In how many distinct ways can flocks be formed?

A problem in sheep combinatorics

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Abstract

In this short paper, we extend the concept of the strict order polynomial \( \Omega_P(n) \), which enumerates the number of strict order-preserving maps \( \phi : P \to n \) for a poset \( P \), to the extended strict order polynomial \( E_P(n,z) \), which enumerates analogous maps for the elements of the power set \( \mathcal{P}(P) \). The problem at hand immediately reduces to the problem of enumeration of linear extensions for the subposets of \( P \). We show that for every \( Q \subset P \) a given linear extension \( v \) of \( Q \) can be associated with a unique linear extension \( w \) of \( P \). The number of such linear extensions \( v \) (of length \( k \)) associated with a given linear extension \( w \) of \( P \) can be expressed compactly as \( \binom{\text{del}_P(w)}{k} \), where \( \text{del}_P(w) \) is the number of deletable elements of \( w \) defined in the text. Consequently the extended strict order polynomial \( E_P(n,z) \) can be represented as follows

\[
E_P(n,z) = \sum_{w \in \mathcal{L}(P)} \sum_{k=0}^{p} \binom{\text{del}_P(w)}{p-k} \binom{n + \text{des}(w)}{k} z^k.
\]

The derived equation can be used for example for solving the following combinatorial problem: Consider a community of \( p \) shepherds, some of whom are connected by a master-apprentice relation (expressed as a poset \( P \)). Every morning, \( k \) of the shepherds go out and each of them herds a flock of sheep. Community tradition stipulates that each of these \( k \) shepherds will herd at least one and at most \( n \) sheep, and an apprentice will always herd fewer sheep than his master (or his master’s master, etc). In how many ways can the flocks be formed? The strict order polynomial answers this question for the case in which all \( p \) shepherds go to work, and the extended strict order polynomial considers also all the situations in which some of the shepherds decide to take a day off.

1 Notation and Definitions

1.1 Standard terminology

The current communication closely follows the poset terminology introduced in Stanley’s book \[1\]. The reader familiar with the terminology can jump directly to Subsection \[1.2\]. A partially ordered set \( P \), or poset for short, is a set together with a binary relation \( <_P \). In this manuscript, we are concerned with finite posets \( P \) consisting of \( p \) elements and with strict partial orders, meaning that the relation \( <_P \) is
irreflexive, transitive and asymmetric. An induced subposet \( Q \subset P \) is a subset of \( P \) together with the order \(<_Q \) inherited from \( P \) which is defined by \( s <_P t \iff s <_Q t \). The symbol \(< \) shall denote the usual relation „larger than‟ in \( N \). The symbol \([n]\) stands for the set \( \{1, 2, \ldots, n\} \), and \((n, m)\) stands for the set \( \{n + 1, n + 2, \ldots, m - 1\} \). The symbol \( n \) represents the chain \( 1 < 2 < 3 < \ldots < n \). A map \( \phi : P \to N \) is a strict order-preserving map if it satisfies \( s <_P t \Rightarrow \phi(s) < \phi(t) \). A natural labeling of a poset \( P \) is an order-preserving bijection \( \omega : P \to [n] \). A linear extension of \( P \) is an order-preserving bijection \( \sigma : P \to p \). A linear extension \( \sigma \) can be represented as a permutation \( \omega \circ \sigma^{-1} \) expressed as a sequence \( w = w_1 w_2 \ldots w_p \) with \( w_i = \omega(\sigma^{-1}(i)) \); the sequence \( w \) shall also be referred to as a linear extension in the following. The set of all such sequences \( w \) is denoted by \( \mathcal{L}(P) \) and is referred to as the Jordan-Hölder set of \( P \). If two subsequent labels \( w_i \) and \( w_{i+1} \) in \( w \) stand in the relation \( w_i > w_{i+1} \), then the index \( i \) is called a descent of \( w \). The total number of descents of \( w \) is denoted by \( \text{des}(w) \). The strict order polynomial \( \Omega^p_\sigma(n) \) of a poset \( P \), which enumerates the strict order-preserving maps \( \phi : P \to [n] \), is given by

\[
\Omega^p_\sigma(n) = \sum_{w \in \mathcal{L}(P)} \binom{n + \text{des}(w)}{p}.
\]  

1.2 Non-standard terminology

We will often construct—by slight abuse of notation—a subposet of \( P \) by specifying a set of labels \( D \subset [p] \); the expression \( P \setminus D \) stands for the induced subposet with the elements \( \{p \in [p] : \omega(p) \notin D\} \). Clearly the subposet constructed in this way has \( p - \#D \) elements; and the full set \( \mathcal{P}(P) \) of subposets of \( P \) stands in direct correspondence to the power set of \( [p] \): \( \mathcal{P}(P) = \{P \setminus D | D \in \mathcal{P}([p])\} \). Similarly, if \( w \) is a sequence in \( \ell(P) \) and \( D \) is a subset of \( [p] \), let us denote by \( w \setminus D \) the subsequence obtained by deleting all the elements of \( D \) from \( w \). For example, \( 13245 \setminus \{1, 4\} = 325 \). Clearly, deleting some arbitrary set \( D \) from two different sequences may produce the same sequence: for example, \( 13245 \setminus \{1, 4\} = 325 = 32154 \setminus \{1, 4\} \). We will later (Lemma[2]) see that deleting deletable elements (see Def.[6]) from two distinct sequences always results in two distinct subsequences.

Further, let us slightly change the representation of linear extensions of subposets: Normally, one would assign to each subposet \( Q = P \setminus D \) a new natural labeling \( \omega^Q : Q \to [q] \), and then express the linear extensions of \( Q \) as sequences of the elements of \( [q] \). Instead, we avoid re-labeling each subposet, and use instead the labeling \( \omega : Q \to [p] \setminus D \) inherited from \( P \). Then, a linear extension \( \sigma \) of \( Q \) is represented by a sequence \( w = w_1 \ldots w_q \) defined in the usual way: \( w_i = \omega(\sigma^{-1}(i)) \). The set of such sequences shall still be denoted by \( \mathcal{L}(Q) \). Using this notation, it is now easy to see (but properly demonstrated later in Lemma[12]) that if \( w \) is a linear extension of \( P \), then \( w \setminus D \) is a linear extension of \( P \setminus D \).

2 Main results

In this short communication, we extend the concept of the strict order polynomial \( \Omega^p_\sigma(n) \) given by Eq.[1] to the extended strict order polynomial \( E^p_\sigma(n, z) \) given by Eq.[2], which enumerates and classifies the totality of strict order-preserving maps \( \phi : Q \to n \) with \( Q \subset P \). We show below in Theorem[3] that there exists a compact combinatorial expression characterizing \( E^p_\sigma(n, z) \). In the following, we shall always assume that \( P \) is a poset with with \( p \) elements, a strict order \(<_P \), and a natural labeling \( \omega \). Subposets of \( P \) are always assumed to be induced.
Definition 1. The extended strict order polynomial $E_P^o(n, z)$ of a poset $P$ is defined as

$$E_P^o(n, z) = \sum_{Q \subset P} \Omega_Q^o(n) z^{\#Q},$$

where the sum runs over all the induced subposets $Q$ of $P$.

Example 2. Consider a family of three shepherds: Fiadh, Fiadh’s father Aidan, and Aidan’s father Lorcan. Every day, some of the shepherds go out and each herd a flock of at least one and at most $n$ sheep. Aidan always herds more sheep than Fiadh, and Lorcan always herds more sheep than both Fiadh and Aidan. How many possible ways are there of assigning flock sizes to the shepherds?

The three shepherds together with the seniority relation form a poset $P$ isomorphic to the chain $3$: Fiadh $<_P$ Aidan $<_P$ Lorcan, see Fig. 1(a). Let us denote the number of sheep in Fiadh’s flock by $n_1$, the size of Aidan’s flock by $n_2$ and the size of Lorcan’s flock by $n_3$; then the above conditions tell us that $1 \leq n_1 < n_2 < n_3 \leq n$. On a day when all three shepherds go to work, such as depicted in Fig. 1(b), the numbers $n_1$, $n_2$ and $n_3$ can be chosen in $\Omega_P^o(n) = \binom{n+2}{3}$ ways. When only Fiadh and Aidan go to work, see Fig. 1(c), i.e. for the subposet $Q = \{\text{Fiadh, Aidan}\}$, we have to choose the two numbers $n_1$ and $n_2$ such that $1 \leq n_1 < n_2 \leq n$; there are $\Omega_Q^o(n) = \binom{n+1}{2}$ ways to do so. The same is true whenever two of the three shepherds are present. Clearly, whenever only one shepherd works, there are $\binom{n}{1}$ ways to choose his flock size, and when all shepherds take the day off, there is only one possibility. Therefore, the extended strict order polynomial $E_P^o(n, z)$ has the form

$$E_P^o(n, z) = \Omega_P^o(n) z^3 + \binom{n+2}{3} + \Omega_{\{\text{Fiadh, Aidan}\}}^o(n) + \Omega_{\{\text{Fiadh, Lorcan}\}}^o(n) + \Omega_{\{\text{Aidan, Lorcan}\}}^o(n) \ z^2$$

$$+ \left( \binom{n+2}{3} + \binom{n}{2} \right) z + \left( \binom{n-1}{1} \right) z^1 + \left( \binom{n}{0} \right) z^0$$

$$= \sum_{k=0}^{3} \binom{3}{k} \binom{n}{k} z^k.$$
applying the following theorem.

**Theorem 3.** The extended strict order polynomial is given by

\[
E_p^\varnothing(n, z) = \sum_{w \in \mathcal{L}(P)} \sum_{k=0}^{p} \binom{\text{del}_P(w)}{p-k} \binom{n + \text{des}(w)}{k} z^k,
\]

where \(\text{del}_P(w)\) denotes the number of deletable labels in \(w\).

Intuitively speaking, deletable labels can be understood to be the entries of \(w\) which are not essential for distinguishing \(w\) from other elements of \(\mathcal{L}(P)\). Theorem 3 is based on the fact that every element of \(\bigcup_{Q \subset P} \mathcal{L}(Q)\) can be uniquely associated with some element of \(\mathcal{L}(P)\). Formally speaking, it is possible to define an equivalence relation \(\sim\) on \(\bigcup_{Q \subset P} \mathcal{L}(Q)\) such that \(\bigcup_{w \in \mathcal{L}(P)} [w]_\sim = \bigcup_{Q \subset P} \mathcal{L}(Q)\). This concept is illustrated below in Examples 4 and 5. The proof of Theorem 3 will be given at the end of this paper after formalizing the concept of deletable labels and proving some technical lemmata.

**Example 4.** Let us consider the lattice \(P = 2 \times 2\), for which \(\mathcal{L}(P) = \{1234, 1324\}\) and \(\bigcup_{Q \subset P} \mathcal{L}(Q) = \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 32, 123, 124, 134, 234, 324, 132, 1234\}\). Our results allow us to partition the set \(\bigcup_{Q \subset P} \mathcal{L}(Q)\) of linear extensions into two equivalence classes \([1234]_\sim\) and \([1324]_\sim\):

\[
\begin{array}{c|c}
\text{des}(w) = 0 & \text{des}(w) = 1 \\
\text{del}_P(w) = 4 & \text{del}_P(w) = 2 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
1234 & 1324 \\
123 & 134 & 234 \\
12 & 13 & 14 & 23 & 24 & 34 \\
\emptyset & 324 & 132 \\
& 32 & \\
\end{array}
\]

The first family, originating from the linear extension 1234, is characterized by zero descents (\(\text{des}(w) = 0\)) and zero non-deletable elements (\(\text{del}_P(w) = 4 - 0\)), and thus contains \(\binom{4-0}{k-0}\) sequences of each length \(k\). The second family, originating from the linear extension 1324, is characterized by one descent (\(\text{des}(w) = 1\)) and two non-deletable elements 3 and 2 (\(\text{del}_P(w) = 4 - 2\)), and thus contains \(\binom{4-2}{k-2}\) sequences of each length \(k\). Consequently, the extended strict order polynomial is given by

\[
E_p^\varnothing(n, z) = \sum_{k=0}^{4} \left( \binom{4}{k} \binom{n}{k} + \binom{4-2}{k-2} \binom{n+1}{k} \right) z^k.
\]

**Example 5.** Let us consider the poset \(P = \{a, b, c\}\) of three non-comparable elements. We have \(\mathcal{L}(P) = \{123, 132, 213, 231, 312, 321\}\) and \(\bigcup_{Q \subset P} \mathcal{L}(Q) = \{\emptyset, 1, 2, 3, 12, 13, 14, 21, 23, 31, 32, 123, 132, 213, 231, 312, 321\}\). Our results allow us to classify the linear extensions in \(\bigcup_{Q \subset P} \mathcal{L}(Q)\) into six families
each of which is characterized by a pair of numbers \((\text{des}(w), \text{del}_P(w))\) specified above. The extended strict order polynomial is given by

\[
E_P(n, z) = 3 \sum_{k=0}^{3} \left( \binom{3}{k} \binom{n}{k} + 3 \left( \binom{3-2}{k-2} \binom{n+1}{k} + \binom{3-3}{k-3} \binom{n+1}{k} + \binom{3-3}{k-3} \binom{n+2}{k} \right) \right) z^k.
\]

3 Classification of linear extensions

**Definition 6.** Consider a sequence \(w = w_1 w_2 \ldots w_p\) which represents a linear extension \(\sigma\) of \(P\). A label \(w_i\) is deletable if

1) neither \(i - 1\) nor \(i\) is a descent, and

2) at least one of the following is true:

   a) \(w_j < w_i\) for any \(j \in (0, i)\), or
   
   b) there exists a label \(w_k\) with \(\omega^{-1}(w_k) < _P \omega^{-1}(w_i)\) such that \(w_j < w_i\) for all \(j \in (k, i)\).

The set of deletable elements of \(w\) is denoted by \(\text{Del}_P(w)\), and its cardinality by \(\text{del}_P(w) = \#\text{Del}_P(w)\). Labels that are not deletable from \(w\) are called fixed in \(w\).

Loosely speaking, a label is fixed if it contributes to a descent, or if it appears after a descent even though it would also be allowed to appear in front of it. An inclined reader might have already noticed that every deletable element appears in a uniquely defined position of \(w\), which may be described as „as early as possible without interfering with the descent pattern“. In other words, removing a deletable element \(w_i\) from \(w\) and reinserting it at any earlier position cannot result in a linear extension with the same labels involved in the descent pairs. This concept constitutes the main idea behind associating a linear extension \(v \in \bigcup_{Q \subset P} \mathcal{L}(Q)\) with a unique linear extension \(w \in \mathcal{L}(P)\) developed in detail later in this communication.

Let us now demonstrate how Definition 6 can be used in a direct manner to identify deletable and fixed labels in linear extensions. A useful, easy-to-use-in-practice graphical reinterpretation of Definition 6 is introduced in Example 9.

**Example 7.** Consider again the poset \(P = 2 \times 2\) shown in Fig. 2 and its two linear extensions \(w = 1234\) and \(w' = 1324\). Let us first determine which of the labels are deletable from 1234. We have no descents in 1234, so the condition 1) of Definition 6 is satisfied for all of the labels. We only need to verify condition 2) of Definition 6. The label \(w_1 = 1\) is deletable because the interval \((0, 1)\) is empty, and therefore condition 2a) is vacuously satisfied. Since \(w_1 < w_2 < w_3 < w_4\), we find also for the remaining labels \(w_i = 2, 3, 4\)
that condition 2a) is satisfied. This shows that all four labels in the linear extension 1234 are deletable: $\text{Del}_P(1234) = \{1, 2, 3, 4\}$ and $\text{del}_P(1234) = 4$. This is not the case for the linear extension 1324. The labels 3 and 2 are fixed by condition 1) because the position $i = 2$ is a descent. The labels 1 and 4 can be shown to be deletable in exactly the same way as for 1234. Consequently, in the linear extension 1324 the set of deletable elements is $\text{Del}_P(1324) = \{1, 4\}$ and $\text{del}_P(1324) = 2$.

**Figure 2:** Hasse diagram of two posets $P = 2 \times 2$ and $P = 3 \times 3$ together with (natural) labelings $\omega$.

**Example 8.** Consider the linear extension $w = 124753689$ of the poset $P = 3 \times 3$ shown in Fig. 2. By condition 1) of Definition 6, the labels 3, 5 and 7 are not deletable; the remaining labels might be deletable if they satisfy condition 2a) or 2b) of Definition 6. Since $w_1 < w_2 < w_3$, the labels $w_1 = 1, w_2 = 2$ and $w_3 = 4$ are deletable by condition 2a); likewise, due to $w_j < w_8 < w_9$ for all $j = 1, \ldots, 7$, the labels 8 and 9 are deletable by condition 2a). Thus the only remaining label for which the deletability (or non-deletability) is not immediately obvious—and hence the machinery of Definition 6 must be fully put to work—is $w_7 = 6$. There exists the label $w_5 = 5$ with $\omega^{-1}(5) \prec_P \omega^{-1}(6)$, and for the only label $w_j$ with $j \in (5, 7) = \{6\}$ we find $w_6 = 3 < w_7 = 6$, so by condition 2b), the label $w_7 = 6$ is deletable. Therefore, $\text{Del}_P(w) = \{1, 2, 4, 6, 8, 9\}$ and $\text{del}_P(w) = 6$.

**Example 9.** Linear extensions and their fixed and deletable elements can be visualized and analyzed graphically in the following way. For a given linear extension $\sigma$, plot the Hasse diagram of $P$ in such a way that each element $t_i$ of $P$ is represented as a point in a Cartesian plane with the coordinates $(\sigma(t_i), \omega(t_i)) = (i, w_i)$, see Fig. 3a). The total order implied by the linear extension $\sigma$ is easily determined by connecting the elements in the resulting diagram from left to right, as shown using red arrows in Fig. 3b). The fixed and deletable labels can now be identified in a graphical way, as illustrated in Fig. 3c): Whenever $i$ is a descent, i.e. whenever an element $t_i$ (in the position $(i, w_i)$) is displayed above $t_{i+1}$ (in the position $(i + 1, w_{i+1})$), both $w_i$ and $w_{i+1}$ are fixed (compare condition 1) of Definition 6); this is marked by coloring the corresponding elements. The line connecting the points $(i, w_i)$ and $(i + 1, w_{i+1})$ then casts a „shadow“ to the right; and all elements in the shadow have fixed labels as long as they are not covered by any elements in the same shadow including $t_i$ and $t_{i+1}$ (if they are covered, their labels are deletable by condition 2b)).
Figure 3: Graphical representation of the linear extension $\sigma$ represented by $w = 124735869$. (a) Hasse diagram plotted in a Cartesian plane with each element $t_i$ of $P$ displayed at the coordinates $(\sigma(t_i), \omega(t_i)) = (i, w_i)$. (b) Red arrows depict the total order implied by the linear extension $\sigma$. (c) Graphical identification of elements with fixed labels (displayed as colored circles): Every descent $i$ fixes the labels $w_i$ and $w_{i+1}$. Further, the line between the two elements $t_i$ and $t_{i+1}$ involved in a descent casts a shadow to the right, and the labels of any elements caught in the shadow—and not covered by other elements inside the same shadow, including $t_i$ and $t_{i+1}$—are also fixed. Elements with deletable labels remain displayed as white circles.

Figure 4: Graphical representation of six linear extensions of the poset $P = 3 \times 3$ shown in Fig. 2. Elements with fixed labels are shown as colored circles.

**Example 10.** Fig. 4 shows six of the linear extensions of the poset $P = 3 \times 3$ shown in Fig. 2. The
introduced graphical representation of each of the linear extensions allows us to identify easily the sets of deletable labels: (a) Del$_P(w) = [9]$, (b) Del$_P(w) = [9] \setminus \{2, 4\}$, (c) Del$_P(w) = [9] \setminus \{3, 7\}$, (d) Del$_P(w) = [9] \setminus \{3, 5, 7\}$, (e) Del$_P(w) = [9] \setminus \{3, 5, 7\}$, (f) Del$_P(w) = \{1, 9\}$. Note that in cases (b) – (e), there are elements in the shadow which are not colored because they are covered by another element in the same shadow – that is, elements that are deletable by condition 2b) of Definition 6. 

We are now ready to investigate formally the relation between the linear extensions of $P$ and the linear extensions of its subposets $P \setminus D$.

### 3.1 Correspondence between linear extensions of a poset and of its subsets

Deleting deletable elements does not affect the number of descents:

**Lemma 11.** Consider a linear extension $w \in \mathcal{L}(P)$ and a subsequence $v = w \setminus D$, where $D \subset \text{del}_P(w)$. Then, $\text{des}(w) = \text{des}(v)$. 

**Proof.** Let us augment the linear extension $w$ with two auxiliary fixed labels $w_0 = 0$ and $w_{p+1} = p + 1$. Then any deletable label of $w$ is located between two fixed labels $w_i$ and $w_j$, which can be selected in such a way that all the labels $w_{i+1}, \ldots, w_{j-1}$ in between are deletable. If there is any $k \in (i, j)$ such that $w_k > w_{k+1}$, $k$ would be a descent in $w$ and $w_k$ and $w_{k+1}$ would be fixed according to condition 1 of Def. 6, contradicting the choice of $w_i$ and $w_j$. Therefore, we have $w_i < w_{i+1} < \ldots < w_{j-1} < w_j$. Every deletable element belongs to such an interval containing monotonously increasing deletable labels flanked by two fixed labels, therefore constructing $v = w \setminus D$ by deleting any deletable elements from $w$ does not remove or introduce any descents.

The following two lemmata establish the correspondence between the elements of $\mathcal{L}(P \setminus D)$ and the elements of $\mathcal{L}(P)$:

**Lemma 12.** Let $D \subset \lfloor p \rfloor$, and let $Q = P \setminus D$ be a subposet of $P$. For every linear extension $v$ of $Q$ there exists exactly one linear extension $w$ of $P$ such that $v = w \setminus D$ and $D \subset \text{del}_P(w)$. 

**Proof.** Denote in the following by $q = \#Q = p - \#D$ the length of $v$. For the sake of brevity of the following expositions, assume during this proof that $v_0 = 0$ and $v_{q+1} = p + 1$. Let us attempt to construct a sequence $w$ of elements of $\lfloor p \rfloor$ such that

- a) $v = w \setminus D$,
- b) $w \in \mathcal{L}(P)$,
- c) $D \subset \text{del}_P(w)$.

A sequence $w$ can only satisfy condition a) if it contains the labels $v_1, \ldots, v_q$ appearing in the same order as in $v$, preceded, interleaved, and/or succeeded by the elements of $D$. Let us denote by $D^0$ the set of elements of $D$ appearing in $w$ before $v_1$, by $D^1$ the set of elements appearing between $v_1$ and $v_2$, and so on. Clearly, $D$ is a disjoint union of the subsets $D^0, D^1, \ldots, D^q$. We can thus construct a sequence $w$ in two steps:

**Step 1:** Partition $D$ into $q + 1$ (possibly empty) subsets $D^0, D^1, \ldots, D^q$ containing $m_0, m_1, \ldots, m_q$ elements, respectively.
Step 2: Arrange the elements of each subset $D^i$ into a subsequence $d_1^i d_2^i \ldots d_{m_i}^i$, and form a sequence $w$ by concatenating the labels in $v$ and the consecutive subsequences $d_1^0 d_2^0 \ldots d_{m_0}^0, \ldots, d_1^n d_2^n \ldots d_{m_n}^n$ in the following way

$$w = d_1^0 d_2^0 \ldots d_{m_0}^0 v_1 d_1^1 d_2^1 \ldots d_{m_1}^1 v_2 \ldots v_q d_1^q d_2^q \ldots d_{m_q}^q.$$  

Obviously, many different sequences $w$ can be constructed in this way by choosing different partitionings of $D$ and by selecting distinct orders of the elements in each $D^i$; we show during the following construction process that the conditions b) and c) restrict this abundance to a single, unique sequence $w$.

Every $d \in D$ must be inserted in such a way that that $w_{i-1} < w_i \equiv d < w_{i+1}$, otherwise, $d$ would—by condition 1) of Definition 6—not be deletable from $w$, thus violating condition c). Therefore, the subsequence $d_1^i d_2^i \ldots d_{m_i}^i$, inserted between $v_i$ and $v_{i+1}$ must satisfy $v_i < d_1^i < d_2^i < \ldots < d_{m_i}^i < v_{i+1}$. This shows that, in Step 2, when we augment $v$ with the elements of a subset $D^i$, the only choice is to arrange these elements into a monotonously increasing sequence before doing so. Moreover, this requirement seriously reduces the number of allowed partitionings of $D$ into subsets $D^i$, as each $d \in D^i$ needs to satisfy the condition $v_i < d < v_{i+1}$.

Consider an element $d \in D$. Let us now narrow down the family of subsets $D^i$ into which the element $d$ may be placed. Let $j_d = \max \{ j \in [q] \mid \omega^{-1}(v_j) <_P \omega^{-1}(d) \}$, or $j_d = 0$ if this set is empty. In order to not violate condition b), $d$ must be in some $D^i$ with $j_d \leq i$. Denote by $I_d = \{ i \mid i \geq j_d, v_i < d < v_{i+1} \}$ the set of possible choices for $i$ limited by the so far derived conditions $i \geq j_d$ and $v_i < d < v_{i+1}$. The set $I_d$ is nonempty: It follows from the order-preserving nature of $\omega$ that $v_{j_d} < d < v_{q+1}$, so there must be at least one value of $i$ with $j_d \leq i \leq q$ such that $v_i < d < v_{i+1}$. Let $i_d = \min I_d$.

We will now show by *reductio ad absurdum* that placing $d$ into a subset other than $D^{i_d}$ leads to a violation of condition c). (Recollect that every deletable element appears in $w$ at least as early as possible.) Assume that $d \in D^i$ with $i \in I_d$ and $i > i_d$. Then, by definition of $I_d$, we know that $d < v_{i_d+1}$. In order for $d$ to be deletable from the sequence $w$, there must be an element $e \in [p]$ such that $\omega^{-1}(e) <_P \omega^{-1}(d)$ and which appears in $w$ between $v_{i_d+1}$ and $d$. Denote by $E$ the set of such elements: $E = \{ e \in [p] \mid \omega^{-1}(e) <_P \omega^{-1}(d), \sigma(\omega^{-1}(v_{i_d+1})) < \sigma(\omega^{-1}(e)) < \sigma(\omega^{-1}(d)) \}$, where $\sigma$ denotes the map $\sigma : P \to p, w_i \mapsto i$ implied by $w$. If there is an $e \in E$ with $e \not\in D$, then $e$ must appear in $v$ at some position $k$, $e \equiv v_k$. Since $\omega^{-1}(e) < \omega^{-1}(d)$, according to the definitions of $j_d$ and $i_d$ we find that $k \leq j_d \leq i_d$, in contradiction with the requirement that $v_{i_d+1}$ precede $e = v_k$ in $v$. Therefore, $e \in D$ and thus $E \subset D$. Consider now the element $c = \min E$. It follows from $\omega^{-1}(e) <_P \omega^{-1}(d)$ that $c < d < v_{i_d+1}$. In order for $c$ to be deletable from the finished sequence $w$, there must be an element $e' \in [p]$ such that $\omega^{-1}(e') <_P \omega^{-1}(c)$ and which appears in $w$ between $v_{i_d+1}$ and $c$. Because of $\omega^{-1}(c) < \omega^{-1}(d)$ and since $c$ must precede $d$ in $w$, the aforementioned element $e'$ must be in $E$, and therefore $\min E = e < e'$. At the same time, since $\omega$ is a natural labeling, $\omega^{-1}(e') <_P \omega^{-1}(c)$ implies $e' < c$. This contradiction shows that $c$ cannot be deletable from $w$, $c \not\in \text{Del}_P(w)$. However we have found before that $c \in D$, which means that the assumption $i > i_d$ leads to a violation of condition c). Therefore, we must have $i \leq i_d$. Since $i_d$ is defined as the minimum allowed value of $i$, we have $i = i_d$.

To summarize, we have shown until now that the only way to construct a sequence $w$ in a way that does not contradict conditions a) – c) is to follow the construction introduced above, which can be described in the following way:
Step 1: For every $d \in D$, let
\[
    j_d = \max \{ \{ j \in [q] | \omega^{-1}(v_j) <_P \omega^{-1}(d) \} \cup \{0\} \} \tag{5}
\]
and
\[
i_d = \min \{ \{ i | j_d \leq i \leq q, v_i < d < v_{i+1} \} \}, \tag{6}
\]
and assign $d$ to the set $D^{i_d}$.

Step 2: For every $0 \leq i \leq q$, insert between $v_i$ and $v_{i+1}$ the elements of $D^i$ in increasing order:
\[
w = d_0^i d_1^i \ldots d_{m_i}^i v_i d_1^i \ldots d_{m_i}^i v_2 \ldots d_0^i d_1^i \ldots d_{m_i}^i,
\]
where $d_q^i \in D^i$ and $v_i < d_1^i < d_2^i < \ldots < d_{m_i}^i < v_{i+1}$.

It remains to be demonstrated that the sequence $w$ uniquely defined in this way indeed satisfies all conditions a) – c). Condition a) is satisfied by construction.

Next, let us verify that $w$ satisfies condition b). Consider two arbitrary elements $s, t \in P$ such that $s <_P t$. Each of their labels $\omega(s)$ and $\omega(t)$ can be in $D$ or in $[p] \setminus D$. For each case, we have to show that $\omega(s)$ precedes $\omega(t)$ in $w$.

- If $\omega(s), \omega(t) \in [p] \setminus D$, then $\omega(s) \equiv v_i$ and $\omega(t) \equiv v_j$ for some $i, j \in [q]$. Since $Q$ is an induced subposet of $P$, we have $s <_Q t$, and therefore we know that $i < j$, i.e., $v_i \equiv \omega(s)$ precedes $v_j \equiv \omega(t)$ in $v$. Then, by construction, $\omega(s)$ precedes $\omega(t)$ also in $w$.

- If $\omega(s) \in [p] \setminus D$ and $\omega(t) \in D$, then $\omega(s) \equiv v_k$ for some $k \in [q]$. Step 1 defines two numbers $j_{\omega(t)}$ and $i_{\omega(t)}$. Since $s <_P t$, $k$ is in $\{ j \in [q] | \omega^{-1}(v_j) <_P t \}$, and thus by Eq. (5) $k \leq j_{\omega(t)}$. From Eq. (6) it is clear that $j_{\omega(t)} \leq i_{\omega(t)}$. Consequently, the label $\omega(t)$ is assigned to $D^{i_{\omega(t)}}$ with $k \leq i_{\omega(t)}$, which means that $\omega(t)$ appears in $w$ after $\omega(s) \equiv v_k$.

- If $\omega(s) \in D$ and $\omega(t) \in [p] \setminus D$, then $\omega(t) \equiv v_k$ for some $k \in [q]$. Any $v_l \in [p] \setminus D$ with $\omega^{-1}(v_l) <_P s$ also satisfies $\omega^{-1}(v_l) <_P s <_P t = \omega^{-1}(v_k)$, and therefore $l < k$. Therefore, application of Step 1 to $\omega(s)$ results in $j_{\omega(s)} < k$ and, due to the fact that $\omega(s) < \omega(t)$, we have $i_{\omega(s)} < k$. Thus, $\omega(s)$ is assigned to a $D^{i_{\omega(s)}}$ with $i_{\omega(s)} < k$, and therefore it appears in $w$ before $\omega(t)$.

- If $\omega(s), \omega(t) \in D$, then for any $v_k$ with $\omega^{-1}(v_k) <_P s$, it follows directly that $\omega^{-1}(v_k) <_P t$. Therefore, in Step 1, we find $j_{\omega(s)} \leq j_{\omega(t)}$, and as a result, in addition to $j_{\omega(s)} \leq i_{\omega(s)}$ (due to Eq. (6)) we also know that $j_{\omega(s)} \leq i_{\omega(t)}$. By construction of $j_{\omega(s)}$ and $i_{\omega(t)}$ as well as the order-preserving nature of $\omega$, we find $v_{j_{\omega(s)}} < \omega(s) < \omega(t) < v_{i_{\omega(t)}+1}$. Therefore, there must be at least one value of $i$ in the interval $j_{\omega(s)}, \ldots, i_{\omega(t)}$ such that $v_i < \omega(s) < v_{i+1}$. It follows that $i_{\omega(s)} \leq i_{\omega(t)}$. If $i_{\omega(s)} < i_{\omega(t)}$ then obviously $\omega(s)$ appears in $w$ before $v_{i_{\omega(t)}}$, which in turn appears before $\omega(t)$. Finally, even if $i_{\omega(s)} = i_{\omega(t)}$, in Step 2 the elements of each $D^i$ are inserted into the sequence $w$ in increasing order, so in any case $\omega(s)$ will be inserted before $\omega(t)$.

We have shown that the constructed sequence $w$ satisfies the condition b), $w \in L(P)$.

Finally let us verify that $w$ satisfies condition c). Consider an element $d \in D$ which is inserted into $w$ at some position $k$, thus $d \equiv w_k$. We have ensured during the construction process that $w_{k-1} < d \equiv w_k < w_{k+1}$, therefore condition 1) of Def. [9] is satisfied. If $w_j < w_k$ for all $j \in (0, k)$, then condition 2a) of Def. [9] is satisfied and $d$ is deletable in $w$. Otherwise, the set $L = \{ i | i < k$ and $w_i > w_k \}$ is nonempty. Let $l = \max L$. It is clear that $w_i < w_k$ for all $i \in (l, k)$ such
that $\omega^{-1}(w_j) <_P \omega^{-1}(w_k)$, then condition 2b) of Def. 6 is satisfied. We show below that indeed this is the case.

It follows directly from the definition of $l$ that $w_l > w_k$ and simultaneously $w_k > w_{l+1}$ (since $l + 1 \in (l, k)$); therefore $w_l > w_{l+1}$, and $l$ is a descent. We already have seen that condition 1) of Def. 6 is satisfied for all the elements of $D$; therefore $w_l$ and $w_{l+1}$ are not in $D$, but appear somewhere in the original sequence $v$, in the form $w_l \equiv v_l$ and $w_{l+1} \equiv v_{l+1}$ for some $l \in [q]$. Since $l + 1 < k$, during Step 1 $d$ must have been assigned to some $D^i$ with $l < i_d$.

Let us assume that $j_d < l$; we will see that this assumption leads to a contradiction. From $\omega^{-1}(v_{j_d}) <_P \omega^{-1}(d)$ it follows that $v_{j_d} < d$. By construction of $\tilde{l}$, we have $d < v_l$. Therefore, if $j_d < l$, then there must be a $i \in (j_d - 1, l)$ such that $v_i < d < v_{i+1}$. Then, by definition of $i_d$, we would have $i_d \leq i < \tilde{l}$, in contradiction with $l < i_d$. Therefore, the assumption $j_d < \tilde{l}$ made at the beginning of this paragraph must be wrong and we have $\tilde{l} \leq j_d$. Since $v_l > d$, we find $\omega^{-1}(v_l) \notin \omega^{-1}(d)$, and therefore (by Eq. (5)) $j_d \neq \tilde{l}$. This reasoning shows that $\tilde{l} < j_d$.

The entry $v_{j_d}$ of $v$ appears in $w$ in the form $v_{j_d} \equiv w_j$ at some position $j \in [p]$. Since $\tilde{l} < j_d \leq i_d$ and elements of $v$ appear in the same order in $w$, we find $l < j < k$. As we have shown earlier, by construction of $l$, we have $w_m < w_k$ for all $m \in (l, k)$ (and thus especially for all $m \in (j, k)$); and by construction of $j$, we have $\omega^{-1}(w_j) <_P \omega^{-1}(w_k)$. Therefore, condition 2b) of Def. 6 is satisfied. It follows that every $d \in D$ is deletable in $w$, meaning that $D \subset \text{Del}_P(w)$.

We have demonstrated that there is exactly one sequence $w$ of elements of $[p]$ that satisfies conditions a) – c) given at the beginning of this proof, i.e., there is exactly one linear extension $w$ such that $v = w \setminus D$ and $D \subset \text{Del}_P(w)$.

**Example 13.** Let us demonstrate the insertion process described in Steps 1&2 during the proof above. Consider the sequence $v = 17536 \in \mathcal{L}(P \setminus D)$ with the poset $P = 3 \times 3$ shown in Fig. 2 and the set of deleted elements $D = \{2, 4, 8, 9\}$. For the labels $d = 2$ and $4$, we find for the set in Eq. (5) $\{j \in [5] | \omega^{-1}(v_j) <_P \omega^{-1}(d) \} = \{j \in [5] | v_j \in \{1\} \} = \{1\}$, and thus $j_2 = j_4 = 1$. Since $v_1 = 1 < 2, 4 < v_2 = 7$, in Eq. (6) we find $i_2 = i_4 = 1$. Therefore, the labels 2 and 4 are assigned to the subset $D^1$, and will be inserted into the sequence $v$ between $v_1 = 1$ and $v_2 = 7$. For the label 8, we find that in Eq. (5) $\{j \in [5] | \omega^{-1}(v_j) <_P \omega^{-1}(8) \} = \{j \in [5] | v_j \in \{5, 7\} \} = \{2, 3\}$, and thus $j_8 = 3$. Since however the label 8 is larger than any of the following labels in $v$, $v_3 = 5$, $v_4 = 3$ and $v_5 = 6$, we find in Eq. (6) that $\{i | j_i = 3 \leq i \leq 5, v_i < 8 < v_{i+1} \} = \{5\}$ and therefore $i_8 = 5$. Finally, for the label 9, we find that in Eq. (5) $\{j \in [5] | \omega^{-1}(v_j) <_P \omega^{-1}(9) \} = \{j \in [5] | v_j \in \{6, 8\} \} = \{5\}$, and thus $j_9 = 5$ and $i_9 = 5$. To summarize, in Step 1, we split the set $D = \{2, 4, 8, 9\}$ into the subsets $D^0 = \emptyset$, $D^1 = \{2, 4\}$, $D^2 = \emptyset$, $D^3 = \emptyset$, $D^4 = \emptyset$ and $D^5 = \{8, 9\}$. In Step 2, the elements of each subset are arranged into a growing sequence, specifically $d_1 d_2^1 = 24$ and $d_1 d_2^2 = 89$, and inserted into $v$:

$$D^0 = \emptyset \quad D^1 = \{2, 4\} \quad D^2 = \emptyset \quad D^3 = \emptyset \quad D^4 = \emptyset \quad D^5 = \{8, 9\}$$

The resulting sequence $w = 124753689$ was previously considered and illustrated in Fig. 3(e).
Lemma 14. Let \( w \) be a linear extension of \( P \) with the set of deletable elements \( \text{Del}_P(w) \). For every set \( D \subseteq \text{Del}_P(w) \), the sequence \( w \setminus D \) is a linear extension of \( P \setminus D \).

Proof. Consider two elements \( s, t \in P \setminus D \) with \( s <_{P \setminus D} t \). In order to show that \( w \setminus D \) is a linear extension of \( P \setminus D \), we have to demonstrate that \( \omega(s) \) appears in \( v \) before \( \omega(t) \). Since \( P \setminus D \) is an induced subposet of \( P \), it follows from \( s <_{P \setminus D} t \) that \( s <_P t \), and since \( w \) is a linear extension, this implies that \( \omega(s) \) precedes \( \omega(t) \) in \( w \). Clearly then, by construction of \( v \), \( \omega(s) \) also precedes \( \omega(t) \) in \( v \). \( \square \)

By combining the previous two lemmata, we find that

Lemma 15. The union of the Jordan-Hölder sets of all subposets of \( P \) is given by

\[
\bigcup_{Q \subseteq P} \mathcal{L}(Q) = \bigcup_{w \in \mathcal{L}(P)} \bigcup_{D \subseteq \text{Del}_P(w)} \{w \setminus D\}.
\]

Proof. Consider first a set \( D \subseteq [p] \) and the corresponding subposet of \( P \) given by \( P \setminus D \). It follows directly from Lemmata 12 and 14 that the collection of linear extensions of \( P \setminus D \) is can be written as

\[
\mathcal{L}(P \setminus D) = \bigcup_{w \in \mathcal{L}(P)} \{w \setminus D\}.
\]

Therefore, the set of linear extensions of subposets of \( P \) is given by

\[
\bigcup_{Q \subseteq P} \mathcal{L}(Q) = \bigcup_{D \subseteq [p]} \mathcal{L}(P \setminus D) = \bigcup_{D \subseteq [p]} \bigcup_{w \in \mathcal{L}(P)} \{w \setminus D\} = \bigcup_{w \in \mathcal{L}(P)} \bigcup_{D \subseteq \text{Del}_P(w)} \{w \setminus D\}.
\]

\( \square \)

4 Proof of Theorem 3

We are now ready to combine the so far derived lemmata into the derivation of the closed form of the extended order polynomial given in the first section.

Proof. (of Theorem 3) By application of the Definitions given in Eqs. (1) and (2) as well as (in line 3) Lemmata 11 and 15, we find

\[
E_p(n, z) = \sum_{Q \subseteq P} \Omega_Q(n) z^{|Q|}
= \sum_{Q \subseteq P} \sum_{v \in \mathcal{L}(Q)} \binom{n + \text{des}(v)}{|Q|} z^{|Q|}
= \sum_{w \in \mathcal{L}(P)} \sum_{D \subseteq \text{Del}_P(w)} \binom{n + \text{des}(w)}{p - |D|} z^{p - |D|}
= \sum_{w \in \mathcal{L}(P)} \sum_{k=0}^{\text{del}_P(w)} \binom{\text{del}_P(w)}{k} \cdot \binom{n + \text{des}(w)}{p - k} z^{p - k}
= \sum_{w \in \mathcal{L}(P)} \sum_{k=0}^{\text{del}_P(w)} \binom{\text{del}_P(w)}{k} \cdot \binom{n + \text{des}(w)}{p - k} z^{p - k}
\]
By inverting the order of summation in the inner sum, we obtain Eq. (4).

5 Perspectives

Our main motivation to develop the extended strict order polynomial \( E^P(n, z) \) introduced in the current communication is its close relation to the Zhang-Zhang polynomial \([4, 5, 6]\) enumerating Clar covers of benzenoid hydrocarbons \([7]\), a topic to which we devoted feverish activity in our laboratory for almost a decade now \([8, 9, 10, 11, 12, 13, 14]\). Our recent contribution, introducing the interface theory of benzenoids \([13, 14]\), demonstrated that enumeration of Clar covers of a benzenoid \( B \) can be efficiently achieved by studying distributions of covered interface edges in interfaces of \( B \). The relative positions of the covered edges can be expressed in a form of a poset. Without giving too many unnecessary details, we can say that for a regular benzenoid strip \( B \) of length \( n \), there exists a poset \( P \) such that the extended strict order polynomial \( E^P(n, z) \) coincides with the Zhang-Zhang polynomial \( \text{ZZ}(B, x) \) of \( B \) (with \( z = x + 1 \)).

A detailed proof of this fact is quite technical and will be announced soon. The announced equivalence between the extended strict order polynomials \( E^P(n, z) \) developed in the current study and the Zhang-Zhang polynomials \( \text{ZZ}(B, x) \) of regular benzenoid strips \( B \) allows us to recognize (currently without a formal proof) a large collection of facts about \( E^P(n, z) \) due to the previously discovered facts about the ZZ polynomials. Among others, the following facts are easy to deduce:

1. The chain \( P = p \) corresponds to a parallelogram \( M(p, n) \)

\[
P = p
\]

for which the ZZ polynomial is given in form of a hypergeometric function, \( \text{ZZ}(M(m, n), x) = 2F1\left[\begin{array}{c}
-m, -n \\
1
\end{array}; x + 1\right] \); consequently, we have

\[
E^p(n, z) = 2F1\left[\begin{array}{c}
-p, -n \\
1
\end{array}; z\right].
\]  

(7)

This result is also directly obvious from Theorem 3. The Jordan-Hölder set of \( p \) consists of only one element, \( \mathcal{L}(p) = \{123 \ldots p\} \), for which \( \text{del}_p(123 \ldots p) = p \) and \( \text{des}_p(123 \ldots p) = 0 \). Thus, Eq. (4) immediately assumes the form of Eq. (7).

2. The poset \( P \) containing \( p \) non-comparable elements corresponds, according to the interface theory of benzenoids, to a prolate rectangle \( Pr(p, n) \)
for which the ZZ polynomial is given by \( \text{ZZ}(P_r(m, n), x) = (1 + n (x + 1))^m \) \( [6, 17, 18] \) consequently, we have

\[
E_{[p]}^p(n, z) = (1 + nz)^p.
\] (8)

3. The poset \( P = 2 \times m \) corresponds to a hexagonal graphene flake \( O(2, m, n) \)

It follows from the ZZ polynomial \( \text{ZZ}(O(2, m, n), x) \) \( [19, 20, 21] \) that the strict order polynomial has the form of a \( 2 \times 2 \) determinant

\[
E_{2 \times m}^p(n, z) = \begin{vmatrix}
\text{2F}_1\left[-m, -n ; -\frac{1}{2}; z\right] & \text{2F}_1\left[1-m, 1-n ; -\frac{3}{2}; z\right] \\
\text{2F}_1\left[1-m, 1-n ; -\frac{3}{2}; z\right] & \text{2F}_1\left[-m, -n ; -\frac{1}{2}; z\right]
\end{vmatrix}
\] (9)

4. The strict order polynomial for the lattice \( P = l \times m \) is unknown, following the fact that this poset corresponds to the hexagonal flake \( O(l, m, n) \)

The ZZ polynomial \( \text{ZZ}(O(l, m, n), x) \) of this structure constitutes the hardest unsolved problem in the theory of ZZ polynomials \( [17, 18, 19, 22] \).

5. The fence \( P = Q(1, m) \) with \( m \) elements corresponds to a zigzag chain \( Z(m, n) \)
The expression for \( ZZ(Z(m, n), x) \)—and consequently, \( E^P_P(n, z) \)—is given by a very lengthy formula \[17, 10, 23\], but the associated generating function has the form of a continued fraction \[23\]

\[
\sum_{m=0}^{\infty} E^P_Q(1,m)(n,z)t^m = \frac{-1}{t + \frac{-1}{zt + \frac{-1}{\ddots + \frac{-1}{zt + \frac{-1}{zt + (z-1)^n}}}}}
\]

(10)

An analogous generating function with respect to \( n \) is unknown.

The extended order polynomial \( E^P_P(n, z) \) can be also computed in an efficient fashion directly from Eq. (4) through an algorithm based on a graph of „compatible” antichains of \( P \). Propagating weights through this graph in a certain way yields the extended order polynomial without ever having to construct the entire set \( L(P) \). This algorithm has been implemented in Maple 16 \[24\] and will be reported later. For the example of the poset \( P = 3 \times 3 \) depicted in Fig. 2 we obtain in this way

\[
E^3_{3 \times 3}(n,z) = \sum_{k=0}^{9} \left(9 \cdot \binom{n}{k} + \binom{9 \cdot 2 - 2}{k - 2} \cdot \binom{9 - 3}{k - 3} \cdot \binom{n + 1}{k} \right.
\]
\[
+ \left(9 \cdot \binom{9 - 3}{k - 3} + 17 \cdot \binom{9 \cdot 4 - 4}{k - 4} + 2 \cdot \binom{9 \cdot 5 - 5}{k - 5} \right) \binom{n + 2}{k}
\]
\[
+ \left(2 \cdot \binom{9 \cdot 5 - 5}{k - 5} + 7 \cdot \binom{9 \cdot 6 - 6}{k - 6} + \binom{9 \cdot 7 - 7}{k - 7} \right) \binom{n + 3}{k}
\]
\[
\left. + \left(9 \cdot 7 \cdot \binom{n + 4}{k} \right) \right) z^k.
\]

We suspect that the coefficients \( e_{l,j}(P) \) appearing in \( E^P_P(n, z) \) in front of the terms \( \binom{9 - 2l - j}{k - 2l - j} \binom{n+l}{k} \) are \#P-complete to compute, in close analogy to the coefficients \( e(P) \) corresponding to the number of linear extensions of \( P \). These coefficients are growing very fast with the size of the poset \( P \). The largest of the coefficients \( e_{l,j}(3 \times 3) \) is only 17 (as can be easily seen from Eq. (11)), but larger \( P \) are characterized by much greater coefficients, e.g., \( \max(e_{l,j} (4 \times 4))=3765, \max(e_{l,j} (4 \times 5))=200440, \max(e_{l,j} (5 \times 5))=61885401, \max(e_{l,j} (5 \times 6))=27950114975 \).

It seems that the introduced here extended strict order polynomial \( E^P_P(n, z) \) can be immediately generalized to the non-strict case using the reciprocity theorem of Stanley (Corollary 3.15.12 of \[1\]), but this problem is not pursued here further.
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