Transition Temperature of Dilute, Weakly Repulsive Bose Gas

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(Dated: March 22, 2022)

Within a quasiparticle framework, we reconsider the issue of computing the Bose-Einstein condensation temperature ($T_c$) in a weakly non-ideal Bose gas. The main result of this and previous investigations is that $T_c$ increases with the scattering length $a$, with the leading dependence being either linear or log-linear in $a$. The calculation of $T_c$ reduces to that of computing the excitation spectrum near the transition. We report two approaches to regularizing the infrared divergence at the transition point. One leads to a $\alpha \sqrt{|\ln a|}$-like shift in $T_c$, and the other allows numerical calculations for the shift.

PACS numbers: 03.75.Fi 05.30.Jp

I. INTRODUCTION

Inter-particle correlations in the weakly repulsive dilute Bose gas have been studied for more than half a century. It is therefore quite remarkable that the dependence of the Bose-Einstein condensation temperature ($T_c$) on the interaction strength has continued to generate controversy until this time. In this paper we present an analysis of the longstanding $T_c$ problem starting from a quasiparticle picture of the system in the un-condensed (high-temperature) side.

The model is described by the Hamiltonian

$$\hat{H} = \sum_k \epsilon_k \hat{b}_k^\dagger \hat{b}_k + \frac{1}{2V} \sum_{p,q,k} U(k) \hat{b}_{p+k}^\dagger \hat{b}_q^\dagger \hat{b}_k \hat{b}_q,$$

where $\epsilon_k = k^2/2m$ is the free-gas spectrum, and $\hat{b}, \hat{b}^\dagger$ are bosonic operators. In principle, one is interested in any possible interaction function $U(k)$, but the simplest possible form is a delta function in real space, so that $U(k) = U$ is momentum-independent. To first order $U$ is related to the $s$-wave scattering length $a$ by $U = 4\pi \hbar^2 a/m$. A dimensionless measure of the interaction is the quantity $an^{1/3}$, where $n = N/V$ is the density. The program is to examine the shift in the critical temperature $T_c$, as compared to a noninteracting gas of the same density, as a function of $an^{1/3}$, for small $an^{1/3}$.

For several decades, the quasiparticle picture has been a central paradigm for studying correlated many-body systems. Even in a highly correlated Bose liquid like liquid helium, a quasiparticle description accounts for the transition temperature extremely well; e.g., the lambda point temperature can be obtained from the ideal gas expression $T_c^{(0)} = (2\pi/m(\zeta(3/2))^{2/3}) n^{2/3}$ by replacing the helium atomic mass $m$ by a quasiparticle effective mass, $m^*$. The effective mass in the liquid exceeds the bare mass, because of the inertia of the medium which has to make way for any atom to move. Correspondingly, the transition temperature is lower for liquid $^4$He than would be the case for an ideal bose gas with the same bare mass.

The question we would like to address is the following: how much information can one get about $T_c$ from a quasiparticle description of the weakly interacting Bose gas? The simplest treatment, a Hartree correction, does not give any contribution to the transition temperature. The exchange contribution at mean field level (Fock), does not have any effect on $T_c$ for a momentum-independent interaction, but gives a negative shift for a momentum-dependent interaction. The Fock contribution to $T_c$ depends on the details of the interaction, and for small interactions is weaker than the leading shift we are examining. The present understanding is that there is no mean field effect on the leading shift, and that higher-order correlations are required to calculate this leading shift in $T_c$.

Interaction effects in the non-ideal Bose gas can be described in terms of the two-body $t$-matrix, i.e., the vacuum scattering amplitude. This quantity describes all possible collisions between two particles, without taking into account the fact that the surrounding medium has an effect on these collisions. Many-body corrections arising from the surrounding gas can be treated by using the many-body $T$-matrix, which is the sum of ladder diagrams.

In this paper, we approach the calculation of $T_c$ for a Bose gas using perturbation theory in the two-body $t$-matrix. This is to be regarded as a first step toward a calculation using the many-body $T$-matrix. The idea is that, if one can regard the system as a weakly interacting gas of quasiparticles, then perturbation theory in the quasiparticle interactions should give a reasonable starting point, and the effects of the surrounding medium can be treated as a correction to these results.
The work in this paper is not meant to be a definitive determination of $T_c$ for the weakly interacting boson gas, but instead pursues the more limited aim of calculating $T_c$ within a quasiparticle framework. As is typical for descriptions of critical-point properties, our approach faces infrared divergences. We elaborate on two ways to regularize these divergences.

During the past several years, work by Baym, Laloë and collaborators\textsuperscript{1-6}, has predicted a positive shift in $T_c$ which, to leading order, is linear in the scattering length: $\Delta T_c = c_1 a^{1/3} + O(a^2 n^{2/3})$. Attempts to calculate the coefficient $c_1$, however, have continued to give fluctuating results. This has prompted the Baym group to predict\textsuperscript{5} a logarithmic contribution at the next order, i.e., $-a^2 \ln a$ contribution. The presence of such a non-analyticity might explain the difficulty in any numerical estimate of $c_1$.

Stoof, from a renormalization-group analysis\textsuperscript{7,8}, has predicted an increase of the transition temperature proportional to $a |\ln a|$, i.e., a non-analyticity at leading order. A leading non-analytic behavior would actually explain even better the widely varying results obtained in attempts to calculate the coefficient $c_1$ based on the assumption of linear shift $\Delta T_c \sim c_1 a^{1/3}$. One should also note that the work of Baym and collaborators relies heavily on power-counting and dimensional arguments. This kind of argument typically does not catch logarithmic corrections. The possibility of leading non-analytic dependence of $\Delta T_c$ on the interaction, therefore, needs to be further examined.

An older attack on the $T_c$ problem is that of Kanno\textsuperscript{1,12}. Using a “quasi-linear” canonical transformation, Kanno calculated the free energy and hence $T_c$ of the Bose gas. Other derivations of the transition temperature include the canonical-ensemble calculation\textsuperscript{13} of Wilkens et al leading to a negative shift, Schakel’s effective-action theory\textsuperscript{15,16} leading to a prediction for a positive linear shift, and Kleinert’s recent five-loop calculation\textsuperscript{17}. In addition, there have been spurious mean-field level predictions\textsuperscript{18,19} for shifts in $T_c$ proportional to the square root of the interaction. One could think of studying the $T_c$ problem experimentally, e.g., in the context of trapped atomic Bose gases. Unfortunately, trap effects dominate in the dependence of $T_c$ on the interaction for these systems. Size effects cause a decrease of $T_c$ at mean-field level, far larger than the intrinsic effect under study in this paper. The $^4$He-Vycor system\textsuperscript{21} might be more suitable for studying the $T_c$ problem, but is yet to reach the accuracies necessary for identifying logarithm-like corrections.

II. THE QUASIPARTICLE APPROACH

The starting points of our analysis is the expansion of the single-particle Green function to leading nontrivial order in quasiparticle interactions, namely:

$$G(k, z) = \frac{1}{z - (\epsilon_k - \mu + \Sigma_R(k, E_k)) - (\Sigma(k, z) - \Sigma_R(k, E_k))}$$

$$\approx \frac{1}{z - E_k} + \frac{[\Sigma(k, z) - \Sigma_R(k, E_k)]}{[z - E_k]^2} + \ldots = \frac{1}{z - E_k} \left[ 1 + \int \frac{d\omega}{\pi} \frac{\Sigma_I(k, \omega)}{(\omega - z)(\omega - E_k)} \right] + \ldots \quad (1)$$

Here $\Sigma_R, I(k, \omega) = \Re \Sigma, \Im \Sigma(k, \omega + i0^+)$ are the real and imaginary parts of the self energy, and $E_k = \epsilon_k - \mu + \Sigma_R(k, E_k)$ is the quasiparticle energy. This expression has a number of attractive properties which would not appear in simple perturbation theory in the inter-particle interactions.

In the quasi-particle approximation (1), the single-particle spectral function $A(k, \omega) \equiv -2\Im G(k, \omega + i0^+)$ is given by:

$$A(k, \omega) \approx 2\pi \delta(\omega - E_k) \left[ 1 + \int \frac{d\omega'}{\pi} \frac{\Sigma_I(k, \omega')}{(\omega' - E_k)^2} \right] - \frac{\Sigma_I(k, \omega)}{(\omega' - E_k)^2} \quad (2)$$

Note that the form (2) automatically satisfies the spectral sum-rule, $\int d\omega A(k, \omega) = 1$.

We proceed by writing the particle number $N$ as a momentum sum over the convolution of the spectral function and the Bose thermal distribution function, $f(\omega) = 1/(e^{\beta \omega} - 1)$.

$$N = \sum_k \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(k, \omega)f(\omega) = N_0 + V \int \frac{d^3k}{(2\pi)^3} f(E_k) + V \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{\pi} \frac{[f(E_k) - f(\omega)]\Sigma_I(k, \omega)}{(\omega - E_k)^2} \quad (3)$$

To proceed further we must resort to an explicit expression for the self-energy. We make the assumption that our quasi-particles interact weakly, so that we can work perturbatively. We limit ourselves to terms up to second order in the interaction.
A. Perturbation Theory

We first perform perturbation theory for the self-energy in the bare potential $U(k)$, retaining explicit momentum-dependence. The diagrams that need to be calculated are shown in figure 1. After performing the Matsubara sums, we get:

$$
\Sigma(k, z) \approx \int \frac{d^3p}{(2\pi)^3} \left[ U(k - p) + U(0) \right] f(\tilde{\epsilon}_p) \\
+ \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \left[ f(\tilde{\epsilon}_q) + f(\tilde{\epsilon}_q) f(\tilde{\epsilon}_{p+q}) + f(\tilde{\epsilon}_q) f(\tilde{\epsilon}_{p-k}) - f(\tilde{\epsilon}_{p+q}) f(\tilde{\epsilon}_{p-k}) \right] \times \frac{z - (\tilde{\epsilon}_{p+q} + \tilde{\epsilon}_{p-k} - \tilde{\epsilon}_q)}{U(p)} [U(-p) + U(k - p - q)] .
$$

(4)

Here $\tilde{\epsilon}_k = \epsilon_k - \mu$. For a momentum-independent $U(k) = U$, the first-order Hartree and Fock terms give equal contributions, as do the two second-order diagrams. However, using $U(k) = U$ directly in (4) causes ultraviolet (UV) problems in the second-order self-energy, due to the term with a single Bose function. The prescription for removal of such divergences is to shift to a description in terms of the vacuum scattering amplitude $t$, which is a more physical quantity than the bare potential. The bare potential $U(k)$ is related to the $t$-matrix by

$$
U(k) = \Re \langle p - k, q + k | t | p, q \rangle - \int \frac{d^3k'}{(2\pi)^3} \Re \langle p - k, q + k | t | p - k', q + k' \rangle [p - k', q + k' | t | p, q] \frac{\epsilon_p + \epsilon_q - \epsilon_{p-k} - \epsilon_{q+k}}{\epsilon_p + \epsilon_q - \epsilon_{p-k} - \epsilon_{q+k}} .
$$

(5)

When eq (5) is used to replace the bare potential by the $t$-matrix in eq (4), we get an $O(t^2)$ term from the first-order self-energy that exactly cancels the UV-divergent term appearing at second order. There is now no problem in using a low-momentum limit in which the $t$-matrix reduces to a momentum-independent constant $t = 4\pi \hbar^2 a/m$. The real (on-shell) and imaginary parts of the self-energy now reduce to:

$$
\Sigma_R(k, \tilde{\epsilon}_k) \approx 2tn + 2t^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{f(\tilde{\epsilon}_q) f(\tilde{\epsilon}_{p+q}) + f(\tilde{\epsilon}_q) f(\tilde{\epsilon}_{p-k}) - f(\tilde{\epsilon}_{p+q}) f(\tilde{\epsilon}_{p-k})}{\epsilon_q + \tilde{\epsilon}_k - \tilde{\epsilon}_{p+q} - \tilde{\epsilon}_{p-k}} ,
$$

(6)

and

$$
\Sigma_I(k, \omega) \approx -2\pi t^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \delta (\omega - [\tilde{\epsilon}_{p+q} + \tilde{\epsilon}_{p-k} - \tilde{\epsilon}_q]) \times \left[ f(\tilde{\epsilon}_q) + f(\tilde{\epsilon}_q) f(\tilde{\epsilon}_{p+q}) + f(\tilde{\epsilon}_q) f(\tilde{\epsilon}_{p-k}) - f(\tilde{\epsilon}_{p+q}) f(\tilde{\epsilon}_{p-k}) \right] .
$$

(7)

B. Self-consistent Perturbation Theory

We will use (6) and (7) to define a self-consistent quasi-particle approximation, formally second order in the vacuum scattering amplitude. This is done by replacing the free-particle dispersion functions ($\epsilon_q$'s) on the right hand sides of
interaction. This is important in light of the continuing appearance of negative-shift predictions in the literature

\[ \Sigma_I(k,\omega) \approx -2\pi\epsilon^{2} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \delta(\omega - [E_{p+q} + E_{p-k} - E_{q}]) \times \left[ f(E_{q}) + f(E_{q})f(E_{p+q}) + f(E_{q})f(E_{p-k}) - f(E_{p+q})f(E_{p-k}) \right]. \]  

When this expression for the imaginary part is used in eq (3), the \( \Sigma \) contribution cancels out exactly, resulting in

\[ n = \int \frac{d^3k}{(2\pi)^3} f(E_k); \]

i.e., the number of quasiparticles is equal to the number of particles. Since we will be using these results above and at the transition, we have neglected the zero-momentum occupancy \( N_0 \) as compared to \( N \). We note that the imaginary part of the self-energy, i.e., effects of quasiparticle broadening, has dropped out of the expression for particle number and hence does not play any role in our determination of the transition temperature.

Finally, making the real part of the self energy (eq 6) self-consistent, we get the following expression for the quasi-particle energy:

\[ E_k = \epsilon_k - \mu + \Sigma_R(k, E_k) \]

\[ = \epsilon_k - (\mu - 2tn) + 2t^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{f(E_{q})f(E_{p+q}) + f(E_{q})f(E_{p-k}) - f(E_{p+q})f(E_{p-k})}{E_q + E_k - E_{p+q} - E_{p-k}}. \]  

III. CRITICAL TEMPERATURE AND SPECTRUM

To derive an expression for the change in the critical temperature due to quasi-particle interactions, we first consider the density of an ideal Bose gas, both at its transition temperature \( T_c(0) = (2\pi/\xi(0)^2)^{2/3}n^{2/3} \), and at temperature \( T_c \) which is the transition point of the interacting Bose gas.

\[ \int \frac{d^3k}{(2\pi)^3} e^{\epsilon_k/k_BT_c} - 1 = n^{(0)}(T_c) \approx \left( \frac{T_c}{T_c(0)} \right)^{3/2} n^{(0)}(T_c(0)) \approx \left( 1 + \frac{3\Delta T_c}{2T_c(0)} \right) n^{(0)}(T_c(0)). \]

Here \( T_c = T_c(0) + \Delta T_c \). We are looking for the difference of transition temperatures of an interacting and a non-interacting gas at the same density, therefore \( n^{(0)}(T_c) = n(T_c) = n \). Evaluating (9) at \( T_c \) and subtracting equation (11) we obtain the leading correction to \( T_c \) due to interactions:

\[ \frac{\Delta T_c}{T_c(0)} \approx -2 \int \frac{d^3k}{(2\pi)^3} \left[ f_c(\xi_k) - f_c(\epsilon_k) \right], \]

Here \( f_c(x) = 1/(e^{x/k_BT_c} - 1) \) is the Bose distribution function evaluated at the critical temperature; and \( \xi_k = E_k(T_c) = \epsilon_k - \mu(T_c) + \Sigma_R(k, \xi_k) = \epsilon_k + \Sigma_R(k, \xi_k) - \Sigma_R(0, 0) \) represents the single particle excitation spectrum at \( T_c \). The problem of calculating the relative shift has thus been reduced to calculating the spectrum at the critical point, \( \xi_k \).

At this stage, we are able to present a simple argument for the increase of \( T_c \) due to the addition of a repulsive interaction. This is important in light of the continuing appearance of negative-shift predictions in the literature\(^{13,19}\). The major contribution to eq (12) comes from the infrared; therefore approximately

\[ \Delta T_c \propto \int_0^{\text{cutoff}} dk \, k^2[\xi_k^{-1} - \xi_k^{-1}]. \]

If the quasiparticle spectrum \( \xi_k \) is “harder” than the bare spectrum \( \epsilon_k = k^2/2m \), i.e., if \( \xi_k \) is sub-quadratic in \( k \) or quadratic with an effective mass \( n^* < m \), then the \( \epsilon_k \) integral dominates and the shift is positive. On the other hand if the spectrum at \( T_c \) were “softened” by the interaction, one would have a decrease in the transition temperature. Since the spectrum \( \xi_k \) at \( T_c \) is known to be intermediate between linear and quadratic, the weakly interacting Bose gas has a positive shift in the transition temperature with the introduction of a weak repulsive interaction.
A. Spectrum from Perturbation Theory

It follows from (10) that \( \xi_k \) satisfies the nonlinear integral equation,

\[
\xi_k = \epsilon_k + 2t^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{f(q,p) f(q) + f(p) f(q-k-p) - f(q) f(p-k)}{\xi_p + \xi_k - \xi_q - \xi_{p-k}} - 2t^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{f(q,p) f(q) + f(p) f(q-p) - f(q) f(p-q)}{\xi_p - \xi_q - \xi_{p-q}}
\]

(13)

In the perturbative quasiparticle approach, equations (12) and (13) completely determine \( \Delta T_c \).

If the quasiparticle dispersion \( \xi_k \) were quadratic at small momenta, the self-energy integral in eq (13) would be divergent in the infrared. A power-counting of this equation shows that, for self-consistency, the dispersion at small momenta should be sub-quadratic, \( \xi_k \sim k^{3/2} \), modulo possible logarithmic factors which power-counting cannot predict. The power-counting arguments were first devised by Patashkii and Pokrovskii.

Unfortunately, this approach using second-order perturbation theory over-modifies the infrared spectrum. The recent Monte Carlo calculation\(^{22} \) has produced the value \( \eta \approx 0.038 \). In contrast, eq (13) gives \( \eta = 0.5 \).

Acknowledging that the present framework over-modifies the spectrum at \( T_c \), we proceed to analyze eqs (12) and (13) in order to extract the shift in the critical temperature due to interactions.

B. Infrared Divergence

There are several ways of dealing with the infrared divergence. One is to treat equation (13) self-consistently and determine the self-consistent \( \xi(k) \) numerically; this procedure is outlined in section V. This approach has also been explored by Baym and collaborators.

In the next section, we take a different approach, motivated by the work of Kanno\(^{12} \). We make an expansion in \( \Sigma_0(k, \xi_k) \) that allows us to use the chemical potential at the critical point, \( \mu(T_c) = \mu_c \), as an infrared cutoff. This approach allows us to express the relative shift in the transition temperature as an expansion in \( \beta_c \mu_c = \mu_c/k_B T_c \), which itself can in turn be related to the scattering length. This leads to a transition temperature shift \( \Delta T_c \) that is \( a \sqrt{K(a)} \) to leading order, where \( K(x) \) is a function approximately like \( \ln(x) \), to be specified in the next section. The appropriateness of this procedure is difficult to gauge, but it provides strong support to the possibility of non-analytic corrections to the transition temperature at leading order.

Another infrared regularization procedure that has appeared in the literature is A.M.J. Schakel’s prescription of analytic continuation\(^{16} \). The connection between Schakel’s treatment and other calculations awaits further study.

IV. NON-SELFCONSISTENT CALCULATION

We expand \( f_c(\xi_k) = f_c(\epsilon_k - \mu_c - \Sigma_R(k, \xi_k)) \) around \( \epsilon_k - \mu_c \):

\[
f_c(\xi_k) \approx f_c(\epsilon_k - \mu_c) + f_c'(\epsilon_k - \mu_c) \Sigma_R(k, \epsilon_k - \mu_c)
\]

(14)

This provides us with an infrared cutoff, \( -\mu_c \), for our integrals. The approximation is non-selfconsistent in the sense that a modification of the spectrum to a non-quadratic form no longer plays any role. Using this expansion, the relative shift in \( T_c \) becomes

\[
\frac{\Delta T_c}{T_c^{(0)}} \approx -\frac{2}{3n} \int \frac{d^3k}{(2\pi)^3} f_c(\epsilon_k - \mu_c) - \frac{2}{3n} \int \frac{d^3k}{(2\pi)^3} f_c'(\epsilon_k - \mu_c) \Sigma_R(k, \epsilon_k - \mu_c)
\]

(15)

Both integrals can be expanded in \( |\beta_c \mu_c| \), as detailed in appendix A, each integral being \( \mathcal{O}(\sqrt{\beta_c \mu_c}) \) at leading order. The result is

\[
\frac{\Delta T_c}{T_c^{(0)}} \approx -\frac{2}{3n} \left[ \frac{\sqrt{\pi}}{\lambda_c^3} \sqrt{-\beta_c \mu_c} \right] + \mathcal{O}(\beta_c \mu_c) = -\frac{2}{3} \left[ \zeta(\frac{3}{2}) \right]^{-1} \sqrt{\pi |\beta_c \mu_c|} + \mathcal{O}(\beta_c \mu_c)
\]

(16)
the width of the quasiparticle peak ($\Sigma$)

open questions. that appear in a perturbative calculation in terms of a momentum-independent bare potential

second-order, and whether this is true for a more accurate treatment in terms of the many-body

relative shift in $T_c$; solid curve is numeric solution and dashed curve is the $\sim a\sqrt{|\ln(a)|}$ approximation.

The infrared cutoff needs to be expressed in terms of the scattering length; this is achieved by using the condition for the transition, $\mu_c = \Sigma_R(0,0)$. The relation is

$$\beta_c \mu_c \approx \frac{16\pi a^2}{\lambda^2} \ln(-\beta_c \mu_c),$$

also derived in appendix A. The functional dependence of $|\beta_c \mu_c|$ on the scattering length is thus similar to $\sim (a/\lambda_c)^2|\ln(a/\lambda_c)|$, and therefore we get a shift approximately of the form $\Delta T_c \sim a\sqrt{|\ln a|}$.

The approximation of eq (17) by $|\beta_c \mu_c| \approx 16\pi(a/\lambda_c)^2|\ln(a/\lambda_c)|$ is displayed in figure 2.

V. SELF-CONSISTENT APPROACH: NUMERICS

A self-consistent treatment of (13) involves a calculation of the (sub-quadratic) quasiparticle dispersion $\xi(k)$. The principal values in (13) make it difficult to treat numerically. A “high-temperature” approximation, obtained by using $f(\xi) \approx 1/\beta \xi$, produces a simpler version without the zeroes of the principal value form:

$$\xi_k = \epsilon_k - 2t^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \left( \frac{1}{\xi_{p} \xi_{p+k-q}} - \frac{1}{\xi_{p} \xi_{p-k+q}} \right).$$

While the same dimensional arguments apply to (18), it is likely to lose information about logarithm-like factors.

Since the self-consistent spectrum has infrared behavior $\xi(k) \sim k^{3/2}$, we look for solutions of the form $\xi(k) = k^{3/2}\sigma(k) + k^2$ where $\sigma(x)$ is some smooth function that decays to zero for large $x$, and is constant or logarithm-like at small $x$. (Here the momenta and $\xi$ have been rescaled to be dimensionless, by factors $\sqrt{\beta/2m}$ and $\beta$ respectively).

Figure 3 shows a numerically determined spectrum $\xi(k)$. Once $\xi(k)$ has been computed for a particular $a/\lambda$, the relative shift in $T_c$ can be computed using our basic eq (12). The $\Delta T_c$ vs. $a$ plot in the inset shows a definite linear-like increase, as opposed to, say, a $2a$ or $a^{3/2}$ dependence. However the existence, or lack thereof, of a logarithm-like factor is difficult to prove from numeric data.

VI. CONCLUDING REMARKS

To summarize, we have outlined the calculation of $T_c$ from a perturbative expansion in the two-body $t$-matrix, or equivalently the scattering length $a$. The transition temperature is expressed in terms of the quasiparticle spectrum at $T_c$, and hence the problem is transformed to one of calculating the critical-point spectrum $\xi_k = E_k(T_c)$.

In developing the formalism, we find that the use of the two-body $t$-matrix removes the ultraviolet divergences that appear in a perturbative calculation in terms of a momentum-independent bare potential $U$. We also find that the width of the quasiparticle peak ($\Sigma_1$) does not affect $T_c$. Whether this is true at all orders, or a peculiarity of second-order, and whether this is true for a more accurate treatment in terms of the many-body $T$-matrix, remain open questions.
FIG. 3: Quasiparticle spectrum, for \((a/\lambda) = 0.0005\). (The “fit” indicates quality of iteration – line shows penultimate, and dots show final, iteration). Crossover from \(k^{3/2}\) to \(k^2\) behavior is clear; dotted line is the noninteracting \(k^2\) spectrum for comparison. Upper inset shows the crossover function \(\sigma(k)\) for several \(a/\lambda\) values (9 \(\times 10^{-5}\), 8 \(\times 10^{-4}\) and 4 \(\times 10^{-3}\)). Lower inset plots relative shift of \(T_c\) against interaction.

The new feature of this work is an unusual approach to dealing with the infrared divergence that appears in treating the spectrum at \(T_c\). This regularization scheme bypasses the issue of spectrum modification, and results in a non-analytic (\(\sqrt{\ln a}\)-like) correction factor to the linear shift of the transition temperature.

The major remaining issue in the \(T_c\) problem seems to be the possible existence of a logarithm-like factor in the leading \(a\)-dependence. The resolution of this problem should lie in a clear treatment of the spectrum at \(T_c\). Of the two major approaches, the perturbative approach alters the quadratic spectrum too strongly, while the RG calculation\(^7\) neglects spectrum alteration.

The reason for perturbation theory over-estimating the spectrum modification remains to be examined in detail. It has been suggested\(^8\) that the reason is that perturbation in \(a\) or \(t\) does not take into account the fact that the effective interaction strength flows to zero under renormalization group flow, at the critical point. An alternate way to view this is to note that the many-body \(T\)-matrix (sum of ladder diagrams) vanishes\(^7,9,23\) at the transition temperature. A perturbative calculation in terms of the many-body \(T\)-matrix, as opposed to one in terms of the two-body \(t\)-matrix that we have considered here, may therefore be expected to give a more realistic modification of the quasiparticle spectrum at \(T_c\). A calculation of \(\xi_k\) in terms of the many-body \(T\)-matrix has not yet appeared in the literature.

Acknowledgments

The junior author (MH) would like to acknowledge informative discussions with Henk Stoof and Franck Laloë, made possible by the generosity of ECT*, Trento (Italy). E-mail responses from M. Holzmann and Adriaan M.J. Schakel were very helpful.

APPENDIX A: DETAILS FOR THE NON-SELFCONSISTENT APPROACH

In this appendix we evaluate the two contributions to \(\Delta T_c/T_c^{(0)}\) in eq (15), and show how eq (17) follows from the criticality condition \(\Sigma(0,0) = \mu\).

The first integral in (15) is \(\int_k \left[ f_c(\epsilon_k - \mu_c) - f_c(\epsilon_k) \right] = \lambda_c^{-3} g_{3/2}(e^{\beta_c \mu_c}) - \lambda_c^{-3} \zeta(\frac{3}{2})\). Here the \(g_{3/2}\)'s are the Bose-Einstein integral functions\(^24\), \(g_n(x) = \sum_i (x^i/i^n)\), and \(\zeta(n) = g_n(1)\). We are using the notation \(\int_k \equiv (2\pi)^{-3} \int d^3k\).

Using the Robinson expansion\(^24\) for \(g_{3/2}\), we get

\[
\int_k \left[ f_c(\epsilon_k - \mu_c) - f_c(\epsilon_k) \right] = \frac{1}{\lambda_c^3} \left[ \frac{1}{2} \sqrt{\pi} \beta_c \mu_c + O(\beta_c \mu_c) \right].
\]

For the second integral in (17), using the second-order self-energy from eq (6), one obtains after some variable...
transformation

$$\int \frac{d^3(\tilde{\epsilon}_k)}{k} \Sigma_R(k, \epsilon_k) \approx -2t^2 \frac{\partial}{\partial \mu_c} \int p \int q \int k \int \frac{f_c(\tilde{\epsilon}_p)f_c(\tilde{\epsilon}_q)f_c(\tilde{\epsilon}_k)}{\epsilon_p + \epsilon_k - \epsilon_q - \epsilon_{p+q}} \approx \frac{1}{\lambda_c^2} 16^{3/2} u^2 \ln(-\beta_c \mu_c) \sqrt{-\beta_c \mu_c}.$$ 

Here $\tilde{\epsilon}_p = \epsilon_p - \mu_c$. In combination with the criticality condition (17), this leads to

$$\int \frac{d^3(\tilde{\epsilon}_k)}{k} \Sigma_R(k, \epsilon_k) \approx \frac{1}{\lambda_c^2} \left[ \sqrt{\pi(-\beta_c \mu_c)} + O(\beta_c \mu_c) \right].$$

The condition (17) for the transition temperature is obtained from $\mu_c = R(0, 0)$ by using the second-order perturbative result for the self-energy (e.g., obtained from eq (6) by setting $k = 0$):

$$\mu_c = 4t^2 \int p \int q \int \frac{f(\tilde{\epsilon}_p)f(\tilde{\epsilon}_q)}{\epsilon_p - \epsilon_q - \epsilon_{p+q} - \mu_c} \approx -\frac{8ma^2}{\pi^2 \beta_c^2} \ln(-\beta_c \mu_c).$$

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