ON THE NEUMANN PROBLEM OF HARDY-SOBOLEV CRITICAL EQUATIONS WITH THE MULTIPLE SINGULARITIES

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Abstract. Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a $C^2$ bounded domain. We study the existence of positive solution $u \in H^1(\Omega)$ of
\[
\begin{cases}
-\Delta u + \lambda u = \frac{|u|^{2^*(s)-2}u}{|x-x_1|^s} + \tau \frac{|u|^{2^*(s)-2}u}{|x-x_2|^s} & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
\]
where $\tau = 1$ or $-1$, $0 < s < 2$, $2^*(s) = \frac{2(N-s)}{N-2}$ and $x_1, x_2 \in \partial \Omega$ with $x_1 \neq x_2$.
First, we show the existence of positive solutions to the equation provided the positive $\lambda$ is small enough. In case that one of the singularities locates on the boundary and the mean curvature of the boundary at this singularity is positive, the existence of positive solutions is obtained for any $\lambda > 0$ and some $s$ depending on $\tau$ and $N$. Furthermore, we extend the existence theory of solutions to the equations for the case of the multiple singularities.

1. Introduction. The Hardy-Sobolev inequality asserts that for all $u \in H^1_0(\mathbb{R}^N)$, there exists a positive constant $C = C(N, s)$ such that
\[
C \left( \int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}|}{|x|^s} \, dx \right)^{\frac{2^*}{2^*-1}} \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx
\]
where $N \geq 3$, $0 < s < 2$ and $2^*(s) = \frac{2(N-s)}{N-2}$. Suppose $\Omega \subset \mathbb{R}^N$, then the Hardy-Sobolev inequality holds for $u \in H^1_0(\Omega)$. The best constant of the Hardy-Sobolev inequality...
inequality is defined as

\[ S_s(\Omega) := \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{(\int_{\Omega} |u|^{2^*(s)} \, dx)^{\frac{s-2}{s}}} \]

It is easy to see, up to a scaling, that the minimizer for \( S_s(\Omega) \) is a least-energy solution of the Euler-Lagrangian equation:

\[
\begin{cases}
-\Delta u = \frac{|u|^{2^*(s)-2} u}{|x|^s}, & u > 0 \quad \text{in } \Omega \\
\quad u = 0 & \quad \text{on } \partial \Omega.
\end{cases}
\] (2)

When \( \Omega = \mathbb{R}^N \), \( S_s(\mathbb{R}^N) \) is attained by

\[ g_a(x) = (a(N-s)(N-2)) \frac{N-2}{s-2} \frac{a + |x|^{2-s}}{s-N}, \]

for some \( a > 0 \). Hence, in case \( 0 \in \Omega \), by a standard scaling invariance argument, it is easy to see \( S_s(\Omega) = S_s(\mathbb{R}^N) \) and \( S_s(\Omega) \) cannot be attained unless \( \Omega = \mathbb{R}^N \). However, if \( \Omega \) is \( C^2 \) smooth bounded domain with \( 0 \in \partial \Omega \), the existence of the minimizer for \( S_s(\Omega) \) is established under the assumption that the mean curvature of \( \partial \Omega \) at 0, \( H(0) \), is negative (see [5]).

Concerning the Dirichlet problem, the second author and his collaborators [9] showed the existence of solutions to the equation

\[-\Delta u = \lambda u \frac{N+2}{s} + \frac{|u|^{2^*(s)-2} u}{|x|^s}, u > 0 \quad \text{in } \Omega \]

for any \( \lambda > 0 \) and some \( s \) (depending on \( N \)) when \( \Omega \) is a \( C^2 \) bounded domain with \( 0 \in \partial \Omega \) and \( H(0) < 0 \). Furthermore, Li-Lin [10] proved the existence of the least energy solution to the equation involving two Hardy-Sobolev critical exponents

\[-\Delta u = \lambda \frac{u^{2^*(s_1)-1}}{|x|^{s_1}} + \frac{u^{2^*(s_2)-1}}{|x|^{s_2}}, u > 0 \quad \text{in } \Omega \]

for \( 0 < s_2 < s_1 < 2 \) and \( 0 \neq \lambda \in \mathbb{R} \) when \( \Omega \) is smooth bounded domain with \( 0 \in \partial \Omega \) and \( H(0) < 0 \). For interested readers, see also [1, 2, 13].

Regarding the Neumann problem

\[
\begin{cases}
-\Delta u + \lambda u = \frac{|u|^{2^*(s)-2} u}{|x|^s}, & u > 0 \quad \text{in } \Omega, \\
\quad \frac{\partial u}{\partial \nu} = 0 & \quad \text{on } \partial \Omega,
\end{cases}
\] (3)

we first notice that if \( \lambda \leq 0 \), then integration of (3) over \( \Omega \) gives

\[ 0 < \int_{\Omega} \frac{|u|^{2^*(s)-2} u}{|x|^s} \, dx = \int_{\Omega} -\Delta u + \lambda u \, dx \leq 0. \]

Hence, there does not exist a positive solution to (3). So, only the case where \( \lambda > 0 \) are addressed in literature. In this case, Ghossoub-Kang [4] showed that (3) has a positive solution if \( \Omega \) is \( C^2 \) bounded domain with \( 0 \in \partial \Omega \) and \( H(0) > 0 \). Furthermore, Chabrowski [3] investigated the solvability of the nonlinear Neumann problem with indefinite weight functions

\[-\Delta u + \lambda u = \frac{Q(x)|u|^{2^*(s)-2} u}{|x|^s}, u > 0 \quad \text{in } \Omega \]

when \( \Omega \) is \( C^2 \) bounded domain with \( 0 \in \partial \Omega \), and gives some sufficient condition on \( Q(x) \) to guarantee the existence of the solution to (4) provided \( H(0) \) is positive.
Recently, concerning the equation (3) the first author investigated the case when $\Omega$ is $C^2$ bounded domain with $0 \in \partial \Omega$ and $H(0) \leq 0$ in [8]. He showed the existence of $\lambda_*$ such that for $\lambda \in (0, \lambda_*)$, a least energy solution of (3) exists, and when $\lambda > \lambda_*$, a least energy solution does not exist. We remark that Zhong-Zou [17] studied the existence, non-existence and regularity of the positive solution to

$$-\Delta u = \frac{|u|^{2^*(\tau)-2}u}{|x-x_1|^\tau} + \tau \frac{|u|^{2^*(\tau)-2}u}{|x-x_2|^\tau} + u^{2^*-1} \text{ in } \mathbb{R}^N$$

for $0 < s_1 < s_2 < 2$.

The main results of this article are as follows

**Theorem 1.1** (Existence of solution to (4) for small positive $\lambda$). Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a $C^2$ bounded domain. Assume $\int_{\Omega} \frac{1}{|x-x_1|^\tau} + \tau \frac{1}{|x-x_2|^\tau} dx > 0$. Then there exists $\Lambda > 0$ such that the equation (4) has a positive solution provided the positive parameter $\lambda < \Lambda$.

**Theorem 1.2** (Existence of solution to (4) with the boundary singularity). Let $s$ a positive number satisfying $0 < s < 2$ for $\tau = 1$, $1 \leq s < 2$ for $\tau = -1$ when $N = 3$ and $0 < s < 2$ when $N \geq 4$. Assume $\int_{\Omega} \frac{1}{|x-x_1|^\tau} + \tau \frac{1}{|x-x_2|^\tau} dx > 0$. Suppose $x_1 \in \partial \Omega$ and the mean curvature of $\partial \Omega$ at $x_1$, $H(x_1)$ is positive. Then there exists a positive solution to (4).

To establish the existence theory, we study the functional

$$J_\lambda(u) = \int_{\Omega} \frac{1}{2}(|\nabla u|^2 + \lambda u^2) - \frac{1}{2^*(\tau)} \left( \frac{u_+^{2^*(\tau)}(s)}{|x-x_1|^s} + \frac{\lambda u_+^{2^*(\tau)}(s)}{|x-x_2|^s} \right) dx$$

defined on $H^1(\Omega)$ where $u_+ = \max(u,0)$. It is not hard to see that $J_\lambda$ is a $C^1$ functional and

$$\langle J_\lambda'(u), \phi \rangle = \int_{\Omega} \nabla u \nabla \phi + \lambda u \phi \left( \frac{u_+^{2^*(\tau)-1}(s)}{|x-x_1|^s} + \frac{\lambda u_+^{2^*(\tau)-1}(s)}{|x-x_2|^s} \right) dx$$

for $\phi \in H^1(\Omega)$. Moreover, by Sobolev embedding theorem, we obtain

- when $\tau = 1$

  $$J_\lambda(u) = \int_{\Omega} \frac{1}{2}(|\nabla u|^2 + \lambda u^2) - \frac{1}{2^*(\tau)} \left( \frac{u_+^{2^*(\tau)}(s)}{|x-x_1|^s} + \frac{\lambda u_+^{2^*(\tau)}(s)}{|x-x_2|^s} \right) dx \geq \int_{\Omega} \frac{1}{2}(|\nabla u|^2 + \lambda u^2) dx - \frac{C_0}{2^*(\tau)} \left( \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} \lambda u^2 dx \right)^{\frac{2^*(\tau)}{2}}.$$
• when $\tau = -1$

$$J_\lambda(u) = \int_\Omega \frac{1}{2} (|\nabla u|^2 + \lambda u^2) - \frac{1}{2s} \frac{u^{2^*(s)}}{|x - x_1|^s} + \frac{1}{2s} \frac{u^{2^*(s)}}{|x - x_2|^s} \, dx$$

$$\quad \geq \int_\Omega \frac{1}{2} (|\nabla u|^2 + \lambda u^2) - \frac{1}{2s} \frac{u^{2^*(s)}}{|x - x_1|^s} \, dx$$

$$\quad \geq \int_\Omega \frac{1}{2} (|\nabla u|^2 + \lambda u^2) - \frac{C_0}{2s} \left( \int_\Omega |\nabla u|^2 \, dx + \int_\Omega \lambda u^2 \, dx \right)^{\frac{s}{2s - 2}} \frac{u^{2^*(s)}}{|x - x_1|^s} \, dx.$$ 

Hence, by a scaling argument, there exist $\alpha > 0$ and $\rho > 0$ such that

$$J_\lambda(u) \geq \alpha \text{ if } \|u\|_{H^1(\Omega)} = \rho.$$ 

The scenario for the proof of the theorems is to apply the mountain pass lemma to attack the existence theory. However, the crux is to decide the threshold of the energy level so that the Palais-Smale condition would hold. We use concentration compactness principle to find this energy level.

**Remark 1.** The existence problem for (4) with $x_1, x_2 \in \Omega$ and $\Lambda \leq \lambda$ is still open.

In section 2, we investigate the threshold of the Palais-Smale condition for $J_\lambda$. In section 3 and 4, we prove the existence of solutions as described in Theorem 1.1 and Theorem 1.2, respectively. In section 5, the positivity of solutions is established. In section 6, regularity of solution is considered. In section 7, we give brief accounts for the Neumann problem with the multiple singularities. Namely, the existence of solutions to

$$\begin{cases}
-\Delta u + \lambda u = \sum_{i=1}^I \tau_i \frac{|u^{2^*(s)-2}u}{|x-x_i|^s} & \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega
\end{cases}$$

where for some $1 < I' < I$, $\tau = 1$ if $1 \leq i \leq I'$, $\tau = -1$ if $I' + 1 \leq i \leq I$ and $x_{i_1} \neq x_{i_2}$ if $i_1 \neq i_2$. Lastly in section 8, we provide short proof for the Neumann problem with the multiple positive singularities.

2. **Palais-Smale Condition.** In this section, we investigate the threshold of the Palais-Smale condition for $J_\lambda$. In what follows, $S_\lambda$ denotes $S_\lambda(\mathbb{R}^N)$. First we recall the Hardy-Sobolev inequality for functions supported on neighborhood of boundary. For the Sobolev inequality, see Lemma 2.1 in [16]. The following lemma is obtained by applying the technique of [16].

**Lemma 2.1** (Proposition 2.3 in [8]). Let $h(x')$ is a $C^1$ function defined in $\{x' \in \mathbb{R}^{n-1}, |x'| < 1\}$ and satisfying $\nabla h(0) = 0$. Denote $\overline{B} = B_1(0) \cap \{x_n > h(x')\}$. Then for any $\phi \in H_0^1(B_1(0))$, we have

1. If $h \equiv 0$, then

$$2^{2-2^*(s)} S_\lambda \left( \int_{\overline{B}} \frac{|\phi|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{s}{2s} - 1} \leq \int_{\overline{B}} |\nabla \phi|^2 \, dx.$$ 

2. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|\nabla h| \leq \delta$, then

$$(2^{2-2^*(s)} S_\lambda - \varepsilon) \left( \int_{\overline{B}} \frac{|\phi|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{s}{2s} - 1} \leq \int_{\overline{B}} |\nabla \phi|^2 \, dx.$$ 

**Proposition 1.** The functional $J_\lambda$ defined in (5) satisfies the $(PS)_\epsilon$ condition for
\[ c < \frac{2^{-s}}{2(N-s)} S^* \frac{(s)}{2} \text{ if } x_1, x_2 \in \Omega \]
\[ c < \frac{2^{-s}}{4(N-s)} S^* \frac{(s)}{2} \text{ if } x_1 \text{ or } x_2 \in \partial \Omega. \]

\[ \text{when } \tau = -1 \]
\[ c < \frac{2^{-s}}{2(N-s)} S^* \frac{(s)}{2} \text{ if } x_1 \in \Omega \]
\[ c < \frac{2^{-s}}{4(N-s)} S^* \frac{(s)}{2} \text{ if } x_1 \in \partial \Omega. \]

**Proof.** The proof is based on P. L. Lions' concentration-compactness principle [12, 14]. Suppose \( \{u_m\} \) be a \( (PS)_c \) sequence. That is
\[
J_\lambda(u_m) = \int_\Omega \frac{1}{2} (|\nabla u_m|^2 + \lambda u_m^2) - \frac{1}{2^*(s)} \left( \frac{(u_m)^{2^*(s)}_+}{|x - x_1|^s} + \tau \frac{(u_m)^{2^*(s)}_+}{|x - x_2|^s} \right) dx \to c
\]
\[
(7)
\]
\[
\langle J'_\lambda(u_m), \phi \rangle = \int_\Omega \nabla u_m \nabla \phi + \lambda u_m \phi - \left( \frac{(u_m)^{2^*(s)-1}_+}{|x - x_1|^s} + \tau \frac{(u_m)^{2^*(s)-1}_+}{|x - x_2|^s} \right) \phi dx \to 0
\]
\[
(8)
\]
as \( m \to \infty \). Plugging \( \phi = u_m \) into (8), we see that
\[
\int_\Omega |\nabla u_m|^2 + \lambda u_m^2 - \left( \frac{(u_m)^{2^*(s)}_+}{|x - x_1|^s} + \tau \frac{(u_m)^{2^*(s)}_+}{|x - x_2|^s} \right) = o(1)\|u_m\|_{H^1}.
\]
\[
(9)
\]
Taking off one-half of (9) from (7), we obtain
\[
\left( \frac{1}{2} - \frac{1}{2^*(s)} \right) \int_\Omega \left( \frac{(u_m)^{2^*(s)}_+}{|x - x_1|^s} + \tau \frac{(u_m)^{2^*(s)}_+}{|x - x_2|^s} \right) dx \leq c + 1 + o(\|u_m\|_{H^1}).
\]
\[
(10)
\]
Hence, we derive from (7) and Young’s inequality that
\[
\frac{1}{2} \int_\Omega (|\nabla u_m|^2 + \lambda u_m^2) dx \leq c + \frac{N - 2}{2 - s} (c + 1 + o(\|u_m\|_{H^1})
\]
\[
\leq C(\varepsilon) + \varepsilon \|u_m\|^2_{H^1}.
\]
Hence \( \{u_m\} \) is a bounded sequence in \( H^1(\Omega) \). So, up to a subsequence, we have the following weak convergence:
\[
u_m \rightharpoonup u \text{ in } H^1(\Omega),
\]
\[
u_m \rightharpoonup u \text{ in } L^\frac{2^*(s)}{2}(\Omega),
\]
\[
u_m \rightharpoonup u \text{ in } L^{2^*(s)}(\Omega, |x - x_1|^{-s}),
\]
\[
u_m \rightharpoonup u \text{ in } L^{2^*(s)}(\Omega, |x - x_2|^{-s}).
\]
Here \( L^{2^*(s)}(\Omega, |x - x_k|^{-s}), k = 1, 2 \) is \( L^{2^*(s)} \) function space equipped with the measure \( |x - x_k|^{-s} dx \).
Then the concentration-compactness principle gives

\[
|\nabla u_m|^2 dx \rightarrow d\mu \geq |\nabla u|^2 dx + \mu_1 \delta_{x_1} + \mu_2 \delta_{x_2} + \sum_{i \in I} \mu_i \delta_{x_i},
\]

\[
|u_m|^{2N} dx \rightarrow |u|^{2N} dx + \nu_1 \delta_{x_1} + \nu_2 \delta_{x_2} + \sum_{i \in I} \nu_i \delta_{x_i},
\]

\[
\frac{|u_m|^{2^*(s)} dx}{|x - x_1|^s} \rightarrow \frac{|u|^{2^*(s)} dx}{|x - x_1|^s} + \nu_1 \delta_{x_1} + \nu_2 \delta_{x_2} + \sum_{i \in I} \nu_i \delta_{x_i},
\]

\[
\frac{|u_m|^{2^*(s)} dx}{|x - x_2|^s} \rightarrow \frac{|u|^{2^*(s)} dx}{|x - x_2|^s} + \nu_1 \delta_{x_1} + \nu_2 \delta_{x_2} + \sum_{i \in I} \nu_i \delta_{x_i},
\]

in the sense of measure where \(\delta_x\) is the Dirac-mass of mass 1 concentrated at \(x \in \mathbb{R}^N\).

Here, \(I\) is at most countable index set and the numbers \(\mu_i, \nu_i, \nu_i \geq 0\).

We will analyse \(\mu_i, \nu_i, \nu_i, \nu_i \) to show that they are all zeros. Let \(\phi\) be a \(C^1\) function such that \(\phi(x) = 1\) on \(B_1(0)\) and \(\phi(x) = 0\) on \(\mathbb{R}^N \setminus B_2(0)\). We define \(\phi^l(x) = \phi(lx)\). Fix \(l > 0\). Then for \(k = 1, 2\), by the weak convergence, we see that

\[
\int_{\Omega} \frac{|u_m|^{2^*(s)} dx}{|x - x_k|^s} (1 - \phi^l(\cdot - x_k)) dx \rightarrow \int_{\Omega} \frac{|u|^{2^*(s)} dx}{|x - x_k|^s} (1 - \phi^l(\cdot - x_k)) dx + \sum_{x_i \in \mathbb{R}^N \setminus B_{\frac{l}{2}}(x_k)} \nu_i (1 - \phi^l(x_i))
\]

as \(m \to \infty\), where \(\nu_i = \nu_i\) or \(\nu_i\) for \(k = 1, 2\), respectively, Since \(2^*(s) < \frac{2N}{N-2}\), we have from strong convergence

\[
\int_{\Omega} \frac{|u_m|^{2^*(s)} dx}{|x - x_k|^s} (1 - \phi^l(\cdot - x_k)) dx \rightarrow \int_{\Omega} \frac{|u|^{2^*(s)} dx}{|x - x_k|^s} (1 - \phi^l(\cdot - x_k)) dx,
\]

as \(m \to \infty\). So we obtain

\[
\nu_i = 0 \text{ if } x_i \in \mathbb{R}^N \setminus B_{\frac{l}{2}}(x_1),
\]

\[
\nu_i = 0 \text{ if } x_i \in \mathbb{R}^N \setminus B_{\frac{l}{2}}(x_2).
\]

Letting \(l \to \infty\), we see that

\[
\frac{|u_m|^{2^*(s)} dx}{|x - x_k|^s} \rightarrow \frac{|u|^{2^*(s)} dx}{|x - x_k|^s} + \nu_1 \delta_{x_1} \text{ and } \frac{|u_m|^{2^*(s)} dx}{|x - x_2|^s} \rightarrow \frac{|u|^{2^*(s)} dx}{|x - x_2|^s} + \nu_2 \delta_{x_2}
\]

where \(\nu_1 := \nu_1\) and \(\nu_2 := \nu_2\).

Now we shall show some relation between \(\nu_k\) and \(\mu_k\) for \(k = 1, 2\). We consider \(v_m = u_m - u\) and

\[
dw_m := \left(\frac{|u_m|^{2^*(s)} dx}{|x - x_k|^s} - \frac{|u|^{2^*(s)} dx}{|x - x_k|^s}\right) dx = \frac{|u_m - u|^{2^*(s)} dx}{|x - x_k|^s} + o(1).
\]

In case of \(x_k \in \Omega\), we have

\[
\int_{\Omega} |\phi^l(\cdot - x_k)|^{2^*(s)} dw_m = \int_{\Omega} \int_{\Omega} \frac{|\phi^l(\cdot - x_k)v_m|^{2^*(s)} dx}{|x - x_k|^s} + o(1)
\]

\[
\leq S_{2^*(s)} \left(\int_{\Omega} |\nabla (\phi^l(\cdot - x_k)v_m)|^2 dx\right)^{\frac{2^*(s)}{2}} + o(1)
\]
For fixed \( l \), we see that \( \phi^l, \nabla \phi^l \in L^\infty(\Omega) \). Moreover, since \( \nabla v_m \to 0 \) weakly in \( L^2 \), we have \( v_m \to 0 \) in \( L^p \) for \( 0 < p < \frac{2N}{N-2} \) by Rellich-Kondrachov Theorem. So we get

\[
\int_\Omega |\nabla (\phi^l \cdot - x_k) v_m|^2 dx
\]
\[
\leq \int_\Omega |\nabla (\phi^l \cdot - x_k)|^2 |v_m|^2 dx + C_l \left( \int_\Omega |v_m|^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega |\nabla v_m|^2 dx \right)^{\frac{1}{2}}
\]
\[
+ \int_\Omega |\phi^l \cdot - x_k|^2 |\nabla v_m|^2 dx
\]
\[
= \int_\Omega |\phi^l \cdot - x_k|^2 |\nabla v_m|^2 dx + o(1)
\]

Hence,

\[
\int_\Omega |\phi^l \cdot - x_k|^{2^*(s)} d\omega_m = S_n^{\frac{2^*(s)}{2}} \left( \int_\Omega (\phi^l \cdot - x_k) |\nabla v_m|^2 dx \right)^{\frac{2^*(s)}{2}} + o(1).
\]

In case of \( x_k \in \partial \Omega \), applying Lemma 2.1, we see that

\[
\int_\Omega |\phi^l \cdot - x_k|^{2^*(s)} d\omega_m
\]
\[
= \int_\Omega \frac{|\phi^l \cdot - x_k| v_m^{2^*(s)}}{|x - x_k|^s} dx + o(1)
\]
\[
\leq \left( \frac{2^*(s)-2}{2} S_n^{\frac{2^*(s)}{2}} + \varepsilon_l \right) \left( \int_\Omega |\nabla (\phi^l \cdot - x_k) v_m|^2 dx \right)^{\frac{2^*(s)}{2}} + o(1)
\]
\[
= \left( \frac{2^*(s)-2}{2} S_n^{\frac{2^*(s)}{2}} + \varepsilon_l \right) \left( \int_\Omega (\phi^l \cdot - x_k) |\nabla v_m|^2 dx \right)^{\frac{2^*(s)}{2}} + o(1)
\]

where \( \varepsilon_l \to 0 \) as \( l \to \infty \). By letting \( m, l \to \infty \), we obtain

\[
\begin{cases}
S_n u_k^{\frac{2^*(s)}{2}} \leq \mu_k & \text{if } x_k \in \Omega \\
2^{2^*(s)/2} S_n u_k^{\frac{2^*(s)}{2}} \leq \mu_k & \text{if } x_k \in \partial \Omega
\end{cases}
\]

for \( k = 1, 2 \).

To complete the proof, we need to show that \( \mu_i = 0 \) for \( i = 1, 2 \) or \( i \in I \). For \( i \in I \), by testing \( u_m(x) \phi^l (x - x_i) \), we have

\[
\langle J_\lambda^*(u_m), u_m \phi^l \cdot - x_i \rangle = \int_\Omega \nabla u_m \nabla (u_m \phi^l \cdot - x_i) + \lambda u_m u_m \phi^l \cdot - x_i
\]
\[
- \left( \frac{(u_m)^{2^*(s)}}{|x - x_1|^s} \phi^l \cdot - x_i + \tau \frac{(u_m)^{2^*(s)}}{|x - x_2|^s} \phi^l \cdot - x_i \right) dx.
\]
One can readily check that
\[
\lim_{l \to \infty} \lim_{m \to \infty} \int_{\Omega} \nabla u_m \nabla u_m \phi^j(\cdot - x_i) dx \geq \mu_i,
\]
\[
\lim_{l \to \infty} \lim_{m \to \infty} \int_{\Omega} \lambda u_m u_m \phi^j(\cdot - x_i) dx = 0,
\]
\[
\lim_{l \to \infty} \lim_{m \to \infty} \int_{\Omega} \frac{(u_m)_+^{2^* (s)}}{|x - x_1|^s} \phi^j(\cdot - x_i) dx = 0,
\]
\[
\lim_{l \to \infty} \lim_{m \to \infty} \int_{\Omega} \frac{(u_m)_+^{2^* (s)}}{|x - x_2|^s} \phi^j(\cdot - x_i) dx = 0.
\]
We claim that
\[
\lim_{l \to \infty} \lim_{m \to \infty} \int_{\Omega} \nabla u_m u_m \nabla \phi^j(\cdot - x_i) dx = 0.
\]
Let \( \Omega'_l := \Omega \cap supp(\nabla \phi^j(\cdot - x_i)) \). First we consider the case where \( x_i \) is not a limit point of \( \{ x_k : k \in I \} \). In this case, we see that
\[ x_i \notin \Omega'_l \] for all \( l \)
and
\[ x_k \notin \Omega'_l \] for \( k \neq i \) as \( l \) is sufficiently large.
Hence we have
\[
\left| \lim_{l \to \infty} \lim_{m \to \infty} \int_{\Omega} \nabla u_m u_m \nabla \phi^j(\cdot - x_i) dx \right|
\]
\[
= \lim_{l \to \infty} \left| \int_{\Omega} \nabla u \cdot \nabla \phi^j(\cdot - x_i) u dx \right|
\]
\[
\leq \lim_{l \to \infty} \left( \int_{\Omega'_l} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega'_l} |u|^{2^* (s)} dx \right)^{\frac{N-2}{2N}} \left( \int_{\Omega'_l} |\nabla \phi^j(\cdot - x_i)|^N dx \right)^{\frac{1}{N}}
\]
\[
\leq \lim_{l \to \infty} C \left( \int_{\Omega'_l} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega'_l} |u|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}}
\]
\[
= 0.
\]
In the case of \( x_i \) is a limit point of \( \{ x_k : k \in I \} \), there is an additional term
\[
\left( \sum_{k \in I, x_k \in \Omega'_l} \mu_k \right)^{\frac{1}{2}} \left( \sum_{k \in I, x_k \in \Omega'_l} \overline{\mu}_k \right)^{\frac{N-2}{2N}} \leq \left( \int_{\Omega'_l} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega'_l} |u|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}} < \infty
\]
which also goes to 0 as \( l \to \infty \). So we get
\[
\mu_i \leq \lim_{l \to \infty} \langle J^*_l(u_m), u_m \phi^j(\cdot - x_i) \rangle = \lim_{l \to \infty} o(\|u_m \phi^j\|_{H^1}) = 0 \text{ for } i \in I.
\]
Using the same argument, we have
\[
\mu_1 \leq \lim_{l \to \infty} \lim_{m \to \infty} \int_{\Omega} \nabla u_m \nabla u_m \phi^j(\cdot - x_1) dx
\]
\[
= \lim_{l \to \infty} \lim_{m \to \infty} \int_{\Omega} \frac{(u_m)_+^{2^* (s)}}{|x - x_1|^s} \phi^j(\cdot - x_1) dx = \nu_1
\]
(12)
Similarly, when \( \tau = 1 \), we get \( \mu_2 \leq \mu_2 \). And when \( \tau = -1 \), we obtain

\[
\mu_2 \leq \lim_{l \to \infty} \lim_{m \to \infty} \int_{\Omega} \nabla u_m \nabla u_m \delta'(\cdot - x_2) dx
\]

\[
= - \lim_{l \to \infty} \lim_{m \to \infty} \int_{\Omega} \frac{(u_m)^{2^*(s)}}{|x - x_2|^s} \delta' \cdot (\cdot - x_2) dx \leq 0.
\]

Hence \( \mu_2 = 0 \).

If we assume \( \mu_1 > 0 \), then

\[
c = \lim_{m \to \infty} J_\lambda(u_m) - \frac{1}{2} \langle J'_\lambda(u_m), u_m \rangle \geq \left( \frac{1}{2} - \frac{1}{2^*(s)} \right) \mu_1.
\]

But from (11) and (12), we have

\[
\begin{cases}
S_s \mu_1^{2^*(s) - 2} \leq \mu_1 \Rightarrow \mu_1 \geq S_s^{2^*(s) - 2} \\
2 - 2^*(s) S_s \mu_1 \leq \mu_1 \Rightarrow \mu_1 \geq \frac{1}{2} S_s^{2^*(s) - 2}
\end{cases}
\]

which is a contradiction. If we assume \( \mu_2 > 0 \) when \( \tau = 1 \), by using same argument, we also meet a contradiction. This proves Proposition 1.

3. **Existence of solution to (4) for small \( \lambda \).** In this section, we show the existence theory of Theorem 1.1. Plugging constant function \( d \) into the functional \( J_\lambda \), we have

\[
J_\lambda(d) = \frac{1}{2} |\Omega| \lambda d^2 - \frac{1}{2^*(s)} C_1 d^{2^*(s)}
\]

where \( C_1 = \int_{\Omega} \frac{1}{|x - x_1|^s} + \tau \frac{1}{|x - x_2|^s} dx \). Since \( 2^*(s) > 2 \), we see that \( J_\lambda(d) < 0 \) for sufficiently large \( d \). From the observation

\[
\frac{d}{dd} (J_\lambda(d)) = 0 \Leftrightarrow d = 0 \text{ or } \left( \frac{\lambda|\Omega|}{C_1} \right)^{\frac{1}{2^*(s) - 2}},
\]

we see that

\[
\max_{d \geq 0} J_\lambda(d) = J_\lambda \left( \frac{\lambda|\Omega|}{C_1} \right)^{\frac{1}{2^*(s) - 2}}
\]

\[
= \frac{1}{2} |\Omega| \lambda \left( \frac{\lambda|\Omega|}{C_1} \right)^{\frac{2}{2^*(s) - 2}} - \frac{1}{2^*(s)} C_1 \left( \frac{\lambda|\Omega|}{C_1} \right)^{\frac{2^*(s)}{2^*(s) - 2}}. \tag{13}
\]

By choosing positive \( \lambda \) small enough, inferring from (13), we see that the min-max number \( c \) of \( J_\lambda \) satisfies

\[
\begin{cases}
c < \frac{2 - s}{2(N - s)} S_s^{2^*(s) - 2} & \text{if } x_1, x_2 \in \Omega \\
c < \frac{2 - s}{4(N - s)} S_s^{2^*(s) - 2} & \text{if } x_1 \text{ or } x_2 \in \partial \Omega.
\end{cases}
\]

Hence, by applying Proposition 1, we prove Theorem 1.1.
4. Existence of solution to (4) with boundary singularity. In this section, we prove the existence of a solution in Theorem 1.2. We shall follow the strategy of [4, 16] to prove Theorem 1.2. We may assume \( x_1 = (0, \cdots, 0) \in \partial \Omega \) and the mean curvature \( H(0) \) is positive. Then, up to rotation, the boundary near the origin can be represented by

\[
x_n = h(x') = \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i x_i^2 + o(|x'|^2)
\]

where \( x' = (x_1, x_2, \cdots, x_{N-1}) \in D_\delta(0) = B_\delta(0) \cap \{x_N = 0\} \) for some \( \delta > 0 \). Here \( \alpha_1, \alpha_2, \cdots, \alpha_{N-1} \) are the principal curvatures of \( \partial \Omega \) at 0 and the mean curvature \( \sum_{i=1}^{N-1} \alpha_i > 0 \). Denote

\[
g(x') = \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i x_i^2.
\]

Consider

\[
U_\varepsilon(x) := \varepsilon \frac{N-2}{2(N+s)} (\varepsilon + |x|^{2-s})^{\frac{2-N}{2}}
\]

for small parameter \( \varepsilon > 0 \). Then, it follows that

\[
\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 dx / (\int_{\mathbb{R}^N} |U_\varepsilon|^{2^*(s)} dx)^{\frac{N-2}{2}} = S_s.
\]

Choose \( \delta \) such that \( x_2 \notin B_{3\delta}(0) \). Set a cut-off function \( \eta \) such that

\[
\eta \in C_c^\infty (\mathbb{R}^N), \quad 0 \leq \eta \leq 1, \quad \eta = 1 \text{ in } B_\delta(0), \quad \eta = 0 \text{ in } \mathbb{R}^N \setminus B_{2\delta}(0).
\]

Note that from \( 2^*(s) > 2 \),

\[
J_\lambda(T\eta U_\varepsilon) = \int_\Omega \frac{T^2}{2} (|\nabla (\eta U_\varepsilon)|^2 + \lambda (\eta U_\varepsilon)^2) - \frac{T^{2^*(s)}}{2^*(s)} \left( \frac{(\eta U_\varepsilon)^{2^*(s)}}{|x-x_1|^s} + \tau \frac{(\eta U_\varepsilon)^{2^*(s)}}{|x-x_2|^s} \right) dx < 0
\]

for sufficiently large \( T \). We define

\[
P = \left\{ p(t) | p(t) : [0, 1] \to H^1(\Omega) \text{ is continuous with } p(0) = 0 \in H^1(\Omega) \text{ and } p(1) = T\eta U_\varepsilon |_{\Omega} \right\}.
\]

Let

\[
c^* = \inf_{p(t) \in P} \sup_{0 \leq t \leq 1} \{ J_\lambda(p(t)) \}.
\]

Then, thanks to Proposition 1, it suffices to show

\[
c^* < \frac{2 - s}{4(N-s) S_s^{s/(s-2)}}.
\]

In the following discussion, we denote

\[
K_0^\varepsilon := \int_\Omega |\nabla (\eta U_\varepsilon)|^2 dx, \quad K_1^\varepsilon := \int_\Omega \frac{(\eta U_\varepsilon)^{2^*(s)}}{|x-x_1|^s} dx, \quad K_2^\varepsilon := \tau \int_\Omega \frac{(\eta U_\varepsilon)^{2^*(s)}}{|x-x_2|^s} dx
\]

and \( K_3^\varepsilon := \int_\Omega (\eta U_\varepsilon)^2 dx \).

First we deal with \( K_0^\varepsilon \). By using Leibniz rule, one has

\[
K_0^\varepsilon = \int_\Omega |\nabla (\eta U_\varepsilon)|^2 dx
\]

\[
= \int_\Omega |\nabla \eta|^2 |U_\varepsilon|^2 dx + 2 \int_\Omega \eta U_\varepsilon \nabla \eta \cdot \nabla U_\varepsilon dx + \int_\Omega |\eta|^2 |\nabla U_\varepsilon|^2 dx.
\]
When $N \geq N$, the last term is more delicate. We consider the case of $N$ separately. When $N = 3$ and the case of $N \geq 4$ separately. When $N = 3$, we have

$$\int_{\Omega} |\eta|^{2s} |U_{\varepsilon}|^{2} \, dx = \int_{\Omega} |\nabla \eta|^{2} \varepsilon^{\frac{N-2}{2}} (\varepsilon + |x|^{2-s}) \frac{2(2-N)}{2-N} \, dx$$

$$\leq \int_{\Omega} |\nabla \eta|^{2} \varepsilon^{\frac{N-2}{2}} (2\delta)^{2(2-N)} \, dx$$

$$\leq C_{1,\delta} \varepsilon^{\frac{N-2}{2}}.$$  

Similarly, it follows that

$$\left| \int_{\Omega} \eta U_{\varepsilon} \nabla \eta \cdot \nabla U_{\varepsilon} \, dx \right|$$

$$= \left| \int_{\Omega} \varepsilon^{\frac{N-2}{2}} (\varepsilon + |x|^{2-s}) \frac{2-N}{2-N} \nabla \eta \cdot \left( (2-N) \varepsilon^{\frac{N-2}{2}} (\varepsilon + |x|^{2-s}) \right) \frac{2-N}{2-N} \, dx \right|$$

$$\leq C_{2,\delta} \varepsilon^{\frac{N-2}{2}}.$$  

The last term is more delicate. We consider the case of $N = 3$ and the case of $N \geq 4$ separately. When $N = 3$, we have

$$\int_{\Omega} |\eta|^{2} |\nabla U_{\varepsilon}|^{2} \, dx = \int_{\Omega} |\nabla U_{\varepsilon}|^{2} \, dx - \int_{D_{\delta}(0)} \int_{0}^{h(x')} |\nabla U_{\varepsilon}|^{2} \, dx' \, dx + o(\varepsilon^{\frac{1}{2-N}}).$$

Since $a|x'|^{2} \leq h(x') \leq A|x'|^{2}$ on $D_{\delta}(0)$ for some $0 < a \leq A < \infty$, we have

$$\int_{D_{\delta}(0)} \int_{0}^{h(x')} |\nabla U_{\varepsilon}|^{2} \, dx' \, dx' \geq C_{3,\delta} \int_{D_{\delta}(0)} (\varepsilon + |x'|^{2-s})^{\frac{N-2}{2-N}} \, dx'$$

$$\geq C_{3,\delta} \varepsilon^{\frac{N-2}{2-N}} |\ln \varepsilon|.$$  

When $N \geq 4$, we have

$$\int_{\Omega} |\eta|^{2} |\nabla U_{\varepsilon}|^{2} \, dx$$

$$= \int_{\Omega} |\nabla U_{\varepsilon}|^{2} \, dx - \int_{D_{\delta}(0)} \int_{0}^{h(x')} |\nabla U_{\varepsilon}|^{2} \, dx' \, dx' + o(\varepsilon^{\frac{N-2}{2-N}})$$

$$= \frac{1}{2} K_{0} - \int_{\Omega} \int_{0}^{g(x')} |\nabla U_{\varepsilon}|^{2} \, dx' \, dx + \int_{D_{\delta}(0)} \int_{g(x')}^{h(x')} |\nabla U_{\varepsilon}|^{2} \, dx' \, dx' + o(\varepsilon^{\frac{N-2}{2-N}})$$

where

$$K_{0} := \int_{\Omega} |\nabla U_{\varepsilon}|^{2} \, dx = (N-2)^{2} \int_{\Omega} \frac{|y|^{2(2-s)}}{1 + |y|^{2}} \, dy.$$
Observe that
\[
I(\varepsilon) := \int_{\mathbb{R}^{N-1}} \int_0^{g(x')} |\nabla U_\varepsilon|^2 dx_N dx'
= (N - 2)^2 \varepsilon^{\frac{N-2}{2}} \int_{\mathbb{R}^{N-1}} \int_0^{g(x')} \frac{|x|^{2-2s}}{(e + |x|^{2-s})^{\frac{2(N-2)}{2-s}}} dx_N dx'
= (N - 2)^2 \int_{\mathbb{R}^{N-1}} \int_0^{g(x') \varepsilon^{\frac{1}{2}}} \frac{|y|^{2-2s}}{(1 + |y|^{2-s})^{\frac{2(N-2)}{2-s}}} dy_N dy'.
\]

So we have
\[
\lim_{\varepsilon \to 0} \varepsilon^{-\frac{1}{2-s}} I(\varepsilon) = (N - 2)^2 \int_{\mathbb{R}^{N-1}} \frac{|x'|^{2-2s} g(x')}{(1 + |x'|^{2-s})^{\frac{2(N-2)}{2-s}}} dx'
= \frac{(N - 2)^2}{2} \int_{\mathbb{R}^{N-1}} \frac{|x'|^{2-2s} \sum_{i=1}^{N-1} \alpha_i |x_i|^2}{(1 + |x'|^{2-s})^{\frac{2(N-2)}{2-s}}} dx'
= \frac{(N - 2)^2}{2} \sum_{i=1}^{N-1} \alpha_i \int_{\mathbb{R}^{N-1}} \frac{|x'|^{2-2s} |x_i|^2}{(1 + |x'|^{2-s})^{\frac{2(N-2)}{2-s}}} dx'
= \left( \sum_{i=1}^{N-1} \alpha_i \right) \frac{(N - 2)^2}{2(N - 1)} \int_{\mathbb{R}^{N-1}} \frac{|x'|^{4-2s}}{(1 + |x'|^{2-s})^{\frac{2(N-2)}{2-s}}} dx'.
\]

which leads to
\[
I(\varepsilon) = O(\varepsilon^{\frac{1}{2-s}}).
\]

The curvature assumption \((H(0) > 0)\) implies
\[
I(\varepsilon) > 0.
\]

Moreover,
\[
I_1(\varepsilon) := \int_{D_\delta(0)} \int_0^{h(x')} |\nabla U_\varepsilon|^2 dx_N dx'
= (N - 2)^2 \varepsilon^{\frac{N-2}{2}} \int_{D_\delta(0)} \int_0^{h(x')} \frac{|x|^{2-2s}}{(e + |x|^{2-s})^{\frac{2(N-2)}{2-s}}} dx_N dx'
\leq \tilde{C}_{4,\delta}(N - 2)^2 \varepsilon^{\frac{N-2}{2}} \int_{D_\delta(0)} \frac{|h(x') - g(x')|}{(e + |x'|^{2-s})^{\frac{2(N-2)}{2-s}} - 1} dx'
\]

Since \(h(x') = g(x') + o(|x'|^2)\), for any \(\sigma > 0\), there exists \(C(\sigma) > 0\) such that
\[
|h(x') - g(x')| \leq \sigma |x'|^2 + C(\sigma) |x'|^{\frac{3}{2}}.
\]

So we have
\[
I_1(\varepsilon) \leq \tilde{C}_{4,\delta} \varepsilon^{\frac{N-2}{2}} \int_{D_\delta(0)} \frac{\sigma |x'|^2 + C(\sigma) |x'|^{\frac{3}{2}}}{(e + |x'|^{2-s})^{\frac{2(N-2)}{2-s}} - 1} dx'
\leq C_{4,\delta} \varepsilon^{\frac{1}{2-s}} (\sigma + C(\sigma) \varepsilon^{\frac{1}{2-s}})
\]

which implies
\[
I_1(\varepsilon) = O(\varepsilon^{\frac{1}{2-s}}).
\]
Thus, we obtain
\[ K_0^\varepsilon = \begin{cases} \frac{1}{2} K_0 - C \varepsilon^{\frac{1}{p^*}} |\ln \varepsilon| + O(\varepsilon^{\frac{1}{p^*}}) & \text{when } N = 3, \\ \frac{1}{2} K_0 - I(\varepsilon) + O(\varepsilon^{\frac{1}{p^*}}) & \text{when } N \geq 4. \end{cases} \tag{17} \]

On the other hand, we have
\[
K_1^\varepsilon = \int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2^*(s)}}{|x|^s} dx - \int_{D_\varepsilon(0)} \int_0^{h(x')} \frac{|U_\varepsilon|^{2^*(s)}}{|x|^s} dx_N dx' + O\left(\varepsilon^{\frac{N}{2s}}\right)
\]
\[
= \frac{1}{2} K_1 - \int_{\mathbb{R}^{N-1}} \int_0^{g(x')} |U_\varepsilon|^{2^*(s)} \left|\frac{\partial}{\partial x'}\right| \frac{dx_N}{|x|^s} dx' - \int_{D_\varepsilon(0)} \int_{g(x')}^{h(x')} |U_\varepsilon|^{2^*(s)} \left|\frac{\partial}{\partial x'}\right| \frac{dx_N}{|x|^s} dx' + O\left(\varepsilon^{\frac{N}{2s}}\right)
\]
where
\[ K_1 = \int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2^*(s)}}{|x|^s} dx = \int_{\mathbb{R}^N} \frac{\varepsilon^{2^*(s)(N-2)} \ln \varepsilon + O(\varepsilon^{\frac{1}{p^*}})}{|x|^s(\varepsilon + |x|^{2-s})^{\frac{2^*(s)(N-2)}{2-s}}} dx
\]
\[ = \int_{\mathbb{R}^N} \frac{1}{|y|^s(1 + |y|^{2-s})^{\frac{2(N-2)}{2-s}}} dy. \]

Observe that
\[
II(\varepsilon) := \int_{\mathbb{R}^{N-1}} \int_0^{g(x')} |U_\varepsilon|^{2^*(s)} \left|\frac{\partial}{\partial x'}\right| \frac{dx_N}{|x|^s} dx'
\]
\[ = \int_{\mathbb{R}^{N-1}} \int_0^{\frac{\varepsilon}{|x'|^s} g(y')} \frac{1}{|y'|^s(1 + |y'|^{2-s})^{\frac{2(N-2)}{2-s}}} dy_N dy'. \tag{18} \]

So, we have
\[
\lim_{\varepsilon \to 0} \varepsilon^{1 - \frac{s}{p^*}} II(\varepsilon) = \int_{\mathbb{R}^{N-1}} \frac{g(y')}{|y'|^s(1 + |y'|^{2-s})^{\frac{2(N-2)}{2-s}}} dy'
\]
\[ = \frac{1}{2} \int_{\mathbb{R}^{N-1}} \left|\sum_{i=1}^{N-1} \alpha_i y_i\right|^2 \frac{1}{|y'|^s(1 + |y'|^{2-s})^{\frac{2(N-2)}{2-s}}} dy'
\]
\[ = \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i \int_{\mathbb{R}^{N-1}} \frac{|y_i|^2}{|y'|^s(1 + |y'|^{2-s})^{\frac{2(N-2)}{2-s}}} dy'
\]
which leads to
\[ II(\varepsilon) = O\left(\varepsilon^{\frac{1}{p^*}}\right). \]

The curvature assumption \((H(0) > 0)\) implies
\[ II(\varepsilon) > 0. \]

Similarly, we can get
\[
\int_{D_\varepsilon(0)} \int_{g(x')}^{h(x')} \frac{|U_\varepsilon|^{2^*(s)}}{|x|^s} dx_N dx' = O\left(\varepsilon^{\frac{1}{p^*}}\right).
\]

Thus, we obtain
\[ K_1^\varepsilon = \frac{1}{2} K_1 - II(\varepsilon) + O\left(\varepsilon^{\frac{1}{p^*}}\right). \tag{19} \]
Then, we are concerned about $K_2^\varepsilon$. Since $x_2 \notin B_{3\delta}(0)$ and $\text{supp}(\eta) \subset B_{2\delta}(0)$, we see that
\[
K_2^\varepsilon = \int_{\Omega} |\eta U_\varepsilon|^{2^*(s)} |x - x_2|^s dx \\
\leq C \int_{\Omega \cap B_{2\delta}(0)} |U_\varepsilon|^{2^*(s)} dx \\
\leq \int_{B_{2\delta}(0)} \left( \varepsilon \frac{N-2}{2^*} \right) \int_{2N-2 \frac{N}{2}} \right) dx \\
= \varepsilon^{\frac{1}{2^*}} \int_{B_{2\delta}(0)} (1 + |y|^{2-s})^{- \frac{2(N-4)}{N-2}} dy = O(\varepsilon^{\frac{1}{2^*}}).
\]

Lastly, direct calculation gives
\[
K_3^\varepsilon = \int_{\Omega} (\eta U_\varepsilon)^2 dx = \begin{cases} 
O(\varepsilon^{\frac{1}{2^*}}), & N = 3, \\
O(|\varepsilon^{\frac{1}{2^*}} \ln \varepsilon|), & N = 4, \\
O(\varepsilon^{\frac{1}{2^*}}), & N \geq 5.
\end{cases}
\]

Actually when $N = 3$, we have
\[
\int_{\Omega} (\eta U_\varepsilon)^2 dx = \int_{\Omega} |\eta|^{2} \varepsilon^{\frac{1}{2^*}} (\varepsilon + |x|^{2-s})^{- \frac{1}{2^*}} dx \\
\leq \int_{B_{2\delta}(0)} \varepsilon^{\frac{1}{2^*}} (\varepsilon + |x|^{2-s})^{- \frac{1}{2^*}} dx \\
\leq C_N \varepsilon^{\frac{1}{2^*}} \int_{0}^{2\delta \varepsilon^{- \frac{1}{2^*}}} (1 + r^{2-s})^{- \frac{1}{2^*}} r^2 dr = O(\varepsilon^{\frac{1}{2^*}}).
\]

When $N = 4$, we see that
\[
\int_{\Omega} (\eta U_\varepsilon)^2 dx = \int_{\Omega} |\eta|^{2} \varepsilon^{\frac{1}{2^*}} (\varepsilon + |x|^{2-s})^{- \frac{1}{2^*}} dx \\
\leq \int_{B_{2\delta}(0)} \varepsilon^{\frac{2}{2^*}} (\varepsilon + |x|^{2-s})^{- \frac{1}{2^*}} dx \\
\leq C_N \varepsilon^{\frac{2}{2^*}} \int_{0}^{2\delta \varepsilon^{- \frac{1}{2^*}}} (1 + r^{2-s})^{- \frac{1}{2^*}} r^{N-1} dr = O(|\varepsilon^{\frac{1}{2^*}} \ln \varepsilon|).
\]

When $N \geq 5$, we have
\[
\int_{\Omega} (\eta U_\varepsilon)^2 dx = \int_{\Omega} |\eta|^{2} \varepsilon^{\frac{N-2}{2^*}} (\varepsilon + |x|^{2-s})^{\frac{2(N-4)}{N-2}} dx \\
\leq \int_{\mathbb{R}^N} \varepsilon^{\frac{N-2}{2^*}} (\varepsilon + |x|^{2-s})^{\frac{2(N-4)}{N-2}} dx = O(\varepsilon^{\frac{1}{2^*}}).
\]

So to prove (15), it suffices to show that
\[
K_0^\varepsilon/(K_1^\varepsilon) = 2^{- \frac{1}{2^*}} S_\varepsilon + O(\varepsilon^{\frac{1}{2^*}}) = \frac{1}{2} K_0/(\frac{1}{2} K_1) + O(\varepsilon^{\frac{1}{2^*}}).
\]

(20)

Now, we are going to prove (15). Let $t_\varepsilon$ be a constant satisfying
\[
J_\lambda(t_\varepsilon \eta U_\varepsilon) = \sup_{t > 0} J_\lambda(t \eta U_\varepsilon) \\
= \sup_{t > 0} \left[ \frac{1}{2} K_0^\varepsilon t^2 - \frac{1}{2^*} (s) K_1^\varepsilon t^{2^*(s)} + \frac{1}{2} K_3^\varepsilon \right].
\]
In case $N = 3$, we see that $K^*_2 > 0$ when $\tau = 1$, $K^*_2 = O(\varepsilon^{2\tau})$ when $\tau = -1$ and $K^*_3 = O(\varepsilon^{2\tau})$. Hence,

$$
J_\lambda(t\varepsilon^\eta U_\varepsilon) \leq \sup_{t > 0} \left\{ \frac{1}{2} K_0^* t^2 - \frac{1}{2\tau(s)} K_1^* t^{2\tau(s)} + O(\varepsilon^{2\tau}) \right\} \\
= \frac{2 - s}{2(N - s)} \left[ \frac{K_0^*}{(K_1^*)^{\frac{N-2}{N-\tau}}} \right]^{\frac{N-2}{N-\tau}} + O(\varepsilon^{\frac{2}{N-\tau}}).
$$

So to prove (15), it suffices to show that

$$
K_0^*/(K_1^*)^{\frac{N-2}{N-\tau}} < 2 - \frac{2}{N-\tau} S_s + O(\varepsilon^{\frac{2}{N-\tau}}) = \frac{1}{2} K_0/(1) (\frac{1}{2} K_1)^{\frac{N-2}{N-\tau}} + O(\varepsilon^{\frac{2}{N-\tau}}). \tag{21}
$$

Taking (14), (17) and (19) into account, (21) is equivalent to

$$
\frac{1}{2} K_0 - C \varepsilon^{\frac{2}{N-\tau}} |\ln \varepsilon| < 2 - \frac{2}{N-\tau} S_s \left[ \frac{1}{2} K_1 - O(\varepsilon^{\frac{2}{N-\tau}}) \right]^{\frac{N-2}{N-\tau}} + O(\varepsilon^{\frac{2}{N-\tau}}) \\
= \frac{1}{2} S_s K_1^{\frac{N-2}{N-\tau}} + O(\varepsilon^{\frac{2}{N-\tau}})
$$

which is true for small $\varepsilon > 0$, because $C > 0$ and

$$
K_0/K_1^{\frac{N-2}{N-\tau}} = S_s.
$$

In case $N \geq 4$, we know that $K^*_2, K^*_3 = O(\varepsilon^{\frac{2}{N-\tau}})$ where $\tilde{s} = \min(1, s)$. Hence,

$$
J_\lambda(t\varepsilon^\eta U_\varepsilon) \leq \sup_{t > 0} \left\{ \frac{1}{2} K_0^* t^2 - \frac{1}{2\tau(s)} K_1^* t^{2\tau(s)} + O(\varepsilon^{2\tau}) \right\} \\
= \frac{2 - s}{2(N - s)} \left[ \frac{K_0^*}{(K_1^*)^{\frac{N-2}{N-\tau}}} \right]^{\frac{N-2}{N-\tau}} + O(\varepsilon^{\frac{2}{N-\tau}}).
$$

So to prove (15), it suffices to show that

$$
K_0^*/(K_1^*)^{\frac{N-2}{N-\tau}} < 2 - \frac{2}{N-\tau} S_s + O(\varepsilon^{\frac{2}{N-\tau}}) = \frac{1}{2} K_0/(1) (\frac{1}{2} K_1)^{\frac{N-2}{N-\tau}} + O(\varepsilon^{\frac{2}{N-\tau}}). \tag{22}
$$

Taking (14), (17) and (19) into account, (22) is equivalent to

$$
\left( \frac{1}{2} K_0 - I(\varepsilon) \right) \left( \frac{1}{2} K_1 \right)^{\frac{N-2}{N-\tau}} \\
< \frac{1}{2} K_0 \left( \frac{1}{2} K_1 - II(\varepsilon) + O(\varepsilon^{\frac{2}{N-\tau}}) \right)^{\frac{N-2}{N-\tau}} + O(\varepsilon^{\frac{2}{N-\tau}}) \tag{23}
$$

$$
= \frac{1}{2} K_0 \left\{ \frac{1}{2} K_1 \right\}^{\frac{N-2}{N-\tau}} - \frac{N - 2}{N - s} \left( \frac{1}{2} K_1 \right)^{\frac{N-2}{N-\tau}} II(\varepsilon) + O(\varepsilon^{\frac{2}{N-\tau}}).
$$

Hence to verify (23), we have to prove

$$
\lim_{\varepsilon \to 0} \frac{II(\varepsilon)}{I(\varepsilon)} < \frac{(N - s)K_1}{(N - 2)K_0}.
$$
By (16), (18) and L’Hôpital’s rule, we obtain
\[
\lim_{\varepsilon \to 0} \frac{II(\varepsilon)}{I(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{II'(\varepsilon)}{I'(\varepsilon)}
\]
\[
= (N - 2)^{-2} \int_{\mathbb{R}^{N-1}} \frac{g(y')}{|y'|^{s} (1 + |y'|^{2-s})^{2(N-s)/(2-s)}} dy' 
\times \left( \int_{\mathbb{R}^{N-1}} \frac{|y'|^{2-2s} g(y')}{(1 + |y'|^{2-s})^{2(N-s)/(2-s)}} dy' \right)^{-1}
\]
\[
= (N - 2)^{-2} \int_{0}^{\infty} \frac{r^{N-s}}{(1 + r^{2-s})^{2(N-s)/(2-s)}} dr \times \left( \int_{0}^{\infty} \frac{r^{N+2-2s}}{(1 + r^{2-s})^{2(N-s)/(2-s)}} dr \right)^{-1}
\]
Integration by parts gives for \(2 \leq \beta \leq 2(N - s) - 1\),
\[
\int_{0}^{\infty} \frac{r^{\beta-2}}{(1 + r^{2-s})^{2(N-s)/(2-s)-1}} dr = \frac{2N - 2 - s}{\beta - 1} \int_{0}^{\infty} \frac{r^{\beta-s}}{(1 + r^{2-s})^{2(N-s)/(2-s)-1}} dr.
\]
Since
\[
\int_{0}^{\infty} \frac{r^{\beta-s}}{(1 + r^{2-s})^{2(N-s)/(2-s)-1}} dr = \int_{0}^{\infty} \frac{r^{\beta-2}}{(1 + r^{2-s})^{2(N-s)/(2-s)-1}} dr - \int_{0}^{\infty} \frac{r^{\beta-2}}{(1 + r^{2-s})^{2(N-s)/(2-s)-1}} dr,
\]
we have
\[
\int_{0}^{\infty} \frac{r^{\beta-s}}{(1 + r^{2-s})^{2(N-s)/(2-s)-1}} dr = \frac{\beta - 1}{2N - \beta - 1 - s} \int_{0}^{\infty} \frac{r^{\beta-2}}{(1 + r^{2-s})^{2(N-s)/(2-s)-1}} dr.
\]
(24)
So, plugging \(\beta = N + 2 - s\) into (24), we obtain
\[
\lim_{\varepsilon \to 0} \frac{II(\varepsilon)}{I(\varepsilon)} = \frac{N - 3}{(N + 1 - s)(N - 2)^2}
\]
and plugging \(\beta = N + 1 - s\) into (24)
\[
\frac{(N - s) K_1}{(N - 2) K_0}
\]
\[
= \frac{N - s}{(N - 2)^3} \left( \int_{0}^{\infty} \frac{r^{N-1-s}}{(1 + r^{2-s})^{2(N-s)/(2-s)}} dr \right) \times \left( \int_{0}^{\infty} \frac{r^{N+1-2s}}{(1 + r^{2-s})^{2(N-s)/(2-s)}} dr \right)^{-1}
\]
\[
= (N - 2)^{-2}.
\]
Therefore we obtain
\[
\frac{II(\varepsilon)}{I(\varepsilon)} < \frac{(N - s) K_1}{(N - 2) K_0}
\]
for sufficiently small \(\varepsilon\) and complete the proof.

5. Positivity of solution. In this section, we establish the positivity of solutions. One first observes that
\[
0 = \langle J'_{\lambda}(u), u_- \rangle = \int_{\Omega} \nabla u_-^2 + \lambda |u_-|^2 dx
\]
where \(u_- = \min(u, 0)\). Since \(\lambda > 0\), we have \(u \geq 0\). Then the interior positivity of \(u\) follows from the maximum principle.

**Proposition 2.** If \(u \in C^1(\Omega \setminus \{x_1, x_2\})\) is a non-negative solution to (4), then \(u > 0\) in \(\Omega\).
Proof. We employ the argument in [15]. If $u$ vanishes somewhere in $\Omega \setminus \{x_1, x_2\}$, then there exists $y_0 \in \Omega \setminus \{x_1, x_2\}$ and a ball $B = B_R(y_0)$ satisfying $u(y_0) = 0$, $B \subset \Omega \setminus \{x_1, x_2\}$, $y_0 \in \partial B$ and $0 < u < a$ in $B$. We observe that $u > 0$ on

$$A = \{x : \frac{R}{2} < |x - y_1| < R\}$$

and

$$c = \inf\{u(x) : |x - y_1| = \frac{R}{2}\}$$

satisfies $0 < c < a$.

For given $k_1, k_2 > 0$, let $v(r)$ be solution to

$$\begin{cases} v'' = k_1 v' + k_2 v \text{ for } 0 < r < \frac{R}{2}, \\ v(0) = 0, v\left(\frac{R}{2}\right) = c. \end{cases}$$

We note that $v'(0) > 0$. Now we consider $u(x) = v(R - |x - y_1|)$. Then

$$\bar{u}(x) = u(x) \text{ on } \partial B_R(y_1) \text{ and } \bar{u}(x) = c \leq u(x) \text{ on } \partial B_R \left(\frac{R}{2}\right)(y_1).$$

Moreover, on $A$, we have

- when $\tau = 1$

$$-\Delta \bar{u} + \lambda \bar{u} = -v''(R - |x - y_1|) - v'(R - |x - y_1|) \left(\frac{1 - N}{|x - y_1|} + \lambda v(R - |x - y_1|)\right)$$

$$\leq \left(\frac{N - 1}{R} - k_1\right)v'(R - |x - y_1|) + (\lambda - k_2)v(R - |x - y_1|)$$

$$\leq 0$$

for sufficiently large $k_1, k_2$.

- when $\tau = -1$

$$-\Delta \bar{u} + \lambda \bar{u} + \frac{|u|^{2^*(s) - 2} \bar{u}}{|x - x_2|^s}$$

$$= -v''(R - |x - y_1|) - v'(R - |x - y_1|) \left(\frac{1 - N}{|x - y_1|} + \lambda v(R - |x - y_1|)\right) + \frac{|u|^{2^*(s) - 2} v(R - |x - y_1|)}{|x - x_2|^s}$$

$$\leq \left(\frac{N - 1}{R} - k_1\right)v'(R - |x - y_1|) + (\lambda - k_2)v(R - |x - y_1|) + \frac{|u|^{2^*(s) - 2} v(R - |x - y_1|)}{|x - x_2|^s}$$

$$\leq 0$$

for sufficiently large $k_1, k_2$.

We claim that $u \geq \bar{u}$ on $A$. Suppose not, there exists $\Omega_1 \subset A$ such that $\bar{u} > u$ on $\Omega_1$. And we have

- when $\tau = 1$

$$-\Delta(\bar{u} - u) + \lambda(\bar{u} - u) \leq 0 \text{ on } \Omega_1.$$
• when $\tau = -1$
\[-\Delta (\pi - u) + \lambda (\pi - u) + \frac{|u|^{2^*(s)-2}(\pi - u)}{|x - x_s|^2} \leq 0 \text{ on } \Omega_1.\]

So, by multiplying $\pi - u$ and integrating over $\Omega_1$, we obtain

• when $\tau = 1$
\[0 < \int_{\Omega_1} |\nabla (\pi - u)|^2 + \lambda |\pi - u|^2 \, dx \leq 0.\]

• when $\tau = -1$
\[0 < \int_{\Omega_1} |\nabla (\pi - u)|^2 + \lambda |\pi - u|^2 \, dx + \frac{|u|^{2^*(s)-2}|\pi - u|^2}{|x - x_s|^2} \leq 0.\]

which is a contradiction.

Since $u(y_0) = \pi(y_0) = 0$, $u \geq \pi$ on $A$ and $u' > 0$, $u'(y_1)$ should be positive which contradicts to $y_0$ is minimum point. \hfill \Box

6. Regularity of solution to (4). In this section, we verify the regularity of solution. By applying the technique of the proof of Proposition 8.1 in [6], one can readily check that the solution of (4) is $C^{2,\alpha}(\Omega), 0 < \alpha < 1$. Moreover, by using the elliptic regularity theory for $\Omega' \subset \subset \Omega \setminus \{x_1, x_2\}$, we can see that this solution is $C_{loc}(\Omega \setminus \{x_1, x_2\})$.

7. Neumann Problem with the multiple singularities. In this section, we deal with the existence theory for the equation

\[
\begin{cases}
-\Delta u + \lambda u = \sum_{i=1}^I \tau_i \frac{|u|^{2^*(s)-2}u}{|x-x_i|^s} & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega
\end{cases}
\]

(25)

where for some $1 < I' < I$, $\tau = 1$ if $1 \leq i \leq I'$, $\tau = -1$ if $I' + 1 \leq i \leq I$ and $\Omega$ is a $C^2$ bounded domain with $x_i \in \Omega$ for $1 \leq i \leq I$. In addition, we assume that $x_i \neq x_i$ if $i \neq j$. Let $B = \{i : 1 \leq i \leq I' \text{ and } x_i \in \partial \Omega\}$. The energy functional is given by

\[J_\lambda(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + \lambda u^2) \, dx - \sum_{i=1}^I \tau_i \frac{1}{2^*(s)} \int_\Omega \frac{u_+^{2^*(s)}}{|x-x_i|^s} \, dx.\]

We see that $J_\lambda$ is $C^1$ and

\[
\langle J'_\lambda(u), \phi \rangle = \int_\Omega (\nabla u \nabla \phi + \lambda u \phi) \, dx - \sum_{i=1}^I \int_\Omega \frac{\tau_i u_+^{2^*(s)-1}}{|x-x_i|^s} \phi \, dx
\]

for $\phi \in H^1(\Omega)$.

In the same fashion as the proof of Proposition 1, we obtain the following proposition:

Proposition 3. The functional $J_\lambda$ satisfies the $(PS)_c$ condition for

\[c < \begin{cases}
\frac{2^* \|s\|^2}{2(N-1)} \frac{2^*(s)}{s} \|s\|^2 |s|^{-2} & \text{if } B = \emptyset \\
\frac{2^* \|s\|^2}{2(N-1)} \frac{2^*(s)}{s} |s|^{-2} & \text{if } B \neq \emptyset.
\end{cases}\]

Using Proposition 3, one can obtain the following theorems by the same method as we prove for Theorem 1.1.
Theorem 7.1 (Existence of solution to (25) for small $\lambda$). Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a $C^2$ bounded domain. Assume $\sum_{1 \leq i \leq I} \int_1^\infty \frac{\tau_i}{|x-x_i|} > 0$. Then there exists $\Lambda > 0$ such that (25) admit a positive solution for $\lambda$ with $0 < \lambda < \Lambda$.

Moreover, under the geometric setting of $x_1 \in \partial \Omega$ and the mean curvature $H(x_1)$ is positive, one can prove the existence of a positive solution to (25). Actually we may assume $x_1 = (0, \cdots, 0) \in \partial \Omega$ and the mean curvature $H(0)$ is positive. Then, up to rotation, the boundary near the origin can be represented by

$$x_n = h(x') = \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i x_i^2 + o(|x'|^2)$$

where $x' = (x_1, x_2, \cdots, x_{N-1}) \in D_\delta(0) = B_\delta(0) \cap \{x_N = 0\}$ for some $\delta > 0$. Here $\alpha_1, \alpha_2, \cdots, \alpha_{N-1}$ are the principal curvature of $\partial \Omega$ at $0$ and the mean curvature $\sum_{i=1}^{N-1} \alpha_i > 0$. Denote

$$g(x') = \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i x_i^2.$$ Consider

$$U_\varepsilon(x) := (\varepsilon | x |^{2-2})^{\frac{2-N}{4}}$$

for small parameter $\varepsilon > 0$. Choose $\delta$ such that $x_2, \cdots, x_I \notin B_{3\delta}(0)$. Set a cut-off function $\eta$ such that

$$\eta \in C^\infty_c(\mathbb{R}^N), \quad 0 \leq \eta \leq 1, \quad \eta = 1 \text{ in } B_\delta(0), \quad \eta = 0 \text{ in } \mathbb{R}^N \setminus B_{3\delta}(0).$$

Note that from $2^*(s) > 2$, $J_\lambda(T\eta U_\varepsilon) < 0$. We define

$$\mathcal{P} = \left\{ p(t) \mid p(t) : [0, 1] \to H^1(\Omega) \text{ is continuous map with } p(0) = 0 \in H^1(\Omega) \text{ and } p(1) = T\eta U_\varepsilon(x)|_\Omega \right\}.$$ Let

$$c^* := \inf_{p(t) \in \mathcal{P}} \sup_{0 \leq t \leq 1} \{ J_\lambda(p(t)) \}.$$ Then, thanks to Proposition 3, it suffices to show

$$c^* < \frac{2-s}{4(N-s)} S_s^2 \sigma^{2^*(s)}.$$ Denote

$$\widetilde{K}_0^\varepsilon := \int_\Omega |\nabla (\eta U_\varepsilon)|^2 \, dx,$$

$$\widetilde{K}_i^\varepsilon := \int_\Omega \tau_i (\eta U_\varepsilon)^{2^*(s)} \, dx \text{ for } 1 \leq i \leq I,$$

$$\widetilde{K}_{I+1}^\varepsilon := \int_\Omega (\eta U_\varepsilon)^2 \, dx.$$ Then by repeating arguments in proof of Theorem 1.2, we get estimates for each $\widetilde{K}_i^\varepsilon$ as follows:

- $\widetilde{K}_0^\varepsilon$

$$\widetilde{K}_0^\varepsilon = \begin{cases} \frac{1}{2} \widetilde{K}_0 - C_\varepsilon \frac{1}{x^{2-2}} |\ln \varepsilon| + O(\varepsilon^{\frac{1}{N}}) & \text{when } N = 3, \\ \frac{1}{2} \widetilde{K}_0 - I(\varepsilon) + O(\varepsilon^{\frac{1}{N-2}}) & \text{when } N \geq 4, \end{cases}$$
where
\[
\bar{I}(\varepsilon) := \int_{\mathbb{R}^{N-1}} \int_{0}^{g(x')} |\nabla U_{\varepsilon}|^2 dx_N dx',
\]
\[
\bar{K}_0 := \int_{\mathbb{R}^N} |\nabla U_{\varepsilon}|^2 dx.
\]

- \(\bar{K}_i^\varepsilon\)
\[
\bar{K}_i^\varepsilon = \frac{1}{2} \bar{K}_1 - \bar{II}(\varepsilon) + O(\varepsilon^{\frac{1}{2}})
\]

where
\[
\bar{II}(\varepsilon) := \int_{\mathbb{R}^{N-1}} \int_{0}^{g(x')} \frac{|U_{\varepsilon}|^2}{|x|^s} dx_N dx',
\]
\[
\bar{K}_1 = \int_{\mathbb{R}^N} \frac{|U_{\varepsilon}|^2}{|x|^s} dx.
\]

- \(\bar{K}_i^\varepsilon\) for \(2 \leq i \leq I\)
\[
\bar{K}_i^\varepsilon = O(\varepsilon^{\frac{1}{2}})
\]

and \(\bar{K}_i^\varepsilon > 0\) if \(2 \leq i \leq I'\)

- \(\bar{K}_{i+1}^\varepsilon\)
\[
\bar{K}_{i+1}^\varepsilon = \begin{cases} 
O(\varepsilon^{\frac{1}{2}}), & N = 3, \\
O(|\varepsilon^{\frac{1}{2}} \ln \varepsilon|), & N = 4, \\
O(\varepsilon^{\frac{1}{2}}), & N \geq 5.
\end{cases}
\]

Therefore using the fact
\[
\frac{\bar{II}(\varepsilon)}{\bar{I}(\varepsilon)} < \frac{(N - s) \bar{K}_1}{(N - 2) \bar{K}_0}
\]

which is verified in the proof of Theorem 1.2, we obtain the following theorem.

**Theorem 7.2** (Existence of solution to (25) with the boundary singularity). *Let \(s\) be a positive number satisfying \(1 \leq s < 2\) when \(N = 3\) and \(0 < s < 2\) when \(N \geq 4\). Assume \(\sum_{1 \leq i \leq I} \int_{\Omega} \frac{1}{|x - x_i|} > 0\). Suppose \(x_1 \in \partial \Omega\) and the mean curvature \(H(x_1)\) is positive. Then there exists a positive solution to (25).*

8. **Neumann Problem with the multiple positive singularities.** In this section, we deal with the existence theory for the equation
\[\begin{aligned}
-\Delta u + \lambda u &= \sum_{i=1}^{I} \frac{|u|^{2^*(s_i)-2} u}{|x - x_i|^s_i} \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}\]

where \(0 < s_i < 2\), \(\Omega\) is a \(C^2\) bounded domain with \(x_i \in \partial \Omega\) for \(1 \leq i \leq I' - 1\) and \(x_i \in \Omega\) for \(I' < i \leq I\). In addition, we assume that \(x_{i_1} \neq x_{i_2}\) if \(i_1 \neq i_2\). The energy functional is given by
\[
J_\lambda(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + \lambda u^2) dx - \sum_{i=1}^{I} \int_{\Omega} \frac{u_{+}^{2^*(s_i)}}{|x - x_i|^{s_i}} dx.
\]
We see that $J_\lambda$ is $C^1$ and
\[
(J_\lambda'(u), \phi) = \int_\Omega (\nabla u \nabla \phi + \lambda u \phi) dx - \sum_{i=1}^l \int_\Omega \frac{u_+^{2^*(s_i)-1}}{|x-x_i|^{s_i}} \phi dx
\]
for $\phi \in H^1(\Omega)$.

In the same fashion as the proof of Proposition 1, we obtain the following proposition:

**Proposition 4.** The functional $J_\lambda$ satisfies the $(PS)_c$ condition for
\[
c < \min \left( \min_{1 \leq i \leq l' - 1} \frac{2 - s_i}{4(N - s_i)} S_{s_i}^{2^*(s_i)-2}, \min_{l' \leq i \leq l} \frac{2 - s_i}{4(N - s_i)} S_{s_i}^{2^*(s_i)-2} \right).
\]

Using Proposition 4, one can obtain the following theorems by the same method as we prove for Theorem 1.1.

**Theorem 8.1** (Existence of solution to (26) for small $\lambda$). There exists $\Lambda > 0$ such that (26) admit a positive solution for $\lambda$ with $0 < \lambda < \Lambda$.

Moreover, under the geometric setting of $x_1 \in \partial \Omega$ and the mean curvature $H(x_1)$ is positive, one can prove the existence of a positive solution to (25) when
\[
\frac{2 - s_1}{4(N - s_1)} S_{s_1}^{2^*(s_1)-2} = \min \left( \min_{1 \leq i \leq l' - 1} \frac{2 - s_i}{4(N - s_i)} S_{s_i}^{2^*(s_i)-2}, \min_{l' \leq i \leq l} \frac{2 - s_i}{4(N - s_i)} S_{s_i}^{2^*(s_i)-2} \right).
\]
Actually we may assume $x_1 = (0, \cdots, 0) \in \partial \Omega$ and the mean curvature $H(0)$ is positive.

We consider the existence of positive solution to the following problem
\[
\begin{aligned}
-\Delta u + \lambda u &= |u|^{2^*(s_1)-2} u \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial \Omega
\end{aligned}
\tag{27}
\]
The corresponding functional is defined by
\[
\widetilde{J}_\lambda(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + \lambda u^2) dx - \frac{1}{2} \int_\Omega \frac{u_+^{2^*(s_1)}}{|x-x_1|^{s_1}} dx.
\]
Then by Theorem 3.1 in [3], (27) has a positive solution $w$ satisfying
\[
\widetilde{J}_\lambda(w) = \min_{u \in H^1(\Omega) \setminus \{0\}} \max_{t > 0} \widetilde{J}_\lambda(tu) < \frac{2 - s_1}{4(N - s_1)} S_{s_1}^{2^*(s_1)-2}.
\]
Therefore
\[
c^* := \min_{u \in H^1(\Omega) \setminus \{0\}} \max_{t > 0} \widetilde{J}_\lambda(tu) = \min_{u \in H^1(\Omega) \setminus \{0\}} \max_{t > 0} \left( \widetilde{J}_\lambda(tu) - \sum_{j \neq 1} \frac{t^{2^*(s_j)}}{2^*(s_j)} \int \frac{|u|^{2^*(s_j)}}{|x-x_j|^{s_j}} dx \right)
\leq \min_{u \in H^1(\Omega) \setminus \{0\}} \max_{t > 0} \widetilde{J}_\lambda(tu) = \widetilde{J}_\lambda(w) < \frac{2 - s_1}{4(N - s_1)} S_{s_1}^{2^*(s_1)-2}.
\]
Hence $c^*$ is a critical value and a positive solution is found.

**Theorem 8.2** (Existence of solution to (26) with the boundary singularity). Suppose $x_1 \in \partial \Omega$ and the mean curvature $H(x_1)$ is positive. If
\[
\frac{2 - s_1}{4(N - s_1)} S_{s_1}^{2^*(s_1)-2} = \min \left( \min_{1 \leq i \leq l' - 1} \frac{2 - s_i}{4(N - s_i)} S_{s_i}^{2^*(s_i)-2}, \min_{l' \leq i \leq l} \frac{2 - s_i}{4(N - s_i)} S_{s_i}^{2^*(s_i)-2} \right),
\]
then there exists a positive solution to \((26)\).

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