw_{i-1}$ contains color 1. When we rotate this color so that edge $b_iw_i$ now has color 1, in the linear ordering of colors at $b_i$, color 1 has moved from the beginning to position $M_i - 1$ from the end. Likewise at $w_{i-1}$, color 1 has moved from the beginning to position $M_{i-1} - 1$ from the end. The net sign change after the face move, which involves all color-1 edges of the face, is $(-1)^{\sum_{i=1}^{k}(M_i+M_{i-1}-2)} = 1$. The sign is thus preserved.

Likewise for the other colors: change the ordering of the colors does not change the sign (since $G$ has an even number of vertices) so any other color can be considered to be the first color, and then the same argument holds.

The global sign can be determined by transforming all dimers covers to the same dimer cover, so that the multiweb is a union of $n$-fold edges. In this case the sign is $+$, since at each vertex the sign is $+$. This completes the proof in the case $n$ is odd.

Now assume $n$ is even. The argument works as for the $n$ odd case if all cilia are in the face $f$. However we also pick up a sign change from rotating the cilia. When we rotate the cilium at a vertex on $f$ to move it into face $f$, we pick up a sign change depending on the parity of the multiplicity of the edges we cross (since $n$ is even, moving it clockwise or counterclockwise results in the same sign change). Then, after rotating color 1 around $f$, we rotate the cilium back to its original position; we get another sign change. However the multiplicities of edges being crossed have changed by exactly 1, so the new sign change is the opposite of the first sign change, and so the net sign change is $-1$. Thus rotating color 1 at face $f$ gives a net sign change of $(-1)^{\text{out cilia}}$, that is, $-1$ per cilia of vertices on $f$ which are not pointing into $f$.

Now given two colorings of the same multiweb, we claim that when we transform one to the other doing face moves, each face move is performed an even number of times, that is, each face is toggled an even number of times. To see this, take a path $\gamma_f$ in the dual graph from $f$ to the exterior face. Compute the $\mathbb{Z}/2\mathbb{Z}$-intersection number $X_f = \langle \gamma_f, m \rangle$ of $\gamma_f$ with the multiweb $m$. Each $f$-face move changes $X_f$ by $1$ mod $2$, and each face move by any other faces changes $X_f$ by $0$ mod $2$. Any sequence of face moves changing $m$ back to $m$ necessarily does not change $X_f$.

This completes the proof.\[\square\]
From the above proof we see that we can arrange the cilia so that (for \( n \) even and the identity connection) all traces are positive:

**Corollary 3.5.** For \( n \) even and the identity connection, if cilia \( L \) are chosen so that each face of \( G \) has an even number of cilia, then for any multiweb \( m \) in \( G \), \( \text{Tr}_L(m) \) is equal to the number of edge-\( n \)-colorings.

We call a choice of cilia *positive* if it yields positive traces, and we write \( \text{Tr}_+ \) instead of \( \text{Tr}_L \). One way to choose cilia so that each face has an even number (and is thus positive) is as follows. Let \( m_1 \) be a dimer cover of \( G \). For each edge of \( m_1 \), choose the cilia at the vertices at its endpoints to be both on the same face containing one of the two sides of \( e \).

### 3.5.1 Finding an edge-\( n \)-coloring of a multiweb

There is a simple algorithm, communicated to us by Charlie Frohman, for finding an explicit edge-\( n \)-coloring of an \( n \)-multiweb of a planar graph, as follows; see also [Fro19]. Define a height function on faces taking values in \( \mathbb{Z}/n\mathbb{Z} \) as follows. On a fixed face \( f_0 \) define \( h(f_0) = 0 \). For any other face \( f \) let \( \gamma \) be a path in the dual graph from \( f_0 \) to \( f \). The height change along every edge of \( \gamma \) is given by the algebraic intersection number of the common edge with \( \gamma \) (considering edges oriented from black to white, and taking multiplicity into account). In other words given two adjacent faces \( f_1, f_2 \) with edge \( bw \) between them of multiplicity \( k \), where \( f_1 \) is on the left, then \( h(f_1) - h(f_2) = k \). Note that \( h \) is well-defined since it is well-defined around each vertex.

Given \( h \), the color set of an edge of multiplicity \( k \) is the interval of colors \( \{a, a+1, \ldots, a+k-1\} \) if the face to its right has height \( a \).

### 4 Kasteleyn matrix

Let \( G \) be a bipartite planar graph, with the same number of black vertices and white vertices. We fix a choice of positive cilia \( L \) for \( G \), in the sense of Corollary 3.5 and its subsequent paragraph. Let \( \Phi = \{\phi_e\}_{e \in E} \) be an \( M_n(\mathbb{R}) \)-connection on \( G \).

Let \( K \in M_N(M_n(\mathbb{R})) \) be the associated Kasteleyn matrix: \( K \) has rows indexing white vertices and columns indexing black vertices, with entries \( K(w, b) = 0_n \) (the \( n \times n \) zero matrix) if \( w, b \) are not adjacent and otherwise \( K(w, b) = \varepsilon_{wb}\phi_{bw} \), where the signs \( \varepsilon_{wb} \in \{-1, +1\} \) are given by the Kasteleyn rule. We let \( \tilde{K} \) be the \( nN \times nN \) matrix obtained from \( K \) by replacing each entry with its \( n \times n \) array of real numbers.

In the case that we have an \( \text{SL}_n \)-local system on \( G \), rather than an \( M_n \)-connection, then \( \tilde{K} \) depends on a particular choice of gauge. The gauge group
\((\text{SL}_n)^V\) acts on \(K\) by left- and right-multiplication by diagonal matrices with entries in \(\text{SL}_n\), or equivalently, acts on \(\tilde{K}\) by left- and right-multiplication by block-diagonal matrices with blocks in \(\text{SL}_n\). These gauge equivalences do not change the determinant of \(\tilde{K}\).

Note there is also the obvious coordinate-free description. Given a \(\text{End}(V)\)-connection \(\Phi\) on \(\mathcal{G}\), the Kasteleyn determinant \(\det \tilde{K}(\Phi)\), depending on a choice of Kasteleyn sign for \(\mathcal{G}\) and an ordering of the black and white vertices, is defined as the determinant of the induced linear endomorphism \(V^{\mid B\mid} \rightarrow V^{\mid W\mid}\).

Recall that \(\Omega_n(\mathcal{G})\) is the set of \(n\)-multiwebs in \(\mathcal{G}\). We define

\[
Z_{nd}(\Phi) := \sum_{m \in \Omega_n(\mathcal{G})} \text{Tr}_L(m).
\]

This generalizes the case of the identity connection from (1).

Our main theorem is

**Theorem 4.1.** Let \(\Phi\) be an \(\text{End}(V)\)-connection on the bipartite graph \(\mathcal{G}\).

For \(n\) even, and a choice of positive cilia \(L\) as discussed above,

\[
\det \tilde{K}(\Phi) = \sum_{m \in \Omega_n(\mathcal{G})} \text{Tr}_L(m);
\]

for \(n\) odd,

\[
\pm \det \tilde{K}(\Phi) = \sum_{m \in \Omega_n(\mathcal{G})} \text{Tr}(m).
\]

Note that the theorem implies that for trivial connections, and \(n\) even, the determinant of the Kasteleyn matrix of a graph admitting a dimer cover is always strictly greater than zero.

Also note that for \(n = 2\) the theorem gives a new and different proof of (a slightly more general version of) Theorem 2 of [Ken14].

For our definition of \(\tilde{K}\) the sign ambiguity is inevitable when \(n\) is odd since the sign of \(\det \tilde{K}\) depends on an arbitrary choice of order for both white vertices and black vertices, and a choice of Kasteleyn signs.

**Proof.** We assume for notational simplicity that \(\mathcal{G}\) is simple. If \(\mathcal{G}\) has multiple edges between pairs of vertices, a slight variation of the following proof will hold.

Let \(\mathcal{G}_n\) be the graph obtained from \(\mathcal{G}\) by replacing each vertex \(v\) with \(n\) copies \(v^1, \ldots, v^n\), and replacing each edge \(bw\) with the complete bipartite graph \(K_{n,n}\) connecting the \(b^j\) and \(w^i\). See Figure 5 below. For edge \(e = bw\) of \(\mathcal{G}\), let \(A_{bw} = \varepsilon_{wb} \phi_{bw}\) be the parallel transport \(\phi_{bw}\) times the Kasteleyn sign \(\varepsilon_{wb}\). For
Figure 5: Example in the case $n = 3$. The graph $\mathcal{G}$ is equipped with a 3-multiweb $m$ (bottom), and one of many possible lifts of $m$ to $\mathcal{G}_n$ is chosen (top). Note that only part of $\mathcal{G}$ is shown. There are two possible lifts of $m$ over this part of the graph; the other lift connects $G$ to $R$, and $B$ to $G$, over the doubled edge.

$i, j \in C$ put weight $A^{ij}_{bw} := (A_{bw})_{ij}$ on the edge $b^i w^j$ of $\mathcal{G}_n$ lying over $e$. If any entry $A^{ij}_{bw}$ is zero, we remove that edge from $\mathcal{G}_n$.

Now when expanded over the symmetric group $S_{nN}$, nonzero terms in $\det \tilde{K}$ are in bijection with single-dimer covers $\sigma$ of $\mathcal{G}_n$: a single dimer cover $\sigma$ is a bijection from black vertices $b^i$ to adjacent white vertices $w^j$, and has “weight” $(\tilde{K})_{\sigma} := (-1)^{\sigma} \prod (A_{bw})_{ij}$ where the product is over dimers in the cover.

Each single-dimer cover $\sigma$ of $\mathcal{G}_n$ projects to an $n$-multiweb $m$ of $\mathcal{G}$. We group all single dimer covers of $\mathcal{G}_n$ according to their corresponding $n$-multiweb $m$. That is

$$\det \tilde{K} = \sum_{m \in \Omega_n(\mathcal{G})} \sum_{\sigma \in m} (\tilde{K})_{\sigma}.$$ 

We claim that the interior sum is, up to a global sign, $Tr_L(m)$; this will complete the proof. That is, we need to prove, for a constant sign $s$ (independent of $m$) and for $m \in \Omega_n(\mathcal{G})$,

$$sTr_L(m) = \sum_{\sigma \in m} (-1)^{\sigma} \prod_{e = \tilde{b} \tilde{w}} (A_{bw})_{\tilde{w} \tilde{b}},$$

where the product is over edges of the dimer cover $\sigma$ of $\mathcal{G}_n$, or in other words, $\tilde{b} = \sigma(\tilde{w})$. In this product, $\tilde{w} \tilde{b} \in \mathcal{G}_n$ lies over an edge $wb$ of the multiweb $m$; the vertex $\tilde{w}$ corresponds to a choice of color of the half edge at $w$ and $\tilde{b}$ corresponds to a choice of color of the half edge at $b$. 

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Now we group the \( \sigma \in S_{nN} \) according to colorings of the edges of \( \mathcal{G} \). An edge \( e = wb \) of \( \mathcal{G} \) of multiplicity \( k \) is colored by two sets \( S_e, T_e \) of colors, both of size \( k \), where \( S_e \) is associated to the white vertex and \( T_e \) to the black vertex. There are then \( k! \) corresponding dimer covers of \( \mathcal{G}_n \) lying over that edge, one for each bijection \( \pi_e \) from \( S_e \) to \( T_e \). We group the \( \sigma \) into colorings \( c \) with the same sets of colors \( S_e, T_e \) on each edge. Each permutation \( \sigma \) corresponds to a choice of such a coloring \( c \) of \( m \) and a choice, for each edge \( e \) of multiplicity \( k \), of a bijection \( \pi_e \) between the sets \( S_e \) and \( T_e \). After this grouping we can write the RHS of (8) as a sum over colorings \( c \) of \( m \):

\[
= \sum_c \sum_{\sigma \in c} (-1)^\sigma \prod_{e=wb} (\phi_{bw})_{\tilde{w}_e \tilde{b}_e} = \sum_c \left( \prod \varepsilon_{\tilde{w}_e \tilde{b}_e} \right) \sum_{\sigma \in c} (-1)^\sigma \prod_{e=wb} (\phi_{bw})_{\tilde{w}_e \tilde{b}_e}
\]

where we used the fact that, once the multiplicities are fixed, \( \prod \varepsilon_{\tilde{w}_e \tilde{b}_e} \) is independent of \( \sigma \). Now \( \sigma \in S_{nN} \) is a composition \( \sigma = (\prod \pi_e)\sigma_0 \) of a permutation \( \sigma_0 \) (which depends only on \( c \), and is the permutation matching each element of each \( S_e \) with the corresponding element of \( T_e \) when both sets are taken in their natural order) and the individual \( \pi_e \). More precisely, we should write \( \sigma = (\prod_e \pi'_e)\sigma_0 \) where \( \pi'_e \) is the bijection from \( S_e \) to \( S'_e \) which, when composed with the natural-order bijection from \( S_e \) to \( T_e \), gives \( \pi_e \). Thus

\[
= \sum_c \left( \prod \varepsilon_{\tilde{w}_e \tilde{b}_e} \right) (-1)^{\sigma_0} \prod_{e=wb} \pi'_e \prod_{s \in S_e} (\phi_{bw})_{s, \pi'_e(s)} = \sum_c \left( \prod \varepsilon_{\tilde{w}_e \tilde{b}_e} \right) (-1)^{\sigma_0} \prod_{e=wb} \det(\phi_{bw})_{S_e, T_e}.
\]

(9)

Recalling (see Proposition 3.2) the definition of trace of a multiweb we have

\[
\text{Tr}_L(m) = \sum_c (-1)^{\sum_v c(v)} \prod_{e=wb} \det(\phi_{bw})_{S_e, T_e}
\]

(10)

where the sum is over colorings \( c \) of the half-edges, and the product is over edges of \( m \).

There is a one-to-one correspondence between the terms of (9) and those of (10); it remains to compare their signs. Let us take a pure dimer cover of \( \mathcal{G}_n \): a dimer covering which matches like colors. Then each \( \pi_e \) is the identity map, and \( \sigma = \sigma_0 \). This dimer cover projects to an edge-\( n \)-coloring of \( \mathcal{G} \). By Kasteleyn’s theorem, \( (-1)^{\sigma_0} \prod \varepsilon_{\tilde{w}_e \tilde{b}_e} \) is constant, that is, the signature \( (-1)^\sigma = (-1)^{\sigma_0} \) of a pure dimer cover exactly cancels the product \( \prod \varepsilon_{\tilde{w}_e \tilde{b}_e} \) of the Kasteleyn signs
(up to a global sign choice). Likewise \((-1)^{\sum v \cdot c(v)}\) is a constant for edge-$n$-colorings, by Proposition 3.4. So we just need to compare the sign of an arbitrary coloring with an edge-$n$-coloring. Suppose we change a coloring by transposing two colors at a single, without loss of generality white, vertex $w$. For the purposes of computing the sign change we can assume $m$ is simple: all multiplicities are 1, by splitting multiple edges into parallel edges. Now in this case under a transposition of colors at $w$ both $\sigma_0$ and $c(w)$ change sign, so both (9) and (10) change sign. Since transpositions at vertices connect the set of all colorings we see that (9) and (10) agree up to a global sign for all colorings.

A small modification of Theorem 4.1 allows us to put positive real edge weights $\{\nu_e\}_{e \in E}$ on the edges of $G$: we simply multiply the entries in $K$ by the corresponding weight $\nu_e$. Then

$$Z_{nd}(\Phi, \nu) := \sum_{m \in \Omega_n(G)} \left( \prod_{e \in m} \nu_e \right) \text{Tr}_L(m) = \pm \det \tilde{K}.$$ 

This allows us to compute $\text{Tr}_L(m)$ in practice for any particular web $m$ as follows. Put variable edge weights $x_e$ on edges of $m$, and extract the coefficient of $\prod_{e \in m} x_e$ from $\det \tilde{K}$:

$$\text{Tr}_L(m) = \pm \left[ \prod_{e \in m} x_e \right] \det \tilde{K}. \quad (11)$$

## 5 Reductions for SL₃

The trace of a large web is mysterious and hard to compute; the method outlined at the end of the previous section is only an exponential-time (in the size of the web) algorithm. While we cannot improve generally on the exponential nature of this computation, we can make it conceptually simpler in the case of SL₃ by applying certain reductions, or skein relations, as described in [Jae92, Kup96], which simplify its computation. (Skein relations also exist for SLₙ for $n > 3$ [Sik01], but it is not clear how to use them to “reduce” a web, as in Section 5.2 below.)

Throughout this section, we assume the (not necessarily planar) graph $G$ is embedded in a surface $\Sigma$.

### 5.1 Skein relations

The skein relations we consider are relations between formal linear combinations of webs. They correspond to and arise from algebraic relations between
Figure 6: Basic skein relation for SL$_3$. Note that we write composition of parallel transports in the natural order, that is, in the composition order, orienting edges from black to white. Each term is part of a larger, possibly nonplanar, web.

the traces of webs. They also work on the level of multiwebs, see Section 5.1.1 below.

Let us start with any SL$_3$-connection $\Phi$ on the graph $G$. The basic skein relation involves replacing two adjacent vertices of a web (adjacent via an edge of multiplicity 1), and their adjacent edges, with either two parallel edges or two crossing edges as shown in Figure 6. This replaces a single web with a linear combination of two webs, one of which is locally nonplanar. The trace of the first web is the signed sum of the traces of the resulting two webs. We have drawn the skein relation along with the parallel transports on its edges, and indicated how they transform under the relation. In our applications however we will only apply the skein relation when the parallel transports on the relevant edges are the identity. (Note that we can locally trivialize the connection on these edges, since the relevant subgraph is contractible).

Using only this basic relation, each web can be reduced to a linear combination of (possibly nonplanar) unions of loops. This shows that the trace of a 3-web is a linear combination of traces of matrices formed from certain loops running through it.

The type-I skein relation, for a connection which is (locally) trivial around a loop, removes the loop and multiplies the coefficient of the web by 3, as shown in Figure 7.

The type-II skein relation, for a connection which is (locally) trivial around a bigon, is depicted in Figure 8 and is a consequence of the basic skein relation. It removes a degree-2 face as shown, and replaces it with a single edge, and multiplying the coefficient of the web by 2. Note this relation preserves planarity.

The type-III skein relation is a relation for a connection which is (locally) trivial around a square. It is shown in Figure 9; this is also a consequence of the basic relation. Note that this relation also preserves planarity.

When the connection is trivial, these skein relations also work at the level of colorings, see Figure 10. That is, there is a bijection between “before”
colorings and “after” colorings. These skein relations are special to $\text{SL}_3$-connections. For instance, one does not obtain a relation for $M_3$-connections by using $C \wedge C$ in place of $C^{-1}$ in Figure 6, in contrast to the phenomenon we saw in Section 3.3.2. This is analogous to the $n = 2$ case, where the Kauffman bracket skein relation corresponds (after twisting by signs) to the classical trace relation $\text{Tr}(A)\text{Tr}(B) = \text{Tr}(AB) + \text{Tr}(AB^{-1})$ specific to matrices $A, B \in \text{SL}_2$.

5.1.1 Multiweb skein relations

We can also define a notion of skein relation for multiwebs in a fixed graph $\mathcal{G}$. A multiweb skein relation is an operation that takes a 3-multiweb $m$ to a formal linear combination of other 3-multiwebs in $\mathcal{G}$.

The type-I, loop-removal, skein relation has a multiweb version which replaces a loop (consisting in an alternating sequence of single and double edges) with a sequence of tripled edges, by increasing the multiplicity of the doubled edges and decreasing the multiplicity of the single edges along the loop; the resulting web has coefficient multiplied by 3.

The skein relations of type-II and type-III above also have multiweb versions as shown in Figure 11,12. For the type-II, we take two vertices which have

![Figure 7: Type-I: loop removal (locally trivial connection).](image)

![Figure 8: Type-II: bigon removal (locally trivial connection).](image)

![Figure 9: Type-III: square removal (locally trivial connection).](image)
Figure 10: The skein relations for edge colorings: if all four diagonal edges of a square are the same color (color 2 in this illustration), there are two ways to color the edges of the central square face with the two remaining colors, and likewise two possible reductions on the right-hand side. If the diagonal edges have two colors, the like colors must be adjacent, as shown; then there is only one way to complete the coloring on the edges of the central square face, and one reduction on the right-hand side. A 2-gon can be colored in two ways, and can be replaced with a single edge (joining its neighboring edges into a single edge as shown) with a factor 2. Finally, a doubled edge can be removed, joining its neighboring edges into a single edge as shown.

Figure 11: Removing a bigon, and replacing with a path in two different ways.
two disjoint paths joining them as shown, not necessarily of the same length, and so that this bigon is topologically trivial. We replace this “bigon” by the two webs shown: each is obtained by increasing the multiplicity on every other edge by 1 and decreasing the multiplicity on the remaining edges. For type-III, we have four vertices $a, b, c, d$ connected into a topologically trivial cycle by four paths (which may have lengths larger than 1, unlike as shown). This is replaced by two webs, each again obtained by increasing the multiplicity of every other edge of the cycle by 1 and decreasing the multiplicity of the other edges of the cycle.

5.2 Reduced webs

A 3-multiweb $m'$ in $\mathcal{G} \subset \Sigma$ is said to be reduced (or nonelliptic) if it has no topologically trivial loops, bigons or quadrilateral faces.

Given an unreduced multiweb $m$ in $\mathcal{G}$, we can apply reductions to $m$ to reduce it to a formal linear combination of reduced multiwebs: $m \mapsto \sum c'm'$. Such a reduction is not unique, due to the type-III skein relation; indeed, it is easy to produce examples of $m$ having more than one square, where different choices of sequences of resolutions of the squares give different end “states” $\sum c'm'$.

For a reduced web $m'$ we denote by $[m']$ its isotopy type as a map from an abstract trivalent graph into $\Sigma$, where the image is reduced as a subset of $\Sigma$. Let $\Lambda_3$ denote the set of isotopy classes of reduced webs $[m']$ coming from multiwebs $m$ in $\mathcal{G}$. It turns out [Jae92, Kup96] that, although the end states depend on the sequence of resolutions, the formal linear combination of isotopy classes $\sum c'[m']$ does not. Thus we can say that the collection of isotopy classes of multiwebs in $m$ is canonical.

Note that if the entire multiweb $m$ is topologically trivial on $\Sigma$, for example if $m$ consists only of tripled edges and contractible loops, then its reduction will be a positive integer times the isotopy class $[\emptyset] \in \Lambda_3$ of the “empty” web consisting only in tripled edges.

At the level of traces, let us assume $\mathcal{G}$ is equipped with a flat $\text{SL}_3$-connection.

Figure 12: Removing a square.
Φ, in the sense of Section 2.1. Then, by the flatness property, we see that this reduction of a multiweb \( m \) into reduced multiwebs applies as well at the level of traces with respect to \( \Phi \). We gather that

\[
\text{Tr}(m) = \sum_\lambda C_{m,\lambda} \text{Tr}(\lambda)
\]

where \( C_{m,\lambda} \) is the number of reduced isotopy classes \( \lambda \) “contained” in \( m \), that is, resulting from the reduction of \( m \).

Now, for \( \mathcal{G} \) planar, Theorem 4.1 writes the determinant \( \det \hat{K} \) as a sum over unreduced multiwebs \( m \). We can further reduce each multiweb to write \( \det \hat{K} \) as a sum of traces of isotopy classes of reduced multiwebs:

**Theorem 5.1.** For a graph \( \mathcal{G} \) embedded in a genus-0 surface minus \( k \) disjoint open disks, \( k \geq 0 \), with a flat \( \text{SL}_3 \) local system, we have

\[
\pm \det \hat{K} = \sum_{\lambda \in \Lambda_3} C_\lambda \text{Tr}(\lambda)
\]

where the sum is over isotopy classes of reduced 3-webs \( \lambda \) of \( \mathcal{G} \), and the \( C_\lambda \) are positive integers.

The most interesting cases are for \( k \geq 2 \). Indeed, when \( k = 0, 1 \) then \( \Lambda_3 \) consists only of the isotopy class of the empty web, \([\emptyset]\).

We describe how to extract the coefficients \( C_\lambda \) in some simple cases, for \( k = 2 \) and \( k = 3 \), in the next section.

Due to the dependence on choices when resolving multiwebs, we do not have (at present) a canonical probability measure \( \mu_3 \) on the set of reduced webs in \( \mathcal{G} \). However, reduction does induce a probability measure \( \mu_3' \) on the set of isotopy classes of reduced webs, \( \Lambda_3 \). It is defined by (recalling (1))

\[
\text{Pr}(\lambda) = \frac{C_\lambda}{Z_3(I)}.
\]

This is in contrast to the case of \( \text{SL}_2 \), where \( \mu_2 \) is defined on the set of 2-multiwebs (loops and doubled edges) in \( \mathcal{G} \). Here, the only planar \( \text{SL}_2 \) skein relation resolves a topologically trivial loop, in two different ways, into a sequence of doubled edges.

### 6 Multiwebs on simple surfaces

We continue studying the case \( n = 3 \). Sikora and Westbury [SW07, Theorem 9.5] showed that reduced webs form a basis for the “\( \text{SL}_3 \)-skein algebra” for any
surface. That is, using skein relations any web can be reduced to a unique linear combination of non-elliptic webs. Thus, in the classical setting [Sik01, Theorem 3.7], the traces for nonelliptic webs form a basis for the algebra of invariant regular functions on the space of flat $\text{SL}_3$-connections modulo gauge (the “$\text{SL}_3$-character variety”).

6.1 Annulus

We consider here the case where the graph $\mathcal{G}$ is embedded on an annulus. The result of [SW07] for the annulus can be stated as follows.

**Proposition 6.1.** By use of skein relations 2 and 3, any web on an annulus with a flat $\text{SL}_3$-local system can be reduced to a unique positive integer linear combination of collections of disjoint noncontractible oriented cycles.

**Proof.** For a bipartite connected trivalent graph on a sphere we have (by Euler characteristic)

$$6 = 2n_2 + n_4 - 1n_8 + \cdots + (3-k)n_{2k} + \ldots$$

(12)

where $n_k$ is the number of faces of degree $k$. When $\mathcal{G}$ is embedded on an annulus at most two faces contain boundary components, so there is at least one contractible degree-2 face or 2 contractible degree-4 faces. We first perform type-2 skein relations to remove any contractible degree-2 faces. Then there remain at least 2 faces of degree 4, upon which we can perform a type-3 skein reduction. This reduces the number of vertices in $\mathcal{G}$ and possibly disconnects $\mathcal{G}$; continue with each component until each component is a loop. \[\square\]

Let $W_{j,k}$ be the set of isotopy classes of reduced webs on $\Sigma$ with $j$ loops of homology class $+1$ and $k$ loops of homology class $-1$. (We orient a 3-multiweb which is a loop so that the single edges are oriented from black to white, and the doubled edges are oriented from white to black.)

Suppose $\Phi$ is a flat connection with monodromy $A \in \text{SL}_3$ around the generator of $\pi_1$. A noncontractible simple loop has trace $\text{Tr}(A)$ or $\text{Tr}(A^{-1})$ depending on orientation. For a general web $m$, by Theorem 6.1, $\text{Tr}(m)$ will be a polynomial in $\text{Tr}(A)$ and $\text{Tr}(A^{-1})$ whose coefficients are nonnegative integers:

$$\text{Tr}(m) = \sum_{j,k \geq 0} M_{j,k} \text{Tr}(A)^j \text{Tr}(A^{-1})^k$$

where $M_{j,k}$ counts reduced subwebs in $W_{j,k}$.

For example for the web of Figure 13 the trace is

$$\text{Tr}(m) = 15 + \text{Tr}(A)\text{Tr}(A^{-1}).$$
Figure 13: A web on an annulus with flat connection having monodromy $A$. We can for example put parallel transports $A, A^{-1}$ as marked and $I$ on the remaining edges.

By Theorem 5.1 for a graph $\mathcal{G}$ on an annulus we have

$$\det \tilde{K}(A) = \sum_{j,k \geq 0} C_{j,k} \operatorname{Tr}(A)^j \operatorname{Tr}(A^{-1})^k.$$

We can compute $C_{j,k}$ concretely as follows. Assume $A$ has eigenvalues $x, y, z$ (with $xyz = 1$). Let $u = \operatorname{Tr}(A), v = \operatorname{Tr}(A^{-1})$. Then $x, y, z$ are roots of

$$x^3 - ux^2 + vx - 1 = 0.$$

We can write

$$\det \tilde{K} = \det K(x) \det K(y) \det K(z) \quad (13)$$

where the $K(\cdot)$ are the corresponding scalar matrices. This is a symmetric polynomial of $x, y, z$ and so can be written as a polynomial in $u, v$.

As a concrete example, take a $2m \times n$ square grid $\mathcal{G}_{2m,n}$ on an annulus (with circumference $2m$), see Figure 15. By Proposition 7.1 in the appendix, for $n$ even,

$$\det K_{2m,n}(x) = \pm \prod_{j=1}^{n/2} \frac{(x + \alpha_j^m)(x + \alpha_j^{-m})}{x}$$

where $\alpha_j = -\cos \theta + \sqrt{1 + \cos^2 \theta}$ and $\theta = \frac{\pi j}{n+1}$. Note $(x + r)(y + r)(z + r) =$
1 + ur + vr^2 + r^3. Hence using (13),

$$\det \tilde{K} = \pm \prod_{j=1}^{n/2} (1 + u\alpha_j^{2m} + v\alpha_j^{4m} + \alpha_j^{6m})(1 + u\alpha_j^{-2m} + v\alpha_j^{-4m} + \alpha_j^{-6m}).$$  \hspace{1cm} (14)

Writing $$\det \tilde{K}(u,v) = \det \tilde{K}(3,3)$$, we can interpret (14) as the probability generating function for an \(n\)-step random walk in \(\mathbb{Z}^2\) starting from \((0,0)\):

$$(X,Y) = (X_1,Y_1) + (X_1,Y_1) + (X_2,Y_2) + \cdots + (X_{n/2},Y_{n/2})$$

where for \(j > 0\) the step \((X_j,Y_j)\) is \((0,0), (1,0), (0,1)\) with probabilities respectively

$$\frac{1 + \alpha_j^{6m}}{1 + 3\alpha_j^{2m} + 3\alpha_j^{4m} + \alpha_j^{6m}}, \frac{3\alpha_j^{2m}}{1 + 3\alpha_j^{2m} + 3\alpha_j^{4m} + \alpha_j^{6m}}, \frac{3\alpha_j^{4m}}{1 + 3\alpha_j^{2m} + 3\alpha_j^{4m} + \alpha_j^{6m}}$$

and a similar formula for \(j < 0\) (with the \((1,0)\) and \((0,1)\) probabilities reversed).

We have

$$\bar{X} = \bar{Y} = \sum_{j=1}^{n/2} \frac{3\alpha_j^{2m} + 3\alpha_j^{4m}}{1 + 3\alpha_j^{2m} + 3\alpha_j^{4m} + \alpha_j^{6m}} = \sum_{j=1}^{n/2} \frac{3\alpha_j^{2m}}{(1 + \alpha_j^{2m})^2}.$$

Now let \(m,n\) tend to \(\infty\) with \(m/n \to \tau\). Let us estimate \(\bar{X}, \bar{Y}\) for \(m,n\) large. We only get a nontrivial contribution if \(|\alpha_j| \approx 1\), that is, when \(j \approx n/2\). We can write (changing \(j\) to \(n/2 - j\))

$$\alpha_j = 1 + \frac{\pi(j + \frac{1}{2})}{n} + O(\frac{j^2}{n^2})$$

and so

$$\alpha_j^{2m} = q^{-2j-1}(1 + o(1))$$

where \(q = e^{-\pi\tau}\). We thus have, up to errors tending to zero as \(m,n \to \infty\),

$$\bar{X} = \bar{Y} = \sum_{j=0}^{\infty} \frac{3q^{2j+1}}{(1 + q^{2j+1})^2}.$$

Now suppose \(\tau\) is large: we have a long thin annulus. Then \(q\) is small. Then from (14) we have

$$\frac{\det \tilde{K}(u,v)}{\det \tilde{K}(3,3)} = 1 + o(1).$$

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With high probability there are zero crossings. The probability of a \((j,k)\)-crossing for \(j + k \geq 0\) is to leading order \(q^{\delta_{j,k}3^{j+k}}\) where the “crossing exponent” is 
\[
\delta_{j,k} = \lceil 2( j^2 + jk + k^2 )/3 \rceil.
\]
This can be seen by expanding (14) and extracting the appropriate term to leading order. (see the calculation in the appendix, Section 8)

A similar computation can be done for a \(n\)-multiweb on an annulus, for \(n > 3\), since [CD] shows that a \(n\)-multiweb on an annulus can be reduced to a collection of loops (of \(n - 1\) types: with multiplicities \((k, n - k)\) for \(k = 1, \ldots, n - 1\)).

### 6.2 Pair of pants

Let \(\Sigma\) be a pair of pants, that is, a sphere with three holes. Let \(\omega = e^{\pi i/3}\) and let \(\eta \in \mathbb{C}\) have the form \(\eta = a + b\omega\) where \(a, b \in \mathbb{Z}_{\geq 0}\). We construct a reduced web \(W_\eta\) on \(\Sigma\) as follows; see Figure 14. We take the rhombus with sides \(\eta \omega^{-1}\) and \(\eta \omega\) in \(\mathbb{C}\), and glue sides as shown, putting punctures at the corners, to form a 3-punctured sphere \(\Sigma\). The image of the standard honeycomb graph (dual to the regular triangulation of \(\mathbb{C}\)) descends to a reduced web on \(\Sigma\).

**Proposition 6.2.** A reduced web on a pair of pants is a union of a collection of loops and at most one of the webs \(W_\eta\).
Proof. If the web has a bigon or quad face not containing a boundary component, it is not reduced. If any boundary face is not a bigon, there must be (by (12)) a non-boundary quad face, so it is not reduced. So if reduced, all boundary faces are bigons, and all other faces are hexagons, again by (12).

The dual of such a web is a triangulation with all vertices of degree 6 except for three vertices of degree 2. Geometrically (replacing triangles with equilateral triangles) it is a $(3,3,3)$-orbifold. Such a space has a 3-fold branched cover (over the vertices of degree 2) which is an equilateral torus. It is a quotient of the regular hexagonal triangulation by a hexagonal sublattice. So these triangulations are indexed by Eisenstein integers \( \eta = a + be^{\pi i/3} \), where \( a, b \in \mathbb{Z}_{\geq 0} \), and \( a^2 + ab + b^2 \) is the number of white vertices (or black vertices).

Note that there are two possible orientations of each \( W_\eta \) (obtained from switching the colors), the other denoted \( W^*_\eta \), but at most one can occur in any reduced web.

Letting \( A, B, C = AB^{-1} \) be the monodromies around the punctures of a flat connection on \( \Sigma \), we have

\[
\det \tilde{K} = \sum_{i_1,i_2,j_1,j_2,k_1,k_2,\eta} C_i \text{Tr}(A)^{i_1} \text{Tr}(A^{-1})^{i_2} \text{Tr}(B)^{j_1} \text{Tr}(B^{-1})^{j_2} \text{Tr}(C)^{k_1} \text{Tr}(C^{-1})^{k_2} \text{Tr}(W_\eta)
\]

for some integers \( C_i = C_{i_1,i_2,j_1,j_2,k_1,k_2,\eta} \).

While extracting the coefficients \( C_i \) can be done in principle (by the result of Sikora-Westbury mentioned at the beginning of this section), in practice it is not easy. Let us only consider one simplified situation.

Suppose a bipartite graph \( G \) is embedded on \( \Sigma \), and two of the three boundary components of the pants are in adjacent faces of \( G \). Then subwebs of \( G \) of type \( W_\eta \) can only occur if \( \eta = 1 \), that is, except for loop components a reduced web \( W \) in \( G \) can only be a \( W_1 \) (or a \( W^*_1 \), depending on the orientation of the edge between the adjacent faces); such a web is homeomorphic to a theta graph, see Section 3.3.1.

For the identity connection we then have \( \det \tilde{K} = Z_0 + Z_1 \text{Tr}(W_1) \), where \( Z_0 \) is the weighted sum of multiwebs not containing \( W_1 \) in their reduction, and \( Z_1 \) is the weighted sum of reduced subwebs containing a component of type \( W_1 \). We can compute \( Z_0, Z_1 \) as follows.

Suppose we impose a flat connection with monodromy \( A, B \) around the generators of \( \pi_1 \), where \( A, B \) are chosen so that traces of simple loops are 3 and the trace of a \( W_1 \) is \( x \). For example \( A = \begin{pmatrix} 1 & a & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -a & -a^2 & 1 \end{pmatrix} \)

so that \( \text{Tr}(AB) = 3 - a^2 \).

Note that \( \text{Tr}(W_1) = \text{Tr}(A)\text{Tr}(B) - \text{Tr}(AB) = 6 + a^2 =: x \). So \( Z_1 \) is the coefficient of \( a^2 \) in \( \det \tilde{K} \).
Figure 15: $G_{6,4}$ is obtained by gluing the left and right sides of this figure. To make a Kasteleyn matrix with flat connection having monodromy $z$ around the annulus, take $\zeta$ such that $\zeta^{2m} = z$ and put connection $\zeta$ on all east-going edges (and 1 on vertical edges).

7 Appendix: annulus determinant

Proposition 7.1. For the grid graph $G_{2m,n}$ on an annulus as in Figure 15 with $n$ even and $m$ odd we have

$$\det K_{2m,n}(z) = \pm \prod_{k=1}^{n/2} \frac{(z + \alpha_k^{2m})(z + \alpha_k^{-2m})}{z}$$

where $\alpha_k = -\cos \theta + \sqrt{1 + \cos^2 \theta}$ with $\theta = \frac{\pi k}{n+1}$. If $n$ is odd and $m$ is odd the result is

$$\det K_{2m,n}(z) = \pm \left(\sqrt{z} + \frac{1}{\sqrt{z}}\right)^{(n-1)/2} \prod_{k=1}^{(n-1)/2} \frac{(z + \alpha_k^{2m})(z + \alpha_k^{-2m})}{z}$$

with $\alpha$ as above.

Proof. Recall that the Kasteleyn matrix $K$ has rows indexing white vertices and columns indexing black vertices. We consider here the large Kasteleyn matrix $\hat{K}$, with rows (and columns) indexing all vertices, both black and white. We have $\hat{K}(\zeta) = \begin{pmatrix} 0 & K(\zeta) \\ K^t(1/\zeta) & 0 \end{pmatrix}$ and, by symmetry, $\det K(\zeta) = \det K(1/\zeta)$ so $\det K(\zeta) = \pm \det \hat{K}(\zeta)^{1/2}$. (The sign depends on choice of gauge and vertex order.)

We put Kasteleyn “signs” $i = \sqrt{-1}$ on vertical edges and 1 on horizontal edges, as in [Ken09].
Indexing vertices by their \(x,y\)-coordinates, where \((x,y) \in [0, 2m-1] \times [1, n]\), the eigenvectors of \(K\) are

\[
f_{k,j}(x,y) = e^{2\pi ijx/(2m)} \sin \frac{\pi ky}{n+1}
\]

where \(j \in \{0, \ldots, 2m-1\}\) and \(k \in \{1, 2, \ldots, n\}\).

The corresponding eigenvalues are

\[
\lambda_{j,k} = \zeta e^{2\pi ij/(2m)} + \zeta^{-1} e^{-2\pi ij/(2m)} + 2i \cos \frac{\pi k}{n+1}.
\]

Thus

\[
det K = \pm \left[ \prod_{j=0}^{2m-1} \prod_{k=1}^{n} \zeta e^{2\pi ij/(2m)} + \zeta^{-1} e^{-2\pi ij/(2m)} + 2i \cos \frac{\pi k}{n+1} \right]^{1/2}
\]

\[
= \pm \left[ \prod_{k=1}^{n} \prod_{\zeta^{2m} = z} \zeta + \zeta^{-1} + 2i \cos \frac{\pi k}{n+1} \right]^{1/2}
\]

\[
= \pm \left[ \prod_{k=1}^{n} \prod_{\zeta^{2m} = z} \frac{(\zeta - \beta_k)(\zeta - \gamma_k)}{\zeta} \right]^{1/2}
\]

\[
= \pm \left[ \prod_{k=1}^{n} \frac{(z - \beta_k^{2m})(z - \gamma_k^{2m})}{z} \right]^{1/2}
\]

where \(\beta_k, \gamma_k\) are roots \(\zeta\) of \(\zeta + \zeta^{-1} + 2i \cos \frac{\pi k}{n+1}\), that is, \(\beta_k, \gamma_k = i(-\cos \theta \pm \sqrt{1 + \cos^2 \theta})\) with \(\theta = \frac{\pi k}{n+1}\). If \(n\) is even we can pair the \(k\) and \(n+1-k\) terms which are identical, to get

\[
det K = \pm \prod_{k=1}^{n/2} \frac{(z - \beta_k^{2m})(z - \gamma_k^{2m})}{z}.
\]

If \(n\) is odd the \(k = (n+1)/2\) term is \(-(z + 1/z + 2)\), yielding

\[
det K = \pm (\sqrt{z} + 1) \prod_{k=1}^{(n-1)/2} \frac{(z - \beta_k^{2m})(z - \gamma_k^{2m})}{z}.
\]

Letting \(\alpha_k = -i\beta_k\) (and noting that \(\gamma_k = \beta_k^{-1}\), and \(m\) is odd) gives the result. \(\square\)
8 Appendix: coefficient extraction

We compute the leading-order term of the coefficient of \( u^j v^k \) in the expansion

\[
\prod_{i=0}^{\infty} (1 + q^{2i+1}u + q^{4i+2}v) (1 + q^{2i+1}v + q^{4i+2}u).
\]

To accomplish this, we need to take the “\( u \)” term from \( j \) factors and the “\( v \)” term from \( k \) factors. Let \( A = \prod_{i=0}^{\infty} (1 + q^{2i+1}u + q^{4i+2}v) \) and \( B = \prod_{i=0}^{\infty} (1 + q^{2i+1}v + q^{4i+2}u) \). From \( A \) we take \( \ell_1 \) of the \( u \) terms, with leading-order coefficients \( q^x \) for \( x \in L_1 \) for some subset \( L_1 \subset \{1, 3, 5, \ldots \} \) of cardinality \( \ell_1 \). Likewise from \( B \) we take \( \ell_2 = j - \ell_1 \) of the \( u \) terms, with coefficients \( q^{2x} \) for \( x \in L_2 \) for some subset \( L_2 \subset \{1, 3, 5, \ldots \} \) of cardinality \( \ell_2 \). Likewise we take \( m_1 \) of the \( v \) terms from \( A \) and \( m_2 = k - m_1 \) of the \( v \) terms from \( B \), corresponding to subsets \( M_1, M_2 \subset \{1, 3, 5, \ldots \} \) of cardinalities \( m_1, m_2 \), and the corresponding coefficients are to leading order \( q^{2x} \) for \( x \in M_1 \) and \( q^x \) for \( x \in M_2 \). We require \( L_1 \cap M_1 = \emptyset = L_2 \cap M_2 \).

Let \( L_1 = \sum_{i \in L_1} i \) and likewise define \( L_2, M_1, M_2 \). We need to make these choices to minimize the exponent of the leading-order term of \( u^j v^k \) which is \( L_1 + 2L_2 + 2M_1 + 2M_2 \). Note that

\[
L_1 + M_1 = 1 + 3 + 5 + \cdots + (2\ell_1 + 2m_1 - 1) = (\ell_1 + m_1)^2
\]

and

\[
L_2 + M_2 = 1 + 3 + 5 + \cdots + (2\ell_2 + 2m_2 - 1) = (\ell_2 + m_2)^2 = (j + k - \ell_1 - m_1)^2.
\]

So

\[
L_1 + 2L_2 + 2M_1 + 2M_2 = (\ell_1 + m_1)^2 + (j + k - \ell_1 - m_1)^2 + L_2 + M_1.
\]

The first two terms here are independent of the choices of individual terms (just depending on \( \ell_1, m_1 \)), so we can just choose the individual terms to minimize \( L_2 \) and \( M_1 \) separately, that is, \( L_2 = \{1, 3, \ldots, (2j - 2\ell_1 - 1)\} \) and \( M_2 = \{1, 3, \ldots, 2m_1 - 1\} \), giving \( L_2 = (j - \ell_1)^2 \) and \( M_2 = m_1^2 \).

We are left with minimizing, for fixed \( j, k \),

\[
L_1 + 2L_2 + 2M_1 + 2M_2 = (\ell_1 + m_1)^2 + (j + k - \ell_1 - m_1)^2 + (j - \ell_1)^2 + m_1^2,
\]

subject to the constraints that \( \ell_1, m_1 \geq 0 \).

Suppose \( j \geq k \). Then the minimum for \( \ell_1 \geq 0 \) occurs when \( m_1 = 0 \), and we have (setting \( m_1 = 0 \) and differentiating with respect to \( \ell_1 \)) \( \ell_1 = \frac{2}{3}j + \frac{1}{3}k \), which yields the minimum exponent of \( \frac{2}{3}(j^2 + jk + k^2) \). This is the minimum for real \( \ell_1 \); taking into account the fact that the minimum must be an integer leads to the exponent \( \left\lfloor \frac{2}{3}(j^2 + jk + k^2) \right\rfloor \).

The argument for \( j < k \) is symmetric.
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