TIME-ENERGY UNCERTAINTY RELATIONS FOR NEUTRINO OSCILLATIONS AND THE MÖSSBAUER NEUTRINO EXPERIMENT

S. M. Bilenky
Joint Institute for Nuclear Research, Dubna, R-141980, Russia

F. von Feilitzsch and W. Potzel
Physik-Department E15, Technische Universität München, D-85748 Garching, Germany

Abstract

Using the Mandelstam-Tamm method we derive time-energy uncertainty relations for neutrino oscillations. We demonstrate that the small energy uncertainty of antineutrinos in a recently considered experiment with recoilless resonant (Mössbauer) production and absorption of tritium antineutrinos is in conflict with the energy uncertainty which, according to the time-energy uncertainty relation, is necessary for neutrino oscillations to happen. Oscillations of Mössbauer neutrinos would indicate a stationary phenomenon where the evolution of the neutrino state occurs in space rather than in time. A Mössbauer neutrino experiment could provide a unique possibility to reveal the true nature of neutrino oscillations.

1 Introduction

The observation of neutrino oscillations in the Super-Kamiokande atmospheric [1], SNO solar [2], KamLAND reactor [3] and other neutrino experiments [4, 5, 6, 7, 8, 9] is one of the most important recent discoveries in particle physics. Small neutrino masses can not naturally be explained by the Standard Higgs mechanism. Their explanation requires a new mechanism of neutrino mass generation beyond the Standard Model. At present, the see-saw mechanism [10], which is based on the assumption of a violation of the total lepton number at a scale which is much larger than the electroweak scale, is considered as the most plausible mechanism of neutrino mass generation. The see-saw mechanism requires for neutrinos with definite
masses to be Majorana particles. The discovery of the neutrino-less double
$\beta$-decay (see [11]), which is allowed only if massive neutrinos are Majorana
particles, would be a strong evidence in favor of the see-saw idea.

Existing neutrino oscillation data are perfectly described if we assume the
three-neutrino mixing (see [12, 13])

$$\nu_{lL}(x) = \sum_{i=1}^{3} U_{li} \nu_{iL}(x) \quad (l = e, \mu, \tau) .$$

(1)

Here $\nu_{lL}(x)$ is the field of the flavor neutrino $\nu_l$, $\nu_i(x)$ is the field of neutrinos
with mass $m_i$ and $U$ is the $3 \times 3$ unitary PMNS \cite{14, 15} mixing matrix which
is characterized by three mixing angles $\theta_{12}, \theta_{23}$ and $\theta_{13}$ and the $CP$ phase $\delta$.

The standard expression for the probability of the transition $\nu_l \rightarrow \nu_{l'}$ has
the form

$$P_{\nu_l \rightarrow \nu_{l'}} = \left| \sum_{i=1}^{3} U_{l'i} e^{-i \Delta m^2_{ll'} \frac{L}{E}} U_{li}^* \right|^2 .$$

(2)

Here $E$ is the neutrino energy, $L$ is the distance between the neutrino source
and the neutrino detector and $\Delta m^2_{kl} = m_i^2 - m_k^2$.

The probability $P(\nu_l \rightarrow \nu_{l'})$ depends on six parameters (two mass-squared
differences, three angles and one phase). From the analysis of the data of
neutrino oscillation experiments follows, however, that the parameter $\Delta m^2_{12}$
is much smaller than $\Delta m^2_{23}:

$$\Delta m^2_{12} \simeq 3 \cdot 10^{-2} \Delta m^2_{23} .$$

(3)

Thus, at $\Delta m^2_{23} \frac{L}{2E} \gtrsim 1$, i.e., in the atmospheric and accelerator long-baseline
(LBL) region, in first approximation we can neglect the small contribution
of $\Delta m^2_{12}$ to the transition probabilities. In this case, the probability of $\nu_l$ to
survive takes the two-neutrino form (see [12])

$$P_{\nu_l \rightarrow \nu_l} = P_{\bar{\nu}_l \rightarrow \bar{\nu}_l} = 1 - \frac{1}{2} B_{ll} \left( 1 - \cos \Delta m^2_{23} \frac{L}{2E} \right) .$$

(4)

Here

$$B_{ll} = 4 |U_{13}|^2 \left( 1 - |U_{13}|^2 \right)$$

(5)

is the amplitude of the oscillations.

From (4) we find the following expression for the $\bar{\nu}_e \rightarrow \bar{\nu}_e$ survival prob-
ability

$$P_{\bar{\nu}_e \rightarrow \bar{\nu}_e} = 1 - \frac{1}{2} \sin^2 2\theta_{13} \left( 1 - \cos \Delta m^2_{23} \frac{L}{2E} \right) .$$

(6)
In the reactor experiment CHOOZ\[16\] no indications for neutrino oscillations driven by $\Delta m^2_{23}$ were found. From the exclusion plot obtained from the data of this experiment, the following upper bound was found for the parameter $\sin^2 \theta_{13}$

$$\sin^2 \theta_{13} \lesssim 5 \cdot 10^{-2}.$$  \hspace{1cm} (7)

Neglecting the small contribution of $\sin^2 \theta_{13}$, we find from (4) and (5) in the atmospheric-LBL region of $L/E$ the following expression for the probability of $\nu_\mu$ to survive:

$$P_{\nu_\mu \rightarrow \nu_\mu} \simeq 1 - \frac{1}{2} \sin^2 2\theta_{23} (1 - \cos \Delta m^2_{23} \frac{L}{2E}).$$  \hspace{1cm} (8)

In the KamLAND region ($\Delta m^2_{12} L/E \gtrsim 1$) the probability of the transition $\bar{\nu}_e \rightarrow \bar{\nu}_e$ is given by (see [17, 12])

$$P_{\bar{\nu}_e \rightarrow \bar{\nu}_e} = |U_{e3}|^4 + (1 - |U_{e3}|^2)^2 \frac{1}{2} \sin^2 2\theta_{12} (1 - \cos \Delta m^2_{12} \frac{L}{2E}).$$  \hspace{1cm} (9)

Thus, in the approximation $|U_{e3}|^2 \rightarrow 0$ we have

$$P_{\bar{\nu}_e \rightarrow \bar{\nu}_e} \simeq 1 - \frac{1}{2} \sin^2 2\theta_{12} (1 - \cos \Delta m^2_{12} \frac{L}{2E}).$$  \hspace{1cm} (10)

The expressions (8) and (10) are widely used in the analysis of the neutrino oscillation data. From the analysis of the data of the atmospheric Super-Kamiokande experiment, the following 90% CL ranges were obtained for the parameters $\Delta m^2_{23}$ and $\sin^2 2\theta_{23}$ [1]

$$1.9 \cdot 10^{-3} \leq \Delta m^2_{23} \leq 3.1 \cdot 10^{-3}\text{eV}^2, \quad \sin^2 2\theta_{23} > 0.9.$$  \hspace{1cm} (11)

The following best-fit values of the parameters were found in [1]

$$\Delta m^2_{23} = 2.5 \cdot 10^{-3}\text{eV}^2, \quad \sin^2 2\theta_{23} = 1.$$  \hspace{1cm} (12)

The results of the Super-Kamiokande experiment have perfectly been confirmed by the accelerator K2K [8] and MINOS [9] long baseline neutrino oscillations experiments. The analysis of the MINOS data gave the result [9]:

$$\Delta m^2_{23} = (2.38^{+0.20}_{-0.16}) \cdot 10^{-3}\text{eV}^2, \quad \sin^2 2\theta_{23} > 0.84 \text{ (90\% CL).}$$  \hspace{1cm} (13)
From the global analysis of the recent data of the reactor experiment KamLAND and the data of the solar neutrino experiments the following values were obtained for the parameters $\Delta m_{12}^2$ and $\tan^2 \theta_{12}$ \cite{3}:

$$\Delta m_{12}^2 = (7.59^{+0.21}_{-0.21}) \cdot 10^{-5} \text{ eV}^2, \quad \tan^2 \theta_{12} = 0.47^{+0.06}_{-0.05} \quad (14)$$

Concerning the study of neutrino oscillations, a new stage of high-precision experiments starts at present. In the future DOUBLE CHOOZ \cite{18} and Daya Bay \cite{19} reactor neutrino experiments, the sensitivities to the parameter $\sin^2 2\theta_{13}$ will be about 10-20 times better than in the CHOOZ experiment. The same sensitivity is planned to be reached in the accelerator T2K experiment \cite{20}. In the latter, the parameters $\Delta m_{23}^2$ and $\sin^2 2\theta_{23}$ will be measured with the accuracies $\delta(\Delta m_{23}^2) \sim 10^{-4}\text{eV}^2$ and $\delta(\sin^2 2\theta_{23}) \sim 10^{-2}$, respectively. High precision neutrino oscillation experiments are planned at the future Super Beam, Beta Beam, and Neutrino Factory facilities \cite{21}.

In spite of the big progress in the investigation of neutrino oscillations and of future prospects there was not so much progress in the understanding of the physics of neutrino oscillations.

Two factors of different origin determine the neutrino transition probability \cite{2}:

- The elements of the mixing matrix $U$ which characterize the mixed states of flavor neutrinos

$$|\nu_i\rangle = \sum_i U_{ii}^* |\nu_i\rangle \quad (15),$$

where $|\nu_i\rangle$ is the state of a neutrino with the mass $m_i$ and momentum $p_i$.

- The oscillation phases

$$\phi_{ik} = \Delta m_{ik}^2 \frac{L}{2E} \quad (16)$$

which are determined by the evolution of the vectors $|\nu_i\rangle$.

Different authors make different assumptions on the neutrino states with definite masses $|\nu_i\rangle$ (same momenta and different energies (see \cite{22, 23}), same energies and different momenta (see \cite{24, 25, 26}), different momenta and different energies (see \cite{27, 28, 29})) and different assumptions on the evolution of
these states (in time or in space and time). These completely different physical assumptions lead to the same standard expression (2) for the transition probability (see, for example, [30]). Thus, the study of neutrino oscillations in usual neutrino oscillation experiments does not allow to distinguish different hypotheses on the states of neutrinos with definite masses and their evolution. We showed in [30] that a new type of neutrino experiment with resonant recoilless (Mössbauer) emission and absorption of monochromatic $\bar{\nu}_e$ from atomic two-body tritium decay, proposed in [31] [32], would allow to discriminate different basic assumptions on the theory of neutrino oscillations.

If neutrino oscillations are a non-stationary phenomenon, the time-energy uncertainty relation holds for such a process (see [33], [34]). In the following section we will obtain the time-energy uncertainty relation for neutrino oscillations using the general Mandelstam-Tamm method [35]. We will discuss the Mössbauer neutrino experiment from the point of view of this relation. We will show that the Mössbauer neutrino experiment could allow to answer the fundamental question: is the time-energy uncertainty relation applicable to neutrino oscillations, i.e. are neutrino oscillations a non-stationary phenomenon?

2 Time-energy uncertainty relation for neutrino oscillations

Uncertainty relations play a fundamental role in quantum theory. They are based on general properties of the theory and manifest its nature. There are two different types of uncertainty relations in quantum theory: the Heisenberg uncertainty relations and the time-energy uncertainty relation.

All uncertainty relations are based on the Cauchy-Schwarz inequality (see, for example, [36])

$$\Delta A \Delta B \geq \frac{1}{2} | \langle \Psi | [A, B] | \Psi \rangle | .$$

(17)

Here $A$ and $B$ are Hermitean operators, $\Delta A$ and $\Delta B$ are the standard deviations and $| \Psi \rangle$ is some state. We have

$$\Delta A = \sqrt{\langle \Psi | (A - \bar{A})^2 | \Psi \rangle} = \sqrt{A^2 - (\bar{A})^2} ,$$

(18)
where
\[ \bar{A} = \langle \Psi | A | \Psi \rangle . \] (19)

The Heisenberg uncertainty relations are a direct consequence of the inequality (17) and the commutation relations for the operators \( A \) and \( B \). For example, from the commutation relation
\[ [p, q] = \frac{1}{i} \] (20)
and the inequality (17), we derive the standard Heisenberg uncertainty relation
\[ \Delta p \Delta q \geq \frac{1}{2} \] (21)
for the operators of the momentum \( p \) and the coordinate \( q \). Let us stress that the Heisenberg uncertainty relations for canonically conjugated quantities have a universal character: the form of these relations does not depend on the state \( |\Psi\rangle \).

The time-energy uncertainty relation has a completely different character. This is connected with the fact that time in quantum theory is a parameter and there is no operator which corresponds to time.

The time-energy uncertainty relation is based on the fact that the dynamics of a quantum system is determined by the Hamiltonian. The most general method of derivation of the time-energy uncertainty relation was given by Mandelstam and Tamm [35].

According to general principles of the quantum field theory, in the Heisenberg representation for any operator \( O(t) \), which does not depend upon time explicitly, we have (see, for example, [37, 38])
\[ i \frac{d}{dt} O(t) = [O(t), H] , \] (22)
where \( H \) is the total Hamiltonian (which does not depend on time). From (17) and (22) we find
\[ \Delta E \Delta O(t) \geq \frac{1}{2} | \frac{d}{dt} \bar{O}(t) | . \] (23)
Here
\[ \bar{O}(t) = \langle \Psi_H | O(t) | \Psi_H \rangle = \langle \Psi(t) | O | \Psi(t) \rangle , \] (24)
where $O$ and $|\Psi(t)\rangle$ are the operator and the vector of the state in the Schrödinger representation.

It is evident from (23) that in the case of a stationary state ($\Delta E = 0$) for any operator $O(t)$ the average $\overline{O}(t)$ does not depend on $t$.

The relation (23) can be written in the form of the time-energy uncertainty relation

$$\Delta E \Delta t \geq \frac{1}{2}, \quad (25)$$

where

$$\Delta t = \frac{\Delta O(t)}{|\frac{d}{dt}O(t)|}. \quad (26)$$

The relation (25), unlike the Heisenberg uncertainty relations, does not have a universal character. For example, for a wave packet, $\Delta t$ is the time interval during which the wave packet passes a fixed space point, for an excited state, $\Delta t$ is the life-time of the state, etc. (see [33, 39]).

From the Mandelstam-Tamm relation (23) we will now obtain the time-energy uncertainty relations for neutrino oscillations. Let us choose $O = P_l$, where

$$P_l = |\nu_l\rangle \langle \nu_l| \quad (27)$$

is the operator of the projection on the flavor neutrino state $|\nu_l\rangle \langle \nu_l' | \nu_l \rangle = \delta_{l'l}$, where $l = e, \mu, \tau$. It is obvious that

$$P^2_l = P_l. \quad (28)$$

The average value of the operator $P_l$ is given by

$$\overline{P}_l(t) = \langle |\nu_l|\Psi(t)\rangle^2. \quad (29)$$

Thus, $\overline{P}_l(t)$ is the probability to find the flavor neutrino $\nu_l$ in the state $|\Psi(t)\rangle$. We will assume that $|\Psi(0)\rangle = |\nu_l\rangle$. In this case we have

$$\overline{P}_l(t) = P_{\nu_l \rightarrow \nu_l}(t), \quad (30)$$

where $P_{\nu_l \rightarrow \nu_l}(t)$ is the probability of the flavor neutrino $\nu_l$ to survive. Obviously we have

$$P_{\nu_l \rightarrow \nu_l}(0) = 1, \quad P_{\nu_l \rightarrow \nu_l}(t) \leq 1, \quad t > 0. \quad (31)$$

Taking into account (28), we have

$$\Delta P_l(t) = \sqrt{P_{\nu_l \rightarrow \nu_l}(t) - P^2_{\nu_l \rightarrow \nu_l}(t)}. \quad (32)$$
The inequality (23) takes the form
\[ \Delta E \geq \frac{1}{2} \frac{|\frac{d}{dt} P_{\nu_\mu \to \nu_l}(t) |}{\sqrt{P_{\nu_\mu \to \nu_l}(t) - P_{\nu_\mu \to \nu_l}^2(t)}}. \tag{33} \]

We will consider the survival probability \( P_{\nu_\mu \to \nu_l}(t) \) in the interval
\[ 0 \leq t \leq t_{1 \text{min}}, \tag{34} \]
where \( t_{1 \text{min}} \) is the time at which the survival probability reaches the first minimum. In this interval \( \frac{d}{dt} P_{\nu_\mu \to \nu_l}(t) \leq 0 \). After the integration of the inequality (33) over the time from \( t = 0 \) to \( t \) we find
\[ \Delta E t \geq \frac{1}{2} \left( \frac{\pi}{2} - \arcsin(2 P_{\nu_\mu \to \nu_l}(t) - 1) \right). \tag{35} \]

The expressions (4), (8) and (10) for the survival probabilities which, as we stressed before, describe all existing experimental data, depend on the distance \( L \) between the neutrino production and detection points. However, for ultrarelativistic neutrinos \(^1\)
\[ L \simeq t. \tag{36} \]

We will assume in accordance with (30) that the transition probabilities depend on time. Let us consider \( P_{\nu_\mu \to \nu_\mu}(t) \) for transitions in the atmospheric-LBL region which are driven by \( \Delta m^2_{23} \). From (8) and (12) we have
\[ P_{\nu_\mu \to \nu_\mu}(t_{1 \text{min}}^{(23)}) \simeq 0, \tag{37} \]
where
\[ t_{1 \text{min}}^{(23)} = 2\pi \frac{E}{\Delta m^2_{23}}. \tag{38} \]

\(^1\)The relation (36) was confirmed by the K2K [8] and MINOS [9] neutrino oscillation experiments. In the K2K experiment neutrinos are produced in 1.1 \( \mu s \) spills. Protons are extracted from the accelerator every 2.2 s. It was found that
\[ -0.2 \leq (t - L/c) \leq 1.3 \mu s. \]

Here \( t = (t_{\text{SK}} - t_{\text{KEK}}) \), where \( t_{\text{KEK}} \) is the time of the neutrino production at the KEK accelerator and \( t_{\text{SK}} \) is the time of the neutrino detection in the SK detector.
From (35), (37) and (38) we obtain the following time-energy uncertainty relation for the neutrino oscillations driven by $\Delta m^2_{23}$

$$\Delta E t_{\text{osc}}^{(23)} \geq \pi,$$  

(39)

where

$$t_{\text{osc}}^{(23)} = 2t_{1\text{min}}^{(23)} = 4\pi \frac{E}{\Delta m^2_{23}}$$

(40)

is the period of neutrino oscillations in the atmospheric-LBL region.

The condition (39) is satisfied if the term

$$\frac{1}{2} (1 - \cos \Delta m^2_{23} \frac{t}{2E})$$

(41)

in (11) is changed in the interval (34) from 0 to 1. In the expression for the survival probability the term (11) is multiplied by the oscillation amplitude which is determined by the mixing angle. In order for neutrino oscillations to be observed not only the term (11) has to be changed significantly in the interval (34) but also the mixing angle must be relatively large. This means that if the mixing angle is small the time-energy uncertainty relation (39) is only a necessary condition for neutrino oscillations driven by $\Delta m^2_{23}$ to be observed.

Driven by $\Delta m^2_{23}$, the probability of $\bar{\nu}_e$ to survive in accordance with the results of the CHOOZ experiment is close to one. Let us apply inequality (35) to this case. From (6) we have

$$P_{\bar{\nu}_e \rightarrow \bar{\nu}_e}(t_{1\text{min}}^{(23)}) = 1 - \sin^2 2\theta_{13},$$

(42)

where $t_{1\text{min}}^{(23)}$ is given by (33). From the results of the CHOOZ experiment [16] follows that

$$\sin^2 2\theta_{13} \lesssim 2 \cdot 10^{-1}$$

(43)

Up to $\sin^3 2\theta_{13}$ terms we have the following expansion

$$\arcsin(2 P_{\bar{\nu}_e \rightarrow \bar{\nu}_e}(t_{1\text{min}}^{(23)}) - 1) \simeq \frac{\pi}{2} - 2\sqrt{\sin^2 2\theta_{13}}.$$  

(44)

From (35) and (44) we find for the $\bar{\nu}_e \rightarrow \bar{\nu}_e$ transition the relation

$$\Delta E t_{\text{osc}}^{(23)} \geq 2 \sin 2\theta_{13},$$

(45)

which is much weaker than the time-energy uncertainty relation (39).
Let us now consider $\bar{\nu}_e \to \bar{\nu}_e$ transitions in the KamLAND region. These transitions are due to $\bar{\nu}_e \leftrightarrow \bar{\nu}_{\mu,\tau}$ oscillations driven by $\Delta m^2_{12}$. From (10) we have

$$P_{\bar{\nu}_e \to \bar{\nu}_e}(t^{(12)}_{1\text{min}}) = 1 - \sin^2 2\theta_{12},$$

(46)

where

$$t^{(12)}_{1\text{min}} = 2\pi \frac{E}{\Delta m^2_{12}}.$$  

(47)

Further from (35) we find

$$\Delta E \ t^{(12)}_{1\text{min}} \geq \frac{1}{2} \left( \frac{\pi}{2} + \arcsin(1 - 2 \left( 1 - \sin^2 2\theta_{12} \right)) \right).$$

(48)

For the best-fit value of the parameter $\tan^2 \theta_{12}$ we have

$$\sin^2 2\theta_{12} = 0.87.$$  

(49)

With this value for $\sin^2 2\theta_{12}$ we obtain from (48) with an accuracy of $\sim 1\%$ the following time-energy uncertainty relation in the KamLAND region

$$\Delta E \ t^{(12)}_{\text{osc}} \geq \left( \pi - 2 \sqrt{(1 - \sin^2 2\theta_{12})} \right),$$

(50)

where

$$t^{(12)}_{\text{osc}} = 4\pi \frac{E}{\Delta m^2_{12}}$$

(51)

is the period of oscillations driven by $\Delta m^2_{12}$. Using the value (49) we have

$$\Delta E \ t^{(12)}_{\text{osc}} \geq (\pi - 0.72).$$

(52)

3 Recoilless creation and resonance absorption of tritium antineutrinos

In \cite{31,32} the possibilities have been considered to perform an experiment on the detection of the tritium $\bar{\nu}_e$ with energy $\simeq 18.6$ keV in the recoilless (Mössbauer) transitions

$$^3\text{H} \to ^3\text{He} + \bar{\nu}_e, \quad \bar{\nu}_e + ^3\text{He} \to ^3\text{H}.$$ 

(53)
It was estimated in [31] that the relative uncertainty of the energy of the antineutrinos produced in (53) is of the order
\[ \frac{\Delta E}{E} \approx 4.5 \cdot 10^{-16}. \] (54)

With such an uncertainty it was estimated [31] that the cross section of the recoilless resonance absorption of antineutrinos in the process \( \bar{\nu}_e + ^3\text{He} \rightarrow ^3\text{H} \) is equal to
\[ \sigma_R \approx 3 \cdot 10^{-33} \text{cm}^2 \] (55)

Such a value is about nine orders of magnitude larger than the normal neutrino cross section.

For the tritium antineutrino with the energy \( \approx 18.6 \text{ keV} \) the length of the oscillations driven by \( \Delta m_{23}^2 \) is given by
\[ L_{\text{osc}}^{(23)} \approx 2.5 \frac{E(\text{MeV})}{\Delta m_{23}^2(eV^2)} m \approx 18.6 \text{ m} \] (56)

It was proposed in [31] to search for neutrino oscillations in a Mössbauer neutrino experiment. Such a measurement would allow to determine the parameter \( \sin^2 \theta_{13} \) (or to improve the CHOOZ bound (7)) in a neutrino experiment with a baseline of about 10 m.

We will now discuss possibilities to observe neutrino oscillations in a Mössbauer neutrino experiment from the point of view of the time-energy uncertainty relations which we obtained in the previous section.

From (39) follows that for neutrino oscillations, which are driven by the "large" atmospheric \( \Delta m_{23}^2 \), the energy uncertainty must satisfy the following condition
\[ \frac{\Delta E}{E} \geq \frac{1}{4} \frac{\Delta m_{23}^2}{E^2} \approx 1.8 \cdot 10^{-12}. \] (57)

Thus, the expected neutrino energy uncertainty [31] in the Mössbauer neutrino experiment is about four orders of magnitude smaller than the minimum energy uncertainty which is required by the time-energy uncertainty relation (39) for neutrino oscillations to occur.

It is of interest to see what energy uncertainty is required by the uncertainty relation in its weak form for neutrino flavour oscillations in the atmospheric-LBL region. From (10) and (15) we have
\[ \frac{\Delta E}{E} \approx \frac{1}{2\pi} \frac{\Delta m_{23}^2}{E^2} \sin 2\theta_{13}. \] (58)
Using the CHOOZ bound for the value of the parameter $\sin 2\theta_{13}$ we have
\[
\frac{\Delta E}{E} \gtrsim 0.5 \cdot 10^{-12} \quad .
\] (59)

In the future reactor and T2K experiments the sensitivity to the parameter $\sin^2 2\theta_{13}$ is planned to be improved by a factor of 20 [20]. If for the value of the parameter $\sin^2 2\theta_{13}$ we take the value $\sin^2 2\theta_{13} = 10^{-2}$ we have from (58)
\[
\frac{\Delta E}{E} \gtrsim 1.2 \cdot 10^{-13} \quad .
\] (60)

Thus, the neutrino energy uncertainty (54) in a M"ossbauer neutrino experiment is several orders of magnitude smaller than the minimum energy uncertainty which is required even by the weak form of the time-energy uncertainty relation (45).

We will now discuss neutrino oscillations in a M"ossbauer neutrino experiment which are driven by the "small" solar-KamLAND neutrino mass-squared difference $\Delta m^2_{12}$ given by (14). In this case, the oscillation length for the tritium neutrinos will be $L_{\text{osc}}^{(12)} \approx 600$ m. Thus, the baseline of the experiment must be about 300-400 meters. This makes such an experiment very difficult. Let us see, however, whether the experiment is possible from the point of view of the time-energy uncertainty relation. From (52) we have
\[
\frac{\Delta E}{E} \gtrsim \frac{(\pi - 0.72)}{4\pi} \frac{\Delta m^2_{12}}{E^2} \approx 4.2 \cdot 10^{-14} \quad ,
\] (61)

Thus, the neutrino energy uncertainty required by the time-energy uncertainty relation for neutrino oscillations driven by $\Delta m^2_{12}$ must also be larger than the estimated energy uncertainty in the M"ossbauer neutrino experiment.

We compared the required minimal neutrino energy uncertainty $\frac{\Delta E}{E}$ with the value (54) estimated in [31]. However, it was stressed in [32] that due to inhomogeneous broadening (impurities, lattice defects and other effects), the real value for $\frac{\Delta E}{E}$ can be still larger than that in (54) and inequality (61) could be satisfied. However, the maximal resonance effect would then be reduced accordingly. Because of the large baseline, such a M"ossbauer neutrino experiment on the investigation of neutrino oscillations driven by the mass-squared difference $\Delta m^2_{12}$ does not look feasible.
4 Conclusion

In spite of neutrino oscillations having been observed in atmospheric, solar, reactor and accelerator neutrino experiments the nature of neutrino oscillations is still an open problem. The Mössbauer neutrino experiment proposed in [31, 32] gives a unique possibility to test the origin of neutrino oscillations [30].

We consider here Mössbauer neutrino experiments from the point of view of the time-energy uncertainty relation, which connects the uncertainty of the neutrino energy with a characteristic time of neutrino oscillations. Using the general Mandelstam-Tamm method, which is based only on the assumption that the evolution of Heisenberg operators in quantum field theory is determined by the Hamiltonian, we derived time-energy uncertainty relations for neutrino oscillations. We conclude that the small energy uncertainty of Mössbauer antineutrinos recoillessly emitted and absorbed in the $^3\text{H}/^3\text{He}$ system is in conflict with the energy uncertainty which, according to the time-energy uncertainty relation, is necessary for neutrino oscillations to happen.

There exist other approaches to neutrino oscillations which do allow oscillations in the Mössbauer neutrino experiment (effectively due to phase differences of the states of neutrinos with different masses, same energy and different momenta; see the recent paper [40], in which the propagation of virtual mixed neutrinos between the source and the detector was considered in detail, and also papers [24, 25, 26]).

In paper [35] it is stated: “From definiteness of the total energy of a system follows the constancy in time of all dynamical variables.” If practically monochromatic Mössbauer neutrinos oscillate this would mean that neutrino oscillations do not follow this general quantum rule. Oscillations of Mössbauer neutrinos would indicate a stationary phenomenon where the evolution of the neutrino state proceeds in space rather than in time. Thus, the search for neutrino oscillations in the Mössbauer neutrino experiment gives us a unique possibility to test fundamentally different approaches to neutrino oscillations. Let us stress that such tests can not be realized in usual neutrino oscillation experiments (see [30]).

In conclusion, we will summarize our main assumptions:

- We consider neutrino oscillations in the framework of QFT.
- We assume that in weak processes, flavor neutrinos are produced and
detected in mixed mass eigenstates. The states of flavor neutrinos are given by vectors (15).

- The Mandelstam-Tamm method, which we use, is based on the assumption that the dynamics of a quantum system is determined by the Schrödinger equation.

- We assume that for ultra-relativistic neutrinos, the average difference between the emission and absorption times is given by the distance between source and detector.

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References

[1] Super-Kamiokande Collaboration, Y. Ashie et al., Phys. Rev. Lett. 93 (2004) 101801; Phys. Rev. D71 (2005) 11205.

[2] SNO Collaboration, S.N. Ahmed et al., Phys. Rev. Lett. 87 (2001) 071301; 89 (2002) 011301; 89 (2002) 011302; 92 (2004) 181301.

[3] KamLAND Collaboration, T. Araki et al., Phys. Rev. Lett. 94 (2005) 081801; S. Abe et al. arXiv:0801.4589.

[4] B. T. Cleveland et al., Astrophys. J. 496 (1998) 505.

[5] GALLEX Collaboration, W. Hampel et al., Phys. Lett. B 447 (1999) 127; GNO Collaboration, M. Altmann et al., Phys. Lett. B 616 (2005) 174.

[6] SAGE Collaboration, J. N. Abdurashitov et al., Nucl. Phys. Proc. Suppl. 118 (2003) 39.

[7] Super-Kamiokande Collaboration, S. Fukuda et al., Phys. Rev. Lett. 86 (2001) 5651.

[8] K2K Collaboration, M.H. Alm et al., Phys. Rev. Lett. 90 (2003) 041801.
[9] MINOS Collaboration, D. G. Michael et al., arXiv: hep-ex/0607088.

[10] M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, p. 315, edited by F. van Nieuwenhuizen and D. Freedman, North Holland, Amsterdam, 1979; T. Yanagida, Proc. of the *Workshop on Unified Theory and the Baryon Number of the Universe*, KEK, Japan, 1979; R.N. Mohapatra and G. Senjanović, Phys. Rev. Lett. 44 (1980) 912, P. Minkovski, Phys. Lett. B67 (1977) 421.

[11] E. Fiorini, Nucl. Phys. Proc. Suppl. 168 (2007) 11.

[12] S.M. Bilenky, C. Giunti, and W. Grimus. Prog. Part. Nucl. Phys. 43 (1999) 1.

[13] M.C. Gonzalez-Garcia and M. Maltene arXiv:0704.1800.

[14] B. Pontecorvo, J. Exptl. Theoret. Phys. 33 (1957) 549. [Sov. Phys. JETP 6 (1958) 429]; J. Exptl. Theoret. Phys. 34 (1958) 247 [Sov. Phys. JETP 7 (1958) 172].

[15] Z. Maki, M. Nakagava, and S. Sakata, Prog. Theor. Phys. 28 (1962) 870.

[16] CHOOZ Collaboration, M. Apollonio et al., Phys. Lett. B 466 (1999) 415; M. Apollonio et al., Eur. Phys. J. C27 (2003) 331; arXiv: hep-ex/0301017.

[17] X.Shi and D.N. Schramm, Phys. Lett. B 283 (1992) 305.

[18] DOUBLE CHOOZ collaboration, F. Ardellier et al., arXiv: hep-ex/0606025.

[19] Daya Bay Collaboration, Xinheng Guo et al., arXiv: hep-ex/0701029.

[20] T2K Collaboration, T. Ishida et al., Nucl.Phys.Proc.Suppl.149 (2005) 154.

[21] See Proceedings of 9th International Workshop On Neutrino Factories, Superbeams and Betabeams, NuFact07, August 6-11, 2007, Okayama University, Japan.

[22] S.M. Bilenky and B. Pontecorvo, Phys. Rep. 41 (1978) 225.
[23] H. Fritzsch and P. Minkowski, Phys. Lett. **B62** (1976) 72.

[24] L. Stodolsky, Phys. Rev. **D58** (1998) 036006.

[25] H. Lipkin, Phys. Lett. **B579** (2004) 355.

[26] B. Kayser, arXiv: hep-ph/0506165

[27] R.G. Winter, Lett. Nuovo Cimento. **30** (1981) 101.

[28] S.M. Bilenky and C. Giunti, Int. J. Mod. Phys. **A16** (2001) 3931.

[29] C. Giunti, Found. Phys. Lett. **17** (2004) 103; arXiv: hep-ph/0409230.

[30] S.M. Bilenky, F. von Feilitzsch, and W. Potzel, J. Phys. **G34** (2007) 987.

[31] R.S. Raghavan, arXiv: hep-ph/0601079.

[32] W. Potzel, Phys. Scr. **T127** (2006) 85.

[33] S.M. Bilenky and M.D. Mateev, Phys. Part. Nucl.38 (2007) 117, arXiv: hep-ph/0604044.

[34] S.M. Bilenky, arXiv: 0708.0260.

[35] L. Mandelstam and I.E. Tamm, J. Phys.(USSR) **9** (1945) 249.

[36] A. Messiah, Quantum Mechanics, Vol.I North Holland, Amsterdam, 1970.

[37] N.N. Bogolubov and D.V. Shirkov, Introduction to the theory of quantized fields, Moscow, Nauka, 1976.

[38] M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley Advanced Book Program, 1995.

[39] P. Busch, arXiv: quant-ph/0105049

[40] E.Kh. Akhmedov, J. Kopp, and M. Lindner, arXiv: 0802.2513.