Behavior of eigenvalues of certain Schrödinger operators in the rational Dunkl setting

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Abstract

For a normalized root system $R$ in $\mathbb{R}^N$ and a multiplicity function $k \geq 0$ let $N = N + \sum_{\alpha \in R} k(\alpha)$. We denote by $d w(x) = \prod_{\alpha \in R} |\langle x, \alpha \rangle|^{k(\alpha)} d x$ the associated measure in $\mathbb{R}^N$. Let $L = -\Delta + V$, $V \geq 0$, be the Dunkl–Schrödinger operator on $\mathbb{R}^N$. Assume that there exists $q > \max(1, N/2)$ such that $V$ belongs to the reverse Hölder class $RH^q(d w)$. For $\lambda > 0$ we provide upper and lower estimates for the number of eigenvalues of $L$ which are less or equal to $\lambda$. Our main tool in the Fefferman–Phong type inequality in the rational Dunkl setting.

Keywords  Rational Dunkl theory · Schrödinger operators · Asymptotic distributions of eigenvalues · Reverse Hölder classes · Fefferman–Phong inequality

Mathematics Subject Classification  35P20 · 35J10 · 42B37 · 35K08 · 42B35

1 Introduction

The current article is devoted to study the estimates for the number of eigenvalues of Schrödinger operators in the Dunkl setting. In the seminal article [8], Charles F. Dunkl defined new commuting differential–difference operators

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}$$
associated with a finite reflection group $G$ which is related to a root system $R$ on a Euclidean space $\mathbb{R}^N$. Here $\xi \in \mathbb{R}^N$, $\sigma_\alpha$ denotes the reflection with respect to the hyperplane orthogonal to the root $\alpha \in R$, and $k : R \rightarrow \mathbb{C}$ is a $G$-invariant function (see Sect. 2 for details). The Dunkl operators are generalizations of the directional derivatives (in fact, they are ordinary partial derivatives for $k \equiv 0$), however, in general, they are non-local operators. They turn out to be a key tool in the study of special functions with reflection symmetries and allow to build up the framework for the theory of special functions and integral transforms in several variables related with reflection groups in [7–11]. Afterwards, the theory was studied and developed by many mathematicians from many different points of view. Beside the special functions and mathematical analysis, the Dunkl theory has deep connections with the other branches of mathematics, for instance probability theory, mathematical physics, and algebra. It is worth to mention that one of a motivation for studying Dunkl operators is their relevance for the analysis of quantum many body systems of Calogero–Moser–Sutherland type. Such models describe algebraically integrable systems in one dimension and have gained considerable interest of theoretical physics, especially in conformal field theory. For more details on Calogero–Moser–Sutherland we refer the reader to [28]. For more information on Dunkl theory and its connections with mathematical physics we refer to the lecture notes [4,19] and the references therein.

In the present paper we consider the Dunkl–Schrödinger operator

$$L = -\Delta + V$$ on $\mathbb{R}^N$, $N \geq 1,$

where $V \in L^2_{\text{loc}}(dw)$ is a non-negative potential and $\Delta = \sum_{j=1}^N T_{e_j}^2$ is the Dunkl Laplacian. Here and subsequently, $\{e_j\}_{1 \leq j \leq N}$ is the canonical orthonormal basis in $\mathbb{R}^N$. Such operators were recently studied by Amri and Hammi in [1,2]. Assume that $V$ is a non-negative function, $V \not\equiv 0$, which satisfies the reverse Hölder inequality:

$$\left( \frac{1}{w(B)} \int_B V(y)^q \, dw(y) \right)^{1/q} \leq \frac{C_{\text{RH}}}{w(B)} \int_B V(y) \, dw(y)$$ for every ball $B$.  \hfill (1)

Here and subsequently,

$$dw(x) = \prod_{\alpha \in R} |\langle x, \alpha \rangle|^{k(\alpha)} \, dx$$  \hfill (2)

is the associated measure. We shall assume that $q > \max(1, \frac{N}{2})$, where $N = N + \sum_{\alpha \in R} k(\alpha)$ is the homogeneous dimension (see Sect. 2 for details). Following [14, p. 146, the assumption of the main lemma] (see also [15, (42)] for its counterpart in the rational Dunkl setting) we define the auxiliary function $m(x)$ by the formula

$$\frac{1}{m(x)} = \sup \left\{ r > 0 : \frac{r^2}{w(B(x, r))} \int_{B(x, r)} V(y) \, dw(y) \leq 1 \right\}.  \hfill (3)$$

The function $m$ is well defined and satisfies $0 < m(x) < \infty$ for every $x \in \mathbb{R}^N$, see [15].
For $\lambda > 0$, we denote by $N(L, \lambda)$ the number of eigenvalues of the operator $L$, counting with their multiplicities, which are less than or equal to $\lambda$.

For $a > 0$ we define

$$\text{(Grid)}_a = \{[0, a]^N + an : n \in \mathbb{Z}^N\}.$$  \hfill (4)

Our goal is to prove the following theorem.

**Theorem 1** Assume that $0 \not\equiv V \in RH^q(dw)$, where $q > \max(1, \frac{N}{2})$, and $V \geq 0$. For $\lambda > 0$ we set

$$E_\lambda = \{x \in \mathbb{R}^N : m(x) \leq \sqrt{\lambda}\}.$$  

Let $M(\lambda)$ denote the number of cubes $K$ from the $(\text{Grid})_{\lambda^{-1/2}}$ [see (4)] such that $K \cap E_\lambda \neq \emptyset$. There are constants $C_1, C_2, C_3 > 0$, which depend on $R, N, q, k$ and the constant $C_{RH}$ [see (1)] such that for all $\lambda > 0$ we have

$$M(C_1^{-1}\lambda) \leq N(L, \lambda) \leq C_2 M(C_3^{-1}\lambda).$$ \hfill (5)

Let $\lambda_0(L)$ denote the smallest eigenvalue of $L$. There is a constant $C_4 > 0$, which depends on $R, N, q, k$ and the constant $C_{RH}$, such that

$$\lambda_0(L) \geq C_4 \min_{x \in \mathbb{R}^N} m(x).$$ \hfill (6)

Actually, one can take $C_4 = (\tilde{C})^{1/2}$, where $\tilde{C}$ is the constant from (23).

For classical Schrödinger operators with reverse Hölder class potentials on $\mathbb{R}^N$ behavior of eigenvalues were studied in the seminal article Fefferman [14] (nowadays, the result is known as ‘Fefferman’s box formula’) and then continued by many authors (see e.g. [16,23–26]). The present article takes inspirations from there.

Let us explain some main difficulties in Dunkl analysis, which distinguish it from the classical Euclidean setting. As it was pointed out in [27], one of the most serious problem in the Dunkl analysis lays in the lack of knowledge about generalized translation $\tau_x$. Let us discuss this phenomenon. The Dunkl translations $\tau_x$ are generalizations of the ordinary translations

$$\tilde{\tau}_x f(-y) = f(x - y) = (e^{(ix, \cdot)} \hat{f}(\cdot))(-y)$$

and reduce to them if the multiplicity function is equal to 0. Replacing in the formula the Fourier and the inverse Fourier transform by the Dunkl transform $\mathcal{F}$ and inverse Dunkl transform $\mathcal{F}^{-1}$ respectively [see (13) and (15)], and the exponential function $e^{(ix, \xi)}$ by $E(i\mathbf{x}, \xi)$ [see (11)], one ends up with the generalized translation operator defined by

$$\tau_x f(-y) = \mathcal{F}^{-1}(E(i\mathbf{x}, \cdot)\mathcal{F} f(\cdot))(-y),$$
which is a contraction on $L^2(dw)$. Although the definition seems to be natural, no explicit useful formula for $\tau_x$ is known in full generality and many properties of $\tau_x$ turn out to be hard to study. For instance, for some root systems the operators $\tau_x$ do not preserve positive functions. The other aspect is that they do not form the group, that means that $\tau_{x+y}$ is not necessarily equal to $\tau_x \circ \tau_y$. Further, the boundedness of $\tau_x$ on $L^p(dw)$-spaces ($p \neq 2$) becomes an open problem in the Dunkl analysis.

The other difficulty when we try to adapt the Fefferman’s ideas in the Dunkl context is the lack of knowledge about the Poincaré’s inequality, which is the main ingredient of the proof in the classical case. To overcome this problem, we use the methods presented in the proof of Fefferman–Phong inequality in the Dunkl setting in [15] together with Fefferman–Phong inequality itself. The idea is inspired by Dziubański [13, Section 9] and it is combined with an appropriate use of the pseudo-Poincaré inequality defined in [29, Section 5].

### 2 Preliminaries and notation

In this section we present necessary definitions and lemmas (with references), which will be used in the proof of Theorem 1.

#### 2.1 Basic definitions of the Dunkl theory

In this subsection we present basic facts concerning the theory of the Dunkl operators. For details we refer the reader to [8,19,20].

We consider the Euclidean space $\mathbb{R}^N$ with the scalar product $\langle x, y \rangle = \sum_{j=1}^{N} x_j y_j$, where $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$, and the norm $\|x\|^2 = \langle x, x \rangle$. For a nonzero vector $\alpha \in \mathbb{R}^N$, the reflection $\sigma_\alpha$ with respect to the hyperplane $\alpha^\perp$ orthogonal to $\alpha$ is given by

$$\sigma_\alpha(x) = x - 2 \frac{\langle x, \alpha \rangle}{\|\alpha\|^2} \alpha.$$

In this paper we fix a normalized root system in $\mathbb{R}^N$, that is, a finite set $R \subset \mathbb{R}^N \setminus \{0\}$ such that $R \cap \alpha \mathbb{R} = \{ \pm \alpha \}$, $\sigma_\alpha(R) = R$, and $\|\alpha\| = \sqrt{2}$ for all $\alpha \in R$. The finite group $G$ generated by the reflections $\sigma_\alpha \in R$ is called the Weyl group (reflection group) of the root system. A *multiplicity function* is a $G$-invariant function $k : R \to \mathbb{C}$ which will be fixed and $\geq 0$ throughout this paper. Let

$$dw(x) = \prod_{\alpha \in R} |\langle x, \alpha \rangle|^{k(\alpha)} \, dx$$

be the associated measure in $\mathbb{R}^N$, where, here and subsequently, $dx$ stands for the Lebesgue measure in $\mathbb{R}^N$. We denote by

$$N = N + \sum_{\alpha \in R} k(\alpha)$$
the homogeneous dimension of the system. Clearly,

\[ w(B(tx, tr)) = t^N w(B(x, r)) \quad \text{for all } x \in \mathbb{R}^N, \ t, r > 0, \]

where \( B(x, r) = \{ y \in \mathbb{R}^N : \| y - x \| < r \} \). Moreover,

\[
\int_{\mathbb{R}^N} f(x) \, dw(x) = \int_{\mathbb{R}^N} t^{-N} f(x/t) \, dw(x) \quad \text{for } f \in L^1(dw) \text{ and } t > 0.
\]

Observe that there is a constant \( C > 0 \) such that

\[ C^{-1} w(B(x, r)) \leq r^N \prod_{\alpha \in R} (|\langle x, \alpha \rangle| + r)^{k(\alpha)} \leq C w(B(x, r)), \tag{7} \]

so \( dw(x) \) is doubling, that is, there is a constant \( C > 0 \) such that

\[ w(B(x, 2r)) \leq C w(B(x, r)) \quad \text{for all } x \in \mathbb{R}^N, \ r > 0. \tag{8} \]

Moreover, there exists a constant \( C \geq 1 \) such that, for every \( x \in \mathbb{R}^N \) and for every \( r_2 \geq r_1 > 0 \),

\[ C^{-1} \left( \frac{r_2}{r_1} \right)^N \leq \frac{w(B(x, r_2))}{w(B(x, r_1))} \leq C \left( \frac{r_2}{r_1} \right)^N. \tag{9} \]

For a measurable subset \( A \) of \( \mathbb{R}^N \) we define

\[ \mathcal{O}(A) = \{ \sigma_\alpha(x) : x \in A, \ \alpha \in R \}. \]

Clearly, by (7), for all \( x \in \mathbb{R}^N \) and \( r > 0 \) we get

\[ w(\mathcal{O}(B(x, r))) \leq |G| w(B(x, r)). \]

For \( \xi \in \mathbb{R}^N \), the Dunkl operators \( T_{\xi} \) are the following \( k \)-deformations of the directional derivatives \( \partial_\xi \) by a difference operator:

\[ T_{\xi} f(x) = \partial_\xi f(x) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}. \tag{10} \]

The Dunkl operators \( T_{\xi} \), which were introduced in [8], commute and are skew-symmetric with respect to the \( G \)-invariant measure \( dw \).

For fixed \( y \in \mathbb{R}^N \) the Dunkl kernel \( E(x, y) \) is the unique analytic solution to the system

\[ T_{\xi} f = \langle \xi, y \rangle f, \quad f(0) = 1. \tag{11} \]
The function $E(x, y)$, which generalizes the exponential function $e^{(x,y)}$, has the unique extension to a holomorphic function on $\mathbb{C}^N \times \mathbb{C}^N$. Moreover, it satisfies $E(x, y) = E(y, x)$ for all $x, y \in \mathbb{C}^N$.

Let $\{e_j\}_{1 \leq j \leq N}$ denote the canonical orthonormal basis in $\mathbb{R}^N$ and let $T_j = T_{e_j}$. As usual, for every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{N}^N_0 = (\mathbb{N} \cup \{0\})^N$, we set $|\alpha| = \sum_{j=1}^{N} \alpha_j$ and
\[ \partial^\alpha = \partial_{e_1}^{\alpha_1} \circ \partial_{e_2}^{\alpha_2} \circ \ldots \circ \partial_{e_N}^{\alpha_N}, \]
where $\{e_1, e_2, \ldots, e_N\}$ is the canonical basis of $\mathbb{R}^N$. The additional subscript $x$ in $\partial^\alpha_x$ means that the partial derivative $\partial^{\alpha}$ is taken with respect to the variable $x \in \mathbb{R}^N$. By $\nabla_x f$ we denote the gradient of the function $f$ with respect to the variable $x$. In our further consideration we shall need the following lemma.

**Lemma 1** For all $x \in \mathbb{R}^N$, $z \in \mathbb{C}^N$ and $\nu \in \mathbb{N}^N_0$ we have
\[ |\partial^\nu_z E(x, z)| \leq \|x\| |\nu| \exp(\|x\| \|\Re z\|). \]

In particular,
\[ |E(i\xi, x)| \leq 1 \quad \text{for all } \xi, x \in \mathbb{R}^N. \]

**Proof** See [17, Corollary 5.3].

**Corollary 1** There is a constant $C > 0$ such that for all $x, \xi \in \mathbb{R}^N$ we have
\[ |E(i\xi, x) - 1| \leq C \|x\| \|\xi\|. \quad (12) \]

The Dunkl transform
\[ \mathcal{F} f(\xi) = c_k^{-1} \int_{\mathbb{R}^N} E(-i\xi, x) f(x) dw(x), \quad (13) \]
where
\[ c_k = \int_{\mathbb{R}^N} e^{-\frac{\|x\|^2}{2}} dw(x) > 0, \]
originally defined for $f \in L^1(dw)$, is an isometry on $L^2(dw)$, i.e.,
\[ \|f\|_{L^2(dw)} = \|\mathcal{F} f\|_{L^2(dw)} \quad \text{for all } f \in L^2(dw), \quad (14) \]
and preserves the Schwartz class of functions $\mathcal{S}(\mathbb{R}^N)$ (see [6]). Its inverse $\mathcal{F}^{-1}$ has the form
\[ \mathcal{F}^{-1} g(x) = c_k^{-1} \int_{\mathbb{R}^N} E(i\xi, x) g(\xi) dw(\xi). \quad (15) \]
Moreover,

\[ \mathcal{F}(T_j f)(\xi) = i \xi_j \mathcal{F} f(\xi). \]  

(16)

The Dunkl translation \( \tau_x f \) of a function \( f \in \mathcal{S}(\mathbb{R}^N) \) by \( x \in \mathbb{R}^N \) is defined by

\[ \tau_x f(y) = c_k^{-1} \int_{\mathbb{R}^N} E(i \xi, x) E(i \xi, y) \mathcal{F} f(\xi) \, dw(\xi). \]

It is a contraction on \( L^2(dw) \), however it is an open problem if the Dunkl translations are bounded operators on \( L^p(dw) \) for \( p \neq 2 \).

The following specific formula was obtained by Rösler [18] for the Dunkl translations of (reasonable) radial functions \( f(x) = \tilde{f}(\|x\|) \):

\[ \tau_x f(-y) = \int_{\mathbb{R}^N} (\tilde{f} \circ A)(x, y, \eta) \, d\mu_x(\eta) \quad \text{for all} \quad x, y \in \mathbb{R}^N. \]  

(17)

Here

\[ A(x, y, \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2 \langle y, \eta \rangle} = \sqrt{\|x\|^2 - \|\eta\|^2 + \|y - \eta\|^2} \]

and \( \mu_x \) is a probability measure, which is supported in the set \( \text{conv}\mathcal{O}(x) \), where \( \mathcal{O}(x) = \{\sigma(x) : \sigma \in G\} \) is the orbit of \( x \). Let

\[ d(x, y) = \min_{\sigma \in G} \|\sigma(x) - y\| \]

be the distance of the orbit of \( x \) to the orbit of \( y \). We have the following elementary estimates (see, e.g., [3]), which hold for \( x, y \in \mathbb{R}^N \) and \( \eta \in \text{conv}\mathcal{O}(x) \):

\[ A(x, y, \eta) \geq d(x, y) \]

and

\[
\begin{align*}
\|\nabla_y [A(x, y, \eta)^2]\| & \leq 2 A(x, y, \eta), \\
|\partial_\beta^x \{A(x, y, \eta)^2\}| & \leq 2 \quad \text{if} \quad |\beta| = 2, \\
\partial_\beta^x \{A(x, y, \eta)^2\} & = 0 \quad \text{if} \quad |\beta| > 2.
\end{align*}
\]

Hence

\[ \|\nabla_y A(x, y, \eta)\| \leq 1. \]  

(18)

The Dunkl convolution \( f \ast g \) of two reasonable functions (for instance Schwartz functions) is defined by

\[ (f \ast g)(x) = c_k \mathcal{F}^{-1}[(\mathcal{F} f)(\mathcal{F} g)](x) = \int_{\mathbb{R}^N} (\mathcal{F} f)(\xi) (\mathcal{F} g)(\xi) E(x, i \xi) \, dw(\xi) \quad \text{for} \quad x \in \mathbb{R}^N. \]  

(19)
or, equivalently, by
\[(f \ast g)(x) = \int_{\mathbb{R}^N} f(y) \tau_x g(-y) \, dw(y) = \int_{\mathbb{R}^N} f(y) g(x, y) \, dw(y)\]
for all \(x \in \mathbb{R}^N\),
where, here and subsequently, \(g(x, y) = \tau_x g(-y)\).

### 2.2 Dunkl Laplacian and Dunkl heat semigroup

The Dunkl Laplacian associated with \(R\) and \(k\) is the differential–difference operator \(\Delta_n = \sum_{j=1}^{N} T_{j}^{2}\), which acts on \(C^2(\mathbb{R}^N)\)-functions by
\[
\Delta f(x) = \Delta_{euc} f(x) + \sum_{\alpha \in \mathbb{R}^k} k(\alpha) \delta_{\alpha} f(x),
\]

\[
\delta_{\alpha} f(x) = \frac{\partial^{2} f(x)}{\|\alpha\|^2} \frac{(f(x) - f(\sigma_{\alpha} x))}{\langle\alpha, x\rangle^2}.
\]

Obviously, \(\mathcal{F}(\Delta f)(\xi) = -\|\xi\|^2 \mathcal{F} f(\xi)\). The operator \(\Delta\) is essentially self-adjoint on \(L^2(dw)\) (see for instance [1, Theorem 3.1]) and generates the semigroup \(H_t\) of linear self-adjoint contractions on \(L^2(dw)\). The semigroup has the form
\[
H_t f(x) = \mathcal{F}^{-1} (e^{-t\|\xi\|^2} \mathcal{F} f(\xi))(x) = \int_{\mathbb{R}^N} h_t(x, y) f(y) \, dw(y),
\]
where the heat kernel
\[
h_t(x, y) = \tau_x h_t(-y), \quad h_t(x) = \mathcal{F}^{-1} (e^{-t\|\xi\|^2}) f(\xi)(x) = c_k^{-1}(2t)^{-N/2} e^{-\|x\|^2/(4t)}
\]
is a \(C^\infty\)-function of all variables \(x, y \in \mathbb{R}^N, t > 0\), and satisfies
\[
0 < h_t(x, y) = h_t(y, x), \quad \int_{\mathbb{R}^N} h_t(x, y) \, dw(y) = 1.
\]

We shall need the following estimates for \(h_t(x, y)\)—their two step proof, which is based on Rösler’s formula (17) for the Dunkl translations of radial functions (see [18]), can be found in [5, Theorem 4.1], [12, Theorem 3.1].

**Theorem 2** There are constants \(C, c > 0\) such that for all \(x, y \in \mathbb{R}^N\) and \(t > 0\) we have
\[
h_t(x, y) \leq C \left(1 + \frac{\|x - y\|}{\sqrt{t}}\right)^{-2} \left(\max(w(B(x, \sqrt{t})), w(B(y, \sqrt{t}))\right)^{-1} \exp\left(-\frac{cd(x, y)^2}{t}\right).
\]

Theorem 2 implies the following Lemma (see [12, Corollary 3.5]).
Lemma 2  Suppose that $\varphi \in C^\infty_c(\mathbb{R}^N)$ is radial and supported by the unit ball $B(0, 1)$. Here and subsequently, set $\varphi_t(x) = t^{-N}\varphi(t^{-1}x)$. Then there is $C > 0$ such that for all $x, y \in \mathbb{R}^N$ and $t > 0$ we have

$$|\varphi_t(x, y)| \leq C \left(1 + \frac{\|x - y\|}{t}\right)^{-2} \left(\max(w(B(x, t)), w(B(y, t)))\right)^{-1} \chi_{[0, 1]}(d(x, y)/t).$$

2.3 Dunkl–Schrödinger operator

Let $V \geq 0$ be a measurable function such that $V \in L^2_{\text{loc}}(dw)$. We consider the following operator on the Hilbert space $L^2(dw)$:

$$\mathcal{L} = -\Delta + V$$

with the domain

$$\mathcal{D}(\mathcal{L}) = \{ f \in L^2(dw) : \|\xi\|^2 \mathcal{F} f(\xi) \in L^2(dw(\xi)) \text{ and } V(x)f(x) \in L^2(dw(x))\}$$

(see [1]). We call this operator the Dunkl–Schrödinger operator. Let us define the quadratic form

$$Q(f, g) = \sum_{j=1}^{N} \int_{\mathbb{R}^N} T_j f(x) T_j g(x) \, dw(x) + \int_{\mathbb{R}^N} V(x)f(x)g(x) \, dw(x) \quad (20)$$

with the domain

$$\mathcal{D}(Q) = \left\{ f \in L^2(dw) : \left(\sum_{j=1}^{N} |T_j f|^2\right)^{1/2}, V^{1/2}f \in L^2(dw) \right\}.$$

The quadratic form is densely defined and closed (see [1, Lemma 4.1]), so there exists a unique positive self-adjoint operator $L$ such that

$$\langle Lf, f \rangle = Q(f, f) \text{ for all } f \in \mathcal{D}(L),$$

moreover,

$$\mathcal{D}(L^{1/2}) = \mathcal{D}(Q) \text{ and } Q(f, f) = \|L^{1/2}f\|_{L^2(dw)},$$

where $L^{1/2}$ is a unique self-adjoint operator such that $(L^{1/2})^2 = L$. It was proved in [1, Theorem 4.6], that $\mathcal{L}$ is essentially self-adjoint on $C^\infty_c(\mathbb{R}^N)$ and $L$ is its closure.
2.4 Auxiliary function \( m \) and the Fefferman–Phong type inequality

The results in this subsection are proved in [15] and they will be used in the proof of Theorem 1. Some of them are inspired by the corresponding results for classical Schrödinger operators (cf. [14,22]).

**Lemma 3** ([15, Lemma 7], see also [22, Lemma 1.2]) Assume that \( V \in \text{RH}^q(dw) \), where \( q > \max(1, \frac{N}{2}) \), and \( V \geq 0 \). There is a constant \( C \geq 1 \) such that for all \( x \in \mathbb{R}^N \) and \( 0 < r_1 < r_2 < \infty \) we have

\[
\frac{r_1^2}{w(B(x, r_1))} \int_{B(x, r_1)} V(y) \, dw(y) \leq C \left( \frac{r_1}{r_2} \right)^\gamma \frac{r_2^2}{w(B(x, r_2))} \int_{B(x, r_2)} V(y) \, dw(y).
\]

**Lemma 4** ([15, Lemma 8], see also [22, Lemma 1.4]) Assume that \( V \in \text{RH}^q(dw) \), where \( q > \max(1, \frac{N}{2}) \), and \( V \geq 0 \). There are constants \( C, \kappa > 0 \) such that for all \( x, y \in \mathbb{R}^N \) we have

\[
C^{-1} m(y) \leq m(x) \leq C m(y) \text{ if } \|x - y\| < m(x)^{-1},
\]

\[
m(y) \leq C m(x)(1 + m(x))\|x - y\|^\kappa,
\]

\[
m(y) \geq C^{-1} m(x)(1 + m(x))\|x - y\|^{-\frac{1}{1+\kappa}}.
\]

For a cube \( Q \subset \mathbb{R}^N \), here and subsequently, let \( d(Q) \) denote the side-length of cube \( Q \). We denote by \( Q^* \) the cube with the same center as \( Q \) such that \( d(Q^*) = 2d(Q) \). We define a collection of dyadic cubes \( Q \) associated with the potential \( V \) by the following stopping-time condition:

\[
Q \in Q \iff Q \text{ is the maximal dyadic cube for which } d(Q)^2 \int_Q V(y) \, dw(y) \leq 1
\]

(see [15, (48)]). It is well-defined (see the comment below [15, (48)] for details) and it forms a covering of \( \mathbb{R}^N \) built from dyadic cubes which have disjoint interiors.

**Fact 3** ([15, Fact 5]) Assume that \( V \in \text{RH}^q(dw) \), where \( q > \max(1, \frac{N}{2}) \), and \( V \geq 0 \). There is a constant \( C > 0 \) such that for any \( Q \in Q \) and \( x \in Q^{**} \) we have

\[
C^{-1} d(Q)^{-1} \leq m(x) \leq C d(Q)^{-1}.
\]

**Proposition 1** ([15, Proposition 3]) Assume that \( V \in \text{RH}^q(dw) \), where \( q > \max(1, \frac{N}{2}) \), and \( V \geq 0 \). The covering \( Q \) defined by (21) satisfies the following finite overlapping condition:

\[
(\exists C_0 > 0) (\forall Q_1, Q_2 \in Q) \ Q_1^{**} \cap Q_2^{**} \neq \emptyset \Rightarrow C_0^{-1} d(Q_1) \leq d(Q_2) \leq C_0 d(Q_1).
\]

(22)
Lemma 5 ([15, Lemma 10]) For all \( j \in \{1, 2, \ldots, N\} \), \( g \in C_c^\infty(\mathbb{R}^N) \), and \( f \in L^2(dw) \) such that its weak Dunkl derivative \( T_j f \) is in \( L^2(dw) \) we have \( T_j (fg) \in L^2(dw) \). Moreover, \[
T_j (fg)(x) = (T_j f)(x)g(x) + f(x)\partial_j g(x) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \alpha_j f(\sigma_\alpha(x)) \frac{g(x) - g(\sigma_\alpha(x))}{(x, \alpha)}
\] in \( L^2(dw) \)-sense.

The following lemma is inspired by its counterpart for fractional laplacian [13, Lemma 9.6].

Lemma 6 ([15, Lemma 12]) Assume that \( V \in RH^q(dw) \), where \( q > \max(1, \frac{N}{2}) \), and \( V \geq 0 \). There is a constant \( C > 0 \) such that for all \( j \in \{1, \ldots, N\} \), \( f \in L^2(dw) \) such that its weak Dunkl derivative \( T_j f \) is in \( L^2(dw) \), and \( Q \in Q \) we have \[
\|T_j (f\phi_Q)\|_{L^2(dw)} \leq C \left( \left( \int_{Q^*} |T_j f(x)|^2 \, dw(x) \right)^{1/2} + \left( \int_{O(Q^*)} |f(x)|^2 m(x)^2 \, dw(x) \right)^{1/2} \right).
\]

The next theorem is a version of Fefferman–Phong inequality ([14, p. 146], see also Shen [21], [22, Lemma 1.9]). It is the main result of [15].

Theorem 4 (Fefferman–Phong type inequality, see [15, Theorem 1]) Assume that \( V \in RH^q(dw) \), where \( q > \max(1, \frac{N}{2}) \), and \( V \geq 0 \). There is a constant \( C > 0 \), which depends on \( R, k, N, q \), and \( C_{RH} \), such that for all \( f \in D(Q) \) we have \[
\int_{\mathbb{R}^N} |f(x)|^2 m(x)^2 \, dw(x) \leq C Q(f, f).
\]

3 Hölder bounds for Dunkl translation of radial function

The next lemma is a version of [5, Theorem 4.1 (b)] with the Dunkl heat kernel replaced by the Dunkl translation of radial \( C_c^\infty(\mathbb{R}^N) \)-function. Its proof is similar to the proof of [5, Theorem 4.1 (b)] and it is based on Rösler’s formula (17).

Lemma 7 Let \( \varphi \in C_c^\infty(\mathbb{R}^N) \) be a radial function supported by the unit ball, that is, \( \varphi(x) = \tilde{\varphi}(\|x\|) \), where \( \tilde{\varphi} \in C_c^\infty(-1, 1) \) is even. There is a constant \( C > 0 \) such that for all \( x, y, z \in \mathbb{R}^N \) and \( t > 0 \) such that \( \|y - z\| < t \) we have \[
|\varphi_t(x, y) - \varphi_t(x, z)| \leq C \frac{\|y - z\|}{t} \left( \max(w(B(x, t)), w(B(y, t))) \right)^{-1} \chi_{[0, 2]}(d(x, y)/t).
\]
The presence of the factor $\chi_{[0,2]}(d(x,y)/t)$ follows by Lemma 2. We may assume that $\|y - z\| < t/8$, otherwise the claim is a consequence of Lemma 2. For $s \in [0, 1]$ we set $y_s = z + s(y - z)$. By (17) we obtain

$$\varphi_t(x, y) - \varphi_t(x, z) = t^{-N} \int_{\mathbb{R}^N} \tilde{\varphi}(A(x, y_s, \eta)/t) - \tilde{\varphi}(A(x, z, \eta)/t) \, d\mu_x(\eta)$$

$$= t^{-N} \int_{\mathbb{R}^N} \int_0^1 \frac{d}{ds} \tilde{\varphi}(A(x, y_s, \eta))/t) \, ds \, d\mu_x(\eta).$$

(24)

Clearly, there is an even continuous function $\tilde{\varphi} : \mathbb{R} \to \mathbb{C}$ supported in $[-3/2, 3/2]$ such that $|\tilde{\varphi}'(x)| \leq \tilde{\varphi}(x)$ for all $x \in \mathbb{R}$. Hence, by Cauchy–Schwarz inequality, (18), and (24), we obtain

$$\left| \frac{d}{ds} \tilde{\varphi}(A(x, y_s, \eta))/t \right| = \left| \tilde{\varphi}(A(x, y_s, \eta))/t \right| \cdot t^{-1} \cdot \left| (\nabla_y A(x, y_s, \eta), (y - z)) \right|$$

$$\leq C \frac{\|y - z\|}{t} \tilde{\varphi}(A(x, y_s, \eta)/t).$$

(25)

Set $\phi(x) = \tilde{\varphi}(|x|)$. Combining (24) and (25) we obtain

$$|\varphi_t(x, y) - \varphi_t(x, z)| \leq Ct^{-N} \frac{\|y - z\|}{t} \int_{\mathbb{R}^N} \int_0^1 \tilde{\varphi}(A(x, y_s, \eta)/t) \, ds \, d\mu_x(\eta)$$

$$= C \frac{\|y - z\|}{t} \int_0^1 \phi_t(x, y_s) \, ds.$$

(26)

Finally, applying Lemma 2 and using the assumption $\|y - z\| < t$ we get

$$\phi_t(x, y_s) \leq C \left(1 + \frac{\|x - y_s\|}{t}\right)^{-2} \left( \max(w(B(x, t)), w(B(y_s, t))) \right)^{-1} \chi_{[0,2]}(d(x, y_s)/t),$$

which, together with (26) and (8), give the claim. 

\[\Box\]

4 Proof of Theorem 1

Definition 1 By the smooth resolution of identity $\{\phi_Q\}_{Q \in \mathcal{Q}}$ associated with $\mathcal{Q}$ [see (21)] we mean the collection of $C^\infty$-functions on $\mathbb{R}^N$ such that $\text{supp} \phi_Q \subseteq Q^\ast$, $0 \leq \phi_Q(x) \leq 1$,

$$|\partial^\beta \phi_Q(x)| \leq C_\beta d(Q)^{-|\beta|} \text{ for all } \beta \in \mathbb{N}_0^N,$$

(27)

and $\sum_{Q \in \mathcal{Q}} \phi_Q(x) = 1$ for all $x \in \mathbb{R}^N$. The collection $\{\phi_Q\}_{Q \in \mathcal{Q}}$ is well-defined thanks to Proposition 1.

The proof of Theorem 1 is partially based on [16, Theorem 4].
Proof of the first inequality in (5) By the min–max principle it is enough to find $M(\lambda)$-dimensional subspace $\mathcal{H}$ of $L^2(dw)$ such that

$$Q(u, u) \leq C_1 \lambda \|u\|_{L^2(dw)}^2$$

for all $u \in \mathcal{H}$. (28)

Fix a small real number $\varepsilon \in (0, 1)$ of the form $\varepsilon = 2^{-s}$ for some $s \in \mathbb{N}$ (it will be chosen latter on). Let

$$\mathcal{K} = \{ K \in (\text{Grid})_{\varepsilon \lambda^{-1/2}} : K \cap E_\lambda \neq \emptyset \}.$$

Then, $\# \mathcal{K} \geq M(\lambda)$. For any cube $K \in \mathcal{K}$ let $\eta_K$ be a nonzero smooth function supported by $K$ such that

$$|\partial^\beta \eta_K(x)| \leq C_{\beta, \varepsilon \varepsilon' \lambda} x_{\beta}^{1 \varepsilon \varepsilon' \lambda}$$

for all $\beta \in \mathbb{N}_0^N$, $x \in \mathbb{R}^N$, (29)

$$|\eta_K(x)| \geq C \lambda^{\frac{N}{2}}$$

for all $x \in K$ (30)

(here $K_\varepsilon$ denotes the cube of the same center as $K$ but two times smaller side-length). Note that there is $M > 0$ independent of $\lambda$ and $\varepsilon$ such that for any $K \in (\text{Grid})_{\varepsilon \lambda^{-1/2}}$ there are at most $M$ cubes $K' \in (\text{Grid})_{\varepsilon \lambda^{-1/2}}$ such that $O(K) \cap O(K') \neq \emptyset$. Moreover, note that $supp \eta_K \subset O(K)$ and, thanks to (10), $supp T_{j\varepsilon} \eta_K \subset O(K)$ for all $j \in \{1, \ldots, N\}$. Hence, by the definition of $Q$ [see (20)], and the Cauchy–Schwarz inequality it is enough to prove (28) for $u = \eta_K$. It is the standard fact that

$$\|T_{j\varepsilon} f\|_{L^\infty} \leq C \|\nabla f\|_{L^\infty}$$

for all $j \in \{1, \ldots, N\}$ and $f \in C^1(\mathbb{R}^N)$ (31)

(see e.g. [15, Lemma 11] for details). Therefore, by (31) and (29) we obtain

$$\sum_{j=1}^N \int_K |T_{j\varepsilon} \eta_K(x)|^2 d\nu(x) \leq C \lambda^{N+1} \varepsilon \nu(K).$$

(32)

Let $x_K$ be the center of $K$. Since $K \cap E_\lambda \neq \emptyset$, by Lemma 4 we get $\varepsilon \sqrt{\lambda}^{-1/2} \leq m(x_K)^{-1}$ for $\varepsilon$ small enough ($\varepsilon$ is independent of $K$ such that $K \cap E_\lambda \neq \emptyset$). Consequently, by the doubling property of the measure $d\nu$ [see (8)], Lemma 3, and the definition of $m$ [see (3)] we obtain

$$\int_K V(x) \eta_K(x)^2 d\nu(x) \leq C \lambda^{N+1} \nu(K) \frac{\lambda^{-1}}{w(B(x_K, \frac{1}{2} \varepsilon \lambda^{-1/2}))} \int_{B(x_K, \varepsilon \sqrt{\lambda}^{-1/2})} V(x) d\nu(x)$$

$$\leq C \lambda^{N+1} \nu(K) \frac{m(x_K)^{-2}}{w(B(x_K, m(x_K)^{-1}))} \left( \frac{\varepsilon \sqrt{\lambda}^{-1/2}}{m(x_K)^{-1}} \right)^\gamma \int_{B(x_K, m(x_K)^{-1})} V(x) d\nu(x)$$

$$\leq C \lambda^{N+1} \nu(K).$$

(33)
By (32) and (33) we get
\[ Q(\eta_K, \eta_K) \leq C \varepsilon \lambda^{N+1} w(K). \tag{34} \]

On the other hand, by (9) and (30) we have
\[ \lambda^{N+1} w(K) \leq C \lambda \int_K \eta_K(x)^2 \, dw(x). \tag{35} \]

Finally, (28) follows by (34) and (35).

\[ \square \]

Proof of the second inequality in (5) By the max–min principle, it suffices to show the existence of a subspace \( \mathcal{H} \) of \( L^2(dw) \) satisfying the following conditions: there exist constants \( C_2, C_3 > 0 \) such that
\[ \dim \mathcal{H} \leq C_2 M(\lambda), \tag{36} \]
\[ Q(u, u) \geq C_3 \lambda \|u\|_{L^2(dw)}^2 \quad \text{for all } u \perp \mathcal{H} \text{ and } u \in D(Q). \tag{37} \]

Let \( \Psi \in C_c^{\infty}(\mathbb{R}^N) \) be a radial function such that \( \int_{\mathbb{R}^N} \Psi(x) \, dw(x) = 1 \) and \( \text{supp } \Psi \subseteq B(0, 1) \). It follows from Corollary 1 that
\[ |c_k \mathcal{F}(\Psi)(\xi) - 1| \leq C \|\xi\| \quad \text{for all } \xi \in \mathbb{R}^N. \tag{38} \]

Set
\[ \Psi^\lambda(x) = \lambda^{N/2} \Psi(\lambda^{1/2} x). \]

Let us consider \( Q \in Q \). If \( Q^{***} \cap E_\lambda^c \neq \emptyset \), then thanks to Fact 3 we have \( m(x) > c \sqrt{\lambda} \) for all \( x \in Q^* \). Consequently, for any \( u \in D(Q) \) we have
\[ \lambda \int_{Q^*} |(u \phi_Q)(x)|^2 \, dw(x) \leq c^{-2} \int_{Q^*} |u(x)|^2 m(x)^2 \, dw(x). \tag{39} \]

If \( Q^{***} \cap E_\lambda^c = \emptyset \), then \( Q^{***} \subseteq E_\lambda \), so \( m(x) \leq \sqrt{\lambda} \) for all \( x \in Q^{***} \), and, by Fact 3, \( d(Q) \geq c \lambda^{-1/2} \). For such a cube \( Q \) we write
\[ \begin{aligned}
\lambda \int_{Q^*} |(u \phi_Q)(x)|^2 \, dw(x) \\
\leq C \lambda \int_{Q^*} |(u \phi_Q)(x) - \Psi^\lambda * (u \phi_Q)(x)|^2 \, dw(x) + C \lambda \int_{Q^*} |\Psi^\lambda * (u \phi_Q)(x)|^2 \, dw(x) \\
=: S_1 + S_2.
\end{aligned} \tag{40} \]
By Plancherel’s formula (14), (16), and (38) we have

\[
S_1 \leq C \lambda \int_{\mathbb{R}^N} \left| \mathcal{F}(u \phi_Q)(\xi) \left( 1 - c_k \mathcal{F}(\Psi^\lambda)(\xi) \right) \right|^2 d\omega(\xi) \\
\leq C \lambda \int_{\mathbb{R}^N} |\mathcal{F}(u \phi_Q)(\xi)|^2 \lambda^{-1} \|\xi\|^2 d\omega(\xi) \\
\leq C \int_{\mathbb{R}^N} \sum_{j=1}^N |T_j(u \phi_Q)(x)|^2 d\omega(x).
\]

(41)

The inequality (41) can be thought as a counterpart of the pseudo-Poincaré inequality (see [13,29]). Using Lemma 6 we get

\[
S_1 \leq C \sum_{j=1}^N \int_{Q^*} |T_j u(x)|^2 d\omega(x) + C \int_{Q(Q^*)} |u(x)|^2 m(x)^2 d\omega(x).
\]

(42)

Fix a small real number \( \varepsilon > 0 \) (it will be chosen latter on). Let

\[
\beta(Q) = \{ K \in (\text{Grid})_{\varepsilon \lambda^{-1/2}} : K \cap Q^* \neq \emptyset \}.
\]

Let \( x_K \) denote the center of \( K \in \beta(Q) \). Set

\[
\mathcal{H}_Q = \text{span}\{ \Psi^\lambda(x_K, \cdot) \phi_Q(\cdot) : K \in \beta(Q) \}.
\]

Clearly,

\[
\dim \mathcal{H}_Q \leq C_N \varepsilon^{-N} \# \{ K \in (\text{Grid})_{\lambda^{-1/2}} \cap Q^* \neq \emptyset \}.
\]

Then, by the definition of the Dunkl convolution [see (19)], for \( u \perp \mathcal{H}_Q \) we have

\[
S_2 \leq C \lambda \int_{Q^*} \left| \int_{Q^*} \sum_{K \in \beta(Q)} \chi_K(x) \Psi^\lambda(x, y) \phi_Q(y) u(y) d\omega(y) \right|^2 d\omega(x) \\
\leq C \lambda \int_{Q^*} \left| \int_{Q^*} \sum_{K \in \beta(Q)} \chi_K(x) \left( \Psi^\lambda(x, y) - \Psi^\lambda(x_K, y) \right) \phi_Q(y) u(y) d\omega(y) \right|^2 d\omega(x).
\]

(43)

Consider the integral kernel \( K_Q(x, y) = \sum_{K \in \beta(Q)} \chi_K(x) |\Psi^\lambda(x, y) - \Psi^\lambda(x_K, y)| \). Then, for fixed \( x \in Q^* \), let \( K' \) be the unique one such that \( x \in K' \in \beta(Q) \). So, by Lemma 7, we have

\[
\int K_Q(x, y) d\omega(y) = \int |\Psi^\lambda(x, y) - \Psi^\lambda(x_{K'}, y)| d\omega(y) \leq C \varepsilon.
\]

(44)
Now fix $y \in Q^*$. Applying once more Lemma 7, we obtain

\[
K_Q(x, y) \leq C \sum_{K \in \beta(Q)} \chi_K(x) \frac{\|x - x_K\| \sqrt{\lambda}}{w(B(y, \lambda^{-1/2}))} \chi_{[0, 2]}(d(x, y)\lambda^{1/2})
\]

Consequently,

\[
\int K_Q(x, y) \, dw(x) \leq C \varepsilon.
\] (46)

Finally, thanks to Schur’s test, (44), and (46) we obtain

\[
S_2 \leq C \lambda \varepsilon^2 \|u\phi_Q\|_{L^2(dw)}^2.
\] (47)

Note that $\dim \left( \bigoplus_{Q \in E_\lambda} \mathcal{H}_Q \right) \leq C_2 M(\lambda)$. Now, if $u$ is orthogonal to the all Hilbert spaces $\mathcal{H}_Q$ for $Q \in Q$ such that $Q^{***} \subseteq E_\lambda$, by (39), (40), (42), (47), and Proposition 1, we conclude

\[
\lambda \|u\|_{L^2(dw)}^2 \leq C \lambda \sum_{Q^{***} \subseteq E_\lambda} \|u\phi_Q\|_{L^2(dw)}^2 + C \lambda \sum_{Q^{***} \subseteq E_\lambda} \|u\phi_Q\|_{L^2(dw)}^2
\]

\[
\leq C \int_{\mathbb{R}^N} |u(x)|^2 m(x)^2 \, dw(x) + C \sum_{j=1}^N \int_{\mathbb{R}^N} |T_j u(x)|^2 d w(x) + C \lambda \varepsilon^2 \|u\|_{L^2(dw)}^2.
\]

Now taking $\varepsilon$ small enough and using the Fefferman–Phong inequality (see Theorem 4) we obtain the claim. \qed

**Proof of (6)** Let $f_0 \in D(L)$ be a nonzero function such that $Lf_0 = \lambda_0 f_0$. Thanks to the Fefferman–Phong inequality we have

\[
\min_{x \in \mathbb{R}^N} m(x)^2 \|f_0\|_{L^2(dw)}^2 \leq \int_{\mathbb{R}^N} |f_0(x)|^2 m(x)^2 \, dw(x) \leq C \mathcal{Q}(f_0, f_0)
\]

\[
= C \langle Lf_0, f_0 \rangle = C \lambda_0^2 \|f_0\|_{L^2(dw)}^2,
\] (48)

so (6) follows. \qed

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