Unsteady mixed convection flow of Casson fluid past an inclined stretching sheet in the presence of nanoparticles

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Abstract. The influence of nanoparticles on the unsteady mixed convection flow of Casson fluid past an inclined stretching sheet is investigated in this paper. The effect of gravity modulation on the flow is also considered. Carboxymethyl cellulose solution (CMC) is chosen as the base fluid and copper as nanoparticles. The basic governing nonlinear partial differential equations are transformed using appropriate similarity transformation and solved numerically using an implicit finite difference scheme by means of the Keller-box method. The effect of nanoparticles volume fraction together with the effect of inclination angle and Casson parameter on the enhancement of heat transfer of Casson nanofluid is discussed in details. The velocity and temperature profiles as well as the skin friction coefficient and the Nusselt number are presented and analyzed.

1. Introduction
The wide range application of non-Newtonian fluids in various industries such as molten plastics, artificial fibers, nuclear waste disposal, food stuffs or slurries, transpiration cooling, petroleum reservoirs has gained attraction to analyze its characteristic behavior. Casson fluid is one type of fluid model for non-Newtonian fluid. Casson fluid has an infinite viscosity at zero rate of shear and zero viscosity at an infinite rate of shear and a yield stress under which no flow occurs. Thus, Casson fluid is defined as a shear thinning liquid or pseudoplastic fluid. A flow characteristics of blood can be accurately described by Casson fluid at low shear rates. Sharada and Shankar [1] analyzed numerically the effect of Soret and Dufour, thermal radiation, chemical reaction on the fluid flow, heat and mass transfers of a Casson fluid over an exponentially stretching surface. Ullah et al. [2] examined the unsteady MHD mixed convection slip flow of Casson fluid towards nonlinearly stretching sheet saturated in a porous medium in the presence of slip and convective boundary conditions. Further, Ullah et al. [3] obtained the numerical solutions for hydromagnetic Falkner-Skan flow of Casson fluid past a moving wedge with heat transfer. Pushpalatha et al. [4] discussed the effects of thermal diffusion and radiation on Casson fluid flow with convective boundary conditions. Recently, the effect of variable thermal conductivity on the mixed convective flow of a Casson fluid at a non-isothermal vertical stretching sheet was studied by Vajravelu et al. [5]. Pushpalatha et al. [6] discussed the thermo diffusion and diffusion thermo effects on Casson fluid over an unsteady stretching surface in presence of thermal radiation and magnetic field. They concluded that Casson parameter and unsteadiness parameter have tendency to depreciate the velocity distribution.
In recent years, the study of convection transport of nanofluid has gained a lot of attention due to their significant applications in engineering and technology. Nanofluid was first introduced by Choi [7] and is defined as a liquid suspension containing nanosized particles in common fluid. Experiment studies show that the nanofluid enhances thermal conductivity of the base fluid enormously. Parasuraman et al. [8] reported the effects of radiation, magnetic field and nanoparticle volume fraction on transient natural convective flow of nanofluid past a semi-infinite vertical plate. Freidoonimehr et al. [9] numerically studied the unsteady MHD free convection boundary-layer flow due to a permeable stretching vertical surface in a nanofluid. They observed that the local Nusselt number is enhanced by increasing the nanoparticle volume fraction but reduces the skin friction coefficient. Further, the mixed convection flow of Casson nanofluid over a stretching surface in presence of thermal radiation, heat source/sink and first order chemical reaction was studied by Hayat et al. [10]. Recently, Ullah et al. [11] investigated the effects of chemical reaction and thermal radiation on electrically conducting natural convection flow of Casson nanofluid caused by nonlinearly stretching sheet through porous medium.

Motivated by the above cited literature survey, the objective of present paper is to examine the influence of nanoparticles on the mixed convection flow of Casson fluid past an inclined stretching sheet with the presence of periodical gravity modulation. In this problem, the non-Newtonian pseudo-plastic fluids carboxymethyl cellulose (CMC) solution is used as the base fluid [12, 13] and the nanoparticles considered is copper (Cu). Based on the experimental studies, CMC solution is one of the common type of time-independent non-Newtonian fluid which exhibited the shear thinning or pseudoplastic rheological behavior [14-16].

2. Governing equations
The unsteady mixed convection flow of a viscous and incompressible Casson fluid past an inclined stretching sheet associating with the presence of copper nanoparticles is considered. In this problem, the \( x \)-axis is extended with inclination angle, \( \alpha \) along the surface to the vertical in the upward direction and \( y \)-axis is normal to the surface. The plate is assumed to have a linear velocity, \( u_x(x) \) moves in \( x \)-direction of the flow. Temperature of the plate varies linearly with the distance \( x \) along the plate, where \( T_w(x) > T_x \), and \( T_w(x) \) being the temperature of the plate and \( T_x \) being the uniform temperature of the ambient Casson fluid. It is assumed that the velocity of the stretching sheet is \( u_x = ax \) and the surface temperature is given as \( T_x = T_c + bx \) where \( a \) and \( b \) are positive constants.

Considering the effect of gravity modulation, following Sharidan et al. [17], the time dependent gravitational field, \( g(t) \) is taken to be \( g(t) = g_0 [1 + \varepsilon \cos(\pi \omega t)] \mathbf{k} \) where \( g_0 \) is the time-averaged value of the gravitational acceleration, \( g(t) \) acting along the direction of the unit vector \( \mathbf{k} \) which is oriented in the upward direction, \( \varepsilon \) is a scaling parameter, which gives the magnitude of the gravity modulation relative to \( g_o \), \( t \) is the time and \( \omega \) is the frequency of oscillation of the gravity driven flow. Under the usual boundary layer and Boussinesq approximations, the basic governing equations of Casson fluid with the present of nanoparticles can be written as

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}
\]

\[
\rho_n \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu_n \left( 1 + \frac{1}{\beta} \frac{\partial^2 u}{\partial y^2} + g(t)(\rho \beta_s)_{nf} (T - T_x) \cos \alpha \right), \tag{2}
\]

\[
(\rho C_p)_{nf} \left[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] = k_{nf} \frac{\partial^2 T}{\partial y^2}, \tag{3}
\]

subjected to the appropriate initial and boundary conditions,
\[ t = 0: u = v = 0, T = T_w \text{ for any } x, y, \]
\[ t > 0: u = u_w, v = 0, T = T_w \text{ at } y = 0, \]
\[ u \to 0, T \to T_w \text{ as } y \to \infty, \tag{4} \]

where \( u \) and \( v \) are the velocity components along \( x \) and \( y \) axes, \( t \) is the time, \( T \) is temperature and \( \beta = \mu_u \sqrt{2\pi_c/p_s} \) is the Casson parameter, where \( \pi_c \) is the critical value of the product based on the non-Newtonian model, \( \mu_a \) is the plastic dynamic of viscosity of the non-Newtonian fluid, and \( p_s \) is the yield stress of the fluid. \( \rho_{nf} \) is the density of nanofluid, \( \mu_s \) is the dynamic viscosity of nanofluid, \( (\beta_f)_{nf} \) is the thermal expansion of nanofluid, \( k_{nf} \) is the effective thermal conductivity of the nanofluid and \( (\rho C_p)_{nf} \) is the heat capacitance of nanofluid, where \( C_p \) is the specific heat at constant pressure, respectively. For nanofluid constant, the following expressions are given as follows,

\[ \rho_{nf} = (1-\phi) \rho_f + \phi \rho_s, (\rho \beta_f)_{nf} = (1-\phi)(\rho \beta_f)_f + \phi (\rho \beta_f)_s, \]

\[ (\rho C_p)_{nf} = (1-\phi)(\rho C_p)_f + \phi (\rho C_p)_s, \mu_{nf} = \mu_f \left(\frac{1-\phi}{1-\phi} \right)^{2/3} k_{nf} = k_f \left(\frac{1}{1-\phi} \right)^{2/3} \left( k_f + 2k_f \right) - 2\phi \left( k_f - k_s \right), \tag{5} \]

where \( \phi \) is the nanoparticles volume fraction of the base fluid. The subscripts \( f \) and \( s \) refer to the base fluid and nanoparticles properties, respectively. The thermophysical properties of base fluid and nanoparticles are given in Table 1 [9, 18].

| Physical properties | \( C_p (\text{J kg}^{-1} \text{K}^{-1}) \) | \( \rho (\text{kg m}^{-3}) \) | \( k (\text{Wm}^{-1} \text{K}^{-1}) \) | \( \beta_f \times 10^5 \text{ (K}^{-1}) \) |
|---------------------|------------------|------------------|------------------|------------------|
| CMC-water (<0.4%)   | 4179             | 997.1            | 0.613            | 21               |
| Nanoparticles (Cu)  | 385              | 8933             | 401              | 1.67             |

Next, the complexity of the problem is reduced by introducing the following non-dimensional variables [17],

\[ \tau = \alpha x, \eta = \left( \frac{C_f}{\upsilon_f} \right)^{1/2} y, \phi = \left( \frac{C_{uf}}{C_{uf}} \right)^{1/2} \phi (\tau, \eta), \theta (\tau, \eta) = \frac{(T-T_w)}{(T_w-T_u)}, g (\tau) = \frac{g(t)}{g_0}, \tag{6} \]

where \( \upsilon_f \) is the effective kinematic viscosity of the base fluid and \( \psi (x, y) \) is the stream function that satisfies (1). Using (6), the governing partial differential equations, (2) and (3) together with the boundary conditions, (4) are reduced to,

\[ \frac{1}{(1-\phi)^{2/3}} \left[ 1 + \frac{1}{\beta} \right] f'' + C_1 \left[ f' \left( f' - f'' \right)^1 + C_2 A \left[ 1 + \varepsilon \cos \left( \pi \tau \right) \right] \theta \cos \alpha = C_3 \Omega \frac{\partial f'}{\partial \tau}, \right. \tag{7} \]

\[ \frac{k_{nf}}{k_f} \theta'' + C_1 Pr \left[ f'' \theta' - f' \theta'' \right] = C_3 Pr \Omega \frac{\partial \theta}{\partial \tau}, \tag{8} \]

with the constants \( C_1, C_2 \) and \( C_3 \) are defined as,

\[ C_1 = (1-\phi) + \phi \frac{\rho_s}{\rho_f}, C_2 = (1-\phi) + \phi \frac{(\rho \beta_f)_s}{(\rho \beta_f)_f}, C_3 = (1-\phi) + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f}, \]

and the transformed boundary conditions are

\[ f = 0, f' = 1, \theta = 1 \text{ at } \eta = 0, \]
\[ f' \to 0, \theta \to 0 \text{ as } \eta \to \infty. \tag{9} \]
where primes denote differentiation with respect to $\eta$, $Pr = \left(C_p \mu/k\right)$ is the Prandtl number, $\Omega = \omega/c$ is the non-dimensional frequency of oscillation and $\lambda = Gr \beta/Re^2$ is the mixed convection parameter with $Gr = g_0\beta\left[T_w(x) - T_s\right]x^3/\nu_0^2$ and $Re_s = u_s(x)/\nu_0$ being the local Grashof and Reynolds numbers, respectively.

3. Results and discussion

The system of the partial differential equations, (7) and (8) together with the boundary conditions (9) are solved numerically using the finite difference scheme known as Keller-box method. This method has been found to be very suitable in dealing with the nonlinear parabolic problems and practically used by many researchers. Table 2 presents the comparison of the present results on the heat transfer rate, $\theta'(0)$ with the numerical results reported by Sharidan et al. [17] and Freidoonimehr et al. [9] for various values of Prandtl number. It is observed that the present work results are in good agreement. Thus, the use of present scheme is validated.

| Table 2. Comparison values of $\theta'(0)$ for various values of $Pr$ when $\varepsilon = \Omega = \lambda = \alpha = 0$ and $\beta \to \infty$. |
|---------------------------------------------------------------|
| $Pr$                                      | $0.72$ | $1.0$  | $3.0$  | $10.0$ |
| Sharidan et al. [17]                             | 0.8086 | 1.0000 | 1.9238 | 3.7225 |
| Freidoonimehr et al. [9]                         | 0.8086 | 1.0000 | 1.9237 | 3.7207 |
| Rawi                                          | 0.8086 | 1.0000 | 1.9239 | 3.7230 |

In order to study the influences of the Casson parameter, $\beta$, nanoparticles volume fraction, $\phi$ and angle of inclination, $\alpha$ on the fluid flow and heat transfer characteristics, the numerical results are graphically presented in figures 1 – 6 and table 3 for the fixed values of Prandtl number, $Pr = 6.2$, amplitude of modulation, $\varepsilon = 0.5$ and frequency of oscillation, $\Omega = 0.2$. It is worth mentioning that, when $\phi = 0$, the governing equations are reduced to the regular Casson fluid problem, where the nanofluid characteristics are eliminated. Figures 1 and 2 are plotted for different values of $\beta$ for both cases of $\phi = 0$ (Casson fluid) and $\phi = 0.1$ (Casson nanofluid). From these figures, it is noticed that, $f'(\eta)$ decreases for the increasing values of $\beta$ with the absence and presence of nanoparticles while an opposite behaviour is observed for $\theta(\eta)$. This is because, an increase in $\beta$ leads to increase the plastic dynamic viscosity which consequently created the resistance in the fluid motion. It is also important to mention that, $f'(\eta)$ are more effected by $\beta$ while compared with $\theta(\eta)$.

Figures 3 and 4 illustrate the effect of inclination angle parameter, $\alpha$ on the distributions of velocity and temperature. It is clear from these figures that, an increase of $\alpha$ enhances the temperature profile but slow down the fluid motion. This phenomenon happen because of increase in the angle of inclination reduces the buoyancy force, which in turn lowers the fluid velocity resulting in higher fluid temperature. As it is obvious from these figures, the similar behaviour is observed for both $f'(\eta)$ and $\theta(\eta)$ even in the presence of nanoparticles volume fraction. The effect of nanoparticles volume fraction, $\phi$, on the velocity and temperature profiles are depicted in the figures 5 and 6 for both $\beta = 0.8$ (Casson fluid) and $\beta \to \infty$ (Newtonian fluid) respectively. It is found that the fluid velocity reduces with an increase of $\phi$ for both fluid while the opposite behaviour is observed for $\theta(\eta)$. Physically, this is due to the fact that an increase in the nanoparticles volume fraction leads to an increase in the thermal conductivity of the nanofluid and hence the thickness of the thermal boundary layer increases.
Table 3 displays the variation of skin friction and heat transfer coefficients for the various values of pertinent parameters. It is interesting to note that, the increasing values of nanoparticles volume fraction enhances the heat transfer coefficient, $\theta'(0)$. In addition, a rise in the values of inclination angle and Casson parameters also increase $\theta'(0)$. However, the increasing values of nanoparticles volume
fraction, inclination angle and Casson parameters decrease the skin friction coefficient, $f'(0)$ significantly.

**Table 3.** The influence of $\phi$, $\beta$ and $\alpha$ on skin friction and heat transfer coefficients.

|     | $\phi = 0$ | $\phi = 0.1$ |
|-----|------------|--------------|
| $f'(0)$ | -0.580419 | -0.591558 |
| $\theta'(0)$ | -2.974093 | -2.527325 |

| $\beta$ | $\theta'(0)$ | $f'(0)$ | $\theta'(0)$ |
|--------|--------------|--------|--------------|
| 0.8    | -2.943793    | -0.596270 | -2.525137    |
| 2      | -0.705852    | -2.971607 | -2.472817    |
| 5      | -0.776262    | -0.605232 | -2.493982    |

| $\alpha$ | $f'(0)$ | $\theta'(0)$ |
|----------|--------|--------------|
| $\pi/4$  | -2.968343 | -0.623108 |
| $\pi/3$  | -2.791607 | -0.605232 |

4. Conclusion

A numerical study has been performed on the unsteady mixed convection flow of Casson fluid with taken into account the effect of nanoparticles volume fraction. With the fixed values of amplitude of modulation and frequency of oscillation, it has been found that, the Casson parameter, inclination angle and nanoparticles volume fraction enhance the temperature profile while reduce the fluid motion. It can also be concluded that, the temperature profile is significantly affected by the nanoparticles volume fraction.

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