Slow-roll inflation in $f(R, T, T_{ab})$ gravity

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Abstract

In the framework of $f(R, T, T_{ab})$ gravity theory, the slow-roll approximation of the cosmic inflation is investigated, where $T$ is the trace of the energy-momentum tensor $T_{ab}$, $R$ and $R_{ab}$ are the Ricci scalar and tensor, respectively. After obtaining the equations of motion of the gravitational field from the action principle in the spatially flat FLRW metric, the fundamental equations of this theory are received by introducing the inflation scalar field as the matter and taking into account only the minimum curvature-inflation coupling term. Remarkably, after taking the slow-roll approximation, the identical equations as in $f(R, T)$ gravity with a $RT$ mixing term are derived. Several potentials of interest in different domains are evaluated individually, calculating the slow-roll parameter and the e-folding number $N$. Finally, we analyze the behavior of the inflation scalar field under perturbation while ignoring the effect of metric perturbations. This research complements the slow-roll inflation in the modified theory of gravity.

1 Introduction

As a concise and elegant theory of gravity, general relativity (GR) has achieved significant success in the field of cosmology. ΛCDM theory in the framework of GR is regarded as the standard model of cosmology and consistent with numerous experimental results in the past decades [1]. However, some traditional cosmological issues, as well as the more refined observational problems brought about by the continuous progress of technological tools, especially the flatness problem and the horizon problem, were unable to find convincing answers within the preexisting framework [2]. A few decades ago, some authors [3, 4, 5, 6] developed techniques to try to address these problems by considering the inclusion of an exponential expansion phase in the early universe, called inflation. The most straightforward and popular method is to introduce a scalar field and consider the behavior of inflation at different potentials [7]. By imposing the slow-roll approximation, i.e., by appropriately neglecting the higher-order terms, it is usually possible to obtain succinct equations of motion. This idea is being examined [8, 9].

The GR has the smallest curvature-matter coupling as observed from the action perspective. The deviation of experimental observations from the theory has led to an interest in the dark sector, including dark matter and dark energy. Einstein’s cosmological constant can be considered as a pioneering attempt, and after that, the search for more complex possibilities has since become one of the main motivations to study various theories of gravity [10, 11]. The $f(R)$ gravity theory [12, 13] is a simple yet effective effort to achieve this by substituting the Ricci scalar in the action under GR with an arbitrary function of it. This model did not enhance the coupling between curvature and matter, which was subsequently remedied by the $f(R, T)$ gravity theory [14, 15, 16]. About a decade ago, $f(R, T, R_{ab}T^{ab})$ gravity theory [17, 18] was proposed, and this model apparently contains a stronger coupling between curvature and matter, further deepening our grasp of spacetime. In contrast to GR, when there is a non-minimal gravitational-matter coupling, the standard conservation of energy-momentum fails and is replaced by an equation containing a curvature part [19], which can be obtained by finding the covariance derivative of the field equation. This implies that there is an energy-momentum exchange between the matter and gravitational fields, which may give rise to interesting phenomena in the investigation of quantum gravity.

The slow-roll inflation in the $f(R)$ and $f(R, T)$ gravitational frames has been research done [20, 21]. Recently, some authors [22] explored the case in $f(R, T)$ gravity with non-minimal coupling, i.e., with a $RT$ mixing term. Inspired by it, we carry out the present work to examines a similar case under

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We start with the general action of $f(R, T, R_{ab}T^{ab})$ gravity. Starting from the action principle, after obtaining the equations of motion for the gravitational field and the inflation scalar field separately and taking the slow-roll approximation, we can obtain the same set of equations as in [22]. This implies that we can study more intricate physical situations without increasing the mathematical difficulty. Various inflation models have been proposed for different problems, which can be distinguished according to different potential functions, a few typical examples such as [23, 24, 25, 26, 27, 28]. We provide a brief introduction to the potentials under consideration and calculate their corresponding slow-roll parameters and e-folding number $N$, which are crucial for the comparison between theoretical results and experimental observations. In the end, we consider a simplified version of the cosmological perturbation theory, which ignores the scalar and tensor perturbations on the metric, and only considers a perturbation that varies with time and space on the spatially homogeneous and isotropic background scalar field, the dynamic equation of this disturbance field is obtained.

The article is organized as follows. In Sec. 2, we review the basic framework of $f(R, T, R_{ab}T^{ab})$ gravity theory, introduce the inflation scalar field, and settle on a particular functional form. In Sec. 3, we introduce the slow-roll approximation and derive the equations of the inflation scalar field in order to give some computational results. The behavior of the model under this approximation. We also carried out a transformation here from the Jordan to the Einstein framework, which led to the definition of the slow-roll parameter. In Sec. 4, we consider several potential functions of interest and give some basic computational results. The behavior of the inflation scalar field under perturbations is taken into consideration in Sec. 5. As a conclusion, Sec. 6 makes a summary. Some computational details are provided in Appendix A.

## 2 Framework of $f(R, T, R_{ab}T^{ab})$ gravity theory

We start with the general action of $f(R, T, R_{ab}T^{ab})$ gravity [17, 18]

$$\mathcal{S} = \mathcal{S}_G + \mathcal{S}_m = \int d^4x \sqrt{-g} \left( f(R, T, R_{ab}T^{ab}) + \int d^4x \sqrt{-g} \mathcal{L}_m, \right)$$

where $\kappa \equiv 8\pi G \equiv 1/M^2_{Pl}$ with $G$ being gravitational constant and $g$ is the determinant of the metric $g_{ab}$. $f$ is an arbitrary function of the Ricci scalar $R$, the trace of the energy-momentum tensor $T$ and the coupling term. $\mathcal{L}_m$ is the matter Lagrangian. We predetermine that the matter field is an inflation scalar field in order to give some computational results.

$$\mathcal{L}_m = -\frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi)$$

Then the energy-momentum tensor can be defined as the variational derivative of $\mathcal{L}_m$

$$T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{ab}} = g_{ab} \mathcal{L}_m - 2 \partial_a \phi \partial_b \phi + g_{ab} \mathcal{L}_m.$$

According to the action principle, equations of motion (EoM) of gravitational field are obtained by varying the action 1 with respect to the metric $g^{ab}$.

$$\delta S = \frac{1}{2\kappa} \int d^4x \left[ \delta(\sqrt{-g}) f(R, T, R_{ab}T^{ab}) + \sqrt{-g} \left( f_R \delta R + f_T \delta T + f_P \delta(R_{ab}T^{ab}) \right) \right] + \int d^4x \delta(\sqrt{-g} \mathcal{L}_m)$$

$R_{ab}T^{ab}$ is written as $P$ and the subscript of $f$ means derivative. The procedures for attributing each term to the variation of the metric $\delta g^{ab}$ are as follows.

$$\delta(\sqrt{-g}) = -\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab}$$

$$f_R \delta R + f_T \delta T = \left[ R_{ab} f_R + (g_{ab} \Box - \nabla_a \nabla_b) f_R + (T_{ab} + \Theta_{ab}) f_T \right] \delta g^{ab}$$

$$f_P \delta(R_{ab}T^{ab}) = f_P \left(T^{ab} \delta R_{ab} + R_{ab} \delta T^{ab}\right)$$

$$f_P T^{ab} \delta R_{ab} = \left[ \frac{1}{2} \Box (f_P T_{ab}) + \frac{1}{2} g_{ab} \nabla_c \nabla_d (f_P T^{cd}) - \nabla_c \nabla_b (f_P g^{cd} T_{ad}) \right] \delta g^{ab}$$

$$f_P R_{ab} \delta T^{ab} = f_P \left[ -G_{ab} \mathcal{L}_m - \frac{1}{2} \left( 2 R_{ab} T_{cd} - 2 R T_{ab} + 2 g^{cd} R_{da} T_{bc} - R_{cd} \right) \left( \frac{\delta^2 \mathcal{L}_m}{\delta g^{cd} \delta g^{ab}} \right) \right] \delta g^{ab} \equiv f_P \Xi_{ab} \delta g^{ab}$$
We define two auxiliary tensors in this process.

\[ \Theta_{ab} \equiv g^{cd} \frac{\delta T_{cd}}{\delta g^{ab}} = g_{ab} \mathcal{L}_m - 2T_{ab} - 2g^{cd} \frac{\delta \mathcal{L}_m}{\delta g^{ab} g^{cd}} = -\partial_a \phi \partial_b \phi - T_{ab} \tag{10} \]

\[ \Xi_{ab} \equiv -G_{ab} \mathcal{L}_m - \frac{1}{2} RT_{ab} + 2g^{cd} R_{d(ab)c} \tag{11} \]

It should be emphasized that, due to the symmetry of metric \( g_{ab} \), there are brackets in the equation that indicate symmetrization. The general form of the gravitational field equation can be produced by summarizing the aforementioned terms.

\[ R_{ab} f_R - \frac{1}{2} g_{ab} f + (g_{ab} \Box - \nabla_a \nabla_b) f_R + (T_{ab} + \Theta_{ab}) f_T + \frac{1}{2} \Box (T_{ab} f_T) + \frac{1}{2} g_{ab} \nabla_c \nabla_d (T^{cd} f_P) - \nabla_c \nabla_d (g^{cd} T_{b|d}) + \Xi_{ab} f_P = \kappa T_{ab} \tag{12} \]

The modified energy-momentum conservation equation can be derived by finding the covariance derivative of the field equation. We will derive it below for specific problems, but do not supply the generic form here.

In the rest part of the present paper, we will just consider the following functional form of \( f(R, T, R_{ab} T^{ab}) \), which is actually the minimal (linear) coupling form in this framework

\[ f(R, T, R_{ab} T^{ab}) = R(1 + \alpha) + \gamma \kappa T + 4\beta \kappa^2 R_{ab} T^{ab}, \quad f_R = 1 + \alpha, \quad f_T = \gamma \kappa, \quad f_P = 4\beta \kappa^2, \tag{13} \]

where \( \alpha, \gamma \) and \( \beta \) are dimensionless constant. Concomitantly, the field equations can be further simplified.

\[ R_{ab}(1 + \alpha) - \frac{1}{2} g_{ab} f + \gamma \kappa (T_{ab} + \Theta_{ab}) + 4\beta \kappa^2 \left[ \frac{1}{2} \Box T_{ab} + \frac{1}{2} g_{ab} \nabla_c \nabla_d T^{cd} - \nabla_c \nabla_d (g^{cd} T_{b|d}) + \Xi_{ab} \right] = \kappa T_{ab} \tag{14} \]

Since we haven’t yet qualified the form of the metric gauge, our analysis is applicable to a spatially non-flat universe. However, since observations seem to indicate that the universe is approximately flat on large scales, we only consider the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric

\[ d^2 s = -d^2 t + a(t)^2 (d^2 x + d^2 y + d^2 z), \tag{15} \]

where \( t \) is the cosmic time and \( a(t) \) is the scale factor. Usually define the Hubble function as \( H \equiv \dot{a}/a \) with \( \dot{\cdot} \) meaning the derivatives with respect to \( t \). Details of the computation of some tensor under this metric can be found in Appendix A. The two independent components of the EoMs are as follows

\[ H^2(1 + \alpha) = \kappa \frac{3}{2} \left[ \frac{\dot{\phi}^2}{2} (1 + \gamma + 18\beta \kappa H^2) + V(\phi) (1 + 2\gamma + 12\beta \kappa H^2) \right] + 4\beta \kappa^2 H \dot{\phi} V'(\phi), \tag{16} \]

\[ \dot{H} [1 + \alpha - 4\beta \kappa^2 V(\phi)] = -\kappa \frac{\dot{\phi}^2}{2} \left[ 1 + \gamma + 12\beta \kappa H^2 - 4\beta \kappa H \right] + 2\beta \kappa^2 \left[ \phi^2 V''(\phi) + \phi V'(\phi) + 2H \phi \ddot{\phi} - H \phi V'(\phi) \right]. \tag{17} \]

The equations of motion of the scalar field, i.e., the modified Klein-Gordon equation, can be obtained by finding the time derivatives of the gravitational field equations \( 16 \& 17 \) and substituting each other, or by finding the covariant derivatives of the general form of the gravitational field equations \( 12 \) or by varying the action \( 1 \) with respect to the scalar field.

\[ \phi [1 + \gamma + 12\beta \kappa H^2] + 3H \ddot{\phi} [1 + \gamma + 12\beta \kappa H^2 + 8\beta \kappa H] + V'(\phi) [1 + 2\gamma + 24\beta \kappa H^2 + 18\beta \kappa H] = 0 \tag{18} \]

\( 16, 17 \) and \( 18 \) are consistent with \( [21] \) when \( \alpha = \beta = 0 \) and naturally return to the case of GR when \( \alpha = \gamma = \beta = 0 \). They will serve as the fundamental foundation for the following research.

### 3 Slow-roll inflation

Considering the time-dependent quantities for quasi-static evolution, the following approximation can be taken:

\[ |\dot{\phi}| \ll |V(\phi)|, \quad |\ddot{\phi}| \ll |H \dot{\phi}|, \quad |\dot{H}| \ll |H \ddot{H}| \ll |H^3|. \tag{19} \]
Immediately thereafter, 16, 17 and 18 can be approximated as

\[ H^2(1 + \alpha) = \frac{k}{3} V(\phi)(1 + 2\gamma + 12\beta \kappa H^2) = 0, \]  
\[ \dot{H} (1 + \alpha - 4\beta \kappa^2 V(\phi)) + k \frac{\phi^2}{2} (1 + \gamma + 12\beta \kappa H^2) + 2\beta \kappa^2 \dot{H} \dot{\phi} V'(\phi) = 0, \]  
\[ 3H \dot{\phi} (1 + \gamma + 12\beta \kappa H^2) + V'(\phi)(1 + 2\gamma + 24\beta \kappa H^2) = 0. \]  

We are surprised to find that this set of equations has the same results as in [22]. Also, the non-approximate equations in [22] have terms with higher order. This can be explained by the fact that \( RT \) in [22] involves the cross terms of the temporal and spatial components of \( R_{ab} \) and \( T^{ab} \), which are not present in \( R_{ab}T^{ab} \) in our action 1. These cross terms suggest a more intricate interchange of energy-momentum between the matter and gravitational fields, which results in a higher order. This consistency suggests that the analysis performed by [22] that we are about to extend can be applied equally to the action

\[ S = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} [R(1 + \alpha) + \gamma \kappa T + 4\beta \kappa^2 R_{ab}T^{ab} + \beta_2 \kappa^2 RT] + \int d^4 x \sqrt{-g} \mathcal{L}_m, \beta_1 + \beta_2 \equiv \beta. \]  

A common step in the study of modified gravity theories is to make the equations under study have the same form as in the standard GR, called the Einstein frame, by means of metric conformal transformations and field redefinitions. This is just a mathematical trick [29], but can facilitate our finding the observable measurements of great interest to observational cosmology, since their form in GR is familiar. Although in general the strict Einstein frame is not solvable in the presence of non-minimal couplings, the slow-roll approximation helps us to solve this obstacle. The first step in the procedure is to define three helper functions as follows

\[ \tilde{g}_{ab} \equiv \Omega_1(\phi)g_{ab}, \text{i.e.} \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = -d\tilde{t}^2 + \tilde{a}(\tilde{t})^2 d\tilde{x}_i d\tilde{x}^i \equiv \Omega_1(\phi)(-dt^2 + a(t)^2 dx_i dx^i), \]  
\[ \Omega_2(\phi) \equiv \left( \frac{d\chi}{d\phi} \right)^2, \]  
\[ \Omega_3(\phi) \equiv \frac{\tilde{V}(\chi(\phi))}{V(\phi)}. \]  

The expected field equation takes the form

\[ 3\dot{\tilde{H}} \frac{d\chi}{dt} + \tilde{V}'(\chi) = 0, \]  
\[ \dot{\tilde{H}}^2 - \frac{k}{3} \tilde{V}(\chi) = 0, \]  
\[ \frac{d\dot{\tilde{H}}}{dt} + \frac{k}{2} \left( \frac{d\chi}{dt} \right)^2 = 0. \]  

These are exactly the slow-roll equations in GR. Substituting 24, 25, 26 into 16, 17, 18 and using care 27, 28, 29, we can invert the solution to 24, 25, 26 as follows

\[ \Omega_1 = 1 + \alpha - 4\beta \kappa^2 V(\phi), \]  
\[ \Omega_2 = \frac{(1 + \alpha)(1 + \gamma) + 4\beta \gamma \kappa^2 V(\phi)}{(1 + \alpha - 4\beta \kappa^2 V(\phi))^2} = \frac{1 + \gamma}{1 + \alpha} + \mathcal{O}(\beta), \]  
\[ \Omega_3 = \frac{1 + 2\gamma}{(1 + \alpha - 4\beta \kappa^2 V(\phi))^2} = \frac{1 + 2\gamma}{(1 + \alpha)^2} + \mathcal{O}(\beta). \]  

For the potential function to be considered, finding the exact solution is difficult, so we will take the approximate transformation

\[ \chi(t) = \sqrt{\frac{1 + \gamma}{1 + \alpha}} \phi(t), \quad \tilde{V}(\chi) = \frac{1 + 2\gamma}{(1 + \alpha)^2} V\left( \sqrt{\frac{1 + \alpha}{1 + \gamma}} \chi \right). \]
In actuality, we can now recast the action 23 as a minimal coupling form of standard GR and a scalar field.

\[
\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left( \frac{\tilde{R}}{2\kappa} - \frac{1}{2} \tilde{g}^{ab} \nabla_a \chi \nabla_b \chi - \tilde{V}(\chi) \right)
\]  

(34)

In standard GR, considering the last equation in 19, we can define

\[
\epsilon = -\frac{\ddot{H}}{H^2} = \frac{1}{2\kappa} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 \equiv \epsilon_V
\]

(36)

\[
\eta = -\frac{\dddot{H}}{2H\dot{H}} = \frac{1}{\kappa} \left( \frac{V''(\phi)}{V(\phi)} \right) - \frac{1}{2\kappa} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 \equiv \eta_V - \epsilon_V
\]

(37)

For the case of \( f(R, T, R_{ab}T^{ab}) \), we directly define

\[
\epsilon_{\tilde{V}} = \frac{1}{2\kappa} \left( \frac{\tilde{V}'(\chi)}{\tilde{V}(\chi)} \right)^2, \quad \eta_{\tilde{V}} = \frac{1}{\kappa} \left( \frac{\tilde{V}''(\chi)}{\tilde{V}(\chi)} \right).
\]

(38)

In principle, it is still possible to start with the definition 35 and obtain parameters solely connected to the potential. However, the process is mathematically complex, and arriving at a usable expression requires a more tedious approximation. Additionally, there are derived quantities that are of interest in light of astronomical observations [30].

\[
n_S = 1 - 6\epsilon_{\tilde{V}} + 2\eta_{\tilde{V}}, \quad n_T = -2\epsilon_{\tilde{V}}, \quad r_+ = 16\epsilon_{\tilde{V}}.
\]

(39)

where \( n_S \) and \( n_T \) are the scalar and tensor spectral index, respectively and \( r_+ \) is the tensor-to-scalar ratio. The e-folding number \( N \), defined by \( N = \ln \left( \frac{a(t_{\text{end}})}{a(t)} \right) \), can be transcribed as

\[
N = \int_{\tilde{t}_i}^{\tilde{t}_{\text{end}}} \tilde{H}(\tilde{t}) \, d\tilde{t} = \int_{\chi}^{\chi_{\text{end}}} \frac{H}{d\chi / dt} \, d\chi = \kappa \int_{\chi_{\text{end}}}^{\chi} \frac{\tilde{V}(\chi)}{\tilde{V}'(\chi)} \, d\chi,
\]

(40)

which reflects the degree of spatio-temporal expansion. At this point, our theory has been constructed. The difference between the various models lies in the difference of potential functions. In the next section, we will discuss various forms of potential functions, which are of interest in different domains, and analyze the implications for slow-rolling inflation.

4 Infaltionary models in \( f(R, T, R_{ab}T^{ab}) \) gravity

4.1 Starobinsky inflation

Alexei Starobinsky from the Soviet Union noted that quantum corrections to GR should be significant for the early universe. These generally result in curvature-squared corrections to the Einstein–Hilbert action and a form of \( f(R) \) modified gravity. When the curvatures are enormous, the solution to Einstein’s equations with curvature-squared terms yields an effective cosmological constant. Therefore, he suggested an inflationary de Sitter era occurred in the early universe[31]. This resolved the cosmological issues and produced precise estimates for the microwave background radiation corrections, which were later meticulously calculated. Starobinsky originally used the semi-classical Einstein equations with free quantum matter fields[3]. However, it soon became clear that the inflation was mostly managed by the effective action’s squared Ricci scalar contribution [32]

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left( R + \frac{R^2}{6M^2} \right),
\]

(41)
where \( M \) is a dimensioned constant. Although this action does not contain a scalar field, in the Einstein framework it corresponds to a coupling of curvature and scalar field [33, 34, 35]

\[
S = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa} - \frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \right), \quad V(\phi) = \frac{3M^2}{4\kappa} \left[ 1 - e^{-\sqrt{\frac{2}{3}} \frac{\phi}{M_{Pl}}} \right]^2,
\]

and \( M = 1.13 \times 10^{-5} m_{Pl} \) is required from the scalar fluctuation analysis [36]. In fact, Higgs inflation with the action

\[
S = \int d^4x \sqrt{-g} \left[ -\frac{M^2 + \xi h^2}{2} R + \frac{\partial^\mu \partial^\nu}{2} - \frac{\lambda}{4} (h^2 - v^2)^2 \right]
\]

(43)
can also be reduced to the form of a scalar field coupled to curvature controlled by a potential of this form [28]. The slow roll parameters can be derived by using the already-obtained approaches

\[
\epsilon_\Phi = \frac{4}{3} \left[ \exp \left( \sqrt{\frac{2}{3}} \frac{\chi}{M_{Pl}} \right) - 1 \right]^2, \quad \eta_\Phi = -\frac{4}{3} \frac{\exp \left( \sqrt{\frac{2}{3}} \frac{\chi}{M_{Pl}} \right) - 2}{\exp \left( \sqrt{\frac{2}{3}} \frac{\chi}{M_{Pl}} \right) - 1^2}, \quad M_{Pl} = \sqrt{\frac{1 + \gamma}{1 + \alpha} M_{Pl}}.
\]

(44)

Further, we can obtain the constraints required by the slow Roll parameter and the e-folding number

\[
\epsilon_\Phi \leq 1, \quad \eta_\Phi \leq 1 \Rightarrow \chi \geq \chi_{end} \equiv \frac{\sqrt{6}}{2} \ln \left[ 1 + \frac{2}{\sqrt{3}} \right] M_{Pl},
\]

(45)

\[
N = \frac{1}{4} \left[ 3e^{\sqrt{\frac{2}{3}} \frac{\chi}{M_{Pl}}} - \sqrt{6} \frac{\chi}{M_{Pl}} \right] - \left( 3 + 2\sqrt{3} - 3 \ln \left[ 1 + \frac{2}{\sqrt{3}} \right] \right).
\]

(46)

However, inflation exhibits a slightly more complex tendency under the appropriate initial conditions. For a discussion of the initial conditions of inflation with Starobinsky and more potential, see [37]. If additional estimates are taken into account, one can take

\[
N = \frac{1}{4} \left[ 3e^{\sqrt{\frac{2}{3}} \frac{\chi}{M_{Pl}}} - \left( 3 + 2\sqrt{3} \right) \right], \quad \frac{\chi}{M_{Pl}} = \sqrt{\frac{3}{2} \ln \left( \frac{4}{3} N + 1 + \frac{2}{\sqrt{3}} \right)}.
\]

(47)

For large \( N \),

\[
\epsilon_\Phi = \frac{12}{(3 - 4N)^2} \approx \frac{3}{4N^2}, \quad \eta_\Phi = \frac{8(3 - 2N)}{(3 - 4N)^2} \approx -\frac{1}{N}.
\]

(48)

This finding is consistent with that in the standard GR [21] because we operate within the Einstein framework and all terms related to \( \beta \) are discarded in the small amount approximation.

### 4.2 Chaotic models with power-law potentials

Power-law potential has a concise form

\[
V(\phi) = \lambda M_{Pl}^4 \left( \frac{\phi}{M_{Pl}} \right)^n, \quad n > 0,
\]

(49)
where \( \lambda, n \) are dimensionless constants. The case of \( n = 4 \) was first presented in [23]. The slow-roll parameters are calculated as follows

\[
\epsilon_\psi = \frac{n^2}{2\kappa^2}, \quad \eta_\psi = \frac{n(n-1)}{\kappa^2}.
\]  

(50)

We find that the results are identical to those in the standard GR and do not exhibit the correction generated by \( f(R, T, R_{ab}T^{ab}) \) gravity. This is because we ignore the higher order terms about \( \beta \) in [33]. Discussing the values of \( n \), we get

\[
\chi_{\text{end}} = \begin{cases} \frac{n}{\sqrt{2\kappa}}, & n \geq 2, \ n < 2 \end{cases}, \quad N = \begin{cases} \frac{\kappa}{2n} \left( \chi^2 - \frac{n^2}{2\kappa} \right), & n \geq 2, \ n < 2 \Rightarrow \chi^2 = \begin{cases} \frac{1}{n}(2nN + n^2/2), & n \geq 2, \\ \frac{1}{n}(2nN + n(n - 1)), & n < 2, \end{cases} \end{cases}
\]

(51)

\[
\epsilon_\psi = \frac{1}{4\kappa n + 1}, \quad \eta_\psi = \frac{(n-1)}{2n(n+1)/\kappa}, \quad n \geq 2, \\
\epsilon_\psi = \frac{1}{2\kappa n + (n-1)}, \quad \eta_\psi = \frac{(n-1)}{2n(n+1)/\kappa}, \quad n < 2.
\]

(52)

To discuss the correction effect introduced by \( \beta \) i.e. \( f(R, T, R_{ab}T^{ab}) \), we set \( \alpha = \gamma = 0 \), so

\[
\Omega_2 = \Omega_3 = \Omega_1^2 = (1 - 4\beta \kappa^2 V(\phi))^{-2} \approx 1 + 8\beta \kappa^2 V(\phi),
\]

(53)

Substituting into 24, 25, 26, we can obtain the field and potential function in Einstein’s framework

\[
d\chi \approx \left[ 1 + 4\beta \kappa^2 \chi \lambda M_{\text{pl}}^4 \left( \frac{\phi}{M_{\text{pl}}} \right)^n \right] d\phi \Rightarrow \chi \approx \phi + \frac{4\beta \lambda M_{\text{pl}}^4}{n+1} \phi^{n+1}, \quad \phi \approx \chi - \frac{4\beta \lambda M_{\text{pl}}^4}{n+1} \chi^{n+1},
\]

(54)

\[
\hat{V}(\chi) \approx \lambda M_{\text{pl}}^4 \left( \frac{\chi}{M_{\text{pl}}} \right)^n \chi \left[ 1 + \frac{4(n+2)}{n+1} \beta \lambda \left( \frac{\chi}{M_{\text{pl}}} \right)^n \right].
\]

(55)

The slow-roll parameters with \( \beta \) corrective items can then be exported

\[
\epsilon_\psi = \frac{n^2}{2} \left( \frac{M_{\text{pl}}}{\chi} \right)^2 + \frac{4n^2(n+2)}{n+1} \beta \lambda \left( \frac{\chi}{M_{\text{pl}}} \right)^n, \quad \eta_\psi = n(n-1) \left( \frac{M_{\text{pl}}}{\chi} \right)^2 + \frac{4n(n+2)(3n-1)}{n+1} \beta \lambda \left( \frac{\chi}{M_{\text{pl}}} \right)^n.
\]

(56)

\[
N = \frac{1}{2n(n+1)} \left[ \left( \frac{\chi}{M_{\text{pl}}} \right)^n \left( n + 1 - 8\beta \lambda \left( \frac{\chi}{M_{\text{pl}}} \right)^n \right) - \left( \chi_{\text{end}} \right)^n \right].
\]

(57)

All of the above approximations are preserved up to the first-order terms of \( \beta \).

### 4.3 Hilltop models

The emergence of hilltop inflation was motivated in part by a predilection for concave potential function [27], which can be written specifically as

\[
V(\phi) = \Lambda^4 \left[ 1 - \left( \frac{\phi}{\mu} \right)^p + \ldots \right],
\]

(58)

where the ellipsis indicates the higher-order terms that make the potential function positive semi-definite. For the convenience of the calculation and the simplicity of the results, it is assumed that \( \phi/\mu \) is small.

For the case of ignoring \( \beta \),

\[
\hat{V}(\chi) = \frac{1+2\gamma}{(1+\alpha)^2} \Lambda^4 \left[ 1 - \left( \frac{\chi}{\hat{\mu}} \right)^p \right], \quad \hat{\mu} = \sqrt{\frac{1+\gamma}{1+\alpha}} \mu,
\]

(59)

\[
\epsilon_\psi \approx \frac{p^2}{2\kappa^2} \left( \frac{\chi}{\mu} \right)^{2p}, \quad \eta_\psi \approx \frac{p(p-1)}{2\kappa} \left( \frac{\chi}{\mu} \right)^{2p},
\]

(60)
When considering $\beta$ alone, $\alpha = \gamma = 0$ is set, and then

$$d\chi = \left[ 1 + 4\beta\kappa^2 A^4 \left( 1 - \left( \frac{\phi}{\mu} \right)^p \right) \right] d\phi \Rightarrow \phi \approx \chi - 4\beta\kappa^2 A^4 \chi \left[ 1 + \frac{1}{p + 1} \left( \frac{\chi}{\mu} \right)^{p+1} \right],$$

(62)

$$\tilde{V}(\chi) \approx A^4 \left[ 1 + 8\beta\kappa^2 A^4 + (4(p - 4)\beta\kappa^2 A^4 - 1) \left( \frac{\chi}{\mu} \right)^p \right],$$

(63)

$$\epsilon_{\tilde{V}} \approx \frac{p^2}{2\kappa\mu^2} - \frac{4p^2(p - 2)\beta\kappa A^4}{\mu^2} \left( \frac{\chi}{\mu} \right)^{2p-2},$$

(64)

$$\eta_{\tilde{V}} \approx \frac{p(1 - p)}{\kappa\mu^2} + \frac{4p(p - 1)(p - 2)\beta\kappa A^4}{\mu^2} \left( \frac{\chi}{\mu} \right)^{p-2},$$

(65)

$$N \approx \frac{\kappa}{p} \left[ \frac{\chi^2}{2} + \left( 1 + 2\beta\kappa^2 A^4 \right) \mu^2 \left( \frac{\chi}{\mu} \right)^{2-p} - \left( \frac{\chi^2}{2} + \left( 1 - 4\beta\kappa^2 A^4 \right) \mu^2 \left( \frac{\chi}{\mu} \right)^{2-p} \right) \right], \quad p \neq 2.$$  

(66)

When $p = 2$, $f(R, T, R_{ab}T^{ab})$ does not generate a correction term for $N$ (up to the first order). The results have been approximated by taking small amounts around zero for $\beta$ and $\phi/\mu$.

### 4.4 Natural inflation

Natural inflation means that the dynamical behavior of a scalar field is controlled by a periodic potential function [24, 25, 26]

$$V(\phi) = A^4 \left[ 1 + \cos \left( \frac{\phi}{f} \right) \right],$$

(67)

whose behavior at the origin is similar to that of hilltop models, but differs on large scales. This potential function is usually reminiscent of spontaneous symmetry breaking and axis-like particles. $A$ and $f$ are dimensional parameters. It has been learned that this potential can motivate inflation when $A \sim M_{\text{GUT}} \sim 10^{16}$GeV and $f \sim M_{\text{Pl}}$. However, in the GR framework it predicts a larger $r$ when $f \geq 10 M_{\text{Pl}}$ and a smaller $n_s$ when $f \sim M_{\text{Pl}}$ compared with Planck observation. As before, we can calculate the slow-roll parameters when $\beta = 0$

$$\epsilon_{\tilde{V}} = \frac{1}{2\kappa f^2} \left[ \frac{\sin(\chi/\tilde{f})}{1 + \cos(\chi/\tilde{f})} \right]^2, \quad \eta_{\tilde{V}} = -\frac{1}{\kappa f^2} \frac{\cos(\chi/\tilde{f})}{1 + \cos(\chi/\tilde{f})}, \quad \tilde{f} \equiv \sqrt{\frac{1 + \gamma}{1 + \alpha}} f,$$

(68)

$$\epsilon_{\tilde{V}} - \eta_{\tilde{V}} = \frac{1}{2\kappa f^2},$$

(69)

$$\exp \left[ \frac{N}{2\kappa f^2} \right] = \frac{\sin(\chi_{\text{end}}/2\tilde{f})}{\sin(\chi/2\tilde{f})}.$$

(70)

The redefinition of the parameter $f$ can be seen as a re-engagement of the mass units.

$$\epsilon_{\tilde{V}}(\chi_{\text{end}}) \equiv 1 \Rightarrow \chi_{\text{end}} = 2\tilde{f} \arctan(\sqrt{2\kappa\tilde{f}}) \Rightarrow \sin \left( \frac{\chi}{2\tilde{f}} \right) = \sqrt{\frac{2\kappa\tilde{f}^2}{1 + 2\kappa\tilde{f}^2}} \exp \left( -\frac{N}{2\kappa f^2} \right).$$

(71)

Therefore the slow-roll parameters can again be expressed in terms of the e-folding number $N$,

$$\epsilon_{\tilde{V}} = \frac{1}{e^{N/\kappa f^2} (1 + 2\kappa f^2) - 2\kappa f^2}, \quad \eta_{\tilde{V}} = \frac{1}{e^{N/\kappa f^2} (1 + 2\kappa f^2) - 2\kappa f^2} - \frac{1}{2\kappa f^2}. $$

(72)
Next we reconsider the case \( \alpha = \gamma = 0 \) to study the nontrivial corrections related to \( \beta \).

\[
\begin{align*}
  d\chi &= d\phi \left[ 1 + 4\beta \kappa^2 \Lambda^4 \left( 1 + \cos \left( \frac{\chi}{f} \right) \right) \right] \Rightarrow \phi \approx \chi - 4\beta \kappa^2 \Lambda^4 \left[ \chi + f \sin \left( \frac{\chi}{f} \right) \right] \\
  \dot{V}(\chi) &\approx \Lambda^4 \left[ 1 + \cos \left( \frac{\chi}{f} \right) \right] \left\{ 1 + 4\beta \kappa^2 \Lambda^4 \left[ 2 + 2 \cos \left( \frac{\chi}{f} \right) + \left( \frac{\chi}{f} + \sin \left( \frac{\chi}{f} \right) \right) \sin \left( \frac{\chi}{f} \right) \right] \right\} \\
  \frac{\chi/f \leq 1}{\Lambda^4} &\approx 2 - \frac{1}{2} \left( \frac{\chi}{f} \right)^2 \left\{ 1 + 4\beta \kappa^2 \Lambda^4 \left[ 4 - \left( \frac{\chi}{f} \right)^2 \right] \right\}
\end{align*}
\]

\[\epsilon_V \approx \frac{2}{\kappa f^2} \left[ \frac{\left( \frac{\chi}{f} \right)^2}{4 - \left( \frac{\chi}{f} \right)^2} + \frac{8\beta \kappa^2 \Lambda^4 \left( \frac{\chi}{f} \right)^2}{4 - \left( \frac{\chi}{f} \right)^2} \right] \frac{\chi/f \leq 1}{8\kappa f^2} (1 + 32\beta \kappa^2 \Lambda^4) \]

\[\eta_V \approx -\frac{2}{\kappa f^2} \left[ 2 + 8\beta \kappa^2 \Lambda^4 \left[ 4 - 5 \left( \frac{\chi}{f} \right)^2 \right] \right] \frac{\chi/f \leq 1}{4 - \left( \frac{\chi}{f} \right)^2} - \frac{8\beta \kappa \Lambda^4}{f^2} \left[ 1 - \left( \frac{\chi}{f} \right)^2 \right]
\]

Some authors \([38, 21, 39, 40, 41, 42, 22]\) are trying to make the results of natural inflation consistent with the Planck results and other new experimental data \([43]\), and the present work provides a complement to such efforts.

### 4.5 Power-law inflation

We consider the situation where the scaling function is power-law connected to cosmic time as our final example of applying our slow-roll equations to describe concrete models,

\[ a(t) \propto t^p \Rightarrow H(t) \propto \frac{p}{t}. \]

At this stage, it is simple to calculate the slow-roll parameters defined by the Hubble parameter and its derivatives \([35]\)

\[ \epsilon = \eta = \frac{1}{p}. \]

For the case of \( p > 1 \), \( \epsilon = \eta < 1 \) is always established, so inflation can go on forever.

In standard GR, the variation of the scalar field with cosmic time and the potential function in this case are expressed in simple form

\[ V(\phi) = V_0 \exp \left( -\sqrt{\frac{2}{p} \frac{\phi}{M_{Pl}}} \right), \quad \phi(t) = \sqrt{2pM_{Pl}} \ln \left( \sqrt{\frac{V_0}{p(3p-1)M_{Pl}}} t \right). \]

For the current \( f(R, T, R_{ab} T^{ab}) \) situation, we can give the dependence of \( V \) and \( \phi \) on \( t \), however, the explicit functional relationship between themselves is different to express in writing.

\[ V(\phi(t)) = \frac{3(1 + \alpha)p^2}{\kappa(1 + 2\gamma)t^2 + 12\beta \kappa p^2} \]

\[ \phi(t) = \frac{(1 + \alpha)\sqrt{1 + 2\gamma}}{6\sqrt{\beta(1 + \gamma)\kappa^{3/2} \gamma^2(1 + 2\gamma)t^2 + 12\beta \kappa p^2}} \left\{ -6\gamma pt \sqrt{\beta(1 + \gamma)(1 + 2\gamma)} \\
+ \sqrt{3}(1 + 2\gamma)t^2 + 12\beta \kappa p^2 \right\} \left[ 1 + \gamma(2 + \gamma) \arccot \left( \frac{2p\sqrt{33\kappa}}{\sqrt{1 + 2\gamma}t} \right) - 2\sqrt{1 + 2\gamma} \arccot \left( \frac{2p\sqrt{33\kappa}}{\sqrt{1 + 2\gamma}t} \right) \right]
\]
5 Perturbations of inflation scalar field

The inflation scalar field we have discussed so far have good spatial translation invariance and isotropy. We attempt to discuss its behavior when it produces a perturbation that varies with time-space position right now. This section frequently appears in discussions of the structure formation of the universe. The perturbed field can be decomposed into a uniform background field and its variation

\[ \phi(t, \vec{x}) = \bar{\phi}(t) + \delta \phi(t, \vec{x}). \]  

Among them, \( \bar{\phi}(t) \) satisfies \( V(\phi) = 0 \) since it is equivalent to the background field we stated previously. The modified Klein-Gordon equation in covariant form, which directly results from the variation of the action \( S \) with respect to the scalar field, serves as our starting point.

\[ (1 + \gamma + 2\beta \kappa R)g^{ab}\nabla_a \nabla_b \phi - 4\beta \kappa R \bar{g}^{ab}\nabla_a \nabla_b \phi - (1 + 2\gamma + 2\beta \kappa R)V'(\phi) = 0 \]  

Substituting \( \phi(t) \) and \( \delta \phi(t, \vec{x}) \) into a Fourier series

\[ \delta \phi(t, \vec{x}) = \sum_k \delta \phi_k(t)e^{i\vec{k} \cdot \vec{x}}. \]  

In Fourier space we have

\[ (1 + \gamma + 12\beta \kappa H^2)(\delta \phi_k)^{-} + (1 + \gamma + 12\beta \kappa H^2)3H(\delta \phi_k)^{-} \left[ (1 + \gamma + 12\beta \kappa H^2) \left( \frac{k}{a} \right)^2 + (1 + 2\gamma + 24\beta \kappa H^2) V''(\phi) \right] \delta \phi_k = 0, \]  

or

\[ H^{-2}(\delta \phi_k)^{-} + 3H^{-1}(\delta \phi_k)^{-} \left[ \left( \frac{k}{aH} \right)^2 + \frac{1 + 2\gamma + 24\beta \kappa H^2 m^2}{1 + \gamma + 12\beta \kappa H^2 H^2} \right] \delta \phi_k = 0, \]  

where \( m^2 \equiv V''(\phi) \).

If \( H \) and \( m^2 \) change slowly during inflation, we make now an approximation where we treat them as constants. Further, by redefining \( m, \delta \phi_k \) can be reduced to the form in GR

\[ H^{-2}(\delta \phi_k)^{-} + 3H^{-1}(\delta \phi_k)^{-} \left[ \left( \frac{k}{aH} \right)^2 + \frac{\dot{m}^2}{H^2} \right] \delta \phi_k = 0, \]  

whose general solution is

\[ \delta \phi_k(t) = a^{-3/2} \left[ A_k J_{-\nu} \left( \frac{k}{aH} \right) + B_k J_{\nu} \left( \frac{k}{aH} \right) \right], \quad \nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}, \]  

where \( A_k \) and \( B_k \) are undetermined constants and \( J \) is the Bessel function of the first kind.

If we ignore \( m/H \) for the cases where \( m \) is small, i.e., \( V(\phi) \) varies smoothly, the aforementioned formula degenerates to the scenario in GR. For small \( \gamma \) and \( \beta \), we keep to their first-order terms

\[ H^{-2}(\delta \phi_k)^{-} + 3H^{-1}(\delta \phi_k)^{-} \left[ \left( \frac{k}{aH} \right)^2 + (1 + \gamma + 12\beta \kappa H^2) \frac{m^2}{H^2} \right] \delta \phi_k = 0. \]  

We only analyzed the spatio-temporal perturbations of the inflation scalar field. Scalar and tensor perturbations of the metric tensor field itself are also of interest, but this content is beyond the scope of this paper.
6 Summary

In this paper, we consider a modified theory of gravity with non-minimal matter-curvature coupling, namely $f(R, T, R_{ab}T^{ab})$, and study the dynamical behavior of cosmological inflation controlled by it. This kind of gravitational theoretical model has been widely used in recent years to study the deep coupling of space-time and matter. We first reviewed the basic framework of $f(R, T, R_{ab}T^{ab})$ gravity, and derived the modified Einstein equation from the action. In the next step, we introduced the slow-roll approximation and obtained the field equation under it. We transform the field equations to the Einstein framework, thus introducing the slow-roll parameter. Afterwards, we focused on the analysis of different forms of potential functions and calculated their deviations from those in GR under the first-order approximation, including Starobinsky inflation, Chaotic models, Hilltop models, Natural inflation, Power-law inflation.

In general, we believe that the various galactic and supergalactic structures in the Universe can be traced back to the earliest non-uniformity of matter density in the early Universe. Therefore, deviations from spatial homogeneity and isotropy are important for the study of the structure formation of the Universe. As an extension, we also derive the behavior of inflation under perturbation.

It is worth pointing out that although the action we initially analyzed did not contain a $RT$ mixing term, after taking the slow-roll approximation it is actually possible to include such a term. This finding extends the applicability of our work.

It should be noted that the analysis in this paper takes a large number of approximations in order to obtain analytical results. However, this makes the results strongly dependent on the validity of the slow-roll approximation taken for the gravitational field equations and the validity of the transformation to the Einstein framework. Near the end of the inflation, the slow parameter can no longer be neglected, which means that the higher order terms in the equations without approximation will become significant.

The present work is not perfect for the study of slow-roll inflation under the $f(R, T, R_{ab}T^{ab})$ theory. There are still many unperformed analyzes that attract people, including numerical prediction of observations, rigorous and complete perturbation analysis, etc. We leave these topics as future work.

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A Calculation details

Non-zero components of Christoffel symbols:

$$\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = a^2 H,$$
$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = H.$$  \hspace{1cm} (93)

Non-zero components of Ricci tensor and scalar:

$$R_{00} = -3\frac{\ddot{a}}{a} = -3(\dot{H} + H^2),$$
$$R_{11} = R_{22} = R_{33} = 2\dot{a}^2 + a\ddot{a} = a^2(\dot{H} + 3H^2),$$
$$R = 6\frac{\dot{a}^2 + a\ddot{a}}{a^2} = 6(\dot{H} + 2H^2).$$  \hspace{1cm} (95)

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Non-zero components of matter tensor:
\[ T_{00} = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad T_{11} = T_{22} = T_{33} = a^2 \left[ \frac{1}{2} \dot{\phi}^2 - V(\phi) \right], \quad (98) \]
\[ T \equiv g^{ab}T_{ab} = \dot{\phi}^2 - 4V(\phi), \quad (99) \]
\[ \Theta_{00} = -\frac{3}{2} \dot{\phi}^2 - V(\phi), \quad \Theta_{11} = \Theta_{22} = \Theta_{33} = -a^2 \left[ \frac{1}{2} \dot{\phi}^2 - V(\phi) \right], \quad (100) \]
\[ \Theta \equiv g^{ab}\Theta_{ab} = 4V(\phi), \quad (101) \]
\[ R_{ab}T^{ab} = 3H^2\dot{\phi}^2 - 6(\dot{H} + 2H^2)V(\phi). \quad (102) \]

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