HIGH ORDER ALGORITHMS FOR FOKKER-PLANCK EQUATION WITH CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE

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Abstract

Based on the continuous time random walk, we derive the Fokker-Planck equations with Caputo-Fabrizio fractional derivative, which can effectively model a variety of physical phenomena, especially, the material heterogeneities and structures with different scales. Extending the discretizations for fractional substantial calculus [Chen and Deng, ESAIM: M2AN, 49, (2015), 373–394], we first provide the numerical discretizations of the Caputo-Fabrizio fractional derivative with the global truncation error $O(\tau^\nu)$ ($\nu = 1, 2, 3, 4$). Then we use the derived schemes to solve the Caputo-Fabrizio fractional diffusion equation. By analysing the positive definiteness of the stiffness matrices of the discretized Caputo-Fabrizio operator, the unconditional stability and the convergence with the global truncation error $O(\tau^2 + h^2)$ are theoretically proved and numerical verified.

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1. Introduction

The Caputo-Fabrizio fractional (CF-fractional) operators have been used to model a variety of applied scientific phenomena, such as in physics, control systems, material science, fluid dynamics. The interest for this new operator with a regular kernel was born from the prospect that there is a class of non-local systems, which can effectively model the
material heterogeneities and structures with different scales compared by the fractional models with singular kernel \[8\]. In fact, based on the continuous time random walk (CTRW), the Fokker-Planck equation with CF-fractional operator can be derived. More concretely, let us consider, for instance, the waiting time probability density function (PDF) with exponential Debye pattern \([13, 20]\) and no-time-taking jumps eliminated

\[
\psi(t) = \sigma \left( 1 + \sigma - \delta(t) \right) \exp(-\sigma t), \quad \sigma = \frac{\gamma}{1 - \gamma}, \quad \gamma \in (0, 1),
\]

and together with a Gaussian jump length PDF

\[
\lambda(x) = (4\pi)^{-1/2} \exp(-x^2/4).
\]

Then, the corresponding Laplace and Fourier transforms are of the forms

\[
\psi(s) = 1 - \frac{s}{\gamma + (1 - \gamma)s} \quad \text{and} \quad \lambda(k) = 1 - k^2 + O(k^4).
\]

Plugging \(\psi(s)\) and \(\lambda(k)\) into the following equation, we get the specific expression of \(u(k, s)\) \([20]\)

\[
u(k, s) = \frac{1 - \psi(s)}{s} \cdot \frac{u_0(k)}{1 - \lambda(k)\psi(s)} \sim \frac{u_0(k)}{s + [\gamma + (1 - \gamma)s] k^2},
\]

where \(u_0(k)\) denotes the Fourier transform of the initial condition \(u_0(x)\). Performing inverse Fourier and Laplace transforms to the above equation leads to

\[
0^\gamma_0 D_t u(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + f(x, t), \quad 0 < \gamma < 1 \quad (1.1)
\]

with the initial condition \(u(x, 0) = u_0(x), \quad x \in \Omega = (0, b)\) and the homogeneous Dirichlet boundary conditions. The CF-fractional operator, for \(0 < \gamma < 1\), is defined through the relation \([8]\)

\[
0^\gamma_0 D_t u(t) = \frac{1}{1 - \gamma} \int_0^t u'(s) e^{-\frac{s}{1 - \gamma}(t-s)} ds. \quad (1.2)
\]

**Remark 1.1.** Let us consider the above waiting times \(\psi(t)\) and a heavy tailed jump length PDF \([17]\)

\[
\lambda(x) = \frac{1}{|x|^{1+\alpha}}, \quad 0 < \alpha < 2,
\]

where the Cauchy example corresponds to \(\alpha = 1\). Thus, in Fourier-Laplace space, the PDF \(u(x, t)\) obeys the algebraic relation

\[
u(k, s) \sim \frac{u_0(k)}{s + [\gamma + (1 - \gamma)s] |k|^\alpha}.
\]
the inverse Laplace and Fourier transform of which leads to the space fractional Fokker-Planck equation with the CF-fractional operator

$$C_0^F D_t^\gamma u(x, t) = K_\alpha \frac{\partial^{\alpha} u}{\partial |x|^{\alpha}}(x, t) + f(x, t), \ 0 < \gamma < 1,$$

where the coefficient $K_\alpha$ depends on $\alpha$. In this work, we mainly focus on the model (1.1), since the model (\ast) can be similarly discussed.

The model of (1.1) which can be described the flow of water at different scales in time within a leaky aquifer [6, 14] and the electro-magneto-hydrodynamic flow of the non-Newtonian behavior of biofluids with heat transfer [1], etc.

Numerical method for solving the CF-fractional diffusion equation (1.1) have been proposed by various authors. For example, numerical approximation of the space-time CF-fractional derivative and application to groundwater model have been discussed in [7, 14]. A type of Fokker-Planck equation with CF-fractional derivative is considered in [15]. In [18], the shifted Legendre polynomials operational matrix together with Tau-method has been successfully applied to solve differential equation with CF-fractional operator. The theoretically observed convergence rates for the new discretization of CF-fractional derivative are numerical verified in [2]. Although the stability analysis of the presented schemes have been considered by Von Neumann method [7, 14], it is still a lack of convergence and stability analysis [2, 15, 18] for the model of (1.1). It seems that achieving a second-order accurate scheme for (1.1) is not an easy task. Based on the Lubich’s operator [19] and the discretized fractional substantial calculus [9, 10, 11], we provide the numerical discretizations of the CF-fractional operator with the global truncation error $O(\tau^\nu), \nu = 1, 2, 3, 4$. By analysing the positive definiteness of the stiffness matrices of the discretized CF-fractional derivative, we theoretically prove and numerically verify that the provided finite difference scheme is unconditionally stable and second order convergence in both space and time directions.

The outline of this paper is as follows. In the next section, we provide the numerical schemes of the CF-fractional derivative with the fourth order approximations and derive the finite difference method for the model (1.1). In Section 3, the unconditionally stability and convergence of the presented numerical schemes are rigourously established. To show the effectiveness of the presented methods, the results of numerical experiments are reported in Section 4. Finally, we conclude the paper with some remarks.
2. High order algorithms for Fokker-Planck equation with Caputo-Fabrizio fractional derivative

In this section, we first provide the numerical discretizations of the CF-fractional derivative (1.2) with the fourth order approximations. Then we use the derived schemes to solve the CF-fractional diffusion equation (1.1).

For designing the numerical scheme of (1.1) and the CF-fractional operator of (1.2), we take the mesh points \(x_i = ih, i = 0, 1, \ldots, M\), and \(t_n = n\tau, n = 0, 1, \ldots, N\), where \(h = b/M\), \(\tau = T/N\) are the uniform space stepsize and time steplength, respectively; and denote \(u_i^n\) as the numerical approximation to \(u(x_i, t_n)\).

2.1. Discretized Caputo-Fabrizio fractional derivative

In this subsection, we provide the high order discretization schemes for the CF-fractional derivative. In fact, we can rewrite (1.2) as the following form

\[
\begin{align*}
\mathcal{C}_0^\gamma D_t^2 u(t) &= \frac{1}{1 - \gamma} \int_0^t u'(\eta) e^{-\gamma (t - \eta)} d\eta \\
&= \frac{1}{1 - \gamma} u(t) - \frac{1}{1 - \gamma} e^{-\sigma t} u(0) - \frac{\gamma}{(1 - \gamma)^2} I_\sigma u(t) \\
&= \frac{1}{1 - \gamma} u(t) - \frac{1}{1 - \gamma} e^{-\sigma t} u(0) - \frac{\gamma}{(1 - \gamma)^2} \int_0^t u(t) e^{-\sigma (t - \eta)} d\eta.
\end{align*}
\]  

with \(\sigma = \gamma/(1 - \gamma)\) and \(I_\sigma u(t) = \int_0^t u(\eta) e^{-\sigma (t - \eta)} d\eta\).

Based on the idea of \([9, 10, 11, 19]\), we use Fourier transform methods to derive the \(\nu\)-th order (\(\nu \leq 4\)) approximations of the CF-fractional derivative by the corresponding coefficients of the generating functions

\[
\kappa^{\nu, \sigma}(\zeta) = \left( \sum_{l=1}^{\nu} \frac{1}{l} \left( 1 - e^{-\sigma \tau} \zeta \right)^l \right)^{-1} = \sum_{m=0}^{\infty} q_m^{\nu, \sigma} \zeta^m, \quad \nu = 1, 2, 3, 4,
\]  

where \(\tau\) is the uniform time stepsize and the coefficients \([9, 10, 11]\)

\[
q_m^{\nu, \sigma} = e^{-\sigma \nu \tau} l_m^{\nu}, \quad \nu = 1, 2, 3, 4.
\]
Here
\[ l_m^1 = 1, \]
\[ l_m^2 = \left( \frac{3}{2} \right)^{-1} \sum_{k=0}^{m} 3^{-k} = 1 - \frac{1}{3^{m+1}}, \]
\[ l_m^3 = \frac{6}{11} \frac{1}{\mu_3 - \mu_3} \left( \frac{\mu_3(1 - \mu_3^{k+1})}{1 - \mu_3^{k+1}} - \frac{\mu_3(1 - \mu_3^k)}{1 - \mu_3^k} \right), \]  \quad (2.6)
\[ l_m^4 = \frac{12}{25} \frac{1}{\mu_4 - \mu_4} \left( \frac{\mu_4^{n+1} - 1}{\nu_4 - 1} - \frac{\mu_4^{n+1} - \mu_4^{n+2}}{\nu_4 - \mu_4} \right) - \frac{12}{25} \frac{1}{\mu_4 - \mu_4} \left( \frac{\nu_4^{n+1} - 1}{\nu_4 - 1} - \frac{\mu_4^{n+1} - \mu_4^{n+2}}{\nu_4 - \mu_4} \right), \]
with \( \mu_3 = \frac{4}{7 + \sqrt{39}} \), \( \mu_3 = \frac{7 - \sqrt{39}}{4} \), and
\[ \nu_4 = \frac{3a}{b - \left( \sqrt{Y_1} - \sqrt{Y_2} \right)}; \]
\[ \mu_4 = \frac{3a}{-b + \frac{a}{2} \left( \sqrt{Y_1} - \sqrt{Y_2} \right) + \frac{a\sqrt{3}}{2} \left( \sqrt{Y_1} + \sqrt{Y_2} \right)i}; \]
\[ \mu_4 = \frac{3a}{-b + \frac{a}{2} \left( \sqrt{Y_1} - \sqrt{Y_2} \right) - \frac{a\sqrt{3}}{2} \left( \sqrt{Y_1} + \sqrt{Y_2} \right)i}, \]
where \( a = -\frac{3}{25} \), \( b = \frac{13}{25} \), \( c = -\frac{23}{25} \), \( d = 1 \); \( A = b^2 - 3ac \), \( B = bc - 9ad \), \( C = c^2 - 3bd \); \( \Delta = B^2 - 4AC > 0 \), \( Y_1 = Ab + \frac{3}{2}a(-B - \sqrt{\Delta}) > 0 \), \( Y_2 = Ab + \frac{3}{2}a(-B + \sqrt{\Delta}) < 0 \), and \( i = \sqrt{-1} \).

**Remark 2.1.** From (2.6), the striking feature is that fourth order schemes generated by \( l_m^4 \) can keep the same computational cost with first order discretization schemes.

**Lemma 2.1.** Let \( 0 < \gamma < 1 \) with \( \sigma = \frac{\gamma}{1 - \gamma} \). Let \( u(t) \), \( I_\sigma u(t) \), \( 0^C\!\!F D_\gamma^\sigma u(t) \) and their Fourier transforms belong to \( L_1(\mathbb{R}) \). Then
\[ \mathcal{F}(I_\sigma u(t)) = \frac{1}{\sigma + i\omega} \hat{u}(\omega), \]
and
\[ \mathcal{F}(0^C\!\!F D_\gamma^\sigma u(t)) = \frac{1}{1 - \gamma} \hat{u}(\omega) - \frac{1}{\sigma + i\omega} \frac{1}{1 - \gamma - i\omega} u(0) - \frac{\gamma}{(1 - \gamma)^2} \frac{1}{\sigma + i\omega} \hat{u}(\omega) = \frac{i\omega \hat{u}(\omega) - u(0)}{i\omega + \gamma(1 - i\omega)}. \]
Here $\mathcal{F}$ denotes Fourier transform operator and $\hat{u}(\omega) = \mathcal{F}(u)$, i.e.,

$$\hat{u}(\omega) = \int_{\mathbb{R}} e^{-i\omega t} u(t) dt.$$ 

**Proof.** From (2.3) and (3.1) of [11], the desired results is obtained. □

**Lemma 2.2.** Let $0 < \gamma < 1$ with $\sigma = \frac{1}{1 - \gamma}$. Let $u(t)$, $I_\sigma u(t)$, $C^\gamma_0 D^\gamma_t u(t)$ and their Fourier transforms belong to $L_1(\mathbb{R})$ and

$$A^{1,\gamma} u(t) = \frac{1}{1 - \gamma} u(t) - \frac{1}{1 - \gamma} e^{-\sigma t} u(0) - \frac{\gamma'}{(1 - \gamma)^2} \tau \sum_{m=0}^{\infty} q_m^{1,\sigma} (t - m\tau).$$

Then

$$C^\gamma_0 D^\gamma_t u(t) = A^{1,\gamma} u(t) + O(\tau).$$

**Proof.** Using Fourier transform and (2.4), we obtain

$$\mathcal{F}(A^{1,\gamma} u)(\omega) = \frac{1}{1 - \gamma} \hat{u}(\omega) - \frac{1}{1 - \gamma} \frac{1 - \gamma'}{(1 - \gamma)^2} \tau \sum_{m=0}^{\infty} q_m^{1,\sigma} (e^{i\omega \tau} - 1) \hat{u}(\omega)$$

$$= \frac{1}{1 - \gamma} \hat{u}(\omega) - \frac{1}{1 - \gamma} \frac{1}{1 - \gamma'} \tau \left(1 - e^{-i\omega \tau - \sigma \tau}\right) \hat{u}(\omega)$$

$$= \frac{1}{1 - \gamma} \hat{u}(\omega) - \frac{1}{1 - \gamma} \frac{1}{1 - \gamma'} \frac{1}{(1 - \gamma)^2} \frac{1}{(\sigma + i\omega)} \varphi(\sigma + i\omega) \hat{u}(\omega)$$

with

$$\varphi(z) = \left(1 - e^{-z\tau}\right)^{-1} = 1 + \frac{1}{2} z \tau + \frac{1}{12} z^2 \tau^2 + O(\tau^4).$$

Therefore, from Lemma [2.1], there exists

$$\mathcal{F}(A^{1,\gamma} u)(\omega) = \mathcal{F}(C^\gamma_0 D^\gamma_t u)(\omega) + \hat{\phi}(\omega),$$

where $\hat{\phi}(\omega) = \frac{1}{2} \frac{\gamma}{(1 - \gamma)^2} \hat{u}(\omega) \cdot \tau + O(\tau^2)$. Then

$$|\hat{\phi}(\omega)| \leq c \left(\frac{1}{2} \frac{\gamma}{(1 - \gamma)^2} |\hat{u}(\omega)|\right) \cdot \tau.$$

With the condition $\hat{u}(\omega) \in L_1(\mathbb{R})$, it leads to

$$\left|C^\gamma_0 D^\gamma_t u(t) - A^{1,\gamma} u(t)\right| = |\phi(t)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{\phi}(\omega)| dt = O(\tau).$$

The proof is completed. □
Lemma 2.3. Let $0 < \gamma < 1$ with $\sigma = \frac{1}{1-\gamma}$. Let $u(t)$, $I_\sigma u(t)$, $\Gamma_0^\gamma D_t^\gamma u(t)$, $u'(t)$ and their Fourier transforms belong to $L_1(\mathbb{R})$ and $A_{2,\gamma} u(t)$ belongs to $L_1(\mathbb{R})$ and $A_{2,\gamma} u(t) = 1$.

Then

$$\Gamma_0^\gamma D_t^\gamma u(t) = A_{2,\gamma} u(t) + O(\tau^2).$$

Proof. Using Fourier transform method, we have

$$\hat{F}(A_{2,\gamma} u)(\omega) = \frac{1}{1-\gamma} \hat{u}(\omega) - \frac{1}{1-\gamma} \frac{1}{\sigma + i\omega} u(0)$$

$$- \frac{\gamma}{(1-\gamma)^2} \left[ \left( 1 - e^{-(\sigma + i\omega)\tau} \right) + \frac{1}{2} \left( 1 - e^{-(\sigma + i\omega)\tau} \right)^2 \right]^{-1} \hat{u}(\omega)$$

$$= \frac{\hat{u}(\omega)}{1-\gamma} - \frac{1}{1-\gamma} \frac{u(0)}{\sigma + i\omega} - \frac{\gamma}{(1-\gamma)^2} \frac{1}{\sigma + i\omega} \varphi(\sigma + i\omega) \psi(\sigma + i\omega) \hat{u}(\omega),$$

where the function $\varphi(z)$ is defined by (2.7) and

$$\psi(z) = \left( 1 + \frac{1}{2} \left( 1 - e^{-z\tau} \right) \right)^{-1}$$

$$= 1 - \frac{1}{2} z\tau + \frac{1}{2} z^2 \tau^2 - \frac{1}{4} z^3 \tau^3 + O(\tau^4).$$

According to the above equation and (2.7), there exists

$$\varphi(z) \psi(z) = \left( \frac{1 - e^{-z\tau}}{z\tau} \right)^{-1} \left( 1 + \frac{1}{2} \left( 1 - e^{-z\tau} \right) \right)^{-1}$$

$$= 1 + \frac{1}{3} z^2 \tau^2 - \frac{1}{4} z^3 \tau^3 + O(\tau^4).$$

From Lemma 2.1 and the above equation, we obtain

$$\hat{F}(A_{2,\gamma} u)(\omega) = \hat{F}(\Gamma_0^\gamma D_t^\gamma u)(\omega) + \hat{\varphi}(\omega),$$

where $\hat{\varphi}(\omega) = \frac{1}{3} \frac{\gamma}{(1-\gamma)^2} (\sigma + i\omega) \hat{u}(\omega) \cdot \tau^2 + O(\tau^3)$. Then

$$|\hat{\varphi}(\omega)| \leq c \cdot \left( \left| \frac{\gamma}{(1-\gamma)^2} (\sigma + i\omega) \hat{u}(\omega) \right| \right) \cdot \tau^2.$$ 

With the condition $\hat{F}(u')(\omega) = i\omega \hat{u}(\omega)$ belong to $L_1(\mathbb{R})$, we have

$$|\Gamma_0^\gamma D_t^\gamma u(t) - A_{2,\gamma} u(t)| = |\varphi(t)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{\varphi}(\omega)| d\omega = O(\tau^2).$$

The proof is completed. \qed
By the similar arguments performed in Lemma 2.3, we further have the following results.

**Theorem 2.1.** Let \(0 < \gamma < 1\) with \(\sigma = \frac{\gamma}{1-\gamma}\). Let \(u(t), I_\sigma u(t), D^\gamma_0 u(t), u^{\nu-1}(t)\) \((\nu = 1, 2, 3, 4)\) and their Fourier transforms belong to \(L_1(\mathbb{R})\) and

\[
A^{\nu,\gamma}u(t) = \frac{1}{1-\gamma}u(t) - \frac{1}{1-\gamma}e^{-\sigma t}u(0) - \frac{\gamma}{(1-\gamma)^2\tau}\sum_{m=0}^{\infty} q^\nu_m u(t - m\tau).
\]

Then

\[
D^\gamma_0 u(t) = A^{\nu,\gamma}u(t) + O(\tau^n), \, \nu = 1, 2, 3, 4.
\]

Assume that the well-defined function \(u(t)\) can be zero extended from the bounded domain \((0, T]\) to \((-\infty, T]\), and satisfy the requirements of the above corresponding theorems; and

\[
\tilde{A}^{\nu,\gamma}u(t) = \frac{1}{1-\gamma}u(t) - \frac{1}{1-\gamma}e^{-\sigma t}u(0) - \frac{\gamma}{(1-\gamma)^2\tau}\sum_{m=0}^{\infty} q^\nu_m u(t - m\tau),
\]

where \(q^\nu_m\) is defined by (2.5). Then

\[
D^\gamma_0 u(t) = \tilde{A}^{\nu,\gamma}u(t) + O(\tau^n), \, \nu = 1, 2, 3, 4
\]

(2.8)

with

\[
\tilde{A}^{\nu,\gamma}u(t_n) = \frac{1}{1-\gamma}u(t_n) - \frac{1}{1-\gamma}e^{-\sigma t_n}u(0) - \frac{\gamma}{(1-\gamma)^2\tau}\sum_{m=0}^{n} q^\nu_m u(t - m\tau).
\]

**2.2. Derivation of the numerical schemes**

For the space direction derivative, we use the simple center difference scheme, i.e.,

\[
\frac{\partial^2 u(x)}{\partial x^2} \bigg|_{x=x_i} = -\frac{1}{h^2}\delta^2_x u(x_i) + O(h^2)
\]

(2.9)

with \(\delta^2_x u(x_i) = -u(x_{i-1}) + 2u(x_i) - u(x_{i+1})\).

Taking \(u = [u(x_1), u(x_2), \ldots, u(x_{M-1})]^T\) with \(u(x_0) = u(x_M) = 0\), and using (2.9), there exists

\[
\delta^2_x u = Lu.
\]

(2.10)

Here the matrix \(L = \text{tridiag}(-1, 2, -1)\) is the \((M-1) \times (M-1)\) one dimensional discrete Laplacian.
According to (2.8) and (2.9), we can write (1.1) as
\[
\frac{1}{1 - \gamma} u(x_i, t_n) - \frac{1}{1 - \gamma} e^{-\sigma t_n} u(x_i, 0) - \frac{\gamma}{(1 - \gamma)^2} \sum_{k=0}^{n} q_k^{\nu, \sigma} u(x_i, t_{n-k})
\]
\[
= - \frac{1}{h^2} \delta_x^2 u(x_i, t_n) + f(x_i, t_n) + r_i^n
\]
with the local truncation error
\[
|r_i^n| \leq C_u (\tau^\nu + h^2), \; \nu = 1, 2, 3, 4,
\]
where the positive constant $C_u$ is independent of $\tau$ and $h$.

Then the resulting discretization of (1.1) can be rewritten as
\[
\frac{1}{1 - \gamma} u_i^n - \frac{\gamma}{(1 - \gamma)^2} \sum_{k=0}^{n} q_k^{\nu, \sigma} u_i^{n-k} = - \frac{1}{h^2} \delta_x^2 u_i^n + \frac{1}{1 - \gamma} e^{-\sigma \tau} u_i^0 + f_i^n.
\]

Without loss of generality, supposing (1.1) with zero initial value [12, 16], then (2.13) reduces to the following second-order scheme
\[
\frac{1}{1 - \gamma} u_i^n - \frac{\gamma}{(1 - \gamma)^2} \sum_{k=0}^{n} q_k^{\nu, \sigma} u_i^{n-k} = \frac{1}{h^2} \delta_x^2 u_i^n = f_i^n.
\]

3. Convergence and Stability Analysis

In this section, we theoretically prove that the above designed scheme is unconditionally stable and second order convergent in both space and time directions.

3.1. A few technical lemmas

First, we introduce some relevant notations and properties of discretized inner product. Denote $u^n = \{u_i^n | 0 \leq i \leq M, n \geq 0\}$ and $v^n = \{v_i^n | 0 \leq i \leq M, n \geq 0\}$, which are grid functions. And
\[
(u^n, v^n) = h \sum_{i=1}^{M-1} u_i^n v_i^n, \; ||u^n|| = (u^n, u^n)^{1/2}.
\]

**Lemma 3.1.** [21 p. 27-28] A matrix $A \in \mathbb{R}^{n \times n}$ is positive definite in $\mathbb{R}^n$ if $(Ax, x) > 0, \forall x \in \mathbb{R}^n, x \neq 0$. A real matrix $A$ of order $n$ is positive definite if and only if its symmetric part $H = \frac{A + A^T}{2}$ is positive definite.

**Lemma 3.2.** Let $L$ be given in (2.10). Then
\[
\frac{1}{h^2} (Lu, u) \geq \frac{1}{b^2} ||u||^2 \text{ with } u \in \mathbb{R}^{M-1}, \; \Omega = (0, b).
Proof. Let the vector \( u = (u_1, u_2, \ldots, u_{M-1})^T \) with \( u_0 = u_M = 0 \) be the associated eigenvector with the discrete Laplace matrix \( L \). It is well known that its eigenvalues are given by \( [23, \text{p. 702}] \)

\[
\lambda_s(L) = 4 \sin^2 \left( \frac{s\pi}{2M} \right), \quad s = 1, 2, \ldots, M-1.
\]

Using \( \sin x > \frac{2}{\pi} x \) with \( 0 \leq x \leq \frac{\pi}{2} \), it yields

\[
\lambda_{\min}(L) = 4 \sin^2 \left( \frac{\pi}{2M} \right) = 4 \sin^2 \frac{\pi h}{2b} \geq \frac{h^2}{b^2},
\]

and

\[
\frac{1}{b^2} (Lu, u) \geq \frac{1}{b^2} \frac{h^2}{b^2} (u, u) = \frac{1}{b^2} ||u||^2.
\]

The proof is completed. \( \square \)

**Lemma 3.3** ([24]). Let

\[
A_{N+1} = \begin{bmatrix}
d - |\alpha| & c & 0 & \cdots & 0 & 0 \\
c & d & c & \cdots & 0 & 0 \\
0 & c & d & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & d & c \\
0 & 0 & 0 & \cdots & c & d
\end{bmatrix}_{(N+1) \times (N+1)}
\]

(3.15)

Then the eigenvalues of \( A_{N+1} \) are given by

\[
\lambda_k(A_{N+1}) = d + 2|c| \cos \frac{2k\pi}{2N+3}, \quad k = 1, 2, \ldots, N+1, \quad \text{if } \alpha = c,
\]

and

\[
\lambda_k(A_{N+1}) = d + 2c \cos \frac{k\pi}{N+2}, \quad k = 1, 2, \ldots, N+1, \quad \text{if } \alpha = 0.
\]

**Lemma 3.4.** Let \( 0 < \gamma < 1 \) with \( \sigma = \frac{\gamma}{1-\gamma} \). Let \( d = b + a^2 + a^2 b, \ c = -\frac{a}{2} - ab \) with \( a = e^{-\sigma \tau}, \ b = \frac{1}{\sigma \tau} - 1 \) and \( \tau = T/N, \ N \geq 1, \ t \in (0, T] \). If

\[
\tau \leq \tau_0, \quad \tau_0 = \frac{2\pi^2}{3(\pi^2 + 10T^2\sigma^2)\sigma},
\]

then the matrix \( A_{N+1} \) of (3.15) is positive definite.

Proof. Using the condition \( \tau_0 \), it yields \( \tau \leq \tau_0 < \frac{1}{\sigma}, \ \sigma \tau < 1 \) and \( b > 0 \). From **Lemma 3.3** the smallest eigenvalues of \( A_{N+1} \) is

\[
\lambda_{N+1} = b + a^2 + a^2 b - (a + 2ab) \cos \theta, \quad \theta = \frac{\pi}{2N+3}.
\]
HIGH ORDER ALGORITHMS FOR CAPUTO-FABRIZIO ... 11

Using Taylor series expansion, we have

\[ \lambda_{N+1} \geq I_1 + I_2 + I_3 \]

with \( I_1 = b(1 - a)^2, I_2 = a(a - 1) \) and \( I_3 = a(1 + 2b)\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!}\right) \).

Next we shall prove \( \lambda_{N+1} > 0 \). By Taylor series expansion, there exists

\[ I_1 \geq \left(\frac{1}{\sigma \tau} - 1\right)\left(\sigma \tau - \frac{(\sigma \tau)^2}{2!} + \frac{(\sigma \tau)^3}{3!} - \frac{(\sigma \tau)^4}{4!}\right)^2 \]

\[ \geq \left(\frac{1}{\sigma \tau} - 1\right)\left(\sigma \tau - (\sigma \tau)^3 + \frac{7(\sigma \tau)^4}{12} - \frac{(\sigma \tau)^5}{4}\right) \]

\[ = \sigma \tau - 2(\sigma \tau)^2 + \frac{19(\sigma \tau)^3}{12} - \frac{5(\sigma \tau)^4}{6} + \frac{(\sigma \tau)^5}{4}; \]

and

\[ I_2 \geq -\left(1 - \sigma + \frac{(\sigma \tau)^2}{2!}\right)\left(\sigma \tau - (\sigma \tau)^2 + \frac{(\sigma \tau)^3}{3!}\right) \]

\[ = -\sigma \tau + \frac{3(\sigma \tau)^2}{2} - \frac{7(\sigma \tau)^3}{6} + \frac{5(\sigma \tau)^4}{12} - \frac{(\sigma \tau)^5}{12}; \]

and

\[ I_3 \geq (1 - \sigma \tau)\left(\frac{2}{\sigma \tau} - 1\right)\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!}\right) \geq \left(\frac{2}{\sigma \tau} - 3 + \sigma \tau\right)\frac{5\theta^2}{12} \]

\[ \geq \left(\frac{2}{\sigma \tau} - 3 + \sigma \tau\right)\left(\frac{5\tau}{2t}\right)^2 \frac{5}{12} \geq \frac{\pi^2}{30T^2} \tau - \frac{\pi^2}{20T^2} \tau^2. \]

According to the above inequality, we get

\[ \lambda_{N+1} \geq I_1 + I_2 + I_3 > \frac{\pi^2}{30T^2} \frac{\tau}{\sigma} - \frac{\pi^2}{20T^2} \tau^2 - \frac{(\sigma \tau)^2}{2} \geq 0 \]

with

\[ \tau \leq \tau_0, \quad \tau_0 = \frac{2\pi^2}{3(\pi^2 + 10T^2 \sigma^2)\sigma}. \]

The proof is completed. \( \Box \)

**Lemma 3.5.** Let \( q_k^{1,\sigma} \) be defined by (2.5) and \( \tau \leq \tau_0 \) in (3.16). Then for any positive integer \( N \) and \( V_i = (v_i^0, v_i^1, \ldots, v_i^N)^T \in \mathbb{R}^{N+1} \), it holds that

\[ \sum_{n=0}^{N} \left(\frac{1}{1 - \gamma} v_i^n - \frac{\gamma}{(1 - \gamma)^2} \tau \sum_{k=0}^{n} q_k^{1,\sigma} v_i^{n-k}\right) v_i^n \geq 0, \quad i = 1, 2, \ldots, M - 1. \]
Proof. By the mathematical induction method, we can prove that

\[ \sum_{n=0}^{N} \left( \frac{1}{1-\gamma} v_i^n - \frac{\gamma}{(1-\gamma)^2} \sum_{k=0}^{n} q_k^{1/\gamma} v_i^{n-k} \right) v_i^n = \frac{\gamma^T}{(1-\gamma)^2} (AV_i, V_i), \]

where

\[ A = \begin{bmatrix} b & -a & -a^2 & \cdots & -a^{N-1} & -a \N \theta b & b \N 0 & b & \cdots & -a^{N-3} & -a^{N-2} \N \vdots & \vdots & \ddots & \vdots & \vdots \N 0 & 0 & \cdots & b & -a \N 0 & 0 & \cdots & 0 & b \end{bmatrix}, \]

with \( b = \frac{1-\gamma}{\gamma^T} - 1, a = e^{-\sigma \tau}. \)

To obtain the desired results, we just need to prove that \( H = \frac{A+A^T}{2} \) is a symmetric positive definite matrix by Lemma 3.1 with

\[ H = \begin{bmatrix} b & -\frac{a}{2} & -\frac{a^2}{2} & \cdots & -\frac{a^{N-1}}{2} & -\frac{a}{2} \N -\frac{a}{2} & b & -\frac{a}{2} & \cdots & -\frac{a^{N-2}}{2} & -\frac{a^2}{2} \N -\frac{a^2}{2} & -\frac{a}{2} & b & \cdots & -\frac{a^{N-3}}{2} & -\frac{a^3}{2} \N \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \N -\frac{a^{N-1}}{2} & -\frac{a^{N-2}}{2} & -\frac{a^{N-3}}{2} & \cdots & b & -\frac{a}{2} \N -\frac{a}{2} & -\frac{a^{N-1}}{2} & -\frac{a^{N-2}}{2} & -\frac{a^2}{2} & -\frac{a}{2} & b \end{bmatrix}, \]

(3.17)

Let \( d = b + a^2 + a^2 b, c = -\frac{a}{2} - ab, \) we obtain

\[ \tilde{H} = P^T HP = \begin{bmatrix} b & c & 0 & \cdots & 0 & 0 \N c & d & c & \cdots & 0 & 0 \N 0 & c & d & \cdots & 0 & 0 \N \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \N 0 & 0 & 0 & \cdots & d & c \N 0 & 0 & 0 & \cdots & c & d \end{bmatrix}, \]

(3.18)

with the nonsingular matrix

\[ P = \begin{bmatrix} 1 & -a & 0 & \cdots & 0 & 0 \N 0 & 1 & -a & \cdots & 0 & 0 \N 0 & 0 & 1 & \cdots & 0 & 0 \N \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \N 0 & 0 & 0 & \cdots & 1 & -a \N 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \]

(3.19)
We can rewrite (3.18) as

\[ \tilde{H} = H_1 + H_2 \]  

(3.19)

with

\[
H_1 = \begin{bmatrix}
  d - |c| & c & 0 & \cdots & 0 & 0 \\
  c & d & c & \cdots & 0 & 0 \\
  0 & c & d & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & d & c \\
  0 & 0 & 0 & \cdots & c & d
\end{bmatrix}_{(N+1) \times (N+1)},
\]

and

\[
H_2 = \begin{bmatrix}
  \frac{a}{2} + ab - (a^2 + a^2b) & 0 & 0 & \cdots & 0 & 0 \\
  0 & 0 & 0 & \cdots & 0 & 0 \\
  0 & 0 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & 0 & 0 \\
  0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}_{(N+1) \times (N+1)}.
\]

From Lemma 3.4, it means that \( H_1 \) is a symmetric positive definite matrix.

On the other hand, using the condition \( \tau_0 \), it yields \( \sigma \tau < 1 \) and

\[
\frac{a}{2} + ab - (a^2 + a^2b) = e^{-\sigma \tau} \left( 1 - \frac{e^{-\sigma \tau}}{\sigma \tau} - \frac{1}{2} \right) \geq e^{-\sigma \tau} \left( \frac{\sigma \tau - (-\sigma \tau)^2}{2} - \frac{1}{2} \right) = e^{-\sigma \tau} \left( \frac{1 - \sigma \tau}{2} \right) > 0,
\]

it implies that \( H_2 \) is a symmetric positive semidefinite matrix.

According to (3.17)-(3.19), we have

\[
(H P x, P x) = (\tilde{H} x, x) = (H_1 x, x) + (H_2 x, x) \geq (H_1 x, x) \quad \forall x \in \mathbb{R}^{N+1},
\]

it means that \( H \) is a symmetric positive definite. From Lemma 3.1, which yields \( A \) also a positive definite. The proof is completed. \( \square \)

**Lemma 3.6.** Let \( q_k^{2,\sigma} = e^{-\sigma k \tau} (1 - \frac{1}{3^{k+1}}) \) be defined by (2.5) and \( \tau \leq \tau_0 \) in (3.16). Then for any positive integer \( N \) and \( V_i = (v_i^0, v_i^1, \ldots, v_i^N)^T \in \mathbb{R}^{N+1} \), it holds that

\[
\sum_{n=0}^{N} \left( \frac{1}{1 - \gamma} v_i^n - \frac{\gamma}{(1 - \gamma)^2} \tau \sum_{k=0}^{n} q_k^{2,\sigma} v_i^{n-k} \right) v_i^n \geq 0, \quad i = 1, 2, \ldots, M - 1.
\]
Proof. By the mathematical induction method, we can prove that
\[
\sum_{n=0}^{N} \left( \frac{1}{1 - \gamma} v_i^n - \frac{\gamma}{(1 - \gamma)^2} \sum_{k=0}^{n} q_k^2 v_i^{n-k} \right) v_i^n = \frac{\gamma^\tau}{(1 - \gamma)^2} (QV_i, V_i),
\]
where \( Q \) is a symmetric matrix, i.e.,
\[
Q = \begin{bmatrix}
a_0 & a_1 & a_2 & \cdots & a_{N-1} & a_N \\
a_1 & a_0 & a_1 & \cdots & a_{N-2} & a_{N-1} \\
a_2 & a_1 & a_0 & \cdots & a_{N-3} & a_{N-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{N-1} & a_{N-2} & a_{N-3} & \cdots & a_0 & a_1 \\
a_N & a_{N-1} & a_{N-2} & \cdots & a_1 & a_0
\end{bmatrix}^{(N+1) \times (N+1)}
\]
with \( b = \frac{1 - \gamma}{\gamma} - 1, a = e^{-\sigma \tau} \) and
\[
a_0 = b + \frac{1}{3}, \quad a_i = - \left( 1 - \frac{1}{3^{i+1}} \right) \times \frac{a}{2}, \quad i = 1, 2, \ldots, N.
\]
We can rewrite (3.20) as \( Q = H + Q_1 \), where the symmetric positive definite matrix \( H \) is given by (3.17) of Lemma 3.5, and
\[
Q_1 = \frac{1}{3} \begin{bmatrix}
1 & \frac{\gamma}{2} & \frac{(\gamma^2)}{2} & \cdots & \frac{(\gamma^{N-1})}{2} & \frac{(\gamma^{N})}{2} \\
\frac{\gamma}{2} & 1 & \frac{\gamma}{2} & \cdots & \frac{(\gamma^{N-2})}{2} & \frac{(\gamma^{N-1})}{2} \\
\frac{(\gamma^2)}{2} & \frac{\gamma}{2} & 1 & \cdots & \frac{(\gamma^{N-3})}{2} & \frac{(\gamma^{N-2})}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{(\gamma^{N-1})}{2} & \frac{(\gamma^{N-2})}{2} & \frac{(\gamma^{N-3})}{2} & \cdots & 1 & \frac{\gamma}{2} \\
\frac{(\gamma^{N})}{2} & \frac{(\gamma^{N-1})}{2} & \frac{(\gamma^{N-2})}{2} & \cdots & \frac{\gamma}{2} & 1
\end{bmatrix}^{(N+1) \times (N+1)}
\]
Then we further obtain
\[
Q_2 = P^T Q_1 P = \frac{1}{3} \begin{bmatrix}
1 & -\frac{\gamma}{2} & 0 & \cdots & 0 & 0 \\
\frac{\gamma}{2} & 1 & -\frac{\gamma}{2} & \cdots & 0 & 0 \\
0 & \frac{\gamma}{2} & 1 & \cdots & -\frac{\gamma}{2} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -\frac{\gamma}{2} \\
0 & 0 & 0 & \cdots & -\frac{\gamma}{2} & 1
\end{bmatrix}^{(N+1) \times (N+1)}
\]
with

\[
P = \begin{bmatrix}
1 & -\frac{a}{3} & 0 & \cdots & 0 & 0 \\
0 & 1 & -\frac{a}{3} & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -\frac{a}{3} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}_{(N+1) \times (N+1)}
\]

Form Lemma 3.3, we know that \( Q_2 \) is a symmetric positive definite matrix, since the smallest eigenvalues of \( Q_2 \) is more than zero. Using \((Q_1 Px, Px) = (Q_2 x, x)\), it means that \( Q_1 \) and \( Q = H + Q_1 \) are the symmetric positive definite matrices. The proof is completed. \( \square \)

### 3.2. Convergence and stability analysis for the model (1.1)

For simplifying the proof of the stability and convergence, we first provide a priori estimate and analyze the numerical stability of the scheme with the zero boundary condition \([12, 16]\).

**Lemma 3.7.** Let \( \{v^n_i\} \) be the solution of the scheme (2.14), i.e.,

\[
\frac{1}{1 - \gamma} v_i^n - \frac{\gamma}{(1 - \gamma)^2} \tau \sum_{k=0}^{n} q^2_{\sigma} v_i^{n-k} + \frac{1}{h^2} \delta_x^2 v_i^n = f_i^n.
\]

\( v_i^0 = \phi_i, \ 1 \leq i \leq M - 1, \)

\( v_i^0 = v_M^n = 0, \ 0 \leq n \leq N. \)

Then for any positive integer \( N \) with \( N\tau \leq T \) and \( \tau \leq \tau_0 \) in (3.16), it holds that

\[
\tau \sum_{n=1}^{N} \|v^n\|^2 \leq \frac{2b^2}{1 - \gamma} \cdot \tau \|v^0\|^2 + b^4 \cdot \tau \sum_{n=1}^{N} \|f^n\|^2
\]

with \( (x, t) \in (0, b) \times (0, T] \).

**Proof.** Multiplying (3.22) by \( hv_i^n \) and summing up for \( i \) from 1 to \( M - 1 \), we obtain

\[
\sum_{i=1}^{M-1} h \left( \frac{1}{1 - \gamma} v_i^n - \frac{\gamma}{(1 - \gamma)^2} \tau \sum_{k=0}^{n} q^2_{\sigma} v_i^{n-k} \right) v_i^n + \frac{1}{h^2} (L v^n, v^n) = (f^n, v^n),
\]

where the matrix \( L \) is given in (2.10) and \( q^2_{\sigma} \) is defined by (2.5) with \( q^0_{\sigma} = \frac{2}{3} \). Further multiplying the above equation by \( \tau \) and summing up
for \( n \) from 1 to \( N \) and adding
\[
\tau \left( \frac{1}{1 - \gamma} - \frac{\gamma}{(1 - \gamma)^2} q \right) ||v^0||^2
\]
on both sides of the obtained results, it yields
\[
\tau h \sum_{i=1}^{M-1} \left\{ \sum_{n=0}^{N} \left( \frac{1}{1 - \gamma} v^0_i - \frac{\gamma}{(1 - \gamma)^2} \tau \sum_{k=0}^{n} q^2 \sigma v^n_{i-k} \right) v^n_i \right\} + \frac{\tau}{h^2} \sum_{n=1}^{N} (L v^n, v^n)
\]
\[
= \tau \sum_{n=1}^{N} (f^n, v^n) + \tau \left( \frac{1}{1 - \gamma} - \frac{\gamma}{(1 - \gamma)^2} q \right) ||v^0||^2.
\]
From Lemmas 3.2 and 3.6 and the Schwarz inequality, Young’s inequality, we obtain
\[
\frac{\tau}{b^2} \sum_{n=1}^{N} ||v^n||^2 \leq \tau \sum_{n=1}^{N} ||f^n|| \cdot ||v^n|| + \frac{\tau}{1 - \gamma} ||v^0||^2
\]
\[
\leq \tau \sum_{n=1}^{N} \left( \epsilon ||v^n||^2 + \frac{||f^n||^2}{4 \epsilon} \right) + \frac{\tau}{1 - \gamma} ||v^0||^2,
\]
where we use \((f^n, v^n) \leq ||f^n|| \cdot ||v^n|| \) with \( \epsilon > 0 \). Taking \( \epsilon = \frac{1}{2b^2} > 0 \) and using the above inequality, there exists
\[
\left( \frac{1}{b^2} - \epsilon \right) \tau \sum_{n=1}^{N} ||v^n||^2 \leq \frac{1}{4 \epsilon} \tau \sum_{n=1}^{N} ||f^n||^2 + \frac{\tau}{1 - \gamma} ||v^0||^2,
\]
which leads to
\[
\tau \sum_{n=1}^{N} ||v^n||^2 \leq \frac{2b^2}{1 - \gamma} \cdot \tau ||v^0||^2 + b^4 \cdot \tau \sum_{n=1}^{N} ||f^n||^2.
\]
The proof is completed.

**Theorem 3.1.** Let \( \tau_0 \) be given in (3.16) with \( \tau \leq \tau_0 \). Then the difference scheme (2.14) is unconditionally stable.

**Proof.** From the Lemma 3.7, the desired results is obtained.

**Theorem 3.2.** Let \( u^n_i \) be the approximate solution of \( u(x_i, t_n) \) computed by the difference scheme (2.13). Denote \( \varepsilon^n_i = u(x_i, t_n) - u^n_i \). Then
\[
\tau \sum_{n=1}^{N} ||\varepsilon^n|| \leq C_u b^{5/2} T \cdot (\tau^2 + h^2),
\]
where \( C_u \) is defined by (2.12) and \((x_i, t_n) \in (0, b) \times (0, T) \) with \( N\tau \leq T \).

**Proof.** Let \( u(x_i, t_n) \) be the exact solution of (1.1) at the mesh point \((x_i, t_n)\), and \( \varepsilon^n_i = u(x_i, t_n) - u^n_i \). Subtracting (2.13) from (2.11) with \( \varepsilon^0_i = 0 \), it yields

\[
\frac{1}{1 - \gamma} \varepsilon^n_i \frac{\gamma}{(1 - \gamma)^2} \tau \sum_{k=0}^{n} q_k^2 \sigma \varepsilon^{n-k}_i + \frac{1}{h^2} \delta_x^2 \varepsilon^n_i = r^n_i. \tag{3.23}
\]

Using Lemma 3.7, it holds that \( \tau \sum_{n=1}^{N} \|\varepsilon^n\|^2 \leq b^4 \cdot \tau \sum_{n=1}^{N} \|r^n\|^2 \leq C_u b^5 T \cdot (\tau^2 + h^2)^2 \).

According to the above inequality and Cauchy-Schwarz inequality, we have

\[
\left( \tau \sum_{n=1}^{N} \|\varepsilon^n\| \right)^2 \leq \left( \tau \sum_{n=1}^{N} 1 \right) \left( \tau \sum_{n=1}^{N} \|\varepsilon^n\|^2 \right) \leq C_u^2 b^5 T^2 \cdot (\tau^2 + h^2)^2,
\]

and

\[
\tau \sum_{n=1}^{N} \|\varepsilon^n\| \leq C_u b^{5/2} T \cdot (\tau^2 + h^2). \]

The proof is completed. \( \square \)

### 4. Numerical results

We numerically verify the above theoretical results including convergence orders and numerical stability. And the \( l_\infty \) norm and the discrete \( L^2 \)-norm, respectively, are used to measure the numerical errors.

**Example 4.1.** To numerically verify the truncation error given in (2.8) in a bounded domain, we utilize the approximation with \( \nu = 4 \) to compute \( \partial^\nu_{D^\gamma} u(t) \), where \( u(t) = t^5 e^{-\sigma t} \) with \( t \in [0, 1] \) and \( \sigma = \frac{\gamma}{1-\gamma} \); and by comparing with the analytical solution \( \frac{1}{1-\gamma} t^5 e^{-\sigma t} (1 - \frac{\sigma t}{\nu}) \), we get the numerical errors and convergence orders in Table 1.

Table 1 shows that the scheme (2.8) is fourth order convergent in time direction.
Table 1. The $l_\infty$ norm and the discrete $L^2$-norm, respectively, are used to measure the numerical errors for (2.8) and convergence orders with $\nu = 4$.

| $\tau$   | $\gamma = 0.1$ | Rate | $\gamma = 0.5$ | Rate | $\gamma = 0.9$ | Rate |
|----------|----------------|------|----------------|------|----------------|------|
| 1/100    | 1.1969e-08     |      | 6.0926e-08     |      | 2.2952e-08     |      |
| 1/200    | 7.6102e-10     | 3.9752 | 3.8852e-09     | 3.9710 | 1.5447e-09     | 3.8932 |
| 1/400    | 4.7972e-11     | 3.9877 | 2.4527e-10     | 3.9856 | 1.0021e-10     | 3.9462 |
| 1/800    | 3.0110e-12     | 3.9938 | 1.5406e-11     | 3.9928 | 6.3816e-12     | 3.9730 |

| $\tau$   | $\gamma = 0.1$ | Rate | $\gamma = 0.5$ | Rate | $\gamma = 0.9$ | Rate |
|----------|----------------|------|----------------|------|----------------|------|
| 1/100    | 5.4396e-09     |      | 3.0445e-08     |      | 1.8999e-08     |      |
| 1/200    | 3.4521e-10     | 3.9780 | 1.9414e-09     | 3.9710 | 1.2839e-09     | 3.8874 |
| 1/400    | 2.1740e-11     | 3.9890 | 1.2256e-10     | 3.9855 | 8.3461e-11     | 3.9433 |
| 1/800    | 1.3640e-12     | 3.9945 | 7.6987e-12     | 3.9928 | 5.3203e-12     | 3.9715 |

Example 4.2. Consider (1.1) on a finite domain with $0 < x < 1$, $0 < t \leq 2$, the forcing function

$$f(x,t) = \frac{1}{1-\gamma} x^2(1-x)^2 \left( \frac{3t^2}{\sigma} - \frac{6t}{\sigma^2} + \frac{6}{\sigma^3} (1-e^{-\sigma t}) \right)$$

$$- (t^3 + 1)(12x^2 - 12x + 2), \quad \sigma = \frac{\gamma}{1-\gamma}$$

with the initial condition $u(x,0) = x^2(1-x)^2$ and the homogeneous Dirichlet boundary conditions. The exact solution of the equation is

$$u(x,t) = (t^3 + 1)x^2(1-x)^2.$$

Table 2 shows that the schemes (2.13) with $\nu = 2$ have the global truncation errors $O(\tau^2 + h^2)$.

5. Conclusions

Based on the CTRW, the Fokker-Planck equations with CF-fractional operator are derived. The properties and numerical discretizations of CF-fractional derivative are detailedly analyzed. Moreover, the unconditional stability and the convergence with the global truncation error $O(\tau^2 + h^2)$ are theoretically proved and numerical verified.

For the future, at least two issues remain alive, that is, the fourth order accurate scheme in both space and time directions for the models (1.1) and (*)& and the high-dimensional case.
Table 2. The $l_\infty$ norm and the discrete $L^2$-norm, respectively, are used to measure the numerical errors for (2.13) and convergence orders with $\nu = 2$, $\tau = h$.

| $h$  | $\gamma = 0.1$ | Rate | $\gamma = 0.5$ | Rate | $\gamma = 0.9$ | Rate |
|------|----------------|------|----------------|------|----------------|------|
| 1/100 | 2.007e-04      | 1.9828e-04 | 1.5138e-03      | 1/200 | 4.953e-05 | 2.0188 | 3.8962e-04 | 1.9580 |
| 1/400 | 1.2058e-05     | 2.0385 | 2.1287e-05 | 2.0667 | 9.8931e-05 | 1.9776 |
| 1/800 | 2.851e-06      | 2.0801 | 2.4926e-05 | 1.9888 |

| $h$  | $\gamma = 0.1$ | Rate | $\gamma = 0.5$ | Rate | $\gamma = 0.9$ | Rate |
|------|----------------|------|----------------|------|----------------|------|
| 1/100 | 1.4713e-04     | 1.4536e-04 | 1.0599e-03      | 1/200 | 3.6331e-05 | 2.0178 | 2.7274e-04 | 1.9583 |
| 1/400 | 8.8567e-06     | 2.0364 | 7.9117e-06 | 2.1362 | 6.9244e-05 | 1.9777 |
| 1/800 | 2.1011e-06     | 2.0756 | 1.7445e-05 | 1.9889 |

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