The Tachyon in a Linear Expanding Universe

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Abstract

We investigate the tachyon coupling in a static Robertson–Walker like metric background. For a tachyon and dilaton field which are only time dependent one can rewrite this model as a SU(2) Wess–Zumino–Witten model and a scalar Feigin–Fuchs theory. In this case the restriction to a real exponential tachyon field fixes the level \( k \) of the Wess–Zumino–Witten model. For a spatially dependent tachyon the world radius and the dilaton are quantized in terms of \( k \) and the tachyon by two integers, i.e. one has a discrete set of fields. The spatial part of the tachyon is given by Chebyshev polynomials of the second kind. An investigation of the tachyon mass shows that the tachyon is massless for \( k = 1 \).

Strings in cosmological background were discussed a lot in the last years. Mainly, there are two different approaches: 1) in terms of (gauged) Wess–Zumino–Witten (WZW) models yielding an exact 2d conformal field theory \([1]\); 2) via the \( \sigma \) model (or effective action) approach in which one gets results in the \( \alpha' \) expansion \([2, 3]\). In this paper we want to make some general remarks about the description of strings in a Robertson–Walker universe especially for \( \epsilon = +1 \) (for \( \epsilon = 0 \) a general solution was found by Mueller \([4]\)). The crucial point in this description is the fact that a Robertson–Walker (RW) space time is conformally flat and therefore it is always possible to find a coordinate system in which:

\[
G_{\mu\nu} = \frac{W(r)}{r^2} \eta_{\mu\nu},
\]

where the scale factor \( W(r) \) describes the original time dependence of the RW metric. One possibility to handle models like this is given by the world sheet \( \sigma \)–model in which one looks for special solutions for the vanishing of the Weyl anomaly.

After some general remarks about the \( \sigma \)–model approach we discuss the description of strings in RW space time. As one example we investigate a linear expanding universe in four space time dimensions. As it is known this model corresponds to a combined SU(2) WZW and Feigin–Fuchs theory \([5, 6]\). Our special interest in this model is to find out what the tachyon field looks like which reproduces in the flat limit the known results from the David–Distler–Kawai (DDK) model \([7]\) (in general supplemented by spatial background charges and by the interpretation of the time as Liouville field \([8]\)). Similar to this model the demand for a real exponential tachyon yields a restriction: if the tachyon is only time dependent the level of the WZW theory must be equal to one. Finally we discuss a spatial dependence and the mass of the tachyon field and find a discrete set of fields where the spatial part is given by the Chebyshev polynomials of the second kind.

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The usual \( \sigma \)-model containing the massless modes and the tachyon field is given by:

\[
Z = \int DX \, e^{-S}, \quad S = \frac{1}{4\kappa^2} \int_M d^2z \sqrt{g} \left( g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \frac{i}{\sqrt{g}} \partial_a X^\mu \partial_a X^\nu B_{\mu\nu} + \alpha' R^{(2)} \phi + \alpha' T \right). \tag{1}
\]

Here \( G_{\mu\nu} \) corresponds to the metric in the target space (space time), \( B_{\mu\nu} \) is the antisymmetric tensor field, \( \phi \) is the dilaton and \( T \) the tachyon field. If one wants to describe non-critical string theory one can e.g. interpret the time as Liouville field. In this case the 2d metric \( g_{ab} \) is a reference metric. The vanishing of the Weyl anomaly gives the known restrictions for the background fields: \( G, B, \phi \) and \( T \): 

\[
\beta_i^T \equiv 0 \quad \forall i \quad ( \text{i numerates all background fields}) , \quad \beta^G_{\mu\nu} = \beta^G_{\mu\nu} + D(\mu)M(\nu) , \quad M_\nu = 2\alpha' \partial_\nu \phi + W_\nu , \\
\beta^B_{\mu\nu} = \beta^B_{\mu\nu} + H_{\mu\nu\lambda} M^\lambda + \partial_\nu K_\nu , \quad M_\nu = 2\alpha' \partial_\nu \phi + W_\nu , \quad M_\nu = 2\alpha' \partial_\nu \phi + W_\nu , \\
\beta^\phi = \beta^\phi + \frac{1}{2} M^\mu \partial_\mu \phi , \quad \beta^T = \beta^T - 2T + \frac{1}{2} M^\mu \partial_\mu T . \tag{2}
\]

Up to the second order in \( \alpha' \) one gets for the \( \beta \) functions:

\[
\beta^T = -\frac{1}{2} \alpha' D^2 T - \alpha'^2 \frac{1}{8} (H^2)^{\mu\nu} D_\mu \partial_\nu T , \\
\beta^G_{\mu\nu} = \alpha' \hat{R}_{(\mu\nu)} + \frac{1}{2} \alpha'^2 \left( \hat{R}^{\alpha\beta\lambda\gamma}_{(\mu} \hat{R}_{\alpha\beta\lambda\gamma)} - \frac{1}{2} \hat{R}^{\alpha\beta\lambda\gamma}_{(\mu} \hat{R}_{\alpha\beta\lambda\gamma)} + \frac{1}{2} \hat{R}_{(\mu\nu)\beta} (H^2)^{\lambda\gamma} \right) , \\
\beta^B_{\mu\nu} = \alpha' \hat{R}_{(\mu\nu)} + \frac{1}{2} \alpha'^2 \left( \hat{R}^{\alpha\beta\lambda\gamma}_{(\mu} \hat{R}_{\alpha\beta\lambda\gamma)} - \frac{1}{2} \hat{R}^{\alpha\beta\lambda\gamma}_{(\mu} \hat{R}_{\alpha\beta\lambda\gamma)} + \frac{1}{2} \hat{R}_{(\mu\nu)\beta} (H^2)^{\lambda\gamma} \right) , \quad (3)
\]

\[
\beta^\phi = \frac{1}{6} (D - 26) - \frac{1}{2} \alpha' D^2 \phi - \frac{1}{8} \alpha'^2 (H^2)^{\mu\nu} D_\mu D_\nu \phi + \\
+ \frac{1}{16} \alpha'^2 \left( R^2_{\mu\nu\alpha\lambda} - \frac{1}{2} RHH + \frac{5}{24} H^4 + \frac{3}{8} (H^2)^2 - \frac{4}{3} D H \cdot D H \right) .
\]

and \( W_\mu = -(\alpha'^2/24) \partial_\mu H^2 , K_\mu = O(\alpha'^2) , H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} , D H \cdot D H \equiv D_\mu H_{\nu\lambda \beta} D_\nu H^{\nu\lambda \beta} \) and \( \hat{R}_{\mu\nu\lambda\beta} \) is the generalized curvature tensor computed in terms of the connection: \( \hat{\Gamma}_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} - \frac{1}{2} H^\mu_{\nu\lambda} \). One should note at this point that some terms depends on the renormalization scheme (see [6]).

Let us give two general remarks about these equations.

1. For \( B_{\mu\nu} = 0 \) and up to \( O(\alpha'^2) \) the tachyon and dilaton \( \beta \) functions are given by:

\[
\beta^T = -\frac{1}{2} \alpha' D^2 T - 2T + \alpha' \partial_\mu \phi \partial_\mu T , \\
\beta^\phi = \frac{1}{6} (D - 26) - \frac{1}{2} \alpha' D^2 \phi + \frac{1}{10} \alpha'^2 R^2_{\mu\nu\alpha\lambda} + \alpha' \partial_\mu \phi \partial_\mu \phi . \tag{4}
\]

* I neglect in my consideration all “non-perturbative” contributions [10].
After the field redefinition: \( T = e^\phi \tilde{T}, \) \( e^{-2\phi} = f \) and using the vanishing of the metric \( \beta \) function the dilaton and tachyon field decouple and we get the equations:

\[
\begin{align*}
-\frac{1}{2} \alpha' D^2 \tilde{T} - \left( \frac{D-2}{12} - \frac{1}{8} \sqrt{\alpha'} R - \frac{1}{32} \alpha' R^2 + ... \right) \tilde{T} &= 0, \\
-\frac{1}{2} \alpha' D^2 f + \left( \frac{26-D}{6} + \frac{1}{16} \alpha'^2 R^2 + ... \right) f &= 0.
\end{align*}
\]

(5)

We see that both fields fulfill a Klein–Gordon equation of motion where the mass depends on the dimension of the space time and on curvature terms. In the flat limit we get just the known result that the tachyon corresponds to a massless field for \( D = 2 \) and the dilaton to a massless field for \( D = 26 \) (if one interprets the time as Liouville field than \( D \) contains the Liouville degree of freedom too).

2. Beside the decoupling of the tachyon and dilaton fields there is another decoupling which one should discuss at this point. The effective action which corresponds to the equations (2) is given by (6):

\[
S_{\text{eff}} \sim \int d^D X \sqrt{G} e^{-2\phi} \left[ \frac{2}{3}(26 - D) + \alpha'(R + 4\alpha' (\partial \phi)^2 - \frac{1}{12} H_{\lambda\mu\nu}^2) + \alpha'^2 \frac{1}{4} R_{\lambda\mu\nu\beta}^2 + ... \right] .
\]

(6)

In this “\( \sigma \)-model parametrization” the graviton and dilaton mix in the propagator. After the Weyl transformation and and rescaling of the dilaton:

\[
\tilde{G}_{\mu\nu} = e^{-\frac{\alpha'}{D-2}\phi} G_{\mu\nu}, \quad \tilde{\phi} = \sqrt{\frac{2}{D-2}} \phi
\]

(7)

we get the effective action in the “\( S \)-matrix parametrization” containing the “right” kinetic terms:

\[
S_{\text{eff}} \sim \int d^D X \sqrt{\tilde{G}} \left[ \alpha' \tilde{R} - \alpha' (\partial \tilde{\phi})^2 + \frac{2}{3}(26 - D)e^{\frac{4}{D-2}\tilde{\phi}} - \alpha' \frac{1}{12} H_{\lambda\mu\nu}^2 e^{-\frac{8}{D-2}\tilde{\phi}} + ... \right] .
\]

(8)

The equations of motion following from this effective action are in the lowest order given by (9):

\[
\begin{align*}
\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{G}_{\mu\nu} \tilde{R} &= T_{\mu\nu}^{\text{matter}}, \\
3\sqrt{2|D-2|} \alpha' D^2 \tilde{\phi} + 4(26 - D)e^{\frac{4}{2|D-2|}\tilde{\phi}} + \alpha' H_{\lambda\mu\nu}^2 e^{-\frac{8}{2|D-2|}\tilde{\phi}} &= 0, \\
D_{\lambda} e^{\frac{8}{2|D-2|}\tilde{\phi}} H_{\lambda\mu\nu}^\mu &= 0
\end{align*}
\]

(9)

where:

\[
T_{\mu\nu}^{\text{matter}} = \frac{1}{4} \left( H_{\mu\nu}^2 - \frac{1}{6} \tilde{G}_{\mu\nu} H^2 \right) e^{-\frac{8}{2|D-2|}\tilde{\phi}} + \left( \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - \frac{1}{2} (\partial \tilde{\phi})^2 \tilde{G}_{\mu\nu} \right).
\]

(10)

In order to get the tachyon \( \beta \) function from the effective action it is necessary to include non-perturbative terms in the \( \beta \) functions which we want to neglect in our consideration (see [10]). The reason is that a (covariant) tachyon coupling in the effective action would (via the equation of motion) influence the metric. But tachyon terms in the metric \( \beta \) function arise only via non-perturbative contributions.
Apparently one obtains in the limit $\alpha' \to 0$ just the general relativity. There is another reason which prefers this parametrization. Due to the mixing of the dilaton and graviton the vertex operators following from the “$\sigma$–model parametrization” are not well defined (see e.g. [12]), e.g. the dilaton vertex operator has not the right conformal behaviour. Therefore it is common to consider not $G_{\mu\nu}$ as the “physical” metric but $\tilde{G}_{\mu\nu}$. Later, in our consideration this reparametrization yields the time evolution of the metric.

To make us more familiar with this framework let us shortly rederive known result in flat space time without Torsion ($G_{\mu\nu} = \eta_{\mu\nu}, B_{\mu\nu} = 0$). In this case one has:

$$\overline{\beta}^{G}_{\mu\nu} = 2\alpha' \partial_{\mu} \partial_{\nu} \phi = 0$$ (11)

and hence

$$\phi = \phi_{0} - \frac{1}{\sqrt{\alpha'}} q_{\mu} X_{\mu} ,$$ (12)

which was already discussed as one example for noncritical strings in arbitrary dimensions [2]. The tachyon field is defined by:

$$\overline{\beta}^{T} = -\frac{1}{2} \alpha' \partial^{2} T - 2T - \sqrt{\alpha'} q_{\mu} \partial^{\mu} T = 0 ,$$ (13)

with the solution:

$$T \sim e^{-\frac{4}{\sqrt{\alpha'}} p_{\mu} X^{\mu} } \quad \text{and:} \quad -\frac{1}{2} p_{\mu} p^{\mu} - 2 + q_{\mu} p^{\mu} = 0 .$$ (14)

For the dilaton $\overline{\beta}$ function one gets:

$$\overline{\beta}^{\phi} = \frac{1}{6} (D - 26) + q_{\mu} q^{\mu} = 0 .$$ (15)

Combining (14) and (15) one obtains finally the known result:

$$q_{0} = \sqrt{\frac{26 - D}{6}} - q^{2} ,$$

$$p_{0} = q_{0} \pm \sqrt{q_{0}^{2} - p^{2} + 2q_{\bar{p}} - 4} = \frac{1}{\sqrt{\alpha'}} \left( \sqrt{26 - D - 6q^{2}} \pm \sqrt{2 - D - 6(q - \bar{p})^{2}} \right) .$$ (16)

In the limit $q = \bar{p} = 0$ and for $D = d + 1$ (spatial degrees of freedom plus Liouville field) we get just the known results from David, Distler and Kawai [7] (it is common to replace: $p_{\mu} \to ip_{\mu}$ and $q_{\mu} \to iq_{\mu}$).

Now we want to investigate a (euclidean) Robertson-Walker space time. This metric describing a (spatial) homogeneous and isotropic universe is given by:

$$(ds)^{2} = (dt)^{2} + \frac{K^{2}(t)}{1 + \frac{4}{\epsilon} \bar{r}^{2}} \left[ (dX^{1})^{2} + (dX^{2})^{2} + ... + (dX^{D-1})^{2} \right] ,$$ (17)

where $\bar{r}^{2} = (X^{1})^{2} + (X^{2})^{2} + ... + (X^{D-1})^{2} , K(t)$ is the so-called world radius and the parameter $\epsilon$ determinates the spatial geometry: flat ($\epsilon = 0$), spherical ($\epsilon = +1$) or hyperbolical ($\epsilon = -1$). Furthermore we restrict ourselves to the compact case $\epsilon = +1$. Since all
RW metrics are conformally flat we can find coordinates in which the metric looks like\[ (ds)^2 = \frac{\tilde{K}^2(r)}{r^2} \left( (dx^0)^2 + (dx^1)^2 + \ldots + (dx^{D-1})^2 \right) . \] where the new radius \( r^2 = (x^0)^2 + |\vec{r}|^2 \) and the new time \( x^0 \) depends on \( \tilde{r} \) and \( t^0 \) via:
\[
|\vec{r}| = e^\eta r_1 \quad \text{and} \quad r_1 = \frac{\tilde{r}}{1 + \frac{\tilde{r}^2}{4}} \quad x^0 = e^\eta \sqrt{1 - r_1^2} \quad \eta = \int \frac{dt}{K}. \]
Hence the new radius \( r \) corresponds just to the former time: \( \log r = \eta \) and one obtains the original world radius \( K \) from \( \tilde{K} \) after the replacement of \( r \) by \( t \) where \( r(t) \) is given by:
\[
\frac{\dot{r}}{r} = \frac{1}{K(r)}. \]
Thus the whole time dependence of the original metric is now controlled (via the radius dependence) by the Weyl factor \( \tilde{K} \). Let us give some simple examples:

1. \( \tilde{K} = \sqrt{Q} = \text{const.} \) \( \rightarrow \) static Einstein universe
2. \( \tilde{K} = Qr \) (local flat space) \( \rightarrow \) \( K = t \) : Milne universe
3. \( \tilde{K} = Qr^n \) \( \rightarrow \) \( K = nt \) : general linear exp. universe
4. \( \tilde{K} \sim \frac{r}{1 + \frac{r^2}{4}} \) \( \rightarrow \) \( K \sim \sin t \) : de Sitter universe

In the following we want to discuss the first case for which an exact solution in four dimensions exists \[3, 4\]. The main point to get the exact result is to rewrite the model (1) in a WZW model and 1-dim. Feigin–Fuchs model. This can be done under the assumption that \( D = 4 \), the tachyon and dilaton field depend on \( r \) only, i.e. in the old coordinates (17) they are only time dependent. In a second step we turn to the investigation of a more general tachyon field in this background.

If:
\[
G_{\mu\nu} = \frac{Q}{r^2} \delta_{\mu\nu} \quad \mu, \nu = 0, 1, 2, 3, \\
H_{\mu\nu\lambda} = \pm \epsilon_{\mu\nu\lambda} \sigma \partial_\sigma \log \frac{\sqrt{Q}}{r} \\
\phi = \phi(r) \quad \text{and} \quad T = T(r)
\]
we can write (1) as:
\[
S = S_{\text{WZW}}(SU(2)) + \frac{1}{4\pi\alpha'} \int \left( (\partial u)^2 Q + \alpha'R^{(2)}(u) + \alpha'T(u) \right)
\]
and
\[
S_{\text{WZW}}(SU(2)) = \frac{Q}{4\pi\alpha'} \int d^2z \tr(\partial g^{-1}\partial g) + \frac{Q}{4\pi\alpha'} \int d^3z tr\epsilon^{abc}((g^{-1}\partial_a g)(g^{-1}\partial_b g)(g^{-1}\partial_c g))
\]
\[\text{c}\]Of course, one can not find a global transformation which transforms the compact space to a non-compact space but for every finite part it is possible.
where \( g = \frac{1}{r}(x^0 1 + x^i \sigma^i) \), \( u = -\log r \) (\( \sigma^i \): Pauli matrices). If one introduces polar coordinates in \( g \) one can see that the special background fields (22) just decouples the dependence on the radius \( r \) from the dependence on the angles. The dependence on the angles is now controlled by the WZW action and the dependence on the radius by the Feigin–Fuchs part. Background fields like (22) are already discussed in the past, e.g. in the context of semi–wormhole or solitons in string theory [13] and the torsion as a special axion field. The WZW theory is well defined and conformal invariant if \([14]\): \( \frac{Q}{\alpha'} = k = 1, 2, 3, \ldots \) is the WZW level and the corresponding central charge for the SU(2) Kac–Moody algebra is given by: \( c_{wzw} = \frac{3k}{k+2} \). In order to get a conformal theory the dilaton and the tachyon field have to satisfy the (corresponding \( \beta \)–function) equations:

\[
\phi''(u) = 0 \\
1 + c_{wzw} - 26 + 6\alpha'(\phi')^2 = 0 \\
-\frac{1}{2}\alpha'T'' - 2T + \alpha'\phi'T' = 0
\]

(26 is the contribution from the ghosts to the central charge). If the tachyon and dilaton satisfy these equations we have an exact conformal theory, i.e. in all (finite) order in \( \alpha' \).

From the first equation (corresponds to the metric \( \beta \)–function) follows that \( \phi \) is at most linear in \( u \), the second equation ensures the vanishing of the total central charge and the last equation defines the tachyon field. As solution one finds:

\[
\phi(u) = \phi_0 + qu \\
k > 1: \ T_{\pm}(u) \sim e^{\alpha_{\pm} u} \\
k = 1: \ T_1(u) \sim e^{2u} ; \ T_2(u) \sim uc^{2u}.
\]

The importance of the second tachyon operator \( T_2 \) was already discussed in \([15]\). Especially, \( T_2 \) plays a crucial role for the explanation of the critical behaviour in comparison to the results coming from random matrix model. Since for \( k > 1 \) \( \alpha_{\pm} \) becomes complex \( T_\pm \) correspond to oscillating (real) fields: \( T(u) \sim e^{\epsilon u} (\frac{3k}{k+2} - 1)^{\frac{1}{2}} \sin(\sqrt{\frac{k}{6}} (\frac{3k}{k+2} - 1)u) \) (and a corresponding solution with \( \cos(\sqrt{\frac{k}{6}} (\frac{3k}{k+2} - 1)u) \)). If one wants to have a real exponential tachyon (and not an oscillating one) motivated by obtaining the known results in non-critical string theory in the flat limit \([7]\) one gets the restriction: \( k = 1 \). In addition, \( k = 1 \) is preferred because this corresponds to a massless tachyon field (see below). Transforming the background fields (25) and (22) back in the coordinate system (17) and denoting the integration constants as a suitable \( t_0 \) we get:

\[
(ds)^2 = dt^2 + \frac{\alpha'}{(1+\epsilon^2 r^2)^2}[(dX^1)^2 + (dX^2)^2 + (dX^3)^2] , \quad H_{\mu\nu\lambda} = \pm \epsilon_{\mu\nu\lambda 0} , \\
\phi(t) = -\frac{2}{\sqrt{\alpha'}} (t - t_0) , \quad T_1 = e^{-\frac{2}{\sqrt{\alpha'}}(t - t_0)} , \quad T_2 \sim (t - t_0)e^{-\frac{2}{\sqrt{\alpha'}}(t - t_0)}.
\]
conformal exact models). After the redefinition (7) and a common time reparametrization we get for the “physical” metric:

\[
\left(\tilde{ds}\right)^2 = (dt)^2 + \frac{2(t - t_0)^2}{\left(1 + \frac{1}{4}\bar{r}^2\right)^2}[(dX^1)^2 + (dX^2)^2 + (dX^3)^2] .
\]

(27)

Thus we have as result a linear expanding universe corresponding to a linear dilaton field and an exponential tachyon with the classical value \(\alpha = 2\). If one does not include the tachyon field in the action (1) or (23) all values of \(k\) are allowed and one has to replace the factor 2 in front of \(t - t_0\) by \(q\) in the dilaton and metric. Then we would have in correspondence to [5] the assertion that the world radius and the dilaton field is quantized by \(k = 1, 2, 3, \ldots\). The incorporation of a (real exponential) tachyon depending only from the time restricted us to \(k = 1\). Unfortunately \(k\) is the perturbation parameter in this theory and if \(k = 1\) the perturbation theory breaks down. Up to this point this does not matter because it is an exact solution of (2). The question is what happens if one has e.g. a more general tachyon field for which the \(\beta\) function is only perturbatively solvable and known. To investigate this question we want now to discuss a tachyon field depending on a angle coordinate too. In this case up to \(O(\alpha'^2)\) we have to solve the equation:

\[
\tilde{\beta}^T = -\frac{1}{2}\alpha'D^2T + \frac{1}{8}\alpha'^2(H^2)^{\mu\nu}D_\mu\partial_\nu T - 2T + \alpha'\partial_\mu\phi\partial^\mu T = 0
\]

(28)

where \(\phi\) and \(H\) are given by (22), (25). The dependence on the angle we separate by a ansatz which gives in the flat limit plane waves (see below):

\[
T = (r^2)^{-a}V\left(\frac{p^\mu X^\mu}{r}\right) ,
\]

(29)

where the coefficient \(a\) corresponds to \(\alpha\) in (25). After using of: \(x^\mu\partial_\mu V = 0\) and \(\frac{p^\mu}{r} = z\) one finds immediately:

\[
(|p|^2 - z^2)V''(z) - 3zV'(z) + \frac{4}{1 - \frac{1}{2k}}(a^2 - qa + k)V = 0 .
\]

(30)

In order to interpret this equation it is useful to look on the flat limit. This can be done by the transformations (see (17), \(K^2 = Q\), for dimensional reasons one has to replace the denominator by \((\alpha' + \frac{1}{4}\bar{r}^2)^2\):

\[
\tilde{X} \rightarrow \lambda \tilde{X} , \quad Q \rightarrow \frac{\alpha'}{\lambda^2} \quad \text{and} \quad \lambda \rightarrow 0
\]

(31)

which corresponds to \(k = \frac{Q}{\alpha'} \rightarrow \frac{1}{\lambda^2}\), i.e. the flat limit corresponds to \(k \rightarrow \infty\). If one looks firstly on the case \(V = const\) corresponding to the solution (25) one finds immediately that \(a = O\left(\frac{1}{k}\right)\), \(q = O\left(\frac{1}{k}\right)\) and in terms of \(z = q^6 + \lambda \frac{p^\mu}{\sqrt{\alpha'}} + O(\lambda^2)\) we get for (30) in the leading order \((O(\frac{1}{\lambda^2}))\):

\[
\alpha'\left|p\right|^2V''(\tilde{p}\tilde{X}) + 4 \left(a^2 - \sqrt{\frac{26 - 4}{6\alpha'}} a + 1\right)V = 0
\]

(32)
with the solution: \( V \sim e^{-\frac{a^2}{\sqrt{\alpha'}}}, \quad a = \frac{1}{\sqrt{2\alpha}} \left( \sqrt{26 - 4} \mp \sqrt{2 - 4 |p|^2} \right) \). The same procedure for \( T \) and \( \phi \) yields: \( T = e^{-\frac{2a|t-t_0|}{\sqrt{\alpha'}}}, \quad \phi = -\frac{1}{\sqrt{\alpha'}}qt \). Therefore we get just the known result (16) if:

\[
t \rightarrow X^0 \quad , \quad q \rightarrow q_0 \quad , \quad 2a = p_0 \quad , \quad \vec{q} = 0 \quad \text{and} \quad D = 4 ,
\]

i.e. \( p_\mu \) corresponds just to the momentum in the flat limit. But it is also possible to get an explicit solution for (30). After the transformations:

\[
z = |p| \cos \theta \quad (\theta \text{ is the angle between } p_\mu \text{ and } x^\mu) \quad \text{and} \quad V(\theta) = \frac{W(\theta)}{\sin \theta}
\]

one gets for (30): \( W''(\theta) + \rho^2 W(\theta) = 0 \) and thus the non-singular solution \( V(\theta) = 1 \) for the tachyon (29) is given by:

\[
T(r, \theta) = r^{p_0} \sin[\rho \theta] \rho \sin \theta , \quad \rho^2 = \frac{1}{1 - \frac{1}{2k}} (p_0^2 - 2qp_0 + 4k - \frac{1}{2k} + 1)
\]

(\( q \) given by (25), \( p_0 \) is the time component of the tachyon momentum). If we now restrict ourselves to a solution which is periodic: \( V(\theta) = V(\theta + 2\pi) \) we get a quantization for the momentum \( p_0 \) \[d\]:

\[
\rho = n + 1 \quad \Leftrightarrow \quad p_0 = \sqrt{\frac{k}{6} \left( 25 - \frac{3k}{k+2} \right) \pm \sqrt{(n^2 + 2n) \left( 1 - \frac{1}{2k} \right) - \frac{k}{6} \left( \frac{3k}{k+2} - 1 \right)}}
\]

where \( n \) is an integer. In order to have a real momentum \( p_0 \) it is necessary that:

\[
|n + 1| \geq \frac{1}{\sqrt{1 - \frac{1}{2k}}} \sqrt{1 - \frac{1}{2k} + \frac{k}{6} \left( \frac{3k}{k+2} - 1 \right)} .
\]

For \( n = 0 \) we have the restriction \( k = 1 \) and we get the same result as in (25) or (26). The final result for a spatially dependent tachyon in the coordinate system (17) is therefore up to the second order in \( \alpha' \) given by:

\[
(ds)^2 = dt^2 + \frac{k \alpha'}{(1 + \frac{1}{4}r^2)^2} [(dX^1)^2 + (dX^2)^2 + (dX^3)^2] , \quad H_{\mu\nu\lambda} = \pm \epsilon_{\mu\nu\lambda} ,
\]

\[
\phi(t) = \frac{k}{6\alpha'} (25 - \frac{3k}{k+2}) (t - t_0) , \quad T(t, \theta) = e^{\frac{p_0(t-t_0)}{\sqrt{\alpha'}}} \frac{\sin[(n+1)\theta]}{(n+1)\sin \theta} ,
\]

with:

\[
|p| \cos \theta = p_0 \frac{\alpha'}{\alpha' + \frac{1}{4}r^2} + \frac{\vec{p} \cdot \vec{X}}{\alpha' + \frac{1}{4}r^2} , \quad k = 1, 2, 3, ... ,
\]

where: \( p_0 \) is given by (36) and the integer \( n \) has to fulfill (37) (the contribution of the second order is given by term \( \frac{1}{2k} \)). We have again a linear dilaton and an exponential time dependence of the tachyon. The spatially (or \( \theta \)) dependent part of the tachyon is given by the Chebyshev polynomials of the second kind which were already discussed as a representation of SU(2) current algebra \[13\]. The main difference to (26) is that we

\[d\]A similar quantization is obtained by a suitable normalization.
have now in all fields a quantization depending on two integers. The world radius and
the dilaton is quantized by \( k \) and the tachyon is also quantized by \( n \). A cosmological
interpretation of a quantized world radius is given in the first ref. of \[3\]. In addition one
should note here that the solution (38) has for even \( n \) the duality symmetry: \( \bar{r} \rightarrow -\frac{4\alpha'}{\bar{r}} \)
which is equivalent to \( \theta \rightarrow \pi - \theta \).

Finally we want to investigate the mass spectrum of the tachyon in (26) and (38). As
we have seen in (5) the tachyon and the dilaton corresponds in the lowest order in \( \alpha' \) to
scalar fields fulfilling a Klein–Gordan equation. After the field redefinition \( T = e^\phi \tilde{T} \) (\( \phi \)
and \( T \) are given by (38)) we get for the tachyon \( \tilde{\beta} \) function:

\[
-\frac{1}{2} \alpha' D^2 \tilde{T} - \frac{1}{12} \left( \frac{3k}{k+2} - 1 \right) \tilde{T} = 0
\]

with the solution:

\[
\tilde{T}(t, \theta) = e^{\sqrt{\alpha'} \tilde{p}_0 (t-t_0) \sin((n+1)\theta)} \frac{\sin(n+1)\theta}{(n+1)\sin\theta}
\]

where \( \tilde{p}_0 = \pm \sqrt{(n^2 + 2n) \left( 1 - \frac{1}{2k} \right) - \frac{k}{6} \left( \frac{3k}{k+2} - 1 \right)} \). Hence, the mass is quantized by \( k \)
and the tachyon is massless if \( k = 1 \). Apart from the argumentation that \( k \) has to be one in
order to obtain in the flat limit the known results from the DDK model the requirement to
get a massless tachyon state is an alternative one (note: the results (25) or (26) for an only
time dependent tachyon correspond to \( n = 0 \) in (38)). But one has to take into account
that this conclusion is only valid in the low energy limit (neglecting of higher order in \( \alpha' \))
because the tachyon \( \beta \) function gets additional terms in the higher orders of \( \alpha' \).

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