Bondi-Metzner-Sachs invariance and electric-magnetic duality

Claudio Bunster, Andrés Gomberoff, and Alfredo Pérez

1Centro de Estudios Científicos (CECs), Casilla 1469, Valdivia, Chile
2Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Av. Diagonal las Torres 2640, Peñalolén, Santiago, Chile

We exhibit a Hamiltonian formulation, both for electromagnetism and gravitation, in which it is not required that the Bondi “news” vanish, but only that the incoming news be equal to the outgoing ones. This requirement is implemented by defining the fields on a two–sheeted hyperbolic surface, which we term “the hourglass”. It is a spacelike deformation of the complete lightcone. On it one approaches asymptotically (null) past and future infinity while remaining at a fixed (hyperbolic) time, by going to large spatial distances on its two sheets. The Hamiltonian formulation and - in particular - a conserved angular momentum, can only be constructed if one brings in both, the electric and magnetic BMS charges, together with their canonically conjugate “memories”. This reveals a close interplay between the BMS and electric-magnetic duality symmetries.

I. INTRODUCTION

The connection between radiation and the existence of an asymptotic infinite dimensional symmetry algebra, discovered by Bondi, Metzner and Sachs (BMS) [1–3] has been a fascinating subject ever since it emerged. In recent years attention on it was revived due first to the work of Barnich et al. [4–6], and later through that of Strominger et al. [7,8]. Barnich employed a classical Hamiltonian defined on future lightcones, and a construction termed by him a “covariantized Regge–Teitelboim method” which needed a spacelike deformation of the lightcone in order to define a Poisson bracket. Strominger, on the other hand, was basically quantum mechanically inclined and also worked on lightcones. Among his many contributions, he took - already at the classical level - an important new step by bringing together future and past-like cones, joined through a spatial inversion that he termed the “antipodal map”. He also brought new light, this time at the quantum level, onto the so called “angular momentum problem”, which is the name that was given in the past to the existence of many angular momenta connected with each other by BMS transformations. His response was to feel at ease with the “problem” by stating that the transformation that mapped one angular momentum to the other, connected different vacua.

The work reported herein takes element from both of the above developments. First it introduces a spatial spacelike deformation, but this time of the complete lightcone, past and future, that we have termed “the hourglass” because of its shape. This two-sheeted surface, which automatically incorporates the antipodal map is, however, not brought in as an auxiliary device to define Poisson brackets, but rather as a fundamental ingredient: it is the surface on which the fields are defined instead of the lightcone. Since it is spacelike, the hourglass has the advantage of enabling one to use the standard, battle tested, Regge–Teitelboim procedure [10] to define the Hamiltonian. On it one approaches asymptotically (null) past and future infinity while remaining at a fixed (hyperbolic) time, by

If one constructs the Hamiltonian by “improving” the generators of different motions so they have well defined functional derivatives, one finds that this can only be done if one brings in both, the electric and magnetic BMS charges together with their canonically conjugate “memories”. This reveals a close interplay between the BMS and electric-magnetic duality symmetries. This interplay becomes specially poignant in connection with angular momentum, which can only be defined so that it is conserved, with the help of electric-magnetic duality invariance.

The construction of the Hamiltonian in the presence of radiation also confirms, in a blatant manner, the crucially different role of “improper gauge transformations”, whose generators involve surface integrals from that of the proper ones whose generators do not. The former are to be regarded as changing the physical state, and are not trivial symmetries, while the latter are just due to redundant counting, and can be factored out by taking a quotient or by fixing the gauge.

In the present case one finds that, already for electromagnetism, what one thought was an “internal symmetry” is inextricably intertwined with spacetime displacements due to the “memory” carried by “the news”. And this implies that for the angular momentum there is no “problem” if one simply accepts what the theory is expressing each and every time it is able to: improper gauge transformations change the physical state. And this is already seen at the classical level.

The plan of the paper is the following. Section II introduces the hourglass and discusses its properties, then section III develops the formalism for the electromagnetic field. Finally section IV is devoted to gravitation.

In the case of gravitation, an electric-magnetic duality invariant description of the linearized theory on the houroglss has not yet been developed; but one can guess by analogy some of its elements. The proposals of that section concerning magnetic BMS charges are, therefore, of a speculative nature.

The results presented in this paper were obtained while improving a manuscript that had been elaborated for a
book in preparation, in honor of Tulio Regge, and are being incorporated in its updated version. This permits us to use that extensive report as an overall reference and review, and concentrate herein on the conceptual issues avoiding technicalities as much as possible.

II. THE HYPERBOLIC HOURGLASS

The hyperbolic hourglass consists of an outgoing hyperboloid with center \( x_\mu^{(0)} = (-\tau_0, 0) \) joined to an incoming one with center \( x_\mu^{(0)} = (\tau_0, 0) \). It obeys,

\[
(x_\mu - x_\mu^{(0)})(x_\mu - x_{(0)\mu}) = -\tau_0^2, \tag{II.1}
\]

and it is defined parametrically from Minkowskian coordinates \((x^0, \vec{x})\) through

\[
x^0 = t + r \sqrt{1 + \frac{\tau_0^2}{r^2} + \tau_0} \quad r \leq 0 \tag{II.2}
\]
\[
x^0 = t + r \sqrt{1 + \frac{\tau_0^2}{r^2} - \tau_0} \quad r \geq 0, \tag{II.3}
\]

\[
\vec{x} = r \hat{r}, \quad -\infty < r < +\infty, \tag{II.4}
\]

where the unit vector \( \hat{r} \) is given by

\[
\hat{r} = (\cos \vartheta \cos \phi, \cos \vartheta \sin \phi, \sin \vartheta), \tag{II.5}
\]

where

\[
0 \leq \vartheta \leq \pi, \\
0 \leq \phi < 2\pi.
\]

The radius \( \tau_0 \) is taken to be positive. The embedding defined by the above equations is continuously differentiable. The tangent vectors are continuous at \( r = 0 \) and the surface has a well defined global orientation.

The hyperbolic hourglass may be regarded as a spacelike deformation of the full (pass and future) lightcone, with an orientation inherited from the propagation of a light front that comes in, goes through itself, and then comes out. Since this wave propagation process is physically smooth, fields defined on the global coordinate system just described should be smooth.

The parametric equations (II.2)-(II.4) automatically incorporate the antipodal map [7] [8], which amounts to rewriting them by using a positive \( r \) for both sheets of the hyperboloid and inverting the orientation of the twosphere at a given \( r \). That is, keeping (II.2)-(II.4) for \( r \geq 0 \) and setting, \( r' = -r, \hat{r}' = -\hat{r} \), for \( r \leq 0 \).

If one considers an incoming wave which is not spherically symmetric, then the spacetime point at which the wavefront goes through itself will be different for different \( \hat{r}' \)'s. But in the present paper we are only interested in the analysis of the asymptotic region and therefore the details of what happens inside are irrelevant. The key aspects are the asymptotic hyperbolic shape and its orientation inherited from that of an incoming wave that goes through itself and becomes outgoing.

Figure 1 shows the embedding in Minkowski space of a single hyperbolic hourglass, figure 2 exhibits the slicing of Minkowski space by a one parameter family of hyperbolic hourglasses and figure 3 shows a sequence of Penrose diagrams with hyperbolic slicings of different radius \( \tau_0 \).

The hourglass foliation consists of hyperboloids of fixed radius and varying center. In contradistinction, hyperbolic foliations used previously by several authors have had fixed center and varying radius.

\[
\text{FIG. 1. The hyperbolic hourglass. The figure shows a two dimensional cut of an incoming hyperboloid and an outgoing one which are joint smoothly at at } r = 0. \quad \text{The arrows show the direction of increasing } r, \text{ which coincide asymptotically with the direction of propagation of a wave that comes in, goes through itself, and then comes out. If the incoming wave is not spherically symmetric, the spacetime point at which the wavefront goes through itself will be different for different } (\vartheta, \phi). \text{ For the analysis of the asymptotic region the details of what happens inside are irrelevant. The key aspects are the asymptotic hyperbolic shape and its orientation inherited from that of an incoming wave that goes through itself and becomes outgoing.}
\]

1 See, for example, [11], [12], [13], and also [14] and references therein. In some of these discussions timelike hyperboloids are employed (in which case \(-\tau_0^2 \) in (II.1) is replaced by \( \lambda_0^2 \)).
differential $\tau$ in the figure, which only depends on the ratios between the surfaces of the hourglass foliation are deformed from nearly light cones to nearly planes as $\tau$ increases from a very small value to a very large one. To better illustrate the effect, the succession of conformal diagrams shows from left to right how the surfaces of the hourglass foliation are deformed from nearly light cones to nearly planes as $\tau_0$ increases from a very small value to a very large one. To better illustrate the effect, different members of the foliation are shown in the different figures of the sequence; but, to keep track of the deformation, the surface at $t = 0$ (shown with a heavy line) in all cases. The Penrose diagram has been doubled to admit negative values of $r$ in the left triangular area. This doubling shows how the curves of constant $t, \vartheta, \varphi$ are smooth spacelike curves that connect asymptotically past and future null infinities. The scale of the length $\tau_0$ is irrelevant for the effect described in the figure, which only depends on the ratios between the different $\tau_0$’s shown.

FIG. 3. **Limits $\tau_0 \to 0$ and $\tau_0 \to \infty$ for Minkowski space.** The succession of conformal diagrams shows from left to right how the surfaces of the hourglass foliation are deformed from nearly light cones to nearly planes as $\tau_0$ increases from a very small value to a very large one. To better illustrate the effect, different members of the foliation are shown in the different figures of the sequence; but, to keep track of the deformation, the surface at $t = 0$ (shown with a heavy line) in all cases. The Penrose diagram has been doubled to admit negative values of $r$ in the left triangular area. This doubling shows how the curves of constant $t, \vartheta, \varphi$ are smooth spacelike curves that connect asymptotically past and future null infinities. The scale of the length $\tau_0$ is irrelevant for the effect described in the figure, which only depends on the ratios between the different $\tau_0$’s shown.

FIG. 2. **Slicing by hyperbolic hourglasses.** A given spacetime point is labeled by two set of coordinates. In the case of the point $P$ shown in the figure, these are $(t = 0, r, \vartheta, \varphi)$ and $(t = 3\tau_0, -r, \vartheta, \varphi + \pi)$. $t, \vartheta, \varphi$ values of $r$ in the left triangular area. This doubling shows how the curves of constant $t, \vartheta, \varphi$ are smooth spacelike curves that connect asymptotically past and future null infinities. The scale of the length $\tau_0$ is irrelevant for the effect described in the figure, which only depends on the ratios between the different $\tau_0$’s shown.

### III. ELECTROMAGNETIC FIELD IN MINKOWSKI SPACE

We will analyze in this section the case of the electromagnetic field on a fixed Minkowskian background. Practically all the features that will be encountered in the gravitational case already appear in this technically simpler context.

The main difference, which does not hinder the analogy, is that, since the background is fixed, its Poincaré symmetry appears as a global symmetry rather than an asymptotic gauge symmetry. There are no constraints associated with the surface deformation $\xi$, which are not varied in the action principle. The Hamiltonian is

$$H_0[\xi] = \int d^3x \left( \epsilon_{\mu}^i H_{\mu}^{(elm)} + \pi^i H_i^{(elm)} \right), \quad (\text{III.6})$$

where the $H_{\mu}$ in (\text{III.6}) are replaced by the energy and momentum densities of the electromagnetic field,

$$H_{\mu}^{(elm)} = \frac{1}{2} \left( g^{-\frac{1}{2}} \pi_i \pi^i + \frac{1}{2} g^{\frac{1}{2}} F_{ij} F_ij \right), \quad (\text{III.7})$$

$$H_i^{(elm)} = F_{ij} \pi^j, \quad (\text{III.8})$$

and $\xi^\perp$ and $\xi^i$ may be taken to be the normal and tangential components of any of the Poincaré Killing vectors.

The only gauge symmetry present in the problem is the electromagnetic one, whose generator is

$$G = -\pi_i^i \approx 0. \quad (\text{III.9})$$

Here $A_i$ is the vector potential, $\pi_i^i$ its conjugate momentum, $g_{ij}$ is the metric on the hourglass, and $g$ denotes its determinant.

If instead of having a fixed background we were considering dynamically coupled electromagnetic and gravitational fields, then expressions (\text{III.7}), (\text{III.8}) would be added to their gravitational counterparts discussed in section \text{IV} and the sum would be constrained to vanish. The asymptotic analysis given below would still hold because at large distances the spacetime would be flat. Then the asymptotic symmetry transformations of the coupled Einstein-Maxwell system would be those discussed here (internal electromagnetic, and Poincaré transformations) and the additional gravitational supertranslations.

We will now discuss the Poincaré and proper and improper gauge transformations for the electromagnetic field on the hourglass slicing. In this case the time equal constant surface is left invariant under the Lorentz group, whereas it is mapped onto a different hourglass by spacetime translations. Thus if one compares the situation with $t = $ constant planes, one sees that the roles of spatial translations and boosts are interchanged.
A. Asymptotic boundary conditions

1. Power expansion near \( r = \pm \infty \)

Starting from the Coulomb field written in hyperbolic coordinates, one is led to the boundary conditions,

\[
A_a = a_a^{(0)} + O(r^{-1}), \quad (III.10)
\]

\[
A_r = a_r^{(2)} r^{-2} + O(r^{-3}), \quad (III.11)
\]

\[
\pi^a = \pi^a_r r^{-2} + O(r^{-3}), \quad (III.12)
\]

\[
\pi^r = \pi^r_{(0)} r^{-1} + O(r^{-1}), \quad (III.13)
\]

\[
\lambda = \lambda_{(0)} + O(r^{-1}), \quad (III.14)
\]

Here \( \lambda \) is the Lagrange multiplier that accompanies the gauge generator \((III.9)\). In addition to the power law it is necessary to introduce parity conditions. This is achieved by splitting some of the variables in longitudinal and transverse parts as follows

\[
a_a^{(0)} = \nabla_a F + \star \nabla_a \bar{G}, \quad (III.15)
\]

\[
h_a = \gamma^2 \left( \nabla_a N + \star \nabla_a \bar{N} \right). \quad (III.16)
\]

Here,

\[
\star \nabla_a = \gamma^2 \epsilon_{ab} \nabla^b, \quad (III.17)
\]

where \( \gamma \) is the determinant of the metric \( \gamma_{ab} \) on the unit two-sphere. The “news” vector \( h^a \) in \((III.15)\), which will play a central role in what follows, is defined by

\[
h^a = \frac{1}{2} \left( \pi^a_r + \gamma \gamma^b \gamma^{ab} f^r_{br} \right), \quad (III.18)
\]

for \( r \to \pm \infty \), where the \( f^r_{br} \) is the leading order coefficient of \( F_{br} \). In Minkowski coordinates the news correspond to an electromagnetic field that decays as \( r^{-1} \), that is to a wave emerging from a confined source \( (r \to +\infty) \), or converging towards an absorber \( (r \to -\infty) \). For an accelerating electric charge \( e \) one has from the Lienard-Wiechert field,

\[
\gamma \gamma^r f_{(2)}^{sr} \bar{a}_a = \pi^r_r \bar{a}_a = -2 e \gamma^\perp \mathbf{r} \times (\mathbf{r} \times \bar{a}),
\]

where \( \bar{a} \) is the acceleration in rest frame of the emitter (outgoing wave) or absorber (incoming wave). See for example, \([15, 16]\).

2. Parity conditions

The parity conditions will be the following,

\[
\left( F, \lambda(0), N, \bar{N} \right) \big|_{r = +\infty} = \left( F, \lambda(0), N, \bar{N} \right) \big|_{r = -\infty}, \quad (III.19)
\]

for each \((\theta, \varphi)\).

Parity conditions play a fundamental role in the Regge-Teitelboim discussion of Poincaré invariance on asymptotic planes. We see that when dealing with Bondi, Metzner, Sachs invariance on hyperboloids, in the presence of news, they again come in\(^2\). It is shown in \([9]\) that the boundary conditions (power expansion and parity requirement) are preserved under Poincaré and improper gauge transformations.

The physical motivation for the parity conditions is very simple. They state that for a closed system (the free electromagnetic field in this case) everything that comes in must come out. That is, one allows for non-vanishing incoming and outgoing fluxes of energy, momentum, and other (BMS) charges; but requires that the net flux should be equal to zero.

This requirement, which physically is a condition connecting the remote past with the remote future, can be formulated as a fixed time statement, because the space-like hyperbolic hourglass is asymptotically tangent to the past and future lightcones. This is the reason for bringing it in to begin with.

B. The hyperbolic hourglass as an unconventional Cauchy surface

When regarded as an initial value surface, the hourglass has the unconventional feature, that a spacetime point, which is not at infinity, lying, say, on the outgoing half of one hourglass at a given time, also lies on the incoming half of another hourglass at a later time. This implies that one cannot give freely initial value data on the complete hourglass but only on half of it, the outgoing half for example. However, the double occurrence of points does not happen at infinity, so if one gives data on the outgoing half one should specify additionally the incoming radiation, that is one should give the news at \( r = -\infty \). But this is precisely what the parity condition does, stating that the incoming news are equal to the outgoing ones. Thus it is sufficient to specify just the data on the outgoing half of the hourglass (or, vice versa, on the incoming one) if the parity condition is imposed.

Therefore one must bring in the complete hourglass in order to deal in Hamiltonian terms with the interrelationship between past and future, but one only gives initial value data on one half of it, together with asymptotic information on the other half. In this sense the hourglass plays the role of a Cauchy surface.

C. Fiber memory

For a pure time translation the equation of motion for the leading order term of \( A_a \) is,

\[
\dot{a}_a^{(0)} = \gamma^{-\frac{1}{2}} h_a. \quad (III.20)
\]

\(^2\) The BMS symmetry has been tamed to fit a foliation by surfaces that are asymptotically planes \([17, 20]\). This has required dexterity, since the symmetry is intimately related to radiation and its natural habitat is an asymptotically null surface, rather than a plane.
for $r \to \pm \infty$. Its longitudinal component is

$$F = N.$$  \hfill (III.21)

Equation (III.21) has a highly non trivial content. It shows that, even when the generator of improper gauge transformations does not act, i.e., when $\lambda(0) = 0$, and one is only moving in the time $t$, there is still a displacement,

$$\delta F = N \delta t \quad \text{(III.22)}$$

along the U(1) fiber at each $(\vartheta, \varphi)$, of amount $N \delta t$, when a time $\delta t$ elapses. That is: (i) If there are no news (and one does not change the gauge frame) $F$ is conserved, (ii) If there are news during a time interval the value of $F$ changes from $F_{\text{before}}$ to $F_{\text{after}}$ according to the integral of (III.21) over the time interval. That is, $F$ “remembers” the news, and for that reason is called the “fiber memory”. Another kind of memory, “charge memory” will be encountered below in section III.

It is important to realize that (III.22) is not just a “re-definition of $\lambda(0)$ by the amount $N$”. This is because $\lambda(0)$ is not in the phase space, and can be held fixed in the variation of the Hamiltonian, whereas $N$ is a dynamical variable, which obeys a (gauge invariant) equation of motion and hence cannot be held fixed.

**D. BMS charges**

1. **Electric BMS charge**

   Taking into account the parity condition on $\lambda(0)$ one finds that the surface integral that must be added to the electromagnetic gauge generator to include improper transformations is given by

   $$\oint \lambda(0)Q$$\hfill (III.23)

   where the gauge charge $Q$ is given by

   $$Q(\vartheta, \varphi) = \pi^{r}(0) \bigg|_{+\infty} - \pi^{r}(0) \bigg|_{-\infty} \equiv Q_{+} + Q_{-}. \quad \text{(III.24)}$$

   It is important to interpret this expression appropriately. The hourglass is a construct that enables one to keep track, within the Hamiltonian formalism, of the incoming and outgoing radiation in an economic manner, that is without introducing separate overlapping incoming and outgoing hyperbolic patches. This brings in a redundancy: one way or another space occurs twice. We just saw one instance of this above in connection with the initial value data. The redundancy strikes again in expression (III.24) for the charge. If one considers the Coulomb field of a particle of charge $e$ at rest at $x' = 0$ one finds,

   $$Q_{+}(\vartheta, \varphi) = \frac{e}{4\pi} \sin \vartheta$$

   and hence

   $$Q(\vartheta, \varphi) = 2 \frac{e}{4\pi} \sin \vartheta \quad \text{(III.25)}$$

   The factor two arises because one is counting twice: $Q_{+}$ is the charge as seen in the outgoing description of space, while $Q_{-}$ is the same charge as seen from its incoming replica. This point will reappear below in connection with radiation rates.

2. **Magnetic BMS charge**

   There is a magnetic analog of (III.24) given by

   $$\tilde{Q} = \epsilon^{ab} \nabla_{a} \tilde{A}_{b}(0) \bigg|_{+\infty} = \gamma^{1/2} \nabla^{2} \tilde{G} \bigg|_{-\infty}, \quad \text{(III.26)}$$

   which is conserved as a consequence of (III.20) and the parity condition for $\tilde{N}$,

   $$\tilde{Q} = 0, \quad \text{(III.27)}$$

   In the electric representation this conservation law appears as an “accidental”, because it does not follow from a symmetry of the action. The formalism becomes complete if one introduces a second potential, so that the electric and magnetic charges are treated on the same footing. This completion of the formalism may be regarded as a matter of elegance and economy, but not of necessity, for questions that can be asked within the electric representation. But, as we will see further below, it becomes essential when one discusses Lorentz transformations. Therefore we recall it right away.

**E. Asymptotic two potential formulation**

One brings in a new, “magnetic” vector potential $\tilde{A}$. For the present purposes it is sufficient to do so only asymptotically. The potential $\tilde{A}$ satisfies,

$$\pi^{r} = -\epsilon^{ab} \partial_{a} \tilde{A}_{b}, \quad \pi^{a} = -\epsilon^{ab} (\partial_{b} \tilde{A}_{r} - \partial_{r} \tilde{A}_{b}). \quad \text{(III.28)}$$

Then, equations (III.10), (III.11) are replaced by

$$\tilde{A}_{a} = \tilde{F}_{a} - \gamma^{-2} \gamma_{ae} \epsilon^{cb} G_{b} + O(r^{-1}), \quad \text{(III.29)}$$

$$\tilde{A}_{r} = \tilde{a}_{r}^{(2)} r^{-2} + O(r^{-3}). \quad \text{(III.30)}$$

It is important to realize that the new potential incorporates with it the additional variable $\tilde{F}$, which was not present in the electric representation and drops out from eqs. (III.28).

There are now also magnetic improper gauge transformations with an associated parameter $\lambda_{(0)}$, which is independent of the “electric” $\lambda_{(0)}$. Under a magnetic BMS transformation $\tilde{F}$ and $G$ transform according to

$$\tilde{F} \to \tilde{F} + \tilde{\lambda}(0), \quad \text{(III.31)}$$
\[
G \to G.
\] (III.32)

The electric and magnetic radial momenta \(\pi^r, \bar{\pi}^r\), are related \(G\) and \(\bar{G}\) through,

\[
\pi^r = \gamma^{\frac{1}{2}} \nabla^2 G, \quad \bar{\pi}^r = \gamma^{\frac{1}{2}} \nabla^2 \bar{G}.
\] (III.33)

If one demands that \(G\) and \(\bar{G}\) be regular on the sphere, there is no room for a zero mode in the electric and magnetic BMS charges. The zero modes must be introduced through Dirac string singularities.

For a magnetic pole of strength \(g\) at the origin, on has

\[
A_\phi = g(1 - \cos \vartheta),
\] (III.34)

\[
\bar{G} = g \log(1 + \cos \vartheta).
\] (III.35)

For an electric pole of strength \(e\), which in the electric representation has

\[
\pi^r_{(0)} = \gamma^{\frac{1}{2}} e,
\] (III.36)

one now writes

\[
\tilde{A}_\phi = e(1 - \cos \vartheta),
\] (III.37)

\[
G = e \log(1 + \cos \vartheta).
\] (III.38)

If one admits Dirac string singularities in \(G\) and \(\bar{G}\) one must also do so for \(F\) and \(\bar{F}\) in order, for example, to be able to implement rotations. This is so because under a rotation the monopole potentials change by a singular gauge transformation.

1. Electric-magnetic duality invariant notation

It is useful to introduce a compact notation that makes electric-magnetic duality invariance of the theory manifest. This is achieved by writing

\[
A^M_a = \partial_a F^M + \epsilon^M_N \ast \nabla_a G^N,
\] (III.39)

\[
A^M = \begin{pmatrix} A \\ \bar{A} \end{pmatrix}, \quad N^M = \begin{pmatrix} N \\ \bar{N} \end{pmatrix},
\] (III.40)

\[
F^M = \begin{pmatrix} F \\ \bar{F} \end{pmatrix}, \quad G^M = \begin{pmatrix} G \\ \bar{G} \end{pmatrix},
\] (III.41)

where

\[
Q^M = \gamma^{\frac{1}{2}} \nabla^2 G^M|^{+\infty}_{-\infty},
\] (III.42)

are the electric and magnetic charges.

F. Time translations: Improved generator

1. Analysis starting from the electric representation

Rather than employing the electric-magnetic invariant formalism ab initio, we prefer to start from the “electric” representation and then use elements of duality to “patch it” in order to cast final results in a duality invariant form. This we do for expediency, but – more importantly – because in the case of gravitation, where the full asymptotic duality invariant formalism has not yet been developed, one can still perform the same steps, starting from the available electric representation.

It will suffice to analyze time translations. Spatial translations are taken care of in the same manner with a bit more of algebra. This is done in [9].

If one considers the Hamiltonian for a motion corresponding to a time translation, the surface term in the variation of the Hamiltonian (III.6) is given, in the electric representation, by

\[
\delta H_0 = -\int h^a \delta a^a(0)|^{\infty}_{-\infty}.
\] (III.43)

Equation (III.43) may be rewritten separating the electric memory and magnetic charge variations as,

\[
\delta H_0 = -\int \gamma^{\frac{1}{2}} \nabla^a N \nabla_a \delta F|^{+\infty}_{-\infty} - \int \gamma^{\frac{1}{2}} \nabla^a \bar{N} \nabla_a \delta \bar{G}|^{+\infty}_{-\infty}.
\] (III.44)

2. Magnetic fiber memory brought in

If the parity conditions are used, the first term on the right hand side on (III.44) vanishes, but the second does not. It may be written as,

\[
\int \bar{N} \delta \bar{Q}.
\] (III.45)

Equation (III.45) shows that in order to improve \(H_0\) one must add to it a term proportional to the magnetic gauge constraint

\[
\int -\bar{\pi}^a_i \bar{N} d^3 x.
\] (III.46)

This means that it is essential to bring in the magnetic sector in order to properly define the spacetime translation generators. The improvement cannot be made solely

\[\text{3 The magnetic gauge generator, } -\bar{\pi}^a_i, \text{ can be treated properly by keeping in Dirac’s “total Hamiltonian” the full constraint } \bar{\pi}_{mag} = 0 \text{ and } \bar{\pi}_{el} + \nabla \times \bar{A} = 0, \text{ whose curl is second class, while their divergence } \nabla \cdot \bar{\pi}_{mag} \text{ is first class. The details of that treatment will not be needed herein.}\]
within the electric sector. In other words, a deformation consisting only of a spacetime translation by itself does not have a well-defined generator. Only when one adds to it a movement along the fiber whose magnitude is \( \vec{N} \), does the generator exist. It is this improved generator which deserves to be called \( P^0 \). Its numerical value is the same as the original \( H_0 \) because the other term (III.46) vanishes weakly.

The need for the addition of the magnetic gauge transformation is simple to understand. It brings in the magnetic fiber memory, that – unlike the magnetic charge – is not present in the purely electric formulation, because only the gauge invariant curl of the magnetic potential appears in it.

The magnetic analog of (III.21) is

\[
\vec{F} = \vec{N},
\]  

(III.47)

It is remarkable how, guided just by the need to have a well defined Hamiltonian, one is compelled to bring in the magnetic sector in full force\(^4\).

Had we have started from the magnetic sector we would have obtained an equation identical to (III.44) but with the electric and magnetic roles reversed. After duality invariance is fully implemented, the variation of the improved generator of time translations will read,

\[
\delta H_0 \approx -\int \gamma \frac{1}{2} \nabla^a N^M \delta (\nabla_a F_M) \Bigl|^{+\infty}_{-\infty}. 
\]  

(III.48)

Here the weak equality means that terms proportional to the electric and magnetic constraints \( -\pi^i_1 \) and \( -\bar{\pi}^i_1 \) have been dropped.

The variation (III.48) vanishes when the parity conditions hold, but it will be useful to know its form even when they do not, when we evaluate emission and absorption rates in section (III).\(^5\)

**G. Lorentz generators. Spin from charge**

We again start from the electric representation and at the end cast the results in a manifestly duality invariant form.

We have

\[
H_{0,\text{Lorentz}}^\text{Improved} = -\int d^3x \xi^i F_{ij} \pi^j,
\]  

(III.49)

where \( \xi \) are the Lorentz Killing vectors. The surface term in its variation reads

\[
\delta H_{0,\text{Lorentz}}^\text{Improved} = -\int \xi^a \delta A_a \pi^r \Bigl|^{+\infty}_{-\infty}. 
\]  

(III.50)

To improve the generator \( H_0 \) we add an electric gauge generator, but this time with the surface term included, namely,

\[
G_{\text{Lorentz}}^\text{Improved} = \int -\pi^i_1 \Lambda_{\text{Lorentz}} d^3x + \int \lambda_{\text{Lorentz}}(\infty) \pi^r \Bigl|^{+\infty}_{-\infty},
\]  

(III.51)

with

\[
\lambda_{\text{Lorentz}}(\infty) = \xi^a_{\text{Lorentz}} A_a.
\]  

(III.53)

One then finds that the variation of

\[
H_{\text{Improved}}^\text{Lorentz} = H_{0,\text{Lorentz}}^\text{Improved} + G_{\text{Lorentz}}^\text{Improved},
\]  

(III.54)

does not have a surface integral.

1. **Lie derivative restored**

The improvement of the Lorentz generator \( H_{0,\text{Lorentz}}^\text{Improved} \) has an important geometrical consequence, in that it restores the Lie derivative at infinity. Indeed, the change in \( A_i \) given by the generator \( H_{0,\text{Lorentz}}^\text{Improved} \) is given by

\[
\delta_0 A_i = \xi^j F_{ji} = L_\xi A_i - \partial_i (\xi^j A_j),
\]

so that

\[
\delta_{\text{Improved}} A_a(\infty) = L_\xi A_a(\infty). 
\]

Therefore, \( H_{\text{Improved}}^\text{Lorentz} \) is the generator that will correctly implement the symmetry algebra given in section (III) below.

2. **Spin from charge**

The numerical value of the generator (III.54) which realizes the improvement of the Lorentz generator is not zero, but it is equal to the surface integral that appears in it. Therefore the numerical value of the angular momentum is not just the volume integral (III.49), but it includes a contribution

\[
\int \xi^a A_a \pi^r \Bigl|^{+\infty}_{-\infty} = \int \pi^r \xi^a \partial_a F \Bigl|^{+\infty}_{-\infty} + S,
\]  

(III.55)

where

\[
S = \int \xi^a \star \nabla_a G \gamma^2 \nabla^2 G \Bigl|^{+\infty}_{-\infty} = \left. -\frac{1}{2} \xi^a \star \nabla_a G \epsilon_{MN} \gamma^2 \nabla^2 G \right|^{+\infty}_{-\infty}. 
\]  

(III.56)
This phenomenon is similar to the modification of the angular momentum which appears in the presence of a magnetic pole in abelian and non-abelian gauge theories. The novelty here is that it occurs already without a magnetic pole.

The spin from charge phenomenon does not happen for energy and momentum because no surface term analogous to the one appearing in (III.55) is included in the translation charge.

3. Duality invariant Lorentz generator

The improved electric Lorentz generator,

\[ H_{el}[\xi] = H_0[\xi] + S + \oint \pi^r \xi^a \partial_a F^r \quad_{-\infty}^{+\infty}, \quad (III.57) \]

is not electric-magnetic duality invariant because, whereas \( H_0 \) and \( S \) have the property, the term proportional to \( \pi^r \) does not. Just as it was discussed for translations, it is evident that the appropriate expression is

\[ H[\xi] = H_{el}[\xi] + \oint \pi^r \xi^a \partial_a F^r \quad_{-\infty}^{+\infty}, \quad (III.58) \]

One may think of \( \oint \xi^a Q^M F_{M,a} \) as the generator of Lorentz transformations at infinity, and \( H_0 + S \) as the “bulk part” (although \( S \) is a surface integral).

It will be shown below, in Sec. III.14 that the duality invariant angular momentum is conserved (the electric part (III.57) is not!). Since this has been an issue in the literature (in the case of gravitation, which will follow this issue in (III.63) it is worth some comment.

First of all one realizes that under improper electric and magnetic gauge transformation, with parameter \( \lambda^M(0) = e^M \), the Lorentz generator changes as,

\[ H[\xi] \rightarrow H[\xi] - \oint e^M \partial_a (\xi^a Q_M), \quad (III.59) \]

and the new angular momentum is also conserved because \( Q_M \) is.

This is just as it happens if one changes the origin for orbital angular momentum, and in our view it is not to be regarded as a difficulty, since the present formalism improper gauge transformations are on the same footing with spacetime translations. All the more so, since a “pure time translation” carries along with it a rotation along the fiber, due to the fiber memory. Corresponding comments will be given below concerning angular momentum radiation rates.

H. Symmetry algebra

The electric and magnetic BMS charges generate improper gauge transformations and therefore commute with the spacetime translation generators which are invariant under them; and also among themselves. The action of the BMS charges on the Lorentz generators is given by (III.59).

Therefore one obtains the algebra

\[ [Q_M, H_{\text{Lorentz}}]^* = \partial_a (Q_M \xi^a_{\text{Lorentz}}), \quad (III.60) \]

\[ [Q_M (\vartheta, \varphi), P_{\mu}]^* = 0, \quad (III.61) \]

\[ [Q_M (\vartheta, \varphi), Q_N (\vartheta', \varphi')]^* = 0, \quad (III.62) \]

for the charges with the Poincaré group and among themselves. The Poincaré generators close according to the Poincaré algebra.

I. Emission and absorption rates. Charge memory

1. General formula for emission rates

Our boundary conditions are appropriate for a closed system, whose Hamiltonian is invariant under Poincaré and improper gauge transformations, and the corresponding conservation laws hold as a consequence of the fact that as much radiation is coming in as going out.

However, the formalism provides expressions for the emission and absorption rates separately. For that purpose one realizes from Eqs. (III.43) and (III.48) that

\[ \dot{Q}_a = -\oint h^a \delta_a \partial_a (0) \quad_{-\infty}^{+\infty}, \]

\[ \approx -\oint \gamma^{\frac{1}{2}} \nabla^a N^M \nabla_a (\delta_a F^M) \quad_{-\infty}^{+\infty}. \quad (III.63) \]

Here \( \delta_a \) is the variation due to the motion generated by the charge \( Q_a \). Thus \( \delta F^M = e^M \) for gauge transformations, \( \delta F^M = N^M \) for time translations, and \( \delta F^M = F^M = F^M_{\alpha} \xi^a \) for rotations. The purely electric form is incomplete for the magnetic charges and the Lorentz charges, because it misses the effect of the magnetic memory. This is not seen by \( P_{\mu} \) or by the electric BMS charge.

Then, the emission rates are read from the upper endpoint in (III.63) and the absorption rates from the lower one. In this way, one obtains the following results.

2. BMS charge

\[ \dot{Q}^M = \partial_a h^a_M \quad_{-\infty}^{+\infty} = \gamma^{\frac{1}{2}} \nabla^2 N^M \quad_{-\infty}^{+\infty}, \quad (III.64) \]
where,
\[ h^a_N = \left( \frac{h^a}{- \star h^a} \right). \]

This equation is to be interpreted as giving either \([+\pi^\ell(\infty)]\), or \([-\pi^\ell(-\infty)]\). These are not to be thought of as the rate of change of two different charges, but rather as the rates of change of one and the same charge, due to outgoing and incoming radiation respectively; which must be calculated using the two replicas of space that form the hourglass. When the parity conditions hold the \(Q^M\) are conserved.

On sees from (III.64), in analogy with (III.21), that the BMS charge also “remembers” the news and that, in this sense, the Laplacian of \(N\) is the “charge memory”. We will see in [21], that when a cosmological constant is introduced the fiber and charge memories are different and that the fiber memory appears to be more fundamental.

3. Energy

Similarly, one finds for the energy
\[
\frac{dP_0}{dt} = - \oint \gamma^{-\frac{3}{2}} \gamma_{ab} h^a h^b \bigg|_{-\infty}^{\infty} \\
= - \oint \gamma^{-\frac{3}{2}} \nabla^a N^M \nabla_a N_M \bigg|_{-\infty}^{\infty}. \tag{III.65}
\]

4. Angular momentum

The equations for the rate of change of the BMS charges and the energy given above can be expressed solely in terms of quantities defined in the electric sector. This is not the case for the angular momentum which as argued before, needs the magnetic sector for its very definition. Therefore, the rate can be read only from the second term on the right hand side of Eq. (III.63), which yields,
\[
\frac{d\vec{J}}{dt} = - \oint \gamma^{-\frac{3}{2}} \nabla^a N^M \nabla_a (\mathcal{L}_F M) \bigg|_{-\infty}^{\infty}, \tag{III.66}
\]
an expression that can be rewritten, with the help of (III.64), as,
\[
\frac{d\vec{J}}{dt} = \oint \dot{Q}^M F_{a\ell}^M \dot{\kappa}^\ell. \tag{III.67}
\]

The last expression shows that when the parity conditions hold, so that \(\dot{Q}^M = 0\), the angular momentum, Eq. (III.58), is conserved, as it was announced and discussed in Sec. IIIC3.

Note that Eq. (III.66) involves the variable \(F\) which does not appear in the electric sector. This is a consequence, in turn, of the fact that the angular momentum changes under the action of the magnetic BMS charge.

The interpretation of these equations is that the left hand sides are the rate of change of one and the same energy and angular momentum due to outgoing and incoming radiation. Therefore, the volume integrals appearing in the definition of \(P_0\) and \(J\) (see Eq. (III.64)), are to be thought of as evaluated on the upper half of the hourglass in the calculation of outgoing radiation and on the lower half in the calculation of incoming radiation.

Just as it was the case with the angular momentum itself, the physical cogency of Eq. (III.66) giving its rate of change, deserves a brief comment. The time rate of change of \(F^M\) is invariant under (improper) gauge transformations. If one agrees to keep the gauge frame fixed, that is, one only moves in the course of time on the fiber as dictated by the fiber memory, then \(F_M(t)\) is determined by the equations of motion – in a gauge invariant manner - once \(F_M(t = 0)\) is given. This means that if one were absorbing angular momentum at infinity so as to, say, make a top start spinning, then one would in principle be able to determine \(F_M(t = 0)\) and thus learn how the BMS origin in (III.66) is shifted from the one arbitrarily chosen on the fiber.

IV. GRAVITATIONAL FIELD

A. Correspondence with electromagnetism

In this section we analyze the gravitational field along the same lines that we analyzed above the electromagnetic field. The parallel between both cases is so close that it permits to make the following discussion succinct. The correspondence is as follows: The \(\ell = 0\) mode of the improper gauge symmetry generated by the total electric charge \(Q\) is the analog of the \(\ell = 0, \ell = 1\) modes of the Bondi-van der Burg-Metzner-Sachs supertranslation, which are the ordinary translations generated by \(P_\mu\). The modes with \(\ell \geq 2\) of the improper gauge symmetry correspond to the modes \(\ell \geq 2\) of the supertranslations. Therefore, altogether, one has the correspondence:
\[
\begin{align*}
\text{electromagnetism} & \leftrightarrow \text{gravitation} \\
(Q (\vartheta, \phi), P_\mu) & \leftrightarrow (\vartheta, \phi). 
\end{align*}
\]

On the other hand, the Lorentz transformations play along side:
\[
\begin{align*}
\text{electromagnetism} & \leftrightarrow \text{gravitation} \\
J_{\mu\nu} & \leftrightarrow J_{\mu\nu}. 
\end{align*}
\]

There is, as emphasized before, the difference that in the gravitational case all the generators are given by surface integrals, whereas in the electromagnetic one since the background was fixed, the spacetime translations and the Lorentz transformations were not. But this is just a
technical point which is easily accounted for and does not hinder at all the close correspondence between both cases.

The important concept of “news” is also present here of course, since it is the context in which it was originally introduced by Bondi [22]. The only difference is that now it is a symmetric traceless tensor \( h_{ab} \), appropriate to describe a gravitational wave, rather than the vector \( h^a \) appropriate for an electromagnetic one. Thus, one has the correspondence:

\[
h_a \quad \leftrightarrow \quad h_{ab} \quad \text{electromagnetism} \quad \text{gravitation}
\]

Keeping this in mind, we will essentially write the corresponding equation without much discussion, because one may translate to gravitation word by word in each case the corresponding comments from electromagnetism.

### B. Asymptotic boundary conditions

For the gravitational field the canonical variables are the spatial metric \( g_{ij} \) and their conjugate \( \pi^{ij} \). The generators of surface deformation are given by,

\[
\mathcal{H}_\perp = \frac{2}{\sqrt{g}} \left( \pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2 \right) - \frac{1}{2} \sqrt{g}^{(3)} R \approx 0,
\]

\[
\mathcal{H}_i = -2\pi_{ij} N^j \approx 0.
\]

Here we have set the cosmological constant equal to zero, and have chosen units such that \( 8\pi G = 1 \). The deformation parameters that multiply \( \mathcal{H}_\perp \) and \( \mathcal{H}_i \) in the Hamiltonian are the lapse \( N^\perp \) and the shift \( N^i \).

#### 1. Power expansion at large distances

Since our spacelike surfaces are asymptotically null, we must take as a starting point a coordinate system for the Schwarzschild metric which incorporates this property. This is provided by the Eddington-Finkelstein coordinates in terms of which the line element reads,

\[
ds^2 = -(dx^0)^2 + dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + \frac{M}{4\pi r} \left( dx^0 - dr \right)^2.
\]

The next step is to pass to hyperbolic coordinates, through the change of variables [11.2], [11.3], extract the asymptotic form of the resulting expression, and proceed by trial and error.

The resulting boundary conditions, in the form of a power law expansion at large distances are given in [9]. We only need to know for the present purposes, that the most general deformation that preserves them is parameterized by a function \( \epsilon_1^{(1)} \) (infinitesimal supertranslation) and two vectors, \( \vec{\omega} \) (infinitesimal rotation) and \( \vec{\beta} \) (infinitesimal boost). The analogs of the asymptotic parts \( A_a^{(0)} \) of the vector potential \( A_a \), and of the news \( n_a \) are now symmetric traceless tensors \( \tilde{f}_{ab}^{(1)} \) and \( \tilde{h}_{ab} \), respectively. They are build out of the leading and subleading terms in the power expansions of \( g_{ij} \) and \( \pi^{ij} \).

#### 2. Parity conditions

In addition to the power law decays it is necessary to introduce parity conditions. This is achieved by splitting \( f_{ab}^{(1)} \) and \( h_{ab} \) in longitudinal and transverse parts as follows,

\[
\tau_0 \tilde{f}_{ab}^{(1)} = \nabla_{ab} F + \star \nabla_{ab} G,
\]

\[
h_{ab} = \frac{1}{2} \left( \nabla_{ab} N + \star \nabla_{ab} \bar{N} \right),
\]

These equations correspond to [III.15]–[III.16] in electromagnetism.

In the above equations the operators \( \nabla_{ab} \) and \( \star \nabla_{ab} \), given by\[
\nabla_{ab} = 2(\nabla_a \nabla_b + \nabla_b \nabla_a - \gamma_{ab} \nabla^2),
\]

\[
\star \nabla_{ab} = 2\sqrt{\gamma} \epsilon_{ac} \nabla_c \nabla_d + \epsilon_{bc} \nabla_a \nabla_d,
\]

are the tensor analogs of the vector gradient, \( \nabla_a \), and curl \( \star \nabla_a \) appearing in [III.15] and [III.16].

These operators were used by Regge and Wheeler in their analysis of the stability of a Schwarzschild singularity [23], and obey the key properties

\[
\nabla^{ab} (\star \nabla_{ab}) = \star \nabla_{ab} (\nabla^{ab}) = 0,
\]

when they act on scalar functions, just as their vector counterparts. Their kernel is spanned by the \( \ell = 0 \) and \( \ell = 1 \) modes of the corresponding scalar functions on which they act.

The parity conditions will be then the following,

\[
\left. \left( F, N, \bar{N} \right) \right|_{+\infty} = \left. \left( F, N, \bar{N} \right) \right|_{-\infty} \quad \text{for each} \quad (\vartheta, \varphi),
\]

in close analogy with Eq. [IV.74] for electromagnetism.

#### 3. Supertranslation memory

Consider a time translation:

\[
\epsilon_1^{(1)} = 1.
\]

Einstein’s equations in Hamiltonian form then yield,

\[
\tilde{f}_{ab}^{(1)} = 2 \tilde{h}_{ab},
\]

which implies

\[
\dot{F} = N.
\]
Therefore, when time $\delta t$ elapses a supertranslation of magnitude
\[
\delta F = N \delta t, \quad \text{(IV.78)}
\]
takes place. This is the supertranslation memory effect, analogous to the fiber memory of electromagnetism discussed in section III.C.

C. Electric and magnetic BMS charges

We saw in the electromagnetic case that it was necessary to employ, asymptotically on the hourglass an electric-magnetic duality invariant formalism, in order to be able to improve the generators. The same will occur in gravitation. In that case we do not possess at the moment an explicit electric-magnetic duality invariant description of the linearized theory on the hourglass, which is what is needed at large distances. However, it is reasonable to assume that such a description exists, and that it can be constructed along lines similar to those employed successfully for asymptotic planes in [24, 25].

Fortunately, it turns out that assuming the existence of the asymptotic electric-magnetic duality invariant description, one can conjecture by analogy some of the elements that are needed. The coherence of the results thus obtained reinforces the hypothesized existence of the electric-magnetic representation. We now pass to discuss those elements.

1. Electric BMS charge

If one varies the Hamiltonian
\[
H_0 = \int d^3x \left( N^\perp \mathcal{H}_\perp + N^i \mathcal{H}_i \right), \quad \text{(IV.79)}
\]
in the electric representation, with the Lorentz parameters $\varpi, \beta$ set equal to zero, one finds
\[
\delta H_0 = - \oint \epsilon^{(1)}_o \left[ \delta P + \frac{1}{2} \epsilon^{ab} \delta \left( \mathcal{T}_{ab}^{(1)} \right) \right] \bigg|_{-\infty}^{+\infty}. \quad \text{(IV.80)}
\]
The explicit expression for $P(\vartheta, \varphi)$ is given in [9]. Equation (IV.80) identifies $Q(\vartheta, \varphi)$
\[
Q = P \bigg|_{-\infty}^{+\infty}, \quad \text{(IV.81)}
\]
as the (electric) supertranslation charge. In the analogy with electromagnetism, the $l = 0$ and $l = 1$ of the charge (IV.81) correspond to spacetime translations whereas those with $l \geq 2$ correspond to the electromagnetic charges with spherical modes $l \geq 1$. The first term on the right hand side of (IV.80) may be compensated in the standard manner by defining a partially improved Hamiltonian $\tilde{H}_0^{\text{elec}}$ through
\[
\tilde{H}_0^{\text{elec}} = H_0 + \oint \epsilon_0^{(1)} (\vartheta, \varphi) Q (\vartheta, \varphi). \quad \text{(IV.82)}
\]
The Hamiltonian (IV.82) is the analog of the Maxwell electric Hamiltonian for spacetime translations and improper gauge transformations, and just as that one it will need to be improved to eliminate the surface term
\[
\oint \frac{1}{2} \epsilon_0^{(1)} \epsilon^{ab} \delta \left( \mathcal{T}_{ab}^{(1)} \right) \bigg|_{-\infty}^{+\infty} = \quad \text{(IV.83)}
\]
\[
\oint -\frac{1}{4} \gamma^{ab} \epsilon_0^{(1)} \left( \mathcal{V}_{ab} N + \star \mathcal{V}_{ab} \bar{N} \right) \left( \mathcal{V}_{ab} \delta F + \star \mathcal{V}_{ab} \delta \bar{G} \right) \bigg|_{-\infty}^{+\infty}. \quad \text{(IV.84)}
\]
The term proportional to $\delta F$ on the right hand side of Eq. (IV.83) vanishes when the parity conditions hold but the one proportional to $\delta G$, which reads
\[
\delta H_0 = - \oint \bar{\eta} \delta \bar{G} \bigg|_{-\infty}^{+\infty}, \quad \text{(IV.84)}
\]
with
\[
\bar{\eta} = \frac{1}{4} \mathcal{V}_{ab} \left[ \epsilon_0^{(1)} \left( \mathcal{V}_{ab} \bar{N} - \star \mathcal{V}_{ab} N \right) \right] \quad \text{(IV.85)}
\]
does not.

2. Magnetic BMS charges

In order to eliminate (IV.84) one should supplement the Hamiltonian acting with the generator of magnetic BMS transformations, whose form we do not know, but which should be such that the surface term in its variation should read
\[
- \oint \epsilon_0^{(1)} \delta \bar{P} \bigg|_{-\infty}^{+\infty}, \quad \text{(IV.86)}
\]
Here, by definition, $\bar{P}$ is the magnetic supertranslation charge and $\epsilon_0^{(1)}$ is the magnetic deformation parameter. So we must have,
\[
\oint \epsilon_0^{(1)} \delta \bar{P} = \oint \bar{\eta} \delta \bar{G}, \quad \text{(IV.87)}
\]
and the question is: what is the relationship between $\bar{P}$ and $\bar{G}$?

This can be established by recalling from electromagnetism that one would like the parameter $\eta$ to bring the magnetic memory. So we set
\[
\epsilon_0^{(1)} = 1, \quad \text{(IV.87)}
\]
in which case the boundary term reads
\[
\delta H_0 = - \oint \mathcal{V}^4 \bar{N} \delta (\gamma^2 \bar{G}) \bigg|_{-\infty}^{+\infty} = - \oint \bar{N} \delta \left( \mathcal{V}^4 (\gamma^2 \bar{G}) \right) \bigg|_{-\infty}^{+\infty}, \quad \text{(IV.88)}
\]
where
\[
\mathcal{V}^4 = \mathcal{V}_{ab} \mathcal{V}^{ab} = \star \mathcal{V}_{ab} \star \mathcal{V}^{ab}
\]
\[
= 8 \left[ (\nabla^2)^2 + 2 \nabla^2 \right]. \quad \text{(IV.89)}
\]
Comparison with the magnetic analog of Eq. (IV.77) then gives
\[ \mathcal{P} = \gamma^{2} \nabla^{4} \mathcal{G} \] (IV.90)
The identification (IV.90) will have a significant consistency check when we discuss angular momentum below.

In order to account for the \( t = 0 \) and \( l = 1 \) modes of \( \mathcal{P} \) (magnetic translations) one would need to bring in Dirac strings into \( \mathcal{G} \) because those modes are in the kernel of \( \nabla^{4} \).

D. Lorentz generators

If one works solely in the electric representation one finds that if one considers the motion corresponding to a Lorentz transformation, with infinitesimal rotation and boost parameters \( \tilde{\omega} \) and \( \tilde{\beta} \), one must improve the Hamiltonian by adding to it the surface term,
\[ \tilde{\omega} \cdot \tilde{J}_{el} + \tilde{\beta} \cdot \tilde{K}_{el}, \]
where \( \tilde{J}_{el} \) and \( \tilde{K}_{el} \) are surface integrals whose expressions are given in [9].

When this is done the generators are well-defined. No additional surface integral containing the news, analogous to (IV.83) appears. This is reasonable because the Lorentz motion lies within the hourglass.

The electric generators thus obtained are the analog of the electromagnetic angular momentum (III.57).

If one takes \( \epsilon_{1}^{(1)} = 1 \), and evaluates the rate of change of \( \tilde{J}_{el} \), one finds, either by direct calculation from Einstein’s equations or, better, by using Eq. (IV.97) below,
\[ \frac{d \tilde{J}_{el}}{dt} = -\oint (\gamma^{2} \nabla^{4} \mathcal{G}) \tilde{N}_{el} \tilde{\xi}_{a} ]^{+\infty}_{-\infty}. \] (IV.91)

Eq. (IV.91) is the analog of (III.66) for electromagnetism. It provides a consistency check of the definition (IV.96) because if we bring in the magnetic analog \( \mathcal{F} \) of \( \mathcal{F} \), and postulate the magnetic memory equation,
\[ \mathcal{F} = \tilde{N}, \] (IV.92)
then,
\[ \tilde{J} = \tilde{J}_{el} + \oint \mathcal{P} \mathcal{F}_{el} \tilde{a} \tilde{\xi}_{a} ]^{+\infty}_{-\infty} \] (IV.93)
is conserved
\[ \frac{d \tilde{J}}{dt} = 0, \] (IV.94)
when the parity conditions hold.

So, by appealing to electric-magnetic duality one can find a conserved angular momentum in general relativity, even in the presence of radiation, but provided the net radiation flux is zero.

For boosts one must include an extra term (see comment at the end of the next subsection). Thus one has in general,
\[ H_{\text{Lorentz}} = H_{el}^{a} + \oint \tilde{Q} \left( \xi^{a} \partial_{a} \mathcal{F} - \frac{3}{2} \nabla_{a} \xi^{a} \mathcal{F} \right). \] (IV.95)
(The second term \( \nabla_{a} \xi^{a} \) vanishes for rotations).

E. Symmetry algebra

The analog of (III.60) and (III.62) for electromagnetism is
\[ \left[ Q^{M} (\vartheta, \varphi), Q^{N} (\vartheta', \varphi') \right]^{*} = 0 \]
\[ \left[ Q^{M} (\vartheta, \varphi), \tilde{J} \right]^{*} = \partial_{a} \left( \tilde{\xi}_{a} Q^{M} (\vartheta, \varphi) \right) \]
\[ \left[ Q^{M} (\vartheta, \varphi), \tilde{K} \right]^{*} = \partial_{a} \left( \tilde{\xi}_{a} Q^{M} (\vartheta, \varphi) \right) \]
\[ + \frac{3}{2} Q^{M} (\vartheta, \varphi) \left( \nabla_{a} \xi_{a}^{b} \right) \] (IV.96)
while the Lorentz generator \( \tilde{K} \) and \( \tilde{J} \) close among themselves in the Lorentz algebra.

We have used Dirac brackets \( [ , ]^{*} \) here because, as explained in the introduction it is only through them that the surface term alone can act as a generator. If one wanted to use Poisson brackets one would have to add to the surface term the weakly vanishing volume part of the generator.

For the electric generators \( Q \) and the Lorentz generators, equations (IV.96) can be obtained directly from the algebra of surface deformations in spacetime [26, 27] (see [9] for details). It is then extended to the magnetic generators by duality.

F. Emission and absorption rates. Charge memory

1. General formula for emission memory

In this case we only possess the formula stemming from electric sector, that is,
\[ \dot{Q}_{a} = \oint \frac{1}{2} \bar{h}_{ab} \delta_{a} \left( \xi_{0} f_{ab}^{(1)} \right) ]^{+\infty}_{-\infty}. \] (IV.97)

The analog of the second expression on the right hand side of (III.63) is not obvious to guess because, this time, under a duality transformation, one must turn the electric time into magnetic time. That is, one would have to compare motions that have \( \epsilon_{1}^{(1)} = 1, \epsilon_{1}^{(1)} = 0 \) with those with \( \epsilon_{1}^{(1)} = 0, \epsilon_{1}^{(1)} = 1 \).

By applying (IV.97) one obtains the following results.
2. Electric BMS charges

\[
\frac{\partial Q}{\partial t} = \left( -\frac{1}{\sqrt{\gamma}} h^{ab} h_{ab} + \nabla_a \nabla_b h^{ab} \right) \bigg|_{-\infty}^{+\infty},
\]
\[
= -\frac{1}{4} \gamma^2 \left( (\nabla^a N)(\nabla_a N) + (\nabla^{ab} \bar{N})(\nabla_{ab} \bar{N}) + 2(\nabla^a N)(\nabla_a \bar{N}) + + \gamma \frac{i}{2} \nabla^4 N \right) \bigg|_{-\infty}^{+\infty}, \quad (\text{IV.98})
\]

(This equation, as well as its relationship with the emission rate, were found previously in [3, 4]: but they interpreted it as meaning that the Hamiltonian cannot be improved if \( h_{ab} \neq 0 \).)

3. Magnetic BMS charge

The (electric) time derivative of the magnetic BMS charge cannot be obtained from the purely electric sector formula (IV.97), although \( \hat{\mathcal{P}} \) does appear in the electric sector. One must resort to its definition (IV.90) and to the equation of motion (IV.76). This yields,

\[
\frac{\partial Q}{\partial t} = \gamma \frac{i}{2} \nabla^4 \bar{N} \bigg|_{-\infty}^{+\infty} \quad (\text{IV.99})
\]

Note that there is no symmetry between the rates of change of \( Q \) and \( \bar{Q} \). That is quite alright because one should not expect any: the duality counterpart of (IV.98) should be the rate of change of \( \bar{Q} \) with respect to a magnetic time displacement with \( \epsilon^{(1)} = 1 \), and \( \epsilon^{(1)} = 0 \).

In the same vein, one could define a variable \( \hat{G} \) thorough

\[
\gamma \frac{i}{2} \nabla^4 \bar{G} = \hat{P}, \quad (\text{IV.100})
\]

Then the derivative of \( \hat{G} \) with respect to electric time would not be equal to \( N \) as one can see from (IV.98). However, one would expect its derivative with respect to magnetic time to be given by \( \bar{N} \), in analogy with (IV.99).

4. Angular momentum

One may write, in analogy with (III.67) in electromagnetism

\[
\frac{d\vec{J}}{dt} = \oint \hat{P}_M F^M_{\alpha \beta} \gamma^\alpha \hat{\xi}^\beta \quad (\text{IV.101})
\]

where \( \hat{P} \) and \( \hat{\mathcal{P}} \) are given by (IV.98) and (IV.99).

It should be stressed that formula (IV.101) has not been proven, but just conjectured by analogy and “informed guess”. Only the conservation of \( \hat{J} \) when \( \hat{P} \) and \( \hat{\mathcal{P}} \) vanish has been proven (once (IV.92) has been postulated!). This is because, in the lack of a complete asymptotic two potential theory, we do not posses an analog of the second expression on the right hand side of (III.63). But the presumption is that (IV.101) will survive the complete development of the asymptotically duality invariant description.

All the comments made for the electromagnetic case in connection with the angular momentum and with its rate of change apply here as well.

G. Taub-NUT and Kerr solutions

To conclude we consider two fundamental solutions of Einstein’s equations for which it is important to verify that they fit into the present treatment. Especially so because of their relationship with magnetic charge and with angular momentum, concepts which have been of central interest throughout this work. They are Taub-NUT space and the Kerr solution respectively.

One can verify (see [9]) that these two solutions can be brought by a change of coordinates to comply to our boundary conditions. One finds that for Taub-NUT,

\[
F = 0, \quad (\text{IV.102})
\]

and

\[
\dot{G} = - N \log (1 + \cos \vartheta), \quad (\text{IV.103})
\]

which are exactly the expressions (III.35) of electromagnetism for a magnetic pole of charge \( q = -N \), with the Dirac string going through the south pole \( \vec{P} \). Here \( N \) is the Taub-NUT parameter.

For the Kerr metric, one finds that the value of the energy is given by

\[
P^0 = \oint \mathcal{P} = M, \quad (\text{IV.104})
\]

and that of the angular momentum (IV.93) by

\[
J_z = aM. \quad (\text{IV.105})
\]

Only the electric part of (IV.93) contributes to (IV.105) because the magnetic part vanishes. \( M \) and \( a \) are the standard mass and specific angular momentum parameters of the Kerr solution.

\[5\] When one discusses Taub-NUT on surfaces which are asymptotically planes, as it was done in [25], one finds, that in order to satisfy the Regge-Teitelboim boundary conditions one must take half of the string to come out of the south pole and the other half to come out from the north pole. No such requirement is present here, where one can take just one string going out through any point on the sphere.
ACKNOWLEDGMENTS

We express our gratitude to Professor Anna Ceresole for her continued encouragement throughout this work. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt. C.B. wishes to thank the Alexander von Humboldt Foundation for a Humboldt Research Award. The work of A.P. is partially funded by Fondecyt Grants No. 1171162 and No. 1181496.

[1] H. Bondi, M.G.J. van der Burg, and A.W.K. Metzner. Gravitational waves in General Relativity. 7. Waves from axisymmetric isolated systems. Proc. Roy. Soc. Lond., A269:21–52, 1962.
[2] R. Sachs. Asymptotic symmetries in gravitational theory. Phys. Rev., 128:2851–2864, 1962.
[3] R. K. Sachs. Gravitational waves in General Relativity. 8. Waves in asymptotically flat space-times. Proc. Roy. Soc. Lond., A270:103–126, 1962.
[4] G. Barnich and F. Brandt. Covariant theory of asymptotic symmetries, conservation laws and central charges. Nucl. Phys., B633:3–82, 2002, hep-th/0111246.
[5] G. Barnich and C. Troessaert. Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited. Phys. Rev. Lett., 105:111103, 2010, 0909.2617.
[6] G. Barnich and C. Troessaert. BMS charge algebra. JHEP, 12:105, 2011, 1106.0213.
[7] A. Strominger. On BMS Invariance of gravitational scattering. JHEP, 07:152, 2014, 1312.2229.
[8] A. Strominger. Lectures on the infrared structure of gravity and gauge theory 2017, 1703.05448, and references therein.
[9] C. Bunster, A. Gomberoff and A. Pérez. Regge-Teitelboim analysis of the symmetries of electromagnetic and gravitational fields on asymptotically null spacelike surfaces, 1805.03728. To appear in the forthcoming volume “Tullio Regge: an eclectic genius, from quantum gravity to computer play,” Eds. L. Castellani, A. Ceresola, R. D’Auria and P. Fré (World Scientific).
[10] T. Regge and C. Teitelboim. Role of surface integrals in the Hamiltonian formulation of General Relativity. Annals Phys., 88:286, 1974.
[11] A. Ashtekar and R. O. Hansen. A unified treatment of null and spatial infinity in general relativity. I - Universal structure, asymptotic symmetries, and conserved quantities at spatial infinity. J. Math. Phys., 19:1542–1566, 1978.
[12] A. Ashtekar and J.D. Romano. Spatial infinity as a boundary of spacetime. Classical and Quantum Gravity, 9(4):1069, 1992.
[13] M. Campiglia and A. Laddha. Asymptotic symmetries of QED and Weinberg’s soft photon theorem. JHEP, 07:115, 2015, 1505.05346.
[14] C. Troessaert. The BMS4 algebra at spatial infinity. 2017, 1704.06223.
[15] F. Rohrlich. Classical charged particles. Addison-Wesley, Boston, 1965.
[16] C. Teitelboim. Splitting of the Maxwell tensor - radiation reaction without advanced fields. Phys. Rev., D1:1572–1582, 1970. [Erratum: Phys. Rev.D2,1763(1970)].
[17] M. Henneaux and C. Troessaert. BMS group at spatial infinity: the Hamiltonian (ADM) approach. JHEP, 03:147, 2018, 1801.03718.
[18] M. Henneaux and C. Troessaert. Asymptotic symmetries of electromagnetism at spatial infinity. 2018, 1803.10194.
[19] M. Henneaux and C. Troessaert, Hamiltonian structure and asymptotic symmetries of the Einstein-Maxwell system at spatial infinity. arXiv:1805.11288 [gr-qc].
[20] M. Henneaux and C. Troessaert, The asymptotic structure of gravity at spatial infinity in four spacetime dimensions. arXiv:1904.04495 [hep-th].
[21] C. Bunster, A. Gomberoff and A. Pérez, “Hamiltonian analysis of the electromagnetic field on asymptotically null spacelike surfaces in the presence of a cosmological constant,” in preparation.
[22] H. Bondi. Gravitational waves in General Relativity. Nature, 186(4724):535–535, 1960.
[23] T. Regge and J. A. Wheeler. Stability of a Schwarzschild singularity. Phys. Rev., 108:1063–1069, 1957.
[24] M. Henneaux and C. Teitelboim. Duality in linearized gravity. Phys. Rev., D71:024018, 2005, gr-qc/0408101.
[25] C. Bunster, S. Cnockaert, M. Henneaux, and R. Portugal. Monopoles for gravitation and for higher spin fields. Phys. Rev., D73:105014, 2006, hep-th/0601222.
[26] C. Teitelboim. The Hamiltonian structure of space-time. Ph.D. Thesis, Princeton, unpublished, 1973.
[27] C. Teitelboim. How commutators of constraints reflect the space-time structure. Annals Phys., 79:542–557, 1973.