QCD sum rules analysis of the rare radiative $B_c \to D_s^* \gamma$ decay

K. Azizi$^1$ *, V. Bashiry$^2$ †

$^1$ Physics Department, Middle East Technical University, 06531 Ankara, Turkey

$^2$ Institute for Studies in Theoretical Physics and Mathematics (IPM), P.O. Box 19395-5531, Tehran, Iran

Abstract

In this work, the radiative $B_c \to D_s^* \gamma$ decay is investigated in the framework of QCD sum rules. The transition form factors responsible for the decay are calculated. The total branching ratio for this decay is estimated to be in the order of $10^{-5}$, so this decay can be measurable at LHC in the near future.

*e-mail: kazizi@newton.physics.metu.edu.tr
†e-mail: bashiry@ipm.ir
1 Introduction

The heavy pseudoscalar meson $B_c$ contains two heavy quarks of different flavor. This meson has been discovered in 1998 via the decay mode $B_c \to J/\psi l^+\nu$ in 1.8 TeV $p\bar{p}$ collisions, using the CDF detector at the Fermi Lab [1]. The $B_c$ meson constitutes a very rich laboratory for studying various decay channels. There are three classes of decays of $B_c$ meson, namely b-quark decay (when c is spectator), c-quark decay (when b is spectator) and the weak annihilation channels. Because of two heavy quark contents, the $B_c$ decay channels are expected to be very rich in comparison with other B mesons, so investigation of this meson is essential from both theoretical and experimental point of view. The $B_c$ meson decays provide windows for reliable determination of the CKM matrix element $V_{cb}$ and can shed light on new physics beyond the standard model.

At LHC with the luminosity values of $L = 10^{34}cm^{-2}s^{-1}$ and $\sqrt{s} = 14$TeV, the number of $B_c^{\pm}$ events is expected to be about $10^8 \sim 10^{10}$ per year, so there are high probability to study not only some rare $B_c$ decays, but also CP violation, T violation and polarization asymmetries. Some possible channels are $B_c \to l\nu\gamma$, $B_c \to \rho^+\gamma$, $B_c \to K^{*+}\gamma$ and $B_c \to B_u^*l^+l^-$, $B_c \to B_u^*\gamma$, which have been studied in the frame of light-cone QCD and three point QCD sum rules [2, 3, 4, 13]. A large set of exclusive nonleptonic and semileptonic decays of the $B_c$ meson, which have been studied within a relativistic constituent quark model can be found in [5]. Another possible decay channel of $B_c$ is $B_c^- \to \eta'\ell^-\bar{\nu}$ decay, which is studied both for decay rate and lepton polarization asymmetry [6]. We analyzed the radiative $B_c \to D_s^{*}\gamma$ decay by using QCD sum rules method. Using our calculations at the end, we also analyze the $B_c \to D^{*}\gamma$ decay by making the necessary changes.

The main quantities in analyzing of $B_c \to D_s^{*}\gamma$ decay are the form factors.
For the calculation of form factors, relevant to this transition, we need some nonperturbative approaches. Among the nonperturbative approaches, QCD sum rules method has received special attention, because this approach is based on QCD lagrangian. This method has been successfully applied to a wide variety of problems (for a review see [2, 3, 4, 7, 8, 9, 10, 11, 12]).

The $B_c \rightarrow D^*_s \gamma$ decay occurs via flavor changing neutral current (FCNC) transition ($b \rightarrow s \gamma$) and weak annihilation channels. The $b$ quark decay (electromagnetic penguin) for $B_c \rightarrow B^*_s \gamma$ has been calculated in [13] (for more details about the electromagnetic penguin diagram see also [14]). We repeated the similar calculations for our problem and found that the corresponding branching ratio contribution was less important than that of the weak annihilation channel. Note that, the $B_c \rightarrow D^*_s \gamma$ decay has been investigated in the perturbative QCD (PQCD) approach [15], relativistic independent quark model (RIQM) [16] and in the framework of perturbative QCD in SM (PQCD), multiscale walking technicolor (MWTCM) and top-color assisted MWTCM (TAMWTCM) models [17]. They found also that the contribution of the weak annihilation was important than that of the electromagnetic penguin in PQCD, RIQM and TAMWTCM. In addition, $B_c \rightarrow D^* \gamma$ decay has also been investigated in the relativistic independent quark model (RIQM) [18].

The paper is organized as follows: In section (2) we construct the transition amplitude for the weak annihilation channel in terms of four relevant form factors, where a photon can be radiated from $B_c$ or $D^*_s$. Two of the relevant form factors for this decay are calculated in [2] in the framework of the light–cone QCD sum rules. In section (3), we calculate the remaining two form factors, when a photon is radiated from $D^*_s$ meson also in light–cone QCD. In section (4), we calculate the transition form factors for electro-
magnetic penguin in the framework of three point QCD sum rules method. Finally, in section (5) numerical analysis, discussion and comparison of our results to those of the other approaches are forwarded and conclusion is presented.

2 Transition amplitude of the weak annihilation for the radiative $B_c \to D_s^*\gamma$ decay

In this section, we concentrated on the main points for obtaining the matrix elements of the radiative decay $B_c \to D_s^*\gamma$ along the lines similar to [18]. The weak annihilation mechanism for this decay is shown in (Fig.1). The transition amplitude for this decay can be written as:

$$M(B_c \to D_s^*\gamma) = \frac{G_F}{\sqrt{2}} V_{cb} V_{cs}^* <D_s^*(p)\gamma(q) \mid \bar{c}\Gamma_{\nu}c)(\bar{s}\Gamma^\nu b) \mid B_c(p+q)^> \quad (1)$$

where $\Gamma_{\nu} = \gamma_{\nu}(1 - \gamma_5)$ and $p$, $q$ and $p+q$ are the momentum of $D_s^*$, photon and $B_c$, respectively. Using factorization hypothesis, The matrix element in Eq. (1) can be written in the following form:

$$< D_s^*(p)\gamma(q) \mid \bar{c}\Gamma_{\nu}c)(\bar{s}\Gamma^\nu b) \mid B_c(p+q)^> = -e\varepsilon^\mu\varepsilon(D_s^*)_\nu f_{D_s^*} m_{D_s^*} T^{(B_c)}_{\mu\nu}$$

Figure 1: The weak annihilation mechanism for $B_c \to D_s^*\gamma$
where, the covariant decomposition of hadronic matrix elements $T_{\mu\nu}^{(B_c)}$ and $T_{\mu\nu}^{(D_s^*)}$ are responsible for the emission of photon from initial and final states, $f_{D_s^*}$, $f_{B_c}$ are the leptonic decay constants of $D_s^*$ and $B_c$ mesons, respectively, and $\epsilon^\mu$ and $\epsilon^{(D_s^*)\nu}$ are the polarization vectors of a photon and $D_s^*$ meson.

The covariant decomposition of hadronic matrix elements are defined by the following two-point correlation functions:

$$T_{\mu\nu}^{(B_c)}(p,q) = i \int d^4x e^{iqx} < 0 | T \{ j_{\mu}^{em} \sigma_{\nu}b(0) \} | B_c(p+q) >$$ \hspace{0.5cm} (3)

and

$$T_{\mu\nu}^{(D_s^*)}(p,q) = i \int d^4x e^{iqx} < D_s^*(p) | T \{ j_{\mu}^{em} \sigma_{\nu}c(0) \} | 0 >$$ \hspace{0.5cm} (4)

where $j_{\mu}^{em}$ stands for electromagnetic current. Our aim is to construct the $T_{\mu\nu}^{(B_c)}$ and $T_{\mu\nu}^{(D_s^*)}$ in terms of form factors and other physical quantities. Let first focus on $T_{\mu\nu}^{(B_c)}$. This quantity can be written in terms of two independent 4-momenta $p$ and $q$ in general as follows:

$$T_{\mu\nu}^{(B_c)}(p,q) = g_{\mu\nu}a + p_\mu q_\nu b + q_\mu p_\nu c + p_\mu p_\nu d + q_\mu q_\nu e + \epsilon_{\nu\mu\lambda\sigma} p^\lambda q^\sigma F_{V}^{(B_c)}$$ \hspace{0.5cm} (5)

where $a$, $b$, $c$, $d$, $e$ and $F_{V}^{(B_c)}$ are invariant amplitudes. Applying the Ward identity for electromagnetic current to Eq. (5) and using the fact that for a real photon $q^2 = 0$, we rewrite Eq. (5) in the following form:

$$T_{\mu\nu}^{(B_c)}(p,q) = (g_{\mu\nu}(p,q) - p_\mu q_\nu) iF_{A}^{(B_c)} + g_{\mu\nu}(p,q) \alpha + p_\mu q_\nu \beta + q_\mu q_\nu c$$

$$+ i\frac{p_\mu p_\nu}{p.q} f_{B_c} + q_\mu q_\nu e + \epsilon_{\nu\mu\lambda\sigma} p^\lambda q^\sigma F_{V}^{(B_c)}$$ \hspace{0.5cm} (6)

where $F_{A}^{(B_c)}$, $\alpha$ and $\beta$ are the new invariant amplitudes. To obtain the relation between $\alpha$ and $\beta$, we compare Eqs. (5) and (6), which leads to

$$g_{\mu\nu}a + p_\mu q_\nu b = (g_{\mu\nu}(p,q) - p_\mu q_\nu) iF_{A}^{(B_c)} + g_{\mu\nu}(p,q) \alpha + p_\mu q_\nu \beta$$ \hspace{0.5cm} (7)
by multiplying both sides of Eq. (7) with $q^\mu$ and using $a + (p.q) = i f_{Bc}$ we get the following relation between $\alpha$ and $\beta$, which is called the Ward identity.

$$\alpha + \beta = \frac{if_{Bc}}{(p.q)}$$  \hspace{1cm} (8)

From Eq. (8), it is clear that $\alpha$ and $\beta$ can take different choices. Within the scope of the present work, in parallel with (18), we set $\beta = 0$ and $\alpha = \frac{if_{Bc}}{(p.q)}$.

Substituting the values of $\epsilon.q$ and $\beta$ of the present work, in parallel with (18), we set $\beta = 0$ and $\alpha = \frac{if_{Bc}}{(p.q)}$.

Using $\epsilon.q = 0$, $\epsilon^{(D^s_2)}_q = 0$ and Eq. (9), we obtain the following expression for the first term in Eq. (2) in terms of two form factors ($F_A^{(Bc)}$, $F_V^{(Bc)}$):

$$T^{(Bc)}_{\mu\nu}(p,q) = (g_{\mu\nu}(p.q) - p_\mu q_\nu)i F_A^{(Bc)} + if_{Bc}(g_{\mu\nu} + q_\mu q_\nu)\epsilon^\nu + i\frac{p_\mu p_\nu}{p.q} f_{Bc} + q_\mu q_\nu + \epsilon_{\nu\mu\lambda\sigma} p^\lambda q^\sigma F_V^{(Bc)}$$  \hspace{1cm} (9)

Using $\epsilon.q = 0$, $\epsilon^{(D^s_2)}_q = 0$ and Eq. (9), we obtain the following expression for the first term in Eq. (2) in terms of two form factors ($F_A^{(Bc)}$, $F_V^{(Bc)}$):

$$e\epsilon^{\mu}(D^s_2) \nu f_{D^s_{\mu\nu}} = e f_{D^s_{\mu\nu}} m_{D^s} T^{(Bc)}_{\mu\nu} \left\{ \{ (\epsilon.D^s_2)(p.q) - (\epsilon.p)(\epsilon.D^s_2) \} i F_A^{(D^s_2)} + \frac{q_\mu q_\nu}{p.q} f_{Bc} + \epsilon_{\nu\mu\lambda\sigma} p^\lambda q^\sigma F_V^{(D^s_2)} \right\}$$  \hspace{1cm} (10)

Omitting the details for the calculation $T^{(D^s_2)}_{\mu\nu}$, we get the following result for the second term in the Eq. (2):

$$ie^{\mu}(p + q) \nu f_{Bc} T^{(D^s_2)}_{\mu\nu} = e f_{Bc} \left\{ \{ (\epsilon.D^s_2)(p.q) - (\epsilon.p)(\epsilon.D^s_2) \} i F_A^{(D^s_2)} + \frac{q_\mu q_\nu}{p.q} f_{Bc} + \epsilon_{\nu\mu\lambda\sigma} p^\lambda q^\sigma F_V^{(D^s_2)} \right\}$$  \hspace{1cm} (11)

where, $F_A^{(D^s_2)}$ and $F_V^{(D^s_2)}$ are two form factors of $D^s_2$. Now, we can write the transition amplitude for the radiative $B_c \rightarrow D^s_2 \gamma$ decay in terms of four form factors $F_A^{(Bc)}$, $F_V^{(Bc)}$, $F_A^{(D^s_2)}$ and $F_V^{(D^s_2)}$, as follows:

$$M(B_c \rightarrow D^s_2 \gamma) = e \frac{G_F}{\sqrt{2}} V_{cb} V_{cs} \left\{ -f_{D^s_{\mu\nu}} T^{(Bc)}_{\mu\nu} \left\{ \{ (\epsilon.D^s_2)(p.q) - (\epsilon.p)(\epsilon.D^s_2) \} i F_A^{(Bc)} + \frac{q_\mu q_\nu}{p.q} f_{Bc} + \epsilon_{\nu\mu\lambda\sigma} p^\lambda q^\sigma F_V^{(Bc)} \right\} - i f_{Bc} \left\{ \{ (\epsilon.D^s_2)(p.q) - (\epsilon.p)(\epsilon.D^s_2) \} i F_A^{(Bc)} + \frac{q_\mu q_\nu}{p.q} f_{Bc} + \epsilon_{\nu\mu\lambda\sigma} p^\lambda q^\sigma F_V^{(Bc)} \right\} \right\}$$  \hspace{1cm} (12)
The form factors $F^{(B_c)}_A$ and $F^{(B_c)}_V$ corresponding to the emission of the photon from b and c quark (see Fig. 1 i, ii), are calculated in [2]. Therefore, we will concentrate on the calculation of the form factors $F^{(D^*_s)}_A$ and $F^{(D^*_s)}_V$ (see Fig. 1 iii, iv).

### 3 Light cone QCD sum rules for the form factors $F^{(D^*_s)}_A$ and $F^{(D^*_s)}_V$

Based on the general idea on QCD sum rules method, we will calculate the transition form factors by equating the representation of a suitable correlator in hadronic and quark–gluon languages. For this aim, we consider the following correlation function:

$$\Pi_{\mu\nu}(p,q) = i \int d^4x e^{iqx} < \gamma(q) | T\{\bar{c}(x)\gamma_\mu(1-\gamma_5)s(x)s(0)\gamma_\nu c(0)\} | 0 >$$

(13)

Here, q and p are the momentum values of photon and $D^*_s$, respectively and $Q = p + q$ is the transferred momentum. Now, we insert the hadronic state $D^*_s(p)$ to Eq. (13). This can be re-written as:

$$\Pi_{\mu\nu}(p,q) = \frac{< \gamma(q) | \bar{c}\gamma_\mu(1-\gamma_5)s \cdot D^*_s(p) >}{m_{D^*_s}^2 - p^2}$$

(14)

The second term in Eq. (14), by definition is:

$$< D^*_s(p) | \bar{s}\gamma_\nu c | 0 > = f_{D^*_s} m_{D^*_s} \varepsilon^{(D^*_s)}_{\nu}$$

(15)

From Fig. 1 (iii, iv) and due to the fact that parity, Lorentz and gauge invariance are musts. We can write the matrix element for the emission of the photon from $D^*_s$ meson as:

$$< \gamma(q) | \bar{c}\gamma_\mu(1-\gamma_5)s \cdot D^*_s(p) > = e\{i\varepsilon_{\mu\alpha\beta\sigma} \varepsilon^{(D^*_s)\alpha} q^\beta F^{(D^*_s)}_V(Q^2) \}

+ [\varepsilon_\mu(\varepsilon^{(D^*_s)\alpha} q^\alpha) - (\varepsilon^{(D^*_s)} q_\alpha)\varepsilon_\mu] F^{(D^*_s)}_A(Q^2) m^2_{D^*_s} \} (16)$$
Substituting Eqs. (15) and (16) to (14), we have

\[
\Pi_{\mu\nu}(p,q) = \frac{ef_{D_s^*}m_{D_s^*}}{m_{D_s^*}^2 - p^2} \left\{ i \varepsilon_{\mu\alpha\beta\gamma} \varepsilon^\alpha q^\gamma \frac{F_{V}^{(D_s^*)}(Q^2)}{m_{D_s^*}^2} \right. \\
+ \left[ \varepsilon_{\mu}^{(D_s^*)} \varepsilon_{\nu}^{(D_s^*)} q - (\varepsilon_{\nu}^{(D_s^*)} \varepsilon_{\mu}^{(D_s^*)} q) \frac{F_{A}^{(D_s^*)}(Q^2)}{m_{D_s^*}^2} \right] \right\} (17)
\]

Summation over polarization of $D_s^*$ meson is performed by using:

\[
\varepsilon_{\mu}^{(D_s^*)} \varepsilon_{\nu}^{(D_s^*)} = - g_{\nu\beta} + \frac{p_{\nu} p_{\beta}}{m_{D_s^*}^2} (18)
\]

After performing the standard calculations for the phenomenological part, we get:

\[
\Pi_{\mu\nu}(p,q) = \frac{ef_{D_s^*}m_{D_s^*}}{m_{D_s^*}^2 - p^2} \left\{ i \varepsilon_{\mu\alpha\beta\gamma} \varepsilon^\alpha q^\gamma \frac{F_{V}^{(D_s^*)}(Q^2)}{m_{D_s^*}^2} \right. \\
+ \left[ q_{\mu} \varepsilon_{\nu}^{(D_s^*)} \varepsilon_{\mu}^{(D_s^*)} q - (\varepsilon_{\nu}^{(D_s^*)} \varepsilon_{\mu}^{(D_s^*)} q) \frac{F_{A}^{(D_s^*)}(Q^2)}{m_{D_s^*}^2} \right] \right\} (19)
\]

The theoretical part (QCD side) of the correlator is calculated by means of OPE up to operators having dimension $d = 5$ in deep Euclidean space, where both $p^2$ and $Q^2$ are large and negative. It is determined by the bare-loop (fig. 2(a, b)) and the power corrections from the operators with $d = 3$, $<\bar{\psi}\psi>$, $d = 4$, $m_s <\bar{\psi}\psi>$, $d = 5$, $m_0^2 <\bar{\psi}\psi>$ (fig. 2(c, d, e)) and the photon interaction with a soft quark line (fig. 2f). In calculating the bare-loop and nonperturbative correction contributions, we first write the Lorentz decomposition of the correlator as:

\[
\Pi_{\mu\nu}(p,q) = i \varepsilon_{\mu\nu\alpha\beta} \varepsilon^\alpha q^\beta \Pi_1 + [q_{\mu} \varepsilon_{\nu} - \varepsilon_{\mu} q_{\nu}] \Pi_2 (20)
\]

and the dispersion representation (Cutkosky method) for the coefficients of corresponding Lorentz structures appearing in the $\Pi_{\mu\nu}(p,q)$ as follows:

\[
\Pi_{1,2}(p,q) = \int ds \frac{\rho_{1,2}(s, p^2)}{s - Q^2} + \text{subtraction terms} (21)
\]
Figure 2: Feynmen diagrams for bare-loop (a, b), power corrections from the operators with 3, 4 and 5 dimensions (c, d, e) and propagation of the soft quark in electromagnetic field (f)

where $\rho_{1,2}(s, p^2)$ are spectral density corresponding to two structures in $\Pi_{\mu\nu}(p, q)$ and subtraction terms stand for corrections. To calculate $\rho_{1,2}(s, p^2)$, we consider Feynmen diagrams in Fig. 2(a, b). For instance, for the contribution of diagram (a) we get

$$T_{\mu\nu} = eN_cQ_s \int \frac{d^4k}{(2\pi)^4} \left\{ Tr \left[ i\frac{\gamma^\mu(1 - \gamma_5)}{k^2 - m_c^2} \right] \right\}$$

(22)

With the help of the above equations, we obtain the following expressions corresponding to the coefficients of the structures $i\varepsilon_{\mu\nu\alpha\sigma}\varepsilon^\alpha q^\sigma$ and $[q_{\mu}\varepsilon_{\nu} - \varepsilon_{\mu} q_{\nu}]:$

$$T_1 = -\frac{eN_cQ_s}{4\pi^2} \left\{ \int_0^1 dx \int_0^1 dy \left[ m_c m_s - m_s^2 x y + p^2 (X_1 y + X_3) - (p \cdot q)(X_2 y^2 - 4 x y) \right] \int_0^\infty \frac{d\alpha e^{-\alpha \Delta}}{\alpha} \right\}.$$
\[ T_2 = \frac{eN_e Q_s}{16\pi^2} \{ \int_0^1 dx \int_0^1 dy [m_c m_s - m_s^2 x y + p^2 (X_1 y - X_3) - (p \cdot q) X_3 (2y - x)] \}
+ 4\pi y \int_0^\infty d\alpha e^{-\alpha \Delta} \]

where \( \overline{x(y)} = 1 - x(y), \Delta = -p^2 x x y - Q_x x x y + m_c^2 x + m_s^2 x \), \( X_1 = x - 2x^2 + x^3 \), \( X_2 = x^2 - 2x^3 \), \( X_3 = -x + x^2 \).

In this calculation, we have also used the exponential representation for the denominator as:
\[ \frac{1}{\Delta^n} = \frac{1}{(n-1)!} \int_0^\infty d\alpha \alpha^{n-1} e^{-\alpha \Delta} \]

Next, we apply the double Borel operator \( \hat{B}(M_1^2) \hat{B}(M_2^2) \) on \( T_{(1,2)} \) and we get:
\[ \tilde{T}_1 = \frac{-eN_e Q_s}{4\pi^2} \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} \int_0^1 dx \int_0^1 \frac{1}{\pi} e^{1/2 \int dx (m_c^2 x + m_s^2 x) (\sigma_1 + \sigma_2) [m_c m_s - m_s^2 x - \sigma_2]} \]
\[ + p^2 ((x^3 - 2x^2) \frac{\sigma_2}{\sigma_1 + \sigma_2} + x^2 - x \frac{\sigma_1}{\sigma_1 + \sigma_2}) - (p \cdot q) (x^2 - 2x^3 + 3x^2) \]
\[ - 4x^3 \frac{\sigma_1^2}{(\sigma_1 + \sigma_2)^2} + (8x^3 - 2x^2 - 4x) \frac{\sigma_1}{\sigma_1 + \sigma_2}) \]

\[ \tilde{T}_2 = \frac{eN_e Q_s}{16\pi^2} \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} \int_0^1 dx \int_0^1 \frac{1}{\pi} e^{1/2 \int dx (m_c^2 x + m_s^2 x) (\sigma_1 + \sigma_2) [m_c m_s - m_s^2 x - \sigma_2]} \]
\[ + p^2 ((x + x^3 - 2x^2) \frac{\sigma_2}{\sigma_1 + \sigma_2} - x^2 + x) - (p \cdot q) (-2x + 2x^2) \frac{\sigma_2}{\sigma_1 + \sigma_2} \]
\[ - 2x^3 \frac{\sigma_2^2}{(\sigma_1 + \sigma_2)^2} + (8x^3 - 4x^2 - 4x) \frac{\sigma_1}{\sigma_1 + \sigma_2}) + (2x^3 - 2x^2) \frac{\sigma_1^2}{(\sigma_1 + \sigma_2)^2} \]

(25)

where \( \sigma_1 = \frac{1}{M_1^2} \) and \( \sigma_2 = \frac{1}{M_2^2} \).

In deriving Eq. (25), we use the definition
\[ \hat{B}(M^2) e^{-\alpha \rho \pi} = \delta(1 - \alpha M^2) \]

(26)

For the determination of the spectral density, we apply the Borel transformations to \( \tilde{T}_1 \) and \( \tilde{T}_2 \) [19] and we obtain:
\[ \varrho_{1,2}(s, t, p^2) = \frac{1}{s t} \hat{B}(\frac{1}{s}, \sigma_1) \hat{B}(\frac{1}{t}, \sigma_2) \frac{T_{1,2}}{\sigma_1 \sigma_2} \]

(27)
In this step, we use the following relations:

$$\sigma^n e^{-\alpha \sigma} = \left(-\frac{d}{d\alpha}\right)^n e^{-\alpha \sigma},$$  \hspace{1cm} (28)

$$\rho_{1,2}(s, p^2) = \int dt \frac{\Theta_{1,2}(s, t, p^2)}{t - p^2},$$  \hspace{1cm} (29)

and

$$\hat{B}(\frac{1}{s}, \sigma_1) \hat{B}(\frac{1}{t}, \sigma_2) e^{-\alpha (\sigma_1 + \sigma_2)} = \delta(1 - \frac{\alpha}{s}) \delta(1 - \frac{\alpha}{t})$$  \hspace{1cm} (30)

Then, we get the following expressions for the two spectral densities, as follows:

$$\rho_1(s, p^2) = \frac{eN_c Q_s}{4\pi^2} \frac{1}{(s - p^2)^3} \int_{x_0}^{x_1} dx \frac{1}{x} \{(m_c m_s - m_s^2 x)(s - p^2)^2 + p^2[(x^3 - x^2)$$

$$(s - p^2)^2 - x(p^2 + \frac{m_c^2}{x} + \frac{m_s^2}{x})(s - p^2) - \frac{1}{2}(m_s^2 - p^2)[(x^2 - 2x^3)$$

$$(s - p^2)^2 - 2(3x^2 - 4x^3)(p^2 + \frac{m_c^2}{x} + \frac{m_s^2}{x})^2 + (8x^3 - 2x^2 - 4x)(p^2$$

$$+ \frac{m_c^2}{x} + \frac{m_s^2}{x})(s - p^2)]\}.$$

$$\rho_2(s, p^2) = \frac{eN_c Q_s}{16\pi^2} \frac{1}{(s - p^2)^3} \int_{x_0}^{x_1} dx \frac{1}{x} \{[m_c m_s - m_s^2 x + p^2(2x - 3x^2 + x^3)]$$

$$(s - p^2)^2 - \frac{1}{2}(m_s^2 - p^2)[(-2x + 2x^2 - 4x^3)(s - p^2)^2 + (8x^2 - 4x^3$$

$$(s - p^2) - 2(2x^3 - 2x^2)(p^2 + \frac{m_c^2}{x} + \frac{m_s^2}{x})^2]\}$$

where the integration region is determined by the following inequality:

$$sx\overline{x} - (m_c^2 \overline{x} + m_s^2 x) \geq 0$$  \hspace{1cm} (32)

Similar to above calculations for diagram (a), we have repeated the entire calculations for diagram (b). Finally, we get the following results for the
spectral densities:

\[
\rho_1(s, p^2) = \frac{eN_c}{48\pi^2} \frac{1}{(s - p^2)^3} \{ Q_s [\lambda \{ 3(-3 + 5\alpha - 5\beta)p^6 + (28 + \alpha(-103 + \alpha(17 + 4\alpha)) + 52\beta - \alpha(61 + 8\alpha)\beta + 4(10 + \alpha)\beta^2 \}p^4s - 3(-2 + 16\alpha^3 + \beta - 31\beta^2 - \alpha^2(75 + 32\beta) + \alpha(39 + 2\beta(41 + 8\beta)) + 7\lambda^2 )p^2s^2 + \alpha(1 - 86\alpha^2 + \beta - 86\beta^2 + \alpha(67 + 172\beta) + 18\lambda^2)s^3 \} + 6\{ p^6 + 2m_c m_s(s - p^2)^2 + (2 - 11\alpha + 4\beta)p^4s(-1 + 26\alpha^2 + 2\beta^2 - 2\alpha(7 + 11\beta))p^2s^2 - \alpha(1 + 18\alpha^2 + 2\beta(1 + \beta) - 4\alpha(4 + 5\beta))s^3 \} \ln \frac{1 + \alpha - \beta - \lambda}{1 + \alpha - \beta + \lambda} \} + Q_c [\lambda \{ 3(-3 + 5\beta - 5\alpha)p^6 + (28 + \beta(-103 + \beta(17 + 4\beta)) + 52\alpha - \beta(61 + 8\beta)\alpha + 4(10 + \beta)\alpha^2 \}p^4s - 3(-2 + 16\beta^3 + \alpha - 31\alpha^2 - \beta^2(75 + 32\alpha) + \beta(39 + 2\alpha(41 + 8\alpha)) + 7\lambda^2 )p^2s^2 + \beta(1 - 86\beta^2 + \alpha - 86\alpha^2 + \beta(67 + 172\alpha) + 18\lambda^2)s^3 \} + 6\{ p^6 + 2m_c m_s(s - p^2)^2 + (2 - 11\beta + 4\alpha)p^4s(-1 + 26\beta^2 + 2\alpha^2 - 2\beta(7 + 11\alpha))p^2s^2 - \beta(1 + 18\beta^2 + 2\alpha(1 + \alpha) - 4\beta(4 + 5\alpha))s^3 \} \ln \frac{1 + \beta - \bar{\alpha} - \lambda}{1 + \beta - \bar{\alpha} + \lambda} \},
\]

\[
\rho_2(s, p^2) = \frac{eN_c}{96\pi^2} \frac{1}{(s - p^2)^3} \{ Q_s [\lambda \{ -(29 + 2\alpha^2 + \beta(17 + 2\beta) - \alpha(19 + 4\beta))p^6 + (4\alpha^3 + \alpha(5 + \beta)(5 + 4\beta) - \alpha^2(23 + 8\beta) + 6(8 + \beta - \beta^2))p^4s + (-23 + 2\alpha^3 - 5\beta(1 + 4\beta) + \alpha(-9 + 2\beta(21 + \beta)) + \alpha^2(-30 - 4\beta + \frac{48}{\lambda^2 - (1 + \alpha - \beta)^2})p^2s^2 + \alpha(13 + \alpha(-5 + 22\alpha) + \beta - 44\alpha\beta + 22\beta^2)s^3 \} + 6\{ -2p^6 + m_c m_s(s - p^2)^2 + (4 + 5\alpha)p^4s - 2(1 + \alpha + 4\alpha^2 - 2\alpha\beta)p^2s^2 + \alpha(1 + 4\alpha(\alpha - \beta))s^3 \} \ln \frac{1 - \alpha + \beta - \lambda}{1 - \alpha + \beta + \lambda} \} + Q_c [\lambda \{ -(29 + 2\beta^2 + \alpha(17 + 2\alpha) - \beta(19 + 4\alpha))p^6 + (4\beta^3 + \beta(5 + \alpha)(5 + 4\alpha) - \beta^2(23 + 8\alpha) + 6(8 + \alpha - \alpha^2))p^4s + (-23 + 2\beta^3 - 5\alpha(1 + 4\alpha) + \beta(-9 + 2\alpha(21 + \alpha)) + \beta^2(-30 \right)
\]
\[-4\alpha + \frac{48}{\lambda^2 - (1 + \beta - \alpha)^2}) s^2 + \beta(13 + \beta(-5 + 22\beta) + \alpha
\]
\[-44\beta \alpha + 22\alpha^2) s^3) + 6\{-2p^6 + m_c m_s (s - p^2)^2 + (4 + 5\beta)p^4 s
\]-2(1 + \beta + 2\beta^2 - 2\beta\alpha) p^2 s^2 + \beta(1 + 4\beta(\beta - \alpha)) s^3\}
\[ln \frac{1 - \beta + \alpha - \lambda}{1 - \beta + \alpha + \lambda}\]\quad (33)

where \(\alpha = \frac{m^2_s}{s}\) and \(\lambda = \sqrt{1 + \alpha^2 + \beta^2 - 2\alpha - 2\beta - 2\alpha\beta}\).

The next step is to calculate contributions coming from the power correction terms. After standard but lengthy calculations for the contributions of the diagrams (c, d, e), we get:

\[
\Pi_1(p,q)^{(c,d,e)} = \frac{m_c}{r'^2 r} < \langle \bar{s}s \rangle - \frac{m_s}{2} < \langle \bar{s}s \rangle + \frac{m_c}{r'^2 r} + \frac{m_s^2}{2} < \langle \bar{s}s \rangle \frac{2m_c^3}{r'^3 r}
\]
\[+ \frac{2m_c^3}{r'^2 r^2} + \frac{m_c}{r'^2 r} + \frac{m_c}{r'^2 r} + \frac{2m_c^3}{r'^3 r} - \frac{m_s^2}{4} < \langle \bar{s}s \rangle \frac{2m_c^3}{r'^3 r} + \frac{m_s^2}{r'^2 r^2}
\]
\[+ \frac{2m_c^3}{r'^3 r} + \frac{m_c}{r'^2 r} + \frac{2m_c^3}{r'^3 r} \quad (34)
\]

where \(r'^2 = p^2 - m_c^2\) and \(r'^2 = Q^2 - m_c^2\). Finally, we calculate the contribution of diagram (f). For the calculation of this diagram corresponding to the propagation of the soft quark in the external electromagnetic field, we use the light-cone expansion for the non–local operators. After contracting the c quark lines in

\[
\Pi_{\mu \nu}(p,q) = i \int d^4 x e^{iQx} < \gamma(q) | T\{\bar{c}(0)\gamma_\mu c(0)\bar{c}(x)\gamma_\nu(1 - \gamma_5) s(x)\} | 0 >
\] (35)
we obtain

\[
\Pi_{\mu\nu}(p, q) = i \int d^4x \frac{d^4k}{(2\pi)^2} \frac{e^{i(Q-k)x}}{m_c^2 - k^2} <\gamma(q) \mid s\gamma_\mu(k + m_c)\gamma_\nu(1 - \gamma_5)s \mid 0 >
\]

(36)

To calculate the matrix element appearing in the above equation, we use the following identities:

\[
\gamma_\mu \gamma_\nu = g_{\mu\nu} + i\sigma_{\mu\nu},
\]

\[
\gamma_\mu \gamma_\nu \gamma_5 = g_{\mu\nu}\gamma_5 - \frac{i}{2}\varepsilon_{\mu\nu\alpha\sigma}\gamma_\alpha\gamma_\sigma,
\]

\[
\gamma_\mu \gamma_\alpha \gamma_\nu = g_{\mu\alpha}\gamma_\nu + g_{\alpha\nu}\gamma_\mu - g_{\mu\nu}\gamma_\alpha + i\varepsilon_{\mu\nu\alpha\sigma}\gamma_\sigma\gamma_5
\]

(37)

and photon distribution amplitudes (DA’s) for twist 2, 3 and 4 [20, 21]:

\[
<\gamma(q) \mid s\gamma_\mu s \mid 0 > = -\frac{Q_s}{2} f_{3\gamma} \int_0^1 du \overline{\psi}_{(V)}(u)x^\theta F_{\theta\nu}(ux)
\]

\[
<\gamma(q) \mid s\gamma_\alpha\gamma_5 s \mid 0 > = -\frac{iQ_s}{4} f_{3\gamma} \int_0^1 du \overline{\psi}_{(A)}(u)x^\theta F_{\theta\alpha}(ux)
\]

\[
<\gamma(q) \mid s\sigma_{\alpha\beta} s \mid 0 > = Q_s <\gamma_s> \int_0^1 du \phi(u) F_{\alpha\beta}(ux)
\]

\[
+ \frac{Q_s}{8} \int_0^1 du \varepsilon_{\alpha\beta}\mathcal{A}(u) F_5(ux)
\]

\[
+ \frac{Q_s}{16} \int_0^1 du \varepsilon_{\alpha\beta}\mathcal{B}(u) x^\theta(x_\beta F_{\alpha\rho}(ux) - x_\alpha F_{\beta\rho}(ux))
\]

(38)

where \( F_{\mu\nu} \) is the field strength tensor of the electromagnetic field, which is defined by

\[
F_{\mu\nu}(x) = i(\varepsilon_{\nu\rho} - \varepsilon_{\mu\rho}) e^{i q x}
\]

\[
\tilde{F}_{\mu\nu}(x) = \frac{1}{2}\varepsilon_{\mu\rho\alpha\beta} F_{\alpha\beta}(x)
\]

(39)

The asymptotic expression for the photon wave function \( \phi(u) \) in terms of magnetic susceptibility of the quark condensate, \( \chi(\mu) \), at a re-normalization
scale \((\mu = 1 \text{ GeV}^2)\) is defined by:

\[
\phi(u) = \chi(\mu)u(1-u)
\] (40)

Other functions used in Eq. (38) are defined by \[20, 21\]

\[
\psi^{(V)}(u) = -20u(1-u)(2u-1) + \frac{15}{16}(\omega_+^4 - 3\omega_+^2)u(1-u)(2u-1)
\]
\[
\times (7(2u-1)^2 - 3),
\]

\[
\psi^{(A)}(u) = (1 - (2u - 1)^2)(5(2u - 1)^2-1) \times \frac{5}{2}(1 + \frac{9}{16}\omega_-^2 - \frac{3}{16}\omega_+^4),
\]

\[
A(u) = 40u(1-u)(3k-k^+ + 1) + 8(\xi_2^+ - 3\xi_2)[u(1-u)(2 + 13u(1-u)) + 2u^3(10 - 15u + 6u^2)lnu + 2(1-u)^3(10 - 15(1-u) + 6(1 - u^2))ln(1-u)],
\]

\[
B(u) = 40 \int_0^u d\alpha (4 - \alpha)(1 + 3k^+)[-\frac{1}{2} + \frac{3}{2}(2\alpha - 1)^2]
\] (41)

where \(k, k^+, \xi_2, \xi_2^+\) and \(f_{3\gamma}\) are constants (see \[20, 21\]). Using the above relations in Eq. (36), we obtain:

\[
\Pi_{\mu\nu}(p, q) = -\int d^4x \int d^4k \frac{e^{i(p-k)\cdot x}}{(2\pi)^2m_c^2 - k^2} \left\{ \frac{Q_s}{2} \int_0^1 d\psi^{(V)}(u)x^\sigma F_{\sigma\mu} - \frac{Q_s}{2} f_{3\gamma} k_\nu \int_0^1 d\psi^{(V)}(u)x^\sigma F_{\sigma\nu} - \frac{Q_s}{2} f_{3\gamma} k_\nu \right\}
\]
\[
\times \left\{ \int_0^1 d\psi^{(V)}(u)x^\sigma F_{\sigma\mu} - \frac{Q_s}{2} f_{3\gamma} g_{\mu\nu} k_\alpha \int_0^1 d\psi^{(V)}(u)x^\sigma F_{\sigma\alpha} + \frac{Q_s}{4} f_{3\gamma} k_\nu \int_0^1 d\psi^{(A)}(u)[-x^\alpha F_{\mu\nu} + x_\mu F_{\alpha\nu} + x_\nu F_{\mu\alpha}] + \frac{iQ_s}{8} f_{3\gamma} [k_\nu \varepsilon_{\sigma\theta\eta\lambda}]
\]
\[
 \times k_\alpha \int_0^1 d\psi^{(A)}(u)x^\theta F_{\eta\lambda} + \frac{iQ_s}{2} f_{3\gamma} k_\alpha \varepsilon_{\mu\nu\alpha\sigma} + \frac{Q_s}{16} \int_0^1 d\psi^{(V)}(u)x^\sigma F_{\theta\sigma} + \frac{m_c}{2} Q_s < \bar{s}s > \int_0^1 d\psi^{(V)}(u)x^\sigma F_{\mu\nu}(ux) + \frac{Q_s}{8} < \bar{s}s > \int_0^1 d\psi^{(V)}(u)x^\sigma F_{\alpha\sigma}(ux)
\]
\[
\times (x_\nu F_{\mu\sigma}(ux) - x_\mu F_{\nu\sigma}(ux)) + \frac{m_c}{2} Q_s < \bar{s}s > \varepsilon_{\mu\nu\sigma}[\int_0^1 d\psi^{(V)}(u)
\]
\[
\times (x_\nu F_{\mu\sigma}(ux) - x_\mu F_{\nu\sigma}(ux))] + \frac{m_c}{2} Q_s < \bar{s}s > \varepsilon_{\mu\nu\sigma}[\int_0^1 d\psi^{(V)}(u)
\]
\[
\times F_{\alpha\sigma}(ux) + \frac{1}{16} \int_0^1 d\psi^{(A)}(u)A(u) F_{\alpha\sigma} + \frac{1}{8} \int_0^1 d\psi^{(A)}(u) A(u) F_{\alpha\sigma} (ux) - x_\alpha F_{\sigma\theta}(ux))\}.
\] (42)
After performing integration over x and k, the following results corresponding to the coefficients of two invariant structures $i\varepsilon_{\mu
u\alpha\sigma}\varepsilon^{\alpha}q^\sigma$ and $[q_{\mu}\varepsilon_{\nu} - \varepsilon_{\mu}q_{\nu}]$ are obtained as follows:

$$\Pi_1(p,q)(f) = \frac{m_cQ_s < \bar{s}s>}{2} \left[ \int_0^1 du \phi(u) \frac{1}{p^2 - m_c^2} - \frac{1}{16} \int_0^1 du A(u) \left( \frac{10}{(p^2 - m_c^2)^2} + \frac{8m_c^2}{(p^2 - m_c^2)^3} \right) \right]$$

$$\Pi_2(p,q)(f) = -\frac{Q_s}{4} f_{3\gamma} \left[ \int_0^1 du \psi^{(A)}(u) \left( \frac{1}{p^2 - m_c^2} + \frac{2m_c^2}{(p^2 - m_c^2)^2} \right) - m_cQ_s < \bar{s}s> \left[ \int_0^1 du \phi(u) \frac{1}{p^2 - m_c^2} - \frac{1}{16} \int_0^1 du A(u) \left( \frac{10}{(p^2 - m_c^2)^2} + \frac{8m_c^2}{(p^2 - m_c^2)^3} \right) \right] \right]$$

These results are the final results of the QCD part (OPE expression) of the correlator. The next step is to equate Eq. (20) and Eq. (19) (the physical or phenomenological side of the correlation function) and perform the Borel transformation, with respect to the momentum of $D_s^*$ meson ($p^2 \rightarrow M_B^2$), in order to suppress the contributions of higher states and continuum. We obtain the following sum rules for the transition form factors, namely:

$$F^{(D_s^*)}_{V,A}(Q^2) = \frac{m_{D_s^*}}{f_{D_s^*}} e^{\frac{m_{D_s^*}}{M_B}} \tilde{B} \left\{ \int_{(m_c+m_s)^2}^{s_0} ds \rho_{1,2}(s,p^2) \frac{s}{s - Q^2} + \Pi_{1,2}^{c+d+e+f} \right\}$$

where V and A correspond to 1 and 2 in r. h. s., respectively. In Eq. (44), in order to subtract the contributions of the higher states and the continuum, quark-hadron duality assumption is used, i.e. it is assumed that

$$\rho^{higher\ states}(s) = \rho^{OPE}(s) \theta(s - s_0)$$

In the calculations, the following rule for the Borel transformation is used:

$$\tilde{B} \left( \frac{1}{(p^2 - s)^n} \right) = (-1)^n \frac{e^{\frac{m_{D_s^*}}{M_B}}}{\Gamma(n)(M_B^2)^{n-1}}$$
4 QCD sum rules for the form factors induced by electromagnetic penguin

The effective Hamiltonian for the $b \rightarrow s\gamma$ transition can be written as follows:

$$H = -\frac{G_F e}{4\pi^2\sqrt{2}} V_{tb} V_{ts}^* C_7(\mu) \bar{s} \sigma_{\mu\nu} [m_b \frac{1 + \gamma_5}{2} + m_s \frac{1 - \gamma_5}{2}] b F_{\mu\nu}$$  \hspace{1cm} (47)$$

In order to obtain the transition amplitude, we need to calculate the following matrix element:

$$< D_s^* | \bar{s} \sigma_{\mu\nu} (1 \pm \gamma_5) q^\nu b | B_c >$$  \hspace{1cm} (48)$$

At $q^2 = 0$, we can write this matrix element in terms of the two gauge invariant form factors $T_1(0)$ and $T_2(0)$

$$< D_s^* (p, \varepsilon^{(D_s^*)}) | \bar{s} \sigma_{\mu\nu} q^\nu b | B_c (Q) > = i \varepsilon_{\mu\alpha\beta\lambda} \varepsilon^{(D_s^*)\alpha} p^\beta Q^\lambda T_1(0),$$

$$< D_s^* (p, \varepsilon^{(D_s^*)}) | \bar{s} \sigma_{\mu\nu} q^\nu \gamma_5 b | B_c (Q) > = [\left( m_{B_c}^2 - m_{D_s}^2 \right) \varepsilon^{(D_s^*)\mu} - (\varepsilon^{(D_s^*)\nu}(p + Q)_\mu] T_2(0)$$  \hspace{1cm} (49)$$

Using the relation

$$\sigma_{\mu\nu} \gamma_5 = -\frac{i}{2} \varepsilon_{\mu\nu\alpha\beta} \sigma^{\alpha\beta}$$  \hspace{1cm} (50)$$

one can immediately obtain that $T_2(0) = \frac{1}{2} T_1(0)$. Then, we need to calculate only the form factor $T_1(0)$. For this aim, we define the following three point correlation function:

$$\Pi_{\mu\alpha} = - \int d^4x d^4y e^{i(Qx - py)} < 0 \big| T\{ \bar{s}(y) \gamma_\alpha s(y) \bar{s}(0) \sigma_{\mu\nu} q^\nu b(0) \bar{c}(x) i \gamma_5 b(x) \}$$  \hspace{1cm} (51)$$

where $\bar{s} \gamma_\alpha s$ and $\bar{c} \gamma_5 b$ are the interpolating currents of $D_s^*$ and $B_c$ mesons, respectively.

After inserting the hadrons full set with quantum numbers of corresponding interpolating currents (see also [12]), we obtain the following expression
for the phenomenological part of the correlation function:

\[
\Pi_{\mu\alpha} = i\frac{f_{B_c}m_{B_c}^2}{(m_b + m_c)(p^2 - m_{D_s}^2)(Q^2 - m_{B_c}^2)}\varepsilon_{\mu\alpha\beta\lambda\rho} Q^\lambda T_1(0) + \text{excited states.} \quad (52)
\]

For the calculation of the QCD part, we write the Lorentz structure in the above correlator as:

\[
\Pi_{\mu\alpha} = i\varepsilon_{\mu\alpha\beta\lambda\rho} Q^\lambda \Pi(p^2, Q^2) \quad (53)
\]

where

\[
\Pi^{\text{per}}(p^2, Q^2) = -\frac{1}{(2\pi)^2} \int d\tilde{s}d s' \frac{\rho^{\text{per}}(p^2, Q^2)}{(\tilde{s} - Q^2)(s' - p^2)} + \text{subtraction terms} \quad (54)
\]

The standard calculations lead to the following result for the perturbative part (bare-loop diagram):

\[
\rho^{\text{per}}(s', \tilde{s}) = 4N_C [m_b m_c (A_1 + A_2 + I_0) - m_{B_c}^2 A_1 - 2A_3] \quad (55)
\]

where

\[
\begin{align*}
A_1 &= \frac{2I_0}{(\tilde{s} - s')^2} [s'(\tilde{s} + m_c^2 - m_b^2) - \frac{1}{2}(\tilde{s} + s')(s' + m_c^2)], \\
A_2 &= \frac{2I_0}{(\tilde{s} - s')^2} \left[ \frac{1}{2}(\tilde{s} + s')(m_b^2 - m_c^2 - \tilde{s}) + \tilde{s}(s' + m_c^2) \right], \\
A_3 &= I_0 \frac{m_b^2 s' + (m_c^2 - s')(\tilde{s} - s')}{2(\tilde{s} - s')^2}, \\
I_0 &= -\frac{1}{4(\tilde{s} - s')}, \quad (56)
\end{align*}
\]

The integration regions over \( \tilde{s} \) and \( s' \) are obtained from the following inequalities:

\[
\begin{align*}
m_c^2 &\leq s' \leq s_0', \\
\frac{s' m_b^2}{m_c^2 - s'} &\leq \tilde{s} \leq \tilde{s}_0
\end{align*} \quad (57)
\]
The quark condensate terms give zero contribution after applying the double Borel transformation, with respect to the $p^2$ ($p^2 \to M_1^2$) and $Q^2$ ($Q^2 \to M_2^2$). Only the gluon condensates can contribute to the form factor. Fig. 3 shows such type of diagrams. After lengthy calculations for the gluon condensates contribution and equating the phenomenological and QCD parts and applying double Borel transformation with respect to the $p^2$ and $Q^2$, we find the following expression for the form factor $T_1(0)$:

$$T_1(0) = -\frac{1}{(2\pi)^2} \frac{(m_b + m_c)}{f_{B_i} n_{B_i} f_{D_j} m_{D_j}} \left[ \int d\bar{s} ds' \rho_{\text{per}}(s', \bar{s}) e^{-\frac{s}{M_1^2}} e^{-\frac{s'}{M_2^2}} \right]$$

$$+ M_1^2 M_2^2 < \frac{\alpha_s}{\pi} G^2 > C_{G^2}$$

where $C_{G^2}$ is the Wilson coefficient of the gluon condensate and we thus have (see Fig. 3):

$$C_{G^2} = C_{G^2}^6 + C_{G^2}^B + C_{G^2}^C + C_{G^2}^d + C_{G^2}^e + C_{G^2}^f$$

(58)
The explicit expressions for $C_{G^2}^a$ are given below as follows:

$$
C_{G^2}^a = 96m_b \{ [m_c(I_0[1, 3, 1] + m_b^2 I_0[1, 4, 1] + I_1[1, 3, 1] + m_b^2 I_1[1, 4, 1])
+ I_2[1, 3, 1] - m_b^2 I_2[1, 4, 1]] + m_b(I_1[1, 3, 1] + m_b^2 I_1[1, 4, 1])
+ 2I_3[1, 4, 1]) \},
$$

(60)

$$
C_{G^2}^b = 16\{ 2I_0[1, 1, 2] + 2m_b m_c I_0[1, 1, 3] + 2I_0[2, 1, 1] + 3m_b m_c I_0[2, 1, 2]
+ 4m_b^2 I_0[2, 1, 2] + 4m_b m_c^2 I_0[2, 1, 3] + 2m_b m_c I_0[3, 1, 1] + 2m_b^2 I_0[3, 1, 1]
+ 6m_b m_c^3 I_0[3, 1, 2] + 2m_c I_0[3, 1, 2] + 2m_b m_c^2 I_0[3, 1, 3] + 2I_1[1, 1, 2]
+ 2m_b^2 I_1[1, 1, 3] + 2m_b m_c I_1[1, 1, 3] + 2I_1[2, 1, 1] - m_b^2 I_1[2, 1, 2]
+ m_b m_c I_2[2, 1, 2] + 4m_b^2 I_2[2, 1, 2] + 4m_b m_c^2 I_2[2, 1, 3] + 4m_b^3 m_c I_1[2, 1, 3]
+ 2m_c^2 I_3[1, 1, 2] + 2m_b^2 I_3[1, 1, 3] - 6m_b^2 m_c I_3[1, 1, 2] + 4m_b m_c^3 I_1[3, 1, 2]
+ 2m_c^3 I_1[3, 1, 2] - 2m_b^2 m_c^4 I_1[3, 1, 3] + 2m_b m_c^5 I_1[3, 1, 3] + 2m_b m_c I_2[1, 1, 3]
+ I_2[2, 1, 1] + 2m_b m_c I_2[2, 1, 2] + 4m_b m_c^2 I_2[2, 1, 3] - 4m_b^2 I_2[3, 1, 1]
+ 4m_b m_c^3 I_2[3, 1, 2] + 2m_c I_2[3, 1, 2] - 4I_3[1, 1, 3] - 4I_3[2, 1, 2]
+ 8m_b^2 I_3[2, 1, 3] - 8I_3[3, 1, 1] - 16m_b^2 I_3[3, 1, 2] - 4m_b I_3[3, 1, 3]
- 32M_2^2 \left( \frac{d}{dM_2^2} \right) \{ M_2^2 I_0[2, 1, 2] + 2m_b m_c I_0[2, 1, 3] + 2m_b m_c I_0[3, 1, 2]
+ m_c I_0[3, 1, 2] + 2m_b m_c I_0[3, 1, 3] + I_1[2, 1, 2] + 2m_b^2 I_1[2, 1, 3]
+ 2m_b m_c I_1[2, 1, 3] - 2m_b^2 I_1[3, 1, 2] + m_b m_c I_1[3, 1, 2] + m_b^2 I_1[3, 1, 2]
+ 2m_b^2 m_c^2 I_1[3, 1, 2] + 2m_b m_c^2 I_1[3, 1, 3] - I_2[2, 1, 2] + 2m_b m_c I_2[2, 1, 3]
- 2I_2[3, 1, 1] + m_b m_c I_2[3, 1, 2] - m_b^2 I_2[3, 1, 2] + 2m_b m_c^3 I_2[3, 1, 3]
- 4I_3[2, 1, 3] - 6I_3[3, 1, 2] - 4m_b I_3[3, 1, 3] \}
- 32M_2^4 \left( \frac{d^2}{dM_2^2} \right)^2 \{ M_2^2 [m_b^2 I_1[3, 1, 3] + 2I_2[3, 1, 2] - m_b m_c (I_0[3, 1, 3]
+ I_1[3, 1, 3] + I_2[3, 1, 3]) + 2I_3[3, 1, 3] \} } \right) ,
$$

(61)

$$
C_{G^2}^c = 96m_c \{ -(m_b^2 m_c I_1[4, 1, 1]) + m_b (I_0[3, 1, 1] + m_b^2 I_0[4, 1, 1]
+ 2I_3[1, 4, 1]) \} \right),
$$

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\[
+ I_1[3, 1, 1] + m_c^2 I_1[4, 1, 1] + I_2[3, 1, 1] + m_c^2 I_2[4, 1, 1]) - 2m_c I_3[4, 1, 1],
\]
(62)

\[
C_{G2}^d = -32 m_b m_c \{ I_0[2, 1, 2] + I_0[3, 1, 1] + m_c^2 I_0[3, 1, 2] + I_1[2, 1, 2] \\
+ m_c^2 I_1[3, 1, 2] + I_2[2, 1, 2] + m_c^2 I_2[3, 1, 2] - 4I_3[3, 1, 2] \\
- M_2^2 \frac{d}{dM_2} [M_2^2 (I_0[3, 1, 2] + I_1[3, 1, 2] + I_2[3, 1, 2])] \} + 16 \{ m_b^2 I_1[2, 1, 2] \\
+ I_2[2, 1, 1] - m_b m_c (I_0[2, 1, 2] + I_1[2, 1, 2] + I_2[2, 1, 2]) + 4I_3[2, 1, 2] \},
\]
(63)

\[
C_{G2}^e = 16 \{ 2I_0[1, 1, 2] + I_0[1, 2, 1] + 2m_b^2 I_0[1, 2, 2] + m_b m_c I_0[1, 2, 2] \\
+ m_c^2 I_0[2, 1, 2] + m_b m_c I_0[2, 2, 1] + m_c^2 I_0[2, 2, 1] + m_b^2 m_c^2 I_0[2, 2, 2] \\
+ m_b m_c I_1[1, 2, 2] + I_1[1, 1, 2] + I_1[1, 2, 1] + m_b^2 I_1[1, 2, 2] \\
+ m_b m_c I_1[1, 2, 1] - I_2[2, 1, 2] + 2m_b m_c I_1[2, 1, 2] - m_b^2 I_1[2, 2, 1] \\
+ m_b^2 I_1[2, 2, 1] - 2m_b^3 m_c I_1[2, 2, 2] + m_b m_c^3 I_1[2, 2, 2] + I_2[1, 1, 2] \\
+ I_2[1, 2, 1] + m_b^2 I_2[1, 2, 1] + m_b m_c I_2[1, 2, 1] - 2m_b m_c I_2[2, 2, 1] \\
+ m_b^2 I_2[2, 2, 1] + m_b m_c^3 I_2[2, 2, 2] - 2I_3[2, 1, 2] - 4I_3[2, 2, 1] \\
- 2m_b^2 I_3[2, 2, 2] - 4m_b m_c I_3[2, 2, 2] \} \\
- 16M_1^2 \frac{d}{dM_1} \{ M_1^2 I_3[2, 2, 1] + 2I_3[2, 2, 2] \} \\
+ 16M_2^2 \frac{d}{dM_2} \{ M_2^2 [-m_b m_c I_0[2, 2, 2] + I_1[2, 1, 2] + m_c^2 I_1[2, 2, 2] \\
- m_b m_c I_1[2, 2, 2] + I_2[2, 2, 1] + I_2[2, 2, 2] + m_b^2 I_2[2, 2, 2] \\
- m_b m_c I_2[2, 2, 2] + 2I_3[2, 2, 2]] \} + 16 \{ I_1[1, 1, 2] + m_b^2 I_1[1, 2, 2] \\
+ I_2[1, 2, 1] - m_b m_c (I_0[1, 1, 2] + I_1[1, 2, 2] + I_2[1, 2, 2]) \\
+ 2I_3[1, 2, 2] \},
\]
(64)

\[
C_{G2}^f = 16 \{ 2I_0[1, 2, 1] + 2I_0[2, 1, 1] + 2m_b^2 I_0[2, 2, 1]
\]
where \( U \) is defined by

\[
U_0(i,j) = B_{-1}^{-1} - B_0 - B_1 y \quad \text{and} \quad U_1(i,j) = B_{-1}^{-1} - B_0 - B_2 y \quad \text{as well as} \quad U_2(i,j) = B_{-1}^{-1} - B_0 - B_3 y
\]

and the for explicit form of the \( I_i[a, b, c] \), we obtain:

\[
\begin{align*}
I_0[a, b, c] &= \frac{(-1)^{a+b+c}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)}(M_1^2)^{2-a-b}(M_2^2)^{2-a-c} \times U_0(a + b + c - 4, 1 - c - b), \\
I_1[a, b, c] &= \frac{(-1)^{a+b+c+1}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)}(M_1^2)^{2-a-b}(M_2^2)^{3-a-c} \times U_0(a + b + c - 5, 1 - c - b), \\
I_2[a, b, c] &= \frac{(-1)^{a+b+c+1}}{32\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)}(M_1^2)^{3-a-b}(M_2^2)^{3-a-c} \times U_0(a + b + c - 6, 2 - c - b)
\end{align*}
\]

The function \( U_0(i, j) \), also, is given by:

\[
U_0(i, j) = \int_0^\infty dy (y + M_1^2 + M_2^2)^i y^j \exp\left[ -\frac{B_{-1}}{y} - B_0 - B_1 y \right],
\]

where

\[
\begin{align*}
B_{-1} &= \frac{m_h^2}{M_1^2} \left[ M_1^2 + M_2^2 \right], \\
B_0 &= \frac{1}{M_1^2 M_2^2} \left[ M_1^2 m_c^2 + M_2^2 (m_c^2 + m_b^2) \right], \\
B_1 &= \frac{m_c^2}{M_1^2 M_2^2}
\end{align*}
\]
5 Numerical analysis

In this section, we present our numerical analysis for the form factors. From the sum rule expressions of these form factors, we see that the condensates, leptonic decay constants of $B_C$ and $D^*_s$ mesons, continuum thresholds $s_0$, $\tilde{s}_0$ and $s'_0$, the relevant parameters in photon distribution amplitudes (DA’s) and Borel parameters $M^2_B$, $M^2_1$ and $M^2_2$ are the main input parameters. In further numerical analysis, we choose the value of the condensates at a fixed renormalization scale of about 1 GeV. The values of the condensates are\[22\]:

\[
\langle \bar{\psi}\psi \mid \mu = 1\text{ GeV} \rangle = -(240 \pm 10\text{ MeV})^3,\quad \langle \bar{s}s \rangle = (0.8 \pm 0.2) < \bar{\psi}\psi >
\]

$m^2_0 = 0.8\text{ GeV}^2$. The quark and mesons masses are taken to be $m_c(\mu = m_c) = 1.275 \pm 0.015\text{ GeV}$, $m_s(1\text{ GeV}) \approx 142\text{ MeV}$\[23\], $m_b = (4.7 \pm 0.1)\text{ GeV}$\[22\], $m_{D^*_s} = 2.112\text{ GeV}$ and $m_{B_C} = 6.258\text{ GeV}$. For the values of the leptonic decay constants of $B_C$ and $D^*_s$ mesons, we use the results obtained from the two-point QCD analysis: $f_{B_C} = 0.35\text{GeV}$\[26, 27, 28\] and $f_{D^*_s} = 266 \pm 32\text{ MeV}$\[24\]. The relevant parameters in photon distribution amplitudes (DA’s) are taken to be $\chi = 3.15 \pm 0.3\text{GeV}^{-2}$, $\kappa = 0.2$, $\kappa^+ = 0$, $\zeta_1 = 0.4$, $\zeta_1^+ = 0$, $\zeta_2 = 0.3$, $\zeta_2^+ = 0$, $f_{3\gamma} = -(4 \pm 2) \times 10^{-3}\text{GeV}^2$, $w^{A}_\gamma = -2.1 \pm 1.0$, $w^{V}_\gamma = 3.8 \pm 1.8$\[20, 21, 25\]. The threshold parameters are also determined from the two-point QCD sum rules: $s_0 = 8\text{ GeV}^2$, $\tilde{s}_0 = 45\text{ GeV}^2$, $s'_0 = 8\text{ GeV}^2$\[2, 24, 29\]. The Borel parameters $M^2_B$, $M^2_1$ and $M^2_2$ are auxiliary quantities and, therefore the results of physical quantities should not depend on them. In the QCD sum rule method, OPE is truncated at finite order, leaving a residual dependence on the Borel parameters. For this reason, the working regions for the Borel parameters should be chosen such that in these regions the form factors are practically independent of them. The working regions for the Borel parameters $M^2_B$, $M^2_1$ and $M^2_2$ can be determined on the condition that, on the one side, the continuum contribution should be
small, and on the other side, the contribution of the operator with the highest dimension should be small. As a result of the above-mentioned requirements, the working regions for this transition are obtained to be:

\[4 \text{ GeV}^2 < M_B^2 < 10 \text{ GeV}^2, \quad 10 \text{ GeV}^2 \leq M_1^2 \leq 25 \text{ GeV}^2, \quad 4 \text{ GeV}^2 \leq M_2^2 \leq 10 \text{ GeV}^2.\] (69)

Now, by calculating the total decay widths and taking \( |V_{cs}| = 0.957 \pm 0.017 \), \( |V_{cb}| = 0.0416 \pm 0.0006 \), \( |V_{tb}| = 0.77^{+0.18}_{-0.24} \), \( |V_{ts}| = (40.6 \pm 2.7) \times 10^{-3} \) [30], \( \langle \frac{2\alpha}{\pi} G^2 \rangle = 0.012 \text{GeV}^4 \) [29], \( C_7(\mu = m_c) = -0.0068 - 0.02i \) [13] and \( \tau_{B_c} = 0.52 \times 10^{-12} \text{s} \) [31], we obtain the numerical results of the electromagnetic penguin (EP), weak annihilation (WA) and total branching ratios for this decay as follows:

\[B^{(EP)}(B_c \to D_s^*\gamma) = 3.468 \times 10^{-6}\]
\[B^{(WA)}(B_c \to D_s^*\gamma) = 1.557 \times 10^{-5}\]
\[B^{(Total)}(B_c \to D_s^*\gamma) = 2.462 \times 10^{-5}\] (70)

From the above results, we see that the weak annihilation contribution to the total branching ratio is about 4.48 times greater than that of the electromagnetic penguin diagram. Here, it is observed that the difference between the total branching ratio with sum of the weak annihilation and electromagnetic penguin branching ratios comes from the cross term in total decay width. Also our result for the total branching ratio shows that the \( B_c \to D_s^*\gamma \) decay can be measured at LHC.

Now, we compare our results of the \( B_c \to D_s^*\gamma \) to the results of the perturbative QCD [15], relativistic independent quark model [16], pertubative QCD in standard model (SM (PQCD)) [17], multi scale walking techni-
color (MWTCM) and topcolor assisted MWTCM (TAMWTCM) for
for $\tau_{B_c} = 0.52 \times 10^{-12} s$ as shown in Table (1).
magnitudes of the input parameters, getting from different references; e.g., we use $m_c(\mu = m_c) = 1.275 \pm 0.015$ GeV for the $c$ quark masses while the authors of [17] use $m_c = 1.6$ GeV and also to the nature of the methods and their accuracy.

In this step, for the analysis of $B_c \rightarrow D^*\gamma$, in the entire calculations we replace the $s$ quark with the $d$ quark. Making $m_{D^*_s} \rightarrow m_{D^*_d}$, $f_{D^*_s} \rightarrow f_{D^*_d}$, $V_{td} \rightarrow V_{cd}$ changes and taking $m_{D^*_d} = 2.010$ GeV, $f_{D^*_d} = 0.23 \pm 0.02$ GeV [32], $V_{cd} = 0.230 \pm 0.011$, $V_{td} = (7.4 \pm 0.8) \times 10^{-3}$ [30] and $m_d = 5$ MeV we obtain the numerical results as below:

$$B^{(EP)}(B_c \rightarrow D^*\gamma) = 1.151 \times 10^{-7}$$
$$B^{(WA)}(B_c \rightarrow D^*\gamma) = 2.162 \times 10^{-6}$$
$$B^{(Total)}(B_c \rightarrow D^*\gamma) = 2.786 \times 10^{-6} \quad (71)$$

These results also enhance the importance of the weak annihilation contribution to the total branching ratio in comparing with the electromagnetic penguin diagram ones for the $B_c \rightarrow D^*\gamma$. Finally, we compare our results to the relativistic independent quark model (RIQM) [16] for $\tau_{B_c} = 0.52 \times 10^{-12}$ s in Table (2).

From the Table 2, it is also seen a good agreement in the order of magnitude

|                   | $B^{EP}(B_c \rightarrow D^*\gamma)$ | $B^{WA}(B_c \rightarrow D^*\gamma)$ | $B^{Total}(B_c \rightarrow D^*\gamma)$ |
|-------------------|-------------------------------------|-------------------------------------|----------------------------------------|
| Present study     | $1.151 \times 10^{-7}$              | $2.162 \times 10^{-6}$              | $2.786 \times 10^{-6}$                 |
| RIQM [16]         | $5.70 \times 10^{-7}$               | $1.33 \times 10^{-6}$               | $3.64 \times 10^{-6}$                 |

Table 2: Comparison of the branching ratio for $B_c \rightarrow D^*\gamma$ decay based on the $\tau_{B_c} = 0.52 \times 10^{-12}$ s.

between the present study and the relativistic independent quark model.
In conclusion, the present study concentrated on the radiative $B_c \to D_s^*\gamma$ and $B_c \to D^*\gamma$ decays in the framework of QCD sum rules. The form factors responsible for these decays were calculated. The branching ratio for this decays were estimated. The results show that the $B_c \to D_s^*\gamma$ case can be measured at LHC in the near future.

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