Deformed Hyperkähler Structure for K3 Surfaces

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ABSTRACT

We apply the method of algebraic deformation to N-tuple of algebraic K3 surfaces. When N=3, we show that the deformed triplet of algebraic K3 surfaces exhibits a deformed hyperkähler structure. The deformation moduli space of this family of noncommutatively deformed K3 surfaces turns out to be of dimension 57, which is three times of that of complex deformations of algebraic K3 surfaces.
I. Introduction

Noncommutative geometry \cite{1} is now an integral part of string/M theory \cite{2}. Since the work of Connes, Douglas, and Schwarz \cite{3} connecting the noncommutative torus \cite{4, 5} and the T-duality in the M theory context, various properties of noncommutative space itself such as noncommutative tori and their varieties have been a subject of intensive study \cite{3, 6, 7, 8, 2}. However, more interesting and complicated structures such as noncommutative orbifolds and noncommutative Calabi-Yau (CY) manifolds have been studied far less \cite{9, 10, 11, 12, 13, 14}. Also, not much has been known about noncommutative spaces with complex structures. Only recently, noncommutative tori with complex structures have been studied \cite{15, 16, 17}.

In investigating the properties of noncommutative space with complex structure, algebraic geometry approach seems to be a good fit. In Ref. \cite{18}, Berenstein, Jejjala, and Leigh initiated an algebraic geometry approach to noncommutative moduli space. Then applying this technique, Berenstein and Leigh \cite{9} studied noncommutative CY threefolds; a toroidal orbifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ and an orbifold of the quintic in $\mathbb{CP}^4$, each with discrete torsion \cite{19, 20, 21, 22, 23}. In their first example, they deformed the covering space in such a way that the center of the deformed algebra corresponds to the commutative classical space, a CY threefold. In that process, the complex structure of the center was also deformed as a consequence of the covering space deformation, and some part of the moduli space of complex deformations was indeed recovered. They could also explain the fractionation of branes at singularities from noncommutative geometric viewpoint under the presence of discrete torsion. There, in order to be compatible with $\mathbb{Z}_2$ discrete torsion the three holomorphic coordinates $y_i$, the defining variables of the three elliptic curves of $T^6$, became to anticommute with each other.

In the commutative K3 case, the moduli space for the K3 space itself has been known already (see for instance \cite{24}), and even the moduli space for the bundles on K3 surfaces has been studied \cite{25}. In Ref. \cite{10}, algebraic deformation of K3 surfaces has been studied in the case of the orbifold $T^4/\mathbb{Z}_2$. There, the work was carried out by considering deformation of the invariants of the K3 itself, unlike the deformation of the variables of the covering space as in Ref. \cite{9}.
In Ref. [26], this method was applied for the algebraic K3 case. Classically, the complete family of complex deformations of K3 surfaces is of 20 dimension inside which that of the algebraic K3 surfaces is of 19 dimension [24]. In [26], a 19 dimensional family of the noncommutative deformations of the general algebraic K3 surfaces was considered. The construction was similar to the Connes-Lott’s “two-point space” construction of the standard model [27]. It was done by deforming a pair of algebraic K3 surfaces and was called “two-point deformation”. It was further generalized to the N-point case by considering the deformation of N-tuple of algebraic K3 surfaces. In the N-point deformation, the dimension of deformation moduli turned out to be $19N(N - 1)/2$ [26].

In this paper, we examine the N-point deformation method in the N=3 case. Considering a 57 dimensional family of noncommutatively deformed K3 surfaces, we show that the N=3 case corresponds to a noncommutative deformation of the hyperkähler structure of K3 surfaces.

In section II, we explain the method of N-point deformation for the algebraic K3 surfaces in detail for N=2. In section III, we show that for N=3 this family of noncommutatively deformed K3 surfaces exhibit deformed hyperkähler structures. In section IV, we conclude with discussion.

II. 2-point deformation

In this section, we explain the method of N-point deformation [26] of algebraic K3 surfaces, specifically for the N=2 case by considering the “two-point space” version of noncommutative deformation for a pair of algebraic K3 surfaces.

The N-point method was carried out by a direct extension of the algebraic deformation done for the $T^4/\mathbb{Z}_2$ case [10]. General algebraic K3 surfaces are given by the following form and with a point added at infinity.

$$y^2 = f(x_1, x_2)$$  \hspace{2cm} (1)

Here $f$ is a function with total degree 6 in $x_1, x_2$.

Now, we compare this with the Kummer surface, the orbifold of $T^4/\mathbb{Z}_2$ case [10]. There
was considered as the product of two elliptic curves, each given in the Weierstrass form

\[ y_i^2 = x_i(x_i - 1)(x_i - a_i) \]  

with a point added at infinity for \( i = 1, 2 \). By the following change of variables, the point at infinity is brought to a finite point:

\[ y_i \rightarrow y_i' = \frac{y_i}{x_i^2}, \]
\[ x_i \rightarrow x_i' = \frac{1}{x_i}. \]  

For algebraic K3 surfaces, we first consider a function with total degree 6 in complex variables \( u, v, w \), for instance

\[ F(u, v, w) = u^2v^3w + u^4v^2. \]

In a patch where the point at infinity of \( w \) can be brought to a finite point, dividing the both sides by \( w^6 \) the above expression can be rewritten as

\[ f(x_1, x_2) = x_1^2x_2^3 + x_1^4x_2^2 \]

where \( x_1 = \frac{u}{w}, x_2 = \frac{v}{w} \). The corresponding algebraic K3 surface is given by

\[ y^2 = f(x_1, x_2) = x_1^2x_2^3 + x_1^4x_2^2. \]  

Similarly, in a patch where the point at infinity of \( u \) can be brought to a finite point, we can reexpress it as

\[ y'^2 = f'(x_1', x_2') = x_1'^3x_2' + x_1'^2 \]

where \( x_1' = \frac{v}{u} = \frac{x_2}{x_1}, x_2' = \frac{w}{u} = \frac{1}{x_1} \). Thus, in the case of the general algebraic K3, a point at infinity in one patch can be brought to a finite point in another patch by the following change of variables

\[ y \rightarrow y' = \frac{y}{x_1^3}, \]
\[ x_1 \rightarrow x_1' = \frac{x_2}{x_1}, \]
\[ x_2 \rightarrow x_2' = \frac{1}{x_1}. \]
We now consider a noncommutative deformation of algebraic K3 surfaces. Following the same reasoning as in Ref. [10], we consider two commuting complex variables $x_1, x_2$ and two noncommuting variables $t_1, t_2$ such that

\begin{align}
  t_1^2 &= h_1(x_1, x_2), \\
  t_2^2 &= h_2(x_1, x_2),
\end{align}

where $h_1, h_2$ are commuting functions of total degree 6 in $x_1, x_2$. To be consistent with the condition that $t_1^2, t_2^2$ belong to the center, one can allow the following deformation for $t_1, t_2$.

\begin{align}
  t_1 t_2 + t_2 t_1 &= P(x_1, x_2) \tag{9}
\end{align}

Here the right hand side should be a polynomial and free of poles in each patch. Thus, under the change of variables (8)

\begin{align}
  x_1 &\longrightarrow x_1' = \frac{x_2}{x_1}, \\
  x_2 &\longrightarrow x_2' = \frac{1}{x_1},
\end{align}

t's should be changed into

\begin{align}
  t_i &\longrightarrow t_i' = \frac{t_i}{x_1'}, \quad \text{for } i = 1, 2. \tag{10}
\end{align}

This is due to the fact that t's transform just like y in (8). Therefore, $P$ transforms as

\begin{align}
  P(x_1, x_2) &\longrightarrow x_1^6 P'\left(\frac{x_2}{x_1}, \frac{1}{x_1}\right). \tag{11}
\end{align}

This implies that $P'$ should be of total degree 6 in $x_1', x_2'$, at most. Interchanging the role of $P$ and $P'$ one can see that $P$ should be also of total degree 6 in $x_1, x_2$.

The above structure was understood as follows. If the condition (9) is not imposed, then there exist two independent commutative K3 surfaces. Once the condition (9) is imposed, these two commutative K3 surfaces become a combined surface in which the two K3 surfaces intertwined each other everywhere on their surfaces and becoming fuzzy. This seems to be similar to the two-point space version of the Connes-Lott model [27]. In the Connes-Lott model, every point of the space becomes fuzzy due to the 1-to-2 correspondence at each point in the space, where the two corresponding points at each classical location are fixed. On the
other hand, the present case is similar to the relation between position $x$ and momentum $p$ in quantum mechanics at every point in the space. However, since the two copies of the classical space are combined to become a noncommutative space just like the Connes-Lott model, this construction was also called two-point deformation though its nature is a little different from the Connes-Lott’s.

To count the dimension of the deformation moduli, one simply needs to count the dimension of the polynomials of degree 6 in three variables from (11) up to constant modulo projective linear transformations of three variables. Namely, $28 - 1 - 8 = 19$, where 28 is the dimension of polynomials of degree 6 in three variables and 1 and 8 correspond to a constant and $PGL(3, \mathbb{C})$, respectively.

III. Deformed hyperkähler structure

In this section, we consider the N-point deformation for N=3. Following the method of the two-point deformation in the previous section, we consider commuting variables $x_1, x_2$ and three noncommuting variables $t_1, t_2, t_3$. Here, each $t_i^2$ should belong to the center and be a function of total degree 6 in $x_1, x_2$, such that

$$
\begin{align*}
t_1^2 &= h_1(x_1, x_2), \\
t_2^2 &= h_2(x_1, x_2), \\
t_3^2 &= h_3(x_1, x_2),
\end{align*}
$$

(12)

where $h_1, h_2, h_3$ are commuting functions of total degree 6 in $x_1, x_2$. To be consistent with the condition that $t_i^2$ belong to the center, we can allow the following deformation for $t_i$’s.

$$
t_it_j + t_jt_i = P_{ij}(x_1, x_2), \quad i, j = 1, 2, 3 \ (i \neq j).
$$

(13)

Here $P_{ij}$ should be polynomials and free of poles in each patch. Thus, when we change from one patch to another, for instance under the change of variables (7) in the previous section;

$$
\begin{align*}
x_1 &\longrightarrow x'_1 = \frac{x_2}{x_1}, \\
x_2 &\longrightarrow x'_2 = \frac{1}{x_1},
\end{align*}
$$
$t_i$ should be changed into

$$t_i \rightarrow t'_i = \frac{t_i}{x_1^3}, \quad \text{for} \quad i = 1, 2, 3. \quad (14)$$

This is due to the fact that $t_i$ transform just like $y$ in (6) in the previous section under the above change of patches. Therefore, under the above change of variables $P_{ij}$ transform as

$$P_{ij}(x_1, x_2) \rightarrow x_1^6 P'_{ij}(\frac{x_2}{x_1}, \frac{1}{x_1}). \quad (15)$$

By the same reasoning as in the two-point deformation case, one can see that each $P_{ij}$ is of total degree 6 in $x_1, x_2$, at most. It is not difficult to show that one can also get the same conclusion for different changes of patches. Here, the condition (12) for $t_i^2$ represents the different complex deformations of K3 surfaces, and its moduli space is of complex dimension 19. The condition (13) provides a characteristic of noncommutativity for otherwise three commutative (algebraic) K3 surfaces given by (12). Since each $P_{ij}$ is a polynomial of total degree 6 in $x_1, x_2$, the condition (13) makes the moduli space of the above noncommutatively deformed K3 surfaces be of complex dimension 57. This is exactly three times of the moduli dimension of complex deformations of the commutative algebraic K3 surfaces that we mentioned above. And it is different from the commutative hyperkähler K3 case, in which the moduli space is of real dimension 58 as we will discuss below. Then, what is the relationship between our newly constructed noncommutatively deformed K3 surfaces and the hyperkähler structure of commutative K3 surfaces?

Before we address this question, we first review the property of the moduli space $\mathcal{M}$ of Ricci flat metrics on a K3 surface $S$. If a given metric $g$ satisfies $g(Jv, Jw) = g(v, w)$ for any tangent vector $v, w$, then we say that the metric $g$ is compatible with the complex structure $J$. If the two form $\Omega(\cdot, \cdot) = g(J\cdot, \cdot)$ is closed, then it is called a Kähler metric and $\Omega$ is called a Kähler form. Any given Ricci-flat metric $g$ induces a Hodge $\ast$ operator on $H^2(S, \mathbb{R}) \cong \mathbb{R}^{3,19}$ by which $H^2(S, \mathbb{R})$ can be decomposed as a direct sum of two eigenspaces, self dual part (eigenvalue 1) of dimension 3 and anti-self dual part (eigenvalue $-1$) of dimension 19. The self dual part is positive definite with the integration on $S$ after wedge product, so that the moduli space of Ricci-flat metrics is locally isomorphic to $(O(3, 19)/O(3) \times O(19)) \times \mathbb{R}^+$. This is because $H^2(S, \mathbb{R})$ has the intersection form $(3,19)$ and the parameter of the scaling of the metric is $\mathbb{R}^+$. So the real dimension of $\mathcal{M}$ is $3 \times 19 + 1 = 58$. 

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We can also understand this in a different setting. Let \( \mathcal{N} = \{ (J, \Omega) \mid \Omega \text{ is a Kähler form in the K3 surface with the complex structure } J \} \). Then the real dimension of \( \mathcal{N} \) is equal to the real dimension of the moduli space of complex structures plus the real dimension of Kähler forms, which is \( 40 + 20 = 60 \). We can define a map \( \Phi \) from \( \mathcal{N} \) to \( \mathcal{M} \) as follows.

\[
\Phi((J, \Omega)) = g, \quad \text{such that } g(\cdot, \cdot) = \Omega(\cdot, J\cdot). 
\]

Then it is onto but not 1-to-1. The inverse image of \( g \) by \( \Phi \) is \( \mathbb{P}^1 \). So,

\[
\dim_{\mathbb{R}} \mathcal{M} = \dim_{\mathbb{R}} \mathcal{N} - 2 = 60 - 2 = 58.
\]

Now, we define the hyperkähler structure on \( S \). In the first setting, for the given Ricci-flat metric \( g \), the self dual part \( \Lambda^+ \) is a 3-dimensional real vector space consisting of vectors whose self intersection is positive. To any compatible complex structure \( J \) to \( g \), we associate \( \Omega \) which is a vector in \( \Lambda^+ \) and is a (1,1) form. Then real \( (2,0) \) and \( (0,2) \) forms in \( \Lambda^+ \) are exactly the orthogonal to \( \Omega \). Different compatible structures \( J \) to \( g \) correspond to different unit vectors in \( \Lambda^+ \), and they form \( S^2 \) isomorphic to \( \mathbb{P}^1 \), inverse of \( \Phi^{-1}(g) \). Here we choose three orthogonal unit vectors \( \Omega_1, \Omega_2, \Omega_3 \) in \( \Lambda^+ \) such that the corresponding complex structures \( J_1, J_2, J_3 \) satisfy the relation \( J_i J_j = \epsilon_{ijk} J_k - \delta_{ij} \) for \( i, j, k = 1, 2, 3 \). This is called a hyperkähler structure on \( S \).

Now we return to our question of the connection between our \( N=3 \) construction and the hyperkähler structure of K3 surfaces. When \( P_{ij} \) all vanish in (\( \mathbb{L}^3 \)), the \( t_i \) (\( i = 1, 2, 3 \)) in (\( \mathbb{L}^3 \)) satisfy the same relation as the complex structures \( J_i \) (\( i = 1, 2, 3 \)) in the case of the commutative hyperkähler K3 surfaces, and \( t_i \)’s actually correspond to the complex structures of the commutative K3 surfaces. We see this as follows. If we consider just one of the \( t_i \)’s and disregard other two \( t_i \)’s for a moment, then the \( t_i \) represents a family of commutative algebraic K3 surfaces whose moduli dimension is of complex dimension 19. On the other hand, when all the three \( t_i \)’s are present but all the \( P_{ij} \) vanish in (\( \mathbb{L}^3 \)), then the \( t_i \)’s are related like the \( J_i \)’s of the commutative hyperkähler K3 case. However, when all \( P_{ij} \) do not vanish and are independent of each other, \( t_i \)’s become all independent and the moduli dimension becomes three times larger than that of each piece represented by one of the \( t_i \)’s. Thus, the space becomes noncommutative under the condition (\( \mathbb{L}^3 \)) provided that all \( P_{ij} \) do not vanish.
and are independent of each other. Therefore, we can regard our new noncommutatively deformed K3 surfaces having a deformed hyperkähler structure for K3 surfaces, since $t_i$’s satisfying (12) and (13) with vanishing $P_{ij}$ do admit the hyperkähler structure $^{[24, 28]}$. \\

This we can see by redefining $t_j$ ($j = 1, 2, 3$) as $t_j = i\sqrt{h_j(x_1, x_2)}\hat{t}_j$ for $j = 1, 2, 3$. Then, in terms of $\hat{t}_j$, (12) and (13) become

\begin{align}
\hat{t}_1^2 &= -1, \\
\hat{t}_2^2 &= -1, \\
\hat{t}_3^2 &= -1, \\
\end{align}

and

\begin{equation}
\hat{t}_i\hat{t}_j + \hat{t}_j\hat{t}_i = -P_{ij}(x_1, x_2)/\sqrt{h_i h_j}, \quad i, j = 1, 2, 3 \ (i \neq j).
\end{equation}

Recall that the quaternionic structure of the hyperkähler structure of K3 surfaces can be expressed as

\begin{align}
J_i^2 &= -1, \\
J_i J_j + J_j J_i &= 0, \quad i, j = 1, 2, 3 (i \neq j).
\end{align}

Comparing with this we see that newly defined $\{\hat{t}_j\}$ exhibit a deformed hyperkähler structure for K3 surfaces.

The above construction of noncommutatively deformed hyperkähler structure for K3 surfaces is a little different from the one constructed in Ref. $^{[28, 29]}$ whose $T_i$ ($i = 1, 2, 3$) operators do the similar role as our $t_i$’s. In Ref. $^{[28, 29]}$, the commutation relation among $T_i$’s was not deformed. However, in their construction there exist extra anticommuting operators which provide holomorphic structures, and we wonder whether these additional anticommuting operators could make the two constructions equivalent.

**IV. Discussion**

In this paper, we deformed the hyperkähler and complex structures of K3 surfaces together. In the deformation of hyperkähler structure, we introduced three noncommuting variables
which correspond to three copies of commutative K3 surfaces and at the same time represent three different complex structures of K3 surfaces. Before deformation, we make these three variables have the same relation as the three complex structures $J_i$ ($i = 1, 2, 3$) of hyperkähler K3 in which $J_i$’s possess the quaternionic structure and anticommute with each other. Here, one may wonder that the deformation condition (13) could also be satisfied with commuting variables when the polynomials $P_{ij}$ do not vanish. That is possible, but the consequences are totally different depending on whether these variables are commuting or noncommuting ones. When they are commuting variables, $P_{ij}$ in (13) are not independent and they all can be expressed in terms of $h_i$ ($i = 1, 2, 3$) functions in (12). Thus, there are only complex deformations and no noncommutative deformations. On the other hand, when these variables are noncommuting ones and $P_{ij}$ are nonvanishing, then $P_{ij}$ in (13) are all independent of $h_i$ functions. Hence, we have both complex deformations from $h_i$ and noncommutative deformations from $P_{ij}$. And since our construction is a deformation from the commutative hyperkähler structure of K3 surfaces in the noncommutative direction, we end up with a noncommutatively deformed hyperkähler structure for K3 surfaces.

About the moduli dimension of our noncommutatively deformed hyperkähler structures for K3, we still do not have a clear understanding of how ours is related with the commutative one. In the commutative case, it has real moduli dimension 58 as we explained before. On the other hand, our noncommutatively deformed hyperkähler structure has complex moduli dimension 57. Apparently, ours is exactly twice of that of the commutative one, once we disregard the parameter of overall scaling in the commutative case. Thus, if we complexify the metric moduli, then it seems that we can fill the gap. In the construction of noncommutative hyperkähler structure for K3 in Ref.[28, 29], there exist a set of anticommuting operators providing the complexification. However, we do not have the corresponding variables in our construction as we mentioned briefly at the end of the last section. Since the commutation relation of the operators representing the complex structures in those works are not deformed unlike our construction, there is some possibility that our noncommuting variables may also possess the property of these anticommuting operators in Ref.[28, 29]. We will leave this investigation for our future work.
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