Optimality of Large MIMO Detection via Approximate Message Passing

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Abstract—Optimal data detection in multiple-input multiple-output (MIMO) communication systems with a large number of antennas at both ends of the wireless link entails prohibitive computational complexity. In order to reduce the computational complexity, a variety of sub-optimal detection algorithms have been proposed in the literature. In this paper, we analyze the optimality of a novel data-detection method for large MIMO systems that relies on approximate message passing (AMP). We show that our algorithm, referred to as individually-optimal (IO) large-MIMO AMP (short IO-LAMA), is able to perform IO data detection given certain conditions on the MIMO system and the constellation set (e.g., QAM or PSK) are met.

I. INTRODUCTION

We consider the problem of recovering the $M_T$-dimensional data vector $s_0 \in \mathbb{C}^{M_T}$ from the noisy multiple-input multiple-output (MIMO) input-output relation $y = Hs_0 + n$, by performing individually-optimal (IO) data detection

$$(IO) \quad \hat{s}_\ell = \arg \max_{\tilde{s}_\ell \in \mathcal{O}} p(\tilde{s}_\ell | y, H).$$

Here, $\hat{s}_\ell$ denotes the $\ell$-th IO estimate, $\mathcal{O}$ is a finite constellation (e.g., QAM or PSK), $p(\tilde{s}_\ell | y, H)$ is a probability density function assuming i.i.d. zero-mean complex Gaussian noise for the vector $n \in \mathbb{C}^{M_R}$ with variance $N_0$ per complex dimension, $M_T$ and $M_R$ denotes the number of transmit and receive antennas, respectively, $y \in \mathbb{C}^{M_T}$ is the receive vector, and $H \in \mathbb{C}^{M_T \times M_R}$ is the (known) MIMO system matrix. In what follows, we assume that the entries of the MIMO system matrix $H$ are i.i.d. zero-mean complex Gaussian with variance $\frac{1}{M_R}$, and we define the so-called system ratio as $\beta = M_T/M_R$.

Although IO detection achieves the minimum symbol error-rate, the combinatorial nature of the (IO) problem requires prohibitive computational complexity, especially in large (or massive) MIMO systems. In order to enable data detection in such high-dimensional systems, a large number of low-complexity but sub-optimal algorithms have been proposed in the literature (see, e.g., [6]–[8]).

A. Contributions

In this paper, we propose and analyze a novel, computationally efficient data-detection algorithm, referred to as IO-LAMA (short for IO large MIMO approximate message passing). We show that IO-LAMA decouples the noisy MIMO system into a set of independent additive white Gaussian noise (AWGN) channels with equal signal-to-noise ratio (SNR); see Fig. 1 for an illustration of this decoupling property. The state-evolution (SE) recursion of AMP enables us to track the effective noise variance $\sigma_t^2$ of each decoupled AWGN channel at every algorithm iteration $t$. Using these results, we provide precise conditions on the MIMO system matrix, the system ratio $\beta$, the noise variance $N_0$, and the modulation scheme for which IO-LAMA exactly solves the (IO) problem.

B. Relevant Prior Art

Initial results for IO data detection in large MIMO systems reach back to [9] where Verdú and Shamai analyzed the achievable rates under optimal data detection in randomly-spread CDMA systems. Tanaka [10] derived expressions for the error-rate performance and the multi-user efficiency for IO detection using the replica method. While Tanaka’s results were limited to BPSK constellations, Guo and Verdú extended his results to arbitrary discrete input distributions [3], [11]. All these results study the fundamental performance of IO data detection in the large-system limit, i.e., for $\beta = M_T/M_R$ with $M_T \to \infty$. Corresponding practical detection algorithms have been proposed for BPSK constellations [12], [13]—to the best of our knowledge, no computationally efficient algorithms for general constellation sets and complex-valued systems have been proposed in the open literature.

Fig. 1. IO-LAMA decouples large MIMO systems (a) into a set of parallel and independent AWGN channels with equal noise variance; (b) equivalent system in the large-system limit, i.e., for $\beta = M_T/M_R$ with $M_T \to \infty$. 

An extended version of this paper including all proofs is in preparation [1].

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Our data-detection method, IO-LAMA, builds upon approximate message passing (AMP) [13]–[16], which was initially developed for the recovery of sparse signals. AMP has been generalized to arbitrary signal priors in [17]–[19] and enables a precise performance analysis via the SE recursion [14], [15]. Recently, AMP-related algorithms have been proposed for data detection [20]–[22]; these algorithms, however, lack of a theoretical performance analysis.

C. Notation

Lowercase and uppercase boldface letters designate vectors and matrices, respectively. For a matrix $H$, we define its conjugate transpose to be $H^H$. The $\ell$-th column of $H$ is denoted by $h_\ell$. We use $\langle \cdot \rangle$ to write $\langle x \rangle = \frac{1}{N} \sum_{k=1}^{N} x_k$. A multivariate complex-valued Gaussian probability density function (pdf) is denoted by $\mathcal{CN}(m, K)$, where $m$ is the mean vector and $K$ the covariance matrix. $\mathbb{E}_X[\cdot]$ and $\mathbb{V}_X[\cdot]$ denote the expectation and variance operator with respect to the pdf of the random variable $X$, respectively.

II. IO-LAMA: LARGE-MIMO DETECTION USING AMP

We now present IO-LAMA and the SE recursion, which is used in Section III for our optimality analysis.

A. The IO-LAMA Algorithm

We assume that the transmit symbols $s_\ell$, $\ell = 1, \ldots, M_T$, of the transmit data vector $s$ are taken from a finite set $O = \{a_j : j = 1, \ldots, |O|\}$ with constellation points $a_j$ chosen, e.g., from a QAM or PSK alphabet. We assume an i.i.d. prior $p(s) = \prod_{\ell=1}^{M_T} p(s_\ell)$, with the following distribution for each transmit symbol $s_\ell$:

$$p(s_\ell) = \sum_{a \in O} p_a \delta(s_\ell - a).$$

Here, $p_a$ designates the (known) prior probability of each constellation point $a \in O$ and $\delta(\cdot)$ is the Dirac delta function; for uniform priors, we have $p_a = |O|^{-1}$.

The IO-LAMA algorithm summarized below is obtained by using the prior distribution in (1) within complex Bayesian AMP. A detailed derivation of the algorithm is given in [1].

Algorithm 1. Initialize $\hat{s}_\ell^1 = \mathbb{E}_S[S]\mathbb{E}_X[Z]$, for $\ell = 1, \ldots, M_T$, $r^1 = y$, and $\tau^1 = \beta \mathbb{V}_{X}[S]^2/\mathbb{E}_{S}[Z]$. Then, for every IO-LAMA iteration $t = 1, 2, \ldots$, compute the following steps:

$$z^t = \hat{s}_\ell^t + H^H r^t\quad s^t_{\ell+1} = \mathbb{F}(z^t, 0)\quad r^{t+1} = y - Hs^t_{\ell+1} + \tau^{t+1} r^t.$$

The functions $F(s_\ell, \tau)$ and $G(s_\ell, \tau)$ correspond to the message mean and variance, and are computed as follows:

$$F(s_\ell, \tau) = \int_{s_\ell} s_\ell f(s_\ell|\hat{s}_\ell, \tau) ds_\ell \quad (2)$$

$$G(s_\ell, \tau) = \int_{s_\ell} |s_\ell|^2 f(s_\ell|\hat{s}_\ell, \tau) ds_\ell - |F(s_\ell, \tau)|^2.$$

Here, $f(s_\ell|\hat{s}_\ell, \tau)$ is the posterior pdf defined by $f(s_\ell|\hat{s}_\ell, \tau) = \frac{1}{Z} p(s_\ell|\hat{s}_\ell, \tau)p(s_\ell)$ with $p(\hat{s}_\ell, \tau) \sim \mathcal{CN}(s_\ell, \tau)$ and a normalization constant $Z$. Both functions $F(s_\ell, \tau)$ and $G(s_\ell, \tau)$ operate element-wise on vectors.

In order to analyze the performance of IO-LAMA in the large-system limit, we next summarize the SE recursion. The SE recursion in the following theorem enables us to track the effective noise variance $\sigma^2_{\ell}$ for the decoupled MIMO system for every iteration $t$ (cf. Fig. 1), which is key for the optimality analysis in Section III. A detailed derivation is given in [1].

Theorem 1. Fix the system ratio $\beta = M_T/M_R$ and the constellation set $O$, and let $M_T \rightarrow \infty$. Initialize $\sigma^2_{1} = N_0 + \beta \mathbb{V}_{S}[S]$. Then, the effective noise variance $\sigma^2_{\ell}$ of IO-LAMA at iteration $t$ is given by the following recursion:

$$\sigma^2_{\ell} = N_0 + \beta \Psi(\sigma^2_{\ell-1}),$$

The so-called mean-squared-error (MSE) function is defined by

$$\Psi(\sigma^2_{\ell-1}) = \mathbb{E}_{S,Z}[|F(S + \sigma_{t-1}Z, \sigma^2_{t-1}) - S|^2],$$

where $F$ is given in (2) and $Z \sim \mathcal{CN}(0, 1)$.

B. IO-LAMA Decouples Large MIMO Systems

In the large-system limit and for every iteration $t$, IO-LAMA computes the marginal distribution of $s_\ell$, $\ell = 1, \ldots, M_T$, which corresponds to a Gaussian distribution centered around the original signal $s_{0\ell}$ with variance $\sigma^2_{\ell}$. These properties follow from [16, Sec. 6], which shows that $z^t = \hat{s}_\ell^t + H^H r^t$ is distributed according to $\mathcal{CN}(s_{0\ell}, \sigma^2_{\ell} I_{M_T})$. Hence, the input-output relation for each transmit stream $z^t_{\ell} = s^t_{\ell} + (h_\ell^t)^H r^t_{\ell}$ is equivalent to the following single-stream AWGN channel:

$$z^t_{\ell} = s_{0\ell} + n^t_{\ell}.$$  

Here, $s_{0\ell}$ is the $\ell$-th original transmitted signal and $n^t_{\ell}$ is AWGN with variance $\sigma^2_{\ell}$ per complex entry. As a consequence, IO-LAMA decouples the MIMO system into $M_T$ parallel and independent AWGN channels with equal noise variance $\sigma^2_{\ell}$ in the large-MIMO limit; see Fig. 1(b) for an illustration.

III. OPTIMALITY OF IO-LAMA

We now provide conditions for which IO-LAMA exactly solves the (IO) problem.

A. Fixed points of IO-LAMA’s State Evolution

For $t \rightarrow \infty$, the SE recursion in Theorem 1 converges to the following fixed-point equation [1], [15]:

$$\sigma^2_{10} = N_0 + \beta \Psi(\sigma^2_{0}),$$

which coincides with the “fixed-point equation” developed for IO detection by Guo and Verdú using the replica method in [3, Eq. (34)]. We note that (4) may have multiple fixed-point solutions. In the case of such non-unique fixed points, Guo and Verdú choose the solution that minimizes the “free energy” [3, Sec. 2-D], whereas IO-LAMA converges, in general, to the fixed-point solution with the largest effective noise variance $\sigma^2_{\ell}$. We note that if the fixed-point solution to (4) is unique, then IO-LAMA recovers the solution with minimal effective noise variance $\sigma^2$ and thus, performs IO detection. However, if there are multiple fixed-points solutions to (4), IO-LAMA is, in general, sub-optimal and does not necessarily converge to the fixed-point solution with the minimal “free energy.”  

1Convergence to another fixed-point solution is possible if IO-LAMA is initialized sufficiently close to such a fixed point; see [1], [23] for the details.
provide conditions for which there is exactly one (unique) fixed point with minimum effective noise variance $\sigma^2$ and—as a consequence—IOLAMA solves the (IO) problem.

B. Exact Recovery Thresholds (ERTs)

We start by analyzing IOLAMA in the noiseless setting. We provide conditions on the system ratio $\beta$ and the constellation set $O$, which guarantee exact recovery of an unknown transmit signal $s_0 \in O^{M_I}$ in the large-system limit, i.e., $\beta$ is fixed and $M_I \to \infty$. In particular, we show that if $\beta < \beta_{O}^{\text{max}}$, where $\beta_{O}^{\text{max}}$ is the so-called exact recovery threshold (ERT), then IOLAMA perfectly recovers $s_0$; for $\beta \geq \beta_{O}^{\text{max}}$, perfect recovery is not guaranteed, in general. To make this behavior explicit, we need the following technical result; the proof is given in Appendix A.

Lemma 2. Fix the constellation set $O$. If $\var_S(S)$ is finite, then there exists a non-negative gap $\sigma^2 - \Psi(\sigma^2) \geq 0$ with equality if and only if $\sigma^2 = 0$. As $\sigma^2 \to 0$, the MSE $\Psi(\sigma^2) \to 0$ and as $\sigma^2 \to \infty$, $\text{MSE } \Psi(\sigma^2) \to \var_S(S)$.

For all $\sigma^2 > 0$, Lemma 2 guarantees that $\Psi(\sigma^2) < \sigma^2$. Suppose that for some $\beta > 1$, $\beta \Psi(\sigma^2) < \sigma^2$ also holds for all $\sigma^2 > 0$. Then, as long as $\beta > 1$ is not too large to also ensure $\beta \Psi(\sigma^2) < \sigma^2$ for all $\sigma^2 > 0$, there will only be a single fixed point at $\sigma^2 = 0$. Therefore, LAMA can still perfectly recover the original signal $s_0$ by Theorem 2 since $\Psi(\sigma^2) = 0$. Leveraging the gap between $\Psi(\sigma^2)$ and $\sigma^2$ will allow us to find the exact recovery threshold (ERT) of LAMA for values of $\beta > 1$. For the fixed (discrete) constellation set $O$, the largest $\beta$ that ensures $\beta \Psi(\sigma^2) = \sigma^2$ is precisely the ERT defined next.

Definition 1. Fix $O$ and let $N_0 = 0$. Then, the exact recovery threshold (ERT) that enables perfect recovery of the original signal $s_0$ using IOLAMA is given by

$$\beta_{O}^{\text{max}} = \min_{\sigma^2 > 0} \left\{ \left( \frac{\Psi(\sigma^2)}{\sigma^2} \right)^{-1} \right\}. \quad (5)$$

With Definition 2, we state Theorem 3, which establishes optimality in the noiseless case; the proof is given in Appendix B.

Theorem 3. Let $N_0 = 0$ and fix a discrete set $O$. If $\beta < \beta_{O}^{\text{max}}$, then IOLAMA perfectly recovers the original signal $s_0$ from $y = Hs_0 + n$ in the large system limit.

Note that for a given constellation set $O$, the ERT $\beta_{O}^{\text{max}}$ can be computed numerically using (5). Furthermore, the signal variance, $\var_S(S)$, has no impact on the ERT as the MSE function $\Psi(\sigma^2)$ and $\sigma^2$ scale linearly with $\var_S(S)$. Table I summarizes ERTs $\beta_{O}^{\text{max}}$ for common QAM and PSK constellation sets.

C. Optimality Conditions for IOLAMA

We now study the optimality of IOLAMA in the presence of noise, where exact recovery is no longer guaranteed. In particular, we provide conditions for which IOLAMA converges to the fixed point with minimal effective noise variance $\sigma^2$, which corresponds to solving the (IO) problem.

Table I

| Constellation | $N_0^{\text{min}}(\beta_{O}^{\text{max}})$ | $N_0^{\text{opt}}(\beta_{O}^{\text{max}})$ | $N_0^{\text{max}}(\beta_{O}^{\text{max}})$ |
|---------------|---------------------------------|---------------------------------|---------------------------------|
| BPSK          | 2.9505                          | 2.999 - 10^{-1}                 | 4.1709                          |
| QPSK          | 1.4752                          | 1.499 - 10^{-1}                 | 2.0855                          |
| 16-QAM        | 0.9830                          | 3.000 - 10^{-2}                 | 1.3629                          |
| 64-QAM        | 0.8424                          | 7.144 - 10^{-3}                 | 1.1573                          |
| 8-PSK         | 1.4576                          | 4.440 - 10^{-2}                 | 1.8038                          |
| 16-PSK        | 1.4782                          | 1.143 - 10^{-2}                 | 1.8005                          |

Note that such a minimum free-energy solution is also the fixed point for the IO detector in [3, Eq. (34)]. We call the fixed point with minimum effective noise variance optimal fixed point; other fixed points are called suboptimal fixed points.

We identify three different operation regimes for IOLAMA depending on the system ratio $\beta$ (see Table II). To make these three regimes explicit, we need the following definition.

Definition 2. Fix the constellation set $O$. Then, the minimum recovery threshold (MRT) $\beta_{O}^{\text{min}}$ is defined by

$$\beta_{O}^{\text{min}} = \min_{\sigma^2 > 0} \left\{ \left( \frac{d\Psi(\sigma^2)}{d\sigma^2} \right)^{-1} \right\}. \quad (6)$$

The definition of MRT shows that for all system ratios $\beta \leq \beta_{O}^{\text{min}}$, the fixed point of (4) is unique. The following lemma establishes a fundamental relationship between MRT and ERT; the proof is given in Appendix C.

Lemma 4. The MRT never exceeds the ERT.

We next define the minimum critical and maximum guaranteed noise variance, $N_0^{\text{min}}(\beta)$ and $N_0^{\text{max}}(\beta)$, that determine boundaries for the optimality regimes when $\beta > \beta_{O}^{\text{min}}$.

Definition 3. Fix $\beta \in (\beta_{O}^{\text{min}}, \beta_{O}^{\text{max}})$. Then, the minimum critical noise $N_0^{\text{min}}(\beta)$ that ensures convergence to the optimal fixed point is defined by

$$N_0^{\text{min}}(\beta) = \min_{\sigma^2 > 0} \left\{ \sigma^2 - \beta \Psi(\sigma^2) : \frac{d\Psi(\sigma^2)}{d\sigma^2} = 1 \right\}. \quad (7)$$

Definition 4. Fix $\beta > \beta_{O}^{\text{min}}$. Then, the maximum guaranteed noise $N_0^{\text{max}}(\beta)$ that ensures convergence to the optimal fixed point is defined by

$$N_0^{\text{max}}(\beta) = \max_{\sigma^2 > 0} \left\{ \sigma^2 - \beta \Psi(\sigma^2) : \frac{d\Psi(\sigma^2)}{d\sigma^2} = 1 \right\}. \quad (7)$$

Table II

| $\beta$ | $\beta_{O}^{\text{max}}$ | $\beta_{O}^{\text{min}}$ | $\beta_0^{\text{opt}}$ | $\beta_0^{\text{submax}}$ |
|---------|-------------------------|-------------------------|-----------------------|-------------------------|
| $\beta$ | $\beta_{O}^{\text{max}}$ | $\beta_{O}^{\text{min}}$ | $\beta_0^{\text{opt}}$ | $\beta_0^{\text{submax}}$ |

For certain constellation sets (e.g., 16-PSK), there exist subintervals in $[N_0^{\text{min}}(\beta), N_0^{\text{max}}(\beta)]$ where IOLAMA is still optimal; see [4] for the details.
correspond to all fixed points of the SE recursion of IO-LAMA; we use this function to study the algorithm’s optimality.

Figure 2 illustrates our optimality analysis for a large-MIMO system with QPSK constellations. We show the curves depending on the effective noise variance $\sigma^2$ and for different system ratios $\beta$. The regimes $\beta < \beta_{\text{min}}^c$, $\beta \in (\beta_{\text{min}}^c, \beta_{\text{max}}^c)$, and $\beta \geq \beta_{\text{max}}^c$ are shown in Fig. 2(a), Fig. 2(b), and Fig. 2(c) respectively. The special case for $\beta = 1$ with $N_0 = 0$ corresponds to the solid blue line, along with the corresponding (unique) fixed point at the origin. In the following three paragraphs, we discuss the three operation regimes of IO-LAMA in detail.

(i) $\beta \leq \beta_{\text{min}}^c$: In this region, the SE recursion of IO-LAMA always converges to the unique, optimal fixed point. For $\beta < \beta_{\text{min}}^c$, the slope of (7) for all $\sigma^2$ is strictly-negative. Hence, as (7) is always decreasing, there exists exactly one unique fixed point of the SE recursion regardless of the noise variance $N_0$. Thus, IO-LAMA converges to the optimal fixed point and consequently, solves the (IO) problem.

We emphasize that we still obtain exactly one fixed point even when $\beta$ is equal to the MRT. Since $\beta = \beta_{\text{min}}^c$, there exists at least one $\sigma^2$ that satisfies $\beta_{\text{min}}^c \frac{d}{d\sigma^2} \Psi(\sigma^2)\big|_{\sigma^2 = \sigma_1^2} = 1$. By definition of $\beta_{\text{min}}^c$, $\Psi(\sigma^2)$ at $\sigma_1^2$ implies that $\sigma_1^2$ is a saddle-point, so (7) has exactly one zero at $\sigma_1^2$. We observe that if $\sigma_1^2$ is unique, then $N_0^\text{min}(\beta_{\text{min}}^c) = N_0^\text{max}(\beta_{\text{min}}^c)$. For all other $\sigma^2 \neq \sigma_1^2$, the construction of $\sigma_1^2$ implies that $\beta_{\text{min}} \frac{d}{d\sigma^2} \Psi(\sigma^2) < 1$, so the fixed point of (7) remains to be unique.

The green, dash-dotted and red, dotted line in Fig. 2(a) shows (7) for $\beta = \beta_{\text{min}}^c$ with $N_0 = 0$ and $N_0 = N_0^\text{min}(\beta_{\text{min}}^c) = N_0^\text{max}(\beta_{\text{min}}^c)$, respectively. In both cases, we see that the SE recursion of IO-LAMA converges to the unique fixed point.

(ii) $\beta_{\text{min}}^c < \beta < \beta_{\text{max}}^c$: In this region, the SE recursion of IO-LAMA converges to the unique, optimal fixed point if $N_0 < N_0^\text{min}(\beta)$ or $N_0 > N_0^\text{max}(\beta)$.

The green, dash-dotted line, cyan, dashed line, and magenta, dotted line in Fig. 2(b) shows (7) for $\beta = (\beta_{\text{min}}^c + \beta_{\text{max}}^c)/2$ with $N_0 = 0$, $N_0 > N_0^\text{min}(\beta^*)$ and $N_0 < N_0^\text{max}(\beta^*)$, respectively. We note that for the three cases, the fixed point is unique, labeled in Fig. 2(b) by a circle. On the other hand, the red, dotted line in Fig. 2(b) shows (7) with $\beta^*$ under noise $N_0 \in [N_0^\text{min}(\beta^*), N_0^\text{max}(\beta^*)]$. In this case, however, we observe that SE recursion of IO-LAMA converges to the rightmost suboptimal fixed point labeled by the crossed circle $\otimes$. Hence, IO-LAMA does not, in general, solve the (IO) problem when $N_0^\text{min}(\beta) \leq N_0 \leq N_0^\text{max}(\beta)$.

(iii) $\beta \geq \beta_{\text{max}}^c$: In this region, the SE recursion of IO-LAMA converges to the unique, optimal fixed point when $N_0 > N_0^\text{max}(\beta)$. As $\beta \rightarrow \beta_{\text{max}}^c$, the low noise $N_0 < N_0^\text{min}(\beta)$ (or high SNR) region of optimality disappears because $N_0^\text{min}(\beta) \rightarrow 0$ as $\beta \rightarrow \beta_{\text{max}}^c$ from (5).

The green, dash-dotted line and red, dotted line in Fig. 2(c) shows (7) for $\beta = \beta_{\text{max}}^c$ with $N_0 = 0$ and $0 < N_0 \leq N_0^\text{max}(\beta)$, respectively. We observe that the SE recursion of IO-LAMA converges to the suboptimal fixed point when $\beta = \beta_{\text{max}}^c$ even with $N_0 = 0$. On the other hand, the cyan, dashed line refers to (7) for $\beta = \beta_{\text{max}}^c$ with $N_0 > N_0^\text{max}(\beta)$. While the noiseless case resulted the SE recursion of IO-LAMA to converge to the suboptimal fixed point, we observe that for strong noise (or equivalently low SNR), the SE recursion of IO-LAMA actually recovers the IO solution. Therefore, when $\beta \geq \beta_{\text{max}}^c$, IO-LAMA solves the (IO) problem when the noise is greater than the maximum guaranteed noise $N_0^\text{max}(\beta)$.

As a final remark, we note that the ERT $\beta^\text{max}$ and MRT $\beta^\text{min}$ in Table I do not depend on $\text{Var}_S[S]$; the critical noise levels $N_0^\text{min}(\beta)$ and $N_0^\text{max}(\beta)$, however, depend on $\text{Var}_S[S]$.

D. ERT, MRT, and Critical Noise Levels

The ERT, MRT, as well as $N_0^\text{min}(\beta)$ and $N_0^\text{max}(\beta)$ for common constellations are summarized in Table I. We assume equally likely priors with the transmit signal normalized to $E_s = \text{Var}_S[S] = 1$.

We note that the calculations of ERT and MRT for the simplest case of BPSK constellations involve computations of a logistic-normal integral for which no closed-form expression is known [24]. Consequently, the following results were obtained via numerical integration for computing the MSE function $\Psi(\sigma^2)$. As noted in Table I.

4The critical noise levels depend linearly on $E_s$. Hence, we assume that $E_s = 1$ without loss of generality.
for a QPSK system under complex-valued noise, the ERT is \( \beta_{\text{QPSK}} \approx 2.0855 \), and the MRT is given as \( \beta_{\text{QPSK}} \approx 1.4752 \).

The MRT's for 16-QAM and 64-QAM indicate that small system ratios \( \beta < 1 \) are required to always guarantee that IO-LAMA solves the (IO) problem in the presence of noise. For instance, we require \( \beta \leq \beta_{\text{64-QAM}} \approx 0.8424 \), i.e. \( M_T \leq 0.8424 M_R \), to ensure that IO-LAMA solves the IO problem for 64-QAM in the large system limit. As \( \beta \to \beta_{\text{64-QAM}} \approx 1.1573 \), IO-LAMA is only optimal for \( N_0 > N_{\max}^{\beta_{\text{64-QAM}}} \approx 5.868 	imes 10^{-3} \). From Table I, we see that IO-LAMA is a suitable candidate algorithm for the detection of higher-order QAM constellations in massive multiuser MIMO systems as one typically assumes \( M_R \gg M_T \) [25].

IV. Conclusions

We have presented the IO-LAMA algorithm along with the state-evolution recursion. Using these results, we have established conditions on the MIMO system matrix, the noise variance \( N_0 \), and the constellation set for which IO-LAMA exactly solves the (IO) problem. While the presented results are exclusively for the large-system limit, our own simulations indicate that IO-LAMA achieves near-optimal performance in realistic, finite-dimensional systems; see [1] for more details.

APPENDIX A

PROOF OF LEMMA 2

Since the variance of \( S \) is finite, denote \( \text{Var}_S[S] = \sigma_S^2. \) By [26] Prop. 5, we have the following upper bound:

\[
\Psi(\sigma^2) \leq \frac{\sigma_S^2}{\sigma_S^2 + \sigma^2} \cdot \frac{1}{1 + \sigma^2/\sigma_S^2}, \quad (8)
\]

This holds for all \( \sigma^2 \) if and only if \( S \) is complex normal with variance \( \sigma_S^2 \) [26]. Note that if \( \sigma^2 = 0 \), then (8) is achieved for any \( \sigma_S^2 \). If \( \sigma^2 > 0 \), then \( \Psi(\sigma^2) < \sigma_S^2 \) by (8).

The first part follows directly from (8) as \( \Psi(\sigma^2) \) is non-negative. The second part requires one to realize that if \( \sigma^2 \to \infty \) also implies \( F(\sigma^2) \to \sum_{a \in \mathcal{A}} \alpha_a = \text{E}_S[S] \), and hence,

\[
\lim_{\sigma^2 \to \infty} \Psi(\sigma^2) \rightarrow \text{E}_S[|S - \text{E}_S[S]|^2] = \text{Var}_S[S].
\]

APPENDIX B

PROOF OF THEOREM 3

We assume the initialization in Algorithm 1. Since \( N_0 = 0 \), if LAMA perfectly recovers the original signal \( s_0 \), then the fixed point in (4) is unique at \( \sigma^2 = 0 \). This happens if the system ratio is strictly less than the ERT \( \beta_{\text{ERT}} \) because otherwise, i.e., \( \beta \geq \beta_{\text{ERT}} \), there exists a non-unique fixed point to (4) for some \( \sigma^2 > 0 \) by Definition 1.

APPENDIX C

PROOF OF LEMMA 4

We show that under a fixed constellation set \( \mathcal{O} \), \( \beta_{\min}^{\text{QPSK}} \leq \beta_{\max}^{\text{QPSK}} \).

The proof is straightforward as,

\[
\beta_{\min}^{\text{QPSK}} = \min_{\sigma^2 \geq 0} \left( \frac{\text{d} \Psi(\sigma^2)}{\text{d} \sigma^2} \right)^{-1} \leq \left( \frac{\text{d} \Psi(\sigma^2)}{\text{d} \sigma^2} \right)^{-1} \bigg|_{\sigma^2 = \beta_{\text{QPSK}} \Psi(\sigma^2)} \leq \beta_{\max}^{\text{QPSK}},
\]

where (a) and (b) follow from the MRT and ERT definitions.