Partially periodic point free self-maps on surfaces, graphs, wedge sums and products of spheres

Jaume Llibre and Víctor F. Sirvent

Department de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, 08193 Barcelona, Catalonia, Spain; Departamento de Matemáticas, Universidad Simón Bolívar, Apartado 89000, Caracas 1086-A, Venezuela

(Received 9 August 2012; final version received 23 January 2013)

Let \((X,f)\) be a topological discrete dynamical system. We say that it is partially periodic point free up to period \(n\), if \(f\) does not have periodic points of periods smaller than \(n+1\). When \(X\) is a compact connected surface, a connected compact graph, or \(S^2 \vee S^m \vee \cdots \vee S^m\), we give conditions on \(X\), so that there exist partially periodic point free maps up to period \(n\). We also introduce the notion of a Lefschetz partially periodic point free map up to period \(n\). This is a weaker concept than partially periodic point free up to period \(n\). We characterize the Lefschetz partially periodic point free self-maps for the manifolds \(S^n \times \cdots \times S^n\), \(S^n \times S^m\) with \(n \neq m\), \(\mathbb{C}P^n\), \(\mathbb{H}P^n\) and \(\mathbb{O}P^n\).

Keywords: periodic point; Lefschetz number; connected compact graph; connected compact surface; wedge sums of spheres; product of spheres

2000 Mathematics Subject Classification: 37C25; 37E15; 37E25

1. Introduction

Let \(X\) be a topological space and let \(f : X \to X\) be a continuous map. A (discrete) topological dynamical system is formed by the pair \((X,f)\).

We say that \(x \in X\) is a periodic point of period \(k\) if \(f^k(x) = x\) and \(f^j(x) \neq x\) for \(j = 1, \ldots, k-1\). We denote by \(\text{Per}(f)\) the set of all periods of \(f\).

The set \(\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}\) is called the orbit of the point \(x \in X\). To study the dynamics of a map \(f\) is to study all the different kinds of orbits of \(f\). If \(x\) is a periodic point of \(f\) of period \(k\), then its orbit is \(\{x, f(x), f^2(x), \ldots, f^{k-1}(x)\}\), and it is called a periodic orbit.

Often the periodic orbits play an important role in the dynamics of a discrete dynamical system, and for studying them we can use topological tools. One of the best-known results in this direction is the result contained in the well-known paper entitled ‘Period three implies chaos’ for continuous self-maps on the interval, see [19].

If \(\text{Per}(f) = \emptyset\) then we say that the map \(f\) is periodic point free. There are several papers studying different classes of periodic point free self-maps on the annulus, see [12,16], or on the two-dimensional torus, see [2,14,18].

If \(\text{Per}(f) \cap \{1, 2, \ldots, n\} = \emptyset\) then we say that the map \(f\) is partially periodic point free up to period \(n\). If \(n = 1\), we say that \(f\) is fixed point free. Different classes of partially periodic point free self-maps are studied in [6,25,28].
Let \( n \) be the topological dimension of a compact polyhedron \( X \). We denote by \( H_k(X, \mathbb{Q}) \), for \( 0 \leq k \leq n \), the homology groups of \( X \) with coefficients over the rational numbers. They are finite dimensional vector spaces over \( \mathbb{Q} \). Given a continuous map \( f : X \to X \), it induces linear maps \( f_* : H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q}) \), for \( 0 \leq k \leq n \). All the entries of the matrices \( f_* \) are integer numbers.

The Lefschetz number \( L(f) \) is defined as

\[
L(f) = \sum_{k=0}^{n} (-1)^k \text{trace}(f_*^k).
\]

One of the main results connecting the algebraic topology with the fixed point theory is the Lefschetz Fixed Point Theorem which establishes the existence of a fixed point if \( L(f) \neq 0 \), see for instance [4]. If we consider the Lefschetz number of \( f^m \), i.e. \( L(f^m) \), then in general it is not true that \( L(f^m) \neq 0 \) implies that \( f \) has a periodic point of period \( m \); it only implies the existence of a periodic point with a periodic divisor of \( m \). The Lefschetz numbers have been used frequently for studying the set of periods of different kinds of maps, see for instance [10,20,22,23] and the references cited therein.

We say that a continuous map \( f : X \to X \) is Lefschetz periodic point free if \( L(f^m) = 0 \) for all \( m \geq 1 \). We say that the map \( f \) is Lefschetz partially periodic point free up to period \( n \) if \( L(f^m) = 0 \) for all \( 1 \leq m \leq n \). If \( n = 1 \), we say that \( f \) is Lefschetz fixed point free. These are weaker notions of periodic point free and partially periodic point free up to period \( n \), since Lefschetz periodic point free is a necessary condition to be a periodic point free, but not sufficient as it is shown by considering the identity map on the circle. The Lefschetz periodic free maps on some connected compact manifolds have been studied in [11,21].

One of the goals of this paper is to study the self-continuous maps on graphs and closed surfaces and other manifolds which are (Lefschetz) partially periodic point free up to period \( n \).

A graph is a union of vertices and edges, which are homeomorphic to the closed interval and have mutually disjoint interiors. The endpoints of the edges are vertices (not necessarily different), and the interiors of the edges are disjoint from the vertices. Some graphs are homotopic to particular cases of wedge sums of spheres, which we shall define later on. However not all graphs can be obtained as particular cases of wedge sums of spheres, which we shall define later on. Some graphs have been used frequently for studying the set of periods of different kinds of graphs, e.g. the interval or the topological space with the shape of the capital letter sigma.

Here a closed surface means a connected compact surface without boundary, orientable or not. More precisely, an orientable connected compact surface without boundary of genus \( g \geq 0 \), \( \mathbb{M}_g \), is homeomorphic to the sphere if \( g = 0 \), to the torus if \( g = 1 \), or to the connected sum of \( g \) copies of the torus if \( g \geq 2 \). An orientable connected compact surface with boundary of genus \( g \geq 0 \), \( \mathbb{M}_g,0 \), is homeomorphic to \( \mathbb{M}_g \) minus a finite number \( b > 0 \) of open discs having pairwise disjoint closure. In what follows \( \mathbb{M}_g,0 = \mathbb{M}_g \).

A non-orientable connected compact surface without boundary of genus \( g \geq 1 \), \( \mathbb{N}_g \), is homeomorphic to the real projective plane if \( g = 1 \), or to the connected sum of \( g \) copies of the real projective plane if \( g > 1 \). A non-orientable connected compact surface with boundary of genus \( g \geq 1 \), \( \mathbb{N}_g,0 \), is homeomorphic to \( \mathbb{N}_g \) minus a finite number \( b > 0 \) of open discs having pairwise disjoint closure. In what follows \( \mathbb{N}_g,0 = \mathbb{N}_g \).

**Theorem 1.** Let \( f : X \to X \) be a continuous map partially periodic point free up to period \( n \). Assume that the induced map in the first homology space \( f_* : H_1(X, \mathbb{Q}) \to H_1(X, \mathbb{Q}) \) is
invertible. Then the following statements hold.

(a) Let $X$ be a connected compact graph such that $\dim H_1(X, \mathbb{Q}) = r$. If $n \geq 2$ then $n < r$.
(b) Let $X = \mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components. Let $d$ be the degree of $f$ if $b = 0$.
(b.1) If $b = 0$, $d \neq 0$ and $n \geq 3$, then $n < 2g$ (consequently $g > 1$).
(b.2) If either $b = d = 0$ or $b > 0$ and $n \geq 2$, then $n < 2g + b - 1$.
(c) Let $X = \mathbb{N}_{g,b}$ be a non-orientable connected compact surface of genus $g \geq 1$ with $b \geq 0$ boundary components. If $n \geq 2$ then $n < g + b - 1$.

All the results stated in this introduction are proved in Section 2.

One of the first results that show the relation between the topology of a compact topological space $M$ and the existence of periodic points of a homeomorphism $f : M \to M$ is due to Fuller [8]. In particular, from Fuller’s result it follows that if $g \geq 1$ and $f : \mathbb{M}_g \to \mathbb{M}_g$, is a homeomorphism then $\text{Per}(f) \cap \{1, 2, \ldots, 2g\} \neq \emptyset$; for more detail see [7].

For homeomorphisms there are results improving Theorem 1 without boundary components by Wang [28], and with boundary components by Chas [5]. In fact, the proof of Theorem 1 uses ideas of [28].

The technique of using Lefschetz numbers to obtain information about the periods of a map is also used in many other papers, see for instance the book of Jezierski and Marzantowicz [17], the article of Gierzkiewicz and Wójcik [9] and the references cited in both.

Given topological spaces $X$ and $Y$ with chosen points $x_0 \in X$ and $y_0 \in Y$, then the 
**wedge sum** $X \vee Y$ is the quotient of the disjoint union $X \cup Y$ obtained by identifying $x_0$ and $y_0$ to a single point (for more detail, see, page 10 of [15]). The wedge sum is also known as ‘one point union’. For example, $\mathbb{S}^1 \vee \mathbb{S}^1$ is homeomorphic to the figure ‘8’, two circles touching at a point. Some graphs can be obtained as particular cases of wedge sums of $\mathbb{S}^1$, and a compact connected graph $X$ such that $\dim (H_1(X, \mathbb{Q})) = r$ is homotopic to $\mathbb{S}^1 \vee \underbrace{\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1}_{r-\text{times}}$.

**Theorem 2.** Let $X = \mathbb{S}^{2m} \vee \mathbb{S}^m \vee \underbrace{\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1}_{s-\text{times}} \vee \mathbb{S}^m$ and $f : X \to X$ be a continuous map partially periodic point free up to period $n$. Assume that the induced map in the $m$-homology space $f_{*m} : H_m(X, \mathbb{Q}) \to H_m(X, \mathbb{Q})$ is invertible.

(a) If $m$ is odd then $n < s$.
(b) If $m$ is even, $s$ odd and the degree of $f$ is $-1$ then $n < s$.

**Theorem 3.** Let $X = \mathbb{S}^n \times \underbrace{\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1}_{k-\text{times}} \vee \mathbb{S}^n$ and $f : X \to X$ be a continuous map.

(a) If $n$ is odd, $f$ is Lefschetz periodic point free if and only if $1$ is an eigenvalue of $f_{*n}$.
(b) If $n$ is even, $f$ is Lefschetz partially periodic point free up to period $m$ if $-1$ is an eigenvalue of $f_{*n}^l$, for $1 \leq l \leq m$. Moreover $f$ is not Lefschetz periodic point free. Additionally, if $m \geq 2$ then $k \geq 3$.

If $X = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ the previous result is well known, see [3,13]. The proof of Theorem 3 in the case $n$ odd is similar to the case $n = 1$.

In [11] the case $k = 2$ is considered. Necessary and sufficient conditions are given for a self-continuous map to be Lefschetz periodic free point. However those conditions are stated in a different manner; they are special cases of the present theorem.
In Proposition 4 we consider continuous self-maps on $\mathbb{S}^n \times \mathbb{S}^m$, with $n \neq m$. We provide conditions in order that they are Lefschetz (partially) periodic point free. The results presented here are generalizations of results contained in [11]. We also consider continuous self-maps on $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{O}P^n$, the $n$-dimensional projective plane over the complex numbers, the quaternions and the octonions. We also provide conditions in order that they are Lefschetz (partially) periodic point free. We present them as examples of the techniques expounded in this article.

**Proposition 4.** Let $X = \mathbb{S}^n \times \mathbb{S}^m$ and $f : X \to X$ be a continuous map with induced maps on homology $f_{\ast n} = a$, $f_{\ast m} = b$ and $f_{\ast n+m} = d$.

(a) If $n$ and $m$ are even and $1 + a + b + d \neq 0$, then $f$ has fixed points. If $1 + a + b + d = 0$, then the map $f$ is Lefschetz fixed point free and it is not Lefschetz partially periodic point free up to period 2.

(b) If both $n$ and $m$ are not even and if $f$ is Lefschetz partially periodic point free up to period 2, then $f$ is Lefschetz periodic point free.

**Example 1.** Let $X = \mathbb{C}P^n$ be the $n$-dimensional complex projective space and $f : X \to X$ be a continuous map with induced map in homology $f_{\ast 2} = a$. Then the map $f$ is Lefschetz fixed point free if $a = -1$ and $n$ odd, otherwise the map $f^m$ has fixed points for all $m \geq 1$.

**Example 2.** Let $X = \mathbb{H}P^n$ ($X = \mathbb{O}P^n$, respectively) be the $n$-dimensional quaternion (octonionic, respectively) projective space and $f : X \to X$ be a continuous map with induced map in homology $f_{\ast 4} = a$, ($f_{\ast 8} = a$, respectively). Then the map $f$ is Lefschetz fixed point free if $a = -1$ and $n$ odd, otherwise the map $f^m$ has fixed points for all $m \geq 1$.

For more information on the complex projective spaces $\mathbb{C}P^n$ or the quaternion projective spaces $\mathbb{H}P^n$ see for instance [27], and for the octonionic projective spaces $\mathbb{O}P^n$, see [1].

2. Proof of theorems and propositions

We separate the proof for the different statements of Theorem 1.

**Proof of statement (a) of Theorem 1.** Let $X$ be a connected compact graph. Since the continuous map $f : X \to X$ does not have periodic points up to period $n$, by the Lefschetz fixed point theorem we have

$$ L(f) = L(f^2) = \cdots = L(f^n) = 0. \tag{1} $$

Due to the fact that $X$ is connected we know that $f_{\ast 0} = (1)$, i.e. $f_{\ast 0}$ is the identity of $H_0(X, \mathbb{Q}) = \mathbb{Q}$, for more detail see [24,27]. From the definition of the Lefschetz number and if $\alpha_j = \text{trace}(f_{\ast j})$, we have

$$ L(f^j) = \text{trace}(f_{\ast 0}^j) - \text{trace}(f_{\ast 1}^j) = 1 - \alpha_j = 0. $$

Therefore, $\alpha_j = 1$ for $1 \leq j \leq n$. Let $r = \dim H_1(X, \mathbb{Q})$ and $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of $f_{\ast 1}$, so

$$ \alpha_j = \sum_{i=1}^{r} \lambda_i. $$
The characteristic polynomial of \( f_{*1} \) is

\[
p(x) = \prod_{j=1}^{r} (x - \lambda_j) = x^r - a_1 x^{r-1} + \cdots + (-1)^r a_r.
\]

Due to Newton’s formulae for symmetric polynomials (see for instance [26]), we have

\[
\begin{align*}
\alpha_1 - a_1 &= 0, \\
\alpha_2 - a_1 \alpha_1 + 2a_2 &= 0, \\
\alpha_3 - a_1 \alpha_2 + 2a_1 \alpha_1 - 3a_3 &= 0, \\
&\vdots \\
\alpha_r + \sum_{i=1}^{r-1} (-1)^i a_{i} \alpha_{r-i} + (-1)^r a_r &= 0.
\end{align*}
\]

So we get \( a_1 = \alpha_1 = 1, \ a_2 = 0, \) and by induction \( a_j = 0 \) for \( 2 \leq j \leq n. \) Since \( a_r = \det(f_{*1}) \neq 0, \) if \( n \geq 2 \) then \( n < r, \) and statement (a) is proved.

\[\Box\]

**Proof of statement (b) of Theorem 1.** Let \( X = M_{g,b} \) be an orientable connected compact surface of genus \( g \geq 0 \) with \( b \geq 0 \) boundary components. Since the continuous map \( f : X \to X \) does not have periodic points up to period \( n, \) by the Lefschetz fixed point theorem we have (1).

Suppose that the degree of \( f \) is \( d \) if \( b = 0. \) We recall the homology groups of \( M_{g,b} \) with coefficients in \( \mathbb{Q}, \) i.e.

\[
H_k(M_{g,b}, \mathbb{Q}) = \mathbb{Q}^{n_0} \oplus \cdots \oplus \mathbb{Q},
\]

where \( n_0 = 1, \ n_1 = 2g \) if \( b = 0, \ n_1 = 2g + b - 1 \) if \( b > 0, \) \( n_2 = 1 \) if \( b = 0, \) and \( n_2 = 0 \) if \( b > 0; \) and the induced linear maps \( f_{*0} = (1), f_{*2} = (d) \) if \( b = 0, \) and \( f_{*2} = 0 \) if \( b > 0 \) (see for additional details [24,27]). In the next computations we must take \( d = 0 \) if \( b > 0. \)

From the definition of the Lefschetz number and if \( \alpha_j = \text{trace}(f_{*1}^j), \) we have

\[
L(f^j) = \text{trace}(f_{*0}^j) - \text{trace}(f_{*1}^j) + \text{trace}(f_{*2}^j) = 1 - \alpha_j + d^j = 0.
\]

Therefore, \( \alpha_j = 1 + d^j \) for \( 1 \leq j \leq n. \) Now the characteristic polynomial of \( f_{*1} \) is (2) with \( r = n_1. \) Using the Newton’s formulae (3) we get \( a_1 = 1 + d, \ a_2 = d, \ a_3 = 0, \) and by induction \( a_j = 0 \) for \( 3 \leq j \leq n. \) Note that if \( b > 0 \) then \( a_j = 0 \) for \( 2 \leq j \leq n. \) Since \( a_{n_1} = \det(f_{*1}) \neq 0, \) if \( n \geq 3 \) then \( n < n_1, \) and statement (b) is proved.

\[\Box\]

**Proof of statement (c) of Theorem 1.** Let \( X = N_{g,b} \) be a non-orientable connected compact surface of genus \( g \geq 1 \) with \( b \geq 0 \) boundary components. Since the continuous map \( f : X \to X \) does not have periodic points up to period \( n, \) by the Lefschetz fixed point theorem we have (1).

We recall the homology groups of \( N_{g} \) with coefficients in \( \mathbb{Q}, \) i.e.

\[
H_k(N_{g,b}, \mathbb{Q}) = \mathbb{Q}^{n_0} \oplus \cdots \oplus \mathbb{Q},
\]
where \( n_0 = 1, n_1 = g + b - 1 \) and \( n_2 = 0 \); and the induced linear map \( f_{*0} = (1) \) (see again for additional details \([24,27]\)).

From the definition of the Lefschetz number and if \( \alpha_j = \text{trace}(f_{j}) \), we have
\[
L(f^j) = \text{trace}(f_{j0}^j) - \text{trace}(f_{j1}^j) = 1 - \alpha_j = 0.
\]
Therefore, \( \alpha_j = 1 \) for \( 1 \leq j \leq n \). Now the characteristic polynomial of \( f_{*1} \) is (2) with \( r = n_1 \). Using the Newton’s formulae (3) we get \( a_1 = 1, a_2 = 0 \), and by induction \( a_j = 0 \) for \( 2 \leq j \leq n \). Since \( a_{n_1} = \det(f_{*1}) \neq 0 \), if \( n \geq 2 \) then \( n < n_1 \), and statement (c) is proved.

**Proof of Theorem 2.** Let \( X = \mathbb{S}^{2m} \vee \mathbb{S}^m \vee s\cdot\text{times} \vee \mathbb{S}^m \). Using the properties of the wedge sum (see page 160 of \([15]\)), the homology spaces of \( X \) with coefficients in \( \mathbb{Q} \) are as follows:
\[
H_k(X, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q},
\]
where \( n_0 = n_2m = 1, n_m = s \) and 0 otherwise. So the non-trivial induced linear maps are \( f_{*0}, f_{*m} \text{ and } f_{*2m} \text{ where } f_{*0} = 1 \text{ and } f_{*2m} = (d), \text{ the degree of } f. \)

We adapt here the argument used in the proof of Theorem 1. Using the fact that \( m \) is odd, the Lefschetz numbers of the iterates of \( f \) are as follows:
\[
L(f^j) = \text{trace}(f_{j0}^j) - \text{trace}(f_{j1}^j) + \text{trace}(f_{j2}^j) = 1 - \alpha_j + d^j,
\]
where \( \alpha_j = \text{trace}(f_{j0}^j) \). If \( f \) is partially periodic point free up to period \( n \). Then \( L(f^j) = 0 \) for \( 1 \leq j \leq n \). Now the characteristic polynomial of \( f_{*m} \) is (2) with \( r = n_m = s \). Using the Newton’s formulae (3) we get \( a_1 = 1 + d, a_2 = d \) and \( a_j = 0 \) for \( 3 \leq j \leq n \). Since \( a_s = \det(f_{*m}) \neq 0 \) then \( n < n_m = s \). This completes the proof of statement (a).

If \( m \) is even, then \( L(f^j) = 1 + \alpha_j + d^j \). Assuming \( L(f^j) = 0 \) for \( 0 \leq j \leq n \), using the Newton’s formulae (3) and induction we get
\[
\alpha_j = (-1)^j(1 + d + \cdots + d^j),
\]
for \( 1 \leq j \leq n \). If \( d = -1 \) and \( j \) odd, then \( \alpha_j = 0 \). So, if \( s \) is odd and \( s \leq n \), then \( a_s = 0 \). This contradicts the hypothesis of \( \det(f_{*m}) \neq 0 \). Therefore \( n < s \). This completes the proof of statement (b).

**Proof of Theorem 3.** Let \( X = \mathbb{S}^n \times \mathbb{S}^n \). According to the Künneth theorem, the homology groups of \( X \) over \( \mathbb{Q} \) are
\[
H_j(X, \mathbb{Q}) = \mathbb{Q} \oplus c(k, j) \mathbb{Q},
\]
where \( c(k, j) = \binom{k}{j} = k!/(j!(k-j)!) \), for \( 0 \leq j \leq k \); and \( H_l(X, \mathbb{Q}) \) is trivial if \( l \) is not of the form \( jn \).

Taking the inclusion of \( \mathbb{Q} \) into \( \mathbb{C} \), we get
\[
H_j(X, \mathbb{C}) = \mathbb{C} \oplus c(k, j) \mathbb{C}.
\]
Using the fact that the cohomology \( H^{jn}(X, \mathbb{C}) \) is the dual of \( H_j(X, \mathbb{C}) \), we get \( \{H^{jn}(X, \mathbb{C})\}_{j=0}^{k} \) is an exterior algebra over \( \mathbb{C} \) with \( k \) generators of dimension 1, for more
detail see [27] (p. 203). Since \( H^Jn(X, \mathbb{C}) \) is torsion free for \( 1 \leq j \leq k \), we compute the Lefschetz numbers over \( \{ H^Jn(X, \mathbb{C}) \}_j \) for \( j = 0 \).

Let \( A = (a_{ij})_{1 \leq i, j \leq k} \) be a complex matrix that represents \( f^m \), the induced map on the cohomology group \( H^Jn(X, \mathbb{C}) \), since \( \mathbb{C} \) is algebraically closed we can suppose that \( A \) is an upper (or lower) triangular matrix. Since \( f^m \) is the dual of \( f^m \), the trace of \( A \), or \( f^m \), is \( \sum_{i=1}^{k} a_{ii} \). By the exterior algebra structure the trace of \( f^m \) is \( \sum_{i<j} a_{ii}a_{jj} \) and the trace of \( f^m \) is

\[
\sum_{i_1<\ldots<i_j} a_{i_1i_1}\ldots a_{i_ji_j}.
\]

So, using the fact that \( n \) is odd, the Lefschetz number of \( f \) is

\[
L(f) = \operatorname{trace}(f_{*0}) + (-1)^{n} \operatorname{trace}(f_{*n}) + \cdots + (-1)^{nk} \operatorname{trace}(f_{*kn})
\]

\[
= 1 + (-1)^{n} \sum_{i=1}^{k} a_{ii} + (-1)^{2n} \sum_{i<j} a_{ii}a_{jj} + \cdots + (-1)^{nk} a_{11}a_{22}\cdots a_{kk}
\]

\[
= (1 - a_{11})(1 - a_{22})\cdots(1 - a_{kk})
\]

\[
= \det(Id - A).
\]

Similarly the trace of \( f^m \) is

\[
\sum_{i_1<\ldots<i_j} d_{i_1i_1}^m\ldots d_{i_ji_j}^m.
\]

So \( L(f^m) = \det(Id - A^m) \). Therefore \( L(f) = \cdots = L(f^m) = 0 \) if and only if 1 is an eigenvalue of \( A^l \) for \( 1 \leq l \leq m \). Hence \( f \) is Lefschetz periodic point free if and only if 1 is an eigenvalue of \( f^m \). This completes the proof of statement (a).

If \( n \) is even

\[
L(f) = \operatorname{trace}(f_{*0}) + (-1)^n \operatorname{trace}(f_{*n}) + \cdots + (-1)^nk \operatorname{trace}(f_{*kn})
\]

\[
= 1 + \sum_{i=1}^{k} a_{ii} + \sum_{i<j} a_{ii}a_{jj} + \cdots + a_{11}a_{22}\cdots a_{kk}
\]

\[
= (1 + a_{11})(1 + a_{22})\cdots(1 + a_{kk})
\]

\[
= \det(Id + A).
\]

Similarly \( L(f^m) = \det(Id + A^m) \). Therefore \( L(f) = \cdots = L(f^m) = 0 \) if and only if \(-1\) is an eigenvalue of \( A^l \) for \( 1 \leq l \leq m \).

The matrix \( A \) is \( k \times k \), if \(-1\) is an eigenvalue of \( A^l \) for \( 0 \leq l \leq k \); then \((-1)^{l/l} \) is an eigenvalue of \( A \). We consider \( A^2 \), with \( s = 2k! \), then its eigenvalues are of the form

\[
(-1)^{2k!/l} = (-1)^{2(k-1)\cdots(l+1)(l-1)\cdots2} = 1.
\]

Hence \(-1\) is the only eigenvalue of \( A^2 \). So \( L(f^2) \neq 0 \), therefore \( f \) is not Lefschetz periodic point free.

If \( f \) is Lefschetz periodic point free up to order \( m \), with \( m \geq 2 \) then \(-1\) and \( \sqrt{-1} \) are roots of the characteristic polynomial of \( A \), so its degree should be greater than or equal to 3. Since the degree of the characteristic polynomial of the matrix \( A \) is \( k \), then \( k \geq 3 \), if \( m \geq 2 \).

Proof of Proposition 4. The homology groups of \( \mathbb{S}^n \times \mathbb{S}^m \) over \( \mathbb{Q} \) are \( H_l(\mathbb{S}^n \times \mathbb{S}^m, \mathbb{Q}) = \mathbb{Q} \), if \( l = 0, n, m \) or \( n + m \) and trivial for the other values of \( l \). If the
induced maps on homology \( f \) are \( f^*_{ln} = a, f^*_{lm} = b \) and \( f^*_{ln+m} = d \); the maps \( f^*_{l} \) are trivial if \( l \) is different from \( 0, n, m \) and \( n + m \), the Lefschetz numbers for the iterates of \( f \) are

\[
L(f^l) = 1 + (-1)^n a^l + (-1)^m b^l + (-1)^{n+m}d^l,
\]

for all \( l \geq 1 \).

If \( n \) and \( m \) are even we get that \( L(f^2) = 1 + a^2 + b^2 + d^2 \), so \( L(f^2) \neq 0 \). So \( f \) is not Lefschetz periodic point free up to period 2 and it is Lefschetz fixed point free if \( 1 + a + b + d = 0 \). This completes the proof of statement (a).

If \( n \) is even and \( m \) odd then the solution of the linear system \( L(f) = L(f^2) = 0 \) is \( a = b \) and \( d = 1 \), or \( a = d \) and \( b = 1 \), so this implies that \( L(f^l) = 0 \) for \( l > 2 \). Similarly for the other cases when \( n \) and \( m \) are not simultaneously even. Therefore \( f \) is Lefschetz periodic point free. This completes the proof of statement (b).

\[ \square \]

**Example 1.** The homology groups of \( \mathbb{C}P^n \) over \( \mathbb{Q} \) are \( H_{2l}(\mathbb{C}P^n, \mathbb{Q}) = \mathbb{Q} \), if \( 0 \leq l \leq n \) and are trivial otherwise. So the induced maps on homology are \( f^*_{2l} = a^l \) if \( 0 \leq l \leq n \), with \( a \in \mathbb{Z} \), and \( f^*_{l} = 0 \) if \( l \) is odd. Therefore \( f^*_{2l} = (a^l)^m \) for \( 0 \leq l \leq n \). Hence the Lefschetz numbers of the iterates of \( f \) are:

\[
\begin{align*}
L(f) &= 1 + a + \cdots + a^n, \\
L(f^2) &= 1 + a^2 + \cdots + a^{2n}, \\
& \quad \vdots \\
L(f^m) &= 1 + a^m + \cdots + a^{mn}.
\end{align*}
\]

If \( |a| \neq 1 \) then \( L(f^m) = (1 - a^{n(n+1)})/(1 - a^m) \neq 0 \) for all \( m \), so \( f^m \) has fixed points. If \( a = 1 \) is clear that \( L(f^m) \neq 0 \), for all \( m \). If \( a = -1 \) the result depends on \( n \) being odd or even, if \( n \) is even then \( L(f^m) \neq 0 \), for all \( m \). If \( a = -1 \) and \( n \) odd then \( L(f) = 0 \) and \( L(f^2) \neq 0 \), so \( f \) is Lefschetz fixed point free.

**Example 2.** The homology groups of \( \mathbb{H}P^n \) over \( \mathbb{Q} \) are \( H_{4l}(\mathbb{H}P^n, \mathbb{Q}) = \mathbb{Q} \), if \( 0 \leq l \leq n \) and trivial otherwise. So the induced maps on homology are \( f^*_{4l} = a^l \) if \( 0 \leq l \leq n \), with \( a \in \mathbb{Z} \), and \( f^*_{l} = 0 \) if \( l \) is not a multiple of 4.

On the other hand the homology groups of \( \mathbb{O}P^n \) over \( \mathbb{Q} \) are \( H_{8l}(\mathbb{O}P^n, \mathbb{Q}) = \mathbb{Q} \), if \( 0 \leq l \leq n \) and otherwise it is trivial. So the induced maps on homology are \( f^*_{8l} = a^l \) if \( 0 \leq l \leq n \), with \( a \in \mathbb{Z} \), and \( f^*_{l} = 0 \) otherwise.

Now the computation follows in a similar manner as in the case of \( \mathbb{C}P^n \).

**Acknowledgements**

The authors would like to thank Matija Cencelj for pointing out the space considered in Theorem 2. The second author would like to thank the Department of Mathematics at the Universitat Autònoma de Barcelona for its hospitality. The research of this article was partially done during a visit of the second author to this institution.

The first author is partially supported by a MICINN/FEDER Grant Number MTM2008–03437, by an AGAUR Grant Number 2009SGR–410, by ICREA Academia and by FP7-PEOPLE-2012-IRSES-316338.
Note
1. Email: jllibre@mat.uab.cat; URL: http://www.gsd.uab.cat/personal/jllibre

References
[1] J.C. Baez, *The octonions*, Bull. Am. Math. Soc. (N.S.) 39(2) (2002), pp. 145–205.
[2] M.M. Barge and R.B. Walker, *Periodic point free maps of tori which have rotation sets with interior*, Nonlinearity 6 (1993), pp. 481–489.
[3] R.B. Brooks, R.F. Brown, J. Pak, and D. Taylor, *Nielsen numbers of maps of tori*, Proc. Am. Math. Soc. 52 (1975), pp. 398–400.
[4] R.F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Company, Glenview, IL, 1971.
[5] M. Chas, *Minimum periods of homeomorphisms of orientable surfaces*, Ph.D. thesis, Universitat Autònoma de Barcelona (1998).
[6] P.E. Conner and E.E. Floyd, *On the construction of periodic maps without fixed points*, Proc. Am. Math. Soc 10 (1959), pp. 354–360.
[7] J. Franks and J. Llibre, *Periods of surface homeomorphisms*, Contemp. Math. 117 (1991), pp. 63–77.
[8] F.B. Fuller, *The existence of periodic points*, Ann. Math. 57 (1953), pp. 229–230.
[9] A. Gierzkiewicz and K. Wójcik, *Lefschetz sequences and detecting periodic points*, Discrete Contin. Dyn. Syst. Ser. A 32 (2012), pp. 81–100.
[10] J.L.G. Guirao and J. Llibre, *Minimal Lefschetz sets of periods for Morse–Smale diffeomorphisms on the n-dimensional torus*, J. Difference Equ. Appl. 16 (2010), pp. 689–703.
[11] J.L.G. Guirao and J. Llibre, *On the Lefschetz periodic point free continuous self-maps on connected compact manifolds*, Topol. Appl. 158 (2011), pp. 2165–2169.
[12] G.R. Hall and M. Turpin, *Robustness of periodic point free maps of the annulus*, Topol. Appl. 69 (1996), pp. 211–215.
[13] B. Halpern, *Periodic points on tori*, Pac. J. Math. 83(1) (1979), pp. 117–133.
[14] M. Handel, *Periodic point free homeomorphism of T^2*, Proc. Am. Math. Soc. 107 (1989), pp. 511–515.
[15] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, UK, 2002.
[16] T. Jäger, *Periodic point free homeomorphisms of the open annulus: From skew products to non-fibred maps*, Proc. Am. Math. Soc. 138 (2010), pp. 1751–1764.
[17] J. Jezierski and W. Marzantowicz, *Homotopy Methods in Topological Fixed and Periodic Points Theory*, Topological Fixed Point Theory and Its Applications, Springer, Dordrecht, 2006.
[18] J. Kwapisz, *A priori degeneracy of one-dimensional rotation sets for periodic point free torus maps*, Trans. Am. Math. Soc. 354 (2002), pp. 2865–2895.
[19] T.Y. Li and J. Yorke, *Period three implies chaos*, Am. Math. Mon 82 (1975), pp. 985–992.
[20] J. Llibre, *A note on the set of periods of transversal homological sphere self-maps*, J. Difference Equ. Appl. 9 (2003), pp. 417–422.
[21] J. Llibre, *Periodic point free continuous self-maps on graphs and surfaces*, Topol. Appl. 159 (2012), pp. 2228–2231.
[22] J. Llibre and V.F. Sirvent, *Minimal sets of periods for Morse–Smale diffeomorphisms on non-orientable compact surfaces without boundary*, to appear in J. Difference Equ. Appl., doi: 10.1080/10236198.2011.647006
[23] J. Llibre and M. Todd, *Periods, Lefschetz numbers and entropy for a class of maps on a bouquet of circles*, J. Difference Equ. Appl. 11 (2005), pp. 1049–1069.
[24] J.R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Englewoods Cliffs, New Jersey, USA, 1984.
[25] P.A. Smith, *Fixed-point theorems for periodic transformations*, Am. J. Math. 63 (1941), pp. 1–8.
[26] J.P. Tignol, *Galois’ Theory of Algebraic Equations*, World Scientific, Singapore, 2001.
[27] J.W. Vick, *Homology Theory. An Introduction to Algebraic Topology*, Springer-Verlag, New York, USA, 1994.
[28] S. Wang, *Free degrees of homeomorphisms and periodic maps on closed surfaces*, Topol. Appl. 46 (1992), pp. 81–87.