This paper is a continuation of \cite{4}, in which we introduced a reduction of the Toda lattice hierarchy (in the limit of infinitesimal lattice spacing), called the equivariant Toda lattice, by imposing the constraints

\[ (\delta_1 - \bar{\delta}_1)L = \nu \partial L, \quad (\delta_1 - \bar{\delta}_1)\bar{L} = \nu \partial \bar{L} \]

on the Lax operators \( L \) and \( \bar{L} \). This reduction is a deformation of the Toda chain, which is the reduction corresponding to the constraint \( L = \bar{L} \).

Seeking an integrable system which would describe the Gromov-Witten invariants of \( \mathbb{C}P^1 \), Eguchi and Yang \cite{2} studied the Toda chain. They conjectured the existence in the limit of infinitesimal lattice spacing of an additional hierarchy of commuting flows: these flows were constructed independently by the author \cite{3} and Zhang \cite{10} using homological perturbation theory, and shown to be bihamiltonian. Recently, Carlet, Dubrovin and Zhang \cite{1} have shown that these additional flows may be described by Lax equations involving the logarithm of the Lax operator \( L \), as conjectured by Eguchi and Yang.

In this paper, borrowing the ideas of Carlet et al., we prove that the equivariant Toda lattice has a Hamiltonian structure which is a deformation of the first Hamiltonian structure of the Toda chain. (We were however unable to find a bihamiltonian structure.)

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1. Notation

In this section, we recall some of the terminology of \cite{4}. All of the commutative algebras which we consider carry an involution \( p \mapsto \bar{p} \). By a differential algebra, we mean a commutative algebra with derivation \( \partial \) such that

\[ \partial \bar{p} = \bar{\partial p}. \]

If \( \mathcal{A} \) is a differential algebra and \( S \) is a set, the free differential algebra \( \mathcal{A}\{S\} \) generated by \( S \) is the polynomial algebra

\[ \mathcal{A}[\partial^n x, \partial^n \bar{x} | x \in S, n \geq 0], \]
with differential $\partial(\partial^n x) = \partial^{n+1} x$. An evolutionary derivation $\delta$ of a differential algebra $\mathcal{A}$ is a derivation such that $[\partial, \delta] = 0$.

Let $\mathcal{A}$ be a differential algebra over $\mathbb{Q}_\varepsilon$, and let $q \in \mathcal{A}$ be a regular element (that is, having no zero-divisors) such that $\bar{q} = q$. The localization $q^{-1}\mathcal{A}$ of $\mathcal{A}$ is a differential algebra, with differential $\partial(q^{-1}) = -q^{-2}\partial q$. Let $\Phi_{\pm}(\mathcal{A}, q)$ be the associative algebras of difference operators

$$
\Phi_+(\mathcal{A}, q) = \left\{ \sum_{k=-\infty}^{\infty} p_k \Lambda^k \mid p_k \in q^{-1} \mathcal{A}, p_k = 0 \text{ for } k \ll 0 \right\},
$$

$$
\Phi_- (\mathcal{A}, q) = \left\{ \sum_{k=-\infty}^{\infty} p_k \Lambda^k \mid p_k \in \mathcal{A}, p_k = 0 \text{ for } k \gg 0 \right\},
$$

with product

$$
\sum_i a_i \Lambda^i \cdot \sum_j b_j \Lambda^j = \sum_k \left( \sum_{i+j=k} \left( E^{-j/2} a_i \right) \left( E^{i/2} b_j \right) \right) \Lambda^k.
$$

Note that $\Phi_-(\mathcal{A}, q)$ is in fact independent of $q$.

Let $A \mapsto A_{\pm}$ be the projections from on $\Phi_{\pm}(\mathcal{A}, q)$ defined by the formulas

$$
\left( \sum_{k=-\infty}^{\infty} p_k \Lambda^k \right)_+ = \sum_{k=0}^{\infty} p_k \Lambda^k, \quad \left( \sum_{k=-\infty}^{\infty} p_k \Lambda^k \right)_- = \sum_{k=-\infty}^{\infty} p_k \Lambda^k.
$$

We see that $A = A_- + A_+$. Define the residue $\text{res} : \Phi_{\pm}(\mathcal{A}, q) \to \mathcal{A}$ by the formula

$$
\text{res} \left( \sum_{k=-\infty}^{\infty} p_k \Lambda^k \right) = p_0.
$$

For $k \in \mathbb{Z}$, let $[k]$ be the isomorphism of $\mathcal{A}$

$$
[k] = \frac{E^{k/2} - E^{-k/2}}{E^{1/2} - E^{-1/2}} = \sum_{j=1}^{k} E^{(k+1)/2-j} = k + O(\varepsilon^2).
$$

Define $q^{[k]}$ by the recursion

$$
q^{[k+1]} = E^k q \cdot E^{-1/2} q^{[k]},
$$

with initial condition $q^{[0]} = 1$. The involution

$$
A = \sum_{k=-\infty}^{\infty} p_k \Lambda^k \mapsto \bar{A} = \sum_{k=1}^{\infty} \bar{p}_k q^{[k]} \Lambda^{-k} + \bar{p}_0 + \sum_{k=1}^{\infty} \bar{p}_{-k} q^{-[k]} \Lambda^k,
$$

defines an anti-isomorphism between the algebras $\Phi_+(\mathcal{A}, q)$ and $\Phi_-(\mathcal{A}, q)$. 

2
2. The dressing operator of the Toda lattice

Let $\mathcal{B}$ be the free differential algebra $\mathbb{Q}_\varepsilon \{ q, w_k \mid k > 0 \}/(q - \bar{q})$, and let $W$ be the universal dressing operator of the Toda lattice

$$W = 1 + \sum_{k=1}^{\infty} w_k \Lambda^{-k} \in \Phi_-(\mathcal{B}, q).$$

The coefficients $w_k^* \in \mathcal{B}$ of $W^{-1}$,

$$W^{-1} = 1 + \sum_{k=1}^{\infty} w_k^* \Lambda^{-k},$$

are characterized by the recursion

$$w_k^* = -w_k - \sum_{j=1}^{k-1} \left( E^{(k-j)/2} w_j \right) \left( E^{-j/2} w_{k-j}^* \right),$$

obtained by extracting the coefficient of $\Lambda^{-k}$ in the equation $WW^{-1} = I$.

The Lax operator of the Toda lattice is the difference operator

$$L = W \Lambda W^{-1} = \Lambda + \sum_{k=1}^{\infty} a_k \Lambda^{-k+1} \in \Phi_-(\mathcal{B}, q).$$

Since $a_k + \varepsilon \nabla w_k$ lies in the differential ideal $(w_1, \ldots, w_{k-1})$ for all $k > 0$, we see that the sequence of elements $a_k$ of $\mathcal{B}$ defines an embedding of differential algebras

$$\mathcal{A} = \mathbb{Q}_\varepsilon \{ q, a_k \mid k > 0 \}/(q - \bar{q}) \hookrightarrow \mathcal{B}.$$

The conjugate Lax operator $\tilde{L}$ is

$$\tilde{L} = W^{-1}(q \Lambda^{-1}) W = q \Lambda^{-1} + \sum_{k=1}^{\infty} \tilde{a}_k q^{-[k-1]} \Lambda^{k-1}. $$

Let $B_n = \varepsilon^{-1} L_n^+$ and $C_n = -\varepsilon^{-1} \tilde{L}_n^-$. We define evolutionary derivations $(\delta_n, \bar{\delta}_n \mid n > 0)$ of $\mathcal{B}$ by the formulas

(2) \quad $\varepsilon \delta_n W + L_n^W = \varepsilon \bar{\delta}_n W + \tilde{L}_n^- W = 0.$

These derivations are called the flows of the Toda lattice. The action of the derivations $\delta_n$ and $\bar{\delta}_n$ on $\mathcal{B}$ restricts to an action on $\mathcal{A}$ such that the derivatives of the Lax operator $L$ are given by the Lax equations $\delta_n L = [B_n, L]$ and $\bar{\delta}_n L = -[C_n, L]$. These flows on $\mathcal{B}$ commute, by the Zakharov-Shabat equations

$$\delta_m B_n - \delta_n B_m = [B_m, B_n], \quad \delta_m C_n - \bar{\delta}_n B_m = [B_m, C_n], \quad \bar{\delta}_m C_n - \bar{\delta}_n C_m = [C_m, C_n],$$

and $\delta_n$ is indeed the conjugate derivation to $\bar{\delta}_n$. 

3
Let $\log(L) = W \log(\Lambda) W^{-1}$, where $\log(\Lambda)$ is a formal symbol for the operator $\varepsilon \partial$: namely, we have the commutation relation

$$\left[\log(\Lambda), f\right] = \varepsilon \partial f, \quad f \in \Phi_-(\mathcal{A}, q).$$

Define $\ell$ to be the difference operator

$$\ell = \log(\Lambda) - \log(L) = \varepsilon(\partial W)W^{-1}$$

$$= \varepsilon \left( \partial w_k + \sum_{j=1}^{k-1} \left( E^{(k-j)/2} \partial w_j \right) \left( E^{-j/2} w_{k-j}^* \right) \right).$$

The following is a result of Carlet, Dubrovin and Zhang [1]. (They work in the context of the Toda chain, so they assume that $a_k = 0$, $k > 2$.)

**Proposition 2.1.** The difference operator $\ell$ is an element of $\Phi_-(\mathcal{A}, q)$.

**Proof.** Write

$$\ell = \sum_{k=1}^{\infty} b_k \Lambda^{-k} \in \Phi_-(\mathcal{B}, q).$$

We show that $b_k \in \mathcal{A}$ for all $k > 0$, by induction on $k$.

Define elements $p_k(n)$ of $\mathcal{A}$ by the formula

$$L^n = \sum_{k=-\infty}^{\infty} p_k(n) \Lambda^k.$$

We have

$$\varepsilon \partial L = \varepsilon \partial(W \Lambda W^{-1}) = \varepsilon(\partial W) W^{-1} - \varepsilon W \Lambda W^{-1}(\partial W) W^{-1} = [\ell, L],$$

hence for each $n > 0$, $\varepsilon \partial L^n = [\ell, L^n]$. Applying the linear map $\text{res} : \Phi_-(\mathcal{A}, q) \rightarrow \mathcal{A}$, we obtain the equation

$$\nabla \left( [n] b_n + \sum_{k=1}^{n-1} [k] (b_k p_k(n)) + P_0(n) \right) = 0. \quad (3)$$

Denote by $\alpha : \mathcal{A} \rightarrow \mathbb{Q}_\varepsilon$ the homomorphism which sends the generators $\{q, a_k, \bar{a}_k\}$ of $\mathcal{A}$ to 0. Since $\alpha \cdot \partial = 0$, we see that $\alpha(\partial W) = 0$, and hence $\alpha(\ell) = 0$. Thus, the constant of integration in (3) vanishes, and we obtain the recursive formula

$$b_n = -\frac{1}{[n]} \sum_{k=1}^{n-1} [k] (b_k p_k(n)) + P_0(n) \quad (4)$$

for the coefficients $b_k$, showing that they are elements of $\mathcal{A}$. \qed
3. Fractional powers of the Lax operator

In this section, we study the fractional powers of the Lax operator $L$; this may be compared with the parallel construction for the KP hierarchy due to Khesin and Zakharevich [5]. The study of these fractional powers is closely related to the operator $\ell$ introduced in the last section.

Let $s$ be a complex number. The fractional power $L^s$ of the Lax operator $L$ is defined by means of the dressing operator:

$$L^s = W \Lambda^s W^{-1} = \Lambda^s + \sum_{k=1}^{\infty} a_k(s) \Lambda^{s-k} \in \Phi_{-}(B, q).$$

The coefficient $a_k(s)$ is given by the explicit formula

$$a_k(s) = E^{-s/2}w_k + \sum_{j=1}^{k-1} \left( E^{(k-j-s)/2}w_j \right) \left( E^{(s-j)/2}w_{k-j} \right) + E^{s/2}w_k^*.$$  

In particular, $a_k(0) = 0$ and $a_k(1) = a_k$. Differentiating the definition (5) of $L^s$ with respect to $s$ and setting $s = 0$, we obtain the formula

$$\frac{dL^s}{ds} \bigg|_{s=0} = -\ell,$$

showing that $a'_k(0) = -b_k$. The following proposition is proved by extending this differential equation to all values $s$.

**Proposition 3.1.** The coefficient $a_{k,i}(s)$ in the expansion

$$a_k(s) = \sum_{i=0}^{\infty} \varepsilon^i a_{k,i}(s)$$

is a polynomial in $s$ of degree $i + 1$ with coefficients in the differential algebra

$$\mathbb{Q}\{q, a_k \mid k > 0\}/(q - \bar{q}).$$

**Proof.** By its definition, the fractional power $L^s$ satisfies the differential equation

$$\frac{dL^s}{ds} = -\frac{1}{2} \left( L^s \ell + \ell L^s \right).$$

Taking the coefficient of $\Lambda^{s-k}$ on both sides, we obtain the differential equation

$$\frac{da_k(s)}{ds} = -\frac{1}{2} \sum_{j=1}^{k-1} \left( E^{(s-j)/2}b_{k-j} E^{(k-j)/2}a_j(s) + E^{(j-s)/2}b_{k-j} E^{(j-k)/2}a_j(s) \right),$$

where we interpret $a_0(s)$ as 1. By an application of Proposition 2.1, the result follows. □
4. Perturbation theory

Let $\Omega(\mathcal{A})$ be the vector space of Kähler differentials of the commutative $\mathbb{Q}_\varepsilon$-algebra $\mathcal{A}$; this is a free module over $\mathcal{A}$ with basis $\{dq, da_k, d\bar{a}_k \mid k > 0\}$. The differential $d : \mathcal{A} \to \Omega(\mathcal{A})$ extends to a morphism

$$d : \Phi_-(\mathcal{A}, q) \to \Phi_-(\mathcal{A}, q) \otimes_\mathcal{A} \Omega(\mathcal{A}).$$

The goal of this section is the calculation of the differentials $dL^s$ and $d\ell$ in terms of the fundamental differential

$$dL = \sum_{k=1}^{\infty} da_k \Lambda^{-k+1}.$$

A basic formula of perturbation theory (Kumar [6]) says that for $f(z)$ an analytic function of $z$,

$$df(L) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \text{ad}(L)^k(f^{(k+1)}(L)dL).$$

For $f(z) = z^s$, this becomes

$$dL^s = \sum_{k=0}^{\infty} (-1)^k \binom{s}{k+1} \text{ad}(L)^k(L^{s-k-1}dL). \tag{7}$$

We will now prove this formula directly.

For $s$ a natural number $n$, the right-hand side of (7) is a finite sum, and the formula is then easily proved by induction on $n$: we have

$$d(L^{n+1}) = dL^n \cdot L + L^n \cdot dL = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \text{ad}(L)^k(L^{n-k-1}dL) \cdot L + L^n \cdot dL$$

$$= \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \left(\text{ad}(L)^k(L^{n-k}dL) - \text{ad}(L)^{k+1}(L^{n-k-1}dL)\right) + L^n \cdot dL$$

$$= \sum_{k=0}^{n} (-1)^k \left(\binom{n}{k} + \binom{n}{k+1}\right) \text{ad}(L)^k(L^{n-k}dL) = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \text{ad}(L)^k(L^{n-k}dL).$$

By analytic continuation, (7) holds for all values of $s$. Indeed, the right-hand side is convergent in the $\varepsilon$-adic topology, since the operation $\text{ad}(L)$ may be split into two terms: $\text{ad}(\Lambda + a_1) = O(\varepsilon)$, and

$$\sum_{k=2}^{\infty} \text{ad}(a_k \Lambda^{-k+1}) = O(\Lambda^{-1}).$$

It only remains to observe that by Theorem 3.1, the coefficient of $\varepsilon^i$ in $da_{k,i}(s)$ is polynomial in $s$. 

6
It is now straightforward to calculate \( d\ell \): taking the derivative of (7) with respect to \( s \) and setting \( s = 0 \), we see that

\[
(8) \quad d\ell = -\sum_{k=0}^{\infty} \frac{1}{k+1} \text{ad}(L)^k(L^{-k-1}dL).
\]

5. The equivariant Toda lattice and \( \ell \)

In this section, we denote the element \( a_1 \in A \) by \( v \). Let \( K \) be the difference operator

\[
K = L_+ + \bar{L}_- = \Lambda + v + q\Lambda^{-1} \in \Phi_+(A, q) \cap \Phi_-(A, q).
\]

In \[4\], we defined the equivariant Toda lattice by the constraints

\[
(9) \quad \varepsilon^{-1}[K, L] = \nu\partial L, \quad \varepsilon^{-1}[K, \bar{L}] = \nu\partial \bar{L},
\]
or equivalently, the constraints (9). We showed that the differential algebra associated to the equivariant Toda lattice is isomorphic to

\[
\tilde{A} = Q_{\varepsilon, \nu}[z_k, \bar{z}_k \mid k > 0] \{q, v, \bar{v}\} / (\nu\partial q - \nabla(v - \bar{v})),
\]
where \( Q_{\varepsilon, \nu} = Q_{\varepsilon}[\nu] \), and the constants of motion \( z_k \) are the images of the elements

\[
p_{-1}(k) - qp_1(k) - \nu Pp_0(k) \in A
\]
under the natural quotient map from \( A \) to \( \tilde{A} \).

Let \( e \) be the derivation \( \partial_v + \partial_{\bar{v}} \) of \( \tilde{A} \); then \( e(K) = 1 \) and

\[
(10) \quad \left(L - \nu + \sum_{k=1}^{\infty} z_k L^{-k}\right)e(L) = L.
\]

**Theorem 5.1.** The constraint (9) defining the equivariant Toda lattice is equivalent to the identity

\[
(11) \quad K = L + \nu\ell - \sum_{k=1}^{\infty} \frac{z_k}{k} L^{-k}.
\]
The vanishing of the constants \( z_k \) is equivalent to the constraint

\[
(12) \quad (\delta_1 - \bar{\delta}_1)W = \nu\partial W,
\]
or equivalently, the equation \( (\delta_1 - \bar{\delta}_1) = \nu\partial \) on the differential algebra \( B \).

**Proof.** Written in terms of \( \ell \), (9) becomes

\[
[K - \nu\ell, L] = 0.
\]
This is equivalent to the statement that

\[
K - \nu\ell \in Q_{\varepsilon, \nu}((L^{-1})).
\]
It is not hard to see that
\[ K - \nu \ell - L = \sum_{k=1}^{\infty} y_k L^{-k} \in \mathbb{Q}_{\epsilon, \nu}[L^{-1}]; \]
the constant term vanishes since, by definition, \( \text{res}(K) \) and \( \text{res}(L) \) equal \( \nu \), while \( \text{res}(\ell) = 0 \).

It remains to identify the constants \( y_k \). If \( \delta \) is an evolutionary derivation of the differential algebra \( \tilde{A} \), (8) implies that
\[ \delta \ell = -\sum_{k=0}^{\infty} \frac{1}{k+1} \text{ad}(L)^k(L^{-k-1}\delta L). \]
In particular, since \( L \) commutes with \( e(L) \), we see that \( e(\ell) = -L^{-1}e(L) \). Likewise, \( e(L^{-k}) = -kL^{-k-1} \). Applying the derivation \( e \) to both sides of (13), we see that
\[ 1 = e(K) = e(L) \left( 1 - \nu L^{-1} - \sum_{k=1}^{\infty} k y_k L^{-k-1} \right). \]
It follows from (10) that \( y_k = -z_k/k \).

We have
\[ (K - L - \nu \ell)W = (L_+ + \tilde{L}_-)W - \varepsilon \nu \partial W \]
\[ = -L_- W + \tilde{L}_- W - \varepsilon \nu \partial W = \varepsilon (\delta_1 - \tilde{\delta}_1 - \nu \partial)W. \]
Thus, the vanishing of the constants \( z_k \) in (11) is equivalent to the constraint (12). \( \square \)

In [4], we conjectured that the equivariant Gromov-Witten invariants of \( \mathbb{C}P^1 \) are described by the equivariant Toda lattice with \( z_k = 0, \ k > 0 \). The results of this section show that this is true. By the work of Okounkov and Pandharipande, the equivariant Gromov-Witten invariants of \( \mathbb{C}P^1 \) are associated with a \( \tau \)-function of the Toda lattice which satisfies \( (\delta_1 - \tilde{\delta}_1)\tau = \nu \partial \tau \). The dressing operator \( W \) corresponding to this \( \tau \)-function is given by the formula
\[ W = \tau^{-1} \exp \left( -\sum_{n=1}^{\infty} \frac{\delta_n}{n!} \right) \tau; \]
it follows that \( W \) satisfies the equation \( (\delta_1 - \tilde{\delta}_1)W = \nu \partial W \).

The other part of the conjecture of [4], relating the equivariant Gromov-Witten flows of \( \mathbb{C}P^1 \) to the flows of the equivariant Toda lattice, is established by Okounkov and Pandharipande. Namely, if \( \partial_k = \partial_{k,Q} \) and \( \tilde{\partial}_k = \partial_{k,Q} - \nu \partial_{k,P} \), then
\[ \sum_{k=0}^{\infty} z^{k+1} \partial_k = \sum_{n=1}^{\infty} \frac{z^n \delta_n}{(1+z\nu)(2+z\nu)\ldots(n+z\nu)}, \]
\[ \sum_{k=0}^{\infty} z^{k+1} \tilde{\partial}_k = \sum_{n=1}^{\infty} \frac{z^n \tilde{\delta}_n}{(1-z\nu)(2-z\nu)\ldots(n-z\nu)}. \]
In particular, we see that the descendent flows $\partial_{k,P}$ of the puncture operator $P$ are given in the non-equivariant limit by the formula

$$
\partial_{k,P} = \lim_{\nu \to 0} \left( \frac{1}{(k+1)!} \nu^{-1}(\delta_{k+1} - \bar{\delta}_{k+1}) - \frac{1}{k!} c_k (\delta_k + \bar{\delta}_k) \right),
$$

where $c_k$ is the harmonic number $c_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

6. Hamiltonian structure

In this section, we use Theorem 5.1 to show that the equivariant Toda lattice has a Hamiltonian structure.

Denote by $\mathcal{R}$ the quotient $\tilde{A}/\partial \tilde{A}$, and denote by $f \mapsto \int f \, dx$ the quotient map from $\tilde{A}$ to $\mathcal{R}$. The idea which this notation is intended to represent is that an element of $\tilde{A}$ is a density $f$, whose associated functional $\int f \, dx$ is obtained by integration with respect to the space variable $x$.

Denote by $\text{Res}$ the trace on $\Phi^{-}(\tilde{A}, q)$ with values in $\mathcal{R}$ given by the formula

$$
\text{Res} \left( \sum_{k=-\infty}^{\infty} f_k \Lambda^k \right) = \int f_0 \, dx.
$$

Clearly, this map vanishes on total derivatives; to see that it vanishes on commutators, we use the formula

$$
\text{Res} \left[ \sum_i a_i \Lambda^i, \sum_j b_j \Lambda^j \right] = \nabla \sum_k [k](a_k b_{-k}).
$$

There is a unique linear map

$$
\text{Res} : \Phi^{-}(\tilde{A}, q) \otimes_{\tilde{A}} \Omega(\tilde{A}) \to \Omega(\tilde{A})/\partial \Omega(\tilde{A})
$$

such that $d \text{Res}(A) = \text{Res}(dA)$.

Associated to the equivariant Toda lattice, we have the basic sequence of functionals

$$
h_n = \frac{1}{n+1} \text{Res}(L^{n+1}), \quad n \geq 0,
$$

with differentials $dh_n = \text{Res}(L^n dL)$. In calculating $h_n$, the following lemma is convenient.

**Lemma 6.1.**

$$
p_0(n + 1) = \sum_{k=0}^{n} [k + 1](a_{k+1} p_k(n))
$$

**Proof.** Applying the operator res to the equations $L^{n+1} = L \cdot L^n$ and $L^{n+1} = L^n \cdot L$, we see that

$$
p_0(n + 1) = E^{1/2} p_{-1}(n) + \sum_{k=0}^{\infty} E^{-k/2} (a_{k+1} p_k(n)),
$$

$$
p_0(n + 1) = E^{-1/2} p_{-1}(n) + \sum_{k=0}^{\infty} E^{k/2} (a_{k+1} p_k(n)).
$$
Taking \( E^{1/2} \) times the second of these equations minus \( E^{-1/2} \) times the first, we see that

\[
\nabla p_0(n + 1) = \nabla \sum_{k=0}^{n} [k + 1](a_{k+1} p_k(n)),
\]

and hence, that

\[
p_0(n + 1) = \sum_{k=0}^{n} [k + 1](a_{k+1} p_k(n)) + \alpha(p_0(n + 1)).
\]

This proves the lemma, since \( \alpha(p_0(n + 1)) = 0. \)

\[\square\]

**Corollary 6.2.**

\[
h_n = \sum_{k=0}^{n} \frac{k+1}{n+1} f(a_{k+1} p_k(n)) \, dx
\]

For example, using the formulas \( a_2 = q + \nu P v + z_1 \) and

\[
a_3 = \nu(P(\frac{1}{4}[2]v^2 + q) - \frac{1}{2}v[2]P v) + \nu^2 P v - z_1 v + \frac{1}{2}z_2,
\]

we see that

\[
h_0 = \int v \, dx,
\]

\[
h_1 = \int \left( \frac{1}{2}v^2 + a_2 \right) \, dx = \int \left( \frac{1}{2}v^2 + q + \nu v + z_1 \right) \, dx,
\]

\[
h_2 = \int \left( \frac{1}{3}v p_0(2) + \frac{2}{3}(a_2 p_1(2)) + a_3 \right) \, dx
\]

\[
= \int \left( \frac{1}{3}v(\nu v^2 + [2]a_2) + \frac{2}{3}(a_2 [2]v) + \nu(\frac{1}{2}v^2 + q - \frac{1}{2}v[2]P v) + \nu^2 v - z_1 v + \frac{1}{2}z_2 \right) \, dx
\]

\[
= \int \left( \frac{1}{3}v^3 + v[2]q + \nu(\frac{1}{2}v^2 + q + \frac{1}{2}v[2]P v) + \nu^2 v + z_1 v + \frac{1}{2}z_2 \right) \, dx.
\]

**Proposition 6.3.** We have \( \text{Res}(L^n \, dK) = dH_n, \) where

\[
H_n = h_n - \nu h_{n-1} + \sum_{k=1}^{n-1} z_k h_{n-k-1}.
\]

**Proof.** From (11), (7) and (8), we see that

\[
dK = dL + \nu d\ell - \sum_{j=1}^{\infty} \frac{z_j}{j} dL^{-j}
\]

\[
= dL + \sum_{k=0}^{\infty} (k + 1)^{-1} \text{ad}(L)^k \left( -\nu + \sum_{j=1}^{\infty} \binom{j+k}{k} z_j L^{-j} \right) L^{-k-1} \, dL.
\]

Multiplying by \( L^n \) and applying Res, all of the terms with \( k > 0 \) drop out, and we obtain

\[
\text{Res}(L^n \, dK) = \text{Res} \left( \left( L - \nu + \sum_{j=1}^{\infty} z_j L^{-j} \right) L^{n-1} \, dL \right),
\]

which equals \( dH_n. \)

\[\square\]

Let \( \delta_v \) and \( \delta_u \) be the variational derivatives with respect to \( v \) and \( u = \log(q). \)
Corollary 6.4. We have \( \delta_v H_n = p_0(n) \), \( \delta_u H_n = q p_1(n) \), \( \delta_v \bar{H}_n = \bar{p}_0(n) \) and 
\[
\delta_u \bar{H}_n = q \bar{p}_1(n) - \nu \bar{p}_0(n).
\]

Proof. The formulas for \( \delta_v H_n \) and \( \delta_u H_n \) follow since \( dK = dv + q du \Lambda^{-1} \). The formulas for \( \delta_v \bar{H}_n \) and \( \delta_u \bar{H}_n \) now follow by taking conjugates, bearing in mind that \( \bar{v} = v - \nu p u \).

For example, we have
\[
H_0 = h_0 = \int v \, dx,
\]
\[
H_1 = h_1 - \nu h_0 = \int (\frac{1}{2} v^2 + q + z_1) \, dx,
\]
\[
H_2 = h_2 - \nu h_1 + \bar{z}_1 h_0 = \int (\frac{1}{2} v^3 + v [2] q + \frac{1}{2} \nu v [2] p v + 2 z_1 v - \nu z_1 + \frac{1}{2} \bar{z}_2) \, dx.
\]

It is now easy to show that the equivariant Toda lattice is Hamiltonian. Applying \( \text{res} \) to the equation \( [K, L^n] = \nu \partial L^n \), we see that
\[
\nabla p_{-1}(n) = \nabla (q p_1(n)) + \nu \partial p_0(n).
\]

It follows that \( \delta_n v = \nabla p_{-1}(n) = \nabla (q p_1(n)) + \nu \partial p_0(n) \). In conjunction with the formula \( \delta_n u = \nabla p_0(n) \), we conclude that
\[
\delta_n \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} \nu \partial & \nabla \\ \nabla & 0 \end{bmatrix} \begin{bmatrix} \delta_v H_n \\ \delta_u H_n \end{bmatrix}.
\]

Since \( \bar{\delta}_n v = \nabla (q \bar{p}_1(n)) \) and \( \bar{\delta}_n u = \nabla \bar{p}_0(n) \), we also conclude that
\[
\bar{\delta}_n \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} \nu \partial & \nabla \\ \nabla & 0 \end{bmatrix} \begin{bmatrix} \delta_v \bar{H}_n \\ \delta_u \bar{H}_n \end{bmatrix}.
\]

In other words, the equivariant Toda lattice is Hamiltonian with respect to the Hamiltonian structure
\[
\{v(x), v(y)\} = \nu \partial \delta(x - y), \quad \{v(x), u(y)\} = \nabla_x \delta(x - y), \quad \{u(x), u(y)\} = 0.
\]

The relationship between the equivariant Toda lattice (with \( z_k = 0 \), \( k > 0 \)) and the equivariant Gromov-Witten invariants of \( \mathbb{CP}^1 \) leads to a new proof of the Toda conjecture for the (non-equivariant) Gromov-Witten invariants of \( \mathbb{CP}^1 \). (See [3] for a discussion of this conjecture and further references.) We see that the descendent flow \( \partial_{k,Q} \) is the limit of the flow \( \frac{1}{(k+1)!} \delta_{k+1} \) as \( \nu \to 0 \), and hence has Hamiltonian
\[
\frac{1}{(k+1)!} \lim_{\nu \to 0} h_{k+1}.
\]

Likewise, by ([4]), the descendent flow \( \partial_{k,P} \) is the limit of the flow
\[
\frac{1}{(k+1)!} \nu^{-1} (\delta_{k+1} - \bar{\delta}_{k+1}) - \frac{1}{k!} c_k (\delta_k + \bar{k})
\]
as \( \nu \to 0 \), and hence has Hamiltonian
\[
\lim_{\nu \to 0} \left( \frac{1}{(k+1)!} \nu^{-1} (H_{k+1} - \bar{H}_{k+1}) - \frac{1}{k!} c_k (H_k + \bar{H}_k) \right).
\]
Let $\ell_0$ equal the limit as $\nu \to 0$ of $\ell$. Since $L = K - \nu \ell = K - \nu \ell_0 + O(\nu^2)$ and

$$\bar{L} = \bar{K} + \nu \bar{\ell} = K + \nu(\ell_0 - \mathcal{P}u) + O(\nu^2),$$

we have

$$\nu^{-1}(H_{k+1} - \bar{H}_{k+1}) = \nu^{-1} \frac{1}{k+2} \text{Res}(L^{k+2} - \bar{L}^{k+2}) - \frac{1}{k+1} \text{Res}(L^{k+1} - \bar{L}^{k+1}) = \text{Res}(K^{k+1}(\mathcal{P}u - 2\ell_0)).$$

It follows that $\partial_{k,P}$ has Hamiltonian $\nu^{-1} \frac{1}{(k+1)!} \text{Res}(K^{k+1}(\mathcal{P}u - 2(\ell_0 + c_k)))$. An equivalent formula was conjectured by Eguchi and Yang \cite{2} and proved by Carlet, Dubrovin and Zhang \cite{1}.

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