SUPERSYMMETRIC FIELD THEORIES ON DEFORMED SPACE-TIME

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Field theories on "quantum" or deformed space-time are considered here. The Moyal-Weyl deformation breaks the Lorentz invariance of the theory, but one can still require invariance under the supertranslation algebra. We investigate some aspects of the Wess-Zumino model, super Yang-Mills theories and analyze the correspondence of the later with the supersymmetric Born-Infeld action.

1 Deformation theory: some generalities

Let $\mathcal{A}$ be the space of $C^\infty$ functions on $\mathbb{R}^m$. A Poisson bracket on $\mathcal{A}$ is a Lie algebra structure that is a bi-derivation with respect to the pointwise multiplication,

$$\{a, b \cdot c\} = b \cdot \{a, c\} + \{a, b\} \cdot c, \quad a, b, c \in \mathcal{A}.$$ 

We consider the standard symplectic structure ($n = 2r$),

$$\{a, b\} = \mathcal{P}_{ij} \partial_i a \partial_j b,$$

with $\mathcal{P}^{ij}$ a constant antisymmetric non degenerate matrix.

If $\mathbb{R}^{2n}$ is the phase space of a hamiltonian system, the quantization map takes (a class of) elements of $\mathcal{A}$ to operators on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. It was proposed by Dirac that the quantization map should be such that the Poisson bracket goes into the Lie bracket of the corresponding operators, that is,

$$D : \mathcal{A} \rightarrow \Theta(\mathcal{H})$$

$$h\{,\} \rightarrow [\, ,\].$$

A map like this with reasonable properties for the operators does not exist (see Ref. for a precise statement of this fact), although the Poisson bracket of the coordinate functions may be preserved in the quantization.

The Weyl quantization consists in associating to any "reasonable" function $a(q, p)$ an operator $A = W(a)$ on $L^2(\mathbb{R}^n)$ defined via the integral

$$A\psi(q) = \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^{2n}} e^{\pm i\mathcal{P}(q-q')} a(\frac{q+q'}{2}, p)\psi(q')dq'dp, \quad \psi \in L(\mathbb{R}^n).$$

$a$ is called the Weyl symbol of the operator $A$. If $a$ a function in the Schwartz space, the operator associated by the Weyl map is a bounded operator. If $a$ is a polynomial, the operator is the one obtained by the symmetric ordering rule.

Moyal wrote the bracket of such operators in the form,

$$W^{-1}([W(a), W(b)]) = h\{a, b\} + \mathcal{O}(\hbar^2).$$
The first term is the Poisson bracket and the remaining terms are necessary corrections to Dirac quantization.

The composition formula gives the star product on $C^\infty(\mathbb{R}^{2n})$. If $A = W(a)$ and $B = W(b)$, then the Weyl symbol of $A \circ B$ is the associative, non commutative star product

$$a \star b = a \cdot b + \sum_{n=1}^{\infty} h^n P^k(a, b),$$

where

$$P^k(a, b) = P^{i_1 j_1} P^{i_2 j_2} \cdots P^{i_n j_n} (\partial_{i_1} \partial_{i_2} \cdots \partial_{i_n}) a \cdot (\partial_{j_1} \partial_{j_2} \cdots \partial_{j_n} b).$$

The star product is given as a series in $h$. There is no guarantee that such series is pointwise convergent, so the star product is well defined only on the space of formal power series in $h$ with values in $C^\infty(\mathbb{R}^{2n})$, namely $C^\infty(\mathbb{R}^{2n})[[h]]$. It was in fact shown in Ref. 4 that no associative, non commutative star product converges for all $C^\infty(\mathbb{R}^{2n})$. One obtains subalgebras that converge for the Schwartz space and polynomials, as we said before. For more general classes of symbols, the star product obeys only an asymptotic convergence (see for example Ref. 5).

For any star product, the associativity condition at first and second orders in $h$ assures that

$$\lim_{h \to 0} \frac{a \star b - b \star a}{h}$$

is a Poisson bracket. Once the Poisson bracket in $\mathbb{R}^{2n}$ is fixed there is, up to isomorphism, only one star product.

2 Deforming superspace

The super space of dimension $(p, q)$ is the affine space $\mathbb{R}^p$ together with a commutative super algebra

$$S^{p,q} = C^\infty(\mathbb{R}^p) \otimes \Lambda(\mathbb{R}^q) = \{a_0(x) + a_i(x) \theta^i + a_{i_1 i_2} \theta^{i_1 i_2} + \cdots a_{i_1 \cdots i_q} \theta^{i_1} \wedge \theta^{i_2} \}. $$

On $S^{p,q}$, left and right, odd and even derivations can be defined. The expression

$$\{\Phi, \Psi\} = P^{ab} \partial_a \Phi \partial_b \Psi + P^{\alpha \beta} \partial^{R}_\alpha \Phi \partial^{L}_\beta \Psi = P^{AB} \partial^R_A \Phi \partial^L_B \Psi,$$

where $P^{\mu \nu}$ and $P^{ij}$ are constant matrices, antisymmetric and symmetric respectively, defines a Poisson bracket on $S^{p,q}$. The superindices $L$ and $R$ denote left and right derivations respectively.

A star product on $S^{p,q}$ is defined as the Weyl quantization of odd variables. The expression of the star product is

$$P^n(\Phi \otimes \Psi) = P^{A_1 B_1} P^{A_2 B_2} \cdots P^{A_n B_n} (\partial^R_{A_1} \partial^R_{A_2} \cdots \partial^R_{A_n}) \Phi \cdot (\partial^L_{B_1} \partial^L_{B_2} \cdots \partial^L_{B_n} \Psi).$$  (1)
If we consider a deformation only of the odd part of the superalgebra, \( \Lambda(\mathbb{R}^q) \), the algebra that one obtains is isomorphic to a Clifford algebra \( \mathbb{C}l_{6,7} \). Clifford algebras are then non commutative superalgebras.

If the manifold that we are deforming is space-time, one can ask for the behaviour of the star product under super Poincaré transformations. Lorentz transformations are not automorphisms of the algebra, while translational invariance is preserved. If \( P^{ij} \) is constant, the star product does not behave well under supertranslations. Using the covariant derivatives \( D^{\alpha L}_\alpha, \bar{D}^{\dot{\alpha} L}_{\dot{\alpha}} \), one can define a new Poisson bracket:

\[
\{ \Phi, \Psi \} = P^{\mu\nu} \partial_\mu \Phi \partial_\nu \Psi + P^{\alpha\beta} D^{\alpha L}_\alpha \Phi \bar{D}^{\dot{\beta} L}_{\dot{\beta}} \Psi.
\]

The star product can be defined via the exponential as in (1). We note that the Poisson bracket is always degenerate in the space of odd variables because of the non trivial commutation relations among \( D \) and \( \bar{D} \). The construction is easily extended to \( N \) supersymmetries by using harmonic superspace.

Finally we notice that chiral superfields \( \bar{D} \Phi = 0 \) are not a subalgebra of the star product unless \( P^{\alpha\beta} = 0 \).

### 3 Supersymmetric deformed field theories

We will consider only deformations only of the even part of the superalgebra. If \( \Phi_1 \) and \( \Phi_2 \) are two superfields we have that

\[
\int d^4 x \phi_1 \star \Phi_2 = \int d^4 x \phi_1 \cdot \Phi_2 = \int d^4 x \phi_2 \star \Phi_1,
\]

if \( \partial_{\mu_1} \cdots \partial_{\mu_n} \Phi \rightarrow 0 \) when \( x \rightarrow \infty \) for all \( n \). Notice that these boundary conditions are not enough to assure the convergence of the star product.

Let \( \Phi \) be a chiral superfield with the expansion

\[
\Phi = A(y) + \sqrt{2} \theta \psi(y) + \theta \theta F(y).
\]

As a first example one can consider the Wess-Zumino model, with action

\[
S_{DWZ} = \int d^4 x d^2 \theta d^2 \bar{\theta} \Phi \Phi + \int d^4 x ( \int d^2 \theta \left( \frac{m}{2} \Phi^2 + \frac{g}{3} \Phi^3 \right) + c. c.).
\]

The auxiliary field \( F \) satisfies algebraic equations

\[
F = -m \bar{A} - g \bar{A} \star \bar{A},
\]

so the quartic potential becomes \((A \star A)(\bar{A} \star \bar{A})\) as opposed to the other possible generalization, \((A \star \bar{A})^2\).

#### 3.1 Rank 1 gauge theory on deformed superspace

Even for the rank 1 theory the gauge symmetry is non abelian, so one has to introduce the formalism of non abelian supersymmetric Yang-Mills theories. The elements of the gauge group in deformed superspace are complex chiral superfields

\[
U = e^{i \Lambda} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\Lambda)^n,
\]
with group law $U_1 \ast U_2 = U_3$.

The connection superfield is $e^* V$ and transforms as

$$e^* V \rightarrow U^\dagger \ast e^* V \ast U$$

while the chiral field strength, $W_\alpha = \bar{D}^2 e^* V \ast D_\alpha e^* V$ transforms as

$$W_\alpha \rightarrow U^{-1} \ast W_\alpha \ast U.$$

The deformed super Yang-mills action is

$$S_{DSYM} = \int d^4 x \int d^2 \theta \, W_\alpha \ast W_\alpha + \text{c.c.}$$

One can choose a Wess-Zumino gauge, $V^* \chi = 0$. It is not preserved by supersymmetry, and in addition, depends explicitly on the deformation parameter. The supersymmetry algebra is then realized in the subset of fields satisfying the Wess-Zumino condition in a way that depends on the deformation parameter, but the supersymmetry algebra is not deformed.

### 3.2 Connection with open string theory

Non commutative gauge fields appear in the context of string theory as a way to incorporate a vacuum expectation value for the $B$ field in the effective theory.

The effective action of open string at low momenta is the Dirac-Born-Infeld action

$$L_{DBI} = \sqrt{\det(g_{\mu \nu} + B_{\mu \nu} + F_{\mu \nu})}.$$ 

It was argued in Ref.~10 that one can use an alternative description in terms of non commutative gauge fields where all the dependence on $B$ is encoded in the star product. In fact, non abelian gauge theories in the canonical formalism have an algebra of first class constrains (which generate the gauge group)

$$\{\phi_i, \phi_j\} = c_{ij}^k \phi_k$$  \hspace{1cm} (3)

where $\{\phi_i, \phi_j\}$ is the Poisson bracket. $\{\phi_i\}$ are constrains defining a submanifold on the phase space. The same submanifold can be described with a different set of constrains $\{\phi_i\}$ and the Poisson bracket relations may not be preserved. In particular, it was shown by Batalin adn Fradkin that when the phase space has a finite number of degrees of freedom, the algebra can be brought to be abelian,

$$\{\phi'_i, \phi'_j\} = 0.$$ 

This was called abelization of the gauge algebra. In field theory the abelization can of course introduce a non local change in the fields.

Seiberg and Witten found an explicit change of variables which performs the abelization of the system. If $A$ is the connection field and $\lambda$ the gauge parameter, the transformation is of the form

$$A \rightarrow \hat{A}(A)$$

$$\lambda \rightarrow \hat{\lambda}(A, \lambda),$$
where the new variables are ordinary U(1) gauge fields. In Ref.\textsuperscript{7} we showed that this change of variables is consistent with supersymmetry.

In the limit $\alpha' \to 0$ one can check some properties of the commutative and the non commutative actions. In Ref.\textsuperscript{10}, the Dirac-Born-Infeld action for commutative and non commutative fields were compared. The supersymmetric actions can be compared using the Cecotti-Ferrara\textsuperscript{12} formalism. The action in terms of the non commutative superfields is quadratic in this limit,

$$\int d^4x \int d^2\theta \, \hat{W}_\alpha \ast \hat{W}^\alpha.$$  

It has a non linear fermionic symmetry despite the fact that the action is not free,

$$\delta W_\alpha = \eta_\alpha.$$  

Since this theory is supposed to be equivalent to the supersymmetric Dirac-Born-Infeld theory in this limit, the symmetry of the non commutative action may correspond to the spontaneously broken supersymmetry that appears in the commutative one ($N = 2$ broken to $N = 1$ supersymmetry).

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