Magnetic-field symmetry breaking in spin glasses

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ABSTRACT

Time-reversal symmetry is spontaneously broken in spin glasses below their glass temperature. Under such conditions, the standard assumption about the equivalence of the most standard protocols (i.e., no big difference between switching the field on or off, as it is sometimes said) is not really justified. In fact, we show here that the spin-glass coherence length behaves differently in the zero-field-cooled (ZFC, magnetic field is turned on) and thermoremanent-magnetization (TRM, magnetic field is turned off) protocols. This conclusion is reached through experiments, carried out on two CuMn single-crystal samples, and through massive simulations on the Janus II dedicated supercomputer. In agreement with the predictions of a simple dynamical model that assumes that the ultrametric tree of states survives the switching-on of the magnetic field, we conclude that (all really justified. In fact, we show here that the spin-glass coherence length behaves differently in

I. INTRODUCTION

Glass formers (spin glasses, fragile molecular glasses, polymers, colloids, etc.) stay perennially out of equilibrium below their glass temperature $T_g$. Indeed, these materials are said to age [1]. As a very counterintuitive consequence, one should be ready to treat time or, more precisely, the thermal history of the sample below $T_g$, on equal footing with thermodynamic control parameters such as temperature, pressure or a external perturbing field (the magnetic field, for instance). Notwithstanding this complexity, two of the main experimental protocols for cooling a glass former below $T_g$, namely the zero-field-cooling (ZFC) and the thermoremanent-magnetization (TRM) protocols, see the discussion below, have been widely regarded as equivalent, see e.g., [2]. Here, we critically assess the long-held assumption of protocol equivalence through high-accuracy experiments (carried out on single-crystal samples of a CuMn spin glass) and simulations of spin-glass dynamics (carried out on the Janus II custom-built supercomputer [3]).
Aging is caused by the expansion of cooperative regions [4] whose linear size $\xi$ defines a glassy coherence length. For temperatures $T < T_\text{g}$, the time growth of $\xi$ is unbounded, albeit very slow. Experimentally, $\xi$ is measured by perturbing the sample with an external field,$^1$ and measuring the time-dependent, non-linear response. This approach was pioneered in spin-glass experiments [5] and later extended to other glass-forming materials, see e.g., [6, 7]. This method to extract $\xi$ has been reproduced in spin-glass simulations, and successfully compared with microscopic determinations of the size of the cooperative regions [8]. Indeed, parallel analysis of laboratory and terminations of the size of the cooperative regions [8].

A problem with the non-linear response method, however, is that, ideally, it should be applied in the limit of a vanishing perturbing field. Yet, real experiments are carried out in finite fields. This fact complicates the interpretation of the results, because of the long-standing controversy on the nature of the $T < T_\text{g}$ phase. On one side of this debate, we find the Replica Symmetry Breaking picture [11–13], which expects the non-linear response to the field to remain anomalous even if the magnetic field $H$ is non-vanishing (provided that $\xi$ grows as large as the sample size). On the other hand, the droplet model [14–17] expects the non-linear response to be regular when $H > 0$ (also in the limit of large $\xi$). So far, experimental analysis of the response seems to indicate a similar behavior for $H > 0$ and for $H = 0$ [18, 19], while analyses of conduction fluctuations in a mesoscopic CuMn spin glass also give some support for Replica Symmetry Breaking [20, 21]. Nevertheless, because $\xi$ will be much smaller than the sample size in our experiments and simulations, the droplet vs. Replica Symmetry Breaking controversy will not enter our analysis.

As explained above, the two thermal protocols compared in this work are the TRM and the ZFC. In both protocols, the system is first let to reach thermal equilibrium at some temperature $T \gg T_\text{g}$, then cooled as fast as possible to the measuring temperature $T_m$,$^2$ where it is let to relax for a waiting time $t_w$. At that point, the magnetic field is varied, and the magnetization is measured at later times $t + t_w$. In the TRM protocol, an external magnetic field $H > 0$ is applied (ant kept constant) from the very beginning, until the field is switched off at time $t_w$. As a consequence, the magnetization $M_{\text{TRM}}(t, t_w)$ decreases with time $t$, see Fig. 1—bottom. On the other hand, in the ZFC protocol one keeps $H = 0$ until the magnetic field is switched-on at time $t_w$. Hence, the magnetization $M_{\text{ZFC}}(t, t_w)$ grows from its vanishing starting value at $t = 0$, see Fig. 1—top. We shall consider $M_{\text{TRM}}(t, t_w)$ as a function of the two times, $t$ and $t_w$, and of the applied magnetic field $H$.

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$^1$ A magnetic field in the case of spin glasses, or an electric field for other glass formers.

$^2$ See Ref. [10] for a discussion of the differences of the sudden cooling in experiments and simulations.
II. A PHENOMENOLOGICAL APPROACH BASED ON THE HAMMING DISTANCE

The beautiful solution of the mean-field Sherrington-Kirkpatrick model [23] is a landmark in the physics of disordered systems and gives rise to a theoretical picture that is known by several names, such as the Replica Symmetry Breaking or the hierarchical models of spin glasses. Unfortunately, connecting this solution to experimental work is not straightforward, because of two major difficulties. First, strictly speaking, the mean-field solution applies only to space dimension higher than six (alas, experiments are carried out in the three-dimensional world we live in). How this hierarchical picture needs to be modified in three dimensions is a much-debated problem (see e.g., Ref. [24] for an updated account). The second, and perhaps more serious, problem is related to the fact that this theory describes systems in thermal equilibrium. Now, below the glass temperature, the coherence length $\xi$ of a system in thermal equilibrium is as large as the system’s size. Unfortunately, because of the extreme slowness of the time growth of $\xi$, the experimental situation is the opposite: the sample size is typically much larger than $\xi$. Clearly, some additional input is needed to connect the hierarchical picture of the spin-glass phase with real experiments.

One such connecting approach is based on generalized fluctuation-dissipation relations [25–32], which have been also investigated experimentally for atomic spin glasses [33, 34]. Unfortunately, these relations focus on the linear response to the magnetic field, while non-linear relations will be crucial to us.

An alternative approach was worked out in Ref. [35], which explores the dynamics in an ultrametric tree of states. A crucial quantity in this approach is the Hamming distance (HD) between the state of the system after the initial preparation at time $t_w$ and the state at the measuring time $t + t_w$. Yet, we still do not know how this Hamming distance should be defined microscopically. We do have a surrogate that can be obtained from a correlation function (this correlation function can be computed, see Sect. V D, and experimentally measured [33]). Unfortunately, the surrogate is not a fully adequate substitute for the Hamming distance of Joh et al. [35]. Nevertheless, the dynamics in the hierarchical tree does provide useful intuition. This is why we briefly recall here its main results and assumptions. The interested reader will find a more complete account in Appendix A. In fact, we shall take a further step because, at variance with Ref. [35], we shall accept the possibility that barrier heights increase faster than linearly with HD (we shall work out the consequences of this possibility as well).

There are many experimental protocols for exploring spin-glass dynamics. As we explained in the Introduction, those that involve the time change of the magnetization are the zero field cooled magnetization (ZFC) and the thermoremanent magnetization (TRM) protocols, generating $M_{ZFC}(t, t_w; H)$ and $M_{TRM}(t, t_w; H)$, respectively. The basic concept in the analysis will be the maximum free-energy barrier $\Delta_{\text{max}}$ between the involved states (see Appendix A). Take, for instance, the TRM protocol. When the magnetic field $H$ is cut to zero, the system remembers its correlations achieved after aging for the time $t_w$. This generates an inflection point in the time decay of $M_{\text{TRM}}(t, t_w; H)$ at $t \approx t_w$. This is exhibited as a peak in the relaxation function [36],

$$S(t, t_w; H) = -\frac{d M_{\text{TRM}}(t, t_w; H)}{d \log t}.$$ (2)

The log of the time at which $S(t, t_w; H)$ peaks, $t_{\text{pe}}^{\text{eff}}$ is thus a measure of $\Delta_{\text{max}}$. The activation energy is set approximately by the maximum barrier height reached in the waiting time $t_w$:

$$t_{\text{pe}}^{\text{eff}} = \tau_0 e^{\Delta_{\text{max}}(t=0, t_w; H)/k_B T},$$ (3)

where $\tau_0$ is an exchange time of the order of $h/k BT_g$, and $\Delta_{\text{max}}(t = 0, t_w; H)$ is the highest barrier created by the growth of $\xi(t = 0, t_w; H)$ in the time $t_w$.

As Bouchaud has shown [37], when a magnetic field is present the barrier heights $\Delta$ are reduced by a Zee-man energy $E_Z$:

$$\Delta(t, t_w; H) = \Delta(t, t_w; 0) + E_Z,$$ (4)

where, [8, 37–39]

$$E_Z = -M_{\text{FC}} H \equiv -\chi_{\text{FC}} N c H^2.$$ (5)

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3 Here, and all over the text, log is the natural logarithm; otherwise, the logarithmic basis is explicitly indicated, i.e. log2.
Here, $\chi_\text{FC}$ is the field-cooled magnetic susceptibility per spin when the spin glass is cooled in a magnetic field to the measurement temperature $T_m$; $N_c$ is the number of correlated spins [spanned by the spin glass correlation length, $\xi(t, t_w; H)$]; and $H^2$ is the square of the applied magnetic field.\footnote{Another view of $E_X$ (Bert et al. \cite{40}) relies on fluctuations in the magnetization of all of the spins. They used $E_X$ linear in $H$, replacing $N_c$ with $\sqrt{N_c}$, and using the free spin value in place of $\chi_\text{FC}$. A very recent investigation of the magnetic field's effect on spin-glass dynamics, Puga et al. \cite{42}, shows that their fit to experiments can also be ascribed to non-linear effects introduced by their use of rather large values of the magnetic field. We therefore shall use our Eq. (5) in our subsequent analysis.} This is the basis underlying the experimental method to determine the coherence length $\xi$.

III. $\xi_{\text{TRM}}(t, t_w; H)$ AND $\xi_{\text{ZFC}}(t, t_w; H)$ FROM EXPERIMENT

The TRM and ZFC experiments were performed on 8 at.$\%$ CuMn samples cut from the same single-crystal boule grown at Ames Laboratory, and characterized in \cite{41}. The TRM experiments were performed both at the Indiana University of Pennsylvania on a home-built SQUID magnetometer, capable of sensitivity roughly an order of magnitude greater than commercial devices, and at The University of Texas at Austin on a Quantum Design commercial SQUID magnetometer. The ZFC experiments were performed at The University of Texas at Austin on the same equipment as the TRM. Both ZFC and TRM protocols use the time at which $S(t, t_w; H)$ peaks to be the measure of $t_{\text{eff}}$ with $S(t, t_w; H)$ defined by Eq. (2)

$$S(t, t_w; H)_{\text{ZFC/TRM}} = (\pm) \frac{dM_{\text{ZFC/TRM}}(t, t_w; H)}{d \log t} ,$$

where $\pm$ sign pertains to TRM experiments and $\pm$ sign to ZFC experiments. The measurements were all made at 37.5 K, or a reduced temperature ($T_r = 41.5$ K) of $T_r = 0.9$. Two waiting times were set at $t_w = 2500$ s and $5000$ s, testing the growth law \cite{42–44}

$$\xi(t_w) = c_1 \left(\frac{t_w}{t_0}\right)^{c_2(T/T_r)} ,$$

where $c_1$ is a constant of order unity and $c_2 \approx 0.104$. The time $t_{\text{eff}}$ at which $S(t, t_w; H)$ peaks is indicative of the largest barrier $\Delta_{\text{eff}}(t_w; H)$ surmounted in the time $t_w$ \cite{45}. Hence, in the presence of a magnetic field, $\log t_{\text{eff}}$ is proportional to $\Delta_{\text{eff}}(t_w; H)$ through the relationship

$$\Delta_{\text{max}}(t_w; H) = (k_B T) \log \left(\frac{t_{\text{eff}}}{t_0}\right) .$$

In the $H \to 0$ limit, $S(t, t_w; H)$ peaks close to $t_w$. The shift of the peak of the relaxation function $S(t, t_w; H)$ from $t_w$ to $t_{\text{eff}}$ as $H$ increases from zero is a direct measure of the reduction of $\Delta_{\text{max}}$ with increasing $H$ [see Eq. (4)]. From Eqs. (4) and (5), one can then extract the number of correlated spins $N_c(t_w)$ and hence the spin-glass correlation length $\xi(t_w)$ through the relation

$$N_c \approx \xi^{3-\theta/2} ,$$

where $\theta$ is the replicon exponent \cite{8}. Thus,

$$\Delta_{\text{max}} - N_c H^2 = k_B T \log t_{\text{eff}} - \log \tau_0 .$$

In this manner, the TRM and ZFC protocols generate $\xi_{\text{TRM}}(t, t_w; H)$ and $\xi_{\text{ZFC}}(t, t_w; H)$, respectively. The hypothesized difference, $\xi_{\text{TRM}}(t, t_w; H) < \xi_{\text{ZFC}}(t, t_w; H)$, can then be tested.

We expect that difference, if any, to be a result of an upward curvature of $\Delta$ as a function of $Hd$, as outlined in the previous Section and in Appendix A. Fig. 2 exhibits experimental values of $\log t_{\text{eff}}$ vs $H^2$ and fits to the data for the ZFC and TRM protocols for waiting times $t_w = 2500$ s and $t_w = 5000$ s at $T = 37.5$ K. Because $T = 37.5$ K is so close to $T_g$, non-linear terms are evident in the data. As a consequence, the fits employ higher-order terms in $H$ than quadratic.

Appendix B presents data taken from another sample cut from a 6 at.$\%$ CuMn single crystal boule with $T_g = 31.5$ K at a measurement temperature of $T_m = 26$ K. The growth of $\xi(t, t_w; H)$ was slower for both ZFC and TRM protocols, leading to smaller values of the correlation lengths and therefore smaller differences between $\xi_{\text{TRM}}(t_w)$ and $\xi_{\text{ZFC}}(t_w)$, as compared to those exhibited here in the main text. Nevertheless, at the largest waiting time, that difference lies well outside the sum of the error bars.

Fitting to the coefficients of the $H^2$ terms for the 8 at.$\%$ CuMn sample described above (see the respective tables in Appendix C for specific values), we are able to extract $\xi_{\text{ZFC}}(t_w)$ and $\xi_{\text{TRM}}(t_w)$.

We find:\n
$$\xi_{\text{ZFC}}(t_w = 2500 \text{ s}) = 220(20),$$

$$\xi_{\text{TRM}}(t_w = 2500 \text{ s}) = 210(16),$$

$$\xi_{\text{ZFC}}(t_w = 5000 \text{ s}) = 270(20),$$

$$\xi_{\text{TRM}}(t_w = 5000 \text{ s}) = 220(30),$$

all in units of the lattice constant $a_0$. As hypothesized, the magnitude of $\xi_{\text{ZFC}}(t_w)$ exceeds $\xi_{\text{TRM}}(t_w)$. It must be noted, however, that the difference lies well within the error bars for $t_w = 2500$ s, while the difference is just inside the sum of the error bars for $t_w = 5000$ s.

Our attempts at larger values of $t_w$ have not been successful, as the $S(t, t_w; H)$ curves broaden so much that it proved too difficult to extract reproducible values for $t_{\text{eff}}$. Smaller values of $t_w$ were not attempted as the difference between the ZFC and TRM correlation lengths would be smaller than for $t_w = 2500$ s and the error bars would obviate any reliable conclusions.

The ratio for the respective values of $\xi(t, t_w; H)$ is
on AgMn (2.6 at.%), developed a quantitative relationship between the change in \(\Delta(t_w)\) as a function of the change in \(H_d\). Writing out their Eq. (13), [see our Eq. (A6)]

\[
\Delta(H_d) - \beta(T)/\alpha(T) = [\Delta(H_d_0) - \beta(T)/\alpha(T)]e^{[\alpha(T)(H_d-H_d_0)]},
\]

where \(\Delta(H_d) - \Delta(H_d_0)\) is the change in barrier heights when \(H_d\) increases from \(H_d_0\) to \(H_d\). The coefficient in the exponent, \(\alpha(T)\), was estimated from experiment to be approximately 38.1 at a reduced temperature of \(T_r = 0.865\). Furthermore \(\alpha(T)\) and \(\beta(T)\) are defined by:

\[
\alpha(T) = -2a(T)/[\delta q_{EA}/\delta T],
\]

\[
\beta(T) = -2b(T)/[\delta q_{EA}/\delta T],
\]

where \(q_{EA}\) is the Edwards-Anderson self-overlap, and \(a(T)\) and \(b(T)\) are defined by an experimental fit,

\[
\langle \Delta/\delta T \rangle |_{T_m} = a(T_m) \Delta + b(T_m),
\]

where \(T_m\) is the measuring temperature. Figs. 11 and 12 of [22] display \(a(T)\) and \(b(T)\), respectively, for four representative values of \(T_m\). For our purposes, we are only interested in the ratio \(\beta/\alpha\), which, from Eq. (12), is independent of \(\delta q_{EA}/\delta T\). Our working temperature is \(T_m = 37.5\), or a reduced temperature \(T_r = 0.90\). From Fig. 10 of Lederman et al. [22], this leads to \(a \approx 29.02\) and \(b \approx 684.03\), generating the ratio,

\[
\beta/\alpha = b/a \approx 23.57.
\]

On the assumption that the ratios for AgMn are relevant to CuMn, we can then address our data. At \(T = 37.5\) K, we fit the time at which \(S(t, t_w; H)\) peaks, \(t_{eff}^H\) by,

\[
\log(t_{eff}^H) = a_0 + a_2 H^2 + a_4 H^4 + a_6 H^6 + O(H^8).
\]

Eq. (15) can be converted to an energy scale by rewriting as

\[
k_B T \log(t_{eff}^H/\tau_0) = k_B T[a_0 - \log(\tau_0)] + k_B T[a_2 H^2 + a_4 H^4 + a_6 H^6 + O(H^8)].
\]

Dividing Eq. (16) by \(k_B T\) gives the energy scale in units of \(k_B T\). We define \(\Delta_0(t_w) = E_0 = (T/T_g)(a_0 - \log(\tau_0))\) as the height of the last barrier encountered during a waiting time \(t_w\) in the absence of a magnetic field, and

\[
E_n(t_w; H) = (T/T_g)a_n(t_w)H^n,
\]

as the nth-order change in the barrier height’s free-energy scale caused by the presence of the external magnetic field at the waiting time \(t_w\). The ZFC experiments for \(t_w = 2500\) s yields \(\Delta_{ZFC}^0(t_w = 2500) \equiv E_0 = 33.55848\) in units of \(k_B T_g\) (see Table VIII in Appendix C). The value for the TRM experiments
is $\Delta_{\text{TRM}}(t_w = 2500 \text{ s}) \equiv E_0 = 33.58718$ (see Table IX), which should be the same as for the ZFC protocol, the slight difference being a result of fits to the data. Likewise, for $t_w = 5000 \text{ s}$, Tables X and XI give $\Delta_{\text{ZFC}}(t_w = 5000 \text{ s}) \equiv E_0 = 34.11658$ and $\Delta_{\text{TRM}}(t_w = 5000 \text{ s}) \equiv E_0 = 34.07752$, respectively.

From these values, and Eqs. (11) and (14) with $\alpha(T_r = 0.90) = 46.97$, we can arrive at $\delta H_d = H_d - H_d(0)$. We find

$$\delta H_d^{\text{TRM}} = 1.02 \times 10^{-3},$$
$$\delta H_d^{\text{ZFC}} = 1.16 \times 10^{-3}. \quad (18)$$

The small values of the difference in Hamming distances for a doubling of the waiting times is an indication of the slow growth of the correlation lengths with waiting times $t_w$. The equilibrium value of the Hamming distance for the ZFC protocol at $q_w \beta = 0$ in Eq. (A3)] is approximately 0.0575 so, even for $t_w = 5000 \text{ s}$, the change in Hamming distance is still tiny. To reach equilibrium would indeed require time scales of the order of the age of the universe.

One can relate the correlation length $\xi(t_w)$ directly to $H_d$. As shown above, the Hamming distance increases by unity for each mutual spin flip, that is, for each reduction in $q_w \beta$ by two. Thus, $\xi(t_w)$ increases by a lattice constant for each mutual spin flip. The volume of real space increases as $[\xi(t_w)]^{D-\theta/2}$, where $D$ is the spatial dimension. This must equal the total number of mutual spin flips, given by $N \times H_d(t_w)$. Tables VIII—XI, see Appendix C, give the correlation lengths $\xi(t_w)$ for $t_w = 2500$ and $t_w = 5000 \text{ s}$. From Eq. (18) the change in $H_d$ is known for both waiting times. One can therefore take the ratio of $\xi(t_w)$ for the two waiting times for each protocol, and establish an absolute value for $H_d(t_w)$ at each value of $t_w$. Expressed numerically,

$$\xi(t_w = 5000 \text{ s})^{[D-(\theta(t_w=5000 \text{ s})/2)]} = \frac{H_d(5000)}{H_d(2500)}$$
$$\xi(t_w = 2500 \text{ s})^{[D-(\theta(t_w=2500 \text{ s})/2)]} = \frac{H_d(2500) + \delta H_d}{H_d(2500)}. \quad (19)$$

In order to evaluate Eq. (19), it is necessary to know the respective values of $\theta$ at $T = 37.5 \text{ K}$. They are

$$\theta_{\text{ZFC}}(T = 37.5 \text{ K}, t_w = 2500 \text{ s}) = 0.354,$$
$$\theta_{\text{TRM}}(T = 37.5 \text{ K}, t_w = 2500 \text{ s}) = 0.356,$$
$$\theta_{\text{ZFC}}(T = 37.5 \text{ K}, t_w = 5000 \text{ s}) = 0.343,$$
$$\theta_{\text{TRM}}(T = 37.5 \text{ K}, t_w = 5000 \text{ s}) = 0.353. \quad (20)$$

Using the values of $\xi(t_w)$ from Tables VIII—XI, and $\delta H_d$ from Eq. (18), one obtains

$$H_d(t_w=2500) = 5.64 \times 10^{-3},$$
$$H_d(t_w=5000) = 6.66 \times 10^{-3},$$
$$H_d^{\text{ZFC}}(t_w=2500) = 1.20 \times 10^{-3},$$
$$H_d^{\text{ZFC}}(t_w=5000) = 2.36 \times 10^{-3}. \quad (21)$$

The value of $q_{\text{EA}}$ at $T_r = 0.90$ is approximately 0.115, so that the full Hamming distance at this temperature, from Eq. (A3) with $q_w \beta = 0$, is 0.0575. The occupied phase space in our experiments from the results exhibited in Eq. (21) therefore spans only about 12% of the available phase space. The slow growth of $\xi(t_w)$ is evidence that true equilibrium in the spin-glass condensed phase can never be accomplished in laboratory time scales, except perhaps at temperatures in the immediate vicinity of $T_s$, where $q_{\text{EA}}$ can become arbitrarily small.

It is also interesting to note from Eq. (21) that $H_d$ for TRM experiments is larger than for ZFC experiments. This is, of course, consistent with our picture of the shift of the beginning of aging from $q_{\text{EA}}$ for ZFC experiments to $q_{\text{EA}} - q(EZ)$ or, equivalently from $H_d = 0$ to $H_d(|E| = E)$ from Eq. (A4) for TRM protocols as compared to ZFC protocols.

An important lesson from this analysis is that a true definition of $\xi(t_w)$ can be extracted only from a ZFC protocol. A TRM protocol, assuming that $\Delta(t_w)$ increases with $H_d$ faster than linearly, will generate a value for $\xi(t_w)$ that is a function of the magnetic field. In that sense, though an average is usually taken, the only meaningful protocol for extraction of $\xi(t_w)$ is ZFC.

In Secs. V-VI, simulations will be used:

1. to establish the microscopic features of a 3D spin glass in the presence of an external magnetic field;

2. to numerically extract the difference between the magnetic response to the thermostimulated magnetization (TRM) and the zero-field-cooled (ZFC) protocols;

3. to investigate the relationship of our simulation results to the Hamming distance, $H_d$, defined in Eq. (A3).

V. $\xi_{\text{TRM}}(t, t_w; H)$ AND $\xi_{\text{ZFC}}(t, t_w; H)$ FROM SIMULATIONS

This section is organized as follows. In Sec. VA we present the details of the simulations carried out on the Janus II supercomputer. Sec. VB will explain the failure of the basic experimental assumption, see Eq. (1). In Sec. VC, we compare the numerical relaxation function, $S(t, t_w; H)$, of the ZFC and the TRM protocols; and in Sec. VD, we display the relationship of our results to the Hamming distance, $H_d$. Finally, we conclude this section with the extraction of an effective correlation length for both the ZFC and TRM protocols using the relationship relied upon through experiment, with a microscopic direct calculation of the correlation length.
A. Details of the simulations

We carried out massive simulations on the Janus II supercomputer [3] studying the Ising-Edwards-Anderson (IEA) model on a cubic lattice with periodic boundary conditions and size $L = 160 \ a_0$, where $a_0$ is the average distance between magnetic moments. The $N = L^D$ Ising spins, $s_x = \pm 1$, interact with their lattice nearest neighbors through the Hamiltonian

$$\mathcal{H} = - \sum_{(x,y)} J_{xy} s_x s_y - H \sum_x s_x, \quad (22)$$

where the quenched disorder couplings are $J_{xy} = \pm 1$, with $50\%$ probability. We name a particular choice of the couplings a sample. In the absence of an external magnetic field ($H = 0$), this model undergoes a spin-glass transition at the critical temperature in simulation units $T_b = 1.102(3)$ [47]. We study the off-equilibrium dynamics of model (22) using a Metropolis algorithm (one lattice sweep roughly corresponds to one picosecond of physical time).

We have studied a single sample, see Ref. [9, 10] for sample variability studies. For each of the considered protocols, we have considered 1024 statistically independent system trajectories (termed replicas), except for Runs 6 and 7 in Table I for which we have simulated 512 replicas. Further simulation details can be found in Table I (the rationale for our choices of temperatures and magnetic fields is explained in Ref. [9]).

In order to simulate the experimental protocols, the following procedures were taken:

- For the TRM protocol, the initial random spin configuration was placed instantaneously at the working temperature $T_m$ in a magnetic field $H$. It was allowed to relax for a time $t_w$ in the presence of $H$, after which the magnetic field was removed, and the magnetization,

$$M_{\text{TRM}}(t, t_w; H) = \frac{1}{160^3} \sum_x s_x(t + t_w; 0), \quad (23)$$

as well as the temporal auto-correlation function,

$$C_{\text{TRM}}(t, t_w; H) = \frac{1}{160^3} \sum_x s_x(t; H) s_x(t + t_w; 0), \quad (24)$$

were recorded.

- For the ZFC protocol, the initial random spin configuration was placed instantaneously at the working temperature $T_m$ and allowed to relax for a time $t_w$ at $H = 0$. At time $t_w$, the magnetic field $H$ was applied and the magnetization,

$$M_{\text{ZFC}}(t, t_w; H) = \frac{1}{160^3} \sum_x s_x(t + t_w; H), \quad (25)$$

as well as the temporal auto-correlation function,

$$C_{\text{ZFC}}(t, t_w; H) = \frac{1}{160^3} \sum_x s_x(t; 0) s_x(t + t_w; H), \quad (26)$$

were recorded.

Note that the auto-correlation function can be obtained as well experimentally in the TRM protocol [33]. Indeed, in the limit $H \to 0$, one has $C_{\text{TRM}}(t, t_w; H) \propto \langle M_{\text{TRM}}(t) M_{\text{TRM}}(t + t_w) \rangle$, where $\langle \ldots \rangle$ indicates the average over the thermal noise. Indeed, although $\langle M_{\text{TRM}}(t_w) M_{\text{TRM}}(t + t_w) \rangle \propto \sum_{x,y} (s_x(t_w; H) s_y(t + t_w; 0))$, the gauge invariance [48] of the Hamiltonian (22), that holds for $H \to 0$, ensures that only terms with $x = y$ are non-vanishing in the double sum.\footnote{The null contribution (in average) of the cross terms $x \neq y$ makes $\langle M_{\text{TRM}}(t_w) M_{\text{TRM}}(t + t_w) \rangle$ rather noisy, as it can be appreciated in Ref. [33].}

B. The superposition principle breaks down for finite magnetic fields

The experimental investigation of spin glasses is based on ZFC and TRM protocols, which are related to each other in the limit $H \to 0^+$ through Eq. (1). The extended superposition principle has been the touchstone of experimental analysis for more than three decades. However, thanks to the massive numerical simulations carried out on Janus II, we have discovered a range of validity for Eq. (1), and the failure of the assumption that the $M_{\text{FC}}(0, t_w + t)$ is always equal to the sum of $M_{\text{ZFC}}(t_w; t) + M_{\text{TRM}}(t_w; t)$ for $H > 0$.

We analyze separately the growth of the left-hand side, $M_{\text{ZFC}}(t_w, t) + M_{\text{TRM}}(t_w, t)$, and the right-hand side, $M_{\text{FC}}(0, t_w + t)$, of Eq. (1). As the reader notices in Fig. 3, when the magnetic field increases, the violation of Eq. (1) increases. Moreover, the field-cooled magnetization, $M_{\text{FC}}(0, t_w + t)$, changes with time.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Run & $t_m$ & $t_w$ & $(\xi(t_m, H = 0))$ & $t_{\text{max}}$ & $\theta(\tilde{x})$ & $C_{\text{peak}}(t_w)$ \\
\hline
1 & 0.9 & $3^2$ & 8.294(7) & 2.30 & 0.455 & 0.533(3) \\
2 & 0.9 & $2^3.5$ & 11.72(2) & 2.30 & 0.436 & 0.515(2) \\
3 & 0.9 & $3^1.25$ & 16.63(5) & 2.32 & 0.415 & 0.493(3) \\
4 & 1.0 & $2^3.75$ & 11.79(2) & 2.28 & 0.512 & 0.422(2) \\
5 & 1.0 & $2^2.625$ & 16.56(5) & 2.32 & 0.498 & 0.400(1) \\
6 & 1.0 & $3^1.75$ & 23.63(14) & 2.34 & 0.484 & 0.386(4) \\
7 & 0.9 & $2^{4}$ & 20.34(6) & 2.34 & 0.401 & 0.481(3) \\
\hline
\end{tabular}
\caption{Parameters for each of our numerical simulations: $T_m$, $t_w$, $\xi(t_m)$, the longest simulation time $t_{\text{max}}$, the replicon exponent $\theta$, and the value of $C_{\text{peak}}(t_w)$ as defined and employed in the zero-field-cooling protocol of Ref. [9, 10]. The replicon exponent $\theta$ is a function of $\tilde{x} = \ell \xi(T)/\xi(t_w)$, where $\ell \xi(T)$ is the Josephson length [41, 46].}
\end{table}
In order to characterize the violation of Eq. (1), let us define the quantity

\[ D(t, t_w; H) = \frac{1}{H} \left( M_{PC}(0, t_w + t) + M_{ZFC}(t_w, t) + M_{TRM}(t_w, t) \right). \]  

(27)

In Fig. 4, we compare the behavior of \( D(t, t_w; H) \) as a function of time for different magnetic fields \( H \) and waiting times \( t_w \). For small magnetic fields, \( H \leq 0.02 \), the quantity \( D(t, t_w; H) \) is consistently equal to zero [Eq. (1) is holding]; for \( H > 0.02 \), \( D(t, t_w; H) \) has an inflection. This behavior suggests to us that the extended superposition principle is valid only when \( H \to 0 \).

Thus, if Eq. (1) is only valid for \( H \to 0 \), we can hypothesize that \( D(t, t_w; H) \) could behave as:

\[ D(t, t_w; H) = a_2(t_w; T)H^2 + a_4(t_w; T)H^4 + O(H^6), \]

(28)

where the coefficients \( a_2(t_w; T) \) and \( a_4(t_w; T) \) are some unknown functions and \( O(H^6) \) represents higher-order terms.

To test Eq. (28), we address the temporal behavior of the rescaled quantity \( D(t, t_w; H)T_m^w/H^2 \) in Fig. 5. We have rescaled the quantity \( D(t, t_w; H) \) by the temperature to compare data at different temperatures.

In Fig. 5, we analyze two aspects:

- For a given waiting time \( t_w \) [i.e., \( \xi(t_w) \)], what is the effect of increasing the magnetic field, \( H \)?
- For a given value of the external magnetic field, \( H = 0.02 \), what is the effect of changing the waiting time, \( t_w \) [i.e., \( \xi(t_w) \)]?

The answer to the first question is straightforward. From Fig. 5, increasing the magnetic field \( H \) the separation between the \( H \) curves increases as well. To the second question, we still do not have a satisfactory answer. Runs at \( T = 1.0 \), namely Run 4, Run 5, and Run 6 (the purple, orange and red points) in Figs. 4 and 5 follow almost the same curve even though they are characterized by a very different correlation length (see Tab. 1).

Thus, the violation of Eq. (1) is caused by the difference between the time developments of \( \xi_{TRM}(t, t_w; H) \) and \( \xi_{ZFC}(t, t_w; H) \). The lack of a dependence on \( t_w \) in the \( a_2(t_w; T) \) and \( a_4(t_w; T) \) coefficients is consistent with the expectation that the only \( t_w \) dependence lies within the \( t_w \) dependence of the correlation lengths themselves. Otherwise, there would be a \( t_w \) dependence even in the \( H^2 \to 0 \) limit.
C. Evaluation of the relaxation function $S(t, t_w; H)$

Exploiting the de-noising method introduced in Refs. [9, 10], we calculate the numerical value for the relaxation function $S_{TRM}(t, t_w; H)$ [5] as

$$S_{TRM}(t, t_w; H) = -\frac{1}{H} \frac{d M_{TRM}(t, t_w; H)}{d \log t}.$$  \hspace{1cm} (29)

In Fig. 6, we exhibit a typical set of results for $S_{TRM}(t, t_w; H)$.

The simulation data strongly suggests that, when $H \rightarrow 0$, the temporal correlation function $C(t, t_w; H)$ approaches a constant value, $C_{\text{peak}}(t_w)$, at the maximum of the relaxation function $S(t, t_w; H)$ [9, 10].

Hence, we define the time $t_{\text{eff}}^H$ in our simulations as the time when $C(t, t_w; H)$ reaches the value $C_{\text{peak}}(t_w)$:

$$C(t_{\text{eff}}^H, t_w; H) = C_{\text{peak}}(t_w).$$  \hspace{1cm} (30)

As shown in Fig. 7, this physical feature holds both for the ZFC and for the TRM protocols. In addition, as seen in Fig. 7, the relaxation functions $S(t, t_w; H)$ peak at the same value for $C_{\text{peak}}(t_w)$ in the two experimental protocols at small magnetic fields. This suggests that the highest free-energy barrier explored in both protocols is the same in the limit that $H \rightarrow 0$.

In the following sub-sections, exploiting the behavior of the Hamming distance, we will show how the value of the effective time, $t_{\text{eff}}^H$, is independent of the value of $C_{\text{peak}}(t_w)$ and unveils the physical meaning of Eq. (30).

D. Hamming distance: Scaling

We extract the Hamming distance, or at least a surrogate of it, from our knowledge of the temporal auto-correlation function $C(t, t_w; H)$:

$$Hd(t, t_w; H) = \frac{1}{2} \left[ 1 - C(t, t_w; H) \right].$$  \hspace{1cm} (31)

A discussion of the connection between the above numerical Hamming distance and the dynamics in the ultrametric tree of states is provided in Appendix D.

In Fig. 8, we exhibit the behavior of the Hamming distance in Eq. (31), $Hd(t, t_w; H = 0)$, as a function of the rescaled time $T_m \log_2(t)$ for $H = 0$. Notice that the $Hd$ curves depart from a simple scaling curve as soon as the time $t$ for each run reaches their respective $t_w$.

FIG. 7. Comparison between the TRM and the ZFC relaxation functions for Run 5, with $T_m = 1.0$ and $t_w = 2^{27.625}$. The empty points are for $S_{TRM}(t, t_w; H)$, while the full dots are for the $S_{ZFC}(t, t_w; H)$. The dashed line displays the value for $C_{\text{peak}}(t_w)$ (see Table I). The ZFC points are taken from Refs. [9, 10].

FIG. 8. Plot of the Hamming distance, $Hd(t, t_w; H = 0)$ see Eq. (31), as a function of the rescaled time, $T_m \log_2(t)$. Notice the scaling for lower times. Runs with the same $T_m$ share the shorter time regime.

If one displays $\log(t/t_{\text{eff}}^H)$ as a function of $Hd(t, t_w; H)$ at the two simulation temperatures, $T_m = 0.9$ and $T_m = 1.0$, a scaling behavior is apparent from Fig. 9, with

$$\log \left\{ t/t_{\text{eff}}^H[C_{\text{peak}}(t_w)] \right\} = \mathcal{F} \left[ C(t, t_w; H), t_w \right].$$  \hspace{1cm} (32)
FIG. 9. Behavior of the rescaled time $\log(t_{\text{eff}}/t_{\text{eff}\to 0+})$ as a function of the Hamming distance $H_\text{d}(t, t_w; H)$. The empty circles represent the ZFC data, while the full triangles and joined with lines represent the TRM data.

The determination of the precise value for $C_{\text{peak}}(t_w)$ is not crucial because $C_{\text{peak}}(t_w)$ changes $\log(t_{\text{eff}}/C_{\text{peak}})$ only by a constant. This implies that $\log(t_{\text{eff}}/C_{\text{peak}})$ does not depend upon $H^2$.

Developing this important concept further, from Eq. (32) we can write,

$$
\log \left[ \frac{t_{\text{eff}}(C)}{t_{\text{eff}}(C_{\text{peak}})} \right] = \log \left[ \frac{t_{\text{eff}}(C, t_w)}{t_{\text{eff}}(C, t_w; H)} \right] = \log \left[ \frac{t_{\text{eff}}(C, t_w; H)}{t_{\text{eff}}(C_{\text{peak}}, t_w; H)} \right] = \log \left[ \frac{t_{\text{eff}}(C_{\text{peak}})}{t_{\text{eff}}(C_{\text{peak}}, t_w; H)} \right],
$$

(33)

implying that the value of the effective time, $t_{\text{eff}}$, is independent of the value of $C_{\text{peak}}(t_w)$.

E. Extraction of the effective response time $t_{\text{eff}}^{H}$

We can extract the effective response time $t_{\text{eff}}^{H}$ using Eq. (30). In Table I we listed the values of $C_{\text{peak}}(t_w)$ for the ZFC protocol from Refs. [9, 10]. The results are displayed in Fig. 10, along with those for the TRM protocol (see below). The data for $\log(t_{\text{eff}}^{H}/t_{\text{eff}}^{H\to 0+})$ are fitted by the function,

$$
f(x) = c_2(t_w; T)x + O(x^2),
$$

(34)

where $x = H^2$. In order to avoid the unphysical wild oscillations at large magnetic fields (recall that $H = 1$ for the IEA model roughly corresponds to $5 \times 10^4$ Oe in physical units [9]), we define a unique fitting range in the small $x$ region, $x = H^2 \in [0, 0.0003]$. Our fitting parameters are displayed in Table II for the ZFC data, and Table III for the TRM data.

FIG. 10. The numerical ratio of $\log(t_{\text{eff}}^{H}/t_{\text{eff}}^{H\to 0+})$ for the seven runs defined in Table I for both the ZFC and TRM protocols. The filled dots refer to the ZFC protocol; the empty squares to the TRM protocol. The coefficients of the $H^2$ fit, $c_2(t_w; T)$ from Eq. (34) are listed in Table II for the ZFC data, and Table III for the TRM data. The continuous lines represent the fit to the ZFC data, while the dashed lines represent the fit to the TRM data. The ZFC data are the same as in Refs. [9, 10].

| $T$ | $t_w$ | Coefficient | Numerical value |
|-----|-------|-------------|-----------------|
| 0.9 | $2^{22}$ | $c_2$ | $-5.01(14) \times 10^2$ |
| 0.9 | $2^{26.5}$ | $c_2$ | $-1.54(2) \times 10^3$ |
| 0.9 | $2^{31.25}$ | $c_2$ | $-4.13(11) \times 10^3$ |
| 0.9 | $2^{34}$ | $c_2$ | $-6.78(13) \times 10^3$ |
| 1.0 | $2^{23.75}$ | $c_2$ | $-1.29(2) \times 10^3$ |
| 1.0 | $2^{27.625}$ | $c_2$ | $-3.25(3) \times 10^3$ |
| 1.0 | $2^{31.75}$ | $c_2$ | $-7.48(17) \times 10^3$ |

Table II. Results for the fit to Eq. (34) for the ZFC data for the time ratio $\log(t_{\text{eff}}^{H}/t_{\text{eff}}^{H\to 0+})$. The fitting range is $0 \leq H^2 \leq 0.0003$.

VI. DIFFERENCE BETWEEN THE ZFC AND THE TRM PROTOCOLS

This section is the core of our numerical results in showing and characterizing the effect of an external magnetic field on a 3D spin glass.

The scaling law, first introduced by Joh et al. [5] and then developed by Refs. [9, 10], solves a decades-old controversy concerning the nature of the Zeeman energy for describing the magnetic response both in
TABLE III. Results for the fit to Eq. (34) for the TRM data for the time ratio $\log(t_{\text{eff}}/t_{H \rightarrow 0})$. The fitting range is $0 \leq H^2 \leq 0.0003$.

| $t_w$ | Coefficient | Numerical value |
|-------|-------------|-----------------|
| 0.9   | $2^{22}$    | $c_2 = -6.77(11) \times 10^2$ |
| 0.9   | $2^{26.5}$  | $c_2 = -1.52(2) \times 10^3$   |
| 0.9   | $2^{31.25}$ | $c_2 = -3.60(14) \times 10^3$  |
| 0.9   | $2^{34}$    | $c_2 = -5.84(16) \times 10^3$  |
| 1.0   | $2^{23.75}$ | $c_2 = -1.06(1) \times 10^3$   |
| 1.0   | $2^{37.625}$| $c_2 = -2.64(3) \times 10^3$   |
| 1.0   | $2^{31.75}$ | $c_2 = -5.65(22) \times 10^3$  |

Numerically, we have easy access to the spin configuration enabling us to calculate the microscopic correlation length $\xi_{\text{micro}}(t_w; H)$ as follows. Let us define the replicon propagator [13, 49]:

$$\mathcal{G}_R(r, t, T) = \frac{1}{V} \sum_w \langle (s_{x, t} s_{x+s, r}) - \langle s_{x, t} \rangle \langle s_{x+s, r} \rangle \rangle T$$

(36)

The replicon correlator $\mathcal{G}_R$ decays to zero in the long-distance limit. We therefore compute $\xi_{\text{micro}}(t_w; H)$ by exploiting the integral estimators [50, 51]:

$$I_k(t; T) = I_{1k}(t; T)$$

(37)

where

$$\xi_{k, k+1}(t; T) = \frac{I_{k+1}(t; T)}{I_k(t; T)}.$$  

(38)

The $\xi_{12}(t_w; T)$ is designated as the microscopic correlation length $\xi_{\text{micro}}(t_w; T)$.

From the experimental point of view, Eq. (35) determines an effective correlation length, $\xi_{\text{eff}}(t_w; H^2 \rightarrow 0)$. See Sec. III and below.

We shall follow two approaches to claim the same result: “the two experimental protocols are not equivalent, and the presence, or absence, of an external magnetic field in the thermal history of a spin glass is not negligible”.

On the one hand, we analyze the effect of the external magnetic field on the microscopic correlation length $\xi_{\text{micro}}$ [directly accessible in simulations]; on the other hand, we follow the same experimental approach, see Sec. III, to evaluate the magnetic response through the lens of the effective time $t_{\text{eff}}^H$.

A. Numerical approach: the effect of an external magnetic field through the lens of the microscopic correlation length $\xi_{\text{micro}}(t_w)$

In the zero-field-cooling protocol, the system is cooled to the working temperature $T_m$ in the absence of an external magnetic field, which is then switched on after a waiting time $t_w$. Thus, for definition, the ZFC protocol can be described by its microscopic correlation length, $\xi_{\text{micro}}(t_w, H = 0)$. The thermoremanent protocol, conversely, brings the system to $T_m$ in the presence of an external magnetic field. This implies that each run has its own $\xi_{\text{TRM}}(t_w; H)$ before $H$ is turned off.

In Fig. 12, we display the $H^2 \rightarrow 0^+$ behavior of $\xi_{\text{TRM}}(t_w; H)$ in order to show the effect of an external magnetic field $H$.

Thus, in Fig. 13 we display the difference between the ZFC and TRM correlation length $\xi_{\text{micro}}(t_w; H)$, defined through Eqs. (36)-(38), against $H^2$.

As can be seen from Figs. 12-13, $\xi_{\text{TRM}}(t_w; H)$ approaches the $H^2 \rightarrow 0^+$ limit with a linear slope and the difference between the ZFC and TRM correlation length seems to have the same behavior for different

FIG. 11. The non-linear parts from the numerical response time data, $\log(t_{\text{eff}}/t_{H \rightarrow 0}) - c_2(t_w, T)H^2$ plotted against $\xi_{\text{eff}}^{(3/2)}$. The abscissa of the main panel is in a linear scale, showing an expanded view for small values of $\xi_{\text{eff}}^{(3/2)}$. The abscissa of the insert is in a log scale in order to report all of our numerical data. The open squares refer to the ZFC data, while the filled squares refer to the TRM data.

One of the main differences between experiments and simulations is access to the microscopic spin configurations. Eq. (35) could be read as a bridge to connect the microscopic observable of the effective time $t_{\text{eff}}^H$ and the microscopic spin rearrangement $\xi_{\text{micro}}$. 

Here, $\xi(t_w)$ is the microscopic correlation length, see below Eq. (38), $S$ is a constant from the Fluctuation-Dissipation Theorem (FDT), $D = 3$ is the spatial dimension and $\theta$ is the replicon exponent [8]. The replicon exponent $\theta$ is a function of $\tilde{x} = \ell_x(T)/\xi(t_w)$, where $\ell_x(T)$ is the Josephson length [41, 46]. For notational simplicity, we have omitted this functional dependence.

In Fig. 11, we show that this scaling law holds for both the ZFC and TRM protocols.

The open squares refer to the ZFC data, while the filled squares refer to the TRM data.

In Fig. 11, we show that this scaling law holds for both the ZFC and TRM protocols.
FIG. 12. Behavior of $\xi_{\text{TRM}}(t_w; H)$ as a function of $H^2 \to 0$.

FIG. 13. The difference between the ZFC and TRM correlation length $\xi_{\text{micro}}(t_w; H)$, defined through Eqs. (36)-(38), is plotted against $H^2$. The main plot scale is log-log; the insert has a linear scale for the ordinate. The ordering of the different runs (see Table I) displays an increase of the difference with increasing $t_w$.

runs (i.e. $t_w$). This enables us to extract a scaling law if the correlation length is rescaled as

$$1 - \frac{\xi_{\text{TRM}}(t_w; H)}{\xi_{\text{ZFC}}(t_w; H = 0)} = A(t_w, T)[\xi_{\text{micro}}(H = 0)]^{D-\theta(\delta)/2} H^2$$

as demonstrated in Fig. 14.

The logic behind the effect of the upward curvature of $\Delta(t_w) \propto H_d$, as suggested in Ref. [22] and discussed in Sec. IV of this paper, requires that the difference $\xi_{\text{ZFC}}(t_w; H) - \xi_{\text{TRM}}(t_w; H)$ increase with increasing waiting time $t_w$. This is because $H_d$ itself increases with $t_w$, and hence the barrier height difference between the ZFC and TRM protocols increases with $t_w$.

B. Experimental approach: evaluation of the magnetic response through the effective times

We now focus on the differences between the two protocols through the lens of the effective time, i.e., following the experimental approach. From Eqs. (34) and (35), the quantities $log t_{H}^{\text{eff}}(\text{ZFC})$ and $log t_{H}^{\text{eff}}(\text{TRM})$ can be written as,

$$log t_{H}^{\text{eff}}(\text{ZFC}) - log t_{H}^{\text{eff}}(\text{TRM}) = [\xi_{\text{ZFC}} - \xi_{\text{TRM}}]_{t_w} - [K^{\text{ZFC}} - K^{\text{TRM}}]_{t_w}$$

$$H^2) = \mathcal{O}(H^4)$$

By definition, $c_{\text{ZFC}} = c_{\text{TRM}} = log t_{H}^{\text{eff}}$ so that,

$$[\xi_{\text{ZFC}} - \xi_{\text{TRM}}]_{t_w} = (\xi_{\text{ZFC}} - \xi_{\text{TRM}})_{t_w} = \mathcal{O}(H^4)$$

$$log t_{H}^{\text{eff}}(\text{ZFC}) - log t_{H}^{\text{eff}}(\text{TRM}) \equiv \delta \log t_{H}^{\text{eff}}$$

We exhibit the rescaled quantity $T(\delta \log t_{H}^{\text{eff}})/[\xi(t_w)]^{D-\theta(\delta)/2}$ as a function of $H^2$ in Fig. 15.

A scaling behavior for $\xi(t_w)$ sufficiently large is observed, as well as support for the principal relationship explored in this paper:

$$K^{\text{ZFC}} > K^{\text{TRM}}$$
Let us consider separately the ZFC and TRM protocols. The numerical data for \( \log\left(\frac{\sigma_H^\alpha}{\sigma_H^\alpha_{H=0}}\right) \) are fitted in the small \( x \) region to the function:

\[
f(x) = c_2(t_w; T)x + O(x^2),
\]

where \( x \equiv H^2 \) and the coefficient \( c_2(t_w; H) \) corresponds to:

\[
c_2^{ZFC}(t_w; H) = K^{ZFC}_c(t_w) D^{-\theta(\hat{x})/2}
\]

\[
c_2^{TRM}(t_w; H) = K^{TRM} \xi(t_w) D^{-\theta(\hat{x})/2}
\]

where \( \xi(t_w) \) is the \( \xi_{micro}(t_w) \) defined in Eq. (38) in the absence of a magnetic field.

According to the scaling law introduced in Refs. \[9, 10\], the coefficient \( c_2^{ZFC}(t_w; T) \) is:

\[
c_2^{ZFC}(t_w; H) = \left[ \frac{\hat{S}}{2T_m} \right] \xi(t_w) D^{-\theta(\hat{x})/2}
\]

\[
\Rightarrow K^{ZFC} = \frac{\hat{S}}{2T_m}. 
\]

Hence, using the fitting data from Table. II, we define a ZFC effective correlation length, \( c_2^{ZFC}(t_w; T) \), as:

\[
c_2^{ZFC}(t_w; T_m) = K^{ZFC}(t_w) \left[ \frac{\xi_{eff}(t_w, T_m)}{\xi_{micro}(t_w^*, T_m)} \right] D^{-\theta(\hat{x})/2}
\]

where \( K^{ZFC}(t_w) = K^{ZFC} \), see Eq. (50), and we have omitted the \( \xi(t_w) \) dependence on the replica \( \theta(\hat{x}) \).

Thus, we define \( c_2^{ZFC}(t_w; T_m) \) as:

\[
\xi_{ZFC}(t_w; T_m) = \left[ \frac{c_2(t_w, T_m)}{c_2(t_w^*, T_m)} \right]^{1/(D-\theta(\hat{x})/2)} \xi_{micro}(t_w^*, T_m).
\]

The quantity \( \xi_{micro}(t_w^*, T_m) \) plays the role of a reference length to avoid having to require a precise determination of the constants in Eq. (49).

Let us now focus on the TRM protocol. Following the results of Sec. VII, the equivalence between the TRM and ZFC protocols is not holding, and we quantified the difference between the energetic barrier in the TRM and ZFC protocol in Fig. 16. Let us formalize this difference as:

\[
K^{ZFC} - K^{TRM} = \bar{B}(t_w).
\]

By manipulating the above expression, we can obtain an expression for \( K^{TRM} \):

\[
K^{TRM}(t_w) = K^{ZFC} - \bar{B}(t_w).
\]

Hence, using the fitting data from Table III, we can rewrite Eq. (51) for the TRM case as:

\[
c_2^{TRM}(t_w, T_m) = \left[ \frac{K^{TRM}(t_w)}{K^{TRM}(t_w^*)} \right] \left[ \frac{\xi_{eff}(t_w, T_m)}{\xi_{micro}(t_w^*, T_m)} \right] D^{-\theta(\hat{x})/2}
\]
Thus, we obtain:

$$
\xi_{\text{eff,TRM}}(t_w; T_m) = \left[ \frac{c_{w,TRM}^2(t_w; T_m)}{c_{w,TRM}^2(t_w^*, T_m)} \right]^{\frac{1}{2}} \frac{c_{w,TRM}^2(t_w^*, T_m)}{c_{w,TRM}^2(t_w^*, T_m)} \times \xi_{\text{micro}}(t_w^*, T_m). 
$$

(56)

In Fig. 17, we report the comparison between $\xi_{\text{eff,TRM}}(t_w; H)$ and $\xi_{\text{micro}}(t_w; H)$ as a function of the waiting time $t_w$ in both the ZFC and TRM cases. By definition, the $t_w^*$ taken as a reference has exactly the same $\xi_{\text{eff}}(t_w^*, T) = \xi_{\text{micro}}(t_w^*, T)$. We used as the reference $t_w^* = 2^{31}$ at $T_m = 0.9$ and $t_w^* = 2^{31.75}$ at $T_m = 1.0$.

FIG. 17. Comparison between $\xi_{\text{micro}}(t_w; H = 0)$ and $\xi_{\text{eff}}(t_w; H^2 \rightarrow 0)$ as a function of the waiting time $t_w$. By definition of $\xi_{\text{eff}}(t_w; H^2 \rightarrow 0)$, see Eqs. (52) and (56), the $t_w^*$ taken as a reference has exactly the same $\xi_{\text{eff}}(t_w^*, T) = \xi_{\text{micro}}(t_w^*, T)$.

Given our lack of statistics, we could simulate only a single sample for each case. The errors are calculated from only thermal fluctuations, and the numerical ansatz of Eq. (45) is confirmed.

VII. CONCLUSION

The fruitful collaboration between the experimental group at Austin and the Janus collaboration exhibits for the first time the consequences of the difference between the two relaxation protocols for spin glasses, ZFC and TRM. The power of the Janus II supercomputer allows us to extend simulation times and length scales to values explored experimentally. The use of single crystals of CuMn enables experiments to exhibit the consequences of very large spin-glass correlation lengths. Both these ingredients were vital for unveiling the difference in the magnetic response of the two experimental protocols, which have been considered equivalent for more than three-decades.

This paper analyzes the effects of magnetic fields on spin-glass dynamics. The scaling law introduced in Refs. [9, 10] has played the role of a touchstone for evaluating the magnetic response of a 3D spin glass. We connected the difference between the ZFC and the TRM protocols to the dynamics of the Hamming distance through the reduction of the free-energy barriers.

In Sec. V, we claim the equivalence between the experimental extraction of the correlation length through Eq. (35) and the microscopic calculation of $\xi$ through the replicon propagator $G_m(r, t_w, T)$.

The unique and extraordinary collaboration between experiments, simulations, and theory has displayed once again its potential for the investigation of complex systems, as the 3D spin glass. We look forward to continued investigation of spin-glass dynamics building on the results of this paper as we examine the microscopic nature of such phenomena as rejuvenation and memory.

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Appendix A: Context for Sect. II

From a phenomenological point of view, Eq. (1) depends upon the relationship between the barrier heights $\Delta(t, t_w)$ between states and their separation, termed the Hamming distance (Hd), defined below in Eq. (A3). If the relationship is linear, Eq. (1) holds. If $\Delta(t, t_w)$ increases more rapidly than linear with Hd, Eq. (1) will not hold for finite magnetic fields $H$. Under such conditions, the decay of the TRM will be slower than the rise of the ZFC, and the departure from Eq. (1) will increase with increasing magnetic field change and increasing $t_w$.

In order to explain these dynamics, it is helpful to employ a simple phenomenological model that utilizes the concept of hierarchy of ancestors of spin-glass states.

One can organize the states, and their ancestors, according to ultrametric symmetry [23]. Immediately after a temperature quench from above $T_g$ to the measurement temperature $T_m$, the spin-glass states have a “self-overlap” $q_{\alpha\alpha}(T_m) \equiv q_{\alpha\alpha}(T_m)$, the Edwards-Anderson order parameter [52]

$$q_{\alpha\alpha}(T_m) \equiv q_{\alpha\alpha}(T_m) = \frac{1}{N} \sum_\alpha \langle s_\alpha(t = 0) \rangle^2$$

where $N$ is the number of (Ising) spins, $\alpha$ is the position coordinate of spin $s$, and $\langle \cdots \rangle_\alpha$ represents a thermal average restricted to a pure state $\alpha$. Now, simple and physically compelling as the above expression might look, we enter into a true mathematical minefield by writing Eq. (A1). There are difficulties of several kinds:

- The decomposition of the Boltzmann weight into a sum over pure phases $\langle \cdots \rangle_\alpha$ [53] holds only in the thermodynamic limit.

- The random couplings make the process of taking the thermodynamic limit highly non-trivial (the construction of the metastate somewhat alleviates this problem [54–58]).

- The construction of the Hamming distance highly non-trivial (the construction of the metastate somewhat alleviates this problem [54–58]).

The state $\alpha$ in Eq. (A1) is not an equilibrium state (as assumed in the two bullet points above), but a dynamic state. The characterization of the metastate in a dynamic context is only incipient [59].

Similar caveats apply to Eq. (A2) below. In fact, this Appendix should be read under the assumption that future mathematical developments will make it possible to give a precise meaning to expressions such as Eqs. (A1) and (A2).

Nevertheless, the self-overlap $q_{\alpha\alpha}(T)$ can be computed without recourse to the pure-state decomposition (see, e.g., Refs. [60, 61]). One finds that, although $q_{\alpha\alpha}(T)$ is zero at $T = T_g$, it rapidly rises to unity as $T \to 0$. As the waiting time $t_w$ grows, the quenched initial state organizes from its paramagnetic structure into a progressively growing spin glass state.

This is expressed through a growing spin glass correlation length $\xi(t, t_w; H)$, which is a function of the time elapsed from quench. Here, $t_w$ is the time from quench to when a change in magnetic field $H$ occurs, and the time $t$ begins just after the change in magnetic field.

As $\xi(t, t_w; H)$ grows, the overlap with the initial state $q_{\alpha\beta}$ diminishes:

$$q_{\alpha\beta} = \frac{1}{N} \sum_\alpha \langle s_\alpha(t = 0) \rangle^2$$

where the state has evolved from its initial quenched state $\alpha(t = 0, t_w = 0; H)$ to $\beta(t, t_w; H)$. This is pictured in Fig. 18 for a random ultrametric tree [62].

The ultrametric structure in Fig. 18 was originally derived for Parisi’s pure states, a small proportion of the total states available to the spin glass system. The barriers between states were infinite, so that no dynamics were present. Experimentally, as developed in Refs. [5, 35], it was not only convenient, but quantitative, to include states with finite barriers obeying the same geometry between the pure states. In fact, experiments over a limited temperature range were shown to display a temperature dependence of barrier heights that could be extrapolated to the pure state limit [63]. A representative temperature-dependent organization of metastable states is exhibited in Fig. 19.

We need some way to account for the distance between the states exhibited in Fig. 18. We express a Hamming distance, Hd, in terms of the departure of the overlap from $q_{\alpha\alpha}$ to $q_{\alpha\beta}$. By this construction, the Hamming distance, Hd, is

$$Hd = \frac{1}{2}(q_{\alpha\alpha} - q_{\alpha\beta})$$

One can intuitively think of a free-energy barrier separating states of different $q_{\alpha\beta}$ as proportional to a distance between them, as shown in Fig. 18. That is,
the barrier heights should increase as some function of decreasing $q_{ab}$ or increasing $\text{Hd}$. A relationship between $\text{Hd}$ and barrier heights was first developed by Vertechi and Virasoro [64]. In Fig. 1 of Ref. [64] it is shown that there is an upward curvature in the relationship between $\Delta$ and $\text{Hd}$ that we shall argue below can be extracted from experiment, and is exhibited in our simulations.

Consider now the Zeeman energy $E_Z$ in Eq. (4). Its effect on barrier heights is treated equivalently by Bouchaud et al. [39] and Joh et al. [5] in terms of a trap model and a barrier model, respectively. One can visualize the effect on the barrier heights using the pictorial description exhibited in Fig. 20 as developed in Ref. [65].

The figure is meant to illustrate the dynamics associated with a TRM protocol. A magnetic field $H$ is applied to the spin glass at a temperature above the condensation temperature $T_g$. Without changing $H$, the system is cooled to the working temperature $T_{w} < T_g$. The lower part of the free energy of the system is then the upper part of Fig. 20. All of the states in Fig. 20 between the barriers are assumed to have the same magnetization. This assumption is consistent with the near constancy in time of the field-cooled magnetization, $M_{\text{FC}}(0, t_{w} + t)$ (see Eq. (1)) [66]. The system is allowed to age at $T_{m}$ for a time $t_{w}$ in the presence of $H$. This results in the growth of the correlation length $\xi(t = 0, t_{w}; H)$ creating ever increasing barriers up to the value $\Delta(t = 0, t_{w}; H)$ at the Hamming distance $\text{Hd}(t_{w})$.

The effect of the magnetic field is expressed through the Zeeman energy $E_Z$, Eq. (5). Both models (trap/barrier) reduce the depth/height of all of the traps/barriers by the same amount, $E_Z$. In the barrier model, those barriers for which $\Delta < |E_Z|$ are assumed to vanish, so that the growth of the correlation length begins at an $\text{Hd}$ set by $\Delta = |E_Z|$. That means that $\text{Hd}(t_{w}; H)$ is larger than it would be at $\text{Hd}(t_{w}; H = 0)$. This point will be crucial when we compare $\xi(t, t_{w}; H)$ for TRM experiments with ZFC experiments.

In a TRM experiment, after aging for $t_{w}$ in the presence of $H$, the field is cut to zero and the remnant magnetization $M_{\text{TRM}}(t, t_{w}; H)$ measured. By construction in Fig. 20, the states with magnetization $M_{\text{FC}}$ now are higher in energy than for those with $M = 0$. There is an instantaneous decay of those states in the $M_{\text{FC}}$ manifold for which $\Delta$ is less than $E_Z$ to the ground $M = 0$ manifold (the “reversible magnetization” change) [65]. The measurement time $t$ begins when $H$ is cut to zero. Diffusion from the remaining occupied states to the ground state ($M = 0$) reduces the magnetization in the (now) higher energy manifold as shown by the arrow in Fig. 20. The exponentially increasing degeneracy exhibited in Fig. 18 leads to the states with the highest barrier dominating...
the decay.

A consequence of Fig. 20 is that, in the presence of a magnetic field $H$, the barriers are quenched for values of $H_d$ such that $\Delta(H_d) \leq |E_z| \equiv \Delta E_z(H_d)$ with $E_z$ defined by Eq. (5). Eq. (A3) can be written alternatively to take into account this cancellation. Take $q_{\alpha\beta}(E_z) = q_{E_z} - q(E_z)$ to be the value of $q_{\alpha\beta}$ at the value of $q$ where $\Delta(q) = |E_z|$. Then, from Eq. (A3), the $H_d$ has the value,

$$H_d(|E_z| = \Delta) = \frac{1}{2}[q_{E_z} - q(E_z)]$$

(A4)

at the value of $q$ for which $|E_z| = \Delta(q)$. The probability of finding a value of $q$ larger than $q(E_z)$ is zero. That is, there are no values of $q$ larger than this value.

As a consequence, $\xi(t = 0, t_w; H)$ grows from zero not from $q = 0$ but rather from $q = q(E_z)$ in a TRM experiment. If $\Delta(t_w)$ would depend linearly on $H_d$, equivalently on $q_{\alpha\beta}$ from Eq. (A3), there would be no difference in $\xi(t = 0, t_w; H)$ between a TRM and a ZFC experiment. In both, the growth of $\xi(t = 0, t_w; H)$ would depend upon $H_d$ in the same manner. However, if instead $\Delta(t = 0, t_w; H)$ behaves as drawn in Fig. 20, that is, experiences an upward curvature as $H_d$ increases, then beginning at a larger value of $q$ in a TRM protocol would mean that the growth of $\Delta(t = 0, t_w; H)$ would be slower for a TRM protocol than for a ZFC protocol.

Lederman et al. [22] found evidence for an upward curvature of $\Delta$ as a function of $H_d$ in a AgMn 2.6 at.% spin glass. Using measurements of the temperature dependence of both $\Delta$ and $q_{E_z}$, they extracted the ratio,

$$\frac{\delta\Delta}{\delta H_d} \bigg|_T = \alpha(T)\Delta - \beta(T),$$

(A5)

where $\alpha(T)$ and $\beta(T)$ are positive constants dependent only on temperature. Integration of Eq. (A5) gives,

$$\Delta(H_d) = C e^{\alpha(T)H_d} + \frac{\beta(T)}{\alpha(T)}.$$

(A6)

where $C$ is an integration constant. Thus, as $H$ increases, the diffusion in the time $t_w$ encounters ever increasing barrier heights for TRM protocols as compared to ZFC protocols. This is the basis for our analysis that $\xi_{ZFC}(t, t_w; H) > \xi_{TRM}(t, t_w; H)$ in the text.

Appendix B: Results from another single crystal

In addition to the 8 at.% CuMn single-crystal sample with $T_m = 37.5$ K ($T_g = 41.5$ K), the properties of which are reported in the text, a 6 at.% CuMn single-crystal sample was measured at $T_m = 26$ K ($T_g = 31.5$ K). The lower measuring temperature resulted in smaller values of $\xi(t, t_w; H)$ because of the attendant slow growth rate, leading to a smaller difference in the difference between $\xi_{ZFC}(t_w; H)$ and $\xi_{TRM}(t_w; H)$ than for the $T_m = 37.5$ K measurements. Nevertheless, at the largest waiting time, $t_w = 10000$ s, the difference lies well outside of the error bars.

The data are exhibited in Fig. 21 for the two values of the waiting time, $t_w$, at $T = 26$ K: $t_w = 3000$ s and $t_w = 10000$ s, for both the ZFC and TRM protocols.

FIG. 21. A plot of data and fit for the log of the effective waiting time, $t_w$, vs $H^2$ for ZFC and TRM measurements on a 6 at.% single crystal at $t_w = 3000$ and 10000 s and $T = 26$ K ($T_g = 31.5$ K). The polynomial fitting parameters for each of the values of $H$, up to and including $H^2$, are given in Tables IV—VII. The value of the correlation length is extracted from the $H^2$ fitting terms.

From Fig. 21,

$$\xi_{ZFC}(t_w = 3000) - \xi_{TRM}(t_w = 3000) = 3.8(4.8)$$

$$\xi_{ZFC}(t_w = 10000) - \xi_{TRM}(t_w = 10000) = 7.7(2.6)$$

taking the sum of the extreme limits of the error bars. Assuming that $\Delta(t_w)$ increases faster than linearly with $H_d$, the larger $t_w$, the slower the growth of $\xi_{TRM}(t_w; H)$, as can be inferred from the shape of $\Delta$ vs $H_d$ exhibited in Fig. 20 of Appendix A. Thus, the difference $\xi_{ZFC}(t_w) - \xi_{TRM}(t_w)$ should increase with increasing $t_w$. The data from Fig. 21 in this Appendix support this prediction: the difference $\xi_{ZFC}(t_w = 10000) - \xi_{TRM}(t_w = 10000)$ is much larger than $\xi_{ZFC}(t_w = 3000) - \xi_{TRM}(t_w = 3000)$, and well beyond the limits of the error bars.

Thus, the measurements on the 6 at.% CuMn single crystal at 26 K provide additional experimental evidence for the assertion that $\xi_{ZFC}(t_w)$ is larger than $\xi_{TRM}(t_w)$ to that exhibited in the main text for an 8 at.% CuMn single crystal.

The tables analogous to Tables VIII—XI for the 6
at.% single crystal sample with $T_m = 26$ K, are listed in Tables IV—VII.

| $H$  | $E_0$     | $E_2$     |
|------|-----------|-----------|
| 22.0 | 30.85298  | -0.04875  |
| 32.0 | 30.85298  | -0.10313  |
| 47.0 | 30.85298  | -0.22249  |
| 59.0 | 30.85298  | -0.35960  |
| 67.0 | 30.85298  | -0.45212  |
| 74.0 | 30.85298  | -0.55153  |

TABLE IV. Converted energy scale for ZFC protocol at $T = 26$ K with $t_w = 3000$ s for various values of $H$ expressed in Oersteds. The $E_n$ are the magnitude of the nth fitting parameter (including the respective $H^n$) expressed in units of $k_B T_s$ (see Eq. (17)). The correlation length $\xi_{ZFC} = 123(2)$ is derived from the coefficients of the $H^2$ term, utilizing Eqs. (8) and (9).

| $H$  | $E_0$     | $E_2$     |
|------|-----------|-----------|
| 22.0 | 30.87475  | -0.04457  |
| 32.0 | 30.87475  | -0.09431  |
| 47.0 | 30.87475  | -0.20344  |
| 59.0 | 30.87475  | -0.32659  |
| 67.0 | 30.87475  | -0.41342  |
| 74.0 | 30.87475  | -0.50492  |

TABLE V. Converted energy scale for TRM protocol at $T_m = 26$ K with $t_w = 3000$ s for various values of $H$ expressed in Oersteds. The $E_n$ are the magnitude of the nth fitting parameter (including the respective $H^n$) expressed in units of $k_B T_s$ (see Eq. (17)). The correlation length $\xi_{TRM} = 120(3)$ is derived from the coefficients of the $H^2$ term, utilizing Eqs. (8) and (9).

| $H$  | $E_0$     | $E_2$     |
|------|-----------|-----------|
| 22.0 | 31.83377  | -0.06386  |
| 32.0 | 31.83377  | -0.13512  |
| 47.0 | 31.83377  | -0.29148  |
| 59.0 | 31.83377  | -0.45932  |
| 67.0 | 31.83377  | -0.59233  |
| 74.0 | 31.83377  | -0.72256  |

TABLE VI. Converted energy scale for ZFC protocol at $T_m = 26$ K with $t_w = 10000$ s for various values of $H$ expressed in Oersteds. The $E_n$ are the magnitude of the nth fitting parameter (including the respective $H^n$) expressed in units of $k_B T_s$ (see Eq. (17)). The correlation length $\xi_{ZFC} = 135(1)$ is derived from the coefficients of the $H^2$ term, utilizing Eqs. (8) and (9).

| $H$  | $E_0$     | $E_2$     |
|------|-----------|-----------|
| 22.0 | 31.82558  | -0.05987  |
| 32.0 | 31.82558  | -0.11418  |
| 47.0 | 31.82558  | -0.24632  |
| 59.0 | 31.82558  | -0.38816  |
| 67.0 | 31.82558  | -0.50056  |
| 74.0 | 31.82558  | -0.61061  |

TABLE VII. Converted energy scale for TRM protocol at $T_m = 26$ K with $t_w = 3000$ s for various values of $H$ expressed in Oersteds. The $E_n$ are the magnitude of the nth fitting parameter (including the respective $H^n$) expressed in units of $k_B T_s$ (see Eq. (17)). The correlation length $\xi_{TRM} = 128(2)$ is derived from the coefficients of the $H^2$ term, utilizing Eqs. (8) and (9).

| $H$  | $E_0$     | $E_2$     |
|------|-----------|-----------|
| 10.0 | 33.55848  | -0.06004  |
| 16.0 | 33.55848  | -0.15371  |
| 22.0 | 33.55848  | -0.29061  |
| 24.9 | 33.55848  | -0.37228  |
| 27.5 | 33.55848  | -0.45408  |
| 29.8 | 33.55848  | -0.53321  |
| 32.0 | 33.55848  | -0.61485  |
| 36.3 | 33.55848  | -0.79119  |
| 40.2 | 33.55848  | -0.97033  |
| 43.7 | 33.55848  | -1.14665  |
| 47.0 | 33.55848  | -1.32637  |

TABLE VIII. Converted energy scale for the ZFC protocol at $T = 37.5$ K for $t_w = 2500$ s for various values of $H$ expressed in Oersteds. The $E_n$ are the magnitude of the nth fitting parameter (including the respective $H^n$) expressed in units of $k_B T_s$ (see Eq. (17)). The correlation length $\xi_{ZFC} = 220(20)$ is derived from the coefficients of the $H^2$ term, utilizing Eqs. (8) and (9).

Appendix D: A discussion of Eq. (31)

In a similar vein to Appendix A, we connect here the dynamics in the abstract ultrametric tree of states with the operational definition of the Hamming distance that we provided in Eq. (31). We rely on an exponential increase in the degeneracy of states with decreasing overlap $q_{\alpha q}$ from their respective initial state as a consequence of an underlying ultrametric topology of overlap space [23].

An initial state (at $t_w = 0$) will evolve into new states as time progresses. It is useful to define a distance, the Hamming distance, in terms of the overlap of states as time progresses. From [23], consider three states $\alpha$, $\beta$, and $\gamma$ and their overlaps $q_{\alpha \beta}$, $q_{\alpha \gamma}$ and $q_{\beta \gamma}$. Order them $q_{\alpha \beta} \geq q_{\alpha \gamma} \geq q_{\beta \gamma}$. The ultrametric topology of the pure states, which we assign to metastable states as well [63], results in $q_{\alpha \beta} \geq q_{\alpha \gamma} = q_{\beta \gamma}$.

Fig. 18 is a pictorial representation of this relationship. A tree is constructed in overlap space, with the level at $T_m$ representing the pure (metastable) states of the system at $T_m$. The states are grouped in such a way that the ultrametric topology is represented by the “branches” that connect the states. Consider the

Appendix C: Tables for Sect. IV

For the reader’s convenience we provide in Tables VIII—XI the numerical values used in the analysis reported in Sect. IV.
the distance along the level at $q$

TABLE IX. Converted energy scale for the TRM protocol

| $H$ | $E_0$ | $E_2$ | $E_4$ | $E_6$ |
|-----|-------|-------|-------|-------|
| 10.0 | 33.58718 | -0.05392 | 0.00043 | |
| 16.0 | 33.58718 | -0.13804 | 0.00284 | |
| 22.0 | 33.58718 | -0.26099 | 0.01014 | |
| 24.9 | 33.58718 | -0.33433 | 0.01664 | |
| 27.5 | 33.58718 | -0.40780 | 0.02175 | |
| 29.8 | 33.58718 | -0.47886 | 0.03413 | |
| 32.0 | 33.58718 | -0.52518 | 0.04058 | |
| 36.3 | 33.58718 | -0.71055 | 0.07515 | |
| 40.2 | 33.58718 | -0.87143 | 0.11303 | |
| 43.7 | 33.58718 | -1.02978 | 0.15784 | |
| 47.0 | 33.58718 | -1.19117 | 0.21120 | |

TABLE XI. Converted energy scale for the TRM protocol at $T_m = 37.5$ K for $t_w = 2500$ s for various values of $H$

| $H$ | $E_0$ | $E_2$ | $E_4$ | $E_6$ |
|-----|-------|-------|-------|-------|
| 10.0 | 34.07752 | -0.06364 | 0.00343 | -0.00400 |
| 16.0 | 34.07752 | -0.16292 | 0.00225 | -0.00400 |
| 22.0 | 34.07752 | -0.30801 | 0.00805 | -0.00003 |
| 24.9 | 34.07752 | -0.39457 | 0.01322 | -0.00007 |
| 27.5 | 34.07752 | -0.48127 | 0.01966 | -0.00112 |
| 29.8 | 34.07752 | -0.56514 | 0.02711 | -0.00019 |
| 32.0 | 34.07752 | -0.65166 | 0.03605 | -0.00030 |
| 36.3 | 34.07752 | -0.83856 | 0.05906 | -0.00063 |
| 40.2 | 34.07752 | -1.02843 | 0.08978 | -0.00116 |
| 43.7 | 34.07752 | -1.21530 | 0.12538 | -0.00192 |
| 47.0 | 34.07752 | -1.40578 | 0.16776 | -0.00297 |
| 50.3 | 34.07752 | -1.61012 | 0.22007 | -0.00446 |
| 53.3 | 34.07752 | -1.80791 | 0.27746 | -0.00631 |
| 56.2 | 34.07752 | -2.00999 | 0.34295 | -0.00867 |

first state (to the left) of the bottom level. Its overlap with itself, $q_{\alpha\alpha} \equiv q_{\alpha\alpha}$, and its value is a function of temperature as illustrated in Fig. 18. The next state, $\beta$, is connected to $\alpha$ by a branch to the first level above the bottom, diminishing $q$ and so on. We represent the distance along the level at $T_m$ by the Hamming distance, $H_d = \sum q_{\alpha\alpha} - q_{\alpha\alpha} \equiv (1/2)q_{\alpha\alpha} - q_{\alpha\alpha}$ where $\phi$ represents some state further along the bottom level.

We interpret the time evolution of the metastable states in terms of diffusion along a level of the tree from the initial state $\alpha$ to states with every diminishing overlap. The ultrametric geometry of the state space leads to an exponential increase in the number (degeneracy) of states encountered in time $t_w + t$, as the overlap $q_{\alpha\alpha}$ diminishes. We associate a barrier height $\Delta_{\alpha\alpha}$ proportional to a function of the reduction in overlap between states $\alpha$ and $\phi$, and hence to a function of $H_d$. Thus, at finite temperatures and times, $t_w + t$, there will be a maximum barrier overcome by the spin-glass system associated with a minimum overlap $q_{\alpha\alpha}$, essentially all of the decrease in the occupation of the initial state can be found in the occupation of the states lying at a Hamming distance associated with states of minimum overlap with state $\alpha$. This allows us to write Eq. (31).

To summarize, the Hamming distance, Eq. (31), expresses the transfer of the population of the states of the spin glass at $t = 0$, the time of the temperature quench to $T_m$, to the population of states with minimum overlap with those states at $t = t_w$ when the magnetic field $H$ is turned off (TRM) or on (ZFC).

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