Magnetic fields in 2D and 3D sphere

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In this note we study the Landau–Hall problem in the 2D and 3D unit sphere, that is, the motion of a charged particle in the presence of a static magnetic field. The magnetic flow is completely determined for any Riemannian surface of constant Gauss curvature, in particular, the unit 2D sphere. For the 3D case we consider Killing magnetic fields on the unit sphere, and we show that the magnetic flowlines are helices with the given Killing vector field as its axis.

Keywords: Magnetic field; Killing field; Riemannian manifold.

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1. Introduction

Helical configurations are structures that occur very often in nature. They appear in microscopic systems (biomolecules, bacterial fibers, nanosprings, protein chains in particular DNA, etc.), as well as in macroscopic phenomena (brussels sprouts, snail shells, coiled springs, vortices, etc.). The helix is usually defined as a curve that makes a constant angle with a given vector. These curves are called generalized helices, and can be characterized by the constancy of the ratio between torsion and curvature (Lancret theorem, [2]). In particular, a curve with constant curvature and torsion is called a helix (circular helix in $\mathbb{R}^3$).

Maxwell’s equations are a set of partial differential equations that, together with the Lorentz force law, form the foundation of classical electrodynamics, classical optics, and electric circuits. In particular, Maxwell’s second equation is Gauss’s Law for magnetism: $\nabla \cdot B = 0$, that is, the magnetic field $B$ is a divergence-free vector field. Physically means that there can be no isolated magnetic poles, which are called magnetic monopoles (magnetic monopoles are the magnetic equivalent of isolated positive or negative electric charges). The Earth’s magnetic field is similar to that of a bar magnet tilted 11 degrees from the spin axis of the Earth. The Earth’s magnetic field helps protect the planet from energetic particles streaking out from deep space and the sun. In fact, it traps charged particles such as electrons and protons as they are forced to execute a spiraling motion back and forth along the field lines.

The Landau–Hall problem is the study of the motion of a charged particle on a Riemannian surface in the presence of a constant and static magnetic field $B$. In this setting, free of any electric field, a particle of mass $m$ and charge $q$ evolves with velocity vector $v(t)$ satisfying the Lorentz force law [6],

Following non-linear second order Landau–Hall differential equation, moves with velocity vector \(v\) the reference frame we write \(B\) stationary, i.e., \(v\) is easy to see that \(\gamma(t) = (x_1(t), x_2(t), x_3(t))\) such that \(\gamma'(t) = v(t)\), and it is given by

\[
\begin{align*}
x_1(t) &= x_1^0 + r \sin(\omega t + \alpha), \\
x_2(t) &= x_2^0 + r \cos(\omega t + \alpha), \\
x_3(t) &= x_3^0 + v_3^0 t.
\end{align*}
\]

where \(\omega = (qhc)/\varepsilon\) is a constant. Therefore, the motion of the particle is described by a curve \(\gamma(t) = (x_1(t), x_2(t), x_3(t))\), satisfies the following conditions,

\[
\frac{d}{dt} v_1(t) = \omega v_2(t), \quad \frac{d}{dt} v_2(t) = -\omega v_1(t), \quad \frac{d}{dt} v_3(t) = 0,
\]

where \(\omega = (qhc)/\varepsilon\) is a constant. Therefore, the motion of the particle is described by a curve \(\gamma(t) = (x_1(t), x_2(t), x_3(t))\) such that \(\gamma'(t) = v(t)\), and it is given by

\[
\begin{align*}
x_1(t) &= x_1^0 + r \sin(\omega t + \alpha), \\
x_2(t) &= x_2^0 + r \cos(\omega t + \alpha), \\
x_3(t) &= x_3^0 + v_3^0 t.
\end{align*}
\]

It is easy to see that \(\gamma\) is a circular helix with radius \(r = ||v||/\omega\) and axis the trajectories of \(B\). In particular, if \(v_3 = 0\) the particle describes a circle in the plane \(x_3 = x_3^0\), with center \((x_1^0, x_2^0, x_3^0)\) and radius \(r\).

Now, we consider this example again from a new point of view: let \(\Pi\) be the plane \(Oxy\) in the Euclidean space \(\mathbb{R}^3\) and define the differential two-form \(F\) on \(\Pi\) as follows:

\[
F(X, Y) = B \wedge Y,
\]

for any vector fields \(X, Y\) on \(\Pi\), and \(<., .>\) is the induced metric on \(\Pi\). It is clear that \(F\) is a closed two-form on this plane and therefore it is a constant multiple of the corresponding area element, indeed \(F = h dx \wedge dy\). Define the skew-symmetric operator \(\Phi\) on \(\Pi\) by \(<\Phi(X), Y> = F(X, Y)\), that is, \(\Phi(X) = B \wedge X\). In terms of this operator, the Lorentz force law can be expressed as

\[
\frac{dv(t)}{dt} = \Phi(v(t)). \tag{1.1}
\]

This approach to the classical picture can be now obviously extended to a more general setting. In fact, it seems natural to define a magnetic field on a \(m(\geq 2)\)-dimensional Riemannian manifold \((M^m, g)\) as a closed 2-form \(F\) on \(M^m\). The Lorentz force of a magnetic background \((M^m, g, F)\) is defined to be the skew-symmetric operator \(\Phi\) given by

\[
g(\Phi(X), Y) = F(X, Y), \tag{1.2}
\]

for any couple of vector fields \(X, Y\) on \(M^m\). Let us remark that \(\Phi\) is metrically equivalent to \(F\), so no information is lost when \(\Phi\) is considered instead \(F\). In classical terminology, it is said that \(\Phi\) is obtained from \(F\) by raising its second index, and \(\Phi\) and \(F\) are then said to be physically equivalent.

A smooth curve \(\gamma\) in \((M^m, g)\) is called a flowline of the dynamical system associated with the magnetic field \(F\), or a magnetic curve of \((M^m, g, F)\), if its velocity vector field, \(\gamma\), satisfies the following non-linear second order Landau–Hall differential equation,

\[
\nabla_\gamma \gamma = \Phi(\gamma'), \tag{1.3}
\]

where \(\nabla\) is the Levi–Civita connection of \(g\) (compare with Eq. (1.1)).
These definitions leads us to the following consequences:

1. For the trivial magnetic field $F = 0 \iff \Phi = 0$, the Landau–Hall equation for the magnetic curves is $\nabla \gamma' = 0$, which means that magnetic curves become the geodesics of $(M^n, g)$. Therefore, on any Riemannian manifold $(M^n, g)$ free of electric and magnetic fields charged particles move along geodesics.

2. Magnetic curves satisfy the following conservation law: particles evolve with constant speed, and so with constant energy along the magnetic trajectories. In fact, from (1.2) we have $g(\Phi(X), Y) = -g(X, \Phi(Y))$ and then

$$\frac{d}{dt} g(\gamma', \gamma') = \nabla g(\gamma', \gamma') = g(\nabla g(\gamma, \gamma'), \gamma') + g(\gamma', \nabla g(\gamma, \gamma')) = 2g(\Phi(\gamma'), \gamma') = 0. \quad (1.4)$$

3. The existence and uniqueness theorem for geodesics remains also true for magnetic curves.

4. The system of differential equations satisfied by geodesics has the following homogeneity property: if $\gamma(t)$ is a geodesic, then for every constant, $c$, the curve $\gamma(ct)$ is also a geodesic. Magnetic curves do not satisfy the homogeneity property. In fact, if $\gamma(t)$ is a magnetic curve of $(M, g, F)$ then $\gamma(ct)$ is a magnetic curve of $(M, g, CF)$ and also of $(M, (1/c)g, F)$ ($c > 0$).

5. As it is well known, geodesics are critical points of the energy functional, and locally they are length-minimizing curves. Magnetic curves locally are also critical points of a functional $[3]$: $\exists U \subset M$ such that $F = d\omega$ in $U$, and if we define $\Gamma = \text{space of curves in } U$ from $p$ to $q, p, q \in U$ then

$$\mathcal{L}(\gamma) = \frac{1}{2} \int_{\gamma} g(\gamma', \gamma') dt + \int_{\gamma} \omega(\gamma') dt, \gamma \in \Gamma,$$

and the Lorentz equation is the Euler–Lagrange equation associated to $\mathcal{L}$.

For a magnetic field $F$ on $(M^n, g)$, the main goal is to determine the corresponding magnetic curves. A first method for finding a magnetic curve through $p \in M^n$ with a given direction $v \in T_p(M^n)$, might be to consider a local chart around $p \in M^n$ $\varphi : U \longrightarrow \mathbb{R}^m$, and then use $\varphi$ to "translate" this problem to $\mathbb{R}^m$. Then, solve the stated nonlinear second order system of differential equations with the given initial conditions in $\mathbb{R}^m$. Now, the pullback of the solution curve to $M^n$ is the required magnetic curve.

A second method (and complementary to the first one) is considering an immersion of the manifold $M^n$ in some Euclidean space $\mathbb{R}^n, m < n$ (Nash embedding theorem [10]). Then we write down the Landau–Hall equation in $\mathbb{R}^n$ and use Gauss–Weingarten formulae to split this equation in two parts, the tangential and the normal components with respect to $M^n$. Now, intrinsic geometry of $M^n$ can be applied to find the curvatures of the magnetic curve. Thus, depending on the particular manifold, the magnetic curve can be completely determined. We will follow this way. In fact, Riemannian geometry offers the necessary tools to have a new insight that will allow us to solve this problem in some 2D and 3D manifolds.

In Section 2 we study magnetic fields on Riemannian surfaces, and in particular, the uniform magnetic fields. The shape of the uniform magnetic curves in the case of constant curvature surfaces is completely determined.

Section 3 deals with 3D Riemannian manifolds, where we first give a review of the one-to-one natural correspondence between vector fields and one-forms. A cross product of vector fields can be also defined on these manifolds that will provide us with a useful formula for the Lorentz force $\Phi$. Furthermore, we introduce Killing magnetic fields which have remarkable properties.
In Section 4 we first find a basis for the 6-dimensional Lie algebra of vector fields of $S^3$. But these vector fields are also unit Killing vector fields, and a useful equation for the Lorentz force of these Killing vector fields is obtained. Then, the curvature and torsion of the corresponding magnetic curves is proved to be constant, and this means that all of them are helices with the given Killing field as its axis.

2. Magnetic fields on Riemannian surfaces

From now on, $M^2$ will be an oriented Riemannian surface with standard complex structure $J$, and area element $\Omega_2$ so that $\Omega_2(X,JX) = 1$ for any unit vector field $X$ in $M^2$. Let $\gamma$ be an arc-length parametrized regular curve in $M^2$ such that $g(\gamma', \gamma') = 1$ and its Frenet apparatus is $\{T = \gamma', N = JT\}$. If $\kappa$ denotes the curvature function, we have the following well-known Frenet equations

$$\nabla_T T = \kappa N, \quad \nabla_T N = -\kappa T.$$ 

Obviously, any differential 2-form on an oriented surface $M^2$ with area element $\Omega_2$ is completely determined by a smooth function, $f$, the strength of $F$, so that $F = f \Omega_2$. We also have that $\Phi(X) = f(JX)$ and $\Phi(JX) = -fX$ and then, the matrix of $\Phi$ with respect to the orthonormal frame $\{X, JX\}$ is given by

$$\Phi \equiv \begin{pmatrix} 0 & -f \\ f & 0 \end{pmatrix}.$$ 

In particular, along a magnetic curve $\gamma$ of $(M^2, g, F)$, and relative to its Frenet frame $\{T = \gamma', N\}$ the Lorentz force $\Phi$ satisfies

$$\Phi(T) = \Phi(\gamma') = \nabla_{\gamma'} \gamma' = \nabla_T T = \kappa N.$$ 

Thus, along the magnetic curve $\gamma$ we have $f = \kappa$, and this gives

$$\Phi \equiv \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix}.$$ 

Therefore, if $\gamma$ is arc-length parametrized we have,

**Theorem 2.1.** The curvature of the magnetic curves on a Riemannian surface $(M^2, g)$ is given by $\kappa = f$.

A magnetic field $F$ with constant strength $f = \mu \in \mathbb{R} - \{0\}$ is called a uniform magnetic field. This class of magnetic fields has been extensively considered in the literature from different points of view ( [1, 7, 12], etc.). The geometric partner of the Landau–Hall problem for uniform magnetic fields in a surface, is nothing but the computation of curves with constant curvature. To be precise, we have,

**Corollary 2.1.** Let $F = \mu \Omega_2$ be a uniform magnetic field, with constant strength $\mu \neq 0$ on a surface $(M^2, g)$. The magnetic curves of $(M^2, g, F)$ are the curves of constant curvature $\kappa = \mu$.

On a Riemannian surface $(M^2, g)$ with constant Gauss curvature $K_0$, the family of magnetic curves is completely determined for any given uniform magnetic field $F = \mu \Omega_2$. In fact, we have the following classification theorem.
Let $F = \mu \Omega_2$, $\mu \neq 0$ a uniform magnetic field on a surface $(M^2,g)$ with constant curvature $K_0$. Then we have,

(1) If $K_0 \equiv 0$ (that is, $(M^2,g)$ is a flat surface), the magnetic curves of $F$ are geodesic circles with geodesic radius $1/|\mu|$.

(2) If $K_0 > 0$, $K_0 = 1/r^2$ then $(M^2,g)$ is the 2D sphere $S^2(r)$ of radius $r$, and the magnetic curves of $F$ are (plane) geodesic circles with plane radius $\rho = r\sqrt{1 + r^2\mu^2} < r$. As a consequence, no great circle of $S^2(r)$ can be a magnetic curve of $F = \mu \Omega_2$ (on the other hand they constitute the whole family of geodesics).

(3) If $K_0 < 0, K_0 = -c, c > 0$, (Poincaré’s upper half-plane $\mathbb{H}^2(-c)$, is the simplest example) then we have: (a) If $|\mu| > \sqrt{c}$, the magnetic field $F$ is strong enough to trap particles that move along closed geodesic circles; (b) If $|\mu| < \sqrt{c}$, the magnetic curves of $F$ are non-closed curves which intersect the boundary line of the surface $\partial \mathbb{H}^2(-c)$; (c) when $|\mu| = \sqrt{c}$ magnetic curves are tangent to this boundary, and so they are horocycles [5].

3. Magnetic fields on 3D Riemannian manifolds

Magnetic fields in dimension three are quite special. In fact, there are several key facts which make their handling easier.

To begin with, two-forms and vector fields are in one-to-one correspondence. In fact, let $(M^3, g)$ be a 3D oriented Riemannian manifold with volume form $\Omega_3$ and consider a differential two-form $F \in \Lambda_2(M^3)$. The Hodge star operator $\ast$ acts on $F$ to produce the 1-form $\omega = \ast F \in \Lambda_1(M^3)$, and the $g$-equivalent (dual) vector field $U = (\omega)^1 \in \mathfrak{X}(M^3)$ is well defined by $g(U, X) = \omega(X)$ for any vector field $X$. Thus we have $F \sim U$. The reverse assignment $U \sim F$ works in the same way: given a vector field $U \in \mathfrak{X}(M^3)$, consider its $g$-equivalent 1-form $\omega = U^g$. Apply the Hodge star operator to get $\ast \omega$, which is a two-form in a 3D manifold. Now, the interior contraction $i_U$ (which is defined by means of $(i_U \Omega_3)(X, Y) = \Omega_3(U, X, Y)$) allows us to write it as $\ast \omega = i_U \Omega_3 = F$. Thus we have a one-to-one map between two-forms and vector fields.

The following proposition remind us that magnetic fields on 3D Riemannian manifolds satisfy Maxwell’s second law.

**Proposition 3.1.** Magnetic fields come from divergence free vector fields.

**Proof.** It is well known that the divergence of a vector field $U$ in a manifold can be defined from the Lie derivative by $\mathcal{L}_U \Omega_3 = d(i_U \Omega_3) = \text{div}(U) \Omega_3$ ([18], p.281) and so, the two-form $\ast \omega = i_U \Omega_3$ is closed if and only if $\text{div}(U) = 0$, i.e., the volume element is invariant by the local flows of $U$. This will allow us to regard the magnetic fields in dimension three as divergence free vector fields.

As a consequence of the correspondence between divergence-free vector fields $U$ and magnetic fields $F_U = i_U \Omega_3$ on 3D Riemannian manifolds, we consider the magnetic field as either $U$ or $F_U$.

**Proposition 3.2.** Uniform magnetic fields correspond to parallel vector fields.

**Proof.** Let $U$ be a parallel vector field in $M^3$, i.e., $\nabla U = 0$. Consider a local coordinate system where $\Omega_3$ is a local volume element and $\{e_1, e_2, e_3\}$ is a local frame field with $\Omega_3(e_1, e_2, e_3) = 1$. We know that $\text{div}(U) = \sum g(\nabla e_i U, e_i) = 0$ ([11], p.196), therefore $F = i_U \Omega_3$ is a closed two-form and hence a magnetic field on $(M^3, g)$. Furthermore, it is clear that $\nabla F = 0$. Conversely, suppose that $F$ is a uniform magnetic field on $(M^3, g)$ and let $U \in \mathfrak{X}(M^3)$ be its corresponding vector field, $F = i_U \Omega_3$. 

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A direct computation shows that
\[ \Omega_3(\nabla_X U, Y, Z) = (\nabla_X F)(Y, Z) = 0, \]
for any \( X, Y, Z \in \mathfrak{X}(M^3) \) which proves that \( \nabla U = 0 \).

A cross product \( X \wedge Y \) of two vector fields \( X, Y \in \mathfrak{X}(M^3) \) on a 3D oriented Riemannian manifold can be defined as follows
\[ g(X \wedge Y, Z) = \Omega_3(X, Y, Z). \]
Now, if \( \{e_1, e_2, e_3\} \) is a local basis and \( X = \sum X^i e_i, Y = \sum Y^j e_j, Z = \sum Z^k e_k \) the following properties of the cross product are easily proved.

**Proposition 3.3.** The cross product on a three-dimensional oriented Riemannian manifold satisfies the following two identities,
\[
X \wedge (Y \wedge Z) = g(X, Z)Y - g(X, Y)Z, \\
g(X \wedge Y, X \wedge Z) = g(X, X)g(Y, Z) - g(X, Y)g(X, Z),
\]
for any \( X, Y, Z \in \mathfrak{X}(M^3) \).

**Theorem 3.1.** The Landau–Hall equation in \((M^3, g)\) can be written as
\[
\nabla_{\gamma'} \gamma' = U \wedge \gamma'.
\] (3.1)

**Proof.** The Lorentz force \( \Phi \) associated with the magnetic field, \( F = i_U \Omega_3 \) satisfies
\[ g(\Phi(X), Y) = F(X, Y) = (i_U \Omega_3)(X, Y) = \Omega_3(U, X, Y) = g(U \wedge X, Y), \]
therefore, we have
\[ \Phi(X) = U \wedge X, \] (3.2)
for any \( X \in \mathfrak{X}(M^3) \) and consequently the Lorentz force equation (1.3) can be written as
\[ \nabla_{\gamma'} \gamma' = \Phi(\gamma') = U \wedge \gamma'. \] (3.3)

As an immediate consequence of Eq. (3.1) we have,

**Corollary 3.1.** An integral curve of a magnetic field is a magnetic trajectory if and only if it is a geodesic.

Equation (3.1) allows one to talk about the spin Hall effect in a doubly extended direction. On one hand, this formula works for any magnetic field even if it is not uniform. On the other hand, the formula also works in spaces with non trivial gravity, i.e., with nonzero curvature.

Let us now consider a divergence-free vector field \( U \) on \((M^3, g)\) and an arc-length parametrized magnetic curve \( \gamma(t) \) of \( U \). Denote by \( \{T = \gamma', N, B\} \) its Frenet frame, and \( \kappa, \tau \) the curvature and
torsion functions, respectively. Then, the Frenet equations of \( \gamma \) are written as follows,

\[
\begin{align*}
\nabla_T T &= \kappa N, \\
\nabla_T N &= -\kappa T + \tau B, \\
\nabla_T B &= -\tau N.
\end{align*}
\]

(3.4) 
(3.5) 
(3.6)

**Proposition 3.4.** The Lorentz force \( \Phi \) of a magnetic field \( U \) satisfies

\[
\begin{align*}
\Phi(T) &= \kappa N, \\
\Phi(N) &= -\kappa T + \omega B, \\
\Phi(B) &= -\omega N.
\end{align*}
\]

(3.7) 
(3.8) 
(3.9)

**Proof.** To prove the first equation notice that \( \kappa N = \nabla_T T = \Phi(T) \), which is (3.7). To prove (3.8) we have \( g(\Phi(N), N) = 0 \) because \( \Phi \) is antisymmetric. The projection of \( \Phi(N) \) on the tangent vector \( T \) is obtained from

\[
g(\Phi(N), T) = -g(N, \Phi(T)) = -g(N, \kappa N) = -\kappa,
\]

and its projection on the binormal vector \( B \) from the product

\[
g(\Phi(N), B) = -g(N, \Phi(B)) = \omega(t),
\]

where \( \omega \) which is a kind of slope of the magnetic curves with respect to the magnetic field. Thus we have \( \Phi(N) = -\kappa T + \omega B \). To prove (3.9) we compute the projections of \( \Phi(B) \) to get

\[
g(\Phi(B), T) = -g(B, \Phi(T)) = g(B, \kappa N) = 0, \quad g(\Phi(B), N) = -g(B, \Phi(N)) = -\omega(t).
\]

Since \( g(\Phi(B), B) = 0 \), equation (3.9) is proved. \( \square \)

**Theorem 3.2.** \( \gamma(t) \) is a magnetic curve of the magnetic field \( U \) if and only if

\[
U(t) = \omega(t) T(t) + \kappa(t) B(t)
\]

(3.10)

along \( \gamma \).

**Proof.** Let us write \( U(t) = a_1(t)T(t) + a_2(t)N(t) + a_3(t)B(t) \), where \( a_i, 1 \leq i \leq 3 \) are functions along \( \gamma \) and assume \( U \) does not vanish on the points of \( \gamma \). Now, from \( \Phi(U) = U \wedge U = 0 \), we obtain \( a_2 = 0 \), otherwise \( \omega(t_0) = \kappa(t_0) = 0 \) for some \( t_0 \) and so \( \Phi = 0 \) at \( \gamma(t_0) \), which implies \( U(t_0) = 0 \). But from (3.7), (3.8) and (3.9) we have

\[
0 = \Phi(U) = a_1 \Phi(T) + a_3 \Phi(B) = (a_1 \kappa - a_3 \omega) N
\]

which means \( a_1 = \omega \) and \( a_3 = \kappa \). \( \square \)

The function \( \omega(s) \) associated with each magnetic curve will be called its quasi-slope, measured with respect to the magnetic field \( U \).

A Killing vector field \( U \) on a Riemannian manifold \( (M^n, g) \), is a vector field that generates local flows of isometries, that is, \( \mathcal{L}_U g = 0 \). But we also have \( L_X Y = [X, Y] \) and \( \nabla_X g = 0 \) for any vector
fields $X, Y$ and it is then clear that

$$0 = (\mathcal{L}_U g)(X, Y) = g(\nabla_X U, Y) + g(\nabla_Y U, X) = 0,$$

(3.11)

which is a useful characterization of Killing fields. But we also have $\text{div}(U) = \sum g(\nabla_e U, e_i) = 0$, that is, any Killing vector field is divergence-free. As a consequence, when $n = 3$ every Killing vector field defines a magnetic field $F_U$ which will be called a Killing magnetic field. In particular, uniform magnetic fields ($\nabla U = 0$) are obviously Killing. Therefore, the class of Killing magnetic fields constitutes an important family of magnetic fields.

Killing fields of constant length are called infinitesimal isometries, and they have been well studied. The following result is well-known ([4], p. 499).

**Proposition 3.5.** A Killing vector field $U$ on a Riemannian manifold $(M^n, g)$ has constant length if and only if every integral curve of $U$ is a geodesic in $(M^n, g)$.

Besides the conservation law (1.4) which asserts that the speed of any magnetic trajectory is a constant, it deserves to be pointed out that the magnetic trajectories of Killing magnetic fields in dimension three satisfy an additional conservation law.

**Theorem 3.3.** A magnetic field $U$ on a 3D Riemannian manifold is Killing if and only if for any magnetic curve $\gamma(t)$ the product $g(U, \gamma')$ is a constant along $\gamma(t)$.

**Proof.** First, we note that if $U$ is Killing, along any magnetic curve $\gamma$ we have $g(\nabla_{\gamma'} U, \gamma') = 0$ and $\nabla_{\gamma'} \gamma' = \Phi(\gamma') = U \wedge \gamma'$. Thus,

$$\frac{d}{dt} g(U, \gamma') = g(\nabla_{\gamma'} U, \gamma') + g(U, \nabla_{\gamma'} \gamma') = 0.$$

Conversely, for $p \in M$ and $v \in T_p M$ let $\gamma$ be a magnetic trajectory of a magnetic field $U$ such that $\gamma(0) = p$, $\gamma'(0) = v$. We have

$$0 = \frac{d}{dt} g(U, \gamma') = g(\nabla_{\gamma'} U, \gamma') + g(U, U \wedge \gamma') = g(\nabla_{\gamma'} U, \gamma').$$

Therefore, $g(\nabla_v U, v) = 0$, which means that $U$ is Killing. \(\square\)

The magnetic flows of Killing magnetic fields also satisfy a certain symmetry property.

**Proposition 3.6.** Let $\{\phi_t\}$ denote a (local) flow of a Killing vector field $U$. We have: (a) If $\gamma$ is any magnetic trajectory of $U$ then $\phi_t \circ \gamma$ is also a magnetic trajectory of $U$; (b) Two magnetic trajectories of $U$ have associated the same conservation law’s constant.

**Proof.** In fact, from (3.1) we have

$$\nabla_{(\phi_t \circ \gamma)'} (\phi_t \circ \gamma)' = d\phi_t (\nabla_{\gamma'} \gamma') = d\phi_t (U \wedge \gamma') = U \wedge (\phi_t \circ \gamma)' .$$

On the other hand, the assertion (b) follows easily because $\phi_t$ is an isometry. \(\square\)

4. Killing magnetic fields on the 3D unit Sphere

Now we consider the 3D unit sphere $S^3 \equiv \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ endowed with its usual metric $g$ induced from the inclusion map $i : S^3 \hookrightarrow \mathbb{R}^4$ in the Euclidean space $(\mathbb{R}^4, g_0)$ provided with the Euclidean metric $g_0$. 

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First we claim that every geodesic $\alpha$ in $S^3$ is a great circle. In fact, let $p, q$ be two points on $\alpha$, close enough together so that a geodesic between them is unique. Then the reflection of $\mathbb{R}^4$ with respect to any three-plane that contains $p, q$ and the origin in $\mathbb{R}^4$ induces an isometry on $S^3$ that fixes $p$ and $q$. Therefore this isometry fixes $\alpha$. Hence $\alpha$ is in every such three-plane, and in the unique two-plane through $p, q$ and the origin in $\mathbb{R}^4$. The intersection of this two-plane and $S^3$ is a curve on $S^3$ which we call a great circle.

The quaternionic structure of $\mathbb{R}^4$ can be used to write $q = x_1 + ix_2 + jx_3 + kx_4$, and the equations $i^2 = j^2 = k^2 = ijk = -1$ determine all the possible products of $i, j, k$. Then, $S^3$ can also be viewed as the set $S^3 = \{ q \in \mathbb{H} : \|q\| = 1 \}$. For any $q \in S^3$, we define the following vectors obtained by rotating $q$,

\[
\begin{align*}
   iq &= -x_2 + ix_1 - jx_4 + kx_3, \\
   jq &= -x_3 - ix_4 + jx_1 - kx_2, \\
   kq &= -x_4 - ix_3 + jx_2 + kx_1.
\end{align*}
\]

Let us now define the six vector fields on $\mathbb{R}^4$ whose components with respect to the respect to the basis of $\mathbb{R}^4$ are the same as those of $iq, jq, kq, iq, jq, kq$ respectively, that is,

\[
\begin{align*}
   (1) \; & U = -x_2 e_1 + x_1 e_2 - x_4 e_3 + x_3 e_4, \\
   (2) \; & V = -x_3 e_1 + x_4 e_2 + x_1 e_3 - x_2 e_4, \\
   (3) \; & W = -x_4 e_1 - x_3 e_2 + x_2 e_3 + x_1 e_4.
\end{align*}
\]

It is clear that all of them are differentiable unit vector fields, and orthogonal to the unit vector field $P = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4$ that represents the position of points in $S^3$ (and therefore the unit normal to $S^3$). This means that $U, V, W, G, H, K$ are unit tangent vector fields on $(S^3, g)$. Besides, it is clear that $\{U, V, W\}$ and $\{G, H, K\}$ are two orthonormal sets of vector fields.

Let us denote by $\tilde{\nabla}, \nabla$ the usual covariant derivatives on $(\mathbb{R}^4, g_0)$ and the induced one in $(S^3, g)$, respectively.

**Proposition 4.1.** The unit vector fields $U, V, W, G, H, K$ are Killing on $S^3$.

**Proof.** It suffices to show the proposition for $U = -x_2 e_1 + x_1 e_2 - x_4 e_3 + x_3 e_4$. In fact, for any $e_i, e_j, i, j = 1, 2, 3, 4$ a direct computation gives

\[
g_0(\tilde{\nabla}_e i, e_j) + g_0(\tilde{\nabla}_j e, e_i) = 0.
\]

We now let $X = \sum X^i e_i, Y = \sum Y^j e_j$ be vector fields on $S^3$. Since $g_0$ is bilinear and $\tilde{\nabla}$ satisfies $\tilde{\nabla}_{\Sigma e_i} = \Sigma \tilde{\nabla}_e$, the result extends to any two vector fields $X = \sum X^i e_i, Y = \sum Y^j e_j$. The Gauss and Weingarten formulas of the theory of submanifolds (in case of a hypersurface) relate derivations on $\mathbb{R}^4$ and $S^3$ as follows ([9], p.15),

\[
\tilde{\nabla}_X Y = \nabla_X + II(X, Y)P, \quad \tilde{\nabla}_X P = -A_P X,
\]

for any vector fields $X, Y \in \mathbb{R}^4$. Here $\nabla_X Y$ is the tangent component with respect to $S^3$ of $\tilde{\nabla}_X Y$, $II(X, Y)P$ is the normal component, $II$ is the second fundamental form of the immersion, and $A_P$ is the shape operator on $S^3$. But $II(X, Y) = g(\tilde{\nabla}_X Y, P) = -g(\tilde{\nabla}_P Y, P) = g(-A_P X, Y) = g(A_P X, Y)$, and $S^3$ is totally umbilical, that is, $A_P X = -X$. Therefore $II(X, Y) = -g(X, Y)$ and then $\tilde{\nabla}_X Y = \nabla_X Y - g(X, Y)P$. In particular, as $U$ is Killing in $(\mathbb{R}^4, g_0)$, from (4.1) we get

\[
0 = g_0(\tilde{\nabla}_X U, Y) + g_0(\tilde{\nabla}_Y U, X) = g(\nabla_X U, Y) + g(\nabla_Y U, X).
\]

Then, equation (3.11) proves the assertion. \[\square\]
As a consequence of this result and Proposition 3.5 we have,

**Corollary 4.1.** The integral curves of $U,V,W,G,H,K$ in $S^3$ are great circles.

It is easy to check that the integral curve of the vector field $U$ in $\mathbb{R}^4$ (and therefore in $S^3$) starting from $(a,b,c,d) \in S^3$ for $t = 0$ is the circle

$$\alpha(t) = (a \cos(t) - b \sin(t), a \sin(t) + b \cos(t), c \cos(t) - d \sin(t), c \sin(t) + d \cos(t)),$$

$t \in (-\infty, \infty)$. Analogous parametrizations can be given for the integral curves of $V,W,G,H,K$.

**Remark 4.1.** On the other hand, it is well known that if $X$ and $Y$ are Killing, the Lie bracket $[X,Y]$ is also Killing. The set of all Killing fields of $S^3$ is a Lie algebra denoted by $i(S^3)$. Even more, the dimension of this algebra must be at most $\frac{1}{2} n(n+1)$ where $n$ is the dimension of the manifold ([8], p. 238). As we have six independent vector fields, $\{U,V,W,G,H,K\}$ is a basis of this Lie algebra, and it is isomorphic with the group $I(S^3)$ of isometries of the sphere.

Now let us pick a Killing vector field in the basis of $i(S^3)$, say $U$, and denote by $F_U$, $\Phi_U$ their associated Killing magnetic field and Lorentz force, respectively. We have the following derivation formula.

**Lemma 4.1.** $\nabla_X U = \Phi_U(X)$ for any $X \in \mathfrak{X}(S^3)$.

**Proof.** First, we can easily find that

$$\tilde{\nabla}_U U = (U(-x_2), U(x_1), U(-x_3), U(x_4)) = (-x_1, -x_2, -x_3, -x_4) = -P,$$

and similarly we get $\tilde{\nabla}_V U = W, \tilde{\nabla}_W U = V$. But from Gauss and Weingarten formulas (4.1) we see that $-P = \tilde{\nabla}_U U = \tilde{\nabla}_U U - g_0(U, U)P$. Then $\tilde{\nabla}_U U = 0$. Once more equations (4.1) with $X = V, Y = U$ give us

$$\tilde{\nabla}_V U = \nabla_V U - g(V, U)P = \tilde{\nabla}_U U, \quad \tilde{\nabla}_W U = \nabla_W U - g(W, U)P = \nabla_W U.$$

Finally we consider any vector field on $S^3$, $X = \lambda U + \mu V + \nu W$ and then it is clear that

$$\nabla_X U = \lambda \nabla_U U + \mu \nabla_V U + \nu \nabla_W U = \mu W - \nu V.$$

But from equation (3.2) we have

$$\Phi_U(X) = \lambda U \wedge U + \mu U \wedge V + \nu U \wedge W = \mu W - \nu V,$$

and this proves the theorem. \hfill \Box

Now we state the theorem that gives us the shape of the magnetic flowlines of the Killing fields.

**Theorem 4.1.** The Killing magnetic curves $\gamma(t)$ of a Killing field $F_U$ in $S^3$ are curves with constant curvature $\kappa(t) = \kappa_0$ and constant torsion $\tau(t) = \tau_0 = 1 - \omega_0$, that is, they are helices with axis the trajectories (great circles) of $U$.

**Proof.** From Theorem 3.2 the vector field $U$ can be written at any point of $\gamma(t)$ as $U(t) = \omega(t)T(t) + \kappa(t)B(t)$, where $T = \gamma'$ and $B$ are the unit tangent and binormal vectors of $\gamma$, respectively. Now, Theorem 3.3 says that $g(U, T) = \omega(t)$ is a constant $\omega(t) = \omega_0$. But $U$ is a unit vector field, $1 = g(U, U) = \omega_0^2 + \kappa^2$, and hence $\kappa^2(t) = 1 - \omega_0^2$ so that $\kappa(t) = \kappa_0 = 1 - \omega_0^2$. The Killing
field $U$ is expressed then as $U(t) = \omega_0T(t) + \kappa_0B(t)$ along $\gamma(t)$. From equation (3.2) and Lemma 4.1 we have $\nabla_{\gamma'}U = \Phi_U(\gamma') = U \wedge \gamma'$. On the one hand if we apply Frenet formulas of $\gamma(t)$ then

$\nabla_{\gamma'}U = \nabla_{\gamma'}(\omega_0\gamma + \kappa_0B) = \omega_0\kappa_0N + \kappa_0(-\tau N) = \kappa_0(\omega_0 - \tau)N$,

where $\tau(t)$ is the torsion of $\gamma$. But we also have $U \wedge \gamma' = (\omega_0T + \kappa_0B) \wedge T = \kappa_0N$, so that $\omega_0 - \tau = 1$. But this means that the torsion $\tau(t)$ of the magnetic curve $\gamma$ is a constant $\tau_0 = \omega_0 - 1$, which proves the theorem.

With respect to this last result, it is worth to mention that in April 1, 2013 a team of astronomers from the International Centre for Radio Astronomy Research (ICRAR) have succeeded in observing the death throws of a giant star in unprecedented detail [13]. In this report we read: “Supernova remnants are like natural particle accelerators, the radio emission we observe comes from electrons spiralling along the magnetic field lines and emitting photons every time they turn”, said Professor Lister Staveley-Smith, Director of ICRAR, the Centre for All-sky Astrophysics. Many newspapers picked up the story.

As a conclusion, we see that no matter your placement in the Universe, on Earth’s surface or on a distant supernova, the Lorentz force law works everywhere in the Universe: charged particles move spiralling along the magnetic field trajectories.

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