SUPPORT OF BORELIAN MEASURES
IN SEPARABLE BANACH SPACES.

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ABSTRACT

We prove in this article that every Borelian measure, for example, the distribution of a random variable, in separable Banach space has a support which is compact embedded Banach subspace; and prove that if the norm of the random variable belongs to some exponential Orlicz space, then the new subspace can be choose such that the norm of this variable in the new space also belongs to other exponential Orlicz space.

Key words: Banach and Orlicz space, Borelian measure, support, compact embedded Banach subspace, norm, rearrangement invariant space, metric entropy, distance, moment, Grand Lebesgue Space, martingale.

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1. Introduction. Notations. Statement of problem.

Let A) $X$ be Separable Banach Space (SBS) with a norm $|·|_X$, briefly: $X = (X, |·|_X) \in SBS$, equipped with Borelian sigma - algebra $B$; 

B) $(\Omega, F, P)$ be probability triple with expectation $E$;

C) $\xi$ be a random variable (r.v.) defined on the our triple with the values in the space $X$:

$$P(\xi \notin X) = 0 \quad (1)$$

with Borelian distribution $\mu_\xi(·)$:

$$\mu_\xi(A) = P(\xi(·) \in A), \ A \in B;$$

D) $\Phi = \Phi(u), \ u \in R$ be an Orlicz function, i.e. convex, even, continuous, twice continuous differentiable in the domain $u \in [2, \infty)$, strictly decreasing in the right side half line, and such that
\[
\Phi(0) = 0; \lim_{u \to \infty} \Phi(u) = \infty.
\]

We will denote the Orlicz space over \((\Omega, F, P)\) with Orlicz function \(\Phi(\cdot)\) as \(Or(\Phi)\) and the correspondent Orlicz norm as \(\| \cdot \|_{Or(\Phi)}\): for all real valued r.v. \(\eta\)

\[
\|\eta\|_{Or(\Phi)} = \inf\{\lambda, \lambda > 0, E\Phi(|\eta|/\lambda) \leq 1\}.
\]

By definition, \(\eta \in Or(\Phi)\) iff \(\|\eta\|_{Or(\Phi)} < \infty\).

It is proved ([1], [2], [5], [6]) that if the condition (1) is satisfied, then there exists a compact embedded SBS in the space \(X\) subspace \(Y = (Y, | \cdot | Y)\) : \(Y \subset X\) such that the distribution of \(\xi\) is concentrated on the subspace \(Y\):

\[
P(\xi \notin Y) = 0. \quad (2)
\]

Recall that the SBS subspace \(Y\) of the space \(X\) : \(Y \subset X\) is called compact embedded, if the unit ball of the space \(Y\) is precompact set in the space \(X\), i.e. that the closure in the \(X\) norm of unit ball of the space \(Y\) is compact set in the space \(X\).

Moreover, if for some Orlicz function \(\Phi\) satisfying the so-called \(\Delta_2\) condition:

\[
\lim_{u \to \infty} \frac{\Phi(2u)}{\Phi(u)} < \infty \quad (3)
\]

the norm of r.v. \(\xi\), belongs to Orlicz space \(Or(\Phi)\):

\[
\| | \xi | X \|_{Or(\Phi)} < \infty, \quad (4)
\]

we can choose the subspace \(Y = (Y, | \cdot | Y)\) such that the new norm of a r.v. \(\xi\) in the space \(Y\), i.e. the variable \(|\xi| Y\) belongs to the space \(Or(\Phi)\):

\[
\| | \xi | Y \|_{Or(\Phi)} < \infty.
\]

The aim of this report is investigate the case when the function \(\Phi\) does not satisfies the \(\Delta_2\) condition (3).

Recall in the end oh this section then the Orlicz function \(\Psi\) is called weaker than \(\Phi\), if for all positive constant \(v; \ v = const > 0\)

\[
\lim_{u \to \infty} \frac{\Psi(vu)}{\Phi(u)} = 0. \quad (5)
\]

Notation: \(\Psi << \Phi\).

2. Main result.

Theorem 1. Let \(\xi\) be a r.v. with Borelian distribution \(\mu_\xi(\cdot)\) in the SBS \((X, | \cdot | X)\) satisfies the condition (1).

Let also \(\Psi(\cdot)\) be other Orlicz function weaker than \(\Phi(\cdot): \Psi << \Phi\).

Then there exists a compact embedded in the space \(X\) the SBS space \(Y = (Y, | \cdot | Y)\), which depended on the \(\mu_\xi\) and on the function \(\Psi\), such that the variable \(\xi\) belongs to the space \(Y\) a.e. and moreover
\[ || \xi \| Y \| Or(\Psi) < \infty. \] 

**Proof.**

1. Note first of all that we can conclude that the space \( X \) coincides with the SBS space \( C[0, 1] \) of all continuous real valued function defined on the closed interval \([0,1]\), as long as the last space is universal in the class of all SBS.

We denote the norm in the space \( C[0, 1] \) as

\[ |f|_{\infty} = \sup_{t \in [0,1]} |f(t)|. \]

2. In detail, the r.v. \( \xi \) can be identified with a continuous a.e. random process \( \xi = \xi(t), t \in [0,1] \).

Since the variable \( |\xi|_{\infty} \) belongs to the Orlicz space \( Or(\Phi) \), we can and will assume that

\[ E \Phi(|\xi|_{\infty}) \leq 1. \]

Let us define as usually for arbitrary continuous on the set \([0,1]\) function \( f = f(t), t \in [0,1] \) the slight modified module of uniform continuity

\[ \omega(f, \delta) = 0.25 \sup_{h, |h| \leq \delta} \sup_{t \in [0,1]} |f(t + h) - f(t)|, \]

where \( \delta \in [0,1], t + h = \min(t + h, 1), h > 0, \) and \( t + h = \max(t + h, 0) \) in the case \( h < 0 \).

Note that

\[ \lim_{\delta \to 0^+} \omega(\xi, \delta) = 0 \]

with probability one and

\[ |\omega(\xi, \delta)| \leq 0.5 |\xi|_{\infty}. \]

It follows from the dominated convergence theorem that

\[ \lim_{\delta \to 0^+} E \Phi(\omega(\xi, \delta)) = 0. \]

3. In the language of the theory of Orlicz spaces (see [3], [4], [7] ) the equality (7) denotes the moment convergence, or convergence in mean the function \( \delta \to \omega(\xi, \delta) \) to zero as \( \delta \to 0^+ \).

If the function \( \Phi \) satisfies the \( \Delta_2 \) condition, the equality (7) means also the Orlicz norm convergence, but in general case from (7) follows only the Orlicz norm \( Or(\Psi) \) convergence:

\[ \lim_{\delta \to 0^+} ||\omega(\xi, \delta)|| Or(\Psi) = 0. \]

4. It follows from (8) then there exists monotonically decreasing tending to zero sequence \( \delta(n), n = 1, 2, \ldots, \) for which
\[ || \omega(\xi, \delta(n)) ||_{Or(\Psi)} \leq 4^{-n}. \] (9)

Let us define the new subspace \( Z = (Z, | \cdot |_{\infty, \omega}) \) of the space \( C[0,1] \) consisting on all the (continuous) function \( \{g\} \) for which

\[
\lim_{n \to \infty} 2^n \omega(g, \delta(n)) = 0,
\] (10)

with the finite norm

\[
|g|_{\infty, \omega} = |g|_{\infty} + \sup_n 2^n \omega(g, \delta(n)).
\] (11)

5. Since the functions from the space \( Z \) satisfies the condition (10), the closed subspace \( Z = (Z, | \cdot |_{\infty, \omega}) \) is SBS. It follows from the famous Arzela theorem that the space \( Z \) is compact embedded in the space \( C[0,1] \).

6. Finally, let us prove that the Orlicz \( Or(\Psi) \) norm of the \( \xi \) in the space \( Z \) is finite. We estimate:

\[
|| |\xi|_{\infty, \omega} ||_{Or(\Psi)} \leq || |\xi|_{\infty} ||_{Or(\Phi)} + 
\sum_{n=1}^{\infty} 2^n ||\omega(\xi, \delta(n))||_{Or(\Psi)} \leq C || |\xi|_{\infty} ||_{Or(\Phi)} + 1 < \infty.
\]

This completes the proof of theorem 1.

3. The case of infinite measure.

In this pilcrow we consider the case when the measure \( \mathbf{P} \) is unbounded, but is sigma - finite and diffuse.

In order to distinct the probabilistic case and the case with unbounded measure, we will write the source triple as \((\Omega, B, Q)\).

We understood here instead the random variable \( \xi \) some measurable function from the triple \((\Omega, B, Q)\) with Borelian distribution

\[ \mu_{\xi}(A) = \mathbf{P}(\xi(\cdot) \in A), \ A \in B. \]

Recall that in this case the Orlicz function \( \Psi \) is called weaker than \( \Phi \), \( \Psi \ll \Phi \) if both the function \( \Psi, \Phi \) satisfy the condition (5) and in addition satisfy the following condition: for all positive constant \( v; \ v = const > 0 \)

\[
\lim_{u \to 0} \frac{\Psi(v u)}{\Phi(u)} = 0.
\] (12)

The next result is proved analogously to the Theorem 1; the important facts for the moment and norm convergence in Orlicz spaces with unbounded measure see in the books [3], [4].

**Theorem 2.** Let \( \xi \) be measurable function \( \xi : \Omega \to X \) with Borelian distribution \( \mu_{\xi}(\cdot) \) in the SBS \((X, | \cdot |_{X})\) satisfies the conditions (1) and (4).
Let also $\Psi(\cdot)$ be Orlicz function weaker than $\Phi(\cdot): \Psi \ll \Phi$.

Then there exists a compact embedded in the space $X$ the SBS space $Y = (Y, |\cdot|_Y)$, which depended on the $\mu_\xi$ and on the function $\Psi$, such that the variable $\xi$ belongs to the space $Y$ a.e. and moreover

$$||\xi||_Y Or(\Psi) < \infty. \quad (13)$$

4. Some generalizations.

A. The assertion of theorems 1 and 2 is true for the family $M = \{\mu_\alpha\}, $ $\alpha \in W$, $W$ is arbitrary set, of Borelian $\sigma$ – finite distributions $\{\mu_\alpha\}$ (i.e. not necessary to be bounded), if the family $M$ is dominated in the Radon - Nikodim sense.

For instance, this is true for countable family $M$.

Indeed, there exists a common compact embedded in the space $X$ subspace $(Y, |\cdot|_Y)$ such that for all $\alpha \in W$

$$\mu_\alpha(X \setminus Y) = 0.$$

The assertion of theorem 1 and 2 about Orlicz uniform integrability of a family $M$ is also true.

Namely, let for some Orlicz function $\Phi = \Phi(u)$

$$\sup_{\alpha \in W} \int_X \Phi(x) \mu_\alpha(dx) < \infty;$$

and let $\Psi = \Psi(u)$ be an Orlicz function weaker than $\Phi: \Psi \ll \Phi$; then the common for all values $\alpha$ compact embedded subspace $Y$ may be constructed such that

$$\sup_{\alpha \in W} \int_Y \Psi(y) \mu_\alpha(dy) < \infty.$$

B. Let $M = \{\mu_m\}, m = 1, 2, \ldots$ be countable family of Borelian distributions in SBS $(X, |\cdot|X)$ which convergent weakly to some distribution $\mu$, i.e. for all continuous bounded functional $g: X \to R$

$$\lim_{n \to \infty} \int_X g(x) \mu_n(dx) = \int_X g(x) \mu(dx).$$

For example, the family $M$ may satisfy the to - called Central Limit Theorem (CLT) in the space $X$.

This means by definition that the measure $\mu$ is Gaussian.

Then the common compact embedded support subspace $(Y, |\cdot|Y)$ can be constructed in addition such that the sequence $M$ convergent weakly to the measure $\mu$ in the space $Y$ : for all continuous bounded functional $h: Y \to R$

$$\lim_{n \to \infty} \int_Y h(y) \mu_n(dy) = \int_Y h(y) \mu(dy).$$

C. V.V.Buldygin [2] proved that in the probabilistic case $\mu(X) = 1$ the subspace $Y$ may be constructed to be reflexive and with continuous differentiable in the Freshe sense norm.
At the same time is true also in general, i.e. in unbounded case $\mu(X) = \infty$.

**D.** For the space $X = C(T)$, where $T$ is compact metric space with distance $d = d(t, s), t, s \in T$, the assertion of Theorem 1 may be formulated as follows.

Let $\xi = \xi(t), t \in T$ be continuous with probability one random field such that for some Orlicz function $\Phi$

$$
E\Phi(|\xi|_\infty) < \infty.
$$

Let also $\Psi$ be another Orlicz function, $\Psi << \Phi$.

Then there exist a non-negative r.v. $\zeta$ and non-random continuous semi-distance $r = r(t, s)$ on the space $T$ such that

$$
|\xi(t) - \xi(s)| \leq \zeta \times r(t, s),
$$

where

$$
E\Psi(\zeta) < \infty.
$$

**DH.** In the connection of the last assertion we dare formulate the following, interest by our opinion, hypothesis. Let $\theta = \theta(t), t \in T$ be arbitrary separable random field, centered: $E\theta(t) = 0$ or not, bounded with probability one:

$$
\sup_{t \in T} |\theta(t)| < \infty \text{ a.e.}
$$

Assume in addition that for some Orlicz function $\Phi(\cdot)$

$$
\sup_{t \in T} ||\theta(t)|| Or(\Phi) < \infty.
$$

Let also $\Psi$ be another Orlicz function such that $\Psi << \Phi$.

Open question: there holds (or not)

$$
|| \sup_{t \in T} |\theta(t)| ||Or(\Psi) < \infty? \quad (14)
$$

The conclusion (14) is true for the centered Gaussian fields [8]; if the field $\theta(\cdot)$ satisfies the so-called entropy or generic chaining conditions [5], [9], [10]; in the case if $\theta$ belongs to the domain of attraction of Law of Iterated Logarithm [11] etc.

Finally, let us consider the following example. Let $\tau$ be Normal (Gaussian) standard distributed r.v.: $Law(\tau) = N(0, 1)$ ant let $T = R = (\infty, \infty)$. We define

$$
\theta(t) = \tau \cdot t - |t|^p/p, \quad p = const \in (1, 2).
$$

Then $\theta(\cdot)$ is upper-bounded Gaussian non-centered random process, but

$$
\sup_{t \in R} \theta(t) = |\tau|^q/q, \quad q = p/(p - 1) > 2.
$$

The tail of distribution of the r.v. $\sup_{t \in R} \theta(t)$ is essentially heavier in comparison to the Gaussian r.v. $\tau$ or to the upper tail of each variable $\theta(t), t \in T$. 
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