ON A CLASS OF ROBUST NONCONVEX QUADRATIC OPTIMIZATION PROBLEMS

F. FLORES-BÁZÁN¹, Y. GARCÍA², AND A. PÉREZ³

ABSTRACT. Let us consider the following robust nonconvex quadratic optimization problem:

\[
\begin{align*}
\min & \quad \frac{1}{2} x^\top Ax + a^\top x \\
\text{s.t.} & \quad \alpha \leq \frac{1}{2} x^\top \left(B_1 + \mu B_2\right)x + (b_1 + \delta b_2)^\top x \leq \beta, \quad \forall \mu \in [\mu_1, \mu_2], \forall \delta \in [\delta_1, \delta_2],
\end{align*}
\]

where \(A, B_1, B_2\) are real symmetric matrices, \(\mu_1, \mu_2, \delta_1, \delta_2, \alpha, \beta \in \mathbb{R}\) satisfying \(\mu_1 \leq \mu_2, \delta_1 \leq \delta_2\) and \(\alpha < \beta\). We establish the robust alternative result; the robust S-lemma and the robust optimality for the above nonconvex problem.

1. INTRODUCTION AND BASIC NOTATION

Robust optimization arises as a deterministic approach when addressing an optimization problem under uncertainty data. This paper revisits the following robust optimization problem:

\[
\begin{align*}
(1.1) \quad \min & \quad \left\{ \frac{1}{2} x^\top Ax + a^\top x : \alpha \leq \frac{1}{2} x^\top Bx + b^\top x \leq \beta, \quad \forall (B, b) \in B_0 \right\},
\end{align*}
\]

where \(B_0 = \{B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2]\} \times \{b_1 + \delta b_2 : \delta \in [\delta_1, \delta_2]\}\), with all the matrices being real symmetric, \(a, b \in \mathbb{R}^n\) and \(\alpha, \beta, \delta_1, \delta_2, \mu_1, \mu_2\) are given real numbers. Optimization problems that can be modeled by quadratic functions appear, for instance, in [2, 8, 9, 10, 12].

Problem (1.1) includes that examined in [7]:

\[
\begin{align*}
(1.2) \quad \min & \quad \left\{ \frac{1}{2} x^\top Ax + a^\top x : \frac{1}{2} x^\top Bx + b^\top x \leq \beta, \quad \forall (B, b) \in B_0 \right\}.
\end{align*}
\]

Theorem 5.1 in [7] provides a characterization of robust optimality for problem (1.2) under the convexity of the set

\[
\left\{ (x^\top H_0 x, x^\top H_1 x, x^\top H_2 x) : x \in \mathbb{R}^{n+1} \right\},
\]

where

\[
H_0 = \begin{pmatrix} A & a \\ a^\top & 2\gamma \end{pmatrix}, \quad H_1 = \begin{pmatrix} B_1 + \mu_1 B_2 & b_1 + \delta_1 b_2 \\ (b_1 + \delta_1 b_2)^\top & -2\beta \end{pmatrix}, \quad H_2 = \begin{pmatrix} B_1 + \mu_2 B_2 & b_1 + \delta_2 b_2 \\ (b_1 + \delta_2 b_2)^\top & -2\beta \end{pmatrix}
\]

with \(\gamma = -f(\overline{x})\). Here, \(\overline{x}\) is a feasible point of the robust optimization problem (1.2), which is either to be supposed becoming an optimal solution, or to be optimal for deriving optimality conditions.

Certainly the presence of the matrices \(H_0, H_1, H_2\) is because the authors in [7] homogenize problem (1.2) in order to apply the Dines convexity theorem.
valid for quadratic forms. Finally, we realize there is a gap in the proof of Theorem 5.1 in [7], but we were unable to find a counterexample to such a result under their assumptions. This is discussed in detail after Theorem 6 in Section 4. Notice that the approach employed in [7] was also applied in [1].

We have to point out that problem (1.1) (and so problem (1.2)) was studied without homogenizing the problem thanks to the convexity result established in [4, Theorem 4.19] valid for inhomogeneous quadratic functions. This allows us to impose the convexity of a set being the image of \( \mathbb{R}^n \) via the inhomogeneous quadratic functions.

Associated to problem (1.1), purposes of the present paper are to establish an alternative robust result (Theorem 5), a robust S-lemma (Theorem 4) and a characterization of robust optimality (Theorem 6), for problem (1.1). Finally, we provide a counterexample (Example 7) to the argument employed in the proof of Theorem 5.1 in [7] related to problem (1.2).

Thus, the structure of the present paper is as follows. Section 2 establishes the convexity of images for quadratic mappings by applying the Ramana-Goldman criterion [11] (see also [5, Theorem 2.1]). The main results are presented in Section 3 and Section 4 revisits problem (1.2) discussed in [7].

2. A PRELIMINARY RESULT: CONVEXITY OF IMAGES

By \( S^n \) we denote the set of symmetric matrices of order \( n \in \mathbb{N} \) with real entries; \( S^n_+ \) denotes the subset of \( S^n \) whose elements are positive semidefinite matrices, and we write \( A \succeq 0 \) if \( A \in S^n_+ \); and \( S^n_+ \) stands for the matrices in \( S^n \) that are positive definite, and in this case we write \( A \succ 0 \) if \( A \in S^n_+ \).

It is our purpose to prove the convexity of images for quadratic mappings under the Ramana-Goldman criterion [11] (see also [5, Theorem 2.1]). To that end, we are given \( A_i \in S^n, b_i \in \mathbb{R}^n, c_i \in \mathbb{R} \) for \( i = 0, 1, \ldots, m \), we set

\[
M_i = \begin{pmatrix} A_i & b_i \\ b_i^\top & 2c_i \end{pmatrix}, f_i(x) = x^\top A_i x + 2b_i^\top x, \quad \overline{f}_i(x) = x^\top A_i x, \quad x \in \mathbb{R}^n.
\]

Furthermore, let us consider the function

\[
G(x, t) = (g_0(x, t), g_1(x, t), \ldots, g_m(x, t)), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R},
\]

where \( g_i \) is defined by \( g_i(x, t) = \begin{pmatrix} x \\ t \end{pmatrix}^\top M_i \begin{pmatrix} x \\ t \end{pmatrix} \).

**Lemma 1.** Let \( A_i \in S^n, c_i \in \mathbb{R}, b_i \in \mathbb{R}^n, i = 0, 1, \ldots, m \) be as above. Set

\[
F(x) = (f_0(x), f_1(x), \ldots, f_m(x)), \quad \overline{F}(x) = (\overline{f}_0(x), \overline{f}_1(x), \ldots, \overline{f}_m(x)).
\]

If \( F(\mathbb{R}^n) \) and \( \overline{F}(\mathbb{R}^n) \) are convex then \( G(\mathbb{R}^{n+1}) \) is convex.

**Proof.** Set \( \Lambda := F(\mathbb{R}^n) \) and \( \overline{\Lambda} := \overline{F}(\mathbb{R}^n) \) and \( \Omega := G(\mathbb{R}^{n+1}) \).

As \( \Lambda \) is convex, by the convexity criterion due to Ramana-Goldman (see also [3, Theorem 2.1]), \( \Lambda + \overline{\Lambda} = \Lambda \). We easily get that for \( (x, t) \in \mathbb{R}^{n+1}, \)

\[
\begin{pmatrix} x \\ t \end{pmatrix}^\top M_i \begin{pmatrix} x \\ t \end{pmatrix} = x^\top A_i x + 2t b_i^\top x + 2c_i t^2, \quad i = 0, 1, \ldots, m.
\]

By setting \( \gamma = 2(c_0, c_1, \ldots, c_m) \), we obtain

\[
(2.3) \quad \Lambda + \gamma \subseteq \Omega.
\]
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Let $z_1 = G(x_1, t_1)$, $z_2 = G(x_2, t_2)$ be any elements in $\Omega$ and let $\lambda \in ]0, 1[$. We distinguish three cases.

(i): $t_1 \neq 0$ and $t_2 \neq 0$. Then, $z_1 = t_1^2 (F(x_1/t_1) + \gamma)$ and $z_2 = t_2^2 (F(x_2/t_2) + \gamma)$.

The convexity of $\Lambda$ implies that

$$\frac{\lambda}{\lambda t_1^2 + (1-\lambda)t_2^2}z_1 + \frac{1-\lambda}{\lambda t_2^2 + (1-\lambda)t_1^2}z_2 = \frac{\lambda t_2^2}{\lambda t_1^2 + (1-\lambda)t_2^2} F(x_1/t_1) + \frac{(1-\lambda)t_1^2}{\lambda t_1^2 + (1-\lambda)t_2^2} F(x_2/t_2) + \gamma \in \Lambda + \gamma.$$  

Taking into account (2.3) and the fact the $\Omega$ is a cone, we obtain

$$\lambda z_1 + (1-\lambda)z_2 \in \mathbb{R}^+ (\Lambda + \gamma) \subseteq \Omega.$$  

(ii): $t_1 = t_2 = 0$. Then, $z_1, z_2 \in \overline{\Lambda}$, and because of the convexity of $\overline{\Lambda}$, we get

$$\lambda (z_1 + (1-\lambda)z_2) \in \overline{\Lambda} \subseteq \Omega.$$  

(iii): $t_1 \neq 0$ and $t_2 = 0$. Then, since $\overline{\Lambda}$ is a cone,

$$\lambda z_1 + (1-\lambda)z_2 \in \lambda t_1^2 (\Lambda + \gamma) + \lambda t_2^2 \overline{\Lambda} \subseteq \lambda t_1^2 (\Lambda + \gamma) \subseteq \Omega.$$  

This completes the proof that $\Omega$ is convex. □ □

Part (a) of the following result is exactly Theorem 2.3 (i) in [2], and (b) is a consequence of the previous lemma.

**Corollary 2.** Let the same hypotheses of Lemma [7] be satisfied. Let $\rho_i \in \mathbb{R}$, for $i = 1, \ldots, m$. If $n \geq m+1$, $A_0 \in S_n^{++}$, $A_i = \rho_i A_0$ for $i = 1, \ldots, m$, then

(a) $F(\mathbb{R}^n)$ and $\overline{F}(\mathbb{R}^n)$ are convex.

(b) $G(\mathbb{R}^{n+1})$ is convex.

**Proof.** By assumption on $A_i$, we can apply [2] Theorem 2.3 (i)) to obtain the convexity of $F(\mathbb{R}^n)$ and $\overline{F}(\mathbb{R}^n)$. Then, (b) follows from Lemma [1] □ □

3. The main results

Denote the function:

$$f(x) = \frac{1}{2} x^\top A x + a^\top x$$

and let us define the following matrices in $S^{n+1}$:

$$H_0 \doteq \begin{pmatrix} A & a \\ a^\top & 2\gamma \end{pmatrix}, \quad W(\delta, \lambda) \doteq \begin{pmatrix} B_1 & b_1 + \delta b_2 \\ (b_1 + \delta b_2)^\top & -2\lambda \end{pmatrix}, \quad W_2 \doteq \begin{pmatrix} B_2 & 0 \\ 0^\top & 0 \end{pmatrix},$$

and set

$$W_{1\beta} = W(\delta_1, \beta); \quad W_{2\beta} = W(\delta_2, \beta); \quad W_{1\alpha} = W(\delta_1, \alpha); \quad W_{2\alpha} = W(\delta_2, \alpha).$$

The following set will play an important role in the following.

$$\Omega_W \doteq \left\{ \frac{1}{2} y^\top H_0 y, \max_{\mu \in [\mu_1, \mu_2]} \frac{1}{2} y^\top (W_{1\beta} + \mu W_2) y, \max_{\mu \in [\mu_1, \mu_2]} \frac{1}{2} y^\top (W_{2\beta} + \mu W_2) y, \right\}.$$
In addition, we observe that
\[
\exists W \text{ where } \mu \text{ is so.}
\]

By Corollary 1 in [8], \( \Omega_W \) is convex if the set
\[
\Omega_\mu = \left\{ \left( y^\top H_0 y, y^\top (W_{1,\alpha} + \mu_1 W_2) y, y^\top (W_{2,\beta} + \mu_1 W_2) y, y^\top (W_{1,\alpha} + \mu_2 W_2) y, y^\top (W_{2,\beta} + \mu_2 W_2) y, -y^\top (W_{1,\alpha} + \mu_1 W_2) y, -y^\top (W_{2,\beta} + \mu_1 W_2) y, -y^\top (W_{1,\alpha} + \mu_2 W_2) y, -y^\top (W_{2,\beta} + \mu_2 W_2) y \right) : y \in \mathbb{R}^{n+1} \right\} + \text{int } \mathbb{R}^3
\]
is so.

**Theorem 3.** (A robust alternative result) Let \( A, B_1, B_2 \in S^n, a, b_1, b_2 \in \mathbb{R}^n \) and \( \gamma, \alpha, \beta, \mu_1, \mu_2, \delta_1, \delta_2 \in \mathbb{R} \), with \( \mu_1 \leq \mu_2, \delta_1 \leq \delta_2 \) and \( \alpha < \beta \). Assume that \( \Omega_W \) is convex. Then, exactly one of the two following assertions hold:

(a) \( \exists x \in \mathbb{R}^n : \frac{1}{2} x^\top A x + a^\top x + \gamma < 0, \alpha < \frac{1}{2} x^\top (B_1 + \mu B_2) x + (b_1 + \delta b_2)^\top x < \beta, \forall \mu \in [\mu_1, \mu_2], \forall \delta \in [\delta_1, \delta_2]. \)

(b) \( \exists (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3 \setminus \{0\}, \exists \mu_\alpha, \mu_\beta \in [\mu_1, \mu_2], \forall \delta_\alpha, \delta_\beta \in [\delta_1, \delta_2] : A x \in \lambda_0 \left( \frac{1}{2} x^\top A x + a^\top x + \gamma \right) + \lambda_1 \left( \frac{1}{2} x^\top (B_1 + \mu_\beta B_2) x + (b_1 + \delta_\beta b_2)^\top x - \lambda_2 (B_1 + \mu_\alpha B_2) x + (b_1 + \delta_\alpha b_2)^\top x \right) \geq 0, \)

where \( \mu_\alpha + \mu_\beta = \mu_1 + \mu_2. \)

In addition, we observe that (b) may be written equivalently as
\[
\lambda_0 A + \lambda_1 (B_1 + \mu_\beta B_2) - \lambda_2 (B_1 + \mu_\alpha B_2) \succeq 0 \quad \text{and} \quad \exists \bar{\pi} \in \mathbb{R}^n : \left( \lambda_0 A + \lambda_1 (B_1 + \mu_\beta B_2) - \lambda_2 (B_1 + \mu_\alpha B_2) \right) \bar{\pi} + \lambda_0 a + \lambda_1 (b_1 + \delta_\beta b_2) + \lambda_2 (b_1 + \delta_\alpha b_2) = 0.
\]

**Proof.** It is obvious that both statements (a) and (b) cannot be fulfilled simultaneously. Thus, we must check that if (a) does not hold, (b) does.

**Step 1: The homogenization system.** If (a) does not hold, then there exists no \( x \in \mathbb{R}^n \) such that for all \( \mu \in [\mu_1, \mu_2] \) and all \( \delta \in [\delta_1, \delta_2] \)
\[
\frac{1}{2} x^\top A x + a^\top x + \gamma < 0, \quad \frac{1}{2} x^\top (B_1 + \mu B_2) x + (b_1 + \delta b_2)^\top x < \beta, \quad \frac{1}{2} x^\top (B_1 + \mu B_2) x + (b_1 + \delta b_2)^\top x < \beta.
\]

By setting \( B_0 = \{ B_1 + \mu B_2 : \mu \in [\mu_1, \mu_2] \} \times \{ b_1 + \delta b_2 : \delta \in [\delta_1, \delta_2] \}, \) the previous is equivalent to the nonexistence of \( x \in \mathbb{R}^n \) such that
\[
\frac{1}{2} x^\top A x + a^\top x + \gamma < 0, \quad \max \left\{ \frac{1}{2} x^\top B x + b^\top x - \beta : (B, b) \in B_0 \right\} < 0, \quad \min \left\{ \frac{1}{2} x^\top B x + b^\top x - \alpha : (B, b) \in B_0 \right\} < 0.
\]
We claim that the following homogeneous system in $\mathbb{R}^{n+1}$:

\[
\begin{align*}
\frac{1}{2}x^\top Ax + ta^\top x + t^2\gamma &< 0, \\
\max \left\{ \frac{1}{2}x^\top Bx + tb^\top x - t^2\beta : (B, b) \in B_0 \right\} &< 0,
\end{align*}
\]

(3.5)

\[
\begin{align*}
- \min \left\{ \frac{1}{2}x^\top Bx + tb^\top x - t^2\alpha : (B, b) \in B_0 \right\} &< 0,
\end{align*}
\]

(3.6)

has no solution. If, on the contrary, there was a solution $(\pi, t) \in \mathbb{R}^{n+1}$ such that

\[
\begin{align*}
\frac{1}{2}\pi^\top A\pi + ta^\top \pi + t^2\gamma &< 0, \\
\max \left\{ \frac{1}{2}\pi^\top B\pi + tb^\top \pi - t^2\beta : (B, b) \in B_0 \right\} &< 0,
\end{align*}
\]

(3.5)\(\ast\) (3.6)

we immediately reach a contradiction in case $t \neq 0$. So, suppose that $t = 0$.

Then, the system (3.5)\(\ast\) (3.6) reduces to

\[
\begin{align*}
\frac{1}{2}\pi^\top A\pi &< 0, \\
\max \left\{ \frac{1}{2}\pi^\top (B_1 + \mu B_2)\pi : \mu \in [\mu_1, \mu_2] \right\} &< 0,
\end{align*}
\]

\[
- \min \left\{ \frac{1}{2}\pi^\top (B_1 + \mu B_2)\pi : \mu \in [\mu_1, \mu_2] \right\} &< 0,
\]

which is impossible to hold. Thus, the claim is proved.

On the other hand, observe that for every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, the minimum and maximum values in (3.5)\(\ast\) (3.6) are achieved in, at least, one of the extreme points of the rectangle $[\mu_1, \mu_2] \times [\delta_1, \delta_2]$, that is, in one of the elements $(B_1 + \mu_1 B_2, b_1 + \delta_1 b_2)$, $(B_1 + \mu_1 B_2, b_1 + \delta_2 b_2)$, $(B_1 + \mu_2 B_2, b_1 + \delta_1 b_2)$ or $(B_1 + \mu_2 B_2, b_1 + \delta_2 b_2)$. Then, the nonexistence of solution to the system (3.5)\(\ast\) (3.6) is equivalent to the nonexistence of $y \in \mathbb{R}^{n+1}$ solution to the system

\[
\begin{align*}
\frac{1}{2}y^\top H_0 y &< 0 \\
\max \left\{ \frac{1}{2}y^\top (W_1 + \mu W_2) y : \mu \in [\mu_1, \mu_2] \right\} &< 0, \\
\max \left\{ \frac{1}{2}y^\top (W_2 + \mu W_2) y : \mu \in [\mu_1, \mu_2] \right\} &< 0, \\
- \min \left\{ \frac{1}{2}y^\top (W_1 + \mu W_2) y : \mu \in [\mu_1, \mu_2] \right\} &< 0, \\
- \min \left\{ \frac{1}{2}y^\top (W_2 + \mu W_2) y : \mu \in [\mu_1, \mu_2] \right\} &< 0.
\end{align*}
\]

This means that $0, 0, 0, 0, 0 \notin \Omega_W$.

**Step 2: A first use of a separation result.** Since $\Omega_W$ is convex, there exist $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}_+^5 \setminus \{0\}$ such that for all $y \in \mathbb{R}^{n+1}$

\[
\begin{align*}
\lambda_0 \left( \frac{1}{2}y^\top H_0 y \right) + \lambda_1 \left( \max_{\mu \in [\mu_1, \mu_2]} \frac{1}{2}y^\top (W_1 + \mu W_2) y \right) + \lambda_2 \left( \max_{\mu \in [\mu_1, \mu_2]} \frac{1}{2}y^\top (W_2 + \mu W_2) y \right) + \\
\lambda_3 \left( - \min_{\mu \in [\mu_1, \mu_2]} \frac{1}{2}y^\top (W_1 + \mu W_2) y \right) + \lambda_4 \left( - \min_{\mu \in [\mu_1, \mu_2]} \frac{1}{2}y^\top (W_2 + \mu W_2) y \right) &\geq 0.
\end{align*}
\]
Thus, there is no \( y \in \mathbb{R}^{n+1} \) solution to the system:

\[
\frac{1}{2} y^\top \left( \lambda_0 H_0 + \lambda_1 (W_1 + \mu_1 W_2) + \lambda_2 (W_2 + \mu_1 W_2) - \lambda_3 (W_1 + \mu_2 W_2) - \lambda_4 (W_2 + \mu_2 W_2) \right) y < 0;
\]

\[
\frac{1}{2} y^\top \left( \lambda_0 H_0 + \lambda_1 (W_1 + \mu_2 W_2) + \lambda_2 (W_2 + \mu_2 W_2) - \lambda_3 (W_1 + \mu_2 W_2) - \lambda_4 (W_2 + \mu_1 W_2) \right) y < 0.
\]

This means that \((0, 0) \notin \Omega_2\), where

\[
\Omega_2 = \left\{ y^\top (\lambda_0 H_0 + \lambda_1 H_{1,1} + \lambda_2 H_{2,2} - \lambda_3 H_{2,1} - \lambda_4 H_{1,2}) y : y \in \mathbb{R}^{n+1} \right\},
\]

with

\[
W_1 + \mu_1 W_2 = H_{1,1}, \quad W_1 + \mu_2 W_2 = H_{2,1},
\]

\[
W_2 + \mu_1 W_2 = H_{1,2}, \quad W_2 + \mu_2 W_2 = H_{2,2},
\]

\[
W_1 + \mu_2 W_2 = H_{1,1}, \quad W_1 + \mu_2 W_2 = H_{2,1},
\]

\[
W_2 + \mu_1 W_2 = H_{1,2}, \quad W_2 + \mu_2 W_2 = H_{2,2}.
\]

**Step 3: A second use of a separation result and conclusion.** By the Dines theorem, \( \Omega_2 \) is convex. Thus, there exists \((\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}\) such that for all \( y \in \mathbb{R}^{n+1} \),

\[
\xi_1 y^\top \left( \lambda_0 H_0 + \lambda_1 H_{1,1} + \lambda_2 H_{2,2} - \lambda_3 H_{2,1} - \lambda_4 H_{1,2} \right) y + \xi_2 y^\top \left( \lambda_0 H_0 + \lambda_1 H_{2,1} + \lambda_2 H_{2,2} - \lambda_3 H_{2,1} - \lambda_4 H_{1,2} \right) y \geq 0.
\]

In particular, for \( y = (x, 1) \) with \( x \in \mathbb{R}^n \), and by setting

\[
\lambda_0 = \lambda_0 (\xi_1 + \xi_2), \quad \lambda_1 = (\lambda_1 + \lambda_2) (\xi_1 + \xi_2), \quad \lambda_2 = (\lambda_3 + \lambda_4) (\xi_1 + \xi_2), \quad \mu_\beta = \frac{\xi_1 \mu_1 + \xi_2 \mu_2}{\xi_1 + \xi_2}, \quad \mu_\alpha = \frac{\xi_1 \mu_2 + \xi_2 \mu_1}{\xi_1 + \xi_2}, \quad \delta_\beta = \frac{\lambda_1 \delta_1 + \lambda_2 \delta_2}{\lambda_1 + \lambda_2}, \quad \delta_\alpha = \frac{\lambda_3 \delta_2 + \lambda_4 \delta_1}{\lambda_3 + \lambda_4},
\]

one gets \((\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3 \setminus \{0\}\), \( \mu_\beta, \mu_\alpha \in [\mu_1, \mu_2], \delta_\beta, \delta_\alpha \in [\delta_1, \delta_2] \) and

\[
\lambda_0 \left( \frac{1}{2} x^\top Ax + a^\top x + \gamma \right) + \lambda_1 \left( \frac{1}{2} x^\top (B_1 + \mu_\beta B_2) x + (b_1 + \delta_\beta b_2)^\top x - \beta \right) + \lambda_2 \left( \alpha - \left( \frac{1}{2} x^\top (B_1 + \mu_\alpha B_2) x + (b_1 + \delta_\alpha b_2)^\top x \right) \right) \geq 0 \quad \forall x \in \mathbb{R}^n,
\]

where \( \mu_\alpha + \mu_\beta = \mu_1 + \mu_2 \). This proves that \((b)\) holds. \(\square\) \(\square\)
**Theorem 4.** (A robust S-lemma) Let $A, B_1, B_2 \in S^n$, $a, b_1, b_2 \in \mathbb{R}^n$ and $\gamma, \alpha, \beta, \mu_1, \mu_2, \delta_1, \delta_2 \in \mathbb{R}$, with $\mu_1 \leq \mu_2$, $\delta_1 \leq \delta_2$ and $\alpha < \beta$. Assume that $\Omega_W$ is convex and that there exists $x_0 \in \mathbb{R}^n$ satisfying (3.7)

$$\alpha < \frac{1}{2} x_0^\top (B_1 + \mu B_2) x_0 + (b_1 + \delta b_2)^\top x_0 < \beta, \quad \forall \mu \in [\mu_1, \mu_2], \forall \delta \in [\delta_1, \delta_2].$$

Then, the following two assertions are equivalent:

(a) $\alpha \leq \frac{1}{2} x^\top (B_1 + \mu B_2) x + (b_1 + \delta b_2)^\top x \leq \beta, \quad \forall \mu \in [\mu_1, \mu_2], \forall \delta \in [\delta_1, \delta_2],$

$$\Rightarrow \frac{1}{2} x^\top Ax + a^\top x + \gamma \geq 0.

(b) $\exists (\lambda_1, \lambda_2) \in \mathbb{R}_+^2, \exists \mu_\alpha, \mu_\beta \in [\mu_1, \mu_2], \exists \delta_\alpha, \delta_\beta \in [\delta_1, \delta_2] : \quad \forall x \in \mathbb{R}^n$

$$\frac{1}{2} x^\top Ax + a^\top x + \gamma + \lambda_1 \left(\frac{1}{2} x^\top (B_1 + \mu_\beta B_2) x + (b_1 + \delta_\beta b_2)^\top x - \beta\right) +$$

$$\lambda_2 \left(\alpha - \left(\frac{1}{2} x^\top (B_1 + \mu_\alpha B_2) x + (b_1 + \delta_\alpha b_2)^\top x\right)\right) \geq 0,$$

where $\mu_\alpha + \mu_\beta = \mu_1 + \mu_2$.

**Proof.** Clearly (b) $\Rightarrow$ (a).
Assume now that (a) is satisfied. Then (a) in Theorem 3 does not hold. Thus (b) of the same theorem fulfills, but then $\lambda_0$ is strictly positive because of (3.7), which implies the desired result.

We are now ready to establish a characterization of optimality for the problem (1.1).

**Theorem 5.** (Characterizing robust optimality) Let $A, B_1, B_2 \in S^n$, $a, b_1, b_2 \in \mathbb{R}^n$ and $\alpha, \beta, \mu_1, \mu_2, \delta_1, \delta_2 \in \mathbb{R}$, with $\mu_1 \leq \mu_2$, $\delta_1 \leq \delta_2$ and $\alpha < \beta$. Let $\overline{x}$ be feasible for problem (1.1) and put $\gamma = -f(\overline{x})$. Assume that $\Omega_W$ is convex, and the Slater-type condition (3.7) holds. Then, $\overline{x}$ is optimal if, and only if there exist $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2, \mu_\alpha, \mu_\beta \in [\mu_1, \mu_2], \delta_\alpha, \delta_\beta \in [\delta_1, \delta_2]$ such that the following statements are satisfied:

(a) $\left(A + \lambda_1(B_1 + \mu_\beta B_2) - \lambda_2(B_1 + \mu_\alpha B_2)\right) \overline{x} = -\left(a + \lambda_1(b_1 + \delta_\beta b_2) - \lambda_2(b_1 + \delta_\alpha b_2)\right);$ 

(b) $\lambda_1 \left(\frac{1}{2} \overline{x}^\top (B_1 + \mu_\beta B_2) \overline{x} + (b_1 + \delta_\beta b_2)^\top \overline{x} - \beta\right) = 0;$

(c) $\lambda_2 \left(\alpha - \left(\frac{1}{2} \overline{x}^\top (B_1 + \mu_\alpha B_2) \overline{x} + (b_1 + \delta_\alpha b_2)^\top \overline{x}\right)\right) \geq 0.$

**Proof.** The necessary condition follows from Theorem 4 where $\gamma$ is substituted by $-f(\overline{x})$. Indeed, if $\overline{x}$ is optimal then (a) in Theorem 4 holds, which means that (b) of the same theorem is also satisfied. This finally implies the desired statements.

The sufficiency part is already standard since (c) implies the convexity of the function

$$h(x) = \frac{1}{2} x^\top Ax + a^\top x - f(\overline{x}) + \lambda_1 \left(\frac{1}{2} x^\top (B_1 + \mu_\beta B_2) x + (b_1 + \delta_\beta b_2)^\top x - \beta\right) +$$
Thus, by Corollary 1 in [6], \( \Omega \) and (4.9) allow us to prove that \( \varpi \) is in fact a solution to problem (1.1). \( \square \ \square \)

4. Revisiting the case \( \alpha = -\infty \)

We consider the problem:

\[
\min \frac{1}{2} x^T A x + a^T x
\]

s.t. \( \frac{1}{2} x^T (B_1 + \mu B_2) x + (b_1 + \delta b_2)^T x \leq \beta, \forall \mu \in [\mu_1, \mu_2], \forall \delta \in [\delta_1, \delta_2] \),

where \( A, B_1, B_2 \) are real symmetric matrices, \( \mu_1, \mu_2, \beta \in \mathbb{R} \) satisfying \( \mu_1 < \mu_2, \delta_1 < \delta_2 \). This problem was also discussed in [7], and actually this paper motivated our study.

By looking at carefully the proof of Theorem 3, we immediately realize that in case there is no lower bound in the inequality constraint, all the terms where \( \alpha \) appears, actually disappear: they are superfluous. Hence, the set \( \Omega^\beta \) reduces to

\[
\Omega^\beta = \left\{ \left( \frac{1}{2} y^T H_0 y, \max_{\mu \in [\mu_1, \mu_2]} \frac{1}{2} y^T (W_{1,\beta} + \mu W_2) y, \max_{\mu \in [\mu_1, \mu_2]} \frac{1}{2} y^T (W_{2,\beta} + \mu W_2) y \right) : y \in \mathbb{R}^{n+1} \right\} + \mathbb{R}_+^3.
\]

Thus, by Corollary 1 in [6], \( \Omega^\beta \) is convex if the set

\[
\Omega^\beta = \left\{ \left( y^T H_0 y, y^T (W_{1,\beta} + \mu_1 W_2) y, y^T (W_{2,\beta} + \mu_1 W_2) y, y^T (W_{1,\beta} + \mu_2 W_2) y, y^T (W_{2,\beta} + \mu_2 W_2) y \right) : y \in \mathbb{R}^{n+1} \right\} + \mathbb{R}_+^5.
\]

**Theorem 6.** Let the data be as described above. Let \( \varpi \) be feasible for problem (4.8) and put \( \gamma = -f(\varpi) \). Assume that \( \Omega^\beta \) is convex, and the Slater-type condition: there exists \( x_0 \in \mathbb{R}^n \) such that

\[
\frac{1}{2} x_0^T (B_1 + \mu B_2) x_0 + (b_1 + \delta b_2)^T x_0 < \beta, \forall \mu \in [\mu_1, \mu_2], \forall \delta \in [\delta_1, \delta_2]
\]

is satisfied. Then, \( \varpi \) is optimal if, and only if there exist \( \lambda \geq 0, \mu \in [\mu_1, \mu_2], \delta \in [\delta_1, \delta_2] \) such that the following statements are satisfied:

(a) \( A + \lambda (B_1 + \mu B_2) \varpi = -\left( a + \lambda (b_1 + \delta b_2) \right) \);

(b) \( \lambda \left( \frac{1}{2} \varpi^T (B_1 + \mu B_2) \varpi + (b_1 + \delta b_2)^T \varpi - \beta \right) = 0 \);

(c) \( A + \lambda (B_1 + \mu B_2) \succeq 0 \).

We recall that

\[
H_0 = \begin{pmatrix} A & a^T \\ a & 2\gamma \end{pmatrix},
\]

with \( \gamma = -f(\varpi) \) as in the previous theorem. Set

\[
H_1 = H_{1,1,\beta}, \quad H_2 = H_{2,2,\beta}, \quad H_3 = H_{1,2,\beta}, \quad H_4 = H_{2,1,\beta}.
\]
The authors in \[7, \text{Theorem 5.1}\] proved the same result expressed in Theorem \(6\) under the convexity of the set
\[(4.11) \quad \{ (y^\top H_0 y, y^\top H_1 y, y^\top H_2 y) : y \in \mathbb{R}^{n+1} \} + \text{int } \mathbb{R}_+^3;\]
whereas ours requires the convexity of
\[\Omega^\beta_\mu = \{ (y^\top H_0 y, y^\top H_1 y, y^\top H_2 y, y^\top H_3 y, y^\top H_4 y) : y \in \mathbb{R}^{n+1} \} + \text{int } \mathbb{R}_+^5.\]

We believe that there is a gap in the proof of Theorem 5.1 in \[7\]. More precisely, the authors assert in page 221 of the same paper that the nonexistence of solution to the system (notice that our \(\beta\) is \(-\beta\) in \[7\])
\[(4.12) \quad \frac{1}{2} x^\top A x + t a^\top x + t^2 \gamma < 0, \quad \max \left\{ \frac{1}{2} x^\top B x + t b^\top x - t^2 \beta : (B, b) \in \mathcal{B}_0 \right\} < 0,\]
implies that
\[(4.13) \quad \frac{1}{2} y^\top H_0 y < 0 \quad \text{and} \quad \forall \mu \in [\mu_1, \mu_2], \quad \frac{1}{2} y^\top (W_1 + \mu W_2) y < 0,\]
has no solution. This is not necessarily true as Example 7 below shows. We recall that
\[W_1 = \begin{pmatrix} B_1 & b_1 + \frac{\delta_1 \mu_2 - \delta_2 \mu_1}{\mu_2 - \mu_1} b_2 \\ (b_1 + \frac{\delta_1 \mu_2 - \delta_2 \mu_1}{\mu_2 - \mu_1} b_2)^\top & -2\beta \end{pmatrix} \quad \text{and} \quad W_2 = \begin{pmatrix} B_2 & \frac{\delta_2 - \delta_1}{\mu_2 - \mu_1} b_2 \\ \frac{\delta_2 - \delta_1}{\mu_2 - \mu_1} b_2^\top & 0 \end{pmatrix}.
\]

**Example 7.** Let \(n \geq 5\). Taking \(A = I_n, B_1 = B_2 = 2I_n, a = b_1 = b_2 = s := (1, 1, \ldots, 1) \in \mathbb{R}^n, \mu_1 = -1, \mu_2 = 1, \delta_1 = -1, \delta_2 = 1\) and \(\beta = 1\), Problem \((4.8)\) takes the form
\[(4.14) \quad \min \frac{1}{2} ||x||^2 + \sum_{i=1}^n x_i = \frac{1}{2} ||x + s||^2 - \frac{n}{2}
\[
\quad \text{s.t. } g(x, \mu, \delta) := (1 + \mu) ||x||^2 + (1 + \delta) \sum_{i=1}^n x_i - 1 \leq 0 \quad \forall \mu, \delta \in [-1, 1].
\]

Let us define the functions
\[f(x) := \frac{1}{2} ||x||^2 + \sum_{i=1}^n x_i = \frac{1}{2} ||x + s||^2 - \frac{n}{2};
\]
\[g(x, \mu, \delta) := (1 + \mu) ||x||^2 + (1 + \delta) \sum_{i=1}^n x_i - 1.
\]
Then, problem (4.14) is equivalent to
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad \|x\|^2 - \frac{1}{2} \leq 0; \\
& \quad \sum_{i=1}^{n} x_i - \frac{1}{2} \leq 0; \\
& \quad \|x\|^2 + \sum_{i=1}^{n} x_i - \frac{1}{2} \leq 0.
\end{align*}

Let \( C \) be the set of constraints, that is, those \( x \) satisfying (4.15)-(4.17). It is clear that \( C \) is convex and compact. Thus, the unique solution to problem (4.14) is the projection of \( -s \) on \( C \), which is \( x = -\frac{1}{\sqrt{2n}}s \). Hence, \( x \) is a robust solution to problem (4.14). In this case, \( \gamma = -f(\mathbf{p}) = \sqrt{\frac{n}{2}} - \frac{1}{4} \).

By identifying the matrices involved in Theorem 4, we get
\begin{align*}
H_0 &= \begin{pmatrix} I_n & s \\ s^\top & 2\gamma \end{pmatrix}; \\
H_1 &= \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}; \\
H_2 &= \begin{pmatrix} 4I_n & 2s \\ 2s^\top & -2 \end{pmatrix}; \\
H_3 &= \begin{pmatrix} 0 & 2s \\ 2s^\top & 2\beta \end{pmatrix}; \\
H_4 &= \begin{pmatrix} 4I_n & 0 \\ 0^\top & 2\beta \end{pmatrix}.
\end{align*}

Since
\begin{align*}
-2H_0 + (-2 - 2\gamma)H_1 + H_2 &= \begin{pmatrix} 2I_n & 0 \\ 0 & 2 \end{pmatrix} > 0,
\end{align*}

applying [10, Theorem 2.1] we have that
\begin{align*}
\left\{ (y^\top H_0y, y^\top H_1y, y^\top H_2y) : y \in \mathbb{R}^{n+1} \right\}
\end{align*}
is convex. This means, according to [7], that (4.14) is regular with respect to \( \mathbf{p} \). Also, by taking \( x_0 = 0 \), we have \( g(x_0, \mu, \delta) < 0 \), for all \( \mu, \delta \in [-1,1] \). So, we have all the conditions of [7, Theorem 5.1] are satisfied. From the first part of the proof of [7] Theorem 5.1, we have that the following homogeneous system in \( \mathbb{R}^{n+1} \), (4.19), has no solution.

Coming back to our example, we obtain
\begin{align*}
W_1 &= \begin{pmatrix} 2I_n & s \\ s^\top & -2 \end{pmatrix} \quad \text{and} \quad W_2 = \begin{pmatrix} 2I_n & s \\ s^\top & 0 \end{pmatrix}.
\end{align*}

Then, the following homogeneous system in \( \mathbb{R}^{n+1} \) (see (4.13))
\begin{align*}
\frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^\top H_0 \begin{pmatrix} x \\ t \end{pmatrix} < 0 \quad \text{and} \quad \forall \mu \in [-1,1], \quad \frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^\top (W_1 + \mu W_2) \begin{pmatrix} x \\ t \end{pmatrix} < 0,
\end{align*}

becomes
\begin{align*}
\frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^\top \begin{pmatrix} I_n & s \\ s^\top & 2\gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} < 0 \quad \text{and}
\end{align*}
\begin{align*}
\forall \mu \in [-1,1], \quad \frac{1}{2} \begin{pmatrix} x \\ t \end{pmatrix}^\top \begin{pmatrix} 2(1 + \mu)I_n \\ (1 + \mu)s^\top \\ (1 + \mu)s^\top & -2 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} < 0.
\end{align*}
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We will see that such a system admits a solution, contradicting the assertion made in page 221 of \cite{7} about the nonexistence of solution to the same system. Indeed, for $(\mathbf{-s}, 1) \in \mathbb{R}^{n+1}$, (4.20) reduces to

\[
\frac{1}{2} \begin{pmatrix} \mathbf{-s} \\ 1 \end{pmatrix}^\top \begin{pmatrix} I_n & s \\ s^\top & 2\gamma \end{pmatrix} \begin{pmatrix} \mathbf{-s} \\ 1 \end{pmatrix} = \frac{1}{2}(-n + 2\gamma) = \frac{1}{2}(-n + \sqrt{\frac{n^2}{4} - 1}) < 0;
\]

whereas (4.21) becomes: for all $\mu \in [-1, 1]$,

\[
\frac{1}{2} \begin{pmatrix} \mathbf{-s} \\ 1 \end{pmatrix}^\top \begin{pmatrix} 2(1 + \mu)I_n & (1 + \mu)s \\ (1 + \mu)s^\top & -2 \end{pmatrix} \begin{pmatrix} \mathbf{-s} \\ 1 \end{pmatrix} = \frac{1}{2}(n(1+\mu)-(1+\mu)n-2) = -1 < 0.
\]

This proves our claim.

Observe that taking in Corollary 2: $m = 4$, $A_0 = I_n$, $\rho_1 = \rho_3 = 4$, $\rho_2 = \rho_4 = 0$, $a_0 = s, a_1 = a_3 = 0$, $a_2 = a_4 = 2s$, $c_0 = \gamma$, $c_1 = c_2 = -1$, $c_3 = c_4 = \beta$ we obtain that

\[
\left\{ \begin{pmatrix} y^\top H_0 y, y^\top H_1 y, y^\top H_2 y, y^\top H_3 y, y^\top H_4 y \end{pmatrix} : \ y \in \mathbb{R}^{n+1} \right\} \subset \mathbb{R}^{5}
\]

is a convex set, which is required in our Theorem 6, providing the characterization of robust optimality for our example. \hfill \Box

Clearly, the previous example only shows there is a gap in the proof of Theorem 5.1 in \cite{7}. We were unable to construct a real counterexample to that result under the convexity either of the set given in (4.11) or in (4.19).

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1, 2 Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile

2 Universidad del Pacífico, Jirón Sánchez Cerro 2050, Jesús María, Lima, Perú

Email address: fflores@ing-mat.udec.cl
Email address: arielperez@udec.cl
Email address: garcia_yv@up.edu.pe