Conformality and self-duality of $N_f = 2$ QED$_3$

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We study the IR phase of three dimensional quantum electrodynamics (QED$_3$) coupled to $N_f = 2$ flavors of two-component Dirac fermions, which has been controversial for decades. This theory has been proposed to be self-dual with symmetry enhancement $(SU(2)_{t} \times U(1)_{t})/\mathbb{Z}_2 \rightarrow O(4)$ at the IR fixed point. We focus on the four-point correlator of monopole operators with unit topological charge of $U(1)_t$. We illustrate the $O(4) \rightarrow SU(2)_f \times U(1)_t$ branching rules based on an $O(4)$ symmetric positive structure in the monopole four-point crossing equations. We use conformal bootstrap method to derive nonperturbative constraints on the CFT data and test the conformality and self-duality of $N_f = 2$ QED$_3$. In particular we find the CFT data obtained from previous lattice simulations can be ruled out by introducing irrelevant assumptions in the spectrum, indicating the IR phase of $N_f = 2$ QED$_3$ is not conformal.

I. INTRODUCTION

Three dimensional $U(1)$ gauge theory coupled to $N_f$ two-component Dirac fermions has fundamental applications in layered condensed matter systems, including high temperature superconductors, topological insulators, etc. The low energy limit of QED$_3$ shows intriguing phenomena reminiscent to the 4D quantum chromodynamics: depending on the flavor number $N_f$, it can realize gauge confinement ($N_f = 0$) [1, 2], chiral symmetry breaking ($N_f \in (0, N_f^*)$) and conformal phase ($N_f \geq N_f^*$). The critical flavor number $N_f^*$ plays a key role in understanding the IR phases of QED$_3$ and its applications.

Various methods have been developed to estimate $N_f^*$. Perturbative approaches generically lead to a small $N_f^*$ in the range $0 < N_f^* < 10$ but it is hard to obtain a reliable estimation due to the strong coupling effect [5–24]. In lattice studies [25–29], the fermion bilinear condensation has been observed with $N_f = 2$ which disappears with higher $N_f \geq 4$, indicating $2 < N_f^* < 4$. In contrast, lattice results from [30–34] suggest the $N_f = 2$ QED$_3$ remains conformal in the low energy limit. The main contradiction among these results is whether the $N_f = 2$ QED$_3$ has an IR fixed point.

The $N_f = 2$ QED$_3$ also plays an important role in the 3D duality web [35]. This theory has been proposed to be self-dual in the IR with an enhanced $O(4)$ symmetry [36]. Such self-duality is also obtained in [37–40] from 3D fermion-fermion particle-vortex duality [41]. Moreover, based on the 3D fermion-boson duality, the $N_f = 2$ QED$_3$ is further suggested to be dual to the easy-plane NCCP$^1$ model, which has been used to describe the deconfined quantum critical point (DQCP) [42–45]. A crucial assumption in this scenario is that the $N_f = 2$ QED$_3$ flows to an IR fixed point, i.e. $N_f^* < 2$. Besides, there could be two possibilities in the duality web [40–41]: the strong version conjectures that the $O(4)$ symmetric fixed point is stable to all the perturbations (except a fermion mass term) allowed by the UV symmetries and all the theories in the duality web flow to the same IR fixed points; while the weak version conjectures that the $O(4)$ symmetric fixed point is unstable to certain perturbations and only the $SU(2)$ symmetric QED$_3$ can flow to the IR fixed point.

Conformal bootstrap [45–46] provides a powerful nonperturbative approach for strongly coupled critical theories. It has been applied to study $O(4)/SO(5)$ symmetric DQCPs [47–50] and conformal QED$_3$ [51–52]. In [50] the author observed a new family of kinks in the $SO(N)$ vector bootstrap bounds for $N \geq 6$, reminiscent to the well-known kinks related to the critical $O(N)$ vector models [53], while the $SO(5)$ symmetric DQCP is just slightly below the conformal window. The kinks show interesting connections with conformal QED$_3$ and indicate a critical flavor number of QED$_3$: $2 < N_f^* < 4$. The bootstrap results provide evidence for the merger and annihilation of fixed points near $N_f^*$ [10–14, 54–56], though the underlying physics is entangled with the bootstrap bound coincidences among different symmetries [57–59].

In this work we use conformal bootstrap to study the $N_f = 2$ QED$_3$, which is the focal point of contradictions among a series of lattice studies in [24–29] and [30–34]. The bootstrap bounds can provide strict and nonperturbative restrictions for the CFT data measured in the lattice simulations [30–34]. The results are in favor of a non-conformal IR phase of $N_f = 2$ QED$_3$. Together with the bootstrap results for the $N_f = 4$ QED$_3$, which support that the $N_f = 4$ QED$_3$ is conformal and can be isolated into a closed region, the bootstrap results can provide a compelling estimation for the critical flavor number of QED$_3$.

II. MONOPOLE FOUR-POINT CORRELATOR

The $N_f = 2$ QED$_3$ has a global symmetry $(SU(2)_f \times U(1)_t)/\mathbb{Z}_2$, which $SU(2)_f$ is the flavor symmetry while $U(1)_t$ is associated with the topological conserved current $j^t_{\mu} = \epsilon_{\mu\nu\rho} F^{\nu\rho}$. Operators charged under the $U(1)_t$ symmetry are tied to configurations of the gauge field with nontrivial topologies, namely the monopole operators [61]. The "bare" charge 1 monopole $\mathcal{M}$ is not gauge invariant and it needs to be attached with a zero
mode of the Dirac fermion $\Psi^i$ to construct gauge invariant monopoles $M_1 \equiv \Psi^i \mathcal{M}$. In consequence the monopoles $M_1$ also form a spin-$1/2$ representation of $SU(2)_f$. They have been studied using perturbative approach \cite{52,53,54} and the bootstrap method \cite{51,52}.

The self-duality of $N_f = 2$ QED$_3$ is derived from 3D particle-vortex duality \cite{38,39,40}, which maps the fundamental fields of one theory to vortices of the other. For $N_f = 2$ QED$_3$, the particle-vortex duality switches the $U(1)_f$ and the Cartan subgroup $U(1)_f \subset SU(2)_f$. The UV symmetry of the dual theory is $(SU(2)_f \times U(1)_f)/\mathbb{Z}_2$. At the IR fixed point the global symmetry is a cover of the symmetries on both sides of the duality $(SU(2)_f \times SU(2)_f)/\mathbb{Z}_2 \cong SO(4)$. Moreover, the self-duality introduces a $\mathbb{Z}_2$ symmetry exchanging $SU(2)_f$ and $SU(2)_{\bar{f}}$, thus the true IR symmetry is $SO(4) \rtimes \mathbb{Z}_2 \cong O(4)$. The monopoles $M_1$ which form a spinor representation of $SU(2)_f$ transform as $\left(\frac{1}{2}, \frac{1}{2}\right)$ under $SU(2)_f \times SU(2)_{\bar{f}}$.

In above $O(4)$ symmetry enhancement, the four real components of monopoles $M_1$ which construct an $SU(2)$ fundamental representations are reset to form an $O(4)$ vector. Curiously the same pattern of “symmetry enhancement” has been found in the conformal four-point crossing equations \cite{58,59} to explain the coincidences of crossing equations \cite{51} and the bootstrap method \cite{51,52}. The correlator (1) satisfies the $SU(2)_f \times SU(2)_{\bar{f}}$ symmetry and anti-symmetry (T) and anti-symmetric (A) \cite{64}. Variables $u, v$ are the conformal invariant cross ratios.

The correlator (1) satisfies the $SU(2)_f \times SU(2)_{\bar{f}}$ symmetric crossing equations \cite{51}

\begin{equation}
M^{O}_{3 \times 3} \cdot 1_{3 \times 1} = 0_{3 \times 1},
\end{equation}

and the square matrix $M^{O}_{3 \times 6}$ is

\begin{equation}
\begin{pmatrix}
0 & 0 & F^{-}_{0,A} & F^{-}_{1,A} & F^{-}_{0,T} & F^{-}_{1,T} \\
-F^{-}_{0,S} & F^{-}_{1,S} & -F^{-}_{0,A} & -F^{-}_{1,A} & 0 & 0 \\
-F^{-}_{0,S} & 3F^{-}_{1,S} & F^{-}_{0,A} & -3F^{-}_{1,A} & -2F^{-}_{0,T} & 6F^{-}_{1,T} \\
0 & 0 & F^{-}_{0,A} & -3F^{-}_{1,A} & F^{-}_{0,T} & -3F^{-}_{1,T} \\
-F^{-}_{0,S} & 3F^{-}_{1,S} & F^{-}_{0,A} & 3F^{-}_{1,A} & 0 & 0 \\
-F^{-}_{0,S} & F^{-}_{1,S} & F^{-}_{0,A} & F^{-}_{1,A} & -2F^{-}_{0,T} & -2F^{-}_{1,T}
\end{pmatrix},
\end{equation}

with correlation functions $F^{\pm}_{n,r}$

\begin{equation}
F^{\pm}_{n,r} \equiv \nu^{\Delta_{M_1}} G_{n,r}(u,v) \pm u^{\Delta_{M_1}} G_{n,r}(v,u).
\end{equation}

The crossing equations of $O(4)$ vector are \cite{65}

\begin{equation}
M^{O}_{3 \times 3} \cdot 1_{3 \times 1} = 0_{3 \times 1},
\end{equation}

where

\begin{equation}
M^{O}_{3 \times 3} = \begin{pmatrix}
0 & F^{-} \cdot F^{-} & F^{-} \\
F^{+} \cdot F^{-} & -3F^{+} \cdot F^{-} & F^{-} \\
-F^{+} \cdot F^{-} & F^{+} \cdot F^{-} & -3F^{+} \cdot F^{-}
\end{pmatrix}.
\end{equation}

Though the crossing equations (2) and (4) are rather different, there is a transformation $\mathcal{F}_{3 \times 6}$

\begin{equation}
\mathcal{F}_{3 \times 6} = \begin{pmatrix}
-\frac{1}{3} & \frac{2}{3} & 0 & -1 & -\frac{2}{3} & \frac{4}{3} \\
\frac{1}{3} & -\frac{2}{3} & 1 & \frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \\
-\frac{1}{3} & -1 & -1 & -\frac{2}{3} & -\frac{2}{3} & \frac{2}{3}
\end{pmatrix},
\end{equation}

which surprisingly connects the two crossing equations

\begin{equation}
\mathcal{F}_{3 \times 6} \cdot M^{O}_{3 \times 6} = \begin{pmatrix}
-\frac{1}{3} F^{-}_{0,S} & \frac{2}{3} F^{-}_{1,S} & 0 & -F^{-}_{1,A} & -\frac{2}{3} F^{-}_{0,T} & \frac{4}{3} F^{-}_{1,T} \\
\frac{1}{3} F^{-}_{0,S} & \frac{2}{3} F^{-}_{1,S} & F^{-}_{0,A} & F^{-}_{1,A} & \frac{2}{3} F^{-}_{0,T} & \frac{2}{3} F^{-}_{1,T} \\
-\frac{1}{3} F^{-}_{0,S} & -F^{-}_{1,S} & F^{-}_{0,A} & -F^{-}_{1,A} & -\frac{2}{3} F^{-}_{0,T} & -2F^{-}_{1,T}
\end{pmatrix}.
\end{equation}

Above matrix is essentially the $M^{O}_{3 \times 3}$ in (5), with replicated columns multiplied by positive rescaling factors. The many-to-one maps from $SU(2)_f \times SO(2)_{\bar{f}}$ to $O(4)$ representations are provided in the Table I. Here we would like to add a few comments about the parity charges in different $SO(2)_{\bar{f}}$ sectors. Operators in the $S/A$ sectors are neutral under the topological $SO(2)_{\bar{f}}$ symmetry. Their parity charges depend on the flavor number $N_f$. As discussed in \cite{51}, the monopole operator $M_1$ in $N_f = 2$ QED$_3$ is pseudo-real with a reality condition

\begin{equation}
(M^{+}_{1})^\dagger \propto \epsilon_{ij} c^{ab} M^{+}_{1b},
\end{equation}

where $i, j$ (a, b) denote the $SU(2)_f$ ($SO(2)_{\bar{f}}$) indices. The monopole operator $M_1$ transforms under the spacetime reflection operation $R$: $RM_1(x)R = M^{+}_{1}(-x)$. From the

**TABLE I.** $SU(2)_f \times SO(2)_{\bar{f}}$ and $O(4)$ representations in the monopole four-point correlator. Operators in the $SO(2)_{\bar{f}}$ $T$ sectors carry charge 2 of the topological $U(1)$ symmetry and do not have definite parity charges, in the sense that operators in this sector may have both even or odd parity charges for different components.

| $SU(2)_f$ | $SO(2)_{\bar{f}}$ | Spin | Parity | DOF | $O(4)$ |
|-----------|----------------|------|--------|-----|--------|
| 0         | $S$            | odd  | –      | 1   | $(1,0) + (0,1)$ |
| 1         | $S$            | even | –      | 3   | $(1,1)$ |
| 0         | $A$            | even | +      | 1   | $(0,0)$ |
| 1         | $A$            | odd  | +      | 3   | $(1,0) + (0,1)$ |
| 0         | $T$            | odd  | –      | 2   | $(1,0) + (0,1)$ |
| 1         | $T$            | even | –      | 6   | $(1,1)$ |
reality condition \([8]\) and reflection positivity restriction \(\langle M_1(x)M_1(-x)^\dagger \rangle > 0\), one can fix the normalization of the monopole operator \(M_1\)

\[
\langle M_1^a(x_1)M_1^b(x_2) \rangle = \frac{\delta^{ab}}{2^2 \Delta M_1}, \tag{9}
\]

According to the above normalization, the \((0, A)\) sector with \(SO(2)_t\) tensor structure \(\epsilon^{ab}\) has even parity, while the \((0, S)\) sector has opposite parity charge. For instances, the unit operator and stress tensor operator appear in the \((0, A)\) sector which are parity even, while the topological \(SO(2)_t\) conserved current \(j_\mu^t\) with odd parity charge appears in the sector \((0, S)\). In the QED\(_3\) with even \(N_f/2\), e.g. \(N_f = 4\), the monopole operators \(M_1^a\) are real instead of pseudo-real, and they satisfy different reality conditions which lead to opposite parity charges of operators in the \(S/A\) sectors. \([66]\) In the \(SO(2)_t\) \(T\) sectors, operators \(O_q\) have charge \(|q| = 2\) of the topological \(SO(2)_t\) symmetry. The parity symmetry transforms the monopole operators \(O_q\) to \(O_{-q}\). These monopole operators do not have definite parity charges, instead different components of the monopoles \(O_q\) transform oppositely under parity symmetry.

Table 1 shows how the \(SU(2)_f \times SO(2)_t\) representations are combined into \(O(4)\) representations. This suggests the \(SU(2)_f \times U(1)_t\) crossing equations actually have an \(O(4)\) symmetric positive structure, which can lead to the \(O(4)\) symmetry enhancement in the bootstrap bounds and \(SO(N)\)-ization of four-point correlators \([58, 59]\). The subtlety in this symmetry enhancement is that the monopoles carrying topological \(U(1)_t \subset SU(2)_t\) charges can be combined with \(U(1)_t\) neutral operators to form \(SU(2)_t\) representations. Specifically, the singlet \((0, A)\) remains singlet of \(O(4)\) \([67]\); the \(O(4)\) conserved current consists of the conserved currents associated with \(SU(2)_f\) and \(U(1)_t\) symmetries, and a charge 2 monopole in the \((0, T)\) sector. The fermion bilinears in the \((1, S)\) sector construct an \(O(4)\) traceless symmetric scalar after combining with a \((1, T)\) monopole, which agrees with the results in \([39]\).

III. Bootstrap Bounds

We bootstrap the monopole four-point correlator \([1]\) to derive constraints on CFTs with an \(SU(2)_f \times U(1)_t\) symmetry. In Fig. 1 we show the upper bound on the scaling dimension of the lowest singlet scalar in the \((0, A)\) sector. The \(SU(2)_f \times SO(2)_t\) singlet bound exactly coincides with the singlet bound from \(O(4)\) vector bootstrap. The kink in the bound near \(\Delta_{M_1} \sim 0.52\) gives the critical \(O(4)\) vector model, whose bootstrap solution was firstly discovered in \([33]\). The coincidence of \(SU(2)_f \times SO(2)_t\) and \(O(4)\) bootstrap bounds can be explained by the hidden relation between their crossing equations given by \([7]\). The transformation \(\mathcal{R}_{\times 6}\) actually maps the \(SU(2)_f \times SO(2)_t\) bootstrap problems to the \(O(4)\) vector bootstrap problems in a way consistent with positive conditions required by bootstrap algorithm. When bootstrapping non-singlet scalars, the \(O(4)\) positive structure in \(M_{\theta \times 6}\) can be broken by the gap conditions in the bootstrap setup, which lead to different bootstrap results, e.g., the bootstrap bounds on \(\Delta_T\) and \(\Delta_{(1, S)}\) in Fig. 2. See Appendix A for more details.

In \(N_f = 2\) QED\(_3\), the lowest singlet scalar in the \((0, A)\) sector is a mixing of the four-fermion operator \((\bar{\Psi} \Psi)^2\) and the gauge kinetic term \(F_{\mu \nu} F^{\mu \nu}\) \([35]\). This operator has dimension 4 in the UV and can receive notable correction in the IR. It has been assumed to be irrelevant in the IR to realize a fixed point of \(N_f = 2\) QED\(_3\) and its dualities. QED\(_3\) like CFTs with a relevant singlet scalar relate to the universality classes of QED\(_3\)-GNY model instead. In the “merger and annihilation” scenario, the two theories merge at the critical flavor \(N_f\), and the lowest singlet scalar approaches marginality condition \(\Delta_S = 3\). According to the conformal perturbation theory, the two fixed points move to complex plane and become non-unitary once the lowest singlet scalar crosses marginality condition \([69]\). In lattice simulations, with a relevant singlet scalar it needs fine tuning to reach the IR fixed point. In particular, the DQCP requires the lowest singlet scalar being irrelevant otherwise the fixed point with enhanced \(O(4)\) symmetry cannot be realized \([40]\). According to the bootstrap bound in Fig. 1 this gap assumption leads to a constraint \(\Delta_{M_1} \geq 0.870\) at \(\Lambda = 31\), and it increases to \(\Delta_{M_1} \geq 0.876\) using higher numerical precision with \(\Lambda = 51\) \([70]\).

Lattice results on the IR phase of \(N_f = 2\) QED\(_3\) re-
main controversial. In [25, 28] the chiral symmetry breaking has been observed which suggests this theory is not conformal. However, in [30, 32] the authors observed conformal behavior in the large distance limit and the scaling dimensions of some low-lying operators have been estimated. Specifically, in [32] the $N_f = 2$ QED$_3$ has been simulated using bilayer honeycomb (BH) model. The results also support the $O(4)$ symmetry enhancement and the duality with the NCCP$_1$ model realized by the easy-plane-J-Q (EPJQ) model. The scaling dimensions of $O(4)$ vector and traceless symmetric scalars are $\Delta_V = 0.550(5)$, $\Delta_T = 1.000(5)$ in the BH model and $\Delta_V = 0.565(15)$, $\Delta_T = 0.955(15)$ in the EPJQ model. According to the bootstrap bound, a unitary CFT with $\Delta_V$ in this range needs a strongly relevant singlet scalar $\Delta_S < 2.0$, which will necessarily break the RG flow to the IR fixed point with an enhanced $O(4)$ symmetry [71]. In a lattice simulation of $N_f = 2$ QED$_3$ [33], the scaling dimension of $M_1$ is estimated in the range $\Delta_{M_1} = 0.826(44)$, and the upper bound is $\Delta_{M_1} \leq 0.87$, which is “conspiratorially” near the lower bound of bootstrap result $\Delta_{M_1} \geq 0.87$ at $\Lambda = 51$ and $\Delta_{M_1} \geq 0.876$ at $\Lambda = 51$ by requiring an irrelevant lowest singlet scalar! Here conformal bootstrap provides quantitatively elaborate consistent check for a unitary CFT. On the other hand, more precise lattice simulations will definitely be helpful in this comparison. Note the two lattice estimations on $\Delta_V$ [32, 33] are significantly different, while the estimations on $\Delta_T$ from [32] are consistent with the estimation in [31] on the fermion bilinears $\Delta_{(1,S)} = 1.0(2)$. The bootstrap bound with condition $\Delta_S \geq 3$ strongly (mildly) excludes the lattice results in [32] (31, 33).

The monopole bootstrap can also generate constraints on the scaling dimension of fermion bilinear mass term $\bar{\Psi}_1\Psi^2 - \bar{\Psi}_2\Psi^2$ in the $(1,S)$ sector shown in Figs. 2 and 3. The results will be particularly useful to test the strong version of the duality web, which conjectures that both the $U(1) \times U(1)$ and $SU(2)$ symmetric QED$_3$ flow to the same IR fixed point. The $U(1) \times U(1)$ symmetric QED$_3$ allows couplings which break $SU(2)_f$ to $U(1)_{f}$, e.g. scalars in $(1,S)$ sector. The lowest scalar in $(1,S)$ sector is the fermion bilinear which is tuned to zero at the critical point, while the second lowest scalar in this sector needs to be irrelevant to realize IR fixed point in the $U(1) \times U(1)$ symmetric theory. This operator has dimension 4 in the UV and is conjectured to be irrelevant in the IR. The $U(1) \times U(1)$ QED$_3$ is interested in the duality web of DQCPs. Firstly the derivation of the $N_f = 2$ QED$_3$ self-duality takes the two fermions separately, and the self-duality is supposed to work for the $U(1) \times U(1)$ symmetric theories [40]. Moreover, the easy-plane NCCP$_1$ model has a $U(1) \times U(1)$ symmetry. According to the strong version of duality web, terms breaking $SU(2)_f$ to $U(1)_{f}$ should also be allowed in the fermionic theory. In terms of the $O(4)$ representations, this suggests the lowest scalar in the $(1,1)$ sector is the tuning parameter while the second lowest scalar is irrelevant. Similar symmetry-breaking terms also appear in the $(2,2)$ representation of $O(4)$. However, they only appear in the OPE of $(1,1)$ scalars instead of $O(4)$ vectors and their bootstrap study is provided in Appendix B.

In Fig. 3 we show the bootstrap allowed region for $\Delta_{M_1} \sim \Delta_{(1,S)}$ (light blue) with assumptions required by the strong version of the dualities, that both the lowest singlet scalar in $(0,A)$ sector and the second lowest scalar in $(1,S)$ sector are irrelevant. If we further impose $O(4)$ symmetry, more regions in the $\Delta_V \sim \Delta_T$ plane can be carved out (dark blue). The bootstrap bounds in Fig. 3 can be applied to the lattice results [31, 32], in which the only tuning parameter is the fermion mass term and the IR phase is stable under other perturbations. This sug-

![FIG. 2. Dashed lines: upper bounds on $\Delta_T$ and $\Delta_{(1,S)}$. Blue shadowed region: bootstrap allowed region of $(\Delta_V, \Delta_T)$ with a gap assumption $\Delta_S > 3$. The red and two green shadow regions denote estimations from lattice simulations.](image1)

![FIG. 3. Light blue: $\Delta_{M_1} \sim \Delta_{(1,S)}$ under assumptions $\Delta_{(0,A)} > 3$ and the second $(1,S)$ scalar is irrelevant. Dark blue: $\Delta_V \sim \Delta_T$ from $O(4)$ vector bootstrap with assumptions $\Delta_S > 3$ and the second traceless symmetric $(T)$ scalar is irrelevant. Green region: lattice results [31, 32] on $\Delta_{M_1} \sim \Delta_{(1,S)}$.](image2)
gests the next operator in the $(1,S)$ or $T$ sector should be irrelevant in their lattice setup. For the $N_f = 2$ QED$_3$ with fermion mass operator $\Delta_{(1,S)} = 1.0(2)$, the bootstrap bound requires $\Delta_{M_1} \geq 0.99$ without imposing the $O(4)$ symmetry, while the lower bound increases to $\Delta_V \geq 1.08$ after introducing $O(4)$ symmetry. In both scenarios, the lattice results [31][33] locate in the regions notably away from the bootstrap bounds and are clearly excluded.

\section*{IV. CONCLUSIONS}

We have provided nonperturbative constraints on the IR phase of $N_f = 2$ QED$_3$ using conformal bootstrap. The results show that the CFT data of $N_f = 2$ QED$_3$ measured from lattice simulations [30][34] are not consistent with bootstrap bounds both with and without an enhanced $O(4)$ symmetry, which suggest the phase transitions observed in these simulations are not truly continuous, resolving the contradiction on the IR phase of $N_f = 2$ QED$_3$ among a series of lattice studies [23][29] and [30][34]. Our results support that the $N_f = 2$ QED$_3$ is not conformal. On the other hand, the bootstrap study of $N_f = 4$ QED$_3$ [60] provides strong evidence for its conformal IR phase, by showing that part of perturbative CFT data of this theory provides a consistent solution to the crossing equations and can be isolated into a closed region. The two results provide a compelling answer for the critical flavor number of QED$_3$: $2 < N_f^c < 4$, consistent with the lattice simulations [25][28] and the observation in the bootstrap results [60].

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\appendix

\section*{Appendix A: $SU(2) \times SO(2)$ and $O(4)$ bootstrap bound coincidence}

In this section we apply the algebraic relation (7) to conformal bootstrap. The relation (7) suggests that the two crossing equations $M_{6,6}^Q$ and $M_{9,3}^Q$ have the same positive structure, which leads to coincidences in certain bootstrap bounds. In particular, the upper bound on the scaling dimension of $SU(2) \times SO(2)$ singlet scalar in the $M_1 \times M_1$ OPE is saturated by the $O(4)$ symmetric four-point correlators, with $O(4) \to SU(2) \times SO(2)$ branching rules consistent with the proposed $N_f = 2$ QED$_3$ self-duality.

To bootstrap the four-point correlator of the monopoles $M_1$, we start from its crossing equation (2).

\[ G_{n,r}(u,v) = \sum_{\mathcal{O} \in (n,r)} \lambda_{M_1,M_1,\mathcal{O}} \Delta_{\mathcal{O},\mathcal{O}}(u,v). \]  

(A1)

Accordingly, the conformal partial wave expansion of the crossing equations (2) is given by $M_{4,6}^Q$ with a replacement

\[ F_{n,r}^{\pm} \to F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} \]  

(A2)

where

\[ F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} = u^{-\Delta_{M_1}+\Delta_{\mathcal{O}}} g_{\Delta_{\mathcal{O}},\mathcal{O}}(u,v) \mp u^{\Delta_{M_1}+\Delta_{\mathcal{O}}} g_{\Delta_{\mathcal{O}},\mathcal{O}}(v,u). \]  

(A3)

Now let us consider the following bootstrap problem: what is the maximum scaling dimension $\Delta_{\mathcal{O}}$ of the lowest non-unit singlet scalar that can appear in the $M_1 \times M_1$ OPE and satisfy above crossing equations and unitarity conditions? According to the standard bootstrap algorithm, this bootstrap problem is equivalent to find linear functionals $\tilde{\beta}$ for a given $\Delta_{M_1}$, which satisfy positive conditions when applied on conformal blocks $F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm}(u,v)$.

\[ \tilde{\beta} \cdot M_{6,6}^Q = (\beta_1 \beta_2 \beta_3 \beta_4 \beta_5 \beta_6). \]

\[ \begin{pmatrix}
 0 & 0 & F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} \\
 -F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} & -F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} & F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} \\
 -F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} & 3F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} & 0 \\
 0 & 0 & F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} \\
 -F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} & F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} & F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} \\
 -F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} & 3F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} & 0 \\
 0 & 0 & F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} \\
 -F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} & F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} & F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} \\
 -F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} & 3F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm} & 0 \\
 0 & 0 & F_{\Delta_{\mathcal{O}},\mathcal{O}}^{\pm}
\end{pmatrix}
\]

\[ = (\beta_{0,S} \beta_{1,S} \beta_{0,A} \beta_{1,A} \beta_{0,T} \beta_{1,T}) \geq 0, \quad \forall \Delta \geq \Delta_{\mathcal{O}} \text{ or unitary bound.} \]  

(A4)

The parameters $(\Delta_{M_1}, \Delta_{\mathcal{O}})$ can be excluded if the actions $\beta_{n,r}$ of linear functionals $\tilde{\beta}$ are strictly positive. In the
optimal case when the conditions $\beta_{n_r} = 0$ are satisfied for a set of spectrum, one obtains extremal solutions to the crossing equations and the boundary of bootstrap bound is reached.

The $O(4)$ vector bootstrap problem can be defined similarly. To obtain the maximum scaling dimension of the lowest non-unit $O(4)$ singlet scalar appearing in the OPE of an $O(4)$ vector $\phi_i$, we aim to find the linear functionals $\vec{\alpha}$ which satisfy

$$\vec{\alpha} \cdot M^{O}_{3 \times 3} = (\alpha_1 \ \alpha_2 \ \alpha_3) \cdot \begin{pmatrix} 0 & F_{\Delta, \ell}^- & F_{\Delta, \ell}^- \\ F_{\Delta, \ell}^- & \frac{1}{2} F_{\Delta, \ell}^- & F_{\Delta, \ell}^- \\ F_{\Delta, \ell}^- & \frac{3}{4} F_{\Delta, \ell}^- & -F_{\Delta, \ell}^- \end{pmatrix} = (\alpha_S \ \alpha_T \ \alpha_A) \supseteq (0 \ 0 \ 0), \ \forall \Delta \geq \Delta^*_S \text{ or unitary bound.} \quad (A5)$$

Now the algebraic relation (7) between $SU(2) \times SO(2)$ and $O(4)$ crossing equations plays a critical role for above two seemingly unrelated bootstrap problems (A4) and (A5). Assuming we have already obtained the $O(4)$ bootstrap linear functionals $\vec{\alpha}^*$ satisfying the positive conditions in (A5), we can construct linear functional $\vec{\beta}^*$ for the $SU(2) \times SO(2)$ bootstrap

$$\vec{\beta}^*_{1 \times 6} = \vec{\alpha}^*_{1 \times 3} \cdot \mathcal{J}_{3 \times 6} \cdot M^O_{6 \times 6}$$

Its action on the crossing equation matrix $M^O_{6 \times 6}$ is

$$\vec{\beta}^*_{1 \times 6} \cdot M^O_{6 \times 6} = \vec{\alpha}^*_{1 \times 3} \cdot \mathcal{J}_{3 \times 6} \cdot M^O_{6 \times 6} = \left( \frac{1}{3} \alpha_A^* \cdot 2 \alpha_T^*, \alpha_S^*, \alpha_A^*, 2 \alpha_A^*, \frac{4}{3} \alpha_T^* \right) \equiv \vec{\alpha}^*_{1 \times 3} \cdot \mathcal{J}_{3 \times 6} \cdot M^O_{6 \times 6}.$$

This is exactly the $O(4)$ vector bootstrap action with $O(4) \to SU(2) \times SO(2)$ branching rules

$$S_{O(4)} \to (0, A), \quad (A8)$$
$$T_{O(4)} \to (1, S) \oplus (1, T), \quad (A9)$$
$$A_{O(4)} \to (0, S) \oplus (1, A) \oplus (0, T), \quad (A10)$$

associated with positive recombination coefficients \(\left(\frac{1}{3}, \frac{2}{3}, 1, 1, \frac{2}{3}, \frac{4}{3}\right)\). Therefore it has exactly the same positive property as the $O(4)$ action \((\alpha_S, \alpha_T, \alpha_A)\). Combining with the bootstrap algorithm, this suggests that any data \((\Delta_{M_1}, \Delta_S^c)\) that is excluded by the $O(4)$ vector bootstrap, is also excluded by the $M_1$ monopole bootstrap. This results to the following relation on the bootstrap upper bound \(\Delta^c_S\)

$$\Delta_S^c \leq \Delta^c_S|_{O(4)}. \quad (A11)$$

On the other hand, on the boundary of bootstrap we obtain extremal solutions to the crossing equations, and any $O(4)$ symmetric solution can be decomposed to the $SU(2) \times SO(2)$ symmetric solutions, therefore we also have

$$\Delta^c_S \leq \Delta^c_S|_{O(4)}. \quad (A12)$$

Relations (A11) and (A12) leads to the identity

$$\Delta^c_S \leq \Delta^c_S|_{O(4)}. \quad (A13)$$

which proves the bootstrap bound coincidence in the singlet sector observed in Fig. 1. Moreover, Eq. (A7) shows

Appendix B: Bootstrap bounds from $O(4)$ (1,1) bootstrap

In the main body of this paper, we have shown the results of $N_f = 2$ QED$_3$ monopole bootstrap. In the self-duality scenario, the charge 1 monopole operator $M_1$ forms a vector representation \(\left(\frac{1}{2}, \frac{1}{2}\right)\) of $O(4)$ symmetry. The monopole bootstrap implementation has access to the \((0,0), (1,0) + (0,1)\) and \((1,1)\) sectors. The results suggest conformal bootstrap can provide nontrivial con-
straints for some fundamental questions on strongly coupled gauge theories. In this part we present the $O(4)$ $(1,1)$ bootstrap results, which provide less conclusive but still interesting constraints on the $O(4)$ DQCP.

$O(4)$ representations that can appear in the $(1,1)$ bootstrap include

$$(1,1) \otimes (1,1) = (0,0) \oplus (1,1) \oplus (2,2) \oplus (1,0) + (0,1) \oplus (1,2) + (2,1) \oplus (2,0) + (0,2). \quad (B1)$$

Crossing equations of the four-point correlator of the $(1,1)$ scalar can be found in [74]. The scalar operators in $(2,2)$ representation carry four $O(4)$ indices $T_{ijkl}$. Some of its components can appear in the easy-plane NCCP model, e.g., $\sum_{i=1}^{2} \sum_{j=1}^{4} T_{i(iii)}$ corresponding to the Néel-VBS anisotropy and $\sum_{i=1}^{2} T_{(iii)}$ in the microscopic models on the square lattice [40]. However, these components are forbidden for the $SU(2)$ symmetric QED$_3$. If the $O(4)$ symmetric IR fixed point has a relevant scalar $T_{ijkl}$, as proposed in the weak version of $O(4)$ duality web, then the easy-plane model is unstable under the perturbations mentioned above, while the $N_f = 2$ QED$_3$ can flow to it. In the $SU(2)$ symmetric QED$_3$, the perturbations in (B1) allowed by the symmetry are $O(4)$ singlet or the $(0,2)$ representation. If there are relevant operators in these sectors, then even the $SU(2)$ symmetric QED$_3$ cannot flow to the IR fixed point and the weak version of $O(4)$ DQCP cannot be realized in the lattice simulation. Generally it is hard to compute scaling dimensions of the operators in these complicated representations. Fortunately, conformal bootstrap can provide useful information on these problems.

In Fig. 4 we show the bootstrap bounds on the lowest scalars in (2,2) and (2,0) + (0,2) representations. The bootstrap bounds suggest the lowest (2,2) scalar is relevant in the range $\Delta_{(1,1)} < 1.1$, while the lowest (2,0) + (0,2) scalar is relevant with $\Delta_{(1,1)} < 0.95$. In the lattice simulation [32], scaling dimension of the (4) (1,1) scalar is estimated at $\Delta_{(1,1)} = 1.000(5)$ in BH model and $\Delta_{(1,1)} = 0.955(15)$ in the EPJQ model. The lowest (2,2) scalar in the theory has to be relevant and the strong version of $O(4)$ duality is excluded, while the lowest (0,2) scalar can be marginally irrelevant. The lattice results in [31] has a more significant error bar on the scaling dimension of (4) (1,1) scalar: $\Delta_{(1,1)} = 1.0(2)$. Bootstrap results suggestion only in the range $\Delta_{(1,1)} > 1.1$ the lowest (2,2) can be irrelevant, which is necessary for the strong version of $O(4)$ duality web.

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It is possible that certain non-singlet scalar crosses marginality condition at critical flavor $N_f$ while the fixed points remain on both sides of $N_f$ and interchange stability, like the cubic and $O(N)$ vector models. See [23] for more comprehensive discussions. [70]

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