How to Detect and Construct N-matrices

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Abstract

N-matrices are real $n \times n$ matrices all of whose principal minors are negative. We provide (i) an $O(2^n)$ test to detect whether or not a given matrix is an N-matrix, and (ii) a characterization of N-matrices, leading to the recursive construction of every N-matrix.

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1 Introduction and Motivation

This work concerns N-matrices, that is, real $n \times n$ matrices, $A \in \mathbb{R}^{n \times n}$, all of whose principal minors are negative.

In prior discussions of N-matrices, their resemblance to P-matrices, which are matrices all of whose principal minors are positive, invariably comes up first. Indeed, P-matrices are widely studied since they contain several prominent classes of matrices, such as the positive definite matrices and the M-matrices; they find applications in mathematical programming, the study of univalence and complexity theory (see e.g., [2, 14, 16]). N-matrices find similar applications and possess properties analogous to P-matrices; they were introduced in [11] and have been studied e.g., in [15, 18, 19], along with special types of N-matrices in [12, 13].

Among the motivating factors for studying N-matrices (see also the conclusions in Section 6) is their connection to univalence (injectivity of differential maps in $\mathbb{R}^n$) and their role in the Linear Complementarity Problem (see Section 2). In addition, as it is evident in the existing theory of N-matrices and will be reinforced by the results herein, it is illuminating to identify and compare the effects of having signed principal minors in the two cases of N-matrices and P-matrices. There are similarities, distinctions, but also some unexpected connections between the two classes. Such instances will surface in our study of how to (i) detect N-matrices efficiently (Section 3), and (ii) construct all the N-matrices (Section 4).

Some background material and basic properties of N-matrices are reviewed in Section 2 which will help us develop and appropriately frame the results. Matlab implementations of algorithms for the detection of N-matrices and P-matrices are included in Section 5 for the reader’s convenience.

2 Background, Notation and Context

For a positive integer $n$, let $\langle n \rangle = \{1, 2, \ldots, n\}$. For $\alpha \subseteq \langle n \rangle$, $|\alpha|$ denotes the cardinality of $\alpha$ and $\overline{\alpha} = \langle n \rangle \setminus \alpha$. For $\alpha \subseteq \langle n \rangle$ with $|\alpha| = k$ and its elements arranged in ascending order, we let $x[\alpha]$ denote the vector in $\mathbb{C}^k$ obtained from the entries of $x \in \mathbb{C}^n$ indexed by $\alpha$. Moreover, we let $A[\alpha, \beta]$ denote the submatrix of $A \in \mathbb{R}^{n \times n}$ whose rows and columns are indexed by $\alpha, \beta \subseteq \langle n \rangle$, respectively; the elements of $\alpha, \beta$ are assumed to be in ascending order. When a row or column index set is empty, the corresponding submatrix is considered vacuous and by convention has determinant equal to 1. We abbreviate $A[\alpha, \alpha]$ by $A[\alpha]$ and refer to it as a principal submatrix of $A$ and its determinant as a principal minor of $A$. 
Given $A \in \mathbb{R}^{n \times n}$ and $\alpha \subseteq \langle n \rangle$ such that $A[\alpha]$ is invertible, $A/A[\alpha]$ denotes the Schur complement of $A[\alpha]$ in $A$, that is,

$$A/A[\alpha] = A[\alpha] - A[\alpha, \alpha] A[\alpha]^{-1} A[\alpha, \alpha].$$

**Definition 2.1.** Matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is

- an **N-matrix** if $\det A[\alpha] < 0$ for all $\alpha \subseteq \langle n \rangle$;
- a **P-matrix** if $\det A[\alpha] > 0$ for all $\alpha \subseteq \langle n \rangle$;
- an **almost P-matrix** if $\det A[\alpha] > 0$ for all proper $\alpha \subseteq \langle n \rangle$ and $\det A < 0$.

We further classify an N-matrix $A \in \mathbb{R}^{n \times n}$ as being

- **of the first category** if there exist $i, j \in \langle n \rangle$ such that $a_{ij} > 0$; or
- **of the second category** if $a_{ij} < 0$ for all $i, j \in \langle n \rangle$;

Array inequalities in the sequel are meant to be entrywise.

For reference and context needed in our further considerations, we gather below some analogous properties of N-matrices and P-matrices. For the definition and background on the Linear Complementarity problem $\text{LCP}(A, q)$, $A \in \mathbb{R}^{n \times n}$, $q \in \mathbb{R}^n$, see [5]. For the definition and properties of the Principal Pivot Transform $\text{ppt}(A, \alpha)$, $A \in \mathbb{R}^{n \times n}$, $\alpha \subseteq \langle n \rangle$, see [20].

**N-matrices:**

- **[N1]** $A \in \mathbb{R}^{n \times n}$ is an N-matrix if and only if $A^{-1}$ is an almost P-matrix [17].
- **[N2]** $A \in \mathbb{R}^{n \times n}$ is an N-matrix of the second category if and only if for every $q > 0$, $\text{LCP}(A, q)$ has exactly 2 solutions [18].
- **[N3]** $A \in \mathbb{R}^{n \times n}$ is an N-matrix of the second category if and only if $A$ does not reverse the sign of any nonzero, unisigned vector $x = [x_i] \in \mathbb{R}^n$; i.e., $(Ax)_i x_i \leq 0$ for all $i \in \langle n \rangle$ implies $x \geq 0$ or $x \leq 0$ [18].
- **[N4]** If $A \in \mathbb{R}^{n \times n}$ is an N-matrix, then $A/A[\alpha]$ is a P-matrix for all proper subsets $\alpha$ of $\langle n \rangle$ [19].
- **[N5]** Let $A \in \mathbb{R}^{n \times n}$ be an N-matrix, $\alpha$ be a proper subset of $\langle n \rangle$ and $B = \text{ppt}(A, \alpha)$. Then $\det(B[\alpha]) < 0$ and all other principal minors of $B$ are positive [19].
• [N6] N-matrices have exactly one real negative eigenvalue [18].

**P-matrices:** See [14, Chapter 3] or [21] for a treatment of P-matrices.

• [P1] $A \in \mathbb{R}^{n \times n}$ is a P-matrix if and only if $A^{-1}$ is a P-matrix.

• [P2] $A \in \mathbb{R}^{n \times n}$ is a P-matrix if and only if for every $q \in \mathbb{R}^n$, LCP($A, q$) has a unique solution.

• [P3] $A \in \mathbb{R}^{n \times n}$ is a P-matrix if and only if any nonzero vector $x = [x_i] \in \mathbb{R}^n$; i.e., $(Ax)_i x_i \leq 0$ for all $i \in \langle n \rangle$ implies $x = 0$.

• [P4] If $A \in \mathbb{R}^{n \times n}$ is a P-matrix, then $A/A[\alpha]$ is a P-matrix for all $\alpha \subseteq \langle n \rangle$.

• [P5] $A \in \mathbb{R}^{n \times n}$ is a P-matrix if and only if ppt($A, \alpha$) is a P-matrix for any (and thus all) $\alpha \subseteq \langle n \rangle$.

• [P6] P-matrices have no real negative eigenvalues.

## 3 Detecting N-matrices

The problem of detecting P-matrices is known to be co-NP-complete [6]. The computation of all the principal minors of $A \in \mathbb{R}^{n \times n}$ via row reduction leads to an $O(n^3 2^n)$ effort. A more efficient (but still exponential) algorithm to compute all principal minors of a square matrix is developed in [9]; the inverse problem of constructing a matrix from a feasible set of principal minors is solved in [10]. In the same vein, an efficient, recursive algorithm to detect P-matrices of $O(2^n)$ time complexity is developed in [22]. These algorithms are based on the following theorem.

**Theorem 3.1.** [22, Theorem 3.1] Let $A \in \mathbb{R}^{n \times n}$ and $\alpha \subseteq \langle n \rangle$ with $|\alpha| = 1$. Then $A$ is a P-matrix if and only if $A[\alpha]$, $A[\overline{\alpha}]$ and $A/A[\alpha]$ are P-matrices.

We can extend the theorem above into the following characterization of N-matrices.

**Theorem 3.2.** Let $A \in \mathbb{R}^{n \times n}$ and $\alpha \subseteq \langle n \rangle$ with $|\alpha| = 1$. Then $A$ is an N-matrix if and only if $A[\alpha]$, $A[\overline{\alpha}]$ are N-matrices and $A/A[\alpha]$ is a P-matrix.

**Proof.** Without loss of generality, let $\alpha = \{1\}$; otherwise, our considerations apply to a permutation similarity of $A$. Suppose that $A$ is an N-matrix. By definition, $A[\alpha]$ and $A[\overline{\alpha}]$ are N-matrices.
By [N4], $A/A[\alpha]$ is a P-matrix.

Conversely, suppose $A[\alpha]$ and $A[\alpha]$ are N-matrices and $A/A[\alpha]$ is a P-matrix. The determinant of any principal submatrix of $A$ without any entries from the first column is a principal minor of $A[\alpha]$ and it is thus negative. Let $B$ be any principal submatrix of $A$ with entries from the first column of $A$. Then $C = A[\alpha]$ is a principal submatrix (diagonal entry) of $B$, and $B/C$ is a principal submatrix of $A/A[\alpha]$. Thus $\det(B/C) > 0$ and so $\det(B) = A[\alpha] \det(B/C) < 0$. Hence $A$ is an N-matrix.

Theorem 3.2 suggests the following recursive algorithm for detecting N-matrices.

**Algorithm N(A)**

1. Input $A = [a_{ij}] \in \mathbb{R}^{n \times n}$
2. If $a_{11} \geq 0$, output “A is not an N-matrix” stop
3. Compute $A/a_{11}$
4. If $A/a_{11}$ is not P-matrix output “A is not an N-matrix” stop
5. Call $N(A[\{1\}])$
6. Output “A is an N-matrix”

A Matlab implementation of algorithm $N(A)$ is found in Section 5 (NTEST). The algorithm needed in (step 4) of NTEST to detect a P-matrix is based on Theorem 3.1 and also provided in Section 5 (PTEST).

## 4 Constructing All N-matrices

Examples of N-matrices, even of special structure and form, are not as easy to generate as is for examples of P-matrices. Some possibilities include the types of N-matrices considered in [8] and [12, 13], as well as the totally negative matrices (all minors are negative) in [3, 7]. In [4, Theorem 7.12], some necessary conditions are presented on the signs of the entries of an N-matrix of the first category.

In this section, using a recursion based on rank-one perturbations of N-matrices, we can reverse the steps of the recursive algorithm $N(A)$ that detects N-matrices and thus construct every N-matrix of either category. This approach is based on the following corollary of Theorem 3.2.
Corollary 4.1. Let $A \in \mathbb{R}^{n \times n}$ be an N-matrix of the second category, $a \in \mathbb{R}$ and let $x, y \in \mathbb{R}^n$. Then the following are equivalent:

(i) $U = \begin{bmatrix} A & x \\ y^T & a \end{bmatrix}$ is an N-matrix of the second category.

(ii) $a, x, y < 0$ and $A - \frac{1}{a} xy^T$ is a P-matrix.

Corollary 4.1 allows us to recursively construct $n \times n$ ($n \geq 2$) N-matrices of the second category as follows.

ALGORITHM NCON2

1. Choose $A_1 < 0$

2. For $i = 1 : n - 1$, given the $i \times i$ N-matrix of the second category $A_i$,

   (a) choose $a_i < 0$ and $x^{(i)}, y^{(i)} \in \mathbb{R}^i$ such that $x^{(i)}, y^{(i)} < 0$ and $A_i - \frac{1}{a_i} x^{(i)} y^{(i)}T$ is a P-matrix

   (b) construct the $(i+1) \times (i+1)$ matrix $A_{i+1} = \begin{bmatrix} A_i & x^{(i)} \\ y^{(i)}T & a_i \end{bmatrix}$

3. Output “$A = A_n$ is an N-matrix of the second category”

Theorem 4.1. Every matrix constructed by NCON2 is an N-matrix of the second category. Conversely, every N-matrix of the second category can be constructed by NCON2.

Proof. By Corollary 4.1, the sequence of matrices $A_{i+1}$ ($i = 1, \ldots, n - 1$) constructed by NCON2, including $A_1$, are N-matrices of the second category. To prove the converse, we proceed by induction on the order of matrices. The statement is trivial for $n = 1$. Let $n \geq 2$ and suppose that every N-matrix of the second category of order smaller than $n$ can be constructed by NCON2. Let $A \in \mathbb{R}^{n \times n}$ be an N-matrix of the second category. Then $A$ can be partitioned as

$$A = \begin{bmatrix} A_{n-1} & u \\ v^T & a \end{bmatrix},$$

where $A_{n-1} \in \mathbb{R}^{(n-1) \times (n-1)}$ is N-matrix of the second category, $u, v \in \mathbb{R}^{n-1}$ and $a \in \mathbb{R}$. By inductive hypothesis, $A_{n-1}$ can be constructed by NCON2. Since $A$ is N-matrix of the second category, by Corollary 4.1, $A_{n-1}/a = A_{n-1} - \frac{1}{a} uv^T$ is a P-matrix and $a, x, y < 0$. Thus $A_n = A$ can be constructed by NCON2 with the following choices:
\[a_{n-1} = a, \ x^{(n-1)} = u \text{ and } y^{(n-1)} = v.\]

To extend our construction methodology to N-matrices of the first category, we recall the following result.

**Theorem 4.2.** [1, Theorem 7.9.4] Let \(A\) be a N-matrix of the first category. Then there exists a permutation matrix \(P\) such that

\[PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\]

(4.1)

where \(A_{11}, A_{22} < 0\) are square matrices and \(A_{12}, A_{21} > 0\).

By Theorem 4.2 in order to construct all N-matrices of the first category of size \(n \geq 2\), it is sufficient to construct them in the form (4.1), where \(A_{11} \in \mathbb{R}^{k \times k}\) \((k < n)\). This can be achieved using the following Corollary of Theorem 3.2.

**Corollary 4.2.** Let \(A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}^{n \times n} \ (n \geq 2)\) be an N-matrix, where \(A_{11} \in \mathbb{R}^{k \times k} \ (k \leq n), A_{22} \in \mathbb{R}^{n-k \times n-k}\) with \(A_{11}, A_{22} < 0\) and \(A_{12}, A_{21} > 0\). Let \(a \in \mathbb{R}\) and let \(x, y \in \mathbb{R}^n\). Then the following are equivalent:

(i) \(U = \begin{bmatrix} A \\ x^T \\ a \end{bmatrix}\) is an N-matrix of the first category.

(ii) \(a < 0, A - \frac{1}{a} xy^T\) is a P-matrix, \(x[\langle k \rangle], y[\langle k \rangle] > 0\) and \(x[\langle k \rangle], y[\langle k \rangle] < 0\).

Using Corollary 4.2 we can construct N-matrices of the first category as follows.

**ALGORITHM NCON1**

1. Construct \(A_k = A_{11}\) using algorithm NCON2

2. For \(i = k : n - 1\), given the \(i \times i\) matrix \(A_i\),

   (a) choose \(a_i < 0, x^{(i)}, y^{(i)} \in \mathbb{R}^i\) such that \(x[\langle k \rangle], y[\langle k \rangle] > 0, x[\langle k \rangle], y[\langle k \rangle] < 0\), and \(A_i - \frac{1}{a_i} x^{(i)} y^{(i)^T}\) is a P-matrix

   (b) construct the \((i + 1) \times (i + 1)\) matrix \(A_{i+1} = \begin{bmatrix} A_i & x^{(i)} \\ y^{(i)^T} & a_i \end{bmatrix}\)

3. Output “\(A = A_n\) is an N-matrix of the first category”
Theorem 4.3. Every matrix constructed by NCON1 is an N-matrix of the first category. Conversely, every N-matrix of the first category can be constructed as a permutational similarity of a matrix constructed by NCON1.

Proof. By Corollary 4.2, the sequence of matrices $A_{i+1}$ ($i = 1, \ldots, n - 1$) constructed by NCON1, are N-matrices of the first category. We use induction on the order of matrices to prove the converse. The base case $n = 2$ is obvious. Let $n > 2$ and suppose that every N-matrix of the first category of order smaller than $n$ can be constructed as a permutational similarity of a matrix constructed by NCON1. Let $A \in \mathbb{R}^{n \times n}$ be an N-matrix of the first category. Then there exist a permutation matrix $P$ such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{k \times k}$ ($k < n$), $A_{22} \in \mathbb{R}^{n-k \times n-k}$ with $A_{11}, A_{22} < 0$ and $A_{12}, A_{21} > 0$. Let

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{n-1} \\ u^T \\ v \end{bmatrix},$$

where $A_{n-1} \in \mathbb{R}^{n-1 \times n-1}$ is N-matrix, $a < 0$, $u, v \in \mathbb{R}^{n-1}$ with $u[k], v[k] > 0$ and $u[k], v[k] < 0$. Now, either $A_{n-1} < 0$ or $A_{n-1}$ is of the form (4.1). By inductive hypothesis, $A_{n-1}$ can be constructed using NCON1. Since $A$ is N-matrix of the first category, by Corollary 4.2, $A_{n-1}/a = A_{n-1} - \frac{1}{a} uv^T$ is a P-matrix. Thus $A_n = P^T AP$ can be constructed by NCON1 with the following choices:

$$a_{n-1} = a, \ x^{(n-1)} = u \text{ and } y^{(n-1)} = v.$$

Remark 4.1.

(1) The implementation of step 2(a) in algorithms NCON1 and NCON2 can be done via random choice of the appropriately signed vectors $x^{(i)}$ and $y^{(i)}$ and judicious choice of the diagonal entries $a_i$. The process of choosing $a_i$ so that $A_i - \frac{1}{a_i} x^{(i)} y^{(i)^T}$ is a P-matrix is developed and its effects explained in the recursive construction of all P-matrices presented in [23, Section 4].

(2) In light of [N1] in Section 2 Algorithms NCON1 and NCON2 may also be viewed as methods to construct almost P-matrices via inversion.

We proceed with two illustrative examples of N-matrices constructed using NCON1 and NCON2.
Example 4.1. We construct $3 \times 3$ N-matrix of the first category. Let $A_1 = [-1]$, $a_1 = -1$, $x^{(1)} = [2]$ and $y^{(1)} = [2]$. Then $A_1 - \frac{1}{a_1} x^{(1)} y^{(1)^T} = [3]$ is a P-matrix.

By NCON1, $A_2 = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ is N-matrix of the first category. Now, let $a_2 = -1$, $x^{(2)} = [2 \ -1]^T$ and $y^{(2)} = [2 \ -2]^T$. Then $A_2 - \frac{1}{a_2} x^{(2)} y^{(2)^T} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$ is a P-matrix. Again, by NCON1, $A_3 = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & -1 \\ 2 & -2 & -1 \end{bmatrix}$ is N-matrix of the first category.

Example 4.2. In this example, we construct a $3 \times 3$ N-matrix of the second category by NCON2. Let $A_1 = [-1]$, $a_1 = -1$, $x^{(1)} = [-1]$ and $y^{(1)} = [-2]$. Then $A_1 - \frac{1}{a_1} x^{(1)} y^{(1)^T} = [1]$ is a P-matrix. By NCON2, $A_2 = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}$ is N-matrix of the second category. Now, we take $a_2 = -1$, $x^{(2)} = [-2 \ -1]^T$ and $y^{(2)} = [-3 \ -2]^T$. Then $A_2 - \frac{1}{a_2} x^{(2)} y^{(2)^T} = \begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}$ is a P-matrix. Hence, by NCON2, $A_3 = \begin{bmatrix} -1 & -1 & -2 \\ -2 & -1 & -1 \\ -3 & -2 & -1 \end{bmatrix}$ is N-matrix of the second category.

5 NTEST and PTEST

We include Matlab code for the detection of P-matrices and N-matrices.

PTEST (detects P-matrices)
function [r] = ptest(A)
    \% Return r=1 if 'A' is a P-matrix (r=0 otherwise).
    n = length(A);
    if ~(A(1,1)>0), r = 0; elseif n==1, r = 1; else
        b = A(2:n,2:n);
        d = A(2:n,1)/A(1,1);
        c = b - d*A(1,2:n);
        r = ptest(b) & ptest(c);
    end
NTEST (detects N-matrices)
function [r] = ntest(A)
% Return r=1 if ‘A’ is a N-matrix (r=0 otherwise).

n = length(A);
if ~(A(1,1)<0), r = 0;
elseif n==1, r = 1;
else
    b = A(2:n,2:n);
    d = A(2:n,1)/A(1,1);
    c = b - d*A(1,2:n);
    r = ntest(b) & ptest(c);
end

Note that the time complexity of PTEST is $O(2^n)$ [22], and so this must also be the case for NTEST as the same binary tree of matrices (of orders recursively reduced by 1) is being created by the two algorithms.

6 Conclusions

N-matrices are challenging to detect and understand their nature. The progress reported allows for their further consideration in applications. One can now detect such matrices by an algorithm that because of its recursive nature can be implemented in parallel. One can also construct generic N-matrices (of either category) for purposes of experimentation and development of new theory and algorithms. Moreover, the work herein contributes in better understanding the role of the signs of the principal minors in the theory of inequalities and in the study of computational complexity (generally and within the confines of complementarity problems).

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