Onset of inflation in inhomogeneous cosmology

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abstract
We study how the initial inhomogeneities of the universe affect the onset of inflation in the closed universe. We consider the model of a chaotic inflation which is driven by a massive scalar field. In order to construct an inhomogeneous universe model, we use the long wavelength approximation (the gradient expansion method). We show the condition of the inhomogeneities for the universe to enter the inflationary phase.
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I. INTRODUCTION

The main objective of inflationary scenario is to explain why the present universe is homogeneous and isotropic on large scales without a fine tuning of initial conditions. It is important to study the generality of initial conditions for the onset of inflation.

Many works on inflation assume homogeneous and flat universe at the pre-inflationary phase. In this case, inflation is generic. The spatial curvature effect on the occurrence of inflation is investigated in the Friedmann-Robertson-Walker model with a massive scalar field by Belinsky et al., and in the Bianchi IX model with a cosmological constant by Wald. They concluded that inflation is not a general property and only the positive spatial curvature can prevent the universe from inflating.

On the other hand, the role of initial inhomogeneities on the occurrence of inflation was studied by a linear perturbation analysis and by a numerical simulation with special symmetries. Goldwirth and Piran concluded that the crucial feature necessary for inflation is a sufficient high average value of the scalar field which drives inflation over a region of several Hubble radius.

We study how the initial inhomogeneities of the universe affect the onset of inflation by use of an alternative approach. In order to treat the inhomogeneities of the universe, we use the long wavelength approximation.

In the long wavelength approximation we take a synchronous reference frame where the line element is of the form

\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu = -dt^2 + \gamma_{ij}(t, x) dx^i dx^j. \]  

Throughout this paper Latin letters will denote spatial indices and Greek letters spacetime indices. In the usual method, we neglect all spatial gradients of \( \gamma_{ij}(t, x) \) in the Einstein equations and construct an approximate solution with the characteristic scale of inhomogeneities larger than the Hubble radius. Furthermore, we consider the quasi-isotropic universe as a special case in the form

\[ ds^2 = -dt^2 + a^2(t) h_{ij}(x) dx^i dx^j, \]
where $h_{ij}(x)$ is an arbitrary function called ‘seed metric’. The universe is assumed to be filled with a perfect fluid characterized by energy density $\rho$ and pressure $p = (\Gamma - 1)\rho$. In this case the Einstein equations reduce to

\[
\frac{\ddot{a}}{a} - \frac{2 - 3\Gamma}{2} \left(\frac{\dot{a}}{a}\right)^2 = 0, \quad (1.3)
\]

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\kappa \rho, \quad (1.4)
\]

where $\kappa \equiv 8\pi G$. These are nothing but the equations which the scale factor $a(t)$ of the flat Friedmann universe should satisfy. We call the approximate inhomogeneous solution given by Eq.(1.2) satisfying Eqs.(1.3) and (1.4) the locally flat Friedmann solution and it is used as a starting point to solve the Einstein equations iteratively [10,11]. In the iteration scheme, the spatial metric can be expanded as a sum of spatial tensors of increasing order in spatial gradients of $h_{ij}(x)$. Thus the approximation is called the gradient expansion method.

Recently, the influence of initial inhomogeneities on the occurrence of inflation is studied by using the gradient expansion method [13–15]. From investigations in the homogeneous universe model, as noted above, the effect of positive curvature is important because it can prevent inflation. So, we should investigate the inhomogeneities of universe with a non-small positive spatial curvature to clarify the generality of inflation. However, the lowest order solution in the gradient expansion method is a locally flat Friedmann universe and the spatial curvature is treated as a small quantity in the expansion scheme.

In ref. [15], the gradient expansion method from the locally closed Friedmann universe is introduced and the inhomogeneous closed universe with a cosmological constant is studied as an application of the scheme. This method is useful to treat the inhomogeneities of non-small curvature associated with a three-space which is conformal to the constant curvature space at lowest order. In this paper we consider inflation which is driven by a massive scalar field in the inhomogeneous universe by use of the first order approximation of the gradient expansion from the locally closed Friedmann universe. We will show the condition of the inhomogeneities of the spatial curvature for the universe to enter the inflationary phase.

This paper is organized as follows. In Sec.2, we derive basic equations by use of the first order approximation in the gradient expansion method from the locally closed Friedmann universe. In Sec.3, we show numerical results. Sec.4 is devoted to conclusion. We use units in which $c = \hbar = 1$.

II. BASIC EQUATIONS IN IMPROVED GRADIENT EXPANSION METHOD

We take the matter in the universe to be a scalar field minimally coupled to gravity. The energy-momentum tensor of a scalar field is given by

\[
T_{\mu\nu} = \phi,_{\mu}\phi,_{\nu} - g_{\mu\nu}\left[\frac{1}{2}\phi,_{\lambda}\phi,^{\lambda} + V(\phi)\right], \quad (2.1)
\]

where $V(\phi)$ is the potential of the scalar field. In the synchronous reference frame Eq.(1.1), the Einstein equations for the gravitational field coupled to the scalar field read

\[
\frac{1}{2}\dot{K} + \frac{1}{4}K^l_{m\,l}K^m_l = \kappa \left[-\dot{\phi}^2 + V(\phi)\right], \quad (2.2)
\]

\[
3\dot{R}^i_l + \frac{1}{2\sqrt{\gamma}} \frac{\partial}{\partial t} \left(\sqrt{\gamma}K^j_i\right) = \kappa \left[\partial_i \phi \partial^j \phi + V(\phi)\delta^j_i\right], \quad (2.3)
\]

\[
\frac{1}{2}(K^l_{ij} - K_{ij}) = \kappa \dot{\phi} \partial_i \phi, \quad (2.4)
\]

where $3\dot{R}^i_l$ is the Ricci tensor associated with $\gamma_{ij}$, $K_{ij} \equiv \dot{\gamma}_{ij}$, $K \equiv \gamma^{ij}K_{ij}$, $\gamma \equiv \det \gamma_{ij}$, and a semicolon denotes the covariant derivative with respect to $\gamma_{ij}$. The equation of motion for the scalar field is

\[
\ddot{\phi} + \frac{1}{2}K \ddot{\phi} - \dot{\phi}^2 + \frac{dV(\phi)}{d\phi} = 0. \quad (2.5)
\]

Now, we assume the metric in the form [15].
where $h_{ij}(x)$ is the metric of three-dimensional sphere whose Ricci curvature is $R_{ij}(h_{kl}) = 2h_{ij}$. The scale factor $a(t,\Omega(x))$ in the spatial metric $\text{Eq.}(2.6)$ has the space-dependence by an arbitrary function $\Omega(x)$. In the usual gradient expansion method, all spatial gradients of $\gamma_{ij}(t,x)$ are assumed to be much smaller than time derivatives. Roughly speaking, the spatial curvature $\gamma_{ij}(t,x)$ is assumed to be smaller than $K^2$ and it is neglected in the lowest approximation. Here, we consider slightly different approximation. The spatial scalar curvature for the metric in the form of $\text{Eq.}(2.6)$ is

$$3\gamma_{ij}(t,x) = a(t,\Omega)\left[ -4\frac{a'}{a} \frac{\nabla_i \nabla_j \Omega}{\Omega} + \frac{2}{a^2} \left( \frac{a''}{a} - 2 \frac{a'}{a^2} \right) \frac{\Omega^2 \nabla_i \nabla_j \Omega}{\Omega^2} \right],$$

where a prime denotes the derivative with respect to $\Omega$ and $\nabla_i$ denotes the covariant derivative with respect to $h_{ij}$. We assume that the second term in the right hand side of $\text{Eq.}(2.7)$ which contains spatial gradients of $\Omega$ is smaller than the first term. If we set $3\gamma_{ij}(t,x) \equiv 6a^{-2}(t,\Omega) \text{Eq.}(2.7)$ is rewritten in the form

$$3\gamma_{ij}(t,x) = 3\gamma_{ij}(t,x) + \left[ \frac{3\gamma_{ij}(t,x)}{a(t,\Omega)} - \frac{3\gamma_{ij}(t,x)}{2} - \frac{3\gamma_{ij}(t,x)}{2} \right].$$

Thus the case, we consider here, is that the spatial variation of the spatial curvature is smaller than the value of itself. All spatial gradients of $\Omega(x)$ is neglected in the first step of our approach. It should be noted that even if the second term in $\text{Eq.}(2.7)$ or $\text{Eq.}(2.8)$ exceeds the first term, the approximation reduces to the usual long wavelength approximation so far as these are smaller than $K^2$.

Though the metric in the form of $\text{Eq.}(2.6)$ has not adequate degree of gravitational freedom, the simple form on which we concentrate is useful to investigate the inhomogeneities of the spatial curvature. It might be very complicated to treat the general form of inhomogeneities of curvature $\gamma_{ij}$.

From now on, we consider that the spatial metric in the lowest order are described by $\text{Eq.}(2.6)$. Substituting $\text{Eq.}(2.6)$ into $\text{Eq.}(2.2)$, and neglecting all spatial derivatives of $\Omega(x)$, we obtain the equation which $a$ and $\phi$ must satisfy:

$$\frac{\dot{a}(t,\Omega(x))}{a(t,\Omega(x))} = \kappa \frac{3}{3} \left[ -\phi^2(t,x) + V(\phi(t,x)) \right].$$

(2.9)

From $\text{Eq.}(2.9)$, it is natural that the scalar field has the space-dependence through $\Omega(x)$. We assume that the form of the scalar field in the lowest order is

$$\phi(t,\Omega(x)) = \phi_0(t,\Omega(x)).$$

(2.10)

Substituting $\text{Eqs.}(2.9)$ and (2.10) into $\text{Eqs.}(2.3)$ and (2.5), and neglecting all spatial derivatives of $\Omega(x)$, we have

$$\frac{\dot{a}(t,\Omega)}{a(t,\Omega)} + 2 \left( \frac{\dot{a}(t,\Omega)}{a(t,\Omega)} \right)^2 + \frac{2}{a^2(t,\Omega)} = \kappa V(\phi_0(t,\Omega)),$$

(2.11)

$$\ddot{\phi}_0(t,\Omega) + 3 \frac{\dot{a}(t,\Omega)}{a(t,\Omega)} \phi_0(t,\Omega) + \frac{dV(\phi_0(t,\Omega))}{d\phi_0(t,\Omega)} = 0.$$  

(2.12)

Equation(2.4) is trivial in the lowest order. Equations(2.9), (2.11) and (2.12) have the same form of the equations for the closed Friedmann universe with a homogeneous scalar field.

At next order, we consider the second order in spatial gradients of $\Omega(x)$. We will take corrections to the metric and the scalar field of the form

$$\gamma_{ij}(t,x) = a(t,\Omega) \left[ \frac{1}{3} F(t,\Omega) \frac{\nabla_i \nabla_j \Omega}{\Omega} + \frac{1}{3} G(t,\Omega) \frac{\nabla_i \nabla_j \Omega}{\Omega^2} \right],$$

$$\phi(t,\Omega) = P(t,\Omega) \frac{\nabla_i \nabla_j \Omega}{\Omega} + Q(t,\Omega) \frac{\nabla_i \nabla_j \Omega}{\Omega^2},$$

(2.13)

(2.14)
where \( \nabla_i \nabla_j \Omega / \Omega \) and \( \nabla_i \nabla_j \nabla_i \Omega / \Omega^2 \) are traceless tensors defined by

\[
\frac{\nabla_i \nabla_j \Omega}{\Omega} = \frac{\nabla_i \nabla_j \Omega}{\Omega} - \frac{1}{3} \frac{\nabla_i \nabla^i \Omega}{\Omega} h_{ij},
\]

\[
\frac{\nabla_i \nabla_j \nabla_i \Omega}{\Omega^2} = \frac{\nabla_i \nabla_j \nabla_i \Omega}{\Omega^2} - \frac{1}{3} \frac{\nabla_i \nabla^i \Omega}{\Omega^2} h_{ij}.
\]

Substituting \( \gamma_{ij} = (0)_{ij} + (1)_{ij} \) and \( \phi = (0) + (1) \) given by Eqs.(2.4), (2.13), (2.10) and (2.14) into the Einstein equations and the equation of motion for the scalar field Eqs.(2.2)–(2.3), and comparing the coefficients of the derivative of \( \Omega \), we get the first order equations which govern the evolution of the variables \( F, \bar{F}, G, \bar{G}, P \) and \( Q \). We should perform tedious calculation to get \( 3\bar{R}^i_j \) associated with \( \gamma_{ij} = (0)_{ij} + (1)_{ij} \). The expression of it in the first order approximation appears in Appendix A.

From Eq.(2.3) we obtain

\[
\ddot{F} + 6H \dot{F} = 6\kappa \left\{ \frac{dV}{d\phi} \right\}_0 P + \frac{4}{a^2} \left[ F - \bar{F} + \frac{2a'}{a} \Omega \right],
\]

\[
\ddot{\bar{F}} + 3H \dot{\bar{F}} = \frac{2a'}{a^3} \Omega,
\]

\[
\ddot{\bar{G}} + 6H \dot{\bar{G}} = 6\kappa \left\{ \frac{dV}{d\phi} \right\}_0 Q + \frac{2\kappa}{a^2} \phi'^2 \Omega^2 + \frac{4}{a^2} \left[ \ddot{\bar{G}} + \dddot{\bar{F}} - \dddot{\bar{F}}' \Omega + \left( 2 \frac{a''}{a} - \frac{a'^2}{a^2} \right) \Omega^2 \right],
\]

\[
\dddot{\bar{G}} + 3H \ddot{\bar{G}} = \frac{2\kappa}{a^2} \phi'^2 \Omega^2 + \frac{2}{a^2} \left[ 2\ddot{\bar{G}} + 2\dddot{\bar{F}} - 2\dddot{\bar{F}}' \Omega - \frac{3}{2} \dddot{\bar{F}}' \Omega + \left( \frac{a''}{a} - \frac{a'^2}{a^2} \right) \Omega^2 \right],
\]

where \( \left\{ \frac{dV}{d\phi} \right\}_0 \) is the lowest order value of \( \frac{dV}{d\phi} \), \( H \equiv \dot{a}/a, \bar{G} \equiv G + \bar{F}^2 \) and \( \bar{G} \equiv \bar{G} - \frac{1}{2} \bar{F}^2 \). Note readers should pay attention to the commutation relation of the covariant derivatives which is not vanishing because the spatial surface in the lowest order solution has a positive curvature. From Eq.(2.5), we have

\[
\ddot{P} + 3H \dot{P} + \frac{d^2V}{d\phi^2} \left\{ \frac{dV}{d\phi} \right\}_0 \left[ 1 - \frac{1}{2a'} \phi' \Omega + \frac{1}{a^2} \phi'' \Omega \right],
\]

\[
\ddot{Q} + 3H \dot{Q} + \frac{d^2V}{d\phi^2} \left\{ \frac{dV}{d\phi} \right\}_0 \left[ -1 + \frac{1}{2a'} \phi' \Omega + \frac{1}{a^2} \phi'' \Omega \right].
\]

From Eqs.(2.2) and (2.4) we obtain

\[
\dddot{\bar{F}} = 2 \left[ \frac{\dot{a'}}{a} - \frac{\dot{a} a'}{a^2} \right] \Omega + \kappa \phi \phi' \Omega,
\]

\[
\dddot{\bar{F}} + 2H \dot{\bar{F}} = 2\kappa \left[ -2\phi' \dot{P} + \left\{ \frac{dV}{d\phi} \right\}_0 P \right],
\]

\[
\dddot{\bar{G}} + 2H \dot{\bar{G}} = 2\kappa \left[ -2\phi' \dot{Q} + \left\{ \frac{dV}{d\phi} \right\}_0 Q + \frac{\dddot{\bar{F}}}{a} \right].
\]

There are nine equations (2.15)–(2.23) for the six variables \( F, \bar{F}, G, \bar{G}, P \) and \( Q \). All of these equations are not independent. Six of these equations govern the evolution and three are constraint conditions. So we find a solution of Eqs.(2.15)–(2.20) with an initial condition which satisfies the constraints. In order to solve these equations, we need to know the time evolution of \( a', a'', \phi', \phi'' \) and \( \bar{F} \). Differentiating Eqs.(2.3), (2.11), (2.12) and (2.21), we obtain the equations for them, Eqs.(2.24)–(2.31). At higher order, we consider the corrections which are constructed by the spatial derivatives of \( \Omega(x) \) and solve the Einstein equations order by order in the gradient expansion. The spatial metric and the scalar field can be expanded as

\[
\gamma_{ij}(t, x) = a^2(t, \Omega) \left[ h_{ij} + \sum_A F^A(t, \Omega)(\nabla^{(2)}\Omega)^A \right],
\]

\[
\phi(t, x) = \phi(t, \Omega) + \sum_A P^A(t, \Omega)(\nabla^{(2)}\Omega)^A + \sum_A P^A(t, \Omega)(\nabla^{(4)}\Omega)^A + \cdots
\]
where the notation \((\nabla^{(2p)}\Omega)^{ij}_A\) denotes symbolically symmetric spatial tensors which contain \(2p\) spatial gradients of \(\Omega(x)\), the suffix \(A\) distinguishes the tensors belonging to the same class, and \(a(t, \Omega), \phi_0(t, \Omega), F^{A}_{(2p)}(t, \Omega)\) and \(P^{A}_{(2p)}(t, \Omega)\) are function of \(t\) and \(\Omega(x)\) which is determined by the Einstein equations. In the case of \(p = 1, 2\), the explicit form of \((\nabla^{(2p)}\Omega)^{ij}_A\) appears in the Appendix III.

This expansion scheme is valid when the characteristic scale of inhomogeneities of the spatial curvature is larger than the Hubble radius \((\dot{a}/a)^{-1}\) or the spatial curvature radius \(\Omega^{\vert_{m=0}}_\Omega^{-1/2}\). If the Hubble radius is smaller than the spatial curvature radius, this expansion scheme reduces to the usual gradient expansion.

### III. THE INFLUENCE OF INHOMOGENEITIES FOR INFLATION

We consider a chaotic inflation model with a massive scalar field, i.e., \(V(\phi) = \frac{1}{2}m^2\phi^2\) as an inflaton field.

When the energy of the scalar field is dominated by a kinetic term at the early stage, the initial behavior of solution without inhomogeneities for Eqs. (2.9), (2.11) and (2.12) is expressed by series expansions in the form

\[
a^2(t) = m^{-2}t^{\frac{2}{3}} \left[ 1 - \frac{9}{7}t^{\frac{4}{3}} + O(t^{\frac{4}{3}}) \right],
\]

\[
\phi_0(t) = \sqrt{\frac{2}{3\kappa}} \left[ \ln t + \frac{81}{56}t^{\frac{2}{3}} + O(t^{\frac{2}{3}}) \right] + \tilde{\phi}_0,
\]

where \(\tilde{\phi}_0\) is the constant which corresponds to the freedom of initial value of the scalar field. The time variable \(t\) is non-dimensionalized by use of \(m\). On assumption that we neglect all spatial derivatives of \(\Omega\), we can generalize the solution Eqs. (3.1) and (3.2) in the form

\[
a^2(t, \Omega(x)) = \Omega^2(x)t^{\frac{2}{3}} \left[ 1 - \frac{9m^{-2}}{7\Omega^2(x)}t^{\frac{4}{3}} + O\left(\frac{m^{-4}}{\Omega^4}t^{\frac{4}{3}}\right) \right],
\]

\[
\phi_0(t, \Omega(x)) = \sqrt{\frac{2}{3\kappa}} \left[ \ln t + \frac{81m^{-2}}{56\Omega^2(x)}t^{\frac{2}{3}} + O\left(\frac{m^{-4}}{\Omega^4}t^{\frac{4}{3}}\right) \right] + \tilde{\phi}_0.
\]

The local scale factor and the scalar field are expressed by series expansions of \((m^{-2}/\Omega^2)t^{\frac{2}{3}}\) in the initial stage. At the beginning of the universe, the local scale factor behaves as \(a(t, \Omega) \propto t^{1/3}\) and after that the time evolution depends on the local value of \(\Omega(x)\). In the initial stage where the kinetic energy of the scalar field dominates the potential energy, the behavior of \(a(t, \Omega)\) does not depend on \(\tilde{\phi}_0\) but in the late stage where the potential energy becomes effective, it depends on \(\tilde{\phi}_0\).

We can see the time evolutions of these variables, which is characterized by the value of \(\Omega(x)\) and \(\tilde{\phi}_0\), by solving Eqs. (2.9), (2.11) and (2.12) numerically. Fig. II shows the inflationary region and the recollapsing one on \((\Omega^2, \tilde{\phi}_0)\) plane. The local scale factor with parameters in the inflationary region enters the accelerating expansion phase: \(\dot{a}/a > 0\) and \(\ddot{a}/a > 0\). On the other hand, the local scale factor with parameters in the recollapsing region enters the recollapsing phase: \(\dot{a}/a < 0\) and \(\ddot{a}/a < 0\). From Fig. II we see that inflation favors large \(\Omega\) and large \(|\tilde{\phi}_0|\) as is expected by the investigation of the homogeneous model III. For \(\Phi_0\) there is a critical value of \(\Omega\), \(\Omega_{cr}\), if \(\Omega > \Omega_{cr}\), inflation occurs. Since the critical value \(\Omega_{cr}\) depends on \(\phi_0\) then the condition for the onset of inflation in the lowest order solution is written in the form

\[
\frac{1}{\Omega^2} < \frac{1}{\Omega^2_{cr}(\tilde{\phi}_0)}.
\]

In the Fig. III \(\Omega_{cr}(\tilde{\phi}_0)\) is drawn as the boundaries between the inflationary region and the recollapsing region.

In the initial stage where the local scale factor \(a(t, \Omega(x))\) is proportional to \(\Omega(x)t^{\frac{2}{3}}\), the spatial scalar curvature behaves as \(\Omega^{-2}t^{-\frac{2}{3}}\). The condition (3.3) is then, translated to the condition on the initial curvature as

\[
3R_{init}(t, \Omega) < 3R_{cr}(t, \tilde{\phi}_0),
\]

where \(3R_{cr} \equiv 6t^{-\frac{2}{3}}\Omega_{cr}^{-2}(\tilde{\phi}_0)\). This condition for the onset of inflation is a restriction on only the local value of \(3R\). Next, we take the first order correction terms into account. We integrate Eqs. (2.13) and (2.20) and Eqs. (3.2) - (3.6) numerically. We restrict ourselves to consider growing solution whose initial asymptotic behavior is
\[ F = \frac{18m^{-2}}{7\Omega^2}t^4, \quad \tilde{F} = \frac{9m^{-2}}{8\Omega^2}t^4, \quad \tilde{G} = \frac{9m^{-2}}{7\Omega^2}t^4, \quad \tilde{G} = -\frac{9m^{-2}}{4\Omega^2}t^4, \quad P = -\sqrt{\frac{3}{2\kappa}} \frac{9m^{-2}}{14\Omega^2}t^4, \quad Q = \sqrt{\frac{3}{2\kappa}} \frac{9m^{-2}}{28\Omega^2}t^4. \]

The time evolution of \( F, \tilde{G}, P \) and \( Q \) is shown in Figs. 2(i)–2(iv).

In the case that the local scale factor in the lowest order will enter the inflationary phase \( (\Omega^2 > \Omega_{cr}^2) \), \( F \) and \( \tilde{G} \) grow in proportion to time, and \( P \) and \( Q \) approach constant values. On the other hand, in the case that the local scale factor in the lowest order will enter the recollapsing phase \( (\Omega^2 < \Omega_{cr}^2) \), \( F, \tilde{G}, P \) and \( Q \) diverge and the approximation breaks down in the course of time. These behavior are consistent with the results by the linear perturbation analysis [3]. The trajectory of scalar field in \((\phi, \dot{\phi})\) space is shown in Figs. 3(i)–3(ii).

At each spatial point, we define an effective local scale factor \( a_{\text{eff}}(t, x) \) by

\[
a_{\text{eff}}(t, x) \equiv \left[ \det \left[ \gamma^{(0)}_{ij}(t, x) + \gamma^{(1)}_{ij}(t, x) \right] \right]^{1/6} = a(t, \Omega) \left[ 1 + \frac{1}{6} F(t, \Omega) \frac{\nabla_\Omega \nabla^\Omega}{\Omega} \frac{\nabla_{\Omega} \nabla^\Omega}{\Omega^2} + \frac{1}{6} \tilde{G}(t, \Omega) \frac{\nabla_\Omega \nabla^\Omega}{\Omega^2} \right], \tag{3.7}
\]

which describes how the universe expands at the spatial point. We can follow the evolution of \( a_{\text{eff}}(t, x) \) by the evolution of \( a(t, \Omega), F(t, \Omega) \) and \( \tilde{G}(t, \Omega) \). The behavior of \( a_{\text{eff}}(t, x) \) is characterized by four parameters \( \tilde{\phi}_0, \Omega, \nabla_\Omega \nabla^\Omega/\Omega \) and \( \nabla_\Omega \nabla^\Omega/\Omega^2 \). In addition to the value of \( \tilde{\phi}_0 \) and \( \Omega \), the spatial derivatives of \( \Omega \) affect the evolution of the local scale factor.

The effective local expansion rate and acceleration rate are given by

\[
\frac{\dot{a}_{\text{eff}}(t, x)}{a_{\text{eff}}(t, x)} = \frac{\dot{a}(t, \Omega)}{a(t, \Omega)} + \frac{1}{6} \left[ \dot{F}(t, \Omega) \frac{\nabla_\Omega \nabla^\Omega}{\Omega} + \frac{\dot{\tilde{G}}(t, \Omega)}{\Omega} \frac{\nabla_\Omega \nabla^\Omega}{\Omega^2} \right],
\]

\[
\frac{\ddot{a}_{\text{eff}}(t, x)}{a_{\text{eff}}(t, x)} = \frac{\ddot{a}(t, \Omega)}{a(t, \Omega)} + \frac{1}{6} \left[ \ddot{F}(t, \Omega) + 2 \frac{\dot{a}(t, \Omega)}{a(t, \Omega)} \dot{F}(t, \Omega) \right] \frac{\nabla_\Omega \nabla^\Omega}{\Omega} + \frac{1}{6} \left[ \frac{\ddot{\tilde{G}}(t, \Omega)}{\Omega} + 2 \frac{\dot{a}(t, \Omega)}{a(t, \Omega)} \frac{\dot{\tilde{G}}(t, \Omega)}{\Omega} \right] \frac{\nabla_\Omega \nabla^\Omega}{\Omega^2}. \]

We can divide the four-dimensional parameter space \((\tilde{\phi}_0, \Omega, \nabla_\Omega \nabla^\Omega/\Omega, \nabla_\Omega \nabla^\Omega/\Omega^2)\) into two regions: inflationary region and recollapsing region by a three dimensional hyper surface. The effective local scale factor with the parameters in the inflationary region enters the accelerating expansion phase: \( \dot{a}_{\text{eff}}/a_{\text{eff}} > 0 \) and \( \ddot{a}_{\text{eff}}/a_{\text{eff}} > 0 \). On the other hand, the effective local scale factor with the parameters in the recollapsing region enters the recollapsing phase: \( \dot{a}_{\text{eff}}/a_{\text{eff}} < 0 \) and \( \ddot{a}_{\text{eff}}/a_{\text{eff}} < 0 \). We assume the approximation is valid while

\[
\left| \frac{\det \left( \gamma^{(0)}_{ij} + \gamma^{(1)}_{ij} \right) - \det \gamma^{(0)}_{ij}}{\det \gamma^{(0)}_{ij}} \right| < 0.5 \quad \text{and} \quad \left| \frac{3^{(1)}_R (\gamma_{ij})}{\max \left\{ H^2, 3^{(0)}_R (\gamma_{ij}) \right\}} \right| < 0.5. \tag{3.8}
\]

We evolve the universe numerically while both of these conditions Eq. (3.3) hold.

Evolving of the effective local expansion rate and acceleration rate numerically, we see how the two spatial gradients, \( \nabla_\Omega \nabla^\Omega/\Omega \) and \( \nabla_\Omega \nabla^\Omega/\Omega^2 \), affect the occurrence of inflation. For fixed \( \tilde{\phi}_0 \), the inflationary region and recollapsing region are shown in Figs. 3(i)–3(ii). Between these two regions, there is a fuzzy region where error becomes large and one of inequality in Eq. (3.8) does not hold before the effective local scale factor enters the accelerating expansion phase or the recollapsing phase. We see that the positive (negative) \( \nabla_\Omega \nabla^\Omega/\Omega \) helps the universe to enter the inflationary (recollapsing) phase. On the other hand, \( \nabla_\Omega \nabla^\Omega/\Omega^2 \) tends to prevent the onset of inflation.

From the numerical results, we obtain the time independent condition for the onset of inflation in the vicinity of \( \Omega/\Omega_{cr} - 1 = 0 \) and \( \nabla_\Omega \nabla^\Omega/\Omega = \nabla_\Omega \nabla^\Omega/\Omega^2 = 0 \) in the parameter space, within the first order of gradient expansion,

\[
\frac{1}{\Omega^2} \left[ 1 + \alpha \left( -\frac{\nabla_\Omega \nabla^\Omega}{\Omega} + \beta \frac{\nabla_\Omega \nabla^\Omega}{\Omega^2} \right) \right] < \frac{1}{\Omega^2_{cr}(\tilde{\phi}_0)}, \tag{3.9}
\]

where \( \alpha \) and \( \beta \) are constants. The parameter \( \alpha \) shows the strength of the effect of inhomogeneities on the occurrence of inflation. In the massive scalar inflaton model, the value of \( \alpha \) is about 2.0. The value of \( \beta \) depends on the initial value of the scalar field \( \tilde{\phi}_0 \): \( \beta = 4.3 \) for \( \tilde{\phi}_0 = 5.0\kappa^{-1/2} \) and \( \beta = 3.3 \) for \( \tilde{\phi}_0 = 10.0\kappa^{-1/2} \). In the early stage where \( \alpha^2(t, \Omega) = \Omega^2 t^4 \), the spatial curvature \( 3R_{init} \) in the first order approximation is
\[ 3R_{\text{init}}(t, \Omega) = \frac{2}{t^2 \Omega^2} \left[ 3 - 2 \frac{\nabla_i \nabla^i \Omega}{\Omega} + \frac{\nabla_i \Omega \nabla^i \Omega}{\Omega^2} \right]. \] (3.10)

By use of Eq.(3.10), the criterion (3.3) is rewritten as

\[ 3R_{\text{init}}(t, \Omega) + \tilde{\alpha} \left\{ \frac{(3R_{\text{init}})_i}{3R_{\text{init}}} \beta + \beta \frac{(3R_{\text{init}})_i (3R_{\text{init}})^j}{3R_{\text{init}}} \right\} < 3R_{\text{cr}}(t, \hat{\phi}_0), \] (3.11)

where \( 3R_{\text{cr}}(t, \hat{\phi}_0) \) is the marginal value of the spatial curvature for inflation in the closed Friedmann model appears in Eq.(3.3) and the parameters \( \tilde{\alpha} \) and \( \tilde{\beta} \) are

\[ \tilde{\alpha} = (3\alpha - 2), \quad \tilde{\beta} = \frac{12\alpha - 3\alpha \beta - 7}{2(3\alpha - 2)}. \] (3.12)

The criterion (3.11) is a condition on the initial value of the spatial curvature and its spatial derivatives. Because \( \tilde{\alpha} \) and \( \tilde{\beta} \) are several numbers when \( \hat{\phi}_0 = (5.0 \sim 10.0)\kappa^{-1/2} \), the onset of inflation is occur if the initial value of spatial curvature \( 3R_{\text{init}} \) is less than the critical value \( 3R_{\text{cr}}(t, \hat{\phi}_0) \) over a region which has several size of the local curvature radius \( (3R_{\text{init}})^{-1/2} \).

Goldwirth and Piran studied how inhomogeneities influence the inflationary epoch numerically in the case of spherical symmetry \[3\]. They conclude that the crucial feature necessary for inflation is a sufficiently high average value of the scalar field ( suitable value for inflation in homogeneous universe ) over a region of several Hubble radius. In order to compare their results with ours, we impose \( \Omega^2(x) \) to spherically symmetric form, \( \Omega^2(\chi, \theta, \varphi) = \Omega^2(\chi) \), in the spherical coordinate where \( h_{ij} = \text{diag}[1, \sin^2 \chi, \sin^2 \chi \sin^2 \theta] \). We consider \( \Omega^2(\chi) \) in the form

\[ \Omega^2(\chi) = \Omega^2_{\text{cr}} + (\Omega^2_0 - \Omega^2_{\text{cr}}) \left[ \frac{1 - \exp \left[-\Delta^2 \cos^2 \chi\right]}{1 - \exp \left[-\Delta^2\right]} \right], \] (3.13)

where \( \Omega^2_0 \) and \( \Omega^2_{\text{cr}} \) are the value of \( \Omega \) at \( \chi = 0 \) and \( \chi = \pi \), respectively, and \( \Delta \) is a parameter. We fix \( \hat{\phi}_0 = 5.0\kappa^{-1/2} \), for example, then \( \Omega^2_{\text{cr}} \approx 0.70m^{-2} \) (see Fig.4). The form of \( \Omega^2 \) for \( (\Omega^2_0 + \Omega^2_{\text{cr}})/2 = \Omega^2_{\text{cr}} \) and \( \Omega^2_0 - \Omega^2_{\text{cr}} = 4 \times 10^{-7}m^{-2} \) is shown in Fig.3. Varying \( \Delta \), we show inflationary and recollapsing regions in the three-dimensional space (see Fig.3).

At the origin \( \chi = 0 \) universe can enter the inflationary phase whenever \( \Delta \) is smaller than about 0.95. From Fig.4, we see that the region of a suitable value of \( \Omega^2 \) is 0 \( \leq \chi \leq 1.85 \). Comparing the physical length \( L \sim a(t, \Omega)\chi \) for this region and the initial curvature radius \( (3R_{\text{init}})^{-1/2} \sim a(t, \Omega)/\sqrt{6} \), we get \( L \sim 4.5(3R_{\text{init}})^{-1/2} \). In this case, inflation occurs at the origin when the initial value of spatial curvature is less than the critical value over the region 4.5 times initial curvature radius.

**IV. CONCLUSION**

We consider a chaotic inflation model which is driven by a massive scalar field. In this model, we study how the initial inhomogeneities of the universe affect the onset of inflation by use of the gradient expansion from a locally closed Friedmann universe.

At lowest order of our approximation, inhomogeneities of the metric and the scalar field are described by an arbitrary function \( \Omega(x) \). At next order, using the effective local scale factor \( a_{\text{eff}}(t, x) \), we investigate the effect of inhomogeneities of the universe for the onset of inflation.

From numerical results, we obtain the condition for the onset of inflation. The spatial curvature in the early stage should be less than the critical curvature value \( 3R_{\text{cr}}(t, \hat{\phi}_0) \) over a region which has several size of the local curvature radius \( (3R_{\text{init}})^{-1/2} \) for the onset of inflation in the future.

We can compare the effect of inhomogeneities in the inflation model which is driven by a massive scalar field with the one in the model driven by a cosmological constant \[12\]. The value of \( \alpha \) in the scalar field model is larger than the one in the cosmological constant model. It means that the inflationary model driven by a massive scalar field is more sensitive to the inhomogeneities than one by a cosmological constant.

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APPENDIX A: EXPRESSION FOR THE SPATIAL CURVATURE IN THE FIRST ORDER

In the gradient expansion from the locally closed Friedmann universe, the spatial metric up to the second order of the spatial gradients is given in the form

\[ \gamma_{ij}(t, x) = (0)^{\gamma_{ij}}(t, x) + (1)^{\gamma_{ij}}(t, x), \]  

where

\[ (0)^{\gamma_{ij}}(t, x) = a^2(t, \Omega)h_{ij}, \]
\[ (1)^{\gamma_{ij}}(t, x) = a^2(t, \Omega) \left[ \frac{1}{3} F(t, \Omega) \frac{\nabla_i \nabla^i \Omega}{\Omega} h_{ij} + \tilde{F}(t, \Omega) \frac{\nabla_i \nabla^i \Omega}{\Omega^2} h_{ij} \right]. \]

The spatial Ricci curvature for the metric is

\[ 3^{(n)} R_{ij}^{(\gamma _{kl})} = - \frac{1}{a^2} \left\{ \frac{2}{3} \left[ F - \dot{F} + \frac{2a'}{a} \frac{\nabla_i \nabla^i \Omega}{\Omega} \right] \frac{\nabla_i \nabla^i \Omega}{\Omega} \delta_{ij} + \frac{2}{3} \left[ G + \dot{F}' \Omega + F' \Omega + \Omega^{2} \left( \frac{2a''}{a} - \frac{a'^2}{a^2} \right) \right] \frac{\nabla_i \nabla^i \Omega}{\Omega^2} \delta_{ij} \right\} + \left( \frac{a}{\Omega} \right)^2 \frac{\nabla_i \nabla^i \Omega}{\Omega^2} \delta_{ij} + \left[ 2\tilde{G} + 2\dot{F} - 2F' \Omega - \frac{5\tilde{F}^2}{2} - 2 \left( \frac{a'}{a} \right)^2 \frac{\nabla_i \nabla^i \Omega}{\Omega^2} \right], \]

APPENDIX B: EQUATIONS FOR a', a'', \phi_0, \phi_0'' AND \overline{F}'

We derive the equation for \( a', a'', \phi_0, \phi_0'' \) and \( \overline{F}' \).

From Eqs. (2.11) and (2.12) we obtain

\[ \dot{a}(t, \Omega) = \frac{\dot{\alpha}^2(t, \Omega)}{a(t, \Omega)} + \frac{1}{a(t, \Omega)} \frac{\kappa}{2} (a(t, \Omega) \phi_0^2(t, \Omega)). \]

Differentiating Eqs. (2.12) and (B1) with respect to \( \Omega \), we find the equations which \( a'(t, \Omega) \) and \( \phi'(t, \Omega) \) must satisfy

\[ \dot{a}' = 2 \frac{\dot{\alpha} a'}{a} - \left[ \frac{\dot{\alpha}^2 + \frac{1}{a^2} + \frac{\kappa \phi_0^2}{2}}{a^2} \right] a' - \kappa \phi_0 \phi_0^2 a, \]

\[ \dot{\phi}_0' = -3 \frac{\dot{\alpha}}{a} \phi_0' - \left( \frac{dV}{d\phi} \right) \bigg|_0 - 3 \phi_0 a a_0' - \dot{a} a' \phi_0. \]

Differentiating Eqs. (B2) and (B3) again, we find

\[ \ddot{a}' = 2 \frac{\dot{\alpha} \ddot{a}'}{a} - \left[ \frac{\dot{\alpha}^2 + \frac{1}{a^2} + \frac{\kappa \phi_0^2}{2}}{a^2} \right] \ddot{a}' + 2 \frac{\dot{\alpha} \dot{a} \ddot{a}'}{a} - 4 \frac{\dot{\alpha} \dot{a}^2 \ddot{a}}{a^2} + 2 \frac{\dot{\alpha}^2 (\ddot{a}^2 + \frac{1}{a^2})}{a^3} - 2 \kappa a' \phi_0 \phi_0^2 - \kappa a \left( \phi_0 \phi_0'' + \phi_0'' \right), \]

\[ \ddot{\phi}_0' = -3 \frac{\dot{\alpha} a'' \phi_0}{a^2} - \left( \frac{dV}{d\phi} \right)^{''} \bigg|_0 - 6 \phi_0 \frac{a a' \ddot{a}' - \dot{a} a'}{a^2} - 3 \phi_0 \left[ \frac{\ddot{a}''}{a} - a \frac{a' \ddot{a}'}{a^2} - \dot{a} \frac{a'' a'}{a^2} + \frac{2 a a'' a'}{a^3} \right]. \]

Similarly from Eq. (2.21), we find that \( \overline{F}' \) must satisfy

\[ \overline{F}' = 2 \frac{\dot{\alpha} \dot{a}' - a' \dot{\alpha}}{a^2} + 2 \Omega \left[ \frac{\dot{\alpha}'' - \frac{\dot{a} \dot{a}'}{a} - \dot{a} \frac{a'' a'}{a^2} + \frac{2 \dot{a} a a''}{a^3}}{a} \right] + \kappa \left( \phi_0 \phi_0' + \phi_0' \phi_0' \Omega + \phi_0 \phi_0'' \Omega \right). \]
APPENDIX C: THE SPATIAL TENSORS

The spatial tensors \( (\nabla^{(2p)}\Omega)^A_{ij} \) which contain \(2p\) spatial gradients of \(\Omega\) are given as follows. For \( (\nabla^{(2)}\Omega)^A_{ij} \)

\[
\frac{\nabla_i\nabla_j\Omega}{\Omega} \quad \text{and} \quad \frac{\nabla_i\Omega\nabla_j\Omega}{\Omega^2}.
\]

For \( (\nabla^{(4)}\Omega)^A_{ij} \)

\[
\frac{\nabla_i\nabla_j\Omega\nabla^4\Omega}{\Omega^2}, \quad \frac{\nabla_i\nabla_j\nabla_i\nabla^4\Omega}{\Omega^2}, \quad \frac{\nabla_i\nabla_j\Omega\nabla_i\nabla^4\Omega}{\Omega^3}, \quad \frac{\nabla_i\nabla_j\nabla_i\nabla^4\Omega}{\Omega^3}, \quad \frac{\nabla_i\Omega\nabla_j(\nabla_i\Omega\nabla^4\Omega)}{\Omega^3}, \quad \frac{\nabla_i\nabla_j(\nabla_i\Omega\nabla^4\Omega)}{\Omega^3}, \quad \frac{\nabla_i\Omega\nabla_j(\nabla_i\nabla^4\Omega)}{\Omega^4}.
\]

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Figure Captions

Fig.1. In the lowest order approximation, an inflationary region and recollapsing region are shown in \((\Omega^2, \tilde{\phi}_0)\) plane.

Figs.2-(i) 2-(iv). Typical examples of the time evolution of (i)\(F(t,\Omega)\), (ii)\(\tilde{G}(t,\Omega)\), (iii)\(P(t,\Omega)\) and (iv)\(Q(t,\Omega)\).
Figs. 3-(i)–3-(ii).
The trajectories in $(\phi, \dot{\phi})$ space.
(i) \( \nabla_l \Omega^4 \nabla_l / \Omega = 0.01(a), 0.001(b), 0(c), -0.001(d) \) and -0.01(e).
(ii) \( \nabla_l \nabla_l \Omega / \Omega = 0 \) and \( \nabla_l \Omega \nabla_l \Omega / \Omega^2 = 0(a), 0.0001(b), 0.001(c) \) and 0.01(d).
At the endpoint of the trajectory the approximation breaks down.

Figs. 4-(i)–4-(iii)
An inflationary region and a recollapsing region for fixed $\tilde{\phi}_0 = 5.0 \kappa^{-1/2}$ ($\Omega^2_{cr} \approx 0.70 m^{-2}$) are shown in the parameter space $(\nabla_l \nabla_l \Omega / \Omega, \nabla_l \Omega \nabla_l \Omega / \Omega^2)$. A meshed region is a region where the approximation breaks down.

Fig. 5.
The shape of $\Omega^2(\chi)$ for varying $\Delta$.

Fig. 6.
An inflationary region and a recollapsing region in the three-dimensional space for fixed $\tilde{\phi}_0 = 5.0 \kappa^{-1/2}$ are shown in the case that $\Omega^2$ is given in Fig. 5.
Fig. 1.
\[
\frac{\Omega^2}{\Omega_{cr}^2} = 0.98
\]

\[
\frac{\Omega^2}{\Omega_{cr}^2} = 1.02
\]

Fig. 2-(i).
\[ \frac{\Omega^2}{\Omega_{cr}^2} = 1.02 \]

\[ \frac{\Omega^2}{\Omega_{cr}^2} = 0.98 \]

Fig. 2-(ii).
\[ P(t) \]

\[ \frac{\Omega^2}{\Omega_{cr}^2} = 1.02 \]

\[ \frac{\Omega^2}{\Omega_{cr}^2} = 0.98 \]

Fig. 2-(iii).
\[
\frac{\Omega^2}{\Omega_{cr}^2} = 0.98
\]

\[
\frac{\Omega^2}{\Omega_{cr}^2} = 1.02
\]

Fig. 2-(iv).
Fig. 3-(i).
Fig. 3-(ii).
\[ \frac{\nabla^2 \Omega}{\Omega} \frac{\Omega^2}{\Omega^2_{cr}} = 1.02 \quad \tilde{\phi}_0 = 5.0 \quad [\kappa^{1/2}] \]

Fig. 4-(i).
\[ \frac{\nabla_l \nabla_l \Omega}{\Omega} \quad \Omega^2 = \Omega_{cr}^2 \quad \phi_0 = 5.0 \quad [\kappa^{1/2}] \]

Fig. 4-(ii).
\[ \frac{\nabla_i \nabla^l \Omega}{\Omega} \quad \frac{\Omega^2}{\Omega^2_{cr}} = 0.98 \quad \tilde{\phi}_0 = 5.0 \quad [\kappa^{-1/2}] \]

Fig. 4-(iii).
\[ \Omega^2 - \Omega_{cr}^2 \times 10^{-7} \text{ m}^2 \]

\[ \Delta = 0.5 \]
\[ \Delta = 2 \]
\[ \Delta = 1.5 \]
\[ \Delta = 0.95 \]

Fig. 5.
$\bar{\phi}_0 = 5.0 \ [\kappa^{1/2}]$

Fig. 6.