Perfect fluids and bad metals: insights from ultracold Fermi gases

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Abstract. In this paper, we examine in a unified-fashion dissipative transport in strongly correlated systems. We thereby demonstrate the connection between ‘bad metals’ (such as high-temperature superconductors) and ‘perfect fluids’ (such as ultracold Fermi gases, near unitarity). One aim of this paper is to communicate to the high-energy physics community some of the central unsolved problems in high-$T_c$ superconductors. Because of the interest in the nearly perfect fluidity of cold gases and because of new tools such as the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, the communication may lead to significant progress in a variety of different fields. A second aim is to draw attention to the great power of transport measurements, which reflect the excitation spectrum more directly than, say, thermodynamics, and therefore strongly constrain microscopic theories of correlated fermionic superfluids. Our calculations show that bad metal and perfect fluid behavior is associated with the presence of a normal state excitation gap that suppresses the effective number of carriers leading to anomalously low conductivity and viscosity above the transition temperature $T_c$. Below $T_c$, we demonstrate that the condensate collective modes (‘phonons’) do not couple to transverse probes, such as shear viscosity or conductivity. As a result, our calculated shear viscosity at low $T$ becomes arbitrarily small, as observed in experiments. In both homogeneous and trap calculations, we do not find the upturn in $\eta$ or $\eta/s$ (where $s$ is the...
entropy density) that is found in most theories. In the process of these studies, we demonstrate compatibility with the transverse sum rule and find reasonable agreement with viscosity experiments.

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1. Introduction

There is widespread interest in the study of ultracold Fermi gases in order to learn about some of the most strongly interacting systems in nature. It has been argued [1] that the atomic Fermi gas near unitarity provides a prototype for the physics of quark-gluon plasmas that are associated with the early stages of the Big Bang. Most remarkable about these two nominally different physical systems is the fact that they correspond to nearly perfect fluids, exhibiting even in their non-superfluid phases extremely small values of the shear viscosity $\eta$. In this paper, we draw attention to a third class of strongly interacting systems, which is the metallic counterpart of the perfect (neutral) fluid, namely the so-called ‘bad metal’ [2]. We show how low conductivity and low viscosity are analogous. Bad metals are known to exist in nature in the most fascinating
class of materials: high-$T_c$ superconductors. We note that it may seem, at first sight, paradoxical that bad metals and perfect fluids have anything in common, since the condensate contribution to conductivity is infinite in the metallic system, whereas the viscosity counterpart in the neutral superfluid is zero. We stress throughout this paper that the dc transport (associated with $\omega \rightarrow 0$) in both cases reflects the normal fluid or excitations of the condensate.

Because the ultracold Fermi gases near the unitary limit are also thought to be related to quark-gluon plasmas [1], much attention has been focused on particle-physics-based calculations of the anomalously low shear viscosity, $\eta$ [3, 4]. However, considerable insight into the thermodynamics [5] and various spectroscopic studies [6, 7] has also been obtained via a condensed matter perspective. This paper belongs to the second school in which Bardeen–Cooper–Schrieffer (BCS) theory is extended to accommodate arbitrarily strong interactions. We apply BCS to Bose–Einstein condensation (BEC) crossover theory [6] to compute $\eta$ in the neutral Fermi gases and the conductivity $\sigma$ (evaluated at $\omega \rightarrow 0$) for the charged counterparts. Our results, which are reasonably consistent with cold gas experiments, apply both above and below $T_c$. Essential to this work is demonstrable consistency with central sum rules and conservation laws [8].

Experimentally, this crossover can be straightforwardly observed in the ultracold Fermi gases by exploiting Feshbach resonances. One can enquire as to whether this crossover is also relevant to cuprates. A detailed summary of this controversy (for and against BCS–BEC crossover) can be found in a review paper by one of us [6]. With new experimental transport data on cold gases providing further insights into the BCS–BEC scenario, it was thought to be useful to take these insights and see what they would imply for the cuprates. Our aim is not to prove one way or the other whether the BCS–BEC scenario applies to the cuprates, but rather to investigate this issue further by firstly making the case that viscosity and conductivity are closely related quantities, and secondly looking into the implications of BCS–BEC crossover theory for metallic conductivity, as in the cuprates. At the simplest level, one would like to use our understanding of transport in the Fermi gases, where we argue that pseudogap effects play an important role, to address the effects of the pseudogap on cuprate transport. At this point, there is no consensus on how to explain the ubiquitous nearly linear resistivity in cuprates, and it is useful to explore what (if anything) the pseudogap might have to do with it. The evidence that the pseudogap should be important for cuprate conductivity is in part based on the claim that the nonzero frequency $\sigma$ has extra weight in the so-called mid-infrared region only when a pseudogap is present [9]. Then, by the optical sum rule, one can presume that dc conductivity takes on a different form, depending on whether or not a pseudogap is present.

In support of this BCS–BEC scenario for cuprates is the argument by Leggett that (i) because of the anomalously short coherence length of these systems, one has to view them as somewhere between BCS and BEC [10]. This is also consistent with (ii) their anomalously high transition temperatures and (iii) the widespread effects of a normal state gap. Experimental groups as well have made this case on the basis of optical [11] and STM [12] experiments. At the same time, there are major concerns about this approach to cuprates. Among the questions being asked are: (a) What is the role of Mott physics? More specifically, what is the role of the strong repulsion that is present in the (undoped) or parent compounds, and what are the effects of proximity to this insulating phase? (b) Does the pseudogap in the cuprates arise from an alternative order parameter, which would then be totally unrelated to the precursor pairing effects associated with BCS–BEC crossover? (c) Since anomalies in various cuprate transport coefficients have different onset temperatures, it is claimed that this may argue against...
a pairing fluctuation approach to cuprate transport. The Nernst effect (which is theoretically discussed in our earlier work [6, 13]), in particular, is singled out in this regard. The spirit of this paper is ‘agnostic’ on the question of whether this scenario applies to cuprates. Rather we use some very beautiful experiments in cold Fermi gases to build our understanding of transport in correlated systems and then address the analogous implications for the cuprates. See an earlier summary [6] for a more lengthy discussion of some of this controversy.

Shear viscosity, like dc conductivity, is a powerful probe for testing microscopic theories because it reflects the normal fluid component and is extremely sensitive to the nature of the excitation spectrum. The low-$T$ normal Fermi liquid phase with scattering lifetime $\gamma^{-1}$ and effective mass $m^*$ helps us to set up the general nomenclature. Here, $\eta = \frac{1}{2} n v_F^2 \gamma^{-1} m^*$. Similarly, $\sigma = \frac{n e^2}{m^*} \gamma^{-1}$. More generally, one can think of $\eta$ and $\sigma$ as characterized by the effective number of normal excitations ($n \rightarrow n_{\text{eff}}(T)$) as well as their lifetime, which, we emphasize here, is a many-body effect. Crucial is gaining an understanding of how $n_{\text{eff}}$ depends on $T$.

Our central conclusion is that both the effects of a fermionic gap (with onset temperature $T^* > T_c$) and of non-condensed pairs act in concert to reduce $n_{\text{eff}}$ and thus lower the shear viscosity and dc conductivity at all $T < T^*$. These non-condensed pairs are associated with the stronger than BCS attraction and are present for $0 < T < T^*$. When quantitatively compared with very recent shear viscosity experiments [14] (we independently infer [15] an estimated lifetime from radio-frequency (RF) data), the agreement is reasonable. Similarly, we compare the ($\omega \rightarrow 0$) dc conductivity (as well as its normal state $\omega \equiv 0$ inverse, $\sigma_{dc}^{-1}(T) = \rho(T)$, the resistivity) with the counterparts in high-$T_c$ superconductors. This allows clarification of the way a precursor-pairing approach to the pseudogap affects transport.

This paper is organized around a second important premise: any theory of transport or scattering in the superfluid Fermi gases (both above and below $T_c$) must be formulated in a fully gauge invariant and sum-rule consistent fashion. One could imagine doing more and more sophisticated many-body theories than standard BCS-related approaches (i.e. summing more and more diagrams), but if sum rules and conservation laws cannot be demonstrated to hold, we would argue that the significance of these calculations is problematic. Indeed, a major theme of our recent [16, 17] work, which is summarized in this overview, is to arrive at a demonstrably consistent theory of this exotic superfluidity by extending the important contributions of Nambu [18] to the case of BCS–BEC crossover. This will serve as a basis for computing gauge invariant and properly conserving response functions that are measured in transport and scattering experiments. It is not as straightforward to demonstrate this consistency in Monte Carlo or other numerically based schemes. Moreover, as theories of BCS–BEC crossover become more and more diagrammatically complicated, it is not straightforward to determine that they are compatible with gauge invariance and conservation laws. Once this compatibility is established, the next step is to assess the physical implications of this consistency in experimental probes. We stress here that our approach to BCS–BEC crossover is semi-analytical and therefore reasonably transparent. Moreover, unlike alternative approaches (more numerical diagrammatic schemes [19]), the normal-to-superfluid transition is properly smooth and second order.

A systematic treatment of transport in Fermi superfluids requires the introduction [20] of inter-dependent fermionic as well as bosonic (or pair) excitations. In the earlier transport literature, there has been a focus on either one [21] or the other [4], but not both. Here we use a Kubo-based formalism that readily accommodates the simultaneous bosonic and fermionic excitations of the normal state and condensate, and thereby addresses $n_{\text{eff}}$ quite accurately, while
Figure 1. Viscosity in helium-3 (left) from [30] and helium-4 from [29] (right). The solid curves on the left represent theoretical predictions that were not verified. The viscosity vanishes for low temperatures in the case of helium-3 because of a suppression of fermionic excitations. The behavior of $\eta$ in helium-4 is governed by the rotons and phonons.

alternative Boltzmann or kinetic theory-based approaches do not naturally incorporate these multiple statistical effects. The Kubo approach includes scattering processes via lifetimes [22] that appear in various Green’s functions, while Boltzmann schemes treat lifetimes via collision integrals. However, because the physics of this dissipation is mainly associated with the many-body processes of boson–fermion inter-conversion (via a parameter $\gamma$ that appears throughout this paper), it should be satisfactorily addressed only in theories that treat the mixed statistics.

We stress in this paper that dissipative transport in the normal phase must also be addressed if one is to fully understand the counterpart below the $T_c$ case. Except in strict BCS theory, this normal phase is affected by the below $T_c$ pairing correlations. As a result, consistency checks need to be applied above $T_c$ and the behavior must necessarily reflect that the transition is second order. We argue here that the associated pseudogap leads to important transport implications, such as ‘nearly perfect fluidity’ (reflecting anomalously small viscosity [1]) or in the analogous charged system ‘bad metal’ [2] behavior (reflecting anomalously small conductivity). Conventionally, ‘bad’ metals are systems in which the estimated mean free path $l$ is shorter than all length scales; along with anomalously low conductivity, this leads to the absence of resistivity saturation. The descriptive ‘perfect’ is also associated with a situation in which $l$ is small compared to physical length scales [23]. In strongly correlated superfluids, we reiterate here that small $l$ does not solely reflect short transport lifetimes $\tau$ but rather a notable suppression of the effective ‘carrier number’. The influence of bad metallicity on superconductivity was studied in [2]. Here we emphasize the converse: the influence of superconductivity on transport [24]. Moreover, as will be discussed in more detail below, pairing fluctuations and the phase fluctuations invoked earlier [2] are clearly distinct.

We have included in section 1.3 a summary of the main results of this paper. For ease in reading this paper, the reader not interested in the technical details can skip sections 2 and 3 and go directly to section 4.
1.1. The experimental overview

To understand viscosity in fermionic superfluids, it is useful to begin with helium-3, which has been successfully described using BCS-based approaches, albeit for the p-wave case [25, 26]. Here, experiments [27] indicate that $\eta$ drops down rapidly to zero in the superfluid phase. This is shown in figure 1 (left). Interestingly, there is a minimum in $\eta$ above $T_c$ that is associated with strict Fermi liquid behavior [28]. In a Fermi liquid, the numbers of carriers and mass are both $T$ independent, while the inter-fermion scattering lifetime varies as $T^{-2}$. The standard interpretation of the data below $T_c$ is that $\eta$ decreases with decreasing $T$ as a result of the suppression of fermionic excitations at low $T$. In BCS-based superfluids, we stress [18] that the Nambu–Goldstone boson effects do not naturally enter into transverse transport properties such as $\eta$. By contrast, in the helium-4 counterpart shown on the right of figure 1, single particle bosonic excitations couple to the collective (Nambu–Goldstone) modes, leading to an upturn [29] in $\eta$ at low $T$, which has also been predicted (but not seen) for atomic Fermi superfluids [4].

For strongly correlated charged systems, the counterpart experiments are summarized in figure 2, here for the high-$T_c$ cuprates, which have d-wave pairing. On the left is the low-$T$, $\omega \to 0$ conductivity for a $T_c \approx 90$ K sample. This shows the fairly generic maximum below $T_c$. The figure on the right is the resistivity or inverse conductivity for different stoichiometries as a function of temperature. Above $T_c$, two crucial points are that $\rho = \sigma^{-1}$ is nearly linear with $T$ and its magnitude seems to decrease from the UND (underdoped) to the OVR (overdoped) samples. It is inferred [33] from similar systematic studies that the effective number of carriers at fixed $T$ is substantially depressed, varying as the doping $x$ rather than the expected $1 + x$. Importantly, for the present purposes, $\sigma(T)$ seems to approach zero at the lowest temperatures. The latter point is consistent with the vanishing $\eta$ shown in the previous figure for the fermionic superfluid. The pronounced maximum is thought to arise from the competition between the decrease in $n_{\text{eff}}$ and the increase in the fermionic lifetime at low $T$. Such a competition is not nearly as apparent in an s-wave system, where $n_{\text{eff}}$ is exponentially suppressed.
Figure 3. Viscosity experiments on unitary Fermi gases from recent experimental results showing the shear viscosity $\eta$ and the ratio of viscosity and entropy density $\eta/s$, [14]. Importantly, the viscosity vanishes at low temperature and this puts a strong constraint on the theories of the unitary Fermi gas.

In figure 3, we summarize recent viscosity experiments from the Duke group [14] on a unitary trapped Fermi gas. One sees here that despite previous predictions, the viscosity and its ratio to entropy density are both strongly suppressed at low $T$. We will return to all of these figures throughout the paper.

1.2. Theoretical overview

The BCS–BEC crossover theory that is adopted here is based on a natural extension of the BCS ground state,

$$ \Psi_0 = \Pi_k \left( u_k + v_k c_{k \uparrow}^\dagger c_{-k \downarrow}^\dagger \right) |0\rangle. $$ (1)

This ground state is often called the ‘BCS–Leggett’ state. The observation that this state is much more general than was originally presumed forms the basis for BCS–BEC crossover theory [34–36]. To implement this generalization of the BCS ground state, all that is required is that one solve for the two variational parameters $u_k$ and $v_k$ in concert with a self-consistent condition on the fermionic chemical potential $\mu$. As the attraction is increased, $\mu$ becomes different from the Fermi energy $E_F$. The two variational parameters $u_k$ and $v_k$ can be converted to two more physically accessible parameters associated with the zero-temperature gap (or equivalently order parameter) and $\mu$. We stress that whereas in the cuprates the system is far from the BEC limit (in large part because of d-wave lattice effects [37]), it is nevertheless quite distinct from the weak-coupling BCS limit due to the anomalously short coherence length [10].

Since this wave function is so closely related to the BCS state, it is natural to ask whether its behavior away from $T = 0$ can be consolidated into as simple a formalism and physical picture as there is for the ground state. In section 2, we answer this question in the affirmative by recasting the equations of conventional BCS theory using a formalism that can then be readily generalized to include BCS–BEC crossover.
1.2.1. Simple physical picture of BCS–BEC crossover scenario. Before introducing the $T \neq 0$ formalism, we present a simple picture of the excitations that ultimately enter into transport. The top row in figure 4 (from left to right) shows the schematic behavior as one passes from the $T = 0$ BCS–Leggett groundstate to the above $T^\ast$ Fermi liquid. The red (dotted circles) pairs are associated with net finite momentum, while the blue (solid circles) pairs correspond to the phase coherent condensate with zero center-of-mass momentum and the lone arrows represent fermionic excitations. The first panel shows that the ground state consists of fully condensed pairs as in equation (1), whereas the second panel shows that below $T_c$ but above $T = 0$ there are both condensed and non-condensed pairs along with fermionic excitations. These non-condensed pairs persist above $T_c$ (third panel) in the form of ‘preformed’ pairs, while the condensed pairs are no longer present. Finally, at temperatures above $T^\ast$, all bosonic-like excitations are absent; the only excitations are fermionic. The second panel with $0 < T < T_c$ is the most interesting from the perspective of the present paper. In cuprates, this is the regime in which the widely discussed \[38\] ‘two-gap’ physics appears. Here the coexistence of the condensate and non-condensed pairs leads to two gap contributions \[39\], one associated with the pseudogap (called $\Delta_{pg}$) and the other associated with the condensate (called $\Delta_{sc}$).

In contrast to this pairing fluctuation picture of the BCS–BEC crossover theory stands the phase fluctuation picture shown in the bottom row of figure 4. In this scenario of the pseudogap \[40\], there exist finite size regions of superconducting order in the normal state. While the amplitude of the pairing gap $\Delta_k$ is fixed in these regions, the superconducting phase, labeled $\Phi$, fluctuates spatially between them. The degree of phase fluctuation increases with temperature, and the Fermi liquid state is reached at $T > T^\ast$. In contrast, the phase is long-range ordered below $T_c$ and the superconducting phase is described by the BCS state consisting only of condensed pairs and unpaired fermions. In this picture, as yet there appears to be no counterpart to the ‘two-gap physics’ below $T_c$ seen in the crossover scenario. In a simplified fashion, it
could be said that the preformed pair theory is concerned with fluctuations in momentum space (in terms of \( q \neq 0 \) pairing) while the phase fluctuation picture focuses on fluctuations of the phase in real space.

In the BCS–BEC crossover scenario, the effects of condensed (i.e. superconducting (sc)) and non-condensed (i.e. pseudo-gap (pg)) pairs are described by two distinct contributions to the fermionic self-energy \( \Sigma(k, \omega) \),

\[
\Sigma(k, \omega) = \Sigma_{pg,k} + \Sigma_{sc,k} = \frac{\Delta_{pg,k}^2}{\omega + \xi_k + i\gamma} + \frac{\Delta_{sc,k}^2}{\omega + \xi_k^*},
\]

(2)

The gap functions \( \Delta_{pg,k} \) and \( \Delta_{sc,k} \) are assumed to follow either a simple s- or d-wave form. The condensed pairs have the usual BCS self-energy contribution, \( \Sigma_{sc} \), whereas the self-energy of the non-condensed pairs \( \Sigma_{pg} \) possesses an additional term, \( \gamma \), with \( \gamma^{-1} \) reflecting the finite lifetime of the non-condensed pairs. This form of \( \Sigma_{pg} \) was derived microscopically in [41] using a \( T \)-matrix approach (see below). It plays a central role in transport, largely through Ward identities, which relate the self-energy to transport properties. At the microscopic level, it is important to stress that the above expression for \( \Sigma_{pg} \) is not generic to all \( T \)-matrix theories, but strongly depends on an underlying BCS-like structure of the ground state associated with the present approach.

1.2.2. Lifetime effects. In our discussion of dissipative transport, a crucial point is to address the origin of finite lifetime effects. Throughout this paper, we will argue that the central dissipation process is associated with the inter-conversion from fermions to pairs. We stress that this is a many-body effect and should not be associated with the two-body scattering length. Nevertheless, at unitarity, these inter-conversion processes are likely to lead to the shortest lifetimes simply because the numbers of fermions and bosons are roughly equal there. This is in contrast to the BCS (BEC) regime in which there are virtually no bosons (fermions). This physical picture is consistent with equation (2), which above \( T_c \) has been rather widely adopted by the high-\( T_c \) community [42–44]. In cold gases, this form of the fermionic self-energy or, equivalently, the spectral function, which is associated with the downward dispersing quasi-particles, was revealed in momentum resolved RF experiments [7, 45]. In cuprates, above \( T_c \), it is the parameter \( \gamma \) (which we associate with the finite lifetime of the non-condensed pairs) that leads to the interesting physics of the ‘Fermi arcs’. These have been interpreted as a blurring of the d-wave nodes. Importantly, below \( T_c \), one sees their sudden collapse to conventional point nodes [44, 46] as a result of the onset of the order parameter \( \Delta_{sc} \).

1.3. Central results of this paper

1.3.1. Summary of central formulae. In the weak dissipation limit (where \( \gamma \) is small), we may write down a combined expression for the shear viscosity and dc conductivity. We have

\[
\tilde{\eta} = \frac{e\sigma}{\sigma} \int_0^\infty dp \frac{p^6}{m^2} \frac{E_p^2 - \Delta_{pg}^2}{E_p^2} \left( u_p^2 - v_p^2 \right)^2 \left( -\frac{\partial f(E_p)}{\partial E_p} \right) \tau_{\eta,\sigma}.
\]

Here, \( E_p \) is the usual BCS dispersion, \( E_p = \sqrt{\xi_p^2 + \Delta^2} \), where \( \Delta^2 = \Delta_{pg}^2 + \Delta_{sc}^2 \), and \( f(E_p) \) is the Fermi function. We define \( c_p = 1/15\pi^2, c_{\sigma} = 1/3\pi^2 \) and the ‘coherence factors’ are \( u_p^2, v_p^2 = \frac{1}{2} \left( 1 \pm \frac{\pm}{E} \right) \). The transport lifetime \( \tau_{\eta,\sigma} \) scales inversely with \( \gamma \). This reduces to the usual BCS results when \( \Delta_{pg}^2 = 0 \). Equation (3) contains two strong correlation effects that reduce the

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size of $\eta$ and $\sigma$: the presence of a gap $\Delta(T)$ in $E$, which persists into the normal state, and the presence of the factor $-\Delta_{pg}^2$, which reflects the reduction in the number of fermions contributing to dc transport as a result of their conversion to bosons. While the gap is fermionic in nature, this term is associated with bosonic degrees of freedom. In the stronger dissipation limit, the shear viscosity and conductivity can be rewritten in terms of generalized Green’s functions (to be defined more precisely in section 3),

$$\eta = -\lim_{\omega \to 0} \lim_{q \to 0} \frac{1}{\omega} \text{Im} \sum_p \frac{2p_x^2 p_y^2}{m^2} \left[ G_{p^+} G_{p^-} F_{sc, p^+} F_{sc, p^-} - F_{pg, p^+} F_{pg, p^-} - 2(u_p^2 - v_p^2)^2 F_{pg, p^+} F_{pg, p^-} \right]_{\omega \to \omega^+},$$

$$\sigma = -\lim_{\omega \to 0} \lim_{q \to 0} \frac{1}{\omega} \text{Im} \sum_p \frac{2p_x^2}{m^2} \left[ G_{p^+} G_{p^-} F_{sc, p^+} F_{sc, p^-} + F_{pg, p^+} F_{pg, p^-} - 2(u_p^2 + v_p^2)^2 F_{pg, p^+} F_{pg, p^-} \right]_{\omega \to \omega^+},$$

where $p^+ = (i(\omega_p + \Omega), p + \frac{q}{2})$, $p^- = (i\omega_p, p - \frac{q}{2})$, and $\omega^+ = \omega + i0^+$. Note that while one can interpret $F_{sc}$ as the usual Gor’kov Green’s function reflecting superconducting order, there must also be a counterpart $F_{pg}$ (discussed in detail later) that reflects non-condensed pairs. In previous works on cold gases [48] and cuprates [49, 50], only the first term involving $GG$ was included and, moreover, this was correctly recognized [48, 50] as inadequate for conservation constraints.

1.3.2. Summary of transport results. We next present summary numerical plots and compare the viscosity and d-wave conductivity calculations based on equations (4). For illustrative purposes, we take a simple Fermi liquid $\gamma(T) \propto T^2$ temperature dependence for both transport properties. That one needs a higher power than linear to arrive at metallic transport is a consequence of the increase in the carrier number with $T$. One might speculate that a $T^2$ dependence in the cuprates is associated with the gapless Fermi arcs. An essential change from s-wave to d-wave pairing is associated with the fact that for the latter the carrier number is no longer exponentially activated. As a consequence, the transport behavior tends to be more metallic.

The four top panels in figure 5 plot (a) the $T_c$-normalized, $\omega \to 0$, d-wave conductivity, $\text{Re} \sigma(T/T_c)$. In figure 5(b), we plot a decomposition of this conductivity with three different contributions as color coded, so that the condensate (sc) is blue and the pg contributions are red, and what makes up the difference is shown in green. Plotted in figure 5(c) is the effective carrier number $(n/m(T))_{\text{eff}}$, which is obtained by multiplying $\sigma$ by $\gamma(T)$ corresponding to the inverse lifetime. Finally in figure 5(d), we plot the resistivity, or inverse conductivity, above $T_c$. The lower panel of figures corresponds to the counterpart plots for the s-wave viscosity, in terms of $\alpha$ defined via $\eta = \alpha n \hbar$.

The figures in the (b) column show that the pg contributions are negative in both cases, while the condensate gives a positive contribution to $\sigma$ and a negative contribution for $\eta$; this can be directly seen from equations (4). The remainder (associated with the addition of the $GG$ terms in equations (4)), which yields the respective totals, is shown in green. That the (pg) contribution from the non-condensed pairs (red) lowers the conductivity and viscosity is because the presence of non-condensed pairs means that fewer fermions are available for dc transport. As plotted in the (c) column, one sees that the effective carrier number $(n/m(T))_{\text{eff}}$ increases more or less monotonically as the temperature is raised. This represents a generic figure, since

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Figure 5. Summary figure of d-wave conductivity (top four panels) and s-wave viscosity (bottom four panels) as a function of temperature. Both panels (a) and the resistivity at the top in panel (d) (or inverse conductivity) can be compared with experiments in figures 3 and 2. Panels (b) are contributions to transport from three components. Here red = pg, blue = sc, while green reflects what makes up the difference. Panels (c) plot the effective carrier number and (d) the inverse transport coefficients.

Here the temperature dependence of the lifetime has essentially been removed. Thus, this figure shows that the effective carrier number associated with the conductivity and its shear viscosity counterpart are increasing functions of temperature, strongly suppressed by both the presence of a fermionic gap and the presence of bosonic degrees of freedom. In figure 5(d), we plot the inverse transport coefficients.

The figure on the lower left of figure 5 seems to capture the qualitative experimental transport features shown in figure 3 for the cold gases. Note that the conductivity exhibits [31] a maximum below $T_c$, as can be seen experimentally from figure 2, whereas the viscosity coefficient monotonically increases, as observed [14]. As shown in panel (d), the conductivity of the normal state is appropriately metallic, but suppressed by the excitation gap. For cold gases, more direct comparison with experiment involves the inclusion of the trap. As is conventional [6], we include trap effects via the local density approximation (LDA), or equivalently Thomas–Fermi approximation. Figure 6 presents a comparison of the viscosity coefficient $\alpha$ between theory (based on the RF-deduced lifetime), as black dots, and experiment [47] (red triangles) as a function of $E$. These calculations can also be compared with more recent experiments, which are summarized in figure 3. Our calculations, which predated the latest experimental data, will be discussed in more detail later. One can see, however, that the trends are compatible.

2. Theoretical framework of Bardeen–Cooper–Schrieffer–Bose–Einstein condensation (BCS–BEC) crossover

2.1. T-matrix derivation of BCS theory

In order to understand how to address BCS–BEC crossover at finite $T$, we now rederive standard BCS theory from a $T$-matrix scheme. Important here is that BCS theory can be considered as
Figure 6. (a) Comparison of shear viscosity $\eta \equiv \alpha n\hbar$ and experiments [47] (red triangles) at unitarity for a trapped gas. In theory plots (black dots), we use the calculated thermodynamics for the trap energy $E$ and entropy density $s$. The inset in panel (a) plots the estimated transport lifetime from RF experiments [15], as compared with the experimentally deduced lifetime needed for an exact fit to the theory. (b) Comparison of $\eta/s$. The blue dashed line labels the quantum lower limit of $\eta/s$ given by string theory [3]. The last panel on the right converts the horizontal axis in panel (b) to temperature.

Figure 7. Characteristic diagrams that enter into the spin and charge response kernels (left). Heavy and light lines correspond to dressed and bare Green’s functions. Here the wavy lines represent the pair propagator $t$. The corrections to the vertex include the MT and AL diagrams. Temperature dependence of $S_-(q)$ (see equation (24) in units of $n/2E_F$) for fixed momentum transfer $q = 1.5k_F$ and frequency $\omega = 2.0E_F$ at unitarity (right). This figure shows that $S_-$ behaves like an order parameter [16].

incorporating virtual non-condensed pairs. Here we consider the general case applicable to both s- and d-wave pairing by incorporating $\varphi_k = [\cos(k_x) - \cos(k_y)]$ for the latter and taking it to be unity for the former. These virtual $Q \neq 0$ pairs are associated with an effective propagator or $t$-matrix, which is constrained to be of the form

$$ t(Q) \equiv \frac{U}{1 + U \sum_K G_K G_{0,-K+Q} \varphi_{K-Q}^2} \quad (5) $$

in order to yield the standard BCS equations. This $t$-matrix is associated with a summation of ladder diagrams in the particle–particle channel (see figure 7(a)) and importantly depends on both $G$ and $G_0$, which represent dressed and non-interacting Green’s functions, respectively. We use $K$ and $Q$ to denote four vectors with $K = (i\omega_n, \mathbf{k})$, $Q = (i\Omega_l, \mathbf{q})$, where $\omega_n$ and $\Omega_l$ are the fermion and boson Matsubara frequencies, respectively. In order to describe pairing
in the \( d_{x^2-y^2} \)-wave channel, we write the attractive fermion–fermion interaction in the form 
\[ U_{k,k'} = U \phi_k \phi_{k'}, \]
where \( U \) is the strength of the pairing interaction. As in bosonic theories, non-condensed pair excitations of the condensate are necessarily gapless below \( T_c \). This means that \( t(Q \to 0) \to \infty \), which is equivalent to the vanishing of the effective pair chemical potential, \( \mu_{\text{pair}} = 0 \), for \( T \leq T_c \). Thus, we have a central constraint on the \( T \)-matrix,

\[ t^{-1}(Q \to 0) = 0 \to \mu_{\text{pair}} = 0, \quad T \leq T_c. \tag{6} \]

In order to show that the above condition is identical to the BCS gap equation, we need an appropriate form of \( G_K \). In the BCS theory, the fermionic self-energy that appears in the fully dressed Green’s function, \( G_K \), is of the form

\[ \Sigma_{sc,K} = \sum_Q t_{sc}(Q)G_{0,-K+Q}\varphi_{k-q/2}^2 = -\sum_Q \Delta_{sc,K}^2 \delta(Q)G_{0,-K+Q} = -\Delta_{sc,K}^2 G_{0,-K}, \tag{7} \]

where \( \Delta_{sc,K}(T) \equiv \Delta_{sc}(T)\varphi_k \) is the superconducting order parameter. The full Green’s function is then obtained via the Dyson equation, \( G_K = [G_{0,K} - \Sigma_{sc,K}]^{-1} \), which when inserted in equation (5) yields the BCS gap equation below \( T_c \),

\[ 1 = -U \sum_k \frac{1 - 2 f(E_{sc,K}^2)}{2 E_{sc,K}^2} \varphi_k^2, \tag{8} \]

with \( E_{sc,K}^2 \equiv \sqrt{\xi_k^2 + \Delta_{sc,K}^2} \). We have thus used equations (5) and (6) to derive the standard BCS gap equation within a \( T \)-matrix language and the result appears in equation (8). Equation (6) can be viewed as representing an extended version of the Thouless criterion of strict BCS, which applies for all \( T \leq T_c \). This derivation leads us to reaffirm the well-known result [51–53] that BCS theory is associated with one bare and one dressed Green’s function in the pair propagator.

### 2.2. Generalization to BCS–BEC crossover

To address BCS–BEC crossover, we presume that the non-condensed \( (Q \neq 0) \) pairs are no longer virtual. Thus the \( T \)-matrix of equation (5) in general possesses two contributions: the \( q = 0 \) contribution that gives rise to the formation of the condensed or superconducting pairs and the \( q \neq 0 \) contribution that describes the correlations associated with the non-condensed pairs. As a result, the fermionic self-energy also possesses two contributions, which are given by

\[ \Sigma(K) = \sum_Q t(Q)G_{0,-K+Q}\varphi_{k-q/2}^2 = \sum_Q [t_{sc}(Q) + t_{pg}(Q)]G_{0,-K+Q}\varphi_{k-q/2}^2 = \Sigma_{sc,K} + \Sigma_{pg,K}. \tag{9} \]

The resulting full Green’s function, \( G^{-1} = G_0^{-1} - \Sigma_{sc} - \Sigma_{pg} \), is illustrated in figure 7(b). While as before, \( \Sigma_{sc,K} = -\Delta_{sc,K}^2 G_{0,-K} \), we find numerically [54, 55] that \( \Sigma_{pg,K} \) is in general of the form

\[ \Sigma_{pg,K} \approx \frac{\Delta_{pg,K}^2}{\omega + \xi_k + i\gamma}, \tag{10} \]

with \( \Delta_{pg,K} = \Delta_{pg}\varphi_k \). That is, the self-energy associated with non-condensed pairs possesses the same structure as their BCS counterparts, albeit with a finite lifetime, \( \gamma^{-1} \). Physically this arises from the fact that \( t_{pg}(Q) \) is strongly peaked around \( Q = 0 \) below \( T_c \) where the pair chemical potential is zero, and for a range of temperatures above \( T_c \) as well where this chemical potential is small.

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Analytic self-consistent equations for $\Delta_{pg}$ and $\Delta_{sc}$ can be obtained microscopically when we consider the small $\gamma$ limit where

$$\Sigma(K) \approx -(\Delta_{sc,k}^2 + \Delta_{pg,k}^2)G_{0,-K} \equiv -\Delta_k^2 G_{0,-K},$$

with

$$\Delta_{pg}^2 \equiv - \sum_Q t_{pg}(Q).$$

Here, we have used the fact that because of the vanishing of $\mu_{pair}$ below $T_c$, the bulk of the contribution to $\Sigma_{pg}$ in the ordered state comes from small $Q$. This then leads to an effective pairing gap $\Delta(T)$ whose square is associated with the sum of the squares of the condensed and non-condensed contributions,

$$\Delta_k^2(T) = \Delta_{sc,k}^2(T) + \Delta_{pg,k}^2(T).$$

Note that the full gap $\Delta_k$ remains relatively $T$ independent, even below $T_c$, as observed, because of the conversion of non-condensed ($\Delta_{pg,k}$) to condensed ($\Delta_{sc,k}$) pairs as the temperature is lowered.

The gap equation for this pairing gap, $\Delta_k(T) = \Delta(T)\varphi_k$, is again obtained from the condition $t_{pg}^{-1}(Q = 0) = 0$, and is given by

$$1 = -U \sum_k \frac{1 - 2f(E_k)}{2E_k} \varphi_k^2.$$

This analysis can be made more explicit after analytical continuation so that [41] $t_{pg}(\omega, \mathbf{q}) \approx [Z(\Omega - \Omega_0^q + \mu_{pair}) + \mathbf{i} \Gamma(q)]^{-1}$, where $Z = (\partial \chi / \partial \Omega)|_{\Omega = 0, q = 0}$, $\Omega_0^q \approx q^2/(2M_k)$ with the effective pair mass $M_k^{-1} = (1/2Z)(\partial^2 \chi / \partial q^2)|_{\Omega = 0, q = 0}$. Near $T_c$, $\Gamma \to 0$ faster than $q^2$ as $q \to 0$ and will be neglected. Then $\Delta_{pg}^2 \approx Z^{-1} \sum_q b(\Omega_0^q - \mu_{pair})$.

Note that one needs to self-consistently determine the chemical potential, $\mu$, by conserving the number of particles, $n = 2 \sum_k G_k$, which leads to

$$n = 2 \sum_k G_k = \sum_k \left[ 1 - \frac{\xi_k}{E_k} + 2 \frac{\xi_k}{E_k} f(E_k) \right].$$

Equations (12)–(14) present a closed set of equations for the chemical potential $\mu$, the pairing gap $\Delta_k(T) = \Delta(T)\varphi_k$, the pseudogap $\Delta_{pg,k}(T) \equiv \Delta_{pg}(T)\varphi_k$, and the superconducting order parameter $\Delta_{sc,k}(T) = \Delta_{sc}\varphi_k$ with $\Delta_{sc}(T) = \sqrt{\Delta^2(T) - \Delta_{pg}^2(T)}$. Following this approximation, $\Delta_{pg}(T)$ essentially vanishes in the ground state where $\Delta = \Delta_{sc}$. This is to be expected from the BCS–Leggett wavefunction in equation (1). In this way, the ‘two-gap’ physics disappears in the ground state. Importantly, numerical studies [37] show that for d-wave pairing, there is no superfluid phase in the bosonic regime where $\mu$ is negative; the pseudogap is, thus, associated with the fermionic regime.

3. Transport theory and gauge invariant approaches to superconductivity and superfluidity

Our transport theory for BCS–BEC crossover is based on linear response theory for both the density (or charge, labeled $C$) and the spin (labeled $S$) degrees of freedom. In this approach,
the U(1) electromagnetic (EM) gauge symmetry and the spin rotational symmetry around the z-axis play important roles and an understanding of transport in strongly correlated superfluids has to incorporate in a central way the related conservation constraints. These enter via (i) the transverse $f$-sum rule. Application of the latter to the conductivity is, in turn, related to (ii) the absence (above $T_c$) and the presence (below $T_c$) of a Meissner effect.

The perturbing Hamiltonian can be written in a compact form $\int d^3r (\lambda_{\sigma,\sigma'}^{C,S} \psi_{\sigma}^\dagger \psi_{\sigma'} + \text{h.c.})$ where $\lambda_{\sigma,\sigma'}^{C,S} \propto \delta_{\sigma,\sigma'}$ and $\lambda_{\sigma,\sigma'}^{S} \propto \delta_{\sigma,\sigma'}$. Here, $\psi^\dagger(\psi)$ are the fermionic creation (annihilation) operators, $\sigma = \uparrow$ or $\downarrow$, $\uparrow = - \downarrow$ and $\bar{\sigma} = -\sigma$. We represent the density–density, current–current and spin correlation functions as $\chi_{\rho\rho}$, $\tilde{\chi}_{JJ}$ and $\chi_{SS}$. Experimentally, the last of these three can be probed by spin-preserving and spin-flip (two-photon) Bragg scattering. The shear ($\eta$) and bulk viscosities ($\xi_2$) and the conductivity may be written in terms of the longitudinal ($\chi_L$) and transverse ($\chi_T$) components of the current–current correlation functions $\tilde{\chi}_{JJ}$,

$$\eta = -m^2 \lim_{\omega \to 0} \lim_{q \to 0} \frac{\omega}{q^2} \text{Im} \chi_T(\omega, q),$$

$$\xi_2 + \frac{4}{3} \eta = -m^2 \lim_{\omega \to 0} \lim_{q \to 0} \frac{\omega}{q^2} \text{Im} \chi_L(\omega, q),$$

$$\sigma(\omega \to 0) = -e^2 \lim_{\omega \to 0} \lim_{q \to 0} \frac{\text{Im} \chi_T(\omega, q)}{\omega},$$

where the longitudinal $\chi_L = \hat{q} \cdot \tilde{\chi}_{JJ} \cdot \hat{q}$ and transverse $\chi_T = (\sum_{\sigma=\uparrow,\downarrow} \chi^{aa}_{\sigma\sigma'} - \chi_L)/2$ susceptibilities satisfy [8]

$$\lim_{q \to 0} \int_{-\infty}^{+\infty} \frac{d\omega}{\pi} \left( -\frac{\text{Im} \chi_T(\omega, q)}{\omega} \right) = \frac{n_n(T)}{m},$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{\pi} \left( -\frac{\text{Im} \chi_L(\omega, q)}{\omega} \right) = \frac{n}{m},$$

$$\int_{-\infty}^{+\infty} d\Omega \text{Re} \sigma(\Omega) = e^2 \frac{n}{m}. $$

Because we simultaneously discuss both neutral and charged systems, it is useful to define $e \equiv 1$ for the neutral case. Here, $n_n$ is the particle number of the normal component in the superfluid and the sum rules are explicitly written for the Fermi gas in which there are no band structure effects. For scattering probes, we define the associated structure factors for spin and charge or density in terms of closely related response functions, $\chi_{\rho\rho}$ and $\chi_{SS}$,

$$S_C(q, \omega) = -\frac{1}{\pi} \coth \left( \frac{\omega}{2T} \right) \text{Im} \chi_{\rho\rho}(q, \omega) \quad \text{and} \quad S_S(q, \omega) = -\frac{1}{\pi} \coth \left( \frac{\omega}{2T} \right) \text{Im} \chi_{SS}(q, \omega).$$

Since the conservation laws for particle number and spin, $\partial^\mu J_\mu = 0$ and $\partial^\mu J_\mu^S = 0$, are satisfied, the following two sum rules must be respected at all temperatures in the whole BCS–BEC crossover regime,

$$\int_{-\infty}^{+\infty} d\omega \omega S_{C,S}(q, \omega) = \frac{n q^2}{m} \quad \text{with} \quad \lim_{q \to 0} S_{C,S}(q, \omega) = 0.$$
The EM kernel is defined by \( \mathbf{J} = -\mathbf{K} \cdot \mathbf{A} \), where \( \mathbf{K} \) is given by \( \kappa_{ij}(Q) = e^2(\bar{n}/m)_{\text{dia}} + \tilde{P}(Q) \), and the paramagnetic contribution, given by \( \tilde{P}(Q) \), is associated with the normal current resulting from fermionic and bosonic excitations [16, 56]. In the superfluid phase, the density correlations that enter into scattering and the current correlations that enter into transport can be schematically written as a sum of three terms where for convenience we drop the \((\omega, \mathbf{q})\) arguments: \[ \tilde{\chi}_{JJ} = (1/e^2) \tilde{P} + \frac{\bar{n}}{m} + \mathbf{Coll}_J \] and \( \chi_{\rho\rho} = (1/e^2)P_{00} + \mathbf{Coll}_\rho \). The counterparts for the spin degrees of freedom are \( \tilde{\chi}_{SS} = \tilde{Q}_S + \frac{\bar{n}}{m} \) and \( \chi_{SS} = Q_{00}^S \). Here, \( \tilde{P} \), \( P_{00} \), and \( \tilde{Q}_S \), \( Q_{00}^S \) represent the ‘bare’ contributions. Collective mode effects in the charge response, which are not present in the spin response, must also be included in the longitudinal response below \( T_c \). These appear in the above equations as \( \mathbf{Coll}_J \) and \( \mathbf{Coll}_\rho \). These collective mode effects are essential to ensure that the sum rules and related conservation laws are satisfied.

In the most general case, the diamagnetic current is expressed in terms of the inverse band mass \( \partial^2 \xi_k/\partial k_\alpha \partial k_\beta \) (with \( \alpha, \beta = x, y, z \)), via \( \langle \bar{n}/m \rangle_{\text{dia}} = 2 \sum_K(\partial^2 \xi_k/\partial k \partial k)G_K \). Importantly, the latter contribution, which is temperature independent, should not be confused with \( \langle n/m(T) \rangle_{\text{eff}} \). We stress that this effective carrier number is sensitive to the pairing gap \( \Delta(T) \), while the diamagnetic contribution is not. We integrate the expression for the diamagnetic contribution by parts and use the self-energy equation and the Generalized Ward identity to obtain (see appendix A) an alternative form above \( T_c \),

\[
\langle \frac{\bar{n}}{m} \rangle_{\text{dia}} = -2 \sum_k \frac{\partial \xi_k}{\partial k} \frac{\partial \xi_k}{\partial k} \left[ G_K^2 + \sum_p t_{pg}(P)G_{0,P,K}^2G_K^2 \right].
\]

This exact \( T \)-matrix-based equation is significant because it has cast the diamagnetic response in the form of a two-particle response function. That there is no Meissner effect in the normal state is related to a precise cancellation between the diamagnetic and paramagnetic terms. Noting that \( \tilde{P}(0) = -e^2(\bar{n}/m)_{\text{dia}} \), we may extend \( \tilde{P}(0) \) to finite \( Q \) to infer that

\[
\tilde{P}(Q) = 2e^2 \sum_k \frac{\partial \xi_k}{\partial k} \frac{\partial \xi_k}{\partial k} \left[ G_K G_{K+Q} + \sum_p t_{pg}(P)G_{0,P,K-Q}G_{0,P,K}G_{K+Q}G_K \right].
\]

So far, our discussion has been quite general, and we have circumvented any mention of specific transport diagrams by building in the absence of a Meissner effect above \( T_c \). One can alternatively [16, 56, 57] introduce the Aslamazov–Larkin (AL) and Maki–Thompson (MT) diagrams to arrive at the above equation (see figure 7), but the former, which involves two factors of \( t_{pg} \), at first sight appears more complicated.

Collective mode effects are not present in the viscosity and conductivity because they represent transverse probes. However, they play an important role in the density–density response. In this regard, it is convenient to define

\[
S_\pm = (S_C \pm S_S)/2 \quad \text{from which it follows that} \quad \int_0^\infty d\omega \omega S_-(\omega, \mathbf{q}) = 0.
\]

This latter is a very unusual sum rule. However, it must be satisfied in any consistent theory of superfluidity, provided spin and charge are conserved. In appendix B, we show how this sum rule is satisfied above \( T_c \) for an alternative BCS–BEC crossover theory introduced by Nozieres and Schmitt-Rink (NSR) [36].
There are other physical consequences that can be deduced once one has a conservation-law consistent theory. The conductivity and the shear viscosity can alternatively be written in terms of the bare response $P(Q)$ so that

$$\sigma(\omega) = -e^2 \lim_{q \to 0} \frac{\text{Im} \, P^{xx}(\omega, q)}{\omega} \quad \text{with} \quad \eta = -m^2 \lim_{q \to 0} \frac{\omega}{q^2} \text{Im} \, P^{xx}(\omega, q). \quad (25)$$

Moreover, in this way, one can see how closely related they are. We will not in this paper address the bulk viscosity, principally because we do not have the same level of theoretical control to satisfy the longitudinal $f$-sum rule, which is also more problematic below $T_c$, where the Goldstone bosons appear [18].

### 3.1. Weak dissipation regime

In the previous section, we set up the general linearized response theory. We now discuss the detailed implementation of this formalism based on equation (2). Consistency with conservation laws requires that transport diagrams assume a specific form, reflecting the behavior of the self energy. We begin with the so-called ‘weak dissipation’ regime, where in equation (2) we presume the quantity $\gamma \approx 0$. The strong dissipation regime is discussed in the next subsection.

We have found that for the viscosity there is very little difference between weak and strong dissipation-based calculations, but this is not true for the d-wave conductivity, because of the easy excitations of fermions around the nodes. We indicate in figure 7 the characteristic diagrams that enter into the generalized correlation functions. As a result, one can show [17] that the ‘bare’ contributions to generalized charge and spin susceptibilities take on an extremely simple form,

$$P^{00}_c(\omega, q) = e^2 \sum_p \left[ \frac{E_p^+ + E_p^-}{E_p^+ E_p^-} (1 - f(E_p^+) - f(E_p^-)) \right], \quad (26)$$

$$\tilde{P}_S(\omega, q) = e^2 \sum_p \frac{p \, p}{m \, m} \left[ \frac{E_p^+ + E_p^-}{E_p^+ E_p^-} (1 - f(E_p^+) - f(E_p^-)) \right], \quad (27)$$

where $\omega$ implicitly has a small imaginary part and we define the quantities $E_p^\pm = E_{p\pm q/2}$ and $\xi_p^\pm = \xi_{p\pm q/2}$. Note that equations (26) and (27) are exactly the same as their BCS counterparts when $\Delta^{2}_{pg} \to 0$. It is quite remarkable that when we compare these two equations, we see that the charge response functions reflect the difference $\Delta^{2}_{sc} - \Delta^{2}_{pg}$, while the spin response functions depend on the total pairing gap $\Delta^{2} = \Delta^{2}_{sc} + \Delta^{2}_{pg}$. We may say that spin correlation functions know only about pairing, whereas their charge counterparts reflect also coherent superfluid order.

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From the definition of $\chi_\ell(\omega, \mathbf{q})$ the shear viscosity is given by

$$\eta = -m^2 \lim_{\omega \to 0} \lim_{\mathbf{q} \to 0} \frac{\pi \omega}{2 q^2} \sum_p \frac{p^2 \sin^2 \theta}{m^2} \left[ (1 - f(E_p^+)) - f(E_p^-) \right] \frac{E_p^+ E_p^- - \xi_p^+ \xi_p^- - \delta \Delta^2}{2 E_p^+ E_p^-} \delta_1(\omega) \right) \right], \quad (28)$$

where we have used the abbreviated notation: $\delta_1(\omega) = \delta(\omega - E_p^+ - E_p^-)$, $\delta_2(\omega) = \delta(\omega - E_p^+ + E_p^-)$. Physically, two types of terms appear in the above equation. Both are well known in standard BCS theory. The first refers to processes that require a minimal frequency of the order of $2\Delta(T)$; they arise from the contribution of fermions that are effectively liberated by the breaking of pairs. The second of these terms, involving $\delta_2$, arises from the scattering of fermionic quasi-particles and is the only surviving contribution to the viscosities, which are defined in the $\omega \to 0$ limit. Note that both contributions involve the difference in the condensate and non-condensate components ($\delta \Delta^2 = \Delta^2_{\sigma c} - \Delta^2_{\sigma p}$) with opposite overall signs. The low-$\omega$ quasi-particle scattering processes are reduced by the presence of non-condensed pairs—because they are associated with a reduction in the number of fermions. By contrast, in the high $\omega \approx 2\Delta$ limit, the number of contributing fermions will be increased by the breaking of pairs. We next take the low $\omega$, $q$ limits in equation (28) and introduce lifetime (or dissipation) effects by writing delta functions as Lorentzians $\delta(\omega \pm q \cdot \nabla E_p) \approx \frac{1}{\pi} \frac{\gamma_p}{(q \cdot \nabla E_p)^2 + \gamma_p^2}$. In this way, we arrive at equation (3), which was presented in the summary section. Note that $\eta$ assumes a form similar to a stress tensor–stress tensor correlation function. The transport expressions summarized earlier in equation (3) correspond to the generally familiar BCS form [25, 26] except for the effects associated with nonzero $\Delta_{\sigma p}$, which appears as a prefactor $1 - \frac{\Delta^2_{\sigma p}}{\Delta^2}$. This deviation from unity can be traced to the AL diagrams and hence may be interpreted as extra ‘bosonic’ contributions that contribute to the normal fluid $n_n$. Their presence is physically required for the sum rule consistency of $\chi_\ell$. Note that these terms enter with an overall reduction coherence-effect prefactor associated with $\xi^2/E^2$. The negative sign for this bosonic term comes from the fact that only fermions contribute directly to transport within a BCS–Leggett-based scheme; the more pairs present, the lower the fermionic contribution to the viscosity. We see from equation (4) that $\sigma$ and $\eta$ are alike except for momentum power-law prefactors and coherence factors. When extended to finite frequency, we note that unlike in BCS theory, there is a pair-creation and pair-breaking contribution to the optical conductivity. This term is necessarily absent in a clean BCS superconductor, but may well be the origin of the widely observed [58] ‘mid-infrared’ contribution to the finite $\omega$ conductivity.

### 3.2. Explicit proof of the transverse f-sum rule

An important check of our microscopic scheme is to show that it satisfies the sum rule for the transverse component. The sum rule that we are going to prove is in equation (18). Our proof here is explicitly for the weak dissipation form of the response functions and for the Fermi gas, where there are no band structure effects. We generalize this later in the paper. The total particle
number is
\[ n = \sum_p \left( 1 - \frac{\xi_p}{E_p} (1 - 2 f(E_p)) \right) = \frac{2}{m} \sum_p \frac{p^2}{E_p^2} \left( \frac{1 - 2 f(E_p)}{2 E_p} \right) \Delta_{1g}^2 - \frac{\xi_p}{m} \frac{\partial f(E_p)}{\partial E_p}. \] (29)

The superfluid density at general temperature \([39]\) is given by
\[ n_s = \frac{2}{3} \Delta_{1g}^2 \sum_p \frac{p^2}{E_p^2} \left( 1 - \frac{2 f(E_p)}{2 E_p} + \frac{\partial f(E_p)}{\partial E_p} \right). \] (30)

Therefore,
\[ n_n = n - n_s = \frac{2}{3} \sum_p \frac{p^2}{E_p^2} \left( \frac{\Delta_{1g}^2}{m} \frac{1 - 2 f(E_p)}{2 E_p} - \frac{E_p^2 - \Delta_{1g}^2}{m} \frac{\partial f(E_p)}{\partial E_p} \right). \] (31)

Using \(\bar{\chi}_{JJ} = (1/e^2) \bar{P} + \frac{\bar{S}}{m}\) yields
\[ \lim_{q \to 0} \int_{-\infty}^{+\infty} \frac{d\omega}{\pi} \left( -\frac{\mathrm{Im} \chi_T}{\omega} \right) = \frac{2}{3m} \sum_p \frac{p^2}{E_p^2} \left( \frac{\Delta_{1g}^2}{m} \frac{1 - 2 f(E_p)}{2 E_p} - \frac{E_p^2 - \Delta_{1g}^2}{m} \frac{\partial f(E_p)}{\partial E_p} \right) = \frac{n_n}{m}. \] (32)

By contrast (except in special cases, such as \(q \to 0\), above \(T_c\)), the longitudinal sum rule requires numerical proof. Following the analysis in \([16]\), we find agreement with the sum rule to within 5–10%. In addition to sum rule consistency, the appropriate diagram set for computing all transport properties must be chosen so that \(n_s(T)\) vanishes at and above \(T_c\). This is somewhat more complicated \([19]\) to ensure than in strict BCS theory because there is a finite excitation gap at the transition. This gap or pseudogap, in turn, reflects the fact that there are bosonic excitations in addition to the fermions that deplete the condensate. In section 3.5, we present a more general argument for establishing the conductivity sum rule whether the dissipation is weak or strong.

3.3. Two-photon Bragg experiments

In a recent paper \([16]\), we used this theory to address the dynamical structure factor and thereby show how two-photon Bragg scattering can be used to establish in situ the presence of coherent order in a superfluid, at any temperature, wavevector and frequency. For the most part, experiments on unitary gases have relied on sweeps to the BEC to find evidence for condensation. Our analysis is based on the definitions and sum rule in equation (24). It thus depends on imposing the current conservation laws, which have been extensively studied and verified \([16]\). Using the characteristic diagrams shown in figure 7, which enter into the density–density correlation functions, we are now led to an important observation: the quantity \(S_{-}(\omega, q)\) for all \(q, \omega, 1/k_F a\) can be used as an indication of in-situ superfluid order, without requiring sweeps to the BEC. We show a plot of this behavior in the right-hand panel of figure 7, where it can be seen that the difference structure factor vanishes in the normal state. An interpretation of this figure is that despite previous claims \([59]\), there is no spin–charge separation in the normal or pseudogap state associated with BCS–BEC crossover. Spin–charge separation is, however, to be found in the superfluid phase. The counterpart normal state calculations can be shown to be valid in an alternative BCS–BEC crossover theory, based on the NSR scheme \([36]\); this is presented in appendix B.
3.4. Strong dissipation approach

We now use the full expression for the self-energy equation (2) to obtain compatible expressions for transport coefficients. The full Green’s function is given by

\[ G_K = \left( i\omega_n - \xi_k + i\gamma - \frac{\Delta_{pg,k}^2}{i\omega_n + \xi_k + i\gamma} - \frac{\Delta_{sc,k}^2}{i\omega_n + \xi_k} \right)^{-1}, \]  

where we have added an extra constant term \( i\gamma \) in order to be consistent with the weak dissipation limit in the case when \( \gamma \) becomes small. To extend \( \tilde{T}(Q) \), which appears in equation (23) below \( T_c \), within a BCS-like formulation one needs to include terms of the form \( F_{sc,k} F_{sc,k+Q} \), which represent the usual Gor’kov functions as a product of one dressed and one bare Green’s function \( (GG^0) \),

\[ F_{sc} \equiv - \frac{\Delta_{sc,k}}{i\omega_n + \xi_k + i\gamma} G_K. \]

Here, as before, \( \Delta_{sc}^2 \equiv \Delta_{pg,k}^2 + \Delta_{sc,k}^2 \). Then, in the same spirit as our derivation of equation (10), we exploit the fact that \( t_{pg}(P) \) is strongly peaked at small \( P \), which leads us to approximate

\[ \tilde{T}(Q) \approx 2e^2 \sum_K \frac{\partial \xi_{k+q/2}}{\partial k} \frac{\partial \xi_{q/2}}{\partial k} \left[ G_K G_{K+Q} + F_{sc,k} F_{sc,k+Q} - F_{pg,k} F_{pg,k+Q} \right], \]  

where \( F_{pg,k} \equiv - \frac{\Delta_{pg,k}}{i\omega_n + \xi_k + i\gamma} G_K. \)\]

From equation (35) and \( \text{Re} \sigma_{\text{para}}(\Omega) \equiv - \text{Im} \tilde{T}(\Omega)/\Omega \), the paramagnetic contribution to the dc conductivity is

\[ \text{Re} \sigma_{\text{para}}(0) \approx - \lim_{q \to 0} \text{Im} \sum_K \left[ 2e^2 \frac{\partial \xi_k}{i\Omega_m} \left( \frac{\partial \xi_k}{\partial k_x} \right)^2 \left( G_K G_{K+Q} - F_{pg,k} F_{pg,k+Q} + F_{sc,k} F_{sc,k+Q} \right) \right] \]  

In order to be consistent, we rewrite equation (22), also adding in the usual BCS condensate terms,

\[ \left( \frac{n_{xx}}{m} \right)_{\text{dia}} \approx - 2 \sum_K \left( \frac{\partial \xi_k}{\partial k_x} \right)^2 \left[ G_K G_{K+Q} - F_{sc,k} F_{sc,k+Q} - F_{pg,k} F_{pg,k+Q} \right]. \]

We can use equation (35) to arrive at equations (4), which were presented earlier in the form of a summary. Note that in previous works [48–50], only the first term involving \( GG \) was included, which was recognized [48, 50] as inadequate.

3.5. Proof of conductivity sum rule

We now revisit the issue of compatibility with the important conductivity or transverse sum rule,

\[ \int_{-\infty}^{\infty} d\Omega \text{Re} \sigma(\Omega) = e^2 (n_{xx}/m)_{\text{dia}}. \]
Figure 8. Theory plots of shear viscosities as functions of $T$ from BCS to BEC, divided by the counterpart unpaired state viscosities $\eta_n$. The red dashed lines are the results in the absence of explicit bosonic degrees of freedom (associated with the AL diagrams). In (a), $T_c = 0.12T_F$, $T^* = 0.13T_F$ and $1/k_Fa = -1$. In (b), $T_c = 0.26T_F$, $T^* = 0.50T_F$ and $1/k_Fa = 0$. In (c), $T_c = 0.21T_F$, $T^* = 1.28T_F$ and $1/k_Fa = 1$. Expanded plots in different $T$ regimes are shown in the various insets.

in a more general fashion. Here we now include band structure effects through the effective mass. Note that we must have two contributions to conductivity, corresponding to the paramagnetic and diamagnetic terms,

$$\text{Re} \sigma(\Omega) = -\text{Im} P_{xx}(\Omega)/\Omega + \pi \delta(\Omega)[\text{Re} P_{xx}(\Omega) + e^2 (n_{xx}/m)_{\text{dia}}].$$ (40)

Integrating the first term over frequency, we obtain the term $-\pi \text{Re} P_{xx}(0)$, whereas the second (delta function) term yields the term $+\pi \text{Re} P_{xx}(0)$, which leaves only the diamagnetic contribution and yields the desired sum rule. Note that this analysis holds both below and above $T_c$ and that this sum rule is intimately connected to the absence (above $T_c$) and the presence (below $T_c$) of a Meissner effect. Importantly, since $(n_{xx}/m)_{\text{dia}}$ can be viewed as essentially independent of temperature, when there are approximations in evaluating the transport diagrams, it is appropriate to evaluate the chemical potential $\mu$ based on the $T$-independence in equation (38). It should be reiterated that this diamagnetic contribution is to be distinguished from $n_{\text{eff}}(T)$, which enters into the dc transport.

4. Results for viscosity

We now summarize some of our numerical calculations, beginning with the shear viscosity. In figure 8, we plot the viscosity divided by a ‘normal state’ value $\eta_n$ as a function of temperature for $1/k_Fa = \pm 1$ and for unitarity. This normal state viscosity is to be associated with the (temperature-dependent) viscosity of the unpaired state $\eta_n = \int_0^{\infty} dp \frac{p^6}{15\pi^4 m^4} (-\frac{d}{dp} |T|) \tau_{\eta}$. In this way, we take the same lifetimes in the numerator and the denominator for the plots, which necessarily cancel out of the ratio. For all $k_Fa$, the ratio drops to zero at low temperatures, reflecting the decrease in the number of condensate excitations. The red dashed line indicates the behavior when the bosonic excitations are removed and the black solid line shows the full calculated viscosity. Thus, the shaded regions correspond to the contribution from the AL diagrams. This contribution is seen to be largest at unitarity, as expanded in the inset of figure 8(b). The ratio $\eta(T_c)/\eta_n$ varies from 0.7 to $4.4 \times 10^{-5}$ as one passes from BCS to BEC.
in these three cases. At unitarity, the viscosity at $T_c$ is reduced by a factor of 10, relative to the unpaired fluid. A weak signature of the transition is largest in the BCS regime, as shown in the inset of figure 8(a).

To incorporate trap effects, we begin by summarizing previous works on the thermodynamical properties of cold Fermi gases. The counterpart thermodynamical experiments have played a role in characterizing the viscosity [1, 47, 60]. Figure 9 presents a comparison of our previous thermodynamical calculations [5, 61] with experiments for both the trapped (left) and the homogeneous (right) case. All data are shown in black, while our theoretical results are shown in red. The agreement is reasonably satisfactory in both cases, particularly with the addition of a very small Hartree adjustment that replaces the dashed lines with solid curves. The first four panels (top and bottom, from the left) correspond to comparisons with the trapped case [60]. The next three are for the homogeneous case, with the black lines corresponding to experiments from Japan [62]. The third column of panels compare the present theory (in red) with the experiments from [62] in the upper panel and with other analytical (light blue) [63] as well as Monte Carlo [64] (dark blue) calculations in the lower panel. The large-scale figure shows that the present approach (red curves) is in quite good agreement with the homogeneous experiments [62] (in black) over a rather wide temperature range. Absent here is the first order transition seen in all other analytic theories.

In figure 10, we show the LDA-calculated viscosity as a function of position in the trap and for various temperatures $T$. The arrows indicate the edge of the superfluid (SF) core. It can be seen that the viscosity is suppressed in the core region. In this way, the nearly free fermions at the trap edge dominate the trap averaged value for $\eta$. As a result, the calculated viscosity in a trap is considerably higher than in the homogeneous case.
Figure 10. Trap profile for viscosity. The various curves show the shear viscosity as a function of trap position for four different indicated temperatures and $R_{TF}$ is the Thomas–Fermi radius.

With these thermodynamical calculations and trap profiles as a backdrop, we now return to an earlier figure, figure 6, which addressed the comparison between theory and experiment regarding the shear viscosity. For a unitary Fermi gas, we estimate the transport lifetime from the characteristic broadening of the single particle fermionic spectral function. The latter, in turn, can be found here by fitting [7, 19] RF cold gas experiments. Recall that this fermionic lifetime is associated with a damping contribution in an otherwise BCS-like self-energy (conventionally parameterized [41, 42] by ‘γ’). The inset of figure 6(a) presents a plot of this RF-deduced lifetime as black circles. The red triangles in this inset represent a plot of the lifetime that one would infer from the data in the case of precise agreement between theory and experiment.

Figure 6(a) presents a comparison of the viscosity coefficient $\alpha$ between theory (based on the RF-deduced lifetime), as black dots, and experiment [47] (red triangles) as a function of $E$. Figure 6(b) shows the comparison of $\eta/s$, where $s$ is the entropy density. We find that $\eta/s$ appears to be relatively $T$ independent at lower temperatures. The last figure on the right represents a transcription of the horizontal axis in figure 6(b), which plots $\eta/s$ as a function of temperature, rather than energy $E$. This is based on using the calculated trap thermodynamics [5] to rescale the various axes. Moreover, our calculations incorporate the same trap averaging procedure as in [47]. One can anticipate that, particularly at lower $T$, the trap-integrated viscosity will be artificially higher than for the homogeneous case, since $\eta$ will be dominated by unpaired fermions at the trap edge. It should be noted that in the homogeneous case, the ratio of the viscosity to its normal counterpart is exponentially activated with $T$. This is to be contrasted with the behavior of the entropy, which reflects bosonic power laws [5] in $T$. Overall, it can be seen that our calculations agree favorably with the experimental data. Interestingly, the observed behavior appears more consistent with previous helium-3 experiments [27] than with those in helium-4 [29], as can be seen from figure 1.

5. Implications of BCS–BEC crossover for conductivity in the cuprates

It is the goal of this section to address the dc conductivity $\sigma(T)$ for a pseudo-gapped superconductor, such as the high $T_c$ cuprates, in a fashion that is analogous to our above
discussion of viscosity for an ultracold Fermi gas. In the cuprate literature, one associates a fixed stoichiometry (hole doping) with a given excitation gap at $T_c$, say. The lower the hole doping ($x$), the larger this gap. Since the parameter $x$ is of no particular interest here, we parameterize instead a given stoichiometry by the size of $\Delta(T_c)$, or alternatively the size of the temperature $T^*$ at which $\Delta(T)$ first vanishes. The two key puzzles of the dc conductivity in the cuprates are the near linearity of the resistivity with temperature and the fact that only the doped holes ($\sigma(T_c) \propto x$) appear to contribute to transport.

A central conclusion of our conductivity study is that, just as for the viscosity, the reduction in the effective carrier number $(n/m(T))_{\text{eff}}$ is revealed to play an important role in both the $T$ and $x$ dependences of transport. This reduction, in turn, is a consequence of the presence of an excitation gap, which persists into the normal phase and increases as $x$ decreases. The suppression of the carrier number is substantial relative to the full diamagnetic or sum rule value in equation (39). Moreover, $(n/m(T))_{\text{eff}}$ rises nearly monotonically with temperature until $T^*$. This contribution leads to a non-metallic tendency with $\sigma$ increasing with $T$, above $T_c$. In order to yield a metallic resistivity (which increases with $T$), the contribution of $(n/m(T))_{\text{eff}}$ must be offset by taking $\gamma(T)$ to be a higher power than linear. Here we illustrate our results for $T > T_c$ with a Fermi liquid-like behavior $\gamma(T/T_c)^2$, which could plausibly be associated with Fermi arcs [46], which are extended gapless regions due to the smearing out of the d-wave nodes above $T_c$. This is the most conventional $T$ dependence for transport processes that involve inter-fermion scattering.

We return to figure 5 (top panel) presented earlier that summarizes the general behavior. Of particular interest is the behavior of the calculated resistivity, illustrated in figure 5(d), which itself is roughly linear, because of the assumed quadratic or Fermi liquid $T^2$ dependence in $\gamma$. This should not be viewed as an in-depth explanation of the mysterious linear resistivity, but rather as clarifying the requirements to find this resistivity in the presence of a pseudogap. An essential component of this transport is the $T$-dependent carrier number. Note that these conductivity calculations are specific to the d-wave case, and for an s-wave counterpart it would be very difficult to find metallic behavior. This can be seen by noting that the inverse viscosity shown in this figure decreases with increasing $T$, reflecting the even more strongly suppressed carrier number.

Figure 11 presents a more detailed plot of these normal state resistivities for dopings that interpolate between the underdoped ($T^*/T_c = 4$) and overdoped cases ($T^* = 1.4T_c$). The resistivities are normalized by the value $\rho_x = \rho(T_c)$ for the case of the highest doping. We may characterize each curve, in order of increasing doping, by the ratio $\gamma/\Delta(T_c) = 0.07, 0.20$ and 0.35. That the size of the resistivities decreases as one increases the hole concentration largely reflects the change in gap size and thus in effective carrier number. It is here that Mott physics is most evident. It would appear that the attractive interaction (via the size of $T^*$) becomes larger with underdoping as the Mott phase is approached. Importantly, the insulator is associated with strong on-site Coulomb repulsion, thus making it unlikely to have s-wave superconductivity. Indeed, the fact that the pairing is in the d-wave channel suggests that it may well be this direct repulsion that indirectly leads to pairing [28, 67] at higher $l$, just as is seen in helium-3.

Although it is subtle, one can see from the figure that there is a change in the nearly linear slope with increased doping from concave to convex bowing, which may be seen in figure 2. In this way, it would appear that the so-called ‘optimal’ doping (where the bowing is minimal) represents a form of midway point, rather than a specific form of ‘strange metal’ [68].
Figure 11. Theoretical plots of resistivity as a function of temperature for three different values of $x$, corresponding to the pairing onset temperature (or strength of attractive interaction). The UND, OPT and OVR labels are the same as those used in figure 2 from [32]. The resistivity decreases as doping increases, deriving from an increase in the effective carrier number. That the plots stop at temperature $T = T^*$ reflects the fact that equation (2) is no longer appropriate when the system is this far from condensation.

6. Comparison with the literature

6.1. Pseudogap theories

Since the present transport calculations emphasize the role of the pseudogap, it is useful to clarify the nature of the pseudogap that is specifically associated with BCS–BEC crossover. One of the earliest observations that a pseudogap may be present in BCS–BEC crossover theories is due to Randeria and co-workers [69]. This first generation analysis focused on ‘spin gap’ effects, so that there were a number of claims that are inconsistent with current understanding. Despite statements to the contrary [69], (i) the pseudogap is not a ‘spin gap’ but is also to be associated, as we have seen in the present transport calculations, with the ‘charge channel’. This mirrors the experimental observations [70] on the nature of the cuprate pseudogap: ‘It is a quasiparticle gap and not just a spin gap’. (ii) The pseudogap is not associated with ‘spin–charge’ separation above $T_c$. In this paper (section 3.5 and appendix B), we have shown that the absence of a Meissner effect leads to the non-separation of spin–charge degrees of freedom. (iii) The pseudogap in crossover theories is quite distinct from the pseudogap associated with the phase fluctuation scenario (as discussed in section 1.2.1).

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Our group was one of the earliest (i) to flesh out an understanding of the nature of this pseudogap and, in particular not only to show [71] that the BCS–BEC crossover pseudogap is associated with spin as was claimed [69], but also, more generally, it represents a quasiparticle gap. This observation was based on calculations of the fermionic spectral function; (ii) to address gauge invariant electrodynamics in the presence of a pseudogap [39, 57]; (iii) to introduce the concept of a pseudogap into the cold gas literature [72]; and (iv) to introduce [41] the currently widely used expression [42] for self-energy (equation (2)).

6.2. Cold Fermi gases

There has been significant effort devoted to achieving a theoretical understanding of the various viscosities associated with highly correlated fermionic superfluids [3]. A detailed overview exploring the relation between cold atomic gases and hot quark-gluon plasmas was presented by Schaefer and Teaney [73]. Taylor and Randeria [74] established sum rules at unitarity for the dynamical counterpart $\eta(\omega)$ and $\zeta^2(\omega)$, which are less general than those in equations (18) and (39). Zwerger and colleagues [75] addressed the behavior of the normal unitary gas predicting a minimum in $\eta$ and $\eta/s$ near but below the superfluid transition.

Bruun and Smith [48, 76] were perhaps the first to emphasize the importance of the (static) shear viscosity in cold Fermi gases using both a high-temperature fermionic Boltzmann approach and a Kubo-stress-tensor-based scheme (above $T_c$), within BCS–BEC theory. They importantly recognized [48] that the introduction of a pseudogap would lower the normal state $\eta$. However, the diagram set that was used was ‘not conserving’ [48].

Rupak and Schafer [4] introduced an alternative (bosonic Boltzmann) transport theory in which the low $T \ll T_c$ shear viscosity is dominated by the Goldstone bosons or phonons. They predicted that $\eta/s$ increases as temperature decreases. In their approach, phonons dominate the low-$T$ transport and lead to the same upturn in $\eta$, as seen in helium-4 and shown in figure 1 (right). Establishing diagrammatic consistency is a central theme of this paper and within a BCS-like theory the shear viscosity, as a transverse response, does not directly couple [18] to the Nambu–Goldstone bosons. This is to be distinguished from the longitudinal current–current correlations, which are the basis for bulk viscosity and necessarily depend on these collective modes.

6.3. Alternative approaches to transport in high-$T_c$ superconductors

Understanding the temperature dependence of the (inverse) conductivity or dc resistivity, particularly near optimal doping, was one of the first puzzles posed by the high-temperature superconductors. It should, nevertheless, be noted that a number of ideas and interpretations about transport were established well before there was a full understanding of how widespread was the pseudogap phase (which extends over most of the phase diagram, including optimal doping). For this reason, one can argue that these earlier theories must be reexamined.

A number of different models from spin–charge separation [68] to marginal Fermi liquid phenomenology [77] were invoked to explain (i) the unusual normal state temperature power laws observed, for example, the linear resistivity. A second notable observation (ii) was that the effective carrier number in transport appeared to scale with the number of (extra) doped holes [68], called $x$, even though the volume of the Fermi surface scaled as expected with $1 + x$. The first of these experimental observations gave rise to the concept of a distinct ‘strange metal’
phase, which was thought to separate the overdoped and underdoped regimes. The second of these gave rise to the concept of ‘spin–charge’ separation. Despite the widespread agreement that the normal state of the cuprates was a non-Fermi liquid phase, there is today a strong belief that the superconducting phase is Fermi liquid based [68].

It must be noted that the BCS–BEC crossover scenario addresses these issues from a very different perspective. It presumes that there is a smooth evolution from over- to underdoped behavior rather than a distinct strange metal phase near optimal doping. We have argued here that the suppression of the number of carriers may be associated with the fact that the magnitude of the excitation gap grows with underdoping. (In ac conductivity, this leads to a very small weight for the $\omega \approx 0$ Drude peak, which requires, by the sum rule, an additional mid-infrared contribution.) As noted above, there is no spin–charge separation in the normal phase. Importantly, the superconducting phase has bosonic excitations representing non-condensed pairs (also present above $T_c$) and therefore this phase is not Fermi liquid based. Finally, the dissipation mechanism here associated with the BCS–BEC crossover scenario arises from the inter-conversion of fermions and pairs, which is distinctly tied to the pseudogap. This is to be distinguished from most transport theories [78, 79], which focus on impurity effects or the dynamics of the pairing boson [80, 81].

There is a shortcoming in our cuprate transport calculations because we have ignored impurity effects in transport altogether. Away from very low temperatures, it is generally accepted [58] that impurity effects are not particularly important. Nevertheless, they may lead to incomplete condensation [31, 79] in the ground state. Indeed, it is difficult to see how to reconcile claims [58] that the $T = 0$ superfluid density scales as $\propto x$, with the transverse $f$-sum rule (equation (39)), without invoking incomplete condensation in the ground state. A deeper understanding of a possibly more inhomogeneous treatment of impurities is needed. Fortunately, this is not an issue in cold gases.

With a growing appreciation of the nature of the pseudogap (and related ‘Fermi arc’ effects), experimentalists have provided some support for our observation that the carrier number is $T$ dependent. In [9], it is stated that ‘This indicates that the functional form of the dc resistivity of cuprates $\rho_{dc}(T)$ is governed not only by the relaxation processes but also by temperature-dependent numbers of carriers’. If, as these and other experimental groups have argued, the pseudogap will lead to a carrier number that increases with temperature, this raises the following important question. How does one find metallic resistivities in the cuprates? This would seem to be a problem even if the pseudogap is to be associated with another order parameter. It is hard to not reach the conclusion that the inverse lifetime should then contain higher powers than linear in $T$. While we proposed here to use a simple $\gamma \propto T^2$ dependence, this is the phenomenological inference that is required to arrive at a linear resistivity in the present case. We end this section by quoting from Ando and co-workers who state [82]: ‘One may notice that a natural extension of the present argument would be that the $T$-linear resistivity usually observed near optimum doping may not necessarily be a sign of a $T$-linear relaxation rate, because $n_{\text{eff}}$ may be changing with $T$.’

7. Conclusions

In summary, this paper has addressed the role of the pseudogap in the $\omega \to 0$ conductivity and in the shear viscosity both above and below $T_c$. We have emphasized the analogy between ‘bad metals’ [2] and ‘perfect fluids’ [1] seen in high-$T_c$ superconductors and the atomic Fermi gases. Both of these phenomena, we argue, may arise from pseudogap effects.
Our approach builds on a consistent gauge invariant treatment of transport (which has not been addressed previously), in which the transverse $f$-sum rule demonstrably holds. As a summary, figure 6 shows semi-quantitative agreement with shear viscosity data from below to above $T_c$. We see no sign of the upturn that others have predicted.

It was our intention in this overview to introduce some of the key challenges in understanding transport in high-$T_c$ superconductors to the wider readership interested in perfect fluidity. It is hoped that an appreciation of this broader context may lead to new breakthroughs in understanding superficially distinct, but quite possibly connected, physical systems.

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Appendix A. Rewriting the number equation

It is useful to fill in a few steps in the line of argument from the text. We wish to show here, for the simpler case of a Fermi gas, how the number equation may be rewritten using a Ward identity,

$$P(Q) = \frac{-2e^2}{3m^2} \sum_k \left( k + \frac{q}{2} \right)^2 \left[ G_K G_{K+Q} + \sum_p t_{pg}(P) G_{0,P-K-Q} G_{0,P-K} G_{K+Q} G_K \right]. \quad (A.1)$$

The number equation

$$n = 2 \sum_k G_K = 2 \sum_k \frac{\partial k_a}{\partial k_a} G_K$$

$$= 2 \sum_k k_a G^2(K) \partial G^{-1}_K / \partial k_a$$

$$= 2 \sum_k k_a G^2_K \left[ \partial G^{-1}_{0,K} / \partial k_a - \partial \Sigma(K) / \partial k_a \right]$$

$$= -2 \sum_k k_a G^2_K \left[ \frac{k_a}{m} - \sum_Q t(Q) \partial G_{0,Q-K}^{-1} / \partial k_a \right]$$

$$= -2 \sum_k k_a G^2_K \left[ \frac{k_a}{m} + \sum_Q t(Q) G^2_{0,Q-K} - \frac{k_a - q_a}{m} \right].$$

$$n = -\frac{2}{3} \sum_k \frac{k^2}{m} \left[ G^2_K + \sum_Q t_{pg}(Q) G^2_{0,Q-K} G^2_K \right], \quad (A.2)$$

where the last step specializes the result to $T > T_c$. The last equation is the central result we wanted to prove; from this equation we have

$$K(0) = \frac{ne^2}{m} + P(0) = 0, \quad \text{above } T_c. \quad (A.3)$$
Appendix B. The Nozieres–Schmitt-Rink (NSR) theory and the normal state structure factor

In recent work [16], we have shown how two-photon Bragg scattering can be used to establish \textit{in situ} the presence of coherent order in a superfluid, at any temperature, wavevector and frequency. For the most part, experiments on unitary gases have relied on sweeps to the BEC to find evidence for condensation. Our analysis is based on the definitions and sum rule in equation (24). It thus depends on imposing the current conservation laws, which have been extensively studied and verified [16].

As an alternative example, here we consider the NSR theory [36] in the normal state. Figure 7 indicates the characteristic class of diagrams. These diagrams enter into the density–density and spin–spin correlation functions. The calculations for the spin response build on an earlier publication [19]. We define the pair susceptibility $\chi_0(Q) = \sum_p G_{0\uparrow\downarrow}(p) G^{-1}_{0\downarrow\uparrow}(p+Q)$, where $G_{0\sigma}$ is the bare Green’s function. The pair propagator is given by $i\eta_0(Q) = U/(1 + U\chi_0(Q))$, where $U$ is the two-body interaction. Here, we present a more systematic discussion of the spin response than what was shown in section 3. The interaction term in the Hamiltonian is given by $H_I \sim \int d^3r J^S_\mu A^{S\mu}$, where $A^{S\mu} = (B_z, \mathbf{m})$ is the ‘effective’ four-vector field and $J^S_\mu = (n^S, J^S)$. Here, $B_z$ is the $z$-component of the magnetic field, $\mathbf{m}$ is the magnetizing field, $n^S$ is the $z$ component of spin and $J^S$ is the magnetization current. The spin rotational symmetry around the $z$-axis leads to the conservation law of spin: $\partial_\mu J^S_\mu = 0$.

From linear response theory, the spin response kernel can be written as

$$Q^{S\lambda}_{\mu\nu}(Q) = \sum_p \lambda^{S\lambda}_{\mu\sigma}(P, P + Q) G_{0\sigma}(P + Q, P) G_{0\sigma}, P + \frac{n}{m} g_{\mu\nu}(1 - g_{\mu\lambda}),$$

where $\lambda^{S\lambda}_{\mu\sigma}(P, P + Q) = \left(\frac{\delta}{m}(p + \frac{q}{2}), S_\sigma\right)$ is the bare vertex function of the spin-external field interaction and $\Lambda^{S\lambda}_{\mu\sigma}$ is the full vertex function. There is an implicit summation over the indices $\sigma$. Note, importantly, that the vertex function has different signs for different spin indices, which is to be contrasted with the charge or equivalently density response functions. In order to satisfy local conservation laws, the vertex must satisfy a Ward identity: $Q \cdot \sum_p \lambda^{S\lambda}_{\mu\sigma}(P, P + Q) = S_\sigma(G_{0\sigma}, P - G^{-1}_{0\sigma}, P + Q)$. As a result, in the NSR theory there are three contributions, $\Lambda^{S\lambda}_{\mu\sigma} = \lambda^{S\lambda}_{\mu\sigma} + \delta\Lambda^{S\lambda}_{\mu\sigma, MT} + \delta\Lambda^{S\lambda}_{\mu\sigma, AL}$, where the subscript MT is associated with the contribution from the MT-like diagrams and the AL diagrams. An important result (for singlet pairing) is that the contribution from the AL diagrams automatically vanishes. Thus, for the spin structure factor, $S_S = S_{C0} + S_{MT^S_{pq}}$ with $S_{MT^S_{pq}} = -S_{MT^C_{pq}}$.

By contrast, in the normal state, the dynamical structure factor in the NSR theory, for the particle density [19], can be written as the sum $S_N = S_{C0} + S_{MT^C_{pq}} + 2S_{AL}$. Here one should note that $S_{C0} = S_{C0}$. The remaining terms denote the corrections from the MT and AL diagrams. Note that the spin and charge contributions from the MT term enter with opposite signs. Moreover, the Ward identity implies a cancellation [6] $S_{MT^C_{pq}} + 2S_{AL} = -S_{MT^C_{pq}}$. Thus

$$S_S = S_{C0} + S_{MT^C_{pq}} = S_C = S_{C0} - S_{MT^C_{pq}},$$

which proves the desired result: \textit{in the normal state} the spin and charge degrees of freedom are indistinguishable. Note that the theoretical proof of this result depends on using a consistent theory of BCS–BEC crossover with full gauge invariance. The same diagrams that are used in the above proof are needed to show that there is no Meissner effect in the normal state. In this

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way we have demonstrated that
\[ S_-(\omega, \mathbf{q}) \equiv 0 \] above the transition. \hfill (B.3)

We are now led to an important observation: the quantity \( S_-(\omega, \mathbf{q}) \) for all \( \mathbf{q}, \omega \), \( 1/k_F a \) can be used as an indication of in-situ superfluid order, without requiring sweeps to the BEC. We show a plot of this behavior on the right side of figure 7, where it can be seen that the difference structure factor vanishes in the normal state. The behavior below \( T_c \) is shown in the figure for the case of the BCS–Leggett theory \[19\].

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