Online Feedback Equilibrium Seeking

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Abstract—This article proposes a unifying design framework for dynamic feedback controllers that track solution trajectories of time-varying generalized equations, such as local minimizers of nonlinear programs or competitive equilibria (e.g., Nash) of noncooperative games. Inspired by the feedback optimization paradigm, the core idea of the proposed approach is to repurpose classic iterative algorithms for solving generalized equations (e.g., Josephy–Newton, forward–backward splitting) as dynamic feedback controllers by integrating online measurements of the continuous-time nonlinear plant. Sufficient conditions for closed-loop stability and robustness of the algorithm-plant cyber-physical interconnection are derived in a sampled-data setting by combining and tailoring results from (monotone) operator, fixed-point, and nonlinear systems theory. Numerical simulations on smart building automation and competitive supply chain management are presented to support the theoretical findings.

Index Terms—Online optimization, game theory, nonlinear systems.

I. INTRODUCTION

ONLINE feedback optimization (FO) [1] is an emerging control paradigm for optimal steady-state operation of complex systems based on their direct closed-loop interconnection with optimization algorithms. FO controllers can handle control objectives beyond set-point regulation, typically tracking (a priori unknown) solution trajectories of time-varying constrained optimization problems. In recent years, FO controllers have been proposed for a wide variety of problem settings [1], [2], [3], [4], [5], [6], [7]. These can be categorized by the type of control objective (e.g., convex or nonconvex) and constraints (e.g., hard or soft), the dynamics of the plant (e.g., nonlinear, linear, or algebraic), the type of algorithm (discrete or continuous-time), and the stability analysis (e.g., continuous-time, discrete-time, or hybrid), see [1] for a comprehensive list. FO has found widespread application in various domains, including power systems (e.g., for optimal power reserve dispatch [2], [3], or frequency regulation in ac grids [4], [5]), communication networks (e.g., for network congestion control [6]), and transportation systems (e.g., for ramp metering control [7]).

These large-scale engineering infrastructures comprise multiple subsystems with local decision authority and preferences, commonly known as agents. These agents are typically self-interested, hence, a more general notion of “efficiency” is needed to model desirable (i.e., safe, locally optimal, and strategically stable) operating points for such systems, motivating the use of game-theoretic equilibria (e.g., Nash, Wardrop) [8]. A timely example are modern supply chain systems, in which suppliers, manufacturers, and retailers compete over the available resources to maximize their profits [9], [10].

There are several classes of control approaches for driving noncooperative multi-agent systems to steady-state configurations given by game-theoretic equilibria. One class uses first-order algorithms and builds on operator-theoretic and passivity arguments [11], [12], [13]. Passivity is leveraged in [11] to design a distributed feedback control law to drive agents with single-integrator dynamics to a steady-state operating point given by a Nash equilibrium (NE). This approach is extended to multi-integrator agents affected by partially known disturbances in [12]. The case of games with coupling constraints and agents with mixed-order integrator dynamics is considered for the first time in [7]. All these works consider agents coupled via their objective functions (and constraints) but with decoupled dynamics and use continuous-time flows. A second class of methods are sampled-data approaches based on the model-free extremum seeking (ES) framework, for finding optima [14], game-theoretic equilibria [15], [16], and solutions to dynamic inclusions (which subsume the two previous cases) [17], [18]. Stankovic et al. [15] designed an ES controller for NE seeking in games in which the agents have nonlinear decoupled dynamics. The extension to games with coupling constraints is presented in [16]. In both cases, practical stability is proven in the disturbance-free case. Poveda and Teel [17] developed a general framework for the design and analysis of a class of ES controllers applicable to optimization as well as noncooperative games. Practical stability is established by relying on the mathematical framework of hybrid dynamic inclusions. To accelerate convergence, Poveda and Teel [18] extended the previous framework to periodic to event-triggered sampled-data control. These methods require the nominal stability of the plant and, typically, time-scale separation assumptions.

A third class of methods are extensions of economic model predictive control to noncooperative systems, including
multiobjective MPC [19] and various flavors of game-theoretic MPC [20], [21]. Unlike the first-order or ES methods, these consider transient operations and can handle unstable systems. In exchange, they require dynamic models of the plant and are typically computationally heavier due to the need to find accurate equilibria of trajectory games at each sampling time.

In this article, we propose feedback equilibrium seeking (FES), an extension of FO that seeks to drive prestabilized dynamical systems to “efficient” operating points encoded by time-varying generalized equations (GEs). GEs contain constrained optimization as a special case and can model a broad range of equilibrium problems (e.g., Nash, Wardrop). FES controllers are most similar to the first class of control approaches discussed above [11], [12], [13]. They are typically based on first-order methods and require an (approximate) steady-state input–output sensitivity of the plant, which is assumed to be stable. As a result, they are faster than model-free ES methods; on the other hand, they are slower but computationally lighter than MPC-based methods, and require only static models. Therefore, FES controllers occupy a unique middle ground between ES and MPC. Their greatest advantage emerges in situations where steady-state input–output sensitivity information is available, but detailed dynamic models are lacking. Such conditions are commonly found in sectors like power systems and process optimization [2], [3], [4], [5]. Unlike [11], [12], and [13], our framework encompasses a broad class of algorithms and systems beyond continuous-time flows and multi-integrator. Moreover, we use discrete-time algorithms, which are especially relevant for embedded distributed implementations, as communication in multi-agent engineering networks is a discrete process.

Some other recent works [22], [23] studied the problem of tracking time-varying generalized NEs (GNEs) with discrete-time algorithms. A distributed algorithm for tracking the solution trajectory of an exogenously varying strongly monotone game is developed in [22] and extended to monotone games in [23]. Agarwal et al. [24] adapted a GNE-seeking algorithm for online operation by integrating measurements from a physical system, however, the plant is treated as an algebraic map. All the above works neglect dynamic interactions between the plant and the algorithm (either the variation is exogenous or the plant is algebraic), while we consider closed-loop stability in the sampled-data case. This is significant as digital control of continuous-time plants is currently the dominant paradigm. Our contribution to this area of research is threefold.

1) We propose a general framework for designing FES controllers for continuous-time prestabilized nonlinear systems by tapping into a broad class of first- and second-order discrete-time algorithms for generalized equations. This class is broad and includes many optimization and game-theoretic algorithms (e.g., sequential convex programming (SCP), proximal-gradient) as special cases.

2) We derive sufficient conditions for the stability and robustness of the sampled-data algorithm–plant interconnection. Specifically, we prove practical input-to-state stability (Def. 1) of the closed loop with respect to exogenous disturbances, under three fundamental conditions: 1) robust stability of the plant; 2) strong regularity of the control objective; 3) robust convergence of the iterative algorithm. Moreover, we prove a sharper notion of input-to-state stability if an additional small-gain condition holds. As a by-product of our analysis, we also demonstrate that some popular classes of algorithms are not suitable for constructing sampled-data FES (or FO) controllers.

3) We showcase the utility of our framework by designing two novel controllers based on a SCP algorithm for nonconvex nonsmooth optimization, and a forward–backward splitting controller for distributed NE seeking in dynamically coupled games. Further, we illustrate their effectiveness through numerical simulations on smart building and supply chain examples.

We build on our preliminary work [25], which addresses the special case of strongly monotone GEs, globally linearly convergent algorithms, and exponentially stable plants. Our results here provide a significant extension that allows nonmonotone GEs, locally stable nonlinear plants and nonlinearly locally convergent algorithms. This algorithmic extension is particularly important since it captures many practically relevant cases, such as algorithms for nonconvex optimization, which are not globally convergent, and primal-dual and distributed algorithms, which typically exhibit sublinear convergence only.

II. PRELIMINARIES

Basic notation: Denote by $I$ and $id$ the identity matrix and operator, respectively; by $\lim$ and $\sup$ the limit and essential suprema, respectively. Given $P = P^\top > 0$ and $x \in \mathbb{R}^n$, $|x|_P := \sqrt{x^\top P x}$ with the convention $|x| = |x|_I$.

Systems theory: Continuous-time signals are denoted by $\mathbb{L}^n = \{f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n\}$ and sequences by $\mathbb{L}^n = \{f : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n\}$. Given $x \in \mathbb{L}^n$ and $y \in \mathbb{L}^m$, we consider the signal/sequence norms $\|x\| = \sup_{t \geq 0}|x(t)|$ and $\|y\| = \sup_{k \geq 0}|y_k|$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $K$ if it is continuous, strictly increasing, and satisfies $\gamma(0) = 0$. If it is also unbounded, then $\gamma \in K_\infty$.

Similarly, a function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $K$ if it is continuous, strictly decreasing, and satisfies $\sigma(s) \to 0$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $K\mathcal{L}$ if $\beta(\cdot, s) \in K$ for each fixed $s \geq 0$ and $\beta(r, \cdot) \in \mathcal{L}$ for fixed $r$. Class $K\mathcal{L}$ functions obey a weak triangle inequality [26, Lemma 10]

$$\forall a, b \geq 0 \text{ and } \alpha \in \mathcal{K}, \quad \alpha(a + b) \leq \alpha(2a) + \alpha(2b). \quad (1)$$

For $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}$, $\gamma_1 \circ \gamma_2$ denotes the composition. Given $\xi \in \mathbb{R}^n$, $u \in \mathbb{L}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $x \in \mathbb{L}^n$ is a solution of the initial value problem

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = \xi. \quad (2)$$

If it is uniformly continuous, obeys the initial condition and satisfies the differential equation almost everywhere.

Definition 1 (LISpS [27]): Consider a system

$$x'(t) = f(x(t), u(t)), \quad x(0) = \xi \quad (3)$$

with solution $x(t, \xi, u)$ (where either $x'(t) = \dot{x}(t)$ and $t \in \mathbb{R}_{\geq 0}$ or $x'(t) = x(t + 1)$ and $t \in \mathbb{Z}_{\geq 0})$ and let $\bar{x}$ be a reference signal. The system (3) is said to be locally input-to-state practically stable (LISpS) about $\bar{x}$ with respect to $u$ if there exists $\epsilon > 0$, $\beta \in K\mathcal{L}$, $\gamma \in \mathcal{K}$, and $b > 0$ such that,

$$\forall t \geq 0, \quad |x(t, \xi, u) - \bar{x}(t)| \leq \beta(|\xi - \bar{x}(0)|, t) + \gamma(\|u\|) + b$$

provided $|\xi| \leq \epsilon$ and $\|u\| \leq \epsilon$. If $b = 0$ then (3) is locally input-to-state stable (LISS) about $\bar{x}$ with respect to $u$. 

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Operator theory [28]: Given a closed convex set $\Omega \subseteq \mathbb{R}^n$, $t_\Omega : \mathbb{R}^n \to \{0, \infty\}$ denotes its indicator function, $N_\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is its normal cone operator, and $\text{proj}_\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is the Euclidean projection onto $\Omega$. A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is $\mu$-strongly monotone, if $(u - v) \cdot (x - y) \geq \mu \|x - y\|^2$ for all $x \neq y \in \mathbb{R}^n$, $u \in F(x)$, $v \in F(y)$, and monotone if $\mu = 0$; $\text{fix}(F) = \{x \in \mathbb{R}^n \mid x \in F(x)\}$ and $\text{zer}(F) = \{x \in \mathbb{R}^n \mid 0 \in F(x)\}$ denote the set of fixed points and of zeros of $F$. For a convex function $f : \mathbb{R}^n \to \mathbb{R}$, $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the subdifferential mapping in the sense of convex analysis.

Definition 2 (Strong regularity [29]): A set-valued mapping $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be strongly regular at $(x, y)$ if and only if $y \in \Psi(x)$ and there exist neighborhoods $U$ of $x$ and $V$ of $y$ such that the truncated inverse mapping $\Psi^{-1} : V \mapsto \Psi^{-1}(V) \cap U$ is a Lipschitz continuous function on $V$.

III. PROBLEM SETTING

We consider the problem of efficiently operating a physical plant described by the following nonlinear state-space system: $$\dot{x}(t) = f(x(t), u(t), w(t))$$ (4a) $$y(t) = g(x(t), u(t))$$ (4b) where $x \in L^{n_x}$ is the state, $y \in L^n$ is the output, $u \in L^{n_u}$ is the control input, with $u(t) \in U$ for all $t \in \mathbb{R}$, and $w \in L^{n_w}$ is a disturbance satisfying $w(t) \in W$ for all $t \in \mathbb{R}$.

We adopt the “stabilize-then-optimize” paradigm, and assume that (4) is stable and has a steady-state map $p : U \times W \to \mathbb{R}^{n_x}$ satisfying $f(p(u, w), u, w) = 0$ for all $u \in U, w \in W$ and a steady-state input–output map: $$h(u, w) = g(p(u, w), w).$$ (5)

Formally, we assume the system (4) satisfies the following properties that ensure its solution trajectories and steady-state mappings are well defined.

Assumption 1 (Robust stability of the Plant (4) with respect to parameter variations): (i) $f$ is locally Lipschitz; (ii) $g$ is globally $L_g$-Lipschitz-continuous; (iii) $w$ is continuously differentiable and $w \in L^{n_w}$ satisfies $\|w\| < \infty$; (iv) $W$ and $U$ are compact and convex; (v) (4) is LISS [27], namely, there exists $\epsilon_x, \epsilon_w, \alpha_3, \alpha_4, \alpha_5 > 0$, a continuously differentiable function $V$, and $\sigma_1 \in K$ such that for any constant $u \in U$:

$$\alpha_3 \|x - p(u, w)\|^2 \leq V(x, u, w) \leq \alpha_4 \|x - p(u, w)\|^2$$

$$\dot{V}(x(t), u, w(t)) \leq -\alpha_3 V(x(t), u, w(t)) + \sigma_1(|w(t)|)$$

if $V(x(t), u, w(t)) \leq \epsilon_x$ and $|w(t)| \leq \epsilon_w$.

Our control objective is to design an output feedback controller that drives (4) and maintain it near efficient operating conditions. We encode “efficiency” using the following structured generalized equation (GE), parameterized by $w \in W$:

$$0 \in G(z, s) + A(z)$$ (efficiency objective) (6a) $$s = h(u, w)$$ (steady-state map) (6b) $$u = g(z)$$ (ctl state-to-input map) (6c) where $z \in \mathbb{R}^{n_z}$ is an auxiliary variable, $s$ is the steady-state output of (4), $G : \mathbb{R}^{n_z} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_z}$ is a single-valued mapping, $A : \mathbb{R}^{n_z} \rightrightarrows \mathbb{R}^{n_z}$ is a set-valued mapping, and $q : \mathbb{R}^{n_z} \to U$ is the output mapping. The notion of efficiency encoded in the GE (6) is flexible and encompasses a wide variety of useful objectives, notably constrained optimization (e.g., minimizing energy consumption as described in VII-A) and generalized Nash equilibria. The auxiliary variable $z$ is the internal state of the controller, and it also allows the modeling of optimality and equilibrium conditions that involve dual variables, such as KKT critical points. Two concrete examples are provided at the end of this section.

Our control objective is to maintain the system (4) near solutions of (6); we must impose some regularity conditions to ensure that this is a well-defined problem. Substituting (6b) and (6c) into (6a) yields the following compact GE:

$$0 \in G(z, w) + A(z)$$ (7)

where $G(z, w) = G(z, h(q(z), w))$. The parameter-to-solution mapping $S : W \to \mathbb{R}^{n_z}$ is defined as

$$S(w) = \{z \in \mathbb{R}^{n_z} \mid 0 \in G(z, w) + A(z)\}$$.

For a given $w \in L^{n_w}$, the set of solution trajectories is

$$S(w) = \{z \in \mathbb{R}^{n_z} \mid \forall t \geq 0, z(t) \in S(w(t))\}.$$ (9)

The following assumption ensures that tracking solution trajectories in $S$ is a well-posed problem.

Assumption 2 (Strong regularity of the GE): (i) $q$ is globally $L_q$-Lipschitz continuous, (ii) $G$ is continuously differentiable, and (iii) the mapping $G(\cdot, w) + A(\cdot)$ in (7) is strongly regular at all points satisfying $z \in S(w)$, for all $w \in W$.

Conditions for strong regularity are context-dependent, as detailed later in this section with two examples. In optimization contexts, it reduces to having a strong local minimizer and unique dual variables, which are followed by second-order sufficient conditions and suitable constraint qualifications.

In general, the set $S(w)$ can be complex and multi-valued; thankfully, Assumption 2 imposes some structure on $S(w)$.

Theorem 1: [30, Th. 3.2] Under Assumption 2, for any $w \in L^{n_w}$, there exist $m$ Lipschitz continuous mappings $\tilde{z}_i \in L^{n_z}$, $i \in \{1, \ldots, m\}$, such that $S(w) = \{\tilde{z}_1, \ldots, \tilde{z}_m\}$.

Remark 1: Theorem 1 demonstrates that the solution map consists of $m$ isolated $w$-parameterized trajectories called “branches”. These branches generalize the notion of a isolated local solution of a GE (in the context of optimization, a strict local minimizer) to a parameterized setting. Our analysis will eventually establish tracking error bounds and the existence of a convergence basin around each branch.

We can now formally state our control problem.

Problem 1: Design an output feedback controller so that the system (4) tracks the input and output trajectories $u^* = q(z^*)$ and $y^* = h(u^*, w)$, for some $z^* \in S(w)$ and $w \in L^{n_w}$.

The GE (6) allows us to naturally express complex control objectives, e.g., constraints can be readily encoded into $A$ using normal cone operators (see Example 1A). Similarly to FO [1], we assume that the dynamic model of the plant (4) is unknown and that the exogenous disturbance $w$ is not measurable. However, the output $y$ can be measured and the input–output steady-state sensitivity $\nabla_h h(u, w)$, or an approximation of it, is available. The robustness of FO controllers to static input–output modeling error is analytically studied in [31] and experimentally in [32]. The inability to measure $w$ is common in practical applications. For example, in power systems, $w$ represents variable microgeneration and uncontrollable loads caused by consumers drawing power from the grid.

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1If the physical plant (4) is not stable, it can be prestabilized by replacing $f(x, u, w)$ with $f(x, k(x, v), w)$, where $v$ is the new input (for example, a reference command) and $k$ is a stabilizing controller.

2For notational simplicity $G$ does not depend on $w$; this is without loss of generality since $w$ can be included in $s$. 

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3With an abuse of notation, we extend the mappings $h$ and $q$ to take signals as arguments, in the sense of $u^*(t) = q(z^*(t))$ for all $t \in \mathbb{R}_{\geq 0}$. 

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**Example 1.A. Nonlinear programming**

Consider the \(w\)-parameterized nonlinear program (NLP)

\[
\min_{\xi, u} \phi(\xi, u) + \varphi(\xi)
\]

(10a)

\[
s.t. \quad \xi = h(u, w), \quad u \in U
\]

(10b)

where \(\phi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}\) and \(h\) are twice continuously differentiable, and \(\varphi : \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}\) is a proper lower semicontinuous convex function [28, Def. 9.12], e.g., an \(\ell_1\)-norm. We assume that (10) is always feasible, namely,

\[
\{\xi, u\} \mid \xi \in \text{Dom } \varphi, u \in U, \xi = h(u, w) \neq \emptyset \quad \forall w \in W.
\]

The associated partial Lagrangian function is

\[
L(\xi, u, \lambda, w) = \phi(\xi, u) + \lambda^\top(h(u, w) - \xi)
\]

(11)

where \(\lambda \in \mathbb{R}_{\geq 0}^m\) is a dual variable, and the KKTs for (10) are

\[
\begin{align*}
0 & \in \nabla_\xi L(\xi, u, \lambda, w) + \partial \varphi(\xi) \\
0 & \in \nabla_u L(\xi, u, \lambda, w) + \nabla h(u) \\
0 & = \nabla_\lambda L(\xi, u, \lambda, w) - h(u, w) - \xi
\end{align*}
\]

(12)

This is a special case of the GE in (6), with \(z = (\xi, u, \lambda)\)

\[
0 \in \begin{bmatrix}
\nabla_\xi L(\xi, u, \lambda, w) & 0 \\
\nabla_u L(\xi, u, \lambda, w) & \nabla h(u) \\
0 & -\xi
\end{bmatrix}
\]

\[G(z,s) + A(z)
\]

(13)

\[0 \in \begin{bmatrix}
F(u, s) \\
0 \\
-A^\top \\
0
\end{bmatrix}
\]

From a game-theoretic perspective, a relevant solution concept for (14) is the generalized NE, where no agent can unilaterally reduce its cost [35].

**Definition 3:** A feasible action profile \(\bar{u} = (\bar{u}_1, \ldots, \bar{u}_N)\) is a generalized NE (GNE) of the game (14) if

\[
\jmath_i(\bar{u}, \bar{u}_i, w) \leq \min_{u_i \in U_i} \{\jmath_i(u_i, \bar{u}_i, w) \mid A(u_i, \bar{u}_i) \leq b\}
\]

holds for all agents \(i \in \mathcal{I}\).

If each \(\jmath_i\) is continuously differentiable and convex with respect to \(u_i\), a GNE can be found by solving a system of coupled KKT conditions of the problems (14) [35, Th. 4.8]

\[
0 \in \nabla_{u_i} \jmath_i(u_i, \bar{u}_i, w) + A_i^\top \lambda + N_i(u_i) \quad \forall i \in \mathcal{I}
\]

(15)

\[0 \in b - A u + N_{\mathbb{R}_+^m}(\lambda)
\]

where \(\lambda \in \mathbb{R}^m\) is a dual variable associated with the coupling constraints \(A u - b \leq 0\). The set of solutions to (15) is a special subclass of GNEs, known as variational GNEs (v-GNEs) [35].

To write (15) in a compact form, we define the pseudogradients \(F(u, w) = F(u, h(u, w)) = [\nabla u_i \jmath_i(u_i, h(u, w))]_{i \in \mathcal{I}}\) obtained by stacking up the partial gradients of the local cost functions \(\jmath_i\), i.e., \(\nabla_{u_i} \jmath_i(u_i, h(u, w)) = F_i(u, h(u, w))\).

The KKT system (15) can then be cast as a special case of the original GE (6) with \(z = (u, \lambda)\), \(q(z) = [I \mid 0] z = u\), and

\[
0 \in \begin{bmatrix}
F(u, s) \\
0 \\
-A^\top \\
0
\end{bmatrix}
\]

(16)

To ensure the problem is well posed, we assume that the game primitives satisfy the following technical conditions:

1) \(F\) is continuously differentiable and, \(\forall w \in W, F(\cdot, w)\) is \(\mu\)-strongly monotone and \(\ell\)-Lipschitz continuous;
2) for all \(u \in U\), \(F(u, \cdot)\) is \(\ell\)-Lipschitz continuous;
3) for all \((\bar{u}, w)\) and \(w \in W\) satisfying (15), it holds that \([\hat{A}]_{i \in \mathcal{I}}(\bar{u})\) has full row rank, where \(\hat{A} = [A^\top B_1^\top \ldots B_N^\top]_{i \in \mathcal{I}}\), \(E(\bar{u}) = \{i \in \mathbb{N} \mid \hat{A}_i \bar{u} = \bar{b}\}\) is the set of active constraints at \(\bar{u}\), \(\bar{b} = (b_1, \ldots, b_N)\), and \(\hat{A}_i\) is the \(i\)th row of \(\hat{A}\).

**Lemma 2:** Under (C1)–(C3), the condensed GE (7) associated with (16) is strongly regular at \((\bar{z}, w)\), with \(\bar{z} \in S(w)\).

**Proof:** See Appendix A1.

Condition (C3) is standard in optimization and is known as the linear independence constraint qualification (LICQ).

### IV. Control Strategy

Our objective is to maintain the system (4) near efficient operating points, namely, the solution trajectories \(s^t = h(u^t, w)\) of the GE in (6). Since (4) is prestabilized, selecting \(u(t) = u^*(t)\) for all \(t \geq 0\) would cause (4) to approximately track the desired steady-state \(s^\ast(t)\). However, computing \(u^*(t)\) requires full knowledge of \(w(t)\) and evaluating the solution mapping \(S(w(t))\), which may be impossible and/or impractical. Instead, we approach the problem by modifying an iterative algorithm for solving the GE (6) with the following form:

\[
s^{k+1} = T(s^k, \lambda^k)
\]

(17b)
where $T : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}^{n_x}$ is the algorithm, i.e., a rule for generating the next iterate. This class of algorithms is abstract and broad, including, e.g., projected-gradient, SCP, and best-response dynamics in strictly convex games. Concrete examples are given are the end of this section and in [25].

By substituting (17a) in (17b), we can compactly cast the algorithmic update rule (17) via the condensed parameterized mapping $\bar{T}(\cdot, w) : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$, defined as

$$
\bar{T}(z, w) := T(z, h(q(z), w)).
$$

(18)

The following ensures that the nominal iteration (17) is locally convergent and well behaved in a parameterized setting.

**Assumption 3 (Robust Convergence of the Algorithm):**

1) For all $w \in W$, $z = \bar{T}(z, w)$ if and only if $z \in S(w)$.
2) There exist a continuous function $W : \mathbb{R}^{n_x} \times W \rightarrow \mathbb{R}$, a constant $\epsilon > 0$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and $\alpha \in \mathcal{K}$ such that for all $w \in W, \bar{z} \in S(w)$, and $z \in \{ \xi : |\xi - \bar{z}| \leq \epsilon \}$

$$
\alpha_1(|z - \bar{z}|) \leq \bar{W}(z, w) \leq \alpha_2(|z - \bar{z}|)
$$

(19a)

3) For all fixed $z$, there exists $L_T > 0$ such that

$$
|T(z, s) - T(z, s')| \leq L_T |y - y'| \quad \forall s, s' \in \mathbb{R}^{n_x}
$$

The function $W$ in Assumption 3 is commonly known as a "merit function," and serves as a Lyapunov function for the algorithmic dynamics. Typical choices include objective functions and weighted distances to the solution set $S(w)$.

If $w(t)$ were fully measurable and the steady-state input–output mapping $h$ in (5) perfectly known, then $u^*(t)$ could be computed using (17). Instead, we construct an output feedback controller by replacing the steady-state input–output model $s^k$ in (17a) with online measurements $y^k$ obtained from the physical system (4). This creates an "online" FES process, where the system is directly integrated into the algorithm, as illustrated in Fig. 1. The incorporation of feedback into the algorithm provides a degree of robustness. In this article, we formally establish robustness solely against the unmeasured disturbances $w$. Evidence of robustness against other factors, including modeling errors, has been shown in [31] analytically, and in [32] empirically.

Since (17) is discrete, we adopt a sampled-data strategy with a zero-order hold. Let $T > 0$ denote the sampling period and let $t_k = kT$ be the sampling instants. Interconnecting (4) with (17) yields the following sampled-data closed-loop system:

$$
\Sigma_1^T : \begin{cases}
\dot{x}(t) = f(x(t), u(t), w(t)) \\
y(t) = g(x(t), w(t))
\end{cases}
$$

(20a)

$$
\Sigma_2^T : \begin{cases}
\dot{z} = T(z, u(t)) \\
u(t) = q(z) \quad \forall t \in [kT, (k+1)T).
\end{cases}
$$

(20b)

Before proceeding to a closed-loop stability analysis, we provide some concrete examples of admissible algorithms.

**Example 1.B. The Josephy–Newton Method**

To solve the GE in (7), the Josephy–Newton (JN) method [36] relies on the implicitly defined iterates

$$
H(z)(z^+ - z) + G(z, w) + A(z^+) \geq 0
$$

(21)

where $H : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x \times n_x}$ is an invertible approximation of the Jacobian $\nabla z G$. The corresponding update rule is

$$
T(z, s) = (H(z) + A)^{-1}(H(z)z - G(z, s)).
$$

(22)

If the mapping $H$ is chosen judiciously, the nominal JN method satisfies Assumption 3, as formalized next.

**Lemma 3:** Given Assumption 2, let $z \in S$ and suppose $\exists \epsilon > 0$ such that for all $w \in W$ and $z$ such that $|z - \bar{z}(w)| \leq \epsilon$

1) $\exists \delta > 0$ such that $|H(z) - \bar{H}(\bar{z}(w), w)| \leq \delta$;
2) the map $\bar{H}(\cdot) + A(\cdot)$ is $M$-Lipschitz continuous.

Then, if $M \delta < 1$ the JN operator satisfies Assumption 3 with $w(z) = |z - \bar{z}(w)|, \alpha_1 = \alpha_2 = \epsilon$, and $\alpha = (1 - \delta)\epsilon$.

**Proof:** See Appendix A2.

When the GE represents the KKT conditions of an NLP, the JN method is equivalent to sequential quadratic programming [34, § 4.2], [37, § 6C]. Specialized to the KKT conditions (13) of the NLP (10), evaluating $T$ is equivalent to letting

$$
T(z, s) = (\xi + d_{\xi}^* - d_{u}^*, u + d_u^*, \lambda^*)
$$

(23)

where $(d_{\xi}^*, d_u^*, \lambda^*)$ is a primal–dual stationary point of the following optimization problem:

$$
\min_{d_{\xi}, d_u} \frac{1}{2} \left[ \begin{array}{c} d_{\xi} \\ d_u \end{array} \right]^T \left[ \begin{array}{cc} Q & S \\ S^T & R \end{array} \right] \left[ \begin{array}{c} d_{\xi} \\ d_u \end{array} \right] + \left[ \nabla_{\xi} \phi(z) \right] d_{\xi} \right] + \phi(\xi + d_{\xi})
$$

(24a)

s.t. $d_{\xi} = \nabla_u h(u, w)d_u + s - \xi$

(24b)

$u + d_u \in \mathcal{U}$

(24c)

The matrices $Q, S, R$ are positive semidefinite approximations of $\nabla^2_{zz} L(z, w), \nabla^2_{\xi z} L(z, w)$, and $\nabla^2_{uu} L(z, w)$, respectively, where $L$ is the Lagrangian function in (11).

**Example 2.B. Semi-decentralized GNE seeking**

Consider the noncooperative game example in Section III. A GNE can be computed by solving the GE in (16) using the forward–backward operator splitting (FBS) algorithm

$$
T(z, s) = (\text{id} + \Phi^{-1}A)^{-1}(z - \Phi^{-1}G(z, s))
$$

(25)
where $\Phi$ is a preconditioning matrix opportunistically designed to distribute the computation among the agents [38], i.e.,
\[
\Phi := \begin{bmatrix} \text{diag}(\gamma_1^{-1}, \ldots, \gamma_N^{-1}) - A^\top & -A \\ \gamma_c^{-1}I & \end{bmatrix}
\]
whose diagonal entries are the step sizes of the algorithm.

With this choice, the update (25) can be performed in a semidecentralized way as $T(z, s) = (u^+, \lambda^+)$ with
\[
u_i^+ = \text{proj}_\Delta(u_i - \gamma_i(F_i(u_i, s) + A_i^\top \lambda)) \quad \forall i \in \mathcal{I}
\]
\[
\lambda^+ = \text{proj}_{\Delta \gamma_c}(\lambda + \gamma_c(2u^+ - u) - b).
\]
Typically, the local action updates $u_i^+$ are performed in parallel by the agents, whereas the dual update $\lambda^+$, associated with the coupling constraints\(^6\) $Au - b \leq 0$, is managed by a central coordinator able to gather the local strategies $u_i$’s and broadcast incentive signals (e.g., prices, dual variables) to all the agents\(^7\) [40], as illustrated in Fig. 2.

In the remainder of this example, we derive conditions under which the update rule in (25) satisfies Assumption 3, and thus, it is a suitable controller choice for the FES scheme [20]. In the offline static case, namely, when $s = h(z, w)$, the convergence analysis of (25) is based on the theory of averaged operators [28, § 4.5], which are a special subclass of strongly nonexpansive operators [28, Prop. 4.35].

Definition 4: Let $\mathcal{D} \subseteq \mathbb{R}^n$ and $R : \mathcal{D} \rightarrow \mathbb{R}^n$. Then, $R$ is strongly quasi-nonexpansive (SQNE) if there exist a norm $\cdot, \cdot $ and a constant $\rho > 0$ such that
\[
|R(z) - z|^2_P \leq |z - z|^2_P - \rho|R(z) - z|^2_P.
\]
for all $z \in \mathcal{D}$, for all $z \in \mathcal{F} R$.

The key result is presented in the next statement, where we show that any parameterized continuous SQNE operator with a unique fixed point admits a merit function as in Assumption 3.

Proposition 1: Let $T(w)$ be SQNE and continuous, and fix $\mathcal{T}(\xi, w) = \{z(w)\}$, for all $w \in \mathcal{W}$. Then, there exist $P > 0$, $\alpha \in \mathbb{K}$, and $\epsilon > 0$ such that for all $z \in \mathcal{F} \{ \xi : |z - z| \leq \epsilon \}$
\[
|T(z) - z|^2_P \leq |z - z|^2_P - \alpha(|z - z|),
\]
(28)

Proof: See Appendix A.

Using Proposition 1 we can show that algorithm (25) satisfies Assumptions 3 and is thus a viable FES controller.

Lemma 4: Consider the generalized game in Section III and the GNE seeking algorithm (25). Assume that (C1)–(C3) hold and set the step sizes/gains in (26) such that $\gamma_i \leq (\|A_i\| + \delta)^{-1}$, for all $i \in \mathcal{I}$, and $\gamma_c \leq (\sum_{i \in \mathcal{I}} \|A_i\| + \delta)^{-1}$, with $\delta > 1/(2\mu)$. Then, the algorithm (25) satisfies Assumption 3.

Proof: See Appendix A.

5 Algorithms based on strongly quasi-nonexpansive operators [28, § 4] include operator splitting methods (such as forward–backward [28, § 26.5], Douglas–Rachford [28, § 26.3]) and other averaged operator-based methods (such as proximal–point algorithm [28, Th. 23.41], alternating projection [28, Ex. 28.11]). By Proposition 1, all these algorithms are valid controllers to form the closed-loop interconnection (20).

V. CLOSED-LOOP STABILITY ANALYSIS

We analyze the closed-loop system (20) by sampling it and forming a discrete-time system, demonstrating LISS of the discretized system, then concluding LISS of (20) by invoking [41, Th. 5]. Periodically sampling (20) yields
\[
\sum_1^d \{ x^{k+1} = \psi^k(x^k, u^k, w^k) \}
\]
(29a)
\[
y^k = g(x^k, u^k)
\]
(29b)
where $\psi^k(x, u, w)$ is the solution of the initial value problem
\[
\begin{align*}
\dot{\xi}(t) &= f(\xi(t), v, w(t)), \\
\xi^{k+1} &= \xi^k, \\
t &\in [t^k, t^{k+1}]
\end{align*}
\]
atime $t = t^{k+1}$ and $\hat{w}^k$ is the restriction of $w$ to $[t^k, t^{k+1})$.

As noted in Remark 1, the solution of the GE (7) consists of finitely many isolated branches. Here, we establish LISpS for each branch, thereby defining their respective convergence basins. The closed-loop system (20) will then track a particular branch, determined implicitly by the initial condition $z(0)$, akin to how an initial condition in gradient-based nonconvex optimization implicitly selects a specific local minimizer.

For the sake of analysis, consider an arbitrary branch $\bar{z} \in \mathcal{S}$ with corresponding solution trajectory $z^* = \bar{z}(w) \in \mathcal{S}$. We introduce the error signals
\[
\delta x(t) = x(t) - p(u(t), w(t))
\]
and shift (29) to the origin to obtain the error dynamics
\[
\sum_1^d \{ \delta x^{k+1} = G_1(\delta x^k, u^k, \Delta u^k, \hat{w}^k) \}
\]
(31a)
\[
\sum_1^d \{ e^{k+1} = G_2(e^k, \delta x^k, \hat{w}^k) \}
\]
(31b)
where $\Delta u^k = u^{k+1} - u^k$, $p$ is the steady-state mapping of the discretized and continuous-time systems, and the expressions for $G_1, G_2, \text{ and } H$ are given in Appendix A. The resulting feedback interconnection is illustrated in Fig. 3.

The main step in the analysis is to show that (31a) is LISpS with respect to the disturbance sequence
\[
d^k = \sum_{t \in [t^k, t^{k+1})} |\hat{w}(t)|.
\]
(32)
We first show that (31a) is LIS with respect to $\Delta u$ and $d$ and then show that (31b) is LIS and locally input-to-output stable with respect to $\delta x$ and $d$. Then we employ a small-gain theorem to derive conditions for LISpS of the feedback interconnection.

The first step is to show LII of the plant.

Theorem 2 (LII of the plant): Let Assumptions 1–3 hold. Then (31a) is LII with respect to $\Delta u$ and $d$, i.e., there exists $\beta_\delta \in K, \gamma_\delta \in L, \gamma_\delta^d \in K, \pi_\delta > 0$, and $\pi_\delta \in K$ such that
\[
V^k \leq \beta_\delta(V^0, \kappa_\delta) + \gamma_\delta^u(t)\|\Delta u\| + \gamma_\delta^d(\|d\|, \tau)
\]
(33)
where $V_k = V(x_k, w_k, u_k)$, provided that $V^0 \leq 0.25\varepsilon_x$, $\|\Delta u\| \leq \varepsilon_x$, $\|d\| \leq \varepsilon_w$.  

**Proof:** See Appendix A6.

This result is a straightforward consequence of the local input-to-state stability of the original continuos-time system (Assumption 1) and states that the discrete-time system (29b) will converge to a steady-state for any given constant input with the changes in $w$ and $u$ acting as disturbances. The impact of input variations on convergence to the steady-state manifold, quantified by $\gamma_w \in L$, decreases as the sampling period $\tau$ increases, as control input update rate decreases.

Next, we show that the equilibrium-seeking algorithm (17) when viewed as a dynamical system (29a) is also LISS.

**Theorem 3 (LIOS of the algorithm):** Let Assumptions 1–3 hold. Then the system (31b) is locally input–output stable (LIOS), i.e., there exist $\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3 > 0, \beta_2, \beta_2 \in K\mathcal{L}, \gamma_{z,x} \in K\mathcal{L},$ and $\gamma_{z,y} \in K$ such that for all $W \leq \bar{\varepsilon}_1, \|V\| \leq \bar{\varepsilon}_2$, $\|d\| \leq \bar{\varepsilon}_3$,

$$W_k \leq \beta_2(W^0, k) + \gamma_{z,x}(\|V\|, \tau) + \gamma_{z,y}(\|d\|)$$

$$\|\Delta u\| \leq \beta_2(W^0, k) + \gamma_{z,x}(\|V\|, \tau) + \gamma_{z,y}(\|d\|)$$

where $\|V\| = \sup_{t \geq 0} V(x_k, u_k, w_k)$ and $W_k = W(x_k, u_k)$.  

**Proof:** See Appendix A7.

In other words, the algorithm (17) is not only convergent whenever the plant is at steady state, as required by Assumption 3, but also robust with respect to transient deviation from this steady state as measured by the ISS Lyapunov function $V$ defined in Theorem 2.

Having established LISS of the subsystems, we derive conditions under which the interconnection is LISS.

**Theorem 4:** Let Assumptions 1–3 hold. Then, the discrete-time closed-loop system (31) is LISS if the small gain condition

$$\gamma_w^x(\tau) \gamma_w^y(s, \tau) < 1 \quad \forall s \in [0, \bar{\varepsilon}]$$

is satisfied where $\varepsilon = \min\{0.25\varepsilon_x, \bar{\varepsilon}_2\}$, and $\gamma_w^x \in \mathcal{L}$, $\gamma_w^y \in \mathcal{K}\mathcal{L}$, $\varepsilon_x$, and $\bar{\varepsilon}_2$ are defined in Theorems 2 and 3.

**Proof:** See Appendix A8.

**Remark 2:** The offset term $b(\tau)$ arises because it is not always possible to upper bound $\gamma_z^2 \in \mathcal{K}\mathcal{L}$ near the origin as required by small-gain theorems. E.g., $\eta(s, \tau) = e^{-\tau} \sqrt{\tau}$, which does not satisfy the small-gain condition $\eta(s, \tau) < 1$ in some neighborhood of the origin no matter how large $\tau$ is. However, the interval where it does not hold can be made arbitrarily small, i.e., $\eta(s, \tau) < 1$ for all $s \in \{e(\tau), \infty\}$ where $e(\tau) \to 0$ as $\tau \to \infty$. The term $b(\tau)$ is a consequence of this interval and also becomes arbitrarily small. Further, $b$ goes exactly to zero in many cases, e.g., in Corollary 1 or if $\gamma_w^y$ is convex.

Due to broad range of system and algorithm combinations allowed under Assumptions 1–3, the small-gain condition can be difficult to verify in practice. Theorem 4 can be sharpened if the algorithm is locally linearly convergent.

**Corollary 1:** Let Assumption 1–3 hold. Additionally, assume that the algorithm (17) is locally q-linearly convergent with convergence rate $\eta \in (0, 1)$, i.e., Assumption 3 holds with $W(z, w) = [z - \bar{z}(w)]^T \alpha(s) = (1 - s)\alpha(s) = \lambda_{\min}(P)s$, and $\alpha(s) = \lambda_{\max}(P)s$. Then $\gamma_w^x(\tau) = c_1 e^{-\alpha s^2}$, $\gamma_w^y(s, \tau) = \frac{1}{\sqrt{1 - e^{-\alpha s^2}}}$ and (35) reduces to the condition

$$\frac{c_1 L V e^{-\alpha s^2}}{1 - e^{-\alpha s^2}} < 1$$

where $c_1 = L q L_T L g \sqrt{\frac{1}{\alpha s}}(1 + \frac{L q \lambda_{\min}(P)}{(1 - s)\lambda_{\max}(P)})$ and $L_V, L_z, L_T$, and $L_y$ are Lipschitz constants for $V, \bar{z}, T$, and $g$ with respect to $u, w, y$, and all arguments, respectively.

Note that the small-gain condition (36) can always be satisfied for large enough $\tau$. Continuous and discrete-time analogs of Corollary 1 can be found in [42] and [43], respectively.

In the general case, the gain $\gamma_w^y$ is nonlinear and one cannot guarantee LISS for sufficiently large finite $\tau$. We can, however, show LISS with a decaying perturbation term.

**Theorem 5:** Let Assumptions 1–3 hold. Then for each branch $\bar{z} \in S$ there exists $\tau \in (0, \infty)$ such that for $\tau > \bar{\tau}$ the sampled-data closed-loop system (20) is LISS, i.e., there exist $\beta \in K\mathcal{L}, \gamma \in \mathcal{K}, v_1, v_2, v_3 > 0$, and $b : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{t \to \infty} b(t) = 0$ such that for all $t \geq 0$

$$\left| \frac{\delta x(t)}{e(t)} \right| \leq \beta \left( \left| \frac{\delta x(0)}{e(0)} \right|, t \right) + \gamma(\|\|w\|\|) + b(t)$$

provided that $|\delta x(0)| \leq v_1, |e(0)| \leq v_2$, and $\|\|w\|\| \leq v_3$.

**Proof:** See Appendix A9.

**Theorem 5** is the main result of the article and states that, if the sampling period $\tau$ is long enough, then the interconnection will become locally practically stable. This makes intuitive sense, since the algorithm is guaranteed to converge if the system remains at steady state (Assumption 3) which longer sampling periods give time to reach (Assumption 1). Input constraints can be satisfied pointwise-in-time via projection operations while all other (e.g., output) constraints are guaranteed to be satisfied asymptotically with bounded violations during transients.

VI. ILLUSTRATIVE EXAMPLES

In this section, we demonstrate that some of the algorithmic preconditions (Assumption 3) in Theorem 5 are sharp, in the
sense that if they are not satisfied, one cannot expect in general the algorithm-plant interconnection (20) to be robust to unmeasured disturbances or even stable. The code for the examples in this section is available in [44].

Consider a single-input single-output dynamic plant governed by the second-order differential equation

\[ \ddot{\xi} + 0.5\dot{\xi} + \xi = u + w \]  

where \( y = \xi \) and \( w \) is a disturbance term. The plant is an LTI system of the form \( \dot{x} = Ax + B(u + w) \) with \( x = (\xi, \dot{\xi}) \) and is asymptotically stable and satisfies Assumption 1 with steady-state mapping \( h(u, w) = u + w \), Lyapunov function \( V(x, u, w) = 1/2|x - (u + w, 0)|^2_P \), where \( P \) satisfies \( AP + PA^T + I = 0 \), and \( \alpha_3 = \lambda_{\min}(P), \alpha_4 = \lambda_{\max}(P) \), and \( \alpha_5 = \lambda_{\min}(P^{-1}) \). The control objective is encoded as

\[ \min_u \frac{1}{2}\|y - y_{ref}\|^2 \text{ s.t. } y = h(u, w), \ u \in [-10, 10] \]  

which models set-point regulation and satisfies Assumption 2.

A. Closing the Loop Can Lead to Instability

The proximal-gradient controller [25, § III.A] for this problem generates control inputs according to the update rule

\[ T(z^k, y) = \text{proj}_{[-10, 10]}(z^k - \gamma(y - y_{ref})) \]  

with \( u^k = z^k \), which satisfies Assumption 3 if \( \gamma \in (0, 1] \), with \( \alpha_1 = \alpha_2 = \text{Id} \), \( W(z, w) = |z - z^*(w)|, z^*(w) = y_{ref} - w, \alpha(s) = (1 - \sqrt{1 - \gamma(2 - \gamma)})s \), and \( L_T = \gamma \).

Based on Theorem 5, we expect that the interconnection of (38) and (40) will only be stable for sufficiently large \( \tau \). Since the system (38) is exponentially stable and the algorithm (40) linearly convergent we can use Corollary 1 to conclude that the interconnection is stable for \( \tau > 5.44 \). The simulation results match our expectations. Fig. 4 demonstrates that the interconnection indeed becomes unstable for small \( \tau \), that the system is stable for sufficiently large \( \tau \), and that Corollary 1 can be conservative as \( \tau = 5 \) leads to a stable interconnection.

B. Vanishing Steps Do Not Track Solutions Trajectories

Finally, consider the algorithm (40), where the step size is time-varying (\( \gamma \) depends on the iteration \( k \)) and such that

\[ \gamma_k \geq 0 \quad \forall k, \quad \sum_{k \in \mathbb{N}} \gamma_k = \infty, \quad \sum_{k \in \mathbb{N}} (\gamma_k)^2 < \infty \]  

Gradient-based algorithms with vanishing steps of this kind are popular in the context of optimization and game theory, e.g., for stochastic approximation and gradient tracking. Here, a vanishing step size results in a vanishing control gain, and thus, unsurprisingly this class of algorithms does not admit a merit function\(^{10} \) in the sense of Assumption 3. Hence, ISS cannot be guaranteed in online settings. This is illustrated in Fig. 5, which shows that whenever the step sizes in (40) vanish to zero, as in (41), the tracking error diverges.

VII. APPLICATION EXAMPLES

A. Temperature Regulation in Smart Buildings

In this section, we illustrate how FES can be applied to smart building automation. Consider the 5-room single-story office building in Fig. 6. Its dynamics are of the form

\[ \dot{x} = Ax + Bu + B_w w + \sum_{i=1}^{n_u} (B_{wu,i} w + B_{xu,i} x) u_i \]  

and are generated using the BRCM toolbox [45]. The state \( x \in L^{115} \) contains the temperatures of the rooms and walls layers, floor layers, etc. The control inputs \( u \in L^5 \) are an air handling unit (AHU) consisting of air flow (0–1 kg/s), cooling/heating power between \( 10^5 \)W and \( 10^3 \)W, and one radiator in each room emitting between 0 and 25 W/m\(^2\). The disturbances \( w \in L^{10} \) include the solar radiation, the ambient outdoor air temperature, the temperature of the ground, and the internal heat gains coming from building occupants. The measurement \( y \in L^5 \) contains the

\[^{10}\text{In fact, for } k \to \infty \text{ the } K\text{-function of (40)} \text{ yields } \lim_{k \to \infty} \alpha_k(\cdot) = (1 - \sqrt{1 - \gamma^2}) = 0 \text{ since } \lim_{k \to \infty} \gamma_k = 0 \text{ by (41).}\]
room, outside air, and ground temperatures. The solar radiation and the heat emitted by the buildings occupants are unmeasured. The nonlinearities are caused by the AHU whose control authority depends on the ambient air temperature and room temperatures. We model 15 building occupants (providing an internal heat gain of 100 W each) by a Markov chain, with a time-dependent probability of being in a given room [46]. The solar radiation and ambient temperature are periodic functions yielding temperatures and solar gains representative of central European springtime.

Our objective is to minimize energy usage while maintaining the room temperatures within a comfortable range \( T = [T_{\min}, T_{\max}] \). This control objective is implicitly encoded via an NLP with composite cost function as in (10), i.e.,

\[
\phi(\xi, u) = \frac{\epsilon}{2}[\xi]^{T}u + \frac{1}{2}d^{T}u + \frac{\eta}{2}\sum_{i=1}^{5}\max\{0, T_{\min,i} - \xi, \xi - T_{\max,i}\}
\]

(43a)

\[
\varphi(\xi) = \frac{\eta}{2}\sum_{i=1}^{5}\max\{0, T_{\min,i} - \xi, \xi - T_{\max,i}\}
\]

(43b)

where \( \eta > 0 \) is a tuning parameter, \( \epsilon \) collects the electricity prices, and \( T_{\min,i}, T_{\max,i} \) are the comfort constraints on the temperature in the \( i \)th room. The quadratic term in \( \phi(u, \xi) \) is a regularizer, with typically small tuning parameter \( \epsilon \), that improves regularity of the minimizers. The purpose of \( \varphi \) is to penalize the comfort constraint violations of the room temperatures. A 1-norm penalty on the violations is used for two reasons. First, the electrical cost of heating is linear in the control input. Second, \( \varphi \) is an exact penalty function, and so for a well-tuned parameter \( \eta \), the (disturbance-free) system can be exactly driven within the temperature bounds [47].

Given the control objective (43) and the steady-state sensitivity associated with (42), we form a FES controller using the JN algorithm, as explained in Section IV. Given the measurements \( y \), the current control input \( u \), and the controller state \( \xi \), the resulting sampled-data SQP controller sets the next control inputs as the solutions (\( d_{u} \)-component) of the QP

\[
\begin{align*}
\min_{d_{u}, d_{w}, \sigma} & \quad \frac{\epsilon}{2}d_{u}^{T}u + \frac{1}{2}d_{w}^{T}u + \frac{\eta}{2}\sum_{i=1}^{5}\sigma_{i} \\
\text{s.t.} & \quad d_{u} = \nabla_{u}h(u, u)d_{u} + y - \xi \\
& \quad u + d_{u} \in \mathcal{U} \\
& \quad \xi + d_{\xi,i} - T_{\max} \leq \sigma_{i}, \quad \forall i = 1, \ldots, 5 \\
& \quad \xi + d_{\xi,i} - T_{\min} \geq -\sigma_{i}, \quad \forall i = 1, \ldots, 5 \\
& \quad \sigma_{i} \geq 0, \quad \forall i = 1, \ldots, 5 \\
\end{align*}
\]

(44)

where \( \sigma_{i} \) are slack variables that allow to reformulate the nonsmooth cost term \( \phi(\xi + d_{\xi}) \) as linear constraints. We set \( \eta = 5 \cdot 10^{4}, \epsilon = 10^{-3} \), and a discrete time step of \( \tau = 3 \) [min]. Simulation results are presented in Fig. 7. The proposed controls are able to keep the rooms between the temperature bounds, with only minor constraint violations. We compare the controller performance to a hysteresis-based thermostat controller, which turns on the radiators and AHU heater ON when \( T_{\text{room}} \leq T_{\min} + \frac{\tau^{P}}{2}T_{\max} - 2 \) and OFF when \( T_{\text{room}} \geq T_{\min} + \frac{\tau^{P}}{2}T_{\max} + 2 \). The AHU cooler is turned on when \( T_{\text{room}} \geq T_{\min} + T_{\max} + \frac{\tau^{P}}{2} \), and OFF when \( T_{\text{room}} \leq T_{\min} + \frac{\tau^{P}}{2}T_{\max} \), with all temperatures in \( ^{\circ}\text{C} \). In our example, the SQP controller provides a 27.84% reduction in constraint violations, and a 32.29% reduction in total cost as measured by (43) compared to the hysteresis controller. The code for this application example is available in [44].

### B. Competitive Supply Chain Management

Consider a supply chain with \( N \) producers \( P_{i} \), labeled by \( i \in \mathcal{I} := \{1, \ldots, N\} \), that are supplied by a supplier \( S \) and a market \( M \). Each producer \( i \) places an order \( o_{i} \) for raw material form \( S \), which is then used to meet the local demand \( d_{i} \). The market \( M \) is a dynamical system whose output, namely, the demands \( d_{i} \), is influenced by the producers’ selected prices \( \sigma_{i} \) and by other exogenous factors \( d^{w} \).

![Fig. 7. Simulations of the SQP controller (44) on the building dynamics (42). The comfort temperature (output) constraints are approximately satisfied throughout the simulations, while heating and cooling effort is minimized.](image)

![Fig. 8. Supply chain network with two producers, \( P_{1} \) and \( P_{2} \), sharing a supplier \( S \) and a market \( M \). Each producers \( i \) places an order \( o_{i} \) for raw material from \( S \), which is then used to meet the local demand \( d_{i} \). The market \( M \) is a dynamical system whose output, namely, the demands \( d_{i} \), is influenced by the producers’ selected prices \( \sigma_{i} \) and by other exogenous factors \( d^{w} \).](image)
insulate against shocks using the inventory control law
\[ a_i = -k_i(s_i - \bar{s}_i) + d_i \]  
(47)
where \( k_i > 0 \) is a tunable control gain.

The common market obeys the linear price/demand curve
\[ \hat{d}_i((\sigma, \sigma_{-i}), d^e_i) = d^e_i - \beta_i \sigma_i + \sum_{j \neq i} \beta_{ij} \sigma_j \quad \forall i \in \mathcal{I} \]  
(48)
where \( \hat{d}_i(d^e_i) \) is the nominal local demand for producer \( P_i \), \( d^e_i \) is a base-line demand, and \( \beta_i, \beta_{ij} \geq 0 \) are market constants [10].

In this model, demand for \( P_i \)'s products drops if \( P_i \) raises its prices \( \sigma_i \) and increases when competitors increase theirs \( \sigma_{-i} \). The market does not respond instantaneously to changes in prices, instead the true demand \( d_i \) evolves according to
\[ \dot{d}_i = -\frac{1}{\tau_M} (d_i - \hat{d}_i(\sigma, d^e_i)) \]  
(49)
where \( \tau_M > 0 \) is the time constant of the market.

The overall dynamics of the supply chain system are
\[
\begin{bmatrix}
\dot{s}_i - \bar{s}_i \\
\dot{l}_i \\
\dot{d}_i
\end{bmatrix}
= 
\begin{bmatrix}
0 & \frac{1}{\tau_i} & \frac{-1}{\tau_i} \\
\frac{1}{\tau_i} & 0 & \frac{1}{\tau_i} \\
0 & \frac{1}{\tau_i} & 0
\end{bmatrix}
\begin{bmatrix}
s_i - \bar{s}_i \\
l_i \\
d_i
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
\hat{d}_i(\sigma, d^e_i)
\end{bmatrix}
\]  
which is a collection of linear time invariant (LTI) systems of the form
\[ \dot{x}_i = A_i x_i + B_i u_i + D w_i, \quad y_i = C_i x_i, \]
with state \( x_i = (s_i - \bar{s}_i, l_i, d_i), y_i = (l_i, d_i), u = (\sigma_i, \sigma_{-i}), \) and \( w_i = d^e_i \).

These LTI systems are stable if the control gains in (47) satisfy \( k_i \geq 0 \). The system-level dynamics are of the same form in (4), where \( f \) collects the LTI dynamics and the outputs \( y_i \) are only locally available. Thus, Assumption 1 is satisfied with steady-state maps \( p_i(\sigma, d^e_i) = (0, \hat{d}_i(\sigma, d^e_i), \hat{d}_i(\sigma, d^e_i)) \) and \( h_i(\sigma, d^e_i) = (\hat{d}_i(\sigma, d^e_i), \hat{d}_i(\sigma, d^e_i)), \) for all \( i \in \mathcal{I} \).

Producers continuously balance local supply and demand using the inventory control law (47) but must choose a price \( \sigma_i \) for their product. The goal of each producer \( P_i \) is to set its prices \( \sigma_i \) so local profit is maximized at steady-state operation while the price regulations are respected, i.e.,
\[ \sigma_i \geq \sigma_{\min}, \quad \min_{\sigma_{\min} \leq \sigma_i \leq \sigma_{\max}} \left[ \frac{c^i_j \hat{d}_i(\sigma, d^e_i) - \sigma_{\max} \hat{d}_i(\sigma, d^e_i)}{\text{sales revenue}} \right] \]  
(50a)
where \( c^i_j \geq 0 \) (50a) is the production price for producer \( P_j \).

The constraints enforce limits on the individuals and average prices, respectively, usually imposed by market regulators (e.g., customers associations). The overall behavior of the supply chain is coupled through the price/demand curve (48) as well as the price-cap regulations (50b).

In practice, producers do not know the market model (48) but can only observe its transient outcome, namely, the local demand \( d_i \). Thus, to guide the supply chain to a “fair” competitive equilibrium (a variational GNE) of the game (50), we use the FBS controller (see Fig. 2) developed in Section IV.

We consider a pricing game played by \( N = 3 \) producers, on a single-product market \((\sigma_i \in \mathbb{R}), \) over a year. All the parameters of the supply chain model and the game are drawn from uniform distributions and fixed over the course of the simulations. The only exception are the baseline demands \( d^e_i \) in (48); these suddenly increased by a factor of 3 at day 120 to simulate a massive (non-price-related) surge in demand due to, for example, a natural disaster, a cultural event, or the introduction of a new product.

At each sampling period \( \tau = 7 \) [days], producers measure their local demand \( d_i \), receive a marginal fee (or loss) \( \lambda \) from the market regulator, and then update their prices \( \sigma_i \) according to the FBS controller in Fig. 2.

We checked numerically that conditions (C1)–(C3) in Section IV hold; thus, we can invoke (i) Lemma 2 to prove that Assumption 2 is satisfied for the game (50), and (ii) Lemma 4 to prove that Assumption 3 holds for the FBS controller. In turn, Theorem 5 guarantees LiSpS of the controlled supply chain system with respect to exogenous demand fluctuations, under an appropriate choice of the sampling time \( \tau \).

In Fig. 9, we illustrate the evolution of the inventory levels, production rates, and local demands. The local controllers (47) keep the inventories at the desired level despite the initial transient and the sudden demand surge. Moreover, the FBS controller guides the prices to track the variational GNE trajectories of the pricing game (50), as illustrated in Fig. 10 (top). Finally, we note that the sudden spike in demand causes the solution trajectories of the game (50) to hit the price capping...
constraints \((50b)\). This can be seen in Fig. 10 (bottom), in which the average price trajectory exceeds its limit and, in turn, the dual variable (internal state) \(\lambda\) of the FBS controller increases to penalize this violation. The code for this application example is available in [44].

VIII. CONCLUSION

Iterative algorithms for solving generalized equations, such as Josephy–Newton, forward–backward splitting, can be used as sample-data robust feedback controllers for guiding complex unknown dynamical systems to constrained and economic equilibria. Under robust stability of the plant, strong regularity of the generalized equation describing the control objective, and robust convergence of the iterative algorithm, the sample-data algorithm–plant cyber-physical interconnection is LISS with respect to unmeasured disturbances affecting the plant, provided that the sampling period is appropriately designed. Illustrative numerical examples in building energy management and supply chain coordination corroborate these theoretical findings. Future research directions include incorporating online model (i.e., input–output sensitivities) learning and real-time constraint satisfaction.

APPENDIX

A1. Proof of Lemma 2

Define the auxiliary GE

\[
\begin{bmatrix}
F(u, s) \\
\tilde{b}
\end{bmatrix}
= \begin{bmatrix}
A^T & 0 \\
-\tilde{A} & 0
\end{bmatrix}
\begin{bmatrix}
u \\
\lambda
\end{bmatrix} + \begin{bmatrix}
0 \\
N_{\mathbb{R}^m}(\lambda)
\end{bmatrix}
\tag{51}
\]

where \(F\) is defined in (15), \(\tilde{A}\) and \(\tilde{b}\) are defined in condition (C3) and \(\tilde{b}\) has \(m\) rows, that results from dualizing all constraints in the game (14). The GE (51) is strongly regular if the LICQ (C3), \(F\) is Lipschitz w.r.t. s, and the strong second-sufficient condition (SSOSC) [34, eq. (1.49)] holds by [34, Prop. 1.27.1,2.8]. Strong monotonicity of \(F\) (C1), linearity of the constraints is sufficient for the SSOSC and thus for strong regularity of (51) [34, Prop. 1.27.1,2.8]. Since the original GE (15) dualizes a subset of the constraints in (51) the solution map of (15) can be constructed by selecting a subset of the dual variables in (51) and is thus also strongly regular.

A2. Proof of Lemma 4

By [33, Th. 10] under these assumptions there exists \(0 < \tilde{e} \leq \epsilon\) such that if \(|z - \tilde{z}(w)| \leq \tilde{e}\) then there exists \(\tilde{q} \in (0, 1)\) such that \(|T(z, w) - \tilde{z}| \leq \tilde{q} |z - \tilde{z}|\) which immediately implies Assumption 3 (i) and (ii). Assumption 3 (iii) follows from the Lipschitz continuity of \((H(z)(\cdot) + A(\cdot))^-1\) and the fact that \(G\) is 1-Lipschitz continuous with respect to \(s\).

A3. Proof of Proposition 1

To prove (28), let us fix \(w \in \mathcal{W}\) and define \(\zeta(z, w) = |T(z, w) - \tilde{z}|^2\), we see that (i) \(\zeta(z, w) = 0\) if and only if \(z \in \text{fix} T(. , w)\), (ii) since \(\tilde{T}(., w)\) is continuous, by assumption, so is \(\zeta(., w)\), and (iii) \(\zeta(., w)\) is positive definite. Thus, by invoking [48, Lemma 4.3], it follows that there exists a \(K\)-function \(\alpha\) such that

\[
\alpha_w(|z - \tilde{z}|) \leq \zeta(z, w), \quad \tilde{z} \in \text{fix} T(. , w). \tag{52}
\]

Now, since \(T(., w)\) is \(\rho\)-SQNE (Definition 4), we have that

\[
|\tilde{T}(z, w) - \tilde{z}|^2_p \leq |z - \tilde{z}|^2_p + \rho|T(., w)(z) - \tilde{z}|^2_p, \tag{53a}
\]

\[
\leq |z - \tilde{z}|^2_p - \rho \lambda_{\text{min}}(P) \alpha_w(|z - \tilde{z}|) \tag{53b}
\]

for some \(P > 0\), where for the second inequality \((53b)\) we used \(|z|_P \geq \lambda_{\text{min}}(P)|z|\) and (52), sequentially. Since \(W\) is compact by assumption, we can define the \(K\)-function \(\alpha(z) = \min_{w \in \mathcal{W}} \alpha_w(z)\) [49, Lemma 1]. Finally, letting \(\alpha\) completes the proof.

A4. Proof of Lemma 4

We show that conditions (i)–(iii) of Assumption 3 hold.

(i) It follows by [28, Prop. 26.1 (iv) (a)] that \(T(., w) = \text{zer}(\Phi^{-1} G(., w) + \Phi^{-1} A) = \text{zer}(\Phi(., w) + A) = S(w)\), for all \(w \in \mathcal{W}\), where \(S(w)\) is a singleton since the pseudogradient \(\tilde{F}(., w)\) is strongly monotone by (C1).

(ii) It follows by [28, Prop. 26.1 (iv) (d)] that \(T(., w) = \tilde{F}(., w)\) is \(\eta\)-averaged with \(\eta = 2\beta \delta/(4\beta \delta - 1)\), for all \(w \in \mathcal{W}\), since \(\Phi^{-1} G(., w)\) is \(\beta\)-cocoercive, with \(\beta = 2\mu/\tilde{\lambda} F\), is maximally monotone, both with respect to the norm \(|\cdot|\_\Phi\), see [40, § III.B]. In turn, every \(\eta\)-averaged operator is continuous, since nonexpansive, and \(\rho\)-SQNE, with \(\rho = 1 - \frac{\tilde{\lambda} F}{\delta}\), by [28, Prop. 4.35]. Moreover, fix \(T(., w) = S(w)\) is a singleton by part (i) of this proof. Hence, we can invoke Proposition 1 to prove that Assumption 3 (ii) is satisfied, with \(W(z, w) = |z - S(w)|^2\_\Phi, \alpha_1 = \lambda_{\text{min}}(\Phi), \alpha_2 = \lambda_{\text{max}}(\Phi), \) and \(\alpha = \rho \lambda_{\text{min}}(\Phi)\lambda_{\text{min}}(\Phi)\), for some \(\alpha \in \mathcal{K}\) and \(\Phi\) as in (26).

(iii) By [28, Prop. 23.8], the resolvent \((id + \Phi^{-1} A)^{-1}\) in (25) is (firmly)nonexpansive, since \(\Phi^{-1} A\) is maximally monotone. By combining this result with the \(\ell\)-Lipschitz continuity of \(F\), i.e., (C2), we can show that Assumption 3 (iii) is satisfied with \(L_T = \ell/\lambda_{\text{min}}(\Phi)\).

A5. Explicit Expressions for the Functions in (31)

\[
G_1(x^k, u^k, \Delta u^k, \tilde{w}^k) = \psi(k^x, x^k, u^k, \tilde{w}^k) - p(u^k + \Delta u^k, w^{k+1})
\]

\[
G_2(x^k, \delta x^k, \tilde{w}^k) = \tilde{T}(x^k, \delta x^k, \tilde{w}^k) - \tilde{z}(w^{k+1})
\]

\[
H(e^k, \delta x^k, \tilde{w}^k) = q(\tilde{T}(e^k, \delta x^k, \tilde{w}^k)) - q(e^k + \tilde{z}(w^k))
\]

\[
\tilde{T}(e^k, \delta x^k, \tilde{w}^k) = T(e^k + \tilde{z}(w^k), q(\psi(\delta x^k + p(q(z^k, w^k), q(e^k + z^k, \tilde{w}^k), w^{k+1})))
\]

A6. Proof of Theorem 2

We begin with a preparatory Lemma.

Lemma 5: Let Assumption 1 hold. Then

\[
V(x^{k+1}, v, w^{k+1}) \leq e^{-\alpha \tau} V(x^k, v, w^k) + \sigma_1(d^k) \tag{54}
\]

where \(x^{k+1} = \psi(t^k, x^k, v, w)\).
Proof: For any \( t \in [t_k, t_{k+1}] \), Assumption 1 and the comparison principle [48, Lemma 3.4] imply that
\[
V(x(t), u(w(t))) \leq e^{-\alpha_5(t-t_k)} V_k + \int_{t_k}^t e^{(s-t)} \sigma_1(|\dot{w}(s)|) \, ds
\]
\[
\leq e^{-\alpha_5(t-t_k)} V_k + \sigma_1 \left( \sup_{s \in [T]} |\dot{w}(s)| \right) \int_{t_k}^t e^{(s-t)} \, ds
\]
\[
= e^{-\alpha_5(t-t_k)} V_k + (1 - e^{-\alpha_5(t-t_k)}) \sigma_1(d)
\]
\[
\leq e^{-\alpha_5(t-t_k)} V_k + \sigma_1(d)
\]
(55)
where \( V_k = V(x_k, u_k, w_k) \). Let \( t = t_{k+1} \) and conclude. □

With this result in hand, we proceed to show LISS of the plant. Let \( V_k = V(x_k, u_k, w_k) \), using Lemma 5 with \( v = u_k+1 \) implies that
\[
V_{k+1} \leq e^{-\alpha_5} V_k + e^{-\alpha_5} L_V v_k + \sigma_1(d_k)
\]
\[
\leq e^{-\alpha_5} V_k + e^{-\alpha_5} |V(x_k, u_k+1, w_k) - V_k| + \sigma_1(d_k)
\]
\[
\leq e^{-\alpha_5} V_k + e^{-\alpha_5} L_V |v_k - u_k| + \sigma_1(d_k)
\]
(56)
where \( L_V > 0 \) is the Lipschitz constant of \( V \) (which exists since \( V \) is continuously differentiable and \( U \) is compact). Continuing, we note that \( e^{-\alpha_5} \in (0, 1) \) for any \( \tau > 0 \), and thus, by the comparison principle, see e.g., [50, Example 3.4]
\[
V_k \leq (e^{-\alpha_5})^k V_0 + (1 - e^{-\alpha_5}) \int_0^t (1 - e^{-\alpha_5}) \sigma_1(||d||) \, ds
\]
\[
\leq e^{-k \alpha_5} V_0 + \gamma_n^u(\tau) \Delta u + \gamma_n^d(||d||, \tau)
\]
(57)
where \( \gamma_n^u(\tau) = \frac{\tau \gamma_n}{1 - e^{-\alpha_5}} \) and \( \gamma_n^d(||d||, \tau) = \frac{\tau \gamma_n}{1 - e^{-\alpha_5}} \sigma_1(s) \).

Assumption 1 requires that \( V(x(t), u(t), w(t)) \leq \epsilon_x \). We first show this holds at the sampling instants, i.e., \( V(x_k, u_k, w_k) \leq \epsilon_x \) then demonstrate intersample satisfaction under the conditions
\[
V_0 \leq 0.25 \epsilon_x, \|u\| \leq 0.125 \gamma_n^u(\tau)^{-1} \epsilon_x, \|d\| \leq \epsilon_w
\]
\[
\gamma_n^d(||d||, \tau) \leq 0.125 \epsilon_x \text{ and } \sigma_1(||d||) \leq 0.5 \epsilon_x.
\]
(58)
We proceed by induction to show that \( V_k \leq 0.5 \epsilon_x \). For \( k = 0 \) it’s clear that \( V_0 \leq 0.25 \epsilon_x \Rightarrow V_0 \leq 0.5 \epsilon_x \). Next assume (56) holds up to \( k - 1 \), it follows that:
\[
V_k \leq e^{-k \alpha_5} V_0 + \gamma_n^u(\tau) \Delta u + \gamma_n^d(d, \tau)
\]
\[
\leq 0.25 \epsilon_x + \gamma_n^u(\tau) \frac{0.125 \gamma_n}{\gamma_n^u(\tau)} \epsilon_x + 0.125 \epsilon_x = 0.5 \epsilon_x
\]
as required. Between sampling instants by Lemma 5, we have that for all \( t \in [t_k, t_{k+1}] \)
\[
V(x(t), u(t), w(t)) \leq e^{-\alpha_5} V_k + \sigma_1(d_k)
\]
\[
\leq \max \{2 V_k, 2 \sigma_1(d_k)\}
\]
\[
\leq \max \{2 \cdot 0.5 \epsilon_x, 2 \cdot 0.5 \epsilon_x\} = \epsilon_x
\]
using (58). Thus, we can define \( \pi_1 = 0.125 \gamma_n^u(\tau)^{-1} \), and \( \pi_2(s) = \max \{s, 8 \gamma_n^d(s, 0), 2 \sigma_1(s)\} \). Letting \( \beta_x(s, r) = e^{-\alpha_5 s} \) completes the proof (see Definition 1).

A7. Proof of Theorem 3

We begin with two preparatory lemmas that are used to bound the disturbances and the distance between steps ON and OFF the steady-state manifold. The first is an immediate consequence the bound on \( ||\dot{w}|| \) in Assumption 1 and the properties of integrals.

Lemma 6: Under Assumption 1 \( |w^{k+1} - w^k| \leq \tau d^k \) \( \forall k \geq 0 \).

Lemma 7: Given Assumptions 1–3, there exists \( \sigma_2 \in K \), \( \sigma_3 \in K \) such that
\[
|\mathcal{T}(z_k, w^{k+1}) - T(z_k, y^{k+1})| \leq \sigma_2(V_k, \tau) + \sigma_3(d^k)
\]
where \( \mathcal{T}(z, w) = T(z, h(q(z), w)) \), provided that \( V_k = V(x_k, u_k, w_k) \leq \epsilon_x \) and \( d \leq \epsilon_w \).

Proof: We adopt the shorthand notation \( x_k = x, x^{k+1} = x^+, \) etc. To begin, note that
\[
|\mathcal{T}(z, w^+) - T(z, y^+)| = |T(z, h(q(z), w^+)) - T(z, g(p(u, w^+))| \leq L_T |g(x^+, w^+) - g(p(u, w^+), w^+)|
\]
\[
\leq L_T |L_g x^+ - p(u, w^+)|
\]
(60)
where we have used \( L_T > 0 \) and \( L_g > 0 \) Lipschitz continuity of \( T \) (Assumption 3) and of \( g \) (Assumption 1), respectively. Next, we invoke Lemma 5 to show that
\[
|x^+ - p(u, w^+)| \leq \alpha_3 \circ V(x, u, w^+)
\]
\[
\leq \alpha_3 \circ (e^{-\alpha_5} V(x, u, w) + \sigma_1(d))
\]
\[
\leq \alpha_3 \circ (2e^{-\alpha_5} V(x, u, w)) + \alpha_3 \circ (2\sigma_1(d))
\]
whenever \( V(x, u, w) \leq \epsilon_x \) and \( d \leq \epsilon_w \). Combining the above inequality with (60) yields
\[
|\mathcal{T}(z, w^+) - T(z, y^+)| \leq \sigma_2(V, \tau) + \sigma_3(d)
\]
(61)
as claimed, with \( \sigma_2(s, \tau) = L_T L_g \alpha_3^{-1}(2e^{-\alpha_5 s}) \) and \( \sigma_3 = L_T L_g \alpha_3^{-1} \circ 2 \sigma_1 \).

Now, we are ready to proceed and show LISS of (31b). We begin with
\[
W^{k+1} = [W(T(z_k, w^{k+1}), w^{k+1})]
\]
\[
+ [W(z^{k+1}, w^{k+1}) - W(T(z_k, w^{k+1}), w^{k+1})]
\]
(62)
Focusing on the first term, by virtue of Assumption 3 we have, if the restriction \( |z_k - z^{k+1}| \leq \epsilon_z \) holds, that
\[
W(T(z_k, w^{k+1}), w^{k+1}) \leq W(z_k, w^{k+1}) - \alpha(|z_k - z^{k+1}|
\]
where \( z^{k+1} = z(w^{k+1}) \). Now consider
\[
\alpha \left( \frac{1}{2} |z - z_k| \right) = \alpha \left( \frac{1}{2} |z - z^{k+1} + z^{k+1} - z_k| \right)
\]
\[
\leq \alpha(|z - z^{k+1}| + \alpha(L_z/2)|w^{k+1} - w^k|)
\]
where we have used the weak triangle inequality and Lipschitz continuity of \( z \). Rearranging the last line yields that
\[
\alpha(|z - z^{k+1}|) \geq \alpha(|z_k|) - \alpha(L_z/\Delta w^k)
\]
(63)
where \( \alpha = \alpha \circ 0.5 \in K_\infty \). Therefore
\[
W(T(z_k, w^{k+1}), w^{k+1}) \leq W(z_k, w^{k+1}) - \alpha(|z - z^{k+1}|
\]
\[
\leq W(z_k, w^{k+1}) - \alpha(|z_k|) + \alpha(L_z/\Delta w^k)
\]
\[
\leq W(z_k, w^{k+1}) - \alpha(|z_k|) + \omega_1 + \alpha L_z \text{id}(|\Delta w^k|)
\]
\[
\leq W(z_k, w^{k+1}) - \alpha(|z_k|) + \omega_1 + \alpha L_z \text{id}(|\Delta w^k|)
\]
\[ \leq W^k - \hat{\alpha}(|e^k|) + ((\omega_1 + \hat{\alpha} \circ L_z \id) \circ \tau \id)(d^k) \]  
where \( \omega_1 \in \mathcal{K} \) exists since \( W \) is uniformly continuous in \( w \) and the last line uses Lemma 6. Uniform continuity holds by the Heine–Cantor theorem as \( W \) is continuous and \( \{ |z - z(w)| \forall w \in \mathcal{W} \} \times \mathcal{W} \) is compact.

Next, we focus on the second term in (62). We see that

\[ \Delta W = |W(z_{k+1}, w^{k+1}) - W(T(z_k, w^{k+1}), w^{k+1})| \]

\[ \leq \omega_1(|z^{k+1} - T(z_k, w^{k+1})) \]

\[ = \omega_1(|T(z_k, y^{k+1}) - T(z_k, w^{k+1})|). \]

Continuing, we invoke Lemma 7 to show that

\[ \Delta W \leq \omega_1(\sigma_2(V^k, \tau) + \sigma_3(d^k)) \]

\[ \leq \omega_1(2\sigma_2(V^k, \tau) + \omega_1(2\sigma_3(d^k)) \]

\[ \leq \omega_2(V^k, \tau) + \omega_3(d^k) \]

where \( \omega_2 = \omega_1 \circ \sigma_2 \in \mathcal{K} \) and \( \omega_3 = \omega_1 \circ \alpha_3 = \mathcal{C} \), subject to the restrictions \( V^k \leq \epsilon_2 \) and \( d^k \leq \epsilon_\omega \).

Combining (64) and (65) to bound (62) we obtain that

\[ W^{k+1} - W^k \leq -\hat{\alpha}(|e^k|) + \omega_2(V^k, \tau) + \omega_3(d^k). \]

Defining \( \omega_1 = (\omega_1 \circ \tau + \hat{\alpha} \circ L_z \tau + \omega_2) \in \mathcal{K} \) we obtain

\[ W^{k+1} - W^k \leq -\hat{\alpha}(|e^k|) + \omega_2(V^k, \tau) + \omega_3(d^k). \]

Next, we focus on the restrictions required for (66) to hold, these are \( V^k \leq \epsilon_2 \), \( d^k \leq \epsilon_\omega \) and \( |z_k - z^{k+1}| \leq \epsilon_\beta \). The first two follow from the assumptions. The third expression can be bounded as follows:

\[ |z_k - z^{k+1}| = |z_k - z + z - z^{k+1}| \]

\[ \leq |z_k - z| + |z - z^{k+1}| \leq |e^k| + L_z \Delta w^k \]

\[ \leq |e^k| + L_z \tau d^k \leq \max\{2|e^k|, 2L_z \tau d^k \} \]

and thus a sufficient condition for \( |z_k - z^{k+1}| \leq \epsilon_2 \) is \( |e^k| \leq 0.5\epsilon_2 \) and \( d^k \leq \frac{0.5\epsilon_2}{L_z \tau} \).

Thus, the dissipation inequality (66) holds in a neighborhood of the origin given by \( |e^k| \leq 0.5\epsilon_2 \), \( d^k \leq \min\{\frac{0.5\epsilon_2}{L_z \tau}, \epsilon_\omega\} \), and \( V^k \leq \epsilon_2 \), and the system is locally asymptotically stable with zero inputs (0-LAS), and thus, LISS [51], i.e., there exists \( \beta_2, \gamma_2, \gamma_2^d, \epsilon_1, \epsilon_2 \) and \( \epsilon_\beta \) such that

\[ W^k \leq \beta_2(W^0, k) + \gamma_2^d(\|V\|, \tau) + \gamma_2^d(\|d\|) \]

for \( W^0 \leq \epsilon_1, \|V\| \leq \epsilon_2, \) and \( \|d\| \leq \epsilon_3 \), proving (34a). That \( \gamma_2^d \)

can be chosen such that \( \gamma_2^d \in \mathcal{K} \) follows from [50, Lemma 3.13] and \( \omega_2 \in \mathcal{K} \). Next, we bound \( \Delta u \).

Lemma 8: Let Assumptions 1–3 hold and define \( \Delta u^k = u^{k+1} - u^k \). Then \( \exists \alpha_4 \in \mathcal{K} \) and \( \sigma_4, \sigma_6 \in \mathcal{K} \) such that

\[ |\Delta u^k| \leq \sigma_4(V^k, \tau) + \sigma_6(d^k) \]

whenever \( V^k = V(x_k, w^k \leq \epsilon_2 \), \( W^k = W(z_k, w^k \leq \alpha_2(\frac{3}{2}\epsilon_2) \), and \( d^k \leq \min(\epsilon_\omega, \frac{0.5\epsilon_2}{L_z \tau}) \) where \( L_z \) is the Lipschitz constant of \( z \).

Proof: We adopt the shorthand notation \( z_k = z, z_k^{k+1} = z^+ \) and so on throughout. We begin by noting that

\[ |\Delta u| = |q(T(z, y^+)) - q(z)| \]

\[ \leq L_q|T(z, y^+) - z| \]

\[ = L_q|T(z, w^+) - z + T(z, y^+) - T(z, w^+)| \]

\[ \leq L_q|T(z, w^+) - z| + L_q|T(z, y^+) - T(z, w^+)| \]

where we have added and subtracted \( \mathcal{T} \) and used that \( q \) is \( L_q \)-Lipschitz (Assumption 2). The first term in (69) can be bounded as follows:

\[ |T(z, w^+) - z| = |T(z, w^+) - z(z(w^+)) + z(z(w^+))| \]

\[ \leq |T(z, w^+) - z(z(w^+))| + |z(z(w^+)) - z(w^+)| \]

\[ \leq \alpha_1^{-1}(W(T(z, w^+), w^+)) + |z(z(w^+)) - z(w^+)| \]

\[ \leq \alpha_1^{-1}(W(z, w^+)) + |z(z(w^+)) - z(w^+)| \]

\[ \leq \alpha_1^{-1}(W(z, w^+)) + \alpha_1^{-1}(W(z, w^+)) \]

\[ = 2\alpha_1^{-1}(W(z, w^+)) \]

\[ = 2\alpha_1^{-1}(W(z, w^+)) \]

\[ \leq 2\alpha_1^{-1}(2W(z, w^+)) + 2\alpha_1^{-1}2\omega_1(|w^+| - w) \]

\[ = \rho_1(W(z, w^+)) + \rho_2(\Delta w) \]

\[ \leq \rho_1(W(z, w^+)) + \rho_2(\Delta w) \]

\[ \leq \rho_1(W(z, w^+)) + \rho_2(\Delta w) \]

where \( \rho_1 = 2\alpha_1^{-1}2\omega_2 + 2\alpha_1^{-1}2\omega_1 \), and \( \omega_1 \in \mathcal{K} \) exists by uniform continuity of \( W \) over the compact set \( \mathcal{W} \).

Using (70) and Lemma 7 to bound the first and second terms in (69), respectively, yields that

\[ |\Delta u| \leq L_q\rho_1(W) + L_q\rho_2(\Delta w) + L_q\sigma_2(V, \tau) + L_q\sigma_4(d) \]

Using that \( |\Delta u| \leq \tau d \) by Lemma 6 and collecting terms, we obtain that

\[ |\Delta u| \leq \sigma_4(V, \tau) + \sigma_5(W) + \sigma_6(d) \]

where \( \sigma_4 = L_q\sigma_2, \sigma_5 = L_q\rho_1 = 2L_q\alpha_1^{-1}2\omega_2, \) and \( \sigma_6 = L_q(\rho_2 + \tau + \sigma_3) = L_q(2\alpha_1^{-1}2\omega_1 + \tau + \sigma_3) \) as claimed.

Regarding restrictions, (71) holds when \( V \leq \epsilon_2, d \leq \epsilon_\omega \) and \( z - z(w^+) \leq \epsilon_2 \). To satisfy the last, we note that

\[ |z - z(w^+)| \leq |z - z(w)| + |z(w^+) - z(w)| \]

\[ \leq |\epsilon| + L_2 \tau d \leq \max\{2|\epsilon|, 2L_2 \tau d\} \]

and thus, a sufficient condition for \( |z - z(w^+) - z(w)| \leq \epsilon_2 \) is \( |\epsilon| \leq 0.5\epsilon_2 \), which is implied by \( W \leq \alpha_2(0.5\epsilon_2) \), and \( d \leq \frac{0.5\epsilon_2}{L_z \tau} \). □

We are now ready to prove LIOS (34b). Combining (67) and Lemma 8 we obtain

\[ |\Delta u^k| \leq \sigma_5(\beta_2(W^0, k) + \gamma_2^d(\|V\|, \tau) + \gamma_2^d(\|d\|) + \sigma_4(V^k, \tau) + \sigma_6(d^k) \]

and, using the weak triangle inequality

\[ |\Delta u^k| \leq \sigma_5(2\beta_2(W^0, k) + \gamma_2^d(\|V\|, \tau) + \gamma_2^d(\|d\|) + \sigma_4(V^k, \tau) + \sigma_6(d^k). \]
Finally, collecting terms we see that
\[
\Delta u_k^k \leq \sigma_5 (2\beta_2 W^0, k) + (\sigma_4 + \sigma_5 \circ 2\gamma^s_\zeta (\|\nu\|, \tau) + (\sigma_5 \circ 2\gamma^s_\zeta + \sigma_6)(\|d\|)
\]
\[
= \beta_6 (W^0, k) + \gamma^s_{\kappa}(\|\nu\|, \tau) + \gamma^d_{\kappa}(\|d\|)
\]
(74)
as claimed, where \(\beta_6 = \sigma_5 \circ 2\beta_2\), \(\gamma^s_{\kappa} = \sigma_4 + \sigma_5 \circ 2\gamma^s_\zeta\), and \(\gamma^d_{\kappa} = \sigma_5 \circ 2\gamma^s_\zeta + \sigma_6\). The restrictions are simply the intersection of the preconditions of (67) and Lemma 8, and thus, \(\bar{\varepsilon}_1 = \min(0.5\varepsilon_1, \alpha_2(0.5\varepsilon_2)), \bar{\varepsilon}_2 = \min(\varepsilon, \varepsilon_3), \bar{\varepsilon}_3 = \min(\varepsilon, \varepsilon_3)
\).}

\[\square\]

A8. Proof of Theorem 4

Let \(V = V(x, u, w, k)\) and \(W = W(z, u, k)\). Thanks to Theorem 2 and [50, Lemma 3.8] we have that
\[
\lim_{k \to \infty} |V_k| \leq \gamma^s_{\kappa}(\|\nu\|, \tau) + \gamma^d_{\kappa}(\|d\|) \leq \tau
\]
(75)
for \(V^0, \|\Delta u\|\), and \(\|d\|\) sufficiently small. Further, using Theorem 3 and [50, Lemma 3.8] we have that
\[
\lim_{k \to \infty} |V_k| \leq \gamma^s_{\kappa}(\|\nu\|, \tau) + \gamma^d_{\kappa}(\|d\|) \leq \tau
\]
(76)
for \(W^0, V_0\), and \(\|d\|\) sufficiently small. Combining (75) and (76) we obtain that
\[
\lim_{k \to \infty} V_k \leq \gamma^s_{\kappa}(\|\nu\|, \tau) + \gamma^d_{\kappa}(\|d\|) + \gamma \leq \tau
\]
(77)
where \(\gamma(s, \tau) = \gamma^s_{\kappa}(\|\nu\|, \tau) + \gamma^d_{\kappa}(\|d\|)\). The combined restrictions on \(V\) from Theorems 2 and 3 are sufficient for \(V \leq \bar{\varepsilon}\). Thus, by the small-gain theorem [51, Th. 1], [52], the coupled system is LISS, if the condition \(\gamma^s_{\kappa}(\|\nu\|, \tau) + \gamma^d_{\kappa}(\|d\|) < s\) for all \(s \in [0, \bar{\varepsilon}]\) holds. \[\square\]

A9. Proof of Theorem 5

We begin with the following preparatory Lemma. The idea is to show LISS with respect to a virtual disturbance representing the problematic supercritical part of the gain \(\gamma^s_{\kappa}\).

\textbf{Lemma 9:} Given Assumptions 1–3, the system (31b) is locally input–output practically stable, i.e., for all \(W^0 \leq \bar{\varepsilon}_1, \|\nu\| \leq \bar{\varepsilon}_2, \|d\| \leq \bar{\varepsilon}_3\)
\[
|\Delta u_k| \leq \beta_6 (W^0, k) + \kappa(\|\nu\|) + \gamma^d_{\kappa}(\|d\|) + a(\tau) \leq \beta_6 (W^0, k) + \kappa(\|\nu\|) + \gamma^d_{\kappa}(\|d\|) + a(\tau)
\]
(78)
where \(\kappa(\tau) = \gamma^s_{\zeta}(\|\nu\|, \tau)/\varepsilon_2, a(\tau) = \max_{s \in [0, \bar{\varepsilon}_2]} \gamma^s_{\zeta}(s, \tau) - \kappa s, \gamma^s_{\zeta}, \gamma^d_{\kappa}\), \(\bar{\varepsilon}_1 \in [1, 2, 3]\) are defined in Theorem 3. Moreover, \(\kappa \in \mathbb{L}, a(\tau) \geq 0, \) and \(a(\tau) \to 0 \text{ as } \tau \to \infty. \)

\textbf{Proof:} By Theorem 3, we have that
\[
|\Delta u_k| \leq \beta_6 (W^0, k) + \gamma^s_{\kappa}(\|\nu\|, \tau) + \gamma^d_{\kappa}(\|d\|)
\]
(79)
for sufficiently small \(W^0, \|\nu\|,\) and \(\|d\|\). We proceed by bounding \(\gamma^s_{\kappa}\). By construction, for any \(\gamma > 0\), the affine function \(\eta(s, \tau) = \kappa(\tau) + a(\tau)\) satisfies \(\eta(s, \tau) \geq \gamma^s_{\zeta}(s, \tau)\) for all \(s \in [0, \bar{\varepsilon}_2]\). Substituting this bound into (79) yields (79). The remaining claims follow from \(\lim_{\tau \to \infty} a(\tau) = 0\).

\[\square\]

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