Generalizations for Schouten-Nijenhuis bracket and for differential analog of special Yang-Baxter equations

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**Abstract:** The Schouten-Nijenhuis bracket is generalized for the superspace case and for the Poisson brackets of opposite Grassmann parities. Quite a number of generalizations for the differential analog of the special Yang-Baxter equations is also proposed.

**Keywords:** Superspaces, Differential and Algebraic Geometry, Integrable Equations in Physics.

* Dedicated to the light memory of Anna Yakovlevna Gelukh (Kalaida)
1. Introduction

Recently a prescription for the construction of new Poisson brackets from the bracket with a definite Grassmann parity was proposed \[1\]. This prescription is based on the use of exterior differentials of diverse Grassmann parities. It was indicated in \[1\] that this prescription leads to the generalizations of the Schouten-Nijenhuis bracket \[2, 3, 4, 5, 6, 7, 8, 9\] on the both superspace case and the case of the brackets with diverse Grassmann parities. In the present paper we give the details of these generalizations. Here we also develop quite a number of generalizations for the differential analog of the special Yang-Baxter equations.

2. Poisson brackets related with the exterior differentials

Let us recall the prescription for the construction from a given Poisson bracket of a Grassmann parity $\epsilon \equiv 0, 1 \pmod{2}$ of another one.

A Poisson bracket, having a Grassmann parity $\epsilon$, written in arbitrary non-canonical phase variables $z^a$

$$\{A, B\}_{\epsilon} = A \partial_{z^a} \omega_{\epsilon}^{ab}(z) \partial_{z^b} B,$$

where $\partial$ and $\bar{\partial}$ are right and left derivatives respectively, has the following main properties:

$$g(\{A, B\}_{\epsilon}) \equiv g_A + g_B + \epsilon \pmod{2},$$

$$\{A, B\}_{\epsilon} = -(-1)^{(g_A + \epsilon)(g_B + \epsilon)} \{B, A\}_{\epsilon}.$$  

\[\text{Concerning terminology see, for example, } [10].\]
\[ \sum_{(ABC)} (-1)^{(g_{A}+\epsilon)(g_{C}+\epsilon)} \{A, \{B, C\}\epsilon} = 0, \]

which lead to the corresponding relations for the matrix \(\omega_{\epsilon}^{ab}\)

\[ g \left( \omega_{\epsilon}^{ab} \right) \equiv g_{a} + g_{b} + \epsilon \quad \text{(mod 2)}, \]  

(2.2)

\[ \omega_{\epsilon}^{ab} = -(-1)^{(g_{a}+\epsilon)(g_{b}+\epsilon)} \omega_{\epsilon}^{ba}, \]  

(2.3)

\[ \sum_{(abc)} (-1)^{(g_{a}+\epsilon)(g_{c}+\epsilon)} \omega_{\epsilon}^{ad} \partial_{\epsilon} \omega_{\epsilon}^{bc} = 0, \]  

(2.4)

where \(\partial_{\epsilon} \equiv \partial/\partial z^{a}\) and \(g_{a} \equiv g(z^{a})\), \(g_{A} \equiv g(A)\) are the corresponding Grassmann parities of phase coordinates \(z^{a}\) and a quantity \(A\) and a sum with a symbol \((abc)\) under it designates a summation over cyclic permutations of \(a, b\) and \(c\). We shall consider the non-degenerated matrix \(\omega_{\epsilon}^{ab}\) which has an inverse matrix \(\omega_{\epsilon}^{ab} = (\omega_{\epsilon}^{ba})^{-1}\) (a grading factor is chosen for the convenience)

\[ \omega_{\epsilon}^{ab} \omega_{\epsilon}^{bc} (-1)^{g_{c}\epsilon} = \delta_{a}^{c} \]

(there is no summation over \(\epsilon\) in the previous relation) with the properties

\[ g(\omega_{\epsilon}^{ab}) \equiv g_{a} + g_{b} + \epsilon \quad \text{(mod 2)}, \]

\[ \omega_{\epsilon}^{ab} = -(-1)^{(g_{a}+1)(g_{b}+1)} \omega_{\epsilon}^{ba}, \]

\[ \sum_{(abc)} (-1)^{(g_{a}+1)g_{c}\epsilon} \partial_{\epsilon} \omega_{\epsilon}^{bc} = 0. \]

The Hamilton equations for the phase variables \(z^{a}\), which correspond to a Hamiltonian \(H_{\epsilon} \ (g(H_{\epsilon}) = \epsilon)\),

\[ \frac{dz^{a}}{dt} = \{z^{a}, H_{\epsilon}\} = \omega_{\epsilon}^{ab} \partial_{\epsilon} H_{\epsilon} \]  

(2.5)

can be represented in the form

\[ \frac{dz^{a}}{dt} = \omega_{\epsilon}^{ab} \partial_{\epsilon} H_{\epsilon} \equiv \omega_{\epsilon}^{ab} \frac{\partial(d_{\zeta} H_{\epsilon})}{\partial(d_{\zeta} z^{b})} \]  

\[ = (z^{a}, d_{\zeta} H_{\epsilon})_{\epsilon + \zeta}, \]  

(2.6)

where \(d_{\zeta} \ (\zeta = 0, 1)\) is one of the exterior differentials \(d_{0}\) or \(d_{1}\), which have opposite Grassmann parities 0 and 1 respectively and following symmetry properties with respect to the ordinary multiplication

\[ d_{0} z^{a} d_{0} z^{b} = (-1)^{g_{a}g_{b}} d_{0} z^{b} d_{0} z^{a}, \]  

(2.7)

\[ d_{1} z^{a} d_{1} z^{b} = (-1)^{(g_{a}+1)(g_{b}+1)} d_{1} z^{b} d_{1} z^{a} \]  

(2.8)
and exterior products
\begin{equation}
 d_0 z^a \wedge d_0 z^b = (-1)^{g_a g_{b+1}} d_0 z^b \wedge d_0 z^a, \tag{2.9}
\end{equation}
\begin{equation}
 d_1 z^a \tilde{\wedge} d_1 z^b = (-1)^{(g_a+1)(g_{b+1})} d_1 z^b \tilde{\wedge} d_1 z^a. \tag{2.10}
\end{equation}

We use different notations $\wedge$ and $\tilde{\wedge}$ for the exterior products of $d_0 z^a$ and $d_1 z^a$ respectively.

By taking the exterior differential $d\zeta$ from the Hamilton equations (2.5), we obtain
\begin{equation}
 d( d\zeta z_a) \frac{dt}{dt} = (d\zeta \omega_{ab}^\epsilon \partial y_b + (-1)^\epsilon (g_a + \epsilon) \omega_{ab}^\epsilon \partial z^b)(d\zeta H_\epsilon) \overset{\text{def}}{=} (d\zeta z_a, d\zeta H_\epsilon)_{\epsilon+\zeta}. \tag{2.11}
\end{equation}

As a result of equations (2.6) and (2.11) we have by definition the following binary composition for functions $F$ and $H$ of the variables $z^a$ and their differentials $d\zeta z_a \equiv \gamma_a$:
\begin{equation}
 (F, H)_{\epsilon+\zeta} = F \left[ \bar{\partial} z^a \bar{\omega}_{\epsilon}^{ab} \bar{\partial} y_b + (-1)^\epsilon (g_a + \epsilon) \bar{\omega}_{\epsilon}^{ab} \bar{\partial} z^b \right. \\
 + \left. \bar{\omega}_{\epsilon}^{bc} (\partial z^a \omega_{\epsilon}^{bc}) \bar{\partial} y_c \right] H. \tag{2.12}
\end{equation}

By using relations (2.2)-(2.4) for the matrix $\omega_{\epsilon}^{ab}$, we can establish the following properties for the binary composition (2.12)
\begin{equation}
 g[(F, H)_{\epsilon+\zeta}] \equiv g_F + g_H + \epsilon + \zeta \pmod{2},
\end{equation}
\begin{equation}
 (F, H)_{\epsilon+\zeta} = - (-1)^{(g_F + \epsilon + \zeta)(g_H + \epsilon + \zeta)} (H, F)_{\epsilon+\zeta},
\end{equation}
\begin{equation}
 \sum_{(EFH)} (-1)^{(g_F + \epsilon + \zeta)(g_H + \epsilon + \zeta)} (E, (F, H)_{\epsilon+\zeta})_{\epsilon+\zeta} = 0,
\end{equation}

which mean that the composition (2.12) satisfies all the main properties for the Poisson bracket with the Grassmann parity equal to $\epsilon + \zeta$. Thus, the application of the exterior differentials of opposite Grassmann parities to the given Poisson bracket results in the brackets of the different Grassmann parities.

By transition to the co-differential variables $y^{a+\zeta}_\epsilon$, related with differentials $y^a_\zeta$ by means of the matrix $\omega_{\epsilon}^{ab}$
\begin{equation}
 y^{a+\zeta}_\epsilon = y^{a+\zeta}_b \omega_{\epsilon}^{ba}, \tag{2.13}
\end{equation}
the Poisson bracket (2.12) takes a canonical form
\begin{equation}
 (F, H)_{\epsilon+\zeta} = F \left[ \bar{\partial} z^a \bar{\partial} y^{a+\zeta}_b - (-1)^{g_b (g_a + \epsilon + \zeta)} \bar{\partial} y^{a+\zeta}_b \bar{\partial} z^a \right] H, \tag{2.14}
\end{equation}
that can be proved with the use of the Jacobi identity (2.4).

\footnote{There is no summation over $\epsilon$ in relation (2.13).}
The bracket \( (2.13) \) is given on the functions of the variables \( z^a, y_a^\zeta \)

\[
F = \sum_p \frac{1}{p!} y_a^{p \zeta} \cdots y_a^{1 \zeta} f_{a_1 \cdots a_p}(z), \quad g(f_{a_1 \cdots a_p}) = g_f + g_{a_1} + \cdots + g_{a_p},
\]

whereas this bracket, rewritten in the form \( (2.14) \), is given on the functions of variables \( z^a \) and \( y_a^\epsilon \)

\[
F = \sum_p \frac{1}{p!} y_a^{p \epsilon} \cdots y_a^{1 \epsilon} f_{a_1 \cdots a_p}(z), \quad g(f_{a_1 \cdots a_p}) = g_f \epsilon + g_{a_1} \epsilon + \cdots + g_{a_p}.
\]

We do not exclude a possibility of the own Grassmann parity \( g_f \equiv g(f) \) for a quantity \( f \). By taking into account relation \( (2.13) \), we have the following rule for the rising of indices:

\[
f_{b_1 \cdots b_p} = (-1)^{p-1} \sum_{k=1}^p [g_{b_1} + \cdots + g_{b_k} + k(\epsilon + \zeta)](g_{b_{k+1}} + g_{b_{k+1} + 1} + \epsilon) \omega_{\epsilon} b_{p a} \cdots \omega_{\epsilon} b_{1 a} f_{a_1 \cdots a_p}.
\]

Note that the quantities \( f_{a_1 \cdots a_p} \) and \( f_{a_1 \cdots a_p}^\epsilon \) have in general the different symmetry and parity properties.

In the case \( \zeta = 1 \), due to relations \( (2.8), (2.10) \), the terms in the decomposition of a function \( F(z^a, y_a^\epsilon) \) into degrees \( p \) of the variables \( y_a^\epsilon \)

\[
F = \sum_p \frac{1}{p!} y_a^{p \epsilon} \cdots y_a^{1 \epsilon} f_{a_1 \cdots a_p}(z)
\]

can be treated as \( p \)-forms and the bracket \( (2.12) \) can be considered as a Poisson bracket on \( p \)-forms so that being taken between a \( p \)-form and a \( q \)-form results in a \((p + q - 1)\)-form.\(^3\) Thus, the bracket \( (2.12) \) is a generalization of the bracket introduced in \([10, 12]\) on the superspace case and on the case of the brackets \( (2.1) \) with arbitrary Grassmann parities \( \epsilon \) \((\epsilon = 0, 1)\).

3. Generalizations of the Schouten-Nijenhuis bracket

If we take the bracket in the canonical form \( (2.14) \), then we obtain the generalizations of the Schouten-Nijenhuis bracket \( [2, 3] \) (see also \([4, 5, 6, 7, 8, 9, 10]\)) onto the cases of superspace and the brackets of diverse Grassmann parities. Indeed, let us consider the bracket \( (2.14) \) between monomials \( F \) and \( H \) having respectively degrees \( p \) and \( q \)

\[
F = \frac{1}{p!} y_a^{p \epsilon} \cdots y_a^{1 \epsilon} f_{a_1 \cdots a_p}(z), \quad g(f_{a_1 \cdots a_p}) = g_f + p\epsilon + g_{a_1} + \cdots + g_{a_p},
\]

\[
H = \frac{1}{q!} y_a^{q \epsilon} \cdots y_a^{1 \epsilon} h_{a_1 \cdots a_q}(z), \quad g(h_{a_1 \cdots a_q}) = g_h + q\epsilon + g_{a_1} + \cdots + g_{a_q}.
\]

\(^3\)Concerning a Poisson bracket between 1-forms and its relation with the Lie bracket of vector fields see in the book \([11]\).
Then as a result we obtain

\[
(F, H)_{\epsilon+\zeta} = \frac{(-1)^{(g_{b_1} + \cdots + g_{b_{q-1}} + (q-1)(\epsilon+\zeta))(g_f + g_l + p\zeta)}}{p!(q-1)!}
\times y_{b_{q-1}}^{\epsilon+\zeta} \cdots y_{b_1}^{\epsilon+\zeta} y_{a_1}^{\epsilon+\zeta} \left(f^{a_1 \cdots a_p} \partial \frac{}{x^l} \right) h^{b_1 \cdots b_{q-1} l}
- \frac{(-1)^{(g_l + \epsilon + \zeta)(g_f + p\epsilon + g_{a_2} + \cdots + g_{a_2} + g_{b_2} + \epsilon + \zeta)))(g_f + \epsilon + (p-1)\zeta)}}{(p-1)! q!}
\times y_{b_2}^{\epsilon+\zeta} \cdots y_{b_1}^{\epsilon+\zeta} y_{a_2}^{\epsilon+\zeta} f^{l a_2 \cdots a_p} \partial \frac{}{x^l} h^{b_1 \cdots b_q}. \tag{3.1}
\]

### 3.1 Particular cases

Let us consider the formula (3.1) for the particular values of \( \epsilon \) and \( \zeta \).

1. We start from the case which leads to the usual Schouten-Nijenhuis bracket for the skew-symmetric contravariant tensors. In this case, when \( \epsilon = 0 \), \( \zeta = 1 \) and the matrix \( \omega_0^{ab}(x) = -\omega_0^{ba}(x) \) corresponds to the usual Poisson bracket for the commuting coordinates \( z^a = x^a \), we have

\[
(F, H)_1 = \frac{(-1)^{(q-1)(g_f + p)}}{p!(q-1)!} \Theta_{b_{q-1}} \cdots \Theta_{b_1} \Theta_{a_1} \left(f^{a_1 \cdots a_p} \partial \right) h^{b_1 \cdots b_{q-1} l}
- \frac{(-1)^{(g_l + \epsilon + \zeta)(g_f + p\epsilon + g_{a_2} + \cdots + g_{a_2} + q(\epsilon + \zeta)))(g_f + \epsilon + (p-1)\zeta)}}{(p-1)! q!}
\times y_{b_2} \cdots y_{b_1} y_{a_2} f^{l a_2 \cdots a_p} \partial \frac{}{x^l} h^{b_1 \cdots b_q}, \tag{3.2}
\]

where \( \Theta_a \equiv y^a \) are Grassmann co-differential variables related owing to (2.13) with the Grassmann differential variables \( \Theta^a \equiv d_1 x^a \)

\[
\Theta^a = \Theta^{b_0} \omega_{ba}. \tag{3.3}
\]

When Grassmann parities of the quantities \( f \) and \( h \) are equal to zero \( g_f = g_h = 0 \), we obtain from (3.2)

\[
(F, H)_1 \overset{\text{def}}{=} (-1)^{(p+1)q+1} \Theta_{a_{p+q}} \cdots \Theta_{a_2} [F, H]^{a_2 \cdots a_{p+q}}, \tag{3.3}
\]

where \([F, H]^{a_2 \cdots a_{p+q}}\) are components of the usual Schouten-Nijenhuis bracket (see, for example, [8]) for the contravariant antisymmetric tensors\(^4\). This bracket has the following symmetry property

\[
[F, H] = (-1)^{pq}[H, F] \tag{3.4}
\]

and satisfies the Jacobi identity

\[
\sum_{(FHE)} (-1)^{ps}[[F, H], E] = 0, \tag{3.5}
\]

\(^4\)Here and below we use the same notation \([F, H]\) for the different brackets. We hope that this will not lead to the confusion.
where $s$ is a degree of a monomial $E$.

2. In the case $\epsilon = \zeta = 0$ and $\omega^a_0(x) = -\omega_0^a(x)$ we obtain the bracket for symmetric contravariant tensors (see, for example, [7])

\[
(F, H)_0 = \frac{1}{p!(q-1)!} y^0_{b_q-1} \cdots y^0_{b_p} (\partial_{x^f} f^{a_1 \cdots a_p} \partial_x) h^{b_1 \cdots b_{q-1} l} - \frac{1}{(p-1)!} q! y^0_{b_q} \cdots y^0_{b_p} f^{a_2 \cdots a_p} \partial_{x^f} h^{b_1 \cdots b_q} \overset{\text{def}}{=} y^0_{a_{p+q}} \cdots y^0_{a_2} [F, H]^{a_2 \cdots a_{p+q}},
\]

where commuting co-differentials $y^0_0$ connected with commuting differentials $y^0_a \equiv d_0 x^a$ in accordance with (2.13)

\[
y^0_a = y^0_{b\omega^b_0}
\]

and the bracket $[F, H]^{a_2 \cdots a_{p+q}}$ has the following symmetry property

\[
[F, H] = -(-1)^{g_f g_h} [H, F]
\]

and satisfies the Jacobi identity

\[
\sum_{(EFH)} (-1)^{g_e g_h} [E, [F, H]] = 0.
\]

3. By taking the Martin bracket [13] $\omega^a_0(\theta) = \omega^a_0(\theta)$ with Grassmann coordinates $z^a = \theta^a (g_a = 1)$ as an initial bracket (2.1), we have in the case $\zeta = 0$ for antisymmetric contravariant tensors on the Grassmann algebra

\[
(F, H)_0 = \frac{(-1)^{(q-1)(g_f+1)}}{p!(q-1)!} \Theta_{b_q-1} \cdots \Theta_{b_1} \Theta_{a_p} \cdots \Theta_{a_1} (f^{a_1 \cdots a_p} \partial_{\theta^f}) h^{b_1 \cdots b_{q-1} l} + \frac{(-1)^{(q-1)g_f+p}}{(p-1)!} q! \Theta_{b_q} \cdots \Theta_{b_1} \Theta_{a_p} \cdots \Theta_{a_2} f^{a_2 \cdots a_p} \partial_{\theta^f} h^{b_1 \cdots b_q} \overset{\text{def}}{=} \Theta_{a_{p+q}} \cdots \Theta_{a_2} [F, H]^{a_2 \cdots a_{p+q}},
\]

where the Grassmann co-differentials $\Theta_a$ related with the Grassmann differentials $\Theta^a$ as

\[
d_0 \theta^a = \Theta^a = \Theta_{b\omega^b_0}.
\]

The bracket $[F, H]$ has the following symmetry property

\[
[F, H] = -(-1)^{g_f g_h} [H, F]
\]

and satisfies the Jacobi identity

\[
\sum_{(EFH)} (-1)^{g_e g_h} [E, [F, H]] = 0.
\]
4. By taking the Martin bracket again, in the case $\zeta = 1$

$$d_i z^a \equiv y_i^a = y_i^b \omega_{ba}^a$$

we obtain for the symmetric tensors on Grassmann algebra

$$(F, H)_1 = \frac{1}{p! (q - 1)!} x_{b_1} \cdots x_{b_l} x_{a_1} \cdots x_{a_p} (f_{a_1 \cdots a_p} \partial_{\theta^l}) h_{b_1 \cdots b_{q-1}}$$

$$= \frac{1}{(p - 1)! q!} x_{b_1} \cdots x_{b_l} x_{a_1} \cdots x_{a_p} f_{a_1 \cdots a_p} \partial_{\theta^l} h_{b_1 \cdots b_q}$$

$$\equiv y_{p+q}^1 \cdots y_{a_2}^1 [F, H]^{a_2 \cdots a_{p+q}},$$

The bracket $[F, H]$ has the following symmetry property

$$[F, H] = -(-1)^{(g_f + p + 1)(g_h + q + 1)} [H, F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{(g_e + s + 1)(g_h + q + 1)} [E, [F, H]] = 0.$$

5. In general, if we take the even bracket in superspace with coordinates $z^a = (x, \theta)$, then in the case $\zeta = 1$ we have

$$(F, H)_1 = \frac{(-1)^{(g_f + g_h + q - 1) + (g_f + g_h + p)}}{p! (q - 1)!} y_{b_1}^1 \cdots y_{b_l}^1 y_{a_1}^1 \cdots y_{a_p}^1 (f_{a_1 \cdots a_p} \partial_{\theta^l}) h_{b_1 \cdots b_{q-1}}$$

$$= \frac{(-1)^{(g_f + g_h + q - 1) + (g_f + g_h + p) + (g_f + g_h + q) (g_f + p - 1)}}{(p - 1)! q!} y_{b_1}^1 \cdots y_{b_l}^1 y_{a_1}^1 \cdots y_{a_p}^1 f_{a_1 \cdots a_p} \partial_{\theta^l} h_{b_1 \cdots b_q}$$

$$\equiv y_{p+q}^1 \cdots y_{a_2}^1 [F, H]^{a_2 \cdots a_{p+q}},$$

where

$$d_i z^a \equiv y_i^a = y_i^b \omega_{ba}^a.$$

The bracket $[F, H]$ has the following symmetry property

$$[F, H] = -(-1)^{(g_f + p + 1)(g_h + q + 1)} [H, F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{(g_e + s + 1)(g_h + q + 1)} [E, [F, H]] = 0.$$

6. In the case of the even bracket in superspace as initial one with $\zeta = 0$ we obtain

$$(F, H)_0 = \frac{(-1)^{g_f + g_h + q - 1)}{p! (q - 1)!} y_{b_1}^0 \cdots y_{b_l}^0 y_{a_1}^0 \cdots y_{a_p}^0 (f_{a_1 \cdots a_p} \partial_{\theta^l}) h_{b_1 \cdots b_{q-1}}$$
\[- \frac{(-1)^{g_f + g_h} \gamma(g_f + g_h + g_{a2, \ldots, a_p})}{(p-1)! q!} y_{b_q}^0 \cdots y_{b_1}^0 \cdots y_{a_p}^0 f^{a_2 \ldots a_p \partial_z} h^{b_1 \ldots b_q} \]

\[= y_{a_{p+q}}^0 \cdots y_{a_2}^0 [F, H]^{a_2 \ldots a_{p+q}}, \]

where

\[d_0 z^a \equiv y_0^a = y_b^0 \omega_b^a.\]

The bracket \([F, H]\) has the following symmetry property

\[\quad [F, H] = (-1)^{g_f + g_h} [H, F]\]

and satisfies the Jacobi identity

\[\sum_{(EFH)} (-1)^{g_e + g_h} [E, [F, H]] = 0.\]

7. Taking as an initial bracket the odd Poisson bracket in superspace with coordinates \(z^a\), for the case \(\zeta = 0\) we have

\[(F, H)_1 = - \frac{(-1)^{g_{b_1} + \cdots + g_{b_{q-1}} + q-1} (g_f + g_t)}{p! (q-1)!} y_{b_q}^1 \cdots y_{b_1}^1 \cdots y_{a_1}^1 (f^{a_1 \ldots a_p} \partial_z) h^{b_1 \ldots b_{q-1}} \]

\[- \frac{(-1)^{g_{b_1} + g_f + g_h} (g_f + g_{a2} + \cdots + g_{a_p}) + (g_f - 1) (g_{a_1} + \cdots + g_{b_q} + q)}{(p-1)! q!} \]

\[\times y_{b_q}^1 \cdots y_{b_1}^1 \cdots y_{a_2}^1 f^{a_2 \ldots a_p \partial_z} h^{b_1 \ldots b_q} \define y_{a_{p+q}}^1 \cdots y_{a_2}^1 [F, H]^{a_2 \ldots a_{p+q}}, \]

where

\[d_0 z^a \equiv y_0^a = y_b^1 \omega_b^a.\]

The bracket \([F, H]\) has the following symmetry property

\[\quad [F, H] = (-1)^{(g_{f+1} + g_{h+1})} [H, F]\]

and satisfies the Jacobi identity

\[\sum_{(EFH)} (-1)^{(g_e + 1) (g_h + 1)} [E, [F, H]] = 0.\]

8. At last for the odd Poisson bracket in superspace, taking as an initial one, we obtain in the case \(\zeta = 1\)

\[(F, H)_0 = (-1)^{g_{b_1} + \cdots + g_{b_{q-1}} + g_f + g_h} \frac{1}{p! (q-1)!} y_{b_q}^0 \cdots y_{b_1}^0 \cdots y_{a_1}^0 (f^{a_1 \ldots a_p} \partial_z) h^{b_1 \ldots b_{q-1}} \]

\[- \frac{(-1)^{g_{b_1} + \cdots + g_{b_{q-1}} + g_f + g_h} (g_f + g_{a2} + \cdots + g_{a_p}) + (g_f - 1) (g_{a_1} + \cdots + g_{b_q} + q)}{(p-1)! q!} \]

\[\times y_{b_q}^0 \cdots y_{b_1}^0 \cdots y_{a_2}^0 f^{a_2 \ldots a_p \partial_z} h^{b_1 \ldots b_q} \define y_{a_{p+q}}^0 \cdots y_{a_2}^0 [F, H]^{a_2 \ldots a_{p+q}}, \]
where
\[ d_iz^a \equiv y_1^a = y_0^a \omega_1^a. \]

The bracket \([F, H]\) has the following symmetry property
\[
[F, H] = -(1)^{(g_f+p)(g_h+q)}[H, F]
\]
and satisfies the Jacobi identity
\[
\sum_{(E,F,H)} (1)^{(g_e+s)(g_h+q)}[E, [F, H]] = 0.
\]

Thus, we see that the formula (3.1) contains as particular cases quite a number of the Schouten-Nijenhuis type brackets.

4. Generalizations for the differential analog of the special Yang-Baxter equations

The bracket (2.12) for the monomials \(F\) and \(H\) having respectively degrees \(p\) and \(q\) and expressed in terms of the variables \(y_e^a\) has the form
\[
(F, H)_{\epsilon+\zeta} = \frac{(-1)^{[g_0+\cdots+g_{q-1}+\epsilon(q-1)+\zeta]}}{p!(q-1)!} \times \frac{g_{n-1}}{z_1^1 \cdots z_1^n} \cdot y_e^{a_1} \cdots y_e^{a_p} \cdot y_e^{a_1} \cdot (\partial_z)^{a_1} \omega_1^{ln} h_{b_1 \cdots b_{q-1}}
\]
\[
+ \frac{(-1)^{[g_0+\cdots+g_{q-1}+\epsilon(q-1)+\zeta]}}{p!(q-1)!} \times \frac{g_{q-1}}{z_1^1 \cdots z_1^n} \cdot y_e^{a_1} \cdots y_e^{a_p} \cdot f_{a_{l-1}} \partial z^{a_1} \omega_1^{ln} h_{b_1 \cdots b_q}
\]
\[
+ \frac{(-1)^{[g_0+\cdots+g_{q-1}+\epsilon(q-1)+\zeta]}}{p!(q-1)!} \times \frac{g_{q-1}}{z_1^1 \cdots z_1^n} \cdot y_e^{a_1} \cdots y_e^{a_p} \cdot f_{a_{l-1}} \partial z^{a_1} \omega_1^{ln} h_{b_1 \cdots b_{q-1}}.
\]

The formulas (3.1) and (4.1) in particular case when \(F = H \equiv S\) and the degree of the monomial \(S\) is equal to two
\[
S = \frac{1}{2} y_e^{a_1} y_e^{a_1} S^{a_1 a_2} = \frac{1}{2} y_e^{a_1} y_e^{a_1} S^{a_1 a_2}
\]
take the following form

\[(S, S)_{\epsilon+\zeta} = \frac{1}{2}(-1)^{(g_s+\epsilon+\zeta)(g_t+g_{a_2}+g_{a_3})+g_s(\epsilon+\zeta)} \left[ (-1)^{g_s} - (-1)^{\epsilon+\zeta} \right] y_{a_3} y_{a_2} y_{a_1} s^{a_1 t} \partial_{z^t} s^{2 a_3} \]

\[= (-1)^{(g_{a_2}+g_{a_3})(g_s+\zeta) + (g_s+g_{a_1})+g_s(\epsilon+\zeta)} y_{a_3} y_{a_2} y_{a_1} \left\{ \frac{(-1)^{g_{a_2}+g_{a_3}}}{2} [(-1)^{g_s+\epsilon+\zeta} + (-1)^{\epsilon+\zeta}] \right\} \]

\[\times s_{a_1 t} \Omega_{a_1 n} \partial_{z^n} s_{a_2 a_3} + (-1)^{g_{a_2}+g_{a_3}} s_{a_1 t} \left( \partial_{z^t} \Omega_{a_1 n} \right) s_{n a_3} \].

(4.2)

In order that the tensor \( s^{ab} \) in (4.2) itself should be a matrix for the Poisson bracket

\[(F, H) = F \partial_{z^a} s^{ab}(z) \partial_{z^b} H \]

and therefore satisfies the Jacobi identity, the bracket (4.2) for \( S \) has to be equal to zero

\[(S, S)_{\epsilon+\zeta} = 0. \]  \( (4.3) \)

4.1 Particular cases

In two cases \( g_s = 0, \epsilon + \zeta = 0 \) and \( g_s = 1, \epsilon + \zeta = 1 \) relation (4.3) is satisfied identically because of the symmetry property of the bracket (4.2).

1. When \( g_s = 0, \epsilon = 0 \) and \( \zeta = 1 \) we have the superspace generalization for the differential analog of the special Yang-Baxter equations

\[(S, S)_1 = (-1)^{g_{a_2}+g_{a_3}} y_1 y_{a_2} y_{a_1} \left\{ s_{a_1 t} \Omega_{a_1 n} \partial_{x^n} s_{a_2 a_3} + (-1)^{g_{a_2}+g_{a_3}} s_{a_1 t} \left( \partial_{x^t} \Omega_{a_1 n} \right) s_{n a_3} \right\} = 0. \]

2. In this case if \( z^a = x^a \) are commuting coordinates and variables \( d_i x^a = y_i \equiv \theta^a \) are Grassmann quantities, we obtain the well-known differential analog of the special Yang-Baxter equations (see, for example, [10])

\[(S, S)_1 = \Theta_{a_3} \Theta_{a_2} \theta^a_1 \left\{ s_{a_1 t} \Omega_{a_1 n} \partial_{x^n} s_{a_2 a_3} + s_{a_1 t} \left( \partial_{x^t} \Omega_{a_1 n} \right) s_{n a_3} \right\} = 0. \]

3. If \( z^a = \theta^a \) are Grassmann coordinates, variables \( y_1^a = d_i \theta^a = x^a \) are commuting and the symmetric matrix \( \Omega_{a_1}^{a_2} \) corresponds to the Martin bracket [8], we have some generalization on Grassmann variables for the differential analog of the special Yang-Baxter equations

\[(S, S)_1 = x^{a_3} x^{a_2} x^{a_1} \left\{ s_{a_1 t} \Omega_{a_1 n} \partial_{\theta^n} s_{a_2 a_3} - s_{a_1 t} \left( \partial_{\theta^t} \Omega_{a_1 n} \right) s_{n a_3} \right\} = 0. \]

4. In the case \( g_s = 0, \epsilon = 1 \) and \( \zeta = 0 \) we obtain

\[(S, S)_1 = (-1)^{g_i} y_0 y_{a_2} y_{a_1} \left\{ (-1)^{g_{a_2}+g_{a_3}} s_{a_1 t} \Omega_{a_1 n} \partial_{x^n} s_{a_2 a_3} \right\} + (-1)^{g_{a_2}+g_{a_3}} s_{a_1 t} \left( \partial_{x^t} \Omega_{a_1 n} \right) s_{n a_3} = 0 \]

that is another generalization of the above mentioned analog on the superspace case.
5. When \( g_s = 1 \) and \( \epsilon = \zeta = 1 \) we have the third superspace generalization for this analog
\[
(S, S)_{0} = -(-1)^{g_{a2} + g_{a3}} y_{1}^{a3} y_{1}^{a2} y_{1}^{a1} \left[ (-1)^{g_{a2} + g_{a3}} \omega_{1}^{ln} \partial_{x^n} s_{1a3} + (-1)^{g_{a2} + g_{a3}} s_{1a1} \left( \partial_{x^{a2}} \omega_{1}^{ln} \right) s_{na3} \right] = 0.
\]

6. At last in the case when \( g_s = 1 \) and \( \epsilon = \zeta = 0 \) we have the fourth superspace generalization
\[
(S, S)_{0} = (-1)^{g_{a2} + g_{a3}} y_{0}^{a3} y_{0}^{a2} y_{0}^{a1} \left[ s_{a1} \omega_{0}^{ln} \partial_{x^n} s_{a2a3} + (-1)^{g_{a2} + g_{a3}} s_{a1} \left( \partial_{x^{a2}} \omega_{0}^{ln} \right) s_{na3} \right] = 0.
\]
In the last case we have two sub-cases.
6a) The first one when we take as an initial bracket (2.1) the usual Poisson bracket
\[
\omega_{ab}^{0} = -\omega_{ba}^{0}, \quad z^{a} \equiv x^{a}
\]
are commuting variables and \( d_{0} z^{a} = d_{0} x^{a} \equiv y_{0}^{a} \) are also commuting
\[
(S, S)_{0} = y_{0}^{a3} y_{0}^{a2} y_{0}^{a1} \left[ s_{a1} \omega_{0}^{ln} \partial_{x^n} s_{a2a3} + (-1)^{g_{a2} + g_{a3}} s_{a1} \left( \partial_{x^{a2}} \omega_{0}^{ln} \right) s_{na3} \right] = 0.
\]

6b) In the second sub-case we take the Martin bracket on the Grassmann algebra as an initial one: \( z^{a} \equiv \theta^{a} \) are Grassmann variables, \( \omega_{ab}^{0} = \omega_{ab}^{0} \) and \( d_{0} z^{a} = d_{0} \theta^{a} = \Theta^{a} \) are also Grassmann quantities. In this sub-case we have
\[
(S, S)_{0} = \Theta^{a3} \Theta^{a2} \Theta^{a1} \left[ s_{a1} \omega_{0}^{ln} \partial_{\theta^n} s_{a2a3} - s_{a1} \left( \partial_{\theta^{a2}} \omega_{0}^{ln} \right) s_{na3} \right] = 0.
\]

Thus, we obtained quite a number of generalizations for the differential analog of the special Yang-Baxter equations.

5. Conclusion

We give the prescription for the construction from a given Poisson bracket of the definite Grassmann parity another bracket. For this construction we use the exterior differentials with different Grassmann parities. We proved that the resulting Poisson bracket essentially depends on the parity of the exterior differential in spite of these differentials give the same exterior calculus \([1]\). The prescription leads to the set of different generalizations for the Schouten-Nijenhuis bracket. Thus, we see that the Schouten-Nijenhuis bracket and its possible generalizations are particular cases of the usual Poisson brackets of different Grassmann parities \((2.14)\). We hope that these generalizations will find their own application for the deformation quantization (see, for example, \([8, 14]\)) as well as the usual Schouten-Nijenhuis bracket.

We also proposed a lot of generalizations for the differential analog of the special Yang-Baxter equations. We also believe that these generalizations can be used for the description of integrable systems as well as the usual Yang-Baxter equations.

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