Constraints on primordial magnetic fields from inflation

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Abstract. We present generic bounds on magnetic fields produced from cosmic inflation. By investigating field bounds on the vector potential, we constrain both the quantum mechanical production of magnetic fields and their classical growth in a model independent way. For classical growth, we show that only if the reheating temperature is as low as $T_{\text{reh}} \lesssim 10^2 \text{MeV}$ can magnetic fields of $10^{-15} \text{G}$ be produced on Mpc scales in the present universe. For purely quantum mechanical scenarios, even stronger constraints are derived. Our bounds on classical and quantum mechanical scenarios apply to generic theories of inflationary magnetogenesis with a two-derivative time kinetic term for the vector potential. In both cases, the magnetic field strength is limited by the gravitational back-reaction of the electric fields that are produced simultaneously. As an example of quantum mechanical scenarios, we construct vector field theories whose time diffeomorphisms are spontaneously broken, and explore magnetic field generation in theories with a variable speed of light. Transitions of quantum vector field fluctuations into classical fluctuations are also analyzed in the examples.

Keywords: inflation, physics of the early universe, primordial magnetic fields, cosmic magnetic fields theory

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1 Introduction

Magnetic fields exist in our universe on various length scales, however their origin remains a mystery. One possibility that has been vigorously pursued is that the magnetic fields were seeded in the early universe, in particular, during the inflationary epoch. The purpose of this paper is to examine the validity of inflationary magnetogenesis in a model independent way, by analyzing field bounds for the vector potential.

There are two ways to have large magnetic fields by the end of cosmic inflation:

- **Classical growth.** Initially tiny magnetic fields (such as those originating from quantum fluctuations of the vector potential) undergo classical growth during inflation.

- **Quantum mechanical production.** Large vector field fluctuations are produced at the quantum level, setting large magnetic fields as an ‘initial condition’ for the subsequent classical evolution.

While many authors have investigated the former scenario since the seminal works [1, 2], the latter is much less studied. Here we note that quantum mechanical scenarios can also be studied in a computable framework. For instance, if the vector field has a non-trivial dispersion relation during inflation then the fields can, in principle, obtain large quantum fluctuations leading to large magnetic fields. This can be achieved in vector field theories with higher derivatives, or with spontaneously broken Lorentz invariance. Alternatively, an
oscillating background can also induce quantum mechanical production of the vector through parametric resonance.

For both classical and quantum mechanical scenarios, in this paper we derive upper bounds on the magnetic fields that can be produced during inflation, independently of the details of the model. To achieve this, we study field range bounds on the vector potential. For classical scenarios, magnetic field generation is supported by the time-evolution of the vector potential. By analyzing the induced kinetic energy of the vector field (which is proportional to the electric field strength squared) and its backreaction to the background cosmology, we constrain the vector field evolution during inflation. The calculations will be carried out independently of the specific dynamics of the vector field.

Quantum mechanical scenarios, on the other hand, do not necessarily rely on time-evolving vector fields, but instead invoke large quantum fluctuations. Nevertheless, magnetogenesis in these scenarios are also found to be severely constrained, as the energy that can be extracted from the inflationary background is limited. By computing the background energy that is available for the quantum fluctuations, we constrain magnetic fields that are set as ‘initial conditions’ for the subsequent classical evolution. Upon discussing quantum mechanical scenarios, we will also study how the quantum vector fluctuations transform into classical fluctuations in some example models.

We will occasionally compare our magnetic field bounds with the lower limit on intergalactic magnetic fields reported by recent gamma ray observations, which is typically of $10^{-15}$ G assuming a correlation length of $\lambda_B \geq 1$ Mpc [4–9]. (If $\lambda_B$ is much smaller than a Mpc, the lower limit of the intergalactic magnetic field strength further improves as $\propto \lambda_B^{-1/2}$.) Our results will show that, within our assumptions, only for a very narrow window of the inflation and reheating scales is it possible to produce magnetic field strengths of $10^{-15}$ G on Mpc scales or larger.

In our calculations for constraining inflationary magnetogenesis, we will assume that the vector field theory has a two-derivative time kinetic term; within this assumption, our bounds apply to generic vector field theories, including models with interactions with other physical degrees of freedom, broken time diffeomorphisms, higher-order spatial derivatives, etc. As an example of quantum mechanical magnetogenesis, we also perform an analysis of vector field theories in backgrounds that break the time diffeomorphisms. The theory is studied using similar techniques to [10, 11] for formulating the effective field theory of inflation; we start by constructing the vector field action in unitary gauge. When the gauge invariant action (i.e. the action whose time diffeomorphisms are restored) is needed for computing quantities such as the energy-momentum tensor, we will use the Stückelberg trick to reintroduce the Nambu-Goldstone boson of the broken time diffeomorphisms. We then discuss magnetic field generation in cases where the broken Lorentz symmetry gives rise to a variable speed of light.

The paper is organized as follows: we first define the electromagnetic fields and set our notation in section 2. Then we derive magnetic field bounds on classical and quantum mechanical scenarios in section 3 and section 4, respectively. In section 5 we summarize our findings and discuss the implications for cosmological magnetogenesis. The appendices are devoted to studies of explicit models. In appendix A, we study $I^2FF$ models as a typical example of the classical scenarios discussed in section 3. In appendix B, we construct and

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1In this sense, our approach can be related to the Lyth bound [3] in inflation. However, unlike the Lyth “bound” which connects observably large gravitational waves to super-Planckian field excursions of the inflaton field, our magnetic field bounds will arise from the requirement that the vector field’s energy density should not significantly backreact on the inflationary background.
study a vector field theory with a general light speed. Here we also analyze how the quantum vector fluctuations transform into classical fluctuations. The model investigated in this appendix will serve as an example of quantum mechanical scenarios discussed in section 4.1.

2 Electromagnetic fields

The theory of electromagnetism could have been quite different in the early universe, e.g. with higher-derivative terms for the vector potential, interactions with beyond the Standard Model particles such as the inflaton. However by the present times, the theory should recover the standard Maxwell theory:

$$S = \int d^4x \sqrt{-g} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \text{(couplings to charged fields)} \right\} ,$$  \hspace{1cm} (2.1)

where $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$. We use Greek letters for the spacetime indices $$\mu, \nu = 0, 1, 2, 3$$, and Latin letters for spatial indices $$i, j = 1, 2, 3$$. Then we can define the electric and magnetic fields such that by the present times, they become the fields measured by a comoving observer with 4-velocity $$u^\mu$$ ($$u^t = 0$$, $$u_\mu u^\mu = -1$$),

$$E_\mu = u^\nu F_{\mu\nu}, \quad B_\mu = \frac{1}{2} \varepsilon_{\mu\nu\sigma} F^{\nu\sigma} ,$$  \hspace{1cm} (2.2)

where $$\varepsilon_{\mu\nu\sigma} = \eta_{\mu\nu\sigma\rho} u^\rho$$, and $$\eta_{\mu\nu\sigma\rho}$$ is a totally antisymmetric tensor with $$\eta_{0123} = -\sqrt{-g}$$. Throughout this paper we consider a flat FRW universe,

$$ds^2 = a(\tau)^2 (-d\tau^2 + d\mathbf{x}^2) .$$  \hspace{1cm} (2.3)

Metric fluctuations will not play an important role in our discussion and have been set to zero.

We assume the electromagnetic theory preserves the gauge invariance $$A_\mu \to A_\mu - \partial_\mu \theta$$ at all times, thus it is possible to always choose a gauge in which the time component of the vector potential vanishes,

$$A_0 = 0 .$$  \hspace{1cm} (2.4)

Then the electromagnetic field amplitudes are written as

$$E^\mu E_\mu = \frac{1}{a^4} A'_i A'_i , \quad B^\mu B_\mu = \frac{1}{a^4} (\partial_i A_j \partial_j A_i - \partial_i A_j \partial_j A_i) ,$$  \hspace{1cm} (2.5)

where a prime denotes a derivative in terms of the comoving time $$\tau$$, and the sum over repeated spatial indices is implied irrespective of their positions.

In the case of the standard Maxwell theory (2.1), when ignoring couplings to other fields and taking the Coulomb gauge $$A_0 = \partial_i A_i = 0$$, the action reduces to (after dropping surface terms),

$$S = \frac{1}{2} \int d\tau d^3x \left( A'_i T^T A'^T_i - \partial_i A'^T_i \partial_j A'_i \right) .$$  \hspace{1cm} (2.6)

Here the “T” superscript denotes the vector potential being transverse. The equations of motion, i.e. Maxwell’s equations, read

$$A'_i T^T - \partial_j \partial_j A'_i = 0 ,$$  \hspace{1cm} (2.7)

yielding plane wave solutions. Thus one sees that both the electric and magnetic field strengths squared (2.5) basically decay as $$\propto a^{-4}$$ in an expanding universe. Furthermore,
Maxwell theory is conformal and therefore the evolution of $A_i$ is equivalent to in flat space, as we can see from (2.6).

In order for large magnetic fields to arise from the early universe, the Maxwell theory has to be modified. Conformal invariance must necessarily be broken to seed the classical primordial fluctuations from the (quantum) vacuum. Furthermore, we must alter fields’ time evolution and/or amplitude of primordial fluctuations in order to leave appreciable magnetic fields at late times. We will discuss each of these possibilities in the following sections and show that, in either case, magnetic field production during the inflationary era is highly restricted.

### 3 Constraints on classical growth

Let us study magnetogenesis scenarios where the initially tiny magnetic fields, such as those originating from quantum fluctuations of the vector potential, experience classical growth during the inflationary epoch. Examples of such scenarios are [1, 2], which considered breaking the conformal invariance of the Maxwell theory during inflation; after the wave modes exit the effective “horizon” induced by the conformal symmetry breaking, the modes become classical and the cosmological background can enhance the super-horizon magnetic fields. However, detailed studies of individual models have revealed that, for the simplest conformal symmetry breaking scenarios, electric fields are also enhanced along with the magnetic fields and tend to become so large as to spoil inflation and/or magnetogenesis [12–16]. In this section, we discuss the electric field production independently of the details of the model, and show that the electric backreaction problem is inherent to the classical growth of magnetic fields during inflation. The approach we take is close to the one presented in [15] for studying $I^2FF$ models, however, we obtain a much stronger result by optimizing the combination of constraints in the theory. For the differences with the calculations in [15], we refer the reader to appendix A.

In order to discuss electromagnetic fields with certain correlation lengths, it is convenient to go to Fourier space,

$$A_i(\tau, x) = \frac{1}{(2\pi)^3} \int d^3k \, e^{i k \cdot x} \tilde{A}_{i,k}(\tau) a_k,$$

where $a_k$ is a stochastic variable that satisfies $\langle a_k a^*_k \rangle = (2\pi)^3 \delta(k - k')$. (In the quantum mechanical case, as in section 4, the stochastic variables $a_k$ and $a^*_k$ are promoted to annihilation and creation operators.) This definition\(^2\) is meant to reflect the idea that the initial amplitude for $\tilde{A}_{i,k}$ arises from a random process. We also remark that since we are interested in an FRW background, the amplitude $\tilde{A}_{i,k}$ is considered to depend only on the magnitude $k = |k|$ of the comoving wave number.

We evaluate the electric and magnetic field strengths (2.5) associated with $k$ in terms of the power spectra defined as

$$\langle B_\mu(\tau, x) B^\mu(\tau, y) \rangle = \int \frac{d^3k}{4\pi k^3} e^{i k \cdot (x-y)} P_B(\tau, k),$$

\(^2\)Given a measurement of a electric or magnetic field, one would use a discrete Fourier transform to describe the field rather than the continuous one that appears here. For the discrete case, we should use $\langle a_i a^*_j \rangle = \delta_{ij}$ where $i$ labels the discrete momentum vectors. This is important only for establishing the correct units for $A_{i,k}$ in each case. We also note that, strictly speaking, we should introduce the stochastic variables for each independent degrees of freedom of the vector field; however the expression (3.1) is sufficient for the purpose of obtaining order-of-magnitude estimates.
While this is equivalent to setting dependent values are preferable for avoiding backreaction issues, as we will soon see. However it should be noted that it is also remarked that, if \( i \) is some arbitrary time before \( \tau \) during inflation, then after canonically normalizing the vector field as \( \bar{A}_i, k(\tau) \bar{A}^i_k(\tau) \), it becomes canonically normalized and the theory connects to the standard Maxwell theory.\(^3\) The vector field action during inflation may contain higher-derivatives and/or interaction terms. However, let us suppose that the dominant time kinetic term of the vector field \( A_i \) is some arbitrary time before \( \tau, \bar{A}_i, k(\tau) \bar{A}^i_k(\tau) \). Noting that \( \bar{A}_i, k(\tau) \bar{A}^i_k(\tau) \) is a function of the inflaton field and thus varies in time as the inflaton field rolls along \( \tau \). Alternatively, we are free to work with the canonically normalized field, \( A^i_\mu \), at all times. This definition implies that \( A_i \) becomes canonically normalized and the theory connects to the standard Maxwell theory (2.6) at late times. For example, in the model proposed in [2], \( I^2 \) is a function of the inflaton field and thus varies in time as the inflaton field rolls along its potential.

\(^3\)This inequality is ill-defined when \( f(x) \) crosses zero. Nevertheless, the integrated inequality still holds provided \( f(x) \) is differentiable.
these terms are small relative to the two derivative kinetic term, then these additional terms do not violate any of our technical assumptions below that the results will hold even if $I \ll 1$ in the past. As a result, $I \ll 1$ is only a meaningful possibility when $|I'|/I$ is sufficiently large that these terms are comparable in size to the standard kinetic term. Of course, even in this case, one has to worry about the strong coupling constraint as well. (See also section 5 for related discussions.)

In the following, to avoid the aforementioned issues, we require

$$I^2 \gtrsim 1$$

(3.8)

to hold during the times $\tau_i \leq \tau \leq \tau_f$. For a classical vector field, the two-derivative kinetic term contributes to the field’s energy density as

$$\rho_{\text{kin}} \sim \frac{I^2}{a^2} \langle \partial^2 A_i \rangle = \int \frac{dk}{k} \frac{k^3}{2\pi^2} \frac{I^2}{a^4} |\tilde{A}'_k|^2$$

(3.9)

which is roughly the electric field strength squared multiplied by the coefficient $I^2$, cf. (3.5). (For a quantum field, this estimate of the energy density is not necessarily meaningful as the contributions from high $k$ modes should be renormalized and absorbed into the definition of the cosmological constant.) Supposing that the kinetic contribution (3.9) to the vector field’s energy density is not cancelled by other vector terms in the action, then $\rho_{\text{kin}}$ should be much smaller than the total energy density of the inflationary universe to avoid the vector fields from spoiling inflation and/or magnetogenesis. Thus, for purely classical modes we require

$$\int \frac{dk}{k} \frac{k^3}{2\pi^2} \frac{I^2}{a^4} |\tilde{A}'_k|^2 \ll 3M_p^2 H_{\text{inf}}^2,$$

(3.10)

where $H_{\text{inf}}$ is the inflationary Hubble rate.

Now, let us assume that the vector field fluctuations on wave modes of interest become classical by the time $\tau_i$.\footnote{In conformal symmetry breaking models such as [1, 2], the vector fluctuations become classical after the modes exit the effective horizon induced by the symmetry breaking. However we also note that modes can become classical much earlier, if, for instance, an oscillatory background gives rise to resonant production of the vector field while the modes are deep inside the horizon. The initial time $\tau_i$ is considered to be taken after the mode is effectively classical. See also the appendices, in particular appendix B.3, where we discuss how modes become classical in more detail.} By ‘becoming classical,’ we mean that the amplitude of the commutator of the vector field and its conjugate momentum $[\tilde{A}_k, \tilde{\Pi}_k]^2$ becomes negligibly tiny compared to $\langle \tilde{A}_k \tilde{A}_h \rangle \langle \tilde{\Pi}_k \tilde{\Pi}_h \rangle$ (see e.g. [18] for a recent discussion). Then from the above discussions, the contribution to the kinetic energy from each classical mode is bounded by the background density as

$$\frac{k^3}{2\pi^2} \frac{I^2}{a^4} |\tilde{A}'_k|^2 \ll 3M_p^2 H_{\text{inf}}^2,$$

(3.11)

where we have assumed that there are modes populating a range of modes, $\Delta k \simeq k$, such that we can approximate the integral as $\int \frac{dk}{k} \to 1$. This condition constrains the time derivative
of the vector potential.\footnote{The condition \eqref{eq:shell} could be violated, if the spectrum of $k^3|\tilde{A}_k|$ has sharp features localized to a very narrow range of $k$ such that $\Delta k \ll k$. In order to achieve such a narrow window, the action would require rapidly time varying coefficients which will typically lead to particle production. It is likely that such scenarios would be characterized by the discussion in section 4.2.1, but in principle there could be room to violate \eqref{eq:shell} and produce large magnetic fields over very narrow $k$ ranges. In such cases, one should instead discuss the magnetic field spectrum integrated over wave numbers of interest in order to relate to measurements of magnetic fields; see also footnote 11.} Therefore the combination of \eqref{eq:vp}, \eqref{eq:epsh}, and \eqref{eq:shell} yields
\begin{equation}
|\tilde{A}_k(\tau_f)| - |\tilde{A}_k(\tau_i)| \lesssim \int_{\tau_i}^{\tau_f} d\tau \frac{\sqrt{6} \pi a^2 M_p H_{\text{inf}}}{k^{3/2}} = \frac{\sqrt{6} \pi M_p (a_f - a_i)}{k^{3/2}}.
\end{equation}
Here, upon obtaining the far right hand side we have used the relationship $d\tau = \frac{da}{a^2 H_{\text{inf}}}$ for the conformal time in a de Sitter universe, and $a_f = a(\tau_f)$, $a_i = a(\tau_i)$. In this section we are interested in cases where the vector potential grows after becoming classical, therefore, taking the time $\tau_i$ to be sufficiently before the end of magnetogenesis such that $a_f \gg a_i$, we can suppose
\begin{equation}
|\tilde{A}_k(\tau_f)| \gg |\tilde{A}_k(\tau_i)|,
\end{equation}
and obtain an upper bound on the absolute value of the vector potential,
\begin{equation}
|\tilde{A}_k(\tau_f)| \lesssim \frac{\sqrt{6} \pi M_p a_f}{k^{3/2}}.
\end{equation}
Since $|\tilde{A}_k|$ becomes time independent after $\tau_f$ and the magnetic field \eqref{eq:epsh} redshifts as $P_B \propto a^{-4}$, we obtain an upper bound on the magnetic field strength in the present universe,
\begin{equation}
P_B(\tau_0, k) = P_B(\tau_f, k) \left( \frac{a_f}{a_0} \right)^4 \lesssim 3 M_p^2 \left( \frac{k}{a_0} \right)^2 \left( \frac{a_f}{a_0} \right)^2 \lesssim 3 M_p^2 \left( \frac{k}{a_0} \right)^2 \left( \frac{a_{\text{end}}}{a_0} \right)^2.
\end{equation}
The subscript “0” denotes values in the present universe, and $a_{\text{end}} (\geq a_f)$ is the scale factor at the end of inflation.

Let us suppose that the universe is effectively matter-dominated after inflation until reheating, i.e.,
\begin{equation}
\left( \frac{H_{\text{reh}}}{H_{\text{inf}}} \right)^2 = \left( \frac{a_{\text{end}}}{a_{\text{reh}}} \right)^3,
\end{equation}
where values at reheating are represented by the subscript “reh.” Then, considering entropy conservation ($s \propto a^{-3}$) after reheating, we have\footnote{To compute the entropy density at reheating}
\begin{equation}
a_{\text{reh}} \approx 6 \times 10^{-32} \left( \frac{M_p}{H_{\text{reh}}} \right)^{1/2},
\end{equation}
which together with \eqref{eq:energy_magn} gives the expansion after inflation,
\begin{equation}
a_{\text{end}} \approx 6 \times 10^{-32} \left( \frac{H_{\text{reh}}}{H_{\text{inf}}} \right)^{1/6} \left( \frac{M_p}{H_{\text{inf}}} \right)^{1/2}.
\end{equation}
Further using $1 \, G \approx 2.0 \times 10^{-20} \, \text{GeV}^2$ (we use the Heaviside-Lorentz units) and $1 \, \text{Mpc} \approx 1.6 \times 10^{28} \, \text{GeV}^{-1}$, then the magnetic field bound (3.15) is rewritten as

$$P_B(\tau_0, k) \lesssim (10^{-15} \, G)^2 \left( \frac{k}{\chi_0} \, \text{Mpc} \right)^2 \left( \frac{H_{\text{reh}}}{H_{\text{inf}}} \right)^{1/3} \frac{10^{-14} \, \text{GeV}}{H_{\text{inf}}}. \quad (3.20)$$

Thus we find that in order to create magnetic fields of $10^{-15} \, G$ on Mpc scales or larger, the inflation scale has to be as low as $H_{\text{inf}} \lesssim 10^{-14} \, \text{GeV}$; otherwise the vector field’s electric backreaction to the inflationary universe would be significant, and/or the theory would have strong couplings. The bound becomes even stronger when there is a hierarchy between the inflation and reheating scales.

Upon obtaining (3.20), we have only required the kinetic energy of the vector field to be smaller than the total energy density of the inflationary universe $\rho_{\text{inf}}$, cf. (3.10). Here we should remark that this condition is not sufficient when taking into account the cosmological perturbations, as the vector field itself sources curvature perturbations of roughly $\zeta_\Lambda \sim 10^{-5}$ on CMB scales, and further since $\epsilon$ is much smaller than unity during inflation, the condition (3.10) is tightened on large scales by at least 5 orders of magnitude. This strengthens the bound (3.20) on the inflation scale to

$$H_{\text{inf}} \lesssim 10^{-19} \, \text{GeV}, \quad (3.22)$$

for producing $10^{-15} \, G$ magnetic fields on Mpc scales. With an inflationary scale of $H_{\text{inf}} = 10^{-19} \, \text{GeV}$, even with the help of instantaneous reheating, the reheate temperature is restricted to

$$T_{\text{reh}} \lesssim 10^2 \, \text{MeV}. \quad (3.23)$$

Since $T_{\text{reh}}$ needs to be higher than about 5 MeV in order not to ruin Big Bang Nucleosynthesis [24, 25], one sees that our upper bound on $H_{\text{inf}}$ leaves a rather narrow window for inflationary magnetogenesis.

We further note that even if the energy density of the vector field is much smaller than $\rho_{\text{inf}}$, the produced electric fields can still prevent the growth of magnetic fields by triggering Schwinger pair creation of charged particles and thus inducing large conductivity to the universe [26]. Extremely strong constraints have been obtained for a class of vector field theories with the Lagrangian $I(\tau)^2 F_{\mu\nu} F^{\mu\nu}$; it would be interesting to systematically study the Schwinger effect constraints on general inflationary magnetogenesis scenarios using the analyses presented in this section.

### 4 Constraints on quantum mechanical production

In the previous section we derived a bound on magnetic fields from a time-evolving classical vector field, cf. (3.13). However the bound can be evaded if large vector fluctuations are produced at the quantum level, as then the vector does not necessarily need to grow after becoming classical. Such large quantum fluctuations can arise, e.g., in theories with modified dispersion relations.
In this section, we derive constraints on quantum mechanical production of the magnetic fields. After the modes become classical, we will assume that the vector fluctuations do not grow significantly in time, i.e., the fluctuations in the present universe are at most comparable to the initial classical amplitude,

\[ |\tilde{A}_k(\tau_0)| \lesssim |\tilde{A}_k(\tau_*)|. \tag{4.1} \]

Here, \( \tau_* \) represents the time during inflation when the mode \( k \) becomes classical, i.e., when quantum uncertainties become negligible. As in section 3, we ignore the component subscripts of the vector and give order-of-magnitude estimates. Then the present magnetic field amplitude obeys

\[ P_B(\tau_0, k) \lesssim \frac{k^5}{2\pi^2} \frac{|\tilde{A}_k(\tau_*)|^2}{a_0^4}. \tag{4.2} \]

In the following we estimate the right hand side by computing the vector fluctuations that can be produced quantum mechanically. The approach we take in section 4.1 is closely related to that used in [27] for studying the Lyth bound in the context of the effective field theory of inflation [10, 11] (see also [28].)

### 4.1 Smooth transition from quantum to classical

Let us suppose that in the asymptotic past, the vector field follows WKB-like solutions with positive frequencies, i.e. \( \tilde{A}_k \propto e^{-i\int \omega_k d\tau} \), up until the time \( \tau_* \) when the mode becomes classical. Here we assume that the WKB solutions smoothly connect to the classical solutions. Then, during \( \tau \leq \tau_* \), a comoving energy \( \omega_k \) of the vector field can be defined as

\[ |\tilde{A}_k'| \sim \omega_k |\tilde{A}_k|. \tag{4.3} \]

We further assume that the dominant time kinetic term of the vector field during inflation is a two-derivative term,

\[ S = \int d\tau d^3x \left( \frac{I^2}{2} A_i' A_i' + \cdots \right), \tag{4.4} \]

where the dimensionless coefficient \( I \) may or may not vary in time, but should not be tiny to avoid strong couplings. (See discussions below (3.7).) Then the conjugate momentum of the vector field is

\[ \Pi_i = \frac{\partial L}{\partial A_i'} = I^2 A_i', \tag{4.5} \]

and thus the commutation relation

\[ [A_i(\tau, x), \Pi_j(\tau, y)] = i\delta^{(3)}(x - y) (\delta_{ij} + \cdots) \tag{4.6} \]

(here \( \cdots \) are additional terms that can show up depending on the gauge choice), sets the normalization of the vector field as

\[ |\tilde{A}_k| \sim \omega_k I^2 |\tilde{A}_k|^2 \sim 1. \tag{4.7} \]

Thus, introducing the phase velocity,

\[ c_p(k) = \frac{\omega_k}{k}, \tag{4.8} \]
the right hand side of (4.2) can be rewritten as
\[ P_B(\tau_0, k) \lesssim \frac{1}{2\pi^2 I^2_{\text{cp}^*}} \frac{(k/a_0)^4}{2} , \tag{4.9} \]
where the subscript * represents quantities at \( \tau = \tau_* \). One can see from (4.7) that, for a fixed wave number \( k \), the quantum fluctuations scale as \( |\tilde{A}_k|^2 \propto c_p^{-1} \), i.e., larger fluctuations for smaller phase velocity. However, the phase velocity at the time the mode becomes classical is actually bounded from below by energy arguments as follows: for a classical vector field, its kinetic energy sourced by the time kinetic term (4.4) cannot exceed the total energy density of the inflationary universe (supposing that the contribution \( \rho_{\text{kin}} \) to the vector field energy density is not cancelled by other terms in the action), and thus we require
\[ \int \frac{dk}{k} \frac{k^3 I^2}{2\pi^2 a^4} |\tilde{A}_k|^2 \ll 3M_p^2 H_{\text{inf}}^2. \tag{4.10} \]
Hence the contribution to the kinetic energy from a classical mode should also be smaller than the inflationary energy (see also footnote 5),
\[ \left( \frac{k^3 I^2}{2\pi^2 a^4} \omega_k^2 |\tilde{A}_k|^2 \right)_* \ll 3M_p^2 H_{\text{inf}}^2, \tag{4.11} \]
where we used \( \Delta k \simeq k \). This, together with the normalization (4.7) at \( \tau_* \), yields a lower bound on the phase velocity,
\[ c_{p^*} \gtrsim \frac{1}{(6\pi^2 M_p^2 H_{\text{inf}}^2)^{1/3}} \left( \frac{\omega_k}{a} \right)_*^{4/3}. \tag{4.12} \]
Combining (4.9), (4.12), and also \( I^2_* \geq 1 \) for avoiding strong couplings, we find
\[ P_B(\tau_0, k) \lesssim \left( \frac{3M_p^2 H_{\text{inf}}^2}{4\pi^4} \right)^{1/3} \left( \frac{a}{\omega_k} \right)_*^{4/3} \left( \frac{k}{a_0} \right)^4 . \tag{4.13} \]
One clearly sees that the magnetic bound is stronger for a higher energy scale \( (\omega_k/a)_* \). Here, since we are interested in magnetogenesis during inflation, we assume the modes become classical at a scale at least of order the inflationary scale,
\[ \left( \frac{\omega_k}{a} \right)_* \gtrsim H_{\text{inf}}. \tag{4.14} \]
Therefore we arrive at
\[ P_B(\tau_0, k) \lesssim \left( \frac{3M_p^2}{4\pi^4 H_{\text{inf}}^2} \right)^{1/3} \left( \frac{k}{a_0} \right)^4 \sim (10^{-43} \text{ G})^2 \left( \frac{10^{-23} \text{ GeV}}{H_{\text{inf}}} \right)^{2/3} \left( \frac{k}{a_0} \text{ Mpc} \right)^4 . \tag{4.15} \]
The reference value \( 10^{-23} \text{ GeV} \) for \( H_{\text{inf}} \) gives a reheat temperature of about 5 MeV with the help of instantaneous reheating, and thus is the lowest possible inflation scale compatible with Big Bang Nucleosynthesis. Thus we find that the present magnetic field amplitude cannot exceed \( 10^{-43} \text{ G} \) on Mpc scales or larger, without invoking classical growth of the vector field. Considerations on the cosmological perturbations can further strengthen the bound, as is discussed below (3.21).
4.2 Violent transition from quantum to classical

In the above discussions, the quantum vector fluctuations were assumed to smoothly connect to classical fluctuations. Here, one may wonder whether large magnetic fields can be produced if this assumption is relaxed. In this subsection, as an example with non-smooth quantum to classical transitions, we study resonant production of the magnetic fields. Then we will give some general remarks on quantum mechanical magnetogenesis.

4.2.1 Resonant production

Let us assume an oscillatory background with a physical frequency \( f \) (which may or may not depend on time), and consider the following situation: the vector fluctuations initially obey WKB solutions with positive comoving frequencies \( \omega_k \), but \( \omega_k/a \) eventually becomes comparable to \( f \) and the mode undergoes parametric resonance. We represent this time by \( \tau^\star \), i.e.,

\[
\left( \frac{\omega_k}{a} \right)^\star = f^\star,
\]

where the values with the subscript \( \star \) are estimated at \( \tau^\star \). After passing through the resonance band, i.e. \( \tau > \tau^\star \), the mode is excited and the vector fluctuation \( \tilde{A}_k \) is a sum of both positive and negative frequency WKB solutions. At this point the mode has become classical. The constraints in this section are inspired by constraints on models of resonant particle production discussed in [29–31].

In such a scenario, the estimation of the vector normalization (4.7) arising from the uncertainty principle can break down after the parametric enhancement, as the positive and negative frequency solutions can cancel each other in the commutation relation. Instead of invoking the uncertainty relation, we can constrain the vector amplitude by considering the fact that the resonant production is induced by the oscillating background; thus the energy density of the produced vector is bounded by the available kinetic energy of the background. Assuming the vector field’s dominant time kinetic term during inflation to be a two-derivative term,

\[
S = \int d\tau d^3x \left( \frac{I^2}{2} \dot{A}_i' \dot{A}_i' + \cdots \right),
\]

we bound the contribution to the vector’s kinetic energy from a mode \( k \) that just underwent parametric enhancement as (we suppose the contribution to the vector energy is not cancelled by other terms in the action),

\[
\left( \frac{k^3}{2\pi^2 a^4} I^2 |\dot{A}_k'|^2 \right)^\star < \epsilon \lambda M_p^2 H_{\text{inf}}^2.
\]

The right hand side denotes the background’s kinetic energy,\(^7\) where \( \epsilon = -H'/(aH^2) \) is typically much smaller than unity during inflation. The left hand side should be considered as an average over some wave number range around \( k \), as the \( |\dot{A}_k'|^2 \) spectrum produced from parametric resonance can be oscillatory. (In cases where the spectrum has spiky features instead of oscillations, one should carry out the analyses as discussed in footnote 11.)\(^8\) The

\(^7\)Here we are imagining the inflationary universe to be dominated by canonical scalars, \( S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} M_p^2 R - \frac{1}{4} (\partial \phi_1)^2 - \frac{1}{4} (\partial \phi_2)^2 - \cdots - V(\phi_1, \phi_2, \cdots) \right\} \), with one of the scalars oscillating and sourcing the resonant production of the vector. Then it is easy to check that the total kinetic energy of the homogeneous background is \( \sum \frac{1}{2} (\dot{\phi}'/a)^2 = \epsilon \lambda M_p^2 H^2 \).

\(^8\)Strictly speaking, the left hand side of (4.18) is the \( k \) mode’s kinetic energy shortly after the resonant production, while the right hand side is the background’s kinetic energy shortly before the resonant production. However we abuse notation and denote both by the subscript \( \star \).
excited vector fluctuation is a linear combination of WKB solutions with frequencies $\pm \omega_k$, however we consider its time derivative to satisfy

$$|\tilde{A}'_k| \sim \omega_k |\tilde{A}_k|$$

(4.19)
in the sense that $\langle |\tilde{A}'_k|^2 \rangle_T \simeq \omega^2_k \langle |\tilde{A}_k|^2 \rangle_T$ where $\langle \rangle_T$ means averaged over one period. We also note that the background’s oscillation frequency should be larger than the Hubble rate for parametric resonance to happen,

$$\left( \frac{\omega_k}{a} \right)_* = f_* > H_{\text{inf}}.$$  

(4.20)

Hence by combining (4.18), (4.19), (4.20), and further using $I_*^2 \gtrsim 1$ to avoid strong couplings, we obtain a bound on the vector fluctuations,

$$|\tilde{A}_k(\tau_*)| \lesssim \sqrt{2\epsilon_* \pi M_p a_* k^{3/2}}.$$  

(4.21)

This field bound can be compared to that derived for classical scenarios in (3.14), except that now the bound is stronger by $\sim \epsilon_*$ due to the energy bound (4.18). Therefore, from (4.2) we obtain an upper bound on the present magnetic field strength,

$$P_{B}(\tau_0, k) \lesssim \epsilon_* M_p^2 \left( \frac{k}{a_0} \right)^2 \left( \frac{a_*}{a_0} \right)^2 \leq \epsilon_* M_p^2 \left( \frac{k}{a_0} \right)^2 \left( \frac{a_{\text{end}}}{a_0} \right)^2,$$  

(4.22)

which is stronger than the bound (3.20) for classical scenarios by a factor of $\epsilon_*/3$.

### 4.2.2 General remarks

The reader will have noticed that the above argument for constraining resonant production also applies for the scenarios with smooth quantum to classical transitions discussed in subsection 4.1, by replacing $\epsilon_*$ with 3. In fact, as long as the vector fluctuation that has become classical satisfies

$$|\tilde{A}'_k(\tau_*)| \gtrsim a_* H_{\text{inf}} |A_k(\tau_*)|,$$  

(4.23)

then since the classical mode’s kinetic energy cannot exceed the background density $3M_p^2 H_{\text{inf}}^2$, the fluctuation amplitude is constrained and thus the magnetic bound is obtained as

$$P_{B}(\tau_0, k) \lesssim 3 M_p^2 \left( \frac{k}{a_0} \right)^2 \left( \frac{a_{\text{end}}}{a_0} \right)^2.$$  

(4.24)

The bound becomes even stronger when considering the cosmological perturbations, as was discussed below (3.21).

Therefore whether or not the quantum to classical transition is smooth, unless (4.23) is violated, bounds on quantum mechanically produced magnetic fields are at least as strong as the bound on classically enhanced magnetic fields.

### 5 Conclusions and outlook

In this work, we presented model independent constraints on magnetic field generation during inflation by focusing on the field bounds of the vector potential. We have classified inflationary magnetogenesis models according to whether large fluctuations of the vector potential
are produced quantum mechanically, or the vector fluctuations grow in time after becoming classical, and derived upper bounds on the resulting magnetic fields for both scenarios.

For classical scenarios we obtained the bound (3.20), which requires an extremely low scale inflation of $H_{\text{inf}} \lesssim 10^{-19}$ GeV (3.22) in order for magnetic fields of $10^{-15}$ G to be produced on Mpc scales or larger without spoiling the cosmological perturbations. Thus unless reheating and baryogenesis happen within a very narrow temperature window of $5$ MeV $\lesssim T_{\text{reh}} \lesssim 10^2$ MeV, classical scenarios cannot produce the intergalactic magnetic fields suggested by gamma ray observations.

Quantum mechanical scenarios were found to be even more restricted; in cases where the quantum vector fluctuations smoothly convert into classical fluctuations, our result (4.15) shows that the produced magnetic fields cannot exceed $10^{-43}$ G on Mpc or larger scales, independently of the inflation scale. Even with non-smooth quantum to classical transitions (such as in resonant production), as long as the condition (4.23) is satisfied, the constraints on quantum mechanically produced magnetic fields are at least as strong as the constraint for classical scenarios.

Our bounds on classically and quantum mechanically produced magnetic fields apply to generic inflationary magnetogenesis models with a two-derivative time kinetic term for the vector field. This was demonstrated explicitly for the particular examples of the $I^2FF$ model, and also in the model with a variable light speed due to spontaneously broken Lorentz invariance. Below, we discuss some possible directions along which our bounds may be ameliorated.

- **Time kinetic terms with less/more than two derivatives.** We have restricted our analyses to vector field theories with two-derivative kinetic terms, however if we relax this condition, it may be possible to produce larger magnetic fields. The kinetic term $I(\tau)^2 A'_i A'_i$ we have studied also incorporates a certain class of one-derivative kinetic terms (as can be seen after canonically normalizing the field by $A^c_i = IA_i$), but it may be interesting to study theories with other forms of one-derivative, or higher-derivative kinetic terms. See also [28] where an example of a one-derivative theory is studied.

- **Suppression of time kinetic terms.** If the kinetic term $I^2 A'_i A'_i$ is allowed to be strongly suppressed during inflation by an extremely tiny coefficient $I^2$, then in principle large magnetic fields can be created while keeping the vector kinetic energy subdominant, as was done in the original work [2]. However, as we mentioned in section 3, there are at least two obvious obstacles to be overcome for taking this option: (i) A tiny $I^2$ gives rise to strong couplings with charged particles. Strong couplings may be evaded if there are other factors that suppress the interactions (such as an additional scalar derivatively coupled to vector-fermion interaction terms [32], though in this case strong couplings may appear in other terms [33]), or if the charged particles are given large mass during inflation (e.g., Higgs inflation [34] can help in this direction.) However we should also note that such (dynamical) fine-tuning of the parameters must happen with an incredible precision, as [26] demonstrated that even if $I^2$ never goes below unity, the Schwinger pair production of charged particles can easily terminate magnetogenesis. (ii) Even if one could avoid strong couplings during magnetogenesis, it is non-trivial to arrange $I^2$ to go back to unity without affecting the produced magnetic fields. (Note that $I^2$ has been defined to become unity by the present epoch.)

- **Negative contribution to the vector energy density.** One may also consider vector terms in the action that contribute negatively to the energy density, such that the vector’s
kinetic energy is cancelled out. See [35] for an attempt along this line, and [15] for discussions on negative interaction energy. However it should also be noted that even if the electric fields are cancelled out from the energy density, they can still give rise to Schwinger production of charged particles and prevent the magnetic fields from growing.

- **Other possibilities for quantum mechanical production.** Upon constraining quantum mechanical scenarios in section 4, we assumed the quantum fluctuations to start in the Bunch-Davies vacuum. However the situation may become different if the initial condition is modified, such as in an \(\alpha\)-vacuum [36, 37], although there is good reason to think such theories are not internally consistent [38, 39]. One may also investigate the possibility of a violent quantum to classical transition such that even the condition (4.23) is violated.

- **Inverse cascade of helical magnetic fields.** Parity violating terms in the action produce helical magnetic fields, which can transfer power from small to large scales in the radiation-dominated era [40–42]. Such an inverse cascade can relax the magnetic field bounds; it would be interesting to systematically analyze the range of possibilities with cascading magnetic fields using the field range bounds presented in this paper.

- **Post-inflationary magnetogenesis.** Although the inflationary epoch is preferable for creating magnetic fields with large correlation lengths, it should also be noted that magnetogenesis can happen in other epochs as well. In particular, by breaking the conformal invariance of the Maxwell theory after inflation, magnetic fields whose wave modes were once inside the horizon during inflation can get enhanced in the post-inflationary era, up until the time of reheating. As was presented in [16], such post-inflationary scenarios of magnetogenesis can produce large magnetic fields on cosmological scales. Along the lines of avoiding the inflationary epoch, one may also consider magnetic field generation in different cosmological backgrounds, such as in bouncing cosmologies [10, 43, 44]. We should also mention the possibility of magnetic field production during phase transitions [45, 46].

These ideas also suggest possible extensions of our formalism. In particular, it should be straightforward to incorporate magnetic field evolution after inflation, or to consider non-inflationary cosmological backgrounds (e.g. matter-dominated, contracting phase, etc.) during the generation of magnetic fields.

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**A Implications for \(I^2 FF\) models**

We perform concrete calculations of the constraints on the classical growth of vectors presented in section 3, in an explicit example with a time-dependent coupling to the vector...
kinetic term,

\[ S = \int d^4x \sqrt{-g} \left( -\frac{I(\tau)^2}{4} F_{\mu\nu} F^{\mu\nu} \right). \]  \hspace{1cm} (A.1)

In this model, the time dependence of the coupling \( I \) can break the conformal invariance of the vector field, and the magnetic fields can grow when the coupling \( I \) decreases in time. In the original work [2], \( I \) was considered to be a function of the inflaton field. Here we analyze this model along the lines discussed in [15], which studied generic bounds on theories of the form (A.1). However we obtain stronger bounds by optimizing the combination of multiple constraints; the differences in the calculations will also be highlighted along the way.

Let us discuss the field range bound of the vector field in terms of its mode functions \( u_k^{(p)} \), where \( p = 1, 2 \) refers to the two polarization modes. Since (A.1) is a subclass of the theories written as (B.1), we refer the reader to appendix B for details of the calculations and here we simply list some relations that are required for the following analyses: the electromagnetic power spectra (cf. (B.23), (B.24)),

\[ \mathcal{P}_B(\tau, k) = \sum_{p=1,2} \mathcal{P}_B^{(p)}(\tau, k), \quad \mathcal{P}_E(\tau, k) = \sum_{p=1,2} \mathcal{P}_E^{(p)}(\tau, k), \]  \hspace{1cm} (A.2)

are expressed in terms of the mode functions as

\[ \mathcal{P}_B^{(p)}(\tau, k) = \frac{k^5}{2\pi^2 a(\tau)} |u_k^{(p)}(\tau)|^2, \quad \mathcal{P}_E^{(p)}(\tau, k) = \frac{k^3}{2\pi^2 a(\tau)} |u_k^{(p)}(\tau)|^2, \]  \hspace{1cm} (A.3)

which also give the energy density of the vector field (given as the \( c = 1 \) limit of (B.38)),

\[ \rho_A = \sum_{p=1,2} \frac{\tau^2}{2} \int \frac{dk}{k} \left( \mathcal{P}_B^{(p)} + \mathcal{P}_E^{(p)} \right). \]  \hspace{1cm} (A.4)

The equation of motion of the mode functions is given in (B.15); in particular, the \( \Gamma'/\Gamma \) term, depending on its sign, can lead to a growth of \( |u_k^{(p)}| \). On the other hand, when \( I \) is constant in time, the mode function is a sum of plane waves and thus \( |u_k^{(p)}| \) is basically constant in time.

Let us now consider the time-evolution of the mode functions between two moments of time \( \tau_i \) and \( \tau_f \) during inflation,

\[ |u_k^{(p)}(\tau_f)| - |u_k^{(p)}(\tau_i)| = \int_{\tau_i}^{\tau_f} d\tau \frac{d|u_k^{(p)}|}{d\tau} \leq \int_{\tau_i}^{\tau_f} d\tau \left| \frac{d|u_k^{(p)}|}{d\tau} \right| = \int_{\tau_i}^{\tau_f} d\tau \left( \frac{2\pi^2 a(\tau)}{k^3} \mathcal{P}_E^{(p)} \right)^{1/2}, \]  \hspace{1cm} (A.5)

where we have used the inequality \( d|f(x)|/dx \leq |df(x)/dx| \) for an arbitrary complex function \( f(x) \) with a real variable \( x \), and also (A.3) upon moving to the far right hand side.\(^9\)

\(^9\)In [15], instead of as (A.5), the time-evolution is constrained as follows,

\[ |u_k^{(p)}(\tau_f)|^2 - |u_k^{(p)}(\tau_i)|^2 \leq 2 \int_{\tau_i}^{\tau_f} d\tau \frac{d|u_k^{(p)}|}{d\tau} \frac{|u_k^{(p)}|}{|u_k^{(p)}|} \]

\[ \leq 2 \int_{\tau_i}^{\tau_f} d\tau \frac{2\pi^2 a(\tau)}{k^3} \left( \mathcal{P}_B^{(p)} + \mathcal{P}_E^{(p)} \right), \]  \hspace{1cm} (A.6)

where \( 2xy \leq x^2 + y^2 \) for real \( x, y \) is used in the second inequality. The integrand in the final expression is proportional to the \( k \) mode contribution to the energy density (A.4), hence one can proceed using the energy density arguments as below (A.7). However, due to the \( 1/k \) factor introduced in the second line of (A.6), the resulting magnetic field bound is weakened on large scales.
We take the final time $\tau_f$ to be at the end of magnetogenesis such that $|u_k^{(p)}|$ becomes time-independent afterwards. The initial time $\tau_i$ is chosen to be before $\tau_f$, but after the $I'/I$ term in the equation of motion (B.15) becomes relevant; thus we consider the vector fluctuations as classical during $\tau_i \leq \tau \leq \tau_f$.

The vector field’s energy density should be subdominant to that of the inflationary universe, i.e.,

$$\rho_A \ll 3M^2_p H_{\text{inf}}^2,$$

and so we further require the electric field contribution to the energy density, from each classical wave mode/polarization mode (cf. (A.4)) to be smaller than the background density,

$$\frac{I^2}{2} P_E^{(p)} \ll 3M^2_p H_{\text{inf}}^2.$$  

(A.8)

This, combined with $I^2 \geq 1$ for avoiding strong couplings, allow one to transform (A.5) into

$$|u_k^{(p)}(\tau_f)| - |u_k^{(p)}(\tau_i)| < \frac{\sqrt{12\pi M_p}}{k^{3/2}} \int_{\tau_i}^{\tau_f} d\tau a^2 H_{\text{inf}} = \frac{\sqrt{12\pi M_p(a_f - a_i)}}{k^{3/2}},$$

(A.9)

where we used the relation $d\tau = da/(a^2 H_{\text{inf}})$ for the conformal time during inflation.

We now focus on cases where the classical vector fluctuations significantly grow during inflation, thus by choosing $\tau_i$ to be well before $\tau_f$ such that $a_f \gg a_i$, we suppose

$$|u_k^{(p)}(\tau_f)| \gg |u_k^{(p)}(\tau_i)|$$

(A.10)

to be satisfied. Then the quantities at $\tau_i$ can be ignored in (A.9), giving

$$|u_k^{(p)}(\tau_f)| < \frac{\sqrt{12\pi M_p a_f}}{k^{3/2}}.$$  

(A.11)

Since $P_B$ scales as $\propto a^{-4}$ after the time $\tau_f$ (cf. (A.3)), we can obtain an upper bound on the magnetic power spectrum in the present epoch,

$$P_B(\tau_f, k) = P_B(\tau_0, k) \left( \frac{a_f}{a_0} \right)^4 < 12M^2_p \left( \frac{k}{a_0} \right)^2 \left( \frac{a_f}{a_0} \right)^2 \leq 12M^2_p \left( \frac{k}{a_0} \right)^2 \left( \frac{a_{\text{end}}}{a_0} \right)^2,$$

(A.12)

where we also used $a_f \leq a_{\text{end}}$. Considering the post-inflationary universe to be effectively matter-dominated until reheating as discussed below (3.15), then (A.12) is rewritten as

$$P_B(\tau_0, k) < (10^{-15} \text{G})^2 \left( \frac{k}{a_0} \text{Mpc} \right)^2 \left( \frac{H_{\text{reh}}}{H_{\text{inf}}} \right)^{1/3} \frac{6 \times 10^{-14} \text{GeV}}{\text{H}_{\text{inf}}}.$$  

(A.13)

Thus we have reproduced our general bound (3.20) in the context of $I^2FF$ models.

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10The notion of “being classical” can be made more precise by computing the $\kappa$ parameter introduced in appendix B.3. For example, in the case of $I \propto (-\tau)^s$ with $s$ being a real constant, if $u_k$ starts in the Bunch-Davies vacuum, then $\kappa \simeq 1$ in the asymptotic past ($-k\tau \rightarrow \infty$) and thus the vector fluctuations are quantum mechanical. In the asymptotic future ($-k\tau \rightarrow 0$), if $|s| > 1/2$, then $\kappa$ decays to zero and the fluctuations can be treated as classical.

11Instead of constraining the magnetic power spectrum on certain wave numbers, [15] focuses on the magnetic power integrated over a range of $k$ that is relevant for interpreting gamma ray observations [4–9]. Considering such an effective magnetic strength would also be useful in cases where the spectrum $P_E^{(p)}$ has spiky features that violate the condition (A.8) in some narrow $k$ ranges.
We study vector field theories in backgrounds that break time diffeomorphisms and give rise to general speed of propagation. As in the previous appendix, we will allow the action to depend explicitly on time but will respect spatial diffeomorphisms. For this purpose, it will prove convenient to invoke methods used in \cite{10, 11} for analyzing the effective field theory of inflation. Later in this appendix, we will also study a simple magnetogenesis model where the vector field’s sound speed (i.e. speed of light) varies in time. This will provide an explicit example of the quantum mechanical production scenarios discussed in section 4.

B.1 Action in unitary gauge

We consider the time diffeomorphisms of the vector field theory to be spontaneously broken by a time-dependent background, such as a time-evolving scalar. Working in the unitary gauge where the time coordinate \( x^0 \) coincides with the preferred time slicing of the background, the theory can include explicit \( x^0 \)-dependences. Thus we can construct theories of a vector field with broken time diffeomorphisms, but unbroken gauge invariance, by considering Lagrangians that consist of \((F_{\mu\nu}, x^0, \nabla_\mu)\). The indices should be contracted using the metric \( g_{\mu\nu} \) or the antisymmetric tensor \( \eta_{\mu\nu\sigma\rho} \), except for that an upper 0 index is allowed to be left free, as indices can always be contracted with \( \nabla_\mu x^0 = \delta_\mu^0 \). We are interested in situations where the vector field does not affect the background universe, and thus the background fluctuations are neglected.

In this appendix, we only consider the following quadratic terms containing two derivatives,

\[
S = \int d^4x \sqrt{-g} \left\{ \mathcal{J}(x^0) F_{\mu\nu} F^{\mu\nu} + \mathcal{K}(x^0) F_0^\mu F^0_\mu \right\},
\]

where \( \mathcal{J}(x^0) \) and \( \mathcal{K}(x^0) \) are arbitrary functions of time \( x^0 \). Note that this theory is invariant under the conformal transformation \( g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \) when the functions also transform as \( \mathcal{J} \to \mathcal{J} \) and \( \mathcal{K} \to \Omega^2 \mathcal{K} \). With a flat FRW background metric (2.3), the action can be rewritten as

\[
S = \int d\tau d^3x a(\tau)^4 \left\{ \frac{I(\tau)^2}{4} \left( 2F_{0i} F^{0i} + c(\tau)^2 F_{ij} F^{ij} \right) \right\},
\]

where we have introduced

\[
I(\tau)^2 \equiv 2 \frac{\mathcal{K}(\tau)}{a(\tau)^2} - 4\mathcal{J}(\tau), \quad c(\tau)^2 \equiv \left( 1 - \frac{\mathcal{K}(\tau)}{2a(\tau)^2 \mathcal{J}(\tau)} \right)^{-1}.
\]

The expression (B.2), and also (B.8) below, clearly show that this is a theory with a variable speed of light, i.e. sound speed of the vector field. Note in particular that the existence of the \( \mathcal{K} \) term deviates the light speed from unity.

Let us decompose the spatial components of the vector field \( A_i \) into irrotational and incompressible parts,

\[
A_\mu = (A_0, \partial_i S + V_i),
\]

where

\[
\partial_i V_i = 0.
\]

Then the action (B.2) can be rewritten as, up to surface terms,

\[
S = \int d\tau d^3x \frac{I(\tau)^2}{2} \left\{ V_i' V_i - c(\tau)^2 \partial_i V_j \partial_j V_i + (\partial_i A_0 - \partial_i S') \left( \partial_i A_0 - \partial_i S' \right) \right\}.
\]

\[\text{– 17 –}\]
Varying the action in terms of the Lagrange multiplier $A_0$ gives a constraint equation, which, by choosing proper boundary conditions, yields

$$A_0 = S'. \tag{B.7}$$

Thus we arrive at

$$S = \int d\tau d^3x \frac{I(\tau)^2}{2} \left\{ V_i' V_i' - c(\tau)^2 \partial_i \partial_j V_j \right\}, \tag{B.8}$$

whose equations of motion, i.e. the modified Maxwell’s equations, take the form of

$$V_i'' + \frac{2}{I} V_i' - c^2 \partial_i^2 V_i = 0, \tag{B.9}$$

where $\partial^2 \equiv \partial_i \partial_i$. Going to Fourier space,

$$V_i(\tau, x) = \frac{1}{(2\pi)^3} \int d^3k \, e^{ik \cdot x} \xi_i(\tau, k), \tag{B.10}$$

then $\xi_i k_i = 0$ should be satisfied from the constraint (B.5). We express $\xi_i$ as a linear combination of two orthonormal polarization vectors $\epsilon_i^{(p)}(k)$ with $p = 1, 2$, that satisfy

$$\epsilon_i^{(p)}(k) \epsilon_i^{(q)}(k) = \delta_{pq}. \tag{B.11}$$

Note that from (B.11) follows

$$\sum_{p=1,2} \epsilon_i^{(p)}(k) \epsilon_j^{(p)}(k) = \delta_{ij} - \frac{k_i k_j}{k^2}, \tag{B.12}$$

where $k \equiv |k|$. We remark that, unlike the spacetime indices, we do not assume implicit summation over the polarization index ($p$).

### B.2 Quantization

In order to quantize the theory, we promote $V_i$ (B.10) to an operator,

$$V_i(\tau, x) = \frac{1}{(2\pi)^3} \sum_{p=1,2} \int d^3k \, \epsilon_i^{(p)}(k) \left\{ e^{ik \cdot x} \sigma^{(p)} k u^{(p)}(\tau) + e^{-ik \cdot x} \sigma^{(p)} k u^{(p)}(\tau) \right\}, \tag{B.13}$$

where $a_k^{(p)}$ and $a_k^{(p)\dagger}$ are respectively annihilation and creation operators satisfying the commutation relations,

$$[a_k^{(p)}, a_{k'}^{(q)\dagger}] = [a_k^{(p)\dagger}, a_{k'}^{(q)}] = 0, \quad [a_k^{(p)}, a_{k'}^{(q)}] = (2\pi)^3 \delta^{pq} \delta^{(3)}(k - h). \tag{B.14}$$

$u_k^{(p)}(\tau)$ is the mode function, in terms of which the equation of motion (B.9) is rewritten as

$$u_k'' + \frac{2}{I} u_k' + c^2 k^2 u_k = 0. \tag{B.15}$$

From the action $S = \int d\tau d^3x \, \mathcal{L}$ in (B.8), the conjugate momentum of $V_i$ is

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial V_i'}, \tag{B.16}$$
We choose the polarization vectors such that
\[
\left[ V_i(\tau, x), V_j(\tau, y) \right] = [\Pi_i(\tau, x), \Pi_j(\tau, y)] = 0, \tag{B.17}
\]
\[
\left[ V_i(\tau, x), \Pi_j(\tau, y) \right] = i\delta^{(3)}(x - y) \left( \delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right).
\tag{B.18}
\]

The relation (B.18) can be rewritten using (B.12) as
\[
\left[ V_i(\tau, x), \Pi_j(\tau, y) \right] = \frac{i}{(2\pi)^3} \sum_{p=1,2} \int d^3k \, e^{ik\cdot(x-y)} \epsilon^{(p)}(k) \epsilon^{(p)}(k).
\tag{B.19}
\]

We choose the polarization vectors such that
\[
\epsilon^{(p)}_i(k) = \epsilon^{(p)}_i(-k),
\tag{B.20}
\]
then one can check that the commutation relations (B.14) are equivalent to (B.17) and (B.19) when the mode function is independent of the direction of \( k \), i.e.,
\[
u^{(p)}_k = u^{(p)}_k,
\tag{B.21}
\]
and obeys the normalization condition
\[
I^2 \left( u^{(p)}_k u^{* (p)}_k - v^{* (p)}_k v^{(p)}_k \right) = i.
\tag{B.22}
\]

Defining the vacuum state by \( u^{(p)}_k |0\rangle = 0 \) for \( p = 1, 2 \) and \( \forall k \), then (B.13) and the commutation relation (B.14) allow us to compute the correlation functions of the electromagnetic fields (2.2) as
\[
\langle B_\mu(\tau, x) B^\mu(\tau, y) \rangle = \int \frac{d^3k}{4\pi k^3} e^{ik\cdot(x-y)} P_B(\tau, k),
\tag{B.23}
\]
\[
\langle E_\mu(\tau, x) E^\mu(\tau, y) \rangle = \int \frac{d^3k}{4\pi k^3} e^{ik\cdot(x-y)} P_E(\tau, k),
\tag{B.24}
\]
where the power spectra are expressed in terms of the mode functions as
\[
P_B(\tau, k) = \frac{k^5}{2\pi^2a(\tau)^4} \sum_{p=1,2} |u^{(p)}_k(\tau)|^2,
\tag{B.25}
\]
\[
P_E(\tau, k) = \frac{k^3}{2\pi^2a(\tau)^4} \sum_{p=1,2} |u^{(p)}_k(\tau)|^2.
\tag{B.26}
\]

### B.3 Quantum or classical

The quantum fluctuations of the vector field can transform into classical fluctuations due to the time-dependent background. To see whether the vector fluctuations are behaving as quantum mechanical or classical, it is useful to compare the size of the commutator \([V, II]\) with the amplitude \( \sqrt{\langle V^2 \rangle \langle II^2 \rangle} \).

For the purpose of discussing the classical or quantum nature of the observable modes, we focus on the Fourier components of the operator \( V_i \) (B.13) and its conjugate momentum,
\[
V_i(\tau, x) = \frac{1}{(2\pi)^3} \sum_{p=1,2} \int d^3k \, e^{ik\cdot x} \epsilon^{(p)}_i(k) \nu^{(p)}_k(\tau),
\tag{B.27}
\]
\[
\Pi_i(\tau, x) = \frac{1}{(2\pi)^3} \sum_{p=1,2} \int d^3k \, e^{ik\cdot x} \epsilon^{(p)}_i(k) \nu^{(p)}_k(\tau).
\]
Here one can check that under (B.20), (B.21), (B.22), the commutation relations (B.14), or (B.17) and (B.19), are rewritten in terms of

\[ v_k^{(p)}(\tau) = a_k^{(p)}u_k^{(p)}(\tau) + a_k^{(p)*}u_k^{(p)*}(\tau), \]
\[ \varpi_k^{(p)}(\tau) = I^2 \left( a_k^{(p)}u_k^{(p)}(\tau) + a_k^{(p)*}u_k^{(p)*}(\tau) \right), \]  

as

\[ [v_k^{(p)}(\tau), v_h^{(q)}(\tau)] = [\varpi_k^{(p)}(\tau), \varpi_h^{(q)}(\tau)] = 0, \quad [v_k^{(p)}(\tau), \varpi_h^{(q)}(\tau)] = i(2\pi)^3 \delta^{pq} \delta^{(3)}(k+h). \]  

As an indicator of whether the vector fluctuation is quantum or classical, we introduce the quantity

\[ \kappa \equiv \frac{|v_k^{(p)}(\tau), \varpi_h^{(q)}(\tau)|^2}{4 |v_k^{(p)} h^{(q)}(\tau), \varpi_h^{(p)}(\tau)|^2} = \frac{1}{4I^4 |v_k^{(p)}|^2 |v_k^{(p)}|^2}. \]  

For quantum vector fluctuations, \( \kappa \) is of order unity. On the other hand, if \( \kappa \ll 1 \), the quantum uncertainty can be neglected and thus the modes can be considered to have ‘become classical.’

### B.4 Energy-momentum tensor

In order to compute the energy-momentum tensor of the vector field, we first invoke the Stückelberg trick to the unitary gauge action (B.1) and restore the time diffeomorphisms by reintroducing a Nambu-Goldstone (NG) boson; then we can vary the covariant action in terms of \( g_{\mu\nu} \) and obtain the energy-momentum tensor in the usual way.

To apply the Stückelberg trick, we start by carrying out a time coordinate transformation to the action:

\[ x^0 \to \tilde{x}^0 = x^0 + \xi^0(x), \quad x^i \to \tilde{x}^i = x^i, \]
\[ A_\mu(x) \to \tilde{A}_\mu(\tilde{x}) = A_\nu(x) \frac{\partial x^\nu}{\partial \tilde{x}^\mu}, \quad \text{etc.} \]  

Then, wherever \( \xi^0(x) \) explicitly appears in the transformed action, we replace \( \xi^0(x) \) by the NG boson \( \pi(x) \). For example,

\[ F^0_\mu(x)F^{0\mu}(x) \to \tilde{F}^0_\mu(\tilde{x})\tilde{F}^{0\mu}(\tilde{x}) \]
\[ = F^0_\rho(x) \frac{\partial \tilde{x}^0}{\partial x^\rho} F^{0\lambda}(x) \frac{\partial \tilde{x}^\lambda}{\partial x^\mu} F^{\rho\lambda}(x) \frac{\partial \tilde{x}^\rho}{\partial x^\nu} \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \]
\[ = F^0_\rho(x)F^{0\rho}(x) \frac{\partial (x^0 + \xi^0(x))}{\partial x^\nu} \frac{\partial (x^0 + \xi^0(x))}{\partial x^\mu} \]
\[ \Rightarrow F^\nu_\rho(x)F^{\nu\rho}(x) \frac{\partial (x^0 + \pi(x))}{\partial x^\nu} \frac{\partial (x^0 + \pi(x))}{\partial x^\mu}, \]  

where in the last line we replaced the \( \xi^0 \)'s by the NG boson \( \pi \). Carrying out this procedure for the entire action (B.1) gives

\[ S_\pi = \int d^4x \sqrt{-g} \left\{ J(x^0 + \pi) F^{\nu\rho} F_{\mu\nu} + K(x^0 + \pi) F^{\mu_\rho} F^{\nu_\rho} \partial^\mu(x^0 + \pi) \partial^\nu(x^0 + \pi) \right\}. \]  

(B.33)
Thus we have restored time-diffeomorphisms; one can check that this action is invariant under the transformation (B.31), given that the NG boson also transforms as

$$\pi(x) \rightarrow \tilde{\pi}(\tilde{x}) = \pi(x) - \xi^0(x).$$  \hspace{1cm} (B.34)

It can also be seen that the unitary gauge action (B.1) corresponds to the gauge choice of $\pi = 0$.

Now that we have obtained the covariant expression (B.33), we can follow the standard procedure and compute the energy-momentum tensor of the vector fields (and the NG boson) from $S_\pi = \int \! d^4x \sqrt{-g} L_\pi$ as

$$T^{\mu\nu}_\pi = g_{\mu\nu} L_\pi - 2 \frac{\partial L_\pi}{\partial g^{\mu\nu}}.$$  \hspace{1cm} (B.35)

After computing (B.35), we go back to the unitary gauge $\pi = 0$, where the vector field’s energy-momentum tensor is obtained as

$$T^{\mu\nu}_\mu = g_{\mu\nu} \left( J F^{\mu\nu} F_{\rho\sigma} + K_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} \right) - 4 J F_{\mu\nu} F^\rho_{\rho} - 2 K \left( F^0_{\mu} F^0_{\nu} + \delta_{\mu}^0 F_{\nu\rho} F^{\rho0} + \delta_{\nu}^0 F_{\mu\rho} F^{\rho0} \right).$$  \hspace{1cm} (B.36)

Moreover, in a flat FRW background universe (2.3), the energy density observed by a comoving observer with 4-velocity $u^\mu$ is

$$T^{\mu\nu}_\mu u^\mu u^\nu = -T^{\mu\nu}_0 = -J F^{\mu\nu} F_{\rho\sigma} + \left( 4 J - \frac{3 K}{a^2} \right) F_{0\mu} F^{0\mu}$$

$$= \frac{I^2}{2} \left\{ c^2 B_\mu B^\mu + (3 - 2 c^2) E_\mu E^\mu \right\},$$  \hspace{1cm} (B.37)

where upon moving to the second line we have used (2.2) and (B.3). Therefore the vacuum expectation value of the vector field’s energy density can be expressed in terms of the electromagnetic power spectra (B.25) and (B.26) as

$$\rho_A = \langle -T^{\mu\nu}_0(\tau, x) \rangle = \frac{I^2}{2} \int \frac{dk}{k} \left\{ c^2 P_B + (3 - 2 c^2) P_E \right\}.$$  \hspace{1cm} (B.38)

### B.5 Case study: inflationary magnetogenesis from variable speed of light

Let us now study magnetic field generation in the above theory. The time dependence of the overall coefficient $I(\tau)$ of the kinetic terms has the effect of sourcing a friction term in the equation of motion (B.15), and in particular, a decreasing $I(\tau)$ gives a negative friction which can enhance the vector field. This model was studied in appendix A.

In this subsection we would like to focus on the effects of the variable light speed $c(\tau)$. For this purpose we fix the coefficient of the kinetic terms to

$$I = 1.$$  \hspace{1cm} (B.39)

This case will provide an explicit example of the quantum mechanical scenarios discussed in section 4.1.

Hereafter we omit the polarization index ($p$), as the computations are identical for either polarizations. The equation of motion (B.15) of the mode function is now

$$u''_k + c^2 k^2 u_k = 0,$$  \hspace{1cm} (B.40)
and the constraint (B.22) is
\[ u_k u_k' - u_k'^* u_k = i. \] (B.41)

We consider an inflationary background and express the conformal time during inflation in terms of the constant Hubble rate \( H_{\text{inf}} \) as
\[ \tau = -\frac{1}{a H_{\text{inf}}}. \] (B.42)

Moreover, to make the discussions concrete, we focus on cases where the light speed scales as a power-law of the conformal time,
\[ c(\tau) = c_1 \left( \frac{-\tau}{-\tau_1} \right)^n, \] (B.43)
where \( c_1, \tau_1, n \) are real constants and \( c_1 \) is further positive. Then (B.40) and (B.41) are solved for the mode function as
\[ u_k(\tau) = (-\tau)^{1/2} \left\{ \alpha H_\nu^{(1)}(-2|\nu|\tau kc(\tau)) + \beta H_\nu^{(2)}(-2|\nu|\tau kc(\tau)) \right\}, \] (B.44)
where the index \( \nu \) of the Hankel functions is
\[ \nu = \frac{1}{2(n + 1)}, \] (B.45)
and \( \alpha, \beta \) are constants satisfying
\[ |\alpha|^2 - |\beta|^2 = \frac{\pi \nu}{2}. \] (B.46)

In the following we restrict our analysis to the case of
\[ -1 < n < -\frac{1}{2} \quad (\text{i.e. } \nu > 1). \] (B.47)

The lower limit \( n > -1 \) is from the requirement that the modes with fixed wave numbers \( k \) can exit the sound horizon \( \sim c/H_{\text{inf}} \). On the other hand, the upper limit \( n < -1/2 \) is required so that the modes become classical after sound horizon exit, as we will soon see.

Under this choice of \( n \), the mode functions are well approximated by WKB solutions in the asymptotic past \( -\tau kc \to \infty \), as is clear from the rate of change of the effective frequency \( ck \) in the equation of motion (B.40),
\[ \left\{ \frac{(ck)'}{(ck)^2} \right\}^2 = \frac{n^2}{(-\tau kc)^2}, \quad \left\{ \frac{(ck)''}{(ck)^3} \right\} = \frac{n(n - 1)}{(-\tau kc)^2}. \] (B.48)

To get the positive frequency solution in the asymptotic past, we set \( \beta = 0 \). Then, in the limit of the modes being well inside the sound horizon, i.e. \( -2\nu \tau kc \to \infty \), the mode function is approximated as
\[ u_k \simeq \alpha(\pi \nu kc)^{-1/2} \exp \left\{ i \left( -2\nu \tau kc - \frac{\pi}{4}(2\nu + 1) \right) \right\}. \] (B.49)

By computing the parameter introduced in (B.30), one sees that the modes are quantum mechanical,
\[ \kappa = \frac{1}{4 |u_k|^2 |u_k'|^2} \simeq 1. \] (B.50)
The electromagnetic power spectra (B.25) and (B.26) are obtained as
\[ P_B \simeq \frac{k^4}{2\pi^2 a^3 c^2}, \quad P_E \simeq \frac{ck^4}{2\pi^2 a^4}. \]  
(B.51)

On the other hand, when the modes are well outside the horizon, i.e. \(-2\nu\tau kc \to 0\), then
\[ u_k \simeq -\frac{i}{\pi} \alpha(-\tau)^{1/2} \Gamma(\nu)(-\nu\tau kc)^{-\nu}, \]  
(B.52)
\[ u'_k = -\alpha(-\tau)^{-1/2}(-\tau kc)H^{(1)}_{\nu-1}(-2\nu\tau kc) \simeq -\frac{i}{\pi} \alpha(-\tau)^{-1/2} \Gamma(\nu - 1)/\nu (-\nu\tau kc)^{-\nu+2}, \]  
(B.53)
giving
\[ \kappa \simeq \frac{\pi^2}{\Gamma(\nu)^2 \Gamma(\nu - 1)^2} (-\nu\tau kc)^4(-\nu+1). \]  
(B.54)

Since this \( \kappa \) decays to zero as \(-2\nu\tau kc \to 0\), one sees that the vector fluctuations become classical outside the horizon. The electromagnetic power spectra are
\[ P_B \simeq \frac{k^4}{2\pi^2 a^3 c} \left( \frac{\Gamma(\nu)^2}{\pi} \right) (-\nu\tau kc)^{-2\nu+1}, \quad P_E \simeq \frac{ck^4}{2\pi^2 a^4} \left( \frac{\Gamma(\nu - 1)^2}{\pi} (-\nu\tau kc)^{-2\nu+3}. \right. \]  
(B.55)

Here note that \((-\nu\tau kc)^{-2\nu+1}/c\) is a constant, and so the magnetic spectrum scales as \( \propto a^{-4}\).

Representing quantities when the mode \( k \) exits the sound horizon by \( * \), i.e.,
\[ -2\nu\tau_* kc_* = 1, \]  
(B.56)

the electromagnetic power spectra outside the horizon can be roughly approximated as,
\[ P_B \sim \frac{k^4}{2\pi^2 a^3 c_*}, \quad P_E \sim \frac{c_* k^4}{2\pi^2 a^4} \left( \frac{a}{a_*} \right)^{-4(n+\frac{1}{2})}. \]  
(B.57)

Here we have ignored time independent factors that are set by \( \nu \). These expressions clearly show that the electromagnetic amplitudes are set by the light speed at horizon exit, and in particular that a small light speed can enhance the magnetic fields while suppressing the electric fields. At first glance, such a feature seems to suggest the possibility of magnetic field generation without having to worry about the electric backreaction or Schwinger effect. However, the magnetic fields are actually severely constrained because of the vector field’s energy density: from (B.38), the contribution to the energy density from a mode \( k \) at its horizon exit is estimated as
\[ \rho_{A_*}(k) \sim \frac{1}{2} \left\{ c_*^2 P_{B_*}(k) + (3 - 2c_*^2)P_{E_*}(k) \right\} \sim \frac{H_{\inf}^4}{\pi^2 c_*^3} \]  
(B.58)

Here, upon moving to the far right hand side, we have assumed \( c_*^2 \ll 1 \), otherwise the fluctuations of the vector potential would not be larger than the quantum vacuum fluctuations in the standard Maxwell theory. \( P_B \) redshifts as \( \propto a^{-4} \) outside the horizon, and supposing the same redshifting to continue after inflation, the magnetic power in the present universe is
\[ P_B(\tau_0, k) \sim \frac{k^4}{2\pi^2 a_0^3 c_*} \sim (10^{-43} \text{G})^2 \left( \frac{k}{a_0} \text{Mpc} \right)^4 \left( \frac{\rho_{A_*}(k)}{3M_p^2 H_{\inf}^2} \right)^{1/3} \left( \frac{10^{-23} \text{GeV}}{H_{\inf}} \right)^{2/3}. \]  
(B.59)

\[ ^{12} \] It is also interesting to analyze the case of \( n > -1/2 \) (i.e. \( 0 < \nu < 1 \)). Here, the modes do not necessarily become classical after horizon exit, as it can be checked that \( \kappa \simeq \sin^2(\pi\nu) \) for \(-2\nu\tau kc \to 0\).
The third factor in the far right hand side is the energy density ratio between the vector field and the inflationary background, and the reference value $10^{-23}$ GeV in the fourth factor gives the lowest possible value for $H_{\text{inf}}$. Therefore one sees that the magnetic fields sourced by a variable light speed during inflation cannot exceed $10^{-43}$ G on Mpc scales or larger in the present universe. Thus we have reproduced our magnetic field bound (4.15) in this explicit model.

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