Finite $W$-superalgebras and quadratic spacetime supersymmetries

E Ragoucy$^1$, L A Yates$^2$ and P D Jarvis$^2$

$^1$ LAPTh, CNRS and USMB, BP 110, F-74941 Annecy-le-Vieux Cedex, France
$^2$ School of Natural Sciences, University of Tasmania, Hobart, Tasmania, Australia

E-mail: ragoucy@lapth.cnrs.fr, Luke.Yates@utas.edu.au and Peter.Jarvis@utas.edu.au

Received 12 May 2020
Accepted for publication 17 August 2020
Published 11 September 2020

Abstract
We consider Lie superalgebras under constraints of Hamiltonian reduction, yielding finite $W$-superalgebras which provide candidates for quadratic space-time superalgebras. These have an undeformed bosonic symmetry algebra (even generators) graded by a fermionic sector (supersymmetry generators) with anti-commutator brackets which are quadratic in the even generators. We analyze the reduction of several Lie superalgebras of type $gl(M|N)$ or $osp(M|2N)$ at the classical (Poisson bracket) level, and also establish their quantum (Lie bracket) equivalents. Purely bosonic extensions are also considered. As a special case we recover a recently identified quadratic superconformal algebra, certain of whose unitary irreducible massless representations (in four dimensions) are ‘zero-step’ multiplets, with no attendant superpartners. Other cases studied include a six dimensional quadratic superconformal algebra with vectorial odd generators, and a variant quadratic superalgebra with undeformed $osp(1|2N)$ singleton supersymmetry, and a triplet of spinorial supercharges.

Keywords: supersymmetry, $W$-algebras, deformed symmetry, quadratic algebras

1. Introduction

The analysis of quadratic algebras as spectrum generating algebras for the solution of quantum models [1], or as a basis of generalized symmetry principles for physical systems, has led to a wide range of applications including special function theory [2] and super-integrable systems [3, 4]. Indeed, extensions of Lie algebra theory underlying exactly solvable models in mathematical physics, including Yangian and $W$-algebras, and related families of $q$-deformation, have been found to provide a rich source of examples of quadratic algebras and associated
superalgebras. At the same time, substantial mathematical underpinnings have been developed for the structure of general quadratic algebras [5].

In this work we wish to introduce certain classes of (finite) quadratic \( W \)-superalgebras by Hamiltonian reduction from standard Lie superalgebras. The embeddings leading to the constraints are arranged in each case so as to leave undeformed, a Lie algebra or Lie superalgebra, together with conserved supercharges (supersymmetry) generators whose (anticommutator) brackets close on combinations quadratic in the even, undeformed generators.

The motivation for this study is to provide constructions of finite \( W \)-superalgebras, which may in appropriate real forms provide alternative models of ‘supersymmetry’ in nature. In particular, we analyze the reduction of several Lie superalgebras of type \( gl(M|N) \) or \( osp(M|2N) \) at the classical (Poisson bracket) level, to finite \( W \)-superalgebras, and also establish their quantum (Lie bracket) equivalents. Purely bosonic extensions are also considered. As a special case we recover by this method a recently identified quadratic conformal superalgebra (first obtained from a first principles construction [6]), whose massless unitary irreducible representations (in four dimensions) have been shown to admit ‘zero step’ multiplets, with no attendant superpartners. Other cases include a six dimensional quadratic conformal superalgebra with undeformed \( osp(1|2N) \) singleton supersymmetry, with partners to the odd generators providing a triplet of spinorial supercharges.

In section 2 below, we provide a brief resumé of the method of Hamiltonian reduction of a Poisson Lie (super) algebra, leading to second class constraints solvable by Dirac brackets at the classical level, yielding (in the quadratic case, via suitable symmetrization) to a quadratic superalgebra at the quantum level (references to the literature are given below). Section 3 implements this for the superalgebra \( gl(N|2) \), recovering (for \( N = 4 \)) the quadratic conformal superalgebra [6] (as well as a purely bosonic equivalent based on \( gl(N + 2) \)). The Casimir operators are also constructed for this case. Section 4 treats cases of reductions of orthosymplectic superalgebras \( osp(M|2N) \), of type \( (M|2) \) and \( (3|2N) \), respectively. Conclusions and further discussion are provided in section 5.

2. Finite \( W \)-algebras and superalgebras

In this section we recall the basic method for the derivation of finite \( W \)-algebras and superalgebras via symplectic reduction (see [7–9] and also [10] for the theory of finite \( W \)-algebras and applications). At the classical level, a finite-dimensional symmetry superalgebra comprises a \( \mathbb{Z}_2 \)-graded super Poisson–Lie algebra of operators \( A, B, \ldots \), with grading \( [\cdot] = 0 \) for even (bosonic) and \( [\cdot] = 1 \) for odd (fermionic) generators, and Poisson brackets that are graded antisymmetric and obey the graded Jacobi identity:

\[
\{ A, B \} = -(-1)^{[A][B]} \{ B, A \}
\]

\[
\{ A, \{ B, C \} \} = \{ \{ A, B \}, C \} + (-1)^{[A][B]} \{ B, \{ A, C \} \}.
\]

In the presence of a system of second class constraints \( \Phi \), with generators \( \{ \varphi_a, a = 1, 2, \ldots \} \), a consistent Poisson bracket structure is built through the Dirac brackets, as follows. One first introduces the matrix \( \Delta \) of pairwise Poisson brackets of all constraint generators:

\[
\Delta_{ab} = \{ \varphi_a, \varphi_b \}.
\]

It is invertible because the constraints are second class, and we define its inverse \( \Delta^{-1} \) with
entries $\Delta_{ab}$. Then the Dirac bracket is defined by
\[
\{A, B\}_\ast \simeq \{A, B\} - \sum_{a,b} \{A, \varphi_a\} \Delta_{ab} \{\varphi_b, B\}
\]
where the symbol $\simeq$ means that one has to apply the constraints once all Poisson brackets on the right-hand side have been computed. The Dirac brackets are well-defined (graded antisymmetric, and obeying the graded Jacobi identity), and being consistent with the constraints in that $\{A, \varphi\}_\ast = 0$, for all operators $A$, and constraints $\varphi \in \Phi$, effect a projection of the symplectic manifold which is the phase space of the system, on to the lower dimensional constraint surface.

For the cases to be treated, the matrix elements $\Delta_{ab}$ become scalars, and not operator (field) dependent, and the constraints are generators, whose Poisson brackets are linear. Hence the resultant $W$-algebra will have at most quadratic Dirac brackets. In this situation there is a simple way to quantize them, using symmetrization. It amounts to replacing all Dirac brackets by (anti) commutators, and all products by their symmetrized version, for example $xy \to \frac{1}{2}(xy + yx)$. Since the Jacobi identities are obeyed at the Poisson bracket level, the symmetrization ensures that they will still be obeyed at the quantum level. A final step is that the Casimirs of the $W$-algebra can be obtained via its embedding in the Lie (super)algebra, with appropriate use of the constraints.

In the standard technique, finite $W$-algebras and superalgebras are constructed by analyzing various embeddings of $\mathfrak{sl}(2)$ in finite-dimensional simple Lie algebras and superalgebras (for example, principal embeddings, in the well-studied $W_n$ cases). They have been first introduced in a physics context, see e.g. [7, 9, 11] and [12] for the supersymmetric version, but then studied at the algebraic level by mathematicians, see for instance [8, 10, 13]. The images of the $\mathfrak{sl}(2)$ generators encapsulate the second class constraints wherein the diagonal Cartan generator is constrained to vanish, and the positive root vector set to unity, leaving a reduced set of nonzero $W$ generators with quadratic Poisson bracket algebra. (Supersymmetric variants involving principal $\mathfrak{osp}(1|2)$ embeddings have also been considered.) For the cases to be examined here, however, the $\mathfrak{sl}(2)$ embeddings are rather regular, and simply involve the identification of an appropriate isomorphic $\mathfrak{sl}(2)$ subalgebra carrying the constraints, augmented by the consistency requirement that certain of the odd generators should also be constrained to vanish. At the same time, a subalgebra of the even generators retains its standard Poisson brackets, so that the final quantum algebra generically becomes that of a Lie algebra graded by odd generators with quadratic anticommutator brackets—and hence, in appropriate real forms, a candidate ‘quadratic spacetime superalgebra’ as described above.

3. Quadratic superalgebras from $\mathfrak{gl}(N|2)$ reduction, and bosonic counterparts

3.1. The $W(\mathfrak{gl}(N|2), \mathfrak{gl}(2))$ superalgebras

The Lie superalgebra has generators $e_{ab}$, $1 \leq a, b \leq N + 2$, and the grading is defined by
\[
[e_{ab}] = [a] + [b] \quad \text{with} \quad \begin{cases} [a] = 0 & \text{for } 1 \leq a \leq N \\ [a] = 1 & \text{for } a = N + 1, N + 2 \end{cases}.
\]

The Poisson brackets are given by
\[
\{e_{ab}, e_{cd}\} = \delta_{bc} e_{ad} - (-1)^{[a]+[b]+[c]+[d]} \delta_{ad} e_{cb}.
\]
We construct the finite $W$-algebra associated to the constraints
\[
\begin{align*}
    e_{j,N+1} &= 0, & 1 \leq j \leq N; \\
    e_{N+2,j} &= 0; \\
    e_{N+2,N+1} &= 1; \\
    e_{N+1,N+1} - e_{N+2,N+2} &= 0.
\end{align*}
\] (3.2)

The $W$-algebra is defined as the vector space generated by the unconstrained generators, and equipped with the Dirac brackets $\{·,·\}^*$ associated to the above second class constraints.

For the case under consideration, the $(2N + 2) \times (2N + 2)$ matrix $\Delta$ reads, using the order $\varphi_1 = e_{1,N+1}, \ldots, \varphi_{N+1} = e_{N+2,1}$, etc (see (3.2)),
\[
\Delta = \begin{pmatrix}
O_N & I_N \\
I_N & O_N \\
(\varphi_1)^T & 0 \\
(\varphi_N)^T & -2 \times 0
\end{pmatrix}
\] (3.3)

where $O_N$ (respectively $I_N$) is a zero (respectively identity) square matrix of size $N$ and $\varphi_i$ is an $N$-vector filled with zeros. Then, it is a matter of calculation to get the Dirac brackets of the $W$-algebra. For ease of reading, we introduce
\[
e_{ij} = e_{ij}, & 1 \leq i, j \leq N; \\
u = \frac{1}{2}(e_{N+1,N+1} + e_{N+2,N+2}); & \bar{\mathbf{3}} = e_{N+1,N+2}; \\
q_i = e_{i,N+2}; & \bar{q}_i = e_{N+1,i}; \\
\q = \sum_{j=1}^{N}q_jq_i; & \bar{\q} = \sum_{j=1}^{N}\bar{q}_j\bar{q}_i;
\] (3.4)

The Poisson brackets of the $W$-algebra read:
\[
\begin{align*}
    \{e_{ij}, e_{kl}\} &= \delta_{kj} e_{il} - \delta_{il} e_{kj}; \\
    \{e_{ij}, u\} &= 0; \quad \{e_{ij}, \bar{\mathbf{3}}\} = 0; \quad \{u, \bar{\mathbf{3}}\} = 0; \\
    \{e_{ij}, q_k\} &= \delta_{kj} q_i; \quad \{e_{ij}, \bar{q}_k\} = -\delta_{ik} \bar{q}_j; \\
    \{u, q_i\} &= -\frac{1}{2} q_i; \quad \{u, \bar{q}_i\} = \frac{1}{2} \bar{q}_i; \\
    \{\bar{\mathbf{3}}, q_k\} &= u q_i + \sum_{k=1}^{N} e_{ik} q_k; \quad \{\bar{\mathbf{3}}, \bar{q}_k\} = -\left( u \bar{q}_i + \sum_{k=1}^{N} \bar{q}_k \bar{q}_i \right); \\
    \{q_i, q_j\} &= 0; \quad \{q_i, \bar{q}_j\} = 0 \\
    \{q_i, \bar{q}_j\} &= \delta_{ij} (\bar{\mathbf{3}} - u^2) - 2u e_{ij} - \sum_{k=1}^{N} e_{ik} e_{kj}.
\end{align*}
\] (3.5)
It is easy to see that
\[ \gamma_1 = u + \frac{1}{2} \langle e \rangle \quad \text{and} \quad \gamma_2 = z + u^2 - \frac{1}{2} \langle e^2 \rangle \] (3.6)
are central generators in the $W$-algebra. Using them to eliminate $u$ and $z$, we get
\[
\begin{align*}
\{e_{ij}, e_{kl}\}_+ &= \delta_{kj} e_{il} - \delta_{il} e_{kj}; \\
\{e_{ij}, q_{kl}\}_+ &= \delta_{kj} q_{il}; \\
\{q_{ij}, q_{kl}\}_+ &= 0; \\
\{q_{ij}, q_{kl}\}_+ &= 0;
\end{align*}
\]
(3.7)
where $e_1 = 2\gamma_1$ and $e_2 = \gamma_2 - 2\gamma_1^2$ are central generators.

In order to implement quantization, note that the two central elements $\gamma_1$ and $\gamma_2$ are already symmetrized. We can thus work\(^3\) at the level of relations (3.7) and (3.8). The first two of these being linear, they remain unchanged (except from the change from Poisson brackets to commutators or anti-commutators). For the last one, one has to symmetrize the two products
\[
\langle e \rangle e_{ij} \rightarrow \frac{1}{2} \left( \langle e \rangle e_{ij} + e_{ij} \langle e \rangle \right) = \langle e \rangle e_{ij};
\]
(3.9)
and
\[
\begin{align*}
\sum_{k=1}^{N} e_{ik} e_{kj} &\rightarrow \frac{1}{2} \left( \sum_{k=1}^{N} e_{ik} e_{kj} + \sum_{k=1}^{N} e_{kj} e_{ik} \right) = \sum_{k=1}^{N} e_{ik} e_{kj} + \frac{1}{2} \left( \delta_{ij} \langle e \rangle - N e_{ij} \right),
\end{align*}
\]
(3.10)
which leads to the anti-commutator
\[
[q_{ij}, q_{kl}]_+ = \delta_{ij} \left( \frac{1}{2} \langle e^2 \rangle - \frac{1}{2} \langle e \rangle^2 + c_1 \langle e \rangle + c_2 \right) - \left( c_1 - \frac{N}{2} - \langle e \rangle \right) e_{ij} - \sum_{k=1}^{N} e_{ik} e_{kj}.
\]
(3.11)
Together with the standard commutations of $q_{ij}, q_{kl}$ and $e_{ij}$ (see equation (3.7)), this superalgebra is precisely that of [6], denoted $g^e_2(N/1)^{1/e}$ (see [6] equation (6)), which was obtained by a first principles construction, and investigated as a potential quadratic spacetime supersymmetry algebra (for further discussion see section 5 below). Explicitly, the correspondence is given by
\[
\begin{align*}
E^l_j &\rightarrow -e_j; \\
Q^l &\rightarrow q_i; \\
\bar{Q}^l &\rightarrow -\bar{q}_i; \\
\alpha &\rightarrow -\left( c_1 + \frac{N}{2} \right); \\
c &\rightarrow c_2.
\end{align*}
\]
(3.12)
Note that the $W$-algebra framework indicates that the parameters $\alpha$ and $c$ should be considered as central generators, a fact that could be of importance in the study of the representations of the superalgebra.

\(^3\) One can check easily that $\gamma_1$ and $\gamma_2$ are indeed central in the quantum/symmetrized version of the algebra (3.5).
To extract the central elements for this W-superalgebra, one considers the \((N + 2) \times (N + 2)\) Gel’fand matrix of reduced generators, derived from the original superalgebra with constraints imposed,

\[
E = \begin{pmatrix}
\epsilon^{11} & \ldots & \epsilon^{1N} & 0 & q^1 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\epsilon^{N1} & \ldots & \epsilon^{NN} & 0 & q^N \\
q^1 & \ldots & q^N & 0 & 1 \\
0 & \ldots & 0 & 1 & u
\end{pmatrix}.
\] (3.13)

The Casimir operators \(\gamma_i, i = 0, 1, 2, \ldots\), are traces of appropriately graded matrix powers of \(E\) (explicit expressions for \(q(H/N)\) are given in [14]). Note that since the \(W\)-algebra is not linear, this result applies only to the classical version. Lower degree terms can occur in the quantum version, see below. Direct calculation for the lowest degrees shows that \(\gamma_1\) and \(\gamma_2\), up to a sign and a factor of \(\frac{1}{2}\), have the expression given in (3.6) (with \(\gamma_0 = 1\)). Their quantum versions remain unchanged. At degrees 3 and 4 we find\(^4\)

\[
\begin{align*}
\gamma_3 &= \langle e^1 \rangle - 3\langle \bar{q}q \rangle + 2u^3 + 6u\bar{u}, \\
\gamma_4 &= \langle e^1 \rangle - 4\langle \bar{q}eq \rangle + 8u\langle \bar{q}q \rangle - 2u^3 + 12u^2 - 2u^4,
\end{align*}
\] (3.14)

where again \(u\) and \(\bar{u}\) may be rewritten in terms of \(\gamma_1\) and \(\gamma_2\) as above. As already stated, \(\gamma_3\) (respectively \(\gamma_4\)) is the classical Casimir of degree 3 (respectively degree 4), i.e. it has vanishing Poisson brackets with any element in the (classical version of the) \(W\)-superalgebra. There are quantum corrections to these classical versions, and we get for the third and fourth Casimir of the quantum \(W\)-superalgebra:

\[
\begin{align*}
c_3 &= \gamma_3 + \frac{N - 6}{2} - \frac{5(N - 2)}{2} u^2 - 5c_1 u - \frac{(N - 1)(N - 2)}{2} u, \\
c_4 &= \gamma_4 + (N - 6)\langle \bar{q}q \rangle + 6(N - 2)u^3 + 2(N + 2)u\bar{u} + 4c_1 \left(3 + 2u^2\right) + \left(4c_2 + 2c_1^2\right)u \\
&\quad + \frac{N^2 - 8}{2} - \frac{(N - 2)(N + 6)}{2} u^2 - \left(3N + 4\right)c_1 u - \frac{(N - 1)(N^2 - 4)}{2} u.
\end{align*}
\] (3.15)

Expressions (3.15) were obtained by direct calculation, demanding that \(c_3\) and \(c_4\) are indeed central in the superalgebra (3.11). We checked that the additional next-to-leading order terms in \(c_3\) exactly correspond to the symmetrization of \(\gamma_3\). This is not enough to get the quantum version of \(\gamma_3\), since we are considering expressions which are not quadratic anymore. The last term, which is a next-to-next-to-leading order term, has to be found by brute calculation. Similarly, the expression for \(c_4\) arises from symmetrization, together with additional lower order terms. When \(N = 4\), which corresponds to a quadratic superconformal algebra, we conjecture they are the only Casimir operators of this algebra.

As above, one can eliminate \(u\) and \(\bar{u}\) using \(\epsilon_1\) and \(\epsilon_2\). For instance, \(\epsilon_3\) can be rewritten as

---

\(^4\)A generating function for them is given by the expansion of the superdeterminant (Berezinian), \(\text{Ber}(E - x I_{N+2})\), with \(I_{N+2}\) the identity matrix [15, 16]. It is related to a supersymmetric version of the Capelli identity.

\(^5\)We omit polynomials in \(\epsilon_1\) and \(\epsilon_2\), since they are already central at the quantum level.
\[c_3 = (e^3) - 3\langle q\bar{q} \rangle - \frac{3}{2}(e^2)\langle e \rangle + \frac{1}{4}(e^3) + \frac{6c_1 + N - 6}{4}(e^2) - \frac{6c_1 + 3N - 8}{4}(e^1) + \frac{6(N - 1)c_1 - 12c_2 + (N - 1)(N - 2)}{4}(e) + \frac{3c_1c_2 + c_1^2 + \frac{N - 6}{2}c_2 - \frac{N + 4}{2}c_1^2 - \frac{(N - 1)(N - 2)}{4}c_1}{N - 6} \]

where the last line can be dropped out, since it is central on its own.

### 3.2. The \(W(gl(N + 2),gl(2))\) algebras

We can construct the ‘bosonic version’ of the finite \(W\)-algebra defined above. We start with the \(gl(N + 2)\) algebra, with Poisson brackets

\[\{e_{ij}, e_{kl}\} = \delta_{ik} e_{lj} - \delta_{il} e_{kj}\] (3.17)

and consider the constraints

\[
\begin{align*}
e_j, N+1 & = 0 \\
e_{N+2,j} & = 0 \\
e_{N+2,N+1} & = 1; \\
e_{N+1,N+1} - e_{N+2,N+2} &= 0.
\end{align*}
\]

The \((2N + 2) \times (2N + 2)\) matrix \(\Delta\) reads, (using the order given in (3.18), viz \(\varphi_1 = e_{1,N+1}, \varphi_{N+1} = e_{N+2,1}, \) etc.)

\[
\Delta = \begin{pmatrix}
O_N & I_N & 0_N & 0_N \\
0_N & O_N & 0_N & 0_N \\
(0_N)^T & (0_N)^T & 0 & 2 \\
(0_N)^T & (0_N)^T & -2 & 0
\end{pmatrix}
\]

(3.19)

where \(O_N\) (respectively \(I_N\)) is a zero (respectively identity) square matrix of size \(N\) and \(0_N\) is an \(N\)-vector filled with zeros.

Again, we introduce

\[
\begin{align*}
e_{ij} & = e_{ij}, \quad 1 \leq i, j \leq N; \\
u & = \frac{1}{2}(e_{N+1,N+1} + e_{N+2,N+2}); \quad \jmath = e_{N+1,N+2}; \\
p_i & = e_{i,N+2}; \quad \bar{p}_i = e_{N+1,i}; \\
\langle e \rangle & = \sum_{i=1}^{N} e_{ii}; \quad \langle e^2 \rangle = \sum_{i,j=1}^{N} e_{ij}e_{ji}.
\end{align*}
\]

(3.20)
The Poisson brackets of the $W$-algebra read:

\[
\{e_{ij}, e_{kl}\} = \delta_{kj} e_{il} - \delta_{il} e_{kj}; \\
\{e_{ij}, \ u\} = 0; \quad \{e_{ij}, j\} = 0; \quad \{u, j\} = 0; \\
\{e_{ij}, p_k\} = \delta_{kj} p_i; \quad \{e_{ij}, \bar{p}_k\} = -\delta_{il} \bar{p}_j; \\
\{u, p_i\} = -\frac{1}{2} p_i; \quad \{u, \bar{p}_i\} = \frac{1}{2} \bar{p}_i; \\
\{j, p_i\} = u p_i - \sum_{k=1}^{N} e_{ik} \bar{p}_k; \quad \{j, \bar{p}_i\} = -\left( u \bar{p}_i - \sum_{k=1}^{N} \bar{p}_k e_{ki} \right) \\
\{p_i, p_j\} = 0; \quad \{\bar{p}_i, p_j\} = 0; \\
\{p_i, \bar{p}_j\} = \delta_{ij} (u^2 - j) - 2u e_{ij} + \sum_{k=1}^{N} e_{ik} e_{kj}.
\]

(3.21)

It is easy to see that

\[
\gamma_1 = u + \frac{1}{2} \langle e \rangle \quad \text{and} \quad \gamma_2 = j + u^2 + \frac{1}{2} \langle e^2 \rangle
\]

(3.22)

are central generators in the $W$-algebra. Using them to eliminate $u$ and $j$, we get

\[
\{e_{ij}, e_{kl}\} = \delta_{kj} e_{il} - \delta_{il} e_{kj}; \\
\{e_{ij}, p_k\} = \delta_{kj} p_i; \quad \{e_{ij}, \bar{p}_k\} = -\delta_{il} \bar{p}_j; \\
\{p_i, p_j\} = 0; \quad \{\bar{p}_i, p_j\} = 0; \\
\{p_i, \bar{p}_j\} = \delta_{ij} \left( \frac{1}{2} \langle e^2 \rangle + \frac{1}{2} \langle e \rangle^2 - c_1 \langle e \rangle - c_2 \right) - \left( c_1 - \langle e \rangle \right) e_{ij} + \sum_{k=1}^{N} e_{ik} e_{kj}
\]

(3.23)

where $c_1 = 2\gamma_1$ and $c_2 = \gamma_2 - 2\gamma_1^2$ are central generators.

Quantization of the quadratic $W$-algebra follows the same procedure as with the fermionic counterpart above, leading to the commutator

\[
[p_i, p_j] = \delta_{ij} \left( \frac{1}{2} \langle e^2 \rangle + \frac{1}{2} \langle e \rangle^2 - \left( c_1 - \frac{1}{2} \langle e \rangle \right) \langle e \rangle - c_2 \right) - \left( c_1 + \frac{N}{2} - \langle e \rangle \right) e_{ij} + \sum_{k=1}^{N} e_{ik} e_{kj}.
\]

The Casimir operators of this bosonic $W$-algebra can be extracted in the same way as for the fermionic counterpart above, using now the traces of standard matrix powers of the Gel’fand generators with constraints (or defining the appropriate generating function using the determinant instead of the Berezinian). We have already defined the two first Casimir operators $c_1$ and $c_2$. At degrees 3 and 4, we find for the classical Casimir operators

\[
\gamma_3 = \langle e^3 \rangle + 3\langle \bar{p}p \rangle + 2u^3 + 6u_3
\]

\[
\gamma_4 = \langle e^4 \rangle + 4\langle \bar{p}ep \rangle + 8u\langle \bar{p}p \rangle + 2\gamma_3^2 + 12\gamma_1^2u^2 + 2u^4
\]

(3.24)

and for their quantum analogs
\[ c_3 = \gamma_3 = \frac{N + 6}{2} + \frac{5(N + 2)}{2} u^2 - 5\epsilon u - \frac{(N + 2)(N + 1)}{2} u, \]
\[ c_4 = \gamma_4 - (N + 6) \langle q \bar{q} \rangle + 6(N + 2) u^3 + 2(N - 2) u_3 - 4\epsilon (\bar{\epsilon} + 2u^2) - (4\epsilon_2 + 2\epsilon_1^2) u \]
\[ - \frac{N^2 - 8}{2}\bar{\epsilon} + \frac{(N + 2)(N - 6)}{2} u^2 - (3N - 4) \epsilon u - \frac{(N - 1)(N^2 - 4)}{2} u. \] (3.25)

### 4. Quadratic superalgebras from \(osp(M|2N)\) reductions

The procedure we have described so far in the context of \(gl(M|N)\) can be applied to other superalgebras. In this section we examine orthosymplectic cases, leading to the construction of quadratic superalgebras with \(so(N)\) or \(sp(2N)\) bosonic subalgebras.

The \(osp(M|2N)\) Lie superalgebra can be constructed as the folding of the \(gl(M|2N)\) algebra \([17,18]\). Recall that the \(gl(M|2N)\) superalgebra has Poisson brackets

\[ \{e_{ab}, e_{cd}\} = \delta_{bc} e_{ad} - (-1)^{|a||b||c||d|} \delta_{ad} e_{cb}, \quad 1 \leq a, b, c, d \leq M + 2N \] (4.1)

and grading

\[ [a] = \begin{cases} 0 & \text{for } 1 \leq a \leq M \\ 1 & \text{for } M < a \leq M + 2N \end{cases} \] (4.2)

The superalgebra admits a morphism \(\varphi\), defined as

\[ \varphi : \begin{align*} gl(M|2N) & \rightarrow gl(M|2N) \\ e_{ab} & \rightarrow -(-1)^{|a||b|+1} \theta_a \theta_b e_{\bar{a} \bar{b}} \end{align*} \] (4.3)

where

\[ \theta_a = \begin{cases} 1 & \bar{a} = M + 1 - a \\ \sigma (M + N + \frac{1}{2} - a) & \bar{a} = 2M + 2N + 1 - a \end{cases} \] (4.4)

\(\varphi\) is a Poisson algebra morphism, viz \(\varphi \{\{A, B\}\} = \{\varphi(A), \varphi(B)\}\). Note the property \([a] = [\bar{a}]\) and \((-1)^{|a|} \theta_a \theta_a = 1\), for all \(a\), which shows that \(\varphi\) is an involution, \(\varphi^2 = \text{id}\). Then the \(gl(M|2N)\) superalgebra can be decomposed as a sum of the two \(\varphi\)-eigenspaces. The \(osp(M|2N)\) algebra can be viewed as the \(\varphi\)-eigenspace corresponding to the eigenvalue \(+1\). Explicitly, the generators are constructed as

\[ s_{ab} = e_{ab} - (-1)^{|a||b|+1} \theta_a \theta_b e_{\bar{a} \bar{b}} = -(-1)^{|a||b|+1} \theta_a \theta_b s_{\bar{a} \bar{b}}. \] (4.5)

leading to the Poisson brackets

\[ \{s_{ab}, s_{cd}\} = \delta_{bc} s_{ad} - (-1)^{|a||b||c||d|} \delta_{ad} s_{cb} + (-1)^{|a||b||c||d|+1} \theta_a \theta_b \delta_{cd} s_{\bar{a} \bar{b}} - (-1)^{|a||b|+1} \theta_a \theta_b \delta_{ac} s_{\bar{b} \bar{d}}. \] (4.6)

For our purpose, we specialize the notation to two particular cases, that we now describe in detail.
4.1. The $W(osp(N|2), sp(2))$ superalgebras

As bosonic generators, one can choose $f_{ij} = e_{ij} - e_{ji}$, $1 \leq i, j \leq N$, with $i = N + 1 - i$, together with $e_+ = e_{N+1,N+2}$, $e_- = e_{2N+1}$. The generators $f_{ij}$ form an $so(N)$ algebra while $e_{i\mu}, \mu = 0, \pm$ generate an $sp(2)$ algebra. The Poisson brackets read

$$\{f_{ij}, f_{kl}\} = \delta_{ij} f_{kl} - \delta_{kl} f_{ij} + \delta_{lj} f_{ik} - \delta_{ik} f_{lj}$$

with $f_{ii} = -f_{ij}$;  
$$\{e_0, e_\pm\} = \pm 2 e_\pm, \quad \{e_+, e_-\} = e_0;$$

$$\{f_{ij}, e_\mu\} = 0, \quad 1 \leq i, j \leq N, \quad \mu = 0, \pm.$$

The fermionic generators are $q^+_i = e_{i,N+2} + e_{N+1,j}$ and $q^-_i = e_{i,N+1} - e_{N+2,j}$, with Poisson brackets

$$\{f_{ij}, q^+_k\} = \delta_{kj} q^+_i - \delta_{ik} q^+_j, \quad \{e_0, q^+_k\} = \pm q^+_k,$n$$

$$\{e_+, q^+_k\} = 0, \quad \{e_\pm, q^+_k\} = -q^\pm_k, $n$$

$$\{q^-_i, q^+_k\} = \pm 2 \delta_{ij} e_\pm, \quad \{q^-_i, q^-_j\} = \delta_{ij} e_0 - f_{ji}.$$

Note that in this case all $\theta$'s are 1, except $\theta_{N+2} = -1$. We impose the constraints

$$q^-_i = 0; \quad e_- = 1; \quad e_0 = 0;$$

leading to a matrix

$$\Delta = \begin{pmatrix}
-2 \lambda N & 0_N & 0_N \\
(0_N)^t & 0 & 2 \\
(0_N)^t & -2 & 0
\end{pmatrix} \quad \text{with} \quad \lambda N = \begin{pmatrix}
0 & \ldots & 0 & 1 \\
\vdots & & \ddots & \vdots \\
0 & 1 & \ldots & 0
\end{pmatrix}.$$ (4.10)

The Dirac brackets associated to these constraints define the $W(osp(N|2), sp(2))$ super-algebra. It has generators

$$\hat{f}_{ij} = f_{ij}; \quad \hat{s} = e_+ = e_{N+1,N+2}; \quad \hat{q}_i = q^+_i = e_{i,N+2} + e_{N+1,j},$$

with Poisson brackets:

$$\{\hat{f}_{ij}, \hat{f}_{kl}\}_s = \delta_{ij} \hat{f}_{kl} - \delta_{kl} \hat{f}_{ij} + \delta_{lj} \hat{f}_{ik} - \delta_{ik} \hat{f}_{lj}$$

with $\hat{f}_{ii} = -\hat{f}_{ij}$;  

$$\{\hat{f}_{ij}, \hat{q}_k\}_s = \delta_{kj} \hat{q}_i - \delta_{ik} \hat{q}_j;$$

$$\{\hat{s}, \hat{q}_i\}_s = \frac{1}{2} \sum_{k=1}^{N} \hat{f}_{ik} \hat{q}_k; \quad \{\hat{s}, \hat{f}_{ij}\}_s = 0;$$

$$\{\hat{q}_i, \hat{q}_j\}_s = 2 \delta_{ij} \hat{s} - \frac{1}{2} \sum_{k=1}^{N} \hat{f}_{ik} \hat{f}_{kj}.$$ (4.12)

Similarly to the $g(N|2)$ case, the generator

$$\gamma_2 = \hat{s} - \frac{1}{8} \hat{f}^2 = \hat{s} - \frac{1}{8} \sum_{k,j=1}^{N} \hat{f}_{k\bar{j}} \hat{f}_{ij}$$

(4.13)
is central, and we use it to eliminate $\mathfrak z$. It leads to the final form for the Poisson brackets of the $W$-superalgebra:

\begin{align}
\{f_{ij}, f_{kl}\}_+ &= \delta_{kj} f_{il} - \delta_{il} f_{kj} + \delta_{ij} f_{lk} - \delta_{lk} f_{ij} \quad \text{with } f_{ji} = -f_{ij}; \\
\{f_{ij}, q_k\}_+ &= \delta_{kj} q_i - \delta_{ik} q_j; \\
\{q_i, q_j\}_+ &= 2 \delta_{ij} \left( \gamma_2 + \frac{1}{8} (\mathfrak f^2) \right) - \frac{1}{2} \sum_{k=1}^{N} f_k f_{kj}.
\end{align}

(4.14)

We perform quantization through symmetrization. In the quantization of the algebra (4.12), two terms have to be symmetrized: $\sum_{k=1}^{N} f_k f_{kj}$ and $\sum_{k=1}^{N} f_k q_k$. We obtain

\begin{align}
[f_{ij}, f_{kl}] &= \delta_{kj} f_{il} - \delta_{il} f_{kj} + \delta_{ij} f_{lk} - \delta_{lk} f_{ij} \quad \text{with } f_{ji} = -f_{ij}; \\
[f_{ij}, q_k] &= \delta_{kj} q_i - \delta_{ik} q_j; \\
[q_i, q_j]_+ &= 2 \delta_{ij} \mathfrak z - \frac{1}{2} \sum_{k=1}^{N} f_k f_{kj} + \frac{N-2}{4} f_{ij}; \quad \text{(4.15)}
\end{align}

\[ [\mathfrak z, q_j] = \frac{1}{2} \sum_{k=1}^{N} f_k q_k - \frac{N-2}{4} q_j. \]

One can check that $c_2 = 1 - \frac{1}{8} (\mathfrak f^2)$ is still central, and we can use it to eliminate $\mathfrak z$ at the quantum level. Of course, the resulting algebra is the same as the quantization through symmetrization of (4.14).

The superalgebra contains as bosonic part, an $so(N)$ algebra, and an $N$-vector of fermions $q^i$, that close quadratically on the orthogonal subalgebra. It should be interesting to look at a non-compact version of the case $N = 6$, corresponding to the conformal $so(4, 2)$ algebra.

4.2. The $W(osp(3|2N), so(3))$ superalgebras

The bosonic generators now are $e^+ = e_{12} - e_{23}$, $e^− = e_{21} - e_{32}$, $e^0 = e_1 - e_3$ which form an $so(3)$ subalgebra, together with $f_{ij} = e_{i+j+3} - \theta_i \theta_j e_{3+i+3}$ (with $i = 2n + 1 - i$) which generate $sp(2N)$, and where we have redefined

$$
\theta_j = \begin{cases} 
+1, & 1 \leq j \leq N \\
-1, & N + 1 \leq j \leq 2n.
\end{cases}
$$

Then the Poisson brackets read

\begin{align}
\{f_{ij}, f_{kl}\} &= \delta_{kj} f_{il} - \delta_{il} f_{kj} + \theta_i \theta_j (\delta_{ij} f_{lk} - \delta_{lk} f_{ij}) \quad \text{with } f_{ji} = -\theta_i \theta_j f_{ij}; \\
\{e_0, e_±\} &= ± e_±; \quad \{e_+, e_−\} = e_0; \\
\{f_{ij}, e_μ\} &= 0, \quad 1 \leq i, j \leq N, \quad μ = 0, ±.
\end{align}

(4.16)

The fermionic generators take now the form $q_{ij}^+ = e_{i+j+3} - \theta_i e_{3+i+3}$, $q_{ij}^0 = e_{2i+3} - \theta_i e_{3+i+2}$ and $q_{ij}^- = e_{3+i+3} - \theta_i e_{i+3}$, with Poisson brackets
\[
\begin{align*}
\{f_{ij}, q_{ij}^\mu\} &= \theta_i \theta_j \left(-\delta_{ik} q_{ij}^\mu + \delta_{jk} q_{ij}^\mu\right) \quad \{e_0, q_k^\mu\} = \mu q_k^\mu \\
\{e^\pm, q_k^\mu\} &= 0; \\
\{q_i^+, q_j^-\} &= 0; \\
\{q_k^0, q_j^\pm\} &= 0; \\
\{q_i^+, q_j^+\} &= \theta_i (\delta_{ij} e_0 - f_{ij}) \\
\{q_i^-, q_j^-\} &= \theta_i \delta_{ij} e_\pm.
\end{align*}
\]

We impose the constraints \(e_- = 1\) and \(e_0 = 0\), leading to the Dirac brackets

\[
\begin{align*}
\{f_{ij}, f_{kl}\}_+ &= \delta_{kl} f_{ij} - \delta_{ij} f_{kl} + \theta_i \theta_j \left(\delta_{kl} f_{ij} - \delta_{ij} f_{kl}\right) \quad \text{with} \quad f_{ij} = -\theta_i \theta_j f_{ij}; \\
\{f_{ij}, e_+\}_+ &= 0, \quad 1 \leq i, j \leq 2n; \\
\{f_{ij}, q_{ij}^\mu\}_+ &= \theta_i \theta_j \left(\delta_{ij} q_{ij}^\mu - \delta_{ik} q_{kj}^\mu\right) \quad \mu = 0, \pm; \\
\{e_+, q_k^0\}_+ &= e^+ q_k^0; \\
\{e^+, q_k^0\}_+ &= q_k^+ - e^+ q_k^0; \\
\{e^+, q_k^-\}_+ &= -q_k^+ q_k^-; \\
\{e^+, q_k^-\}_+ &= -q_k^+ q_k^- + \theta_i \delta_{ij} e_+; \\
\{q_i^+, q_j^-\}_+ &= q_i^+ q_j^- - \theta_i f_{ij}; \\
\{q_i^0, q_j^\pm\}_+ &= -q_i^0 q_j^- - \theta_j \delta_{ij}; \\
\{q_i^-, q_j^-\}_+ &= \theta_i f_{ij} \\
\{q_i^0, q_j^0\}_+ &= 0.
\end{align*}
\]

The superalgebra is quadratic in fermions, and thus is a variant on the standard construction from \(W(gl(N + 2|2), gl(2))\) in section 3 above. Moreover, it contains an undeformed \(osp(1|2N)\) singleton superalgebra (generated by \(q_0^0\) and \(f_{ij}\)), and so in real forms is a candidate for higher dimensional quadratic spacetime supersymmetry. We leave the quantization step to be completed by (anti)symmetrization, as in the cases already examined above.

5. Conclusion

In this paper we have provided explicit constructions for a class of candidate quadratic spacetime supersymmetries as \(W\)-superalgebras, via systematic application of Hamiltonian reduction to certain simple Lie superalgebras.

In particular we recover as \(W(gl(N + 2|2), gl(2))\), the two-parameter family of quadratic superalgebras analysed in [6] there denoted \(gl_2(N/1)^{\alpha, \beta}\) (see also [19, 20]). These were first obtained from a first principles construction, and their consistency as algebras of PBW type established via the formal theory of abstract quadratic algebras [5]. For the \(N = 4\) real form with even part \(u(2, 2) \cong so(4, 2) + u(1)\), a Kac module-type construction led to the observation that for certain parameter choices, there are ‘zero step’ unitary irreducible representations which comprise a single module of the even subalgebra, namely, a multiplet corresponding to one of the standard physical massless conformal fields of spin 0, 1/2 or 1, with the odd generators being identically zero (the standard conformal supersymmetry Lie superalgebra is a contraction limit of \(gl_2(N/1)^{\alpha, \beta}\)). This scenario of ‘ultra-short supermultiplets’ can thus be likened to an extreme form of unbroken supersymmetry, where a (degenerate) ground state is annihilated by the supercharges, but with no other (paired) states present in the spectrum. This scenario has potential implications for phenomenology in the context of particle symmetries, and the absence of super-partners.

With the identification of this and other potential candidate quadratic spacetime supersymmetry algebras as \(W\)-superalgebras, which have been intensively studied in recent years, formal tools from that work are available for their investigation. These include direct results on Casimir
operators (as discussed above), and also methods for constructing representations, and providing a more general theory for the existence of special ‘zero step’, no-superpartner cases. On this point, the tuning of the parameters identifying which of the family of quadratic superalgebras, admit the existence of such ‘hyper-atypical’ representations [6], can be seen instead, in the context of the W construction, as the selection of representations of the primary superalgebra (which is to undergo Hamiltonian reduction), which have fixed values of the lowest Casimir invariants.

In this work we have also used the Hamiltonian reduction formalism to construct new finite quadratic W-superalgebras, beyond the above superconformal case, which are equally candidates for quadratic spacetime superalgebras. If appropriate real forms exist, their unitary representations may supply a tool kit of interesting variants on (quadratic) ‘extended supersymmetries’. This applies to the W-superalgebras derived from osp(M|2N) reductions in section 4 above, which include a case with 6 dimensional orthogonal (conformal) group as bosonic symmetry graded by vectorial supergenerators (see also [11]), as well as a case with sp(2N) singleton bosonic symmetry, accompanied by triplet of spinorial supergenerators (whose bracket relations are no longer pure anticommutators), but with an undeformed osp(1|2N) subalgebra. A subsequent issue is to establish a relation with (super)Yangian truncations, as has been done for certain classes of finite W-algebras and superalgebras [12, 13, 21].

The existence of field theoretical realizations of the quadratic supersymmetries as discussed here, perhaps as supersymmetric systems with constraints, is of course an open question. While infinite dimensional W-algebras arise as spectrum generating algebras in certain lattice models [22], and as higher spin current algebras in conformal field theories, there exists no concrete field theoretical construction of W-algebras beyond dimension 1 or 2. It is to be hoped however, that the rich theoretical understandings of W-superalgebras and Yangian superalgebras, which can be brought to bear on quadratic spacetime supersymmetries with their transcription into this formalism, will be able to inform further progress on these questions.

Acknowledgments

ER thanks the SMRI International Visitor program at the University of Sydney, and the University of Tasmania, Discipline of Mathematics, for their support and their warm hospitality during his visit to Hobart in October 2019.

ORCID iDs

E Ragoucy https://orcid.org/0000-0002-9350-8254
L A Yates https://orcid.org/0000-0002-1685-3169
P D Jarvis https://orcid.org/0000-0002-5330-6789

References

[1] Sklyanin E K 1985 On an algebra generated by quadratic relations Usp. Mat. Nauk 40 214
[2] Zhedanov A S 1991 ‘Hidden symmetry’ of Askey–Wilson polynomials Theor. Math. Phys. 89 1146–57
[3] Kalnins E G, Miller W and Post S 2009 Models of quadratic quantum algebras and their relation to classical superintegrable systems Phys. Atom. Nucl. 72 801–8
[4] Genest V X, Vinet L and Zhedanov A 2014 The Racah algebra and superintegrable models J. Phys.: Conf. Ser. 512 012011
[5] Polishchuk A and Positselski L 2005 Quadratic Algebras vol 37 (Providence, RI: American Mathematical Society)
[6] Yates L A and Jarvis P D 2018 Hidden supersymmetry and quadratic deformations of the space-time conformal superalgebra J. Phys. A: Math. Theor. 51 145203
[7] Tjin T 1992 Finite W-algebras Phys. Lett. B 292 60–6
[8] de Boer J and Tjin T 1993 Quantization and representation theory of finite W algebras Commun. Math. Phys. 158 485–516
[9] de Boer J, Harmske F and Tjin T 1996 Non-linear finite W-symmetries and applications in elementary systems Phys. Rep. 272 139–214
[10] Brown J, Brundan J and Goodwin S 2013 Principal W-algebras for GL(m|n) Algebra Number Theory 7 1849–82
[11] Barbarin F, Ragoucy E and Sorba P 1997 W-realization of Lie algebras: application to so(4, 2) and Poincaré Algebras Commun. Math. Phys. 186 393–411
[12] Briot C and Ragoucy E 2003 W-superalgebras as truncations of super-Yangians J. Phys. A: Math. Gen. 36 1057–81
[13] Peng Y-N 2014 Finite W-superalgebras and truncated super Yangians Lett. Math. Phys. 104 89–102
[14] Jarvis P D and Green H S 1979 Casimir invariants and characteristic identities for generators of the general linear, special linear and orthosymplectic graded Lie algebras J. Math. Phys. 20 2115–22
[15] Nazarov M L 1991 Quantum Berezinian and the classical Capelli identity Lett. Math. Phys. 21 123–31
[16] Molev A and Retakh V 01 2004 Quasideterminants and Casimir elements for the general linear Lie superalgebra Int. Math. Res. Not. 2004 611–9
[17] Frappat L, Sciarrino A and Sorba P 1996 Dictionary on Lie superalgebras (arXiv:hep-th9607161)
[18] Jarvis P D and Murray M K 1983 Casimir invariants, characteristic identities, and tensor operators for ‘strange’ superalgebras J. Math. Phys. 24 1705–10
[19] Jarvis P D, Rudolph G and Yates L A 2011 A class of quadratic deformations of Lie superalgebras J. Phys. A: Math. Theor. 44 235205
[20] Jarvis P D 2012 Hidden supersymmetry-a ‘no superpartner’ theorem for representations of deformed conformal superalgebra J. Phys. A: Math. Theor. 45 322001
[21] Ragoucy E and Sorba P 1998 Yangians and finite W-algebras Czechoslovak J. Phys. 48 1483–7
[22] Deluc F, Ragoucy E and Sorba P 1992 Towards a classification of W algebras arising from nonabelian Toda theories Phys. Lett. B 279 319–25