Estimating a density near an unknown manifold: a Bayesian nonparametric approach

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Abstract

We study the Bayesian density estimation of data living in the offset of an unknown submanifold of the Euclidean space. In this perspective, we introduce a new notion of anisotropic Hölder for the underlying density and obtain posterior rates that are minimax optimal and adaptive to the regularity of the density, to the intrinsic dimension of the manifold, and to the size of the offset, provided that the latter is not too small — while still allowed to go to zero. Our Bayesian procedure, based on location-scale mixtures of Gaussians, appears to be convenient to implement and yields good practical results, even for quite singular data.

1 Introduction

1.1 Manifold density estimation

In many high dimensional statistical problems it is common to consider that the data has an intrinsic low dimensional structure. More precisely, statistics and computer sciences have seen a growing interest in the so-called manifold hypothesis where the data is believed to be supported (or near supported) on a low dimensional submanifold $M$ of an ambient space (see [44] for an introduction).

There are good intuitive reasons to believe that real world data (such as natural images, sounds, texts, etc) belong to the vicinity of a low dimensional submanifold, often due to physical constraints, see for instance [42] or [23]. Empirical evidence has also been shown in a number of important cases such as texts data sets [5], sounds [40, 5], images and videos [63, 39] or more recently in Covid data [48]. Analysing such data sets is often called manifold learning (see [44] for an introduction). Manifold learning deal with either nonlinear dimension reduction techniques, manifold estimation or the construction of generative models and the estimation of the distribution on or near an unknown manifold. These problems are strongly

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connected. Dimension reduction consists in finding low dimensional representations of the data. This is typically done by constructing mappings as in Kernel PCA [58] or graph based methods such as Isomap [61], Locally Linear Embeddings [57] or Laplacian Eigenmaps [6]. Instead of estimating an embedding, the problem of reconstructing the manifold is another popular aspect of manifold learning, see [24, 1, 18] or [21] among others. Finally the estimation of the distributions on or near manifolds and the construction of generative models have received recent wide interests in the statistics and machine learning communities, specially with the developments of deep learning algorithms. There is a growing literature on generative models under the manifold hypothesis with many methodological developments around variational autoencoders (see [32, Sec 14.6] or [39]), Generative adversarial networks (see [33, 3, 4] among others) or recent versions of normalizing flows (see [36]). The theoretical results associated to these approaches control the error between the true generative process and the estimated generative models typically under adversarial losses such as the Wasserstein distance, as in [60], since the focus is more on generating interesting samples than on estimating the distribution per se.

In this paper we study the estimation of the density in the vicinity of an unknown submanifold $M$ of the ambient space $\mathbb{R}^D$. Density estimation is an important class of problems in statistics and machine learning and in addition to being of interest in itself can be used as an intermediate steps in many tasks of unsupervised learning such as clustering, prediction or in ridge estimation ([25, 13]). Density or distribution estimation under the exact manifold hypothesis (assuming that the data belong to a submanifold) has been studied theoretically for instance in [56] or [19] under Wasserstein losses and in [7] under the pointwise loss. Assuming that the data belong exactly to a smooth manifold may be too restrictive since signals are often corrupted by noise. Hence in this paper we assume that the data belong to a neighbourhood $M^\delta = \{x \in \mathbb{R}^D; d(x, M) \leq \delta\}$, where neither $M$ nor $\delta$ or the density $f$ are known. This problem is studied in [12] in the special case of data corrupted with Gaussian noise and [49] proposes a Bayesian nonparametric method to estimate a density on $M^\delta$ based on mixtures of Fisher - Gaussian distributions for which they prove consistency under the assumption that the width $\delta$ of the tube $M^\delta$ is fixed.

1.2 Our approach and contributions

As far as we are aware there is no theoretical results on convergence rates - either from a frequentist or a Bayesian approach - for estimating a density on tubes $M^\delta$ when $M$ and $\delta$ are unknown and $\delta$ is possibly small. In this paper we bridge this gap and we propose a Bayesian nonparametric method based on specific families of location-scale mixtures of Gaussian distributions. We study the posterior concentration rates associated to these priors, i.e. the smallest possible $\varepsilon_n$ such that

$$\Pi(d(f_0, f) \leq \varepsilon_n | X_1, \ldots, X_n) \rightarrow 1,$$

in probability when the data $X_1, \ldots, X_n$ are a $n$ sample from $f_0$ and where $\Pi(\cdot | X_1, \ldots, X_n)$ denotes the posterior distribution, see [27]. As is well known, when the distance $d(\cdot, \cdot)$ is the Hellinger or the $L_1$ metric, this posterior concentration rates induces also a convergence rate
\( \varepsilon_n \) for the posterior mean \( \hat{f} \), see for instance [27]. Typically the rate \( \varepsilon_n \) depends on regularity properties of the density \( f_0 \) and on the prior.

To do so we first define a general mathematical framework describing regularity properties of densities defined on possibly small neighbourhoods \( M^\delta \) of submanifolds \( M \), with the idea that the density has a given smoothness \( \beta_0 \) along the manifold \( M \) and another smoothness \( \beta_\perp \) along the normal to the manifold. This manifold driven anisotropic smoothness is defined in Section 2.2 and is an extension to anisotropic Hölder functions along coordinate axes. Building on that we show that the posterior concentration rate depends on \( \beta_0, \beta_\perp \) together with the dimension \( d \) of \( M \) and the width \( \delta \) of the tube. Interestingly the prior \( \Pi \) does not need to depend on \( \beta_0, \beta_\perp, d, \delta \) or \( M \) which makes the approach fully adaptive and the rate we obtain, at least when \( \delta \) is not too small is of order \( n^{-\gamma} \) with

\[
\gamma = \frac{\beta_0}{2\beta_0 + d + (D - d)\beta_0/\beta_\perp},
\]

and is minimax.

Nonparametric location mixtures of Gaussians are known to be flexible models for densities, and adaptive minimax rates of convergence on Hölder types spaces have been obtained using Bayesian or frequentist estimation procedures based on location mixtures of normals, see [41, 59], and [28] for Bayesian methods and [46] for a penalized likelihood approach. However, location mixtures are not versatile enough since the covariance matrix remains fixed across the components, so we instead take advantage of the flexibility of location-scale mixtures of Gaussians. In [11] the authors derive a suboptimal posterior concentration rate for isotropic positive Hölder densities on \( \mathbb{R}^D \), while [46] obtained minimax convergence rates for penalized maximum likelihood methods based on the same type of location-scale mixtures and [51] obtained also minimax posterior concentration rates using a hybrid location-scale mixture prior in the regression model. These results thus indicate that one has to be careful in designing the prior in nonparametric location-scale mixtures of Gaussians. The priors we consider in this paper are variants of location-scale mixture priors, see Section 2.3, which are flexible enough to adapt to the non linear or manifold driven smoothness of the class of densities studied here. This prior construction can also be seen as tiling the manifold by low-rank Gaussian pancakes, a method that is similar to mixtures of factors analyzers [26, 20] or manifold Parzen windows [62] where, however, no theoretical guarantees on the estimation of the density were proven.

Hence our contributions are both methodological and theoretical. From a methodological point of view, we provide with a family of versatile priors (see Section 2) that are shown empirically and theoretically to perform very well in modelling data that are singularly supported near submanifolds. In particular we show empirically that these variants of location-scale mixtures of Gaussian priors behave much better than the standard conjugate location-scale mixture of Gaussian prior, see Section 4.

From a theoretical point of view, we introduce a new notion of Hölder smoothness along a submanifold (see Section 2.2) which is proving to be adequate for the study of such almost-
degenerated densities, and we derive posterior concentration rates for this new model (Section 3.1). The rates are optimal if the data do not collapse too quickly towards the manifold. These results rely on an intermediate result in approximation theory which has interests in its own right and is provided in Section 3.2.

1.3 Organisation of the paper

In Section 2, we define manifold-anisotropic Hölder function, together with the families of priors we consider in the paper. Section 3 contains the main theoretical results and Section 4 the empirical, numerical results. We provide in Section 5 proofs of the main results, namely the contraction rate Theorem 3.1 and the approximation Theorem 3.3. Some useful facts on manifolds are presented in Appendix A. The other Appendices contain additional proofs and lemmata, as well as details on the numerical setting of Section 4.

1.4 Notations

For a multi-index \( k = (k_1, \ldots, k_D) \in \mathbb{N}^D \), we set \(|k| = k_1 + \cdots + k_D\) and \(k! = k_1! \cdots k_D!\). For \( x \in \mathbb{R}^D \), we write \( x^k = x_1^{k_1} \cdots x_D^{k_D} \in \mathbb{R} \) and \( x_{\max} \) (resp. \( x_{\min} \)) to be the maximal (resp. minimal) value of its entries. For any two indices \( i, j \in \{1, \ldots, D\} \) with \( i < j \), we set \( x_{ij} = (x_i, \ldots, x_j) \in \mathbb{R}^{j-i+1} \). Finally, for a sufficiently regular function \( f : \mathbb{R}^D \to \mathbb{R} \), we define its \( k \)-th partial derivative as

\[
D^k f(x) = \frac{\partial^{|k|} f}{\partial x_1^{k_1} \cdots \partial x_D^{k_D}}(x).
\]

If \( M \subset \mathbb{R}^D \) is a measurable subset with Hausdorff dimension \( d \), one denote \( \mu_M \) for the Borel measure \( \mu_M = \mathcal{H}^d(\cdot \cap M) \) where \( \mathcal{H}^d \) is the \( d \)-dimensional Hausdorff measure on \( \mathbb{R}^D \). For \( r > 0 \) and \( x \in \mathbb{R}^D \), one write \( B_M(x, r) = B(x, r) \cap M \) where \( B(x, r) \) is the usual Euclidean ball of \( \mathbb{R}^D \). If \( M \) is closed, then \( \text{pr}_M \) defines the (possible multi-valued) orthonormal projection from \( \mathbb{R}^D \) to \( M \).

We will denote by \( \| \cdot \| \) the usual Euclidean norm of \( \mathbb{R}^k \) for any \( k \in \mathbb{N}^* \). When \( \mathcal{L} \) is a linear map between such spaces, we write \( \| \mathcal{L}\|_{\text{op}} \) for the operator norm associated with the Euclidean norms. The notation \( \| \cdot \|_1 \) (resp. \( \| \cdot \|_\infty \)) will refer to both the \( L^1 \)-norm (resp sup-norm) for vectors of \( \mathbb{R}^k \) for any \( k \in \mathbb{N}^* \), and to the \( L^1 \)-norm (resp sup-norm) for measurable functions from \( \mathbb{R}^k \) to \( \mathbb{R} \) for any \( k \in \mathbb{N}^* \). The brackets \( \langle \cdot, \cdot \rangle \) will be used to denote the usual Euclidean product in \( \mathbb{R}^k \) for any \( k \in \mathbb{N}^* \). For any matrix \( A \in \mathbb{R}^{k \times k} \), the notation \( \| \cdot \|_A \) will refer to the quadratic form over \( \mathbb{R}^k \) defined by \( x \mapsto \langle Ax, x \rangle \), which is a norm if \( A \) is positive. The set of orthogonal transform of \( \mathbb{R}^D \) will be denoted by \( \mathcal{O}(D, \mathbb{R}) \), or sometimes simply \( \mathcal{O}(D) \).

For two positive functions \( f, g : \mathbb{R}^D \to \mathbb{R} \) we write the Hellinger distance as

\[
\text{d}_H(f, g) = \left\{ \int_{\mathbb{R}^D} (\sqrt{f(x)} - \sqrt{g(x)})^2 dx \right\}^{1/2}.
\]

In this paper, \( M \) will designate a closed submanifold of \( \mathbb{R}^D \) of dimension \( 1 \leq d \leq D - 1 \). For any point \( x \in M \), the tangent and normal spaces of \( M \) at \( x \) will be denoted \( T_xM \) and \( N_xM \), and
the corresponding bundles $TM$ and $NM$. We write $\exp_x : (TxM, 0) \to (M, x)$ for the exponential map of $M$ at point $x$. We let $d_M(x, y)$ denote the intrinsic distance between $x$ and $y$ in $M$.

Finally we will use throughout the symbols $\simeq$, $\preceq$ and $\succeq$ to denote equalities or inequalities up to a constant, when the constant is not important.

2 Model : distributions concentrated near manifolds

We assume that we observe $X_1, \ldots, X_n$ independent and identically distributed from $P_0$ on $\mathbb{R}^D$ with density $f_0$ with respect to Lebesgue measure. We assume that there is a low dimensional structure underlying our observations, i.e. that $f_0$ has support concentrated near a low dimensional manifold $M$ which is unknown. More precisely there exists $\delta > 0$ unknown and typically small such that $P_0(M^\delta) = 1$, where $M^\delta$ is the $\delta$-offset of $M$: it is the set of points that are at distance less than $\delta$ from $M$,

$$M^\delta := \bigcup_{x \in M} B(x, \delta) = \{ z \in \mathbb{R}^D \mid d(z, M) \leq \delta \}.$$ 

A typical example is when the observations are noisy versions of data whose support is $M$: $X = Y + Z$ with $Y \in M$ and $|Z| \leq \delta$ almost surely. When the noise $Z$ has a density smoother than the density of $Y$ on $M$ (with respect to the Hausdorff measure), the density of $X$ is anisotropic with a smoothness along the manifold $M$ smaller than that along the normal directions. In this paper we thus aim at constructing priors which are flexible enough to lead to good estimation of $f_0$ in situations where the density has a complex anisotropic structure in that it has an unknown smoothness $\beta_0$ along an unknown manifold $M$ and a different (larger) smoothness $\beta_\perp$, also unknown, along the normal spaces of the manifold. In this context, since the anisotropy varies spatially, it is therefore important to consider priors which adapt spatially to such non linear smoothness. In Section 2.3 below we consider certain families of location-scale mixtures with a careful modelling of the prior on the variance of the components and we show in Section 3 that these priors allow for manifold driven smoothness.

To begin with, we define what we think is a new notion of anisotropic Hölder spaces on the Euclidean space $\mathbb{R}^D$, and which happens to be a natural extension of the usual notion of (isotropic) Hölder smoothness. We are aware that there exist various notions of anisotropic smoothness, see for instance [38, 35, 15, 30, 31], with most of them stemming from the anisotropic smoothness as defined in [54]. In all the aforementioned references, the anisotropy was consistently defined as a control of the variations of the partial derivatives along each axis separately, with no control of the cross-derivatives (and no guarantee that they, in fact, exist). While this is enough in a Euclidean framework, we argue that, to the best of our effort, we could not make such assumptions sufficient in our non-linear setting, as the proofs presented in Section 5 or in the Appendices might highlight. Instead, we come out with a new notion of Hölder anisotropy, in the footsteps of what [59] already sketched in their paper, that handles cross-derivatives in the same way that the usual notion of (isotropic) Hölder smoothness does, and which in fact coincides with the latter when the anisotropy vector is isotropic. This new class is defined in the following subsection and its main properties
reviewed in Section B.1. We would also like to mention the notion of mixed smoothness as introduced in [14], which is a stronger notion of regularity than ours: the classes defined below can be seen as nested between the ones of [54] and [14].

2.1 General anisotropic Hölder functions

An anisotropic Hölder functions $f : \mathbb{R}^D \to \mathbb{R}$ is, informally, a function whose smoothness is different along each axis of $\mathbb{R}^D$. Letting $\beta = (\beta_1, \ldots, \beta_D) \in (\mathbb{R}_+^*)^D$, which will represent the regularity indices along each axis, we define

$$\alpha = (\alpha_1, \ldots, \alpha_D) \quad \text{where} \quad \alpha_i = \beta_i / \beta \in [0, D] \quad \text{and} \quad \beta^{-1} = \frac{1}{D} \sum_i \beta_i^{-1}.$$ 

The coefficient $\beta$ acts as the effective smoothness of the function $f$. Notice that $\alpha_1 + \cdots + \alpha_D = D$.

In this section, we define the spaces of anisotropic functions over bounded open subset of $\mathbb{R}^D$.

We defer to Section B.1 the introduction of the same class over general open subsets. We let $U \subset \mathbb{R}^D$ be a bounded open subset and $L : U \to \mathbb{R}$ be any non-negative function.

**Definition 2.1.** The anisotropic Hölder spaces $\mathcal{H}_an^\beta(U, L)$ is the set of all functions $f : U \to \mathbb{R}^D$ satisfying:

i) For any multi-index $k \in \mathbb{N}^D$ such that $(k, \alpha) < \beta$, the partial derivative $D^k f$ is well defined on $U$ and $|D^k f(x)| \leq L(x)$ for all $x \in U$;

ii) For any multi-index $k \in \mathbb{N}^D$ such that $\beta - \alpha_{\max} \leq (k, \alpha) < \beta$, there holds

$$|D^k f(y) - D^k f(x)| \leq L(x) \sum_{i=1}^D |y_i - x_i|^{\beta - (k, \alpha)} / \alpha_i \wedge 1 \quad \forall x, y \in U. \tag{1}$$

See Figure 1 for a graphical representation of the quantities at stake. The function $L$ acts as an upper-bound for the localized and anisotropic version of the usual Hölder-norm:

$$\left\{ \max_{(k, \alpha) < \beta} |D^k f(x)| \right\} \vee \max_{\beta - \alpha_{\max} \leq (k, \alpha) < \beta} \sup_{y \in U} \frac{|D^k f(y) - D^k f(x)|}{\sum_{i=1}^D |y_i - x_i|^{\beta - (k, \alpha)} / \alpha_i \wedge 1}.$$

Note that the constraint on the intermediate derivatives $D^k f(x)$ for $0 < (k, \alpha) < \beta$ may seem superfluous since some Kolmogorov-Landau type inequalities would yield some bounds on these derivatives, but we add them nonetheless to our functional class to simplify some notations. We list and prove in Section B.1 various useful properties of functions in the anisotropic Hölder class. Also the function $L$ in the definition of $\mathcal{H}_an^\beta(U, L)$ can be constant, in which case we will typically denote it $C$, to make it more explicit (leading to $\mathcal{H}_an^\beta(U, C)$).

**Remark 1.** The usual isotropic Hölder spaces are special cases of our definition of $\mathcal{H}_an^\beta(U, L)$ corresponding to $\beta = (\beta, \ldots, \beta)$ with $\beta > 0$. In this case we write

$$\mathcal{H}_iso^\beta(U, L) := \mathcal{H}_an^\beta(U, L) \quad \text{for} \quad \beta = (\beta, \ldots, \beta).$$
Figure 1: An example in dimension $D = 2$. The vector $\alpha$ is the only vector of 1-norm $D$ which has positive coordinates and which is orthogonal to the simplex of vertices $\{\beta_i e_i\}_{1 \leq i \leq D}$. In black are the points $k$ of $\mathbb{N}^2$ such that $(k, \alpha) < \beta$.

As a final remark, we will use the same notations for the spaces of multivalued functions when their coordinate functions are all in the corresponding space. For instance, if $\Psi : U \to \mathbb{R}^D$, then

$$
\Psi = (\Psi_1, \ldots, \Psi_D) \in \mathcal{H}_{an}^\beta(U, L) \iff \Psi_i \in \mathcal{H}_{an}^\beta(U, L) \text{ for all } i \in \{1, \ldots, D\},
$$

and the same holds for the other spaces defined in this subsection.

2.2 Manifold anisotropic Hölder functions

We now consider functions whose smoothness directions at point $x \in \mathbb{R}^D$ are dependant on the position of $x$ with respect to a given submanifold $M \subset \mathbb{R}^D$ of dimension $1 \leq d \leq D - 1$. More specifically, we extend the above notions of anisotropy to functions with a given regularity in the tangential directions of $M$, and of another regularity in the normal directions of $M$. We call such functions manifold-anisotropic Hölder, or sometimes simply $M$-anisotropic. To define such a class of function, we assume that $M$ is a closed submanifold with reach bounded from below by $\tau > 0$ (see Appendix A for definition and properties of the reach) and we consider local parametrizations at any $x_0 \in M$

$$
\Psi_{x_0} : \mathcal{V}_{x_0} \to M,
$$

where $\mathcal{V}_{x_0}$ is a neighborhood of 0 in $T_{x_0}M$. The maps $\Psi_{x_0}$ can be taken in a wide class of parametrizations of $M$. For instance, one could consider taking $\Psi_{x_0}$ to be (close to) the inverse projection over $M \to T_{x_0}M$ where $T_{x_0}M$ is seen as an affine subspace of $\mathbb{R}^D$ going through $x_0$, see for instance [1] or [19]. For purely practical matter, we choose $\Psi_{x_0}$ to be the exponential map $\exp_{x_0}$, although the results in this paper could be carried out with other well-behaved parametrizations, such as the one mentioned above. In particular, in the case of the exponential maps, we can define the domain of $\Psi_{x_0}$ to be $B_{T_{x_0}M}(0, \pi \tau)$, see Appendix A. In the rest of this paper, we set

$$
\mathcal{V}_{x_0} := B_{T_{x_0}M}(0, \tau/8),
$$
with factor 1/8 being there for technical reasons. If all the maps $\Psi_{x_0}$ are of regularity $\beta_M > 1$, meaning that there exists a constant $C_M > 0$ such that

$$\Psi_{x_0} \in \mathcal{H}_\text{iso}^{\beta_M}(V_{x_0}, C_M), \quad \forall x_0 \in M$$

(in particular $M$ is at least $C^k$ with $k = \lceil \beta_M - 1 \rceil$), then one can construct a map

$$\overline{\Psi}_{x_0} : \begin{cases} V_{x_0} \times N_{x_0} M & \to \mathbb{R}^D \\ (v, \eta) & \mapsto \Psi_{x_0}(v) + N_{x_0}(v, \eta). \end{cases}$$

where $N_{x_0}(v, \cdot)$ is an isometry from $N_{x_0} M$ to $N_{\Psi_{x_0}(v)} M$ and where $v \mapsto N_{x_0}(v, \cdot) \in \mathcal{H}_\text{iso}^{\beta_M-1}(V_{x_0}, C_M^+)$. We refer to Appendix A for further details concerning the construction of $\overline{\Psi}_{x_0}$ and the proof of its regularity. When restricting the latter map, one get a local parametrization of the offset $M^{\tau/2}$ around $x_0$

$$\overline{\Psi}_{x_0} : V_{x_0} \times B_{N_{x_0} M}(0, \tau/2) \to M^{\tau/2}$$

as shown in Lemma A.2. This parametrization is such that $\text{pr}_M(\overline{\Psi}_{x_0}(v, \eta)) = \Psi_{x_0}(v)$ for any $(v, \eta) \in V_{x_0} \times B_{N_{x_0} M}(0, \tau/2)$ and $\overline{\Psi}_{x_0}$ is a diffeomorphism from $V_{x_0} \times B_{N_{x_0} M}(0, \tau/2)$ to its image which satisfies

$$\overline{\Psi}_{x_0} \in \mathcal{H}_\text{iso}^{\beta_M-1}(V_{x_0} \times B_{N_{x_0} M}(0, \tau/2), C_M^+)$$

for some $C_M^+ > 0$ depending on $C_M$, $\tau$ and $\beta_M$. See Figure 2 for a visual interpretation of this parametrizations.

![Figure 2: A visual interpretation of the parametrization $\overline{\Psi}_{x_0}$.](image)

For any $\delta > 0$, we define $\overline{\Psi}_{x_0,\delta}(v, \eta) := \overline{\Psi}_{x_0}(v, \delta \eta)$ to be the rescaled version of $\overline{\Psi}_{x_0}$ in the normal directions. It is a well defined parametrization of $M^{\tau/2}$ on the set $\mathcal{W}_{x_0,\delta} := V_{x_0} \times B_{N_{x_0} M}(0, \tau/2\delta)$. We let $\beta_0, \beta_\perp$ be two positive real numbers, and define the vector

$$\beta_{0,\perp} = (\beta_0, \ldots, \beta_0, \beta_\perp, \ldots, \beta_\perp) \in \mathbb{R}^D.$$ 

Now for any function $L : \mathbb{R}^D \to \mathbb{R}_+$, we define:

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**Definition 2.2.** Let $L : \mathbb{R}^D \to \mathbb{R}_+$ be a function; the class $\mathcal{H}^{\beta_0,\beta_1}_\delta(M,L)$ is the set of all functions $f : \mathbb{R}^D \to \mathbb{R}$ which satisfy:

i) $f$ is supported on $M^\delta$;

ii) For any $x_0 \in M$, set

$$\tilde{f}_{x_0,\delta} := \delta^{D-d} f \circ \Psi_{x_0,\delta}$$

and

$$L_{x_0,\delta} := \delta^{D-d} L \circ \Psi_{x_0,\delta},$$

then

$$\tilde{f}_{x_0,\delta} \in \mathcal{H}^{\beta_0,\beta_1}_\delta(W_{x_0,\delta}, L_{x_0,\delta}).$$

(4)

Informally, such a function is $\beta_0$-Hölder along the manifold $M$, and $\beta_1$-Hölder normal to the manifold $M$. The normalization $\delta^{D-d}$ accounts for the scaling $\eta \mapsto \delta\eta$ along the normal spaces (which are of dimension $D-d$) in the definition of $\Psi_{x_0,\delta}$. Its presence is natural and can be understood as follows: when $f$ is a density supported on $M^\delta$, the typical magnitude of its values is of order $1/\delta^{D-d}$, and the absence of normalization would whence make the above functional class irrelevant to describe the regularity of such densities.

M-anisotropic functions happen to be a convenient way to describe the regularity of a number of densities that are naturally supported around $M$. To illustrate this, take $f_* : M \to \mathbb{R}$ to be a $\beta_0$-Hölder density, meaning that there exists $L_0 : M \to \mathbb{R}$ such that for any $x_0 \in M$,

$$f_* \circ \Psi_{x_0} \in \mathcal{H}^{\beta_0}_\delta(\mathcal{V}_{x_0}, L_0 \circ \Psi_{x_0}).$$

Now take $K : \mathbb{R}^D \to \mathbb{R}$ to be a normalized positive smooth isotropic kernel supported on $\mathcal{B}(0,1)$. We introduce $c_1^{-1} = \int K(\varepsilon) \, d\mu_E(\varepsilon)$ where $E$ is any (through isotropy) $(D-d)$-dimensional subspaces of $\mathbb{R}^D$. We also assume that $K \in \mathcal{H}^{\beta_1}_\delta(\mathbb{R}^D, L_1)$ for some function $L_1$ which is also rotationally invariant.

**Proposition 2.3.** Let $f$ be the density of a random variable $Z = X + \delta \mathcal{E}$ where $X \sim f_*(x) \, \mu_M(dx)$ and $0 < \delta < \tau$. Then,

1. (Orthonormal noise) If $\beta_1 < \beta_M - 1$, and if $\mathcal{E}|X \sim c_1 K(\varepsilon) \mu_{N \times M}(d\varepsilon)$, then

$$f \in \mathcal{H}^{\beta_0,\beta_1}_\delta(M,L), \quad \text{with,} \quad L(x) := C \delta^{-(D-d)} L_0(\text{pr}_M x) \times L_1(x - \text{pr}_M x), \quad C > 0.$$

2. (Isotropic noise) If $\delta < \tau/32$ and $\beta_0 \leq \beta_1 \leq \beta_M - 1$, and if $\mathcal{E} \sim K(\varepsilon) \, d\varepsilon$, independently of $X$, then

$$f \in \mathcal{H}^{\beta_0,\beta_1}_\delta(M,L), \quad \text{with} \quad L(x) := C \delta^{-D} \int_M L_1 \left( \frac{x-y}{\delta} \right) L_0(y) \mu_M(dy), \quad C > 0.$$

In both cases $C$ depends on $C_M$, $\tau$, $\beta_M$, $\beta_0$, $\beta_1$.

See Section B.2 for a proof of this result.

**Remark 2.** Throughout the paper we assume that the true density $f_0 \in \mathcal{H}^{\beta_0,\beta_1}_\delta(M,L)$, which implies that $P_0(M^\delta) = 1$. However it is enough to assume that $P_0(M^\delta) \geq 1 - o(1/n)$ where $n$ is the number of observations. This makes no difference in terms of the results presented in Section 3. The weaker assumption $P_0(M^\delta) \geq 1 - o(1/n)$ is for instance fulfilled in the additive noise model (see Proposition 2.3) with $Z = X + \frac{\delta}{\sqrt{\mathcal{E} \log n}} \mathcal{E}$ with the Gaussian kernel $K(x) := (2\pi)^{D/2} \exp(-\|x\|^2/2)$.

In the following section we describe the family of priors which we use to estimate the above family of densities.
2.3 Location-scale mixtures of normal priors

We model the manifold-anisotropic Hölder densities using location-scale mixtures of normals. We parametrize the covariances of the components by $\Sigma = O^T \Lambda O$ where $O$ is a unitary matrix and $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_D)$ is diagonal. Location-scale mixtures can then be written as:

$$f_P(x) = \int_{\mathbb{R}^D} \varphi_{O^T \Lambda O}(x - \mu) \, dP(\mu, O, \Lambda), \quad P = \sum_{k=1}^{K} p_k \delta((\mu_k, O_k, \Lambda_k)), \quad K \in \mathbb{N} \cup \{+\infty\},$$

(5)

where, for any positive definite matrix $\Sigma$,

$$\varphi_{\Sigma}(z) := \frac{1}{\det^{1/2}(2\pi \Sigma)} \exp\left\{ -\frac{1}{2} \|z\|^2_{\Sigma^{-1}} \right\},$$

is the density of a centered Gaussian with covariance matrix $\Sigma$. The two most well known families of priors on $P$ are Dirichlet process priors and mixtures with random number of components, also known as mixtures of finite mixtures. Recall that if $P$ follows a Dirichlet process priors with parameters $A$ and $H$ where $A > 0$ and $H$ is a probability measure on some measurable space $\Theta$, then

$$P = \sum_{k=1}^{\infty} p_k \delta_{\theta_k} \quad \text{with} \quad p_k = V_k \prod_{i < k} (1 - V_i), \quad V_i \overset{iid}{\sim} \text{Beta}(1, A) \quad \text{and} \quad \theta_k \overset{iid}{\sim} H.$$

If $P$ follows a mixture of finite mixtures prior of parameters $\alpha_K$ and $\pi_K$ where $\alpha_K > 0$ and $\pi_K$ is a probability measure on $\mathbb{N}$, then

$$P = \sum_{k=1}^{K} p_k \delta_{\theta_k} \quad \text{with} \quad K \sim \pi_K, \quad (p_1, \ldots, p_K) | K \sim \mathcal{D}(\alpha_K, \ldots, \alpha_K) \quad \text{and} \quad \theta_k \overset{iid}{\sim} H.$$

In both cases (Dirichlet process and mixture of finite mixtures) we call $H$ the base probability measure. Obviously in the case of mixtures of finite mixtures the conditional prior on $(p_1, \ldots, p_K)$ and $\theta_1, \ldots, \theta_K$ could be different but we consider this setup for the sake of simplicity.

As shown empirically by [50] location-scale Dirichlet process mixtures with base measure constructed from the conjugate prior of the Gaussian model are not well adapted to the problem at hand. We show however that if particular care is given to the choice of $H$, the posterior on manifold-anisotropic density is well behaved. In particular we consider the two following types of location-scale mixtures:

- **Partial location-scale mixtures**: The eigenvalues $\Lambda$ of the covariance of the Gaussians are common across components,

$$f_{\Lambda, P}(x) = \int_{\mathbb{R}^D} \varphi_{O^T \Lambda O}(x - \mu) \, dP(\mu, O), \quad P = \sum_{k=1}^{K} p_k \delta((\mu_k, O_k)), \quad K \in \mathbb{N} \cup \{+\infty\}$$

(6)

where $P$ is a probability distribution on $\mathbb{R}^D \times \mathcal{O}(D)$ (where $\mathcal{O}(D)$ is the set of unitary matrices in $\mathbb{R}^D$) and is either a Dirichlet process prior or a mixture of finite mixtures.
• **Hybrid location-scale mixtures**: The density $f_P$ is written as (5) where $P$ conditionally on a probability $Q_2$ on $\mathbb{R}^D$ follows a Dirichlet process mixture or a mixture of finite mixtures with base measure $H_0(d\mu, dO, d\lambda) = H_1(d\mu, dO) \otimes Q_2(d\lambda)$, and $Q_2$ follows a distribution $\tilde{\Pi}_\Lambda$.

We denote by $\Pi$ the prior on the parameter and we consider the following assumptions on $\Pi$. These conditions differ whether $\Pi$ is assumed to come from a partial location-scale mixture prior or a hybrid location-scale mixture prior.

**Conditions on the partial location-scale mixtures.** $f$ is modelled as in (6) and $P$ follows either a Dirichlet process with base measure $H$ or a mixture of finite mixtures with base measure $H$ and prior on $K$ satisfying

$$-\log \Pi_K(K = x) \approx x (\log x)^r, \quad r = 0, 1.$$  

(7)

Here $r = 0$ corresponds to the geometric prior on $K$ and $r = 1$ to the Poisson one. The base measure $H(d\mu, dO) = h(\mu, O) \, d\mu \, dO$ where $d\mu$ designates the Lebesgue measure on $\mathbb{R}^D$ and $dO$ the Haar measure on $O(D)$. And we further assume that there exist $c_1, b_1 > 0$ and $b_2 > 2D - 1$ such that

$$e^{-c_1\|\mu\|^{b_1}} \leq h(\mu, O) \leq (1 + \|\mu\|)^{-b_2} \quad \text{with} \quad \forall \mu, O. \quad (8)$$

We also assume that $\Lambda$ is drawn from a probability measure $\Pi_\Lambda$ that has a density $\pi_\Lambda$ with respect to Lebesgue measure on $\mathbb{R}^D$, and that this density satisfies: there exist $c_2, c_3, b_3 > 0$ and $b_4 > 2D - 1$ such that

$$e^{-c_2 \sum_{i=1}^D \frac{1}{\lambda_i}} \leq \Pi_\Lambda(\lambda_1, \ldots, \lambda_D) \quad \text{for small} \quad \lambda_1, \ldots, \lambda_D \in (\mathbb{R}_+)^D,$n

$$\Pi_\Lambda\left(\min_{1 \leq i \leq D} \lambda_i < x\right) \leq e^{-c_3 x^{-b_3}} \quad \text{for small} \quad x > 0,$$

(9)

$$\Pi_\Lambda\left(\max_{1 \leq i \leq D} \lambda_i > x\right) \leq x^{-b_4} \quad \text{for large} \quad x > 0.$$

Condition (8) is weak and is for instance satisfied as soon as $\mu$ and $O$ are independent under $H$ with positive and continuous density for $O$ and positive density for $\mu$ with weak tail assumptions. Condition (9) is also weak and common in the case of location Gaussian mixtures and is verified in particular if the $\sqrt{\lambda_i}$’s are independent inverse Gammas under $\Pi_\Lambda$, or if the $\lambda_i$’s are independent inverse Gammas and $d \geq 2$.

**Conditions on the hybrid location-scale mixtures.** $H_1$ satisfies (8) and $Q_2$ is random with distribution $\tilde{\Pi}_\Lambda$ which satisfies: for all $b > 0$ there exists $B_0, c_2 > 0$ such that for $2x_1 \leq x_2$ both small,

$$\tilde{\Pi}_\Lambda\left[Q_2\left([x_1, x_1(1 + x_1^b)]^D \times [x_2, x_2(1 + x_2^b)]^{D-d}\right) \geq x_1^{B_0}\right] \geq e^{-c_2 x_1^{-d/2}}.$$

(10)

Moreover we assume that for some positive constant $c_3, b_3, c_4, b_4 > 0$ such that for $x > 0$ small,

$$\mathbb{E}_{\tilde{\Pi}_\Lambda}\left[Q_2\left(\min_{1 \leq i \leq D} \lambda_i \leq x\right)\right] \leq e^{-c_3 x^{-b_3}} \quad \text{for small} \quad x,$n

$$\mathbb{E}_{\tilde{\Pi}_\Lambda}\left[Q_2\left(\max_{1 \leq i \leq D} \lambda_i > x\right)\right] \leq e^{-c_4 x^{b_4}} \quad \text{for large} \quad x.$$  

(11)
Remark 3. One can view the partial location-scale mixture as a special instance of the hybrid location-scale mixture defined above: take $Q_2$ to be the Dirac mass at a value $\Lambda$ where $\Lambda \sim \Pi_\Lambda$.

Conditions (10) and (11) are in particular satisfied if $Q_2$ comes from a Dirichlet process. More precisely, if $Q_2$ is of the form

$$Q_2(d\lambda) = \prod_{i=1}^{D} Q_0(d\lambda_i) \quad \text{with} \quad Q_0 \sim \text{DP}(BH_\lambda),$$

with $B > 0$, then (10) and (11) are satisfied for reasonable choices of probability distribution $H_\lambda$ on $\mathbb{R}_+$. We show in the next proposition that this is in particular true when $H_\lambda$ is a square-root- inverse Gamma.

Proposition 2.4. Assume that $Q_2 = Q_0^{\otimes D}$ where $Q_0 \sim \text{DP}(BH_\lambda)$, where $B > 0$ and $H_\lambda$ has density $h_\lambda$ verifying

$$e^{-c_2\lambda^{-1/2}} \mathbb{1}_{\lambda \leq 1} \leq h_\lambda(\lambda) \leq e^{-c_3\lambda^{-b_3}} e^{-c_4\lambda^{b_4}},$$

then conditions (10) and (11) are satisfied.

A proof of Proposition 2.4 can be found in Appendix B.4. For instance if $\sqrt{\lambda}$ follows an inverse Gamma truncated on $[0,R]$ for arbitrarily large $R$ under $H_\lambda$, then Proposition 2.4 holds.

Remark 4. Although the conditions on the prior: (9) and (10) depend on $d$, they are satisfies for all $d$ by setting $d = 1$ and in particular they are verified for all $d \geq 1$ if $\sqrt{\lambda_i}$ follow an inverse Gamma under the base measure, which is agnostic to $d$.

| Conditions | MFM | DPM | Partial | Hybrid |
|------------|-----|-----|---------|--------|
| (7)+(8) | (8) | (9) | (10)+(11) |

Table 1: Summary table of the required conditions depending on the type of mixture and the type of scale sampling.

3 Main results

3.1 Posterior contraction rates

Recall that $X_1, \ldots, X_n$ is an $n$ sample drawn from a distribution $P_0$ with density $f_0$. This density is concentrated around a submanifold $M$, with a a given smoothness $\beta_0$ along the manifold and a typically much larger smoothness $\beta_\perp$ along the normal spaces. More precisely, we will assume:

- **Conditions on $M$:** the submanifold $M$ is of dimension $d$ and has a reach greater than $\tau > 0$. Furthermore, there exists $\beta_M > 2$ and $C_M > 0$ such that $\Psi_{x_0} \in \mathcal{H}_{\text{iso}}^{\beta_M}(V_{x_0}, C_M), \forall x_0 \in M$. In particular, $M$ also satisfies (3).
• **Conditions on** $f_0$: the density $f_0$ is in $H_{\delta_0, \beta_1}(M, L)$. Furthermore, there exists $c_5, \kappa > 0$, 

$$f_0(x) \leq e^{-c_5|x|^\kappa} \quad \forall x \in \mathbb{R}^D,$$

and for some $\omega > 6\beta$ and $C_0 < \infty$,

$$\int_{W_{x_0, \delta}} \left| \frac{D^k \tilde{f}_{x_0, \delta}}{\tilde{f}_{x_0, \delta}} \right|^{\omega/(k, \alpha)} \tilde{f}_{\delta, x_0} \leq C_0 \quad \text{and} \quad \int_{W_{x_0, \delta}} \left| \frac{L_{x_0, \delta}}{\tilde{f}_{x_0, \delta}} \right|^{\omega/\beta} \tilde{f}_{\delta, x_0} \leq C_0. \quad (13)$$

for all $\delta$ small, $x_0 \in M$ and all $0 \leq (k, \alpha) < \beta$

• **Conditions on** $\Pi$: the prior $\Pi$ is originating from a Partial / Hybrid location-scale mixture of finite mixtures / Dirichlet process mixture satisfying the conditions displayed in the Table 1.

**Remark 5.** Note that the conditions regarding the prior $\Pi$ do not involve $M, \delta, L, \beta, \tau$: they are regarded as unknown in this framework. In fact, the only feature of $M$ or $f_0$ from which $\Pi$ seems to depend is the intrinsic dimension $d$, through (9) or (10). However, as noted in Remark 4 we can choose priors which do not depend on $d$ and such that these assumptions are verified for all $d \geq 1$.

In the rest of this paper the symbols $\equal, \preceq$ and $\succeq$ denote equalities or inequalities up to a constant depending on $D, d, \tau, \beta_M, \beta_0, \beta_1$ and all the other parameters appearing in conditions (7) to (13).

**Theorem 3.1.** Let $X_1, \ldots, X_n$ be a $n$-sample from $f_0$ satisfying (12) and (13) with $\omega > 6\beta + (2\beta + D)(D - d) \log(1/\delta)/\log n$ and that $\beta_0 \leq \beta_1 \leq \beta_M - 3$. Then, under the conditions stated above,

$$\Pi(d_H(f_P, f_0) \geq \varepsilon_n \mid X_n) \rightarrow 0 \quad \text{in $\mathbb{P}_0^n$-probability}$$

where

$$\varepsilon_n \simeq \log^p n \times \left\{ \frac{1}{\sqrt{n \delta^{\beta_0/\alpha_1}}} \lor \frac{1}{n^{-(\beta_0/2\beta_1 + d)}} \right\}, \quad (14)$$

with $p > 0$ depending on $D, \kappa$ and $\beta$.

**Remark 6.** The case where $\beta_1$ is infinite is particularly of interest. Then, $\beta \rightarrow \beta_0$, $\alpha_0 \rightarrow D/d$, $\alpha_1 \rightarrow 0$ and the rate $\varepsilon_n$ becomes

$$\varepsilon_n^\infty = \log^p(n) \times \left\{ \frac{1}{\sqrt{n \delta^{\beta_0/\alpha_1}}} \lor \frac{1}{n^{-(\beta_0/2\beta_0 + d)}} \right\},$$

which is, when $\delta$ is not too small (i.e. $\delta \succeq n^{-1/(2\beta_0 + d)}$), the minimax rate for estimating a $\beta_0$ Hölder density in $\mathbb{R}^d$, up to a log term. Here the strength of our result lies in that the manifold (and thus the support of $f_0$) is unknown and the prior depends neither on $\beta$ nor on $\delta$ or $d$ (or $M$).
Since the class of densities contains the case where $M$ is a $d$ dimensional subspace of $\mathbb{R}^D$, when $\delta \geq n^{-(\alpha_0-\alpha_1)/(2\beta+D)}$ the rate $\varepsilon_n$ is the minimax estimation rate (up to a log $n$ term). Although no proof of a minimax bound exists in our framework, a careful look at the proof of [31, Thm 4 (ii)] for $p = 1, r = (\infty, \ldots, \infty), s = \infty$ and $\theta = 1$ (tail dominance from (12)) show that the lower-bound translates in our context. Indeed the densities used to derived the lower bound are obtained from a smooth density with additive perturbations of the form $x \mapsto h^\beta \prod g(x_i/h^{\alpha_i})$ where $h > 0$ and $g$ is a smooth and compactly supported function of zero-mean. Such a perturbations belong to anisotropic Hölder classes defined in Section 2.1.

**Remark 7.** Note that following our proof technics, a refinement of [59] shows that location mixture priors are able to achieve the rate of $n^{-\frac{\beta_0}{2\beta_0+d+D}}$ (up to a polylog factor) in the models of Proposition 2.3 when the manifold $M$ is in fact a $d$-dimensional subspace of $\mathbb{R}^d$, thanks to the simple euclidian anisotropic smoothness property of the underlying density (up to an orthogonal change of basis adapted to the linear subspace). Our result is thus in particular a nonlinear extension of this work, for which location mixture priors are not sufficiently flexible to approximate the densities of interest.

**Remark 8.** Because the approximation results of Subsection 3.2 are stable under finite mixtures, so do the results of Theorem 3.1. In particular, the support of $f_0$ can be a finite union of submanifolds $M_i$ with non trivial intersections and with each $M_i$ fulfilling (2). See Figure 3 for a diagram of such a situation. Hence the assumption on the lower bound on the reach can be significantly weakened. We consider such an example in the simulations of Section 4.

![Figure 3](image.png)

**Figure 3:** (Left) An example of smooth submanifolds with a reach constrained to be greater than $\tau$ and (Right) a finite union of such manifolds. Both subsets are admissible as the (near) support for a density $f_0$ satisfying the contraction rates displayed in Theorem 3.1, as explained in Remark 8.

Assumptions (12) and (13) are common assumptions in density estimation based on mixtures of Gaussians, see for instance [41] or [59] for the Bayesian approaches and [46] for the frequentist approaches. They are rather weak assumptions. The difficulty with (13) is that it is expressed on $\tilde{f}_{x_0,\delta}$, which is natural in our context since the smoothness assumption on $f_0$ is expressed in terms of $\tilde{f}_{x_0,\delta}$, but is not so intuitive. However, a careful examination of (13) show that this assumption is for instance implied by the stronger, but chart-independent assumption:

$$\int_{\mathbb{R}^D} \left\{ \frac{L(x)}{f_0(x)} \right\}^{\omega_*} f_0(x) \, dx \leq C_0,$$
for some large $\omega^\ast$. Moreover, to understand better what (13) means in terms of $f_0$, we illustrate it in our archetypal model where $f_0$ is the density of $X = Y + \delta \mathcal{E}$ as in Proposition 2.3. We then have the following result:

**Lemma 3.2.** Under the conditions of Proposition 2.3 and if

$$
\int_M \left\{ \frac{L_0(x)}{f_\ast(x)} \right\}^{\omega^\ast} f_\ast(x) \, d\mu_M(x) < \infty \quad \text{and} \quad \int_{B(0,1)} \left\{ \frac{L_1(\eta)}{K(\eta)} \right\}^{\omega^\ast} K(\eta) \, d\eta < \infty,
$$

for some $\omega^\ast$ large. Then (13) is satisfied in both the orthonormal noise model and in the isotropic noise model.

The proof of Lemma 3.2 can be found in Section B.3. Note that the conditions in Lemma 3.2 are fulfilled by a number of natural kernels or densities such as $K(\eta) \approx (1 - \|\eta\|^2)_+^p$ and $L_1 \approx (1 - \|\eta\|^2)_+^{p - \beta_1}$ with large $p$ or $K(\eta) \approx \exp(-(1 - \|\eta\|^2)_+^{-1})$ and $L_1(\eta) \approx (1 - \|\eta\|^2)^{-\beta M} K(\eta)$, see Lemma A.5 for further details.

### 3.2 Approximating M-anisotropic densities

Theorem 3.1 is proved using the approach of [28], with a control on the prior mass of Kullback-Leibler neighbourhoods of $f_0$ and on the entropy of the support of the prior. The main difficulty in our setup is in proving the Kullback-Leibler prior mass condition. To do so, we need to construct an efficient approximation of $f_0$ by mixtures of Gaussian. This construction is of interest in its own as it sheds light on the behaviour of such mixtures and on the geometry of M-anisotropic densities.

To explain the construction, we denote, for any $x \in M^\tau$, $T_x = T_{\text{pr}_M(x)} M$ and $N_x = N_{\text{pr}_M(x)} M$. We also write $\Sigma(x) = O_x^T \Delta^2_{\sigma,\delta} O_x$ where $O_x$ is the matrix in the canonical basis of $\mathbb{R}^D$ and in arbitrary orthonormal basis of $T_x$ and $N_x$ of the linear map $z \mapsto (\text{pr}_{T_x} z, \text{pr}_{N_x} z)$ and where

$$
\Delta_{\sigma,\delta} = \begin{pmatrix} \sigma^{\alpha_0} \text{Id}_d & 0 \\ 0 & \delta \sigma^{\alpha_1} \text{Id}_{D - d} \end{pmatrix}.
$$

Note that $\Sigma_x$ does not depend on the choice of an orthonormal basis of $T_x$ and $N_x$ since for any orthonormal base change $P$ that preserves $T_x$, one have $P^T \Delta_{\sigma,\delta} P = \Delta_{\sigma,\delta}$. For any function $f : \mathbb{R}^D \to \mathbb{R}$, we define

$$
K_{\Sigma} f(x) := \int_{M^\tau} \varphi_{\Sigma(y)}(x - y) f(y) \, dy,
$$

where we recall that $\varphi_{\Sigma(y)}$ is the density of a centered Gaussian with variance $\Sigma(y)$. The idea behind the construction of the approximation of $f_0$ by a mixture of gaussians is to show that $K_{\Sigma} f_0(x)$ is close to $f_0$ and then to define a perturbation $f_1$ of $f_0$ such that $K_{\Sigma} f_1(x) - f_0(x) = O(L(x) \sigma^\beta)$. Compared to the construction proposed by [41] in the univariate case or [59] in the multivariate case where $K_{\Sigma} f = \varphi_{\Sigma} * f$, in our construction $\Sigma$ varies with the location $y$. Note that in particular $\int \varphi_{\Sigma(y)}(x - y) \, dy$ may be different from 1. This dependence in $y$ is crucial to adapt to the geometry of the manifold but considerably complicates the proof as the underlying kernel integral operator can no longer be written as a convolution. We first show that $f_0$ can be efficiently approximated pointwise.
The function $g$ and $\delta \sigma$ where $z$ and $\alpha$'s being fixed ahead.

We present two different inference approaches: Gibbs sampling (using [53, Algo 8]) and the aforementioned priors to describe in a relevant way data that can be very singular. More specifically, we consider the hierarchical model:

$$(y_i)_{i=1}^n \mid (\mu_i, O_i)_{i=1}^n, \Lambda \sim \bigotimes_{i=1}^n \mathcal{N}(\mid \mu_i, O_i \Lambda O_i^T) \quad \text{with} \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_D),$$

$$(\mu_i, O_i)_{i=1}^n \mid P \sim P^{\otimes n}, \quad P \sim \text{DP}(\alpha P_0), \quad P_0 = \mathcal{N}(\mid \mu_0, \Sigma_0) \otimes \text{Unif}(O(D))$$

$$(\lambda_j)_{j=1}^D \mid (b_j)_{j=1}^D \sim \bigotimes_{j=1}^D \text{InvGamma}(a_j, b_j), \quad (b_j)_{j=1}^D \sim \bigotimes_{j=1}^D \text{Exp}(\kappa_j)$$

We set in our experiments the value of the hyperparameters as $\alpha = 1$, $a_j = \kappa_j = 1$ for all $1 \leq j \leq D$ and $\mu_0 = 0$, $\Sigma_0 = I_D$. Alternatively, we also consider the previous model with the $b_j$’s being fixed ahead.

We present two different inference approaches: Gibbs sampling (using [53, Algo 8]) and stochastic approximation of the Maximum A Posteriori estimator (MAP) using the Python package pyro [8]. The former is an asymptotically exact MCMC sampler while the latter
is an approximate (but way easier to compute) point estimator. We refer the reader to the Section E for further details of the implementation.

The manifolds we use for the experiments are a 2D spiral, two intersecting circles, a 3D spiral and a torus, see Figures 4 to 7. Precise parametrizations are given in Section E.1. As for the data generating distributions, we use the additive isotropic noise model (as described in Proposition 2.3), with noise kernel $K_{\beta_j}(x) \propto (1 - \|x\|^2)^{\beta_j}$, and base densities chosen as follows: we take the uniform distribution for the two-circles and for the torus, whereas for the 2D and 3D spiral, we push-forward through the embeddings (described in Section E.1) the probability distribution $g_{\beta_0}$ on $[0, 1]$ given by $g_{\beta_0}(t) \propto (1 - (1 - 2t)^{\beta_0})1_{t \in [0,1/2]} + (1 - (2t - 1)^{\beta_0+1})1_{t \in [1/2,1]}$. In the experiments, the regularities were set to $\beta_0 = 2$ and $\beta_1 = 6$.

We test different values for $\delta$ and challenge the model (16) for various choices of $b_j$'s against the conjugate Normal Inverse-Wishart prior on these datasets (for which we choose to use the R package dirichletprocess [45]). Since the Gibbs sampling algorithm is computationally very intensive, we compare this algorithm with the much lighter approximate MAP on the 2-dimensional datasets, with a sample of size $n = 300$, see Figures 4 and 5.

As seen in these figures, the hierarchical prior (16) seem to perform much better than the prior with fixed $b_j$ (unless these are chosen very carefully) and much better than the Normal Inverse-Wishart, which is also consistent with the empirical results of [49]. Our theoretical results together with our empirical results strongly suggest that this conjugate DP mixture is suboptimal is this case.

Thanks to the low computational cost of approximating the MAP, we were able to conduct experiments on 3D datasets, the 3D-spiral and the torus. See Figures 6 and 7.

![Figure 4](image-url) (First column) We observe $n = 300$ points drawn under $P_0$ for the 2D-spiral with $\delta = 0.1$ (Top row) and $\delta = 0.01$ (Bottom row). We then sample the same amount of points using the Gibbs sampler with (Second column) Inverse-Wishart prior, (Third column) model (16) with $b_j = 1$, (Forth column) model (16) with $b_j = 0.001$ and (Fifth column) model (16) with the exponential hyperprior on $b_j$. Finally, we used in (Sixth column) the approximate MAP distribution.
Figure 5: Same experiment as in Figure 4 but for the two overlapping circles.

Figure 6: We observe \( n = 1000 \) points (blue plots) and sample the same amount through the approximate MAP distribution (orange plots). The width \( \delta \) is set to (Left two plots) 0.1 and (Right two plots) to 0.01.

Figure 7: Same experiment as in Figure 6 for the torus with \( n = 10000 \) points observed and sampled and with respectively (Left two plots) \( \delta = 0.5 \) and (Right two plots) \( \delta = 0.05 \).

5 Proofs of the main results

5.1 Proof of Theorem 3.1

Theorem 3.1 is proved using [28, Thm 5], which relies on two things: making sure that the prior probability distribution puts enough mass around the true density \( f_0 \) (Kullback-Leibler condition), and ensuring that most of its probability mass is concentrated on a subset of manageable entropy (entropy condition).

Proof of Theorem 3.1. The most challenging part of the proof is the kullback Leibler condition. To verify it we define for any \( \varepsilon > 0 \),

\[
\mathcal{B}(f_0, \varepsilon) = \left\{ f : \mathbb{R}^D \to \mathbb{R} \ \big| \ P_0 \left( \log \frac{f_0}{f} \right) \leq \varepsilon^2 \ \text{and} \ \ P_0 \left( \log^2 \frac{f_0}{f} \right) \leq \varepsilon^2 \right\},
\]

(17)
and introduce
\[ \tilde{\varepsilon}_n := \delta^{\sigma_0} n^{-\beta} \quad \text{and} \quad \varepsilon_n := \left\{ \frac{C_{1/2}}{\sqrt{n}} \varepsilon_n^{-D/2\beta} \log^{1/2}(1/\varepsilon_n) \right\} \vee \{ \tilde{\varepsilon}_n \log^s(1/\tilde{\varepsilon}_n) \} \]

where \( s, t \) and \( C \) are introduced in Lemma 5.2. The sequence \( \tilde{\varepsilon}_n \) goes to 0 and is such that \( \tilde{\varepsilon}_n^s < \delta^2 \varepsilon_n^s \) so that \( B(f_0, \tilde{\varepsilon}_n \log^s(1/\tilde{\varepsilon}_n)) \subseteq B(f_0, \varepsilon_n) \). The control of \( \Pi(f_P \in B(f_0, \varepsilon_n)) \) relies first on Theorem 3.3 which constructs explicitly an approximation of \( f_0 \) by a continuous location scale mixture of Gaussians \( K_{f_1} \) with \( f_1 \) close to \( f_0 \) and then on a discrete approximation of \( K_{f_1} \); it is formally stated in Lemma 5.2 below and implies that \( \Pi(f_P \in B(f_0, \tilde{\varepsilon}_n \log^s(1/\tilde{\varepsilon}_n)))) \geq \exp \left( -C \varepsilon_n^{-D/\beta} \log^s(1/\varepsilon_n) \right) \geq \exp(-n \varepsilon_n^2) \).

To verify the entropy condition of [28, Thm 5], we define, for any sequence \( \varepsilon_n \) going to 0, \( F_n = F_n(\varepsilon_n, R_0, H_0, \sigma_0, \sigma_1) \) to be the set of all probability density function \( f_P \) with \( P = \sum_{h \geq 1} \pi_h \delta_{\mu_h, \Lambda_h} \) such that
\[
\sum_{h \geq H_n} \pi_h \leq \varepsilon_n, \quad \forall h \leq H_n, \mu_h \in B(0, R_n) \quad \text{and} \quad \forall h \leq H_n, \Lambda_h \in \mathcal{Q}_n = [\sigma_n^2, \sigma_n^2]^D, \quad U_h \in \mathcal{O}(D)
\]

where \( R_n = \exp(R_0 n \varepsilon_n^2), \; H_n = \lfloor H_0(n \varepsilon_n^2)/\log n \rfloor, \; \sigma_n^2 = \sigma_n^2(n \varepsilon_n^2)^{-1/\beta_3} \) for some positive constant \( R_0, H_0, \sigma_0 \) and \( \sigma_1 \) and where, for some \( \sigma_1 > 0 \),

- \( \sigma_n^2 = \exp(\sigma_1^2 n \varepsilon_n^2) \) in the case of the Partial location-scale mixture

- \( \sigma_n^2 = \sigma_1^2(n \varepsilon_n^2)^{1/\beta_4} \) in the case of the Hybrid location-scale mixture.

We then show in Lemma 5.1 below that \( \Pi(F_n(\varepsilon_n)) \leq \exp(-c_n n \varepsilon_n^2) \) as soon as \( n \varepsilon_n^2 \geq n^\omega \) for some \( \omega > 0 \).

We now define, for \( j = (j_h, h \leq H_n) \in \mathbb{N}^{H_n} \),
\[
F_{n,j} := \{ f_P \in F_n | \forall h \leq h_n, \; j_h \sqrt{n} \leq \| \mu_h \| \leq (j_h + 1) \sqrt{n} \}
\]

along with the following refinement
\[
F_{n,j,0} := \{ f_P \in F_{n,j} | \max_i \lambda_i \leq n \}
\]

and \( \forall \ell \geq 1, \; F_{n,j,\ell} := \{ f_P \in F_{n,j} | \sqrt{n^\ell} \leq \max_i \lambda_i \leq n^\ell \} \).

In Lemma 5.1 below we show that for the partial location scale prior
\[
\sum_{j,\ell} \sqrt{\Pi(F_{n,j,\ell}) N(\varepsilon_n, F_{n,j,\ell}, \| \cdot \|)} e^{-M_0 n \varepsilon_n^2} = o(1);
\]

while for the hybrid one the same holds true with \( F_{n,j} \).

Hence [28, Thm 5] implies that the posterior contracts at rate \( \varepsilon_n \), so long as \( n \varepsilon_n^2 \geq n^{t_0} \) for some \( t_0 > 0 \). We now distinguish three cases. Denoting \( p = s \vee t/2 \), there holds:

1. If first \( \delta^{\sigma_0} n^{-\beta} \leq n^{-1} \) then the results of Theorem 3.1 is trivial and there is nothing to show (because the contraction rate goes to \( \infty \) instead of 0);
2. If \( \delta_{a_{0}^{-1}}^{D/\delta_{a_{0}^{-1}}} \geq n^{-\frac{D}{2\beta + D}} \), then \( \tilde{\varepsilon}_n = n^{-\frac{\beta}{2\beta + D}} \) and one easily get that \( \varepsilon_n \simeq n^{-\frac{\beta}{2\beta + D}} \log^p n \). In particular, \( n \varepsilon_n^2 \gg n^\delta \) for \( \delta = D/(2\beta + D) \).

3. If finally \( n^{-1} < \delta_{a_{0}^{-1}}^{D/\delta_{a_{0}^{-1}}} < n^{-\frac{D}{2\beta + D}} \), then \( \tilde{\varepsilon}_n = \delta_{a_{0}^{-1}}^{D/\delta_{a_{0}^{-1}}} \) and \( \log(1/\delta) \simeq \log n \) so that
\[
\sqrt{\varepsilon_n} \geq C \tilde{\varepsilon}_n^{D/\beta} \log^t (1/\tilde{\varepsilon}_n) \simeq \delta_{a_{0}^{-1}}^{D/\delta_{a_{0}^{-1}}} \log^t (n) \gg n^{\frac{D}{2\beta + D}}.
\]

Finally to understand how \( \varepsilon_n \) depends on \( \delta \) in this last case, note that by assumption, \( \log(1/\tilde{\varepsilon}_n) \leq \log(1/\delta) \leq \log(n) \) and \( \tilde{\varepsilon}_n < n^{-\beta/(2\beta + D)} \), so that
\[
\varepsilon_n \approx \log^p n \times \left( \frac{\tilde{\varepsilon}_n^{-D/2\beta}}{\sqrt{n}} \vee \tilde{\varepsilon}_n \right) = \log^p n \times \frac{1}{\sqrt{n} \delta^{D/(\alpha_{0}^{-1})}}.
\]

We end this section with the presentations of the technical lemmata that are keys to the proof of Theorem 3.1.

**Lemma 5.1.** Under assumptions of Table 1, for any sequence \( \varepsilon_n \to 0 \) such that \( n \varepsilon_n^2 \simeq n^\delta \), for some \( \omega > 0 \), and for all \( c_1 > 0 \), if \( R_0, H_0, \sigma_1^2 \) are large enough and \( \sigma_0^2 \) is small enough, there exists \( M_0 > 0 \) such that,

i) \( \Pi(F_n^c) \lesssim \exp(-c_1 n \varepsilon_n^2) \);

ii) In the case of the partial location-scale prior
\[
\sum_j \sqrt{\Pi(F_{n,j})} N(\varepsilon_n, F_{n,j}, \|\cdot\|_1) e^{-M_0 \varepsilon_n^2} = o(1);
\]

iii) In the case of the hybrid location-scale prior
\[
\sum_j \sqrt{\Pi(F_{n,j})} N(\varepsilon_n, F_{n,j}, \|\cdot\|_1) e^{-M_0 \varepsilon_n^2} = o(1).
\]

The proof of Lemma 5.1 can be found in Section D.1. The last elementary brick in the proof of Theorem 3.1 is the control of the probability of small balls around \( \mathbb{P}_0 \), which is stated below.

**Lemma 5.2.** Let \( \varepsilon > 0 \) and assume that it is small enough so that \( \varepsilon_{a_{0}^{-1}}^{1} \leq \delta^{1} \varepsilon_{a_{1}^{-1}}^{1} \) and that \( \varepsilon_{a_{1}^{-1}}^{1} \ll \log^{-1}(1/\varepsilon) \). Then, in the context of Theorem 3.1, there holds
\[
\Pi \left( f \in B(f_0, \varepsilon) \right) \lesssim \exp \left\{ -C \varepsilon^{D/\beta} \log^t (1/\varepsilon) \right\},
\]
where \( B(f_0, \cdot) \) is defined in (17), where the constant \( C \) depends on the parameters, where \( \tilde{\varepsilon} \approx \varepsilon \log^s (1/\varepsilon) \) and with \( s, t > 0 \) depending on \( D, \beta \) and \( \kappa \).

The complete proof is given in Section D.2 and is sketched as follow: the first step is the Hellinger approximation of \( f_0 \) by \( K_{2} \tilde{h} \) as expressed in Corollary 3.4. Then we exhibit an \( \varepsilon \)-approximation of \( K_{2} \tilde{h} \) by a discrete location scale mixture with a controlled number of atoms through the use of Lemma 5.3 below. The result then follows from similar arguments as in [59, 41] or [52].
Lemma 5.3. Let \( \varepsilon > 0 \) such that \( \delta \sigma^a \log(1/\varepsilon) \ll 1 \). For any density \( g \) on \( \mathbb{R}^d \) satisfying (12), there exists a discrete probability measure \( G \) on \( \mathbb{R}^D \) with at most \( N \approx \sigma^{-D} \log^D (1/\varepsilon) \) atoms such that
\[
\| K_G - K_{\Sigma}g \|_\infty \leq \frac{\varepsilon}{\sigma^D \delta^D \cdot \delta^d} \quad \text{and} \quad \| K_G - K_{\Sigma}g \|_1 \leq \varepsilon \log^{D/2} (1/\varepsilon).
\]
The atoms of \( G \) are in \( M^\delta \) and are \( \sigma^{2\alpha_0} \varepsilon \)-apart.

The proof of Lemma 5.3 can be found in Section D.3. We underline that although it uses similar ideas to [29, Lem 3.1], it is not a straightforward adaptation of it, since in \( K_G \) the covariances depend on the locations of the mixture in a complicated way.

5.2 Proof of Theorem 3.3

As explained in Section 3.2 a key ingredient of the proof of Theorem 3.1 is the pointwise approximation of \( f_0 \) by \( K_G \hat{g} \) where \( g \) is close to \( f_0 \) and is explicited in the proof of Theorem 3.3 below.

Proof of Theorem 3.3. Let \( x_0 \in M \) and define
\[
\mathcal{W}_{x_0}^j := B_{T_{x_0}} \left( 0, \frac{2 + j}{16} \tau \right) \times B_{N_{x_0}} \left( 0, \frac{6 + j}{8} \tau \right) \quad \text{for} \quad j \in \{0, 1, 2\},
\]
and \( \mathcal{O}_{x_0}^j = \Psi_{x_0}(\mathcal{W}_{x_0}^j) \) for \( j \in \{0, 1, 2\} \). We have \( \mathcal{W}_{x_0}^0 \subset \mathcal{W}_{x_0}^1 \subset \mathcal{W}_{x_0}^2 \) and \( \mathcal{O}_{x_0}^0 \subset \mathcal{O}_{x_0}^1 \subset \mathcal{O}_{x_0}^2 \). Furthermore, the sets \( \mathcal{O}_{x_0}^0 \) for \( x_0 \in M \) forms a covering of \( M^{3\tau/4} \), see Section A.3 for more details.

We now drop the \( x_0 \) from the notation. Let \( f : \mathbb{R}^D \to \mathbb{R} \) be in \( \mathcal{H}_{2,3}^{\beta_0,\beta_1}(M,L) \) and supported on \( \mathcal{O}^0 \) — it is to be thought of as \( f_0 \) multiplied by a smooth function supported on \( \mathcal{O}^0 \). Take \( x \in \mathcal{O}^1 \) and compute
\[
K_{\Sigma}f(x) := \int_{\mathbb{R}^D} \varphi_{\Sigma(u)}(x - u) f(u) \, du = \int_{M^{\delta \tau} \cap \mathcal{O}^0} \varphi_{\Sigma(u)}(x - u) f(u) \, du.
\]
We first prove that we can construct a function \( g \) such that \( K_{\Sigma}g \) is close to \( f \) and we then apply this result to \( f = f_0 \chi_j \) with \( \chi_j \) the partition of unity defined in Lemma A.5.

The idea is to use the fact that \( \Sigma(u) = o(1) \) and the smoothness of \( u \mapsto \Sigma(u) \) so that
\[
K_{\Sigma}f(x) \approx \int_{M^{\delta \tau} \cap \mathcal{O}^0} \varphi_{\Sigma(x)}(x - u) f(u) \, du \approx f(x).
\]
We now write down the approximation rigorously and quantify the error, taking into account the geometry of the manifold \( M \). In all that follows, we use the notation \( z = (v, \eta) \) for points belonging to \( B_{T_{x_0}M}(0, \tau/16) \times B_{N_{x_0}M}(0, \tau/2) \), while throughout \( w = (v, \delta \eta) \in B_{T_{x_0}M}(0, \tau/16) \times B_{N_{x_0}M}(0, \delta \tau/2) \). We first make the change of variable \( w = \Psi^{-1}(u) \), yielding
\[
K_{\Sigma}f(x) = \int_{\Psi^{-1}(M^{\delta \tau} \cap \mathcal{O}^0)} \varphi_{\Sigma(\Psi(w))}(x - \Psi(w)) f(\Psi(w)) | \det d\Psi(w) | \, dw.
\]
Then, denoting by \( w_x = \Psi^{-1}(x) \), we write
\[
w = \Delta_{\sigma, \delta} z + w_x = \Delta_{\sigma, \delta} z + \Delta_{1, \delta} z_x, \quad w = (v, \delta \eta, \delta), \quad z_x = \Delta_{1, \delta}^{-1} w_x,
\]
in the integral above, giving
\[ K_{\Sigma}f(x) = \frac{1}{(2\pi)^{D/2}\delta^{D-d}} \int_{\Delta_{x,1}^{-1}(\mathcal{W}^0-z_x)} e^{-B_\sigma(x,z)} \tilde{f}_\delta(\Delta_{\sigma,1}z + z_x) \zeta(\Delta_{\sigma,\delta}z + w_x) \, dz, \]  
(18)

with
\[ B_\sigma(x,z) := \frac{1}{2} \|x - \Psi(\Delta_{\sigma,\delta}z + w_x)\|_{\Sigma^{-1}}^2(\Psi(\Delta_{\sigma,\delta}z + w_x)) \]

and
\[ \zeta(\Delta_{\sigma,\delta}z + w_x) := |\det d\Psi(\Delta_{\sigma,\delta}z + w_x)|, \]

and where we used the fact that $|\det \Sigma(u)|$ is constantly $\sigma^D \delta^{D-d}$ for $u \in M^\tau$. Since $z \mapsto \tilde{f}_\delta(\Delta_{\sigma,1}z + z_x)$ is zero outside of $\Delta_{\sigma,1}^{-1}(\mathcal{W}^0-z_x)$, we can replace the latter set with $T_{x_0}M \times N_{x_0}M \approx \mathbb{R}^D$ in the integral above. We now develop each term separately. First
\[ \zeta(\Delta_{\sigma,\delta}z + w_x) = |\det d\Psi(w_x)| + \sum_{1 \leq |k| < \beta_M - 2} \frac{(\Delta_{\sigma,\delta}z)^k}{k!} D^k \zeta(w_x) + R_x^\zeta(x,z) \]

with $R_x^\zeta(x,z) \lesssim \|\Delta_{\sigma,\delta}z\|^{\beta_M-2}$, up to a constant that depends on $C_M$. Secondly, there holds
\[ \tilde{f}_\delta(\Delta_{\sigma,1}z + z_x) = \tilde{f}_\delta(z_x) + \sum_{0 < |(k,\alpha)| < \beta} \frac{z^k}{k!} \sigma^{(k,\alpha)} D^k \tilde{f}_\delta(z_x) + R_\sigma(x,z) \]  
(19)

with $R_\sigma(x,z) \lesssim D \delta^{D-d}f(x)$.

It remains to understand $B_\sigma(x,z)$. Notice that
\[ \Phi(\Delta_{\sigma,\delta}z + w_x) = x + d\Phi(w_x)[\Delta_{\sigma,\delta}z] + \sum_{2 \leq |k| < \beta_M - 1} \frac{(\Delta_{\sigma,\delta}z)^k}{k!} D^k \Phi(w_x) + R_\Phi(x,z). \]  
(20)

with again $R_\Phi(x,z) \lesssim \|\Delta_{\sigma,\delta}z\|^{\beta_M-1}$. Let write $k = (k_0, k_1)$ with $k_0 \in \mathbb{N}^d$ and $k_1 \in \mathbb{N}^{D-d}$. We know that $\Phi$ is affine with respect to its second variable, so that
\[ D^{(k_0,k_1)} \Phi = 0 \quad \text{for} \quad |k_1| \geq 2. \]  
(21)

The sum in the RHS of (20) thus rewrites
\[ \sum_{k_1=0}^{k_0|\alpha_0|} \frac{z^{k_1}}{k_1!} D^{k_1} \Phi(w_x) + \sum_{|k_1|=1} \frac{z^{k_1}}{k_1!} D^{k_1} \Phi(w_x) \]  
(22)

and $R_\sigma^\Phi$ is bounded by the more precise quantity
\[ \|R_\sigma^\Phi(x,z)\| \lesssim \sum_{|k|\geq|\beta_M-2|} |\Delta_{\sigma,\delta}z|^k \|\Delta_{\sigma,\delta}z\|^{\beta_M-1-|\beta_M-2|}. \]
Furthermore, the first differential of $\bar{\Psi}$ reads
\[
d\bar{\Psi}(w_x)[\Delta_{\sigma, \delta} z] = \sigma^{-\alpha_0} d\Psi(v_x)[v] + \sigma^{-\alpha_0} \text{pr}_{T_x} d^v N(w_x)[v] \epsilon_{T_x} + \sigma^{-\alpha_0} \text{pr}_{N_x} d^v N(w_x)[v] + \delta \sigma^{\alpha_1} N(v_x, \eta).
\]

Now recall that $u = \bar{\Psi}(\Delta_{\sigma, \delta} z + w_x)$ and that
\[
B_\sigma(x, z) = \frac{1}{2} \|x - u\|^2_{\Sigma^{-1}(u)} = \frac{1}{2\sigma^{2\alpha_0}} \|\text{pr}_v (x - u)\|^2 + \frac{1}{2\sigma^{2\alpha_0}} \|\text{pr}_{N_x} (x - u)\|^2
\]

Using the development of $u$ that we have in (20,22), we find that, noting again $k = (k_0, k_1)$, for any $y \in \mathbb{R}^D$,
\[
\text{pr}_{T_x}(y) = \text{pr}_{T_x}(y) + \sum_{1 \leq |k| < \beta_M - 1 \atop |k_0| \geq 1} (\Delta_{\sigma, \delta} z)^k \Phi_k^T(w_x)[y] + R^T_\sigma(x, y, z)
\]

with
\[
|R^T_\sigma(x, y, z)| \leq \|y\| \sum_{|k| = \beta_M - 2 \atop |k_0| \geq 1} |\Delta_{\sigma, \delta} z|^k \|\Delta_{\sigma, \delta} z\|^{\beta_M - 1 - \beta_M _{-2}}.
\]

and for some $(\beta_M - 1 - |k|)$-Hölder functions $\Phi_k^T(w_x)$ (which are smooth functions of $D^f\Psi(w_x)$ for $1 \leq |\ell| \leq |k|$, see for instance Lemma A.2). Here the important fact is that the sums in both displays above start at $|k_0| \geq 1$. This is because:

i) $s \mapsto \text{pr}_M(x + s)$ is constant over the fiber $N_x$, so there is no sole contribution of the normal part of $u - x$ (only mixed contribution with the tangential part) in (24);

ii) The tangential part of $d\bar{\Psi}(w_x)[\Delta_{\sigma, \delta} z]$ does not inherit the contribution of the normal displacement in $\delta \alpha^1$, as shown in (23);

iii) In (22), neither sums contain any term of type $(0, k_1)$.

Plugging $y = u - x = \bar{\Psi}(\Delta_{\sigma, \delta} z + w_x) - x$ in (24) and using (23) yields
\[
\sigma^{-\alpha_0} \text{pr}_{T_x}(x - u) = d\Psi(v_x)[v] + \text{pr}_{T_x} d^v N(w_x)[v] + \sum_{2 \leq |k| < \beta_M - 1 \atop |k_0| \geq 1} \sigma^{-\alpha_0} (\Delta_{\sigma, \delta} z)^k \bar{\Phi}_k^T(w_x) + R^T_\sigma(x, z)
\]

with $\bar{\Phi}_k^T$ being a again smooth functions of $D^f\bar{\Psi}(w_x)$ for $|\ell| \leq |k|$, and where
\[
\|R^T_\sigma(x, z)\| \leq \sigma^{-\alpha_0} \sum_{|k| = \beta_M - 2 \atop |k_0| \geq 1} |\Delta_{\sigma, \delta} z|^k \|\Delta_{\sigma, \delta} z\|^{\beta_M - 1 - \beta_M _{-2}} \leq \delta^{\beta_M - 2} \sigma^{\alpha_1} \|z\|^{\beta_M - 1}
\]

up to a constant that depends on $C_M$. Now notice that
\[
\text{pr}_{N_x}(z) = z - \text{pr}_{T_x} z = \text{pr}_{N_x} z - \sum_{1 \leq |k| < \beta_M - 1 \atop |k_0| \geq 1} (\Delta_{\sigma, \delta} z)^k \Phi_k^T(w_x)[z] - R^T_\sigma(x, y, z)
\]
whence again plugging \( z = u - x \) and using (23)

\[
\delta^{-1} \sigma^{-\alpha_1} \text{pr}_{N_u}(x - u) = N(v_x, \eta) + \sigma^{\alpha_0 - \alpha_1} \delta^{-1} \text{pr}_{N_u} d^v N(w_x)[v] + \sum_{2 \leq |k| < \beta M - 1 \atop |k_0| > 1} \sigma^{-\alpha_1} \delta^{-1} (\Delta_{\sigma, \delta} z)^k \Phi_k^N(w_x) + R^N_\sigma(x, z)
\]

where again \( \Phi_k^N(w_x) \) is polynomial in \( D^k \Psi(x) \) for \( \ell \leq k \) and

\[
\| R^N_\sigma(x, z) \| \lesssim \sigma^{-\alpha_1} \delta^{-1} \sum_{|k| < \beta M - 1} \| \Delta_{\sigma, \delta} z^k \| \Delta_{\sigma, \delta} z \|^{\beta M - 1 - \lfloor \beta M - 2 \rfloor} \leq \delta^{\beta M - 3} \sigma^{-\alpha_1} (\beta M - 3) \| z \|^{\beta M - 1}.
\]

Recall that by assumption \( \sigma^{\alpha_0} \lesssim \delta \sigma^{\alpha_1} \) so that \( R_N \) is of greater order than \( R_T \). There thus exists functions \( \Phi_k \) similar to \( \Phi_k^T \) and \( \Phi_k^N \) such that

\[
B_\sigma(x, z) = \frac{1}{2} \| A(w_x)[z] \|^2 + \sum_{1 \leq |k| < \beta M - 1 \atop |k_0| > 1} \sigma^{-\alpha_1} \delta^{-1} (\Delta_{\sigma, \delta} z)^k \Phi_k(w_x) + R_B^B(x, z)
\]

with \( R_B^B(x, z) \lesssim \delta^{\beta M - 2} \sigma^{(\beta M - 2) \alpha_1} \| z \|^{\beta M - 1} \) and where

\[
A(w_x)[v, \eta] := d\Psi(v_x)[v] + \text{pr}_{T_x} d^v N(w_x)[v] + N(v_x, \eta).
\]

We can rewrite the development of \( B_\sigma \), up to a slight modification of the \( \Phi_k \) and of \( R_B \), which we write again \( \Phi_k \) and \( R_B \) with a slight abuse of notation, in the following form

\[
-B_\sigma(x, z) + \frac{1}{2} \| A(w_x)[z] \|^2 = \log \left\{ 1 + \sum_{1 \leq |k| < \beta M - 1 \atop |k_0| > 1} \sigma^{-\alpha_1} \delta^{-1} (\Delta_{\sigma, \delta} z)^k \Phi_k(w_x) + R_B^B(x, z) \right\}
\]

or again

\[
e^{-B_\sigma(x, z)} = e^{-\frac{1}{2} \| A(w_x)[z] \|^2} \left\{ 1 + \sum_{1 \leq |k| < \beta M - 1 \atop |k_0| > 1} \sigma^{-\alpha_1} \delta^{-1} (\Delta_{\sigma, \delta} z)^k \Phi_k(w_x) + R_B^B(x, z) \right\}
\]

All in all, we obtain a development of \( K_\Sigma f(x) \) of the form

\[
K_\Sigma f(x) = \sum_{0 \leq (k, \alpha) < \beta} \sum_{0 \leq |m| < \beta M - 2} \sigma^{(k + m, \alpha)} \delta^{|m|} \xi_{k,0,m}(x) + \sum_{0 \leq (k, \alpha) < \beta} \sum_{1 \leq |\ell_1| < \beta M - 1} \sum_{0 \leq |m| < \beta M - 2} \sigma^{(k + m, \alpha)} \delta^{|m|} \xi_{k,\ell,m}(x) + R_\sigma(x)
\]

(26)
with \(|R_\sigma(x)| \leq \{\sigma^\beta + \sigma^\beta(\beta M - 2)/\beta, \delta^\beta M - 2\} L(x) \leq \sigma^\beta L(x)\) as soon as \(\beta M - 2 \geq \beta_1\). The term \(L(x)\) appears in the control of \(R_\sigma\), because every term in the remainder is multiplied by one term of the development (19) of \(f_\delta\), and each one of this term is upper bounded by \(\beta_1\). The term \(\xi_{k,\ell,m}(x)\) (for both \(\ell = 0\) and \(\ell > 0\) with the convention \(\Phi_0 = 1\)) is exactly

\[
D^k f_\delta(z_x) \times \Phi_\ell (w_x) \times D^m \zeta (w_x) \times \frac{1}{(2\pi)^{D/2} D - d} \int_{R_D} e^{-\frac{1}{2} ||A(w_x)[z]||^2} x^{(k+\ell+m)} \text{d}z.
\]

The zero-th order term is equal to

\[
\bar{f}_\delta(z_x) \times \zeta (w_x) \times \frac{1}{(2\pi)^{D/2} D - d} \int_{R_D} e^{-\frac{1}{2} ||A(w_x)[z]||^2} \text{d}z = f(x) \times |\det \Psi (w_x)| \times \frac{1}{|\det A(w_x)|}
\]

But recall that

\[
d\Psi (w_x)[z] = d\Psi (v_x)[v] + \text{pr}_{T_x} d^\nu N(w_x)[v] + \text{pr}_{N_x} d^\nu N(w_x)[v] + N(v_x, \eta)
\]

which, written from an orthonormal basis concatenated from orthonormal bases of \(T_{x_0}\) and \(N_{x_0}\) to an orthonormal basis concatenated from orthonormal bases of \(T_x\) and \(N_x\) leads to a matrix which is block triangular inferior with diagonal blocks corresponding to \(d\Psi (v_x)[v] + \text{pr}_{T_x} d^\nu N(w_x)[v]\) and \(N(v_x, \eta)\) so that \(|\det d\Psi (w_x)| = |\det A(w_x)|\). Now for the higher-order terms, notice that the map

\[
w \mapsto D^m \zeta (w) \times \Phi_\ell (w) \times \frac{1}{(2\pi)^{D/2}} \int_{R_D} e^{-\frac{1}{2} ||A(w)[\eta]||^2} \eta^{(k+\ell+m)} \text{d}\eta
\]

belongs, in application of Proposition B.6 and Proposition B.1, to \(\mathcal{H}_{\beta_0,\beta_1}^{\beta M - 2 - |m|\nu |\ell|} (\mathcal{W}_{x_0,\delta}, C)\) for some \(C\) depending on \(C_M\) and \(\tau\). Likewise, \(z \mapsto D^k f_\delta(z)\) belongs to \(\mathcal{H}_{\beta M}^{\beta(k)} (\mathcal{W}_{x_0,\delta}, L)\) according to Proposition B.1. Using Proposition B.6 once again and the definition of manifold-driven Hölder spaces, one get that

\[
\xi_{k,\ell,m} \in \mathcal{H}_{\beta_0,\beta_1}^{\beta M} (M, CL) \quad \text{with} \quad \left\{\begin{array}{ll}
\beta_0 &= \left\{\beta_0 - \langle k, \alpha \rangle/\alpha_0\right\} \land \left\{\beta_M - 2 - |m| \nu |\ell|\right\}
\beta_1 &= \left\{\beta_1 - \langle k, \alpha \rangle/\alpha_1\right\} \land \left\{\beta_M - 2 - |m| \nu |\ell|\right\}
\end{array}\right.
\]

Looking at (26), if we were to prove that

\[
\left\{\begin{array}{ll}
\langle k + m, \alpha \rangle + \beta_0 \geq \beta & \text{if } \ell = 0
\langle k + \ell + m, \alpha \rangle + \beta_0 \geq \beta & \text{if } |\ell_0| > 1
\end{array}\right.
\]

then we would get, by induction, that there exists a function \(g\), supported on \(O^0\), of the form

\[
g(x) = f(x) + \frac{1}{\delta^{D-d}} \sum_{0 < \langle k, \alpha \rangle < \beta} \sigma^{(k, \alpha)} d_k (x, \sigma, \delta) D^k f_\delta (z_x)
\]

with \(d_k\) uniformly bounded, such that

\[
|K_{\Sigma} g(x) - f(x)| \leq L(x) \sigma^\beta \quad \forall x \in O^1.
\]

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It only remains to prove (27). We start with $\ell = 0$. Then $|m|\sqrt{|\ell|} = |m|$ and since $\beta_0 - \langle k, \alpha \rangle/\alpha_0 \leq \beta_1 - \langle k, \alpha \rangle/\alpha_1$, there are three cases to cover: case 1.: $\beta_M - 2 - |m| \geq \beta_1 - \langle k, \alpha \rangle/\alpha_1$; case 2.: $\beta_M - 2 - |m| \leq \beta_0 - \langle k, \alpha \rangle/\alpha_0$ and case 3.: $\beta_0 - \langle k, \alpha \rangle/\alpha_0 \leq \beta_M - 2 - |m| \leq \beta_1 - \langle k, \alpha \rangle/\alpha_1$.

We start with case 1: in this case, $\bar{\beta} = \beta - \langle k, \alpha \rangle$ and (27) follows immediately. In both case 2 and case 3, notice that denoting $\bar{\alpha}_1 := \bar{\beta}/\bar{\beta}_1$, we either have $\bar{\alpha}_1 = 1 \geq \alpha_1$ (case 2) or $\bar{\alpha}_1 = \bar{\beta}/\bar{\beta}_1 > \alpha_1$ because $\bar{\beta}_1/\bar{\beta}_0 \leq \beta_1/\beta_0$ by assumption (case 3). This yields that

$$
\bar{\beta} = (\beta_M - 2 - |m|\sqrt{|\ell|})\bar{\alpha}_1 \geq (\beta_M - 2 - |m|)\alpha_1 \geq (\beta_M - 2)\alpha_1 - \langle m, \alpha \rangle
$$

ending the proof of (27) for $\ell = 0$ since $\beta_M - 2 \geq \beta_1$. Now for the case $|\ell| > 1$, if $|\ell| \leq |m|$, the previous reasoning still holds and we get

$$
\langle k + \ell + m, \alpha \rangle + \bar{\beta} \geq \langle \ell, \alpha \rangle + \langle k + m, \alpha \rangle + \bar{\beta} \geq \alpha_0 + \beta.
$$

If finally $|\ell| > |m|$, then we can write $\ell = \tilde{\ell} + \epsilon_i$ for some $i \in \{1, \ldots, d\}$ so that $|\ell| = |\tilde{\ell}| + 1$ and

$$
\langle k + \ell + m, \alpha \rangle + \bar{\beta} \geq \alpha_0 + \langle k + \tilde{\ell}, \alpha \rangle + \bar{\beta}.
$$

Now noticing that $\beta_M - 2 - |\ell| = \beta_M - 3 - |\tilde{\ell}|$, we obtain, using the same reasoning as above, that $\langle k + \tilde{\ell}, \alpha \rangle + \bar{\beta} \geq \beta$ as soon as $\beta_M - 3 \geq \beta_1$, ending the proof of (27).

We are now ready to prove Theorem 3.3. We let

$$
R = \{ H \log(1/\sigma) \}^{1/\kappa}
$$

and $\chi_1, \ldots, \chi_j$ be the functions defined at Lemma A.5 from a $\tau/64$-packing of $M \cap \mathbf{B}(0, R)$. Recall that we can always choose $J$ of order less than $R^D$. In light of the point (iv) of Lemma A.5, the function $f_j := \chi_j f_0$ is still in $\mathcal{H}^{\beta_0, \beta_1}(M, CL)$ for some constant $C$ depending on $\tau$ and $\beta_1$. Since $\text{supp} \ f_j \subset O^0_{x_j}$ (point (i) of Lemma A.5), the first part of this proof yields that there exists some functions $g_j$ supported on $O^1_{x_j}$, such that

$$
|K_{x_0} g_j(x) - f_j(x)| \leq L(x)\sigma^\beta
$$

uniformly on $O^1_{x_j}$. Now notice that for $x$ outside of $O^1_{x_j}$, we have $d(x, O^0_{x_j}) > (\sigma^{a_0} \vee \delta^a \sigma^{-a}) \sqrt{(H + D) \log(\sigma)}$ so that

$$
|K_{x_0} g_j(x)| \leq \int_{\mathbf{O}^0_{x_j}} |\varphi_{x_0}(u) (x - u) g_j(u)| \, du \leq \frac{\sigma^H}{(2\pi)^{D/2} \delta^D d} \int_{\mathbf{O}^0_{x_j}} |g_j(u)| \, du \leq \sigma^H \sup_{\mathbf{O}^0_{x_j}} L(x)
$$

and the equality (28) extends to the whole set $\mathbb{R}^D$ with the bound $\sigma^H \|L\|_{\infty}$ on $\mathbb{R}^D \setminus O^1_{x_j}$. Using the linearity of $K_{x_0}$, we thus find that for $g = \sum_{j=1}^J g_j$, and for any $x \in \mathbb{R}^D$, there holds,

$$
|K_{x_0} g(x) - f_0(x)| \leq \sum_{j=1}^J |K_{x_0} g_j(x) - f_j(x)| \leq \sum_{j \in J(x)} |K_{x_0} g_j(x) - f_j(x)| + \sum_{j \notin J(x)} |K_{x_0} g_j(x)|
$$

$$
\leq |J(x)| \times L(x)\sigma^\beta + (J - |J(x)|) \times \sigma^H \|L\|_{\infty}
$$

$$
\leq \sigma^\beta L(x) \times \{ H \log(1/\sigma) \}^{D/\kappa} \sigma^H \|L\|_{\infty}
$$

where we denoted $J(x) = \{ 1 \leq j \leq J \mid x \in O^1_{x_j} \}$, and used the fact that $J(x)$ is bounded from above by something depending on $D$ and $\tau$ only, ending the proof. \qed
6 Discussion

With the aim of developing Bayesian procedures in the framework of manifold learning, we exhibited a new family of priors based on location-scales of Dirichlet mixture of Gaussians, and described a general setting for studying density supported near a submanifold. The latter relies on two things: first, a parametrization of the offset of the manifold and second, an anisotropic class of Hölder functions. In this model, we obtained concentration rates in Theorem 3.1 for the associated posterior distribution that are adaptive to the regularity of the underlying density while being totally agnostic of the underlying submanifolds and their main features. Our procedure is therefore fully adaptive.

An interesting feature of our theoretical framework is that it allows to express the rate in terms of the smoothnesses of the density and the manifold together with the thickness $\delta$ of the support around the manifold. When $\delta$ is fixed, our results can be viewed as an extension of minimax rates for regular anisotropic densities to manifold driven anisotropic densities. But we are also considering the regime where $\delta = o(1)$ which corresponds to the manifold learning problem. The rates obtained in Theorem 3.1 have two regimes: one when $\delta$ is not too small with rate $n^{-\beta/(2\beta+D)}$ (up to log $n$ terms) and the concentrated regime where $\delta$ is very small ($\delta = o(n^{-1/(2\beta+D)}))$ where we obtain the rate $(n\delta^{D/(\alpha_0-\alpha_1)})^{-1/2}$. It is not clear if this latter rate is optimal or not. The case $\delta = 0$ would correspond to the observations belonging to the manifold $M$ (and for which we would expect the rate $n^{-\beta_0/(2\beta_0+d)}$) but cannot be thought as a limiting case of our problem since then the distribution has density with respect to the Hausdorff measure on $M$ and not with respect to the Lebesgue measure on $\mathbb{R}^D$. When $M$ is unknown the model is not dominated and our approach is not applicable. The problem of posterior contraction rates when the distribution lives on an unknown manifold remains open, although some interesting ideas in [60] or [10] could be used to address it.

Another interesting output of our results is that if nonparametric mixtures of normal densities define a versatile and flexible model for smooth densities, the structure of the prior on the mixing distribution is crucial. In this paper we propose two classes of priors which we believe enjoy many strong theoretical properties while remaining reasonably simple to implement. Moreover variational Bayes algorithms using pyro can be easily implemented and for which the same theoretical guarantees hold. Moreover, in order to make the MCMC algorithm more scalable to the dimension $D$ and to the number of observations $n$, we could use a variant based on parallel computing for Dirichlet process mixtures (see for instance [47]). It is quite possible that other nonparametric mixture models such as the Fisher-Gaussian kernels of [50] would enjoy the same theoretical guarantees and we believe that our approximation result can be useful to study the theoretical properties of mixtures of Fisher-Gaussian kernels which are strongly related to Gaussian kernels.

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A Some facts on submanifolds with bounded reach

A.1 The geometry of submanifolds with bounded reach

The reach $\tau_K$ of a closed subset $K \subset \mathbb{R}^D$, initially introduced by [22, Def 4.1 p.432], is the supremum of all the $r > 0$ such that the orthogonal projection from the $r$-offset $K^r = \{x \in \mathbb{R}^D \mid d(x, K) \leq r\}$ to $K$ is well-defined, namely

$$\tau_K := \sup \{r \geq 0 \mid \forall x \in K^r, \forall y, z \in K, d(x, K) = \|x - y\| = \|x - z\| \Rightarrow y = z\}.$$ 

When the reach of a closed submanifold $M \subset \mathbb{R}^D$ is bounded away from zero, $M$ enjoys a number useful properties, that we list and prove below. In all the results stated hereafter, the reach of $M$ is bounded from below by some $\tau > 0$. Lemma A.1 provide already existing results from the literature, sometimes slightly rephrased to better suit our needs. We start off with a control of the exponential map over submanifold with bounded curvature together with a comparison between the intrinsic distance on $M$, denoted by $d_M(.,.,)$, and the ambient Euclidean distance. Recall that for any $x, y \in M$, $d_M(x, y)$ is the infimum of the length of all continuous path between $x$ and $y$ in $M$ and if $x$ and $y$ are in two separate path-connected components, then $d_M(x, y) = \infty$.

**Lemma A.1.** The following facts hold true

i) For any $x \in M$, the exponential map $\exp_x$ is a diffeomorphism from $B_{T_xM}(0, \pi \tau)$ to $\exp_x \{B_{T_xM}(0, \pi \tau)\}$.

ii) It is double Lipschitz from $B_{T_xM}(0, \tau/4)$ to its image with

$$\forall v, w \in B_{T_xM}(0, \tau/4), \; \frac{11}{16} \|v - w\| \leq \|\exp_x(v) - \exp_x(w)\| \leq \frac{21}{16} \|v - w\|. \tag{29}$$

iii) Letting $\kappa : \gamma \mapsto 2(1 - \sqrt{1 - \gamma})/\gamma$. If $\|x - y\| \leq \gamma \tau/2$ with $\gamma \leq 1$, then

$$\|x - y\| \leq d_M(x, y) \leq \kappa(\gamma) \|x - y\|. \tag{30}$$

iv) Finally, if $\|x - y\| \leq \tau/2$, there holds

$$\|\pr_{T_xM} - \pr_{T_yM}\|_{op} \leq \frac{d_M(x, y)}{\tau} \leq \frac{2}{\tau} \|x - y\|. \tag{31}$$

**Proof.** The first result on $\exp_x$ is an application of [2, Thm 1.3]. For ii), denoting $R_x(v) = \exp_x(v) - x - v$, there holds that,

$$\|\exp_x(v) - \exp_x(w)\| - \|v - w\| \leq \|R_x(v) - R_x(w)\| \leq \frac{5}{16} \|v - w\|$$

where we used [1, Lem 1]. Finally iii) comes from the monotonicity of $\kappa$ and [55, Prp 6.3], and using [9, Lem 6], there holds $|d_M(x, y)/\tau$, which, together with iii) for $\gamma = 1$, leads to iv).
We now wish to define the parametrization of the $\tau/2$-offset of $M$ that we introduced in Subsection 2.2. This requires to identify in a non-ambiguous way every normal fiber $N_x M$ to the base fiber $N_{x_0} M$ for $x$ in the vicinity of $x_0$. A natural way to do that would be to use parallel transport, and define

$$N_{x_0}(v, \eta) := t_\gamma(\eta)$$

where $t_\gamma : N_{x_0} M \to N_{\exp_{x_0}(v)} M$ is the parallel transport along the path $\gamma(s) := \exp_{x_0}(sv)$. We refer to [43, Sec 4] for a formal introduction to parallel transport.

In order to make things more comprehensible for the reader who is unfamiliar with parallel transport, and in order to have clear and quantitative controls and the quantity at stake, we suggest another, more elementary approach. We assert that the two approaches yields similar regularity classes as introduced in Subsection 2.2. We start off with a few notations. For a matrix $A \in \mathbb{R}^{D \times D}$ and $1 \leq k \leq D$, we let $V_k(A)$ be the vector space spanned by the first $k$ columns of $A$. We denote by $\text{Norm} : x \mapsto x/\|x\|$. We let

$$G : GL(D, \mathbb{R}) \to O(D, \mathbb{R})$$

be the Gram-Schmidt process, defined recursively on the columns of any invertible matrix $A = (A_1, \ldots, A_D)$ as $G(A) = (G_1(A), \ldots, G_D(A))$ with

$$G_1(A) := \text{Norm}(A_1) \quad \text{and} \quad \forall 1 \leq j < D - 1, \quad G_{j+1}(A) := \text{Norm}(G_{j+1}(A))$$

where $G_{j+1}(A) := A_{j+1} - \sum_{1 \leq i < j} (A_{j+1, i} G_i(A)) G_i(A)$. Because $G$ is such that $V_k(A) = V_k(G(A))$ for every $1 \leq k \leq D$, there holds that $G_{k+1}(A) = \text{pr}_{V_{k}(G(A))}(A_{k+1})$, and that $G_j(A)$ is thus non zero everywhere, so that $G$ is a well-defined, smooth application. In order to bound its derivatives, we need to control how $G_k$ is far away from zero. We introduce

$$GL_{\varepsilon}(D, \mathbb{R}) := \{ A \in GL(D, \mathbb{R}) \mid d(A_{k+1}, V_k(A)) > \varepsilon \quad \forall 1 \leq k \leq D \},$$

so that $\|G_k(A)\| > \varepsilon$ for every $k$ and any $A \in GL_{\varepsilon}(D, \mathbb{R})$, and thus straightforwardly all the derivatives, up to any order, of $G$ are bounded on $GL_{\varepsilon}(D, \mathbb{R})$. We let $B_0$ be an arbitrary basis of $T_{x_0} M$, $B_1$ be an arbitrary basis of $N_{x_0} M$ and let $B = (B_0, B_1)$. We define, for $v \in B_{T_{x_0} M}(0, \tau/4)$, $A_{x_0}(B, v) := (d\Psi_{x_0}(v)[B_0], B_1)$. Note that since $\Psi_{x_0}$ is a diffeomorphism, there holds

$$V_d(A_{x_0}(B, v)) = \text{Vect}(d\Psi_{x_0}(v)[B_0]) = T_{\Psi_{x_0}(v)} M.$$  \hspace{1cm} (32)

Set

$$N^B_{x_0}(v, \cdot) := G(A_{x_0}(B, v))[\cdot], \quad \Psi^B_{x_0}(v, \eta) := \Psi_{x_0}(v) + N^B_{x_0}(v, \eta).$$

We show in Lemma A.2 that $N^B_{x_0}$ and $\Psi^B_{x_0}$ are well defined and smooth, which combined with (32) yields in particular that $N^B_{x_0}(v, \cdot)$ is an isometry between $N_{x_0} M$ and $N_{\Psi_{x_0}(v)} M$.

**Lemma A.2.** For any $v \in B_{T_{x_0} M}(0, \tau/4)$, there holds that $A_{x_0}(B, v) \in GL_{\varepsilon}(D, \mathbb{R})$ with $\varepsilon = 1/2^d$. Consequently, the map $v \mapsto N^B_{x_0}(v, \cdot)$ is in $\mathcal{H}^{\beta M}_{iso} (B_{T_{x_0} M}(0, \tau), C)$ for some constant $C_M$, $\tau$, $\beta_M$ and $D$ (and not on $B$).

Moreover for any basis $B$, it holds that $|d \Psi^B_{x_0}(v, \eta)| \geq (3/16)^d$ for any $v \in B_{T_{x_0} M}(0, \tau/4)$ and $\eta \in B_{N_{x_0} M}(0, \tau/2)$.  

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Proof. First notice that for \( d \leq k \leq D - 1 \) and \( v \in B_{T_0 M}(0, \tau / 4) \), there holds, letting \( A = A_{x_0}(B, v) \), and because of (32) and the fact that \( B_\perp \) is an orthonormal frame of \( N_{x_0} M \),

\[
d(A_{k+1}, V_k(A)) = d(A_{k+1}, T_{\Psi_{x_0}(v)} M) \geq 1 - \| \text{pr}_{T_0 M} - \text{pr}_{T_{\Psi_{x_0}(v)} M} \|_{op} \geq 3/4
\]

where we used (31). Now we let \( 1 \leq k \leq d - 1 \) and let \( V_0(A) = \{0\} \). Letting \( Q = (A_1, \ldots, A_d) \), here holds that

\[
\prod_{k=0}^{d-1} d(A_{k+1}, V_k(A))^2 = \det Q = \det d\Psi_{x_0}(v) d\Psi_{x_0}(v).
\]

Using [1, Lem 1], it holds \( \| d\Psi_{x_0}(v) - \iota \|_{op} \leq 5/16 \) for \( v \in B_{T_0 M}(0, \tau / 4) \) where \( \iota : T_{x_0} M \to \mathbb{R}^D \) is the inclusion so that for any \( h \in T_{x_0} M \), \( \| d\Psi_{x_0}(v)[h] \| \geq (1 - 5/16)\| h \| = 11/16\| h \| . \) In particular \( \det \Psi_{x_0}(v)^T d\Psi_{x_0}(v) \geq (11/16)^2d \). Since now

\[
d(A_{k+1}, V_k(A))^2 \leq \| A_{k+1} \|^2 \leq \| d\Psi_{x_0}(v) \|^2_{op} \leq (21/16)^2
\]

there holds that for any \( 1 \leq k \leq d \),

\[
d(A_{k+1}, V_k(A))^2 \geq (16/21)^2(11/16)^2d \geq (11/21)^2d \geq 1/2^{2d}.
\]

Using that the Gram-Schmidt transform is smooth on \( GL_\varepsilon(D, \mathbb{R}) \) with \( \varepsilon = 1/2^d \) and Proposition B.7, we obtain that the map \( v \mapsto N^B_{x_0}(v, \cdot) \) is in \( H^1_{\text{iso}}(B_{T_0 M}(0, \tau), C) \).

Also, in \( B \) and in orthonormal bases of \( T_{\Psi_{x_0}(v)} M \) and \( N_{\Psi_{x_0}(v)} \), the Jacobian of \( \tilde{\Psi}^B_{x_0}(v, \eta) \) write

\[
\begin{pmatrix}
    d\Psi_{x_0}(v) + \text{pr}_{T_{\Psi_{x_0}(v)} M} \circ d_v N^B_{x_0}(v, \eta) & 0 \\
    \text{pr}_{N_{\Psi_{x_0}(v)} M} \circ d_v N^B_{x_0}(v, \eta) & N^B_{x_0}(v, \cdot)
\end{pmatrix}
\]

so that \( | \det \tilde{\Psi}^B_{x_0}(v, \eta) = | \det \{ d\Psi_{x_0}(v) + \text{pr}_{T_{\Psi_{x_0}(v)} M} \circ d_v N(v, \eta) \} | \). We saw earlier in the proof that \( \| d\Psi_{x_0}(v) \|_{op} \geq 11/16 \). Furthermore, using (31), we find that for any small \( w \in T_{x_0} M \),

\[
\| \text{pr}_{T_{\Psi_{x_0}(v)} M} \{ N^B_{x_0}(v + w, \eta) - N^B_{x_0}(v, \eta) \} \| = \| (\text{pr}_{T_{\Psi_{x_0}(v + w)} M} - \text{pr}_{T_{\Psi_{x_0}(v)} M} ) N^B_{x_0}(v + w, \eta) \|
\leq \| w \|_{\eta},
\]

and consequently \( \| \text{pr}_{T_{\Psi_{x_0}(v)} M} \circ d_v N^B_{x_0}(v, \eta) \|_{op} \leq \| \eta \| / \tau \leq 1/2 \). Thus, for any \( h \in T_{x_0} M \), \( \| d\Psi_{x_0}(v)[h] + \text{pr}_{T_{\Psi_{x_0}(v)} M} \circ d_v N(v, \eta)[h] \|_{op} \geq (11/16 - 1/2)\| h \| \geq 3/16\| h \| \) and thus

\[
| \det \tilde{\Psi}^B_{x_0}(v, \eta) | \geq (3/16)^d.
\]

An important feature of the parametrizations \( \tilde{\Psi}^B_{x_0} \) is that the subsequent Hölder classes as defined in Definition 2.2 do not depend, up to a universal constant, on the choice of the collection of basis \( (B_{x_0})_{x_0 \in M} \). This is shown in Section A.2.
A.2 Stability by a change of basis

We now show that a change of basis does not interfere with the anisotropic regularity of a map seen through $\Psi_{x_0}^B$.

Lemma A.3. For any orthonormal basis $B' = (B'_{0}, B'_1)$ subordinated to $T_{x_0}M$ and $N_{x_0}M$, and for any $\delta > 0$, it holds that

$$\left(\Psi_{x_0}^B\right)^{-1} \circ \Psi_{x_0,\delta}^{B'}(v, \eta) = (v, C_{B,B'}(v)\eta) \quad (33)$$

where $C_{B,B'}$ is independent of $\delta$ and is in $\mathcal{H}^{-1}_{iso}(B_{T_{x_0}M}(0, \tau), C)$ for some constant $C$ depending on $C_M$, $\tau$, $\beta_M$ and $D$ (and not on $B$ and $B'$).

Proof. Short and simple computations shows that $C_{B,B'}(v) := N_{x_0}^B(v, \cdot)^T N_{x_0}^{B'}(v, \cdot)$ so that an application of Lemma A.2 with Proposition B.6 immediately yields the result. □

Corollary A.4. In the context of Subsection 2.2, assume that $\beta_0 \leq \beta_1 \leq \beta_M - 1$. Then, there exists a constant $C$ depending on $C_M$, $\tau$, $\beta_M$ and $D$ such that, if there exists a basis $B$ such that $f \circ \Psi_{x_0,\delta}^B \in \mathcal{H}^\beta_{an}(W_{x_0,\delta}, L_{x_0,\delta})$, then, for any other orthonormal basis $B'$, $f \circ \Psi_{x_0,\delta}^{B'} \in \mathcal{H}^\beta_{an}(W_{x_0,\delta}, C L_{x_0,\delta}^{B_{x_0,\delta}})$.

Proof. Using the lemma above, there holds

$$f \circ \Psi_{x_0,\delta}^{B'} = f \circ \Psi_{x_0,\delta}^{B'}(v, C_{B,B}(v)\eta) = f \circ \Psi_{x_0,\delta}^B \circ J(v, \eta)$$

where $J$ is defined through (33). We denote for short $f^{B'} = f \circ \Psi_{x_0,\delta}^{B'}$, $f^B = f \circ \Psi_{x_0,\delta}^B$ so that $f^{B'} = f^B \circ J$. Taking $(k, \alpha) < \beta$ and using the multivariate Faa di Bruno formula [16], we find that $D^k f^{B'}$ is a sum of products of the form

$$D^\ell(f^B) \circ J \times \prod(D^{k^{(j)}} J)^{\ell^{(j)}}$$

subject to $|\ell| \leq |k|$, $\sum_j \ell^{(j)} = \ell$ and $\sum_j |\ell^{(j)}| k^{(j)} = k$ with $k^{(j)} \neq 0$. Now notice that $(D^{k^{(j)}} J)_i = 0$ for $1 \leq i \leq d$ as soon as $k^{(j)} \neq 0$. For a configuration of $\ell, \ell^{(j)}$ and $k^{(j)}$ such that the above product is not zero, there thus holds

$$|\ell_0| = \sum_j |\ell^{(j)}_0| = \sum_{k^{(j)}_0=0} |\ell^{(j)}_0| \leq \sum_{k^{(j)}_0=0} \sum_{k^{(j)}_0=0} |\ell^{(j)}_0| k^{(j)}_0 \leq |k_0|$$

which, together with $|\ell| \leq |k$ and $\alpha_0 \geq \alpha_1$, yields that $(\ell, \alpha) < (k, \alpha)$ whenever the above product is non zero. We conclude with a telescopic argument with Lemma A.3. □

A.3 Partitions of unity and packings

The approximation result uses a particular covering of an offset of the manifold $M$, which we describe here. Take $x_0 \in M$ and define

$$\mathcal{W}_{x_0}^j := B_{T_{x_0}} \left(0, \frac{2 + j}{16} \tau \right) \times B_{N_{x_0}} \left(0, \frac{6 + j}{8} \tau \right)$$

for $j \in \{0, 1, 2\}$,
and $O^j_{x_0} = \bar{\Psi}_{x_0}(O^j_{x_0})$ for $j \in \{0, 1, 2\}$. We have

$$W^0_{x_0} \subset W^1_{x_0} \subset W^2_{x_0} \quad \text{and} \quad O^0_{x_0} \subset O^1_{x_0} \subset O^2_{x_0}.$$  

Furthermore, the sets $O^0_{x_0}$ for $x_0 \in M$ forms a covering of $M^{3\pi/4}$. See Figure 8 for an illustration of these open sets.

In what follows, we will need the notion of packing. An $\varepsilon$-packing of a subset $A \subset \mathbb{R}^D$ is a set $\{y_1, \ldots, y_J\}$ of points of $A$ such that $\|x_j - x_k\| > \varepsilon$ for any $1 \leq i \neq j \leq J$ and such that no set of $J + 1$ points has this property. We denote by $\text{pk}(A, \varepsilon) := J$ the $\varepsilon$-packing number of $A$. By maximality of $J$, it is straightforward to see that $A$ is covered by the union of the balls $B(x_j, \varepsilon)$. Furthermore, the balls $B(x_j, \varepsilon/2)$ must be disjoint by definition of a packing so that

$$\text{pk}(A, \varepsilon) \times \min_{x \in A} \text{vol}(A \cap B(x, \varepsilon/2)) \leq \text{vol}\left\{A \cap \bigcup_{1 \leq j \leq J} B(x_j, \varepsilon/2)\right\} \leq \text{vol}(A).$$

In the case where $A$ is a ball of radius $R$ with $R$ large before $\varepsilon$, it is straightforward to see that $\text{vol}(A \cap B(x, \varepsilon/2)) \geq \varepsilon^D$ for any $x \in A$ so that $\text{pk}(A, \varepsilon) \leq (R/\varepsilon)^D$. We now let

$$\rho(x) := \exp\left\{\frac{-1}{(1 - \|x\|^2)^+}\right\}$$

which is an infinitely differentiable, radially symmetric function from $\mathbb{R}^D$ to $[0, 1]$ supported on $B(0, 1)$. For any $x_0 \in M$, we define

$$\rho_{x_0}(x) := \rho\left(32\frac{x-x_0}{\tau}\right)$$

For any large $R > 0$, one can take a $\tau/64$-packing of $M \cap B(0, R)$, say $\{x_1, \ldots, x_J\}$ with $J$ of order less than $\text{pk}(B(0, R), \tau/64) \leq R^D$. We can define

$$\chi_j(x) := \frac{\rho_{x_j}(x)}{\sum_{i=1}^J \rho_{x_i}(x)}.$$
In a similar fashion as what is done in [19], we first review a few properties satisfied by the maps \{\chi_j\}_{1 \leq j \leq J}, which forms a partition of unity associated with the covering \{B(x_j, \tau/32)\}_{1 \leq j \leq J} of \(M \cap B(0, R)\). See Figure 9 for a geometric interpretation of the situation.

**Lemma A.5.** The following assertions hold true:

i) For any \(1 \leq j \leq J\), \(\text{supp} \chi_j \subset O_{x_j}\);

ii) There exists a numeric constant \(\gamma > 0\) such that \(M^{\tau} \cap B(0, R) \subset \text{supp} \sum_j \chi_j\);

iii) There exists a numeric constant \(\nu > 0\), such that for any \(x \in M^{\tau} \cap B(0, R)\), there holds \(\sum_{j=1}^J \rho_{x_j}(x) \geq \nu\);

iv) For any \(|k| \leq K\), there holds that \(\|D^k \chi_j\|_\infty \leq C < \infty\) with \(C\) depending on \(K\) and \(\tau\);

v) For any \(|k| \leq K\), there exists a non-negative function \(I_K\) such that, for any \(1 \leq j \leq J\),

\[
|D^k \chi_j(x)| \leq I_K(x - x_j) \chi_j(x), \quad \forall x \in M^{\tau} \cap B(0, R)
\]

with \(I_K\) being such that for any \(\omega > 0\)

\[
\sup_{x \in M^{\tau}} (I_K(x - x_j))^{\omega} \chi_j(x) \leq C < \infty
\]

where \(C\) depends on \(K\), \(\tau\) and \(\omega\).

We stress out that the constants appearing in Lemma A.5 do not depend on \(R\) or \(J\) at all, so that it can be applied for a family of covering \(\{x_1, \ldots, x_J\}\) indexed by \(R \to \infty\), as it will be the case in the proofs below.

![Figure 9: A visual interpretation of the framework and results of Lemma A.5.](image)

**Proof of Lemma A.5.** We start with proving i). Let \(x \in B(x_j, \tau/32)\) and let \(z = \text{pr}_M x\). There holds that

\[
\|z - x_j\|^2 = \|z - x\|^2 + \|x - x_j\|^2 + 2(z - x, x - x_j) = \|x - x_j\|^2 - \|z - x\|^2 + 2(z - x, z - x_j).
\]

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But now using [22, Thm 4.7 (8)], there holds that $2(z - x, z - x_j) \leq \|z - x_j\|^2 \|z - x\|/\tau$ so that in the end
\[
\|z - x_j\|^2 \leq \frac{\|z - x\|^2}{1 - \|z - x\|/\tau} \leq \left(\sqrt{\frac{32}{31}} \frac{\tau}{32} \right)^2,
\]
where we used the fact that $\|z - x\| \leq \|x_j - x\| \leq \tau/32$. In particular, using (30) in Lemma A.1, we obtain
\[
d_M(z, x_j) \leq \kappa \left(\sqrt{\frac{32}{31}} \frac{1}{16} \right) \|z - x_j\| \leq \kappa \left(\sqrt{\frac{32}{31}} \frac{1}{16} \right) \sqrt{\frac{32}{32} \tau} < \tau/16
\]
where the last inequality was checked with a calculator, so that $z = \Psi_{x_j}(v)$ with $v \in B_{T_{x_j}, M}(0, \tau/16)$. Finally $\|x - z\| \leq \tau/32$ so that $x - z = N_{x_j}(v, \eta)$ for some $\eta \in B_{N_{x_j}, M}(0, \tau/32)$. In the end $x = \Psi_{x_j}(v, \eta)$ with $(v, \eta) \in W^0_{x_j}$ so $x \in C^0_{x_j}$, ending the proof of the assertion i).

We continue with proving ii) and iii) with $\gamma = 1/128$ and $\nu = \exp\{-16/7\}$. Take $x \in M^{\gamma, \tau} \cap B(0, R)$ and $z = pr_M(x)$. There exists $x_j$ such that $\|z - x_j\| \leq \tau/64$, leading to $\|x - x_j\| \leq 3\tau/128$ and thus $x \in \text{supp} \chi_{x_j}$. Furthermore,
\[
\rho_{x_j}(x) = \exp\left\{-\frac{1}{(1 - (32\|x - x_j\|/\tau)^2)}\right\} \geq \exp\left\{-\frac{1}{(1 - (3/4)^2)}\right\} = \nu,
\]
so that points ii) and iii) are proven.

We finish with the proof of iv) and v). We let $R(x) = (\sum_{i=1}^q \rho_{x_i}(x))^{-1}$, which is well defined on $M^{\gamma, \tau} \cap B(0, R)$ and bounded from above by $\nu^{-1}$ so that $\chi_j(x) = R(x) \rho_{x_j}(x)$ and
\[
D^k \chi_j(x) = \sum_{|\ell| \leq k} \binom{k}{\ell} D^{k-\ell} R(x) D^\ell \rho_{x_j}(x).
\]

Outside of $B(x_j, \tau/32)$ one can take $I_K = 0$. Now if $x \in B(x_j, \tau/32)$, one get that for any $|\ell| < k$, using the Faa Di Bruno formula yields that $D^\ell \rho_{x_j}(x)$ is a sum of terms of the form
\[
(-1)^r \prod_{i=1}^r D^r \Upsilon(x) \times \rho_{x_i}(x) \quad \text{with} \quad \Upsilon(x) := \frac{1}{1 - \frac{32}{32} \|x - x_j\|^2} \quad \text{and} \quad \sum_{i=1}^r \ell_i = \ell.
\]
In particular, we get that $D^\ell \rho_{x_j}(x)$ is bounded from above by some constant depending on $K$ and $\tau$ on $B(x_j, \tau/32)$. Furthermore, for any $|\ell| < k$, the derivative $D^\ell R(x)$ is a sum of terms of the form
\[
R(x) \prod_{i=1}^q D^r \rho_{x_i}(x) \quad \text{with} \quad \sum_{i=1}^q r_i \leq \ell \quad \text{and} \quad r + \sum_{i=1}^q |r_i| = |\ell| + 2,
\]
and $D^r \rho_{x_i}(x) = \sum_{q} D^r \rho_{x_q}(x)$ which is bounded from above by something depending on $\tau$ and $K$. All in all, we get the uniform boundedness of $D^k \chi_j$ and that
\[
|D^k \chi_j(x)| \leq I_K(x - x_j) \chi_j(x)
\]
with $I_K(x) := \begin{cases} C_{K, \tau} \left(1 - \frac{32}{\tau^2} \|x\|^2\right)^{(K+1)} & \text{if } x \in B(0, \tau/32), \\
0 & \text{otherwise}, \end{cases}$

where $C_{K, \tau}$ depends on $K$ and $\tau$, ending the proof. \qed
B Appendix to Section 2

B.1 Auxiliary results on general anisotropic Hölder functions

We first extend Definition 2.1 to functions defined on general open sets of $\mathbb{R}^D$. For any open set $\mathcal{U} \subset \mathbb{R}^D$, any function $L : \mathcal{U} \to \mathbb{R}_+$ and any positive real number $\zeta > 0$, we define the anisotropic Hölder spaces $\mathcal{H}_an^\beta(\mathcal{U}, L, \zeta)$ as the set of functions $f : \mathcal{U} \to \mathbb{R}^D$ satisfying that

i) For any multi-index $k \in \mathbb{N}^D$ such that $\langle k, \alpha \rangle < \beta$ the partial derivative $D^k f$ is well defined on $\mathcal{U}$ and $|D^k f(x)| \leq L(x)$ for all $x \in \mathcal{U}$;

ii) For any multi-index $k \in \mathbb{N}^D$ such that $\beta - \max_{\alpha} \langle k, \alpha \rangle < \beta$, there holds

$$|D^k f(y) - D^k f(x)| \leq L(x) \sum_{i=1}^D |y_i - x_i|^{\frac{\beta - \langle k, \alpha \rangle}{\alpha_i}} \quad \forall x, y \in \mathcal{U}, \|x - y\| \leq \zeta. \quad (34)$$

Like in Subsection 2.1, we define in a similar fashion $\mathcal{H}_iso^\beta(\mathcal{U}, L, \zeta)$, $\mathcal{H}_an^\beta(\mathcal{U}, C, \zeta)$ and $\mathcal{H}_iso^\beta(\mathcal{U}, C, \zeta)$ for some constant $C > 0$. We now list a number of useful results which hold for our definition of anisotropy. The proofs of these results are provided below.

**Proposition B.1.** Let $f \in \mathcal{H}_an^\beta(\mathcal{U}, L, \zeta)$. Then for any $k \in \mathbb{N}^D$ such that $\langle k, \alpha \rangle < \beta$, the partially differentiated function $D^k f$ is in $\mathcal{H}_an^{\beta(k)}(\mathcal{U}, L, \zeta)$ with

$$\beta^{(k)} = \left\{ 1 - \frac{\langle k, \alpha \rangle}{\beta} \right\} \beta.$$

Anisotropic Hölder functions enjoy the same convenient Taylor expansion as usual Hölder function.

**Proposition B.2.** Let $f \in \mathcal{H}_an^\beta(\mathcal{U}, L, \zeta)$. Then, for any $x, y \in \mathcal{U}$ with $\|x - y\| \leq \zeta$,

$$f(y) = f(x) + \sum_{0 < \langle k, \alpha \rangle < \beta} \frac{(y - x)^k}{k!} D^k f(x) + R(x, y),$$

where the remainder $R$ satisfies the following bound

$$|R(x, y)| \leq L(x) \sum_{\beta - \max_{\alpha} \langle k, \alpha \rangle < \beta} \frac{|y - x|^k}{k!} \sum_{i=1}^D |y_i - x_i|^{\frac{\beta - \langle k, \alpha \rangle}{\alpha_i}}.$$

**Proposition B.3.** Let $f \in \mathcal{H}_an^\beta(\mathcal{U}, L, \zeta)$. Then, for any $k \in \mathbb{N}^D$ such that $\langle k, \alpha \rangle < \beta$, and any $x, y \in \mathcal{U}$ with $\|x - y\| \leq \zeta$,

$$|D^k f(y) - D^k f(x)| \leq CL(x) \left\{ \sum_{i=1}^D |y_i - x_i|^{\frac{\beta - \langle k, \alpha \rangle}{\alpha_i}} \right\}.$$

with $C$ depending on $k, \zeta$ and $D$.

**Proposition B.4.** Let $f \in \mathcal{H}_an^\beta(\mathcal{U}, L, \zeta)$. Then, for any $\beta' < \beta$, $f \in \mathcal{H}_an^{\beta'}(U, CL, \zeta)$ with $C$ depending on $\beta, \zeta$ and $D$. 

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Proposition B.5. Let \( f \in \mathcal{H}^{\beta_1}(U_1, L_1, \zeta_1) \) and \( g \in \mathcal{H}^{\beta_2}(U_2, L_2, \zeta_2) \) with \( U_1 \subset \mathbb{R}^{D_1} \) and \( U_2 \subset \mathbb{R}^{D_2} \). Then
\[
f \otimes g \in \mathcal{H}^{(\beta_1, \beta_2)}(U_1 \times U_2, CL_1 \otimes L_2, \zeta),
\]
where \( f \otimes g(x, y) = f(x)g(y) \) and \( \zeta = \zeta_1 \wedge \zeta_2 \), and where \( C \) depends on \( D_1, D_2, \zeta, \beta_1 \) and \( \beta_2 \).

Proposition B.6. Let \( f \in \mathcal{H}^{\beta_1}(U, L_1, \zeta) \) and \( g \in \mathcal{H}^{\beta_2}(U, L_2, \zeta) \). Then \( f \times g \in \mathcal{H}^{\beta}(U, L, \zeta) \) with \( \beta = \beta_1 \wedge \beta_2 \), and where, for some constant \( C \) depending on \( \zeta, \beta \) and \( D \),
\[
L(x) = CL_1(x)L_2(x).
\]

Proposition B.7. Let \( f \in \mathcal{H}^{\beta_1}(U_1, L_1) \) and \( g \in \mathcal{H}^{\beta_0}(U_0, C_0) \) where \( g \) takes its value in \( U_1 \), and where \( U_0 \) is bounded. Assume furthermore that \( \beta_0 \geq 1 \). Then \( f \circ g \in \mathcal{H}^{\beta}(U_0, L_1 \circ g) \) where \( \beta = \beta_0 \wedge \beta_1 \) and where \( C \) depends on \( C_0, C_1 \) and diam\(U_0\).

Proof of Proposition B.1. Let \( k \in \mathbb{N}^D \) such that \( (k, \alpha) < \beta \). Note that
\[
\alpha = \beta \frac{1}{\beta} = \beta(k) \frac{1}{\beta(k)}.
\]
Let \( \ell \in \mathbb{N}^D \) such that \( (\ell, \alpha) < \beta(k) \). Then \( (k + \ell, \alpha) < \beta \) so that by definition, \( D^\ell D^k f = D^{k+\ell} f \) exists and is bounded from above by \( L \). If now \( \ell \) is such that \( \beta(k) - \alpha_{\max} \leq (\ell, \alpha) < \beta(k) \), then \( \beta - \alpha_{\max} \leq (k + \ell, \alpha) < \beta \) and for any \( x, y \in \mathcal{U} \) that are at most \( \zeta \)-apart,
\[
|D^\ell D^k f(y) - D^\ell D^k f(x)| \leq L(x) \sum_{i=1}^D |y_i - x_i|^{\frac{\beta(k+\ell, \alpha)}{\alpha_i} \wedge 1} = L(x) \sum_{i=1}^D |y_i - x_i|^{\frac{\beta(k, \alpha) - (\ell, \alpha)}{\alpha_i} \wedge 1},
\]
ending the proof.

Proof of Proposition B.2. We let for any \( 0 \leq i \leq D \),
\[
z^{(i)} = (y_1, \ldots, y_i, x_{i+1}, \ldots, x_D) \in \mathbb{R}^D
\]
such that \( z^{(0)} = x \) and \( z^{(D)} = y \). We prove the results recursively on the integer \( N = ||\beta - 1|| \). If \( N = 0 \), then every coefficient \( \beta_i \) are strictly less than 1, and the results follow immediately from the definition of being \( \beta \)-Hölder (there are no \( k \) that satisfies \( (k, \alpha) < \beta \) except \( k = 0 \)). If \( N \geq 1 \), we can order without loss of generality and for ease of notations
\[
\beta_1 \leq \ldots \leq \beta_k < 1 \leq \beta_{k+1} \leq \ldots \leq \beta_D
\]
with \( k \leq D - 1 \) because \( N \geq 1 \). We write
\[
f(y) - f(x) = f(z^{(k)}) - f(x) + \sum_{i=k}^{D-1} f(z^{(i+1)}) - f(z^{(i)}).
\]
The first term is simply bounded from above by
\[
|f(z^{(k)}) - f(x)| \leq L(x) \sum_{i=1}^k |y_i - x_i|^{|\beta|/\alpha_i}.
\]
Recall we write \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) with 1 at the \( i \)-th position. For the other terms, we do the Taylor expansion with integral remaining term:

\[
f(z^{(i)}) - f(z^{(i-1)}) = \sum_{\ell=1}^{L_i-1} \frac{(y_i - x_i)\ell}{\ell!} D^{\epsilon_i\ell} f(z^{(i-1)}) + R_i(y, x), \quad L_i = \lceil \beta_i - 1 \rceil \geq 1
\]

\[
R_i(y, x) = \frac{(y_i - x_i)^{L_i}}{(L_i-1)!} \int_0^1 (1 - t)^{L_i-1} D^{L_i\epsilon_i} f(z^{(i)}(t)) \, dt,
\]

\[
\text{with } z^{(i)}(t) = t z^{(i)} + (1 - t) z^{(i-1)}.
\]

Note that \( L_i \alpha_i = \frac{[\beta_i - 1]}{\beta_i} \beta \) and for all \( k = (k_{1:i}, 0) \neq 0 \), \( L_i \alpha_i + \langle k, \alpha \rangle \geq \beta \). Indeed, because the coefficient of \( \beta \) are ordered, there holds

\[
L_i \alpha_i + \langle k, \alpha \rangle = \frac{\beta}{\beta_i} \left( [\beta_i - 1] + \sum_{j \leq i} k_j \beta_i / \beta_j \right) \geq \frac{\beta}{\beta_i} ([\beta_i - 1] + 1) \geq \beta.
\]

Therefore

\[
|D^{L_i\epsilon_i} f(z^{(i)}(t)) - D^{L_i\epsilon_i} f(x)| \leq L(x) \sum_{j=1}^i |y_j - x_j| \frac{\beta - L_i \alpha_i}{\alpha_j}
\]

and in particular

\[
R_i(x, y) = \frac{(y_i - x_i)^{L_i}}{L_i!} D^{L_i\epsilon_i} f(x) + \widetilde{R}_i(x, y)
\]

where \( |\widetilde{R}_i(x, y)| \leq L(x) \frac{|y_i - x_i|^{L_i}}{L_i!} \sum_{j=1}^i |y_j - x_j| \frac{\beta - L_i \alpha_i}{\alpha_j} \).

Now we use the induction hypothesis on \( f_{i, \ell} = D^{\epsilon_i \ell} f \) which belongs to \( H^{\beta(\epsilon_i)}(U, L, \zeta) \) (according to Proposition B.1):

\[
f_{i, \ell}(z^{(i-1)}) - f_{i, \ell}(x) = \sum_{0 < \langle k, \alpha \rangle < \beta - \ell \alpha_i \atop k_i, D = 0} \frac{(y - x)^k}{k!} D^{\ell \epsilon_i + k} f(x) + R_{i, \ell}(x, y),
\]

where the remainder \( R \) satisfies the following bound

\[
|R_{i, \ell}(x, y)| \leq L(x) \sum_{\beta - \alpha_{\max} \langle k + \ell \epsilon_i, \alpha \rangle < \beta \atop k_i, D = 0} \frac{|y - x|^k}{k!} \sum_{i=1}^D |y_j - x_j| \frac{\beta - (k + \ell \epsilon_i, \alpha)}{\alpha_j}.
\]
All in all, gathering all the developments yields
\[
f(z^{(i)}) - f(z^{(i-1)}) = \sum_{\ell=1}^{L_i} \left( \frac{y_i - x_i}{\ell!} \left( y - x \right)^k \right) \sum_{\ell \in \mathbb{Z}, \ell > 0} D^{\ell e_i} f(x) + R_i(y, x)
\]
\[
= \sum_{\ell=1}^{L_i} \left( \frac{y_i - x_i}{\ell!} \left( y - x \right)^k \right) D^{\ell e_i} f(x) + R_i(y, x)
\]
\[
= \sum_{\ell=1}^{L_i} \left( \frac{y_i - x_i}{\ell!} \left( y - x \right)^k \right) D^{\ell e_i} f(x) + R_i(y, x)
\]

so that
\[
f(y) - f(x) = \sum_{\ell=1}^{L_i} \left( \frac{y_i - x_i}{\ell!} \left( y - x \right)^k \right) D^{\ell e_i} f(x) + R_i(y, x)
\]

with
\[
R_i(y, x) = f(z^{(k)}) - f(x),
\]
and with \( R(y, x) \) being exactly bounded from above by
\[
|R(y, x)| \leq L(x) \sum_{\beta - \alpha_{\max} \leq (k, \alpha)} |y - x|^k \sum_{j=1}^{D} |y_j - x_j|^\alpha_j
\]

and Proposition B.2 is proved.

**Proof of Proposition B.3.** Either \((k, \alpha) \geq \beta - \alpha_{\max}\) and then there is nothing to show, or \((k, \alpha) < \beta - \alpha_{\max}\), in which case \((k + e_i, \alpha) < \beta\) for any \(i\) and thus \(D^k f\) is well defined. There thus exist \(z \in [x, y]\) such that
\[
|D^k f(y) - D^k f(x)| = |D^{k+e_i} f(z)| \times |y_i - x_i|.
\]

Using an induction argument, there exists \(C_i > 0\) such that
\[
|D^{k+e_i} f(z)| \leq C_i L(x) \sum_{i=1}^{D} |y_i - x_i|^\beta \frac{\alpha}{\alpha_i} \leq C_i L(x) \sum_{i=1}^{D} |y_i - x_i|^{\beta - \alpha_{\max}/\alpha_i} \leq C_i D(1 + \zeta) L(x)
\]

leading the the right results with
\[
C = 1 + D(1 + \zeta) \max_{1 \leq i \leq D} C_i.
\]

**Proof of Proposition B.4.** Let \(\beta'\) be the harmonic mean of \(\beta'\) and \(\alpha' = \beta' / \beta\). For any \(k \in \mathbb{N}^D\) such that \((k, \alpha') < \beta',\) we have \((k, \alpha) < \beta\) so that \(D^k f\) is well defined and bounded from above by \(L\). What’s more, using Proposition B.3, we have
\[
|D^k f(y) - D^k f(x)| \leq C L(x) \sum_{i=1}^{D} |y_i - x_i|^{\beta - \alpha_{\max}/\alpha_i}.
\]
Now notice that
\[
\frac{\beta - \langle k, \alpha \rangle}{\alpha_i} = \beta_i \{1 - \langle k, 1/\beta \rangle\} \geq \beta'_i \{1 - \langle k, 1/\beta' \rangle\} = \frac{\beta' - \langle k, \alpha' \rangle}{\alpha'_i}
\]
so that
\[
|y_i - x_i| \frac{\beta - \langle k, \alpha \rangle}{\alpha_i} \leq (1 \vee \zeta) \times |y_i - x_i| \frac{\beta' - \langle k, \alpha' \rangle}{\alpha'_i}
\]
yielding the result. \hfill \square

**Proof of Proposition B.5.** Let \( k = (k_1, k_2) \in \mathbb{N}^{D_1 + D_2} \). Let \( \beta = (\beta_1, \beta_2) \) and \( \beta_1, \beta_2 \) be the harmonic means of \( \beta, \beta_1 \), and \( \beta_2 \). Let also \( \alpha = \beta/\beta \) and \( \alpha_i = \beta_i/\beta \), for \( i \in \{1, 2\} \). If \( \langle k, \alpha \rangle < \beta \), then
\[
\langle k_1, 1/\beta_1 \rangle + \langle k_2, 1/\beta_2 \rangle < 1
\]
so that both term is strictly less than one \( D^k f \) and \( D^k g \) are well defined and \( D^k (f \otimes g) = D^k f \otimes D^k g \) is well defined as well. Furthermore, if \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \), is such that \( \|x - y\| \leq \zeta \), then \( \|x_1 - y_1\| \leq \zeta \leq \zeta_1 \) and \( \|x_2 - y_2\| \leq \zeta \leq \zeta_2 \). Using Proposition B.3, one find that
\[
|D^k(f \otimes g)(x) - D^k(f \otimes g)(y)| \\
\leq |D^k g(x_2)| \times |D^k f(x_1) - D^k f(y_1)| + |D^k f(y_1)| \times |D^k g(x_2) - D^k g(y_2)| \\
\leq C L_2(x_2) L_1(x_1) \left\{ \sum_{i=1}^{D_1} |y_{1,i} - x_{1,i}| \frac{\beta_i - \langle k_1, \alpha_i \rangle}{\alpha_i} \frac{\lambda_{\zeta_1}}{} \right\} \\
+ C |D^k f(y_1)| L_2(x_2) \left\{ \sum_{i=1}^{D_2} |y_{2,i} - x_{2,i}| \frac{\beta_i - \langle k_2, \alpha_i \rangle}{\alpha_i} \frac{\lambda_{\zeta_2}}{} \right\}.
\]
for some constant \( C > 0 \) depending on \( D_1, D_2, \zeta \) and \( \beta \). Now notice first that
\[
|D^k f(y_1)| \leq |D^k f(y_1) - D^k f(x_1)| + |D^k f(x_1)| \leq (CD(1 \vee \zeta) + 1) L_1(x_1),
\]
and notice furthermore that of either \( j \in \{1, 2\} \),
\[
\frac{\beta_j - \langle k_j, \alpha_j \rangle}{\alpha_{j,i}} = \beta_{j,i} \{1 - \langle k_j, 1/\beta_j \rangle\} \geq \beta_{j,i} \{1 - \langle k, 1/\beta \rangle\} = \frac{\beta - \langle k, \alpha \rangle}{\alpha_i}
\]
so that
\[
|D^k(f \otimes g)(x) - D^k(f \otimes g)(y)| \\
\leq (1 \vee \zeta) C L_2(x_2) L_1(x_1) \left\{ \sum_{i=1}^{D_1} |y_{1,i} - x_{1,i}| \frac{\beta - \langle k, \alpha \rangle}{\alpha_i} \frac{\lambda_{\zeta_1}}{} \right\} \\
+ (1 \vee \zeta) C CD(1 \vee \zeta) + 1 \left\{ \sum_{i=1}^{D_1} |y_{1,i} - x_{1,i}| \frac{\beta - \langle k, \alpha \rangle}{\alpha_i} \frac{\lambda_{\zeta_1}}{D_1} \right\} \\
\leq (1 \vee \zeta) C CD(1 \vee \zeta) + 1 \left( L_1 \otimes L_2 \right)(x) \left\{ \sum_{i=1}^{D_1 + D_2} |y_{1,i} - x_{1,i}| \frac{\beta - \langle k, \alpha \rangle}{\alpha_i} \frac{\lambda_{\zeta_1}}{D_1 + D_2} \right\}
\]
ending the proof. \hfill \square
**Proof of Proposition B.6.** We let \( \alpha = \beta/\beta \) with \( \beta \) the harmonic mean of \( \beta \). Now for any multi-index \( k \) such that \(( k, \alpha ) < \beta \), we have \(( \ell, \alpha ) < \beta \) for any \( \ell \leq k \) so that \( D^k f g \) is indeed well-defined and

\[
D^k f g(x) = \sum_{\ell \leq k} \binom{k}{\ell} D^\ell f(x) D^{k-\ell} g(x)
\]

so that \( |D^k f g(x)| \leq 2^k L_1(x) L_2(x) \) and

\[
|D^k f g(y) - D^k f g(x)| \leq L_2(x) \sum_{\ell \leq k} \binom{k}{\ell} |D^\ell f(y) - D^\ell f(x)| + L_1(x) \sum_{\ell \leq k} \binom{k}{\ell} |D^\ell g(y) - D^\ell g(x)|.
\]

Using now Proposition B.3, we find that there exists \( C > 0 \) depending on \( \beta, D \) and \( \zeta \) such that

\[
|D^k f g(y) - D^k f g(x)| \leq 2C L_1(x) L_2(x) \sum_{\ell \leq k} \binom{k}{\ell} \sum_{i=1}^D y_i - x_i|^{\beta-(k, \alpha) \wedge 1}
\]

\[
\leq 2^k (1 \lor \zeta) C L_1(x) L_2(x) \sum_{i=1}^D |y_i - x_i|^{\beta-(k, \alpha) \wedge 1}
\]

where we used again that

\[
|y_i - x_i|^{\beta-(k, \alpha) \wedge 1} \leq (1 \lor \zeta) \times |y_i - x_i|^{\beta-(k, \alpha) \wedge 1},
\]

for \( \ell \leq k \), ending the proof. \( \square \)

**Proof of Proposition B.7.** A simple use of the multivariate Faa Di Bruno formula \([16]\) yields that, for any \( k \in \mathbb{N}^D \) with \( |k| \leq \beta \), \( D^k (f \circ g)(x) \) for \( x \in U_0 \) is a sum of a term of the form

\[
D^\ell f(g(x)) \prod_{j=1}^s (D^{k(j)} g(x))^{\ell(j)}
\]

with \( |\ell| \leq |k|, s \leq |k|, \sum_j \ell(j) = \ell \), and \( \sum_j |\ell(j)| k^{j(j)} = k \). In particular, it is bounded by \( C L_1(g(x)) \) where \( C \) depends on \( C_0 \) and \( \beta \). Likewise, a telescopic argument would yield that

\[
|D^k (f \circ g)(y) - D^k (f \circ g)(x)| \leq C L_1(g(x)) \prod_{i=1}^D |x - y|^{(\beta-k \wedge 1)}
\]

where \( C \) depends on \( C_0, \beta \) and \( \text{diam} U_0 \), ending the proof. \( \square \)

### B.2 Taylor expansion of M-anisotropic Hölder functions

In this section, we derive a Taylor expansion for manifold-anisotropic Hölder functions. Recall from Section 3 that for any \( \sigma, \delta > 0 \),

\[
\Delta_{\sigma, \delta} = \begin{pmatrix}
\sigma^{\alpha_0} \text{Id}_d & 0 \\
0 & \delta^{\alpha_1} \text{Id}_{D-d}
\end{pmatrix}.
\]
Corollary B.8. Let \( f \in \mathcal{H}_{b_0,b_1}(M,L) \). Then, for any \( x_0 \in M \), any \( w \in \mathcal{W}_{x_0,\delta} \), and any \( z \in T_{x_0}M \times N_{x_0}M \) such that \( w + \Delta_{\sigma,1}z \in \mathcal{W}_{x_0,\delta} \), there holds

\[
\bar{f}_{x_0,\delta}(w + \Delta_{\sigma,1}z) = \bar{f}_{x_0,\delta}(w) + \sum_{0<k,\alpha<\beta} \sigma^{(k,\alpha)} \frac{z^k}{k!} D^k \bar{f}_{\delta,x_0}(w) + R(w, z),
\]

where the remainder \( R \) satisfies the following bound

\[
|R(w, z)| \leq D\sigma^\beta \|1 \vee z\|_{1} \beta_{\max} L_{x_0,\delta}(w).
\]

Proof of Corollary B.8. Simply applying Proposition B.2 to \( \bar{f}_{x_0,\delta} \) yields

\[
\bar{f}_{x_0,\delta}(w + \Delta_{\sigma,1}z) = \bar{f}_{x_0,\delta}(w) + \sum_{0<k,\alpha<\beta} (\Delta_{\sigma,1}z)^k \frac{k!}{k!} D^k \bar{f}_{\delta,x_0}(w) + R_{x_0,\delta}(w, w + \Delta_{\sigma,1}z)
\]

where \( R_{x_0,\delta}(w, \Delta_{\sigma,1}z) \) satisfies

\[
|R_{x_0,\delta}(w, w + \Delta_{\sigma,1}z)| \leq L_{x_0,\delta}(w) \sum_{\beta - \alpha \leq \Delta_{\sigma,1}z < \beta} \|((\Delta_{\sigma,1}z)^k \frac{k!}{k!} D^k \bar{f}_{\delta,x_0}(w) + R_{x_0,\delta}(w, w + \Delta_{\sigma,1}z)
\]

\[
= L_{x_0,\delta}(w) \sum_{\beta - \alpha \leq \Delta_{\sigma,1}z < \beta} \sigma^{(k,\alpha)} \frac{1}{k!} \sum_{j=1}^{D} \sigma^{\beta-(k,\alpha)} |\frac{z_j}{\alpha_j}| \beta_{\max}
\]

where we used the fact that \( k_j + \frac{\beta-(k,\alpha)}{\alpha_j} \leq \beta_j \leq \beta_{\max} \) for any \( 1 \leq j \leq D \). \( \square \)

B.3 Proofs associated to the examples of Proposition 2.3

Proof of Proposition 2.3. Take \( x_0 \in M \). In the orthonormal noise model, the density \( f \) takes the very simple form

\[
\bar{f}_{x_0,\delta}(v, \eta) = \delta^{D-d} f \circ \Psi_{x_0,\delta}(v, \eta) = \delta^{D-d} \times f_{\hat{0}}(\Psi_{x_0}(v)) \times \delta^{-(D-d)} c_1 K \left( \frac{1}{\delta} N_{x_0}(v, \delta \eta) \right)
\]

\[
= f_{x_0}(v) \times c_1 K(\eta)
\]

where we used the isotropy of \( K \). Thus, \( \bar{f}_{x_0,\delta} \) lies in \( \mathcal{H}_{b_0}(\mathcal{W}_{x_0,\delta}, L_{x_0,\delta}) \) where \( L \) was defined in the statement of Proposition 2.3, through Proposition B.5 and the isotropy of \( L_1 \).

In the isotropic noise model, one can write

\[
\bar{f}_{x_0,\delta}(v, \eta) = \delta^{D-d} \int_M \delta^{-D} K\left( \frac{\Psi_{x_0,\delta}(v, \eta) - x}{\delta} \right) f_{\hat{0}}(x) d\mu(x).
\]

If \( \|\eta\| \geq 1 \), the integrand is trivially 0, so one may focus on \( \|\eta\| \leq 1 \). Now if \( x \in M \) is at least \( 2\delta \) apart from \( \Psi_{x_0}(v) \), there holds

\[
\|x - \Psi_{x_0,\delta}(v, \eta)\| \geq \|x - \Psi_{x_0}(v)\| - \delta \geq \delta
\]
so that the integrand in the integral above is zero for \( x \) outside of \( B(\Psi_{x_0}(v), 2\delta) \). Doing the variable change \( x = \Psi_{x_0}(w) \), we can write

\[
\tilde{f}_{x_0, \delta}(v, \eta) = \frac{1}{\delta^d} \int_{\exp_{x_0}^{-1}B(\Psi_{x_0}(v), 2\delta)} K \left( \frac{\Psi_{x_0}(v) - \Psi_{x_0}(w)}{\delta} \right) f_{x_0}(w) \det d\Psi_{x_0}(w) \, dw
\]

\[
= \int_{\mathcal{Z}_\delta} K \left( \frac{\Psi_{x_0}(v) - \Psi_{x_0}(v + \delta s)}{\delta} + N_{x_0}(v, \eta) \right) f_{x_0}(v + \delta s) \det d\Psi_{x_0}(v + \delta s) \, ds
\]

\[
= h_{\delta, s}(v)
\]

where

\[
\mathcal{Z}_\delta = \frac{1}{\delta} \{ \exp_{x_0}^{-1}B(\Psi_{x_0}(v), 2\delta) - v \}.
\]

Now notice that, using the first inequality of (30), we have \( B(\Psi_{x_0}(v), 2\delta) \subset B(x_0, \tau/8 + 2\delta) \subset B(x_0, 3\tau/16) \), so that, using the second inequality of (30) with \( \gamma = 3/8 \), we find

\[
\exp_{x_0}^{-1}B(\Psi_{x_0}(v), 2\delta) \subset B_{T_{x_0}M}(0, \kappa(3/8)3\tau/16) \subset B_{T_{x_0}M}(0, \tau/4)
\]

so that in particular, according to (29), \( \exp_{x_0}^{-1} \) is \( 16/11 \)-Lipschitz on \( B(\Psi_{x_0}(v), 2\delta) \) and consequently, \( \mathcal{Z}_\delta \subset B_{T_{x_0}M}(0, 32/11) \). Furthermore, notice that

\[
v + \delta B_{T_{x_0}M}(0, 32/11) \subset B_{T_{x_0}M}(0, 19\tau/88)
\]

with \( 19\tau/88 \) being smaller than the injectivity radius, so that finally

\[
\tilde{f}_{x_0, \delta}(v, \eta) = \int_{B_{T_{x_0}M}(0, 32/11)} K \circ g_{\delta, s}(v, \eta) \times h_{\delta, s}(v) \, ds
\]

(37)

It is straightforward to see that

\[
v \mapsto \frac{\Psi_{x_0}(v) - \Psi_{x_0}(v + \delta s)}{\delta}
\]

has derivatives bounded from above, in virtue of Proposition B.3, by something depending on \( C_M \) and \( \beta_M \) only (and not on \( \delta \)) and up to order \( [\beta_M - 1] \geq \beta_\perp \). Furthermore, \( (v, \eta) \mapsto N_{x_0}(v, \eta) \) is \( (\beta_M - 1) \)-Hölder by construction. It so happens that \( g_{\delta, s} \in \mathcal{H}_\text{iso}(\mathcal{W}_{x_0, \delta}, C_M) \) for some \( g \) depending on \( C_M \) and \( \beta_M \). Using Proposition B.7, we get that \( K \circ g_{\delta, s} \in \mathcal{H}_\text{iso}(\mathcal{W}_{x_0, \delta}, C_1 L_1 \circ g_{\delta, s}) \) for some constant \( C_1 \) depending on \( C_g, C_K \) and \( \beta_\perp \). Similar reasoning and the use and Proposition B.6 would lead to \( h_{\delta, s} \in \mathcal{H}_\text{iso}(\mathcal{W}_{x_0, \delta}, C_2 L_0 \circ \Psi_{x_0} \circ \tau_{\delta s}) \) where \( \tau_{\delta s}(v) = v + \delta s \) and for some constant \( C_2 \) depending on \( C_M, \tau \) and \( \beta_0 \). Now seeing \( h_{\delta, s} \) as a function of \( (v, \eta) \) would trivially lead to \( h_{\delta, s} \in \mathcal{H}_\text{an}(\mathcal{W}_{x_0, \delta}, C_2 L_0 \circ \Psi_{x_0} \circ \tau_{\delta s} \circ \pi_1) \) with \( \beta = (\beta_0, \ldots, \beta_0, \beta_\perp, \ldots, \beta_\perp) \) and \( \pi_1(v, \eta) = v \), and an application of Proposition B.6 together with the observation that \( \beta_0 \leq \beta_\perp \) yields

\[
K \circ g_{\delta, s} \times h_{\delta, s} \in \mathcal{H}_\text{an}(\mathcal{W}_{x_0, \delta}, CL_1 \circ g_{\delta, s} \times L_0 \circ \Psi_{x_0} \circ \tau_{\delta s} \circ \pi_1)
\]
for some constant C depending on C_2, C_1 and \( \beta_0, \beta_1 \). We conclude by using the linearity of the integral, so that integrating (37) gives \( \tilde{f}_{x_0,\delta} \in \mathcal{H}_{an}^\beta(\mathcal{W}_{x_0,\delta}, L) \), where

\[
\tilde{L}(v, \eta) = C \int_{B_{Tx_0}(0,32/11)} L_\perp \circ g_{\delta,s}(v, s) \times L_0 \circ \Psi_{x_0}(v + \delta s) \, ds
\]

\[
\leq \int_{B_{Tx_0}(0,32/11)} L_\perp \circ g_{\delta,s}(v, s) \times L_0 \circ \Psi_{x_0}(v + \delta s) \times |\det d\Psi_{x_0}(v + \delta s)| \, ds
\]

\[
= \delta^{-d} \int_M L_\perp \left( \frac{\Psi_{x_0,\delta}(v, \eta) - x}{\delta} \right) \times L_0(x) \, d\mu_M(x)
\]

so that

\[
L(y) = C \delta^{-D} \int_M L_\perp \left( \frac{y - x}{\delta} \right) \times L_0(x) \, d\mu_M(x)
\]

where \( L \) was defined in the statement of Proposition 2.3, and where we used Lemma A.2 to justify the introduction of \( |\det d\Psi_{x_0}(v + \delta s)| \). This ends the proof.

**Proof of Lemma 3.2.** In the orthogonal noise model, recall (see proof of Proposition 2.3) that

\[
\begin{cases}
f_0(x) = f_s(\text{pr}_M(x)) \times c_1 \delta^{-(D-d)} \times K((x - \text{pr}_M x)/\delta) \\
L(x) = C \delta^{-(D-d)} L_0(\text{pr}_M(x)) \times L_\perp((x - \text{pr}_M x)/\delta),
\end{cases}
\]

for some \( C > 0 \). There thus holds, letting \( O_{x_0} = \Psi_{x_0,1}(\mathcal{W}_{x_0,1}) \), and using Cauchy-Schwartz inequality

\[
\left\{ \int_{O_{x_0}} \left[ \frac{L(x)}{f_0(x)} \right]^{\omega^*/2} f_0(x) \, dx \right\}^2 \leq \int_{O_{x_0}} \left[ \frac{L_0(\text{pr}_M x)}{f_s(\text{pr}_M x)} \right]^\omega^* f_0(x) \, dx \int_{O_{x_0}} \left[ \frac{L_\perp((x - \text{pr}_M x)/\delta)}{K((x - \text{pr}_M x)/\delta)} \right]^\omega^* \, f_0(x) \, dx.
\]

The first integral in the RHS is simply, up to the constant \( c_\perp \),

\[
\int_{\Psi_{x_0}(\mathcal{W}_{x_0})} f_s(z) \frac{L_0(z)}{f_s(z)} \omega^* f_s(z) \, d\mu_M(z),
\]

which is bounded by assumption. The second integral is exactly

\[
\int_{\Psi_{x_0}(\mathcal{W}_{x_0})} f_s(z) \, d\mu_M(z) \times \int_{B_{R_{D-d}}(0,1)} \left[ \frac{L_\perp(\eta)}{K(\eta)} \right]^\omega^* K(\eta) \, dx.
\]

which is also bounded by assumption. Hence (13) holds true in this case.

In the isotropic noise model, there holds this time

\[
\begin{cases}
f_0(x) = \delta^{-D} \int_{M \cap B(x,\delta)} K((y - x)/\delta) f_s(y) \, d\mu_M(y) \\
L(x) = C \delta^{-D} \int_{M \cap B(x,\delta)} L_\perp((y - x)/\delta) L_0(y) \, d\mu_M(y),
\end{cases}
\]

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for some $C > 0$. Using the fact that for any integrable function $\phi, \psi \geq 0$ and any $\omega \geq 1$, there holds
\[
\left\{ \int \phi \right\}^\omega \leq \left\{ \int \phi^{\omega_1} \right\} \times \left\{ \int \psi \right\}^{\omega_1},
\]
(which is simple consequence of the Hölder inequality), one find that
\[
\int_{Q_{x_0}} \left\{ \frac{L(x)}{f_0(x)} \right\}^{\omega^*/2} f_0(x) \, dx
\leq \int_{Q_{x_0}} \int_M \left\{ \frac{L_1((y - x)/\delta)L_0(y)}{K((y - x)/\delta)f_*(y)} \right\}^{\omega^*/2} \delta^{-D} K((y - x)/\delta)f_*(y) \, d\mu_M(y) \, dx
\leq \delta^{-D} I_1^{1/2} I_2^{1/2},
\]
where
\[
I_1 := \int_{Q_{x_0}} \int_M \left\{ \frac{L_0(y)}{f_*(y)} \right\}^{\omega^*} K((y - x)/\delta)f_*(y) \, d\mu_M(y) \, dx
\leq \delta^D \int_M \left\{ \frac{L_0(y)}{f_*(y)} \right\}^{\omega^*} f_*(y) \, d\mu_M(y) \leq \delta^D,
\]
by assumption, and where
\[
I_2 := \int_{Q_{x_0}} \int_M \left\{ \frac{L_1((y - x)/\delta)}{K((y - x)/\delta)} \right\}^{\omega^*} K((y - x)/\delta)f_*(y) \, d\mu_M(y) \, dx
\leq \delta^D \int_M f \, d\mu_M \int_{B(0,1)} \left\{ \frac{L_1(\eta)}{K(\eta)} \right\}^{\omega^*} K(\eta) \, d\eta,
\]
so that indeed $\delta^{-D} I_1^{1/2} I_2^{1/2} \leq 1$, which ends the proof.

**B.4 Proof of Proposition 2.4**

If $Q_2 = Q_0^{D_0}$ with $Q_0$ a DP$(BH_\lambda) := \Pi_{DP}$ then, for $0 < x_1 < x_2/2$ and $x_1 < 1$, letting $A_1 = [x_1, x_1(1 + x_1^d)]$, $A_2 = [x_2, x_2(1 + x_1^d)]$ and $A_3 = (A_1 \cup A_2)^c$, we have
\[
(Q_0(A_1), Q_0(A_2), Q_0(A_3)) \sim D(BH_\lambda(A_1), BH_\lambda(A_2), BH_\lambda(A_3)),
\]
and
\[
Q_2(A_1^d \times A_2^{D_0-d}) = Q_0(A_1)^d Q_0(A_2)^{D_0-d}.
\]
We let \( \gamma = \Gamma(B)/(\Gamma(BH_\lambda(A_1))\Gamma(BH_\lambda(A_2))\Gamma(BH_\lambda(A_3)) \). Noticing that when \( x_1, x_2 \) are small, \( H_\lambda(A_3) = 1 - o(1) \), there holds

\[
\widetilde{\Pi}_X[Q_2(A_1^{d_1}A_2^{d_2}) > x_0^{B_0}] \geq \Pi_{DP}\left(Q_0(A_1) > x_1^{B_0/D}, Q_0(A_2) > x_1^{B_0/D}\right)
\]

\[
\geq \frac{\gamma}{2(B-1)} \int_0^{1/\lambda} x^{B_0/D} x^{BH_\lambda(A_1)+1} dx \int_x^{1/\lambda} x^{BH_\lambda(A_2)+1} dx
\]

\[
\geq \frac{\Gamma(B)}{(1 + o(1))2(B-1)} \left[ 4^{BH_\lambda(A_1)} - \frac{BH_\lambda(A_1)B_0}{\lambda} \right] \left[ 4^{BH_\lambda(A_2)} - \frac{BH_\lambda(A_2)B_0}{\lambda} \right],
\]

where we used the fact that for \( x_1, x_2 \) small, \( H_\lambda(A_1) \leq H_\lambda(A_2) \) and were both small. Under the assumption on \( H_\lambda \) near 0,

\[
e^{-a_2x_1^{-3/2}} x_1^{-a_1} x_1^{1+b} \leq H_\lambda(A_1) \leq e^{-a_2x_1^{-3/2}/\lambda} x_1^{-a_1} x_1^{1+b} \leq e^{-a_2x_1^{-3/2}/2}
\]

so that for small \( x_1 \),

\[
1 - e^{-\frac{BH_\lambda(A_1)B_0}{\lambda} \log(4D/B_0 x_1)} \geq e^{-a_2x_1^{-3/2}/\lambda} x_1^{-a_1} x_1^{1+b} \leq e^{-2a_2x_1^{-3/2}/2}
\]

so that (10) holds. Condition (11) is verified similarly. First note that \( Q_2(\min_{i \leq D} \lambda_i \leq x) \leq DQ_0((0, x]) \) and using the fact that \( Q_0((0, x]) \) follows a Beta variable with parameters \( BH_\lambda((0, x]), B(1 + o(1)) \) we obtain that

\[
\mathbb{E}_{\Pi}_X \left[ Q_2 \left( \min_{i \leq D} \lambda_i \leq x \right) \right] \leq H_\lambda((0, x]) \leq e^{-a_2x_1^{-3/2}/2},
\]

for small \( x \). Similar computation terminates the proof of (11)

### C Appendix to Section 5.2: proof and framework of Theorem 3.3

#### C.1 Technical Lemmata

We let \( \chi_1, \ldots, \chi_J \) be the partition of unity defined in Section A.3 associated with a \( \tau/64 \)-packing of \( \{x_1, \ldots, x_J\} \) of \( M \cap B(0, R) \). We recall that \( J \leq R^D \) and we take \( R = (H \log(1/\sigma))^{1/\kappa} \).

We define

\[
f_j := c_j^{-1}\chi_j \times f_0 \quad \text{where} \quad c_j = \int_{R^D} \chi_j(x) f_0(x) \, dx.
\]

Before we give the proof of Corollary 3.4 we need a few technical Lemmata.

**Lemma C.1.** For all \( 1 \leq j \leq J \), \( f_j \in H_\delta^{\beta_0, \beta_j}(M, L_j) \) with \( L_j(x) := Cc_j^{-1}I_K(x - x_j)\chi_j(x)L(x) \) where \( C > 0 \) is a constant depending on \( \tau \) and where we took \( K = [\beta_1] \).

**Proof.** The proof is a direct consequence of Proposition B.6 and Proposition B.7 applied to \( \chi_j(x) \), along with the assertion iv) from Lemma A.5. \( \square \)
We let $L_{j,\delta} := \delta^{D-d}L_j \circ \Psi_{x_j,\delta}$ and
\[
\bar{f}_{j,\delta} := f_j \circ \Psi_{x_j,\delta} : \mathcal{W}_{x_j,\delta} \to \mathbb{R}^D.
\]
and also introduce the functions $g_j : \mathbb{R}^D \to \mathbb{R}$ from the proof of Theorem 3.3. Recall that these functions are of the form
\[
g_j(x) := f_j(x) + \frac{1}{\delta^{D-d}} \sum_{0 < \langle k, \alpha \rangle < \beta} \sigma^{\langle k, \alpha \rangle} d_{j,k}(x, \sigma, \delta) D^k \bar{f}_{j,\delta}(z_x)
\]
where $z_x := \Delta_{1,\delta}^{-1} \Psi_{x_j,\delta}^{-1}(x)$ and
\[
\sigma \text{ up to a constant depending on the parameters.}
\]
and satisfy

1. They are supported on $O^0_{x_j}$;
2. The functions $d_{j,k}$ are uniformly bounded by a constant $C$ depending on $C_M$;
3. $|K_{S}g_j(x) - f_j(x)| \lesssim \sigma^\beta L_j(x)$ on $O^1_{x_j}$;
4. For all $H > 0$, $|K_{S}g_j(x) - f_j(x)| \lesssim \sigma^H \|L_j\|_\infty$ outside of $O^1_{x_j}$.

**Lemma C.2.** Under (13), there holds, for any $\varepsilon \in (0, \omega - 2\beta)$, any $\langle k, \alpha \rangle < \beta$ and any $1 \leq j \leq J$,
\[
\int \left( \frac{|D^k \bar{f}_{j,\delta}(z)|^{2/\beta + \varepsilon}}{\bar{f}_{j,\delta}(z)} \right) \bar{f}_{j,\delta}(z) \, dz \lesssim \sigma^{-1} \quad \text{and} \quad \int \left( \frac{|L_{j,\delta}(z)|^{2/\beta}}{\bar{f}_{j,\delta}(z)} \right) \bar{f}_{j,\delta}(z) \, dz \lesssim \sigma^{-1},
\]
up to a constant depending on the parameters.

**Proof.** We denote by $\bar{X}_{j,\delta} = \chi_j \circ \Psi_{x_j,\delta}$ and $\bar{I}_{j,\delta} := I_K(\Psi_{x_j,\delta}(\cdot) - x_j)$. Writing that
\[
\left| \frac{D^k \bar{f}_{j,\delta}}{\bar{f}_{j,\delta}} \right| \leq \sum_{\ell \leq k} \binom{k}{\ell} \left| \frac{D^{k-\ell} \bar{X}_{j,\delta}}{\bar{X}_{j,\delta}} \right| \times \left| \frac{D^\ell \bar{f}_{x_j,\delta}}{\bar{f}_{x_j,\delta}} \right| \leq \bar{I}_{j,\delta} \sum_{\ell \leq k} \left| \frac{D^\ell \bar{f}_{x_j,\delta}}{\bar{f}_{x_j,\delta}} \right|
\]
we easily see that
\[
\int \left| \frac{D^k \bar{f}_{j,\delta}}{\bar{f}_{j,\delta}} \right|^{2/\beta + \varepsilon} \bar{f}_{j,\delta} \lesssim \sigma^{-1} \sum_{\ell \leq k} \int \left( \bar{I}_{j,\delta} \left| \frac{D^\ell \bar{f}_{x_j,\delta}}{\bar{f}_{x_j,\delta}} \right|^{2/\beta + \varepsilon} \bar{X}_{j,\delta} \right) \bar{f}_{x_j,\delta} \lesssim \sigma^{-1} \sum_{\ell \leq k} \int \left| \frac{D^\ell \bar{f}_{x_j,\delta}}{\bar{f}_{x_j,\delta}} \right|^{2/\beta + \varepsilon} \bar{f}_{x_j,\delta} \lesssim \sigma^{-1}
\]
where we used both Lemma A.5 iv) and (13) with the fact that $\langle \ell, \alpha \rangle \leq \langle k, \alpha \rangle$ for $\ell \leq k$. The bound on the second integral follows from the same line of reasoning. \hfill $\square$

**Lemma C.3.** We let $A_{j,\sigma}$ be the event of all $x \in O^2_{x_j}$ such that
\[
\forall 0 < \langle k, \alpha \rangle < \beta, \quad \left| \frac{D^k \bar{f}_{j,\delta}(z_x)}{\bar{f}_{j,\delta}(z_x)} \right| \lesssim \sigma^{-\langle k, \alpha \rangle} \log(1/\sigma)^{-(\langle k, \alpha \rangle)/2} \quad \text{and} \quad \frac{L_{j,\delta}(z_x)}{\bar{f}_{j,\delta}(z_x)} \lesssim \sigma^{-\beta} \log(1/\sigma)^{-\beta/2}.
\]

Then, the following assertions hold true for $\sigma$ small enough

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i) \( g_j(x) \geq f_j(x)/2 \) for all \( x \in \mathcal{A}_{j,\sigma} \);

ii) \( P_{f_j}(\mathcal{A}_{j,\sigma}^c) \lesssim c_j^{-1} \{ \sigma \log(1/\sigma) \}^{2\beta+\varepsilon} \);

iii) For any \( (k, \alpha) < \beta \), \( \int_{\mathcal{A}_{j,\sigma}^c} |D^k \tilde{f}_j,\delta(z_x)| \, dx \lesssim c_j^{-1} \delta^{D-d} \{ \sigma \log(1/\sigma) \}^{2\beta+\varepsilon-(k,\alpha)} \).

Proof. We start by proving i). For any \( x \in \mathcal{A}_{j,\sigma} \), there holds, using the fact that the functions \( d^j_{j,k} \) are uniformly bounded,

\[
|g_j(x) - f_j(x)| = \left| \frac{1}{\delta^{D-d}} \sum_{0 < (k,\alpha) < \beta} \sigma^{(k,\alpha)} d_k(y, \sigma, \delta) D^k \tilde{f}_j,\delta(z_x) \right| \lesssim \frac{\tilde{f}_j,\delta(z_x)}{\delta^{D-d}} \sum_{0 < (k,\alpha) < \beta} \sigma^{(k,\alpha)} \left| \frac{D^k \tilde{f}_j,\delta(z_x)}{\tilde{f}_j,\delta(z_x)} \right| \lesssim f_j(x) \sum_{0 < (k,\alpha) < \beta} \log(1/\sigma)^{-(k,\alpha)/2} \lesssim \log^{-\alpha/2}(1/\sigma) f_j(x) < \frac{1}{2} f_j(x)
\]

provided that \( \sigma \) is chosen small enough. For ii), notice that

\[
P_{f_j}(\mathcal{A}_{j,\sigma}^c) \lesssim \sum_{0 < (k,\alpha) < \beta} P_{f_j} \left( |D^k \tilde{f}_j,\delta(z_x)| > \sigma^{-(k,\alpha)} \log(1/\sigma)^{-(k,\alpha)/2} \right)
\lesssim \sum_{0 < (k,\alpha) < \beta} \sigma^{2\beta+\varepsilon} \log(1/\sigma)^{2\beta+\varepsilon} \int_{\mathcal{O}^\beta_j} \left( \frac{|D^k \tilde{f}_j,\delta(z_x)|}{\tilde{f}_j,\delta(z_x)} \right)^{2\beta+\varepsilon} f_j(x) \, dx
\lesssim c_j^{-1} \sigma^{2\beta+\varepsilon} \log(1/\sigma)^{2\beta+\varepsilon}
\]

where we used Lemma C.2 after a variable change \( z = z_x \). Finally, for iii), there holds

\[
\int_{\mathcal{A}_{j,\sigma}^c} |D^k \tilde{f}_j,\delta(z_x)| \, dx = \int \left( \frac{|D^k \tilde{f}_j,\delta(z_x)|}{\tilde{f}_j,\delta(z_x)} \right)^{2\beta+\varepsilon} \tilde{f}_j,\delta(z_x) \, d\mathcal{A}_{j,\sigma}^c(x) \, dx
\lesssim \left\{ \int \left( \frac{|D^k \tilde{f}_j,\delta(z_x)|}{\tilde{f}_j,\delta(z_x)} \right)^{2\beta+\varepsilon} \tilde{f}_j,\delta(z_x) \, d\mathcal{A}_{j,\sigma}^c(x) \, dx \right\}^{(k,\alpha)} \times \{ \delta^{D-d} P_{f_j}(\mathcal{A}_{j,\sigma}^c) \}^{(k,\alpha)}
\lesssim \{ c_j^{-1} \delta^{D-d} \} \times \{ c_j^{-1} \delta^{D-d} \} \{ \sigma \log(1/\sigma) \}^{2\beta+\varepsilon}
\lesssim c_j^{-1} \delta^{D-d} \{ \sigma \log(1/\sigma) \}^{2\beta+\varepsilon-(k,\alpha)}.
\]

ending the proof. \( \square \)

C.2 Proof of Corollary 3.4

Let define

\[
\tilde{h}_j := g_j \mathbb{I}_{g_j > f_j/2} + \frac{1}{2} f_j \mathbb{I}_{g_j < f_j/2}
\]

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and $h_j = \tilde{h}_j / I_j$ where $I_j = \int \tilde{h}_j$. The function $h_j$ is a probability measure on $\mathcal{O}^0_{x_j}$ and furthermore, using the convexity of $f \mapsto d^2_H(f, f_0)$, there holds

$$d^2_H(K_{\Sigma}(h), f_0) \leq \sum_{j=1}^J c_j d^2_H(K_{\Sigma}(h_j), f_j) \quad \text{where} \quad h = \sum_{j=1}^J c_j h_j.$$ 

We will control each term separately. First notice that

$$\sum_{j=1}^J c_j d^2_H(K_{\Sigma}(h_j), f_j) \leq J \sigma^{2\beta} + \sum_{c_j > \sigma^{2\beta}} c_j d^2_H(K_{\Sigma}(h_j), f_j).$$

Now take $1 \leq j \leq J$ such that $c_j > \sigma^{2\beta}$ and define $\mathcal{U}_{j, \sigma} = \{x \in \mathbb{R}^D \mid f_j(x) \geq \sigma^{H_1}\}$ for some $H_1 > 0$ to be specified later. Then there holds

$$d^2_H(K_{\Sigma} h_j, f_j) \leq \int_{\mathcal{U}_{j, \sigma} \cap \mathcal{O}_{x_j}^1} \left(\sqrt{K_{\Sigma}(h_j)} - \sqrt{f_j}\right)^2 + \int_{\mathcal{O}_{x_j}^1 \setminus \mathcal{U}_{j, \sigma}^c} K_{\Sigma}(h_j) + \int_{\mathcal{U}_{j, \sigma}^c} [K_{\Sigma}(h_j) + f_j]$$

and

$$\int_{\mathcal{U}_{j, \sigma} \cap \mathcal{O}_{x_j}^1} \left(\sqrt{K_{\Sigma}(h_j)} - \sqrt{f_j}\right)^2 \leq \int_{\mathcal{U}_{j, \sigma} \cap \mathcal{O}_{x_j}^1} \frac{(K_{\Sigma}(h_j) - f_j)^2}{K_{\Sigma}(h_j) + f_j}$$

$$\leq \int_{\mathcal{U}_{j, \sigma} \cap \mathcal{O}_{x_j}^1} \frac{K_{\Sigma}(h_j - \tilde{h}_j)^2}{K_{\Sigma}(h_j) + f_j} + \int_{\mathcal{U}_{j, \sigma} \cap \mathcal{O}_{x_j}^1} \frac{K_{\Sigma}(\tilde{h}_j - g_j)^2}{K_{\Sigma}(h_j) + f_j} + \int_{\mathcal{U}_{j, \sigma} \cap \mathcal{O}_{x_j}^1} \frac{(K_{\Sigma}(g_j) - f_j)^2}{K_{\Sigma}(h_j) + f_j}$$

and therefore

$$d^2_H(f_j, K_{\Sigma}(h_j)) \leq \int_{\mathcal{O}_{x_j}^1 \setminus \mathcal{U}_{j, \sigma}^c} K_{\Sigma}(h_j) \tag{38}$$

$$+ \int_{\mathcal{U}_{j, \sigma}^c} [K_{\Sigma}(h_j) + f_j] \tag{39}$$

$$+ (1 - I_j)^2 \tag{40}$$

$$+ \int_{\mathcal{U}_{j, \sigma} \cap \mathcal{O}_{x_j}^1} \frac{K_{\Sigma}(f_j/2 - g_j)^2 I_{g_j < f_j/2}}{K_{\Sigma}(h_j) + f_j} \tag{41}$$

$$+ \int_{\mathcal{U}_{j, \sigma} \cap \mathcal{O}_{x_j}^1} \left(\frac{K_{\Sigma}(g_j) - f_j}{f_j}\right)^2 f_j \tag{42}$$

where each term will be bounded independently.

1. For (38), notice that for any $y \in \mathcal{O}_{x_j}^0$ and any $x \notin \mathcal{O}_{x_j}^1$, there holds that $\|x - y\|_{\Sigma_{y}^1}^2$
\( C/(\delta^2 \sigma^{2\alpha_1}) \) for some \( C \) depending on \( \tau \). We can thus write

\[
\int_{(\Omega_j)^c} K_\Sigma(h_j) = \int_{(\Omega_j)^c} h_j(y) \varphi_\Sigma(y)(x - y) \, dy \, dx
\]

\[
= \int_{(\Omega_j)^c} \int_{\sigma_{x_j}^0} h_j(y) \exp\left(-\frac{1}{2} \|x - y\|_\Sigma^{-1}(y)\right) \, dy \, dx
\]

\[
\leq \exp\left\{ -\frac{C^2}{4\delta^2 \sigma^{2\alpha_1}} \right\} \int_{(\Omega_j)^c} \int_{\sigma_{x_j}^0} h_j(y) \exp\left(-\frac{1}{2} \|x - y\|_\Sigma^{-1}(y)\right) \, dy \, dx
\]

\[
\leq \exp\left\{ -\frac{C^2}{4\delta^2 \sigma^{2\alpha_1}} \right\} \int_{(\Omega_j)^c} K_{2\Sigma}(h_j) \leq \exp\left\{ -\frac{C^2}{4\delta^2 \sigma^{2\alpha_1}} \right\} \lesssim \sigma^{2\beta},
\]

because \( \delta \sigma^4 = o(\log(\sigma)^{1/2}) \) by assumption.

2. For (42), there holds

\[
\int_{U_j, \sigma \subset \Omega_j^1} \left( \frac{K_{\Sigma}(g_j) - f_j}{f_j} \right)^2 \lesssim \int_{\sigma_{x_j}^1} \left( \frac{L_j \sigma^\beta}{f_j} \right)^2 f_j \lesssim \sigma^{2\beta} \int_{\sigma_{x_j}^1} \left( \frac{L_j}{f_j} \right)^{\frac{2\beta + \varepsilon}{\beta}} f_j \]

\[
\lesssim \sigma^{2\beta} \left\{ \int_{W_j^1} \left( \frac{L_j}{f_j} \right)^{\frac{2\beta + \varepsilon}{\beta}} f_j \right\} \]

\[
\lesssim \sigma^{2\beta} \left\{ \int_{W_j^1} \left( \frac{L_j}{f_j} \right)^{\frac{2\beta + \varepsilon}{\beta}} \right\} \lesssim c_j^{-\frac{\beta}{2\beta + \varepsilon}} \sigma^{2\beta} \lesssim c_j^{-1/2} \sigma^\beta,
\]

where we used that \( |\det d\Psi_{x_j,\delta}| \lesssim \delta^{D-d} \) and Lemma C.2.

3. For (40), notice that

\[
I_j = \int g_j \mathbb{1}_{\{f_j < 2g_j\}} + \frac{f_j}{2} \mathbb{1}_{\{f_j \geq 2g_j\}} = \int g_j + \frac{f_j}{2} \mathbb{1}_{\{f_j > 2g_j\}}
\]

and that \( \int g_j = \int K_\Sigma(g_j) \). Moreover

\[
\int_{\sigma_{x_j}^1} K_\Sigma(g_j) = \int_{\sigma_{x_j}^1} f_j + \int_{\sigma_{x_j}^1} \left( K_\Sigma(g_j) - f_j \right) = 1 + \int_{\sigma_{x_j}^1} \left( K_\Sigma(g_j) - f_j \right)
\]

and since

\[
\left| \int_{\sigma_{x_j}^1} K_\Sigma(g_j) - f_j \right| \lesssim \int_{\sigma_{x_j}^1} L_j \sigma^\beta = \sigma^\beta \int_{W_j^1} L_j \delta \left| \frac{\det d\Psi_{x_j,\delta}}{\delta^{D-d}} \right| ^{\frac{\beta}{2\beta + \varepsilon}}
\]

\[
\lesssim \sigma^\beta \left\{ \int_{W_j^1} \left( \frac{L_j}{f_j} \right)^{\frac{2\beta + \varepsilon}{\beta}} \right\} \lesssim c_j^{-\frac{\beta}{2\beta + \varepsilon}} \sigma^\beta \lesssim c_j^{-1/2} \sigma^\beta.
\]
where we again used that \(|\det d\bar{\Psi}_{x,\delta}| \leq \delta^{D-d}\) and Lemma C.2, we have that
\[
\int_{\Omega_{x,j}^c} K_{\Sigma}(g_j) = 1 + O(c_j^{-1/2}\sigma^2),
\]
We also have, as in (38) that
\[
\int_{\Omega_{x,j}^c} K_{\Sigma}([g_j]) \leq e^{-\frac{c^2}{4\pi^{d-2}}} \int_{\Omega_{x,j}^c} |g_j|(y)dy \leq e^{-\frac{c^2}{4\pi^{d-2}}} \|L_j\|_\infty = o(c_j^{-1}\sigma^2) = o(c_j^{-1/2}\sigma^2),
\]
where the two last inequality comes from \(\delta^2\sigma^2 = o(1/\log(1/\sigma))\) and \(c_j \geq \sigma^2\). Therefore \(|1 - \int g_j| \leq c_j^{-1/2}\sigma^2\). Moreover, using this time Lemma C.3
\[
\int_{J_j > 2g_j} (f_j - 2g_j) \leq P_{f_j}(f_j > 2g_j) + \sum_{0 < (k,\alpha) < \delta^{D-d}} \int_{J_j > 2g_j} |D^k\bar{f}_{j,\delta}(z_x)|dx \leq c_j^{-1}\sigma^{2\beta + \varepsilon}.
\]
All in all, we find that
\[
(1 - I_j)^2 \leq \exp \left\{-C^2/(4\delta^2\sigma^2)\right\} + \left\{c_j^{-1}\sigma^{2\beta}\right\} \cup \left\{c_j^{-2}\left\{\sigma \log(1/\sigma)\right\}^{4\beta + 2\varepsilon}\right\} \leq \exp \left\{-C^2/(4\delta^2\sigma^2)\right\} + c_j^{-1}\sigma^{2\beta}
\]
where the last inequality again holds because \(c_j \geq \sigma^{2\beta}\).

4. For (39), we start with taking \((k,\alpha) < \beta\). Notice that, thanks to Proposition B.3, there exists a constant \(C > 0\) such that for any \(x, y \in \Omega_{x,j}^c\), \(|D^k\bar{f}_{j,\delta}(z_x) - D^k\bar{f}_{j,\delta}(z_y)| \leq C\delta^{D-d} L_j(x)\). Then, using (18), together with \(|\det d\bar{\Psi}(w)| \leq C\),
\[
|K_{\Sigma}(D^k\bar{f}_{j,\delta}(z_0))|(x) \leq \int_{\Delta^{-1}_{\sigma,1}(W_{0-z_x})} e^{-B_{2\beta}(x,z)}|D^k\bar{f}_{j,\delta}(\Delta_{\sigma,1}\bar{z} + z_x)|dz \leq \delta^{D-d} L_j(x) \int_{\Delta^{-1}_{\sigma,1}(W_{0-z_x})} e^{-B_{2\beta}(x,z)}dz \leq \delta^{D-d} L_j(x).
\]
Now notice that
\[
\int_{U_{j,\sigma}} |D^k\bar{f}_{j,\delta}(z_x)|dx \leq \left\{\int_{U_{j,\sigma}} \left\{|D^k\bar{f}_{j,\delta}(z_x)| \right\}^{2\beta + \varepsilon}_{(k,\alpha)} \bar{f}_{j,\delta}(z_x) dx \right\}^{(k,\alpha)}_{2^\beta + \varepsilon} \times \left\{\int_{U_{j,\sigma}} \bar{f}_{j,\delta}(z_x) dx \right\}^{(k,\alpha)}_{2^\beta + \varepsilon} \leq c_j^{-1}\delta^{D-d} \sigma^{H_1 \times (1 - (k,\alpha)/(2\beta + \varepsilon))} \leq c_j^{-1}\delta^{D-d} \sigma^{H_1/2}
\]
and likewise, \(\int_{U_{j,\sigma}} L_j(x) dx \leq c_j^{-1}\delta^{D-d} \sigma^{H_1 \times (1 - \beta/(2\beta + \varepsilon))} \leq \delta^{D-d} \sigma^{H_1/2}\). We thus have shown that
\[
\int_{U_{j,\sigma}} |K_{\Sigma}(D^k\bar{f}_{j,\delta}(z_0))|(x) \leq c_j^{-1}\delta^{D-d} \sigma^{H_1/2}.
\]
Coming back to (39), there immediately holds that $\int_{U_{j,\sigma}} f_j \leq \sigma^{H_1}$, and furthermore, noticing that

$$h_j \leq \tilde{h}_j \leq f_j + \frac{1}{\delta^{D-d}} \sum_{0 \leq (k,\alpha) < \beta} \sigma^{(k,\alpha)} |D^k \tilde{f}_{j,\delta}(z_\partial)|$$

and using (43) and the monotonicity of $K_\Sigma$, we find that $\int_{U_{j,\sigma}} K_\Sigma(h_j) \leq c_j^{-1} \sigma^{H_1/2}$.

5. Finally, for (41), notice that $f_j/2 - g_j$ is a sum of terms of the form $\sigma^{(k,\alpha)} \delta-2(D-d) K_\Sigma(D^k \tilde{f}_{j,\delta}(z_\partial) \mathbb{1}_{g_j \in f_j/2})^2$. Now there holds

$$\int_{U_{j,\sigma} \cap O_{j,\varepsilon}} \frac{K_\Sigma(D^k \tilde{f}_{j,\delta}(z_\partial) \mathbb{1}_{g_j \in f_j/2})^2}{K_\Sigma(h_j) + f_j} \leq \sigma^{-H_1} \|K_\Sigma(D^k \tilde{f}_{j,\delta}(z_\partial) \mathbb{1}_{g_j \in f_j/2})\|_\infty \times \int K_\Sigma(\|D^k \tilde{f}_{j,\delta}(z_\partial) \mathbb{1}_{g_j \in f_j/2}) dy \leq \frac{\sigma^{-H_1}}{c_j^2} \delta^{D-d} \left(\sigma \log(1/\sigma)\right)^{2\beta+\varepsilon(k,\alpha)}$$

where we used Lemma C.3. Summing all the bounds in $k$, we obtain that (41) is bounded from above, up to constant, by

$$\max_k c_j^{-2} \sigma^{2\beta+\varepsilon-H_1+2(k,\alpha)} \delta^{(D-d)} \log(1/\sigma)^{2\beta+\varepsilon} \leq c_j^{-1} \delta^{(D-d)} \sigma^{\varepsilon-H_1} \log(1/\sigma)^{2\beta+\varepsilon}$$

where we used that $c_j \geq \sigma^{2\beta}$.

Collecting all the bounds on (38-42) together with $\delta^2 \sigma^{2\alpha} = o(1/\log(1/\sigma))$, we get that

$$c_j d_H^2(K_\Sigma(h_j), f_j) \leq c_j \left[\sigma^{2\beta} \log(1/\sigma)^{4\beta+2\varepsilon} + \sigma^{H_1/2} + \sigma^{\varepsilon-H_1} \delta^{(D-d)} \log(1/\sigma)^{2\beta+\varepsilon}\right].$$

Choosing $H_1 = 4\beta$ and $\varepsilon \geq 6\beta + \varepsilon_1$ where $\varepsilon_1 \leq \delta^{D-d}$, we obtain the result.

## D Appendix to Section 5.1: proof of Theorem 3.1

For any probability distribution $P$ on $\mathbb{R}^D \times \mathcal{S}^{++}(D, \mathbb{R})$, one defines the probability density function on $\mathbb{R}^D$

$$f_P(x) := \int \varphi_\Sigma(x-y) dP(y, \Sigma).$$

Note that when $g$ is a probability distribution on $\mathbb{R}^D$, then

$$K_\Sigma g(x) = f_P(x) \quad \text{with} \quad dP(y, \Sigma) = \delta_{\Sigma(y)}(\Sigma) g(y) dy.$$

**Lemma D.1.** Let $V_0, \ldots, V_N$ be a partition of $\mathbb{R}^D$ and let $P = \sum_{j=1}^N \pi_j \delta_{\Sigma_j, \pi_j}$ with $z_j \in V_j$. Then, for any probability measure $Q$ on $\mathbb{R}^D \times \mathcal{S}^{++}(D, \mathbb{R})$,

$$\|Q - P\|_1 \leq 2 \sum_{j=1}^N \|Q_1(V_j) - \pi_j\| + \frac{1}{2} \sup_{1 \leq j \leq N} \|\Sigma_j^{-1/2}\|_{\text{op}} \text{diam} V_j + \frac{3}{2} \sup_{1 \leq j \leq N} \text{sup} \sqrt{\text{tr}(\Sigma_j^{-1} \Sigma - \text{Id})^2},$$

where $Q_1$ is the first marginal of $Q$ and $S_j$ is the support of the second marginal of $\mathbb{1}_{V_j}(y) dQ(y, \Sigma)$.  

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Proof. We write that $f_Q(x) - f_P(x)$ is

$$
\int_{V_0} \varphi_\Sigma(x-y) \ dQ(y, \Sigma) + \sum_{j=1}^N \int_{V_j} \{ \varphi_\Sigma(x-y) - \varphi_{\Sigma_j}(x-z_j) \} \ dQ(y, \Sigma) \\
+ \sum_{j=1}^N (Q_1(V_j) - \pi_j) \varphi_{\Sigma_j}(x-z_j),
$$

The last term can be readily bounded in $L^1$-norm by

$$
\sum_{j=1}^N |Q_1(V_j) - \pi_j| \int \varphi_{\Sigma_j}(x-z_j) \ dx = \sum_{j=1}^N |Q_1(V_j) - \pi_j|,
$$

and the first one by $Q_1(V_0) = 1 - \sum_{j=1}^N Q_1(V_j) \leq \sum_{j=1}^N |Q_1(V_j) - \pi_j|$. For the second term, notice that each term of the sum is upper-bounded in $L^1$-norm by

$$
\int_{V_j} \| \varphi_\Sigma(\cdot-y) - \varphi_{\Sigma_j}(\cdot-z_j) \|_1 \ dQ(y, \Sigma) \\
\leq \int_{V_j} \| \varphi_{\Sigma_j}(\cdot-y) - \varphi_{\Sigma_j}(\cdot-z_j) \|_1 \ dQ(y, \Sigma) + \int_{V_j} \| \varphi_\Sigma(\cdot-y) - \varphi_{\Sigma_j}(\cdot-y) \|_1 \ dQ(y, \Sigma) \\
\leq \frac{1}{2} \int_{V_j} \| \Sigma_j^{-1/2} (y-z_j) \| \ dQ(y) + \frac{3}{2} \int_{V_j} \sqrt{\text{tr}(\Sigma_j^{-1} \Sigma - \text{Id})^2} \ dQ(y, \Sigma) \\
\leq \frac{1}{2} Q_1(V_j) \| \Sigma_j^{-1/2} \|_{\text{op}} \text{diam} V_j + \frac{3}{2} Q_1(V_j) \sup_{\Sigma E \Sigma_j} \sqrt{\text{tr}(\Sigma_j^{-1} \Sigma - \text{Id})^2},
$$

where we used Prp 2.1 and Thm 1.1 of [17].

**D.1 Proof of Lemma 5.1**

Throughout the proof, $C$ denotes a generic constant whose value depends only on $D$. We start with bounding the probability measure of $\mathcal{F}_n^c$. There holds

$$
\Pi(\mathcal{F}_n^c) \leq \Pi \left\{ \exists h \in H_n, \mu_h \notin B(0, R_n) \right\} + \Pi \left\{ \sum_{h \in H_n} \pi_h \geq \epsilon_n \right\} + \Pi(\exists h \in H_n, \Lambda_h \notin \left[ \sigma_n^2, \bar{\sigma}_n^2 \right]^D).
$$

The first mass is bounded using (8): 

$$
\Pi \left\{ \exists h \in H, \mu_h \notin B(0, R_n) \right\} \leq H_n \int \| \mu \|^{-b_2} \chi_{\| \mu \| \geq R_n} \ d\mu \lesssim H_n R_n^{-b_2} \lesssim e^{-c_1 n \epsilon_n^2},
$$

as soon as $b_2 R_0 \geq 2 c_1$. The second term is bounded in [59, p. 15] by

$$
\Pi \left\{ \sum_{h \in H_n} \pi_h \geq \epsilon_n \right\} \lesssim \left\{ \frac{eB}{H_n} \log(1/\epsilon) \right\}^{H_n} \lesssim e^{-c_1 n \epsilon_n^2},
$$

as soon as $H_0$ is large enough. Finally, to bound the last probability, we consider separately the partial and hybrid location-scale priors.

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• Partial location-scale: \( \Lambda_h = \Lambda_1 \) for all \( h \) and using (9),

\[
\Pi(\Lambda_1 \notin \mathcal{Q}_n) \leq \Pi_A \left\{ \min_i \Lambda_i \leq \sigma_0^2 \right\} + \Pi_A \left\{ \max_i \Lambda_i \geq \sigma_n^2 \right\} \leq e^{-c_3 \sigma_0^{-2\beta_3} n \varepsilon_n^2} + \sigma_0^{-2b_4} \leq e^{-c_1 n \varepsilon_n^2},
\]

as soon as \( \sigma_n^2 \) is large enough and \( \sigma_0^2 \) is small enough.

• Hybrid location-scale prior

\[
\Pi(3h \in H_n, \Lambda_h \notin \mathcal{Q}_n) \leq H_n \mathbb{E}_{\Pi_A} \left[ Q_2(\min_i \Lambda_i \leq \sigma_n^2) \right] + H_n \mathbb{E}_{\Pi_A} \left[ Q_2(\max_i \Lambda_i > \sigma_n^2) \right] \\
\leq H_n e^{-c_3 \sigma_0^{-2\beta_3} n \varepsilon_n^2} + H_n e^{-c_4 \sigma_1^{-2b_4} n \varepsilon_n^2} \leq e^{-c_1 n \varepsilon_n^2}.
\]

We then turn on bounding the entropy of the partitions of \( \mathcal{F}_n \). Note that on \( \mathcal{F}_n \) \( \min_i \lambda_i \geq \sigma_n^2 \) so that

\[
\frac{\max_i \lambda_i}{\min_i \lambda_i} \leq \max_i \lambda_i \sigma_n^{-2} \leq \sigma_0^{-2} n^{2t_0/b_3} \times \max_i \lambda_i.
\]

From [11], the covering number of \( \mathcal{F}_{n,j,\ell} \) is bounded by

\[
N(\varepsilon_n, \mathcal{F}_{n,j,\ell}, \| \cdot \|_1) \leq \exp \left( CH_n [\log n + \log(1/\varepsilon_n)] + (D - 1) \sum_{h \in H_n} \log(j_h + 1) + D(D - 1)2^{\ell-2}H_n \log n \right),
\]

and the covering number of \( \mathcal{F}_{n,j} \) is bounded by

\[
N(\varepsilon_n, \mathcal{F}_{n,j}, \| \cdot \|_1) \leq \exp \left( CH_n [\log n + \log(1/\varepsilon_n)] + \sum_{h \in H_n} \log(j_h + 1) + DH_n \log \sigma_n^2 \right)
\leq \exp \left( CH_n [\log n + \log(1/\varepsilon_n)] + \sum_{h \in H_n} \log(j_h + 1) \right).
\]

We bound \( \Pi(\mathcal{F}_{n,j,\ell}) \) in the case of the partial location-scale prior, for \( \ell \geq 1 \):

\[
\Pi(\mathcal{F}_{n,j,\ell}) \leq \prod_{h \in H_n} (j_h \sqrt{n})^{-(b_2 - D)1}_{j_h > 1} \Pi_A \left\{ \max_i \lambda_i > \sigma_0^2 n^{2\ell-1-2t_0/b_3} \right\}
\leq \prod_{h \in H_n} (j_h \sqrt{n})^{-(b_2 - D)1}_{j_h > 1} n^{-b_4(2\ell-1-2t_0/b_3)}
\leq \exp \left\{ -(b_2 - D) \sum_{h \in H_n} 1_{j_h > 1} \log(j_h) - \frac{1}{2} (b_2 - D) n H_n - b_4(2\ell-1-2t_0/b_3) H_n \log n \right\}.
\]

We bound \( \Pi(\mathcal{F}_{n,j}) \) in the case of the hybrid location-scale prior:

\[
\Pi(\mathcal{F}_{n,j}) \leq \prod_{h \in H_n} (j_h \sqrt{n})^{-(b_2 - D)1}_{j_h > 1}
\leq \exp \left\{ -(b_2 - D) \sum_{h \in H_n} 1_{j_h > 1} \log(j_h) - \frac{1}{2} (b_2 - D) n H_n \log n \right\}.
\]
This implies in particular that for the partial location-scale prior
\[
\sum_{j, \ell} \sqrt{\Pi(F_{n,j,\ell})} N(\varepsilon_n, F_{n,j,\ell}, \| \cdot \|_1) \lesssim \exp \left( \frac{1}{2} CH_n \log n + \log(1/\varepsilon_n) \right)
\]
\[
\times \sum_{j, \ell} \exp \left( \frac{1}{2} \sum_{h} [(D - 1) \log(j_h + 1) - (b_2 - D)(\log(j_h) \Pi_{j_h > 1} - 1)]
\right.
\]
\[
+ \parallel D \parallel_{(h)} 2^{\ell - 1} \log n [D(D - 1)/2 - b_4]
\]
\[
\lesssim \exp \left( CH_n \log n + \log(1/\varepsilon_n) \right) / 2
\],

since \( b_4 > D(D - 1)/2 \) and \( b_2 > 2D - 1 \). Therefore, by choosing \( M_0 > 0 \) large enough
\[
\sum_{j, \ell} \sqrt{\Pi(F_{n,j,\ell})} N(\varepsilon_n, F_{n,j,\ell}, \| \cdot \|_1) e^{-\lambda n^2} = o(1).
\]

We obtain a similar result for the hybrid location-scale prior.

### D.2 Proof of Lemma 5.2

We let again \( R = (H \log(1/\varepsilon))/C_\beta \) and define \( \sigma := \varepsilon^{1/\beta} \). Thanks to Corollary 3.4, we know there exists a density function \( g \) supported on \( M^q \) such that \( d_\Pi^2(K_{\Sigma G}, f_0) \leq \sigma^{2q} \log^q(1/\sigma) \) for some \( a > 0 \). We can in turn, thanks to Lemma 5.3, find a discrete probability measure \( G \) on \( M^q \cap B(0, R) \) with \( N \) atoms at least \( \sigma^{a_1, \varepsilon^{2}} \)-apart such that
\[
\| K_{\Sigma G} - K_{\Sigma G} \|_1 \leq 2 \sigma^2 \log(D)(1/\varepsilon) \quad \text{and} \quad N \leq \sigma^{-D} \log^D(1/\varepsilon).
\]

We thus have
\[
d_\Pi^2(K_{\Sigma G}, f_0) \leq \sigma^{2q} \log^q(1/\sigma) + 2 \sigma^{2q} \log^q(1/\varepsilon) \leq 2 \sigma^{2q} \log^q(1/\varepsilon)
\]
with \( r := q/2 \vee D/4 \). We let \( z_1, \ldots, z_N \) be the atoms of \( G \) and denote by \( p_j = G(z_j) \). We let \( V_j \) be the ball centered around \( z_j \) with radius \( \sigma^{2a_0, \varepsilon^2}/2 \). We complete \( V_1, \ldots, V_N \) with sets \( V_{N+1}, \ldots, V_J \) that forms a partition of \( M^r \cap B(0, R) \) with \( V_j \) included in balls of the form \( \{ x \in \mathbb{R}^D | \| x - z_j \|_{\Sigma^{-1}(z_j)} \leq 1 \} \) for some \( z_j \in M^r \cap B(0, R) \), so that we can take \( J \leq N + (R/\sigma)^D \leq \sigma^{-D} \log^D(1/\varepsilon) \). We then set \( V_0 \) to be the complementary set of the reunion of the \( V_j \) and set further \( p_j = 0 \) for \( j \) greater than \( N + 1 \).

We write under the prior \( \Pi, P = \sum_{h=1}^\infty \pi_h \delta_{\mu_h, U_h, A_h} \) and \( \Sigma_h = U_h^T A_h U_h \). We use the convention that \( A_h = A \) for all \( h \) in the case of the Partial location scale prior and \( \pi_h = 0 \) for \( h \geq K \) for the mixture of finite mixtures prior. Set \( \tilde{N} = \log(1/\varepsilon) \times N \), we consider the following events
\[
P_J = \left\{ \sum_{j=1}^J |p_j - P_\mu(V_j)| \leq \varepsilon^2 \quad \text{and} \quad \min_{1 \leq j \leq J} P_\mu(V_j) \geq \varepsilon^4 \right\},
\]
\[
F_{\tilde{N}} = \left\{ \sum_{h \in \tilde{N}} \pi_h \geq 1 - \varepsilon^4 \right\},
\]
\[
\mathcal{O}_{\tilde{N}} = \left\{ \forall 1 \leq h \leq \tilde{N}, \| U_h O_{\mu_h}^T - \text{Id} \|_{op} \leq \sigma^{2a_0, \varepsilon^2} \right\},
\]
and \( \mathcal{L}_{\tilde{N}} = \left\{ \forall 1 \leq h \leq \tilde{N}, A_h \in S(\sigma^{a_0})^D \times S(\delta \sigma^{a_1})^{D-d} \right\} \) where \( S(t) = \{ s \mid t^2 \leq s \leq t^2(1 + \sigma^{2\beta}) \} \).
We first show that if $P \in \mathcal{P}_J \cap \mathcal{F}_N \cap \mathcal{O}_N \cap \mathcal{L}_N$, then

$$d_H(f_P, f_0) \leq \varepsilon \log^\tau(1/\varepsilon).$$

Indeed, we have

$$d_H(f_P, f_0) \leq d_H(f_P, K \Sigma G) + d_H(K \Sigma G, f_0) \leq d_H(f_P, f_{\overline{G}}) + d_H(f_{\overline{P}}, f_P) + \varepsilon \log^\tau(1/\varepsilon),$$

where $d_{\overline{G}}(y, \Sigma) = \delta_{\Sigma(y)} dG(y)$ and $\overline{P} = \sum_{h \geq 1} \pi_h \delta_{\mu_h, \Sigma_h}$ with

$$\overline{\Sigma}_h = \begin{cases} \Sigma_h & \text{if } h \leq \overline{N} \\
\Sigma(z_h) & \text{if } h > \overline{N} \text{ and } \mu_h \in V_j \text{ with } j \leq N; \\
0 & \text{otherwise}, \end{cases}$$

where, conventionally, $\varphi_{\Sigma}(z - x) dz = \delta_z$ when $\Sigma = 0$. Since $P \in \mathcal{F}_N$, there easily holds $\|f_P - f_{\overline{P}}\|_1 \leq 2\varepsilon^2$ and, using Lemma D.1, we find that

$$\|f_{\overline{P}} - f_{\overline{G}}\|_1 \leq 2 \sup_{j=1}^N |P_j(V_j) - p_j| + \frac{1}{2} \sigma_{\alpha_0} \sup_{1 \leq j \leq N} \text{diam } V_j + \frac{3}{2} \sup_{1 \leq j \leq N} \sup_{\mu_h \in V_j} \text{tr}(\Sigma(z_j)^{-1} \overline{\Sigma}_h - \text{Id})^2,$$

$$\leq \varepsilon^2 + \sup_{1 \leq j \leq N} \sup_{\mu_h \in V_j} \|\Sigma(z_j)^{-1} \overline{\Sigma}_h - \text{Id}\|_{op},$$

where we used that $P \in \mathcal{P}_J$ and that $\text{diam } V_j \leq \sigma_{\alpha_0} \varepsilon^2$ for $j \leq N$. In the last supremum, if $h > \overline{N}$ and $\mu_h \in V_j$, then $\Sigma_h = \Sigma(z_j)$ so we only need to handle the case when $h \leq \overline{N}$. Moreover,

$$\Sigma(z_j)^{-1} \overline{\Sigma}_h - \text{Id} = \sum_{j=1}^N |O_{z_j} \Delta_{\sigma, \delta}^2 O_{z_j}^t \Lambda_h U_h - \text{Id}|$$

$$= \sum_{j=1}^N |O_{z_j} \Delta_{\sigma, \delta}^2 (O_{z_j} U_h^t - \text{Id}) \Lambda_h U_h + O_{z_j} \Delta_{\sigma, \delta}^2 \Lambda_h U_h - \text{Id}|$$

$$= \sum_{j=1}^N |O_{z_j} \Delta_{\sigma, \delta}^2 (O_{z_j} U_h^t - \text{Id}) \Lambda_h U_h + O_{z_j} (\Delta_{\sigma, \delta}^2 \Lambda_h - \text{Id}) U_h + O_{z_j} U_h - \text{Id}|,$$

so that

$$\|\Sigma(z_j)^{-1} \overline{\Sigma}_h - \text{Id}\|_{op} \leq \|\Delta_{\sigma, \delta}^2\|_{op} \|\Lambda_h\|_{op} \|O_{z_j} U_h^t - \text{Id}\|_{op} + \|\Delta_{\sigma, \delta}^2 \Lambda_h - \text{Id}\|_{op} + \|O_{z_j} U_h - \text{Id}\|_{op} \leq \varepsilon^2 + \sigma^{2\beta} \varepsilon^2,$$

where we used both that $h \leq \overline{N}$ and $P \in \mathcal{O}_N \cap \mathcal{L}_N$.

Using [59, Lem B2], we find that for $\lambda > 0$ small enough

$$P_0 \log \frac{f_0}{f_P} \leq d_H^2(f_0, f_P) (1 + \log(1/\lambda)) + P_0 \left\{ \log \frac{f_0}{f_P} 1_{f_P / f_0 < \lambda} \right\},$$

$$P_0 \left( \log \frac{f_0}{f_P} \right)^2 \leq d_H^2(f_0, f_P) (1 + \log^2(1/\lambda)) + P_0 \left( \left( \log \frac{f_0}{f_P} \right)^2 1_{f_P / f_0 < \lambda} \right),$$

up to numeric constant. Notice that by assumption, $f_0$ is bounded from above by $\|L\|_{\infty}$. Let $x \in M^r \cap B(0, R)$, and let $1 \leq j \leq J$ be such that $x \in V_j$. Notice that, since $P \in \mathcal{F}_N \cap \mathcal{P}_J$, there holds

$$\varepsilon^4 \leq P_\mu(V_j) = \sum_{\mu_h \in V_j} \pi_h \leq \sum_{\mu_h \in V_j \atop h \leq \overline{N}} \pi_h + \varepsilon^8 \quad \text{so that} \quad \sum_{\mu_h \in V_j \atop h \leq \overline{N}} \pi_h \geq \varepsilon^4/2.$$
Now we can write
\[ f_P(x) \geq \sum_{\mu_h \in V_j, h \in N} \pi_h \varphi_{\Sigma_h}(x - \mu_h) \geq \frac{\varepsilon^4}{\delta^{D-d} \sigma^D}, \]
where we used that \(|\det \Sigma_h|^{1/2} \leq (1 + \sigma^{2\beta})^{D/2} \delta^{-d} \sigma^D \leq \delta^{D-d} \sigma^D\) and that \(\|x - y\|_{\Sigma^{-1}(y)} \leq 1\) for any \(x, y \in V_j\) with a similar line of reasoning as in the proof of Lemma 5.3, relying on \(P \in \mathcal{O}_N \cap \mathcal{L}_N\). Furthermore, since \(f_P(x) \leq \varepsilon^8 + \sum_{h \in N} \pi_h \varphi_{\Sigma_h}(x - \mu_h)\), there must be some \(h \in \tilde{N}\) such that both \(\pi_h \geq \varepsilon^2\) and \(\mu_h \in B(0, R)\) holds, otherwise one would get a contradiction looking at the mass of \(f_0\) since one would get
\[
\|f_0\|_1 = \|f_0 \mathbb{1}_{B(0,R/2)}\|_1 + \|f_0 \mathbb{1}_{B(0,R/2)^c}\|_1 \\
\leq \|f_P \mathbb{1}_{B(0,R/2)}\|_1 + \|f_P \mathbb{1}_{B(0,R/2)^c}\|_1 + \|f_0 \mathbb{1}_{B(0,R/2)^c}\|_1,
\]
where the inequalities occurs up to log-term. For \(x \notin B(0, R)\), notice that, for this particular \(h \leq \tilde{N}\), \(f_P(x) \geq \varepsilon^2 \varphi_{\Sigma_h}(x - \mu_h)\). Taking \(\lambda = C\varepsilon^4/(\delta^{D-d} \sigma^D)\) for some small constant \(C\), one get for any \(\ell \geq 1\),
\[
P_0 \left( \left\{ \log \frac{f_0}{f_P} \left| \mathbb{1}_{f_P/f_0 < \lambda} \right\} \right\} \leq P_0 \left( \left\{ \log \frac{f_0}{f_P} \left| \mathbb{1}_{B(0,R)} \right\} \right\} \right)
\leq \log^\ell \frac{\varepsilon^2}{\sigma^D \delta^{D-d}} P_0(B(0,R)^c) + \int_{B(0,R)^c} \|x - \mu_h\|_{\Sigma_h}^2 f_0(x) \, dx
\leq \varepsilon^H \log^\ell \frac{\varepsilon^2}{\sigma^D \delta^{D-d}} + \sigma^{-2\alpha_0} \left\{ \int \|x\|_{4\ell}^4 f_0(x) \, dx \right\}^{1/2} P_0(B(0,R)^c)^{1/2}
\leq \varepsilon^{2 \log^2(1/\varepsilon)},
\]
where the last inequality holds for \(\ell \in \{1, 2\}\), provided that we chose \(H \geq 8\alpha_0 + 4\beta\). This shows that \(f_P \in B(f_0, \varepsilon)\) for \(\varepsilon \approx \varepsilon \log^s(1/\varepsilon)\) with \(s = r \vee 1\). It only remains to lower bound the prior mass of the event \(P_J \cap F_N \cap \mathcal{O}_N \cap \mathcal{L}_N\).

For this, one can use (8) and the fact that the scales are drawn independently to rest to find that
\[ \Pi(\mathcal{P}_J \cap F_N \cap \mathcal{O}_N \cap \mathcal{L}_N) \geq \Pi(\mathcal{P}_J \cap F_N) \times c_0^N \prod_{h \in N} O\left\{ \|U_{O_{\mu_h}} - \text{Id}\|_{op} \leq \sigma^{2\alpha_0} \varepsilon^2 \right\} \times \Pi(\mathcal{L}_N). \]

We easily get that \(O\left\{ \|U_{O_{\mu_h}} - \text{Id}\|_{op} \leq \sigma^{2\alpha_0} \varepsilon^2 \right\} \geq (\sigma^{2\alpha_0} \varepsilon^2)^{D(D-1)/2}\). For \(\Pi(\mathcal{L}_N)\), we use (9) or (10) along with a simple Markov inequality to find that
\[ \Pi(\mathcal{L}_N) = \mathbb{E}_\Pi \left\{ Q_2 \left( \left[ \sigma^{2\alpha_0}, (1 + \sigma^{2\beta}) \sigma^{2\alpha_0} \right]^d \times \left[ \delta^2 \sigma^{2\alpha_1}, (1 + \sigma^{2\beta}) \delta^2 \sigma^{2\alpha_1} \right]^{D-d} \right)^N \right\}
\geq \exp\left\{ -2\alpha_0 \beta_0 \log(1/\sigma) \tilde{N} \right\} \exp\left\{ -c_2 D \sigma^{-D} \right\}. \]

For \(\Pi(\mathcal{P}_J \cap F_N)\), we write
\[ \Pi(\mathcal{P}_J \cap F_N) \geq \Pi(\mathcal{P}_J) - \Pi(\mathcal{F}_N^c) \geq e^{-C_1 \log(1/\varepsilon)} - e^{-\tilde{N} \log(\tilde{N})}, \]

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Let \( \bar{\Psi} \) be a density supported on \( \Omega_0 \) and \( \vec{\Psi} \) is less than order \( \kappa \). We can write

\[
K_{\Sigma} g(x) = K_{\Sigma}(g \mathbb{1}_{B^c(0,R)})(x) + \sum_{i \in \mathcal{I}} K_{\Sigma}(\chi_i g \mathbb{1}_{B^c(0,R)})(x) = K_{\Sigma}(g \mathbb{1}_{B^c(0,R)})(x) + \sum_{i \in \mathcal{I}} c_i K_{\Sigma} g_i(x)
\]

where \( c_i = \| \chi_i g \mathbb{1}_{B^c(0,R)} \|_1 \) and \( g_i = \chi_i g \mathbb{1}_{B^c(0,R)}/c_i \) is a density supported on \( \Omega^0_{x_i} \). Notice that for any \( x \in \mathbb{R}^D \)

\[
K_{\Sigma}(g \mathbb{1}_{B^c(0,R)})(x) \leq C_1 \varepsilon^H K_{\Sigma}(\mathbb{1}_{B^c(0,R)})(x) \leq C_1 \varepsilon^H
\]

and that

\[
\| K_{\Sigma}(g \mathbb{1}_{B^c(0,R)}) \|_1 = \| g \mathbb{1}_{B^c(0,R)} \|_1 \leq \int_{|x| > R} C_1 e^{-C_2 |x|^\kappa} \lesssim \int_{\tau > R} e^{-C_2 \tau^\kappa + D-1} d\tau
\]

where we used a NP-bound on the incomplete Gamma function in the last inequality. Consequently, the term \( K_{\Sigma}(g \mathbb{1}_{B^c(0,R)}) \) need not be discretized. Take now \( i \in \mathcal{I} \). As in the proof of Theorem 3.3, there holds that

\[
\varphi_{\Sigma}(y)(x-y) \preceq \frac{\varepsilon^H}{\sigma^D \delta^D d} \quad \forall x \in \Omega^1_{x_i}, y \in \Omega^0_{x_i},
\]

provided that \( \tau \gtrsim R \delta \sigma^{\alpha} \). This means that \( |K_{\Sigma} g_i(x)| \preceq \frac{\varepsilon^H}{\sigma^D \delta^D d} \) for all \( x \notin \Omega^1_{x_i} \). If now \( x \in \Omega^1_{x_i} \), there holds

\[
K_{\Sigma} g_i(x) = \int_{\mathcal{W}_{x_i}} \varphi_{\Sigma}(\vec{\Psi}_{x_i}(w)) (x - \vec{\Psi}_{x_i}(w)) \tilde{g}_i(w) \, dw
\]

\[
= \sum_{j \in J_i} \tilde{c}_{i,j} \int_{\mathcal{W}_{x_i}} \varphi_{\Sigma}(\vec{\Psi}_{x_i}(w)) (x - \vec{\Psi}_{x_i}(w)) \tilde{g}_{i,j}(w) \, dw
\]

where \( \tilde{g}_i(w) = g_i(\vec{\Psi}_{x_i}(w)) \times |\det d\vec{\Psi}_{x_i}(w)| \), \( \tilde{c}_{i,j} = \int_{\mathcal{W}_{x_i}} \tilde{g}_i \) and \( \tilde{g}_{i,j} = \tilde{g}_i/\tilde{c}_{i,j} \). The sets \( \mathcal{W}_{x_i} \) form a partition of \( \mathcal{W}_{x_i}^{0} \) that are included in

\[
\prod_{\ell=1}^{d} [w_{\ell}^0, w_{\ell+1}^0] \times \prod_{\ell=d+1}^{D} [w_{\ell}^0, w_{\ell+1}^0] \cap \mathcal{W}_{x_i}^{0},
\]
for \((u^0_{i_1}, \ldots, u^0_{i_d}, u^1_{i_{d+1}}, \ldots, u^1_{i_D})\) vary on a grid of size \((\sigma^\alpha_0, \ldots, \sigma^\alpha_0, \delta^\alpha_{i_1}, \ldots, \delta^\alpha_{i_1})\). Notice that

\[
\text{Card}(\mathcal{I}) = \frac{1}{\sigma^{\alpha_d}} \times \left(\frac{\delta}{\delta^\alpha_{i_1}}\right)^{D-d} = \sigma^{-D}.
\]

Let \(i \in \mathcal{I}\) and \(j \in \mathcal{J}_i\) be fixed and denote for short \(\bar{\Psi} = \bar{\Psi}_{x_i}\) and \(\bar{\Sigma} = \Sigma \circ \bar{\Psi}\). We let \(\Gamma > 0\) and we distinguish two cases: inf \(w \in \mathcal{W}_{i,j} \| x - \bar{\Psi}(w) \|^2_{\Sigma^{-1}(w)} \geq \Gamma \log(1/\varepsilon) \) and inf \(w \in \mathcal{W}_{i,j} \| x - \bar{\Psi}(w) \|^2_{\Sigma^{-1}(w)} < \Gamma \log(1/\varepsilon)\). In the former,

\[
\int_{\mathcal{W}_{i,j}} \varphi_{\Sigma(w)}(x - \bar{\Psi}(w)) \bar{g}_{i,j}(w) \, dw \lesssim \frac{\varepsilon^{\Gamma/2}}{\sigma \delta^{D-d}}.
\]

While if \(\inf_{w \in \mathcal{W}_{i,j}} \| x - \bar{\Psi}(w) \|^2_{\Sigma^{-1}(w)} \leq \Gamma \log(1/\varepsilon)\), we first show that \(\sup_{w \in \mathcal{W}_{i,j}} \| x - \bar{\Psi}(w) \|^2_{\Sigma^{-1}(w)} \leq \Gamma' \log(1/\varepsilon)\) for some \(\Gamma' > \Gamma\). We let \(w_0 \in \mathcal{W}_{i,j}\) such that \(\| x - \bar{\Psi}(w_0) \|^2_{\Sigma^{-1}(w)} \leq 2\Gamma \log(1/\varepsilon)\) and take \(w \in \mathcal{W}_{i,j}\). Using (31), we get that there \(\| \text{pr}_{T_{\bar{\Psi}(w)}} - \text{pr}_{T_{\bar{\Psi}(w_0)}} \| \leq 2 \tau \sigma^{\alpha_0}\) and the same holds for \(\text{pr}_{N_{\bar{\Psi}(w)}} - \text{pr}_{N_{\bar{\Psi}(w_0)}}\). Furthermore, using (20) and (23) with the same set of notations yields

\[
\| \text{pr}_{T_{\bar{\Psi}(w_0)}}(\bar{\Psi}(w) - \bar{\Psi}(w_0)) \| \leq C' \sigma^{\alpha_0} \quad \text{and} \quad \| \text{pr}_{T_{\bar{\Psi}(w_0)}}(\bar{\Psi}(w) - \bar{\Psi}(w_0)) \| \leq C' \delta \sigma^{\alpha_0},
\]

for some other constant \(C' > 0\). All in all, there holds

\[
\| x - \bar{\Psi}(w) \|^2_{\Sigma^{-1}(w)} = \frac{1}{\sigma^{2\alpha_0}} \| \text{pr}_{T_{\bar{\Psi}(w)}}(x - \bar{\Psi}(w)) \|^2 + \frac{1}{\delta^{2\alpha_0}} \| \text{pr}_{N_{\bar{\Psi}(w)}}(x - \bar{\Psi}(w)) \|^2
\]

\[
\leq 4C^2 + \frac{2}{\sigma^{2\alpha_0}} \| \text{pr}_{T_{\bar{\Psi}(w_0)}}(x - \bar{\Psi}(w)) \|^2 + \frac{2}{\delta^{2\alpha_0}} \| \text{pr}_{N_{\bar{\Psi}(w_0)}}(x - \bar{\Psi}(w)) \|^2
\]

\[
\leq 4C^2 + 8C'\delta + \frac{4}{\sigma^{2\alpha_0}} \| \text{pr}_{T_{\bar{\Psi}(w_0)}}(x - \bar{\Psi}(w_0)) \|^2 + \frac{4}{\delta^{2\alpha_0}} \| \text{pr}_{N_{\bar{\Psi}(w_0)}}(x - \bar{\Psi}(w_0)) \|^2
\]

\[
= 4\| x - \bar{\Psi}(w) \|^2_{\Sigma^{-1}(w_0)}
\]

\[
\leq 4C^2 + 8C'\delta + 8 \Gamma \log(1/\varepsilon) \leq 9 \Gamma \log(1/\varepsilon),
\]

for \(\varepsilon\) small enough. Denote \(R_T(u) = \exp(u) - \sum_{t=0}^{T-1} u^t / t!\), then \(|R_T(u)| \leq e^u |u|^T / T!\) and

\[
\exp \left\{ -\frac{1}{2} \| x - \bar{\Psi}(w) \|^2_{\Sigma^{-1}(w)} \right\} = \sum_{t=0}^{T-1} \frac{(-1)^t}{2^t t!} \| x - \bar{\Psi}(w) \|^2_{\Sigma^{-1}(w)} + R_T \left( -\| x - \bar{\Psi}(w) \|^2_{\Sigma^{-1}(w)}/2 \right).
\]

Note that \(R_T(x, w)\) is uniformly bounded by

\[
|R_T(x, w)| \leq e^{5 \Gamma \log(1/\varepsilon)} \left( \frac{5 \Gamma \log(1/\varepsilon) / T!}{T!} \right) \approx \frac{e^{5 \Gamma} \log^T(1/\varepsilon)}{T!}.
\]

Set,

\[
A_{i,j}(w) := \langle e_i, \Sigma^{-1}(w) e_j \rangle, \quad B_i(w) := \langle e_i, \Sigma^{-1}(w) \bar{\Psi}(w) \rangle, \quad \text{and} \quad C(w) := \| \bar{\Psi}(w) \|^2_{\Sigma^{-1}(w)};
\]

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so that all functions $A_{i,j}$, $B_i$ and $C$ are continuous functions of $w$ and if $x = (x_1, \ldots, x_D)$,
\[
\|x - \bar{\Psi}(w)\|_{\Sigma^{-1}(w)}^t = \left\{ \|x\|_{\Sigma^{-1}(w)}^t - 2(x, \Sigma^{-1}(w)\bar{\Psi}(w)) + \|\bar{\Psi}(w)\|_{\Sigma^{-1}(w)}^2 \right\}^t
\]
\[
= \sum_{|k|=t} \binom{t}{k}(-2)^{k_2} \left\{ \sum_{1 \leq i,j \leq D} x_ix_j A_{i,j}(w) \right\}^{k_1} \left\{ \sum_{i=1}^D x_i B_i(w) \right\}^{k_2} C(w)^{k_3}
\]
\[
= \sum_{|k|=t} \binom{t}{k}(-2)^{k_2} C(w)^{k_1} \left\{ \sum_{|\ell|=k_1} \binom{k_1}{\ell} \prod_{1 \leq i,j \leq D} (x_i x_j)\ell_{i,j} A_{i,j}(w) \right\} \sum_{|m|=k_2} \binom{k_2}{m} \prod_{i=1}^D x_i^m B_i^m(w)
\]
\[
= \sum_{(p,\ell,m) \in \mathcal{G}_t} P_{p,\ell,m}(x) \times C(w)^p \prod_{1 \leq i,j \leq D} A_{i,j}(w) \prod_{1 \leq i \leq D} B_i^m(w)
\]
\[
\quad\quad =: Q_{p,\ell,m}(w)\]

where $P_{p,\ell,m}(x)$ are polynomial functions of $x$, $Q_{p,\ell,m}(w)$ are continuous functions of $w$, and where $\mathcal{G}_t$ is the set \{ $(p, \ell, m) \mid p + |\ell| + |m| = t \} \subset \mathbb{N}^{D+D+1}$. According to [29, Lem 3.1], one can always find an atomic probability measure $G_{i,j}$ such that
\[
\int Q_{p,\ell,m}(w)\bar{g}_{i,j}(w)\,dw = \int Q_{p,\ell,m}(w)G_{i,j}(dw)
\]
for all $p, \ell, m$ such that $p + |\ell| + |m| \leq T - 1$. Since there are less than $T^{D+D+1}$ such triplets, the probability measure $G_{i,j}$ can be taken to have less than $T^{D+D+1}$ atoms. Note that then, this measure satisfies that
\[
\left| \int_{W_{i,j}} \varphi_{\Sigma(w)}(x - \bar{\Psi}(w))(\bar{g}_{i,j}(w)\,dw - G_{i,j}(dw)) \right|
\]
\[
= \frac{1}{(2\pi)^{D/2} \delta_D \delta_{D-d}} \left| \int_{W_{i,j}} R_T(x, w)(\bar{g}_{i,j}(w)\,dw - G_{i,j}(dw)) \right| \leq \frac{\varepsilon^{-5} \log^T(1/\varepsilon)}{\delta^{D-d} \sigma^D T!}.
\]

All in all, $G_{i,j}$ is such that
\[
\left| \int_{W_{i,j}} \varphi_{\Sigma(w)}(x - \bar{\Psi}(w))\bar{g}_{i,j}(w)\,dw - \int_{W_{i,j}} \varphi_{\Sigma(w)}(x - \bar{\Psi}(w))G_{i,j}(dw) \right|
\]
\[
\leq \left\{ \begin{array}{ll}
\frac{1}{\sigma^D \delta_D \delta_{D-d}} \varepsilon^H & \text{if } x \notin \mathcal{O}_{x_i}^1; \\
\frac{1}{\sigma^D \delta_D \delta_{D-d}} \varepsilon^{1/2} & \text{if } x \in \mathcal{O}_{x_i}^1, \inf_{w \in J} \|x - \bar{\Psi}(w)\|_{\Sigma^{-1}(w)} \geq \Gamma \log(1/\varepsilon); \\
\frac{1}{\sigma^D \delta_D \delta_{D-d}} \varepsilon^{-5} \log^T(1/\varepsilon) & \text{if } x \in \mathcal{O}_{x_i}^1, \inf_{w \in J} \|x - \bar{\Psi}(w)\|_{\Sigma^{-1}(w)} \leq \Gamma \log(1/\varepsilon).
\end{array} \right.
\]

Taking $H \geq 1$, $\Gamma \geq 2$ and $T \geq 5\Gamma \log(1/\varepsilon)$ yields a bound of order $\varepsilon/\sigma^D \delta_D \delta_{D-d}$ in every case. Then the probability measure
\[
G = \sum_{i \in \mathcal{I}} \alpha_i \sum_{j \in J_i} \bar{c}_{i,j}(\bar{\Psi}_{x_i}) \# G_{i,j},
\]
is a discrete measure on $M^\delta$ with at most
\[
N_\varepsilon = T^{D^2+D+1} \sum_{i \in \mathcal{I}} \text{Card } J_i \approx \log^{D^2+D+1}(1/\varepsilon) R^D \sigma^{-D} \leq \sigma^{-D} \log D^{D+D}(\kappa^2)(1/\varepsilon)
\]
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The above inequalities also imply that \( \|K\Sigma G - K\Sigma q\|_\infty \leq \varepsilon/(\sigma^D \delta^{D-d}) \).

We now turn to bounding the \( L^1 \) norm. Let again pick \( i \in I \) and \( j \in J_i \). We let \( x_{i,j} = \Phi x_i(w_{i,j}) \) for some \( w_{i,j} \in W_{i,j} \) and for \( T > 0 \) we define
\[
\mathcal{H}_A = \{ x \in \mathbb{R}^D \mid \|x - x_{i,j}\|_{\Sigma^{-1}(x_{i,j})} \leq A \}.
\]

There holds that \( \text{Leb} \, \mathcal{H}_A \approx \delta^{D-d} \sigma^D T^D \). Furthermore, we have, denoting respectively \( P_{i,j} \) and \( Q_{i,j} \) the push-forwards of \( \Phi_{i,j} \) and \( g_{i,j} \) through \( \Phi_{x_i} \),
\[
\|K\Sigma P_{i,j} - K\Sigma Q_{i,j}\|_1 = \int_{\mathcal{H}_A} |K\Sigma P_{i,j} - K\Sigma Q_{i,j}| + \int_{\mathcal{H}_A^c} |K\Sigma P_{i,j} - K\Sigma Q_{i,j}|
\leq \delta^{D-d} \sigma^D A^D \|K\Sigma P_{i,j} - K\Sigma Q_{i,j}\|_\infty \int_{\mathcal{H}_A} [K\Sigma Q_{i,j} + K\Sigma P_{i,j}]
\leq A^D \varepsilon + \int_{\mathcal{H}_A} [K\Sigma P_{i,j} + K\Sigma Q_{i,j}].
\]

Recall that the support of \( P_{i,j} \) and \( Q_{i,j} \) are in \( \Phi_{x_i}(W_{i,j}) \). Furthermore, since
\[
diam \text{pr}_{M_i} \Phi_{x_i}(W_{i,j}) \leq \sigma^{\alpha_0},
\]
there holds, using (31), that \( \|\text{pr}_{T_y} - \text{pr}_{T'_{y'}}\|_{op} \leq \sigma^{\alpha_0} \) for any two \( y, y' \in \Phi_{x_i}(W_{i,j}) \). Let \( y \in \Phi_{x_i}(W_{i,j}) \) and \( x \in \mathcal{H}_A^c \), we now show that \( \|x - y\|_{\Sigma^{-1}(y)} \geq \zeta A \) is \( A \) is chosen large enough, where \( \zeta a > 0 \) is a fixed constant. We have
\[
\|x - y\|_{\Sigma^{-1}(y)} = \left\| \frac{1}{\sigma^{\alpha_0}} \text{pr}_{T_y}(x - y) + \frac{1}{\sigma^{\alpha_1}} \text{pr}_{N_y}(x - y) \right\|
= \left\| \frac{1}{\sigma^{\alpha_0}} \text{pr}_{T_{x_{i,j}}}(x - y) + \frac{1}{\sigma^{\alpha_1}} \text{pr}_{N_{x_{i,j}}}(x - y) \right\| + O(\|x - y\|)
= \left\| \frac{1}{\sigma^{\alpha_0}} \text{pr}_{T_{x_{i,j}}}(x - x_{i,j}) + \frac{1}{\sigma^{\alpha_1}} \text{pr}_{N_{x_{i,j}}}(x - x_{i,j}) \right\| + O(\|x - y\|) + O(1),
= \|x - x_{i,j}\|_{\Sigma^{-1}(x_{i,j})} + O(\|x - y\|) + O(1). \quad (45)
\]
where the term \( O(1) \) comes from noticing that
\[
\|\text{pr}_{T_{x_{i,j}}}(y - x_{i,j})\| \leq \|\text{pr}_{T_{x_{i,j}}}(\Phi_{x_i}(v) - \Phi_{x_i}(v'))\| + \|\text{pr}_{T_{x_{i,j}}}(N_{x_i}(v, \eta))\|
\leq |v - v_{i,j}| + \|\text{pr}_{T_{x_{i,j}}} - \text{pr}_{T_y}\|_{op} \times \|\eta\| \leq \sigma^{\alpha_0},
\]
and
\[
\|\text{pr}_{N_{x_{i,j}}}(y - x_{i,j})\| \leq \|\text{pr}_{N_{x_{i,j}}}(\Phi_{x_i}(v) - \Phi_{x_i}(v'))\| + \|\text{pr}_{N_{x_{i,j}}}(N_{x_i}(v, \eta) - N_{x_i}(v_{i,j}, \eta_{i,j}))\|
\leq |v - v_{i,j}| + |\eta - \eta_{i,j}| \leq \sigma^{\alpha_1} \delta.
\]
The above inequalities also imply that \( \|y - x_{i,j}\| = O(1) \) so that \( \|x - y\| = \|x - x_{i,j}\| + O(1). \) Decomposing \( x - x_{i,j} \) into \( \text{pr}_{T_{x_{i,j}}} \) and \( \text{pr}_{N_{x_{i,j}}} \) show that \( \|x - x_{i,j}\| = o(\|x - x_{i,j}\|_{\Sigma^{-1}(x_{i,j})}) \) so that finally
\[
\|x - y\|_{\Sigma^{-1}(y)} = \|x - x_{i,j}\|_{\Sigma^{-1}(x_{i,j})}(1 + o(1)) + O(1)
\]
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uniformly for \( x \in \mathcal{H}_A^c \) and \( y \in \mathcal{W}_{i,j} \). By choosing \( A \) larger than a fixed constant we then obtain that
\[
\| x - y \|_{\Sigma^{-1}(y)} \geq A/2 \quad \text{uniformly on } x \in \mathbb{R}^D \setminus \mathcal{H}_A \text{ and } y \in \Psi_{x_i}(\mathcal{W}_{i,j}).
\]
This yields that
\[
\int_{\mathcal{H}_A^c} K_\Sigma P_{i,j} = \int_{\mathcal{W}_{i,j}} \int_{\mathcal{H}_A} \varphi_{\Sigma}(y) (x - y) \, dx P_{i,j}(dy)
= \frac{1}{(2\pi)^{D/2} \sigma^D \delta^{-d}} \int_{\mathcal{W}_{i,j}} \exp \left\{ -\frac{1}{2} \| x - y \|_{\Sigma^{-1}(y)}^2 \right\} \, dx P(dy)
\leq \frac{1}{(2\pi)^{D/2}} \int_{|z| \geq A/2} \exp \left\{ -\frac{1}{2} \| z \|_2^2 \right\} \, dz \leq \exp(-A^2/8),
\]
where we made the variable change \( z = \Sigma^{-1/2}(y)(x - y) \) in the second to last inequality. The same holds for \( K_\Sigma Q_{i,j} \). Setting \( A = (8 \log(1/\varepsilon))^{1/2} \) and combining with (44), \( \| K_\Sigma P_{i,j} - K_\Sigma Q_{i,j} \|_1 \leq \log^{D/2}(1/\varepsilon) \varepsilon \). We finally obtain that
\[
\| K_\Sigma g - K_\Sigma G \|_1 \leq \sum_i \alpha_i \sum_{j \neq i} \tilde{c}_{i,j} \log^{D/2}(1/\varepsilon) \varepsilon \leq \log^{D/2}(1/\varepsilon) \varepsilon.
\]

Also we can choose the atoms of \( G \) to be \( \sigma^{\alpha_0} \varepsilon \) apart, thanks to Lemma D.1 together with the following bound on \( \Sigma(z)^{-1}\Sigma(y) - \text{Id} \) when \( \| z - y \| \leq \sigma^{\alpha_0} \varepsilon \):
\[
\Sigma(z)^{-1}\Sigma(y) - \text{Id} = O_z^T \Delta_{\sigma,\delta}^{-2} O_y \Delta_{\sigma,\delta}^2 O_y - \text{Id} = O_z^T (O_y^T \Delta_{\sigma,\delta}^2 O_y - \text{Id}) \Delta_{\sigma,\delta}^{-2} O_y + O_z O_y^T - \text{Id},
\]
so that
\[
\sqrt{\text{tr}(\Sigma(z)^{-1}\Sigma(y) - \text{Id})^2} \lesssim \sqrt{D} \| \Sigma(z)^{-1}\Sigma(y) - \text{Id} \|_{op} \leq \| \Delta_{\sigma,\delta}^{-2} \|_{op} \| O_y^T \Delta_{\sigma,\delta}^{-2} O_y - \text{Id} \|_{op} \| \Delta_{\sigma,\delta}^2 \|_{op} + \| O_z O_y^T - \text{Id} \|_{op}
\lesssim \sigma^{-2\alpha_0} \| O_z - O_y \|_{op} \leq \sigma^{-2\alpha_0} \| \text{pr}_{T_z} - \text{pr}_{T_y} \|_{op} \lesssim \sigma^{-2\alpha_0} \| z - y \| \lesssim \varepsilon,
\]
where we used (31) in the second to last inequality, ending the proof.

E Appendix to Section 4

E.1 Additional details on the numerical setting

The manifolds used in the numerical experiments are given through the following equations:

- **The two circles**: it is the union of \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) with equation \((x - x_i)^2 + (y - y_i)^2 = r_i^2 \) for \( i \in \{1, 2\} \). In the experiments, we chose \((x_1, y_1) = (0, 0), (x_2, y_2) = (2, 0)\) and \( r_1 = r_2 = 2 \).

- **The 2D-spiral**: it is given by the parametric embedding
\[
\varphi_2 : t \in [0, 1] \mapsto \begin{cases} R(\omega t + \theta_0) \cos(\omega t + \theta_0) \\ R(\omega t + \theta_0) \sin(\omega t + \theta_0). \end{cases}
\]

In the numerical studies, we chose \( R = 1/2\pi \), \( \omega = 7\pi/2 \) and \( \theta_0 = \pi/2 \).
The 3D-spiral: it is given by the parametric embedding

\[
\varphi_3 : t \in [0, 1] \mapsto \begin{cases} 
R(\omega t + \theta_0) \cos(\alpha t + \theta_0) \\
R(\omega t + \theta_0) \sin(\alpha t + \theta_0) \\
\nu t.
\end{cases}
\]

In the numerical studies, we chose \( R = 1/2\pi \), \( \omega = 7\pi/2 \), \( \theta_0 = \pi/2 \) and \( \nu = 2 \).

The torus: it is the set \( T \) given by the equation \((\sqrt{(x-x_0)^2 + (y-y_0)^2} - R)^2 + (z-z_0)^2 = r^2\). In the experiments, we chose \((x_0,y_0,z_0) = (0,0,0), R = 3 \) and \( r = 1 \).

### E.2 Details on the Gibbs sampling algorithm

This first section presents the implementation of the partial location-scale prior (6) with Gibbs sampling for a particular choice of measure \( H \).

In order to simplify the notations we will write indifferently \( \Lambda \) as a \( D \)-dimensional diagonal matrix or a \( D \)-dimensional vector.

We use the latent cluster representation of the \((\mu_iO_i)\) and denote by \( c_i \) the cluster allocation of observation \( i \):

\[
c_i = c_j \iff y_i \text{ and } y_j \text{ are in the same cluster}
\]

and let \( N \) be the number of clusters, so that \( 1 \leq c_i \leq N \). We write \( \phi_c = (\mu_c^*, O_c^*)_{1 \leq c \leq N} \) for the unique values of the cluster parameters.

**Proposition E.1.** In the model (16), we have :

\[
p(\Lambda|Y, (\mu_i, O_i), b) = \bigotimes_{j=1}^{D} \text{Inv} \Gamma \left( a_j + n/2, b_j + \frac{1}{2} \sum_{i=1}^{n} (O_i^j|y_i - \mu_i)^2 \right) \\
p(b|Y, (\mu_i, O_i), \Lambda) = \bigotimes_{j=1}^{D} \text{Exp}(\kappa_j + \lambda_j^{-1})
\]

and :

\[
p(\mu_c^*|O_c^*, Y, (c_i), \Lambda) = \mathcal{N}(\mu_c^*, A^{-1}), c \leq N \\
p(O_c^*|\mu_c^*, Y, (c_i), \Lambda) = p_{BMF}(1/2, A^{-1}, M_0)
\]

where

\[
A = \Sigma_0^{-1} + n\mu_c^* A^{-1} (O_c^*)^T, \quad n_c = \text{Card} \{ i : c_i = c \}, \\
\hat{\mu}_c = A^{-1} [\Sigma_0^{-1} \mu_0 + O_c^* A^{-1} (O_c^*)^T \sum_{c_i=c} y_i] \quad \text{and} \quad S = \sum_{c_i=c} (y_i - \mu_c^*)(y_i - \mu_c^*)^T.
\]

**Proof of Proposition E.1.** • For the variances \( \Lambda \) we have

\[
p(\Lambda|Y, (\mu_i, O_i), b) \propto p(\Lambda|b) \prod_i \mathcal{N}(y_i|\mu_i, O_i \Lambda O_i^T) \propto \left[ \prod_{j=1}^{D} \lambda_j^{-(\alpha+1)} e^{-b_j/\lambda_j} \right] \prod_{i=1}^{n} \mathcal{N}(y_i|\mu_i, O_i \Lambda O_i^T).
\]
But by orthogonality:

\[
\mathcal{N}(y_i|\mu_i, O_i\Lambda O_i^T) \propto \prod_{j=1}^{D} \lambda_j^{-1/2} \exp \left( -\frac{1}{2} (y_i - \mu_i) O_i \Lambda^{-1} O_i^T (y_i - \mu_i) \right) \\
\propto \prod_{j=1}^{D} \lambda_j^{-1/2} \exp \left( -\frac{1}{2} (O_i^T (y_i - \mu_i) \Lambda^{-1} O_i^T (y_i - \mu_i)) \right) \\
\propto \prod_{j=1}^{D} \lambda_j^{-1/2} \exp \left( -\frac{1}{2} \sum_{j=1}^{D} \lambda_j^{-1} (O_i^j |y_i| \mu_i)^2 \right) \\
\propto \prod_{j=1}^{D} \left[ \lambda_j^{-1/2} \exp \left( -\frac{1}{2} \lambda_j^{-1} (O_i^j |y_i| \mu_i)^2 \right) \right],
\]

where \( O_i^j \) is the j-th column of \( O_i \). Therefore,

\[
p(\Lambda|Y, (\mu_i, O_i), b) \propto \prod_{j=1}^{D} \lambda_j^{-(a+1)} e^{-b_j/\lambda_j} \lambda_j^{-n/2} \exp \left( -\frac{1}{2} \lambda_j^{-1} \sum_{i=1}^{n} (O_i^j |y_i| \mu_i)^2 \right) \\
\propto \prod_{j=1}^{D} \left[ \lambda_j^{-(a+1+n/2)} \exp \left( -\lambda_j^{-1} \left[ b_j + \frac{1}{2} \sum_{i=1}^{n} (O_i^j |y_i| \mu_i)^2 \right] \right) \right],
\]

i.e conditionally on the rest of the variables, the \((\lambda_j)_j\) are independent with distribution \( \text{Inv}\Gamma(a + n/2, b_j + \frac{1}{2} \sum_{i=1}^{n} (O_i^j |y_i| \mu_i)^2) \).

- For the hyperparameters \((b_j)_j\) we have

\[
p(b|Y, (\mu_i, O_i), \Lambda) \propto p(b) p(\Lambda|b) \propto \prod_{j=1}^{D} e^{-b_j (\kappa_j + \lambda_j^{-1})}
\]

i.e conditionally on the rest of the variables, the \( b_j \)'s are independent with distribution \( \text{Exp}(\kappa_j + \lambda_j^{-1}) \).

- For the cluster locations \((\mu_c^*)_{1 \leq c \leq K}\) we have

\[
p(\mu_c^*|O_c^*, Y, (c_i), \Lambda) \propto p(\mu_c^*) \prod_{c_i = c} \mathcal{N}(y_i|\mu_c^*, O_c^* \Lambda (O_c^*)^T) \\
\propto \exp \left( -\frac{1}{2} \left\{ (\mu_c^* - \mu_0) \Sigma_0^{-1} (\mu_c^* - \mu_0) \right\} + \sum_{c_i = c} (y_i - \mu_c^*) (O_c^* \Lambda^{-1} (O_c^*)^T (y_i - \mu_c^*)) \right),
\]

and since the prior \( p(\mu_c^*) \) is Gaussian with mean \( \mu_0 \) and variance \( \Sigma_0 \) the above conditional posterior distribution is a Gaussian distribution with mean \( \hat{\mu}_c = \Lambda^{-1}[\Sigma_0^{-1} \mu_0 + (O_c^*) \Lambda^{-1} (O_c^*)^T \sum_{c_i = c} y_i] \) and variance \( A^{-1} = [\Sigma_0^{-1} + n_c (O_c^*) \Lambda^{-1} (O_c^*)^T]^{-1} \).
Remark 9. Here we can see that when \( \sum_{i=1}^{n}(O_i^j | y_i - \mu_i)^2 = o(n^{3/2}) \), \( \text{Inv}\Gamma(a + n/2, b + \frac{1}{2} \sum_{i=1}^{n}(O_i^j | y_i - \mu_i)^2) \) is tightly concentrated around its mean

\[
\frac{b_j + \frac{1}{2} \sum_{i=1}^{n}(O_i^j | y_i - \mu_i)^2}{a + n/2 - 1} \geq \frac{b_j}{a + n/2 - 1};
\]

and if we fix \( b_j \) this may induce a rather strong penalization on small values of \( \lambda_j \) for finite \( n \).

Even though to the best of our knowledge no direct samplers are available for \( p_{BMF} \), it is still possible to perform a Gibbs sampling update over the columns of \( O_c^* \) (cf \[34\] and the associated package \texttt{rstiefel}). Another more involved (but more efficient) option would be to perform Hamiltonian Monte Carlo via polar expansion as suggested in \[37\]. With a slight abuse of notation we will write

\[
O_c^* \sim p_{BMF, \text{Gibbs}}(\cdot | A, B, C, O_c^*)
\]

for a Gibbs sampling scan on the column of \( O_c^* \) starting from \( O_c^* \). All in all, this leads us to Algorithm 1 below. It uses the conditional allocation distributions:

\[
P(c_i = l | c_{-i}, y_i, (\phi_l)_{1 \leq l \leq h}) \propto \begin{cases} 
\frac{1}{m} N(y_i | \mu_l, O_l \Lambda O_l^T) & \forall 1 \leq l \leq k \\
\frac{1}{m} N(y_i | \mu_l, O_l \Lambda O_l^T) & \forall k < l \leq h
\end{cases}
\]

(46)
Algorithm 1: Gibbs sampling algorithm for Partial location scale DP mixture

**Input**: a dataset $Y = (y_i)_{i=1}^n$, a current partition $c = (c_i)_{i=1}^n$, a current state $\phi = (\mu_i^*, O_i^*)_i^K$ (with $K$ the current number of clusters), current variances $\Lambda = (\lambda_j)_{j=1}^D$, current hyperparameters $(b_j)_{j=1}^D$, a number of auxiliary parameters $m \in \mathbb{N}^*$, $\alpha > 0, \mu_0 \in \mathbb{R}^D, \Sigma_0 \in \mathcal{S}_{+}^D(\mathbb{R})$, $M_0 \in \mathcal{M}_D(\mathbb{R})$ the parameters for the base distribution of the Dirichlet process, $(a_j)_{j=1}^D$ the prior parameters for the variances

**Output**: an updated state $(c, \phi, \Lambda, b)$

for $i = 1, \ldots, n$ do

$k \leftarrow \# \{j \neq i : c_j = c_i\}; h \leftarrow k + m$; and label $(c_j)_{j=i}^n$ with $\{1, \ldots, k\}$

if $\exists j \neq i : c_i = c_j$ then

$(\phi_i)_{k \leq h} \overset{i.i.d.}{\sim} G_0$;

else

if $\forall j \neq i : c_i \neq c_j$ then

$c_i \leftarrow k + 1; (\phi_i)_{k+1 \leq h} \overset{i.i.d.}{\sim} G_0$;

end

end

$n_{-i,c} \leftarrow \# \{c_j : j \neq i\}$; draw $c_i$ from $\{1, \ldots, h\}$ with probability (46);

discard the $(\phi_i)$ not associated with any observations;

for $l \in \{c_1, \ldots, c_n\}$ do

$n_l \leftarrow \# \{i : c_i = l\}; A = \Sigma_0^{-1} + n_l O_l^* \Lambda^{-1} (O_l^*)^T$;

$\hat{\mu}_l = A^{-1} [\Sigma_0^{-1} \mu_0 + O_l^* \Lambda^{-1} (O_l^*)^T \sum_{c_i=l} y_i]$;

$\mu_l \sim \mathcal{N}(\hat{\mu}_l, A^{-1})$; $S \leftarrow \sum_{c_i=l} (y_i - \mu_l^*) (y_i - \mu_l^*)^T$;

$O_l^* \sim \text{pBMFGibbs}(O_l^* | S, -\frac{1}{2} \Lambda^{-1}, M_0, O_l^*)$;

end

$(\lambda_j)_{j=1}^D \leftarrow (\lambda_j)_{j=1}^D \overset{\text{ind}}{\sim} p(\lambda_j | (y_i), c, (\phi_l)) \propto \text{InvGamma}(a + n/2, b + \frac{1}{2} \sum_{i=1}^n ((O_i^*)^j | y_i - \mu_i^*)^2)$;

$b \leftarrow (b_j)_{j=1}^D \overset{\text{ind}}{\sim} \mathcal{E}(\kappa_j + \lambda_j^{-1})$;

end