INDEPENDENCE OF $\ell$ IN LAFFORGUE'S THEOREM

CheeWhye Chin

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Abstract. Let $X$ be a smooth curve over a finite field of characteristic $p$, let $\ell \neq p$ be a prime number, and let $L$ be an irreducible lisse $\mathbb{Q}_\ell$-sheaf on $X$ whose determinant is of finite order. By a theorem of L. Lafforgue, for each prime number $\ell' \neq p$, there exists an irreducible lisse $\mathbb{Q}_{\ell'}$-sheaf $L'$ on $X$ which is compatible with $L$, in the sense that at every closed point $x$ of $X$, the characteristic polynomials of Frobenius at $x$ for $L$ and $L'$ are equal. We prove an “independence of $\ell$” assertion on the fields of definition of these irreducible $\ell'$-adic sheaves $L'$: namely, that there exists a number field $F$ such that for any prime number $\ell' \neq p$, the $\mathbb{Q}_{\ell'}$-sheaf $L'$ above is defined over the completion of $F$ at one of its $\ell'$-adic places.

Introduction

In the recent spectacular work [L], L. Lafforgue has proved the Langlands Correspondence and the Ramanujan-Petersson Conjecture for $\text{GL}_r$ over function fields. As a consequence, he has also established the following fundamental result concerning irreducible lisse $\ell$-adic sheaves on curves over finite fields.

Theorem (L. Lafforgue, [L] Théorème VII.6). Let $X$ be a smooth curve over a finite field of characteristic $p$. Let $\ell \neq p$ be a prime number, and let $L$ be a lisse $\mathbb{Q}_\ell$-sheaf on $X$, which is irreducible, of rank $r$, and whose determinant is of finite order.

1. There exists a number field $E \subset \overline{\mathbb{Q}_\ell}$ such that for every closed point $x$ of $X$, the polynomial
   
   $\det(1 - T \text{Frob}_x, L)$
   
   has coefficients in $E$.

2. Let $x$ be a closed point of $X$, and let $\alpha \in \overline{\mathbb{Q}_\ell}$ be an eigenvalue of Frobenius at $x$ acting on $L$, i.e. $1/\alpha$ is a root of the polynomial
   
   $\det(1 - T \text{Frob}_x, L)$.

Then:

(a) $\alpha$ is an algebraic number;
(b) for every archimedean absolute value $|\cdot|$ of $E(\alpha)$, one has $|\alpha| = 1$;

(c) for every non-archimedean valuation $\lambda$ of $E(\alpha)$ not lying over $p$, $\alpha$ is a $\lambda$-adic unit, i.e. one has $\lambda(\alpha) = 0$;

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(d) for every non-archimedean valuation $\nu$ of $E(\alpha)$ lying over $p$, one has
\[ \frac{\nu(\alpha)}{\nu(\#(x))] \leq (r-1)^2. \]

(3) For any place $\lambda'$ of $E$ lying over a prime number $\ell' \neq p$, and for any algebraic closure $\overline{E}_{\ell'}$ of the completion $E_{\lambda'}$ of $E$ at $\lambda'$, there exists a lisse $\overline{E}_{\ell'}$-sheaf $\mathcal{L}'$ on $X$, which is irreducible, of rank $r$ such that for every closed point $x$ of $X$, one has
\[ \det(1 - T \text{Frob}_x, \mathcal{L}') = \det(1 - T \text{Frob}_x, \mathcal{L}) \quad \text{(equality in } E[T]). \]

Moreover, the sheaf $\mathcal{L}'$ is defined over a finite extension of $E_{\lambda'}$.

In part (3) of Lafforgue’s theorem, it is not a priori clear that the number field $E$ may be replaced by a finite extension (in $\overline{E}_{\ell}$) so that the various $\overline{E}_{\ell}$-sheaves $\mathcal{L}'$ form an $(E, \Lambda)$-compatible system in the sense of Katz (cf. [K], pp. 202–203, “The notion of $(E, \Lambda)$-compatibility”), or equivalently, that they form an $E$-rational system of $\lambda$-adic representations in the sense of Serre (cf. [Se], §2.3 and §2.5). The existence of a number field with this property may be interpreted as an “independence of $\ell$” assertion on the fields of definition of these irreducible $\ell'$-adic sheaves $\mathcal{L}'$. We shall prove that this is indeed the case.

**Theorem.** With the notation and hypotheses of Lafforgue’s Theorem, the following assertion holds.

(3') There exists a finite extension $F$ of $E$ in $\overline{E}_{\ell}$ such that for any place $\lambda'$ of the number field $F$ lying over a prime number $\ell' \neq p$, there exists a lisse $F_{\lambda'}$-sheaf $\mathcal{L}'$ on $X$ (i.e. a lisse $\overline{E}_{\ell'}$-sheaf defined over $F_{\lambda'}$), which is absolutely irreducible, of rank $r$, such that for every closed point $x$ of $X$, one has
\[ \det(1 - T \text{Frob}_x, \mathcal{L}') = \det(1 - T \text{Frob}_x, \mathcal{L}) \quad \text{(equality in } E[T]). \]

According to a conjecture of Deligne (cf. [D] Conjecture (1.2.10)), all four assertions (1),(2),(3),(3') should also hold in the general case when $X$ is a normal variety of arbitrary dimension over a finite field. Our proof of assertion (3') uses assertions (1) and (3) of Lafforgue’s Theorem only as “black boxes”; so assertion (3') will hold for higher dimensional varieties if parts (1) and (3) of Lafforgue’s theorem hold for these varieties. To state this more precisely, we make assertions (1) and (3) into hypotheses, as follow.

**Definition.** Let $\mathbb{F}_q$ be a finite field of characteristic $p$, and let $\ell \neq p$ be a prime number. Let $Y$ be a normal variety over $\mathbb{F}_q$, and let $\mathcal{F}$ be a lisse $\mathbb{F}_q$-sheaf on $Y$, which is irreducible, and whose determinant is of finite order. We shall say that hypothesis (1) holds for $(Y, \mathcal{F})$ if:

(1) there exists a number field $E \subset \overline{E}_{\ell}$ such that for every closed point $y$ of $Y$, the polynomial
\[ \det(1 - T \text{Frob}_y, \mathcal{F}) \]
has coefficients in $E$.

When hypothesis (1) holds for $(Y, \mathcal{F})$, we shall say that hypothesis (3) holds for $(Y, \mathcal{F})$ if:

(3) for any place $\lambda'$ of $E$ lying over a prime number $\ell' \neq p$, and for any algebraic closure $\overline{E}_{\ell'}$ of the completion $E_{\lambda'}$ of $E$ at $\lambda'$, there exists a lisse $\overline{E}_{\ell'}$-sheaf $\mathcal{F}'$ on $Y$, which is irreducible, such that for every closed point $y$ of $Y$, one has
\[ \det(1 - T \text{Frob}_y, \mathcal{F}') = \det(1 - T \text{Frob}_y, \mathcal{F}) \quad \text{(equality in } E[T]). \]

With this definition, our goal is to prove:
Main Theorem. Let \( \mathbb{F}_q \) be a finite field of characteristic \( p \), and let \( \ell \neq p \) be a prime number. Let \( X \) be a normal variety over \( \mathbb{F}_q \). Assume that:

for any normal variety \( Y \) over \( \mathbb{F}_q \) which is finite etale over \( X \), and for any lisse \( \overline{\mathbb{Q}}_\ell \)-sheaf \( \mathcal{F} \) on \( Y \), which is irreducible, and whose determinant is of finite order, hypotheses (1) and (3) hold for the pair \((Y, \mathcal{F})\).

Let \( \mathcal{L} \) be a lisse \( \overline{\mathbb{Q}}_\ell \)-sheaf on \( X \), which is irreducible, of rank \( r \), and whose determinant is of finite order. Let \( E \) denote the number field given by hypothesis (1) applied to \((X, \mathcal{L})\). Then:

\[(3') \quad \text{There exists a finite extension } F \text{ of } E \text{ in } \overline{\mathbb{Q}}_\ell \text{ such that for any place } \lambda' \text{ of the number field } F \text{ lying over a prime number } \ell' \neq p, \text{ there exists a lisse } F_{\lambda'} \text{-sheaf } \mathcal{L}_{\lambda'} \text{ on } X, \text{ which is absolutely irreducible, of rank } r, \text{ such that for every closed point } x \text{ of } X, \text{ one has}
\]

\[
\det(1 - T \text{ Frob}_x, \mathcal{L}_{\lambda'}) = \det(1 - T \text{ Frob}_x, \mathcal{L}) \quad \text{(equality in } E[T] \text{).}
\]

We shall prove this theorem by exploiting properties of the monodromy groups associated to these irreducible lisse sheaves. The proof begins in §4, after a discussion of the preliminary results we need: propositions 1 and 2 of §1, corollary 6 of §2, and propositions 7 and 9 of §3.

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§1. Monodromy Groups

In this section, we recall some basic properties of monodromy groups of lisse \( \ell \)-adic sheaves on varieties over a finite field; see [D] §1.1 and §1.3 for details.

Let \( X \) be a normal, geometrically connected variety over a finite field \( \mathbb{F}_q \) of characteristic \( p \). Let \( \bar{\eta} \to X \) be a geometric point of \( X \), and let \( \overline{\mathbb{F}}_q \) be the algebraic closure \( \mathbb{F}_q \) in \( \kappa(\bar{\eta}) \); we regard \( \bar{\eta} \) also as a geometric point of \( X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q \). The profinite groups \( \pi_1(X, \bar{\eta}) \) and \( \pi_1(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \bar{\eta}) \) are respectively called the arithmetic fundamental group of \( X \) and the geometric fundamental group of \( X \). They sit in a short exact sequence

\[
1 \to \pi_1(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \bar{\eta}) \to \pi_1(X, \bar{\eta}) \overset{\text{deg}}{\longrightarrow} \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to 1.
\]

The group \( \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \) has a canonical topological generator \( \text{Frob}_{\overline{\mathbb{F}}_q} \) called the geometric Frobenius, which is defined as the inverse of the arithmetic Frobenius automorphism \( a \mapsto a^q \) of the field \( \overline{\mathbb{F}}_q \). We have the canonical isomorphism

\[
\widehat{\mathbb{Z}} \overset{\cong}{\longrightarrow} \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), \quad \text{sending } 1 \text{ to } \text{Frob}_{\overline{\mathbb{F}}_q}.
\]

For a prime number \( \ell \neq p \), the functor

\[
\{ \text{lisse } \overline{\mathbb{Q}}_\ell \text{-sheaves on } X \} \longrightarrow \{ \text{finite dimensional continuous } \overline{\mathbb{Q}}_\ell \text{-representations of } \pi_1(X, \bar{\eta}) \}
\]

\[
\mathcal{L} \longmapsto \mathcal{L}_{\bar{\eta}}
\]

is an equivalence of categories; a similar statement holds with \( X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q \) in place of \( X \). Via this equivalence, standard notions associated to representations (e.g. irreducibility, semisimplicity, constituent, etc.) are also applicable to lisse sheaves.
Let $\mathcal{L}$ be a lisse $\mathbb{Q}_l$-sheaf on $X$, corresponding to the continuous monodromy representation

$$\pi_1(X, \bar{\eta}) \to \text{GL}(\mathcal{L}_{\bar{\eta}})$$

of the arithmetic fundamental group of $X$. The arithmetic monodromy group $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$ of $\mathcal{L}$ is the Zariski closure of the image of $\pi_1(X, \bar{\eta})$ in $\text{GL}(\mathcal{L}_{\bar{\eta}})$. The inverse image $\mathcal{L} \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ of $\mathcal{L}$ on $X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ is a lisse $\mathbb{Q}_l$-sheaf on $X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$, corresponding to the continuous monodromy representation

$$\pi_1(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \bar{\eta}) \hookrightarrow \pi_1(X, \bar{\eta}) \to \text{GL}(\mathcal{L}_{\bar{\eta}})$$

of the geometric fundamental group of $X$, obtained by restriction. The geometric monodromy group $G_{\text{geom}}(\mathcal{L}, \bar{\eta})$ of $\mathcal{L}$ is the Zariski closure of the image of $\pi_1(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \bar{\eta})$ in $\text{GL}(\mathcal{L}_{\bar{\eta}})$.

Both $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$ and $G_{\text{geom}}(\mathcal{L}, \bar{\eta})$ are linear algebraic groups, and it is clear that $G_{\text{geom}}(\mathcal{L}, \bar{\eta})$ is a closed normal subgroup of $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$. Both $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$ and $G_{\text{geom}}(\mathcal{L}, \bar{\eta})$ are given with a faithful representation on $\mathcal{L}_{\bar{\eta}}$ corresponding to their realizations as subgroups of $\text{GL}(\mathcal{L}_{\bar{\eta}})$. Thus, if $\mathcal{L}$ is semisimple (as a representation of $\pi_1(X, \bar{\eta})$, and therefore as a representation of $\pi_1(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \bar{\eta})$), then both $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$ and $G_{\text{geom}}(\mathcal{L}, \bar{\eta})$ are (possibly non-connected) reductive algebraic groups.

**Proposition 1.** Let $\mathcal{L}$ be a lisse $\mathbb{Q}_l$-sheaf on $X$.

(i) If $\mathcal{L}$ is semisimple, then $G_{\text{geom}}(\mathcal{L}, \bar{\eta})$ is a (possibly non-connected) semisimple algebraic group.

(ii) If $\mathcal{L}$ is irreducible, and its determinant is of finite order, then $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$ is a (possibly non-connected) semisimple algebraic group, containing $G_{\text{geom}}(\mathcal{L}, \bar{\eta})$ as a normal subgroup of finite index.

Assertion (i) is [D] Corollaire (1.3.9). For the proof of assertion (ii), we shall make use of the construction in [D] (1.3.7), which we summarize below.

Recall that the Weil group $W(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ of $\mathbb{F}_q$ is the subgroup of $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ consisting of integer-powers of $\text{Frob}_{\mathbb{F}_q}$; it is considered as a topological group given with the discrete topology, and we have the canonical isomorphism

$$\mathbb{Z} \cong W(\bar{\mathbb{F}}_q/\mathbb{F}_q), \quad \text{sending} \ 1 \ \text{to} \ \text{Frob}_{\mathbb{F}_q}.$$

The Weil group $W(X, \bar{\eta})$ of $X$ is the preimage of $W(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ in $\pi_1(X, \bar{\eta})$ by the degree homomorphism $\pi_1(X, \bar{\eta}) \xrightarrow{\text{deg}} \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$; it is considered as a topological group given with the product topology via the isomorphism

$$W(X, \bar{\eta}) \cong \pi_1(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \bar{\eta}) \rtimes_{\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)} W(\bar{\mathbb{F}}_q/\mathbb{F}_q),$$

where $\pi_1(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \bar{\eta})$ retains its profinite topology, and is an open and closed subgroup of $W(X, \bar{\eta})$. These groups sit in the following diagram:

$$
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \bar{\eta}) & \longrightarrow & W(X, \bar{\eta}) & \xrightarrow{\text{deg}} & \mathbb{Z} \cong W(\bar{\mathbb{F}}_q/\mathbb{F}_q) & \longrightarrow & 1 \\
\| & \quad & \quad & \quad & \quad & \quad & \|
\end{array}
$$

$$
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \bar{\eta}) & \longrightarrow & \pi_1(X, \bar{\eta}) & \xrightarrow{\text{deg}} & \hat{\mathbb{Z}} \cong \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) & \longrightarrow & 1 \\
\end{array}
$$

where the right two vertical arrows are inclusion homomorphisms with dense images. (Note that the topologies of $W(X, \bar{\eta})$ and $W(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ are not the ones induced by the right two vertical arrows!)
Given a lisse $\mathbb{Q}_p$-sheaf $\mathcal{L}$ on $X$, the push-out construction of \cite{D} (1.3.7) produces an algebraic group $G(\mathcal{L}, \bar{\eta})$, which is locally of finite type, but not quasi-compact; it is characterized by the fact that it sits in a diagram:

$$
\begin{array}{ccc}
1 & \longrightarrow & \pi_1(X \otimes_{F_q} \overline{F_q}, \bar{\eta}) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & G_{\text{geom}}(\mathcal{L}, \bar{\eta})
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\deg & \cong & \cong \\
\downarrow & & \downarrow \\
G(\mathcal{L}, \bar{\eta}) & \longrightarrow & \text{GL}(\mathcal{L}, \bar{\eta})
\end{array}
\begin{array}{ccc}
Z & \cong & Z \\
\cong & & \cong \\
\downarrow & & \downarrow \\
1 & \longrightarrow & 1
\end{array}
$$

such that the composite of the two continuous homomorphisms

$$W(X, \bar{\eta}) \rightarrow G(\mathcal{L}, \bar{\eta}) \rightarrow \text{GL}(\mathcal{L}, \bar{\eta})$$

is equal to the continuous representation of $W(X, \bar{\eta})$ on $\mathcal{L}_{\bar{\eta}}$ obtained via restriction:

$$W(X, \bar{\eta}) \rightarrow \pi_1(X, \bar{\eta}) \rightarrow \text{GL}(\mathcal{L}, \bar{\eta}).$$

**Proof of Proposition 1 (ii).** From assertion (i), we already know that the group $G_{\text{geom}}(\mathcal{L}, \bar{\eta})$ is a semisimple closed normal subgroup of $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$. Hence, to prove assertion (ii), it suffices for us to show that $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$ contains $G_{\text{geom}}(\mathcal{L}, \bar{\eta})$ as a subgroup of finite index, for then both groups will have the same identity component, which is a connected semisimple algebraic group.

Since $W(X, \bar{\eta}) \rightarrow \pi_1(X, \bar{\eta})$ is an inclusion with dense image, $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$ can also be described as the Zariski closure of the image of $W(X, \bar{\eta})$ in $\text{GL}(\mathcal{L}, \bar{\eta})$; likewise, since $W(X, \bar{\eta}) \rightarrow G(\mathcal{L}, \bar{\eta})$ is an inclusion with dense image, $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$ is also equal to the Zariski closure of the image of $G(\mathcal{L}, \bar{\eta})$ in $\text{GL}(\mathcal{L}, \bar{\eta})$. Let

$$\rho : G(\mathcal{L}, \bar{\eta}) \rightarrow \text{GL}(\mathcal{L}, \bar{\eta})$$

denote the canonical homomorphism from $G(\mathcal{L}, \bar{\eta})$ into $\text{GL}(\mathcal{L}, \bar{\eta})$; then the composite map

$$G_{\text{geom}}(\mathcal{L}, \bar{\eta}) \rightarrow G(\mathcal{L}, \bar{\eta}) \xrightarrow{\rho} \text{GL}(\mathcal{L}, \bar{\eta})$$

is just the identity map on $G_{\text{geom}}(\mathcal{L}, \bar{\eta})$. We are thus reduced to showing that $\rho^{-1}(G_{\text{geom}}(\mathcal{L}, \bar{\eta}))$ is a subgroup of $G(\mathcal{L}, \bar{\eta})$ of finite index.

The fundamental fact we need about $G(\mathcal{L}, \bar{\eta})$ is \cite{D} Corollaire (1.3.11), which asserts that because $\mathcal{L}$ is irreducible (hence semisimple) by hypothesis, there exists some element $g$ in the center of $G(\mathcal{L}, \bar{\eta})$ whose degree is $> 0$ (i.e. $g$ maps to a positive integer under $G(\mathcal{L}, \bar{\eta}) \xrightarrow{\deg} \mathbb{Z} \cong W(\overline{F_q}/F_q)$). Therefore, $\rho(g)$ is an element of $\text{GL}(\mathcal{L}, \bar{\eta})$ which centralizes $\rho(G(\mathcal{L}, \bar{\eta}))$, and so it centralizes $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$. Since $\mathcal{L}$ is irreducible as a representation of $\pi_1(X, \bar{\eta})$ and hence as a representation of $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$, it follows that $\rho(g)$ must be a scalar.

By hypothesis, the determinant of $\mathcal{L}$ is of finite order, which means that the 1-dimensional representation of $\pi_1(X, \bar{\eta})$ on the determinant $\det(\mathcal{L}, \bar{\eta})$ of $\mathcal{L}_{\bar{\eta}}$ is given by a character of finite order, say $d$. The same is therefore true for $\det(\mathcal{L}, \bar{\eta})$ as a representation of $W(X, \bar{\eta})$ and of $G(\mathcal{L}, \bar{\eta})$. From this it follows that, if $\mathcal{L}$
has rank $r$, then $\rho(g)$ is a scalar which is a root of unity of order dividing $dr$, and so $g^{dr} \in G(L, \bar{\eta})$ lies in the kernel of $\rho$. Hence $\rho^{-1}(G_{\text{geom}}(L, \bar{\eta}))$ contains $\deg^{-1}(\deg(g^{dr}))$ in $G(L, \bar{\eta})$, which is of finite index in $G(L, \bar{\eta})$. □

Let $\mathcal{L}$ be a lisse $\mathbb{Q}_\ell$-sheaf on $X$. Its arithmetic monodromy group $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$ contains the identity component $G_{\text{arith}}(\mathcal{L}, \bar{\eta})^0$ as an open normal subgroup; $G_{\text{arith}}(\mathcal{L}, \bar{\eta})^0$ is a connected algebraic group. The faithful representation

$$G_{\text{arith}}(\mathcal{L}, \bar{\eta}) \hookrightarrow \text{GL}(\mathcal{L}_\bar{\eta})$$

of $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$, when restricted to the subgroup $G_{\text{arith}}(\mathcal{L}, \bar{\eta})^0$ of $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$, gives a faithful representation

$$G_{\text{arith}}(\mathcal{L}, \bar{\eta})^0 \hookrightarrow G_{\text{arith}}(\mathcal{L}, \bar{\eta}) \hookrightarrow \text{GL}(\mathcal{L}_\bar{\eta})$$

of $G_{\text{arith}}(\mathcal{L}, \bar{\eta})^0$ on $\mathcal{L}_\bar{\eta}$. We say that the lisse sheaf $\mathcal{L}$ is Lie-irreducible if $\mathcal{L}_\bar{\eta}$ is irreducible as a representation of $G_{\text{arith}}(\mathcal{L}, \bar{\eta})^0$. It is clear that Lie-irreducibility implies irreducibility.

**Proposition 2.** Let $\mathcal{L}$ be a lisse $\mathbb{Q}_\ell$-sheaf on $X$, which is Lie-irreducible, and whose determinant is of finite order. Then there exist $\alpha \in \mathbb{Q}_\ell$ and a closed point $x_0$ of $X$, such that $\alpha$ is an eigenvalue of multiplicity one of Frob$_{x_0}$ acting on $\mathcal{L}$; i.e. $1/\alpha$ is a root of multiplicity one of the polynomial $\det(1 - T \text{Frob}_{x_0}, \mathcal{L})$.

**Proof of Proposition 2.** First, we claim that it is a Zariski-open condition for an element of $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$ to have an eigenvalue of multiplicity one on $\mathcal{L}_\bar{\eta}$; in other words, we claim that the set

$$U := \{ g \in G_{\text{arith}}(\mathcal{L}, \bar{\eta}) : \text{g acting on } \mathcal{L}_\bar{\eta} \text{ has an eigenvalue of multiplicity one in } \mathbb{Q}_\ell \}$$

is a Zariski-open subset of $G_{\text{arith}}(\mathcal{L}, \bar{\eta})$. We show this as follows. For an element $g \in G_{\text{arith}}(\mathcal{L}, \bar{\eta})$, let $\text{ch}(g) \in \mathbb{Q}_\ell[T]$ denote the characteristic polynomial of $g$; then the set $U$ can also be described as

$$U = \{ g \in G_{\text{arith}}(\mathcal{L}, \bar{\eta}) : \text{ch}(g) \in \mathbb{Q}_\ell[T] \text{ has a root of multiplicity one in } \mathbb{Q}_\ell \}.$$

Let $r$ be the rank of $\mathcal{L}_\bar{\eta}$; then $\text{ch}$ gives rise to a morphism of $\mathbb{Q}_\ell$-varieties

$$\text{ch} : G_{\text{arith}}(\mathcal{L}, \bar{\eta}) \to \mathbb{Q}_\ell[T]_{\text{monic}}^{\text{deg } r}, \quad g \mapsto \text{ch}(g),$$

where $\mathbb{Q}_\ell[T]_{\text{monic}}^{\text{deg } r}$ denotes the affine space of monic polynomials in $T$ of degree $r$. For $g \in G_{\text{arith}}(\mathcal{L}, \bar{\eta})$, the polynomial $\text{ch}(g)$ has a root of multiplicity one in $\mathbb{Q}_\ell$ if and only if it does not divide the square $\text{ch}(g)^{r^2}$ of its derivative $\text{ch}(g)'$ in $\mathbb{Q}_\ell[T]$. Thus it suffices for us to show that the set

$$Z := \{ f \in \mathbb{Q}_\ell[T]_{\text{monic}}^{\text{deg } r} : f \text{ divides } f^{r^2} \text{ in } \mathbb{Q}_\ell[T] \}$$

is Zariski-closed in $\mathbb{Q}_\ell[T]_{\text{monic}}^{\text{deg } r}$. But for $f \in \mathbb{Q}_\ell[T]_{\text{monic}}^{\text{deg } r}$, the Euclidean division algorithm shows that the remainder of dividing $f^{r^2}$ by $f$ is a polynomial of degree $< r$ whose coefficients are given by certain (universal) $\mathbb{Z}$-polynomial expressions in terms of the coefficients of $f$; as the set $Z$ above is precisely the zero-set of these polynomial expressions, it is Zariski-closed.
Next, we claim that the set \( U \) above is in fact Zariski-open and non-empty in \( G_{\text{arith}}(\mathcal{L}, \bar{\eta}) \). Indeed, by part (ii) of proposition 1, \( G_{\text{arith}}(\mathcal{L}, \bar{\eta})^0 \) is a connected semisimple algebraic group; the representation \( \mathcal{L}_{\bar{\eta}} \) of \( G_{\text{arith}}(\mathcal{L}, \bar{\eta})^0 \) is irreducible by hypothesis, and so by the representation theory of connected semisimple algebraic groups, it is classified by its highest weight, which occurs with multiplicity one. Thus a generic element of any maximal torus of \( G_{\text{arith}}(\mathcal{L}, \bar{\eta})^0 \) lies in \( U \).

Finally, by Čebotarev’s density theorem, there exist infinitely many closed points \( x \) of \( X \) whose Frobenius conjugacy classes \( \text{Frob}_x \subset \pi_1(X, \bar{\eta}) \) are mapped into \( U \) under the monodromy representation of \( \pi_1(X, \bar{\eta}) \) on \( \mathcal{L}_{\bar{\eta}} \). Thus we can pick \( x_0 \) to be any one of these closed points of \( X \), and pick \( \alpha \in \overline{\mathbb{Q}}_\ell \) to be an eigenvalue of multiplicity one of \( \text{Frob}_{x_0} \) acting on \( \mathcal{L} \).

Remark. In proposition 2, it is not enough to just assume that the lisse \( \mathbb{Q}_\ell \)-sheaf \( \mathcal{L} \) is irreducible; the assumption that it is Lie-irreducible is necessary. If \( \mathcal{L} \) is irreducible but not Lie-irreducible, it may happen that every element of \( G_{\text{arith}}(\mathcal{L}, \bar{\eta}) \) acting on \( \mathcal{L}_{\bar{\eta}} \) has repeated eigenvalues, which is to say that the set \( U \subset G_{\text{arith}}(\mathcal{L}, \bar{\eta}) \) in the proof of the proposition is empty. For a specific example, we may take \( G_{\text{arith}}(\mathcal{L}, \bar{\eta}) \) to be the finite symmetric group on 6 letters, and take \( \mathcal{L}_{\bar{\eta}} \) to be the 16-dimensional irreducible representation of this finite group; such a situation can arise geometrically.

§2. Dévissage of Representations

Let \( k \) be an algebraically closed field of characteristic 0 — such as \( \overline{\mathbb{Q}}_\ell \). In this section, we consider (possibly non-connected) reductive groups over \( k \) and their finite dimensional \( k \)-rational representations. If \( G \) is such a reductive group, any \( k \)-rational representation of \( G \) is semisimple (a direct sum of irreducible representations), since \( k \) is of characteristic 0. By the quasi-compactness of \( G \), a subgroup \( H \) of \( G \) is (Zariski-) open if and only if it is (Zariski-) closed of finite index, in which case \( H \) necessarily contains the identity component \( G^0 \) of \( G \).

The following two results are proved in [I] for representations of finite groups. The same proofs, with minor modifications, work for representations of reductive groups. We reproduce the (modified) arguments below for the sake of completeness.

Lemma 3 (I. M. Isaacs, [I] Theorem 6.18). Let \( G \) be a reductive group, and let \( K \) and \( L \) be open normal subgroups of \( G \), with \( L \subseteq K \). Suppose that \( K/L \) is abelian, and that there does not exist a normal subgroup \( M \) of \( G \) with \( L \subseteq M \subseteq K \). Let \( \pi \) be an irreducible representation of \( K \) whose isomorphism class is invariant under \( G \)-conjugation. Then one of the following holds:

(i) \( \text{Res}_L^K(\pi) \) is isomorphic to a direct sum \( \sigma_1 \oplus \cdots \oplus \sigma_t \) of \( t := [K:L] \) many irreducible representations \( \sigma_1, \ldots, \sigma_t \) of \( L \) which are pairwise non-isomorphic;
(ii) \( \text{Res}_L^K(\pi) \) is an irreducible representation of \( L \);
(iii) \( \text{Res}_L^K(\pi) \) is isomorphic to \( \sigma^{\oplus e} \), where \( \sigma \) is an irreducible representation of \( L \), and \( e^2 = [K:L] \).

Proof of Lemma 3. Since \( L \) is normal in \( K \), the irreducible constituents of \( \text{Res}_L^K(\pi) \) are \( K \)-conjugate to one another, and each of these constituents occurs in \( \text{Res}_L^K(\pi) \) with the same multiplicity. Choose any irreducible constituent \( \sigma \) of \( \text{Res}_L^K(\pi) \), and let

\[ I := \{ g \in G : g \sigma \cong \sigma \ \text{as representations of} \ L \} \]

be the open subgroup of \( G \) (containing \( L \)) which stabilizes the isomorphism type of \( \sigma \) under \( G \)-conjugation. Since \( \pi \) is invariant under \( G \)-conjugation, every \( G \)-conjugate of \( \sigma \) is a constituent of \( \text{Res}_L^K(\pi) \), and so
every $G$-conjugate of $\sigma$ is $K$-conjugate to $\sigma$. It follows that $[G : I] = [K : K \cap I]$, and hence $KI = G$. Since $K/L$ is abelian, $K \cap I$ is normal in $K$; since $K$ is normal in $G$, $K \cap I$ is normal in $I$. As $KI = G$, we see that $K \cap I$ is normal in $G$. From the hypothesis of the proposition, it follows that $K \cap I$ is either $L$ or $K$.

Suppose $K \cap I = L$. Then there are $t = [K : L]$ many pairwise non-isomorphic irreducible constituents $\sigma = \sigma_1, \ldots, \sigma_t$ of $\text{Res}^K_L(\pi)$, and so we have

$$\text{Res}^K_L(\pi) \cong (\sigma_1 \oplus \cdots \oplus \sigma_t)^{\oplus e}$$

for some multiplicity $e \geq 1$. The constituents $\sigma_j$ of $\text{Res}^K_L(\pi)$ are $K$-conjugate to one another, and so they have the same rank as $\sigma$. Hence

$$\text{rk}(\pi) = \text{rk}(\text{Res}^K_L(\pi)) = et \text{rk}(\sigma).$$

But $\pi$ is a constituent of $\text{Ind}^K_L(\sigma)$, so

$$\text{rk}(\pi) \leq \text{rk}(\text{Ind}^K_L(\sigma)) = t \text{rk}(\sigma).$$

Thus $e = 1$, and this is case (i).

Henceforth suppose $K \cap I = K$. Then $\sigma$ is invariant under $K$-conjugation, so we have

$$\text{Res}^K_L(\pi) \cong \sigma^{\oplus e}$$

for some multiplicity $e \geq 1$. Let $\chi_1, \ldots, \chi_t$ be the distinct linear characters of the abelian group $K/L$. Then $\chi_1 \otimes \pi, \ldots, \chi_t \otimes \pi$ are irreducible representations of $K$, each having the same rank as $\pi$, and we have

$$\text{Res}^K_L(\chi_j \otimes \pi) \cong \sigma^{\oplus e} \quad \text{for each } j = 1, \ldots, t.$$

Suppose $\chi_1 \otimes \pi, \ldots, \chi_t \otimes \pi$ are pairwise non-isomorphic representations of $K$. Then we obtain an inclusion

$$\bigoplus_{j=1}^t (\chi_j \otimes \pi)^{\oplus e} \subseteq \text{Ind}^K_L(\sigma).$$

Comparing ranks, we get

$$et \text{rk}(\pi) \leq \text{rk}(\text{Ind}^K_L(\sigma)) = t \text{rk}(\sigma),$$

and so

$$e \text{rk}(\pi) \leq \text{rk}(\sigma).$$

But

$$e \text{rk}(\sigma) = \text{rk}(\text{Res}^K_L(\pi)) = \text{rk}(\pi).$$

Thus $e = 1$, and this is case (ii).

In the remaining situation, at least two of the representations $\chi_1 \otimes \pi, \ldots, \chi_t \otimes \pi$ are isomorphic; this implies that $\pi \cong \chi \otimes \pi$ for some non-trivial linear character $\chi$ of $K/L$. Let $M = \text{Ker}(\chi)$; we have $L \subseteq M \subseteq K$. First, consider the representation $\pi$, with trace-function

$$\text{Tr} \circ \pi : K \to k, \quad x \mapsto \text{Tr}(\pi(x)).$$
On $K - M$, the linear character $\chi$ takes values different from 1; since $\text{Tr} \circ \pi = \text{Tr} (\pi \otimes \pi) = \chi \cdot (\text{Tr} \circ \pi)$, it follows that $\text{Tr} \circ \pi$ vanishes on $K - M$. Since the representation $\pi$ is invariant under $G$-conjugation, it follows that $\text{Tr} \circ \pi$ vanishes on $K - gMg^{-1}$ for all $g \in G$. The normal subgroup $\bigcap_{g \in G} gMg^{-1}$ of $G$ contains $L$ and is properly contained in $K$, so it must be equal to $L$ by hypothesis. Thus $\text{Tr} \circ \pi$ vanishes on $K - L$. Next, consider the representation $\text{Ind}^K_L (\text{Res}^K_L (\pi)) \cong \text{Ind}^K_L (1) \otimes \pi$, with its trace-function

$$\text{Tr} \circ \text{Ind}^K_L (\text{Res}^K_L (\pi)) : K \to k, \quad x \mapsto \text{Tr}(\text{Ind}^K_L (1)(x)) \text{Tr}(\pi(x)).$$

Since the trace-function of $\text{Ind}^K_L (1)$ is 0 on $K - L$ and is $t$ on $L$, it follows that the trace-function of $\text{Ind}^K_L (\text{Res}^K_L (\pi))$ vanishes on $K - L$, and its values on $L$ are $t$ times those of $\text{Tr} \circ \pi$. Comparing the trace-functions of $\pi$ and $\text{Ind}^K_L (\text{Res}^K_L (\pi))$, we see that

$$\text{Tr} (\pi \oplus t) = \text{Tr} \circ \text{Ind}^K_L (\text{Res}^K_L (\pi)).$$

By the trace comparison theorem of Bourbaki (cf. [B] §12, no. 1, Prop. 3), this implies

$$\pi \oplus t \cong \text{Ind}^K_L (\text{Res}^K_L (\pi))$$

as representations of $K$. Hence

$$e^2 = \text{dim} \text{Hom}_L (\text{Res}^K_L (\pi), \text{Res}^K_L (\pi)) = \text{dim} \text{Hom}_K (\pi, \text{Ind}^K_L (\text{Res}^K_L (\pi))) = t = [K : L]$$

and this is case (iii).

\begin{proof}[Proposition 4 (I. M. Isaacs, [I] Theorem 6.22)]
Let $G$ be a reductive group, and let $N$ be an open normal subgroup of $G$ such that $G/N$ is a nilpotent finite group. Let $\rho$ be an irreducible representation of $G$. Then there exists an open subgroup $H$ of $G$ with $N \subseteq H \subseteq G$, and an irreducible representation $\sigma$ of $H$, such that $\rho \cong \text{Ind}^G_H (\sigma)$, and such that $\text{Res}^H_N (\sigma)$ is an irreducible representation of $N$.

Remark. The proposition holds in slightly greater generality: we need only to assume that $G/N$ is a solvable finite group whose chief factors are of square-free orders; see [I]. This technical condition is automatically verified when $G/N$ is nilpotent or supersolvable.

Proof of Proposition 4. The theorem is clear when $G = N$. We proceed by induction on $\#(G/N)$; hence assume that the theorem holds for any proper subgroup of $G$ containing $N$. If $\text{Res}^G_N (\rho)$ is irreducible, then the theorem holds with $H = G$ and $\sigma = \rho$. Hence suppose $\text{Res}^G_N (\rho)$ is reducible.

Since $G/N$ is finite, we can find an open normal subgroup $K$ of $G$ which is minimal for the conditions that $N \subseteq K$ and $\text{Res}^K_G (\rho)$ is irreducible. Then $N \subseteq K$ necessarily, and so we can find an open normal subgroup $L$ of $G$ which is maximal for the conditions that $N \subseteq L \subseteq K$. Since $G/N$ is nilpotent, it follows that $K/L$ is cyclic of prime order, say $t$.

The isomorphism class of the irreducible representation $\pi = \text{Res}^G_K (\rho)$ of $K$ is invariant under $G$-conjugation, since $\pi$ is the restriction of an irreducible representation $\rho$ of $G$. Thus we may apply lemma 3 to the representation $\pi$ of $K$. By the choice of $L$ and $K$, $\text{Res}^K_L (\pi)$ is not irreducible, so case (ii) cannot occur; since $t = [K : L]$ is a prime number, case (iii) cannot occur. Hence we are in case (i), and it follows that $\text{Res}^G_L (\rho)$ is isomorphic to a direct sum $\sigma_1 \oplus \cdots \oplus \sigma_t$ of $t$ many irreducible representations $\sigma_1, \ldots, \sigma_t$ of $L$ which are pairwise non-isomorphic.
Let
\[ I := \{ g \in G : \sigma_1 \cong g \sigma_1 \text{ as representations of } L \} \]
be the open subgroup of \( G \) (containing \( L \)) which stabilizes the isomorphism type of \( \sigma_1 \) under \( G \)-conjugation. Thus \([G : I] = t > 1\), and \( \rho \cong \text{Ind}_I^G(\rho') \) for some irreducible representation \( \rho' \) of \( I \).

Applying the induction hypothesis to \( I \), we obtain an open subgroup \( H \) of \( I \) with \( N \subseteq H \subseteq I \), and an irreducible representation \( \sigma \) of \( H \), such that \( \rho' \cong \text{Ind}_H^I(\sigma) \) and \( \text{Res}_H^I(\sigma) \) is an irreducible representation of \( N \). Then \( \rho \cong \text{Ind}_H^G(\sigma) \), which completes the proof of the proposition. \( \square \)

If \( G \) is a reductive group over \( k \), we let \( K(G) \) denote the Grothendieck group of the abelian category of finite dimensional \( k \)-rational representations of \( G \). It is clear that \( K(G) \) as a \( \mathbb{Z} \)-module is freely generated by the irreducible representations of \( G \). The tensor product of representations gives rise to a commutative ring structure on \( K(G) \), whose unit element is the class 1 of the trivial representation of \( G \). If \( H \subseteq G \) is an open subgroup, then induction of representations from \( H \) to \( G \) gives rise to a homomorphism of \( \mathbb{Z} \)-modules
\[ \text{Ind} : K(H) \to K(G). \]
The projection formula shows that the \( \text{Ind} \)-image of \( K(H) \) in \( K(G) \) is an ideal.

Recall that, for \( p \) a prime number, a finite group \( G \) is called \( p \)-elementary if it is isomorphic to a direct product \( A \times B \), where \( A \) is a cyclic group of order prime to \( p \), and \( B \) is a \( p \)-group. A finite group \( G \) is called elementary if it is \( p \)-elementary for some prime number \( p \). It is clear that an elementary finite group is nilpotent.

Let \( G \) be a reductive group, and \( N \) be an open normal subgroup of \( G \). We say that, for a prime number \( p \), an open subgroup \( H \) of \( G \) is \( p \)-elementary modulo \( N \) if one has the inclusions \( N \subseteq H \subseteq G \) and furthermore the finite quotient \( H/N \) is \( p \)-elementary; we say that \( H \) is elementary modulo \( N \) if it is \( p \)-elementary modulo \( N \) for some prime number \( p \).

**Proposition 5 (R. Brauer).** Let \( G \) be a reductive group, and let \( N \) be an open normal subgroup of \( G \). Then the \( \mathbb{Z} \)-homomorphism
\[ \text{Ind} : \bigoplus_{H \subseteq G \text{ elem. mod } N} K(H) \to K(G) \]
is surjective (the direct sum is over all subgroups \( H \) of \( G \) which are elementary modulo \( N \)).

**Proof of Proposition 5.** Recall that Brauer’s theorem on induced characters for finite groups (see [I] Theorem 8.4 or [H] Theorem 34.2 for instance) states that if \( G \) is a finite group, then the \( \mathbb{Z} \)-homomorphism
\[ \text{Ind} : \bigoplus_{H \subseteq G \text{ elem.}} K(H) \to K(G) \]
is surjective; the key point is that the unit element 1 of \( K(G) \) lies in the ideal generated by the \( \text{Ind} \)-images of \( K(H) \) where \( H \) runs over all elementary subgroups of \( G \). Therefore, the proposition follows from applying Brauer’s theorem to the finite group \( G/N \). \( \square \)
Corollary 6. Let \( G \) be a reductive group, and let \( N \) be an open normal subgroup of \( G \). Let \( \rho \) be a representation of \( G \). Then there exist a finite list of pairs:

\[
(H_1, \sigma_1), \ldots, (H_s, \sigma_s),
\]

where, for each \( i = 1, \ldots, s \),

(a) \( H_i \) is an open subgroup of \( G \) with \( N \subseteq H_i \subseteq G \),
(b) \( \sigma_i \) is an irreducible representation of \( H_i \), and in fact,
(c) \( \text{Res}_{H_i}^N(\sigma_i) \) is an irreducible representation of \( N \),

such that one has an isomorphism of representations of \( G \) of the form

\[
\rho \oplus \left( \bigoplus_{i=1}^{t} \text{Ind}_{H_i}^G(\sigma_i) \right) \cong \left( \bigoplus_{j=t+1}^{s} \text{Ind}_{H_j}^G(\sigma_j) \right)
\]

for some \( t \) with \( 1 \leq t \leq s \).

Remark. If one takes \( N \) to be the identity component \( G^0 \) of \( G \), then property (c) asserts that each \( \sigma_i \) is Lie-irreducible. This is the situation which we shall encounter later in \( \S 4 \).

Proof of Corollary 6. Proposition 5 tells us that we can find a finite list of pairs as in (\*\*), such that an isomorphism of the form (\*\*) holds, such that properties (a) and (b) are verified, and such that each \( H_i \) is elementary modulo \( N \). Since each \( H_i/N \) is then a nilpotent finite group, proposition 4 allows us to replace each \( H_i \) by a subgroup containing \( N \) and each \( \sigma_i \) by an irreducible representation of the corresponding subgroup, so that, furthermore, property (c) is also verified. This proves the corollary. \( \square \)

\[ \section 3. Descent of Representations} \]

Let \( \Gamma \) be a group, let \( k_0 \) be a field of characteristic zero, and let \( k \) be a field extension of \( k_0 \). In this section, we prove two criteria (propositions 7 and 9) for descending a \( k \)-representation of \( \Gamma \) to a \( k_0 \)-representation.

Proposition 7. Let \( \rho \) be a finite-dimensional \( k \)-representation of \( \Gamma \), which is absolutely irreducible (i.e. irreducible over an algebraic closure of \( k \)). Assume:

(i) \( \rho \) is defined over a finite Galois extension \( K \) of \( k_0 \) in \( k \); 
(ii) for every \( \gamma \in \Gamma \), the trace \( \text{Tr}(\rho(\gamma)) \) of \( \gamma \) with respect to \( \rho \) lies in \( k_0 \); 
(iii) there exists some \( \alpha \in k_0 \) and some \( \gamma_0 \in \Gamma \) such that \( \alpha \) is an eigenvalue of multiplicity one of \( \gamma_0 \) with respect to \( \rho \).

Then \( \rho \) is defined over \( k_0 \).

Proof of Proposition 7. By (i), we may assume that \( \rho \) is given as a \( K \)-matrix representation of \( \Gamma \):

\[
\rho : \Gamma \to \text{GL}_r(K),
\]
and we let \( \Sigma = \text{Gal}(K/k_0) \) be the finite Galois group. According to (iii), we may choose an eigenvector \( v \in K^{\otimes r} \) of \( \rho(\gamma_0) \) with eigenvalue \( \alpha \). By changing basis, we may assume that \( v \) is the first basis vectors of \( K^{\otimes r} \); thus the matrix \( \rho(\gamma_0) \) has the form

\[
\begin{pmatrix}
\alpha & * & \ldots & * \\
0 & * & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \ldots & *
\end{pmatrix}.
\]

Each \( \sigma \in \Sigma \) defines a \( K \)-representation

\[
\sigma \rho : \Gamma \xrightarrow{\rho} \text{GL}_r(K) \xrightarrow{\text{GL}_r(\sigma)} \text{GL}_r(K).
\]

Since \( \alpha \in k_0 \) is invariant under \( \Sigma \), the matrices \( \sigma \rho(\gamma_0) \) also have the same form as \( \rho(\gamma_0) \) above; thus \( v \) is also an eigenvector with eigenvalue \( \alpha \) of each \( \sigma \rho(\gamma_0) \), \( \sigma \in \Sigma \).

Assumption (ii) and the invariance of \( k_0 \) under \( \Sigma \) gives the equality in \( k_0 \):

\[
\text{Tr}(\sigma \rho(\gamma)) = \text{Tr}(\rho(\gamma)) \quad \text{for any } \sigma \in \Sigma, \text{ any } \gamma \in \Gamma.
\]

Therefore, by the trace comparison theorem of Bourbaki (cf. [B] \S 12, no. 1, Prop. 3), the \( K \)-representations \( \sigma \rho \) of \( \Gamma \), for various \( \sigma \in \Sigma \), are all isomorphic over \( K \) to \( \rho \). Choose such isomorphisms over \( K \):

\[
a(\sigma) : (\sigma \rho, K^{\otimes r}) \xrightarrow{\cong} (\rho, K^{\otimes r}), \quad \sigma \in \Sigma.
\]

Since \( \rho \) is absolutely irreducible by hypothesis, any automorphism of it must be a scalar in \( K \). It follows that each \( a(\sigma) \in \text{GL}_r(K) \) is determined up to a \( K \)-scalar multiple. For any \( \sigma, \sigma' \in \Sigma \), the two different ways of expressing \( \sigma' \sigma \rho \) in terms of \( \rho \) then gives

\[
a(\sigma' \sigma) = \text{scalar in } K \cdot a(\sigma') \cdot \sigma' a(\sigma).
\]

We shall now rigidify the situation. For each \( \sigma \in \Sigma \), we have the equality

\[
a(\sigma) \cdot \sigma \rho(\gamma_0) = \rho(\gamma_0) \cdot a(\sigma),
\]

and the fact that \( v \in K^{\otimes r} \) is an eigenvector of \( \sigma \rho(\gamma_0) \) with eigenvalue \( \alpha \); it follows that \( a(\sigma) v \in K^{\otimes r} \) is an eigenvector of \( \rho(\gamma_0) \) with eigenvalue \( \alpha \). Thanks to the multiplicity-one hypothesis (iii) on \( \alpha \), \( a(\sigma) v \) is necessarily a \( K \)-scalar multiple of \( v \) itself. Since we are free to adjust \( a(\sigma) \in \text{GL}_r(K) \) by any \( K \)-scalar multiple, we may and do assume that each \( a(\sigma) \) maps \( v \) to itself. Thus the matrices \( a(\sigma) \), for \( \sigma \in \Sigma \), have the form

\[
\begin{pmatrix}
1 & * & \ldots & * \\
0 & * & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \ldots & *
\end{pmatrix},
\]

and it follows that the matrices \( \sigma' a(\sigma) \), for \( \sigma, \sigma' \in \Sigma \), also have the same form above, which implies that each \( \sigma' a(\sigma) \) also maps \( v \) to itself. Therefore, we now have

\[
a(\sigma' \sigma) = a(\sigma') \cdot \sigma' a(\sigma) \quad \text{for any } \sigma, \sigma' \in \Sigma.
\]
By Hilbert Theorem 90 for $\text{GL}_r$, there exists some $b \in \text{GL}_r(K)$ such that
\[ a(\sigma) = b \cdot \sigma b^{-1} \quad \text{for each } \sigma \in \Sigma. \]
Using $b^{-1} \in \text{GL}_r(K)$ for a change of basis, we obtain the $K$-representation $\tilde{\rho} := b^{-1} \rho b$ defined by
\[ \tilde{\rho} : \Gamma \to \text{GL}_r(K), \quad \gamma \mapsto b^{-1} \rho(\gamma) b, \]
which is isomorphic over $K$ to $\rho$. A straightforward computation now shows that the matrices $\tilde{\rho}(\gamma) \in \text{GL}_r(K)$, for $\gamma \in \Gamma$, are all fixed under the action of the Galois group $\Sigma$; in other words, $\sigma \tilde{\rho} = \tilde{\rho}$ for any $\sigma \in \Sigma$. Thus the representation $\tilde{\rho}$ factorizes as
\[ \Gamma \to \text{GL}_r(k_0) \hookrightarrow \text{GL}_r(K). \]
So $\tilde{\rho}$ is defined over $k_0$, and the same is therefore true for $\rho$. □

Lemma 8. Let $M, N$ be $k_0$-representations of $\Gamma$.

(i) The canonical homomorphism of $k$-vector spaces
\[ k \otimes_{k_0} \text{Hom}_{k_0 \Gamma}(M, N) \to \text{Hom}_{k \Gamma}(k \otimes_{k_0} M, k \otimes_{k_0} N) \]

is injective; it is surjective if $M$ is finitely generated as a left $k_0 \Gamma$-module.

(ii) The canonical homomorphism of $k$-vector spaces
\[ k \otimes_{k_0} \text{Ext}^1_{k_0 \Gamma}(M, N) \to \text{Ext}^1_{k \Gamma}(k \otimes_{k_0} M, k \otimes_{k_0} N) \]

is injective if $M$ is finitely generated as a left $k_0 \Gamma$-module.

Remarks. a) If $M$ is finitely presented as a left $k_0 \Gamma$-module, the lemma follows from the well-known “change of rings” isomorphisms applied to $k_0 \Gamma \hookrightarrow k \Gamma$ (see [R] Th. 2.39 for instance). Of course, if $M$ is a finite-dimensional $k_0$-representation of $\Gamma$, then it is automatically a finitely generated left $k_0 \Gamma$-module; however, it need not be finitely presented as a left $k_0 \Gamma$-module.

b) When $\Gamma$ is a finite group, the group ring $k_0 \Gamma$ is left-noetherian, so a finite dimensional $k_0$-representation $M$ of $\Gamma$ is finitely presented as a left $k_0 \Gamma$-module, and the lemma follows from a) above. But since we will use the lemma when $\Gamma$ is a profinite group, and we could not identify a satisfactory reference for the corresponding result, we find it prudent to give a complete proof here.

c) The proof below actually shows that the lemma holds in slightly greater generality: it suffices to assume that $k_0$ is any commutative ring, and that $k$ is a $k_0$-algebra which is free as a $k_0$-module.

Proof of Lemma 8. We first show that the canonical homomorphism
\[ k \otimes_{k_0} \text{Hom}_{k_0 \Gamma}(M, N) \to \text{Hom}_{k \Gamma}(k \otimes_{k_0} M, k \otimes_{k_0} N) \]

is injective. Choose a basis $\{e_i \in k : i \in I\}$ of $k$ as a $k_0$-vector space. Then the $k_0 \Gamma$-module $k \otimes_{k_0} N$ is the direct sum of the $k_0 \Gamma$-submodules $e_i \otimes N$:
\[ k \otimes_{k_0} N \cong \bigoplus_{i \in I} e_i \otimes N; \]
likewise, the $k_0$-vector space $k \otimes_{k_0} \text{Hom}_{k_0 \Gamma}(M,N)$ is the direct sum of the corresponding $k_0$-subspaces $e_i \otimes \text{Hom}_{k_0 \Gamma}(M,N)$:

$$k \otimes_{k_0} \text{Hom}_{k_0 \Gamma}(M,N) \cong \bigoplus_{i \in I} e_i \otimes \text{Hom}_{k_0 \Gamma}(M,N).$$

Any $\phi \in k \otimes_{k_0} \text{Hom}_{k_0 \Gamma}(M,N)$ is therefore equal to a sum

$$\phi = \sum_{i \in I} e_i \otimes \phi_i$$

for some uniquely determined $\phi_i \in \text{Hom}_{k_0 \Gamma}(M,N), \ i \in I$, all but finitely of which are the zero-map. Suppose $\phi$ lies in the kernel of the canonical homomorphism. Then for any $m \in M$, one has

$$\sum_{i \in I} e_i \otimes \phi_i(m) = 0 \quad \text{in} \quad k \otimes_{k_0} N \cong \bigoplus_{i \in I} e_i \otimes N,$$

so $\phi_i(m) = 0$ in $N$ for each $i \in I$. It follows that $\phi = 0$, which is what we want.

If $M$ is finite free as a left $k_0 \Gamma$-module, then it follows from the functorial properties of $\text{Hom}$ and $\otimes$ that the canonical homomorphism is an isomorphism. In general, if $M$ is finitely generated as a left $k_0 \Gamma$-module, let

$$0 \to K \to F \to M \to 0$$

be a short exact sequence of left $k_0 \Gamma$-modules with $F$ finite free. Then

$$0 \to \text{Hom}_{k_0 \Gamma}(M,N) \to \text{Hom}_{k_0 \Gamma}(F,N) \to \text{Hom}_{k_0 \Gamma}(K,N)$$

is an exact sequence of $k_0$-vector spaces. From this and the fact that $k$ is flat over $k_0$, we obtain the following commutative diagram with exact columns:

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
 k \otimes_{k_0} \text{Hom}_{k_0 \Gamma}(M,N) & \longrightarrow & \text{Hom}_{k \Gamma}(k \otimes_{k_0} M, k \otimes_{k_0} N) \\
\downarrow & & \downarrow \\
 k \otimes_{k_0} \text{Hom}_{k_0 \Gamma}(F,N) & \cong & \text{Hom}_{k \Gamma}(k \otimes_{k_0} F, k \otimes_{k_0} N) \\
\downarrow & & \downarrow \\
 k \otimes_{k_0} \text{Hom}_{k_0 \Gamma}(K,N) & \longrightarrow & \text{Hom}_{k \Gamma}(k \otimes_{k_0} K, k \otimes_{k_0} N) \\
\downarrow & & \downarrow \\
 & & \\
\end{array}
\]

where the middle horizontal arrow is an isomorphism and the bottom horizontal arrow is injective, by what we have already shown. A diagram chase shows that the top horizontal arrow is surjective. This proves part (i).
Proposition 9 (E. Noether – M. Deuring). Let $\rho$, $\tau$ and $\pi$ be semisimple finite-dimensional $k$-representations of $\Gamma$ such that

$$\rho \oplus \tau \cong \pi.$$ 

Suppose $\tau$ and $\pi$ are defined over $k_0$. Then $\rho$ is also defined over $k_0$.

Proof of Proposition 9. Our argument here is adapted from that given for representations of finite groups (see [H] Theorem 37.6 for instance). The proposition is clear when $\tau = 0$. We proceed by induction on the rank $\text{rk}(\tau)$ of $\tau$; hence assume that $\text{rk}(\tau) \geq 1$. By hypothesis, there exist $k_0$-representations $\tau_0$, $\pi_0$ of $\Gamma$ such that

$$\tau \cong k \otimes_{k_0} \tau_0, \quad \pi \cong k \otimes_{k_0} \pi_0.$$ 

For any finite-dimensional $k_0$-representations $M, N$ of $\Gamma$, we have the canonical inclusion:

$$\text{Ext}^1_{k_0\Gamma}(M, N) \hookrightarrow k \otimes_{k_0} \text{Ext}^1_{k_0\Gamma}(M, N) \xrightarrow{\text{Lemma 8}} \text{Ext}^1_{k\Gamma}(k \otimes_{k_0} M, k \otimes_{k_0} N);$$

this fact and the hypothesis that $\tau, \pi$ are semisimple as $k$-representations of $\Gamma$ imply that $\tau_0, \pi_0$ are semisimple as $k_0$-representations of $\Gamma$.

Let $\sigma_0 \subseteq \tau_0$ be an irreducible constituent of the $k_0$-representation $\tau_0$ of $\Gamma$. Then

$$k \otimes_{k_0} \text{Hom}_{k_0\Gamma}(\sigma_0, \pi_0) \xrightarrow{\cong \text{Lemma 8}} \text{Hom}_{k\Gamma}(\sigma, \pi) \cong \text{Hom}_{k\Gamma}(\sigma, \rho \oplus \tau)$$

contains

$$\text{Hom}_{k\Gamma}(\sigma, \tau) \xleftarrow{\cong \text{Lemma 8}} k \otimes_{k_0} \text{Hom}_{k_0\Gamma}(\sigma_0, \tau_0) \neq 0,$$

whence $\text{Hom}_{k_0\Gamma}(\sigma_0, \tau_0) \neq 0$. Thus $\sigma_0$ is also an irreducible constituent of the $k_0$-representation $\pi_0$ of $\Gamma$. Therefore,

$$\tau_0 \cong \tau'_0 \oplus \sigma_0, \quad \pi_0 \cong \pi'_0 \oplus \sigma_0,$$

for some semisimple $k_0$-representations $\tau'_0$ and $\pi'_0$ of $\Gamma$. Letting

$$\tau' := k \otimes_{k_0} \tau'_0, \quad \pi' := k \otimes_{k_0} \pi'_0, \quad \sigma := k \otimes_{k_0} \sigma_0,$$

we obtain

$$\tau' \neq 0, \quad \pi' \neq 0, \quad \sigma \neq 0.$$
we obtain an isomorphism
\[ \rho \oplus \tau' \oplus \sigma \cong \pi' \oplus \sigma \]
of semisimple \( k \)-representations of \( \Gamma \), and hence an equality of their \( k \)-valued trace functions:
\[ \text{Tr}(\rho(g)) + \text{Tr}(\tau'(g)) + \text{Tr}(\sigma(g)) = \text{Tr}(\pi'(g)) + \text{Tr}(\sigma(g)) \quad \text{for every } g \in \Gamma. \]
Applying the trace comparison theorem of Bourbaki (cf. [B] §12, no. 1, Prop. 3) to the equality
\[ \text{Tr}(\rho(g)) + \text{Tr}(\tau'(g)) = \text{Tr}(\pi'(g)) \quad \text{for every } g \in \Gamma, \]
we obtain an isomorphism
\[ \rho \oplus \tau' \cong \pi' \]
of semisimple \( k \)-representations of \( \Gamma \). Since \( \text{rk}(\tau') < \text{rk}(\tau) \), our induction hypothesis shows that \( \rho \) is defined over \( k_0 \). \( \square \)

§4. Proof of Main Theorem

We shall now prove the main theorem stated in the introduction.

Thus, let \( F_q \) be a finite field of characteristic \( p \), let \( \ell \neq p \) be a prime number, let \( X \) be a normal variety over \( F_q \), and let \( \mathcal{L} \) be a lisse \( \overline{\mathbb{Q}}_\ell \)-sheaf on \( X \), which is irreducible, and whose determinant is of finite order. Let \( E \subset \overline{\mathbb{Q}}_\ell \) denote the number field given by hypothesis (1) applied to \( (X, \mathcal{L}) \); thus for every closed point \( x \) of \( X \), the polynomial
\[ \det(1 - T \text{Frob}_x, \mathcal{L}) \]
has coefficients in \( E \). We may replace the finite field \( F_q \) by its algebraic closure in the function field \( \kappa(X) \) of \( X \), and hence assume that \( X \) is geometrically connected over \( F_q \); this allows us to use the results in §1. Let \( \bar{\eta} \to X \) be a geometric point of \( X \), and set
\[ \Gamma := \pi_1(X, \bar{\eta}), \quad G := G_{\text{arith}}(\mathcal{L}, \bar{\eta}). \]
Let
\[ \rho : G \to \text{GL}(\mathcal{L}_{\bar{\eta}}) \]
denote the faithful representation of \( G_{\text{arith}}(\mathcal{L}, \bar{\eta}) \) on \( \mathcal{L}_{\bar{\eta}} \).

By proposition 1 (ii), \( G \) is a (possibly non-connected) semisimple algebraic group. We apply corollary 6 to the representation \( \rho \) of \( G \), with \( N := G^0 = G_{\text{arith}}(\mathcal{L}, \bar{\eta})^0 \), to obtain a finite list of pairs as in \((*)\), satisfying the properties (a), (b) and (c) listed there, such that an isomorphism of representations of \( G \) of the form \((**\)) holds.

Consider any pair \((\mathcal{H}_i, \sigma_i)\) in \((*)\). By property (a), the identity component \( \mathcal{H}_i^0 \) of \( \mathcal{H}_i \) is a connected semisimple algebraic group (in fact it is \( G_{\text{arith}}(\mathcal{L}, \bar{\eta})^0 \)), which is therefore equal to its own commutator subgroup; hence the 1-dimensional representation \( \det(\sigma_i) \) of \( \mathcal{H}_i \), given by the determinant of \( \sigma_i \), factors
through $H_i/H_i^0$, and so is given by a character of $H_i$ of finite order. This and properties (b) and (c) show that each $\sigma_i$ is a Lie-irreducible representation of $H_i$, and its determinant is of finite order.

Set

$$\Gamma_i := (\rho_{\ell})^{-1}(H_i) \subseteq \Gamma.$$ 

Then $\Gamma_i$ is an open subgroup of $\Gamma$, corresponding to a finite etale cover $X_i \to X$ of $X$ by a connected variety $X_i$ pointed by the geometric point $\bar{\eta}$; we identify $\Gamma_i$ with the arithmetic fundamental group $\pi_1(X_i, \bar{\eta})$ of $X_i$. If $V_i$ is the representation space of $\sigma_i$, then the composite homomorphism

$$\sigma_{F_i} : \Gamma_i \xrightarrow{\rho_{\ell}} H_i \xrightarrow{\sigma_i} \text{GL}(V_i)$$

is a $\overline{\mathbb{Q}}_\ell$-representation of $\Gamma_i$ which corresponds to a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $F_i$ on the variety $X_i$. It follows from the corresponding properties of $\sigma_i$ that $F_i$ is Lie-irreducible, and its determinant is of finite order. By hypothesis (1) applied to $(X_i, F_i)$, there is a number field $E_i \subset \overline{\mathbb{Q}}_\ell$ such that for every closed point $x$ of $X_i$, the polynomial

$$\det(1 - T \text{Frob}_x, F_i)$$

has coefficients in $E_i$; and by proposition 2, there is some $\alpha_i \in \overline{\mathbb{Q}}_\ell$ and some closed point $x_0^{(i)}$ of $X_i$ such that $\alpha_i$ is an eigenvalue of multiplicity one of $\text{Frob}_{x_0^{(i)}}$ acting on $F_i$. It follows that $\alpha_i$ is algebraic over the number field $E_i$.

Let

$$\rho_{\ell,i} := \text{Ind}^\Gamma_{\Gamma_i}(\sigma_{F_i})$$

be the $\overline{\mathbb{Q}}_\ell$-representation of $\Gamma$ induced from $\sigma_{F_i}$, and let

$$F := \text{composite of } E_1(\alpha_1), \ldots, E_s(\alpha_s) \text{ and } E \text{ in } \overline{\mathbb{Q}}_\ell.$$ 

It is clear that $F$ is a finite extension of $E$ in $\overline{\mathbb{Q}}_\ell$. The isomorphism (**) implies that for any closed point $x$ of $X$, one has

$$\text{(**) } \sum_{i=1}^t \text{Tr}(\rho_{\ell,i}(\text{Frob}_x)) + \sum_{j=t+1}^s \text{Tr}(\rho_{\ell,j}(\text{Frob}_x)) = \sum_{j=t+1}^s \text{Tr}(\rho_{\ell,j}(\text{Frob}_x)) \quad (\text{equality in } F \subset \overline{\mathbb{Q}}_\ell).$$

We shall now show that the number field $F$ satisfies the conclusion of assertion (3’).

To that end, pick a place $\lambda'$ of $F$ lying over a prime number $\ell' \neq p$, and choose an algebraic closure $\overline{\mathbb{Q}}_{\ell'}$ of $F_{\lambda'}$. By hypothesis (3) applied to $(X, \mathcal{L})$ and each $(X_i, F_i)$, there exist irreducible lisse $\overline{\mathbb{Q}}_{\ell'}$-sheaves $\mathcal{L}'$ on $X$ and $\mathcal{F}_i'$ on $X_i$, which are compatible with $\mathcal{L}$ and $\mathcal{F}_i$ respectively; i.e. for each closed point $x$ of $X$, one has

$$\text{(1) } \det(1 - T \text{Frob}_x, \mathcal{L}') = \det(1 - T \text{Frob}_x, \mathcal{L}) \quad (\text{equality in } F[T]),$$

and for each $i = 1, \ldots, s$ and each closed point $x$ of $X_i$, one has

$$\text{(2) } \det(1 - T \text{Frob}_x, \mathcal{F}_i') = \det(1 - T \text{Frob}_x, \mathcal{F}_i) \quad (\text{equality in } F[T]).$$

It follows that $\mathcal{L}'$ has the same rank as $\mathcal{L}$ (and each $\mathcal{F}_i'$ has the same rank as $\mathcal{F}_i$). It also follows from these compatibility relations that

$$\text{(3) } \alpha_i \in F \subset F_{\lambda'} \subset \overline{\mathbb{Q}}_{\ell'} \text{ is an eigenvalue of multiplicity one of } \text{Frob}_{x_0^{(i)}} \text{ acting on } \mathcal{F}_i'.$$
Let $\rho_{L'}$ denote the irreducible monodromy $\mathbb{Q}_{L'}$-representation of $\Gamma$ corresponding to $L'$, and let $\sigma_{F'_i}$ denote the irreducible monodromy $\mathbb{Q}_{L'}$-representation of $\Gamma_i$ corresponding to $F'_i$. Let

$$\rho_{L'_i} := \text{Ind}^\Gamma_{\Gamma_i}(\sigma_{F'_i})$$

be the $\mathbb{Q}_{L'}$-representation of $\Gamma$ induced from $\sigma_{F'_i}$. From (1) and (2), we deduce that for each closed point $x$ of $X$, one has

$$\text{Tr}(\rho_{L'}(\text{Frob}_x)) = \text{Tr}(\rho_{L}(\text{Frob}_x)) \quad \text{(equality in } F' \text{),}$$

and for each $i = 1, \ldots, s$ and each closed point $x$ of $X_i$, one has

$$\text{Tr}(\sigma_{F'_i}(\text{Frob}_x)) = \text{Tr}(\sigma_{F_i}(\text{Frob}_x)) \quad \text{(equality in } F' \text{),}$$

whence for each $i = 1, \ldots, s$ and each closed point $x$ of $X$, one has

$$\text{Tr}(\rho_{L'_i}(\text{Frob}_x)) = \text{Tr}(\rho_{L_i}(\text{Frob}_x)) \quad \text{(equality in } F' \text{).}$$

Combining the equalities (4), (6) with (**'), we see that for any closed point $x$ of $X$, one has

$$\text{Tr}(\rho_{L'}(\text{Frob}_x)) + \sum_{i=1}^t \text{Tr}(\rho_{L'_i}(\text{Frob}_x)) = \sum_{j=t+1}^s \text{Tr}(\rho_{L'_j}(\text{Frob}_x)) \quad \text{(equality in } F \subset \mathbb{Q}_{L'}.$$

By Čebotarev’s density theorem, this equality of traces, as an equality in $\mathbb{Q}_{L'}$, holds for every element of $\Gamma$. Therefore, by the trace comparison theorem of Bourbaki (cf. [B] §12, no. 1, Prop. 3), we obtain an isomorphism of semisimple $\mathbb{Q}_{L'}$-representations of $\Gamma$:

$$\rho_{L'} \oplus \left( \bigoplus_{i=1}^t \rho_{L'_i} \right) \cong \left( \bigoplus_{j=t+1}^s \rho_{L'_j} \right).$$

Consider the (absolutely) irreducible $\mathbb{Q}_{L'}$-representation $\sigma_{F'_i}$ of $\Gamma_i$. We wish to apply proposition 7 to this representation; so let us check that the hypotheses there are verified.

(i) By the definition of lisse $\mathbb{Q}_{L'}$-sheaves (cf. [D] (1.1.1) — alternatively, apply [KSa] Remark 9.0.7), the $\mathbb{Q}_{L'}$-representation $\sigma_{F'_i}$ is defined over a finite extension of $\mathbb{Q}_{L'}$, which we may of course assume to be finite Galois over $F_{\lambda'}$.

(ii) From (5), we see that for every closed point $x$ of $X_i$, the trace $\text{Tr}(\sigma_{F'_i}(\text{Frob}_x))$ of $\text{Frob}_x \subset \Gamma_i$ with respect to $\sigma_{F'_i}$ lies in $F_{\lambda'}$; so it follows from Čebotarev’s density theorem that the trace $\text{Tr}(\sigma_{F'_i}(\gamma))$ of every element $\gamma \in \Gamma_i$ with respect to $\sigma_{F'_i}$ lies in $F_{\lambda'}$.

(iii) Finally, from (3), we know that $\alpha_i \in F_{\lambda'}$ is an eigenvalue of multiplicity one of $\text{Frob}_{z_i \lambda_{10}} \subset \Gamma_i$ with respect to $\sigma_{F'_i}$.

Hence proposition 7 shows that $\sigma_{F'_i}$ is defined over $F_{\lambda'}$. Then each $\rho_{L'_i}$, being induced from $\sigma_{F'_i}$, is also defined over $F_{\lambda'}$. Therefore, in (7), the two representations in parentheses are defined over $F_{\lambda'}$. Proposition 9 now shows that $\rho_{L'}$ is also defined over $F_{\lambda'}$, and hence the lisse $\mathbb{Q}_{L'}$-sheaf $L'$ is defined over $F_{\lambda'}$; in other words, there exists a lisse $F_{\lambda'}$-sheaf $L_{\lambda'}$ on $X$ such that $L' \cong L_{\lambda'} \otimes_{F_{\lambda'}} \mathbb{Q}_{L'}$. The asserted properties of $L_{\lambda'}$ follow from this isomorphism and (1).

This completes the proof of our main theorem.
INDEPENDENCE OF $\ell$ IN LAFFORGUE’S THEOREM

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Department of Mathematics, Princeton University, Princeton, NJ 08544, U.S.A.

Current address: Department of Mathematics, University of California, Berkeley, CA 94720, U.S.A.

E-mail address: cchin@math.princeton.edu, cheewhye@math.berkeley.edu