Formulation of a constrained system in terms of extended Lagrangian and its local symmetries

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Abstract
It is shown that an arbitrary singular Lagrangian theory (with first and second class constraints up to $N$-th stage presented in the Hamiltonian formulation) can be reformulated as a theory with at most third-stage constraints. The corresponding Lagrangian $\tilde{L}$ can be obtained by pure algebraic methods, its manifest form in terms of quantities of the initial formulation is find. Local symmetries of $\tilde{L}$ are obtained in closed form. All the first class constraints of the initial Lagrangian turn out to be gauge symmetry generators for $\tilde{L}$.

1 Introduction

Conventional way to describe a relativistic theory is to formulate it in terms of a singular Lagrangian. In turn, analysis of the singular theory can be carried out in a Hamiltonian formalism. In this framework, possible motions of the singular system are restricted to lie on some surface of a phase space. Algebraic equations of the surface (constraints) can be revealed in the course of a Dirac procedure, the latter in general case requires a number of stages. According to the order of appearance, the constraints are called primary, second-stage, ..., $N$-th stage constraints. All the constraints, beside the primary ones are called higher-stage constraints below. Whenever are appeared, the higher-stage constraints represent perceptible problems for analysis of the theory. In particular, search for
local symmetries of the Lagrangian action, which is main subject of the present work, turns out to be rather nontrivial issue in a general case [1-5]. So, it may be reasonable to adopt a different approach to the problem. Namely, instead of looking for properties of the initial Lagrangian $L$ (provided all its constraints are known), we work out an equivalent Lagrangian $\tilde{L}$, the latter implies more transparent structure of constraints (in fact, all the higher-stage constraints of the original formulation enter into $\tilde{L}$ in a manifest form, see the last term in Eq. (17) below). It allows one to find infinitesimal local symmetries of the formulation in closed form, in terms of the constraints of the initial formulation. For some particular examples, such a kind possibility has been tested in the recent work [6]. Here we develop the formalism for an arbitrary theory, with first and second class constraints up to $N$-th stage presented in the original formulation $L$. $\tilde{L}$ is called an extended Lagrangian, since the corresponding complete Hamiltonian turns out to be closely related with an extended Hamiltonian of the original formulation\footnote{By definition, the extended Hamiltonian is obtained from the complete one by addition of the higher-stage constraints with corresponding Lagrangian multipliers. It is known \cite{8} that the two formulations are equivalent.}. So, in this work we also clarify a relation among the complete and the extended Hamiltonian formulations of a given theory.

The work is organized as follows. With the aim to fix our notations, we outline in Section 2 the Hamiltonization procedure for an arbitrary singular Lagrangian theory. In Section 3 we formulate pure algebraic recipe for construction of the extended Lagrangian. All the higher-stage constraints of $L$ appear as secondary constraints for $\tilde{L}$. Besides, we demonstrate that $\tilde{L}$ is a theory with at most third-stage constraints. Then it is proved that $\tilde{L}$ and $L$ are equivalent\footnote{Popular physical theories usually do not involve more than third-stage constraints (example of a theory with third-stage constraints is the membrane, in the formulation with world-volume metric). Our result can be considered as an explanation of this fact.}. Since the original and the reconstructed formulations are equivalent, it is matter of convenience to use one or another of them for description of a theory under investigation\footnote{Let us point out that the higher stage constraints usually appear in a covariant form. So one expects manifest covariance of the extended formulation.}. In Section 4 we demonstrate one of advantages of the extended formulation by finding its complete irreducible set of local symmetries. Properties of the extended formulation for some particular cases of original gauge algebra are discussed in the Conclusion.
2 Initial formulation with higher-stage first and second class constraints

Let $L(q^A, \dot{q}^B)$ be Lagrangian of singular theory: $\text{rank} \frac{\partial^2 L}{\partial q^A \partial q^B} = [i] < [A]$, defined on configuration space $q^A, A = 1, 2, \ldots, [A]$. From the beginning, it is convenient to rearrange the initial variables in such a way that the rank minor is placed in the upper left corner. Then one has $q^A = (q^i, q^\alpha), i = 1, 2, \ldots, [i], \alpha = 1, 2, \ldots, [\alpha] = [A] - [i]$, where $\det \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \neq 0$.

Let us construct the Hamiltonian formulation for the theory. To fix our notations, we carry out the Hamiltonization procedure in some details. One introduces conjugate momenta according to the equations $p_i = \frac{\partial L}{\partial \dot{q}^i}, p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}$. They are considered as algebraic equations for determining velocities $\dot{q}^A$. According to the rank condition, the first $[i]$ equations can be resolved with respect to $\dot{q}^i$, let us denote the solution as

$$\dot{q}^i = v^i(q^A, p_j, \dot{q}^\alpha). \quad (1)$$

It can be substituted into remaining $[\alpha]$ equations for the momenta. By construction, the resulting expressions do not depend on $\dot{q}^A$ and are called primary constraints $\Phi_\alpha(q^A, p^A)$ of the Hamiltonian formulation. One finds

$$\Phi_\alpha \equiv p_\alpha - f_\alpha(q^A, p_j) = 0, \quad (2)$$

where

$$f_\alpha(q^A, p_j) \equiv \frac{\partial L}{\partial \dot{q}^\alpha} \bigg|_{\dot{q}^i = v^i(q^A, p_j, \dot{q}^\alpha)}. \quad (3)$$

The original equations for the momenta are thus equivalent to the system (1), (2). By construction, one has the identities

$$\frac{\partial L(q, \dot{q})}{\partial \dot{q}^i} \bigg|_{\dot{q}^i = v^i(q^A, p_j, \dot{q}^\alpha)} \equiv p_i, \quad v^i(q^A, p_j, \dot{q}^\alpha) \bigg|_{p_j = \frac{\partial L}{\partial p_j}} \equiv \dot{q}^i. \quad (4)$$

Next step of the Hamiltonian procedure is to introduce an extended phase space parameterized by the coordinates $q^A, p_A, \dot{q}^\alpha$, and to define a complete Hamiltonian $\tilde{H}$ according to the rule

$$\tilde{H}(q^A, p_A, \dot{q}^\alpha) = H_0(q^A, p_j) + v^\alpha \Phi_\alpha(q^A, p_B), \quad (5)$$
where
\[
H_0 = (\dot{p}_i \dot{q}^i - L + \dot{q}^\alpha \frac{\partial L}{\partial \dot{q}^\alpha})_{\dot{q}^i \rightarrow v^i(q^A, p_j, \dot{q}^\alpha)}.
\] (6)

Then the following system of equations on this space
\[
\dot{q}^A = \{q^A, H\}, \quad \dot{p}_A = \{p_A, H\}, \quad \Phi_\alpha(q^A, p_B) = 0,
\] (7)
is equivalent to the Lagrangian equations following from \(L\), see [8]. Here \(\{,\}\) denotes the Poisson bracket.

From Eq. (7) it follows that all the solutions are confined to lie on a surface of the extended phase space defined by the algebraic equations \(\Phi_\alpha = 0\). It may happen, that the system (7) contains in reality more then \([\alpha]\) algebraic equations. Actually, derivative of the primary constraints with respect to time implies, as algebraic consequences of the system (7), the so called second stage equations:
\[
\{\Phi_\alpha, H\} \equiv \{\Phi_\alpha, \Phi_\beta\} v^\beta + \{\Phi_\alpha, H_0\} = 0.
\] They can be added to Eq. (7), which gives an equivalent system. Let on-shell one has \(\text{rank}\{\Phi_\alpha, \Phi_\beta\} = [\alpha'] \leq [\alpha]\). Then \([\alpha']\) equations of the second-stage system can be used to represent some \(v^{\alpha'}\) through other variables. It can be substituted into the remaining \([\alpha''] = [\alpha] - [\alpha']\) equations, the resulting expressions do not contain \(v^{\alpha}\) at all. Thus the second-stage system can be presented in the equivalent form
\[
v^{\alpha'} = v^{\alpha'}(q^A, p_j, v^{\alpha''}), \quad T_{\alpha''}(q^A, p_j) = 0.
\] (8)

Functionally independent equations among \(T_{\alpha''} = 0\), if any, represent secondary Dirac constraints. Thus all the solutions of the system (7) are confined to the surface defined by \(\Phi_\alpha = 0\) and by the equations (8).

The secondary constraints may imply third-stage constraints, and so on. We suppose that the theory has constraints up to \(N\)-th stage, \(N \geq 2\). Higher stage constraints are denoted by \(T_{\alpha}(q^A, p_j) = 0\). Then the complete constraint system is \(G_I \equiv (\Phi_\alpha, T_{\alpha})\), while all the solutions of Eq. (7) are confined to the surface defined by the equations \(\Phi_\alpha = 0\) as well as by
\[
\{G_I, H\} = 0.
\] (9)

\footnote{It is known [8], that the procedure reveals all the algebraic equations presented in the system (7). Besides, surface of solutions of Eq. (7) coincides with the surface \(\Phi_\alpha = 0\), \(\{G_I, H\} = 0\).}
By construction, after substitution of the velocities determined during the Dirac procedure, these equations vanish on the complete constraint surface $G_J$.

Suppose that $\{G_I, G_J\} = \triangle_{IJ}(q^A, p_j)$, where $\text{rank}\triangle_{IJ}|_{G_I=0} = [I_2] < [I]$, that is both first and second class constraints are presented. It will be convenient to separate them. According to the rank condition, there exist $[I_1] = I - [I_2]$ independent null-vectors $\vec{K}_{I_1}$ of the matrix $\Delta$ on the surface $G_I = 0$, with the components $K_{I_1}^J(q^A, p_j)$. Then bracket of the constraints $G_{I_1} \equiv K_{I_1}^J G_J$ with any $G_I$ vanishes, hence $G_{I_1}$ represent first class subset. Let $K_{I_2}^J(q^A, p_j)$ be any completion of the set $K_{I_1}^J$ up to a basis of $[I]$-dimensional vector space. By construction, the matrix

$$K_I^J = \begin{pmatrix} K_{I_1}^J \\ K_{I_2}^J \end{pmatrix},$$

is invertible. So the system $\tilde{G}_I \equiv (G_{I_1} \equiv K_{I_1}^J G_J, G_{I_2} \equiv K_{I_2}^J G_J)$ is equivalent to the initial system of constraints $G_I$. The constraints $G_{I_2}$ form the second class subset of the complete set. In arbitrary theory, the constraints obey the following Poisson bracket algebra:

$$\{\tilde{G}_I, \tilde{G}_J\} = \triangle_{IJ}(q^A, p_B), \quad \{G_{I_1}, G_J\} = c_{I_1}^K(q^A, p_B)G_K, \quad \{G_{I_1}, H_0\} = b_{I_1}^J(q^A, p_B)G_J, \quad \{G_{I_2}, G_{J_2}\} = \triangle_{I_2J_2}(q^A, p_B),$$

where

$$\text{rank}\triangle_{IJ}|_{G_I=0} = [I_2], \quad \text{det}\triangle_{I_2J_2}|_{G_I=0} \neq 0.$$  

3 Construction of the extended Lagrangian and its properties

Starting from the theory described above, we construct here a Lagrangian $\tilde{L}(q^A, \dot{q}^A, s^a)$ defined on the configuration space $q^A, s^a$. In the Hamiltonian formalism, it leads to the Hamiltonian $H_0 + s^aT_a$, and to the primary constraints $\Phi_a = 0$, $\pi_a = 0$, where $\pi_a$ represent conjugate momenta for $s^a$. Due to special form of the Hamiltonian, preservation in time of the primary constraints implies, that all the

\[\text{Let us stress once again, that in our formulation the variables } s^a \text{ represent a part of the configuration-space variables.}\]
higher stage constraints \( T_a \) of initial theory appear as secondary constraints for the theory \( \tilde{\mathbb{L}} \). Moreover, the Dirac procedure stops on third stage: \( \tilde{\mathbb{L}} \) turns out to be a theory with at most third-stage constraints presented. Besides, we demonstrate that the formulations \( \mathbb{L} \) and \( \tilde{\mathbb{L}} \) are equivalent.

To construct the extended Lagrangian for \( \mathbb{L} \), let us consider the following equations for the variables \( q^A, \omega_j, s^a \):

\[
\dot{q}^i - v^i(q^A, \omega_j, \dot{q}^\alpha) - s^a \frac{\partial T_a(q^A, \omega_j)}{\partial \omega_i} = 0. \tag{13}
\]

Here the functions \( v^i(q^A, \omega_j, \dot{q}^\alpha), \quad T_a(q^A, \omega_j) \) are taken from the initial formulation. The equations can be resolved algebraically with respect to \( \omega_i \) in a neighborhood of the point \( s^a = 0 \). Actually, Eq. (13) with \( s^a = 0 \) coincides with Eq. (1) of the initial formulation, the latter can be resolved, see Eq. (4). Hence \( \det \frac{\partial (\text{Eq. (13)})}{\partial \omega_i} \neq 0 \) at the point \( s^a = 0 \). Then the same is true in some vicinity of this point, and Eq. (13) thus can be resolved. Let us denote the solution as

\[
\omega_i = \omega_i(q^A, \dot{q}^A, s^a). \tag{14}
\]

By construction, one has the identities

\[
\omega_i(q, \dot{q}, s) \bigg|_{\dot{q}^i \to v^i(q^A, \omega_j, \dot{q}^\alpha) + s^a \frac{\partial T_a(q^A, \omega_j)}{\partial \omega_i}} \equiv \omega_i,
\]

\[
\left( v^i(q^A, \omega_j, \dot{q}^\alpha) + s^a \frac{\partial T_a(q^A, \omega_j)}{\partial \omega_i} \right) \bigg|_{\omega_i(q, \dot{q}, s)} \equiv \dot{q}^i, \tag{15}
\]

as well as the following property of the function \( \omega \)

\[
\omega_i(q^A, \dot{q}^A, s^a) \bigg|_{s^a = 0} = \frac{\partial L}{\partial \dot{q}^i}. \tag{16}
\]

Now, the extended Lagrangian for \( \mathbb{L} \) is defined according to the expression

\[
\tilde{\mathbb{L}}(q^A, \dot{q}^A, s^a) = L(q^A, v^i(q^A, \omega_j, \dot{q}^\alpha), \dot{q}^\alpha) + \omega_i(\dot{q}^i - v^i(q^A, \omega_j, \dot{q}^\alpha)) - s^a T_a(q^A, \omega_j), \tag{17}
\]

where the functions \( v^i, \omega_i \) are given by Eqs. (1), (14). As compare with the initial Lagrangian, \( \tilde{\mathbb{L}} \) involves the new variables \( s^a \), in a number equal to the number of higher stage constraints \( T_a \). Let us enumerate some properties of \( \tilde{\mathbb{L}} \)

\[
\tilde{\mathbb{L}}(s^a = 0) = L, \tag{18}
\]
\[
\frac{\partial \tilde{L}}{\partial \omega} \bigg|_{\omega(q, \dot{q}, s)} = 0, \quad (19)
\]
\[
\frac{\partial \tilde{L}}{\partial \dot{q}^\alpha} = \frac{\partial L(q^A, v^i, \dot{q}^\alpha)}{\partial \dot{q}^\alpha} \bigg|_{v^i(q, \dot{q}, \dot{q}^\alpha)} \bigg|_{\omega(q, \dot{q}, s)} = f_\alpha(q^A, \omega_j(q, \dot{q}, s)). \quad (20)
\]

Eq. (18) follows from Eqs. (16), (4). Eq. (19) is a consequence of the identities (4), (15). Eq. (19) will be crucial for discussion of local symmetries in the next section. At last, Eq. (20) is a consequence of Eqs. (19), (4).

Following to the standard prescription [7, 8], let us construct the Hamiltonian formulation for \(\tilde{L}\). By using of Eqs. (19), (20), one finds the conjugate momenta for \(q^A, s^a\)
\[
\tilde{p}_i = \frac{\partial \tilde{L}}{\partial \dot{q}^i} = \omega_i(q^A, \dot{q}^A, s^a), \quad \tilde{p}_\alpha = \frac{\partial \tilde{L}}{\partial \dot{q}^\alpha} = f_\alpha(q^A, \omega_j), \quad \pi_a = \frac{\partial \tilde{L}}{\partial \dot{s}^a} = 0. \quad (21)
\]
Due to the identities (15), these expressions can be rewritten in the equivalent form
\[
\dot{q}^i = v^i(q^A, \tilde{p}_j, \dot{q}^\alpha) + s^a \frac{\partial T_a(q^A, \tilde{p}_j)}{\partial \tilde{p}_i}, \quad \tilde{p}_\alpha - f_\alpha(q^A, \tilde{p}_j) = 0, \quad \pi_a = 0. \quad (22)
\]
Thus the velocities \(\dot{q}^i\) have been determined. There are presented trivial constraints \(\pi_a = 0\), in a number equal to the number of all the higher stage constraints of the initial formulation, as well as all the primary constraints \(\Phi_\alpha = 0\) of the initial theory. Using the definition (6), one obtains the Hamiltonian \(\tilde{H}_0 = H_0 + s^a T_a\), so the complete Hamiltonian for \(\tilde{L}\) is given by the expression
\[
\tilde{H} = H_0(q^A, \tilde{p}_j) + s^a T_a(q^A, \tilde{p}_j) + v^\alpha \Phi_\alpha(q^A, \tilde{p}_B) + v^a \pi_a, \quad (23)
\]
where \(v^\alpha, v^a\) are multipliers corresponding to the primary constraints. Note that, if one discards the constraints \(\pi_a = 0\), \(\tilde{H}\) coincides with the extended Hamiltonian for \(L\) after identification of configuration space variables \(s^a\) with the Lagrangian multipliers for higher stage constraints of the original formulation.

Further, preservation in time of the primary constraints \(\pi_a\) implies the equations \(T_a = 0\). Hence all the higher stage constraints
of the initial formulation appear now as the secondary constraints. Preservation in time of the primary constraints $\Phi_\alpha$ leads to the equations 
\[
\{\Phi_\alpha, \tilde{H}\} = \{\Phi_\alpha, H_0\} + \{\Phi_\alpha, \Phi_\beta\} v^\beta + \{\Phi_\alpha, T_b\} s^b = 0.
\]
In turn, preservation of the secondary constraints $T_a$ leads to the similar equations 
\[
\{T_a, \tilde{H}\} = \{T_a, H_0\} + \{T_a, \Phi_\beta\} v^\beta + \{T_a, T_b\} s^b = 0.
\]

To continue the analysis, it is convenient to unify them as follows:
\[
\{G_I, H_0\} + \{G_I, G_J\} S^J = 0.
\]

Here $G_I$ are all the constraints of the initial formulation and it was denoted $S^J \equiv (v^\alpha, s^a)$. Using the matrix (10), the system (24) can be rewritten in the equivalent form
\[
\{G_{I_1}, H_0\} + O(G_I) = 0,
\]
\[
\{G_{I_2}, H_0\} + \{G_{I_2}, G_J\} S^J = O(G_I).
\]

Eq. (25) does not contain any new information, since the first class constraints commute with the Hamiltonian, see Eq. (11). So, let us analyze the system (26). First, one notes that due to the rank condition $\text{rank}\{G_{I_2}, G_J\}_{G_I} = [I_2] = \text{max}$, exactly $[I_2]$ variables among $S^I$ can be found from the equations. According to the Dirac prescription, one needs to find maximal number of $v^\alpha$. To make this, let us restore $v$-dependence in Eq. (26): 
\[
\{G_{I_2}, \Phi_\alpha\} v^\alpha + \{G_{I_2}, H_0\} + \{G_{I_2}, T_b\} s^b = 0.
\]
Since the matrix $\{G_{I_2}, \Phi_\alpha\}$ is the same as in the initial formulation, from these equations one determines some group of variables $v^{\alpha_2}$ through the remaining variables $v^{\alpha_1}$, where $[\alpha_2]$ is number of primary second-class constraints among $\Phi_\alpha$. After substitution of the result into the remaining equations of the system (26), the latter acquires the form
\[
v^{\alpha_2} = v^{\alpha_2}(q, p, s^a, v^{\alpha_1}), \quad Q_{a_2b}(q, p)s^b + P_{a_2}(q, p) = 0,
\]
where $[a_2]$ is the number of higher-stage second class constraints of the initial theory. It must be $P \approx 0$, since for $s^b = 0$ the system (26) is a subsystem of (9), but the latter vanish after substitution of the multipliers determined during the procedure, see discussion after Eq. (9). Besides, one notes that $\text{rank} Q = [a_2] = \text{max}$. Actually, suppose that $\text{rank} Q = [a'] < [a_2]$. Then from Eq. (26) only $[\alpha_2] + [a'] < [I_2]$ variables among $S^I$ can be determined, in contradiction with the conclusion made before. In resume, the system (24) for determining
the second-stage and the third-stage constraints and multipliers is equivalent to the following one

$$v^{a_2} = v^{a_2}(q, p, s^{a_1}, v^{a_1}),$$  \hspace{1cm} (28)$$

$$s^{a_2} = Q^{a_2 b_1}(q, p)s^{b_1}. \hspace{1cm} (29)$$

Conservation in time of the constraints (29) does not produce new constraints, giving equations for determining the multipliers

$$v^{a_2} = \{Q^{a_2 b_1}(q, p)s^{b_1}, \tilde{H}\}, \hspace{1cm} (30)$$

The Dirac procedure for $\tilde{L}$ stops on this stage. All the constraints of the theory have been revealed after completing the third stage.

Now we are ready to compare the theories $\tilde{L}$ and $L$. Dynamics of the theory $\tilde{L}$ is governed by the Hamiltonian equations

$$\dot{q}^A = \{q^A, H\} + s^a \{q^A, T_a\}, \quad \dot{p}_A = \{p_A, H\} + s^a \{p_A, T_a\}, \quad \dot{s}^a = v^a, \quad \dot{\pi}_{a_1} = 0, \quad \dot{\pi}_{a_2} = 0, \hspace{1cm} (31)$$

as well as by the constraints

$$\Phi_{a_1} = 0, \quad T_a = 0, \hspace{1cm} (32)$$

$$\pi_{a_1} = 0, \hspace{1cm} (33)$$

$$\pi_{a_2} = 0, \quad s^{a_2} = Q^{a_2 b_1}(q, p)s^{b_1}. \hspace{1cm} (34)$$

Here $H$ is complete Hamiltonian of the initial theory (5), and the Poisson bracket is defined on the phase space $q^A, s^a, p_A, \pi_a$. The constraints $\pi_{a_1} = 0$ can be replaced by the combinations $\pi_{a_1} - \pi_{a_2}Q^{a_2 a_1}(q, p) = 0$, the latter represent first class subset. Let us make partial fixation of a gauge by imposing the equations $s^{a_1} = 0$ as a gauge conditions for the subset. Then $(s^a, \pi_a)$-sector of the theory disappears, whereas the equations (31), (32) coincide exactly with those of the initial theory $\tilde{L}$. Let us remind that $\tilde{L}$ has been constructed in some vicinity of the point $s^a = 0$. The gauge $s^{a_1} = 0$ implies $s^a = 0$ due to the homogeneity of Eq. (29). It guarantees a self consistency of the construction. Thus $\tilde{L}$ represents one of the gauges for $\tilde{L}$, which proves equivalence of the two formulations.

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6In more rigorous treatment, one writes Dirac bracket corresponding to the equations $\pi_{a_1} - \pi_{a_2}Q^{a_2 a_1} = 0$, s^{a_1} = 0, and to the second class constraints (34). After that, the equations used in construction of the Dirac bracket can be used as strong equalities. For the case, they reduce to the equations $s^a = 0, \pi_a = 0$. For the remaining phase-space variables $q^A, p_A$, the Dirac bracket coincides with the Poisson one.
4 Local symmetries of the extended Lagrangian

Since the initial Lagrangian is one of gauges for $\tilde{L}$, physical system under consideration can be equally analyzed by using of the extended Lagrangian. In contrast to $L$, the extended Lagrangian contains the higher-stage constraints $T_a$ of $L$ in the manifest form, see Eq. (17). Moreover, while $T_a$ appear as the secondary constraints of the formulation $\tilde{L}$, they are also presented in the manifest form in the complete Hamiltonian $\tilde{H}$. Here we demonstrate one of consequences of this property: all the infinitesimal local symmetries of $\tilde{L}$ can be found in closed form.

According to the analysis made in the previous section, the primary constraints of the extended formulation are $\Phi_\alpha = 0, \pi_a = 0$. Among $\Phi_\alpha = 0$ there are presented first class constraints, in a number equal to the number of primary first class constraint of $L$. Among $\pi_a = 0$, we have find the first class constraints \( \pi_a - \pi_{a_2} Q_{a_1}(q, p) = 0 \), in a number equal to the number of all the higher-stage first class constraints of $L$. Thus the number of primary first class constraints of $\tilde{L}$ coincide with the number \([I_1]\) of all the first class constraints of $L$. Hence one expects \([I_1]\) local symmetries presented in the formulation $\tilde{L}$. Now we demonstrate that the action $\tilde{S}_L = \int d\tau \tilde{L}$ is invariant (modulo to a surface term) under the following infinitesimal transformations:

$$\delta_{I_1} q^A = \epsilon^{I_1} \{ q^A, G_{I_1}(q^A, p_B) \} \bigg|_{p_i \rightarrow \omega_i(q, \dot{q}, s), p_a \rightarrow f_a(q, \omega(q, \dot{q}, s))},$$

$$\delta_{I_1} s^a = \left( \epsilon^{I_1} K_{I_1}^a + \epsilon^{I_1} \left( b_{I_1}^a + s^b c_{I_1} b^a + \dot{q}^\beta c_{I_1} a^\beta \right) \right) \bigg|_{p_i \rightarrow \omega_i(q, \dot{q}, s), p_a \rightarrow f_a(q, \omega(q, \dot{q}, s))} . \tag{35}$$

Here $\epsilon^{I_1}(\tau), I_1 = 1, 2, \ldots, [I_1]$ are the local parameters, and $K$ is the conversion matrix, see Eq. (10). Eq. (35) gives the symmetries of $\tilde{L}$ in closed form in terms of the first class constraints $G_{I_1}$ of the initial formulation. One should note that the transformations of $q^A$ represent Lagrangian version of canonical transformations with the generators being $G_{I_1}$.

In the subsequent computations we omit all the terms which are total derivatives. Besides, the notation $A \bigg|_{A}$ implies the substitution indicated in Eq. (35).

To make a proof, it is convenient to represent the extended Lagrangian (17) in terms of the initial Hamiltonian $H_0$, instead of the
initial Lagrangian \( L \). With help of Eq. (5) one writes

\[
\tilde{L}(q^A, \dot{q}^A, s^a) = \omega_i \dot{q}^i + f_a(q^A, \omega_j)q^a - H_0(q^A, \omega_j) - s^a T_a(q^A, \omega_j),
\]

where the functions \( \omega_i(q, \dot{q}, s) \), \( f_a(q, \omega) \) are defined by Eqs. (14), (3). Using the identity (19), variation of this expression under the transformation (35) can be presented in the form

\[
\delta \tilde{L} = -\omega_i(q, \dot{q}, s) \frac{\partial G_{I_1}}{\partial p_i} |_{\epsilon^{I_1}} - f_a(q, \omega(q, \dot{q}, s) \frac{\partial G_{I_1}}{\partial p_a} |_{\epsilon^{I_1}}
\]

\[
- \left( \frac{\partial H_0(q^A, p_j)}{\partial q^A} + q^a \frac{\partial \Phi_a(q^A, p_B)}{\partial q^A} + s^a \frac{\partial T_a(q^A, p_j)}{\partial q^A} \right) \{q^A, G_{I_1}\} |_{\epsilon^{I_1}}
\]

\[-\delta_{I_1}s^a T_a(q^A, \omega_j).
\]

(37)

To see that \( \delta \tilde{L} \) is total derivative, we add the following zero

\[
0 \equiv \left[ \frac{\partial \tilde{L}}{\partial \omega_i} \right] \{p_i, G_{I_1}\}
\]

\[-\left( \frac{\partial H_0}{\partial p_j} + q^a \frac{\partial \Phi_a}{\partial p_j} + s^a \frac{\partial T_a}{\partial p_j} \right) \{p_j, G_{I_1}\} + q^a \{p_a, G_{I_1}\} \right] |_{\epsilon^{I_1}},
\]

(38)

to r.h.s. of Eq. (37). It gives the expression

\[
\delta \tilde{L} = \left[ \epsilon^{I_1} G_{I_1} - \epsilon^{I_1} (\{H_0, G_{I_1}\} + q^a \{\Phi_a, G_{I_1}\} + s^a \{T_a, G_{I_1}\}) \right]
\]

\[-\delta_{I_1}s^a T_a(q^A, \omega_j) = \left[ \epsilon^{I_1} G_{I_1} + \epsilon^{I_1} \left( b_{I_1}^a + q^a c_{I_1a} + s^b c_{I_1b} \right) G_{I_1} \right] - \delta_{I_1}s^a T_a(q^A, \omega_j),
\]

(39)

where \( b, c \) are coefficient functions of the constraint algebra (11). Using the equalities \( G_{I_1} = (0, T_a(q^A, \omega_j)) \), \( G_{I_1} = K_{I_1} a T_a(q^A, \omega_j) \), one finally obtains

\[
\delta \tilde{L} = \left[ \epsilon^{I_1} K_{I_1}^a + \epsilon^{I_1} \left( b_{I_1}^a + q^a c_{I_1a} + s^b c_{I_1b} \right) - \delta_{I_1}s^a \right] |_{p_i=\omega_i} T_a.
\]

(40)

Then the variation of \( s^a \) given in Eq. (35) implies \( \delta \tilde{L} = \text{div} \), as it has been stated.

5 Conclusion

In this work we have presented a relatively simple way for finding the local symmetries in a singular theory of a general form. Instead of
looking for the symmetries of initial Lagrangian, one can construct an equivalent Lagrangian $\tilde{L}$ given by Eq. (17), the latter implies at most third-stage constraints in the Hamiltonian formulation. Due to special structure of $\tilde{L}$ (all the higher-stage constraints $T_a$ of the original formulation enter into $\tilde{L}$ in a manifest form, see the last term in Eq. (17)), local symmetries of $\tilde{L}$ can be immediately written according to Eq. (35). The latter gives the symmetries in terms of the first class constraints $G_I$ of the initial formulation, moreover, transformations of $q^A$ represent Lagrangian version of canonical transformations with the generators being $G_I$. In contrast to a situation with symmetries of $L$ [2-5], the transformations (35) do not involve the second class constraints.

The extended formulation can be appropriate tool for development of a general formalism for conversion of second class constraints into the first class ones according to the ideas of the work [10]. To apply the method proposed in [10], it is desirable to have a formulation with some configuration space variables entering into the Lagrangian without derivatives. It is exactly what happens in the extended formulation.

To conclude with, we discuss properties of the extended formulation for some particular cases of the original gauge algebra (11).

Suppose that all the original constraints $G_I$ are first class. It implies the extended formulation with at most secondary constraints. One obtains the primary constraints $\Phi_\alpha = 0$, $\pi_a = 0$ and the secondary constraints $T_a = 0$, all of them being the first class. An appropriate gauge for $\pi_a = 0$ is $s^a = 0$. For the case, Eq. (35) reduces to the result obtained in [6].

Suppose that all the original constraints $G_I$ are second class (that is there are no local symmetries in the theory). It implies the extended formulation with at most third-stage constraints, all of them being the second class: $\Phi_\alpha = 0$, $T_a = 0$, $\pi_a = 0$, $s^a = 0$.

Suppose that the original $L$ represents a formulation with at most second-stage first and second class constraints. It implies the extended formulation with at most third-stage constraints. Nevertheless, namely for the extended formulation the local symmetries can be find in a manifest form according to Eq. (35).

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7In the recent work [9] it was demonstrated that the primary constraints, while are convenient, turn out to be not necessary for the Hamiltonization procedure. So, one can said that for any theory there exists a formulation with secondary and tertiary constraints.
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