ON THE NOTION OF HYPERCYCLICITY FOR UNBOUNDED LINEAR OPERATORS

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Abstract. We provide a Rolewicz-type example of hypercyclic unbounded linear operators in the (real or complex) sequence spaces $l_p$ $(1 \leq p < \infty)$, $c_0$, which can be easily modified to fit the (real or complex) spaces $L_p(0, \infty)$ $(1 \leq p < \infty)$, $C_0[0, \infty]$.

An old thing becomes new if you detach it from what usually surrounds it.

Robert Bresson

1. Introduction

We give a Rolewicz-type example of hypercyclic unbounded linear operators in the sequence spaces $l_p$ $(1 \leq p < \infty)$, $c_0$, which can be easily modified to fit the spaces $L_p(0, \infty)$ $(1 \leq p < \infty)$, $C_0[0, \infty]$.

The notion of hypercyclicity, traditionally considered and well studied for continuous linear operators on Fréchet spaces, in particular for bounded linear operators on Banach spaces, and shown to be a purely infinite-dimensional phenomenon (see, e.g., [9, 10, 15]), in [1, 2] is extended to unbounded linear operators in Banach spaces, where also found are sufficient conditions for unbounded hypercyclicity and certain examples of hypercyclic unbounded linear differential operators.

Observe that, for an unbounded linear operator

$$A : X \supseteq D(A) \to X$$

(where $D(\cdot)$ is the domain of an operator) in a Banach space $(X, \| \cdot \|)$, of primary concern, in particular, becomes the issue of how substantial is the subspace

$$C^\infty(A) := \bigcap_{n=0}^\infty D(A^n)$$

($A^0 := I$, $I$ is the identity operator on $X$) of all elements $x \in X$ whose orbit

$$\{A^n x\}_{n \in \mathbb{Z}_+}$$

($\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ is the set of nonnegative integers) is well defined.

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In view of the fact that, for many important unbounded operators, including the scalar type spectral in a complex Banach space, in particular normal ones in a complex Hilbert space, (see, e.g., [3–7, 14]) such a subspace is dense in the underlying space [13], following [1, 2], we naturally define unbounded hypercyclicity (cf. [13]).

Definition 1.1 (Hypercyclicity). Let
\[ A : X \supseteq D(A) \to X \]
be a (bounded or unbounded) linear operator in a (real or complex) Banach space \( (X, \| \cdot \|) \). A nonzero vector \( x \in C^\infty(A) \) is called hypercyclic if its orbit under \( A \)
\[ \{ A^n x \}_{n \in \mathbb{Z}_+} \]
is dense in \( X \).

Operators possessing hypercyclic vectors are said to be hypercyclic.

More generally, a collection \( \{ T(t) \}_{t \in I} \) (\( I \) is a nonempty indexing set) of linear operators in \( X \) is called hypercyclic if it possesses hypercyclic vectors, i.e., such nonzero vectors \( x \in \bigcap_{t \in I} D(T(t)) \), whose orbit
\[ \{ T(t) x \}_{t \in I} \]
is dense in \( X \).

Remarks 1.1.

- Obviously, in the definition of hypercyclicity for an operator, the underlying space is necessarily separable. Generally, for a collection of operators, this need not be so.
- It is noteworthy that, if a linear operator \( A \) is hypercyclic, then the subspace \( C^\infty(A) \), which contains the dense orbit of a hypercyclic vector, is also dense.

In [15], S. Rolewicz gave the first example of a hypercyclic bounded linear operator on a Banach space (see also [9, 10]), which on (real or complex) sequence space \( l_p \) (\( 1 \leq p < \infty \)) or \( c_0 \) (of vanishing sequences), the latter equipped with the supremum norm
\[ c_0 \ni x := (x_n)_{n \in \mathbb{N}} \mapsto \| x \|_\infty := \sup_{n \in \mathbb{N}} |x_n| \]
(\( \mathbb{N} := \{1, 2, \ldots \} \) is the set of natural numbers), is the following weighted backward shift:
\[ A(x_n)_{n \in \mathbb{N}} := \lambda (x_{n+1})_{n \in \mathbb{N}}, \]
where \( \lambda \in \mathbb{R} \) (or \( \lambda \in \mathbb{C} \)) with \( |\lambda| > 1 \) is arbitrary.

2. Examples of Unbounded Hypercyclic Operators

In [13], (bounded or unbounded) scalar type spectral operators in a complex Banach, in particular normal ones in a complex Hilbert space, (see, e.g., [3–7, 14]) and certain collections of their exponentials, which, in particular, form strongly continuous semigroups or groups of bounded linear operators (see, e.g., [8, 11, 12]), are proven to be non-hypercyclic. Here, we are to give a natural example of a hypercyclic
unbounded linear operator by modifying the celebrated Rolewicz’s example [15] and its refined proof given for a particular case in [9, Example 2.18].

**Theorem 2.1.** In the (real or complex) sequence space \( X := l_p \) (1 ≤ p < ∞) or \( X := c_0 \), the weighted backward shift

\[
D(A) \ni x := (x_n)_{n \in \mathbb{N}} \mapsto Ax := (λ^n x_{n+1})_{n \in \mathbb{N}},
\]

where \( λ \in \mathbb{R} \), \(|λ| > 1\) is arbitrary, with the domain

\[
D(A) := \{ (x_n)_{n \in \mathbb{N}} \in X \mid (λ^n x_{n+1})_{n \in \mathbb{N}} \in X \}
\]

is a hypercyclic unbounded linear operator.

**Proof.** We designate the norm on \( X \) by \( \| \cdot \| \).

For each \( n \in \mathbb{N} \), the operator \( A^n \) is densely defined, the domain \( D(A^n) \) containing the dense subspace

\[
c_{00} := \{ (x_n)_{n \in \mathbb{N}} \mid \exists N \in \mathbb{N} : x_n = 0, \; n \geq N \}
\]

of eventually zero sequences, and, as is easily seen, is unbounded and closed. Hence, the subspace

\[
C^\infty(A) := \bigcap_{n=0}^{\infty} D(A^n)
\]

of all possible initial values for the orbits under \( A \) is dense in \((X, \| \cdot \|)\).

Observe that

\[
A^n(x_m)_{m \in \mathbb{N}} = (λ^{(n+m-1)+(n+m-2)+\cdots+m}x_{m+n})_{m \in \mathbb{N}}, \; n \in \mathbb{N}, (x_m)_{m \in \mathbb{N}} \in D(A^n),
\]

with

\[
D(A^{n+1}) \subseteq D(A^n), \; n \in \mathbb{N},
\]

and, for the bounded right inverse of \( A \):

\[
B(x_m)_{m \in \mathbb{N}} := (0, λ^{-1}x_1, λ^{-2}x_2, \ldots), \; (x_m)_{m \in \mathbb{N}} \in X,
\]

we have:

\[
B^n(x_m)_{m \in \mathbb{N}} = (0, \ldots, 0, λ^{-[n+(n-1)+\cdots+1]}x_1, \lambda^{-[(n+1)+n+\cdots+2]}x_2, \ldots), \; n \in \mathbb{N}.
\]

The subspace \( c_{00} \) contains a countable dense in \((X, \| \cdot \|)\) subset

\[
Y := \left\{ y^{(k)} := \left( y^{(k)}_m \right)_{m \in \mathbb{N}} \right\}_{k \in \mathbb{N}}
\]

of all eventually zero sequences with rational (or complex rational) terms.

For each \( k \in \mathbb{N} \), let

\[
m_k := \max \left\{ m \in \mathbb{N} \mid y^{(k)}_m \neq 0 \right\}.
\]

Inductively, one can choose an increasing sequence \((n_k)_{k \in \mathbb{N}}\) of natural numbers such that, for all \( j, k \in \mathbb{N} \) with \( k > j \),

\[
n_k \geq n_j + m_j \text{ and } |λ|^{n_k+(n_k-1)+\cdots+1} \geq |λ|^{n_k+(n_k-1)+\cdots+(n_k-n_j+1)+k} \| y^{(k)} \|.
\]
Let us show that the vector
\[ x := \sum_{j=1}^{\infty} B^{n_j} y^{(j)} \]
is hypercyclic for \( A \), the above series converging in \((X, \| \cdot \|)\) since, for any \( j = 2, 3, \ldots \), in view of (2.5),
\[ \left\| B^{n_j} y^{(j)} \right\| \leq |\lambda|^{-n_j + [n_j^2 + (n_j - 1) + \cdots + 1]} \left\| y^{(j)} \right\| \leq |\lambda|^{-j}. \]

Further, for each \( k \in \mathbb{N} \),
\[
\sum_{j=1}^{\infty} A^{n_k} B^{n_j} y^{(j)} = \sum_{j=1}^{k-1} A^{n_k-n_j} y^{(j)} + y^{(k)} + \sum_{j=k+1}^{\infty} B^{n_j-n_k} y^{(j)}
\]

since, by (2.5), for \( j = 1, \ldots, k-1 \), \( n_k - n_j \geq m_j \), by (2.4), \( A^{n_k-n_j} y^{(j)} = 0 \);
\[ = y^{(k)} + \sum_{j=k+1}^{\infty} B^{n_j-n_k} y^{(j)}. \]

Since, for all \( k, j \in \mathbb{N} \) with \( j \geq k+1 \), in view of (2.5),
\[ \left\| B^{n_j-n_k} y^{(j)} \right\| \leq |\lambda|^{-(n_j-n_k) + (n_j - n_k - 1) + \cdots + 1} \left\| y^{(j)} \right\| \leq |\lambda|^{-j}. \]

This implies that, for any \( k \in \mathbb{N} \), the series
\[ \sum_{j=1}^{\infty} A^{n_k} B^{n_j} y^{(j)} \]
converges in \((X, \| \cdot \|)\), and hence, by the closedness of \( A^{n_k} \) and in view of inclusion (2.2),
\[ x \in \bigcap_{k=1}^{\infty} D(A^{n_k}) = C^\infty(A) \]
and
\[ A^{n_k} x = y^{(k)} + \sum_{j=k+1}^{\infty} B^{n_j-n_k} y^{(j)}, \quad k \in \mathbb{N}. \]

Furthermore, since, by (2.7), for each \( k \in \mathbb{N} \),
\[ \left\| A^{n_k} x - y^{(k)} \right\| \leq \sum_{j=k+1}^{\infty} \left\| B^{n_j-n_k} y^{(j)} \right\| \leq \sum_{j=k+1}^{\infty} |\lambda|^{-j} = \frac{|\lambda|^{-(k+1)}}{1 - |\lambda|^{-1}}, \]
with \( Y \) (see (2.3)) being a dense set in \((X, \| \cdot \|)\), we infer that the orbit \( \{A^n x\}_{n \in \mathbb{Z}_+} \) of \( x \) under \( A \) is dense in \((X, \| \cdot \|)\) as well. This implies that the element \( x \) defined by (2.6) is hypercyclic for \( A \), and hence, hypercyclic is the operator \( A \), which completes the proof. \( \square \)

**Remark 2.1.** For the complex case, one can take an arbitrary \( \lambda \in \mathbb{C} \) with \( |\lambda| > 1 \).
Considering that, in the (real or complex) separable Banach spaces $L_p(0, \infty)$ ($1 \leq p < \infty$) and $C_0[0, \infty)$ (of continuous on $[0, \infty)$ and vanishing at infinity functions), the latter equipped with the supremum norm

$$C_0[0, \infty) \ni x \mapsto \|x\|_{\infty} := \sup_{t \geq 0} |x(t)|,$$

there similarly exist countable dense subsets of eventually zero elements (see, e.g., [5]), we immediately obtain the following

**Theorem 2.2.** In the (real or complex) space $L_p(0, \infty)$ ($1 \leq p < \infty$) and $C_0[0, \infty)$, the weighted backward shift

$$[Ax](t) := \lambda^t x(t + a), \quad t \geq 0,$$

where $\lambda > 1$ and $a > 0$ are arbitrary, with the domain

$$D(A) := \{x(\cdot) \in X | \lambda x(\cdot + a) \in X\}$$

is a hypercyclic unbounded linear operator.

**Remark 2.2.** In the case of $L_p(0, \infty)$ ($1 \leq p < \infty$), the notations $x(\cdot)$ and $\lambda x(\cdot + a)$ are used to designate both the equivalence classes of functions and their corresponding representatives.

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