INVERSE LIMITS OF FINITE RANK FREE GROUPS

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Abstract. We will show that the inverse limit of finite rank free groups with surjective connecting homomorphism is isomorphic either to a finite rank free group or to a fixed universal group. In other words, any inverse system of finite rank free groups which is not equivalent to an eventually constant system has the universal group as its limit. This universal inverse limit is naturally isomorphic to the first shape group of the Hawaiian earring. We also give an example of a homomorphic image of a Hawaiian earring group which lies in the inverse limit of free groups but is neither a free group nor a Hawaiian earring group.

One of the first to consider the inverse limits of finite rank free groups was Higman. In [8], he studies the inverse limit of finite rank free groups which he calls the unrestricted free product of countably many copies of \( \mathbb{Z} \). There he proves that this group is not a free group and that each of its free quotients has finite rank. He considers a subgroup \( P \) of the unrestricted product which turns out to be a Hawaiian earring group but does not prove it there. In [5], de Smit gives a proof that the Hawaiian earring group embeds in an inverse limit of free groups and gives a characterization of the elements of the image. Daverman and Venema in [4] showed that a one-dimensional Peano continuum either has the shape of a finite bouquet of circles or of a Hawaiian earring. Hence any inverse limit of finite rank free groups that arises from the inverse system of a one-dimensional Peano continuum is either a finite rank free group or the standard Hawaiian inverse limit. In section 2, we will show that the result of Daverman and Venema can be generalized in the following way. Every inverse limit of finite rank free groups with surjective connecting homomorphisms is isomorphic to a free group or the standard Hawaiian inverse limit. Hence being the shape group of a one-dimensional Peano continuum is not necessary.

In [3], Conner and Eda show that the fundamental group of a one-dimensional Peano continuum which is not semilocally simply connected at any point determines the homotopy type of the space. This was done using uncountable homomorphic images of Hawaiian earring groups. It was believed that any uncountable homomorphic image of a Hawaiian earring group which embedded in an inverse limit of free groups was itself a Hawaiian earring group. In Section 3, we will show that there exists an uncountable homomorphic image of a Hawaiian earring group which embeds in an inverse limit of free groups but is not a Hawaiian earring group or a free group. This is done by using two propositions which were originally proved by Higman in [8] to construct a homomorphism with our desired image.

1. Definitions

A Hawaiian Earring group, which we will denote by \( \mathbb{H} \), is the fundamental group of the one-point compactification of a sequence of disjoint open arcs. The Hawaiian earring group is uncountable and locally free. Cannon and Conner in [1] and [2] showed that the Hawaiian earring group is generated in the sense of infinite products by a countable sequence of loops corresponding to the disjoint arcs, where an infinite product is legal if each loop is transversed only finitely many times. (For more information on infinite products, see [2].) The Hawaiian earring can be realized in the plane as the union of circles centered at \( (0, \frac{1}{n}) \) with radius \( \frac{1}{n} \). We will use \( E \) to denote this subspace of the plane and \( a_n \) to denote the circle centered at \( (0, \frac{1}{n}) \) with radius \( \frac{1}{n} \).
The group $\mathbb{H}$ is generated, in the sense of infinite products, by an infinite set of loops which correspond to the circles $\{a_n\}$. When there is no chance of confusion, we will refer to this infinite generating set for the fundamental group of $E$ as $\{a_n\}$, i.e. $a_n$ represents the loop which transverses counterclockwise one time the circle of radius $\frac{1}{n}$ centered at $(0, \frac{1}{n})$. We will frequently denote the base point $(0,0)$ of $E$ by just $0$.

An inverse system of groups is a collection of groups $F_\alpha$ indexed by a partially ordered set $J$ along with a collection of homomorphisms $\{\varphi_{\alpha,\beta} : F_\alpha \to F_\beta \mid \alpha \geq \beta\}$, which are called connecting homomorphisms. The connecting homomorphisms must satisfy the following condition. For every triple $\{\alpha, \beta, \gamma\}$ such that $\alpha \geq \beta \geq \gamma$, the connecting homomorphisms satisfy $\varphi_{\alpha,\beta} \circ \varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$. An inverse limit of an inverse system is the subgroup of the direct product which consists of functions $f : J \to \bigcup_{\alpha \in J} F_\alpha$ such that $f(\beta) \in F_\beta$ and $f(\beta) = \varphi_{\alpha,\beta}(f(\alpha))$, for every $\alpha, \beta$ such that $\beta \leq \alpha$.

2. Inverse Limits of Finite Rank Free Groups

We will now describe the inverse system of finite rank free groups constructed by Higman. Let $A_i = \langle a_1, \ldots, a_i \rangle$ be the free group on $i$ generators with the natural inclusions $A_i \subset A_{i+1} \subset \cdots$. For $i \geq j$, the connecting homomorphisms $P_{i,j} : A_i \to A_j$ sends $a_k$ to $a_j$ for $k \leq j$ and $a_k$ to $1$ for $k > j$. We will denote this inverse limit of this system by $G$. We will use $P_i : G \to A_i$ to denote the standard projection homomorphism.

Eda [6] pointed out to the authors that Proposition 2.1 doesn’t hold in the case that the rank of $F$ is unknown. Otherwise by passing to a cofinal subsequence, we may assume that the rank of $F_n$ is a strictly increasing sequence.

Lemma 2.2. If $G$ is an inverse limit of finite rank free groups, $F_n$, with surjective connected homomorphisms indexed over the natural numbers. Then $G$ is isomorphic to $G'$.

Lemma 2.3. Any morphism of inverse systems which consists of isomorphisms induces an isomorphism of limits.

Proof of Proposition 2.1. Let $\pi_{i,j}$ be the connecting homomorphisms of $G$. If the rank of $F_n$ is eventually constant then the connecting homomorphisms must eventually be isomorphisms (see Proposition 2.12 in [10]). Hence $G$ is a finite rank free group. Otherwise by passing to a cofinal subsequence, we may assume that the rank of $F_n$ is a strictly increasing sequence.

Let $B_1$ be a basis for $F_1$. By induction, suppose that for all $m < n$, $B_m \cup K_m$ is a basis for $F_m$ such that $\langle K_m \rangle \subset \ker(\pi_{m,m-1})$ and $\pi_{m,m-1}$ maps $B_m$ bijectively onto $B_{m-1} \cup K_{m-1}$.

There exists a free basis $B'_n \cup K_n$ of $F_n$ with the property that $\langle K_n \rangle \subset \ker(\pi_{n,n-1})$ and $\pi_{n,n-1}$ restricted to $\langle B'_n \rangle$ is an isomorphism (again by Proposition 2.12 in [10]). Now restricting $\pi_{n,n-1}$ to this set we may define $B_n = (\pi_{n,n-1}|_{\langle B'_n \rangle})^{-1}(B_{n-1})$. Since $\pi_{n,n-1}$ restricted to $\langle B'_n \rangle$ is an isomorphism; $B_n$ is a free basis for $\langle B'_n \rangle$. It is a simple exercise to show that $B_n \cup K_n$ is still a free basis for $F_n$.

It is now trivial to find isomorphisms between the groups $F_n$ and $A_{|B_n \cup K_n|}$ which commute with the connecting homomorphisms. The result follows from the Lemma 2.3 and Lemma 2.2.

Remark 2.4. Given a system of countable rank free groups $(F_n, \pi_{n,n-1})$ with inverse limit $G$, one can pass to the inverse system $(\pi(n)(G), \pi_{n,n-1})$ without changing isomorphism types of the
inverse limit where \( \pi_n \) is the canonical projection of \( G \) to \( F_n \). The new system then has surjective connecting homomorphisms. However, \( \pi_n(G) \) is a free group of possibly infinite rank. Then as in the proof of Theorem 2.1, we can find a basis \( B_n \cup K_n \) of \( \pi_n(G) \) with the same properties as before. After passing to a cofinal inverse system, we may assume that \( B_n, K_n \) are eventually trivial, finite, or countably infinite. Then the isomorphism type of the inverse limit is determined by the cardinalities of \( B_n, K_n \). This gives five possible isomorphism types. Eda [6] shows that these five types can all be realized and are distinct.

3. Images of the Hawaiian earring group

Note that \( A_i \) embeds in \( G \) by sending \( a_i \) to the element which is 1 in the first \( i - 1 \) coordinates and \( a_i \) in all other coordinates. The proof of de Smit in [5] shows the embedding of \( \mathbb{H} \) into \( G \) sends \( a_i \) to \( a_i \).

We will give \( G \) a metric such that \( G \) under the induced topology is a topological group. In [8], Higman defines a topology which is equivalent to our metric topology. Let \( d_i \) be the \((0,1)\)-metric on \( B_i \), i.e. \( d_i(x,y) = 0 \) if \( x = y \) and 1 otherwise. For \( (g_n), (h_n) \in G \), let \( d((g_n), (h_n)) = \sum \frac{1}{n} d_n(g_n, h_n) \).

**Remark 3.1.** Under this topology, \( g_i = (g_i') \) converges to \( h = (h_n) \) if and only if for each \( n \) there exists an \( M(n) \) such that \( g_i' h_n^{-1} = 1 \) for all \( i \geq M(n) \). It follows that if \( g_i \) converges \( g_i g_{i+1}^{-1} \) must converge to 1. Suppose that \( g_i g_{i+1}^{-1} \) converges to 1. Then \( g_i' \) considered as a sequence in \( i \) is eventually constant for every \( n \). Hence \( g_i \) converges.

We will leave it to the reader to verify that this topology makes \( G \) into a topological group. An interesting note is that under this metric \( G \) is complete and all three of the sets \( \mathbb{H}, G - \mathbb{H} \), and the free group generated by \( \{ a_i \mid i \in \mathbb{N} \} \) which we will denote by \( \langle a_1, a_2, \cdots \rangle \) are dense.

The following two propositions of Higman demonstrate the elegance of this topology. For completeness and to make the proof readily accessible to the reader, we will include their proofs here.

**Proposition 3.2.** Any endomorphism of \( G \) is continuous.

**Proof.** Let \( \varphi : G \to G \) be an endomorphism. Then \( d((g_n), (h_n)) \leq \frac{1}{n} \) if and only if \( g_n = h_n \) for all \( n \leq i \). Higman in [8] showed that \( P_i \circ \varphi \) factors through some \( P_{n(i)} \). (see Theorem 1) Cannon and Conner have shown that the same holds true for any \( \varphi : \mathbb{H} \to F \) where \( F \) is a free group. (see Theorem 4.4 in [2]) Hence, \( P_i \circ \varphi = \varphi \circ P_{n(i)} \) for some \( n(i) \) which depends on \( i \). Thus \( d((g_n), (h_n)) \leq \frac{1}{2^{n(i)}} \) implies that \( d(\varphi((g_n)), \varphi((h_n))) \leq \frac{1}{2^i} \). \( \square \)

**Proposition 3.3.** Any set function \( \varphi \) from \( \{ a_1, a_2, \cdots \} \) to \( G \) such that \( d(\varphi(a_i), 1) \) converges to 0 extends to an endomorphism of \( G \).

**Proof.** Let \( \varphi : \{ a_1, a_2, \cdots \} \to G \) be a set function such that \( d(\varphi(a_i), 1) \) converges to 0. Then \( \varphi \) will extend to the free group \( \langle a_1, a_2, \cdots \rangle \leq G \). We want to be able to extend \( \varphi \) to all of \( G \).

Suppose that \( g_i \in \langle a_1, a_2, \cdots \rangle \) converges in \( G \). We must show that \( \varphi(g_i) \) also converges. By Remark 3.1, it is enough to show that \( \varphi(g_i) \varphi(g_{i+1})^{-1} \) converges. Since \( \varphi \) is a homomorphism on \( \langle a_1, a_2, \cdots \rangle \), \( \varphi(g_i) \varphi(g_{i+1})^{-1} = \varphi(g_i g_{i+1}^{-1}) \). Hence it is sufficient to show that if \( g_i \to 1 \) then \( \varphi(g_i) \to 1 \).

Let \( \varphi(a_i) = (\varphi(a_i))_n \). Suppose that \( g_i = (g_i') \) converges to 1. Fix \( n \). We will show that \( \varphi(g_i)_n \) when considered as a sequence in \( i \) is eventually trivial. Fix \( M \) such that \( \varphi(a_i)_n = 1 \) for all \( i \geq M \). Fix \( M' \) such that \( g_i^n = 1 \) for all \( i \geq M' \). By construction, \( g_i \) is a word \( w_i(\{ a_j \}) \) in \( \langle a_1, a_2, \cdots \rangle \). Then \( \varphi(g_i) = w_i(\{ \varphi(a_j) \}) \); hence, \( \varphi(g_i)_n = w_i(\{ \varphi(a_j)_n \}) \). Then for all \( i \geq \max\{ M, M' \}, \varphi(g_i)_n = w_i(\{ \varphi(a_j)_n \}) = 1 \).
Suppose that \( g_i, g'_i \in \langle a_1, a_2, \ldots \rangle \) converge to the same element of \( G \). Then \( g'_i g_i^{-1} \) converges to 1. Then \( \varphi(g'_i) \varphi(g_i^{-1}) = \varphi(g'_i g_i^{-1}) \) converges to 1. Thus \( \varphi \) extends to a well defined continuous function \( \overline{\varphi} : G \to G \) which is independent of the chosen sequence.

Suppose that \( g, h \in G \). Then there exists \( g_i, h_i \in \langle a_1, a_2, \ldots \rangle \) such that \( g_i \to g \) and \( h_i \to h \). Since \( \overline{\varphi} \) is independent of the chosen sequence, \( \varphi(g_i) \varphi(h_i) \to \overline{\varphi}(g) \overline{\varphi}(h) \) and \( \varphi(g_i) \varphi(h_i) = \varphi(g_i h_i) \to \overline{\varphi}(gh) \). Hence \( \varphi \) extends to a homomorphism \( \overline{\varphi} \). \( \square \)

For a path \( \alpha : [0, t] \to X \), we will also use \( \overline{\alpha} \) to represent the path where \( \overline{\alpha}(s) = \alpha(t - s) \). We will also use the following theorem.

**Theorem 3.4.** [Eda [7]] Let \( \psi : \mathbb{H} \to \pi_1(X, x_0) \) a homomorphism into the fundamental group of a one-dimensional Peano continuum \( X \). Then there exists a continuous function \( f : (E, 0) \to (X, x) \) and a path \( \alpha : (I, 0, 1) \to (X, x_0, x) \), with the property that \( f_* = \hat{\alpha} \circ \varphi \). Additionally, if the image of \( \psi \) is uncountable the \( \alpha \) is unique up to homotopy rel endpoints.

Another proof, as well as a proof for a planar version, of this theorem can be found in the Masters Thesis of the second author (see [9]). Cannon and Conner in [2] showed that in one dimensional spaces there exists a unique (up to reparametrization) reduced representative for each path class. We will use \([ \cdot ]_r \) (or \( \varphi(\cdot)_r \)) to represent the unique reduced representative for the path class \([ \cdot ] \) (or \( \varphi(\cdot) \)).

We are now ready to give our counterexample. We will begin by defining a set function \( \varphi : \{a_1, a_2, \ldots \} \to G \) by

\[
\varphi(a_i) = \begin{cases} 
    a_i & \text{if } i \text{ is odd} \\
    a_1 a_i a_1^{-1} & \text{if } i = 2 \pmod{4} \\
    a_{i-2} & \text{if } i = 0 \pmod{4}
\end{cases}
\]

Then \( \varphi \) extends to an endomorphism of \( G \) which we may then restrict to the naturally embedded \( \mathbb{H} \). Thus after extending and restricting, \( \varphi : \mathbb{H} \to G \) is a homomorphism with uncountable image. Suppose there existed an isomorphism \( \psi : \varphi(\mathbb{H}) \to \pi_1(E, 0) \). Then \( \psi \circ \varphi : \mathbb{H} \to \pi_1(E, 0) \) is a homomorphism from \( \mathbb{H} \) to a one-dimensional Peano continuum which by Theorem 3.4 must be conjugate to a homomorphism induced by a continuous function. Let \( T \) be the path such that \( T \circ \varphi \) is induced by a continuous function. Then by construction 

\[
[T \ast \psi \circ \varphi(a_{4i-2})_r \ast T] = [T \ast (\psi(a_1)^{-1} \psi(a_{4i-2}) \psi(a_1))_r \ast T] = [T \ast (\psi(a_1)^{-1} \psi(a_{4i-2}) \psi(a_1))_r \ast T]
\]

By our choice of \( T \), \([T \ast \psi \circ \varphi(a_{4i-2})_r \ast T]_r \) and \([T \ast \psi \circ \varphi(a_{4i})_r \ast T]_r \) are null sequences of loops in \( E \). However, the second sequence of loops is conjugate to the first by a non-trivial loop \([T \ast \psi(a_1)_r \ast T]_r \), which is a contradiction.

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