THE STRINGY E-FUNCTION OF THE MODULI SPACE OF HIGGS BUNDLES WITH TRIVIAL DETERMINANT

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Abstract. Let \( M \) be the moduli space of semistable rank 2 Higgs pairs \((V, \phi)\) with trivial determinant over a smooth projective curve \( X \) of genus \( g \geq 2 \). We compute the stringy E-function of \( M \) and prove that there does not exist a symplectic desingularization of \( M \) for \( g \geq 3 \).

1. Introduction

Let \( X \) be a smooth projective curve of genus \( g \geq 2 \). Let \( M \) be the moduli space of semistable Higgs pairs \((V, \phi)\) over \( X \) with \( V \) a rank 2 vector bundle with \( \det V \cong \mathcal{O}_X \) and \( \phi \in H^0(\text{End}_0 V \otimes K_X) \). Then \( M \) is a singular quasi-projective irreducible normal variety of dimension \( 6g - 6 \). The locus \( M^s \) of stable pairs in \( M \) is an open dense subset which is equipped with a symplectic form\(^1\) \([8, 16, 19]\). In this paper we give an explicit formula for the stringy E-function \( E_{st}(M) \) of \( M \) as defined in \([1]\) which retains useful information about the singularities (Theorem 5.2).

The moduli space \( M \) of Higgs pairs can be also thought of as the moduli space of sheaves of certain topological type on the symplectic surface \( T^*X \) \([19, \S 6]\). If a surface is equipped with a symplectic form, it induces a natural symplectic form on the smooth part of a moduli space of sheaves on the surface \([15]\). A natural question raised by O’Grady \([17]\) asks whether there exists a desingularization of such a moduli space on which the symplectic form extends everywhere without degeneration. It was shown in \([9, 10, 12, 13]\) that except for the two 10-dimensional moduli spaces studied by O’Grady \([17, 18]\), there does not exist a symplectic desingularization when the surface is K3 or Abelian.

In this paper we study the Kirwan desingularization of \( M \) by using O’Grady’s analysis of the K3 surface case and show the following.

**Theorem 1.1.** (Theorem 4.1 and Corollary 5.4.)
1. When \( g = 2 \), there is a symplectic desingularization of \( M \).
2. When \( g \geq 3 \), there does not exist a symplectic desingularization of \( M \).

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\(^1\)In this paper, a *symplectic form* is always a holomorphic 2-form which is nondegenerate everywhere.
If there exists a symplectic desingularization of $M$, then it has to be a crepant resolution as the canonical bundle $K_M$ is trivial. Hence the stringy E-function of $M$ has to be equal to the Hodge-Deligne polynomial of the symplectic desingularization which is a polynomial with integer coefficients (Theorem 2.1). For the non-existence result (2), it suffices to prove that the stringy E-function $E_{st}(M)$ of $M$ is not a polynomial for $g \geq 3$. Because we have an explicit formula of $E_{st}(M)$ (Theorem 5.2), it is a simple matter to prove this.

To compute $E_{st}(M)$ we consider all possible types of semistable pairs, and find a description of the locally closed subvariety of $M$ corresponding to each type. Then we compute the Hodge-Deligne polynomials of the subvarieties explicitly.

In §2, we recall basic facts about stringy E-function and Higgs pairs. In §3, we compute the Hodge-Deligne polynomial of the stable locus $M^s$. In §4, we construct the Kirwan desingularization of $M$ along the line of O’Grady’s analysis in [17] and show that $M$ admits a symplectic desingularization when $g = 2$. In §5, we complete the computation of $E_{st}(M)$ and prove non-existence of symplectic desingularization.

In the context of Mirror Symmetry, Hausel and Thaddeus computed the stringy E-function of the moduli space of Higgs pairs of odd degree [7]. All the varieties in this paper are defined over the complex number field.

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2. Preliminaries

In this section we collect some facts that we shall use in this paper.

2.1. Stringy E-function

Stringy E-function introduced in [1] is a new invariant of varieties which retains useful information about singularities. We recall the definition and basic facts about stringy E-functions from [1][6]. Let $W$ be a normal irreducible variety with at worst log-terminal singularities, i.e.

1. $W$ is $\mathbb{Q}$-Gorenstein;
2. for a resolution of singularities $\rho : V \to W$ such that the exceptional locus of $\rho$ is a divisor $D$ whose irreducible components $D_1, \cdots, D_r$ are smooth divisors with only normal crossings, we have

$$K_V = \rho^* K_W + \sum_{i=1}^{r} a_i D_i$$

with $a_i > -1$ for all $i$, where $D_i$ runs over all irreducible components of $D$. The divisor $\sum_{i=1}^{r} a_i D_i$ is called the discrepancy divisor.
For each subset \( J \subset I \), define \( D_J = \bigcap_{j \in J} D_j \), \( D_0 = V \) and \( D_J^0 = D_J - \bigcup_{i \in I - J} D_i \). Then the stringy E-function of \( W \) is defined by

\[
E_{st}(W; u, v) = \sum_{J \subseteq I} E(D_J^0; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{d_j+1} - 1}
\]

where

\[
E(Z; u, v) = \sum_{p,q,k \geq 0} (-1)^k h^{p,q}(H^k_c(Z; \mathbb{C})) u^p v^q
\]

is the Hodge-Deligne polynomial for a variety \( Z \). We will also use the alias, \( E\)-polynomial, for the Hodge-Deligne polynomial and often use the abbreviation \( E(Z) \) for \( E(Z; u, v) \). Note that the Hodge-Deligne polynomials have

1. the additive property: \( E(Z; u, v) = E(U; u, v) + E(Z - U; u, v) \) if \( U \) is a smooth open subvariety of \( Z \);
2. the multiplicative property: \( E(Z; u, v) = E(B; u, v)E(F; u, v) \) if \( Z \) is a Zariski locally trivial \( F \)-bundle over \( B \).

By [11, Theorem 6.27], the function \( E_{st} \) is independent of the choice of a resolution (Theorem 3.4 in [11]) and the following holds.

**Theorem 2.1.** ([11, Theorem 3.12]) Suppose \( W \) is a \( \mathbb{Q} \)-Gorenstein algebraic variety with at worst log-terminal singularities. If \( \rho : V \to W \) is a crepant desingularization (i.e. \( \rho^*K_W = K_V \)) then \( E_{st}(W; u, v) = E(V; u, v) \). In particular, \( E_{st}(W; u, v) \) is a polynomial.

### 2.2. Higgs pairs

Let \( X \) be a smooth projective curve of genus \( g \geq 2 \). A Higgs pair, more precisely an \( SL(2) \)-Higgs pair, is a pair of a rank 2 vector bundle \( V \) with trivial determinant and a section \( \phi \) of \( \text{End}_0 V \otimes K_X \) where \( K_X \) is the canonical bundle of \( X \) and \( \text{End}_0 V \) denotes the traceless part of \( \text{End}V \). To construct the moduli space of Higgs pairs, a stability condition has to imposed and Hitchin introduced the following.

**Definition 2.2.** (1) A Higgs pair \((V, \phi)\) with \( \det V \cong \mathcal{O}_X \) is stable (resp. semistable) if for any nonzero proper subbundle \( W \) satisfying \( \phi(W) \subset W \otimes K \), we have \( \deg W < 0 \) (resp. \( \deg W \leq 0 \)).

(2) A Higgs pair \((V, \phi)\) with \( \det V \cong \mathcal{O}_X \) is polystable if it is either stable or a direct sum \((L, \psi) \oplus (L^{-1}, -\psi)\) where \( L \in \text{Pic}^0(X) \) is a line bundle of degree 0 and \( \psi \) is a section of \( \text{Hom}(L, L) \otimes K_X \cong K_X \).

If a bundle \( V \) is stable (resp. semistable), then the Higgs pair \((V, \phi)\) is stable (resp. semistable) for any choice of \( \phi \in H^0(\text{End}_0 V \otimes K_X) \).

The set of isomorphisms classes of polystable pairs \((V, \phi)\) admits a structure of quasi-projective variety of dimension \( 6g - 6 \) \([8, 16, 19]\) and we denote

\[2A \text{ vector bundle } V \text{ with trivial determinant is stable (resp. semistable) for any nonzero proper subbundle } W, \text{ we have } \deg W < 0 \text{ (resp. } \deg W \leq 0). \text{ A vector bundle is strictly semistable if it is semistable but not stable.}\]
it by \( M \). Furthermore, it is known [3, 19] that \( M \) is an irreducible normal variety. The locus \( M^s \) of stable pairs \((V, \phi)\) is a smooth open dense subvariety and its complement is precisely the locus of singularities, isomorphic to \( T^*J/\mathbb{Z}_2 \) where \( J = \text{Pic}^0(X) \) is the Jacobian and \( \mathbb{Z}_2 \) acts on the cotangent bundle \( T^*J = J \times H^0(K_X) \) by \((L, \psi) \mapsto (L^{-1}, -\psi)\).

The stable locus \( M^s \) is homeomorphic to the space of irreducible representations of the fundamental group of \( X \) into \( SL(2) \) and the complex structures of \( X \) and \( SL(2) \) induce two integrable complex structures on \( M^s \). Therefore, \( M^s \) admits a hyperkähler metric and thus there is a symplectic form on \( M^s \). In particular the canonical bundle of \( M^s \) is trivial and \( M^s \) is Gorenstein.\(^3\) We will see in Theorem 4.1 that \( M^s \) has only log terminal singularities. Therefore, the stringy E-function of \( M^s \) is a well-defined rational function which we intend to compute.

3. Stable pairs

In this section we compute the E-polynomial of the stable part \( M^s \). Let \((V, \phi) \in M^s \). There are three possibilities for \( V \):

1. \( V \) is stable
2. \( V \) is strictly semistable
3. \( V \) is unstable (= not semistable)

We deal with these cases separately in the subsequent subsections.

3.1. Stable case

Let \( N^s \) be the moduli space of rank 2 stable bundles with trivial determinant over \( X \). \( N^s \) is a \( 3g - 3 \) dimensional quasi-projective variety and the Hodge-Deligne polynomial of \( N^s \) is

\[
E(N^s) = \frac{(1 - u^2v)^g(1 - uv^2)^g - (uv)^{g+1}(1 - u)^g(1 - v)^g}{(1 - uv)(1 - (uv)^2)} - \frac{1}{2} \frac{(1 - u)^g(1 - v)^g}{1 - uv} + \frac{(1 + u)^g(1 + v)^g}{1 + uv}
\]

from [11 (18)] or [13 §6.2].

If \( V \) is stable, a pair \((V, \phi)\) is stable for any \( \phi \in H^0(\text{End}_0V \otimes K_X) \). It is well-known that such pairs are parameterized by the cotangent bundle \( T^*N^s \) which is embedded in \( M^s \) as an open subvariety [3]. Hence the E-polynomial of the locus of stable pairs \((V, \phi)\) with \( V \) stable is

\[
E(T^*N^s) = (uv)^{3g-3} \frac{(1 - u^2v)^g(1 - uv^2)^g - (uv)^{g+1}(1 - u)^g(1 - v)^g}{(1 - uv)(1 - (uv)^2)}
\]

\(^3\)A normal variety \( M \) is Gorenstein if the canonical divisor \( K_M \) is Cartier [11]. In our case, the existence of symplectic form guarantees \( K_M^s = 0 \) and hence \( K_M = 0 \) since \( \text{codim}(M - M^s) \geq 2 \). Obviously 0 is Cartier.
3.2. Strictly semistable case

When \( V \) is a strictly semistable rank 2 bundle with trivial determinant, there are four possibilities:

Type I: \( V = L \oplus L^{-1} \) for \( L \in \text{Pic}^0(X) \) with \( L \not\cong L^{-1} \)

Type II: \( V \) is a nontrivial extension of \( L^{-1} \) by \( L \) for \( L \in \text{Pic}^0(X) \) with \( L \not\cong L^{-1} \)

Type III: \( V = L \oplus L^{-1} \) for \( L \in \text{Pic}^0(X) \) with \( L \cong L^{-1} \)

Type IV: \( V \) is a nontrivial extension of \( L^{-1} \) by \( L \) for \( L \in \text{Pic}^0(X) \) with \( L \cong L^{-1} \)

We consider the loci of stable pairs \((V, \phi)\) for the above four cases separately.

3.2.1. Type I

Let \( J = \text{Pic}^0(X) \) and \( J_0 \) be the locus of \( L \in J \) with \( L \cong L^{-1} \) so that \( J_0 \cong \mathbb{Z}_2^g \). Let \( J^0 = J - J_0 \) be the complement of \( J_0 \) in \( J \). Then the bundle \( V \) is parameterized by \( J^0/\mathbb{Z}_2 = J/\mathbb{Z}_2 - J_0 \) where \(-1 \in \mathbb{Z}_2 \) acts as \( L \mapsto L^{-1} \).

When \( V = L \oplus L^{-1} \) for \( L \in J^0 \), we have the decomposition

\[
H^0(\text{End}_0(V) \otimes K_X) = H^0(K_X) \oplus H^0(L^2K_X) \oplus H^0(L^{-2}K_X).
\]

For \( \phi \in H^0(\text{End}_0(V) \otimes K_X) \), write \( \phi = (a, b, c) \) or

\[
\phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}
\]

with \( a \in H^0(K_X), b \in H^0(L^2K_X) \) and \( c \in H^0(L^{-2}K_X) \). Then a pair \((V, \phi)\) is stable if and only if \( L \) and \( L^{-1} \) are not preserved by \( \phi \), i.e. \( b \neq 0 \) and \( c \neq 0 \).

Because the automorphism group of \( V = L \oplus L^{-1} \) with \( L \not\cong L^{-1} \) is \( \mathbb{C}^* \times \mathbb{C}^* \), \((V, \phi_1) \cong (V, \phi_2)\) for \( \phi_i = (a_i, b_i, c_i) \) if and only if

\[
\phi_1 = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \phi_2 \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}
\]

for some \( t \in \mathbb{C}^* \), i.e. \( a_1 = a_2, b_1 = t^2b_2 \) and \( c_1 = t^{-2}c_2 \). Therefore, for fixed \( V = L \oplus L^{-1} \) of Type I, the isomorphism classes of stable pairs \((V, \phi)\) are parameterized by

\[
H^0(K_X) \times \frac{(H^0(L^2K_X) - 0) \times (H^0(L^{-2}K_X) - 0)}{\mathbb{C}^* \times (\mathbb{C}^* - 0)}.
\]

It is well-known that the quotient \( \mathbb{P}(\mathbb{C}^g - 1 \times \mathbb{C}^g - 1)/\mathbb{C}^* \) is \( \mathbb{P}^{g-2} \times \mathbb{P}^{g-2} \) and \( \mathbb{C}^g - 1 \times \mathbb{C}^g - 1 / \mathbb{C}^* \) is the affine cone over \( \mathbb{P}^{g-2} \times \mathbb{P}^{g-2} \) while \((0 \times \mathbb{C}^g - 1) \cup \)
\((\mathbb{C}^g - 1 \times 0)\) is the inverse image in \(\mathbb{C}^g - 1 \times \mathbb{C}^g - 1\) of the vertex of the affine cone via the quotient map. Hence
\[
(\mathbb{C}^g - 1 \times 0) \times (\mathbb{C}^g - 1 \times 0) \cong \mathbb{C}^*(u \cdot v)
\]
is the line bundle \(\mathcal{O}_{\mathbb{P}g - 2 \times \mathbb{P}g - 2}(-1, -1)\) minus the zero section and the E-polynomial of \((\mathbb{C}g - 1 - 0) \times (\mathbb{C}g - 1 - 0)\) is
\[
E((\mathbb{C}g) \cdot E(\mathbb{C}^*) \cdot E(\mathbb{P}g - 2 \times \mathbb{P}g - 2)) = (uv)^g(uv - 1) \left(\frac{(uv)^g}{uv - 1} - 1\right)^2.
\]

Now we let \(V\) vary. Let \(\mathcal{L} \rightarrow J^0 \times X\) be a universal bundle of degree 0 line bundles and let \(\pi_1, \pi_2\) be the projections of \(J^0 \times X\) to \(J^0\) and \(X\) respectively. Let
\[
W_j = \pi_1^*(\mathcal{L}^{2j} \otimes \pi_2^*K_X)\quad \text{for } j = 0, 1, -1
\]
and let \(W = W_0 \oplus W_1 \oplus W_{-1}\). Since \(H^1(L^{\pm 2}K_X) \cong H^0(L^{\mp 2}) = 0\), \(W_0, W_{\pm 1}\) are vector bundles over \(J^0\) of rank \(g\) and \(g - 1\) respectively whose fibers over \(L \in J^0\) are \(H^0(K_X)\) and \(H^0(L^{\pm 2}K_X)\).

There is an obvious family of Higgs pairs \((V, \Phi)\) parameterized by \(W\) with \(V = \tilde{\mathcal{L}} \oplus \tilde{\mathcal{L}}^{-1}\) where \(\tilde{\mathcal{L}}\) is the pull-back of \(\mathcal{L}\) to \(W \times X\) and this family restricted to
\[
W_0 \oplus (W_1 - 0) \oplus (W_{-1} - 0)
\]
parameterizes stable pairs where 0 denotes the zero section. Hence we have a morphism
\[
W_0 \oplus (W_1 - 0) \oplus (W_{-1} - 0) \to M^g
\]
Since the action of \(\mathbb{C}^*\) on \(W_j\) with weight \(2j\) for \(j = 0, 1, -1\) preserves the isomorphism classes of stable pairs, \((3.5)\) factors through
\[
W_0 \oplus \frac{(W_1 - 0) \oplus (W_{-1} - 0)}{\mathbb{C}^*} \to M^g.
\]
Furthermore, the \(\mathbb{Z}_2\)-action on \(J^0\) which interchanges \(L\) with \(L^{-1}\) obviously extends to \(W\) interchanging \(W_1\) and \(W_{-1}\). Thus we get a morphism
\[
\left[\frac{W_0 \oplus (W_1 - 0) \oplus (W_{-1} - 0)}{\mathbb{C}^*}\right]/\mathbb{Z}_2 \to M^g.
\]
From construction, it is clear that this is a bijection onto the locus of stable pairs \((V, \phi)\) with \(V\) of Type I.

**Lemma 3.1.** The morphism \((3.7)\) is an isomorphism onto a locally closed subvariety of \(M^g\).

**Proof.** First observe that the locus of \((V, \phi)\) with \(V\) of Type I is locally closed. Indeed, given a family of stable pairs \((V, \phi)\) on \(X\) parameterized by a variety \(T\), the locus in \(T\) of semistable \(V\) is open and the locus of decomposable semistable \(V\) is closed in this open set. The condition \(L \not\equiv L^{-1}\) determines an open subset of this locally closed set.
Let \((\tilde{V}, \tilde{\Phi})\) be a family of stable pairs \((V, \phi)\) with \(V\) of Type I parameterized by a variety \(T\). Then there is a morphism \(T \to J^0/\mathbb{Z}_2\) which sends \(V = L \oplus L^{-1}\) to the \(\mathbb{Z}_2\)-orbit \((L, L^{-1})\). Indeed, as \(\tilde{V}\) is a family of semistable bundles, there is a morphism \(T \to N\) whose image is obviously the Kummer variety \(J/\mathbb{Z}_2\) minus \(J_0\), i.e. \(J^0/\mathbb{Z}_2\).

Moreover, the section \(\tilde{\Phi}\) of \(\text{End}_0(\tilde{V} \otimes K_X)\) induces a lifting of the morphism \(T \to J^0/\mathbb{Z}_2\) to a morphism

\[
T \to \left[ W_0 \oplus \frac{(W_1 - 0) \oplus (W_{-1} - 0)}{\mathbb{C}^*} \right] / \mathbb{Z}_2.
\]

Indeed, if we let \(\hat{T}\) be the fiber product of \(T\) and \(J^0\) over \(J^0/\mathbb{Z}_2\) and \(f : \hat{T} \to J^0\) be obvious map, then the pull-back \(\hat{V}\) of \(\tilde{V}\) to \(\hat{T} \times X\) is \((f \times 1_X)^* (L \oplus L^{-1})\) and from the commutative square

\[
\begin{array}{ccc}
\hat{T} \times X & \xrightarrow{f \times 1} & J^0 \times X \\
\pi & & \pi' \\
\hat{T} & \xrightarrow{f} & J^0
\end{array}
\]

where the vertical maps are the projections, \(\tilde{\Phi}\) induces a section of

\[
\pi_* (\text{End}_0(\tilde{V}) \otimes K_X) = \pi_* (f \times 1_X)^* (\text{End}_0(L \oplus L^{-1}) \otimes K_X)
\]

\[
= f^* \pi'_* \text{End}_0(L \oplus L^{-1}) \otimes K_X = f^* W.
\]

Therefore, \(\tilde{\Phi}\) gives us a morphism

\[
\hat{T} \to W_0 \oplus (W_1 - 0) \oplus (W_{-1} - 0)
\]

as \(\hat{V}\) is a family of stable pairs. So we obtain a morphism

\[
T \to \left[ W_0 \oplus \frac{(W_1 - 0) \oplus (W_{-1} - 0)}{\mathbb{C}^*} \right] / \mathbb{Z}_2
\]

after taking quotients. Obviously this gives us the inverse of \((3.7)\). \(\square\)

Consequently, the locus of stable pairs \((V, \phi)\) with \(V\) of Type I is

\[
\left[ W_0 \oplus \frac{(W_1 - 0) \oplus (W_{-1} - 0)}{\mathbb{C}^*} \right] / \mathbb{Z}_2
\]

which is a fiber bundle over \(J^0/\mathbb{Z}_2\) with fiber \((3.3)\). The E-polynomial of the fiber bundle is the E-polynomial of the \(\mathbb{Z}_2\)-invariant part of

\[
H^*_e \left( W_0 \oplus \frac{(W_1 - 0) \oplus (W_{-1} - 0)}{\mathbb{C}^*} \right).
\]

From the Leray spectral sequence, this is precisely \((3.8)\)

\[
(uv)^g \cdot (uv - 1) \cdot \left[ E(\mathbb{P}^{g-2} \times \mathbb{P}^{g-2})^+ \cdot E(J^0)^+ + E(\mathbb{P}^{g-2} \times \mathbb{P}^{g-2})^- \cdot E(J^0)^- \right]
\]
where $E(Z)$ denote the E-polynomials of the $\mathbb{Z}_2$-invariant and anti-invariant part of the compact support cohomology of a variety $Z$. From the computation of $E(\mathcal{D}_2^{(2)})$ in [13] p516, we deduce that (3.8) is

\[
(uv)^g(uv-1)\left[\frac{1}{2}(1-u)^g(1-v)^g+\frac{1}{2}(1+u)^g(1+v)^g-2^g\right]\frac{(uv)^g-1)((uv)^g-1)}{(uv-1)((uv)^g-1)}
\]

\[
+(\frac{1}{2}(1-u)^g(1-v)^g-\frac{1}{2}(1+u)^g(1+v)^g)uv\frac{(uv)^g-1)((uv)^g-1)}{(uv-1)((uv)^g-1)}]
\]

This is the E-polynomial of the locus of stable pairs with $V$ of Type I.

### 3.2.2. Type II

Now we consider the locus of stable pairs $(V, \phi)$ with $V$ of Type II. Let $\mathcal{L} \to J^0 \times X$ be a universal line bundle on $J^0$ and let $\pi : J^0 \times X \to J^0$ be the projection. Then $R^1\pi_*\mathcal{L}^2$ is a vector bundle of rank $g-1$ by Riemann-Roch. Let $\Lambda = \mathbb{P}R^1\pi_*\mathcal{L}^2$ be the projectivization of $R^1\pi_*\mathcal{L}^2$ and let $\mathcal{L}^\#$ be the pull-back of $\mathcal{L}$ by $\Lambda \times X \to J^0 \times X$. It is well-known that there is a universal extension bundle

\[
0 \to \mathcal{L}^\# \otimes \mathcal{O}_\Lambda(1) \to \mathcal{V}^\# \to (\mathcal{L}^\#)^{-1} \to 0
\]

over $\Lambda \times X$ where $\mathcal{O}_\Lambda(1)$ is the hyperplane bundle over the projective bundle $\Lambda$. Then the family $\mathcal{V}^\#$ over $\Lambda \times X$ parameterizes all isomorphism classes of rank 2 vector bundles $V$ of Type II.

Next we consider the bundle $\mathcal{E}nd_{\Lambda}\mathcal{V}^\# \otimes K_X$ where $K_X$ denotes the pull-back of the canonical bundle of $X$ by abuse of notation. This fits into a short exact sequence [3] (3.7)

\[
0 \to \mathcal{H}om(\mathcal{V}^\#, K_X\mathcal{L}^\#) \to \mathcal{E}nd_{\Lambda}\mathcal{V}^\# \otimes K_X \to K_X(\mathcal{L}^\#)^{-2} \to 0
\]

which gives rise to an exact sequence

\[
0 \to p_*\mathcal{H}om(\mathcal{V}^\#, K_X\mathcal{L}^\#) \to p_*(\mathcal{E}nd_{\Lambda}\mathcal{V}^\# \otimes K_X) \to p_*K_X(\mathcal{L}^\#)^{-2} \to R^1p_*\mathcal{H}om(\mathcal{V}^\#, K_X\mathcal{L}^\#)
\]

where $p : \Lambda \times X \to \Lambda$ is the projection. From (3.10) we also have an exact sequence

\[
R^1p_*K_X(\mathcal{L}^\#)^2 \to R^1p_*\mathcal{H}om(\mathcal{V}^\#, K_X\mathcal{L}^\#) \to R^1p_*K_X\mathcal{O}_\Lambda(-1) \to 0.
\]

By the Serre duality, $R^1p_*K_X(\mathcal{L}^\#)^2 = 0$ and $R^1p_*K_X \otimes \mathcal{O}_\Lambda(-1)$ is a line bundle over $\Lambda$ since $X$ is irreducible projective of dimension 1. Hence the last map in (3.12) gives us

\[
p_*K_X(\mathcal{L}^\#)^{-2} \to R^1p_*\mathcal{H}om(\mathcal{V}^\#, K_X\mathcal{L}^\#) \cong R^1p_*K_X\mathcal{O}_\Lambda(-1).
\]

Over a point $s \in \Lambda$ lying over $[L] \in J^0$, the fibers are

\[
H^0(K_X L^{-2}) \to H^1(V^* \otimes K_X L) \cong H^1(K_X) \cong \mathbb{C}
\]

which is the multiplication by the extension class of $V$, a nonzero representative in $H^1(L^2)$ of $s$. In particular, this is surjective and hence the kernel
of the last map in (3.12) is a vector bundle of rank $g - 2$ because the rank of $p_\ast K_X(\mathcal{L}^\#)^{-2}$ is $g - 1$.

From (3.10) again, we have an exact sequence

(3.13)

$$0 \to p_\ast K_X(\mathcal{L}^\#)^2 \to p_\ast \mathcal{H}om(\mathcal{V}^\#, K_X \mathcal{L}^\#) \to p_\ast K_X \mathcal{O}_\Lambda(-1) \to R^1 p_\ast K_X(\mathcal{L}^\#)^2 = 0.$$ 

The first term in (3.13) is a vector bundle over $\Lambda$ of rank $g - 1$ and the last term is a vector bundle of rank $g$. Therefore, $p_\ast(\mathcal{E}nd_0 \mathcal{V}^\# \otimes K_X)$ is a vector bundle over $\Lambda$ of rank $3g - 3$ from (3.12).

Let $(\mathcal{V}^\dagger, \Phi^\dagger)$ be the obvious family of Higgs pairs over $p_\ast(\mathcal{E}nd_0 \mathcal{V}^\# \otimes K_X) \times X$ where $\mathcal{V}^\dagger$ is the pull-back of $\mathcal{V}^\#$ via the bundle projection $p_\ast(\mathcal{E}nd_0 \mathcal{V}^\# \otimes K_X) \times X \to \Lambda \times X$. A pair $(V, \phi)$ in this family lying over $[L] \in J^0$ is stable if and only if $L$ is not preserved by $\phi$. This amounts to saying that the image of this point by the middle map in (3.12)

$$p_\ast \mathcal{E}nd_0 \mathcal{V}^\# \otimes K_X \to p_\ast K_X(\mathcal{L}^\#)^{-2}$$

is nonzero. Therefore the stable locus in $p_\ast(\mathcal{E}nd_0 \mathcal{V}^\# \otimes K_X)$ is precisely the complement of the subbundle $p_\ast \mathcal{H}om(\mathcal{V}^\#, K_X \mathcal{L}^\#)$. For a bundle $V$ in $\mathcal{V}^\#$, the automorphism group is trivial. So we obtain the following.

**Lemma 3.2.** The locus of stable pairs $(V, \phi)$ in $M^g$ with $V$ of Type II is locally closed and isomorphic to

(3.14)  

$$p_\ast(\mathcal{E}nd_0 \mathcal{V}^\# \otimes K_X) - p_\ast \mathcal{H}om(\mathcal{V}^\#, K_X \mathcal{L}^\#).$$

In particular the E-polynomial of this locus is

(3.15)  

$$[(uv)^{3g-3}-(uv)^{2g-1}] E(\Lambda) = [(uv)^{3g-3}-(uv)^{2g-1}](uv)^{g-1} - 1 \quad \frac{uv^g - 1}{uv - 1}.$$ 

The proof is similar but much easier than Lemma 3.1 and we omit it.

### 3.2.3. Type III

We turn now to the Type III case. Let $L \cong L^{-1} \in J_0 \cong \mathbb{Z}_2^{2g}$. By tensoring $L \in J_0$, we may restrict our concern to the case $L = \mathcal{O}_X$ so that $V = \mathcal{O}_X \oplus \mathcal{O}_X$.

Since $H^0(\mathcal{E}nd_0 V \otimes K_X) = H^0(K_X) \otimes \mathfrak{sl}(2) \cong \mathbb{C}^g \otimes \mathfrak{sl}(2)$, a Higgs field is of the form

$$\phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

for $a, b, c \in H^0(K_X)$.

If the pair $(V, \phi)$ is not stable, there must be an injective map

$$\iota: \mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{O}_X = V$$

and $\phi$ should preserve the image of $\iota$. Let $p$ (resp. $q$) in $\mathbb{C}$ be the composition of $\iota$ and the projection $V \to \mathcal{O}_X$ onto the first (resp. second) component. If $\phi$ preserves the image of $\iota$, there exists $\lambda \in H^0(K_X)$ such that

$$p\lambda = pa + qb, \quad q\lambda = pc - qa.$$
hold at the same time. It is an elementary exercise to show that this condition is equivalent to saying that $\phi$ is conjugate to an upper triangular matrix $\begin{pmatrix} a' & b' \\ 0 & -a' \end{pmatrix}$. By the Hilbert-Mumford criterion, we conclude that $(V, \phi)$ is stable if and only if $\phi$ is nonzero and the line $[\phi] \in \mathbb{P}(\mathbb{C}^g \otimes \mathfrak{sl}(2))$ is stable with respect to the adjoint action of $SL(2)$. The automorphism group of $V$ is $GL(2)$ which acts on $H^0(\text{End}_0 V \otimes K_X) = \mathbb{C}^g \otimes \mathfrak{sl}(2)$ by conjugation.

Note that the center $\mathbb{C}^*$ of $GL(2)$ acts trivially. So we obtain the following.

Lemma 3.3. The locus of stable pairs $(V, \phi)$ in $M^s$ with $V$ of Type III is the disjoint union of $2^{2g}$ locally closed subvarieties, each isomorphic to $(\mathbb{C}^g \otimes \mathfrak{sl}(2))^{st}/SL(2)$ where $(\mathbb{C}^g \otimes \mathfrak{sl}(2))^{st}$ is the set of nonzero $\phi \in \mathbb{C}^g \otimes \mathfrak{sl}(2)$ with $[\phi] \in \mathbb{P}(\mathbb{C}^g \otimes \mathfrak{sl}(2))$ stable.

In particular, the E-polynomial of the locus of Type III is

$$2^{2g}(uv - 1) \cdot E(\mathbb{P}(\mathbb{C}^g \otimes \mathfrak{sl}(2))^{st}/SL(2)).$$

We can compute $E(\mathbb{P}(\mathbb{C}^g \otimes \mathfrak{sl}(2))^{st}/SL(2))$ by Kirwan’s algorithm [14] as follows. (See [11], §4.) We start with the $SL(2)$-equivariant cohomology of $\mathbb{P}(\mathbb{C}^g \otimes \mathfrak{sl}(2))$ whose Hodge-Deligne series is

$$\frac{1}{1 - (uv)^2} \cdot \frac{1 - (uv)^{3g}}{1 - uv}$$

and subtract out the Hodge-Deligne series of the unstable part

$$(uv)^{2g-1} \cdot \frac{1 + uv + \cdots + (uv)^{g-1}}{1 - uv}.$$

Next we blow up along $SL(2)\mathbb{P}(\mathbb{C}^g \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ and delete the unstable part. For the Hodge-Deligne series, we have to add

$$\frac{1 + uv + \cdots + (uv)^{g-1}}{1 - (uv)^2} \cdot (uv + \cdots + (uv)^{2g-3})$$

$$- (uv)^{g-1} \cdot \frac{1 + uv + \cdots + (uv)^{g-2}}{1 - uv} \cdot (1 + uv + \cdots + (uv)^{g-1}).$$

The $SL(2)$-quotient of this blow-up is Kirwan’s partial desingularization whose E-polynomial\footnote{Note that this is projective and the compact support cohomology is the same as the ordinary cohomology which is isomorphic to the equivariant cohomology of the stable part.} is

$$\frac{(1 - (uv)^{g-1})(1 - (uv)^g)(1 - (uv)^{g+1})}{(1 - uv)^2(1 - (uv)^2)}.$$
from \(3.17\), \(3.18\), and \(3.19\). The quotient of the exceptional divisor of the blow-up is a Zariski locally trivial bundle over \(\mathbb{P}^{g-1}\) with fiber \(\mathbb{P}^{g-2} \times \mathbb{Z}_2\). The E-polynomial is

\[
\frac{1 - (uv)^g}{1 - uv} \cdot \frac{1}{2} \left[ \frac{1 - (uv)^{g-1}}{1 - uv} \right]^2 + \frac{1 - (uv)^{2g-2}}{1 - (uv)^2}
\]

(3.21)

Upon subtracting (3.21) from (3.20), we deduce that

\[
E(\mathbb{P}(\mathbb{C}^g \otimes \text{sl}(2))^{st}/\text{SL}(2)) = \frac{(uv)^g(1 - (uv)^{g-1})(1 - (uv)^g)}{(1 - uv)(1 - (uv)^2)}.
\]

From (3.10), the E-polynomial of the locus of Type III is finally

\[
2^{2g} \cdot \frac{(uv)^g((uv)^{g-1} - 1)((uv)^g - 1)}{(uv)^2 - 1}.
\]

(3.22)

3.2.4. Type IV

As in the Type III case, we may assume \(L \cong \mathcal{O}_X\) and \(V\) is a nontrivial extension of \(\mathcal{O}_X\) by \(\mathcal{O}_X\). The isomorphism classes of such bundles \(V\) are parameterized by

\[
\Gamma = \mathbb{P} \text{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) = \mathbb{P} H^1(\mathcal{O}_X) \cong \mathbb{P}^{g-1}.
\]

There is a universal extension bundle

(3.23)

\[
0 \to \mathcal{O}_\Gamma(1) \to V \to \mathcal{O}_\Gamma \to 0
\]

where \(\mathcal{O}_\Gamma(1)\) is the hyperplane bundle over \(\Gamma\). Let \(p : \Gamma \times X \to \Gamma\) be the projection. Exactly as in the Type II case, we have an exact sequence of vector bundles

(3.24)

\[
0 \to p_* \mathcal{H}om(V, K_X) \to p_*(\mathcal{E}nd_0 V \otimes K_X) \to p_* K_X \to R^1 p_* \mathcal{H}om(V, K_X) \cong R^1 p_* K_X \otimes \mathcal{O}_\Gamma(-1)
\]

and the last map is surjective because the extensions \(V\) are nontrivial. Hence the kernel of the last map is a vector bundle over \(\Gamma\) of rank \(g - 1\). From (3.23), we also have an exact sequence of vector bundles

(3.25)

\[
0 \to p_* K_X \to p_* \mathcal{H}om(V, K_X) \to p_* K_X \otimes \mathcal{O}_\Gamma(-1) \to R^1 p_* K_X
\]

whose last map is surjective because the extensions \(V\) are nontrivial. So the rank of the vector bundle \(p_* \mathcal{H}om(V, K_X)\) over \(\Gamma\) is \(2g - 1\).

As in the Type II case, if we consider the obvious family of Higgs pairs over \(p_*(\mathcal{E}nd_0 V \otimes K_X) \times X\), the locus of stable pairs in \(p_*(\mathcal{E}nd_0 V \otimes K_X)\) is precisely the complement of the subbundle \(p_* \mathcal{H}om(V, K_X)\).

Finally the automorphisms of \(V\) should be taken into account. For a nontrivial extension \(V\) of \(\mathcal{O}_X\) by \(\mathcal{O}_X\), the automorphism group is the additive group \((\mathbb{C}, +)\) and locally \(q \in \mathbb{C}\) acts by

(3.26)

\[
q \cdot \phi = \begin{pmatrix} a + qc & b - 2qa - q^2c \\ c & -a - qc \end{pmatrix} \quad \text{for} \quad \phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.
\]
Lemma 3.4. The locus of stable pairs $(V, \phi)$ with $V$ of Type IV is the disjoint union of $2^{2g}$ locally closed subvarieties, each isomorphic to a $\mathbb{C}^g$-bundle over a $\mathbb{C}^{g-2}$-bundle over a $(\mathbb{C}^{g-1} - 0)$-bundle over $\mathbb{P}^{g-1}$. All the bundles are Zariski locally trivial.

Proof. Let $A$ be the kernel of the last map in (3.24) minus the zero section. Then $A$ is a $(\mathbb{C}^{g-1} - 0)$-bundle over $\Gamma \cong \mathbb{P}^{g-1}$. We think of $p^*(\text{End}_0V \otimes K_X) - p^*\text{Hom}(V, K_X)$ as a vector bundle of rank $2g - 1$ over $\mathbb{A}$.

The kernel of the last map in (3.25) gives rise to a vector bundle $\mathbb{A}$ over $\mathbb{A}$ of rank $g - 1$ and the second map in (3.25) lifts to a $\mathbb{C}$-equivariant map $\left[p^*(\text{End}_0V \otimes K_X) - p^*\text{Hom}(V, K_X)\right] \to \mathbb{A}$ of vector bundles over $\mathbb{A}$ whose kernel is of rank $g$. The action of $\mathbb{C}$ on $\mathbb{A}$ is linear $a \mapsto a + qc$ as the first entry in (3.26). Hence the quotient $\mathbb{A}/\mathbb{C}$ is equivariant and $\mathbb{C}$ acts freely on $\mathbb{A}$,

$$\left[p^*(\text{End}_0V \otimes K_X) - p^*\text{Hom}(V, K_X)\right]/\mathbb{C}$$

is a vector bundle of rank $g$ over $\mathbb{A}/\mathbb{C}$. □

Consequently the E-polynomial of the locus of stable pairs of Type IV is

$$2^{2g} \cdot (uv)^{2g-2} \cdot ((uv)^{g-1} - 1) \frac{(uv)^2 - 1}{uv - 1}.$$ (3.28)

3.3. Unstable case

Suppose $V$ is an unstable rank 2 bundle with trivial determinant. Then there is a unique line subbundle $L$ of $V$ with maximal degree and an exact sequence

$$0 \to L \to V \to L^{-1} \to 0 \quad \text{deg } L = d > 0.$$ (3.29)

Our goal in this subsection is to find the subvariety of $M^s$ parameterizing stable pairs $(V, \phi)$ with $V$ as in (3.29) for each $d > 0$. If $d > g - 1$, then deg$(K_X L^{-1}) < 0$ and Hom$(L, K_X L^{-1}) = 0$. This means that $L$ is preserved by any $\phi \in H^0(\text{End}_0V \otimes K_X)$ and hence $(V, \phi)$ is never stable. From now on, we let $1 \leq d \leq g - 1$.

Proposition 3.5. For $1 \leq d \leq g - 1$, the Hodge-Deligne polynomial of the locus of stable pairs $(V, \phi)$ with $V$ as in (3.29) is

$$(uv)^{3g-3} \cdot E(\tilde{S}^{2g-2-2d}X)$$

where $\tilde{S}^{2g-2-2d}X$ is a $2^{2g}$-fold covering of the symmetric product $S^{2g-2-2d}X$ of $X$. 
Let $\text{Pic}^d(X) \rightarrow \text{Pic}^{2g-2-2d}(X)$ be the map $L \mapsto K_X L^{-2}$. This is obviously a $2^{2g_d}$-fold covering. Let $P^r_d$ be the locus in $\text{Pic}^{2g-2-2d}(X)$ of line bundles $\xi$ satisfying $h^0(\xi) = r + 1$ and $\tilde{P}^r_d$ be the inverse image of $P^r_d$ in $\text{Pic}^d(X)$. Then $\tilde{P}^r_d$ parameterizes line bundles $L$ of degree $d$ with $h^1(L^2) = h^0(K_X L^{-2}) = r + 1$. Let $S^{2g-2-2d}_r X$ be the inverse image of $P^r_d$ by the Abel-Jacobi map $S^{2g-2-2d}_r X \rightarrow \text{Pic}^{2g-2-2d} X$ and $S^{2g-2-2d}_r X$ be the fibre product of $\tilde{P}^r_d$ and $S^{2g-2-2d}_r X$ over $P^r_d$. Then we have a commutative square

\[
\begin{array}{ccc}
S^{2g-2-2d}_r X & \xrightarrow{\tilde{g}} & \tilde{P}^r_d \\
\downarrow f & & \downarrow f \\
S^{2g-2-2d}_r X & \xrightarrow{g} & P^r_d.
\end{array}
\]

The vertical maps are $2^{2g_d}$-fold coverings and the horizontal maps are $\mathbb{P}^r$-bundles.

Let $\mathcal{L} \rightarrow \tilde{P}^r_d \times X$ be the restriction of a universal line bundle over $\text{Pic}^d(X) \times X$. Then there is a line bundle $\mathcal{M} \rightarrow P^r_d \times X$ such that

\[
(f \times 1_X)^* \mathcal{M} \cong K_X \mathcal{L}^{-2}
\]

by the universal property of $\text{Pic}(X)$. If we let $\overline{\pi} : P^r_d \times X \rightarrow P^r_d$ be the projection, then $S^{2g-2-2d}_r X = \mathbb{P}(\overline{\pi}_* \mathcal{M})$ and hence we have an injection

\[
\mathcal{O} \rightarrow \tilde{g}^* \overline{\pi}_* \mathcal{M} \otimes \mathcal{O}(1)
\]

over $S^{2g-2-2d}_r X$. Note that

\[
\tilde{f}^* \tilde{g}^* \overline{\pi}_* \mathcal{M} = \tilde{g}^* \tilde{f}^* \overline{\pi}_* \mathcal{M} = \tilde{g}^* \pi_* (f \times 1_X)^* \mathcal{M} = \tilde{g}^* \pi_* K_X \mathcal{L}^{-2}
\]

by (3.31), where $\pi : \tilde{P}^r_d \times X \rightarrow \tilde{P}^r_d$ is the projection. The sheaf $R^1 \pi_* \mathcal{L}^2$ is a vector bundle of rank $r + 1$ and there is a perfect pairing

\[
\tilde{g}^* R^1 \pi_* \mathcal{L}^2 \otimes \tilde{g}^* \pi_* K_X \mathcal{L}^{-2} = \tilde{g}^* (R^1 \pi_* \mathcal{L}^2 \otimes \pi_* K_X \mathcal{L}^{-2}) \rightarrow \tilde{g}^* (R^1 \pi_* K_X) \cong \mathcal{O}
\]

by the Serre duality. Combining (3.32), (3.33) and (3.31), we get a surjective map of vector bundles

\[
\tilde{g}^* R^1 \pi_* \mathcal{L}^2 \rightarrow \mathcal{O}(1).
\]

Let $A$ be the kernel of this map which is a vector bundle over $S^{2g-2-2d}_r X$ of rank $r$. Let $\mathcal{L}^\#$ be the pull-back of $\mathcal{L}$ to $A \times X$ via the composition $A \rightarrow S^{2g-2-2d}_r X \rightarrow \tilde{P}^r_d$. It is well-known that there is a universal extension bundle over $R^1 \pi_* \mathcal{L}^2$ of $\mathcal{L}^{-1}$ by $\mathcal{L}$ and hence we have a rank 2 vector bundle $\mathcal{V}^\#$ which fits into an exact sequence

\[
0 \rightarrow \mathcal{L}^\# \rightarrow \mathcal{V}^\# \rightarrow (\mathcal{L}^\#)^{-1} \rightarrow 0
\]

over $A \times X$. 
To incorporate the Higgs field, we consider Hitchin’s sequence (3.7)[1]

\[ 0 \to \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X) \to \mathcal{E}nd_0 \mathcal{V}^\# \otimes K_X \to K_X(\mathcal{L}^\#)^{-2} \to 0 \]

over \( A \times X \). Let \( p : A \times X \to A \) be the projection. Then we have an exact sequence

\[ 0 \to p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X) \to p_* \mathcal{E}nd_0 \mathcal{V}^\# \otimes K_X \to p_* K_X(\mathcal{L}^\#)^{-2} \to R^1 p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X). \]

As \( \mathcal{L}^\# \) is the pull-back of \( \mathcal{L} \to \tilde{P}_d^r \times X \), \( p_* K_X(\mathcal{L}^\#)^{-2} \) is the pull-back of \( \tilde{g}^* \pi_*(K_X\mathcal{L}^{-2}) \). By (3.32) and (3.33), there is an injection \( \mathcal{O}(-1) \to p_* K_X(\mathcal{L}^\#)^{-2} \).

From (3.35) we have an exact sequence

\[ 0 \to K_X(\mathcal{L}^\#)^2 \to \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X) \to K_X \to 0 \]

and a long exact sequence

\[ 0 \to p_* K_X(\mathcal{L}^\#)^2 \to p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X) \to p_* K_X \to R^1 p_* K_X(\mathcal{L}^\#)^2 \to R^1 p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X) \to R^1 p_* K_X \to 0. \]

By the Serre duality, \( R^1 p_* K_X(\mathcal{L}^\#)^2 = 0 \) and thus

\[ R^1 p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X) \cong R^1 p_* K_X \cong 0. \]

By the definition of \( A \), the composition of (3.35) with the last map in (3.37) is zero. Let us consider the commutative diagram

\[ \begin{array}{ccc}
p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X) & \to & p_* \mathcal{E}nd_0 \mathcal{V}^\# \otimes K_X \\
p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X) & \to & p_* K_X(\mathcal{L}^\#)^{-2} & \to & R^1 p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X) \\
 & & B & \to & \mathcal{O}(-1) & \to & 0
\end{array} \]

where \( B \) is the fiber product of \( \mathcal{O}(-1) \) and \( p_* \mathcal{E}nd_0 \mathcal{V}^\# \otimes K_X \) over \( p_* K_X(\mathcal{L}^\#)^{-2} \). From (3.39), \( p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X) \) is a vector bundle of rank \( h^0(L^2 K_X) + h^0(K_X) = 2g + 2d - 1 \) for \( L \in \tilde{P}_d^r \) and hence \( B \) is a vector bundle over \( A \) of rank \( 2g + 2d \).

There is an obvious family \( (\mathcal{V}^\dagger, \Phi^\dagger) \) of Higgs pairs parameterized by \( B \), namely the restriction of the tautological family of Higgs pairs parameterized by \( p_* (\mathcal{E}nd_0 \mathcal{V}^\# \otimes K_X) \). A member \( (\mathcal{V}, \phi) = (\mathcal{V}^\dagger, \Phi^\dagger)|_s \) for \( s \in B \) lying over \( L \) is stable if and only if \( \phi \) does not preserve \( L \), i.e. \( s \) is not mapped to the zero section of \( \mathcal{O}(-1) \) via the bottom right horizontal map in (3.40). Hence the locus of stable pairs in \( B \) is

\[ B - p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X) \]

which is a \( \mathbb{C}^{2g+2d-1} \)-bundle over a \( \mathbb{C}^* \times \mathbb{C}^* \)-bundle over \( \tilde{S}_r^{2g-2-2d}X \). By construction, it is clear that the bundles are all Zariski locally trivial.

Finally we consider the isomorphism classes of stable pairs in the family parameterized by \( B - p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\#K_X) \). Fix \( L \in \tilde{P}_d^r \) and \( D \in S_r^{2g-2-2d}X \).
such that \( K_X L^{-2} \cong \mathcal{O}(D) \), i.e. \((L, D) \in \bar{S}_r^{2g-2-2d} X\). There is an action of \( \mathbb{C}^* \) on the fiber of 
\[
[B - p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\# K_X)] \rightarrow \bar{S}_r^{2g-2-2d} X
\]
over \((L, D)\) as follows. Let \((U_i)\) be a sufficiently fine open cover of \(X\). Then for any stable pair \((V, \phi)\) in the fiber, the transition matrices for \(V\) and \(\phi\) can be written as 
\[
T_{ij} = \begin{pmatrix} \lambda_{ij} & \rho_{ij} \\ 0 & \lambda_{ij}^{-1} \end{pmatrix} \quad \text{and} \quad \phi|_{U_i} = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix}
\]
with \(c = (c_i) \in H^0(K_X L^{-2})\) whose divisor is \(\text{div}(c) = D\). Then for \(t \in \mathbb{C}^*\), the diagonal matrix with diagonal entries \((t, t^{-1})\) acts on \(T_{ij}\) and \(\phi|_{U_i}\) by conjugation:
\[
t \cdot T_{ij} = \begin{pmatrix} \lambda_{ij} & t^2 \rho_{ij} \\ 0 & \lambda_{ij}^{-1} \end{pmatrix}, \quad t \cdot \phi|_{U_i} = \begin{pmatrix} a_i & t^2 b_i \\ t^{-2} c_i & -a_i \end{pmatrix}
\]
Hence \(\mathbb{C}^*\) acts on the fiber of \(\mathcal{O}(-1)\) and \(A\) over \(\bar{S}_r^{2g-2-2d} X\) with weights \(-2\) and \(2\) respectively. But the quotient of \(\mathbb{C}^* \times \mathbb{C}^r\) by the action of \(\mathbb{C}^*\) with weights \(-2\) and \(2\) is exactly \(\mathbb{C}^r\). Hence
\[
[B - p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\# K_X)] / \mathbb{C}^* \cong p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\# K_X)
\]
which is a \(\mathbb{C}^{2g+2d-1}\)-bundle over \(A\).

Next the additive group \((H^0(L^2), +)\) acts on \(p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\# K_X)\) as follows. Let \((\mu_i)\) be a cocycle representing a class in \(H^0(L^2)\). Then \(\mu_i\) acts on \(T_{ij}\) and \(\phi|_{U_i}\) by conjugation:
\[
\begin{pmatrix} 1 & \mu_i \\ 0 & 1 \end{pmatrix} T_{ij} \begin{pmatrix} 1 & -\mu_i \\ 0 & 1 \end{pmatrix} = T_{ij}
\]
(3.41) \[
\begin{pmatrix} 1 & \mu_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \begin{pmatrix} 1 & -\mu_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_i - c_i \mu_i & b_i + 2a_i \mu_i - c_i \mu_i^2 \\ c_i \mu_i - a_i \end{pmatrix}.
\]

Because \(\text{div}(c) = D\), we have an exact sequence
\[
0 \rightarrow L^2 \xrightarrow{c} K_X \rightarrow K_X|_D \rightarrow 0
\]
and hence \(c : H^0(L^2) \rightarrow H^0(K_X)\) is injective. Exactly as in the proof of Lemma 3.34 for the Type IV case, the quotient of \(p_* K_X\) by the free linear action \(a \mapsto a - c \mu\) of \(H^0(L^2)\) as in the first entry of (3.41) is a vector bundle of rank \(g - h^0(L^2)\) and the second nonzero map in (3.39)
\[
p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\# K_X) \rightarrow p_* K_X
\]
is equivariant. Hence the quotient of \(p_* \mathcal{H}om(\mathcal{V}^\#, \mathcal{L}^\# K_X)\) by \(H^0(L^2)\) is a vector bundle of rank \(h^0(K_X L^2) = g + 2d - 1\) over a vector bundle of rank
\[
g - h^0(L^2) = g - (2d - g + 1 + r + 1) = 2g - 2d - r - 2
\]
over $A$ because $h^1(L^2) = r + 1$. Since $A$ is a Zariski locally trivial $\mathbb{C}^*$-bundle over $\mathbb{S}^{2g-2-2d}_r$, the E-polynomial of the locus of stable pairs $(V, \phi)$ with $V$ as in (3.29) and $h^1(L^2) = r + 1$, is
\[
(uv)^{g+2d-1}(uv)^{g-2d-r-2}(uv)^r E(\mathbb{S}^{2g-2-2d}_r) = (uv)^{3g-3} E(\mathbb{S}^{2g-2-2d}_r).
\]
Summing up for all $r$, we get
\[
(uv)^{3g-3} E(\mathbb{S}^{2g-2-2d}_r).
\]
This completes the proof of Proposition 3.5.

**Corollary 3.6.** The E-polynomial of the locus of stable pairs with $V$ unstable is
\[
(3.42) \quad (uv)^{3g-3} \cdot \sum_{d=1}^{g-1} E(\mathbb{S}^{2g-2-2d}_r).
\]

By mimicking Hitchin’s computation in [8, §7], we see that
\[
E(\mathbb{S}^n X) = E(S^n X) + (2g^2 - 1) \sum_{r+s=n} (-1)^{r+s} \binom{g-1}{r} \binom{g-1}{s} u^r v^s
\]
\[
(3.43) = \text{Coeff}_{x^n} \left[ \frac{(1-ux)^g(1-vx)^g}{(1-x)(1-uxx)} + (2g^2 - 1)(1-ux)^{g-1}(1-vx)^{g-1} \right]
\]
\[
= \text{Coeff}_{x^n} (1-ux)^{g-1}(1-vx)^{g-1} \left[ \frac{x(1-u)(1-v)}{(1-x)(1-uxx)} + 2g \right].
\]
Observe that
\[
(3.44) \quad \sum_{d=1}^{g-1} \text{Coeff}_{x^{2g-2-2d}} (1-ux)^{g-1}(1-vx)^{g-1}
\]
\[
= \frac{1}{2} \left[ (1-u)^{g-1}(1-v)^{g-1} + (1+u)^{g-1}(1+v)^{g-1} - 2(uv)^{g-1} \right].
\]

We keep using Hitchin’s method of calculation. We also have
\[
(3.45) \quad \sum_{d=1}^{g-1} \text{Coeff}_{x^{2g-2-2d}} \frac{x(1-ux)^{g-1}(1-vx)^{g-1}}{(1-x)(1-uxx)}
\]
\[
= \sum_{d=1}^{g-1} \text{Res}_{x=0} \frac{(1-ux)^{g-1}(1-vx)^{g-1}}{x^{2g-2-2d}(1-x)(1-uxx)}
\]
\[
= \sum_{d=1}^{g-1} \text{Res}_{x=0} \frac{(1-ux)^{g-1}(1-vx)^{g-1}}{x^{2g-2-2d}(1-x)(1-uxx)}
\]
\[
= \text{Res}_{x=0} \frac{(1-ux)^{g-1}(1-vx)^{g-1}}{x^{2g-4}(1-x^2)(1-x)(1-uxx)}.
\]
As $x \to \infty$, the function is close to $(uv)^{g-2}/x^2$. By Cauchy’s residue theorem,
\[
\text{Res}_{x=0} = - \left( \text{Res}_{x=1} + \text{Res}_{x=-1} + \text{Res}_{x=(uv)^{-1}} \right).
\]
The residue at the simple pole $x = -1$ is
\[
\frac{(1 + u)^{g-1}(1 + v)^{g-1}}{4(1 + uv)}.
\]
The residue at the simple pole $x = (uv)^{-1}$ is
\[
\frac{(uv)^{g-1}(1 - u)^{g-1}(1 - v)^{g-1}}{(uv - 1)^2(1 + uv)}.
\]
The residue at the double pole $x = 1$ is
\[
\frac{g - 1}{2} \left\{ \frac{u + v - 2uv(1 - u)^{g-2}(1 - v)^{g-2}}{1 - uv} \right. \\
- \frac{4g - 7}{4} \frac{(1 - u)^{g-1}(1 - v)^{g-1}}{1 - uv} + \frac{uv(1 - u)^{g-1}(1 - v)^{g-1}}{2(1 + uv)} \right.
\]
Therefore the E-polynomial of the locus of stable pairs $(V, \phi)$ with $V$ unstable is
\[
(3.46) \quad (uv)^{3g-3} \cdot \sum_{d=1}^{g-1} E(S^{2g-2-2d} X)
= 2^{2g-4}(uv)^{3g-3} \left[ (1 - u)^{g-1}(1 - v)^{g-1} + (1 + u)^{g-1}(1 + v)^{g-1} - 2(uv)^{g-1} \right] \\
+ (uv)^{3g-3}(1 - u)(1 - v) \left[ -\frac{(1 + u)^{g-1}(1 + v)^{g-1}}{4(1 + uv)} + \frac{(uv)^{g-1}(1 - u)^{g-1}(1 - v)^{g-1}}{(uv - 1)^2(1 + uv)} \right] \\
+ \frac{g - 1}{2} \left\{ \frac{u + v - 2uv(1 - u)^{g-2}(1 - v)^{g-2}}{1 - uv} \right. \\
- \frac{4g - 7}{4} \left( (1 - u)^{g-1}(1 - v)^{g-1} \right) \right.
\]
from (3.41), (3.42), and (3.43).

3.4. Hodge-Deligne polynomial of $M^s$

So far we considered all possible types of stable pairs $(V, \phi)$ and computed the E-polynomials of the corresponding loci in $M^s$. By adding up (3.2), (3.9), (3.15), (3.22), (3.28) and (3.46), we obtain the following.

Theorem 3.7.

\[
E(M^s) = (uv)^{3g-3} \left\{ (1 - u^2v)^g(1 - uv)^2 - (uv)^{g+1}(1 - u)^g(1 - v)^g \right\} \\
\frac{1}{1 - (uv)^2} \left\{ \frac{(1 - u)^g(1 - v)^g}{1 - uv} + \frac{(1 + u)^g(1 + v)^g}{1 + uv} \right\} \\
+ (uv)^g \cdot \left( \frac{1}{2} (1 - u)^g(1 - v)^g + \frac{1}{2} (1 + u)^g(1 + v)^g \right) \frac{((uv)^g - 1)((uv)^{g-1} - 1)}{(uv)^2 - 1} \\
+ (uv)^{g+1} \cdot \left( \frac{1}{2} (1 - u)^g(1 - v)^g - \frac{1}{2} (1 + u)^g(1 + v)^g \right) \frac{((uv)^{g-1} - 1)((uv)^{g-2} - 1)}{(uv)^2 - 1}
\]
In this section, we show that $M$ is desingularized by three blow-ups by Kirwan’s algorithm for desingularizations [14]. We will see that the singularities of $M$ are identical to those of the moduli space of rank 2 semistable sheaves with Chern classes $c_1 = 0$ and $c_2 = 2g$ on a K3 surface with generic polarization as studied in [17]. O’Grady constructed the Kirwan desingularization by three blow-ups and we use his arguments.

We first collect some of Simpson’s results. Let $N$ be a sufficiently large integer and $p = 2N + 2(1 - g)$. Then we have the following.

1. [19, Theorem 3.8]
   
   There is a quasi-projective scheme $Q$ representing the moduli functor which parameterizes the isomorphism classes of triples $(V, \phi, \alpha)$ where $(V, \phi)$ is a semistable Higgs pair with $\det V \cong \mathcal{O}_X$, $\text{tr} \phi = 0$ and $\alpha$ is an isomorphism
   
   $$\alpha : \mathbb{C}^p \to H^0(X, V \otimes \mathcal{O}(N)).$$

2. [19, Theorem 4.10]
   
   Fix $x \in X$. Let $\tilde{Q}$ be the frame bundle at $x$ of the universal bundle restricted to $x$. Then the action of $GL(p)$ lifts to $\tilde{Q}$ and $SL(2)$ acts on the fibers of $\tilde{Q} \to Q$ in an obvious fashion. Every point of $\tilde{Q}$ is stable with respect to the free action of $GL(p)$ and
   
   $$R = \tilde{Q}/GL(p)$$
   
   represents the moduli functor which parameterizes triples $(V, \phi, \beta)$ where $(V, \phi)$ is a semistable Higgs pair with $\det V \cong \mathcal{O}_X$, $\text{tr} \phi = 0$ and $\beta$ is an isomorphism
   
   $$\beta : V|_x \to \mathbb{C}^2.$$
3. \[19\] Theorem 4.10
Every point in $R$ is semistable with respect to the (residual) action of $SL(2)$. The closed orbits in $R$ correspond to polystable pairs, i.e. $(V, \phi)$ is stable or

$$(V, \phi) = (L, \psi) \oplus (L^{-1}, -\psi)$$

for $L \in \text{Pic}^0(X)$ and $\psi \in H^0(K_X)$. The set $R^s$ of stable points with respect to the action of $SL(2)$ is exactly the locus of stable pairs.

4. \[19\] Theorem 4.10
The good quotient $R//SL(2)$ is $M$.

5. \[19\] Theorem 11.1
$R$ and $M$ are both irreducible normal quasi-projective varieties.

6. \[19\] §10
Let $A^i$ (resp. $A^{i,j}$) be the sheaf of smooth $i$-forms (resp. $(i,j)$-forms) on $X$. For a polystable Higgs pair $(V, \phi)$, consider the complex

\[ 0 \rightarrow \text{End}_0 V \otimes A^0 \rightarrow \text{End}_0 V \otimes A^1 \rightarrow \text{End}_0 V \otimes A^2 \rightarrow 0 \]

whose differential is given by $D'' = \overline{\partial} + \phi$. Because $A^1 = A^{1,0} \oplus A^{0,1}$ and $\phi$ is of type $(1,0)$, we have an exact sequence of complexes with \[11\] in the middle

\[ 0 \rightarrow \text{End}_0 V \otimes A^0 \rightarrow \text{End}_0 V \otimes A^1 \otimes A^{1,0} \rightarrow \text{End}_0 V \otimes A^{1,1} \rightarrow 0 \]

This gives us a long exact sequence

\[ 0 \rightarrow T^0 \rightarrow H^0(\text{End}_0 V) \xrightarrow{[\phi,-]} H^0(\text{End}_0 V \otimes K_X) \rightarrow \]

\[ \rightarrow T^1 \rightarrow H^1(\text{End}_0 V) \xrightarrow{[\phi,-]} H^1(\text{End}_0 V \otimes K_X) \rightarrow T^2 \rightarrow 0 \]

where $T^i$ is the $i$-th cohomology of \[11\]. The Zariski tangent space of $M$ at polystable $(V, \phi)$ is isomorphic to $T^1$. 
7. [19, Theorem 10.4] 
Using the above notation, let $C$ be the quadratic cone in $T^1$ defined by the map

$$T^1 \to T^2$$

which sends a $\text{End}_0 V$-valued 1-form $\eta$ to $[\eta, \eta]$. Let $y = (V, \phi, \beta) \in R$ be a point with closed orbit. Then the formal completion $(R, y)^\wedge$ is isomorphic to the formal completion $(C \times h^\perp, 0)^\wedge$ where $h^\perp$ is the perpendicular space to the image of $T^0 \to H^0(\text{End}_0 V) \to \mathfrak{sl}(2)$. Furthermore, if we let $Y$ be the étale slice at $y$ of the $SL(2)$-orbit in $R$, then

$$(Y, y)^\wedge \cong (C, 0)^\wedge.$$ 

8. [19, Lemma 10.7] 
The dimension of the Zariski tangent space to $R$ at $y = (V, \phi, \beta)$ is

$$\dim T^1 + 3 - \dim T^0$$

and $\dim T^0 = \dim T^2$ by the Serre duality and (4.3).

By Riemann-Roch and [13], we have

(4.5) \[ \dim T^1 = \chi(\text{End}_0 \otimes K_X) - \chi(\text{End}_0 V) \]

+ \[ \dim T^0 + \dim T^2 = 6g - 6 + 2 \dim T^0. \]

Therefore, $R$ is smooth at $y = (V, \phi, \beta)$ if and only if $T^0 = 0$. When $(V, \phi)$ is stable, there is no section of $H^0(\text{End}_0 V)$ commuting with $\phi$ and so $T^0 = 0$. Hence the stable locus $R^s$ is smooth and so is the orbit space $M^s$.

The complement of $R^s$ in $R$ consists of 4 subvarieties parameterizing strictly semistable pairs $(V, \phi)$ of the following 4 types respectively:

(i) $(L, 0) \oplus (L, 0)$ for $L \cong L^{-1}$
(ii) nontrivial extension of $(L, 0)$ by $(L, 0)$ for $L \cong L^{-1}$
(iii) $(L, \psi) \oplus (L^{-1}, -\psi)$ for $(L, \psi) \not\cong (L^{-1}, -\psi)$
(iv) nontrivial extension of $(L^{-1}, -\psi)$ by $(L, \psi)$ for $(L, \psi) \not\cong (L^{-1}, -\psi)$

where $L \in \text{Pic}^0(X) =: J$ and $\psi \in H^0(K_X)$. Higgs pairs of type (ii) and (iv) are not polystable and their orbits do not appear in $M$.

Let us first consider the locus of type (i). It is obvious from definition that the loci of type (i) in $M$ and in $R$ are both isomorphic to the $\mathbb{Z}_2$-fixed point set $J_0 \cong \mathbb{Z}_2^{2g}$ in the Jacobian $J$ by the involution $L \mapsto L^{-1}$. From (4.3) we have isomorphisms

$$T^0 \cong H^0(\text{End}_0 V) \cong \mathfrak{sl}(2)$$

and an exact sequence

(4.6) \[ 0 \to H^0(\text{End}_0 V \otimes K_X) \to T^1 \to H^1(\text{End}_0 V) \to 0. \]
Choose a splitting of $T^1 \cong H^0(\text{End}_0 V \otimes K_X) \oplus H^1(\text{End}_0 V) \cong H^0(K_X) \otimes \mathfrak{sl}(2) \oplus H^1(\mathcal{O}_X) \otimes \mathfrak{sl}(2)$.

The quadratic map (4.4) is just the Lie bracket of $\mathfrak{sl}(2)$ coupled with the perfect pairing $H^0(K_X) \otimes H^1(\mathcal{O}_X) \to H^1(K_X)$.

It is easy to check that this coincides with the quadratic map $\Upsilon$ in [17, p.65] and thus the singularity of $R$ along the locus $\mathbb{Z}_2^{2g}$ of type (i) is the same as the deepest singularity in O’Grady’s case [17]. Moreover, the actions of $SL(2)$ on the quadratic cones are identical. In particular, the singularity of the locus $\mathbb{Z}_2^{2g}$ in $M$ of Higgs pairs $(L,0) \oplus (L,0)$ with $L \cong L^{-1}$ is the hyperkähler quotient

$$\mathbb{H}^g \otimes \mathfrak{sl}(2)//SL(2)$$

where $\mathbb{H}$ is the division algebra of quaternions.

Next let us look at the locus of type (iii). It is clear that the locus of type (iii) in $M$ is isomorphic to

$$J \times \mathbb{Z}_2 H^0(K_X) - \mathbb{Z}_2^{2g} \cong T^* J/\mathbb{Z}_2 - \mathbb{Z}_2^{2g}$$

where $\mathbb{Z}_2$ acts on $J$ by $L \mapsto L^{-1}$ and on $H^0(K_X)$ by $\psi \mapsto -\psi$. The locus of type (iii) in $R$ is a $\mathbb{P}SL(2)/\mathbb{C}^*$-bundle over $T^* J/\mathbb{Z}_2 - \mathbb{Z}_2^{2g}$ and in particular it is smooth. As in the type (i) case, by using (4.3) and (4.4), it is straightforward to show that the singularity along the locus of type (iii) in $R$ is the same as the singularity along $\Sigma^0_Q$ in O’Grady’s case [17, §1.4]. In particular, the singularity along the locus in $M$ of type (iii) is the hyperkähler quotient

$$\mathbb{H}^{g-1} \otimes T^* \mathbb{C}///\mathbb{C}^*$$

and we have a stratification of $M$;

$$M = M^s \sqcup (T^* J/\mathbb{Z}_2 - \mathbb{Z}_2^{2g}) \sqcup \mathbb{Z}_2^{2g}.$$

Since we have identical singularities and stratification as in O’Grady’s case in [17], we can copy his arguments almost line by line to construct the Kirwan desingularization $\hat{M}$ of $M$ and study its blow-downs. So we skip the details but give only a brief outline.

We blow up $R$ first along the locus of type (i) and then along the locus of type (iii). Then we delete the unstable part with respect to the action of $SL(2)$. After these two blow-ups the loci of types (ii) and (iv) become unstable by [14] (or more explicitly by [17, Lemma 1.7.4]) and thus they are deleted anyway. Let $S^{ss}$ (resp. $S^s$) be the open subset of semistable (resp. stable) points after the two blow-ups. Then we have

(a) $S^{ss} = S^s$,

(b) $S^s$ is smooth.

In particular, $S^s/SL(2)$ has at worst orbifold singularities. When $g = 2$, this is already smooth. When $g \geq 3$, by blowing up $S^s$ one more time along
the locus of points with stabilizers larger than the center \( \mathbb{Z}_2 \) of \( SL(2) \), we obtain a variety \( \hat{S} \) whose orbit space
\[
\hat{M} := \hat{S}/SL(2)
\]
is a smooth variety obtained by blowing up \( M \) first along \( \mathbb{Z}_2^{2g} \), second along the proper transform of \( T^*J/\mathbb{Z}_2 \) and third along a nonsingular subvariety lying in the proper transform of the exceptional divisor of the first blow-up. Let
\[
\pi : \hat{M} \rightarrow M
\]
be the composition of the three blow-ups. We call \( \hat{M} \) the Kirwan desingularization of \( M \). Along the line of O’Grady’s \[17\], we obtain the following.

**Theorem 4.1.**  
(1) For \( g = 2 \), \( \hat{M} \) can be blown down to give us a symplectic desingularization of \( M \).
(2) For \( g \geq 3 \), \( \hat{M} \) can be blown down twice to give us another smooth model of \( M \), which we call the O’Grady desingularization.
(3) The three exceptional divisors \( D_1, D_2, D_3 \) of \( \pi : \hat{M} \rightarrow M \) coming from the three blow-ups are smooth and normal crossing. The discrepancy divisor is
\[
(4.8) \quad K_M = K_{\hat{M}} - \pi^*K_M = (6g - 7)D_1 + (2g - 4)D_2 + (4g - 6)D_3.
\]

Note that \( K_M = 0 \) since \( M \) is hyperkähler \[5\].

It is possible to extract explicit descriptions of the divisors \( D_1, D_2, D_3 \) from \[17\] as follows. (See also \[3, Proposition 3.2\], \[4, Proposition 3.6\].) Let \( \hat{\mathbb{P}}^5 \) be the blow-up of \( \mathbb{P}^5 \) (projectivization of the space of \( 3 \times 3 \) symmetric matrices) along \( \mathbb{P}^2 \) (the locus of rank 1 matrices). Let \( (\mathbb{C}^{2g}, \omega) \) be a symplectic vector space and let \( Gr^\omega(k, 2g) \) be the Grassmannian of \( k \)-dimensional subspaces of \( \mathbb{C}^{2g} \), isotropic with respect to the symplectic form \( \omega \) (i.e. the restriction of \( \omega \) to the subspace is zero). Let \( I_{2g-3} \) denote the incidence variety given by
\[
I_{2g-3} = \{(p, H) \in \mathbb{P}^{2g-3} \times \mathbb{P}^{2g-3} | p \in H \}.
\]

Then we have the following.

**Proposition 4.2.** Let \( g \geq 3 \).
(1) \( D_1 \) is the disjoint union of \( 2^{2g} \) copies of a \( \hat{\mathbb{P}}^5 \)-bundle over \( Gr^\omega(3, 2g) \).
(2) \( D_2 \) is a free \( \mathbb{Z}_2 \)-quotient of a \( I_{2g-3} \)-bundle over \( T^*J - J_0 \).
(3) \( D_3 \) is the disjoint union of \( 2^{2g} \) copies of a \( \mathbb{P}^{2g-4} \)-bundle over a \( \mathbb{P}^2 \)-bundle over \( Gr^\omega(2, 2g) \).
(4) \( D_{22} \) is the disjoint union of \( 2^{2g} \) copies of a \( \mathbb{P}^2 \)-bundle over \( Gr^\omega(2, 2g) \).
(5) \( D_{23} \) is the disjoint union of \( 2^{2g} \) copies of a \( \mathbb{P}^{2g-4} \)-bundle over a \( \mathbb{P}^1 \)-bundle over \( Gr^\omega(3, 2g) \).
(6) \( D_{13} \) is the disjoint union of \( 2^{2g} \) copies of a \( \mathbb{P}^2 \)-bundle over a \( \mathbb{P}^2 \)-bundle over \( Gr^\omega(3, 2g) \).
(7) $D_{123}$ is the disjoint union of $2^{2g}$ copies of a $\mathbb{P}^1$-bundle over a $\mathbb{P}^2$-bundle over $\text{Gr}^\omega(3, 2g)$.

All the bundles above are Zariski locally trivial.

5. STRINGY E-FUNCTION OF $\mathcal{M}$

In this section we compute the stringy E-function of $\mathcal{M}$ by using Proposition 4.2 and (4.8) and show that there does not exist a symplectic desingularization of $\mathcal{M}$ for $g \geq 3$.

By Simpson’s theorem, $\mathcal{M}$ is an irreducible normal variety with Gorenstein singularities because $K_\mathcal{M} = 0$. By (4.8), the singularities are canonical and the stringy E-function of $\mathcal{M}$ is a well-defined rational function of $u, v$. From (2.1) and (4.8), the stringy E-function $E_{st}(\mathcal{M}; u, v)$ of $\mathcal{M}$ is

\begin{equation}
E(\mathcal{M}^s; u, v) + E(D^0_1; u, v) \frac{1-uv}{1-(uv)^{g-\sigma}_0} + E(D^0_2; u, v) \frac{1-uv}{1-(uv)^{g-\sigma}_2} + E(D^0_3; u, v) \frac{1-uv}{1-(uv)^{g-\sigma}_3} + E(D^0_{12}; u, v) \frac{1-uv}{1-(uv)^{g-\sigma}_{12}} + E(D^0_{123}; u, v) \frac{1-uv}{1-(uv)^{g-\sigma}_{123}}.
\end{equation}

From Proposition 4.2 and the identity ([3, Lemma 3.1], [4, Lemma 4.1])

\[ E(\text{Gr}^\omega(k, 2g); u, v) = \prod_{1 \leq i \leq k} \frac{1 - (uv)^{2g-2k+2i}}{1 - (uv)^i}, \]

we obtain the following. (See [3 Corollary 3.3] or [4 Corollary 4.2].)

Proposition 5.1.

\[ E(D_1; u, v) = 2^{2g} \cdot \left( \frac{1-(uv)^6}{1-uv} - \frac{1-(uv)^3}{1-uv} + \left( \frac{1-(uv)^3}{1-uv} \right)^2 \right) \cdot \prod_{1 \leq i \leq 3} \frac{1-(uv)^{2g-6+2i}}{1-(uv)^i}, \]

\[ E(D_3; u, v) = 2^{2g} \cdot \frac{1-(uv)^{2g-3}}{1-uv} \cdot \frac{1-(uv)^3}{1-uv} \cdot \prod_{1 \leq i \leq 2} \frac{1-(uv)^{2g-4+2i}}{1-(uv)^i}, \]

\[ E(D_{12}; u, v) = 2^{2g} \cdot \left( \frac{1-(uv)^3}{1-uv} \right)^2 \cdot \prod_{1 \leq i \leq 3} \frac{1-(uv)^{2g-6+2i}}{1-(uv)^i}, \]

\[ E(D_{23}; u, v) = 2^{2g} \cdot \frac{1-(uv)^{2g-3}}{1-uv} \cdot \frac{1-(uv)^2}{1-uv} \cdot \prod_{1 \leq i \leq 2} \frac{1-(uv)^{2g-4+2i}}{1-(uv)^i}, \]

\[ E(D_{13}; u, v) = 2^{2g} \cdot \frac{1-(uv)^3}{1-uv} \cdot \frac{1-(uv)^{2g-4}}{1-uv} \cdot \prod_{1 \leq i \leq 2} \frac{1-(uv)^{2g-4+2i}}{1-(uv)^i}, \]

\[ E(D_{123}; u, v) = 2^{2g} \cdot \frac{1-(uv)^2}{1-uv} \cdot \frac{1-(uv)^{2g-4}}{1-uv} \cdot \prod_{1 \leq i \leq 2} \frac{1-(uv)^{2g-4+2i}}{1-(uv)^i}. \]
For $D^0_2$, observe that $H^*_c(D^0_2)$ is the $\mathbb{Z}_2$-invariant part of

\[ H^*(I_{2g-3}) \otimes H^*_c(T^*J - J_0) \]

and hence we have

\[ E(D^0_2; u, v) = E(I_{2g-3}; u, v)^+ \cdot (E(T^*J; u, v)^+ - 2^{2g}) + E(I_{2g-3}; u, v)^- \cdot E(T^*J; u, v)^- \]

where $E(Z; u, v)^+$ (resp. $E(Z; u, v)^-$) denotes the E-polynomial of the $\mathbb{Z}_2$-invariant (resp. anti-invariant) part of $H^*_c(Z)$. By elementary computation ([3] §5 or [4] Lemma 4.3), we have

\[ E(I_{2g-3}; u, v)^+ = \frac{(1 - (uv)^{2g-2})(1 - (uv)^{2g-3})}{(1 - uv)(1 - (uv)^2)} \]

\[ E(I_{2g-3}; u, v)^- = uv \cdot \frac{(1 - (uv)^{2g-2})(1 - (uv)^{2g-3})}{(1 - uv)(1 - (uv)^2)}. \]

It is also elementary that

\[ E(T^*J; u, v)^+ = \frac{1}{2} (uv)^g \cdot [(1 - u)^g(1 - v)^g + (1 + u)^g(1 + v)^g] \]

\[ E(T^*J; u, v)^- = \frac{1}{2} (uv)^g \cdot [(1 - u)^g(1 - v)^g - (1 + u)^g(1 + v)^g]. \]

Therefore we have

\[(5.2) \quad E(D^0_2; u, v) = (uv)^g \cdot \frac{(1 - (uv)^{2g-2})(1 - (uv)^{2g-3})}{(1 - uv)(1 - (uv)^2)} \times \left[ \frac{1}{2} (1 + uv)(1 - u)^g(1 - v)^g + \frac{1}{2} (1 - uv)(1 + u)^g(1 + v)^g - 2^{2g} \right]. \]

By the additive property of the Hodge-Deligne polynomials we have

\[ E(D^0_1; u, v) = 2^{2g} \cdot (uv)^2 \cdot ((uv)^5 - (uv)^2) \cdot \prod_{1 \leq i \leq 3} \frac{1 - (uv)^{2g-6+2i}}{1 - (uv)^i} \]

\[ E(D^0_3; u, v) = 2^{2g} \cdot (uv)^2 \cdot \prod_{1 \leq i \leq 2} \frac{1 - (uv)^{2g-4+2i}}{1 - (uv)^i} \]

\[ E(D^0_{12}; u, v) = 2^{2g} \cdot (uv)^2 \cdot (1 + uv + (uv)^2) \cdot \prod_{1 \leq i \leq 3} \frac{1 - (uv)^{2g-6+2i}}{1 - (uv)^i} \]

\[ E(D^0_{23}; u, v) = 2^{2g} \cdot (uv)^2 \cdot (1 + uv) \cdot \prod_{1 \leq i \leq 2} \frac{1 - (uv)^{2g-4+2i}}{1 - (uv)^i} \]

\[ E(D^0_{13}; u, v) = 2^{2g} \cdot (uv)^2 \cdot (1 + uv + (uv)^2) \cdot \prod_{1 \leq i \leq 3} \frac{1 - (uv)^{2g-6+2i}}{1 - (uv)^i} \]

\[ E(D^0_{123}; u, v) = 2^{2g} \cdot (1 + uv) \cdot (1 + uv + (uv)^2) \cdot \prod_{1 \leq i \leq 3} \frac{1 - (uv)^{2g-6+2i}}{1 - (uv)^i}. \]
By direct computation with (5.1), (5.2) and the above, we obtain that $E_{st}(M) - E(M^g)$ is equal to

(5.3)

\[
(\nu \omega)^g \frac{1 - (\nu \omega)^{2g-2}}{1 - (\nu \omega)^2} \left[ \frac{1}{2} (1 + \nu \omega)(1 - \nu \omega)^g (1 - \nu)^g + \frac{1}{2} (1 - \nu \omega)(1 + \nu \omega)^g (1 + \nu)^g - 2^{2g} \right] \\
+ 2^{2g} \cdot \frac{(1 - (\nu \omega)^{2g-2})(1 - (\nu \omega)^2)}{1 - (\nu \omega)^{4g-5}} \cdot \left[ \frac{1 - (\nu \omega)^{8g-10}}{(1 - (\nu \omega)^{2g-3})(1 - (\nu \omega)^{6g-6})} \right] \\
+ \frac{(\nu \omega)^2(1 - (\nu \omega)^{2g-4})(1 - (\nu \omega)^{6g-8})}{(1 - (\nu \omega)^2)(1 - (\nu \omega)^{2g-3})(1 - (\nu \omega)^{6g-6})} + \frac{(\nu \omega)^{2g-2}}{1 - (\nu \omega)^2}. 
\]

By Theorem 5.2 and the above, we proved the following.

**Theorem 5.2.**

\[
E_{st}(M) = (\nu \omega)^{3g-3} \frac{(1 - \nu \omega)^g (1 - \nu \omega)^g - (\nu \omega)^{g+1}(1 - \nu)^g (1 - \nu)^g}{(1 - \nu \omega)(1 - (\nu \omega)^2)} \\
- (\nu \omega)^{3g-3} \frac{1}{2} (1 - \nu \omega)^g (1 - \nu)^g + (1 + \nu \omega)(1 + \nu)^g}{1 + \nu \omega} \\
+ (\nu \omega)^g \cdot \frac{1}{2} (1 - \nu \omega)^g (1 - \nu)^g + \frac{1}{2} (1 + \nu \omega)(1 + \nu)^g}{(\nu \omega)^{g-1}((\nu \omega)^{g-1} - 1)} \\
+ (\nu \omega)^{g+1} \cdot \frac{1}{2} (1 - \nu \omega)^g (1 - \nu)^g - \frac{1}{2} (1 + \nu \omega)(1 + \nu)^g}{((\nu \omega)^{g-1} - 1)((\nu \omega)^{g-2} - 1)} \\
+ (\nu \omega)^{2g-1} \frac{(\nu \omega)^{g-2} - 1)((\nu \omega)^{g-1} - 1)}{(\nu \omega)^2 - 1} \\
+ (\nu \omega)^{2g-2} \cdot \frac{(\nu \omega)^{g-1} - 1)((\nu \omega)^{g-1} - 1)}{uv - 1} \\
+ 2^{2g-1}(uv)^{3g-3} \left[ (1 - \nu \omega)^{g-1} (1 - \nu)^{g-1} + (1 + \nu \omega)^{g-1} (1 + \nu)^{g-1} - 2(\nu \omega)^{g-1} \right] \\
+ (\nu \omega)^{3g-3} (1 - \nu)(1 - \nu)^g \left[ - \frac{(1 + \nu \omega)^{g-1}(1 + \nu)^{g-1}}{4(1 + \nu \omega)} + \frac{(\nu \omega)^{g-1}(1 - \nu)^{g-1}(1 + \nu)^{g-1}}{(uv - 1)^2(\nu \omega + 1)} \right] \\
+ \frac{g - 1}{2} \frac{(1 + \nu \omega)(1 - \nu)^{g-2}(1 - \nu)^{g-2}}{1 - \nu \omega} + \frac{4g - 7}{4} \frac{(1 - \nu)^{g-1}(1 - \nu)^{g-1}}{1 - \nu \omega} \\
- \frac{(uv)^{g-1}(1 - \nu)^{g-1}}{2(\nu \omega - 1)^2} \\
+ (\nu \omega)^g \cdot \frac{1 - (\nu \omega)^{2g-2}}{1 - (\nu \omega)^2} \left[ \frac{1}{2} (1 + \nu \omega)(1 - \nu \omega)^g (1 - \nu)^g + \frac{1}{2} (1 - \nu \omega)(1 + \nu \omega)^g (1 + \nu)^g - 2^{2g} \right] \\
+ 2^{2g} \cdot \frac{(1 - (\nu \omega)^{2g-2})(1 - (\nu \omega)^2)}{1 - (\nu \omega)^{4g-5}} \cdot \left[ \frac{1 - (\nu \omega)^{8g-10}}{(1 - (\nu \omega)^{2g-3})(1 - (\nu \omega)^{6g-6})} \right] \\
+ \frac{(\nu \omega)^2(1 - (\nu \omega)^{2g-4})(1 - (\nu \omega)^{6g-8})}{(1 - (\nu \omega)^2)(1 - (\nu \omega)^{2g-3})(1 - (\nu \omega)^{6g-6})} + \frac{(\nu \omega)^{2g-2}}{1 - (\nu \omega)^2}. 
\]
By taking the limit $u, v \to 1$, we obtain the stringy Euler number

$$e_{st}(M) = \lim_{u, v \to 1} E_{st}(M; u, v)$$

as follows.

**Corollary 5.3.**

$$e_{st}(M) = 2^{2g} \frac{3g - 3}{2g - 3}.$$

In particular, the stringy Euler number is never an integer for $g \geq 4$. When $g = 3$, one can check directly that $E_{st}(M)$ (or (5.3)) is not a polynomial.

**Corollary 5.4.** There does not exist a symplectic desingularization of $M$ for $g \geq 3$.

**Proof.** Since $M^s$ is hyperkähler, the canonical bundle $K_M$ is trivial. If there were a symplectic desingularization $\mathcal{M}$ of $M$, then $K_{\mathcal{M}} = 0$ by definition and hence the resolution would be crepant. But in that case, $E_{st}(M)$ has to be equal to the Hodge-Deligne polynomial of $\mathcal{M}$ which is a polynomial with integer coefficients. This contradicts the fact that $e_{st}(M)$ is not an integer for $g \geq 4$ and that $E_{st}(M)$ is not a polynomial for $g = 3$. \qed

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