Research Article

Ju Myung Kim*

Approximation properties of tensor norms and operator ideals for Banach spaces

Abstract: For a finitely generated tensor norm $\alpha$, we investigate the $\alpha$-approximation property ($\alpha$-AP) and the bounded $\alpha$-approximation property (bounded $\alpha$-AP) in terms of some approximation properties of operator ideals. We prove that a Banach space $X$ has the $\lambda$-bounded $g_{p,q}$-AP if it has the $\lambda$-bounded $g_{p}$-AP. As a consequence, it follows that if a Banach space $X$ has the $\lambda$-bounded $g_{p}$-AP, then $X$ has the $\lambda$-bounded $w_{p}$-AP.

Keywords: approximation property, tensor norm, Banach operator ideal

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1 Introduction

The main subjects of this paper originate from the classical approximation properties for Banach spaces, which was systematically investigated by Grothendieck [1]. A Banach space $X$ is said to have the approximation property (AP) if

$$\text{id}_X \in \mathcal{F}(X, X)^c,$$

where $\text{id}_X$ is the identity map on $X$, $\mathcal{F}$ is the ideal of finite rank operators and $\tau_c$ is the topology of uniform convergence on compact sets.

Let $X$ and $Y$ be Banach spaces. We denote by $X \otimes Y$ the algebraic tensor product of $X$ and $Y$. The normed space $X \otimes Y$ equipped with a norm $\alpha$ is denoted by $X \circlearrowleft_{\alpha} Y$ and its completion is denoted by $X \otimes_{\alpha} Y$. The basic two norms on $X \otimes Y$ are the injective norm $\varepsilon$ and the projective norm $\pi$ which are defined as follows.

$$\varepsilon(u; X, Y) = \sup \left\{ \sum_{j=1}^{n} x_j^* y_j^* : x_j^* \in B_{X^*}, y_j^* \in B_{Y^*} \right\},$$

where $\sum_{j=1}^{n} x_j \otimes y_j$ is any representation of $u$ and we denote by $B_Z$ the closed unit ball of a normed space $Z$.

$$\pi(u; X, Y) = \inf \left\{ \sum_{j=1}^{n} \| x_j \| \| y_j \| : u = \sum_{j=1}^{n} x_j \otimes y_j, n \in \mathbb{N} \right\}.$$

It is well known that a Banach space $X$ has the AP if and only if for every Banach space $Y$, the natural map $J_{\pi} : Y \otimes_{\pi} X \to Y \otimes_{c} X$ is injective (cf. [2, Theorem 5.6]). This equivalent statement can be naturally extended to tensor norms. For basic definitions and general background of the theory of tensor norms, we refer to [2,3]. For a finitely

* Corresponding author: Ju Myung Kim, Department of Mathematics and Statistics, Sejong University, Seoul 05006, Korea, e-mail: kjm21@sejong.ac.kr

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generated tensor norm \( \alpha \), a Banach space \( X \) is said to have the \( \alpha \)-AP if for every Banach space \( Y \), the natural map 

\[
I_\alpha : Y \hat{\otimes}_\alpha X \to Y \otimes_\varepsilon X
\]

is injective (cf. [2, Section 21.7]). It is well known that if a Banach space \( X \) has the AP, then it has the \( \alpha \)-AP for every finitely generated tensor norm \( \alpha \) (cf. [2, Proposition 21.7(1)]).

Some of the well-known tensor norms can be obtained from the tensor norm \( \alpha_{p,q} \) (1 \( \leq p, q \leq \infty \), \( 1/p + 1/q \geq 1 \)), which was introduced by Lapresté [4]. For \( 1 \leq p < \infty \), \( \ell_p^w(X) \) stands for the Banach space of all \( X \)-valued weakly \( p \)-summable sequences endowed with the norm \( \| \cdot \|_p^w \). Let \( 1 \leq r \leq \infty \) with \( 1/r = 1/p + 1/q - 1 \). For \( u \in X \otimes Y \), let

\[
\alpha_{p,q}(u) = \inf \left\{ \| (\lambda_j)_{j=1}^n \|_p \| (x_j)_{j=1}^n \|_q \| (y_j)_{j=1}^n \|_p : u = \sum_{j=1}^n \lambda_j x_j \otimes y_j, n \in \mathbb{N} \right\},
\]

where \( p^* \) is the conjugate index of \( p \). Then \( \alpha_{p,q} \) is a finitely generated tensor norm and the transposed tensor norm \( \alpha_{q,p}^* = \alpha_{q,p} \) (cf. [2, Proposition 12.5]). The special cases \( g_p = \alpha_{p,1} \) and \( d_p = \alpha_{1,p} \) are called the Chevet-Saphar tensor norms [5,6] and \( a_1 = \pi \). The tensor norm \( w_p = \alpha_{p,p^*} \) is also well known. Díaz et al. [7] studied the \( \alpha_{p,q} \)-AP in terms of certain approximation properties of operator ideals. As a consequence, it was shown that a Banach space \( X \) has the \( \alpha_{p,q} \)-AP if it has the \( \alpha_{p,1} \)-AP.

Let \( \lambda \geq 1 \). A Banach space \( X \) is said to have the \( \lambda \)-bounded AP if 

\[
\text{id}_X \in \{ S \in \mathcal{L}(X,X) : \| S \| \leq \lambda \}.
\]

It is well known that a Banach space \( X \) has the \( \lambda \)-bounded AP if and only if for every Banach space \( Y \), the natural map 

\[
I_\lambda : Y \otimes_\lambda X \to (Y^* \otimes_\varepsilon X^*)^*
\]

satisfies \( \pi(u; Y, X) = \lambda \| I_\lambda(u) \|_{Y^* \otimes_\omega X^*} \) for every \( u \in Y \otimes X \) (cf. [2, Corollary 16.3.2]). More generally, for a finitely generated tensor norm \( \alpha \), a Banach space \( X \) is said to have the \( \lambda \)-bounded \( \alpha \)-AP if for every Banach space \( Y \), the natural map 

\[
I_\lambda : Y \otimes_\alpha X \to (Y^* \otimes_\omega X^*)^*
\]

satisfies \( \alpha(u; Y, X) = \lambda \| I_\lambda(u) \|_{Y^* \otimes_\omega X^*} \) for every \( u \in Y \otimes X \) (cf. [2, Section 21.7]), where \( \omega' \) is the dual tensor norm (cf. [2]) of \( \alpha \). Note that \( \pi' = \varepsilon \).

The main goal of this paper is to study the \( \alpha \)-AP and the \( \lambda \)-bounded \( \alpha \)-AP in terms of operator ideals. In Section 2, we extend the result of Díaz et al. [7], and in Section 3, we obtain some bounded versions of the results obtained in Section 2. As an application, it is shown that a Banach space \( X \) has the \( \lambda \)-bounded \( \alpha_{p,q} \)-AP if it has the \( \lambda \)-bounded \( \alpha_{p,1} \)-AP. Consequently, if \( X \) has the \( \lambda \)-bounded \( \alpha_{p,1} \)-AP, then \( X \) has the \( \lambda \)-bounded \( \alpha_{p,p^*} \)-AP.

## 2 The \( \alpha \)-approximation property

We denote by \( [\mathcal{L}, \| \cdot \|] \) the ideal of all operators and refer to [2,8–10] for operator ideals and their some information. A tensor norm \( \alpha \) is said to be associated with a Banach operator ideal \( [\mathcal{A}, \| \cdot \|_\mathcal{A}] \) if the canonical map \( (\mathcal{A}(M, N), \| \cdot \|_\mathcal{A}) \to M^* \otimes_\alpha N \) is an isometry for all finite-dimensional normed spaces \( M \) and \( N \). Let \( X \) and \( Y \) be Banach spaces. For \( T \in \mathcal{L}(X, Y) \), let

\[
\| T \|_\alpha = \sup \{ \| q_T^Y T I_{\mathcal{M}}^X \| : \dim M, \dim Y/L < \infty \},
\]

where \( I_{\mathcal{M}}^X: M \to X \) is the inclusion map and \( q_T^Y : Y \to Y/L \) is the quotient map, and

\[
\mathcal{A}^\text{max}(X, Y) = \{ T \in \mathcal{L}(X, Y) : \| T \|_\alpha < \infty \}.
\]
We call \([\mathcal{A}^{\max}, \| \cdot \|_{\mathcal{A}^{\max}}] \) the maximal hull of \([\mathcal{A}, \| \cdot \|_{\mathcal{A}}] \). If \([\mathcal{A}, \| \cdot \|_{\mathcal{A}}] = [\mathcal{A}^{\max}, \| \cdot \|_{\mathcal{A}^{\max}}] \), then \([\mathcal{A}, \| \cdot \|_{\mathcal{A}}] \) is called maximal. If \(\alpha \) is a finitely generated tensor norm, then its associated maximal Banach operator ideal is uniquely determined (cf. [2, Sections 17.1, 17.2 and 17.3]). For a finitely generated tensor norm \(\alpha \), the adjoint ideal \([\mathcal{A}^{\text{adj}}, \| \cdot \|_{\mathcal{A}^{\text{adj}}}] \) is the maximal Banach operator ideal associated with the adjoint tensor norm \(\alpha^* = (\alpha')' = (\alpha'^{*})' \).

**Lemma 2.1.** [2, Theorem 17.5] Let \([\mathcal{A}, \| \cdot \|_{\mathcal{A}}] \) be the maximal Banach operator ideal associated with a finitely generated tensor norm \(\alpha \). Then for all Banach spaces \(X \) and \(Y \), \(\mathcal{A}(X, Y^*) \) is isometric to \((X \otimes_{\alpha} Y^*)^* \) and \(\mathcal{A}(X, Y) \) is isometrically imbedded in \((X \otimes_{\alpha} Y)^* \) by the natural dual actions.

Let \(\alpha \) be a finitely generated tensor norm. According to [2, Proposition 21.7(4)], a Banach space \(X \) has the \(\alpha \)-AP if and only if for every Banach space \(Y \), the natural map

\[
J_{\alpha} : Y^* \otimes_{\alpha} X \rightarrow Y^* \otimes_{\alpha} X \hookrightarrow \mathcal{L}(Y, X)
\]

is injective.

**Theorem 2.2.** Let \([\mathcal{A}, \| \cdot \|_{\mathcal{A}}] \) be the maximal Banach operator ideal associated with a finitely generated tensor norm \(\alpha \). Then the following statements are equivalent for a Banach space \(X \).

(a) \(X \) has the \(\alpha \)-AP.

(b) For every Banach space \(Y \), \(\mathcal{F}(X, Y) \) is dense in \(\mathcal{A}^{\text{adj}}(X, Y) \) with the weak* topology on \((X \otimes_{\alpha} Y^*)^* \).

(c) For every Banach space \(Y \), \(\mathcal{F}(X, Y^*) \) is dense in \(\mathcal{A}^{\text{adj}}(X, Y^*) \) with the weak* topology on \((X \otimes_{\alpha} Y^*)^* \).

**Proof.** (a) \( \Rightarrow \) (b): Let \(Y \) be a Banach space. Since \([\mathcal{A}^{\text{adj}}, \| \cdot \|_{\mathcal{A}^{\text{adj}}}] \) is associated with \(\alpha^* \) and \((\alpha')' = \alpha^* \), by Lemma 2.1, \(\mathcal{A}^{\text{adj}}(X, Y) \) is isometrically imbedded in \((X \otimes_{\alpha^*} Y^*)^* \). Let \(T \in \mathcal{A}^{\text{adj}}(X, Y) \). Suppose that \(T \notin \mathcal{F}(X, Y)^{\text{weak}} \). Then by the separation theorem, there exists a \(u \in X \otimes_{\alpha^*} Y^* \) such that for every \(S \in \mathcal{F}(X, Y) \),

\[
\langle S, u \rangle = 0 \text{ but } \langle T, u \rangle \neq 0,
\]

where \(\langle \cdot, \cdot \rangle \) is the dual action on \((X \otimes_{\alpha^*} Y^*)^* \). We will show that \(u = 0 \) in \(X \otimes_{\alpha^*} Y^* \), which is a contradiction. Let

\[
J_{\alpha'} : X \otimes_{\alpha^*} Y^* \rightarrow X \otimes_{\alpha} Y^* \hookrightarrow \mathcal{L}(X^*, Y^*)
\]

be the natural map. To show that \(J_{\alpha'}u = 0 \) in \(X \otimes_{\alpha} Y^* \), let \(x^* \in X^* \) and \(y \in Y \). For every \(v = \sum m k=1, n k=1 x_k \otimes y_k^* \in X \otimes Y^* \),

\[
\langle x^* \otimes y, v \rangle = \sum m k=1 x^*(x_k) y_k^*(y) = (J_{\alpha'}v)x^*(y).
\]

Let \((u_n)\) be a sequence in \(X \otimes Y^* \) such that \(\lim_{n \to \infty} \alpha^* (u_n - u) = 0 \). Then

\[
\lim_{n \to \infty} \langle x^* \otimes y, u_n \rangle = \langle x^* \otimes y, u \rangle.
\]

Since

\[
|\langle J_{\alpha'}u_n x^* \rangle(y) - \langle J_{\alpha'}u x^* \rangle(y) | \leq \|x^*\| \|y\| \|t_{\alpha'}(u_n - u)\| X, Y^* \leq \|x^*\| \|y\| \|\alpha^*(u_n - u)\| \rightarrow 0
\]

as \(n \to \infty \), and for every \(n \),

\[
\langle x^* \otimes y, u_n \rangle = \langle J_{\alpha'}u_n x^* \rangle(y),
\]

\[
0 = \langle x^* \otimes y, u \rangle = \langle J_{\alpha'}u x^* \rangle(y).
\]

Thus, \(J_{\alpha'}u = 0 \) in \(X \otimes_{\alpha} Y^* \).

The aforementioned argument also shows that

\[
x^*(J_{\alpha'}u) = \langle J_{\alpha'}u x^* \rangle(y)
\]

for every \(x^* \in X^* \) and \(y \in Y \), where \(J_{\alpha} : Y^* \otimes_{\alpha} X \rightarrow Y^* \otimes_{\alpha} X \hookrightarrow \mathcal{L}(Y, X) \) is the natural map. Consequently, \(J_{\alpha}u^t = 0 \) in \(Y^* \otimes_{\alpha} X \). Since \(X \) has the \(\alpha \)-AP, \(u^t = 0 \) in \(Y^* \otimes_{\alpha} X \) and so \(u = 0 \) in \(X \otimes_{\alpha^*} Y^* \).
(b) ⇒ (c): Let $Y$ be a Banach space. By Lemma 2.1, $\mathcal{A}^{\text{adj}}(X, Y)$ is isometric to $\left( X \hat{\otimes}_d Y \right)^*$. Let $T \in \mathcal{A}^{\text{adj}}(X, Y^*)$. Then by (b),

$$T \in \mathcal{F}(X, Y^*)^{\text{weak}'} \cap \left( X \hat{\otimes}_d Y^* \right)^*.$$ 

Since the canonical imbedding from $X \hat{\otimes}_d Y$ to $X \hat{\otimes}_d Y^{**}$ is an isometry,

$$T \in \mathcal{F}(X, Y^*)^{\text{weak}'} \cap \left( X \hat{\otimes}_d Y \right)^*.$$  

(c) ⇒ (a): Let $Y$ be a Banach space. We show that the natural map

$$J_a : Y \hat{\otimes}_a X \to Y \hat{\otimes}_a X \hookrightarrow \mathcal{L}(Y^*, X)$$

is injective. Assume that $J_a u = 0$ in $Y \hat{\otimes}_a X$. To show that $u = 0$ in $Y \hat{\otimes}_a X$, we will show that $u' = 0$ in $X \hat{\otimes}_d Y$, that is, $\langle T, u' \rangle = 0$ for every $T \in \mathcal{A}^{\text{adj}}(X, Y^*)$. Let $T \in \mathcal{A}^{\text{adj}}(X, Y^*)$ be fixed. Since $J_a u = 0$ in $Y \hat{\otimes}_a X$, for every $x^* \in X^*$ and $y^* \in Y^*$,

$$y'(J_a u') x^* = x'(J_a u) y^* = 0,$$

where $J_a' : X \hat{\otimes}_d Y \to X \hat{\otimes}_a Y \hookrightarrow \mathcal{L}(X^*, Y)$ is the natural map. As in the proof of (a) ⇒ (b), we see that

$$\langle x^* \otimes y^*, u' \rangle = y'(J_a u') x^* = 0$$

for every $x^* \in X^*$ and $y^* \in Y^*$, and so

$$\langle S, u' \rangle = 0$$

for every $S \in \mathcal{F}(X, Y^*)$. Since $T \in \mathcal{F}(X, Y^*)^{\text{weak}'} \cap \left( X \hat{\otimes}_d Y \right)^*$, this implies

$$\langle T, u' \rangle = 0. \quad \square$$

Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \leq 1$ and let $1 \leq r \leq \infty$ with $1/p + 1/q + 1/r^* = 1$, where $1/r + 1/r^* = 1$. A linear map $T : X \to Y$ is called $(p, q)$-dominated if there exists a $C > 0$ such that

$$\|y^*_n(Tx_n)\| \leq C \|(x_n)\|_p \|y^*_n\|_q$$

for every $(x_n) \in \ell^p(X)$ and $(y^*_n) \in \ell^q(Y)$. We denote by $\mathcal{D}_{p,q}(X, Y)$ the collection of all $(p, q)$-dominated operators from $X$ to $Y$ and for $T \in \mathcal{D}_{p,q}(X, Y)$, let $\|T\|_{\mathcal{D}_{p,q}}$ be the infimum $C$ satisfying all such inequalities. Then $[\mathcal{D}_{p,q}, \|\cdot\|_{\mathcal{D}_{p,q}}]$ is a Banach operator ideal (cf. [2, Section 19]). $\mathcal{D}_p = \mathcal{D}_{p,\infty}$ is well known as the ideal of absolutely $p$-summing operators (cf. [2,8–10]) and $\mathcal{D} = \mathcal{D}_{p,p^*}$ is the ideal of $p$-dominated operators. For $1/p + 1/q \geq 1$, let $[\mathcal{L}_{p,q}, \|\cdot\|_{\mathcal{L}_{p,q}}]$ be the maximal Banach operator ideal associated with the tensor norm $\alpha_{p,q}$. $\mathcal{L}_{p,q}$ is well known as the ideal of $(p, q)$-factorable operators. Then

$$[\mathcal{L}_{p,q}, \|\cdot\|_{\mathcal{L}_{p,q}}]^{\text{adj}} = [\mathcal{D}_{p',q'}, \|\cdot\|_{\mathcal{D}_{p',q'}}]$$

(see [2, Section 17.12] and [9, Section 17.4]).

Theorem 2.2 applied to the tensor norm $\alpha_{p,q}$ covers [7, Theorem 1].

**Corollary 2.3.** Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \geq 1$. The following statements are equivalent for a Banach space $X$.

(a) $X$ has the $\alpha_{p,q}$-AP.

(b) For every Banach space $Y$, $\mathcal{F}(X, Y)$ is dense in $\mathcal{D}_{p',q'}(X, Y)$ with the weak* topology on $\left( X \hat{\otimes}_{a,q} Y^* \right)^*$.

(c) For every Banach space $Y$, $\mathcal{F}(X, Y^*)$ is dense in $\mathcal{D}_{p',q'}(X, Y^*)$ with the weak* topology on $\left( X \hat{\otimes}_{a,q} Y \right)^*$.

Recall that a Banach space $X$ has the AP if and only if $X$ has the $\pi$-AP. Then the most special case of Corollary 2.3 is the following.
Corollary 2.4. The following statements are equivalent for a Banach space $X$
(a) $X$ has the $\text{AP}$.
(b) For every Banach space $Y$, $\mathcal{F}(X, Y)$ is dense in $\mathcal{L}(X, Y)$ with the weak$^*$ topology on $(X \hat{\otimes}_\pi Y)^*$.
(c) For every Banach space $Y$, $\mathcal{F}(X, Y')$ is dense in $\mathcal{L}(X, Y')$ with the weak$^*$ topology on $(X \hat{\otimes}_\pi Y)^*$.

Proof. It is well known that $\pi$ is associated with the ideal $I$ of integral operators and $I^\text{adj} = \mathcal{L}$ holds isometrically (cf. [2]). Since $\pi' = \pi$, we have the conclusion. \hfill $\square$

Theorem 2.5. Let $[\mathcal{A}, \|\cdot\|_\mathcal{A}]$ be the maximal Banach operator ideal associated with a finitely generated tensor norm $a$. Then a Banach space $X$ has the $\alpha^t$-AP if and only if for every Banach space $Y$, $\mathcal{F}(Y, X')$ is dense in $\mathcal{A}^\text{adj}(Y, X')$ with the weak$^*$ topology on $(Y \hat{\otimes}_\alpha X)^*$.

Proof. Assume that $X$ has the $\alpha^t$-AP. Let $Y$ be a Banach space. By Lemma 2.1, $\mathcal{A}^\text{adj}(Y, X')$ is isometric to $(Y \hat{\otimes}_\alpha X)^*$. Let $T \in \mathcal{A}^\text{adj}(Y, X')$. Suppose that $T \notin \overline{\mathcal{F}(Y, X')}^\text{weak}$. Then there exists a $u \in Y \hat{\otimes}_\alpha X$ such that for every $S \in \mathcal{F}(Y, X')$,
\[
\langle S, u \rangle = 0 \text{ but } \langle T, u \rangle \neq 0.
\]
Then as in the proof of Theorem 2.2, we can show that the natural map $J_{\alpha^t} : Y \hat{\otimes}_\alpha X \to Y \hat{\otimes}_\varepsilon X$ is not injective. This contradicts the assumption that $X$ has the $\alpha^t$-AP.

To show the converse, let $Y$ be a Banach space. We want to show that the natural map
\[
J_{\alpha^t} : Y \hat{\otimes}_\alpha X \to Y \hat{\otimes}_\varepsilon X \hookrightarrow \mathcal{L}(Y^*, X)
\]
is injective. Assume that $J_{\alpha^t}u = 0$ in $Y \hat{\otimes}_\varepsilon X$. Let $T \in \mathcal{A}^\text{adj}(Y, X')$. Since $J_{\alpha^t}u = 0$ in $Y \hat{\otimes}_\varepsilon X$, we see that for every $y^* \in Y^*$ and $x^* \in X^*$,
\[
\langle y^* \otimes x^*, u \rangle = x^*((J_{\alpha^t}u)y^*) = 0.
\]
Thus, $\langle S, u \rangle = 0$ for every $S \in \mathcal{F}(Y, X')$. Since $T \notin \overline{\mathcal{F}(Y, X')}^\text{weak}$, $\langle T, u \rangle = 0$. Hence, $u = 0$ in $Y \hat{\otimes}_\alpha X$. \hfill $\square$

Corollary 2.6. Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \geq 1$. Then a Banach space $X$ has the $\alpha_{p,q}$-AP if and only if for every Banach space $Y$, $\mathcal{F}(Y, X')$ is dense in $\mathcal{D}_{p,q}(Y, X')$ with the weak$^*$ topology on $(Y \hat{\otimes}_{\alpha_{p,q}} X)^*$.

3 The bounded $\alpha$-approximation property

Let $\tilde{\alpha}$ be a tensor norm and let $X$ and $Y$ be Banach spaces. Recall from [2, 12.4] that for every $u \in X \otimes Y$, let
\[
\tilde{\alpha}(u; X, Y) = \inf \{ \alpha(u; M, N) : u \in M \otimes N, \dim M, \dim N < \infty \}
\]
and
\[
\widetilde{\alpha}(u; X, Y) = \sup \{ \alpha((q_K^R \otimes q_L^T)(u); X/K, Y/L) : \dim X/K, \dim Y/L < \infty \}.
\]
It follows that $\tilde{\alpha} \leq \alpha \leq \widetilde{\alpha}$. A tensor norm $\alpha$ is called totally accessible if $\tilde{\alpha} = \widetilde{\alpha}$.

From [2, Proposition 21.7(2)], a Banach space $X$ has the $\lambda$-bounded $\alpha$-AP if and only if for every Banach space $Y$,
\[
\alpha(u; Y, X) \leq \lambda \tilde{\alpha}(u; Y, X)
\]
for every $u \in Y \otimes X$. Since $\tilde{\alpha}^t = (\tilde{\alpha})^t$, it follows that a Banach space $X$ has the $\lambda$-bounded $\alpha^t$-AP if and only if for every Banach space $Y$,
\[
\alpha(u; Y, X) \leq \lambda \tilde{\alpha}(u; Y, X)
\]
for every $u \in X \otimes Y$. 

\[\]
Lemma 3.1. [2, Theorem 15.5] For all Banach spaces $X$ and $Y$, and a tensor norm $\alpha$, the natural maps
\[ I_\alpha : X \otimes_\alpha Y \to (X^* \otimes_{\alpha^*} Y^*)^*, \]
\[ I_\alpha^* : X^* \otimes_\alpha Y^* \to (X \otimes_{\alpha^*} Y)^* \]
are isometries.

The following lemma is a reformulation of [2, Lemma 16.2].

Lemma 3.2. Let $\alpha$ be a tensor norm and let $X$ and $Y$ be Banach spaces. Let $\lambda \geq 1$. Then $\alpha \leq \lambda \alpha$ on $X \otimes Y$ if and only if for every $\phi \in B_{(X_0,Y)^*}$, there exists a net $(T_\eta)$ in $\lambda B_{X_0^*Y^*}$ such that for every $x \in X$ and $y \in Y$,
\[ \lim_{\eta} (T_\eta x) (y) = \langle \phi, x \otimes y \rangle. \]

Proof. Suppose that $\alpha \leq \lambda \alpha$ on $X \otimes Y$. Let $\phi \in B_{(X_0,Y)^*}$. By Lemma 3.1, we can choose a Hahn-Banach extension $\hat{\phi} \in (X^* \otimes_{\alpha^*} Y^*)^*$ of $\phi$. By Goldstine's theorem, there exists a net $(T_\eta)$ in $\lambda B_{X_0^*Y^*}$ such that
\[ \lim_{\eta} \langle \hat{\phi}, f \rangle = \langle \phi, f \rangle \]
for every $f \in (X^* \otimes_{\alpha^*} Y^*)^*$. Thus, for every $x \in X$ and $y \in Y$,
\[ \lim_{\eta} (T_\eta x) (y) = \lim_{\eta} \langle x \otimes y, T_\eta \rangle = \langle \hat{\phi}, x \otimes y \rangle = \langle \phi, x \otimes y \rangle. \]

Also, since
\[ \| \hat{\phi} \|_{X_0^*Y^*} = \| \phi \|_{X_0^*Y^*} \leq \lambda \| \phi \|_{X_0^*Y^*} \leq \lambda, \]
the net $(T_\eta)$ is in $\lambda B_{X_0^*Y^*}$.

To show the converse, let $u = \sum_{k=1}^m x_k \otimes y_k \in X \otimes Y$. Then there exists $\phi \in B_{(X_0,Y)^*}$ such that $\phi(u; X, Y) = \langle \phi, u \rangle$. By assumption, there exists a net $(T_\eta)$ in $\lambda B_{X_0^*Y^*}$ such that
\[ \lim_{\eta} \langle \phi, u \rangle = \lim_{\eta} \sum_{k=1}^m \langle T_\eta x_k, y_k \rangle = \langle \phi, u \rangle. \]

Hence,
\[ \alpha(u; X, Y) = \langle \phi, u \rangle \leq \lambda \sup \{ \| \langle u, v \rangle \| : v \in B_{X_0^*Y^*} \} = \lambda \| u \|_{X_0^*Y^*} = \lambda \alpha(u; X, Y). \]


Lemma 3.3. [2, Proposition 21.8] Let $[\mathcal{A}, \| \cdot \|_\pi]$ be the maximal Banach operator ideal associated with a finitely generated tensor norm $\alpha$ and let $\lambda \geq 1$. Let $X$ and $Y$ be Banach spaces. Then $\alpha \leq \lambda \alpha$ on $X \otimes Y$ if and only if for every $T \in B_{\mathcal{A}^0(\mathcal{L}(X,Y^*))}$, there exists a net $(T_\eta)$ in $\lambda B_{X_0^*Y^*}$ such that for every $x \in X$ and $y^* \in Y^*$,
\[ \lim_{\eta} y^*(T_\eta x) = (Tx)(y^*). \]

We denote the strong operator topology and the weak operator topology on $\mathcal{L}$, respectively, by $\tau_{\text{so}}$ and $\tau_{\text{wo}}$. For a net $(T_\eta)$ in $\mathcal{L}(X,Y^*)$, we say that $T_\eta \to 0$ in the weak* operator topology if
\[ \lim_{\eta} (T_\eta x)(y) \to 0 \]
for every $x \in X$ and $y \in Y$. We denote the weak* operator topology by $\tau_{\text{w}^*}$. 

Theorem 3.4. Let $[\mathcal{A}, \| \cdot \|_\pi]$ be the maximal Banach operator ideal associated with a finitely generated tensor norm $\alpha$ and let $\lambda \geq 1$. Then the following statements are equivalent for a Banach space $X$. 

(a) $X$ has the $\lambda$-bounded $\alpha$-AP.
(b) For every Banach space $Y$ and every $T \in \mathcal{A}^{\text{adj}}(X, Y)$,

$$T \in \left\{ S \in \bar{F}(X, Y) : \alpha'(S; X', Y) \leq \lambda \|T\|_{\mathcal{A}} \right\}^{\text{tna}}.$$ 

(c) For every Banach space $Y$ and every $T \in \mathcal{A}^{\text{adj}}(X, Y^*)$,

$$T \in \left\{ S \in \bar{F}(X, Y^*) : \alpha'(S; X', Y^*) \leq \lambda \|T\|_{\mathcal{A}} \right\}^{\text{tna}}.$$ 

\textbf{Proof.}

(b) $\Rightarrow$ (c) is trivial.

(a) $\Rightarrow$ (b): This proof is essentially due to [11, Theorem 4.1]. Let $Y$ be a Banach space and let $T \in \mathcal{A}^{\text{adj}}(X, Y)$. Consider $i_T T \in \mathcal{A}^{\text{adj}}(X, Y^*)$, where $i_T : Y \to Y^*$ is the canonical isometry. Since $X$ has the $\lambda$-bounded $\alpha$-AP, by Lemma 3.3, there exists a net $(T_{\eta})$ in $\mathcal{F}(X, Y)$ with $\alpha'(T_{\eta}; X^*, Y) \leq \lambda$ such that for every $x \in X$ and $y^* \in Y^*$,

$$\lim_{\eta} y^*(\|i_T T_{\eta} x\|) = (i_T T x)(y^*) = y^*(Tx).$$

Since $\alpha'(\|i_T T_{\eta} x\|) \leq \lambda \|i_T T_{\eta} x\|$, we have

$$T \in \left\{ S \in \bar{F}(X, Y) : \alpha'(S; X', Y) \leq \lambda \|T\|_{\mathcal{A}} \right\}^{\text{tna}} = \left\{ S \in \bar{F}(X, Y) : \alpha'(S; X', Y) \leq \lambda \|T\|_{\mathcal{A}} \right\}^{\text{tna}}.$$ 

(c) $\Rightarrow$ (a): Let $Y$ be a Banach space. Since $\alpha = (\alpha')^*$, $\alpha \leq \lambda \alpha^*$ on $Y \otimes X$ if and only if $\alpha^* \leq \lambda \alpha^*$ on $X \otimes Y$. So, in order to show that $X$ has the $\lambda$-bounded $\alpha$-AP, we will show that $\alpha^* \leq \lambda \alpha^*$ on $X \otimes Y$ using Lemma 3.2.

Now, let $\phi \in B_{X_{\alpha^*}, Y^*}$. By Lemma 2.1, we can choose the representation $T_\phi \in \mathcal{A}^{\text{adj}}(X, Y^*)$ of $\phi$ with $\|T_\phi\|_{\mathcal{A}} \leq 1$. Then by (c), there exists a net $(S_{\eta})$ in $\lambda B_{X_{\alpha^*}, Y^*}$ such that for every $x \in X$ and $y \in Y$,

$$\lim_{\eta} (S_{\eta} x)(y) = (T_\phi x)(y) = \langle \phi, x \otimes y \rangle.$$ 

Hence by Lemma 3.2, we complete the proof.

\textbf{Corollary 3.5.} Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \geq 1$ and let $\lambda \geq 1$. The following statements are equivalent for a Banach space $X$.

(a) $X$ has the $\lambda$-bounded $\alpha_{p,q}$-AP.
(b) For every Banach space $Y$ and every $T \in \mathcal{D}_{p',q'}(X, Y)$,

$$T \in \left\{ S \in \bar{F}(X, Y) : \|S\|_{\mathcal{D}_{p',q'}} \leq \lambda \|T\|_{\mathcal{D}_{p',q'}} \right\}^{\text{tna}}.$$ 

(c) For every Banach space $Y$ and every $T \in \mathcal{D}_{p',q'}(X, Y^*)$,

$$T \in \left\{ S \in \bar{F}(X, Y^*) : \|S\|_{\mathcal{D}_{p',q'}} \leq \lambda \|T\|_{\mathcal{D}_{p',q'}} \right\}^{\text{tna}}.$$ 

\textbf{Proof.} If $\mathcal{A}$ is the maximal Banach operator ideal associated with a totally accessible finitely generated tensor norm $\alpha$, then by Lemmas 2.1 and 3.1, $a = \|\alpha\|$ on $\mathcal{F}$. Since $\alpha_{p,q}$ is totally accessible (cf. [2, Theorem 21.5]), by Theorem 3.4, we have the conclusion. The equivalence (a) $\iff$ (b) is also a consequence of [11, Theorem 4.1].

For the following result, we will need [11, Corollary 2.14], which can be reformulated as follows.

\textbf{Lemma 3.6.} Let $1 \leq p \leq \infty$ and let $\lambda \geq 1$. The following statements are equivalent for a Banach space $X$.

(a) For every Banach space $Y$ and every $T \in \mathcal{P}_p(X, Y)$,

$$T \in \left\{ S \in \bar{F}(X, Y) : \|S\|_{\mathcal{P}_p} \leq \lambda \|T\|_{\mathcal{P}_p} \right\}^{\text{tna}}.$$ 

(b) For every Banach space $Y$ and every $T \in \mathcal{P}_p(X, Y)$,

$$\text{id}_X \in \left\{ S \in \bar{F}(X, X) : \|TS\|_{\mathcal{P}_p} \leq \lambda \|T\|_{\mathcal{P}_p} \right\}^{\text{tna}}.$$
According to [11, Definition 1.2], for a Banach operator ideal $\mathcal{A}$, a Banach space $X$ is said to have the weak $\lambda$-BAP for $\mathcal{A}$ if for every Banach space $Y$ and every $T \in \mathcal{A}(X, Y)$,

$$id_X \in \{ S \in \mathcal{F}(X, X) : \| TS \|_{\mathcal{A}} \leq \lambda \| T \|_{\mathcal{A}} \}^{\text{w}}.$$ 

**Theorem 3.7.** Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \geq 1$ and let $\lambda \geq 1$. If a Banach space $X$ has the $\lambda$-bounded $g_p$-AP, then $X$ has the weak $\lambda$-BAP for $D_{p', q'}$.

**Proof.** By Corollary 3.5 and Lemma 3.6, if $X$ has the $\lambda$-bounded $g_p$-AP, then for every Banach space $Z$ and every $T \in \mathcal{P}_{p'}(X, Z)$,

$$id_X \in \{ S \in \mathcal{F}(X, X) : \| TS \|_{\mathcal{P}_{p'}} \leq \lambda \| T \|_{\mathcal{P}_{p'}} \}^{\text{w}}.$$ 

Now, let $Y$ be a Banach space and let $T \in D_{p', q'}(X, Y)$. Let $\delta > 0$. Then by Kwapień's factorization theorem (cf. [2, Theorem 19.3]), there exist a Banach space $Z$, $R \in \mathcal{P}_{p'}(X, Z)$ and $U^* \in \mathcal{P}_{q'}(Y', Z')$ with $\| U^* \|_{p'} \| R \|_{p'} \leq (1 + \delta) \| T \|_{\mathcal{D}_{p', q'}}$ such that the following diagram is commutative.

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{R} & & \downarrow{U} \\
Z & & \\
\end{array}
\]

By the aforementioned statement, for every finite $x_1, ..., x_m \in X$ and every $\varepsilon > 0$, there exists an $S \in \mathcal{F}(X, X)$ with $\| RS \|_{p'} \leq \lambda \| R \|_{p'}$, such that

$$\| Sx_i - x_i \| \leq \varepsilon$$

for every $i = 1, ..., m$. Since

$$\| TS \|_{\mathcal{D}_{p', q'}} \leq \| U^* \|_{p'} \| RS \|_{p'} \leq (1 + \delta) \lambda \| T \|_{\mathcal{D}_{p', q'}},$$

we have shown that for every $\delta > 0$,

$$id_X \in \{ S \in \mathcal{F}(X, X) : \| TS \|_{\mathcal{D}_{p', q'}} \leq (1 + \delta) \lambda \| T \|_{\mathcal{D}_{p', q'}} \}^{\text{w}}.$$ 

Let $x_1, ..., x_m \in X$ and let $\varepsilon > 0$. Choose a $\delta > 0$ so that

$$\left( \delta \lambda / (1 + \delta) \lambda \right) \max_{1 \leq k \leq m} \| x_k \| \leq \varepsilon / 2.$$ 

Then, there exists an $S \in \mathcal{F}(X, X)$ with $\| TS \|_{\mathcal{D}_{p', q'}} \leq (1 + \delta) \lambda \| T \|_{\mathcal{D}_{p', q'}}$ such that for every $i = 1, ..., m$, $\| Sx_i - x_i \| \leq \varepsilon / 2$. Consider

$$\left( \lambda / (1 + \delta) \lambda \right) S \in \{ S \in \mathcal{F}(X, X) : \| TS \|_{\mathcal{D}_{p', q'}} \leq \lambda \| T \|_{\mathcal{D}_{p', q'}} \}.$$ 

Then for every $i = 1, ..., m$,

$$\left\| \frac{\lambda}{(1 + \delta) \lambda} Sx_i - x_i \right\| \leq \frac{\lambda}{(1 + \delta) \lambda} \| Sx_i - x_i \| + \frac{\lambda}{(1 + \delta) \lambda} \max_{1 \leq k \leq m} \| x_k \| \leq \varepsilon.$$ 

Hence, $id_X \in \{ S \in \mathcal{F}(X, X) : \| TS \|_{\mathcal{D}_{p', q'}} \leq \lambda \| T \|_{\mathcal{D}_{p', q'}} \}^{\text{w}}$. \hfill $\Box$

In [7, Proposition 2], it was shown that if a Banach space $X$ has the $g_p$-AP, then $X$ has the $a_{p', q'}$-AP. From Theorem 3.7 and Corollary 3.5, we have:

**Corollary 3.8.** Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \geq 1$ and let $\lambda \geq 1$. If a Banach space $X$ has the $\lambda$-bounded $g_p$-AP, then $X$ has the $\lambda$-bounded $a_{p, q}$-AP.
Theorem 3.9. Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be the maximal Banach operator ideal associated with a finitely generated tensor norm $\alpha$ and let $\lambda \geq 1$. Then a Banach space $X$ has the $\lambda$-bounded $\alpha^t$-AP if and only if for every Banach space $Y$ and every $T \in \mathcal{A}^{\alpha t}(Y, X)$,

$$T \in \{S \in \mathcal{F}(Y, X) : \alpha^t(S; Y^*, X^*) \leq \lambda \|T\|_{\mathcal{A}^{\alpha t}}^{\tau^{\|\cdot\|_{\mathcal{A}}}}\}.$$ 

Proof. Let

$$i : X \otimes_{\alpha} Y \rightarrow Y \otimes_{\alpha'} X$$

be the isometry defined by $i(u) = u'$. 

Suppose that $X$ has the $\lambda$-bounded $\alpha^t$-AP. Let $Y$ be a Banach space and let $T \in \mathcal{A}^{\alpha t}(Y, X)$. By Lemma 2.1, we can choose the representation $\phi_T \in (Y \otimes_{\alpha'} X)^*$ of $T$. Consider $\phi_T i \in (X \otimes_{\alpha} Y)^*$. Since $X$ has the $\lambda$-bounded $\alpha^t$-AP, $\alpha \leq \lambda^{\alpha}$ on $X \otimes Y$. Then by Lemma 3.2, there exists a net $(u_{\eta})_{\eta}$ in $\lambda B_{X \otimes_{\alpha} Y}$ such that for every $x \in X$ and $y \in Y$,

$$\lim_{\eta} \langle x \otimes y, u_{\eta} \rangle = \left( \frac{\phi_T i}{\|\phi_T i\|_{(X \otimes_{\alpha} Y)^*}} \right) x \otimes y.$$

Let us consider the net $(\|\phi_T i\|_{(X \otimes_{\alpha} Y)^*} u_{\eta}^T)_{\eta}$ in $Y^* \otimes X^* = \mathcal{F}(Y, X^*)$. Then

$$\alpha^t(\|\phi_T i\|_{(X \otimes_{\alpha} Y)^*} u_{\eta}^T; Y^*, X^*) = \|\phi_T i\|_{(X \otimes_{\alpha} Y)^*} \alpha^t(u_{\eta}; X^*, Y^*) \leq \lambda \|T\|_{\mathcal{A}^{\alpha t}}^{\tau^{\|\cdot\|_{\mathcal{A}}}},$$

and for every $y \in Y$ and $x \in X$,

$$\lim_{\eta} (\|\phi_T i\|_{(X \otimes_{\alpha} Y)^*} u_{\eta}^T)(y) = \lim_{\eta} \langle x \otimes y, u_{\eta} \rangle = \langle \phi_T i, x \otimes y \rangle = (Ty)(x).$$

Hence, $T \in \{S \in \mathcal{F}(Y, X^*) : \alpha^t(S; Y^*, X^*) \leq \lambda \|T\|_{\mathcal{A}^{\alpha t}}^{\tau^{\|\cdot\|_{\mathcal{A}}}}\}$. 

To show the converse, we also use Lemma 3.2. Let $Y$ be a Banach space and let $\phi \in B_{(X \otimes_{\alpha} Y)^*}$. Consider $\phi T^{-1} \in B_{(Y \otimes_{\alpha'} X)^*}$. By Lemma 2.1, we can choose the representation $T^{-1}_{\phi^t} \in \mathcal{A}^{\alpha t}(Y, X)$ of $\phi T^{-1}$ with $\|T^{-1}_{\phi^t}\|_{\mathcal{A}^{\alpha t}} \leq 1$. By assumption, there exists a net $(S_{\eta})_{\eta}$ in $\lambda B_{X \otimes_{\alpha} Y}$ such that for every $y \in Y$ and $x \in X$,

$$\lim_{\eta} (T_{\phi^t} S_{\eta})(y) = (T_{\phi^t} y)(x).$$

Consider the net $(S_{\eta}^T)_{\eta}$ in $X^* \otimes Y^*$. Then $\alpha(S_{\eta}^T; X^*, Y^*) = \alpha^t(S_{\eta}; Y^*, X^*) \leq \lambda$ and for every $x \in X$ and $y \in Y$,

$$\lim_{\eta} (S_{\eta}^T)(x) = \lim_{\eta} (S_{\eta})(y) = (T_{\phi^t} y)(x) = \langle \phi, x \otimes y \rangle.$$

Thus by Lemma 3.2, $\alpha \leq \lambda^{\alpha}$ on $X \otimes Y$. Hence, $X$ has the $\lambda$-bounded $\alpha^t$-AP. \qed

From Theorem 3.9, we have:

Corollary 3.10. Let $1 \leq p, q < \infty$ with $1/p + 1/q \geq 1$ and let $\lambda \geq 1$. A Banach space $X$ has the $\lambda$-bounded $a_{q, p^*}$-AP if and only if for every Banach space $Y$ and every $T \in \mathcal{D}_{a_{q, p^*}}(Y, X)$,

$$T \in \{S \in \mathcal{F}(Y, X^*) : \|S\|_{\mathcal{D}_{a_{q, p^*}}} \leq \lambda \|T\|_{\mathcal{D}_{a_{q, p^*}}}^{\tau^{\|\cdot\|_{\mathcal{A}}}}\}.$$ 

4 Open problems

The following question is a well-known problem (cf. [2, Section 21.12]).

Problem 1
Is the tensor norm $w_p$ ($1 < p < \infty$, $p \neq 2$) totally accessible?
Since a finitely generated tensor norm $\alpha$ is totally accessible if and only if every Banach space has the $1$-bounded $\alpha$-AP, the problem can be reformulated as follows.

**Problem 1**
Does every Banach space have the $1$-bounded $w_p$-AP ($1 < p < \infty$, $p \neq 2$)?

According to Corollaries 3.5 and 3.10, a Banach space $X$ has the $1$-bounded $w_p$-AP if and only if for every Banach space $Y$ and every $T \in \mathcal{D}_p(X, Y)$,
\[ T \in \{ S \in \mathcal{F}(X, Y) : \| S \|_{\mathcal{D}_p} \leq \| T \|_{\mathcal{D}_p} \}^{w_0} \]
if and only if for every Banach space $Y$ and every $T \in \mathcal{D}_p(Y, X^*)$,
\[ T \in \{ S \in \mathcal{F}(Y, X^*) : \| S \|_{\mathcal{D}_p} \leq \| T \|_{\mathcal{D}_p} \}^{w_0}. \]

Therefore, the problem can be reformulated as follows.

**Problem 1**
Let $1 < p < \infty$, $p \neq 2$. For all Banach spaces $X$ and $Y$ is dense in $\mathcal{D}_p(X, Y)$ with the weak* topology on $(X \hat{\otimes}_{w_p} Y)^*$ for every Banach space $Y$ if and only if $\mathcal{F}(Y, X^*)$ is dense in $\mathcal{D}_p(Y, X^*)$ with the weak* topology on $(Y \hat{\otimes}_{w_p} X)^*$ for every Banach space $Y$. We ask:

**Problem 2**
Let $1 < p < \infty$, $p \neq 2$. For all Banach spaces $X$ and $Y$ is dense in $\mathcal{D}_p(X, Y)$ with the weak* topology on $(X \hat{\otimes}_{w_p} Y^*)$? 

Or, is the space $\mathcal{F}(Y, X^*)$ dense in $\mathcal{D}_p(Y, X^*)$ with the weak* topology on $(Y \hat{\otimes}_{w_p} X)^*$?

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