New Relations for Coefficients of Fractional Parentage—the Redmond Recursion Formula with Seniority

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Abstract

We find a relationship between coefficients of fractional parentage (cfp) obtained on the one hand from the principal parent method and on the other hand from a seniority classification. We apply this to the Redmond recursion formula which relates $n \to n + 1$ cfp’s to $n - 1 \to n$ cfp’s where the principal-parent classification is used. We transform this to the seniority scheme. Our formula differs from the Redmond formula inasmuch as we have a sum over the possible seniorities for the $n \to n + 1$ cfp’s, whereas Redmond has only one term. We show that there are useful applications of both the principal parent and the seniority classification.

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I. INTRODUCTION

The wave function of a system of identical particles is antisymmetric in space and spin. It is, however, often convenient to single out a given particle. This can be done by means of a fractional parentage expansion. For \( n \) identical particles in a single \( j \) shell, the expansion is as follows

\[
\Psi(1 \cdot \cdot \cdot n)^{I_{\alpha}} = \sum_{J_{1}\alpha_{1}}[j^{n-1}J_{1}\alpha_{1}] J^{n} I_{\alpha} \{ \Psi(1 \cdot \cdot \cdot n - 1)^{J_{1}\alpha_{1}} \cdot \Psi(n)^{\alpha} \}^{I_{\alpha}},
\]

where the curly brackets designate a Clebsch–Gordan coupling and the quantities in square brackets are the coefficients of fractional parentage. Although the total wave function is antisymmetric, each term in the expansion is not. A given term is antisymmetric in the first \( n - 1 \) particles, but not in the \( n \)-th. In some sense, with cfp's we have our cake and we eat it too.

Such an expansion enables one to single out a certain particle despite the fact that the total wave function is antisymmetric. A simple example for the use of a cfp involves transfer reactions. The cross section for the pickup of a neutron from a shell which has \( n \) neutrons and no protons is proportional to a spectroscopic factor, the value of which is

\[
S = n[j^{n-1}(J_{f}\alpha_{f}) J^{n} J_{0}^2],
\]

where \((J_{i}\alpha_{i})\) refer to the \((n+1)\)-neutron system and \((J_{f}\alpha_{f})\) to the \(n\)-neutron system. The summed pickup strength over all \((J_{f}\alpha_{f})\) is equal to \(n\), the number of particles available to be picked up.

One method of calculating coefficients of fractional parentage is by the principal-parent technique. An explicit example for a system of three identical particles is given by de Shalit and Talmi [2] (see Eq. 26.11 on page 271) and will be repeated in this introduction in order to establish notation.

To get a cfp for three identical particles, one first combines two of them to a total angular momentum \( J_{0} \): \([jj]^{J_{0}}\). We call this the principal parent. We then add a third particle and, after antisymmetrizing and normalizing, the resulting wave function is

\[
\Psi^{J}[J_{0}] = N[J_{0}](1 - P_{12} - P_{13}) [[12]^{J_{0}3}]^{J},
\]
where \( N[J_0] \) is the normalization factor. We then expand (3) as per Eq. (1), obtaining

\[
\begin{align*}
j^2(J_1)jJ\}j^3[J_0]J &= N[J_0] \left[ \delta_{J_0,J_1} + 2\sqrt{(2J_0+1)(2J_1+1)} \left\{ \frac{j}{J} \right\}_{J_0,J_1} \right],
\end{align*}
\]

where

\[
N[J_0] = \left[ 3 + 6(2J_0+1) \left\{ \frac{j}{J} \right\}_{J_0,J} \right]^{-1/2}.
\]

Note the relationship between the cfp and the normalization factor:

\[
[j^2(J_0)jJ]\}j^3[J_0]J = \frac{1}{3N[J_0]}.
\]

A recursion formula for cfp’s due to Redmond [1] is presented in the books of Talmi and de Shalit [2] on page 528, and Talmi [3] on page 274. It can be written as follows

\[
(n+1)[j^n(\alpha_0J_0)jJ]\}j^{n+1}[\alpha_0J_0]\}j^n(\alpha_1J_1)jJ]\}j^{n+1}[\alpha_0J_0]J = \\
\delta_{\alpha_1\alpha_0}\delta_{J_1J_0} + n(-1)^{J_0+J_1} \sqrt{(2J_0+1)(2J_1+1)} \sum_{\alpha_2J_2} \left\{ \frac{J_2}{J} \right\}_{J_0,J_1} \times \left[ j^{n-1}(\alpha_2J_2)jJ_0]\}j^n(\alpha_0J_0)\}j^{n-1}(\alpha_2J_2)jJ_1]\}j^n\alpha_1J_1].
\]
construct the cfp’s for the unique $I = 7/2$ state of the $f_{7/2}^3$ configuration, using first $J_0 = 2$ and then $J_0 = 4$ as principal parents, we get exactly the same cfp’s for $J_1 = 0, 2, 4, 6$ in the two cases. There is a redundancy.

For the $j^3$ configuration with $j = 9/2$, $I = 9/2$, there are two states. In the Bayman–Lande scheme [4], the states are classified by the seniority quantum number. Of the two $I = 9/2$ states above, one has seniority 1 and the other has seniority 3. However, in the principal parent scheme, there are five sets of cfp’s corresponding to $J_0 = 0, 2, 4, 6, 8$. This is clearly an overcomplete set.

In Table I we show the results of the two schemes for the example above: $I = j = 9/2$.

Of course, one can have more than one state of a given seniority. For example, for $j = 15/2$, $(j^3)^I=15/2$, there is one state of seniority 1 and two of seniority 3.

The discussion of seniority (a topic introduced by G. Racah [5, 6]) is given extensively in text books [2, 3, 7], so we will be very brief on this. To simplify the discussion, let us consider a closed shell of protons and focus only on the open-shell system of neutrons, i.e., deal only with identical particles. For an even number of neutrons, there is a tendency for their spins to be paired. This corresponds to a seniority $\nu = 0$ state with total angular momentum 0 (note that all even–even nuclei have angular momentum 0). To form a $2^+$ state, one must break at least one pair. As noted by Lawson, the seniority $\nu$ of a nuclear state “is the number of unpaired nucleons in the eigenfunction describing the state” [7]. He also mentions that the delta-function potential conserves seniority in a single $j$ shell. For an even number of neutrons, the seniority $\nu$ must be an even integer; for an odd number of neutrons, it must be an odd integer. In the case of a semimagic nucleus with an open shell of, say, neutrons, whereas the $I = 0$ ground state has dominantly $\nu = 0$, the first $2^+$ state is dominantly $\nu = 2$. However, for the $I = 4$ state in $^{44}$Ca ($f_{7/2}^4$), the seniority $\nu = 4$ state is slightly lower than the seniority $\nu = 2$ state. One can understand this by noting that the $\nu = 2$ state consists of one broken pair with $J = 4$, while the seniority $\nu = 4$ state can be constructed from two $J = 2$ pairs. For a two-particle system, the $J = 2$ pair energy is lower than the $J = 4$ pair energy often by a factor of two or more.
TABLE I: Coefficients of fractional parentage in the principal-parent scheme and in the seniority scheme (results from Bayman–Lande) for the $I = 9/2$ states of the $g_{9/2}^3$ configuration.

| Principal-parent scheme | Seniority scheme |
|--------------------------|------------------|
| $J_0 = 0$                | $v = 1$          |
| $J_1 = 0$                | $J_1 = 0$        |
| 0.516398                | $J_1 = 0$        |
| $J_1 = 2$                | $J_1 = 2$        |
| $-0.288675$             | $0.288675$       |
| $J_1 = 4$                | $J_1 = 4$        |
| $-0.387298$             | $0.387298$       |
| $J_1 = 6$                | $J_1 = 6$        |
| $-0.465475$             | $0.465475$       |
| $J_1 = 8$                | $J_1 = 8$        |
| $-0.532291$             | $0.532291$       |
| $J_0 = 2$                | $J_0 = 4$        |
| $J_1 = 0$                | $J_1 = 0$        |
| $-0.437384$             | $-0.262994$      |
| $J_1 = 2$                | $J_1 = 2$        |
| $0.340825$              | $-0.008910$      |
| $J_1 = 4$                | $J_1 = 4$        |
| $-0.019881$             | $0.760475$       |
| $J_1 = 6$                | $J_1 = 6$        |
| $0.764610$              | $-0.362496$      |
| $J_1 = 8$                | $J_1 = 8$        |
| $0.327887$              | $0.470138$       |
| $J_0 = 6$                | $J_0 = 8$        |
| $J_1 = 0$                | $J_1 = 0$        |
| $-0.286884$             | $-0.473618$      |
| $J_1 = 2$                | $J_1 = 2$        |
| $0.311026$              | $0.192553$       |
| $J_1 = 4$                | $J_1 = 4$        |
| $-0.329013$             | $0.616034$       |
| $J_1 = 6$                | $J_1 = 6$        |
| $0.837866$              | $0.149270$       |
| $J_1 = 8$                | $J_1 = 8$        |
| $0.103396$              | $0.580370$       |

II. RELATION BETWEEN PRINCIPAL PARENT CFP’S AND THOSE IN THE SENIORITY SCHEME

We here note a relationship between the overcomplete set of principal-parent coefficients of fractional parentage and those with the seniority classification:
\[ [j^n(v_0J_0)jJ]|j^{n+1}[v_0J_0]J][j^n(v_1J_1)jJ]|j^{n+1}[v_0J_0]J] = \sum_v [j^n(v_0J_0)jJ]|j^{n+1}Jv][j^n(v_1J_1)jJ]|j^{n+1}Jv]. \tag{8} \]

In the left-hand side above, the first principal parent is formed by adding the \((n+1)\)-th nucleon to an \(n\)-nucleon antisymmetric system with good seniority and angular momentum \((v_0 J_0)\), then coupling the combined system to a total angular momentum \(J\), and then antisymmetrizing and normalizing the total wave function. On the right-hand side, the sum over \(v\) is a sum over all the possible seniorities of the combined \((n+1)\) system and, for a given seniority, over all states with that seniority.

A proof of the above result will be given in Appendix A.

We can verify the result of Eq. (8) for specific examples. Consider first a system of three identical particles in a \(j = 15/2\) shell with total angular momentum \(J = 15/2\). Take the principal-parent angular momentum \(J_0\) to be equal to 2, and also \(J_1 = 2\). Using the explicit formulae of Eqs. (4) and (5), we obtain

\[
[j^2(J_0)jJ]|j^3[J_0]J] = \frac{1}{3} \left[ 1 + 2(2J_0 + 1) \left( \begin{array}{c} j \\ j \\ J_0 \end{array} \right) \right] = 0.153945. \tag{9} \]

From Bayman and Lande \cite{4} we find

\[
[j^2(2)jJ = 15/2]|j^3v = 1] = 0.172516, \tag{10} \]
\[
[j^2(2)jJ = 15/2]|j^3v = 3, \alpha = 1] = 0.153452, \tag{11} \]
\[
[j^2(2)jJ = 15/2]|j^3v = 3, \alpha = 2] = 0.317231. \tag{12} \]

We easily verify that the sum of the squares is 0.153945.

As a second example, consider the case \(j = 9/2, J = 9/2\), with \(J_0 = [J_0] = 2\) and \(J_1 = 4\). The left-hand side of Eq. (8) is given by

\[
\text{lhs} = \frac{4}{3} \sqrt{45} \left( \begin{array}{ccc} 9/2 & 9/2 & 2 \\ 9/2 & 9/2 & 4 \end{array} \right) = -0.00677596. \tag{13} \]

The right-hand side has contributions from \(v = 1\) and \(v = 3\). Using the Bayman–Lande
tables, we find

\[ v = 1 \quad 0.288675 \times 0.387598 \]  
\[ v = 3 \quad -0.181166 \times 0.654463 \]  
\[ \text{Total} \quad -0.006776 \]

(14a)  
(14b)  
(14c)

### III. AN EXAMPLE OF THE USE OF THE OVERCOMPLETE CFP’S

Ironically, one can get the most useful information from principal parent cfp’s by calculating them for states which do not exist. We use Eq. (4) to illustrate this point. As noted by Racah [5, 6], de Shalit and Talmi [2] and Talmi [3], there are no states of the \( j^3 \) configuration with total angular momentum \( J = 3j - 4 \). If in Eq. (4) we choose the principal parent \( J_0 = 2j - 1 \) and \( J_1 = 2j - 3 \), then the fact that the cfp does not exist leads to the relation

\[
\begin{bmatrix}
  j & j & (2j - 1) \\
  (3j - 4) & j & (2j - 3)
\end{bmatrix} = 0 .
\]

(15)

For a different choice, \( J_0 = J_1 \), one gets

\[
\begin{bmatrix}
  j & j & J_0 \\
  J & j & J_0
\end{bmatrix} = -\frac{1}{2(2J_0 + 1)}
\]

(16)

for certain states \( J \) that do not exist in the \( j^3 \) configuration. Note that, for these select \( J \) values, this \( 6j \) symbol does not depend on what \( J \) is. For example, for \( j = 7/2 \), Eq. (16) holds for \( J = 1/2, 13/2, 17/2, \) and \( 19/2 \), but not for the allowed \( (f^3_{7/2}) \) states mentioned previously.

An interesting use of these \( 6j \)-symbol relations has been found by Robinson and Zamick [8] for a system of two neutrons and one proton (or two protons and one neutron), e.g., \( ^{43}\text{Sc} (^{43}\text{Ti}) \) for \( j = 7/2 \). To perform a shell model calculation, one uses as input two-body matrix elements \((j_1j_2)^J | V | (j_3j_4)^J \) , where \( J \) is the total two-particle angular momentum and \( T \) is the isospin. Of course, \( T \) can only be either zero or one for a two-particle system. The resulting wave function for \( ^{43}\text{Sc} \) in the single \( j \) shell can be written as

\[
\Psi^I = \sum_{J_N} D^I(j_\pi, J_N) [j_\pi(j^2)^J_N]^I ,
\]

(17)
where, for a state of total angular momentum $I$, $D^I(j_\pi,J_N)$ is the probability amplitude that the neutrons couple to $J_N$ ($J_N = 0, 2, 4, \text{ or } 6$).

Without going into detail (these are given in Ref. [8]), the authors considered a model in which the two-body matrix elements with isospin $T = 0$ were set equal to zero. Only the $T = 1$ two-body matrix elements entered into the calculation. When this was done, an interesting partial dynamical symmetry was found for the previously mentioned angular momenta $I$ which cannot occur for a $j^3$ configuration of identical particles, namely $I = 1/2, 13/2, 17/2$, and 19/2. It was found for these states that $J_N$ was a good quantum number for the wave functions, i.e., a given state wave function was of the form $[j_\pi \ J_N]^I$. As an example, for $I = 13/2$ the matrix element $\langle j_\pi 13/2 | V | j_\pi 6 \rangle$ was zero. This is explained by the vanishing of the $6j$ symbol of Eq. (5)

$$\begin{bmatrix} 7/2 & 7/2 & 6 \\ 3/2 & 7/2 & 4 \end{bmatrix} = 0,$$

which we remember was obtained by completely different considerations.

There were also degenerate states, such as $I = 1/2^-$ and $13/2^-$, whose wave functions were of the form $[j_\pi 7/2, J_N = 4]$. Likewise, $13/2^- , 17/2^-$, and $19/2^-$ were all degenerate with wave functions $[j_\pi 7/2, J_N = 6]$. These degeneracies follow from Eq. (16).

We call the above a partial dynamical symmetry because it applies only to states of angular momentum $I$ which can occur for a system of two neutrons and a proton, but cannot occur for a system of three neutrons (or three protons).

IV. THE REDMOND RECURSION RELATION IN THE SENIORITY SCHEME

We here present the equivalent of the Redmond recursion relation, but for cfp’s classified by the seniority quantum number $v$ and for which there are no redundacies. Here is our formula

$$(n + 1) \sum_{v_s} [j^n(v_0 J_0) j I_s | j^{n+1} v_s I_s] [j^n(v_1 J_1) j I_s | j^{n+1} v_s I_s] =$$

$$= \delta_{j_0 J_1} \delta_{v_0 v_1} + n(-1)^{J_0 + J_1} \sqrt{(2J_0 + 1)(2J_1 + 1)} \sum_{v_2 J_2} \begin{bmatrix} J_2 & j & J_1 \\ I_s & j & J_0 \end{bmatrix}$$

$$\times [j^{n-1}(v_2 J_2) j J_0] j^n v_0 J_0 [j^{n-1}(v_2 J_2) j J_1] j^n v_1 J_1].$$

(19)
This differs from the Redmond formula inasmuch as there is now a sum on the left-hand side of the equation over $v_s$. Note that $I_s$ is fixed. Basically, then, the sum is over all states that are present which have angular momentum $I_s$ for the $(n+1)$-particle system.

Of course, the fixed values of $(v_0 J_0)$ and $(v_1 J_1)$ will lead to some restrictions on the possible values of $v_s$.

We give now an example. Consider the case $n = 3$, $j = 9/2$; for the three-particle systems, take $J_0 = 9/2, v_0 = 3$ and $J_1 = 11/2, v_1 = 3$; and for the angular momentum of the four-particle system, take $I_s = 2$. Taking into account that, in this case, we have two values for the seniority $v_s$, the result of the left-hand side of Eq. (19) is

$$I_s = 2, v_s = 2 \quad 4 \cdot (-0.128118) \cdot 0.320983 = -0.164495$$

$$I_s = 2, v_s = 4 \quad 4 \cdot (-0.265908) \cdot 0.666200 = -0.708592$$

Sum (lhs) $-0.873087$

For the right-hand side of Eq. (19), we obtain

$$J_2 = 2 \quad -0.119633$$
$$J_2 = 4 \quad -0.789733$$
$$J_2 = 6 \quad +0.199694$$
$$J_2 = 8 \quad -0.163415$$

Sum (rhs) $-0.873087$

As can be seen, we get the same result.

For the same case as above but with $v_0 = 1$ and $v_s = 3$, we find that

$$\text{lhs} = \text{rhs} = 0.78674.$$

(20)

V. THE SPECIAL CASE $n = 2$. APPLICATION OF THE SENIORITY REDMOND RELATION TO THE NUMBER OF STATES OF ANGULAR MOMENTUM $I_s$ FOR THREE IDENTICAL PARTICLES IN A SINGLE $j$ SHELL

For $n = 2$ the two cfp’s on the right-hand side of Eq. (19) are equal to 1 and $J_2 = j$, i.e., the sum over $J_2$ consists of only one term. We find
\[
2 \begin{pmatrix}
 j & j & J_1 \\
 I_s & j & J_0
\end{pmatrix} (-1)^{J_0+J_1} \sqrt{(2J_0+1)(2J_1+1)} = \\
= -\delta_{J_0,J_1} \delta_{\text{even}} + 3 \sum_{v_s} [j^2(v_0J_0)jI_s][j^3v_sI_s][j^2(v_1J_1)jI_s][j^3v_sI_s].
\]

For \( J_1 = J_0 \) we get
\[
\frac{2}{3} \begin{pmatrix}
 j & j & J_0 \\
 I_s & j & J_0
\end{pmatrix} (2J_0+1) + \frac{1}{3} = \sum_{v_s,\alpha_s} [j^2(v_0J_0)jI_s][j^3v_sI_s]^2.
\]

If we sum over \( J_0 \) (even) on the right-hand side, we obtain
\[
\sum_{J_0} [j^2(v_0J_0)jI_s][j^3v_sI_s]^2 = 1.
\]

And then, the sum over \( v_s \) gives us the number of states with total angular momentum \( I_s \).

For \( I_s = j \) we get a result previously obtained by Rosensteel and Rowe using a quasispin formulation
\[
\frac{1}{3} \left[ \frac{2j+1}{2} + 2 \sum_{J_0 \text{ even}} (2J_0+1) \begin{pmatrix}
 j & j & J_0 \\
 j & j & J_0
\end{pmatrix} \right] = \# \text{ of states with } I_s = j.
\]

Ginocchio and Haxton showed this quantity to be equal to \([ (2j+3)/6 ]\), where the square brackets mean the largest integer less than what is inside them.

For \( I_s = j + 1 \) we get the Zhao–Arima result
\[
\frac{1}{3} \left[ \frac{2j-1}{2} - 2 \sum_{J_0 \text{ even}} (2J_0+1) \begin{pmatrix}
 j & j & J_0 \\
 j & j+1 & J_0
\end{pmatrix} \right] = \# \text{ of states with } I_s = j + 1,
\]

which can be shown to be \([ j/3 ]\). The present authors have presented an alternative derivation of the above two results by using an \( m \) scheme. A recent preprint by Talmi also uses the \( m \) scheme to go beyond the above two examples and to prove many conjectures of Zhao and Arima.

10
VI. ISOSPIN CONSIDERATIONS. APPLICATION OF THE PRINCIPAL-PARENT REDMOND RELATION TO PROBLEMS INVOLVING NEUTRONS AND PROTONS

In a previous work, “Interrelationship of isospin and angular momentum” \[14\], we considered the following simple interaction in a single \( j \) shell of neutrons and protons

\[
\langle (j^2)^J^A V (j^2)^J^A \rangle = a \frac{1 - (-1)^J^A}{2}.
\]  

(26)

Since in a single \( j \) shell when \( J^A \) is even the isospin \( T^A \) is 1, and when \( J^A \) is odd \( T^A \) is 0, we see that this interaction acts only for \( T^A = 0 \) states, i.e., only for the neutron–proton interaction in the \( T^A = 0 \) channel. The interaction vanishes for two neutrons or for two protons—they have isospin 1.

When applied to the \( I = 0 \) states of the even–even Ti isotopes with configuration \([ (j^2)^I^n (j^n)^J^ν f^I^f = 0 \), the authors found the following expression for the interaction matrix elements:

\[
\langle [J^0 J^0] H [JJ^0] \rangle / a = n\delta_{JJ^0} - n\sqrt{(2J + 1)(2J^0 + 1)}
\]

\[
\times \sum_{J^0} \left[ \begin{array}{c} j^n - 1 J_0 j \end{array} \right] \left[ \begin{array}{c} J_0 j J \end{array} \right] \left[ \begin{array}{c} j^n - 1 J_0 j \end{array} \right] \left[ \begin{array}{c} J_0 j J^0 \end{array} \right] \left\{ \begin{array}{c} J_0 j J^0 \end{array} \right\}.
\]  

(27)

By using the principal-parent Redmond formula [Eq. 7 of this work], one obtains

\[
\langle [J^0 J^0] H [JJ^0] \rangle = (n + (-1)^J^0 J^0)' \delta_{JJ^0}
\]

\[
- (n + (-1)^J^0 J^0) [j^n J^0 j] [j^n + 1 J^0 j] [j^n J^0 j] [j^n + 1 J^0 j].
\]  

(28)

However, this can be simplified because, if we have a system of 2 protons, then both \( J \) and \( J' \) must be even.

The above result was coupled with the fact that one could also write the same interaction in the isospin space as \( a(1/4 - t(1) \cdot t(2)) \). This also vanishes for \( T = 1 \) and is equal to a constant \( a \) for \( T = 0 \).

From the isospin point of view, it is trivial to obtain the eigenvalues for a system of 2 protons and \( n \) neutrons:
\[ \langle V \rangle = (n + 1)a \quad \text{for} \quad T = T_{\text{min}} = |N - Z|/2, \]
\[ \langle V \rangle = 0 \quad \text{for} \quad T = T_{\text{min}} + 2. \]

The angular momentum expression did not involve seniority. From what we have seen in the previous sections, the generalization is not too difficult
\[ \langle J'(J'v') \rangle^0 H \ [Jv]\rangle^0 = (n + 1)\delta_{JJ'}\delta_{vv'} \]
\[ - (n + 1) \sum_{v_f} [j^n(Jv)j]\} j^{n+1}(jv_f)] [j^n(J'v')j]\} j^{n+1}(jv_f)]. \]

The eigenvalue equation for this Hamiltonian is
\[ (n + 1)D(J, Jv) - (n + 1) \sum_{v_f} [j^n(Jv)j]\} j^{n+1}(jv_f)] \times \sum_{J'} [j^n(J'v')j]\} j^{n+1}(jv_f)]D(J', J'v') = \lambda D(J, Jv). \]

However, from the isospin point of view, the eigenvalue \( \lambda \) for \( T = T_{\text{min}} \) is equal to \((n + 1)\). Hence, for \( T = T_{\text{min}} \) we obtain
\[ \sum_{v_f} [j^n(Jv)j]\} j^{n+1}(jv_f)] \sum_{J'} [j^n(J'v')j]\} j^{n+1}(jv_f)]D(J', J'v') = 0. \]

We can multiply by \([j^n(Jv)j]\} j^{n+1}(jv_x)] \) and sum over \( v \). Thus, using the property
\[ \sum_{v} [j^n(Jv)j]\} j^{n+1}(jv_x)] [j^n(Jv)j]\} j^{n+1}(jv_f)] = \delta_{vfvx}, \]

we find
\[ \sum_{J'} [j^n(J'v')j]\} j^{n+1}(jv_x)]D^{T_{\text{min}}}(J', J'v') = 0 \]
for each \( v_x \) state.

What is the significance of Eq. (33)? We will now show, by a generalization of a result of Zamick and Devi, that this equation expresses the fact that states with isospin \( T = T_{\text{min}} + 2 \) are orthogonal to states with isospin \( T = T_{\text{min}} \).

States of 2 protons and \( n \) neutrons with isospin \( T_{\text{max}} = T_{\text{min}} + 2 \) are double analogs of states of \((n + 2)\) identical particles. This leads to the fact that the values of the wave-function components for \( T = T_{\text{max}} \) are two-particle coefficients of fractional parentage.

\[ D^{T_{\text{max}}=0,v_f}(J, Jv) = [j^n(Jv)j^2(J)]\} j^{n+2}I = 0 v_f]. \]
For a system of \( n + 2 \) identical particles (neutrons), we can write

\[
|j^{n+2}\rangle_I v_f = \sum_{J_0 v_0 v_1} [j^n(J_0 v_0) j^2(J_0 v_1)] j^{n+2} I v_f \langle (j^n(J_0 v_0) j^2(J_0 v_1))^{I v_f}.
\]  

We can, however, reach this result in two stages with successive one-particle cfp’s:

\[
|j^{n+2}\rangle_I v_f = \sum_{J_0 v_0} \sum_{v_3} [j^{n+1}(j v_3) j] j^{n+2} I v_f [j^n(J_0 v_0) j] j^{n+1} j v_3 \times U(J_0,j I j; j J_0) \langle (j^n(J_0 v_0) j^2(J_0))^{I v_f}.
\]

In the above, \( U \) is a unitary Racah coefficient. For \( I = 0 \) the value of \( U \) is 1.

So far we have

\[
[j^n(J_0 v_0) j^2(J_0 v_1)] j^{n+2} I = 0 v_f = \sum_{v_3} [j^n(J_0 v_0) j] j^{n+1}(j v_3)] [j^{n+1}(j v_3) j] j^{n+2} I = 0 v_f.
\]

However, it can be shown, e.g. in Bayman and Lande [4], that the \((n + 1) \rightarrow (n + 2)\) cfp above to a final state with \( I = 0 \) is 1 for \( v_3 = v_f - 1 \) and 0 otherwise, except when \( v_f = 0 \), in which case \( v_3 = 1 \). So we have

\[
[j^n(J_0 v_0) j^2(J_0 v_1)] j^{n+2} I = 0 v_f = \begin{cases} [j^n(J_0 v_0) j] j^{n+1}(j, v_f - 1) & , \text{ for } v_f > 0, \\ [j^n(J_0 v_0) j] j^{n+1}(j, v = 1) & , \text{ for } v_f = 0. \end{cases}
\]

Thus, in Eq. (37) we can replace the one-particle cfp \( n \rightarrow (n + 1) \) by the two-particle cfp \( n \rightarrow (n + 2) \). The latter is more obviously identified with the wave function of a state with \( I = 0 \) and \( T = T_{\text{max}} \).

VII. CLOSING REMARKS

We have here discussed both principal-parent coefficients of fractional parentage and those obtained by seniority schemes. The former are easier to calculate, but form an overcomplete set; while the latter form a complete orthonormal set. The original Redmond formulation gives \( n \rightarrow n + 1 \) cfp’s which are obtained by a principal-parent classification via the \( n-1 \rightarrow n \) cfp’s with no clear classification. Our main result in this work was to obtain Redmond-type relations (see Eqs. (8) and (19)) in which we have seniority cfp’s on both sides of the
equation. A new feature is that on the left-hand side of Eq. \((19)\) we have, for fixed final angular momentum, a sum over all possible final seniorities.

We have noted that both principal parent and seniority cfp’s have their uses. For the former, we noted earlier works which showed that, by constructing cfp’s to non-existent states, e.g., \((f_{7/2}^3), J = 13/2\), we obtain conditions on \(6j\) symbols. In turn, the vanishing of these cfp’s was put to use in a completely different problem by Robinson and Zamick \([8]\), namely to explain a partial dynamical symmetry for a system of two neutrons and one proton (likewise two protons and one neutron) when the \(T = 0\) two-body interaction is set to zero and only \(T = 1\) two-body matrix elements are used.

We applied our new Redmond relation to the problem of the number of states of a given angular momentum in a \((j^3)\) configuration. Previously, we had results only up to \(j \leq 7/2\), but with the new Redmond relation we have it for all \(j\). We also used this relation to generalize a relation by Zamick, Mekjian and Lee \([14]\), again for \(j \leq 7/2\), to higher \(j\) values, where states of a given angular momentum occur more than once for a three-particle system. We thereby obtain conditions on the wave functions of states of mixed neutrons and protons which boil down to the fact that states of higher isospin are orthogonal to states of lower isospin.

While our new Redmond relation at first sight appears more complicated than the original one, because of the sum over final seniorities on the left-hand side, we find that this sum can be used to obtain closure and ultimately can lead to simple results.

**APPENDIX A**

We here offer a proof of Eq. \((8)\). We will do it for a \((g_{9/2}^3)\) three-particle system. The proof for any number of particles and other configurations is essentially the same.

The wave function for the three-particle system with a principal parent \([J_0]\) for the two particles is [see Eq. \((3)\)]

\[
\Psi^J[J_0] = N[J_0](1 - P_{12} - P_{13}) \left[\begin{array}{c} 12 \\ J_0 \end{array}\right]^J,
\]

where \(J\) is the total angular momentum of the three particles. These \(\Psi^J[J_0]\)'s are an overcomplete set; e.g., for \(J = 9/2\) there are five \(\Psi^J[J_0]\)'s, but only two independent wave functions. Although not necessary, we can separate the two into states with definite seniority \(v = 1\)
and \( v = 3 \).

The following relation must hold between the principal parent wave functions and the seniority wave functions

\[
\Psi^J[J_0] = C[J_0] \Psi^J(v = 1) + D[J_0] \Psi^J(v = 3),
\]

(A2)

or in more detail

\[
\sum_{J_1} [j^2(J_1)jJ]\} j^3[J_0,J] \left[ [12]^J_1 3 \right] =
\]

\[
\sum_{J_1} \left\{ C[J_0][j^2(J_1)jJ]\} j^3v = 1, J] \left[ [12]^J_1 3 \right] +
\]

\[
D[J_0][j^2(J_1)jJ]\} j^3v = 3, J] \left[ [12]^J_1 3 \right].
\]

(A3)

This leads to the following relation between cfp's

\[
[j^2(J_1)jJ]\} j^3[J_0,J] = C[J_0][j^2(J_1)jJ]\} j^3v = 1, J] + D[J_0][j^2(J_1)jJ]\} j^3v = 3, J],
\]

(A4)

with \( C \) and \( D \) independent of \( J_1 \).

By taking overlaps, we see

\[
C[J_0] = N[J_0] \langle \Psi^J(v = 1) | (1 - P_{12} - P_{13} [12]^J_0 3] \rangle.
\]

(A5)

Since \( \Psi(v = 1) \) is totally antisymmetric, this leads to

\[
C[J_0] = 3N[J_0] \langle \Psi(v = 1) | [12]^J_0 3] \rangle
\]

\[
= 3N[J_0][j^2(J_0)jJ]\} j^3v = 1, J].
\]

(A6)

Likewise

\[
D[J_0] = 3N[J_0][j^2(J_0)jJ]\} j^3v = 3, J].
\]

(A7)

Thus, we have

\[
[j^2(J_1)jJ]\} j^3[J_0,J] = 3N[J_0] \sum_{v=1,3} [j^2(J_0)jJ]\} j^3vJ][j^2(J_1)jJ]\} j^3vJ.
\]

(A8)

But, from Eq. (6), we see that

\[
3N[J_0] = \frac{1}{[j^2(J_0)jJ]\} j^3[J_0,J]}.
\]

(A9)

By cross multiplication, we get the result we are after—Eq. (8).

Once this has been shown, the Redmond relation of Eq. (19) follows because, as discussed in the text after Eq. (8), the cfp’s on the right-hand side of Eq. (19) for the \( n \)-particle system (from \( n - 1 \) to \( n \)) have been constructed with definite seniority.
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