CLASS OF THE INFINITESIMAL GENERATOR OF A
DIFFEOMORPHISM IN \((\mathbb{C}^m, 0)\)

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Abstract. Let \(F\) be an analytic diffeomorphism in \((\mathbb{C}^m, 0)\) tangent to the identity of order \(n\). The infinitesimal generator of \(F\) is the formal vector field \(X\) such that \(\text{Exp} \, X = F\). In this paper we provide an elementary proof of the fact that \(X\) belongs to the Gevrey class of order \(1/n\).

1. Introduction

For each couple of integers \(m \geq 1\) and \(n \geq 2\), let us denote \(\hat{X}_n(\mathbb{C}^m, 0)\) the module of formal vector fields of order \(\geq n\) in \((\mathbb{C}^m, 0)\) and \(\hat{\text{Diff}}_n(\mathbb{C}^m, 0)\) the group of formal diffeomorphisms in \((\mathbb{C}^m, 0)\) if and only if \(\nu(F) := \min\{\nu_0(x_i \circ F - x_i) | i = 1, \ldots, m\} - 1 \geq n\). For any \(X \in \hat{X}_n(\mathbb{C}^m, 0)\), the exponential operator of \(X\) is the application \(\text{exp} \, X : \mathbb{C}[[x]] \rightarrow \mathbb{C}[[x]]\) defined by the formula

\[
\text{exp} \, X(g) = \sum_{j=0}^{\infty} \frac{1}{j!} X^j(g)
\]

where \(X^0(g) = g\) and \(X^{j+1}(g) = X(X^j(g))\). It is a classical result (for instance, see [4]) that the application

\[
\text{Exp} : \hat{X}_n(\mathbb{C}^m, 0) \rightarrow \hat{\text{Diff}}_{n-1}(\mathbb{C}^m, 0)
\]

\[
X \mapsto (\text{exp} \, X(x_1), \ldots, \text{exp} \, X(x_m))
\]

is a bijection. The formal vector field \(X\) such that \(F = \text{Exp}(X)\) is called the infinitesimal generator of \(F\).

Let \(x = (x_1, \ldots, x_m)\) and for any \(s \in \mathbb{R}\) let \(\mathbb{C}[[x]]_s\) denote the subset of elements of \(\mathbb{C}[[x]]\) that satisfy the \(s\)-Gevrey condition, i.e.

\[
f(x) = \sum_{k=0}^{\infty} f_k(x) \in \mathbb{C}[[x]]_s \quad \text{if and only if} \quad \sum_{k=0}^{\infty} \frac{f_k(x)}{k!^s} \in \mathbb{C}\{x\},
\]

where \(f_k(x)\) is homogeneous of degree \(k\). Let us observe that \(0\)-Gevrey condition means analyticity, and \(\mathbb{C}\{x\} \subset \mathbb{C}[[x]]_s \subset \mathbb{C}[[x]]_t\) if \(0 < s < t\). Let \(\hat{X}_n(\mathbb{C}^m, 0)_s \subseteq \hat{X}_n(\mathbb{C}^m, 0)\) be the set of \(s\)-Gevrey vector fields \(X = \sum_{k=1}^{m} X(x_k) \frac{\partial}{\partial x_k}\) with \(X(x_k) \in \mathbb{C}[[x]]_s\) and \(\hat{\text{Diff}}_n(\mathbb{C}^m, 0)_s = \hat{\text{Diff}}_n(\mathbb{C}^m, 0) \cap (\mathbb{C}[[x]]_s)^m\) the set of \(s\)-Gevrey diffeomorphisms tangent to the identity of order \(\geq n\).

We will prove the following result

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Theorem 1.1. For any $s \geq \frac{1}{n-1}$ the application $\text{Exp}$ gives a bijection

$$\text{Exp} : \mathcal{X}_n(\mathbb{C}^m, 0)_s \to \text{Diff}_{n-1}(\mathbb{C}^m, 0)_s.$$ 

In particular, the infinitesimal generator of any tangent to the identity analytic diffeomorphism $F$ is $\frac{1}{\nu(F)}$-Gevrey.

In general, $X$ may be divergent for a convergent $F$, for instance, Szekeres [8] and Baker [2] proved that every entire holomorphic function tangent to the identity of order $k$ in dimension 1 has a non-convergent infinitesimal generator, Ahern and Rosay [1] proved that this kind of diffeomorphisms cannot be the time-1 map of a $C^{k+3}$-vector field, and finally J. Rey [7] showed that it cannot be the time-1 map of a $C^{k+1}$-vector field, which is the best possible bound. Thus, the map $\text{Exp} : \mathcal{X}_n(\mathbb{C}^m, 0)_0 \to \text{Diff}_{n-1}(\mathbb{C}^m, 0)_0$ is not surjective for any couple of positive integers $m, n$. In addition, in dimension 1, using resummation arguments, it is proved that if an analytic diffeomorphism $f(x) = x + a_{k+1}x^{k+1} + \cdots$ with $a_{k+1} \neq 0$ has a divergent infinitesimal generator $X$, then $X$ is $k$-summable, so $X$ is Gevrey of order $\frac{1}{k}$, but not smaller (see [2, 3] and [9]). Therefore, the condition $s \geq \frac{1}{n-1}$ is necessary.

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2. Technical estimations

In this paper, we take the following notations:

- $h_k(x)$ will denote the homogeneous polynomial $\sum_{\alpha \in \mathbb{N}^m} x^\alpha$.
- $H_{s,n}(x)$ the series $\sum_{q=n}^{\infty} (q + m - n)!^s h_q(x)$.
- $\frac{\partial}{\partial x}$ the differential operator $\sum_{k=1}^{m} \frac{\partial}{\partial x_k}$.

For formal series $f(x) = \sum_{\alpha} f_\alpha x^\alpha$ and $g(x) = \sum_{\alpha} g_\alpha x^\alpha$, we say that $f \preceq g$ if $|f_\alpha| \leq |g_\alpha|$ for any $\alpha \in \mathbb{N}^m$. We get in this way a partial order in $\mathbb{C}[[x]]$, and also in $\mathcal{X}_n(\mathbb{C}^m, 0)$ and $\text{Diff}_n(\mathbb{C}^m, 0)$, working on the component function. From the definition of Gevrey condition, it can be seen that $X \in \mathcal{X}_n(\mathbb{C}^m, 0)_s$ if and only if there exists $a \in \mathbb{R}^+$ such that

$$\text{Coef}_q(X) \leq (q + m - n)!^s a^q h_q(x) \frac{\partial}{\partial x},$$

where $\text{Coef}_q(X)$ denotes the homogeneous term of $X$ of degree $q$. Thus $X \in \mathcal{X}_n(\mathbb{C}^m, 0)_s$ if and only if there exists $a \in \mathbb{R}^+$ such that $X \preceq H_{s,n}(ax) \frac{\partial}{\partial x}$.

We need the following technical lemmas:

Lemma 2.1. For every $k, l \in \mathbb{N}^*$

$$h_k \frac{\partial}{\partial x} h_l \leq (l + m - 1) \min \left\{ \left( \binom{k + m - 1}{m - 1} \right), \left( \frac{l + m - 2}{m - 1} \right) \right\} h_{k+l-1}.$$
Proof: Observe that
\[
\frac{\partial}{\partial x} h_l = \sum_{k=1}^{m} \frac{\partial}{\partial x_k} \sum_{|\alpha| = l} x^{\alpha} = \sum_{k=1}^{m} \sum_{|\alpha| = l} \alpha_k x^{\alpha}
\]
\[
= \sum_{\beta \in \mathbb{B}_m} \sum_{|\beta| = l-1} (\beta_k + 1) x^\beta = (l + m - 1) h_{l-1}
\]

Now, the coefficient of \(x^n\) in the product \(h_k(x) h_{l-1}(x)\) is less than or equal to the minimum between the number of monomials of \(h_k\) and the number of monomials of \(h_{l-1}\), and the number of monomials of \(h_j\) is \((j + m - 1)\), that corresponds to the number of ordered partitions of \(j\) in \(m\) parts; therefore,
\[
h_k \frac{\partial}{\partial x} h_l = (l + m - 1) h_k h_{l-1} \leq (l + m - 1) \left( \min \{k, l-1\} + m - 1 \right) h_{k+l-1}. \quad \square
\]

**Lemma 2.2.** Let \(\Theta(y) = \sum_{j=n}^{\infty} (m-1+j) y^{j-n}\). Then \(\Theta(y)\) converges for any \(|y| < 1\).

Proof: Since \(\sum_{j=n}^{\infty} y^{m-1+j} = \frac{y^{m-1}}{1-y} \) converges for any \(|y| < 1\) then
\[
\Theta(y) = \frac{1}{(m-l)!} \left( \frac{1}{1-y} \right) y^{m+n-1}
\]
converges for any \(|y| < 1\). \quad \square

**Lemma 2.3.** For any \(s > 0\) and integers \(m \geq 1\) and \(n \geq 2\), the sequence \(\{b_q\}_{q \geq 2^n - 1}\) given by
\[
b_q = \sum_{j=n}^{\lfloor s + 1 \rfloor} \left( \frac{(j + m - n)! (q-j+1+m-n)!}{m!(q+m-n)!} \right) (q-j+m)^{n-1} \left( j + m - 1 \right),
\]
is bounded.

Proof: Observe that
\[
\frac{(q-j+m)^{n-1}}{(q-j+2+m-n) \cdots (q-j+m)} < \left( \frac{q-j+m}{q-j+2+m-n} \right)^{n-1} \leq \left( \frac{q-1+m}{q-1+2+m-n} \right)^{n-1} \leq \left( \frac{m+n-1}{m+1} \right)^{n-1}
\]
then
\[
b_n \leq \left( \frac{m+n-1}{m+1} \right)^{s(n-1)} \sum_{j=n}^{\lfloor s + 1 \rfloor} \left( \frac{(j + m - n)! (q-j+m)!}{m!(q+m-n)!} \right) s \left( j + m - 1 \right).
\]
In addition
\[
\frac{m+1}{q+m-j+1} < \frac{m+2}{q+m-j+2} < \cdots < \frac{j+m-n}{q+m-n}
\]
and
\[
\frac{j+m-n}{q+m-n} \leq \frac{\lfloor q+1 \rfloor + m-n}{q+m-n} \leq \max \left\{ \frac{1}{2}, \frac{m}{m+n-1} \right\} = C_{m,n} < 1;
\]
from lemma 2.2,
\[ b_q < \left( \frac{m + n - 1}{m + 1} \right)^{s(n-1)} \Theta(C_{m,n}^s). \]

Proposition 2.4. Let \( s \geq \frac{1}{n-1} \), \( X \in \hat{X}_n(\mathbb{C}^m, 0) \) and \( a \in \mathbb{R}^+ \) such that
\[ \text{Coef}_q(X) \preceq (q + m - n)!^s a^q h_q(x) \frac{\partial}{\partial x} \]
for all \( n \leq q \leq N \), and let us denote \( A = 2m!^s \left( \frac{m + n - 1}{m + 1} \right)^{s(n-1)} \Theta(C_{m,n}^s) \). For every \( q, k \) with \( n \leq q \leq N + k - 1 \),
\[ \text{Coef}_q(X^k) \preceq (aA)^{k-1}(q + m - n)!^s a^q h_q(x) \frac{\partial}{\partial x}, \]

Proof: Since \( X^k = \sum_{i=1}^{m} X^k(x_i) \frac{\partial}{\partial x} \), it is enough to prove the affirmation for \( X^k(x_i) \), where \( i \in \{1, 2, \ldots, m\} \). Let us write \( X = \sum_{j=n}^{\infty} X_j \), where \( X_j \) is homogeneous of degree \( j \). We will proceed by induction on \( k \); if \( k = 1 \), by hypothesis
\[ X_q(x_i) \preceq (q + m - n)!^s a^q h_q(x) \quad \text{for every} \; n \leq q \leq N. \]
Suppose that the lemma is true for every \( k \leq p \), then, since the order of \( X^j \) is greater than or equal to \((n-1)j+1\), \( \text{Coef}_q(X^{p+1}) = 0 \) for \( 2 \leq q \leq (n-1)p + n - 1 \) and for \((n-1)p + n \leq q \leq N + p \) we have
\[ \text{Coef}_q(X^{p+1}(x_i)) = \text{Coef}_q(X(X^{p}(x_i))) = \text{Coef}_q \left( \sum_{j=n}^{\infty} X_j(X^p(x_i)) \right) \]
\[ = \sum_{j=n}^{q-(n-1)p} X_j \text{Coef}_{q+1-j}(X^p(x_i)) \]
\[ \preceq \sum_{j=n}^{q-(n-1)p} (j + m - n)!^s a^q h_q(x) \frac{\partial}{\partial x} \left( (aA)^{p-1}(q - j + 1 + m - n)!^s a^{q+1-j} h_{q+1-j}(x) \right) \]
\[ \preceq \sum_{j=n}^{q-(n-1)p} (j + m - n)!^s (q - j + 1 + m - n)!^s (q - j + m) \left( \min_{1 \leq k \leq \infty} \left( A^{p-1} a^{q+p} h_q, \right) \right) \]
\[ \preceq 2 \sum_{j=n}^{q-(n-1)p} (j + m - n)!^s (q - j + 1 + m - n)!^s (q - j + m)^{n-1} \left( A^{p-1} a^{q+p} h_q. \right) \]

Now, observe that
\[ b_j m^s (q + m - n)!^s = \sum_{j=n}^{q-(n-1)p} (j + m - n)! (q - j + 1 + m - n)! (q - j + m)^{n-1} \left( \frac{j + m - 1}{m - 1} \right), \]
where \( \{b_q\} \) is the sequence defined in lemma 2.3 it follows that
\[ \text{Coef}_q(X^{p+1}(x_i)) \preceq 2b_q m^s (q + m - n)!^s A^{p-1} a^{q+p} h_q \]
\[ \preceq (q + m - n)!^s (aA)^p a^q h_q \quad \square \]

3. Proof of theorem 1.1

To prove that the application \( \text{Exp} : \mathcal{X}_n(\mathbb{C}^m, 0)_s \to \text{Diff}_{s-1}(\mathbb{C}^m, 0)_s \) is well defined for \( s \geq \frac{1}{n-1} \), let \( X \in \mathcal{X}_n(\mathbb{C}^m, 0)_s \), \( a > 0 \) be such that \( X \preceq H_{s,n}(ax) \), and \( A \) as in
Suppose that the claim is true for any integer between $n$ and $q$, we will by induction on $X$ such that its infinitesimal generator is not of degree $q$. Then by proposition 2.4 we have

$$\text{Coef}_q(\exp X(x_j)) = \sum_{k=1}^{\infty} \frac{1}{k!} \text{Coef}_q(X^k(x_j))$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k!}(aA)^{k-1}(q + m - n)!^q h_q(x)$$

therefore $\exp(X) \leq \sum_{k=1}^{\infty} \frac{(aA)^{k-1}}{k!} H_{s,n}(ax)$. Now, to prove that $\exp$ is surjective, let us consider a diffeomorphism $F(x) = (x_1 + f_1(x), \ldots, x_m + f_m(x)) \in \text{Diff}_{n-1}(\mathbb{C}^m, 0)_s$ where $f_j(x) = \sum_{q=s}^{\infty} f_{j,q}(x) \in \mathbb{C}[[x]]_s$ and $f_{j,q}(x)$ is an homogeneous polynomial of degree $q$. Then there exists $a > 0$ such that $f_{j,q}(x) \leq (q + m - n)!^q h_q(x)$. Observe that, making a linear change of coordinates, we can suppose that $a$ is small enough such that $\sum_{k=2}^{\infty} \frac{1}{k!}(2aA)^{k-1} \leq \frac{1}{3}$. If $X = \sum_{q=s}^{\infty} X_q$ is the infinitesimal generator of $F(x)$, we will by induction on $q$ that

$$X_q \leq (q + m - n)!^q (2a)^q h_q(x) \frac{\partial}{\partial x}.$$ 

For $q = n$

$$X_n(x_j) = f_{j,n}(x) \leq m!^n a^n h_n(x) \leq m!^s (2a)^n h_n(x).$$

Suppose that the claim is true for any integer between $n$ and $q$, it follows that

$$f_{j,q+1}(x) = \text{Coef}_{q+1}(\sum_{k=1}^{\infty} \frac{1}{k!} X^k(x_j)) = X_{q+1}(x) + \sum_{k=1}^{q} \frac{1}{k!} \text{Coef}_{q+1}(X^k(x_j)),$$

using proposition 2.4

$$X_{q+1}(x_j) \leq (q + 1 + m - n)!^q a^{q+1} h_{q+1}(x)$$

$$+ \sum_{k=2}^{\infty} \frac{1}{k!}(2aA)^{k-1}(q + 1 + m - n)!^q (2a)^q h_{q+1}(x)$$

$$\leq \left( \frac{1}{2a} + \sum_{k=2}^{\infty} \frac{1}{k!}(2aA)^{k-1} \right) (q + 1 + m - n)!^q (2a)^q h_{q+1}(x)$$

$$\leq (q + 1 + m - n)!^q (2a)^q h_{q+1}(x),$$

in other words $X \leq H_{s,n}(2a) \frac{\partial}{\partial x}$. 

\[\square\]

4. Case $0 < s < \frac{1}{n-1}$

As we indicated in the introduction, in this case, there exists $F \in \text{Diff}_{n-1}(\mathbb{C}^m, 0)_s$ such that its infinitesimal generator is not $s$-Gevrey, but the reciprocal is true, i.e.

**Proposition 4.1.** Let $0 \leq s \leq \frac{1}{n-1}$, and $X \in \mathcal{X}_n(\mathbb{C}^m, 0)_s$. Then $\exp(X) \in \text{Diff}_{n-1}(\mathbb{C}^m, 0)_s$.

Observe that the case $s = 0$ is a classical result about the existence of solution of an analytic differential equation. To prove this proposition in the case $s > 0$ we need the following lemma.
Lemma 4.2. Let $t, r \in \mathbb{R}$ such that $0 < t < 1$ and $1 - t < r < 1$. Let $\{a_k\}$ be the sequence defined by $a_1 = a > 0$ and for $k \geq 1$, $a_{k+1} = \sup_{q \in \mathbb{N}^*} \sqrt[q]{(q+m)^{1-r}} a_k$. Then $\{a_k\}$ is increasing and convergent.

Proof: Taking $q \gg k$ it clear that $\sqrt[q]{(q+m)^{1-r}} > 1$, and then $a_{k+1} > a_k$. Now, we know by Bernoulli inequality that

$$q^{rac{q + m}{(k + 1)^{1-r}}} < 1 + \frac{1}{q + k} \left( \frac{q + m}{(k + 1)^{1-r}} - 1 \right) < 1 + \frac{1}{(k + 1)^{1-r}},$$

for $k > m$, so

$$a_{k+1} < \left( 1 + \frac{1}{(k + 1)^{1-r}} \right)^{1-t} a_k < \left( \prod_{j=m+1}^{k+1} \left( 1 + \frac{1}{j^{1-r}} \right) \right)^{1-t} a_m,$$

and since $\frac{1}{j^{1-r}} > 1$ it follows that $\{a_k\}$ is bounded, thereby it is convergent. \qed

Proof of proposition 4.1: If $s \in (0, \frac{1}{n})$, $X \in \mathfrak{X}_n(\mathbb{C}^m, 0)_s$ and $a \in \mathbb{R}^+$ such that $X \leq H_{s,n}(ax)\frac{\partial}{\partial x}$ then for $t = s(n-1)$, $r \in (1-t, 1)$ and $\{a_k\}$ as in lemma 4.2 using the arguments of proposition 2.3 and the fact that $k^r a_k^{k+q-1} \geq (q + m)^{1-t} a_k^{k+q-1}$ for every $q \geq 2$, we can prove that

$$X^k \leq (a_k A)^{k-1} k! H_{s,n}(a_k x) \frac{\partial}{\partial x},$$

where $A = 2m! \left( \frac{m+n-1}{m+1} \right)^{s(n-1)} \Theta(C_{m,n})$. Let $c = \lim_{k \to \infty} a_k$. Therefore we have

$$\text{Coeff}_q(\exp(X)(x_j)) = \sum_{k=1}^{\infty} \frac{1}{k!} \text{Coeff}_q(X^k(x_j)) \leq \sum_{k=1}^{\infty} \frac{(cA)^{k-1}}{k^{1-r}} (m + q - n)! q^q r_q(x)$$

Thus $\text{Exp}(X) \leq \sum_{k=1}^{\infty} \frac{(cA)^{k-1}}{k!^{1-r}} H_{s,n}(cx) \frac{\partial}{\partial x}$. \qed

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