Inherit Differential Privacy in Distributed Setting: Multiparty Randomized Function Computation

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Abstract. How to achieve differential privacy in the distributed setting, where the dataset is distributed among the distrustful parties, is an important problem. We consider in what condition a protocol inherit the differential privacy property of a function it computes. The heart of the problem is the secure multiparty computation of randomized function. A notion obliviousness is introduced, which captures the key security problems when computing a randomized function from a deterministic one in the distributed setting. By this observation, a sufficient and necessary condition about computing a randomized function from a deterministic one is given. The above result can not only be used to determine whether a protocol computing differentially private function is secure, but also be used to construct secure one. Then we prove that the differential privacy property of a function can be inherited by the protocol computing it if the protocol privately computes it. A composition theorem of differentially private protocols is also presented. We also construct some protocols to generate random variate in the distributed setting, such as the uniform random variates and the inversion method. By using these fundamental protocols, we construct protocols of the Gaussian mechanism, the Laplace mechanism and the Exponential mechanism. Importantly, all these protocols satisfy obliviousness and so can be proved to be secure in a simulation based manner. We also provide a complexity bound of computing randomized function in the distribute setting. Finally, to show that our results are fundamental and powerful to multiparty differential privacy, we construct a differentially private empirical risk minimization protocol.

Keywords: multiparty differential privacy, random variate generation, secure multiparty computation, randomized function, obliviousness

1 Introduction

Nowadays, a lot of personal information are collected and stored in many databases. Each database is owned by a particular autonomous entity, e.g., financial data by banks, medical data by hospitals, online shopping data by e-commerce companies, online searching records by search engine companies, income data by
tax agencies. Some entities may want to mine useful information among these databases. For example, insurance companies may want to analyze the insurance risk of some group by mining both the bank’s database and the hospital’s database, or several banks may want to aggregate their databases to estimate the loan risk in some area, or, more generally, one may want to learn a classifier among these private databases [1]. However, due to privacy consideration, data integrating or data mining among these databases should be conducted in a privacy-preserving way: First, one must perform computations on database that must be kept private and there is no single entity that is allowed to see all the databases on which the analysis is run; Second, it is not a priori clear whether the analysis results contain sensitive information traceable back to particular individuals [23]. The first privacy problem is the research field of secure MultiParty Computation (MPC) [4]. However, since standard MPC does not analyze and prevent what is (implicitly) leaked by the analysis results [56], the second privacy problem can not be treated by MPC. Fortunately, the second privacy problem could be analyzed by differential privacy (DP) [78], which is a mathematically rigorous privacy model that has recently received a significant amount of research attention for its robustness to known attacks, such as those involving side information [23]. Therefore, solving the above privacy problems needs the combination of MPC and DP as a tool.

There is a misunderstanding that the above problem can easily be solved without using MPC: Each party first locally analyzes and perturbs the local data using the appropriate differentially private algorithm and then outputs the result. These results are then synthesized to obtain the final result. Obviously, the final result satisfies differential privacy. However, the above method will either add more noise to the final result, such as in the noise mechanism [8], or need redesign of the related algorithm, such as in the exponential mechanism [9], which would be a more hard work.

We now present the considered problem in a more formal way. Let a dataset \( x = (x_1, \ldots, x_n) \) be distributed among the mutually distrustful parties \( P_1, \ldots, P_n \), where \( x_i \) is owned by \( P_i \). We call the above dataset owning condition by the distributed setting. The parties want to implement differentially private analyses in the distributed setting by the following way: First choose what to compute, i.e., a differentially private function \( M(x) \); Then decide how to compute it, i.e., construct an MPC protocol to compute \( M(x) \). In the paper we only treat the second step. That is, we assume that there has been a differentially private algorithm \( M(x) \) in the client-server setting. Our task is to construct an MPC protocol \( \pi \) to compute \( M(x) \) in the distributed setting. Furthermore, it is vital that \( \pi \) should ‘inherit’ the differential privacy property of \( M(x) \). That is, in executing \( \pi \), each party’s view (or each subgroup of the parties’ views) should be differentially private to other parties’ private data. However, constructing such protocol is challenging. To see that, we consider two examples appeared in the related works to construct differentially private protocols.

**Example 1 (Gaussian mechanism).** The party \( P_i \) has the math score list \( x_i \) of Class \( i \) for \( i = 1, 2 \). \( P_1, P_2 \) are willing to count the total number of the students
where \( f \) is the counting function. We use Gaussian mechanism to achieve differential privacy, i.e., adding Gaussian noise to \( f(x_1, x_2) \). Note that the sensitivity of \( f \) is \( \Delta f = 1 \). Therefore, we can add a random number \( N \sim \mathcal{N}(0, \sigma^2) \) to achieve \( (\epsilon, \delta) \)-differential privacy, where \( \sigma > \sqrt{2 \ln \frac{1.25}{\delta/\epsilon}} [4] \). There are two intuitive protocols to achieve the task:

1. Each \( P_i \) generates a random number \( N_i \sim \mathcal{N}(0, \sigma^2/2) \) and computes \( o_i = f(x_i) + N_i \) locally. \( P_1, P_2 \) then compute \( o_1 + o_2 \) using an MPC protocol and output the result \( o \). Note that \( o = f(x_1, x_2) + (N_1 + N_2) \) since \( f(x_1, x_2) = f(x_1) + f(x_2) \) and that \( (N_1 + N_2) \sim \mathcal{N}(0, \sigma^2) \) due to the infinitely divisibility of Gaussian noise.

2. Each \( P_i \) generates a random number \( N'_i \sim \mathcal{N}(0, \sigma^2) \) locally. \( P_1, P_2 \) then compute and output \( o = f(x_1) + f(x_2) + \text{LT}(N'_1, N'_2) \) using an MPC protocol, where \( \text{LT}(N'_1, N'_2) \) outputs the smaller one in \( N'_1, N'_2 \).

Intuitively, both of the two protocols in Example 1 satisfy \( (\epsilon, \delta) \)-differential privacy since both of them add noises drawn from \( \mathcal{N}(0, \sigma^2) \) to \( f(x_1, x_2) \). However, to the first protocol, if \( P_1 \) computes \( o - N_1 \) it obtains the value of \( f(x_1, x_2) + N_2 \). Since \( N_2 \sim \mathcal{N}(0, \sigma^2/2) \) but not \( N_2 \sim \mathcal{N}(0, \sigma^2) \), \( P_1 \) obtains an output not satisfying \( (\epsilon, \delta) \)-differential privacy. To the second protocol, either \( N'_1 = \text{LT}(N'_1, N'_2) \) or \( N'_2 = \text{LT}(N'_1, N'_2) \). Without loss of generality, assuming \( N'_1 = \text{LT}(N'_1, N'_2) \), \( P_1 \) can then compute the value of \( o - N'_1 \) to obtain \( f(x_1, x_2) \), which obviously violates differential privacy. A similar protocol, which has the similar drawback as the second protocol, is used to generate Laplace noise in the distributed setting in [10]. Therefore, both of the two protocols in Example 1 do not inherit the \( (\epsilon, \delta) \)-differential privacy property of the function they compute.

**Example 2 (Laplace mechanism).** The same as Example 1, \( P_1, P_2 \) want to output \( f(x_1, x_2) \). In this time, they use Laplace mechanism to achieve differential privacy, i.e., adding Laplace noise to \( f(x_1, x_2) \). Since \( \Delta f = 1 \), they can add a random number \( N \sim \text{Lap}(1/\epsilon) \) to achieve \( \epsilon \)-differential privacy. They construct a protocol as follows: Each party \( P_i \) generates two random numbers \( Y_{i1}, Y_{i2} \) drawn from \( \mathcal{N}(0, 1/2\epsilon) \) locally. The parties then use an MPC protocol to compute \( o = f(x_1, x_2) + N \) and output \( o \), where \( N = \sum_i (Y_{i1}^2 - Y_{i2}^2) \).

The above protocol is shown in [10][11][12]. However, we conclude that it does not inherit the \( \epsilon \)-differential privacy property of the function it computes. The reason is that \( P_1 \) can obtain the value of \( f(x_1, x_2) + (Y_{i1}^2 - Y_{i2}^2) \) by subtracting \( (Y_{i1}^2 - Y_{i2}^2) \) from \( o \). However, since the distribution function of \( (Y_{i1}^2 - Y_{i2}^2) \) is not \( \text{Lap}(1/\epsilon) \) the value of \( f(x_1, x_2) + (Y_{i1}^2 - Y_{i2}^2) \) will not satisfy \( \epsilon \)-differential privacy.

From Example 1 and Example 2 we see that it is difficult to construct a protocol that can inherit the differential privacy property of the function it computes. The crux of the difficulty is that differentially private function is a kind of randomized function, whose output is a random element drawn from a prescribed distribution function (please see Definition 1 in Section 2.2) and that
the result about computing randomized function in MPC is rare. In the paper we will develop some theoretical results about computing randomized function in the distributed setting and then treat the above inheritance problem. Note that differentially private function and random variate are two kinds of randomized function: with constant inputs for the second one.

1.1 Contribution

Our contributions are as follows.

First, we provide a special security definition of computing randomized function in the distributed setting, in which a new notion obliviousness is introduced. Obliviousness captures the key security problems when computing a randomized function from a deterministic one. By this observation, we provide a sufficient and necessary condition (Theorem 3) about computing a randomized function from a deterministic one. The above result can not only be used to determine whether a protocol computing a randomized function (and therefore computing a differentially private function) is secure, but also be used to construct secure one. To the best of our knowledge, ours (Theorem 3) is the first to provide a sufficient and necessary condition about this problem.

Second, we prove that a differentially private algorithm can preserve differential privacy property in the distributed setting if the protocol computing it is secure (Theorem 4), i.e., the inheritance problem. We also introduce the composition theorem of differential privacy in the distributed setting (Theorem 5). To the best of our knowledge, the paper is the first to present these results in differential privacy.

Third, we construct some fundamental protocols to generate random variate in the distributed setting, such as Protocol 4 and Protocol 7. By using these fundamental protocols, we construct protocols of the Gaussian mechanism (Protocol 8), the Laplace mechanism (Protocol 9) and the Exponential mechanism (Protocol 10 and Protocol 11). To the best of our knowledge, Protocol 11 is the first exponential mechanism to treat high-dimensional continuous range in the distributed setting. Importantly, all these protocols satisfy obliviousness and, therefore, can be proved to be secure in a simulation based manner by using the conclusion of Theorem 3. Furthermore, The later four protocols inherit the differential privacy property of the function they compute.

Forth, we provide a complexity bound of multiparty computation of randomized function, which show the intrinsic complexity of the method the paper use to achieve obliviousness, i.e., bits XOR.

Finally, to show that the protocols in Section 5 are powerful and fundamental, we constructed a differentially private empirical risk minimization (ERM) protocol in the distributed setting by using the protocols in Section 5.

1.2 Outline

The rest of the paper is organized as follows: Section 2 briefly reviews the Shamir’s secret sharing scheme, differential privacy definition and non-uniform
random variate generation. Section 3 discusses the security of the protocol computing randomized function. Section 4 mainly discusses how a protocol inherit the differential privacy property of a function it computes. The composition theorem of differentially private protocols is also given. Section 5 constructs some fundamental protocols to generate random variates in the distributed setting. It also provides the Gaussian mechanism, the Laplace mechanism and the exponential mechanism in the distributed setting. Section 5.5 applies the protocols in Section 5 to solve the empirical risk minimization problem. Section 6 presents related works. Finally, concluding remarks and a discussion of future work are presented in Section 7.

2 Preliminary

2.1 Secure Multiparty Computation Framework

MPC enables $n$ parties $P_1, \ldots, P_n$ jointly evaluate a prescribed function on private inputs in a privacy-preserving way. We assume that the $n$ parties are connected by perfectly secure channels in a synchronous network. We employ the $(t, n)$-Shamir’s secret sharing scheme for representation of and secure computation on private values, by using which the computation of a function $f(\cdot)$ can be divided into three stages. Stage I: Each party enters his input $x_i$ to the computation using Shamir’s secret sharing. Stage II: The parties simulate the circuit computing $f(x_1, \ldots, x_n)$, producing a new shared secret $T$ whose value is $f(x_1, \ldots, x_n)$. Stage III: At least $t+1$ shares of $f(x_1, \ldots, x_n)$ are sent to one party, who reconstructs it. All operations are assumed to be performed in a prime field $F_p$. When treating fixed point and floating point number operations, we borrow the corresponding protocols in [13-15]. By using these protocols we can treat the real number operations in a relatively precise way. Therefore, in the paper we assume there are some fundamental real number operations in MPC: addition, multiplication, division, comparison, exponentiation etc. For more formal and general presentation of this approach please see [16-17].

2.2 Differential Privacy

Differential privacy of a function means that any change in a single individual input may only induce a small change in the distribution on its outcomes. A differentially private function is a kind of randomized function. The related definitions follow from the book [7].

**Definition 1 (Randomized Function).** A randomized function $\mathcal{M}$ with domain $A$ and discrete range $B$ is associated with a mapping $\mathcal{M} : A \rightarrow \Delta(B)$, where $\Delta(B)$ denotes the set of all the probability distribution on $B$. On input $x \in A$, the function $\mathcal{M}$ outputs $\mathcal{M}(x) = b$ with probability $(\mathcal{M}(x))_b$ for each $b \in B$. The probability space is over the coin flips of the function $\mathcal{M}$. 
Definition 2 (Differential Privacy [8,7]). A randomized function $M$ gives $(\epsilon, \delta)$-differential privacy if for all datasets $x$ and $y$ differing on at most one element, and all $S \subseteq \text{Range}(M)$,
\[
\Pr[M(x) \in S] \leq \exp(\epsilon) \times \Pr[M(y) \in S] + \delta,
\]
where the probability space is over the coin flips of the function $M$. If $\delta = 0$, we say that $M$ is $\epsilon$-differentially private.

There are mainly two ways to achieve differential privacy: noise mechanism [8] and exponential mechanism [9]. Noise mechanism computes the desired function on the data and then adds noise proportional to the maximum change than can be induced by changing a single element in the data set.

Definition 3 ([7]). The exponential mechanism $M(x, u, \mathcal{R})$ outputs an element $r \in \mathcal{R}$ with probability proportional to $\exp(\frac{u(x, r)}{2\Delta u})$. The Gaussian mechanism $M(x, f)$ generates a random vector $r = (r_1, \ldots, r_n)$, where each $r_i \sim \mathcal{N}(f_i(x), \sigma^2)$, $\sigma > \sqrt{2\ln 1.25/\delta \Delta f/\epsilon}$. The Laplace mechanism $M(x, f)$ generates a random vector $r = (r_1, \ldots, r_n)$, where each $r_i \sim \text{Lap}(f_i(x), \Delta f/\epsilon)$, $\text{Lap}(f_i(x), \Delta f/\epsilon)$ denotes the Laplace distribution with variance $2(\Delta f/\epsilon)^2$ and mean $f_i(x)$.

Both the exponential mechanism and the Laplace mechanism satisfy $\epsilon$-differential privacy. The Gaussian mechanism satisfies $(\epsilon, \delta)$-differential privacy.

Any sequence of computations that each provide differential privacy in isolation also provide differential privacy in sequence.

Lemma 1 (Sequential composition of DP [18]). Let $M_i$ is $(\epsilon_i, \delta_i)$-differentially private. Then their combination, defined to be $M_1 \cdots n(x) = (M_1, \ldots, M_n)$, is $(\sum \epsilon_i, \sum \delta_i)$-differentially private.

Note that Lemma 1 is true not only when $M_1, \ldots, M_n$ are run independently, but even when subsequent computations can incorporate the outcomes of the preceding computations [18].

2.3 Non-Uniform Random variate Generation

Non-uniform random variate generation studies how to generate random variates drawn from a prescribed distribution function. In general, it assumes that there exists a random variate, called it a seed, to generate randomness for the random variates needed.

The Inversion Method [19] is an important method to generate random variates, which is based upon the following property:
Theorem 1. Let $F$ be a continuous distribution function on $\mathbb{R}$ with inverse $F^{-1}$ defined by

$$F^{-1}(u) = \inf\{x : F(x) = u, 0 < u < 1\}.$$ 

If $U$ is a uniform $[0,1]$ random variate, then $F^{-1}(U)$ has distribution function $F$. Also, if $X$ has distribution function $F$, then $F(X)$ is uniformly distributed on $[0,1]$.

Theorem [19, Theorem 2.1] can be used to generate random variates with an arbitrary univariate continuous distribution function $F$ provided that $F^{-1}$ is explicitly known. Formally, we have

Algorithm 1: The inversion method

- **input**: None
- **output**: A random variate drawn from $F$

1. Generate a uniform $[0,1]$ random variate $U$;
2. return $X \leftarrow F^{-1}(U)$.

The Gibbs Sampling [20] is one Markov Chain Monte Carlo (MCMC) algorithm, a kind of algorithms widely used in statistics, scientific modeling, and machine learning to estimate properties of complex distributions. For a distribution function $F$, the Gibbs sampling generates a Markov chain $\{Y_m\}_{m\geq0}$ with $F$ as its stationary distribution.

Let $f(r_1, \ldots, r_k)$ be the density function of $F$ and let $(R_1, \ldots, R_k)$ be a random vector with distribution $F$. For $r = (r_1, r_2, \ldots, r_k)$, let $r_i = (r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_k)$ and $p_i(\cdot | r_i)$ be the conditional density of $R_i$ given $R_i = r_i$. Algorithm 2 generates a Markov chain $\{Y_m\}_{m\geq0}$.

Algorithm 2: The Gibbs sampling algorithm

- **input**: Set the initial values $[R_{0j}] \leftarrow [r_{0j}], j = 1, 2, \ldots, k - 1$
- **output**: A random vector $[Y]$ with density $f(r)$

1. Generate a random variate $[R_{0k}]$ from the conditional density $p_k(\cdot | R_i = r_i, \ell = 1, 2, \ldots, k - 1)$;
2. for $i := 1$ to $m$ do
3. for $j := 1$ to $k$ do
4. Generate a random variate $[R_{ij}]$ from the conditional density $p_j(\cdot | R_{i\ell} = s_{i\ell}, \ell \in \{1, \ldots, k\} \setminus \{j\}$, where $s_{i\ell} = r_{i\ell}$ for $1 \leq \ell < j$ and $s_{i\ell} = r_{(i-1)\ell}$ for $j < \ell \leq k$;
5. The parties output the random vector $[Y_m] = ([R_{m1}], \ldots, [R_{mk}])$. 

2.4 Notations

Throughout the paper, let \([x]\) denote that the value \(x\) is secretly shared among the parties by using Shamir’s secret sharing. Let \(s \sim F\) denote the random variate \(s\) follows the distribution function \(F\).

3 The Security of Computing Randomized Function

In the section, we study the security of computing randomized function in the distributed setting. We focus on in what condition can the computation of a randomized function be reduced to a deterministic one. The results of the section is vital to construct differentially private protocols.

We first give the definition of (statistically) indistinguishability.

Definition 4 (Indistinguishability \cite{4}). Two probability ensembles \(X \triangleq \{X_w\}_{w \in S}\) and \(Y \triangleq \{Y_w\}_{w \in S}\) are called (statistically) indistinguishable, denoted \(X \equiv Y\), if for every positive polynomial \(p(\cdot)\), every sufficiently large \(k\), and every \(w \in S \cap \{0, 1\}^k\), it holds that

\[
\sum_{\alpha \in \{0, 1\}^k} |\Pr[X_w = \alpha] - \Pr[Y_w = \alpha]| < \frac{1}{p(k)}.
\]

The security definition of protocols computing randomized functions mainly follows from \cite{4} Definition 7.5.1.

Definition 5. Let \(\mathcal{M}(x)\) be an \(n\)-ary randomized function and let \(\pi(x)\) be an \(n\) party protocol to compute \(\mathcal{M}(x)\), where \(\mathcal{M}_i(x)\) denotes the \(i\)-th element of \(\mathcal{M}(x)\). The view of the party \(P_i\) during an execution of \(\pi\) on \((x, s)\), denoted \(\text{VIEW}^\pi_i(x, s)\), is \((x_i, s_i, m_1, \ldots, m_i)\), where \(s_i \sim F_i\) is a random variate \(P_i\) inputs, and \(m_j\) represents the \(j\)-th message it has received. The output of \(P_i\) after an execution of \(\pi\) on \((x, s)\), denoted \(\text{OUTPUT}^\pi_i(x, s)\), is implicit in the party’s own view of the execution, and \(\text{OUTPUT}^\pi(x, s) = (\text{OUTPUT}^\pi_1(x, s), \ldots, \text{OUTPUT}^\pi_n(x, s))\).

For \(I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}\), let \(\mathcal{M}_I(x)\) denote the subsequence \(\mathcal{M}_{i_1}(x), \ldots, \mathcal{M}_{i_k}(x)\).

Let \(\text{VIEW}^\pi_I(x, s) \triangleq \{\text{VIEW}^\pi_{i_1}(x, s), \ldots, \text{VIEW}^\pi_{i_k}(x, s)\}\). We say that \(\pi\) privately computes \(\mathcal{M}\) if there exists an algorithm \(S\), such that for every \(I \subseteq \{1, \ldots, n\}\), it holds that

\[
\{S(I, x_I, F_I, \mathcal{M}_I(x)), \mathcal{M}(x)\}_x \equiv \{\text{VIEW}^\pi_I(x, s), \text{OUTPUT}^\pi(x, s)\}_x,
\]

where \(x = (x_1, \ldots, x_n)\), \(x_I = (x_{i_1}, \ldots, x_{i_k})\), \((x, s) = ((x_1, s_1), \ldots, (x_n, s_n))\) and \(F_I = (F_{i_1}, \ldots, F_{i_k})\).

Throughout the paper, we assume that \(\mathcal{M}_1(x) = \cdots = \mathcal{M}_n(x)\) and that \(\text{OUTPUT}^\pi_1(x, s) = \cdots = \text{OUTPUT}^\pi_n(x, s)\). That is, each party obtains the same output.
We remark that the above definition is slightly different from Definition 7.5.1 in [4] that a private random variate \( s = (s_1, \ldots, s_n) \) is input during the execution of \( \pi \). The role of \( s \) is to generate randomness in order to compute \( M(x) \) (since \( M(x) \) is a randomized function). We call \( s \) a seed to compute the randomized function \( M(x) \). By providing the seed \( s \), Definition 5 try to capture the vital characteristic of the process of computing randomized function in the distributed setting, such as Example 1 and Example 2.

We define the notion of private reduction and cite a corresponding composition theorem. We refer the reader to [4,21] for further details.

**Definition 6 (Privacy Reductions).** An oracle aided protocol using an oracle functionality \( f \) privately computes \( M \) if there exists a simulator \( S \) for each \( I \) as in Definition 5. The corresponding views are defined in the natural manner to include oracle answers. An oracle-aided protocol privately reduces \( M \) to \( f \) if it privately computes \( M \) when using oracle functionality \( f \).

**Theorem 2 (Composition Theorem for the Semi-Honest Model[4]).** Suppose \( M \) is privately reducible to \( f \) and there exists a protocol for privately computing \( f \). Then, the protocol defined by replacing each oracle-call to \( f \) by a protocol that privately computes \( f \) is a protocol for privately computing \( M \).

### 3.1 Reducing Computation of Randomized Function to Deterministic One

Given a randomized function \( M \), let \( M(x, s') \) denote the value of \( M(x) \) when using a random seed \( s' \) drawn from a distribution function \( F \). That is, \( M(x) \) is the randomized process consisting of selecting \( s' \sim F \), and deterministically computing \( M(x, s') \). Let \( f \) be a deterministic function such that

\[
f((x_1, s_1), \ldots, (x_n, s_n)) \overset{\text{def}}{=} M(x, g(s)),
\]

where \( g \) is a deterministic function such that \( s' = g(s) \), \( s = (s_1, \ldots, s_n) \) and the random variate \( s_i \sim F_i \). That is, in the distributed setting, we reduce computing the randomized function \( M \) to computing the deterministic function \( f \). In the section, we consider the security problem induced by the reduction.

We now introduce the notion of obliviousness, which is important to privately reduce the computation of randomized function to deterministic one.

**Definition 7 (Obliviousness).** With the notation denoted as Definition 5, the seed \( s \) is said to be oblivious to \( \pi \) if for every \( I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \) and every \( s'_{I_1} \), there is

\[
\{\text{OUTPUT}^s(x, s)|s_I = s'_I\}_x \equiv \{\text{OUTPUT}^s(x, s)\}_x,
\]

where \( s_I = (s_{i_1}, \ldots, s_{i_k}) \) and \( s'_{I_1} \) is one admissible assignment to \( s_I \).

**Lemma 2.** With the notation denoted as Definition 5, if \( s \) is not oblivious to \( \pi \), then \( \pi \) is not secure to compute \( M \).
Proof. Assume that $s$ is not oblivious to $\pi$. There then exist one $I$ and one $s'_I$ such that
\[
\{\text{OUTPUT}^\pi(x,s)|s_I = s'_I\} \not\equiv \{\text{OUTPUT}^\pi(x,s)\}_{x}.
\]
Now imaging the following execution of $\pi$: The parties $P_I$ input fixed value $s'_I$ for $s_I$. For any simulator with input $(x_I,M_I,F_I)$, who does not know $s_I = s'_I$, it is unable to get the distribution function $\{\text{OUTPUT}^\pi(x,s)|s_I = s'_I\}$.

Therefore, there exist one $I$ and one $s'_I$ such that $\{\text{OUTPUT}^\pi(x,s)|s_I = s'_I\}$ is unable to be simulated by any simulator. However, the above distribution function is known to $P_I$ since they know the value $s_I = s'_I$, which implies that $(\text{OUTPUT}^\pi(x,s)|s_I = s'_I) \in \text{VIEW}_I(x,s)$. Therefore, Equation (1) does not hold for $I$ and $s_I = s'_I$, for any simulator. Hence, $\pi$ is not secure to compute $M$. The claim is proved.

Obliviousness, which is a (non trivial) generalization of the notion “obliviously” in [22], says that the seed (of each party or each proper subgroup of the parties) should be independent to the protocol’s output. In other words, the execution of the protocol should be “oblivious” to the seed. One can verify that both the (not secure) two protocols in Example 1 to generate Gaussian noise and the (not secure) protocol in Example 2 to generate Laplace noise do not satisfy the property of obliviousness.

Lemma 2 gives a necessary condition to the security of protocol computing randomized function. Therefore, in order to reducing the computation of a randomized function to deterministic one, the seed should not only be secret among the parties but also be oblivious to the protocol’s output. In the following, we give it a sufficient condition.

Lemma 3. Let $M$, $s$ and $f$ be defined as in Equation (2). Suppose that the following protocol, denoted $\pi$, is oblivious to $s$. Then it privately reduces $M$ to $f$.

**Protocol 3:** privately reducing a randomized function to a deterministic one

| input  | $P_i$ gets $x_i$ |
|--------|------------------|
| output | Each party outputs the oracle’s response |

1. Step 1: $P_i$ selects $s_i \sim F_i$;
2. Step 2: $P_i$ invokes the oracle of $f$ with query $(x_i,s_i)$, and records the oracle response.

Proof. Clearly, this protocol computes $M$. To show that $\pi$ privately computes $M$, we need to present a simulator $S_I$ for each group of parties $P_i, \ldots, P_k$’s view. For notational simplicity, we only prove that there exists a simulator $S_I$ for each party $P_i$. On input $(x_i,v_i)$, where $x_i$ is the local input to $P_i$ and $v_i$ is its local output, the simulator selects $s_i \sim F_i$, and outputs $(x_i,s_i,v_i)$. The
main observation underlying the analysis of this simulator is that for every fixed
\(x = (x_1, \ldots, x_n)\) and \(s'\), we have \(v = \mathcal{M}(x, s')\) if and only if \(v = f(x, s)\), for
every \(s\) satisfying \(s' = g(s)\). Now, let \(\xi_i\) be a random variable representing the
random choice of \(P_i\) in Step 1, and \(\xi'_i\) denote the corresponding choice made by
the simulator \(S_i\). Then, referring to Equation 1, we show that for every fixed \(x, s_i\) and \(v = (v_1, \ldots, v_n)\), it holds that

\[
\Pr[\text{VIEW}^\pi_i(x, s) = (x_i, s_i, v_i) \land \text{OUTPUT}^\pi(x, s) = v] = \Pr[(\xi_i = s_i) \land \text{OUTPUT}^\pi(x, \xi) = f(x, \xi) = v]
\]

\[
= \Pr[(\xi_i = s_i)] \Pr[\mathcal{M}(x) = v]
\]

\[
= \Pr[(\xi'_i = s_i)] \Pr[\mathcal{M}(x) = v]
\]

\[
= \Pr[\xi'_i = s_i] \cap \mathcal{M}(x) = v]
\]

\[
= \Pr[S_i(x_i, F_i, M_i(x)) = (x_i, s_i, v_i) \land \mathcal{M}(x) = v]
\]

where the equalities are justified as follows: the 1st by the definition of \(\pi\), the
2nd by the obliviousness of \(\pi\) to \(\xi\) and the definition of \(f\), the 3rd by definition
of \(\xi_i\) and \(\xi'_i\), the 4th by the independence of \(\xi'_i\) and \(\mathcal{M}\), and the 5th by definition
of \(S_i\). Thus, the simulated view (and output) is distributed identically to the
view (and output) in a real execution.

Similarly, for each group of parties \(P_{i_1}, \ldots, P_{i_k}\)’s view, there exists a simulator
\(S\) such that Equation 1 holds.

The proof is complete.

We remark that the proof technique in Lemma 3 is borrowed from [4, Proposition 7.3.4].

By combining Theorem 2, Lemma 3, and Lemma 3, we have the following
theorem.

**Theorem 3.** Let Equation 2 hold and let \(\pi(x, s)\) be a secure protocol to compute
\(f(x, s)\). Then \(\pi(x, \cdot)\) privately compute \(\mathcal{M}(x)\) if and only if \(\pi(x, s)\) is oblivious
to \(s\).

Theorem 3 holds for differentially private functions since the later is a kind of
randomized functions. Therefore, Theorem 3 gives a necessary and sufficient
condition about how to privately compute differentially private functions. Theorem
3 can not only be used to determine whether a protocol computing differentially
private function is secure, such as the protocols in Example 1 and Example 2
but also be used to construct secure one, such as those protocols in Section 5.

4 Multiparty Differential Privacy

For an \((\epsilon, \delta)\)-differentially private function and a protocol computing it in the
distributed setting, we are willing to see that the protocol has inherited the \((\epsilon, \delta)\)-
differential privacy property of the function it computes. It is intuitive that if the
protocol privately compute the function it will inherit the property naturally. In
the section, we will prove that this is the fact.
We first introduce the notion of differential privacy in the distributed setting, which says that the view of each party (or each subgroup of the parties) is differentially private in respect to other parties’ inputs.

**Definition 8 (Multiparty differential privacy [23]).** Let the notations be denoted as Definition 5. We say that \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) differ on at most one element if there exists \( i_0 \) such that \( x_i = y_i \) for all \( i \in \{1, \ldots, n\} \setminus \{i_0\} \) and that \( x_{i_0}, y_{i_0} \) differ on at most one element. The protocol \( \pi \) is said to be \((\epsilon, \delta)\)-differentially private if for all datasets \( x, y \) differing on at most one element, for all \( S \), and for all \( I \subseteq \{1, 2, \ldots, n\} \setminus \{i_0\} \),

\[
\Pr[(\text{VIEW}_{\pi I}(x, s), \text{OUTPUT}_{\pi I}(x, s)) \in S] \leq \exp(\epsilon) \times \Pr[(\text{VIEW}_{\pi I}(y, s), \text{OUTPUT}_{\pi I}(y, s)) \in S] + \delta.
\]

**Theorem 4.** Assume that \( M \) is an \((\epsilon, \delta)\)-differentially private algorithm and that \( \pi \) is a protocol to privately compute \( M \) in the distributed setting. Then \( \pi \) is \((\epsilon, \delta)\)-differentially private.

**Proof.** For notational simplicity, we only prove the case of \( I \in \{1, 2, \ldots, n\} \). The general case can be treated similarly. We inherit the notations from Definition 8.

Since \( M(x) \) is \((\epsilon, \delta)\)-differentially private, we have

\[
\Pr[M(x) \in \bar{S}] \leq \exp(\epsilon) \times \Pr[M(y) \in \bar{S}] + \delta.
\]

Then for all \( S' \) and for all \( i \in \{1, \ldots, n\} \setminus \{i_0\} \),

\[
\Pr[(x_i, M(x)) \in S'] \leq \exp(\epsilon) \times \Pr[(y_i, M(y)) \in S'] + \delta,
\]

since \( x_i = y_i \).

Therefore, for all (post-processing [7, page 229]) algorithm \( S_i \) and all domain \( S'' \),

\[
\Pr[S_i(x_i, M(x)) \in S''] \leq \exp(\epsilon) \times \Pr[S_i(y_i, M(y)) \in S''] + \delta.
\]

On the other hand, since \( \pi(x) \) is a protocol to privately compute \( M(x) \), there exists an algorithm \( \bar{S}_i \) such that

\[
\{\bar{S}_i(x_i, M(x))\}_x \equiv \{\text{VIEW}_{\pi I}^x(x, s)\}_x.
\]

Combining the last two formulas, we have

\[
\Pr[\text{VIEW}_{\pi I}(x, s) \in S''] \leq \exp(\epsilon) \times \Pr[\text{VIEW}_{\pi I}(y, s) \in S''] + \delta.
\]

Moreover, since \( \text{OUTPUT}_{\pi I}(x, s) \) is implicit in \( \text{VIEW}_{\pi I}(x, s) \) (see Definition 5), the later can be seen as a post-processing of the former. Therefore, for all \( x \),

\[
\Pr[(\text{VIEW}_{\pi I}(x, s), \text{OUTPUT}_{\pi I}(x, s)) \in S] = \Pr[\text{VIEW}_{\pi I}(x, s) \in S']
\]

Inputting Equation (4) into Equation (5), we have Equation (3).

The proof is complete.
The following theorem provides the sequential composition property to differentially private protocols.

**Theorem 5 (Composition theorem).** Assume that the protocol \( \pi_i \) privately computes \((\epsilon_i, \delta_i)\)-differentially private algorithm \( M_i \) for \( 1 \leq i \leq n \). Then their composition, defined to be \( \pi_1 \cdots \pi_n = (\pi_1, \ldots, \pi_n) \), is \((\sum_i \epsilon_i, \sum_i \delta_i)\)-differentially private.

**Proof.** Since each \( \pi_i \) is secure to compute \( M_i \), then their combination \( \pi_1 \cdots \pi_n \) is secure to compute \( M_1 \cdots M_n \) by Theorem 2. By Theorem 4 and Lemma 1 we have \( \pi_1 \cdots \pi_n \) is \((\sum_i \epsilon_i, \sum_i \delta_i)\)-differentially private.

Note that, by Lemma 1 Theorem 5 is true not only when \( \pi_1, \ldots, \pi_n \) are run independently, but even when subsequent computations can incorporate the outcomes of the preceding computations.

## 5 Protocol Construction

In this section, we use the results in Section 3 to construct secure protocols to compute randomized functions. We first design a protocol to generate the uniform random variate and a protocol to implement the inversion method in the distributed setting. Then we construct secure protocols to implement the Laplace mechanism and the exponential mechanism. Importantly, all of these protocols satisfy the property of obliviousness.

Recall that, we let \([x]\) denote that the value \(x\) is secretly shared among the parties by using Shamir’s secret sharing.

### 5.1 Multiparty Inversion Method

We first provide Protocol 4 to generate random variate \( X \) drawn from the Bernoulli \( \text{Bern}(1/2) \) distribution in the distributed setting, where \( X \) takes on only two values: 0 and 1, both with probability 1/2. Protocol 4 uses the fact that the XOR of two Bernoulli \( \text{Bern}(1/2) \) random variates is also a Bernoulli \( \text{Bern}(1/2) \) random variate.

**Protocol 4:** Multiparty generation of Bernoulli \( \text{Bern}(1/2) \) random variate

| input  | None |
|--------|------|
| output | The parties obtain a random variate \([X]\) drawn from \( \text{Bern}(1/2) \) |

1. The party \( P_i \) generates a random bit \( s_i \) drawn from the Bernoulli \( \text{Bern}(1/2) \) distribution by flipping an unbiased coin and shares it among the parties, for \( 1 \leq i \leq n \);

2. The parties compute \([X] \leftarrow \oplus_{i=1}^{n}[s_i]\) and output it, where \( \oplus \) denote XOR operation.
We give Protocol 5 to generate random variate drawn from the standard Gaussian distribution $\mathcal{N}(0, 1)$ in the distributed setting. The protocol approximates the Gaussian distribution $\mathcal{N}(0, 1)$ by using the central limit theorem.

**Protocol 5:** Multiparty generation of Gaussian $\mathcal{N}(0, 1)$ random variate

- **input:** None
- **output:** The parties obtain a random variate $X$ drawn from $\mathcal{N}(0, 1)$

1. The parties generate $k$ independent random variates $[s_1], \ldots, [s_k]$ drawn from the Bernoulli $\text{Bern}(1/2)$ distribution by invoking Protocol 4;
2. The parties compute $Y \leftarrow \sum_{i=1}^{k} [s_i]$;
3. The parties compute $X \leftarrow (Y - k/2)/(\sqrt{k}/2)$.

We now use Protocol 5 to design Protocol 6 to generate random variate drawn from the uniform distribution $U(0, 1)$ in the distributed setting. Protocol 6 uses the result in Theorem 1.

**Protocol 6:** Multiparty generation of Uniform $U(0, 1)$ random variate

- **input:** None
- **output:** The parties obtain a random variate $X$ drawn from $U(0, 1)$

1. The parties generate a random variate $\xi$ drawn from $\mathcal{N}(0, 1)$ by using Protocol 5;
2. The parties compute $X \leftarrow \lfloor G(\xi) \rfloor$, where $G(x)$ is the distribution function of $\mathcal{N}(0, 1)$. Note that $\lfloor G(\xi) \rfloor = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{0}^{\xi} \exp \left( -\frac{t^2}{2} \right) dt$ where the second summand can be evaluated as follows by using the composite trapezoidal method:

   - Set $f(t) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{t^2}{2} \right)$ and $Y = \int_{0}^{\xi} f(t) dt$.
   - The parties negotiate a step length $h$ and a positive integer $k$ such that $kh = 1$;
   - Each party computes $t_i = hi$ for $i \in \{0, \ldots, k\}$ separately;
   - The parties compute $[t'_i] = t_i[\xi]$ for $i \in \{0, \ldots, k\}$;
   - The parties compute $f([t'_i])$ for $i \in \{0, \ldots, k\}$;
   - The parties compute $[Y] \leftarrow h[\xi]([f(0) + f([\xi])] + 2 \sum_{i=1}^{k-1} f([t'_i]))/2$.

The inversion method presented in Algorithm 1 is an important method to generate univariate random variable. We now give its new edition in the distributed setting as shown in Protocol 7. Protocol 7 is a powerful and fundamental protocol to construct other complex protocols, such as protocols of the Gaussian mechanism, the Laplace mechanism and the exponential mechanism as shown in the followings.
Protocol 7: Multiparty inversion method

**input**: The univariate continuous distribution function \(F(t)\)

**output**: The parties obtain a random variate \([X]\) drawn from \(F(t)\)

1. The parties generate a random number \([\xi]\) drawn from \(U(0, 1)\) by using Protocol 6;
2. The parties compute \(X \leftarrow F^{-1}([\xi])\).

Note that \(F^{-1}([\xi])\), if it has an explicit expression, can be computed by using the non-decreasing property of \(F(t)\) and the comparison operator. When \(F^{-1}([\xi]) = t\) only has implicit integral expression, i.e., \([\xi] = \int_{-\infty}^{t} f(s)ds\), it can be computed as follows.

1. The parties compute \([\xi'] \leftarrow [\xi] - \int_{0}^{\xi} f(s)ds = \int_{\xi}^{t} f(s)ds\);
2. The parties choose two values \([a], [b]\) such that \(\int_{a}^{0} f(s)ds \leq [\xi'] \leq \int_{b}^{0} f(s)ds\) by using the non-decreasing property of \(f(s)ds\) and the comparison operator;
3. The parties evaluate \([t]\) in the equation \([\xi'] = \int_{0}^{t} f(s)ds\) by using the bisection method [24] over the initial interval \([a, b]\);
4. The parties set \([X] \leftarrow [t]\);

5.2 Multiparty Differentially Private Protocols

We now use the protocols of generating random variates to construct protocols of the Laplace mechanism and the exponential mechanism.

Multiparty Gaussian Mechanism We give Protocol 5 to generate random variate drawn from the Gaussian distribution \(N(f(x), \sigma^2)\) in the distributed setting, which achieves Gaussian mechanism. The protocol approximates the Gaussian distribution \(N(0, \sigma^2)\) by using the central limit theorem [20, Corollary 11.1.3].

Protocol 8: Multiparty Gaussian Mechanism

**input**: Each party \(P_i\) shares his input \(x_i\) among the parties

**output**: The parties obtain a random variate random vector \([X] = ([X_1], \ldots, [X_n])\) drawn from \(\prod_{j=1}^{n} N(f_j(x), \sigma^2)\)

1. for \(j := 1 \text{ to } n\) do
   2. The parties generate \(k\) independent random variates \([s_1], \ldots, [s_k]\) drawn from the Bernoulli \(Bern(1/2)\) distribution by invoking Protocol 4;
   3. The parties compute \([Y_j] \leftarrow \sigma [s_i]\) for \(1 \leq i \leq k\);
   4. The parties compute \([Y] \leftarrow \sum_{i=1}^{k} [Y_j]/k\);
   5. The parties compute \([X_j] \leftarrow \sqrt{k}(Y - \sigma/2)\);
   6. The parties set \([X_j] \leftarrow [f_j(x)] + [X_j]\).
**Multiparty Laplace Mechanism** The Laplace mechanism in the distributed setting is shown in Protocol 9.

### Protocol 9: Multiparty Laplace Mechanism

**input**: Each party $P_i$ secretly shares his input $x_i$ among the parties

**output**: The parties obtain a random vector $[X] = ([X_1], \ldots, [X_n])$ drawn from $\prod_{j=1}^n \text{Lap}(f_j(x), \Delta f/\epsilon)$

1. For $j := 1$ to $n$
   1. The parties generate a random variate $[\xi_j]$ drawn from $\text{Lap}(\Delta f/\epsilon)$ by using Protocol 7($F(t)$), where $F(t) = \frac{\epsilon^2}{2\Delta f} \int_{-\infty}^t \exp\left(-\frac{\epsilon|s|}{\Delta f}\right)ds$;
   2. The parties set $[X_j] \leftarrow [f_j(x)] + [\xi_j]$.

**Multiparty Exponential mechanism** When the range $\mathcal{R}$ is a finite set, we set $\mathcal{R} = \{r_1, \ldots, r_{|\mathcal{R}|}\}$. The aim of the exponential mechanism is to draw a random element $r \in \mathcal{R}$ with probability $\exp\left(\frac{\epsilon u(x, r)}{2\Delta u}\right)$. Protocol 10 achieves the aim whose main idea is the sequential search algorithm in [19, page 85]. In Protocol 10 the comparison function $\text{LT}([S], [\xi]) = 1$ if $S < \xi$ and $\text{LT}([S], [\xi]) = 0$ if $S \geq \xi$.

### Protocol 10: Multiparty discrete Exponential mechanism

**input**: Each party $P_i$ secretly shares his input $x_i$ among the parties

**output**: The parties obtain a random variate $[X]$ on $\mathcal{R}$ with probability mass function $\exp\left(\frac{\epsilon u(x, R)}{2\Delta u}\right)$, where $x = (x_1, \ldots, x_n)$ and $\mathcal{R} = \{1, 2, \ldots, |\mathcal{R}|\}$

1. The parties compute $[p_i] \leftarrow \left[\exp\left(\frac{\epsilon u(x, R)}{2\Delta u}\right)\right]$ for each $i \in \mathcal{R}$;
2. The parties generate a random variate $[U]$ drawn from $U(0, 1)$ by using Protocol 6;
3. The parties compute $[\xi] \leftarrow [U] \times \left[\sum_{i \in \mathcal{R}} p_i\right]$;
4. The parties set $[X] \leftarrow 1$, $[S] \leftarrow [p_1]$;
5. For $k := 2$ to $|\mathcal{R}|$ do
   6. The parties compute $[X] \leftarrow [X] + \text{LT}([S], [\xi])$;
   7. The parties compute $[S] \leftarrow [S] + [p_i]$;
8. The parties output $[X]$.

When $\mathcal{R}$ is a set of high dimensional continuous random vectors, we can use the Gibbs sampling algorithm.

Let $\mathcal{R} = \{(r_1, \ldots, r_k) : r_i \in R \text{ for } 1 \leq i \leq k\}$. Setting $\alpha = \int_{r \in \mathcal{R}} \exp\left(\frac{\epsilon u(x, r)}{2\Delta u}\right)dr$, then $f(r) = \frac{1}{\alpha} \exp\left(\frac{\epsilon u(x, r)}{2\Delta u}\right)$ is a density function on $\mathcal{R}$. The Gibbs sampling method generates a Markov chain $\{R_m\}_{m \geq 0}$ with $f(r)$ as its stationary density. Let
\[ p_i(\cdot | r_{(i)}) = \frac{f(r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_k)}{\int_{x \in \mathbb{R}} f(r_1, r_2, \ldots, r_{i-1}, x, r_{i+1}, \ldots, r_k) \, dx}. \]

Note that \( p_i(\cdot | r_{(i)}) \) is a univariate density function. Protocol 11 outputs a random vector \( R \) drawn from the density \( f(r) \), which uses the multiparty edition of Algorithm 2.

### Protocol 11: The multiparty high dimensional exponential mechanism

**input**: Each party \( P_i \) secretly shares his input \( x_i \) among the parties; The parties obtain the initial values \( [R_{0j}] \leftarrow [r_0], j = 1, 2, \ldots, k-1 \)

**output**: A random vector \( R \) drawn from \( f(r) \)

1. The parties generate a random variate \( [R_{0j}] \) from the conditional density \( p_j(\cdot | X_\ell = r_{0\ell}, \ell = 1, 2, \ldots, k-1) \);
2. for \( i := 1 \) to \( m \) do
   3. for \( j := 1 \) to \( k \) do
      4. The parties generate a random variate \( [R_{ij}] \) from the conditional density \( p_j(\cdot | X_\ell = s_{i\ell}, \ell \in \{1, \ldots, k\} \setminus \{j\}) \) by using Protocol 7 where \( s_{i\ell} = r_{i\ell} \) for \( 1 \leq \ell < j \) and \( s_{i\ell} = r_{i(\ell-1)} \) for \( j < \ell \leq k \);
5. The parties output the random vector \( [R_m] = ([R_{m1}], \ldots, [R_{mk}]). \)

### 5.3 Security and Privacy Analysis

The security of the protocols in Section 5 can be analyzed by Lemma 3. Given a randomized function \( M \), we first select \( s' \sim F \), and deterministically compute \( M(x, s') \). Let \( f \) be a deterministic function satisfying Equation 2. By Lemma 3 if there is a protocol \( \pi \) privately computing \( f \) and that \( \pi(\cdot, s) \) is oblivious to \( s \sim F \), then \( \pi(x, \cdot) \) is secure to compute \( M(x) \) by inputting the seed \( s \sim F \).

Since the paper focuses on computing randomized functions in the distributed setting and in order to keep the readability, the protocols in the section are presented in an algorithmic manner but not explicitly presented in the mathematical operations like [15]. The involved sub-protocols to compute some fundamental operations, e.g., addition, multiplication, XOR, comparison and exponentiation etc., and the sub-protocols to compute some fundamental algorithms, e.g., the bisection method and the composite trapezoidal method, can be achieved by the works in [14,13,15], which would be as one future work. Therefore, in the paper, we assume that each deterministic function can be privately computed. Hence, to prove the security of the protocols in Section 5 we only need to prove the correctness and obliviousness of these protocols.

**Semi-Honest Model Obliviousness**: All the protocols in Section 5 satisfy the property of obliviousness. This is because of seeds needed in these protocols.
are all input through invoking Protocol 4. However, it can be easily verified that Protocol 4 satisfies the property of obliviousness. Therefore, other protocols inherit the obliviousness of Protocol 4.

Correctness: Protocol 4 is due to the fact that the XOR of two Bernoulli Bern(1/2) random variates is also a Bernoulli Bern(1/2) random variate. Therefore, $\oplus_{i=1}^n b_i$ is a Bern(1/2) random variate since each $b_i$ is a Bern(1/2) random variate. The correctness of Protocol 5 is due to the central limit theorem [20, Corollary 11.1.3]. The correctness of Protocol 6 is due to Theorem 1: If the random variate $\xi$ is drawn from $N(0, 1)$, then $G(\xi)$ is drawn from $U(0, 1)$, where $G(x) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{x} \exp \left(-\frac{t^2}{2}\right) dt$ is the distribution function of $N(0, 1)$. The correctness of Protocol 7 is due to the classical inversion method, i.e., Algorithm 1. In Protocol 8, Step 2 to Step 5 generate a random variate $X' \sim N(0, \sigma^2)$ by using the central limit theorem [20, Corollary 11.1.3]. Then $f_j(x) + X' \sim N(f_j(x), \sigma^2)$. Protocol 9 is due to the Laplace mechanism in Definition 3. Protocol 10 is due to the sequential search algorithm in [19, page 85]. The correctness of Protocol 11 uses the correctness of Algorithm 2.

Corollary 1. Protocol 8 is $(\epsilon, \delta)$-differentially private. Protocol 9, Protocol 10 and Protocol 11 are all $\epsilon$-differentially private.

Proof. This is a direct corollary of Theorem 4.

Malicious Model By forcing parties to behave in an effectively semi-honest manner, we can transform the above protocols in the semi-honest model into protocols secure in the malicious-behavior model. The above process needs some preliminaries: the commitment schemes, zero-knowledge proof techniques and the Verifiable Secret Sharing (VSS) scheme. In the paper we do not intend to give it a detailed construction but as a future work. Besides of these, we consider the malicious behavior in computing seeds. Seeds are generated bit by bit by invoking Protocol 4 in which a malicious party may input either a non-bit random element or a non-uniform random bit. The first malicious behavior can be avoided by verifying the input $x$ satisfies $x^2 = x$. The second malicious behavior can be solved by first generating a public random variate drawn from Bern(1/2) and then XOR it with the output of Protocol 4 by the fact that the XOR of two random bits is uniform so long as one of which is uniform.

5.4 Optimal Complexity

By Section 3, each party $P_i$ should input a seed $s_i$, a random variate, to the protocol $\pi$ for computing a randomized function $M$ to generate the randomness of the final output. We call $s = (s_1, \ldots, s_n)$ a seed of $\pi$ for computing $M$ and call the number of bit in $s$, denoted $|s|$, the length of $s$. Each protocol in Section 5 takes independent random bit sequence as its seed. Note that the length of the seed is an important indicator of the complexity of the protocol, the minimum length of the seed is of special interest.
We now discuss the minimum length of the seeds of all the protocols for generating independent random bits.

**Theorem 6.** Let \( \pi \) be a protocol to privately compute the randomized function \( \mathcal{M} \) of generating random vector \( v = (v_1, \ldots , v_k) \), where \( v_1, \ldots , v_k \sim \text{i.i.d} \ Bern(1/2) \). Let \( s_i = (s_{i1}, \ldots , s_{i\ell_i}) \) be the seed of \( P_i \), where each \( s_{ij} \) denotes a bit. Then \( \ell_i \geq k \) for \( 1 \leq i \leq n \).

Therefore, the protocol \( \pi' \) of independent \( k \) times execution of Protocol 4 has the shortest seed among all the protocols for privately computing \( \mathcal{M} \).

**Proof.** Let notations be denoted as Equation 2 and Definition 7. Since \( \pi \) should satisfy obliviousness, then for each \( i \in \{1, \ldots , n\} \) and each admissible value \( s_{i\bar{i}} \) of \( s_{\bar{i}} \), we have

\[
\{ \text{OUTPUT}^\pi (x, s) \mid s_{\bar{i}} = s_{\bar{i}}' \} \equiv \{ \text{OUTPUT}^\pi (x, s) \}.
\]

where \( s_{\bar{i}} = (s_{1}, \ldots , s_{i-1}, s_{i+1}, \ldots , s_{n}) \).

For the \((n + 1)\)-ary deterministic function \( \mathcal{M}(x, s) \), let \( g_i(s_i) := \mathcal{M}(x, s|\bar{x} = x', s_i = s_{i}') \) denote the univariate deterministic function about \( s_i \) when \( x = x' \), \( s_i = s_{i}' \). Then \( g_i : \{0, 1\}^{\ell_i} \to \{0, 1\}^k \). Assume that \( \ell_i < k \) and, without loss of generality, set \( \ell_i = k - 1 \). Set \( S = \{ g_i(y) : y \in \{0, 1\}^{k-1} \} \). Since \( |S| \leq 2^{k-1} < 2^k \), there would have at least \( 2^k - 1 \) elements of \( \{0, 1\}^k \) not contained in \( S \). Letting \( g_i(s_i) = (Y_1 \cdots Y_k) \), where \( Y_1, \ldots , Y_k \sim \text{i.i.d} \ Bern(1/2) \), for any \( k \)-bit sequence \( y_1 \cdots y_k \notin S \), we have

\[
\prod_{j=1}^{k} \Pr[Y_j = y_j] = \Pr[Y_1 \cdots Y_k = y_1 \cdots y_k] = 0.
\]

Therefore, there exists one \( j \in \{1, \ldots , k\} \), such that \( \Pr[Y_j = y_j] = 0 \), which is contrary to the assumption that \( (Y_1 \cdots Y_k) \) are random bits. Therefore, \( \ell_i \geq k \) for all \( 1 \leq i \leq n \).

On the other hand, the length of the seed of \( \pi' \) is \( nk \). Therefore, it has the shortest seed among all the protocols privately compute \( \mathcal{M} \).

The claim is proved.

**Theorem 6** shows one intrinsic bound on optimizing the complexity of those protocols for computing randomized functions by invoking Protocol 4 and shows that our protocols in the section reach the bound.

### 5.5 Application to Empirical Risk Minimization

Our protocols are fundamental and powerful to construct other complex differentially private protocols. We now use our protocols to construct a differentially private empirical risk minimization (ERM) protocol in the distributed setting.

We consider a differentially private (ERM) algorithm [25, Algorithm 1]. For Algorithm 1 in [25], we can add a noise vector to the output of \( \arg \min_f \mathcal{M}(f, \mathcal{D}) \)
in order to achieve differential privacy, where
\[
J(f, D) = \frac{1}{k} \sum_{i=1}^{k} \ell(f(x_i), y_i) + AN(f).
\]
If the added noise vector \(b\) is drawn from \(\frac{1}{\alpha} \exp(-\lambda ||b||)\) the output satisfies differential privacy, where \(\lambda = \frac{2\epsilon}{\sqrt{\alpha}}\).

In the distributed setting, let the dataset \(D = \{(x_j, y_j)\}\) is partitioned into \(n\) parts \(D_1, \ldots, D_n\), where the party \(P_i\) owns \(D_i\). Each party \(P_i\) first shares its dataset \(D_i\) among the parties. Then the parties approximately compute a share \([f]\) of the minimizer of \(J(f, ([D_1], \ldots, [D_n]))\) by using a deterministic function evaluation protocol. (Since the paper focuses on randomized function evaluation protocols, we omit to construct the protocol of computing \([f]\).) The parties now use Protocol 12 to generate a random vector \([X]\) drawn from \(\frac{1}{\alpha} \exp(-\lambda ||b||)\), where Protocol 12 is a multiparty edition of the polar method in [19, page 225]. The parties then compute \([X + f]\). Finally, the parties recover and output \(X + f\), which would be a differentially private ERM in the distributed setting.

**Protocol 12:** Multiparty generation of random variate drawn from 
\[
f(x_1, \ldots, x_d) = \frac{1}{\alpha} e^{-\lambda\sqrt{x_1^2 + \cdots + x_d^2}}
\]

| input | output: A random variate \([X]\) drawn from \(f(x_1, \ldots, x_d) = \frac{1}{\alpha} e^{-\lambda\sqrt{x_1^2 + \cdots + x_d^2}}\) |
|-------|---------------------------------------------------------------|
| 1     | The parties generate i.i.d normal randoms \([N_1], \ldots, [N_d]\) by Protocol 5 |
| 2     | The parties compute a share of random vector \([X']\) drawn from \(f(x_1, \ldots, x_d) = \frac{1}{\alpha} e^{-\lambda\sqrt{x_1^2 + \cdots + x_d^2}}\), where \(S \leftarrow \sqrt{N_1^2 + \cdots + N_d^2}\) |
| 3     | The parties generate a random variate \([R]\) drawn from the density \(dV_{d/2}^{d-1} g(r)\) \((r \geq 0)\) by using Protocol 7, where \(V_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}\), and \(g(x) = \frac{1}{\alpha} e^{-\lambda x}\) |
| 4     | The parties compute \([X]\) drawn from \([X']\). |

### 6 Related Work

Secure multiparty computation [14][13][21][1] studies how to privately compute functions in the distributed setting. The computation of randomized function, such as random variate generation, is seldom studied in MPC. Until recently, the development of DP in the distributed setting makes the study of the computation of randomized functions necessary in MPC. Except the works mentioned in Section 1, other former works are presented as follows.

Proposition 7.3.4 in [4] privately reduces computing randomized function to a deterministic one. However, it does not give criterion about what kind of seed, which is used to generates the randomness, is secure. That is, the criterion for how to determine a protocol computing a randomized function is secure is not
given. Our conclusion gives a sufficient and necessary condition (Theorem 3) about it and therefore gives the criterion, i.e., obliviousness. Furthermore, obliviousness gives some clue on finding more (efficient) reduction protocols except the one in [3, Proposition 7.3.4]. Note also that the randomized functions the paper considers are confided to be \( n \)-ary functions having the same value for all components.

The notion of obliviousness can be seen as a (non-trivial) generation of the notion Obliviously in [22]. However, they have one major difference: Obliviously emphasises on the independence of the seed to the execution of the protocol computing the randomized function, where as obliviousness focus on the independence of the seed to the output of the protocol computing the deterministic function, to which the randomized function is privately reduced. The advantage of the later is that it separates the choosing of the seed from the execution of the protocol computing the deterministic function, which makes the design and the analysis of the protocol computing randomized function easy to do.

[26] gives two protocols to generate Gaussian random variate and Laplace random variate in the distributed setting, which are used to compute differentially private summation functions. Although Protocol 5 in our paper is similar with the one in [26] to generate Gaussian random variate, our work focus mainly on the fundamental theory and fundamental tools to compute randomized functions in the distributed setting and is therefore different from theirs.

Random Value Protocol [22] is a two-party protocol to generate uniform random integers from \( Z_N \) while keeping \( N \) secret, which is used to approximately generate uniform random variate [11,27] following \( U(0,1) \) in the two-party setting. It satisfies obliviousness but is too complicated that we can not see a way to extend it to a multiparty one. Furthermore, the distributed exponential mechanism protocols in [11,27] are two special instantiations of Protocol 10.

[15] presents a protocol to implement exponential mechanism, in which a sub-protocol is needed to generate uniform random variate drawn from the uniform distribution \( U(0,1) \). In order to generate such uniform random variate, the parties first secretly generate a uniform \((\gamma + 1)\)-bit integer using the protocol RandInt(\( \gamma + 1 \)). Then this integer is considered to be fractional part of fixed point number, whose integer part is 0. Afterwards, the fixed point number is converted to floating point by a secure protocol, which is output as the final result. The above protocol to generate uniform random number has two drawbacks. First, the invoked protocol RandInt(\( \gamma + 1 \)), borrowed from [28,29], generates a uniform random element in \( Z_{\gamma} \) by the modular sum of the uniform random elements in \( Z_p \) generated by each of the parties. Note that the modular sum of two uniform random elements in \( Z_p \) is, in general, not a uniform random elements in \( Z_p \) [22]. Therefore, RandInt(\( \gamma + 1 \)) (most probably) generates a non-uniform random \((\gamma + 1)\)-bit integer, which in turn leads to the non-uniformity of the one in [15]. Second, since \( \gamma \) is predetermined, the random number generated may not get value from many sub-intervals of \([0,1]\), such as the sub-interval \((0, 2^{-\gamma - 1})\). Therefore, strictly speaking, the above method may not generate a
random number with range $[0, 1]$. Of course, the uniform property in the range $[0, 1]$ of the generated random number will be not satisfied.

[23, 30] studies the accuracy difference in computing Boolean functions between the client-server setting and the distributed setting. [31] introduces the notion of computational differential privacy in the two-party setting. [5] studies the influence to the accuracy of computing binary sum, gap threshold etc., when both of differentially private analyses and the construction of protocol are considered simultaneously, which is contrary to the paradigm we use in which we first analyze a problem using differentially private algorithm and then construct corresponding protocol to compute it.

Differential privacy is a rigorous and promising privacy model. Much works have been done in differentially private data analysis [32, 33, 34, 25, 35, 36, 37, 38, 39]. Our work tries to extend these algorithms to the distributed setting. It constructs fundamental theory, such as Theorem 4 and Theorem 5, and fundamental tools, such as the protocols in Section 5, about it.

Non-uniform Random variate generation [19] is a well developed field in computer science and statistics. It studies how to generate non-uniform random variate drawn from the prescribed distribution function. Some work of the paper studies secure random variate generation in the distributed setting. It redesigns the traditional random variate generation protocols to adapt to the distributed setting. Note that most powerful algorithms, such as the rejection method, are not fit for the distributed setting.

7 Conclusion and Future Work

The paper tried to answer in what condition can a protocol inherit the differential privacy property of a function it computes and how to construct such protocol. We proved that the differential privacy property of a function can be inherited by the protocol computing it if the protocol privately computes it. Then a theorem provided the sufficient and necessary condition of privately computing a randomized function (and so differentially private function) from a deterministic one. The above result can not only be used to determine whether a protocol computing differentially private function is secure, but also be used to construct secure one. In obtaining these results, the notion obliviousness plays a vital role, which captures the key security problems when computing a randomized function from a deterministic one in the distributed setting. However, we can not prove the assertion that a protocol can not inherit the differential privacy property of the function it computes if the protocol does not satisfy obliviousness. We tend to a negative answer to the assertion.

The theoretical results in Section 3 and Section 4 is fundamental and powerful to multiparty differential privacy. By using these results, some fundamental differentially private protocols, such as protocols for Gaussian mechanism, Laplace mechanism and Exponential mechanism, are constructed in Section 5. By using these fundamental protocols, differentially private protocols for many complex problems, such as the empirical risk minimization problem, can be constructed.
with little effort. Therefore, our results can be seen as a foundation and a pool of necessary tools for multiparty differential privacy.

Furthermore, obliviousness is of independent interest to MPC. The deep meaning of it in the security of computing randomized function is still needed to be explored. Theorem 6 shows the intrinsic complexity of the method the paper use to achieve obliviousness, i.e., bits XOR. Finding other efficient method to achieve obliviousness is therefore an important topic to reduce protocols’ complexity.

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