A RAINBOW MATCHING IN A BIPARTITE GRAPH

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Abstract. A recent conjecture of Aharoni, Charbit and Howard states that \( n \) matchings, each of size \( n + 1 \), in a bipartite graph have a rainbow matching of size \( n \). The same authors proved that if the size of the matchings is \( \lceil \frac{7}{4} n \rceil \) then a rainbow matching of size \( n \) exists. In this work we apply a different method to improve the bound to \( \lceil \frac{5}{3} n \rceil \).

1. Introduction

Let \( G(V, E) \) be a graph with a set of vertices \( V \) and a set of edges \( E \). A matching in \( G \) is a set of pairwise disjoint edges from \( E \). Let \( \mathcal{F} = \{F_1, \ldots, F_k\} \) be a family of \( k \) subsets, where \( F_i \subseteq E \) for all \( i \) (the \( F_i \)’s are not necessarily disjoint). A set \( R \) of edges is a (partial) rainbow matching in \( \mathcal{F} \) if \( R \) is a matching consisting of at most one edge from each \( F_i \). During the last decade the problem of finding conditions for large rainbow matchings in a graph was extensively explored. To mention a few, see [1, 7, 9, 10]. Many results and conjectures on the subject were influenced by the well-known conjectures of Ryser [11], asserting that every Latin square of odd order \( n \) has a transversal of order \( n \), and Brualdi [5] (see also [4] p. 255), asserting that every Latin square of even order \( n \) has a partial transversal of size \( n - 1 \). Brualdi’s conjecture may also be casted into the form of a rainbow matching problem:

**Conjecture 1.1.** Let \( \mathcal{F} \) be a partition of the edges of the complete bipartite graph \( K_{n,n} \) into \( n \) matchings, each of size \( n \). Then \( \mathcal{F} \) has a rainbow matching of size \( n - 1 \).

A far reaching generalization of Conjecture 1.1 was posed by Stein [12]:

**Conjecture 1.2.** Let \( \mathcal{F} \) be a partition of edges of the complete bipartite graph \( K_{n,n} \) into \( n \) subsets, each of size \( n \). Then \( \mathcal{F} \) has a rainbow matching of size \( n - 1 \).

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Recently, Aharoni, Charbit and Howard [2] presented another generalization of Conjecture 1.1.

**Conjecture 1.3.** A family of \( n \) matchings, each of size \( n \), in a bipartite graph has a rainbow matching of size \( n - 1 \).

Along this paper we assume that the bipartite graph has \( 2n \) vertices with \( n \) vertices in each side. Under this assumption Conjecture 1.3 may also be considered as a special case of Conjecture 1.2. As noted in [2], a modification of results of Woolbright [13] and Brower, de Vries and Wieringa [3] about large transversals in Latin squares yields the following theorem:

**Theorem 1.1.** A family of \( n \) matchings, each of size \( n \), in a bipartite graph has a rainbow matching of size \( n - \sqrt{n} \).

When no edge is allowed to appear in more than one matching, a tighter bound was achieved by Shor and Hatami [8]:

**Theorem 1.2.** Let \( F \) be a partition of edges of the complete bipartite graph \( K_{n,n} \) into \( n \) matchings, each of size \( n \). Then \( F \) has a rainbow matching of size \( n - O(\log^3 n) \).

If one insists on finding a rainbow matching of size \( n \) in a family of matchings, each of size \( n \), then the size of the family must be raised dramatically, as proved by Drisko [6]:

**Theorem 1.3.** Let \( F = \{F_1, \ldots, F_{2n-1}\} \) be a family of \( 2n - 1 \) matchings, each of size \( n \), in a bipartite graph. Then \( F \) has a rainbow matching of size \( n \).

Drisko provided an example showing that the bound \( 2n - 1 \) is tight.

Aharoni, Charbit and Howard [2] looked at the problem from a different angle: They fixed the size of the family but allowed to increase the size of each of the \( n \) matchings and posed the following conjecture:

**Conjecture 1.4.** Any \( n \) matchings, each of size \( n + 1 \), in a bipartite graph possess a rainbow matching of size \( n \).

These authors achieved a bound as follows:

**Theorem 1.4.** A family of \( n \) matchings, each of size \( \lceil \frac{7}{4} n \rceil \), in a bipartite graph has a rainbow matching of size \( n \).

In this work (Theorem 2.1) we use a different method to improve the bound in Theorem 1.4.

2. A Rainbow Matching of Size \( n \)

Let \( G \) be a bipartite graph consisting of two parts \( U \) and \( W \) such that \( U \cup W = V(G) \).

**Proposition 2.1.** Let \( \mathcal{A} = \{A_1, \ldots, A_n\} \) be a family of \( n \) matchings in \( G \), where \( |A_i| = \lceil \frac{7}{4} n \rceil \), \( i = 1, \ldots, n \). Then, \( \mathcal{A} \) has a rainbow matching of size \( n - 1 \).
Proof. Assume by contradiction that a maximal rainbow matching $R$ has size $|R| \leq n - 2$. Without loss of generality we may also assume that $R \cap A_{n-1} = \emptyset$ and $R \cap A_n = \emptyset$. Let $X$ and $Y$ be the subsets of $U$ and $W$, respectively, that are not covered by $R$ (see Figure 1). Since $R$ is maximal the whole set $X$ is matched by $A_{n-1}$ with some subset $W' \subset W \setminus Y$ and the whole set $Y$ is matched by $A_n$ with some subset $U' \subset U \setminus X$. Since $|R| \leq n - 2$ we have $|U'| + |W'| > |R|$. It follows that there exist edges $e_1 \in A_{n-1}$, $e_2 \in A_n$ and $e \in R$ such that $e_1 \cap e \cap W \neq \emptyset$ and $e_1 \cap X \neq \emptyset$, and similarly, $e_2 \cap e \cap U \neq \emptyset$ and $e_2 \cap Y \neq \emptyset$. Clearly, $(R \setminus \{e\}) \cup \{e_1, e_2\}$ is a rainbow matching, contradicting the maximality of $R$.

Theorem 2.1. Let $A = \{A_1, \ldots, A_n\}$ be a family of $n$ matchings in $G$, each of size $\lfloor \frac{5}{3}n \rfloor$. Then, $A$ has a rainbow matching of size $n$.

Proof. Assume, by contradiction, that a maximal rainbow matching $R$ has size $|R| \leq n - 1$. By Proposition 2.1, $|R| = n - 1$. Without loss of generality we may assume that $R \cap A_n = \emptyset$. Let $X \subset U$ and $Y \subset W$ be the vertices of $G$ not covered by $R$. Since $R$ is maximal, all the vertices in $Y$ are matched by $A_n$ with some subset $Z \subset U \setminus X$. Let $R'$ be the subset of $R$ that matches the elements in $Z$ (see Figure 2). We have

$$|X| = |Y| = |R'| = \lfloor \frac{2n}{3} \rfloor + 1$$

Define,

$$A' = \{A_i \in A|A_i \cap R' \neq \emptyset\}.$$

Claim 1. Any matching $A_i \in A'$ has at most one edge between $X$ and $Y$.

Proof of Claim 1. Suppose $A_i \cap R' = \{e\}$ and $A_i$ has two edges $e_1$ and $e_2$ between $X$ and $Y$. Let $f$ be the edge of $A_n$ such that $f \cap e \neq \emptyset$ and $f \cap Y \neq \emptyset$. Without loss of generality we may assume that $f$ does not meet $e_1$ (it may or may not meet $e_1$). Thus, $(R \setminus \{e\}) \cup \{f, e_2\}$ is a rainbow matching of size $n$ (Figure 2), contradicting the maximality of $R$.

The next claim follows directly from (2.1) and Claim 1
Claim 2. Each $A_i \in A'$ has at least $|2n/3|$ edges with one endpoint in $X$ and the other endpoint in $W \setminus Y$.

Now, let $W' \subset W$ be the set of endpoints of the edges in $R'$.

Claim 3. Each $A_i \in A'$ has at least $|n/3|$ edges with one endpoint in $X$ and the other endpoint in $W'$.

Proof of Claim 3. By (2.1), $|R \setminus R'| = n - 1 - (|2n/3| + 1) = |n/3| - 2$. By Claim 2 the set of edges with one endpoint in $X$ and the other endpoint in $W \setminus Y$ meet at least $|2n/3| - (|n/3| - 2) \geq |n/3|$ edges of $R'$.

Without loss of generality we assume that $A_1 \in A'$. Let $A_1 \cap R' = \{r_1\}$ and let $e_1 \in A_1$ be such that $e_1 \cap X \neq \emptyset$ and $e_1 \cap W' \neq \emptyset$ (by Claim 3 such $e_1$ exists). Let $r_2 \in R' \setminus \{r_1\}$ be an edge such that $r_2 \cap e_1 \neq \emptyset$. We may assume $r_2 \in A_2$. Let $e_2 \in A_2$ be an edge satisfying $e_2 \cap X \neq \emptyset$, $e_2 \cap W' \neq \emptyset$ and $e_2 \cap e_1 = \emptyset$. If $e_2 \cap r_1 \neq \emptyset$, then we can augment $R$ in the following way: Let $f \in A_n$ be the edge satisfying $f \cap r_1 \cap U \neq \emptyset$ and $f \cap Y \neq \emptyset$. Then $(R \setminus \{r_1, r_2\}) \cup \{e_1, e_2, f\}$ is a rainbow matching of size $n$. If $e_2 \cap r_1 = \emptyset$, let $r_3 \in R' \setminus \{r_1, r_2\}$ be such that $e_2 \cap r_3 \neq \emptyset$. We may assume $r_3 \in A_3$.

We proceed in this manner to obtain disjoint edges $E = \{e_1, e_2, \ldots, e_k\}$ each with endpoints in $X$ and in $W'$ and a set of edges $R'' = \{r_1, r_2, \ldots, r_k, r_{k+1}\} \subset R'$ such that $e_i \cap r_{i+1} \neq \emptyset$ for $i = 1, \ldots, k$ (Figure 3), and for each $i$, $e_i$ and $r_i$ belong to the same matching. Without loss of generality we assume that $e_i, r_i \in A_i$ for $i = 1, \ldots, k$ and $r_{k+1} \in A_{k+1}$. The process will end in one of two ways:

**Case 1:** There exists an edge $e_{k+1} \in A_{k+1}$ such that $e_{k+1} \cap X \neq \emptyset, e_{k+1} \cap r_t \neq \emptyset$ for some $t \in \{1, \ldots, k\}$.

**Case 2:** There is no edge $e_{k+1} \in A_{k+1}$ such that $e_{k+1} \cap X \neq \emptyset, e_{k+1} \cap e_i = \emptyset$ for $i = 1, \ldots, k$, and $e_{k+1} \cap W' \neq \emptyset$.

We show that in either case a rainbow matching of size $n$ must exist.

**Case 1:** Suppose $e_{k+1} \cap r_t \cap W' \neq \emptyset$ for some $t \in \{1, \ldots, k\}$. Let $f \in A_n$ be such that $f \cap Y \neq \emptyset$ and $f \cap r_t \neq \emptyset$ for some $l \in \{t, \ldots, k+1\}$. Then $(R \setminus \{r_t, \ldots, r_{k+1}\}) \cup \{e_t, \ldots, e_{k+1}, f\}$ is a rainbow matching of size $n$ (Figure 3).
Figure 3. Construction of the sets $E$ and $R''$ and augmenting the rainbow matching $R$ in Case 2. Solid edges are omitted, dashed and dotted edges are added.

Figure 4. Augmenting the rainbow matching $R$ in Case 1. Solid dark edges are omitted. Dashed and dotted dark edges are added.

Case 2: By Claim 3

(2.2) $|R''| > \lceil n/3 \rceil$

Now, the set of edges $(R \setminus R'') \cup E$ forms a partial rainbow matching of size $n - 2$. It excludes the matchings $A_{k+1}$ and $A_n$. By (2.2), we have $|R \setminus R''| < n - 1 - \lceil n/3 \rceil = \lceil 2n/3 \rceil - 1 < \lceil 2n/3 \rceil$. By Claim 3 there are at least $\lceil 2n/3 \rceil$ edges of $A_{k+1}$ with endpoints in $Y$ and $U \setminus X$. Thus, there exists an edge $e \in A_{k+1}$ such that $e \cap Y \neq \emptyset$ and $e \cap r_i \neq \emptyset$ for some $r_i \in R''$. Let $f \in A_n$ be such that $f \cap Y \neq \emptyset$ and $f \cap r_j \neq \emptyset$ for some $r_j \in R'' \setminus \{r_i\}$. Such an edge must exist since $|R''| > 1$ and all the vertices in $R''$ are matched by $A_n$ to vertices in $Y$. Then, $(R \setminus R'') \cup E \cup \{e, f\}$ is a rainbow matching of size $n$ (Figure 3). This completes the proof.

□

References

[1] R. Aharoni and E. Berger, Rainbow matchings in $r$-partite $r$-graphs, Electron. J. Combin 16 (2009), no. 1, R119.
[2] R. Aharoni, P. Charbit, and D. Howard, On a generalization of the Ryser-Brualdi-Stein conjecture, manuscript.
[3] A.E. Brouwer, A.J. de Vries, and R.M.A. Wieringa, A lower bound for the length of partial transversals in a latin square, Nieuw Arch. Wisk. 24 (1978), no. 3, 330–332.
[4] R.A. Brualdi and H.J. Ryser, Combinatorial matrix theory, Cambridge University Press, 1991.
[5] J. Dénes and A. D. Keedwell, *Latin squares and their applications*, Academic Press, New York, 1974.

[6] A. A. Drisko, *Transversals in row-Latin rectangles*, J. Combin. Theory Ser. A **84** (1998), 181–195.

[7] A. Gyárfás and G. N. Sárközy, *Rainbow matchings and partial transversals of latin squares*, Arxiv preprint [arXiv:1208.5670](arXiv:1208.5670) (2012).

[8] P. Hatami and P. W. Shor, *A lower bound for the length of a partial transversal in a Latin square*, J. Combin. Theory A **115** (2008), 1103–1113.

[9] A. Kostochka and M. Yancey, *Large rainbow matchings in edge-colored graphs*, Combinatorics, Probability and Computing **21** (2012), no. 1-2, 255–263.

[10] T.D. LeSaulnier, C. Stocker, P.S. Wenger, and D.B. West, *Rainbow matching in edge-colored graphs*, Electron. J. Combin **17** (2010), N26.

[11] H.J. Ryser, *Neuere probleme der kombinatorik*, Vorträge über Kombinatorik, Oberwolfach, Matematisches Forschungsinstitute (Oberwolfach, Germany), July 1967, pp. 69–91.

[12] S.K. Stein, *Transversals of Latin squares and their generalizations*, Pacific Journal of Mathematics **59** (1975), no. 2, 567–575.

[13] D.E. Woolbright, *An n × n Latin square has a transversal with at least n − √n distinct elements*, J. Combin. Theory A **24** (1978), 235–237.