Two-loop gluon-condensate contributions to heavy-quark current correlators: exact results and approximations

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Abstract The coefficient functions of the gluon condensate \( \langle G^2 \rangle \), in the correlators of heavy-quark vector, axial, scalar and pseudoscalar currents, are obtained analytically, to two loops, for all values of \( z = q^2/4m^2 \). In the limiting cases \( z \to 0 \), \( z \to 1 \), and \( z \to -\infty \), comparisons are made with previous partial results. Approximation methods, based on these limiting cases, are critically assessed, with a view to three-loop work. High accuracy is achieved using a few moments as input. A single moment, combined with only the leading threshold and asymptotic behaviours, gives the two-loop corrections to better than 1% in the next 10 moments. A two-loop fit to vector data yields \( \langle \frac{\alpha_s}{\pi} G^2 \rangle \approx 0.021 \text{ GeV}^4 \).
1 Introduction

Whilst considerable progress on multi-loop diagrams was made between the AI-HENP 92 [1] and AI-HENP 93 [2] workshops, there is still a pressing need for methods that exploit hard-won analytical results by efficient techniques of numerical approximation. Progress in this direction was made, independently, in [3] and [4], whose authors combine in this present work, which derives new analytical two-loop results for heavy-quark current correlators and uses them to refine the phenomenological extraction [4] of the gluon condensate and to assess the numerical methods of [3, 4].

In Section 2, we obtain analytically, to two loops, the coefficient functions of the gluon condensate, $\langle G^2 \rangle$, in the correlators of heavy-quark vector, axial, scalar and pseudoscalar currents, for all values of $z = q^2/4m^2$. This work is therefore an extension of that in [5, 6], where the first 7 moments of these coefficient functions were analytically computed, by reducing the problem to the evaluation of vacuum scalar two-loop diagrams with one massless and two massive lines and arbitrary integer indices. Here we apply the more general strategy of [4, 7], reducing the problem to evaluation of two-point diagrams, of the type needed for the two-loop photon propagator [8], and using the programs developed for [4] to obtain explicit analytical expressions for such diagrams, in $d = 4 - 2\varepsilon$ spacetime dimensions. As input to these programs, we use the intermediate results of [5, 6], obtained after the calculation of traces and the action of the projector for the operator $G^2$.

Renormalization of the resulting bare two-loop terms amounts to no more than mass renormalization of the $d$-dimensional one-loop contributions, which we perform in the on-shell scheme, expressing our results in terms of the pole-mass, $m$. Doing this, we discover that a term was missed in [4, 6], when performing the limit $\varepsilon \to 0$ in the $\overline{\text{MS}}$ scheme, with the effect that the results given in [4, 6] do not hold in the $\overline{\text{MS}}$ scheme (nor in any scheme that respects the Ward identity relating axial and pseudoscalar correlators). After correcting this inconsistency, we find agreement between our new results and the partial results of [4, 6] for the first 7 moments. In addition to studying this $z \to 0$ limit in Section 3, we also compare our results with those obtained in [8], as $z \to -\infty$, and in [10], as $z \to 1$, finding agreement with the leading term, in each case.

In Section 4, we assess numerical approximations, developed in [2, 3, 4, 11, 12], by comparing predictions, made on the basis of previously available input [4, 8, 10], with exact new two-loop results. The outcome is most satisfactory: with 7 moments as input, the methods achieve very high accuracy; with 4 moments, they fare almost as well; most remarkably, we shall show that a single moment, combined with only the leading behaviours as $z \to 1$ and $z \to -\infty$, gives the two-loop terms in the next 10 moments to better than 1% accuracy.

Section 5 gives our conclusions, concerning existing phenomenology and future calculations, both analytical and numerical. One significant motivation for studying numerical approaches with limited analytical input concerns the feasibility of approximating the three-loop photon propagator from a few of its moments, obtainable by the methods of [13], and then using it for applications such as the muon anomaly [14]. We suggest that such an approach to ambitious calculations in both QED and QCD is indeed feasible.
2 Exact results

We evaluate contributions to the correlators, $i \int e^{ixz} \langle T(J(x)J(0)) \rangle dx$, of the vector (V) current $J^V_\mu = \overline{\psi} \gamma_\mu \psi$, the axial (A) current $J^A_\mu = \overline{\psi} \gamma_\mu \gamma_5 \psi$, the scalar (S) current $J^S = 2m \overline{\psi} \psi$, and the pseudoscalar (P) current $J^P = 2m \overline{\psi} \gamma_5 \psi = \partial^\mu J^A_\mu$, where $\psi$ is a heavy-quark field of mass $m$. We choose to work in the on-shell (OS) scheme, where the results are simplest. Since the currents have no anomalous dimensions, one may translate the results to any other scheme (e.g. the $\overline{\text{MS}}$ scheme) by making a one-loop transformation from the pole mass, $m$, to the renormalized mass of that scheme (e.g. by using $m = \overline{m}(\mu)(1 + (\ln(\mu^2/m^2) + \frac{1}{3})\alpha_s/\pi + O(\alpha_s^2))$, where $\overline{m}(\mu)$ is the $\overline{\text{MS}}$ mass, at scale $\mu$).

We denote the correlators of $J^{S,P}$ by $\Pi^{S,P}(q^2)$ and decompose the tensor structure of the vector and axial correlators as follows:

$$i \int e^{ixz} \langle T(J^V_\mu(x)J^V_\nu(0)) \rangle dx = \Pi^V(q^2)(q_\mu q_\nu - q^2 g_{\mu\nu}),$$

$$i \int e^{ixz} \langle T(J^A_\mu(x)J^A_\nu(0)) \rangle dx = \Pi^A(q^2) \left( \frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right) + \frac{\Pi^P(q^2) - \Pi^P(0)}{q^4} q_\mu q_\nu,$$

where $\Pi^P(0) = -4(\overline{m} \overline{\psi} \psi)$ enters $\Pi$ as an equal-time commutator \cite{13}. Finally, we define dimensionless coefficients of the non-perturbative gluon condensate by writing the dimension-4 contributions to the correlators as

$$\Pi^J_{np}(q^2) = \frac{\langle (\alpha_s/\pi) G^{a_\mu\nu} G_{a_\mu \nu} \rangle}{(2m)^{n_J}} \left( C^J(z) + O(\alpha_s^2) \right); \quad C^J(z) = C^J_1(z) + \frac{\alpha_s}{\pi} C^J_2(z),$$

in the channels $J = V, A, S, P$, with exponents $n_J = 4, 2, 0, 0$, respectively, making $C^J(z)$ a dimensionless function of $z \equiv q^2/4m^2$, with one- and two-loop contributions $C^J_{1,2}(z)$. Since the currents are not renormalized, and $\alpha_s G^2$ is not renormalized at one loop, the renormalization scale of $\alpha_s$ in the two-loop term is irrelevant here. It should, however, be taken as $O(m)$, to suppress large logarithms at three-loop order \cite{13}.

Using REDUCE \cite{17} for the trace calculations of \cite{3, 6} and for the recursion of the resulting scalar integrals of \cite{4}, we obtained the following on-shell results in the 4 channels:

$$48z^2(1 - z)^2 C^V(z) = -3 + 4z - 4z^2 + 3(1 - 2z)G(z) + \frac{\alpha_s}{\pi} P^V(z),$$

$$8z(1 - z) C^A(z) = 1 - 2z - G(z) + \frac{\alpha_s}{\pi} P^A(z),$$

$$16z(1 - z) C^S(z) = -1 - 2z + (1 - 4z)G(z) + \frac{\alpha_s}{\pi} P^S(z),$$

$$48(1 - z)^2 C^P(z) = 7 - 10z + 3(3 - 4z)G(z) + \frac{\alpha_s}{\pi} P^P(z),$$

$$P^J(z) = P^J_1(z) + P^J_2(z)G(z) + P^J_3(z)G^2(z) + (1 - z)P^J_4(z)H(z),$$

with polynomials, $P^J_i(z)$, given in Table 1. As $d \to 4$, two basic integrals are encountered:

$$G(z) = \frac{2y}{y^2 - 1} \ln y; \quad y = \frac{\sqrt{1 - 1/z - 1}}{\sqrt{1 - 1/z + 1}},$$

$$H(z) = \frac{2y}{y^2 - 1} \left( \text{Li}_2(y^2) - \text{Li}_2(y) + (2 \ln(1 + y) + \ln(1 - y) - \frac{3}{4} \ln y) \ln y \right)\text{,}$$

with dilogarithms $\text{Li}_2(y^p) = \sum_{n \geq 0} y^{pn}/n^2, p = 1, 2$, giving Clausen’s integral if $z \in [0, 1]$. 

2
3 Limiting cases

For $z \to 0$, one may approximate the coefficients by truncation of their Taylor series

$$C^J(z) = \sum_{n=0}^{\infty} \left( a_n^J + \frac{a_n^J}{\pi} c_n^J \right) z^n; \quad b_n^J \equiv c_n^J / a_n^J,$$

with one-loop moments given by [18]

$$a_n^V = -\frac{2n + 2}{15} \left( \frac{4}{z} \right)_n, \quad a_n^A = -\frac{1}{3} \left( \frac{6}{z} \right)_n, \quad a_n^S = -\frac{3n + 4}{12} \left( \frac{5}{z} \right)_n, \quad a_n^P = -\frac{n - 4}{12} \left( \frac{3}{z} \right)_n,$$

where $(a)_n = \Gamma(a + n)/\Gamma(a)$. The two-loop corrections, $b_n^J$, are easily obtained from Table 1, using the expansions [16]

$$G(z) = \sum_{n=0}^{\infty} g_n z^n, \quad (1 - z)G^2(z) = \sum_{n=0}^{\infty} g_n z^n / n + 1, \quad H(z) = \sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} \frac{1}{2r + 1} - \sum_{r=1}^{n} \frac{3}{2r} \right) g_n z^n,$$

with $g_n = n!/(3/2)_n$. Results for $n \leq 10$ are given in Table 2, which may be extended up to $n = 200$ in a few minutes of CPU time. Since $a_n^V = 0$, $b_n^V$ is undefined. We find that $c_n^P = -3266/4725$. Two features of Table 2 are notable: the relation $b_n^V = b_n^A$ ensures the absence of a singularity in (2) as $q^2 \to 0$; the scheme-independent value $b_n^P = b_n^S = 11/4$ gives, via a Ward identity [15, 16], the new two-loop term in the heavy-quark expansion

$$\langle m\bar{\psi}\psi \rangle = -\frac{1}{4} \Pi_{np}(0) = -\frac{\langle \alpha_s G^2 \rangle}{12\pi} \left( 1 + \frac{11 \alpha_s}{4\pi} + O(\alpha_s^2) \right) + O(1/m^2),$$

whose one-loop term was used in [19]. In the MS scheme, at $\mu = m$, one obtains two-loop corrections $\Pi_n^V = b_n^V - \frac{4}{3}(2n + n)$. Comparing Table 2 with the corresponding MS tables in [3], we find that the latter give $(b_n^V - 2)$, instead of $\Pi_n^V$, for the reasons given in the introduction.

Next we consider the limit $z \to -\infty$, which yields the asymptotic corrections

$$C_n^V / C_V^1 \to 5, \quad C_n^A / C_A^1 \to \frac{34}{9}, \quad C_n^S / C_S^1 \to 6 + 2 \ln \left( \frac{m^2}{-q^2} \right), \quad C_n^P / C_P^1 \to 4 + 2 \ln \left( \frac{m^2}{-q^2} \right).$$

To relate these results to previous work, we use the methods of [14]. The vector and axial light-quark results of [3], combined with our heavy-quark result (14), give

$$q^4 \Pi_{np}(q^2) \to \frac{\langle \alpha_s G^2 \rangle}{12\pi} \left( 1 + \frac{7 \alpha_s}{6\pi} \right) + 2 \langle m\bar{\psi}\psi \rangle \left( 1 + \frac{1}{3\pi} \right) = -\frac{\langle \alpha_s G^2 \rangle}{12\pi} \left( 1 + \frac{5 \alpha_s}{4\pi} \right),$$

$$q^2 \Pi_{np}(q^2) \to \frac{\langle \alpha_s G^2 \rangle}{12\pi} \left( 1 + \frac{7 \alpha_s}{6\pi} \right) - 2 \langle m\bar{\psi}\psi \rangle \left( 1 + \frac{7 \alpha_s}{3\pi} \right) = \frac{\langle \alpha_s G^2 \rangle}{12\pi} \left( 1 + \frac{34 \alpha_s}{9\pi} \right).$$

The (pseudo)scalar results of [3] similarly agree with (14,15). Note, in (16), the change in sign between one-loop light- and heavy-quark $G^2$ terms. (The authors of [3] regret that a one-loop sign error in their previous work obscured the need to use [19].)

Finally, for the threshold behaviour in the vector channel, as $z \to 1$, we obtain

$$C_n^V(z) = -\frac{197}{2304 \pi} \left( \frac{4}{1 - z} \right)^2 + \frac{65}{768 \pi} \left( \frac{1}{1 - z} \right)^{5/2} - \frac{413}{6912 \pi} \left( \frac{1}{1 - z} \right)^2 + \frac{17}{72} \pi \ln(1 - z) + \frac{17}{72} \pi \ln(1 - z) + \frac{1}{1 - z^{3/2}},$$

whose first term agrees with [14]. (An input used in [3] is ruled out by the second term.)
4 Approximations

We now assess previous numerical methods [2, 3, 4, 12], by testing the accuracy to which they predict new features of the exact result for $C^N_2(z)$. As input for the methods we take the following ‘old’ data: the leading asymptotic behaviour, from [3]; the leading term in the threshold expansion, from [10]; the radiative corrections $\{b^N_n| n < 7\}$ to the first 7 moments. (This is the input used in [3], corrected for errors in [3, 5, 6].)

First we consider the approximate spectral ansatz of [3], which is identical to using the following (exact) one-loop and (approximate) two-loop threshold expansions:

$$C^N_1(z) = -\frac{1}{6}F_2(z) + \frac{1}{30}F_6(z), \quad C^N_2(z) = \sum_{k=1}^{N+3} f_k F_k(z); \quad F_k(z) = 2F_1(3, 1; \frac{k+1}{2}; z), \quad (19)$$

with hypergeometric basis functions, $F_k(z)$, giving polynomials in $1/(1-z)$, for $k = 1, 3, 5$, and terms involving $G(z)$ or $\ln(1-z)$, otherwise. The $(N+3)$ two-loop coefficients, $f_k$, are determined by the first $N$ moments and by 3 further constraints [3]:

$$f_1 = -\frac{197\pi^2}{2304}, \quad \sum_{k=2}^{N+3} (k-1)f_k = 0, \quad \sum_{k=2}^{N+3} k(k-1)f_k = \frac{10}{3}, \quad (20)$$

giving the leading threshold singularity, derived from [10], and $z^2C^N_2(z) \rightarrow -5/12$, derived from [3], as $z \rightarrow -\infty$. We remark that ansatz (19) of [3] cannot reproduce the form of the 4th term in the true threshold expansion (18). We assess it by comparing the values of $f_{2,3}$, required by the input, with those required by the second and third terms in (18), namely $f_2 = 65/144$ and $f_3 = -413\pi^2/3456$, and also by comparing the output for $\{b^N_n| n = N \ldots 10\}$ and $B(z) = C^N_2(z)/C^N_1(z)$ with exact results. Table 3 shows the very high accuracy achieved with 7 moments, in all but the $f_{2,3}$ tests. The results using only 4 moments are almost as good, except for $z \sim 3$. Remarkably, the $N = 1$ column shows that just 3 input numbers, namely $b^N_0$, $f_1$ and $B(-\infty)$, give 10 additional moments to better than 1% accuracy, which is most encouraging for three-loop applications.

For the last 3 columns of Table 3, we mapped [12] and Padé-approximated [2, 4, 12]

$$z(1-z)^2C^N_2(z) + \frac{5z}{12} \frac{f_1}{1-z} = \frac{D(\omega)}{1-\omega}; \quad z = \frac{4\omega}{(1+\omega)^2}, \quad (21)$$

with the $z$-plane, cut along $z \in [1, \infty]$, mapped to the unit disk, $|\omega| < 1$. By construction, $D(\omega)$ is finite at $\omega = 1$ (i.e. $z = 1$) and diverges only logarithmically as $\omega \rightarrow -1$ (i.e. $z \rightarrow -\infty$). Its value at $\omega = 0$ (i.e. $z = 0$) is $-f_1$. With $N = 1, 4, 7$ moments as input, we computed the $[0/1], [2/2], [3/4]$ Padé approximants to $D(\omega)$, which were expanded to give the further moments of Table 3. From the approximations to $D(1)$ and $D'(1)$, we obtained the second and third terms in the Padé-approximated threshold expansion and compared them with the exact ones, in (18). Finally, we computed Padé-approximated values of $B(z) = C^N_2(z)/C^N_1(z)$, using [12] $\omega = \exp(2i \arccos z^{-1/2})$ in the cases with $z \in [1, \infty]$.

Table 3 clearly demonstrates that both (19) and (21) are highly effective predictors of additional moments, even with very limited input. More input is needed to achieve high accuracy on the cut, since neither method performs well in the $f_{2,3}$ tests. With $N = 10$ moments, the Padé method gives the modulus of $C^N_2(z)$ to 0.05% and the phase to 0.04°, on the entire cut, whilst the figures for ansatz (19) are 0.5% and 0.2°, respectively.
Summary and conclusions

Our main analytical results are in [4–10] and Table 1, which give the two-loop coefficient functions of the gluon condensate in the correlators of heavy-quark currents. As suggested in [1, 4], computer algebra in $d$ dimensions proves to be a most efficient way of obtaining new 4-dimensional results, whose analytical complexity enters only at the final stage, in this case via the dilogarithms of (10).

By combining and refining the REDUCE programs developed for [4, 5], we obtained a complete analytical result in the vector channel in about 2 hours of CPUtime on a 486DX2-50 PC, whose timing in the standard REDUCE test is 3 s. In comparison, it took 20 hours to obtain merely the first 7 moments in [5], using a machine with a benchmark of 11 s. Our improvement in generality and speed entailed considerable programming effort to produce efficient code, of the type used in [4], in order to process the many scalar integrals in the 0.7 Mbyte of output from the programs used in [5]. This involved systematic implementation of recurrence relations for $5\,F_4$ hypergeometric functions, which was exhaustively checked by independent FORM code, written for [4].

We have found full agreement with previous results in three limiting cases: $z \to 0$, $z \to -\infty$, and $z \to 1$, after correcting errors in [3, 5, 6]. Using a Ward identity, we have obtained the new two-loop term in the heavy-quark expansion (14).

Combining our new results with the procedure used in [3], to estimate the gluon condensate from empirical data [20], we arrive at $\langle (\alpha_s/\pi)C_\mu^a C^{\mu\nu}_a \rangle \approx 0.021$ GeV$^4$, as compared with 0.025 GeV$^4$ in [3]. Our two-loop extraction thus gives a value that is still about twice as large as the effective one-loop value [4].

We have critically assessed two methods of numerical estimation: the approximate spectral [3] ansatz (19); and Padé approximation [4, 12] after the mapping (21). Table 3 shows that each performs rather well, even with limited input. Despite its analytical simplicity (rational approximation to $D(\omega)$ in (21)) the Padé method is as good as (19) for moment prediction and better for achieving high accuracy on the cut. We recommend both methods for future applications, with agreement between them providing a useful cross check. Note that the Padé method is more general, as it does not rely on the specific form of a lower-loop result. Most importantly, one should use information from all 3 limiting cases, in each method. We find that 4-figure accuracy can be obtained on the cut using only 10 moments in the Padé method of (21), whilst 17 moments are needed to achieve comparable accuracy using only the methods of [12], without the limiting values as $z \to 1$ and $z \to -\infty$.

In conclusion, we believe that our analytical progress and accuracy of numerical approximation offer real prospects of extending the two-loop approaches of [4, 11, 12] to three loops. Future work can be based on calculation of a few three-loop terms as $z \to -\infty$ [14] and $z \to 0$ [13, 14], for both QED and QCD two-point functions. In both theories, the leading three-loop term at threshold is already known [10]. We anticipate being able to obtain an accurate three-loop approximation for the photon propagator, for use in multi-loop QED [14], and accurate estimates of three-loop corrections to QCD sum rules. Finally we note that mapped Padé methods are well suited to a very wide variety of standard-model applications, as indicated by their successful application to two-loop three-point functions [2, 12].
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Table 1: The polynomial coefficients of (8), for use in (1), (4)

\[
P^V(z) = -\frac{1}{22} (613 + 8828z - 22092z^2 + 13248z^3) \\
P^V(z) = \frac{1}{72} (659 - 3774z + 30184z^2 - 55344z^3 + 26496z^4) \\
P^V(z) = -\frac{1}{36} (23 + 1456z - 41064z^2 + 195024z^3 - 350688z^4 + 275328z^5 - 79488z^6) \\
P^V(z) = \frac{1}{9}z(473 - 3614z + 6504z^2 - 3312z^3) \\
P^P(z) = \frac{1}{216} (1241 - 8390z + 6624z^2) \\
P^P(z) = -\frac{1}{216} (1339 + 4128z - 17656z^2 + 13248z^3) \\
P^P(z) = \frac{1}{108} (49 - 2404z + 32328z^2 - 98112z^3 + 107616z^4 - 39744z^5) \\
P^P(z) = \frac{4}{9}z(121 - 632z + 552z^2) \\
P^P(z) = -\frac{1}{24} (3213 - 9542z + 6624z^2) \\
P^P(z) = \frac{1}{24} (2739 + 9428z - 26104z^2 + 13248z^3) \\
P^P(z) = -\frac{1}{24} (99 - 11470z + 73008z^2 - 151776z^3 + 130080z^4 - 39744z^5) \\
P^P(z) = \frac{4}{9} (54 - 1299z + 2936z^2 - 1656z^3) \\
\]

Table 2: Values of $b^j_n = c^j_n/a^j_n$ for $J = V, A, S, P$ and $n \leq 10$

| $n$ | $V$ | $A$ | $S$ | $P$ |
|-----|-----|-----|-----|-----|
| 0   | 1469.62 | 131.24 | 11.4 | 11.4 |
| 1   | 135779 | 3193 | 2447 | 2447 |
| 2   | 12960 | 486 | 501 | 501 |
| 3   | 1969 | 7263 | 2813 | 2813 |
| 4   | 168 | 972 | 140 | 140 |
| 5   | 546421 | 124847 | 160591 | 160591 |
| 6   | 42525 | 15120 | 21840 | 21840 |
| 7   | 661687433 | 36991619 | 400895 | 400895 |
| 8   | 476280000 | 488400 | 488400 | 488400 |
| 9   | 1808293669 | 604286927 | 1457 | 1457 |
| 10  | 121080960 | 62868960 | 160 | 160 |
| 11  | 180657583657 | 5566166237 | 1672757 | 1672757 |
| 12  | 11442150720 | 544864320 | 1698840 | 1698840 |
| 13  | 7294752403 | 2643899563 | 30924999 | 30924999 |
| 14  | 431887550 | 2318866440 | 7857900 | 7857900 |
| 15  | 377046256819 | 943766186719 | 32147183403 | 32147183403 |
| 16  | 21549939840 | 833624296900 | 28817268480 | 28817268480 |
| 17  | 22322605695461 | 9817563210793 | 3051820292371 | 3051820292371 |
| 18  | 1220106888000 | 8296728838400 | 2597964969600 | 2597964969600 |
| 19  | 16257042450882 | 38349293613443 | 2191394981453 | 2191394981453 |
| 20  | 8527479553593 | 3111272564400 | 1780863804800 | 1780863804800 |
| 21  | 1496529600 | 1496529600 | 1496529600 | 1496529600 |
Table 3: Assessment of ansatz (19) and mapping (21), using $N$ moments

| test feature | exact result | ansatz $N = 7$ | ansatz $N = 4$ | ansatz $N = 1$ | mapped $N = 7$ | mapped $N = 4$ | mapped $N = 1$ |
|--------------|--------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $b^V_0$      | 9.06790      | input          | input          | input          | input          | input          | input          |
| $b^V_1$      | 10.4768      | input          | input          | 10.4484        | input          | input          | 10.4778        |
| $b^V_2$      | 11.7202      | input          | input          | 11.6675        | input          | input          | 11.7281        |
| $b^V_3$      | 12.8494      | input          | input          | 12.7761        | input          | input          | 12.8666        |
| $b^V_4$      | 13.8928      | input          | 13.8928        | 13.8019        | input          | 13.8927        | 13.9203        |
| $b^V_5$      | 14.8685      | input          | 14.8684        | 14.7625        | input          | 14.8682        | 14.9065        |
| $b^V_6$      | 15.7888      | input          | 15.7885        | 15.6695        | input          | 15.7881        | 15.8369        |
| $b^V_7$      | 16.6625      | 16.6625        | 16.6620        | 16.5315        | 16.6625        | 16.6614        | 16.7204        |
| $b^V_8$      | 17.4964      | 17.4964        | 17.4955        | 17.3549        | 17.4964        | 17.4946        | 17.5636        |
| $b^V_9$      | 18.2956      | 18.2956        | 18.2944        | 18.1446        | 18.2956        | 18.2931        | 18.3715        |
| $b^V_{10}$   | 19.0643      | 19.0643        | 19.0626        | 18.9046        | 19.0643        | 19.0609        | 19.1485        |
| $-f_1$       | 0.84388      | input          | input          | input          | input          | input          | input          |
| $f_2$        | 0.45139      | 0.45868        | 0.50439        | 0.57116        | 0.47972        | 0.53134        | 0.40795        |
| $-f_3$       | 1.17944      | 1.49727        | 2.09688        | 2.23783        | 1.86522        | 2.39296        | 1.62672        |

$B(-\infty)\ 5.00000$  | input          | input          | input          | input          | input          | input          | input          |
$B(-9)\ 5.58641$   | 5.58639        | 5.58495        | 5.62888        | 5.58733        | 5.58208        | 5.61983        |
$B(-3)\ 6.28843$  | 6.28843        | 6.28800        | 6.33614        | 6.28847        | 6.28730        | 6.31060        |
$B(-0.9)\ 7.41565$ | 7.41565        | 7.41561        | 7.44826        | 7.41565        | 7.41557        | 7.42198        |
$B(-0.3)\ 8.29735$ | 8.29735        | 8.29735        | 8.31290        | 8.29735        | 8.29734        | 8.29824        |
$B(0.3)\ 10.3704$ | 10.3704        | 10.3704        | 10.3449        | 10.3704        | 10.3704        | 10.3745        |
$B(0.9)\ 25.6939$ | 25.6939        | 25.6835        | 25.4851        | 25.6928        | 25.6745        | 25.8290        |
$|B(3)|\ 1.81071$   | 1.80980        | 1.91045        | 2.47868        | 1.80718        | 2.00465        | 2.21432        |
$|B(9)|\ 4.47078$   | 4.47366        | 4.44784        | 4.46713        | 4.46202        | 4.43571        | 4.69021        |
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