Asymptotic decomposition for nonlinear damped Klein-Gordon equations

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Abstract In this paper, we proved that if the solution to damped focusing Klein-Gordon equations is global forward in time, then it will decouple into a finite number of equilibrium points with different shifts from the origin. The core ingredient of our proof is the existence of the “concentration-compact attractor” which yields a finite number of profiles. Using damping effect, we can prove all the profiles are equilibrium points.

1 Introduction

In this paper, we consider the following damped focusing Klein-Gordon equation:

\[
\begin{aligned}
&u_{tt} - \Delta u + u + 2\alpha u_t - |u|^{p-1}u = 0, \\
u(0) = u_0, \quad \partial_t u(0) = u_1 \in \mathcal{H},
\end{aligned}
\]

where \( \mathcal{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \), \( \alpha \geq 0 \). The energy is given by

\[
E(f, g) = \int_{\mathbb{R}^d} \left( \frac{1}{2} |\nabla f|^2 + \frac{1}{2} |f|^2 + \frac{1}{2} |g|^2 - \frac{1}{p+1} |f|^{p+1} \right) dx.
\]

Dispersive equations such as Klein-Gordon equations, wave equations, Schrödinger equations have been intensively studied for decades. For \( \alpha = 0 \), namely nonlinear Klein-Gordon equation, T. Cazenave [3] gave the following dichotomy: solutions either blow up at finite time or are global forward in time and bounded in \( \mathcal{H} \), provided \( 1 < p < \infty \), when \( d = 1, 2 \) and \( 1 < p < \frac{d}{d-2} \) if \( d \geq 3 \). For \( \alpha > 0 \), E. Feireisl [13] gave an independent proof of the boundedness of the trajectory to global solutions, for \( 1 < p < 1 + \min(\frac{d}{d-2}, \frac{4}{d}) \) when \( d \geq 3 \), and in his paper [12], the case \( d = 1 \) is considered. N. Burq, G. Raugel, W. Schlag [2] studied the long time behaviors of solutions to nonlinear damped Klein-Gordon equations in radial case. They proved that radial global solutions will converge to equilibrium points as time goes infinity. A natural problem is what happens for non-radial solutions? It is widely conjectured that the solutions will decouple into the superposition of equilibrium points. A positive result given by E. Feireisl [13] implied there exists a global solution which decouples into a finite number of equilibrium points with different shifts from origin. Indeed, this problem is closely related to the soliton resolution conjecture.
in dispersive equations. The (imprecise sense) soliton resolution conjecture states that for “generic” large global solutions, the evolution asymptotically decouples into the superposition of divergent solitons, a free radiation term, and an error term tending to zero as time goes to infinity. For more expression and history, see A. Soffer [24].

There are a lot of works devoted to the verification of the soliton resolution conjecture. T. Duyckaerts, C. Kenig, and F. Merle [8] first make a breakthrough on this topic. For radial data to three dimensional focusing energy-critical wave equations, they proved the solution with bounded trajectory is in fact a superposition of a finite number of rescalings of the ground state plus a radiation term which is asymptotically a free wave. One of the key ingredients of their arguments is the novel tool, called “channels of energy” introduced by [8] [9]. The method developed by them has been applied to many other situations, such as [6] [7] [16] [18] [17] for wave maps, [5] [11] [14] [24] for semilinear wave equations. By a weak version of outer energy inequality, the soliton resolution along a sequence of times was proved by R. Cote, C. Kenig, A. Lawrie and W. Schlag [5] for four dimensional critical wave equations in radial case, by R. Cote [4] for equivariant wave maps, and by H. Jia, and C. Kenig [15] for semilinear wave equations, wave maps.

It is known that (1.1) admits a radial positive stationary solution with the minimized energy among all the non-zero stationary solutions. Besides the ground state, (1.1) also has an infinite number of nodal solutions which owns zero points. (see for instance H. Berestycki, and P.L. Lions [1]). Hence it seems that subcritical problems need different techniques. The dynamics of solutions below and slightly above the ground state is known. If \( \alpha = 0 \), for initial data with energy below the ground state, I.E. Payne, and D.H. Sattinger [22] proved that the solution either blows up in finite time or scatters to zero. K. Nakanishi, and W. Schlag [19] described the asymptotics of the solutions with energy slightly larger than the ground state. In fact they proved the trichotomy forward in time: the solution (1) either blows up at finite time (2) or globally exists and scatters to zero (3) or globally exists and scatters to the ground states. In radial setting, the above trichotomy was obtained in K. Nakanishi, and W. Schlag [20], followed by K. Nakanishi, and W. Schlag [21] in non-radial case. The main technical ingredient of their papers is the “one pass” theorem which excludes the existence of (almost) homoclinic orbits between the ground state and (almost) heteroclinic orbits connecting ground state \( Q \) with \(-Q\). N. Burq, G. Raugel, and W. Schlag [2] studied the longtime dynamics for damped Klein-Gordon equations in radial case. By developing some dynamical methods especially invariant manifolds, they proved the \( \omega \)-limit set of the trajectory is just one single point, hence they showed the dichotomy in forward time (1) the solution either blows up at finite time, (2) or converges to some equilibrium point.

In this paper, we aim to study the long time behaviors of damped Klein-Gordon equations
without radial assumptions. Let
\[\lambda(d) = \begin{cases} 
\infty, & d = 1, 2 \\
1 + \frac{4}{d-2}, & d = 3, 4.
\end{cases}\]

Then, we have the main theorem as follows:

**Theorem 1.1.** Let \(\alpha > 0\), \(1 \leq d \leq 4\), \(1 < p < \lambda(d)\). For any data \((u_0, u_1) \in \mathcal{H}\). Then
(i) either the solution of (1.1) blows up at finite positive time,
(ii) or it is global forward in time with unbounded trajectory;
(iii) or for any time sequence \(t_n \to \infty\), up to a subsequence, there exist \(0 \leq J < \infty\), \(x_{j,n} \in \mathbb{R}^d\) for \(j = 1, 2, ..., J\) and equilibrium points \(\{Q^j\}\) such that
\[u(t_n) = \sum_{j=1}^{J} Q^j(x - x_{j,n}) + o_{\mathcal{H}^1}(1),\]
and \(\lim_{t \to \infty} \partial_t u(t) = 0\), in \(L^2\), where \(\{x_{j,n}\}\) satisfies the separation property:
\[\lim_{n \to \infty} |x_{j,n} - x_{i,n}| = 0, \text{ for } i \neq j.\]

An adaptation of arguments in T. Cazenave [3] shows for \(1 < p < \infty\), when \(d=1,2\), \(1 < p < \frac{d}{d-2}\), when \(d \geq 3\), every global solution has bounded trajectory. Therefore, for this range of \(p\), we have the following dichotomy:

**Corollary 1.2.** For \(1 < p < \infty\), when \(d=1,2\), \(1 < p < \frac{d}{d-2}\), when \(d \geq 3\), either the solution to (1.1) blows up at finite time ((i) in Theorem 1.1) or decouples into the superposition of equilibriums ((iii) in Theorem 1.1).

**Remark 1.1.** For \(3 \leq d \leq 6\), \(1 < p < \frac{d}{d-2}\), the proof in E. Feireisl [13] is sufficient to give Corollary 1.2. Thus, in this paper, when \(d \geq 3\), we always assume \(p \geq \frac{d}{d-2}\).

In order to describe our proof, the following notions are needed:

**Definition 1.1.** Given any \(h \in \mathbb{R}^d\), let \(\tau_h : \mathcal{H} \to \mathcal{H}\) be the shift operator \(\tau_h f(x) = f(x - h)\), and we denote the translation group by \(G = \{\tau_h : h \in \mathbb{R}^d\}\). Given any \(K \subseteq \mathcal{H}\), we denote the orbit of \(K\) by \(GK = \{gf : g \in G, f \in K\}\). If \(GK = K\), then we call \(K\) \(G\)-invariant. Suppose that \(J \geq 0\) is an integer, we let
\[JK \equiv \{f_1 + ... + f_J : f_1, f_2, ..., f_J \in K\}.
We say \(E \subseteq \mathcal{H}\) is \(G\)-precompact with \(J\) components if \(E \subseteq J(GK)\) for some compact \(K \subseteq \mathcal{H}\) and \(J \geq 1\).
Our proof is divided into three parts. In the first step, we prove the trajectory of \( u(t) \) is attracted by a G-precompact set with \( J \) components, namely the existence of concentration-compact attractor. The key ingredient in this step is frequency localization and spatial localisation. The idea of “concentration compact” attractor was introduced by T. Tao \[25\]. In the second step, for any sequence going to infinity, we prove up to a subsequence there exist a finite number profiles. Then by applying perturbation theorem, we obtain a nonlinear profile decomposition. Using damping effect of (1.1), we can show all the profiles are exactly equilibriums. Finally we prove the convergence for all time.

Our paper is organized as follows: In Section 2, we recall some preliminaries, such as Strichartz estimates, local wellposedness, perturbation theorem. In Section 3, we prove the frequency localization and spatial localization. In Section 4, we prove the existence of concentration-compact attractor. In Section 5, we extract the profiles and finish our proof by using damping.

**Notation and Preliminaries** We will use the notation \( X \lesssim Y \) whenever there exists some positive constant \( C \) so that \( X \leq CY \). Similarly, we will use \( X \sim Y \) if \( X \lesssim Y \lesssim X \). We define the Fourier transform on \( \mathbb{R}^d \) to be

\[
F(f)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) \, dx,
\]

\( P_N \) is the usual Littlewood-Paley decomposition operator with frequency truncated in \( N \). Similarly, we use \( P_{\leq N} \) and \( P_{\geq N} \). Sometimes, we denote \( P_{<\mu} u \) by \( u_{<\mu} \). \( \|u(t)\|_H \) means \( \|(u(t), \partial_t u(t))\|_H \).

All the constants are denoted by \( C \) and they can change from line to line.

### 2 Preliminaries

As explained in Remark 1.1 we only need to consider

\[
\begin{cases}
1 < p < \infty, & d = 1, 2 \\
\frac{d}{2} \leq p < 1 + \frac{4}{d-2}, & d = 3, 4.
\end{cases}
\]

In this section we give the Strichartz estimates, local wellposedness and perturbation theorem, we closely follow notations in [2]. Consider the linear equation,

\[
u_{tt} + 2\alpha u_t - \Delta u + u = G, \quad (u(0, x), u_t(0, x)) = (u_0, u_1) \in H, \tag{2.1}
\]

then by Duhamel principle,

\[
u(t) = e^{-\alpha t} \left[ \cos(t\sqrt{-\Delta + 1 - \alpha^2}) + \alpha \frac{\sin(t\sqrt{-\Delta + 1 - \alpha^2})}{\sqrt{-\Delta + 1 - \alpha^2}} \right] u_0
\]
\[ + e^{-\alpha t^2 \sin(t^\Delta (1 - \alpha^2))} u_1 + \int_0^t \frac{\sin((t - s)^2 \Delta (1 - \alpha^2))}{\sqrt{-\Delta + 1 - \alpha^2}} e^{-\alpha(t-s)} G(s) ds \]
\[ \triangleq S_{1, \alpha}(t) u_0 + S_{2, \alpha}(t) u_1 + \int_0^t S_{2, \alpha}(t-s) G(s) ds. \]

Define \( S_L(t)(u_0, u_1) = S_{1, \alpha} u_0 + S_{2, \alpha} u_1. \)

The Strichartz estimates are given by the following lemma. We emphasize that since we only need local Strichartz estimates in this paper, it is possible to get \( L^p_t(I; L^q_x) \) estimates for non-admissible pair by Hölder inequality (see (2.2) below).

**Lemma 2.1.** ([2]) Let \( t_0 > T > 0, u \) be a solution to (2.1) on \([t_0 - T, t_0 + T] \times \mathbb{R}^d\). For \( d \geq 3 \), \( \theta^* = \frac{4 + 2}{d - 2} \), we have

\[
\sup_{t \in [-T, T]} \|(u(t), \partial_t u(t))\|_{\mathcal{H}} + \|u\|_{L^\theta_t([-T, T]; L^\theta_x)} \lesssim e^{T \alpha} \left[ \|(u_0, u_1)\|_{\mathcal{H}} + \int_{-T}^T \|G(s)\|_2 ds \right].
\]

If \( d = 3, 4 \), then

\[
\|u\|_{L^\theta_t([-T, T]; L^\theta_x)} \lesssim e^{T \alpha} \left[ \|(u_0, u_1)\|_{\mathcal{H}} + \int_{-T}^T \|G(s)\|_2 ds \right].
\]

If \( d = 1, 2 \), it holds that

\[
\sup_{t \in [-T, T]} \|(u(t), \partial_t u(t))\|_{\mathcal{H}} \lesssim e^{T \alpha} \left[ \|(u_0, u_1)\|_{\mathcal{H}} + \int_{-T}^T \|G(s)\|_2 ds \right].
\]

Define \( Q = p + 1 \), when \( d = 1, 2 \), \( p < Q < \frac{2p}{q(d - 2) - d} \), when \( 3 \leq d \leq 4 \), \( \frac{d}{d - 2} \leq p < 1 + \frac{4}{d - 2} \), we have

\[
\|u\|_{L^Q_t([-T, T]; L^Q_x)} \lesssim C(T) \left[ \|(u_0, u_1)\|_{\mathcal{H}} + \int_{-T}^T \|G(s)\|_2 ds \right]. \tag{2.2}
\]

As a corollary of Strichartz estimates, we have the perturbation theorem.

**Lemma 2.2.** Let \( M > 0 \), there exists \( \varepsilon_0 = \varepsilon_0(M) \) satisfying the following: Let \( I \subset \mathbb{R}^+ \) is a finite interval containing \( t_0 \), \( \tilde{u} \) is defined on \( I \times \mathbb{R}^d \), and satisfies

\[
\sup_{t \in I} \| (\tilde{u}, \partial_t \tilde{u})(t) \|_{\mathcal{H}} \leq M.
\]

Suppose that \( v \) is a solution to (1.1) with initial data \((v(t_0), \partial_t v(t_0))\) at time \( t_0 \). Let \( \varepsilon \in (0, \varepsilon_0) \), suppose that

\[
\|e\|_{L^1_t(I; L^2_x)} + \|S_{1, \alpha}(t-t_0)(\tilde{u} - v)(t_0)\|_{L^Q_t(I; L^Q_x)} + \|S_{2, \alpha}(t-t_0)(\partial_t \tilde{u} - \partial_t v)(t_0)\|_{L^Q_t(I; L^Q_x)} \leq \varepsilon.
\]
Then
\[ \| \ddot{u} - v - S_L(t - t_0)(\ddot{u} - v)(t_0) \|_{L^\infty(I; H)} + \| \ddot{u} - v \|_{L^2_t(I; L^2_y)} \leq C(M) \varepsilon. \]

Combining T. Cazenave [3] and N. Burq, G. Raugel, W. Schlag [2], we obtain the local wellposedness theorem as follows:

**Proposition 2.3.** For \((u_0, u_1) \in H\), there exists \(T > 0\) such that \((1.1)\) is well-defined in \([0, T]\), with \(T\) depending on \(\| (u_0, u_1) \|_H\). Furthermore, if \(\| (u_0, u_1) \|_H < \varepsilon\) with \(\varepsilon\) sufficiently small, then there exists \(\gamma > 0\) such that
\[ \| (u(t), \partial_t u(t)) \|_H \leq C e^{-\gamma t} \| (u_0, u_1) \|_H. \]

Moreover, if the solution \(u(t)\) is globally defined, then we have
\[ \int_0^\infty \| \partial_t u(s) \|_2^2 ds < \infty. \]

## 3 Frequency localization and Spatial localization

Since we focus on bounded solution throughout the paper, we assume
\[ \sup_{t \in [0, \infty)} \| (u, \partial_t u)(t) \|_H \leq E. \]

In the first step, we prove the localization of frequency, namely

**Lemma 3.1.** For any \(\mu_0 > 0\) there exists \(c(\mu_0) > 0\) depending on \(E\) such that
\[ \limsup_{t \to \infty} \| P_{\geq \frac{1}{\mu_0}} u(t) \|_{H^1} \leq \mu_0, \]
\[ \limsup_{t \to \infty} \| P_{\geq \frac{1}{\mu_0}} \partial_t u(t) \|_{L^2} \leq \mu_0. \]

**Proof.** From Duhamel principle,
\[ P_{\geq \frac{1}{\mu}} u(t) = S_{1, \alpha} P_{\geq \frac{1}{\mu}} u_0 + S_{2, \alpha} P_{\geq \frac{1}{\mu}} u_1 + \int_0^t S_{2, \alpha}(t - s) P_{\geq \frac{1}{\mu}} \left( |u|^{p-1} u \right)(s) ds. \]

Since
\[ \left\| S_{1, \alpha} P_{\geq \frac{1}{\mu}} u_0 \right\|_{H^1} \leq e^{-\alpha t} \| u_0 \|_{H^1}, \quad \left\| S_{2, \alpha} P_{\geq \mu^{-1}} u_1 \right\|_{H^1} \leq e^{-\alpha t} \| u_1 \|_{L^2}, \]
for \(\mu\) sufficiently small, we have
\[ \| P_{\geq \mu^{-1}} u(t) \|_{H^1} \leq C e^{-\alpha t} \| (u_0, u_1) \|_H + \int_0^t e^{-\alpha(t-s)} \left( \| u \|_{p-1}^2 \right) ds. \]
Let \( h(u) = |u|^{p-1}u \), split \( u = P_{\leq \mu^{-1}}u + P_{\geq \mu^{-1}}u \), then
\[
h(u) = h(P_{\leq \mu^{-1}}u) + P_{\geq \mu^{-1}}uO(|u|^{p-1}).
\]

**Case 1.** \( 1 < p < \frac{d}{d-2} \) for \( d \geq 3 \)
Bernstein’s inequality and Hölder’s inequality imply
\[
\| P_{\geq \mu^{-1}}h(u) \|_2 \leq \| P_{\geq \mu^{-1}}h(P_{\leq \mu^{-1}}u) \|_2 + \| P_{\geq \mu^{-1}} \left( P_{\geq \mu^{-1}}uO(|u|^{p-1}) \right) \|_2
\leq \mu \| \nabla h(P_{\leq \mu^{-1}}u) \|_2 + \| P_{\geq \mu^{-1}}uO(|u|^{p-1}) \|_2
\leq \mu \| \nabla P_{\leq \mu^{-1}}u \|_2 \| P_{\leq \mu^{-1}}u \|^{p-1}_\infty \leq \mu^{\frac{d(p-1)}{2}} \| \nabla u \|_2 \| P_{\leq \mu^{-1}}u \|^{p-1}_2.
\]
where \( \frac{1}{m} + \frac{p-1}{2} = \frac{1}{2} \). By Bernstein’s inequality, we have
\[
\mu \| \nabla P_{\leq \mu^{-1}}u \|_2 \| P_{\leq \mu^{-1}}u \|^{p-1}_\infty \leq \mu^{\frac{d(p-1)}{2}} \| \nabla u \|_2 \| P_{\leq \mu^{-1}}u \|^{p-1}_2.
\]
Since \( 1 < p < \frac{d}{d-2} \), we conclude for some \( \kappa > 0 \),
\[
\| P_{\geq \mu^{-1}}h(P_{\leq \mu^{-1}}u) \|_2 \leq \mu^\kappa \| u \|_{H^1}.
\tag{3.3}
\]
Applying Bernstein’s inequality, we have
\[
\| P_{\geq \mu^{-1}}u \|_m \leq \left( \sum_{N \geq \mu^{-1}} \| P_N u \|_m^2 \right)^{1/2} \leq \left( \sum_{N \geq \mu^{-1}} N^{2d\left(\frac{1}{d}-\frac{1}{m}\right)-2} N^2 \| P_N u \|_2^2 \right)^{1/2}
\leq \mu^{-d\left(\frac{1}{d}-\frac{1}{m}\right)+1} \left( \sum_{N \geq \mu^{-1}} N^2 \| P_N u \|_2^2 \right)^{1/2}.
\]
which combined with (3.3) gives (3.1) by \( 1 < p < \frac{d}{d-2} \). Next, we bound \( \partial_t u \). From Duhamel principle, we have
\[
\partial_t u(t) = -\alpha u(t) + e^{-\alpha \delta} \left[ -\sqrt{\Delta + 1 - \alpha^2} \sin \left( t \sqrt{-\Delta + 1 - \alpha^2} \right) + \alpha \cos \left( t \sqrt{-\Delta + 1 - \alpha^2} \right) \right] u(t - \delta)
+ e^{-\alpha \delta} \cos \left( t \sqrt{-\Delta + 1 - \alpha^2} \right) \partial_t u(t - \delta) + \int_{t-\delta}^t \cos \left( (t-s) \sqrt{-\Delta + 1 - \alpha^2} \right) e^{-\alpha(t-s)} \left( |u|^{p-1} u \right)(s) ds.
\]
For \( \mu_1 \ll \mu_0 \), (3.1) implies that there exist \( \eta > 0 \) and \( T_0 > 0 \) such that
\[
\| P_{\geq \mu^{-1}}u(t) \|_{H^1} < \mu_1,
\]
for $t > T_0$. Taking $\delta$ large such that $e^{-\alpha \delta} < \mu_1$, then for $t > T_0 + \delta$, it suffices to prove

$$
\|P_{\geq \mu^{-1}} h(t)\|_2 \leq \eta^\lambda,
$$

for some $\lambda > 0$. The rest of the proof of (3.2) is the same as (3.1).

*Case 2.* $1 < p < \infty$ for $d = 1$.

By Bernstein’s inequality, Hölder’s inequality, Sobolev embedding theorem,

$$
\|P_{\geq \mu^{-1}} h(u)\|_2 \leq \mu \|\nabla P_{\leq \mu^{-1}} u\|_2 \|P_{\leq \mu^{-1}} u\|^{p-1}_\infty + \|P_{\geq \mu^{-1}} u\|_2 \|u\|^{p-1}_\infty \leq \mu.
$$

The remaining proof is the same as Case 1.

*Case 3.* $1 < p < \infty$ for $d = 2$.

The proof is also similar to Case 1, we omit it.

*Case 4.* $\frac{d}{d-2} \leq p < 1 + \frac{d}{d-2}$ for $d = 3, 4$.

Choosing $\delta$ sufficiently large, such that $e^{-\alpha \delta} \ll \mu_0$. Fix $t_0 > \delta$, consider the interval $I \equiv [t_0 - \delta, t_0]$. By Duhamel principle, for $t \in I$, we have

$$
u(t) = S_{1,\alpha}(t - t_0 + \delta)u(t_0 - \delta) + S_{2,\alpha}(t - t_0 + \delta)u(t_0 - \delta) + \int_{t_0 - \delta}^t S_{2,\alpha}(t - s)h(u(s))ds.
$$

Fix $\varepsilon$ sufficiently small, divide $I$ into subintervals $I_1, I_2, \ldots, I_n$, such that $|I_j| \sim \varepsilon$, then $n \sim \frac{\delta}{\varepsilon}$.

Taking $p - 1 < R < \frac{1}{d-2}$, Hölder’s inequality and Strichartz estimates give

$$
\|u(t)\|_{L^R(I_j; L_x^{d/(d-2)})} \leq |I_j|^{\frac{1}{R} - \frac{d-2}{4d}} \|u(t)\|_{L^4(I_j; L_x^{d/(d-2)})} \lesssim |I_j|^{\frac{1}{R} - \frac{d-2}{4}} E + |I_j|^{\frac{1}{R} - \frac{d-2}{4}} \|h(u(s))\|_{L^1(I_j; L_x^2)}.
$$

By Hölder’s inequality, Sobolev embedding theorem,

$$
\|h(u(s))\|_{L^1(I_j; L_x^2)} \leq \int_{I_j} \|u|^{p-1}\|_{L^{2(p-1)(d-2)}} \|u\|_{L^6} dt \leq C(E) |I_j|^{-\frac{p-2}{4} + \frac{1}{2}} \|u\|_{L^1(I_j; L^4d/(d-2))}^{p-2},
$$

where $\frac{1}{\sigma} + \frac{(p-1)(d-2)}{4d} = \frac{1}{2}$. Thus

$$
\|u(t)\|_{L^2(I_j; L_x^{d/(d-2)})} \leq C(E) \varepsilon^{\frac{1}{R} - \frac{d-2}{4}} (1 + |\varepsilon|^{-\frac{p-2}{4} + \frac{1}{2}} \|u\|_{L^1(I_j; L^4d/(d-2))}^{p-2}).
$$

Since $p \geq 2$ in Case 3, by continuity method, we have

$$
\|u(t)\|_{L^2(I_j; L_x^{d/(d-2)})} \leq 8C(E) \varepsilon^{\frac{1}{R} - \frac{d-2}{4}}.
$$
Summing up all the intervals, we get
\[
\|u(t)\|_{L_t^4(I; L_x^{4d/(d-2)})} \lesssim C(\delta, E),
\] (3.4)
where \(C(E, \delta)\) is independent of \(t\). Again by Duhamel principle,
\[
\|P_{\geq \mu^{-1}} h(t_0)\|_{H^1} \leq C(E) e^{-\alpha \delta} + \int_{t_0}^{t_0 - \delta} e^{-\alpha(t-s)} \|P_{\geq \mu^{-1}} h(u(s))\|^2 \, ds.
\]
Hence it suffices to bound \(\|P_{\geq \mu^{-1}} h(u)\|_2\). Using similar arguments in Case 1, by Bernstein’s inequality and Hölder’s inequality, we have
\[
\|P_{\geq \mu^{-1}} h(u)\|_2 \leq \mu \|\nabla P_{\leq \mu^{-1}} u\|_2 \|P_{\leq \mu^{-1}} u|^{p-1}\|_\infty + \|P_{\leq \mu^{-1}} u\|_\theta \|u|^{p-1}\|_\frac{4d}{(p-1)(d-2)},
\]
where \(\frac{1}{p} + \frac{(p-1)(d-2)}{4d} = \frac{1}{2}\). Applying Bernstein’s inequality, we get
\[
\|P_{\leq \mu^{-1}} u|^{p-1}\|_\infty \leq \mu^{-(p-1)\frac{d-2}{4d}} \|u|^{p-1}\|_\frac{4d}{(p-1)(d-2)},
\]
which combined with Hölder’s inequality and (3.4) give
\[
\int_{t_0 - \delta}^{t_0} e^{-\alpha(t-s)} \mu \|\nabla P_{\leq \mu^{-1}} u(s)\|_2 \|P_{\leq \mu^{-1}} u(s)|^{p-1}\|_\infty \, ds \\
\leq C(E) \mu^{1-(p-1)\frac{d-2}{4d}} \int_{t_0 - \delta}^{t_0} e^{-\alpha(t-s)} \|u|^{p-1}\|_\frac{4d}{(d-2)} \, ds \\
\leq C(E) \mu^{1-(p-1)\frac{d-2}{4d}} \|u|^{p-1}\|_\frac{4d}{(p-1)(d-2)} L_t^4(I; L_x^{4d/(d-2)}) \\
\leq C(E, \delta) \mu^{1-(p-1)\frac{d-2}{4d}}.
\]
This bound is acceptable since \(1 < p < 1 + \frac{4}{d-2}\). The same arguments as Case 1 yield the desired bound for \(\|P_{\geq \mu^{-1}} u\|_\theta\) by \(1 < p < 1 + \frac{4}{d-2}\). Therefore, we finish our proof.

Now, we prove the spatial localization, namely the following proposition:

**Proposition 3.2.** Let \(u\) be a global solution to (1.1) with \(H\) norm at most \(E > 0\). Then there exist \(J = J(E)\) depending only on \(E\), and functions \(x_1(t), \ldots, x_J(t) : \mathbb{R}^+ \to \mathbb{R}^d\), such that for any \(\mu > 0\) there exist \(\eta = \eta(E, \mu) > 0\) such that
\[
\limsup_{t \to \infty} \int_{\text{dist}(\{x, (x_1(t), \ldots, x_J(t))\}) > \eta^{-1}} |\nabla u|^2 + |u|^2 + |\partial_t u|^2 \leq \mu.
\]
Before proving Proposition 3.2, we first prove a weaker proposition:
Proposition 3.3. Let $u$ be a global solution to (1.1) with $\mathcal{H}$ norm at most $E > 0$. Then for $\mu_0 > 0$, there exits $J = J(E, \mu_0)$ and functions $\tilde{x}_1(t), ..., \tilde{x}_J(t) : \mathbb{R}^+ \to \mathbb{R}^d$, and $\eta = \eta(E, \mu_0) > 0$ such that

$$\limsup_{t \to \infty} \int_{\text{dist}(x, \{\tilde{x}_1(t), ..., \tilde{x}_J(t)\}) > \eta^{-1}} |u|^2 \leq \mu_0.$$ 

Proof. The whole proof is divided into five parts. Fix $E > 0$ and $\mu_0$, choose parameters $\mu_0 \gg \mu_1 \gg \mu_2 \gg \mu_3 \gg \mu_4 > 0$.

Step One. Selecting a “good” time sequence For any $t_0 > T_0$, consider the time interval $[t_0 - \mu_1^{-1}, t_0 + \mu_1^{-1}]$. Since

$$\lim_{t_0 \to \infty} \int_{t_0 - \mu_1^{-1}}^{t_0 + \mu_1^{-1}} \|\partial_t u(s)\|^2 ds = 0,$$

there exists $T_1$ sufficiently large such that for $t_0 > T_1$,

$$\int_{t_0 - \mu_1^{-1}}^{t_0 + \mu_1^{-1}} \|\partial_t u(s)\|^2 ds \leq \mu_2.$$ 

Thus there exits good time $t_* \in [t_0 - \mu_1^{-1}, t_0 + \mu_1^{-1}]$, such that

$$\|\partial_t u(t_*)\|_2 \leq \mu_2^2. \quad (3.5)$$

Step Two. $L^\infty_x$ spatial localization at fixed time. From Lemma 3.1, for any $\mu_2 > 0$ there exists $c(\mu_2) > 0$, such that for $T > T_0$,

$$\|u > c(\mu_2)^{-1}\|_{H^1} \leq \mu_2^2. \quad (3.6)$$

As step one, we fix time $t > T_1$. Now we claim there exist $J(E, \mu_2, \mu_3)$ and $x_1(t), ..., x_J(t) : \mathbb{R}^+ \to \mathbb{R}^d$, such that

$$|u_{< c(\mu_2)^{-1}}(t, x)| < \mu_3, \text{ whenever } \text{dist}(x, \{x_1(t), ..., x_J(t)\}) \geq 2\mu_3^{-1}. \quad (3.7)$$

Indeed, let $x_1(t), ..., x_J(t)$ be a maximal $2\mu_3^{-1}$-separated set of points in $\mathbb{R}^d$ such that

$$|u_{< c(\mu_2)^{-1}}(t, x_j(t))| \geq \mu_3 \text{ for all } 1 \leq j \leq J(t).$$

It is easy to verify

$$|u_{< c(\mu_2)^{-1}}(t, x_j(t))| \lesssim c(\mu_2)^{d/2} \int_{|x - x_j(t)| \leq \mu_3^{-1}} |u|^2 dx + \mu_3^d \|u\|_2.$$
Claim 2. From the rapid decay of the convolution kernel of (3.6), it suffices to prove 

\[ |u_{<c(\mu_2)^{-1}}(t, x_j(t))| \lesssim c(\mu_2)^{d/2} \int_{|x-x_j(t)| \leq \mu_3^{-1}} |u|^2 dx. \]

Since \( x_j(t) \) are \( 2\mu_3^{-1} \)-separated, thus \( J \) is finite depending on \( \mu_2, \mu_3 \). By the maximal property of the set \( \{ x_1, \ldots, x_J \} \), we conclude

\[ |u_{<c(\mu_2)^{-1}}(t, x_j(t))| < \mu_3, \text{ whenever } \text{dist}(x, \{ x_1, \ldots, x_J \}) \geq 2\mu_3^{-1}. \]

Step Three. \( L_\infty^\infty \) spatial localization on an interval centered at good time. For \( t > T_1 \), consider good time \( t_\ast \) in \([t - \mu_1^{-1}, t + \mu_1^{-1}]\). Then \([t - \mu_1^{-1}, t + \mu_1^{-1}] \subset [t_\ast - 4\mu_1^{-1}, t_\ast + 4\mu_1^{-1}] \equiv I \). Define the distance function \( D(x) = \text{dist}(x, \{ x_1(t_\ast), x_2(t_\ast), \ldots, x_j(t_\ast) \}) \). Let \( \chi : \mathbb{R}^d \to \mathbb{R}^+ \) be a smooth cutoff function which equals 1 for \( D(x) \leq 2\mu_3^{-1} \), vanishes for \( D(x) \geq 3\mu_3^{-1} \), and \( \nabla^k \chi = O_k(\mu_3^k) \) for \( k \geq 0 \). Then we have

Claim 1. \( \| S_{1, \alpha}[(1 - \chi)u(t_\ast)] \|_{L_t^Q(I; L_x^{2p})} + \| S_{2, \alpha}[(1 - \chi)u(t_\ast)] \|_{L_t^Q(I; L_x^{2p})} \lesssim_{\mu_1} \mu_2^2. \)

By Strichartz estimates,

\[ \| S_{2, \alpha}[(1 - \chi)u(t_\ast)] \|_{L_t^Q(I; L_x^{2p})} \leq C e^{C\mu_1^{-1}} \| \partial_t u(t_\ast) \|_2, \]

which combined with \( (3.5) \) yields the desired bounds for \( S_{2, \alpha} \). Since high frequency is small by \( (3.6) \), it suffices to prove

\[ \| S_{1, \alpha}[(1 - \chi)P_{<c(\mu_2)^{-1}}u(t_\ast)] \|_{L_t^Q(I; L_x^{2p})} \lesssim_{\mu_1} \mu_2^2. \]

From the rapid decay of the convolution kernel of \( P_{<c(\mu_2)^{-1}} \) and the support of \( 1 - \chi \), we see that \( (1 - \chi)P_{<c(\mu_2)^{-1}}(1_{D_{>\mu_3^{-1}}P_{<c(\mu_2)^{-1}}}u) \) can be absorbed by \( \mu_2^2 \), it suffices to prove

\[ \| S_{1, \alpha} \left[ (1 - \chi)P_{<c(\mu_2)^{-1}} \left( 1_{D_{>\mu_3^{-1}}P_{<c(\mu_2)^{-1}}}u(t_\ast) \right) \right] \|_{L_t^Q(I; L_x^{2p})} \leq \mu_2^2. \]

Indeed, stationary phase shows that the operator \( S_{1, \alpha} (1 - \chi)P_{<c(\mu_2)^{-1}} \) have an operator norm of \( C_1(\mu_2) \) on \( L_x^{2p} \), then thanks to \( (3.7) \), for some \( \delta > 0 \), we have

\[ \| S_{1, \alpha} \left[ (1 - \chi)P_{<c(\mu_2)^{-1}} \left( 1_{D_{>\mu_3^{-1}}P_{<c(\mu_2)^{-1}}}u(t_\ast) \right) \right] \|_{L_t^Q(I; L_x^{2p})} \leq C_1(\mu_2) \left[ 1_{D_{>\mu_3^{-1}}P_{<c(\mu_2)^{-1}}}u(t_\ast) \right] \|_{L_t^Q(I; L_x^{2p})} \]

\[ \leq C_1(\mu_2) \left[ 1_{D_{>\mu_3^{-1}}P_{<c(\mu_2)^{-1}}}u(t_\ast) \right] \delta \|_{L_t^Q(I; L_x^{2p})} \left[ 1_{D_{>\mu_3^{-1}}P_{<c(\mu_2)^{-1}}}u(t_\ast) \right] \|_{L_t^Q(I; L_x^{2p})}^{1-\delta} \]

\[ \leq C_1(\mu_2, \mu_1) \mu_3^\delta \lesssim \mu_2^2. \]

Claim 2. \( \| 1_{D_{>\mu_3^{-1}}}u \|_{L_t^Q(I; L_x^{2p})} \lesssim_{\mu_1} \mu_2. \)
This claim can be proved by perturbation theorem and Strichartz estimates. Indeed, let $v$ be a solution to (1.1) on $I$ with initial data $v(t_*) = \chi u(t_*)$, $\partial_t v(t_*) = \partial_t u(t_*)$. Then by perturbation theorem, Claim 1, (3.5), we have

$$\|u - v\|_{L^q_t(I; L^p_x)} \lesssim \mu_1 \mu_2^2.$$  

It suffices to prove

$$\|1_{D > \mu_4^{-2}} v\|_{L^q_t(I; L^p_x)} \lesssim \mu_1 \mu_2^2. \quad (3.8)$$

Choosing another weight function $W : \mathbb{R}^d \to \mathbb{R}^+$ comparable to $1 + \mu_4 D$ which obeys the bounds $\nabla W, \nabla^2 W = O(\mu_4)$. Since $W\chi = O(1)$, we have

$$\|W v(t_0)\|_2 \lesssim 1.$$  

Since $v$ solves (1), we have

$$\partial_t (Wv) + 2\alpha \partial_t (Wv) - \Delta (Wv) + Wv = W|v|^{p-1} v + O(\mu_4 |v|) + O(\mu_4 |\nabla v|).$$

Strichartz estimates imply

$$\|W v\|_{L^q_t(I; L^p_x)} + \|(Wv, \partial_t (Wv))\|_{L^\infty_t(I; \mathcal{H})} \lesssim \|(Wv(t'), \partial_t Wv(t'))\|_{\mathcal{H}} + \|W|u|^p\|_{L^1_t(I; L^2_x)} + \mu_4,$$

for any subinterval $I'$ of $I$ and any $t' \in I'$. Denote the left side by $X(I')$, Hölder’s inequality and Sobolev embedding theorem reveal that

$$\|W|u|^p\|_{L^1_t(I'; L^2_x)} \leq C |I'|^{\frac{d+2}{d+2}} X(I').$$

Chopping $I$ up to sufficiently small intervals, we have

$$X(I') \leq C(\mu_1);$$

particularly, we conclude

$$\|1_{D > \mu_4^{-2}} v\|_{L^q_t(I; L^p_x)} \leq C(\mu_1) \mu_4,$$

which yields (3.8) by interpolation inequality thus finishing the proof of Claim 2.

Claim 3. $\|1_{D > \mu_4^{-3}} \int_I S_{2,\alpha}(t-s) \left(|u|^{p-1} u\right)(s) ds\|_{L^2(\mathbb{R}^d)} \lesssim \mu_1 \mu_2.$

From finite speed of propagation, $S_{2,\alpha}(t-s) \left(|u|^{p-1} u\right)(s)$ is supported in $D \leq \mu_4^{-2} + 4\mu_4^{-1}$. 

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Therefore $\int_{I} S_{2,\alpha}(t-s)(|u|^{p-1}u(s))d\mu_{I} = 0$. Thus it suffices to prove

$$\left\| \int_{I} S_{2,\alpha}(t-s)|u|^{p-1}u(s)1_{D>\mu_{I}^{-1}}ds \right\|_{L^{2}} \lesssim \mu_{1} \mu_{2}. \quad (3.9)$$

By Strichartz estimates, the left is bounded by

$$\left\| |u|^{p}1_{D>\mu_{I}^{-1}} \right\|_{L^{1}L^{2}}.$$ 

Then (3.9) follows from Hölder’s inequality and Claim 2.

**Step Four. $L^{2}$ localization of good times.** In this step, we prove the $L^{2}$ localization of $u(t_{*})$, namely for $T_{1}$ sufficiently large, $t > T_{1}$,

$$\|1_{D>\mu_{I}^{-3}}u(t_{*})\|_{2} = O_{L^{2}}(\mu_{1}). \quad (3.10)$$

The proof is based on the decay of linear part and Claim 3 in step three. Indeed, from Duhamel principle and Claim 3,

$$1_{D>\mu_{I}^{-3}}u(t_{*}) = 1_{D>\mu_{I}^{-3}}S_{1,\alpha}(\mu_{I}^{-1})u(t_{*} - \mu_{1}^{-1}) + 1_{D>\mu_{I}^{-3}}S_{2,\alpha}(\mu - 1^{-1})u(t_{*} - \mu_{1}^{-1}) + O_{L^{2}}(\mu_{2}).$$

Then since $S_{1,\alpha}$ and $S_{2,\alpha}$ have an exponential decay, we obtain

$$\left\| 1_{D>\mu_{I}^{-3}}u(t_{*}) \right\|^{2}_{2} = \left\langle 1_{D>\mu_{I}^{-3}}u(t_{*}), S_{1,\alpha}(\mu_{I}^{-1})u(t_{*} - \mu_{1}^{-1}) \right\rangle + \left\langle 1_{D>\mu_{I}^{-3}}u(t_{*}), S_{2,\alpha}(\mu_{1}^{-1})\partial_{t}u(t_{*} - \mu_{1}^{-1}) \right\rangle + O(\mu_{2})$$

$$\leq e^{-\mu_{1}^{-1}\alpha}\|u(t_{*})\|_{2}^{2} + O(\mu_{2}) \lesssim \mu_{1}.$$ 

Thus (3.10) follows.

**Step Five. $L^{2}$ localization of all time.** First from Duhamel principle and similar arguments as step four, it is easy to verify,

$$\|1_{D>\mu_{I}^{-3}}u(t)\|_{2} \leq \mu_{1},$$

for $t \in (t_{*} - 4\mu_{1}^{-1})$. Indeed, from Duhamel principle, finite speed of propagation, Claim 2 and Claim 3, we have

$$\left\| 1_{D>\mu_{I}^{-3}}u(t) \right\|_{2} \leq \left\| 1_{D>\mu_{I}^{-3}}S_{1,\alpha}(t-t_{*})u(t_{*}) \right\|_{2} + \left\| 1_{D>\mu_{I}^{-3}}S_{2,\alpha}(t-t_{*})u(t_{*}) \right\|_{2}$$

$$\quad + \left\| 1_{D>\mu_{I}^{-3}} \int_{t_{*}}^{t} e^{-\alpha(t-s)} S_{2,\alpha}(t-s)\left(|u|^{p-1}u\right)(s)ds \right\|_{2}$$

$$\lesssim \left\| 1_{D>\mu_{I}^{-3}}u(t_{*}) \right\|_{2} + \left\| 1_{D>\mu_{I}^{-3}}u(t_{*}) \right\|_{2} + \mu_{2} \lesssim \mu_{1}.$$ 

Splitting the whole interval $[T_{1}, \infty)$ into subintervals with length $2\mu_{1}^{-1}$, denote these subintervals as $I_{1}, I_{2}, I_{3}, \ldots$. Denote $t_{*} \in I_{j}$ by $t_{*}^{j}$. It is obvious that $I_{j+1}$ is covered by $(t_{*}^{j}, t_{*}^{j} + 4\mu_{1}^{-1})$, for $j = \ldots$
1, 2, ..., Now let’s define \( \tilde{x}_j(t) \) for each \( t \) by the following rule: For \( t \in I_{j+1} \), take \( \tilde{x}_j(t) = x_j(t^*_j) \). It is direct to see \( \tilde{x}_j(t) \) defined above satisfies Proposition 3.3 for all \( t > T_1 + 2\mu_1^{-1} \). \( \square \)

**Proposition 3.4.** Let \( u \) be a global solution to \( (1.1) \) with \( H \) norm at most \( E > 0 \). Then for \( \mu_0 > 0 \), there exits \( J = J(E, \mu_0) \) and functions \( \tilde{x}_1(t), ..., \tilde{x}_J(t) : \mathbb{R}^+ \to \mathbb{R}^d \), and \( \eta = \eta(E, \mu_0) > 0 \) such that

\[
\limsup_{t \to \infty} \int_{\text{dist}(x, \{\tilde{x}_1(t), ..., \tilde{x}_J(t)\}) > \eta^{-1}} |\nabla u|^2 + |u|^2 + |\partial_t u|^2 \leq \mu_0.
\]

**Proof.** Choose \( \mu_4 \leq \mu_3 \leq \mu_2 \leq \mu_1 \leq \mu_0 \) as Proposition 3.3. Suppose that \( t \in I_{j+1} \), then \( 0 < t - t^*_j < 4\mu_1^{-1} \). Define \( D(t) = \text{dist}(x, \{\tilde{x}_1(t), ..., \tilde{x}_J(t)\}) \). Then the proof of Proposition 3.3 implies that for all \( t \in I_{j+1} \),

\[
1_{D(t)} = 1_{D(t^*_j)}.
\] (3.11)

Let \( \chi_1 \) be a cutoff function supported in \( D(t^*_j) > \mu_4^{-1} \), which equals one in \( D > \mu_4^{-5} \), with bound \( |\nabla \chi_1| \leq \mu_4 \). Therefore we have

\[
\int_{D(t^*_j) > \mu_4^{-5}} |\nabla u(t)|^2 \, dx \leq \|u(t)\|_{H^1} + \mu_4.
\]

Duhamel principle and finite speed of propagation give

\[
u(t)\chi_1 = S_{1, \alpha}(\mu_1^{-1})1_{D(t^*_j) > \mu_4^{-3}}u(t - \mu_1^{-1}) + S_{2, \alpha}(\mu_1^{-1})1_{D(t^*_j) > \mu_4^{-3}}\partial_t u(t - \mu_1^{-1})
+ \int_{t - \mu_1^{-1}}^t S_{2, \alpha}(t - s)|u|^{p-1}u1_{D(t^*_j) > \mu_4^{-3}}(s) \, ds.
\]

By Strichartz estimates and exponential decay of \( S_{1, \alpha}, S_{2, \alpha} \), we get

\[
\|u(t)\chi_1\|_{H^1} \leq C(E)e^{-\alpha\mu_1^{-1}} + \left\|\|u|^{p-1}u1_{D(t^*_j) > \mu_4^{-3}}\|_{L^1_t(L^p_x((t-\mu_1^{-1}, t); L^2_x))}\right\|.
\]

Then Claim 2, \( (t - \mu_1^{-1}, t) \subset (t^*_j - 4\mu_1^{-1}, t^*_j + 4\mu_1^{-1}) \), and (3.11) imply,

\[
\int_{D(t) > \mu_4^{-5}} |\nabla u(t)|^2 \, dx \leq \mu_1,
\] (3.12)

which gives us the desired bound for \( |\nabla u(t)| \).

Next, we prove the desired bound for \( \partial_t u \). By Duhamel principle, and finite speed of propagation, we obtain

\[
1_{D(t^*_j) > \mu_4^{-3}}u(t) = S_{1, \alpha}(t - t^*_j)1_{D(t^*_j) > \mu_4^{-3}}u(t^*_j) + S_{2, \alpha}(t - t^*_j)1_{D(t^*_j) > \mu_4^{-3}}\partial_t u(t^*_j)
+ \int_{t^*_j}^t e^{-\alpha(t-s)}S_{2, \alpha}(t - s)|u(s)|^{p-1}u(s) \, ds.
\]
Direct calculations yield
\[
\left\| \partial_t \left[ 1_{D(t'_j) > \mu_1^{-4} u(t)} \right] \right\|_2 \leq \left\| 1_{D(t'_j) > \mu_1^{-4} u(t)} \right\|_{H^1}^2 + \left\| 1_{D(t'_j) > \mu_1^{-4} \partial_t u(t)} \right\|_2^2 + \left\| 1_{D(t'_j) > \mu_1^{-4} |u|^{p-1} u} \right\|_{L^1(I;L^2)}^2,
\]
where \( I = [t'_j - \mu_1^{-1}, t'_j + \mu_1^{-1}] \). Then Claim 2, (3.5) and (3.12) imply
\[
\left\| \partial_t \left[ 1_{D(t'_j) > \mu_1^{-4} u(t)} \right] \right\|_2 \lesssim \mu_1.
\]
Then the desire bound follows from (3.11).

From the proof of Proposition 10.1 in T. Tao [25], Proposition 3.2 is a corollary of Proposition 3.4 and the following lemma.

**Lemma 3.5.** Let \( u \) be a global solution with \( \mathcal{H} \) norm at most \( E \). Suppose that we have the energy concentration bound
\[
\int_{|x-x_0|<R} |u(t_0, x)|^2 + |\nabla u(t_0, x)|^2 + |\partial_t u(t_0, x)|^2 dx \geq \eta_1^2
\]
for some \( x_0 \in \mathbb{R}^d, t_0 \in \mathbb{R}^+, R > 0, \) and sufficiently small \( \eta_1 > 0 \). Then, if \( t_0 \) is sufficiently large depending on \( u, E, x_0, R, \eta_1 \), we have the improved energy concentration
\[
\int_{|x-x_0|<R'} |u(t_0, x)|^2 + |\nabla u(t_0, x)|^2 + |\partial_t u(t_0, x)|^2 dx \geq \beta(E),
\]
for some \( \beta(E) > 0 \) independent of \( \eta_1 \) and some \( R' \) depending on \( E, R, \eta_1 \).

The proof of Lemma 3.5 can be reduced to the following lemma.

**Lemma 3.6.** Given \( E > 0, \eta_1 > 0 \) sufficiently small, there exists \( \beta > 0 \) with the following property: Suppose that we have the energy concentration bound
\[
\eta_1^2 \leq \int_{|x-x_0|<R} |\nabla u(t_0, x)|^2 + |u(t_0, x)|^2 + |\partial_t u(t_0, x)|^2 \leq \beta
\]
for some \( x_0 \in \mathbb{R}^d, t_0 \in \mathbb{R}^+, R > 0, \) and some global solution \( u \) with \( \mathcal{H} \) norm at most \( E \). Then, if \( t_0 \) is sufficiently large depending on \( E, x_0, R, \eta_1 \), we have
\[
\int_{|x-x_0|<R'} |\nabla u(t_0, x)|^2 + |u(t_0, x)|^2 + |\partial_t u(t_0, x)|^2 \geq \int_{|x-x_0|<R} |\nabla u(t_0, x)|^2 + |u(t_0, x)|^2 + |\partial_t u(t_0, x)|^2 + \eta_4^2,
\]
for some \( \eta_4(E, \eta_1) > 0 \) and \( R'(E, R, \eta_1, \eta_4) \).
Proof. For simplicity, define $e = |\nabla u(t_0, x)|^2 + |u(t_0, x)|^2 + |\partial_t u(t_0, x)|^2$. Fix $E > 0$, let $\beta > 0$ be a sufficiently small quantity to be determined. Choose parameters $\eta_1 \gg \eta_2 \gg \eta_3 \gg \eta_4 > 0$. Let $R_0 > \max(32R, \frac{32}{\eta_4})$. Suppose by contradiction that there exists $R' > R_0$ such that our claim fails, then
\[
\int_{R <|x-x_0|<R'} e(t_0, x) dx \lesssim \eta_1^2,
\]
especially, we have
\[
\int_{|x|<R'} e(t_0, x) \lesssim \beta.
\]
Choose $\beta < \epsilon$, where $\epsilon$ is the constant in Proposition 2.3. Denote the solution of (1.1) with initial data $1_{|x|<R'} u(t_0, x)$ at $t_0$. Proposition 2.3 implies
\[
\|\tilde{u}\|_H \leq e^{-\gamma(t-t_0)} \|\tilde{u}(t_0)\|_H \lesssim \beta. \tag{3.13}
\]
Finite speed of propagation implies
\[
u(x, t) = \tilde{u}(x, t) \text{ in } \{(x, t) : |x| < R' - |t - t_0|\}.
\]
Consider a time interval $I = [t_0, t_0 + \eta_3^{-1}]$. Then for $t \in I$, we have
\[
\int_{|x|<R'/2} e(t, x) dx = \int_{|x|<R'/2} \tilde{e}(t, x) dx.
\]
Combining with (3.13), we have verified
\[
\int_{|x|<R'/2} e(t + \eta_3^{-1}, x) dx \lesssim e^{-\eta_3^{-1}} \beta \lesssim \eta_3^3.
\]
If we have obtained
\[
\inf_{t \in I} \int_{|x|<R'/2} e(t, x) \geq \eta_1^2, \tag{3.14}
\]
then contradiction follows. Hence, it suffices to prove (3.14). By Lemma 3.1, there exists $\mu > 0$, $T_0 > 0$, such that for any $t > T_0$,
\[
\|P_{>\mu^{-1}} u\|_{H^1} + \|P > \mu^{-1} \partial_t u\|_{L^2} \leq \eta_1^6.
\]
Hence it suffices to prove
\[
\inf_{t \in I} \int_{|x|<R'/2} \left|P_{\leq \mu^{-1}} u \right|^2 + \left|\nabla P_{\leq \mu^{-1}} u \right|^2 + \left|\partial_t P_{\leq \mu^{-1}} u \right|^2 dx \gtrsim \eta_1^2.
\]
Let \( \psi(x) \) be a smooth cutoff function which equals 1 in \( \{|x| < R'/4\} \), vanishes when \( |x| > R'/2 \), with bound \( |\nabla \psi(x)| = O(R'^{-1}) \), then
\[
\frac{d}{dt} \int_{\mathbb{R}^d} \psi(x) \left[ |P_{\leq \mu^{-1}} u|^2 + |\nabla P_{\leq \mu^{-1}} u|^2 + |P_{\leq \mu^{-1}} u_t|^2 \right] dx
= 2 \int_{\mathbb{R}^d} \psi(x) (P_{\leq \mu^{-1}} u_t) P_{\leq \mu^{-1}} h(u) - 4\alpha \int_{\mathbb{R}^d} \psi(x) |P_{\leq \mu^{-1}} u_t|^2 dx - 2 \int_{\mathbb{R}^d} \nabla \psi(x) (\nabla P_{\leq \mu^{-1}} u) (P_{\leq \mu^{-1}} u_t) dx.
\]
Define
\[
e_{\mu}(t) = |P_{\leq \mu^{-1}} u| + |\nabla P_{\leq \mu^{-1}} u|^2 + |\partial_t P_{\leq \mu^{-1}} u|^2.
\]
Hölder’s inequality yields,
\[
\left| \int_{\mathbb{R}^d} \psi(x)e_{\mu}(t)dx - \int_{\mathbb{R}^d} \psi(x)e_{\mu}(t_0)dx \right|
\leq \int_{t_0}^{t_0+1/\eta_3} \left| \frac{d}{dt} \int_{\mathbb{R}^d} \psi(x)e_{\mu}dx \right| dt
\leq C \int_{t_0}^{t_0+1/\eta_3} \left( \|u_t\|_2 + \|u_t\|_2 \right) dt.
\]
For \( d \geq 3, 1 < p \leq \frac{d}{d-2} \), Sobolev embedding theorem implies \( \|P_{\leq \mu^{-1}} h(u)\|_2 \leq C(E) \), thus
\[
\int_{\mathbb{R}^d} \psi(x)e_{\mu}(t)dx - \int_{\mathbb{R}^d} \psi(x)e_{\mu}(t_0)dx \leq C(E, \mu) \int_{t_0}^{t_0+1/\eta_3} \left( \|u_t\|_2 + \|u_t\|_2 \right) dt.
\] (3.15)
For \( d \geq 3, \frac{d}{d-2} < p < 1 + \frac{4}{d-2} \), by Bernstein’s inequality,
\[
\|P_{\leq \mu^{-1}} h(u)\|_2 \leq \mu^{-d(\frac{d}{2}-\frac{1}{2})} \|P_{\leq \mu^{-1}} h(u)\|_p \leq \mu^{-d(\frac{d}{2}-\frac{1}{2})} C(E),
\]
which yields (3.15) again. For \( d = 1, 2 \), (3.15) can be obtained directly by Sobolev embedding theorem. Since \( \int_{0}^{\infty} \|u_t\|_2^2 dt < \infty \), choose \( t_0 \) sufficiently large such that
\[
\int_{t_0}^{t_0+1/\eta_3} \|u_t\|_2^2 dt \leq \sqrt{\eta_3} \mu^d(\frac{d}{2}-\frac{1}{2}) \eta_1^6,
\]
then
\[
\int_{|x|<R'/2} e_{\mu}(t)dx \geq \int_{\mathbb{R}^d} \psi(x)e_{\mu}(t)dx \geq \int_{\mathbb{R}^d} \psi(x)e_{\mu}(t_0)dx - \sqrt{\eta_3} \eta_1^3 \eta_1^2,
\]
thus proving (3.14), from which our lemma follows.

### 4 Concentration compact attractor

In this section, we first derive the global attractor, then we prove Theorem 1.1 immediately.
4.1 Concentration-compactness attractor

We recall the following criterion for compact attractors proved by Proposition B.2 in Tao [25].

**Proposition 4.1.** Let \( U \) be a collection of trajectories \( u : \mathbb{R}^+ \to H \). If \( U \) is bounded in \( H \), and for any \( \mu_0 > 0 \) there exists \( \mu_1 > 0 \) such that

\[
\limsup_{t \to \infty} \| P_{1/\mu_1} u(t) \|_H \leq \mu_0,
\]

\[
\limsup_{t \to \infty} \int_{|x| > 1/\mu_1} |u(x,t)|^2 + |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \, dx \leq \mu_0.
\]

Then there exists a compact set \( K \subset H \) such that \( \lim_{t \to \infty} \text{dist}_H(u(t), K) = 0 \).

**Proposition 4.2.** Let \( U \) be a collection of trajectories \( u : \mathbb{R}^+ \to H \), and let \( J \geq 1 \). If \( U \) is bounded in \( H \), and for any \( \mu_0 > 0 \) there exists \( \mu_1 > 0 \) such that for every \( u \in U \) we have \( x_1, \ldots, x_J : \mathbb{R}^+ \to \mathbb{R}^d \) for which

\[
\limsup_{t \to \infty} \| P_{1/\mu_1} u(t) \|_H \leq \mu_0,
\]

\[
\limsup_{t \to \infty} \int_{\text{dist}(x, \{x_1(t), x_2(t), \ldots, x_J(t)\}) > 1/\mu_1} |u(x,t)|^2 + |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \, dx \leq \mu_0.
\]

Then there exists a G-precompact set \( K \subset H \) with \( J \) components such that \( \lim_{t \to \infty} \text{dist}_H(u(t), K) = 0 \).

**Proof.** Although the proof is almost the same as proposition B.3 of T. Tao [25], for reader’s convenience, we give a sketch here. We use the partition of unity

\[
1 = \sum_{j=1}^J \psi_{j,t}(x),
\]

where

\[
\psi_{j,t}(x) = \frac{|x - x_j(t)|^{-1}}{\sum_{l=1}^J |x - x_l(t)|^{-1}}.
\]

Split \((u(x,t), \partial_t u(x,t))\) as

\[
u(t) = \sum_{j=1}^J \tau_{x_j(t)} w_j(t), \quad \partial_t u(t) = \sum_{j=1}^J \tau_{x_j(t)} v_j(t),
\]

where

\[
w_j(t) = \tau_{-x_j(t)} \psi_{j,t}(x) u(t), \quad v_j(t) = \tau_{-x_j(t)} \psi_{j,t}(x) \partial_t u(t).
\]
The localization of $u$ and $\partial_t u$ implies for any $\mu_0 > 0$, there exists $\eta > 0$ such that
\[
\limsup_{t \to \infty} \|P_{> \eta} w_j\|_{H^1} + \|P_{> \eta} v_j\|_{L^2} \leq \mu_0,
\limsup_{t \to \infty} \int_{|x| > \eta} |w_j|^2 + |\nabla w_j|^2 + |v_j|^2 \leq \mu_0.
\]

From Proposition 4.1 there exist a compact set $K_1 \subset H^1$ and a compact set $K_2 \subset L^2$, such that
\[
\lim_{t \to \infty} \text{dist}(w_j(t), K_1) = 0, \quad \lim_{t \to \infty} \text{dist}(v_j(t), K_2) = 0,
\]
for all $j = 1, 2, ..., J$. Combining with (4.1), we obtain
\[
\text{dist}_\mathcal{H}(u, J(GK)) = 0,
\]
where $K = K_1 \times K_2$. \hfill \Box

As a corollary of Proposition 4.2, Proposition 3.2, Lemma 3.1, we have

**Corollary 4.3.** There exists a compact set $K \subset \mathcal{H}$ and $0 \leq J < \infty$, such that
\[
\lim_{t \to \infty} \text{dist}_\mathcal{H}(u(t), J(GK)) = 0.
\]

5 Proof of theorem 1.1

**Step one.** Combining Corollary 4.3 with Lemma B.7 in Tao [25], we have for any $t_n \to \infty$, up to a subsequence there exits $J_1, J_2, ..., J_M$ and $w_m \in J_m(GK)$ such that
\[
\begin{align*}
u(t_n) &= \sum_{m=1}^{M} \tau_{x_{m,n}} w_m + o_{H^1}(1) \\
\partial_t u(t_n) &= \sum_{m=1}^{M} \tau_{x_{m,n}} v_m + o_{L^2}(1),
\end{align*}
\]
where $x_{m,n} \in \mathbb{R}^d$ and they satisfies $\lim_{n \to \infty} |x_{m,n} - x_{k,n}| = \infty$, for $k \neq m$.

**Step two.** By linear energy decoupling property, we have $\sup_m \|(w_m, v_m)\|_\mathcal{H} < C$, by the local theory, there exists $T > 0$ such that the solution $W_j$ to (1.1) with initial data $(w_j, v_j)$ is wellposed on $[0, T]$. From perturbation theorem and separation of $x_{m,n}$, we obtain
\[
\partial_t u(t_n + t) = \sum_{j=1}^{M} \partial_t W_j(x - x_{j,n}, t) + o_{L^2}(1).
\]
Since \( \lim_{n \to \infty} \int_0^T \| \partial_t u(t_n + t) \|_2^2 dt = 0 \), by the separation of linear energy, we conclude

\[
\int_0^T \| \partial_t W_j(t) \|_2^2 dt = 0.
\]

Therefore, \( W_j \) is an equilibrium, the same holds for \( w_j \), thus we have proved there exists a finite number of equilibrium points \( Q_m \) such that for any sequence \( t_n \to \infty \), there exists \( x_{m,n} \) for which

\[
u(t_n) = \sum_{m=1}^M Q_m(x - x_{m,n}) + o_{H^1}(1), \quad \partial_t u(t_n) = o_{L^2}(1).
\]

By contradiction arguments, we can prove our theorem.

References

[1] H. Berestycki, P.L. Lions, Nonlinear scalar field equations. II. Existence of infinite many solutions, Arch. Rational Mech. Anal. 82 (1983), 347-375.

[2] N. Burq, G. Raugel, W. Schlag, Long time dynamics for damped Klein-Gordon equations, \texttt{arXiv:1505.05981}.

[3] T. Cazenave, Uniform estimates for solutions of nonlinear Klein-Gordon equations, J. Functional Analysis, 60 (1985), 36-55.

[4] R. Cote, Soliton resolution for equivariant wave maps to the sphere, \texttt{arXiv:1305.5325}, to appear in Comm. Pure. Appl. Math.

[5] R. Cote, C. Kenig, A. Lawrie, W. Schlag, Profiles for the radial focusing 4d energy-critical wave equation, Preprint arXiv: 1402.2307.

[6] R. Cote, C. Kenig, A. Lawrie, W. Schlag, Characterization of large energy solutions of the equivariant wave map problem: II, Amer. J. Math. 137 (2015), no. 1, 209-250, see also \texttt{arXiv:1209.3684v2}.

[7] R. Cote, C. Kenig, A. Lawrie, W. Schlag, Characterization of large energy solutions of the equivariant wave map problem: I, American Journal of Mathematics, 137 (2015), no.1, 139-207, see also \texttt{arXiv:1209.3682v2}.

[8] T. Duyckaerts, C. Kenig, F. Merle, Classification of radial solutions of the focusing, energy-critical wave equation, Cambridge Journal of Mathematics 1 (2013), no. 1, 75-144.

[9] T. Duyckaerts, C. Kenig, F. Merle, Universality of blow-up profile for small radial type II blow-up solutions of the energy-critical wave equation, J. Eur. Math. Soc. (JEMS) 13 (2011), no. 3, 533-599.
[10] T. Duyckaerts, C. Kenig, F. Merle, Universality of blow-up profile for small radial type II blow-up solutions of the energy-critical wave equation: the nonradial case, J. Eur. Math. Soc. (JEMS) 14 (2011), 1389-1454.

[11] T., Duyckaerts, C., Kenig, F., Merle, Scattering for radial, bounded solutions of focusing supercritical wave equations, Int. Math. Res. Not. IMRN 2014, no. 1, 224-258.

[12] E. Feireisl, Convergence to an equilibrium for semilinear wave equations on unbounded intervals, Dynam. Syst. Appl. 3 (1994), 423-434.

[13] E. Feireisl, Finite energy travelling waves for nonlinear damped wave equations, Quarterly Journal of Applied mathematics LVI(1998), 55-70.

[14] H., Jia, B.P., Liu, G.X., Xu, Long time dynamics of defocusing energy critical 3 + 1 dimensional wave equation with potential in the radial case, to appear in in Communications in Mathematical Physics, see also arXiv:1403.5696

[15] H. Jia, C. Kenig, Asymptotic decomposition for semilinear wave and equivariant wave map equations, arXiv. 1503.06715.

[16] C., Kenig, A., Lawrie, W., Schlag, Relaxation of wave maps exterior to a ball to harmonic maps for all data, arXiv:1301.0817.

[17] C., Kenig, A., Lawrie, B.P., Liu, W., Schlag, Stable soliton resolution for exterior wave maps in all equivariance classes, arXiv:1409.3644.

[18] C., Kenig, A., Lawrie, B.P., Liu, W., Schlag, Channels of energy for the linear radial wave equation, arXiv:1409.3643.

[19] K. Nakanishi, W. Schlag, Invariant manifolds and dispersive Hamiltonian Evolution Equations, Zurich Lectures in Advanced mathematicas. European Mathematical Society (EMS), Zürich, 2011.

[20] K. Nakanishi, W. Schlag, Global dynamics above the ground state energy for the focusing nonlinear Klein-Gordon equation, J. Differetnial Equations, 250 (2011), 2299-2333.

[21] K. Nakanishi, W. Schlag, Global dynamics above the ground state energy for the nonlinear Klein-Gordon equation without a radial assumption, Arch. Rational Mech. Anal. 203 (2012) 809-851.

[22] I. E. Payne, D.H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, Israel J. math. 22(1975), 273-303.
[23] C. Rodriguez, Profiles for the radial focusing energy-critical wave equation in odd dimensions, arXiv:1412.1388

[24] A. Soffer, Soliton dynamics and scattering, http://www.icm2006.org/proceedings/VolIII/contents/ICM Vol3, 24.pdf

[25] T. Tao, A (concentration-) compact attractor for highdimensional nonlinear Schrödinger equations, Dynamics of Partial Differential Equations, 4 (2007) 1-53.