REPLICABILITY OF 1-D SCHRÖDINGER EQUATION WITH UNBOUNDED OSCILLATION PERTURBATIONS

Z. LIANG AND Z. WANG

ABSTRACT. We build a new estimate for the normalized eigenfunctions of the operator $-\partial_x^2 + V(x)$ based on the oscillatory integrals and Langer’s turning point method, where $V(x) \sim |x|^{2\ell}$ at infinity with $\ell > 1$. From it and an improved reducibility theorem we show that the equation

$$i\partial_t \psi = -\partial_x^2 \psi + V(x)\psi + \epsilon(x)^m W(\nu x, \omega t)\psi, \quad \psi = \psi(t, x), \ x \in \mathbb{R}, \ \nu < \ell - 1 + \frac{1}{2\ell + 1},$$

can be reduced in $L^2(\mathbb{R})$ to an autonomous system for most values of the frequency vector $\nu$ and $\nu$, where $W(\varphi, \phi)$ is a smooth map from $T^d \times T^n$ to $\mathbb{R}$ and odd in $\varphi$.

1. Introduction of the Main Results

1.1. Main Results. In this paper we study the problem of reducibility of the time dependent Schrödinger equation

$$\mathcal{H}(t)\psi(x, t) = i\partial_t \psi(x, t), \ x \in \mathbb{R};$$

$$\mathcal{H}(t) := -\frac{d^2}{dx} + V(x) + \epsilon(x)^m W(\nu x, \omega t), \ \epsilon \in \mathbb{R},$$

where $V(x) \sim |x|^{2\ell}$ at infinity with $\ell > 1$ and $W$ is a smooth function on $\mathbb{T}^d \times \mathbb{T}^n$. In order to state the results we need to introduce some notations and spaces. We define the weight $\lambda(x, \xi) = (1 + |x|^2 + |\xi|^{2\ell})^\frac{1}{2\ell}$. For $x, y \in \mathbb{R}$, define $\rho := \sqrt{1 + x^2}$ and $x \vee y := \max\{x, y\}$ and $x \sim y$ means that there exist some positive constants $C, \tilde{C}$ such that $\tilde{C}y \leq x \leq Cy$. As [2] we define the following.

Symbol. The space $S^{m_1, m_2}$ is the space of the symbols $g \in C^\infty(\mathbb{R}^2)$ such that $\forall k_1, k_2 \geq 0$, there exists $C_{k_1, k_2}$ with the property that

$$|\partial_{k_1}^\xi \partial_{k_2}^\xi g(x, \xi)| \leq C_{k_1, k_2}(\lambda(x, \xi))^{m_1-k_1\ell}(x)^{m_2-k_2}.$$  (1.2)

The best constants $C_{k_1, k_2}$ such that (1.2) holds form a family of semi-norms for that space $S^{m_1, m_2}$.

Quantization. To a symbol $g \in S^{m_1, m_2}$, we associate its Weyl quantization, namely the operator $g^w(x, -i\partial_x)$, defined by

$$g^w(x, -i\partial_x)\psi(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{(x-y)\cdot \xi} g\left(\frac{x+y}{2}, \xi\right) \psi(y) dy d\xi.$$  

We use the symbol $\lambda(x, \xi)$ to define, for $s \geq 0$ the spaces $\mathcal{H}^s = D([\lambda^{w}(x, -i\partial_x)]^{s(\ell+1)})$ (domain of the $(s(\ell + 1))$th-power of the operator operator $\lambda^{w}(x, -i\partial_x)$, endowed by the graph norm. For negative $s$, the space $\mathcal{H}^s$ is the dual of $\mathcal{H}^{-s}$. We will denote by $B(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, where $\mathcal{H}_1, \mathcal{H}_2$ are Banach spaces. In particular, $B(\mathcal{H}_1, \mathcal{H}_1)$ is usually abbreviated as $B(\mathcal{H}_1)$. As [3] in what follows we will identify $L^2$ with $\ell_2$ by introducing the basis denoted by $\{h_j(x)\}_{j \geq 1}$ of the eigenvector of $\mathcal{H}_0 := -\partial_{xx} + V(x)$. Similarly we will identify $\mathcal{H}^s$ with the space $\ell_2^s$ of the sequences $\psi_j$ such that $\sum_{j \geq 1} |\mathcal{H}^s| \psi_j^2 < \infty$.

Now we can state our main results below. Consider the time dependent Schrödinger equation
Remark 1.1 We denote by $s$ the sequence of the eigenvalues of $H_0$ labeled in increasing order.

As [2, 19] one can show that $\lambda_j \sim c_j \frac{2\ell}{j}$ as $j \to \infty$ and $\lambda_1 > 0$.

Our purpose is to prove the following.

Theorem 1.2 Assume A1-A3 and $\mu < \ell - 1 + \frac{1}{2\ell+1}$. Fix a $\gamma > 0$ small, there exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ there exists a closed set $\Pi \subset \Pi := [0, 1]^n$ and $\forall \omega \in \Pi$, the linear Schrödinger equation (1.1) reduces to a linear equation with constant coefficients in $L^2$.

More precisely, for a $\gamma > 0$, there exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ there exists a closed set $\Pi \subset \Pi$ satisfying $\text{meas}(\Pi \setminus \Pi) \leq C\gamma$, and for $\omega \in \Pi$, there exists a unitary (in $L^2$) time quasiperiodic operator $\Psi_{\omega, \epsilon}(\phi)$ such that $t \mapsto \psi(t, \cdot) \in L^2$ satisfies (1.1) if and only if $t \mapsto u(t, \cdot) = \Psi_{\omega, \epsilon}^{-1}\psi(t, \cdot)$ satisfies the equation
\[
i u = \mathcal{H}_\infty u
\]
with $\mathcal{H}_\infty = \text{diag}(\lambda_j^\infty)$ and $|\lambda_j^\infty - \lambda_j| \leq C\epsilon \frac{2\ell}{j^{\frac{1}{\ell+1}}(2\ell+1)\gamma^0}$.

Furthermore one has:
1. $\lim_{\gamma \to 0} \text{meas}(\Pi \setminus \Pi) = 0$;
2. $\Psi_{\omega, \epsilon}(\phi)$ is analytic in the norm $\| \cdot \|_{B(L^2)}$ on $|\text{Im}\phi| < \frac{\bar{\gamma}}{2}$;
3. $\|\Psi_{\omega, \epsilon}(\omega t) - \text{Id}\|_{B(L^2)} \leq C\epsilon^{\frac{2\ell}{j}}$

Remark 1.3 $s$ satisfies $0 < s < \rho$.

Remark 1.4 In [1] Bambusi and Graffi first proved the reducibility of 1d Schrödinger equation with an unbounded time quasiperiodic perturbation. They assumed a similar potential as (1.1) and the perturbation operator is $\epsilon W(x, \omega t)$ with $|W(x, \omega)| \sim |x|^\beta$ as $|x| \to \infty$, where $\beta < \ell - 1$. The reducibility in the limiting case $\beta = \ell - 1$ was obtained by Liu and Yuan in [24]. Comparing with [1] and [24], we improve the boundedness for $\beta$ from $\ell - 1$ to $\ell - 1 + \frac{1}{2\ell+1}$ for $\ell > 1$ when the perturbation terms have the oscillatory terms as (1.1).

Remark 1.5 In [2] Bambusi studied the reducibility of 1d Schrödinger equations
\[
\mathcal{H}_0(t) \psi(x, t) = i\partial_t \psi(x, t), \quad x \in \mathbb{R};
\]
\[
\mathcal{H}_0(t) := -\frac{d^2}{dx^2} + V(x) + \epsilon W(\omega t), \quad \epsilon \in \mathbb{R},
\]
where $V(x) \sim |x|^{2\ell}$ at infinity and satisfies a similar condition as A1. The perturbation operator $W(\omega t)$ belongs to a class of unbounded symbols, in which the oscillatory perturbation terms $\epsilon(x)^\mu W(x, \omega t)$ in (1.1) are clearly excluded by [2], [3] and [4]. See Remark 2.7 in [2] for further explanations.
Remark 1.6 See [21] for 1d quantum harmonic oscillators with similar perturbation terms as (1.1). We remark that the reducibility results in [17], [21] and [22] were proved only in $\mathcal{H}^1$. The reason partly lies in (iii) of Lemma 2.1 in [17]. In our notations, when the perturbation operator $P$ belongs to $\mathcal{M}_\beta$, one can only deduce that $P \in \mathcal{B}(\ell^2_1; \ell^2_1)$ for $t > 2\beta + 1$ from Lemma 2.2, where $0 \leq 2\beta < \ell - 1$.

Similarly, we consider the Schrödinger equation
$$\mathcal{H}_1(t)\psi(x,t) = i\partial_t\psi(x,t), \quad x \in \mathbb{R};$$
$$\mathcal{H}_1(t) := -\frac{d^2}{dx^2} + V(x) + \epsilon(x)^\mu X(x,\omega t),$$
where $X(x, \phi) = \sum_{k \in \Lambda} a_k(\phi) \sin kx + b_k(\phi) \cos kx$ with $k \in \Lambda \subset \mathbb{R}\setminus\{0\}$ with $|\Lambda| < \infty$, $a_k(\phi)$ and $b_k(\phi)$ are analytic on $T^n_\rho$ and continuous on $\overline{T^n_\rho}$.

Theorem 1.7 Assume A1 and $\mu < \ell - 1 + \frac{1}{2\beta + 1}$. For a $\gamma > 0$ small, there exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ there exists a closed set $\Pi_* \subset \Pi := [0, 1]^n$ and $\forall \omega \in \Pi_*$, the linear Schrödinger equation (1.3) reduces to a linear equation with constant coefficients in $L^2$.

We consider the Schrödinger equation
$$\mathcal{H}_2(t)\psi(x,t) = i\partial_t\psi(x,t), \quad x \in \mathbb{R};$$
$$\mathcal{H}_2(t) := -\frac{d^2}{dx^2} + V(x) + \epsilon(x)^\mu g(x,\omega t),$$
where $g(x, \phi)$ is continuous on $x \in \mathbb{R}$ and analytic on $T^n_\rho$ and there exists a positive constant $C$ such that for any $(x, \phi) \in \mathbb{R} \times T^n_\rho$, $|g(x, \phi)| \leq C$.

Corollary 1.8 Assume A1 and $\mu < \ell - 1$. For a $\gamma > 0$ small, there exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ there exists a closed set $\Pi_* \subset \Pi := [0, 1]^n$ and $\forall \omega \in \Pi_*$, the Schrödinger equation (1.4) reduces to a linear equation with constant coefficients in $L^2$.

Remark 1.9 The set $\Pi_*$ in Theorem 1.7 and Corollary 1.8 is similar as that in Theorem 1.2.

Remark 1.10 This result was first proved by Bambusi & Graffi [4] except that they assumed $V(x, \phi) \sim |x|^{\mu}$ with $\mu < \ell - 1$ (page 477, line 6). Here we only assume that $V(x, \phi)$ has the form $(x)\psi g(x, \phi)$ with a bounded $g(x, \phi)$, which can include the oscillatory terms such as $(x)\psi \sin x \cdot f(\phi)$ and etc.

A consequence of the above theorems and corollary is that in the considered range of parameters all the Sobolev norms, i.e., the $\mathcal{H}^s$ norms of the solutions are bounded forever and the spectrum of the Floquet operator is pure point.

In the end we recall some relevant results. See [18], [34] and [35] for the reducibility results for 1d harmonic oscillators with bounded perturbations. We remark that the pseudodifferential calculus is used for checking the assumption B3 in this paper. More applications of pseudodifferential calculus can be found in the following papers (e.g. [7, 8, 9, 13, 15, 25, 30]). We mention that some higher dimensional results have been recently obtained [6, 14, 17, 22, 28].

As we mentioned before, the reducibility implies the boundedness of the solutions in some Sobolev norms for all the time. There are many literatures relative with the upper boundedness of the solution in some Sobolev space (e.g. [5], [10], [27], [29]). There are not too much papers to study the lower boundedness of the PDEs. See the interesting examples given by Bourgain for a Klein-Gordon and Schrödinger equation on $T(10)$, by Delort for the harmonic oscillator on $\mathbb{R}$ ([11]). Combining the ideas in [6] and [12], Z. Zhao, Q. Zhou and the first author [23] build some lower boundedness estimates for 1d harmonic oscillators with quadratic time-dependent perturbations. We remark that the result in [11] was reproved in [26] by exploiting the idea in [16].
1.2. A new oscillatory integral estimation. The following oscillatory integral estimations are critical for us to establish Theorem 1.2 and 1.7.

**Assumption 1.1** The potential $V(x)$ is real-valued and of $C^3$-class. There exists a positive constant $R_0$ such that the following conditions are satisfied for $V(x)$ when $|x| \geq R_0$:

(i). $V''(x) \geq 0$. \hspace{1cm} (1.5)

(ii). For $j = 1, 2, 3$, $|xV^{(j)}(x)| \leq C_1|V^{(j-1)}(x)|$, where $C_1 \geq 1$. \hspace{1cm} (1.6)

(iii). For $\ell > 1$, $D_1|x|^{2\ell} \leq V(x) \leq D_2|x|^{2\ell}$, where $0 < D_1 \leq 1 \leq D_2 < \infty$.

**Assumption 1.2** The function $f(x)$ is real-valued and of $C^1$-class. There exist positive constant $R_0$ and $C_2 > 0$ such that for $|x| \geq R_0$, $|f(x)| \leq C_2|x|^\mu$ and $|f'(x)| \leq C_2|x|^\mu^{-1}$ where $0 \leq \mu < \ell - 1 + \frac{1}{m+1}$.

**Lemma 1.11** Assume $V(x)$ satisfies Assumption 1.1. Let $h_n(x)$ be the normalized eigenfunction of $H = -\partial_{xx} + V(x)$ with the eigenvalue $\lambda_n$ and $f(x)$ satisfies Assumption 1.2, then for any $k \neq 0$, one has

$$\left| \int_{\mathbb{R}} f(x)e^{ikx}h_m(x)\overline{h_n(x)}dx \right| \leq C(|k| \vee |k|^{-1})(\lambda_m\lambda_n)^{\frac{m}{m+1}},$$

(1.7)

where $C$ depends on $(\mu, \ell)$ and $0 \leq \mu < \ell - 1 + \frac{1}{m+1}$.

**Remark 1.12** See [36] for $L^p$ estimate of $h_n(x)$ based on Langer’s turning point method when $n$ is large enough. For a complete introduction of Langer’s turning point method refer to the contents in Chapter 22.27 of [33]. Refer to [35] for a weighted $L^2$ estimate of the eigenfunctions of $-\partial_{xx}^2 + x^2$ on $\mathbb{R}$, which is another application of this method.

**Remark 1.13** In [21], Luo and the first author proved the following: for any $k \neq 0$ and for any $m, n \geq 1$,

$$\left| \int_{\mathbb{R}} \langle x \rangle^{\mu}e^{ikx}f_m(x)\overline{f_n(x)}dx \right| \leq C(|k| \vee |k|^{-1})m^{-\frac{\mu}{2}}n^{\frac{\mu}{2}+\frac{1}{2}},$$

(1.8)

where $C$ is an absolute constant and $0 \leq \mu < \frac{1}{2}$ and $(-\partial_{xx}^2 + x^2)f_j = (2j - 1)f_j$ with $\|f_j\|_{L^2(\mathbb{R})} = 1$ with $j \geq 1$.

**Lemma 1.14** Assume $V(x)$ satisfies Assumption 1.1 and $h_n(x)$ is the same as in Lemma 1.11. If $f(x)$ is continuous and satisfies $|f(x)| \leq C_2|x|^\mu$ for $|x| \geq R_0$ with $0 \leq \mu < \ell - 1$, then

$$\left| \int_{\mathbb{R}} f(x)h_m(x)\overline{h_n(x)}dx \right| \leq C(\lambda_m\lambda_n)^{\frac{m}{m+1}}.$$ 

(1.9)

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2. A Reducibility Theorem

Before give the proof of main theorem we present a reducibility theorem in a more abstract setting.
2.1. Setting. Following [17], we will introduce some spaces and norms and discuss some algebraic properties.

**Linear Space.** Let \( s \in \mathbb{R} \), we define the complex weighted-\( \ell^2 \)-space
\[
\ell^2_s = \{ \xi = (\xi_j) \in C, j \in \mathbb{Z}_+ \mid \| \xi \|_s < \infty \}, \quad \text{where } \| \xi \|_s^2 = \sum_{j \in \mathbb{Z}_+} j^s |\xi_j|^2.
\]

**Infinite Matrices.** We denote by \( \mathcal{M}_\beta \) the set of infinite matrix \( A : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C} \) that satisfy
\[
|A|_\beta := \sup_{i,j \geq 1} |A|^{(ij)}_{\beta} \leq \infty. \quad \text{We will also need the space } \mathcal{M}_\beta^+ \text{ the following subspace of } \mathcal{M}_\beta:\n
\text{an infinite matrix } A \text{ is in } \mathcal{M}_\beta^+ \text{ if } |A|_\beta^+ := \sup_{i,j \geq 1} A^{(ij)}_{\beta} \leq (1+i+j)(i^{-1}+j^{-1}) < \infty, \text{ where } 0 \leq 2\beta < \iota - 1 \text{ and } \iota > 1.
\]

**Lemma 2.1** For \( \iota > 1 \), \(|k^\iota - j^\iota| \geq \frac{1}{2}|k-j|(k^{-1} + j^{-1}) \) (see [20]).

**Lemma 2.2** If \( 0 \leq 2\beta < \iota - 1 \), there exists a constant \( C > 0 \) such that
(i). Let \( A \in \mathcal{M}_\beta \) and \( B \in \mathcal{M}_\beta^+ \), then \( AB \) and \( BA \) belong to \( \mathcal{M}_\beta \) and \(|AB|_\beta, |BA|_\beta \leq C|A|_\beta|B|_\beta^+ \).

The proof is by the definition.

(ii). Let \( A, B \in \mathcal{M}_\beta^+ \), then \( AB \) belongs to \( \mathcal{M}_\beta^+ \) and \(|AB|_\beta \leq C|A|_\beta^+|B|_\beta^+ \).

**Proof.** Since \( A, B \in \mathcal{M}_\beta^+ \), then
\[
|AB|^{(ij)}_\beta = |A|^{(ij)}_\beta|B|^{(ij)}_\beta \leq C|A|^{(ij)}_\beta|B|^{(ij)}_\beta,
\]

where \( C \) is a constant.

(iii). Let \( A \in \mathcal{M}_\beta \), then for any \( t > 2\beta + 1 \), \( A \in \mathcal{B}(\ell^2_\iota; \ell^2_{-\iota}) \) and \( \|A\xi\|_{-\iota} \leq C|A|_\beta\|\xi\|_{\iota}. \)

**Proof.** Let \( A\xi = \eta \), \( \eta_k = \sum_{i \geq k} A^{(i)}_k \xi_i \), then
\[
\|\eta\|_{-\iota}^2 = \sum_{i \geq 1} i^{-\iota} |\eta_i|^2 = \sum_{i \geq 1} i^{-\iota} \sum_{k \geq 1} A^{(i)}_k \xi_k |^2 \leq |A|^{(i)}_\beta \sum_{i \geq 1} \frac{1}{i^{-2\beta}} \sum_{k \geq 1} k^{2\beta-i} (\sum_{k \geq 1} k^{i} \xi_k |^2).
\]

Note \( t > 1 + 2\beta \), then we obtain \( \|A\xi\|_{-\iota} \leq C|A|_\beta\|\xi\|_{\iota}. \)

(iv). Let \( A \in \mathcal{M}_\beta^+ \), then for any \( s \in [0, 2\iota - 2\beta - 1] \), \( A \in \mathcal{B}(\ell^2_\iota) \) and satisfies
\[
\|A\|_{\mathcal{B}(\ell^2_\iota)} \leq C(\beta, \iota, s)|A|_{\beta}^+,
\]

where \( 0 \leq 2\beta < \iota - 1 \).

(v). Let \( A \in \mathcal{M}_\beta^+ \), then for any \( s \in [0, 2\iota - 2\beta - 1] \), \( A \in \mathcal{B}(\ell^2_{-\iota}) \) and satisfies
\[
\|A\|_{\mathcal{B}(\ell^2_{-\iota})} \leq C(\beta, \iota, s)|A|_{\beta}^+,
\]

where \( 0 \leq 2\beta < \iota - 1 \).

The proof for (iv) and (v) is a little long and we will delay it in section 4.

**Lemma 2.3** \((\mathcal{M}_\beta, | \cdot |_\beta) \) and \((\mathcal{M}_\beta^+, | \cdot |_{\beta}^+) \) are Banach spaces.
Parameter. In the paper $\omega$ will play the role of a parameter belonging to $\Pi = [0, 1]^n$. All the constructed operators or matrices will depend on $\omega$ in Lipschitz sense which will be clear in the following.

Let $D \subset \Pi$ and $\sigma > 0$. We denote by $\mathcal{M}_\beta(D, \sigma)$ the set of mappings at $T_\sigma^n \times D \ni (\phi, \omega) \mapsto Q(\phi, \omega) \in \mathcal{M}_\beta$ which is real analytic on $\phi \in T_\sigma^n = \{ \phi \in \mathbb{C}^n \mid \| \text{Im} \phi \| < \sigma \}$ and Lipschitz continuous on $\omega \in D$. This space is equipped with the norm $\|Q\|_{D, \sigma} = \|Q\|_{D, \sigma} + \|Q\|_{D, \sigma}$, where we define $\|Q(\cdot, \omega)\|_{D, \sigma} = \sup_{\omega \in D} \|Q(\phi, \omega)\|_{\beta, \sigma}$ and

$$\|Q\|_{D, \sigma} = \sup_{\omega \in D} \frac{\|Q(\phi, \omega) - Q(\phi, \omega')\|_{\beta, \sigma}}{|\omega - \omega'|}.$$ 

Similarly, we can define the subspace of $\mathcal{M}_\beta(D, \sigma)$, named by $\mathcal{M}^+_\beta(D, \sigma)$, the set of mappings at $T_\sigma^n \times D \ni (\phi, \omega) \mapsto R(\phi, \omega) \in Q(\phi, \omega)$ which is real analytic on $\phi \in T_\sigma^n$ and Lipschitz continuous on $\omega \in D$. This space is equipped with the norm $\|R\|_{D, \sigma} = \|R\|_{D, \sigma} + \|R\|_{D, \sigma}$, where we define $\|R\|_{D, \sigma}$ and $\|R\|_{D, \sigma}$ similarly. Generally, for a Banach space $(\mathcal{B}, \| \cdot \|)$, we define the parameterized space $\mathcal{B}(D, \sigma)$, the set of mappings at $T_\sigma^n \times D \ni (\phi, \omega) \mapsto F(\phi, \omega) \in \mathcal{B}$ which is real analytic on $\phi \in T_\sigma^n$ and Lipschitz continuous on $\omega \in D$. This space is equipped with the norm $\|F\|_{D, \sigma} = \|F(\cdot, \omega)\|_{D, \sigma} + \|F(\cdot, \omega)\|_{D, \sigma}$, where we define $\|F(\cdot, \omega)\|_{D, \sigma} = \sup_{|\text{Im} \phi| < \sigma} \|F(\phi, \omega)\|_{\beta, \sigma}$

$$\|F\|_{D, \sigma} = \sup_{\omega \in D} \|F(\phi, \omega)\|_{\beta, \sigma}$$

and

$$\|F\|_{D, \sigma} = \sup_{\omega \in D} \frac{\|F(\phi, \omega) - F(\phi, \omega')\|_{\beta, \sigma}}{|\omega - \omega'|}.$$ 

In addition, for convenience we abbreviate $\mathcal{B}(D, 0)$ as $\mathcal{B}(D)$.

**Lemma 2.4** If $0 \leq 2\beta < \iota - 1$ and $\iota > 1$, there exists a constant $C > 0$ such that

(i). If $A \in \mathcal{M}_\beta(D, \sigma)$ and $B \in \mathcal{M}^+_\beta(D, \sigma)$, then $AB$ and $BA$ belong to $\mathcal{M}_\beta(D, \sigma)$ and

$$\|AB\|_{D, \sigma} \leq C \|A\|_{D, \sigma} \|B\|_{D, \sigma}.$$ 

Moreover, if $A, B \in \mathcal{M}^+_\beta(D, \sigma)$, then $AB$ belongs to $\mathcal{M}^+_\beta(D, \sigma)$ and

$$\|AB\|_{D, \sigma} \leq C \|A\|_{D, \sigma} \|B\|_{D, \sigma}.$$ 

(ii). If $B \in \mathcal{M}^+_\beta(D, \sigma)$ then $e^B - I$ and $e^B - I - B$ belong to $\mathcal{M}^+_\beta(D, \sigma)$ and

$$\|e^B - I\|_{D, \sigma} \leq C \|B\|_{D, \sigma} + \|B\|_{D, \sigma}.$$ 

Moreover, if $A \in \mathcal{M}_\beta(D, \sigma)$, then $Ae^B$, $e^B A \in \mathcal{M}_\beta(D, \sigma)$ and

$$\|Ae^B\|_{D, \sigma} \leq C \|A\|_{D, \sigma} \|B\|_{D, \sigma} e^{C \|B\|_{D, \sigma}}.$$ 

**Proof.** If $A \in \mathcal{M}_\beta(D, \sigma)$, $B \in \mathcal{M}^+_\beta D, \sigma)$, note $e^B - I = \sum_{n=1}^{\infty} \frac{B^n}{n!}$, thus for any $\omega \in D$ and $\phi \in T_\sigma^n$,

$$|e^B - I|_{\beta, \sigma} \leq \sum_{n=1}^{\infty} \frac{|B^n|_{\beta, \sigma}}{n!} \leq \frac{1}{C} (e^{C|B|_{\beta, \sigma}} - 1) \leq \|B\|_{D, \sigma} e^{C \|B\|_{D, \sigma}}.$$ 

It follows

$$\|e^B - I\|_{D, \sigma} \leq \|B\|_{D, \sigma} e^{C \|B\|_{D, \sigma}}.$$ 

(2.3)
In the following we estimate \( \|e^B - I\|_{\beta,\sigma}^{D,\text{lip}^+} \). In fact, \( e^B(\omega) - e^B(\omega') = \sum_{n=1}^{\infty} \frac{(B(\omega))^n}{n!} - \frac{(B(\omega'))^n}{n!} \). By induction for \( n \geq 1 \), we have \( \| (B(\omega))^n - (B(\omega'))^n \|_{\beta,\sigma}^+ \leq n \| C \| B \|^{D,\text{lip}^+}_{\beta,\sigma} \| B(\omega) - B(\omega') \|_{\beta,\sigma}^+ \). It follows for \( \omega \neq \omega' \), \( B, \omega' \in D \),

\[
\| e^B(\omega) - e^B(\omega') \|_{\beta,\sigma}^+ \leq \sum_{n=1}^{\infty} n \| C \| B \|^{D,\text{lip}^+}_{\beta,\sigma} \| B(\omega) - B(\omega') \|_{\beta,\sigma}^+.
\]

Thus, \( \frac{\| e^B(\omega) - e^B(\omega') \|_{\beta,\sigma}^+}{\| \omega - \omega' \|} \leq \| B \|_{\beta,\sigma}^{D,\text{lip}^+} e \| B \|^{D,\text{lip}^+}_{\beta,\sigma} \). Together with (2.3) we have \( \| e^B - I \|_{\beta,\sigma}^{D,\text{lip}^+} \leq \| B \|_{\beta,\sigma}^{D,\text{lip}^+} e \| B \|^{D,\text{lip}^+}_{\beta,\sigma} \). Similarly, we obtain (2.1). The proof of (2.2) is similar.

(iii) If \( B \in M_{\beta}^+(D, \sigma) \) and \( P \in M_{\beta}(D, \sigma) \), then \( e^{-B} P e^B \) belongs to \( M_{\beta}(D, \sigma) \) and

\[
\| e^{-B} P e^B - P \|_{\beta,\sigma}^{D,\text{lip}^+} \leq C e \| B \|^{D,\text{lip}^+}_{\beta,\sigma} \| B \|_{\beta,\sigma}^{D,\text{lip}^+},
\]

\[
\| e^{-B} P e^B - P \|_{\beta,\sigma}^{D,\text{lip}} \leq C e \| B \|^{D,\text{lip}}_{\beta,\sigma} \| B \|_{\beta,\sigma}^{D,\text{lip}}.
\]

Proof. From (2.3), if \( B \in M_{\beta}^+(D, \sigma) \), then \( \| e^B - I \|_{\beta,\sigma}^{D,\text{lip}^+} \leq \| B \|_{\beta,\sigma}^{D,\text{lip}^+} e \| B \|^{D,\text{lip}^+}_{\beta,\sigma} \). Similarly, \( \| e^{-B} - I \|_{\beta,\sigma}^{D,\text{lip}^+} \leq \| B \|_{\beta,\sigma}^{D,\text{lip}^+} e \| B \|^{D,\text{lip}^+}_{\beta,\sigma} \). Since \( e^{-B} P e^B - P = (e^{-B} - I) P (e^B - I) + (e^-B - 1) P + P (e^B - 1) \), the proof is clear by a straightforward computation.

2.2. A Reducibility Theorem. Before present the new reducibility theorem we give a rough introduction of the proof and what is new here. In fact the equation (1.1) can be written as

\[
i \dot{x} = (A + \epsilon P(\phi))x,
\]

where \( A = \text{diag}\{\lambda_j\}_{j \geq 1} \) and \( \lambda_j \sim j^i (j \to \infty) \) and \( P = (P_j(\phi)) \) with \( \ell = \frac{2\delta + 1}{\ell + 1} \) and

\[
P_j(\phi) = P_j^i(\phi) = \int_{\mathbb{R}} (x)^j W(\nu x, \phi) h_1(x) h_2(x) dx.
\]

From Lemma 2.46 we can show that the map \( T^\phi \ni \phi \mapsto P(\phi) \in B(\ell^2_0, \ell^2_{2\delta}) \) is analytic on \( T^\phi \) if \( 0 \leq \mu \leq \delta(\ell + 1) \). If \( \mu < \ell - 1 \), then one can choose \( \delta = \frac{\mu}{\ell + 2} \) and furthermore, \( P(\phi) \in B^\delta \) with \( \delta < \frac{\ell + 2}{\ell + 1} \) (we use the notation from [1]). From the reducibility Thm. in [1] and Lemma 2.46 we can prove the reducibility result for the equation (1.1) when \( \mu < \ell - 1 \).

The main improvement is that we can deal with the case when \( \ell - 1 \leq \mu < \ell - 1 + \frac{1}{2\ell + 1} \) for the equation (1.1) from Theorem 2.5. In the new reducibility theorem we assume that \( 0 \leq \delta < \frac{2\delta + 1}{\ell + 1} - 2 \beta - 1, 0 \leq 2\beta < \ell - 1, 2\beta \leq \delta \) and \( 0 \leq \mu \leq \delta(\ell + 1) \). In fact, when \( \mu < \ell - 1 + \frac{1}{2\ell + 1} \), from Lemma 1.1 one can show that \( P(\phi) \in M_{\beta} \) with \( \beta = \frac{\mu}{\ell + 1} - \frac{1}{2\ell + 1} \). If we choose \( \delta = \frac{\ell + 1}{2\ell + 1} \), it follows 0 \leq 2\beta \leq \frac{\ell + 1}{2\ell + 1} and all the assumptions in Theorem 2.5 are easily checked, from which we can prove Theorem 1.2. Comparing with the proof in [1], we need to control \( \| P^- \|_{\beta,s}^L + \| P^- \|_{B(\ell^2_0, \ell^2_{2\delta})}^L \) in every step, while in [1] only \( \| P^- \|_{B(\ell^2_0, \ell^2_{2\delta})}^L \) was controlled.

In the end we explain a little bit about the equivalence of the two equations \( i \dot{x} = (A + \epsilon P)x \) and \( i \dot{y} = A^\infty(\omega t)y \), which comes from the reducibility equality

\[
U(\omega t)A(t) = -i \frac{d}{dt} U(\omega t) + (A^0 + P^0(\omega t))U(\omega t) \quad \text{in} \quad B(\ell^2_0, \ell^2_{2\delta})(\Pi_s).
\]

We remark that from \( U((\omega t) - I) \in M_{\beta}^+ \) and Lemma 2.2 we have \( U((\omega t) - I) \in B(\ell^2_s) \) with \( s \in [0, 2\ell - 2\beta - 1) \). But it usually makes no sense for \( U(\omega t)A(\omega t)x_0 \in \ell^2_{2\delta} \) when \( x_0 \in \ell^2_0 \). The proof seriously depends on the homological equation (2.13). See Lemma 2.31, 2.33 and 2.34.
for details. Now we present the new reducibility theorem.

Consider the non-autonomous, linear differential equation in a separable Hilbert space $\ell^2_0$

$$i\dot{x}(t) = (A + \epsilon P(\omega_1 t, \omega_2 t, \ldots, \omega_n t))x(t), \quad \epsilon \in \mathbb{R}, \quad (2.4)$$

under the following conditions:

B1: $A = \text{diag}(\lambda_j)_{j \geq 1}$ with $0 < \lambda_1 < \lambda_2 < \cdots$. There exists a $\ell > 1$ such that $\lambda_j \sim cj^\ell$ as $j \to \infty$.

B2: The map $\mathbb{T}^n \ni \phi \mapsto P(\phi) := (P^i_j(\phi))_{i,j \geq 1} \in \mathcal{M}_\beta$ is analytic on $\mathbb{T}_s$ and $P^i_j(\phi) = P^i_j(\phi)$ for $\phi \in \mathbb{T}^n$, where $0 \leq 2\beta < \ell - 1$.

B3: The map $\mathbb{T}^n \ni \phi \mapsto P(\phi) \in B(\ell^2_0, \ell^2_{-2\delta})$ is analytic on $\mathbb{T}_s$, where $0 \leq \delta < \ell - \frac{1}{2}$ and $2\beta \leq \delta$.

**Theorem 2.5** Assume that B1 – B3 are satisfied. Then for a $\gamma > 0$ small, there exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ there exists $\Pi_* \subset \Pi := [0, 1]^n$ satisfying $\text{meas}(\Pi \setminus \Pi_*) \leq C\gamma$, such that for all $\omega \in \Pi_*$, the equation (2.4) reduces to a linear equation

$$i\dot{y}(t) = A^\infty(\omega)t)y(t), \quad A^\infty(\omega) := \text{diag}\{\lambda_1^\infty, \lambda_2^\infty, \cdots\},$$

where $\{\lambda_j^\infty\}_{j \geq 1} \in \mathbb{R}$ and the function $\mu^\infty(\phi) : \mathbb{T}^n \to \mathbb{R}$ is analytic on $\mathbb{T}_s$ with zero average.

More precisely, for $\gamma > 0$ small, there exists $\epsilon_*$ such that for all $0 \leq \epsilon < \epsilon_*$ there exists $\Pi_* \subset \Pi := [0, 1]^n$ satisfying $\text{meas}(\Pi \setminus \Pi_*) \leq C\gamma$, and for $\omega \in \Pi_*$, there exists a linear unitary transformation $U^\infty(\phi)$ in $\ell^2_0$ which analytically depends on $\phi \in \mathbb{T}_s$ such that $t \mapsto y(t) \in C^0(\mathbb{R}, \ell^2_0) \cap C^1(\mathbb{R}, \ell^2_{-2\delta})$ satisfies the equation (2.5) if and only if

$$t \mapsto x(t) = U^\infty(\omega)t)y(t) \in C^0(\mathbb{R}, \ell^2_0) \cap C^1(\mathbb{R}, \ell^2_{-2\delta})$$

satisfies the equation (2.4), where there exists a positive constant $C$ such that

$$\|U^\infty(\omega)t - I\|_{B(\ell^2_0)} \leq C\epsilon^\frac{2}{\ell}, \quad |\lambda_j^\infty - \lambda_j| \leq Cj^{2\beta}\epsilon, \quad |\mu_j^\infty(\omega)| \leq Cj^{2\beta}\epsilon. \quad (2.7)$$

**Remark 2.6** The assumption $2\beta \leq \delta$ is not necessary but it can simplify the proof.

**Corollary 2.7** Assume that B1 – B3 are satisfied. Then for a $\gamma > 0$ small, there exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ there exists $\Pi_* \subset \Pi := [0, 1]^n$ satisfying $\text{meas}(\Pi \setminus \Pi_*) \leq C\gamma$, such that for all $\omega \in \Pi_*$, there is a unitary transformation $U_F(\omega)$ in $\ell^2_0$, quasiperiodic with frequency $\omega$ and such that $\|U_F(\omega)t - I\|_{B(\ell^2_0)} \leq C\epsilon^\frac{2}{\ell}$, which transforms (2.4) into the equation

$$i\dot{z}(t) = A_Fz(t), \quad A_F = \text{diag}\{\lambda_1^\infty, \lambda_2^\infty, \cdots\}.$$

Moreover, if $x(0) \in \ell^2_0$, then $x(t) = U_F(\omega)t\text{diag}(e^{-i\lambda_j^\infty t})U_F^{-1}(0)x(0)$ is the solution of (2.4) in the sense of (2.6).

### 2.3. Squaring the order of the Perturbation

Let $\mathbb{T}^n_s$ be the complexified torus with $|\text{Im}\phi| < s$ and $\Pi^-$ be a closed nonempty subset of $\Pi$ of positive measure. If the map $f : \mathbb{T}^n_s \times \Pi^- \to B(\ell^2_{s_1}, \ell^2_{s_2})$ is analytic on $\phi \in \mathbb{T}^n_s$ and Lipschitz continuous on $\omega \in \Pi^-$, we define $\|f\|_{B(\ell^2_{s_1}, \ell^2_{s_2})} = \|f\|_{B(\ell^2_{s_1}, \ell^2_{s_2})} + \|f\|_{B(\ell^2_{s_1}, \ell^2_{s_2})}$. Similarly, if the map $f : \mathbb{T}^n_s \times \Pi^- \to \mathcal{M}_\beta$ is analytic on $\phi \in \mathbb{T}^n_s$ and Lipschitz continuous on $\omega \in \Pi^-$, we define $\|f\|_{\mathcal{M}_\beta} = \|f\|_{\mathcal{M}_\beta} + \|f\|_{\mathcal{M}_\beta}$. For convenience we omit the symbol $\Pi^-$ here.

Now we consider the equation in $\ell^2_0$

$$i\dot{x}(t) = (A^- + P^-(\omega))x$$

under the following conditions

**H1)**

$$A^- = \text{diag}\{\lambda_{1^-}(\omega) + \mu_{1^-}(\omega, \omega), \lambda_{2^-}(\omega) + \mu_{2^-}(\omega, \omega), \cdots\}. \quad (2.9)$$

Here:

H1.a) $\forall i = 1, \cdots, \lambda_{i^-}(\omega)$ is positive and Lipschitz continuous w.r.t $\omega \in \Pi^-$ and satisfies $C_0^- i^t \leq \lambda_i^\infty \leq C_1^- i^t$ with $C_0^- > 0$. We also assume that there is $C^-_{\lambda^\infty} > 0$ independent of $\omega$ such that
\[ |\lambda^-_i - \lambda^-_j| \geq C^-_i |i' - j'|. \]

H1.b) There is \( C^-_i > 0 \) suitably small and \( 0 \leq 2\beta < \iota - 1 \) such that
\[
\sup_{\omega, \omega' \in \Pi^-} \frac{|\lambda^-_i (\phi, \omega) - \lambda^-_j (\phi, \omega')|}{|\omega - \omega'|} \leq C^-_i 2^{\beta}. 
\]

H1.c) \( \forall i, \cdots, |\mu^-_i (\phi, \omega) : \mathbb{T}_n^\omega \times \Pi^- \rightarrow \mathbb{R} \) is analytic w.r.t \( \phi \), Lipschitz continuous w.r.t \( \omega \), and has zero average, i.e. \( \int_{\mathbb{T}_n^\omega} \mu^-_i (\phi, \omega) d\phi = 0 \). Moreover it fulfills the estimates
\[
\|\mu^-_i\| \leq C^-_i 2^{\beta}, \quad \sup_{\omega, \omega' \in \Pi^-} \frac{\|\mu^-_i (\phi, \omega) - \mu^-_i (\phi, \omega')\|}{|\omega - \omega'|} \leq C^-_i 2^{\beta},
\]
where we denote \( \|\mu^-_i (\cdot, \omega)\| = \sup_{\|\mu\| \leq s} |\mu^-_i (\phi, \omega)| \).

H2) The map \( P^- : \mathbb{T}_n^\omega \times \Pi^- \rightarrow \mathbb{X} \) is analytic w.r.t \( \phi \in \mathbb{T}_n^\omega \) in the norm of \( |\cdot|_\beta \) or \( \|\cdot\|_{B(\ell^2_0, \ell^2_{-2\beta})} \) and Lipschitz continuous w.r.t \( \omega \in \Pi^- \) uniformly in \( \phi \in \mathbb{T}_n^\omega \), where \( \mathbb{X} = \mathcal{M}_\beta \) or \( B(\ell^2_0, \ell^2_{-2\beta}) \).

H3) There exist \( \gamma^- > 0 \) and \( \iota > n + \frac{1}{2\tau} \) such that, for any \( \omega \in \Pi^- \), one has
\[
|\langle k, \omega \rangle| \geq \frac{\gamma^-}{|k|^\iota}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\},
\]
\[
|\chi^-_k - \chi^-_j + \langle k, \omega \rangle| \geq \frac{\gamma^- |i' - j'|}{1 + |k|^\iota}, \quad \forall k \in \mathbb{Z}^n, i \neq j.
\]

Let now
\[
B : \mathbb{T}_n^\omega \ni (\phi_1, \ldots, \phi_n) \mapsto B(\phi_1, \ldots, \phi_n) \in \mathcal{M}_\beta^+
\]
be an analytic map with \( B(\phi_1, \ldots, \phi_n) \) anti-self-adjoint for each real value of \( (\phi_1, \ldots, \phi_n) \). Consider the corresponding unitary operator \( e^{B(\phi_1, \ldots, \phi_n)} \) and, for any \( \omega \in \Pi^- \) consider the unitary transformation of basis \( x = e^{B(wt)} y \). Substitution in equation (2.8) yields
\[
i\bar{y} = (A^+ + P^+ (\omega t)) y, \quad A^+ := A^- + \text{diag}(P^-).
\]

In fact, \( A^+ = \text{diag}(\lambda^+_i + \mu^+_i (\omega t)) \), where \( \lambda^+_i = \bar{\lambda}^-_i + P^-_{ii} (\phi) \) (the overline denotes angular average) and \( \mu^+_i (\omega t) = \mu^-_i (\omega, \omega) + P^+_{ii} (\phi) - P^-_{ii} (\phi) \). Hence the functions \( \mu^+_i (\phi) \) have zero average and \( \text{diag}(P^-) := \text{diag}(P^+_{11} (\omega t) + P^+_{22} (\omega t), \ldots) \). The new perturbation \( P^+ \) is given by
\[
P^+ := ([A^- - B] - i\bar{B} + (P^- - \text{diag}(P^-)) + (e^{-B} A^- e^B - A^- - [A^-, B]) + (e^{-B} P^- e^B - P^-) - i(e^{-B} \frac{d}{dt} e^B - \bar{B}).
\]

The main step of the proof is to construct \( B \) such that the following vanish, i.e. to solve for the unknown \( B \) the equation \( [A^- - B] - i\bar{B} + (P^- - \text{diag}(P^-)) = 0 \). The construction is based on a lemma by Kuksin and a method from Bambusi & Graffi [1]. We also use the same notation as Bambusi & Graffi [1] for reader’s convenience. The proof of Lemma 2.9 is similar as Lemma 3.2 in [1] and we will concentrate the difference with the proof in [1]. In the following we introduce Kuksin’s lemma for completeness.

On the \( n \)-dimensional torus consider the equation
\[
-\sum_{k=1}^{n} \omega_k \frac{\partial}{\partial \phi_k} \chi (\phi) + E_1 \chi (\phi) + E_2 h (\phi) \chi (\phi) = b (\phi).
\]
Here \( \chi \) denotes the unknown, while \( b, h \) denote given analytic functions on \( \mathbb{T}_n^\omega \). \( h \) has zero average, \( E_1, E_2 \) are positive constants and \( \|h\|_s \leq 1 \). Concerning the frequency vector \( \omega = (\omega_1, \ldots, \omega_n) \) Assumptions are:
\[
|\langle k, \omega \rangle| \geq \frac{\gamma_2}{|k|^\iota}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\},
\]
\[
|\langle k, \omega \rangle + E_1| \geq \frac{\gamma_1}{|k|^\iota + 1}, \quad \forall k \in \mathbb{Z}^n.
\]
The important hypothesis is an order assumption, namely: given $0 < \theta < 1$ and $C_0 > 0$ we assume
\begin{equation}
E_1^\theta \geq C_0 E_2.
\end{equation}

**Lemma 2.8 (Kuksin)** Under the above assumptions, equation (2.10) has a unique analytic solution $\chi$ which for any $0 < \sigma < s$ fulfills the
\[ \|\chi\|_{s-\sigma} \leq \frac{C_1}{\gamma_1 \sigma^{a_4}} \exp \left( \frac{C_2}{\gamma_2 \sigma^{a_3}} \right) \|b\|_s. \]

Here $a_3, a_4, a_5, C_1, C_2$ are constants independent of $E_1, E_2, \sigma, s, \gamma_1, \gamma_2, \omega$.

By Kuksin’s lemma as [1] we have

**Lemma 2.9** Let
\[ \gamma^- \geq \gamma_0/2, 0 < \sigma \leq C(\gamma_0, n, \tau) < 1, 0 \leq C^- \leq 1, \]
and $\theta = \frac{2\theta}{1-\theta}$ in (2.11). For any $0 < \sigma < s$, equation
\begin{equation}
[A^-, B] - iB + (P^- - \text{diag}(P^-)) = 0
\end{equation}
has a unique solution $B \in \mathcal{M}^+_{\beta, s-\sigma}$ analytic on $\mathbb{T}^n_{s-\sigma}$, fulfilling the estimate
\begin{equation}
\|B\|_{\beta, s-\sigma} \leq \exp \left( \frac{C}{\sigma^{a_2}} \right) \|P^-\|_{\beta, s} + \|P^-\|_{B(\beta, s, 2\beta, 1, s)},
\end{equation}
where $C = C(\gamma_0, \beta, \iota, n, \tau), a_3 = n + \tau + \theta(n + \tau + 2)/1 - \theta$.

**Proof.** The equation equals to
\[ -i \sum_{k=1}^{n} \omega_k \frac{\partial}{\partial \phi_k} B_{ij} + (\lambda_i^- - \lambda_j^-) B_{ij} + (\mu_i^- (\phi) - \mu_j^- (\phi)) B_{ij} = -P_{ij}, \quad i \neq j, \omega \in \Pi^- .\]

Assume $E_1 = (\lambda_i^- - \lambda_j^-) \geq 0$ and $h_{i,j}(\phi) = \frac{\mu_i^- (\phi) - \mu_j^- (\phi)}{\|\mu_i^- (\phi) - \mu_j^- (\phi)\|_{s+1}}, E_2 = \|\mu_i^- (\phi) - \mu_j^- (\phi)\|_s + 1$, also denote $\gamma_1 = \gamma^- |i^\prime - j^\prime|$ and $\gamma_2 = \gamma^-$. We can choose $\theta = \frac{2\theta}{1-\theta}$ and a suitable constant $C_0$ such that (2.11) holds. In fact, since $|\lambda_i^- - \lambda_j^-| \geq C_3 |i^\prime - j^\prime|$ and $\|\mu_i^- (\phi) - \mu_j^- (\phi)\|_s \leq C_3 (2\beta + 2\beta)$, As [1], one has $E_1^\theta \geq C_0 E_2$ with $\theta$ defined above. Then a direct application of Kuksin’s lemma yields
\[ \|B_{ij}\|_{s-\sigma} \leq \frac{C_1}{\gamma^- |i^\prime - j^\prime|^{a_4}} \exp \left( \frac{C_2}{\gamma^- \sigma^{a_3}} \right) \|P_{ij}\|_s, \quad \text{for } i \neq j. \]

Note our assumptions for $P^-$, it results in
\[ \|B_{ij}\|_{s-\sigma} (i^\prime - j^\prime)^{-\beta} (1 + |i - j|)(i^\prime - 1 + j^\prime - 1) \leq \frac{4C_1}{\gamma^- \sigma^{a_4}} \exp \left( \frac{C_2}{\gamma^- \sigma^{a_3}} \right) \|P^-\|_{\beta, s}, \quad \forall i \neq j. \]

When $i = j, B_{ij}(\phi) = 0$. Therefore, $B(\phi) \in \mathcal{M}^+_{\beta}$ for any $|\text{Im} \phi| < s - \sigma$ and
\[ \|B(\phi)\|_{\beta, s-\sigma} \leq \frac{4C_1}{\gamma^- \sigma^{a_4}} \exp \left( \frac{C_2}{\gamma^- \sigma^{a_3}} \right) \|P^-\|_{\beta, s}. \]

A similar computation follows for $\omega \neq \omega', i \neq j,$
\[ \frac{\|\Delta B_{ij}\|_{s-2\sigma}}{|\omega - \omega'|} (i^\prime - j^\prime)^{-\beta} (1 + |i - j|)(i^\prime - 1 + j^\prime - 1) \leq \frac{C}{(\gamma^-)^{2\sigma^{a_4}} + 1} \exp \left( \frac{2C_2}{\gamma^- \sigma^{a_3}} \right) \|P^-\|_{\beta, s}. \]
Thus,
\[
\|B(\phi)\|_{\beta,s-2\sigma}^{\xi,+} \leq \frac{C(n)}{\sigma} \|B\|_{\beta,s-\sigma}^{\xi,+},
\]
where \( a_4 = n + \tau,\ a_5 = \frac{1}{\sqrt{\gamma}},\ a_3 = n + \tau + \frac{\theta(n+\tau+2)}{1-\gamma} > 1.\) So if we denote \( m = [2a_4 + 3] \) and choose \( 3\sigma \leq \min\{\frac{1}{mC(\gamma_0,n,\tau)}\}, \) then \( \frac{C(n,\tau)}{(3\sigma)^{2a_4+1}} \leq \frac{1}{(3\sigma)^m} \leq \exp\left(\frac{1}{3\sigma}\right).\) Redefine \( 3\sigma \) as \( \sigma \) one obtains
\[
\|B(\phi)\|_{\beta,s-\sigma}^{\xi,+} \leq \exp\left(\frac{C}{\sigma^{m^2}}\right) \|P^-\|_{\beta,s}^{\xi},
\]
(2.15)

where \( C = C(\gamma_0,\beta,\iota,n,\tau).\)

In fact by Cauchy’s estimate and (ii) of Lemma 2.4 and (2.15) we have

**Lemma 2.10**
\[
\left\| B \right\|_{\beta,s-2\sigma}^{\xi,+} \leq \frac{C(n)}{\sigma} \left\| B \right\|_{\beta,s-\sigma}^{\xi,+},
\]
\[
\left\| \frac{d}{dt} e^B \right\|_{\beta,s-2\sigma}^{\xi,+} \leq \frac{C(\beta,\iota)}{\sigma} e^{C(\beta)} \left\| B \right\|_{\beta,s-\sigma}^{\xi,+},
\]
\[
\left\| \frac{d}{dt} (e^B - I - B) \right\|_{\beta,s-2\sigma}^{\xi,+} \leq \frac{C(\beta,\iota)}{\sigma} e^{C(\beta)} \left\| B \right\|_{\beta,s-\sigma}^{\xi,+} \left( \left\| B \right\|_{\beta,s-\sigma}^{\xi,+} \right)^2.
\]

**Lemma 2.11** For any \(|s| \leq 1\), if \( \|B\|_{\beta,s-\sigma}^{\xi,+} \ll 1, \) then
\[
\left\| e^{s_1 B} \right\|_{B(\ell_0^2),s-2\sigma}^{\xi} \leq 2,
\]
\[
\left\| e^{s_1 B} \right\|_{B(\ell_0^2),s-2\sigma}^{\xi} \leq 2\left(1 + C(\beta,\iota) e^{s_1 B} - I \right) \left\| B \right\|_{\beta,s-\sigma}^{\xi,+} \leq 2.
\]

Similarly, by \( 0 \leq \delta < \iota - \beta - \frac{1}{2}, \) (v) of Lemma 2.2 and Lemma 2.4, if \( \|B\|_{\beta,s-\sigma}^{\xi,+} \ll 1, \) then
\[
\left\| e^{s_1 B} \right\|_{B(\ell_0^2),s-2\sigma}^{\xi} \leq 2.
\]

**Lemma 2.12** Consider the system
\[
i\dot{x} = \left(A^- + P^- (\omega t)\right)x
\]
(2.16)
with the stated assumptions. Assume furthermore that also (2.12) holds. Then there exists an anti-self-adjoint operator \( B \in \mathcal{M}_{\beta,s-\sigma}^+ \) analytically depending on \( \phi \in \mathcal{T}_{\alpha,s-\sigma}^\alpha \), and Lipschitz continuous in \( \omega \in \Pi^- \) such that
(1). \( B \) fulfills the estimate (2.14);
(2). For any \( \omega \in \Pi^- \) the unitary operator \( e^{B(\omega t)} \) transforms the system (2.16) into the system
\[
iy = \left(A^+ + P^+ (\omega t)\right)y;
\]
(3). The new perturbation \( P^+ \) fulfills the estimate
\[
\left\| P^+ \right\|_{\beta,s-2\sigma}^{\xi} \leq \exp\left(\frac{2C}{\sigma^{m^2}}\right) \left( \left\| P^- \right\|_{\beta,s}^{\xi} + \left\| P^- \right\|_{B(\ell_0^2,\ell_{-2\sigma})}^{\xi}\right)^2,
\]
where we assume that \( \left\| B \right\|_{\beta,s-\sigma}^{\xi} \ll \frac{1}{C(\sigma)}, 0 < \sigma \ll \frac{1}{C(\sigma)} < 1 \) and \( \tau > 1; \)
(4). For any positive \( K \) such that \( 2\left( \left\| P^- \right\|_{\beta,s}^{\xi} + \left\| P^- \right\|_{B(\ell_0^2,\ell_{-2\sigma})}^{\xi}\right) (1 + K\tau) < \gamma^- - \gamma^+, \) there exists a closed set \( \Pi^+ \in \Pi^- \) fulfilling
\[
\left| \Pi^- \setminus \Pi^+ \right| \leq \frac{C\gamma_0}{K^{\tau^2 - \frac{n-1}{\tau^2}}},
\]
under the condition (2.22), (2.23), (2.24) and (2.25):
(5). If \( \omega \in \Pi^+ \) assumptions \( H1 - H3 \) above are fulfilled by \( A^+ \) and \( P^+ \) provided that the constants are replaced by the new ones defined by
\[
\gamma^+ = \gamma^- - 2(\|P^-\|_{\beta,s} + \|P^-\|_{B(t_0, t_2), s})(1 + K^+),
\]
\[
C^+_{\lambda} = C_\lambda^- - 2(\|P^-\|_{\beta,s} + \|P^-\|_{B(t_0, t_2), s}),
\]
\[
C^+_{\omega} = C_\omega^- + 2(\|P^-\|_{\beta,s} + \|P^-\|_{B(t_0, t_2), s}),
\]
\[
C^+_{\mu} = C_\mu^- + 2(\|P^-\|_{\beta,s} + \|P^-\|_{B(t_0, t_2), s}).
\]
(2.17)

**Remark 2.13** If \( \|P^-\|_{\beta,s} + \|P^-\|_{B(t_0, t_2), s} \leq \nu \), we can choose
\[
\gamma^+ = \gamma^- - 2\epsilon (1 + K^+), \quad C^+_{\lambda} = C_\lambda^- - 2\epsilon, \quad C^+_{\omega} = C_\omega^- + 2\epsilon, \quad C^+_{\mu} = C_\mu^- + 2\epsilon
\]
in place of (2.17).

**Proof.** Similar as [1], we can prove that \( B \) is anti-self-adjoint operator and \( e^{B(\omega t)} \) is a unitary operator and (1) and (2) follow easily. For (3), we write the new perturbation \( P^+ := (I) + (II) + (III) \), where \((I) = e^{-B}A^+e^{-B}A^+ - A^+, (II) = e^{-B}P^-e^{-B}P^- \) and \((III) = -ie^{-B} \frac{d}{dt}e^{-B}B\). We first estimate (I). By (2.13) one has
\[
[[A^- , B], B] = [\dot{B} - (P^- - \text{diag}(P^-)), B] = [\dot{B}, B] - [P^- - \text{diag}(P^-), B].
\]
For any \( |s_2| \leq 1 \), from (iii) of Lemma 2.4 we have
\[
\|e^{-s_2B}[[A^- , B], B]e^{s_2B}\|_{\beta,s-2\sigma} \leq C(\beta)\|\dot{B}, B\|_{\beta,s-2\sigma} + C(\beta)\|P^- - \text{diag}(P^-), B\|_{\beta,s-2\sigma} \leq C(\beta, n) \exp \left( \frac{C}{\sigma a_3} \right) \|P^-\|_{\beta,s-2\sigma}^2.
\]
Since
\[
(I) = e^{-B}A^+e^{-B}A^+ - A^+ = \int_0^1 ds_1 \int_0^{s_1} e^{-s_2B}[[A^- , B], B]e^{s_2B}ds_2,
\]
it follows
\[
\|I\|_{\beta,s-2\sigma} \leq \int_0^1 ds_1 \int_0^{s_1} \|e^{-s_2B}[[A^- , B], B]e^{s_2B}\|_{\beta,s-2\sigma} ds_2 \leq C(\beta, n) \exp \left( \frac{C}{\sigma a_3} \right) \|P^-\|_{\beta,s-2\sigma}^2.
\]
If \( B \) is \( \|B\|_{\beta,s-\sigma} \leq \frac{1}{C(\beta)} \), by (iii) of Lemma 2.4 one has \( \|I\|_{\beta,s-2\sigma} \leq C_1(\beta) \exp \left( \frac{C}{\sigma a_3} \right) \|P^-\|_{\beta,s}^2 \) and
\[
\|I\|_{\beta,s-2\sigma} \leq C(\beta,n) \exp \left( \frac{C}{\sigma a_3} \right) \|P^-\|_{\beta,s}^2. \]
Combining all the above estimates, if \( |B|_{\beta,s-\sigma} \leq \frac{1}{C(\beta)} \) and \( 0 < \sigma < 1 \), then
\[
\|P^+\|_{\beta,s-2\sigma} \leq C(\beta, n) \exp \left( \frac{C}{\sigma a_3} \right) \|P^-\|_{\beta,s-2\sigma}^2,
\]
(2.18)

where \( a_3 = n + \tau + \frac{\theta(\tau+2)}{1-\theta} \) and \( \tau > 1 \). In the following we turn to the estimate on
\[\|P^+\|_{B(t_0, t_2), s-2\sigma}.\]

By (2.13), Cauchy’s estimate and (iv) of Lemma 2.2 we have
\[
\|[A^-, B]\|_{B(t_0, t_2), s-2\sigma} \leq \|[\dot{B}, B]\|_{B(t_0, t_2), s-2\sigma} + 2\|P^-\|_{B(t_0, t_2), s} \leq \frac{C(\beta,n,\ell)}{\sigma} \|B\|_{\beta,s-\sigma} + 2\|P^-\|_{B(t_0, t_2), s}^\epsilon \]
together with (2.15), (iv), (v) of Lemma 2.2 we obtain
\[
\|[A^-, B]\|_{B(t_0, t_2), s-2\sigma} \leq \|[A^-, B]\|_{B(t_0, t_2), s-2\sigma} + \|B[A^-, B]\|_{B(t_0, t_2), s-2\sigma}
\]
Thus, if \( H_1.a \) and then from Lemma 2.2, By (2.11), we have

\[
\|P^{-1}\|_{\|\cdot\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s-2\sigma} \leq C(\beta, n, \iota) \exp \left( \frac{C_{\sigma \alpha_3}}{\sigma \alpha_3} \right) \|P^{-1}\|_{\|\cdot\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s-2\sigma} + \|P^{-1}\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s-2\sigma}
\]

and from Lemma 2.11, one can draw

\[
\|((I))\|_{B(\ell^2_0, \ell^2_{-2\sigma}), s-2\sigma} = \left\| \int_0^1 e^{-s_1 B} [P^{-1}, B] e^{s_1 B} ds_1 \right\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s-2\sigma}
\]

Similarly, with (iv) of Lemma 2.2, one obtains

\[
\|((III))\|_{B(\ell^2_0, \ell^2_{-2\sigma}), s-2\sigma} \leq \|((III))\|_{B(\ell^2_0, \ell^2_{-2\sigma}), s-2\sigma} \leq C(\beta, n, \iota) \|((III))\|_{\|\cdot\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s-2\sigma}
\]

Thus, if \( \|B\|_{\|\cdot\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s-2\sigma} \ll 1 \) and \( 0 < \sigma < 1 \), we have

\[
\|P^{-1}\|_{\|\cdot\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s-2\sigma} \leq C(\beta, n, \iota) \exp \left( \frac{C_{\sigma \alpha_3}}{\sigma \alpha_3} \right) \|P^{-1}\|_{\|\cdot\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s-2\sigma} + \|P^{-1}\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s-2\sigma}
\]

where \( \alpha_3 = n + \tau + \frac{2(q+\tau+2)}{1-\theta} \) and \( \tau > 1 \). Combining with (2.18) and (2.20), we obtain (3). In the following we turn to prove (4) and (5).

For H1.a) As [1], if choose \( C^+_\lambda = C^-_\lambda - 2(\|P^{-1}\|_{\|\cdot\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s} + \|P^{-1}\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s} \), then \( |\lambda^+_\lambda - \lambda^+_j| \geq C^+_\lambda |i-j| \).

H1.b) Choose \( C^+_\omega = C^-_\omega + 2(\|P^{-1}\|_{\|\cdot\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s} + \|P^{-1}\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s} \), then \( \sup_{\omega, \omega' \in \Pi \omega \neq \omega'} \frac{|\lambda^+_\omega - \lambda^+_\omega'|}{|\omega - \omega'|} \leq C^+_\omega i^{2\beta} \)

holds true.

H1.c) Similarly, if \( C^+_\mu = C^-_\mu + 2(\|P^{-1}\|_{\|\cdot\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s} + \|P^{-1}\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s} \), then \( \mu^+_\mu \|s \leq C^+_\mu i^{2\beta} \). If choose \( C^+_\omega = C^-_\omega + 2(\|P^{-1}\|_{\|\cdot\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s} + \|P^{-1}\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s} \), we have \( \sup_{\omega, \omega' \in \Pi \omega \neq \omega'} \frac{|\mu^+_\omega - \mu^+_\omega'|}{|\omega - \omega'|} \leq C^+_\omega i^{2\beta} \).

H3: To check H3 for next step, one need to throw away suitable parameter sets. This step is very similar as [1] and we only give a sketch here. In fact, if choose \( \gamma^+ = \gamma^- - 2(\|P^{-1}\|_{\|\cdot\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s} + \|P^{-1}\|_{B(\ell^2_0, \ell^2_{-2\sigma})}, s} + (1 + K)^\tau) \), then for any \( |k| \leq K \), \( i \neq j \)

\[
|\lambda^+_i - \lambda^+_j + \langle k, \omega \rangle| \geq \frac{\gamma^+ |i-j|}{1 + |k|^\tau},
\]

and

\[
|\langle k, \omega \rangle| \geq \frac{\gamma^+}{|k|^\tau}, \quad \forall \ k \in \mathbb{Z} \setminus \{0\}.
\]
For $|k| > K$ and $i \neq j$, we need to throw away a suitable parameter set in $\Pi^-$ to guarantee (2.21) holds true. Clearly, a standard procedure shows us

$$
\Pi^\perp \Pi^+ := \bigcup_{|k| > K} \bigcup_{i \neq j, i, j \geq 1} \mathcal{R}_{ij}^k(\omega) := \bigcup_{|k| > K} \bigcup_{i \neq j, i, j \geq 1} \left\{ \omega \in \Pi^- : |\lambda_i^+ - \lambda_j^+ + \langle k, \omega \rangle| < \frac{\gamma^+|i^j - j^i|}{1 + |k|} \right\}
$$

\begin{align*}
&\subset \bigcup_{|k| > K} \bigcup_{i \neq j, i, j \geq 1} \left\{ \omega \in \Pi^+ : |\lambda_i^+ - \lambda_j^+ + \langle k, \omega \rangle| < \frac{\gamma^+|i^j - j^i|}{1 + |k|} \right\} := \bigcup_{|k| > K} \bigcup_{i \neq j, i, j \geq 1} \mathcal{R}_{ij}^k(\omega).
\end{align*}

Next we will estimate the measure for $\bigcup_{|k| > K} \bigcup_{i \neq j, i, j \geq 1} \mathcal{R}_{ij}^k(\omega)$. Similar as [1], if

$$
C_{\lambda, l} \geq \frac{1}{2} C_{\lambda}, \quad \text{for any } l,
$$

and

$$
\gamma_l < \frac{1}{4} C_{\lambda}, \quad \text{for any } l,
$$

and

$$
C_{\omega}^+ < \frac{C_{\lambda}}{16n},
$$

and

$$
\tau > n + \frac{2}{l - 1}, \quad \gamma^+ \leq \gamma_0 \quad \text{and } K \geq 2
$$

are fulfilled, then

$$
\text{meas}(\Pi^\perp \setminus \Pi^+) \leq \frac{C_{\gamma_0}}{K^{\tau - n - \frac{2}{l - 1}}}
$$

\[\square\]

2.4. Iteration. In this section we set up the iteration. First we preassign the value of the various constants. Hence we keep $\epsilon_0, K, s_0$ and $\gamma_0$ fixed which satisfy

$$
0 < s_0 < 1, \quad \epsilon_0 \leq \min\{C_{\epsilon}, \exp(-a_6 s_0^{-a_3})\}, \quad \gamma_0 < \frac{1}{4} C_{\lambda}, \quad K = 4^3,
$$

where $C_{\epsilon}$ and $a_6$ depend on $\beta, C_{\lambda}, n, \tau, \gamma_0$, and $a_3 \sim n, \tau, \beta, \epsilon$, and $l \geq 1$. For $l \geq 1$ we define

$$
\epsilon_l = \epsilon_{l-1}^4, \quad \sigma_l = \left( \frac{3C}{|\ln \epsilon_{l-1}|} \right)^{\frac{1}{2}}, \quad s_l = s_{l-1} - 2\sigma_l, \quad K_l = lK, \quad \gamma_l = \gamma_{l-1} - 2\epsilon_{l-1}(1 + K_l),
$$

$$
C_{\lambda,l} = C_{\lambda,l-1} - 2\epsilon_{l-1}, \quad C_{\omega,l} = C_{\omega,l-1} + 2\epsilon_{l-1}, \quad C_{\mu,l} = C_{\mu,l-1} + 2\epsilon_{l-1}.
$$

The initial values of the sequences are chosen as follows:

$$
\gamma_0 := \gamma, \quad s_0 = s, \quad C_{\lambda, 0} := C_{\lambda}, \quad C_{\omega, 0} := 0, \quad C_{\mu, 0} := 0.
$$

From these settings, we can obtain that for any $l \geq 0$, a). $0 < \sigma_l \leq \gamma_l \leq \gamma_0 < \frac{1}{4} C_{\lambda}$, b). $0 < \sigma_l \leq \min\{C(\gamma_0, n, \tau), \frac{C_{\gamma_0}}{C_{\lambda, 0}}\} < 1$, c). $0 \leq C_{\omega,l} \leq \min\{\frac{C_{\gamma_0}}{10}, 1\}$, d). $C_{\lambda,l} \geq \frac{1}{4} C_{\lambda}$, e). $K_l \geq 2$ with $l \geq 1$, and f). $0 \leq \epsilon_l \leq \frac{1}{C_{\lambda}}$.

**Proposition 2.14** There exist $\epsilon_\ast = \epsilon_\ast(\gamma, s) > 0$ and, for any $l \geq 1$, a closed set $\Pi^l \subset \Pi$ such that, if $0 \leq \epsilon < \epsilon_\ast$, one can construct for $\omega \in \Pi_l^\perp$ a unitary transformation $U^l$, analytic and quasi-periodic in $t$ with frequencies $\omega$, mapping the system $i\dot{x} = (A + \epsilon P(\omega t))x$, into the system

$$
i\dot{x} = (A^l + P^l(\omega t))x,
$$

where

(1). $U^l(\omega t)$ is as follow: $U^l(\omega t) = e^{B^l(\omega t)} e^{B^2(\omega t)} \ldots e^{B^1(\omega t)}$, and the anti-self-adjoint operator $B^l \in \mathcal{M}_\beta$ depending analytically on $\phi \in T^n_{s_{l-1} - \sigma_l}$, are Lipschitz continuous in $\omega \in \Pi_l^\perp$ and
fulfilling (2.14) with \( P^{j-1}, s_{j-1}, \sigma_j \) in place of \( P^-, s, \sigma \), respectively.\\n
(2). \( A^l \) has the form of (2.9) with the upper index “minus” replaced by \( l \), i.e.
\[
A^l := \text{diag}\{\lambda^l_1(\omega) + \mu^l_1(\omega, \omega), \lambda^l_2(\omega) + \mu^l_2(\omega, \omega), \ldots\}.
\]

(3). The corresponding \( \lambda^l_i \) and \( \mu^l_i \) fulfill conditions H1, H3 of the previous section, provided \( \lambda^l_i, \mu^l_i \) are replaced by \( \lambda^l_1, \mu^l_1 \), respectively.

(4). \( P^l \) fulfills condition H2 with the upper index “minus” replaced by \( l \) and the following estimates hold:
\[
\|P^l\|_{\beta,s_i} + \|P^l\|_{\beta,[\ell_0, \ell_2(\beta)]} \leq \epsilon_i, \\
\|B^l\|_{\beta,s_i-\sigma_i} \leq \epsilon_i, \\
\|A^l\|_{\beta,s_i} \leq \epsilon_i.
\]

Proof. We proceed by induction applying Lemma 2.12. First we apply it to the original system to obtain the system (2.27) for \( l = 1 \). To this end we notice that all assumptions are satisfied except for the non-resonance conditions H3 on the frequencies. We define
\[
\Pi_0 := \{\omega : |\langle k, \omega \rangle| < \gamma |k| \tau\},
\]
and
\[
\Pi_1 := \Pi_0 \cup \{\omega : |\langle k, \omega \rangle| \geq \gamma |k| \tau\}.
\]

and all assumptions are satisfied for Theorem 2.12. Then there exists an anti-self-adjoint operator \( B^{l+1} \) which satisfies
\[
\|B^{l+1}\|_{\beta,s_i-\sigma_i+1} \leq \epsilon_i - \epsilon_i + \epsilon_i = \epsilon_i,
\]
where \( B^{l+1} \) is analytically depending on \( \phi \in T^{n}_{\beta,s_i-\sigma_i+1} \) and Lipschitz continuous in \( \omega \in \Pi_i^\gamma \). By the unitary operator \( e^B^{l+1}(\omega t) \) transforms the system (2.28) into the system \( i\dot{x} = (A^l + P^{l+1}(\omega t))x \), the new perturbation \( P^{l+1} \) fulfills
\[
\|P^{l+1}\|_{\beta,s_i+1} + \|P^{l+1}\|_{\beta,[\ell_0, \ell_2(\beta)]} \leq \epsilon_i - \epsilon_i + \epsilon_i = \epsilon_i.
\]

Moreover, there exists a closed set \( \Pi_{\ell+1}^\gamma \subset \Pi_i^\gamma \) and \( b_1 = \tau - n - \frac{2}{\ell_1} > 1 \) fulfilling \( \Pi_{\ell+1}^\gamma \backslash \Pi_{\ell+1}^\gamma \leq \frac{C_\gamma}{K_{\ell+1}} \).

If \( \omega \in \Pi_{\ell+1}^\gamma \), then assumptions H1-H3 are fulfilled by \( A^{l+1} \), \( P^{l+1} \) provided that the constants are replaced by the new ones defined by
\[
\gamma_{\ell+1} = \gamma - 2\epsilon_i(1 + K_{\ell+1}), \\
C_{\lambda,i+1} = C_{\lambda,i} - 2\epsilon_i, \\
C_{\omega,i+1} = C_{\omega,i} + 2\epsilon_i, \\
C_{\mu,i+1} = C_{\mu,i} + 2\epsilon_i.
\]

In the following we will prove Theorem 2.5, but we first need a series of preparation lemmas.

Lemma 2.15 \( A^l(\phi) \to A^\infty(\phi) \) in \( H^*_{\text{c}}(\Pi^\gamma, \frac{\gamma}{\tau}) \) and \( B([\ell_0, \ell_2(\beta)])(\Pi^\gamma, \frac{\gamma}{\tau}) \), if \( \epsilon_0 \ll 1 \), then
\[
\|A^\infty\|_{[\ell_0, \ell_2(\beta)]} \leq \|A^\infty\|_{\frac{\gamma}{\tau}}, \\
\|A^\infty - A^0\|_{[\ell_0, \ell_2(\beta)]} \leq \|A^\infty - A^0\|_{\frac{\gamma}{\tau}} \leq 6\epsilon_0.
\]
Proof. From the iteration, for \( l \geq 0 \) one has

\[
|\lambda_{i}^{l+1} - \lambda_{i}^{l}| \leq \|P_{l}\|_{\beta, s_{l}} t^{2\beta} \leq \epsilon_{l} t^{2\beta}
\]

and

\[
|\mu_{i}^{l+1} - \mu_{i}^{l}| \leq 2\|P_{l}\|_{\beta, s_{l}} t^{2\beta} \leq 2\epsilon_{l} t^{2\beta}.
\]

Thus, for given \( i \in \mathbb{Z}_{+} \), \( \{\lambda_{i}^{l}\} \) and \( \{\mu_{i}^{l}\} \) are Cauchy sequences. Define \( \lambda_{i}^{\infty} := \lambda_{i} + \sum_{l=0}^{\infty} \lambda_{i}^{l+1} - \lambda_{i}^{l} \) and \( \mu_{i}^{\infty} := \sum_{l=0}^{\infty} \mu_{i}^{l+1} - \mu_{i}^{l} \). For \( l \geq 0 \), we have \( |\lambda_{i}^{\infty} - \lambda_{i}^{l}| \leq 2\epsilon_{l} t^{2\beta} \) and \( |\mu_{i}^{\infty} - \mu_{i}^{l}| \leq 4\epsilon_{l} t^{2\beta} \). Note \( 2\beta < \iota - 1 < \iota \), then \( (|\lambda_{i}^{\infty} + \mu_{i}^{\infty}) - (\lambda_{i}^{l} + \mu_{i}^{l})| \leq 6\epsilon_{l} t^{\iota} \). It follows \( \|A^{\infty} - A^{l}\|_{\beta, \frac{1}{2}} \leq 6\epsilon_{l} \), together with Lemma 4.1 and thus,

\[
\|A^{\infty} - A^{l}\|_{B(\ell_{0}^{2}, \ell_{2}^{2})}, \frac{1}{2} \leq \|A^{\infty} - A^{l}\|_{\beta, \frac{1}{2}} \leq 6\epsilon_{l} \to 0 \quad \text{as} \quad l \to \infty.
\]

It follows that \( A^{l} \to A^{\infty} \) in \( M_{\frac{1}{2}}(\Pi_{s}, \frac{1}{2}) \) and \( B(\ell_{0}^{2}, \ell_{2}^{2})(\Pi_{s}, \frac{1}{2}) \). Set \( l = 0 \), we have

\[
\|A^{\infty} - A^{0}\|_{B(\ell_{0}^{2}, \ell_{2}^{2})}, \frac{1}{2} \leq \|A^{\infty} - A^{0}\|_{\beta, \frac{1}{2}} \leq 6\epsilon_{0}.
\]

Then for any \( (\phi, \omega) \in T_{l}^{\frac{1}{2}} \times \Pi_{s} \), one obtains \( A^{\infty}(\phi, \omega) \in M_{\frac{1}{2}} \bigcap B(\ell_{0}^{2}, \ell_{2}^{2}) \) and \( \|A^{\infty}\|_{B(\ell_{0}^{2}, \ell_{2}^{2})}, \frac{1}{2} \leq \|A^{\infty}\|_{\beta, \frac{1}{2}} \).

**Remark 2.16** From the iteration as [1] one can also have \( |\lambda_{i}^{l+1} - \lambda_{i}^{l}| \leq \epsilon_{l} t^{\delta} \) and \( |\mu_{i}^{l+1} - \mu_{i}^{l}| \leq 2\epsilon_{l} t^{\delta} \). But note the assumption \( \delta \geq 2\beta \), these estimates are weaker than (2.29) and (2.30).

Recall \( \|P_{l}\|_{\beta, s_{l}} + \|P_{l}\|_{B(\ell_{0}^{2}, \ell_{2}^{2})}, s_{l} \leq \epsilon_{l} \), and \( s_{l} \geq \frac{\lambda}{2} \) and \( \Pi_{s} = \bigcap_{l \geq 0} \Pi_{l} \), it follows

**Lemma 2.17** \( P^{l}(\phi) \to 0 \) in \( M_{\beta}(\Pi_{s}, \frac{1}{2}) \) and \( B(\ell_{0}^{2}, \ell_{2}^{2})(\Pi_{s}, \frac{1}{2}) \).

In the sequel, let \( B_{0}^{0}(\phi) = 0 \) and \( \epsilon_{-1} = 0 \). From Lemma 2.4 and the induction, we have

**Lemma 2.18** For \( l \geq 0 \), \( U^{l}(\phi) = e^{B_{l}^{1}(\phi)} e^{B_{l}^{2}(\phi)} \cdots e^{B_{l}^{1}(\phi)} \), if \( \epsilon_{0} \ll 1 \), then

\[
\|U^{l}(\phi) - I\|_{\beta, s_{l}} \leq 3(\epsilon_{0} + \cdots + \epsilon_{l-1}).
\]

Similarly,

**Lemma 2.19** For \( 0 \leq l_{1} < l_{2} \), if \( \epsilon_{0} \ll 1 \), \( \|e^{B_{l_{1}}^{1}(\phi)} \cdots e^{B_{l_{2}}^{1}(\phi)} - I\|_{\beta, s_{l_{2}-1}} \leq 3(\epsilon_{l_{1}} + \cdots + \epsilon_{l_{2}-1}) \).

**Lemma 2.20** For \( 0 \leq l_{1} < l_{2} \), if \( \epsilon_{0} \ll 1 \), then \( \|U^{l_{1}}(\phi) - U^{l_{2}}(\phi)\|_{\beta, s_{l_{2}-1}} \leq 8\epsilon_{l_{1}} \).

**Proof.** Note \( U^{l_{1}}(\phi) = e^{B_{l_{1}}^{0}(\phi)} e^{B_{l_{1}}^{1}(\phi)} \cdots e^{B_{l_{1}}^{1}(\phi)} \), \( U^{l_{2}}(\phi) = U^{l_{1}}(\phi) e^{B_{l_{2}+1}^{1}(\phi)} \cdots e^{B_{l_{2}}^{1}(\phi)} \), it follows

\[
U^{l_{1}}(\phi) - U^{l_{2}}(\phi) = U^{l_{1}}(\phi) (I - e^{B_{l_{2}}^{1}(\phi)} \cdots e^{B_{l_{2}}^{1}(\phi)})
\]

\[
= (U^{l_{1}}(\phi) - I)(I - e^{B_{l_{2}}^{1}(\phi)} \cdots e^{B_{l_{2}}^{1}(\phi)}) + (I - e^{B_{l_{2}}^{1}(\phi)} \cdots e^{B_{l_{2}}^{1}(\phi)}).
\]

Then by Lemma 2.4, Lemma 2.18 and Lemma 2.19, we have

\[
\|U^{l_{1}}(\phi) - U^{l_{2}}(\phi)\|_{\beta, l_{2}} \leq C(\beta)
\]

\[
\leq C(\beta)\|U^{l_{1}}(\phi) - I\|_{\beta, s_{l_{2}-1}}\|I - e^{B_{l_{2}}^{1}(\phi)} \cdots e^{B_{l_{2}}^{1}(\phi)}\|_{\beta, l_{2}} + \|I - e^{B_{l_{2}}^{1}(\phi)} \cdots e^{B_{l_{2}}^{1}(\phi)}\|_{\beta, l_{2}}
\]

\[
\leq 4(\epsilon_{l_{1}} + \cdots + \epsilon_{l_{2}-1}) \leq 8\epsilon_{l_{1}}.
\]

\( \square \)
For $l_2 > l_1 \geq 0$, if $\epsilon_0 \ll 1$, by Lemma 2.20 one has
\[
\|U^{l_1} - U^{l_2}\|_{\beta, \frac{2}{2}} \leq \|U^{l_1} - U^{l_2}\|_{\beta, \frac{2}{2}}^{+} \leq 8\epsilon_0^2 \to 0 \quad \text{(as } l_1 \to \infty) .
\] (2.31)

It follows that \{U^l\} is a Cauchy sequence in $M_{\beta}(P_\epsilon, \frac{2}{2})$. Define $U^\infty = \lim_{l \to \infty} U^l$, then by (2.31), we have
\[
\|U^\infty - U^l\|_{\beta, \frac{2}{2}}^{+} \leq 8\epsilon_0^2 .
\] (2.32)

The following lemma is clear by (iv), (v) of Lemma 2.2 and $\delta \in [0, \omega_0 - \frac{1}{2})$. 

**Lemma 2.21** $U^l(\phi) \to U^\infty(\phi)$ in $M_{\beta}(P_\epsilon, \frac{2}{2})$, $B(\ell_0^2)(P_\epsilon, \frac{2}{2})$ and $B(\ell_2^{2n})(P_\epsilon, \frac{2}{2})$, if $\epsilon_0 \ll 1$, then
\[
\|U^\infty - 1\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} , \quad \|U^\infty - 1\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} \leq C\|U^\infty - 1\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} \leq C\epsilon_0^2 ,
\]
\[
\|U^\infty\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} , \quad \|U^\infty\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} \leq 1 + C\epsilon_0^2 ,
\]
where $U^\infty(\phi)$ is real analytic on $P_\epsilon$ and Lipschitz continuous on $\omega \in P_\epsilon$.

In the following we denote $V^l(\phi) = e^{-B^l(\phi)e^{-B^{l-1}(\phi)}\ldots e^{-B^0(\phi)}}$ for $l \geq 0$.

**Lemma 2.22** For $l \geq 0$, if $\epsilon_0 \ll 1$, then $\|V^l(\phi) - 1\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} \leq 3(\epsilon_0^2 + \ldots + \epsilon_l^2)$.

Similarly, we have

**Lemma 2.23** For $0 \leq l_1 < l_2$, if $\epsilon_0 \ll 1$, then $\|V^{l_1}(\phi) - V^{l_2}(\phi)\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} \leq 8\epsilon_0^2$.

It follows that \{V^l\} is a Cauchy sequence in $M_{\beta}(P_\epsilon, \frac{2}{2})$. Define $V^\infty = \lim_{l \to \infty} V^l$, then we have

**Lemma 2.24** $V^l(\phi) \to V^\infty(\phi)$ in $M_{\beta}(P_\epsilon, \frac{2}{2})$, $B(\ell_0^2)(P_\epsilon, \frac{2}{2})$ and $B(\ell_2^{2n})(P_\epsilon, \frac{2}{2})$, if $\epsilon_0 \ll 1$, then
\[
\|V^\infty - 1\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} , \quad \|V^\infty - 1\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} \leq C\|V^\infty - 1\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} \leq C\epsilon_0^2 ,
\]
\[
\|V^\infty\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} , \quad \|V^\infty\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} \leq 1 + C\epsilon_0^2 ,
\]
where $V^\infty(\phi)$ is real analytic on $P_\epsilon$ and Lipschitz continuous on $\omega \in P_\epsilon$.

The following is clear by (2.32) and Cauchy’s estimate.

**Lemma 2.25** For $l \geq 0$, if $\epsilon_0 \ll 1$, then $\|\partial t^l(\phi) - \partial t^l(\phi)\|_{\beta, \frac{2}{2}}^{\frac{l}{2}} \leq C\epsilon_0^2 \to 0$, $\forall j \in \{1, 2, \ldots, n\}$.

By Lemma 2.25, Cauchy estimate and (iv) and (v) of Lemma 2.2, we have

**Lemma 2.26** $\frac{d}{dt}U^l(\omega t) \to \frac{d}{dt}U^\infty(\omega t)$ in $M_{\beta}(P_\epsilon, B(\ell_0^2)(P_\epsilon)$ and $B(\ell_2^{2n})(P_\epsilon)$ when $l \to \infty$.

From Lemma 2.26, we have

**Corollary 2.27** $\frac{d}{dt}U^l(\omega t) \to \frac{d}{dt}U^\infty(\omega t)$ in $B(\ell_0^2, \ell_2^{2n})(P_\epsilon)$ when $l \to \infty$.

**Lemma 2.28** $A^0U^l(\phi) \to A^0U^\infty(\phi)$ in $B(\ell_0^2, \ell_2^{2n})(P_\epsilon, 2)$ when $l \to \infty$.

**Proof.** From Lemma 4.1, one has $\|A_0\|_{B(\ell_0^2, \ell_2^{2n})} \leq C$. For any $\phi \in P_\epsilon$, by Lemma 2.21 we have
\[
\|A^0U^l(\phi) - A^0U^\infty(\phi)\|_{B(\ell_0^2, \ell_2^{2n})} \leq \|A^0\|_{B(\ell_0^2, \ell_2^{2n})} \|U^l(\phi) - U^\infty(\phi)\|_{B(\ell_0^2)} \to 0 \quad \text{as } l \to \infty.
\]
\[\square\]
Lemma 2.29 \( P^0U^l(\phi) \to P^0U^{\infty}(\phi) \) in \( B(\ell^2_0, \ell^2_{-2d})(\Pi_\phi, \tilde{\tau}) \) and \( B(\ell^2_0, \ell^2_{-2d})(\Pi_\phi, \tilde{\tau}) \) when \( l \to \infty \).

Proof. For any \( \phi \in \mathbb{T}_2^\phi \), by Lemma 2.21 one has
\[
\| P^0(\phi)U^l(\phi) - P^0(\phi)U^{\infty}(\phi) \|_{B(\ell^2_0, \ell^2_{-2d}), \tilde{\tau}} \leq \| P^0(\phi)U^l(\phi) - P^0(\phi)U^{\infty}(\phi) \|_{B(\ell^2_0, \ell^2_{-2d}), \tilde{\tau}}
\]
\[
\leq \| P^0(\phi)U^l(\phi) - U^l \|_{B(\ell^2_0, \ell^2_{-2d}), \tilde{\tau}} \leq C\epsilon_0^2 \to 0 \quad \text{as} \ l \to \infty.
\]
\( \square \)

Lemma 2.30 \( U^lP^l(\phi) \to 0 \) in \( B(\ell^2_0, \ell^2_{-2d})(\Pi_\phi, \tilde{\tau}) \) and \( B(\ell^2_0, \ell^2_{-2d})(\Pi_\phi, \tilde{\tau}) \).

Proof. For any \( \phi \in \mathbb{T}_2^\phi \), by Lemma 2.18 and (v) of Lemma 2.2 we have
\[
\| U^l \|_{B(\ell^2_0, \ell^2_{-2d}), \tilde{\tau}} \leq 1 + C\| U^l - I \|_{B(\ell^2_0, \ell^2_{-2d}), \tilde{\tau}} \leq 1 + C\epsilon_0^2,
\]
then one obtains
\[
\| U^l(\phi)P^l(\phi) \|_{B(\ell^2_0, \ell^2_{-2d}), \tilde{\tau}} \leq \| U^l(\phi)P^l(\phi) \|_{B(\ell^2_0, \ell^2_{-2d}), \tilde{\tau}} \leq \| U^l \|_{B(\ell^2_0, \ell^2_{-2d}), \tilde{\tau}} \| P^l \|_{B(\ell^2_0, \ell^2_{-2d}), \tilde{\tau}} \leq C\epsilon_l \to 0 \quad \text{as} \ l \to \infty.
\]
\( \square \)

Lemma 2.31 For any \( l \geq 1 \), \( U^lA^l \in B(\ell^2_0, \ell^2_{-2d}) \).

Proof. When \( l = 0 \), we have \( U^0A^0 = A^0 \in B(\ell^2_0, \ell^2_{-2d}) \). Suppose \( U^lA^l \in B(\ell^2_0, \ell^2_{-2d}) \), we consider \( U^{l+1}A^{l+1} \). From \( U^{l+1}A^{l+1} = U^l e^{B^l+1} A^{l+1} \) and \( A^{l+1} = A^l + \text{diag}(P^l) \), it results in
\[
e^{B^l+1} A^{l+1} = e^{B^l+1} A^l + e^{B^l+1} \text{diag}(P^l).
\]
Besides,
\[
e^{B^l+1} A^l e^{-B^{l+1}} - A^l = [B^{l+1}, A^l] + \int_0^1 ds_1 \int_0^{s_1} e^{s_2 B^{l+1}} [B^{l+1}, A^l] e^{-s_2 B^{l+1}} ds_2 := \tilde{F}^l.
\]
and
\[
[A^l, B^{l+1}] - i\tilde{F}^{l+1} + P^l - \text{diag}(P^l) = 0. \tag{2.33}
\]
Thus, we have
\[
e^{B^l+1} A^l = A^l e^{B^l+1} + \tilde{F}^l e^{B^l+1},
\]
\[
U^{l+1}A^{l+1} = U^l e^{B^l+1} (A^l + \text{diag}(P^l)) = U^l e^{B^l+1} A^l + U^l e^{B^l+1} \text{diag}(P^l)
\]
\[
= U^l A^l e^{B^l+1} + U^l \tilde{F}^l e^{B^l+1} + U^{l+1} \text{diag}(P^l). \tag{2.34}
\]
From \( U^lA^l \in B(\ell^2_0, \ell^2_{-2d}) \), Lemma 2.2, we have \( U^lA^l e^{B^{l+1}} \in B(\ell^2_0, \ell^2_{-2d}) \). From (2.33) and assumption B3, we have \( [A^l, B^{l+1}] \in B(\ell^2_0, \ell^2_{-2d}) \). From \( B^{l+1} \in \mathcal{M}^+ \) and Lemma 2.2, we have \( \tilde{F}^l \in B(\ell^2_0, \ell^2_{-2d}) \). Note \( e^{B^{l+1}} \in B(\ell^2_0, \ell^2_{-2d}) \), we obtain \( \tilde{F}^l e^{B^{l+1}} \in B(\ell^2_0, \ell^2_{-2d}) \). From above, \( U^l \in B(\ell^2_{-2d}) \), it follows \( U^l \tilde{F}^l e^{B^{l+1}} \in B(\ell^2_0, \ell^2_{-2d}) \). Similarly, \( U^{l+1} \text{diag}(P^l) \in B(\ell^2_0, \ell^2_{-2d}) \). From above and (2.34), one can draw \( U^{l+1}A^{l+1} \in B(\ell^2_0, \ell^2_{-2d}) \).
\( \square \)

Remark 2.32 If we only assume B1, B2 without B3, one can’t prove \( [A^l, B^{l+1}] \in B(\ell^2_0, \ell^2_{-2d}) \) without any further assumption.

Lemma 2.33 For any \( l \geq 1 \), if \( \epsilon_0 \ll 1 \), then one has
\[
\| U^l A^l \|_{B(\ell^2_0, \ell^2_{-2d}), n_l} \leq C \prod_{i=0}^{l-1} (1 + \epsilon_i^{2}) + \sum_{i=0}^{l-1} \epsilon_i^{2}.
\]
Proof. From the prove of Lemma 2.12, one has \( \| \tilde{P} \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} < C\epsilon^2 \) (see (2.19)).

When \( l = 0 \), we have \( U^0 A^0 = A^0 \) and \( \| A^0 \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq C \). When \( l = 1 \), by (2.34) we have

\[
U^1 A^1 = A^0 e^{B^1} + \tilde{P}^0 e^{B^1} + e^{B^1} \text{diag}(P^0),
\]

where \( \| \tilde{P}^0 \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq C\epsilon^2 \) and \( \| \text{diag}(P^0) \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq \| P^0 \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq \epsilon_0 \). Combining (iv) of Lemma 2.2 with (ii) of Lemma 2.4, one has \( \| e^{B^1} \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq 1 + \| e^{B^1} - 1 \|_{\beta,s_0,s_1} \leq 2 \). Similarly, we have \( \| e^{B^1} \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq 2 \). Hence,

\[
\| U^1 A^1 \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq \| A^0 e^{B^1} \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} + \| \tilde{P}^0 e^{B^1} \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} + \| e^{B^1} \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \text{diag}(P^0) \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq C(1 + \epsilon^2_0) + \epsilon^2_0.
\]

By induction we finish the proof. \( \square \)

From (2.34) and Lemma 2.33, we have

\[
\| U^{l+1} A^{l+1} - U^l A^l \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq \| U^l A^l \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \| e^{B^1} - 1 \|_{\beta,s_0,s_1} + \| \tilde{P}^l e^{B^1} \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} + \| e^{B^1} \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \text{diag}(P^1) \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq C \epsilon^2_1 + 4 C \epsilon^2_1 + 2 \epsilon_1 \leq 6 C \epsilon^2_1.
\]

It follows \( \{ U^l A^l \} \) is a Cauchy sequence in \( \mathcal{B}(\ell^p_0,\ell^{p_2}_{-2})(\Pi_\star, \frac{2}{2}) \) and thus define \( U^\infty A^\infty := \lim_{l \to \infty} U^l A^l \) in the norm of \( \| \cdot \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \) below. It is clear that

\[
\| U^\infty A^\infty - U^l A^l \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq \sum_{l=1}^\infty \| U^{l+1} A^{l+1} - U^l A^l \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq C \epsilon^2_1.
\]

Thus we have the following.

**Lemma 2.34** When \( l \to \infty \), \( U^l A^l(\phi) \to U^\infty A^\infty(\phi) \) in \( \mathcal{B}(\ell^p_0,\ell^{p_2}_{-2})(\Pi_\star, \frac{2}{2}) \) and \( U^\infty A^\infty \) satisfies

\[
\| U^\infty A^\infty - A^0 \|_{\mathcal{B}(\ell^p_0,\ell^{p_2}_{-2}))} \leq C \epsilon^2_0
\]

for some positive constant \( C \).

From the construction we can prove the reducibility identity

\[
U^l(\omega t)A^l(\omega t) + P^l(\omega t) = -i \frac{d}{dt} U^l(\omega t) + (A^0 + P^0) U^l(\omega t) \quad \text{in } \mathcal{B}(\ell^p_0,\ell^{p_2}_{-2})(\Pi_\star).
\]

From the above lemmas and let \( l \to \infty \) in (2.35), one has

\[
U^\infty(\omega t)A^\infty(\omega t) = -i \frac{d}{dt} U^\infty(\omega t) + (A^0 + P^0(\omega t)) U^\infty(\omega t), \quad \omega \in \Pi_\star,
\]

where the identity holds in \( \mathcal{B}(\ell^p_0,\ell^{p_2}_{-2})) \).

Proof of Theorem 2.5. The measure estimate for \( \Pi_\star \) is similar as \([1]\). The estimate (2.7) is clear from Lemma 2.21 and the proof of Lemma 2.15. We only need to prove the equivalence of two relative equations.

If \( t \mapsto y(t) \in C^0(\mathbb{R},\ell^p_0) \cap C^1(\mathbb{R},\ell^{p_2}_{-2})) \) satisfies the equation (2.5), define \( x(t) = U^\infty(\omega t)y(t) \). By a straightforward computation, we have

\[
i\dot{x} = i\left( \frac{d}{dt} U^\infty(\omega t) \right) y(t) + U^\infty(\omega t) i y
\]

by (2.5) \( = i\left( \frac{d}{dt} U^\infty(\omega t) \right) y(t) + U^\infty(\omega t) A^\infty(\omega t) y(t)
\]

by (2.36) \( = (A^0 + P^0(\omega t)) U^\infty(\omega t) y(t)
\)
\[ (A^0 + P^0(\omega t))x(t). \]

From \( y(t) \in C^0(\mathbb{R}, \ell^2_0) \cap C^1(\mathbb{R}, \ell^2_{-2}) \), we can draw \( x(t) \in C^0(\mathbb{R}, \ell^2_0) \cap C^1(\mathbb{R}, \ell^2_{-2}) \). On the contrary, if \( C^0(\mathbb{R}, \ell^2_0) \cap C^1(\mathbb{R}, \ell^2_{-2}) \supseteq x(t) \) satisfies (2.4), we define \( y(t) = V^\infty(\omega t)x(t) \). By Lemma 2.24 one has \( y(t) \in C^0(\mathbb{R}, \ell^2_0) \cap C^1(\mathbb{R}, \ell^2_{-2}) \). Since \( V^\infty(\omega t)U^\infty(\omega t) = I \), it follows

\[
\frac{d}{dt}V^\infty(\omega t)U^\infty(\omega t) = -V^\infty(\omega t)\frac{d}{dt}U^\infty(\omega t),
\]

and

\[
iy(t) = i\left(\frac{d}{dt}V^\infty(\omega t)\right)x(t) + V^\infty(\omega t)i\dot{x}(t)
\]

by (2.4)

\[
= i\left(\frac{d}{dt}V^\infty(\omega t)\right)U^\infty(\omega t)y(t) + V^\infty(\omega t)(A^0 + P^0(\omega t))x(t)
\]

by (2.37)

\[
= -iV^\infty(\omega t)\left(\frac{d}{dt}U^\infty(\omega t)\right)y(t) + V^\infty(\omega t)(A^0 + P^0(\omega t))U^\infty(\omega t)y(t)
\]

by (2.36)

\[
= V^\infty(\omega t)U^\infty(\omega t)A^\infty(\omega t)y(t)
\]

\[
= A^\infty(\omega t)y(t).
\]

\[ \square \]

**Remark 2.35** Lemma 2.34 and 4.5 prove that \( U^\infty(\omega t)A^\infty(\omega t)y \) and \( V^\infty(\omega t)A^0x \) belong to \( \ell^2_{-2} \) if \( x, y \in \ell^2_0 \).

The proof of Corollary 2.7, see [1].

### 2.5. Proof of Main Theorems.

We first prove Theorem 1.2 based on Theorem 2.5. All the assumptions B1- B3 should be checked. Define \( H_0 = -\frac{d^2}{dx^2} + V(x) \). \( H_0 \) is self-adjoint in \( L^2(\mathbb{R}) \) and \( \text{spec}(H_0) \) is discrete, and all eigenvalues \( 0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \) are simple, and \( \lambda_j \sim cj^{\frac{2\beta}{\mu}} \) when \( j \to \infty \) and all eigenfunctions \( \{h_j(x)\}_{j \geq 1} \) form a complete basis in \( L^2 \). As [1], the equation (1.1) can be written as (2.4), where \( A = \text{diag}\{\lambda_1, \lambda_2, \cdots\} \) and \( P(\phi) = \langle P_j(\phi) \rangle_{j \geq 1} \) with \( P_j(\phi) = \int_{\mathbb{R}} \langle x \rangle^\nu W(\nu x, \phi)h_j(x)dx \) and \( P_j(\phi) = P_j(\phi) \) for \( \phi \in \mathbb{T}^n \). As [1] and [2], the assumption B1 is satisfied. In the following we will show the assumptions B2 - B3 are fulfilled for the equation (2.4).

**Lemma 2.36** The map \( P(\phi) \) is analytic from \( \mathbb{T}^n_\delta \) into \( M_\beta \), where \( s = \rho - \delta_0 \) with \( s > 0 \) and

\[
\beta = \left\{ \begin{array}{ll}
\frac{1}{2\ell + 1} & \frac{1}{2\ell + 1} \leq \mu < \ell - 1 + \frac{1}{2\ell + 1}, \\
0 & \mu < \frac{1}{2\ell + 1}.
\end{array} \right.
\]

*Proof.* We discuss the case when \( \frac{1}{2\ell + 1} \leq \mu < \ell - 1 + \frac{1}{2\ell + 1} \). In this case, if \( |\text{Im}\phi| < \rho - \delta_0 \), then

\[
|P_j(\phi)| = \left| \int_{\mathbb{R}} \langle x \rangle^\mu \sum_{k \in \mathbb{Z}^d} \langle k \rangle^\mu \sum_{l \in \mathbb{Z}^d} \mathcal{W}(k, l)e^{ik \cdot \nu x}e^{i\phi h_i(x)}h_j(x)dx \right|
\]

\[
\leq \sum_{l \in \mathbb{Z}^d} e^{ijkl(\rho - \delta_0)} \sum_{k \in \mathbb{Z}^d} \left| \mathcal{W}(k, l) \right| \left| \int_{\mathbb{R}} e^{ik \cdot \nu x} \langle x \rangle^\mu h_i(x)h_j(x)dx \right|
\]

\[
W(-\varphi, \phi) = -W(\varphi, \phi) \leq \sum_{l \in \mathbb{Z}^d} e^{ijkl(\rho - \delta_0)} \sum_{0 \neq k \in \mathbb{Z}^d} \left| \mathcal{W}(k, l) \right| \left| \int_{\mathbb{R}} e^{ik \cdot \nu x} \langle x \rangle^\mu h_i(x)h_j(x)dx \right|
\]

**Lemma 1.11** \( \leq C \sum_{l \in \mathbb{Z}^d} e^{ijkl(\rho - \delta_0)} \sum_{k \neq 0} \left| \mathcal{W}(k, l) \right| (|k \cdot \nu| \vee |k \cdot \nu|^{-1})(ij)^\beta \)
\[ A2 \leq C \frac{1}{\gamma} \sum_{l \in \mathbb{Z}^n} e^{\|l\|_2} \sum_{k \neq 0} |\hat{W}(k, l)| |k|^{1/\tau_1} (ij)^{\beta} \]

\[ A3 \leq C \frac{1}{\gamma} \sum_{l \in \mathbb{Z}^n} e^{\|l\|_2} \sum_{k \neq 0} e^{-\|l\|_2} |k|^{1/\tau_1} (ij)^{\beta} \]

\[ \exists i_0 \in \{1, \ldots, d\}, \ |k_{i_0}| \geq \frac{|k|}{d} \leq C \frac{1}{\gamma} (ij)^{\beta}. \]

It follows \( P(\phi) \) is an analytic map from \( T^m \) into \( M_\beta \) with \( 0 < 2\beta < \frac{1}{1+1} \). The rest is similar. \( \square \)

**Remark 2.37** From Assumption A1 we can show that when \( |x| \geq R_0 > 0 \) large enough, Assumption 1.1 in Lemma 1.11 is satisfied for the potential \( V(x) \) and thus, we can apply it in the above proof. From Lemma 2.36 we prove that B2 is satisfied for the equation (2.4).

In the following we will show that B3 is satisfied. Following [2, 4], we have

**Lemma 2.38** Let \( g \in S^{m_1, m_2} \), then one has

\[ g^w(x, -i\partial_x) \in \mathcal{B}(\mathcal{H}^{s+1}, \mathcal{H}^\gamma), \ \forall s \in \mathbb{R}, \ \forall s_1 \geq m_1 + |m_2| \ \text{with} \ |m_2| := m_2 \lor 0. \]

**Definition 2.39** An operator \( G \) will be said to be pseudodifferential of class \( OPS^{m_1, m_2} \) if there exists a symbol \( g \in S^{m_1, m_2} \) such that \( G = g^w(x, -i\partial_x) \).

**Definition 2.40** An operator \( F \) will be said to be a pseudodifferential operator of class \( O^{m_1, m_2} \) if there exists a sequence \( f \in S^{m_1, m_2} \) with \( m_1 + |m_2| \leq m_1 + |m_2| \) and, for any \( \alpha \) there exist \( N \) and an operator \( R_N \in \mathcal{B}(\mathcal{H}^{s-\frac{\alpha}{2}}, \mathcal{H}^{s}) \) for any \( s \) such that \( F = \sum_{j=1}^{N} f_j^w + R_N. \)

**Lemma 2.41** If \( a(x, \xi) \in S^{m_1, m_2} \) and \( b(x, \xi) \in S^{m_1', m_2'} \), then \( a \cdot b \in S^{m_1 + m_1', m_2 + m_2'}. \)

Given a symbol \( g \in S^{m_1, m_2} \) we will write

\[ g \sim \sum_{j \geq 0} g_j, \ g_j \in S^{m_1, m_2}, \ m_1 + |m_2| \leq m_1 + |m_2| \]

if \( \forall \alpha \) there exist \( N \) and \( r_N \in S^{-\alpha, 0} \) such that \( g = \sum_{j=0}^{N} g_j + r_N. \) The following lemma is from [3].

**Lemma 2.42** Given a couple of symbols \( a \in S^{m_1, m_2} \) and \( b \in S^{m_1', m_2'} \), then there exists a symbol \( c \), denoted by \( c = a \# b \) such that

\[ (a \# b)^w(x, D_x) = a^w(x, D_x) b^w(x, D_x), \]

furthermore one has \( a \# b \sim \sum_{j \geq 0} c_j \) with

\[ c_j = \sum_{k_1 + k_2 = j} \frac{1}{k_1! k_2!} \frac{k_1}{2} (-\frac{1}{2}) k_2 (\partial_\xi^{k_1} D_x^{k_2} a)(\partial_\xi^{k_2} D_x^{k_1} b) \in S^{m_1 + m_1', jl, m_2 + m_2', j} \]

where \( D_x = -i\partial_x. \)

From Lemma 2.42, we have

**Corollary 2.43** If \( A \in OPS^{m_1, m_2}, \ B \in OPS^{m_1', m_2'}, \) then \( AB \in OPS^{m_1 + m_1', m_2 + m_2'}. \)

**Lemma 2.44** If \( 0 \leq \mu \leq \delta(\ell + 1), \) then \( (\lambda^w(x, D_x))^{-\delta(\ell + 1)} \langle x \rangle^\mu \in \mathcal{B}(\mathcal{H}^\delta). \)

**Proof.** It is clear that

\[ (\lambda^w(x, D_x))^{-\delta(\ell + 1)} \in \mathcal{O}^{-\delta(\ell + 1), 0}. \]
From the definition, there exists $N$ and an operator $R_N \in \mathcal{B}(\mathcal{H}^{-\mu}, \mathcal{H}^0)$ and $(\lambda^w(x, D_x))^{-\delta(\ell + 1)} = \sum_{j=0}^{N} f_j^w + R_N$. From (2.38) we have $\sum_{j=0}^{N} f_j^w \in OPS^{-\delta(\ell + 1)}_\mu$. On the other hand, $(\langle x \rangle^\mu) \in OPS^{\mu, 0}$ since $\mu \geq 0$(see [4]). From Lemma 2.38, it follows $(\langle x \rangle^\mu) \in \mathcal{B}(\mathcal{H}^0, \mathcal{H}^{-\mu})$ and thus $\left( \sum_{j=0}^{N} f_j^w \right) \cdot (\langle x \rangle^\mu) \in OPS^{\mu - \delta(\ell + 1), 0}_\mu \subset OPS^{0, 0}$ by $\mu \leq \delta(\ell + 1)$. Therefore, $\left( \sum_{j=0}^{N} f_j^w \right) \cdot (\langle x \rangle^\mu) \in \mathcal{B}(\mathcal{H}^0)$. For the second part it is easy to check that $R_N(\langle x \rangle^\mu) \in \mathcal{B}(\mathcal{H}^0)$. Combining with the two parts we finish the proof. □

**Lemma 2.45** If $0 \leq \mu \leq \delta(\ell + 1)$, then the multiplication operator $(\langle x \rangle^\mu) \in \mathcal{B}(\mathcal{H}^0, \mathcal{H}^{-\delta})$.

**Proof.** From the self-adjointness, Hölder inequality and Lemma 2.44 one has

$$\|\langle x \rangle^\mu\|_{\mathcal{B}(\mathcal{H}^{\delta}, \mathcal{H}^{-\delta})} = \sup_{\|f\|_{\mathcal{H}^0} = 1} \|\langle x \rangle^\mu f\|_{\mathcal{H}^{-\delta}} = \sup_{\|\langle x \rangle^\mu f\|_{\mathcal{H}^0} = 1} \|\langle x \rangle^\mu f\|_{\mathcal{H}^0} \leq \sup_{\|\langle x \rangle^\mu f\|_{\mathcal{H}^0} = 1} \|\langle x \rangle^\mu f\|_{\mathcal{H}^0} \leq \sup_{\|\langle x \rangle^\mu f\|_{\mathcal{H}^0} = 1} \|\langle x \rangle^\mu f\|_{\mathcal{H}^0} < \infty.$$ 

□

**Lemma 2.46** Suppose that $g(x, \phi)$ is continuous on $x \in \mathbb{R}$ and analytic on $\phi \in \mathbb{T}^n_\delta$ and there exists a positive constant $C > 0$ such that $|g(x, \phi)| \leq C$ on $\mathbb{R} \times \mathbb{T}^n_\delta$, then if $0 \leq \mu \leq \delta(\ell + 1)$, for any $\phi \in \mathbb{T}^n_\delta$, $(\langle x \rangle^\mu)g(x, \phi)$ is an analytic map from $\mathbb{T}^n_\delta$ to $\mathcal{B}(\mathcal{H}^0, \mathcal{H}^{-\delta})$. On the other hand, if $\mu < 0$, then $(\langle x \rangle^\mu)g(x, \phi)$ is an analytic map from $\mathbb{T}^n_\delta$ to $\mathcal{B}(\mathcal{H}^0)$.

**Proof.** From the boundedness of $g(x, \phi)$ on $\phi \in \mathbb{T}^n_\delta$ and the definition, we can draw that the multiplication operator $g(x, \phi) \in \mathcal{B}(\mathcal{H}^0)$ on $\mathbb{T}^n_\delta$. Together with Lemma 2.45 and $0 \leq \mu \leq \delta(\ell + 1)$, one has the multiplication operator $(\langle x \rangle^\mu)g(x, \phi) \in \mathcal{B}(\mathcal{H}^0, \mathcal{H}^{-\delta})$ for $\phi \in \mathbb{T}^n_\delta$. The rest is clear. □

From Lemma 2.46 it follows the map $\mathbb{T}^n_\delta \ni \phi \mapsto P(\phi) \in \mathcal{B}(\mathcal{L}^2_0, \mathcal{L}^{2\delta}_{23})$ is analytic on $\mathbb{T}^n_\delta$ if $0 \leq \mu \leq \delta(\ell + 1)$. As above we discuss two cases. When $\frac{1}{2\ell+1} \leq \mu < \ell - \frac{1}{2\ell+1}$, one has $\beta = \frac{\mu}{2(\ell + 1)} - \frac{1}{2(\ell + 1)}$. If choose $\delta = \frac{\ell}{\ell+1}$, we have $\mu \leq \delta(\ell + 1)$, $0 \leq \delta < \frac{2\ell}{\ell+1} - \frac{1}{2}$ and $2\beta \leq \delta$. When $0 \leq \mu < \frac{1}{2\ell+1}$, one has $\beta = 0$ and $\delta = \frac{\ell}{\ell+1}$. If $\mu < 0$, set $\beta = \delta = 0$. This confirms the assumption B3.

Proof of Theorem 1.2. As we mentioned above, the equation (1.1) can be written as (2.4). Since all the assumptions B1 - B3 are checked, we can use Theorem 2.5 and Corollary 2.7 to finish the proof. For details, see [1].

Proof of Theorem 1.7. It is similar.

Proof of Corollary 1.8. As above $P_i^x(\phi) = \int_{\mathbb{R}} \langle x \rangle^\mu g(x, \phi)h_i(x)h_j(x)dx$. Clearly, for all $\phi \in \mathbb{T}^n_\mu$, there exists some positive constant $C$ such that $|\langle x \rangle^\mu g(x, \phi)| \leq C(\mu)|x|^\mu$ is satisfied for $|x| \geq 1$ and $\mu \geq 0$. By Lemma 1.14 one has

$$|P_i^x(\phi)| \leq C \int_{\mathbb{R}} \langle x \rangle^\mu g(x, \phi)\overline{h_i(x)h_j(x)}dx \leq C(\mu)\overline{|x|^\mu}, \quad 0 \leq \mu \leq \ell - 1.$$
Thus, we have $0 \leq 2\beta = \frac{\mu}{\ell + 1} < \frac{\ell}{\ell + 1}$. By Lemma 2.46, B3 is satisfied if we choose $\delta = \frac{\mu}{\ell + 1}$. The following is similar as above.

If $\mu < 0$, we set $\beta = 0$ and $\delta = 0$. The rest is similar. \hfill \Box

3. Estimates on eigenfunctions

In this section we will prove Lemma 1.11 and 1.14 based on Langer’s turning point method and oscillatory integrals. For the proof the rough idea is that we first rewrite the eigenfunction into the sum of two different functions, and then use Lemma 4.11 to estimate the relative integrals, if necessary.

3.1. Langer’s turning point and the new form of the eigenfunctions. Consider the function

$$h''_n(x) + (\lambda_n - V(x))h_n(x) = 0, \quad x \geq 0,$$

where $V(x)$ satisfies Assumption 1.1. From Lemma 4.7 there exists a positive constant $R \geq 2\bar{R} \geq 2R_0$ such that the following conditions are satisfied:

(i). $V(x) \leq xV'(x)$, for $x \in \left[\frac{R}{2}, \infty\right)$,
(ii). $|V(x)| < V(R)$, for $x \in [0, R)$.

Let $n_0 := \min \{n \in \mathbb{Z}_+ | \lambda_n \geq V(R) \}$ and $\lambda_n = V(X_n)$ for $n \geq n_0$. From the above, $X_n$ is unique when $n \geq n_0$ as the figure 1 below.

![Figure 1. Potential Function](image)

**Lemma 3.1** For $n \geq n_0$, if $x \geq 0$, we have

$$h_n(x) = \psi_1^{(n)}(x) + \psi_2^{(n)}(x) \quad \text{with} \quad |\psi_2^{(n)}(x)| \leq \frac{C}{X_n-\pi} |\psi_1^{(n)}(x)|,$$

where

$$\psi_1^{(n)}(x) := C_n(\lambda_n - V(x))^{-\frac{1}{4}} \left(\frac{\pi \zeta_n}{2}\right)^{\frac{1}{8}} H^{(1)}_{\frac{1}{8}}(\zeta_n)$$

and $C_n \sim \frac{X_n}{\zeta_n}$, $\zeta_n(x) = \int_{X_n}^{x} (\lambda_n - V(t))^{\frac{1}{2}} dt$ with $\arg \zeta_n(x) = \begin{cases} \frac{\pi}{2}, & x > X_n, \\ -\pi, & x < X_n. \end{cases}$
For the proof see section 4.

Lemma 3.2 For $1 \leq n < n_0$, if $x \geq 2R$, we have
\[ h_n(x) = \psi_1^{(n)}(x) + \psi_2^{(n)}(x) \quad \text{with} \quad |\psi_2^{(n)}(x)| \leq \frac{C}{xV^2(x)}|\psi_1^{(n)}(x)|, \quad (3.6) \]
where $\zeta_n(x) = \int_{X_n}^{x} (\lambda_n - V(t)) \frac{x}{t} dt$ with $	ext{arg} \zeta_n(x) = \frac{\pi}{2}$ and $X_n = R$.

3.2. Proof of Lemma 1.11: Part 1.

3.2.1. the integral on $|0, +\infty)$ in the three cases. In the following lengthy proof we first estimate the integral on $[0, +\infty)$ under Assumption 1.2. The rest integral estimation on $(-\infty, 0)$ can be obtained by a coordinate transformation. In this way we prove the estimate (1.7) of Lemma 1.11.

Corollary 3.3 Given $n \in \mathbb{Z}_+$, $h_n(x)$ is bounded on $[0, \infty)$.

Proof. If $n \geq n_0$, by (3.5), (3.4) and Lemma 4.10 we obtain $|h_n(x)| \leq C$ for $x \geq 0$. If $1 \leq n < n_0$ and $x \in [0, 2R]$ one has $|h_n(x)| \leq C$. For $x \geq 2R$ one also has $|h_n(x)| \leq C$ by (3.5), (3.6) and Lemma 4.10. □

As in [36], we have

Lemma 3.4 Assume $V(x)$ satisfies Assumption 1.1, then there exists constants $a_1, a_2, A_1, A_2$ such that the following estimates are satisfied uniformly for $n \geq n_0$:
\[ a_1 X_n^{2\ell-1}(X_n - x) \leq \lambda_n - V(x) \leq a_2 X_n^{2\ell-1}(X_n - x), \quad \text{for} \quad 0 \leq x < X_n, \]
\[ V(x) - \lambda_n \geq a_1 X_n^{2\ell-1}(x - X_n), \quad \text{for} \quad x \geq X_n \quad (3.7) \]
and
\[ A_1 X_n^{\ell-\frac{1}{2}}(X_n - x)^2 \leq -i\zeta_n(x) \leq A_2 X_n^{\ell-\frac{1}{2}}(X_n - x)^2, \quad \text{for} \quad 0 \leq x < X_n, \]
\[ -i\zeta_n(x) \geq A_1 X_n^{\ell-\frac{1}{2}}(x - X_n)^2, \quad \text{for} \quad x \geq X_n, \quad (3.8) \]
where $0 < a_1 \leq 1 \leq a_2$ and $0 < A_1 \leq 1 \leq A_2$.

For the proof see section 4.

For the integral on $[0, +\infty)$, we have the following lemma.

Lemma 3.5 If $f(x)$ satisfies Assumption 1.2, then
\[ \left| \int_0^{+\infty} f(x)e^{ikx}h_m(x)\overline{h_n(x)}dx \right| \leq C(|k|^{-1} \vee |k|)(\lambda_m\lambda_n)^{\frac{\ell}{2}-\frac{1}{4}+\frac{1}{4}} , \quad \forall \, k \neq 0, \]
where $C$ only depends on $(\mu, \ell)$.

Define $n_1 := \min \{ n > n_0 : X_n^\frac{1}{2} \geq 2X_{n_0} \}$. Assume $m \leq n$ in the following. We now prepare to prove Lemma 3.5 in three different cases, which are $m, n < n_1$, $m < n_0$ and $n \geq n_1$ and $m, n \geq n_0$. For the first case we have

Lemma 3.6 If $f(x)$ satisfies Assumption 1.2, then one has
\[ \left| \int_0^{+\infty} f(x)e^{ikx}h_m(x)\overline{h_n(x)}dx \right| \leq C(X_mX_n)^{\frac{\ell}{2}-\frac{1}{4}}, \quad \forall \, k \in \mathbb{R}, \]
where $C$ only depends on $(\mu, \ell)$ and $1 \leq m \leq n < n_1$. 

Proof. By Lemma 3.1 and 3.2, for $x \geq 2X_n$, there exists a positive constant $C$ such that

$$h_n(x) = \psi_1^{(n)}(x) + \psi_2^{(n)}(x) \quad \text{with} \quad |\psi_2^{(n)}(x)| \leq C|\psi_1^{(n)}(x)|.$$ 

When $x \geq 2X_n$, there exists a $C_0 > 0$ such that $V(x) - \lambda_n \geq C_0$ and $|\zeta_n(x)| \geq C_0(x - X_n)$, then

$$\left| \int_{2X_n}^{+\infty} f(x) e^{ikx} h_n(x) h_n(x) dx \right| \leq C \int_{2X_n}^{\infty} x^\mu (V(x) - \lambda_n)^{-\frac{1}{4}} e^{-\left(\frac{1}{2} |\zeta_n(x)| + |\zeta_n(x)|\right)} dx \leq C \int_{2X_n}^{\infty} (x - X_n)^{\mu} e^{-C_0(x - X_n)} dx \leq C(X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}.$$ 

Besides, by Hölder inequality we have

$$\left| \int_{0}^{2X_n} f(x) e^{ikx} h_n(x) h_n(x) dx \right| \leq C(X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}. \quad \square$$

For the second case we have

**Lemma 3.7** If $f(x)$ satisfies Assumption 1.2, then

$$\left| \int_{0}^{+\infty} f(x) e^{ikx} h_m(x) h_n(x) dx \right| \leq C(X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}, \quad \forall k \in \mathbb{R},$$

where $C$ only depends on $(\mu, \ell)$ and $1 \leq m < n_0$, $n \geq n_1$.

Proof. By Corollary 3.3 one has $h_m(x)$ are bounded in $[0, \infty)$ uniformly for $m < n_0$, then

$$\left| \int_{0}^{X_n^\frac{1}{2}} f(x) e^{ikx} h_m(x) h_n(x) dx \right| \leq C X_n^{\frac{\mu}{2} + \frac{1}{4}} \int_{0}^{X_n^\frac{1}{2}} (\lambda_n - V(x))^{-\frac{1}{4}} dx \leq C(X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}.$$ 

Note that $X_n^\frac{1}{2} \geq 2X_n$ from $n \geq n_1$. For any $m \leq n_0$, there exists a $C_0 > 0$ such that

$$V(x) - \lambda_m \geq C_0, \quad |\zeta_m(x)| \geq C_0(x - X_m) \geq \frac{C_0}{2} x, \quad \forall x \geq X_n^\frac{1}{2}.$$ 

By Hölder inequality we obtain

$$\left| \int_{X_n^\frac{1}{2}}^{+\infty} f(x) e^{ikx} h_m(x) h_n(x) dx \right| \leq C \left( \int_{X_n^\frac{1}{2}}^{\infty} x^{2\mu} h_m^2(x) dx \right)^{\frac{1}{2}} \leq C \left( \int_{X_n^\frac{1}{2}}^{\infty} (x - X_m)^{2\mu} e^{-2C_0(x - X_m)} dx \right)^{\frac{1}{2}} \leq C e^{-\frac{C_0}{4} X_n^\frac{1}{2}} \leq C(X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}. \quad \square$$

For the third case, we have

**Lemma 3.8** Assume $f(x)$ satisfies Assumption 1.2, then one has

$$\left| \int_{0}^{+\infty} f(x) e^{ikx} h_m(x) h_n(x) dx \right| \leq C(|k|^{-1} \vee |k|)(X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}} \frac{1}{|k|}, \quad \forall k \neq 0,$$

where $C$ only depends on $(\mu, \ell)$ and $n_0 \leq m \leq n$.

To prove above Lemma 3.8, we split the integral into two parts which is delayed in the following.
3.2.2. the Integral on \([X_n, +\infty)\). For the following part in this section, we will denote \(F(x) := f(x)e^{ikx}\psi_1^{(m)}(x)\psi_1^{(n)}(x)\) for simplicity. Our main result in this part is the following.

**Lemma 3.9** If \(f(x)\) satisfies Assumption 1.2, then

\[
\left| \int_{X_n}^{+\infty} f(x)e^{ikx}h_m(x)\overline{h_n(x)}dx \right| \leq C(X_mX_n)^{\frac{1}{2} - \frac{k}{2}}, \quad \forall \, k \in \mathbb{R},
\]

where \(C\) only depends on \((\mu, \ell)\) and \(n_0 \leq m \leq n\).

**Lemma 3.9** is the direct corollary of the following two lemmas.

**Lemma 3.10** If \(f(x)\) satisfies Assumption 1.2, then

\[
\left| \int_{2X_n}^{+\infty} f(x)e^{ikx}h_m(x)\overline{h_n(x)}dx \right| \leq Ce^{-c_0X_n^{\ell+1}}, \quad \forall \, k \in \mathbb{R},
\]

where \(C_0\), \(C\) only depend on \((\mu, \ell)\) and \(n_0 \leq m \leq n\).

**Proof.** Recall that \(\psi_1^{(n)}(x) = C_n(\lambda_n - V(x))^{-\frac{1}{2}}(\zeta_n)H_1^{(1)}(\zeta_n)\) with \(C_n \sim X_n^{-\frac{1}{2}}\).

From Lemma 3.4, for any \(n \geq n_0\), if \(x \geq 2X_n\), we have \(V(x) - \lambda_n \geq a_1X_n^{2\ell}\) and \(|\zeta_n(x)| \geq A_1X_n^{\ell+1}\) and thus

\[
\left| \int_{2X_n}^{+\infty} F(x)dx \right| \leq CX_n^{\ell-1}X_n^{-\frac{1}{2}}\int_{2X_n}^{+\infty} x^{\mu}(V(x) - \lambda_n)^{-\frac{1}{2}}(V(x) - \lambda_n)^{-\frac{1}{2}}e^{-|\zeta_n(x)|}e^{-|\zeta_n(x)|}dx
\]

\[
\leq CX_n^{\ell-1}\int_{2X_n}^{+\infty} x^{\mu}e^{-\frac{2}{X_n}(x-X_n)}dx
\]

\[
\leq Ce^{-\frac{4}{X_n}X_n^{\ell+1}}(x-X_n)^{\mu}e^{-c_0(x-X_n)}dx \leq Ce^{-c_0X_n^{\ell+1}},
\]

where \(C_0 > 0\) only depends on \((\mu, \ell)\). Note that \(h_n(x) = \psi_1^{(n)}(x) + O(X_n^{-\ell+1})\psi_1^{(n)}(x)\), then we obtain

\[
\left| \int_{2X_n}^{+\infty} f(x)e^{ikx}h_m(x)\overline{h_n(x)}dx \right| \leq Ce^{-c_0X_n^{\ell+1}}.
\]

**Lemma 3.11** If \(f(x)\) satisfies Assumption 1.2, then

\[
\left| \int_{X_n}^{2X_n} f(x)e^{ikx}h_m(x)\overline{h_n(x)}dx \right| \leq C(X_mX_n)^{\frac{1}{2} - \frac{k}{2}}, \quad \forall \, k \in \mathbb{R},
\]

where \(C\) only depends on \(\mu, \ell\) and \(n_0 \leq m \leq n\).

**Proof.** Since \(X_n \geq 1\) from \(n \geq n_0\), then \(X_n + X_n^{-\frac{1}{2}} \leq 2X_n\). We split the integral into two parts as:

\[
\int_{X_n}^{2X_n} f(x)e^{ikx}h_m(x)\overline{h_n(x)}dx = \left( \int_{X_n}^{X_n + X_n^{-\frac{1}{2}}} + \int_{X_n + X_n^{-\frac{1}{2}}}^{2X_n} \right) dx.
\]

From Lemma 3.4, if \(x \geq X_n + X_n^{-\frac{1}{2}}\), we have

\[
V(x) - \lambda_n \geq a_1X_n^{2\ell-1}(x-X_n), \quad |\zeta_n(x)| \geq A_1X_n^{\ell+\frac{1}{2}}(x-X_n)^{\frac{1}{2}} \geq A_1X_n^{\ell+1}.
\]
Then
\[ \left| \int_{X_n \mp X_n \frac{\ell}{2}}^{2X_n} F(x) \, dx \right| \leq C X_m^{\ell-1} X_n^{\ell-1} \int_{X_n \mp X_n \frac{\ell}{2}}^{2X_n} x^\mu (V(x) - \lambda_m)^{-\frac{\mu}{2}} (V(x) - \lambda_n)^{-\frac{\ell}{2}} e^{-|\zeta_n(x)|} \, dx \]
\[ \leq C X_m^{\mu+\ell-1} \int_{X_n \mp X_n \frac{\ell}{2}}^{2X_n} (V(x) - \lambda_n)^{-\frac{\mu}{2}} e^{-|\zeta_n(x)|} \, dx \]
\[ \leq C e^{-A_1 X_m^{\ell-1}} X_n^\mu \int_{X_n \mp X_n \frac{\ell}{2}}^{2X_n} (x - X_n)^{-\frac{\mu}{2}} \, dx \]
\[ \leq C e^{-A_1 X_m^{\ell-1}} X_n^\mu \leq C e^{-C_0 X_n^{\ell-1}}, \]

where \( C_0 \) only depends on \((\mu, \ell)\). Then we estimate the remainder integral under two cases. If \( X_m \leq X_n < 2X_m \), we have
\[ \left| \int_{X_n \mp X_n \frac{\ell}{2}}^{2X_n} F(x) \, dx \right| \leq C X_m^{\ell-1} X_n^{\ell-1} \int_{X_n \mp X_n \frac{\ell}{2}}^{2X_n} (V(x) - \lambda_n)^{-\frac{\mu}{2}} (V(x) - \lambda_n)^{-\frac{\ell}{2}} \, dx \]
\[ \leq C X_m^{\mu+\ell-1} \int_{X_n \mp X_n \frac{\ell}{2}}^{2X_n} (V(x) - \lambda_n)^{-\frac{\mu}{2}} \, dx \]
\[ \leq C X_m^\mu \leq C (X_m X_n)^{\frac{\mu}{2} - \frac{\ell}{2}}. \]

Otherwise, \( X_n \geq 2X_m \). From the Lemma 3.4, for any \( m \geq n_0 \), if \( x \geq X_n \), we have
\[ V(x) - \lambda_m \geq a_1 X_m^{2\ell-1} (x - X_m) \geq a_1 X_m^{2\ell}, \quad |\zeta_n(x)| \geq A_1 X_m^{\ell-1} (x - X_m) \geq A_1 X_m^{\ell}(x - X_m). \]

It follows
\[ \left| \int_{X_n \mp X_n \frac{\ell}{2}}^{2X_n} F(x) \, dx \right| \leq C X_m^{\ell-1} X_n^{\ell-1} \int_{X_n \mp X_n \frac{\ell}{2}}^{2X_n} (x - X_m)^{\mu} (V(x) - \lambda_n)^{-\frac{\mu}{2}} (V(x) - \lambda_n)^{-\frac{\ell}{2}} e^{-|\zeta_n(x)|} \, dx \]
\[ \leq C X_m^{\mu+\ell-1} \int_{X_n \mp X_n \frac{\ell}{2}}^{2X_n} (x - X_m)^{\mu} (x - X_n)^{-\frac{\mu}{2}} e^{-A_1 X_m^{\ell}(x - X_m)} \, dx \]
\[ \leq C X_m^{\mu} X_n^{-\frac{\mu}{2}} \int_{X_n}^{X_n \pm X_n \frac{\ell}{2}} (x - X_n)^{-\frac{\mu}{2}} \, dx \leq C (X_m X_n)^{\frac{\mu}{2} - \frac{\ell}{2}}. \]

Note \( h_n(x) = \psi_1^{(n)}(x) + O(X_m^{-(\ell+1)} \psi_1^{(n)}(x)) \), we obtain
\[ \left| \int_{X_n}^{2X_n} f(x) e^{ikx} h_n(x) \overline{h_n(x)} \, dx \right| \leq C (X_m X_n)^{\frac{\mu}{2} - \frac{\ell}{2}}. \]

\( \square \)

3.3. Proof of Lemma 1.11: Part 2.

3.3.1. the Integral on \([0, X_n)\): preparations. In this part we will prove the following.

**Lemma 3.12** If \( f(x) \) satisfies Assumption 1.2, then
\[ \left| \int_0^{X_n} f(x) e^{ikx} h_n(x) \overline{h_n(x)} \, dx \right| \leq C (|k|^{-1} \vee |k|) (X_m X_n)^{\frac{\mu}{2} - \frac{\ell + 1}{2}}, \quad \forall k \neq 0, \]

where \( C \) only depends on \((\mu, \ell)\) and \( n_0 \leq m \leq n \).
For the simplicity we will introduce the following notations. For \( m \geq n_0 \) we denote \( f_m(x) := \int_{0}^{\infty} e^{-t - \frac{1}{4} x} \left( 1 + \frac{it}{2\zeta_m} \right)^{-\frac{1}{2}} dt \). Then for \( x \in [0, X_m) \), one has
\[
\psi^{(m)}_1(x) = C_m(\lambda_m - V(x))^{-\frac{1}{4}} \left( \frac{\pi \zeta_m}{2} \right)^{\frac{1}{2}} H^{(1)}_{\frac{1}{2}}(\zeta_m)
= C_m(\lambda_m - V(x))^{-\frac{1}{4}} \left( \frac{\pi \zeta_m}{2} \right)^{-\frac{1}{2}} \int_{0}^{\infty} e^{-t} \left( 1 + \frac{it}{2\zeta_m} \right)^{-\frac{1}{2}} dt
= C_m(\lambda_m - V(x))^{-\frac{1}{4}} e^{i\zeta_m(x)} f_m(x),
\]
where \( C_m \sim X_m^{\frac{1}{2}} \). Similarly for \( x \in [0, X_m) \) we have \( \psi^{(m)}_1(x) = C_n(\lambda_n - V(x))^{-\frac{1}{4}} e^{-i\zeta_n(x)} f_n(x) \) with \( C_n \sim X_n^{\frac{1}{2}} \). For \( x \in [0, X_m) \), denote
\[
g(x) := (\zeta_n(x) - \zeta_m(x) - kx)' = (\lambda_m - V(x))^{-\frac{1}{2}} - (\lambda_m - V(x))^{-\frac{1}{2}} - k,
\]
\[
\Psi(x) := (\lambda_m - V(x))^{-\frac{1}{2}} (\lambda_n - V(x))^{-\frac{1}{2}} f_m(x) f_n(x).
\]
By a straightforward computation we have \( g'(x) = \frac{V'(x)}{2} ((\lambda_m - V(x))^{-\frac{1}{2}} - (\lambda_n - V(x))^{-\frac{1}{2}}) \) and
\[
\Psi'(x) = \frac{1}{4} V'(x) (\lambda_m - V(x))^{-\frac{1}{2}} (\lambda_n - V(x))^{-\frac{1}{2}} f_m(x) f_n(x)
+ \frac{1}{4} V'(x) (\lambda_m - V(x))^{-\frac{1}{2}} (\lambda_n - V(x))^{-\frac{1}{2}} f_m(x) f_n(x)
+ (\lambda_m - V(x))^{-\frac{1}{2}} (\lambda_n - V(x))^{-\frac{1}{2}} \left( f'_m(x) f_n(x) + f_m(x) f'_n(x) \right).
\]
From \( x \in [0, X_m) \) it is easy to obtain \( |f_m(x)| \leq \Gamma(\frac{3}{2}) \) and \( |f_n(x)| \leq \Gamma(\frac{3}{2}) \). From Lemma 3.4 we have the following.

**Corollary 3.13** For \( x \in [0, X_m) \) and \( n_0 \leq m \leq n \), one has
\[
|\Psi(x)| \leq C(\lambda_m - V(x))^{-\frac{1}{2}} \leq CX_m^{-\ell + \frac{1}{2}} (X_m - x)^{-\frac{1}{2}}
\]
and
\[
|\Psi'(x)| \leq C(J_1 + J_2 + J_3 + J_4) \leq C(J_1 + J_3),
\]
where
\[
J_1 := \langle x \rangle^{2\ell - 1} (\lambda_m - V(x))^{-\frac{1}{2}} (\lambda_n - V(x))^{-\frac{1}{2}},
J_2 := \langle x \rangle^{2\ell - 1} (\lambda_m - V(x))^{-\frac{1}{2}} (\lambda_n - V(x))^{-\frac{1}{2}},
J_3 := \frac{(\lambda_m - V(x))^{-\frac{1}{2}} (\lambda_n - V(x))^{-\frac{1}{2}}}{X_m^{-\ell} (X_m - x)^3},
J_4 := \frac{(\lambda_m - V(x))^{-\frac{1}{2}} (\lambda_n - V(x))^{-\frac{1}{2}}}{X_n^{2\ell} (X_n - x)^3}.
\]

Lemma 3.12 is a direct corollary of Lemma 3.14 and 3.18 in which we suppose \( X_n > 4X_m \) and \( X_m \leq X_n \leq 4X_m \) respectively.

3.3.2. the Integral on \([0, X_n] \) when \( X_n > 4X_m \). In this case, from (3.2) and Lemma 4.7 we have
\[
\lambda_m = V(X_m) \leq V(\frac{1}{4} X_n) \leq \frac{1}{4} V(X_n) = \frac{1}{4} \lambda_n,
\]
for \( m \geq n_0 \).

**Lemma 3.14** If \( f(x) \) satisfies Assumption 1.2 and \( X_n > 4X_m \), then
\[
\left| \int_{0}^{X_n} f(x) e^{ikx} h_m(x) h_n(x) dx \right| \leq C(|k|^{\frac{1}{2}} + 1) (X_m X_n)^{\frac{1}{2} - \frac{1}{2}}, \quad \forall k \neq 0,
\]
where \( C \) only depend on \( (\mu, \ell) \) and \( n_0 \leq m \leq n \).
Lemma 3.14 is a direct corollary from Lemma 3.15, 3.16 and 3.17.

Lemma 3.15  If \( f(x) \) satisfies Assumption 1.2 and \( X_n > 4X_m, \) then
\[
\left| \int_0^{X_m - X_m^{\frac{1}{2}}} f(x)e^{ikx}h_m(x)\overline{h_n(x)}dx \right| \leq C(|k|^\frac{1}{2} + 1)(X_mX_n)^{-\frac{1}{2}}, \quad \forall \ k \neq 0,
\]
where \( C \) only depends on \((\mu, \ell)\) and \( n_0 \leq m \leq n. \)

Proof. Clearly, we have
\[
\int_0^{X_m - X_m^{\frac{1}{2}}} F(x)dx = C_mC_n \int_0^{X_m - X_m^{\frac{1}{2}}} f(x)e^{i(\zeta_m - \zeta_n + kx)}\Psi(x)dx,
\]
where \( C_m \sim X_m^{\frac{1}{2}}, \) \( C_n \sim X_n^{\frac{1}{2}}. \) In the following discussion we always suppose \( k \neq 0. \) We estimate it under two cases: \( k < \frac{\sqrt{2D_1}}{8}X_n^\ell \) and \( k \geq \frac{\sqrt{2D_1}}{8}X_n^\ell. \)

Case 1): \( 0 \neq k < \frac{\sqrt{2D_1}}{8}X_n^\ell. \) When \( x \in [0, X_m - X_m^{\frac{1}{2}}], \) by (3.3) we have \( V(x) \geq -V(R). \) Together with \( V(R) \leq V(X_n) \leq \lambda_m \leq \lambda_n \) and (3.9), one has
\[
g(x) = \frac{\lambda_n - \lambda_m}{\sqrt{\lambda_n + \lambda_m} - \sqrt{\lambda_n - \lambda_m}} - k \geq \frac{\lambda_n - \lambda_m}{\sqrt{2}\lambda_m + \sqrt{2}\lambda_m} - k
\]
\[= \frac{\sqrt{2}}{2}(\sqrt{\lambda_n} - \sqrt{\lambda_m}) - k \geq \frac{\sqrt{2}}{4}\sqrt{\lambda_n} - k \geq \frac{\sqrt{2}}{4}\lambda_n^\ell - k \geq \frac{\sqrt{2}}{8}\lambda_n^\ell.
\]
Thus, by Lemma 4.11 we have
\[
\left| \int_0^{X_m - X_m^{\frac{1}{2}}} f(x)e^{i(\zeta_m - \zeta_n + kx)}\Psi(x)dx \right| \leq CX_n^{-\ell}\left( X_m^n \left| \Psi(x) \right| - \int_0^{X_m - X_m^{\frac{1}{2}}} (J_1 + J_3)dx \right) + \int_0^{X_m - X_m^{\frac{1}{2}}} \langle x \rangle^{\mu - 1}d\left( \Psi(x) \right)dx.
\]

By Corollary 3.13 we have \( \left| \Psi(x) \right| \leq CX_m^{\ell + \frac{2}{3}} \) and
\[
\int_0^{X_m - X_m^{\frac{1}{2}}} \langle x \rangle^{\mu - 1}d\left( \Psi(x) \right)dx \leq CX_m^{\ell + \frac{2}{3}} \int_0^{X_m - X_m^{\frac{1}{2}}} (x)^{\mu - 1}dx \leq CX_m^{\ell + \frac{2}{3}}.
\]

Besides, one has
\[
\int_0^{X_m - X_m^{\frac{1}{2}}} J_1 dx \leq C \int_0^{X_m - X_m^{\frac{1}{2}}} (\langle x \rangle)^{2\ell - 1}(\lambda_n - V(x))^{-\frac{1}{2}}dx
\]
\[\leq CX_m^{2\ell - 1}X_m^{-3\ell + \frac{2}{3}} \int_0^{X_m - X_m^{\frac{1}{2}}} (x)^{-\frac{1}{2}}dx \leq CX_m^{-\ell + \frac{2}{3}}
\]
and \( \int_0^{X_m - X_m^{\frac{1}{2}}} J_3 dx \leq CX_m^{-\ell + \frac{2}{3}}. \) It follows that
\[
\left| \int_0^{X_m - X_m^{\frac{1}{2}}} F(x)dx \right| \leq CX_m^{-\frac{1}{2} + \ell + \mu - \frac{2}{3}} X_m^{-\frac{2\ell - 1}{3}} \leq C(X_mX_n)^{\frac{1}{2} - \frac{1}{2}}.
\]

For the remainder terms, since \( \lambda_m \leq \frac{1}{4}\lambda_n, \) then
\[
\left| \int_0^{X_m - X_m^{\frac{1}{2}}} f(x)e^{ikx}\psi_2^{(m)}(x)\psi_1^{(n)}(x)dx \right|
\]
\[
\leq C X_m^{\ell - 1} X_n^{\ell - 1} X_m^\mu (\ell + 1) \int_{0}^{X_m - X_n^\beta} (\lambda_m - V(x))^{-1} (\lambda_n - V(x))^{-1} dx \leq C (X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}.
\]

In the same way we have
\[
\left| \int_{0}^{X_m - X_n^\beta} f(x) e^{ikx} \psi^{(m)}(x) \overline{\psi^{(n)}(x)} dx \right| \leq C (X_m X_n)^{\frac{\mu}{2} - \frac{3}{4}}
\]
and
\[
\left| \int_{0}^{X_m - X_n^\beta} f(x) e^{ikx} \psi^{(m)}(x) \overline{\psi^{(n)}(x)} dx \right| \leq C (X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}.
\]

Therefore, for the first case we have
\[
\left| \int_{0}^{X_m - X_n^\beta} f(x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq C (X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}.
\]

Case 2): \( k \geq \sqrt{\frac{2dL}{n}} X_l^\xi \). By Hölder inequality we have
\[
\left| \int_{0}^{X_m - X_n^\beta} f(x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq C X_m^{n} \leq C (|k|^{\frac{1}{2}} \vee 1) (X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}.
\]

Combining above estimations, we obtain
\[
\left| \int_{0}^{X_m - X_n^\beta} f(x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq C (|k|^{\frac{1}{2}} \vee 1) (X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}, \quad \forall \ k \neq 0.
\]

\( \square \)

**Lemma 3.16** If \( f(x) \) satisfies Assumption 1.2 and \( X_n > 4X_m \), then
\[
\left| \int_{X_m - X_n^{\beta}}^{X_m} f(x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq C (X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}, \quad \forall \ k \neq 0,
\]
where \( C \) only depend on \( (\mu, \ell) \) and \( n_0 \leq m \leq n \).

**Proof.** Recall that \( \lambda_m \leq \frac{1}{4} \lambda_n \), then we have
\[
\left| \int_{X_m - X_n^{\beta}}^{X_m} F(x) dx \right| \leq C X_m^{\ell - 1} X_n^{\ell - 1} \int_{X_m - X_n^{\beta}}^{X_m} x^\mu (\lambda_m - V(x))^{-\frac{1}{4}} (\lambda_n - V(x))^{-\frac{1}{4}} dx
\]
\[
\leq C X_m^{\ell + \mu} X_n^{\frac{\ell}{2}} (\lambda_n - \lambda_m)^{-\frac{1}{4}} \int_{X_m - X_n^{\beta}}^{X_m} (X_m - x)^{-\frac{1}{4}} dx \leq C (X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}.
\]

Similarly, we have
\[
\left| \int_{X_m - X_n^{\beta}}^{X_m} f(x) e^{ikx} \psi^{(m)}(x) \overline{\psi^{(n)}(x)} dx \right| \leq C (X_m X_n)^{\frac{\mu}{2} - \frac{3}{4}}, \quad \text{for } j_1, j_2 \in \{1, 2\} \text{ and } j_1 + j_2 \geq 3.
\]

Hence,
\[
\left| \int_{X_m - X_n^{\beta}}^{X_m} f(x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq C (X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}.
\]

\( \square \)

**Lemma 3.17** If \( f(x) \) satisfies Assumption 1.2 and \( X_n > 4X_m \), then
\[
\left| \int_{X_m}^{X_n} f(x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq C (X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}, \quad \forall \ k \neq 0,
\]
where \( C \) only depends on \( (\mu, \ell) \) and \( n_0 \leq m \leq n \).

For the proof see section 4.
3.3.3. the Integral on \([0, X_n]\) when \(X_m \leq X_n \leq 4X_m\). We will prove the following lemma in this part.

**Lemma 3.18** If \(f(x)\) satisfies Assumption 1.2 and \(X_m \leq X_n \leq 4X_m\), then

\[
\left| \int_0^{X_n} f(x)e^{ikx}h_m(x)\overline{h_n(x)}\,dx \right| \leq C(|k|^{-1} \vee |k|)(X_mX_n)^{\frac{3}{2} - \frac{1}{m+1}}, \quad \forall \, k \neq 0,
\]

where \(C\) only depends on \((\mu, \ell)\) and \(n_0 \leq m \leq n\).

If \(R \geq 8, X_m \geq 8\) from \(m \geq n_0\) and \(X_m - X_m^{-\frac{1}{4}} \geq X_m^{\frac{3}{4}}\). Hence, we can split the integral into three parts as:

\[
\int_0^{X_n} f(x)e^{ikx}h_m(x)\overline{h_n(x)}\,dx = \left( \int_0^{X_m^{\frac{3}{4}}} + \int_{X_m^{\frac{3}{4}}}^{X_m} + \int_{X_m}^{X_n} \right) dx.
\]

Lemma 3.18 comes from Lemma 3.19, 3.20 and 3.27. For the first part of the above integral, we have

**Lemma 3.19** If \(f(x)\) satisfies Assumption 1.2 and \(X_m \leq X_n \leq 4X_m\), then

\[
\left| \int_0^{X_m^{\frac{3}{4}}} f(x)e^{ikx}h_m(x)\overline{h_n(x)}\,dx \right| \leq C(X_mX_n)^{\frac{3}{2} - \frac{1}{m}}, \quad \forall \, k \neq 0,
\]

where \(C\) only depends on \((\mu, \ell)\) and \(n_0 \leq m \leq n\).

**Proof.** By Lemma 3.4 we have

\[
\left| \int_0^{X_m^{\frac{3}{4}}} F(x)\,dx \right| \leq CX_m^{\frac{1}{2} - \frac{1}{4}}X_m^{\frac{1}{4} - \frac{1}{2}}X_m^{\frac{3}{4}}\int_0^{X_m^{\frac{3}{4}}} (\lambda_m - V(x))^{-\frac{1}{4}}(\lambda_n - V(x))^{-\frac{1}{4}}\,dx \leq CX_m^{\frac{1}{2} - \frac{1}{4}}.
\]

Similarly, we have

\[
\left| \int_0^{X_m^{\frac{3}{4}}} f(x)e^{ikx}\psi_j^{(m)}(x)\psi_j^{(n)}(x)\,dx \right| \leq CX_m^{\frac{1}{2} - \frac{1}{4}}, \quad \text{for } j_1, j_2 \in \{1, 2\} \text{ and } j_1 + j_2 \geq 3.
\]

Thus, we obtain

\[
\left| \int_0^{X_m^{\frac{3}{4}}} f(x)e^{ikx}h_m(x)\overline{h_n(x)}\,dx \right| \leq C(X_mX_n)^{\frac{3}{2} - \frac{1}{m}}. \quad \square
\]

Next we estimate the integral on \([X_m^{\frac{3}{4}}, X_m - X_m^{\frac{1}{4}}]\), and obtain the following lemma.

**Lemma 3.20** If \(f(x)\) satisfies Assumption 1.2 and \(X_m \leq X_n \leq 4X_m\), then one has

\[
\left| \int_{X_m^{\frac{1}{4}}}^{X_m} f(x)e^{ikx}h_m(x)\overline{h_n(x)}\,dx \right| \leq C(|k|^{-1} \vee |k|)(X_mX_n)^{\frac{3}{2} - \frac{1}{m+1}}, \quad \forall \, k \neq 0,
\]

where \(C\) only depends on \((\mu, \ell)\) and \(n_0 \leq m \leq n\).

If \(k > X_m^{\frac{1}{4}}\), then \(X_n^{\frac{1}{4}} \leq Ck\). By Hölder inequality we have

\[
\left| \int_{X_m^{\frac{1}{4}}}^{X_m} f(x)e^{ikx}h_m(x)\overline{h_n(x)}\,dx \right| \leq CX_m^{\mu} \leq Ck(X_mX_n)^{\frac{3}{2} - \frac{1}{4}}. \quad (3.10)
\]

Thus we prove Lemma 3.20 when \(k > X_m^{\frac{1}{4}}\). In the following we turn to the case when \(0 < k \leq X_m^{\frac{1}{4}}\). From Lemma 3.21 to Lemma 3.25 we always suppose the following assumptions: 1. \(f(x)\) satisfies Assumption 1.2; 2. \(X_m \leq X_n \leq 4X_m\); 3. \(0 < k \leq X_m^{\frac{1}{4}}\).
Lemma 3.21  If \( 0 \leq \lambda_n - \lambda_m < \sqrt{m} k \lambda^{-\frac{1}{2}} \), then
\[
\left| \int_{X_m^{-\frac{1}{2}}}^{X_m^{\frac{1}{2}}} f(x) e^{ikx} h_m(x) dx \right| \leq C(k^{-1} \vee 1)(X_m X_n)^{\frac{\ell}{4} - \frac{1}{8}}, \quad \forall \ k \in (0, X_m^{\frac{1}{2}}],
\]
where \( C \) only depends on \((\mu, \ell)\) and \(n_0 \leq m \leq n\).

Proof. Write \( I := \int_{X_m^{-\frac{1}{2}}}^{X_m^{\frac{1}{2}}} \mathcal{F}(x) dx \) and \( I = C_m C_n \int_{X_m^{-\frac{1}{2}}}^{X_m^{\frac{1}{2}}} f(x) e^{i(\zeta_m - \zeta_n + kx)} \Psi(x) dx \) with \( C_m \sim X_m^{\frac{\ell}{2}} \) and \( C_n \sim X_n^{\frac{\ell}{2}} \). Since \( \lambda_m - V(X_m - X_m^{\frac{1}{2}}) \geq \alpha_1 X_m^{2\ell - \frac{3}{2}} \), then
\[
g(x_m - x_m^{\frac{1}{2}}) = \frac{\lambda_m - \lambda_m}{\sqrt{\lambda_m - \lambda_m} + \lambda_m - V(X_m - X_m^{\frac{1}{2}})} - k \leq \frac{\sqrt{\alpha_1} k X_m^{\ell - \frac{1}{2}}}{2 \sqrt{\lambda_m - V(X_m - X_m^{\frac{1}{2}})}} - k \leq \frac{k}{2}.
\]
By \( g'(x) \geq 0 \) one obtains \(|g(x)| \geq \frac{k}{2} \) for \( x \in [X_m^{\frac{1}{2}}, X_m - X_m^{\frac{1}{2}}] \) as the figure 2 below.

![Figure 2. Phase in Lemma 3.21](image)

Then by Lemma 4.11 one obtains
\[
\left| \int_{X_m^{-\frac{1}{2}}}^{X_m^{\frac{1}{2}}} f(x) e^{i(\zeta_m - \zeta_n + kx)} \Psi(x) dx \right| \leq \frac{C}{k} \left( X_m^{\mu} \left| \Psi(x_m - x_m^{\frac{1}{2}}) \right| + \int_{X_m^{-\frac{1}{2}}}^{X_m^{\frac{1}{2}}} \left| \Psi'(x) \right| dx + \int_{X_m^{-\frac{1}{2}}}^{X_m^{\frac{1}{2}}} x^{\mu - 1} |\Psi(x)| dx \right).
\]
By corollary 3.13 we have \(|\Psi(x_m - x_m^{\frac{1}{2}})| \leq C X_m^{-\ell + \frac{1}{2}} \) and \( \int_{X_m^{-\frac{1}{2}}}^{X_m^{\frac{1}{2}}} x^{\mu - 1} |\Psi(x)| dx \leq C X_m^{\mu - \ell + \frac{1}{2}} \).

Besides, one has
\[
\int_{X_m^{-\frac{1}{2}}}^{X_m^{\frac{1}{2}}} J_1 dx \leq C \int_{X_m^{-\frac{1}{2}}}^{X_m^{\frac{1}{2}}} x^{2\ell - 1} (\lambda_m - V(x))^{-\frac{\ell}{4}} (\lambda_n - V(x))^{-\frac{1}{4}} dx \leq C \int_{X_m^{-\frac{1}{2}}}^{X_m^{\frac{1}{2}}} x^{2\ell - 1} (\lambda_m - V(x))^{-\frac{3}{4}} dx \leq C X_m^{-\ell + \frac{1}{4}}.
\]
and

\[
\int_{X_m^\frac{1}{2}}^{X_m^{-\frac{1}{2}}} \left| f(x) \right| dx \leq \frac{1}{2} \int_{X_m^{-\frac{1}{2}}}^{X_m^{\frac{1}{2}}} (\lambda_m - V(x)) \left( \lambda_m - V(x) \right)^{-\frac{1}{2}} dx \leq C X_m^{-\ell + \frac{1}{2}}.
\]

It follows that

\[
\int_{X_m^\frac{1}{2}}^{X_m^{-\frac{1}{2}}} F(x) dx \leq \frac{C}{2} X_m^{-\frac{3}{2}}.
\]

Similarly,

\[
\int_{X_m^\frac{1}{2}}^{X_m^{-\frac{1}{2}}} f(x) e^{ikx} \frac{\psi_2^{(m)}}{\psi_1^{(n)}} dx \leq C X_m^{-\frac{3}{2}} \int_{X_m^\frac{1}{2}}^{X_m^{-\frac{1}{2}}} (\lambda_m - V(x))^{-\frac{1}{2}} dx \leq C (X_m X_n)^{\frac{1}{2} - \frac{1}{2} - \frac{1}{2}}.
\]

The other two terms have the same estimates. Therefore,

\[
\int_{X_m^\frac{1}{2}}^{X_m^{-\frac{1}{2}}} f(x) e^{ikx} h_m(x) \overline{h_n(x)} dx \leq C (k^{-1} \vee 1)(X_m X_n)^{\frac{-1}{2} - \frac{1}{2}}.
\]

Lemma 3.22 If \( \sqrt{\alpha_1 k X_m^{\ell - \frac{1}{2}}} \leq \lambda_n - \lambda_m < \sqrt{\alpha_1 k X_m^{\ell + \frac{1}{2}}} \), then

\[
\int_{X_m^\frac{1}{2}}^{X_m^{-\frac{1}{2}}} f(x) e^{ikx} h_m(x) \overline{h_n(x)} dx \leq C (k^{-1} \vee 1)(X_m X_n)^{\frac{1}{2} - \frac{1}{2}}.
\]

where \( C \) only depends on \( (\mu, \ell) \) and \( n \leq m \leq n \).

We put the proof of Lemma 3.22 to Lemma 3.25 into section 4.

Lemma 3.23 If \( \sqrt{\alpha_1 k X_m^{\ell - \frac{1}{2}}} \leq \lambda_n - \lambda_m < \sqrt{\alpha_1 k X_m^{\ell + \frac{1}{2}}} \), then

\[
\int_{X_m^\frac{1}{2}}^{X_m^{-\frac{1}{2}}} f(x) e^{ikx} h_m(x) \overline{h_n(x)} dx \leq C (1 + 1)(X_m X_n)^{\frac{1}{2} - \frac{1}{2}}.
\]

where \( C \) only depends on \( (\mu, \ell) \) and \( n \leq m \leq n \).

Lemma 3.24 If \( \sqrt{\alpha_1 k X_m^{\ell - \frac{1}{2}}} \leq \lambda_n - \lambda_m < 3 \sqrt{D_2} k X_m^{\ell}, \) then

\[
\int_{X_m^\frac{1}{2}}^{X_m^{-\frac{1}{2}}} f(x) e^{ikx} h_m(x) \overline{h_n(x)} dx \leq C (k^{-1} \vee 1)(X_m X_n)^{\frac{1}{2} - \frac{1}{2} - \frac{1}{2}}.
\]

where \( C \) only depends on \( (\mu, \ell) \) and \( n \leq m \leq n \).

Lemma 3.25 If \( \lambda_n - \lambda_m \geq 3 \sqrt{D_2} k X_n^{\ell} \), then

\[
\int_{X_m^\frac{1}{2}}^{X_m^{-\frac{1}{2}}} f(x) e^{ikx} h_m(x) \overline{h_n(x)} dx \leq C (k^{-1} \vee 1)(X_m X_n)^{\frac{1}{2} - \frac{1}{2} - \frac{1}{2}}.
\]

where \( C \) only depends on \( (\mu, \ell) \) and \( n \leq m \leq n \).

Lemma 3.26 If \( f(x) \) satisfies Assumption 1.2 and \( X_m \leq X_n \leq 4 X_m \), then one has

\[
\int_{X_m^\frac{1}{2}}^{X_m^{-\frac{1}{2}}} f(x) e^{ikx} h_m(x) \overline{h_n(x)} dx \leq C (k^{-1} \vee 1)(X_m X_n)^{\frac{1}{2} - \frac{1}{2} - \frac{1}{2}}.
\]

where \( C \) only depends on \( (\mu, \ell) \) and \( n \leq m \leq n \).

Proof. Clearly, if \( x \in [X_m^{\frac{1}{2}}, X_m^{-\frac{1}{2}}] \), then

\[
g(x) = \sqrt{\lambda_n - V(x)} - \sqrt{\lambda_m - V(x)} - k \geq -k = \left| k \right|.
\]

The proof is similar as Lemma 3.21. \( \square \)
Combining Lemma 3.21–3.26 with (3.10), one completes the proof of Lemma 3.20. Then we turn to the last term and obtain the following lemma.

**Lemma 3.27** If \( f(x) \) satisfies Assumption 1.2 and \( X_m \leq X_n \leq 4X_m \), then

\[
\left| \int_{X_m-X_n^\frac{1}{4}}^{X_n} f(x)e^{ikx}h_m(x)h_n(x)dx \right| \leq C(X_mX_n)^{\frac{\mu}{4}-\frac{1}{4}}, \quad \forall \ k \neq 0,
\]

where \( C \) only depends on \((\mu, \ell)\) and \( n_0 \leq m \leq n \).

**Proof.** By Lemma 3.4 we have \( \lambda_m - V(x) \geq a_1 X_m^{2\ell-1} (X_m - x) \) for \( x \in [X_m - X_m^{\frac{1}{4}}, X_m] \), then

\[
\left| \int_{X_m-X_n^\frac{1}{4}}^{X_n} F(x)dx \right| \leq CX_m^\frac{\ell-1}{2} X_n^\frac{\ell-1}{2} \int_{X_m-X_n^\frac{1}{4}}^{X_n} x^\mu (\lambda_m - V(x))^{-\frac{\mu}{4}}(\lambda_n - V(x))^{-\frac{\mu}{4}}dx
\]

\[
\leq C(X_mX_n)^{\frac{\mu}{4}-\frac{1}{4}}.
\]

For the integral on \([X_m, X_n]\), we discuss it under two cases.

1). \( X_n - X_n^\frac{1}{4} \geq X_m + X_m^{\frac{1}{4}} \). We split the integral into three parts as:

\[
\int_{X_m}^{X_n} f(x)e^{ikx} \psi_1^{(m)}(x)\psi_1^{(n)}(x)dx = \left( \int_{X_m}^{X_m+X_m^{\frac{1}{4}}} + \int_{X_m+X_m^{\frac{1}{4}}}^{X_m+X_n^{\frac{1}{4}}} + \int_{X_n-X_n^{\frac{1}{4}}}^{X_n} \right)dx.
\]

For the first part, since \( \lambda_n - V(X_m + X_m^{\frac{1}{4}}) \geq \lambda_n - V(X_n - X_n^{\frac{1}{4}}) \geq a_1 X_n^{2\ell-\frac{3}{4}} \), then

\[
\left| \int_{X_m}^{X_m+X_m^{\frac{1}{4}}} F(x)dx \right| \leq CX_m^\frac{\ell-1}{2} X_m^{\frac{\ell-1}{2}} \int_{X_m}^{X_m+X_m^{\frac{1}{4}}} (V(x) - \lambda_m)^{-\frac{\mu}{4}}(\lambda_n - V(x))^{-\frac{\mu}{4}}dx
\]

\[
\leq CX_m^{\frac{\ell-1}{2}} X_m^{\frac{\ell-1}{2}} \left( \lambda_n - V(X_m + X_m^{\frac{1}{4}}) \right)^{-\frac{\mu}{4}} \int_{X_m}^{X_m+X_m^{\frac{1}{4}}} (x - X_m)^{-\frac{\mu}{4}}dx \leq C(X_mX_n)^{\frac{\mu}{4}-\frac{1}{4}}.
\]

By lemma 4.7 we have \( |\zeta_m(x)| \geq A_1 X_m^{\ell-\frac{3}{4}} (x - X_m)^{\frac{1}{4}} \geq A_1 X_m^{\ell-\frac{3}{4}} (x - X_m) \) for \( x \geq X_m + X_m^{\frac{1}{4}} \). Thus,

\[
\left| \int_{X_m+X_m^{\frac{1}{4}}}^{X_m+X_n^{\frac{1}{4}}} F(x)dx \right| \leq CX_m^\frac{\ell-1}{2} X_n^{\frac{\ell-1}{2}} \int_{X_m+X_m^{\frac{1}{4}}}^{X_m+X_n^{\frac{1}{4}}} (V(x) - \lambda_m)^{-\frac{\mu}{4}}(\lambda_n - V(x))^{-\frac{\mu}{4}} e^{-|\zeta_m|}dx
\]

\[
\leq CX_m^\frac{\ell-1}{2} X_n^{\frac{\ell-1}{2}} \left( \lambda_n - V(X_m^{\frac{1}{4}}) \right)^{-\frac{\mu}{4}} \int_{X_m+X_m^{\frac{1}{4}}}^{X_m+X_n^{\frac{1}{4}}} (x - X_m)^{-\frac{\mu}{4}} e^{-C_0(x-X_m)}dx
\]

\[
\leq CX_m^\frac{\ell-1}{2} X_n^{\frac{\ell-1}{2}} \mu \int_{0}^{\infty} t^{-\frac{\mu}{4}}e^{-t}dt \leq C(X_mX_n)^{\frac{\mu}{4}-\frac{1}{4}}.
\]

For the last part, since \( V(X_n - X_n^{\frac{1}{4}}) - \lambda_m \geq V(X_m + X_m^{\frac{1}{4}}) - \lambda_m \geq a_1 X_m^{2\ell-\frac{3}{4}} \), then

\[
\left| \int_{X_n-X_n^{\frac{1}{4}}}^{X_n} F(x)dx \right| \leq CX_n^{\frac{\ell-1}{2}} X_n^{\frac{\ell-1}{2}} \int_{X_n-X_n^{\frac{1}{4}}}^{X_n} (V(x) - \lambda_m)^{-\frac{\mu}{4}}(\lambda_n - V(x))^{-\frac{\mu}{4}}dx
\]

\[
\leq CX_n^{\frac{\ell-1}{2}} X_n^{\frac{\ell-1}{2}} \left( V(X_n - X_n^{\frac{1}{4}}) - \lambda_m \right)^{-\frac{\mu}{4}} X_n^{\frac{\ell-1}{2}} \int_{X_n-X_n^{\frac{1}{4}}}^{X_n} (X_n - x)^{-\frac{\mu}{4}}dx
\]

\[
\leq CX_n^{\frac{\ell-1}{2}} X_n^{\frac{\ell-1}{2}} \mu \leq C(X_mX_n)^{\frac{\mu}{4}-\frac{1}{4}}.
\]

Thus, for the first case we have \( \left| \int_{X_m}^{X_n} F(x)dx \right| \leq C(X_mX_n)^{\frac{\mu}{4}-\frac{1}{4}} \).

2). \( X_n - X_n^{\frac{1}{4}} < X_m + X_m^{\frac{1}{4}} \). Note \( X_m \geq 8 \), it follows \( X_n - X_n^{\frac{1}{4}} \leq X_n^{\frac{1}{4}} + X_m^{\frac{1}{4}} \leq 1 \). Hence,

\[
\left| \int_{X_n}^{X_m} F(x)dx \right| \leq CX_n^{\frac{\ell-1}{2}} X_n^{\frac{\ell-1}{2}} \mu \int_{X_n}^{X_m} (V(x) - \lambda_m)^{-\frac{\mu}{4}}(\lambda_n - V(x))^{-\frac{\mu}{4}}dx
\]
\[ \leq CX_n^{\frac{1}{4}}X_n^{\frac{3}{4}+\mu} \int_{X_m}^{X_n} (x - X_m)^{-\frac{1}{4}}(X_n - x)^{-\frac{1}{4}} dx \leq C(X_mX_n)^{\frac{3}{8} - \frac{1}{4}}. \]

Thus, in the second case we have \( |F(X_n)dx| \leq C(X_mX_n)^{\frac{3}{8} - \frac{1}{4}}. \) Since the other three integrals have better estimates, we finish the proof. \( \square \)

3.4. **Proof of Lemma 1.11 and Lemma 1.14.** Proof of Lemma 1.11. If denote \( \widetilde{V}(x) = V(-x), \ h_n(x) = h_n(-x) \) and \( \tilde{f}(x) = f(-x), \) we have

\[ \left( -\frac{d^2}{dx^2} + \tilde{V}(x) \right) \tilde{h}_n(x) = \lambda_n \tilde{h}_n(x), \quad x \in \mathbb{R}. \]

Clearly, \( \tilde{V}(x) \) satisfies Assumption 1.1 and \( \tilde{f}(x) \) satisfies Assumption 1.2. Applying Lemma 3.5, we obtain

\[ \left| \int_{0}^{+\infty} \tilde{f}(y)e^{-iky} \tilde{h}_m(y) \tilde{h}_n(y) dy \right| \leq C(|k|^{-1} \vee |k|)(\lambda_m\lambda_n)^{\frac{1}{8} - \frac{1}{4(n+1)}}, \quad \forall k \neq 0. \]

Note that \( \int_{0}^{+\infty} \tilde{f}(y)e^{iky} \tilde{h}_m(y) \tilde{h}_n(y) dy = \int_{-\infty}^{0} f(x)e^{ikx} \lambda_m \lambda_n \tilde{h}_n(x) dx, \) it follows

\[ \left| \int_{-\infty}^{-\infty} f(x)e^{ikx} \lambda_m \lambda_n \tilde{h}_n(x) dx \right| \leq C(|k|^{-1} \vee |k|)(\lambda_m\lambda_n)^{\frac{1}{8} - \frac{1}{4(n+1)}}, \quad \forall k \neq 0. \quad (3.11) \]

Combining Lemma 3.5 with (3.11), we finish the proof. \( \square \)

Proof of Lemma 1.14. Similar as the above proof, we only need to estimate the integral on \([0, +\infty).\)

As Lemma 3.6, 3.7 and 3.9 one has

\[ \left| \int_{0}^{+\infty} f(x) \lambda_m \lambda_n \tilde{h}_n(x) dx \right| \leq C(X_mX_n)^{\frac{3}{8} - \frac{1}{4}} \leq C(X_mX_n)^{\frac{3}{8}}, \quad \text{for } m \leq n < n_1; \]

\[ \left| \int_{0}^{+\infty} f(x) \lambda_m \lambda_n \tilde{h}_n(x) dx \right| \leq C(X_mX_n)^{\frac{3}{8} - \frac{1}{4}} \leq C(X_mX_n)^{\frac{3}{8}}, \quad \text{for } m < n_0, n \geq n_1; \]

\[ \left| \int_{X_n}^{+\infty} f(x) \lambda_m \lambda_n \tilde{h}_n(x) dx \right| \leq C(X_mX_n)^{\frac{3}{8} - \frac{1}{4}} \leq C(X_mX_n)^{\frac{3}{8}}, \quad \text{for } n \geq m \geq n_0. \]

Next we estimate the integral on \([0, X_n]\) for \(n \geq n_0.\) If \(X_m \leq X_n \leq 4X_m, \) by Hölder inequality we have

\[ \left| \int_{0}^{X_n} f(x) \lambda_m \lambda_n \tilde{h}_n(x) dx \right| \leq CX_n^{\mu} \leq C(X_mX_n)^{\frac{3}{8}}. \]

When \(X_n > 4X_m\) we have \(X_n - X_m^{-\frac{1}{4}} \geq \frac{1}{4}X_n \geq 2X_m, \) and thus we split the integral into three parts as:

\[ \int_{0}^{X_n} f(x) \lambda_m \lambda_n \tilde{h}_n(x) dx = \left( \int_{0}^{2X_m} + \int_{2X_m}^{X_n - X_m^{-\frac{1}{4}}} + \int_{X_n - X_m^{-\frac{1}{4}}}^{X_n} \right) f(x) \lambda_m \lambda_n \tilde{h}_n(x) dx. \]

By Hölder inequality one has \( \left| \int_{0}^{2X_m} f(x) \lambda_m \lambda_n \tilde{h}_n(x) dx \right| \leq C(X_mX_n)^{\frac{3}{8}}. \)

By Lemma 3.4 we have \(\lambda_n - V(X_m - X_n^{-\frac{1}{4}}) \geq a_1 X_n^{2\epsilon + \frac{1}{4}} \) and \( |\zeta_n(x)| \geq A_1 X_n^{\mu} \) for \(x \geq 2X_m.\)

It follows

\[ \left| \int_{2X_m}^{X_n - X_m^{-\frac{1}{4}}} \mathcal{F}(x) dx \right| \leq C X_n^{\frac{3}{8} - \frac{1}{4}} \int_{2X_m}^{X_n - X_m^{-\frac{1}{4}}} x^{\mu}(V(x) - \lambda_m)^{-\frac{3}{4}}(\lambda_n - V(x))^{-\frac{3}{4}e^{-|\zeta_n|}} dx \]

\[ \leq C X_n^{\frac{3}{8} - \frac{1}{4}}(\lambda_n - V(X_n - X_m^{-\frac{1}{4}}))^{-\frac{1}{4}} \int_{2X_m}^{X_n - X_m^{-\frac{1}{4}}} (x - X_m)^{-\frac{1}{4}+\mu} e^{-C_0(x - X_m)} dx \]
\[ \leq CX_m^{-\frac{1}{2}}X_n^{-\frac{1}{4}} \int_0^\infty t^{-\frac{1}{4}+\mu}e^{-t}dt \leq CX_m^{-\frac{1}{2}}X_n^{-\frac{1}{4}}. \]

Similarly, from Lemma 3.4 we have \( V(X_n - X_n^{-\frac{1}{4}}) - \lambda_m \geq a_1X_m^{2\epsilon_1} \) and \( |\zeta_m(x)| \geq C X_m^\ell X_n \) for \( x \geq X_n - X_n^{-\frac{1}{4}} \). Therefore,

\[
\left| \int_{X_n - X_n^{-\frac{1}{4}}}^{X_n} F(x)dx \right| \leq CX_m^{\frac{1}{4} - \frac{\ell}{2}} X_n^{\frac{1}{2} - \frac{\ell}{2}} \int_{X_n - X_n^{-\frac{1}{4}}}^{X_n} x^{\mu}(V(x) - \lambda_m)^{-\frac{1}{4}}(\lambda_n - V(x))^{-\frac{1}{4}}e^{-|\zeta_m|}dx
\]
\[
\leq CX_m^{\frac{1}{4} - \frac{\ell}{2}}(V(X_n - X_n^{-\frac{1}{4}}) - \lambda_m)^{-\frac{1}{4}}e^{C\lambda_m} \int_{X_n - X_n^{-\frac{1}{4}}}^{X_n} (x - X_m)^{\mu}e^{-C(\lambda - X_m)}(X_n - x)^{-\frac{1}{4}}dx
\]
\[
\leq CX_m^{-\frac{1}{2}}X_n^{-\frac{1}{4}}.
\]

Since the other integrals have better estimates, we obtain \( \left| \int_0^{X_n} f(x)h_m(x)\overline{|h_n(x)|}dx \right| \leq C(X_mX_n)^{\frac{3}{4}}. \)

Thus we finish the proof. \( \square \)

4. Appendix

4.1. Lemma 2.2 (iv) and (v). Proof of Lemma 2.2 (iv). Since \( A \in M_j^\alpha \), then \( |A_j^\alpha| \leq |A_j^\alpha(ij)^{\beta}(1+i-j-1)^{(\mu-1+j-1)^{-1}} \) for any \( i, j \in \mathbb{Z}_+ \). For a conjugated pair \((p, q)\), i.e. \( \frac{1}{p} + \frac{1}{q} = 1 \) with \( p, q \geq 1 \), given \( i \in \mathbb{Z}_+ \), \( \sum_{j \geq 1} |A_j^\alpha(ij)^{\beta} \leq |A_j^\alpha|_\beta \sum_{j \geq 1} (1 + |i-j|)(i-1+j-1)^{\beta} \). We first consider the sum

\[ I_j(i, j) := \sum_{j \geq 1} \left( \frac{1}{(1 + |i-j|)(i-1+j-1)^{\beta}} \right) \leq \sum_{j \geq 1} \left( \frac{1}{1 + |i-j|} \right) \leq \sum_{j \geq 1} \frac{1}{1 + |i-j|} \leq \sum_{j \geq 1} \frac{1}{1 + |i-j|} \leq \sum_{j \geq 1} \frac{1}{1 + |i-j|} \leq C(\beta, s). \]

For \( I_1 \), by \( \frac{\beta}{p} \geq 0 \), we have

\[ I_1 \leq \sum_{j \geq i} \frac{j^{2\beta}}{(1 + |i-j|)(i-1)^{\beta}} \leq \sum_{j \geq i} \left( \frac{1}{1 + |i-j|} \right)^{\beta} \leq \sum_{j \geq i} \frac{1}{1 + |i-j|} \leq C(\beta, \ell). \]

For \( I_2 \), we have

\[ I_2 \leq C(\beta, s) \sum_{j \geq 1} \frac{j^{2\beta}}{(1 + |i-j|)^{\beta}} \leq C(\beta, s) \sum_{j \geq 1} \left( \frac{1}{1 + |i-j|} \right)^{\beta} \leq C(\beta, \ell, s). \]

For \( I_3 \), if \( \ell - \beta - \frac{s}{p} \geq 0 \), then

\[ I_3 \leq \sum_{j \geq \frac{s}{p}} \frac{2(ij)^{\beta}}{i^{\beta}} \leq \sum_{j \geq \frac{s}{p}} \frac{2}{i^{\beta} j^{\beta - \frac{s}{p}}} \leq \sum_{j \geq \frac{s}{p}} \frac{2}{j^{\beta - \frac{s}{p}}} \leq C(\beta, \ell). \]

Hence, we obtain \( I \leq C(\beta, \ell, s) \) provided

\[ s, \ell - \beta - \frac{s}{p} \geq 0. \]

For given \( j \in \mathbb{Z}_+ \), since \( \sum_{i \geq 1} |A_i^\alpha(ij)^{\beta} \leq |A_j^\alpha|_\beta \sum_{i \geq 1} \left( \frac{1}{(1 + |i-j|)(i-1+j-1)^{\beta}} \right) \), as above we consider the sum

\[ J := \sum_{i \geq 1} \left( \frac{1}{1 + |i-j|} \right)^{\beta} \leq \sum_{j \geq 1} \left( \frac{1}{1 + |i-j|} \right)^{\beta} \leq \sum_{j \geq 1} \left( \frac{1}{1 + |i-j|} \right)^{\beta} \leq C(\beta, s). \]
We discuss two cases in order to obtain the estimates of $I$ and $J$.

Case 1: $s \in [\beta, 2\ell - 2\beta - 1)$. For $J_1$, we choose $q \geq 1$ such that $\frac{q}{q} \geq \beta$ and then

$$J_1 \leq \sum_{i<j} \frac{(ij)^2}{(1 + |i - j|)^{\nu - 1}} \leq \sum_{i<j} \frac{1}{(1 + |i - j|)^{\nu - 1 - 2\beta}} \leq \sum_{i \geq 1} \frac{1}{(1 + |i - j|)^{\nu - 1 - 2\beta}} \leq C(\beta, \iota).$$

For $J_2$, we have

$$J_2 \leq C(\beta, s) \sum_{j \leq 2j} \frac{q^{2\beta}}{(1 + |j - j|)^{\nu - 1}} \leq C(\beta, s) \sum_{i \geq 1} \frac{1}{(1 + |i - j|)^{\nu - 1 - 2\beta}} \leq C(\beta, \iota, s).$$

For $J_3$, if we choose $q$ such that $\iota - \beta - \frac{s}{q} > 1$, $\frac{s}{q} - \beta > 0$, then

$$J_3 \leq \sum_{i \geq 1} \frac{2(ij)^2}{\nu^{2\beta}} \leq \sum_{i \geq 1} \frac{2}{(1 + |i - j|)^{\nu - 1 - 2\beta}} \leq C(\beta, \iota, s).$$

Hence, when $s \in [\beta, 2\ell - 2\beta - 1)$, we obtain $I, J \leq C(\beta, \iota, s)$ provided

$$\iota - \beta - \frac{s}{p} \geq 0, \quad \iota - \beta - \frac{s}{q} > 1, \quad \frac{s}{q} - \beta \geq 0. \quad (4.2)$$

More precisely, one needs to choose $q \geq 1$ such that $\frac{1}{q} \in \left\{ \frac{\beta}{s}, \frac{s - 1 + \beta}{s - 1}, \frac{\iota - \beta - 1}{s - 1}, \frac{s - \beta}{s - 1} \right\}$, $\beta \leq s < \iota$, $\iota - \beta - \frac{s}{q} > 1$, $\frac{s}{q} - \beta \geq 0$.

In fact one can choose $q(s) = \left\{ \frac{\beta}{s}, \frac{s - \beta}{s - 1}, \iota - \beta - \frac{s - 1 + \beta}{s - 1}, \frac{s - \beta}{s - 1} \right\}$ such that (4.2) holds true.

Case 2: $s \in [0, \beta]$. In this case we estimate $J_1$ to $J_3$ again. In fact, For $J_1$, we choose $q = \infty$ and then

$$J_1 \leq \sum_{i<j} \frac{(ij)^2}{(1 + |i - j|)^{\nu - 1}} \leq \sum_{i \geq 1} \frac{1}{(1 + |i - j|)^{\nu - 1 - 2\beta}} \leq \sum_{i \geq 1} \frac{1}{(1 + |i - j|)^{\nu - 1 - 2\beta}} \leq C(\beta, \iota).$$

For $J_2$, the proof is the same. For $J_3$, if $q = \infty$, then $J_3 \leq C \sum_{i \geq 1} \frac{1}{(i + 1)^{\nu - 1}} \leq C(\beta, \iota)$. We remark that this case (4.1) also holds since $\iota > 2\beta + 1$ when $q = \infty$. Hence, when $s \in [0, \beta]$, we still obtain $I, J \leq C(\beta, \iota, s)$. Thus, for any $s \in [0, 2\ell - 2\beta - 1)$, one can choose $p, q \geq 1$ such that $1/p + 1/q = 1$ and $\sum_j |A_j^1|(\frac{s}{j})^{\iota} = C(\beta, \iota, s)|A_j^1|^{\iota}$. By Hölder inequality, we have

$$\|A\|_{L^2(B_t^2)}^2 = \sum_{i \geq 1} \|A_j^1|^2 \leq \sum_{i \geq 1} \left( \sum_{j \geq 1} |A_j^1|^2 \right)^{\iota} \leq C(\beta, \iota, s)|A_j^1|^{\iota} \sum_{i \geq 1} \|A_j^1|^{\iota} \leq C^2(\beta, \iota, s)|A_j^1|^{\iota} \leq C(\beta, \iota, s)|A_j^1|^{\iota} \|\xi\|_{L^2}^2.$$
From the proof in (iv) we have for any \( s \in [0, 2t - 2\beta - 1) \), \( \sum_{j \geq 1} |A_j^1|((\frac{1}{q})^\frac{s}{2}) \), \( \sum_{i \geq 1} |A_i^1|((\frac{1}{q})^\frac{s}{2}) \leq C(\beta, \iota, s) |A|_\beta^+ \), i.e.

\[
\sum_{i \geq 1} |A_i^1|((\frac{1}{q})^\frac{s}{2}), \sum_{j \geq 1} |A_j^1|((\frac{1}{q})^\frac{s}{2}) \leq C(\beta, \iota, s) |A|_\beta^+ . \tag{4.3}
\]

By Hölder inequality, (4.3) and a similar method as (iv), we obtain

\[
\|A\xi\|_s^2 \leq \sum_{i \geq 1} \left( \sum_{j \geq 1} |A_i^1|((\frac{1}{q})^\frac{s}{2}) |A_j^1|((\frac{1}{q})^\frac{s}{2}) |\xi_j| j^{-\frac{s}{2}} \right)^{\frac{2}{s}} \leq \sum_{i \geq 1} \left( \sum_{j \geq 1} |A_i^1|((\frac{1}{q})^\frac{s}{2}) \right)^{\frac{2}{s}} \left( \sum_{j \geq 1} |A_j^1|((\frac{1}{q})^\frac{s}{2}) |\xi_j| j^{-s} \right) \leq (C(\beta, \iota, s) |A|_\beta^+)^2 \|\xi\|_s^2 .
\]

It results in \( \|A\|_{B(\ell^2, s)} \leq C(\beta, \iota, s) |A|_\beta^+ . \)

\[\square\]

4.2. some lemmas for section 2.

**Lemma 4.1** For \( s \geq 0 \), if \( A \in M_{\ell^2} \) is a diagonal matrix, then \( A \in B(\ell^2, \ell^2_{-2s}) \) and satisfies

\[
\|A\|_{B(\ell^2, \ell^2_{-2s})} \leq \|A\|_{\ell^2} .
\]

**Proof.** Since \( A \in M_{\ell^2} \) is a diagonal matrix, then \( |A_i^1| \leq |A|_\ell (ii)^{\frac{s}{2}} = |A|_\ell \iota^s \). Given \( \iota \in \ell^2_0 \), one has

\[
\|A\xi\|_{-2s}^2 = \sum_{i \geq 1} i^{-2s} \sum_{j \geq 1} A_i^1 \xi_j^2 \leq \sum_{i \geq 1} i^{-2s} \sum_{j \geq 1} |A_i^1|^2 |\xi_j|^2 \leq |A|_{\ell^2}^2 \sum_{i \geq 1} |\xi_i|^2 = |A|_{\ell^2}^2 \|\xi\|_\ell^2 .
\]

It follows that \( \|A\|_{B(\ell^2, \ell^2_{-2s})} \leq |A|_{\ell^2} \).

\[\square\]

We present the following lemmas to prove Lemma 4.5 and complete the proof of Theorem 2.5. The following proof is similar to Lemma 2.31 to Lemma 2.34 and we don’t give the details here.

**Lemma 4.2** For \( l \geq 1 \), if \( \ell_0 \ll 1 \), then \( V^l A^0 = A^l V^l + Q_l \), where \( Q_l \in B(\ell^2, \ell^2_{-2s}) \) and satisfies

\[
\|Q_l\|_{B(\ell^2, \ell^2_{-2s})} \leq \prod_{i=0}^{l-1}(1 + \ell_i^{2}) + \sum_{i=0}^{l-1} \ell_i^{s} .
\]

From a straightforward computation, we have \( \|Q_{l+1} - Q_l\|_{B(\ell^2, \ell^2_{-2s})} \leq 3C\ell_i^{2} \). It follows \{\(Q_l\)\} is a Cauchy sequence in \( B(\ell^2, \ell^2_{-2s})(\Pi, \ell^2) \) and \( B(\ell^2, \ell^2_{-2s})(\Pi, \ell^2) \).

Define \( Q_\infty := Q_1 + \sum_{l=1}^{\infty} Q_{l+1} - Q_l \). We have the following

**Lemma 4.3** For \( l \to \infty \), \( Q_l(\phi) \to Q_\infty(\phi) \) in \( B(\ell^2, \ell^2_{-2s})(\Pi, \ell^2) \) and \( B(\ell^2, \ell^2_{-2s})(\Pi, \ell^2) \), where

\[
\|Q_{\infty}\|_{B(\ell^2, \ell^2_{-2s})} \leq \|Q_{\infty}\|_{B(\ell^2, \ell^2_{-2s})} \leq C\ell_0^{2} ,
\]

\[
\|Q_{\infty} - P^0\|_{B(\ell^2, \ell^2_{-2s})} \leq \|Q_{\infty} - P^0\|_{B(\ell^2, \ell^2_{-2s})} \leq C\ell_0^{2} .
\]

for some positive constant \( C \).

From \( A^{l+1} V^{l+1} - A^l V^l = A^l (e^{-B^{l+1}} - 1) V^l + \text{diag}(P^l) V^{l+1} \) and Lemma 2.15, 2.22, one obtains

\[
\|A^{l+1} V^{l+1} - A^l V^l\|_{B(\ell^2, \ell^2_{-2s})} \leq 2C\ell_i^{2} .
\]

Thus, \{\(A^l V^l\)\} is a Cauchy sequence in \( B(\ell^2, \ell^2_{-2s}) \). We now define \( A^\infty V^\infty = A_0 + \sum_{l=0}^{\infty} A^{l+1} V^{l+1} - A^l V^l \), it follows \( \|A^\infty V^\infty - A^0\|_{B(\ell^2, \ell^2_{-2s})} \leq 4C\ell_0^{2} .
\]

Therefore, we have

**Lemma 4.4** When \( l \to \infty \), \( A^l V^l(\phi) \to A^\infty V^\infty(\phi) \) in \( B(\ell^2, \ell^2_{-2s})(\Pi, \ell^2) \), where

\[
\|A^\infty V^\infty - A^0\|_{B(\ell^2, \ell^2_{-2s})} \leq C\ell_0^{2} ,
\]

where \( C > 0 \).
Combining Lemma 4.2, 4.3 with 4.4 one has $V^l A^0 = A^l V^l + Q_l$, where $\{A^l V^l\}$, $\{Q_l\}$ are Cauchy sequences in $B(\ell^2, \ell^2_2)(\Pi_\ast, \frac{\pi}{2})$ and $\|A^\infty V^\infty - A^0\|_{B(\ell^2, \ell^2_2), \frac{\pi}{2}} \leq C\varepsilon_0^4$ and $\|Q_\infty\|_{B(\ell^2, \ell^2_2), \frac{\pi}{2}} \leq C\varepsilon_0^4$ with some $C > 0$. Thus, $\{V^l A^0\}$ is a Cauchy sequence in $B(\ell^2, \ell^2_2)(\Pi_\ast, \frac{\pi}{2})$. Define $\lim_{l \to \infty} V^l A^0 := V^\infty A^0 = A^\infty V^\infty + Q_\infty$, then one has

$$\|V^\infty A^0 - A^0\|_{B(\ell^2, \ell^2_2), \frac{\pi}{2}} \leq \|A^\infty V^\infty - A^0\|_{B(\ell^2, \ell^2_2), \frac{\pi}{2}} + \|Q_\infty\|_{B(\ell^2, \ell^2_2), \frac{\pi}{2}} \leq 2C\varepsilon_0^4.$$ 

It can be written as the following.

**Lemma 4.5** When $l \to \infty$, $V^l A^0(\phi) \to V^\infty A^0(\phi)$ in $B(\ell^2, \ell^2_2)(\Pi_\ast, \frac{\pi}{2})$, where

$$\|V^\infty A^0 - A^0\|_{B(\ell^2, \ell^2_2), \frac{\pi}{2}} \leq C\varepsilon_0^4, \quad C > 0.$$

### 4.3. some lemmas for section 3.

**Proof of Lemma 3.1.** The proof is almost from [33](see [36], Lemma 2.1). As in [33] we set $\eta(x) = (\lambda - V(x))^{\frac{1}{2}} h(x)$ and $\zeta(x) = \int_x^\infty (\lambda - V(t))^{\frac{1}{2}} dt$, where arg $\zeta(x) = \begin{cases} \frac{\pi}{2}, & x > X, \\ -\pi, & x < X. \end{cases}$ Then the equation (3.1) is transformed into $\frac{d^2\eta}{dx^2} + \eta + \frac{V''(x)}{4(\lambda - V(x))} + \frac{5V'(x)^2}{16(\lambda - V(x))^2}]\eta = 0$ and it can be rewritten as

$$\frac{d^2\eta}{dx^2} + (1 + \frac{5}{36\xi^2})\eta = f(x)\eta, \quad (4.4)$$

where $f(x) = \frac{5}{36\xi^2} - \frac{V''(x)}{4(\lambda - V(x))^2} - \frac{5V'(x)^2}{16(\lambda - V(x))^3}$. As we know, Bessel equation $\frac{d^2\eta}{dx^2} + (1 + \frac{5}{36\xi^2})G = 0$ has two linearly independent solutions $(\frac{5}{2})^{\frac{1}{2}} J_{\frac{1}{2}}(\xi)$ and $(\frac{5}{2})^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)}(\xi)$, where $J_{\frac{1}{2}}(x)$ and $H_{\frac{1}{2}}^{(1)}(x)$ are the first kind Bessel function and one of the third kind Bessel function, respectively. By the property of Bessel function that $x(J_{\nu}(x)H_{\nu}^{(1)}(x) - J_{\nu}'(x)H_{\nu}^{(1)}(x)) = \frac{\pi}{\xi}$, then (4.4) is formally equivalent to the integral equation

$$\eta = (\frac{1}{2})^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)}(\xi) + \frac{\pi}{2} \int_x^\infty \left(H_{\frac{1}{2}}^{(1)}(\xi) J_{\frac{1}{2}}(\theta) - J_{\frac{1}{2}}(\xi) H_{\frac{1}{2}}^{(1)}(\theta)\right) \xi^{\frac{1}{2}} \theta^{\frac{1}{2}} f(t)(\lambda - V(t))^{\frac{1}{2}} \eta(t) dt,$$

where we write $\zeta(\xi) = \zeta(x)$ and $\theta = \zeta(t)$ for convenience. Set

$$\alpha(x) = e^{-i\xi} (\frac{5}{2})^{\frac{1}{2}} H_{\frac{1}{2}}^{(1)}(\xi), \quad \beta(x) = e^{i\xi} (\frac{5}{2})^{\frac{1}{2}} J_{\frac{1}{2}}(\xi), \quad \chi(x) = e^{-i\xi} \eta(x).$$

then

$$\chi(x) = \alpha(x) + i \int_x^\infty \left(\alpha(x)\beta(t) - e^{2i(\theta - \xi)} \beta(x)\alpha(t)\right) f(t)(\lambda - V(t))^{\frac{1}{2}} \chi(t) dt.$$ 

From [33], $\alpha(x)$, $\beta(x)$ are bounded, and $\text{Im}(\theta - \xi) = \text{Im}(\int_0^t (\lambda - q(u))^{\frac{1}{2}} du) \geq 0$ together with Lemma 4.8 and 4.9, we can prove that the iteration converges. In fact, if we denote $\int_0^\infty |f(t)||\lambda - V(t)|^{\frac{1}{2}} dt = M_0 = O\left(\frac{1}{X\lambda^{\frac{1}{2}}}\right)$, and $|\alpha(x)\beta(t) - e^{2i(\theta - \xi)} \beta(x)\alpha(t)| \leq M$ uniformly, then $|\chi_0(x)| = |\alpha(x)| \leq C$, $|\chi_1(x) - \chi_0(x)| \leq C M_0$, and generally, if $|\chi_n(x) - \chi_{n-1}(x)| \leq C M^n M_0^n$, then

$$|\chi_{n+1}(x) - \chi_n(x)|$$

$$= \left|\int_x^\infty \left(\alpha(x)\beta(t) - e^{2i(\theta - \xi)} \beta(x)\alpha(t)\right) f(t)(\lambda - V(t))^{\frac{1}{2}} (\chi_n(t) - \chi_{n-1}(t)) dt\right|$$

$$\leq C M^{n+1} M_0^n \int_x^\infty |f(t)(\lambda - V(t))^{\frac{1}{2}}| dt \leq C M^{n+1} M_0^{n+1}.$$ 

Thus,

$$|\chi_n(x)| \leq |\chi_0(x)| + |\chi_1(x) - \chi_0(x)| + \cdots + |\chi_n(x) - \chi_{n-1}(x)|$$
If $n \geq n_0$, then $X_n \geq R$ and $X_n \sim \lambda^{\beta^2}$. Thus, choose $n \geq n_0$ large enough such that $MM_0 < 1$, and by the theorem of dominated convergence, when $n \to \infty$, $\chi_n(x) \to \chi(x) = \alpha(x) + O(\frac{1}{X^{\lambda^2}})$ uniformly w.r.t $x$, which means that $\chi(x)$ is bounded.

Next we show that

$$\chi(x) = \alpha(x) \left(1 + O\left(\frac{1}{X^{\lambda^2}}\right)\right).$$

From Remark 4.6, if $\zeta(x) < -c_0$ or $i\zeta(x) < -c_0$, where $c_0$ are arbitrary two positive constants, then we can prove that $|\alpha(x)| > C$ and (4.5) holds. While for $0 < |\zeta(x)| \leq c_0$ we have $|\beta(x)| \leq C|\alpha(x)|$. Thus, $|\chi(x) - \alpha(x)| = \left|\int_{x}^{\infty} \left(\alpha(x)\beta(t) - e^{2i(\theta-C)}\beta(x)\alpha(t)\right)f(t)(\lambda - V(t))^{\frac{1}{2}}\chi(t)dt\right| \leq \frac{C|\alpha(x)|}{X^{\lambda^2}}$.

From the above proof when $\lambda > c_0 > 0$ large enough such that

$$M \int_{0}^{\infty} |f(t)||\lambda - V(t)|^{\frac{1}{2}}dt \leq \frac{CM}{X^{\lambda^2}} < \frac{C_1}{\lambda^2 + \frac{\tau^2}{\beta}} < 1,$$

the solution of (3.1) can be written as $h(x) = (\lambda - V(x))^{-\frac{1}{2}}(\frac{\zeta(x)}{2})^{\frac{1}{2}}H^{(1)}(\zeta)(1 + O(\frac{1}{X^{\lambda^2}}))$. It follows that $h_n(x) = (\lambda_n - V(x))^{-\frac{1}{2}}(\frac{\zeta(x)}{2})^{\frac{1}{2}}H^{(1)}(\zeta)(1 + O(\frac{1}{X^{\lambda^2}}))$ when $n \geq n_0$ large enough. Note $h_n(x)$ is also the solution of (3.1). Titchmarsh([32, 33]) shows $C_n \sim \frac{\ell}{X^{\beta^2}}$ and thus finish the proof. □

**Remark 4.6** For (4.6) we let $R$ large enough. It follows when $n \geq n_0$, $\lambda_n \geq \lambda_{n_0} \geq V(R) \geq c_0 > 0$.

Proof of Lemma 3.4. From Remark 4.6 we obtain $X_n \geq R$, then we discuss it under 3 cases.

Case 1. When $x \geq X_n$, by (3.2) one obtains

$$V(x) - \lambda_n = V'(\xi)(x - X_n) \geq V'(X_n)(x - X_n) \geq \frac{V(X_n)}{X_n}(x - X_n) \geq D_1X_n^{2\ell - 1}(x - X_n), \quad \xi \in (X_n, x).$$

From a straightforward integral estimation, we obtain the second one in (3.8).

Case 2. When $\frac{\lambda_n}{2} < x < X_n$, we have

$$\lambda_n - V(x) = V'(\xi)(X_n - x) \leq V'(X_n)(X_n - x) \leq \frac{C_1V(X_n)}{X_n}(X_n - x) \leq C_1D_2X_n^{2\ell - 1}(X_n - x), \quad \xi \in (X_n, x).$$

Similarly, combining (3.2) with Lemma 4.7 we have $V(\frac{\lambda_n}{2}) \geq 2^{-C_1}V(X_n)$, then

$$\lambda_n - V(x) \geq V'(\frac{\lambda_n}{2})(X_n - x) \geq \frac{2V(\frac{\lambda_n}{2})}{X_n}(X_n - x) \geq 2^{1-C_1}D_1X_n^{2\ell - 1}(X_n - x).$$

Case 3. When $0 < x < \frac{\lambda_n}{2}$, then $\frac{\lambda_n}{2} < X_n - x < X_n$. We discuss it under 2 subcases as follows. Subcase 3.1: $X_n \geq 2R$. By (3.3) and Lemma 4.7 we have $|V(x)| \leq V(\frac{\lambda_n}{2}) \leq \frac{1}{2}\lambda_n$, then

$$\lambda_n - V(x) \geq \frac{1}{2}\lambda_n \geq \frac{D_1}{2}X_n^{2\ell} \geq \frac{D_1}{2}X_n^{2\ell - 1}(X_n - x), \quad \forall \, x \in [0, \frac{\lambda_n}{2})$$

$$\lambda_n - V(x) \leq \frac{3}{2}\lambda_n \leq \frac{3D_2}{2}X_n^{2\ell} \leq 3D_2X_n^{2\ell - 1}(X_n - x), \quad \forall \, x \in [0, \frac{\lambda_n}{2}).$$

Subcase 3.2: $R \leq X_n < 2R$. Denote $\mathcal{N} = \{n \in \mathbb{Z}_+: R \leq X_n < 2R\}$, then $|\mathcal{N}| \ll \infty$. From (3.3) one has $|V(x)| \leq \lambda_n$, then $\lambda_n - V(x) \leq 2\lambda_n \leq 4D_2X_n^{2\ell - 1}(X_n - x), \quad \forall \, x \in [0, \frac{\lambda_n}{2})$. On the other side, consider the function $\lambda_n - V(x)$ on $x \in [0, \frac{\lambda_n}{2})$. Since $\frac{\lambda_n}{2} < R$, then from (3.3) one
has $\lambda_n - V(x) > \lambda_n - V(R) \geq 0$. By continuity of $V(x)$ there exists a positive $c_n > 0$ such that $\lambda_n - V(x) \geq c_n$ for any $x \in [0, \frac{2\pi}{n}]$. Define $c_0 := \min_{n \in \mathbb{N}} \{c_n\}$, then for any $n \in \mathbb{N}$ one obtains

$$\lambda_n - V(x) \geq \frac{c_0}{X_n^{2\ell - 1}(X_n - x)} X_n^{2\ell - 1}(X_n - x) \geq \frac{c_0}{(2R)^{2\ell}} X_n^{2\ell - 1}(X_n - x), \quad \forall x \in [0, \frac{2\pi}{n}].$$

Combining all the above cases we obtain the first estimate in (3.7). The rest is clear. □

Proof of Lemma 3.17. Since $X_m + X_m^{-\frac{1}{2}} \leq 2X_m < \frac{1}{2}X_n \leq X_n - X_n^{-\frac{1}{2}}$, then we split the integral into five parts as:

$$\int_{X_m}^{X_n} f(x)e^{ikx}h_m(x)h_n(x)dx = \left( \int_{X_m}^{X_m + X_m^{-\frac{1}{2}}} + \int_{X_m + X_m^{-\frac{1}{2}}}^{2X_m} + \int_{2X_m}^{X_n} + \int_{X_n}^{X_n + X_n^{-\frac{1}{2}}} + \int_{X_n + X_n^{-\frac{1}{2}}}^{X_n} \right)dx.$$

By Lemma 3.4 we have $\lambda_n - V(X_m + X_m^{-\frac{1}{2}}) \geq \lambda_n - V(\frac{2\pi}{m}) \geq \frac{c_0}{X_n^{2\ell}}$, then

$$\left| \int_{X_m}^{X_m + X_m^{-\frac{1}{2}}} \mathcal{F}(x)dx \right| \leq C\frac{X_m^{-\ell + 1}}{X_n^{2\ell - \ell + 1}X_m^\ell} \int_{X_m}^{X_m + X_m^{-\frac{1}{2}}} (V(x) - \lambda_m)^{-\frac{1}{4}}(\lambda_n - V(x))^{-\frac{1}{4}}dx$$

$$\leq C\frac{X_m^{-\ell + 1}}{X_n^{2\ell - \ell + 1}X_m^\ell} (\lambda_n - V(X_m + X_m^{-\frac{1}{2}}))^{-\frac{1}{4}} \int_{X_m}^{X_m + X_m^{-\frac{1}{2}}} (x - X_m)^{-\frac{1}{4}}dx \leq C(X_m X_n)^{-\frac{1}{2} - \frac{1}{4}}.$$

Then we turn to the second part. By Lemma 3.4 we have $\lambda_n - V(2X_m) \geq \lambda_n - V(\frac{2\pi}{m}) \geq \frac{c_0}{X_n^{2\ell}} X_n^{2\ell}$ and $|\zeta_m(x)| \geq A_1X_m^{\ell - \frac{1}{2}}(x - X_m) \geq A_1X_m^{\ell - \frac{1}{2}}$ for $x \geq X_m + X_m^{-\frac{1}{2}}$. Therefore,

$$\left| \int_{X_m + X_m^{-\frac{1}{2}}}^{2X_m} \mathcal{F}(x)dx \right| \leq C\frac{X_m^{-\ell + 1}}{X_n^{2\ell - \ell + 1}X_m^\ell} \int_{X_m + X_m^{-\frac{1}{2}}}^{2X_m} (V(x) - \lambda_m)^{-\frac{1}{4}}(\lambda_n - V(x))^{-\frac{1}{4}}e^{i\zeta_m(x)}dx$$

$$\leq C\frac{X_m^{-\ell + 1}}{X_n^{2\ell - \ell + 1}X_m^\ell} (\lambda_n - V(2X_m))^{-\frac{1}{4}} e^{-C_0X_m^{\ell - 1}} \int_{X_m + X_m^{-\frac{1}{2}}}^{2X_m} (x - X_m)^{-\frac{1}{4}}e^{-C_0(x - X_m)}dx$$

$$\leq C\frac{X_m^{-\ell + 1}}{X_n^{2\ell - \ell + 1}X_m^\ell} \int_0^\infty t^{-\frac{1}{4}}e^{-t}dt \leq C(X_m X_n)^{-\frac{1}{2} - \frac{1}{4}}.$$

For the third part, from $x \geq 2X_m$ we have $\frac{2\pi}{2} \leq X_m \leq x$ and $|\zeta_m(x)| \geq A_1X_m^{\ell}(x - X_m) \geq A_1X_m^{\ell + \frac{1}{2}}$. It follows

$$\left| \int_{2X_m}^{X_n - X_n^{-\frac{1}{2}}} \mathcal{F}(x)dx \right| \leq C\frac{X_m^{-\ell + 1}}{X_n^{2\ell - \ell + 1}X_m^\ell} \int_{2X_m}^{X_n - X_n^{-\frac{1}{2}}} x^{\ell}(V(x) - \lambda_m)^{-\frac{1}{4}}(\lambda_n - V(x))^{-\frac{1}{4}}e^{i\zeta_m(x)}dx$$

$$\leq C\frac{X_m^{-\ell + 1}}{X_n^{2\ell - \ell + 1}X_m^\ell} (\lambda_n - V(2X_n))^{-\frac{1}{4}} e^{-C_0X_n^{\ell - 1}} \int_{2X_m}^{X_n - X_n^{-\frac{1}{2}}} (x - X_m)^{-\frac{1}{4}}e^{-C_0(x - X_m)}dx \leq C(X_m X_n)^{-\frac{1}{2} - \frac{1}{4}}.$$

Then we turn to the fourth part, from $x \geq \frac{2\pi}{m}$ we have $x - X_m \geq \frac{2\pi}{m} - \frac{2\pi}{4} = \frac{2\pi}{4m}$ and $\lambda_n - V(X_n - X_n^{-\frac{1}{2}}) \geq A_1X_m^{2\ell - \frac{1}{2}}$ and $|\zeta_m(x)| \geq A_1X_m^{\ell}(x - X_m) \geq \frac{2\pi}{4m}X_m^{\ell}$. It follows

$$\left| \int_{\frac{2\pi}{m}}^{X_n - X_n^{-\frac{1}{2}}} \mathcal{F}(x)dx \right| \leq C\frac{X_m^{-\ell + 1}}{X_n^{2\ell - \ell + 1}X_m^\ell} \int_{\frac{2\pi}{m}}^{X_n - X_n^{-\frac{1}{2}}} (V(x) - \lambda_m)^{-\frac{1}{4}}(\lambda_n - V(x))^{-\frac{1}{4}}e^{i\zeta_m(x)}dx$$

$$\leq C\frac{X_m^{-\ell + 1}}{X_n^{2\ell - \ell + 1}X_m^\ell} (\lambda_n - V(X_n - X_n^{-\frac{1}{2}}))^{-\frac{1}{4}} e^{-C_0X_n^{\ell}} \int_{\frac{2\pi}{m}}^{X_n - X_n^{-\frac{1}{2}}} (x - X_m)^{-\frac{1}{4}}e^{-C_0(x - X_m)}dx$$

$$\leq C\frac{X_m^{-\ell + 1}}{X_n^{2\ell - \ell + 1}X_m^\ell} \int_0^\infty t^{-\frac{1}{4}}e^{-t}dt \leq C(X_m X_n)^{-\frac{1}{2} - \frac{1}{4}}.$$
For the last part, from $X_n - X_n^{-\frac{1}{4}} \geq \frac{1}{2} X_n \geq 2 X_m$, we have $V(X_n - X_n^{-\frac{1}{4}}) - \lambda_m \geq V(2X_m) - V(X_m) \geq a_1 X_m^{2\ell} + |\xi_m| \geq a_1 X_m^{\frac{\ell}{4}} X_n$. It follows

$$
\left| \int_{X_n - X_n^{-\frac{1}{4}}}^{X_n} \mathcal{F}(x) dx \right| \leq C X_m^{\ell} X_n \int_{X_n - X_n^{-\frac{1}{4}}}^{X_n} \mu(V(x) - \lambda_m)^{-\frac{1}{4}} (\lambda_m - V(x))^{-\frac{1}{4}} e^{-|\xi_m(x)|} dx
$$

$$
\leq C X_m^{\ell} X_n^{-\frac{1}{4}+\mu} e^{-\epsilon X_n} \left( \frac{(V(X_n - X_n^{-\frac{1}{4}}))}{\lambda_m} \right)^{-\frac{1}{4}} \int_{X_n - X_n^{-\frac{1}{4}}}^{X_n} (X_n - x)^{-\frac{1}{4}} dx
$$

$$
\leq C X_n^{-\frac{1}{4}} X_n^{-\frac{1}{4}} \int_{X_n - X_n^{-\frac{1}{4}}}^{X_n} (X_n - x)^{-\frac{1}{4}} dx \leq C (X_m X_n)^{\frac{\ell}{2} - \frac{1}{4}}.
$$

Similarly, we obtain

$$
\left| \int_{X_m}^{X_n} f(x) e^{i k x} \psi_j^{(m)}(x) \psi_j^{(n)}(x) dx \right| \leq C (X_m X_n)^{\frac{\ell}{2} - \frac{1}{4}}, \text{ for } j_1, j_2 \in \{1, 2\} \text{ and } j_1 + j_2 \geq 3.
$$

Hence, we have

$$
\left| \int_{X_m}^{X_n} f(x) e^{i k x} h_m(x) \eta(x) dx \right| \leq C (X_m X_n)^{\frac{\ell}{2} - \frac{1}{4}}.
$$

Proof of Lemma 3.22. Let $b = X_m - \frac{a_1}{16 \alpha_2} X_m^\frac{\ell}{2}$, then

$$
\left| \int_{X_n - X_n^{-\frac{1}{4}}}^{X_n} \mathcal{F}(x) dx \right| \leq C X_m^{\ell+\ell - 1} \int_{X_n - X_n^{-\frac{1}{4}}}^{X_n} (\lambda_m - V(x))^{-\frac{1}{4}} dx \leq C X_m^{\frac{\ell}{2} - \frac{1}{4}}.
$$

For the remainder integral on $[X_n^\frac{1}{2}, b]$, since

$$
\lambda_m - V(b) \leq a_2 X_m^{2\ell - 1} (X_m - b) - \frac{a_1}{16} X_m^{2\ell - \frac{1}{4}}, \quad \lambda_m - V(X_m - X_m^\frac{1}{2}) \geq a_1 X_m^{2\ell - \frac{1}{4}},
$$

then

$$
g(b) \geq \frac{\sqrt{a_1 k X_m^{\ell - \frac{1}{4}}}}{\sqrt{\lambda_m - V(b)} + \sqrt{\lambda_m - V(b)}} - k \geq \frac{k X_m^{\ell - \frac{1}{4}}}{\sqrt{\lambda_m - \lambda_m + \frac{a_1}{16} X_m^{2\ell - \frac{1}{4}}} + \sqrt{\alpha_2 X_m^{2\ell - \frac{1}{4}}}} - k \geq -\frac{1}{3},
$$

$$
g(X_n - X_n^\frac{1}{2}) \leq \frac{\sqrt{a_1 k X_m^{\ell - \frac{1}{4}}}}{2 \sqrt{\lambda_m - V(X_m - X_n^\frac{1}{2})}} - k \leq \frac{k \sqrt{a_1 X_m^{\ell - \frac{1}{4}}}}{2 \sqrt{\alpha_2 X_m^{\ell - \frac{1}{4}}}} - k = -\frac{k}{2}.
$$

Let $g(a) = -k X_n^\frac{1}{2}$, by the monotonicity of $g(x)$ we have $X_m - X_n^\frac{1}{2} < a < b$ as the figure 3 below. In the following we first estimate the integral on $[a, b]$. Note that $X_m \geq 8$, then $g(a) \geq -\frac{k}{2}$. Since

$$
\lambda_m - V(a) \leq a_2 X_m^{2\ell - 1} (X_m - a) \leq a_2 X_m^{2\ell - \frac{1}{4}}
$$

and $g'(x) > 0$, then

$$
g'(x) \geq g'(a) \geq \frac{V'(a)(a + k)}{2 \sqrt{\lambda_m - V(a)}} \geq C k X_m^{2\ell - 1} X_m^{\frac{1}{2} - \frac{1}{4}} \geq C k X_m^{\frac{1}{4}}, \quad \text{for } x \in [a, b].
$$

By Lemma 4.11, one obtains

$$
\left| \int_a^b f(x)^{i(k - \zeta_m + kx) x} \Psi(x) dx \right| \leq C k X_m^{\frac{1}{4}} \left( X_m \left( |\Psi(b)| + \int_a^b |\Psi(x)| dx \right) + \int_a^b x^{\mu - \frac{1}{4}} |\Psi(x)| dx \right).
$$

By Corollary 3.13 we have $|\Psi(b)| \leq C X_m^{\frac{\ell}{4} + \frac{1}{4}}$ and $\int_a^b x^{\mu - \frac{1}{4}} |\Psi(x)| dx \leq C X_m^{\frac{\ell}{4} + \frac{1}{4}}$. Besides, one has $f_a^b J_1 dx \leq C f_a^b x^{2\ell - 1} (\lambda_m - V(x))^{-\frac{1}{4}} (\lambda_m - V(x))^{-\frac{1}{4}} dx \leq C X_m^{\frac{\ell}{4} + \frac{1}{4}}$ and $f_a^b J_3 dx \leq C X_m^{\frac{\ell}{4} + \frac{1}{4}}$. It
follows that $\left| \int_{a}^{b} F(x)dx \right| \leq Ck^{-\frac{1}{2}}X_{m}^{-\frac{1}{2}}$. Next we estimate the integral on $[X_{m}^{2}, a]$, where $|g(x)| \geq kX_{m}^{-\frac{1}{4}}$. By Lemma 4.11 one has

$$\left| \int_{X_{m}^{2}}^{a} f(x)e^{i(\zeta_{n} - \zeta_{n} + kx)}\Psi(x)dx \right| \leq Ck^{-1}X_{m}^{\frac{1}{2}} \left( X_{m}^{\frac{1}{2}} \left| \Psi(a) \right| + \int_{X_{m}^{2}}^{a} |\Psi'(x)|dx \right) + \int_{X_{m}^{2}}^{a} x^{\mu - 1} |\Psi(x)|dx.$$  

Similarly, we have $\left| \int_{X_{m}^{2}}^{X_{m}^{2}} F(x)dx \right| \leq Ck^{-1}X_{m}^{-\frac{1}{2}}$. It follows that

$$\left| \int_{X_{m}^{2}}^{X_{m}^{2}} F(x)dx \right| \leq C(k^{-1} \lor 1)(X_{m}X_{n})^{\frac{1}{2} - \frac{1}{4}}.$$  

From a straightforward computation, we obtain

$$\left| \int_{X_{m}^{2}}^{X_{m}^{2}} f(x)e^{ikx} \psi_{j_{1}}^{(m)}(x) \psi_{j_{2}}^{(n)}(x)dx \right| \leq C(X_{m}X_{n})^{\frac{1}{2} - \frac{1}{4}},$$

for $j_{1}, j_{2} \in \{1, 2\}$ and $j_{1} + j_{2} \geq 3$.

Hence, we have $\left| \int_{X_{m}^{2}}^{X_{m}^{2}} f(x)e^{ikx} h_{m}(x) \overline{h_{n}(x)}dx \right| \leq C(k^{-1} \lor 1)(X_{m}X_{n})^{\frac{1}{2} - \frac{1}{4}}$.  

---

**Figure 3. Phase in Lemma 3.22**

**Figure 4. Phase in Lemma 3.23**

**Proof of Lemma 3.23.** Let $b = X_{m} - \frac{\alpha}{16\ell_{0}}X_{m}^{2}$. By Lemma 3.4 we have $\lambda_{m} - V(b) \leq \frac{\alpha}{16\ell_{0}}X_{m}^{2}\lambda_{m}$, $\alpha \frac{1}{2}X_{m}^{2} \leq \lambda_{m} - V(\frac{X_{m}}{2}) \leq \alpha \frac{1}{2}X_{m}^{2}$. Similar as before, we have $g(b) \geq \frac{k}{3}$ and $g(\frac{X_{m}}{2}) \leq \frac{k}{3}$. Let $g(a) = -\frac{k}{3}$, then $\frac{\alpha}{2} < a < b$ as the figure 4 above. We first estimate the integral on $[a, b]$. Since

$$g'(a) = \frac{V'(a)(g(a) + k)}{2\sqrt{\alpha} - V(a)\sqrt{\alpha} - V(a)} \geq \frac{CkX_{m}^{2\ell - 1}}{X_{m}^{2\ell}} \geq CkX_{m}^{-1},$$

then $g'(x) \geq CkX_{m}^{-1}$, for $x \in (a, b)$. By Lemma 4.11 we have

$$\left| \int_{a}^{b} f(x)e^{i(\zeta_{n} - \zeta_{n} + kx)}\Psi(x)dx \right| \leq Ck^{-\frac{1}{2}}X_{m}^{\frac{1}{2}} \left( X_{m}^{\frac{1}{2}} \left| \Psi(b) \right| + \int_{a}^{b} |\Psi'(x)|dx \right) + \int_{a}^{b} x^{\mu - 1} |\Psi(x)|dx.$$  

By corollary 3.13 we have $|\Psi(b)| \leq CX_{m}^{-\ell + \frac{1}{2}}$ and $\int_{a}^{b} x^{\mu - 1} |\Psi(x)|dx \leq CX_{m}^{-\ell + \frac{1}{2}}$. Besides, $\int_{a}^{b} J_{1}dx \leq CX_{m}^{-\ell + \frac{1}{2}}$ and $\int_{a}^{b} J_{3}dx \leq CX_{m}^{-\ell + \frac{1}{2}}$. It follows that $\left| \int_{a}^{b} F(x)dx \right| \leq Ck^{-\frac{1}{2}}X_{m}^{-\frac{1}{2}}$. Next we estimate the integral on $[b, X_{m} - \frac{\alpha}{16\ell_{0}}X_{m}^{2}]$. From $|g(x)| \geq \frac{k}{3}$ and Lemma 4.11, we have

$$\left| \int_{b}^{X_{m} - \frac{\alpha}{16\ell_{0}}X_{m}^{2}} f(x)e^{i(\zeta_{n} - \zeta_{n} + kx)}\Psi(x)dx \right|$$
3.24

Proof of Lemma 3.24. Let \( a = X_{m+1}^{2/3} \), \( b = (1 - \frac{1}{\sqrt{4m}})X_m \), then by Lemma 3.4 we have \( \lambda_m - V(b) \leq a_2X^{2\ell-1}(X_m - b) = \frac{a_1}{16}X_m^2 \). Hence, \( g(b) \geq \frac{\sqrt{\lambda_n - \lambda_m} + 2\sqrt{kX_m^2}}{2\sqrt{kX_m^2} + \sqrt{\frac{a_1}{16}}} - k \geq \frac{k}{3} \) as the figure 5 below. We first estimate the integral on \([b, X_m - \mathcal{X}]\). By Lemma 4.11 one has

\[
\left| \int_b^{X_m - \mathcal{X}} f(x)e^{i(\zeta_n - \zeta_n + kx)}\Psi(x)dx \right| \leq Ck^{-1}\left( X_m^\mu \left| \Psi(x) \right| + \int_b^{X_m - \mathcal{X}} |\Psi'(x)|dx + \int_b^{X_m - \mathcal{X}} x^{\mu-1} |\Psi(x)|dx \right).
\]

From similar computations, we obtain \( \left| \int_a^{X_m - \mathcal{X}} \mathcal{F}(x)dx \right| \leq Ck^{-1}X_m^{\mu - \frac{1}{2}} \). Next we estimate the integral on \([a, b]\). For \( x \in [a, b] \), we have \( \lambda_m - V(x) \leq \lambda_m - V(a) \leq a_2X_m^{2\ell-1} \), then \( g'(x) \geq \frac{CkX_m^{2\ell(2\ell-1)-kX_m^2}}{X_m^{2\ell+1}} \). By Lemma 4.11, one has

\[
\left| \int_a^b f(x)e^{i(\zeta_n - \zeta_n + kx)}\Psi(x)dx \right| \leq Ck^{-\frac{1}{2}}X_m^{\frac{3\ell}{2}} \left( X_m^\mu \left| \Psi(b) \right| + \int_a^b |\Psi'(x)|dx + \int_a^b x^{\mu-1} |\Psi(x)|dx \right).
\]

By similar procedures, we have \( \left| \int_a^{X_m - \mathcal{X}} \mathcal{F}(x)dx \right| \leq CX_m^{\mu - \frac{3\ell}{2}} \). From a straightforward computation, we obtain \( \left| \int_a^{X_m - \mathcal{X}} \mathcal{F}(x)dx \right| \leq CX_m^{\mu - \frac{3\ell}{2}} \). It follows that

\[
\left| \int_{X_m - \mathcal{X}}^{X_m - \mathcal{X}} \mathcal{F}(x)dx \right| \leq C(k^{-1} \lor 1)(X_mX_n)^{\frac{\mu-1}{4}}.
\]

Since the other three integrals have better estimates, we finish the proof. \( \square \)

Figure 5. Phase in Lemma 3.24  
Figure 6. Phase in Lemma 3.25
Proof of Lemma 3.25. Since \( \sqrt{\lambda_n} + \sqrt{\lambda_m} \leq 2\sqrt{\lambda_n} \leq 2\sqrt{D_2 X_n^t} \), then for \( x \in [X_m^{\frac{3}{2}}, X_m^{\frac{1}{2}}] \),
\[
g(x) = \frac{\lambda_n - \lambda_m}{\sqrt{\lambda_n - V(x)} + \sqrt{\lambda_m - V(x)}} - k \geq \frac{3\sqrt{D_2 k X_n^t}}{2 \sqrt{\lambda_n + \sqrt{\lambda_m}} - k} - \frac{3\sqrt{D_2 k X_n^t}}{2 \sqrt{D_2 X_n^t}} - k = \frac{k}{2}
\]
as the figure 6 above. By Lemma 4.11, one has
\[
\left| \int_{X_m^{\frac{3}{2}}}^{X_m^{\frac{1}{2}}} e^{i(\zeta_m - \zeta_m + k x)} \Psi(x) dx \right| ^2 \leq \frac{C}{k} \left( X_m^{\mu} \left( |\Psi(x_m - X_m^{\frac{1}{2}})|^2 + \int_{X_m^{\frac{3}{2}}}^{X_m^{\frac{1}{2}}} |\Psi(x)|^2 dx \right) + \int_{X_m^{\frac{3}{2}}}^{X_m^{\frac{1}{2}}} |\Psi(x)| x^{-\mu - 1} dx \right).
\]
The remaining proof is similar as Lemma 3.21. \( \square \)

**Lemma 4.7** Assume \( V(x) \) satisfies Assumption 1.1 and \( \theta \in (0, 1) \), there exists a positive constant \( \bar{R} \geq R_0 \) such that \( \theta^{C_1} V(x) \leq \theta V(x) \leq \theta^{C_1} V(x) \) for \( \theta x \geq \bar{R} \).

**Proof.** By Assumption 1.1, we have \( \lim_{x \to +\infty} V'(x) = +\infty \). Define
\[
\bar{R} := \min \left\{ x \in [R_0, +\infty) : V'(x) \geq \frac{V(R_0)}{R_0} \right\},
\]
then by (1.5) in Assumption 1.1, one has

\[
V(x) = V(R_0) + \int_{R_0}^x V'(t) dt \leq V(R_0) + V'(x)(x - R_0) \leq V'(x)x, \quad \forall x \geq \bar{R}.
\]
Together with (1.6) in Assumption 1.1, we have \( \frac{1}{x} \leq \frac{V'(x)}{V(x)} \leq \frac{C}{x} \) for all \( x \geq \bar{R} \). Given \( \theta \in (0, 1) \), if \( \theta x \geq \bar{R} \), then \( \log \frac{V'(x)}{V(x)} = \int_{\theta x}^{\bar{R}} d \log V(t) = \int_{\theta x}^{\bar{R}} \frac{V'(t)}{V(t)} dt \geq \int_{\theta x}^{\bar{R}} \frac{C}{t} dt = \log \theta^{C_1} \). Hence \( V(\theta x) \geq \theta^{C_1} V(x) \). Similarly, we have \( V(\theta x) \leq \theta V(x) \). \( \square \)

**Lemma 4.8** ([33]) For fixed \( \lambda \), if \( x > 2X \), then \( \int_x^\infty |f(t)(\lambda - V(t))^{\frac{1}{2}}| dt \leq \frac{C}{x(V(x))^\frac{1}{2}} \), here \( C \) is a constant independent of \( x \) and \( \lambda \).

**Lemma 4.9** ([33]) \( \int_0^\infty |f(x)||\lambda - V(x)|^{\frac{1}{2}} dx = O \left( \frac{1}{x \lambda^{\frac{1}{2}}} \right), \quad \lambda \to \infty. \)

The following lemma is from [21].

**Lemma 4.10** Bessel function of third kind \( H^{(1)}_\frac{1}{2}(z) \) satisfies the following:
\[
\left| \sqrt{\frac{\pi z}{2}} H^{(1)}_\frac{1}{2}(z) \right| \leq 1, \quad z \in (-\infty, -c_1),
\]
\[
\left| \sqrt{\frac{\pi z}{2}} H^{(1)}_\frac{1}{2}(z) \right| \leq \frac{20}{d_1} |z|^\frac{3}{4}, \quad z \in [-c_2, 0),
\]
\[
\left| \sqrt{\frac{\pi z}{2}} H^{(1)}_\frac{1}{2}(z) \right| \leq \frac{C e^{c_3}}{d_1} \max \{|z|^{\frac{1}{2}}, |z|^{\frac{3}{2}}\}, \quad z \in (0, c_3 i],
\]
\[
\left| \sqrt{\frac{\pi z}{2}} H^{(1)}_\frac{1}{2}(z) \right| \leq e^{-|z|}, \quad z \in (c_4, \infty)i,
\]
where \( c_1 > 0, c_2 \in (0, 1], c_3, c_4 \) can be arbitrary positive numbers and \( C \) is a positive constant.

The last lemma is from [31].
Lemma 4.11 Suppose $\phi$ is real-valued and smooth in $(a, b)$, $\psi$ is complex-valued, and that $|\phi^{(k)}(x)| \geq 1$ for all $x \in (a, b)$. Then
\[
\left| \int_a^b e^{i\lambda \phi(x)} \psi(x) dx \right| \leq c_k \lambda^{-1/k} \left( |\psi(b)| + \int_a^b |\psi'(x)| dx \right),
\]
holds when:
(i) $k \geq 2$, or (ii) $k = 1$ and $\phi(x)$ is monotonic.
The bound $c_k$ is independent of $\phi$, $\psi$ and $\lambda$.

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