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Geodesic X-ray tomography for piecewise constant functions on nontrapping manifolds.
(English) Zbl 1440.53082
Math. Proc. Camb. Philos. Soc. 168, No. 1, 29-41 (2020).

The authors define a piecewise constant function on a manifold \( M \) as follows:

Definition 2.5 (Piecewise constant function). Let \( M \) be a manifold with or without boundary, with a \( C^1 \)-structure. We say that a function \( f: M \to \mathbb{R} \) is piecewise constant if there is a regular tiling \( \{ \Delta_i; i \in I \} \) of \( M \) so that \( f \) is constant in the interior of each regular \( n \)-simplex \( \Delta_i \), and vanishes on \( \bigcup_{i \in I} \partial \Delta_i \).

They quote the definition of strictly convex foliation given in an earlier paper of the third author [G. P. Paternain et al., Invent. Math. 193, No. 1, 229–247 (2013; Zbl 1275.53067)] as below:

Definition 2.8 (Strictly convex foliation). Let \( M \) be a smooth Riemannian manifold with boundary. We say that: (i) the manifold \( M \) satisfies the foliation condition if there is a smooth strictly convex function \( \phi: M \to \mathbb{R} \); (ii) a connected open subset \( U \) of \( M \) satisfies the foliation condition if there is a smooth strictly convex exhaustion function \( \phi: U \to \mathbb{R} \), in the sense that the set \( \{ x \in U : \phi(x) \geq c \} \) is compact for any \( c > \inf U \phi \).

In a recent paper, G. P. Paternain et al. [Invent. Math. 193, No. 1, 229–247 (2013; Zbl 1275.53067)] conjectured that the geodesic X-ray transform on compact nontrapping Riemannian manifolds with strictly convex boundary is injective. The authors of the paper under review prove the veracity of this conjecture of Paternain et al. [loc. cit.] for two-dimensional nontrapping manifolds for the case of piecewise constant functions by establishing the following core result of this paper:

Theorem 1.1. Let \( (M, g) \) be a compact nontrapping Riemannian manifold with strictly convex smooth boundary, and let \( f: M \to \mathbb{R} \) be a piecewise constant function. Let either

- (a) \( \dim (M) = 2 \), or
- (b) \( \dim (M) \geq 3 \) and \( (M, g) \) admits a smooth strictly convex function.

If \( f \) integrates to zero over all geodesics joining boundary points, then \( f \equiv 0 \).

The result of the above theorem for the case \( \dim (M) = 2 \) appears to be new, while the result for the case \( \dim (M) \geq 3 \) is a special case of the much more general result applicable to the functions in \( L^2(M) \) proven recently by G. Uhlmann and A. Vasy [Invent. Math. 205, No. 1, 83–120 (2016; Zbl 1350.53098)].

For proving the result of Theorem 1.1, the authors prove a number of ancillary lemmas and theorems in Sections 4, 5 and 6. Two representative results which deserve to be mentioned in this connection are:

Lemma 5.1. Let \( M \) be a \( C^2 \)-smooth Riemannian surface and assume that \( f: M \to \mathbb{R} \) is a piecewise constant function in the sense of Definition 2.5. Fix \( x \in \text{int}(M) \) and let \( \Sigma \) be a 1-dimensional hypersurface (curve) through \( x \). Suppose that \( V \) is a neighbourhood of \( x \) so that:

- (i) \( V \) intersects only simplices containing \( x \);
- (ii) \( \Sigma \) is strictly convex in \( V \);
- (iii) \( f|_{V^-} = 0 \); and
- (iv) \( f \) integrates to zero over every maximal geodesic in \( V \) having endpoints on \( \Sigma \).

Then \( f|_{V^-} = 0 \) (where, \( V^- \) is an open set of \( V \backslash \Sigma \)).

Lemma 6.2. Let \( M \) be a \( C^2 \)-smooth Riemannian manifold and \( f: M \to \mathbb{R} \) be a piecewise constant function in the sense of Definition 2.5. Fix \( x \in \text{int}(M) \) and let \( \Sigma \) be an \((n-1)\)-dimensional hypersurface through \( x \). Suppose that \( V \) is a neighbourhood of \( x \) so that:

- (i) \( V \) intersects only simplices containing \( x \);
- (ii) \( \Sigma \) is strictly convex in \( V \);
- (iii) \( f|_{V^-} = 0 \); and
- (iv) \( f \) integrates to zero over every maximal geodesic in \( V \) having endpoints on \( \Sigma \).
Then $f|_V = 0$.

With the help of these two crucial lemmas and a number of other auxiliary lemmas and theorems proven in the Sections 3 to 6, the authors prove their key theorem (a support theorem from which ‘the classical support theorem of S. Helgason [The Radon transform. 2nd ed. Boston, MA: Birkhäuser (1999; Zbl 0932.43011)] for the X-ray transform in the case of piecewise constant functions follows’) in the sixth section of the paper as:

Theorem 6.4. Let $M$ be a smooth Riemannian manifold with strictly convex boundary. Assume that $\dim(M) \geq 2$. Suppose that there is a strictly convex foliation of an open subset $U \subset M$ in the sense of Definition 2.8. Let $f : M \to \mathbb{R}$ be a piecewise constant function in the sense of Definition 2.5. If $f$ integrates to zero over all geodesics in $U$, then $f|_U = 0$.

Three corollaries are deduced from the above support theorem which ultimately lead to the proof of the core result of the Theorem 1.1 above. The authors fully deserve the commendation of the reviewer for proving this deep result of far reaching consequences for the case of piecewise constant functions in geodesic X-ray transforms in an elementary manner as stated by them in the abstract and introduction of this paper. The reviewer most optimistically thinks that these types of fundamental results being enunciated by the leading mathematicians of our generation and our times today will go a long way in deciding the course of the future technological developments of the advanced medical imaging and diagnostic tools and equipments in the coming many decades for which these and related current developments in the mathematical literature in the field of geodesic X-ray transform and the allied fields undoubtedly provide a firm mathematical and engineering foundation.

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**MSC:**

- 53C65 Integral geometry
- 44A12 Radon transform
- 53C22 Geodesics in global differential geometry
- 57R30 Foliations in differential topology; geometric theory
- 57R32 Classifying spaces for foliations; Gelfand-Fuks cohomology
- 92C50 Medical applications (general)
- 92C55 Biomedical imaging and signal processing
- 94A08 Image processing (compression, reconstruction, etc.) in information and communication theory
- 94A12 Signal theory (characterization, reconstruction, filtering, etc.)

**Keywords:**

geodesic X-ray transform; piecewise constant function; strictly convex foliation; compact nontrapping Riemannian manifold; regular tiling; regular simplex; tangent function; tangent cone of a regular simplex; tangent space of a regular simplex

**Full Text:** DOI arXiv

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