Abstract

Berwald metrics are particular Finsler metrics which still have linear Berwald connections. Their complete classification is established in an earlier work, [Sz1], of this author. The main tools in these classification are the Simons-Berger holonomy theorem and the Weyl-group theory. It turns out that any Berwald metric is a perturbed-Cartesian product of Riemannian, Minkowski, and such non-Riemannian metrics which can be constructed on irreducible symmetric manifolds of rank > 1. The existence of these metrics are well established by the above theories.

The present paper has several new features. First, the Finsler functions of Berwald manifolds are explicitly described by the Chevalley polynomials. New results are also the complete lists of reversible ($d(x, y) = d(y, x)$) resp. irreversible ($d(x, y) \neq d(y, x)$) Berwald metrics. The Cartan symmetric Finsler manifolds are also completely determined. The paper is concluded by proving that a Berwald metric is uniquely determined by the Minkowski metric induced on an arbitrarily fixed maximal totalgeodesic flat submanifold (Cartan flat). Moreover, any two Cartan flats are isometric.

1 Introduction

Loosely speaking, Finsler metrics generalize the Riemannian metrics like the Banach spaces do the Hilbert spaces. On Riemannian manifolds the norm of tangent vectors is defined by a field, $\langle \cdot, \cdot \rangle_p$, of inner products, while on a Finsler manifold, $(M, L)$, the norm is defined by an appropriate Banach-Minkowski norm $\|X\|_p = L_p(X)$ on each tangent space $T_p(M)$. The Finsler functions, $L$, correspond to the Lagrange functions in the more general Lagrange-Hamilton theory. Analogously to the Riemannian connections,
Berwald introduced the so-called Berwald connections on Finsler manifolds, which are, however, non-linear in general. This implies that the parallel transports $\tau : T_p(M) \to T_q(M)$ defined along the curves of $M$ are just non-linear homogeneous maps preserving the norm $L_p(X)$.

A Finsler metric is called Berwald metric if the Berwald connection is still a linear connection. This concept is the closest possible to the concept of Riemannian metrics. The prototype of Berwald metrics are the Minkowski metrics defined by an appropriate norm on the affine space $\mathbb{R}^n$. In this case the Berwald connection is nothing but the natural flat linear connection on the affine space.

The classification of Berwald metrics is established in an earlier paper ([Sz1]) of this author. The basic tools used in this classification are the Simons-Berger holonomy theorem (asserting that the holonomy group $\mathcal{H}_p$ at a point $p \in M$ of an irreducible Riemannian manifold is transitive on the unit sphere $I_p$ (indicatrix), unless the metric is symmetric of rank $\geq 2$), the Weyl group theory on symmetric spaces, and the deRham decomposition theory.

It turned out that the linear Berwald connection $\nabla$ on a Berwald manifold is always Riemann metrizable, furthermore, the holonomy group, $\mathcal{H}_p$, leaves the indicatrix defined by the Finsler function $L_p(X)$ invariant. Therefore, it remained the question that which Riemannian connections are metrizable also by non-Riemannian Finsler metrics. This metrization problem can be decided at an arbitrarily fixed point $p \in M$ by means of the holonomy group $\mathcal{H}_p$, since it boils down to the determination of the Riemannian holonomy groups which leave also non-Euclidean Finsler norms $\|X\|_p = L_p(X)$ invariant. Then the pursued Berwald metrics are nothing but the parallel extensions of these norms onto the whole manifold $M$.

From the Simons-Berger theorem we get that for irreducible linear connections only the Riemannian connections of symmetric spaces with rank $\geq 2$ can be Berwald connections of non-Riemannian Berwald metrics.

The existence of the desired Berwald metrics on symmetric manifolds is established by showing that the above $\mathcal{H}_p$-invariant norms are uniquely determined by their restrictions $L_a$ onto an arbitrarily fixed Cartan subalgebra $a \subset T_p(M)$, furthermore, these restricted norms are nothing but the Finsler norms which are invariant under the action of the Weyl group $W_a$ acting on $a$. More precisely, exactly these Weyl group invariant norms $L_a$ have unique extensions, first, to $\mathcal{H}_p$-invariant norms $L_p$ on the tangent space $T_p(M)$, and then, by parallel displacement, to the desired Berwald metrics defined on the whole manifold $M$. Thus, the question of existence is traced back to the existence of the Weyl group invariant Finsler norms on an ar-
bitrarily fixed Cartan subalgebra. Since the Weyl groups are finite, there is an infinite dimensional variety of the metrics appropriate to the problem.

The general reducible cases can be treated by the deRham decomposition theorem. Let it also be mentioned that for non-Riemannian Berwald metrics this decomposition means Cartesian product regarding the manifold $M$ and the connection $\Delta$, however, it is not the usual Cartesian product with respect to the metric. The usual Cartesian product of Finsler metrics leads to singular (non-strictly convex) metrics in general. However, one can easily overcome these difficulties by certain perturbation of the metric. This technique is used both in [Sz1] and this paper.

Besides the new results, also the results accomplished in [Sz1] are reestablished in this paper in a novel form. The key concepts used in these considerations are the rank, the Cartan subspace, the Cartan flat, and the Weyl group acting on a Cartan subspace. These concepts will be introduced for general Riemannian connections.

The rank of an irreducible locally symmetric Riemannian metric (or a Riemann metrizable linear connection) is defined in the usual way and the rank of an irreducible locally non-symmetric Riemannian metric is defined by 1. For reducible Riemannian metrics the rank is defined by the sum of ranks defined by the irreducible factors in the de Rahm decomposition. Though it strongly relates to them, this rank-concept is different from those introduced in [B, BBE, BBS].

With respect to the holonomy group $H_p$, consider the irreducible decomposition

$$T_p(M) = V_0 \oplus S_1 \oplus \ldots \oplus S_l \oplus U_1 \oplus \ldots \oplus U_k$$

of the tangent space at $p$, where $V_0$ is the maximal subspace fixed pointwise, and the subspaces $S_i$ resp. $U_i$ belong to irreducible symmetric resp. non-symmetric spaces (cf. (12)). Then a Cartan flat, $c$, is defined by the direct sum

$$c = V_0 \oplus c_{s1} \oplus \ldots \oplus c_{sl} \oplus c_{u1} \oplus \ldots \oplus c_{uk},$$

where $c_{si}$ is a Cartan subalgebra in $S_i$ and $c_{ui}$ is an arbitrary 1-dimensional subspace in $U_i$ (cf. (13)).

The Weyl group $W_0$ on $V_0$ consists only of the identity map and the Weyl group $W_{si}$ acting on the Cartan subalgebra $c_{si}$ of a symmetric Lie algebra is defined by the standard definition. The Weyl group $W_{ui}$ acting on $c_{ui}$ contains only two elements, the identity map and the halfturn about the origin. Then the Weyl group $W_c$ acting on $c$ is defined by the direct product (cf. (14)):

$$W_c = W_0 \times W_{s1} \times \ldots \times W_{sl} \times W_{u1} \times \ldots \times W_{uk}.$$
The main results of this paper are as follows. The basic question about characterizing the non-Riemannian Berwald metrization is answered by

**Theorem 1.1 (Abstract Main Theorem 1)** A torsion-free linear connection, $\nabla$, is Berwald metrizable by a non-Riemannian Berwald metric if and only if the closure, $\overline{\mathcal{H}_p}$, of the holonomy group is compact (i.e., it is Riemann metrizable) and the rank of $\nabla$ is greater than 1.

There are two important corollaries to this theorem:

**Theorem 1.2 (Abstract Main Theorem 2)** (1) In the irreducible case, exactly the Riemannian connections of symmetric spaces of rank $\geq 2$ are Berwald metrizable by non-Riemannian Berwald metrics.

(2) Every reducible Riemannian connection is metrizable by non-Riemannian Berwald metrics.

These statements can be reformulated as follows

**Theorem 1.3 (Abstract Structure Theorem)** Let $(M, \nabla, L)$ be a connected, simply connected, and complete Berwald manifold. Then the affine manifold $(M, \nabla)$ decomposes into the Cartesian product $(M = M_0 \times M_1 \times \ldots \times M_k, \nabla = \nabla_0 \times \nabla_1 \times \ldots \times \nabla_k)$, where $(M_0, \nabla_0, L_0)$ is the maximal Minkowskian factor and the irreducible factors $(M_i, \nabla_i, L_i)$ are either Riemannian manifolds or non-Riemannian affine symmetric Berwald manifolds. The metric $L$ is a holonomy-invariant (perturbed Cartesian) product of the metrics $L_j$ defined on the factor manifolds.

By these theorems, the non-Riemannian factors in the deRham decomposition of a Berwald manifold are either Minkowski spaces or irreducible affine symmetric Berwald spaces of rank $\geq 2$. Therefore, the Cartan classification of Riemannian symmetric spaces provides classification opportunity also for classifying Berwald manifolds. The complete list of irreducible simply connected globally symmetric spaces can be found in Table 3.2. One should exclude only the rank-one symmetric spaces (listed in (10)) in order to get the complete list of simply connected complete irreducible Riemannian spaces whose Riemannian connection can be metrized also by non-Riemannian Berwald metric. The groups involved are the unit components of the isometry groups also for the Berwald metrics.

One of the new features in this paper is the explicit constructions of Berwald metrics. As we have seen, all these metrics can be constructed by picking a $W_c$-invariant Finsler norm $L_c$ on an arbitrarily fixed Cartan subspace $c$. The theory of polynomials invariant under the action of Weyl
groups (finite reflection groups) was developed by Chevalley [Ch]. According to this theory, this polynomial ring is finitely generated and all these invariant polynomials extend to unique $\mathcal{H}_p$-invariant polynomials on the tangent space $T_p(M)$.

The explicit construction of the Berwald metrics is achieved by means of these Chevalley polynomials. This approach describes the Berwald metrics in a much more precise form. In the earlier paper the investigations focused mainly on proving the existence of non-Riemannian Berwald metrics. In **Constructive Main Theorem 1;2** both the symmetric (reversible) and non-symmetric (irreversible) B-metrics are completely classified.

These classifications are followed by two new topics. The first one is the construction of all **Cartan-symmetric Finsler manifolds**, defined by those manifolds where there exists an involutive isometry $\sigma_p$, for any point $p \in M$, such that $p$ is an isolated fix point of $\sigma_p$. Then we prove:

**Theorem 1.4** A Finsler manifold is Cartan-symmetric if and only if it is a Berwald manifold with a symmetric Berwald connection and absolute homogeneous metric.

This theorem along with the classification of Berwald spaces provides the classification of Cartan-symmetric Finsler manifolds.

In the other new topic the maximal totally geodesic flats (called Cartan flats) on Berwald manifolds are investigated. These flats are defined by the surfaces inscribed by the geodesics tangent to a Cartan subspace $c$. We show that each vector is tangent to a Cartan flat and the induced Minkowski metrics are locally isometric on these flats.

By the above construction we get that the whole Berwald metric is uniquely determined by the metric on an arbitrarily fixed Cartan flat.

Then we prove that the Berwald connection of a Berwald space is symmetric if and only if the isometries from the unit component $J^0$ of isometries act transitively on the set of pairs $(p, t)$, where $t$ is a Cartan flat and $p \in t$. One can read this statement as follows: The k-flat homogeneous Berwald spaces are nothing but the symmetric Berwald spaces of rank $k$.

Note that this theorem generalizes the characterization of irreducible 2-point homogeneous spaces as being exactly the rank-one-symmetric spaces.

Berger’s holonomy theorem is one of the central issues today, both in mathematics and mathematical physics. This theorem along with the Simon-Berger transitivity theorem was introduced to me, at the time when the first paper [Sz1] was written, by János Szenthe. It is my pleasure to thank also Ádám Korányi for the recent helpful conversations.
2 Technicalities

Berwald connections. A Finsler manifold is denoted by \((M, L)\), where \(M\) is the underlying manifold and \(L\) is the smooth Finsler function defined on the punctured tangent bundle \(T(M)\setminus 0\). A coordinate neighborhood \(x = (x^1, \ldots, x^n)\) on \(M\) defines a coordinate neighborhood \((x = (x^i), y = (y^i))\) on \(T(M)\) such that \(y^i = \dot{x}^i\) holds. Then, with respect to the variable \(y\), the \(L(x, y)\) is a positive homogeneous function of degree 1 (called in short \(+1\)-homogeneous), furthermore, the fundamental tensor \(g_{ij} = (1/2)\dot{x}^i\dot{x}^jL^2\) is positive definite. This latter assumption guarantees that the indicatrix \(I_p\), defined by vectors of unit norm in \(T_p(M)\), is strongly convex.

Finsler functions having the symmetricity condition \(L(x, y) = L(x, -y)\) are called absolute homogeneous. They define symmetric metrics satisfying \(d(x, y) = d(y, x)\). The more general metrics defined by positive homogeneous Finsler functions are non-symmetric, \(d(x, y) \neq d(y, x)\), in general. A geodesic \(\sigma(t), 0 \leq t \leq r\), on a Finsler manifold is called reversible if the reversed curve \(\sigma(r - t)\) is also a geodesic. The non-reversible geodesics are called also irreversible ones. It is well known that all the geodesics are reversible if the metric is defined by an absolute homogeneous Finsler function, or, it is a Berwald metric [BCS]. In the following reversible resp. irreversible metrics are the synonyms for the symmetric resp. non-symmetric ones defined by absolute resp. proper positive homogeneous Finsler functions.

The connection coefficients \(G^i, G^i_j, G^i_{jk}\) of the Berwald connection are defined by

\[
G^i = \frac{1}{4} g^{ik} (y^r \partial_k \partial_r L^2 - \partial_k L^2), \quad G^i_j = \partial_j G^i, \quad G^i_{jk} = \partial_j G^i_k = G^i_{kj},
\]

which are derived from the Euler-Lagrange equation \(\ddot{x}^i + 2G^i(x(t), \dot{x}(t)) = 0\) for a geodesic \(x(t) = (x^i(t))\). The coefficients \(G^i_{jk}(x, y)\) correspond to the Christoffel symbols on Riemannian manifolds. On general Finsler manifolds these symbols are \(+0\)-homogeneous functions, which non-trivially depend on the \(y\)-variable in general.

The differential equation for a parallel vector field \((y^i(t))\) along a curve \((x^i(t))\) is

\[
\frac{dy^i}{dt} + G^i_k(x(t), y(t)) \frac{dx^k}{dt} = 0.
\]

This parallel transport keeps the Finsler-norm of the vectors, however, it is non-linear if the functions \(G^i_{jk}\) depend also on the \(y\)-variables. If \(\nabla_k\) stands for the covariant derivative with respect to the Berwald connection,
then the norm-keeping property is equivalent to the fundamental equation
$\nabla_k L = L_{|k} = 0$ of Finsler geometry.

One can give also an invariant description of Berwald connections by means of non-linear connections. A general non-linear connection is defined by a smooth field (distribution), $H(x, y)$, of n-dimensional horizontal subspaces on the tangent bundle $T(M)$, such that the subspace $H(x, y)$ is complement to the vertical subspace at each point $(x, y) \in T(M) \setminus 0$. The connection coefficients, $G_i^j(x, y)$, on a coordinate system $(x^i, y^i)$ are defined such that the field $E_j = \partial_j - G_i^j \partial_i$ is horizontal. Among these general non-linear connections the Berwald connection of a Finsler manifold $(M, L)$ can be pinned down by the following three properties:

**Proposition 2.1** On any Finsler manifold there exists a uniquely determined non-linear, so called Berwald connection, which is characterized by the following properties:

1. The functions $G_i^j(x, y)$ are $+1$-homogeneous,
2. it is torsion free (i.e., $G_i^{jk} = G_i^{kj}$), and
3. the parallel transport defined by it is norm preserving (i.e., $\nabla_j L = E_j(L) = 0$).

This theorem is well known (cf. one of its demonstrations in [Sz1]).

**Basic tool applied in the constructions.** A Finsler metric is said to be Berwaldian if the Berwald connection is a linear connection, i.e., the connection coefficients $G_i^j$ depend only on the $x$-variable. Let $(M, L, \nabla)$ be a Berwald manifold. Assume that $M$ is both arc-wise and simply connected. Then, at an arbitrary point $p \in M$, the holonomy group $\mathcal{H}_p$ of the linear connection $\nabla$ has only one component.

By the above Proposition, the $\mathcal{H}_p$ is a subgroup of the compact group, $G$, of linear transformations leaving the indicatrix $I_p$ invariant. Let $dg$ be the normalized invariant Haar measure on $G$ and consider an arbitrary inner product $\langle , \rangle^*$ on the tangent space $T_p(M)$. Then the averaged inner product

$$\langle X, Y \rangle = \int_G \langle g(X), g(Y) \rangle^* dg$$

(1)

is invariant under the action both of $G$ and $\mathcal{H}_p$. Therefore, by the parallel transports defined by $\nabla$, it can be extended into a Riemannian metric, $g(X, Y)$, for which the Riemannian connection is $\nabla$. Thus the linear connection $\nabla$ of a Berwald metric is Riemann metrizable [Sz1]. By the above considerations we also have: A torsion-free linear connection is Riemann metrizable if and only if the closure $\overline{\mathcal{H}}_p$ of the holonomy group is compact.
Thus the actual question is this: *Which Riemann connections can be metrized also by non-Riemannian Berwald metrics?* This question can be decided at an arbitrary fixed point $p$ by means of the holonomy group. A subtler reformulation of the problem is this: *Which Riemannian holonomy groups $H_p$ leave also non-Euclidean Finsler norm $\|X\|_p = L_p(X)$ invariant?* After finding these norms, the parallel extension of $L_p$ onto the whole manifold provides the desired Berwald metric.

The *rank* of an irreducible locally symmetric Riemannian metric is defined in the usual way and the rank of an irreducible locally non-symmetric Riemannian metric is defined by 1. For reducible Riemannian metrics the rank is defined by the sum of ranks of the irreducible factors appearing in the deRahm decomposition. One of the main results of this paper is the following

**Theorem 2.2 (Abstract Main Theorem 1.)** A torsion-free linear connection, $\nabla$, is Berwald metrizable by a non-Riemannian Berwald metric if and only if the closure, $\overline{H}_p$, of the holonomy group is compact (i.e., the $\nabla$ is Riemannian metrizable) and the rank of $\nabla$ is greater than 1.

There are two important corollaries to this theorem:

**Corollary 2.3 (Abstract Main Theorem 2.)** (1) In the irreducible case, exactly the Riemannian connections of symmetric spaces of rank $\geq 2$ are Berwald metrizable by non-Riemannian Berwald metrics.

(2) Every reducible Riemannian connection is metrizable by non-Riemannian Berwald metrics.

The name of the above statements indicates that they will be established in a much stronger form, namely, by explicit constructions described in the next sections.

### 3 Constructing the irreducible Berwald metrics

**Basic constructions.** By the Berger-Simons theorem, the only candidates for irreducible Riemann connections which can be metrized also by non-Riemannian Berwald metrics are the irreducible symmetric connections of rank $\geq 2$. Next, besides pointing out the existence of these Berwald metrics a technique is developed by which all these metrics can be constructed.

In [Sz1], the construction was established by Lemmas 2, 3, and 4. The proofs are carried out by considering the *symmetric Lie algebra* $T_p \oplus h_p$, at an arbitrarily fixed point $p \in M$ (cf. [KN], chapter 11). This decomposition of
the Lie algebra corresponds to the usual Cartan decomposition \( g = p \oplus k \), i.e., the horizontal subspace is identified with the tangent space \( T_p(M) \) and the vertical subspace is identified with the Lie algebra, \( h_p \), of the holonomy group \( \exp(h_p) = \mathcal{H}_p \). (On irreducible symmetric spaces the group \( \mathcal{H}_p \) is identified with the unit component, \( K \), of isometries. Thus also the identification \( h_p = k \) is well defined.)

The Lie bracket \([T_p, T_p] \rightarrow h_p\) is defined by \([X, Y] = -R_p(X, Y)\), where \( R_p(X, Y)Z \) denotes the Riemannian curvature at \( p \). On \( h_p \) we keep the original Lie bracket, furthermore, \([h_p, T_p] = h_p(T_p)\).

Let \( a \) be a maximal Abelian subalgebra (Cartan subalgebra) in \( T_p \). On geometric level they correspond to the tangent spaces of the maximal totally geodesic flats. It is well known that any two Cartan subalgebras can be transformed to each other by isometries represented in \( \mathcal{H}_p \). Furthermore, every vector \( v \in T_p \) is contained in at least one of the Cartan subspaces. More precisely, there exist an everywhere dense open subset of the so called regular vectors which belong exactly to one of the Cartan subspaces. The others (called singular vectors) are covered by more Cartan subspaces. The rank is defined by the common dimension of the Cartan subspaces.

Denotation \( \mathcal{H}_p(X) \) stands for the orbit of the holonomy group passing through \( X \in T_p \). Then the tangent space to this orbit at \( X \) is \( h_p(X) \). The following statements are standard whose proofs can be found also in [Sz1].

**Lemma 3.1** (A) Any Cartan subalgebra \( a \) is intersected by any above orbit at finite many points: \( a \cap \mathcal{H}_p(X) = \{q_1, \ldots, q_l\} \neq \emptyset \).

(B) For every pair \( q_i, q_j \) of intersection points there exist some transformations \( g \in \mathcal{H}_p \) leaving \( a \) invariant \((g(a) = a)\) and satisfying \( g(q_i) = q_j \).

(C) Let \( W \) be the subgroup of \( \mathcal{H}_p \), consisting of transformations leaving a invariant. The \( W \) stands for the restriction of \( W \) onto \( a \). Then \( W \) is a finite irreducible group acting transitively on the intersection points \( \{q_1, \ldots, q_l\} \), with respect to any orbit.

The \( W \) is nothing but the Weyl group defined on a Cartan subspace \( a \).

According to the construction technique described in the previous section, we need to construct all the \( \mathcal{H}_p \)-invariant Finsler norms, \( L_p(X) \), defined on the tangent space \( T_p(M) \) at a fixed point \( p \in M \). The above statements clearly reveal the following

**Corollary 3.2** (Basic Observation) Any \( \mathcal{H}_p \)-invariant F-norm, \( L_p(X) \), is uniquely determined by its restriction, \( L_{/a}(V) \), to an arbitrary Cartan subalgebra \( a \subset T_p(M) \). More precisely, the restricted (Finsler!) norms \( L_{/a} \) are invariant under the action of the Weyl group \( W \), furthermore, \( L_p(X) = \)
$L_{/a}(V_X)$ holds, where $V_X$ is an arbitrary vector from the intersection $a \cap H_p(X)$.

The question remained after this Observation is that which $W$-invariant Finsler norms (defined only on $a$) extend to $H_p$-invariant Finsler norms (defined on $T_p(M)$). Keep in mind that the extended norm should be both smooth on the punctured space $T_p(M) \setminus 0$ and strongly convex. The most simple technique providing all the desired $H_p$-invariant Finsler norms is the averaging of Finsler norms, $L^*_p(X)$, by the holonomy group $H_p$. This averaged norm,

$$L_p(X) = \int_{H_p} L^*_p(g(X)) dg,$$

is obviously a Finsler norm. This construction does not use the above Basic Observation and it gives only vague idea about the size of the class of $H_p$-invariant Finsler norms. In order to discover the wide range of the solutions (in fact, we prove that the non-equivalent solutions form an infinite dimensional variety) we turn to the technique offered by Observation 3.2. Namely, we try to reach the solutions by extending suitable $W$-invariant norms on an arbitrarily fixed Cartan subalgebra $a \subset T_p(M)$ to the whole tangent space $T_p(M)$. This subtler construction technique exploits the theory of Chevalley’s polynomials.

**Constructing by the Chevalley extension.** If one considers an arbitrary $W$-invariant Finsler norm, $L_{/a}$, then only the continuity and convexity of the extended norm $L_p(X)$ can be established easily. Surprisingly enough, the smoothness and strong convexity lead to difficult problems. These hardships are clearly indicated in the following review of Weyl group invariant functions on a subalgebra $a$.

The $W$-invariant polynomials on a Cartan subspace $a$ were investigated by Chevalley [Ch]. In one of his theorems he proved: *If a finite group $W$ of linear transformations of an $n$-dimensional vector space is generated by reflections (the Weyl groups fall into this category!) then the algebra of polynomials invariant under the action of $W$ is generated by $n$ algebraically independent homogeneous polynomials and the identity.* In an other theorem he proved that any $W$-invariant polynomial $P$ on $a$ can be extended to $Ad(K) = Ad(H_p)$-invariant polynomial $\tilde{P}$ on $p = T_p(M)$ (unpublished, cf. [He], p. 430).

By the invariant theory developed in [Gl], [Sch], [Ma], every $C^\infty W$-invariant function, $f$, on $a$ can be written as

$$f(x) = g(\sigma_1(x), \ldots, \sigma_n(x)), \quad g \in C^\infty(R^n),$$
where \( \sigma_1, \ldots, \sigma_n \) generate the \( W \)-invariant polynomials, therefore, the \( f \) extends to the \( C^\infty \), \( H_p \)-invariant function \( \tilde{f} \) on \( T_p(M) \). However, this statement is false if \( C^\infty \) is replaced by \( C^k \). This does not terminates, of course, the possibility for the corresponding \( C^k \)-extension, however, as far the author knows, this \( C^k \)-extension is still an open problem. Let it be also mentioned that a direct elementary proof of the \( C^\infty \)-extension theorem can be found in [Da]. First we describe a simple technique using this \( C^\infty \)-extension.

The whole set of \( W \)-invariant \( C^\infty \) Finsler norms on \( a \) can be constructed by the averaging \( L/H = \sum_{\varphi \in W} L^*_{/a}(\varphi(H))/|W| \) of arbitrary \( C^\infty \) Finsler norms \( L^*_{/a} \) defined on \( a \). Let \( L/H \) be such an invariant norm. The \( C^\infty \) Chevalley extension can not be directly applied to this function, since it is of class \( C^\infty \) only on the punctured space \( a\setminus0 \). The function \( e^{-1/|H|^2} L/H \), however, is \( C^\infty \) everywhere and its Chevalley extension is of the form \( e^{-1/|X|^2} \tilde{L}/a(X) \), where \( \tilde{L}/a(X) \) is the Chevalley extension of \( L/H \) onto \( T_p(M) \).

The next problem arising about the extended function is that it is not strongly convex in general. One can control this problem by applying an appropriate correction (perturbation) term by considering the norm

\[
L(X) = (\gamma|X|^2 + \tilde{L}^2(X))^{1/2},
\]

where \( |X| \) is a \( H_p \)-invariant Euclidean norm and the constant \( \gamma > 0 \) is chosen such that \( L \) should be strongly convex. Since the matrix field \( \tilde{g}_{ij} = (1/2)\partial_i \partial_j \tilde{L}^2 \) is +0-homogeneous, the existence \( \gamma \) is obvious. In fact, there exists a \( \delta > 0 \) such that all the constants satisfying \( \gamma > \delta \) are appropriate to the problem. One can construct all the \( C^\infty \) solutions by this technique.

**Remark.** A deeper insight reveals that any positive \( \gamma \) is suitable to the problem since the matrix field is positive semi-definite everywhere. (The proof of this latter statement is omitted).

**Chevalley polynomials.** In this section we briefly review the theory describing Chevalley’s polynomials explicitly. We are particularly interested in cases when there are also skew-symmetric polynomials among the \( W \)-invariant polynomials. One can construct non-symmetric (irreversible) Berwald metrics exactly in these cases.

The Chevalley polynomials are described with the help of the root systems of symmetric Lie algebras. Bourbaki [Bou] defines an abstract root system, \( \Delta \), in a real inner product space \( V \) by a finite set of non-zero elements of \( V \) satisfying the following properties:

(i) \( \Delta \) spans \( V \),

(ii) the orthogonal transformations \( s_\alpha(\varphi) = \varphi - \frac{2(\varphi, \alpha)}{\langle \alpha, \alpha \rangle} \alpha \), for all \( \alpha \in \Delta \),
leave \( \Delta \) stable, and

(iii) \( 2(\beta, \alpha)/|\alpha|^2 \) is an integer number for all \( \alpha, \beta \in \Delta \).

It is well known that \( -\alpha \in \Delta \) if \( \alpha \in \Delta \). Furthermore, the only possible members of \( \Delta \) proportional to \( \alpha \) are \( \pm \alpha \), or \( \pm 2\alpha \), or \( \pm \alpha \) and \( \pm \frac{1}{2} \alpha \).

An abstract root system is called \textit{reduced} if \( \alpha \in \Delta \) implies that \( 2\alpha \) is not in \( \Delta \). The irreducible reduced root systems are classified by means of the Dynkin diagram \([\text{Bou}], [\text{Wa}]\). According to this classification, the different classes are

\[
A_l, B_l, C_l, (BC)_l, D_l, E_6, E_7, E_8, F_4, G_2,
\]

where the indexes are always equal to \( \dim(V) \).

The restricted root system, \( \Delta_p \), on the dual space \( V = a^* \) of a Cartan subalgebra \( a \subset p \) of a Riemannian symmetric Lie algebra \( g = p \oplus k \) is defined by the real non-zero functionals \( \alpha(H) \) for which there exist algebra elements of the form \( E_\alpha = X_\alpha + Z_\alpha \), where \( X_\alpha \in p \) and \( Z_\alpha \in k \), such that \([H, X_\alpha] = \alpha(H)Z_\alpha\), \([H, Z_\alpha] = \alpha(H)X_\alpha\), and therefore \([H, E_\alpha] = \alpha(H)E_\alpha\) hold. Vectors \( E_\alpha \) span the subspaces \( g_\alpha \) in the root-space decomposition \( g^C = a^C \oplus \sum_{\alpha \in \Delta} g_\alpha \) resp. \( g = a \oplus \sum_{\alpha \in \Delta} g_\alpha \) corresponding to the compact resp. non-compact cases. These restricted root systems are always reduced and, in the irreducible cases, each of them falls into one of the categories described in (4).

The Weyl group is generated by the reflections \( s_\alpha \), where \( \alpha \in \Delta_p \), in the hyperplanes defined by \( \alpha(H) = 0 \) in \( a \). Therefore, it is determined by the root system. It is well known \([\text{Bou}], [\text{Wa}]\) that the systems \( B_l, C_l, (BC)_l \) determine the same Weyl group \( B_l \) and the complete list of distinct irreducible Weyl groups is

\[
A_l, B_l, C_l, E_6, E_7, E_8, F_4, G_2.
\]

The Weyl group is transitive on a complete set of roots having the same length. Thus there exists an isometry \( w_\alpha \in W \) sending \( \alpha \) to \( -\alpha \), for all \( \alpha \in \Delta \). This property is not satisfied for the other elements of \( a^* \) in general. More precisely, it is satisfied for all vector \( v \in a^* \) if and only if the function space invariant under the action of the Weyl group consists only even (reversible) functions. This case can be characterized also as follows.

\textbf{Proposition 3.3} For any \( v \in a \) there exist \( w_v \in W \) reversing \( v \) to \( -v \), i.e. \( w_v(v) = -v \), if and only if \( -id \in W \).

There exist non-trivial skew (odd) functions, satisfying \( \varphi(-H) = -\varphi(H) \), among the \( W \)-invariant functions if and only if \( W \) excludes the central symmetry \( -id \).
This statement can be established by considering the Weyl chambers, which are the connected components of the regular vectors \( v \) of \( a \) (recall that \( v \) is regular if \( \alpha(v) \neq 0 \) holds for all \( \alpha \in \Delta \)). By a standard statement of the theory, the \( W \) is simply transitive on the Weyl chambers. Fix a Weyl chamber and let \( w \in W \) be the group element transforming this Weyl chamber to the opposite one. If \( -id \not\in W \) then there exist vectors \( v \) in the chamber such that \( w(v) \neq -v \) hold. By the simply transitivity of the Weyl group on the Weyl chambers, relation \( \varphi(v) \neq -v \) must hold for all \( \varphi \in W \). This proves the existence of irreversible \( v \)'s, if \( -id \not\in W \). If \( -id \in W \), then each \( v \in a \) is obviously reversible. This proves the statement completely.

The fundamental weights \( \{\lambda_1, \ldots, \lambda_l\} \) with respect to a basis \( \{\alpha_1, \ldots, \alpha_l\} \) of the root system are defined by \( 2\left\langle \lambda_i, \alpha_j \right\rangle / \left\langle \alpha_j, \alpha_j \right\rangle = \delta_{ij} \) on \( a^* \). The \( W \)-invariant polynomials are described by means of these weights as follows.

**Proposition 3.4** (\[Bou\], p. 188) The \( W \)-invariant functions \( q_1, \ldots, q_l \) defined by

\[
q_i(H) = \sum_{\varphi \in W} e^{\sqrt{-1} \lambda_i(\varphi(H))}
\]

generate the set \( P(a)^W \) of \( W \)-invariant trigonometric polynomials.

Real valued generating set, \( r_1, \ldots, r_l \), can be extracted by taking real or imaginary parts of \( q_i \). The real parts are even functions and the imaginary parts are odd functions. \( W \)-invariant homogeneous polynomials can be found by expanding each trigonometric polynomial, \( r_i \), into its Taylor series

\[
r_i = \sum_{m=0}^{\infty} p_{i}^m \text{ about the origin, where the } p_{i}^m \text{ is homogeneous of degree } m.
\]

One can find the independent generators of the set \( S(a)^W \) of \( W \)-invariant homogeneous polynomials as follows \[Da\].

By formula (8) of \[Da\], there exist positive integers \( m_1, \ldots, m_l \) such that

\[
dp_1^{m_1} \land \ldots \land dp_l^{m_l} = c(\Pi_{\alpha \in \Delta^+} \alpha) dH,
\]

where the set \( \Delta^+ \) is defined by those roots which are positive on a fixed Weyl chamber, furthermore, \( dH \) is the \( l \)-form giving the Euclidean volume element on \( k \). Then one has:

**Proposition 3.5** \[Da\] The polynomials \( p_1^{m_1}, \ldots, p_l^{m_l} \) are algebraically independent and generate \( S(a)^W \). Furthermore, \( p_i^{m_i}(H) = \sum_{\varphi \in W} \lambda_i^{m_i}(\varphi(H)) \) and \( \sum m_i \) is equal to the number of positive roots.

The degrees \( m_i \) of the generating Chevalley polynomials are well known (cf. \[Wa\], p. 144, \[Bou\], \[Ch\], \[BoCh\], \[Ko1\], \[Ko2\]). Their complete list is as follows.
Table 3.1
Irreducible Weyl groups and their Chevalley degrees

| diagram | rank | $W$  | $m_i$                              |
|---------|------|------|-----------------------------------|
| $A_l$   | $l$  | $A_l$| $2, 3, 4, \ldots, l + 1$          |
| $B_l$   | $l$  | $B_l$| $2, 4, \ldots, 2l$                |
| $C_l$   | $l$  | $B_l$| $2, 4, \ldots, 2l$                |
| $BC_l$  | $l$  | $B_l$| $2, 4, \ldots, 2l$                |
| $D_l$   | $l$  | $D_l$| $2, 4, \ldots, 2l - 2, l$         |
| $E_6$   | 6    | $E_6$| $2, 5, 6, 8, 9, 12$               |
| $E_7$   | 7    | $E_7$| $2, 6, 8, 10, 12, 14, 18$         |
| $E_8$   | 8    | $E_8$| $2, 8, 12, 14, 18, 20, 24, 30$    |
| $F_4$   | 4    | $F_4$| $2, 6, 8, 12$                     |
| $G_2$   | 2    | $G_2$| $2, 6$                            |

Though the degrees $m_i$ are known, it is not clear that which $\lambda_i$ do they belong to. From this point of view this description of the generators can be considered only as a semi-explicit description. More explicit description can be given by considering the individual Weyl groups separately. We do not go into the details of such individual investigations here. An other imperfection is that, up to the knowledge of the author, no explicit description of the extended polynomials is known in the literature so far.

**Constructing by Chevalley’s polynomials.** Next, the above $W$-invariant homogeneous polynomials are used to construct irreducible Berwald metrics. First pick a Cartan subspace $a$ and a $H_p$-invariant inner product $\langle , \rangle$ on $T_p(M)$. The latter is uniquely determined up to a constant factor in the considered irreducible cases. Choose also a rectangular coordinate system $(y_1, \ldots, y_n)$ defined by an orthonormal basis on $T_p(M)$. In what follows, the constructions are described for absolute- and proper positive-homogeneous norms separately.

(A) **Constructing the absolute homogeneous solutions.** In this case we choose a $W$-invariant homogeneous polynomial $P(H)$ of even order, $2k$, on $a$. These polynomials are required to be strictly positive on the punctured space $a \setminus 0$. If $P(H)$ is not strictly positive yet, this requirement can be furnished by adding the term $c \langle H, H \rangle^k$ to $P(H)$, where $c > \max |P|_S$ and $P|_S$ means the restriction of the negative part $P^-(H) = (1/2)(P(H) - |P(H)|)$ of $P(H)$ onto the unit sphere $S$ around the origin of the Euclidean space $a$. Then, by the Chevalley Theorem, the polynomial $Q(H) = c \langle H, H \rangle^k + P(H)$ extends to a $H_p$-invariant homogeneous polynomial $\tilde{Q}(X)$ of order $2k$ on $T_p(M)$ such that $\tilde{Q}^{1/k}$ is a $C^\infty(a \setminus 0)$, homogeneous function of order 2.

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As in (3), consider the 0-homogeneous matrix field $\frac{1}{2} \partial_i \partial_j \tilde{Q}_{ij}^{1/k}$. If this field is not positive definite and $\delta$ is the minimum of negative eigenvalues of this field then the norm

$$L(X) = (\gamma |X|^2 + \tilde{Q}^{1/k}(X))^{\frac{1}{2}},$$

(6)

where $\gamma > |\delta|$, is obviously a $\mathcal{H}_p$-invariant Finsler norm of class $C^\infty$.

This construction can be generalized by considering the norms

$$L(X) = (\gamma |X|^2 + \sum c_i \tilde{Q}_i^{1/k_i}(X))^{\frac{1}{2}},$$

(7)

where $\gamma > |\delta|$ are defined with respect to the function $\sum c_i \tilde{Q}_i^{1/k_i}$. If this latter sum consists only finite many terms, the constants can be arbitrary positive or negative numbers. The $L$ is always a function of class $C^\infty$.

By using infinite sums, one can construct solutions of class $C^m$ by choosing constants $c_i$ such that $\sum c_i \tilde{Q}_i^{1/k_i}$ is convergent with respect to the $C^m$-norm on the unit sphere $S$. This technique reveals a considerably wider class of solutions, however, the examples introduced so far have absolute homogeneous Finsler functions.

**Constructing the positive homogeneous Finsler functions.** Now let $k = 2s + 1$ be an odd number and $P_{2k}$ resp. $P_k$ be $W$-invariant homogeneous polynomials of degree $2k$ resp. $k$. By the discussions given below about the explicit form of $W$-invariant polynomials, such odd numbers and polynomials exist exactly on those symmetric spaces where the Weyl group does not contain the reflection $-id$ in the origin. In this case there exist $W$-invariant homogeneous polynomials $P_k$ of odd order $k = 2s + 1$. The polynomial $P_{2k}$ can be chosen, for instance, as the product of two such polynomials.

By adding a suitable correction term $c(H, H)^k$, the polynomial $R_{2k}(H) = c(H, H)^k + P_{2k}(H)$ becomes strictly positive on $a \setminus 0$ and the extended function $\tilde{R}_{2k}^{1/2}$ is a positive, $C^\infty$, and absolute homogeneous function of degree $k$ on $T_p(M)$.

In the next step consider the positive homogeneous function

$$Q^* = (\tilde{R}_{2k}^{\frac{1}{2}} + \tilde{P}_k)^2 = (\tilde{R}_{2k}^{\frac{1}{2}} + \tilde{P}_k^2) + 2\tilde{R}_{2k}^{\frac{1}{2}} \tilde{P}_k$$

(8)

of even degree $2k$. Note that the last term is a skew-symmetric function while the first two terms are absolute homogeneous. If the $Q^*$ is not strictly positive on $T_p(M) \setminus 0$, then $\tilde{Q}(X) = d(X, X)^k + Q^*(X)$ will have this property for all $d > 0$.

The construction is completed by the Finsler norm

$$L(X) = (\gamma |X|^2 + \tilde{Q}^{1/k}(X))^{\frac{1}{2}}.$$

(9)
These solutions are just positive- but not absolute-homogeneous Finsler functions.

The most general solutions can be defined by formula (7), where in the sum one can use both the absolute and positive homogeneous functions constructed above.

**Classification theorems.** These theorems can be established by combining the previous statements with the well known Cartan-list of irreducible symmetric manifolds. The table below provides the complete list along with the restricted Weyl groups belonging to these manifolds (cf. [He], p. 532-534, [KN], p. 714, [Wa], p. 30-32).

| Table 3.2 | Irreducible symmetric manifolds and their restricted Weyl groups |
|-----------|---------------------------------------------------------------|
| non-compact | compact | $W_l$ | dim($M$) |
| $SL(n, R)/SO(n)$ | $SU(n)/SO(n)$ | $A_{n-1}$ | $\frac{1}{2}(n-1)(n+2)$ |
| $SU^*(2n)/Sp(n)$ | $SU(2n)/Sp(n)$ | $A_{n-1}$ | $(n-1)(2n+1)$ |
| $SU(p \geq q)/SU_U$ | $SU(p + q)/SU_U$ | $B_l$ | $2pq$ |
| $SO_0(p, R)/SO(p)$ | $SO(p + q)/SO(p)$ | $D_0$ | $p^2$ |
| $SO_0(p > q)/SO(p) \times SO(q)$ | $SO(p + q)/SO(p) \times SO(q)$ | $B_q$ | $pq$ |
| $SO^*(2n)/U(n)$ | $SO(2n)/U(n)$ | $B_0$ | $n(n-1)$ |
| $Sp_0(n, R)/U(n)$ | $Sp(n)/U(n)$ | $B_q$ | $4pq$ |
| $Sp(p \geq q)/Sp(p) \times Sp(q)$ | $Sp(p + q)/Sp(p) \times Sp(q)$ | $A_{n+1}$ | $n(n+2)$ |
| $SL(n+1, C)/SU(n+1)$ | $x_2SU(n+1)/SU(n+1)$ | $B_{n+2}$ | $n(2n+1)$ |
| $SO(2n+1, C)/SO(2n+1)$ | $x_2SO(2n+1)/SO(2n+1)$ | $B_{n+3}$ | $n(2n+1)$ |
| $Sp(n, C)/Sp(n)$ | $Sp(n)/Sp(n)$ | $D_{n+4}$ | $n(2n-1)$ |
| $SO(2n, C)/SO(2n)$ | $SO(2n) \times SO(2n)/SO(2n)$ | $E_0$ | 42 |
| $E_6(6)/Sp(4)$ | $E_6(−78)/Sp(4)$ | $F_1$ | 40 |
| $E_6(2)/SU(6) \times SU(2)$ | $E_6(−78)/SU(6) \times SU(2)$ | $B_2$ | 32 |
| $E_6(−14)/SU(10) \times R$ | $E_6(−78)/SU(10) \times R$ | $A_2$ | 26 |
| $E_6(−26)/F_4$ | $E_6(−78)/F_4$ | $E_7$ | 70 |
| $E_{7(7)}/SU(8)$ | $E_{7(−133)}/SU(8)$ | $E_{7(−133)}/SO(12) \times SU(2)$ | 64 |
| $E_{7(−25)}/E_6 \times R$ | $E_{7(−133)}/E_6 \times R$ | $B_3$ | 54 |
| $E_{8(8)}/SO(16)$ | $E_{8(−240)}/SO(16)$ | $E_8$ | 128 |
| $E_{6(−24)}/E_7 \times SU(2)$ | $E_{6(−240)}/E_7 \times SU(2)$ | $F_1$ | 112 |
| $F_4(4)/Sp(3) \times SU(2)$ | $F_4(−52)/Sp(3) \times SU(2)$ | $F_2$ | 28 |
| $F_4(−20)/SO(9)$ | $F_4(−52)/SO(9)$ | $A_1$ | 16 |
| $G_2(2)/SU(2) \times SU(2)$ | $G_2(−14)/SU(2) \times SU(2)$ | $G_2$ | 8 |
| $\xi_6^C/\xi_6$ | $\xi_6 \times \xi_6/\xi_6$ | $E_6$ | 78 |
| $\xi_7^C/\xi_7$ | $\xi_7 \times \xi_7/\xi_7$ | $E_7$ | 133 |
| $\xi_8^C/\xi_8$ | $\xi_8 \times \xi_8/\xi_8$ | $E_8$ | 248 |
| $F_4^C/F_4$ | $F_4 \times F_4/F_4$ | $F_4$ | 52 |
| $G_2^C/G_2$ | $G_2 \times G_2/G_2$ | $G_2$ | 14 |

In this table formula $SU(p \geq q)$ denotes the group $SU(p, q)$ such that $p \geq q$ holds. The groups $SO_0(p \geq q)$ and $Sp(p \geq q)$ are similarly defined. The Cartesian product $G \times G$ is sometimes written in the form $\times_2 G$. Formula
$D_{n \geq 4}$ indicates that in the line of the formula the relation $n \geq 4$ is supposed to be satisfied. The meaning of formulas $A_{n \geq 1}, B_{n \geq k}$ is similar.

The second table describes the complete list of irreducible globally symmetric Riemannian manifolds (cf. [He], p. 516-518). Non-Riemannian Berwald metrics can be constructed exactly on those having rank $\geq 2$. The symmetric spaces of rank 1 are exactly those having the restricted Weyl group $A_1$ or $B_1$. By this principle one can sort out the well known list

$$SU(p,1)/S(U_p \times U_1), SU(p+1)/S(U_p \times U_1),$$

$$SO_0(p,1)/SO(p), SO_0(p+1)/SO(p),$$

$$Sp(p,1)/Sp(p) \times Sp(1), Sp(p,1)/Sp(p) \times Sp(1),$$

$$F_{4(-20)}/SO(9), F_{4(-52)}/SO(9)$$

of simply connected and complete symmetric spaces of rank 1.

Finally, Table 3.1 reveals that there exist skew-symmetric $W$-invariant polynomials if and only if $W$ is either $A_p$ with $p \geq 2$, or, it is one of the groups $D_{2k+1}$ and $E_6$. An irreducible Riemannian connection can be metrized by irreversible ($d(x,y) \neq d(y,x)$) Berwald metric if and only if it is symmetric and its Weyl group is one of these groups. By summing up we get

**Theorem 3.6 (Constructive Main Theorem 1)** (A) An irreducible connection of Riemannian type is Berwald metrizable by non-Riemannian Finsler metrics if and only if it is symmetric of rank $\geq 2$. Table 3.2 lists all the simply connected, complete irreducible Riemannian symmetric spaces. By leaving out from this list the rank 1 symmetric cases listed in (11), one gets the complete list of those simply connected, complete irreducible Riemannian manifolds whose Riemannian connection can be metrized also by non-Riemannian Berwald metrics.

The Berwald metrics on these manifolds are explicitly constructed in formulas (3)-(9).

(B) The only simply connected, complete irreducible Riemannian manifolds whose Riemannian connection can be metrized also by irreversible Berwald metrics are the following ones:

$$SL(n,\mathbb{R})/SO(n), SU(n)/SO(n), \quad n \geq 3,$$

$$SU^*(2n)/Sp(n), SU(2n)/Sp(n), \quad n \geq 3,$$

$$SO_0(p,p)/\times_2 SO(p), SO_0(2p)/\times_2 SO(p), \quad p = 2k + 1,$$

$$SL(n+1,\mathbb{C})/SU(n+1), \times_2 SU(n+1)/SU(n+1), \quad n \geq 2,$$

$$SO(2n,\mathbb{C})/SO(2n), \times_2 SO(2n)/SO(2n), \quad n = 2k + 1,$$
Explicit proper positive homogeneous Finsler functions on these manifolds are described in formulas (8) and (9). The Riemannian connections on these manifolds allow also reversible (d(x, y) = d(y, x)) non-Riemannian Berwald metrics. The other manifolds mentioned in (A) allow only reversible Berwald metrics.

### 4 Reducible Berwald metrics

For reducible holonomy groups the above construction is combined with the deRham decomposition of Riemannian connections. Let

\[ T_p(M) = V_0 \oplus S_1 \oplus \ldots \oplus S_l \oplus U_1 \oplus \ldots \oplus U_k \]  

be the irreducible decomposition of the tangent space at \( p \), with respect to the holonomy group \( \mathcal{H}_p \), where \( V_0 \) is the maximal point-wise fixed subspace, and the subspaces \( S_i \) resp. \( U_i \) belong to irreducible symmetric resp. to non-symmetric spaces. The integral manifolds \( M_0, M_{S_i}, M_{U_i} \) corresponding to the deRham decomposition are globally defined for simply connected and complete \( M \). Otherwise they are only locally defined. By the Berger-Simons holonomy theorem, a Berwald metric induces a Minkowski metric on \( M_0 \), the above described metrics on \( M_{S_i} \), and Riemannian metrics on the manifolds \( M_{U_i} \).

In order to construct all the Berwald metrics in the case of reducible Berwald connections, the concepts of Cartan subalgebras and Weyl groups are introduced for arbitrary Riemannian connections. A Cartan subspace, \( c \subset T_p(M) \), is defined by the direct sum

\[ c = V_0 \oplus c_{s1} \oplus \ldots \oplus c_{sl} \oplus c_{u1} \oplus \ldots \oplus c_{uk}, \]  

where \( c_{si} \) is a Cartan subalgebra on \( S_i \) and \( c_{ui} \) is an arbitrary 1-dimensional subspace in \( U_i \). The Weyl group \( W_0 \) on \( V_0 \) consists only of the identity map and the Weyl group \( W_{ui} \) acting on \( c_{ui} \) contains only two elements, the identity map and the central symmetry with respect to the origin. Then the Weyl group, \( W \), acting on \( c \) is defined by the direct product

\[ W = W_0 \times W_{s1} \times \ldots \times W_{sl} \times W_{u1} \times \ldots \times W_{uk}, \]  

where \( W_{si} \) is the usual Weyl group acting on the Cartan subalgebra \( a_{si} = c_{si} \) of the corresponding symmetric Lie algebra.
This Weyl group is obviously a finite reflection group and both the theory of Chevalley and the invariant theory applies to it. By summing up we have

**Theorem 4.1 (Constructive Main Theorem 2)**

(A) Any orbit \( \mathcal{H}_p(X) \) intersects any Cartan flat \( c \) at a non-empty set of finite many points. The finite Weyl group \( W \) acts transitively on these intersection points.

(B) All the \( \mathcal{H}_p \)-invariant Finsler norms, \( L_p(X) \), defined on \( T_p(M) \) can be represented by averaging, \( L_p(X) = \int_{\mathcal{H}_p} L_p^*(g(X)) \, dg \), of Finsler norms \( L_p^* \).

Much more explicit technique is offered by the above statements, however, since the norms \( L_p(X) \) in the question can be constructed by extending \( W \)-invariant Finsler norms \( L_c \), defined on a fixed Cartan subspace \( c \), onto \( T_p(M) \).

Any \( W \)-invariant norm \( L_c \) can be uniquely extended to a continuous \( \mathcal{H}_p \)-invariant Banach norm defined on \( T_p(M) \) by the formula \( L_p(X) = L_c(Y) \), where \( Y \) is an arbitrary vector from \( c \cap \mathcal{H}_p(X) \). There arise difficult problems regarding the smoothness and strong convexity of the extended norm, however. One can avoid these difficulties and be able to construct appropriate Finsler norms by means of the Chevalley polynomials, or in general, using \( C^\infty \) and \( W \)-invariant Finsler norms. Then the extended Finsler norms appear in the form

\[
L_p(X) = (\gamma |X|^2 + \sum c_i \tilde{Q}_i^{1/k_i}(X))^{1/2},
\]

where the functions \( Q_i \) can be even or odd, like the functions used in formulas (7) and (9). The \( C^\infty \) Finsler norms can be represented in the form

\[
L_p(X) = (\gamma |X|^2 + \tilde{L}_i^{1/2}(X))^{1/2},
\]

where \( L_i \) is a \( C^\infty \) Finsler norm defined on \( c \) and \( \tilde{L}_i \) is its extension to \( T_p(M) \).

In both formulas the correction term \( \gamma |X|^2 \) makes up the norm to a strongly convex one.

The parallel extension of \( L_p \) defines a Berwald metrization of the considered linear connection.

(C) The Riemannian connection of a reducible Riemannian manifold

\[
M = M_0 \times M_1 \times \ldots M_k
\]

can be metrized by irreversible Berwald metrics if and only if the Euclidean factor \( M_0 \) is non-trivial or at least one of the other factors is equal to one of the manifolds listed in (11).

For a reducible Riemann connection belonging to a Riemann metric \( g \), the most simple non-Riemannian Berwald metrics are those having the
Finsler function

\[ L(X) = \sqrt{c_1 |X|^2 + c_2 (|X_0|^{2p} + |X_1|^{2p} + \ldots + |X_{1+s+k}|^{2p})^{1/p}}, \quad (17) \]

where \( c_1, c_2 > 0 \), \( p > 1 \) and \( X = X_0 + \ldots X_{1+s+k} \) corresponds to the de Rham decomposition. Arbitrary positive scaling factors can be applied also before the terms \( |X_i|^{2p} \).

Notice that the function \( (|X_0|^{2p} + |X_1|^{2p} + \ldots + |X_{1+s+k}|^{2p})^{1/p} \) defines a norm for which the tensor \( g^{(p)}_{ij} \) is positive semi definite (proved by Minkowski), therefore, the norm \( (17) \) is a Finsler norm because of the correction term \( |X|^2 = g(X, X) \). This latter statement proves that any reducible Riemannian connection can be metrized by non-Riemannian Berwald metrics.

**Remark.** The only explicit non-Riemannian Berwald metrics mentioned in [Sz1] are the metrics described in (17) (more precisely, they are introduced with the specific constants \( c_1 = c_2 = 1 \)). Notice that the correction term \( |X|^2 \) adjusts the convex Minkowski norm determined by the second term to a strongly convex one.

One should be precautious about the definition of deRham decomposition of Berwald manifolds. The Cartesian product concerns only the manifold \( (M = M_0 \times M_1 \times \ldots \times M_k) \) and the Berwald connection \( (\nabla = \nabla_0 \times \nabla_1 \times \ldots \times \nabla_k) \), the metric, however, may not be the Cartesian product of the metrics defined on the factor manifolds. It is defined by the weaker requirement demanding that the Finsler function should be invariant with respect to the holonomy group.

This requirement allows infinitely many inequivalent solutions for defining Finsler functions on the product manifold \( M \) by means of the Finsler functions \( L_i \) defined on the factor manifolds \( M_i \). The Cartesian product is just one of the options. In order to make a clear distinction between Cartesian product and the product applied in this paper, the latter is called holonomy-invariant, or, perturbed Cartesian product of the metrics. Then the deRham decomposition of Berwald spaces can be stated as follows.

**Theorem 4.2 (Structure Theorem)** Let \( (M, \nabla, L) \) be a connected, simply connected, and complete Berwald manifold. Then the affine manifold \( (M, \nabla) \) decomposes into the Cartesian product \( (M = M_0 \times M_1 \times \ldots \times M_k, \nabla = \nabla_0 \times \nabla_1 \times \ldots \times \nabla_k) \), where \( (M_0, \nabla_0, L_0) \) is the maximal Minkowskian factor and the irreducible factors \( (M_i, \nabla_i, L_i) \) are either Riemannian manifolds or non-Riemannian affine symmetric Berwald manifolds. The metric \( L \) is a holonomy-invariant product of the metrics \( L_j \) defined on the factor manifolds.
Finally, we describe a straightforward generalization of Hano’s Theorem concerning the decomposition of group of isometries, induced by the de Rham decomposition.

**Theorem 4.3 (Generalized Hano Theorem [Sz1])** Let \( M = M_0 \times \ldots \times M_{1+s+k} \) be the de Rham decomposition of a complete simply connected Berwald manifold. If \( U_0(M_i) \) and \( J_0(M_i) \) denote the unit component of the affine and the isometry group on \( M_i \) respectively, then \( U_0(M) = U_0(M_0) \times \ldots \times U_0(M_{1+s+k}) \) resp. \( J_0(M) = J_0(M_0) \times \ldots \times J_0(M_{1+s+k}) \).

5 Cartan-symmetric Finsler manifolds

A diffeomorphism, \( \varphi : M \to M \), is an isometry on a Finsler manifold \((M, L)\) if it preserves the Finsler function. By the classical Dantzig-van der Waerden Theorem (cf. [KN], vol. I, chapter I, Theorem 4.7) asserting: “The group \( G \) of isometries of a connected, locally compact metric space \( M \) is locally compact with respect to the compact-open topology”, and the Montgomery-Zippin Theorem (cf. [KN], vol. I, chapter I, Theorem 4.6) asserting: “A locally compact effective transformation group \( G \) of \( C^1 \)-diffeomorphisms acting on a connected manifold \( M \) of class \( C^k \) is a Lie group and the mapping \( G \times M \to M \) is of class \( C^k \)”, the group of isometries on a connected Finsler manifold form a Lie group. Strictly speaking, these theorems prove the statement for absolute homogeneous Finsler functions. For positive homogeneous Finsler functions consider the metric, \( d^* \), defined by the function \( L^*(X) = L(X) + L(-X) \). Then the \( G \) is a closed subgroup of \( G^* \) defined for \( d^* \). Thus both groups are Lie groups.

A Finsler manifold is called **Cartan-symmetric** if there exist involutive isometry \( \sigma_p \), for each point \( p \in M \), for which the \( p \) is an isolated fixpoint. It is obvious that such \( \sigma_p \) induces \( -id \) on the tangent space \( T_p(M) \), therefore, Cartan-symmetric Finsler manifolds have reversible metrics and geodesics.

The geodesic symmetries \( s_p : \exp(y) \to \exp(-y) \) on Berwald manifolds having symmetric Riemannian connections are investigated in [DH1, DH2]. They are isometries (Cartan-symmetries) if and only if the symmetric B-manifold has absolute homogeneous Finsler function. A general Berwald manifold with symmetric Riemannian connection is called **affine symmetric Berwald manifold**, since such a manifold with proper positive homogeneous Finsler function is not Cartan-symmetric. Note that the Cartan symmetric manifolds are defined among the most general Finsler manifolds and not just among the Berwald manifolds. Next we prove, however, that these general
Cartan-symmetric Finsler manifolds are Berwald manifolds, having affine
symmetric connection $\nabla$ and an absolute homogeneous Finsler function.

In the following considerations we suppose that the symmetries $\sigma_p$ are
globally defined. By passing to the universal covering space, one can also
suppose that the $M$ is simply connected. The above statement is established
via several steps.

(A) Cartan-symmetric manifolds are homogeneous. Indeed, for two points
$p$ and $q$ which can be connected by a geodesic segment $s(p, q)$, the central
symmetry $\sigma_m$, where $m$ is the midpoint of $s(p, q)$, corresponds
$p$ and $q$ to each other. Note that this argument exploits that Cartan symmetric
manifolds have reversible geodesics. Considering two arbitrary points $p$ and
$q$, connect them by a sequence of geodesics. Then the composition of the
central symmetries defined by the midpoints of these geodesics sends the
starting point $p$ to the endpoint $q$. This proves the statement completely.

(B) The group of isometries is a Lie group by the argument described at the
very beginning of this section. Furthermore, the isotropy group $H_p$ of isome-
tries fixing a point $p$ is compact. (This latter statement can be established
by observing that the indicatrix in the tangent space $T_p(M)$ is invariant
under the actions of the induced maps $\varphi_\ast$, for all $\varphi \in H_p$.) It follows that
globally Cartan-symmetric manifolds are complete ones.

(C) The homogeneous space $J(M)/H_p$ is symmetric. In fact, let $G_\sigma \subset J$
be the subgroup of isometries generated by both algebraic and topological
closure of the set of Cartan-symmetries. Because of closedness, it is a Lie
group, moreover, it is a normal subgroup of $J$. Let $G_{\sigma 0}$ be the subgroup
generated by those isometries which can be expressed as products of even
number of Cartan-symmetries. This subgroup is in the unit component
of $G_\sigma$. For let $c_1(t), \ldots, c_{2k}(t)$ be continuous curves connecting the point
$p = c_1(0) = \ldots = c_{2k}(0)$ with the points $p_i = c_i(1)$. Then the curve
$S(t) = \sigma_{c_1(t)} \circ \ldots \circ \sigma_{c_{2k}(t)}$ in $G_{\sigma 0}$ connects $S(1) = \sigma_{p_1} \circ \ldots \circ \sigma_{p_{2k}}$ with $id = S(0)$.
For any two points $p$ and $q$ can be connected by even number of geodesics,
the group $G_{\sigma 0}$ is a transitive and normal subgroup of the isometries.

Let $H_{\sigma^p 0} \subset G_{\sigma 0}$ be the subgroup whose elements fix a point $p \in M$. Then
$M$ is a homogeneous space of the form $M = G_\sigma / H_{\sigma p} = G_{\sigma 0} / H_{\sigma p 0}$. One can
construct all of the $G_\sigma$-invariant Cartan-symmetric Finsler metrics on $M$
by extending the $H_{\sigma p}$-invariant Finsler norms, defined on $T_p(M)$, by the $G_\sigma$
on to $M$. Since the $H_{\sigma p}$ is compact, there exists also a $H_{\sigma p}$-invariant inner
product on $T_p(M)$ which extends into a $G_\sigma$-invariant Riemannian metric of
$M$. (See more about these constructions in [Sz3] and/or Theorems 1.3 and
1.4 in [DH1].)

Since the isometry group of this Riemannian metric contains the cen-
tral symmetries $\sigma_q$ for all $q \in M$, it defines a symmetric Riemannian space [KN]. (The standard proof of this statement is as follows. Let $G$ be the unit component of isometries on this Riemannian manifold and $H_p \subset G$ be the isotropy subgroup fixing the point $p$. Then $M = G/H_p$. Consider the involutive automorphism $\sigma(g) = \sigma_p \circ g \circ \sigma_p$ on $G$. Then $K_{\sigma_0} \subset H_p \subset K_\sigma$, where $K_\sigma$ is the set of fixpoints of the $\sigma$ and $K_{\sigma_0}$ is its unit component. Therefore, the $(G, H_p)$ is a Riemannian symmetric pair, proving the statement completely.)

On irreducible affine symmetric Berwald manifolds the unit component, $J_0$, of the isometries is a simple group (cf. Cartan’s theorem), in which the $G_{\sigma_0} \subset J_0$ is a normal subgroup. Thus, $G_{\sigma_0} = J_0$, $H_{\sigma p_0} = H_{p_0}$ hold. Furthermore, there is a natural identification between the holonomy group at a point $p$ and the isotropy group $H_{p_0}$. These arguments and the de Rham decomposition described earlier establish also the Generalized Hanno Theorem completely.

Consider the Riemannian connection $\nabla$ of this Riemannian symmetric manifold. If $\tau_c : T_p(M) \to T_q(M)$ is the parallel transport along a curve $c$ joining $p$ and $q$ then there exist $\varphi \in G_{\sigma_0}$ satisfying $\varphi(p) = q$ such that the induced map $\varphi_* : T_p(M) \to T_q(M)$ is nothing but $\tau$ (cf. Theorem 3.2 of chapter XI in [KN], vol. II). Therefore $\nabla L = 0$ holds for any $G_\sigma$-invariant Finsler metric. This proves that a Cartan-symmetric Finsler metric is always an affine symmetric Berwald metric with an absolute homogeneous Finsler function. Thus also statement (C) is established.

(D) The converse statement, asserting that any affine symmetric Berwald metric having reversible Finsler function is Cartan-symmetric, is also true. In this case the Finsler function is defined by parallel transports of a Finsler function $L_p$, defined for a fixed point $p$ of $M$, to the other points of the manifold. This construction insures, by the theorem quoted from [KN] vol. II, that the maps from $G_{\sigma_0}$ act as isometries on the manifold. In order to establish this property for any Cartan symmetry $\sigma_q$, we have to prove yet that for any two points $p_1$ and $p_2 = \sigma_q(p_1)$ the induced map $\sigma_{q*}$ transports $L_{p_1}$ to $L_{p_2}$.

First note that $G_\sigma = G_{\sigma_0} \cup \sigma_p G_{\sigma_0}$ and $H_p = H_{p_0} \cup \sigma_p H_{p_0}$ hold, which statements follow from $\sigma_p \circ \cdots \circ \sigma_{2k-1} = \sigma_p \circ \sigma_p \circ \cdots \circ \sigma_{2k+1}$ immediately. Furthermore, there exist maps $\alpha_1, \alpha_2 \in G_{\sigma_0}$ such that $\alpha_1(p) = p_1, \alpha_2(p) = p_2$ hold (cf. (C)), therefore, the map $\varphi = \alpha_2^{-1} \circ \sigma_q \circ \alpha_1$ fixes $p$. Moreover, the $\alpha_{i*}$ transports $L_p$ to $L_{p_1}$, thus, $\sigma_{q*}$ transports $L_{p_1}$ to $L_{p_2}$ if and only if $L_p$ is invariant under the action of $\varphi_{ps} : T_p(M) \to T_p(M)$. Reversible Berwald-Finsler functions, $L_p$, are invariant under the action of $\sigma_{ps}$ and $h_{p_0\alpha}$, for all $h_{p_0} \in H_{p_0}$, which statement proves this invariance and the converse
statement (D) completely.

By summing up (A) through (D), we have

**Theorem 5.1** The Cartan-symmetric Finsler metrics on simply connected and complete manifolds are exactly the affine symmetric Berwald metrics having absolute homogeneous (reversible) Finsler functions.

By this theorem, the classification of Berwald metrics gives a complete list also for Cartan-symmetric Finsler metrics.

### 6 Cartan flats on Berwald manifolds

Let $T_c$ be the surface described by the geodesics tangent to the Cartan subspace $c = V_0 \oplus c_i \subset T_p(M)$ at a $p \in M$ of the Berwald manifold $M$. Considering it as an affine submanifold with the induced affine connection $\tilde{\nabla}$, the $T_c$ is a Cartesian product of the totally geodesic flats $M_0 = T_{V_0}$ and $T_{c_i}$. (Note that on an irreducible non-symmetric manifold the surface $T_{c_i}$ is one-dimensional defined by a geodesic.) Therefore, the $T_c$ is a totally geodesic flat surface on which the induced metric is a Minkowski metric.

These flats are called *Cartan flats*. The uniquely determined dimension of the flats is equal to the rank of the Berwald manifold.

Since the parallel transports $\tau : T_p(M) \to T_q(M)$ along the curves joining $p$ and $q$ corresponds Cartan subspaces to each other such that they keep also the Finsler function $L/c$, any two Cartan flats with the induced Minkowski metrics are locally isometric. Thus we have

**Theorem 6.1** Any tangent vector $v$ is contained in at least one of the Cartan flats. Any two Cartan flats of a Berwald manifold are locally isometric. By the above construction technique, the Minkowskian metric on a single Cartan flat uniquely determines the metric on the whole Berwald manifold.

We conclude the paper by a characterization of affine symmetric Berwald manifolds, which reminds the theorem asserting that rank-one symmetric spaces are exactly the 2-point homogeneous Riemannian metrics. (The precise formulation of this latter theorem is as follows: An irreducible Riemannian manifold of $\text{dim} > 1$ is rank-one symmetric if and only if the isometries from the unit component $J_0$ act transitively on the set of pairs $(p, t)$, where $t$ is a geodesic and ($p \in t$) is a point on it.) The corresponding characterization is:
Theorem 6.2 A Berwald metric is affine symmetric if and only if the
isometries $J_0(M)$ act transitively on the set of pairs $(p, T_c)$, where $T_c$ is
a totally geodesic Cartan flat with a distinguished point $p \in T_c$.

The first unified proof, working both in the compact and non-compact
cases, of symmetry of 2-point homogeneous spaces is given in [Sz2]. The
proof is elementary, using only simple topological ideas. The following “point
version” of the theorem is established there: If the isotropy group $J_p$ of
isometries fixing the point $p$ are transitive on the unit sphere in the tangent
space $T_p(M)$, then $(\nabla R)/p = 0$ at $p$.

Theorem 6.2 immediately follows from the Generalized Hano Theorem
and from this theorem. In fact, the transitivity in the theorem implies
that also the rank-one-factors in the deRham decomposition are symmetric
spaces of rank one. The irreducible factors of higher rank are automatically
symmetric spaces.

The above type of characterization of symmetric spaces appeared first
in [HPTT], where the compact symmetric spaces of rank $k$ are identified
with the so called $k$-flat homogeneous manifolds. We should emphasize,
however, that the $k$-flats are introduced in that paper by much weaker as-
sumptions, namely, by the connected closed $k$-dimensional totally geodesic
flat submanifolds of Riemannian manifolds. Note that this concept does
not refer to Cartan subalgebras. The $k$-flat homogeneity is introduced on
those manifolds where any tangent vector $v$ is in at least one $k$-flat $\sum$
and is defined by the property that the isometries are transitive on the set of
pairs $(p, \sum)$, where $p$ is a point on the $k$-flat $\sum$. The proof under such
weak conditions is much more challenging and difficult. In [HPTT] the
characterization is established only on compact manifolds, by using even
the classification of symmetric spaces. Still on compact manifolds but using
weaker assumptions, the above characterization is established also in [EO].
Proofs established both in the compact and non-compact cases are due to
Samiou [Sa].

The concept of rank introduced in this paper has an apparent relation
also to the rank-concept intensely investigated in [BBE, BBS, E, BS]. In
these papers the rank of a geodesic in a Riemannian manifold is defined
by the dimension of the vector space spanned by the parallel Jacobi fields
along the geodesic. The rank of the Riemannian manifold is defined by the
minimum of the ranks of the geodesic. This is even a much more subtle con-
sideration of the rank-concept. There is proved in these papers that any lo-
ally irreducible, compact Riemannian manifold with non-positive sectional
curvature and higher rank is a locally symmetric space.
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