Pre-service Teachers’ Use of Technology as Reorganizer to Promote Differing Levels of Conceptual Understanding

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Accepted: 9 July 2022 / Published online: 18 August 2022
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Abstract
Research suggests that tasks that use technology as a reorganizer (technology is used to produce dynamic representations) are linked with the development of students’ conceptual knowledge, yet many secondary mathematics textbooks in the USA predominantly include tasks that use technology as an amplifier (technology is used to produce static images). Thus, if teachers wish to incorporate tasks that use technology as a reorganizer, they must locate these resources elsewhere or construct them themselves. This study took place in a technology, pedagogy, content course where each prospective teacher (PT) engaged in at least one component of an ideation, rehearsal/refinement, enactment, and reflection intervention, where they were asked to adapt traditional textbook lessons to promote students’ rich conceptual understandings of function transformations via technology as reorganizer tasks. A total of 15 PTs agreed to participate, 8 seeking a grade K-8 certification and 7 seeking a grade 6–12 certification. On the initial lesson plan involving linear function transformations, the majority of PTs (13 out of 15) used technology as an amplifier, but on the final lesson plan involving absolute value function transformations, 13 PTs used technology as a reorganizer. There were also increases in the incidence of lesson elements that promoted moderate or rich forms of conceptual knowledge between the initial and final lesson plans. We discuss the implications of these results and introduce readers to dynamic conceptual components as manipulable bridges between different mathematical representations that hold the potential to develop students’ richer forms of conceptual knowledge.

Keywords Conceptual understanding · Technology · Dynamic technology as reorganizer · Pre-service teachers

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Today’s mathematics teachers must be adept at using a variety of resources to provide students with opportunities to engage in challenging mathematics. Two important resources to which teachers have increasing access are mathematical action technologies (MATs)\(^1\) which perform mathematical actions (e.g., graphing calculator) and curriculum materials.\(^2\) Indeed, these two components are guiding principles for school mathematics in the National Council of Teachers of Mathematics’ *Principles to Actions* (NCTM, 2014), and the more recent Association of Mathematics Teacher Educators teaching standards (AMTE, 2017) have emphasized teachers’ proficiency with these tools. One of the most powerful affordances of technology is its ability to enable users to construct mathematical representations that can be dynamically manipulated to reveal invariant relationships which have the potential to lead to conceptual change (Roschelle et al., 2017).

Many teachers in the USA continue to use textbooks to guide their daily instruction (Banilower et al., 2018). Yet research by Sherman et al., (2020) suggests that many of these secondary mathematics textbooks only rarely use the dynamic capabilities of technology. Consequently, teachers need either to adapt their curriculum materials themselves or to locate these resources elsewhere, in order to provide students with opportunities to engage with dynamic technology.

In the present study, we describe the effects of a technology, pedagogy, content (TPC) course for prospective teachers (PTs) of middle and high school mathematics to learn how to adapt curriculum materials where technology is either not present or is focused solely on procedural knowledge into curriculum materials where technology plays a central role in developing students’ rich conceptual knowledge around function transformations. This article examines the effect of a technology, pedagogy, content (TPC) course on prospective teachers’ (PTs) effectiveness to infuse conventional curriculum materials with dynamic manipulation technology that develop students’ conceptual understandings of function transformations.

**Background Literature**

**Reconsidering Conceptual Knowledge**

Star (2005, 2007) argued that initial descriptions of procedural and conceptual knowledge provided by Hiebert and Lefevre (1986) conflate knowledge quality and knowledge type. They define procedural knowledge as consisting of two types, one involving an understanding of individual symbols and how those symbols can be arranged, while the second involves knowledge of procedures by which mathematical problems can be solved. Conceptual knowledge is defined as an interconnected collection of discrete knowledge chunks. Thus, we see that procedural knowledge is defined as a

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\(^1\) We use MAT, MATs, and technology interchangeably for the remainder of this article to refer to mathematical action technologies.

\(^2\) We follow Stein, Remillard and Smith’s (2007) definition of curriculum materials to consist of the textbook designed for students, as well as accompanying materials developed to support the teacher’s implementation of the curriculum, such as the teacher’s edition of the textbook.
knowledge type, while the definition for conceptual knowledge involves quality as indicated by its connected nature. When examining research regarding technology to develop students’ conceptual understanding, the latter, when defined, is usually considered as a rich connection of different mathematical ideas (e.g., Vahey et al., 2020) or connections across multiple representations (e.g., Garofalo et al., 2000). As a result, there is a need to examine pre-service teachers’ abilities to use technology to craft lessons involving different gradations of conceptual knowledge.

Nilsson (2020) addressed the issues pointed out by Star by developing a framework involving a $2 \times 2$ grid with a knowledge dimension consisting of procedural and conceptual knowledge types and a quality dimension involving limited and rich. In addition to conceptual understanding, the National Research Council (NRC, 2001) includes procedural fluency as two of five interconnected strands comprising mathematical proficiency, describing procedural fluency as involving more than the efficient execution of mathematical procedures, since it includes estimation, an understanding of procedures that can solve entire sets of mathematical problems, and the use of flexible thinking around numbers to solve mathematical calculations such as realizing that $199 + 67 = 267 - 1 = 266$. While procedural fluency does not include the entire description of Star’s rich procedural knowledge, it does admit the importance of richer forms of procedural knowledge.

Despite the presence of richer forms of procedural knowledge in the form of procedural fluency, we were unable to locate any research studies that involved the use of technology to promote students’ procedural fluency. In the present study, we follow Anderson et al., (2001), considering conceptual knowledge to consist of knowledge of classifications, structures, and principles, while procedural knowledge consists of algorithms, methods, and calculation processes. We agree with Nilsson (2020) that both forms of knowledge can exist on a continuum ranging from superficial to rich. We provide more details in the “Methods and Frameworks” section of this article.

**Using MATs to Learn and Teach Mathematical Concepts**

Drijvers (2012) organized technology in mathematics education to have three pedagogical purposes: doing mathematics, practicing skills, and developing concepts. Due to historical struggles that students experience in learning concepts in the USA (NRC, 2001), we focus this sub-section and our study on PTs’ abilities to use technology to promote students’ conceptual understanding of mathematics. MATs have positive effects on students’ conceptual understandings in a variety of different mathematics content areas, such as rational number (Olive, 2000), function (Brown et al., 1997), geometry (Clements et al., 2008), and calculus (Heid, 1988).

Technological dynamism first appeared in geometry during the 1990s, whereby geometric representations such as a triangle could be transformed by a user in different ways, such as the dragging of a vertex using software (e.g., *The Geometer’s Sketchpad*, Jackiw, 1989). Dynamic manipulation of mathematical representations has moved beyond geometry to early number (Sinclair & Crespo, 2006), algebra (Olive, 1998), trigonometry (Shaffer, 1995), and calculus (Gorini, 1997).
Furthermore, as Vahey et al., (2020) point out, there are two types of dynamism: *intra-representational dynamism* and *inter-representational dynamism*.

The former occurs when users can manipulate some component of a representation and the updated changes only occur to this representation as in the dragging of a vertex of a triangle resulting in another triangle. In the case of the latter, individuals make changes to one representation, and this appears simultaneously in at least one other linked representation. For example, students can enter $y = mx$ into the Desmos graphing calculator with $m$ as a slider. Changing this slider can result in simultaneous updates both to the equation representation and to the graphical representation of this linear function.

This ability to manipulate a virtual object dynamically has been shown to be more effective in enabling users to identify invariant properties when compared to examining static images (Battista, 2008). In summarizing research in this area, Roschelle et al., (2017) state these dynamic manipulation capabilities have the potential to spark conceptual change in students’ understandings across a variety of different mathematical areas. Moreover, Orrill and Polly (2013) argue that dynamic representations can play the following roles in school mathematics classrooms: connecting multiple approaches; testing conjectures; socially negotiating meaning; uncovering limitations of learning.

Despite these positive effects, research suggests that pre-service teachers struggle in constructing lessons that draw upon the dynamic capabilities of MAT to develop students’ conceptual understanding. For instance, in examining the lesson plans of 14 pre-service elementary teachers, Johnston and Suh (2009) found that these teachers integrated technology based on whether it was fun or engaging, rather than whether it promoted conceptual understanding. In addition, Backfisch et al., (2020) examined the instructional quality and technology exploitation of the technology-enhanced lesson plans of 92 German mathematics teachers, 28 at the pre-service level (undergraduate students with a major in mathematics teaching and no teaching experience), 42 trainee teachers who had an average of half a year of teaching experience, and 22 in-service teachers who had an average of 14 years of teaching experience.

The teachers were asked to create a lesson plan involving the use of technology to introduce students to the Pythagorean theorem. The instructional quality of the lesson plans involved seven criteria, one of which was initiating conceptual change. The lesson plans of the trainee and in-service teachers were of higher instructional quality than those of the pre-service teachers: $F(1, 89) = 16.70, p < 0.001$. There were no statistically significant differences in the instructional quality of the lesson plans for trainee and in-service teachers: $F(1, 89) = 0.68, p = 0.411$. In terms of technological exploitation, trainee teachers and in-service teachers were more likely to use MATs that enabled users to manipulate mathematical representations dynamically than pre-service teachers, and this difference was statistically significant: $\chi^2(2) = 10.89, p = .004$. There were no statistically significant differences in the quality of technological exploitation between trainee and in-service teachers: $F(1, 89) = 0.09, p = 0.769$.

The results of Backfisch and colleagues regarding in-service teachers are tempered by those of Vahey et al., (2020). The latter examined the implementation of
a curricular activity system in US middle school classrooms involving dynamic representations targeted to develop students’ conceptual understanding of ratios, proportionality, algebraic expressions, and geometric transformations across 24 lessons. Ten of the lessons were categorized as level 1, in which teachers only minimally integrated dynamic representations into their instruction. These teachers asked students to use technology to check their predictions and projected the outcome of technologically based dynamic representations, but classroom discourse did not involve the comparison of student predictions to dynamic representations and instead focused on procedures.

Ten of the lessons were categorized as level 2, in which dynamic representations were used to check predictions for correctness, without using technology to scaffold student understanding of concepts. The remaining four lessons were categorized at the highest level (level 3), where dynamic representations were used to facilitate student understanding of concepts. In sum, when experienced mathematics teachers are provided with a curriculum and professional development involving that curriculum, they still struggle to use intra-representational dynamism to promote students’ conceptual understanding.

Dynamism and its Connections to Pea’s Amplification/Reorganization

Bruner (1966) described technologies such as written language as cultural amplifiers. Others, such as Brown (1984), have described their electronic technologies as “idea amplifiers” because they enabled students to offload paper-and-pencil calculations to computers and focus instead on processes associated with problem solving. Pea (1985), citing Cole and Griffin (1980), stated that the word amplification means to increase intensity without changing its structure or nature. Cole and Griffin provide an example of the use of a pencil to assist a student in remembering a long list of words. The pencil did not amplify the student’s memory, but instead, they argue, caused a restructuring of the individual’s process for remembering. Pea introduced the term reorganization to account for how different technologies may cause a restructuring of the thinking process.

Sherman (2014), building on the work of Pea (1985), operationalized tasks involving MAT as amplifier or reorganizer in the following ways. A task involving technology as amplifier is one in which the MAT is used to execute tasks that lie within the set of skills that the user possesses: that is, the technology enables the task to be completed much faster, more efficiently, and with less error. Tasks that have the potential to reorganize student thinking are difficult or impossible for users to complete without the use of the MAT. Sherman considered its use as to create static mathematical representations or to finish calculations that students could complete on their own as using technology as amplifier, while he believed the use of the dynamic capabilities of technology, such as dragging a triangle and noting that the sum of the angles remained 180 degrees, to be a use of technology as a reorganizer. Sherman also argued that a task could use technology both as an amplifier and as a reorganizer.
Sherman examined the implementation of 63 mathematics tasks by three secondary and one middle school teacher, and found that tasks that had low cognitive demand were associated with technology as amplifier. High-level tasks, on the other hand, were correlated with technology as reorganizer. The four teachers recognized that technology has the potential to engage students in high cognitive demand, as tasks that were set up as a high level frequently involved technology, yet many of these tasks diminished to a low level during implementation involving technology as amplifier.

Sherman et al. (2017) used a 3-week intervention with 22 PTs and 1 practicing teacher to assist them in designing tasks involving technology. The intervention used Sherman and Cayton’s (2015) interactive geometry software (IGS) framework (described in more detail in the next section) to guide PTs in the analysis, revision, and design of tasks where technology promoted the reorganization instead of the amplification of student thinking. They found that 15 of the PTs were able to create a task that used technology as a reorganizer. In addition, the authors found that 21 PTs were able to design a task that was rated as high cognitive demand (Smith & Stein, 1998).

Uzan (2017) examined the lesson plans of a group of six pre-service secondary mathematics teachers (PSTs) for their use of technology. He found that 3 of the 12 tasks only used conveyance technologies, which were coded as low cognitive demand. Of the remaining nine tasks, four used technology as amplifier and five used technology as reorganizer. These nine tasks were all coded as high cognitive demand. In the three tasks that were coded as doing mathematics (the highest cognitive demand level), technology was used as a reorganizer. He also found that, when PSTs used pre-created files or previously used tasks, technology appeared as a reorganizer, but when PSTs constructed tasks themselves, they used technology as amplifier. Thus, this group of PSTs tended to struggle in developing activities that involved the use of technology as reorganizer.

Looking across these studies, we find that, in the case of Sherman (2014), technology was already a part of the tasks that teachers were implementing. In the Sherman et al., (2017) intervention, the tasks that the PTs and teacher revised already included the use of technology, albeit at an amplification level. In this article, they provided the actual textbook tasks as appendices in their study. Uzan (2017) investigated the processes the PSTs used to infuse technology into existing textbook resources, but no information was provided concerning the actual textbook lessons. We turn to research involving technology and textbooks next.

**Curriculum Materials, Cognitive Demand, and Technology**

Research suggests that mathematics teachers in the USA continue to use district-adopted curriculum materials (Banilower et al., 2018), and these resources play a strong role in shaping classroom instruction (Choppin et al., 2020; Grouws et al., 2013; Senk & Thompson, 2003). Research by Banilower and colleagues has found that the vast majority of curriculum materials in use in the USA are what
Stein et al. (2007) describe as conventional, and they report that these materials tend to promote the direct instruction of lower-level skills, such as procedures.

Recently, Sherman et al., (2020) investigated the use of technology in different US secondary mathematics curriculum materials. Their research involving 10,100 tasks appearing in 20 different secondary mathematics textbooks revealed that only 8.21% (829) of tasks required students to use technology to some extent. Moreover, of these 829 tasks, only one third used technology to reorganize student thinking. These tasks are associated with high cognitive demand (Sherman, 2014) and have the potential to develop students’ conceptual understanding. Consequently, if teachers in the USA are to enact tasks that foster students’ conceptual understanding of important mathematical ideas and use technology as a reorganizer, then they must engage in adapting their district-adopted curriculum materials or creating such materials themselves.

**Situating this Study**

The present study builds on the work of Sherman et al. (2017) in terms of examining PTs’ abilities to create tasks that use technology as amplifier or reorganizer, but, instead of examining the cognitive demand of these tasks, we focus on whether these tasks involve procedural or conceptual knowledge. Specifically, this study uses an intervention including components such as field experience (Meagher et al., 2011) and collaborative learning (Polly et al., 2010) that have been found to be advantageous in preparing PTs to develop and teach lessons utilizing the capabilities of MAT to support school students’ understanding of important mathematical ideas.

In addition, we feel that our intervention is much more closely tied to the actual work of teachers, by considering the conventional textbook resources that many US practicing teachers are provided with and supporting them in adapting these resources to use MAT to promote students’ conceptual understanding. Furthermore, our study provides more detail regarding the nature of the understanding promoted with the use of technology provided by using Star’s (2005, 2007) critiques of how conceptual knowledge has been characterized historically to examine the level of conceptual and procedural knowledge in PTs’ lesson plans and student activity sheets. Specifically, this study was designed to respond to the following two research questions:

1. What is the effect of a TPC course with a Technology as Reorganizer Curriculum Adaptation Process (TRCAP) component on a group of PTs’ abilities to infuse technology into two conventional curriculum lessons involving function transformations?
2. What is the effect of a TPC course with a Technology as Reorganizer Curriculum Adaptation Process (TRCAP) component on a group of PTs’ abilities to use technology to promote different levels of procedural or conceptual knowledge of function transformations?
Methods and Frameworks

Technology as Amplifier/Reorganizer

We draw on the work of Pea (1985) and Sherman and Cayton (2015) with regard to tasks that use technology to either amplify or reorganize user’s mathematical thinking. Tasks that use technology as an amplifier enable users to complete a task more efficiently by offloading calculations or representation generation to technology, but how users think about the mathematics embedded in those tasks remains unchanged. Sherman and Cayton assert that tasks where technology is used as an amplifier can be accomplished without the use of technology. Tasks that use technology as a reorganizer are difficult or impossible to achieve without the use of technology and therefore have the potential to reshape or reorganize users’ thinking.

Sherman and Cayton described characteristics of mathematical tasks in which technology is used as amplifier or reorganizer. Technology as amplifier tasks include using multiple static examples; asking students to make observations or generalizations based upon the values in a table without referencing the static sketches; containing the same examples for students to explore; not providing feedback to students to explore their conjectures; asking students to test conjectures. Technology as reorganizer tasks involve enabling students to drag objects within a sketch and to see measurements and relationships update in real time; supporting student exploration via alternative pathways; examining measurements dynamically; asking students to create conjectures using feedback provided during dynamic manipulations of sketches.

We used this designation to distinguish between technology as amplifier tasks and technology as reorganizer tasks in this study. However, we also built on this categorization to include the movement between different function representations (e.g., equation to graph), as well as other components of the Sherman and Cayton framework such as exploration.

Conceptual/Procedural Knowledge Type and Quality

To address the conflation of knowledge type and quality identified by Star (2005, 2007), we use (Anderson et al., 2001) definition of conceptual knowledge as consisting of knowledge of classifications, structures, and principles. In our study, we considered conceptual knowledge to consist of knowing that a change to the equation representation of a linear or absolute value function will result in a change to the graphical representation of that function. Instead of using Hiebert and Lefevre’s (1986) two types of procedural knowledge, we simplify this categorization and consider procedural knowledge to consist of knowledge of algorithms, methods, and calculation processes (Anderson et al., 2001).

Furthermore, we agree with Star (2005, 2007) that both types of knowledge can be held by individuals on a continuum ranging from superficial to rich. We build on Nilsson’s (2020) limited/rich quality dimensions to include a third dimension lying
in the middle, which we refer to as “moderate.” Like Nilsson, we discern different levels of knowledge quality by the presence of an increasing number of inferences. In our study, we considered inferences to involve understanding why changes to the equation representation of a function result in certain changes to its graphical representation. In the data analysis sub-section, we define and provide examples of these different types and quality dimensions regarding the mathematical foci of this study: transformations of linear functions and transformations of absolute value functions.

**Distinguishing Between Digital Mathematical Representations and Curricula**

We begin this sub-section by introducing readers to two types of technology-infused resources: technological application resources (TARs) and technological curricular resources (TCRs). TARs consist of virtual objects that students can view or manipulate and are designed to embody one or more mathematical ideas, but with no supporting curricular materials (e.g., questions for students to consider as they interact with the TAR). Users can create these applications in the process of interacting with a technology (e.g., constructing a graph in the Desmos graphing calculator—[www.desmos.com](http://www.desmos.com)) or they may be pre-constructed.

In a geometry context, Hollebrands and Smith (2009) refer to the pre-constructed TAR as either a drawing or a construction, while Sinclair (2003) uses the term *sketch*. We use TAR to highlight that the electronic representations and calculations can occur within any subject area, not necessarily geometry, and emphasize our alignment with a resource approach to mathematics (Trouche et al., 2019). TCRs consist of one or more tasks that are linked together to form a lesson, as well as the supporting materials that connect to some type of TAR. Sinclair (2003) uses the term *task* to refer to the TCR itself. We prefer the inclusion of curriculum in this description to emphasize that not only do tasks need to be provided to students, but that teachers also need guidance in implementing them.

In addition, “task” can refer to any type of question or instructions directed toward users, while we seek to emphasize that these supporting materials are linked to technologically generated representations which differ from those produced on paper. In this study, PTs constructed TARs on their own, used pre-constructed TARs, or asked school students to construct TARs as part of the lesson. The TCRs consisted of a student activity sheet and a lesson plan. In the post-intervention lesson (described in more detail below), the student activity sheet existed virtually as a collection of slides or screens in Desmos Activity Builder ([www.teacher.desmos.com](http://www.teacher.desmos.com)).

For this study, we found TARs and TCRs served as a useful unit of analysis for determining to what extent PTs were able to leverage the dynamic affordances of technology as a means to supporting students’ progress toward meeting a conceptually oriented lesson goal. With respect to TARs, we saw using technology as reorganizer as an indication of more support for developing conceptual knowledge, but

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3 For an example of a pre-constructed TAR, see: https://media.pearsoncmg.com/curriculum/math/cmp3/math_tools/A82378/index.html.
this often required further support in the form of prompts and questions on TCRs to encourage an understanding of why the relevant relationships exist.

Participants

The study took place within a technology, pedagogy, content (TPC) course at a large university situated in the mid-western region of the USA during the spring 2020 semester. There were a total of 27 class meetings, 17 of which involved face-to-face instruction, while the remaining 10 occurred in a synchronous, on-line learning environment due to the pandemic. A total of 15 PTs (10 female and 5 male) agreed to participate in the study. Eight of the PTs were working toward a grade K–8 mathematics certification, which we will refer to as pre-service elementary teachers or PSETs, while the remaining seven PTs (referred to as pre-service secondary teachers or PSTs) were working towards a grade 6–12 mathematics education certification.

Context: Creating a Collaborative Environment for the Infusion of Technology into Conventional Curriculum Materials

The intervention occurring in the TPC course where this study took place consisted of four phases: ideation, refinement, enactment, and reflection (see Fig. 1). A total of 11 PTs went through the entire intervention process due to the COVID-19 pandemic disrupting the field experience (see Table 1). Two PTs (Chloe and Ella) did not experience the enactment and reflection phases of the intervention. Two PTs’ (Olivia and Mei) field experiences occurred in the course with other PTs acting as students: consequently, they did not experience the rehearsal (described in more detail below) phase of the intervention.

The intervention lasted 3 weeks for each participant, with the first PT beginning at the end of January and the last PT finishing in March. Ten PTs were given a copy of the target lesson from the conventional curriculum materials currently being used by a target local middle or high school mathematics classroom. Oliver taught a lesson involving the locker problem (Kimani et al., 2016), but this content did not align with a particular classroom textbook lesson. Following Land and Drake (2014), we

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All teacher names are pseudonyms.

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conjectured that PTs engaged with the curriculum materials by reading, evaluating, and adapting them.

Ideation involved these three processes with regard to the curriculum materials: reading the curriculum materials for conceptual understanding; evaluating how the curriculum materials could be leveraged to develop conceptual understanding; adapting the curriculum materials to use technology to focus on developing students’ conceptual understanding. In cases where TARs could be easily infused into the lesson, we asked PTs to develop the TAR that they wanted to use in the lesson. In other cases, where it was more difficult to develop the TAR, the instructor (first author) either developed the TAR for the PT or located a pre-constructed TAR for PTs to use in the lesson.

Given the positive effects of learning by design (Agyei & Voogt, 2012; Hu & Fyfe, 2010; Johnson, 2014; Koehler & Mishra, 2005; Polly et al., 2010) on PTs’ abilities to develop lessons involving technology, we incorporated this process in two components of the intervention. The first component was during the ideation phase when each of the PTs met with the instructor to discuss and co-develop the TCR and TAR for the target lesson. The second component was during the rehearsal/refinement phase which we describe in more detail below.

Given the difficulty of teaching for understanding, encouraging student contributions, reacting to student responses, and the added complexity of incorporating technology, we felt that rehearsals (Kazemi et al., 2016) would be an important tool in helping to prepare the PTs to create and enact a lesson involving the use of MAT to develop students’ conceptual understanding. All PTs had experienced rehearsals as part of the methods course preceding this one. A total of 11 PTs chose first to

| Pre-service teacher | Ideation | Rehearsal/refinement | Enactment | Reflection |
|---------------------|----------|-----------------------|-----------|------------|
| Kiara               | Yes      | Rehearsal             | Yes       | Yes        |
| Oliver              | No       | Rehearsal             | Yes       | Yes        |
| Ella                | Yes      | Refinement            | No        | Yes        |
| Noah                | Yes      | Rehearsal             | Yes       | Yes        |
| Avery               | Yes      | Rehearsal             | Yes       | Yes        |
| Chloe               | Yes      | Refinement            | No        | Yes        |
| Mei                 | Yes      | Refinement            | Yes       | Yes        |
| Ruby                | Yes      | Refinement            | Yes       | Yes        |

| PSTs                |          |                       |           |            |
| Rebecca             | Yes      | Refinement            | Yes       | Yes        |
| Olivia              | Yes      | Refinement            | Yes       | Yes        |
| Amari               | Yes      | Rehearsal             | Yes       | Yes        |
| Emma                | Yes      | Rehearsal             | Yes       | Yes        |
| Liam                | Yes      | Rehearsal             | Yes       | Yes        |
| Lucas               | Yes      | Rehearsal             | Yes       | Yes        |
| Sophie              | Yes      | Rehearsal             | Yes       | Yes        |
rehearse the lesson, with the other PTs acting as students, followed by the community providing suggestions for improving the lesson. Two PTs (Rebecca and Ruby) simply shared the proposed lesson and student activity sheet with the rest of the classroom community, in order to refine the student activity sheet before enacting in a middle or high school classroom.

The refinement component of the model consisting either of a rehearsal or of a lesson/activity sheet sharing occurred within a week of the actual classroom enactment of the lesson. The graduate student assisting with the planning and development of the course (second author) played the role of student along with the other PTs enrolled in the course during the rehearsals. The course instructor took notes and used time-outs (Waage & Fauskanger, 2021) to stop the lesson, in order to focus the teacher of the lesson and classmates on important enactment issues or to engage the teacher and classmates in thinking through enactment issues identified by the instructor.

After the rehearsal, PTs, the course instructor and the PT acting as the teacher would engage in a reflection and debriefing of the teaching focusing on the enactment, as well as the design of the tasks students with which students engaged. During the enactment phase, each PT taught the lesson in an area middle school or high school mathematics classroom. Two PSETs (Chloe and Ella) did not have the opportunity to teach their lesson in an actual classroom environment due to the pandemic closing area schools. They were not able to teach their lessons to the other PTs, as their lessons involved the iPad and their classmates did not have access to the technology in their homes. Olivia (PST) and Mei (PSET) taught their respective lessons to the other PTs in a distance learning environment using WebEx and the Desmos Activity Builder. Chloe, Ella, Olivia, and Mei all engaged in a refinement of their lessons when they met with the instructor and graduate student during office hours. The complete list of PTs, MAT technology involved, and the content focus of their lessons is in the Appendix.

The instructor of the course (first author) video-recorded each lesson enactment, focusing the camera on the PT and following the PT as he/she interacted with students in small co-operative group settings. As the name suggests, the reflection phase involved the PT watching the video recording of the lesson to focus on the similarities and differences between the intended lesson and the enacted lesson, as well as how the lesson might change if implemented in the future. All PTs engaged in reflection, even if they did not teach the lesson. In those cases where the lesson was not taught (Ella and Chloe), we asked the PTs to imagine the lesson being taught and how that hypothetical enactment may influence their adaptations of the lesson in the future.

Data Collection

During the first week of the semester, PTs were asked to create a lesson plan and student activity sheet for Sects. 3.6: Transformations of graphs of linear functions from Big Idea Math: Algebra 1 (Larson & Boswell, 2019), an example of a lesson from a set of conventional curriculum materials. To prepare PTs to develop a lesson
geared toward conceptual understanding, we read and discussed Simon’s (2018) article on conceptual understanding during the first week of the course. This discussion focused on defining conceptual understanding as why a mathematical idea was true, which we considered to be either a moderate or a rich form of conceptual knowledge.

We provided all PTs with a copy of the teacher’s edition of the lesson Sect. 3.6 involving translations, reflections, stretches, and shrinks of linear functions. The original lesson contained an exploration in the beginning of the lesson, which directed students to use technology as an amplifier to check their sketches of translations, stretches, and shrinks. We removed this page from the student textbook and its accompanying page from the teacher resource materials given to PTs, so they would not use this approach: instead, we were interested in the approach that PTs would take without any external influence. During the last week of class, all PTs were asked to construct TCRs consisting of a student activity sheet and lesson plan and TARs for Sect. 3.7: Graphing absolute value functions from Big Idea Math: Algebra 1 (Larson & Boswell, 2019).

Data Analysis

Following Sherman et al., (2020), we considered tasks as using technology as an amplifier if users were not able to move one or more of the mathematical representations dynamically or if the technology were used to produce individual mathematical representations (e.g., graph of an absolute value function) or calculations that students could produce on their own. Tasks that used technology as a reorganizer involved what Vahey et al., (2020) refer to as inter-representational dynamism: that is, one representation (e.g., a parameter within a function’s equation representation) could be manipulated by the user, resulting in changes to other representations of the function (e.g., graph).

We coded the inferred purpose of the technological use, whether technology was used as amplifier or reorganizer, whether the initial function representation was given, and whether the representation was produced by the technology. We used the purposes listed in the IGS framework (Sherman & Cayton, 2015): observe\_g (make general observations); observe\_s (make specific observations); explore; conjecture test (to make a conjecture about a relationship before being exposed to the mathematics content).

The following purpose codes were generated using an open coding process: optional (technological use is optional for students); practice (students were asked to use technology to generate representations to practice content that students had previously been exposed to); check (students were asked to name a transformation on the basis of a graph or equation and later asked to use technology to check their answers); assess (technology was used to produce a graph from an equation to assess student understanding).

For example, if students were asked to make general observations of a static graph produced by entering an equation, this was given a code of \text{amp}_e \rightarrow \text{gr} (\text{observe}_g). We only counted unique instances of a particular code for a given PT. For instance,
| Pre-service teacher | Pre-intervention | Post-intervention |
|---------------------|------------------|-------------------|
| **PSETs**           |                  |                   |
| Kiara               | PSETs            | Ro_{id} (assess)  |
|                     | Kiara Ampeq      | Ro_{eq}→gr (observe) |
|                     | Oliver           | Ro_{eq}→gr (observe) |
|                     | Ella             | Amp_{eq}→gr (observe) |
|                     | Noah             | Ro_{eq}→gr (observe) |
|                     | Avery            | Amp_{eq}→gr (practice) |
|                     | Chloe            | Amp_{eq}→gr (check) |
|                     | Mei              | Ro_{eq}→gr (observe) |
|                     | Ruby             | Amp_{eq}→gr (check) |
| **PSTs**            |                  |                   |
| Rebecca             | Amp_{eq}→t, gr (practice) | Ro_{eq}→gr (explore) |
|                     | Olivia           | Ro_{eq}→gr (conjecture test) |
|                     | Amari            | Ro_{eq}→gr (observe) |
|                     | Emma             | Ro_{eq}→gr (observe) |
|                     | Liam             | Ro_{eq}→gr (observe) |
|                     | Lucas            | Ro_{eq}→gr (observe) |
|                     | Sophie           | Amp_{eq}→gr (practice) |

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if a PT-constructed student activity sheet included multiple instances of \( \text{amp}_\text{eq} \rightarrow \text{gr} \) (observe), this was only listed once in Table 2. After coding using the categories we described above, we noticed that some PTs used technology as amplifier or reorganizer across several categories (e.g., observing and testing conjectures), while other PTs only focused on one category. We wanted to capture this variability and defined a technological variability score for each set of lessons, by adding together all the different types of technology used in the pre-intervention in Table 2 and dividing by 15 the total number of PTs. A similar technological variability score was calculated for the post-intervention lessons in Table 2.

Second, we analyzed three components of the lesson plan (goal statement/objectives, student–teacher sample dialogue, and remaining support materials), student activity sheet, and TAR for the type of knowledge (procedural or conceptual) they were designed to promote in students, as well as the level of that knowledge (limited, moderate, and rich). We defined limited conceptual knowledge, CK (limited), in both lessons as recognizing that a change to an equation representation resulted in predictable changes to the graphical representation of a linear function or absolute value function. Rich conceptual knowledge, CK (rich), in the case of the pre-intervention lesson, would consist of knowing that adding a constant to a linear function \( f(x) \) as in \( f(x) + a \) results in a vertical translation because the \( x \)-co-ordinate remains the same while the \( y \)-co-ordinate is changing by \( a \), resulting in points to move upward if \( a > 0 \) and downward if \( a < 0 \).

Moderate conceptual knowledge, CK (moderate), was defined as involving some, but not all, of the justifications described under the rich conceptual knowledge description. Initial examinations of our data identified an additional code, which we termed CK (richns), representing rich conceptual knowledge that was not supported in the TAR or student activity sheet. That is, questions appeared in the student activity sheet that specifically asked students why a certain change to an equation resulted in a change to a graphical representation, but there existed no scaffolds or support to assist students in answering these questions.

We defined limited procedural knowledge as involving the procedure of how to graph a transformation of a function by substituting values of \( x \) into \( f(x) + a \) following the procedure described by \( f(x) \) to yield a given output then adding “\( a \)” amount to that output to determine the \( y \) co-ordinate associated with the initial \( x \) co-ordinate. Rich procedural knowledge for linear functions would involve realizing that to graph \( f(x) + a \) one simply must add \( a \) to two \( y \) co-ordinates of \( f(x) \), keep the \( x \) co-ordinates the same, and then sketch in the remaining points in order to graph \( f(x) + a \). Moderate procedural knowledge for either linear functions or absolute value functions would involve some instances of point-by-point graphing, characteristics of a limited procedural knowledge with a use of moving whole sections of the graph, reminiscent of rich conceptual knowledge.

All PTs were asked to construct a lesson plan that contained a conceptually oriented goal statement, sample hypothetical dialogue between the teacher and students involving the conceptual goal, a student activity sheet, and either directions for using MAT, such as a graphing calculator or a technological file containing the TAR.

During the last week of class, all of the PTs were asked to construct a TAR and TCR (post-intervention lesson) focusing on conceptual knowledge associated
with absolute value function transformations. They were provided with the teacher resources and student textbook lesson for Sect. 3.7: Graphing absolute value functions (Larson & Boswell, 2019, pp. 154–162). PTs were asked to construct a TCR either with technological directions or with a TAR similar to the first lesson they constructed. For both textbook sections, PTs were told to construct a lesson that involved either the goals of the textbook lesson or a subset of those goals, so that they could focus on conceptual knowledge.

The first author analyzed the goal statement, sample dialogue, student activity sheet, and technology in the pre-intervention lessons for the presence of procedural/conceptual knowledge and the level of those knowledge types. The second author used these codings and descriptions to assist the first author in the coding of the post-intervention lesson. We analyzed the TAR for knowledge type and level by examining how the technology was being deployed by the hypothetical student. For limited procedural knowledge, this involved knowing or following steps in order to accurately graph absolute value functions.

We did not code any instances of moderate procedural knowledge, but we would expect it to appear when there is ambiguity with respect to what the directions given to students are. For example, perhaps there are questions on a document asking students to “make observations about what they see,” in reference to the virtual mathematical representations produced by the technology. This could result in productive observations leading to generalizations, but it could also be interpreted in such a way as to lead to non-mathematical observations pertaining to the appearance of a graph (e.g., its color, what it resembles). For rich procedural knowledge, there was no explicit mention of the related mathematical principles, but the ability to make observations leading to insights about the principles was enabled or some decision-making with respect to which procedure is most efficient in this case. For example, asking students to “describe how the value of c affected the graph” tries to make the connection across representations clear, but does not necessitate referring to relevant mathematical principles.

For limited conceptual knowledge, this involved making observations about the connection between the symbolic and graphical representation of the transformed absolute value function, but not necessarily about the why behind the observations. For example, providing students with an opportunity to compare the graphs of a transformed graph with its “parent” function’s graph to observe that the function shifts up is a much different experience than providing students with support to observe why the function shifts up when a positive constant is added to the value.

For moderate conceptual knowledge, we observed this in instances of potentially rich conceptual understanding, but, due to ambiguities or sometimes idiosyncratic vocabulary usage, we could not clearly infer how rich a connection was intended. For example, using the terms “noticings” and “change” in problems lend themselves to be interpreted in many different ways without a clear direction to “notice about” an important aspect of the problem. This often leads to ambiguity with respect to how students were expected to engage with the rich conceptual knowledge. Finally, rich conceptual knowledge was seen when the goal was clearly observable as requiring students to make connections across representations in order to explore,
conjecture, and make general arguments about the mathematical principles that cause the transformations.

Findings

Technology Use

As can be seen in Table 2, technology as amplifier predominated in pre-intervention lessons on linear function transformations with 13 out of 15 PTs using MAT in this way. In addition, there was little variation in how technology was used during these pre-intervention lessons as the technological variability score was 1.5: that is, the majority of PTs only used technology in one way, typically to check work that was completed with paper and pencil. All these technology uses asked students to enter an equation to produce a graph and involved some form of graphing calculator (e.g., TI-Nspire or Desmos or an unnamed generic calculator).

These results suggest that PTs’ experiences in K-12 schooling align with the findings of Sherman et al., (2020) regarding the use of technology in secondary mathematics textbooks: that is, teachers most likely used technology, such as graphing calculators, to produce graphs from an equation representation. In addition, the results also suggest two possibilities with regard to their college level mathematics classes. First, it might have been the case that none of the PTs had experienced technology-as-reorganizer in their college level mathematics courses. Second, if their instructors did use technology as reorganizer, they chose not to use it in their lesson plans.

In the case of Kiara, she struggled on the pre-intervention lesson even to determine a way for students to use technology. Although she listed technology in her materials list in her lesson plan, we saw no specific directions for students to offload their work to technology. In fact, Kiara asked students to construct a table from an equation, in order to produce the graph of a parent function and its transformation, suggesting that technology was not needed at all in order to complete the lesson.

The majority of PTs did not use technology to promote students’ active learning via investigations of mathematical ideas during the pre-intervention lesson, despite being asked to do so. Only four PTs (two PSETs and two PSTs) used technology as a tool to help students develop mathematical content, as seen by the presence of observe or conjecture test codes. The other seven PTs used direct instruction to teach the different types of linear function transformations, then asked students to demonstrate knowledge of these transformations with technology being used either to check their answers or to produce a parent graph and its transformation to practice identifying the transformation that occurred.

Two PTs used technology as a reorganizer in the pre-intervention lesson. Noah used sliders to represent $h$ and $k$ in $f(x+h)$ and $f(x)+k$ and asked students to make observations of how the graph changed when $h$ and $k$ changed. Liam asked students to graph a point on the TI-Nspire, to reflect this point across the $y$-axis and the $x$-axis, to drag the point and, lastly, to identify invariants in the co-ordinates of the reflected points. Thus, in both of these cases, the presence of technology as reorganizer was aligned with student investigation of mathematical ideas.
On the post-intervention lesson involving transformations of absolute value functions, there were many more different uses of technology, as seen in a technological variability score of 2.7. There was a predominance of technology as reorganizer in these lessons, with a total of thirteen out of 15 PTs using technology in this way. Six PTs used a mixture of technology as amplifier and technology as reorganizer tasks. Not surprisingly, most of the PTs worked along what we describe as the equation-graphical representational corridor. They either asked students to enter an equation to produce a graph with technology as amplifier or used dynamic equation parameters to generate graphical representations.

Some other types of inter-representational dynamism were also created by the PTs. For instance, Amari asked students to move a point on a line that resulted in a change to a graph and its slope. In order to understand the concept of absolute value, Mei asked students to manipulate a point along the $x$-axis, resulting in changes to its co-ordinates, as well as to its distance from the origin. Two PTs (Chloe and Emma) moved beyond the equation to graphical corridor. Chloe, for instance, created a TAR whereby students could change an equation parameter that resulted in changes to a graphical representation, a vertical line segment connecting the parent function with its transformation, and co-ordinates of a point.

The point could also be dragged, resulting in a change to the position of the vertical line segment, the other point, as well as a change to the co-ordinates of the points (see Fig. 2). We used the term *dynamic conceptual component* to refer to the vertical line segment and the co-ordinates. We use the word *dynamic* in it, as these components could be manipulated, as well as the word *conceptual*, as we felt that they had the potential to support students’ rich conceptual understanding.

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**Fig. 2** Chloe’s TAR using technology as reorganizer and a collection of interconnected representations
All the PTs drew on technology as a tool to help students to learn mathematics content using an active learning approach during their post-intervention lessons, as opposed to using direct instruction, which is how technology was predominately used by teachers during their pre-intervention lesson. Technology as either amplifier or reorganizer was used to provide students with opportunities to make general observations, specific observations, and test conjectures, and to explore mathematical representations. PTs also used technology as amplifier or reorganizer to assess student understanding of previously learned content at the beginning of the lesson in the form of a warm-up or to assess their understanding of ideas associated with the lesson as an exit ticket.

Procedural and Conceptual Understanding

Procedural and conceptual understanding associated with the pre-intervention and post-intervention for PSETs’ goal statements, sample dialogues, other supporting material in the lesson plan, and student activity sheet can be seen in Table 3. Despite backgrounds steeped in procedural knowledge, the PSETs were successful in creating lessons that promoted students’ conceptual knowledge. However, that conceptual knowledge was primarily limited in nature and tended to focus either on seeing connections between equation and the graph or between the equation and the name of a transformation. Also, there were two instances of rich conceptual knowledge opportunities (Oliver and Ela) that were not supported. For example, these two PSETs asked students why certain changes to an equation resulted in changes to a graph, but they did not provide support to assist students in answering these questions. Such support could have come in the form of dynamic conceptual components (see Fig. 2).

Moderate levels of conceptual knowledge often appeared in the sample student–teacher dialogue in the pre-intervention lessons. There were more instances of higher forms of conceptual knowledge in the post-intervention lessons when compared with the pre-intervention ones. For instance, there were a total of five instances of moderate/rich conceptual knowledge in the pre-intervention lessons, while there were 12 such instances in post-intervention lessons. Moreover, in the post-intervention lessons, these moderate/rich conceptual knowledge elements occurred in other components of the lesson, such as the activity sheet or the goal statement.

Examined another way, out of eight PSETs, five incorporated moderate/rich conceptual knowledge in their pre-intervention lessons. In the post-intervention lessons, this had risen to seven PSETs. Kiara and Ella’s moderate/rich conceptual knowledge components were restricted to the sample dialogue sections of the lesson plan. Only Mei did not include moderate/rich conceptual knowledge descriptions or opportunities either in the pre-intervention lesson or in the post-intervention lesson.

The procedural and conceptual understanding associated with the pre-intervention and post-intervention for PSTs’ goal statements, sample dialogues, other supporting material in the lesson plan, and student activity sheet can be seen in Table 4. Like the PSETs, the PSTs’ pre-intervention lessons tended to focus on conceptual understanding, with two PSTs (Olivia and Emma) including
| Pre-service teacher | Category               | Pre-intervention | Post-intervention |
|---------------------|------------------------|------------------|-------------------|
| Kiara               | Goal statement         | CK (limited)     | CK (limited)      |
|                     | Sample dialogue        | CK (limited)     | CK (rich)         |
|                     | Activity sheet         | PK (limited)     | CK (rich<sub>ns</sub>) |
|                     | Other supporting       | CK (limited)     | CK (rich)         |
| Oliver              | Goal statement         | CK (limited)     | CK (limited)      |
|                     | Sample dialogue        | CK (limited)     | CK (moderate)     |
|                     | Activity sheet         | PK (limited)     | CK (rich<sub>ns</sub>) |
|                     | Other supporting       | CK (limited)     | CK (limited)      |
| Ella                | Goal statement         | CK (limited)     | CK (limited)      |
|                     | Sample dialogue        | CK (limited)     | CK (rich)         |
|                     | Activity sheet         | CK (limited)     | CK (rich<sub>ns</sub>) |
|                     | Other supporting       | CK (limited)     | CK (moderate)     |
| Noah                | Goal statement         | PK (limited)     | CK (rich)         |
|                     | Sample dialogue        | CK (limited)     | CK (moderate)     |
|                     | Activity sheet         | CK (limited)     | CK (moderate)     |
|                     | Other supporting       | CK (limited)     | CK (rich)         |
| Avery               | Goal statement         | CK (limited)     | CK (limited)      |
|                     | Sample dialogue        | CK (limited)     | CK (moderate)     |
|                     | Activity sheet         | CK (limited)     | CK (moderate)     |
|                     | Other supporting       | CK (limited)     | CK (moderate)     |
| Chloe               | Goal statement         | CK (limited)     | CK (rich)         |
|                     | Sample dialogue        | CK (rich)       | CK (moderate)     |
|                     | Activity sheet         | CK (limited)     | CK (rich)         |
|                     | Other supporting       | CK (limited)     | CK (moderate)     |
| Mei                 | Goal statement         | CK (limited)     | CK (limited)      |
|                     | Sample dialogue        | Not present     | Not present       |
|                     | Activity sheet         | CK (limited)     | CK (limited)      |
|                     | Other supporting       | CK (limited)     | CK (limited)      |
| Ruby                | Goal statement         | CK (limited)     | CK (limited)      |
|                     | Sample dialogue        | CK (moderate)    | CK (limited)      |
|                     | Activity sheet         | CK (limited)     | PK (limited)      |
|                     | Other supporting       | CK (limited)     | CK (limited)      |
This contrasts with Kiara, Oliver, and Noah (all PSETs), who focused on limited procedural knowledge. Recall that rich procedural knowledge involved using transformations to move an entire graph, instead of using a table to graph a collection of points that had been transformed in some way.

### Table 4 Procedural and conceptual knowledge across pre-intervention and post-intervention lessons (PSTs)

| Pre-service teacher | Category      | Pre-intervention          | Post-intervention          |
|---------------------|---------------|---------------------------|----------------------------|
| Rebecca             | Goal statement| Unable to code PK (limited)| PK (limited)               |
|                     | Sample dialogue| CK (limited)               | CK (limited)               |
|                     | Activity sheet | CK (limited)               | CK (limited)               |
|                     | Other supporting| CK (limited)               | CK (limited)               |
| Olivia              | Goal statement| CK (limited)               | CK (moderate)              |
|                     | Sample dialogue| CK (limited)               | CK (moderate)              |
|                     | Activity sheet | PK (rich)                  | CK (rich<sub>n</sub>)      |
|                     | Other supporting| CK (limited)               | CK (rich<sub>n</sub>)      |
| Amari               | Goal statement| CK (limited)               | CK (limited)               |
|                     | Sample dialogue| CK (limited)               | Not present                |
|                     | Activity sheet | CK (limited)               | CK (rich<sub>n</sub>)      |
|                     | Other supporting| CK (limited)               | CK (moderate)              |
| Emma                | Goal statement| Unable to code PK (limited)| CK (rich)                 |
|                     | Sample dialogue| CK (limited)               | CK (rich)                 |
|                     | Activity sheet | PK (rich)                  | CK (rich)                 |
|                     | Other supporting| CK (moderate)              | CK (rich)                 |
| Liam                | Goal statement| CK (limited)               | CK (rich)                 |
|                     | Sample dialogue| CK (limited)               | CK (rich)                 |
|                     | Activity sheet | CK (limited)               | CK (moderate)              |
|                     | Other supporting| CK (limitation)            | CK (rich)                 |
| Lucas               | Goal statement| Unclear PK (limited)       | PK (limited)               |
|                     | Sample dialogue| Not present                | Not present               |
|                     | Activity sheet | CK (limited)               | CK (limited)               |
|                     | Other supporting| CK (limited)               | CK (limited)               |
| Sophie              | Goal statement| CK (limited)               | CK (moderate)              |
|                     | Sample dialogue| CK (moderate)              | CK (moderate)              |
|                     | Activity sheet | CK (moderate)              | CK (moderate)              |
|                     | Other supporting| CK (moderate)              | CK (moderate)              |
Different from the PSETs, the majority of PSTs did not include any moderate forms of conceptual knowledge in their pre-intervention lesson, except for Sophie who included them in the sample dialogue and the activity sheet. A total of six out of seven PSTs included moderate to rich forms of conceptual knowledge in at least one component of their post-intervention lessons. For Rebecca, this occurred only in the sample dialogue, but the remaining PSTs included moderate/rich forms of conceptual knowledge in other lesson components as well. Rebecca also struggled to hand her lesson over to conceptual understanding completely, as it still included limited procedural knowledge goals.

Given the importance of objectives or goals, it was reassuring to see that four of the PSTs included moderate/rich forms of conceptual knowledge in this component of their plans on the post-intervention lesson. Similar to the PSET Mei, Lucas showed no instances of moderate or rich forms of conceptual knowledge either on the pre-intervention lesson or on the post-intervention lesson. Only Olivia included why questions in the student activity sheet that were not supported, giving rise to a CK (rich n) code. Moderate forms of conceptual knowledge connected co-ordinate or table changes to alterations in the graph, providing some support for why the graph became altered in predictable ways. Rich forms of conceptual knowledge typically made connections across equation representations, graphical representations, and co-ordinate/table representations.

Table representations have the potential to assist students in developing a deeper conceptual understanding of function transformations, as they help students see the effect of a change to an equation on a change to the graph via the change in co-ordinates. Yet tables are often depicted in textbook materials as a tool to graph different functions. Therefore, with the introduction of technology to produce graphs, we were interested in seeing if PTs would still include table representations and, if so, how they would be used.

Both textbook sections include table representations, but the purpose of these representations is not transparent and, given their proximity to graphs, they could be interpreted as a tool to graph a parent function and its transformed function. The book also includes a unique table representation that includes a row of x-values, a row for the y-values associated with the parent function \( f(x) \), and a third row for the transformed function \(-f(x)\) (see Fig. 3).

The summary of table use across all PTs for the pre-intervention and post-intervention lessons is in Table 5. Four PTs (Mei, Olivia, Amari, and Lucas) did not include table representations on their pre-intervention or post-intervention lessons, possibly highlighting the role that technology played in decreasing the importance of tables. Many PTs left the production of tables up to students to assist them in understanding if they were struggling, as seen in the optional codes in Table 4. Overall, only five PTs used tables in the pre-intervention lesson, but this increased to 11 PTs using them in the post-intervention lesson.

Thus, while technology was used more frequently as a reorganizer in the post-intervention lesson, PTs still included tables at a higher rate despite the use of technology to produce graphs without the use of tables. While tables appeared more
| $x$  | $f(x)$ | $-f(x)$ |
|------|--------|---------|
| -4   | -1     | 1       |
| -2   | 0      | 0       |
| 0    | 1      | -1      |

Fig. 3 Use of a unique multiple-row table in the textbook along with the graph of the function and its transformation (Larson & Boswell, 2019, p. 147; reprinted with permission)
frequently during the post-intervention lessons, these tables tended to focus on understanding the shape of the absolute value functions. Tables were conspicuously missing from understanding absolute value function transformations. The tables that students were either given or asked to produce tended to focus on understanding the absolute value parent graph and not on transformations of this graph.

### Discussion

The present study contributes to our growing understanding of the effects of a TPC course on PTs’ abilities to plan lessons and develop lesson materials involving the infusion of technology. Specifically, the study shows that the TPC course, with a focus on technology as reorganizer incorporated throughout the semester, was successful in increasing PTs’ abilities to adapt conventional curriculum lessons to include technology as reorganizer. We considered this effect to be a success, given that PTs have struggled to construct their own lessons involving technology as reorganizer (Uzan, 2017). These findings are especially encouraging, given that a total of 11 PTs engaged in all four components of the intervention.

There was also a modest improvement in their ability to design TCR and TAR that involved either moderate or rich forms of conceptual understanding. The success of the course can be noted by the change from most students only using technology for amplification in the pre-intervention lesson to a majority of the students being able to create TARs which utilized technology as reorganizer in their

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**Table 5** Use of table representations during pre-intervention and post-intervention lessons

| PT   | Pre-intervention | Post-intervention |
|------|------------------|-------------------|
| **PSETs** |                  |                   |
| Kiara | Optional         | Limited for students |
| Oliver| Optional         | Optional          |
| Ella  | No use           | Optional          |
| Noah  | Optional         | Produced by students |
| Avery | No use           | Produced by students |
| Chloe | Optional         | Dynamic           |
| Mei   | No use           | No use            |
| Ruby  | No use           | Limited           |
| **PSTs** |                  |                   |
| Rebecca| Technology produced | Limited for students |
| Olivia | No use           | No use            |
| Amari  | No use           | No use            |
| Emma  | No use           | Dynamic           |
| Liam  | No use           | Limited for/by students |
| Lucas | No use           | No use            |
| Sophie | Optional         | Produced by students |
post-intervention lesson. The intervention at the center of this TPC course involved
the analysis of textbook lessons, learning by design, making static technology rep-
resentations dynamic, lesson rehearsal/critique, field experience, and reflection, in
order to provide prospective teachers with opportunities to infuse conventional text-
book lessons with technology as reorganizer that focuses on conceptual knowledge.

Overall, this study found that, in the pre-intervention lesson, 13 out of the 15
PTs used technology as amplifier, but, in the post-intervention lesson, we had the
same number use technology as reorganizer. On the pre-intervention lesson, six PTs
included moderate/rich conceptual knowledge components in their lessons, but this
doubled up to 12 on the post-intervention lessons.

The findings of this study underscore the importance of alignment across dif-
ferent lesson components. This includes both across components within the TCR
and alignment between TARs and TCRs. During the pre-intervention lessons, PTs
included moderate/rich conceptual elements in the student dialogue, but other com-
ponents of the lesson, such as the goal statement, the activity sheet, or the technol-
ogy, were focused on limited forms of conceptual knowledge. Thus, while PTs made
inroads in developing lessons that focused on moderate/rich conceptual elements,
many of the post-intervention lessons also demonstrated this lack of alignment
among different elements.

This lack of alignment was also seen in the use of tables. Five PTs mentioned
tables either in the student activity or in the lesson plan, but not both on the pre-
intervention lesson and post-intervention lesson. An especially egregious case
of this was when tables were mentioned as a tool to help students understand
absolute value transformations, but students were never asked to produce tables
in the Desmos activity. Thus, future implementations of the intervention need to
highlight the importance of alignment between the student activity (whether it
exists in written form or electronically as a Desmos Activity) and the lesson plan
elements.

It is unclear why this occurred or what the effect of it might be on students, but
perhaps it could contribute to a lowering of the cognitive demand during implemen-
tation, since a lack of conceptual knowledge being brought out during a classroom
discussion might suggest a lack of conceptual knowledge being accessed by the
students during implementation. Future studies regarding the relationship between
TARs and TCRs may look at how an alignment or lack of it in pursuing a conceptual
goal may play out during implementation.

Overall, the majority of PTs were able to incorporate at least one element
of moderate/rich conceptual knowledge in their post-intervention lessons, but
there were two PTs, Mei (PSET) and Lucas (PST), who did not incorporate
any conceptual knowledge elements, neither in their pre-intervention nor in
their post-intervention lessons. Both PTs struggled with mathematics content
during the course, as measured through their homework and test scores. This
highlights the importance of mathematics content knowledge in the develop-
ment of activities that provide students with opportunities to develop con-
ceptual understanding. It may be the case that a well-developed conceptual
understanding contributes to a clear articulation of conceptually oriented goal
statements, which may guide the development of conceptual understanding during implementation.

In addition to unclear articulations of the concept in a goal statement, it may also be the case that considering how dialogue may support the development of conceptual understanding is also challenging when teachers lack a clear articulation of the concept for themselves. Future directions on analyzing TCRs may contain comparisons between conceptually oriented goals and sample dialogues as windows into the teacher’s understanding of the concept, as well as the efficacy of the lesson materials for meeting the lesson goal.

Although there was an increase from the pre-intervention lesson to the post-intervention lesson in the inclusion of conceptual understanding that moved beyond limited forms, a total of ten PTs did not provide students with opportunities to learn rich conceptual understanding in the student activity sheet, which provides students with the majority of their interactions with the lesson content. Despite working with the instructor in constructing a student activity sheet that focused on rich conceptual understanding, and being exposed to such activities while their classmates rehearsed their lessons, the PTs still seemed to focus just on the connection between an equation change, resulting in a predictable change to the graph, namely a limited conceptual understanding of function transformations.

We see an issue of alignment here as well: that is, just as alignment between the student activity sheet and lesson plan components are important, so is alignment between the TAR and TCR that comprise the student activity. For instance, students need to be asked why certain changes to an equation result in predictable changes to a graph, but they need to be supported with the tools to be able to answer this question. One potentially impactful support may come in the form of dynamic conceptual components. In our future iterations of this course, we intend to provide PTs with opportunities to craft dynamic conceptual components, and thereby make the technological content knowledge needed to construct these elements more transparently.

In a similar vein, we intend to provide PTs with opportunities to craft a conceptually oriented goal statement for an activity involving dynamic conceptual components. Although the impact of this lack of alignment between TARs and TCRs is unclear from this study, it perhaps supports the notion that technology itself cannot reorganize student thinking. Of vital importance for meaningful interactions with technology is the environment in which the technology lies. We intend to research the effects of these methods course changes on PTs’ lessons. Two more important research areas suggested by this study are whether PTs are able to enact lessons involving technology as reorganizer to promote a rich conceptual understanding in the classroom and whether these lessons effect change to students’ conceptual understanding of important mathematics.
### Appendix

| Pre-service teacher | Lesson content focus                          | MAT            |
|---------------------|-----------------------------------------------|----------------|
| Rebecca             | Solving equations using graphing               | Desmos         |
| Kiara               | Effect of $c$ on the graph of $f(x) = ax^2 + c$| Desmos activity builder |
| Olivia              | Area of a parallelogram                       | GeoGebra       |
| Amari               | Polygon interior angle sum                    | GeoGebra       |
| Oliver              | Locker problem                                | Locker problem |
|                     | application from Connected Mathematics        |                |
| Emma                | Effect of $a$ on the graph of $f(x) = ax^2$    | Desmos activity builder |
| Liam                | Trig ratios                                   | GeoGebra       |
| Lucas               | Solving systems of linear inequalities        | TI-Nspire CX CAS |
| Ella                | Composition/decomposition of numbers          | TouchCounts    |
| Noah                | Parallel lines, transversals, and angle sums  | TI-Nspire CX CAS |
| Avery               | Solving equations                             | TI-Nspire CX CAS |
| Chloe               | Skip counting                                 | TouchCounts    |
| Mei                 | Triangle inequality theorem                   | Virtual polystrips |
|                     | application from Connected Mathematics        |                |
| Sophie              | Special polynomial products                   | TI-Nspire CX CAS |
| Ruby                | Perimeter and area of similar figures         | GeoGebra       |

**Data Availability**  De-identified data will be made available to interested readers upon demand.

**Declarations**

**Conflict of Interest**  The authors declare no competing interests.

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