The minimum width in relativistic quantum mechanics

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Abstract
We challenge the widespread belief, originated by Newton and Wigner (1949 Rev. Mod. Phys. 21 400) that the incorporation of special relativity into quantum mechanics implies that a massive particle cannot be localized within an arbitrarily small spatial extent, that there is a minimum width approximately equal to the Compton wavelength. Our argument is in four parts. First, the scalar function used by Newton and Wigner as a measure of localization is not a position probability amplitude. The correct relativistic position probability amplitude becomes a delta function for a state vector localized according to the criteria of Newton and Wigner. Second, the possibility of Lorentz contraction as observed from a boosted frame means that the wavepacket width in the boost direction can take arbitrarily small values. Third, we refer to the work of Almeida and Jabs (1984 Am. J. Phys. 52 921) who show that the long time wavepacket spreading rate for relativistic position probability amplitudes is always less than the speed of light no matter how small the initial width of the wavepacket. Lastly, we show that it is a simple matter to construct scalar amplitudes with spatial widths smaller than the supposed minimum.

Keywords: relativistic quantum mechanics, quantum theory, Lorentz transformations

1. Introduction

The aim of this paper is to challenge a widely held belief, expressed first by Newton and Wigner [1] and repeated elsewhere [2–5]. This belief is that the incorporation of special relativity into quantum mechanics implies that a massive, spinless, particle localized according to the Newton–Wigner criteria has a nonzero spatial extent, a minimum width approximately equal to its Compton wavelength. In support of this conclusion, it has been argued [3, 6] that attempting to localize an electron to within a Compton wavelength could use energy more than twice the rest energy of the electron, so that electron–positron pair creation would result. We will deal with this argument, but note first that it is puzzling why a theory to describe a free particle would have anything to say about pair creation.

The argument against this position presented here has four parts. First, the quantity used by Newton and Wigner to characterize the spatial extent of the localized particle is not a position probability amplitude. Its modulus-squared does not have the properties required of a position probability density. In contrast, such a position probability amplitude can be defined in the relativistic context [7–10] and its modulus-squared satisfies all the conditions required of a position probability density. The second argument against a minimum width is that such a statement is not consistent with the possibility of Lorentz contraction of the wavepacket. In section 3 we demonstrate the Lorentz contraction of a position probability amplitude and compare with the case of a scalar amplitude. The result is that a particle can be localized in an arbitrarily small volume as observed from a boosted frame, with no pair creation taking place. Thirdly, we refer to the work of Almeida and Jabs [11], who calculated the relativistic spreading of position probability amplitudes for initial extents small compared to the Compton wavelength. They found that the initial wavepacket could have an arbitrarily small spatial extent without causing the long time spreading rate to exceed the speed of light. Lastly, it is a simple matter to construct well-behaved scalar amplitudes with spatial widths smaller than the supposed minimum.

The organization of this paper is as follows. In section 2 we review the properties of relativistic probability amplitudes and consider the results of Newton and Wigner. In section 3 we calculate the Lorentz contraction of a position probability amplitude and of a scalar amplitude. In section 4 we review the results of Almeida and Jabs on the spreading of relativistic...
probability amplitudes. In section 5 we construct scalar amplitudes with spatial widths less than the Compton wavelength. Conclusions follow.

Throughout this paper, we use Heaviside–Lorentz units, in which $\hbar = c = e_0 = \mu_0 = 1$.

2. Position probability amplitudes and the Newton–Wigner result

The wavefunctions that we call relativistic probability amplitudes have been studied before [7–10]. They are called Newton–Wigner–Foldy wavefunctions by Rosenstein [8]. Here we briefly review their definition. For a fuller account, including their transformation properties, we refer the reader to a paper in preparation by this author [7].

The set of basis vectors we use for a free, massive, spinless particle is $\{| p \rangle ; p \in \mathbb{R}^3, p^0 = \omega = \sqrt{p^2 + m^2}\}$, eigenvectors of the four-momentum operator $\hat{p}^\mu$ with eigenvalues $p^\mu$, for components with $\mu = 0, 1, 2, 3$. Only positive energies are considered. These basis vectors carry the unitary, irreducible representations of the Poincaré group. We choose to use the invariant normalization,

$$\langle p_1 | p_2 \rangle = \omega_1 \delta^3(p_1 - p_2),$$

with $\omega_1 = \sqrt{p_1^2 + m^2}$, as this makes the transformation properties of the basis vectors take the simplest forms. A normalized state vector can be written

$$| \psi \rangle = \int \frac{d^3p}{\sqrt{\omega}} \langle p \rangle \Psi(p).$$

In this form, $\Psi(p)$ can be interpreted as a momentum probability amplitude, since its normalization condition is

$$\int d^3p \, |\Psi(p)|^2 = 1$$

and the expectation of four-momentum is

$$\langle \psi | \hat{p}^\mu | \psi \rangle = \int d^3p \, |\Psi(p)|^2 \, p^\mu.$$  

The Newton–Wigner (improper) state vector for a particle localized at the origin at $t = 0$ is [1]

$$| 0, 0 \rangle = \int \frac{d^3p}{\sqrt{\omega}} \langle p \rangle \frac{1}{(2\pi)^2}.$$  

A spacetime translation then gives

$$| x \rangle = \int \frac{d^3p}{\sqrt{\omega}} \langle p \rangle \frac{e^{ip \cdot x}}{(2\pi)^2}.$$  

These obey the localization condition at equal times

$$\langle t, x | t, y \rangle = \delta^3(x - y).$$

Then we form the amplitudes, functions of position and time $(x^\mu = (t, x^i))$,

$$\psi(x) = \langle x | \psi \rangle = \int \frac{d^3p}{(2\pi)^2} e^{-ip \cdot x} \Psi(p),$$

as in the nonrelativistic theory. The position operator at $t = 0$,

$$\hat{x}(0) = \int d^3x \, | 0, x \rangle \langle 0, x |,$$

can be represented by

$$\hat{x}(0) = i \frac{\partial}{\partial p}$$

acting on wavefunctions $\Psi(p)$, such that

$$\langle \psi | \hat{x}(0) | \psi \rangle = \int d^3p \, \Psi^*(p) \left( i \frac{\partial}{\partial p} + \frac{p}{\omega} \right) \Psi(p)$$

$$= \int d^3x \, | \psi(0, x) |^2 x.$$  

The result at arbitrary times is

$$\langle \psi | \hat{x}(t) | \psi \rangle = \int d^3p \, \Psi^*(p) \left( i \frac{\partial}{\partial p} + \frac{p}{\omega} \right) \Psi(p)$$

$$= \int d^3x \, | \psi(t, x) |^2 x.$$  

So this amplitude, $\psi(x)$, can be interpreted as a position probability amplitude. From the Parseval theorem, it is normalized at all times:

$$\int d^3x \, | \psi(t, x) |^2 = \int d^3p \, |\Psi(p)|^2 = 1.$$  

Newton and Wigner [1] claim that the position operator at $t = 0$ is

$$\hat{x}_{NW}(0) = i \frac{\partial}{\partial p} - i \frac{p}{2\omega^2},$$

but theirs is defined to act on scalar amplitudes $\Phi(p) = \sqrt{\omega} \, \Psi(p)$, not momentum probability amplitudes $\Psi(p)$. It is, in fact, the same operator when acting on the same wavefunctions. The apparent difference is resolved in the identity

$$\int d^3p \, \Phi^{(1)*}(p) \left\{ i \frac{\partial}{\partial p} - i \frac{p}{2\omega^2} \right\} \Phi^{(2)}(p)$$

$$= \int d^3p \, \Psi^{(1)*}(p) \left\{ i \frac{\partial}{\partial p} - i \frac{p}{2\omega^2} \right\} \Psi^{(2)}(p).$$  

This fact was noted in [7, 9].

Both $\Psi(p)$ and $\psi(x)$ have well-defined transformations under translations, rotations, boosts, space inversion and time reversal. These are given in the general massive case for particles with spin in [7]. We note that the boost transformation of $\psi(x)$ is nonlocal, as it must be to preserve total probability. As stated in [7], special relativity does not require that all physically relevant quantities transform as scalars, four-vectors and tensors. It merely requires that all such transformation properties be well-defined and depend only on the translation, rotation and boost parameters. In quantum mechanics, these transformations must be unitary for Lorentz
transformations and space inversion, antiunitary for time reversal.

Newton and Wigner use a function that transforms as a scalar function as their measure of localization:

\[
\varphi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega}} e^{-ip \cdot x} \sqrt{\omega} \psi(p).
\]

(16)

Since the momentum components of this quantity are multiplied by \(1/\sqrt{\omega}\) compared to those of \(\psi(x)\), \(\varphi(x)\) is smeared in position space compared to \(\psi(x)\). Hence, the position probability amplitude for the particle localized at the origin \((\psi_0(p) = 1/(2\pi)^3)\) is

\[
\psi_0(0, x) = \delta^4(x),
\]

(17)

while the scalar function is

\[
\varphi_0(0, x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega}} e^{ip \cdot x} \sqrt{2\omega} \int \frac{d^3p}{(2\pi)^3} e^{-ip \cdot x} \sqrt{\omega} \psi(p).
\]

(18)

This latter has the asymptotic form \(\exp(-mr)\), so the relevant spatial width is the Compton wavelength, \(\lambda_C = 1/m\). In contrast, the position probability amplitude, a delta function with vanishing width, measures the localization correctly. So we see the origin of the misconception: analysis of the spatial distribution of a particle must use the relativistic probability amplitudes.

### 3. Lorentz contraction

We consider the normalized state vector

\[
|\psi\rangle = \int \frac{d^3p}{\sqrt{\omega}} |p\rangle \psi(p) = \int \frac{d^3p}{\sqrt{2\omega}} |p\rangle e^{-|p|^2/4\sigma^2} (2\pi\sigma^2)^{3/2},
\]

(19)

representing a spinless particle with vanishing average momentum. The position probability amplitude at \(t = 0\) is

\[
\psi(0, x) = \frac{e^{-ix^2/4\sigma^2}}{(2\pi\sigma^2)^{3/4}},
\]

(20)

with \(\sigma, \sigma_p = 1/2\). So the wavepacket is minimal and localized around the origin with a spatial width \(\sigma_x\) in all directions at this time.

Using the transformation results from [7], a boost by velocity \(p_0\) produces the momentum wavefunction

\[
\Psi'(p) = \sqrt{\gamma_0(1 - p_0 \cdot \beta)} \Psi(\Lambda^4 p),
\]

(21)

with \(\gamma_0 = 1/\sqrt{1 - \beta^2}\) and \(\beta = p/\omega\). The boosted position wavefunction at \(t = 0\) is then

\[
\psi'(0, x) = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot x} \sqrt{\gamma_0(1 - p_0 \cdot \beta)} \Psi(\Lambda^4 p).
\]

(22)

With \(p' = \Lambda^{-1} p\), we have

\[
p' = p_\perp + \gamma_0(p_\parallel - \beta_0 \omega),
\]

(23)

where we have separated the momentum into parts parallel \((p_\parallel)\) and perpendicular \((p_\perp)\) to the boost velocity. So the exponent contains a factor

\[
|\psi'(0)|^2 = |p_\perp|^2 + \gamma_0^2 |p_\parallel - \beta_0 \omega|^2.
\]

(24)

We find that this vanishes where

\[
p_\perp = 0 \quad \text{and} \quad p_\parallel = m\gamma_0\beta_0.
\]

(25)

so the modulus-squared of the wavefunction will have its peak there.

We expand equation (24) in powers of \(p_\perp\) and \(p_\parallel - m\gamma_0\beta_0\), to find

\[
|\psi'(0)|^2 \approx |p_\perp|^2 + \frac{1}{\gamma_0^2} |p_\parallel - m\gamma_0\beta_0|^2.
\]

(26)

The correction terms are of third order in the expansion quantities. Note that the width in momentum in the boost direction will be enlarged by the gamma factor. We choose the momentum width, \(\sigma_p\), so that the boosted wavefunction will be narrow in momentum:

\[
\frac{\gamma_0 \sigma_p}{m\gamma_0\beta_0} = \frac{\sigma_p}{\beta_0} \ll 1.
\]

(27)

Then the third order terms we neglected in equation (26) will produce higher powers of this small ratio, so are justifiably negligible.

Also we take \(\beta \to \beta_0\), its peak value, in the slowly varying factor in equation (21), giving

\[
\sqrt{\gamma_0(1 - \beta_0 \cdot \beta)} \to \frac{1}{\sqrt{\gamma_0}}.
\]

(28)

The position wavefunction at \(t = 0\) is then, defining \(x_\parallel\) and \(x_\perp\) as components of the position parallel and perpendicular to \(\beta_0\), respectively,

\[
\psi'(0, x) \approx \int \frac{d^3p}{(2\pi)^3} e^{i(p \cdot x + p_\perp x_\perp)} \frac{1}{\sqrt{\gamma_0}}
\]

\[
\times e^{-|p|^2/4\sigma^2} e^{-|p_\perp|^2/4\sigma_p^2} (2\pi\sigma^2)^{3/2}.
\]

(29)

We evaluate the integrals using [12] (their formula 3.32.3.2) in the form

\[
\int_{-\infty}^{\infty} dz e^{-z^2/\sigma^2} e^{i z/\sigma} = (\pi \sigma^2)^{1/2} e^{-z^2/4}.
\]

(30)

This gives

\[
\psi'(0, x) \approx e^{im\gamma_0\beta_0 x_\parallel} e^{-|x|^2/4\sigma^2} e^{-|x_\parallel|^2/4\sigma_p^2} (2\pi\sigma^2)^{3/2} (2\pi\sigma^2)^{1/2} (2\pi\sigma^2)^{1/2}.
\]

(31)

Thus we see the Lorentz contraction of the spatial width in the boost direction,

\[
\alpha_\parallel \to \frac{\sigma_p}{\gamma_0}.
\]

(32)
Since $\gamma_0$ can take arbitrarily large values, the particle can be localized in an arbitrarily small volume.

If high energy photons were used to localize an electron in a small volume, electron–positron pair creation would indeed occur. The method of strong localization considered here also requires a large amount of energy, but only from the propellant used to boost the observer. Both observers would confirm that no pair creation was taking place.

We perform a very similar calculation to find the boost behaviour of the scalar amplitude for the same choice of initial state vector

$$\psi'(0, x) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x} + \gamma_0(1 - \beta_0 \cdot \beta)} \Psi(X^{-1}p).$$  \hspace{1cm} (33)

We replace the slowly varying factor $1/\omega$ by its value at the momentum wavefunction peak,

$$\frac{1}{\omega} \rightarrow \frac{1}{\sqrt{m^2 + m^2\gamma_0^2/\beta_0^2}} = \frac{1}{m\gamma_0}. \hspace{1cm} (34)$$

So the result is (with the correct dimension)

$$\psi'(0, x) \approx \frac{1}{m\gamma_0} e^{i\omega\mathbf{p} \cdot \mathbf{x}} e^{-|\mathbf{x}|^2/4\sigma_0^2} e^{-\gamma_0^2|\mathbf{p}|^2/4\sigma_0^2} \left(\frac{2\pi\sigma_0^2}{2\pi\sigma_0^2/\gamma_0^2}\right). \hspace{1cm} (35)$$

So we see that the scalar amplitude also Lorentz contracts, another confirmation of our assertion that the Compton wavelength is not a minimum width. (Note that the overall normalization, which is not conserved under boosts for this function, also changes.)

4. Wavepacket spreading

Almeida and Jabs [11] treat the spreading of relativistic wavepackets of the form of equation (8) in generality for the spinless case. (Naumov [13] treats the spreading of position probability amplitudes and of the scalar amplitude, but does not consider the small $\sigma_i$ limit.) They consider the positive energy solution of the Klein–Gordon equation can be written

$$\omega(p) = \sqrt{p^2 + m^2}, \hspace{1cm} (36)$$

so that the relativistic velocity is

$$\beta = \frac{\partial \omega}{\partial p} = \frac{p}{\omega}. \hspace{1cm} (37)$$

They use a general result due to Bradford [14] that the average position and total variance in position are given by

$$\langle x(t) \rangle = \langle x(0) \rangle + \langle \beta \rangle t,$$

$$\sigma^2(t) = \sigma^2(0) + [\langle \beta^2 \rangle - \langle \beta \rangle^2]t^2, \hspace{1cm} (38)$$

respectively, where $\sigma^2(t)$ is the total variance

$$\sigma^2(t) = \sigma_x(t)^2 + \sigma_y(t)^2 + \sigma_z(t)^2. \hspace{1cm} (39)$$

Almeida and Jabs assume that the initial average position is the origin $\langle x(0) \rangle = 0$, accomplished just with a phase change to $\Psi(p)$) and that the average velocity vanishes $\langle \beta \rangle = 0$, accomplished with a translation of $\Psi(p)$ in momentum space. Then they require that the momentum probability distribution, $|\Phi(p)|^2$, be such that it becomes wider and flatter as the variance of momentum increases. This eliminates the case of two peaks that become further apart as the variance of momentum increases. Under only these assumptions, they demonstrate that what they define as the spreading velocity,

$$v_{sp} = \frac{\partial \sigma(t)}{\partial t},$$

remains less than the speed of light but approaches it asymptotically for large times.

We note that in three dimensions, $\sigma(t)$ is not a good measure of the spreading rate. In the spherically symmetric case, for example, it is $\sqrt{3}$ times any of the three individual widths, such as $\sigma_x(t)$. Thus we find a stronger bound in the case of the spherically symmetric Gaussian wavefunction (equation (19)), by evaluating the expectation values in equation (38),

$$\sigma_i(t) \rightarrow \sqrt{\sigma_i^2(0) + \frac{1}{3}t^2} \hspace{1cm} \text{as } \sigma_i(0) \rightarrow 0,$$

with limiting spreading rate $\partial \sigma_i/\partial t \rightarrow 1/\sqrt{3}$ as $t \rightarrow \infty$.

Thus this causality requirement does not place any lower limit on the spatial extent of a wavepacket.

5. Sub-minimal scalar amplitudes

We have written the time-dependent position probability amplitude as equation (8). Since $\Psi(p)$ is a momentum probability amplitude, we require it to be square-integrable for physical states, to make equation (3) finite. A general positive energy solution of the Klein–Gordon equation can be written

$$\psi(x) = \int \frac{d^3p}{\omega} e^{-i\mathbf{p} \cdot \mathbf{x}} \Phi(p). \hspace{1cm} (40)$$

We require square-integrability of $\Phi(p)/\sqrt{\omega}$. Otherwise $\Psi(p)$ and $\Phi(p)$ are arbitrary. The only differences between these two expressions are the physical interpretation given to the quantities involved and the intended Lorentz transformation properties of the functions of position or momentum.

So we may choose

$$\Phi(p) = N \left(\frac{\sigma_p}{m}\right)^{\frac{\omega}{\sqrt{m}}} e^{-|\mathbf{p}|^2/4\sigma_p^2} \left(\frac{2\pi\sigma_p^2}{2\pi\sigma_p^2/\gamma_0^2}\right), \hspace{1cm} (41)$$

with $\Phi(p)/\sqrt{\omega}$ square-integrable, where $N(\sigma_p/m)$ is a dimensionless normalization factor. This gives

$$\psi(0, x) = N \left(\frac{\sigma_p}{m}\right)^{1/\sqrt{m}} e^{-|\mathbf{p}|^2/4\sigma_p^2} \left(\frac{2\pi\sigma_p^2}{2\pi\sigma_p^2/\gamma_0^2}\right). \hspace{1cm} (42)$$

The result is a function intended to transform as a scalar function. In this one frame at one time, its spatial width can
take any nonzero value, \( \sigma_x = 1/2\sigma_p \), including values smaller than the Compton wavelength.

6. Conclusions

We have provided four arguments against the claim that relativistic quantum mechanics implies a minimum width for a particle, of order its Compton wavelength. We find that the origin of the misconception is the unjustified use of the scalar amplitude as a measure of localization, instead of the relativistic probability amplitude. In relativity, a minimum length scale is meaningless since observation from a boosted frame will see Lorentz contraction. We reviewed the results of Almeida and Jabs [11] on the spreading of wavepackets. They demonstrated that a causality condition was always satisfied: the long time spreading rate of relativistic wavepackets remained less than the speed of light no matter how small the initial wavepacket. Lastly, we constructed an infinite set of scalar amplitudes that violate the lower bound on wavepacket size.

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