Quantum States from Tangent Vectors

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Abstract

We argue that tangent vectors to classical phase space give rise to quantum states of the corresponding quantum mechanics. This is established for the case of complex, finite-dimensional, compact, classical phase spaces \( \mathcal{C} \), by explicitly constructing Hilbert–space vector bundles over \( \mathcal{C} \). We find that these vector bundles split as the direct sum of two holomorphic vector bundles: the holomorphic tangent bundle \( T(\mathcal{C}) \), plus a complex line bundle \( N(\mathcal{C}) \). Quantum states (except the vacuum) appear as tangent vectors to \( \mathcal{C} \). The vacuum state appears as the fibrewise generator of \( N(\mathcal{C}) \). Holomorphic line bundles \( N(\mathcal{C}) \) are classified by the elements of \( \text{Pic}(\mathcal{C}) \), the Picard group of \( \mathcal{C} \). In this way \( \text{Pic}(\mathcal{C}) \) appears as the parameter space for nonequivalent vacua. Our analysis is modelled on, but not limited to, the case when \( \mathcal{C} \) is complex projective space \( \mathbb{C}P^n \).

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1 Introduction

Fibre bundles are powerful tools to formulate the gauge theories of fundamental interactions and gravity [1]. The question arises whether or not quantum mechanics may also be formulated using fibre bundles. Important physical motivations call for such a formulation.

In quantum mechanics one aims at constructing a Hilbert–space vector bundle over classical phase space. In geometric quantisation this goal is achieved in a two–step process that can be very succinctly summarised as follows. One first constructs a certain holomorphic line bundle (the quantum line bundle) over classical phase space. Next one identifies certain sections of this line bundle as defining the Hilbert space of quantum states. Alternatively one may skip the quantum line bundle and consider the one–step process of directly constructing a Hilbert–space vector bundle over classical phase space. Associated with this vector bundle there is a principal bundle whose fibre is the unitary group of Hilbert space.

Standard presentations of quantum mechanics usually deal with the case when this Hilbert–space vector bundle is trivial. Such is the case, e.g., when classical phase space is contractible to a point. However, it seems natural to consider the case of a nontrivial bundle as well. Beyond a purely mathematical interest, important physical issues that go by the generic name of dualities [2] motivate the study of nontrivial bundles.

Triviality of the Hilbert–space vector bundle implies that the transition functions all equal the identity of the structure group. In passing from one coordinate chart to another on classical phase space, vectors on the fibre are acted on by the identity. Since these vectors are quantum states, we can say that all observers on classical phase space are quantised in the same way. This is no longer the case on a nontrivial bundle, where the transition functions are different from the identity. As opposed to the previous case, different neighbourhoods on classical phase space are quantised independently and, possibly, differently. The resulting quantisation is only local on classical phase space, instead of global. This reflects the property of local triviality satisfied by all fibre bundles.

Given a certain base manifold and a certain fibre, the trivial bundle over the given base with the given fibre is unique. This may mislead one to conclude that quantisation is also unique, or independent of the observer on classical phase space. In fact the notion of duality points precisely to the opposite conclusion, i.e., to the nonuniqueness of the quantisation procedure and to its dependence on the observer [2].

Clearly a framework is required in order to accommodate dualities within quantum mechanics [2]. Nontrivial Hilbert–space vector bundles over classical phase space provide one such framework. They allow for the possibility of having different, nonequivalent quantisations, as opposed to the uniqueness of the trivial bundle. However, although nontriviality is a necessary condition, it is by no means sufficient. A flat connection on a nontrivial bundle would still allow, by parallel transport, to canonically identify the Hilbert–space fibres above different points on classical phase space. This identification would depend only on the homotopy class of the curve joining the base-
points, but not on the curve itself. Now flat connections are characterised by constant
transition functions [3], this constant being always the identity in the case of the triv-
ial bundle. Hence, in order to accommodate dualities, we will be looking for nonflat
connections. We will see presently what connections we need on these bundles.

This article is devoted to constructing nonflat Hilbert–space vector bundles over
classical phase space. In motivating the subject we have dealt with unitary groups as
structure groups and linear fibres such as Hilbert spaces. However quantum states are
rays rather than vectors. Therefore it is more precise to consider the corresponding
projective spaces and projective unitary groups, as we will do from now on.

Throughout this article, C will denote a complex n–dimensional, connected, com-
 pact classical phase space, endowed with a symplectic form \(\omega\) and a complex structure
\(J\). We will assume that \(\omega\) and \(J\) are compatible, so holomorphic coordinate charts on
\(C\) will also be Darboux charts. We will mostly concentrate on the case when \(C\)
 is projective space \(\mathbb{CP}^n\). Its holomorphic tangent bundle will be denoted \(T(\mathbb{CP}^n)\).
The following line bundles over \(\mathbb{CP}^n\) will be considered: the trivial line bundle \(\mathcal{E}\), the
tautological line bundle \(\mathcal{T}^{-1}\) and its dual \(\mathcal{T}\). The Picard group of \(C\) will be denoted
\(\text{Pic}(C)\). \(\mathcal{H}\) will denote the complex, \((N + 1)\)–dimensional Hilbert space of quantum
states \(\mathbb{C}^{N+1}\), with unitary group \(U(N + 1)\). They projectivise to \(\mathbb{CP}^N\) and \(PU(N)\),
respectively.

Our analysis will deal mostly with the case when \(C = \mathbb{CP}^n\). In section 2 we
summarise its useful properties as a classical phase space. In section 2 we recall some
well–known facts from geometric quantisation. They concern the dimension of the
space of holomorphic sections of the quantum line bundle on a compact, quantisable
Kähler manifold. This dimension is rederived in section 3 using purely quantum–
mechanical arguments, by constructing the Hilbert–space bundle of quantum states
over \(\mathbb{CP}^n\). For brevity, the following summary deals only with the case when the
Hilbert space is \(\mathbb{C}^{n+1}\) (see sections 3, 4, 5 for the general case). The fibre \(\mathbb{C}^{n+1}\) over a
given coordinate chart on \(\mathbb{CP}^n\) is spanned by the vacuum state \(|0\rangle\), plus \(n\) states \(A_j^\dagger|0\rangle\),
\(j = 1, \ldots, n\), obtained by the action of creation operators. We identify the transition
functions of this bundle as jacobian matrices plus a phase factor. The jacobian matrices
account for the transformation (under coordinate changes on \(\mathbb{CP}^n\)) of the states \(A_j^\dagger|0\rangle\),
while the phase factor corresponds to \(|0\rangle\). This means that all quantum states (except
the vacuum) are tangent vectors to \(\mathbb{CP}^n\). In this way the Hilbert–space bundle over
\(\mathbb{CP}^n\) splits as the direct sum of two holomorphic vector bundles: the tangent bundle
\(T(\mathbb{CP}^n)\), plus a line bundle \(N(\mathbb{CP}^n)\) whose fibrewise generator is the vacuum.

All complex manifolds admit a Hermitian metric, so having tangent vectors as
quantum states suggests using the Hermitian connection and the corresponding curv-
iture tensor to measure flatness. Now \(T(\mathbb{CP}^n)\) is nonflat, so it fits our purposes.
The freedom in having different nonflat Hilbert–space bundles over \(\mathbb{CP}^n\) resides in the dif-
ferent possible choices for the complex line bundle \(N(\mathbb{CP}^n)\). Such choices are 1–to–1
with the elements of the Picard group \(\text{Pic}(\mathbb{CP}^n) = \mathbb{Z}\).

The previous picture of quantum states (except the vacuum) as tangent vectors re-
 mains substantially correct in the case of an arbitrary, compact, complex manifold \(C\)
whose complex and symplectic structures are compatible; this is proved in section 3.
Flatness of the resulting Hilbert–space bundle depends on whether or not the holomor-
phic tangent bundle \(T(C)\) is flat. We continue to have the Picard group \(\text{Pic}(C)\) as the
parameter space for different Hilbert–space bundles over \( \mathcal{C} \). Finally section 5 discusses our results.

Topics partially overlapping with ours are dealt with in refs. [4, 5, 6, 7, 8].

2 Properties of \( \mathbb{C}P^n \) as a classical phase space

We will consider a classical mechanics whose phase space \( \mathcal{C} \) is complex, projective \( n \)-dimensional space \( \mathbb{C}P^n \). The following properties are well known [3].

Let \( Z_1, \ldots, Z_{n+1} \) denote homogeneous coordinates on \( \mathbb{C}P^n \). The chart defined by \( Z^k \neq 0 \) covers one copy of the open set \( U_k = \mathbb{C}^n \). On the latter we have the holomorphic coordinates \( z^j_{(k)} = Z^j/Z^k, j \neq k \); there are \( n+1 \) such coordinate charts. \( \mathbb{C}P^n \) is a Kähler manifold with respect to the Fubini–Study metric. On the chart \((U_k, z^j_{(k)})\) the Kähler potential reads

\[
K(z^j_{(k)}, \bar{z}^j_{(k)}) = \log \left( 1 + \sum_{j=1}^{n} z^j_{(k)} \bar{z}^j_{(k)} \right).
\]

The singular homology ring \( H_* (\mathbb{C}P^n, \mathbb{Z}) \) contains the nonzero subgroups

\[
H_{2k} (\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}, \quad k = 0, 1, \ldots, n,
\]

while

\[
H_{2k+1} (\mathbb{C}P^n, \mathbb{Z}) = 0, \quad k = 0, 1, \ldots, n - 1.
\]

We have \( \mathbb{C}P^n = \mathbb{C}^n \cup \mathbb{C}P^{n-1} \), with \( \mathbb{C}P^{n-1} \) a hyperplane at infinity. Topologically, \( \mathbb{C}P^n \) is obtained by attaching a (real) \( 2n \)-dimensional cell to \( \mathbb{C}P^{n-1} \). \( \mathbb{C}P^n \) is simply connected,

\[
\pi_1 (\mathbb{C}P^n) = 0,
\]

it is compact, and inherits its complex structure from that on \( \mathbb{C}^{n+1} \). It can be regarded as the Grassmannian manifold

\[
\mathbb{C}P^n = U(n+1)/(U(n) \times U(1)) = S^{2n+1}/U(1).
\]

Let \( \tau^{-1} \) denote the tautological bundle on \( \mathbb{C}P^n \). We recall that \( \tau^{-1} \) is defined as the subbundle of the trivial bundle \( \mathbb{C}P^n \times \mathbb{C}^{n+1} \) whose fibre at \( p \in \mathbb{C}P^n \) is the line in \( \mathbb{C}^{n+1} \) represented by \( p \). Then \( \tau^{-1} \) is a holomorphic line bundle over \( \mathbb{C}P^n \). Its dual, denoted \( \tau \), is called the hyperplane bundle. For any \( l \in \mathbb{Z} \), the \( l \)-th power \( \tau^l \) is also a holomorphic line bundle over \( \mathbb{C}P^n \). In fact every holomorphic line bundle \( L \) over \( \mathbb{C}P^n \) is isomorphic to \( \tau^l \) for some \( l \in \mathbb{Z} \); this integer is the first Chern class of \( L \).

In the framework of geometric quantisation it is customary to consider the case when \( \mathcal{C} \) is a compact Kähler manifold. In this context one introduces the notion of a quantisable, compact, Kähler phase space \( \mathcal{C} \), of which \( \mathbb{C}P^n \) is an example. This means that there exists a quantum line bundle \((\mathcal{L}, g, \nabla)\) on \( \mathcal{C} \), where \( \mathcal{L} \) is a holomorphic line bundle, \( g \) a Hermitian metric on \( \mathcal{L} \), and \( \nabla \) a covariant derivative compatible with the...
complex structure and $g$. Furthermore, the curvature $F$ of $\nabla$ and the symplectic 2–form $\omega$ are required to satisfy
\[ F = -2\pi i\omega. \] (6)
It turns out that quantisable, compact Kähler manifolds are projective algebraic manifolds and vice versa [9]. After introducing a polarisation, the Hilbert space of quantum states is given by the global holomorphic sections of $\mathcal{L}$.

Recalling that, on $\mathbb{C}P^n$, $\mathcal{L}$ is isomorphic to $\tau^l$ for some $l \in \mathbb{Z}$, let $\mathcal{O}(l)$ denote the sheaf of holomorphic sections of $\mathcal{L}$ over $\mathbb{C}P^n$. The vector space of holomorphic sections of $\mathcal{L} = \tau^l$ is the sheaf cohomology space $H^0(\mathbb{C}P^n, \mathcal{O}(l))$. The latter is zero for $l < 0$, while for $l \geq 0$ it can be canonically identified with the set of homogeneous polynomials of degree $l$ on $\mathbb{C}^{n+1}$. This set is a vector space of dimension $\binom{n+l}{n}$:
\[ \dim H^0(\mathbb{C}P^n, \mathcal{O}(l)) = \binom{n+l}{n}. \] (7)
We will give a quantum–mechanical derivation of eqn. (7) in section 3.

Equivalence classes of holomorphic line bundles over a complex manifold $\mathcal{C}$ are classified by the Picard group $\text{Pic}(\mathcal{C})$. The latter is defined [10] as the sheaf cohomology group $H^1_{\text{sheaf}}(\mathcal{C}, \mathcal{O}^*)$, where $\mathcal{O}^*$ is the sheaf of nonzero holomorphic functions on $\mathcal{C}$. When $\mathcal{C} = \mathbb{C}P^n$ things simplify because the above sheaf cohomology group is in fact isomorphic to a singular homology group,
\[ H^1_{\text{sheaf}}(\mathbb{C}P^n, \mathcal{O}^*) = H^2_{\text{sing}}(\mathbb{C}P^n, \mathbb{Z}), \] (8)
and the latter is given in eqn. (2). Thus
\[ \text{Pic}(\mathbb{C}P^n) = \mathbb{Z}. \] (9)
The zero class corresponds to the trivial line bundle $\epsilon = \tau^0$; all other classes correspond to nontrivial bundles. As the equivalence class of $\mathcal{L}$ varies, so does the space $\mathcal{H}$ of its holomorphic sections vary.

3 Quantum Hilbert–space bundles over $\mathbb{C}P^n$

As discussed in section 1 in quantum mechanics one skips the quantum line bundle $\mathcal{L}$ of geometric quantisation and proceeds directly to construct Hilbert–space bundles over classical phase space. We will therefore analyse such vector bundles (that we will call quantum Hilbert–space bundles, or $\mathcal{QH}$–bundles for short), their principal unitary bundles and, finally, their projectivisations. Our aim is to demonstrate that there are different nonequivalent choices for the nonflat $\mathcal{QH}$–bundles, to study how the corresponding quantum mechanics varies with each choice, and to provide a physical interpretation. Although we will be able to reproduce the results that geometric quantisation derives from $\mathcal{L}$, our approach will be based on the $\mathcal{QH}$–bundles instead. In particular, triviality of the quantum line bundle $\mathcal{L}$ does not imply, nor is implied by, triviality of the $\mathcal{QH}$–bundle; the same applies to flatness.
Our analysis will be modelled on the case when $\mathcal{C} = \mathbb{C}P^n$. An example of a classical dynamics on $\mathbb{C}P^n$ is given by the projective oscillator. On the coordinate chart $U_k$ of eqn. $\mathcal{H}$, the classical Hamiltonian equals the Kähler potential $\mathcal{H}$. Compactness of $\mathbb{C}P^n$ implies that, upon quantisation, the Hilbert space $\mathcal{H}$ is finite–dimensional, and hence isomorphic to $\mathbb{C}^{N+1}$ for some $N$. This property follows from the fact that the number of quantum states grows monotonically with the symplectic volume of $\mathcal{C}$; the latter is finite when $\mathcal{C}$ is compact. We are thus led to considering principal $U(N+1)$–bundles over $\mathbb{C}P^n$ and to their classification. Equivalently, we will consider the associated holomorphic vector bundles with fibre $\mathbb{C}^{N+1}$. The corresponding projective bundles are $\mathbb{C}P^n$–bundles and principal $PU(N)$–bundles. Each choice of a different equivalence class of bundles will give rise to a different quantisation. How many such equivalence classes are there? For the moment let us observe that there is more than one. For example one can consider the class of the trivial bundle $\mathbb{C}P^n \times U(N)$, or the class of a nontrivial bundle over $\mathbb{C}P^n$ such as the Hopf bundle. For the same reasons we can expect more than one equivalence class of projective bundles to exist. That this is actually true will also be proved later on.

So far we have left $N$ undetermined. In order to fix it we first pick the symplectic volume form $\omega$ on $\mathbb{C}P^n$ such that $\int_{\mathbb{C}P^n} \omega = n + 1$. (10) Next we set $N = n$, so $\dim \mathcal{H} = n + 1$. This normalisation corresponds to 1 quantum state per unit of symplectic volume on $\mathbb{C}P^n$. Thus, e.g., when $n = 1$ we have the Riemann sphere $\mathbb{C}P^1$ and $\mathcal{H} = \mathbb{C}^2$. The latter is the Hilbert space of a spin $s = 1/2$ system, and the counting of states is correct. There are a number of further advantages to this normalisation. In fact eqn. (10) is more than just a normalisation, in the sense that the dependence of the right–hand side on $n$ is determined by physical consistency arguments. This will be explained in section 3.1. Normalisation arguments can enter eqn. (10) only through overall numerical factors such as $2\pi$, $i\hbar$, or similar. It is these latter factors that we fix by hand in eqn. (10).

The right–hand of our normalisation (10) differs from that corresponding to eqn. (9). Up to numerical factors such as $2\pi$, $i\hbar$, etc, it is standard to set $\int_{\mathbb{C}P^n} F = n$ $\mathbb{Z}$. However we will find our normalisation (10) more convenient. Indeed we will make no use of the quantum line bundle $\mathcal{L}$, while we will be able to reproduce quantum–mechanically the results of geometric quantisation.

### 3.1 Computation of $\dim H^0(\mathbb{C}P^n, \mathcal{O}(1))$

Next we present a quantum–mechanical computation of $\dim H^0(\mathbb{C}P^n, \mathcal{O}(1))$ without resorting to sheaf cohomology. That is, we compute $\dim \mathcal{H}$ when $l = 1$ and prove that it coincides with the right–hand side of eqn. (10). The case $l > 1$ will be treated in section 3.3.

Starting with $\mathcal{C} = \mathbb{C}P^0$, i.e., a point $p$ as classical phase space, the space of quantum rays must also reduce to a point. Then the corresponding Hilbert space is $\mathcal{H}_1 = \mathbb{C}$. The only state in $\mathcal{H}_1$ is the vacuum $|0\rangle_{l=1}$. Henceforth, for brevity, we drop the Picard class index from the vacuum.
Next we pass from $\mathcal{C} = \mathbb{C}P^0$ to $\mathcal{C} = \mathbb{C}P^1$. Regard $p$, henceforth denoted $p_1$, as the point at infinity with respect to a coordinate chart $(U_1, z_{(1)})$ on $\mathbb{C}P^1$ that does not contain $p_1$. This chart is biholomorphic to $\mathcal{C}$ and supports a representation of the Heisenberg algebra in terms of creation and annihilation operators $A^+_1(1)$, $A(1)$. On the chart $U_1$, the Hilbert space $\mathcal{H}_2 = \mathbb{C}^2$ is the linear span of the vacuum $|0(1)\rangle$ and its excitation $A^+_1(1)|0(1)\rangle$.

On $\mathbb{C}P^1$ we have the charts $(U_1, z_{(1)})$ and $(U_2, z_{(2)})$. Point $p_1$ is at infinity with respect to $(U_1, z_{(1)})$, while it belongs to $(U_2, z_{(2)})$. Similarly, the point at infinity with respect to $(U_2, z_{(2)})$, call it $p_2$, belongs to $(U_1, z_{(1)})$ but not to $(U_1, z_{(1)})$. On $U_2$, the fibre is the linear span of $|0(2)\rangle$ and $A^+_1(2)|0(2)\rangle$, $A^+_1(2)$ and $|0(2)\rangle$ respectively being the creation operator and the vacuum on $U_2$. On the common overlap $U_1 \cap U_2$, the coordinate transformation between $z_{(1)}$ and $z_{(2)}$ is biholomorphic. This implies that, on $U_1 \cap U_2$, the fibre $\mathbb{C}^2$ can be taken in either of two equivalent ways: either as the linear span of $|0(1)\rangle$ and $A^+_1(1)|0(1)\rangle$, or as the linear span of $|0(2)\rangle$ and $A^+_1(2)|0(2)\rangle$. Indeed, biholomorphicity of the coordinate change between $(U_1, z_{(1)})$ and $(U_2, z_{(2)})$ implies that the vectors $|0(1)\rangle$ and $A^+_1(1)|0(1)\rangle$ transform bijectively into the vectors $|0(2)\rangle$ and $A^+_1(2)|0(2)\rangle$, and vice versa.

When $n > 1$ we proceed by analogy with the case $n = 1$. Topologically we have $\mathbb{C}P^n = \mathbb{C}^n \cup \mathbb{C}P^{n-1}$, with $\mathbb{C}P^{n-1}$ a hyperplane at infinity; we also need to describe the coordinate charts and their overlaps. There are coordinate charts $(U_j, z_{(j)})$, $j = 1, \ldots, n+1$ and nonempty $f$–fold overlaps $\cap_{j=1}^f U_j$ for $f = 2, 3, \ldots, n+1$. Each chart $(U_j, z_{(j)})$ is biholomorphic with $\mathbb{C}^n$ and has a $\mathbb{C}P^{n-1}$–hyperplane at infinity; the latter is charted by the remaining charts $(U_k, z_{(k)})$, $k \neq j$. Over $(U_j, z_{(j)})$ the Hilbert–space bundle $Q\mathcal{H}_{n+1}$ has a fibre $\mathcal{H}_{n+1} = \mathbb{C}^{n+1}$ spanned by

$$|0(j)\rangle, \quad A^+_i(j)|0(j)\rangle, \quad i = 1, 2, \ldots, n. \tag{11}$$

On every nonempty $f$–fold overlap $\cap_{j=1}^f U_j$, the fibre $\mathbb{C}^{n+1}$ can be taken in $f$ different, but equivalent ways, as the linear span of $|0(j)\rangle$ and $A^+_i(j)|0(j)\rangle$, $i = 1, 2, \ldots, n$, for every choice of $j = 1, \ldots, f$. This is proved by analyticity arguments analogous to those above, but let us spell out the details in the simple case when $f = 2$ (the case $f > 2$ involves no novelty with respect to $f = 2$). Assume that $U_{j_1} \cap U_{j_2}$ is nonempty for some indices $j_1, j_2$. On the chart $(U_{j_1}, z_{(j_1)})$, the fibre $\mathbb{C}^{n+1}$ is the linear span of the vectors $|0(j_1)\rangle$, $A^+_i(j_1)|0(j_1)\rangle$, for $i_1 = 1, 2, \ldots, n$. Similarly, on $(U_{j_2}, z_{(j_2)})$, the fibre $\mathbb{C}^{n+1}$ is the linear span of $|0(j_2)\rangle$, $A^+_i(j_2)|0(j_2)\rangle$, for $i_2 = 1, 2, \ldots, n$. The coordinate transformation between $(U_{j_1}, z_{(j_1)})$ and $(U_{j_2}, z_{(j_2)})$ is biholomorphic. This implies that, on the overlap $U_{j_1} \cap U_{j_2}$, the vectors $|0(j_1)\rangle$, $A^+_i(j_1)|0(j_1)\rangle$, $i_1 = 1, 2, \ldots, n$, transform bijectively into the vectors $|0(j_2)\rangle$, $A^+_i(j_2)|0(j_2)\rangle$, $i_2 = 1, 2, \ldots, n$, and vice versa.

A complete description of this bundle requires the specification of the transition functions. This will be done in section 3.4 where transition functions will be identified with jacobian matrices (for the coordinate transformations on $\mathbb{C}P^n$), plus a phase factor. Two properties will follow from this fact. The first one is the cocycle condition, which the transition functions will certainly satisfy. The second one is the independence of the bundle with respect to the specific coordinates chosen on $\mathbb{C}P^n$, as long
as the coordinates are holomorphic. In other words, although we have found it con-
venient to use the particular holomorphic coordinates \((U_j, z_{(j)})\) described in section 2 any other holomorphic atlas consisting of charts \((V_j, w_{(j)})\) would have produced the same results. In particular, none of the above results depends on the fact that the charts \(U_j\) are biholomorphic to the whole of \(C^n\). If the new charts \(V_j\) were biholomorphic to open subsets of \(C^n\) not identical to all of \(C^n\), the previous arguments would continue to hold just as well. The key property is the biholomorphicity of coordinate transformations on overlapping charts, something that is guaranteed by the definition of a complex manifold. Thus our construction of the \(QH\)–bundle is functorial, in the sense that it is coordinate–independent.

3.2 Representations

The \((n + 1)\)–dimensional Hilbert space of eqn. (11) may be regarded as a kind of defining representation, in the sense of the representation theory of \(SU(n + 1)\) when \(n > 1\). To make this statement more precise we observe that one can replace unitary groups with special unitary groups in eqn. (5). Comparing our results with those of section 2 we conclude that the quantum line bundle \(L\) now equals \(\tau\),

\[
    L = \tau, \quad (12)
\]

because \(l = 1\). This is the smallest value of \(l\) that produces a nontrivial \(\mathcal{H}\), as eqn. (4) gives a 1–dimensional Hilbert space when \(l = 0\). So our \(\mathcal{H}\) spans an \((n + 1)\)–dimensional representation of \(SU(n + 1)\), that we can identify with the defining representation. There is some ambiguity here since the dual of the defining representation of \(SU(n + 1)\) is also \((n + 1)\)–dimensional. This ambiguity is resolved by convening that the latter is generated by the holomorphic sections of the dual quantum line bundle

\[
    L^* = \tau^{-1}. \quad (13)
\]

On the chart \(U_j, j = 1, \ldots, n + 1\), the dual of the defining representation is the linear span of the covectors

\[
    \langle (j)0 |, \quad \langle (j)0 | A_i(j), \quad i = 1, 2, \ldots, n. \quad (14)
\]

These conclusions must be slightly modified in the limiting case when \(n = 1\), since all \(SU(2)\) representations are selfdual. This point will be explained in section 3.4.

Taking higher representations is equivalent to considering the principal \(SU(n+1)\)–bundle (associated with the vector \(C^{n+1}\)–bundle) in a representation higher than the defining one. We will see next that this corresponds to having \(l > 1\) in our choice of the line bundle \(\tau^l\).

3.3 Computation of \(\dim H^0(\mathbb{CP}^n, \mathcal{O}(l))\)

We extend now our quantum–mechanical computation of \(\dim H^0(\mathbb{CP}^n, \mathcal{O}(l))\) to the case \(l > 1\). As in section 3.1 we do not resort to sheaf cohomology. The values \(l = 0, 1\) respectively correspond to the trivial and the defining representation of \(SU(n + 1)\).
The restriction to nonnegative $l$ follows from our convention of assigning the defining representation to $\tau$ and its dual to $\tau^{-1}$. Higher values $l > 1$ correspond to higher representations and can be accounted for as follows. Let us rewrite eqn. (5) as

$$\mathbb{C}P^{n+l} = SU(n+l+1)/ (SU(n+l) \times U(1)), \quad (15)$$

where now $SU(n+l+1)$ and $SU(n+l)$ act on $\mathbb{C}^{n+l+1}$. Now $SU(n+l)$ admits \( n+1 \)-dimensional representations (Young tableaux with a single column of $n$ boxes) that, by restriction, are also representations of $SU(n+1)$. Letting $l > 1$ vary for fixed $n$, this reproduces the dimension of eqn. (7).

By itself, the existence of $SU(n+1)$ representations with the dimension of eqn. (7) does not prove that, picking $l > 1$, the corresponding quantum states lie in those \( n+1 \)-dimensional representations. We have to prove that no other value of the dimension fits the given data. In order to prove it the idea is, roughly speaking, that a value of $l > 1$ on $\mathbb{C}P^n$ can be traded for $l' = 1$ on $\mathbb{C}P^{n+l}$. That is, an $SU(n+1)$ representation higher than the defining one can be traded for the defining representation of $SU(n+1)$ in this way. The $\mathcal{QH}$-bundle on $\mathbb{C}P^n$ with the Picard class $l' = l$ equals the $\mathcal{QH}$-bundle on $\mathbb{C}P^{n+l}$ with the Picard class $l' = 1$. On the latter we have $n+l$ excited states (i.e., other than the vacuum), one for each complex dimension of $\mathbb{C}P^{n+l}$. We can sort them into unordered sets of $n$, which is the number of excited states on $\mathbb{C}P^n$, in \( n+1 \) different ways. This selects a specific dimension for the $SU(n+1)$ representations and rules out the rest. More precisely, it is only when $n > 1$ that some representations are ruled out. When $n = 1$, i.e., for $SU(2)$, all representations are allowed, since their dimension is $l+1 = (1+1)/1$. However already for $SU(3)$ some representations are thrown out. The number \( \binom{2+l}{2} \) matches the dimension $d(p, q) = (p+1)(q+1)(p+q+2)/2$ of the $(p, q)$ irreducible representation if $p = 0$ and $l = q$ or $q = 0$ and $l = p$, but arbitrary values of $(p, q)$ are in general not allowed.

To complete our reasoning we have to prove that the quantum line bundle $\mathcal{L} = \tau$ on $\mathbb{C}P^{n+l}$ descends to $\mathbb{C}P^n$ as the $l$-th power $\tau^l$. For this we resort to the natural embedding of $\mathbb{C}P^n$ into $\mathbb{C}P^{n+l}$. Let $(U_1, z(1)), \ldots, (U_{n+l+1}, z(n+l+1))$ be the coordinate charts on $\mathbb{C}P^n$ described in section 2 and let $(\tilde{U}_1, \tilde{z}(1)), \ldots, (\tilde{U}_{n+1}, \tilde{z}(n+1)), (\tilde{U}_{n+2}, \tilde{z}(n+2)), \ldots, (\tilde{U}_{n+l+1}, \tilde{z}(n+l+1))$ be charts on $\mathbb{C}P^{n+l}$ relative to this embedding. This means that the first $n+1$ charts on $\mathbb{C}P^{n+l}$, duly restricted, are also charts on $\mathbb{C}P^n$; in fact every chart on $\mathbb{C}P^n$ is contained $l$ times within $\mathbb{C}P^n$. Let $t_{jk}(\tau)$, with $j, k = 1, \ldots, n+l+1$, be the transition function for $\tau$ on the overlap $\tilde{U}_j \cap \tilde{U}_k$ of $\mathbb{C}P^{n+l}$. In passing from $U_j$ to $\tilde{U}_k$, points on the fibre are acted on by $t_{jk}(\tau)$. Due to our choice of embedding, the overlap $\tilde{U}_j \cap \tilde{U}_k$ on $\mathbb{C}P^{n+l}$ contains $l$ copies of the overlap $U_j \cap U_k$ on $\mathbb{C}P^n$. Thus points on the fibre over $\mathbb{C}P^n$ are acted on by $(t_{jk}(\tau))^l$, where now $j, k$ are restricted to $1, \ldots, n+1$. This means that the line bundle on $\mathbb{C}P^n$ is $\tau^l$ as stated, and the vacuum $|0\rangle_{\nu=1}$ on $\mathbb{C}P^n$ equals the vacuum $|0\rangle_{\nu=1}$ on $\mathbb{C}P^{n+l}$. Hence there are on $\mathbb{C}P^n$ as many inequivalent vacua as there are elements in $\mathbb{Z} = \text{Pic} (\mathbb{C}P^n)$ (remember that sign reversal $l \rightarrow -l$ within $\text{Pic} (\mathbb{C}P^n)$ is the operation of taking the dual representation, i.e., $\tau \rightarrow \tau^{-1}$).
3.4 Transition functions

At each point \( p \in \mathbb{C}P^n \) there is an isomorphism between the holomorphic cotangent space \( T_p^*(\mathbb{C}P^n) \) and a complex \( n \)-dimensional subspace of \( \mathcal{H} = \mathbb{C}^{n+1} = \mathbb{C}^n \oplus \mathbb{C} \), where \( \mathbb{C}^n \) is cotangent to \( \mathbb{C}P^n \) and \( \mathbb{C} \) is normal to it. As \( p \) varies over \( \mathbb{C}P^n \) we have the following holomorphic bundles: the quantum Hilbert–space bundle \( \mathcal{Q}\mathcal{H} \) (with fibre \( \mathbb{C}^n \)), the cotangent bundle \( T^*(\mathbb{C}P^n) \) (with fibre \( \mathbb{C}^n \)), and the normal bundle \( N(\mathbb{C}P^n) \) (with fibre \( \mathbb{C} \)). Modulo a choice of representation for \( T^*(\mathbb{C}P^n) \), which will be done below, next we prove that

\[
\mathcal{Q}\mathcal{H}(\mathbb{C}P^n) = T^*(\mathbb{C}P^n) \oplus N(\mathbb{C}P^n). \tag{16}
\]

Eqn. (16) follows from the fact that, in the dual (14) of the defining representation, the operators \( A_i(j) \) act as \( \partial/\partial z^i_{(j)} \), i.e., as tangent vectors. Correspondingly, in the defining representation (11), their adjoints \( A_i^\dagger(j) \) in \( \mathcal{H} \) act as multiplication by \( z^i_{(j)} \). Since adjoints in \( \mathcal{H} \) transform as duals on tangent space, the \( A_i^\dagger(j) \) transform as differentials \( dz^i_{(j)} \), or cotangent vectors. In what follows we will identify the cotangent and the tangent bundles, so we can write

\[
\mathcal{Q}\mathcal{H}(\mathbb{C}P^n) = T(\mathbb{C}P^n) \oplus N(\mathbb{C}P^n), \tag{17}
\]

where \( T(\mathbb{C}P^n) \) and \( N(\mathbb{C}P^n) \) are subbundles of \( \mathcal{Q}\mathcal{H}(\mathbb{C}P^n) \). It follows that tangent vectors to \( \mathbb{C}P^n \) are quantum states in (the defining representation of) Hilbert space. In eqn. (11) we have given a basis for these states in terms of creation operators acting on the vacuum \( |0\rangle \). The latter can be regarded as the basis vector for the fibre \( \mathbb{C} \) of the line bundle \( N(\mathbb{C}P^n) \).

As a holomorphic line bundle, \( N(\mathbb{C}P^n) \) is isomorphic to \( \tau^l \) for some \( l \in \text{Pic}(\mathbb{C}P^n) = \mathbb{Z} \). Now the bundle \( T(\mathbb{C}P^n) \oplus N(\mathbb{C}P^n) \) has \( SU(n+1) \) as its structure group, which we consider in the representation \( \rho_l \) corresponding to the Picard class \( l \in \mathbb{Z} \):

\[
\mathcal{Q}\mathcal{H}_l(\mathbb{C}P^n) = \rho_l(T(\mathbb{C}P^n)) \oplus \tau^l, \quad l \in \mathbb{Z}. \tag{18}
\]

The importance of eqn. (15) is that it classifies \( \mathcal{Q}\mathcal{H} \)-bundles over \( \mathbb{C}P^n \): holomorphic equivalence classes of such bundles are in \( 1 \)-to-\( 1 \) correspondence with the elements of \( \mathbb{Z} = \text{Pic}(\mathbb{C}P^n) \). The class \( l = 1 \) corresponds to the defining representation of \( SU(n+1) \),

\[
\mathcal{Q}\mathcal{H}_{l=1}(\mathbb{C}P^n) = T(\mathbb{C}P^n) \oplus \tau, \tag{19}
\]

and \( l = -1 \) to its dual. The quantum Hilbert–space bundle over \( \mathbb{C}P^n \) is generally nontrivial, although particular values of \( l \) may render the direct sum (18) trivial. The separate summands \( T(\mathbb{C}P^n) \) and \( N(\mathbb{C}P^n) \) are both nontrivial bundles. Nontriviality of \( N(\mathbb{C}P^n) \) means that, when \( l \neq 0 \), the state \( |0\rangle \) transforms nontrivially (albeit as multiplication by a phase factor) between different local trivialisations of the bundle. When \( l = 0 \) the vacuum transforms trivially.

The preceding discussion also answers the question posed in section 3.1: what are the transition functions \( t(\mathcal{Q}\mathcal{H}_l) \) for \( \mathcal{Q}\mathcal{H}_l \)? According to eqn. (18), they decompose as a direct sum of two transition functions, one for \( \rho_l(T(\mathbb{C}P^n)) \), another one for \( \tau^l \):

\[
t(\mathcal{Q}\mathcal{H}_l(\mathbb{C}P^n)) = t(\rho_l(T(\mathbb{C}P^n))) \oplus t(\tau^l). \tag{20}
\]
If the transition functions for $\tau$ are $t(\tau)$, those for $\tau^l$ are $(t(\tau))^l$. On the other hand, the transition functions $t(\rho_l(T\mathbb{C}P^n))$ are the jacobian matrices (in representation $\rho_l$) corresponding to coordinate changes on $\mathbb{C}P^n$. Then all the $Q\mathcal{H}(\mathbb{C}P^n)$–bundles of eqn. (13) are nonflat because the tangent bundle $T(\mathbb{C}P^n)$ itself is nonflat.

Knowing the transition functions $t(Q\mathcal{H}(\mathbb{C}P^n))$ we can also answer the question posed in section 3.2 concerning the selfduality of the $SU(2)$ representations. It suffices to consider the defining representation. The latter is 2–dimensional. By eqn. (20), the corresponding transition functions, which are $2 \times 2$ complex matrices, split block–diagonally into $1 \times 1$ blocks, with zero off–diagonal entries. Hence these matrices are symmetric, i.e., invariant under transposition, which is the operation involved in passing from a representation to its dual. No complex conjugation is involved, since $z \mapsto \bar{z}$ would involve creation and annihilation operators with respect to the antiholomorphic coordinate $\bar{z}$. The notations $A, A^\dagger$ indicate that, if the latter acts as multiplication by a holomorphic coordinate $z$, the former acts by differentiation with respect to the same holomorphic coordinate $z$.

4 Tangent vectors as quantum states

We have seen in section 3.4 that (co)tangent vectors to $\mathbb{C}P^n$ are quantum states. The converse is not true, as exemplified by the vacuum. Let us generalise and replace $\mathbb{C}P^n$ with an arbitrary classical phase space $C$. We would like to write, as in eqn. (17),

$$Q\mathcal{H}(C) = T(C) \oplus N(C),$$

(21)

where $N(C)$ is a holomorphic line bundle on $C$, whose fibre is generated by the vacuum state, and $T(C)$ is the holomorphic tangent bundle. Does eqn. (21) hold in general?

The answer is trivially affirmative when $C$ is an analytic submanifold of $\mathbb{C}P^n$. Such is the case, e.g., of the embedding of $\mathbb{C}P^n$ within $\mathbb{C}P^{n+l}$ considered in section 3.3. Grassmann manifolds provide another example. The answer is also affirmative provided that $C$ is a complex $n$–dimensional, compact, symplectic manifold, whose complex and symplectic structures are compatible. Notice that $C$ is not required to be Kähler; examples of Hermitian but non–Kähler spaces are Hopf manifolds. Let $\omega$ denote the symplectic form. Then $\int_C \omega^n < \infty$ thanks to compactness; this ensures that $\dim \mathcal{H} < \infty$. Assuming that the vacuum is nondegenerate, as was the case with $\mathbb{C}P^n$, we can adopt a normalisation similar to that of eqn. (10).

$$\int_C \omega^n = n + 1,$$

(22)

Let us cover $C$ with a finite set of holomorphic coordinate charts $(W_k, w(k))$, $k = 1, \ldots, r$; the existence of such an atlas follows from the compactness of $C$. We can pick an atlas such that $r$ is minimal; compactness implies that $r \geq 2$.

The construction of the $Q\mathcal{H}(C)$–bundle proceeds along the same lines of section 3.1. The chart $W_k$ is biholomorphic to an open subset of $C^n$. The $n$ components of the holomorphic coordinates $w^j(k)$, $j = 1, \ldots, n$ give rise to creation and annihilation operators $A_j(k), A^\dagger_m(k)$, $j, m = 1, \ldots, n$, for every fixed value of $k = 1, \ldots, r$. The
vacuum $|0(k)\rangle_t$ corresponding to $l \in \Pic(C)$, plus the $n$ states $A_{m}^{l}(k)|0(k)\rangle_t$, span the fibre $C^{n+1}$ of the Hilbert–space bundle over the chart $W_k$. On overlaps $W_j \cap W_k$, the fibre can be taken in either of two equivalent ways. $C^{n+1}$ is either the linear span of $|0(j)\rangle_t$, plus the $n$ states $A_{m}^{j}(j)|0(j)\rangle_t$, or the linear span of $|0(k)\rangle_t$ plus the $n$ states $A_{m}^{l}(k)|0(k)\rangle_t$. Indeed, since the coordinate transformation between $W_j$ and $W_k$ is biholomorphic on $W_j \cap W_k$, the states $|0(j)\rangle_t$, $A_{m}^{j}(j)|0(j)\rangle_t$ transform bijectively into the states $|0(k)\rangle_t$, $A_{m}^{l}(k)|0(k)\rangle_t$.

Analyticity is central to the above construction of the $\mathcal{QH}(C)$–bundle. On the contrary, whether the charts $W_k$ are biholomorphic to the whole of $C^n$, or only to an open subset strictly contained within $C^n$, is immaterial. It suffices that $C$ be a complex manifold. Then coordinate transformations are biholomorphic on overlapping coordinate charts, and it follows that the above construction of the $\mathcal{QH}(C)$–bundle is functorial in the sense of coordinate–independence. Such was already the case with $\mathbb{CP}^n$. Additional parts of section 4 concerning $\mathbb{CP}^n$ carry over to $C$. Choosing $l \in \Pic(C)$ we determine a holomorphic line bundle $N_l(C)$ as in eqn. (21). and the latter holds (with a subindex $l$ on the left–hand side) under the assumptions made above. By eqn. (21) we can write for the transition functions

$$t(\mathcal{QH}_l(C)) = t(T(C)) \oplus t(N_l(C)),$$

as we did in eqn. (20). Transition functions for $T(C)$ are jacobian matrices, and tangent vectors are quantum states. Holomorphic line bundles such as $N_l(C)$ are classified by the Picard group $\Pic(C)$, although the latter need not be $\mathbb{Z}$. Now $T(C)$ may or may not be trivial. If both $T(C)$ and $N_l(C)$ are trivial, then the full quantum Hilbert–space bundle is trivial. A nontrivial $\mathcal{QH}_l(C)$–bundle arises if $T(C)$ is nontrivial and this nontriviality cannot be compensated by a nontrivial $N_l(C)$, or viceversa. On the other hand $\mathcal{QH}_l(C)$ is flat if, and only if, both $T(C)$ and $N_l(C)$ are flat.

All these analogies with $\mathbb{CP}^n$ notwithstanding, it is worth stressing that the construction of the $\mathcal{QH}(C)$–bundle by no means relies on them. Rather, the existence of the $\mathcal{QH}(C)$–bundle is a general result that holds for all complex manifolds $C$ satisfying the requirements stated at the beginning of this section. In fact there may also be differences with respect to $\mathbb{CP}^n$. One would like to identify $N_l(C)$ (for some class $l \in \Pic(C)$) with the quantum line bundle $L$, but $C$ need not be quantisable and/or $N_l(C)$ need not possess holomorphic sections. Another potential difference is the possible degeneracy of the vacuum. While all vacua on $\mathbb{CP}^n$ were nondegenerate, this need not be the case on a general $C$. We will analyse these issues in a forthcoming article.

5 Discussion

Quantum mechanics is defined on a Hilbert space of states whose construction usually assumes a global character on classical phase space. Under globality we understand, as explained in section 4, the property that all coordinate charts on classical phase space are quantised in the same way. A novelty of our approach is the local character of the Hilbert space: there is one on top of each Darboux coordinate chart on classical phase space. The patching together of these Hilbert–space fibres on top of each chart may
be global (trivial bundle) or local (nontrivial bundle). In order to implement duality transformations we need a nonflat bundle (hence nontrivial). Flatness would allow for a canonical identification, by means of parallel transport, of the quantum states belonging to different fibres.

Given a classical phase space as a base manifold and a Hilbert space as a fibre, the trivial bundle corresponding to these data is unique. On the contrary, there may be more than one (equivalence class of) nonflat (and hence nontrivial) bundles possessing the given base and fibre. This means that, considering nonflat bundles, the choice of a quantum mechanics need not be unique, even if the corresponding classical mechanics is kept fixed. The freedom in choosing different Hilbert–space bundles is parametrised by the Picard group of classical phase space. This group parametrises (equivalence classes of) holomorphic line bundles. The corresponding 1–dimensional fibre is spanned by the vacuum state. The remaining quantum states are obtained by the action of creation operators on the vacuum chosen. The quantum states so obtained can be identified with tangent vectors to classical phase space. When the Picard group is trivial, there exists just one Hilbert–space bundle (though not necessarily trivial). A nontrivial Picard group means that there is more than one equivalence class of Hilbert–space bundles. Any two different choices of a Hilbert–space bundle correspond to two different choices of a line bundle on which the vacuum state lies. The previous conclusions are valid on an arbitrary complex, compact classical phase space whose complex structure is kept fixed and is compatible with the symplectic structure, and assuming nondegeneracy of the vacuum.

In the presence of a nontrivial Picard group, each choice of a line bundle carries with it the choice of a representation for the unitary structure group of the Hilbert–space bundle. This may lead to the wrong conclusion that duality transformations are just different choices of a representation for the unitary group of Hilbert space. A choice of representation is not a duality transformation. The choice of a representation for the unitary group is subordinate to the choice of a class in the Picard group. Picking a class in the latter, one determines a representation for the former. In other words, in eqn. (18), one does not vary the representation \( \rho_l \) independently of the Picard class \( l \).

A duality thus arises as the possibility of having two or more, apparently different, quantum–mechanical descriptions of the same physics. Mathematically, a duality arises as a nonflat, quantum Hilbert–space bundle over classical phase space. This notion implies that the concept of a quantum is not absolute, but relative to the quantum theory used to measure it [2]. That is, duality expresses the relativity of the concept of a quantum. In particular, classical and quantum, for long known to be deeply related [11] are not necessarily always the same for all observers on phase space.

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