MODERATE DEVIATION PRINCIPLES FOR UNBOUNDED ADDITIVE FUNCTIONALS OF DISTRIBUTION DEPENDENT SDES

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Abstract. By comparing the original equations with the corresponding stationary ones, the moderate deviation principle (MDP) is established for unbounded additive functionals of several different models of distribution dependent SDEs, with non-degenerate and degenerate noises.

1. Introduction. To characterise long time behaviours of stochastic systems, various limit theorems, including LLN(law of large numbers), CLT(central limit theorems), and LDP(large deviation principle) have been intensively investigated in the literature of Markov processes and random sequences, see for instance [1, 2, 3, 4, 5, 6, 9, 11, 19, 20, 21]. On the other hand, less is known for limit theorems on nonlinear systems, where a typical model is the distribution dependent SDE (also called McKean-Vlasov or mean-field SDE), which arises from characterizations on nonlinear Fokker-Planck equations and mean-filed particle systems, see [10] and references within. Recently, the Donsker-Varadhan LDP for path-distribution dependent SDEs was investigated in [13] for empirical measures, which in particular implies LDP for bounded continuous additive functionals. In this paper, we investigate the MDP(moderate deviation principle) for unbounded additive functionals.

Below, we first briefly recall the notions of LDP and MDP, then introduce the model studied in the present paper.

Let $E$ be a Polish space, and let $(X_t)_{t\geq 0}$ be a right continuous Markov process on $E$ with infinitesimal generator $\mathcal{A}$. For a measurable space $(E, \mathcal{B})$, let $\mathcal{P}(E)$ denote the set of all probability measures on $E$ with weak topology. Consider the empirical measures of $(X_t)_{t\geq 0}$:

$$L_t := \frac{1}{t} \int_0^t \delta_{X_s} \, ds, \quad t > 0.$$
The following Donsker-Varadhan type long time LDP for $L_t$ has been studied in [7]:

$$\mathbb{P}_x(L_t \in M) \approx \exp\{-t \inf_{\nu \in M} J(\nu)\}, \ M \subset \mathcal{P}(E), \quad (1.1)$$

where $J(\nu) = \sup_{U(\nu) > 0, U \in \mathcal{D}(\nu)} \int -\frac{dU}{U} d\nu$, $\mathbb{P}_x$ denotes the probability of Markov process starting from $x$.

In general, when the Markov process $(X_t)_{t \geq 0}$ is ergodic, in order to describe the convergence of the empirical distribution $L_t$ to the unique invariant probability measure $\bar{\mu}$ as $t \to \infty$, a standard way is to look at the convergence rate of

$$L^A_t := \int_E A dL_t = \frac{1}{t} \int_0^t A(X_s) ds \to \bar{\mu}(A) \text{ as } t \to \infty$$

for $A$ in a class of reference functions. This leads to the study of the LDP (MDP) for the additive functional $L^A_t$. When $A$ is bounded and continuous, $(1.1)$ and the Contraction Principle imply the LDP of $L^A_t$, that is, for $M \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{P}_x(L^A_t \in M) \approx \exp\{-t \inf_{z \in M} J^A(z)\}, \ A \subset \mathbb{R},$$

where $J^A(z) = \inf\{J(\nu); \int A d\nu = z\}$. But this approach does not apply when $A$ is unbounded. So, we consider the MDP for $L^A_t$ with unbounded $A$, which is equivalent to LDP for the modified additive functional

$$l^A_t := \frac{t}{a(t)} \left(L^A_t - \bar{\mu}(A)\right) = \frac{1}{a(t)} \int_0^t (A(X_s) - \bar{\mu}(A)) ds,$$

where $a(t)$ is a positive function satisfying

$$\lim_{t \to \infty} \frac{\sqrt{t}}{a(t)} = 0, \ \lim_{t \to \infty} \frac{a(t)}{t} = 0. \quad (1.2)$$

**Definition 1.1.** (1) $L^A_t$ is said to satisfy the upper bound MDP with a rate function $I$, denoted by $L^A_t \in MDP_u(I)$, if for any $a$ satisfying $(1.2)$,

$$\limsup_{t \to \infty} \frac{t}{a^2(t)} \log \mathbb{P}(l^A_t \in F) \leq -\inf_F I, \quad F \subset \mathbb{R} \text{ is closed.}$$

(2) $L^A_t$ is said to satisfy the lower bound MDP with a rate function $I$, denoted by $L^A_t \in MDP_l(I)$, if for any $a$ satisfying $(1.2)$,

$$\liminf_{t \to \infty} \frac{t}{a^2(t)} \log \mathbb{P}(l^A_t \in G) \geq -\inf_G I, \quad G \subset \mathbb{R} \text{ is open.}$$

(3) $L^A_t$ is said to satisfy MDP with a rate function $I$, denoted by $L^A_t \in MDP(I)$, if $L^A_t \in MDP_u(I)$ and $L^A_t \in MDP_l(I)$.

The MDP has been established for non-degenerate SDEs by using Wang’s Harnack inequality [16]:

$$dX_t = b(X_t) dt + \sigma(X_t) dB_t, \ X_0 = x \in \mathbb{R}^d.$$  \hspace{1cm} (1.3)

The assumptions in [9] was further simplified and improved in [19], so that degenerate situations are also included.

In this paper, we investigate MDP for unbounded additive functionals of the following distribution dependent SDE (DDSDE for short) on $\mathbb{R}^d$:

$$dX_t = b(X_t, \mathcal{L}X_t) dt + \sigma(X_t, \mathcal{L}X_t) dB_t, \quad (1.4)$$
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where \( b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \), \( \sigma : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times d} \), \((B_t)\) is a \( d \)-dimensional Brownian motion, \( \mathcal{L}_X \) is the law of \( X_t \) under the reference probability space.

Let \( \mathcal{P}_2(\mathbb{R}^d) \) be the space of all probability measures \( \mu \) on \( \mathbb{R}^d \) such that

\[
\|\mu\|_2 := \left( \int_{\mathbb{R}^d} |x|^2 \mu(dx) \right)^{\frac{1}{2}} < \infty.
\]

It is well known that \( \mathcal{P}_2(\mathbb{R}^d) \) is a Polish space under the Wasserstein distance

\[
W_2(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy) \right)^{\frac{1}{2}},
\]

where \( \mathcal{C}(\mu, \nu) \) is the set of all couplings for \( \mu \) and \( \nu \).

As in [13], to establish MDP for DDSDE (1.3), we choose a reference SDE whose solution is Markovian so that existing results on the MDP apply. By comparing the original equation with the reference one in the sense of MDP, we establish the MDP for the DDSDE. We will state the main results in Section 2, and present complete proofs of main results in Section 3.

2. Main results. We consider several different situations.

2.1. Lipschitz Continuous. A. We consider DDSDE (1.3) and make the following assumptions:

(H1) \( b \) is continuous, and \( \sigma \) is Lipschitz continuous on \( \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \) such that

\[
2(b(x, \mu) - b(y, \nu), x - y) + \|\sigma(x, \mu) - \sigma(y, \nu)\|_{HS}^2
\]

\[
\leq \lambda_2 W_2(\mu, \nu)^2 - \lambda_1 |x - y|^2, \quad x, y \in \mathbb{R}^d; \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d),
\]

holds for some constants \( \lambda_1 > \lambda_2 \geq 0 \).

(H2) There exist constants \( 0 < \kappa_1 \leq \kappa_2 < \infty \) such that

\[
\kappa_1^2 I \leq \sigma(x, \mu)\sigma(x, \mu)^\ast \leq \kappa_2^2 I, \quad x \in \mathbb{R}^d, \mu \in \mathcal{P}_2(\mathbb{R}^d),
\]

where \( \sigma^\ast \) denotes the transpose of the matrix \( \sigma \), \( I \) denotes the identity matrix.

According to [18, Theorem 2.1], assumption (H1) implies that for any \( X_0 \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P}) \), the equation (1.3) has a unique solution. We write \( P_t^\ast \nu = \mathcal{L}_{X_t} \), if \( \mathcal{L}_{X_t} = \nu \). By [18, Theorem 3.1(2)], \( P_t^\ast \) has a unique invariant probability measure \( \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d) \) such that

\[
W_2(P_t^\ast \nu, \bar{\mu})^2 \leq W_2(\nu, \bar{\mu})^2 e^{-(\lambda_1 - \lambda_2)t}, \quad t \geq 0, \quad \nu \in \mathcal{P}_2(\mathbb{R}^d).
\]

Consider the stationary reference SDE:

\[
d\bar{X}_t = b(\bar{X}_t, \bar{\mu})dt + \sigma(\bar{X}_t, \bar{\mu})dB_t, \quad \mathcal{L}_{\bar{X}_0} = \bar{\mu}.
\]

Under (H1), the equation (2.2) has a unique solution \( \bar{X}_t \) for any starting point \( x \in \mathbb{R}^d \), and \( \bar{\mu} \) is the unique invariant probability measure of the associated Markov semigroup

\[
\bar{P}_t f(x) := \mathbb{E}[f(\bar{X}_t)], \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad f \in \mathcal{B}_b(\mathbb{R}^d),
\]

where \( \bar{P}_t \) is generated by

\[
\mathcal{A} := \frac{1}{2} \sum_{i,j=1}^d \{\sigma\sigma^\ast\}_{ij}(x, \bar{\mu})\partial_i \partial_j + \sum_{i=1}^d b_i(x, \bar{\mu})\partial_i.
\]

According to [9] and [19], under assumptions (H1) and (H2), \( \bar{P}_t \) is \( \bar{\mu} \)-hypercontractive and strong Feller, i.e., \( \|\bar{P}_t\|_{L^2(\bar{\mu}) \to L^4(\bar{\mu})} = 1 \) for large \( t > 0 \) and \( \bar{P}_t \mathcal{B}_b(\mathbb{R}^d) \subset
\(C_b(\mathbb{R}^d)\) for \(t > 0\). In particular, the hypercontractivity implies that there exists \(\lambda > 0\) such that
\[
\mu(|P_tf - \mu(f)|^2) \leq e^{-\lambda t} \mu(|f - \mu(f)|^2), \quad t \geq 0, \quad f \in L^2(\mu),
\]
so, for any \(f \in L^2(\mu),\)
\[
V(f) := \int_0^\infty \mu(|P_tf - \mu(f)|^2)dt < \infty. \quad (2.3)
\]
We have the following result:

**Theorem 2.1.** Assume (H1) and (H2). If \(\mathbb{E}[|\delta X_0|^2] < \infty\) for some constant \(\delta > 0\), then for any Lipschitz continuous function \(A\) on \(\mathbb{R}^d\), \(L^A_1 \in MDP(I)\) for \(I(y) = y^2/(8\bar{V}(A)), y \in \mathbb{R}\).

### 2.2. Hölder continuous \(A\)
When \(A\) is Hölder continuous, we need to assume that \(\sigma(x, \mu) = \sigma(\mu)\) does not depend on \(x\). In this case, the DDSDE becomes
\[
dX_t = b(X_t, \mathcal{L}X_t)dt + \sigma(\mathcal{L}X_t)dB_t, \quad (2.4)
\]
and the reference SDE reduces to
\[
\hat{d}X_t = b(\hat{X}_t, \hat{\mu})dt + \sigma(\hat{\mu})dB_t. \quad (2.5)
\]
Below we give the main result of this subsection.

**Theorem 2.2.** Assume (H1), (H2), and let \(\sigma(x, \mu) = \sigma(\mu)\) does not depend on \(x\). If there exists a constant \(\delta > 0\) such that \(\mathbb{E}[|\delta X_0|^2] < \infty\), then for any function \(A\) such that
\[
\sup_{x\neq y} \frac{|A(x) - A(y)|}{|x - y|^\alpha (1 + |x| + |y|)^{2-\alpha}} < \infty, \quad x, y \in \mathbb{R}^d
\]
holds for some \(\alpha \in (0, 1)\), \(L^A_1 \in MDP(I)\) for \(I(y) = y^2/(8\bar{V}(A)), y \in \mathbb{R}\).

### 2.3. Non-Hölder continuous \(A\)
In this part, we consider non-Hölder continuous \(A\) for which we need to further strengthen the assumption that \(\sigma\) is constant matrix. So, the DDSDE and the reference SDE reduce to
\[
dX_t = b(X_t, \mathcal{L}X_t)dt + \sigma dB_t, \quad (2.6)
\]
and
\[
\hat{d}X_t = b(\hat{X}_t, \hat{\mu})dt + \sigma dB_t. \quad (2.7)
\]

**Theorem 2.3.** Assume (H1), (H2) and let \(\sigma\) be constant. If \(\mathbb{E}[|\delta X_0|^2] < \infty\) for some \(\delta > 0\), then for any function \(A\) such that
\[
\sup_{x\neq y} \frac{|A(x) - A(y)|\log(e + |x|^2 + |y|^2)[\log(e + |x - y|^{-1})]^p}{K(1 + |x|^2 + |y|^2)} < \infty, \quad x, y \in \mathbb{R}^d
\]
holds for some \(p > 1\) and \(K > 0\), \(L^A_1 \in MDP(I)\) for \(I(y) = y^2/(8\bar{V}(A)), y \in \mathbb{R}\).
2.4. The degenerate case. In this section, we consider the distribution dependent stochastic Hamiltonian system for $X_t = (X^{(1)}_t, X^{(2)}_t)$ on $\mathbb{R}^{m+d}$:

\[
\begin{align*}
\frac{dX^{(1)}_t}{dt} &= (AX^{(1)}_t + BX^{(2)}_t)dt, \\
\frac{dX^{(2)}_t}{dt} &= Z(X_t, X_t)dt + MdB_t,
\end{align*}
\]

where $A, B$ and $M$ are $m \times m$, $m \times d$ and $d \times d$ matrices respectively, $B_t$ is $d$ dimensional Brownian motion. Define

\[
W_2(\nu_1, \nu_2) := \inf_{\pi \in \mathcal{P}(\nu_1, \nu_2)} \left( \int_{\mathbb{R}^{m+d} \times \mathbb{R}^{m+d}} (|\xi_1^{(1)} - \xi_2^{(1)}|^2 + |\xi_1^{(2)} - \xi_2^{(2)}|^2) \pi(d\xi_1, d\xi_2) \right)^{\frac{1}{2}}.
\]

We assume

(D1) $M$ is invertible and $\text{Rank}[B, AB, \ldots, A^{m-1}B] = m$.

(D2) $Z : \mathbb{R}^{m+d} \times \mathcal{P}_2(\mathbb{R}^{m+d}) \to \mathbb{R}^d$ is Lipschitz continuous.

(D3) There exist constants $r > 0$, $\theta_1 > \theta_2 > 0$ and $r_0 \in (-\|B\|^{-1}, \|B\|^{-1})$ such that

\[
\begin{align*}
&\langle x^{(1)} - y^{(1)}, x^{(2)} - y^{(2)} \rangle - r\| B(x^{(2)} - y^{(2)}) \| + \langle Z(x, \mu) - Z(y, \nu), x^{(2)} - y^{(2)} \rangle + r_0B^*(x^{(1)} - y^{(1)}) \\
&\leq -\theta_1 \| x^{(1)} - y^{(1)} \|^2 + \| x^{(2)} - y^{(2)} \|^2 + \theta_2 \nu_1 \circ \nu_2.
\end{align*}
\]

Theorem 2.4. Assume (D1)-(D3), and let $A$ be a Lipschitz continuous function on $\mathbb{R}^{m+d}$. Then

1. For any $\mu_0, \nu_0 \in \mathcal{P}_2(\mathbb{R}^{m+d})$, there exists a constant $C$ such that

\[
W_2(P_t^\mu \mu_0, P_t^\nu \nu_0)^2 \leq Ce^{-\frac{\theta_1}{\theta_2}t}W_2(\mu_0, \nu_0)^2,
\]

2. $P_t^\mu$ has an invariant probability measure $\bar{\nu} \in \mathcal{P}_2(\mathbb{R}^{m+d})$ such that

\[
W_2(P_t^\mu \mu_0, \bar{\nu})^2 \leq Ce^{-\frac{\theta_1}{\theta_2}t}W_2(\mu_0, \bar{\nu})^2,
\]

3. If there exists a constant $\delta > 0$ such that $\mathbb{E}[e^{\delta|X_t|^2}] < \infty$, then $L_t^A \in \text{MDP}(I)$ for $I(y) = y^2/(8V(A))$, $y \in \mathbb{R}^{m+d}$.

3. Proofs of main results.

3.1. Proof of Theorem 2.1. To prove the Theorem 2.1, we will compare $l_t^A$ with the additive functional for $X_t$, let

\[
\tilde{l}_t^A := \frac{1}{t} \int_0^t A(X_s)ds,
\]

and

\[
\tilde{L}_t^A := \frac{t}{a(t)}(\tilde{l}_t^A - \bar{\nu}(A)) = \frac{1}{a(t)} \int_0^t (A(X_s) - \bar{\nu}(A))ds,
\]

where $a(t)$ is a positive function satisfying (1.2).

Define the Cramér functional of $\tilde{l}_t^A$:

\[
\Lambda(z) := \lim_{t \to +\infty} \frac{t}{a^2(t)} \log \mathbb{E} \left[ \exp \left\{ \frac{a^2(t)}{t} z \tilde{l}_t^A \right\} \right] = \lim_{t \to +\infty} \frac{t}{a^2(t)} \log \mathbb{E} \left[ \exp \left\{ \frac{a(t)}{t} z \int_0^t (A(X_s) - \bar{\nu}(A))ds \right\} \right],
\]

(3.1)
where $z$ is a constant. The Legendre transformation of $\Lambda(z)$ is defined by

$$\Lambda^*(y) = \sup_{z \in \mathbb{R}^d} \{zy - \Lambda(z)\},$$

which is related to the rate function. According to the Gärtner-Ellis Theorem and [9, Theorem 1.3], $L^A_t \in \text{MDP}(I)$ for $I(y) = y^2/(8V(A))$, $y \in \mathbb{R}$.

Below we introduce the following exponential approximation lemma which is useful in applications, see for instance [8, Theorem 4.2.16] and [14, Theorem 3.2].

**Lemma 3.1.** (Exponential approximation) If $L^A_t \in \text{MDP}_u(I)$ (respectively $\text{MDP}_l(I)$) and for any $a$ satisfying (1.2),

$$\lim_{t \to \infty} \frac{t}{a^2(t)} \log \mathbb{P}(|L^A_t - \tilde{L}^A_t| > \varepsilon) = -\infty, \quad \forall \varepsilon > 0,$$

then $L^A_t \in \text{MDP}_u(I)$ (respectively $\text{MDP}_l(I)$).

Using this Lemma, we prove the following result, which is crucial in the present study.

**Theorem 3.2.** If $L^A_t \in \text{MDP}_u(I)$ (respectively $\text{MDP}_l(I)$) and there exists a constant $\delta_0 > 0$ such that

$$\mathbb{E} \left[ \exp \left\{ \delta_0 \int_0^\infty |X_s - \bar{X}_s| ds \right\} \right] < \infty,$$

(3.2)

then $L^A_t \in \text{MDP}_u(I)$ (respectively $\text{MDP}_l(I)$).

**Proof.** Since $A$ is Lipschitz continuous, there exists a constant $K > 0$ such that

$$|A(x) - A(y)| \leq K|x - y|, \quad x, y \in \mathbb{R}^d.$$

Due to that $\lim_{t \to \infty} \frac{\alpha(t)}{t} = 0$, there exists $t_0 > 0$ such that $\sqrt{\frac{\alpha(t)}{t}} \leq \delta_0$ when $t \geq t_0$. Below, we assume that $t \geq t_0$ and we have

$$\mathbb{P}(|L^A_t - \tilde{L}^A_t| > \varepsilon) \leq \mathbb{P} \left( \sqrt{\frac{a(t)}{t}} \int_0^t |X_s - \bar{X}_s| ds > \frac{\alpha(t) a(t) \varepsilon}{K} \right),$$

by Chebyshev’s inequality, we obtain

$$\mathbb{P}(|L^A_t - \tilde{L}^A_t| > \varepsilon) \leq \frac{\mathbb{E} \left[ \exp \left\{ \sqrt{\frac{a(t)}{t}} \int_0^t |X_s - \bar{X}_s| ds \right\} \right]}{\exp \left\{ \frac{\alpha(t) a(t) \varepsilon}{K} \right\}},$$

then (1.2) and (3.2) imply that for any $\varepsilon > 0,$

$$\lim_{t \to \infty} \frac{t}{a^2(t)} \log \mathbb{P}(|L^A_t - \tilde{L}^A_t| > \varepsilon)$$

$$\leq \lim_{t \to \infty} \frac{t}{a^2(t)} \left( \log \mathbb{E} \left[ \exp \left\{ \sqrt{\frac{a(t)}{t}} \int_0^t |X_s - \bar{X}_s| ds \right\} \right] - \sqrt{\frac{\alpha(t) a(t) \varepsilon}{K}} \right)$$

$$\leq \lim_{t \to \infty} \frac{t}{a^2(t)} \log \mathbb{E} \left[ \exp \left\{ \delta_0 \int_0^t |X_s - \bar{X}_s| ds \right\} \right] - \lim_{t \to \infty} \frac{t \varepsilon}{a(t) K} = -\infty.$$

Then the desired assertion follows from Lemma 3.1. \qed
Proof of Theorem 2.1. Let \( \mathcal{L}_{X_0} = \nu \) and \( \mathcal{L}_{\bar{X}_0} = \bar{\mu} \). According to [9, Theorem 1.1-1.3], \( \bar{L}_t \in \text{MDP}(I) \). So, it suffices to show (3.2) for some \( \delta > 0 \).

Condition (H1) implies that the reference SDE (2.2) is well-posed and the solution is a Markov process, \( \bar{\mu} \) is the unique invariant probability measure of \( P_t^* \).

Simply denote \( X_t = X_t^\nu, \bar{X}_t = \bar{X}_t^\mu \) and \( P_t^* \nu = \mathcal{L}_{X_t} \) for \( \nu \in \mathcal{P}_2(\mathbb{R}^d) \). By Itô’s formula and (H1),

\[
d|X_t - \bar{X}_t| ^ 2 \leq \left\{ \lambda_2 W_2(P_t^* \nu, \bar{\mu})^2 - \lambda_1 |X_t - \bar{X}_t|^2 \right\} dt + 2\langle X_t - \bar{X}_t, (\sigma(X_t, P_t^* \nu) - \sigma(\bar{X}_t, \bar{\mu})) dB_t \rangle.
\]

Let \( \xi_t = (e^{-\lambda t} + |X_t - \bar{X}_t|^2)^{\frac{1}{2}} \), where \( \lambda : = \lambda_1 - \lambda_2 \). Then we have

\[
d\xi_t \leq -\frac{\lambda_1}{2} e^{-\lambda t} \frac{1}{\xi_t} dt + \frac{1}{2 \xi_t} d|X_t - \bar{X}_t|^2.
\]

Combining this with the last inequality and (2.1), we find a constant \( C > 0 \) such that

\[
d\xi_t \leq (-\lambda.e^{-\lambda t} + \lambda_2 W_2(\nu, \bar{\mu})^2 e^{-\lambda t} - \lambda_1 |X_t - \bar{X}_t|^2) dt + dM_t,
\]

where \( dM_t = \frac{1}{\xi_t} \langle X_t - \bar{X}_t, (\sigma(X_t, P_t^* \nu) - \sigma(\bar{X}_t, \bar{\mu})) dB_t \rangle \). Therefore, for some \( \delta > 0 \), by the Hölder’s inequality, we obtain that

\[
\mathbb{E} e^{\delta \int_0^t \xi_s ds} \leq e^{\frac{\delta^2 \lambda_1}{21}} \mathbb{E} \left[ e^{\frac{25 \lambda_1}{24} f_0^0 dM_s} \right] = e^{\frac{\delta^2 \lambda_1}{21}} \mathbb{E} \left[ e^{\frac{25 \lambda_1}{24} f_0^0 dM_s} | F_0 \right]
\]

\[
= e^{\frac{\delta^2 \lambda_1}{21}} \mathbb{E} \left[ e^{\frac{25 \lambda_1}{24} f_0^0 dM_s} | F_0 \right] \leq e^{\frac{\delta^2 \lambda_1}{21}} \left( \mathbb{E} \left[ e^{\frac{25 \lambda_1}{24} f_0^0 dM_s} \right] \right)^\frac{1}{2}, \quad t > 0.
\]

By the exponential martingale inequality, we have

\[
\mathbb{E} \left[ e^\frac{\lambda_1}{2} \int_0^t dM_s \right] = \mathbb{E} \left[ e^\frac{\lambda_1}{2} \int_0^t \langle X_s - \bar{X}_s, (\sigma(X_s, P_s^* \nu) - \sigma(\bar{X}_s, \bar{\mu})) dB_s \rangle \right]
\]

\[
\leq \left( \mathbb{E} \left[ e^\frac{\lambda_1}{2} \int_0^t \frac{|X_s - \bar{X}_s|^2}{\xi_s^2} | \sigma(X_s, P_s^* \nu) - \sigma(\bar{X}_s, \bar{\mu}) |^2 ds \right] \right)^\frac{1}{2},
\]

since that \( \sigma \) is Lipschitz continuous, there exist constants \( \delta_1, \delta_2 > 0 \) such that

\[ |\sigma(x, \mu) - \sigma(y, \nu)| \leq \delta_1 |x - y| + \delta_2 W_2(\mu, \nu) \]

holds for \( x, y \in \mathbb{R}^d \) and \( \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d) \). This together with (H2) implies that

\[
\mathbb{E} e^{\delta \int_0^t \xi_s ds} \leq e^{\frac{\delta^2 \lambda_1}{21}} \left( \mathbb{E} \left[ e^{\frac{25 \lambda_1}{24} f_0^0 2 \delta_2 \sqrt{\delta_1} |X_s - \bar{X}_s| + \delta_2 W_2(P_s^* \nu, \bar{\mu}) | ds | \right] \right)^\frac{1}{2}
\]

\[
\leq C(\delta) \left( \mathbb{E} \left[ e^{\frac{25 \lambda_1}{24} f_0^0 |X_s - \bar{X}_s| ds | \right] \right)^\frac{1}{2}, \quad t > 0,
\]

where \( C(\delta) = \exp \left\{ \frac{4 \delta \lambda_1 \lambda_2 + 32 \delta \lambda_2 \sqrt{\delta_1} W_2(\nu, \bar{\mu})}{\lambda_1 (\lambda_1 - \lambda_2)} \right\} \). Therefore, we obtain that

\[
\mathbb{E} e^{\delta \int_0^t |X_s - \bar{X}_s| ds \leq \mathbb{E} e^{\delta \int_0^t \xi_s ds} < \infty
\]

for \( 0 < \delta \leq \frac{\lambda_2}{64 \kappa_2 \delta_1 \nu d} \) and (3.2) holds.
3.2. Proof of Theorem 2.2. In order to prove the Theorem 2.2, we need the following result.

**Theorem 3.3.** If $\tilde{L}^A_t \in MDP_u(I)$ (respectively $MDP_p(I)$) and there exists a constant $\delta > 0$ such that

$$E \left[ \exp \left\{ \delta \int_0^\infty |X_s - \bar{X}_s|^{\alpha} (1 + |X_s| + |\bar{X}_s|)^{2-\alpha} ds \right\} \right] < \infty, \quad (3.3)$$

then $L^A_t \in MDP_u(I)$ (respectively $MDP_p(I)$).

The proof is similar to that of Theorem 3.2, so we omit to save space.

Proof of Theorem 2.2. Let $\mathcal{L}X_0 = \nu$ and $\mathcal{L}\bar{X}_0 = \bar{\mu}$. According to [19, Theorem 2.1], $\tilde{L}^A_t \in MDP(I)$ for $I(y) = y^2/(8V(A))$, $y \in \mathbb{R}$. Therefore, it suffices to show (3.3) for some $\delta > 0$.

The assumption (H1) and (2.1) yield

$$\|\sigma(P_t^* \nu) - \sigma(\bar{\mu})\|^2_{H^S} \leq \lambda_2 W_2(P_t^* \nu, \bar{\mu})^2 \leq \lambda_2 e^{- (\lambda_2 - \lambda_1)t} W_2(\nu, \bar{\mu})^2.$$  

By Young’s inequality and $C_r$ inequality, for any $\lambda > 0$

$$|X_s - \bar{X}_s|^{\alpha} (1 + |X_s| + |\bar{X}_s|)^{2-\alpha} = |X_s - \bar{X}_s|^{\alpha} e^{\lambda t} (1 + |X_s| + |\bar{X}_s|)^{2-\alpha} \leq \alpha \frac{e^{\lambda t}}{2} |X_s - \bar{X}_s|^2 + \frac{2 - \alpha}{2} e^{- \frac{\alpha}{2}} 3(1 + |X_t|^2 + |\bar{X}_t|^2).$$

Below, for simplicity, we take $\lambda < \frac{\alpha (\lambda_1 - \lambda_2)}{2}$. By Hölder’s inequality, we have

$$E \left[ e^{\delta \int_0^\infty |X_s - \bar{X}_s|^{\alpha} (1 + |X_s| + |\bar{X}_s|)^{2-\alpha} ds} \right] \leq \left( E \left[ e^{\alpha \delta \int_0^\infty e^{\frac{2\lambda}{\alpha}} |X_s - \bar{X}_s|^2 ds} \right] \right)^{\frac{1}{2}} \left( E \left[ e^{3(2-\alpha)\delta \int_0^\infty e^{- \frac{2\lambda}{\alpha}} (1 + |X_s|^2 + |\bar{X}_s|^2) ds} \right] \right)^{\frac{1}{2}}.$$  

Below search for a constant $\delta > 0$ such that

$I_1 := E \left[ e^{\alpha \delta \int_0^\infty e^{\frac{2\lambda}{\alpha}} |X_s - \bar{X}_s|^2 ds} \right] < \infty,$

and

$I_2 := E \left[ e^{3(2-\alpha)\delta \int_0^\infty e^{- \frac{2\lambda}{\alpha}} (1 + |X_s|^2 + |\bar{X}_s|^2) ds} \right] < \infty.$

By Itô’s formula and (H1), we have

$$d|X_t - \bar{X}_t|^2 \leq (\lambda_2 W_2(P_t^* \nu, \bar{\mu})^2 - \lambda_1 |x - y|^2) dt + 2\langle X_t - \bar{X}_t, (\sigma(P_t^* \nu) - \sigma(\bar{\mu})) dB_t \rangle.$$  

By the chain rule and (H1), we obtain

$$d\{ e^{\frac{2\lambda}{\alpha}} |X_t - \bar{X}_t|^2 \} \leq \{ - (\lambda_1 - \frac{2\lambda}{\alpha}) |X_t - \bar{X}_t|^2 + \lambda_2 W_2(P_t^* \nu, \bar{\mu})^2 \} dt + 2e^{\frac{2\lambda}{\alpha}} \langle X_t - \bar{X}_t, (\sigma(P_t^* \nu) - \sigma(\bar{\mu})) dB_t \rangle,$$

then we have

$$\int_0^t e^{\frac{2\lambda}{\alpha}} |X_s - \bar{X}_s|^2 ds \leq \frac{|X_0 - \bar{X}_0|^2}{\lambda_1 - \frac{2\lambda}{\alpha}} + \frac{\lambda_2 W_2(\nu, \bar{\mu})^2}{\lambda_1 - \frac{2\lambda}{\alpha}} \int_0^t e^{- (\lambda_1 - \lambda_2 + \frac{2\lambda}{\alpha}) s} ds + \frac{2}{\lambda_1 - \frac{2\lambda}{\alpha}} \int_0^t e^{\frac{2\lambda}{\alpha}} \langle X_s - \bar{X}_s, (\sigma(P_s^* \nu) - \sigma(\bar{\mu})) dB_s \rangle.$$
Therefore, by the Hölder’s inequality, exponential martingale inequality and (H1), we obtain that
\[
\begin{align*}
\mathbb{E}
\left[
\int_0^t e^{\alpha \delta} \frac{e^{\frac{2\lambda s}{\alpha}}}{X_s - \bar{X}_s} |X_s - \bar{X}_s|^2 ds
\right]
\leq C_1(\delta) \mathbb{E}
\left[
\int_0^t e^{\alpha \delta} \frac{e^{\frac{2\lambda s}{\alpha}}}{X_s - \bar{X}_s} |X_s - \bar{X}_s|^{\alpha} (\mathbb{E}
\left[
\frac{e^{\frac{2\lambda s}{\alpha}}}{X_s - \bar{X}_s} |X_s - \bar{X}_s|^{\alpha} |X_s - \bar{X}_s|^{\alpha} ds
\right])^{\frac{2}{\alpha}}
\right]
\leq C_1(\delta) \mathbb{E}
\left[
\int_0^t e^{\alpha \delta} \frac{e^{\frac{2\lambda s}{\alpha}}}{X_s - \bar{X}_s} |X_s - \bar{X}_s|^{\alpha} ds
\right]
\leq C_1(\delta) \mathbb{E}
\left[
\int_0^t e^{\alpha \delta} \frac{e^{\frac{2\lambda s}{\alpha}}}{X_s - \bar{X}_s} |X_s - \bar{X}_s|^{2} ds
\right]
\leq C_1(\delta) \mathbb{E}
\left[
\int_0^t e^{\alpha \delta} \frac{e^{\frac{2\lambda s}{\alpha}}}{X_s - \bar{X}_s} |X_s - \bar{X}_s|^{2} ds
\right]
\leq C_1(\delta) \mathbb{E}
\left[
\int_0^t e^{\alpha \delta} \frac{e^{\frac{2\lambda s}{\alpha}}}{X_s - \bar{X}_s} |X_s - \bar{X}_s|^{2} ds
\right]
\leq C_1(\delta) \mathbb{E}
\left[
\int_0^t e^{\alpha \delta} \frac{e^{\frac{2\lambda s}{\alpha}}}{X_s - \bar{X}_s} |X_s - \bar{X}_s|^{2} ds
\right]
\end{align*}
\]

where \( C_1(\delta) := e^{\max(0, \frac{\alpha \lambda_2 W_2(\nu, \mu)^2}{\lambda_1 - 2\alpha \lambda_2 / \alpha})} \). We choose \( \delta \) such that \( \delta \leq \frac{(\lambda_1 - 2\lambda_2 \alpha)^2}{32\alpha \lambda_2 W_2(\nu, \mu)^2} \). This leads to \( I_2 < \infty \).

Next we need to prove that \( I_2 < \infty \). By (H1), there exist constants \( c_1, c_2 > 0 \) such that
\[
\begin{align*}
d|X_t|^2 &\leq (c_2 - c_1 |X_t|^2 + c_2 W_2(P_t^* \nu, \bar{\mu})^2)dt + 2\langle X_t, \sigma(P_t^* \nu)dB_t \rangle, \quad (3.4)
d|\bar{X}_t|^2 &\leq (c_2 - c_1 |\bar{X}_t|^2 + c_2 |\mu|^2)^2 dt + 2\langle \bar{X}_t, \sigma(\bar{\mu})dB_t \rangle, \quad (3.5)
\end{align*}
\]
and
\[
\|\sigma(P_t^* \nu)\|_{\text{HS}}^2 \leq c_2 (1 + W_2(P_t^* \nu, \bar{\mu})^2).
\]

Recall that
\[
I_2 := \mathbb{E}
\left[
\int_0^t e^{3(2-\alpha)\delta} f_0 e^{-\frac{2\lambda s}{\alpha}} (1 + |X_s|^2 + |\bar{X}_s|^2)^2 ds
\right]
\leq \mathbb{E}
\left[
\int_0^t e^{3(2-\alpha)\delta} f_0 e^{-\frac{2\lambda s}{\alpha}} (1 + |X_s|^2 + |\bar{X}_s|^2)^2 ds
\right]
\leq \frac{3(2-\alpha)^2}{2\lambda} \left( \mathbb{E}
\left[
\int_0^t e^{6\delta(2-\alpha)} f_0 e^{-\frac{2\lambda s}{\alpha}} |X_s|^2 ds
\right]
\right)^{\frac{1}{2}} \left( \mathbb{E}
\left[
\int_0^t e^{6\delta(2-\alpha)} f_0 e^{-\frac{2\lambda s}{\alpha}} |\bar{X}_s|^2 ds
\right]
\right)^{\frac{1}{2}}.
\]
Thus, for \( I_2 < \infty \), it suffices to show
\[
\begin{align*}
I_2' := \mathbb{E}
\left[
\int_0^t e^{6\delta(2-\alpha)} f_0 e^{-\frac{2\lambda s}{\alpha}} |X_s|^2 ds
\right] < \infty,
\end{align*}
\]
and
\[
\begin{align*}
I_2'' := \mathbb{E}
\left[
\int_0^t e^{6\delta(2-\alpha)} f_0 e^{-\frac{2\lambda s}{\alpha}} |\bar{X}_s|^2 ds
\right] < \infty.
\end{align*}
\]
By the chain rule and (3.4), we have
\[
\begin{align*}
\mathbb{E}
\left[
\int_0^t e^{6\delta(2-\alpha)} f_0 e^{-\frac{2\lambda s}{\alpha}} |X_s|^2 ds
\right]
\leq e^{-\frac{2\lambda s}{\alpha}} \left( - \frac{2\lambda}{2 - \alpha} - c_1 \right) |X_t|^2 + c_2 + c_2 W_2(P_t^* \nu, \bar{\mu})^2
\end{align*}
\]
Then
\[
\begin{align*}
\mathbb{E}
\left[
\int_0^t e^{6\delta(2-\alpha)} f_0 e^{-\frac{2\lambda s}{\alpha}} |X_s|^2 ds
\right]
\leq C_2(\delta) \mathbb{E}
\left[
\int_0^t e^{\frac{2\lambda s}{\alpha}} (X_s, \sigma(P_t^* \nu)dB_s)
\right]
\end{align*}
\]
We conclude, there exists 10 PANPAN REN AND SHEN WANG

\[ (2^\beta - 3) = \sup |X|, \]

this implies that \( \lambda \geq \beta \log(1 + e) \lambda_1 \). So, when \( \lambda < e^{\lambda_2} \), we have \( I_2 < \infty \).

On the other hand, the same argument gives

\[ E\left[ e^{(2\beta - 3)\lambda_1 |X|^2} \right] \]

where \( C_3(\delta) = \exp\left\{ \dfrac{6(2 - \alpha)(c_2 + c_2 \| \mu \|^2)}{4(2 - \alpha) - 3} \right\} \). So, when \( \lambda \leq \dfrac{(2 + c_2(2 - \alpha))^2}{4(2 - \alpha) - 3} \), we have \( I_2'' < \infty \).

Finally, we take \( \delta \leq \min\left\{ \dfrac{(2 + c_2(2 - \alpha))^2}{2\lambda_1 \lambda^2 + W_2(\nu, \mu)^2}, \dfrac{\lambda_2(2 + c_2(2 - \alpha))^2}{48(2 - \alpha)^2 \| \sigma(\mu) \|^2_{\mathbb{H}}}, \dfrac{48(2 - \alpha)^2}{48(2 - \alpha)^2} \right\} \).

We conclude, there exists \( \delta > 0 \) such that \( I_2 < \infty \), which together with \( I_1 < \infty \) finishes the proof.

3.3. Proof of Theorem 2.3. Let \( L_{X_0} = \nu \) and \( L_{X_0} = \bar{\mu} \). According to \[ \text{[19, Theorem 2.1]}, \bar{L}_t \in MDP(I) \text{ for } I(y) = y^2/8V(A), y \in \mathbb{R}. \]

Therefore, by the Lemma 3.1 (see also \[ \text{[8, Theorem 4.2.16]} \text{ or [14, Theorem 3.2]}, \), it suffices to prove

\[ E\left[ \exp \left\{ \delta \int_0^\infty K(1 + |X|^2 + |\bar{X}|^2) \log(e + |X|^2 + |\bar{X}|^2)[\log(e + |X| - |\bar{X}|)]^{-1} \right\} \right] < \infty \]

for some constant \( \delta > 0 \).

By the chain rule and (H1), we have

\[ d(\lambda^{1/4} |X_t - \bar{X}_t|^2) = e^{\lambda t/4} \{ \lambda_1 |X_t - \bar{X}_t|^2 dt + d|X_t - \bar{X}_t|^2 \} \leq \lambda_2 e^{\lambda t} W_2(\nu, \bar{\mu})^2 dt. \]

Then we obtain

\[ |X_t - \bar{X}_t|^2 \leq e^{\lambda t - \lambda_2 t} W_2(\nu, \bar{\mu})^2, \]

this implies that

\[ |X_t - \bar{X}_t|^{-1} \geq e^{\lambda t - \lambda_2 t} (W_2(\nu, \bar{\mu})^{-1}. \]

Let \( \alpha = \sup_{t \geq 0} (|X_t|^2 + |\bar{X}_t|^2), \beta = W_2(\nu, \bar{\mu}), \lambda = \lambda_1 - \lambda_2, \) then we have

\[ \int_0^\infty |A(X_t) - A(\bar{X}_t)| dt \leq K(e + \alpha) \int_0^\infty \frac{dt}{\log(e + \beta^{-1} e^{\lambda t})}. \]

Let \( \beta^{-1} e^{\lambda t} = s \), then we have \( dt = \frac{ds}{\lambda s} \), so that

\[ \int_0^\infty \frac{dt}{\log(e + \beta^{-1} e^{\lambda t})} \]

\[ = \frac{1}{\lambda} \int_{\beta^{-1} s}^\infty \frac{ds}{s \log(e + s)} p \]

\[ \leq \frac{1}{\lambda} \int_{\beta^{-1} s}^{\beta^{-1} s + 1} \frac{ds}{s} + \frac{1}{\lambda} \int_{\beta^{-1} s + 1}^\infty \left( 1 + \frac{e}{1 + \beta^{-1}} \right) \frac{d\log(e + s)}{\log(e + s)} \]

\[ \leq \frac{\log(1 + \beta)}{\lambda} + \frac{\log(1 + e)}{\lambda (p - 1)}. \]
Thus,
\[
\int_0^\infty |A(X_t) - A(\bar{X}_t)| dt \leq K \left\{ \frac{e + \alpha}{\log(e + \alpha)} \cdot \frac{1 + e}{p - 1} + J \right\},
\]
where \( J := \frac{e + \alpha}{\log(e + \alpha)} \cdot \log(e + \beta) \). Let
\[
h(\alpha) = \frac{e + \alpha}{\log(e + \alpha)} \cdot \log(e + \beta) - \alpha.
\]
When \( \alpha \geq \beta \), we have \( h'(\alpha) \leq 0 \), which implies that \( h \) decreases with respect to \( \alpha \), and we obtain that \( J \leq \alpha + e \).

When \( 0 < \alpha < \beta \), let \( g(\alpha) = \frac{e + \alpha}{\log(e + \alpha)} \), we have \( g'(\alpha) \geq 0 \), which implies that \( g \) increases in \( \alpha \), so that \( \sup_{\alpha \in (0, \beta)} h(\alpha) \leq \frac{e + \beta}{\log(e + \beta)} \cdot \log(e + \beta) = e + \beta \).

Combining this with (3.7), we find a constant \( C_0 > 0 \) such that
\[
\int_0^\infty |A(X_t) - A(\bar{X}_t)| dt \leq C_0(e + \alpha + \beta)
\]
\[
= C_0 \left\{ \sup_{t > 0} \{|X_t|^2 + |\bar{X}_t|^2\} + W_2(\nu, \bar{\mu}) + e \right\}.
\]
Since \( E[e^{\delta|\bar{X}_0|^2}] + \bar{\mu}(e^{\delta|\cdot|^2}) < \infty \) for some \( \delta > 0 \) and \( \mathcal{L}_{\bar{X}_t} = \bar{\mu} \), (3.6) follows if
\[
E\left[ \sup_{t > 0} e^{\delta|X_t|^2} \right] < \infty
\]
holds for some \( \delta > 0 \).

Indeed, (3.8) holds also for \( \bar{X}_t \) replacing \( X_t \), since when \( \mathcal{L}_{\bar{X}_0} = \bar{\mu} \), we have \( \mathcal{L}_{(X_t)_{t \geq 0}} = \mathcal{L}_{(\bar{X}_t)_{t \geq 0}} \).

By \( \text{(H1)} \), there exists a constant \( C_1 > 0 \) such that
\[
d\left( e^{(\lambda_1 - \lambda_2)t} |X_t|^2 \right) \leq C_1 e^{(\lambda_1 - \lambda_2)t} dt + 2e^{(\lambda_1 - \lambda_2)t} \langle X_t, \sigma dB_t \rangle.
\]
So,
\[
\delta|X_t|^2 \leq \frac{\delta C_1}{\lambda} + \delta e^{-\lambda_1} |X_0|^2 + 2\delta e^{-\lambda_2} \int_0^t e^{\lambda s} \langle X_s, \sigma dB_s \rangle,
\]
where \( \lambda := \lambda_1 - \lambda_2 \). Therefore, we obtain
\[
E\left[ \sup_{0 \leq s \leq t} e^{\delta|X_s|^2} \right] \leq e^{\delta C_1/\lambda} E\left[ \sup_{0 \leq s \leq t} e^{\delta|X_0|^2} \cdot e^{2\delta e^{-\lambda_1} f_0' e^{\lambda s} \langle X_s, \sigma dB_s \rangle} |\mathcal{F}_0 \right] \]
\[
\leq e^{\delta C_1/\lambda} \left( E\left[ \sup_{0 \leq s \leq t} e^{2\delta|X_0|^2} \cdot e^{2\delta e^{-\lambda_1} f_0' e^{\lambda s} \langle X_s, \sigma dB_s \rangle} \right] \right)^{1/2}.
\]
By the BDG inequality, there exists a constant \( C_2 > 0 \), such that
\[
\tilde{J} := E\left[ \sup_{0 \leq s \leq t} e^{4\delta e^{-\lambda_2} f_0' e^{\lambda s} \langle X_s, \sigma dB_s \rangle} \right]
\]
\[
\leq C_2 \left( E\left[ e^{16\delta^2 e^{-2\lambda t} f_0' e^{2\lambda s} \langle X_s, \sigma dB_s \rangle} \right] \right)^{1/2}
\]
\[
= C_2 \left( E\left[ e^{16\delta^2 f_0' \|\sigma\|^2 |X_s|^2 \frac{1-e^{-2\lambda t}}{2\lambda} e^{-2\lambda (t-s)} ds} \right] \right)^{1/2}.
\]
Since $\lambda_t(ds) := \frac{2\lambda}{1-e^{-2\lambda t}} e^{-2\lambda t} ds$ is a probability measure on $[0,t]$, therefore by the Jensen's inequality, we obtain

$$J \leq C_2 \left( E \left[ e^{16\lambda^2(1-e^{-2\lambda t})} \int_0^t \|\sigma\| |X_t|^2 \lambda_t(ds) \right] \right)^{\frac{1}{2}} \leq C_2 \left( E \left[ \int_0^t e^{8\lambda^2|\sigma|^2|X_t|^2} \lambda_t(ds) \right] \right)^{\frac{1}{2}}. $$

When $t \geq 1$, we have

$$J \leq \frac{C_2}{4\lambda} + \lambda \int_0^t E \left[ e^{\delta|X_t|^2 - 2\lambda(t-s)} \right] ds.$$

Substituting into (3.9) and applying the Gronwall’s lemma, we obtain

$$E \left[ \sup_{0 \leq s \leq t} e^{\delta|X_s|^2} \right] \leq C_3 \left( 1 + E \left[ \sup_{0 \leq s \leq 1} e^{\delta|X_s|^2} \right] \right)$$

(3.10) for some constant $C_3 > 0$. Finally,

$$E \left[ \sup_{0 \leq s \leq 1} e^{\delta|X_s|^2} \right] \leq \left( E \left[ e^{2\delta|X_0|^2} \right] \right)^{\frac{1}{2}} \cdot e^{\delta \int_0^1 \left( 1 + W_2(P^*\sigma,\nu)^2 \right) dt} \cdot \left( E \left[ \sup_{0 \leq s \leq 1} e^{4\delta \int_0^1 (X_s,\sigma dB_s) \right] \right)^{\frac{1}{2}}.$$

By the exponential inequality and the Jensen’s inequality, we obtain that there exists a constant $C_3$ such that

$$E \left[ \sup_{0 \leq s \leq t} e^{\delta|X_s|^2} \right] \leq C_3 \sqrt{C} \left( \int_0^1 E \left[ e^{16\lambda^2\|\sigma\|^2|X_t|^2} \right] ds \right)^{\frac{1}{2}}.$$

Taking $\delta \leq 1/(16\|\sigma\|^2)$ and we obtain that $E[\sup_{0 \leq s \leq t} e^{\delta|X_s|^2}] < \infty$. This together with (3.10) imply that $E[\sup_{0 \leq s \leq t} e^{\delta|X_s|^2}] < \infty$.

3.4. Proof of Theorem 2.4. Let

$$\rho(\xi_1, \xi_2) := \left( |\xi_1^{(1)} - \xi_2^{(1)}|^2 + |\xi_1^{(2)} - \xi_2^{(2)}|^2 \right)^{1/2}.$$

We take $X_0, Y_0 \in L^2(\Omega \rightarrow \mathbb{R}^{m+d}, \mathcal{F}_0, \mathbb{P})$ such that $\mathbb{L}X_0 = \mu_0, \mathbb{L}Y_0 = \nu_0$ and $W_2(\mu_0, \nu_0)^2 = \mathbb{E}\rho(X_0, Y_0)^2$.

Let $X_t = (X_t^{(1)}, X_t^{(2)})$ and $Y_t = (Y_t^{(1)}, Y_t^{(2)})$ solve (2.4) with initial values $X_0$ and $Y_0$ respectively. Obviously, $X_t^{(1)} - Y_t^{(1)}$ and $X_t^{(2)} - Y_t^{(2)}$ solve the ODE

\begin{align*}
\frac{d(X_t^{(1)} - Y_t^{(1)})}{dt} &= \left( A(X_t^{(1)} - Y_t^{(1)}) + B(X_t^{(2)} - Y_t^{(2)}) \right) dt, \\
\frac{d(X_t^{(2)} - Y_t^{(2)})}{dt} &= \left( Z(X_t, \mathbb{L}X_t) - Z(Y_t, \mathbb{L}Y_t) \right) dt.
\end{align*}

(3.11)

Since $r_0 \in (-\|B\|^{-1}, \|B\|^{-1})$, for any $r > 0$ there exists a constant $C > 1$ such that

$$\frac{1}{C} (|X_t^{(1)} - Y_t^{(1)}|^2 + |X_t^{(2)} - Y_t^{(2)}|^2) \leq \Psi_t := r_0 \left( \frac{r_0}{2} |X_t^{(1)} - Y_t^{(1)}|^2 + \frac{1}{2} |X_t^{(2)} - Y_t^{(2)}|^2 + rr_0 (X_t^{(1)} - Y_t^{(1)}, B(X_t^{(2)} - Y_t^{(2)})) \right) \leq C (|X_t^{(1)} - Y_t^{(1)}|^2 + |X_t^{(2)} - Y_t^{(2)}|^2).$$

Combining this with (3.11) and (D3), we obtain

$$d\Psi_t \leq -\theta_1 (|X_t^{(1)} - Y_t^{(1)}|^2 + |X_t^{(2)} - Y_t^{(2)}|^2) + \theta_2 W_2(P^{*\mu}_t, \nu_0)^2,$$
by the chain rule, we have
\[ d(e^{\lambda t} \Psi_t) \leq e^{\lambda t} \left\{ \lambda \Psi_t - \theta_1 \left( |X_t^{(1)} - Y_t^{(1)}|^2 + |X_t^{(2)} - Y_t^{(2)}|^2 \right) + \theta_2 W_2(P_t^* \mu_0, P_t^* \nu_0)^2 \right\} dt, \]
thus we obtain
\[ \mathbb{E} \Psi_t \leq e^{-\lambda t} \mathbb{E} \Psi_0 - e^{-\lambda t} \int_0^t e^{\lambda s}(\theta_1 - \theta_2 - \lambda C)\mathbb{E} \left[ |X_s^{(1)} - Y_s^{(1)}|^2 + |X_s^{(2)} - Y_s^{(2)}|^2 \right] ds, \]
we take \( \lambda = \frac{\theta_1 - \theta_2}{2C} \) and we obtain
\[ \mathbb{E} \Psi_t \leq e^{-\frac{\theta_1 - \theta_2}{2C} t} \mathbb{E} \Psi_0, \]
and we deduce that
\[ W_2(P_t^* \mu_0, P_t^* \nu_0)^2 \leq \mathbb{E}[\rho(X_t, Y_t)^2] \leq C e^{-\frac{\theta_1 - \theta_2}{2C} t} W_2(\mu_0, \nu_0)^2, \]
Consequently, \( P_t^* \) has a unique invariant probability measure \( \bar{\mu} \) such that (2.9) holds.
Next, let \( Z_{X_t} = \bar{\mu} \), consider the reference Stochastic Hamiltonian System for \( \bar{X}_t = (\bar{X}_t^{(1)}, \bar{X}_t^{(2)}) \) on \( \mathbb{R}^{m+d} \):
\[
\begin{align*}
\frac{d\bar{X}_t^{(1)}}{dt} &= (A\bar{X}_t^{(1)} + B\bar{X}_t^{(2)})dt, \\
\frac{d\bar{X}_t^{(2)}}{dt} &= Z(\bar{X}_t, \bar{\mu})dt + MdB_t.
\end{align*}
\]
According to [19, Theorem 3.1], \( \bar{X}_t \in \text{MDP}(I) \) for \( I(y) = y^2/(8\bar{V}(A)) \). Since \( A \) is Lipschitz continuous, by (2.9), we can find some small \( \delta > 0 \) such that (3.2) holds. Therefore, the proof is finished by Theorem 3.2.

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