Thompson Sampling with Approximate Inference

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Abstract

We study the effects of approximate inference on the performance of Thompson sampling in the $k$-armed bandit problems. Thompson sampling is a successful algorithm for online decision-making but requires posterior inference, which often must be approximated in practice. We show that even small constant inference error (in $\alpha$-divergence) can lead to poor performance (linear regret) due to under-exploration (for $\alpha < 1$) or over-exploration (for $\alpha > 0$) by the approximation. While for $\alpha > 0$ this is unavoidable, for $\alpha \leq 0$ the regret can be improved by adding a small amount of forced exploration even when the inference error is a large constant.

1 Introduction

The stochastic $k$-armed bandit problem is a sequential decision making problem where at each time-step $t$, a learning agent chooses an action (arm) among $k$ possible actions and observes a random reward. Thompson sampling (Russo et al., 2018) is a popular approach in bandit problems based on sampling from a posterior in each round. It has been shown to have good performance both in term of frequentist regret and Bayesian regret for the $k$-armed bandit problem under certain conditions.

This paper investigates Thompson sampling when only an approximate posterior is available. This is motivated by the fact that in complex models, approximate inference methods such as Markov Chain Monte Carlo or Variational Inference must be used. Along this line, Lu & Van Roy (2017) propose a novel inference method – Ensemble sampling – and analyze its regret for linear contextual bandits. To the best of our knowledge this is the only work with theoretical analysis of Thompson sampling with approximate inference.

This paper analyzes the regret of Thompson sampling with approximate inference. Rather than considering a particular inference algorithm, we parameterize the error using the $\alpha$-divergence, a typical measure of inference accuracy. Our contributions are as follows:

- Even small inference errors can lead to linear regret with naive Thompson sampling. Given any error threshold $\epsilon > 0$ and any $\alpha$ we show that approximate posteriors with error at most $\epsilon$ in $\alpha$-divergence at all times can result in linear regret (both frequentist and Bayesian). For $\alpha > 0$ and for any reasonable prior, we show linear regret due to over-exploration by the approximation (Theorem[1] Corollary[1]). For $\alpha < 1$ and for priors satisfying certain conditions, we show linear regret due to under-exploration by the approximation, which prevents the posterior from concentrating (Theorem[2] Corollary[2]).

- Forced exploration can restore sub-linear regret. For $\alpha \leq 0$ we show that adding forced exploration to Thompson sampling can make the posterior concentrate and restore sub-linear regret (Theorem[3]) even when the error threshold is a very large constant. We illustrate this effect by showing that the performances of Ensemble sampling (Lu & Van Roy 2017)

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We consider the $k$-armed bandit problem parameterized by the mean reward vector $m = (m_1, ..., m_k) \in \mathbb{R}^k$, where $m_i^*$ denotes the mean reward of arm (action) $i$. At each round $t$, the learner chooses an action $A_t$ and observes the outcome $Y_t$, which, conditioned on $A_t$, is independent of the history up to and not including time $t$, $H_{t-1} = (A_1, Y_1, ..., A_{t-1}, Y_{t-1})$. For a time horizon $T$, the goal of the algorithm $\pi$ is to maximize the expected cumulative reward up to time $T$.

Let $\Omega \subseteq \mathbb{R}^k$ be the domain of the mean and $\Omega_i \subseteq \Omega$ denote the region where the $i$th arm has the largest mean. Let the function $A^* : \Omega \rightarrow \{a_1, ..., a_k\}$ denoting the best action be defined as: $A^*(m) = i$ if $m \in \Omega_i$.

In the frequentist setting we assume that there exists a true mean $m^*$ which is fixed and unknown to the learner. Therefore, a policy $\pi^*$ that always chooses $A^*(m^*)$ will get the highest reward. The performance of policy $\pi$ is measured by its expected regret compared to an optimal policy $\pi^*$, which is defined as:

$$\text{Regret}(T, \pi, m^*) = Tm^*_{A^*(m^*)} - \mathbb{E} \sum_{t=1}^{T} m^*_{A_t}.$$ (1)

On the other hand, in the Bayesian setting, we assume that the mean is a random variable $M = (M_1, ..., M_k)$ distributed according to a known prior denoted $\Pi_0$. The Bayesian regret is the expectation of the regret over the prior of parameter $M$:

$$\text{BayesRegret}(T, \pi) = \mathbb{E}_{\Pi_0} \text{Regret}(T, \pi, M)$$ (2)

### 2.2 Thompson Sampling with Approximate Inference

In the frequentist setting, in order to perform Thompson sampling we define a prior which is only used in the algorithm. On the other hand, in the Bayesian setting the prior is given.

Let $\Omega_t$ be the posterior distribution of $M|H_{t-1}$ with density function $\pi_t(m)$. Thompson sampling obtains a sample $\tilde{m}_t$ from $\Pi_t$ and then selects arm $A_t$ as follow: $A_t = i$ if $\tilde{m}_t \in \Omega_i$. In each round, we assume an approximate sampling method is available that generates sample from an approximate distribution $Q_t$. We use $q_t$ to denote the density function of $Q_t$.

Popular approximate sampling methods include Markov Chain Monte Carlo (MCMC) (Andrieu et al., 2003), Sequential Monte Carlo (Doucet & Johansen, 2009) and Variational Inference (VI) (Blei et al., 2017). There are packages that conveniently implement VI and MCMC methods, such as Stan (Carpenter et al., 2017), Edward (Tran et al., 2016), PyMC (Salvatier et al., 2016) and infer.NET (Minka et al., 2018).

To provide a general analysis of approximate sampling methods, we will use the $\alpha$-divergence (Section 2.3) to quantify the distance between the posterior $\Pi_t$ and the approximation $Q_t$.

### 2.3 The Alpha Divergence

The $\alpha$-divergence between two distributions $P$ and $Q$ with density functions $p(x)$ and $q(x)$ is defined as:

$$D_\alpha(P, Q) = \frac{1 - \int p(x)^\alpha q(x)^{1-\alpha} dx}{\alpha(1-\alpha)}.$$ (3)

$\alpha$-divergence generalizes many divergences, including $KL(Q, P)$ ($\alpha \to 0$), $KL(P, Q)$ ($\alpha \to 1$), Hellinger distance ($\alpha = 0.5$) and $\chi^2$ divergence ($\alpha = 2$) and is a common way to measure errors in inference methods. MCMC errors are measured by the Total Variation distance, which can be upper
bounded by the KL divergence using Pinsker’s inequality ($\alpha = 0$ or $\alpha = 1$). Variational Inference tries to minimize the reverse KL divergence between the target distribution and the approximation ($\alpha = 0$). Ensemble sampling (Lu & Van Roy, 2017) provides error guarantees using reverse KL divergence ($\alpha = 0$). Expectation Propagation tries to minimize the KL divergence ($\alpha = 1$) and $\chi^2$. Variational Inference tries to minimize the $\chi^2$ divergence ($\alpha = 2$).

![Figure 1: The Gaussian $Q$ which minimizes $D_\alpha(P, Q)$ for different values of $\alpha$ where the target distribution $P$ is a mixture of two Gaussians. Based on Figure 1 from (Minka, 2005)](image)

When $\alpha$ is small, the approximation fits the posterior’s dominant mode. When $\alpha$ is large, the approximation covers the posterior’s entire support (Minka, 2005) as illustrated in Figure 1. Therefore changing $\alpha$ will affect the exploration-exploitation trade-off in bandits problem.

2.4 Problem Statement.

Problem Statement. For the $k$-armed bandit problem, given $\alpha$ and $\epsilon > 0$, if at all time-step $t$ we sample from an approximate distribution $Q_t$ such that $D_\alpha(\Pi_t, Q_t) < \epsilon$, will the regret be sub-linear in $t$?

3 Motivating Example

In this section we present a simple example to show the effects of inference errors on the frequentist regret.

Example. Consider a 2-armed bandit problem where the reward distributions are $\text{Norm}(0.6, 0.2^2)$ and $\text{Norm}(0.5, 0.2^2)$ for arm 1 and 2 respectively. The prior $\Pi_0$ is $\text{Norm}(\mu_0^T, 0.5^2 I)$ where $\mu_0 = [0.1, 0.9]$ is the prior means of arm 1 and 2 respectively, and $I$ denotes the identity matrix.

Let $\Pi_t = \text{Norm}(\mu_t, \Sigma_t)$ be the posterior at time $t$. Approximations $Q_t$ and $Z_t$ are calculated such that $\text{KL}(\Pi_t, Q_t) = 2$ and $\text{KL}(Z_t, \Pi_t) = 1.5$ by multiplying the co-variance $\Sigma_t$ by a constant: $Q_t = \text{Norm}(\mu_t, 4.5^2 \Sigma_t)$ and $Z_t = \text{Norm}(\mu_t, 0.3^2 \Sigma_t)$. The KL divergence between 2 Gaussian distributions is provided in Appendix F.

We perform the following simulations 1000 times and plot the mean cumulative regret up to time $T = 100$ in Figure 2b:

1. At each time-step $t$, sample from the true posterior $\Pi_t$.
2. At each time-step $t$, compute $Q_t$ from $\Pi_t$ and sample from $Q_t$.
3. At each time-step $t$, compute $Z_t$ from $\Pi_t$ and sample from $Z_t$.

The regret of sampling from the approximations $Q_t$ and $Z_t$ are both larger than that of exact Thompson sampling. Intuitively, the regret of $Q_t$ is larger because $Q_t$ explores more than the true posterior (Figure 2a). In Section 4 we show that when $\alpha > 0$ the approximation can incur this type of error, leading to linear regret. On the other hand, the regret of $Z_t$ is larger because $Z_t$ explores less than the exact Thompson sampling algorithm and therefore commits to the sub-optimal arm (Figure 2a). In Section 5 we show that when $\alpha < 1$ the approximation can change the posterior concentration rate, leading to linear regret. We also show that adding a uniform sampling step can help the posterior to concentrate when $\alpha \leq 0$, and make the regret sub-linear.
Over-dispersed (approximation $Q_t$) and under-dispersed sampling (approximation $Z_t$) yield different posteriors after $T = 100$ time-steps. $m_1$ and $m_2$ are the means of arms 1 and 2. $Q_t$ picks arm 2 more often than exact Thompson sampling and $Z_t$ mostly picks arm 2. The posteriors of exact Thompson sampling and $Q_t$ concentrate mostly in the region where $m_1 > m_2$ while $Z_t$’s spans both regions.

The regret of sampling from the approximations $Q_t$ and $Z_t$ are both larger than that of exact Thompson sampling from the true posterior $\Pi_t$. Shaded regions show 95% confidence intervals.

Figure 2: Approximation $Q_t$ (with high variance) and approximation $Z_t$ (with small variance) are defined in Section 3 where $D_1(\Pi_t, Q_t) = 2$ and $D_0(\Pi_t, Z_t) = 1.5$. Arm 1 is the true best arm.

4 Regret Analysis When $\alpha > 0$

In this section we analyze the regret when $\alpha > 0$. Our result shows that the approximate method might pick the sub-optimal arm with constant probability in every time-step, leading to linear regret.

**Theorem 1** (Frequentist Regret). Let $\alpha > 0$, the number of arms be $k = 2$ and $m_1^* > m_2^*$. Let $\Pi_0$ be a prior where $P_{\Pi_0}(M_2 > M_1) > 0$. For any error threshold $\epsilon > 0$, there is a deterministic mapping $f(\Pi)$ such that for all $t \geq 0$:

1. Sampling from $Q_t = f(\Pi_t)$ chooses arm 2 with a constant probability.
2. $D_\alpha(\Pi_t, Q_t) < \epsilon$.

Therefore sampling from $Q_t$ for $T/10$ time-steps and using any policy for the remaining time-steps will cause linear frequentist regret.

At every time-step, the mapping $f$ constructs the approximation $Q_t$ from the posterior $\Pi_t$ by moving probability mass from the region $\Omega_1$ where $m_1 > m_2$ to the region $\Omega_2$ where $m_2 > m_1$. Then $Q_t$ will choose arm 2 with a constant probability at every time-step. The constant average regret per time-step is discussed in Appendix A.4.

Therefore, if we sample from $Q_t = f(\Pi_t)$ for $0.1T$ time steps and use any policy in the remaining $0.9T$ time steps, we will still incur linear regret from the $0.1T$ time-steps. On the other hand, when $\alpha \leq 0$, we show in Section 5.1 that sampling an arm uniformly at random for $\log T$ time-steps and sampling from an approximate distribution that satisfies the divergence constraint for $T - \log T$ time-steps will result in sub-linear regret.

Agrawal & Goyal (2013) show that the frequentist regret of exact Thompson sampling is $O(\sqrt{T})$ with Gaussian or Beta priors and bounded rewards. Theorem 1 implies that when the assumptions in (Agrawal & Goyal, 2013) are satisfied but there is a small constant inference error at every time-step, the regret is no longer guaranteed to be sub-linear.

If the assumption $m_1^* > m_2^*$ in Theorem 1 is satisfied with a non-zero probability ($P_{\Pi_0}(M_1 > M_2) > 0$), the Bayesian regret will also be linear:

**Corollary 1** (Bayesian Regret). Let $\alpha > 0$ and the number of arms be $k = 2$. Let $\Pi_0$ be a prior where $P_{\Pi_0}(M_1 > M_2) > 0$ and $P_{\Pi_0}(M_2 > M_1) > 0$. Then for any error threshold $\epsilon > 0$, there is a deterministic mapping $f(\Pi)$ such that for all $t \geq 0$ the 2 statements in Theorem 1 hold.
Therefore sampling from $Q_t$ for $T/10$ time-steps and using any policy for the remaining time-steps will cause linear Bayesian regret.

[2] Russo & Roy (2016) prove that the Bayesian regret of Thompson sampling for $k$-armed bandits with sub-Gaussian rewards is $O(\sqrt{T})$. Corollary 1 implies that even when the assumptions in [2] are satisfied, under certain conditions and with approximation errors, the regret is no longer guaranteed to be sub-linear.

5 Regret Analysis When $\alpha < 1$

In this section we analyze the regret when $\alpha < 1$. Our result shows that for any error threshold, if the posterior $\Pi_t$ places too much probability mass on the wrong arm then the approximation $Q_t$ is allowed to avoid the optimal arm. If the sub-optimal arms do not provide information about the arms’ ranking, the posterior $\Pi_{t+1}$ does not concentrate. Therefore $Q_{t+1}$ is also allowed to be close in $\alpha$-divergence while avoiding the optimal arm, leading to linear regret in the long term.

**Theorem 2** (Frequentist Regret). Let $\alpha < 1$, the number of arms be $k = 2$ and $m_*^1 > m_*^2$. Let $\Pi_0$ be a prior where $M_2$ and $M_1 - M_2$ are independent. There is a deterministic mapping $f(\Pi)$ such that for all $t \geq 0$:

1. Sampling from $Q_t = f(\Pi_t)$ chooses arm 2 with probability 1.
2. For any $\epsilon > 0$, there exists $0 < z \leq 1$ such that if $P_{\Pi_0}(M_2 > M_1) = z$ and arm 2 is chosen at all times before $t$ then $D_\alpha(\Pi_t, Q_t) < \epsilon$.  
   For any $0 < z \leq 1$, there exists $\epsilon > 0$ such that if $P_{\Pi_0}(M_2 > M_1) = z$ and arm 2 is chosen at all times before $t$ then $D_\alpha(\Pi_t, Q_t) < \epsilon$.

Therefore sampling from $Q_t$ at all time-steps results in linear frequentist regret.

We discuss why the above results are not immediately obvious. When $\alpha \to 0$, the $\alpha$-divergence becomes $\text{KL}(Q_t, \Pi_t)$. We might believe that the regret should be sub-linear in this case because the posterior $\Pi_t$ becomes more concentrated, and so the total variation between $Q_t$ and $\Pi_t$ must decrease. For example, [Ordentlich & Weinberger (2004)] show the distribution-dependent Pinsker’s inequality between $\text{KL}(Q, P)$ and the total variation $TV(P, Q)$ for discrete distributions $P$ and $Q$ as follows:

$$\text{KL}(Q, P) \geq \phi(P) \cdot TV(P, Q)^2.$$  (4)

Here, $\phi(P)$ is a quantity that will increase to infinity if $P$ becomes more concentrated. However, the algorithm in Theorem 2 constructs an approximation distribution that never pick the optimal arm, so the posterior $\Pi_t$ cannot concentrate and the regret is linear. The error threshold $\epsilon$ causing linear frequentist regret is correlated with the probability mass the prior places on the true best arm (Appendix B.3).

With some assumptions on the rewards, [Gopalan et al. (2013)] show that the problem-dependent frequentist regret is $O(\log T)$ for finitely-supported, correlated priors with $\pi_0(m^*) > 0$. Theorem 2 implies that when the assumptions in [1] are satisfied, if $M_2$ and $M_1 - M_2$ are independent and there are approximation errors, the regret is no longer guaranteed to be sub-linear.

If the assumption $m_*^1 > m_*^2$ in Theorem 2 is satisfied with a non-zero probability ($P_{\Pi_0}(M_1 > M_2) > 0$), the Bayesian regret will also be linear:

**Corollary 2** (Bayesian Regret). Let $\alpha < 1$ and the number of arms be $k = 2$. Let $\Pi_0$ be a prior where $P_{\Pi_0}(M_1 > M_2) > 0$ and $M_2$ and $M_1 - M_2$ are independent. There is a deterministic mapping $f(\Pi)$ such that for all $t \geq 0$ the 2 statements in Theorem 2 hold.

Therefore sampling from $Q_t$ at all time-steps results in linear Bayesian regret.

[Russo & Roy (2016)] prove that the Bayesian regret of Thompson sampling for $k$-armed bandits with sub-Gaussian rewards is $O(\sqrt{T})$. Corollary 2 implies that even when the assumptions in [Russo & Roy (2016)] are satisfied, under certain conditions and with approximation errors, the regret is no longer guaranteed to be sub-linear.

We note that, unlike the case when $\alpha > 0$, if we use another policy in $o(T)$ time-steps to make the posterior concentrate and sample from $Q_t$ for the remaining time-steps, the regret can be sub-linear. We provide a concrete algorithm in Section 5.1 for the case when $\alpha \leq 0$.  

5 Regret Analysis When $\alpha < 1$
5.1 Algorithms with Sub-linear Regret for $\alpha \leq 0$

In the previous section, we see that when $\alpha < 1$, the approximation has linear regret because the posterior does not concentrate. In this section we show that when $\alpha \leq 0$, it is possible to achieve sub-linear regret even when $\epsilon$ is a very large constant by adding a simple exploration step to force the posterior to concentrate (the case of $\alpha > 0$ cannot be improved according to Theorem 1). We first look at the necessary and sufficient condition that will make the posterior concentrate, and then provide an algorithm that satisfies it. Russo (2016) and Qin et al. (2017) both show the following result under different assumptions:

**Lemma 1** (Lemma 14 from Russo (2016)). Let $m^* \in \mathcal{R}^k$ be the true parameter and let $a^* = A^*(m^*)$ be the true best arm. If for all arms $i$, $\sum_{t=1}^{\infty} P(A_t = i | H_{t-1}) = \infty$, then

$$\lim_{t \to \infty} P(A^*(M) = a^* | H_{t-1}) = 1 \text{ with probability } 1.$$  (5)

If there exists arm $i$ such that $\sum_{t=1}^{\infty} P(A_t = i | H_{t-1}) < \infty$, then $\lim_{t \to \infty} P(A^*(M) = i | H_{t-1}) > 0$ with probability 1.

Russo (2016) make the following assumptions, which allow correlated priors:

**Assumption 1.** Let the reward distributions be in the canonical one dimensional exponential family with the density: $p(y|m) = b(y) \exp(mT(y) - A(m))$ where $b, T$ and $A$ are known function and $A(m)$ is assumed to be twice differentiable. The parameter space $\Omega = (\overline{m}, \underline{m})$ is a bounded open hyper-rectangle, the prior density is uniformly bounded with $0 < \inf_{m \in \Omega} \pi_0(m) < \sup_{m \in \Omega} \pi_0(m) < \infty$ and the log-partition function has bounded first derivative with $\sup_{\theta \in [\overline{m}, \underline{m}]} |A'(m)| < \infty$.

Qin et al. (2017) make the following assumptions:

**Assumption 2.** Let the prior be an uncorrelated multivariate Gaussian. Let the reward distribution of arm $i$ be $\text{Norm}(m_i, \sigma^2)$ with a common known variance $\sigma^2$ but unknown mean $m_i$.

Even though we consider the error in sampling from the posterior distribution, the regret is a result of choosing the wrong arm. We define $\tilde{\Pi}_t$ as the posterior distribution of the best arm and $\tilde{Q}_t$ as the approximation of $\tilde{\Pi}_t$ with the density functions

$$\pi_t(i) = P(A^* = i | H_{t-1}) \text{ and } \overline{\pi}_t(i) = P(A_t = i | H_{t-1}).$$

We now define an algorithm where each arm will be chosen infinitely often, satisfying the condition of Lemma 1.

**Theorem 3** (Bayesian and Frequentist Regret). Consider the case when Assumption 1 or 2 is satisfied. Let $\alpha \leq 0$ and $p_t = o(1)$ be such that $\sum_{t=1}^{\infty} p_t = \infty$. For any number of arms $k$, any prior $\Pi_0$ and any error threshold $\epsilon > 0$, the following algorithm has $o(T)$ frequentist regret: at every time-step $t$,

- with probability $1 - p_t$, sample from an approximate posterior $Q_t$ such that $D_\alpha(\Pi_t, Q_t) < \epsilon$,
- with probability $p_t$, sample an arm uniformly at random.

Since the Bayesian regret is the expectation of the frequentist regret over the prior, for any prior if the frequentist regret is sub-linear at all points the Bayesian regret will be sub-linear.

The following lemma shows that the error in choosing the arms is upper bounded by the error in choosing the parameters. Therefore whenever the condition $D_\alpha(\Pi_t, Q_t) < \epsilon$ is satisfied, the condition $D_\alpha(\tilde{\Pi}_t, \tilde{Q}_t) < \epsilon$ will be satisfied and Theorem 3 is applicable.

**Lemma 2.**

$$D_\alpha(\Pi_t, Q_t) \leq D_\alpha(\tilde{\Pi}_t, \tilde{Q}_t).$$

We also note that we can achieve sub-linear regret even when $\epsilon$ is a very large constant. We revisit Eq. 4 to provide the intuition: $\text{KL}(Q, P) \geq \phi(P) \cdot \text{TV}(P, Q)^2$. Here, $\phi(P)$ is a quantity that will increase to infinity if $P$ becomes more concentrated. Hence, if $KL(\tilde{Q}_t, \tilde{\Pi}_t) < \epsilon$ for any constant $\epsilon$ and $\tilde{\Pi}_t$ becomes concentrated, the total variation $\text{TV}(\tilde{Q}_t, \tilde{\Pi}_t)$ will decrease. Therefore, $\tilde{Q}_t$ will become concentrated, resulting in sub-linear regret.
Application. Lu & Van Roy (2017) propose an approximate sampling method called Ensemble sampling where they maintain a set of $M$ models to approximate the posterior and analyze its regret for the linear contextual bandits when $M$ is $\Omega(\log(T))$. For the $k$-armed bandit problem and when $M$ is $\Theta(\log(T))$, Ensemble sampling satisfies the condition $KL(Q_t, \Pi_t) < \epsilon$ in Theorem 3 with high probability. In this case, Lu & Van Roy (2017) show a regret bound that scales linearly with $T$.

We discuss in Appendix E how to apply Theorem 3 to get sub-linear regret with Ensemble sampling when $M$ is $\Theta(\log(T))$.

6 Simulations

For each approximation method we repeat the following simulations for 1000 times and plot the mean cumulative regret. At each time-step $t$:

1. Perform exact Thompson sampling (for reference).
2. Sample from the approximation to choose an action and update the posterior.
3. Sample from the approximation to choose an action but do not update the posterior. The posterior is updated by exact Thompson sampling at all time-steps. We do this to understand how the approximation affects the posterior (discussed in Section 6.3).
4. With a probability (the exploration rate), choose an action uniformly at random and update the posterior. Otherwise, sample from the approximation to choose an action and update the posterior.

6.1 Adding Forced Exploration to the Motivating Example

In this section we revisit the example in Section 3. We apply $Q_t$, $Z_t$ and Ensemble sampling with $M = 2$ models to the bandit problem described in the example. We set the exploration rate at time $t$ to be $1/t$, $T = 100$, show the results in Figure 3a and discuss them in Section 6.3.

6.2 Simulations of Ensemble Sampling and Variational Inference for 50-armed bandits

Now we add forced exploration to mean-field Variational Inference (VI) and Ensemble Sampling with $M = 5$ models for a 50-armed bandit instance. We generate the prior and the reward distribution as follows: the prior is $\text{Norm}(0, \Sigma_0)$. To generate a positive semi-definite matrix $\Sigma_0$, we generate a random matrix $A$ of size $(k, k)$ where entries are uniformly sampled from $[0, 1)$ and set $\Sigma_0 = A^T A / k$.

The true mean $m^*$ is sampled from the prior. The reward distribution of arm $i$ is $\text{Norm}(m^*_i, 1)$.

Mean-field VI approximates the posterior by finding an uncorrelated multivariate Gaussian distribution $Q_t$ that minimizes $KL(\Pi_t, Q_t)$. If the posterior is $\Pi_t = \text{Norm}(\mu_t, \Sigma_t)$ then $Q_t$ has the closed-form solution $Q_t = \text{Norm}(\mu_t, \text{Diag}(\Sigma_t^{-1})^{-1})$, which we used to perform the simulations. We set the exploration rate at time $t$ to be $50/t$, $T = 3000$, show the results in Figure 3b and discuss them in Section 6.3.

6.3 Discussion

We observe in Figure 3a that the regret of $Q_t$ calculated from the posterior updated by exact Thompson sampling does not change significantly, which implies that $Q_t$ has the same effect on the posterior as exact Thompson sampling. Therefore adding forced exploration is not helpful.

On the other hand, in Figures 3a and 3b the regrets of $Z_t$, Ensemble sampling and mean-field VI calculated from the posterior updated by exact Thompson sampling decrease significantly, suggesting that these methods do not explore enough for the posterior to concentrate. Therefore adding forced exploration is helpful, which is compatible with the result in Theorem 3.

7 Related Work

There have been many works on sub-linear Bayesian and frequentist regrets for exact Thompson sampling. We discussed relevant works in details in Section 4 and Section 5.

To the best of our knowledge, Ensemble sampling (Lu & Van Roy 2017) is the only work that provides a theoretical analysis of Thompson sampling with approximate inference. Lu & Van Roy (2017)
(a) Applying approximations \( Q_t \), \( Z_t \) and Ensemble Sampling to the motivating example (Section 6.1).

(b) Applying mean-field Variational Inference (VI) and Ensemble sampling on a 50-armed bandit (Section 6.2).

Figure 3: Updating the posterior by exact Thompson sampling or adding forced exploration does not help the over-explored approximation \( Q_t \), but lowers the regrets of the under-explored approximations \( Z_t \), Ensemble sampling and mean-field VI. Shaded regions show 95% confidence intervals.

There has been a number of empirical works using approximate methods to perform Thompson sampling. Riquelme et al. (2018) implement variational inference, MCMC, Gaussian processes and other methods on synthetic and real world data sets and measure the regret. Urteaga & Wiggins (2018) derive a variational method for contextual bandits. Kawale et al. (2015) use particle filtering to implement Thompson sampling for matrix factorization.
8 Conclusion

In this paper we analyzed the performance of approximate Thompson sampling when at each time-step \( t \), the algorithm obtains a sample from an approximate distribution \( Q_t \) such that the \( \alpha \)-divergence between the true posterior and \( Q_t \) remains at most a constant \( \epsilon \) at all time-steps.

Our results have the following implications. To achieve a sub-linear regret, we can only use \( \alpha > 0 \) for \( o(T) \) time-steps. Therefore we should use \( \alpha \leq 0 \) with forced exploration to make the posterior concentrate. This method theoretically guarantees a sub-linear regret even when \( \epsilon \) is a large constant.
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A Proof of Theorem 1 and Corollary 1

First we will prove Theorem 1. Let $\Omega_i \subseteq \Omega$ denote the region where arm $i$ is the best arm. Let $\Pi_{t,i}$ denote $\Pi_t(\Omega_i)$, the posterior probability that arm $i$ is the best arm. For $r > 1$, We construct the pdf of $Q_t$’s as follows:

$$ q_t(m) = \begin{cases} \frac{1}{T} \pi_t(m), & \text{if } m_1 > m_2 \\ \frac{1 - \Pi_{t,1}/r}{1 - \Pi_{t,1}} \pi_t(m), & \text{otherwise.} \end{cases} \tag{6} $$

We will prove the theorem by the following steps:

- In Lemma 3, we show that $Q_t$’s are valid distributions.
- In Lemma 4, we show that when $\alpha > 0$ the $\alpha$-divergence between $Q_t$ and $\Pi_t$ can be arbitrarily small.
- In Lemma 5, we show that sampling from $Q_t$ for $\Theta(T)$ time-steps will generate linear frequentist regret, and lower bound the regret.

Since the regret is linear, in Appendix A.4 we discuss the constant average regret per time-step as a function of $\epsilon$ and $\alpha$. In Appendix A.5 we provide the Bayesian regret proof for Corollary 1.

**Lemma 3.** $q_t(m)$ in Eq. 6 is well-defined and if $\int \pi_t(m)dm = 1$ then:

$$ \int q_t(m)dm = 1. $$

**Lemma 4.** When $\alpha > 0$, for all $\epsilon > 0$, for all $\Pi_t$, there exists $r > 1$ such that when $Q_t$’s are constructed from $r$ as shown in Eq. 6 $D_\alpha(\Pi_t, Q_t) < \epsilon$.

**Lemma 5.** The expected frequentist regret of the policy that constructs $Q_t$’s as in Eq. 6 and sample from $Q_t$ for $T' = \Theta(T)$ time-steps is linear and the lower bound of the average regret per time-step is

$$ L = \begin{cases} c\Delta(1 - (1 - \epsilon\alpha(1 - \alpha))^{\frac{1}{1 - \alpha}}), & \text{when } \alpha > 1 \text{ and } 0 < \epsilon \\ c\Delta(1 - \frac{1}{\epsilon}), & \text{when } \alpha = 1 \text{ and } 0 < \epsilon \\ c\Delta(1 - (1 - \epsilon\alpha(1 - \alpha))^{\frac{1}{1 - \alpha}}), & \text{when } 0 < \alpha < 1 \text{ and } 0 < \epsilon \leq \frac{1}{\alpha(1 - \alpha)}. \end{cases} $$

where $c = \frac{T'}{T}$ is $\Theta(1)$.

A.1 Proof of Lemma 3

**Proof.** First we will show that $\Pi_{t,2} = 1 - \Pi_{t,1} > 0$ for all $t \geq 0$, so that $q_t(m)$ is well-defined. We have $\Pi_{0,2} = P(M_2 > M_1) > 0$ by assumption. Let $S_t = \{m \in \Omega_2 : \pi_t(m) > 0\}$ be the support of $\Pi_t$ in $\Omega_2$. If $\pi_0(m) > 0$, then $\pi_t(m) > 0$ because $\pi_t(m)$ is the product of $\pi_0(m)$ and non-zero likelihoods. Therefore $S_0 \subseteq S_t$.

Since $P(M_2 > M_1) = \int_{S_0} \pi_0(m)dm > 0, \int_{S_0} dm > 0$, $\int_{S_t} \pi_t(m)dm > 0$ since $S_0 \subseteq S_t, \int_{S_t} dm > 0$. Therefore $\int_{S_t} \pi_t(m)dm > 0$ since $S_t = \{m \in \Omega_2 : \pi_t(m) > 0\}$ by definition. Then $\Pi_{t,2} = \int_{\Omega_2} \pi_t(m)dm = \int_{S_t} \pi_t(m)dm > 0$.
Assume that \( \int \pi_t(m)dm = 1 \), we will show that \( \int q_t(m)dm = 1 \):

\[
\int q_t(m)dm \\
= \int_{\Omega_1} q_t(m)dm + \int_{\Omega_2} q_t(m)dm \\
= \int_{\Omega_1} \frac{1}{r} \pi_t(m)dm + \int_{\Omega_2} \frac{1 - \Pi_{t,1}/r}{1 - \Pi_{t,1}} \pi_t(m)dm \\
= \frac{1}{r} \Pi_{t,1} + \frac{1 - \Pi_{t,1}/r}{1 - \Pi_{t,1}} \Pi_{t,2} \\
= \frac{1}{r} \Pi_{t,1} + \frac{1 - \Pi_{t,1}/r}{1 - \Pi_{t,1}} (1 - \Pi_{t,1}) \\
= 1
\]

\( \square \)

A.2 Proof of Lemma 4

\( \text{Proof.} \) First we calculate the \( \alpha \)-divergence between \( \Pi_t \) and \( Q_t \) constructed in Eq. 6. Let \( \Omega_1 \subseteq \Omega \) denote the region where \( m_1 > m_2 \) and \( \Omega_2 \subseteq \Omega \) denote the region where \( m_2 \geq m_1 \).

When \( \alpha > 0, \alpha \neq 1 \) we have:

\[
D_\alpha(\Pi_t, Q_t) \\
= 1 - \int \left( \frac{\pi_t(m)}{q_t(m)} \right)^\alpha q_t(m)dm \\
= 1 - \int_{\Omega_1} \left( \frac{\pi_t(m)}{q_t(m)} \right)^\alpha q_t(m)dm - \int_{\Omega_2} \left( \frac{\pi_t(m)}{q_t(m)} \right)^\alpha q_t(m)dm \\
= \frac{1 - \int_{\Omega_1} (r)^\alpha q_t(m)dm - \int_{\Omega_2} \left( \frac{1 - \Pi_{t,1}}{1 - \Pi_{t,1}/r} \right)^\alpha q_t(m)dm}{\alpha(1 - \alpha)} \\
= \frac{1 - Q_t(\Omega_1) (r)^\alpha - Q_t(\Omega_2) \left( \frac{1 - \Pi_{t,1}}{1 - \Pi_{t,1}/r} \right)^\alpha}{\alpha(1 - \alpha)} \\
= \frac{1 - \frac{\Pi_{t,1}}{r} (r)^\alpha - \left( 1 - \frac{\Pi_{t,1}}{r} \right) \left( \frac{1 - \Pi_{t,1}}{1 - \Pi_{t,1}/r} \right)^\alpha}{\alpha(1 - \alpha)} \\
= \frac{1}{\alpha(1 - \alpha)} \left( 1 - \Pi_{t,1}r^{-1+\alpha} - (1 - \Pi_{t,1})^\alpha \left( 1 - \frac{\Pi_{t,1}}{r} \right)^{1-\alpha} \right) \tag{7}
\]

When \( \alpha = 1 \):

\[
D_\alpha(\Pi_t, Q_t) \\
= \int \pi_t(m) \log \left( \frac{\pi_t(m)}{q_t(m)} \right) dm \\
= \int_{\Omega_1} \pi_t(m) \log \frac{\pi_t(m)}{q_t(m)} dm + \int_{\Omega_2} \pi_t(m) \log \frac{\pi_t(m)}{q_t(m)} dm \\
= \int_{\Omega_1} \pi_t(m) \log(r)dm \\
+ \int_{\Omega_2} \pi_t(m) \log \frac{1 - \Pi_{t,1}}{1 - \Pi_{t,1}/r} dm \\
= \Pi_{t,1} \log(r) + (1 - \Pi_{t,1}) \log \frac{1 - \Pi_{t,1}}{1 - \Pi_{t,1}/r}
\]

We will now upper bound the above expression. Consider 2 cases
• $\alpha = 1$: We have
\[
D_\alpha(\Pi_t, Q_t) = \Pi_{t,1} \log(r) + (1-\Pi_{t,1}) \log \frac{1-\Pi_{t,1}}{1-\Pi_{t,1}/r} \\
\leq \Pi_{t,1} \log(r) + (1-\Pi_{t,1}) \log(r) \text{ because } r > 1 \\
\leq \log(r)
\]

• $\alpha > 0, \alpha \neq 1$: The following inequality is true by simple calculations when $0 < \alpha < 1$ or $\alpha > 1$:
\[
\frac{\left(\frac{1-\Pi_{t,1}}{1-\Pi_{t,1}/r}\right)^{\alpha-1}}{\alpha(\alpha-1)} \leq \frac{r^{\alpha-1}}{\alpha(\alpha-1)}.
\]
(8)

Then we will have:
\[
D_\alpha(\Pi_t, Q_t) = \Pi_{t,1} r^{\alpha-1} + (1-\Pi_{t,1}) \left(\frac{1-\Pi_{t,1}}{1-\Pi_{t,1}/r}\right)^{\alpha-1} - 1 \\
\leq \frac{1}{\alpha(\alpha-1)} (\Pi_{t,1} r^{\alpha-1} + (1-\Pi_{t,1}) r^{\alpha-1} - 1) \\
= \frac{1}{\alpha(\alpha - 1)} (r^{-1+\alpha} - 1).
\]

Therefore $D_\alpha(\Pi_t, Q_t)$ is upper bounded by:
\[
\begin{cases} 
\frac{1-r^{\alpha-1}}{\alpha(1-\alpha)}, & \text{if } 0 < \alpha < 1 \text{ or } \alpha > 1 \\
\log(r), & \text{if } \alpha = 1
\end{cases}
\]
(9)

Since $\lim_{r \to 1} \log(r) = 0$ and $\lim_{r \to 1} \frac{1-r^{-1+\alpha}}{\alpha(1-\alpha)} = 0$, for any $\epsilon > 0$, there exists $r > 1$ such that
\[
D_\alpha(\Pi_t, Q_t) \leq \epsilon.
\]

A.3 Proof of Lemma 5

Proof. We will now lower bound the regret as a function of $\epsilon$.

For any posterior $\Pi_t$, since the approximate algorithm sampling from $Q_t$ picks the optimal arm with probability at most $1/r$ then it picks arm 2 with probability at least $1 - 1/r$.

Since we sample from $Q_t$ for $T'$ time steps, the lower bound of the average expected regret per time step is:
\[
L = \frac{T'}{T} (m_1^* - m_2^*)(1 - 1/r) = c\Delta(1 - 1/r)
\]
where $\Delta = m_1^* - m_2^*$ and $c = \frac{T'}{T}$ is $\Theta(1)$.

We calculate $\epsilon$ as a function of $r$ from Eq. 9:
\[
\epsilon = \begin{cases} 
\frac{1-r^{-1+\alpha}}{\alpha(1-\alpha)}, & \text{if } \alpha \neq 1 \\
\log(r), & \text{if } \alpha = 1
\end{cases}
\]

The functions are continuous when $r > 1$. Then by direct calculations when $r \to \infty$ and $r \to 1$, the domain of $\epsilon$ is:

$0 < \epsilon$ when $\alpha \geq 1$.

$0 < \epsilon < \frac{1}{\alpha(1-\alpha)}$ when $0 < \alpha < 1$.  

Then
\[ r = \begin{cases} 
(1 - \epsilon \alpha (1 - \alpha))^{\frac{1}{1+\alpha}} & \text{when } \alpha > 1 \text{ and } 0 < \epsilon \\
\epsilon^\alpha & \text{when } \alpha = 1 \text{ and } 0 < \epsilon \\
(1 - \epsilon \alpha (1 - \alpha))^\frac{1}{1+\alpha} & \text{when } 0 < \alpha < 1 \text{ and } 0 < \epsilon \leq \frac{1}{\alpha (1 - \alpha)}. 
\end{cases} \]

Therefore we can calculate the lower bound of the regret per time-step as:
\[ L = \begin{cases} 
\epsilon \Delta (1 - (1 - \epsilon \alpha (1 - \alpha))^{\frac{1}{1+\alpha}}) & \text{when } \alpha > 1 \text{ and } 0 < \epsilon \\
\epsilon \Delta (1 - \frac{1}{\alpha}) & \text{when } \alpha = 1 \text{ and } 0 < \epsilon \\
\epsilon \Delta (1 - (1 - \epsilon \alpha (1 - \alpha))^\frac{1}{1+\alpha}) & \text{when } 0 < \alpha < 1 \text{ and } 0 < \epsilon \leq \frac{1}{\alpha (1 - \alpha)}. 
\end{cases} \]

We plot the lower bound of the average regret per time step when \( \Delta = 0.1 \) as a function of \( \epsilon \) in Fig. 4.

**A.4 The Average Regret per Time-step**

To understand how the constant average regret per time-step depends on \( \epsilon \) and \( \alpha \), we plot in Figure 4 the lower bound of the average regret per time-step in Lemma 5 as a function of \( \epsilon \) in the following setting of the example constructed in the proof of Theorem 1. The algorithm samples from \( Q_t \) at \( T/2 \) time-steps and \( \Delta = 0.1 \). In this case the average regret per time step is upper bounded by \( \Delta/2 = 0.05 \). The formula and proof are detailed in Lemma 5 in Appendix A. When \( \alpha \) is around 1, the lower bound, and therefore the average regret per time-step, converges the fastest to \( \Delta/2 \) as \( \epsilon \) increases. When \( \alpha \) is very large or close to 0, the lower bound grows slowly as \( \epsilon \) increases.

**A.5 Proof of Corollary 1**

Since \( \mathbb{P}(M_1 > M_2) > 0 \), there exist constants \( \Delta > 0, \gamma > 0 \) such that \( \mathbb{P}(M_1 - M_2 \geq \Delta) = \gamma \). The probability that the assumption \( m_1^* > m_2^* \) in Theorem 1 is satisfied is at least \( \gamma > 0 \). Therefore the expected regret over the prior is at least \( \gamma \) times the frequentist regret in Theorem 1 which is linear.

**B Proof of Theorem 2 and Corollary 2**

First we will prove Theorem 2. Let \( \Pi_{t,i} \) denote \( \Pi_t(\Omega_i) \). We construct the pdf of \( Q_t \)'s as follows:
\[ q_t(m) = \begin{cases} 
\frac{1}{n_t^i} \pi_t(m), & \text{if } m_2 > m_1 \\
0, & \text{otherwise}. 
\end{cases} \] (10)

We will prove the theorem by the following steps:
In Lemma 6 we show that $Q_t$’s are valid distributions.
In Lemma 7 we show that $Q_t$ has linear frequentist regret, and calculate the constant average regret per time-step.
In Lemma 8 we show that there exists a bad prior such that the $\alpha$-divergence between $Q_t$ and $\Pi_t$ can be arbitrarily small.

In Appendix B.3 we discuss the prior-dependent error threshold $\epsilon$ that will cause linear regret. In Appendix B.5 we provide the Bayesian regret proof for Corollary 2.

**Lemma 6.** $q_t(m)$ in Eq. 10 is well-defined and if $\int \pi_t(m) dm = 1$ then:
\[
\int q_t(m) dm = 1.
\]

**Lemma 7.** $Q_t$ constructed in Eq. 10 chooses arm 2 at all time-steps. The average frequentist regret per time-step is $\Delta = m_1^* - m_2^*$.

**Lemma 8.** Let $\alpha < 1$, $M_1 - M_2$ and $M_2$ are independent and arm 2 be chosen at all time-steps before $t$.

For any $\epsilon > 0$, there exists $0 < z \leq 1$ such that if $\Pi_{0,2} = z$ then $D_{\alpha}(\Pi_t, Q_t) < \epsilon$ where $Q_t$ is constructed in Eq. 10.

For any $0 < z \leq 1$, there exists $\epsilon > 0$ such that if $\Pi_{0,2} = z$ then $D_{\alpha}(\Pi_t, Q_t) < \epsilon$ where $Q_t$ is constructed in Eq. 10.

**B.1 Proof of Lemma 6**

**Proof.** Similar to the proof of Lemma 3 we have that $\Pi_{t,2} > 0$ for all $t \geq 0$.

Assume that $\int \pi_t(m) dm = 1$, we will show that $\int q_t(m) dm = 1$:
\[
\int q_t(m) dm = \int_{\Omega_1} q_t(m) dm + \int_{\Omega_2} q_t(m) dm
= 0 + \int_{\Omega_2} \frac{1}{\Pi_{t,2}} \pi_t(m) dm
= \frac{1}{\Pi_{t,2}} \int_{\Omega_2} \pi_t(m) dm
= 1.
\]

**B.2 Proof of Lemma 7**

**Proof.** Under the approximate distribution, arm 2 is chosen with probability 1 at all times. Clearly this approximate distribution has linear regret, with $\Delta = m_1^* - m_2^*$ being the average regret per time-step.

**B.3 Proof of Lemma 8**

**Proof.** Let $D = M_1 - M_2$ which is independent of $M_2$ by the assumption. Let $f$ denote the pdf.
Since the algorithm always pick arm 2, $H_{t-1}$ and $M_1$ are independent given $M_2$. Therefore for all $m_1, m_2$ and $h$, $f_{M_1|M_2,H_{t-1}}(m_1|m_2, h) = f_{M_1|M_2}(m_1|m_2)$.

Since $D = M_1 - M_2$, we have $f_{D|M_2,H_{t-1}}(m_1 - m_2|m_2, h) = f_{M_1|M_2,H_{t-1}}(m_1|m_2, h)$. Therefore for all $d, m_2$ and $h$:
\[
f_{D|M_2,H_{t-1}}(d|m_2, h) = f_{M_1|M_2,H_{t-1}}(m_2 + d|m_2, h) = f_{M_1|M_2}(m_2 + d|m_2) = f_{D|M_2}(d|m_2).
\]
Since \( f_{D|M_2,H_{t-1}}(d|m_2,h) = f_{D|M_2}(d|m_2) \) for all \( d, m_2 \) and \( h, D \) and \( H_{t-1} \) are independent given \( M_2 \). Then

\[
f_{D|M_2,H_{t-1}}(d|m_2,h) = f_{D|M_2}(d|m_2) \text{ because } D \text{ and } H_{t-1} \text{ are independent given } M_2
\]

\[
= f_D(d) \text{ because } D \text{ and } M_2 \text{ are independent.}
\]

Now we will show that \( D \) and \( H_{t-1} \) are independent. For all \( d \) and \( h \):

\[
f_{D|H_{t-1}}(d|h) = \int f_{D,M_2|H_{t-1}}(d,m_2|h)dm_2
\]

\[
= \int f_{D|M_2,H_{t-1}}(d,m_2,h)f_{M_2|H_{t-1}}(m_2|h)dm_2
\]

\[
= \int f_D(d)f_{M_2|H_{t-1}}(m_2|h)dm_2
\]

\[
= f_D(d)
\]

Since \( D \) and \( H_{t-1} \) are independent, at all time \( t \) the posterior does not concentrate:

\[
\Pi_{t,2} = \mathbb{P}(M_1 - M_2 < 0|H_{t-1}) = \mathbb{P}(M_1 < M_2)
\]

For simplicity let

\[
z := \mathbb{P}(M_1 < M_2)
\]

We will show that \( D(\Pi_t,Q_t) \) is small if \( z \) is large enough. First we calculate the \( \alpha \)-divergence between \( \Pi_t \) and \( Q_t \) constructed in Eq [10].

When \( \alpha < 1, \alpha \neq 0 \):

\[
D_\alpha(\Pi_t,Q_t)
\]

\[
= 1 - \int \left( \frac{q_t(m)}{\pi_t(m)} \right)^{1-\alpha} \pi_t(m)dm
\]

\[
= 1 - \int_{\Omega_1} \left( \frac{q_t(m)}{\pi_t(m)} \right)^{1-\alpha} \pi_t(m)dm - \int_{\Omega_2} \left( \frac{q_t(m)}{\pi_t(m)} \right)^{1-\alpha} \pi_t(m)dm
\]

\[
= 1 - \int_{\Omega_1} \left( \frac{1}{\Pi_{t,2}} \right)^{1-\alpha} \pi_t(m)dm - \int_{\Omega_2} \left( \frac{1}{\Pi_{t,2}} \right)^{1-\alpha} \pi_t(m)dm
\]

\[
= 1 - \left( \frac{1}{\Pi_{t,2}} \right)^{1-\alpha} \Pi_{t,2} = 1 - \left( \frac{1}{\Pi_{t,2}} \right)^{1-\alpha} \Pi_{t,2} \alpha(1-\alpha)
\]

\[
= 1 - \left( \frac{1}{z} \right)^{1-\alpha} \Pi_{t,2} \alpha(1-\alpha)
\]

\[
= 1 - \frac{z^{\alpha}}{\alpha(1-\alpha)}
\]
When $\alpha = 0$:

\[
D_\alpha(\Pi_t, Q_t) = \int q_t(m) \log \frac{q_t(m)}{\pi_t(m)} dm
\]

\[
= \int_{\Omega_1} q_t(m) \log \frac{q_t(m)}{\pi_t(m)} dm
+ \int_{\Omega_2} q_t(m) \log \frac{q_t(m)}{\pi_t(m)} dm
\]

\[
= \int_{\Omega_1} 0 \log(0) dm + \int_{\Omega_2} q_t(m) \log \frac{1}{\Pi_{t,2}} dm
\]

\[
= 0 + \log \frac{1}{\Pi_{t,2}} = \log \frac{1}{\Pi_{t,2}} = \log \frac{1}{z}
\]

Note that if we don’t have the condition on the prior such that picking arm 2 does not help to learn which arm is the better one, $\Pi_{t,2}$ may converge to 0, making $D_\alpha(\Pi_t, Q_t)$ go to $\infty$ when $\alpha \leq 0$. But since $\Pi_{t,2} = z$, we will now show that for any $\alpha < 1$, for any $\epsilon > 0$, there exists $z(0 < z < 1)$ such that

\[
D_\alpha(\Pi_t, Q_t) < \epsilon
\]

Consider the 2 cases

- When $\alpha < 1, \alpha \neq 0$: Since

\[
\lim_{z \to 1} \frac{1 - z^\alpha}{\alpha(1 - \alpha)} = 0
\]

Then for any $\epsilon > 0$ there exists $0 < z < 1$ such that $D_\alpha(\Pi_t, Q_t) < \epsilon$. For any $0 < z < 1$ there exists $\epsilon > 0$ such that $D_\alpha(\Pi_t, Q_t) < \epsilon$

- When $\alpha = 0$:

\[
D_\alpha(\Pi_t, Q_t) = \log \frac{1}{z}
\]

Since $\lim_{z \to 1} \log(1/z) = 0$, for any $\epsilon > 0$ there exists $0 < z < 1$ such that $D_0(\Pi_t, Q_t) < \epsilon$. For any $z < 1$ there exists $\epsilon > 0$ such that $D_\alpha(\Pi_t, Q_t) < \epsilon$

\[
B.4 \text{ Prior-dependent Error Threshold for Linear Frequentist Regret}
\]

In the example constructed in the previous sections, the $\alpha$-divergence between $\Pi_t$ and $Q_t$ can be calculated as: $\epsilon = \begin{cases} \frac{1 - z^\alpha}{\alpha(1 - \alpha)}, & \text{if } 0 < \alpha < 1 \text{ or } \alpha < 0 \\ \log \frac{1}{z}, & \text{if } \alpha = 0 \end{cases}$.

In Figure 5 we show the values of $\epsilon$ as a function of $z$ that will make the regret linear for different values of $\alpha$. We can see that for both cases when $\alpha \leq 0$ and $0 \leq \alpha < 1$, and $z$ is not too small, there is a threshold of $\epsilon$ for each value of $z$ that make the regret linear. For each value of $z$, if the error is smaller than the threshold we hypothesize that the regret might become sub-linear. However even if that is the case, it is not possible to calculate the exact threshold for more complicated priors. Therefore in Section 5.1 we propose an algorithm that is guaranteed to have sub-linear regret for any $\epsilon$ and any $z$ when $\alpha \leq 0$.

\[
B.5 \text{ Proof of Corollary 2}
\]

Since $\mathbb{P}(M_1 > M_2) > 0$, there exist constants $\Delta > 0, \gamma > 0$ such that $\mathbb{P}(M_1 - M_2 \geq \Delta) = \gamma$. The probability that the assumption $m_1^* > m_2^*$ in Theorem 2 is satisfied is at least $\gamma > 0$. Therefore the expected regret over the prior is at least $\gamma$ times the frequentist regret in Theorem 2, which is linear.
(a) $D_\alpha(\Pi_t, Q_t) = \epsilon$ as a function of $z$ when $\alpha \leq 0$. When $z$ is very small and $\alpha$ is small, $\epsilon$ needs to be very large. When $z > 0.2$, there is a threshold of $\epsilon$ which is less than 8 that can cause linear regret.

(b) $\epsilon$ as a function of $z$ when $0 \leq \alpha < 1$. There is a threshold of $\epsilon$ which is less than 8 for each value of $z$ that can cause linear regret.

Figure 5: $\epsilon$ as a function $z$ that makes the regret linear for different values of $\alpha$ for the example constructed in the proof of Theorem 2.

C Proof of Lemma 2

To convert between $D_\alpha(\Pi_t, Q_t)$ and $D_\alpha(\Pi_t, Q_t)$ we first prove the following lemma:

**Lemma 9** (Jensen’s Inequality). Let $f : \mathcal{R}^2 \to \mathcal{R}$ be a convex function. Let $P : \mathcal{R}^k \to \mathcal{R}$ and $Q : \mathcal{R}^k \to \mathcal{R}$ be 2 functions. Let $S$ is a subset of $\mathcal{R}^k$, the domain of $x$ and $|S|$ denote the volume of $S$. Then

$$\frac{1}{|S|} \int_S f(P(x), Q(x))dx \geq f\left(\frac{1}{|S|} \int_S P(x)dx, \frac{1}{|S|} \int_S Q(x)dx\right)$$

(11)

**Proof.** Multivariate Jensen’s Inequality states that if $X$ is a n-dimensional vector random variable and $f : \mathcal{R}^n \to \mathcal{R}$ is a convex function then

$$\mathbb{E}(f(X)) \geq f(\mathbb{E}(X)).$$

To use the multivariate Jensen’s Inequality we define the 2-dimensional vector random variable $X : S \to \mathcal{R}^2$ by $X(x) := (P(x), Q(x))$ and a probability distribution over $S$ such that for all $x \in S$: $P(x) = \frac{1}{|S|}$.

Then the left-hand side of Eq. [11] becomes $\mathbb{E}(f(X))$, while the right-hand side becomes $f(\mathbb{E}(X))$, and Eq. [11] follows from the multivariate Jensen’s Inequality.

Now we will prove Lemma 2.

**Proof of Lemma 2.** Since $D_\alpha(p, q)$ is convex [Cichocki & Amari, 2010], the following functions:

$$f(p, q) = q \log \frac{q}{p}$$
$$f(p, q) = p \log \frac{p}{q}$$
$$f(p, q) = \frac{p^\alpha q^{1-\alpha}}{\alpha(\alpha - 1)}$$

are convex, and we can apply Lemma 9.
• When $\alpha = 0$:
\[ D_\alpha(\Pi_t, Q_t) \]
\[ = \int q_t(m) \log \frac{q_t(m)}{\pi_t(m)} dm \]
\[ = \sum_i \int_{\Omega_i} q_t(m) \log \frac{q_t(m)}{\pi_t(m)} dm \]
\[ \geq \sum_i |\Omega_i| \frac{1}{|\Omega_i|} \int_{\Omega_i} q_t(m) dm \log \frac{1}{|\Omega_i|} \int_{\Omega_i} q_t(m) dm \] by applying Lemma 9
\[ = \sum_i Q_{t,i} \log \frac{Q_{t,i}}{\Pi_{t,i}} \]
\[ = D_\alpha(\Pi_t, Q_t) \]

• When $\alpha = 1$:
\[ D_\alpha(\Pi_t, Q_t) \]
\[ = \int \pi_t(m) \log \frac{\pi_t(m)}{q_t(m)} dm \]
\[ = \sum_i \int_{\Omega_i} \pi_t(m) \log \frac{\pi_t(m)}{q_t(m)} dm \]
\[ \geq \sum_i |\Omega_i| \frac{1}{|\Omega_i|} \int_{\Omega_i} \pi_t(m) dm \log \frac{1}{|\Omega_i|} \int_{\Omega_i} q_t(m) dm \] by applying Lemma 9
\[ = \sum_i \Pi_{t,i} \log \frac{\Pi_{t,i}}{Q_{t,i}} \]
\[ = D_\alpha(\Pi_t, Q_t) \]

• When $\alpha \neq 0, \alpha \neq 1$:
\[ D_\alpha(\Pi_t, Q_t) \]
\[ = \int \pi(x) \alpha q(x)^{1-\alpha} - 1 \, dx \]
\[ = -\frac{1}{\alpha(1-\alpha)} + \sum_i \int_{\Omega_i} \pi(x) \alpha q(x)^{1-\alpha} \, dx \]
\[ \geq -\frac{1}{\alpha(1-\alpha)} + \sum_i |\Omega_i| \frac{(\Pi_{t,i})^\alpha (Q_{t,i})^{1-\alpha}}{\alpha(1-\alpha)} \] by applying Lemma 9
\[ = -\frac{1}{\alpha(1-\alpha)} + \sum_i \Pi_{t,i}^\alpha Q_{t,i}^{1-\alpha} \]
\[ = D_\alpha(\Pi_t, Q_t) \]

\[ \square \]

D Proof of Theorem 3

We will prove that the frequentist regret is sub-linear for any $m^*$. If the algorithm has sub-linear frequentist regret for all values $M = m^*$, the Bayesian regret (which is the expected value over $M$) will also be sub-linear.

Without loss of generalization, let arm 1 be the best arm. From Lemma 1 since $\sum_{t=1}^\infty p_t = \infty$, we have for all arms $i$, $\sum_{t=1}^\infty P(A_t = i|H_{t-1}) = \infty$ and therefore with probability $1$:
\[ \lim_{t \to \infty} \Pi_{t,1} = \lim_{t \to \infty} P(A^* = 1|H_{t-1}) = 1, \] (12)
which means that the posterior probability that arm 1 is the best arm converges to 1.

We will prove the theorem by proving the following steps:

- In Lemma 10 we show that if the probability that the posterior chooses the best arm tends to 1, then the probability that the approximation chooses the best arm also tends to 1.
- In Lemma 11 and Lemma 12 we show that if the probability that the approximation chooses the best arm also tends to 1 almost surely, then it has sub-linear regret with probability 1. Therefore it has sub-linear regret in expectation over the history.

**Lemma 10.** Let $\alpha \leq 0$ and arm 1 be the true best arm. Let $\Omega_i = \{m | m_i = max(m_1, ..., m_k)\}$ be the region where arm $i$ is the best arm. If the posterior probability that arm 1 is the best arm converges to 1:

$$\lim_{t \to \infty} \Pi_{t,1} = 1$$

and for all $t \geq 0$:

$$D_{\alpha}(\Pi_t, Q_t) < \epsilon,$$

then the sequence $\{Q_{t,1}\}_t$ where $Q_{t,1} = \int_{\Omega_1} q_t(m) dm$ converges and

$$\lim_{t \to \infty} Q_{t,1} = 1.$$

Next we show that if the approximate distribution concentrates, then the probability that it chooses the wrong arm decreases as $T$ goes to infinity.

**Lemma 11.** If

$$\lim_{t \to \infty} Q_{t,1} = 1$$

then

$$\lim_{T \to \infty} \frac{\sum_{t=1}^{T} (1 - Q_{t,1})}{T} = 0.$$

From Lemma 10 and Lemma 11 since $\lim_{t \to \infty} \Pi_{t,1} = 1$ with probability 1, we have $\lim_{T \to \infty} \frac{\sum_{t=1}^{T} (1 - Q_{t,1})}{T} = 0$ with probability 1. We will now show that the expected regret is sub-linear:

**Lemma 12.** Let $p_t = o(1)$ be such that $\sum_{t=1}^{\infty} p_t = \infty$. For any number of arms $k$, any prior $\Pi_0$ and any error threshold $\epsilon > 0$, the following algorithm has $o(T)$ regret: at every time-step $t$,

- with probability $1 - p_t$, sample from an approximate posterior $Q_t$ such that $\lim_{T \to \infty} \frac{\sum_{t=1}^{T} (1 - Q_{t,1})}{T} = 0$ with probability 1, and
- with probability $p_t$, sample an arm uniformly at random.

**D.1 Proof of Lemma 10**

**Proof.** Let $Q_{t,i} = \int_{\Omega_i} q_t(m) dm$ and $\Pi_{t,i} = \int_{\Omega_i} \pi_t(m) dm$. Then

$$\lim_{t \to \infty} \Pi_{t,1} = 1$$

and we want to show that $\{Q_{t,1}\}_t$ converges and

$$\lim_{t \to \infty} Q_{t,1} = 1.$$

Since $D_{\alpha}(\Pi_t, Q_t) < \epsilon$ and $\lim \Pi_{t,1} = 1$ we want to show that $\lim \sup Q_{t,1} = 1$. By contradiction, assume that:

$$\lim \sup Q_{t,1} = \epsilon < 1.$$
Then there exists a sub-sequence of \( \{Q_{t,1}\}_t \), denoting \( Q_{t_1,1}, Q_{t_2,1}, \ldots, Q_{t_n,1}, \ldots \) such that
\[
\lim_{n \to \infty} Q_{t_n,1} = c. \tag{13}
\]
which implies
\[
0 < 1 - c = \lim_{n \to \infty} \sum_{i=2}^{k} Q_{t_n,i} \leq \sum_{i=2}^{k} \limsup_{n \to \infty} Q_{t_n,i}.
\]
Therefore there exists \( j \in [2, k] \) such that:
\[
\limsup_{n \to \infty} Q_{t_n,j} = d > 0
\]
Then there exists a sub-sequence of \( \{Q_{t_n,j}\}_n \), denoting \( Q_{t_{n_1},j}, Q_{t_{n_2},j}, \ldots, Q_{t_{n_m},j}, \ldots \) such that
\[
\lim_{m \to \infty} Q_{t_{n_m},j} = d.
\]
We consider the 2 cases:

- When \( \alpha = 0 \):
  \[
  D_0(\Pi_t, \overline{Q}_t) = \sum_{i=1}^{k} Q_{t,i} \log \frac{Q_{t,i}}{\Pi_{t,i}}
  \]
  Then we have:
  \[
  \epsilon = \lim_{m \to \infty} D_0(\Pi_{t_{n_m}}, \overline{Q}_{t_{n_m}})
  \geq \lim_{m \to \infty} Q_{t_{n_m},1} \log \frac{Q_{t_{n_m},1}}{\Pi_{t_{n_m},1}} + \lim_{m \to \infty} Q_{t_{n_m},j} \log \frac{Q_{t_{n_m},j}}{\Pi_{t_{n_m},j}}
  = c \log \frac{c}{1} + d \log \frac{d}{0}
  = \infty \text{ since } d > 0,
  \]
  which is a contradiction. Therefore \( c = 1 \).

- When \( \alpha < 0 \):
  \[
  D_{\alpha}(\Pi_t, \overline{Q}_t) = \frac{\sum_{i=1}^{k} \Pi_{t,i}^\alpha Q_{t,i}^{1-\alpha} - 1}{\alpha(\alpha - 1)}
  \]
  Then we have:
  \[
  \epsilon = \lim_{m \to \infty} D_{\alpha}(\Pi_{t_{n_m}}, \overline{Q}_{t_{n_m}})
  \geq \lim_{m \to \infty} \Pi_{t_{n_m},1}^{\alpha} Q_{t_{n_m},1}^{1-\alpha} + \Pi_{t_{n_m},j}^{\alpha} Q_{t_{n_m},j}^{1-\alpha} - 1
  = \frac{1^{\alpha} c^{1-\alpha} + d^{1-\alpha}}{(0)^{\alpha}} - 1
  = \frac{\alpha(\alpha - 1)}{\alpha(\alpha - 1)}
  = \infty, \text{ since } d > 0 \text{ and } \alpha < 0,
  \]
  which is a contradiction. Therefore \( c = 1 \).

Similarly we will show that:
\[
\liminf Q_{t,1} = 1.
\]
By contradiction, assume that:
\[
\liminf Q_{t,1} = c' < 1.
\]
Then there exists a sub-sequence of \(\{Q_{t,1}\}_t\), denoting \(Q_{t_1,1}, Q_{t_2,1}, \ldots, Q_{t_n,1}, \ldots\) such that
\[
\lim_{n \to \infty} Q_{t_n,1} = c'.
\]
Using the same argument following Eq. [13] we will have \(c' = 1\). Since \(\lim \inf Q_{t,1} = \lim \sup Q_{t,1} = 1\), we have that \(\{Q_{t,1}\}_t\) converges and
\[
\lim Q_{t,1} = 1.
\]

D.2 Proof for Lemma 11

For simplicity let \(x_t\) denote \(1 - Q_{t,1}\). We want to show that if a sequence \(\{x_t\}\) satisfies \(x_t \geq 0 \forall t\) and:
\[
\lim_{t \to \infty} x_t = 0,
\]
then
\[
\lim_{T \to \infty} S_T = 0,
\]
where
\[
S_T = \sum_{t=1}^{T} x_t.
\]
Since \(\lim_{t \to \infty} x_t = 0\) and \(x_t \geq 0 \forall t\), for any \(\epsilon > 0\) there exists \(T_0\) such that for all \(t > T_0\):
\[
x_t < \frac{\epsilon}{2}
\]
Then for all \(T > T_0\):
\[
S_T = \frac{x_1 + \ldots + x_{T_0}}{T} + \frac{x_{T_0+1} + \ldots + x_T}{T}
\]
\[
\leq \frac{x_1 + \ldots + x_{T_0}}{T} + \frac{\frac{\epsilon}{2} T}{T}
\]
\[
\leq \frac{x_1 + \ldots + x_{T_0}}{T} + \frac{\epsilon}{2}
\]
Choose \(T_1\) large enough such that \(\frac{x_1 + \ldots + x_{T_0}}{T_1} < \frac{\epsilon}{2}\). Let \(T_2 = \max(T_0, T_1)\). Then for all \(T > T_2\):
\[
S_T = \frac{x_1 + \ldots + x_{T_0}}{T} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
Therefore for any \(\epsilon > 0\), there exists \(T_2\) such that for all \(T > T_2\), \(S_T < \epsilon\). Since \(S_T \geq 0 \forall T\), we have:
\[
\lim_{T \to \infty} S_T = 0
\]

D.3 Proof of Lemma 12

Without loss of generalization, let arm 1 be the true best arm. Let \(\Delta = m^*_1 - \max(m^*_2, \ldots, m^*_k)\) be the gap between the highest mean \(m^*_1\) and the next highest mean of the arms.
Since \(p_t = o(1), \sum_{t=1}^{T} p_t = o(T)\). Therefore the regret from the uniform sampling steps is \(o(T)\).
Since \(1 - Q_{t,1}\) is the probability of choosing a sub-optimal arm by sampling from \(Q_t\), the regret of sampling from \(Q_t\) is upper bounded by:
\[
E \sum_{t=1}^{T} \Delta(1 - Q_{t,1})
\]
Since \( \lim_{T \to \infty} \sum_{t=1}^{T} (1 - Q_{t,1}) \) = 0 with probability 1, we have

\[
\lim_{T \to \infty} \frac{\sum_{t=1}^{T} \Delta(1 - Q_{t,1})}{T} = 0
\]

with probability 1. Therefore

\[
\lim_{T \to \infty} E\sum_{t=1}^{T} \Delta(1 - Q_{t,1}) = 0,
\]

which means that the regret of sampling from \( Q_t \) is sub-linear. Since both the expected regrets of the uniform sampling steps and of sampling from \( Q_t \) is sub-linear, the total expected regret is sub-linear.

### E Ensemble Sampling and Uniform Exploration

To the best of our knowledge, (Lu & Van Roy, 2017) is the only work that provides a theoretical analysis of Thompson sampling when the sampling step is approximate. Lu & Van Roy (2017) propose an approximate sampling method called Ensemble sampling where they maintain a set of \( M \) models to approximate the posterior, and analyze its regret for linear contextual bandits. When the model is \( k \)-armed bandit, the regret bound is as follow:

**Lemma 13** (implied by (Lu & Van Roy, 2017)). Let \( \pi_{TS} \) and \( \pi_{ES} \) denote the exact Thompson sampling and Ensemble sampling policies. Let \( \Delta = \max(m_1^*, ..., m_k^*) - \min(m_1^*, ..., m_k^*) \). For all \( \epsilon > 0 \), if

\[
\mathcal{M} \geq \frac{2k}{\epsilon^2} \log \frac{2kT}{\epsilon^2 \delta},
\]

then

\[
\text{Regret}(T, \pi_{ES}) \leq \text{Regret}(T, \pi_{TS}) + \epsilon \Delta T + \delta \Delta T
\]

(Lu & Van Roy, 2017) prove the regret bound by only using the following property of the Ensemble sampling method: at time \( t \), with probability \( 1 - \delta \), Ensemble sampling satisfies the following constraint:

\[
\text{KL}(\bar{Q}_t, \bar{\Pi}_t) < \epsilon^2,
\]

(15)

where \( \epsilon \) is a constant if \( \mathcal{M} = \Theta(\log(T)) \). If \( \epsilon \) is a constant the regret will be linear because of the term \( \epsilon \Delta T \).

At time \( t \), with probability \( 1 - \delta \), \( \text{KL}(\bar{Q}_t, \bar{\Pi}_t) < \epsilon^2 \). The first 2 terms in the right hand side of Eq. 14 comes from the time-steps when \( \text{KL}(\bar{Q}_t, \bar{\Pi}_t) < \epsilon^2 \), and the last term comes from the other case with probability \( \delta \).

Theorem 3 shows that applying an uniform sampling step will make the posterior concentrate. Moreover, Lemma 10 implies that if Eq. 15 is satisfied at a sub-set of times \( T_0 \subseteq [0, 1, ..., T] \), the approximation \( Q_t \) will also concentrate when \( t \in T_0 \). Therefore the regret from the time-steps in \( T_0 \) will be sub-linear in \( T_0 \), which is sub-linear in \( T \).

So if we want to maintain a small number of model \( M = \Theta(\log(T)) \) and achieve sub-linear regret, we can apply Theorem 3 as follow. First we choose \( \delta \) to be small such that the last term in Eq. 14 \( \delta \Delta T \) is \( o(T) \). Then we apply the uniform sampling step as shown in Theorem 3 so that the first 2 terms in the right hand side of Eq. 14 become sub-linear. We can then achieve sub-linear regret with Ensemble sampling with a \( \Theta(\log T) \) number of models.

### F KL Divergence between 2 Gaussian Distributions

The KL divergence between 2 Gaussian distributions are:

\[
\text{KL}(\text{Norm}(\mu_1, \Sigma_1), \text{Norm}(\mu_2, \Sigma_2))
\]

\[
= \frac{1}{2}(\text{trace}(\Sigma_2^{-1}\Sigma_1) - k) + (\mu_2 - \mu_1)^T \Sigma_2^{-1}(\mu_2 - \mu_1) + \ln \frac{\det \Sigma_2}{\det \Sigma_1}
\]
G Posterior Calculation

In our simulations, when both the prior and the reward distributions are Gaussian, we calculate the true posterior using the following closed-form solution.

Let the posterior at time $t$ be multivariate Gaussian distribution $\text{Norm}(\mu_t, \Sigma_t)$ where $\mu_t$ is a $k \times 1$ vector and $\Sigma_t$ is a $k \times k$ co-variance matrix. Let the reward distribution of arm $i$ be $\text{Norm}(m^*_i, \sigma^2)$ where $\sigma$ is known and $m^*_i$’s are unknown.

Let $A_t \in \{0, 1\}^k$ be a 0/1 vector where $A_t(i) = 1$ if arm $i$ is chosen at time $t$, and 0 otherwise. Let $r_t \in \mathbb{R}$ be the reward of the arm chosen at time $t$.

Then the posterior at time $t + 1$ is $\text{Norm}(\mu_{t+1}, \Sigma_{t+1})$ where:

\[
\Sigma_{t+1} = (\Sigma_t^{-1} + A_t A_t^T / \sigma^2)^{-1}
\]

\[
\mu_{t+1} = \Sigma_{t+1}(\Sigma_t^{-1} \mu_t + A_t r_t / \sigma^2)
\]