Diagonalization of boundary transfer matrix
for the $U_{q,p}(\hat{sl}(3, \mathbb{C}))$ ABF model

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Abstract

We construct a free field realization of the ground state of the boundary transfer matrix
for the $U_{q,p}(\hat{sl}(3, \mathbb{C}))$ ABF model. Using this ground state and type-II vertex operator, we
have a diagonalization of the boundary transfer matrix.

1 Introduction

The vertex operator approach [1, 2] provides a powerful method to study solvable models. In [1] authors diagonalized the XXZ-transfer matrix on infinite spin chain by vertex operators. Sklyanin [3] begin systematic approach to boundary condition generalization in the framework of the algebraic Bethe ansatz. In periodic boundary condition, we construct a family of commuting transfer matrix from a solution of the Yang-Baxter equation. Sklyanin showed that similar construction is possible with the aid of a solution of the boundary Yang-Baxter equation. In open boundary condition, we construct a family of commuting transfer matrix from both a solution of the Yang-Baxter equation and a solution of the boundary Yang-Baxter equation.

the authors diagonalized the boundary XXZ-transfer matrix on semi-infinite spin chain by vertex operators. The vertex operator approach was extended to higher-rank boundary XXZ-model in [9].

The vertex operator approach to solvable model was originally formulated for vertex type model such as the XXZ-model [1], and then extended to the face type model such as the Andrews-Baxter-Forrester (ABF) model [4, 10]. The ABF model is described by the elliptic algebra $U_{q,p}(\hat{sl}(2, \mathbb{C}))$. The $U_{q,p}(\hat{sl}(3, \mathbb{C}))$ ABF model we are going to study in this paper were introduced in [5], as the higher-rank generalization of the ABF model introduced in [4]. In this paper we diagonalize the boundary transfer matrix for the $U_{q,p}(\hat{sl}(3, \mathbb{C}))$ ABF model by vertex operator approach. We construct a free field realization of the ground-state of the commuting boundary transfer matrix. Using this groundstate and type-II vertex operators for the elliptic algebra, we get a diagonalization of the boundary transfer matrix of the $U_{q,p}(\hat{sl}(3, \mathbb{C}))$ ABF model. The result of this paper gives a higher-rank generalization of the ABF model [7].

The text is organized as follows. In section 2 we recall the boundary $U_{q,p}(\hat{sl}(3, \mathbb{C}))$ ABF model, and introduce the boundary transfer matrix. In section 3 we recall the free field realization of the vertex operators. In section 4 we construct a free field realization of the ground-state of the boundary transfer matrix, and give a diagonalization of the boundary transfer matrix by using the type-II vertex operators.

## 2 Boundary $U_{q,p}(\hat{sl}(3, \mathbb{C}))$ ABF model

We recall the boundary $U_{q,p}(\hat{sl}(3, \mathbb{C}))$ ABF model.

### 2.1 Bulk Boltzmann weights

The $U_{q,p}(\hat{sl}(3, \mathbb{C}))$ ABF model has two parameters $x$ and $r$. We assume $0 < x < 1$ and $r \geq 5$ ($r \in \mathbb{Z}$). We set the elliptic theta function $[u]$ by

$$[u] = x^{\frac{x^2}{2}} u_2^{-\nu} \Theta_{x^2}(x^{2u}). \quad (2.1)$$

Here we have used

$$\Theta_q(z) = (q, q)_\infty(z; q)_\infty(q/z; q)_\infty, \quad (2.2)$$

$$\quad (z; q_1, q_2, \cdots, q_m)_\infty = \prod_{j_1, j_2, \cdots, j_m = 0}^{\infty}(1 - q_1^{j_1} q_2^{j_2} \cdots q_m^{j_m} z). \quad (2.3)$$
Let $\epsilon_\mu(1 \leq \mu \leq 3)$ be the orthonormal basis of $\mathbb{R}^3$ with the inner product $(\epsilon_\mu | \epsilon_\nu) = \delta_{\mu,\nu}$. Let us set $\bar{\epsilon}_\mu = \epsilon_\mu - \epsilon$ where $\epsilon = \frac{1}{3} \sum_{\nu=1}^{3} \epsilon_\nu$. The type $\widehat{sl}(3,\mathbb{C})$ weight lattice is the linear span of $\bar{\epsilon}_\mu$.

$$P = \sum_{\mu=1}^{3} \mathbb{Z}\bar{\epsilon}_\mu. \quad (2.4)$$

Note $\sum_{\mu=1}^{3} \bar{\epsilon}_\mu = 0$. Let the simple root $\alpha_\mu = \bar{\epsilon}_\mu - \bar{\epsilon}_{\mu+1}$ ($\mu = 1, 2$). For $a \in P$ we set

$$a_{\mu,\nu} = a_\mu - a_\nu, \quad a_\mu = (a + \rho | \bar{\epsilon}_\mu), \quad (2.5)$$

where $\rho = 2\bar{\epsilon}_1 + \bar{\epsilon}_2$. The Boltzmann weights $W\left(\begin{array}{cc} a & b \\ c & d \end{array} \middle| u \right)$ are given by

$$W\left(\begin{array}{cc} a + 2\bar{\epsilon}_\mu & a + \bar{\epsilon}_\mu \\ a + \bar{\epsilon}_\mu & a \end{array} \middle| u \right) = r_1(u), \quad (2.6)$$

$$W\left(\begin{array}{cc} a + \bar{\epsilon}_\mu & a + \bar{\epsilon}_\mu \\ a + \bar{\epsilon}_\nu & a \end{array} \middle| u \right) = r_1(u) \frac{|[a_{\mu,\nu} - 1]|}{|u - 1||a_{\mu,\nu}|}, \quad (2.7)$$

$$W\left(\begin{array}{cc} a + \bar{\epsilon}_\mu & a + \bar{\epsilon}_\nu \\ a + \bar{\epsilon}_\nu & a \end{array} \middle| u \right) = r_1(u) \frac{|u - a_{\mu,\nu}|}{|u - 1||a_{\mu,\nu}|}. \quad (2.8)$$

Otherwise are zero. The function $r_1(u)$ is given by

$$r_1(u) = z^{-\frac{1}{2}} \frac{h_1(z^{-1})}{h(z)}, \quad h_1(z) = \frac{(x^2 z; x^{2r} z^2; x^6)^\infty (x^{2r+1} z; x^{2r} z^6)^\infty}{(x^{2r+1} z; x^{2r} z^6)^\infty (x^6 z; x^{2r} z^6)^\infty}. \quad (2.9)$$

Here we have used $z = x^{2u}$. The Boltzmann weights satisfy the following relations.

1) Yang-Baxter equation:

$$\sum_g W\left(\begin{array}{cc} d & e \\ c & g \end{array} \middle| u_1 \right) W\left(\begin{array}{cc} c & g \\ b & a \end{array} \middle| u_2 \right) W\left(\begin{array}{cc} e & f \\ g & a \end{array} \middle| u_1 - u_2 \right) = \sum_g W\left(\begin{array}{cc} g & f \\ b & a \end{array} \middle| u_1 \right) W\left(\begin{array}{cc} d & e \\ g & f \end{array} \middle| u_2 \right) W\left(\begin{array}{cc} d & g \\ c & b \end{array} \middle| u_1 - u_2 \right). \quad (2.10)$$

2) The first inversion relation:

$$\sum_g W\left(\begin{array}{cc} c & g \\ b & a \end{array} \middle| -u \right) W\left(\begin{array}{cc} c & d \\ g & a \end{array} \middle| u \right) = \delta_{b,d}. \quad (2.11)$$

3) The second inversion relation:

$$\sum_g G_g W\left(\begin{array}{cc} g & b \\ d & c \end{array} \middle| 3-u \right) W\left(\begin{array}{cc} g & d \\ b & a \end{array} \middle| u \right) = \delta_{a,c} \frac{G_b G_d}{G_a}. \quad (2.12)$$

where $G_a = [a_{1,2}][a_{1,3}][a_{2,3}]$. The Boltzmann weights are related to the elliptic algebra $U_{q,p}(\widehat{sl}(3,\mathbb{C}))$. 

3
2.2 Boundary Boltzmann weights

In [8] the boundary Boltzmann weights $K \begin{pmatrix} a & b \\ c & u \end{pmatrix}$ are given by

$$K \begin{pmatrix} k + \epsilon_\mu \\ k \end{pmatrix} = z^{2(-\frac{\epsilon_1}{4} + \epsilon_{1|k})} \frac{\epsilon}{h(z)} \frac{[c - u][k_{1,\mu} + c + u]}{[c + u][k_{1,\mu} + c - u]}, \quad (c \in \mathbb{R}; \mu = 1, 2, 3). \quad (2.13)$$

Otherwise are zero. The function $h(z)$ is given in (4.18). They satisfy the boundary Yang-Baxter equation,

$$\sum_{f,g} W \begin{pmatrix} c & f \\ b & a \end{pmatrix} u - v \quad W \begin{pmatrix} c & d \\ f & g \end{pmatrix} u + v \quad K \begin{pmatrix} g & u \\ f & u \end{pmatrix} K \begin{pmatrix} e & v \\ d & g \end{pmatrix} = \sum_{f,g} W \begin{pmatrix} c & d \\ f & e \end{pmatrix} u - v \quad W \begin{pmatrix} c & f \\ b & g \end{pmatrix} u + v \quad K \begin{pmatrix} g & u \\ f & u \end{pmatrix} K \begin{pmatrix} e & v \\ d & g \end{pmatrix}. \quad (2.14)$$

2.3 Vertex operator

Following the general scheme of algebraic approach in solvable lattice models, we give the type-I vertex operators. Let us consider the corner transfer matrices $A(z), B(z), C(z), D(z)$ which represent NW, SW, SE, NE quadrants, respectively. The space $\mathcal{H}_{l,k}$ of the eigenvectors of $A(z)$ is parametrized by $l, k \in P$. Let us introduce the type-I vertex operator $\Phi^{(a,b)}(z)$. We denote by $\Phi^{(k+\epsilon_j,k)}(z^{-1})$ the half-infinite transfer matrix extending to infinity in the north. This is an operator

$$\Phi^{(k+\epsilon_j,k)}(z^{-1}) : \mathcal{H}_{l,k} \rightarrow \mathcal{H}_{l,k+\epsilon_j}.$$ 

The operator $\Phi^{(a,b)}(z) = 0$ for $a - b \neq \epsilon_j$. Similarly, we introduce dual type-I vertex operator $\Phi^{*^{(a,b)}}(z)$. We denote by $\Phi^{*(k,k+\epsilon_j)}(z)$ the half-infinite transfer matrix extending to infinity in the west. This is an operator

$$\Phi^{*(k,k+\epsilon_j)}(z) : \mathcal{H}_{l,k+\epsilon_j} \rightarrow \mathcal{H}_{l,k}.$$ 

The operator $\Phi^{(a,b)}(z) = 0$ for $b - a \neq \epsilon_j$. They satisfy the following relations.

(1) Commutation relation

$$\Phi^{(c,b)}(z_1)\Phi^{(b,a)}(z_2) = \sum_d W \begin{pmatrix} c & d \\ b & a \end{pmatrix} u_1 - u_2 \quad \Phi^{(c,d)}(z_2)\Phi^{(d,a)}(z_1). \quad (2.15)$$
(2) Inversion relation
\[ \sum_g \Phi^{*(a,g)}(z) \Phi^{(g,a)}(z) = 1, \quad \Phi^{(a,b)}(z) \Phi^{*(b,c)}(z) = \delta_{a,c}. \quad (2.16) \]
Later we give a free field realization of the vertex operators acting on the bosonic Fock space.

2.4 Boundary transfer matrix

We define the boundary transfer matrix
\[ T_B^{(k)}(z) = \sum_{j=1,2,3} \Phi^{*(k,k+\bar{\epsilon}_j)}(z) K \begin{pmatrix} k + \bar{\epsilon}_j & k \end{pmatrix} u \Phi^{(k+\bar{\epsilon}_j,k)}(z^{-1}). \quad (2.17) \]
The boundary Yang-Baxter equation implies the commutativity.
\[ [T_B^{(k)}(z_1), T_B^{(k)}(z_2)] = 0. \quad (2.18) \]
Our problem of this paper is to diagonalize the boundary transfer matrix \( T_B^{(k)}(z) \).

3 Free field realization

We give a free field realization of the vertex operators [11, 12].

3.1 Boson

We set the bosonic oscillators \( \beta_m^i, (i = 1, 2; m \in \mathbb{Z}) \) by
\[ [\beta_m^i, \beta_n^j] = \begin{cases} m [r-1]_x [2m]_x [3m]_x \delta_{m+n,0} & (j = k) \\ -m x^3 \exp(j-k) [r-1]_x [m]_x [3m]_x \delta_{m+n,0} & (j \neq k). \end{cases} \quad (3.1) \]
Here the symbol \( [a]_x = \frac{e^a - x^a}{x-1} \). Let us set \( \beta_m^3 \) by \( \sum_{j=1}^3 x^{-2jm} \beta_m^j = 0 \). The above commutation relations are valid for all \( 1 \leq j, k \leq 3 \). We also introduce the zero-mode operators \( P_\alpha, Q_\alpha, (\alpha \in P) \) by
\[ [iP_\alpha, Q_\beta] = (\alpha|\beta), \quad (\alpha, \beta \in P). \quad (3.2) \]
In what follows we deal with the bosonic Fock space \( \mathcal{F}_{l,k}(l, k \in P) \) generated by \( \beta_m^j(m > 0) \) over the vacuum vector \( |l, k\rangle \):
\[ \mathcal{F}_{l,k} = \mathbb{C}[\{\beta_{-1}^i, \beta_{-2}^j, \cdots \}_{j=1,2,3}]|l, k\rangle. \quad (3.3) \]
where
\[
\beta_m^j |k, l\rangle = 0, \quad (m > 0),
\]
\[
P_\alpha |l, k\rangle = \left( \alpha \left( \sqrt{\frac{r}{r - 1}} - \sqrt{\frac{r - 1}{r}} \right) \right) |l, k\rangle,
\]
\[
|l, k\rangle = e^{i \sqrt{\frac{r - 1}{r}} Q_l - \sqrt{\frac{r}{r - 1}} Q_k} |0, 0\rangle.
\]

### 3.2 Vertex operator

We give a free field realization of the type-I vertex operators [11], associated with the elliptic algebra \( U_{q,p}(\hat{sl}(3,\mathbb{C})) \) [13, 14]. Let us set \( P(z), Q(z), R_j^i(z), S_j^i(z) (j = 1, 2) \) by
\[
P(z) = \sum_{m>0} \frac{1}{m} \beta_{-m}^1 z^m, \quad Q(z) = - \sum_{m>0} \frac{1}{m} \beta_{-m}^1 z^{-m},
\]
\[
R_j^i(z) = - \sum_{m>0} \frac{1}{m} (\beta_{-m}^j - \beta_{-m}^{j+1}) x^{jm} z^m, \quad S_j^i(z) = \sum_{m>0} \frac{1}{m} (\beta_{-m}^j - \beta_{-m}^{j+1}) x^{-jm} z^{-m}.
\]

Let us set the basic operators \( U(z), F_{\alpha_1}(z), F_{\alpha_2}(z) \) on the Fock space \( \mathcal{F}_{l,k} \).
\[
U(z) = z^{r-1} e^{-i \sqrt{\frac{r - 1}{r}} Q_{\alpha_1} z - \sqrt{\frac{r}{r - 1}} P_{\alpha_1} e^{P(z)} e^{Q(z)}},
\]
\[
F_{\alpha_j}(z) = z^{r-1} e^{i \sqrt{\frac{r}{r - 1}} Q_{\alpha_j} z - \sqrt{\frac{r - 1}{r}} P_{\alpha_j} e^{R_j^i(z)} e^{S_j^i(z)}},
\]

In what follows we set
\[
\pi_{\mu} = \sqrt{r(r - 1)} P_{\mu}, \quad \pi_{\mu\nu} = \pi_{\mu} - \pi_{\nu}.
\]

Then \( \pi_{\mu\nu} \) acts on \( \mathcal{F}_{l,k} \) as an integer \((\epsilon_{\mu} - \epsilon_{\nu}) (rl - (r - 1)k)\). We give the free field realization of the type-I vertex operators.
\[
\Phi_1(z) = U(z),
\]
\[
\Phi_2(z) = \oint_{C_1} \frac{dw_1}{w_1} U(z) F_{\alpha_1}(w_1) \frac{[v_1 - u + \frac{1}{2} - \pi_{1,2}]}{[v_1 - u - \frac{1}{2}]},
\]
\[
\Phi_3(z) = \oint_{C_2} \oint_{C_2} \frac{dw_1}{w_1} \frac{dw_2}{w_2} U(z) F_{\alpha_1}(w_1) F_{\alpha_2}(w_2) \frac{[v_1 - u + \frac{1}{2} - \pi_{1,3}]}{[v_1 - u - \frac{1}{2}]},
\]

Here we set \( z = x^{2u}, w_j = x^{2v_j} (j = 1, 2) \). We take the integration contours to be simple closed curves around the origin satisfying
\[
C_1: |z| < |w_1| < x^{|z|},
\]
\[
C_2: |z| < |w_1| < x^{-1}|z|, \quad x|w_1| < |w_2| < x^{-1}|w_1|.
\]
We identify $\Phi^{(k+\epsilon_j,k)}(z) = \Phi_j(z)$.

Let us introduce the type-II vertex operators [12], associated with the elliptic algebra $U_{q,p}(\hat{sl}(3,\mathbb{C}))$. The type-II vertex operators represents the excitations. Let us set $r^* = r - 1$. Let us set the elliptic theta function $[u]^*$ by

$$[u]^* = x^{u^2} \Theta_{x^2r^*}(x^{2u}). \quad (3.12)$$

Let us set $P^*(z), Q^*(z), R^j_+(z), S^j_+(z) (j = 1, 2)$ by

$$P^*(z) = -\sum_{m>0} \frac{[rm]_x}{m[r^*m]_x} \beta^1_m z^m, \quad Q^*(z) = \sum_{m>0} \frac{[rm]_x}{m[r^*m]_x} \beta^{-1}_m z^{-m}, \quad (3.13)$$

$$R^j_+(z) = \sum_{m>0} \frac{[rm]_x}{m[r^*m]_x} (\beta^j_m - \beta^{-j+1}_m) x^{jm} z^m, \quad S^j_+(z) = -\sum_{m>0} \frac{[rm]_x}{m[r^*m]_x} (\beta^j_m - \beta^{-j+1}_m) x^{-jm} z^{-m}. \quad (3.14)$$

Let us set the basic operators $V(z), E_{a_1}(z), E_{a_2}(z)$ on the Fock space $\mathcal{F}_{l,k}$.

$$V(z) = z^{\sum l_i} e^{i\sum Q_{l_1} z \sqrt{\sum P_{l_1}}} e^{P^*(z)} e^{Q^*(z)}, \quad (3.15)$$

$$F_{a_j}(z) = z^{\sum l_i} e^{-i\sum Q_{l_1} z \sqrt{\sum P_{l_1}}} e^{P^*(z)} e^{S^j_+(z)}. \quad (3.16)$$

We give a free field realization of the type-II vertex operators.

$$\Psi^1(z) = V(z), \quad (3.17)$$

$$\Psi^2_2(z) = \oint \frac{dw_1}{w_1} V(z) E_{a_1}(w_1) \frac{[v_1 - u - \frac{1}{2} + \pi_{1,2}]^*}{[v_1 - u + \frac{1}{2}]^*}, \quad (3.18)$$

$$\Psi^3_3(z) = \oint \oint \frac{dw_1}{w_1} \frac{dw_2}{w_2} V(z) E_{a_1}(w_1) E_{a_2}(w_2) \frac{[v_1 - u - \frac{1}{2} + \pi_{1,3}]^* [v_2 - v_1 - \frac{1}{2} + \pi_{2,3}]^*}{[v_1 - u + \frac{1}{2}]^* [v_2 - v_1 + \frac{1}{2}]^*}. \quad (3.19)$$

The integration contours $C_1$ encloses the poles at $w_1 = x^{-1+2sr} z, (s = 0, 1, 2, \cdots)$, but not the poles at $w_1 = x^{1-2sr} z, (s = 0, 1, 2, \cdots)$. The integration of $\Psi^3_3(z)$ is carried out in the order $w_2, w_1$ along the contours $C_2$ which encloses the poles the poles at $w_{j+1} = x^{-1+2sr} w_j, (s = 0, 1, 2, \cdots ; j = 1, 2)$, but not the poles at $w_{j+1} = x^{1-2sr} w_j, (s = 0, 1, 2, \cdots ; j = 1, 2)$. Here we set $w_0 = z$. Let us set the type-II Boltzmann weights $W^* \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) u$ are given by

$$W^* \left( \begin{array}{cc} a + 2\epsilon_\mu & a + \bar{\epsilon}_\mu \\ a + \bar{\epsilon}_\mu & a \end{array} \right) u = r^*_1(u), \quad (3.20)$$

$$W^* \left( \begin{array}{cc} a + \epsilon_\mu + \epsilon_\nu & a + \bar{\epsilon}_\mu \\ a + \bar{\epsilon}_\nu & a \end{array} \right) u = r^*_1(u) \frac{[u]^*[a_{\mu,\nu} - 1]^*}{[u - 1]^*[a_{\mu,\nu}]^*}. \quad (3.21)$$
The type-I and type-II vertex operators commute modulo the "energy function" $G$ where

\[ W^*(a + \bar{e}_{\mu} + \bar{e}_{\nu} \mid a + \bar{e}_{\nu}) = \frac{r_1^*(u)[u - a_{\mu,\nu}]^*[1]^*}{[u - 1]^*[a_{\mu,\nu}]^*}. \] (3.22)

Otherwise are zero. The function $r_1^*(u)$ is given by

\[ r_1^*(u) = \frac{z^2 \bar{h}_1^*(z^{-1})}{h_1^*(z)}, \quad h_1^*(z) = \frac{(z; x^{2\nu}, x^{6})_{\infty}(x^4 z; x^{2\nu}, x^{6})_{\infty}}{(x^{2\nu}; z; x^{2\nu}, x^{6})_{\infty}(x^6 z; x^{2\nu}, x^{6})_{\infty}}. \] (3.23)

The Boltzmann weights satisfy the following relations.

1. Yang-Baxter equation:

\[ \sum_g W^*(d \mid c \mid e \mid g \mid u_1) W^*(c \mid b \mid a \mid g \mid u_2) W^*(e \mid a \mid f \mid g \mid u_1 - u_2) = \sum_g W^*(g \mid b \mid a \mid u_1) W^*(d \mid c \mid f \mid g \mid u_2) W^*(d \mid a \mid g \mid c \mid u_1 - u_2). \] (3.24)

2. The first inversion relation:

\[ \sum_g W^*(c \mid g \mid b \mid a \mid -u) W^*(c \mid d \mid g \mid a \mid u) = \delta_{b,d}. \] (3.25)

3. The second inversion relation:

\[ \sum_g G_g^* W^*(g \mid b \mid d \mid c \mid 3 - u) W^*(g \mid d \mid b \mid a \mid u) = \delta_{a,c} \frac{G_d^* G_g^*}{G_a^*}. \] (3.26)

where $G_g^* = [a_{1,2}]^*[a_{1,3}]^*[a_{2,3}]^*$. The type-II vertex operators satisfy the following relations.

\[ \Phi_{\mu_1}^*(z_1) \Phi_{\mu_2}^*(z_2) = \sum_{\epsilon_{\mu_1'} + \epsilon_{\mu_2'} = \epsilon_{\mu_1} + \epsilon_{\mu_2}} W^*(k + \bar{e}_{\mu_1} + \bar{e}_{\mu_2} \mid k + \bar{e}_{\mu_1'} \mid k + \bar{e}_{\mu_2'} \mid k + \bar{e}_{\mu_2} \mid u_2 - u_1) \Phi_{\mu_2'}^*(z_2) \Phi_{\mu_1'}^*(z_1). \] (3.27)

The type-I and type-II vertex operators commute modulo the "energy function" $\chi(z)$.

\[ \Phi_{\mu_1}(z_1) \Psi_{\mu_2}^*(z_2) = \chi(z_1/z_2) \Psi_{\mu_2}(z_2) \Phi_{\mu_1}(z_1). \] (3.28)

Here we have set

\[ \chi(z) = \frac{z^2}{(x^{-3}; x^6)_{\infty}(x^5 z; x^6)_{\infty}}. \] (3.29)
4 Boundary state

In this section we construct the free field realization of the ground state of the boundary transfer matrix of the $U_{q,p}(\mathfrak{sl}(3, \mathbb{C}))$ ABF model, which give higher-rank generalization of [7]. We construct the free field realization of the ground state on the Fock space $\mathcal{F}_{l,k}$. Precisely the Fock space $\mathcal{F}_{l,k}$ is different from the space of the state $\mathcal{H}_{l,k}$. The space of state $\mathcal{H}_{l,k}$ is obtained by the cohomological argument from the Fock space $\mathcal{F}_{l,k}$ by using the so-called screening operators. In this paper we omit the detailed of the screening operators and consider the operators acting on the Fock space $\mathcal{F}_{l,k}$.

4.1 Bogoliubov transformation

The commutation relations of bosons $\hat{\beta}_m^j$ are not symmetric. Hence it is convenient to introduce new generators of bosons $\alpha_m^1, \alpha_m^2$ whose commutation relations are symmetric.

$$\alpha_m^1 = x^{-m}(\beta_m^1 - \beta_m^2), \quad \alpha_m^2 = x^{-2m}(\beta_m^2 - \beta_m^3).$$

They satisfy the following commutation relations.

$$[\alpha_m^j, \alpha_n^k] = \left\{ \begin{array}{ll}
m \frac{[r-1]m}{[r]m} \frac{[2m]_x}{[m]_x} \delta_{m+n,0} & (j = k) \\
-m \frac{[r-1]m}{[r]m} \delta_{m+n,0} & (j \neq k).
\end{array} \right.$$  \hspace{1cm} (4.2)

Let us set

$$F_0 = -\sum_{m>0} \frac{1}{m} \frac{[r]m}{[r-1]m} \frac{[2m]_x}{[3m]_x} (2m) \alpha_m^1 \alpha_m^1 + 2m \alpha_m^1 \alpha_m^2 + 2m \alpha_m^2 \alpha_m^2).$$  \hspace{1cm} (4.3)

The adjoint action of $e^{F_0}$ has the effect of a Bogoliubov transformation,

$$e^{-F_0} \alpha_m^j e^{F_0} = \alpha_m^j - \beta_m^j \quad (m > 0, j = 1, 2),$$

and

$$e^{-F_0} \beta_m^j e^{F_0} = -\beta_m^j \quad (m > 0),$$

$$e^{-F_0} \beta_m^j e^{F_0} = (x^{2m-1}) \beta_m^j - x^{2m} \beta_m^j \quad (m > 0),$$

$$e^{-F_0} \beta_m^j e^{F_0} = \beta_m^j \quad (m > 0, j = 1, 2).$$

Proposition 4.1

$$e^{-F_0} e^{Q(z)} e^{F_0} = \sqrt{(x^{2r+4z-2}; x^{2r}, x^6)_\infty (x^{2r}, x^6)_\infty (x^{2r-2}; x^{2r}, x^6)_\infty} e^{P(1/z) e^{Q(z)}},$$  \hspace{1cm} (4.9)

$$e^{-F_0} e^{S_j(z)} e^{F_0} = \sqrt{(1 - z^{-2}) (x^{2r-2} z^{-2}; x^{2r})_\infty} e^{R_j(1/z) e^{S_j(z)}} \quad (j = 1, 2).$$  \hspace{1cm} (4.10)
4.2 Boundary state

Let us set the bosonic operator $F$ by

$$F = F_0 + F_1,$$  \hspace{1cm} (4.11)

where $F_0$ is given in (4.3) and $F_1$ is given by

$$F_1 = \sum_{m>0} (D_1(m)\beta^+_{-m} + D_2(m)\beta^-_{-m}).$$  \hspace{1cm} (4.12)

Here we set

$$D_1(m) = \frac{x^{-m}[(r - 2c + 2 - 2\pi_{1,3})m]_x - [(r - 2c - 1)m]_x + x^{(r - 2c - 2\pi_{1,2})m} [m]_x}{m[(r - 1)m]_x},$$

$$D_2(m) = \frac{x^m([(r - 2c + 2 - 2\pi_{1,3})m]_x - [(r - 2c - 2\pi_{1,2})m]_x)}{m[(r - 1)m]_x} - \theta_m \left( \frac{x^m [m/2]_x^+] [m/2]_x^-}{m[(r - 1)m]_x} \right),$$

where

$$[a]_x^+ = ax + a^{-x}, \quad \theta_m(x) = \begin{cases} x & (m : even) \\ 0 & (m : odd) \end{cases}.$$

Let us set the vector

$$|B\rangle_{l,k} = e^F |l, k\rangle.$$  \hspace{1cm} (4.15)

We call this vector $|B\rangle_{l,k}$ the boundary state.

**Proposition 4.2** The boundary state $|B\rangle_{l,k}$ has the following properties.

$$e^{Q(z)} |B\rangle_{l,k} = h(z) e^{P(1/z)} |B\rangle_{l,k},$$  \hspace{1cm} (4.16)

$$e^{S_j(w)} |B\rangle_{l,k} = g_j(w) e^{R_j(1/w)} |B\rangle_{l,k}, \quad (j = 1, 2).$$  \hspace{1cm} (4.17)

Here the functions $h(z), g_j(w)(j = 1, 2)$ are given by

$$h(z) = \frac{(x^{2r+4} z^{-2}; x^{2r}, x^{12})_{\infty} (x^{8} z^{-2}; x^{2r}, x^{12})_{\infty}}{(x^{12} z^{-2}; x^{2r}, x^{12})_{\infty} (x^{2r} z^{-2}; x^{2r}, x^{12})_{\infty}} \times \frac{(x^{2r+6-2c+2\pi_{1,2}} z^{-1}; x^{2r}, x^{12})_{\infty} (x^{2c+2} z^{-1}; x^{2r}, x^{12})_{\infty}}{(x^{2r-2c+2\pi_{1,2}} z^{-1}; x^{2r}, x^{12})_{\infty} (x^{2r+6} z^{-1}; x^{2r}, x^{12})_{\infty}} \times$$

$$\times \frac{(x^{2r+4-2c-2\pi_{1,2}} z^{-1}; x^{2r}, x^{12})_{\infty} (x^{2c+2+2\pi_{1,2}} z^{-1}; x^{2r}, x^{12})_{\infty}}{(x^{2r+4-2c-2\pi_{1,2}} z^{-1}; x^{2r}, x^{12})_{\infty} (x^{2c+2+2\pi_{1,2}} z^{-1}; x^{2r}, x^{12})_{\infty}}.$$
Theorem 4.3

The boundary state

\[
\varphi \in (x^{2r+6-2c-2\pi_{1,3} z^{-1}}; x^{2r}, x^{12})_\infty (x^{2c+2\pi_{1,3} z^{-1}}; x^{2r}, x^{12})_\infty,
\]
(4.18)

\[
g_1(w) = (1 - 1/w^2) (x^{2c+1} + 1/w; x^{2r})_\infty (x^{2r-2c-2\pi_{1,2}} + 1/w; x^{2r})_\infty,
\]
(4.19)

\[
g_2(w) = (1 - 1/w^2) (x^{2c+2\pi_{1,2}} + 1/w; x^{2r})_\infty (x^{2r-2c-2\pi_{1,3}} + 1/w; x^{2r})_\infty.
\]
(4.20)

The parameters \( l, k \in P \) are determined by the boundary conditions. The parameter \( k \) represents the central height. The parameter \( l \) represents the asymptotic boundary height. In what follows we consider the case \( k = l \in P \) for simplicity. For more general \( l, k \in P \), there exist similar boundary state. We construct the eigenvector of the commuting boundary transfer matrix, \([T_B^{(k)}(z_1), T_B^{(k)}(z_2)] = 0\). The following is the main result of this paper.

**Theorem 4.3**

The boundary state \( |B\rangle_{k,k} \) is the eigenvector of boundary transfer matrix \( T_B^{(k)}(z) \).

\[
T_B^{(k)}(z)|B\rangle_{k,k} = |B\rangle_{k,k}.
\]
(4.21)

**Corollary 4.4**

Using the type-II vertex operators, we get general eigenvectors of the boundary transfer matrix \( T_B^{(k)}(z) \).

\[
T_B^{(k)}(z) \cdot \Psi_{\mu_1}^* (\xi_1) \cdots \Psi_{\mu_M}^* (\xi_M) |B\rangle_{k,k} = \prod_{j=1}^M \chi(1/z\xi_j) \chi(\xi_j/z) \cdot \Psi_{\mu_1}^* (\xi_1) \cdots \Psi_{\mu_M}^* (\xi_M) |B\rangle_{k,k},
\]
(4.22)

where \( \chi(z) \) is given by (3.29).

Here we sketch proof of main theorem 4.3. Acting the vertex operator \( \Phi^{(k+\ell_j,k)}(z) \) to the condition \( T_B^{(k)}(z)|B\rangle_{k,k} = |B\rangle_{k,k} \) from the left, we get the following necessary and sufficient condition.

\[
z^{\frac{1}{2}}(-\tau^{-1} + (\ell_j 1)) h(z) [c - u] \tau_{1,j} + c + u] \Phi_j(z^{-1}) |B\rangle_{k,k} = (z \leftrightarrow z^{-1}) (j = 1, 2, 3).
\]
(4.23)

Here we used the inversion relation of the type-I vertex operators. In the case of \( j = 1 \), after some calculation, LHS becomes the following

\[
h(z)h(z^{-1}) [c - u] [c + u] e^{P(z) + P(1/z)} |B\rangle_{k,k},
\]
which is invariant $z \leftrightarrow z^{-1}$. Hence the relation (4.23) for $j = 1$ holds. In the case of $j = 2, 3$, after some calculation as similar as [9], the relation (4.23) are reduced to the following theta identity.

$$\frac{[v + k + c - \frac{1}{2}] [v - c - \frac{1}{2}]}{[-v + k + c - \frac{1}{2}] [-v - c - \frac{1}{2}]} = \frac{[c - u] [k + c + u] [u - v + \frac{1}{2} - k] [v - u + \frac{1}{2}] - [c + u] [k + c - u] [-u - v + \frac{1}{2} - k] [-v + u + \frac{1}{2}]}{[c - u] [k + c + u] [u + v + \frac{1}{2} - k] [v - u + \frac{1}{2}] - [c + u] [k + c - u] [-u + v + \frac{1}{2} - k] [v + u + \frac{1}{2}].}$$ (4.24)

Later the author will write complete proof of this theorem, $U_{q,p}(\hat{sl}(N, \mathbb{C}))$ version and generalization of asymptotic boundary condition $l \in P$ in the another place [15]. Our proof is different from those given in [7] even for $U_{q,p}(\hat{sl}(2, \mathbb{C}))$ case.

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**A Some formulae**

$$h(z) = \exp \left( - \sum_{m > 0} \frac{1}{2m} \left[ (r - 1) m \right]_{x} \left[ 2m \right]_{x} z^{-2m} - \sum_{m > 0} \left( \frac{2m}{3m} \right)_{x} \left( D_{1}(m) - \frac{x^{-3m}}{3m} D_{2}(m) \right) z^{-m} \right),$$

$$g_{1}(z) = \exp \left( - \sum_{m > 0} \frac{1}{2m} \left[ (r - 1) m \right]_{x} \left( x^{m} + x^{-m} \right) z^{-2m} + \sum_{m > 0} \left( \frac{2m}{3m} \right)_{x} (D_{1}(m) - x^{-2m} D_{2}(m) ) z^{-m} \right),$$

$$g_{2}(z) = \exp \left( - \sum_{m > 0} \frac{1}{2m} \left[ (r - 1) m \right]_{x} \left( x^{m} + x^{-m} \right) z^{-2m} + \sum_{m > 0} \left( \frac{2m}{3m} \right)_{x} D_{2}(m) x^{-m} z^{-m} \right).$$

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