Degrees of Freedom in Wireless Interference Networks with Cooperative Transmission and Backhaul Load Constraints

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Abstract

Degrees of freedom (DoF) gains are studied in wireless networks with cooperative transmission under a backhaul load constraint that limits the average number of messages that can be delivered from a centralized controller to the basestation transmitters. The backhaul load is defined as the sum of all the messages available at all the transmitters per channel use, normalized by the number of users. For Wyner’s linear interference network, where each transmitter is connected to the receiver having the same index as well as one succeeding receiver, the per user DoF is characterized and the optimal scheme is presented. When the backhaul load is constrained to an integer level $B$, the asymptotic per user DoF is shown to equal $\left\lfloor \frac{4B-1}{4B} \right\rfloor$. Furthermore, it is shown that the optimal assignment of messages to transmitters is asymmetric and satisfies a local cooperation constraint, and that the optimal coding scheme relies only on one-shot zero-forcing transmit beamforming. The results are then extended to locally connected linear interference networks and the more practical hexagonal sectored cellular networks. For locally connected networks where each transmitter is connected to the receiver with the same index as well as $L$ succeeding receivers, we show that an asymptotic per user DoF of $\frac{L}{2}$ is achievable for $L \leq 6$ with simple zero-forcing (interference avoidance) and without incurring overall load on the backhaul ($B = 1$). For hexagonal cellular networks, we also show that an asymptotic per user DoF of $\frac{3}{7}$ is achievable with simple zero-forcing and the backhaul load constraint $B = 1$. Further, we show that $\frac{3}{7}$ is the optimal value if each message can only be available at a single transmitter (no cooperative transmission), and that $\frac{3}{7}$ is an upper bound on the asymptotic per user DoF achievable through interference avoidance if no cooperative transmission is allowed. Hence, our main conclusion for all the studied network models is that by allowing for cooperative transmission and a flexible message assignment that is constrained only by an average backhaul load, one can deliver the rate gains promised by information-theoretic upper bounds with practical one-shot schemes that incur little or no additional load on the backhaul.

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I. INTRODUCTION

Managing interference in wireless networks has emerged as a challenging and important task over the past decade. We explore the potential degrees of freedom gains of cooperative transmission in wireless networks through different models for the interference, and under average backhaul load constraints. In particular, we show that cooperative transmission can be used to achieve significant DoF gains without requiring extra backhaul capacity.

We begin by studying the degrees of freedom (DoF) in Wyner’s linear interference network, introduced in [3], where interference is modeled by assuming that the transmission of each transmitter is heard only by the receiver that has the same index as well as one succeeding transmitter. Wyner’s model, while being simple, allows us to obtain rigorous conclusions about the optimal schemes for interference management. Further, as we show in this work, the insights obtained through analyzing linear networks such as Wyner’s network can often be carried over to more complex network models that better approximate practical wireless networks.

Our focus on the DoF criterion is justified by the fact that it is useful to capture roughly the available capacity as a fraction of the capacity of an interference-free network consisting of point-to-point links. Two major advantages of the DoF criterion are as follows: (i) it is easy to analyze, and in many cases, the problem of finding an information theoretic upper bound or converse reduces to a straightforward combinatorial problem; and (ii) it captures the effect of interference, while circumventing the difficulties in analysis introduced by the additive Gaussian noise at the receivers. The DoF of a point-to-point link with white Gaussian noise is unity, and this is the reference benchmark for any given user’s rate in an interference network, i.e., the per user DoF is at most one.

The DoF gain offered by cooperative transmission\(^1\) in Wyner’s linear interference networks was studied in [5], for the special case where each message is available at the transmitter with the same index as well as \(M - 1\) succeeding transmitters. The asymptotic limit of the per user DoF as the number of users goes to infinity was shown to be \(\frac{N}{M+1}\). An asymptotic per user DoF of \(\frac{2M-1}{2M}\) was achieved using a smarter message assignment in [6]. In the proposed scheme of [6], each message is assigned to the transmitter with the same index as well as \(M - 1\) other transmitters. However, unlike the assignment of [5], in [6] the choice of the \(M - 1\) other transmitters is not simply the succeeding \(M - 1\) transmitters. In [7], it is shown that under a cooperation order constraint that limits the number of transmitters at which each message can be available by \(M\), the asymptotic per user DoF is \(\frac{2M}{2M+1}\) and is achieved by a flexible assignment of messages to transmitters where it is not necessary to assign each message to the transmitter with the same index. The DoF gains discussed in [5], [6] and [7] are achieved by a simple signaling scheme that relies only on zero-forcing transmit beamforming.

The maximum transmit set size constraint of \(M\) is not met tightly for all messages in the optimal message assignment scheme presented in [7]. In this work, we therefore consider a cooperation constraint that is more general and relevant to many scenarios of practical significance. In particular, we define the backhaul load constraint \(B\) as the ratio between the sum of the transmit set sizes for all the messages and the number of users. In other words, we allow the transmit set size to vary across the messages, while maintaining a constraint on the average transmit set size of \(B\). We establish in this paper that the asymptotic per user DoF in this new

\(^1\)Also called Coordinated Multi-Point (CoMP) transmission [4].
setting is $\frac{4B-1}{4M}$, which is larger than the per user DoF of $\frac{2M}{2M+1}$ obtained with the more stringent per message transmit set size constraint of $M = B$. The identified optimal scheme relies only on zero-forcing beamforming at the transmitters, and an asymmetric or unbalanced assignment of messages, with some messages being assigned to more than $B$ transmitters and others being assigned to fewer than $B$ transmitters.

We apply these insights to more general channel models such as general linear interference models and the hexagonal sectored cellular model. In particular, we show that with cooperative transmission that is based on zero-forcing beamforming (interference avoidance) under a backhaul load constraint of unity, it is possible to achieve an asymptotic per user DoF of $\frac{1}{2}$ for: (i) linear locally connected networks where each transmitter is connected to at most six receivers other than the receiver with the same index; and (ii) hexagonal cellular networks. On the other hand, we show that if cooperative transmission in not allowed ($M = 1$), then the $\frac{1}{2}$ per user DoF is the optimal value, and cannot be obtained by simple interference avoidance. This shows that simple one-shot zero-forcing beamforming combined with non-uniform message assignments can be used to achieve significant gains in the per user DoF, while maintaining a low value for the average backhaul load.

A. Related Work

A major advance in the theoretical analysis of interference management in large wireless networks took place with the introduction of asymptotic interference alignment in [8] (IA). IA beamforming relies on signaling over a number of time slots (symbol extension) that goes to infinity in order to enable the achievability of a per user DoF of $\frac{1}{2}$ in a fully connected interference network. However, the gains offered by IA are considered to be infeasible in practice, and a major reason for the infeasability is the excessive requirement on the length of symbol extension, which would lead to impractical delays. An important aspect of this work is that we show that the promised gains of interference alignment can be achieved with one-shot coding schemes that do not require symbol extension, if we consider more practical network models than the fully connected model and allow for cooperative transmission, even without requiring additional overall load on the supporting backhaul.

Degrees of freedom gains in the hexagonal cellular downlink using cooperative transmission was considered in [9], where the transmitting basestations cooperate by exchanging quantized dirty paper coded signals. However, implementing such a scheme can face practical challenges as each transmitter gets its message only after a series of preceding transmitters have encoded their messages; this will require either significant delay or coding over multiple time slots. Further, under this setting, the only way for messages to be delivered to transmitters through a centralized controller, is for the controller to be aware of the channel state information.

B. Document Organization

We describe the system model in Section II. In Section III, we outline the arguments that we use throughout the paper for deriving Degrees of Freedom lower and upper bounds. We then characterize the degrees of freedom for the Wyner linear network in Section IV. We extend the results to the linear locally connected channel model in Section V, and to the hexagonal cellular network in Section VI. Finally, we provide concluding remarks in Section VII.
II. SYSTEM MODEL AND NOTATION

We use the standard model for the $K$-user interference channel with single-antenna transmitters and receivers,

$$Y_i = H_{i,i}X_i + \sum_{j \in \mathcal{N}_i} H_{i,j}X_j + Z_i$$  \hspace{1cm} (1)

where $X_j$ denotes the signal transmitted by transmitter $j$ under an average transmit power constraint, $Z_i$ denotes the additive white Gaussian noise at receiver $i$, $H_{i,j}$ denotes the channel gain coefficient from transmitter $j$ to receiver $i$, and $\mathcal{N}_i$ denotes the set of interferers at receiver $i$ (neighbors in the connectivity graph). All channel coefficients that are not identically zero are assumed to be drawn from a continuous joint distribution. Finally, it is assumed that global channel state information is available at all transmitters and receivers.

A. Linear Interference Networks

1) Wyner assymmetric model: In this channel model, each transmitter is connected to its corresponding receiver as well as one following receiver, and the last transmitter is only connected to its corresponding receiver. More precisely,

$$H_{i,j} \equiv 0 \text{ iff } i \notin \{j, j+1\}, \forall i, j \in [K],$$  \hspace{1cm} (2)

where $[K]$ denotes the set $\{1, 2, \ldots, K\}$.

2) Locally connected channels: This is a more general linear network defined in [7], where each receiver sees interference from $L$ neighboring transmitters. More precisely, for the following channel model,

$$H_{i,j} \equiv 0 \text{ iff } i \notin \left[ j - \left\lfloor \frac{L}{2} \right\rfloor, j + \left\lceil \frac{L}{2} \right\rceil \right], \forall i, j \in [K].$$  \hspace{1cm} (3)

B. Two-dimensional Wyner network

Consider the two-dimensional network depicted in Figure 1 where each transmitter is connected to four cell edge receivers. The precise channel model for a $K$-user channel is as follows,

$$H_{i,j} \text{ is not identically } 0, \text{ if and only if }$$

$$i \in \{j, j + 1, j + \left\lfloor \sqrt{K} \right\rfloor, j + \left\lceil \sqrt{K} \right\rceil + 1\},$$  \hspace{1cm} (4)

C. Hexagonal Cellular Network

This is a sectored $K$ user cellular network with three sectors per cell as shown in Figure 2(a). We assume a local interference model, where the interference at each receiver is only due to the basestations in the neighboring sectors in adjacent cells. It is assumed that the sectors belonging to the same cell do not interfere with each other, the justification being that the interference power due to sectors in the same cell is usually far lower than the interference from out-of-cell users located in the sector’s line of sight.
1) Connectivity graph: The cellular model is represented by an undirected connectivity graph $G(V, E)$ shown in Figure 2(b) where each vertex $u \in V$ corresponds to a transmitter-receiver pair. For any node $a$, the transmitter, receiver and intended message (word) corresponding to the node are denoted by $T_a$, $R_a$ and $W_a$, respectively. An edge $e \in E$ between two vertices $u, v \in V$ corresponds to a channel existing between the transmitter at $u$ and the receiver at $v$, and vice-versa. The dotted lines denote interference between sectors that belong to the same cell, and is ignored in our model. To simplify the presentation, without much loss of generality, we consider only $K$-user networks where $\sqrt{K}$ is an integer, and nodes are numbered as in Figure 2(b). (In the figure, $\sqrt{K} = 6$). Since we are studying the performance in the asymptotic limit of the number of users, the assumption is not restrictive.

We formally define the connectivity graph $G(V, E)$ using Eisenstein integers similar to [9].

**Definition 1:** (Eisenstein integers) : Eisenstein integers $\mathbb{Z}[\omega]$ are numbers of the form $a + b\omega$ where $\omega = \frac{1}{2}(-1 + i\sqrt{3})$ and $a, b \in \mathbb{Z}$.

Let $B_r = \{z \in \mathbb{C} : \text{Re}(z) \leq r, \text{Im}(z) \leq \frac{\sqrt{3}r}{2}\}$. The set $B_r$ denotes the Eisenstein integers enclosed in the rectangle centered at the origin with the real part bounded by $r$ and the imaginary part bounded by $\sqrt{3}r/2$. 

May 24, 2017 DRAFT
Consider the following one-to-one mapping \( g : V \to \mathbb{Z}[\omega] \cap \mathbb{B}_r \) between vertices of the graph and Eisenstein integers. For each \( v \in V \), \( g(v) \) denotes the corresponding vertex in the Eisenstein graph. Note that

\[
V = \{ g^{-1}(z) : z \in \mathbb{Z}[\omega] \cap \mathbb{B}_r \}. \tag{5}
\]

Consider the function \( f(a + b\omega) = (a + b) \mod 3 \). This partitions the space of Eisenstein integers into three cosets represented by \( \Omega_{sq}, \Omega_{cir}, \Omega_{dia} \) corresponding to \( f(z) = 0, f(z) = 1 \) and \( f(z) = 2 \) for all \( z \in \mathbb{Z}[\omega] \). The subscripts of \( \Omega_{sq}, \Omega_{cir}, \Omega_{dia} \) correspond to the squares, circles and diamonds which are used to represent the respective cosets in Figure 3. For any \( z \in \mathbb{Z}[\omega] \cap \mathbb{B}_r \), we define the following set of edges \( \Delta(z) \) forming a triangle between the vertices \( z, z + \omega \) and \( z + \omega + 1 \),

\[
\Delta(z) = \{(z, z + \omega), (z, z + \omega + 1), (z + \omega, z + \omega + 1)\}.
\]

Let us assume that the edges \( \Delta(z) \) where \( z \in \Omega_{sq} \) correspond to the intra-cell interference. We ignore intra-cell interference in our model, and hence we define the set of interfering edges in the graph as

\[
E = \{(u, v) : v \in V \text{ and } (g(u), g(v)) \in D\}, \tag{6}
\]

where,

\[
D = \{\Delta(z) : z \in \{\Omega_{cir} \cup \Omega_{dia}\}\}.
\]

Thus the interference graph is \( G(V, E) \) where \( V \) is given by (5) and the set of edges \( E \) is given by (6).

![Fig. 3: Eisenstein integers.](image)

**D. Message Assignment**

For each \( i \in [K] \), let \( W_i \) be the message intended for receiver \( i \), and \( T_i \subseteq [K] \) be the transmit set of receiver \( i \), i.e., those transmitters with the knowledge of \( W_i \). The transmitters in \( T_i \) cooperatively transmit the message \( W_i \) to the receiver \( i \). Let \( X_{i,j} \) represent the transmit signal at transmitter \( i \) corresponding to the encoding of the message \( W_j \) for some node \( j \), where \( i \in T_j \). A particular message assignment is denoted by \( \{T_i\}_{i \in [K]} \). For
a particular message assignment, $M$ denotes the maximum transmit set size and $B$ denotes the backhaul load or the average transmit set size,

\begin{equation}
M = \max_i |T_i|,
\end{equation}

\begin{equation}
B = \frac{\sum |T_i|}{K}.
\end{equation}

In this work, we allow for flexible association of messages, i.e., we only restrict the size of transmit sets, without constraints on the specific set of transmitters that each message is assigned to. The case $M = 1$ corresponds to the case of no cooperation, but with possibly a flexible association of cells. The case $B = 1$ corresponds to an average backhaul load of one message per transmitter, i.e., no extra backhaul load due to cooperation.

\section*{E. Local Cooperation}

We say that the local cooperation constraint is satisfied, if and only if there exists a function $r(K)$ such that $r(K) = o(K)$, and for every $K \in \mathbb{Z}^+$, the transmit sets used for the $K$-user channel satisfy the following:

\begin{equation}
T_i \subseteq \{i - r(K), i - r(K) + 1, \ldots, i + r(K)\}, \forall i \in [K].
\end{equation}

\section*{F. Zero-forcing (Interference Avoidance) Schemes}

We consider in this work the class of interference avoidance schemes, where each message is either not transmitted or allocated one degree of freedom. Accordingly, every receiver is either active or inactive. An active receiver does not observe any interfering signals.

\section*{G. Degrees of Freedom}

Let $P$ be the average transmit power constraint at each transmitter, and let $\mathcal{W}_i$ denote the alphabet for message $W_i$. Then the rates $R_i(P) = \frac{\log |\mathcal{W}_i|}{n}$ are achievable if the decoding error probabilities of all messages can be simultaneously made arbitrarily small for a large enough coding block-length $n$, and this holds for almost all channel realizations. The degrees of freedom (DoF) $d_i$, $i \in [K]$, is defined as $d_i = \lim_{P \to \infty} \frac{R_i(P)}{\log P}$. The DoF region $\mathcal{D}$ is the closure of the set of all achievable DoF tuples. The total DoF ($\eta$) is the maximum value of the sum of the achievable degrees of freedom, $\eta = \max_{\mathcal{D}} \sum_{i \in [K]} d_i$.

For a $K$-user channel, we define $\eta(K, M)$ and $\eta^{\text{avg}}(K, B)$ as the maximum achievable $\eta$ over all possible message assignments satisfying the constraints (7) and (8) respectively. We define the following asymptotic quantities which capture how $\eta$ scales with $K$.

\begin{equation}
\tau(M) = \lim_{K \to \infty} \frac{\eta(K, M)}{K}
\end{equation}

\begin{equation}
\tau^{\text{avg}}(B) = \lim_{K \to \infty} \frac{\eta^{\text{avg}}(K, B)}{K}
\end{equation}

We use the superscript $zf$ to indicate a further restriction to zero-forcing or interference avoidance schemes. We denote the asymptotic per user DoF for locally connected networks with connectivity parameter $L$ with subscript $L$, and for the hexagonal cellular network with subscript $c$. 

May 24, 2017 DRAFT
III. PROOF TECHNIQUES

Before discussing the results we have for the above introduced network models, we provide in this section a brief overview of the main arguments used in the achievability and converse proofs throughout this work.

A. Message Assignments and Coding Schemes

For the considered system model, a proof of achievability involves a choice of assigning messages to transmitters, and a transmission scheme that indicates coding and scheduling decisions. We employ interference-aware message assignments that divide the network into subnetworks of optimal size, where each subnetwork consists of a fixed number of transmitter-receiver pairs. The messages destined for receivers in a subnetwork can only be assigned to transmitters within the same subnetwork. Hence, all of our message assignments satisfy the local cooperation constraint in (9). We then employ a zero-forcing scheme that guarantees complete interference cancellation for all active receivers within a subnetwork. Because we assume that all channel coefficients are drawn from a continuous joint distribution, thereby ensuring that the probability of any specific realization is zero, cancelling a message’s interference at a number $n$ of undesired receivers would require assigning this message at $n$ transmitters other than the transmitter delivering the message. Hence, the backhaul load constraint induces a constraint on the number of receivers at which each message’s interference can be canceled, and accordingly, a constraint on the size of each subnetwork.

In [7], a cooperation constraint the limits the maximum transmit set size was considered. The asymptotic per user DoF was then characterized for Wyner’s linear asymmetric network and achievable puDoF values were presented for other network models. Here, we employ the results obtained in [7] by using convex combinations of the schemes that are optimal under a maximum transmit set size constraint, in order to obtain a scheme that is optimal under the considered average transmit set size constraint. By a convex combination of schemes, we refer to employing each of the schemes in a part of the network that consists of a number of successive transmitter-receiver pairs, and that number equals a fraction of the total number of users, with the sum of these fractions equaling unity.

It is interesting to observe that local cooperation combined with one-shot zero-forcing schemes can be used to achieve significant scalable DoF gains in large networks. Further, these gains can be achieved with no or minimal extra load on the backhaul, since deactivating few transmitters not only helps with avoiding interference and splitting the network into small subnetworks, but also releases backhaul resources that could be used to assign active messages at more than one transmitter. Finally, it is worth noting that even though we only capture the sum rate through the asymptotic per user DoF criterion, fairness between the users could be achieved through a fractional reuse mechanism where user indices are shifted across time or frequency slots.

B. Converse Proofs

The converse proofs presented in this work rely on two fundamental results, that were proved in [7]. First, we use Lemma 4 from [7], which we restate below. For any set of receiver indices $A \subseteq [K]$, define $U_A$ as the set of indices of transmitters that exclusively carry the messages for the receivers in $A$, and its complement $\bar{U}_A$. More precisely, $U_A = \cup_{i \notin A} T_i$. 

May 24, 2017 DRAFT
Lemma 1 ([7, Lemma 4]): If there exists a set $\mathcal{A} \subseteq [K]$, a function $f_1$, and a function $f_2$ whose definition does not depend on the transmit power constraint $P$, and $f_1(\mathcal{Y}_A, \mathcal{X}_U_A) = X_{U_A} + f_2(Z_A)$, then the sum DoF $\eta \leq |\mathcal{A}|$.

What Lemma 1 implies is that if there is a centralized decoder that has access to all the received signals $\mathcal{Y}_A$, and a reliable communication scheme is used, then this decoder would be able to decode all the $K$ messages, and hence, the DoF is bounded by the number of signals used for decoding $|\mathcal{A}|$. First, the centralized decoder would be able to decode the messages $W_A$ because we assumed that the communication scheme is reliable. The transmit signals $X_{U_A}$ can then be reconstructed, since their reconstruction solely relies on the messages $W_A$. Using the functions $f_1$ and $f_2$ in the statement of the lemma, the remaining transmit signals can then be reconstructed. Finally, using the knowledge of all the transmit signals, all the messages can be recovered with a vanishing error probability.

The second concept that we borrow from [7] is that of irreducible message assignments. By reducibility, we refer to the possibility of removing a message assignment to one or more transmitters without affecting the sum rate, regardless of the choice of the coding scheme. A message assignment to a transmitter can only be useful either for delivering the message to its intended receiver, or for aiding in cancelling the message’s interference at an unintended receiver. If it guaranteed that neither functions can be achieved through a given assignment, then this assignment can be removed without affecting the sum rate. As a simple example for this argument, when cooperation is not allowed, any irreducible message assignment will have each message assigned only to one of the transmitters connected to its intended destinations.

When deriving a converse under the backhaul load (average transmit set size) constraint, we combine the above two concepts with concentration inequalities that allow us to infer from the backhaul load constraint facts about the existence of a number of messages whose transmit set sizes can be bounded by a maximum value, and have guarantees on that number. This step simplifies the combinatorial aspect of the problem by allowing us to restrict our attention to a narrower class of possible message assignments.

IV. WYNER INTERFERENCE NETWORK

In [7], each transmit set size was bounded by a cooperation order constraint $M$, i.e., $|T_i| \leq M, \forall i \in [K]$. The DoF achieving coding scheme was then characterized for every value of $M$. We now consider the problem with an average transmit set size constraint $B$ and show that the per user DoF $\tau_{\text{asy}}(B)$ can be achieved using a combination of the schemes that are characterized as optimal in [7] for the cases of $M = 2B - 1$ and $M = 2B$.

We now understand from this result that even though the maximum transmit set size constraint may not reflect a physical constraint, the solutions in [7] provide a set of tools that can be used to achieve the optimal per user DoF value under the more natural constraint on the total backhaul load that is considered in this work.

A. Example: $B = 1$

Before introducing the main result, we illustrate through a simple example how the potential flexibility in the backhaul design according to the constraint in (8) can offer DoF gains over a traditional design where all messages are assigned to the same number of transmitters. We know from [7] that an asymptotic per user DoF greater than $\frac{2}{3}$ cannot be achieved through assigning each message to one transmitter. We now show
that \( \tau_{\text{avg}}(B = 1) \geq \frac{3}{4} \), by allowing few messages to be available at more than one transmitter at the cost of not transmitting other messages. Consider the following assignment of the first four messages, \( T_1 = \{1, 2\}, T_2 = \{2\}, T_3 = \emptyset, \) and \( T_4 = \{3\} \). Note that the backhaul load constraint \( B = 1 \) is respected, because \( \sum_{i=1}^{4} |T_i| = 1 \). Message \( W_1 \) is transmitted through \( X_1 \) to \( Y_1 \) without interference. Since the channel state information is known at the second transmitter, the transmit beam for \( W_1 \) at \( X_2 \) can be designed to cancel the interference caused by \( W_1 \) at \( Y_2 \), and then \( W_2 \) can be transmitted through \( X_2 \) to \( Y_2 \) without interference. Finally, \( W_4 \) is transmitted through \( X_3 \) to \( Y_4 \) without interference. It follows that the sum DoF for the first four messages \( \sum_{i=1}^{4} d_i \geq 3 \). Since the fourth transmitter is inactive, the subnetwork consisting of the first four users does not interfere with the rest of the network, and hence, we can see that \( \tau_{\text{avg}}(B = 1) \geq \frac{3}{4} \) through a similar assignment of messages in each consecutive 4-user subnetwork. We illustrate this example in Figure 4.

![Figure 4: Achieving 3/4 per user DoF with a backhaul load constraint \( B = 1 \). The figure shows only signals corresponding to the first subnetwork in a general \( K \)-user network. The signals in the dashed boxes are deactivated.](image)

### B. Main Result

We now characterize the asymptotic per user DoF \( \tau_{\text{avg}}(B) \) for any integer value of the backhaul load constraint.

**Theorem 1:** The asymptotic per user DoF \( \tau_{\text{avg}}(B) \) is given by,

\[
\tau_{\text{avg}}(B) = \frac{4B - 1}{4B}, \forall B \in \mathbb{Z}^+.
\]  

**Proof:** We provide the proof for the lower bound here; the proof of the upper bound is presented in the appendix. We treat the network as a set of subnetworks, each consisting of consecutive 4\( B \) transceivers. The last transmitter of each subnetwork is deactivated to eliminate *inter-subnetwork* interference. It then suffices to show that a DoF of \( 4B - 1 \) can be achieved in each subnetwork. Without loss of generality, consider the cluster of users with indices in the set \([4B]\). This is illustrated for \( B = 2 \) in Figure 5. We define the following subsets of \([4B]\),

\[
S_1 = [2B], \quad S_2 = \{2B + 2, 2B + 3, \ldots, 4B\}.
\]
We next show that each user in $S_1 \cup S_2$ achieves one degree of freedom, while message $W_{2B+1}$ is not transmitted. Let the message assignments be as follows,

$$T_i = \begin{cases} 
{i, i+1, \ldots, 2B}, & \forall i \in S_1, \\
{2B+1, 2B+2, \ldots, i-1}, & \forall i \in S_2,
\end{cases}$$

and note that $\sum_{i=1}^{4B} |T_i| = B$, and hence, the constraint in (8) is satisfied. Now, due to the availability of channel state information at the transmitters, the transmit beams for message $W_i$ can be designed to cancel its effect at receivers with indices in the set $C_i$, where,

$$C_i = \begin{cases} 
{i+1, i+2, \ldots, 2B}, & \forall i \in S_1, \\
{2B+2, 2B+3, \ldots, i-1}, & \forall i \in S_2.
\end{cases}$$

Note that both $C_{2B}$ and $C_{2B+2}$ equal the empty set, as both $W_{2B}$ and $W_{2B+2}$ do not contribute to interfering signals at receivers in the set $Y_{S_1} \cup Y_{S_2}$. The above scheme for $B = 2$ is illustrated in Figure 5. We conclude that each receiver whose index is in the set $S_1 \cup S_2$ suffers only from Gaussian noise, thereby enjoying one degree of freedom. Since $|S_1 \cup S_2| = 4B - 1$, it follows that $\sum_{i=1}^{4B} d_i \geq 4B - 1$. Using a similar argument for each following subnetwork and noting that the last transmitter in each subnetwork is inactive to eliminate inter-subnetwork interference, we establish that $\tau_{\text{avg}}(B) \geq \frac{4B-1}{4B}$, thereby proving the lower bound in Theorem 1.

![Fig. 5: Achieving 7/8 per user DoF with a backhaul load constraint $B = 2$. The figure shows only signals corresponding to the first subnetwork in a general $K$-user network. The signals in the dashed boxes are deactivated.](image)

We note that the local cooperation constraint of (9) is satisfied, when we use the illustrated message assignment. In other words, the network can be split into subnetworks, each of size $4B$, and the messages corresponding to users in a subnetwork can only be assigned to transmitters with indices in the same subnetwork. Few remarks are now in order.
Remark 1: Although we assume availability of all channel coefficients at every transmitter in the network, the achievable schemes used require only local channel state information.

Remark 2: In the proposed achievable schemes, some messages are being sent interference free at the expense of other messages not being transmitted. Fairness can be maintained in the allocation of the available DoF over all users through fractional reuse by deactivating different sets of receivers in different sessions, e.g., in different time or frequency slots.

Remark 3: The result of Theorem 1 can be achieved by a convex combination of the schemes in [7] for \( M = 2B - 1 \) and \( M = 2B \) in the ratio \( 4B - 1 : 4B + 1 \). Hence, even though the maximum transmit set size constraint that is used in the analysis of [7] may not reflect a practical setting, but the obtained optimal schemes can be used to obtain the optimal scheme in the considered setting, where the more practical backhaul load constraint is considered.

Remark 4: The coding scheme used to prove the lower bound part of Theorem 1 is similar to the scheme introduced in [5], as it relies on deactivating appropriately selected transmitters to maximize the number of interference free links. Mitigating interference among the remaining users is carried out through dedicating each assignment of a message to a transmitter either for message delivery at its intended destination, or for cancelling the message’s interference at a single unintended destination. However, the scheme in [5] relies on the dirty paper coding scheme introduced in [10], while here we are using zero-forcing transmit beamforming. We note that replacing zero-forcing transmit beamforming with dirty paper coding in the presented scheme would lead to the same DoF result.

V. LOCALLY CONNECTED LINEAR NETWORKS

We saw in Section III that the use of a convex combination of the schemes that are optimal under the maximum transmit set size constraint can lead to optimal schemes under backhaul load constraints. We now show that this approach can also provide good achievable schemes for the more general locally connected channel model.

We first present the schemes for locally connected channels from [7, Theorem 4] under a maximum transmit set size constraint \( M \), along with the backhaul load required by these schemes.

Theorem 2: [7, Theorem 4] Under the maximum transmit set size constraint \( M \), the following lower bound holds for the asymptotic per user DoF,

\[
\tau_L(M) \geq \frac{2M}{2M + L},
\]

with corresponding backhaul load

\[
B = \frac{M(M + 1)}{2M + L}.
\]

Proof: Let the network be divided into subnetworks consisting of \( 2M + L \) consecutive tranceivers. The last \( L \) transmitters in each subnetwork are deactivated to eliminate inter-subnetwork interference. It suffices to show that the sum DoF of \( 2M \) can be achieved in each subnetwork. We explain the scheme only for the first subnetwork and it follows similarly for the remaining subnetworks. In the scheme, the messages \( W_{M+1}, W_{M+2}, \ldots, W_{M+L} \) are not transmitted and the remaining messages are sent interference free. The users
in the subnetwork are further divided into two clusters. Cluster $S_1$ consists of the first $M$ users, and cluster $S_2$ consists of the last $M$ users. Denote them as the following subsets of $[2M + L]$:

$$S_1 = [M],$$
$$S_2 = \{L + M + 1, L + M + 2, ..., L + 2M\}.$$  

The message assignments are as follows.

$$T_i = \begin{cases} 
\{i, i + 1, ..., M\}, & \forall i \in S_1, \\
\{M + 1, M + 2, ..., i - L\}, & \forall i \in S_2,
\end{cases}$$

and hence there is no interference between clusters $S_1$ and $S_2$. The strategy for transmitting messages of users in $S_1$ interference free is demonstrated for $W_1$. The first receiver does not observe interference from any transmitter and hence the message $W_1$ is encoded into $X_{1,1}$. The interference seen at receiver with index 2 due to $W_1$ needs to be cancelled, and this is achieved by setting $X_{2,1} = \frac{-H_{2,1}}{H_{1,1}}X_{1,1}$. Similarly the beams $X_{i,1}$ where $i \in \{3, ..., M\}$ are successively designed with respect to order of the index $i$ such that the received signal due to $X_{1,1}$ at the $i^{th}$ receiver cancels the interference caused by $W_1$.

In general, the availability of channel state information at the transmitters allows a design for the transmit beams for message $W_i$ that delivers it to the $i^{th}$ receiver with a capacity achieving point-to-point code and simultaneously cancels its effect at receivers with indices in the set

$$C_i = \begin{cases} 
\{i + 1, i + 2, ..., M\}, & \forall i \in S_1, \\
\{L + M + 1, L + M + 2, ..., i - 1\}, & \forall i \in S_2.
\end{cases}$$

Note that both $C_M$ and $C_{M+1}$ equal the empty set, because both $W_M$ and $W_{L+M+1}$ do not contribute to interfering signals at receivers with indices in the set $S_1 \cup S_2$. Thus each receiver with index in the set $S_1 \cup S_2$ enjoys one degree of freedom.

We now consider the average backhaul load incurred by this scheme. In each subset of $[2M + L]$, from the message assignments considered, $|T_i| = M - i + 1$ when $i \in S_1$, and $|T_i| = i$ when $i \in S_2$ and the remaining messages are not assigned. Hence for each $M$, this scheme has an average backhaul load of $B = \frac{2(1+...,M)}{2M+L} = \frac{M(M+1)}{2M+L}$.

Recall that $\tau_L^{avg}(B)$ is the asymptotic per user DoF for a locally connected channel defined in (3) with connectivity parameter $L$. We use a convex combination of the schemes that are characterized as optimal in [7] and described in the above theorem to achieve the inner bounds as stated in corollary 1.

In [7], it was shown for $L \geq 2$, that $\tau_L(M = 1) \leq 1/2$. We now show that this upper bound is achievable with a one-shot scheme if we allow for a flexible assignment.

**Corollary 1:** Under the average backhaul constraint $B = 1$, the following lower bound holds for the asymptotic per user DoF, when $2 \leq L \leq 6$

$$\tau_L^{avg}(B = 1) \geq 1/2.$$  

**Proof:** We show that $\tau_L^{avg}(B = 1) \geq 1/2$ for $2 \leq L \leq 6$, with exact inner bounds given in Table I. We use a convex combination of the schemes described in Theorem 2 for the maximum transmit set size constraint $M$ to achieve the corresponding per user DoF while satisfying an average backhaul load of unity ($B = 1$).
For $L = 2$, the scheme described in Theorem 2 for the maximum transmit set size constraint $M = 2$ achieves a per user DoF of $2/3$ while satisfying an average backhaul load of unity. For $L = 3$, a convex combination of the scheme with $M = 2$ and $B = 6/7$ achieving a per user DoF of $4/7$ and the scheme with $M = 3$ and $B = 4/3$ achieving a per user DoF of $2/3$ in the ratio $7 : 3$, achieves a per user DoF of $3/5$ in the network while maintaining $B = 1$. For $L = 4$, a convex combination of the scheme with $M = 2$ and $B = 3/4$ achieving a per user DoF of $1/2$ and the scheme with $M = 3$ and $B = 6/5$ achieving a per user DoF of $3/5$ in the ratio $4 : 9$, achieves a per user DoF of $5/9$ in the network while maintaining $B = 1$. For $L = 5$, a convex combination of the scheme with $M = 2$ and $B = 2/3$ achieving a per user DoF of $4/9$ and the scheme with $M = 3$ and $B = 12/11$ achieving a per user DoF of $6/11$ in the ratio $3 : 11$, achieves a per user DoF of $11/21$ in the network while maintaining $B = 1$. For $L = 6$, the scheme described in Theorem 2 for the maximum transmit set size constraint $M = 3$ achieves a per user DoF of $1/2$ while satisfying $B = 1$.

| Scheme 1 | Scheme 2 | Ratio | $\tau_L(B = 1) \geq$ |
|----------|----------|-------|------------------|
| $L = 2$  | $\tau_L(M = 2) \geq 2/3$, $\tau_L^{\text{avg}}(B = 1) \geq 2/3$ | -     | -               |
| $L = 3$  | $\tau_L(M = 2) \geq 2/3$, $\tau_L(B = 6/7) \geq 4/7$ | $\tau_L(M = 3) \geq 4/7$, $\tau_L^{\text{avg}}(B = 4/3) \geq 7/3$ | $7 : 3$ |
| $L = 4$  | $\tau_L(M = 2) \geq 2/3$, $\tau_L(B = 3/4) \geq 2/3$ | $\tau_L(M = 3) \geq 2/3$, $\tau_L^{\text{avg}}(B = 6/5) \geq 4/3$ | $4 : 9$ |
| $L = 5$  | $\tau_L(M = 2) \geq 2/3$, $\tau_L(B = 2/3) \geq 2/3$ | $\tau_L(M = 3) \geq 2/3$, $\tau_L^{\text{avg}}(B = 12/11) \geq 6/11$ | $3 : 11$ |
| $L = 6$  | $\tau_L(M = 3) \geq 2/3$, $\tau_L(B = 1) \geq 2/3$ | -     | -               |

TABLE I: Achievable per user DoF values for locally connected channels with a backhaul constraint $\sum_{i=1}^{K} |T_i| \leq K$. For each $L$, the schemes used and the convex combination in which they are used are shown.

Now, we note that the lower bounds stated in Table I are achieved through the use of only zero-forcing transmit beamforming. In other words, there is no need for the symbol extension idea required by the asymptotic interference alignment scheme of [8]. In [7, Theorem 8], it was shown that for $L \geq 2$, by allowing each message to be available at one transmitter, the asymptotic per user DoF is $1/2$; it was also shown in [7, Theorem 6] that the $1/2$ per user DoF value cannot be achieved through zero-forcing transmit forming for $L \geq 3$. In contrast, it can be seen in Table I that for $L \leq 6$, the $1/2$ per user DoF value can be achieved through zero-forcing transmit beamforming and a flexible design of the backhaul links, without incurring additional overall load on the backhaul ($B = 1$).

VI. CELLULAR NETWORK

For linear networks with $L = 1, 2, 3, 4, 5, 6$, we have shown that compared to the case of no cooperation ($M = 1$), a larger per user DoF can be achieved with delay-free one-shot schemes under an average backhaul load $B = 1$, i.e., without incurring additional backhaul load. We now investigate whether these results hold for the hexagonal sectored cellular model introduced in Section II-C. Our goal is to highlight the advantage of cooperative transmission that is based on flexible cell associations for cellular networks, by first showing that the asymptotic per user DoF is at most $1/2$ for the case when each message can be available at a single transmitter. Further, we show for this case that interference avoidance schemes can only be used to achieve an asymptotic per user DoF of at most $3/7$. On the other hand, when cooperative transmission is allowed, but the
overall load on the backhaul is not increased \((B = 1)\), interference avoidance schemes can be used to achieve the \(\frac{1}{2}\) asymptotic per user DoF value.

A. Example: Two-dimensional Wyner network

Through this example, we illustrate that the insights for linear interference networks, may apply in denser networks by treating the denser network as a set of interfering linear networks. We consider the two-dimensional network described in Section II-B.

Under the backhaul load constraint \(\sum_{i=1}^{\sqrt{K}} |T_i| / K \leq 1\), a per user DoF of \(\frac{5}{9}\) can be achieved using only zero-forcing transmit beamforming. This can be done by deactivating every third row of transmitters, and splitting the rest of the network into non-interfering linear subnetworks (see Figure 6). In each subnetwork, a backhaul load constraint of \(\frac{3}{2}\) is imposed. For example, the following constraint is imposed on the first row of users, \(\sum_{i=1}^{\sqrt{K}} |T_i| / \sqrt{K} \leq \frac{3}{2}\). A convex combination of the schemes described in Theorem 2 with \(M = 2, B = 6/5\) and per user DoF of 4/5, and with \(M = 3, B = 12/7\) and per user DoF of 6/7, is then used to achieve a per user DoF of \(\frac{5}{6}\) in each active subnetwork while satisfying a backhaul load constraint of \(\frac{3}{2}\). Since \(\frac{2}{3}\) of the subnetworks are active, a per user DoF of \(\frac{5}{6}\) is achieved while satisfying a backhaul load constraint of unity. We observe that a per user DoF greater than 1/2 is achieved by using simple zero-forcing schemes, with an average backhaul load of one without the need for interference alignment.

B. Flexible Cell Association for Hexagonal Cellular Network

Here, we impose the maximum transmit set size constraint of \(M = 1\) in the network, i.e., a message of a cell edge mobile receiver can be assigned to any single basestation transmitter, thus leading to a flexible cell association in the cellular downlink.

We present the following lemma from [7] for the case of \(M = 1\), which gives a relation between the DoF of the message being delivered by a transmitter and the DoF corresponding to the messages of the users connected to that transmitter. Here, \(R_j\) denotes the set of receivers that are connected to transmitter \(X_j\). The lemma serves as a building block for the proof of \(\tau_c(M = 1) \leq \frac{1}{2}\).

**Lemma 2 ([7, Lemma 5]):** If \(T_i = \{X_j\}\), then \(d_i + d_k \leq 1, \forall k \in R_j\).

![Fig. 6: Two dimensional interference network. We show an example coding scheme where dashed red boxes and lines represent inactive nodes and edges. The signals \(\{X_1, \ldots, X_{\sqrt{\mathcal{P}}}\}\) and \(\{Y_1, \ldots, Y_{\sqrt{\mathcal{P}}}\}\) form a linear subnetwork. Similarly, the signals \(\{X_{\sqrt{\mathcal{P}}+1}, \ldots, X_{2\sqrt{\mathcal{P}}}\}\) and \(\{Y_{2\sqrt{\mathcal{P}}+1}, \ldots, Y_{3\sqrt{\mathcal{P}}}\}\) form a linear subnetwork.](image-url)
Each transmitter-receiver pair in the network is referred to as a node. If \( a \) and \( b \) are two nodes such that they are connected in the connectivity graph, and the transmitter of node \( a \) has the message for node \( b \), i.e., \( a \in \mathcal{T}_b \), we denote this by \( a \rightarrow b \).

**Theorem 3:** For the considered hexagonal cellular network model, the following bound holds for the case of no-cooperation,

\[
\tau_c(M = 1) \leq \frac{1}{2}.
\]

**Proof:** Consider the division of the network into triangles \( \mathcal{D} = \{\Delta(z) : z \in \Omega_{cir}\} \) as shown in Figure 7.

For any \( z \in \Omega_{cir} \), triangle \( \Delta(z) \) consists of vertices \( z, z + \omega, z + \omega + 1 \). Note that each triangle contains one vertex from each of the cosets, \( \Omega_{cir}, \Omega_{sq} \) and \( \Omega_{dia} \).

We say that a triangle is in state \( S_i \) if exactly \( i \) of the messages of the triangle are assigned to transmitters within the triangle, \( 0 \leq i \leq 3 \). Let \( S_i \) denote the set of all triangles in state \( S_i \) and \( S_S \) denote the set of all self serving nodes belonging to triangles in state \( S_1 \). We refer to a node as a self serving node if the message to the receiver corresponding to the node is assigned to its own transmitter. We refer to a node as an outsider node if no message within its triangle is assigned to its transmitter, and also its message is not assigned within its triangle. Let \( O \) denote the set of all outsider nodes. Note that every triangle in state \( S_0 \) consists of three outsider nodes, every triangle in state \( S_1 \) has at least one outsider node, and a triangle in state \( S_2 \) may contain an outsider node. We also define a middle triangle which is formed by the connected nodes of three different neighboring triangles.

Middle triangles are triangles of the form \( \{\Delta(z) : z \in \Omega_{dia}\} \). We say that a triangle is associated with a node if the node belongs to the triangle. If \( z \in \Omega_{dia} \), the middle triangle associated with vertex \( z \) is \( \Delta(z) \). If \( z \in \Omega_{sq} \), the middle triangle associated with vertex \( z \) is \( \Delta(z - \omega) \). If \( z \in \Omega_{cir} \), the middle triangle associated with vertex \( z \) is \( \Delta(z - \omega - 1) \). For any vertex \( a \), we denote the middle triangle associated with it as \( M_a \).

Note that each vertex is associated with exactly one main triangle and one middle triangle. We note that the definition of an outsider node is with respect to the main triangle associated with the node and not the middle triangle associated with it.

Let \( \tau_S \) denote the per user DoF for the messages with indices in some set \( S \). We present Algorithm 1, to define a strategy for including nodes in a set \( S \), such that at any stage, the per user DoF of the nodes already included in \( S \) is upper bounded by \( 1/2 \) i.e., \( \tau_S \leq 1/2 \). Note that at the end of the algorithm, all nodes are included in \( S \). To facilitate the understanding of Algorithm 1, we observe the following:
Algorithm 1

1: Initialize $S \leftarrow \phi$

2: while $SS_1 \setminus S \neq \phi$ do

3: for $a \in SS_1$ where $a \in \Delta(z)$ for some $z \in \Omega_{cir}$ do

4: $S \leftarrow S \cup \{a, j\}$ where $j = \min_{x \in \Delta(z) \setminus \{a\}} \Re(x)$

5: end for

6: end while

7: while $O \setminus S \neq \phi$ do

8: for $a \in O \setminus S$ where $a \in \Delta(z)$ for some $z \in \Omega_{cir}$ and the associated middle triangle $M_a$ contains nodes $b$ and $c$ apart from $a$. do

9: if $M_a \setminus S$ contains 3 outsider nodes then

10: $S \leftarrow S \cup \{a, b, c\}$

11: else if $M_a \setminus S$ contains 2 outsider nodes $a$ and $j$ where $j \in \{b, c\}$ then

12: $S \leftarrow S \cup \{a, j\}$

13: else if $M_a \setminus S$ contains $a$ as the only outsider node and message for $a$ is assigned within $M_a \setminus S$ at $j \in \{b, c\}$, i.e., $j \rightarrow a$ then

14: $S \leftarrow S \cup \{a, j\}$

15: else if $M_a \setminus S$ contains $a$ as the only outsider node and message for $a$ is not assigned within $M_a \setminus S$ then

16: $S \leftarrow S \cup \{a\}$

17: end if

18: end for

19: end while

20: while $S_1 \cup S_2 \cup S_3 \setminus S \neq \phi$ do

21: for triangle $T \in S_1 \cup S_2 \cup S_3$ do

22: $S \leftarrow S \cup T \setminus S$

23: end for

24: end while

- If $a \in SS_1$, then $a$ is a self-serving node and since the triangle $T$ associated with it is in state $S_1$, the other nodes in the triangle $b, c$ are outsider nodes. We have $d_a + d_b \leq 1$ and $d_a + d_c \leq 1$, according to Lemma 2. Without loss of generality, we include the node with minimum real value among the two nodes $b, c$ in the set $S$ as in line 4.

- If $M_a \setminus S$ where $M_a$ is a middle triangle, contains 3 outsider nodes, we include the nodes of that middle triangle $a, b, c$ in the set $S$ as in line 10. If $M_a \setminus S$ contains only two outsider nodes $a, j$, where $j \in \{b, c\}$, we include them in the set $S$ as in line 12.

This follows because for any middle triangle with nodes $a, b, c$, containing at least two outsider nodes, we show that $d_a + d_b \leq 1$, $d_b + d_c \leq 1$, $d_a + d_c \leq 1$ and hence $d_a + d_b + d_c \leq 3/2$. Without loss of generality,
let the two outsider nodes be a and b. For each of these nodes, we have the following possibilities.

- The message $W_a$ is not available at either b or c. From our assumption, $W_a$ is not available at neighboring nodes outside the triangle. Hence, $W_a$ cannot be transmitted and we have $d_a = 0$.
- The message $W_a$ is available at one vertex in b or c. From lemma 2, we have $d_a + d_c \leq 1$ and $d_a + d_b \leq 1$.

Considering the same for b, we either have $d_b = 0$ or $d_a + d_b + d_c \leq \frac{3}{2}$.

Although for any middle triangle with at least two outsider nodes, the per user DoF is upper bounded by $1/2$, we do not include the third node in the set $S$ in line 12 in order to simplify the cases considered later.

- Let a be the only outsider node in $M_a \setminus S$, where $M_a$ is the middle triangle. If its message $W_a$ is available at neighboring node $j \in M_a \setminus S$ where $j \in \{b, c\}$, i.e., $j \rightarrow a$, then we have $d_j + d_a \leq 1$ and include nodes $a, j$ in the set $S$ as in line 14.
- In the middle triangle $M_a$, if $W_a$ is not assigned within nodes b, c, we have $d_a = 0$ and we include a in the set $S$ as in line 16.

We now consider the case where the message $W_a$ is assigned to a node in the set $M_a \cap S$ and show that $\tau_S \leq 1/2$ when we add only the node a in the set $S$. Suppose $j \rightarrow a$ where $j \in \{b, c\}$ but $j \in S$. This case is shown in Figure 8 for $j = c$. So far, we have only added all outsider nodes in a few middle triangles and nodes from self-serving triangles. Hence this is possible only when $j$ was included in $S$ according to line 4 in the algorithm. Without loss of generality, let $j$ be the self serving node and $m$ be the outsider node which was included in line 4. We have $d_j + d_a \leq 1$, $d_m + d_a \leq 1$ and we have $d_j + d_m \leq 1$ from before. Note that we have $d_m + d_a \leq 1$ from Theorem 1 since $T_a = \{j\}$ and $m \in R_j$. Hence a can be included without any increase in the per user DoF. The same argument holds even if $j$ was the outsider node and $m$ the self serving node included in line 4. Note that $j$ and $m$ could both contain messages for the only remaining outsider nodes a and k in their respective middle triangles. In that case we see that $d_j + d_a \leq 1$, $d_k + d_m \leq 1$ and $\tau_S \leq 1/2$ when $k$ is added later according to line 16.

![Fig. 8](image)

**Fig. 8:** We illustrate the case when a is the only outsider node in its middle triangle and its message is available at c where $c \in S$. The node c is a self-serving node and node m has been included according to line 4. The node m contains the message for the only outsider node k in the middle triangle containing m and k.

Consider all triangles in $S_1 \cup S_2 \cup S_3$. If T denotes such a triangle with nodes a, b, c, let t denote the set of nodes in T but not included in $S$ by line 19. For triangles in $S_2, S_3$ with nodes a, b, c, we have $d_a + d_b \leq$
$1, d_b + d_c \leq 1, d_a + d_c \leq 1$ and hence $d_a + d_b + d_c \leq 3/2$ from Lemma 2. Consider the following cases for any triangle $T$ that has one or more nodes in the set $t = T \setminus S$:

- The set $t$ contains only one node $a$. We first find two nodes $b, j$ where $b \rightarrow j$ that were previously added to $S$ according to line 14 and show that $d_j + d_a + d_b \leq 3/2$ holds. We then show that nodes $b$ and $j$ do not appear in any other such combination, and hence $\tau_S \leq 1/2$ after adding $a$ to $S$.

Note that by definition, a triangle in state $S_2$ or $S_3$ has at least two messages assigned within the triangle and thus has at least two non-outsider nodes. Hence, if $T \in S_2 \cup S_3$, there exists at least one node say $b$ such that $b$ is a non-outsider node and $d_a + d_b \leq 1$. We have the same conclusion if $T \in S_1$, since all the self serving nodes and outsider nodes have already been included in $S$. Hence, it is either the case that $a \rightarrow b$ or $b \rightarrow a$.

Since $b$ was a non-outsider node that was previously considered, it must have been added according to line 14. Hence, there is an assignment $b \rightarrow j$ where $j$ is an outsider node in the middle triangle $M_b$, $j \in \{a, c\}$ and $d_b + d_j \leq 1$ was considered. We also have $d_b + d_j \leq 1$ and $d_a + d_j \leq 1$ from Lemma 2 since $T_j = \{b\}$ and $b, a \in R_b$. Hence we have $d_j + d_a + d_b \leq 3/2$.

Note that neither $j$ nor $b$ is part of any other such combination. This is true for $b$ because all the nodes in its triangle have already been considered. Since $b \rightarrow j$ and $j$ has been added to the set $S$ according to line 14, outsider node $j$ cannot be part of any such combination that does not involve $b$. Thus, we include $t = \{a\}$ in the set $S$ as in line 22 while $\tau_S \leq 1/2$.

- The set $t$ contains two nodes say $a, c$. If $T \in S_1$, then either $a \rightarrow c$ or $c \rightarrow a$ and we have $d_a + d_b \leq 1$. If $T \in S_2 \cup S_3$, we have $d_a + d_b \leq 1$ and we include $t = \{a, b\}$ in the set $S$ as in line 22.

- The set $t$ contains three nodes $a, b, c$. This can happen only when $T \in S_2 \cup S_3$. In this case, we have $d_a + d_b + d_c \leq 3/2$ and we include $t = \{a, b, c\}$ in the set $S$ as in line 22.

\begin{figure}[h]
\centering
\includegraphics{diagram}
\caption{Fig. 9}
\end{figure}

1) Illustrative Example: We demonstrate the algorithm on a simple network shown in Figure 9. The message assignment is given by $a_1 \rightarrow a_2, b_3 \rightarrow b_1, b_2 \rightarrow a_3, c_3 \rightarrow c_2, c_1 \rightarrow c_1$ and the remaining messages are not assigned. Initially, $S = \emptyset$. Note that $SS_1 = \{c_1\}$ and $O = \{a_3, b_2\}$. According to line 4, we have $S = \{c_1, c_2\}$. From Lemma 2, we have $d_{c_1} + d_{c_2} \leq 1$ and hence $\tau_S \leq 1/2$. We start the for loop in line 8 with $a_3$ and consider the middle triangle $M_{a_3}$. Note that $M_{a_3} \setminus S = \{a_3, b_2\}$ contains two outsider nodes $a_3, b_2$ and according to line 11, $S = \{a_3, b_2, c_1, c_2\}$. Since $b_2 \rightarrow a_3$, we have $d_{a_3} + d_{b_2} \leq 1$ from Lemma 2, and hence $\tau_S \leq 1/2$. For triangles $\Delta(a_2), \Delta(b_2) \in S_1$ and $\Delta(c_3) \in S_2$, we have $\Delta(a_2) \setminus S = \{a_1, a_2\},$
\(\Delta(b_2)\backslash S = \{b_1, b_3\}\) and \(\Delta(c_2)\backslash S = \{c_3\}\). We start the for loop in line 21 with \(\Delta(a_2)\). According to line 22, we have \(S = \{a_1, a_2, a_3, b_2, c_1, c_2\}\). Since \(a_1 \rightarrow a_2\), we have \(d_{a_1} + d_{a_2} \leq 1\) from Lemma 2, and \(\tau_{S} \leq 1/2\). Similarly for \(\Delta(b_2)\), according to line 22, we have \(S = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2\}\). Since \(b_3 \rightarrow b_1\), we have \(d_{b_1} + d_{b_2} \leq 1\) from Lemma 2, and \(\tau_{S} \leq 1/2\). Finally, for \(\Delta(c_2)\), we have \(S = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}\) according to line 22. Since the message for \(c_3\) is not assigned, \(d_{c_3} = 0\) and thus, all nodes are included in the set \(S\) in the end, and \(\tau_{S} \leq 1/2\).

This shows that using a traditional approach for interference management, the maximum asymptotic per user DoF for the considered hexagonal cellular network model is \(\frac{1}{2}\). Further, the only known way this DoF value can be approached is in the limit as the length of symbol extension goes to infinity as in the asymptotic interference alignment scheme of [8].

2) Zero-forcing schemes: We now restrict ourselves to the class of zero-forcing schemes and characterize lower and upper bounds for the maximum achievable per user DoF.

![Image](https://example.com/image.png)

**Fig. 10:** Division of network into triangular subnetworks in (a). In (b), we note that by deactivating square and diamond nodes, a per user DoF of \(\frac{1}{3}\) is achieved.

**Theorem 4:** The following bounds hold under restriction to zero-forcing schemes for the asymptotic per user DoF of hexagonal cellular networks with no cooperation,

\[
\frac{1}{3} \leq \tau_c^Z(M = 1) \leq \frac{3}{7}.
\]

**Proof:**

**Lower Bound:** Consider the division of the network into triangles \(D = \{\Delta(z) : z \in \Omega_{cir}\}\) as shown in Figure 10. For any \(z \in \Omega_{cir}\), triangle \(\Delta(z)\) consists of vertices \(z, z + \omega, z + \omega + 1\). By deactivating the nodes \(\{z : z \in \Omega_{sqr} \cup \Omega_{dia}\}\), i.e., the square and diamond nodes in each triangle, the network decomposes into \(K/3\) isolated nodes \(\{z : z \in \Omega_{cir}\}\) that each achieves a DoF of one, thus achieving a per user DoF of \(\frac{1}{3}\) in the network.

**Upper Bound:** Consider the division of the network discussed for the lower bound shown in Figure 10. For zero forcing (interference avoidance) schemes we note that if a transmitter is active, then only the intended receiver among the set of neighboring interferers can be active. And similarly if a receiver is active, then only the appropriate transmitter among its neighboring interferers can be active. Hence if there is a transmission within a triangle, exactly one transmitter and receiver can be active. However, if the transmitter of triangle \(A\) serves the
receiver of a neighboring triangle $B$, the remaining receivers and transmitters of triangles $A$ and $B$ respectively are inactive. Note that this transmission does not impose any constraints on the remaining transmitters and receivers of triangles $A$ and $B$ respectively. Thus, for any zero forcing scheme, a fully connected triangle in the network is in one of the following states.

State 0 (inactive triangle): All transmitters and receivers in the triangle are inactive.

State 1 (self-serving triangle): Exactly one transmitter in the triangle sends a message to exactly one receiver within the triangle. None of the other transmitters or receivers can be active in this triangle.

State 2 (serving triangle): At least one transmitter in the triangle is activated to serve a receiver in another triangle and there are no active receivers in the considered triangle.

State 3 (served triangle): At least one receiver in the triangle is activated as it is being served by a transmitter in another triangle and there are no active transmitters within the considered triangle.

For the triangles in state 1 and state 0, the number of active receivers is bounded by the number of triangles, i.e., a fraction of $\frac{1}{3}$ of the number of users.

For every transmitter $b$ that is active in a triangle, say $T_1$, that is in state 2, there exists a neighboring receiver $c$ in a different triangle say $T_2$ in state 3 that is being served by it and a neighboring node $a$ in a third triangle $T_3$, whose transmitter and receiver are both inactive. We have $d_a + d_b + d_c \leq 1$, because receivers with indices $a$ and $b$ are inactive and hence the per user DoF among the nodes $a, b, c$ is $\frac{1}{3}$. We now consider the following cases for the state of triangle $T_3$.

Case 1: $T_3$ is in state 2 or 3. We have $d_a + d_b + d_c \leq 1$ and the remaining neighbors of $a, b, c$ are the nodes in their own triangles. Further, because none of the nodes $a, b$ and $c$ have other neighbors except in their own triangles, there is no overcounting when we repeat this procedure to obtain DoF bounds on other similar sets of users.

Case 2: $T_3$ is in state 1. Suppose in $T_3$, there is a node $a_2$ which serves itself. Then there is another inactive node in $T_3$ which may form a group similar to $a, b, c$ with its neighbors from different triangles, say $b_1, c_1$. We note that these two groups are disjoint. This scenario is illustrated in Figure 11 for $b_1 \rightarrow c_1$. So among the seven nodes ($\{a_1, a_2, a, b, c, b_1, c_1\}$), there are at most three active receivers. Suppose $T_3$ does not contain a self-serving node. Then $a$ is the only node with inactive transmitter and receiver in $T_3$, and among the five nodes ($\{a, b, c, a_1, a_2\}$), we attain a sum DoF of at most two.

![Fig. 11](image)

Case 3: $T_3$ is in state 0. Then in the set of the five nodes ($T_3 \cup \{b, c\}$), we attain a sum DoF of at most one.
For any scheme, the network can be rearranged into a combination of disjoint groups of three, five and seven users, and the per user DoF for each group is at most $\frac{3}{7}$. It follows that $\tau \leq \frac{3}{7}$ holds asymptotically for any choice of cell associations and interference avoidance schemes.

\section{C. Flexible Message Assignment with Cooperation}

We now show through the result in Theorem 5, how a smart choice for assigning messages to transmitters, aided by cooperative transmission, can achieve scalable DoF gains through an interference avoidance coding scheme. We achieve this by treating the hexagonal model as interfering locally connected linear networks with $L = 2$. In particular, we show a lower bound on the achievable per user DoF that is greater than the $\frac{2}{3}$ upper bound of the case without cooperation, that was proved in Theorem 4.

\textbf{Theorem 5:} Under the average backhaul constraint $B = 1$, the following lower bound holds for the asymptotic per user DoF,

$$\tau_{\text{avg, zf}}^\text{avg, zf}(B = 1) \geq \frac{1}{2}.$$  \hspace{1cm} (18)

\textbf{Proof:} Consider a division of the network formed by deactivating a third of the nodes as shown in Figure 12a. We note that the remaining network consists of non-interfering locally connected subnetworks with connectivity parameter $L = 2$. In each subnetwork, we use the scheme in [7] for $M = 3$ that considers a division of the subnetwork into non-interfering blocks of eight nodes. The message assignment is shown in Figure 12b. This scheme achieves a per user DoF of $3/4$ with $B = 3/2$ in the locally connected linear subnetwork. Since the linear subnetworks only account for $2/3$ of the network, we obtain a per user DoF of $1/2$ with $B = 1$ in the entire network.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image.png}
\caption{(a) Division of cellular network into subnetworks, and (b) the message assignment in each subnetwork. The nodes, transmitters and receivers in red indicate that they are inactive.}
\end{figure}
VII. CONCLUSION

We studied the potential gains offered by cooperative transmission in the downlink of cellular networks, under an average backhaul load constraint. We first characterized the asymptotic per user DoF in the linear interference network and showed that the optimal coding scheme relies only on zero-forcing transmit beamforming. The optimal schemes rely on an asymmetric assignment of messages, such that the backhaul constraint is satisfied, where some messages are assigned to more than $B$ transmitters, others are assigned to fewer than $B$ transmitters, and the remaining messages are not assigned at all. Thus, the average backhaul constraint allows for higher degree of freedom gains compared to the maximum transmit set size constraint and hence we have $\tau_{avg}(B) > \tau(M)$. We then extended these results to more general and practically relevant networks, such as linear interference networks with higher connectivity and hexagonal sectored cellular networks. We showed that in locally connected linear networks $\tau_{avg, zf}^L(B = 1) > \tau_L(M = 1)$ for $L \leq 5$, and in hexagonal cellular networks $\tau_{avg, zf}^c(B = 1) > \tau_c(M = 1)$. The proposed schemes are simple zero-forcing schemes with a flexible message assignment that achieve the information theoretic upper bound of the per user DoF for the case of no-cooperation with an average backhaul load of one message per transmitter, i.e., with no extra backhaul load, without the need for interference alignment.

It is important to note that the conclusions in this work, rely on the assumption that accurate channel state information (CSIT) is available at the transmitters. Recently, the problem of interference management through cooperative transmission has been studied with weak and no CSIT in [11]-[16]. In [13], it was shown that significant gains could be achieved through a flexible cell association strategy that does not constrain availability of the $i^{th}$ message to only the $i^{th}$ transmitter. In [17], it was shown that cooperative transmission cannot lead to a per user DoF gain in large Wyner’s linear networks with no CSIT, when restricted to linear cooperation schemes.

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We prove the upper bound in Theorem 1 in two steps. First, we provide an information theoretic argument in Lemma 3 to prove an upper bound on the DoF of any network that has a subset of messages whose transmit set sizes are bounded. We then finalize the proof with a combinatorial argument that shows the existence of such a subset of messages in any assignment of messages satisfying the backhaul constraint of (8).

In order to establish the information theoretic argument in Lemma 3, we use Lemma 1, that is introduced in Section III. We also need [7, Corollary 3] in the proof of Lemma 3; we restate it for the considered system model.

**Corollary 2 ([7, Corollary 3]):** For any $K$-user linear interference channel, if the size of the transmit set $|T_i| \leq M, i \in [K]$, then any element $k \in T_i$ such that $k \notin \{i-M, i-M+1, \ldots, i+M-1\}$ can be removed from $T_i$, without decreasing the sum rate.
We now make the following definition to use in the proof of the following lemma. For any set $S \subseteq [K]$, let $g_S : S \to \{1, 2, \ldots, |S|\}$ be a function that returns the ascending order of any element in the set $S$, e.g., $g_S(\min \{i : i \in S\}) = 1$ and $g_S(\max \{i : i \in S\}) = |S|

Lemma 3: For any $K$-user linear interference channel with DoF $\eta$, if there exists a subset of messages $S \subseteq [K]$, such that each message in $S$ is available at a maximum of $M$ transmitters, i.e., $|T_i| \leq M, \forall i \in S$, then the DoF is bounded by,

$$\eta \leq K - \frac{|S|}{2M+1} + C_K,$$

where $\lim_{K\to\infty} \frac{C_K}{K} = 0$.

Proof: We use Lemma 1 with a set $A$ such that the size of the complement set $|\bar{A}| = \frac{|S|}{2M+1} - o(K)$. We define the set $A$ such that $A = \{i : i \in S, g_S(i) = (2M + 1)(j - 1) + M + 1, j \in \mathbb{Z}^+\}$.

Now, we let $s_1, s_2$ be the smallest two indices in $\bar{A}$. We see that $g_S(s_1) = M + 1, g_S(s_2) = 3M + 2$. Note that $X_1 + \frac{Z_1}{H_{1,1}} = Y_1$, and

$$X_2 + \frac{Z_2 - \frac{H_2}{H_{1,1}}Z_1}{H_{2,2}} = \frac{Y_2 - \frac{H_2}{H_{1,1}}Y_1}{H_{2,2}}.$$  

Similarly, it is clear how the first $s_1 - 1$ transmit signals $X_1, X_2, \ldots, X_{s_1 - 1}$ denoted as $X_{[s_1-1]}$ can be recovered from the received signals $Y_{[s_1-1]}$ and linear combinations of the noise signals $Z_{[s_1-1]}$. In what follows, we show how to reconstruct a noisy version of the signals $\{X_{s_1}, X_{s_1 + 1}, \ldots, X_{s_2 - 1}\}$, where the reconstruction noise is a linear combination of the signals $Z_A$. Then it will be clear by symmetry how the remaining transmit signals can be reconstructed.

We now notice that it follows from Corollary 2 that message $W_{s_1}$ can be removed from any transmitter in $T_{s_1}$ whose index is greater than $s_1 + M - 1$, without affecting the sum rate. Similarly, there is no loss of generality in assuming that $\forall s_i \in S, s_i \neq s_1, T_{s_i}$ does not have an element with index less than $s_i - M$. Since $s_1 - s_1 \geq g_S(s_1) - g_S(s_1) \geq 2M + 1$, it follows that $X_{s_1 + M} \in X_{U_A}$. The signal $X_{s_1 + M + 1} + \frac{Z_{s_1 + M + 1}}{H_{s_1 + M + 1, s_1 + M + 2}}$ can be reconstructed from $Y_{s_1 + M + 1}$ and $X_{s_1 + M}$. Then, it can be seen that the transmit signals $\{X_{s_1 + M + 2}, X_{s_1 + M + 3}, \ldots, X_{s_2 - 1}\}$ can be reconstructed from $\{Y_{s_1 + M + 1}, Y_{s_1 + M + 2}, \ldots, Y_{s_2 - 1}\}$ and linear combinations of the noise signals $\{Z_{s_1 + M + 1}, Z_{s_1 + M + 2}, \ldots, Z_{s_2 - 1}\}$. Similarly, since $X_{s_1 + M}$ is known, the transmit signals $\{X_{s_1 + M - 1}, X_{s_1 + M - 2}, \ldots, X_{s_1}\}$ can be reconstructed from $\{Y_{s_1 + M}, Y_{s_1 + M - 1}, \ldots, Y_{s_1 + 1}\}$, and linear combinations of the noise signals $\{Z_{s_1 + M}, Z_{s_1 + M - 1}, \ldots, Z_{s_1 + 1}\}$. By following a similar argument to reconstruct all transmit signals from the signals $Y_{U_A}$, and linear combinations of the noise signals $Z_A$, we can show the existence of functions $f_1$ and $f_2$ of Lemma 1 to complete the proof.

We now explain how Lemma 3 can be used to prove that $\tau^{\text{avg}}(B = 1) \leq \frac{3}{4}$. For any message assignment satisfying (8) for a $K$-user channel, let $R_j$ be defined as follows for every $j \in \{0, 1, \ldots, K\}$,

$$R_j = \frac{|\{i : i \in [K], |T_i| = j\}|}{K}.$$  

$R_j$ is the fraction of users whose messages are available at exactly $j$ transmitters. Now, if $R_0 + R_1 \geq \frac{3}{4}$, then Lemma 3 can be used directly to show that $\eta \leq \frac{3K}{4} + o(K)$. Otherwise, more than $\frac{K}{4}$ users have their messages at two or more transmitters, and it follows from (8) that $R_0 \geq \sum_{j=0}^{K} R_j \geq \frac{1}{4}$, and hence, $\eta \leq (1 - R_0)K \leq \frac{3K}{4}$. We generalize the above argument to complete the proof that $\tau^{\text{avg}}(B = 1) \leq \frac{3B - 1}{4B}, \forall B \in \mathbb{Z}^+$. More specifically, we show that for any message assignment satisfying (8) for a $K$-user channel with an average transmit set size

May 24, 2017 DRAFT
constraint $B$, there exists an integer $M \in \{0, 1, \ldots, K\}$, and a subset $S \subseteq [K]$ whose size $|S| \geq \frac{2M+1}{4B}K$, such that each message in $S$ is available at a maximum of $M$ transmitters, i.e., $|T_i| \leq M, \forall i \in S$. Fix any message assignment satisfying (8) for a $K$-user channel with backhaul constraint $B$, and let $R_j, j \in \{0, 1, \ldots, K\}$ be defined as in (20). If $\sum_{j=2B}^K R_j \leq \frac{1}{4B}$, then more than $\frac{4B-1}{4B}K$ users have a transmit set whose size is at most $2B-1$, and the statement follows with $M = 2B-1$. It is then possible to assume that $\sum_{j=2B}^K R_j > \frac{1}{4B}$. In what follows, we show by contradiction that there exists an integer $M \in \{0, \ldots, 2B-2\}$ such that $\sum_{j=0}^M R_j > \frac{2M+1}{4B}$.

Define $R_j^*, j \in \{0, 1, \ldots, 2B\}$ such that $R_0^* = R_2B = \frac{1}{4B}$, and $R_j^* = \frac{1}{4B}, \forall j \in \{1, \ldots, 2B-1\}$. Now, note that $\sum_{j=0}^{2B} R_j^* = 1$, and $\sum_{j=0}^{2B} jR_j^* = B$. It follows that if $R_j = R_j^*, \forall j \in \{0, \ldots, 2B\}$, and $R_j = 0, \forall j \geq 2B+1$, then the constraint in (8) is tightly met, i.e., $\sum_{j=0}^M R_j^* = B$. We will use this fact in the rest of the proof.

Assume that $\sum_{j=2B}^K R_j > R_{2B}^*$, and that $\forall M \in \{0, 1, \ldots, 2B-2\}, \sum_{j=0}^M R_j \leq \sum_{j=0}^M R_j^* = \frac{2M+1}{4B}$. We know from (8) that $\sum_{j=0}^K jR_j \leq \sum_{j=0}^{2B} jR_j^* = B$. Also, since $\sum_{j=0}^K R_j = \sum_{j=0}^{2B} R_j^* = 1$ and $\sum_{j=2B}^K R_j > R_{2B}^*$, it follows that there exists an integer $M \in \{0, 1, \ldots, 2B-1\}$ such that $R_M > R_M^*$; let $m$ be the smallest such integer. Since $\sum_{j=0}^M R_j \leq \sum_{j=0}^m R_j^*$, and $\forall j \in \{0, 1, \ldots, m-1\}, R_j \leq R_j^*$, we can construct another message assignment by removing elements from some transmit sets whose size is $m$, such that the new assignment satisfies (8), and has transmit sets $T_i^*$ where $\forall j \in \{0, 1, \ldots, m\}, |\{i : i \in [K], T_i^* = j\}| \leq R_j^*$. By successive application of the above argument, we can construct a message assignment that satisfies (8), and has transmit sets $T_i^*$ where $\forall j \in \{0, 1, \ldots, 2B-1\}, |\{i : i \in [K], T_i^* = j\}| \leq R_j^*$ and $|\{i : i \in [K], T_i^* \geq 2B\}| \geq R_{2B}^*$. Note that the new assignment has to violate (8) since $\sum_{j=0}^{2B} jR_j^* = B$, and we reach a contradiction.

We now know from Lemma 3 that under the backhaul load constraint of (8), the DoF for any $K$-user channel is upper bounded by $\frac{4B-1}{4B}K + o(K)$. It follows that the asymptotic per user DoF $\tau^{\text{avg}}(B) \leq \frac{4B-1}{4B},$ thereby proving the upper bound in Theorem 1.