Lie Symmetry Analysis on Benjamin-Ono Equation

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Abstract. Benjamin-Ono Equation are significantly important in describes the one-dimensional internal waves in deep water. Because of its significance, Lie symmetry reduction were chosen to reduce the equation and hence solve the equation. Lie symmetry analysis is one of the powerful methods to solve partial differential equation. Due to its effectiveness, this method is widely applied in solving equation in various field. In this paper, calculation of symmetry of the equation was first present, followed by reduction of the equation. The equation are reduced from non-linear partial differential equation (PDE) to ordinary differential equation (ODE) and hence analytic solution of the partial differential equation was obtained by solving the reduced ODE.

1. Introduction

Differential equations are widely applied in various field which includes physics, biology, mathematics, engineering as well as economics. It is widely used to model real world situations that are continuously changing since derivatives describe the rate of change mathematically. The nonlinear partial differential equation has been used in a variety of physical theories including dynamics to generate canonical transformation, continuum mechanics to record conservation of mass, momentum and so on as well as optics to describe wave fronts. This paper is to analysis Benjamin-Ono Equation. Benjamin-Ono equation is one type of nonlinear partial integro-differential equation. It is a well-known interval waves equation which is used to determine the one-dimensional internal waves in deep water. This equation introduced by Thomas Brooke Benjamin in 1967 and H. Ono 1975 [1, 2]. Benjamin-Ono equation is given as below

\[ u_\xi + H u_{\xi\xi} + uu_\xi = 0, \]

where \( H \) is the Hilbert transform,

\[ H(u)(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\tau)}{\xi - \tau} \, d\tau \]

Hilbert Transform is a specific linear operator for the improper integral are easily understood in the principle value sense. In 1977, by using analytical method, Joseph obtained a two-soliton solution. In 1978, through found a new conservation quantity, Meiss and Pereira surmised that two-soliton and three-soliton solution for the equation. Besides, Hirota had transform the equation to a bilinear equation form and hence obtained the multi-soliton or N-soliton solution. By using N-Soliton method in solving this equation, obtained the following:
In this paper, because of the equation’s significance on describe wave motion in deep water, Lie symmetry method been applied to solve the equation to obtain more solutions of the equation, hence one can observe the properties of the wave.

2. Symmetry of Benjamin-Ono Equation

Symmetry analysis is one of useful method to deal with partial differential equation (PDE) and ordinary differential equation (ODE) [3]. This can be seen from the past research regarding applied symmetry analysis to solve differential equation [4, 5]. Because of its efficiency in solving equation, it is widely applied to solve complicated equation. For example, Kumar & Kumar [6] used Lie symmetry to solve shallow wave equation, Zhang [7] had use Lie symmetry to analysis longitudinal wave motion equation. In order apply symmetry analysis, one have to find symmetry of the equation. The general infinitesimal operator (symmetry) is defined as

\[ G = \varphi \frac{\partial}{\partial \zeta} + \tau \frac{\partial}{\partial \xi} + \phi \frac{\partial}{\partial u} \]

The derivation of first order prolongation of infinitesimal operator

\[ G^{[1]} = \varphi \frac{\partial}{\partial \zeta} + \tau \frac{\partial}{\partial \xi} + \phi \frac{\partial}{\partial u} + \phi_{\xi} \frac{\partial}{\partial u_{\xi}} + \phi_{\xi \xi} \frac{\partial}{\partial u_{\xi \xi}} + \phi_{\xi \xi \xi} \frac{\partial}{\partial u_{\xi \xi \xi}} \]

whereas the second order prolongation is

\[ G^{[2]} = \varphi \frac{\partial}{\partial \zeta} + \tau \frac{\partial}{\partial \xi} + \phi \frac{\partial}{\partial u} + \phi_{\xi} \frac{\partial}{\partial u_{\xi}} + \phi_{\xi \xi} \frac{\partial}{\partial u_{\xi \xi}} + \phi_{\xi \xi \xi} \frac{\partial}{\partial u_{\xi \xi \xi}} + \phi_{\xi \xi \xi} \frac{\partial}{\partial u_{\xi \xi \xi}} \]

with coefficients

\[ \phi_{\xi} = D_{\xi}(\varphi) - u_{\xi} D_{\xi}(\phi) - u_{\xi} D_{\xi}(\tau), \]
\[ \phi_{\xi \xi} = D_{\xi}(\phi_{\xi}) - u_{\xi} D_{\xi}(\varphi) - u_{\xi} D_{\xi}(\tau), \]
\[ \phi_{\xi \xi \xi} = D_{\xi}(\phi_{\xi \xi}) - u_{\xi} D_{\xi}(\phi_{\xi}) - u_{\xi} D_{\xi}(\tau), \]
\[ \phi_{\xi \xi \xi \xi} = D_{\xi}(\phi_{\xi \xi \xi}) - u_{\xi} D_{\xi}(\phi_{\xi \xi}) - u_{\xi} D_{\xi}(\tau). \]

In order to calculate symmetry of equation (1), applied

\[ G^{[2]}(u_{\xi} + H u_{\xi} + u_{\zeta}) = 0, \]

After simplified, one can get equation (2) as

\[ \phi_{\xi} + H \phi_{\xi \xi} + \phi_{\xi \xi} u + \phi_{u} u_{\zeta} = 0 \]

Since \( \phi_{\xi} \), \( \phi_{\xi \xi} \) and \( \phi_{\xi \xi \xi} \) are present, these three terms are to be derived as follow:

\[ \phi_{\xi} = \phi_{\xi} + \phi_{u} u_{\xi} - \phi_{\xi} u_{\xi} - \phi_{u} u_{\xi} u_{\xi} - \tau_{\xi} u_{\xi} - \tau_{u} u_{\xi}^{2} \]
\[ \phi_{\xi \xi} = \phi_{\xi \xi} + 2 \phi_{u} u_{\xi} - \tau_{\xi} u_{\xi} - \phi_{\xi} u_{\xi} + \phi_{u u} u_{\xi}^{2} - 2 \tau_{\xi} u_{\xi} - 2 \phi_{u} u_{\xi} u_{\xi} - \tau_{u u} u_{\xi} u_{\xi}^{2} - \phi_{u u} u_{\xi} u_{\xi}^{2} + \phi_{u} u_{\xi \xi} - 2 \tau_{\xi} u_{\xi \xi} - 2 \phi_{u} u_{\xi} u_{\xi} - 3 \tau_{u} u_{\xi} u_{\xi} - \phi_{u} u_{\xi} u_{\xi} - 2 \phi_{u} u_{\xi} u_{\xi} \]
\[ \phi_{\xi \xi \xi} = \phi_{\xi \xi \xi} + 2 \phi_{u} u_{\xi} - \phi_{\xi \xi} u_{\xi} - \tau_{\xi} u_{\xi} + \phi_{u u} u_{\xi}^{2} - 2 \phi_{u} u_{\xi} u_{\xi} - 2 \tau_{\xi} u_{\xi} - \phi_{u u} u_{\xi}^{2} - 3 \tau_{u} u_{\xi} u_{\xi} - \phi_{u} u_{\xi} u_{\xi} - 2 \tau_{u} u_{\xi} u_{\xi} \]

Substitute \( \phi_{\xi} \), \( \phi_{\xi \xi} \) and \( \phi_{\xi \xi \xi} \) and equation (1) into equation (3).
The equation is then expanded and categorized by the derivatives of $u$.

$$0 = \phi_\xi - \phi_\zeta(-Hu_\zeta - uu_\zeta) + (\phi_u - \tau_u)(-Hu_\zeta - uu_\zeta) - \phi_\xi u_\zeta u_\xi - \tau_u(-Hu_\zeta - uu_\zeta)^2 + H\phi_\zeta$$

$$+ Hu_\xi(2\phi_{\zeta u} - \phi_\zeta) - H\tau_{\zeta u}(-Hu_\zeta - uu_\zeta) + H\phi_\zeta(\phi_u - 2\phi_\zeta)$$

$$- 2H\tau_{\zeta u}u_\zeta(-Hu_\zeta - uu_\zeta) - H\phi_u u_\zeta^3 + Hu_\zeta(\phi_u - 2\phi_\zeta) - 2H\tau_u u_\zeta - 3H\phi_u u_\zeta u_\zeta$$

$$- H\tau_{\zeta u}u_\zeta(-Hu_\zeta - uu_\zeta) - 2H\tau_u u_\zeta u_\zeta - H\tau_{\zeta u}u_\zeta^2(-Hu_\zeta - uu_\zeta) + u\phi_\zeta$$

$$+ uu_\zeta(\phi_u - \phi_\zeta) - u\tau_u(-Hu_\zeta - uu_\zeta) - u\phi_u u_\zeta^2 - uu_\zeta u_\zeta(-Hu_\zeta - uu_\zeta) + u\phi_\zeta$$

Remainder: $\phi_\tau + H\phi_0 + u\phi_\xi = 0$

After simplified all the equation above, obtain

$$\tau(\zeta, \xi, u) = -2c_1^2 + c_3, \quad \phi(\zeta, \xi, u) = -c_1^2 + c_2^2 + c_4, \quad \phi(\zeta, \xi, u) = c_1^2 + c_2.$$

Where, $c_2, c_3, c_4, \text{ and } c_5$ are arbitrary constants. Hence, the symmetry of equation (1) are

$$X_1 = \frac{\partial}{\partial \zeta}, \quad X_2 = \frac{\partial}{\partial \xi}, \quad X_3 = \frac{\partial}{\partial u} + \xi \frac{\partial}{\partial \zeta}, \quad X_4 = \frac{\partial}{\partial \tau} - 2\xi \frac{\partial}{\partial \zeta} - \zeta \frac{\partial}{\partial \zeta}$$

3. Symmetry Reduction of Benjamin-Ono Equation

By taking different combination of symmetry, two reduction were conducted in order to obtain exact solutions of equation (1).

3.1. Reduction By $X_1$ and $X_2$

By taking combination of $X_1$ and $X_2$,

$$X = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \xi}$$

To find the invariants, the corresponding characteristic equation is

$$\frac{d\zeta}{1} = \frac{d\xi}{1} = \frac{du}{0}.$$ 

Hence, similarity equation obtained as

$$\beta = \zeta - \xi, \quad \alpha = u.$$ 

Derivatives as below

$$u_\zeta = -\alpha_\beta, \quad u_{\zeta\xi} = \alpha_\beta, \quad u_\zeta = \alpha_\beta, \quad u_{\zeta\zeta} = \alpha_\beta.$$
Substitute all the derivatives of $u$ into equation (1) to obtain a first order ordinary differential equation after the second order derivatives have been cancelled off.

$$H \alpha_{\beta \beta} + \alpha_{\beta} (\beta - 1) = 0$$

This is a second order ordinary differential equation (ODE), to reduce this equation into first order ODE, Lie symmetry reduction been used again, take $M = \alpha$ and $N = \alpha_{\beta}$, compute the following:

$$\frac{dN}{dM} = \frac{\frac{dN}{d\alpha}}{\frac{dM}{d\alpha}} = \frac{\frac{dN}{d\alpha}}{1} = \frac{dN}{d\alpha} = \frac{\frac{dN}{d\beta}}{\frac{d\alpha}{d\beta}} = \frac{\alpha_{\beta \beta}}{N}$$

$$N \frac{dN}{dM} = \alpha_{\beta \beta} \cdot$$

Substitute these variables into the second order ODE and get

$$HN \frac{dN}{dM} + N(M - 1) = 0.$$ 

This implies

$$\frac{dN}{dM} = \frac{1 - M}{H}$$

Integrate both sides,

$$N = \frac{M}{H} - \frac{M^2}{H} + k,$$

where $k$ is integrate constant. Which means

$$\alpha_{\beta} = \frac{\alpha}{H} - \frac{\alpha^2}{H} + k,$$

Solve the 1st ODE

$$\alpha(\beta) = 1 + \tanh \frac{1}{2} \sqrt{2k H + 1} (\beta + c) \frac{\beta}{H} \sqrt{2Hk + 1}$$

and transform back, following solution of equation (1) are obtained

$$u(\zeta, \xi) = 1 + \tanh \frac{1}{2} \sqrt{2k H + 1} (\zeta - \xi + c) \frac{\zeta}{H} \sqrt{2Hk + 1}.$$ 

3.2. Reduction By $X_4$

By taking symmetry $X_4$,

$$x_4 = u \frac{\partial}{\partial u} - 2 \xi \frac{\partial}{\partial \xi} - \zeta \frac{\partial}{\partial \zeta}$$

To find the invariants, the corresponding characteristic equation is

$$\frac{d\zeta}{-\zeta} = \frac{d\xi}{-2\xi} = \frac{du}{U}.$$
Hence, similarity equation obtained as
\[ \alpha = \frac{\zeta}{\sqrt{\xi}}, \quad \beta = u\zeta, \]

Derivatives as below
\[ u_\xi = -\frac{1}{2} \beta_a \xi^{\frac{3}{2}}, \quad u_\zeta = -\frac{\beta}{\xi^2} + \frac{\beta_a}{\xi^{\frac{1}{2}}}, \quad u_{\zeta\zeta} = \frac{2\beta}{\xi^3} + \frac{\beta_{aa}}{\xi^{\frac{3}{2}}} - \frac{2\beta_a}{\xi^{\frac{5}{2}}} \]

Substitute all the derivatives of \( u \) into equation (1) to obtain a first order ordinary differential equation after the second order derivatives have been cancelled off.
\[-\frac{\beta_a}{2} + \frac{2H\beta}{\alpha^2} + \frac{\beta_{aa}}{\alpha} - \frac{2\beta_a}{\alpha^2} + \frac{\beta^2}{\alpha^3} + \frac{\beta\beta_a}{\alpha^2} = 0 \]

This is a second order ordinary differential equation (ODE). So, let \( M = \alpha, N = \alpha\beta \) and \( N \frac{dN}{dM} = \alpha\beta \). Now rewrite the ordinary differential equation
\[-\frac{N}{2} + \frac{2HM}{\alpha^2} + \frac{NN'}{\alpha} - \frac{2N}{\alpha^2} + \frac{M^2}{\alpha^3} + \frac{MN}{\alpha^2} = 0 \]

After simplified,
\[ N' = \frac{1}{2} - \frac{2HM}{\alpha^2N} + \frac{2}{\alpha} - \frac{M^2}{\alpha^2N} - \frac{M}{\alpha} \]

This is a Chini Equation, which only admits under certain condition. For the equation we obtain, does not lies under the condition, hence for this moment, no possible solutions were obtained here. However, from a non-linear PDE to a 1st order ODE.

4. Discussion And Conclusion
In this paper, two set of reductions are done. First reduction is done by combination of \( X_1 \) and \( X_2 \). One solution is successfully obtained.
\[ u(\zeta, \xi) = 1 + \tanh \frac{1}{2} \sqrt{2Hk + 1} \left( \zeta - \xi + c \right) \sqrt{2Hk + 1}. \]

For the second reduction, the equation (1) is reduce by using \( X_4 \). For this reduction, Chini equation was obtained. This equation was unable to solve. However, the equation (1) had been reduced from a second order nonlinear PDE into a first order ODE. One can observe the properties of the equation from the ODE. Furthermore, comparison between the results yielded by this project and other researches could not be done to check for its precision as the results are in different forms compare to research done before. For further research, author wish to find conservation laws of the equation by using Lie symmetry. Conservation laws are significant in deciding the integrability of the equation. [8]

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