The field-theoretical methods in Lovelock gravity

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Abstract. The field-theoretical methods are used to construct conserved currents and related superpotentials for perturbations on arbitrary backgrounds in the Lovelock gravity. The perturbations are considered as a dynamic field configuration propagating in a given spacetime. The field-theoretical formalism is exact (without approximations) and equivalent to the original metric theory. As Lagrangian based formalism, it allows us to apply the Noether theorem. As a result, we construct conserved currents and superpotentials, where we use arbitrary displacement vectors, not only the Killing ones or other special vectors. The developed formalism is checked in calculating mass of the Schwarzschild-anti-de Sitter (AdS) black hole. The new formalism is adopted to the case of a so-called pure Lovelock gravity, where in the Lagrangian only a one polynomial in Riemannian tensor presents. We construct conserved charges and currents for static and dynamic black holes of the Vaidya type with AdS, dS and flat asymptotics. New properties of the solutions under consideration have been found. The more results are discussed.

1. Introduction

1.1. Motivation

The Lovelock gravity [1] becomes very popular in the last years. The researchers develop its theoretical foundations, find various solutions and analyze them, see, for example [2,3] and references therein. Many methods have been suggested to construct conserved quantities in the Lovelock theory. However, by our opinion, these approaches do not cover all the arising problems. In particular, construction of conserved quantities for perturbations is restricted, as a rule, to maximally symmetric backgrounds, not arbitrary ones; one chooses usually Killing vectors as vectors of displacements, not other vectors.

1.2. Goals

Keeping in mind the above, we plan to close the mentioned gap, at least, particularly. The theoretical goal of the present paper is to develop a so-called field-theoretical formalism (as a more universal one) in metric theories [4-7] in the framework of the Lovelock gravity. We check it, calculating the mass of the Schwarzschild-anti-de Sitter (AdS) black hole. The other goal is to adopt the new general formalism to the case of a so-called pure Lovelock gravity, see, for example [8] and references therein. Then, using powerful possibilities of the formalism, we construct conserved quantities for Vaidya-type black holes (static and dynamic) in the pure Lovelock gravity, solutions for which have been suggested in [9,10]. We plan to construct both global and quasi-local charges (energy expressions) presented both on the backgrounds of asymptotic solutions and backgrounds of related static black holes. Besides, our formalism is valid to calculate energy densities and densities of energy fluxes measured by freely falling
observers on the backgrounds of static black holes. The presentation is based on our papers [11,12] and one of the goals here is to unite their results and attract to them the attention of readers.

2. The field-theoretical formalism in an arbitrary metric theory

Let us recall the history of development of the field-theoretical methods. Many solutions in general relativity are considered when perturbations are propagating on various fixed backgrounds. Then equations are reformulated in a perturbed form. A part linear in metric perturbations is placed at the left hand side, whereas all the other terms are transferred to the right hand side and interpreted as an effective energy-momentum. For example, choosing as a background Minkowski space with the metric \( \eta^{\mu \nu} \), defining the metric perturbations as \( h_{\mu \nu} = g_{\mu \nu} - \eta_{\mu \nu} \), one has:

\[
G_{\mu \nu}^L (h) = \kappa t_{\nu}^{\nu}.
\]

Here and below, Greek indices mean spacetime components; Latin indices mean space components. Because the divergence of the left hand side of (1) disappears identically, \( \partial_{\mu} G_{\mu \nu}^L (h) = 0 \), one obtains the conservation law for the effective energy-momentum \( \partial_{\mu} t_{\nu}^{\mu} = 0 \).

In 1950-th and 1960-th the above picture has been provided by iterations and approximations. In the famous work [13], Deser has united the previous results in the exact and closed form:

\[
L_{\text{tot}} (h, \Gamma, \phi^A) = L_{\text{tot}}^L (h, \Gamma, \phi^A)
\]

It is obtained in the framework of the Lagrangian based method by varying the related action with respect to \( h_{\mu \nu} \); the total energy-momentum is obtained by varying with respect to the background metric; it was used the first order formalism with variables \( h_{\mu \nu} \) and \( \Gamma_\mu^{\nu \rho} \); \( \phi^A \) is a generalized matter variable. However, Deser’s formalism has been presented for the Minkowski background only. We [4] have generalized his approach in general relativity for the case of arbitrary curved backgrounds including a background matter. Later we have developed this approach for the case of many modified metric theories and examined various solutions of these theories [5-7].

Because the field-theoretical methods are quite universal, we outline them below on the example of an arbitrary covariant metric theory in \( D \) dimensions. A unique requirement is that its field equations are not more than of second order in derivatives, like in the Lovelock gravity. Let the Lagrangian of such a theory be

\[
L = \frac{1}{2\kappa} L_s (g_{\mu \nu}) + L_m (g_{\mu \nu}, \Phi^A),
\]

with metric \( g_{\mu \nu} \) and generalized matter \( \Phi^A \) variables. Here and below, \( \kappa = 2\Omega_{D-2} G_D \), where \( G_D \) is \( D \)-dimensional Newton’s constant, \( \Omega_{D-2} \) is a square of \( (D - 2) \)-dimensional sphere with unit radius. The system (3) is considered as a perturbed one with respect to the background system:

\[
\bar{L} = \frac{1}{2\kappa} L_s (\bar{g}_{\mu \nu}) + L_m (\bar{g}_{\mu \nu}, \bar{\Phi}^A) \]

where \( \bar{L} = L(\bar{g}_{\mu \nu}, \bar{\Phi}^A) \), \( \bar{L}_s = L_s (\bar{g}_{\mu \nu}) \), \( \bar{L}_m = L_m (\bar{g}_{\mu \nu}, \bar{\Phi}^A) \); \( \bar{g}_{\mu \nu} \) and \( \bar{\Phi}^A \) present a known (fixed) solution of the theory. Next we define perturbations:

\[
\kappa_{\mu \nu} = g_{\mu \nu} - \bar{g}_{\mu \nu},
\]

\[
\phi^A = \Phi^A - \bar{\Phi}^A
\]

and interpret them as a configuration of dynamic variables. Generalizing the Deser prescription [13], we define the Lagrangian for \( \kappa_{\mu \nu} \) and \( \phi^A \) in the form [5]:

\[
L = \frac{1}{2\kappa} L_s (\kappa_{\mu \nu}) + L_m (\kappa_{\mu \nu}, \phi^A).
\]
\[ L_{\text{kin}} = L(\mathcal{G} + \kappa, \Phi + \phi) - \kappa_{\mu\nu} \frac{\delta L}{\delta \mathcal{G}_{\mu\nu}} + \phi^A \frac{\delta L}{\delta \Phi^A} - \tilde{L}. \] (6)

Varying this Lagrangian with respect to \( \kappa_{\mu\nu} \) one obtains the field equations in the form:
\[ G^L_{\mu\nu} + F^L_{\mu\nu} = \kappa \left( t^g_{\mu\nu} + t^m_{\mu\nu} \right) = \kappa t_{\mu\nu}, \] (7)
where the left hand side is linear in \( \kappa_{\mu\nu} \) and \( \phi^A \). The right hand side of (7) is the metric energy-momentum tensor density defined for the system (6): \( t_{\mu\nu} = 2 \frac{\partial L_{\text{kin}}}{\partial \mathcal{G}^\mu \mathcal{G}^\nu} \). In the case of a vacuum background, \( \Phi^A = 0 \), one has \( F^L_{\mu\nu} = 0 \) and \( \tilde{\nabla}^\nu (G^L_{\mu\nu} + F^L_{\mu\nu}) = 0 \). Then, one has again the conservation law for the energy-momentum: \( \tilde{\nabla}^\nu t_{\mu\nu} = 0 \). Here and below \( \tilde{\nabla}^\nu \) means a covariant derivative in the background spacetime. In the case of a non-vacuum background one has \( F^L_{\mu\nu} \neq 0 \) and \( \tilde{\nabla}^\nu (G^L_{\mu\nu} + F^L_{\mu\nu}) \neq 0 \). As a result, there is no a conservation for the energy-momentum in general. The problem is resolved only when one constructs a conserved current including \( t_{\mu\nu} \) together with terms, which interact with a complicated background.

To obtain conserved currents one can apply the Noether procedure directly to the Lagrangian (6). However, in this case one meets a very complicated calculations. There is another more economical way. Let us consider an auxiliary Lagrangian:
\[ L_1 = -\frac{1}{2\kappa} \kappa_{\mu\nu} \frac{\delta L_g}{\delta \mathcal{G}_{\mu\nu}} \] (8)
that is linear in \( \kappa_{\mu\nu} \) and is a part of (6). Because \( L_1 \) is a scalar density one can use the diffeomorphism invariance and apply the Noether theorem to (8). As a result, one obtains the identities:
\[ \tilde{\nabla}_\mu i^\mu_1 = \partial_\mu i^\mu_1 = 0 \] (9)
for the current \( i^\mu_1 \) (vector density) that is expressed through a divergence of a superpotential \( i^\mu_{1\nu} \) (antisymmetrical tensor density) as
\[ i^\mu_1 \equiv \tilde{\nabla}_\nu i^\mu_{1\nu} \equiv \partial_\nu i^\mu_{1\nu}. \] (10)
However, (9) and (10) are the identities only. To transform them to physically sensible conservation laws we use the field equations (7). In the result one obtains:
\[ \tilde{\nabla}_\mu I^\mu = \partial_\mu I^\mu = 0, \] (11)
\[ I^\mu = \tilde{\nabla}^\nu I^{\mu\nu} = \partial_\nu I^{\mu\nu}. \] (12)
Thus, we have a differential conservation law (11) for the current \( I^\mu \).

It is important to consider its structure \( I^\mu = \tau^{\alpha\mu} \xi_\alpha - z^\mu_1(\xi) \). The generalized energy-momentum has the form:
\[ \tau_{\mu\nu} = t_{\mu\nu} - \frac{1}{\kappa} F^L_{\mu\nu} + \frac{1}{2\kappa} \frac{\delta L_g}{\delta g^{\alpha\beta}} \left( \kappa_{\alpha\beta}{\mathcal{G}^\mu \mathcal{G}^\nu} - \kappa_{\alpha\beta}{\mathcal{G}^\mu \mathcal{G}^\nu} - \kappa_{\alpha\beta} \mathcal{G}^\mu \mathcal{G}^\nu \right). \] (13)
Above we have remarked that \( t_{\mu\nu} \) itself is not conserved on complicated backgrounds with a background matter. Formula (13) shows this explicitly. Indeed, only with interacting terms in (13) we have a conservation law for the current in (11). The term \( z^\mu_1(\xi) \) disappears if the displacement vector \( \xi^\mu \) is a Killing vector of the background spacetime. The structure of the superpotential is important as well:
\[ I^{\mu\nu} = \frac{3}{2} \left( 2 z^{\alpha \sigma} \tilde{\nabla}_{\lambda} \omega_{\sigma |\mu\nu|\lambda} - \omega_{\alpha |\mu\nu|\lambda} \tilde{\nabla}_{\lambda} z^{\sigma} \right). \] (14)
The key quantity here that is valid for an arbitrary background is defined as \( \omega^{\alpha\beta\mu\nu} \equiv \delta L / \delta g_{\alpha\beta\mu\nu} \) with the symmetry properties \( \omega^{\alpha\beta\mu\nu} = \omega^{\beta\alpha\mu\nu} = \omega^{\alpha\beta\nu\mu} = \omega^{\nu\beta\mu\alpha} \).

Differential conservation laws give a possibility to define integral conserved quantities. Thus, the integration of (11) leads to constructing a quantity

\[
P(\xi) = \int_{\Sigma} d^{D-1}x I^0(\xi) \tag{15}
\]
defined on a spacelike section \( \Sigma := x^0 = \text{const} \); interpretation of \( P(\xi) \) is determined by a choice of \( \xi^\mu \). The conservation law (12) allows us to represent (15) in the form of a surface integral:

\[
P(\xi) = \oint_{\partial \Sigma} ds I^0(\xi), \tag{16}
\]
where \( \partial \Sigma \) is a boundary of \( \Sigma \), and \( ds \) is a coordinate volume element at \( \partial \Sigma \).

3. Conserved quantities for perturbations in Lovelock gravity

3.1. The Lovelock formulae in general

All the formulae from the above section can be used for concrete calculations in the Lovelock gravity. We consider the Lagrangian (3), where the pure gravitational part presents the Lovelock Lagrangian in \( D \) dimensions [1]:

\[
L_\ell = \sqrt{-g} \sum_{p=0}^{m} \frac{\alpha_p}{2^p} \delta^{\nu}_{\{\nu_1...\nu_{2p}\}} \mathcal{R}_{\nu_1\nu_2}^{\nu_3...\nu_{2p}} R_{\nu_1\nu_2}^{\nu_3...\nu_{2p}...\nu_{2p}}.
\]

Then the auxiliary Lagrangian (8) acquires the explicit form:

\[
L_{1\ell} = -\sqrt{-g} \kappa^\rho \sum_{p=0}^{m} \frac{\alpha_p}{2^{p+1}} \delta^{\nu}_{\{\nu_1...\nu_{2p}\}} \mathcal{R}_{\nu_1\nu_2}^{\nu_3...\nu_{2p}} - \mathcal{R}_{\nu_1\nu_2}^{\nu_3...\nu_{2p}} \frac{\alpha_p}{2^{p-1}} g^{\nu_{2p+1}...\nu_{2p}} g^{\nu_{2p+1}...\nu_{2p}} D_{\sigma\tau\kappa} \omega^{\alpha\beta\mu\nu},
\]

where \( \omega^{\alpha\beta\mu\nu} \), with the use of which one can construct both superpotential (14) and current (12), is:

\[
\omega^{\alpha\beta\mu\nu} = \sqrt{-g} \kappa^\rho \sum_{p=0}^{m} \frac{p\alpha_p}{2^{p+1}} \delta^{\nu}_{\{\nu_1...\nu_{2p}\}} \mathcal{R}_{\nu_1\nu_2}^{\nu_3...\nu_{2p}} - \mathcal{R}_{\nu_1\nu_2}^{\nu_3...\nu_{2p}} \frac{\alpha_p}{2^{p-1}} g^{\nu_{2p+1}...\nu_{2p}} g^{\nu_{2p+1}...\nu_{2p}} D_{\sigma\tau\kappa} \omega^{\alpha\beta\mu\nu}, \tag{19}
\]

The key quantity \( \omega^{\alpha\beta\mu\nu} \), with the use of which one can construct both superpotential (14) and current (12), is:

\[
\frac{\delta L_\ell}{\delta g_{\sigma\tau\kappa}} = \sqrt{-g} \sum_{p=0}^{m} \frac{p\alpha_p}{2^{p+1}} \delta^{\nu}_{\{\nu_1...\nu_{2p}\}} \mathcal{R}_{\nu_1\nu_2}^{\nu_3...\nu_{2p}} - \mathcal{R}_{\nu_1\nu_2}^{\nu_3...\nu_{2p}} \frac{\alpha_p}{2^{p-1}} g^{\nu_{2p+1}...\nu_{2p}} g^{\nu_{2p+1}...\nu_{2p}} D_{\sigma\tau\kappa} \omega^{\alpha\beta\mu\nu}. \tag{20}
\]

3.2. Perturbations on AdS background

To check the formalism one has to apply it to calculate conserved quantities for known models. One of them is the Schwarzschild-AdS black hole, and we calculate a global mass for it. From the start we adopt our formulae for perturbations propagating on AdS backgrounds. Thus, let us derive vacuum background equations in the Lovelock gravity:

\[
\frac{\delta L_\ell}{\delta g_{\rho\sigma\tau}} = \sqrt{-g} \sum_{p=0}^{m} \frac{p\alpha_p}{2^{p+1}} \delta^{\nu}_{\{\nu_1...\nu_{2p}\}} \mathcal{R}_{\nu_1\nu_2}^{\nu_3...\nu_{2p}} - \mathcal{R}_{\nu_1\nu_2}^{\nu_3...\nu_{2p}} \frac{\alpha_p}{2^{p-1}} g^{\nu_{2p+1}...\nu_{2p}} g^{\nu_{2p+1}...\nu_{2p}} D_{\sigma\tau\kappa} \omega^{\alpha\beta\mu\nu} = 0.
\]

The AdS space has to satisfy these equations with the Riemannian tensor \( \mathcal{R}_{\mu\nu} = -\ell^2 \omega^{[\mu\nu]} \), where the effective AdS radius \( \ell_{\text{eff}} \) is to be the solution to the equation:

\[
V(x)|_{x=\ell_{\text{eff}}} = \sum_{p=0}^{m} \frac{(D-3)!}{(D-2p-1)!} \alpha_p (p-1)! \left( \ell_{\text{eff}}^2 \right)^p = 0. \tag{21}
\]
The AdS metric in a more popular form is
\begin{equation}
\begin{aligned}
\tilde{ds}^2 &= -\tilde{f}(r)dt^2 + \tilde{f}^{-1}(r)dr^2 + r^2 \sum_{a,b} q_{ab}dx^a dx^b
\end{aligned}
\end{equation}
where \(\tilde{f}(r) = 1 + r^2 f_{\text{eff}}^{-2}\) and \(q_{ab}\) is the metric at the \((D-2)\)-dimensional unit sphere. Then the quantity (19) for the metric perturbations (5) on the background (22) becomes
\begin{equation}
\alpha_{\mu\nu}^{\text{eff}} = \alpha_{\mu\nu}^{\text{eff}} V'(x) \bigg|_{x = \xi_0} = \alpha_{\mu\nu}^{\text{eff}} \left[ \sum_{p=0}^{m} \frac{(D-3)!}{(D-2p-1)!} \alpha_p (\ell_{\text{eff}})^{p-1} \right],
\end{equation}
where \(\alpha_{\mu\nu}^{\text{eff}}\) is the quantity for the Einstein theory in \(D\) dimensions:
\begin{equation}
\alpha_{\mu\nu}^{\text{eff}} = -\frac{\sqrt{-g}}{4\kappa} \left[ g^{\mu\nu} \kappa^{\rho\lambda} + \bar{g}^{\rho\lambda} k^{\mu\nu} - \bar{g}^{\rho(\mu} k^{\nu)\lambda} - \bar{g}^{\rho(\mu} k^{\nu)} - k(\bar{g}^{\mu\nu} \bar{g}^{\rho\lambda} - \bar{g}^{\rho(\mu} \bar{g}^{\nu)\lambda}) \right].
\end{equation}
Finally, superpotential on the background (22) acquires a concrete form:
\begin{equation}
I^{\mu\nu}(\xi) = \frac{\sqrt{-g}}{\kappa} V'(\ell_{\text{eff}}) \left[ \bar{\xi}_{\rho} \bar{V}^{\rho(\mu} \bar{V}^{\nu)\lambda} - \xi^{[\mu} \bar{V}^{\nu] \lambda} + \xi^{[\mu} \bar{V}^{\nu]} \kappa + \kappa^{[\mu} \bar{V}^{\nu]} \xi^{\lambda} + \frac{1}{2} \kappa \bar{V}^{[\mu} \bar{V}^{\nu]} \right].
\end{equation}

### 3.3. Mass of the Schwarzschild-AdS black hole
Now, we will check the formula (25) to calculate the mass of the Schwarzschild-AdS black hole. After substituting its metric in the form:
\begin{equation}
\begin{aligned}
\tilde{ds}^2 &= -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 \sum_{a,b} q_{ab}dx^a dx^b
\end{aligned}
\end{equation}
into the vacuum equations (20) and integrating their \(rr\)-component one obtains
\begin{equation}
\begin{aligned}
\sum_{p=0}^{m} \frac{\alpha_p}{(D-2p-1)!} \left( \frac{1 - f}{f^2} \right)^p &= \frac{m_0}{(D-3)!} r^{D-3},
\end{aligned}
\end{equation}
where \(m_0\) is a constant of integration (mass parameter). In general case it is impossible to derive \(f\), However, we assume that there is a black hole solution. Then one can define asymptotics of \(f\). In this case we have a possibility to define the asymptotics of perturbations:
\begin{equation}
\Delta f = f - \tilde{f} \sim \frac{1}{V'(\ell_{\text{eff}})} \frac{m_0}{r^{D-3}}, \quad \kappa_{00} \sim -\Delta f, \quad \kappa_{11} \sim -\Delta f \tilde{f}^{-2}.
\end{equation}
To calculate the mass we choose a displacement vector as a timelike Killing vector \(\xi^\mu = (-1,0,...,0)\), take perturbations (28) and the background metric (22). Then we substitute the superpotential (25) into (16) and obtain:
\begin{equation}
M = \lim_{r \to \infty} \frac{1}{2\kappa} \int_{\Sigma_{\rho}} dx^{D-2} I^{01} = \frac{D-2}{2\kappa} m_0 \frac{1}{V'(\ell_{\text{eff}})} \int_{\Sigma_{\rho}} dx^{D-2} \sqrt{\det q_{ij}} = \frac{D-2}{4\kappa G_D} m_0.
\end{equation}
It is the standard result that has been obtained in many researches, see, for example, [14-16].

### 4. The pure Lovelock gravity and conserved quantities for the Vaidya type black holes
The Lovelock theory of the general type (17) has \(p+1\) connection constants. In this case, a physical interpretation of them becomes unclear. Therefore, in many researches authors reduce the number of these constants by various ways. One of the very popular ways is to preserve \(\alpha_0\) (it can be \(\alpha_0 = 0\) or \(\alpha_0 \neq 0\)) and the unique \(\alpha_p \neq 0\) from other constants in (17). Such a theory is called a pure Lovelock gravity. The key expression (19), which totally determines the superpotential (14), is simplified
correspondently to this assumption, namely, one has the term with \( \alpha \), and the other term with \( \alpha_p \neq 0 \) without summing \( \Sigma \). In our calculations to simplify expressions we use the redefinition of the connection constants: \( \alpha_0(D - 1)^{-1}(D - 2)^{-1} \equiv - \ell^{-2} \) and \( \alpha_p(D - 3)![(D - 2p - 1)!]^{-1} \equiv \ell^{2p-2} \).

4.1. Generalized Vaidya solutions

Below, applying the field-theoretical formalism, we calculate conserved quantities both for static [9] and for dynamic black holes of the Vaidya type constructed in [10]. These solutions are presented in the general form:

\[
ds^2 = -f(r,v)dv^2 + 2dvdr + r^2 q_\alpha dx^\alpha dx^\beta,
\]

where the null coordinate \( v = x^0 \), and \( r = x^1 \). In this subsection we outline main properties of these solutions. It is known that for the classic Vaidya solution one considers the energy-momentum tensor of the null fluid: \( T_{\alpha\beta} = \mu n_\alpha n_\beta \), where \( n_\alpha \) are ingoing unit null vectors. The authors of [10] have generalized this energy-momentum in the pure Lovelock gravity to

\[
T_{\alpha\beta} = \mu n_\alpha n_\beta + P_r (n_\alpha l_\beta + l_\alpha n_\beta) + P_\sigma r^2 q_{\alpha\beta}.
\]

Here, \( l^\alpha \) are outgoing unit null vectors; \( P_r = -C(v)r^{-(D-1)(1-\sigma)} \) is a radial pressure of the null fluid with undetermined up to now function \( C(v) \); \( P_\sigma = -\sigma P_r \) is a tangential pressure of the null fluid with a constant \( \sigma \). The energy density is defined as \( \mu = \kappa^{-1} \left( \dot{m}(v) + \dot{C}(v) \Theta(r) \right) r^{-(D-2)} \). The dominant energy conditions lead to the next restrictions: \( \mu \geq 0 \), \( C(v) \leq 0 \) and \(-1 \leq \sigma \leq 0 \). More details for outline of \( \Theta(r) \) are as follows: first, \( \sigma = -1/(D-2) \rightarrow \Theta(r) = \ln(r) \), second, \(-1 \leq \sigma \leq 0 \) when \( \sigma \neq -1/(D-2) \rightarrow \Theta(r) = r^{(D-2)\sigma+1}/[(D-2)\sigma+1] \). The authors of [10] have found the general solutions in the pure Lovelock gravity with the matter presented by the energy-momentum (31). Among these solutions we chose only the solutions which present black holes and we do not consider naked singularities and other exotic objects. These black hole solutions 1) with \( 1/\ell^2 < 0 \) have AdS asymptotics; 2) with \( 1/\ell^2 > 0 \) have dS asymptotics and 3) with \( 1/\ell^2 = 0 \) have a flat asymptotics. In below subsections we derive concrete formulae for aforementioned solutions.

4.2. General formulae

Here, before considering the aforementioned variants of black holes we give the general formulae, which are valid for all the cases. The background is chosen in the Eddington-Finkelstein coordinates in the form:

\[
ds^2 = -\bar{f}(r)dr^2 + 2dvdr + r^2 q_\alpha dx^\alpha dx^\beta.
\]

Non-zero components of metric perturbations in the general form are \( \kappa_{00} = -\bar{f} + \bar{f} = -\Delta f \) and \( \kappa^1_0 = \kappa^{11} = \kappa_{00} \). The displacement vector we chose in the form preserving the spherical symmetry, but without other restrictions: \( \xi^\alpha = (\xi^0(v,r),\xi^1(v,r),0,...,0) \). Then, keeping in mind all the above, we found out that the superpotential (25) has the unique non-zero component:

\[
I^{01}(\xi) = -I^{10}(\xi) = (D-2) \frac{p \sqrt{\det q_{ij}}}{2\kappa} r^{D-3} \left[ \frac{\alpha^2}{r^2} \left( 1 - \frac{\alpha}{r} \right) \right]^{p-1} \Delta f \xi_1.
\]

Then the components of the current (12) are only: \( I^0(\xi) = \partial_1 I^{01}(\xi) \) and \( I^1(\xi) = \partial_0 I^{10}(\xi) \).
For the background (32) it is convenient to consider sections $\Sigma := v = \text{const}$. If one chooses a timelike Killing vector $\vec{\xi}^\alpha = (-1, 0, \ldots, 0)$ the charge (16) with (33) presents the quasi-local energy when $r = r_0$ or global energy when $r \to \infty$:

$$P(\vec{\xi}) = \int_{\partial \Sigma} dx^{D-2} f_0^1(\vec{\xi})$$  \hspace{1cm} (34)

where $\partial \Sigma$ is the boundary of $\Sigma$ and is an intersection of $\Sigma$ with the surface $r = \text{const}$; the quantity (34) is considered at each of the moment $v$. The flux of the energy (34) through $\partial \Sigma$ is defined by

$$\dot{P}(\vec{\xi}) = -\int_{\partial \Sigma} dx^{D-2} I^1(\vec{\xi})$$  \hspace{1cm} (35)

that can be obtained by differentiating (34) with respect to $v$. The minus sign in (35) is chosen because the direction of coordinate $r$ is opposite to a contraction of black hole.

The field-theoretical formalism allows us to use arbitrary curved backgrounds. For studying the black holes with the use of (34) and (35) we can consider the background metric (32), first, as a metric of asymptotic maximally symmetric spacetime with respect to both dynamic and static black holes. Second, the background metric (32) can be considered as a metric of static black holes (not maximally symmetric) with respect to the metric of dynamic black holes. The other advantage of the field-theoretical methods is that one can use arbitrary displacement vectors. Together with the Killing vectors $\vec{\xi}$ we use proper vectors of freely falling observers $\vec{\xi}$ on the background of static black holes. As perturbations we consider solutions for related dynamic black holes and construct for them components $I^0(\vec{\xi})$ (energy density) and $I^1(\vec{\xi})$ (density of energy flux) measured by the observers. Below we have a possibility only briefly to outline these results, for a more detail see [12].

4.3. Black holes with AdS asymptotics. The case $1/\ell^2 < 0$

For black holes with AdS asymptotics it is natural to choose the AdS space as a background:

$$\tilde{f}(r) = 1 + \frac{r^2}{\alpha^{2-2/p} \left| \frac{1}{\ell^2} \right|}.$$  \hspace{1cm} (36)

The static black hole is describe by

$$f(r) = 1 + \frac{r^2}{\alpha^{2-2/p} \left( \left| \frac{1}{\ell^2} \right| - \frac{m_0}{r^{D-1}} \right)^{\frac{1}{p}}}.$$  \hspace{1cm} (37)

whereas the dynamic of Viadya type black hole is presented by

$$f(r, v) = 1 + \frac{r^2}{\alpha^{2-2/p} \left| \frac{1}{\ell^2} \right| - \frac{m_0 + m(v)}{r^{D-1}} - \frac{C(v)\Theta(r)}{r^{D-1}}}.$$  \hspace{1cm} (38)

It was calculated the global mass for (37) on the background (36) that exactly coincides with the standard result (29). The mass of the dynamic black hole (38) on the background (36) is $P(\vec{\xi}) \big|_{v=0} = (D-2)(m_0 + m(v)) / 4G_P$. Besides, our formalism allows us to calculate the mass of (38) on the background (37), it is $P(\vec{\xi}) \big|_{v=0} = (D-2)m(v) / 4G_P$. First, the calculation of global mass in the AdS case restricts the parameters of the null fluid. The requirement of the finiteness leaves only: $-1 \leq \sigma < -1/(D-2) \to \Theta(r) = r^{(D-2)\sigma+1}/[(D-2)\sigma+1]$. Second, the pressure does not contribute to the global expressions. Third, the global energy of the dynamic black holes turns out additive as calculated on the backgrounds (36) and (37).

Analogously to the global mass it was calculated the quasi-local mass at the finite radius $r = r_0$ in the three aforementioned cases. This gives a possibility for the next conclusions. First, all the expressions depend on the radial pressure, although one has to preserve the previous restrictions for the null fluid:
$-1 \leq \sigma < -1/(D-2) \rightarrow \Theta(r) = r^{(D-2)\sigma+1}/[(D-2)\sigma + 1]$. Second, there is no additivity discussed above. Third, the quasi-local energy expressions calculated outside of the horizons allows us to conclude that the density of the total (energy perturbations and null matter) energy is negative. This result generalizes the textbook [17] conclusions in general relativity on the negative energy density of the potential gravitational field.

4.4. Black holes with dS asymptotics. The case $1/\ell^2 > 0$

For black holes with dS asymptotics it is natural to choose the dS space as a background:

$$\bar{f}(r) = 1 - \frac{r^2}{\alpha^{2-2/p}} \frac{1}{\ell^2}.$$  \hspace{1cm} (39)

In this case one cannot consider the asymptotics at $r \rightarrow \infty$ because there is a cosmological horizon. For the static solution

$$f(r) = 1 - \frac{r^2}{\alpha^{2-2/p}} \left( \frac{1}{\ell^2} + \frac{m_0}{r^{D-1}} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (40)

we consider only the situation when the parameters allow the existence of the cosmological horizon and the black hole event horizon. We do not consider other variants leading to exotic objects. Concerning the dynamic solution

$$f(r, v) = 1 - \frac{r^2}{\alpha^{2-2/p}} \left( \frac{1}{\ell^2} + \frac{m_0 + m(v)}{r^{D-1}} + \frac{C(v)\Theta(r)}{r^{D-1}} \right)^{\frac{1}{2}},$$  \hspace{1cm} (41)

we restrict the time interval of falling the null matter to preserve the relation of parameters necessary for the existence of the mentioned two horizons.

Again both the solutions (40) and (41) can be considered on the background of (39). Besides, the dynamic black hole (41) can be considered on the background of the related static black hole (40). By the impossibility to provide the asymptotics at $r \rightarrow \infty$ we cannot construct global conserved quantities. As a result we cannot introduce additional restrictions for parameters of null matter. Thus, we construct only quasi-local conserved energy at finite $r = r_0$ in all the three cases. We have found the next results.

First, the same as in the AdS case, all the expressions depend on the radial pressure. Second, there is no additivity for the quasi-local charges. Third, the quasi-local energy expressions calculated outside of the event horizons allows us to conclude that the density of the total (metric perturbations and null matter) energy is positive. It is unusual result. However, it is not surprising. Indeed, analogous exotic conclusions frequently appear in literature when black holes with dS asymptotics are considered, see, for example, [18,19].

4.5. Black holes with flat asymptotics. The case $1/\ell^2 = 0$

For the black holes with a flat asymptotics one has

$$\bar{f}(r) = 1.$$  \hspace{1cm} (42)

The related static black holes are defined by

$$f(r) = 1 - \frac{r^2}{\alpha^{2-2/p}} \left( \frac{m_0}{r^{D-1}} \right)^{\frac{1}{2}},$$  \hspace{1cm} (43)

whereas for the dynamic black hole one has

$$f(r, v) = 1 - \frac{r^2}{\alpha^{2-2/p}} \left( \frac{m_0 + m(v)}{r^{D-1}} + \frac{C(v)\Theta(r)}{r^{D-1}} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (44)

It is impossible to calculate the charges on the background (42). The reason is that the superpotential (33) vanishes for (42) when $p \neq 1$. However, a special trick, see [12], allows us to calculate the global
mass only, not a quasi-local mass. Thus for the static solution (43) we have again the standard result (29), and for the dynamic solution (44) one has \( P(\bar{\xi}) \bigg|_{v \to \infty} = (D - 2)(m_0 + m(v))/4G_D \) at each of the moment \( v \). At the same time, we calculate without difficulties the global and quasi-local mass for (44) on the background (44). Finally, we find that there is no the additivity of both global and quasi-local energies. Considering the quasi-local energy the density of the total (metric perturbations and null matter) energy is negative. But it is determined only by the pressure of the null fluid, when the pressure is absent one has a zero total energy density. In this case the energy density of the gravitational field and null matter are compensated exactly.

5. Concluding remarks
In the present format, there is no a possibility to include the results giving fluxes of energy of various kinds noted in (35). It is impossible to give the very complicated expressions for energy density and density of energy flux measured by freely falling observers on the geometry of static black holes. For a detail see [12].

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