LAZARSFELD-MUKAI BUNDLES AND APPLICATIONS

MARIAN APRODU

Abstract. We survey the development of the notion of Lazarsfeld-Mukai bundles together with various applications, from the classification of Mukai manifolds to Brill-Noether theory and syzygies of $K3$ sections. To see these techniques at work, we present a short proof of a result of M. Reid on the existence of elliptic pencils.

Introduction

Lazarsfeld–Mukai bundles appeared naturally in connection with two completely different important problems in algebraic geometry from the 1980s. The first problem, solved by Lazarsfeld, was to find explicit examples of smooth curves which are generic in the sense of Brill-Noether-Petri. The second problem was the classification of prime Fano manifolds of coindex 3. More recently, Lazarsfeld–Mukai bundles have found applications to syzygies and higher-rank Brill–Noether theory.

The common feature of all these research topics is the central role played by $K3$ surfaces and their hyperplane sections. For the Brill–Noether–Petri genericity, Lazarsfeld proves that a general curve in a linear system that generates the Picard group of a $K3$ surface satisfies this condition. For the classification of prime Fano manifolds of coindex 3, after having proved the existence of smooth fundamental divisors, one uses the geometry of a two-dimensional linear section which is a very general $K3$ surface.

The idea behind this definition is that the Brill–Noether theory of smooth curves on a $K3$ surface, also called $K3$ sections, is governed by higher-rank vector bundles on the surface. To be more precise, consider $S$ a $K3$ surface (considered always to be smooth, complex, projective), $C$ a smooth curve on $S$ of genus ≥ 2, and $|A|$ a base-point-free pencil on $C$. If we attempt to lift the linear system $|A|$ to the surface $S$, in most cases, we will fail. For instance, $|A|$ cannot lift to a pencil on $S$ if $C$ generates Pic($S$) or if $S$ does not contain any elliptic curve at all. However, interpreting a general divisor in $|A|$ as a zero-dimensional subscheme of $S$, it is natural to try and find a rank-two bundle $E$ on $S$ and a global section of $E$ whose scheme of zeros coincides with the divisor in question. Varying the divisor, one should exhibit in fact a two-dimensional space of global sections of $E$. The effective construction of $E$ is realized through elementary modifications, see Sect. 1 and this is precisely a Lazarsfeld–Mukai bundle of rank two. The passage to higher ranks is natural, if we start with a complete, higher-dimensional, base-point-free
linear system on $C$. At the end, we obtain vector bundles with unusually high number of global sections, which provide us with a rich geometric environment.

The structure of this chapter is as follows. In the first section, we recall the definition of Lazarsfeld–Mukai bundles and its first properties. We note equivalent conditions for a bundle to be Lazarsfeld–Mukai in Sect. 1.1 and we discuss simplicity in the rank-two case in Sect. 1.2. The relation with the Petri conjecture and the classification of Mukai manifolds, the original motivating problems for the definition, are considered in Sects. 1.3 and 1.4 respectively. In Sect. 2 we treat the problem of constancy of invariants in a given linear system. For small gonality, Saint-Donat and Reid proved that minimal pencils on $K^3$ sections are induced from elliptic pencils on the $K^3$ surface; we present a short proof using Lazarsfeld–Mukai bundles in Sect. 2.1. Harris and Mumford conjectured that the gonality should always be constant. We discuss the evolution of this conjecture, from Donagi–Morrison’s counterexample, Sect. 2.1 to Green–Lazarsfeld’s reformulation in terms of Clifford index, Sect. 2.2 and to Ciliberto–Pareschi’s results on the subject, Sect. 2.3. The works around this problem emphasized the importance of parameter spaces of Lazarsfeld–Mukai bundles. We conclude the section with a discussion of dimension calculations of these spaces, Sect. 2.4 which are applied afterwards to Green’s conjecture. Sect. 3 is devoted to Koszul cohomology and notably to Green’s conjecture for $K^3$ sections. After recalling the definition and the motivations that led to the definition, we discuss the statement of Green’s conjecture, and we sketch the proof for $K^3$ sections. Voisin’s approach using punctual Hilbert schemes, which is an essential ingredient, is examined in Sect. 3.3. Lazarsfeld–Mukai bundles are fundamental objects in this topic, and their role is outlined in Sect. 3.4. The final step in the solution of Green’s conjecture for $K^3$ sections is tackled in Sect. 3.5. We conclude this chapter with a short discussion on Farkas–Ortega’s new applications of Lazarsfeld–Mukai bundles to Mercat’s conjecture (which belongs to the rapidly developing higher-dimensional Brill–Noether theory), Sect. 4.

Notation. The additive and the multiplicative notation for divisors and line bundles will be mixed sometimes. If $E$ is a vector bundle on $X$ and $L \in \text{Pic}(X)$, we set $E(-L) := E \otimes L^*$; this notation will be used especially when $E$ is replaced by the canonical bundle $K_C$ of a curve $C$.

1. Definition, Properties, the First Applications

1.1. Definition and First Properties. We fix $S$ a smooth, complex, projective $K^3$ surface and $L$ a globally generated line bundle on $S$ with $L^2 = 2g - 2$. Let $C \in |L|$ be a smooth curve and $A$ be a base-point-free line bundle in $W^r_d(C) \setminus W^{r+1}_d(C)$. As mentioned in the Introduction, the definition of Lazarsfeld–Mukai bundles emerged from the attempt to lift the linear system $A$ to the surface $S$. Since it is virtually impossible to lift it to another linear system, a higher-rank vector bundle is constructed such that $H^0(C, A)$ corresponds to an $(r + 1)$-dimensional space of global sections. Hence $|A|$ lifts to a higher-rank analogue of a linear system.

The kernel of the evaluation of sections of $A$

$$0 \to F_{C,A} \to H^0(C, A) \otimes \mathcal{O}_S \xrightarrow{ev} A \to 0$$

is a vector bundle of rank $(r + 1)$. 
**Definition 1.1** (Lazarsfeld [18], Mukai [23]). The Lazarsfeld–Mukai bundle $E_{C,A}$ associated to the pair $(C, A)$ is the dual of $F_{C,A}$.

By dualizing the sequence (1) we obtain the short exact sequence

$$0 \to H^0(C, A)^* \otimes O_S \to E_{C,A} \to K_C(-A) \to 0,$$

and hence $E_{C,A}$ is obtained from the trivial bundle by modifying it along the curve $C$ and comes equipped with a natural $(r+1)$-dimensional space of global sections as planned.

We note here the first properties of $E_{C,A}$:

**Proposition 1.2** (Lazarsfeld). The invariants of $E$ are the following:

1. $\det(E_{C,A}) = L$.
2. $c_2(E_{C,A}) = d$.
3. $h^0(S, E_{C,A}) = h^0(C, A) + h^1(C, A) = 2r - d + 1 + g$.
4. $h^1(S, E_{C,A}) = h^2(S, E_{C,A}) = 0$.
5. $\chi(S, E_{C,A} \otimes F_{C,A}) = 2(1 - \rho(g, r, d))$, where $\rho(g, r, d) = g - (r+1)(g-d+r)$.
6. $E_{C,A}$ is globally generated off the base locus of $K_C(-A)$; in particular, $E_{C,A}$ is globally generated if $K_C(-A)$ is globally generated.

It is natural to ask conversely if given $E$ a vector bundle on $S$ with $\text{rk}(E) = r + 1$, $h^1(S, E) = h^2(S, E) = 0$, and $\det(E) = L$, $E$ is the Lazarsfeld–Mukai bundle associated to a pair $(C, A)$. To this end, note that there is a rational map

$$h_E : G(r + 1, H^0(S, E)) \dashrightarrow |L|$$

defined in the following way. A general subspace $A \in G(r + 1, H^0(S, E))$ is mapped to the degeneracy locus of the evaluation map: $\text{ev}_A : A \otimes O_S \to E$. If the image $h_E(A)$ is a smooth curve $C \in |L|$, we set $\text{Coker}(\text{ev}_A) := K_C(-A)$, where $A \in \text{Pic}(C)$ and $\deg(A) = c_2(E)$, and observe that $E = E_{C,A}$. Indeed, since $h^1(S, E) = 0$, $A$ is globally generated, and from $h^2(S, E) = 0$ it follows that $A \cong H^0(C, A)^*$. The conclusion is that:

**Proposition 1.3.** A rank-$(r+1)$ vector bundle $E$ on $S$ is a Lazarsfeld–Mukai bundle if and only if $H^1(S, E) = H^2(S, E) = 0$ and there exists an $(r+1)$-dimensional subspace of sections $A \subset H^0(S, E)$, such that the degeneracy locus of the morphism $\text{ev}_A$ is a smooth curve. In particular, being a Lazarsfeld–Mukai vector bundle is an open condition.

Note that there might be different pairs with the same Lazarsfeld–Mukai bundles, the difference being given by the corresponding spaces of global sections.

1.2. Simple and Non-simple Lazarsfeld–Mukai Bundles. We keep the notation from the previous subsection. In the original situation, the bundles used by Lazarsfeld [18] and Mukai [23] are simple. The non-simple Lazarsfeld–Mukai bundles are, however, equally useful [3] [5]. For instance, Lazarsfeld’s argument is partly based on an analysis of the non-simple bundles.

Proposition 1.2 already shows that for $\rho(g, r, d) < 0$ the associated Lazarsfeld–Mukai bundle cannot be simple. The necessity of making a distinction between simple and non-simple bundles for nonnegative $\rho$ will become more evident in the next sections.

In the rank-two case, one can give a precise description [6] of non-simple Lazarsfeld–Mukai bundles, see also [5] Lemma 2.1:
Lemma 1.4 (Donagi–Morrison). Let $E_{C,A}$ be a non-simple Lazarsfeld–Mukai bundle. Then there exist line bundles $M, N \in \text{Pic}(S)$ such that $h^0(S, M), h^0(S, N) \geq 2$, $N$ is globally generated, and there exists a locally complete intersection subscheme $\xi$ of $S$, either of dimension zero or the empty set, such that $E_{C,A}$ is expressed as an extension

$$0 \to M \to E_{C,A} \to N \otimes I_\xi \to 0.$$ 

Moreover, if $h^0(S, M \otimes N^*) = 0$, then $\xi = \emptyset$ and the extension splits.

One can prove furthermore that $h^1(S, N) = 0$, Remark 3.6. We say that (3) is the Donagi–Morrison extension associated to $E_{C,A}$. This notion makes perfect sense as this extension is uniquely determined by the vector bundle, if it is indecomposable [3]. Actually, a decomposable Lazarsfeld–Mukai bundle $E$ cannot be expressed as an extension (3) with $\xi \neq \emptyset$, and hence a Donagi–Morrison extension is always unique, up to a permutation of factors in the decomposable case. Moreover, a Lazarsfeld–Mukai bundle is decomposable if and only if the corresponding Donagi–Morrison extension is trivial.

In the higher-rank case, we do not have such a precise description. However, a similar sufficiently strong statement is still valid [18, 19, 20].

Proposition 1.5 (Lazarsfeld). Notation as above. If $E_{C,A}$ is not simple, then the linear system $|L|$ contains a reducible or a multiple curve.

In the rank-two case, this statement comes from the decomposition $L \cong M \otimes N$.

1.3. The Petri Conjecture Without Degenerations. A smooth curve of genus $g$ is said to satisfy Petri’s condition, or to be Brill–Noether–Petri generic, if the multiplication map (the Petri map)

$$\mu_{0,A} : H^0(C, A) \otimes H^0(C, K_C(-A)) \to H^0(C, K_C),$$

is injective for any line bundle $A$ on $C$. One consequence of this condition is that all the Brill–Noether loci $W^d_d(C)$ have the expected dimension and are smooth away from $W^{d+1}_d(C)$; recall that the tangent space at the point $[A]$ to $W^d_d(C)$ is naturally isomorphic to the dual of $	ext{Coker}(\mu_{0,A})$. The Petri conjecture, proved by degenerations by Gieseker, states that a general curve satisfies Petri’s condition. Lazarsfeld [18] found a simpler and elegant proof without degenerations by analyzing curves on very general $K3$ surfaces.

Lazarsfeld’s idea is to relate the Petri maps to the Lazarsfeld–Mukai bundles; this relation is valid in general and has many other applications. Suppose, as in the previous subsections, that $S$ is a $K3$ surface and $L$ is a globally generated line bundle on $S$. For the moment, we do not need to assume that $L$ generates the Picard group. E. Arbarello and M. Cornalba constructed a scheme $W^r_d(|L|)$ parameterizing pairs $(C, A)$ with $C \in |L|$ smooth and $A \in W^r_d(C)$ and a morphism

$$\pi_S : W^r_d(|L|) \to |L|.$$ 

Assume that $A \in W^r_d(C) \setminus W^{r+1}_d(C)$ is globally generated, and consider $M_A$ the vector bundle of rank $r$ on $C$ defined as the kernel of the evaluation map

$$0 \to M_A \to H^0(C, A) \otimes \mathcal{O}_C \xrightarrow{\cdot A} A \to 0.$$ 

\[ \text{In fact, we do have a Harder–Narasimhan filtration, but we cannot control all the factors.} \]
Twisting \( E \) with \( K_C \otimes A^* \), we obtain the following description of the kernel of the Petri map:

\[
\text{Ker}(\mu_{0,A}) = H^0(C, M_A \otimes K_C \otimes A^*).
\]

There is another exact sequence on \( C \)

\[
0 \to \mathcal{O}_C \to F_{C,A}|C \otimes K_C \otimes A^* \to M_A \otimes K_C \otimes A^* \to 0,
\]

and from the defining sequence of \( E_{C,A} \) one obtains the exact sequence on \( S \)

\[
0 \to H^0(C, A)^* \otimes F_{C,A} \to E_{C,A} \otimes F_{C,A} \to F_{C,A}|C \otimes K_C \otimes A^* \to 0.
\]

From the vanishing of \( h^0(C, F_{C,A}) \) and of \( h^1(C, F_{C,A}) \), we obtain

\[
H^0(C, E_{C,A} \otimes F_{C,A}) = H^0(C, F_{C,A}|C \otimes K_C \otimes A^*).
\]

Suppose that \( \mathcal{W} \subset \mathcal{W}_d^r(|L|) \) is a dominating component and \((C, A) \in \mathcal{W} \) is an element such that \( A \) is globally generated and \( h^0(C, A) = r + 1 \). A deformation-theoretic argument shows that if the Lazarsfeld–Mukai bundle \( E_{C,A} \) is simple, then the coboundary map \( H^0(C, M_A \otimes K_C \otimes A^*) \to H^1(C, \mathcal{O}_C) \) is zero \([26]\), which eventually implies the injectivity of \( \mu_{0,A} \).

By reduction to complete base-point-free bundles on the curve \([18, 26]\) this analysis yields:

**Theorem 1.6** (Lazarsfeld). Let \( C \) be a smooth curve of genus \( g \geq 2 \) on a K3 surface \( S \), and assume that any divisor in the linear system \(|C|\) is reduced and irreducible. Then a generic element in the linear system \(|C|\) is Brill–Noether–Petri generic.

A particularly interesting case is when the Picard group of \( S \) is generated by \( L \) and \( \rho(g, r, d) = 0 \). Obviously, the condition \( \rho = 0 \) can be realized only for composite genera, as \( g = (r + 1)(g - d + r) \), for example, \( r = 1 \) and \( g \) even. Under these assumptions, there is a unique Lazarsfeld–Mukai bundle \( E \) with \( c_1(E) = L \) and \( c_2(E) = d \), and different pairs \((C, A) \) correspond to different \( A \in G(r + 1, H^0(S, E)) \); in other words the natural rational map \( G(r + 1, H^0(S, E)) \to \mathcal{W}_{d}^r(|L|) \) is dominating. Note that \( E \) must be stable and globally generated.

1.4. Mukai Manifolds of Picard Number One. A Fano manifold \( X \) of dimension \( n \geq 3 \) and index \( n - 2 \) (i.e., of coindex 3) is called a *Mukai manifold*\(^2\) In the classification, special attention is given to prime Fano manifolds: note that if \( n \geq 7 \), \( X \) is automatically prime as shown by Wisniewski; see, for example, \([10]\).

Assume that the Picard group of \( X \) is generated by an ample line bundle \( L \), and let the sectional genus \( g \) be the integer \((L^n)/2 + 1 \). Mukai and Gushel used vector bundle techniques to obtain a complete classification of these manifolds. A first major obstacle is to prove that the fundamental linear system contains indeed a smooth element, aspect which is settled by Shokurov and Mella; see, for example, \([16]\). Then the \((g + n - 2)\)-dimensional linear system \(|L|\) is base-point-free, and a general linear section with respect to the generator of the Picard group is a K3 surface. More precisely, if \( \text{Pic}(X) = \mathbb{Z} \cdot L \), then for \( H_1, \cdots, H_{n-2} \) general elements in the fundamental linear system \(|L|\), \( S := H_1 \cap \cdots \cap H_{n-2} \) is scheme-theoretically a K3 surface. Note that if \( n \geq 4 \) and \( i \geq 3 \), the intersection \( H_1 \cap \cdots \cap H_{n-i} \) is again a Fano manifold of coindex 3.

\(^2\)This ingenious procedure is an efficient replacement of the base-point-free pencil trick; “it has killed the base-point-free pencil trick,” to quote Enrico Arbarello.

\(^3\)Some authors consider that Mukai manifolds have dimension four or more.
Mukai noticed that the fundamental linear system either is very ample, and the image of \( X \) is projectively normal or is associated to a double covering of \( \mathbb{P}^n \) \( (g = 2) \) or of the hyper-quadratic \( Q^n \subset \mathbb{P}^{n+1} \) \( (g = 3) \). The difficulty of the problem is thus to classify all the possible cases where \( |L| \) is normally generated, called of the first species. Taking linear sections one reduces (not quite immediately) to the case \( n = 3 \) [10] p.110.

For simplicity, let us assume that \( X \) is a prime Fano 3-fold of index 1. If \( g = 4 \) and \( g = 5 \), \( X \) is a complete intersection; hence the hard cases begin with genus 6. A hyperplane section \( S \) is a \( K^3 \) surface, and, by a result of Moishezon, \( \text{Pic}(S) \) is generated by \( L|_S \).

Let us denote by \( \mathcal{F}_g \) the moduli space of polarized \( K^3 \) surfaces of degree \( 2g - 2 \), by \( \mathcal{P}_g \) the moduli space of pairs \((K^3 \text{ surface}, \text{curve})\) and \( \mathcal{M}_g \) the moduli space of genus-\( g \) curves. There are two nice facts in Mukai’s proof involving these two moduli spaces. His first observation is that if there exists a prime Fano 3-fold \( X \) of the first species of genus \( g \geq 6 \) and index 1, the rational map \( \phi_g : \mathcal{P}_g \rightarrow \mathcal{M}_g \) is not generically finite [24]. The second nice fact is that \( \phi_g \) is generically finite if and only if \( g = 11 \) or \( g \geq 13 \) [24]. Hence, one is reduced to study the genera \( 6 \leq g \leq 12 \) with \( g \neq 11 \). At this point, Lazarsfeld–Mukai bundles are employed. It has already been noticed that the bundle \( E \) is stable and globally generated. Moreover, the determinant map

\[
\det : \wedge^{r+1} H^0(S, E) \rightarrow H^0(S, L)
\]

is surjective [23], and hence it induces a linear embedding

\[
\mathbb{P} H^0(S, L)^* \hookrightarrow \mathbb{P}(\wedge^{r+1} H^0(S, E)^*).
\]

Following [23], we have a commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\phi_E} & G \\
\downarrow{\phi|_L} & & \downarrow{\text{Pluecker}} \\
\mathbb{P} H^0(L)^* & \xrightarrow{\phi} & \mathbb{P}(\wedge^{r+1} H^0(E)^*)
\end{array}
\]

where \( G := G(r + 1, H^0(S, E)^*) \) and \( \phi_E \) is given by \( E \). This diagram shows that \( S \) is embedded in a suitable linear section of the Grassmannian \( G \). Moreover, this diagram extends over \( X \): by a result of Fujita, \( E \) extends to a stable vector bundle on \( X \), and the diagram over \( X \) is obtained for similar reasons. Hence \( X \) is a linear section of a Grassmannian. By induction on the dimension, \( X \) is contained in a maximal Mukai manifold, which is also a linear section of the Grassmannian. A complete list of maximal Mukai manifolds is given in [23]. Notice that in genus 12, the maximal Mukai manifolds are threefold already.

2. Constancy of Invariants of \( K^3 \) Sections

2.1. Constancy of the Gonality. I. In his analysis of linear systems on \( K^3 \) surfaces Saint–Donat [28] shows that any smooth curve which is linearly equivalent to a hyperelliptic or trigonal curve is also hyperelliptic, respectively trigonal. The idea was to prove that the minimal pencils are induced by elliptic pencils defined
on the surface. This result was sensibly extended by Reid [27] who proved the following existence result:

**Theorem 2.1** (Reid). Let $C$ be a smooth curve of genus $g$ on a $K3$ surface $S$ and $A$ be a complete, base-point-free $g^1_d$ on $C$. If

$$\frac{d^2}{4} + d + 2 < g,$$

then $A$ is the restriction of an elliptic pencil on $S$.

It is a good occasion to present here, as a direct application of techniques involving Lazarsfeld–Mukai bundles, an alternate shorter proof of Reid’s theorem.

**Proof.** We use the notation of previous sections. By the hypothesis, the Lazarsfeld–Mukai bundle $E$ is not simple, and hence we have a unique Donagi–Morrison extension

$$0 \to M \to E \to N \otimes I_\xi \to 0,$$

with $\xi$ of length $\ell$. Note that $M \cdot N = d - \ell \leq d$. By the Hodge index theorem, we have $(M^2) \cdot (N^2) \leq (M \cdot N)^2 \leq d^2$, whereas from $M + N = C$ we obtain $(M^2) = 2(g - 1 - d) - (N^2)$, hence

$$(N^2) \leq \frac{d^2}{2(g - 1 - d) - (N^2)}.$$

Therefore, the even integer $x := (N^2)$ satisfies the following inequality $x^2 - 2x(g - 1 - d) + d^2 \geq 0$. The hypothesis shows that the above inequality fails for $x \geq 2$, and hence $N$ must be an elliptic pencil. \hfill \Box

In conclusion, for small values, the gonality\footnote{The gonality $\text{gon}(C)$ of a curve $C$ is the minimal degree of a morphism from $C$ to the projective line.} is constant in the linear system. Motivated by these facts, Harris and Mumford conjectured that the gonality of $K3$-sections should always be constant [14].

This conjecture is unfortunately wrong as stated: Donagi and Morrison [6] gave the following counterexample:

**Example 2.2.** Let $S \to \mathbb{P}^2$ be a double cover branched along a smooth sextic and $L$ be the pull-back of $O_{\mathbb{P}^2}(3)$. The curves in $|L|$ have all genus 10. The general curve $C \in |L|$ is isomorphic to a smooth plane sextic, and hence it is pentagonal. On the other hand, the pull-back of a general smooth plane cubic $\Gamma$ is a double cover of $\Gamma$, and thus it is tetragonal.
and is denoted by $\text{Cliff}(C)$. The Clifford index is related to the gonality by the following inequalities

$$\text{gon}(C) - 3 \leq \text{Cliff}(C) \leq \text{gon}(C) - 2,$$

and curves with $\text{gon}(C) - 3 = \text{Cliff}(C)$ are very rare: typical examples are plane curves and Eisenbud–Lange–Martens–Schreyer curves \cite{EisenbudLangeMartensSchreyer1996, Knutsen2006}.

From the Brill–Noether theory, we obtain the bound $\text{Cliff}(C) \leq \left\lfloor \frac{(g-1)}{2} \right\rfloor$ (and, likewise, $\text{gon}(C) \leq \left\lfloor \frac{(g+3)}{2} \right\rfloor$), and it is known that the equality is achieved for general curves. The Clifford index is in fact a measure of how special a curve is in the moduli space.

The precise statement obtained by Green and Lazarsfeld is the following \cite{GreenLazarsfeld1985}:

**Theorem 2.3** (Green–Lazarsfeld). Let $S$ be a $K3$ surface and $C \subset S$ be a smooth irreducible curve of genus $g \geq 2$. Then $\text{Cliff}(C') = \text{Cliff}(C)$ for every smooth curve $C' \in |C|$. Furthermore, if $\text{Cliff}(C)$ is strictly less than the generic value $\left\lfloor \frac{(g-1)}{2} \right\rfloor$, then there exists a line bundle $M$ on $S$ whose restriction to any smooth curve $C' \in |C|$ computes the Clifford index of $C'$.

The proof strategy is based on a reduction method of the associated Lazarsfeld–Mukai bundles. The bundle $M$ is obtained from the properties of the reductions; we refer to \cite{GreenLazarsfeld1985} for details.

From the Clifford index viewpoint, Donagi–Morrison’s example is not different from the other cases. Indeed, all smooth curves in $|L|$ have Clifford index 2. We shall see in the next subsection that Donagi–Morrison’s example is truly an isolated exception for the constancy of the gonality.

### 2.3. Constancy of the Gonality. II.

As discussed above, the Green–Lazarsfeld proof of the constancy of the Clifford index was mainly based on the analysis of Lazarsfeld–Mukai bundles. It is natural to try and explain the peculiarity of Donagi–Morrison’s example from this point of view. This was done in \cite{CilibertoPareschi2006}. The surprising answer found by Ciliberto and Pareschi \cite{CilibertoPareschi2006} (see also \cite{CilibertoPareschi2006}) is the following:

**Theorem 2.4** (Ciliberto–Pareschi). Let $S$ be a $K3$ surface and $L$ be an ample line bundle on $S$. If the gonality of the smooth curves in $|L|$ is not constant, then $S$ and $L$ are as in Donagi–Morrison’s example.

Theorem 2.4 was refined by Knutsen \cite{Knutsen2006} who replaced ampleness by the more general condition that $L$ be globally generated. The extended setup covers also the case of exceptional curves, as introduced by Eisenbud, Lange, Martens, and Schreyer \cite{EisenbudLangeMartensSchreyer1996}.

The proof of Theorem 2.4 consists of a thorough analysis of the loci $W^1_d(|L|)$, where $d$ is the minimal gonality of smooth curves in $|L|$, through the associated Lazarsfeld–Mukai bundles. The authors identify Donagi–Morrison’s example in the following way:

**Theorem 2.5** (Ciliberto–Pareschi). Let $S$ be a $K3$ surface and $L$ be an ample line bundle on $S$. If the gonality of smooth curves in $|L|$ is not constant and if there is a pair $(C, A) \in W^1_d(|L|)$ such that $h^1(S, EC_A \otimes FC_A) = 0$, then $S$ and $L$ are as in Donagi–Morrison’s example.

\[6\text{It is conjectured that the only other examples should be some half-canonical curves of even genus and maximal gonality} \text{;} \text{ however, this conjecture seems to be very difficult.} \]
To conclude the proof of Theorem 2.4 Ciliberto and Pareschi prove that non-constancy of the gonality implies the existence of a pair \((C, A)\) with \(h^1(S, E_{C,A} \otimes F_{C,A}) = 0\); see \([5]\) Proposition 2.4.

It is worth to notice that, in Example 2.2 if \(C\) is the inverse image of a plane cubic and \(A\) is a \(g^1_3\) (the pull-back of an involution), then \(E_{C,A}\) is the pull-back of \(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)\) \([5]\), and hence the vanishing of \(h^1(S, E_{C,A} \otimes F_{C,A})\) is guaranteed in this case.

2.4. Parameter Spaces of Lazarsfeld–Mukai Bundles and Dimension of Brill–Noether Loci. We have already seen that the Brill–Noether loci are smooth of expected dimension at pairs corresponding to simple Lazarsfeld–Mukai bundles. It is interesting to know what is the dimension of these loci at other points as well. Precisely, we look for a uniform bound on the dimension of Brill–Noether loci of general curves in a linear system.

A first step was made by Ciliberto and Pareschi \([5]\) who proved, as a necessary step in Theorem 2.4 that an ample curve of gonality strictly less than the generic value, general in its linear system, carries finitely many minimal pencils. This result was extended to other Brill–Noether loci \([3]\), proving a phenomenon of linear growth with the degree; see below. Let us mention that, for the moment, the only results in this direction are known to hold for pencils \([3]\) and nets \([20]\).

As before, we consider \(S\) a \(K3\) surface and \(L\) a globally generated line bundle on \(S\). In order to parameterize all pairs \((C, A)\) with non-simple Lazarsfeld–Mukai bundles, we need a global construction. We fix a nontrivial globally generated line bundle \(N\) of length \(\ell\), and set \(\mathcal{P}_{N,\ell}\) to be the family of vector bundles of rank 2 on \(S\) given by nontrivial extensions

\[
0 \to M \to E \to N \otimes I_{\ell} \to 0,
\]

where \(\xi\) is a zero-dimensional locally complete intersection subscheme (or the empty set) of \(S\) of length \(\ell\), and set

\[
\mathcal{P}_{N,\ell} := \{ [E] \in \bar{\mathcal{P}}_{N,\ell} : h^1(S, E) = h^2(S, E) = 0 \}.
\]

Equivalently (by Riemann–Roch), \([E] \in \mathcal{P}_{N,\ell}\) if and only if \(h^0(S, E) = g - c_2(E) + 3\) and \(h^1(S, E) = 0\). Note that any non-simple Lazarsfeld–Mukai bundle on \(S\) with determinant \(L\) belongs to some family \(\mathcal{P}_{N,\ell}\), from Lemma 1.4.

The family \(\mathcal{P}_{N,\ell}\), which, a priori, might be the empty set, is an open Zariski subset of a projective bundle of the Hilbert scheme \(S^{[\ell]}\).

Assuming that \(\mathcal{P}_{N,\ell} \neq \emptyset\), we consider the Grassmann bundle \(\mathcal{G}_{N,\ell}\) over \(\mathcal{P}_{N,\ell}\) classifying pairs \((E, A)\) with \([E] \in \mathcal{P}_{N,\ell}\) and \(A \in \mathbb{G}(2, H^0(S, E))\). If \(d := c_2(E)\) we define the rational map \(h_{N,\ell} : \mathcal{G}_{N,\ell} \dashrightarrow W_d^1([L])\), by setting \(h_{N,\ell}(E, A) := (C_A, A_A)\), where \(A_A \in \text{Pic}^d(C_A)\) is such that the following exact sequence on \(S\) holds:

\[
0 \to A \otimes \mathcal{O}_S \xrightarrow{ev} E \to K_{C_A} \otimes A_A^* \to 0.
\]

One computes \(\dim \mathcal{G}_{N,\ell} = g + \ell + h^0(S, M \otimes N^*)\). If we assume furthermore that \(\mathcal{P}_{N,\ell}\) contains a Lazarsfeld–Mukai vector bundle \(E\) on \(S\) with \(c_2(E) = d\) and consider \(W \subset W_d^1([L])\) the closure of the image of the rational map \(h_{N,\ell} : \mathcal{G}_{N,\ell} \dashrightarrow W_d^1([L])\), then we find \(\dim W = g + d - M \cdot N = g + \ell\).
On the other hand, if $C \in |L|$ has Clifford dimension one and $A$ is a globally generated line bundle on $C$ with $h^0(C, A) = 2$ and $[E_{C,A}] \in \mathcal{P}_{N,E}$, then $M \cdot N \geq \text{gon}(C)$.

These considerations on the indecomposable case, together with a simpler analysis of decomposable bundles, yield finally \[3\].

**Theorem 2.6.** Let $S$ be a K3 surface and $L$ a globally generated line bundle on $S$, such that general curves in $|L|$ are of Clifford dimension one. Suppose that $\rho(g, 1, k) \leq 0$, where $L^2 = 2g - 2$ and $k$ is the (constant) gonality of all smooth curves in $|L|$. Then for a general curve $C \in |L|$, we have

\[
\dim W^1_{g-k+2}(C) = g - 2k + 2.
\]

The condition \(\rho\) is called the *linear growth condition*. It is equivalent to

\[
\dim W^1_{g-k+2}(C) = \rho(g, 1, g - k + 2) = g - 2k + 2.
\]

Note that the condition that $C$ carry finitely many minimal pencils, which is a part of \[3\], appears explicitly in \[5\]. It is directly related to the constancy of the gonality discussed before.

3. **Green’s Conjecture for Curves on K3 Surfaces**

3.1. **Koszul Cohomology.** Let $X$ be a (not necessarily smooth) complex, irreducible, projective variety and $L \in \text{Pic}(X)$ globally generated. The Euler sequence on the projective space $\mathbb{P}(H^0(X, L)^*)$ pulls back to a short exact sequence of vector bundles on $X$

\[
0 \to M_L \to H^0(X, L) \otimes O_X \to L \to 0.
\]

After taking exterior powers in the sequence \(7\), twisting with multiples of $L$ and going to global sections, we obtain an exact sequence for any nonnegative $p$ and $q$:

\[
0 \to H^0(\wedge^{p+1} M_L \otimes L^q) \to \wedge^{p+1} H^0(L) \otimes H^0(L^q) \to H^0(\wedge^p M_L \otimes L^q).
\]

The finite-dimensional vector space $K_{p,q}(X, L) := \text{Coker}(\delta)$ is called the *Koszul cohomology space* \[4\] of $X$ with values in $L$ \[19, \ 11, \ 13\]. Observe that $K_{p,q}$ can be defined alternatively as:

\[
K_{p,q}(X, L) = \text{Ker}(H^1(\wedge^{p+1} M_L \otimes L^q) \to \wedge^{p+1} H^0(L) \otimes H^1(L^q)),
\]

description which is particularly useful when $X$ is a curve.

Several versions are used in practice, for example, replace $H^0(L)$ in \(\delta\) by a subspace that generates $L$ or twist \[5\] by $F \otimes L^{q-1}$ where $F$ is a coherent sheaf. For our presentation, however, we do not need to discuss these natural generalizations.

Composing the maps

\[
\wedge^{p+1} H^0(L) \otimes H^0(L^q) \xrightarrow{\delta} H^0(\wedge^p M_L \otimes L^q) \to \wedge^p H^0(L) \otimes H^0(L^q)
\]

we obtain, by iteration, a complex

\[
\wedge^{p+1} H^0(L) \otimes H^0(L^{q-1}) \to \wedge^p H^0(L) \otimes H^0(L^q) \to \wedge^{p-1} H^0(L) \otimes H^0(L^{q+1})
\]

whose cohomology at the middle is $K_{p,q}(X, L)$, and this is the definition given by Green \[11\].

\[7\] The indices $p$ and $q$ are usually forgotten when defining Koszul cohomology.
An important property of Koszul cohomology is upper-semicontinuity in flat families with constant cohomology; in particular, vanishing of Koszul cohomology is an open property in such families. For curves, constancy of $h^1$ is a consequence of flatness and of constancy of $h^0$, as shown by the Riemann–Roch theorem.

The original motivation for studying Koszul cohomology spaces was given by the relation with minimal resolutions over the polynomial ring. More precisely, if $L$ is very ample, then the Koszul cohomology computes the minimal resolution of the graded module

$$R(X, L) := \bigoplus_q H^0(X, L^q)$$

over the polynomial ring [11, 13]; see also [7, 2], in the sense that any graded piece that appears in the minimal resolution is (non-canonically) isomorphic to a $K_{p,q}$. If the image of $X$ is projectively normal, this module coincides with the homogeneous coordinate ring of $X$. The projective normality of $X$ can also be read off Koszul cohomology, being characterized by the vanishing condition $K_{0,q}(X, L) = 0$ for all $q \geq 2$. Furthermore, for a projectively normal $X$, the homogeneous ideal is generated by quadrics if and only if $K_{1,q}(X, L) = 0$ for all $q \geq 2$.

The phenomenon continues as follows: if $X$ is projectively normal and the homogeneous ideal is generated by quadrics, then the relations between the generators are linear if and only if $K_{2,q}(X, L) = 0$ for all $q \geq 2$, whence the relation with syzygies [11].

Other notable application of Koszul cohomology is the description of Castelnuovo–Mumford regularity, which coincides with [11, 2]

$$\min_q \{K_{p,q}(X, L) = 0, \text{ for all } p\}.$$}

Perhaps the most striking property of Koszul cohomology, discovered by Green and Lazarsfeld [11, Appendix], is a consequence of a nonvanishing result:

**Theorem 3.1** (Green–Lazarsfeld). Suppose $X$ is smooth and $L = L_1 \otimes L_2$ with $r_i := h^0(X, L_i) - 1 \geq 1$. Then $K_{r_1+r_2-1,1}(X, L) \neq 0$.

Note that the spaces $K_{p,1}$ have the following particular attribute: if $K_{p,1} \neq 0$ for some $p \geq 1$ then $K_{p',1} \neq 0$ for all $1 \leq p' \leq p$. This is obviously false for $K_{p,q}$ with $q \geq 2$.

Theorem 3.1 shows that the existence of nontrivial decompositions of $L$ reflects onto the existence of nontrivial Koszul classes in some space $K_{p,1}$. Its most important applications are for curves, in particular for canonical curves, case which is discussed in the next subsection. In the higher-dimensional cases, for surfaces, for instance, the meaning of Theorem 3.1 becomes more transparent if it is accompanied by a restriction theorem which compares the Koszul cohomology of $X$ with the Koszul cohomology of the linear sections [11]:

**Theorem 3.2** (Green). Suppose $X$ is smooth and $h^1(X, L^q) = 0$ for all $q \geq 1$. Then for any connected reduced divisor $Y \in |L|$, the restriction map induces an isomorphism

$$K_{p,q}(X, L) \cong K_{p,q}(Y, L|_Y),$$

for all $p$ and $q$.

---

8The dimension of $K_{1,q}$ indicates the number of generators of degree $(q + 1)$ in the homogeneous ideal.
The vanishing of $h^1(X, \mathcal{O}_X)$ suffices to prove that the restriction is an isomorphism between the spaces $K_{p,1}$.

In the next subsections, we shall apply Theorem 3.2 for $K3$ sections.

**Corollary 3.3.** Let $C$ be a smooth connected curve on a $K3$ surface $S$. Then

$$K_{p,q}(S, \mathcal{O}_S(C)) \cong K_{p,q}(C, K_C)$$

for all $p$ and $q$.

One direct consequence is a duality theorem for Koszul cohomology of $K3$ surfaces. It shows the symmetry of the table containing the dimensions of the spaces $K_{p,q}$, called the Betti table.

### 3.2. Statement of Green’s Conjecture

Let us particularize Theorem 3.1 for a canonical curve. Consider $C$ a smooth curve and choose a decomposition $K_C = A \otimes K_C(-A)$. Theorem 3.1 applies only if $h^0(C, A) \geq 2$ and $h^1(C, A) \geq 2$, i.e., if $A$ contributes to the Clifford index. The quantity $r_1 + r_2 - 1$ which appears in the statement equals $g - \text{Cliff}(A) - 2$, and hence, if $A$ computes the Clifford index, we obtain the following:

**Theorem 3.4** (Green–Lazarsfeld). For any smooth curve $C$ of genus $g$ Clifford index $c$ we have $K_{g-c-2,1}(C, K_C) \neq 0$.

It is natural to determine whether or not this result is sharp, question which is addressed in the statement Green’s conjecture:

**Conjecture 3.5** (Green). Let $C$ be a smooth curve. For all $p \geq g - c - 1$, we have $K_{p,1}(C, K_C) = 0$.

For the moment, Green’s conjecture remains a hard open problem. At the same time, strong evidence has been discovered. For instance, it is known to hold for general curves [31, 32], for curves of odd genus and maximal Clifford index [32, 15], for general curves of given gonality [31, 30, 29], for curves with small Brill–Noether loci [1], for plane curves [21], for curves on $K3$ surfaces [31, 32, 3], etc.; see also [2] for a discussion.

We shall consider in the sequel the case of curves on $K3$ surfaces with emphasis on Voisin’s approach to the problem and the role played by Lazarsfeld–Mukai bundles. It is interesting to notice that Green’s conjecture for $K3$ sections can be formulated directly in the $K3$ setup, as a vanishing result on the moduli space $\mathcal{F}_g$ of polarized $K3$ surfaces. However, in the proof of this statement, as it usually happens in mathematics, we have to exit the $K3$ world, prove a more general result in the extended setup, and return to $K3$ surfaces. The steps we have to take, ordered logically and not chronologically, are the following. In the first, most elaborated step, one finds an example for odd genus [32, 31]. At this stage, we are placed in the moduli space $\mathcal{F}_{2k+1}$. Secondly, we exit the $K3$ world, land in $\mathcal{M}_{2k+1}$, and prove the equality of two divisors [15, 31]. The first step is used, and the identification of the divisors extends to their closure over the component $\Delta_0$ of the boundary [1]. In the third step, we jump from a gonality stratum $\mathcal{M}^1_{g,d}$ in a moduli space $\mathcal{M}_g$ to the boundary of another moduli space of stable curves $\overline{\mathcal{M}}_{2k+1}$, where $k = g - d + 1$ [1]. The second step reflects into a vanishing result on an explicit open subset of $\overline{\mathcal{M}}_{2k+1}$. For higher-dimensional manifolds, some supplementary vanishing conditions are required [11, 13].

---

9Duality for Koszul cohomology of curves follows from Serre’s duality. For higher-dimensional manifolds, some supplementary vanishing conditions are required [11, 13].

10Voisin’s and Teixidor’s cases complete each other quite remarkably.
Finally one goes back to K3 surfaces and applies the latter vanishing result \cite{la} on $\mathcal{F}_a$. In the steps concerned with K3 surfaces (first and last), the Lazarsfeld–Mukai bundles are central objects.

3.3. **Voisin’s Approach.** The proof of the generic Green conjecture was achieved by Voisin in two papers \cite{voisin1, voisin2}, using a completely different approach to Koszul cohomology via Hilbert scheme of points.

Let $X$ be a complex connected projective manifold and $L$ a line bundle on $X$. It is obvious that any global section $\sigma$ is uniquely determined by the collection $\{\sigma(x)\}_x$, where $\sigma(x) \in L|_x \cong \mathbb{C}$ and $x$ belongs to a nonempty open subset of $X$. One tries to find a similar fact for multisections in $\bigwedge^n H^0(X, L)$.

Let $\sigma_1 \wedge \cdots \wedge \sigma_n$ be a decomposable element in $\bigwedge^n H^0(X, L)$ with $n \geq 1$. By analogy with the case $n = 1$, we have to look at the restriction $\sigma_1|_\xi \wedge \cdots \wedge \sigma_n|_\xi \in \bigwedge^n L|_\xi$ where $\xi$ is now a zero-dimensional subscheme, and it is clear that we need $n$ points for otherwise this restriction would be zero. Note that a zero-dimensional subscheme of length $n$ defines a point in the punctual Hilbert scheme $X^{[n]}$. For technical reasons, we shall restrict to curvilinear subschemes\footnote{A curvilinear subscheme is defined locally, in the classical topology, by $x_1 = \cdots = x_{n-1} = x_n^k = 0$; equivalently, it is locally embedded in a smooth curve.} which form a large open subset $X_c^{[n]}$. For any $n$ points $(\xi_i)_{i=1}^n$ that give rise to a decomposable element $\sigma_1 \wedge \cdots \wedge \sigma_n$, choosing one of them, we denote by $\sigma_{i,n} = \sigma_i \wedge \cdots \wedge \sigma_n$ a collection of sections $\sigma_{i,n} \in \bigwedge^n H^0(X, L)|_{\xi_i} \cong L|_{\xi_i}$. In conclusion, the collection $\{\sigma_{i,n}\}_{i=1}^n$ defines a section in the line bundle $\det(L^{[n]})$. The map we are looking at $\bigwedge^n H^0(L) \to H^0(\det(L^{[n]}))$ is deduced from the evaluation map $\ev_n : H^0(L) \otimes \mathcal{O}_{X^{[n]}} \to L^{[n]}$, taking $\bigwedge^n \ev_n$ and applying $H^0$. It is remarkable that \cite{voisin1, voisin2, lei}.

**Theorem 3.6** (Voisin, Ellingsrud–Göttsche–Lehn). The map

$$H^0(\bigwedge^n \ev_n) : \bigwedge^n H^0(X, L) \to H^0(X^{[n]}, \det(L^{[n]}))$$

is an isomorphism.

Since the exterior powers of $H^0(L)$ are building blocks for Koszul cohomology, it is natural to believe that the isomorphism above yields a relation between the Koszul cohomology and the Hilbert scheme. To this end, the Koszul differentials must be reinterpreted in the new context.

There is a natural birational morphism\footnote{The connectedness of $X^{[n]}$ follows from the observation that a curvilinear subscheme is a deformation of a reduced subscheme.}

$$\tau : \Xi_{n+1} \to X^{[n]} \times X, \ (\xi, x) \mapsto (\xi - x, x)$$

presenting $\Xi_{n+1}$ as the blowup of $X^{[n]} \times X$ along $\Xi_n$. If we denote by $D_\tau$ the exceptional locus, we obtain an inclusion \cite{la}

$$q^* \det(L^{[n+1]}) \cong \tau^*(\det(L^{[n]}) \boxtimes L)(-D_\tau) \hookrightarrow \tau^*(\det(L^{[n]}) \boxtimes L)$$

We see one advantage of working on $X^{[n]}$: subtraction makes sense only for curvilinear subschemes.
whence
\[ H^0 \left( X^{[n+1]}_c, \det(L^{[n+1]}) \right) \hookrightarrow H^0 \left( X^{[n]}_c \times X, \det(L^{[n]}) \boxtimes L \right), \]
identifying the left-hand member with the kernel of a Koszul differential \[31\]. A version of this identification leads us to \[31\] [32]:

**Theorem 3.7** (Voisin). For any integers \( m \) and \( n \), \( K_{n,m}(X, L) \) is isomorphic to the cokernel of the restriction map:

\[ H^0 \left( X^{[n+1]}_c \times X, \det(L^{[n+1]}) \boxtimes L^{m-1} \right) \rightarrow H^0 \left( \Xi_{n+1}, \det(L^{[n+1]}) \boxtimes L^{m-1}|_{\Xi_{n+1}} \right). \]

The vanishing of Koszul cohomology is thus reduced to proving surjectivity of the restriction map above. In general, it is very hard to prove surjectivity directly, and one has to make a suitable base-change \[31\].

### 3.4. The Role of Lazarsfeld–Mukai Bundles in the Generic Green Conjecture and Consequences

In order to prove Green’s conjecture for general curves, it suffices to exhibit one example of a curve of maximal Clifford index, which verifies the predicted vanishing. Afterwards, the vanishing of Koszul cohomology propagates by semicontinuity. Even so, finding one single example is a task of major difficulty. The curves used by Voisin in \[31, 32\] are \( K \)-sections, and the setups change slightly, according to the parity of the genus. For even genus, we have \[31\]:

**Theorem 3.8** (Voisin). Suppose that \( g = 2k \). Consider \( S \) a \( K \) surface with \( \text{Pic}(S) \cong \mathbb{Z} \cdot L, L^2 = 2g - 2, \) and \( C \in |L| \) a smooth curve. Then \( K_{k,1}(C, K_C) = 0 \).

For odd genus, the result is \[32\]:

**Theorem 3.9** (Voisin). Suppose that \( g = 2k + 1 \). Consider \( S \) a \( K \) surface with \( \text{Pic}(S) \cong \mathbb{Z} \cdot L \oplus \mathbb{Z} \cdot \Gamma, L^2 = 2g - 2, \) \( \Gamma \) a smooth rational curve. \( L \cdot \Gamma = 2 \) and \( C \in |L| \) a smooth curve. Then \( K_{k,1}(C, K_C) = 0 \).

Note that the generic value for the Clifford index in genus \( g \) is \( [(g-1)/2] \), and hence, in both cases, the prediction made by Green’s conjecture for general curve \( C \) is precisely \( K_{k,1}(C, K_C) = 0 \).

There are several reasons for making these choices: the curves have maximal Clifford index, by Theorem \[27, 28\] (and the Clifford dimension is one), the Lazarsfeld–Mukai bundles associated to minimal pencils are \( L \)-stable, the hyperplane section theorem applies, etc.

We outline here the role played by Lazarsfeld–Mukai bundles in Voisin’s proof and, for simplicity, we restrict to the even-genus case. By the hyperplane section Theorem \[37, 2\] the required vanishing on the curve is equivalent to \( K_{k,1}(S, L) = 0 \). From the description of Koszul cohomology in terms of Hilbert schemes, Theorem \[3.7\] adapting the notation from the previous subsection, one has to prove the surjectivity of the map

\[ q^* : H^0 \left( S^{[n+1]}_c, \det(L^{[n+1]}) \right) \rightarrow H^0 \left( \Xi_{n+1}, q^* \det(L^{[n+1]})|_{\Xi_{n+1}} \right). \]

The surjectivity is proved after performing a suitable base-change.

We are in the case \( \rho(g, 1, k + 1) = 0 \); hence there is a unique Lazarsfeld–Mukai bundle \( E \) on \( S \) associated to all \( g^1_{k+1} \) on curves in \( |L| \). The uniqueness yields an alternate description of \( E \) as extension

\[ 0 \rightarrow \mathcal{O}_S \rightarrow E \rightarrow L \otimes I_\xi \rightarrow 0, \]
where $\xi$ varies in $S_c^{[k+1]}$.

There exists a morphism $\mathbb{P}H^0(S, E) \to S^{[k+1]}$ that sends a global section $s \in H^0(S, E)$ to its zero set $Z(s)$. By restriction to an open subset $\mathbb{P} \subset \mathbb{P}H^0(S, E)$, we obtain a morphism $\mathbb{P} \to S_c^{[k+1]}$, inducing a commutative diagram

$$
\begin{array}{ccc}
\mathbb{P}' = \mathbb{P} \times S_c^{[k+1]} & \longrightarrow & S_c^{[k+1]} \\
\downarrow q' & \quad & \downarrow q \\
\mathbb{P} & \longrightarrow & S_c^{[k+1]}
\end{array}
$$

Set-theoretically

$$\mathbb{P}' = \{(Z(s), x) | s \in H^0(S, E), x \in Z(s)\}.$$

Unfortunately, this very natural base-change does not satisfy the necessary conditions that imply the surjectivity of $q^*$, [31]. Voisin modifies slightly this construction and replaces $\mathbb{P}$ with another variety related to $\mathbb{P}$ which parameterizes zero-cycles of the form $Z(s) - x + y$ with $[s] \in \mathbb{P}$, $x \in \text{Supp}(Z(s))$ and $y \in S$. It turns out, after numerous elaborated calculations using the rich geometric framework provided by the Lazarsfeld–Mukai bundle, that the new base-change is suitable and the surjectivity of $q^*$ follows from vanishing results on the Grassmannian [31].

In the odd-genus case, Voisin proves first Green’s conjecture for smooth curves in $|L + \Gamma|$, which are easily seen to be of maximal Clifford index. The situation on $|L + \Gamma|$ is somewhat close to the setup of Theorem 4.8 and the proof is similar.

The next hard part is to descend from the vanishing of $K_{k+1,1}(S, L \otimes \mathcal{O}_S(\Gamma))$ to the vanishing of $K_{k,1}(S, L)$. This step uses again intensively the unique Lazarsfeld–Mukai bundle associated to any $g_{k+2}^1$ on curves in $|L + \Gamma|$.

The odd-genus case is of maximal interest: mixed with Hirschowitz-Ramanan result [15]. Theorem 3.9 gives a solution to Green’s conjecture for any curve of odd genus and maximal Clifford index:

**Theorem 3.10** (Hirschowitz–Ramanan, Voisin). Let $C$ be a smooth curve of odd genus $2k + 1 \geq 5$ and Clifford index $k$. Then $K_{k,1}(C, K_C) = 0$.

Note that Theorem 3.10 implies the following statement:

**Corollary 3.11.** A smooth curve of odd genus and maximal Clifford index has Clifford dimension one.

The proof of Theorem 3.10 relies on the comparison of two effective divisors on the moduli space of curves $\mathcal{M}_{2k+1}$, one given by the condition $\text{gon}(C) \leq k + 1$, which is known to be a divisor from [14], and the second given by $K_{k,1}(C, K_C) \neq 0$. By duality $K_{k,1}(C, K_C) \cong K_{k-2,2}(C, K_C)$. Note that $K_{k-2,2}(C, K_C)$ is isomorphic to

$$\text{Coker} \left( \wedge^k H^0(K_C) \otimes H^0(K_C) / \wedge^{k+1} H^0(K_C) \to H^0(\wedge^{k-1} M_{K_C} \otimes K_C^2) \right)$$

and the two members have the same dimension. The locus of curves with $K_{k,1} \neq 0$ can be described as the degeneracy locus of a morphism between vector bundles of the same dimension, and hence it is a virtual divisor. Theorem 3.9 implies that this locus is not the whole space, and in conclusion it must be an effective divisor. Theorem 3.11 already gives an inclusion between the supports of two divisors in question, and the set-theoretic equality is obtained from a divisor class calculation [15].
3.5. Green’s Conjecture for Curves on K3 Surfaces. We have already seen that general K3 sections have a mild behavior from the Brill–Noether theory viewpoint. In some sense, they behave like general curves in any gonality stratum of the moduli space of curves.

As in the previous subsections, fix a K3 surface $S$ and a globally generated line bundle $L$ with $L^2 = 2g - 2$ on $S$, and denote by $g$ the gonality of a general smooth curve in the linear system $|L|$. Suppose that $\rho(g, 1, k) \leq 0$ to exclude the case $g = 2k - 3$ (when $\rho(g, 1, k) = 1$). If in addition the curves in $|L|$ have Clifford dimension one, Theorem 2.6 shows that
\[
\dim W_{g-k+2}^1(C) = \rho(g, 1, g - k + 2) = g - 2k + 2,
\]
property which was called the linear growth condition.

This property appears in connection with Green’s conjecture [1] for a much larger class of curves:

**Theorem 3.12.** If $C$ is any smooth curve of genus $g \geq 6$ and gonality $3 \leq k < [g/2] + 2$ with $\dim W_{g-k+2}^1(C) = \rho(g, 1, g - k + 2)$, then $K_{g-k+1}(C, K_C) = 0$.

One effect of Theorems 3.12 and 3.1 is that an arbitrary curve that satisfies the linear growth condition then $X$ has maximal gonality, i.e., $X$ lies outside the closure of the divisor $\mathcal{M}_{2(g-k+1)+1, g-k+2}$ consisting of curves with a pencil $g_{g-k+2}$. The class of the failure locus of Green’s conjecture on $\mathcal{M}_{2(g-k+1)+1, g-k+2}$ is a multiple of the divisor $\mathcal{M}_{2(g-k+1)+1, g-k+2}$; hence Theorem 3.12 extends to irreducible stable curves of genus $2(g - k + 1) + 1$ of maximal gonality $(g - k + 3)$. In particular, $K_{g-k+1}(X, \omega_X) = 0$, implying $K_{g-k+1}(C, K_C) = 0$.

Coming back to the original situation, we conclude from Theorems 3.12 and 2.6 and Corollary 3.3 that Green’s conjecture holds for a K3 section $C$ having Clifford dimension one. If $\text{Cliff}(C) = \text{gon}(C) - 3$, either $C$ is a smooth plane curve or else there exist smooth curves $D, \Gamma \subset S$, with $\Gamma^2 = -2, \Gamma \cdot D = 1$ and $D^2 \geq 2$, such that $C \equiv 2D + \Gamma$ and $\text{Cliff}(C) = \text{Cliff}(\mathcal{O}_C(D))$. The linear growth condition is no longer satisfied, and this case is treated differently, by degeneration to a reduced curve with two irreducible components [3].

The outcome of this analysis of the Brill–Noether loci is the following [31, 32, 3]:

**Theorem 3.13.** Green’s conjecture is valid for any smooth curve on a K3 surface.

Applying Theorem 3.13, Theorem 3.2 and the duality, we obtain a full description of the situations when Koszul cohomology of a K3 surface is zero [3]:

The gonality for a singular stable curve is defined in terms of admissible covers [14].
Theorem 3.14. Let $S$ be a K3 surface and $L$ a globally generated line bundle with $L^2 = 2g - 2 \geq 2$. The Koszul cohomology group $K_{p,q}(S,L)$ is nonzero if and only if one of the following cases occurs:

1. $q = 0$ and $p = 0$, or
2. $q = 1$, $1 \leq p \leq g - c - 2$, or
3. $q = 2$ and $c \leq p \leq g - 1$, or
4. $q = 3$ and $p = g - 2$.

The moral is that the shape of the Betti table, i.e., the distribution of zeros in the table, of a polarized K3 surface is completely determined by the geometry of hyperplane sections; this is one of the many situations where algebra and geometry are intricately related.

4. Counterexamples to Mercat’s Conjecture in Rank Two

Starting from Mukai’s works, experts tried to generalize the classical Brill–Noether theory to higher-rank vector bundles on curves. Within these extended theories\textsuperscript{15} we note the attempt to find a proper generalization of the Clifford index. H. Lange and P. Newstead proposed the following definition. Let $E$ be a semistable vector bundle of rank $n$ on a smooth curve $C$. Put

$$\gamma(E) := \mu(E) - \frac{h^0(E)}{n} + 2.$$  

Definition 4.1 (Lange–Newstead). The Clifford index of rank $n$ of $C$ is

$$\text{Cliff}_n(C) := \min \{ \gamma(E) : \mu(E) \leq g - 1, h^0(E) \geq 2n \}.$$  

From the definition, it is clear that $\text{Cliff}_1(C) = \text{Cliff}(C)$ and $\text{Cliff}_n(C) \leq \text{Cliff}(C)$ for all $n$\textsuperscript{16}.

Mercat conjectured\textsuperscript{22} that $\text{Cliff}_n(C) = \text{Cliff}(C)$. In rank two, the conjecture is known to hold in a number of cases: for general curves of small gonality, i.e., corresponding to a general point in a gonality stratum $\mathcal{M}_{g,k}^{1}$ for small $k$ (Lange–Newstead), for plane curves (Lange–Newstead), for general curves of genus $\leq 16$ (Farkas–Ortega), etc. However, even in rank two, the conjecture is false. It is remarkable that counterexamples are found for curves of maximal Clifford index\textsuperscript{10}:

Theorem 4.2 (Farkas–Ortega). Fix $p \geq 1$, $a \geq 2p + 3$. Then there exists a smooth curve of genus $2a + 1$ of maximal Clifford index lying on a smooth K3 surface $S$ with $\text{Pic}(S) = \mathbb{Z} \cdot C \oplus \mathbb{Z} \cdot H$, $H^2 = 2p + 2$, $C^2 = 2g - 2$, $C \cdot H = 2a + 2p + 1$, and there exists a stable rank-two vector bundle $E$ with $\det(E) = \mathcal{O}_S(H)$ with $h^0(E) = p + 3$, $\gamma(E) = a - \frac{1}{2} < a = \text{Cliff}(A)$, and hence Mercat’s conjecture in rank two fails for $C$.

The proof uses restriction of Lazarsfeld–Mukai bundles. However, it is interesting that the bundles are not restricted to the same curves to which they are associated. More precisely, the genus of $H$ is $2p + 2$ and $H$ has maximal gonality $p + 2$. Consider $A$ a minimal pencil on $H$, and take $E = E_{H,A}$ the associated Lazarsfeld–Mukai bundle. The restriction of $E$ to $C$ is stable and verifies all the required properties.

A particularly interesting case is $g = 11$. In this case, as shown by Mukai\textsuperscript{26}, a general curve $C$ lies on a unique K3 surface $S$ such that $C$ generates $\text{Pic}(S)$.

\textsuperscript{15}Higher-rank Brill–Noether theory is a major, rapidly growing research field, and it deserves a separate dedicated survey.

\textsuperscript{16}For any line bundle $A$, we have $\gamma(A^\otimes n) = \text{Cliff}(A)$.
It is remarkable that the failure locus of Mercat’s conjecture in rank two coincides with the Noether-Lefschetz divisor
\[ \mathcal{N}L_{1,13} := \{ [C] \in \mathcal{M}_{11} : C \text{ lies on a } K3 \text{ surface } S, \text{ Pic}(S) \supset Z \cdot C \oplus Z \cdot H, \}
\]
inside the moduli space \( \mathcal{M}_{11} \). We refer to \([10]\) for details.

**References**

[1] Aprodu, M.: Remarks on syzygies of d-gonal curves. Math. Res. Lett. **12**(3), 387–400 (2005)
[2] Aprodu, M., Nagel, J.: Koszul cohomology and algebraic geometry. University Lecture Series, vol. 52. American Mathematical Society, Providence (2010)
[3] Aprodu, M., Farkas, G.: Green’s conjecture for curves on arbitrary K3 surfaces. Compositio Math. **147**, 839–851 (2011)
[4] Arbarello, E., Cornalba, M., Griffiths, P.A., Harris, J.: Geometry of algebraic curves, vol. I. Grundlehren der Mathematischen Wissenschaften, vol. 267. Springer, New York (1985)
[5] Ciliberto, C., Farkas, G.: Green’s conjecture for curves on arbitrary K3 surfaces. Internat. Math. Res. Nat. Acad. Sci. USA **96**, 3000–3002 (1999)
[6] Donagi, R., Morrison, D.R.: Linear systems on K3 sections. Diff. J. Geom. **29**, 49–64 (1989)
[7] Eisenbud, D.: Geometry of syzygies. Graduate Texts in Mathematics, vol. 229. Springer, New York (2005)
[8] Eisenbud, D., Lange, H., Martens, G., Schreyer, F.-O.: The Clifford dimension of a projective curve. Compositio Math. **72**, 173–204 (1989)
[9] Ellingsrud, G., Göttsche, L., Lehn, M.: On the cobordism class of the Hilbert scheme of a surface. Alg. J. Geom. **10**, 81–100 (2001)
[10] Farkas, G., Ortega, A.: Higher-rank Brill–Noether theory on sections of K3 surfaces. Internat. Math. J. **23**, 1250075, 18 pp (2012)
[11] Green, M.: Koszul cohomology and the geometry of projective varieties. Diff. J. Geom. **19**, 125–171 (1984)
[12] Green, M., Lazarsfeld, R.: Special divisors on curves on a K3 surface. Inventiones Math. **89**, 73–90 (1987)
[13] Green, M.: Koszul cohomology and geometry. In: Cornalba, M., et al. (eds.) Proceedings of the first conference on Riemann Surfaces held in Trieste, Italy, November 1987, pp. 177–200. World Scientific, Singapore (1989)
[14] Harris, J., Mumford, D.: On the Kodaira dimension of the moduli space of curves. Inventiones Math. **67**, 23–86 (1982)
[15] Hirschowitz, A., Ramanan, S.: New evidence for Greens conjecture on syzygies of canonical curves. Ann. Sci. École Norm. Sup. **31**, 145–152 (1998)
[16] Iskovskih, V.A., Prokhorov, Yu.: Fano varieties. In: Parshin, A.N., Shafarevich, I.R. (eds.) Algebraic Geometry V. Encyclopedia of Mathematical Science. Springer, New York (1999)
[17] Knutsen, A.: On two conjectures for curves on K3 surfaces. Internat. Math. J. **20**, 1547–1560 (2009)
[18] Lazarsfeld, R.: Brill–Noether–Petri without degenerations. Diff. J. Geom. **23**, 299–307 (1986)
[19] Lazarsfeld, R.: A sampling of vector bundle techniques in the study of linear series. In: Cornalba, M., et al. (eds.) Proceedings of the first conference on Riemann surfaces held in Trieste, Italy, November 1987, pp. 500–559. World Scientific, Singapore (1989)
[20] Lelli-Chiesa, M.: Stability of rank-3 Lazarsfeld–Mukai bundles on K3 surfaces. Proc. Lond. Math. Soc. **107**, 451–479 (2013)
[21] Loose, F.: On the graded Betti numbers of plane algebraic curves. Manuscr. Math. **64**, 503–514 (1989)
[22] Mercat, V.: Clifford’s theorem and higher rank vector bundles. Internat. Math. J. **13**, 785–796 (2002)
[23] Mukai, S.: Biregular classification of Fano 3-folds and Fano manifolds of coindex 3. Proc. Natl. Acad. Sci. USA **86**, 3000–3002 (1989)
[24] Mukai, S.: Fano 3-folds. London Math. Soc. Lect. Note Ser. **179**, 255–263 (1992)
[25] Mukai, S.: Curves and K3 surfaces of genus eleven. In: Moduli of Vector Bundles. Lecture Notes in Pure Applied Mathematics, vol. 179, pp. 189–197. Dekker, New York (1996)
[26] Pareschi, G.: A proof of Lazarsfeld’s theorem on curves on K3 surfaces. Alg. J. Geom. 4, 195–200 (1995)
[27] Reid, M.: Special linear systems on curves lying on K3 surfaces. J. London Math. Soc. (2) 13, 454–458 (1976)
[28] Saint-Donat, B.: Projective models of K3 surfaces. American J. Math. 96(4), 602–639 (1974)
[29] Schreyer, F.-O.: Green’s conjecture for general p-gonal curves of large genus. Algebraic curves and projective geometry (Trento, 1988), pp. 254–260. Lecture Notes in Mathematics, vol. 1389. Springer, Berlin (1989)
[30] Teixidor i Bigas, M.: Green’s conjecture for the generic r-gonal curve of genus $g \geq 3r - 7$. Duke Math. J. 111, 195–222 (2002)
[31] Voisin, C.: Green’s generic syzygy conjecture for curves of even genus lying on a K3 surface. J. European Math. Soc. 4, 363–404 (2002)
[32] Voisin, C.: Green’s canonical syzygy conjecture for generic curves of odd genus. Compositio Math. 141, 1163–1190 (2005)