MEIXNER MATRIX ENSEMBLES

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Abstract. We construct a family of matrix ensembles that fits Anshelevich’s regression postulates for “Meixner laws on matrices”, namely the distribution with an invariant property of \(X\) when \(E(X^2|X + Y) = a(X + Y)^2 + b(X + Y) + cI_n\) where \(X\) and \(Y\) are iid on symmetric matrices of order \(n\). We show that the Laplace transform of a general \(n \times n\) Meixner matrix ensemble satisfies a system of partial differential equations which is explicitly solvable for \(n = 2\). We rely on these solutions to identify the six types of \(2 \times 2\) Meixner matrix ensembles.

1. Introduction

The classical Meixner laws are the one dimensional binomial, Poisson, negative binomial, gamma, normal, and hyperbolic Meixner laws that include hyperbolic secant law. These laws make their appearance as the orthogonality measures of a certain family of orthogonal polynomials [32], as the laws characterized by quadratic regression property [25], and as the exponential families corresponding to the quadratic variance functions [33], see also [19]. The Laha-Lukacs [23, 25] characterization is the description of the possible distributions of a square-integrable random variable \(X\) such that there exist real numbers \(a, b, c\) satisfying \(E(X^2|S) = aS^2 + bS + c\) where \(S = X + Y\), where \(Y \sim X\) and where \(Y\) is independent of \(X\). Another convenient form of this condition is to postulate the existence of real numbers \(A, B, C\) such that

\[
E((X - Y)^2|S) = AS^2 + BS + C
\]

(use \((X - Y)^2 = 2X^2 + 2Y^2 - S^2\) to see that \((A, B, C) = (4a - 1, 4b, 4c)\)).

In 2006 M. Anshelevich [4, pages 22 and 25] observes that (1.1) makes sense when the real random variable \(X\) is replaced by a random square matrix \(X\). He then raises the question of defining analogues of Meixner distributions on matrices. In other words, he asks for the matrix version of Laha-Lukacs [25] result. This question on random matrices is so general that it is suitable to restrict this problem to random variables \(X\) valued in the space \(H_{n,1}\) of all symmetric matrices and such that such that \(UXU^* \sim X\) for matrices \(U\) in the orthogonal group \(O(n) = K_{n,1}\). On the other hand, it is natural to extend this simpler framework to the space \(H_{n,2}\) of Hermitian complex matrices for random variables \(X\) such that \(UXU^* \sim X\) for matrices \(U\) in the unitary group \(U(n) = K_{n,2}\) and even to
the space $\mathbb{H}_{n,4}$ of Hermitian-quaternionic matrices, again for random variables $X$ such that $UXU^* \sim X$ for matrices $U$ in the symplectic group $\text{Sp}(n) = \mathcal{K}_{n,4}$. There are also connections with free probability (see [9] and Section 6.3 below), and connections with Jordan algebras (see Section 6.4 below).

A probability law $\mu$ on $\mathbb{H}_{n,\beta}$ (where $\beta = 1, 2$ or 4), or a random variable $X$ with law $\mu$, is said to be rotation invariant if the law of $UXU^*$ is also $\mu$ for all $U$ in $\mathcal{K}_{n,\beta}$. In such case, following a physicists tradition we shall call $\mu$ an ensemble.

**Definition 1.1.** We will say that the probability law $\mu$ on $\mathbb{H}_{n,\beta}$ is a Meixner ensemble with parameters $A, B, C \in \mathbb{R}$ if $\mu$ is rotation invariant, the second moments exist, and $\mu$ has the following property: if $X, Y$ are independent with the same law $\mu$ and $S = X + Y$, then

$$
\mathbb{E}((X - Y)^2|S) = AS^2 + BS + CI_n.
$$

This is a different concept of a Meixner matrix ensemble than the one introduced in [15] under the name “multivariate Meixner classes of invariant distributions of random matrices”. In particular, we note that the oldest known ensemble, namely the Wishart distribution with mean proportional to $I_n$ is not a Meixner ensemble in the sense of Definition 1.1, since if $X, Y$ are i.i.d. with Wishart distribution, then there exist two real numbers $A, B$ such that

$$
\mathbb{E}((X - Y)^2|S) = AS^2 + BS \text{tr} S,
$$

which is quadratic in $S$ but not of the desired form (1.2). For a proof of (1.3), see [27, page 582].

To complete this introduction, let us make a number of simple remarks about the Meixner ensembles. It is easy to check that if the law of $X$ is a Meixner ensemble with parameters $A, B, C$, then an affine transformation $\tilde{X} = \alpha X + t$ with real $\alpha, t$ is also Meixner, with parameters

$$
\tilde{A} = A, \quad \tilde{B} = \alpha B - 4At, \quad \tilde{C} = \alpha^2 C + 4At^2 - 2Bat.
$$

In particular, since passing to $-X$ changes only the sign of $B$, without loss of generality we will consider only $B \geq 0$. Suppose now that $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$ are given; in this case the constants $A, B, C$ satisfy an equation with coefficients that depend on $\mathbb{E}(X)$, $\mathbb{E}(X^2)$. The equation is obtained from taking the expected values of both sides of (1.2). Finally, rotation invariance implies that both $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$ are constant multiples of $I_n$, so in the non-degenerate case $\mathbb{E}(X^2) \neq 0$ we can normalize the matrices so that $\mathbb{E}(\tilde{X}) = 0$ and $\mathbb{E}(\tilde{X}^2) = I_n$, and express relation (1.2) in terms of two real constants $a, b$ as

$$
\mathbb{E}((\tilde{X} - \tilde{Y})^2|\tilde{S}) = \frac{2}{1 + 2a} \left( I_n + b\tilde{S} + a\tilde{S}^2 \right).
$$

In this normalized setting one can classify the Meixner ensembles into

(i) the elliptic case $b^2 > 4a$, which in Laha-Lukacs theorem on $\mathbb{R}$ corresponds to negative binomial ($a > 0$), Poisson ($a = 0$), or binomial ($a < 0$) laws.

(ii) the parabolic case $b^2 = 4a$, which corresponds to gamma law ($a > 0$) or Gaussian law ($a = 0$)
(iii) the hyperbolic case $b^2 < 4a$, which corresponds to the family of hyperbolic Meixner laws that include hyperbolic secant law.

Of course, for non-degenerate ensembles (1.2) and (1.4) are equivalent. If (1.2) holds with constants $A \neq 1, B, C$, then the standardization $\tilde{X}$ of $X$ with mean $\mu I_n$ and variance $\text{Var}(X) := E(X^2) - (E(X))^2 = \sigma^2 I_n$ satisfies (1.4) with

$$
(1.5) \quad a = \frac{A}{2(1 - A)}, \quad b = \frac{B + 4A\mu}{2\sigma(1 - A)}.
$$

What we call trivial Meixner ensembles are random multiples of the identity, $X = \xi I_n$ where $\xi$ is $\mathbb{R}$-valued with one of the six Meixner laws. The aim of the paper is the study of non-trivial Meixner ensembles for all $n$. We will be able to find all of them for $n = 2$, we will give examples of Meixner ensembles for $n \geq 3$ and we will write a system of $n$ linear partial differential equations (PDEs) of the second order satisfied by the Laplace transform of a Meixner ensemble of given parameters $(a, b)$ as defined by (1.4). Let us emphasize the fact that several different ensembles correspond to one set of parameters $a, b$ in (1.4).

The zoo of the known Meixner ensembles includes obviously the Gaussian ensembles which are described by their cumulant transform $c_1 \text{tr} \theta + \frac{c_2}{2} (\text{tr} \theta)^2 + \frac{c_3}{2} \text{tr} (\theta^2)$ with the familiar GOE, GUE and GSE (see [2] or [31]) corresponding to $c_3 = 1$ and $c_1 = c_2 = 0$ among them. The Gaussian ensembles yield a family of Meixner ensembles corresponding to $a = b = 0$ in (1.4), but the existence of non Gaussian ensembles for $a = b = 0$ and $n \geq 3$ has not been proved or disproved. Another important example is a Bernoulli ensemble: if $\mu_m$ is the distribution of the projection on a uniformly random $m$ dimensional space of the Euclidean space of dimension $n$, a Bernoulli ensemble is any mixing of $\mu_0, \ldots, \mu_n$; for $n = 1$ we get back the ordinary Bernoulli distribution. The Bernoulli ensembles can be seen as the only distributions of random projections $\mathbf{P}$ such that $U \mathbf{P} U^\ast \sim \mathbf{P}$ for $U \in \mathcal{K}_{n,\beta}$. The convolution of $N$ identical Bernoulli ensembles provides the binomial ensemble, and randomizing $N$ by Poisson or negative binomial distribution yields the Poisson and the negative binomial ensembles. The construction of the full family of Meixner ensembles with $b^2 = 4a$ or with $b^2 < 4a$ is available only for $n = 2$ and their extension to $n \geq 3$ is a challenge.

The paper is organized as follows. In Section 2 we describe Laplace transforms of Meixner ensembles. In Section 3 we give examples of Meixner ensembles. In Section 4 we derive the system of PDEs that determine the Laplace transforms of the general $n \times n$ Meixner ensembles. The general solution of this system is elusive for $n \geq 3$. In Section 5 we use the system of PDEs to show that under a natural integrability assumption the examples of Meixner ensembles from Section 3 exhaust all $2 \times 2$ Meixner ensembles. In Section 6 we collected additional material: Proposition 6.1 answers in negative a question raised in [9, Remark 5.8] for $2 \times 2$ random matrices with finite Laplace transform. Proposition 6.3 gives a series expansion for a Laplace transform of a random projection which can be used to derive (some) solutions to the system of PDEs. Section 6.4 makes some comments about the Jordan algebra context of the problem.
2. LAPLACE TRANSFORMS

In this section we rewrite Meixner property \[1.2\] in terms of the Laplace transform. We will work simultaneously with symmetric, unitary and symplectic ensembles, thus with probability laws on \( \mathbb{H}_{n,\beta} \) which are invariant under the action \( X \mapsto UXU^* \) where \( U \) is in the group \( \mathcal{K}_{n,\beta} \). We call \( \beta = 1, 2, 4 \) the Peirce constant (see \[16\], page 97, where \( \beta \) is denoted as \( d \) and \( n \) is denoted by \( r \)). The space \( \mathbb{H}_{n,\beta} \) is a real vector space equipped with the Euclidean structure defined by the real inner product \((x, y) \mapsto \Re(\text{tr}(xy))\) (the real part of the trace is needed for the symplectic case \( \beta = 4 \) only). For an \( \mathbb{H}_{n,\beta} \)-valued random variable \( X \), the Laplace transform is a mapping from \( \mathbb{H}_{n,\beta} \) into \((0, \infty]\), defined by

\[ L(\theta) = \mathbb{E}(\exp(\langle \theta | X \rangle)) \]

Let

\[ (x, y) \mapsto \langle x | y \rangle := \Re(\text{tr}(xy)) \]

(2.1) \( \Theta_X = \text{int}\{\theta : L(\theta) < \infty\} \).

(We will also use notation \( \Theta_\mu \) when \( \mu \) is the law of random variable \( X \).) Throughout this paper we consider only random matrices \( X \) with values in \( \mathbb{H}_{n,\beta} \) with non-empty \( \Theta_X \).

We need a result of linear algebra that we are not going to prove. Consider the space \( L_s(\mathbb{H}_{n,\beta}) \) of symmetric endomorphisms of the Euclidean space \( \mathbb{H}_{n,\beta} \). To each \( y \in \mathbb{H}_{n,\beta} \) we associate the elements \( y \otimes y \) and \( \mathbb{P}_y \) of \( L_s(\mathbb{H}_{n,\beta}) \) defined respectively by

\[ h \mapsto (y \otimes y)(h) = y \text{tr}(yh), \quad h \mapsto \mathbb{P}_y(h) = yhy. \]

They provide important examples of elements of \( L_s(\mathbb{H}_{n,\beta}) \). Now, \( L_s(\mathbb{H}_{n,\beta}) \) is itself a linear space, and the result that we are going to admit as a black box is the following (see \[14\] Lemma 6.3 and \[27\], Proposition 3.1 for a proof):

**Proposition A.** There exists a unique endomorphism \( \Psi \) of \( L_s(\mathbb{H}_{n,\beta}) \) such that for all \( y \in \mathbb{H}_{n,\beta} \) one has \( \Psi(y \otimes y) = \mathbb{P}_y \). Furthermore

\[ \Psi(\mathbb{P}_y) = \frac{\beta}{2} y \otimes y + \left(1 - \frac{\beta}{2}\right) \mathbb{P}_y. \]

With this notation we have the simple but crucial result. (Note that this result could remove the hypothesis of invariance under rotations.)

**Proposition 2.1.** Let \( \mu \) be an ensemble on \( \mathbb{H}_{n,\beta} \) such that its Laplace transform \( L \) is finite on an open non-empty set. The following are equivalent:

(i) The ensemble \( \mu \) is Meixner with real parameters \( A, B, C \), i.e. \[1.2\] holds for \( X \sim \mu \).

(ii) With \( k(\theta) = \ln L(\theta) \), we have

\[ 2(1 - A)\Psi(k''(\theta))(I_n) = 4A(k'(\theta))^2 + 2Bk'(\theta) + CI_n \]

for all \( \theta \in \Theta_X \).
Proposition 2.3.
Suppose \( \Theta \) generated by all the \( \mathcal{P}_L \) \( \mathcal{E} \) Definition 2.1.
A \( \mu \) that its Laplace transform is finite on a non empty open set. The Jørgensen set \( \Lambda(\mu) \).

For showing (i) \( \Rightarrow \) (ii) we first observe that
\[
(2.4) \quad E \left( (X - Y) \otimes (X - Y) e^{\theta(S)} \right) = 2L''(\theta)L(\theta) - 2L'(\theta) \otimes L'(\theta).
\]
To see this, note that since \( X, Y \) are i.i.d, the left hand side of \( (2.4) \) is
\[
2L(\theta)E \left( X \otimes X e^{\theta(X)} \right) - 2E \left( X e^{\theta(X)} \right) \otimes E \left( Y e^{\theta(Y)} \right).
\]
The same reasoning gives
\[
(2.5) \quad E \left( S \otimes S e^{\theta(S)} \right) = 2L''(\theta)L(\theta) + 2L'(\theta) \otimes L'(\theta).
\]
Since \( E \left( S e^{\theta(S)} \right) = 2L(\theta)L'(\theta) \), applying \( \Psi \) to \( (2.4) \) and \( (2.5) \), from \( (1.2) \) we get
\[
(2.6) \quad 2(1 - A)\Psi(L''(\theta))(I_n)L(\theta) = 2(1 + A)(L'(\theta))^2 + 2BL(\theta)L'(\theta) + C(L(\theta))^2 I_n.
\]
Since \( L' = k'L \) and \( L'' = (k'' + k' \otimes k)L \), this implies \( (2.3) \).

The proof of the converse implication is omitted. (Compare [23, Section 1.1.3].)

In the above proposition, the use of the endomorphism \( \Psi \) may seem surprising. For a random variable \( \hat{X} \) valued in a linear space \( E \), the link between \( E(X \otimes X) \) and the Laplace transform \( L \) of \( X \) is easy through the second differential of \( L \). However, if \( E \) is an algebra, like the space of square real matrices, relating \( E(X^2) \) to \( L \) is difficult. Here our reasoning was based on \( P_X(A) = XAX \) and thus \( P_X(I_n) = X^2 \). Some other facts about \( \Psi \) are known; it is a symmetric operator on \( L(H_{\alpha,\beta}) \) with only two eigenspaces: the one generated by all the \( \frac{\beta}{2} y \otimes y + P_y \) for the eigenvalue 1 and the one generated by all the \( \frac{\beta}{2} y \otimes y + P_y \) for the eigenvalue \(-\beta/2 \). Details are in [29, Section 5] where the dimensions of these two subspaces are also computed.

Remark 2.2. Note that if \( A = 1 \) then \( X \) is degenerate. Indeed, from \( (2.3) \) we see that \( A = 1 \) implies \( k'(\theta) = \text{const} \).

We now show that Meixner ensembles are preserved under convolution power.

Definition 2.1. Let \( \mu \) be a probability measure on a finite dimensional linear space such that its Laplace transform is finite on a non empty open set. The Jørgensen set \( \Lambda(\mu) \) of \( \mu \) is the set of \( \alpha > 0 \) such that there exists a probability measure \( \mu_\alpha \) with \( L_{\mu_\alpha} = L_\mu^\alpha \) and \( \Theta_{\mu_\alpha} = \Theta_\mu \), see e.g. [14, page 767].

For instance \( \Lambda(\mu) = (0, \infty) \) if \( \mu \) is infinitely divisible and \( \Lambda(\mu) \) is the set of positive integers if \( \mu \) is the Bernoulli distribution on \( \{0, 1\} \).

Proposition 2.3. Suppose \( \Theta_\mu \neq \emptyset \). If \( \mu \) is a Meixner ensemble with parameters \( A \neq 1, B, C \) and if \( \alpha \in \Lambda(\mu) \), then \( \mu_\alpha \) is a Meixner ensemble with parameters
\[
A_\alpha = A/(A + \alpha(1 - A)), \quad B_\alpha = \alpha B/(A + \alpha(1 - A)), \quad C_\alpha = \alpha^2 C/(A + \alpha(1 - A)).
\]

Proof. Since \( k_\alpha(\theta) := k_{\mu_\alpha}(\theta) = \alpha k_\mu(\theta) \), multiplying \( (2.3) \) by \( \alpha(1 - A_\alpha)/(1 - A) \) we get
\[
2(1 - A_\alpha)\Psi(k''_\alpha(\theta))(I_n) = 4\frac{A(1 - A_\alpha)}{\alpha(1 - A)}\frac{(k'_\alpha(\theta))^2}{1 - A} + \frac{2B(1 - A_\alpha)}{1 - A} k'_\alpha(\theta) + \frac{C\alpha(1 - A_\alpha)}{1 - A}.
\]
Solving the equation
\[ A_\alpha = \frac{A(1-A_\alpha)}{\alpha(1-A)}, \]
we get the formulas for \( A_\alpha, B_\alpha, C_\alpha \) such that (2.3) holds.

It will be convenient to have a version of Proposition 2.1 for standardized ensembles.

**Corollary 2.4.** Suppose \( E(\tilde{X}) = 0, E(\tilde{X}^2) = I_n \). Denote \( k(\theta) = \ln E(e^{\langle \theta | \tilde{X} \rangle}) \). If (1.4) holds, then
\[ \Psi(k''(\theta))(I_n) = I_n + 2b k'(\theta) + 4a(k'(\theta))^2. \]

Conversely, if the logarithm of the Laplace transform of \( \tilde{X} \) satisfies (2.7) (together with the initial conditions) then (1.4) holds, and \( E(\tilde{X}) = 0, E(\tilde{X}^2) = I_n \).

**Proof.** We apply (2.3) with \( A = 2a/(1+2a), B = 2b/(1+2a), C = 2/(1+2a) \). □

### 3. Examples of Meixner Ensembles

In this section we give examples of Meixner ensembles. In Theorem 5.1 we will show that the examples exhaust the family of all Meixner ensembles on \( H_{2,\beta} \) with \( \Theta_\mu \neq \emptyset \).

**3.1. Bernoulli and binomial ensembles.** The Bernoulli ensembles (which are different than the Bernoulli random matrices of independent two-valued entries) will be our basic building blocks for more complicated ensembles. Suppose \( P_m \) is the orthogonal projection onto the random and uniformly distributed \( m \)-dimensional subspace, i.e. \( P_m = U I_{m,n} U^* \) where \( I_{m,n} \) is the \( n \times n \) matrix with \( m \) ones on the diagonal followed by \( n-m \) zeroes, and \( U \) is uniformly distributed on the group \( K_{n,\beta} \).

**Definition 3.1.** Denote by \( \mu_m \) the distribution of \( P_m \). A Bernoulli ensemble with parameters \( q_1, \ldots, q_n \geq 0 \) that add to at most one, is the law
\[ \mu = q_0 \delta_0 + q_1 \mu_1 + \cdots + q_{n-1} \mu_{n-1} + q_n \delta_{I_n}, \]
where \( q_0 = 1 - (q_1 + \cdots + q_n) \).

In terms of random variables, Bernoulli ensemble can be realized by the random variable
\[ X = \begin{cases} 0 & \text{with probability } q_0 = 1 - (q_1 + \cdots + q_n), \\ P_1 & \text{with probability } q_1, \\ \vdots \\ I_n & \text{with probability } q_n. \end{cases} \]

**Proposition 3.1.** A Bernoulli ensemble is a Meixner ensemble with parameters \( A = -1, B = 2, C = 0 \).
\textbf{Proof.} We note that for any pair of projections
\begin{equation}
(P - Q)^2 = 2(P + Q) - (P + Q)^2.
\end{equation}
Applying this algebraic identity to random projections \(X\) and \(Y\), where \(X\) is given by (3.1) and \(Y\) is its independent copy, we see that (1.2) holds. \(\square\)

We note that for the Bernoulli ensemble formula (2.6) takes a particularly simple form: its Laplace transform \(L_B(\theta) = \mathbb{E}(\exp(\langle \theta \vert X \rangle))\) satisfies
\begin{equation}
\Psi(L_B''(\theta))(I_n) = L_B'(\theta).
\end{equation}
The corresponding version of (2.3) is
\begin{equation}
\Psi(k''_B(\theta))(I_n) = k'_B(\theta)(I_n - k'_B(\theta))
\end{equation}
The fact that all Bernoulli ensembles share the same A, B, C will be convenient for calculations. But it will be easier to classify the resulting Meixner laws into the familiar families from the standardized form that appears in formula (1.4). To do so, we find the moments of \(X\). We first note that by rotation invariance, \(\mathbb{E}(P_k) = cI_n\), so \(\mathbb{E}(\text{tr } (P_k)) = nc\).
\begin{equation}
\mathbb{E}(P_k) = \mathbb{E}(P_k^2) = \frac{k}{n} I_n.
\end{equation}
Therefore, with \(\bar{q} = \frac{1}{n}(q_1 + 2q_2 + \cdots + nq_n)\), we have \(\mathbb{E}(X^2) = \mathbb{E}(X) = \bar{q}I_n\).

If \(\bar{q} \neq 0, 1\), the standardized matrix is \(\widetilde{X} = \frac{1}{\sqrt{\bar{q}(1 - \bar{q})}}(X - \bar{q}I_n)\), and we get
\begin{equation}
\mathbb{E}((\widetilde{X} - \widetilde{Y})^2|\widetilde{S}) = 4 \left( I_n + \frac{(1 - 2\bar{q})\widetilde{S}}{2\sqrt{\bar{q}(1 - \bar{q})}} - \frac{\widetilde{S}^2}{4} \right).
\end{equation}
So (1.4) holds with parameters \(a = -1/4\) and \(b = (1/2 - \bar{q})/\sqrt{\bar{q}(1 - \bar{q})}\).

An intrinsic characterization of Bernoulli ensembles with an elementary proof is as follows.

\textbf{Proposition 3.2.} If \(X\) with values in \(H_{n,\beta}\) has a law invariant under rotations and \(X = X^2\) then its law is a Bernoulli ensemble.

\textbf{Proof.} Since \(X\) is symmetric, it is an orthogonal projection. Denote by \(d(X)\) the dimension of its image. Invariance under rotations means that for any bounded measurable function \(f\), we have
\begin{equation}
\mathbb{E}(f(UXU^*) - f(X)) = 0
\end{equation}
for all \(U \in K_{n,\beta}\). If \(d = 0, 1, \ldots, n\) and if \(f\) is zero outside of matrices of trace \(d\), then
\begin{equation}
\mathbb{E}(f(UXU^*) - f(X)|d(X) = d) = \mathbb{E}(f(UXU^*) - f(X)) = 0.
\end{equation}
This implies that the conditional law \(\mathcal{L}(X|d(X) = d)\) is invariant under rotations, so it is a law of \(P_d\). Then \(X\) is a Bernoulli ensemble with parameters \(q_m = \Pr(d(X) = m)\). \(\square\)

This implies the following converse of Proposition 3.1.
Proposition 3.3. Suppose that a law \( \mu \) belongs to a Meixner ensemble with parameters \( A = -1, B = 2, C = 0 \), and that the first four moments are finite. Then \( \mu \) is a Bernoulli ensemble.

Proof. Let \( X, Y \) be independent random variables with law \( \mu \), and let \( S = X + Y \). Since

\[
(X - Y)^2 = 2X^2 + 2Y^2 - S^2,
\]

and since the law of \((X, S)\) is the same as \((Y, S)\) from [12] we get

\[
E(X^2|S) = \frac{1}{2}S.
\]

From \( E(X|S) = \frac{1}{2}S \) we see that

\[
E(X^2|S) = E(X|S).
\]

From this, we deduce that \( E(X) = E(X^2) \), and that

\[
E(X^2S) = E(XS),
\]

which implies that \( E(X^3) = E(X) \).

Using (3.7) again, we have

\[
\text{tr } E(X^2S^2) = \text{tr } E(XS^2).
\]

By rotation invariance, the moments \( E(Y^j) = M_j \mathbf{1}_n \) are given by real numbers, so

\[
E(\text{tr } (X^*Y^s)) = nM_1, M_s = \text{tr } (E(X^*)E(Y^s)).
\]

Therefore, (3.8) implies that

\[
\text{tr } E(X^1) + 2 \text{tr } EX^2EX + \text{tr } ((EX^2)^2) = \text{tr } E(X^3) + 2 \text{tr } (EX^2)^2 + \text{tr } (EX^2EX).
\]

Since we already proved that \( M_1 = M_2 = M_3 \), we see that \( nM_4 = \text{tr } E(X^4) = \text{tr } E(X^3) = nM_1. \)

Let \( \Lambda_1, \Lambda_2, ..., \Lambda_n \) be the (random) eigenvalues of \( X \). From the equality of the first four moments of \( X \), we see that

\[
\sum_{j=1}^{n} E(\Lambda_j(1-\Lambda_j))^2 = \text{tr } E(X^2(\mathbf{1}_n - X)^2) = n(M_2 + M_4 - 2M_3) = 0.
\]

Therefore, all eigenvalues of \( X \) are either 0 or 1, and \( X^2 = X \). So by Proposition 3.2, the law \( \mu \) of \( X \) is a Bernoulli ensemble. \( \square \)

The Jørgensen set of a Bernoulli ensemble is of interest due to connections with free probability which allows continuous values for the analog of parameter \( N \). For \( n = 2 \), we will show that the Jørgensen set is \( \mathbb{N} \). The following sufficient condition shows that this is a ”generic case”.

Proposition 3.4. If \( \alpha \in \Lambda(\mu) \) and \( \mu \) is a Bernoulli ensemble with parameters \( q_1, ..., q_n \) on \( H_{n,0} \), then \((q_0 + q_1z + ... + q_nz^n)^\alpha \) is a polynomial in variable \( z \). In particular, \( n\alpha \) is an integer.

Proof. For \( \theta_s = \text{diag}(s, s, ..., s) \), the Laplace transform is \( L(\theta_s) = \sum_{r=0}^{n} q_re^{rs} = \prod_j (e^s - z_j)^{m_j} \), where \( z_j \in \mathbb{C} \) are the distinct roots of the polynomial \( q_0 + q_1z + ... + q_nz^n \) taken with their multiplicities \( m_j \). Since the Laplace transform must be an analytic function on its domain, we see that \( \alpha \) must be rational, and that if \( \alpha = p/q \) with relatively prime
$p, q \in \mathbb{N}$, then $q$ must be a common divisor of all multiplicities $m_1, m_2, \ldots$ of the roots. So $q$ divides also their sum $m_1 + m_2 + \cdots = n$. \hfill \Box

3.1.1. Binomial ensemble. The real, complex or quaternionic binomial ensembles are measures on $H_{n,\beta}$ which are parametrized by integer $N$ and a discrete probability law on \{0, 1, \ldots, n\}. We will follow the tradition of not listing the probability of 0 among the parameters.

Fix integer $N$ and non-negative numbers $q_1, \ldots, q_n$ with $q_1 + \cdots + q_n \leq 1$. Let $X_1, \ldots, X_N$ be independent random matrices with the same Bernoulli distribution (3.1).

**Definition 3.2.** The binomial ensemble $\text{Bin}(N, q_1, \ldots, q_n)$ with parameters $N = 1, 2, \ldots$ and $q_1, \ldots, q_n$, is the law of $X = \sum_{j=1}^{N} X_j$.

**Proposition 3.5.** A binomial ensemble with parameter $N$ is a Meixner ensemble with parameters $A = -1/(2N - 1)$, $B = 2N/(2N - 1)$, $C = 0$.

**Proof.** This is a special case of Proposition 2.3, applied to the law $\mu$ of $X_1$ with parameters described in Proposition 3.1, and to $\alpha = N$. \hfill \Box

For standardization, we will need to know that

$$E(X) = N\bar{q} \quad \text{and} \quad \text{Var}(X) = N\bar{q}(1 - \bar{q}).$$

From (1.5) we then get that (1.4) holds with $a = -1/(4N)$ and $b = (1/2 - \bar{q})/\sqrt{N\bar{q}(1 - \bar{q})}$.

3.2. Poisson ensemble. Our Poisson ensembles have parameters $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ with $\lambda = \lambda_1 + \cdots + \lambda_n > 0$.

**Definition 3.3.** Let $N$ be a Poisson real random variable, $\Pr(N = j) = e^{-\lambda} \lambda^j / j!$, $j = 0, 1, \ldots$, and let $q_1 = \lambda_1/\lambda, \ldots, q_n = \lambda_n/\lambda \geq 0$. Let $X_1, X_2, \ldots$ be independent Bernoulli matrices with the same parameters $q_1, \ldots, q_n$, and $X_0 = 0$. The Poisson ensemble is the law of $X = \sum_{k=0}^{N} X_k$.

We will use notation Poiss($\lambda_1, \ldots, \lambda_n$).

**Proposition 3.6.** The Poisson ensemble is a Meixner ensemble, as it satisfies (1.2) with parameters $A = 0$, $B = 1$, $C = 0$.

**Proof.** Let $L_B(\theta)$ be the Laplace transform of the corresponding Bernoulli ensemble. Then the log of the Laplace transform for the Poisson ensemble is $k(\theta) = \lambda(L_B(\theta) - 1)$. From (3.3) we get

$$\Psi(k''(\theta))(I_n) = k'(\theta),$$

which is (2.3) with $A = 0$, $B = 1$, $C = 0$. Therefore (1.2) holds by Proposition A. \hfill \Box

For standardization, we will need to know that, with $\bar{\lambda} = (\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n)/n$,

$$E(X) = \bar{\lambda} I_n \quad \text{and} \quad \text{Var}(X) = \bar{\lambda} I_n.$$

Indeed, $E(X) = E(E(X|N))$ and

$$\text{Var}(X) = E(\text{Var}(X|N)) + \text{Var}(E(X|N)) = E(N\bar{q}(1 - \bar{q}))I_n + \text{Var}(N\bar{q})I_n.$$

So $q$ must be a common divisor of all multiplicities $m_1, m_2, \ldots$ of the roots. So $q$ divides also their sum $m_1 + m_2 + \cdots = n$. \hfill \Box

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Fix integer $N$ and non-negative numbers $q_1, \ldots, q_n$ with $q_1 + \cdots + q_n \leq 1$. Let $X_1, \ldots, X_N$ be independent random matrices with the same Bernoulli distribution (3.1).

**Definition 3.2.** The binomial ensemble $\text{Bin}(N, q_1, \ldots, q_n)$ with parameters $N = 1, 2, \ldots$ and $q_1, \ldots, q_n$, is the law of $X = \sum_{j=1}^{N} X_j$.

**Proposition 3.5.** A binomial ensemble with parameter $N$ is a Meixner ensemble with parameters $A = -1/(2N - 1)$, $B = 2N/(2N - 1)$, $C = 0$.

**Proof.** This is a special case of Proposition 2.3, applied to the law $\mu$ of $X_1$ with parameters described in Proposition 3.1, and to $\alpha = N$. \hfill \Box

For standardization, we will need to know that

$$E(X) = N\bar{q} \quad \text{and} \quad \text{Var}(X) = N\bar{q}(1 - \bar{q}).$$

From (1.5) we then get that (1.4) holds with $a = -1/(4N)$ and $b = (1/2 - \bar{q})/\sqrt{N\bar{q}(1 - \bar{q})}$.

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We will use notation Poiss($\lambda_1, \ldots, \lambda_n$).

**Proposition 3.6.** The Poisson ensemble is a Meixner ensemble, as it satisfies (1.2) with parameters $A = 0$, $B = 1$, $C = 0$.

**Proof.** Let $L_B(\theta)$ be the Laplace transform of the corresponding Bernoulli ensemble. Then the log of the Laplace transform for the Poisson ensemble is $k(\theta) = \lambda(L_B(\theta) - 1)$. From (3.3) we get

$$\Psi(k''(\theta))(I_n) = k'(\theta),$$

which is (2.3) with $A = 0$, $B = 1$, $C = 0$. Therefore (1.2) holds by Proposition A. \hfill \Box

For standardization, we will need to know that, with $\bar{\lambda} = (\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n)/n$,

$$E(X) = \bar{\lambda} I_n \quad \text{and} \quad \text{Var}(X) = \bar{\lambda} I_n.$$

Indeed, $E(X) = E(E(X|N))$ and

$$\text{Var}(X) = E(\text{Var}(X|N)) + \text{Var}(E(X|N)) = E(N\bar{q}(1 - \bar{q}))I_n + \text{Var}(N\bar{q})I_n.$
One can then check that (1.4) holds with $a = 0$ and $b = \frac{1}{2\sqrt{\lambda}}$. It is also easy to see that if the law of $X$ is Poisson with parameters $\lambda_1, \ldots, \lambda_n > 0$, then $X = \sum_{m=1}^{n} X_m$ is the sum of independent random variables $X_m$ from Poisson ensembles with parameters $(0, \ldots, 0, \lambda_m, 0 \ldots, 0)$. Furthermore, real random variable $\text{tr}(X)$ has the compound Poisson law with the law of summands given as $\sum_{k=0}^{n} \frac{\lambda_k}{\lambda} \delta_k$.

We remark that the Poisson model with exactly one parameter $\lambda_1 \neq 0$ appears explicitly in [12, page 638], as part of the construction of matrix models for all-free-infinitely divisible laws; see also [7]. However, regression properties of the model were not analyzed.

3.3. Negative binomial ensemble. We now use Bernoulli ensembles to construct the negative binomial ensemble. Let $N$ be a negative binomial random variable with parameters $r > 0$, $0 < p < 1$, i.e.

$$P(N = j) = \frac{\Gamma(r + j)}{\Gamma(r)j!} p^r q^j, \quad j = 0, 1, 2, \ldots, q = 1 - p.$$ (3.11)

Definition 3.4. The negative binomial ensemble $\text{NB}(r, q_1, \ldots, q_n)$ with parameters $r > 0$ and $q_1, \ldots, q_n \geq 0$ such that $q_1 + \cdots + q_n < 1$, is the law of the random sum

$$X = \sum_{k=0}^{N} X_k,$$

where $N$ has distribution (3.11) with $p = 1 - (q_1 + \cdots + q_n)$, $X_1, X_2, \ldots,$ are independent Bernoulli ensembles with parameters $q_1/(1 - p), \ldots, q_n/(1 - p)$, and $X_0 = 0$.

Proposition 3.7. The negative binomial ensemble is a Meixner ensemble, as it satisfies (1.2) with parameters $A = \frac{1}{2r+1}$, $B = \frac{2r}{2r+1}$, $C = 0$.

Proof. Let $L_B(\theta)$ be the Laplace transform of the corresponding Bernoulli ensemble. The generating function $\mathbb{E}(z^N) = p^r/(1 - qz)^r$ gives

$$L(\theta) = \mathbb{E}(e^{\theta X}) = p^r (1 - qL_B(\theta))^{-r},$$ (3.12)

so

$$1 - qL_B(\theta) = p(L(\theta))^{-1/r}.$$  

Differentiating, we get

$$qL_B'(\theta) = \frac{p}{r} L'(\theta)(L(\theta))^{-(1+r)/r}$$

and

$$qL_B''(\theta) = \frac{p}{r} L''(\theta)(L(\theta))^{-(1+r)/r} - \frac{p(1+r)}{r^2} L'(\theta) \otimes L'(\theta)(L(\theta))^{-(1+2r)/r}.$$  

From (3.3), we get

$$\frac{1}{r} \Psi(L''(\theta)(I_n))L(\theta) = \frac{1 + r}{r^2} (L'(\theta))^2 + \frac{1}{r} L'(\theta)L(\theta).$$  

We re-write this as

$$\frac{2r}{2r+1} \Psi(L''(\theta)(I_n))L(\theta) = \frac{2(r+1)}{2r+1} (L'(\theta))^2 + \frac{2r}{2r+1} L'(\theta)L(\theta),$$
which is (2.6) with \( A = \frac{1}{2r+1} \), \( B = \frac{2r}{2r+1} \), \( C = 0 \). Therefore (1.2) holds by Proposition \( \square \)

For standardization, we will need to know that with \( \bar{q} = (q_1 + 2q_2 + \cdots + nq_n)/n \),

\[
E(X) = r \frac{\bar{q}}{p} I_n \quad \text{and} \quad \text{Var}(X) = \frac{\bar{q}}{p^2} (p + \bar{q}) I_n.
\]

Indeed, \( E(X) = E(E(X|N)) = \bar{q}/(1-p)E(N)I_n = \bar{q}r/p I_n \) and

\[
\text{Var}(X) = E(\text{Var}(X|N)) + \text{Var}(E(X|N)) = E(Nq(1-\bar{q}))I_n + \text{Var}(N\bar{q})I_n
= \bar{q}(1-\frac{\bar{q}}{1-p}) \frac{r}{p} I_n + \frac{\bar{q}^2}{(1-p)p^2} I_n.
\]

Then from (1.3) we see that (1.4) holds with \( a = \frac{1}{4r} \) and \( b = \frac{p+2\bar{q}}{2\sqrt{r(p+\bar{q})}} \).

3.4. Gaussian ensemble. Since \( X - Y \) and \( S \) are independent for any Gaussian independent identically distributed pair \( X, Y \), formula (1.2) holds with \( A = B = 0 \) and \( C = 2 \). The requirement of rotational invariance reduces the choices of the Gaussian law to the three-parameter family:

\[
k(\theta) = c_1 \text{tr} \theta + c_2 (\text{tr} \theta)^2/2 + c_3 \text{tr} (\theta^2)/2
\]

with \( c_3 \geq 0 \) and \( nc_2 + c_3 \geq 0 \). To see this, without loss of generality we take \( c_1 = 0 \). Let \( \theta_1, \ldots, \theta_n \) be the eigenvalues of \( \theta \), and consider the matrix of the quadratic form \( c_2 (\theta_1 + \cdots + \theta_n)^2 + c_3 (\theta_1^2 + \cdots + \theta_n^2) \) whose characteristic polynomial is \((z-(nc_2 + c_3))(z-c_3)^{n-1}\). This shows that \( c_3 \geq 0 \) and \( nc_2 + c_3 \geq 0 \).

The explicit construction is to start with auxiliary GUE/GOE/GSE matrix \( Z \) and independent real standard normal \( \zeta \).

Definition 3.5. For \( c_1 \in \mathbb{R} \), \( c_3 \geq 0 \) and \( c_2 \in \mathbb{R} \) such that \( nc_2 + c_3 \geq 0 \), the Gaussian Meixner ensemble is the law of

\[
X = \sqrt{c_3} \left( Z - \frac{1}{n} \text{tr}(Z) I_n \right) + \sqrt{c_2 + \frac{c_3}{n}} \zeta I_n + c_1 I_n.
\]

Proposition 3.8. The logarithm of the Laplace transform of the Gaussian Meixner ensemble is given by (3.14).

Proof. Since \( \langle \theta | Z - a \text{tr} Z \rangle = \langle \theta - a \text{tr} \theta | Z \rangle \) and \( \text{tr} ((\theta - \frac{1}{n} \text{tr} \theta)^2) = \text{tr} \theta^2 - \frac{1}{n} (\text{tr} \theta)^2 \), we see that the answer follows from the well known GUE/GOE/GSE formula \( E \exp(\text{tr} (\theta Z)) = \exp(\text{tr} \theta^2/2) \). \( \square \)

3.5. Gamma ensemble. The remaining types of Meixner ensembles will be constructed only for matrices of size \( n = 2 \). One difficulty we encounter is lack of continuity, which we now explain.

It is natural to expect that gamma ensemble arises as \( \lim_{p \to 0} pX_p \) of a sequence of negative binomial ensembles with varying parameter \( p \) while \( r \) and the ratios \( q_i/(1-p) \)
are kept fixed. However from (3.12) we see that
\[ \lim_{p \to 0} E e^{\langle \theta | p X_p \rangle} = \lim_{p \to 0} p^r (1 - qL_B(\theta))^{-r} = (1 - \text{tr} (\theta L_B'(0)))^{-r} = (1 - q \text{tr} \theta)^{-r}. \]
So this is a trivial ensemble of the form $\xi I_n$ with real gamma-distributed $\xi$.

We now show that there are non-trivial gamma ensembles for $n = 2$. This construction is based on a more detailed analysis of the system of PDEs that arises from (2.7).

**Definition 3.6.** The Gamma ensemble on $H_{2,\beta}$ with parameters $p > \beta/2, c > 1$ is defined by its Laplace transform
\[ (3.15) \quad E(e^{\langle \theta | X \rangle}) = \left(1 - 4c \sqrt{1 + \beta \text{tr} \theta + \beta \text{tr}^2 \theta + 4 \text{det} \theta} \right)^{-p}, \]
defined on $\Theta_X = \{ \theta \in H_{2,\beta} : 1 - 4c \sqrt{1 + \beta \text{tr} \theta + \beta \text{tr}^2 \theta + 4 \text{det} \theta} > 0 \}$.

Of course, this definition requires a proof that the required law on $H_{2,\beta}$ exists. The Gamma ensemble on $H_{2,\beta}$ is constructed by choosing the appropriate law on $R^{\beta+2}$, and arranging the corresponding real random variables into the random matrix. The construction is based on the laws analyzed by Letac and Wesolowski [28, Theorem 3.1]. According to this result, for $p > \beta/2$ and $c > 1$ there is a probability measure $\nu_{p,c}(dx)$ on the open Lorentz cone $\Omega_\beta = \{ x \in R^{\beta+2} : x_0 > \sqrt{\sum_{j=1}^{\beta+1} x_j^2} \}$ with the Laplace transform
\[ (3.16) \quad \int_{\Omega_\beta} e^{\sum s_j x_j} \nu_{p,c}(dx) = \left(1 - 2cs_0 + s_0^2 - \sum_{j=1}^{\beta+1} s_j^2 \right)^{-p}. \]

**Proposition 3.9.** Let $(\xi_0, \ldots, \xi_{\beta+1})$ have joint distribution $\nu_{p,c}$. The gamma ensemble with parameters $p, c$ is:
(i) for $\beta = 1$,
\[ (3.17) \quad X = \begin{bmatrix} \sqrt{1 + \beta \xi_0 + \xi_1} & \xi_2 \\ \xi_2 & \sqrt{1 + \beta \xi_0 - \xi_1} \end{bmatrix}, \]
(ii) for $\beta = 2$,
\[ (3.18) \quad X = \begin{bmatrix} \sqrt{1 + \beta \xi_0 + \xi_1} & \xi_2 + i\xi_3 \\ \xi_2 - i\xi_3 & \sqrt{1 + \beta \xi_0 - \xi_1} \end{bmatrix}, \]
where $i = \sqrt{-1}$.
(iii) for $\beta = 4$,
\[ (3.19) \quad X = \begin{bmatrix} \sqrt{1 + \beta \xi_0 + \xi_1} & \xi_2 + i\xi_3 + j\xi_4 + k\xi_5 \\ \xi_2 - i\xi_3 - j\xi_4 - k\xi_5 & \sqrt{1 + \beta \xi_0 - \xi_1} \end{bmatrix}, \]
where $i, j, k$ are the standard quaternion basis.

**Proof.** We give the proof for $\beta = 4$. Writing
\[ (3.20) \quad \theta = \begin{bmatrix} s_0 + s_1 & s_2 + is_3 + js_4 + ks_5 \\ s_2 - is_3 - s_4 - ks_5 & s_0 - s_1 \end{bmatrix}, \]
we have \( \langle \theta | X \rangle = 2\sqrt{5} s_{0} \xi_{0} + 2 \sum_{j=1}^{5} s_{j} \xi_{j} \), so (3.19) follows from (3.16). \[ \square \]

**Proposition 3.10.** A gamma ensemble on \( H_{2, \beta} \) with parameters \( p > \beta/2, c > 1 \) is a Meixner ensemble with parameters

\[ A = \frac{1}{1 + 2p}, \quad B = 0, \quad C = 0. \]

The mean and the variance are \( \mathbb{E}(X) = 2pc \sqrt{1 + \beta} I_{2}, \) \( \text{Var}(X) = 4pc^{2}(1 + \beta)I_{2}. \) For the standardized version, (1.4) holds with \( a = 1/(4p) \) and \( b = 1/\sqrt{p}. \)

The proof relies on the system of PDEs derived in Theorem 4.1; it appears after Proposition 5.6.

**Remark 3.11.** Somewhat more generally, one can define gamma ensembles on \( H_{n, \beta} \) as the ensembles with Laplace transform

\[ L(\theta) = (1 - c \text{tr} \theta + (\beta(n - 1) + 2)(\text{tr} \theta)^{2} - 2 \text{tr} \theta^{2})^{-p}. \]

Such an ensemble is a Meixner ensemble, as it follows from Theorem 4.1 that (1.4) holds with \( a = 1/(4p) \) and \( b = 1/\sqrt{p}. \) Since this is only a one-parameter subset of the possible solutions, we do not pursue this construction further.

3.6. **Hyperbolic ensemble.** Recall that the Bessel functions and the modified Bessel functions are [38, $3.1 (8), $3.7 (2)]

\[ J_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n}(z/2)^{\nu+2n}}{n! \Gamma(n+\nu+1)}, \quad I_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{n! \Gamma(n+\nu+1)}. \]

Following [24], we work with differently normalized Bessel functions

\[ J_{\nu}(x) := \Gamma(\nu + 1) (2/x)^{\nu} J_{\nu}(x), \]
\[ I_{\nu}(x) := \Gamma(\nu + 1) (2/x)^{\nu} I_{\nu}(x). \]

which are entire functions of \( z \in \mathbb{C} \) for \( \nu > -1. \)

**Definition 3.7.** Fix \( \alpha > 0, 0 \leq \lambda < 1, \rho \in \mathbb{R}. \) Define \( \varphi \in (-\pi, \pi) \) by \( \sin \varphi = \rho/\sqrt{(1 - \lambda)^{2} + \rho^{2}}. \) Let \( \ell_{\beta} \) be the first positive zero of \( J_{(\beta-1)/2}(x) \) i.e., \( \ell_{1} \approx 2.40483, \ell_{2} \approx 3.14159, \) and \( \ell_{4} \approx 4.49341. \)

Let

\[ \Theta_{X} = \{ \theta \in H_{2, \beta} : |\theta_{1} + \theta_{2} + \varphi| < \pi/2, |\theta_{1} - \theta_{2}| < \ell_{\beta} \}, \]

where \( \theta_{1}, \theta_{2} \) are the eigenvalues of \( \theta. \)

The hyperbolic ensemble on \( H_{2, \beta} \) with parameters \( \alpha, \lambda, \rho \) is defined by its Laplace transform:

\[ \mathbb{E}(\exp(\theta|X|)) = \left((1 - \lambda) \cos \text{tr} \theta - \rho \sin \text{tr} \theta + \lambda J_{(\beta-1)/2}(\sqrt{\text{tr}^{2} \theta - 4 \det \theta})\right)^{-\alpha} \]

on the set \( \Theta_{X}. \) (When \( \rho = 0 \) we also allow \( \lambda = 1. \))
These ensembles are constructed again by finding an appropriate law on $\mathbb{R}^{\beta+2}$ and arranging the real random variables into a random matrix in $H_{2,\beta}$. We begin with an auxiliary fact about Laplace transforms of probability measures on product of Euclidean spaces.

**Proposition 3.12.** Let $E$ and $F$ two Euclidean spaces, and $S \subset E$, $T \subset F$ be open non-empty sets. If $f : S \to \mathbb{R}$, $g : T \to \mathbb{R}$ are such that

$$
\frac{1}{f^\alpha(s)}, \quad \frac{1}{g^\alpha(t)}
$$

are Laplace transforms of probability measures for some $\alpha > 0$ and also for $\alpha = 1$, then for $0 < \lambda < 1$, and $(s, t) \in S \times T$, function

$$
\frac{1}{((1 - \lambda)f(s) + \lambda g(t))^\alpha}
$$

is a Laplace transform of a probability measure $H_\alpha(dx, dy)$ on $E \times F$.

**Proof.** We denote by $\gamma_\alpha$ the gamma distribution on $(0, \infty)$ with shape parameter 1 and scale parameter $\alpha > 0$. Fix $0 < \lambda < 1$, and let $\mu_\alpha$ and $\nu_\alpha$ be two probabilities on $E$ and $F$ such that the integrals

$$
\frac{1}{f^\alpha(s)} = \int_E e^{\langle s \mid x \rangle} \mu_\alpha(dx), \quad \frac{1}{g^\alpha(t)} = \int_F e^{\langle t \mid y \rangle} \nu_\alpha(dy)
$$

are finite on the maximal open sets $S \subset E$ and $T \subset F$ respectively. (Denote $\mu = \mu_1$ and $\nu = \nu_1$.)

Consider the set of independent variables $U$, $A$, $B$, $X_1, X_2, \ldots$, $Y_1, Y_2, \ldots$, $N_X$, $N_Y$ where $U \sim \gamma_\alpha$, $A \sim \mu_\alpha$, $B \sim \nu_\alpha$, $X_j \sim \mu$, $Y_j \sim \nu$, and $t \mapsto N_X(t)$ and $t \mapsto N_Y(t)$ are two standard Poisson processes. Define the random variables $X$ and $Y$ on $E$ and $F$ respectively by

$$
X = A + \sum_{j=1}^{N_X(U)} X_j, \quad Y = B + \sum_{j=1}^{N_Y((1-\lambda)U)} Y_j.
$$

We have

$$
\mathbb{E}(e^{\langle s | X \rangle + \langle t | Y \rangle}) = \frac{1}{f^\alpha(s)g^\alpha(t)} e^{-\lambda U(\frac{1}{\alpha}-1)} e^{-(1-\lambda)U(\frac{1}{\gamma}-1)}.
$$

Thus

$$
\mathbb{E}(e^{\langle s | X \rangle + \langle t | Y \rangle}) = \frac{1}{f^\alpha(s)g^\alpha(t)} \int_0^\infty e^{-\lambda u} \frac{(1-\lambda)u}{\gamma(t)} u^{\alpha-1} \, du
$$

$$
= \frac{1}{((1 - \lambda)f(s) + \lambda g(t))^\alpha}.
$$

This shows that the distribution $H_\alpha(dx, dy)$ of $(X, Y)$ satisfies on $S \times T$

$$
\iint_{E \times F} e^{\langle s | x \rangle + \langle t, y \rangle} H_\alpha(dx, dy) = \frac{1}{((1 - \lambda)f(s) + \lambda g(t))^\alpha}.
$$

$\square$
We now apply this result to the following Laplace transform, with the notation of Definition 3.7.

Proposition 3.13. For every \( \alpha > 0, m \geq 1, \beta > -1, \rho \in \mathbb{R}, \) and \( 0 < \lambda < 1 \) there exist a unique probability measure \( H = H_{\alpha,\beta,\lambda,\rho,m} \) on \( \mathbb{R} \times \mathbb{R}^m \) with the Laplace transform

\[
\int_{\mathbb{R} \times \mathbb{R}^m} \exp(sx + \langle t, y \rangle) H(dx, dy) = \frac{1}{((1-\lambda)(\cos(s) - \rho \sin(s)) + \lambda J_{(\beta-1)/2}(\|t\|))^{\alpha}}
\]

for \( s \in \mathbb{R}, t \in \mathbb{R}^m \) such that \( |s + \phi| < \pi/2 \) and \( \|t\| < \ell_\beta \).

Proof. According to [38] $15.27$, Bessel function \( J_{(\beta-1)/2} \) has simple real zeros that come in opposite pairs. Arranging the positive zeros in increasing order, \( 0 < j_1 < j_2 < \ldots \), from [38] $15.41 (3)$ we get

\[
J_{(\beta-1)/2}(z) = \prod_{k=1}^{\infty} (1 - z^2/j_k^2),
\]

so for \( 0 < r < j_1 = \ell_\beta \),

\[
\frac{1}{(J_{(\beta-1)/2}(\sqrt{r}))^{\alpha}} = \prod_{k=1}^{\infty} \frac{1}{(1 - rj_k^2)^{\alpha}}
\]

is a Laplace transform (of the convergent series of independent symmetrized Gamma random variables). Therefore, by Schoenberg’s theorem [36, Theorem 2], for every \( m \) there exists a measure \( \nu_\alpha \) on Borel sets of \( \mathbb{R}^m \) such that for \( \|t\| < \ell_\beta \),

\[
\frac{1}{(J_{(\beta-1)/2}(\|t\|))^{\alpha}} = \int_{\mathbb{R}^m} \exp(t, y) \nu_\alpha(dy).
\]

Similarly, since \( \cos \phi = (1 + \rho^2)^{-1/2} \), we have

\[
\cos(s) - \rho \sin s = \frac{\cos(s + \phi)}{\cos \phi} = \prod_{k=1}^{\infty} \frac{1 - 4(s + \phi)^2/(\pi(2k-1))^2}{1 - 4\phi^2/(\pi(2k-1))^2},
\]

so there is a probability measure \( \mu_\alpha \) such that for \( |s + \phi| < \pi/2 \),

\[
\frac{1}{(\cos s - \rho \sin s)^{\alpha}} = \int_{\mathbb{R}} e^{sx} \mu_\alpha(dx).
\]

Proposition 3.12 shows that there is a probability measure \( H_{\alpha,\beta,\lambda,\rho,m} \) on \( \mathbb{R} \times \mathbb{R}^m \) such that (3.27) holds. \( \square \)

Proposition 3.14. Let \( (\xi_0, \ldots, \xi_{\beta+1}) \) have joint distribution \( H_{\alpha,\beta,\lambda,\rho/(1-\lambda),\beta+1} \). The hyperbolic ensemble on \( \mathbb{H}_{2,\beta} \) with parameters \( \alpha, \lambda, \rho \) is:

(i) for \( \beta = 1 \),

\[
X = \begin{bmatrix} \xi_0 + \xi_1 & \xi_2 \\ \xi_2 & \xi_0 - \xi_1 \end{bmatrix}.
\]
(ii) for $\beta = 2$,

$$X = \begin{bmatrix} \xi_0 + \xi_1 & \xi_2 + i\xi_3 \\ \xi_2 - i\xi_3 & \xi_0 - \xi_1 \end{bmatrix},$$

where $i = \sqrt{-1}$.

(iii) for $\beta = 4$, (3.28)

$$X = \begin{bmatrix} \xi_0 + \xi_1 & \xi_2 + i\xi_3 + j\xi_4 + k\xi_5 \\ \xi_2 - i\xi_3 - j\xi_4 - k\xi_5 & \xi_0 - \xi_1 \end{bmatrix},$$

where $i, j, k$ is the standard basis of quaternions.

Proof. To get the Laplace transform, we use (3.20) (or its appropriate modifications for $\beta < 4$) with $\langle \theta | X \rangle = 2s_0\xi_0 + 2\sum_{j=1}^{\beta+1} s_j\xi_j$. From (3.27) with $s = 2s_0$ and $t = (2s_1, 2s_2, \ldots, 2s_{\beta+1})$ we get

(3.29) $E(e^{\sum_{j=0}^{\beta+1} 2s_j\xi_j}) = ((1 - \lambda)(\cos(2s_0) - \rho \sin(2s_0)) + \lambda J_{(\beta-1)/2}(|t|))^{-\alpha}$,

which gives (3.25). $\Box$

Proposition 3.15. The hyperbolic ensemble with parameters $\alpha > 0$, $0 \leq \lambda < 1$, $\rho \in \mathbb{R}$ is a Meixner ensemble with parameters

$$A = \frac{1}{1 + 2\alpha}, \ B = 0, \ C = \frac{2\alpha^2(\rho^2 + 2)}{2\alpha + 1}.$$

The mean and the variance are $E(X) = \rho\alpha I_2$, $\text{Var}(X) = \alpha(1 + \rho^2)I_2$. For the standardized version, (1.4) holds with $a = 1/(4\alpha)$ and $b = \rho/\sqrt{\alpha(1 + \rho^2)}$.

We remark that the “exceptional” hyperbolic ensemble with parameters $\alpha > 0$, $\lambda = 1$, $\rho = 0$ is also a Meixner ensemble with parameters $A = \frac{1}{1+2\alpha}$, $B = 0$, $C = \frac{4\alpha^2}{2\alpha + 1}$, with mean $E(X) = 0$ and with variance $\text{Var}(X) = \alpha I_2$. The proof relies on the system of PDEs derived in Theorem 4.1; it appears after Proposition 5.6.

4. The system of PDEs

In this section, we derive a system of PDEs for the Laplace transform of general Meixner ensembles. As previously, we consider simultaneously the real, complex, and quaternionic cases.

We introduce some notation. For $\theta \in \mathbb{H}_{n,\beta}$ and $j = 0, 1, \ldots$, we consider $\sigma_j(\theta)$ defined by

(4.1) $\det(I_n + z\theta) = \sum_{j=0}^{n} \sigma_j(\theta)z^j$.

Recall that for $\beta = 4$, the determinant if quaternionic matrices are defined only for hermitian matrices, see [10] page 29 and [13]. Note that

$$\sigma_0 = 1, \ \sigma_1(\theta) = \text{tr } \theta, \ \sigma_2(\theta) = \frac{1}{2}(\text{tr } \theta)^2 - \frac{1}{2} \text{tr } (\theta^2), \ \sigma_n(\theta) = \det \theta,$$
and that $\sigma_j = 0$ for $j > n$. We also adopt the convention that $\sigma_j = 0$ for $j < 0$. In general, $\sigma_j(\theta) = e_j(\theta_1, \theta_2, \ldots)$ is the $j$-th elementary symmetric function of the eigenvalues $\theta_1, \theta_2, \ldots$ of $\theta \in \mathbb{H}_{n,\beta}$; this notation and most of our calculations do not depend on the dimension $n$. Denote by $U$ the open subset of $\mathbb{R}^n$ such that if $(\sigma_1, \ldots, \sigma_n) \in U$ then the polynomial in $x$ defined by $\sum_{i=0}^n (-1)^{n-i} \sigma_i x^{n-i}$ has only distinct real roots.

The main result of this section is the system of PDEs that determines the Laplace transform of a Meixner ensemble on a nonempty open set $\Theta_X \subset \mathbb{H}_{n,\beta}$. This system is written in terms of an auxiliary function $g(\sigma_1, \ldots, \sigma_n)$ which is defined on an appropriate non-empty open set $U_X \subset U$. The link between $L$ and this auxiliary function $g$ varies according to the fact that $a$ or $b$ are zero or not. To define the set $U_X$, we need $\Theta_X$ to be closed under conjugation ($\theta \in \Theta_X$ implies $U\theta U^* \in \Theta_X$ for all $U \in \mathbb{K}_{n,\beta}$), and to have $0$ in its closure. Consider the $n$-dimensional real subspace $D_n \subset \mathbb{H}_{n,\beta}$ consisting of diagonal matrices, and let $D_0^n \subset D_n$ denote the set of matrices with distinct eigenvalues. Since $\Theta_X$ is invariant under rotations, $\Theta_X \cap D_n$ is a nonempty open set by removing a finite number of hyperplanes, so it is also non-empty, and has $0$ in its closure.

Denote by $\sigma : \mathbb{H}_{n,\beta} \to \mathbb{R}^n$ the mapping $\theta \mapsto (\sigma_1(\theta), \ldots, \sigma_n(\theta))$. If $\theta \in D_0^n$ then by the Vandermonde’s determinant, $B_\theta = \{I_n, \theta, \theta^2, \ldots, \theta^{n-1}\}$ is a basis of $D_n$. From formula (4.10) below, we see that in that basis, the matrix representation of the derivative of $\sigma$ restricted to $D_n$ is triangular at $\theta \in D_0^n$, with $\sigma_0 = 1$ on the diagonal,

$$
[\sigma'(\theta)]_{B_\theta} = \\
\begin{bmatrix}
\sigma_0 & 0 & 0 & \cdots & 0 \\
-\sigma_1(\theta) & \sigma_0 & 0 & \cdots & 0 \\
\sigma_2(\theta) & -\sigma_1(\theta) & \sigma_0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
(-1)^{n-1}\sigma_{n-1}(\theta) & (-1)^{n-2}\sigma_{n-2}(\theta) & \cdots & \cdots & \sigma_0
\end{bmatrix}
$$

Therefore, the Jacobian of $\sigma$ is 1 and $\sigma : D_0^n \to \mathbb{R}^n$ is an open mapping. Thus, the set

$$
U_X = \sigma(\Theta_X \cap D_0^n)
$$

is open, non-empty, and since $\sigma(0) = 0$, it has $0 \in \mathbb{R}^n$ in its closure.

To shorten the formulas, for $i, j = 1, 2, \ldots, n$ we write

$$
g_i = \frac{\partial g}{\partial \sigma_i} \quad \text{and} \quad g_{ij} = \frac{\partial^2 g}{\partial \sigma_i \partial \sigma_j}.
$$

Recall the convention that $\sigma_j = 0$ for $j < 0$ or $j > n$. 


For \( g : U_X \to \mathbb{R} \) consider the vector
\[
\mathbb{D}(g) = \begin{bmatrix}
g_{11} - \sum_{r,s=2}^{n} \sigma_{r+s-2}g_{rs} - (n-1)\frac{\beta}{2}g_2 \\
g_{12} + 2g_{12} - \sum_{r,s=3}^{n} \sigma_{r+s-3}g_{rs} - (n-2)\frac{\beta}{2}g_3 \\
\vdots \\
\sum_{r,s=1}^{n-1} \sigma_{r+s-n}g_{rs} - \sigma_ng_{nn} - \frac{\beta}{2}g_n \\
\sum_{r,s=1}^{n} \sigma_{r+s-n-1}g_{rs}
\end{bmatrix}
\]
(4.3)

**Theorem 4.1.** Let \( X \) be a random variable with values in \( H_{n,\beta} \) with the law invariant under rotations. Let \( L(\theta) = e^{\beta(\theta)} = \mathbb{E}(e^{\theta X}) \) be its Laplace transform, and assume that \( \Theta_X \neq 0 \). Suppose that \( \mathbb{E}(X) = 0, \mathbb{E}(X^2) = I_n \) and that \( X \) satisfies (1.4) with some \( a \) and \( b \). Let \( U_X \) be the associated open subset of \( \mathbb{R}^n \), see (1.2).

(i) If \( a \neq 0 \), define \( g : U_X \to \mathbb{R} \) by \( g(\sigma_1(\theta), \ldots, \sigma_n(\theta)) = \exp(-4ak(\theta) - b\text{tr}\theta) \). Then \( g \) satisfies the system of PDEs
\[
\mathbb{D}(g) = \begin{bmatrix}
(b^2 - 4a)g \\
0 \\
\vdots \\
0
\end{bmatrix}
\]
Furthermore,
\[
g(\sigma_1, \ldots, \sigma_n) \to 1, \quad \frac{\partial g(\sigma_1, \ldots, \sigma_n)}{\partial \sigma_1} \bigg|_{\sigma_1=0} \to -b
\]
as \( (\sigma_1, \ldots, \sigma_n) \to 0 \) over \( U_X \).

(ii) If \( a = 0, b \neq 0 \), define \( g : U_X \to \mathbb{R} \) by \( g(\sigma_1(\theta), \ldots, \sigma_n(\theta)) = k(\theta) + \frac{1}{2b} \text{tr}\theta \). Then \( g \) satisfies the system of PDEs
\[
\mathbb{D}(g) = 2b \begin{bmatrix}
g_1 \\
g_2 \\
\vdots \\
g_n
\end{bmatrix}
\]
Furthermore,
\[
g(\sigma_1, \ldots, \sigma_n) \to 0, \quad \frac{\partial g(\sigma_1, \ldots, \sigma_n)}{\partial \sigma_1} \to \frac{1}{2b}
\]
as \( (\sigma_1, \ldots, \sigma_n) \to 0 \) over \( U_X \).

(iii) If \( a = 0, b = 0 \), define \( g : U_X \to \mathbb{R} \) by \( g(\sigma_1(\theta), \ldots, \sigma_n(\theta)) = k(\theta) \). Then \( g \) satisfies the system of PDEs
\[
\mathbb{D}(g) = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]
Furthermore,

\[(4.6) \quad g(\sigma_1, \ldots, \sigma_n) \to 0, \quad \frac{\partial g(\sigma_1, \ldots, \sigma_n)}{\partial \sigma_1} \to 0\]

as \((\sigma_1, \ldots, \sigma_n) \to 0\) over \(U_X\).

Conversely, let \(\mu\) be an ensemble on \(H_{n,\beta}\) such that \(\Theta_\mu \neq \emptyset\). If one can find constants \(a, b\) that lead to one of the equations in (i), (ii) or (iii), then the corresponding version of equation (2.7) holds for all \(\theta \in \Theta_\mu\), and hence \(\mu\) is a Meixner ensemble with parameters \(a, b\).

The left hand side \(D(g)\) of our systems of the PDE’s resembles the system of PDE’s for the Bessel function of matrix argument given in \([21, \text{Eqtn (18)}]\) when \(\beta = 1\), see also \([34, \text{page 275 Eqtn (31)}]\). However, as we will see in Proposition 5.6, in contrast to \([21]\) our system has multiple analytic solutions.

4.1. Proof of Theorem 4.1. The Cayley Hamilton theorem (for the case \(\beta = 4\), see \([16, \text{Proposition II.2.1}]\)) implies

\[(4.7) \quad \theta^n = \sum_{i=1}^{n} (-1)^{i-1} \sigma_i(\theta) \theta^{n-i}.\]

We use also without proof the Newton formula

\[(4.8) \quad m\sigma_m(\theta) = \sum_{i=0}^{m-1} (-1)^{m-1-i} \sigma_i(\theta) \text{tr}(\theta^{m-i});\]

we remark that under the convention that \(\sigma_m = 0\) for \(m > n\), formula (4.8) holds true for all \(m \geq 1\), i.e. also for \(m > n\). Since \(\text{tr} I_n = n\), it can be rewritten as

\[(4.9) \quad (m-n)\sigma_m(\theta) = \sum_{i=0}^{m} (-1)^{m-1-i} \sigma_i(\theta) \text{tr}(\theta^{m-i}).\]

We also need a differentiation formula.

Proposition 4.2. Fix \(n \geq 1\). Then for all \(m \geq 1\),

\[(4.10) \quad \sigma'_m(\theta) = \sum_{i=0}^{m-1} (-1)^{m-1-i} \sigma_i(\theta) \theta^{m-1-i}.\]

Proof. We prove (4.10) by induction with respect to \(m\). Since \(\sigma'_1 = I_n = \sigma_0 \theta^n\), the formula holds true for \(m = 1\). Suppose (4.10) holds for some \(m \geq 1\). With \(t_j = \text{tr}(\theta^j)\), we differentiate Newton’s formula (4.8) written for \(\sigma_{m+1}\) and use the induction assumption.
We get

\[
(4.11) \quad (m + 1)\sigma'_{m+1} = (-1)^m(m + 1)\theta^m + \sum_{j=1}^{m}(-1)^{m-j}(m + 1 - j)\sigma_j\theta^{m-j} + t_{m+1-j} \sum_{s=0}^{j-1}(-1)^{j-1-s}\sigma_s\theta^{j-1-s} \\
= (-1)^m(m + 1)\theta^m + (m + 1)\sum_{j=1}^{m}(-1)^{m-j}\sigma_j\theta^{m-j} + R,
\]

where

\[
R = -\sum_{j=1}^{m}(-1)^{m-j}j\sigma_j\theta^{m-j} + \sum_{j=1}^{m}(-1)^{m-j}t_{m+1-j} \sum_{s=0}^{j-1}(-1)^{j-1-s}\sigma_s\theta^{j-1-s} \\
= -\sum_{j=1}^{m}(-1)^{m-j}j\sigma_j\theta^{m-j} + \sum_{s=1}^{m}(-1)^{m-s}\theta^{m-s} \sum_{j=0}^{s-1}(-1)^{s-1-j}\sigma_j\theta^{s-j}.
\]

Using Newton’s formula again, we see that \(\sum_{j=1}^{s-1}(-1)^{s-1-j}\sigma_j\theta^{s-j} = s\sigma_s\), so \(R = 0\) and the formula follows from (4.11).

We now consider separately the following cases, in which we re-write (2.7) by the indicated substitutions:

(i) If \(a \neq 0\), then \(f(\theta) = \exp(-4ak(\theta) - b \text{tr}(\theta))\) solves

\[
(4.12) \quad \Psi(f''(\theta))(I_n) = (b^2 - 4a)f(\theta)I_n.
\]

The limits as \(\theta \to 0\) in \(\Theta_X\) are:

\[
(4.13) \quad f(\theta) \to 1, \quad f'(\theta) \to -bI_n.
\]

(ii) If \(a = 0, b \neq 0\), then \(f(\theta) = k(\theta) + \frac{1}{2b} \text{tr}(\theta)\) solves

\[
(4.14) \quad \Psi(f''(\theta))(I_n) = 2bf'(\theta)I_n.
\]

The limits as \(\theta \to 0\) in \(\Theta_X\) are:

\[
(4.15) \quad f(\theta) \to 0, \quad f'(\theta) \to \frac{1}{2b}I_n.
\]

(iii) If \(a = 0, b = 0\), then \(f(\theta) = k(\theta)\) solves

\[
(4.16) \quad \Psi(f''(\theta))(I_n) = I_n.
\]

The limits as \(\theta \to 0\) in \(\Theta_X\) are:

\[
(4.17) \quad f(\theta) \to 0, \quad f'(\theta) \to 0.
\]

The remaining part of the proof consists of finding a suitable expression for \(f\).

Now the fact that the law of \(X\) is invariant under the rotations is equivalent to \(k\) being invariant under the substitution \(\theta \mapsto U\theta U^*\) for all \(U \in K_{n,\beta}\), and is equivalent to saying
that in each of the three cases mentioned above, there exists a real analytic function \( g \) on \( U_X \) such that

\[
f(\theta) = g(\sigma_1(\theta), \ldots, \sigma_n(\theta))
\]

for all \( \theta \in \Theta_X \) with distinct eigenvalues. (This is Weyl’s formula, see [31, Lemma 2.6.1].)

Using (4.10) we get

\[
f'(\theta) = \sum_{m=1}^{n} g_m \sigma'_m(\theta) = \sum_{m=1}^{n} g_m \sum_{i=0}^{m-1} (-1)^{m-1-i} \sigma_i(\theta) \theta^{m-1-i}.
\]

The second derivative of \( f \) is trickier: we write \( f'' = A + B \) with

\[
A = \sum_{m=1}^{n} \sum_{i=1}^{n} g_{mi} \sigma'_m(\theta) \otimes \sigma'_i(\theta),
\]

\[
B = \sum_{m=1}^{n} g_m \sigma''_m(\theta).
\]

The calculation of \( \Psi(A)(I_n) \) is quick and gives

\[
\Psi(A)(I_n) = \sum_{m=1}^{n} \sum_{i=1}^{n} g_{mi} \sigma'_m \sigma'_i.
\]

For calculating \( \Psi(B)(I_n) \) we prove the surprising formula:

**Proposition 4.3.** \( \Psi(\sigma''_m)(I_n) = \frac{\beta}{2}(m - 1 - n)\sigma'_{m-1} \).

**Proof.** We split \( \sigma''_m(\theta) \) into \( \sigma''_m(\theta) = C_m + D_m \) where

\[
C_m = \sum_{i=0}^{m-1} (-1)^{m-1-i} \sum_{j=0}^{i-1} (-1)^{i-1-j} \sigma_j \theta^{i-1-j} \otimes \theta^{m-1-i}
\]

(4.19)

\[
= \sum_{j=0}^{m-2-j} (-1)^{m-j} \sigma_j \sum_{s=0}^{m-2-j} \theta^s \otimes \theta^{m-2-j-s},
\]

(4.20)

\[
D_m = \sum_{j=0}^{m-1-j} (-1)^{m-1-j} \sigma_j [\theta^{m-1-j}]'.
\]

Using Proposition A we get

\[
\Psi \left( \sum_{s=0}^{m-2-j} \theta^s \otimes \theta^{m-2-j-s} \right)(I_n) = (m - j - 1)\theta^{m-j-2},
\]

which leads to the calculation of \( \Psi(C_m)(I_n) \):

\[
\Psi(C_m)(I_n) = \sum_{j=0}^{m-2} (-1)^{m-j} \sigma_j (m - j - 1)\theta^{m-j-2}.
\]
Having in mind the calculation of $\Psi(D_m)(I_n)$ we write

$$\theta_s'(h) = \sum_{i=0}^{s-1} \theta^i h \theta^{s-1-i} = \frac{1}{2} \sum_{i=0}^{s-1} [P_{\theta^i+\theta^{s-1-i}} - P_{\theta^i} - P_{\theta^{s-1-i}}](h).$$

From Proposition A we have $\Psi(P_x) = (1 - \frac{\beta}{2}) P_x + \frac{\beta}{2} x \otimes x$ which leads to $\Psi(P_x)(I_n) = (1 - \frac{\beta}{2}) x^2 + \frac{\beta}{2} x \text{tr}(x)$ and to

$$\Psi(P_{x+y} - P_x - P_y)(I_n) = \left(1 - \frac{\beta}{2}\right)(xy + yx) + \beta \frac{x \text{tr}(y) + y \text{tr}(x)}{2}.$$

Thus we can compute

$$\Psi(\theta^s')(I_n) = \frac{1}{2} \left(1 - \frac{\beta}{2}\right) \sum_{i=0}^{s-1} [\theta^i \theta^{s-i-1} + \theta^{s-i-1} \theta^i] + \frac{\beta}{4} \sum_{i=0}^{s-1} \theta^i \text{tr}(\theta^{s-1-i}) + \theta^{s-1-i} \text{tr}(\theta^i)]
= \left(1 - \frac{\beta}{2}\right) s \theta^{s-1} + \frac{\beta}{2} \sum_{i=0}^{s-1} \theta^i \text{tr}(\theta^{s-1-i}).$$

We apply this to $s = m - 1 - j$ and compute $\Psi(D_m)(I_n)$:

$$\Psi(D_m)(I_n) = \left(1 - \frac{\beta}{2}\right) \sum_{j=0}^{m-2} (-1)^{m-1-j} \sigma_j (m - 1 - j) \theta^{m-2-j}$$
$$+ \frac{\beta}{2} \sum_{j=0}^{m-2} (-1)^{m-1-j} \sigma_j \sum_{i=0}^{m-2-j} \theta^i \text{tr}(\theta^{m-2-j-i}).$$
Note that we replaced $m - 1$ by $m - 2$ in the first sum of the right hand side. We obtain

$$
\Psi(\sigma''_m(I_n) = \frac{\beta}{2} \sum_{j=0}^{m-2} (-1)^{m-j} \sigma_j (m - 1 - j) \theta^{m-2-j} 
+ \frac{\beta}{2} \sum_{i=0}^{m-2} (-1)^i \theta^i \sum_{j=0}^{m-2-i} (-1)^{m-1-j-i} \sigma_j \text{tr}(\theta^{m-2-i-j})
= \frac{\beta}{2} \sum_{j=0}^{m-2} (-1)^{m-j} \sigma_j (m - 1 - j) \theta^{m-2-j}
+ \frac{\beta}{2} \sum_{i=0}^{m-2} (-1)^i \theta^i (m - 2 - i - n) \sigma_{m-2-i} 
(4.24)
= \frac{\beta}{2} \sum_{j=0}^{m-2} (-1)^{m-j} \sigma_j (m - 1 - j) \theta^{m-2-j}
+ \frac{\beta}{2} \sum_{j=0}^{m-2} (-1)^{m-j} \theta^{m-2-j} (j - n) \sigma_j 
= \frac{\beta}{2} \sum_{j=0}^{m-1-n} (-1)^{m-j} \theta^{m-2-j} \sigma_j 
(4.25)
= \frac{\beta}{2} \sum_{j=0}^{m-1-n} \sigma_j 
(4.26)
$$

In this chain of equalities, the first is obtained by combining (4.22) and (4.23) and reversing the summation in $i$ and $j$ in the last sum; (4.24) is obtained by applying (4.9) (replacing $m$ by $m - 2 - i$); (4.25) is obtained by changing the variable $j = m - 2 - i$. The last equality (4.26) uses the equality (4.10). Thus the Proposition 4.3 is proved.

Gathering all terms, using the change of variable $m = j - 1$ in $\Psi(B)$ and the fact that $\sigma'_0 = 0$, the expression $\Psi(f''(I_n) = \Psi(A + B)(I_n)$ becomes,

$$
\Psi(f''(I_n) = \sum_{m=1}^{n} \sum_{i=1}^{n} g_{mi} \sigma'_m \sigma'_i + \frac{\beta}{2} \sum_{j=1}^{n-1} (j - n) g_{j+1} \sigma'_j 
$$

Since $\sigma'_1(\theta) = I_n$, the three equations (4.12), (4.14) and (4.16) respectively become

$$
\sum_{m=1}^{n} \sum_{i=1}^{n} g_{mi} \sigma'_m \sigma'_i + \frac{\beta}{2} \sum_{j=1}^{n-1} (j - n) g_{j+1} \sigma'_j = (b^2 - 4a) g \sigma'_1(\theta), 
(4.27)
$$

$$
\sum_{m=1}^{n} \sum_{i=1}^{n} g_{mi} \sigma'_m \sigma'_i + \frac{\beta}{2} \sum_{j=1}^{n-1} (j - n) g_{j+1} \sigma'_j = 2b \sum_{m=1}^{n} g_{m} \sigma'_m(\theta), 
(4.28)
$$
\begin{equation}
\sum_{m=1}^{n} \sum_{i=1}^{n} g_{mi} \sigma'_m \sigma'_i + \frac{\beta}{2} \sum_{j=1}^{n-1} (j - n) g_{j+1} \sigma'_j = \sigma'_i(\theta).
\end{equation}

Noting that \(\sigma'_1, \sigma'_2, \ldots, \sigma'_n\) is a basis of the algebra generated by \(\theta\) with distinct eigenvalues, the corresponding systems of PDEs arise by comparing the coefficients at \(\sigma'_j\) in (4.27) \(\ldots\) (4.29), respectively. To end the proof, we only need to find the coefficients \(P_j(m, i)\) in the basis expansion

\begin{equation}
\sigma'_m \sigma'_i = \sum_{j=1}^{n} P_j(m, i) \sigma'_j.
\end{equation}

This is accomplished by the following formula.

**Proposition 4.4.** For all \(r, s \geq 1\),

\begin{equation}
\sigma'_i(\theta) \sigma'_s(\theta) = \sum_{j=0}^{r-1} [\sigma_j, \sigma_{r+s-1-j}],
\end{equation}

where

\([f, g] := f(\theta)g'(\theta) - f'(\theta)g(\theta)\).

*Proof.* From (4.10) we get \(\sigma'_{j+1} = \sigma_j \theta^0 - \theta \sigma'_j\). Since the algebra generated by \(\theta\) is commutative, \(\sigma'_{j+1} \sigma'_n - \sigma'_j \sigma'_{n+1} = (\sigma_j \theta^0 - \theta \sigma'_j) \sigma'_n - \sigma'_j (\sigma_n \theta^0 - \theta \sigma'_n)\) simplifies to

\begin{equation}
\sigma'_{j+1} \sigma'_n - \sigma'_j \sigma'_{n+1} = [\sigma_j, \sigma_n].
\end{equation}

We now prove the proposition by induction. We assume that (4.31) holds for some fixed \(r\) and all \(s \geq 1\). This is trivially true for \(r = 1\) since \(\sigma'_1 \sigma'_s = \sigma'_s = [\sigma_0, \sigma_s]\). If (4.31) holds true for some \(r \geq 1\) and all \(s \geq 1\), then by (4.32) and the induction assumption,

\[
\sigma'_{r+1} \sigma'_s = \sigma'_r \sigma'_{s+1} + [\sigma_r, \sigma_s] = \sum_{j=0}^{r-1} [\sigma_j, \sigma_{r+s-j}] + [\sigma_r, \sigma_s],
\]

which shows that (4.31) holds for \(r + 1\) and all \(s \geq 1\). \(\square\)

*Proof of Theorem 4.1.* Cases (i)-(iii) correspond to equations (4.12) \(\ldots\) (4.16). The left hand sides of the resulting three equations (4.27) \(\ldots\) (4.29) are then re-written using (4.30). From (4.31) we see that coefficients \(P_j(m, i)\) are linear with respect to \(\sigma_1, \sigma_2, \ldots, \sigma_n\). Rewriting (4.31) as

\[
\sum_j P_j(r, s) \sigma'_j = \sum_{j=s}^{r+s-1} \sigma_{r+s-1-j} \sigma'_j - \sum_{j=1}^{r-1} \sigma_{r+s-1-j} \sigma'_j,
\]

we get the explicit formula

\begin{equation}
P_j(r, s) = \begin{cases}
\sigma_{r+s-1-j} & \text{if } \max\{r, s\} \leq j \leq r + s - 1 \leq n, \\
-\sigma_{r+s-1-j} & \text{if } 1 \leq j < \min\{r, s\} \text{ and } j \leq r + s - 1 \leq n, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}
With this notation equations (4.27)-(4.29) become the equations asserted in Theorem 4.1. To derive the corresponding boundary limits, we use (4.18) and the respective limits (4.13), (4.15), or (4.17).

It is also clear that these calculations can be reversed. From the equations listed in (i)-(iii), we see that (2.7) holds for all $\theta \in \Theta_X$ with distinct eigenvalues. By uniqueness of analytic extension, (2.7) extends to the whole open set $\Theta_X$. \qed

**Remark 4.5.** Note that to have a solution of the system in each of the three cases is not a guarantee of existence of an ensemble: the function $L(\theta)$ still have to be a Laplace transform of a probability measure. For example, for $n=2$ there are solutions of the system for all $a < 0$, but according to Theorem 5.1(i) they are the Laplace transforms of a matrix ensemble only if $-1/(4a) \in \mathbb{N}$.

**Remark 4.6.** Particular Meixner ensemble can be found from the system of PDEs by restricting $g$ to be a function of $\sigma_1, \sigma_2$ only. In other worlds, we look for an ensemble such that the Laplace transform $L(\theta)$ is a function of $\text{tr} \theta, \text{tr} \theta^2$. Such strategy leads us to the gamma ensemble on $\mathbb{H}_{n,\beta}$ mentioned in Remark 3.11 and to Definition 3.5 of the Gaussian ensemble. This strategy leads only to trivial ensembles for $b^2 \neq 4a$.

## 5. Meixner ensembles on $2 \times 2$ matrices

The goal of this section is to use Theorem 4.1 to prove that the constructions from Section 3 exhaust all Meixner ensembles on $\mathbb{H}_{2,\beta}$.

**Theorem 5.1.** Up to affine transformations, the only Meixner ensembles on $\mathbb{H}_{2,\beta}$ with the Laplace transform defined on a non-empty open subset of $\mathbb{H}_{2,\beta}$ are of the six types, depending on the values of $a, b$ from (1.4):

- (i) If $\mu$ is a Meixner ensemble on $\mathbb{H}_{2,\beta}$ with $a < 0$ and $\Theta_\mu \neq \emptyset$, then $-1/(4a) = N \in \mathbb{N}$ and $\mu$ is a binomial ensemble $\text{Bin}(N, q_1, q_2)$ from Definition 3.2 for some $q_1, q_2$.
- (ii) If $\mu$ is a Meixner ensemble on $\mathbb{H}_{2,\beta}$ with $a = 0$, $b > 0$ and $\Theta_\mu \neq \emptyset$, then $\mu$ is a Poisson ensemble from Definition 3.3 for some $\lambda > 0$.
- (iii) If $\mu$ is a Meixner ensemble on $\mathbb{H}_{2,\beta}$ with $b^2 > 4a > 0$ and $\Theta_\mu \neq \emptyset$, then $\mu$ is one of the negative binomial ensembles from Definition 3.4.
- (iv) If $\mu$ is a Meixner ensemble on $\mathbb{H}_{2,\beta}$ with $a = b = 0$ and $\Theta_\mu \neq \emptyset$, then $\mu$ is one of the Gaussian ensembles from Definition 3.5.
- (v) If $\mu$ is a Meixner ensemble on $\mathbb{H}_{2,\beta}$ with $b^2 = 4a > 0$ and $\Theta_\mu \neq \emptyset$, then $\mu$ is one of the gamma ensembles from Definition 3.6.
- (vi) If $\mu$ is a Meixner ensemble on $\mathbb{H}_{2,\beta}$ with $b^2 < 4a$ and $\Theta_\mu \neq \emptyset$, then $\mu$ is one of the hyperbolic Meixner ensembles from Definition 3.7.

(The seventh case of non-random ensembles arises from degenerate affine transformations.)

### 5.1. Laplace transforms of binomial, negative binomial and Poisson ensembles.
Lemma 5.2. If $P$ is a random projection in $\text{Bin}(1,1,0)$, then with $\sigma_j = \sigma_j(\theta)$, we have

$$
E(e^{\theta P}) = e^{\sigma_1/2}I_{(\beta-1)/2} \left( \sqrt{\frac{\sigma_1^2}{4} - \sigma_2} \right),
$$

where $I_\nu$ is a Bessel function \(^{(3.24)}\).

Proof. To construct $P$, we choose a direction $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ at random by taking two independent standard normal random variables as the components of the vector. Note that this means that $X_1 = Z_1 + iZ_2$ in the case $\beta = 2$ or $X_1 = Z_1 + iZ_2 + kZ_4$ in the case $\beta = 4$, where $Z_1, Z_2, Z_3, Z_4$ are the standard real-valued independent normal random variables. The matrix representation of $P$ is

$$
P = \frac{1}{|X_1|^2 + |X_2|^2} \begin{bmatrix} |X_1|^2 & \bar{X}_2X_1 \\ X_1X_2 & |X_2|^2 \end{bmatrix}.
$$

Due to invariance under rotations, without loss of generality we take diagonal $\theta = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}$, so that

$$
\langle \theta | P \rangle = \text{tr}(\theta P) = \frac{1}{|X_1|^2 + |X_2|^2} (\theta_1|X_1|^2 + \theta_2|X_2|^2) = \theta_1 U + \theta_2 (1 - U).
$$

Since $|X_1|^2, |X_2|^2$ are independent gamma distributed with shape parameter $\beta/2$ and scale parameter $1/2$, the distribution of $U = |X_1|^2/(|X_1|^2 + |X_2|^2)$ is Beta($\beta/2, \beta/2$). Denoting by $R = I_n - 2P$ the corresponding reflection, this gives

$$
E(e^{\theta R}) = e^{\theta_1 + \theta_2} E(e^{-2\theta P})
$$

$$
= e^{\theta_1 + \theta_2} \frac{\Gamma(\beta)}{\Gamma^2(\beta/2)} \int_0^1 e^{-2(u\theta_1 + (1-u)\theta_2)} u^{\beta/2-1}(1 - u)^{\beta/2-1} du
$$

$$
= e^{\theta_1 - \theta_2} \frac{\Gamma(\beta)}{\Gamma^2(\beta/2)} \int_0^1 e^{2u(\theta_2 - \theta_1)} u^{\beta/2-1}(1 - u)^{\beta/2-1} du.
$$

Substituting $u = (1 + v)/2$ we get

$$
L_R(\theta) = \frac{\Gamma(\beta)}{2^{\beta-1}\Gamma^2(\beta/2)} \int_{-1}^1 e^{(\theta_1 - \theta_2)v} (1 - v^2)^{\beta/2-1} dv.
$$

The integral is expressed in terms of the Bessel functions in \(^{(38\text{, }3.71\text{, }9)}\). \qed

Remark 5.3. Denoting by $\theta_1, \theta_2 \in \mathbb{R}$ the eigenvalues of $\theta$,

$$
E(e^{\theta R}) = \begin{cases} 
I_0(\theta_1 - \theta_2) & \text{if } \beta = 1, \\
\frac{\sinh(\theta_1 - \theta_2)}{\theta_1 - \theta_2} & \text{if } \beta = 2, \\
3 \left( \frac{\cosh(\theta_1 - \theta_2)}{(\theta_1 - \theta_2)^2} - \frac{\sinh(\theta_1 - \theta_2)}{(\theta_1 - \theta_2)^3} \right) & \text{if } \beta = 4.
\end{cases}
$$
Combining this with the Laplace transforms derived in the proofs of Propositions 3.5, 3.6, and 3.7, we get.

**Corollary 5.4.**  
(i) If $X$ has $\text{Bin}(N, q_1, q_2)$ law, then its Laplace transform is well defined for all $\theta \in \mathbb{H}_\beta$, and is given by

$$E(e^{\langle \theta | X \rangle}) = e^{N\sigma_1/2} \left( q_0 e^{-\sigma_1/2} + q_1 I_{(\beta-1)/2} \left( \sqrt{\frac{\sigma_1^2}{4} - \sigma_2} + q_2 e^{\sigma_1/2} \right) \right)^N,$$

where $q_0 = 1 - q_1 - q_2$.

(ii) If $X$ has $\text{Poiss}(\lambda_1, \lambda_2)$ law, then its Laplace transform is well defined for all $\theta \in \mathbb{H}_\beta$, and is given by

$$E(e^{\langle \theta | X \rangle}) = \exp \left( \lambda_1 e^{\sigma_1/2} I_{(\beta-1)/2} \left( \sqrt{\frac{\sigma_1^2}{4} - \sigma_2} \right) + \lambda_2 e^{\sigma_1} - \lambda \right),$$

where $\lambda = \lambda_1 + \lambda_2$.

(iii) If $X$ has $\text{NB}(r, q_1, q_2)$ law, then its Laplace transform is well defined for $\theta$ in a neighborhood of 0 in $\mathbb{H}_{2,\beta}$, and is given by

$$E(e^{\langle \theta | X \rangle}) = \frac{1 - (q_1 + q_2)^r}{1 - (q_1 e^{\sigma_1/2} I_{(\beta-1)/2} \left( \sqrt{\frac{\sigma_1^2}{4} - \sigma_2} \right) + q_2 e^{\sigma_1})}.$$

**Remark 5.5.** A calculation shows that for a given $b^2 > 4a > 0$, parameter $p = 1 - q_1 - q_2$ in (5.5) can range over the set

$$\{p \in (0, 1) : p/(2 - p) \leq \kappa/b \leq p\},$$

while $q_2 = (bp - \kappa)/\kappa \in [0, 1 - p]$ and $q_1 = 1 - p - q_2$ are uniquely determined by $p$. (Here $\kappa = \sqrt{b^2 - 4a}$, $b > 0$.)

5.2. **Solutions of the system of PDEs for** $n = 2$. We consider in more detail the case $n = 2$ with

$$U = \{(\sigma_1, \sigma_2) : 4\sigma_2 < \sigma_1^2\}.$$

We require that the solutions extend by continuity to $\sigma_1 = \sigma_2 = 0$.

The system of PDEs simplifies to:

$$g_{11} - \sigma_2 g_{22} = \frac{\beta}{2} g_2 + (b^2 - 4a) g,$$

$$2g_{12} + \sigma_1 g_{22} = 0.$$

We will use as the initial conditions

$$g(0, 0) = 1, \quad \frac{\partial g(\sigma_1, 0)}{\partial \sigma_1} \bigg|_{\sigma_1 = 0} = -b.$$

Denote $\kappa = \sqrt{|b^2 - 4a|}$.

**Proposition 5.6.** Consider the system (5.7), (5.8) with initial conditions (5.9).
(i) For \( b^2 = 4a > 0 \) all solutions are
\[
g(\sigma_1, \sigma_2) = 1 - b\sigma_1 + C(\beta \sigma_1^2 + 4\sigma_2),
\]
where \( C \) is an arbitrary constant. Accordingly,
\[
(5.10) \quad L(\theta) = \frac{\exp(-\sigma_1(\theta)/b)}{(1 - b\sigma_1(\theta) + C(\beta \sigma_1^2(\theta) + 4\sigma_2(\theta)))^{1/b^2}}.
\]

(ii) For \( b^2 > 4a \) all solutions are
\[
g(\sigma_1, \sigma_2) = C_1 e^{\kappa \sigma_1} + C_2 e^{-\kappa \sigma_1} + C_3 \mathcal{I}(\beta - 1)/2(\kappa \sqrt{\sigma_1^2/4 - \sigma_2}),
\]
where \( \mathcal{I}(\beta - 1)/2 \) is defined in (3.24), and \( C_1, C_2, C_3 \) are arbitrary real numbers such that \( C_1 + C_2 + C_3 = 1 \) and \( C_2 - C_1 = b/\kappa \). Accordingly,
\[
(5.11) \quad L(\theta) = e^{-b\sigma_1/(4a)} \left(C_1 e^{\kappa \sigma_1} + C_2 e^{-\kappa \sigma_1} + C_3 \mathcal{I}(\beta - 1)/2(\kappa \sqrt{\sigma_1^2/4 - \sigma_2})\right)^{-1/(4a)},
\]
which can also be written as
\[
(5.12) \quad L(\theta) = e^{-b\sigma_1/(4a)} \left((1 - \lambda) \cosh(\kappa \sigma_1) - \frac{b}{\kappa} \sinh(\kappa \sigma_1) + \lambda \mathcal{I}(\beta - 1)/2(\kappa \sqrt{\sigma_1^2/4 - \sigma_2})\right)^{-1/(4a)},
\]
where \( \lambda \in \mathbb{R} \) is an arbitrary constant.

(iii) For \( b^2 < 4a \) all solutions are
\[
g(\sigma_1, \sigma_2) = (1 - \lambda) \cos \kappa \sigma_1 - \frac{b}{\kappa} \sin \kappa \sigma_1 + \lambda \mathcal{I}(\beta - 1)/2(\kappa \sqrt{\sigma_1^2/4 - \sigma_2}),
\]
where \( \lambda \) is an arbitrary real constant. Accordingly,
\[
(5.14) \quad L(\theta) = e^{-b\sigma_1/(4a)} \left((1 - \lambda) \cos \kappa \sigma_1 - \frac{b}{\kappa} \sin \kappa \sigma_1 + \lambda \mathcal{I}(\beta - 1)/2(\kappa \sqrt{\sigma_1^2/4 - \sigma_2})\right)^{-1/(4a)}.
\]

**Proof.** The solutions that depend only on \( \sigma_1 \) satisfy equation \( g_{11} = (b^2 - 4a)g \) with two linearly independent solutions:
\[
(5.15) \quad e^{\kappa \sigma_1}, e^{-\kappa \sigma_1} \quad \text{if } b^2 - 4a > 0,
\]
\[
(5.16) \quad \cos(\kappa \sigma_1), \sin(\kappa \sigma_1) \quad \text{if } b^2 - 4a < 0,
\]
\[
(5.17) \quad 1, \sigma_1 \quad \text{if } b^2 - 4a = 0.
\]
To find the solutions that depend also on \( \sigma_2 \), we first note that (5.8) implies that
\[
(5.18) \quad 2g_1 + \sigma_1 g_2 = C(\sigma_1),
\]
so
\[
(5.19) \quad 2g_{11} + g_2 + \sigma_1 g_{12} = C'(\sigma_1).
\]
We now eliminate \( g_{11} \) and \( g_{12} \) from the equations. Subtracting from (5.19) the linear combination of 2 times equation (5.7) and \( \sigma_1/2 \) times equation (5.8) we get
\[
(\sigma_1^2/4 - \sigma_2)g_{22} - \frac{1 + \beta}{2} g_2 - (b^2 - 4a)g = -C'(\sigma_1)/2.
\]
For fixed \( \sigma_1 \), we consider (5.20) as a differential equation with respect to \( \sigma_2 \) in the interval \( \sigma_2 < \sigma_1^2/4 \), since we need \( g \) in this domain, see (5.6). For this reason, we denote \( t = \sigma_1^2/4 - \sigma_2 \) and, assuming \( b^2 \neq 4a \), we denote
\[
y(t) = g(\sigma_1, \sigma_2) - \frac{C'(\sigma_1)}{2(b^2 - 4a)}.
\]
Thus \( g_2 = -y' \), \( g_{22} = y'' \) and (5.20) becomes
\[
2ty'' + (1 + \beta)y' \pm 2\kappa^2 y = 0,
\]
where the sign is chosen according to the sign of \( 4a - b^2 \). Substitution \( y(t) = t^{(1-\beta)/4} u(x) \) with \( x = 2\kappa \sqrt{t} \) converts (5.21) into the Bessel equation
\[
x^2 u'' + xu' + \left( \pm x^2 - \left( \frac{\beta - 1}{2} \right)^2 \right) u = 0.
\]
Using [1 9.6.18] or [1 9.1.20], after some calculation one verifies that all solutions of (5.21) which are bounded in a neighborhood of 0 are proportional to \( \mathcal{I}_{(\beta-1)/2}(\kappa \sqrt{t}) \) when \( b^2 > 4a \) and to \( \mathcal{J}_{(\beta-1)/2}(\kappa \sqrt{t}) \) when \( b^2 < 4a \), see (3.24) and (3.23). Substituting \( g(\sigma_1, \sigma_2) = K(\sigma_1) \mathcal{I}_{(\beta-1)/2}(\kappa \sqrt{\sigma_1^2/4 - \sigma_2}) \) back into (5.8) we get that (5.8) holds if \( K' = 0 \). So we deduce that \( K \) is constant, and a direct verification shows that \( g \) solves also (5.7). (We repeat the same reasoning with \( \mathcal{J}_{(\beta-1)/2} \) instead of \( \mathcal{I}_{(\beta-1)/2} \) if \( b^2 < 4a \).)

Similarly, one works out the solutions for \( b^2 = 4a > 0 \). In this case, after taking \( y(t) = g(\sigma_1, \sigma_2) + 2\sigma_2 C'(\sigma_1)/(1 + \beta) \) we get equation (5.21) with \( \kappa = 0 \).After some work this leads elementary solutions proportional to \( g(\sigma_1, \sigma_2) = \frac{\beta \sigma_1^2}{4} + \sigma_2 \) and to additional solutions \( g(\sigma_1, \sigma_2) = (\sigma_1^2 - 4\sigma_2)^{(1-\beta)/2} \) for \( \beta > 1 \) or \( g(\sigma_1, \sigma_2) = \log(\sigma_1^2 - 4\sigma_2) \) for \( \beta = 1 \), which are unbounded at the origin.

To conclude the proof of (ii), we note that the general solution that is defined at \( \sigma_1 = \sigma_2 = 0 \) is
\[
g(\sigma_1, \sigma_2) = C_1 + C_2 \sigma_1 + C_3 (\beta \sigma_1^2 + 4\sigma_2).
\]
The initial conditions determine \( C_1 = 1 \), \( C_2 = -b \), which gives (5.10).

To conclude the proof of (iii), we note that the general solution bounded at the origin is (5.11), and the initial conditions are satisfied when \( C_1 + C_2 + C_3 = 1 \) and \( C_2 - C_1 = b/\kappa \). This gives (5.12).

The initial conditions similarly imply (5.14).

We are now ready to prove that gamma and hyperbolic Meixner ensembles are indeed Meixner. □
Proof of Proposition 3.15. To calculate the moments, we differentiate (3.16) to get $E(\xi_0) = 2\rho$, $E(\xi_1) = 0$, $\text{Var}(\xi_0) = 2\rho(2\alpha^2 - 1)$, $E(\xi_1^2) = 2\rho$, and $E(\xi_\ell, \xi_{\ell+2}) = 0$ for $\ell_1 \neq \ell_2$.

For the standardized gamma ensemble $X$ with $E(X) = 0$ and $E(X^2) = 1$, from (3.16) we get
\begin{equation}
E(e^{\theta X}) = e^{-\sqrt{\pi} \text{tr} \theta} (1 - \text{tr} \theta / \sqrt{\theta} + D(\sqrt{\theta} \text{ tr} \theta^2 + 4 \text{ det} \theta))^{-p},
\end{equation}
where $D = 1/(4\rho^2(1 + \beta))$. This is (5.10) with $p = 1/b^2$, so property (1.2) follows from the converse part of Theorem 4.1. For the standardized version, (1.4) holds with $a = 1/(4p)$ and $b = 1/\sqrt{\theta}$. Formula for the parameters $A, B, C$ is now re-calculated from formula (1.5).

Note that the admissible ranges of parameters are $p > \beta / 2$ and $0 \leq D < 1/(4p(1 + \beta))$.

Proof of Proposition 3.10. We compute the moments of $X$ entrywise: differentiating (3.29) we get $E\xi_0 = \rho\alpha$, $E\xi_0^2 = \alpha(1 - \lambda + \rho^2(1 + \alpha))$ and for $j \geq 1$, $E\xi_j = 0$, $E\xi_j^2 = \alpha\lambda/(1 + \beta)$, $E(\xi_\ell, \xi_{\ell+j}) = 0$ for $\ell_1 \neq \ell_2$.

For the standardized hyperbolic Meixner ensemble $X$ with $E(X) = 0$ and $E(X^2) = 1$, from (3.25) we get
\begin{equation}
E(e^{\theta X}) = e^{-\rho \sqrt{\alpha/(1 + \rho^2)}} \left[(1 - \lambda) \cos(\text{tr} \theta / \sqrt{\alpha(1 + \rho^2)}) - \rho \sin(\text{tr} \theta / \sqrt{\alpha(1 + \rho^2)})
+ \lambda J_{(\beta - 1)/2}(\sqrt{\theta \text{ tr} \theta^2 - 4 \text{ det} \theta} / \sqrt{\alpha(1 + \rho^2)})\right]^{-\alpha}.
\end{equation}
Substituting $\rho = b/\kappa = b/\sqrt{4a - b^2}$, and $1/\alpha = 4a$, we get (5.14). So from the converse part of Theorem 4.1 we see that (1.4) holds with $a = 1/(4\alpha)$ and $b = \rho / \sqrt{\alpha(1 + \rho^2)}$. Formula for the parameters $A, B, C$ is now re-calculated from formula (1.5).

5.2.1. Poisson case: $a = 0, b \neq 0$. The system of PDEs simplifies to:
\begin{align}
g_{11} - \sigma_2 g_{22} &= \frac{\beta}{2} g_2 + 2bg_1, \\
g_{21} + \sigma_1 g_{22} &= 2bg_2.
\end{align}
We seek solutions such that
\begin{equation}
g(0, 0) = 0, \quad \frac{\partial g(\sigma_1, 0)}{\partial \sigma_1} \big|_{\sigma_1 = 0} = \frac{1}{2b}.
\end{equation}

Proposition 5.7 (Poisson ensemble). Consider the system (5.24), (5.24) with initial condition (5.25). The solution is
\begin{equation}
g(\sigma_1, \sigma_2) = \frac{1}{2b^2} \left((1 - C) e^{2b\sigma_1} + (2C - 1) e^{b\sigma_1} I_{(\beta - 1)/2} \left(b \sqrt{\sigma_1^2 - 4\sigma_2} - C\right)\right),
\end{equation}
where $C$ is an arbitrary constant. Thus
\begin{equation}
L(\theta) = \exp\left(-\frac{\sigma_1}{2b} + \frac{1}{2b^2} \left((1 - C) e^{2b\sigma_1} + (2C - 1) e^{b\sigma_1} I_{(\beta - 1)/2} \left(b \sqrt{\sigma_1^2 - 4\sigma_2} - C\right)\right)\right).
\end{equation}
Proof. The solution that depends on $\sigma_1$ only is in [25]; in our notation,
\[ g(\sigma_1) = C_1 + C_2 e^{2b\sigma_1}. \]
To find the solutions that depend on both variables, we first note that (5.24) implies
\[ 2g_1 + \sigma_1 g_2 - 2ag = C(\sigma_1), \tag{5.27} \]
which we differentiate with respect to $\sigma_1$ to get
\[ 2g_{11} + g_2 + \sigma_1 g_{12} - 2ag_1 = C'(\sigma_1). \tag{5.28} \]
We now eliminate $g_{11}$, $g_{12}$ and $g_1$ from the equations. To do so, from equation (5.28) we subtract 2 times equation (5.23), $b$ times equation (5.27), and $\sigma_1/2$ times equation (5.24). This gives
\[ (\sigma_2^2/4 - \sigma_2)g_{22} - \frac{1 + \beta}{2}g_2 - b^2 g = -(C'(\sigma_1) + aC(\sigma_1))/2. \tag{5.29} \]
Noting similarity with (5.20), we again we consider (5.29) as a differential equation with respect to $\sigma_2$ in the interval $\sigma_2 < \sigma_1^2/4$. With $b \neq 0$, $t = \sigma_2^2/4 - \sigma_2$, we take
\[ y(t) = g(\sigma_1, \sigma_2) - \frac{C'(\sigma_1) + aC(\sigma_1)}{2b^2}, \]
and we get
\[ 2ty'' + (1 + \beta)y' - 2b^2 y = 0. \]
Therefore, the solution bounded at the origin is $g(\sigma_1, \sigma_2) = K(\sigma_1)I_{(\beta-1)/2}(b\sqrt{\sigma_1^2/4 - \sigma_2})$. Substituting this back into (5.24), we get $K' = bK$, i.e., $K(\sigma_1) = e^{b\sigma_1}$, and the resulting $g(\sigma_1, \sigma_2)$ solves (5.23), too.

Thus the general solution bounded at the origin is
\[ g(\sigma_1, \sigma_2) = C_1 + C_2 e^{2b\sigma_1} + C_3 e^{b\sigma_1}I_{(\beta-1)/2}(b\sqrt{\sigma_1^2/4 - \sigma_2}). \]
The initial conditions are satisfied when $C_1 + C_2 + C_3 = 0$ and $2C_2 + C_3 = 1/(2b^2)$. After some calculations, we get the answer.

5.2.2. Gaussian case: $a = 0$, $b = 0$. The system of PDEs in Theorem 4.1(ii) simplifies to:
\[ g_{11} - \sigma_2 g_{22} = \frac{\beta}{2} g_2 + 1, \tag{5.30} \]
\[ 2g_{12} + \sigma_1 g_{22} = 0. \tag{5.31} \]

Proposition 5.8. Consider the system (5.30), (5.31) with zero initial conditions. The solution is
\[ C\sigma_2^2/2 - (1 - C)\frac{2}{\beta} \sigma_2, \tag{5.32} \]
where $C \in \mathbb{R}$ is arbitrary. Thus
\[ L(\theta) = \exp \left( C\sigma_2^2/2 - (1 - C)\frac{2}{\beta} \sigma_2 \right). \tag{5.33} \]
Since the homogeneous system is the same as in the case $b^2 - 4a = 0$ of Proposition 5.6(i), we omit the details.

We recall the following facts about univariate Laplace transforms, which belong to the folklore:

**Proposition B.** Suppose $L(t)$ is a Laplace transform of a real random variable $X$. If $L(t)$ is defined on an open interval $\Theta$, then $L$ has analytic extension to $\Theta + i\mathbb{R}$. If $z_0 \in \Theta + i\mathbb{R}$ is a zero of $L$ of order $m \in \mathbb{N}$, then $L^\alpha$ cannot be a Laplace transform of a probability measure unless $m\alpha \in \mathbb{N}$.

**Proof of Theorem 5.1.** Without loss of generality, we may assume that $X$ is non-degenerate, and therefore that it is standardized with mean $0$ and variance $1$. Therefore, (1.4) holds with some constants $a, b$, as explained in the introduction, replacing $X$ by $-X$ if needed, without loss of generality we may assume $b \geq 0$. Therefore the Laplace transform of $X$ must be given by one of the six formulas from Proposition 5.6(i-iii), Proposition 5.7 or Proposition 5.8. It remains therefore to show that these functions are the Laplace transforms of probability measures only when the parameters satisfy the conditions listed in the corresponding definitions of our Meixner ensembles. The solutions, and our definitions, depend on the constraints satisfied by parameters $a, b$, so we need to consider each case separately.

In the proof, we repeatedly compute $L(\theta)$ on diagonal matrices

$$\theta^\pm_t = \begin{bmatrix} t/2 & 0 \\ 0 & \pm t/2 \end{bmatrix}.$$ 

(i) Case $b^2 > 4a$ with $a < 0$. In this case, all potential Laplace transforms for the standardized ensembles are given by formula (5.12), and (5.3) shows that $L(\theta)$ is indeed the Laplace transform if $a = -1/(4N)$ and $C_1, C_2, C_3 \geq 0$. It remains to show that $L(\theta)$ is not a Laplace transform if $1/(4a) \notin \mathbb{N}$ or if one of the constants is negative.

Denote $\alpha = -1/(4a) > 0$. If $C_3 = 0$, then $L(\frac{1}{\nu}f^+_{\theta} = (C_1 e^t + C_2 e^{-t})^\alpha$, and it is a classical fact that $\alpha \in \mathbb{N}$ and $C_1, C_2$ are the binomial probabilities, compare Proposition 3.4. Therefore, through the remainder of the proof we assume $C_3 > 0$. We first show that $\alpha \in \mathbb{N}$; the same reasoning shows that the Jørgensen set of a Bernoulli ensemble on $\mathbb{H}_{2,\beta}$ is \{1, 2, \ldots\}. We write $L(\theta^-) = (f(s))^{\alpha}$, where

$$f(s) = 1 - C_3 + C_3 \mathcal{I}(\beta_{-1})/2(s).$$

To show that $\alpha \in \mathbb{N}$, we will apply Proposition 13. We first observe that (3.24) implies

$$I_\nu(\sqrt{z}) = \sum_{n=0}^{\infty} a_n z^n,$$

where

$$a_n = \frac{1}{2^{2n} n! \Gamma(n + \nu + 1)}.$$
This shows that \( \mathcal{I}_\nu(\sqrt{z}) \) is an entire function of order

\[
\rho = \limsup_{n \to \infty} \frac{\log n}{n \log a_n} = 1/2.
\]

Therefore by a refinement of the little Piccard’s theorem \([30, \text{Theorem 9.11}]\), the equation \( \mathcal{I}_\nu(\sqrt{z}) = (C_3 - 1)/C_3 \) has an infinite number of roots.

We also observe that

(5.35) \[
\frac{d}{dz} \mathcal{I}_\nu(\sqrt{z}) = \frac{1}{4} \mathcal{I}_{\nu+1}(\sqrt{z}),
\]

see \([38, \text{page 479}]\).

Coming back to \( f(s) \), we see that the function \( s \mapsto f(s) \) has an infinite set \( Z_0 \) of roots. We claim that \( Z_0 \) has at least one simple root. If not, \( Z_0 \) would be included in the set \( Z_1 \) of roots of the derivative \( f'(s) \). Now from (5.35) we see that \( Z_1 \) is the set of roots of \( \mathcal{I}_{(\beta+1)/2} \) which by Lommel’s Theorem, lies on the imaginary axis \([38, 15.25]\). From \([38, \text{page 199}]\) we see that

\[
\lim_{t \to \pm \infty} \mathcal{I}_\nu(it) = 0,
\]

therefore \( \mathcal{I}_\nu(it) \) cannot be equal to \((C_3 - 1)/C_3\) on an infinite subset of \( Z_1 \). Therefore \( f \) has at least one simple zero, and from Proposition \([13]\) we deduce that \( \alpha \) must be an integer.

The final step is to show that \( C_1, C_2, C_3 \geq 0 \). Since the second derivative of \( \log L(\frac{1}{\kappa} \theta_t^-) = \alpha \log(1 - C_3 + C_3 \mathcal{I}_{(\beta-1)/2}(t)) \) is \( \alpha C_3 / (\beta + 1) \), we see that \( C_3 \geq 0 \).

Next we compute the second derivative of \( \frac{1}{\alpha} \log L(\frac{1}{\kappa} \theta_t^-) \), which is

\[
(C_2C_3 + e^{2t}C_1C_3 + 4C_1C_2e^t) \frac{e^t}{(C_2 + e^t (e^t C_1 + C_3))^2}.
\]

Since \( \alpha \in \mathbb{N} \), the Laplace transform \( L(\frac{1}{\kappa} \theta_t^-) \) is well defined for all real \( t \). The sign of the second derivative for large \( t \) is determined by \( C_1C_3 \), showing that \( C_1 \geq 0 \). The sign of the second derivative as \( t \to -\infty \) is determined by \( C_2C_3 \), showing that \( C_2 \geq 0 \). (Recall that we consider the case \( C_3 \neq 0 \) only.)

Thus Definition 3.2 indeed covers all examples of Meixner ensembles on \( \mathbb{H}_{2,\beta} \) with \( a < 0 \).

(ii) Case \( b^2 > 0, a = 0 \). In this case, all potential Laplace transforms for the standardized ensembles are given by formula \([5.26]\). From (5.4), after passing to standardized \( \bar{X} \) we confirm that (5.26) is indeed a Laplace transform when \( 1/2 \leq C \leq 1 \), \( b \neq 0 \).

To verify that \( L(\theta) \) fails to be a Laplace transform of an ensemble in all other cases, we compute

\[
\frac{d^2}{dt^2} \log L \left( \theta_t^- / b \right) \bigg|_{t=0} = 2C - 1.
\]
Thus we must have $C \geq 1/2$. Next, we note that in order for
\[ \frac{d^2}{dt^2} \log L \left( \theta^+ / b \right) = e^t \left( 4e^t(1-C) + 2C - 1 \right) \]
to be positive for large $t$, we must have $C \leq 1$.

This shows that Definition 3.3 covers all examples of Meixner ensembles on $H_{2,\beta}$ with $b \neq 0$, $a = 0$.

(iii) Case $b^2 > 4a$, $a > 0$. In this case, all potential Laplace transforms for the standardized ensembles are given by formula (5.13). To see which values of $\lambda$ correspond to the Laplace transform (5.5) of the negative binomial ensemble, we take $r = 1/(4a)$ and define parameters $q_1, q_2$ and $p = 1 - q_1 - q_2$ as follows
\[ \frac{2}{p} = 1 - \lambda + b/\kappa, \quad q_2 = (p-1)p\lambda, \quad q_1 = 1 - p - q_1. \]

From the admissible range of $p$ given in Remark 5.5 we see that (5.13) is a Laplace transform of an ensemble when $1 \leq 1 - \lambda \leq b/\kappa$.

It remains to show that no other values of $\lambda$ are allowed. The second derivative of
\[ 4a \log L(\theta^+ / \kappa) = -\log(1 - \lambda + \lambda I_{(\beta-1)/2}(t)) \]
at $t = 0$ is $-\lambda/(1 + \beta)$, proving that $\lambda \leq 0$.

Next suppose $1 - \lambda > b/\kappa$. Then, with $\rho = b/(\kappa(1 - \lambda)) < 1$ we see that
\[ (1 - \lambda) \cosh t - b/\kappa \sinh(t) = (1 - \lambda)(\cosh t - \rho \sinh t) \]
is an increasing function of $t > 0$. So $L(\theta^+ / \kappa)$ is well defined for all $t > 0$. We now compute the second derivative of
\[ (5.36) \quad 4a \log L(\theta^+ / \kappa) = -\log(\lambda + (1 - \lambda) \cosh(t) - b/\kappa \sinh t). \]

We get
\[ \frac{b^2 - \kappa^2(\lambda - 1)^2 + \kappa \lambda(\lambda - 1) \cosh(t) + b \sinh(t)}{(-\kappa \lambda + \kappa(\lambda - 1) \cosh(t) + b \sinh(t))^2}. \]

Since $\kappa(\lambda - 1) + b < 0$, this expression fails to be positive for large $t$, contradicting that $L$ is a Laplace transform of a probability measure. Thus we must have $1 - \lambda \leq b/\kappa$.

This shows that Definition 3.4 covers all examples of Meixner ensembles on $H_{2,\beta}$ with $a > 0$, $b^2 > 4a$.

(iv) Case $a = 0$, $b = 0$ (Proposition 5.8). It is clear that Definition 3.5 covers all examples of Meixner ensembles on $H_{2,\beta}$ with Laplace transform (5.33).

(v) Case $a > 0$, $b^2 = 4a$ (Proposition 5.6(i)). From [28] it follows that Definition 3.6 covers all examples of Meixner ensembles on $H_{2,\beta}$ with $a > 0$, $b^2 = 4a$.

(vi) Case $b^2 < 4a$ (Proposition 5.6(iii)). From Proposition 3.14 we see that (5.14) is a Laplace transform of a probability measure when $0 \leq \lambda < 1$, and also when $\lambda = 1, b = 0$. It remains to show that (3.25) fails to be a Laplace transform in the remaining cases.
Denote $\alpha = 1/(4a) > 0$. The second derivative of
\[
\log L(\theta^-) = -\alpha \log(1 - \lambda + \lambda J_{(\beta-1)/2}(t))
\]
at $t = 0$ is $\alpha \lambda/(1 + \beta)$, proving that $\lambda \geq 0$.

Next, for $\lambda \neq 1$ let $\phi \in (-\pi/2, \pi/2)$ be such that $\sin \phi = \rho/\sqrt{(1 - \lambda)^2 + \rho^2}$. Then the second derivative of
\[
\log L(\theta^+_{\ell}) = -\alpha \log(\lambda + (1 - \lambda)(\cos t - \rho/(1 - \lambda) \sin t))
\]
at $t = -\phi$ is
\[
\alpha \frac{1 - \lambda}{\cos \phi - 1}\lambda + 1.
\]
Since $\lambda > 0$, this shows that $\lambda \leq 1$.

It remains to show that (5.14) is not a Laplace transform of a probability measure when $\lambda = 1$ and $b \neq 0$. In this case, consider
\[
g(s, t) = \log L\left(\frac{1}{\kappa}(\theta^+_{\ell} + \theta^-_{\ell})\right) = -bs/(4a) - \alpha \log \left(J_{(\beta-1)/2}(t) - b \sin(s)/\kappa\right).
\]
Then $g(s, t)$ is well defined on $(s, t) \in [0, \pi) \times (-\ell, \ell)$. We calculate the Hessian
\[
H(s, t) = \det \begin{bmatrix}
\frac{\partial^2}{\partial s^2}g(s, t) & \frac{\partial^2}{\partial s \partial t}g(s, t) \\
\frac{\partial^2}{\partial s \partial t}g(s, t) & \frac{\partial^2}{\partial t^2}g(s, t)
\end{bmatrix}
\]
at $s = 0$ and we get
\[
H(0, t) = -\frac{b^2 \alpha^2 J''_{(\beta-1)/2}(t)}{\kappa^2 (J_{(\beta-1)/2}(t))^3}.
\]
Using Mathematica, we verify that $J''_{(\beta-1)/2}(0)J''_{(\beta-1)/2}(\ell) < 0$, so for $b \neq 0$ the Hessian must change sign over the domain $(s, t) \in [0, \pi) \times (-\ell, \ell)$. Thus $\log L(\theta)$ cannot be a convex function, i.e., when $\lambda = 1$, $b \neq 0$, function $L(\theta)$ cannot be a Laplace transform of a probability measure for any $\alpha \neq 0$.

So Definition 3.7 exhausts all the Meixner ensembles on $H_{\beta, \gamma}$ with $b^2 < 4a$.

\[\square\]

6. Additional observations

6.1. Independence of $S$ and $S^{-1}X^2S^{-1}$. Bożejko and Bryc \[9, \text{Remark 5.8}\] raise the question whether there exists a non-trivial law on positive $n \times n$ matrices such that
\[
Z := S^{-1}X^2S^{-1}, \quad S := X + Y
\]
are independent when $X$ and $Y$ are i. i. d. matrices with this law. Such laws could provide matrix models for the "free gamma" law.

Here we answer this question in negative, at least for the laws on positive $2 \times 2$ matrices which are invariant under orthogonal/unitary/symplectic group. We show that in the case...
of $2 \times 2$ positive random matrices, $S^{-1}X^2S^{-1}$ and $S$ are independent only if $X$ arises as a gamma random variable multiplied by $I_2$.

**Proposition 6.1.** If $X, Y \in H_{2,\beta}$ are i.i.d. square-integrable non-degenerate positive random matrices such that $S = X + Y$ and $S^{-1}X^2S^{-1}$ are independent and rotation invariant, then $X = \xi I_2$ and $\xi$ has a univariate gamma law.

**Lemma 6.2.** Suppose $X, Y \in H_{2,\beta}$ are positive i.i.d. random matrices with the Laplace transform

$$L(\theta) = (1 + A \operatorname{tr} \theta + B \operatorname{tr} \theta^2 + C \operatorname{tr} \theta^3)^{-p}.$$  

Let $S = X + Y$. If real random variables $\det(X)/\det(S)$ and $\det S$ are independent, then one of the following cases occurs:

(i) $A = B = C = 0$. (Then $X = I$.)

(ii) $A = \pm 2\sqrt{B}$, $C = 0$. (Then $X = \frac{1}{2} \xi I$ where $\xi$ is univariate gamma with shape parameter $2p$.)

(iii) $A = \pm \sqrt{2}B$, $C = -B$. (Then $X = AY$ where $Y$ has a Wishart law with shape parameter $p$, i.e., with the Laplace transform $L(\theta) = (1 - \operatorname{tr} (\theta) + \det \theta)^{-p} = \det(I - \theta)^{-p}$.)

**Proof.** The independence of determinants implies

$$E(\det X \exp \operatorname{tr} (\theta S)) = c_0 E(\det S \exp \operatorname{tr} (\theta S)),$$

where

$$c_0 = E \left( \frac{\det X}{\det S} \right) > 0.$$  

With $\theta = \begin{bmatrix} \theta_1 & \theta_{12} \\ \theta_{12} & \theta_2 \end{bmatrix}$, this becomes

$$L(\theta) \left( \frac{\partial^2}{\partial \theta_1 \partial \theta_2} - \frac{\partial^2}{4 \partial^2 \theta_{12}} \right) L(\theta) = c_0 \left( \frac{\partial^2}{\partial \theta_1 \partial \theta_2} - \frac{\partial^2}{4 \partial^2 \theta_{12}} \right) L^2(\theta).$$

Writing (6.2) as $L(\theta) = 1/(f(\theta))^p$, when $c_0 \neq \frac{p+1}{2(2p+1)}$ we get

$$\frac{\partial f}{\partial \theta_1} \frac{\partial f}{\partial \theta_2} - \frac{1}{4} \left( \frac{\partial f}{\partial \theta_{12}} \right)^2 = c_1 f(\theta) \left( \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2} - \frac{1}{4} \frac{\partial^2 f}{\partial^2 \theta_{12}} \right).$$

where

$$c_1 = \frac{2c_0 - 1}{2(2p+1)c_0 - p - 1}.$$  

Solving the resulting equations for $A, B, C, c_1$ we verify that the only non-zero solutions correspond either to $c_1 = 2$, $A = \pm 2\sqrt{B}$, $C = 0$ with $f(\theta) = (1 + A \operatorname{tr} (\theta)/2)^2$, or to $c_1 = 2/3$ with $A = \pm \sqrt{2}B$, $C = -B$.

When $c_0 = \frac{p+1}{2(2p+1)}$, the equations give $C = 2B$, so $f(\theta) = 1 + A \operatorname{tr} \theta + B(\operatorname{tr} \theta^2 + 2 \operatorname{tr} \theta^3)$. Using [28, Theorem 3.1] one can verify that in this case $L(\theta)$ is not a Laplace transform.

**Proof of Proposition 6.1.** Denote $Z = S^{-1}X^2S^{-1}$. Note that this is a positive-definite matrix. First we note that for positive matrices the Laplace transform is defined on $\theta \in H$ when $-\theta$ is positive.
Next we note that rotation invariance of the law of $X$ implies that $E(X) = \alpha I$ with $\alpha > 0$, $E(X^2) = (\alpha^2 + \beta^2)I$ with $\beta^2 > 0$, and $E(Z) = c_0 I$. Therefore,
\[ E(X^2|S) = E(SZS|S) = c_0 S^2. \]
Taking the expected value of both sides, $c_0 = \frac{\beta^2}{2(\beta^2 + 2\alpha^2)}$. We also note that by exchangeability, $E(X^2|S) = E(Y^2|S)$. So
\[ E((X - Y)^2|S) = E(2X^2 + 2Y^2 - S^2|S) = (4c_0 - 1)S^2. \]
Passing to standardized matrices $X' = (X - \alpha I)/\beta$, we see that (1.4) holds with $b^2 = 4a = \alpha^2/\beta^2$. Thus by Proposition 6.3(i),
\[ E(\exp \text{tr}(\theta X)) = e^{\alpha \text{tr} \theta} E(\exp \text{tr}(\beta \theta X')) \]
and equation (6.2) holds with $p = \beta^2/\alpha^2 > 1$.

Since $\det(X)/\det(S) = \sqrt{\det Z}$ is independent of $\det S$, by Lemma 6.2, there are only two non-degenerate choices for the law of $X$. It is known that (1.4) does not hold for the Wishart law, see [27, pg. 582], which excludes the law from Lemma 6.2(iii). Thus $X$ is the identity matrix multiplied by a Gamma random variable.

### 6.2. Some series solutions of the general system of PDEs

The Laplace transform of Bernoulli ensembles can be computed readily. For example, if $P$ is the orthogonal projection onto a randomly rotated line through the origin in $\mathbb{C}$ for $P$ of Bernoulli ensembles can be computed readily. For example, if
\[ (6.4) \]
\[ \text{some solutions of the system of PDEs, see e.g., Corollary 6.4.} \]
\[ \text{combined with properties of Meixner ensembles from Section 3, this can be used to produce} \]
\[ \text{E} \]
\[ (6.3) \]
\[ \text{the elementary symmetric functions} \]
\[ \text{However, it does not seem to be obvious how to express this Laplace transform in terms of} \]
\[ \text{the following result gives the expansion for rank 1 projections in $H_{n,\beta}$ for all $n \geq 2$. When} \]
\[ \text{combined with properties of Meixner ensembles from Section 3 this can be used to produce} \]
\[ \text{some solutions of the system of PDEs, see e.g., Corollary 6.4} \]
\[ \text{Recall the Pochhammer symbol} \]
\[ (6.4) \]
\[ (b)_k = \frac{\Gamma(b + k)}{\Gamma(b)} = b(b + 1) \ldots (b + k - 1). \]

### Proposition 6.3

If $P_1$ is an random projection of $H_{n,\beta}$ invariant by rotation with trace 1, then $E e^{\theta|P_1|} = L_n(\theta)$ where
\[ (6.5) \]
\[ L_n(\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n^2)_k} \sum_{\nu_1 + \nu_2 + \nu_3 + \cdots = k} (-1)^{\nu_1 + \nu_2 + \nu_3 + \cdots} \frac{(n^2)!}{\nu_1! \nu_2! \nu_3! \cdots} \sigma_1^{\nu_1}(\theta) \sigma_2^{\nu_2}(\theta) \sigma_3^{\nu_3}(\theta) \ldots \]

Since $P_2 = I_3 - P_1$ for $n = 3$, we have the following.

### Corollary 6.4

The Laplace transform of the Bernoulli ensemble on $H_{3,\beta}$ with parameters $q_1, q_2, q_3$ is
\[ L(\theta) = q_0 + q_1 L_3(\theta) + q_2 e^{\sigma_2} L_3(-\theta) + q_3 e^{\sigma_3}. \]
For the proof of the proposition we use facts about Jack polynomials (spherical polynomials) which require additional notation.

6.2.1. **Partitions.** A partition \( \kappa = (k_1, k_2, \ldots, k_r) \) of \( k \) into at most \( r \) parts is the sequence \( k_1 \geq k_2 \geq \cdots \geq k_r \geq 0 \) such that \( k_1 + k_2 + \cdots + k_r = k \). We write \( \kappa \vdash k \). Other standard notation is \( |\kappa| = k_1 + k_2 + \cdots \) (which is just \( k \)). We denote the length (the number of parts) of a partition \( \kappa \) by \( \ell(\kappa) = \#\{j : k_j > 0\} \).

6.2.2. **Pochhammer symbols.** For a partition \( \lambda \vdash k \), we need the \( \beta \)-Pochhammer symbol:

\[
(b)_\lambda = \prod_{j=1}^{\ell(\lambda)} \left( b - \frac{\beta}{2}(j - 1) \right).
\]

6.2.3. **Jack polynomials.** For a not necessarily symmetric \( n \times n \) matrix \( \theta \) with eigenvalues \( \theta_1, \theta_2, \ldots, \theta_n \), the Jack polynomials \( \{ J_\kappa(\theta; \beta) \} \) are the symmetric polynomials in \( \theta_1, \theta_2, \ldots, \theta_n \), indexed by the partitions \( \kappa \) of \( k = |\kappa| \) into \( r = \ell(\kappa) \leq n \) parts. In \([20]\) formula (1) the Jack polynomial \( J_\kappa(\theta; \beta) \) is defined as the coefficient at \( t_1 t_2 \cdots t_r \) in the expansion of

\[
(2/\beta)^r \det(I_n - t_1 \theta^{k_1} - t_2 \theta^{k_2} - \cdots - t_r \theta^{k_r})^{-\beta/2}.
\]

It is convenient to set \( J_\kappa(\theta; \beta) = 0 \) if the partition \( \kappa \) has more than \( n \) parts. (The usual notation for \( J_\kappa(x; \beta) \) is \( C_\kappa(x; \alpha) \) with \( \alpha = 2/\beta \).)

We need several properties of Jack polynomials.

**Proposition C.**

(i) If \( \ell(\kappa) \leq n \) and \( A, B \) are \( n \times n \) matrices, then

\[
\int_{K_n} J_\kappa(UA^*U; \beta) dU = \frac{J_\kappa(A; \beta) J_\kappa(B; \beta)}{J_\kappa(I_n; \beta)},
\]

where the integral is over the normalized Haar measure on the orthogonal group \( O_n \) (\( \beta = 1 \)), on the unitary group \( U_n \) (\( \beta = 2 \)), or on the symplectic group \( K_n \) (\( \beta = 4 \)).

(ii) If \( A \) is an \( n \times n \) matrix and \( k \geq 0 \), then

\[
(\Re \text{tr} (A))^k = \sum_{\kappa \vdash k} J_\kappa(A; \beta).
\]

(iii) If \( \ell(\lambda) \leq \min\{m,n\} \), then

\[
\frac{J_\lambda(I_{m,n}; \beta)}{J_\lambda(I_n; \beta)} = \frac{(\frac{n^2}{2})_{\lambda;\beta}}{(\frac{m^2}{2})_{\lambda;\beta}}.
\]

(If \( \ell(\lambda) > m \) then \( J_\lambda(I_{m,n}; \beta) = 0 \).)

(iv) For a one-part partition \( \lambda = (k) \vdash k \), we have

\[
J_{(k)}(A; \beta) = \frac{(-1)^k k!}{(\frac{\beta}{2})_k} \sum_{\nu_1 + 2\nu_2 + 3\nu_3 + \cdots = k} \frac{(-1)^{\nu_1+\nu_2+\cdots}(\frac{\beta}{2})_{\nu_1+\nu_2+\cdots}}{\nu_1!\nu_2!\cdots} \sigma_1^{\nu_1} \sigma_2^{\nu_2} \cdots
\]
Formula (6.7) appears in [18], where it is attributed to [17, (45)], see also [16, Corollary XI.3.2]. Formula (6.8) appears in [37, Proposition 2.3], see also [16, page 234]. Ref. [22, (21)] gives $J_\kappa(I_{m}; 1)$. Formula (6.10) is [37, Proposition 2.2(c)].

We are now ready to prove Proposition 6.3.

**Proof of Proposition 6.3.** Recall that $P_1 = U^*I_{1,n}U$, where $I_{1,n}$ is the identity matrix padded with zeros to make it into the $n \times n$ matrix. From (6.8),

$$E \exp\langle \theta | P_1 \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} E \text{tr}^k(U \theta U^* I_{1,n}) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa-k \vdash} J_\kappa(U \theta U^* I_{1,n}; \beta).$$

By (6.7), this becomes

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\kappa-k \vdash} \frac{J_\kappa(\theta; \beta) J_\kappa(I_{1,n}; \beta)}{J_\kappa(I_{1,n}; \beta)}.$$  

Since $J_\kappa(I_{1,n}; \beta) = 0$ when $\ell(\kappa) > 1$, this gives

$$E \exp\langle \theta | P_1 \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{J_{(k)}(\theta; \beta) J_{(k)}(I_{1,n}; \beta)}{J_{(k)}(I_{1,n}; \beta)}.$$  

Now by (6.9) this becomes

$$E \exp\langle \theta | P_1 \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(\frac{\beta}{2})_k}{(n^2 \beta^2)_k} J_{(k)}(\theta; \beta),$$

and formula (6.10) gives the answer.

\[ \square \]

### 6.3. Connection with free probability

The free probability analog of Meixner family was introduced by Anshelevich, Saitoh and Yoshida [3, 35] as the orthogonality measure of certain orthogonal polynomials. These free Meixner laws include the Kesten law (a special case of the free binomial law, which is the free additive convolution of the Bernoulli law), the Marchenko-Pastur law (also called the free-Poisson law), and the Wigner semi-circle law (a free probability analog of the normal law). The free Meixner family of laws again appeared as the laws characterized by a quadratic regression property in free probability [9], in a class of Markov processes, and as generating measures of Cauchy-Stieltjes kernel families with quadratic variance function [10]. See also [5], [6] and [11, Theorem 4.3].

Except for the free binomial law, the remaining free-Meixner laws are infinitely divisible with respect to free additive convolution and appear as limit laws of large dimensional random matrices [7, 12]. However, this correspondence is based on the Bercovici-Pata bijection [8], which, as observed by Anshelevich [3, page 241] is different from the correspondence based on the kernel families or regression properties.

### 6.4. Connection with Euclidean Jordan algebras

The natural framework of the Anshelevich question is rather the one of the Euclidean Jordan algebras well described in the book of Faraut and Koranyi [16]. The spaces $H_{n,\beta}$ for $\beta = 1, 2, 4$ are the 3 more important types of (irreducible) Euclidean Jordan algebras. But there are two more types to be considered: the exceptional 27 dimensional Albert algebra which can be roughly be
considered as a space of $(3,3)$ Hermitian matrices on octonions, and the Lorentz algebra on $\mathbb{R}^2 \times E$ where $E$ is a Euclidean space of dimension $d - 2$. In this algebra, the product of $(x_1, x_2, \vec{v})$ and $(y_1, y_2, \vec{w})$ is

$$(x_1y_1 + \langle \vec{v}, \vec{w} \rangle, x_2y_2 + \langle \vec{v}, \vec{w} \rangle, \frac{1}{2}(y_1 + y_2)\vec{v} + \frac{1}{2}(x_1 + x_2)\vec{w})$$

which leads to the square of $(x_1, x_2, \vec{v})$ as $$(x_1^2 + \|\vec{v}\|^2, x_2^2 + \|\vec{v}\|^2, 2(x_1 + x_2)\vec{v}).$$ A good way to memorize this product is to write formally $(x_1, x_2, \vec{v})$ as a $2 \times 2$ matrix $X = \begin{bmatrix} x_1 & \vec{v} \\ \vec{v}^t & x_2 \end{bmatrix}$, doing the same with $Y$ for $(y_1, y_2, \vec{w})$ and to consider the Jordan product

$$X \cdot Y = \frac{1}{2}(XY + YX).$$

If $d = \beta + 2$ with $\beta = 1, 2, 4$ this algebra is $\mathbb{H}_{2,\beta}$. For a Jordan algebra $V$, rotational invariance in $\mathbb{H}_{2,\beta}$ is generalized to the invariance by a certain compact group $K$ acting on $V$, as described in Faraut Koranyi [16, page 6 and 55]. The Anshelevich problem can be solved for the Lorentz algebra for any $d \geq 1$ since the rank is 2, and the preceding study for $\mathbb{H}_{2,\beta}$ extends easily to this case. Since the consideration of the Lorentz and Albert algebras can be seen as rather academic, we have refrained to place our paper in this framework.

6.5. **Unresolved questions.**

6.1. For fixed $A, B, C$ the research of the set $M(A, B, C)$ of the ensembles on $\mathbb{H}_{n,\beta}$ which satisfies $\mathbb{E}((X - Y)^2|S) = AS^2 + BS + CI_n$ led for $n = 1$ to the Laha Lukacs study which shows $M(A, B, C)$ is a natural exponential family. For $n \geq 2$ the structure of the $M(A, B, C)$ is not really understood: is it possible to pass easily from the knowledge of one element of $M(A, B, C)$ to a knowledge of the whole set as it is the case for an exponential family? The answer is hidden in the structure of the PDE system.

6.2. The dimension of the set of solutions of the PDE systems from Theorem 4.1 is equal 4 if $n = 2$ and is unknown if $n \geq 3$. The subset of solutions that are analytic at zero has dimension 3 for $n = 2$, and is unknown for $n \geq 3$. A related system of PDEs in [21], has a one-dimensional set of solutions analytic at 0.

6.3. Since the free binomial law is well defined for all real values of $N \geq 1$, it would be interesting to determine the Jørgensen set of the Bernoulli ensembles for $n > 2$.

6.4. For $n \geq 3$, it is not known whether a Meixner ensemble with parameters $A = -1/(2N - 1), B = 2N/(2N - 1), C = 0$ is a binomial ensemble from Definition 3.2. Similarly, the converses to Propositions 3.6 and 3.7 are not known beyond dimension $n = 2$ which is known from Theorem 5.1.

6.5. For $n \geq 3$, do we have non-Gaussian ensembles on $\mathbb{H}_{n,\beta}$ such that if $X, Y$ are i.i.d. then $\mathbb{E}((X - Y)^2|X + Y) = 2I_n$?

6.6. The distribution of eigenvalues as the dimension $n \to \infty$ was studied in [12] for a special case of Poisson ensembles. It is interesting to study this limit for other Meixner ensembles.
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