Some Properties Associated with Clifford-Fourier Transform

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Abstract. Several useful properties of the Clifford-Fourier transform have been studied recently and many are being investigated at present. In this paper, we explore more properties of the Clifford-Fourier transform. We find that the properties are extensions of corresponding properties of the classical Fourier transform.

1. Introduction

In recent years, the work related the Clifford-Fourier transform (CFT) has grown rapidly. At first, the CFT was introduced by Brackx et al. \cite{1} who proposed to extend the classical Fourier transformation \cite{2} to Clifford analysis $\mathbb{C}l_{0,n}$. Several fundamental properties of this generalized transformation were studied. Further, the application of the CFT to vector fields and the behaviour of vector-valued filters has been investigated by Ehling and Scheuermann \cite{3}. As far, there are various types of the Clifford-Fourier transforms have been proposed by the researcher. In \cite{4, 5, 6, 7} the authors developed the CFT of the reference \cite{3} to higher dimensions and obtained fundamental properties like convolution, correlation and uncertainty principle. The CFT of this approach has used the authors \cite{8, 9, 10} to constructed windowed Fourier and wavelet transformations in the setting of Clifford algebra. The different approach of the CFT has been proposed the authors \cite{11}. Some results related to this CFT has been published in \cite{12, 13, 14}. Again in \cite{15} Hitzer has proposed the new type of the CFT which it can be considered as a general form of the double-sided quaternion Fourier transformation \cite{16, 17, 18}. Some important result related to new transformation such as convolution and correlation were investigated in detail.

The main purpose of this work is to investigate the several results of the CFT which do not have been published in the literature. To achieve this we first introduce the definition of the CFT. Some results related to the properties of the CFT kernel are presented. We then derive the duality property associated with the CFT. We finally show that under certain condition the Clifford function related to the CFT is continuous and bounded.

The remainder of the present paper is organized as follows. In Section 2, some preliminary results related to Clifford algebra are discussed. The results will be used in the sequel. In Section 3 we introduce the definition the CFT and obtain some results related to the CFT

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kernel. Section 4 derive in detail the Clifford function related to the CFT is continuous and bounded.

2. Clifford Algebra

Let \( \{e_1, e_2, e_3, \cdots, e_n\} \) be an orthonormal basis of the real \( n \)-dimensional space \( \mathbb{R}^{(r,t)} \) with \( r + t = n \). The real Clifford algebra (see [19, 20]) over \( \mathbb{R}^{(r,t)} \) is denoted by \( Cl(r,t) \) such that

\[
\{1, e_1, e_2, \cdots, e_n, e_1e_2, e_31, e_23, \cdots, e_1e_2 \cdots e_n\},
\]

where \( I \) is an unit oriented pseudoscalar. The product of the above basis vectors fulfills the following rules:

\[
e^k e^l = -e^l e^k \quad \text{for} \quad k \neq l, \quad 1 \leq k, l \leq n,
\]

\[
e^k e^k = 1 \quad \text{for} \quad 1 \leq k \leq r
\]

\[
e^k e^k = -1 \quad \text{for} \quad r + 1 \leq k \leq n.
\]

The elements of Clifford algebra are called multivectors. It means that every \( g \in Cl(r,t) \) may be expressed as

\[
g = \sum_C g^C e^C,
\]

where \( g^C \in \mathbb{R} \), \( e^C = e^{\alpha_1}e^{\alpha_2} \cdots e^{\alpha_k} = e_{\alpha_1}e_{\alpha_2} \cdots e_{\alpha_k} \), and \( 1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq n \) with \( \alpha_j \in \{1, 2, \cdots n\} \). For simplicity, we write \( \langle g \rangle_l = \sum_{|C|=l} f^C e^C \) to denote \( l \)-vector part of \( g \) \((l = 0, 1, 2, \cdots, n) \), then

\[
g = \sum_{l=0}^{n} \langle g \rangle_l = \langle g \rangle_0 + \langle g \rangle_1 + \langle g \rangle_2 + \cdots + \langle g \rangle_n,
\]

where \( \langle \ldots \rangle_0 = \langle \ldots \rangle \).

The reverse \( \bar{g} \) of a multivector \( g \) is given by

\[
\bar{g} = \sum_{l=0}^{n} (-1)^{(l-1)/2} \langle g \rangle_l,
\]

which satisfies \( \bar{g} h = \bar{h} \bar{g} \) for every \( g, h \in Cl(r,t) \).

The scalar product of multivectors \( f, \bar{g} \) is defined as the scalar part of the geometric product \( f \bar{g} \) of multivectors

\[
\langle \bar{g} h \rangle = g \star \bar{h} = \sum_C g^C h^C.
\]

Notice that if \( g = h \) in (6), then we get the modulus \( |g| \) of a multivector \( g \in Cl(r,t) \) given by

\[
|g|^2 = g \star \bar{g} = \sum_C g^C \bar{g}^C.
\]

It is not difficult to see that for \( g, h \in Cl(r,t) \) one can obtain

\[
|g h|^2 \leq 2^n |g|^2 |h|^2.
\]

**Definition 2.1.** A multivector \( g \in Cl(r,t) \) is called vectorial if it takes the form

\[
g = g_0 + e_1 g_1 + e_2 g_2 + \cdots + e_n g_n.
\]
We define the inner product for multivector functions $g, h : \mathbb{R}^{(r,t)} \rightarrow Cl_{(r,t)}$ as follows:

$$
(g, h) = \int_{\mathbb{R}^{(r,t)}} g(x) \overline{h(x)} \, d^n x
= \sum_{C,D} e_C e_D \int_{\mathbb{R}^{(r,t)}} g_C(x) h_D(x) \, d^n x,
\quad d^n x = dx_1 dx_2 \cdots dx_n.
$$

Thus for $g = h$ we get

$$
\|g\|^2 = \int_{\mathbb{R}^{(r,t)}} \sum_C g_C^2(x) \, d^n x.
$$

### 3. Clifford-Fourier Transform (CFT)

In what follows, we provide the definition of the Clifford-Fourier transform (CFT) and its inverse. We also demonstrate an important property of the CFT kernel.

**Definition 3.1.** Let $I \in Cl_{(r,t)}$ be a square root of $-1$ such that $I^2 = -1$. The CFT of $g \in L^1(\mathbb{R}^{(r,t)}; Cl_{(r,t)})$ is given by

$$
\mathcal{F}_{Cl}\{g\}(u) = \hat{g}(u) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{(r,t)}} g(x) e^{-I\nu(u,x)} \, d^n x,
$$

with $x, u \in \mathbb{R}^{(r,t)}$ and $\nu : \mathbb{R}^{(r,t)} \times \mathbb{R}^{(r,t)} \rightarrow \mathbb{R}$.

In the remainder of the paper, we always assume that

$$
v(u, x) = u_1 x_1 + u_2 x_2 + \cdots + u_n x_n.
$$

**Lemma 3.1.** [13] For any $g \in Cl_{(r,t)}$ and $I \in Cl_{(r,t)}$, one can get

$$
|e^{-I\nu(u,x)}| \leq (1 + |I|^2)^{\frac{1}{2}},
$$

and

$$
|ge^{-I\nu(u,x)}| \leq 2^n |g| (1 + |I|^2)^{\frac{1}{2}}.
$$

**Definition 3.2.** For arbitrary $g \in L^1(\mathbb{R}^{(r,t)}; Cl_{(r,t)})$ we define the inverse of the CFT as

$$
\mathcal{F}_{Cl}^{-1}\{g\}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{(r,t)}} g(u) e^{I\nu(u,x)} \, d^n u.
$$

**Theorem 3.2.** For $f \in L^1(\mathbb{R}^{(r,t)}; Cl_{(r,t)})$, it holds

$$
\mathcal{F}_{Cl}^{-1}[\mathcal{F}_{Cl}\{g\}](u) = g(u).
$$

### 4. New results for CFT

We first investigate a number of useful properties of the CFT, which can be regarded as extensions of the results of the classical Fourier transformation [2].

**Theorem 4.1.** For $h, g \in L^2(\mathbb{R}^{(r,t)}; Cl_{(r,t)})$, it holds that

$$
(\mathcal{F}_{Cl}\{h\}, g) = (h, \mathcal{F}_{Cl}^{-1}\{g\}).
$$
Proof. In fact, one has

\[
\langle \mathcal{F}_{Cl}\{h\}, g \rangle = \int_{\mathbb{R}^{(r,t)}} \mathcal{F}_{Cl}\{h\}(u)\overline{g(u)} \, du \\
= \int_{\mathbb{R}^{(r,t)}} \left[ \int_{\mathbb{R}^{(r,t)}} h(x)e^{-I\psi(u,x)} \, d^n x \right] \overline{g(u)} \, d^n u \\
= \int_{\mathbb{R}^{(r,t)}} \int_{\mathbb{R}^{(r,t)}} h(x)e^{-I\psi(u,x)} \overline{g(u)} \, d^n x \, d^n u \\
= \int_{\mathbb{R}^{(r,t)}} \int_{\mathbb{R}^{(r,t)}} h(x)\overline{g(u)}e^{I\psi(u,x)} \, d^n u \, d^n x \\
= \int_{\mathbb{R}^{(r,t)}} h(x)\mathcal{F}_{Cl}^{-1}\{g\}(x) \, d^n x \\
= \langle h, \mathcal{F}_{Cl}^{-1}\{g\} \rangle,
\]

which is the desired result. \(\square\)

**Definition 4.1.** Let \(\mathcal{F}_{Cl}\) be the Clifford-Fourier transformation. The adjoint of \(\mathcal{F}_{Cl}\) is denoted by \(\mathcal{F}_{Cl}^*\) and is defined by

\[
\langle \mathcal{F}_{Cl}\{h\}, g \rangle = (h, \mathcal{F}_{Cl}^*\{g\}). \tag{20}
\]

The following result demonstrates the relationship between adjoint of CFT and its inverse.

**Theorem 4.2.** Let \(h, g \in L^2(\mathbb{R}^{(r,t)}; Cl(r,t))\). The adjoint of the CFT is its inversion formula, that is,

\[
\langle \mathcal{F}_{Cl}\{h\}, g \rangle = (h, \mathcal{F}_{Cl}^{-1}\{g\}). \tag{21}
\]

*Proof. By combining (18) and (20) we can finish the proof.* \(\square\)

**Theorem 4.3** (Parseval’s formula for \(\mathcal{F}_{Cl}\)). If \(h, g \in L^2(\mathbb{R}^{(r,t)}; Cl(r,t))\), then the following is satisfied

\[
\langle \mathcal{F}_{Cl}^*\{h\}, \mathcal{F}_{Cl}^*\{g\} \rangle = (h, g). \tag{22}
\]

*Proof. By (21) we obtain

\[
\langle \mathcal{F}_{Cl}^*\{h\}, \mathcal{F}_{Cl}^*\{g\} \rangle = (\mathcal{F}_{Cl}^{-1}\{h\}, \mathcal{F}_{Cl}^{-1}\{g\}) \\
= \int_{\mathbb{R}^{(r,t)}} \mathcal{F}_{Cl}^{-1}\{h\}(x)\overline{\mathcal{F}_{Cl}^{-1}\{g\}(x)} \, d^n x \\
= \int_{\mathbb{R}^{(r,t)}} \left( \int_{\mathbb{R}^{(r,t)}} h(u)e^{I\psi(u,x)} \, d^n u \right) \overline{\mathcal{F}_{Cl}^{-1}\{g\}(x)} \, d^n x \\
= \int_{\mathbb{R}^{(r,t)}} \left( \int_{\mathbb{R}^{(r,t)}} \mathcal{F}_{Cl}\{h\}(u)e^{I\psi(u,x)} \, d^n u \right) \overline{\mathcal{F}_{Cl}^{-1}\{g\}(x)} \, d^n x \\
= \int_{\mathbb{R}^{(r,t)}} \mathcal{F}_{Cl}\{h\}(u) \left( \int_{\mathbb{R}^{(r,t)}} \overline{g(x)e^{I\psi(u,x)}} \, d^n x \right) \, d^n u \\
= \int_{\mathbb{R}^{(r,t)}} \mathcal{F}_{Cl}\{h\}(u)\overline{\mathcal{F}_{Cl}\{g\}(u)} \, d^n u \\
= \langle \mathcal{F}_{Cl}\{h\}, \mathcal{F}_{Cl}\{g\} \rangle \\
= (h, g),
\]

which gives the required result. \(\square\)
**Theorem 4.4.** Given \( h \in L^2(\mathbb{R}^{(r,t)}; Cl(\mathbb{R}^{r,t})) \) and \( g = F_{Cl}\{h\} \). If we assume that

\[
\langle h, F_{Cl}\{g\} \rangle = \langle F_{Cl}\{h\}, g \rangle.
\]  

(24)

Then we have

\[
h = F_{Cl}\{g\}.
\]

(25)

**Proof.** From the hypothesis of the theorem and the Parseval’s theorem for the CFT, we see that

\[
\langle h, F_{Cl}\{g\} \rangle = \langle F_{Cl}\{h\}, g \rangle = \|F_{Cl}\{h\}\|_2^2 = \|h\|^2.
\]

Applying Parseval’s formula results in

\[
\|F_{Cl}\{g\}\|_2^2 = \|g\|_2^2 = \|F_{Cl}\{h\}\|_2^2 = \|h\|_2^2.
\]

(26)

Therefore we get

\[
\|h - F_{Cl}\{g\}\|^2 = \langle h - F_{Cl}\{g\}, h - F_{Cl}\{g\} \rangle
\]

\[
= \|h\|^2 - \langle h, F_{Cl}\{g\} \rangle - \langle F_{Cl}\{g\}, h \rangle + \|F_{Cl}\{g\}\|^2
\]

\[
= \|h\|^2 - \|F_{Cl}\{h\}\|^2 - \|h\|^2 + \|F_{Cl}\{g\}\|^2
\]

\[
= 0.
\]

(27)

This proves the theorem.

**Theorem 4.5** (CFT duality). Let \( F_{Cl}\{g\} \) be a CFT of Clifford function \( g \). Then we get

\[
F_{Cl}\{\hat{g}(u)\} = g(-u).
\]

(29)

**Proof.** It directly follows from (12) that

\[
F_{Cl}\{\hat{g}(u)\} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{(r,t)}} F_{Cl}\{g\}(u) e^{-i\nu(u,x)} d^n x
\]

\[
= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{(r,t)}} F_{Cl}\{g\}(x) e^{-i\nu(u,x)} d^n u
\]

\[
= g(-u).
\]

(30)

Then one has

\[
F_{Cl}\{\hat{g}(-u)\} = g(u).
\]

(31)

Now observe that

\[
F_{Cl}\{F_{Cl}\{g(u)\}\} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{(r,t)}} g(-u) e^{-i\nu(u,x)} d^n u
\]

\[
= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{(r,t)}} g(z) e^{i\nu(z,x)} d^n z
\]

\[
= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{(r,t)}} g(t) e^{i\nu(t,u)} d^n t
\]

\[
= F_{Cl}\{g\}(-u).
\]

(32)

This is the desired result.
**Theorem 4.6** (Continuity). If $g \in L^1(\mathbb{R}^{(r,t)}; Cl\langle r,t \rangle)$ then $\mathcal{F}_{Cl}\{g\}$ is continuous and bounded on $\mathbb{R}^{(r,t)}$.

**Proof.** For every $\xi, u \in \mathbb{R}^{(r,t)}$ we obtain

$$|\mathcal{F}_{Cl}\{g\}(u) - \mathcal{F}_{Cl}\{g\}(u)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{(r,t)}} |g(x)e^{-Iv(u,x)}| \, d^n x$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{(r,t)}} g(x)e^{-Iv(u,x)} \, d^n x$$

$$\leq \left( \frac{2}{\pi} \right)^{\frac{n}{2}} (1 + |I|^2) \int_{\mathbb{R}^{(r,t)}} |g(x)| \, d^n x. \quad (33)$$

This gives $\mathcal{F}_{Cl}\{g\}$ is bounded. Next, using the CFT definition (12), we have

$$|\mathcal{F}_{Cl}\{g\}(u + \xi) - \mathcal{F}_{Cl}\{g\}(u)|$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{(r,t)}} |g(x)e^{-Iv(u+\xi,x)}| \, d^n x - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{(r,t)}} g(x)e^{-Iv(u,x)} \, d^n x$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{(r,t)}} g(x)e^{-Iv(u,x)} \, d^n x - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{(r,t)}} g(x)e^{-Iv(u,x)} \, d^n x$$

$$\leq \left( \frac{2}{\pi} \right)^{\frac{n}{2}} (1 + |I|^2) \int_{\mathbb{R}^{(r,t)}} |g(x)| \, d^n x. \quad (34)$$

Since $g(x)$ is integrable, then we can use the Lebesgue dominated convergence theorem to get

$$\lim_{\xi \to 0} |\mathcal{F}_{Cl}\{g\}(u + \xi) - \mathcal{F}_{Cl}\{g\}(u)| = 0,$$

which shows that $\mathcal{F}_{Cl}\{g\}$ is continuous on $\mathbb{R}^{(r,t)}$. \qed

**References**

[1] Brackx F, Delanghe R, and Sommen F 1982 *Clifford Analysis* 76 of Research Notes in Mathematics Pitman Advanced Publishing Program

[2] Bracewell R 2000 *The Fourier Transform and its Applications* McGraw-Hill Book Company

[3] Ebling J and Scheuermann G 2005 Clifford Fourier Transform on Vector Fields *IEEE Transactions on Visualization and Computer Graphics* 11 (4) 469-79

[4] Mawardi B and Hitzer E 2006 Clifford Fourier transformation and uncertainty principle for the Clifford geometric algebra $Cl_{n,0}$ *Advances in Applied Clifford Algebras* 16 (1) 41-61

[5] Hitzer E and Mawardi B 2006 Uncertainty principle for the Clifford geometric Algebra $Cl_{n,0}$, $n=3$(mod 4) based on Clifford Fourier transform in the *Springer (SCI) book series Applied and Numerical Harmonic Analysis* p 45-54

[6] Bahri M, Ashino R and Vailancourt R 2014 Convolution theorems for Clifford fourier transform and properties *Journal of the Indonesian Mathematical Society* 20 (2) 125-40

[7] Hitzer E and Mawardi B 2008 Clifford Fourier Transform on Multivector Fields and Uncertainty Principle for Dimensions $n=2$ (mod 4) and $n=3$ (mod 4) *Advances in Applied Clifford Algebras* 18 (3-4) 715-36
[8] Bahri M, Adji S and Zhao J 2011 Real Clifford windowed Fourier transform Acta Mathematica Sinica, English Series 27 (3) 505-18
[9] Bahri M, Adji S and Zhao J 2011 Clifford algebra-valued wavelet transform on multivector fields Advances in Applied Clifford Algebras 21 (1) 13-30
[10] Bahri M and Ashino R 2012 Two-dimensional quaternion Fourier transform of type II and quaternion wavelet transform 2012 International Conference on Wavelet Analysis and Pattern Recognition Xian China p 359-64
[11] Brackx F, Schepper N D, and Sommen F 2005 The Clifford-Fourier transform Journal of Fourier Analysis and Applications 11 (6) 669-81
[12] Rim J 2018 Heisenberg’s and Hardy’s uncertainty principles in real Clifford algebras Integral Transforms and Special Function 29 (8) 663-77
[13] Haoui Y and Fahlaoui S 2019 Donoho-Stark’s Uncertainty Principles in Real Clifford Algebras arXiv:1902.08465 [math.CA]
[14] Kamel J E and Rim J 2017 Uncertainty Principles for the Clifford-Fourier Transform Advances in Applied Clifford Algebras 27 (3) 2429-43
[15] Hitzer E 2017 General steerable two-sided Clifford Fourier transform, convolution and Mustard convolution Advances in Applied Clifford Algebras 27 (3) 2215-34
[16] Bahri M 2016 A modified uncertainty principle for two-sided quaternion Fourier transform Advances in Applied Clifford Algebras 26 (2) 513-27
[17] Bahri M and Ashino R 2017 A variation on uncertainty principle and logarithmic uncertainty principle for continuous quaternion wavelet transform Abstract and Applied Analysis 217 Article ID 3795120 11 pages
[18] Hitzer E 2017 Quaternion Fourier transform on quaternion fields and generalizations Advances in Applied Clifford Algebras 17 (3) 497-517
[19] Hestenes D 1986 New Foundations for Classical Mechanics D. Reidel Publishing Company
[20] Hestenes D and Sobczyk G 1984 Clifford Algebra to Geometric Calculus Kluwer Academic Publishers