Open Gauged Sigma Models, Equivariant Branes, and Equivariant Homological Mirror Symmetry

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Abstract: We describe supersymmetric A-branes and B-branes in open \( \mathcal{N} = (2,2) \) dynamically gauged nonlinear sigma models (GNLSM), placing emphasis on toric manifold target spaces. For a subset of toric manifolds, these equivariant branes have a mirror description as branes in gauged Landau-Ginzburg models with neutral matter. We then study correlation functions in the topological A-twisted version of the GNLSM, and identify their values with open Hamiltonian Gromov-Witten invariants. Supersymmetry breaking can occur in the A-twisted GNLSM due to nonperturbative open symplectic vortices, and we canonically BRST quantize the mirror theory to analyze this phenomenon.
1 Introduction

D-branes are crucial to the nonperturbative dynamics of string theory, and their importance has been well-understood since Polchinski [1] recognized them as the source of BPS states carrying RR charge, leading to their identification with black p-brane solutions of supergravity. From the point of view of mathematics, D-branes are essential objects of homological mirror symmetry, first conjectured by Kontsevich [2].

In this paper, we investigate D-branes of dynamically gauged nonlinear sigma models (GNLSMs) with $\mathcal{N} = (2, 2)$ supersymmetry, which we shall refer to as equivariant branes. One motivation for this is that GNLSMs with target space $X$ and gauge group $G$ flow
in the IR limit to nonlinear sigma models (NLSMs) with target space $X//G$, and hence, we will obtain new descriptions of D-branes in $\mathcal{N} = (2,2)$ NLSMs, including those with Calabi-Yau targets useful for physical compactifications of string theory. A more mathematically oriented motivation is furnishing an equivariant generalization of homological mirror symmetry. As we shall see, describing equivariant branes will also allow us to define an open version of the mathematical theory of Hamiltonian Gromov-Witten invariants \[29, 31, 32\].

The $\mathcal{N} = (2,2)$ dynamically gauged nonlinear sigma model (GNLSM) governing maps from a closed Riemann surface into a Kähler manifold $X$ with Hamiltonian isometry group $G$ was studied in depth by Baptista \[3, 12\]. In particular, it was shown that the $A$-twisted GNLSM localizes to the moduli space of symplectic vortices, and its correlation functions compute the Hamiltonian Gromov-Witten invariants of $X$. Moreover, for abelian $G$, Baptista used mirror symmetry (as proven by Hori and Vafa \[9\]) to describe the quantum equivariant cohomology ring for toric $X$.

Also, D-branes in $\mathcal{N} = (2,2)$ NLSMs on open Riemann surfaces have been studied by Hori, Iqbal and Vafa \[10, 11\], with the mirrors of these D-branes being identified. We are thus led to attempt an understanding of equivariant branes by combining the insights described above, that is, by analyzing $\mathcal{N} = (2,2)$ GNLSMs on open Riemann surfaces and their mirrors. Since only a subset of the $\mathcal{N} = (2,2)$ supersymmetry can be preserved at the boundaries of these open Riemann surfaces, we are led to two types of equivariant branes, namely equivariant A-branes and equivariant B-branes.

Equivariant B-branes have been previously studied by Kapustin et al. \[14\] within the context of topologically B-twisted GNLSMs, although the mirrors of these branes were not elucidated. On the other hand, equivariant A-branes have only been studied for $G = U(1)$ by Setter \[16\], using a specialized topologically A-twisted non-dynamical $U(1)$-GNLSM (where the gauge field is identified with the worldsheet spin connection); in this case, their mirrors were not elucidated either. We shall study both types of equivariant branes, and provide the description of their mirrors in a subset of toric target spaces, hence defining equivariant homological mirror symmetry in these contexts. Other proposals for equivariant homological mirror symmetry of equivariant B-branes have appeared in the mathematical literature \[23, 24\].

In addition, understanding equivariant A-branes allows us to define open Hamiltonian Gromov-Witten invariants, which can be understood as integrals over the moduli spaces of open symplectic vortices that describe a map from an open Riemann surface $\Sigma$ to a Kähler and Hamiltonian $G$-manifold $X$, whereby the boundaries of $\Sigma$ correspond to equivariant A-branes in $X$. We note that closed Hamiltonian Gromov-Witten invariants have been studied extensively in the mathematical literature \[29, 31, 32\]. However, the open invariants have been largely unexplored, with the exception of the work of Xu \[35\], which concerns the compactification of the moduli space of open symplectic vortices on the disk for $G = U(1)$, as well as the work of Wang and Xu \[34\] on the relationship between open symplectic vortices for $X$ and open worldsheet instantons for $X//G$ (the open quantum Kirwan map).
An outline of the paper

In Section 2, we introduce the $\mathcal{N} = (2, 2)$ supersymmetric dynamically gauged nonlinear sigma model on the infinite strip, focusing on its gauge symmetry and supersymmetry. In Section 3, we first review the mirror symmetry between abelian GNLSMs and gauged Landau-Ginzburg (LG) models with neutral matter. We then derive the explicit reduction of open gauged linear sigma models (GLSMs) to open GNLSMs, making use of the example of $\mathbb{C}P^{N-1}$. In Section 4, we study equivariant B-branes, paying particular attention to abelian equivariant B-branes in toric manifolds $X$, as well as the LG mirrors of these branes when $X$ is Fano. Nonabelian equivariant B-branes in general $G$-manifolds are also analyzed. In Section 5, abelian equivariant A-branes in toric manifolds $X$ are introduced, and their LG mirror description is shown for toric manifolds $X$ with $c_1(X) \geq 0$. We also explore nonabelian equivariant A-branes for general $G$-manifolds. In Section 6, we use the data of equivariant A-branes to study open Hamiltonian Gromov-Witten invariants. The open gauged A-model is first introduced, together with its bulk and boundary observables. The path integrals of these observables are given by classical integrals over the moduli spaces of open symplectic vortices on a Riemann surface with boundaries, and these integrals are identified with the open Hamiltonian Gromov-Witten invariants. Then, we compute the dimension of these moduli spaces, as well as the related boundary axial R-anomaly. For abelian invariants, we use mirror symmetry to compute the $\hat{Q}_A^2 \neq 0$ anomaly, which implies supersymmetry breaking, and indicates an obstruction to integration over the moduli spaces. We shall find the condition whereby the anomaly vanishes and supersymmetry is manifest. Finally, we show how mirror symmetry can be used to compute the abelian invariants themselves.

The reader who is interested in equivariant B-branes should read Sections 2, 3 and 4, whereas the reader who is interested in equivariant A-branes and open Hamiltonian Gromov-Witten invariants should read Sections 2, 3, 5 and 6.

2 The Gauged Nonlinear Sigma Model with Boundaries

The $\mathcal{N} = (2, 2)$ supersymmetric nonlinear sigma model (NLSM) has the isometry group, $G$ of its Kähler target space, $X$, as a global symmetry. This global symmetry can be gauged, by allowing the isometry transformations of the target space to depend on the local coordinates of the worldsheet. As in the case of Yang-Mills theory, invariance under this local symmetry requires the introduction of a gauge field, $A_\mu$. Supersymmetry then requires the introduction of gaugino and scalar fields, denoted as $\lambda$ and $\sigma$. In addition, we can introduce kinetic terms for the gauge field and its superpartners, and as a result we have a dynamically gauged supersymmetric nonlinear sigma model, or GNLSM.

We shall take the worldsheet $\Sigma$ to be the infinite strip $I \times \mathbb{R}$ (where the interval is $I = [0, \pi]$) equipped with a flat Minkowski metric $\eta = \text{diag}(-1, 1)$. The spatial coordinate along the interval will be denoted $x^1$ and the time coordinate parametrizing $\mathbb{R}$ will be denoted $x^0$. The main fields of the GNLSM are a connection, $A$, on a principal $G$-bundle

$$P \to \Sigma,$$  \hspace{1cm} (2.1)
and a section
\[ \phi : \Sigma \rightarrow E \] (2.2)
of the associated bundle \( E := P \times_G X \), where \( X \) is a Kähler manifold. Locally on \( \Sigma \), \( E \) looks like the product \( \Sigma \times X \), which implies that locally the section \( \phi \) looks like a map \( \phi : \Sigma \rightarrow X \),\(^1\) as one finds in non-gauged NLSMs.

The other fields of the GNLSM are sections of the following bundles:

\[ \psi_\pm \in \Gamma(\Sigma; S_\pm \otimes \phi^* \ker d\pi_E) \quad F \in \Gamma(\Sigma; \phi^* \ker d\pi_E) \] (2.3)
\[ \sigma \in \Gamma(\Sigma; g_P^\mathbb{C}) \quad D \in \Gamma(\Sigma; g_P^\mathbb{C}) \]
\[ \lambda_\pm \in \Gamma(\Sigma; S_\pm \otimes g_P^\mathbb{C}) \]

Here, \( S_\pm := K^{\pm 1/2} \) are the spinor bundles of \( \Sigma \), whereby \( K := \Lambda^{1,0} \Sigma \) is the canonical bundle of \( \Sigma \) (the bundle of one-forms of type \((1,0)) \); \( \ker d\pi_E \rightarrow E \) is a bundle which locally looks like \( TX \) and is the sub-bundle of \( TE \rightarrow E \) corresponding to the kernel of the derivative \( d\pi_E : TE \rightarrow T\Sigma \) of the projection \( \pi_E : E \rightarrow \Sigma \); \( \phi^*(\ker d\pi_E) \rightarrow \Sigma \) is the pullback of \( \ker d\pi_E \) with respect to the section \( \phi \) and \( g_P := P \times_{Ad} g \) is the associated adjoint bundle (where \( g \) is the Lie algebra of \( G \)) with \( g_P^\mathbb{C} \) denoting its complexification.

The GNLSM action is
\[ S = \frac{1}{2\pi}(S_{\text{matter}} + S_{\text{gauge}} + S_B + S_\theta), \] (2.4)

where
\[ S_{\text{matter}} = \int_{\Sigma} d^2 x \left( -g_{jk} \partial^A \phi^j \partial^A \phi - \frac{i}{2} g_{jk} \overline{\psi}_- (\phi^* \nabla^A)_+ \psi^j_+ + \frac{i}{2} g_{jk} \overline{\psi}_+ (\phi^* \nabla^A)_- \psi^j_- - \frac{1}{2} g_{jk} \sigma^a \psi^j_+ \psi^j_- - \frac{1}{2} g_{jk} \sigma^a \psi^j_+ \psi^j_- + i g_{jk} (\nabla_i \tilde{e}^j_k) (\sigma^a \psi^j_- \psi^j_+ + \overline{\sigma}^a \psi^j_+ \psi^j_-) + g_{jk} (\tilde{\lambda}^a_j \psi^j_- \psi^j_+ + \tilde{\lambda}^a_j \psi^j_- \psi^j_+) + R_{ijkl} \psi^i_+ \psi^j_- \psi^k_- \psi^l_+ + g_{jk} (F^j - \Gamma^j_{il} \psi^l_+ \psi^l_-) (\overline{F^k} - \Gamma^k_{lm} \psi^l_- \psi^l_+ \overline{\psi}_- \overline{\psi}_+) \right), \]
(2.5)

\[ S_{\text{gauge}} = \frac{1}{2e^2} \int_{\Sigma} d^2 x \left( F^a_{01} F_{01a} - \nabla^a \sigma^a \overline{\nabla} \overline{\sigma}_a + \frac{1}{4} [\sigma, \overline{\sigma}]^a [\sigma, \overline{\sigma}]_a + D^a D_a - 2 e^2 \phi^* \mu_a D^a \right. \]
\[ + \frac{i}{2} (\overline{\lambda}^-)_a \overline{\nabla}^A \lambda^a_+ + \frac{i}{2} (\overline{\lambda}^-)_a \overline{\nabla}^A \lambda^a_+ + i \overline{\lambda}^- \nabla^a \lambda^a_+ + i \overline{\lambda}^- \nabla^a \lambda^a_+ \right) + \left( \overline{\lambda}^- \nabla^a \lambda^a_+ + \overline{\lambda}^- \nabla^a \lambda^a_+ \right), \] (2.6)

\[ S_B = - \int_{\Sigma} \phi^* B + \int_{\partial \Sigma} \phi^* C_a A^a, \] (2.7)

and
\[ S_\theta = \int_{\Sigma} d^2 x \ (\theta, F_{01}), \]
(2.8)

\(^1\)When \( P \) is the trivial \( G \)-bundle, this becomes true \textit{globally}. 

\[ -4 - \]
with $d^2 x = dx^1 \wedge dx^0$. \(^2\) Here, $F^j$ and $D^a$ are auxiliary fields. The covariant derivatives induced by the connection $A$ on the bundles $E$, $\phi^* \ker d\pi_E$ and $g_P$ over $\Sigma$ appear above in their local forms, which are given explicitly as

\[
\begin{align*}
\partial^A \phi^k &= \partial_\mu \phi^k + A_\mu^a \tilde{e}^k_a \\
(\phi^* \nabla^A)_{\mu} \psi^b &= \partial_\mu \psi^k + A_\mu^a \psi^j \partial_j \tilde{e}^k_a + \Gamma^k_{ji}(\partial^j_\mu \phi^l)\psi^l \\
\nabla_\mu \sigma^a &= \partial_\mu \sigma^a + [A_\mu, \sigma]^a \\
\nabla_\mu \lambda^a &= \partial_\mu \lambda^a + [A_\mu, \lambda]^a
\end{align*}
\]

(2.9)

where $\phi$ and $\sigma$ are locally regarded as maps $\Sigma \rightarrow X$ and $\Sigma \rightarrow g$; $\psi$ and $\lambda$ are locally regarded as (fermionic) maps $\Sigma \rightarrow \phi^* TX$ and $\Sigma \rightarrow g$; and $A$ is regarded as a local $1$-form on $\Sigma$. We have also used the notations

\[
\begin{align*}
\nabla_+ &= \nabla_0 + \nabla_1 \\
\nabla_- &= \nabla_0 - \nabla_1
\end{align*}
\]

(2.10)

as well as

\[
\begin{align*}
A \hat{\nabla} B &= A \nabla B - \nabla BA.
\end{align*}
\]

(2.11)

The notation (2.11) appears in all the fermionic kinetic terms, since they must have this symmetrized form in order to preserve the reality of the action in the presence of worldsheet boundaries.

Let us now explain the quantities that appear in the action. Firstly, note that the components of the Killing vector fields

\[
\tilde{e}_a = \tilde{e}_a^i \partial_\phi^i + \tilde{e}_a^\alpha \partial_\phi^\alpha
\]

(2.12)

which generate the $G$-isometry of $X$ appear in the action. These components are holomorphic/antiholomorphic,

\[
\frac{\partial \tilde{e}_a^i}{\partial \phi^i} = \frac{\partial \tilde{e}_a^\alpha}{\partial \phi^\alpha} = 0,
\]

(2.13)

and this constraint can be shown to be a consequence of the complex structure of $X$ being $G$-invariant, i.e., $\mathcal{L}_{\tilde{e}} J = 0$. Furthermore, they realize an antihomomorphism of the Lie algebra, $g$, via their Lie bracket

\[
\begin{align*}
[\tilde{e}_a, \tilde{e}_b] &= -f_{ab}^c \tilde{e}_c, \\
[\tilde{e}_a, \tilde{e}_b] &= -f_{ab}^c \tilde{e}_c
\end{align*}
\]

(2.14)

(the generators of $g$ satisfy $[T_a, T_b] = f_{ab}^c T_c$, where $f_{ab}^c$ are the real structure constants of $g$). Also note that the covariant derivative of the holomorphic component, $\nabla_1 \tilde{e}_a^i$ appears in the action.

\(^2\)We only consider the case where $G$ is compact, in order to ensure positive-definiteness of the gauge multiplet kinetic terms.
The moment map \( \mu_a \), which is a map \( X \to \mathfrak{g}^* \) (where \( \mathfrak{g}^* \) is the dual of \( \mathfrak{g} \)) also appears in (2.6), and obeys
\[
\partial_k \mu_a = ig_{jk} \tilde{e}^j_a
\]
or
\[
d\mu_a = \iota_{\tilde{e}^a} \omega,
\]
where \( \omega = \frac{i}{2} \omega_{IJ} d\phi^I \wedge d\phi^J = ig_{\sigma\tau} d\phi^\sigma \wedge d\phi^\tau \) is the Kähler form.\(^3\) This indicates that the \( G \)-isometry of \( X \) is Hamiltonian.

The fact that only the derivative of \( \mu_a \) enters (2.16) ostensibly implies that the moment map \( \mu = \mu_a T^a \) is only defined up to a constant in \( g^* \). In fact, the moment map is defined up to a constant in \([g, g]^0\), the subspace of \( g^* \) that annihilates commutators;\(^4\) as in [3], we shall follow the convention of ([4], page 164) where the definition of a Hamiltonian \( G \)-action includes the additional condition
\[
\rho^* \mu = \text{Ad}^* \circ \mu
\]
for all elements \( g \in G \). Here, \( \rho \) is the \( G \)-action on \( X \) and \( \text{Ad}^* \) is the coadjoint representation on \( \mathfrak{g}^* \). This then implies ([5], page 190) that the moment map is only defined up to a constant, \( r \), in \([g, g]^0\). In fact, this freedom to redefine the moment map as
\[
\mu \to \mu + r
\]
is manifest in the action; we may add the term
\[
S_r = -\frac{1}{12} r_a D^a = \frac{1}{12} \int d^2 x (r_a D^a) = \frac{1}{12} \int d^2 x r_a D^a
\]
to (2.4), which results in
\[
\mu_a \to \mu_a + r_a
\]
in (2.6). From (2.19), we see that the constant \( r \) now plays the role of a \([g, g]^0\)-valued Fayet-Iliopoulos (FI) parameter.

In (2.7), the \( B \)-field action is given. Here, \( B \) is an arbitrary \( G \)-invariant (i.e., \( \mathcal{L}_\xi B = 0 \)) and closed 2-form on \( X \), and \( \phi^* B \) denotes the pullback of \( B \). Explicitly, the \( B \)-field term is denoted as
\[
-\phi^* B = -\frac{1}{2} B_{IJ} (\partial_1 \phi^I \partial_0 \phi^J - \partial_0 \phi^I \partial_1 \phi^J) dx^1 \wedge dx^0,
\]
where \( B = \frac{1}{2} B_{IJ} d\phi^I \wedge d\phi^J \), and \( \phi^I \) are real coordinates on \( X \). A boundary term is also included in the \( B \)-field action, with
\[
\phi^* C_a A^a = \phi^* C_a A^a_0 dx^0,
\]
\(^3\)In [3], the moment map equation is given as \( 2\partial_\sigma \mu_a = ig_{\sigma\tau} \tilde{e}^\tau_a \), since the Kähler form is defined as \( \omega = \frac{i}{2} g_{\sigma\tau} d\phi^\sigma \wedge d\phi^\tau \).
\(^4\)\([g, g]^0\) can be further identified with the centre of \( g \), via the identification \( \mathfrak{g}^* \cong g \) provided by the inner product \( \mathfrak{g}^* \times g \to \mathbb{R} \).
where \( C \) is a map \( X \to g^* \) which obeys
\[
dC_a = \iota_{\tilde{e}_a} B, \tag{2.23}
\]
as well as
\[
\rho_g^* C = \text{Ad}_g^* \circ C \tag{2.24}
\]
for all elements \( g \in G \). The necessity of this boundary term will be explained below when we investigate the gauge invariance of the \( B \)-field action.

Finally, the \( \theta \)-term is given in (2.8), where \( \theta \) is a constant in \([g, g]^0 \subset g^*\), while \((\cdot, \cdot)\) is the inner product \( g^* \times g \to \mathbb{R} \). We can then combine \( r \) with \( \theta \) as
\[
r - i\theta = t \in [g, g]^0_C, \tag{2.25}
\]
to obtain the complex FI-theta parameter, \( t \), valued in \([g, g]^0\) (the complexification of \([g, g]^0\)).

**Gauge and Supersymmetry Invariance**

The action is invariant under the following gauge symmetry transformations (where the parameter \( \alpha^a \) is a local function on the worldsheet)\(^5\)
\[
\begin{align*}
\delta \phi^k &= \alpha^a \tilde{e}^k_a \\
\delta \phi^k &= \alpha^a \tilde{e}^k_a \\
\delta \psi^k_\pm &= \alpha^a \psi^i_\pm \partial_i \tilde{e}^k_a \\
\delta \psi^k_\pm &= \alpha^a \psi^i_\pm \partial_i \tilde{e}^k_a \\
\delta F^k &= \alpha^a F^i \partial_i \tilde{e}^k_a \\
\delta F^k &= \alpha^a F^i \partial_i \tilde{e}^k_a \\
\delta A^a_\mu &= [\alpha, A_\mu]^a - \partial_\mu \alpha^a = -\nabla_\mu \alpha^a \\
\delta \sigma^a &= [\alpha, \sigma]^a \\
\delta \sigma^a &= [\alpha, \sigma]^a \\
\delta \lambda^a_\pm &= [\alpha, \lambda_\pm]^a \\
\delta \lambda^a_\pm &= [\alpha, \lambda_\pm]^a \\
\delta D^a &= [\alpha, D]^a
\end{align*}
\tag{2.26}
\]

Let us first explain how gauge invariance requires the boundary term in the \( B \)-field action (2.7). For a closed worldsheet, the term containing the closed two-form \( \phi^* B \) is gauge invariant. However, on an open worldsheet, a gauge transformation of this term generates a boundary term, and in order to restore gauge invariance, the boundary term containing \( \phi^* C \) must be added to the action.\(^6\)

\(^5\)Note that one needs to use various identities in order to show gauge invariance, including the Jacobi identity and the Killing equation \( \mathcal{L}_{\tilde{e}_i} g_\tau = 0 \), which implies \( \mathcal{L}_{\tilde{e}_i} \Gamma^j_{ik} = 0 \) and \( \mathcal{L}_{\tilde{e}_i} R^j_{\tau \sigma} = 0 \) ([6], page 52).\(^6\)Note that the proof of this requires the use of the identities (2.23) and (2.24), with the latter implying that \( \alpha^a \mathcal{L}_{\tilde{e}_i} C_a = [\alpha, C]_a \).
With the exception of this B-field action (2.7), the gauge symmetry of the action is insensitive to the presence of boundaries. To understand this, note that for a global $G$-isometry of $X$, the corresponding symmetry variation of the (non-gauged) NLSM scalar kinetic term is

$$\int g_{ij} \partial_{\mu} \phi^i \partial^\mu \phi^j$$

$$\int \partial_{\mu} (g_{ij} \phi^j \partial^\mu \phi^i)$$

$$\sum \delta \left( g_{ij} \partial_{\mu} \phi^i \partial^\mu \phi^j \right)$$

$$\int \sum \delta \left( g_{ij} \partial_{\mu} \phi^i \partial^\mu \phi^j \right)$$

where we have used the Killing equation $\mathcal{L}_{\tilde{g}} g_{ij} = 0$. In the computation above, we have not used integration-by-parts (which would introduce a nonzero boundary term), so the presence of boundaries is inconsequential for global $G$-symmetry of NLSMs.

A crucial step in (2.28) is that the worldsheet derivative of $\phi^i$ transforms under the global $G$-symmetry as a target space vector field, i.e.,

$$\delta (\partial_{\mu} \phi^i) = \alpha^a \partial_{\mu} \phi^i \partial_j \tilde{e}^i_a.$$  \hspace{1cm} (2.29)

When the $G$-symmetry is local, it is the covariant derivative that transforms in the above manner, i.e.,

$$\delta (\partial^A \phi^i) = \partial_{\mu} (\alpha^a \tilde{e}^i_a) - \nabla^A \alpha^a \tilde{e}^i_a + A^a_\mu \alpha^b \tilde{e}^j_a \partial_j \tilde{e}^i_b + A^a_\mu [\alpha, \tilde{e}^i_a]$$

$$= \alpha^a \partial^A \phi^i \partial_j \tilde{e}^i_a.$$  \hspace{1cm} (2.30)

where we have used (2.14). Then, the same steps in the computation (2.28) hold, with $g_{ij} \partial_{\mu} \phi^i \partial^\mu \phi^j$ replaced by $g_{ij} \partial^A \phi^i \partial^A \phi^j$, and the scalar kinetic term of the GNLSM is gauge invariant without using integration-by-parts. In a similar manner, all the other terms in (2.5) are gauge invariant without the generation of nonzero boundary terms via integration-by-parts. Furthermore, the gauge action (2.6), the $\theta$-term in (2.8), and the FI term (2.19) are also gauge invariant, without using integration-by-parts.\(^7\) In this way, the GNLSM action ((2.4)+(2.19)) is gauge invariant, and no nonzero boundary terms are generated via gauge transformations.

However, this is not the case for supersymmetry. For a closed worldsheet, the action ((2.4)+(2.19)) would be invariant under the following $N = (2, 2)$ supersymmetry transfor-
continuation to a Minkowski worldsheet following the Appendix in loc. cit., since our theory is defined on \( \mathbb{R}^8, \) rescaling the theta parameter as 
\[
\epsilon \to \epsilon \left( \frac{g}{g_0} \right) \quad \text{and} \quad \sigma \to \sigma \left( \frac{g}{g_0} \right) \].
\( \text{Here,} \ g \) is the conformal group of \( \mathbb{R}^8, \) and \( g_0 \) is the conformal group of \( \mathbb{R}^4. \) Our action (2.4) is also given an overall factor of \( \frac{g}{g_0}. \)

Note that if \( r \) was a constant in \( \mathfrak{g}^* \) instead of \( [g, \mathfrak{g}]^0, \) the supersymmetry invariance of (2.19) on a closed worldsheet would not hold. This is another reason we need the condition (2.17).
and we find
\[ \delta S_{\text{matter}} = \frac{1}{2\pi} \int_{\partial \Sigma} dx^0 \left\{ -\epsilon_+ \left( \frac{1}{2} g_\sigma (\partial_0^\sigma + \partial_1^\sigma) \partial_0^\sigma \psi^i_+ + \frac{1}{2} g_\sigma \sigma^a e_a^\varepsilon \partial_1^\sigma \psi_i^a \right) + \epsilon_+ \left( \frac{1}{2} g_\sigma (\partial_0^\sigma + \partial_1^\sigma) \partial_0^\sigma \psi^i_+ + \frac{1}{2} g_\sigma \sigma^a e_a^\varepsilon \partial_1^\sigma \psi_i^a \right) \right\} \]
\[ \delta(S_{\text{gauge}} + S_r + S_\theta) = \]
\[ \frac{1}{2\pi} \int_{\partial \Sigma} dx^0 \left\{ \epsilon_+ \left( \frac{i}{2} \lambda_-^a (\nabla_1^A + \nabla_1^\dag) \sigma^a - \lambda_+^a \left( \frac{i}{4} F_1^{0\sigma} + \frac{i}{4} [\sigma, \sigma]^a \right) + \epsilon_+ (\phi^a \mu - r - s) \lambda_+^a \right) \right\} + \epsilon_+ \left( \frac{i}{2} \lambda_-^a (\nabla_1^A + \nabla_1^\dag) \sigma^a - \lambda_+^a \left( \frac{i}{4} F_1^{0\sigma} + \frac{i}{4} [\sigma, \sigma]^a \right) + \epsilon_+ (\phi^a \mu - r - s) \lambda_+^a \right) \right\} \]
\[ \delta S_B = \frac{1}{2\pi} \int_{\partial \Sigma} dx^0 \left\{ \left( B_{ij} \delta_0^i \phi^j + B_{ij} \delta_1^i \phi^j \right) (\epsilon_+ \psi^i_+ - \epsilon_- \psi^i_-) + \left( B_{ij} \delta_0^i \phi^j + B_{ij} \delta_1^i \phi^j \right) (-\epsilon_+ \psi^i_+ + \epsilon_- \psi^i_-) \right\} + \frac{i}{2} \left( \epsilon_+ \lambda_-^0 + \epsilon_- \lambda_+^0 + \epsilon_+ \lambda_0^0 + \epsilon_- \lambda_0^0 \right) \epsilon^a C^a \right\} \]
\[ \text{(2.36)} \]

In deriving the above, we have used various identities from Kähler geometry, as well as the Killing equation in the form
\[ g_{jk} \nabla^k \phi^a + g_{kl} \nabla^k \phi^c = 0. \quad \text{(2.37)} \]

We have also used the identity (2.23) to arrive at the form of \( \delta S_B \) given in (2.36).

In order to restore supersymmetry at the boundaries, we need to choose an appropriate set of boundary conditions on the fields, and these conditions must themselves be supersymmetric. In fact, only an \( \mathcal{N} = 2 \) subset of the four supersymmetries can be preserved at the boundaries, because of the following reasons. Translation symmetry on the worldsheet is broken at the boundaries, where the worldsheet momentum is no longer conserved. Since the \( \mathcal{N} = (2, 2) \) supersymmetry algebra includes
\[ \{ Q_\pm, Q_\pm \} = H \pm P, \quad \text{(2.38)} \]
the previous statement implies that on an open worldsheet, some of the supersymmetries are broken at the boundaries. In particular, only certain linear combinations of the original supersymmetries are preserved at the boundaries, i.e., those whose algebra do not include the worldsheet momentum. There are two such combinations ([7], Chapter 39)
\[ Q_A = \overline{Q}_+ + e^{i\beta} Q_- \quad \text{and} \quad Q_A^\dagger = Q_+ + e^{-i\beta} \overline{Q}_-, \quad \text{(2.39)} \]
or
\[
Q_B = Q_+ e^{i\beta} Q_-, \quad Q_B^\dagger = Q_+ + e^{-i\beta} Q_-,
\] (2.40)
where $\beta$ is a real parameter, and these satisfy
\[
\{Q, Q^\dagger\} = 2H,
\] (2.41)
which is in fact the supersymmetry algebra of supersymmetric quantum mechanics. These combinations are known as A-type and B-type supersymmetry.\footnote{The supercharges $Q_A$ and $Q_B$ also happen to correspond to the scalar supercharges preserved in the A-twisted and B-twisted topological sigma models, when $\beta = 0$.}

Now, note that one can further generalize the action \((2.4)+(2.19)\) by considering quiver GNLSMs, i.e., GNLSMs gauged by a direct product $G_1 \times G_2 \times G_3 \ldots$. This corresponds to having several copies of \((2.6)+(2.19)+(2.8)\), each corresponding to one gauge group $G_i$ together with its own coupling constant $e_i$, as well as coupling the matter in \((2.5)+(2.7)\) to all of these gauge groups. In Sections 4 and 5 of this paper, we shall focus on finding the A-type and B-type supersymmetric boundary conditions for quiver abelian GNLSMs on toric manifolds, that is, GNLSMs with gauge group $G = U(1)^N$. These boundary conditions will ensure that \((2.34), (2.35)\) and \((2.36)\) vanish, restoring A-type or B-type supersymmetry at the boundaries. Furthermore, we will identify boundary interactions compatible with these conditions. These boundary conditions and boundary interactions shall correspond to equivariant generalizations of D-branes, which we shall refer to as equivariant A-branes and equivariant B-branes.

Before ending this section, we note that the action $S + S_r \ (2.4) + (2.19)$ and supersymmetry variations \((2.34), (2.35)\) and \((2.36)\) reduce to the familiar formulae given in \cite{10} for GLSMs and NLSMs with boundary, in the appropriate limits. We can recover the $\mathcal{N} = (2, 2)$ NLSM by setting the group $G$ to be trivial. Alternatively, by choosing $X = \mathbb{C}^N$, we find the usual formulae for a $\mathcal{N} = (2, 2)$ GLSM, with the gauge group $G$ being an isometry group of $\mathbb{C}^N$. For example, the $U(1)^N$ model with target space $\mathbb{C}^N$, i.e., an abelian quiver GLSM, corresponds to the action $S + S_r$ with the $B$-field and all Lie algebra commutators vanishing, the moment map
\[
\mu_a = -\left(\sum_i^N Q_i a |\phi_i|^2\right),
\] (2.42)
and the Killing vector fields
\[
\tilde{e}_a = i Q_i a \phi_i,
\]
\[
\tilde{e}_a = -i Q_i a \bar{\phi}_i
\] (2.43)
where we have chosen the flat metric $g_{ij} d\phi^i \otimes d\bar{\phi}^j = \delta_{ij} d\phi_i \otimes d\bar{\phi}_j$, and where $Q_{ia}$ are the $U(1)$ charges of $\phi_i$ with $a = 1, \ldots, N$.\footnote{The supercharges $Q_A$ and $Q_B$ also happen to correspond to the scalar supercharges preserved in the A-twisted and B-twisted topological sigma models, when $\beta = 0$.}
3 GNLSMs from GLSMs and Mirror Symmetry

It is well-known that $\mathcal{N} = (2, 2)$ NLSMs with target spaces being toric manifolds can be obtained in the IR limit of $\mathcal{N} = (2, 2)$ quiver abelian GLSMs [8]. Hori and Vafa [9] made use of this to prove the mirror symmetry of manifolds with non-negative first Chern class in terms of Landau-Ginzburg theories. This proof of mirror symmetry was later applied to worldssheets with boundaries, whereby the Landau-Ginzburg mirrors of B-branes [10] and A-branes [11] were found.

As shown by Baptista [12], it is also possible to obtain quiver abelian GNLSMs on closed worldssheets with toric target spaces by taking a different limit of quiver abelian GLSMs. Moreover, Baptista found the mirror Landau-Ginzburg theories of these GNLSMs. This then suggests a natural generalization of Baptista’s proof to worldsheets with boundaries, in order to find equivariant A-branes and B-branes in abelian GNLSMs, as well as the Landau-Ginzburg mirrors of these branes. We shall pursue this line of thought in Sections 4 and 5.

Before doing so, let us review Baptista’s generalization of mirror symmetry for GNLSMs on closed manifolds. In superfield language, the action of a $U(1)^N$-GLSM with target space $\mathbb{C}^N$ is

$$S_{GLSM} = \frac{1}{2\pi} \int d^2 x \int d^4 \theta \left\{ \sum_{j=1}^{N} \Phi_j (e^{\tilde{Q}^c_j \tilde{V}_c} + \tilde{Q}^c_j \tilde{V}_c) \Phi_j - \sum_{b=1}^{N-k} \left( \frac{1}{2 e^c_b} \tilde{\Sigma}_b \hat{\Sigma}_b \right) - \sum_{c=1}^{k} \left( \frac{1}{2 e^c_c} \tilde{\Sigma}_c \hat{\Sigma}_c \right) \right\}$$

$$+ \frac{1}{2\pi} \frac{1}{2} \int d^2 x \left( \int d^2 \tilde{\theta} \left( \hat{t}^b \tilde{\Sigma}_b - \tilde{\Sigma}_c \hat{t}^c \right) + \text{c.c.} \right),$$

(3.1)

where $U(1)^N = U(1)^{(N-k)} \times U(1)^k$, with the indices $b = 1, \ldots, N-k$ and $c = 1, \ldots, k$. The mirror of this theory is the following Landau-Ginzburg sigma model with twisted chiral superfields $Y^j$ (whose imaginary parts are periodic, with period $2\pi$) and action

$$S_{dual} = \frac{1}{2\pi} \int d^2 x \int d^4 \theta \left\{ - \frac{1}{2} \sum_{j=1}^{N} (Y^j + \bar{Y}^j) \log(Y^j + \bar{Y}^j) - \sum_{b=1}^{N-k} \left( \frac{1}{2 e^c_b} \tilde{\Sigma}_b \hat{\Sigma}_b \right) - \sum_{c=1}^{k} \left( \frac{1}{2 e^c_c} \tilde{\Sigma}_c \hat{\Sigma}_c \right) \right\}$$

$$+ \frac{1}{2\pi} \frac{1}{2} \int d^2 x \left( \int d^2 \tilde{\theta} \left( \hat{Q}^j_0 Y^j - \tilde{t} \hat{\Sigma}_b \right) + \left( \hat{Q}^j_0 Y^j - \tilde{t} \hat{\Sigma}_c \right) + \sum_{j=1}^{N} e^{-Y^j} \right) + \text{c.c.} \right\}.$$

(3.2)

It may seem from the logarithm in this action that the real part of $Y^j$ is positive-definite, implying a boundary in the target space where the metric becomes singular. However, quantum effects remove this boundary via the field renormalization $Y^j_0 = Y^j + \log(\Lambda_{UV}/\mu)$.
The corresponding Kähler metric for this renormalized field is

\[ ds^2 = \sum_{i=1}^{N} \frac{|dy^i|^2}{2(2\log(\Lambda_{UV}/\mu) + y^i + \overline{y}^i)} \approx \frac{1}{4\log(\Lambda_{UV}/\mu)} \sum_{i=1}^{N} |dy^i|^2 \]  

(3.3)

(where \( y^i \) is the lowest component of \( Y^i \)), which becomes flat in the continuum limit \( \Lambda_{UV} \to \infty \). Taking into account the periodicity of the imaginary part of \( Y^i \), this implies that the target space for this Landau-Ginzburg sigma model is the algebraic torus \((\mathbb{C}^\times)^N\).

The duality between these actions is shown as follows. The GLSM action (3.1) can be obtained by integrating out \( Y^j \) from the following action

\[
S = \frac{1}{2\pi} \int d^2x \int d^4\theta \left\{ \sum_{j=1}^{N} \left( e^{\hat{Q}_j^b \hat{V}_b + \hat{Q}_j^c \hat{V}_c + B^j} - \frac{1}{2} (Y^j + \overline{Y}^j) B^j \right) - \sum_{b=1}^{N-k} \left( \frac{1}{2\epsilon_b^2} \hat{\Sigma}_b \hat{\Sigma}_b \right) - \sum_{c=1}^{k} \left( \frac{1}{2\epsilon_c^2} \hat{\Sigma}_c \hat{\Sigma}_c \right) \right\} + \frac{1}{2\pi} \int d^2\tilde{\theta} \left( \int d^2\tilde{\theta} \left( -\hat{\Sigma}_b - \hat{\Sigma}_{\tilde{C}} \right) + c.c. \right).
\]

(3.4)

Alternatively, one can integrate out \( B^j \), to obtain (3.2), modulo the term

\[
\sum_{j=1}^{N} e^{-Y^j} + c.c.
\]

(3.5)

This term appears when we take into account quantum effects, and is generated by vortices ([7], page 469). The \( \mathcal{N} = (2, 2) \) supersymmetric actions (3.1) and (3.2) are mirror theories, since the former is in terms of chiral superfields and the latter is in terms of twisted chiral superfields. Furthermore, by comparing the different expressions one obtains for \( B^j \) when deriving the two mirror theories, it can be shown that

\[
Y^j + \overline{Y}^j = 2\overline{\Phi}_j e^{\hat{Q}_j^b \hat{V}_b + \hat{Q}_j^c \hat{V}_c} \Phi_j,
\]

(3.6)

which is an explicit relationship between the fields of the mirror theories.

Note that we have neglected the boundary terms which appear when integrating out \( Y^j \), as well as the boundary terms that come about when deriving (3.2) from (3.4). The above technique of proving mirror symmetry can be generalized to the situation whereby the worldsheet has boundaries, where the fields obey boundary conditions and where additional boundary interactions could occur. We shall see this in Sections 4 and 5, when we use the above technique to find mirrors of abelian equivariant branes.

---

11The field renormalization of \( Y^i \) arises due to the presence of the twisted superpotential in the action (3.2), which can be written (modulo the exponential term) as

\[
\tilde{W} = \sum_{a}^{N} \left( Q_{a} Y_{a} - t_{a} \right) \Sigma_{a}
\]

where \( Y_{a0} \) and \( t_{a0} = r_{a0} - i\theta_{a0} \) are bare quantities. Taking into account the renormalization of the FI parameter \( (r_{a0} = r_{a} + \sum_{b}^{N} Q_{a} \log(\Lambda_{UV}/\mu)) \) ([7] (where \( \Lambda_{UV} \) is the UV cutoff, and \( \mu \) is a finite energy scale), in order for the superpotential to be finite, we must renormalize the bare fields \( Y_{a0}^i \) as \( Y_{a}^i = Y_{a0}^i + \log(\Lambda_{UV}/\mu) \).
To obtain a GNLSM from (3.1), we take the limit where \( \hat{\epsilon}_b \to \infty \). The \( \hat{\Sigma}_b \) kinetic terms vanishes, and the remaining fields belonging to \( \hat{V}_c \) become auxiliary fields, and are integrated out. The resulting sigma model has \( \mathbb{C}^N//U(1)^{N-k} \) as target space, but is still gauged, since the vector superfields \( \hat{V}_c \) are still present in the action. However, note that in order to obtain a Kähler target space with \( k \) complex dimensions, the FI parameters \( \hat{r}_b \) must be within a Kähler cone. In this way, we obtain the \( \mathcal{N} = (2,2) \) U(1)\(^k\)-gauged nonlinear sigma model (GNLSM) with Kähler \( \mathbb{C}^N//U(1)^{N-k} \) target space.

Taking the same limit \( \hat{\epsilon}_b \to \infty \) in the dual Landau-Ginzburg sigma model (3.2) makes the \( \hat{\Sigma}_b \) kinetic terms vanish. \( \hat{\Sigma}_b \) is then an auxiliary superfield, which we can integrate out to impose the constraints

\[
\hat{Q}^b_j Y^j - \hat{t}^b = 0. \tag{3.7}
\]

These constraints have the solution

\[
Y^j = \hat{s}^j + \sum_{c=1}^k \nu^j_c \Theta^c, \tag{3.8}
\]

where \( \Theta^1, \ldots, \Theta^k \) are new twisted chiral fields, the complex constants \( \hat{s}^1, \ldots, \hat{s}^N \in \mathbb{C} \) are any solution of the algebraic relation \( \hat{Q}^b_j \hat{s}^j = \hat{t}^b \), and \( \nu^j_c \) are \( N \) primitive vectors \( v^1, \ldots, v^N \in \mathbb{Z}^k \) (which generate the regular fan associated with \( \mathbb{C}^N//U(1)^{N-k} \)) that span \( \mathbb{Z}^k \) and satisfy \( \sum_j \nu^j_c v^j = 0 \). Thus, the \( \hat{\epsilon}_b \to \infty \) limit gives the following U(1)\(^k\)-gauged Landau-Ginzburg sigma model

\[
\hat{S}_{\text{dual}} = \frac{1}{2\pi} \int d^2 x \int d^4 \theta \left[ -\frac{1}{2} \sum_{j=1}^N (\hat{s}^j + \hat{s}^j) + (\hat{v}^j, \Theta + \overline{\Theta}) \right] \log (\hat{s}^j + \hat{s}^j + (\hat{v}^j, \Theta + \overline{\Theta})) - \sum_{c=1}^k \left( \frac{1}{2\epsilon^2_c} \hat{\Sigma}_c \overline{\hat{\Sigma}}_c \right)
\]

\[+ \frac{1}{2\pi} \int d^2 x \left\{ \int d^2 \theta \left( \langle \hat{\Sigma}, \hat{Q}^j \rangle (v^j, \Theta) + \hat{s}^j \right) - \langle \hat{\Sigma}, \hat{\ell} \rangle + \sum_{j=1}^N e^{-(v^j, \Theta) - \hat{s}^j} \right\} + \text{c.c.}, \tag{3.9}
\]

where \( \langle \cdot, \cdot \rangle \) is the canonical inner product on \( \mathbb{R}^k \). This gauged Landau-Ginzburg theory has the holomorphic twisted superpotential

\[
\overline{W}(\Theta, \hat{\Sigma}) = \langle \hat{\Sigma}, \hat{Q}^j \rangle (v^j, \Theta) + \hat{s}^j \rangle - \langle \hat{\Sigma}, \hat{\ell} \rangle + \sum_{j=1}^N e^{-(v^j, \Theta) - \hat{s}^j}. \tag{3.10}
\]

The parametrization (3.8) implies that the target space of this gauged Landau-Ginzburg sigma model, which is mirror to the GNLSM on the \( k \)-complex dimensional toric manifold \( \mathbb{C}^N//U(1)^{N-k} \), is the \( k \)-complex dimensional algebraic torus, \((\mathbb{C}^\times)^k\).

It is important to recognize that even though the mirror action (3.9) has U(1)\(^k\) gauge symmetry, its matter kinetic terms do not contain the vector superfields \( \hat{V}_c \), and therefore, the components of the twisted chiral superfields \( \Theta^c \) are not coupled to those in the vector superfields, except in the superpotential. In other words, the matter fields in the gauged Landau-Ginzburg theory (3.9), including the scalar fields which parametrize the \((\mathbb{C}^\times)^k\) target space, are neutral under the U(1)\(^k\) gauge symmetry.
The above calculation, due to Baptista, is essentially the same as Hori and Vafa’s. The only difference in Hori and Vafa’s case is that they start with a $U(1)^{N-k}$ GLSM with $\mathbb{C}^N$ target space instead of one with $U(1)^{N-k} \times U(1)^k$ gauge symmetry, and in the $\hat{e}_b \to \infty$ limit, the gauge symmetry completely disappears in both the GLSM and its mirror, leaving us with an NLSM with target space $\mathbb{C}^N // U(1)^{N-k}$ mirror to a (non-gauged) Landau-Ginzburg sigma model with target space $(\mathbb{C} \times)^k$ and holomorphic twisted superpotential

$$\bar{W} = \sum_{j=1}^{N} e^{-\langle \psi^j, \Theta \rangle} - \hat{s}^j. \tag{3.11}$$

Put differently, the Kähler quotient $\mathbb{C}^N // U(1)^{N-k}$ has a $U(1)^k$ isometry which descends from the residual $U(1)^k$ isometry of $\mathbb{C}^N$ ([7], page 362), and this can manifest either as a local or global symmetry in a $\mathbb{C}^N // U(1)^{N-k}$ sigma model. Baptista started with a GLSM fully gauged by the $U(1)^N$ abelian isometry group of $\mathbb{C}^N$, and ended up with a local $U(1)^k$ symmetry when taking $\hat{e}_b \to \infty$, whereas Hori and Vafa started with a GLSM gauged by only $U(1)^{N-k}$, with the remaining $U(1)^k$ isometry forming a global symmetry of the GLSM, which descends to a global symmetry of the NLSM in the $\hat{e}_b \to \infty$ limit.

Baptista’s technique of obtaining GNLSMs from GLSMs is an extremely powerful one, as it allows us to obtain multiple GNLSMs from a single GLSM, by choosing which coupling constants we wish to send to infinity. This implies the equivalence of several GNLSMs with different Kähler target manifolds, as well as the equivalence of equivariant branes contained in these manifolds. These branes will be the main objective of our study in the following sections. Furthermore, once a particular GNLSM is obtained, even its gauge group can be modified, by demoting some of its $U(1)$ gauge symmetries to global symmetries. These points shall be useful to keep in mind when reading the following sections, where we attempt to study abelian equivariant branes in as much generality as possible. This shall be achieved by generalizing Baptista’s technique to obtain GNLSMs on open worldsheets with toric target spaces, while studying both boundary conditions and boundary interactions.

**Explicit reduction of GLSMs to GNLSMs in the case with boundaries**

We have only discussed the method of obtaining GNLSMs from GLSMs in superfield language for closed worldsheets thus far. We will now derive this reduction in component form for an open worldsheet, so that we may eventually find explicit boundary actions and boundary conditions in GNLSMs. The $U(1)^N = U(1)^{N-k} \times U(1)^k$ GLSM action with boundaries is given explicitly as

$$S_{\text{GLSM}}$$

\footnote{It is important to keep in mind that the superfield action (3.1) is only equal to this action upon integration by parts, which give rise to boundary terms. However, we shall be concerned with this action, (which has the standard kinetic terms), as well as the GNLSMs we can obtain from it, and their mirrors.}
\[
\frac{1}{2\pi}\sum_{i}^{N}\int d^{2}x\left\{ -D_{\mu}\overline{\phi}_{i}D^{\mu}\phi_{i} + \frac{i}{2}\overline{\psi}_{i}(\overline{D}_{0} + \overline{D}_{1})\psi_{i} + \frac{i}{2}\overline{\psi}_{+i}(\overline{D}_{0} - \overline{D}_{1})\psi_{+i}
\right.
\]
\[
- \left(\sum_{a}Q_{ia}\sigma_{a}\right)\left(\sum_{a}Q_{ia}\overline{\sigma}_{a}\right)\overline{\phi}_{i}\phi_{i} - \sum_{a}Q_{ia}(\sigma_{a}\overline{\psi}_{+i}\psi_{i} + \sigma_{a}\overline{\psi}_{-i}\psi_{i})
\right)
\]
\[
- \sum_{a}iQ_{ia}\phi_{i}(\overline{\lambda}_{-a}\overline{\psi}_{+i} - \overline{\lambda}_{+a}\overline{\psi}_{-i}) - \sum_{a}iQ_{ia}\overline{\phi}_{i}(\psi_{-i}\lambda_{+a} - \psi_{+i}\lambda_{-a}) + |F_{i}|^{2}\right\}
\]
\[
+ \frac{1}{2\pi}\sum_{a}^{N}\frac{1}{2\epsilon_{a}^{2}}\int d^{2}x\left\{ (F_{01a})^{2} - \partial_{\mu}\sigma_{a}\partial^{\mu}\sigma_{a} + (D_{a})^{2} + 2\epsilon_{a}^{2}D_{a}\left(\sum\overline{Q}_{ia}\overline{\phi}_{i} - r_{a}\right)\right\}
\]
\[
+ \frac{i}{2}\overline{\lambda}_{+a}(\overline{D}_{0} - \overline{D}_{1})\lambda_{+a} + \frac{i}{2}\overline{\lambda}_{-a}(\overline{D}_{0} + \overline{D}_{1})\lambda_{-a} + 2\epsilon_{a}^{2}\theta_{a}F_{01a}\right\}
\]
\[
= \frac{1}{2\pi}\sum_{i}^{N}\int d^{2}x\left\{ -D_{\mu}\overline{\phi}_{i}D^{\mu}\phi_{i} + \frac{i}{2}\overline{\psi}_{i}(\overline{D}_{0} + \overline{D}_{1})\psi_{i} + \frac{i}{2}\overline{\psi}_{+i}(\overline{D}_{0} - \overline{D}_{1})\psi_{+i}
\right.
\]
\[
- \left(\sum_{b}^{N-k}Q_{ib}\overline{\phi}_{b} + \sum_{c}^{k}\overline{Q}_{ic}\sigma_{c}\right)\left(\sum_{b}^{N-k}Q_{ib}\overline{\sigma}_{b} + \sum_{c}^{k}\overline{Q}_{ic}\overline{\phi}_{c}\right)\overline{\phi}_{i}
\left.ight.
\]
\[
- \sum_{b}^{N-k}\overline{Q}_{ib}(\overline{\sigma}_{b}\overline{\psi}_{+i}\psi_{i} - \overline{\sigma}_{b}\overline{\psi}_{-i}\psi_{+i}) - \sum_{c}^{k}\overline{Q}_{ic}(\overline{\sigma}_{c}\overline{\psi}_{+i}\psi_{i} - \overline{\sigma}_{c}\overline{\psi}_{-i}\psi_{+i})
\left.ight.
\]
\[
- \sum_{b}^{N-k}iQ_{ib}\phi_{i}(\overline{\lambda}_{-b}\overline{\psi}_{+i} - \overline{\lambda}_{+b}\overline{\psi}_{-i}) - \sum_{c}^{k}iQ_{ic}\overline{\phi}_{i}(\psi_{-i}\overline{\lambda}_{+c} - \psi_{+i}\overline{\lambda}_{-c}) + |F_{i}|^{2}\right\}
\]
\[
+ \frac{1}{2\pi}\int d^{2}x\left\{ \sum_{b}^{N-k}(\overline{D}_{b}(\sum_{i}^{N}Q_{ib}\overline{\phi}_{i} - \overline{r}_{b}) + \theta_{b}\overline{F}_{01b}) + \sum_{c}^{k}(\overline{D}_{c}(\sum_{i}^{N}Q_{ic}\overline{\phi}_{i} - \overline{r}_{c}) + \theta_{c}\overline{F}_{01c})\right\}
\]
\[
+ \frac{1}{2\pi}\sum_{b}^{N-k}\frac{1}{2\epsilon_{b}^{2}}\int d^{2}x\left\{ (\overline{F}_{01b})^{2} - \partial_{\mu}\sigma_{b}\partial^{\mu}\sigma_{b} + (\overline{D}_{b})^{2} + \frac{i}{2}\overline{\lambda}_{+b}(\overline{D}_{0} - \overline{D}_{1})\lambda_{+b} + \frac{i}{2}\overline{\lambda}_{-b}(\overline{D}_{0} + \overline{D}_{1})\lambda_{-b}\right\}
\]
\[
+ \frac{1}{2\pi}\sum_{c}^{k}\frac{1}{2\epsilon_{c}^{2}}\int d^{2}x\left\{ (\overline{F}_{01c})^{2} - \partial_{\mu}\sigma_{c}\partial^{\mu}\sigma_{c} + (\overline{D}_{c})^{2} + \frac{i}{2}\overline{\lambda}_{+c}(\overline{D}_{0} - \overline{D}_{1})\lambda_{+c} + \frac{i}{2}\overline{\lambda}_{-c}(\overline{D}_{0} + \overline{D}_{1})\lambda_{-c}\right\},
\]
(3.12)
where the covariant derivatives are

\[
D_\mu \phi_i = (\partial_\mu + i \sum_b Q_{ib} \hat{A}_{\mu b} + i \sum_c \tilde{Q}_{ic} \hat{A}_{\mu c}) \phi_i,
\]

\[
D_\mu \bar{\phi}_i = (\partial_\mu - i \sum_b Q_{ib} \hat{A}_{\mu b} - i \sum_c \tilde{Q}_{ic} \hat{A}_{\mu c}) \bar{\phi}_i,
\]

\[
D_\mu \psi_{\pm i} = (\partial_\mu + i \sum_b Q_{ib} \hat{A}_{\mu b} + i \sum_c \tilde{Q}_{ic} \hat{A}_{\mu c}) \psi_{\pm i},
\]

\[
D_\mu \bar{\psi}_{\pm i} = (\partial_\mu - i \sum_b Q_{ib} \hat{A}_{\mu b} - i \sum_c \tilde{Q}_{ic} \hat{A}_{\mu c}) \bar{\psi}_{\pm i}.
\]

In (3.12), we have first written the GLSM in its familiar form, and then split the terms containing vector multiplet components as in (3.1). We shall now take the \( \hat{e}_b \to \infty \) limit in (3.12), whereby the vector multiplet kinetic terms as well as the term \((\hat{D}_b)^2\) vanish. This means that all the components of the vector superfields \( \hat{V}_b = \{\hat{A}_{\mu b}, \hat{\sigma}_b, \hat{\lambda}_b, \hat{D}_b\} \) become auxiliary.

Consequently, the equations of motion of \( \hat{A}_{\mu b} \) and \( \hat{\sigma}_b \) give the following constraints on themselves\(^{13}\)

\[
\sum_i Q_{ib} [i(\bar{\phi}_i D_0 \phi_i - \phi_i D_0 \bar{\phi}_i) + \bar{\psi}_{-i} \psi_{-i} + \bar{\psi}_{+i} \psi_{+i}] = 0
\]

\[
\sum_i \tilde{Q}_{ib} [i(\bar{\phi}_i D_1 \phi_i - \phi_i D_1 \bar{\phi}_i) - \bar{\psi}_{-i} \psi_{-i} + \bar{\psi}_{+i} \psi_{+i}] = 0,
\]

\[
\sum_i [-\hat{Q}_{ib}(\sum_b Q_{id} \hat{\sigma}_d + \sum_c \tilde{Q}_{ic} \hat{\sigma}_c) \phi_i \bar{\phi}_i - \hat{Q}_{ib} \bar{\psi}_{+i} \psi_{-i}] = 0
\]

\[
\sum_i [-\hat{Q}_{ib}(\sum_d Q_{id} \hat{\sigma}_d + \sum_c \tilde{Q}_{ic} \hat{\sigma}_c) \phi_i \bar{\phi}_i - \hat{Q}_{ib} \bar{\psi}_{-i} \psi_{+i}] = 0,
\]

while integrating out the gauginos \( \tilde{\lambda}_b \) gives the constraints

\[
\sum_i \hat{Q}_{ib} \bar{\phi}_i \psi_{\pm i} = 0,
\]

and finally, integrating out \( \hat{D}_b \) gives

\[
\sum_i \hat{Q}_{ib} \bar{\phi}_i \phi_i - \hat{r}_b = 0.
\]

\(^{13}\)To be precise, the equation of motion for \( \hat{A}_{\mu b} \) (which is the first equation in (3.14)) will be modified by a boundary term proportional to \( \tilde{\theta}_b \), unless appropriate boundary conditions and/or boundary terms are used. We shall assume that this is the case for now, and in the following sections we will study boundary actions whereby the \( \hat{A}_{\mu b} \) equation of motion in (3.14) is precise both in the bulk and at the boundaries.
Thus, we should begin with the GLSM (with charges $\hat{Q}$), a good example is that of case, the constraint ($\psi$) is before taking the $\hat{c}_i \to \infty$ limit. Next, we need to find parametrizations for the scalar fields $\phi_i$ which satisfy (3.17), as well as parametrizations for $\psi$ which satisfy (3.16). Then, $\hat{A}_{\mu b}$ and $\hat{\sigma}_b$ must be integrated out of the action using (3.14) and (3.15). Finally, we need to replace the matter auxiliary field term

$$\sum_i |F_i|^2$$

with the matter auxiliary field term of the GNLSM in (2.5).

This procedure is simplest for the case of $N - k = 1$, where (3.14) and (3.15) reduce to

$$\hat{A}_0 = \sum_i \hat{Q}_i [i(\bar{\phi}_i \tilde{D}_0 \phi_i - \phi_i \tilde{D}_0 \bar{\phi}_i) + \bar{\psi}_{-i} \psi_{-i} + \bar{\psi}_{+i} \psi_{+i}]$$

$$\hat{A}_1 = \sum_i \hat{Q}_i [i(\bar{\phi}_i \tilde{D}_1 \phi_i - \phi_i \tilde{D}_1 \bar{\phi}_i) - \bar{\psi}_{-i} \psi_{-i} + \bar{\psi}_{+i} \psi_{+i}]$$

and

$$\hat{\sigma} = \sum_i \hat{Q}_i [i(\bar{\phi}_i \tilde{D}_1 \phi_i - \phi_i \tilde{D}_1 \bar{\phi}_i) - \bar{\psi}_{-i} \psi_{-i} + \bar{\psi}_{+i} \psi_{+i}]$$

where

$$\tilde{D}_\mu \phi_i = (\partial_\mu + i \sum_c \tilde{Q}_{ic} \tilde{A}_{\mu c}) \phi_i,$$

$$\tilde{D}_\mu \bar{\phi}_i = (\partial_\mu - i \sum_c \tilde{Q}_{ic} \tilde{A}_{\mu c}) \bar{\phi}_i.$$  

A good example is that of $X = \mathbb{C}P^{N-1}$, which corresponds to the quotient

$$\mathbb{C}^N / / U(1)$$

with charges $\hat{Q}_i = 1$, and the FI parameter $\hat{r} > 0$, which is the Kähler cone of $\mathbb{C}P^{N-1}$. Thus, we should begin with the GLSM (3.12) with $N - k = 1$, $\hat{Q}_i = 1$ and $\hat{r} > 0$. In this case, the constraint (3.17) is

$$\sum_i \bar{\phi}_i \phi_i = \hat{r},$$

These constraints are consistent with the $N = (2, 2)$ supersymmetry of the GLSM action.

In fact choosing the FI parameters $\hat{r}_b$ to be in a Kähler cone actually triggers spontaneous symmetry breaking of $U(1)^{N-k}$ via a Higgs mechanism. This can be seen by integrating out the auxiliary fields $\tilde{D}_b$. Then, the $U(1)^{N-k}$ vector multiplets and the scalar fields transverse to the Kähler manifold defined by this Kähler cone become massive, with masses given in terms of $\hat{c}_b$. Then, taking the $\hat{c}_b \to \infty$ limit decouples these massive modes and gives us a theory of massless fields, without $U(1)^{N-k}$ symmetry.
and is solved by
\[
\phi_i = \frac{Z^i \sqrt{\epsilon} e^{it}}{\sqrt{1 + \sum_k^{N-1} |Z^k|^2}}, \quad i = 1, \ldots, N - 1
\]
\[
\phi_N = \frac{\sqrt{\epsilon} e^{it}}{\sqrt{1 + \sum_k^{N-1} |Z^k|^2}},
\]
where
\[
Z^i = \frac{\phi_i}{\phi_N}
\]
(3.25)
correspond to the inhomogeneous coordinates which parametrize a local patch of \(CP^{N-1}\). Furthermore, the fermionic constraint (3.16) can be solved by
\[
\psi_{i\pm} = \frac{\psi_{Z^i} \sqrt{\epsilon} e^{it}}{(1 + \sum_k^{N-1} |Z^k|^2)^{1\over 2}} - \frac{Z^i (\sum_j^{N-1} \psi_{Z^j} \overline{Z}^j) \sqrt{\epsilon} e^{it}}{(1 + \sum_k^{N-1} |Z^k|^2)^{1\over 2}}, \quad i = 1, \ldots, N - 1
\]
\[
\psi_{N\pm} = - \frac{\sum_j^{N-1} (\psi_{Z^j} \overline{Z}^j) \sqrt{\epsilon} e^{it}}{(1 + \sum_k^{N-1} |Z^k|^2)^{1\over 2}}
\]
where \(\psi_{Z^i}\) correspond to Grassmann-valued vector fields defined on the aforementioned patch of \(CP^{N-1}\).

Using (3.19) and (3.24), we can show that the terms containing only bosonic fields in the scalar kinetic term of the GLSM \((- \sum_i^N D_\mu \phi_i D^\mu \phi_i\) are
\[
\frac{\hat{\ell} (\sum_j^{N-1} \partial_\mu^{A^j} \partial_\nu Z^j) (\sum_k^{N-1} \overline{Z}^k \partial_\mu^{A^k} Z^k)}{(1 + \sum_k^{N-1} |Z^k|^2)^2},
\]
where
\[
\partial_\mu^{A^j} Z^j = \partial_\mu Z^j + i \sum_c^{N-1} (\hat{Q}_{jc} - \tilde{Q}_{Nc}) \tilde{A}_\mu Z^j, \quad \text{No sum over } j.
\]
(3.28)
In (3.27), we find the scalar kinetic term given in (2.5) for \(X = CP^{N-1}\) and \(G = U(1)^{N-1}\), with the metric on \(CP^{N-1}\) being the standard Fubini-Study metric.\(^{16}\) Comparing the covariant derivative (3.28) with the general form given in (2.9), we find that the holomorphic Killing vector fields corresponding to the \(U(1)^{N-1}\) isometry on \(CP^{N-1}\) are given by\(^{17}\)
\[
\tilde{e}_c = i (\hat{Q}_{jc} - \tilde{Q}_{Nc}) Z^j.
\]
(3.29)
The term proportional to \(\hat{D}_c\) in the GLSM is found to contain the moment map for the \(U(1)^{N-1}\) isometry of \(CP^{N-1}\),
\[
\tilde{\mu}_c = \frac{-\hat{\ell} (\sum_i^{N-1} \hat{Q}_{ic} |Z|^2 + \hat{Q}_{Nc})}{(1 + \sum_k^{N-1} |Z|^2)},
\]
(3.30)
\(^{16}\)Here, the FI parameter plays the role of the modulus which parametrizes the size of \(CP^{N-1}\).
\(^{17}\)The fact that the \(U(1)^{N-1}\) charges of the inhomogeneous coordinates \(Z^i\) are given by \((\hat{Q}_{jc} - \tilde{Q}_{Nc})\) can also be deduced from (3.25).
via (3.24), and thereby we retrieve the moment map term of (2.6). Similarly, using (3.19) and (3.24), we find that the \( \hat{\theta} \) term gives rise to the the \( B \)-field and \( C \)-field terms of the \( \text{GNLSM} \), with

\[
B = -\frac{\hat{\theta}}{r} \omega, \quad (3.31)
\]

(where \( \omega \) is the Fubini-Study Kähler form) and

\[
C_c = -\frac{\hat{\theta}}{r} \mu_c, \quad (3.32)
\]

as well as the boundary term

\[
-\frac{1}{2\pi} \int_{\partial \Sigma} \frac{\hat{\theta}}{2r} \sum_{i} (\overline{\psi}_{-i} \psi_{-i} + \overline{\psi}_{+i} \psi_{+i}), \quad (3.33)
\]

where we have maintained its GLSM form for convenience. We shall comment more on this term below.

In a similar manner, we may continue the procedure explained below (3.17) with the help of (3.19), (3.20), (3.24) and (3.26) in order to obtain the complete \( \text{GNLSM} \) action given in (2.4)+(2.19) for \( X = \mathbb{C}P^{N-1} \) and \( G = U(1)^{N-1} \). For conciseness, we shall only write out the resulting action for \( N = 2 \), i.e., the \( U(1) \)-\( \text{GNLSM} \) with \( \mathbb{C}P^1 \) target:

\[
S = \frac{1}{2\pi} \int d^2x \left\{ -\hat{\theta}^A Z \partial^A \overline{Z} + \frac{i}{2} \left( \overline{\psi}_+ \nabla_+ \psi_+ + \overline{\psi}_- \nabla_- \psi_+ \right) \nabla Z \overline{Z} \right\} \left( \hat{F}_0 \right)^2 + \frac{1}{2} \left( \overline{\psi}_+ \psi_- \hat{F}_0 \overline{F}_0 + \overline{\psi}_- \psi_+ \hat{F}_0 \overline{F}_0 \right) + \frac{1}{2} \int d^2x \left\{ \phi^* C_a A^a - \frac{\theta}{2r} \sum_{i} (\overline{\psi}_{-i} \psi_{-i} + \overline{\psi}_{+i} \psi_{+i}) \right\}
\]

\[
+ \frac{1}{2\pi} \int d^2x \left( \tilde{\Phi} + \frac{1}{2\pi} \int_{\partial \Sigma} \left\{ \phi^* C_a A^a - \frac{\theta}{2r} \sum_{i} (\overline{\psi}_{-i} \psi_{-i} + \overline{\psi}_{+i} \psi_{+i}) \right\} \right)
\]

where the holomorphic Killing vector field is

\[
\tilde{e}^Z = i(\tilde{Q}_1 - \tilde{Q}_2)Z \quad (3.35)
\]

with covariant derivative

\[
\nabla_Z \tilde{e}^Z = i(\tilde{Q}_1 - \tilde{Q}_2) \left( \frac{1 - |Z|^2}{1 + |Z|^2} \right), \quad (3.36)
\]
and the moment map
\[ \tilde{\mu} = -\hat{r}(\tilde{Q}_1|Z|^2 + \tilde{Q}_2) \left(1 + |Z|^2\right). \]

The action we have obtained for the \( U(1)^{N-1} \)-GNLSM with \( \mathbb{C}P^{N-1} \) target space corresponds exactly to the general action (2.4)+-(2.19), with the exception of the (spurious) boundary term (3.33). This term takes the form
\[ \frac{1}{2\pi} \int_{\partial \Sigma} \frac{\hat{\theta}}{2} \sum_{i=1}^{N} \hat{Q}_i(\bar{\psi}_{-i}\psi_{-i} + \bar{\psi}_{+i}\psi_{+i})\sum_j \hat{Q}_j^2|\phi_j|^2, \]
when we start with GLSMs with arbitrary \( \hat{Q}_i \), for which the reduction procedure gives GNLSMs with toric target space \( X = \mathbb{C}^N//U(1) \). This boundary term also occurs in the reduction of GLSMs to NLSMs \( [13] \), and can be removed in several ways, including the addition of a boundary term to the GLSM we start with \( [13] \), or by a judicious choice of boundary conditions on the fermionic fields. In the following sections, we shall explain how this term is removed when investigating the cases of A-type and B-type supersymmetry at the boundaries.

Before ending this section, we would like to point out that the classical procedure of obtaining GNLSMs from GLSMs which we have explained above is valid at the quantum level, since the \( \hat{e}_b \to \infty \) limit can be taken for the path integral of the GLSM, and functional integration over the auxiliary components of \( \hat{V}_b \) is equivalent to imposing their algebraic equations of motion as constraints. However, taking renormalization of the FI parameters into account, it can be shown that we may only obtain quantum GNLSMs for Kähler targets with \( c_1(X) \geq 0 \). This is because the RG flow at the one-loop level of the bare FI parameters \( \hat{r}_0b \) is
\[ \hat{r}_{0b} = \hat{r}_b(\mu) + \sum_{i=1}^{N} \hat{Q}_i b \log \left( \frac{\Lambda_{UV}}{\mu} \right), \]
(where \( \Lambda_{UV} \) is an ultraviolet cut-off and \( \mu \) is a finite energy scale). As shown in [7], for a basis \( e_b \) of \( H_2(X,\mathbb{Z}) \), we have \( \sum_i^{N} \hat{Q}_{ib} = c_1(X) \cdot e_b \). Then, for a holomorphic curve \( m = \sum_b^{N-k} m_b e_b \) in \( X \) (i.e., an element of the Mori cone of \( X \))
\[ \sum_b^{N-k} m_b \hat{r}_{0b} = \sum_b^{N-k} m_b \hat{r}_b(\mu) + \sum_b^{N-k} m_b (c_1(X) \cdot e_b) \log \left( \frac{\Lambda_{UV}}{\mu} \right). \]
For the bare FI parameters to be in the Kähler cone of \( X \), the LHS of (3.40) ought to be greater than zero. In the continuum limit (\( \Lambda_{UV} \to \infty \)), this is impossible if \( c_1(X) \geq 0 \) is not satisfied.

4 Equivariant B-branes and their Mirrors

We shall apply the techniques discussed above to find boundary actions and boundary conditions in abelian GNLSMs with toric target spaces, \( X \), as well as their mirror descriptions. These boundary actions and boundary conditions will correspond to branes in \( X \), which
we refer to as equivariant branes. We first study the case where B-type supersymmetry is preserved at the boundaries of the \( I \times \mathbb{R} \) worldsheet, since this leads us to make contact with a result found previously by Kapustin et al. [14, 15], while in the next section, we shall use similar techniques to study equivariant A-branes. After gaining insights from the study of abelian equivariant B-branes, we shall then proceed to analyze equivariant B-branes for nonabelian GNLSMs.

The combination of supercharges which define B-type supersymmetry are given by (2.40). In the following, we shall set \( \beta = 0 \) for simplicity, though it is straightforward to study the \( \beta \neq 0 \) generalization by the same techniques. In other words, we assume that the supercharges conserved at the boundaries are

\[ Q_B = \overline{Q}_+ + \overline{Q}_-, \quad Q_B^\dagger = Q_+ + Q_- . \]  

(4.1)

From (2.33), we find that the corresponding relations among the supersymmetry transformation parameters are

\[ \epsilon = \epsilon_+ = -\epsilon_- \]

\[ \overline{\epsilon} = \overline{\epsilon}_+ = -\overline{\epsilon}_- . \]

(4.2)

We shall also make use of superfields when discussing boundary conditions, and to this end, the concept of ‘boundaries’ in superspace [11] is useful. For the case at hand, the relevant boundary in superspace is known as ‘B-boundary’, and corresponds to

\[ \theta = \theta^+ = \theta^- \]

\[ \overline{\theta} = \overline{\theta}^+ = \overline{\theta}^- . \]

(4.3)

Let us first review what is known of ordinary B-branes. For \( \mathcal{N} = (2, 2) \) NLSMs, the boundary condition needed to preserve B-type supersymmetry at the boundaries maps each boundary to a holomorphically embedded complex submanifold of the target space [10]. In addition, we may include the following boundary action

\[ S_{\partial \Sigma} = \int dx^0 \frac{1}{\partial \Sigma} \left\{ A_X^N \partial_0 X^N - \frac{i}{4} F_{MN}^{X} (\psi^M + \bar{\psi}^M)(\psi^N + \bar{\psi}^N) \right\} , \]

(4.4)

which is B-type supersymmetric if \( F_{mn}^{X} = F_{mn}^{\bar{X}} = 0 \) (here, we use \( (M, N, \ldots) \) as coordinate indices on the holomorphically embedded branes), where \( A_X^M \) corresponds to a connection of a line \( (U(1)) \) bundle on each B-brane, and \( F_{MN}^{X} \) the corresponding curvature. The conditions \( F_{mn}^{X} = F_{mn}^{\bar{X}} = 0 \) indicate that each line bundle is holomorphic [17]. This boundary action is in fact a supersymmetric Wilson line, and since we have two boundary components, we actually have two different Wilson lines along each boundary, corresponding to two different B-branes supporting holomorphic line bundles, each with different connections and curvatures ([10], page 21).

An alternative formulation of B-branes exists [10], where mixed Dirichlet-Neumann boundary conditions are imposed on some of the target space coordinates, and the boundary action is

\[ S_{\partial \Sigma} = \int dx^0 A_X^M \partial_0 X^M , \]

(4.5)
which is an ordinary Wilson line along each boundary component. This formulation leads us to the same spacetime theory as (4.4) [18]. Moreover, B-type supersymmetry is preserved at the boundaries if \( F^X_{mn} = F^X_{nm} = 0 \), indicating a holomorphic line bundle on each B-brane.

We are interested in the generalizations of (4.4) and (4.5) (and their corresponding boundary conditions) for GNLSMs. One method of obtaining such a generalization would be to replace ordinary worldsheet derivatives by covariant ones, and attempt to maintain supersymmetry and gauge symmetry by adding additional terms. However, it is known that (4.4) and (4.5) and their corresponding boundary conditions can be obtained from GLSM boundary actions and boundary conditions [10]. This suggests the more elegant method of obtaining the GNLSM generalizations from GLSM boundary conditions and a GLSM boundary action, using the methods of Section 3. In the following, we shall attempt to generalize the boundary action (4.4) to the case of \( U(1)^k \)-GNLSMs with Kähler toric target space, before proceeding to do the same for the boundary action (4.5).

**B-branes on \( C^N//U(1) \) from GLSM**

Let us first recall how B-type supersymmetric boundary conditions and the boundary action (4.4) for an NLSM with \( C^N//U(1) \) target space can be obtained from boundary conditions and the boundary action of a \( U(1) \)-GLSM with \( C^N \) target [10]. We shall focus on obtaining NLSM boundary conditions corresponding to space-filling branes. To this end, we must impose B-type supersymmetric boundary conditions at the GLSM level which include Neumann boundary conditions on the chiral superfields.

Using the language of superfields, these conditions are [10, 11]

\[
\hat{D}_+ \Phi_i = \hat{D}_- \Phi_i \\
\hat{\Sigma} = \overline{\Sigma}
\]

at B-boundary,\(^19\) where \( \hat{D}_\pm = e^{-\hat{Q}_i \hat{V}} D_\pm e^{\hat{Q}_i \hat{V}} \); while in components, they are given as

\[
\begin{align*}
\psi_{+i} - \psi_{-i} &= 0 \\
F_i &= 0 \\
\hat{D}_1 \phi_i &= 0 \\
\hat{D}_1 (\psi_{+i} + \psi_{-i}) &= 0
\end{align*}
\]

and

\[
\begin{align*}
\text{Im}(\hat{\sigma}) &= 0 \\
\hat{\lambda}_+ + \hat{\lambda}_- &= 0 \\
\partial_1 \text{Re}(\hat{\sigma}) + \hat{F}_{01} &= 0
\end{align*}
\]

\(^18\)In general, the B-type NLSM also admits boundary conditions which correspond to lower dimensional branes, the only restriction is that such a brane corresponds to a complex submanifold holomorphically embedded in the target space [10, 19]. These boundary conditions are obtainable in the \( \hat{e} \to \infty \) limit from GLSM boundary terms which effectively impose Dirichlet boundary conditions on some of the chiral multiplets [11].

\(^19\)A boundary condition imposed on a superfield automatically implies that its components obey a set of boundary conditions which are closed under supersymmetry.
where the covariant derivative of the scalar fields is

$$\hat{D}_\mu \phi_i = \partial_\mu \phi_i + i \hat{Q}_i \hat{A}_\mu \phi_i.$$  \hfill (4.9)

However, the boundary conditions of some of the vector multiplet fields remain to be specified, and therefore we impose \[10, 11\]

$$\hat{F}_{01} = - e^2 \hat{\theta},$$  \hfill (4.10)

which further implies

$$\partial_1 (\hat{\lambda}_+ - \hat{\lambda}_- ) = 0$$
$$\partial_1 (\hat{D} + \partial_1 \text{Im}(\hat{\sigma})) = 0$$  \hfill (4.11)

via B-type supersymmetry. These conditions are also invariant under $U(1)$ gauge transformations. We also need to add a boundary term to cancel the B-type supersymmetry variation of the bulk theta term, i.e., the expression

$$\frac{\hat{\theta}}{2\pi} \int_{\Sigma} d^2 x \hat{F}_{01} + \frac{\hat{\theta}}{2\pi} \int_{\partial \Sigma} d x^0 \frac{(\hat{\sigma} + \hat{\bar{\sigma}})}{2}$$  \hfill (4.12)

is B-type supersymmetry invariant. Having imposed the above boundary conditions and the boundary term in (4.12), the GLSM action is B-type supersymmetric at the boundaries.

In addition, we include the boundary action

$$S_{\partial \Sigma} = \frac{\hat{\theta}}{4\pi} \int_{\partial \Sigma} d x^0 \sum_i \left( i \hat{D}_0 \bar{\phi}_i \phi_i - i \phi_i \hat{D}_0 \bar{\phi}_i + (\psi_{+i} + \psi_{-i}) (\bar{\psi}_{+i} + \bar{\psi}_{-i}) - \hat{Q}_i (\hat{\sigma} + \hat{\bar{\sigma}}) |\phi_i|^2 \right).$$  \hfill (4.13)

which is B-type supersymmetric on its own.\footnote{If we were to discard the expression (4.13), the boundary conditions would ensure the locality of the equations of motion of the worldsheet fields. Its inclusion renders the equations of motion for $\phi_i$, $\psi_i$, $\hat{A}_\mu$, and $\hat{\sigma}$ nonlocal \[10\], i.e., they contain boundary terms.} Its inclusion is necessary to obtain the boundary action (4.4), which plays the role of elucidating the geometry of the branes.

Now, recall from \[10\] that the bulk theta term can be converted into a boundary term in some circumstances. In particular, we have

$$\frac{\hat{\theta}}{2\pi} \int_{\Sigma} d^2 x \hat{F}_{01} = - \frac{\hat{\theta}}{2\pi} \int_{\partial \Sigma} d x^0 \hat{A}_0,$$  \hfill (4.14)

via Stoke’s theorem, but this violates gauge invariance. The violation is

$$\frac{\hat{\theta}}{2\pi} \int_{\partial \Sigma} d x^0 \partial_0 \alpha = \frac{\hat{\theta}}{2\pi} \int_{\partial \Sigma} d \alpha = \frac{\hat{\theta}}{2\pi} 2\pi m$$  \hfill (4.15)

where $m \in \mathbb{Z}$. However, if $\hat{\theta} \in 2\pi \mathbb{Z}$, then (4.15) implies that $\exp(-i \frac{\hat{\theta}}{2\pi} \int_{\partial \Sigma} \hat{A}_0)$ is gauge invariant, and hence, the path integral remains gauge invariant. We shall assume that $\hat{\theta} \in 2\pi \mathbb{Z}$ hereon, by setting $\hat{\theta} = 2\pi n$, where $n \in \mathbb{Z}$.\footnote{If we were to discard the expression (4.13), the boundary conditions would ensure the locality of the equations of motion of the worldsheet fields. Its inclusion renders the equations of motion for $\phi_i$, $\psi_i$, $\hat{A}_\mu$, and $\hat{\sigma}$ nonlocal \[10\], i.e., they contain boundary terms.}
Doing so, we may then write (4.12) and (4.13) as

$$S'_{\partial \Sigma} = \frac{n}{2\hat{r}} \int d\Sigma^0 \left( \sum_i^N i\hat{\partial}_0 \phi_i - i\overline{\phi}_i \hat{\partial}_0 \phi_i + \sum_i^N (\psi_{+i} + \psi_{-i})(\overline{\psi}_{+i} + \overline{\psi}_{-i}) \right)$$

$$+ 2\hat{A}_0 \left( \sum_i^N \hat{Q}_i |\phi_i|^2 - \hat{r} \right) - (\hat{\sigma} + \overline{\sigma})(\sum_i^N \hat{Q}_i |\phi_i|^2 - \hat{r}) \right)$$

(4.16)

Taking the $\hat{e} \to \infty$ (NLSM) limit, the components of the vector multiplet become auxiliary. Integrating $\hat{D}$ out of the bulk action enforces the constraint

$$\sum_i^N \hat{Q}_i |\phi_i|^2 - \hat{r} = 0$$

(4.17)

and consequently, the second line of (4.16) vanishes. Integrating out the rest of the vector multiplet gives several more constraints, the one relevant to the boundary action being

$$\sum_i^N \hat{Q}_i \phi_i \psi_{\pm i} = 0.$$  

(4.18)

Thus, the boundary action (4.16) reduces to

$$S'_{\partial \Sigma} = \frac{n}{2\hat{r}} \int d\Sigma^0 \left( \sum_i^N \left( (i\hat{\partial}_0 \phi_i - i\overline{\phi}_i \hat{\partial}_0 \phi_i) + (\psi_{+i} + \psi_{-i})(\overline{\psi}_{+i} + \overline{\psi}_{-i}) \right) \right)$$

with (4.17) and (4.18) strictly imposed. As explained in [10], the first term in (4.19) is nothing but the hermitian connection

$$A^X_I dX^I = -n \frac{i}{2} \sum_{i=1}^N \overline{\phi}_i \frac{\partial}{\partial \phi_i}$$

(4.20)

of $\mathcal{O}_X(-n)$ on the toric manifold $X = \mathbb{C}^N//U(1)$, since it transforms under $U(1)$ gauge transformations ($\phi_i \to e^{i\hat{Q}_i e^{\alpha}} \phi_i$) as

$$A^X_I dX^I \to A^X_I dX^I - (-n) d\alpha.$$  

(4.21)

Here, $\mathcal{O}_X(-n)$ is the holomorphic line bundle on $X$ with $\int_X c_1(\mathcal{O}_X(-n)) = -n$.

Using parametrizations for $\phi_i$ and $\psi_{\pm i}$ which satisfy (4.17) and (4.18), the explicit NLSM boundary action (4.4) is obtained, with $F^X_{jk} = F^X_{jk} = 0$, together with the compatible boundary conditions. For example, for $X = \mathbb{C}P^{N-1}$, we may use the parametrizations (3.24) and (3.26) to satisfy (4.17) and (4.18), and with the help of the constraints which come from integrating out the vector multiplet, the boundary conditions (4.7) become

$$\psi^Z_i - \psi^Z_i = 0$$

$$F_i^Z = 0$$

$$\partial_1 Z^i = 0$$

$$\partial_1 (\psi^Z_i + \psi^{-Z}_i) = 0,$$

(4.22)
where the purely Neumann boundary conditions on \( Z^i \) indicate that the B-brane is a space-filling brane; while the boundary action (4.19) is reexpressed as

\[
S_{\partial \Sigma}^i = \frac{n}{2\ell} \int dx^0 \left\{ \frac{-i\hat{\sigma} \sum_{j}^{N-1} \left( \frac{Z_j^i \partial_0 Z_j^i - Z_j^i \partial_0 Z_j^i}{1 + \sum_k^{N-1} |Z^k|^2} \right)}{1 + \sum_k^{N-1} |Z^k|^2} + 2i\hat{\sigma} \partial_0 t \right. \\
- \frac{i\hat{\sigma} \sum_{j}^{N-1} \left( \frac{Z_j^i + Z_j^i}{1 + \sum_k^{N-1} |Z^k|^2} \right)}{1 + \sum_k^{N-1} |Z^k|^2} \left. \right\},
\]

\[
= \int dx^0 \left\{ A^X_j \partial_0 X^j + A^X_j \partial_0 X^j + n\partial_0 t - i \frac{F_{\bar{X}}(\psi^i + \psi^i)(\bar{\psi}^+ + \bar{\psi}^-)}{2} \right\}, \tag{4.23}
\]

where \( X^j = Z^j, \) \( \psi^i = \psi^i, \)

\[
A^X_j = -\frac{n}{2} \frac{i Z_j^i}{1 + \sum_k^{N-1} |Z^k|^2}, \quad A^X_j = \frac{n}{2} \frac{Z_j^i}{1 + \sum_k^{N-1} |Z^k|^2}, \tag{4.24}
\]

are the components of the connection of \( \mathcal{O}_{\mathbb{C}P^{N-1}}(-n) \), while its curvature is

\[
F_{\bar{X}} = n\omega_{\bar{X}}, \tag{4.25}
\]

where \( \omega_{\bar{X}} = ig_{\bar{X}} \) are the components of the normalized\(^{21} \) Fubini-Study Kähler form of \( \mathbb{C}P^{N-1}, \)

\[
\omega = i \frac{(\sum_j^{N-1} dZ^j \wedge d\bar{Z}^j)}{1 + \sum_k^{N-1} |Z^k|^2} - i \frac{(\sum_j^{N-1} \bar{Z}^j dZ^j) \wedge (\sum_j^{N-1} Z^j d\bar{Z}^j)}{1 + \sum_k^{N-1} |Z^k|^2}. \tag{4.26}
\]

The expression (4.23) agrees with (4.4) up to the term \( n\partial_0 t. \) This term merely reflects the fact that the \( U(1) \) gauge symmetry is broken at the boundaries, and it can be removed via the gauge transformation (4.21), with \( \alpha = -t. \) At the path integral level, we may instead use \( \int_{\partial \Sigma} dx^0 n\partial_0 t = 2\pi nm \) (for some \( m \in \mathbb{Z} \)) to remove the term. Thus, the boundary conditions (4.22) and the boundary action (4.23) indicate that the worldsheet boundaries are mapped to a space-filling B-brane on \( \mathbb{C}P^{N-1}, \) which supports the holomorphic line bundle \( \mathcal{O}_{\mathbb{C}P^{N-1}}(-n). \)

### 4.1 Equivariant B-branes on \( \mathbb{C}^N//U(1) \) from GLSM

Having recalled how the B-type supersymmetric boundary action and boundary conditions for NLSMs with toric target spaces of the form \( \mathbb{C}^N//U(1) \) are obtained from a GLSM, we shall now proceed to obtain the B-type supersymmetric boundary action and boundary conditions for abelian GNLSMs with the same target spaces.

Firstly, we impose the following B-type supersymmetric boundary conditions, which are invariant under \( U(1)^N \) gauge symmetry, and which include Neumann boundary conditions on the chiral superfields, i.e.,

\[
D_+ \Phi_i = D_- \Phi_i \\
\Sigma_a = \bar{\Sigma}_a \tag{4.27}
\]

\(^{21}\)Unlike (4.26), the Fubini-Study metric that appears in the bulk NLSM action contains the FI parameter, \( \hat{\sigma}, \) which is the size modulus of the Fubini-Study metric; see (3.27).
at B-boundary, where $D_{\pm} = e^{-\sum_a Q_a V_a} D_{\pm} e^{\sum_a Q_a V_a}$, as well as

$$F_{01a} = -e_a^2 \theta_a. \quad (4.28)$$

In components, these boundary conditions are

$$\begin{align*}
\psi_{+i} - \psi_{-i} &= 0 \\
F_i &= 0 \\
D_1 \phi_i &= 0 \\
D_1 (\psi_{+i} + \psi_{-i}) &= 0.
\end{align*} \quad (4.29)$$

and

$$\begin{align*}
\text{Im}(\sigma_a) &= 0 \\
\lambda_{+a} + \lambda_{-a} &= 0 \\
\partial_1 \text{Re}(\sigma_a) &= e_a^2 \theta_a \\
F_{01a} &= -e_a^2 \theta_a \\
\partial_1 (\lambda_{+a} - \lambda_{-a}) &= 0 \\
\partial_1 (D_a + \partial_1 \text{Im}(\sigma_a)) &= 0.
\end{align*} \quad (4.30)$$

It is crucial to note that these boundary conditions are compatible with the constraints (3.19) and (3.20) which are imposed when taking the $\hat{e} \to \infty$ limit to reduce the GLSM to a GNLSM. We also ought to supersymmetrize the bulk theta terms as in (4.12), which gives

$$\begin{align*}
\hat{\theta} \int_{\Sigma} F_{01} d^2 x + \hat{\theta} \int_{\partial \Sigma} \frac{(\hat{\sigma} + \overline{\sigma})}{2} d x^0 + N \sum_{i} \int_{\Sigma} (\hat{\sigma}_i + \overline{\sigma}_i) d x^0 + \hat{\theta} \int_{\partial \Sigma} \frac{(\hat{\sigma}_c + \overline{\sigma}_c)}{2} d x^0.
\end{align*} \quad (4.31)$$

With the above boundary conditions and boundary terms, the GLSM action is B-type supersymmetric at the boundaries.

In addition, we must generalize the $U(1)$-GLSM boundary action (4.13) to a boundary action for the $U(1)^N$-GLSM given in (3.12), with $N - k = 1$. This is given by

$$S_{\partial \Sigma} = \frac{\hat{\theta}}{4\pi^2} \int_{\partial \Sigma} d x^0 \sum_i \left( i D_0 \phi_i \phi_i - i \phi_i D_0 \phi_i + (\psi_{+i} + \psi_{-i})(\overline{\psi}_{+i} + \overline{\psi}_{-i}) - \sum_a Q_{ia} (\sigma_a + \overline{\sigma}_a) |\phi_i|^2 \right). \quad (4.32)$$

(where the covariant derivatives of the scalar fields are given by (3.13)) and is B-type supersymmetric on its own.

Next, we set $\hat{\theta} = 2\pi n$, which allows us to write (4.31) and (4.32) in a form which
generalizes (4.16), i.e.,

\[ S'_{\partial \Sigma} = \frac{n}{2r} \int d\sigma^0 \left( \sum_{i} (i \partial_0 \phi_i - i \bar{\phi}_i \partial_0 \phi_i) + \sum_{i} (\psi_{+i} + \psi_{-i})(\bar{\psi}_{+i} + \bar{\psi}_{-i}) \right) \]

\[ + 2 \hat{A}_0 \left( \sum_{i} \hat{Q}_i |\phi_i|^2 - \hat{r} \right) - (\hat{\sigma} + \bar{\sigma}) \left( \sum_{i} \hat{Q}_i |\phi_i|^2 - \hat{r} \right) \]

\[ + 2 \sum_{c} \hat{A}_{0c} \sum_{i} \hat{Q}_{ic} |\phi_i|^2 - \sum_{c} \left( \hat{\sigma}_c + \bar{\sigma}_c \right) \sum_{i} \hat{Q}_{ic} |\phi_i|^2 \]

\[ + 2 \sum_{c} \left( \hat{\theta}_c \int_{\Sigma} \hat{F}_{01c} dx + \frac{\hat{\theta}_c}{2} \int_{\partial \Sigma} (\hat{\sigma}_c + \bar{\sigma}_c) dx \right). \]

(4.33)

As in the NLSM case, taking the \( \hat{e} \to \infty \) limit results in the vector multiplet components becoming auxiliary (see Section 3). Integrating \( \hat{D} \) out of the bulk action imposes the condition (4.17), and this results in the second line of (4.33) vanishing. Integrating out the rest of the vector multiplet components imposes (4.18), as well as (3.19) and (3.20) (the latter are no longer relevant to the boundary action once the second line of (4.33) vanishes).\(^{22}\)

Next, to find the explicit B-type GNLSM boundary conditions and boundary action, we must find parametrizations which satisfy (4.17) and (4.18). Let us study our usual example of \( \mathbb{C}P^{N-1} \). Using the parametrizations (3.24) and (3.26), as well as the constraints (3.19) and (3.20), the boundary conditions become

\[
\begin{align*}
\psi^Z_+ - \psi^Z_- &= 0 \\
F^Z &= 0 \\
\partial_1 \psi^Z_i &= 0 \\
\phi^A_i \nabla^A_1 (\psi^Z_+ + \psi^Z_-) &= 0
\end{align*}
\]

(4.34)

and

\[
\begin{align*}
\text{Im}(\tilde{\sigma}_c) &= 0 \\
\bar{\lambda}_{+c} + \bar{\lambda}_{-c} &= 0 \\
\partial_1 \text{Re}(\bar{\sigma}_c) &= \tilde{e}_c^2 \tilde{\theta}_c \\
\tilde{F}_{01c} &= -\tilde{e}_c^2 \tilde{\theta}_c \\
\partial_1 (\bar{\lambda}_{+c} - \bar{\lambda}_{-c}) &= 0 \\
\partial_1 (\tilde{D}_c + \partial_1 \text{Im}(\bar{\sigma}_c)) &= 0,
\end{align*}
\]

(4.35)

which are invariant under the \( U(1)^{N-1} \) gauge symmetry, and satisfy the B-type supersymmetry transformations obtained from (2.31) and (2.32). Moreover, these boundary conditions result in the vanishing of the expressions (2.34) and (2.35), thus ensuring the

\(^{22}\)It is important to integrate out \( \hat{D} \) before integrating out \( \hat{A}_0 \), otherwise the algebraic equation of motion of \( \hat{A}_0 \) given in (3.19) will be modified by a boundary term, see footnote 13.
preservation of B-type supersymmetry at the boundaries. Note that the expression (2.36) does not occur when performing a supersymmetry variation, since the B-field and C-field terms do not appear in the action of the GNLSM, as we have used the bulk $\tilde{\theta}$ term of the corresponding GLSM in the construction of our boundary action via (4.14). The spurious boundary term (3.33) also does not occur, for the same reason. In analogy with the NLSM case, the Neumann boundary conditions on $Z_i$ imply that the equivariant B-brane wraps the entire target space, $\mathbb{C}P^{N-1}$, i.e., it is space-filling.

Next, let us find the explicit form of the boundary action. The parametrizations (3.24) and (3.26) give

$$S'_{\partial\Sigma} = \int dx^0 \left\{ \frac{\theta}{2\pi} \int_{\Sigma} -\sum_{c} i R_c \tilde{A}_c + n \partial_0 t -\frac{i}{2} F_{01c} (\psi_+ + \psi_-) (\overline{\psi}_+ + \overline{\psi}_-) \right\} ,$$

(4.36)

where, as in (4.23), $X^j = Z^j$, $\psi^j_\pm = \psi^{Z^j}_\pm$, and the components of the connection $A$ and curvature $F$ of $O_{\mathbb{C}P^{N-1}}(-n)$ are given by (4.24) and (4.25), respectively. Besides the supersymmetrized $\theta$ terms, the only other new term (with respect to (4.23)) is

$$-\sum_{c} i R_c \tilde{A}_c ,$$

(4.37)

where

$$\tilde{A}_c = -i (\tilde{A}_{0c} - (\tilde{\sigma}_c + \overline{\sigma}_c)) ,$$

(4.38)

and

$$R_c = -\frac{n (\sum_{i} \tilde{Q}_{|Z|^2} + \tilde{Q}_{Nc})}{(1 + \sum_{k} \tilde{Q}_{Nc})} .$$

(4.39)

The expression (4.36) is gauge invariant under the unbroken $U(1)^{N-1}$ symmetry. Now, we must remove the $n \partial_0 t$ term as we did in the NLSM case, since $t$ is not a coordinate of the $\mathbb{C}P^{N-1}$ target space, but rather locally parametrizes the Hopf fiber over $\mathbb{C}P^{N-1}$ which gives rise to the sphere $S^{N+1}$ defined by equation (3.23). Furthermore, it is not a field which appears in the bulk theory, and has no supersymmetry transformation, leaving us unable to test the supersymmetry of the boundary action. Thus, we shall remove $n \partial_0 t$. However, doing so will break the classical $U(1)^{N-1}$ symmetry of our GNLSM at the boundaries. Now, this $U(1)^{N-1}$ gauge symmetry is not broken if we only require that it holds at the path integral level. Nevertheless, attempting to restore the classical symmetry will help make the geometric properties of the brane obvious.

\[ {\text{Note that, from the local parametrizations (3.24) and (3.26), we can see that the U(1)\textsuperscript{N-1} gauge transformation of t is } \delta t = \tilde{Q}_{Nc} \alpha, \text{ since we know that the U(1)\textsuperscript{N-1} charge of } Z^i \text{ is } \tilde{Q}_{1c} - \tilde{Q}_{Nc}.} \]
The cure to this broken symmetry is via the supersymmetrized $\tilde{\theta}$ terms, as follows. Setting $\tilde{\theta}_c = 2\pi n \tilde{Q}_{Nc}$, we have

$$
\sum_{c}^{N-1} \left( \frac{\tilde{\theta}_c}{2\pi} \int_{\Sigma} F_{01c} d^2x + \frac{\tilde{\theta}_c}{2\pi} \int_{\partial\Sigma} \left( \tilde{\sigma}_c + \overline{\tilde{\sigma}_c} \right) dx^0 \right)
= \sum_{c}^{N-1} \left( -n \tilde{Q}_{Nc} \int_{\partial\Sigma} \tilde{A}_0 dx^0 + n \tilde{Q}_{Nc} \int_{\partial\Sigma} \left( \tilde{\sigma}_c + \overline{\tilde{\sigma}_c} \right) dx^0 \right)
= \sum_{c}^{N-1} \left( -in \tilde{Q}_{Nc} \int_{\partial\Sigma} \tilde{A}_c dx^0 \right)
$$

(4.40)

since both $n$ and the charge $\tilde{Q}_{Nc}$ are integers, as explained below (4.14). Then, the final boundary action takes the form

$$
S'_{\partial\Sigma} = \int_{\partial\Sigma} dx^0 \left\{ A_j^X \partial_0 X^j + A_j^X \partial_0 X^\overline{j} - \sum_{c}^{N-1} i \tilde{R}_c \tilde{A}_c - \frac{i}{2} F^X_{jk}(\psi^j_+ + \psi^j_-)(\overline{\psi}_+^T + \overline{\psi}_-^T) \right\},
$$

(4.41)

where

$$
\tilde{R}_c = \frac{-n(\sum_{i}^{N-1} (\tilde{Q}_{i} - \tilde{Q}_{Nc})|Z^i|^2)}{(1 + \sum_{k}^{N-1} |Z^k|^2)}
= -A_j^X \tilde{c}_c + A_j^X \tilde{\overline{c}}_c
= -i\epsilon_c A^X.
$$

(4.42)

Invariance of the boundary action (4.41) under the B-type supersymmetry transformations (given by (2.31) and (2.32) for $\epsilon_+ = -\epsilon_-)$ holds since

$$
d \tilde{R} = i\epsilon F^X.
$$

(4.43)

This is known as the the equivariant Bianchi identity, and implies that the line bundle $\mathcal{O}_{CP^{N-1}}(-n)$ has $U(1)^{N-1}$-equivariant structure, for which $\tilde{R}_c$ is the moment [20, 21].

The equivariant Bianchi identity is in fact a restatement of the $U(1)^{N-1}$-invariance of the connection,

$$
\mathcal{L}_{\tilde{\omega}} A^X = 0,
$$

(4.44)

---

24 The boundary conditions (4.35) and the boundary action (4.41) result in equations of motion which are modified by boundary terms, for some of the fields.

25 The $G$-equivariant Bianchi identity is equivalent to the $G$-invariance of the connection, $A$, of the bundle (equation (4.44)), which implies that the covariant derivative $d + A$ is $G$-invariant, and this defines a $G$-equivariant bundle, see [21], Section 3.2.

26 For equivariant bundles with abelian connection, the equivariant Bianchi identity takes the same form as the moment map equation (2.16). For the present case of $\mathcal{O}_{CP^{N-1}}(-n)$, it is in fact proportional; the curvature is $F = n\omega$, which means that $\tilde{R} = n\bar{\mu}$, where $\bar{\mu}$ is the normalized moment map of the $U(1)^{N-1}$ isometry of $CP^{N-1}$ (the discrepancy with (3.30) is because the moment map for an abelian $G$-action is only defined up to the addition of a constant, as explained below equation (2.17)).
Now, rewriting the boundary action (4.41) as

\[ S'_{\partial \Sigma} = \int dx^0 \left\{ A_j^X \partial_0 A_j^X + A_j^X \partial_0 \bar{X}^\gamma + \sum_{c} N \bar{R}_c - \frac{i}{2} F^{X}_{jk}(\psi_+^j + \psi_-^j)(\bar{\psi}_+ + \bar{\psi}_-) \right\} \]

facilitates the proof that it is invariant under the gauge transformations given in (2.26) and (2.27) for \( G = U(1)^{N-1} \). The variation is

\[ \delta S'_{\partial \Sigma} = \sum_a \alpha_a \int dx^0 \left\{ \mathcal{L}_{\tilde{e}_a} A_j^X \partial_0 A_j^X + \mathcal{L}_{\tilde{e}_a} A_j^X \partial_0 \bar{X}^\gamma + \sum_{c} \tilde{e}_a d \tilde{R}_c - \frac{i}{2} \mathcal{L}_{\tilde{e}_a} F^{X}_{jk}(\psi_+^j + \psi_-^j)(\bar{\psi}_+ + \bar{\psi}_-) \right\} \]

which vanishes using (4.44), as well as the identities \( \mathcal{L}_{\tilde{e}} F^X = 0 \) and \( \mathcal{L}_{\tilde{e}} \tilde{R} = \tilde{e} d \tilde{R} = 0 \).

It may seem that we have picked a random value for \( \tilde{\theta}_c \) in the derivation above. If we only required \( U(1)^{N-1} \) gauge invariance of the path integral, then we would have been free to choose \( \tilde{\theta}_c = 2\pi m_c \) for any integer \( m_c \), and we would still derive a boundary action which is B-type supersymmetry invariant, as well as gauge invariant mod \( 2\pi \mathbb{Z} \). This freedom is merely a reflection of the fact that the moment in the equivariant Bianchi identity (4.43) is only defined up to a constant.

We have thus found B-type supersymmetric and \( U(1)^{N-1} \) gauge invariant boundary conditions and boundary interactions corresponding to an equivariant B-brane in \( \mathbb{C}P^{N-1} \), which is a space-filling brane supporting the holomorphic line bundle \( \mathcal{O}_{\mathbb{C}P^{N-1}}(-n) \) with \( U(1)^{N-1} \)-equivariant structure. We may follow a procedure analogous to that presented above for \( \mathbb{C}P^{N-1} \) in order to describe an equivariant B-brane in a toric manifold \( X = \mathbb{C}^N//U(1) \) (by choosing different values for \( \tilde{Q}_i \)), which would be a space-filling brane supporting the holomorphic line bundle \( \mathcal{O}_X(-n) \) with \( U(1)^{N-1} \)-equivariant structure.

The GNLSM boundary action (4.41) that we have derived from the GLSM expressions (4.31) and (4.32) is a special case of the more general boundary Wilson line found by Kapustin et al. [14, 15], using a B-twisted topological nonabelian GNLSM, with gauge group \( G \) and target space \( X \), i.e., a gauged B-model. When the worldsheet is the Euclidean strip \( I \times \mathbb{R} \), this Wilson line takes the form of the path integral insertion

\[ W = \text{Str}(P(e^{iN})) \]  

with

\[ N = \int dx^2 \left\{ A_j^X \partial_2 X^j + A_j^X \partial_2 \bar{X}^\gamma - \sum_c R_c \tilde{A}_{2c} - \frac{1}{2} F^{X}_{jk}(\psi_+^j + \psi_-^j)(\bar{\psi}_+ + \bar{\psi}_-) + \frac{1}{2} (\psi_+^j + \psi_-^j) \nabla_j E T \right\} \]

where \( x^2 \) is the direction along the boundaries,

\[ \tilde{A}_{2c} = \tilde{A}_{2c} + i(\tilde{\sigma}_c + \tilde{\sigma}_c)^2 \]

(4.49)
is a complexified gauge field valued in $G_{C}$, and where quantities with subscript ‘2’ are components of one-forms along the boundaries, while the quantity with subscript ‘s’ is a scalar. Here, $A$ is the superconnection of a graded $G$-equivariant holomorphic vector bundle, $E$, with the covariant derivative $\nabla^{E} = d + A$; while $T$ is a holomorphic degree-1 endomorphism $T : E \rightarrow E$, which satisfies $T^{2} = 0$. The holomorphic transition functions of $E$ are valued in a structure supergroup. Moreover, $A$, $F$, $\tilde{R}$, and $T$ are valued in the Lie superalgebra of this structure supergroup, and obey the equivariant Bianchi identity

$$\nabla^{E} \tilde{R} = \iota_{c} F^{X},$$

as well as the identity

$$\partial_{i} \nabla^{E}_{i} T + [\tilde{R}, T] = 0.$$  

The expression (4.48) agrees with our result (4.41) when we take $E$ to be an ungraded holomorphic line bundle with $U(1)$ structure group. To see this, we need to analytically continue the Minkowski strip to Euclidean signature ($x^{0} = -ix^{2}$) and B-twist the fields in (4.41). Then, the expression (4.38) becomes the complexified gauge field (4.49), and the fermionic fields become scalars or one-forms along the boundaries. Finally, we note that the $\nabla^{E}_{j} T$ term in (4.48) does not occur in (4.41) since an ungraded bundle does not admit a degree-1 endomorphism; hence, $T = 0$. As explained by Kapustin et al. [14, 15], in some cases, the category of branes defined by (4.48) is equivalent to $D_{b}^{0}(G_{C}(\text{Coh}(X)))$, the bounded, derived category of $G_{C}$-equivariant coherent sheaves on the target space, $X$. This occurs if $X$ has a $G$-resolution property, i.e., any $G$-equivariant coherent sheaf on $X$ has a $G$-equivariant resolution by $G$-equivariant holomorphic vector bundles. This property, however, does not hold for general complex manifolds. Nevertheless, even for such spaces where it does not hold, it is believed that the full category of equivariant B-branes is still $D_{b}^{0}(G_{C}(\text{Coh}(X)))$, where the GNLSMs for these spaces require more general boundary actions corresponding to differential graded (DG) modules over the Dolbeault DG-algebra of $X$, instead of holomorphic bundles [14, 15]. In our construction, we have found abelian equivariant B-branes which wrap toric manifolds given by the quotient $X = \mathbb{C}^{N}/U(1)$, and which support the $U(1)^{N-1}$-equivariant holomorphic line bundle $\mathcal{O}_{X}(-n)$. In the language of algebraic geometry, $\mathcal{O}_{X}(-n)$ is a locally-free sheaf of rank 1, and is in fact one of the simplest objects of $D_{b}(\text{Coh}(X))$ ([19], page 56). The additional $U(1)^{N-1}$-equivariant structure then implies that the equivariant B-branes we have found are objects in $D_{(\mathbb{C}^{\times})^{N-1}}^{b}(\text{Coh}(X))$, the bounded, derived category of $(\mathbb{C}^{\times})^{N-1}$-equivariant coherent sheaves on $X$.\(^{27}\) Of course, we have not constructed all the objects in the category.

In particular, we have not constructed non-space-filling equivariant B-branes. The latter, i.e., equivariant B-branes of lower dimension, should exist, in analogy with the NLSM case (see footnote 18), although we shall not attempt to derive them from GLSMs here. The path to doing so is via Hori’s construction of non-space-filling ordinary B-branes from GLSMs [11]. Using the same GLSM used there, but with gauge group generalized to $U(1)^{N}$, we should be able to derive the relevant GNLSM boundary action and boundary conditions, as we have done for space-filling equivariant B-branes in this section.

\(^{27}\)The algebraic torus $(\mathbb{C}^{\times})^{N-1}$ is the complexification of $U(1)^{N-1}$.
4.2 Equivariant B-branes on $\mathbb{C}^N//U(1)^{N-k}$ from GLSM

The prior discussion can be generalized to the case of general Kähler toric manifolds, i.e., $X = \mathbb{C}^N//U(1)^{N-k}$. We impose the B-type supersymmetric boundary conditions (4.27) and (4.28) on the GLSM (for $N - k > 1$), which include the purely Neumann boundary conditions on $\phi_i$, while also supersymmetrizing the GLSM theta terms

$$
\sum_b \left( \frac{\hat{\theta}_b}{2\pi} \int_{\Sigma} \bar{F}_{01b} d^2x + \frac{\hat{\theta}_b}{2\pi} \int_{\partial\Sigma} \frac{(\bar{\sigma}_b + \bar{\sigma}_b)}{2} dx^0 \right) + \sum_c \left( \frac{\tilde{\theta}_c}{2\pi} \int_{\Sigma} \bar{F}_{01c} d^2x + \frac{\tilde{\theta}_c}{2\pi} \int_{\partial\Sigma} \frac{(\bar{\sigma}_c + \bar{\sigma}_c)}{2} dx^0 \right).
$$

(4.52)

This preserves B-type supersymmetry at the boundaries. In addition, the B-type supersymmetric GLSM boundary action needed is

$$
S_{\partial\Sigma} = \frac{\theta'}{4\pi r'} \int_{\partial\Sigma} dx^0 \left( \sum_i (i\partial_0 \bar{\phi}_i \phi_i - i\bar{\phi}_i \partial_0 \phi_i) + \sum_i (\psi_{+i} + \psi_{-i})(\bar{\psi}_{+i} + \bar{\psi}_{-i}) - \sum_a N_a (\sigma_a + \overline{\sigma}_a) |\phi_i|^2 \right),
$$

(4.53)

where $\theta' = 2\pi n'$ ($n' \in \mathbb{Z}$) and $r' \in \mathbb{R}$. In addition, we ought to set $\hat{\theta}_b = 2\pi \hat{n}_b$, where $\hat{n}_b \in \mathbb{Z}$, and we need to impose the condition

$$
\frac{\theta'}{r'} = \frac{\hat{\theta}_b}{\hat{r}_b}
$$

(4.54)

for all values of $b$.

This allows us to write (4.53) and (4.52) as

$$
S'_{\partial\Sigma} = \frac{n'}{2r'} \int_{\partial\Sigma} dx^0 \left( \sum_i (i\partial_0 \bar{\phi}_i \phi_i - i\bar{\phi}_i \partial_0 \phi_i) + \sum_i (\psi_{+i} + \psi_{-i})(\bar{\psi}_{+i} + \bar{\psi}_{-i}) - \sum_a N_a (\sigma_a + \overline{\sigma}_a) |\phi_i|^2 \right)
$$

$$
\quad + 2 \sum_b A_{0b} \left( \sum_i N_i |\phi_i|^2 - \hat{r}_b \right) - \sum_{b^2} (\bar{\sigma}_b + \bar{\sigma}_b) \left( \sum_i N_i |\phi_i|^2 - \hat{r}_b \right)
$$

$$
\quad + 2 \sum_c \tilde{A}_{0c} \left( \sum_i \tilde{N}_i |\phi_i|^2 - \sum_c (\bar{\sigma}_c + \bar{\sigma}_c) \sum_i \tilde{N}_i |\phi_i|^2 \right)
$$

$$
\quad + \sum_{c^2} \left( \frac{\tilde{\theta}_c}{2\pi} \int_{\Sigma} \bar{F}_{01c} d^2x + \frac{\tilde{\theta}_c}{2\pi} \int_{\partial\Sigma} \frac{(\bar{\sigma}_c + \bar{\sigma}_c)}{2} dx^0 \right)
$$

(4.55)

Taking the $\hat{r}_b \to \infty$ limit allows us to integrate $\tilde{D}_b$ out of the action, which imposes the constraints (3.17), and the second line in (4.55) vanishes. Integrating out the other components of the vector multiplets, $\tilde{V}_b$, then imposes (3.16) on the entire action, as well (3.14) and (3.15) on the bulk action. Then, to find the explicit boundary action, one needs to use parametrizations which satisfy (3.16) and (3.17). The explicit boundary conditions are also found using these parametrizations, together with (3.14) and (3.15).

We would then be able to identify the first term in (4.55) as the Hermitian connection of the holomorphic line bundle $\mathcal{O}(k_1, \ldots, k_{N-k})$ over $\mathbb{C}^N//U(1)^{N-k}$, where $k_1, \ldots, k_{N-k} \in \mathbb{Z}$ would be integers related to $\hat{n}_1, \ldots, \hat{n}_{N-k}$. Both supersymmetry invariance and gauge
invariance under the residual $U(1)^k$ gauge symmetry of the GNLSM would then require that this line bundle has $U(1)^k$-equivariant structure. Moreover, we would be able to identify the equivariant B-branes we have found as objects in $D^b_{(\mathbb{C}^\times)^k}(\text{Coh}(X))$, the bounded, derived category of $(\mathbb{C}^\times)^k$-equivariant coherent sheaves on $X$.

The simplest example would be that of the $U(1)^2$-equivariant holomorphic line bundle $\mathcal{O}_X(\hat{n}_1, -\hat{n}_2)$ over $X = \mathbb{C}P^1 \times \mathbb{C}P^1$, which just corresponds to two copies of the boundary action given in (4.41), with $N = 2$.\textsuperscript{28} One can even consider equivariant B-branes on fibrations of $\mathbb{C}P^1$ over $\mathbb{C}P^1$ known as Hirzebruch surfaces, using GLSMs with appropriately charged scalar fields \cite{7}. It is worth noting that the derived categories of $\mathbb{C} \times$-equivariant coherent sheaves over $\mathbb{C}P^1$, Hirzebruch surfaces, $\mathbb{C}P^1$ fibered over Hirzebruch surfaces etc. provide a construction of Khovanov homology \cite{22}.

### 4.3 Alternative Formulation

We shall now derive the alternative formulation of abelian equivariant B-branes, in terms of a boundary action which generalizes (4.5), as well as the relevant boundary conditions.

Let us first recall the derivation of the NLSM boundary action (4.5) for a space-filling B-brane on $X = \mathbb{C}^N//U(1)$ \cite{10}. The $U(1)$-GLSM boundary action from which (4.5) can be derived is

\[
S_{\partial \Sigma} = \frac{\hat{\theta}}{4\pi \hat{r}} \int d\sigma^0 \sum_i \left( i\hat{D}_0 \bar{\phi}_i \phi_i - i\bar{\phi}_i \hat{D}_0 \phi_i \right). \tag{4.56}
\]

To preserve B-type supersymmetry at the boundaries, we must impose \cite{10}

\[
e^{-i\hat{\gamma}} \hat{D}_+ \phi_i = e^{i\hat{\gamma}} \hat{D}_- \phi_i
\]

\[
e^{i\hat{\gamma}} \hat{\Sigma} = e^{-i\hat{\gamma}} \Sigma
\]

at B-boundary, where $\hat{D}_\pm = e^{-\hat{Q}_1 \hat{V}} D_\pm e^{\hat{Q}_1 \hat{V}}$, and where $\hat{\gamma}$ is the phase of $\hat{t} = \hat{r} - i\hat{\theta} = |\hat{t}| e^{i\hat{\gamma}}$. In components, these are

\[
e^{-i\hat{\gamma}} \psi_{+i} - e^{i\hat{\gamma}} \psi_{-i} = 0
\]

\[
F_i = 0
\]

\[
\cos(\hat{\gamma}) \hat{D}_1 \phi_i - i\sin(\hat{\gamma}) \hat{D}_0 \phi_i = 0
\]

\[
\cos(\hat{\gamma}) \hat{D}_1 (\psi_{+i} + \psi_{-i}) - i\sin(\hat{\gamma}) \hat{D}_0 (\psi_{+i} + \psi_{-i}) - \cos(\hat{\gamma}) (\lambda_+ + \lambda_-) \phi_i = 0
\]

and

\[
\text{Im}(e^{i\hat{\gamma}} \hat{\sigma}) = 0
\]

\[
e^{-i\hat{\gamma}} \lambda_+ + e^{i\hat{\gamma}} \lambda_- = 0
\]

\[
\partial_1 \text{Re}(e^{i\hat{\gamma}} \hat{\sigma}) + \cos(\hat{\gamma}) \tilde{F}_{01} - \sin(\hat{\gamma}) \tilde{D} = 0
\]

which includes the mixed Dirichlet-Neumann boundary condition on the scalar fields $\phi_i$. These conditions are not sufficient for B-type supersymmetry of the GLSM action at the

\textsuperscript{28}In fact, for toric manifolds which are Cartesian products like $X = \mathbb{C}P^1 \times \mathbb{C}P^1$, the complete decoupling of the two boundary actions means that we no longer need the constraint (4.54).
boundaries, and in addition, we must impose the boundary condition
\[ \hat{F}_{01} = -\tilde{\theta} + \sum_{i} N \hat{Q}_{i} |\phi_{i}|^{2}, \]
(4.60)
as well as integrate \( \hat{D} \) out of the action to obtain its algebraic equation of motion
\[ \frac{\hat{D}}{\tilde{r}} = \hat{r} - \sum_{i} N \hat{Q}_{i} |\phi_{i}|^{2}, \]
(4.61)
which holds on the entire worldsheet.\(^{29}\)

Note that the B-type supersymmetry transformations of (4.60) further implies the boundary conditions
\[ \hat{e}^{-\hat{F}_{01}}(\partial_{0} - \partial_{1}) X_{+} + (\partial_{0} + \partial_{1}) X_{-}) = \hat{r} \sum_{i} N \hat{Q}_{i} (\psi_{-i} + \psi_{+i}) \phi_{i}, \]
\[ \frac{1}{\hat{e}^{\hat{F}_{01}}}(\partial_{1}(\hat{D} + \partial_{1} \text{Im}(\hat{\sigma}))) + \partial_{0}(i\hat{F}_{01} - \partial_{0} \text{Im}(\hat{\sigma}))) = \hat{r} \sum_{i} N \hat{Q}_{i} (2iD_{0}\phi_{i}\overline{\phi}_{i} + 2\text{Re}(\hat{\sigma})\hat{Q}_{i} |\phi_{i}|^{2}) \]
(4.62)
\[ - (\psi_{+i} + \psi_{-i})(\overline{\psi}_{+i} + \overline{\psi}_{-i}) \].

Unlike the formulation presented earlier, the boundary conditions given above ensure the locality of the equations of motion derived from the (bulk+boundary) action.

Converting the theta term to a boundary term as in (4.14), with \( \hat{e}^{-\hat{A}_{0}} \) proportional to \( \hat{e}^{\hat{A}_{0}} \) vanishes in the \( \hat{e} \to \infty \) limit, whereby (4.17) is strictly imposed via (4.61), and the remaining term is just the hermitian connection (4.20) of the holomorphic line bundle \( O_{X}(-n) \) on the toric manifold \( X = C^{N}//U(1) \), and therefore we obtain (4.5) for a space-filling B-brane. Integrating out the rest of the vector multiplet components imposes additional constraints which only affect the bulk action but not the boundary action. These constraints, together with the appropriate parametrizations for \( \phi_{i} \) and \( \psi_{i} \), are useful for finding the corresponding NLSM boundary conditions.

Now, to derive the boundary action for a GNLSM with \( X = C^{N}//U(1) \), we start with the \( U(1)^{N} \)-GLSM boundary action
\[ S_{\partial \Sigma} = \frac{n}{2\tilde{r}} \int d^{2}x \left( \sum_{i} N \hat{Q}_{i} (\partial_{0} \phi_{i} \phi_{i} - \partial_{i} \phi_{i} \phi_{i}) + 2\hat{A}_{0} (\sum_{i} N \hat{Q}_{i} |\phi_{i}|^{2} - \hat{r}) \right), \]
(4.63)
The term proportional to \( \hat{A}_{0} \) vanishes in the \( \hat{e} \to \infty \) limit, whereby (4.17) is strictly imposed via (4.61), and the remaining term is just the hermitian connection (4.20) of the holomorphic line bundle \( O_{X}(-n) \) on the toric manifold \( X = C^{N}//U(1) \), and therefore we obtain (4.5) for a space-filling B-brane. Integrating out the rest of the vector multiplet components imposes additional constraints which only affect the bulk action but not the boundary action. These constraints, together with the appropriate parametrizations for \( \phi_{i} \) and \( \psi_{i} \), are useful for finding the corresponding NLSM boundary conditions.

To derive the boundary action for a GNLSM with \( X = C^{N}//U(1) \), we start with the \( U(1)^{N} \)-GLSM boundary action
\[ S_{\partial \Sigma} = \frac{\hat{r}}{4\pi i} \int d^{2}x \sum_{i} N \hat{Q}_{i} (\partial_{0} \phi_{i} \phi_{i} - \partial_{i} \phi_{i} \phi_{i}), \]
(4.64)
where the covariant derivatives of the scalar fields are given by (3.13). B-type supersymmetry invariance of the \( U(1)^{N} \)-GLSM at the boundaries of the worldsheet firstly requires that we impose
\[ e^{-i\hat{F}_{i}^{\pm}} \partial_{+} \Phi_{i} = e^{i\hat{F}_{i}^{\pm}} \partial_{-} \Phi_{i} \]
(4.65)
\[ 29 \text{The constraints (4.60) and (4.61) result in the third equation of (4.59) becoming } \partial_{1} \text{Re}(e^{i\hat{F}_{01}}) = 0. \]
\[ e^{i\gamma_a} \Sigma_a = e^{-i\gamma_a} \Sigma_a \]  
(4.66)

at B-boundary, where \( D_{\pm} = e^{-\sum_a Q_i V_a} D_{\pm} e^{\sum_a Q_i V_a} \), while \( \hat{\gamma} \) and \( \gamma_a \) are the phases of \( \hat{t} = |\hat{t}| e^{i\hat{\gamma}} \) and \( t_a = |t_a| e^{i\gamma_a} \) respectively. Secondly, we also ought to impose

\[ \frac{\hat{\theta}}{\hat{r}} = \frac{\theta_a}{r_a} \]  
(4.67)

and

\[ \hat{\gamma} = \gamma_a. \]  
(4.68)

Then, in components, (4.65) and (4.66) become

\[ e^{-i\hat{\gamma}} \psi_{+i} - e^{i\hat{\gamma}} \psi_{-i} = 0 \]
\[ F_i = 0 \]
\[ \cos(\hat{\gamma}) D_1 \phi_i - i \sin(\hat{\gamma}) D_0 \phi_i = 0 \]  
(4.69)

\[ \cos(\hat{\gamma}) D_1 (\psi_{+i} + \psi_{-i}) - i \sin(\hat{\gamma}) D_0 (\psi_{+i} + \psi_{-i}) = \cos(\hat{\gamma}) \sum_a Q_{ia}(\lambda_{+a} + \lambda_{-a}) \phi_i = 0 \]

and

\[ \text{Im}(e^{i\gamma_a}) = 0 \]
\[ e^{-i\hat{\gamma}} \lambda_{+a} + e^{i\hat{\gamma}} \lambda_{-a} = 0 \]  
(4.70)

which includes the mixed Dirichlet-Neumann boundary condition on the scalar fields \( \phi_i \). Finally, for complete boundary B-type supersymmetry invariance, we must impose the boundary condition

\[ \frac{F_{01a}}{e_a^2} = -\theta_a + \hat{\theta} \sum_i Q_{ia}|\phi_i|^2 \]  
(4.71)

as well as integrate \( D_a \) out of the action to obtain its algebraic equation of motion

\[ \frac{D_a}{e_a^2} = r_a - \sum_i Q_{ia}|\phi_i|^2, \]  
(4.72)

which holds on the entire worldsheet.\(^{30}\) The condition (4.71) further implies two more boundary conditions via B-type supersymmetry. As expected, all the boundary conditions above ensure the locality of the equations of motion derived from the action.

Now, setting \( \hat{\theta} = 2\pi n \), the relevant action which consists of (4.64) together with the theta terms is

\[ S'_{\partial \Sigma} = \frac{n}{2\pi} \int_{\partial \Sigma} dx^0 \left( \sum_i (i\partial_0 \overline{t}_i \phi_i - i\overline{t}_i \partial_0 \phi_i) + 2 \hat{A}_0 \left( \sum_i \hat{Q}_i |\phi_i|^2 - \hat{r} \right) \right) \]
\[ + 2 \sum_{c=0}^{N-1} \tilde{A}_{0c} \sum_i (\tilde{Q}_{ic} |\phi_i|^2) - \int_{\Sigma} \frac{\hat{\theta}}{2\pi} \int \tilde{F}_{01c} d^2 x \]  
(4.73)

\(^{30}\)The constraints (4.68), (4.71) and (4.72) result in the third equation of (4.70) becoming \( \partial_1 \text{Re}(e^{i\gamma_a}) = 0. \)
The term proportional to $\hat{A}_0$ vanishes in the $\hat{e} \to \infty$ limit using the equation of motion for $\hat{D}$ given in (4.72), while the constraints that arise from subsequently integrating out the rest of the vector multiplet $\hat{V}$ do not affect the boundary action. For $X = \mathbb{C}P^{N-1}$, we can use the parametrizations (3.24), and (4.73) becomes

$$S'_{\partial \Sigma} = \int dx^0 \left\{ A_j^X \partial_0 X^j + A_j^X \partial_0 \bar{X}^j - \sum_{c}^{N-1} \bar{I}_c \hat{A}_c + n \partial_0 t \right\} + \sum_{c}^{N-1} \left( \frac{\tilde{\theta}_c}{2\pi} \int_{\Sigma} \tilde{F}_{01c} d^2 x \right),$$

(4.74)

with $A_I$ given in (4.24), and $R_c$ given in (4.39). Then, gauging away the $n \partial_0 t$ term, and setting $\tilde{\theta}_c = 2\pi n Q_{Nc}$, we arrive at the boundary action

$$S'_{\partial \Sigma} = \int dx^0 \left\{ A_j^X \partial_0 X^j + A_j^X \partial_0 \bar{X}^j - \sum_{c}^{N-1} i \tilde{R}_c \bar{A}_c \right\},$$

(4.75)

where $\tilde{R}_c$ is the moment given by (4.42). The boundary action can be rewritten concisely as

$$S'_{\partial \Sigma} = \int dx^0 \left\{ A_j^X \partial_0 A^j + A_j^X \partial_0 \bar{A}^j \right\},$$

(4.76)

and gauge invariance follows since (4.44) is obeyed, which implies that the line bundle $O_{\mathbb{C}P^{N-1}}(-n)$ supported by the equivariant B-brane has $U(1)^{N-1}$-equivariant structure.

The boundary conditions for the GNLSM with $\mathbb{C}P^{N-1}$ target can similarly be found; for the $U(1)^{N-1}$ vector multiplets, the boundary conditions follow from (4.70) and (4.71), while for the matter fields, the boundary conditions are

$$g_\mathcal{B}(\psi^Z_i^+ - \psi^Z_i^-) + 2\pi F_{\mathcal{B}}^X (\psi^Z_i^+ + \psi^Z_i^-) = 0,$$

$$g_\mathcal{B} \partial_0^X Z^i - 2\pi F_{\mathcal{B}}^X \partial_0^X Z^i = 0,$$

(4.77)

and their B-type supersymmetric completions, where $g$ is the Fubini-Study metric and $F$ is the curvature of $O_{\mathbb{C}P^{N-1}}(-n)$ given in (4.25).

An alternative formulation also exists for $U(1)^k$-GNLSMs with $X = \mathbb{C}^N // U(1)^{N-k}$, i.e., general Kähler toric manifolds. The boundary action for the $U(1)^{N-k} \times U(1)^k$ GLSM is

$$S_{\partial \Sigma} = \frac{\theta'}{4\pi r'} \int dx^0 \sum_{i}^{N} \left( iD_0 \bar{\phi}_i \phi_i - i\bar{\phi}_i D_0 \phi_i \right),$$

(4.78)

where $\theta' = 2\pi n'$ ($n' \in \mathbb{Z}$) and $r' \in \mathbb{R}$, together with the theta terms

$$\sum_{b}^{N-k} \left( \frac{\hat{\theta}_b}{2\pi} \int_{\Sigma} \hat{F}_{01b} d^2 x \right) + \sum_{c}^{k} \left( \frac{\tilde{\theta}_c}{2\pi} \int_{\Sigma} \tilde{F}_{01c} d^2 x \right).$$

(4.79)

Setting

$$\frac{\theta'}{r'} = \frac{\hat{\theta}_b}{\hat{r}_b} = \frac{\tilde{\theta}_c}{\tilde{r}_c},$$

(4.80)
and
\[ \gamma' = \tilde{\gamma}_b = \tilde{\gamma}_c, \] (4.81)
the relevant boundary conditions are (4.69), (4.70) and (4.71), with \( \frac{\partial}{\partial t} \) replaced by \( \frac{\partial}{\partial t'} \) and \( \dot{\gamma} \) replaced by \( \gamma' \). In addition, the \( D_a \) equation of motion is also necessary for complete B-type supersymmetry at the boundaries.

By taking the \( \hat{e}_b \to \infty \) limit and repeating the familiar procedure, we can obtain the GNLSM boundary action which includes the Hermitian connection of a \( U(1)^k \)-equivariant holomorphic line bundle over \( \mathbb{C}^N / U(1)^{N-k} \), as well as the relevant GNLSM boundary conditions.

An important advantage of the alternative formulation of equivariant B-branes over the first one is that because of the constraints (4.67) and (4.80), the form of the GLSM boundary action does not depend on which gauge symmetries we are breaking to obtain the GNLSM. This implies the equivalence of equivariant B-branes in different toric targets of GNLSMs obtained from a single GLSM. In order to ensure that the first formulation also does not depend on which gauge symmetries we are breaking, we can impose the same constraints for it.

### 4.4 Quantum Corrections

We have heretofore analyzed the boundary conditions of the classical \( U(1)^{N-k} \times U(1)^k \) GLSM, and the respective GNLSM limits of these conditions, in two equivalent formulations. We shall now investigate quantum effects for the alternative formulation of equivariant B-branes given in Section 4.3,\(^{31}\) since we shall use this formulation for the proof of mirror symmetry in the following section.\(^{32}\)

There are two quantum effects of the \( U(1)^{N-k} \times U(1)^k \) GLSM with \( \sum_{i=1}^{N} Q_{ia} \neq 0 \) which are important. The first of these is the running of the FI parameters
\[
\begin{align*}
\lambda_{0a} & = \lambda_a(\mu) + \sum_{i=1}^{N} Q_{ia} \log \left( \frac{\Lambda_{UV}}{\mu} \right),
\end{align*}
\] (4.82)
where \( \lambda_{0a} \) denotes bare parameters, \( \Lambda_{UV} \) is an ultraviolet cut-off, and \( \mu \) is a finite energy scale. By integrating the beta functions of the FI parameters, \( \beta_a = \mu \frac{d \lambda_a}{d \mu} \), the \( \mu \)-dependence is found to be
\[
\lambda_a(\mu) = \sum_{i=1}^{N} Q_{ia} \log \left( \frac{\mu}{\Lambda} \right),
\] (4.83)
where \( \Lambda \) is the renormalization group invariant dynamical scale. The running of \( \lambda_a \) implies that the phase, \( e^{i \gamma_a} = t_a / |t_a| \), which appears in the boundary conditions we have used, changes with the renormalization group flow. The second quantum effect is the anomaly of the \( U(1) \) axial R-symmetry, whereby axial R-rotations \( \psi_{\pm i} \to e^{\pm i \beta / 2} \psi_{\pm i} \), \( \sigma_a \to e^{-i \beta} \sigma_a \)

\(^{31}\) We shall not study the quantum effects for the first formulation, since the main quantum correction is the running of the FI parameters, and the FI parameters do not enter the boundary conditions in that formalism.

\(^{32}\) The following is a generalization of the analysis given in Section 6 of \cite{10} to the case of multiple \( U(1) \) gauge groups.
and \(\lambda_{\pm a} \rightarrow e^{\pm i\beta/2} \lambda_{\pm a}\) no longer leave the action invariant, but result in a shift of the theta angles, i.e.,

\[
\theta_a \rightarrow \theta_a + \sum_{i=1}^{N} Q_{ia} \beta. \tag{4.84}
\]

These effects should be apparent in a quantum effective description, whereby the lowest components \(\sigma_a\) of the superfields \(\Sigma_a\) are chosen to be slowly varying and to be large compared to the energy scale \(\mu\) at which we look at the effective theory. This imparts large masses to the charged matter superfields \(\Phi_i\), which can then be integrated out as long as we are studying the theory at some finite energy scale \(\mu\). From a path integral computation \([7]\), the superpotential of the effective action, which corresponds to a Landau-Ginzburg model,\(^{33}\) is

\[
\tilde{W}_{(\text{eff})} = - \sum_{a=1}^{N} \left[ \sum_{i}^{N} Q_{ia} \left( \log \left( \frac{\sum_{a'}^{N} Q_{ia'} \Sigma_{a'}}{\mu} \right) - 1 \right) \right] \Sigma_a - \sum_{a=1}^{N} t_a(\mu) \Sigma_a, \tag{4.85}
\]

wherefrom the effective FI-Theta parameter

\[
t_{(\text{eff})a} = t_a(\mu) + \sum_{i}^{N} Q_{ia} \left( \log \left( \frac{\sum_{a'}^{N} Q_{ia'} \Sigma_{a'}}{\mu} \right) \right), \tag{4.86}
\]

is obtained. Now, by performing an ordinary axial R-rotation \(\Sigma_a \rightarrow e^{-i\beta} \Sigma_a\) in (4.86), we can retrieve the shift (4.84).

Now, it is known from \([10]\) that a D-brane which preserves the B-type supercharges \(Q_B = \overline{Q}_+ + \overline{Q}_-\) and \(Q_B^\dagger = Q_+ + Q_-\) is a Lagrangian submanifold of the space \(\mathbb{C}^N\) defined by the fields \(\sigma_a\). In addition, this D-brane ought to be the preimage of a horizontal straight line in the \(\tilde{W}_{(\text{eff})}\)-plane, i.e., \(\text{Im}(\tilde{W}_{(\text{eff})}(\sigma)) = \text{constant}\). If we were to solve these constraints in terms of \(\sigma_a\), then we will obtain the quantum corrected boundary condition for \(\sigma_a\). In general, these constraints are difficult to solve. However, when the parameters \(\theta_a = 0\), then there is the solution \(\sigma_a = |\sigma_a|\), which satisfies \(\text{Im}(\sigma_a) = 0\) and \(\text{Im}(\tilde{W}_{(\text{eff})}(\sigma)) = 0\).

In order to obtain a less trivial solution, we can perform an axial R-rotation, which includes the shift of \(\theta_a = 0\) to \(\theta_a = \sum_{i=1}^{N} Q_{ia} \beta\), due to the aforementioned anomaly. Then, we obtain the solution \(\sigma_a = e^{i\beta}|\sigma_a|\), which satisfies \(\text{Im}(e^{-i\beta} \sigma_a) = 0\) and the straight line equation \(\text{Im}(e^{-i\beta} \tilde{W}_{(\text{eff})}(\sigma)) = 0\). These conditions are compatible with the constraints of the B-type supercharges

\[
Q_B = \overline{Q}_+ + e^{i\beta} \overline{Q}_- \tag{4.87}
\]

and \(Q_B^\dagger = Q_+ + e^{-i\beta} Q_-\) found in \([10]\), i.e., the D-brane ought to be a Lagrangian submanifold of the field space \(\mathbb{C}^N\), and it ought to be the preimage of a straight line in the \(\tilde{W}_{(\text{eff})}\)-plane with slope \(\tan(\beta)\), i.e., \(\text{Im}(e^{-i\beta} \tilde{W}_{(\text{eff})}(\sigma)) = \text{constant}\).

Hence, we find that there is a family of explicit solutions which include

\[
\sigma_a = e^{i\beta}|\sigma_a|,
\]

\[
e^{i\beta/2} \lambda_{+a} + e^{-i\beta/2} \lambda_{-a} = 0, \quad \text{at } \partial \Sigma, \tag{4.88}
\]

\[
e^{-i\beta/2} \overline{\lambda}_{+a} + e^{i\beta/2} \overline{\lambda}_{-a} = 0,
\]

\(^{33}\text{To be precise, the theory involves a gauge field, whose only effect is a vacuum energy [7].}\)
parametrized by $\beta = \theta_a / \sum_{i=1}^{N} Q_{ia}$,\(^{34}\) which preserve the B-type supercharges $Q_B = \overline{Q}_+ + e^{i\beta}Q_-$ and $Q_B^\dagger = Q_+ + e^{-i\beta}Q_-$. Other solutions, including those with $\beta \neq \theta_a / \sum_{i=1}^{N} Q_{ia}$, should exist, but in these cases the quantum corrections are non-trivial, and therefore they are difficult to determine, and we shall not consider them.

Now, note that we have $\beta = \theta_a / \sum_{i=1}^{N} Q_{ia}$ for all $a = 1, \ldots, N$. Using (4.82) and (4.83), we have $r_0 = \sum_{i=1}^{N} Q_{ia} \log \left( \frac{\Lambda_{UV}}{\Lambda} \right)$, which implies

\[
\frac{\theta_a}{r_0} = \frac{\sum_{i=1}^{N} Q_{ia} \beta}{\sum_{i=1}^{N} Q_{ia} \log \left( \frac{\Lambda_{UV}}{\Lambda} \right)} = \frac{\beta}{\log \left( \frac{\Lambda_{UV}}{\Lambda} \right)},
\]

i.e., we find that $\theta_a/r_0$ are equal for all values of $a$.\(^{35}\) This agrees with the constraints (4.67) and (4.80). In other words, we find that these constraints, which we previously imposed by hand at the classical level, emerge naturally as a result of quantum effects.

### 4.5 Mirrors of Equivariant B-branes

In this section, we shall use the alternative formulation for equivariant B-branes, given in Section 4.3, to derive the Landau-Ginzburg mirrors of equivariant B-branes, following the exposition in Section 3, as well as the results of [10]. We shall assume in the following that

\[
b_{1a} = \sum_{i}^{N} Q_{ia} > 0.
\]

In particular, $\hat{b}_{1b} = \sum_{i}^{N} \hat{Q}_{ib} > 0$ implies that we are studying the mirrors of GNLSMs with Fano target spaces.

Let us start with the mirrors of equivariant B-branes on Fano manifolds of the form $X = \mathbb{C}^N // U(1)$. We focus on the family of boundary conditions (4.88). The corresponding boundary conditions of the matter fields include

\[
\begin{align*}
\cos(\gamma_0) D_1 \phi_i - i \sin(\gamma_0) D_0 \phi_i &= 0, \\
e^{-i\gamma_0 + i\beta/2} \psi_{+i} &= e^{i\gamma_0 - i\beta/2} \psi_{-i}, \\
e^{i\gamma_0 - i\beta/2} \overline{\psi}_{+i} &= e^{-i\gamma_0 + i\beta/2} \overline{\psi}_{-i},
\end{align*}
\]

where the axial R-rotations on the fermionic fields have been taken into account. These boundary conditions preserve the B-type supercharge $Q_B = \overline{Q}_+ + e^{i\beta}Q_-$ and its conjugate.

Now, in the continuum limit $\Lambda_{UV} \to \infty$ whereby $\hat{r}_0 = \hat{b}_1 \log(\Lambda_{UV}/\Lambda) \to \infty$, we have $\hat{\gamma}_0 \to 0$. As a result, the mixed Dirichlet-Neumann boundary conditions on $\phi_i$ reduce to pure Neumann boundary conditions.

\(^{34}\) $\sigma_a = e^{i\beta} |\sigma_a|$ implies the boundary condition $\text{Im}(e^{-i\beta} \sigma_a) = 0$.

\(^{35}\) Naively, it may seem that the boundary action (4.64) vanishes in the continuum limit ($\Lambda_{UV} \to \infty$) due to (4.89). However, this is not the case, at least for $\sum_{i=1}^{N} Q_{ia} > 0$, as we shall see in the next section.
With these facts in mind, let us shift our attention to the boundary action

\[ S_{\partial \Sigma} = \frac{\hat{\theta}}{4\pi \hat{r}_0} \int_{\partial \Sigma} \sum_{i=1}^{N} \left( iD_0 \phi_i \phi_i - i \bar{\phi}_i D_0 \phi_i \right) dx^0 \]

\[ = \frac{\hat{\theta}}{2\pi \hat{r}_0} \int_{\partial \Sigma} \sum_{i=1}^{N} |\phi_i|^2 (\partial_0 \phi_i + \sum_a Q_{ia} A_{0a}) \, dx^0. \]  

(4.92)

Now, by integrating over the modes of \( \phi_i \) in the frequency range \( \mu \leq |k| \leq \Lambda_{UV} \) in the path integral, \( |\phi_i|^2 \) is replaced by \( \langle |\phi_i|^2 \rangle = \log(\Lambda_{UV}/\mu) \). Since \( \hat{r}_0/\hat{b}_1 = \log(\Lambda_{UV}/\mu) + \hat{r}/\hat{b}_1 \), taking the continuum limit \( \Lambda_{UV} \to \infty \) gives us \( |\phi_i|^2 \approx \hat{r}_0/\hat{b}_1 \), which implies that

\[ S_{\partial \Sigma} = \frac{\hat{\theta}}{2\pi} \int_{\partial \Sigma} \left( \frac{1}{b_1} \sum_{i=1}^{N} \partial_0 \phi_i + \hat{A}_0 + \sum_{c} \frac{\hat{b}_{1c}}{b_1} \hat{A}_{0c} \right) dx^0. \]  

(4.93)

The relevant portion of the action with regard to the dualization of mirror symmetry is then

\[ S_{\varphi} = \frac{1}{2\pi} \int_{\Sigma} \sum_{i=1}^{N} \frac{\hat{r}_0}{b_1} d\varphi_i + \sum_a Q_{ia} A_{0a}^2 - \frac{i\hat{\theta}}{2\pi} \int_{\Sigma} (\frac{1}{b_1} \sum_{i=1}^{N} d\varphi_i + \hat{A} + \sum_{c} \frac{\hat{b}_{1c}}{b_1} \hat{A}_{0c}), \]  

(4.94)

where we have considered Euclidean signature on the worldsheet for simplicity,\(^{36}\) and where the terms with fermionic fields which are not essential in the present analysis have been ignored. Let us consider another action with one-form fields \( B_i = B_{i\mu} dx^\mu \) given by

\[ S' = \sum_{i=1}^{N} \left[ \frac{\hat{b}_1}{8\pi \hat{r}_0} \int_{\Sigma} B_i \wedge *B_i + \frac{i}{2\pi} \int_{\Sigma} (B_i \wedge (d\varphi_i + \sum_a Q_{ia} A_{0a}) \right] - \frac{i\hat{\theta}}{2\pi} \int_{\partial \Sigma} \left( \frac{1}{b_1} \sum_{i=1}^{N} d\varphi_i + \hat{A} + \sum_{c} \frac{\hat{b}_{1c}}{b_1} \hat{A}_{0c} \right). \]  

(4.95)

The one-form fields \( B_i \) have the boundary condition

\[ B_i|_{\partial \Sigma} = 0, \]  

(4.96)

i.e., their inner products with tangent vectors of the boundaries vanish. If we were to first integrate out \( B_i \), the constraint \( B_i = i2(\hat{r}_0/\hat{b}_1) * (d\varphi_i + \sum_a Q_{ia} A_{0a}) \) is obtained (whereby the boundary condition (4.96) is consistent with the boundary condition \( D_1 \phi_i = 0 \) obtained in the continuum limit) and the original action (4.94) is obtained. Alternatively, if we were to first integrate out \( \varphi_i \), the constraint

\[ B_i = d\varphi_i \]  

(4.97)

\(^{36}\)In the following derivation, we use the notation \( \langle |A|^2 \rangle = A \wedge *A \).
is obtained, where the fields \( \vartheta_i \) are periodic with period \( 2\pi \). The boundary conditions (4.96) then imply that \( \vartheta_i \) are constants at the boundaries of the worldsheet. The boundary terms containing \( \partial_2(\delta \varphi_i) \) obtained when integrating out \( \varphi_i \) cancel if these constants are

\[
\vartheta_i = \hat{\vartheta}/b_1 \quad \text{at} \quad \partial \Sigma, \tag{4.98}
\]

for all \( i \), where \( \hat{\vartheta}/b_1 = \beta = \theta_a/b_{1a} \). Now, using the constraint (4.97) in (4.95), the mirror action

\[
S_\vartheta = \sum_{i=1}^{N} \int_{\Sigma} \left[ \frac{\hat{b}_1}{4\pi \theta} \left( |d\vartheta_i|^2 + i \left( \vartheta_i \wedge (\sum_{a}^{N} Q_{ia} A_a) \right) \right) - \frac{i\hat{\vartheta}}{2\pi} \left( \hat{A} + \sum_{c}^{N-1} \hat{b}_{1c} \hat{A}_c \right) \right]
\]

\[
= \sum_{i=1}^{N} \int_{\Sigma} \left[ \frac{\hat{b}_1}{4\pi \theta} \left( |d\vartheta_i|^2 - i \left( \sum_{a}^{N} Q_{ia} \vartheta_i dA_a \right) \right) \right] + \frac{i\hat{\vartheta}}{2\pi} \left( \sum_{i=1}^{N} \hat{Q}_{ic} \vartheta_i - \sum_{c}^{N-1} \hat{b}_{1c} \hat{\vartheta} \hat{A}_c \right), \tag{4.99}
\]

is obtained. Finally, the boundary term in this action vanishes when we use the boundary condition (4.98), and the dualization process ends with only a bulk action.

In particular, the relationship (reviewed in Section 3) between the fields of the mirror theories, i.e.,

\[
Y_i + \overline{Y}_j = 2\Phi e^{\Sigma_0} Q_{ia} V_a \Phi_i, \tag{4.100}
\]

holds, and we have the following relationships between superfield components:

\[
\begin{align*}
Y_i & = \vartheta_i - i\varphi_i, \\
\chi_{i+} & = \psi_{i+} + \psi_{i+} \varphi_i, \\
\chi_{i-} & = \psi_{i-} + \psi_{i-} \varphi_i, \\
E_i & = -2\psi_{i-} \psi_{i+} - 2|\varphi_i|^2 \sum_{a}^{N} Q_{ia} \varphi_a,
\end{align*}
\]

where \( \vartheta_0 = 0 \) \( \equiv \vartheta_1 \), \( \vartheta_i = Y_i + \theta^+ \chi_{i+} + \theta^+ \overline{\chi_{i-}} + \theta^+ \overline{\chi_{i-}} + \theta^+ \overline{E_i} \). The relationship between the periodic fields \( \vartheta_i \) and \( \varphi_i \) is in fact evidence that mirror symmetry of the two theories stems from T-duality on the phase of the charged chiral superfields \( \Phi_i \), whereby the neutral twisted chiral superfields \( Y_i \) are periodic, i.e., \( Y_i \equiv \vartheta_i + 2\pi i \)[7].

Furthermore, the Kähler metric of the target space of the mirror Landau-Ginzburg sigma model is given by

\[
d\hat{s}^2 = \frac{\hat{b}_1}{4\pi \theta} \sum_{i=1}^{N} \left( (d\vartheta_i)^2 + (d\varphi_i)^2 \right), \tag{4.102}
\]

\[\footnote{For details on why \( \vartheta_i \) ought to be periodic, see ([7], page 250).} \]
which is the flat cylinder metric on $(\mathbb{C}^\times)^N$. As in Section 3, taking the $\hat{e} \to \infty$ limit allows us to integrate $\hat{\Sigma}$ out of the action, and imposes the constraint
\[
\sum_{j}^{N} \hat{Q}_j Y_j - \hat{t} = 0, \tag{4.103}
\]
giving us the gauged Landau-Ginzburg theory with holomorphic twisted superpotential
\[
\tilde{W} = \sum_{c}^{N-1} \left( \sum_{j=1}^{N} \hat{Q}_{j,c} Y_j - \hat{\tau}_c \right) \tilde{\Sigma}_c + \sum_{j=1}^{N} e^{-Y_j}. \tag{4.104}
\]
We recall that the constraint (4.103) fixes the target space of the gauged Landau-Ginzburg theory to be the algebraic torus $(\mathbb{C}^\times)^{N-1}$.

The boundary conditions (4.98) imply that $e^{-y_i}$ have a common phase which is fixed. In other words, the boundaries of the worldsheet are mapped by $e^{-y_i}$ to a cycle $\gamma_{\hat{\theta}}$ in $(\mathbb{C}^\times)^{N-1}$ which has $N - 1$ real dimensions. This cycle is given by
\[
(e^{-y_1}, \ldots, e^{-y_N}) = (e^{-\varrho_1 + i\hat{\theta}/b_1}, \ldots, e^{-\varrho_N + i\hat{\theta}/b_1}), \tag{4.105}
\]
where $\varrho_i$ are constrained by $\sum_{i=1}^{N} \hat{Q}_i \varrho_i = \hat{\tau}$. In the continuum limit, the pure Neumann boundary condition we obtain for $\varphi_i$ from (4.91), implies the Neumann boundary condition
\[
\partial_1 \varrho_i = 0 \tag{4.106}
\]
for the coordinates $\varrho_i$ tangent to $\gamma_{\hat{\theta}}$. Using (4.101) and (4.91), we may also obtain boundary conditions on the fermionic dual fields, which are
\[
e^{-i\beta/2} \chi_+ + e^{i\beta/2} \chi_- = 0, \\
e^{i\beta/2} \chi_+ + e^{-i\beta/2} \chi_- = 0. \tag{4.107}
\]
These boundary conditions correspond to a D-brane wrapped on the cycle $\gamma_{\hat{\theta}}$.

The cycle $\gamma_{\hat{\theta}}$ is a Lagrangian submanifold of $(\mathbb{C}^\times)^{N-1}$. The A-brane wrapping this Lagrangian submanifold is the mirror of the space-filling B-brane supporting the holomorphic line bundle $\mathcal{O}_X(-n)$ with $U(1)^{N-1}$-equivariant structure, where $X$ is a Fano toric manifold of the form $\mathbb{C}^N/\mathbb{U}(1)$.

Let us investigate this A-brane further, by studying the image of the cycle $\gamma_{\hat{\theta}}$ in the $\tilde{W}$-plane. In particular, we would like to find the mirror of the $U(1)^{N-1}$-equivariant structure on the B-brane. The twisted superpotential (4.104) can be rewritten as
\[
\tilde{W} = \tilde{W}_{\text{equiv}} + \tilde{W}_X, \tag{4.108}
\]
where the first and second term of (4.104) correspond respectively to the first and second term of (4.108). The image of $\gamma_{\hat{\theta}}$ in the $\tilde{W}_X$-plane is
\[
\tilde{W}_X|_{\partial \Sigma} = e^{i\beta} \sum_{i=1}^{N} |e^{-y_i}|, \tag{4.109}
\]
which is the mirror condition found in [10] when studying the mirrors of B-branes without equivariant structure. In particular, it is a straight line which makes an angle $\beta = \hat{\theta}/\hat{b}_1$ with respect to the real axis.

Shifting our focus to the boundary value of $\tilde{W}_{\text{equiv}}$, we find that it is given by

$$\tilde{W}_{\text{equiv}}|_{\partial \Sigma} = \sum_{c}^{N-1} \text{Re}(e^{-i\beta \tilde{x}_c}) e^{i\beta} \left( \sum_{j}^{N} \tilde{Q}_{jc} \tilde{\theta}_j - \tilde{r}_c \right)$$

where we have used the boundary conditions $\theta_i = \hat{\theta} / \hat{b}_1 = \beta$ and $\text{Im}(e^{-i\beta \tilde{x}_c}) = 0$ as well as the identity $\beta = \tilde{\theta}_c / \sum_{i=1}^{N} \tilde{Q}_{ic}$. Here, $E_{q_c}$ is a complex-valued map

$$\text{Eq}: \gamma_{\hat{\theta}} \rightarrow u(1)^{N-1},$$

where $u(1)^{N-1}$ is the Lie algebra of $U(1)^{N-1}$. In particular, for a given value of $c$, $\gamma_{\hat{\theta}}$ is mapped to a straight line in the $E_{q_c}$-plane, which makes an angle $\beta$ with respect to the real axis. Thus, this map $\text{Eq}$ from the cycle $\gamma_{\hat{\theta}}$ (on which the A-brane is wrapped) to $u(1)^{N-1}$ is the mirror of $U(1)^{N-1}$-equivariant structure on the B-brane. In addition, we note that the boundary value of the total twisted superpotential is

$$\tilde{W}|_{\partial \Sigma} = e^{i\beta} \left( \sum_{c=1}^{N} e^{-\tilde{\theta}_c} + \sum_{c}^{N-1} \text{Re}(e^{-i\beta \tilde{x}_c}) \left( \sum_{j}^{N} \tilde{Q}_{jc} \tilde{\theta}_j - \tilde{r}_c \right) \right),$$

which is a map from $\text{Re}(e^{-i\beta \tilde{x}_c})$ and $\tilde{\theta}_c$ to a straight line in the $\tilde{W}$-plane which makes an angle $\beta$ with respect to the real axis. Since we have set $\hat{\theta} = 2\pi n$ earlier, and $\int_{X} c_1(\mathcal{O}_X(-n)) = -n$, the slope of this straight line depends on the first Chern class of the holomorphic line bundle $\mathcal{O}_X(-n)$ supported by the B-brane.

The mirrors of equivariant B-branes on Fano toric manifolds of the form $X = \mathbb{C}^N / / U(1)^{N-k}$ can similarly be found using the above method. These mirror A-branes correspond to Lagrangian submanifolds $\gamma_{\theta'}$ of the cylinder $(\mathbb{C}^N)^k$ which is defined by

$$\sum_{j}^{N} \tilde{Q}_{jc} Y_j - \tilde{r}_c = 0,$$

with the additional data of the superpotential

$$\tilde{W} = \sum_{c=1}^{k} \left( \sum_{j=1}^{N} \tilde{Q}_{jc} Y_j - \tilde{r}_c \right) \tilde{\Sigma}_c + \sum_{j=1}^{N} e^{-Y_j}.$$

The first term on the right hand side of (4.114), when restricted to its boundary value, contains the mirror data of equivariant structure on the holomorphic line bundle (which is supported by the space-filling B-brane), which is a map

$$\text{Eq}: \gamma_{\theta'} \rightarrow u(1)^k.$$
4.6 Nonabelian Equivariant B-branes

Although Kapustin et al. [14] introduced the nonabelian equivariant B-brane boundary Wilson line via a gauged B-model, the compatible boundary conditions were not derived explicitly. This motivates us to derive boundary conditions corresponding to nonabelian equivariant B-branes in our untwisted GNLSM. We shall use the insights obtained from studying abelian equivariant B-branes to find the complete description of nonabelian equivariant B-branes. This will be achieved by generalizing the first formulation studied in this section for abelian gauge groups (c.f. Sections 4.1-4.2) to nonabelian gauge groups. Note that the GNLSM notation of Section 2 is used in this subsection. For simplicity, we shall only consider the case where the $B$-field, $C$-field and $\theta$-parameter of the GNLSM given in (2.7) and (2.8) are zero.

We shall first investigate the boundary conditions required for B-type supersymmetry, before proceeding to discuss the admissible boundary action. Now, note that all the terms in $\delta(S_{\text{gauge}} + S_r)$ (equation (2.35)) vanish using the following boundary conditions

\begin{align*}
\text{Im}(\sigma_a) &= 0, \\
\lambda_+ + \lambda_- &= 0, \\
\partial_1 \text{Re}(\sigma_a) &= 0, \\
A_{1a} &= 0, \\
\partial_1 A_{0a} &= 0, \\
\partial_1 (\lambda_- - \lambda_+) &= 0, \\
\partial_1 (D_a + \partial_1 \text{Im}(\sigma_a)) &= 0.
\end{align*}

These conditions are a generalization of the conditions given in (4.35) for the example of $\mathbb{C}P^{N-1}$, except that the boundary condition for $F_{01a}$ is replaced by the stricter conditions $A_{1a} = 0$ and $\partial_1 A_{0a} = 0$, and the boundary condition for $\text{Re}(\sigma_a)$ becomes $\partial_1 \text{Re}(\sigma_a) = 0$. These stricter conditions are necessary since we now require that the boundary conditions preserve the locality of the relevant equations of motion when no additional boundary action is added, and because the supersymmetry transformations now contain nonabelian terms, which causes B-type supersymmetry invariance of the set of boundary conditions to not hold unless we use the stricter conditions on the gauge fields.\textsuperscript{38} The boundary conditions in fact imply that gauge transformations have to be restricted such that the transformation parameter $a^a$ has vanishing derivative with respect to $x^1$ at the boundaries, in order for these boundary conditions to be gauge invariant.

Next, we turn to the boundary conditions for the matter fields. Let us first consider the $\mathcal{N} = 1$ subalgebra of B-type supersymmetry, which corresponds to $\epsilon_+ = i\tilde{\epsilon}$, $\tilde{\tau}_+ = -i\tilde{\epsilon}$, $\epsilon_- = -i\tilde{\epsilon}$ and $\tilde{\tau}_- = i\tilde{\epsilon}$, where $\tilde{\epsilon}$ is a real parameter. In this case, after integrating out the

\textsuperscript{38}If we relax the requirement of locality of equations of motion, then the boundary conditions on $A_{0a}$ and $\text{Re}(\sigma_a)$ become $\partial_1 A_{0a} = \tau_a$ and $\partial_1 \text{Re}(\sigma_a) = \tau_a$, where $\tau$ is a constant valued in the centre of $g$. 

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auxiliary fields $F^a$ and $\overline{F}^a$, we find that (2.34) is

$$
\delta S_{\text{matter}} = -\frac{1}{2\pi} \overline{e} \int d\Sigma \left\{ g_{IJ} \partial^A \phi^I (\psi_J^L - \psi_J^L) + g_{IJ} \partial^A \phi^J (\psi_I^L + \psi_I^L) + g_{IJ} (\psi_I^L - \psi_J^L) \text{Re}(\sigma^a) e^a_I + \omega_{IJ} (\psi_I^L + \psi_J^L) \text{Im}(\sigma^a) e^a_I \right\},
$$

where $g_{I,J} X^I Y^J = g_{\sigma}(X^I Y^J + X^J Y^I)$ and $\omega_{I,J} X^I Y^J = ig_{\sigma}(X^I Y^J - X^J Y^I)$, and where $(I,J,K,\ldots)$ are indices corresponding to real coordinates on $X$. In addition, if we insist on locality of the matter equations of motion, we require that

$$
g_{IJ} \delta \phi^I \partial^A \phi^J = 0 \quad \text{(4.118)}
$$

at the boundaries, where $\delta \psi^J = \delta \psi^J + \Gamma^J_{KL} \delta \phi^K \psi^L$. An equivariant B-brane shall wrap a submanifold (denoted as $\gamma$) of $X$, to which a boundary of the worldsheet is mapped via $(\phi^I, \overline{\phi}^I)$. Now, any allowed variation of $\phi$ (denoted $\delta \phi^I$ for the real coordinate $\phi^I$) along the boundary, and the derivative along the boundary, $\partial_\phi \phi^I$, ought to be tangent to $\gamma$. The first constraint of (4.118) then implies that $\partial_\phi \phi^I$ is normal to $\gamma$, since $A_{1a} = 0$ at the boundaries. Then, taking into account the facts that $\text{Im}(\sigma_a) = 0$ and $A_{1a} = 0$ at the boundaries, we find that (4.117) if $\psi^I_+ - \psi^I_+$ and $\psi^I_+ + \psi^I_+$ are respectively normal and tangent to $\gamma$, and $e^a_i$ is tangent to $\gamma$, which implies that $\gamma$ is $G$-invariant. In addition, we note that $\psi^I_+ - \psi^I_+$ being normal to $\gamma$ and $\psi^I_+ + \psi^I_+$ being tangent to $\gamma$ implies that

$$
\begin{align*}
\psi^I_+ - \psi^I_+ &= 0, \quad I : \text{tangent to } \gamma, \\
\psi^I_+ + \psi^I_+ &= 0, \quad I : \text{normal to } \gamma,
\end{align*}
$$

(4.119)

(for a choice of coordinates which separates the normal and tangent directions) which satisfies the second constraint of (4.118).

Next, the $\mathcal{N} = (2,2)$ supersymmetry transformation of $\phi^I$ is

$$
\delta \phi^I = i(\epsilon_+ \psi^I_+ - \epsilon_+ J^J_K \psi^K_+ - \epsilon_- \psi^I_+ + \epsilon_- J^J_K \psi^K_+),
$$

where $\epsilon_+ = \epsilon_{+1} + i \epsilon_{+2}$ and $\epsilon_- = \epsilon_{-1} + i \epsilon_{-2}$, and where $J$ is the almost complex structure of $X$ locally given by $J^i_k = i \delta^i_k$ and $J^i_{\overline{k}} = -i \delta^i_{\overline{k}}$. B-type supersymmetry corresponds to $\epsilon_{+1} = - \epsilon_{-1}$ and $\epsilon_{+2} = - \epsilon_{-2}$, whereby

$$
\delta \phi^I = i(\epsilon_+ (\psi^I_+ + \psi^I_+) - \epsilon_+ J^J_K (\psi^K_+ + \psi^K_+)).
$$

Hence, $\psi^I_+ + \psi^I_+$ and $J^{J}_K (\psi^K_+ + \psi^K_+)$ are tangent to $\gamma$, which implies that the application of the almost complex structure, $J$, preserves the tangent space of $\gamma$. Therefore, $\gamma$ is a holomorphically embedded complex submanifold of $X$. This complex submanifold also happens to be $G$-invariant, which we know from the previous paragraph.
Indeed, (2.34) vanishes under this boundary condition; integrating out the auxiliary fields \( F^\alpha \) and \( \overline{F}^\alpha \), (2.34) can be rewritten (for \( \epsilon_+ = -\epsilon_- = \epsilon \)) as

\[
\delta S_{\text{matter}} = \frac{1}{2\pi} \frac{1}{4} \int_{\partial \Sigma} dx^0 \left\{ \epsilon \left( -g(\partial_0^4 \phi, \psi_- - \psi_+) - i\omega(\partial_0^4 \phi, \psi_- - \psi_+) \\
- g(\partial_1^4 \phi, \psi_- + \psi_+) - i\omega(\partial_1^4 \phi, \psi_- + \psi_+) \\
- \text{Re}(\sigma^a)g(\overline{e}_a, \psi_+ - \psi_-) - i\text{Re}(\sigma^a)\omega(\overline{e}_a, \psi_+ - \psi_-) \\
- i\text{Im}(\sigma^a)g(\overline{e}_a, \psi_+ + \psi_-) + \text{Im}(\sigma^a)\omega(\overline{e}_a, \psi_+ + \psi_-) \right) + c.c. \right\}
\]

(4.122)

(where \( g(X, Y) = g_{IJ} X^I Y^J \) and \( \omega(X, Y) = \omega_{IJ} X^I Y^J \), which vanishes using \( \text{Im}(\sigma_a) = 0 \) and \( A_{1a} = 0 \) as well as the conditions that \( \partial_0 \phi^I, \psi_-^I + \psi_+^I \) and \( \overline{e}_a^I \) are tangent to \( \gamma \) while \( \partial_1 \phi^I \) and \( \psi_-^I - \psi_+^I \) are normal to \( \gamma \)).

We may add the B-type supersymmetric boundary action

\[
S_{\partial \Sigma} = \int_{\partial \Sigma} dx^0 \left\{ A_m^X \partial_0^4 \phi^m + A_m^X \overline{\partial_0^4 \phi} + \overline{R}_a \left( \frac{\sigma^a + \overline{\sigma}^a}{2} \right) - \frac{i}{2} F_{mn}^X (\psi_+^m + \psi_-^m) (\psi_+^n + \psi_-^n) \right\}
\]

\[
= \int_{\partial \Sigma} dx^0 \left\{ A_m^X \partial_0^4 \phi^m + A_m^X \overline{\partial_0^4 \phi} - i\overline{R}_a A^a - \frac{i}{2} F_{mn}^X (\psi_+^m + \psi_-^m) (\psi_+^n + \psi_-^n) \right\},
\]

(4.123)

where we use \( (m, \overline{m}, n, \overline{n}) \) as coordinate indices on the B-branes, where the curvature of \( A^X \) satisfies \( F_{mn}^X = F_{nm}^X = 0 \), and where \( A_a = -i(A_{0a} - \frac{i(\sigma_a + \overline{\sigma}_a)}{2}) \) and

\[
\overline{R}_a = -A_m^X \overline{\sigma}_a^m - A_m^X \overline{\sigma}_a^m
\]

\[
= -i\overline{e}_a^I A^X.
\]

(4.124)

B-type supersymmetry invariance and gauge invariance of this action require the equivariant Bianchi identity

\[
d\overline{R} = \epsilon \overline{e} F^X,
\]

(4.125)

and this implies that each B-brane supports a \( G \)-equivariant holomorphic line bundle (c.f. footnote 25), for which \( \overline{R}_a \) is the moment.\(^{40}\) The inclusion of this boundary action results in some of the equations of motion being modified by boundary terms. One may generalize this even further (at least in the twisted case), as shown by Kapustin et al. \cite{14} (c.f. Section 4.1), by instead including a Wilson line which represents a \( G \)-equivariant graded holomorphic vector bundle.

In conclusion, we find that in general,

**Equiavariant B-branes are \( G \)-invariant holomorphically embedded complex submanifolds of \( X \), which support \( G \)-equivariant holomorphic vector bundles (which may be graded).**

As discussed in Section 4.1, at least in some cases, this implies that they are objects in the bounded, derived category of \( G_\mathbb{C} \)-equivariant coherent sheaves on \( X \).

\(^{39}\) Recall that for a tangent vector, \( T \), and normal vector, \( N \), of a holomorphically embedded complex submanifold, \( \gamma \), of the Kähler manifold \( X \), we have \( \omega(T, N) = g(JT, N) = 0 \).

\(^{40}\) Note that gauge invariance of the boundary action requires the use of the identity \( \alpha^b \mathcal{L}_{\overline{e}_a} \overline{R}_a = [\alpha, \overline{R}_a] \).
5 Equivariant A-branes and their Mirrors

In this section, we study the A-type supersymmetric boundary actions and boundary conditions in abelian GNLSMs on $I \times \mathbb{R}$ with toric target spaces, $X$, as well as their mirror descriptions. These boundary actions and boundary conditions correspond to equivariant A-branes wrapping submanifolds of $X$. Then, with the insights we find from analyzing these abelian equivariant A-branes, we shall proceed to study equivariant A-branes for nonabelian GNLSMs.

A-type supersymmetry is defined by the combination of supercharges given by (2.39). In what follows, we shall set $\beta = 0$ for simplicity, though it is straightforward to study the $\beta \neq 0$ generalization using the same techniques. In other words, we assume that the supercharges conserved at the boundaries are

$$Q_A = Q_+ - Q_-,$$

$$Q_A^\dagger = Q_+ + Q_-.$$  \hspace{1cm} (5.1)

From (2.33), it can be seen that the corresponding relations among the supersymmetry transformation parameters are

$$\epsilon = \epsilon_+ = \epsilon_-,$$
$$\bar{\epsilon} = \epsilon_+ = \epsilon_-.$$ \hspace{1cm} (5.2)

We shall also make use of superfields when discussing boundary conditions and boundary actions, and to this end, we shall make use of the concept of ‘boundaries’ in superspace [11]. For A-type supersymmetry, the relevant boundary in superspace is known as ‘A-boundary’, and corresponds to

$$\theta = \theta^+ = -\bar{\theta}^-,$$
$$\bar{\theta} = \bar{\theta}^+ = -\theta^-.$$ \hspace{1cm} (5.3)

Let us first review what is known of ordinary A-branes. For $\mathcal{N} = (2,2)$ NLSMs, the boundary condition needed to preserve A-type supersymmetry at the boundaries maps each boundary to a middle-dimensional Lagrangian submanifold of the target space $X$. With target space coordinates ($X_I$) chosen appropriately, this is expressed as Dirichlet boundary conditions on half of the fields $X_I$ with Neumann boundary conditions for the rest. Here, we have used real coordinates for the Kähler target space $X$. In addition, we may include the following boundary action

$$S_{\partial \Sigma} = \int_{\partial \Sigma} dx^0 A^X_M \partial_0 X^M = \int_{x^1 = 0} dx^0 \partial_0 \phi^M A^{X(a)}_M - \int_{x^1 = \pi} dx^0 \partial_0 \phi^M A^{X(b)}_M,$$ \hspace{1cm} (5.4)

where $A^{X(a)}_M$ and $A^{X(b)}_M$ are the connections of $U(1)$ line bundles on the A-branes $\gamma_a$ and $\gamma_b$ on which the boundaries $x_1 = 0$ and $x_1 = \pi$ end (we shall use $(M,N,\ldots)$ as coordinate indices on the Lagrangian submanifold branes). This boundary action is A-type supersymmetric if $F^{X(a)}_{MN} = \partial_M A^{X(a)}_N - \partial_N A^{X(a)}_M = 0$. This condition on the curvature of each $U(1)$ bundle indicates that it is flat. This boundary action takes the form of a Wilson line, and we see that since we have two boundary components, we actually have two different Wilson
lines along each boundary, corresponding to two different A-branes supporting flat $U(1)$ bundles, each with different connections \[10\].

We are interested in the generalizations of Lagrangian boundary conditions and the boundary action (5.4) for GNLSMs. A possible method of obtaining such a generalization would be to replace ordinary worldsheet derivatives by covariant ones, and to attempt to maintain supersymmetry and gauge symmetry by adding additional terms, if necessary. However, it is known that the boundary conditions and the boundary action (5.4) can be obtained from a GLSM boundary action \[11\]. We are then led to attempt the more elegant method of obtaining the GNLSM boundary conditions and boundary action from a GLSM boundary action, using the methods of Section 3. In the following, we shall attempt to generalize the NLSM Lagrangian boundary conditions and boundary action (5.4) to the case of $U(1)^k$-GNLSMs with Kähler toric target space.

Let us elaborate further on how NLSM boundary data is obtained from GLSM boundary data. The essential idea of \[11\] is to obtain Lagrangian boundary conditions and the boundary action (5.4) from a GLSM using boundary conditions given by

\[
\Phi_i e^{\hat{V}} \Phi_i = c_i \tag{5.5}
\]

which includes the Dirichlet boundary condition

\[
|\phi_i|^2 = c_i \tag{5.6}
\]

and Neumann boundary conditions on $\varphi_i$ (where $\varphi_i$ is defined by $\phi_i = |\phi_i| e^{i \varphi_i}$), as well as the boundary Wilson line

\[
S_a = N \sum_i \int_{\partial \Sigma} \frac{a_i}{2\pi} \partial_0 \varphi_i \, dx^0, \tag{5.7}
\]

(where $a_i$ is a constant).\[41\] These indicate that at the GLSM level, the A-brane is wrapped on a torus parametrized by the $\varphi_i$’s, which is a Lagrangian submanifold of $\mathbb{C}^N$, and where $c_i$ measures the size of the circle parametrized by $\varphi_i$, while $a_i$ parametrizes the holonomy of the $U(1)$ connection on the D-brane. However, instead of imposing the boundary condition (5.6) and its A-type supersymmetric completions at the GLSM level before taking the NLSM ($\hat{\epsilon} \to \infty$) limit, one uses ‘boundary superfields’ \[11\], whereby no boundary conditions are imposed by hand at the GLSM level, but rather they are understood as being derived through boundary interactions involving these boundary superfields. The advantage of this formulation is that the geometric parameters of the NLSM D-brane enter a ‘boundary F-term’, and this aids our understanding of quantum corrections \[11\].

We shall follow this method in deriving the GNLSM generalization of the boundary conditions and the boundary action (5.4). To this end, let us first briefly review the concept of boundary superfields \[11\], in particular, those living in A-boundary superspace. The coordinates of A-boundary superspace are $x^0, \theta, \overline{\theta}$, and boundary superfields are simply functions of these coordinates, and transform under A-type supersymmetry. Boundary superfields can be of both bosonic and fermionic nature.

\[41\]Although $a_i$ is a constant, Stoke’s theorem does not result in the vanishing of (5.7), since $\varphi_i$ is a periodic, multi-valued function \[11\].
The important differential operators on A-boundary superspace are $\partial_0$ and
\[
D = D_+ + D_- = \frac{\partial}{\partial \theta} - i \theta \partial_0, \\
\overline{Q} = \overline{Q}_+ + Q_- = -\frac{\partial}{\partial \theta} - i \theta \partial_0, \\
Q = Q_+ + \overline{Q}_- = \frac{\partial}{\partial \theta} + i \theta \partial_0.
\]

(5.8)

Boundary superfields in A-boundary superspace transform under A-type supersymmetry by $\delta = \epsilon \overline{Q} - \epsilon Q$. Bulk superfields restricted to A-boundary are boundary superfields. Furthermore, boundary chiral superfields obey
\[
\overline{D} \Phi = 0,
\]

(5.9)

and are expanded as
\[
\Phi = \phi(x^0) + \theta \psi(x^0) - i \theta \partial_0 \phi(x^0).
\]

(5.10)

Following the convention of [11], fermionic boundary chiral superfields shall be referred to as boundary Fermi superfields. The integral
\[
\int dx^0 d\theta \Phi \overline{J}(F_i),
\]

(5.11)

over a function $J(F_i)$ of boundary superfields $F_i$ is invariant under A-type supersymmetry transformations. In addition, the integral
\[
\int dx^0 d\theta \Psi (\Phi_i) \bigg|_{\theta=0},
\]

(5.12)

where $\Psi$ is a boundary Fermi superfield and $V(\Phi_i)$ is a holomorphic function of boundary chiral superfields $\Phi_i$, is also invariant under A-type supersymmetry. Expressions of the form (5.11) and (5.12) are known as boundary D-terms and boundary F-terms respectively.

Finally, we recall that only the axial R-symmetry of the bulk is preserved at the boundaries, since vector R-rotations do not leave the A-boundary (defined by (5.3)) invariant.

**A-branes on $\mathbb{C}^N // U(1)$ from GLSM**

We shall first recall from [11] how the boundary conditions and the boundary action (5.4) for an NLSM with $\mathbb{C}^N // U(1)$ target space can be obtained from an A-type supersymmetric boundary action of a $U(1)$-GLSM with $\mathbb{C}^N$ target. The boundary action consists of two parts
\[
S_1 = \frac{1}{2\pi} \int d\Sigma \left[ \frac{1}{2} \sum_i |\partial_i |\phi_i|^2 + \frac{i}{2} \sum_i (\overline{F}_i \phi_i - \overline{\phi}_i F_i) + \frac{i}{4e^2} (\lambda_+ \partial_0 - \lambda_0 \partial_+ - \overline{\lambda}_+ \overline{\lambda}_0 + \overline{\lambda}_0 \overline{\lambda}_+) \right]
\]

(5.13)
and
\[
S_2 = \frac{1}{2\pi} \sum_i N \int dx_0 \left[ \int d\theta d\bar{\theta} e^{V} \Phi_i(U_i - \text{Im} \log \Phi_i) + \text{Re} \int d\theta s_i \Upsilon_i \right],
\tag{5.14}
\]
where \(U_i\) is a real, bosonic, boundary auxiliary superfield expanded as
\[
U_i = u_i + \theta \overline{\chi}_i - \overline{\theta} \chi_i + \theta \overline{\theta} E_i
\tag{5.15}
\]
(with the lowest component \(u_i\) being a periodic (multivalued) scalar field defined on the boundaries), \(\Upsilon_i = \overline{D} U_i\) is the ‘field-strength’ of \(U_i\), expanded as
\[
\Upsilon_i := \overline{D} U_i = \chi_i + \theta (E_i + i \partial_0 u_i) - i \overline{\theta} \partial_0 \chi_i,
\tag{5.16}
\]
and is a boundary Fermi superfield satisfying \(\overline{D} \Upsilon_i = 0\), while the parameter
\[
s_i = c_i - i a_i
\tag{5.17}
\]
is the boundary analogue of the complex FI-theta parameter \(\hat{t}\). It is important to note that although both \(u_i\) and \(\varphi_i\) are periodic, multi-valued functions, the presence of the term \(\int d\theta d\overline{\theta} e^{V} \Phi_i(U_i - \varphi_i)\) in (5.14) requires that \(u_i - \varphi_i\) is single-valued.

The first part (5.13), together with the bulk GLSM action, \(S\), have the simple supersymmetry transformation
\[
\delta(S + S_1) = \frac{\hat{\tau}}{4\pi} \int dx^0 \left\{ \epsilon (\overline{\lambda}_+ + \lambda_-) - \overline{\epsilon} (\overline{\lambda}_- + \lambda_+) \right\}.
\tag{5.18}
\]

The second part, (5.14), includes interactions which effectively impose boundary conditions on the components of the chiral multiplets. Let us recall how it transforms under supersymmetry. Gauge invariance of the first term in (5.14), requires that \(U_i\) transforms under gauge transformations as
\[
U_i \rightarrow U_i + \frac{\hat{Q}_i}{2} (A + \overline{A}),
\tag{5.19}
\]
in order to cancel the gauge variation of \(\text{Im} \log \Phi_i\). This implies the following modification of the supersymmetry transformations of the components of \(U_i\) in order to preserve the Wess-Zumino gauge
\[
\delta u_i = \epsilon \chi_i - \overline{\epsilon} \overline{\chi}_i,
\delta \chi_i = -\overline{\epsilon} (E_i + i (\partial_0 u_i + \hat{Q}_i \hat{A}_0)) - i \epsilon \hat{Q}_i \hat{\sigma},
\delta \overline{\chi}_i = -\epsilon (E_i - i (\partial_0 u_i + \hat{Q}_i \hat{A}_0)) + i \epsilon \hat{Q}_i \overline{\sigma},
\delta E_i = i \epsilon \partial_0 \chi_i + \overline{\epsilon} \partial_0 \overline{\chi}_i - \frac{1}{2} \epsilon \hat{Q}_i (\overline{\lambda}_- + \lambda_+) + \frac{1}{2} \overline{\epsilon} \hat{Q}_i (\overline{\lambda}_+ + \lambda_-).
\tag{5.20}
\]
Under these supersymmetry transformations, the boundary superpotential term
\[
\frac{1}{2\pi} \sum_i N \int dx_0 \text{Re} \int d\theta s_i \Upsilon_i = \frac{1}{2\pi} \sum_i N \int dx_0 \left( c_i E_i + a_i \partial_0 u_i \right)
\tag{5.21}
\]
is not invariant,\textsuperscript{42} but rather varies as
\[
\delta \left[ \frac{1}{2\pi} \sum_i^N \int_{\partial \Sigma} dx^0 \Re \int d\theta \, s_i \Upsilon_i \right] = -\frac{\sum_i^N \hat{Q}_i c_i}{4\pi} \int_{\partial \Sigma} dx^0 \left\{ \epsilon(\bar{\lambda}_+ + \bar{\lambda}_-) - \bar{\epsilon}(\bar{\lambda}_- + \bar{\lambda}_+) \right\}.
\]
\begin{equation}
(5.22)
\end{equation}

Supersymmetry invariance of the entire action then requires that (5.18) and (5.22) cancel, which is possible if and only if
\[
\sum_i^N \hat{Q}_i c_i = \hat{r}.
\]
\begin{equation}
(5.23)
\end{equation}

Similarly, the first part (5.13) of the boundary action is not gauge invariant, but varies as
\[
\delta S_1 = \frac{1}{2\pi} \int_{\partial \Sigma} dx^0 \hat{\theta}(-\partial_0 \alpha),
\]
\begin{equation}
(5.24)
\end{equation}

while (5.21) varies under gauge transformations as
\[
\delta \left[ \frac{1}{2\pi} \sum_i^N \int_{\partial \Sigma} dx^0 \Re \int d\theta \, s_i \Upsilon_i \right] = \frac{\sum_i^N \hat{Q}_i a_i}{2\pi} \int_{\partial \Sigma} dx^0 \partial_0 \alpha,
\]
\begin{equation}
(5.25)
\end{equation}

since the residual gauge transformation \( A = \alpha(x) \) of the Wess-Zumino gauge shifts \( u_i \rightarrow u_i + \hat{Q}_i \alpha \), while leaving \( \mathcal{X}_i \) and \( E_i \) invariant. Thus, gauge invariance of the boundary action follows if\textsuperscript{43}
\[
\sum_i^N \hat{Q}_i a_i = \hat{\theta}.
\]
\begin{equation}
(5.26)
\end{equation}

Combining (5.23) and (5.26), we find that we need
\[
\sum_i^N \hat{Q}_i s_i = \hat{t}
\]
\begin{equation}
(5.27)
\end{equation}

for gauge invariance and A-type supersymmetry invariance of the action.

\textsuperscript{42}The reason for this nonzero variation is that the boundary Fermi superfield \( \Upsilon \) is not invariant under the gauge transformation (5.19).

\textsuperscript{43}To be precise, (5.26) only needs to hold up to the additional term \( 2\pi m \), where \( m \in \mathbb{Z} \), since the path integral remains gauge invariant in such cases \cite{11}. However, we shall set \( m = 0 \) in the following for simplicity.
To analyze the $\epsilon \to \infty$ limit, it is advantageous to write the boundary action explicitly\textsuperscript{44}

$$S_{\partial \Sigma} = S_1 + S_2$$

$$= \frac{1}{2\pi} \int_{\partial \Sigma} dx^0 \left( \frac{i}{4e^2} (\bar{\lambda}_- \dot{\lambda}_+ - \bar{\lambda}_+ \dot{\lambda}_-) + \dot{\theta} \dot{A}_0 \right)$$

$$+ \frac{1}{2\pi} \sum_i \int_{\partial \Sigma} dx^0 \left( (i \bar{\phi}_i \overset{\rightarrow}{D}_1 \phi_i + \bar{\psi}_+ i \psi_{+i} - \bar{\psi}_{-i} \psi_{-i} + \bar{F}_i \phi_i + \bar{\phi}_i F_i) u'_i \right)$$

\begin{equation}
(5.28)
\end{equation}

$$+ (\bar{\phi}_i \psi_{-i} + \bar{\psi}_{+i} \phi_i) \pi_i + \dot{X}_i (\bar{\psi}_{-i} \phi_i + \bar{\phi}_i \psi_{+i}) - i \frac{3}{2} \bar{\phi}_i \psi_{+i} \psi_{-i} + i \frac{3}{2} \bar{\phi}_i \bar{\psi}_{-i} \bar{\psi}_{+i}$$

$$- (|\phi_i|^2 - c_i) E_i + a_i \partial_0 u_i \right)$$

where $u'_i = u_i - \varphi_i$, and where the covariant derivative of the scalar field is $\overset{\rightarrow}{D}_1 \phi_i = \partial_0 \phi_i + i \overset{\rightarrow}{Q}_i \overset{\rightarrow}{A}_1 \phi_i$. Firstly, we note that Stoke’s theorem implies

$$\frac{a_i}{2\pi} \int_{\partial \Sigma} \partial_0 u_i dx^0 = \frac{a_i}{2\pi} \int_{\partial \Sigma} \left\{ \partial_0 \bar{\varphi}_i + \partial_0 (u_i - \varphi_i) \right\} dx^0 = \frac{a_i}{2\pi} \int_{\partial \Sigma} \partial_0 \bar{\varphi}_i dx^0,$$

\begin{equation}
(5.29)
\end{equation}

since $u_i - \varphi_i$ is single-valued, and we find that one of the terms in the boundary action is the expression (5.7). Then, taking $\epsilon \to \infty$, and integrating out the boundary auxiliary superfields, we are left with

$$\frac{1}{2\pi} \int_{\partial \Sigma} dx^0 \left( \sum_i^N a_i \partial_0 \bar{\varphi}_i + \dot{\theta} \dot{A}_0 \right),$$

\begin{equation}
(5.30)
\end{equation}

with the boundary conditions defined by (5.5) imposed at the boundaries. These boundary conditions are given explicitly as

$$|\phi_i|^2 = c_i,$$

$$\bar{\phi}_i \psi_{-i} + \bar{\psi}_{+i} \phi_i = 0,$$

$$\bar{\psi}_{-i} \phi_i + \bar{\phi}_i \psi_{+i} = 0,$$

\begin{equation}
(5.31)
\end{equation}

Integrating the vector multiplet out of the bulk action sets

$$\dot{A}_0 = \frac{1}{2} \sum_{i=1}^N \overset{\rightarrow}{Q}_i (i \bar{\phi}_i \overset{\rightarrow}{\partial}_0 \phi_i + \bar{\psi}_{-i} \phi_{-i} + \bar{\psi}_{+i} \phi_{+i}) = - \sum_{i=1}^N \overset{\rightarrow}{Q}_i c_i \bar{\partial}_0 \varphi_i,$$

\begin{equation}
(5.32)
\end{equation}

at the boundaries, where (5.31) has been used in the last step.\textsuperscript{45} Thus, the final boundary action is

$$S_{\partial \Sigma} = \frac{1}{2\pi} \int_{\partial \Sigma} \left[ \sum_{i=1}^N a_i d\bar{\varphi}_i - \dot{\theta} \sum_{i=1}^N \overset{\rightarrow}{Q}_i c_i d\varphi_i \right].$$

\begin{equation}
(5.33)
\end{equation}

\textsuperscript{44}Note that some terms in (5.13) cancel terms in (5.14).

\textsuperscript{45}The presence of the boundary term proportional to $\dot{\theta} \dot{A}_0$ ensures that the algebraic equation of motion for $\dot{A}_0$ does not contain a boundary term, see footnote 13.
It will be useful for us to analyze Hori’s results (5.31) and (5.33) for $X = \mathbb{C}P^{N-1}$, which corresponds to $\hat{Q}_i = 1$. Firstly, the inhomogeneous coordinates (3.25) which parametrize a local patch of $\mathbb{C}P^{N-1}$ can be written as

$$|Z^i|^2 = \frac{c_i}{c_N}$$

(5.34)

In other words, the argument of $Z^i$ is

$$\gamma^i = \varphi_i - \varphi_N.$$ (5.35)

The A-type supersymmetric boundary conditions of the NLSM which can be obtained from (5.31) using the parametrizations (3.24) and (3.26) are\textsuperscript{46}

$$\text{c}\left(\overline{Z}^i \psi Z^i + Z^i \overline{\psi}_+ = 0 \right)$$

(5.36)

$$\text{c}\left(\overline{Z}^i \psi Z^i + Z^i \overline{\psi}_- = 0 \right)$$

$$\text{i}\left(\overline{Z}^i \partial_1 Z^i - Z^i \partial_1 \overline{Z}^i \right) + \overline{\psi}_+ \psi Z^i - \overline{\psi}_- \psi Z^i + F^i Z^i = 0,$$

where the last condition is in fact a Neumann boundary condition on $\gamma^i$, since $\overline{Z}^i \partial_1 Z^i - Z^i \partial_1 \overline{Z}^i = 2|Z^i|^2 i\partial_1 \gamma^i$. Thus, the Neumann boundary condition on $\gamma^i$ and Dirichlet boundary condition on $|Z^i|$ in (5.36) implies that the A-brane wraps a torus $T^{N-1}$ parametrized by $\gamma^i$.\textsuperscript{47} Moreover, this torus is a Lagrangian submanifold of $\mathbb{C}P^{N-1}$ with respect to the Fubini-Study Kähler form given by (4.26).

We can rewrite the boundary action (5.33) with the help of (5.35) as

$$S_{\partial \Sigma} = \frac{1}{2\pi} \int_{\partial \Sigma} \sum_{i=1}^{N} \left[ a_i - \hat{\theta} \frac{c_i}{\sum_{j=1}^{N} c_j} \right] d\varphi_i$$

$$= \frac{1}{2\pi} \int_{\partial \Sigma} \left[ \sum_{i=1}^{N-1} \left( a_i - \left( \sum_{k=1}^{N} a_k \right) \frac{c_i}{\sum_{j=1}^{N} c_j} \right) d\varphi_i + \left( a_N - \left( \sum_{k=1}^{N} a_k \right) \frac{c_N}{\sum_{j=1}^{N} c_j} \right) d\varphi_N \right]$$

(5.37)

$$= \frac{1}{2\pi} \int_{\partial \Sigma} \left[ \sum_{i=1}^{N-1} \left( a_i - \left( \sum_{k=1}^{N} a_k \right) \frac{c_i}{\sum_{j=1}^{N} c_j} \right) d\left( \varphi_i - \varphi_N \right) \right]$$

$$= \int_{\partial \Sigma} \sum_{i=1}^{N-1} A^X d\gamma^i,$$

\textsuperscript{46}To be precise, the last condition of (5.31) is actually trivialized using the algebraic equation of motion of $\hat{A}_1$. The last condition of (5.36) is in fact obtained via A-type supersymmetry transformations of the fermionic boundary conditions.

\textsuperscript{47}Here, both boundaries are mapped to the same A-brane. If the boundaries are assigned unique parameters $s^i_1$ and $s^i_0$ in (5.14), then each boundary is mapped to a different A-brane. However, for simplicity, in most of what follows in this section, we shall assume that both boundaries are assigned the same parameter $s_i$. 

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where we have also used (5.26), and defined the constant

\[ A^X_i = \frac{1}{2\pi} \left( a_i - \sum_{k=1}^{N} a_k \frac{c_i}{\sum_{j=1}^{N} c_j} \right), \tag{5.38} \]

which is understood to be the connection of a flat $U(1)$ bundle on the Lagrangian torus $T^{N-1}$ parametrized by $\gamma_i$. In other words, the A-brane, defined by the boundary conditions (5.36) and boundary action (5.37), wraps a Lagrangian submanifold of $X = CP^{N-1}$ and supports a flat $U(1)$ bundle. This corresponds to the data of an object in the Fukaya category of $CP^{N-1}$ [25]. Supersymmetry invariance of the boundary action (5.37) follows since the fermionic superpartners of $\gamma^i$ are not periodic nor multivalued ([7], page 307), hence, the supersymmetry variation of (5.37) vanishes via Stoke’s theorem.

We have thus derived the boundary action (5.4) and the pertinent boundary conditions for $X = CP^{N-1}$ from a GLSM. Similarly, for other toric manifolds $X = C^N//U(1)$, we may use the same techniques shown above to find that the A-brane wraps a torus $T^{N-1}$ which is a Lagrangian submanifold of $X$, and supports a flat $U(1)$ bundle [11].

### 5.1 Equivariant A-branes on $C^N//U(1)$ from GLSM

We shall now proceed to obtain the A-type supersymmetric boundary conditions and boundary action for abelian GNLSMs with toric target spaces of the form $C^N//U(1)$. In order to do so, we must generalize the $U(1)$-GLSM boundary action consisting of (5.13) and (5.14) to a boundary action for the $U(1)^N$-GLSM given in (3.12), with $N - k = 1$.

The first step would be the obvious generalization of the terms with vector multiplet fields, i.e., from $U(1)$ to $U(1)^N$. Next, we note that in obtaining a $U(1)^N$-GNLSM from the $U(1)^{N-1}$-GLSM, we do not integrate out all vector multiplets, unlike in the procedure of obtaining the NLSM. However, the boundary action (5.14) only imposes boundary conditions on the matter fields in the $\hat{e} \to \infty$ limit. This implies that we ought to include additional boundary interactions at the GLSM level, which impose boundary conditions on the remaining vector multiplet fields in the $\hat{e} \to \infty$ limit. We claim that the $U(1)^N$-GLSM boundary action consists of

\[
S_1 = \frac{1}{2\pi} \int_{\partial \Sigma} d\sigma^0 \left[ \frac{1}{2} \sum_i \partial_1 |\phi_i|^2 + \frac{i}{2} \sum_{i,j} (\Phi^*_i \phi_j - \phi_i \Phi^*_j) + \sum_a \frac{i}{4e_a^2} (\lambda_a \lambda_a - \bar{\lambda}_a \bar{\lambda}_a) + \sum_a \theta_a A_{0a} \right]
\tag{5.39}
\]

and

\[
S_2 = \frac{1}{2\pi} \sum_i \int_{\partial \Sigma} d\sigma^0 \left[ \int d\theta d\bar{\theta} \text{Tr} \Sigma^a \Phi^*_i \Sigma^a \Phi_i (U_i - \text{Im} \log \Phi_i) + \text{Re} \int d\theta s_i \Upsilon_i 
\right.
+ \sum_a \frac{1}{2e_a^2} \int d\theta d\bar{\theta} \text{Re} \left[ \Xi_a (D_+ \Sigma_a - \bar{D}_- \Sigma_a) \right] \right],
\tag{5.40}
\]

\[ - 55 - \]
where we have introduced an A-type supersymmetry invariant boundary D-term for the vector superfields, which contains the complex boundary Fermi superfields

\[ \Xi_a = \xi_a + \theta G_a + \theta H_a + \theta \theta \kappa_a \]
\[ \Xi_a = \bar{\xi}_a + \theta \bar{G}_a + \theta \bar{H}_a + \theta \theta \bar{\kappa}_a, \]

(5.41)

where \( \xi_a \) and \( \kappa_a \) are fermionic auxiliary fields while \( G_a \) and \( H_a \) are bosonic auxiliary fields, all defined along the boundaries. The A-type supersymmetry transformations of these fields may be found using the differential operator \( \delta = \epsilon \bar{Q} - \bar{\epsilon} Q \) defined in (5.8) on the superfields \( \Xi_a \) and \( \Xi_a \). In addition, they are defined to be invariant under gauge transformations. The form of (5.40) is chosen such that the boundary conditions

\[ \Phi_i e^{\sum_a Q_a V_a} \Phi_i = c_i \]
\[ D_+ \Sigma_a = \overline{D}_- \Sigma_a \]
\[ D_+ \Sigma_a = \overline{D}_- \Sigma_a \]

(5.42)

are effectively imposed via boundary interactions. In components, these are

\[ |\phi_i|^2 = c_i, \]
\[ \bar{\phi}_i \psi_{-i} + \bar{\psi}_{+i} \phi_i = 0, \]
\[ \bar{\psi}_{-i} \phi_i + \bar{\phi}_i \psi_{+i} = 0, \]
\[ \bar{\phi}_i \overline{D}_1 \phi_i + \bar{\psi}_{+i} \psi_{-i} - \bar{\psi}_{-i} \psi_{+i} + \bar{F}_i \phi_i + \bar{\phi}_i F_i = 0 \]

(5.43)

and

\[ \lambda_{+a} - \bar{\lambda}_{-a} = 0, \]
\[ \partial_1 \sigma_a = 0, \]
\[ F_{01a} = 0, \]
\[ \Sigma_a = 0, \]
\[ \partial_1 (\lambda_{+a} + \bar{\lambda}_{-a}) = 0, \]

(5.44)

and the complex conjugates of the conditions in (5.44).

The supersymmetry transformation of the bulk GLSM action together with (5.39) is

\[ \delta (S + S_1) = \sum_a^N \frac{r_a}{4\pi} \int d\Sigma^0 \left\{ \epsilon (\lambda_{+a} + \lambda_{-a}) - \bar{\epsilon} (\bar{\lambda}_{-a} + \lambda_{+a}) \right\}. \]

(5.45)

Now, \( U(1)^N \) gauge invariance of the first term in (5.40), requires that \( U_i \) transforms under \( U(1)^N \) gauge transformations as

\[ U_i \rightarrow U_i + \sum_a^N \frac{Q_{ia}}{2} (A_a + \bar{A}_a), \]

(5.46)

in order to cancel the gauge variation of Im log \( \Phi_i \). This implies the following modification of the supersymmetry transformations of the components of \( U_i \) in order to preserve the
Wess-Zumino gauge
\[ \delta u_i = \epsilon X_i - \bar{\epsilon} X_i, \]
\[ \delta X_i = -\epsilon (E_i + i(\partial_0 u_i + \sum_a N Q_{ia} A_{0a})) + i\epsilon \sum_a N Q_{ia} \sigma_a, \]
\[ \delta E_i = i\epsilon \partial_0 X_i + \bar{\epsilon} \partial_0 X_i - \frac{1}{2} \epsilon \sum_a N Q_{ia} (\lambda_{-a} + \lambda_{+a}) + \frac{1}{2} \frac{1}{2} \epsilon \sum_a N Q_{ia} (\lambda_{-a} + \lambda_{+a}). \] (5.47)

The boundary superpotential term in (5.40) is not invariant under supersymmetry,\(^{48}\) but rather varies as
\[ \delta \left[ \frac{1}{2\pi} \sum_i \int d\sigma^0 \sum_i \int d\theta s_i \Upsilon_i \right] = -\sum_a N \sum_i N Q_{ia} \frac{\theta_a(-\partial_0 \alpha_a)}{4\pi} \int d\sigma^0 \frac{1}{\partial \Sigma} \{ \epsilon (\lambda_{+a} + \lambda_{-a}) - \bar{\epsilon} (\lambda_{-a} + \lambda_{+a}) \}. \] (5.48)

Hence, supersymmetry invariance of the entire action requires that (5.45) and (5.48) cancel, which is possible if and only if
\[ \sum_i N Q_{ia} \theta_i = r_a. \] (5.49)

Likewise, the first part (5.39) of the boundary action is not \(U(1)^N\)-gauge invariant, but varies as
\[ \delta S_1 = \frac{1}{2\pi} \sum_i \int d\sigma^0 \sum_a \theta_a(-\partial_0 \alpha_a), \] (5.50)
while the boundary superpotential term varies under gauge transformations as
\[ \delta \left[ \frac{1}{2\pi} \sum_i \int d\sigma^0 \sum_i \int d\theta s_i \Upsilon_i \right] = \sum_a N \sum_i N Q_{ia} \frac{\theta_a(-\partial_0 \alpha_a)}{2\pi} \int d\sigma^0 \frac{1}{\partial \Sigma} \alpha_a, \] (5.51)
since the residual gauge transformation \(A_a = \alpha_a(x)\) of the Wess-Zumino gauge shifts \(u_i \rightarrow u_i + \sum_a N Q_{ia} \alpha_a\), while leaving \(X_i\) and \(E_i\) invariant. Therefore, gauge invariance of the boundary action follows if\(^{49}\)
\[ \sum_i N Q_{ia} a_i = \theta_a. \] (5.52)

Combining (5.49) and (5.52), we find that we need
\[ \sum_i N Q_{ia} s_i = t_a \] (5.53)

\(^{48}\)This nonzero variation occurs because the boundary Fermi superfield \(\Upsilon\) is not invariant under the gauge transformation (5.46).

\(^{49}\)As noted in footnote 43, (5.52) only needs to hold up to the additional term \(2\pi m\), but we shall set \(m = 0\) in the following for simplicity.
for gauge invariance and A-type supersymmetry invariance of the action. Expanding the boundary action in components, we have

\[ S_{\partial \Sigma} = S_1 + S_2 \]

\[ = \frac{1}{2\pi} \int dx^0 \left( \sum_{a}^{N} \frac{i}{4e_a^2} (\lambda_{-a} \lambda_{-a} - \overline{\lambda}_{-a} \overline{\lambda}_{-a}) + \sum_{a}^{N} \theta_a A_{0a} \right) \]

\[ + \frac{1}{2\pi} \sum_{i}^{N} \int dx^0 \left( (i\phi_i \overline{D}_1 \phi_i + \overline{\psi}_{+i} \psi_{+i} - \overline{\psi}_{-i} \psi_{-i} + \overline{F}_i \phi_i + \overline{\phi}_i F_i)u_i' \right. \]

\[ + (\overline{\phi}_i \psi_{-i} + \overline{\psi}_{+i} \phi_i) \xi_a + \lambda_a (\overline{\psi}_{-i} \phi_i + \overline{\phi}_i \psi_{+i}) - i \frac{3}{2} \left( \overline{\phi}_i \psi_{+i} \psi_{-i} + i \frac{3}{2} \overline{\phi}_i \overline{\psi}_{-i} \overline{\psi}_{+i} \right) \]

\[ - (|\phi_i|^2 - c_i) E_i + a_i \partial_0 u_i \right) \]

\[ + \sum_{a}^{N} \frac{1}{2e_a^2} \int dx^0 \frac{1}{2} \left( \xi_a \left( \partial_0 (\lambda_{-a} - \overline{\lambda}_{+a}) + 2 \partial_1 (\lambda_{-a} + \overline{\lambda}_{+a}) \right) \right. \]

\[ + i2G_a (\partial_1 \sigma_a) - 2H_a (D_a - i F_{01a}) + iK_a (\lambda_{-a} - \overline{\lambda}_{+a}) + c.c. \right) , \quad (5.54) \]

where the covariant derivative of the scalar fields is given in (3.13). Performing the manipulation given in (5.29), taking the \( \hat{e} \to \infty \) limit, and subsequently integrating out the boundary auxiliary fields, we obtain the boundary action

\[ S_{\partial \Sigma} = \frac{1}{2\pi} \int dx^0 \left( \sum_{i}^{N} a_i \partial_0 \varphi_i + \hat{\theta} \hat{A}_0 + \sum_{c}^{N-1} \overline{\tilde{\theta}}_c \tilde{A}_{0c} \right) , \quad (5.55) \]

together with boundary conditions

\[ |\phi_i|^2 = c_i , \]

\[ \overline{\phi}_i \psi_{-i} + \overline{\psi}_{+i} \phi_i = 0 , \]

\[ \psi_{-i} \phi_i + \overline{\phi}_i \psi_{+i} = 0 , \quad (5.56) \]

\[ i \phi_i \overline{D}_1 \phi_i + \overline{\psi}_{+i} \psi_{+i} - \overline{\psi}_{-i} \psi_{-i} + \overline{F}_i \phi_i + \overline{\phi}_i F_i = 0 \]

on the matter fields, as well as boundary conditions

\[ \tilde{\lambda}_{+c} - \overline{\lambda}_{-c} = 0 , \]

\[ \partial_1 \tilde{\sigma}_c = 0 , \]

\[ \overline{F}_{01c} = 0 , \]

\[ \tilde{D}_c = 0 , \]

\[ \partial_1 (\tilde{\lambda}_{+c} + \overline{\lambda}_{-c}) = 0 , \quad (5.57) \]

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on vector multiplet fields, and their complex conjugates. In superfield notation, the latter
are
\[ \Phi_i e^{\sum a_i Q_i a} \Phi_i = c_i \]
\[ D_+ \Sigma_c = \overline{D}_- \Sigma_c \]
\[ D_+ \Sigma_c = \overline{D}_- \Sigma_c. \]  
(5.58)

Before proceeding, we note that the boundary conditions on the matter fermion fields in
(5.56) ensure that the spurious boundary term (3.33) vanishes.

Now, we shall rewrite (5.55) as
\[ \frac{1}{2\pi} \int d\Sigma^0 \left( \sum_i a_i \tilde{D}_0 \psi_i + \hat{\theta} \tilde{A}_0 \right), \]
where we have used \( \sum_i \check{Q}_i a_i = \hat{\theta} \), and where the covariant derivative of \( \psi_1 \)
is
\[ \tilde{D}_0 \psi_i = \partial_0 \psi_i + \sum_c \check{Q}_c \tilde{A}_0 c, \]
(5.60)
which agrees with the general definition for scalar fields given in (2.9). By integrating the
vector multiplet out of the bulk action (c.f. (3.19)), we obtain
\[ \hat{A}_0 = \frac{1}{2} \sum_{i=1}^N \check{Q}_i \left( i \tilde{D}_0 \check{\psi}_i + \bar{\psi}_- \bar{\psi}_+ + \bar{\psi}_+ \bar{\psi}_- \right) \]
\[ \sum_{j=1}^N \check{Q}_j^2 |\phi_j|^2 \]
at the boundaries,\(^{50}\) where (5.56) has been used in the last step. Hence, the final boundary
action is\(^{51}\)
\[ S_{\partial\Sigma} = \frac{1}{2\pi} \int d\Sigma^0 \left[ \sum_{i=1}^N a_i \tilde{D}_0 \psi_i - \hat{\theta} \sum_{i=1}^N \check{Q}_i c_i \tilde{D}_0 \psi_i \right], \]
(5.62)

Now, let us investigate the example of \( X = \mathbb{C}P^{N-1} \). We can derive the A-type supersymmetric boundary conditions of the GNLSM matter fields from (5.56) using the parametrizations (3.24) and (3.26)\(^{52}\)

\[ |Z|^2 = \frac{c_i}{c_N} \]
\[ \bar{Z}^i \bar{\psi}_+^Z + Z \bar{\psi}_+^Z = 0 \]
\[ \bar{Z}^i \bar{\psi}_-^Z + Z \bar{\psi}_-^Z = 0 \]
\[ i(\bar{Z}^i \partial_1 \bar{Z}^j - Z^i \partial_1 \bar{Z}^j) + \bar{\psi}_+^Z \bar{\psi}_-^Z - \bar{\psi}_-^Z \bar{\psi}_+^Z + F^Z \bar{Z}^i + \bar{F}^Z Z^i = 0, \]
(6.3)

\(^{50}\)As in the NLSM case, the presence of the boundary term proportional to \( \hat{\theta} \hat{A}_0 \) ensures that the algebraic
equation of motion for \( \hat{A}_0 \) does not contain a boundary term, see footnote 13.
\(^{51}\)To be precise, the complete boundary action includes the \( C \)-field term given in (2.7). However, to
simplify the following arguments, we shall consider the \( C \)-field term to be part of the bulk action, by using
Stoke’s theorem to promote it to a bulk term.
\(^{52}\)Analogous to the NLSM case, the last condition of (5.56) is trivialized using the algebraic equation of
motion of \( \hat{A}_1 \) in (3.19). The last condition of (6.3) is obtained via A-type supersymmetry transformations
of the fermionic boundary conditions.
where the last condition is in fact a Neumann boundary condition on $\gamma^i$, since $\nabla^i \partial_A Z^i - Z^i \partial_A \nabla^i = 2|Z|^2 i \partial_A \gamma^i$, where

$$\partial_A \gamma^i = \partial_\mu \gamma^i + \sum_c (\tilde{Q}_{ic} - \tilde{Q}_{Nc}) \tilde{A}_{\mu c} = \partial_\mu \gamma^i + \sum_c \tilde{e}_c^i \tilde{A}_{\mu c},$$  \hspace{1cm} (5.64)$$

with $\tilde{e}_c^i$ being the Killing vector field which generates the $U(1)^{N-1}$ isometry of the torus, $T^{N-1}$, parametrized by $\gamma_i$. The Neumann boundary condition on $\gamma^i$ together with the Dirichlet boundary condition on $|Z|^i$ implies that the equivariant A-brane wraps this torus. Furthermore, this torus is a Lagrangian submanifold of $\mathbb{C}P^{N-1}$ with respect to the Fubini-Study Kähler form given by (4.26). The remaining boundary conditions, i.e., for the fields in the vector multiplet of the GNLSM, are given by (5.57). The complete set of GNLSM boundary conditions is invariant under the $U(1)^{N-1}$ gauge symmetry, and satisfy the supersymmetry transformations given in (2.31) and (2.32) for $\epsilon_+ = 0$. In addition, the boundary conditions also ensure the locality of the classical equations of motion, i.e., that they contain no boundary terms.

Next, with the aid of (5.35), we can rewrite the boundary action (5.62) as

$$S_{\partial \Sigma} = \frac{1}{2\pi} \int_{\partial \Sigma} d\vec{\Gamma} \sum_{i=1}^N \left[ a_i - \hat{\theta} \frac{c_i}{\sum_{j=1}^N c_j} \right] \hat{D}_0 \phi_i - \sum_{i=1}^N \left( a_i - \sum_{k=1}^N a_k \frac{c_i}{\sum_{j=1}^N c_j} \right) \hat{D}_0 \phi_N + \sum_{i=1}^N \left( \partial_\mu (\phi_i - \phi_N) + \sum_{c=1}^N (\tilde{Q}_{ic} - \tilde{Q}_{Nc}) \tilde{A}_{0c} \right),$$

or

$$S_{\partial \Sigma} = \frac{1}{2\pi} \int_{\partial \Sigma} d\vec{\Gamma} \sum_{i=1}^N \left( a_i - \sum_{k=1}^N a_k \frac{c_i}{\sum_{j=1}^N c_j} \right) \left( \partial_\mu (\phi_i - \phi_N) + \sum_{c=1}^N (\tilde{Q}_{ic} - \tilde{Q}_{Nc}) \tilde{A}_{0c} \right),$$  \hspace{1cm} (5.65)$$

or

$$S_{\partial \Sigma} = \int_{\partial \Sigma} d\vec{\Gamma} \left( \sum_{i=1}^{N-1} A_{X i} \hat{D}_0 \gamma^i - \sum_{c=1}^{N-1} \tilde{R}_c \tilde{A}_{0c} \right),$$  \hspace{1cm} (5.66)$$

where we have also used (5.26), and where $A_{X i}$ is the constant given in (5.38), which is the connection of a flat $U(1)$ bundle on the Lagrangian torus $T^{N-1}$ parametrized by $\gamma_i$, and where

$$\tilde{R}_c = - \sum_{i=1}^{N-1} (\tilde{Q}_{ic} - \tilde{Q}_{Nc}) A_{X i} \hspace{1cm} (5.67)$$

As we explain below, A-type supersymmetry invariance holds since

$$d\tilde{R} = i\tau F^X,$$  \hspace{1cm} (5.68)$$
which is equal to zero because $F^X = 0$. This is known as the equivariant Bianchi identity, and implies that the flat $U(1)$ bundle has $U(1)^{N-1}$-equivariant structure, for which $\tilde{R}_c$ is the moment [20, 21].

Now, the boundary action (5.66) is not invariant under the supersymmetry transformations (2.31) and (2.32) for $\epsilon_+ = \tau_-$. Instead, the total action $S + S_{\partial \Sigma}$ is invariant under these transformations at the boundaries, using the boundary conditions (5.63) and (5.57), and therefore the sum of the expressions (2.34), (2.35) and (2.36) with the supersymmetry variation of the boundary action vanishes. The proof of this involves the supersymmetry invariance of the constant moment $\tilde{R}_c$, which is essentially the equivariant Bianchi identity (5.68), as well as the boundary constraint

$$\tilde{\mu}_c = -\tilde{\tau}_c,$$

(5.69)
on the moment map, which can be derived from (5.23) using $c_i = |\phi_i|^2$ and the parametrization (3.24). Furthermore, the nonzero supersymmetry variation of the boundary action (5.66) is cancelled by the $C$-term in (2.36) and the $\tilde{\theta}_c$-term in (2.35) via

$$2\pi \tilde{R}_c = -\tilde{\theta}_c + C_c,$$

(5.70)
which can be shown to hold via (3.32), (5.69), and (5.53). Finally, the $B$-field terms in (2.36) (where the $B$-field is proportional to the Kähler form), vanish using the boundary conditions given in (5.63).

Next, writing the boundary action as

$$S_{\partial \Sigma} = \int d\Sigma^0 \left( \sum_{i=1}^{N-1} A_i^X \partial_0^i \gamma^i \right),$$

(5.71)
it becomes obvious that it is invariant under the gauge transformations given in (2.26) and (2.27), since $A_i^X$ is a constant and the expression $\partial_0^i \gamma^i$ is invariant under gauge transformations.

We have thus found A-type supersymmetric and $U(1)^{N-1}$ gauge invariant boundary conditions and boundary interactions corresponding to an equivariant A-brane in $\mathbb{C}P^{N-1}$, which wraps a Lagrangian submanifold $T^{N-1}$ which supports a $U(1)^{N-1}$-equivariant flat $U(1)$ bundle. We may follow a procedure analogous to that presented above for $\mathbb{C}P^{N-1}$ in order to describe an equivariant A-brane in a toric manifold $X = \mathbb{C}^N // U(1)$ (by choosing different values for $\hat{Q}_i$), which would again be a Lagrangian submanifold $T^{N-1}$ supporting a flat $U(1)$ bundle with $U(1)^{N-1}$-equivariant structure.

### 5.2 Equivariant A-branes on $\mathbb{C}^N // U(1)^{N-k}$ from GLSM

We can generalize further, since the examples above have been solely for equivariant A-branes on $X = \mathbb{C}^N // U(1)^{N-k}$ where $N-k = 1$. For general values of $N-k$, we may derive

---

The G-equivariant Bianchi identity is equivalent to the G-invariance of the connection, $A$, of the bundle $(\mathcal{L}_c A = 0)$, which implies that the covariant derivative $d + A$ is G-invariant, and this defines a G-equivariant bundle, see [21], Section 3.2.
the relevant boundary conditions and boundary action from the GLSM boundary action (5.54), but instead of taking the $\hat{e} \to \infty$ limit for a single gauge group, we take $\hat{e}_b \to \infty$, where $b = 1, \ldots, N - k$. Integrating out auxiliary fields, and using parametrizations analogous to (3.24) and (3.26), we will be able to derive the $U(1)^k$-GNLSM boundary conditions and boundary action which represent an equivariant A-brane wrapping a Lagrangian torus $T^k$, which supports a flat $U(1)$ bundle with $U(1)^k$-equivariant structure.

Kapustin et al. ([14], page 58) have conjectured that the category of $G$-equivariant A-branes is some sort of $G$-equivariant version of the Fukaya category (which includes Lagrangian submanifolds which support flat unitary vector bundles as objects). Indeed, if we generalize the definition of the equivariant Fukaya category given for finite groups by Cho and Hong ([26], page 68) to $G = U(1)^k$, we see that the equivariant A-branes which we have found are objects in the $U(1)^k$-equivariant Fukaya category, and therefore, we have partially verified the conjecture of Kapustin et al. The other objects in the category which we have not constructed correspond to Lagrangian submanifolds which support equivariant flat unitary vector bundles.

5.3 Quantum Corrections

There are two important quantum effects of the bulk $U(1)^{N-k} \times U(1)^k$ GLSM, which affect the FI parameters $r_a$ and theta angles $\theta_a$ [7]. The first effect is the renormalization of the FI parameters,

$$r_{0a} = r_a(\mu) + \sum_{i=1}^{N} Q_{ia} \log \left( \frac{\Lambda_{UV}}{\mu} \right),$$

(5.72)

where $r_{0a}$ denotes bare parameters, $\Lambda_{UV}$ is an ultraviolet cut-off, and $\mu$ is a finite energy scale. Via integration of the beta functions of the FI parameters,

$$\beta_a = \mu \frac{dr_a}{d\mu},$$

the $\mu$-dependence is found to be

$$r_a(\mu) = \sum_{i=1}^{N} Q_{ia} \log \left( \frac{\mu}{\Lambda} \right),$$

(5.73)

where $\Lambda$ is the renormalization group invariant dynamical scale. The second quantum effect is the anomaly of the bulk $U(1)$ axial R-symmetry, whereby axial R-rotations $\psi_{\pm i} \rightarrow e^{\mp i \beta} \psi_{\pm i}$, $\sigma_a \rightarrow e^{2i\beta} \sigma_a$ and $\lambda_{\pm a} \rightarrow e^{\mp i \beta} \lambda_{\pm a}$ no longer leave the action invariant, but result in a shift of the theta angles, i.e.,

$$\theta_a \rightarrow \theta_a - 2 \sum_{i=1}^{N} Q_{ia} \beta.$$

(5.74)

The FI parameters are closely related to the boundary parameters $c_i$, via (5.49), and the latter undergo similar renormalization to that of (5.72) [11], i.e., the parameters $c_i$ run as

$$c_i(\mu) = \log \left( \frac{\mu}{\Lambda} \right).$$

(5.75)

Note that this quantum effect is nontrivial even when $\sum_{i=1}^{N} Q_{ia} = 0$, unlike the running of $r_a(\mu)$. In particular, (5.75) implies that the size of the equivariant A-brane in the toric
manifold $X$ could depend on the energy scale $\mu$. However, for $\mathbb{CP}^{N-1}$, this is not the case, because the Dirichlet boundary condition is $|Z|^2 = c_i/c_N$, and hence the equivariant A-brane stays the same size regardless of the energy scale. On the other hand, when $\sum_{i=1}^N Q_{ia} > 0$, the manifold $X$ becomes large at high energies due to (5.73), since $\hat{r}_b$ are the size moduli of $X$ (for $\mathbb{CP}^{N-1}$, this is obvious from (3.27)). Finally, it is expected that in addition to the bulk axial R-anomaly, a boundary axial R-anomaly also occurs [7].

### 5.4 Mirrors of Equivariant A-branes

Having described equivariant A-branes in toric manifolds, we shall now use mirror symmetry to find the Landau-Ginzburg mirrors of these branes, following the exposition in Section 3, as well as the results of [11]. We shall obtain the mirrors of branes in toric manifolds which obey $c_1(X) \geq 0$, since mirror symmetry is a quantum duality (which holds after taking all pertubative and nonpertubative quantum effects into account), and we can only obtain quantum GNLSMs for Kähler targets with $c_1(X) \geq 0$ from GLSMs (c.f. Section 3.1).

The boundary action of the $U(1)^N$ GLSM which we wish to dualize is given by (5.54), with $\sum_{i=1}^N a_i\partial_0\varphi_i$ replaced by $\sum_{i=1}^N a_i\partial_0\varphi_i$ via (5.29). The terms in the full $U(1)^N$ GLSM action relevant for the dualization are those which involve $\varphi_i$:

$$S_\varphi = -\frac{1}{2\pi} \sum_{i=1}^N \int_{\Sigma} |\phi_i|^2 (\partial_\mu \varphi_i + \sum_a Q_{ia} A_{\mu a})(\partial^\mu \varphi_i + \sum_a Q_{ia} A^\mu_a) d^2 x$$

$$+ \frac{1}{2\pi} \int_{\partial \Sigma} \sum_{i=1}^N \left( -2u_i |\phi_i|^2 (\partial_\mu \varphi_i + \sum_a Q_{ia} A_{1a}) + a_i \partial_0 \varphi_i \right) + \sum_a \theta_{a0} A_{0a} d x^0,$$

where $-2|\phi_i|^2(\partial_\mu \varphi_i + \sum_a Q_{ia} A_{1a}) = \chi_{\phi_i} \overline{D}_1 \phi_i$. Here, the boundary theta term

$$\sum_a \frac{\theta_{a}}{2\pi} \int_{\partial \Sigma} A_{0a} d x^0$$

has been included in order to maintain the gauge invariance, i.e., the gauge transformations $\varphi_i \rightarrow \varphi_i + \sum_a Q_{ia} \alpha_a$, $A_{\mu a} \rightarrow A_{\mu a} - \partial_\mu \alpha_a$ leave the expression (5.76) invariant (as long as (5.52) holds). All other terms, including those involving fermions, have been suppressed for simplicity.

Now, let us consider a system of $N$ one form fields $(B_i)_\mu$, as well as $N + N$ periodic scalar fields consisting of $\vartheta_i$ and $\tilde{u}_i$ with the action

$$S' = \frac{1}{2\pi} \sum_{i=1}^N \int_{\Sigma} \left( -|\phi_i|^2 B_{i\mu} B^{\mu i} d^2 x - B_i \wedge d \vartheta_i + \sum_a Q_{ia} \vartheta_i F_a \right) + \int_{\partial \Sigma} (a_i - \vartheta_i) \partial_0 \tilde{u}_i d x^0,$$

\[\text{In the following analysis, we shall take } |\phi_i|^2 \text{ to be non-zero, and } \varphi_i = \text{Im log } \phi_i \text{ is understood to be well-defined, permitting us to set } U_i = \varphi_i + U'_i \text{ whereby } U'_i \text{ is a boundary superfield which is single-valued.}\]

\[\text{The dualization portion of the subsequent analysis follows from that given in [11].}\]
where $F_a$ is the curvature of $A_a = A_{\mu a} dx^\mu$, $F_a = dA_a$. In addition, the boundary condition

$$ (B_i)_1 = 0 \quad (5.78) $$

is imposed. Integrating out $\vartheta_i$ gives rise to the constraints

$$ dB_i = \sum_a Q_{ia} F_a \quad \text{on } \Sigma \quad (5.79) $$

$$ (B_i)_0 = \partial_0 \tilde{u}_i \quad \text{along } \partial \Sigma. $$

The first of these constraints is solved by $B_i = d\varphi_i + \sum_a Q_{ia} A_a$, where $\varphi_i$ is a periodic scalar field of period $2\pi$. Then, the second constraint together with the boundary condition $(5.78)$ implies the relations

$$ \partial_0 \varphi_i + \sum_a Q_{ia} A_{0a} = \partial_0 \tilde{u}_i, \quad (5.80) $$

$$ \partial_1 \varphi_i + \sum_a Q_{ia} A_{1a} = 0. $$

on the boundaries. Inserting the first expression of $(5.80)$ into $(5.77)$ we obtain the action $(5.76)$ without the $u'_i$-dependent terms (using $\sum_{i=1}^N Q_{ia} a_i = \theta_a$). The second condition in $(5.80)$ is equivalent to the presence of the $u'_i$-dependent terms, since integrating out $u'_i$ imposes the second equation of $(5.80)$.

Alternatively, integrating out the fields $B_i$ imposes

$$ (B_i)_0 = -\frac{\partial_1 \vartheta_i}{2|\varphi_i|^2}, \quad (5.81) $$

$$ (B_i)_1 = -\frac{\partial_0 \vartheta_i}{2|\varphi_i|^2}, \quad (5.82) $$

and we obtain

$$ S_\vartheta = \frac{1}{2\pi} \sum_{i=1}^N \left[ \int_\Sigma \left( -\frac{1}{4|\varphi_i|^2} \partial_\mu \vartheta_i \partial^\mu \vartheta_i \, d^2x + \sum_a Q_{ia} \partial_\mu F_a \right) + \int_{\partial \Sigma} (a_i - \vartheta_i) \partial_0 \tilde{u}_i dx^0 \right]. \quad (5.83) $$

Following Hori [11], the bulk portion of the full mirror action is given by $(3.2)$ (modulo boundary terms that arise from putting the scalar kinetic terms in $(3.2)$ in their standard form), while the mirror boundary action takes the form\(^{57}\)

$$ S_{\partial \Sigma} = \frac{1}{2\pi} \sum_{i=1}^N \int dx^0 \text{Re} \int d\theta \left( s_i - Y_i \right) \tilde{Y}_i $$

$$ + (\text{Additional boundary terms required to cancel bulk SUSY variation}) \quad (5.84) $$

$$ + \sum_a \frac{1}{2\epsilon_a^2} \int d\theta d\bar{\theta} \text{Re} \left[ \Xi_a (D_+ \Sigma_a - \bar{D}_- \Sigma_a) \right], $$

\(^{56}\)For details on why $\varphi_i$ ought to be periodic, see [7], page 250.

\(^{57}\)As explained in [11], unlike the bulk superpotential $\sum_{i=1}^N e^{-Y_i}$ which is generated by vortices, no boundary F-terms can be generated by such effects.
where the boundary term in (5.83) is contained in the first term.

Here, \( \mathring{Y}_i \) is the \('field strength'\) \( \mathring{D}\mathring{U}_i \) of the boundary superfield \( \mathring{U}_i \), whose only difference from \( U_i \) is that its lowest component is \( \mathring{u}_i \). Integrating out \( \mathring{Y}_i \), we find the boundary condition

\[
Y_i = s_i, \quad (5.85)
\]
at A-boundary, which is

\[
y_i = s_i
\]
\[
\mathfrak{Y}_{+i} - \chi_{-i} = 0
\]
in components. In fact, integrating out all the boundary auxiliary fields in (5.84) imposes the boundary conditions (5.85) and (5.44), which result in the entire boundary action vanishing.

As in Section 3, taking the \( \mathring{e}_b \to \infty \) limit allows us to integrate \( \mathring{\Sigma}_b \) out of the action, and imposes the constraint

\[
\sum_j^N \mathring{Q}_{j b} Y_j - \mathring{t}_b = 0, \quad (5.87)
\]
giving us the gauged Landau-Ginzburg theory with holomorphic twisted superpotential

\[
\mathring{W} = \sum_c^k \left( \sum_{j=1}^N \mathring{Q}_{j c} Y_j - \mathring{t}_c \right) \mathring{\Sigma}_c + \sum_{j=1}^N e^{-Y_j}. \quad (5.88)
\]

We recall that the constraint (5.87) fixes the target space of the gauged Landau-Ginzburg theory to be the algebraic torus \((\mathbb{C}^\times)^k\). It is solved (c.f. Section 3) by

\[
Y_j = \mathring{s}_j + \sum_{c=1}^k v_{c j} \Theta_c, \quad (5.89)
\]
where \( \mathring{s}_j \) is any solution of \( \mathring{Q}_{j b} \mathring{s}_j = \mathring{t}_b \). Note that with \( \Theta_c = \theta_c + \theta^+ \chi^{\theta}_c + \mathring{\theta} \chi^{-\theta}_c + \theta^+ \mathring{\theta} E^{\theta}_c \), the full mirror action, expanded in components is

\[
S = \frac{1}{2\pi} \int d^2 x \left[ \sum_c^k \sum_d^k \left( -g_{c d} \partial_{\mu} \partial_{\sigma} \tilde{\sigma}_{c d} + \frac{i}{2} g_{c c} \bar{\chi}^{\theta}_c (\bar{\partial}_{\downarrow} + \bar{\partial}_{\uparrow}) \chi^\theta_c + \frac{i}{2} g_{c c} \bar{\chi}_c (\bar{\partial}_{\downarrow}) \chi^\theta_c + g_{c c} E^\theta_c \right) \right]
\]
\[
+ \sum_c^k \left( \mathring{\bar{F}}_{01 c} \right)^2 - \partial_{\mu} \tilde{\sigma}_c \partial_{\sigma} \tilde{\sigma}_c + (\mathring{D}_c)^2 + \frac{i}{2} \mathring{\chi}^{\theta}_c (\partial_{\downarrow}) \mathring{\chi}_c + \frac{i}{2} \mathring{\chi}^{\theta}_c (\partial_{\uparrow}) \mathring{\chi}_c
\]
\[
+ \frac{1}{2} \left( \sum_{j}^N \sum_c^k \sum_{d}^k \mathring{Q}_{j c} \mathring{v}_d (\sigma_c E^\theta_{c d} - i \chi^\theta_{c + d} - i \mathring{\chi}_{c + d} E^\theta_{c d} + (\mathring{D}_c - \mathring{t}_{01 c}) \theta_d ) \right)
\]
\[
+ \sum_c^k \left( \sum_{j}^N \sum_{d}^k \mathring{Q}_{j c} \mathring{v}_d (\sigma_c E^\theta_{c d} - i \chi^\theta_{c + d} - i \mathring{\chi}_{c + d} E^\theta_{c d} + (\mathring{D}_c - \mathring{t}_{01 c}) \theta_d ) \right)
\]
\[
+ \sum_{c}^k \sum_{j}^N \mathring{Q}_{j c} \mathring{v}_d (\mathring{D}_c - \mathring{t}_{01 c}) + \sum_{j}^N e^{-\sum_{c}^k \mathring{v}_c \theta_c - \mathring{s}_j} \left( - \sum_{c}^k \mathring{v}_c \mathring{\chi}^{\theta}_c + \sum_{d}^k \mathring{v}_d \chi^\theta_d - \sum_{c}^k \mathring{v}_c E^\theta_c ) \right) + c.c.), \quad (5.90)
\]
where \((\theta_c, \theta_d)\) (the lowest components of \((\Theta_c, \Theta_d)\)) parametrize the mirror target space \((\mathbb{C}^\times)^k\), on which the flat Kähler metric is

\[
d s^2 = \sum_c \sum_d \sum_j 1 \frac{v_j v_d^j}{4 \log(A_{UV}/\mu)} d\theta_c d\bar{\theta}_d = \sum_c \sum_d g_{cd} d\theta_c d\bar{\theta}_d.
\]  

(5.91)

Now, the boundary condition (5.85) on \(Y_i\) implies the boundary condition

\[
\sum_{c=1}^k v_c \Theta_c = s_j - \hat{s}_j,
\]

(5.92)

and this means that the \(U(1)^k\)-equivariant A-brane in \(X = \mathbb{C}^N / U(1)^N - k\) is mapped to a B-brane which is a D0-brane in the mirror Landau-Ginzburg model located at \(\theta_c\), where \(\theta_c\) is a solution of \(\sum_{c=1}^k v_c \theta_c = s_j - \hat{s}_j\). Let us investigate this D0-brane further, by studying how it is described in the \(\tilde{W}\)-plane. In particular, we would like to find the mirror of the \(U(1)^k\)-equivariant structure on the A-brane.

Firstly, we note that the twisted superpotential (5.88) can be rewritten as

\[
\tilde{W} = \tilde{W}_{\text{equiv}} + \tilde{W}_X.
\]

(5.93)

where the first and second term of (5.88) correspond respectively to the first and second term of (5.93). The image of the D0-brane in the \(\tilde{W}_X\)-plane is

\[
\tilde{W}_X = \sum_{i=1}^N e^{-s_i},
\]

(5.94)

which is the mirror condition found when studying the mirrors of A-branes without equivariant structure. However, turning to \(\tilde{W}_{\text{equiv}}\), we find that the boundary condition (5.85) implies that the image of the D0-brane in the \(\tilde{W}_{\text{equiv}}\)-plane is \(\tilde{W}_{\text{equiv}} = 0\), and thus we require further analysis to identify the mirror of the \(U(1)^k\)-equivariant structure on the A-brane.

Now, for the D0-brane mirrors of ordinary A-branes, there is an additional requirement which is necessary to prevent spontaneous supersymmetry breaking, that is the D0-brane should be at a critical point of the twisted superpotential \(\tilde{W}_X(\theta) = \sum_{j=1}^N e^{-s_j} - \sum_{c=1}^k v_c \theta_c\) \([7, 11, 27, 28]\). This condition is necessary for the potential energy of the mirror Landau-Ginzburg model (with twisted superpotential \(\tilde{W}_X\)) to have a vanishing vacuum expectation value. We shall generalize this analysis to the gauged Landau-Ginzburg model with neutral matter (5.90) which we are presently concerned with. Here, the twisted superpotential

\[\text{In the case where the two boundaries of the strip are mapped to different equivariant A-branes, labelled by } s_i^+ \text{ and } s_i^-, \text{ the positions of the mirror D0-branes are determined by } \sum_{c=1}^k v_c \theta_c = s_i^+ - \hat{s}_j \text{ and } \sum_{c=1}^k v_c \theta_c = s_i^- - \hat{s}_j \text{ respectively.}\]
terms can be expanded as

\[
\frac{1}{2\pi} \int d^2x \frac{1}{2} \left( \int d^2\bar{W}(\Theta, \bar{\Sigma}) + c.c. \right)
= \frac{1}{2\pi} \int d^2x \frac{1}{2} \left( \sum_{c} E_c^\theta \frac{\partial \bar{W}}{\partial \theta_c} + (\bar{D}_c - i\bar{F}_{0c}) \frac{\partial \bar{W}}{\partial \bar{\sigma}_c} + c.c. \right)
+ \sum_{c} \sum_{d} \left( \chi^\theta_{-c} \chi^\theta_{+d} \frac{\partial^2 \bar{W}}{\partial \theta_c \partial \theta_d} + \lambda^\theta_{-c} \lambda^\theta_{+d} \frac{\partial^2 \bar{W}}{\partial \bar{\sigma}_c \partial \bar{\sigma}_d} + i\chi^\theta_{-c} \lambda^\theta_{+d} \frac{\partial^2 \bar{W}}{\partial \theta_c \partial \bar{\sigma}_d} + i\chi^\theta_{-c} \lambda^\theta_{+d} \frac{\partial^2 \bar{W}}{\partial \bar{\sigma}_c \partial \theta_d} + c.c. \right)
\]

(5.95)

where \( \bar{W} \) is given by (3.10). Taking into account the presence of the auxiliary field terms

\[
\frac{1}{2\pi} \int d^2x \left( \sum_{c} \sum_{d} g_{cd} E_c^\theta E_d^\sigma + \sum_{c} \frac{1}{2\pi c} \bar{D}_c \bar{D}_c \right)
\]

(5.96)
in the action, upon integrating out the auxiliary fields \( \bar{D}_c \) and \( E_c^\theta \), the potential energy becomes

\[
V = \frac{1}{2\pi} \int dx \left( \frac{1}{4} g^{cd} \frac{\partial \bar{W}}{\partial \theta_c} \frac{\partial \bar{W}}{\partial \theta_d} + \frac{1}{2} \sum_{c} \bar{\epsilon}_c \text{Re} \left( \frac{\partial \bar{W}}{\partial \bar{\sigma}_c} \right) \text{Re} \left( \frac{\partial \bar{W}}{\partial \sigma_c} \right) \right).
\]

(5.97)

Now, in the non-gauged case, \( \frac{\partial \bar{W}_x}{\partial \theta_c} \) is a constant at the boundaries, and therefore supersymmetry would be broken for any classical configuration unless the D0-brane is located at the critical point \( \frac{\partial \bar{W}_x}{\partial \theta_c} = 0 \). However, in (5.97),

\[
\frac{\partial \bar{W}}{\partial \theta_c} = \sum_{j}^{N} \langle \bar{\sigma}, \bar{Q}_j \rangle v^j_c - \sum_{j}^{N} v^j_c e^{-(v^j, \theta) - i^j},
\]

(5.98)
is not a constant at the boundaries (since \( \bar{\sigma}_c \) obeys a Neumann boundary condition \( \partial_1 \bar{\sigma}_c = 0 \), unlike \( \theta_c \)), and hence classical configurations where \( \frac{\partial \bar{W}}{\partial \theta_c} = 0 \) at the boundaries can be achieved without any additional constraint on the position of the D0-brane. Next, the second term in (5.97) implies that \( \text{Re} \left( \frac{\partial \bar{W}}{\partial \sigma_c} \right) \) ought to vanish at each boundary in order to prevent spontaneous breaking of supersymmetry. Indeed,

\[
\text{Re} \left( \frac{\partial \bar{W}}{\partial \sigma_c} \right) = \text{Re} \left( \sum_{j}^{N} \bar{Q}_j s^j - \bar{t}_c \right)
\]

(5.99)
at the boundaries, which is identically zero because it is the real part of the condition \( \sum_{j}^{N} \bar{Q}_j s^j - \bar{t}_c = 0 \), which is implied by \( \sum_{j}^{N} Q_{ja} s^j - t_a = 0 \). The latter holds since it was necessary for the A-type supersymmetry and gauge symmetry of the \( U(1)^{N-k} \times U(1)^k \) GLSM (see (5.53)). Therefore, spontaneous supersymmetry breaking does not occur in the mirror theory, since zero-energy classical configurations can always be achieved at the boundaries. The condition \( \sum_{j}^{N} \bar{Q}_j s^j - \bar{t}_c = 0 \) is a new condition which did not appear in the non-gauged case, and in fact constrains the position of the D0-brane (defined by \( s^j \) via (5.92)). In conclusion, unlike the mirrors of ordinary A-branes, we have found that
The mirrors of $U(1)^k$-equivariant A-branes on $\mathbb{C}^N//U(1)^{N-k}$ do not need to satisfy the critical point condition $\frac{\partial W}{\partial c} = 0$, but instead their position must be further constrained by $\sum_j^N \tilde{Q}_{j,c} s^j - \tilde{t}_c = 0$.

In this section, we have restricted ourselves to equivariant A-branes whose mirrors are D0-branes. However, there are A-branes whose mirrors are higher-dimensional branes holomorphically embedded in the mirror target space. In Hori’s construction [11], these can be studied by promoting the parameter $s_i$ to a superfield $S_i$. It would be interesting to study equivariant structure on these branes.

5.5 Nonabelian Equivariant A-branes

We may use the insights obtained from analyzing the equivariant A-branes for abelian groups which we have found thus far to find the description of equivariant A-branes for general nonabelian groups. We shall use the GNLSM notation of Section 2 in this subsection.

Firstly, the terms in (2.35) (except the terms proportional to $(\phi^* \mu + r_a)$ and $\theta_a$) vanish using the boundary conditions

$$
\begin{align*}
\lambda_+ - \overline{\lambda}_- &= 0, \\
\partial_1 \sigma_a &= 0, \\
A_{1a} &= 0, \\
\partial_1 A_{0a} &= 0, \\
D_a &= 0, \\
\partial_1 (\lambda_- + \overline{\lambda}_+) &= 0.
\end{align*}
$$

(5.100)

Note that these conditions are a direct generalization of the conditions given for the example of $\mathbb{C}P^{N-1}$, except that $F_{01a} = 0$ is replaced by the stricter conditions $A_{1a} = 0$ and $\partial_1 A_{0a} = 0$. This is necessary since the supersymmetry transformations now contain nonabelian terms, and this causes A-type supersymmetry invariance of the set of boundary conditions to not hold unless we use the stricter conditions. The boundary conditions in fact imply that gauge transformations have to be restricted such that the transformation parameter $\alpha^a$ has vanishing derivative with respect to $x^1$ at the boundaries, in order for these boundary conditions to be gauge invariant.

Next, we turn to the boundary conditions for the matter fields. We first recall that for $\mathbb{C}P^{N-1}$, the equivariant A-brane corresponded to a Lagrangian torus $T^{N-1}$, which was invariant under the $U(1)^{N-1}$ isometry of $\mathbb{C}P^{N-1}$. Let us consider the $\mathcal{N} = 1$ subalgebra of A-type supersymmetry, which corresponds to $\epsilon_+ = i\tilde{\epsilon}$, $\tau_+ = -i\tilde{\epsilon}$, $\epsilon_- = -i\tilde{\epsilon}$ and $\tau_- = i\tilde{\epsilon}$, where $\tilde{\epsilon}$ is a real parameter. In this case, after integrating out the auxiliary fields $F^i$ and
$\mathbf{F}$, we find that (2.34) and the $B$-field terms in (2.36) are

$$-\frac{1}{2\pi} \frac{\hat{\kappa}}{2} \int_{\partial \Sigma} \delta \phi' \left\{ (g_{IJ}(\psi'_J - \psi'_I) - B_{IJ}(\psi'_J + \psi'_I)) \partial^A \phi'^I + \left( g_{IJ} \partial^A \phi'^I + B_{IJ} \partial^0 \phi'^I \right)(\psi'_J + \psi'_I) + g_{IJ}(\psi'_I - \psi'_J) \Re(\sigma^a) \bar{c}_a + \omega_{IJ}(\psi'_I + \psi'_J) \Im(\sigma^a) \bar{c}_a \right\},$$  \hspace{1cm} (5.101)

where $g_{IJ}X^IY^J = g_{\sigma}(X^IY^J + \bar{X}Y^J)$ and $\omega_{IJ}X^IY^J = ig_{\sigma}(X^IY^J - \bar{X}Y^J)$, and where $(I, J, K, \ldots)$ are indices corresponding to real coordinates on $X$. In addition, if we insist on locality of the matter equations of motion (like in the $\mathbb{C}P^{N-1}$ case), we require that

$$\delta \phi'(g_{IJ} \partial^A \phi'^I + B_{IJ} \partial^0 \phi'^I) = 0$$

(5.102)

at the boundaries, where $\delta \psi' = \delta \psi' + \Gamma'_{KL} \delta \phi' \psi'^L$. An equivariant A-brane shall wrap a submanifold (denoted as $\gamma$) of $X$, to which a boundary of the worldsheet is mapped via $(\phi', \bar{\phi'})$. Now, any allowed variation of $\phi$ (denoted $\delta \phi'$ for the real coordinate $\phi'$) along the boundary, and the derivative along the boundary, $\partial_\tau \phi'$, ought to be tangent to $\gamma$. Hence, taking into account the fact that $A_{1\alpha} = 0$ at the boundaries, we find that (5.101) vanishes while satisfying the first constraint of (5.102) if $\partial_\tau \phi'$ is normal to $\gamma$, $\psi'_L - \psi'_I$, and $\psi'_L + \psi'_I$ are respectively normal and tangent to $\gamma$, $\bar{c}_a$ is tangent to $\gamma$, the Kähler form vanishes against tangent vectors of $\gamma$, and the $B$-field vanishes against tangent vectors of $\gamma$. These last three conditions respectively imply that $\gamma$ is $G$-invariant, that it is an isotropic submanifold of $X$, and that the restriction of the two-form $B$ to $\gamma$ vanishes. In addition, we note that $\psi'_L - \psi'_I$ being normal to $\gamma$ and $\psi'_L + \psi'_I$ being tangent to $\gamma$ implies that

$$\psi'_L - \psi'_I = 0, \ I : \text{tangent to } \gamma, \hspace{1cm} \psi'_L + \psi'_I = 0, \ I : \text{normal to } \gamma,$$

(5.103)

(for a choice of coordinates which separates the normal and tangent directions) which satisfies the second constraint of (5.102).

Next, the $\mathcal{N} = (2, 2)$ supersymmetry transformation of $\phi'$ is

$$\delta \phi' = i(\epsilon_+ \psi'^I - \epsilon_{-1} J'^I_K \psi'^K - \epsilon_{-2} \psi'^I + \epsilon_{-1} J'^I_K \psi'^K),$$

(5.104)

where $\epsilon_+ = \epsilon_{_1} + i \epsilon_{_2}$ and $\epsilon_- = \epsilon_{-1} + i \epsilon_{-2}$, and where $J$ is the almost complex structure of $X$ locally given by $J^I_K = i \delta^I_K$ and $J'^I_K = -i \delta'^I_K$. A-type supersymmetry corresponds to $\epsilon_{_1} = \epsilon_{-1} = 1$ and $\epsilon_{_2} = \epsilon_{-2} = 0$, whereby

$$\delta \phi' = i(\epsilon_+ (\psi'^I + \psi'^I) - \epsilon_{-1} J'^I_K (\psi'^K - \psi'^K)).$$

(5.105)

Hence, $\psi'_L + \psi'_I$ and $J'^I_K (\psi'^K - \psi'^K)$ are tangent to $\gamma$. However, from the previous paragraph, we know that $\psi'_L - \psi'_I$ is normal to $\gamma$. In addition, $J'^I_K J'^I_K = -\delta'^I_K$. Hence, the application of the almost complex structure, $J$, converts normal vectors of $\gamma$ into tangent vectors of $\gamma$, and vice versa. Thus, $\gamma$ is a middle-dimensional Lagrangian submanifold of $X$. This
Lagrangian submanifold also happens to be $G$-invariant, which we know from the previous paragraph.

Indeed, (2.34) and the $B$-field terms in (2.36) vanish under this boundary condition; integrating out the auxiliary fields $F^a$ and $\overline{F^a}$, (2.34) and the $B$-field terms in (2.36) can be rewritten (for $\epsilon_+ = \epsilon_- = \epsilon$) as

$$
\frac{1}{2\pi i} \int d\Omega \left\{ \epsilon \left( -g(\partial_0^A \phi, \psi_- - \psi_+) - i\omega(\partial_0^A \phi, \psi_- + \psi_+) 
- g(\partial_1^A \phi, \psi_- + \psi_+) - i\omega(\partial_1^A \phi, \psi_- - \psi_+) 
- \Re(\sigma^a)g(\tilde{e}_a, \psi_+ - \psi_-) - i\Re(\sigma^a)\omega(\tilde{e}_a, \psi_+ + \psi_-) 
- i\Im(\sigma^a)g(\tilde{e}_a, \psi_+ - \psi_-) + \Im(\sigma^a)\omega(\tilde{e}_a, \psi_+ + \psi_-) 
+ 2B(\partial_0^A \phi, \psi_- + \psi_+) - 2i\omega^{-1}(g(\psi_- - \psi_+), B\partial_0^A \phi) \right) + c.c. \right\}
$$

(5.106)

(where $g(X, Y) = g_{IJ}X^I Y^J$, $\omega(X, Y) = \omega_{IJ}X^I Y^J$, $B(X, Y) = B_{IJ}X^I Y^J$ and $\omega^{-1}(X, Y) = \omega^{IJ}X_I Y_J$), which vanishes using $A_{1a} = 0$ as well as the conditions that $\partial_0 \phi^I$, $\psi_+^I + \psi_-^I$ and $\tilde{e}_a^I$ are tangent to $\gamma$ while $\partial_1 \phi^I$ and $\psi_+^I - \psi_-^I$ are normal to $\gamma$, together with the condition that $B|_\gamma = 0$.)

Next, we consider the terms proportional to $(\phi^* \mu_a + r_a)$ and $\theta_a$ in (2.35), as well as the term proportional to $\phi^* C_a$ in (2.36). Now, on a $G$-invariant Lagrangian submanifold, we have $\omega_{IJ} \tilde{e}_a^I T^J = 0$ for any tangent vector $T$. Using (2.16), this implies $\partial_1 \mu_a T^J = 0$, i.e., $\mu$ ought to be a constant along $\gamma$ [29]. Moreover, gauge invariance of the pull-back of this condition to $\partial \Sigma$ requires that the constant be an element of $[g, g]^0$, via the identity $\alpha^b \mathcal{L}_{\tilde{e}_a} \mu_a = [\alpha, \mu]_a$ [29]. Choosing the constant to be

$$
\mu_a = -r_a,
$$

we find that the terms proportional to $(\phi^* \mu_a + r_a)$ in (2.35) vanish. Analogously, the fact that $B_{IJ} \tilde{e}_a^I T^J = 0$ along $\gamma$ implies that $C$ ought to be a constant element of $[g, g]^0$ along $\gamma$. Choosing the constant to be

$$
C_a = \theta_a,
$$

we find that the remaining term in (2.35) and the remaining term in (2.36) cancel. Note that the boundary conditions (5.100) together with the constraint (5.108) preserve the locality of the equations of motion for vector multiplet components.

Finally, we consider a boundary action. We note that the boundary action (5.71) for $\mathbb{CP}^{N-1}$ is an example of the GNLSM generalization of the NLSM boundary Wilson line.

\footnote{Recall that for a tangent vector, $T$, and normal vector, $N$, of a Lagrangian submanifold, $\gamma$, of the Kähler manifold $X$, we have $\omega(T, T) = \omega(N, N) = 0$. Also, $\omega^{-1}(gN, BT) = 0$ means that the restriction of $B$ to $(T\gamma)^{\circ} \times T\gamma$ vanishes, where $(T\gamma)^{\circ}$ is the subspace of $TX$ orthogonal to $T\gamma$ with respect to $\omega$. When $\gamma$ is a Lagrangian submanifold, then $(T\gamma)^{\circ} = T\gamma$, and $B$ vanishes when restricted to $\gamma$.}
(5.4). Hence, for general GNLSMs the boundary action ought to be

\[ S_{\partial \Sigma} = \int_{\partial \Sigma} dx^0 A^X_\theta \partial_0 \phi^M \]

\[ = \int_{\partial \Sigma} dx^0 \left( A^X_\theta \partial_0 \phi^M - \tilde{R}_a A^a_0 \right) \tag{5.109} \]

where \( A^X \) corresponds to a \( G \)-invariant \( (\mathcal{L}_a A^X = 0) \) connection of a flat \( (F^X_{MN} = 0) \) \( U(1) \) bundle on each A-brane, and where \( \tilde{R}_a = -\iota_{\tilde{e}_a} A^X \) (we shall use \( (M, N, \ldots) \) as coordinate indices on the A-branes).\(^{60}\) Gauge invariance of this boundary action follows from the equivariant Bianchi identity

\[ d\tilde{R} = \iota_{\tilde{e}} F^X, \tag{5.110} \]

and this implies that each A-brane supports a flat, \( G \)-equivariant \( U(1) \) bundle, for which \( \tilde{R}_a \) is the moment.\(^{61}\) Its supersymmetry variation is

\[ \delta S_{\partial \Sigma} = - \int_{\partial \Sigma} dx^0 \tilde{R}_a \left( \frac{i}{2} (\epsilon(\lambda^a_+ + \lambda^a_-) + \tau(\lambda^a_+ + \lambda^a_-)) \right), \tag{5.111} \]

where we have used (5.110). Just like in the \( \mathbb{C}P^{N-1} \) case, we require that this cancels the \( C \)-term in (2.36) and the \( \theta_a \)-term in (2.35), i.e., we require that

\[ 2\pi \tilde{R}_a = -\theta_a + C_a \tag{5.112} \]

on \( \gamma \), the pull-back of which is a gauge invariant condition on \( \partial \Sigma \). This modification of (5.108) (together with the boundary conditions (5.100)) also preserves the locality of the equations of motion for vector multiplet components, just like in the example of \( \mathbb{C}P^{N-1} \).

In conclusion, we find that in general,

\begin{center}
Equivariant A-branes are \( G \)-invariant Lagrangian submanifolds of \( X \), which support \( G \)-equivariant flat \( U(1) \) bundles, and on which the restriction of the \( B \)-field vanishes.
\end{center}

This implies that they are objects in the \( G \)-equivariant Fukaya category of \( X \), by generalizing the definition of the equivariant Fukaya category for finite groups ([26], page 68) to any compact Lie group \( G \). As mentioned in Section 5.2, Kapustin et al. ([14], page 58) have conjectured that the category of \( G \)-equivariant A-branes is some sort of \( G \)-equivariant version of the Fukaya category. Hence, we have further verified their conjecture for non-abelian \( G \). Fully proving their conjecture would require constructing the other objects in the category, which correspond to Lagrangian submanifolds that support equivariant flat unitary vector bundles, and these should correspond to the insertion of certain \( G \)-invariant Wilson lines in the path integral.

\(^{60}\)Note that the inclusion of this boundary action does not modify the constraints (5.102), since it vanishes under arbitrary variations of \( \phi^X \) because \( F_{MN} = 0 \).

\(^{61}\)Note that gauge invariance of the boundary action requires the use of the identity \( \alpha^b \mathcal{L}_{\tilde{e}_a} \tilde{R}_a = [\alpha, \tilde{R}]_a \).
6 Open Hamiltonian Gromov-Witten Invariants

In this section, we shall use equivariant A-branes to define open Hamiltonian Gromov-Witten invariants. We shall first study the nonabelian open Hamiltonian Gromov-Witten invariants via the open topological gauged A-model, using the boundary conditions and boundary term we have found in Section 5.5. In the final two subsections, we shall focus on investigating the abelian open Hamiltonian Gromov-Witten invariants via mirror symmetry.

6.1 Open Topological Gauged A-model

The closed topological gauged A-model was introduced by Baptista [3], and in the following we shall generalize it to the case with boundaries, i.e., the open topological gauged A-model. This involves analytically continuing the Minkowski strip to the Euclidean one, subsequently twisting the fields in the action (2.4) + (2.19) as well as its supercharges using their vector R-charges, and imposing the appropriate boundary conditions (found in Section 5.5) which are supersymmetric with respect to the scalar supercharge $Q_A = Q_- + \overline{Q}_+$. We also include the gauge invariant boundary term (5.109).

The twisted fields are redefined as follows:

\[
\begin{align*}
\chi^k &= \sqrt{2} \psi^k \\
\overline{\chi} &= \sqrt{2} \overline{\psi}_+ \\
\varphi^a &= -i 2 \sigma^a \\
\xi^a &= \overline{\sigma} / 4 \\
\eta^a &= -i (\overline{\lambda}_-^a + \lambda_+^a) / (2 \sqrt{2}) \\
\mathcal{H}^a &= 4 id\phi^k \psi^k + 2 (F^k - \Gamma^k_{ij} \psi^i \psi^j) \\
\rho^a &= \sqrt{2} \overline{\psi}_+ \\
\kappa^a &= i (\overline{\lambda}_-^a - \lambda_+^a) / \sqrt{2} \\
C^a &= 2 (F_A)^{a}_{12} + 2 D^a,
\end{align*}
\]

where the fields are now sections of the following bundles

\[
\begin{align*}
\chi &\in \Omega_0^- (\Sigma; \phi^* \ker d\pi_E) \\
\varphi, \xi, C &\in \Omega_+^0 (\Sigma; g^p) \\
\rho &\in \Omega_-^{0,1} (\Sigma; \phi^* \ker d\pi_E) \\
\eta, \kappa &\in \Omega_0^- (\Sigma; g^p) \\
\mathcal{H} &\in \Omega_+^{0,1} (\Sigma; \phi^* \ker d\pi_E) \\
\psi &\in \Omega_+^1 (\Sigma; g^p),
\end{align*}
\]

with the remaining ‘barred’ fields being interpreted as the local complex conjugates of the
ones above. The action of the open gauged A-model is then\(^{62}\)

\[
S_A = \frac{1}{2\pi} \int_\Sigma \left\{ \frac{1}{2e^2} F_A^2 + |d^A\phi|^2 + \frac{1}{2} e^2 [\mu \circ \phi + r]^2 + \frac{i}{e^2} (\nabla^A \varphi, \nabla^A \xi) + \frac{1}{2e^2} |[\varphi, \xi]|^2 \\
+ \frac{1}{2e^2} [\varphi, \eta]^a [\eta]^a - \frac{1}{2e^2} [\varphi, \kappa]^a [\kappa]^a - \frac{1}{2e^2} \frac{1}{2} \mathcal{C} - *F_A - e^2 (\mu \circ \phi + r)^2 \\
+ \frac{1}{4} |\mathcal{H} - 4i\partial^A \phi|^2 + ig_{j\xi}(\varphi^a \xi^b + \varphi^b \xi^a)\tilde{\epsilon}^a_{\xi^b} + 2ig_{j\xi}(\nabla_j \xi^a)\tilde{\epsilon}^a_{\xi^b} \right\} \text{vol}_\Sigma \tag{6.5}
\]

where \(F_A\) is the curvature two-form of the connection \(A = A_\mu dx^\mu\), and the measure on the worldsheet is \(\text{vol}_\Sigma = dx \wedge dr = \frac{i}{2} dz \wedge d\bar{z}\).\(^{53,64}\) The fields \(C\) and \(\mathcal{H}\) are auxiliary fields, which can be integrated out of the action using their equations of motion

\[
\mathcal{C}^a = 2 * F_A^a + 2e^2 (\mu^a \circ \phi + r^a) \tag{6.6}
\]

\[
\mathcal{H}_\xi^k = 4id^A \phi^k. \tag{6.7}
\]

The supersymmetry transformations generated by the scalar supercharge \(Q_A = Q_- + \bar{Q}_+\) on the new fields follow from the supersymmetry transformations (2.31) and (2.32), with \(\epsilon_+ = \bar{\epsilon}_- = \sqrt{2}\epsilon\) and \(\epsilon_- = \bar{\epsilon}_+ = 0\), which gives

\[
\begin{align*}
Q_A \phi^k & = \chi^k \\
Q_A \chi^k & = \varphi^a e_a^k \\
Q_A \xi & = \eta \\
Q_A \eta & = [\varphi, \xi] \\
Q_A \rho^j & = H^k - \Gamma_{ij}^k \rho^i \phi^j \\
Q_A \mathcal{H}^k & = -R_{ijkl} \chi^i \chi^j \chi^k \rho^l - \Gamma_{ij}^k \mathcal{H}^l \chi^i + \varphi^a (\nabla_j e_a^k) \rho^j.
\end{align*}
\]

\(^{62}\)Here, we follow the notation of [3].\(^{63}\) We recall at this point that when \(G = G_1 \times G_2 \times G_3 \ldots\), each factor \(G_i\) has its own coupling constant, \(e_i\). In the case \(G = U(1)^k\) that we have focused on in the previous section, each \(U(1)\) factor had its own coupling constant, previously denoted \(\tilde{e}_i\).\(^{64}\)Note that we have performed integration by parts to undo the symmetrized form of the fermionic kinetic terms present in (2.4), which is no longer necessary since a Euclidean action is not real. The resulting boundary terms vanish using boundary conditions found in Section 5.5. For the case of \(CP^{N-1}\) which we studied extensively in the previous section, the relevant boundary conditions (given in (5.63) and (5.74)) are \(\kappa_a = 0\), \(\psi_+^a + \psi_-^a = 0\), \(\bar{\chi} = \bar{\xi} = 0\), \(\bar{\mathcal{H}} = 0\) and \(\bar{\rho} = 0\).
The action (6.5) is in fact $Q_A$-exact up to topological terms,

\[ S_A = Q_A \Psi + \frac{1}{2\pi} \int_{\Sigma} \phi^* (\eta_\omega + i [\eta_B]) + \frac{1}{2\pi} i \int_{\Sigma} (\theta, F_A) - i \int_{\partial \Sigma} A_M \chi^A \phi^M \]  

(6.9)

with gauge fermion

\[ \Psi = \frac{1}{2\pi} \int_{\Sigma} \left\{ \frac{1}{2e^2} \kappa_a (\ast F_A + e^2 (\mu \circ \phi + r))^a - \frac{1}{8e^2} \kappa_a C^a + \frac{1}{2e^2} \eta_a [\phi, \xi]^a + ig_j \xi^a (\tilde{e}_a \chi^j + e_a \chi^j) \right\} \text{vol}_{\Sigma} + \frac{1}{2\pi} \int_{\Sigma} \left\{ \frac{1}{e^2} \xi_a (\nabla^A \ast \psi^a) - \frac{i}{16} g_{jk} \rho^j \wedge (H - 8i \overline{\partial}^A \phi)^j + \frac{i}{16} g_{jk} \rho^j \wedge (\mathcal{H} - 8i \overline{\partial}^A \phi)^k \right\}, \]

where we have performed integration by parts in (6.5) such that $\nabla^A \phi_a \nabla^A \xi^a$ becomes $-\xi_a \nabla^A \nabla^A \phi^a$. The resulting boundary term vanishes using the boundary conditions $\partial_1 \varphi_a = 0$ and $A_{1a} = 0$ found in Section 5.5.

Let us elucidate the first topological term of (6.9). Here, $[\eta_\omega]$ and $[\eta_B]$ are the cohomology classes in $H^2(E)$ represented by the two-forms

\[ \eta_\omega (A) = \omega - d((\mu_a + r_a) A^a) \in \Omega^2 (P \times X), \]

\[ \eta_B (A) = B - d(C_a A^a) \in \Omega^2 (P \times X), \]

(6.10)

both of which descend to $E = P \times_G X$. In particular, this term is topological since $\int_{\Sigma} \phi^* [\eta_\omega]$ and $\int_{\Sigma} \phi^* [\eta_B]$ do not change under deformations of the map $\phi$, since the pull-back map is always homotopy invariant. In addition, the cohomology classes $[\eta_\omega]$ and $[\eta_B]$ are the pull-backs of the equivariant cohomology classes in $H^2_G(X)$ represented by $\omega - (\mu + r)$ and $B - C$.\textsuperscript{65}

The open gauged A-model is topological as a \textit{quantum} theory,\textsuperscript{66} and in order to consistently quantize such a gauge theory, one ought to perform BRST gauge-fixing, which involves the inclusion of Faddeev-Popov ghost fields in the action. This can be done straightforwardly, and we shall not write down the gauge-fixing action, $S_{BRST}$, explicitly. However, the open gauged A-model is anomalous, and in Section 6.4, we shall compute this anomaly by canonically quantizing the gauged Landau-Ginzburg mirror of the abelian open gauged A-model. Notably, in the process we shall describe $S_{BRST}$ for abelian gauge groups in detail.

### 6.2 Observables and Open Hamiltonian Gromov-Witten Invariants

A canonical set of bulk observables of the closed gauged A-model were described in [3], with the path integrals over these observables eventually argued to be equal to the Hamiltonian Gromov-Witten invariants. In this section, we shall recall the description of these bulk observables, as well as introduce \textit{boundary} observables which are defined with respect to the topology of the equivariant A-branes.

\textsuperscript{65}This follows because $H^2_G(X) = H^2(EG \times_G X)$, and since $P \times_G X \to \Sigma$ is the pull-back bundle of $EG \times_G X \to BG$ via a map $U : \Sigma \to BG$, where $EG \to BG$ is the universal bundle [30].

\textsuperscript{66}In particular, the correlation functions of the theory are invariant under diffeomorphisms of the world-sheet, e.g., transforming it from a strip to a disk.
In the ordinary open A-model, one can construct bulk observables from the de Rham cohomology classes of the target $X$, as well as construct boundary observables from the de Rham cohomology classes of the A-branes, which wrap the subspaces of $X$ (i.e., Lagrangian submanifolds) to which boundaries of the worldsheet are mapped. For the open gauged A-model, one uses the $G$-equivariant cohomology classes of $X$ to define bulk operators, as well as the $G$-equivariant cohomology classes of the equivariant A-branes to define boundary operators.

The $G$-equivariant cohomology classes of a manifold, $M$, are defined using the $G$-equivariant complex $\Omega^*_G(M)$, which is the set of $G$-invariant elements in the tensor product $S^*(g^*) \otimes \Omega^*(M)$, with $S^*(g^*)$ being the symmetric algebra of the dual of $g$.

For $M = X$, an equivariant form, $\alpha$, can be written on a local patch of $X$ as

$$\alpha = \alpha_{a_1 \cdots a_r, k_1 \cdots k_p, l_1 \cdots l_q}(w) \xi^{a_1} \cdots \xi^{a_r} dw^{k_1} \wedge \cdots \wedge dw^{k_p} \wedge d\overline{w}^{l_1} \wedge \cdots \wedge d\overline{w}^{l_q},$$

(6.11)

where $(w^{k_i}, \overline{w}^{l_i})$ are the coordinates on the patch. The coefficients $\alpha_{a_1 \cdots a_r, k_1 \cdots k_p, l_1 \cdots l_q}$ are symmetric with respect to the indices $a_i$, and antisymmetric with respect to the indices $k_i$ and $l_i$. Such a local form can be associated with a bulk operator $O_{\alpha}$ in the open gauged A-model,

$$O_{\alpha} = (\alpha_{a_1 \cdots a_r, k_1 \cdots k_p, l_1 \cdots l_q} \circ \phi) \left( \prod_{j=1}^r (\varphi + \psi - F_A)^{a_j} \right) \left( \prod_{i=1}^p (\chi^{k_i} - d^A \phi^{k_i}) \right) \left( \prod_{i=1}^q (\overline{\chi}^{l_i} - d^A \overline{\phi}^{l_i}) \right).$$

(6.12)

This correspondence holds globally on $X$. Moreover, we have

$$(d_\Sigma + Q_A) O_{\alpha} = O_{d_G \alpha},$$

(6.13)

where $d_\Sigma$ is the exterior derivative on the open worldsheet, $\Sigma$, while $d_G = 1 \otimes d + e^a \otimes \iota_{\varepsilon_a}$ is the Cartan operator defined on $\Omega^*_G(X)$. $O_{\alpha}$ can be decomposed with respect to the form degree on the worldsheet,

$$O_{\alpha} = O_{\alpha}^{(0)} + O_{\alpha}^{(1)} + O_{\alpha}^{(2)},$$

where, in particular,

$$O_{\alpha}^{(0)} = (\alpha_{a_1 \cdots a_r, k_1 \cdots k_p, l_1 \cdots l_q} \circ \phi) \left( \prod_{j=1}^r \varphi^{a_j} \right) \left( \prod_{i=1}^p \chi^{k_i} \right) \left( \prod_{i=1}^q \overline{\chi}^{l_i} \right),$$

(6.14)

is a local operator.

If we assume that $d_G \alpha = 0$, then (6.13) splits into the descent equations

$$d_\Sigma O_{\alpha}^{(2)} = 0,$$

(6.15)

$$d_\Sigma O_{\alpha}^{(1)} = - Q_A O_{\alpha}^{(2)}$$

(6.16)

$$d_\Sigma O_{\alpha}^{(0)} = - Q_A O_{\alpha}^{(1)}$$

(6.17)

$$Q_A O_{\alpha}^{(0)} = 0.$$
For a closed worldsheet, $\Sigma$, if $\beta$ is a \( j \)-dimensional homology cycle in $\Sigma$, one could define the $Q_A$-invariant operators

$$W(\alpha, \beta) := \int_{\beta} O^{(j)}_{\alpha}.$$ 

However, there are no 2-cycles on an open worldsheet. Hence, $O^{(2)}_\alpha$ ought to be integrated over the entire open worldsheet, and it is necessary for $Q_A$-invariance of $O^{(2)}_\alpha$ that $O^{(1)}_\alpha = 0$ at the boundaries.

For $M = L$, where $L$ is an equivariant A-brane (to which a boundary component $\partial \Sigma_L$ is mapped), an equivariant form, $\zeta$, can be written on a local patch of $L$ as

$$\zeta = \zeta_{a_1 \ldots a_r} u^{m_1} \wedge \ldots \wedge u^{m_s},$$

(6.19)

where $u^{m_i}$ are the coordinates on the patch. The coefficients $\zeta_{a_1 \ldots a_r m_1 \ldots m_s}$ are symmetric with respect to the indices $a_i$, and antisymmetric with respect to the indices $m_i$. Such a local form can be associated with a boundary operator $O_\zeta|_{\partial \Sigma_L}$ in the open gauged A-model,

$$O_\zeta|_{\partial \Sigma_L} = (\zeta_{a_1 \ldots a_r m_1 \ldots m_s} \circ \gamma) \left[ \prod_{j=1}^r (\phi + \psi|_{\partial \Sigma_L})^{a_j} \right] \left[ \prod_{i=1}^s ((Q_A \gamma)^{m_i} - d^A \gamma^{m_i}) \right],$$

(6.20)

where $\gamma$ is a section $\gamma: \partial \Sigma_L \to E_L$ of the associated bundle $E_L = P_{\partial \Sigma_L} \times_G L$ (which looks like a map $\gamma: \partial \Sigma_L \to L$ locally on $\partial \Sigma_L$, where $P_{\partial \Sigma_L}$ is the principal $G$-bundle over $\partial \Sigma_L$, and where $\psi|_{\partial \Sigma_L}$ is the restriction of $\psi$ to the boundary in question. In particular, we have

$$(d_{\partial \Sigma_L} + Q_A) O_\zeta = O_{dG \zeta},$$

(6.21)

where $d_{\partial \Sigma_L}$ is the exterior derivative on the worldsheet boundary $\partial \Sigma_L$, while $d_G$ is the Cartan operator defined on $\Omega^*_G(L)$. Just like bulk operators, $O_\zeta$ can be decomposed with respect to the form degree on the worldsheet boundary,

$$O_\zeta = O^{(0)}_\zeta + O^{(1)}_\zeta,$$

If it is assumed that $d_G \zeta = 0$, then (6.21) splits into the descent equations

$$d_{\partial \Sigma_L} O^{(1)}_\zeta = 0,$$

(6.22)

$$d_{\partial \Sigma_L} O^{(0)}_\zeta = - Q_A O^{(1)}_\zeta,$$

(6.23)

$$Q_A O^{(0)}_\zeta = 0,$$

(6.24)

where, for example,

$$O^{(0)}_\zeta = (\zeta_{a_1 \ldots a_r m_1 \ldots m_s} \circ \gamma) \left( \prod_{j=1}^r \varphi^{a_j} \right) \left( \prod_{i=1}^s (Q_A \gamma)^{m_i} \right).$$

(6.25)

Thus, if $\nu$ is a \( j \)-dimensional homology cycle in $\partial \Sigma_L$,\(^{67}\) one can then define the $Q_A$-invariant operators

$$W_{\partial \Sigma_L}(\zeta, \nu) := \int_{\nu} O^{(j)}_\zeta.$$

(6.26)

---

\(^{67}\)For \( j = 1 \), $\nu$ is taken to be $\partial \Sigma_L$, which is also the appropriate choice for noncompact boundaries.
The most general correlation function based on the above bulk and boundary operators can then be written down (for $\Sigma = I \times \mathbb{R}$) as the following path integral

$$\int \mathcal{D}(A, \phi, \varphi, \zeta, \eta, \kappa, \psi, \chi, b, c) \ e^{-(S_A + S_{BRST})} \prod_i W(\alpha_i, \gamma_i) \prod_j W_{\partial \Sigma_0}(\zeta_j, \nu_j) \prod_k W_{\partial \Sigma_\pi}(\zeta'_k, \nu'_k) ,$$

(6.27)

where $b$ and $c$ are ghost fields which appear in $S_{BRST}$.

Before proceeding, we shall return to our example of $X = \mathbb{C}P^{N-1}$, where any equivariant A-brane wraps a Lagrangian submanifold $T^{N-1}$. Here, we note that

$$Q_A \gamma^i = Q_A \frac{(\log Z^i - \log \bar{Z}^i)}{2i} = \frac{1}{2i} \left( \frac{\chi^i}{Z^i} - \frac{\bar{\chi}^i}{\bar{Z}^i} \right) = \frac{1}{2i |Z^i|^2} \left( \chi^i Z^i - \bar{\chi}^i \bar{Z}^i \right)$$

(6.28)

is well-defined (since $|Z^i|^2 = c_i/c_N$ at the boundaries), and is nonzero since the only fermionic Dirichlet boundary condition involving $\chi^i$ and $\bar{\chi}^i$ is $(\bar{Z}^i \chi^i + Z^i \bar{\chi}^i) = 0$ (c.f. (5.63)). In addition, the boundary conditions (5.63) imply that $d^A \gamma^i$ is nonzero at the boundaries. Moreover, the boundary condition $(\lambda_{i+} - \lambda_{i-}) = 0$ translates to $(\psi_{zc} + \psi_{zc}) = \psi_{1c} = 0$, and $\psi_c|_{\partial \Sigma} = i(\psi_{zc} - \psi_{zc}) dx^2 = \psi_{2c} dx^2$ is nonzero and well-defined. Finally, the boundary condition for $\varphi_c$ is $\partial_{\bar{1}} \varphi_c = 0$ (which comes from $\partial_{\bar{1}} \bar{\sigma}_c = 0$), and thus, $\varphi_c$ is nonzero at the boundaries. Hence, boundary operators of the form (6.20) are nonzero and well-defined.

Now, any supersymmetric path integral localizes to the bosonic field configurations that are fixed points of the supersymmetry $\mathcal{J}$. For the open gauged A-model, these field configurations can be read from the $Q_A$ variations of the fermionic fields in (6.20), after integrating out the auxiliary fields. They correspond to the solutions of

$$\mathcal{J}^A \phi = 0 \quad (6.29)$$

$$* F_A + e^2 (\mu \circ \phi + r) = 0$$

$$\nabla^A \varphi = \varphi^a (\tilde{\epsilon}_a \circ \phi) = 0 .$$

The first two equations are known as the symplectic vortex equations on an infinite strip, and were introduced by Cieliebak et al. in [29], and are a generalization of the typical Nielsen-Olsen vortex equations on a strip. In what follows, we shall refer to them as the open symplectic vortex equations. The last two equations are non-trivial, but in most interesting cases that we will consider have the trivial solution $\varphi = 0$ [3], and therefore we can ignore them in these cases. For the first two equations of (6.29), the boundary condition used by Cieliebak et al. on the strip was that each boundary component of the strip was mapped to a $G$-invariant Lagrangian submanifold of $X$, and this is precisely the boundary condition we found in Section 5.5. In addition, for the second equation, we have found the boundary conditions $A_{1a} = 0$, $\partial_{\bar{1}} A_{0a} = 0$ and $\mu_a = -r_a$. For the example of $X = \mathbb{C}P^{N-1}$, the open symplectic vortex equations read

$$\partial^A_i Z^i = 0$$

$$* \tilde{F}_A + \tilde{e}_c^2 (\tilde{\mu}_c \circ Z + \tilde{r}_c) = 0 .$$

(6.30)
Recall that the boundary conditions in this case are Lagrangian boundary conditions for \( Z^i \) which map each boundary to a \( U(1)^{N-1} \)-invariant Lagrangian torus \( T^{N-1} \) as well as \( \tilde{\mu}_c = -\tilde{\gamma}_c \) and \( *F_A c = 0 \).

The localization of supersymmetric path integrals of the form (6.27) thus reduce them to ordinary integrals of differential forms over the moduli spaces of open symplectic vortices, which are the spaces of solutions to the open symplectic vortex equations up to gauge equivalence. These moduli spaces are finite-dimensional, though they may be noncompact and contain singularities. The (infinite-dimensional) path integrals thus reduce to finite-dimensional integrals, which are well-defined mathematically (modulo issues related to the aforementioned noncompactness and singularities of the moduli spaces). These finite-dimensional integrals give us numbers which can be identified with the open version of the Hamiltonian Gromov-Witten invariants of \( X \) in [29, 31, 32].

We note that in the limit where \( \epsilon^2 \to +\infty \), a dynamically gauged sigma model with target \( X \) flows to an ordinary sigma model with target \( X//G \) [3]. Hence, in analogy with the closed case [33], it is predicted that there is a relationship between the open Hamiltonian Gromov-Witten invariants of \( X \) and the open Gromov-Witten invariants of \( X//G \) [34].

### 6.3 Dimension of Moduli Space of Open Symplectic Vortices and R-anomaly

The boundary axial R-anomaly has been previously used to compute the dimension of moduli spaces of holomorphic maps from an open Riemann surface to a Kähler manifold whereby the boundaries are mapped to Lagrangian submanifolds [7]. One can also compute the dimension of moduli spaces of symplectic vortices on a closed Riemann surface [3]. Using insights from these results, we may attempt to compute the boundary axial R-anomaly for the open gauged A-model and find the dimension of a moduli space of open symplectic vortices on an open Riemann surface. In what follows, we shall assume that we have a compact open Riemann surface, \( \Sigma \), with arbitrary genus and an arbitrary number of boundary circles.

The axial R-anomaly can be deduced by investigating the zero-modes of the fermionic fields via their kinetic terms

\[
\frac{1}{2\pi} \int_{\Sigma} dz \wedge d\bar{z} \left( \frac{i}{\epsilon^2 \sqrt{2}} \lambda_a \nabla^A \psi^a_z + \frac{i}{\epsilon^2 \sqrt{2}} \bar{\lambda}_a \nabla^A \psi^a_{\bar{z}} - \frac{1}{2} g_{jk} \bar{\rho}^j \phi^*(\phi^* \nabla^A) \bar{\chi}_j + \frac{1}{2} g_{jk} \rho^j \phi^*(\phi^* \nabla^A) \chi_j \right),
\]

where we have defined the fields

\[
\lambda_a = \frac{i2\sqrt{2}\eta_a + i\sqrt{2}\kappa_a}{2} \quad \text{and} \quad \bar{\lambda}_a = \frac{i2\sqrt{2}\eta_a - i\sqrt{2}\kappa_a}{2}. \tag{6.31}
\]

In order to evaluate the anomaly, we ought to double the open worldsheet as well as the bundles on it, as in [7], in order to form a closed worldsheet, on which the indices of the relevant operators can be evaluated. This is done by taking the metric on the worldsheet
close to each component of \( \partial \Sigma \) to be that of a flat cylinder, and gluing \( \Sigma \) with its orientation reversal, \( \Sigma^* \). The resulting closed Riemann surface is denoted \( \Sigma \# \Sigma^* \).

The corresponding bundles over \( \Sigma \) and \( \Sigma^* \) shall be glued using the relevant boundary conditions. To demonstrate this, let us first consider the index of the twisted Dirac operator \( \tilde{\mathcal{D}} \) which acts on the fermionic fields in the first two terms in the parantheses of (6.31),

\[
\text{Index } \tilde{\mathcal{D}} = \#[(\psi_z, \overline{\psi}_z) \text{ zero modes}] - \#[(\overline{\lambda}, \lambda) \text{ zero modes}].
\] (6.33)

The boundary conditions (c.f. Section 5.5) for these fermionic fields are

\[
\psi^a_z = -\psi^a_{\overline{z}} \quad \overline{\lambda}_a = \lambda_a.
\] (6.34)

Now, let us consider \( \overline{\lambda}_a \) and \( \psi^a_z \) as fields on \( \Sigma \), and \( \lambda_a \) and \( -\psi^a_{\overline{z}} \) as fields on \( \Sigma^* \). By the boundary conditions above, \( \overline{\lambda}_a \) on \( \Sigma \) and \( \lambda_a \) on \( \Sigma^* \) continuously glue along \( \partial \Sigma \), and define a continuous section of \( \mathfrak{g}_P \# \mathfrak{g}_P \) which we denote \( \overline{\lambda}_a \# \lambda_a \). Likewise, \( \psi^a_z \) and \( -\psi^a_{\overline{z}} \) define a continuous section \( \psi^a_z \# (-\psi^a_{\overline{z}}) \) of \( (\mathfrak{g}_P \# \mathfrak{g}_P) \otimes K_{\Sigma \# \Sigma^*} \). Then, if \( \nabla^A_z \lambda_a = 0, \overline{\lambda}_a \# \lambda_a \) is holomorphic on \( \Sigma \subset \Sigma \# \Sigma^* \). If \( \nabla^A_z \lambda_a = 0, \lambda_a \) is holomorphic on \( \Sigma^* \) due to orientation reversal, and hence \( \overline{\lambda}_a \# \lambda_a \) is holomorphic on \( \Sigma^* \subset \Sigma \# \Sigma^* \). Thus, if \( \nabla^A_z \lambda_a = 0 \) on \( \Sigma \) and \( \nabla^A_{\overline{z}} \lambda_a = 0 \) on \( \Sigma^* \), then \( \overline{\lambda}_a \# \lambda_a \) is entirely holomorphic on \( \Sigma \# \Sigma^* \).

Similarly, if \( \nabla^A_z \psi^a_z = 0 \) on \( \Sigma \) and \( \nabla^A_{\overline{z}} \psi^a_{\overline{z}} = 0 \) on \( \Sigma^* \), then \( \psi^a_z \# (-\psi^a_{\overline{z}}) \) is entirely holomorphic on \( \Sigma \# \Sigma^* \). This implies that the index of \( \tilde{\mathcal{D}} \) is the index of the Dolbeault operator of \( \mathfrak{g}_P \# \mathfrak{g}_P \), i.e.,

\[
\text{Index } \tilde{\mathcal{D}} = \dim H^0(\Sigma \# \Sigma^*, \mathfrak{g}_P \# \mathfrak{g}_P) - \dim H^0(\Sigma \# \Sigma^*, (\mathfrak{g}_P \# \mathfrak{g}_P) \otimes K_{\Sigma \# \Sigma^*})
\]

\[
= c_1(\mathfrak{g}_P \# \mathfrak{g}_P) + \dim(G)(2 - 2g - h)
\] (6.35)

where \( g \) is the genus and \( h \) is the number of boundary circles of the worldsheet \( \Sigma \). For compact \( G \), \( c_1(\mathfrak{g}_P \# \mathfrak{g}_P) = 0 \).

The index for the twisted Dirac operator \( \tilde{\mathcal{D}}' \) which acts on the fermionic fields in the last two terms in the parantheses of (6.31),

\[
\text{Index } \tilde{\mathcal{D}}' = \#[(\chi, \overline{\chi}) \text{ zero modes}] - \#[(\overline{\rho}_z, \rho_z) \text{ zero modes}],
\] (6.36)

can analogously be determined.\(^{68}\) Before A-twisting, we found boundary conditions (c.f. Section 5.5) which map each boundary to a \( G \)-invariant Lagrangian submanifold, \( L \). In what follows, we shall assume each boundary component is mapped to the same Lagrangian submanifold. In other words, at \( \partial \Sigma \), the scalar fields \( \phi^I \) constitute a section \( \phi : \partial \Sigma \to E_L \) of the associated bundle \( E_L = P_{\partial \Sigma} \times_G L \). We shall fix a bosonic background of the open Riemann surface \( \Sigma \), i.e., a map

\[
\phi : (\Sigma, \partial \Sigma) \to (P \times_G X, P_{\partial \Sigma} \times_G L).
\] (6.37)

\(^{68}\) Setzer [16] studied a specialized version of the non-dynamical open \( U(1) \)-gauged A-model, where the gauge field is the spin connection on the open Riemann surface, and performed a similar anomaly computation. In comparison, we are considering a dynamically gauged theory with arbitrary compact nonabelian gauge group.
Next, recall from Section 2 that before A-twisting, the matter fermionic fields are the sections $\Psi_\pm \in \Gamma(\Sigma; S_\pm \otimes \phi^* \ker \pi_E)$.\textsuperscript{69} The boundary conditions for the fermion fields were stated in Section 5.5 for a Minkowski worldsheet as $(\Psi^I_+ + \Psi^I_-)$ is tangent to $L$ and $(\Psi^I_+ - \Psi^I_-)$ is normal to $L$. This can be restated for a Euclidean worldsheet. First, as in [7], the spin bundles of opposite chirality $(S_+ \text{ and } S_-)$ can be identified at each boundary. Then, the boundary condition can be written succinctly as

$$\tau(\Psi_-) = \Psi_+,$$  \hspace{1cm} (6.38)

where the map $\tau: \ker \pi_E|_L \to \ker \pi_E|_L$ is the identity on $\ker \pi_E|_L$ (which looks like $TL$ on a local patch of $\partial \Sigma$), and is $(-1)\times$ the identity on $\ker \pi_E|_{\Sigma^\perp} \to E_L$ (which looks like $NL$ on a local patch of $\partial \Sigma$).\textsuperscript{70}

Next, $\Psi^I_+$ is decomposed into $\psi^I_+ \text{ and } \overline{\psi}^I_+$, which are valued in $\phi^* \ker \pi_E(1,0)$ (which looks like $\phi^* T(1,0) \chi$ locally on $\Sigma$) and $\phi^* \ker \pi_E(0,1)$ (which looks like $\phi^* T(0,1) \chi$ locally on $\Sigma$), respectively.\textsuperscript{71} The map $\tau$ acts linearly on $\ker \pi_E(1,0)|_L \oplus \ker \pi_E(0,1)|_L$, whereby the $(1,0)$ and $(0,1)$ components are exchanged. The reason for this is that if $t \in \ker \pi_E|_{\Sigma^\perp}$, then $Jt \in \ker \pi_E|_{\Sigma^\perp}$, and therefore $\tau: (t - iJt) \in \ker \pi_E(1,0)|_L \to (t + iJt) \in \ker \pi_E(0,1)|_L$ (this follows from the definition of $\tau$ below (6.38)). Hence,

$$\tau: \phi^* \ker \pi_E(1,0)|_{\partial \Sigma} \to \phi^* \ker \pi_E(0,1)|_{\partial \Sigma}.$$  \hspace{1cm} (6.39)

The boundary condition (6.38) can then be written as

$$\tau(\psi_-) = \overline{\psi}_+, \tau(\overline{\psi}_-) = \psi_+.$$  \hspace{1cm} (6.40)

After A-twisting, these become

$$\tau(\chi) = \overline{\chi}, \tau(\overline{\rho}_z) = \rho_z.$$  \hspace{1cm} (6.41)

These boundary conditions are analogous to those given in (6.34), and can be used to continuously glue $\phi^* \ker \pi_E(1,0)$ over $\Sigma$ with $\phi^* \ker \pi_E(0,1)$ over the orientation reversal $\Sigma^*$, by considering $\chi$ and $\overline{\rho}_z$ as fields on $\Sigma$ and $\overline{\chi}$ and $\rho_z$ as fields on $\Sigma^*$. In this way, we obtain a continuous section of $\phi^* \ker \pi_E(1,0) \otimes \phi^* \ker \pi_E(0,1)$ (denoted $\chi \# \overline{\chi}$) and a continuous section of $(\phi^* \ker \pi_E(1,0) \# \phi^* \ker \pi_E(0,1))^\ast \otimes K_{\Sigma^\# \Sigma^*}$ (denoted $\overline{\rho}_z \# \rho_z$).

Now, if $\phi^* \nabla^A_\chi \overline{\chi} = 0$, $\chi \# \overline{\chi}$ is holomorphic on $\Sigma \subset \Sigma^\# \Sigma^*$, and if $\phi^* \nabla^A_\chi \overline{\chi} = 0$, $\chi \# \overline{\chi}$ is holomorphic on $\Sigma^* \subset \Sigma^\# \Sigma^*$ due to orientation reversal. Hence, if $\phi^* \nabla^A_\chi \overline{\chi} = 0$ and $\phi^* \nabla^A_\overline{\chi} \chi = 0$, then $\chi \# \overline{\chi}$ is entirely holomorphic on $\Sigma^\# \Sigma^*$. Analogously, if $\phi^* \nabla^A_\overline{\chi} \rho_z = 0$ and $\phi^* \nabla^A_\chi \overline{\chi} = 0$, then $\chi \# \overline{\chi}$ is entirely holomorphic on $\Sigma^\# \Sigma^*$.\textsuperscript{69} For this subsection, we shall follow the real notation of [7] whereby $\psi^I_-$ is written as $\Psi^I_-$.

\textsuperscript{70}The vector bundle $\ker \pi_E|_{\Sigma^\perp} \to E_L$ is the kernel of the derivative $\pi_E: TE_L \to T\partial \Sigma$, while the vector bundle ker $\pi_E|_{\Sigma^\perp} \to E_L$ is the orthogonal complement of ker $\pi_E$ in ker $\pi_E|_{\Sigma^\perp}$, i.e., ker $\pi_E|_{\Sigma^\perp} = \ker \pi_E|_{\Sigma^\perp} \oplus \ker \pi_E|_{\Sigma^\perp}$, which looks like $TX|_L = TL \oplus NL$ locally on $\partial \Sigma$. The orthogonal complement is defined with respect to the metric on ker $\pi_E$, which is inherited from the metric on $X$ (Appendix C).

\textsuperscript{71}The vector bundle ker $\pi_E$ inherits a complex structure from that of $X$ (Appendix C), and therefore its complexification can be decomposed into holomorphic and antiholomorphic subbundles as ker $\pi_E \otimes C = \ker \pi_E(1,0) \oplus \ker \pi_E(0,1)$.\n
\( \phi^* \nabla^A_z \rho_z = 0 \), \( \nabla_z \# \rho_z \) is entirely holomorphic on \( \Sigma \# \Sigma^* \). This implies that the index of \( \tilde{D}' \) is the index of the Dolbeault operator of \( \phi^* \ker d\pi_{E}^{(1,0)} # \phi^* \ker d\pi_{E}^{(0,1)} \), i.e.,

\[
\text{Index } \tilde{D}' = \dim H^0(\Sigma \# \Sigma^*, \mathcal{E} \# \mathcal{E}^*) - \dim H^0(\Sigma \# \Sigma^*, (\mathcal{E} \# \mathcal{E}^*)^* \otimes K_{\Sigma \# \Sigma^*})
\]

\[
= c_1(\phi^* \ker d\pi_{E}^{(1,0)} # \phi^* \ker d\pi_{E}^{(0,1)}) + \text{rank}(\phi^* \ker d\pi_{E}^{(1,0)})(2 - 2g - h),
\]

(6.42)

where \( \mathcal{E} = \phi^* \ker d\pi_{E}^{(1,0)} \).

In addition, we note that the map \( \tau \) is associated with the orthogonal decomposition

\[
(\phi^* \ker d\pi_{E}^{(1,0)})|_L = \phi^*[\ker d\pi_{EL}^{(1,0)}] + i\phi^*[\ker d\pi_{EL}^{(0,1)}],
\]

(6.43)

which is obtained from \( (\phi^* \ker d\pi_{E})|_L = \phi^* \ker d\pi_{EL} \oplus \phi^* \ker d\pi_{EL}^\perp \) via the projection \( (1 - i\mathcal{J}) : \phi^* \ker d\pi_{E} \to \phi^* \ker d\pi_{E}^{(0,1)} \). Following the general argument in [7], this allows us to identify \( c_1(\phi^* \ker d\pi_{E}^{(1,0)} # \phi^* \ker d\pi_{E}^{(0,1)}) \) with \( \mu(\phi^* \ker d\pi_{E}^{(1,0)}, \phi^*[\ker d\pi_{EL}^{(1,0)}]) \) which is known as the Maslov index of the pair \( (\phi^* \ker d\pi_{E}^{(1,0)}, \phi^*[\ker d\pi_{EL}^{(1,0)}]) \). Thus, we obtain

\[
\text{Index } \tilde{D}' = \mu(\phi^* \ker d\pi_{E}^{(1,0)}, \phi^*[\ker d\pi_{EL}^{(1,0)}]) + \text{dim}_\mathbb{C}(X)(2 - 2g - h),
\]

(6.44)

where we have used \( \text{rank}(\phi^* \ker d\pi_{E}^{(1,0)}) = \text{dim}_\mathbb{C}(X) \).

We find from (6.35) and (6.44) that for compact \( G \), the axial R-anomaly is

\[
\mu(\phi^* \ker d\pi_{E}^{(1,0)}, \phi^*[\ker d\pi_{EL}^{(1,0)}]) + (\text{dim}_\mathbb{C}(X) + \text{dim}(G))(2 - 2g - h).
\]

(6.45)

Hence, in order for correlation functions to be nonzero, an appropriate number of boundary operators with suitable axial R-charges should be inserted into the path integral, such that axial R-symmetry is preserved at the boundaries.

The virtual real dimension of the moduli space of open symplectic vortices is given by the difference of (6.44) and (6.35) for compact \( G \), which is

\[
\mu(\phi^* \ker d\pi_{E}^{(1,0)}, \phi^*[\ker d\pi_{EL}^{(1,0)}]) + (\text{dim}_\mathbb{C}(X) - \text{dim}(G))(2 - 2g - h).
\]

(6.46)

The reason for this is that the linearized operator (whose index is the dimension of the moduli space [4]) one derives from the symplectic vortex equations is a compact perturbation of the direct sum of the operator \( \phi^* \nabla^A_z \) (which has the same index as \( \tilde{D}' \)) and an operator whose index can be evaluated to be \( -\chi_{\Sigma} \text{dim } G \) [32], where \( \chi_{\Sigma} = 2 - 2g - h \) is the Euler characteristic of \( \Sigma \). The Maslov index in (6.45) and (6.46) can be regarded as the equivariant Maslov index for the pair \( (X, L) \), since for trivial \( G \) it reduces to \( \mu(\phi^* T^{(1,0)}X, \phi^*[TL]^{(1,0)}) \).

### 6.4 \( \dot{Q}_A^2 \neq 0 \) Anomaly

We have previously defined open Hamiltonian Gromov-Witten invariants as integrals over the moduli spaces of open symplectic vortices. However, as mentioned, we have in fact ignored problems related to singularities in such a moduli space. In particular, we have ignored the singular boundary strata which have codimension one in the moduli space, which occur due to disk bubbling [35]. This phenomenon obstructs integration over the moduli space.
This is also a problem for ordinary open Gromov-Witten invariants, since disk bubbling also causes singular codimension one boundary strata in the moduli spaces of open worldsheet instantons of the non-gauged open A-model [36]. Disk bubbling manifests itself in the open A-model as a nonpertubative instanton effect which causes the violation of the nilpotency of the scalar supercharge, i.e., \( Q_A^2 \neq 0 \) ([7], page 833). Moreover, this anomaly of the supersymmetry algebra also spoils the cohomological structure of the space of supersymmetric ground states of the open A-model, which are identified with elements of the Floer cohomology group for a pair of intersecting Lagrangian submanifolds. In fact, the anomaly implies that there are no supersymmetric ground states, and therefore supersymmetry is broken.

Now, the fact that open symplectic vortices are open worldsheet instantons when \( G \) is trivial means that open symplectic vortices cause \( Q_A^2 \neq 0 \) and therefore supersymmetry breaking, for trivial \( G \). Thus, for nontrivial \( G \), we expect that open symplectic vortices will cause an analogous effect in the open \( G \)-gauged A-model, i.e., \( \hat{Q}_A^2 \neq 0 \) (where \( \hat{Q}_A = Q_A + Q_{BRST} \), with \( Q_{BRST} \) being the BRST charge), indicating singular codimension one boundary strata in the moduli spaces of open symplectic vortices, and implying that the supersymmetric ground states of the open gauged A-model (which we expect to be elements of the vortex Floer cohomology group [29, 37] for a pair of \( G \)-invariant Lagrangian submanifolds) would not only lose their cohomological structure, but would cease to be supersymmetric, implying supersymmetry breaking.

For the non-gauged open A-model, it is difficult to directly compute the violation of \( Q_A^2 = 0 \) in general; one can only do so for specific examples, e.g., \( X = S^2 \) [27]. Fortunately, at least for toric manifolds with \( c_1(X) \geq 0 \), one is able to use the mirror theory to compute this violation in general (and identify the condition whereby it vanishes) via canonical quantization, as shown by Hori [7, 27]. The condition found was that for a pair of Lagrangian submanifolds supporting flat \( U(1) \) bundles, the \( Q_A^2 \neq 0 \) anomaly vanishes if and only if the value of the superpotential on the mirror B-branes match each other, and in such a case supersymmetry is manifest.

It is thus natural to investigate the \( \hat{Q}_A^2 \neq 0 \) anomaly due to open symplectic vortices in the open gauged A-model via canonical quantization of its mirror theory. We shall do this for toric target spaces \( X = \mathbb{C}^N//U(1)^{N-k} \), i.e., by topologically A-twisting the \( U(1)^k \)-GNLSM on an infinite strip whose boundaries are mapped to different equivariant
A-branes in $X$,\footnote{These equivariant A-branes are labelled by the GLSM parameters $s_1^0$ and $s_2^0$, which determine the position of their respective mirror D0-branes (see footnote 58).} whose mirror (c.f. Section 5.4) has the (Euclidean) action

$$S_E = \frac{1}{2\pi} \int d^2x \left[ \sum_{k} \sum_{c} \sum_{d} (g_{cd} \partial_v \theta_c \partial_v \overline{\theta}_d - \frac{i}{2} g_{cd} \overline{\chi}_{-c} (\overline{D}_c) \chi_{-d}^\theta - \frac{i}{2} g_{cd} \overline{\chi}_{+c} (\overline{D}_c) \chi_{+d}^\theta - g_{cd} \overline{E}_d E_d^\theta) \right] + \sum_{c} \frac{1}{2e_c^2} \left( (\overline{F}_{12c})^2 + \partial_v \overline{\sigma}_c \partial_v \overline{\sigma}_c - (\overline{D}_c)^2 - \frac{i}{2} \overline{\chi}_{+c} (\overline{D}_c) \chi_{+c}^\theta - \frac{i}{2} \overline{\chi}_{-c} (\overline{D}_c) \chi_{-c}^\theta \right)$$

$$- \frac{1}{2} \left( \sum_{j} \sum_{c} \sum_{d} \tilde{Q}_{jc} v_d^j (\overline{\sigma}_c E_d^\theta) - i \overline{\lambda}_{+c} \chi_{-d}^\theta - i \overline{\lambda}_{-c} \chi_{+d}^\theta + (\overline{D}_c - \overline{F}_{12c}) \theta_d \right) + \sum_{j} (\sum_{c} \sum_{N} \tilde{Q}_{jc} \overline{\theta}_j - \tilde{t}_c) (\overline{D}_c - \overline{F}_{12c})$$

$$+ \sum_{j} e^{-\sum_{c} v_j^c \theta_c - \sum_{d} v_j^d \lambda_d} ( - \sum_{c} v_j^c \lambda_{c}^\theta \sum_{d} v_j^d \lambda_{d}^\theta - \sum_{c} v_j^c \overline{E}_c^\theta)$$

$$+ \sum_{j} \sum_{c} \sum_{d} \tilde{Q}_{jc} v_d^j (\overline{\sigma}_c E_d^\theta) - i \overline{\lambda}_{+c} \chi_{-d}^\theta - i \overline{\lambda}_{-c} \chi_{+d}^\theta + (\overline{D}_c + \overline{F}_{12c}) \overline{\theta}_d \right) + \sum_{j} (\sum_{c} \sum_{N} \tilde{Q}_{jc} \overline{\theta}_j - \tilde{t}_c) (\overline{D}_c + \overline{F}_{12c})$$

$$+ \sum_{j} e^{-\sum_{c} v_j^c \sigma_c - \sum_{d} v_j^d \lambda_d} ( - \sum_{c} v_j^c \lambda_{c}^\theta \sum_{d} v_j^d \lambda_{d}^\theta - \sum_{c} v_j^c \overline{E}_c^\theta) \right],$$

(6.47)

where $d^2x = dx^1 dx^2$ and $\partial_\pm = i \partial_2 \pm \partial_1$. Performing the topological A-twist for the mirror theory amounts to the following field redefinitions:

$$\tilde{\sigma}_c = -i 2 \overline{\sigma}_c \quad \tilde{\xi}_c = \overline{\sigma}_c / 4$$

$$\tilde{\chi}_c = \overline{\chi}_{+c} \quad \tilde{\chi}_c = \overline{\chi}_{-c}$$

$$\tilde{\psi}_c = \frac{2i}{\sqrt{2}} \overline{\lambda}_{+c} \quad \tilde{\psi}_c = \frac{2i}{\sqrt{2}} \overline{\lambda}_{-c}$$

$$\chi^{0c} = \overline{\chi}_{+c} \quad \chi^{0c} = \overline{\chi}_{-c}$$

$$\lambda^{0c} = 2 \overline{\lambda}_{+c} \quad \lambda^{0c} = 2 \overline{\lambda}_{-c}$$

(6.48)

where $\overline{\lambda}_c$, $\overline{\chi}_c$, $\chi^{0c}$ and $\lambda^{0c}$ are scalars, while $\tilde{\psi}_c = i \tilde{\psi}_2 \pm \tilde{\psi}_1$ and $\chi^{0c} = i \chi_2^{0c} \pm \chi_1^{0c}$ are one-forms. Hence, the mirror action of the open $U(1)^k$-gauged A-model with toric target
$X = \mathbb{C}^N / \text{U}(1)^{N-k}$ is

\[ S_E = \frac{1}{2\pi} \int d^2 x \left[ \sum_{c} \sum_{k} \left( g_{cd} \partial_\mu \theta_c \partial^\mu \overline{\phi}_d - \frac{i}{4} g_{cd} \overline{\chi}^\theta_c (\overline{\phi}_+)^\theta (\overline{\phi}_-)^\theta_d - \frac{i}{4} g_{cd} \chi^\theta_c (\overline{\phi}_+)^\theta_d - g_{cd} E^\theta_c F^\theta_d \right) \right] \\
+ \sum_{c} \frac{1}{2e_c^2} \left( (\overline{F}_{12c})^2 + 2i \partial_\mu \overline{\psi}_c \partial^\mu \overline{\xi}_c - (\overline{D}_c)^2 - \frac{\sqrt{2}}{e} \psi_+ \overline{\phi}_+ c^\theta \overline{\xi}_c - \frac{\sqrt{2}}{e} \psi_- \overline{\phi}_- c^\theta \overline{\xi}_c + (\overline{D}_c - \overline{F}_{12c}) \partial_\mu \overline{\psi}_c \right) \\
- \frac{1}{2} \left( \sum_{j=1}^{N} \sum_{c} \sum_{d} \overline{\bar{Q}}_{jc} v_d^j (\frac{i}{2} \overline{\phi}_c E_d^\theta - \frac{\sqrt{2}}{e} \psi_+ \overline{\phi}_+ c^\theta \overline{\xi}_c - \frac{\sqrt{2}}{e} \psi_- \overline{\phi}_- c^\theta \overline{\xi}_c + (\overline{D}_c - \overline{F}_{12c}) \partial_\mu \overline{\psi}_c \right) \\
+ \sum_{c} \left( \sum_{j} \overline{\bar{Q}}_{jc} \delta^j - \overline{\bar{e}}_c \right) (\overline{D}_c - \overline{F}_{12c}) + \sum_{j} \overline{\phi}_c X^\theta_j + \sum_{c} \sum_{d} \overline{\phi}_c X^\theta_d - \sum_{c} \sum_{j} \overline{\phi}_c E^\theta_j \right],
\]

which is invariant under the supersymmetry transformations

\[ \delta_{Q_A} \tilde{A}_{1c} = \frac{\sqrt{2}}{2} \overline{\psi}_1 c \\
\delta_{Q_A} \tilde{A}_{2c} = \frac{\sqrt{2}}{2} \overline{\psi}_2 c \\
\delta_{Q_A} \overline{\phi}_c = 0 \\
\delta_{Q_A} \overline{\xi}_c = - \frac{i \epsilon (\overline{\lambda}_c + \overline{\lambda}_c)}{4} \\
\delta_{Q_A} \overline{D}_c = - \frac{\sqrt{2}}{2} \epsilon (\partial_1 \overline{\psi}_2 c - \partial_2 \overline{\psi}_1 c) \\
\delta_{Q_A} \theta^c = 0 \\
\delta_{Q_A} \overline{\eta} = \epsilon (\overline{\chi}^\theta - \overline{\chi}^\theta c) \\
\delta_{Q_A} E^{\theta c} = i \epsilon \overline{\phi}_c X^\theta \\
\delta_{Q_A} F^{\theta c} = 0
\]

(6.50)

generated by the supercharge $Q_A$. However, this mirror theory is in fact a gauge theory, and any consistent quantization procedure should include gauge fixing, in order to remove unphysical degrees of freedom.

To this end, we shall choose the Lorentz gauge

\[ \langle \psi | \partial_\mu \tilde{A}_\mu^1 | \psi \rangle = 0 \]

(6.51)
(where $|\psi\rangle$ and $|\psi'\rangle$) are physical states), and include the following BRST gauge fixing action

$$S_{BRST} = \frac{1}{2\pi} \sum_{c} \frac{1}{2\epsilon_c^2} \int d^2x (-i B_c \partial_\mu \tilde{A}_c^\mu - (B_c)^2 + \partial_\mu \tilde{b}_c \partial^\mu \tilde{c}_c - i \sqrt{2} \partial_\mu \tilde{b}_c \tilde{\psi}_c^\mu),$$

(6.52)

where $\tilde{b}_c$ and $\tilde{c}_c$ are fermionic ghost fields, while $B_c$ is a bosonic auxiliary field. As expected, the first term explicitly breaks the $U(1)^k$ gauge symmetry of the gauged LG model.

Now, note that the gauge fixing action (6.52) can be rewritten as

$$S_{BRST} = \epsilon^{-1}(\delta_{QA} + \delta_{BRST}) \left\{ \sum_{c} \frac{1}{2\epsilon_c^2} \int d^2x (-i \tilde{b}_c \partial_\mu \tilde{A}_c^\mu + \tilde{b}_c B_c) \right\},$$

(6.53)

where we have performed integration by parts, and used the boundary condition $\tilde{\lambda}_{-c} - \tilde{\lambda}_{+c} = 0$ (which is equivalent to $\tilde{\psi}_{1c} = 0$) that we have previously imposed, as well as the boundary condition

$$\partial_1 \tilde{c}_c = 0,$$

(6.54)

which we impose at present. Here, $\delta_{BRST}$ is the standard BRST symmetry variation given by

$$\delta_{BRST} \tilde{A}_c^\mu = i\epsilon \partial_\mu \tilde{c}_c$$
$$\delta_{BRST} \tilde{b}_c = \epsilon B_c$$
$$\delta_{BRST} \tilde{c}_c = 0$$
$$\delta_{BRST} B_c = 0,$$

(6.55)

with the BRST variations of all other fields being equal to zero. For the unphysical fields used for gauge fixing, the supersymmetry transformations are $\delta_{QA} \tilde{b}_c = 0$ and $\delta_{QA} B_c = 0$ while$^{73}$

$$\delta_{QA} \tilde{c}_c = -\frac{i}{2} \tilde{\varphi}_c.$$  

(6.56)

Now, $\delta_{QA}^2 \propto \delta_{G}(\tilde{\varphi})$ and $\delta_{BRST}^2 = 0$ on all fields.$^{74}$ In addition, we can show that

$$(\delta_{QA} + \delta_{BRST})^2 = 0$$

(6.57)

on all fields. This implies that the BRST gauge fixing action (6.53) is in fact invariant under $\delta = \delta_{QA} + \delta_{BRST}$. Since the physical action (6.49) is also invariant under $\delta_{BRST}$, this further implies that the entire action $S_E + S_{BRST}$ is invariant under $\delta = \delta_{QA} + \delta_{BRST}$. This suggests that the relevant symmetry of the action after gauge fixing is that which is generated by $\tilde{Q}_A = Q_A + Q_{BRST}$.

The conjugate momentum for any field, denoted $X$, is defined as

$$\pi_X = \frac{\partial L_E}{\partial (\partial_2 X)},$$

(6.58)

---

$^{73}$Note that with respect to (6.66), the boundary condition (6.54) obeys A-type supersymmetry, since the boundary condition on $\tilde{\varphi}_c$ is $\partial_1 \tilde{\varphi}_c = 0$.

$^{74}$The transformation $\delta_{G}(\tilde{\varphi})$ is a $U(1)^k$ gauge transformation whose local parameter is $\tilde{\varphi}_c$. 

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with the convention that derivatives are taken from the right for fermionic fields. The canonical conjugate momenta are\(^{75}\)

\[
\begin{align*}
\pi_{\theta^c} &= \frac{1}{2\pi} g_{cd} \partial_d \theta^d \\
\pi_{\bar{\theta}^c} &= \frac{1}{2\pi} g_{cd} \partial_d \bar{\theta}^d \\
\pi_{\chi^{bc}} &= \frac{1}{4\pi} g_{cd} \chi^{bd}_+ \\
\pi_{\bar{\chi}^{bc}} &= \frac{1}{4\pi} g_{cd} \chi^{bd}_- \\
\pi_{\bar{\varphi}_c} &= \frac{1}{2\pi} g_{2c} \partial_c \bar{\varphi}_c \\
\pi_{\bar{\xi}_c} &= \frac{1}{2\pi} g_{2c} \partial_c \bar{\xi}_c \\
\pi_{\bar{\lambda}_c} &= \frac{1}{2\pi} \sqrt{2} \bar{\psi}^c \\
\pi_{\bar{\psi}^c} &= \frac{1}{2\pi} \sqrt{2} \bar{\lambda}_c \\
\pi_{A_{1c}} &= -\frac{1}{2\pi} \left( \frac{1}{e^2_c} F_{12c} + i \text{Im} \left( \sum_{j} N \bar{Q}_{jc} \left( (\psi^j, \theta) + \bar{\psi}^j \right) - \bar{\chi}^c \right) \right) \\
\pi_{A_{2c}} &= \frac{1}{2\pi} \frac{1}{e^2_c} (-i B_c) \\
\pi_{\bar{b}_c} &= \frac{1}{2\pi} \frac{1}{e^2_c} \left( \partial_2 \bar{c}_c - \frac{\sqrt{2}}{4} (\bar{\psi}_{+c} + \bar{\psi}_{-c}) \right) \\
\pi_{\bar{e}_c} &= \frac{1}{2\pi} \frac{1}{e^2_c} \left( \partial_2 \bar{b}_c \right).
\end{align*}
\]

The equal-time canonical commutation relations are

\[
[X_i(x^1), \pi_{X_j}(y^1)] = \delta_{ij} \delta(x^1 - y^1) \\
[X_i(x^1), X_j(y^1)] = 0 \\
[\pi_{X_i}(x^1), \pi_{X_j}(y^1)] = 0
\]

for \(X_i = \{\theta^c, \bar{\theta}^c, \chi^{bc}, \bar{\chi}^{bc}, \bar{\varphi}_c, \bar{\xi}_c, \bar{\lambda}_c, \bar{\psi}_c, \bar{b}_c, \bar{e}_c\}\), and the equal-time canonical anticommutation relations are

\[
\{X_i(x^1), \pi_{X_j}(y^1)\} = \delta_{ij} \delta(x^1 - y^1) \\
\{X_i(x^1), X_j(y^1)\} = 0 \\
\{\pi_{X_i}(x^1), \pi_{X_j}(y^1)\} = 0
\]

for \(X_i = \{\chi^{bc}, \bar{\chi}^{bc}, \bar{\lambda}_c, \bar{\psi}_c, \bar{b}_c, \bar{e}_c\}\). In addition, all commutators between a bosonic operator and a fermionic operator vanish, and the commutation relations obeyed by \(E_c^\theta\) and \(\bar{D}_c\) are

\(^{75}\text{Note that in order to derive a consistent set of anticommutation relations, we convert the fermionic kinetic terms with \(\partial_2\) derivatives to a form identical to that found in closed theories via integration by parts, e.g. } \frac{1}{2} g_{cd} \chi^{cd}_+ i (\partial_2 \chi^{cd}_+) \lambda^{cd}_d \rightarrow \frac{1}{2} g_{cd} \chi^{cd}_+ (i \partial_2 \lambda^{cd}_d). \text{ The symmetric form of the kinetic terms are in fact not necessary because a Lagrangian on a Euclidean worldsheet is not real.}
specified by their equations of motion, which hold as operator equations due to Ehrenfest’s theorem. Then, the non-vanishing canonical commutation and anticommutation relations are

\[
\begin{align*}
[\theta^c(x^1), \phi_d \partial_2 \bar{\theta}^{c}(y^1)] &= 2\pi \delta_{cd}(x^1 - y^1) \\
[\bar{\theta}^{c}(x^1), \phi_d \partial_2 \theta^c(y^1)] &= 2\pi \delta_{cd}(x^1 - y^1) \\
\{\chi^{de}(x^1), \lambda^d(y^1)\} &= 4\pi g^{cd}\delta(x^1 - y^1) \\
\{\lambda^d(x^1), \chi^d(y^1)\} &= 4\pi g^{cd}\delta(x^1 - y^1) \\
[\tilde{\varphi}_c(x^1), \frac{1}{2\epsilon_c^2}2\partial_2 \tilde{\xi}_d(y^1)] &= 2\pi \delta_{cd}(x^1 - y^1) \\
[\tilde{\xi}_c(x^1), \frac{1}{2\epsilon_c^2}2\partial_2 \tilde{\varphi}_d(y^1)] &= 2\pi \delta_{cd}(x^1 - y^1) \\
\{\tilde{\lambda}_c(x^1), \frac{1}{2\epsilon_c^2} \sqrt{2}\tilde{\psi}_d(y^1)\} &= 2\pi \delta_{cd}(x^1 - y^1) \\
\{\tilde{\psi}_c(x^1), \frac{1}{2\epsilon_c^2} \sqrt{2}\lambda_d(y^1)\} &= 2\pi \delta_{cd}(x^1 - y^1) \\
[\tilde{A}_1c(x^1), -(\frac{1}{\epsilon_c} \tilde{F}_{12c}(y^1))] &= 2\pi \delta_{cd}(x^1 - y^1) \\
[\tilde{A}_2c(x^1), \frac{1}{2\epsilon_c^2}(-i\tilde{B}_d(y^1))] &= 2\pi \delta_{cd}(x^1 - y^1) \\
\{\tilde{b}_c(x^1), \frac{1}{2\epsilon_c^2}(\partial_2 \tilde{c}_d(y^1))\} &= 2\pi \delta_{cd}(x^1 - y^1) \\
\{\tilde{c}_c(x^1), \frac{1}{2\epsilon_c^2}(\partial_2 \tilde{b}_d(y^1))\} &= 2\pi \delta_{cd}(x^1 - y^1).
\end{align*}
\] (6.62)

The integral form of the supercharge \( \hat{Q}_A \) is

\[
\hat{Q}_A = \int dx^1 (\hat{J}_A^2),
\] (6.63)

where \( \hat{J}_A^\mu \) is the supercurrent defined via

\[
\delta S_E = \int d^2 x \partial_\mu \epsilon \hat{J}_A^\mu.
\] (6.64)
Its explicit form is\footnote{Charge conservation follows from $\partial_2 Q_A = \int dx^1 \partial_2 J^A_\lambda = - \int dx^1 \partial_1 J^A_\lambda$, which can be shown to be zero using the boundary conditions we have previously imposed, as well as the boundary condition $\partial_1 b_c = 0$, which we impose at present.}

$$\hat{Q}_A = \frac{1}{2\pi} \int dx^1 \left( g_{\alpha\beta} \partial_2 \theta^\alpha (\chi^{\beta d} - \bar{\chi}^{\beta d}) - i g_{\alpha\beta} \partial_1 \theta^\alpha (\chi^{\beta d} + \bar{\chi}^{\beta d}) - k \frac{\sqrt{7}}{4\epsilon_c^2} \tilde{F}_{12c}(\bar{\psi}_{+c} - \bar{\psi}_{-c}) - \sum_c \frac{i}{4\epsilon_c^2} \tilde{F}_{12c} \partial_1 \tilde{c}_c + \sum_c \frac{1}{4\epsilon_c^2} (\bar{\lambda}_c + \bar{\lambda}_c) \partial_2 \bar{\varphi}_c + \sum_c \frac{i}{4\epsilon_c^2} (\bar{\lambda}_c - \bar{\lambda}_c) \partial_1 \bar{\varphi}_c + \frac{1}{4} \sum_c (\chi_+^{\phi c} + \chi_-^{\phi c}) \frac{\partial \tilde{W}}{\partial \theta^c} + \sum_c \frac{\sqrt{7}}{4} (\bar{\psi}_{+c} - \bar{\psi}_{-c}) \left( \frac{2}{i} \frac{\partial \tilde{W}}{\partial \bar{\varphi}_c} + \frac{1}{4} \frac{\partial \tilde{W}}{\partial \xi_c} \right) \right),$$

where

$$\frac{\partial \tilde{W}}{\partial \theta^c} = \frac{i}{2} \sum_{j=1}^N \langle \bar{\varphi}, \bar{Q}_j \rangle \nu^j_c - \sum_{j=1}^N \nu^j_c e^{-(\nu^j \theta) - \hat{s}^j},$$

$$\frac{2}{i} \frac{\partial \tilde{W}}{\partial \bar{\varphi}_c} = \sum_{j=1}^N \tilde{Q}_{jc} \left( \langle \nu^j \theta \rangle + \hat{s}^j \right) - \tilde{t}_c, \quad \left(6.65\right)$$

$$\frac{1}{4} \frac{\partial \tilde{W}}{\partial \xi_c} = \sum_{j=1}^N \tilde{Q}_{jc} \left( \langle \nu^j \bar{\theta} \rangle + \hat{s}^j \right) - \tilde{t}_c. \quad \left(6.66\right)$$

Then,

$$\hat{Q}_A^2 = \frac{1}{2} \{ Q_A, \hat{Q}_A \} = \frac{1}{2\pi} \int dx^1 \left( \langle -i \rangle \sum_c \partial_1 \theta^c \frac{\partial \tilde{W}}{\partial \theta^c} + \langle -i \rangle \sum_c \partial_1 \varphi_c \frac{\partial \tilde{W}}{\partial \bar{\varphi}_c} - \sum_c \frac{i}{4\epsilon_c^2} \partial_2 \bar{\varphi}_c B_c \right)$$

$$- \sum_c \sum_d \frac{\partial^2 \tilde{W}}{\partial \theta^c \partial \xi_c} (\chi^{\beta d} - \bar{\chi}^{\beta d})(\langle \frac{\sqrt{7}}{4i} (\bar{\psi}_{+c} - \bar{\psi}_{-c}) + \partial_1 \tilde{c}_c \rangle) \right), \quad \left(6.67\right)$$

where we have used the boundary conditions $\tilde{F}_{12c} = 0$ and $\sum_{c=1}^k v_{cj} \Theta_c = s_j - \hat{s}_j$ (where $s_j = s_j^\alpha$ at $x^1 = \pi$ and $s_j = s_j^\beta$ at $x^1 = 0$), as well as the constraint $\sum_{j=1}^N \tilde{Q}_{jc} s_j - \tilde{t}_c = 0$.

The terms with first-order derivatives of the superpotential can be written as

$$\frac{(-i)}{2\pi} \int dx^1 \left( \sum_c \partial_1 \theta^c \frac{\partial \tilde{W}}{\partial \theta^c} + \sum_c \partial_1 \varphi_c \frac{\partial \tilde{W}}{\partial \bar{\varphi}_c} \right) \right), \quad \left(6.68\right)$$

$$\frac{(-i)}{2\pi} \int dx^1 \partial_1 \tilde{W}(\theta, \bar{\varphi})$$

$$\frac{(-i)}{2\pi} \left( \tilde{W}(\theta, \bar{\varphi}) - \tilde{W}(\theta, \bar{\varphi})_0 \right).$$
From the analysis below (5.93), we know that this is equal to \( \frac{(-i)}{2\pi} \left( \sum_{i=1}^{N} e^{-s_i^2} - \sum_{i=1}^{N} e^{-s_i^0} \right) \).

The remaining terms to consider are then

\[
\frac{1}{2\pi} \int dx^1 \left( - \sum_{c} \left( \frac{i}{4e_c} \right) \partial_2 \tilde{\varphi}_c B_c - \sum_{c} \sum_{d} \left( \frac{\partial^2 \tilde{W}}{\partial \theta^d \partial \xi_c} \right) \left( \chi^d - \tilde{\chi}^d \right) \left( \frac{i}{8} \right) \left( \frac{\sqrt{7}}{4i} \left( \tilde{\psi}_{+c} - \tilde{\psi}_{-c} \right) + \partial_1 \tilde{c}_c \right) \right).
\]

Unlike (6.68), these cannot be written in terms of boundary data, and hence are bulk terms which occur even for closed worldsheets. However, as in the non-anomalous closed case, these bulk terms ought to be equal to zero. The vanishing of these terms can also be understood as follows. The auxiliary field \( B_c \) obeys its equation of motion \( B_c = -\frac{i}{2} \partial_\mu \tilde{A}_c^\mu \) as an operator equation due to Ehrenfest’s theorem, and hence the matrix elements of the first term in the integrand with respect to the physical Hilbert space vanish due to the Lorentz gauge condition (6.51).\(^{77}\) Next, note that we are dealing with an A-twisted theory, whose topological correlation functions are invariant under \( \hat{Q}_A \)-exact deformations of the action. Therefore, we should be able to deform the action such that the second term in (6.69) vanishes. Indeed, this can be achieved by adding the following term which is \( \hat{Q}_A \)-exact to the action, i.e.,

\[
\epsilon^{-1} \delta_{Q_\Lambda} \left[ \frac{1}{2\pi} \int d^2x \frac{1}{2} \left( \sum_{c} \left( \frac{i}{4} \right) \partial_2 \tilde{\varphi}_c \left( \sum_{j} \tilde{Q}_{jc} (\sum_{d} v_{d}^j \tilde{\theta}^d + \tilde{s}_d^j) - \tilde{t}_c \right) - \sum_{c} 4 \tilde{c}_c \left( \sum_{j} \sum_{d} \tilde{Q}_{jc} v_{d}^j \tilde{\chi}_d \right) \right) \right]
\]

\[
= \frac{1}{2\pi} \int d^2x \frac{1}{2} \left( \sum_{j} \sum_{c} \sum_{d} \tilde{Q}_{jc} \tilde{v}_{d}^j (4 \tilde{\xi}_c \tilde{\theta}_d - i \tilde{\lambda}_c \tilde{\chi}_d - i \tilde{\lambda}_c \chi_d) + (\tilde{D}_c + \tilde{F}_{12c}) \tilde{\theta}_d \right)
\]

\[
+ \sum_{c} \sum_{j} \tilde{Q}_{jc} \tilde{s}_d^j - \tilde{t}_c (\tilde{D}_c + \tilde{F}_{12c}) \right),
\]

which upon doing so, the terms proportional to \( \partial \tilde{W}(\tilde{\theta}, \tilde{\varphi})/\partial \tilde{\xi}_c \) in (6.65) vanish, whence the second term in (6.69) also vanishes.

Hence, we find that

\[
\hat{Q}_A^2 = \frac{(-i)}{2\pi} (\tilde{W}(\theta, \tilde{\varphi})_\pi - \tilde{W}(\theta, \tilde{\varphi})_0),
\]

i.e., the \( \hat{Q}_A^2 \neq 0 \) anomaly (which occurs due to the nonperturbative quantum effects of open symplectic vortices in the open gauged A-model) vanishes when the value of the superpotential \( \tilde{W}(\theta, \tilde{\varphi}) \) is equal on both boundaries, i.e., \( \sum_{i=1}^{N} e^{-s_i^2} = \sum_{i=1}^{N} e^{-s_i^0} \). In other words, there is no anomaly when each boundary ends on a D0-brane such that both D0-branes are mapped to the same value of \( \tilde{W}(\theta, \tilde{\varphi}) \). One way this can occur is when the boundaries end on coincident D0-branes. Although the condition \( \sum_{i=1}^{N} e^{-s_i^2} = \sum_{i=1}^{N} e^{-s_i^0} \) seems identical to the condition (found by Hori [7, 27]) for the vanishing of

\(^{77}\)This statement follows from the fact that the Lorentz gauge condition can equivalently be written as \( \partial_\nu (\tilde{A}_c^\nu)^+ \psi = 0 \) or \( \langle \psi | \partial_\nu \tilde{A}_c^\nu \rangle^- = 0 \) (where \( \tilde{A}_c^\nu = (\tilde{A}_c^\nu)^+ + (\tilde{A}_c^\nu)^- \) is the decomposition with respect to positive and negative momenta), as well as the fact that \( \partial_\nu \tilde{A}_c^\nu \) commutes with \( \partial_2 \tilde{\varphi}_c \).
the $Q_A^2 \neq 0$ anomaly of the open A-model, this is in fact not true, as the D0-branes do not have to be located at a critical point where $\partial_\theta \tilde{W}_X = 0$ in our case (where $\tilde{W}_X$ is the superpotential in the non-gauged case, which only depends on $\theta$ in the bulk), and the position of each D0-brane (defined by $s_i$ via (5.92)) is in our case constrained by $\sum_i N_i^a s_i - t_a = 0$ instead of just $\sum_i N_i^a \hat{Q}_b s_i - \hat{t}_b = 0$. In conclusion, for abelian $G$, we have found that for a pair of $G$-invariant Lagrangian tori of a toric manifold supporting flat $G$-equivariant $U(1)$ bundles, the quantum anomaly of $Q_A^2 \neq 0$ (which indicates an obstruction to integration over the moduli spaces of open symplectic vortices) vanishes if and only if the values of the superpotential $\tilde{W}(\theta, \vec{\varphi})$ on the mirror B-branes are the same, and in this case, supersymmetry is manifest.

6.5 Mirror Computation of Abelian Invariants

In principle, it is simpler to use the mirror gauged Landau-Ginzburg description of the open gauged A-model to compute open Hamiltonian Gromov-Witten invariants for abelian gauge groups and toric target spaces with $c_1(X) \geq 0$, since there are no open symplectic vortices in this gauged LG model.

We shall focus on the mirror computation of invariants that come from path integrals over the $Q_A$-invariant local observables associated with equivariant cohomology classes, i.e., those given by (6.14) (where $d_G \alpha = 0$) and (6.25) (where $d_G \zeta = 0$). After integrating out the auxiliary fields, the supersymmetry transformations (generated by $\hat{Q}_A$) of the physical fields of the mirror theory on a Euclidean worldsheet parametrized by complex coordinates $(z, \bar{z})$ are (with $\epsilon = \sqrt{2}$)

$$
\begin{align*}
\delta_{Q_A} \tilde{A}_Z &= \tilde{\psi}_{Zc} + i \sqrt{2} \partial_z \tilde{\varphi}_c \\
\delta_{Q_A} \tilde{A}_{Zc} &= \tilde{\psi}_{Zc} + i \sqrt{2} \partial_z \tilde{\varphi}_c \\
\delta_{Q_A} \tilde{\varphi}_c &= 0 \\
\delta_{Q_A} \tilde{c}_c &= \tilde{\eta}_c \\
\delta_{Q_A} \theta^c &= 0 \\
\delta_{Q_A} \overline{\tilde{\varphi}} &= \sqrt{2}(\chi^{bc} - \overline{\chi}^{bc}) \\
\delta_{Q_A} (\chi^{bc} - \overline{\chi}^{bc}) &= 0 \\
\delta_{Q_A} \left[g_{cd}(\chi^{bd} + \overline{\chi}^{bd})\right] &= -\sqrt{2} \partial_\theta \tilde{W}. \\
\end{align*}
$$

The bulk physical operators of this theory were studied by Baptista [12], where he showed that the bulk chiral ring is given by

$$
\mathbb{C}[\tilde{\varphi}^1, \ldots, \tilde{\varphi}^k, (x^1)^{\pm 1}, \ldots, (x^k)^{\pm 1}] / D(\tilde{W}) ,
$$

---

Note that the $G$-invariance of these physical observables implies that they are invariant under $Q_{BRST}$, and therefore also invariant under $Q_A$.

The fermionic fields $\tilde{\eta}_c$ and $\tilde{\kappa}_c$ are related to the fields $\tilde{\lambda}_c$ and $\overline{\lambda}_c$ defined in the previous section via a field redefinition of the form given in (6.32).
i.e., holomorphic functions of \( \varphi_c \) and \((x^i)^{\pm 1} := \exp(\mp \theta^c)\), modulo the ideal \( D(\bar{W}) \), where \( D(\bar{W}) \) is generated by the derivatives

\[
\partial_\theta \bar{W} = -x^c \partial_x \bar{W} = \sum_{j=1}^n \bar{Q}^j \left[ \frac{i}{2} \langle \bar{\varphi}, \nu^j \rangle - e^{-\delta_j} \prod_{d=1}^k (x^d \nu^j)^2 \right].
\]  

(6.74)

In addition, one ought to restrict the bulk physical operators to finite-degree polynomials, since in the equivariant de Rham complex one only considers finite-degree forms and polynomials in the Lie algebra.

Let us now find the elements of the boundary chiral ring, concentrating first on boundary physical operators which come from the matter multiplets. Now, at each boundary, we know that \( \theta^c \) and \( \bar{\theta}^c \) are constants which determine the position of the mirror D0-brane (see (5.92) and footnote 58). Then, via (6.72), we find that \( (\chi^{\theta c} - \bar{\chi}^{\theta c}) = 0 \) on each boundary, and thus \( \chi^\theta c - \bar{\chi}^{\theta c} \) cannot be an operator in the boundary chiral ring. Even \( \theta^c \) and \( \bar{\theta}^c \) cannot be elements of the ring since they are not fields along each boundary; rather, they are constants. On the other hand, \( \chi^{\theta c} + \bar{\chi}^{\theta c} \) is a nonzero field at each boundary. For the mirror of the non-gauged open A-model, due to the critical point condition \( \theta^c \bar{W}_X = 0 \) at the boundaries, the \( k \) fermionic fields \( \chi^{\theta c} + \bar{\chi}^{\theta c} \) are \( Q_A \)-invariant at each boundary and in fact form the boundary chiral ring [11]. However, recall from Section 5.4 that we do not have such a critical point condition, implying that \( \chi^{\theta c} + \bar{\chi}^{\theta c} \) is not \( \bar{Q}_A \)-invariant at the boundaries, and therefore is not an element of the boundary chiral ring for the mirror of the open gauged A-model.

Since there are no \( \bar{Q}_A \)-invariant boundary operators which can be obtained from the matter fields, let us now turn to the vector multiplet fields. From the boundary condition \( \langle \lambda_c - \bar{\lambda}_{+c} \rangle = 0 \), we know that \( \bar{\kappa}_c = 0 \) at each boundary, so it cannot be such an operator. The operator \( \bar{\kappa}_c \) is nonzero at each boundary, and is \( \bar{Q}_A \)-invariant, but it was not included in (6.25) for abelian \( G \) since its anticommuting behaviour implies that it cannot be associated with equivariant cohomology classes, and hence it should not be included as a mirror boundary observable. On the other hand, \( \bar{\varphi}_c \) is \( \bar{Q}_A \)-invariant, and obeys a Neumann boundary condition, and as such is a valid bosonic boundary operator. Thus, the boundary chiral ring at a particular boundary component \( \partial \Sigma_L \) is given by

\[
\mathbb{C}[\varphi^1, \ldots, \varphi^k] / (D(\bar{W})|_{\partial \Sigma_L}).
\]  

(6.75)

Here, we have taken into account the fact that \( \partial_\theta \bar{W} \) is a \( \bar{Q}_A \)-exact function of \( \bar{\varphi}_c \) at the boundaries. Moreover, we ought to restrict the boundary physical operators to finite-degree polynomials, as we did for the bulk physical operators.

Denoting an arbitrary element of the bulk chiral ring (6.73) as \( W_{\text{mirror}} \), and an arbitrary element of a boundary chiral ring (6.75) as \( W_{\partial \Sigma_L}^{\text{mirror}} \), the most general correlation function of local bulk and boundary observables in the gauged Landau-Ginzburg model is therefore written (for \( \Sigma = I \times \mathbb{R} \)) as

\[
\int D(\tilde{A}, \theta, \tilde{\varphi}, \xi, \tilde{\eta}, \tilde{\kappa}, \tilde{\psi}, \chi^\theta, \tilde{b}, \tilde{c}) \ e^{-(S_A + S_{BRST})} \prod_i W_i^{\text{mirror}} \prod_j \left( \frac{\partial}{\partial \Sigma_L[j]} W_{\text{mirror}}^{\partial \Sigma_L} \prod_k W_{\text{mirror}}^{\partial \Sigma_L[k]} \right).
\]  

(6.76)
This is the mirror correlation function which computes (6.27) for local observables.

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