Multi-tiling sets, Riesz bases, and sampling near the critical density in LCA groups

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Abstract
We prove the existence of sampling sets and interpolation sets near the critical density, in Paley Wiener spaces of a locally compact abelian (LCA) group $G$. This solves a problem left by Gröchenig, Kutyniok, and Seip in the article: ‘Landau’s density conditions for LCA groups’ (J. of Funct. Anal. 255 (2008) 1831-1850). To achieve this result, we prove the existence of universal Riesz bases of characters for $L^2(\Omega)$, provided that the relatively compact subset $\Omega$ of the dual group $\hat{G}$ satisfies a multi-tiling condition. This last result generalizes the Fuglede’s theorem, and extends to LCA groups setting recent constructions of Riesz bases of exponentials in bounded sets of $\mathbb{R}^d$.

Keywords: Sampling; Interpolation; Beurling’s densities; Riesz bases; Multi-tiling; Dyadic cubes; Locally compact abelian groups.

1 Introduction
The problem of the existence of a Riesz basis of exponentials on $L^2(\Omega)$, for some $\Omega \subseteq \mathbb{R}^d$, has its origins in the Fuglede’s Conjecture or Spectral Set Conjecture [7]. Recall that a set $\Lambda$ is called spectrum of $\Omega$ if the system
$$\{e^{2\pi i \lambda \cdot x} \chi_{\Omega}\}_{\lambda \in \Lambda},$$
constitutes an orthogonal basis for $L^2(\Omega)$. Then, the conjecture states that a domain $\Omega$ admits a spectrum if and only if it tiles $\mathbb{R}^d$ when translated along the set $\Lambda \subset \mathbb{R}^d$. A set $\Omega$ tiles the space $\mathbb{R}^d$ with translation set $\Lambda$ if the sets $\Omega + \lambda$, with $\lambda \in \Lambda$, form a partition of $\mathbb{R}^d$ up to a set of zero measure, i.e.,
$$\Delta_\Omega(x) := \sum_{\lambda \in \Lambda} \chi_{\Omega}(x - \lambda) = 1, \ a.e.$$

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When $\Lambda$ is a lattice, the Fuglede’s Conjecture follows by the Poisson summation formula. The conjecture still holds in several important special cases such as compact convex domains in the plane [13]. However, it has been disproved in its generality (for more details see [3], [4], [12], [18], [19], [20], and [27]).

Since there exist sets with no spectra, the next best option is to look for $\Lambda$ such that $
{e^{2\pi i \lambda x} \chi_\Omega}_{\lambda \in \Lambda}$ is a Riesz basis for $L^2(\Omega)$ (see [15], [22], [23], [24] and the references therein). A (countable) family $\{f_\lambda\}_{\lambda \in \Lambda}$ is a Riesz basis for a Hilbert space $\mathcal{H}$ if the mapping
\[
g \mapsto \{ \langle g, f_\lambda \rangle \}_{\lambda \in \Lambda}
\]
is bounded and invertible from $\mathcal{H}$ to $\ell^2(\Lambda)$. This mapping is not necessarily unitary as in the case of orthonormal bases.

There exist sufficient conditions that guarantee the existence of Riesz basis of exponentials in terms of a multi-tiling condition defined as follows. A set $\Omega$ $k$-tiles the space $\mathbb{R}^d$ with translation set $\Lambda \subset \mathbb{R}^d$ if
\[
\Delta_\Omega(x) = k, \quad \text{a.e.,}
\]
for some nonnegative integer $k$. If $k = 1$, it coincides with the definition of tiling. In [8] Grepstad and Lev pointed out the relation between multi-tiling sets and the existence of Riesz bases. More precisely, they proved that a bounded Riemann integrable Borel set $\Omega \subseteq \mathbb{R}^d$ admits a Riesz basis of exponentials if it multi-tiles $\mathbb{R}^d$ with translation set a lattice $\Lambda$. Later on, in [17], Kolountzakis gave a simpler proof of this result (in a slightly more general form). Important special cases had been proved by Lyubarskii and Seip [23], and Marzo [24], (see also [22] and [26]). It should be mentioned that the constructed Riesz basis depends only on the translation lattice $\Lambda$ and the tiling multiplicity $k$, and not on the specific structure of the set $\Omega$.

The construction of such basis and the study of its geometry in the context of locally compact groups (LCA groups) is one of the main goals of this paper. Consider an LCA group $G$, and let $\hat{G}$ denote its dual group. Fix a lattice $H$ in $G$, and let $\Lambda$ denote its dual lattice. We prove that there exist $a_1, \ldots, a_k \in G$ such that for any relatively compact Borel subset $\Omega$ of $\hat{G}$ satisfying
\[
\Delta_\Omega(\omega) = \sum_{\lambda \in \Lambda} \chi_\Omega(\omega - \lambda) = k, \quad \text{a.e. } \omega \in \hat{G},
\]
the set $\{e_{a_j \cdot h} \chi_\Omega : h \in H, \ j = 1, \ldots, k\}$ is a Riesz basis for $L^2(\Omega)$. We show that almost any choice of $a_1, \ldots, a_k$ can be used (see Theorem 3.2). Moreover, we prove a variant of the above result, replacing the lattice by slightly more general translation sets (see Theorem 3.7). Also, our approach to these problems allows us to answer negatively a question left by Kolountzakis in [17] (see Subsection 3.2).

The other main goal of this work is to use the multi-tiling results to construct sampling sets and interpolation sets on Paley Wiener spaces near the critical density. This solves an open problem raised by Gröchenig, Kutyniok and Seip in [9]. The Paley Wiener space
$PW_{\Omega}$ consists on all square integrable functions with Fourier transform supported on a fixed relatively compact Borel set $\Omega \subseteq \hat{G}$. For this space, $\Lambda$ is a sampling set if there exist constants $A, B > 0$ such that for any $f \in PW_{\Omega}$,

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq B\|f\|_2^2.$$  

Equivalently, $\Lambda$ is a sampling set if the set $\Lambda$ as characters on $\hat{G}$ and restricted to $\Omega$ is a frame for $L^2(\Omega)$. Recall also that $\Lambda$ is an interpolation set for $PW_{\Omega}$ if the interpolation problem $f(\lambda) = c_\lambda$, has a solution $f \in PW_{\Omega}$ for every $\{c_\lambda\}_{\lambda \in \Lambda} \in \ell^2(\Lambda)$. A set that is at the same time a sampling and interpolation set is a complete interpolation set.

Some geometric information of these sets is encoded in the so called upper and lower Beurling’s densities, denoted by $D^+$ and $D^-$ respectively. Roughly speaking, these densities compare the concentration of the points of a given discrete set with some reference lattice (see Definition 4.2). For instance, in the case of $R^d$ the reference lattice is $Z^d$. This comparison is motivated by the classical sampling theorem which states that $Z^d$ is a complete interpolation set for $PW[{-1/2,1/2}^d]$. A result, proved by Landau in [21] for $R^d$, and later on extended to LCA groups in [9] gives the following necessary conditions:

(i) A sampling set $\Lambda$ for $PW_{\Omega}$ satisfies $D^-(\Lambda) \geq m_{\hat{G}}(\Omega)$;

(ii) An interpolation set $\Lambda$ for $PW_{\Omega}$ satisfies $D^+(\Lambda) \leq m_{\hat{G}}(\Omega)$.

Now, the aforementioned open problem is whether or not there exist sampling sets and interpolation sets for $PW_{\Omega}$ with densities arbitrarily close to the critical density $m_{\hat{G}}(\Omega)$. If $G = R^d$, it had been proved positively by Marzo in [24], adapting a construction of Lyubarskii and Seip [23] and Kohlenberg [16]. The strategy of their proof is to approximate in measure the set $\Omega$ by a set $\Omega_\varepsilon$ that is a disjoint union of equal side cubes, and then show that $PW_{\Omega_\varepsilon}$ admits a complete interpolation set. The main obstacle to accomplish the same approach in the general framework was to obtain those sets $\Omega_\varepsilon$ such that each $PW_{\Omega_\varepsilon}$ admits a complete interpolation set.

Given a subset $\Omega$ of an LCA group $G$, we construct such approximating sets $\Omega_\varepsilon$ as a finite unions of building blocks. These building blocks copy the geometry of dyadic cubes in $R^d$ in terms of inclusions and generations. For this reason, we call them quasi-dyadic cubes. Since each quasi-dyadic cube tiles the dual group $\hat{G}$ with some common translation set, the approximating sets $\Omega_\varepsilon$ satisfy a multi-tiling condition. Hence, by the multi-tiling results, we get a complete interpolation set for $PW_{\Omega_\varepsilon}$. This leads to the proof of the existence of sampling sets and interpolation sets for $PW_{\Omega}$ with densities arbitrarily close to the critical one (see Theorem 4.10).

Finally, we would like to mention that the quasi-dyadic cubes used in the construction of the approximating sets may be also a useful tool to extrapolate results from Harmonic Analysis on $R^d$ or $T^d$ to other LCA groups.

The paper is organized as follows. In section 2, we introduce preliminary results on LCA groups. Section 3 is devoted to the results relating the multi-tiling condition with the
existence of Riesz bases of characters. Finally, in section 4 we study the sampling sets and interpolation sets near the critical density. With this aim, we introduce the quasi-dyadic cubes, and we prove the existence of the approximating sets \( \Omega \) mentioned above.

# 2 Preliminaries

Throughout this section we review basic facts on locally compact abelian groups, (for more details see [10], [11], [25]), setting in this way the notation we need for the following sections. Then, we introduce \( H \)-invariant spaces that generalize the concept of shift invariant spaces in the context of these groups (see [1]).

## 2.1 LCA Groups

Let \( G \) denote a Hausdorff locally compact abelian group, and \( \hat{G} \) its dual group, that is:

\[
\hat{G} = \{ \gamma : G \to \mathbb{C}, \text{ and } \gamma \text{ is a continuous character of } G \},
\]

where a character is a function satisfying the following properties.

(i) \(|\gamma(x)| = 1, \forall x \in G\);

(ii) \(\gamma(x + y) = \gamma(x)\gamma(y), \forall x, y \in G\).

Thus, the characters generalize the exponential functions \( \gamma(x) = \gamma_t(x) = e^{2\pi itx} \) in the case \( G = (\mathbb{R}, +) \). On every LCA group \( G \) there exists a Haar measure. It is a non-negative, regular Borel measure \( m_G \) that is non-identically zero and translation-invariant, which means:

\[
m_G(E + x) = m_G(E),
\]

for every element \( x \in G \) and every Borel set \( E \subset G \). This measure is unique up to a constant. Analogously to the Lebesgue spaces, we can define the \( L^p(G) = L^p(G, m_G) \) spaces associated to the group \( G \) and the measure \( m_G \):

\[
L^p(G) := \left\{ f : G \to \mathbb{C}, \text{ } f \text{ is measurable and } \int_G |f(x)|^p \, dm_G(x) < \infty \right\}.
\]

**Theorem 2.1.** Let \( G \) be an LCA group and \( \hat{G} \) its dual. Then

(i) The dual group \( \hat{G} \), with the operation \( (\gamma + \gamma')(x) = \gamma(x)\gamma'(x) \) is an LCA group. The topology in \( \hat{G} \) is the one induced by the identification of the characters of the group with the characters of the algebra \( L^1(G) \).

(ii) The dual group of \( \hat{G} \) is topologically isomorphic to \( G \), that is, \( \hat{\hat{G}} \approx G \), with the identification \( g \in G \leftrightarrow e_g \in \hat{\hat{G}} \), where \( e_g(\gamma) := \gamma(g) \).

(iii) \( G \) is discrete (resp. compact) if and only if \( \hat{G} \) is compact (resp. discrete).
As a consequence of (ii) of the previous theorem, we could use the notation \((x, \gamma)\) for the complex number \(\gamma(x)\), representing either the character \(\gamma\) applied to \(x\) or the character \(x\) applied to \(\gamma\).

Taking \(f \in L^1(G)\) we define the Fourier transform of \(f\), as the function \(\hat{f} : \hat{G} \rightarrow \mathbb{C}\) given by

\[
\hat{f}(\gamma) = \int_{G} f(x)(-\gamma) \, dm_G(x), \quad \gamma \in \hat{G},
\]

If the Haar measure of the dual group \(\hat{G}\) is normalized conveniently, we obtain the inversion formula

\[
f(x) = \int_{\hat{G}} \hat{f}(\gamma)(x,\gamma) \, dm_{\hat{G}}(\gamma),
\]

for a specific class of functions. In the case that the Haar measures \(m_G\) and \(m_{\hat{G}}\) are normalized such that the inversion formula holds, the Fourier transform on \(L^1(G) \cap L^2(G)\) can be extended to a unitary operator from \(L^2(G)\) onto \(L^2(\hat{G})\). Thus the Parseval formula holds:

\[
\langle f, g \rangle = \int_{G} f(x)\overline{g(x)} \, dm_G(x) = \int_{\hat{G}} \hat{f}(\gamma)\overline{\hat{g}(\gamma)} \, dm_{\hat{G}}(\gamma) = \langle \hat{f}, \hat{g} \rangle
\]

for \(f, g \in L^2(G)\). We conclude this subsection with the next classical result.

**Proposition 2.2.** If \(G\) is a compact group, then the characters of \(G\) form an orthonormal basis for \(L^2(G)\).

### 2.2 H-invariant spaces

In this subsection we will review some basic aspects of the theory of shift invariant spaces in LCA groups. We will specially focus on the Paley Wiener spaces, that constitute an important family of shift invariant spaces in which we are particularly interested. The reader is referred to [1], where he can find the results in full generality, as well as other results related to shift invariant spaces in LCA groups. Let \(G\) be an LCA group, and let \(H\) be a uniform lattice on \(G\), i.e., a discrete subgroup of \(G\) such that \(G/H\) is compact. Recall that a Borel section of \(G/H\) is a set of representatives of this quotient, that is, a subset \(A\) of \(G\) containing exactly one element of each coset. Thus, each element \(x \in G\) has a unique expression of the form \(x = a + h\) with \(a \in A\) and \(h \in H\). Moreover, it can be proved that there exists a relatively compact Borel section of \(G/H\), which will be called fundamental domain (see [6] and [14]).

**Definition 2.3.** We say that a closed subspace \(V \subset L^2(G)\) is \(H\)-invariant if

\[
f \in V \quad \text{then} \quad \tau_h f \in V, \quad \forall h \in H,
\]

where \(\tau_h f(x) = f(x - h)\).
As we have mentioned, Paley-Wiener spaces are important examples of $H$-invariant spaces, which in this context are defined by

$$PW_\Omega = \{ f \in L^2(G) : \text{supp} \hat{f} \subset \Omega \},$$

where $\Omega \subset \hat{G}$ is a relatively compact Borel set (see [9]). Actually, this space is invariant by any translation. In particular, it is $H$-invariant for any lattice $H$. Let $\Lambda$ be the dual lattice of $H$; that is, the annihilator of $H$ defined by

$$\Lambda = \{ \gamma \in \hat{G} : (h, \gamma) = 1, \text{ for all } h \in H \}.$$ 

Suppose that $\Omega$ tiles $\hat{G}$ by translations of $\Lambda$, i.e.

$$\Delta_\Omega(x) := \sum_{\lambda \in \Lambda} \chi_\Omega(x - \lambda) = 1, \text{ a.e.}$$

In this case, it is well known that $\{e_h\}_{h \in H}$ is an orthonormal basis of $L^2(\Omega)$. Indeed, since $\Lambda$ is also a uniform lattice, in particular, $\hat{G}/\Lambda$ is compact. So, as we recall in Proposition 2.2, $H \simeq (\hat{G}/\Lambda)^\sim$ is an orthonormal basis of $L^2(\hat{G}/\Lambda)$. On the other hand, this space is isometrically isomorphic to $L^2(\Omega)$, because 1-tiling sets are Borel sections of the quotient group $\hat{G}/\Lambda$ up to a zero measure set.

In order to deal with multi-tiling sets, we recall that a set $\Omega$ multi-tiles, or more precisely $k$-tiles $\hat{G}$ by translations of $\Lambda$ if

$$\Delta_\Omega(x) := \sum_{\lambda \in \Lambda} \chi_\Omega(x - \lambda) = k, \text{ a.e.}$$

For example, if $\Omega$ is a disjoint union of 1-tiling sets then the previous condition is satisfied. Next lemma shows that the reverse also holds, not only in $\mathbb{R}^n$ (see Lemma 1 in [17]), but also in the context of the LCA groups.

**Lemma 2.4.** Let $G$ be an LCA group and $H \subset G$ a countable discrete subgroup. A measurable set $\Omega \subset G$, $k$-tiles $G$ under the translation set $H$, if and only if

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_k \cup R,$$

where $R$ has measure zero, and the sets $\Omega_j$, $1 \leq j \leq k$ are disjoint and each of them tiles $G$ by translation with elements of $H$.

**Proof.** If $\Omega$ is a disjoint union of $k$ sets of representatives of $G/H$ up to measure zero then clearly $\sum_{h \in H} \chi_\Omega(x - h) = k$, a.e.

For the converse, consider $D$ to be a Borel section of $G/H$ and let $\{h_j\}_{j \in \mathbb{N}}$ be an enumeration of the elements of $H$. We have $\Delta_\Omega(d) = k$ for almost all $d \in D$. If $E$ denotes the set of the exceptions, define for $d \in D \setminus E$,

$$i_j(d) = \min \{ n \in \mathbb{N} : \sum_{s=1}^{n} \chi_\Omega(d + h_s) = j \}, \quad j = 1, \ldots, k.$$
and the measurable sets,

\[ E_{j,n} = \{ d \in D \setminus E : i_j(d) = n \}, \quad n \in \mathbb{N}. \]

Finally for \( j = 1, \ldots, k \), let \( \Omega_j = \bigcup_{n \in \mathbb{N}} (E_{j,n} + h_n) \). It is straightforward to see that \( \Omega = \bigcup_{j=1}^{k} \Omega_j \cup R \), is the desired decomposition. Here the remaining set \( R = \Omega \setminus (\Omega_1 \cup \ldots \cup \Omega_k) \) has measure zero because it is contained in \( E + H \). \[ \square \]

Let us recall now the following simple but useful proposition, that in the case of LCA-groups is a direct consequence of Parseval identity and Weil’s formula.

**Proposition 2.5.** Let \( D \) be a Borel section of \( \hat{G}/\Lambda \). The map \( T : L^2(G) \rightarrow L^2(D, \ell^2(\Lambda)) \) defined by

\[ Tf(\omega) = \{ \hat{f}(\omega + \lambda) \}_{\lambda \in \Lambda}, \]

is an isometric isomorphism. Moreover, for each element \( h \in H \)

\[ T(\tau_h f)(\omega) = e_h(\omega) \{ \hat{f}(\omega + \lambda) \}_{\lambda \in \Lambda}, \]

for almost every \( \omega \in D \).

**Remark 2.6.** It is not difficult to see that if \( \hat{f} \) and \( \hat{g} \) are equal almost everywhere, then for almost every \( \omega \in D \)

\[ \{ \hat{f}(\omega + \lambda) \}_{\lambda \in \Lambda} = \{ \hat{g}(\omega + \lambda) \}_{\lambda \in \Lambda}. \]

This guarantees that \( T \) is well defined, and justifies the evaluation of elements of \( L^2(G) \). With respect to the second part of Proposition 2.5, roughly speaking, it says that \( T \) diagonalizes the \( H \)-translations. \[ \blacksquare \]

When the \( H \)-invariant space is finitely generated, Proposition 2.5 allows to translate a problem in (infinite dimensional) \( H \)-invariant spaces, to simpler linear algebra problems in finite dimensional Hilbert spaces. Suppose for instance that \( \Omega \) is a measurable subset of \( \hat{G} \) that \( k \)-tiles \( \hat{G} \) by translations of \( \Lambda \). Then, for almost every \( \omega \in D \) there exist precisely \( k \) elements \( \lambda_1, \ldots, \lambda_k \), of \( \Lambda \) such that \( \omega + \lambda_j \in \Omega \), where \( \lambda_j = \lambda_j(\omega) \). Otherwise we would contradict that \( \Omega \) is a \( k \)-tiling set. This implies that the subspace

\[ J_\Omega(\omega) := \{ \{ \hat{f}(\omega + \lambda) \}_{\lambda \in \Lambda} : f \in PW_\Omega \} \]

\[ \cong \{ \{ \hat{f}(\omega + \lambda_j) \}_{j=1,\ldots,k} : f \in PW_\Omega \} \]

has dimension at most \( k \). This remark together with Proposition 2.5 lead to the following result.
Theorem 2.7. Let $\Omega$ be a relatively compact, $k$-tiling subset of $\hat{G}$. Given $\phi_1, \ldots, \phi_k \in PW_\Omega$, we define

$$T_\omega = \left( \begin{array}{ccc} \hat{\phi}_1(\omega + \lambda_1) & \cdots & \hat{\phi}_k(\omega + \lambda_1) \\ \vdots & \ddots & \vdots \\ \hat{\phi}_1(\omega + \lambda_k) & \cdots & \hat{\phi}_k(\omega + \lambda_k) \end{array} \right)$$

where the $\lambda_j = \lambda_j(\omega)$ for $j = 1, \ldots, k$ are the $k$ values of $\lambda \in \Lambda$ such that $\omega + \lambda \in \Omega$. Then, the following statements are equivalent:

(i) The set $\Phi_H = \{ \tau_h \phi_j : h \in H, \; j = 1, \ldots, k \}$ is a Riesz basis for $PW_\Omega$.

(ii) There exist $A, B > 0$ such that for almost every $\omega \in D$,

$$A ||x||^2 \leq ||T_\omega x||^2 \leq B ||x||^2,$$

for every $x \in \mathbb{C}^k$.

Moreover, in this case the constants of the Riesz basis are

$$A = \inf_{\omega \in D} ||T_\omega^{-1}||^{-1} \quad \text{and} \quad B = \sup_{\omega \in D} ||T_\omega||.$$

For a sake of completeness, we will give a proof of this result adapted to our setting. For the proof in more general $H$-invariant spaces see [1].

Proof. Let $D$ be a fundamental domain of $\hat{G}/\Lambda$. Consider a family $\{a_{j,h}\}$ with finitely many non-zero terms, where $j = 1, \ldots, k$ and $h \in H$. Using the Fourier transform and a $\Lambda$-periodization argument we get

$$\left\| \sum_{j,h} a_{j,h} \tau_h \phi_j \right\|^2_{L^2(\hat{G})} = \int_D \sum_{j,t=1}^k m_j(\omega) \left( \sum_{\lambda \in \Lambda} \hat{\phi}_j(\omega + \lambda) \overline{\hat{\phi}_t(\omega + \lambda)} \right) m_t(\omega) \; dm_{\hat{G}}(\omega).$$

where $m_j = \sum_{h \in H} a_{j,h} e^{-h}$. For each $j$, the vector $\{\hat{\phi}_j(\omega + \lambda)\}$ has at most $k$ non-zero coordinates. More precisely, the only coordinates that can be different from zero are those corresponding to the elements $\lambda_j(\omega) \in \Lambda$ considered in the matrix $T_\omega$. So, if $m = (m_1, \ldots, m_k)$ then

$$\left\| \sum_{j=1}^k \sum_{h \in H} a_{j,h} \tau_h \phi_j \right\|^2_{L^2(\hat{G})} = \int_D \langle T_\omega^* T_\omega m(\omega), m(\omega) \rangle_{\mathbb{C}^k} \; dm_{\hat{G}}(\omega)$$

$$= \int_D \|T_\omega m(\omega)\|^2_{\mathbb{C}^k} \; dm_{\hat{G}}(\omega). \quad (1)$$

On the other hand

$$\int_D \|m(\omega)\|^2_{\mathbb{C}^k} \; dm_{\hat{G}}(\omega) = \sum_{j=1}^k \int_D |m_j(\omega)|^2 \; dm_{\hat{G}}(\omega) = \sum_{j=1}^k \sum_{h \in H} |a_{j,h}|^2. \quad (2)$$
Combining (1), (2) and standard arguments of measure theory we get that (i) $\Rightarrow$ (ii).

For the other implication, note that from (1) and (2) we immediately get that the family $\Phi_H$ is a Riesz sequence for $PW_\Omega$. So, it only remains to prove that condition (ii) also implies that $\Phi_H$ is complete. With this aim, let $f \in PW_\Omega$, and suppose that $\langle f, \tau_h \phi_j \rangle = 0$ for every $h \in H$ and $j = 1, \ldots, m$. By a $\Lambda$-periodization argument we get

$$0 = \langle \hat{f}, e_{-h} \hat{\phi}_j \rangle = \int_D \left( \sum_{\lambda \in \Lambda} \hat{f}(\omega + \lambda) \overline{\hat{\phi}_j(\omega + \lambda)} \right) e_h(\omega) \ dm_G(\omega).$$

Since $\{e_h\}_{h \in H}$ is an orthonormal basis for $L^2(\hat{G}/\Lambda)$, then

$$\sum_{\lambda \in \Lambda} \hat{f}(\omega + \lambda) \overline{\phi_j(\omega + \lambda)} = 0, \ a.e. \ \omega \ m_{\hat{G}}.$$

This implies that $T_w^w(\{f(\omega + \lambda_j)\}_{j=1}^k) = 0$, where $\lambda_j = \lambda_j(\omega)$. Thus, $T(f) = 0$ a.e $m_{\hat{G}}$. ■

2.3 Standing hypothesis

Since we will work with relatively compact sets $\Omega$, throughout this paper we will assume that $G$ is an LCA group such that its dual $\hat{G}$ is compactly generated (but not compact to avoid trivialities). By the standard structure theorems, $\hat{G}$ is isomorphic to $\mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{K}$, where $\mathbb{K}$ is a compact subgroup of $T^\omega$. Consequently, $G$ is isomorphic to $\mathbb{R}^d \times T^m \times \mathbb{D}$, where $\mathbb{D}$ is a countable discrete group. In particular, note that both $G$ and $\hat{G}$ are second countable.

This is not a serious restriction, as the following lemma shows (see [5] or [9]).

\textbf{Lemma 2.8.} Assume that $\Omega \subseteq \hat{G}$ is relatively compact, and let $H$ be the open subgroup generated by $\Omega$. Then $H$ is compactly generated and there exists a compact subgroup $K \subseteq G$ such that every $f \in PW_\Omega$ is $K$-periodic. Furthermore, the quotient $G/K$ is homeomorphic to $\mathbb{R}^d \times T^m \times \mathbb{D}$, where $\mathbb{D}$ is a countable discrete abelian group, and $\hat{G}/\hat{K} \simeq H$.

Therefore, given a relatively compact set $\Omega \subseteq \hat{G}$, this lemma shows that the space $PW_\Omega$ essentially lives in $L^2(G/K)$, and $\hat{G}/\hat{K} \simeq H$ is compactly generated.

3 Multi-tiling and Riesz basis in the context of LCA groups

In [8] and [17], the authors found sufficient conditions on a subset $\Omega$ of $\mathbb{R}^d$ in order to have a Riesz basis of exponentials in $L^2(\Omega)$. In this section we generalize this result to the setting of LCA groups. We first assume that the group $G$ admits a uniform lattice $H$. In the following subsection we relax this condition, since there exist groups that do
not possess lattices. Note that since $G$ and $\hat{G}$ are second countable the lattices in these groups are countable.

**Theorem 3.1.** Let $H$ be a uniform lattice of $G$, $\Lambda$ its dual lattice, and $k \in \mathbb{N}$. Then, there exist $a_1, \ldots, a_k \in G$, depending only on the lattice $\Lambda$, such that for any relatively compact Borel subset $\Omega$ of $\hat{G}$ satisfying

$$\Delta_\Omega(\omega) := \sum_{\lambda \in \Lambda} \chi_\Omega(\omega - \lambda) = k, \quad \text{a.e. } \omega \in \hat{G},$$

the set

$$\{e_{a_j - h} \chi_\Omega : h \in H, j = 1, \ldots, k\}$$

is a Riesz basis for $L^2(\Omega)$.

We would like to emphasize that, in the previous theorem, the same set $\{a_1, \ldots, a_k\}$ can be used for any $k$-tiling set $\Omega$. If we call such a $k$-tuple $(a_1, \ldots, a_k)$ $H$-universal, then we also get the following result:

**Theorem 3.2.** Let $H$ be a uniform lattice of $G$ and $k \in \mathbb{N}$. Then, there exists a Borel set $N \subseteq G^k$ such that $m_{G^k}(N) = 0$ and for every $k$-tuple $(a_1, \ldots, a_k) \in G^k \setminus N$ is $H$-universal.

**Remark 3.3.** Note that, if we fix a fundamental domain $D$, given any universal $k$-tuple $(a_1, \ldots, a_k)$ there exist a unique $k$-tuple $(d_1, \ldots, d_k) \in D^k$ such that

$$\{e_{a_j - h} \chi_\Omega : h \in H, j = 1, \ldots, k\} = \{e_{d_j - h} \chi_\Omega : h \in H, j = 1, \ldots, k\}$$

So, we can restrict out attention to universal $k$-tuples belonging to $D^k$. In this case, consider the “uniform” probability measure on $D^k$ given by the restriction of the Haar measure of $G^k$ to $D^k$ (conveniently normalized). Then another way to state Theorem 3.2 is that a $k$-tuple in $D^k$ is almost surely $H$-universal.

### 3.1 Proofs of Theorem 3.1 and 3.2

We begin with two technical lemmas. Following Rudin’s book [25], we will say that a function $p$ is a **trigonometric polynomial** on $G$ if it has the form

$$p(g) = \sum_{j=0}^{n} c_j \gamma_j(g)$$

for some $n \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $\gamma_j \in \hat{G}$.

**Lemma 3.4.** The zero set of a trigonometric polynomial $p$ on $G$ has zero Haar measure.
Proof. By the standing hypothesis, we can identify $G$ with the group $\mathbb{R}^d \times \mathbb{T}^m \times \mathbb{D}$, for some countable discrete LCA group $\mathbb{D}$. Hence, given $(x, \omega, d) \in \mathbb{R}^d \times \mathbb{T}^m \times \mathbb{D}$, the polynomial $p$ can be written as

$$p(x, \omega, d) = \sum_{j=0}^{n} c_j \rho_j(x) \tau(\omega) \delta(d)$$

where $\rho \in \hat{\mathbb{R}}^d$, $\tau \in \hat{\mathbb{T}}^m$, and $\delta \in \hat{\mathbb{D}}$. Let $C_p = \{(x, \omega, d) : p(x, \omega, d) = 0\}$, and suppose by contradiction that $m_G(C_p) > 0$. Since

$$C_p = \bigcup_{d \in D} C_p \cap (\mathbb{R}^d \times \mathbb{T}^m \times \{d\}),$$

there exists $d_0 \in D$ such that

$$m_G(C_p \cap (\mathbb{R}^d \times \mathbb{T}^m \times \{d_0\})) > 0$$

If we restrict $p$ to $\mathbb{R}^d \times \mathbb{T}^m \times \{d_0\}$ we get the trigonometric polynomial $q$ on $\mathbb{R}^d \times \mathbb{T}^m$

$$q(x, \omega) = \sum_{j=0}^{n} (c_j \delta(d)) \rho_j(x) \tau(\omega)$$

that is non-trivial and its zero set has positive measure. This is a contradiction, and therefore $m_g(C_p) = 0$. \[\blacksquare\]

Lemma 3.5. Let $K_1$ and $K_2$ be compact subsets of $\hat{G}$. If

$$\Gamma = \{\lambda \in \Lambda : (\lambda + K_1) \cap K_2 \neq \emptyset\},$$

then $\#\Gamma < \infty$.

Proof. Note that $\Gamma \subset \Lambda \cap (K_1 - K_2)$, where $K_1 - K_2 = \{k_1 - k_2 : k_j \in K_j, j = 1, 2\}$. Since $\Lambda$ is a discrete set and $(K_1 - K_2)$ is compact, $\Gamma$ should be necessarily a finite set. \[\blacksquare\]

Proof of Theorems 3.1 and 3.2. Given $a_1, \ldots, a_k \in G$, define the functions $\phi_1, \ldots, \phi_k$ by their Fourier transform in the following way:

$$\hat{\phi}_j := e_{a_j} \chi_\Omega, \quad j \in \{1, \ldots, k\}. \tag{3}$$

We will show that under the hypothesis on $\Omega$, there exist $a_1, \ldots, a_k$ such that $\phi_1, \ldots, \phi_k$ translated by $H$ form a Riesz basis for $PW_\Omega$.

Choose a fundamental domain $D$ of $\hat{G}/\Lambda$. Since $\Omega$ is a set that $k$-tiles $\hat{G}$, for almost every $\omega \in D$, the vectors $\hat{\phi}_j(\omega)$ have at most $k$ entries different from zero. These entries are
those that correspond to the (different) elements $\lambda_j = \lambda_j(\omega) \in \Lambda$, $1 \leq j \leq k$, such that $\omega + \lambda_j \in \Omega$. For $\omega \in D$ consider the matrix

$$T_\omega = \begin{pmatrix} \hat{\phi}_1(\omega + \lambda_1) & \cdots & \hat{\phi}_k(\omega + \lambda_1) \\ \vdots & \ddots & \vdots \\ \hat{\phi}_1(\omega + \lambda_k) & \cdots & \hat{\phi}_k(\omega + \lambda_k) \end{pmatrix}$$

By Theorem 2.7, the $H$-translations of $\phi_1, \ldots, \phi_k$ form a Riesz basis for $PW_\Omega$ if and only if there exist $A, B > 0$ such that

$$A||x||^2 \leq ||T_\omega x||^2 \leq B||x||^2,$$

for every $x \in \mathbb{C}^k$ and almost every $\omega \in D$. The rest of the proof follows ideas of [17] suitably adapted to our setting. Firstly, note that

$$T_\omega = \begin{pmatrix} \hat{\phi}_1(\omega + \lambda_1) & \cdots & \hat{\phi}_k(\omega + \lambda_1) \\ \vdots & \ddots & \vdots \\ \hat{\phi}_1(\omega + \lambda_k) & \cdots & \hat{\phi}_k(\omega + \lambda_k) \end{pmatrix} = \begin{pmatrix} e_{a_1}(\omega + \lambda_1) & \cdots & e_{a_k}(\omega + \lambda_1) \\ \vdots & \ddots & \vdots \\ e_{a_1}(\omega + \lambda_k) & \cdots & e_{a_k}(\omega + \lambda_k) \end{pmatrix} = (a_1, \omega) \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & (a_2, \omega) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & (a_{k-1}, \omega) \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (a_k, \omega) \end{pmatrix} = E_\omega U_\omega.$$

Since $U_\omega$ is unitary, to prove the inequalities in (4) is equivalent to show that

$$A||x||^2 \leq ||E_\omega x||^2 \leq B||x||^2,$$

for every $x \in \mathbb{C}^k$, and almost every $\omega \in D$. By Lemma 3.5, applied to $K_1 = D$ and $K_2 = \Omega$, when $\omega$ runs over (a full measure subset of) the Borel section $D$, only a finite number of different matrices $E_\omega$ appear in (5), say $E_1, \ldots, E_N$. Thus, it is enough to prove that they are all invertible. Note that the determinants of the $E_\omega$ are polynomials of the form

$$d(x_1, \ldots, x_k) = \sum_{\pi \in S_k} \text{sgn}(\pi) \prod_{j=1}^k (x_{\pi(j)}, \lambda_j(\omega)),$$

evaluated in $(a_1, \ldots, a_k) \in G \times \cdots \times G = G^k$, where $S_k$ denotes the permutation group on $1, \ldots, k$. Since $\Lambda$ is countable, the set of trigonometric polynomials on $G^k$

$$\mathcal{P}_k = \left\{ p(x_1, \ldots, x_k) = \sum_{\pi \in S_k} \text{sgn}(\pi) \prod_{j=1}^k (x_{\pi(j)}, \lambda_j) : \text{for any } (\lambda_1, \ldots, \lambda_k) \in \Lambda^k \right\}$$

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is countable. This set contains the trigonometric polynomials $d(x_1, \ldots, x_k)$ associated to the determinants of the matrices $E_j$. Note that it also contains the polynomials associated to matrices $E_j'$ coming from any other $k$-tiling set. Therefore, the universal $k$-tuple $(a_1, \ldots, a_k)$ that we are looking for, is any $k$-tuple such that

$$p(a_1, \ldots, a_k) \neq 0 \quad \forall p \in \mathcal{P}_k.$$ 

To prove that such a $k$-tuple exists, we will use a measure theoretical argument based on Lemma 3.4. Note that $G^k$ is a compactly generated LCA-group, and $\Lambda^k$ is the dual lattice of the uniform lattice $H^k$ in $G^k$. Hence, using Lemma 3.4 with $G^k$ instead of $G$, we get that the union of the zero sets corresponding to these polynomials has zero Haar measure in $G^k$. Therefore, there exists $a_1, \ldots, a_N \in G$ so that $(a_1, \ldots, a_N)$ does not belong to any of these zero sets. In particular, for these values of $a_j$, the matrices $E_1, \ldots, E_N$ are invertible. Then, by Theorem 2.7, the $H$-translations of the functions $\phi_1, \ldots, \phi_k$ form a Riesz basis for $PW_{\Omega}$. This is equivalent to say that

$$\{e^{a_j - h} \chi_{\Omega} : h \in H, j = 1, \ldots, k\},$$

is a Riesz basis on $L^2(\Omega)$. The same holds for any other $k$-tiling set $\Omega'$ by construction of $\mathcal{P}_k$ and the $k$-tuple $(a_1, \ldots, a_k)$. ■

3.2 A counterexample for unbounded sets in $\mathbb{R}$

The same scheme cannot be applied if $\Omega$ is not relatively compact, as the following example shows. This example also gives a negative answer to the open problem left by Kolountzakis in [17].

Example 3.6. Let $G = \mathbb{R}$, and consider the following subset of $\mathbb{R} \simeq \mathbb{R}$:

$$\Omega_0 = [0, 1) \cup \bigcup_{n=2}^{\infty} [n - 2^{-(n-2)}, n - 2^{-(n-1)}].$$

This is a 2-tiling set with respect to the lattice $\mathbb{Z}$ (see Figure 1).

![Figure 1: The set $\Omega_0$.](image)

However, if we consider functions $\phi_1$ and $\phi_2$ defined through their Fourier transform by

$$\hat{\phi}_j(\omega) = e^{2\pi i a_j \omega} \chi_{\Omega},$$

for $j = 1, 2$, then integer translations of $\phi_1$ and $\phi_2$ are not a Riesz basis for $PW_{\Omega}$ for any choice of $a_1, a_2 \in \mathbb{R}$. In other words, $(a_1 + \mathbb{Z}) \cup (a_2 + \mathbb{Z})$ is not a complete interpolation sets for $PW_{\Omega}$ for any pair $a_1, a_2 \in \mathbb{R}$. 13
To prove this, recall that in the proof of Theorem 3.1 we proved that the integer translations of \( \phi_1 \) and \( \phi_2 \) form a Riesz basis for \( PW_\Omega \) if and only if the matrices

\[
E_\omega = \begin{pmatrix}
1 & 0 \\
e^{2\pi i a_1 \lambda_1(\omega)} & e^{2\pi i a_2 \lambda_1(\omega)} \\
e^{2\pi i a_1 \lambda_2(\omega)} & e^{2\pi i a_2 \lambda_2(\omega)}
\end{pmatrix},
\]

and their inverses are uniformly bounded for almost every \( \omega \) in the fundamental domain, that for simplicity we choose the interval \([0, 1)\). For this particular \( \Omega_0 \), \( \lambda_1(\omega) \) is always equal to zero, while \( \lambda_2(\omega) = n \) if \( \omega \in [1 - 2^{-(n-1)}, 1 - 2^{-n}] \), for \( n \in \mathbb{N} \). Therefore

\[
E_\omega = \begin{pmatrix} 1 & 0 \\
e^{2\pi i a_1 \lambda_2(\omega)} & e^{2\pi i a_2 \lambda_2(\omega)} \end{pmatrix},
\]

which can be rewritten as

\[
E_\omega = \begin{pmatrix} 1 & 0 \\
e^{2\pi i a_1 \lambda_2(\omega)} & e^{2\pi i (a_2 - a_1) \lambda_2(\omega)} \end{pmatrix}.
\]

So, if \( a_2 - a_1 \in \mathbb{Q} \), there exists a set of positive measure such that the matrices \( T_\omega \) are not invertible for \( \omega \) in that set. On the other hand, if \( a_2 - a_1 \notin \mathbb{Q} \), as the set \( \{e^{2\pi i (a_2 - a_1)n}\}_{n \in \mathbb{N}} \) is dense in \( \mathbb{T} \), the matrices \( E_\omega^{-1} \) are uniformly bounded.

Since the set \( \Omega_0 \) also multi-tiles \( \mathbb{R} \) with other lattices too, a natural question is whether or not we can obtain Riesz basis using these lattices. The answer is no, and the idea of the proof is essentially the same. For this reason, we only make some comments on the main differences, and we leave the details to the reader. First of all, recall that the (uniform) lattices of \( \mathbb{R} \) have the form \( \Lambda_\alpha = \alpha \mathbb{Z} \), for some \( \alpha \in \mathbb{R} \). It is not difficult to prove that \( \Omega_0 \) multi-tiles \( \mathbb{R} \) only for lattices corresponding to \( \alpha = k^{-1} \), for some \( k \in \mathbb{N} \). Moreover, with respect to the lattice \( \Lambda_{k-1} \), the set \( \Omega_0 \) is \( 2k \)-tiling. Given \( a_1, \ldots, a_{2k} \in \mathbb{R} \), as we already mentioned, in the proof of Theorem 3.1 we show that \( (a_1 + \mathbb{Z}) \cup \ldots \cup (a_{2k} + \mathbb{Z}) \) is a complete interpolation sets for \( PW_\Omega \) if and only if the matrices

\[
E_\omega = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
e^{2\pi i a_1 \lambda_1(\omega)} & e^{2\pi i a_2 \lambda_1(\omega)} & \cdots & e^{2\pi i a_{2k-1} \lambda_1(\omega)} \\
e^{2\pi i a_1 \lambda_2(\omega)} & e^{2\pi i a_2 \lambda_2(\omega)} & \cdots & e^{2\pi i a_{2k-1} \lambda_2(\omega)} \\
e^{2\pi i a_1 \lambda_{2k}(\omega)} & e^{2\pi i a_2 \lambda_{2k}(\omega)} & \cdots & e^{2\pi i a_{2k-1} \lambda_{2k}(\omega)} \\
e^{2\pi i a_1 \lambda_{2k-1}(\omega)} & e^{2\pi i a_2 \lambda_{2k-1}(\omega)} & \cdots & e^{2\pi i a_{2k-1} \lambda_{2k-1}(\omega)}
\end{pmatrix}
\]

and their inverses are uniformly bounded for almost every \( \omega \in [0, k^{-1}] \). By construction of \( \Omega_0 \), for each \( m \in \mathbb{N} \) we can find an interval \( I \subseteq [0, k^{-1}] \) of positive measure such that for every \( \omega \in I \) there exists \( j \in \{1, \ldots, 2k\} \) so that \( \lambda_j(\omega) = m \). On the other hand, as in the case studied before, either the orbit

\[
\{(e^{2\pi i a_1 m}, e^{2\pi i a_2 m}, \ldots, e^{2\pi i a_{2k-1} m}, e^{2\pi i a_{2k} m})\}_{m \in \mathbb{Z}},
\]

is periodic or there exists elements of the orbit as close as we want to the first row vector \((1, \ldots, 1)\). Therefore, the matrices \( E_\omega^{-1} \) can not be uniformly bounded in a full measure subset of \([0, 1)\).
3.3 Multi-tiling properties in quotient spaces

In Theorem 3.1 we constructed a Riesz basis of characters for \( L^2(\Omega) \) assuming that \( \Omega \) multi-tiles the space with some lattice. In this subsection, we will show a simple way to replace the lattice by a slightly more general translation set. This will be also useful in the next section, where we study the sampling sets and interpolation sets near the critical density.

**Theorem 3.7.** Let \( K \) be a compact subgroup of \( \hat{G} \) such that \( \hat{G}/K \) has a lattice \( \Xi \), and let \( \pi : \hat{G} \to \hat{G}/K \) be the canonical projection. Then, given \( k \in \mathbb{N} \), there exists a countable set \( L \subseteq G \) such that for every relatively compact Borel subset of \( \hat{G} \) satisfying \( \Omega + K = \Omega \), and \( \Delta_{\pi(\Omega)}(\omega) = \sum_{\xi \in \Xi} \chi_{\pi(\Omega)}(\omega - \xi) = k \), a.e \( \omega \in \hat{G}/K \),

\[
\text{(7)}
\]

the set \( \{ e_g \chi_{\Omega} : g \in L \} \) is a Riesz basis for \( L^2(\Omega) \).

The hypothesis (7) guarantees the existence of a Riesz basis in the quotient. Hence, in order to prove Theorem 3.7 we need a way to lift this basis to \( L^2(\Omega) \). This is provided by the next proposition, which has its own interest, and so we will state it for a general groups \( G \), without any assumptions on \( \hat{G} \).

**Proposition 3.8.** Let \( K \) be a compact subgroup of an LCA group \( G \) such that \( \hat{K} \) is countable. Suppose that there exists a subset \( Q \) of \( G/K \) such that \( L^2(Q) \) admits a Riesz basis of characters of \( G/K \). If \( \pi : G \to G/K \) denotes the canonical projection, and \( \tilde{Q} = \pi^{-1}(Q) \), then \( L^2(\tilde{Q}) \) also admits a Riesz basis of characters.

**Proof.** On the one hand, note that \( \hat{G}/K \cong K^\perp \subseteq \hat{G} \), where \( K^\perp \) denotes the annihilator of \( K \). So, the Riesz basis for \( L^2(Q) \) can be identified with some elements \( \{ \gamma_n \} \) in \( \hat{G} \). On the other hand, the elements \( \hat{K} \) form an orthonormal basis for \( L^2(K) \) endowed with the normalized Haar measure \( m_K \). Moreover, since \( \hat{K} \) can be identified with the quotient group \( \hat{G}/K^\perp \), the orthonormal basis for \( L^2(K) \) can be identified with a system of representatives \( \{ \kappa_m \} \) of \( \hat{G}/K^\perp \).

Now, we will prove that \( \{ \gamma_n + \kappa_m \} \) is a Riesz basis for \( L^2(\tilde{Q}) \). First of all, we will prove that it is complete. Let \( F \in L^2(Q) \) such that \( \langle F, \gamma_n + \kappa_m \rangle = 0 \) for every \( n \) and \( m \). By the Weil’s formula, \( m_G = m_K \times m_{G/K} \) provided we renormalize conveniently the Haar measure on \( G/K \). So, using this formula and the fact that \( (k, \gamma_n) = 1 \) for every \( k \in K \) we get for every \( m \) and every \( n \)

\[
0 = \int_{\tilde{Q}} F(g)(g, \gamma_n + \kappa_m) \, dm_G(g)
= \int_{Q} \left( \int_{K} F(g + k)(g + k, \gamma_n + \kappa_m) \, dm_K(k) \right) \, dm_{G/K}(\pi(g))
= \int_{Q} \left( \int_{K} F(g + k)(g, \kappa_m) (k, \kappa_m) \, dm_K(k) \right) (\pi(g), \gamma_n) \, dm_{G/K}(\pi(g)).
\]
Fix \( m \). Then, using that \( \{\gamma_n\} \) is a Riesz basis for \( L^2(Q) \) we get that
\[
(g, \kappa_m) \int_K F(g + k) \overline{(k, \kappa_m)} dm_K(k) = 0 \quad m_{G/K} - a.e.
\]
So, since \( \{\kappa_m\} \) is a (countable) orthonormal basis of \( K \), we get that
\[
\int_K |F(g + k)|^2 dm_K(k) = \sum_m \left| \int_K F(g + k) \overline{(k, \kappa_m)} dm_K(k) \right|^2 = 0 \quad m_{G/K} - a.e.
\]
So, by the Weil’s formula we get:
\[
\|F\|_{L^2(\widetilde{Q})}^2 = \int_Q \left( \int_K |F(g + k)|^2 dm_K(k) \right) dm_{G/K}(\pi(g)) = 0.
\]
Therefore \( \{\gamma_n + \kappa_m\} \) is complete.

Now, in order to prove that it is also a Riesz sequence, consider a sequence \( \{c_{n,m}\} \) with finitely many non-zero terms. Then
\[
\left\| \sum_{n,m} c_{n,m} (\gamma_n + \kappa_m) \right\|_{L^2(\widetilde{Q})}^2 = \int_Q \left| \sum_{n,m} c_{n,m} (g, \gamma_n + \kappa_m) \right|^2 dm_G(g)
\]
\[
= \int_Q \left( \int_K \left| \sum_{n,m} c_{n,m} (g + k, \gamma_n + \kappa_m) \right|^2 dm_K(k) \right) dm_{G/K}(\pi(g)).
\]
Since \( (k, \gamma_n) = 1 \) for every \( k \in K \), the sum inside the integrals can be rewritten as
\[
\sum_{n,m} c_{n,m} (g + k, \gamma_n + \kappa_m) = \sum_{n,m} \left( (g, \kappa_m) \sum_n c_{n,m} (\pi(g), \gamma_n) \right) (k, \kappa_m).
\]
Therefore, using that \( \{\kappa_m\} \) is an orthonormal basis of \( L^2(K) \) we get
\[
\int_K \left| \sum_{n,m} c_{n,m} (g + k, \gamma_n + \kappa_m) \right|^2 dm_K(k) = \sum_m \left| \sum_n c_{n,m} (\pi(g), \gamma_n) \right|^2.
\]
So, putting all together
\[
\left\| \sum_{n,m} c_{n,m} (\gamma_n + \kappa_m) \right\|_{L^2(\widetilde{Q})}^2 = \sum_m \int_Q \left| \sum_n c_{n,m} (\pi(g), \gamma_n) \right|^2 dm_{G/K}(\pi(g)).
\]
Finally, since \( \{\gamma_n\} \) as a Riesz basis for \( L^2(Q) \), there exist \( A, B > 0 \) such that
\[
A \sum_{m,n} |c_{n,m}|^2 \leq \left\| \sum_{n,m} c_{n,m} (\gamma_n + \kappa_m) \right\|_{L^2(\widetilde{Q})}^2 \leq B \sum_{m,n} |c_{n,m}|^2,
\]
and this concludes the proof.
Remark 3.9. Although our version is for Riesz bases, it also holds for orthonormal basis or frames with minor changes. For instance, a proof for orthonormal basis is contained in the proof of Lemma 3 of [9].

Now, we are ready to prove Theorem 3.7.

Proof. By hypothesis (7) and Theorem 3.1, there exist $a_1, \ldots, a_k$ depending on $\Xi$ such that
\[
\{ e_{a_j - h} \chi_{\pi(\Omega)} : h \in \Xi^\perp \}
\]
is a Riesz basis for $L^2(\pi(\Omega))$. By Proposition 3.8 we obtain that $L^2(\Omega)$ admits a Riesz basis of characters.

\[\blacksquare\]

Remark 3.10. In Theorem 3.1 we give a concrete description of the Riesz basis as a finite union of translations of a lattice. In the case of Theorem 3.7, it is also possible, but the lattice are replaced by a quasi-lattice.

Recall that a subset set $\Gamma$ of an LCA group $G$ is a quasi-lattice if the following holds. There is a compact subgroup $K$ of $\hat{G}$ and a uniform lattice $H$ in $K^\perp$ such that $\Gamma = \{ \hat{k} + h \}$, where $h \in H$ and $\hat{k}$ runs over a set of representatives of $G/K^\perp \simeq \hat{K}$ in $G$ (for further details see [9]). Note that quasi-lattices always exist in LCA groups, even though there may be no lattices. A classical example, that may be useful to keep in mind is the following: Let $G = \mathbb{R} \times \mathbb{Z}$, and take $K = \{0\} \times \mathbb{T}$ in $\hat{G}$. Then, $K^\perp \simeq \mathbb{R} \times \{0\}$. For any choice of a set of representatives $\{(x_j, j) : j \in \mathbb{Z}, x_j \in \mathbb{R}\}$ of the quotient $G/K^\perp$, the set $\{(n + x_j, j) : n, j \in \mathbb{Z}\}$ is a quasi-lattice, whereas it is a lattice only when $x_j = jq$ for some fixed rational number $q$.

Following the proofs of Theorem 3.7 and Proposition 3.8, we get that there exist $a_1, \ldots, a_k \in G$ such that
\[
g \in \mathcal{L} \iff g = a_j - (g_n + k_m),
\]
for some $g_n \in \Xi^\perp \subseteq \hat{G}/K \simeq K^\perp$, and some $k_m$ in a set of representatives of $\hat{K}$. So, the set $\mathcal{L}$ in Theorem 3.7 is the union of a finite number of translations of the quasi-lattice in $G$ given by $\{g_n + k_m\}$.

4 Beurling’s densities in LCA groups

In this section we study sampling sets, and interpolation sets on $PW_\Omega$, when $\Omega$ is a relatively compact subset of $\hat{G}$. In [9], Gröchenig, Kutyniok, and Seip introduced two notions of densities that suitably generalize the Beurling’s densities defined in $\mathbb{R}^d$.

Our main goal is to prove that there exist sampling sets, and interpolation sets whose densities are arbitrarily close to the critical one, answering a question raised by Gröchenig et. al. in [9].

Since $\Omega$ is relatively compact, unless otherwise is specified, we will consider an LCA group $G$ such that $\hat{G}$ is compactly generated (see Section 2.3). In particular, recall that by the
structure theorems, $\hat{G}$ is isomorphic to $\mathbb{R}^d \times \mathbb{Z}^m \times K$, where $K$ is a compact subgroup of $\mathbb{T}^\omega$. Consequently, $G$ is isomorphic to $\mathbb{R}^d \times \mathbb{T}^m \times D$, where $D$ is a countable discrete group. In what follows we will use these identifications, as well as the others that will appear later on, without mentioning it explicitly.

4.1 Beurling-type densities in LCA groups

To begin with, recall that a subset $\Lambda$ of $G$ is called uniformly discrete if there exists an open set $U$ such that the sets $\lambda + U$ ($\lambda$ in $\Lambda$) are pairwise disjoints. In some sense, the densities in $\mathbb{R}^d$ compare the concentration of the points of a given discrete set with that of the integer lattice $\mathbb{Z}^d$. In a topological group, this comparison is done by means of the following relation:

**Definition 4.1.** Given two uniformly discrete sets $\Lambda$ and $\Lambda'$ and non-negative numbers $\alpha$ and $\alpha'$, we write $\alpha \Lambda \preceq \alpha' \Lambda'$ if for every $\varepsilon > 0$ there exists a compact subset $K$ of $G$ such that for every compact subset $L$ we have

$$(1 - \varepsilon)\alpha \# (\Lambda \cap L) \leq \alpha' \# (\Lambda' \cap (K + L)).$$

Now, we have to fix a reference lattice in the group $G$. As we mentioned at the beginning of this section, since $\hat{G}$ is compactly generated, $G$ is isomorphic to $\mathbb{R}^d \times \mathbb{T}^m \times D$, where $D$ is a countable discrete group. So, a natural reference lattice is $H_0 = \mathbb{Z}^d \times \{e\} \times D$. Using this reference lattice, and the above transitive relation, we have all what we need to recall the definitions of upper and lower densities:

**Definition 4.2.** Let $\Lambda$ be a uniformly discrete subset of $G$. The lower uniform density of $\Lambda$ is defined as

$$D^- (\Lambda) = \sup \{ \alpha \in \mathbb{R}^+ : \alpha H_0 \preceq \Lambda \}.$$

On the other hand, its upper uniform density is

$$D^+ (\Lambda) = \inf \{ \alpha \in \mathbb{R}^+ : \Lambda \preceq \alpha H_0 \}.$$

These densities always satisfy that $D^- (\Lambda) \leq D^+ (\Lambda)$, and they are finite. Moreover, it can be shown that the infimum and the supremum are actually a minimum and a maximum. In the case that both densities coincide, we will simply write $D (\Lambda)$. It should be also mentioned that in the case of $\mathbb{R}^d$, these densities coincide with the Beurling’s densities when the reference lattice is $\mathbb{Z}^d$.

Using these densities, Gröchenig, Kutyniok, and Seip obtained in [9] the following extension of the classical result of Landau to LCA-groups.

**Theorem 4.3.** Suppose $\Lambda$ is a uniform discrete subset of $G$. Then

S) If $\Lambda$ is a sampling set for $PW_\Omega$, then $D^- (\Lambda) \geq m_{\hat{G}} (\Omega)$;

I) If $\Lambda$ is an interpolation set for $PW_\Omega$, then $D^+ (\Lambda) \leq m_{\hat{G}} (\Omega)$. 

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A natural question is whether there exist sampling sets and interpolation sets near the critical density. In the case $G = \mathbb{R}^d$ a positive answer was given by Marzo in [24]. In what follows we will show that this holds in the general LCA groups setting. The strategy of the proof will be similar to the one used in [23] and [24] (see also [16]). However, the extension is far from being straightforward. Indeed, in our context, one of the main difficulties is to construct a set $\Omega_{\varepsilon}$ close in measure to $\Omega$, and such that $L^2(\Omega_{\varepsilon})$ admits a Riesz basis consisting of characters of $\hat{G}$. The approximating set $\Omega_{\varepsilon}$ will be constructed as a union of some building blocks, that are the analogs to the small dyadic cubes of $\mathbb{R}^d$ (with side length smaller or equal to one).

### 4.2 Quasi-dyadic cubes

The main obstacle to accomplish the construction of dyadic cubes in general LCA-groups is to find a right substitute for the dilations. Note that, in the classical case of $\mathbb{R}^d$, the dyadic cubes of side length equal to $2^{-n}$ are fundamental domains for the lattice $2^{-n}\mathbb{Z}$. Hence, the dilation of the cubes is reflected in the refinement of the lattices. So, we can consider a fundamental domain of a given lattice, and define its dilations by means of refinements of that lattice. From this point of view, in the factorization $\mathbb{R}^d \times \mathbb{Z}^m \times K$ of the group $\hat{G}$, only the compact factor present a difficulty, because it could be difficult (or impossible) to find a right nested family of lattices in $K$. The key to overcome this difficulty is given by the following classical result (see [10]).

**Lemma 4.4.** Given a neighborhood $U$ of $e$ in $\hat{G}$, there exists a compact subgroup $K$ included in $U$ and such that $\hat{G}/K$ is elemental, that is

$$\hat{G}/K \simeq \mathbb{R}^d \times \mathbb{Z}^m \times T^\ell \times F,$$

where $F$ is a finite group.

**Remark 4.5.** The group $K$ in the previous lemma has the form $\{0\} \times \{0\} \times K_0$ for some compact subgroup $K_0$ of $K$. In particular, as $K_0$ is also a closed subgroup of $T^\omega$, its dual group $\hat{K}$ not only is discrete but also it is countable. This fact will be used in the next subsection.

In $\mathbb{R}^d \times \mathbb{Z}^m \times T^\ell \times F$, the above mentioned strategy to obtain dilations by means of refinements of lattices can be done without any problem. More precisely, we consider

$$\Lambda_n = \Lambda_n(d, m, \ell) = (2^{-n}\mathbb{Z})^d \times \mathbb{Z}^m \times \mathbb{Z}_2^\ell \times F \subseteq \hat{G}/K,$$

$$Q^{(n)}_0 = [-2^{-n-1}, 2^{-n-1})^d \times \{0\} \times [-2^{-n-1}, 2^{-n-1})^\ell \times \{e\}.$$  

This leads to the following definition of quasi-dyadic cubes.

**Definition 4.6.** Let $K$ be a compact subgroup of $\hat{G}$ such that $\hat{G}/K$ is elemental, and let $\pi$ the canonical projection from $\hat{G}$ onto the quotient. Identifying the quotient $\hat{G}/K$ with the
group $\mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{T}^\ell \times F$, the family of **quasi-dyadic cubes of generation $n$ associated to $K$**, denoted by $\mathcal{Q}^{(n)}_K$, are defined by

$$Q^{(n)}_\lambda = \pi^{-1}(Q^{(n)}_\lambda)$$

where $Q^{(n)}_\lambda = \lambda + Q^{(n)}_0$ for $\lambda \in \Lambda_n$.

Note that in order to distinguish the cubes in the quotient from the cubes in $\hat{G}/K$ we use calligraphic letters. Note also that the quasi-dyadic cubes $Q^{(n)}$ are relatively compact. Indeed, if $S^{(n)}_\lambda$ is a relatively compact Borel section of $Q^{(n)}_\lambda$ in the group $\hat{G}$, then $Q^{(n)}_\lambda = S^{(n)}_\lambda + K$.

The main difference with the classical case, is that the quasi-dyadic cubes are parametrized not only by a (dyadic) lattice, but also by some compact subgroups. This family of quasi-dyadic cubes clearly satisfies many of the arithmetical and combinatorial properties of the classical dyadic cubes. However, from the analysis point of view, maybe the following approximation result is the most important.

**Proposition 4.7.** Let $C$ be a compact set and $V$ an open set such that $C \subset V \subset \hat{G}$. There exists a compact subgroup $K$ of $\hat{G}$ such that $\hat{G}/K$ is an elemental LCA group, and $Q^{(m)}_{\lambda_1}, \ldots, Q^{(m)}_{\lambda_k} \in \mathcal{Q}^{(m)}_K$ for $m \in \mathbb{N}$ large enough such that

$$C \subseteq \bigcup_{j=1}^k Q^{(m)}_{\lambda_j} = \pi^{-1} \left( \bigcup_{j=1}^k \lambda_j + Q^{(m)}_0 \right) \subseteq V.$$

**Proof.** Let $U$ be a compact neighborhood of $e$ in $\hat{G}$ such that $C + U \subseteq V$. Take a compact subgroup $K$ contained in $U$ which satisfies that

$$\hat{G}/K \simeq \mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{T}^\ell \times F,$$

for some integers $d, m, \ell \geq 0$. Let $\pi : \hat{G} \rightarrow \hat{G}/K$ denote the canonical projection. By our assumptions on the open set $U$ we have for $n$ large enough that

$$C \subseteq C + Q^{(2n)}_0 \subseteq C + Q^{(n)}_0 \subseteq V.$$

On the other hand, by the compactness of $C$, there exist $\gamma_1, \ldots, \gamma_j \in C$ such that

$$C \subseteq \bigcup_{i=1}^j (\gamma_i + Q^{(2n)}_0).$$

Consider the lattice $\Lambda_{4n}$, and $\lambda_{i,1}, \ldots, \lambda_{i,s_i} \in \Lambda_{4n}$ such that

$$\pi(\gamma_i + Q^{(2n)}_0) = \pi(\gamma_i) + Q^{(2n)}_0 \subseteq \bigcup_{h=1}^{s_i} (\lambda_{i,h} + Q^{(4n)}_0) \subseteq \pi(\gamma_i) + Q^{(n)}_0.$$
Let \( \{\lambda_1, \ldots, \lambda_k\} \) an enumeration of the elements of \( \Lambda_{4n} \) used to cover all the sets \( \pi(\gamma_i) + Q_0^{(2n)} \). Then, the above inclusions imply that

\[
C \subseteq \bigcup_{i=1}^j (\gamma_i + Q_0^{(2n)}) \subseteq \bigcup_{j=1}^k \pi^{-1}(\lambda_j + Q_0^{(4n)}) \subseteq \bigcup_{i=1}^j (\gamma_i + Q_0^{(n)}) \subseteq V
\]

Thus, we can take \( m = 4n \), and the proof is complete.

Another good property of the quasi-dyadic cubes is the following.

**Proposition 4.8.** Let \( \Omega \) be finite union of quasi-dyadic cubes in \( \mathcal{D}_K^{(n)} \). Then, the space \( L^2(\Omega) \) admits a Riesz basis of characters (restricted to \( \Omega \)).

**Proof.** Let \( \pi \) the canonical projection from \( \hat{G} \) onto \( \hat{G}/K \). Then, \( \pi(\Omega) \) multi-tiles the quotient space with the lattice \( \Lambda_n \) defined in (9). Therefore, by Theorem 3.7, the space \( L^2(\Omega) \) admits a Riesz basis of characters.

**Remark 4.9.** The above result generalizes Lemma 3 of [9].

### 4.3 Sampling and interpolation near the critical density

Now, we turn back to the main problem of this section, that is, to find sampling sets and interpolation sets, whose densities is near to the critical one. Recall that a subset \( \Omega \) of \( \hat{G} \) is called Riemann integrable if the Haar measure of its boundary is zero.

Now, we state the main result of this section.

**Theorem 4.10.** Let \( \Omega \) be a compact subset of \( \hat{G} \), and let \( \varepsilon > 0 \). Then, the following statements hold:

(i) There exists a sampling set \( J_\varepsilon \) for \( \text{PW}_\Omega \) such that

\[
\mathcal{D}(J_\varepsilon) \leq m_{\hat{G}}(\Omega) + \varepsilon.
\]

(ii) If \( \Omega \) is Riemann integrable, then there exists an interpolation set \( J^\varepsilon \) for \( \text{PW}_\Omega \) such that

\[
\mathcal{D}(J^\varepsilon) \geq m_{\hat{G}}(\Omega) - \varepsilon.
\]

The proof of this result will be obtained by a combination of the main tools developed in this paper: Theorem 3.7, the lifting Proposition 3.8, and the properties of the quasi-dyadic cubes.
Proof of Theorem 4.10.

(i) Since the Haar measure is regular, there exist an open subset \( V \) of \( \hat{G} \) such that \( \Omega \subseteq V \) and \( m_{\hat{G}}(V \setminus \Omega) \leq \varepsilon \). By Lemma 4.7, there exists a compact subgroup \( K \) of \( \hat{G} \) such that \( \hat{G}/K \) is elemental, \( m \in \mathbb{N} \) large enough, and \( Q_{\lambda_1}, \ldots, Q_{\lambda_k} \in \mathcal{D}_K^{(m)} \) such that

\[
\Omega \subseteq \bigcup_{j=1}^{k} Q_{\lambda_j}^{(m)} \subseteq V,
\]

Let \( \Omega_\varepsilon \) be the union of these \( k \) quasi-dyadic cubes. Then, by Proposition 4.8, the space \( L^2(\Omega_\varepsilon) \) admits a Riesz basis consisting of characters of \( \hat{G} \) (restricted to \( \Omega_\varepsilon \)). Let \( \{ e_{b_n}, \chi_n \} \) denote this basis, and let \( J_\varepsilon = \{ b_n \} \subseteq G \). Using Theorem 4.3 we get that \( D(J_\varepsilon) = m_{\hat{G}}(\Omega_\varepsilon) \leq m_{\hat{G}}(\Omega) + \varepsilon \). Note that \( \{ e_{b_n}, \chi_n \} \) is a frame for \( L^2(\Omega) \), because it is a projection onto \( L^2(\Omega) \) of a Riesz basis in the bigger space \( L^2(\Omega_\varepsilon) \). So, \( J_\varepsilon \) is a sampling set for \( PW_\Omega \).

(ii) Since \( \Omega \) is Riemann integrable, without loss of generality we can assume that it is open. Let \( C \) be a compact subset of \( \Omega \) such that \( m_{\hat{G}}(\Omega \setminus C) \leq \varepsilon \). Again by Lemma 4.7, there exists a compact subgroup \( K \) of \( \hat{G} \) such that \( \hat{G}/K \) is elemental, \( m \in \mathbb{N} \) large enough, and \( Q_{\lambda_1}, \ldots, Q_{\lambda_k} \in \mathcal{D}_K^{(m)} \) such that

\[
C \subseteq \bigcup_{j=1}^{k} Q_{\lambda_j}^{(m)} \subseteq \Omega,
\]

As before, if \( \Omega_\varepsilon \) be the union of the \( k \) quasi-dyadic cubes, then the space \( L^2(\Omega_\varepsilon) \) admits a Riesz basis consisting characters of \( \hat{G} \) (restricted to \( \Omega_\varepsilon \)). In this case, the set \( J_\varepsilon \) consisting of these characters, forms a Riesz sequence in \( L^2(\Omega) \). This is equivalent to say that, as points of \( G \), they form an interpolation set of \( PW_\Omega \). Since \( D(J_\varepsilon) = m_{\hat{G}}(\Omega_\varepsilon) \geq m_{\hat{G}}(\Omega) - \varepsilon \), which concludes the proof. ■

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