NEW FAMILIES OF IRREDUCIBLE WEIGHT MODULES OVER $\mathfrak{sl}_3$

VYACHESLAV FUTORNY, GENQIANG LIU, RENCAI LU, KAIMING ZHAO

Abstract. Let $n > 1$ be an integer, $\alpha \in \mathbb{C}^n$, $b \in \mathbb{C}$, and $V$ a $\mathfrak{gl}_n$-module. We define a class of weight modules $F^\alpha_b(V)$ over $\mathfrak{sl}_{n+1}$ using the restriction of modules of tensor fields over the Lie algebra of vector fields on $n$-dimensional torus. In this paper we consider the case $n = 2$ and prove the irreducibility of such 5-parameter $\mathfrak{sl}_3$-modules $F^\alpha_b(V)$ generically. All such modules have infinite dimensional weight spaces and lie outside of the category of Gelfand-Tsetlin modules. Hence, this construction yields new families of irreducible $\mathfrak{sl}_3$-modules.

Keywords: Witt algebra, $\mathfrak{gl}_n$, $\mathfrak{sl}_{n+1}$, weight module, irreducible module

2010 Math. Subj. Class.: 17B10, 17B20, 17B65, 17B66, 17B68

1. Introduction

Representation theory of infinite-dimensional Lie algebras has been attracting extensive attentions of many mathematicians and physicists. These Lie algebras include in particular the Witt algebra $\mathcal{W}_n$ which is the derivation algebra of the Laurent polynomial algebra $A_n = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. The algebra $\mathcal{W}_n$ is a natural higher rank generalization of the Virasoro algebra, which has many applications to different branches of mathematics and physics. It can also be defined as the Lie algebra of polynomial vector fields on $n$-dimensional torus. Weight representations of Witt algebras was recently studied by many authors; see [B, E1, E2, BMZ, GLZ, L3, L4, L5, MZ2, Z, BF1, BF2, BF3]. In particular, in [BF3] a classification of weight irreducible $\mathcal{W}_n$-modules with finite weight multiplicities was given.

In 1986, Shen [Sh] defined a class of modules $F^\alpha_b(V)$ over the Witt algebra $\mathcal{W}_n$ for $\alpha \in \mathbb{C}^n$, $b \in \mathbb{C}$, and an irreducible module $V$ over the special linear Lie algebra $\mathfrak{sl}_n$. This construction was also considered by Larsson in [L3]. They are known as modules of tensor fields when $V$ is finite dimensional and they have a geometric origin. These modules play essential part in the classification of weight irreducible $\mathcal{W}_n$-modules with finite weight multiplicities [BF3]. In 1996, Eswara Rao determined the necessary and sufficient conditions for modules of tensor fields to be irreducible [E1] (see also [GZ]). When $V$ is infinite dimensional, the $\mathcal{W}_n$-module $F^\alpha_b(V)$ is always irreducible, see [LZ].
With this paper we begin a systematic study of the functors 
\[ V \rightarrow F_\alpha^a(V), \quad V \rightarrow F_\beta^b(V)|_{\mathfrak{sl}_{n+1}} \]
from the category of weight \( \mathfrak{sl}_n \)-modules to the the category of weight \( \mathcal{W}_n \)-modules and \( \mathfrak{sl}_{n+1} \)-modules respectively. Our goal is to construct irreducible weight modules over \( \mathfrak{sl}_{n+1} \) with infinite dimensional weight spaces.

At the moment classification of all irreducible weight \( \mathfrak{sl}_n \)-modules is only known for \( n = 1 \). The largest subcategory of weight modules with infinite weight multiplicities where classification problem can be handled is the category of Gelfand-Tsetlin modules \([DFO]\). A classification of irreducible modules in this category is complete for \( n = 2 \) \([FGR1]\) and is known up to some finiteness \([Ov]\), \([FO]\) in general. Recently, new families of irreducible weight modules for \( \mathfrak{sl}_{n+1} \) were constructed by Futorny, Grantcharov and Ramirez \([FGR2]\), \([FGR3]\), \([FGR4]\).

It is not easy to construct examples of irreducible weight modules beyond the category of Gelfand-Tsetlin modules. In the present paper we treat the case \( n = 2 \) and construct new families of 5-parameter irreducible modules over the Witt algebra \( \mathcal{W}_2 \) with infinite dimensional weight spaces and over the Lie algebra \( \mathfrak{sl}_3 \). For the latter algebra, generic weight modules lie outside of the category of Gelfand-Tsetlin modules. We conjecture that the same holds for an arbitrary \( n \geq 2 \).

**Acknowledgements.** V.F. is supported in part by the CNPq grant (301320/2013-6) and by the Fapesp grant (2014/09310-5); G.L. is partially supported by NSF of China (Grant 11301143) and the school fund of Henan University (2012YBZR031, yqpy20140044); K.Z. is supported in part by the Fapesp grant (2015/08615-0), by NSF of China (Grant 11271109) and NSERC. He gratefully acknowledges the hospitality and excellent working conditions at the S˜ao Paulo University where part of this work was done.

2. \( \mathfrak{sl}_{n+1} \)-modules from \( \mathfrak{gl}_n \)-modules

We denote by \( \mathbb{Z} \), \( \mathbb{Z}_+ \), \( \mathbb{N} \) and \( \mathbb{C} \) the sets of all integers, nonnegative integers, positive integers and complex numbers, respectively.

For a positive integer \( n > 1 \), \( \mathbb{Z}^n \) denotes the direct sum of \( n \) copies of \( \mathbb{Z} \). For any \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n_+ \) and \( m = (m_1, \ldots, m_n) \in \mathbb{C}^n \), we denote \( ma = m_1 a_1 m_2 a_2 \cdots m_n a_n \). Let \( \mathfrak{gl}_n \) be the Lie algebra of all \( n \times n \) complex matrices, \( \mathfrak{sl}_n \) the subalgebra of \( \mathfrak{gl}_n \) consisting of all traceless matrices. For \( 1 \leq i, j \leq n \) we denote by \( \hat{E}_{ij} \) the matrix units of \( \mathfrak{gl}_n \) and by \( \hat{E}_{\bar{i}, \bar{j}} \) the matrix units of \( \mathfrak{gl}_{n+1} \). For a matrix \( X \) we will denote by \( X' \) its transpose.

2.1. Witt algebras \( \mathcal{W}_n \). We denote by \( \mathcal{W}_n \) the Lie algebra of derivations of the Laurent polynomial algebra \( A_n = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}] \), see
For $i \in \{1, 2, \ldots, n\}$, denote $\partial_i = t_i \frac{\partial}{\partial t_i}$; and for any $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ set $t^a = t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$.

Denote the standard basis by $\{e_1, e_2, \ldots, e_n\}$ of the vector space $\mathbb{C}^n$. For any $u \in \mathbb{C}^n$ we view $u'$ as a column vector. Let $(\cdot | \cdot)$ be the standard symmetric bilinear form such that $(u|v) = uv' \in \mathbb{C}$. For $u \in \mathbb{C}^n$ and $r \in \mathbb{Z}^n$, we set $D(u, r) = t^r \sum_{i=1}^n u_i \partial_i$. Then we have

$$[D(u, r), D(v, s)] = D(w, r + s), \quad u, v \in \mathbb{C}^n, r, s \in \mathbb{Z}^n,$$

where $w = (u|s)v - (v|r)u$. Note that for any $u, v, z, y \in \mathbb{C}^n$, both $u'v$ and $x'y$ are $n \times n$ matrices, and

$$(u'v)(z'y) = (v|z)u'y.$$

A subalgebra $\mathfrak{h} = \text{span}\{\partial_1, \partial_2, \ldots, \partial_n\}$ is the Cartan subalgebra of $\mathcal{W}_n$.

The extended Witt algebra $\tilde{\mathcal{W}}_n$ is the semidirect sum of $\mathcal{W}_n$ and the abelian Lie algebra $A_n$ with intertwining brackets

$$[D(u, r), t^s] = (u|s)t^{r+s}, \quad u \in \mathbb{C}^n, r, s \in \mathbb{Z}^n.$$

For any $\alpha \in \mathbb{C}^n, b \in \mathbb{C}$ and a $\mathfrak{gl}_n$-module $V$ on which the identity matrix acts as the scalar $b$, set $F^a_b(V) = V \otimes A_n$. For simplicity we write $v(n) = v \otimes x^n$ for any $v \in V, n \in \mathbb{Z}^n$. Then $F^a_b(V)$ becomes a $\mathcal{W}_n$-module with respect to the following action

$$D(u, r)v(n) = \left((u | n + \alpha)v + (r' u)v\right)(n + r),$$

where $u \in \mathbb{C}^n, v \in V, n, r \in \mathbb{Z}^n$ see [L1] and [Sh]. It is easy to see that the module $F^a_b(V)$ obtained from any $\mathfrak{gl}_n$-module $V$ is always a weight module over $\mathcal{W}_n$. The following result is well-known [E1] [Ru] [GZ] [Z].

**Theorem 2.1.** Let $\alpha \in \mathbb{C}^n, b \in \mathbb{C}$, and let $V$ be an irreducible finite dimensional module over $\mathfrak{gl}_n$ on which the identity matrix acts as the scalar $b$. Then $F^a_b(V)$ is irreducible $\mathcal{W}_n$-module unless it appears in the de Rham complex of differential forms

$$t^a \Omega^0 \to t^a \Omega^1 \to \cdots \to t^a \Omega^n.$$

The middle terms in this complex are reducible $\mathcal{W}_n$-modules, while the modules $t^a \Omega^0$ and $t^a \Omega^n$ are reducible whenever $\alpha \in \mathbb{Z}^n$. Here $t^a \Omega^i \simeq F^a_b(\Lambda^i(\mathbb{C}^n))$.

**2.2. Defining $\mathfrak{sl}_{n+1}$-modules.** Recall a standard embedding of $\mathfrak{sl}_{n+1}$ into $\mathcal{W}_n$. Set

$$E_{ij} = t_i t_j^{-1} \partial_j, \quad 1 \leq i, j \leq n;$$

$$E_{i,n+1} = -t_i \sum_{j=1}^n \partial_j, \quad E_{n+1,i} = t_i^{-1} \partial_i, \quad 1 \leq i \leq n;$$

$$E_{n+1,n+1} = -\sum_{j=1}^n \partial_j.$$
Then it is well-known that the linear span over $\mathbb{C}$ of the set
\[ \{ E_{ij} : 1 \leq i \neq j \leq n + 1 \} \cup \{ E_{ii} - E_{i+1,i+1} : 1 \leq i \leq n \} \]
is isomorphic to the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$, see for example [M]. So $\mathfrak{sl}_{n+1}(\mathbb{C})$ can be regarded as a subalgebra of $\mathcal{W}_n$ and each $\mathcal{W}_n$-module can be seen as a $\mathfrak{sl}_{n+1}(\mathbb{C})$-module by restriction. We fix the following Cartan subalgebra of $\mathfrak{sl}_{n+1}(\mathbb{C})$:
\[ \mathfrak{h} = \bigoplus_{i=1}^{n-1} \mathbb{C}(\partial_i - \partial_{i+1}) + \mathbb{C}(\sum_{j=1}^{n} \partial_j + \partial_n) = \bigoplus_{i=1}^{n} \mathbb{C} \partial_i. \]

Also, set
\[ n_+ = \bigoplus_{1 \leq i < j \leq n+1} \mathbb{C} E_{ij}, \quad n_- = \bigoplus_{1 \leq j < i \leq n+1} \mathbb{C} E_{ij}. \]

Then $\mathfrak{sl}_{n+1}(\mathbb{C})$ has a triangular decomposition $\mathfrak{sl}_{n+1}(\mathbb{C}) = n_- \oplus \mathfrak{h} \oplus n_+$. Restricting a $\mathcal{W}_n$-module $F_b^n(V)$ onto $\mathfrak{sl}_{n+1}$ we obtain an $\mathfrak{sl}_{n+1}$-module which we call Witt module and denote again as $F_b^n(V)$. It is easy to see that all $F_b^n(V)$ are weight (with respect to $\mathfrak{h}$) modules over $\mathfrak{sl}_{n+1}$.

More precisely, we have
\begin{align*}
E_{ij} \cdot v(r) &= ((r_j + \alpha_j)v + (E_{ij} - E_{jj})v)(r + e_i - e_j); \\
E_{i,n+1} \cdot v(r) &= -(\sum_{j=1}^{n} (\alpha_j + r_j)v + \sum_{j=1}^{n} E_{ij}v)(r + e_i); \\
E_{n+1,i} \cdot v(r) &= ((r_i + \alpha_i)v - E_{ii}v)(r - e_i); \\
E_{n+1,n+1} \cdot v(r) &= -\sum_{j=1}^{n} (\alpha_j + r_j)v(r),
\end{align*}
(2.2)

for all $v \in V$ and $1 \leq i, j \leq n$.

3. From $\mathfrak{gl}_2$-modules to $\mathfrak{sl}_3$-modules

In this section we consider Witt modules over $\mathfrak{sl}_3$ obtained from infinite dimensional weight $\mathfrak{gl}_2$-modules. Let $\alpha_1, \alpha_2, \lambda, b, c \in \mathbb{C}$ with $c \pm \lambda \notin \mathbb{Z}$. Let $V = \text{span}\{v_i | i \in \mathbb{Z}\}$ be an irreducible cuspidal (i.e., with injective action of generators $E_{12}, E_{21}$) $\mathfrak{gl}_2$-module with 1-dimensional weight subspaces and with the following module structure:
\begin{align*}
E_{11}v_i &= (b + i')v_i, \\
E_{22}v_i &= (b - i')v_i, \\
E_{12}v_i &= (c + i'')v_{i+1}, \\
E_{21}v_i &= (c - i'')v_{i-1},
\end{align*}
(3.1)

where $i'' = \lambda + i$. It is well known that these modules exhaust all irreducible cuspidal weight $\mathfrak{gl}_2$-modules. For convenience, we set
\[ r'_i = r_i + \alpha_i, \forall r_i \in \mathbb{Z}, i = 1, 2. \]
We will consider the $\mathfrak{sl}_3$-module

$$F_{2b}^\alpha(V) = \text{span}\{v_i(r_1, r_2) | i, r_1, r_2 \in \mathbb{Z}\}.$$ 

Then we have

\begin{align*}
\bar{E}_{11}v_i(r_1, r_2) &= r'_1v_i(r_1, r_2), \\
\bar{E}_{22}v_i(r_1, r_2) &= r'_2v_i(r_1, r_2), \\
\bar{E}_{33}v_i(r_1, r_2) &= -(r'_1 + r'_2)v_i(r_1, r_2), \\
\bar{E}_{12}v_i(r_1, r_2) &= \left((i'' - b + r'_2)v_i + (c + i'')v_{i+1}\right)(r_1 + 1, r_2 - 1), \\
\bar{E}_{21}v_i(r_1, r_2) &= \left((c - i'')v_{i-1} + (-i'' - b + r'_1)v_i\right)(r_1 - 1, r_2 + 1), \\
\bar{E}_{13}v_i(r_1, r_2) &= \left((r'_1 + r'_2 + b + i'')v_i + (c + i'')v_{i+1}\right)(r_1 + 1, r_2), \\
\bar{E}_{31}v_i(r_1, r_2) &= (r'_1 - b - i'')v_i(r_1 - 1, r_2), \\
\bar{E}_{23}v_i(r_1, r_2) &= \left((c - i'')v_{i-1} + (r'_1 + r'_2 + b - i'')v_i\right)(r_1, r_2 + 1), \\
\bar{E}_{32}v_i(r_1, r_2) &= (r'_2 - b + i'')v_i(r_1, r_2 - 1).
\end{align*}

**Lemma 3.1.** Let $\lambda, b, c, \alpha_1, \alpha_2 \in \mathbb{C}$ such that $c \pm \lambda, \alpha_1 - b - \lambda, \alpha_2 - b + \lambda, \alpha_1 + 2b, \alpha_2 + 2b, \alpha_1 + \alpha_2 + b \pm c \not\in \mathbb{Z}$, and $V$ be the $\mathfrak{gl}_3$-module defined by (3.1). Then the $\mathfrak{sl}_3$-module $F_{2b}^\alpha(V)$ can be generated by any single vector $v_i(r_1, r_2)$ where $i, r_1, r_2 \in \mathbb{Z}$.

**Proof.** Let $W$ be the $\mathfrak{sl}_3$-submodule of $F_{2b}^\alpha(V)$ generated by $v_i(r_1, r_2)$ where $i, r_1, r_2 \in \mathbb{Z}$ are fixed.

**Claim.** We have $v_i(j, r_1 + r_2 - j) \in W$ for all $j \in \mathbb{Z}$.

Indeed, from

\begin{align*}
\bar{E}_{12}v_i(r_1, r_2) &= \left((i'' - b + r'_2)v_i + (c + i'')v_{i+1}\right)(r_1 + 1, r_2 - 1), \\
\bar{E}_{13}\bar{E}_{32}v_i(r_1, r_2) &= (r'_2 - b + i'')\bar{E}_{13}(v_i(r_1, r_2 - 1)), \\
&= -(r'_2 - b + i'')(r'_1 + r'_2 - 1 + b + i'')v_i \\
&+ (c + i'')v_{i+1}(r_1 + 1, r_2 - 1),
\end{align*}

and noting that $\alpha_1 + 2b, \alpha_2 - b + \lambda \not\in \mathbb{Z}$ we see that

$$v_i(r_1 + 1, r_2 - 1), v_{i+1}(r_1 + 1, r_2 - 1) \in W.$$
Now, from
\[ \bar{E}_{21} v_i(r_1, r_2) = \left( (c - i)v_{i-1} + (-i'' - b + r'_1)v_i \right)(r_1 - 1, r_2 + 1), \]
\[ \bar{E}_{23} \bar{E}_{31} v_i(r_1, r_2) = (r'_1 - b - i'') \bar{E}_{23}(v_i(r_1 - 1, r_2)) \]
\[ = - (r'_1 - b - i'') \left( (c - i'')v_{i-1} + (r'_1 + r'_2 + b - i'' - 1)v_i \right)(r_1 - 1, r_2 + 1) \]
and using the fact that \( \alpha_2 + 2b, \alpha_1 - b - \lambda \notin \mathbb{Z} \) we obtain
\[ v_i(r_1 - 1, r_2 + 1), v_{i-1}(r_1 - 1, r_2 + 1) \in W. \]
In this manner we deduce the claim.

For integer \( s, r \in \mathbb{Z} \) set \( V(s, r) = \text{span}\{v(s, r) | v \in V\} \).
Repeatedly using Claim, (3.3) and (3.4), we deduce that \( V(j, r_1 + r_2 - j) \subseteq W, \forall j \in \mathbb{Z} \).
Applying \( \bar{E}_{31} \) we obtain that
\[ V(j_1, j_2) \subseteq W, \forall j_1, j_2 \in \mathbb{Z}, j_1 + j_2 \leq r_1 + r_2. \]
Since
\[ \bar{E}_{13} v_i(j_1, j_2) = - \left( (j'_1 + j'_2 + b + i'')v_i + (c + i'')v_{i+1} \right)(j_1 + 1, j_2), \]
\[ \bar{E}_{23} v_{i+1}(j_1 + 1, j_2 - 1) = - \left( (c - i'' - 1)v_i + (j'_1 + j'_2 + b - i'' - 1)v_{i+1} \right)(j_1 + 1, j_2), \]
and \( \alpha_1 + \alpha_2 + b \pm c \notin \mathbb{Z} \), we see that
\[ v_i(j_1 + 1, j_2), v_{i+1}(j_1 + 1, j_2) \in W. \]
Repeatedly using the claim and the above equation we deduce that \( W = F_{2b}^a(V) \), and hence the lemma follows.

**Theorem 3.2.** Let \( \lambda, b, c, \alpha_1, \alpha_2 \in \mathbb{C} \) such that \( c \pm \lambda, \alpha_1 - b - \lambda, \alpha_2 - b + \lambda, \alpha_1 + 2b, \alpha_2 + 2b, \alpha_1 + \alpha_2 + b \pm c \), and \( c \pm 3b \) are not integers. If \( V \) is a \( \mathfrak{gl}_2 \)-module define by (3.1) then the \( \mathfrak{s}l_3 \)-module \( F_{2b}^a(V) \) is irreducible.

**Proof.** To the contrary, we assume that the \( \mathfrak{s}l_3 \)-module \( F_{2b}^a(V) \) is reducible. Take a nonzero proper submodule \( W \) of \( F_{2b}^a(V) \). For
\[ w_i(r_1, r_2) = \sum_{j=0}^{s} a_{ij}(r_1, r_2)v_{i+j}(r_1, r_2) \in F_{2b}^a(V) \]
where \( a_{ij}(r_1, r_2) \in \mathbb{C} \) with \( a_{i0}(r_1, r_2) = 1, a_{is}(r_1, r_2) \neq 0 \), we say that the length of \( w_i(r_1, r_2) \) is \( s + 1 \). We may assume that \( s + 1 \) is the
minimal length of nonzero homogeneous elements in \( W \), in particular, \( w_i(r_1, r_2) \in W \) for some fixed \( i, r_1, r_2 \in \mathbb{Z} \). We know that \( s \geq 1 \).

We will first prove that all vectors in (3.6) form a basis of \( W \). Then analyze the coefficients \( a_{ij}(r_1, r_2) \) to deduce some contradictions.

**Claim:** The submodule \( W \) is spanned by elements of length \( s + 1 \), i.e., \( W \) is spanned by all \( w_j(s_1, s_2) \) for \( j, s_1, s_2 \in \mathbb{Z} \), which are defined as in (3.6).

To prove this claim, we essentially follow the proof of Lemma 3.1. Since \( \alpha_1 + 2b, \alpha_2 - b + \lambda \notin \mathbb{Z} \), the following two elements in \( W \):

\[
\tilde{E}_{12}w_i(r_1, r_2) = \sum_{j=0}^{s} a_{ij}(r_1, r_2)(t'' + j - b + r'_2)v_{i+j}(r_1 + 1, r_2 - 1)
\]

\[
+ \sum_{j=0}^{s} a_{ij}(r_1, r_2)(c + t'' + j)v_{i+j+1}(r_1 + 1, r_2 - 1),
\]

\[
\tilde{E}_{13}\tilde{E}_{32}w_i(r_1, r_2) = \sum_{j=0}^{s} a_{ij}(r_1, r_2)(r'_2 - b + t'' + j)\tilde{E}_{13}(v_{i+j}(r_1, r_2 - 1))
\]

\[
= -\sum_{j=0}^{s} a_{ij}(r_1, r_2)(r'_2 - b + t'' + j)(r'_1 + r'_2 - 1 + b + t'' + j)v_{i+j}
\]

\[
+ (c + t'' + j)v_{i+j+1}(r_1 + 1, r_2 - 1),
\]

are linearly independent which can be seen by looking at the coefficients of \( v_i(r_1, r_2) \) and \( v_{i+s+1}(r_1, r_2) \). By taking linear combinations of the above two elements we obtain two homogeneous elements of the form (3.6):

\[
(3.7) \quad w_i(r_1 + 1, r_2 - 1), w_{i+1}(r_1 + 1, r_2 - 1) \in W.
\]

Also, since \( \alpha_2 + 2b, \alpha_1 - b - \lambda \notin \mathbb{Z} \), the following two elements in \( W \):

\[
\tilde{E}_{21}w_i(r_1, r_2) = \sum_{j=0}^{s} a_{ij}(r_1, r_2)((c - t' - j)v_{i+j-1} + (-t'' - j - b + r'_1)v_{i+j})(r_1 - 1, r_2 + 1),
\]

\[
\tilde{E}_{23}\tilde{E}_{31}w_i(r_1, r_2) = -\sum_{j=0}^{s} a_{ij}(r_1, r_2)(r'_1 - b - t'' - j)((c - t' - j)v_{i+j-1} + (r'_1 + r'_2 - b - t'' - j - 1)v_{i+j})(r_1 - 1, r_2 + 1)
\]

are linearly independent which can be seen by looking at the coefficients of \( v_{i-1}(r_1, r_2) \) and \( v_{i+s}(r_1, r_2) \). By taking linear combinations of the above two elements we obtain two homogeneous elements of the form
We also see that
\[ w_k(j, r_1 + r_2 - j) \in W, \forall k, j \in \mathbb{Z}. \]

Applying \( E_{31} \), we deduce that
\[ w_k(j_1, j_2) \in W, \forall k, j_1, j_2 \in \mathbb{Z}, j_1 + j_2 \leq r_1 + r_2. \]

From
\[
E_{13}w_k(j_1, j_2) = -\sum_{j=0}^{s} a_{kj}(j_1, j_2) \left( (j_1' + j_2' + b + k'' + j) v_{k+j} \right) + (c + k'') v_{k+j+1} \right) (j_1 + 1, j_2) \in W,
\]
\[
E_{23}w_{k+1}(j_1 + 1, j_2 - 1) = -\sum_{j=0}^{s} a_{kj}(j_1, j_2) \left( (c - k'' - 1) v_{k+j} \right) + (j_1' + j_2' + b - k'' - j - 1) v_{k+j+1} \right) (j_1 + 1, j_2) \in W,
\]
and the fact that \( \alpha_1 + \alpha_2 + b \pm c \not\in \mathbb{Z} \), we obtain
\[
w_k(j_1+1, j_2), w_{k+1}(j_1+1, j_2) \in W, \forall k, j_1, j_2 \in \mathbb{Z}, j_1+j_2 \leq r_1+r_2.
\]
Repeating this process, we deduce the Claim.

Now we have
\[
E_{31}w_i(r_1, r_2) = \sum_{j=0}^{s} (r_1' - b - i'' - j) a_{ij}(r_1, r_2) v_{i+j}(r_1 - 1, r_2).
\]
Hence,
\[
a_{ij}(r_1 - 1, r_2) = \frac{r_1' - b - i'' - j}{r_1' - b - i''} a_{ij}(r_1, r_2),
\]
for \( i, r_1, r_2 \in \mathbb{Z} \) and \( j = 1, 2, \ldots, s. \) Since
\[
E_{32}w_i(r_1, r_2) = \sum_{j=0}^{s} (r_2' - b + i'' + j) a_{ij}(r_1, r_2) v_{i+j}(r_1, r_2 - 1),
\]
we also see that
\[
a_{ij}(r_1, r_2 - 1) = \frac{r_2' - b + i'' + j}{r_2' - b + i''} a_{ij}(r_1, r_2),
\]
for \( i, r_1, r_2 \in \mathbb{Z} \) and \( i, r_1, r_2 \in \mathbb{Z} \) and \( j = 1, 2, \ldots, s. \)

Cancelling the term \( v_{i+s+1}(r_1+1, r_2-1) \) in the formulas of \( E_{12}w_i(r_1, r_2) \) and \( E_{13}E_{32}w_i(r_1, r_2) \), we have
\[
\left( (r_2' - b + i'' + s) \bar{E}_{12} + \bar{E}_{13} \bar{E}_{32} \right) w_i(r_1, r_2) = (r_2' - b + i'') (s + 1 - 2b - r_1') w_i(r_1 + 1, r_2 - 1),
\]
i.e.,

\[
(r'_2 - b + i'' + s) \left( (r'_2 - b + i'' + j) a_{i,j}(r_1, r_2) \\
+ (c + i'' + j - 1) a_{i,j-1}(r_1, r_2) \right) \\
- (r'_2 - b + i'' + j)(r'_1 + r'_2 - 1 + b + i'' + j) a_{i,j}(r_1, r_2) \\
- (r'_2 - b + i'' + j - 1)(c + i'' + j - 1) a_{i,j-1}(r_1, r_2) \\
= (r'_2 - b + i'')(s + 1 - 2b - r'_1) a_{ij}(r_1 + 1, r_2 - 1), \\
= (s + 1 - r'_1 - 2b) \frac{(r'_1 - b - i'' + 1)(r'_2 - b + i'' + j)}{(r'_1 - b - i'' - j + 1)} a_{ij}(r_1, r_2),
\]

where in the last step we have used (3.10) and (3.11). So we have

\[
(r'_2 - b + i'' + j)(s + 1 - r'_1 - 2b - j) a_{i,j}(r_1, r_2) \\
+ (c + i'' + j - 1)(s + 1 - j) a_{i,j-1}(r_1, r_2) \\
= \frac{(r'_2 - b + i'' + j)(s + 1 - r'_1 - 2b)(r'_1 - b - i'' + 1)}{(r'_1 - b - i'' - j + 1)} a_{ij}(r_1, r_2),
\]
i.e.,

\[
(s + 1 - j)(c + i'' + j - 1) a_{i,j-1}(r_1, r_2) \\
= \frac{(r'_2 - b + i'' + j)(s + 2 - 3b - i'' - j)}{(r'_1 - b - i'' - j + 1)} a_{ij}(r_1, r_2).
\]

or

(3.13)

\[
a_{i,j-1}(r_1, r_2) = \frac{(r'_2 - b + i'' + j)(s + 2 - 3b - i'' - j)}{(r'_1 - b - i'' - j + 1)(s + 1 - j)(c + i'' + j - 1)} a_{ij}(r_1, r_2),
\]

for \(i, r_1, r_2 \in \mathbb{Z}\) and \(j = 1, 2, \cdots, s\). From this formula we see that \(\lambda + 3b \notin \mathbb{Z}\).

Cancelling the term \(v_{i-1}(r_1 - 1, r_2 + 1)\) in the formulas of \(\bar{E}_{21} w_i(r_1, r_2)\) and \(\bar{E}_{23} E_{31} w_i(r_1, r_2)\), we see that

(3.14)

\[
\left( (r'_1 - b - i'') E_{21} + \bar{E}_{23} E_{31} \right) w_i(r_1, r_2) \\
= \frac{(r'_1 - b - i'' - s)(s + 1 - 2b - r'_2) a_{is}(r_1, r_2)}{a_{is}(r_1 - 1, r_2 + 1)} w_i(r_1 - 1, r_2 + 1),
\]
i.e.,

\[
(r_1' - b - i'') \left( (-i'' - j - b + r_1') a_{i,j}(r_1, r_2) 
+ (c - i'' - j - 1) a_{i,j+1}(r_1, r_2) \right) 
- (r_1' - b - i'' - j)(r_1' - 1 + r_2' + b - i'' - j) a_{i,j}(r_1, r_2) 
- (r_1' - b - i'' - j - 1)(c - i'' - j - 1) a_{i,j+1}(r_1, r_2) 
= \frac{(r_1' - b - i)(s + 1 - 2b - r_2') a_{i,j}(r_1, r_2)}{a_{i,j}(r_1 - 1, r_2 + 1)}
\]

\[
= \frac{(s + 1 - r_2' - 2b)(r_2' - b + i'' + s + 1)(r_1' - b - i'' - j)}{(r_2' - b + i'' + j + 1)} a_{i,j}(r_1, r_2),
\]

where in the last step we have used (3.10) and (3.11). Hence,

\[
(r_1' - b - i'' - j)(j + 1 - r_2' - 2b) a_{i,j}(r_1, r_2) 
+ (c - i'' - j - 1)(j + 1) a_{i,j+1}(r_1, r_2) 
= \frac{(s + 1 - r_2' - 2b)(r_2' - b + i'' + s + 1)(r_1' - b - i'' - j)}{(r_2' - b + i'' + j + 1)} a_{i,j}(r_1, r_2).
\]

i.e.,

\[
(c - i'' - j - 1)(j + 1) a_{i,j+1}(r_1, r_2) 
= \frac{(r_1' + b - i'' - j)(s - j)(s + 2 - 3b + i'' + j)}{r_2' - b + i'' + j + 1} a_{i,j}(r_1, r_2).
\]

or

\[
(c - i'' - j) j a_{i,j}(r_1, r_2) 
= \frac{(r_1' - b - i'' - j + 1)(s + 1 - j)(s + 1 - 3b + i'' + j)}{r_2' - b + i'' + j} a_{i,j-1}(r_1, r_2),
\]

for \(i, r_1, r_2 \in \mathbb{Z}\) and \(j = 1, 2, \ldots, s\). From this formula we see that \(\lambda - 3b \not\in \mathbb{Z}\). So we have

\[
\quad a_{i,j-1}(r_1, r_2) = \frac{(c - i'' - j) j(r_2' - b + i'' + j)}{(r_1' - b - i'' - j + 1)(s + 1 - j)(s + 1 - 3b + i'' + j)} a_{i,j}(r_1, r_2),
\]

for \(i, r_1, r_2 \in \mathbb{Z}\) and \(j = 1, 2, \ldots, s\). Compare this with (3.13) we see that

\[
\frac{(c - i'' - j) j(r_2' - b + i'' + j)}{(r_1' - b - i'' - j + 1)(s + 1 - j)(s + 1 - 3b + i'' + j)} 
= \frac{(r_2' - b + i'' + j) j(s + 2 - 3b - i'' - j)}{(r_1' - b - i'' - j + 1)(s + 1 - j)(c + i'' + j - 1)},
\]

i.e.,

\[
\frac{c - i'' - j}{s + 1 - 3b + i'' + j} = \frac{s + 2 - 3b - i'' - j}{c + i'' + j - 1}, \forall i \in \mathbb{Z}.
\]
It implies
\[ c + 3b - s - 2 = 0 \text{ or } c - 3b + s + 3 = 0. \]
This is impossible since we have assumed that \( c \pm 3b \notin \mathbb{Z} \). Thus the \( \mathfrak{sl}_3 \)-module \( F^\alpha_{2b}(V) \) is irreducible. \( \square \)

Theorem 3.2 provides a large family of weight modules over \( \mathfrak{sl}_3 \) with infinite dimensional weight spaces with explicit basis and the action of generators of the Lie algebra. We will call a Witt module \( F^\alpha_{2b}(V) \) generic if its parameters satisfy conditions of Theorem 3.2. For example, it is the case if the five parameters \( \lambda, b, c, \alpha_1, \alpha_2 \in \mathbb{C} \) and 1 are linearly independent over \( \mathbb{Q} \).

Remark 3.3. Assume \( \alpha_1 - b - \lambda = k_1 \in \mathbb{Z} \) and \( \alpha_2 - b + \lambda = k_2 \in \mathbb{Z} \). Then \( X = \{ v \in F^\alpha_{2b}(V) \mid E_{31}v = E_{32}v = 0 \} \neq 0 \) and it generates a proper \( \mathfrak{sl}_3 \)-submodule of \( F^\alpha_{2b}(V) \). This submodule is a generalized Verma module over \( \mathfrak{sl}_3 \). This shows that generalized Verma modules can be recovered from Witt modules.

4. Witt modules versus Gelfand-Tsetlin modules

In this section we show that irreducible generic Witt modules are not Gelfand-Tsetlin modules. Let us first recall Gelfand-Tsetlin modules for \( \mathfrak{gl}_n \) which were introduced in [DFO]. Let \( U = U(\mathfrak{gl}_n) \). For each \( m \leq n \) let \( \mathfrak{gl}_m \) be the Lie subalgebra of \( \mathfrak{gl}_n \) spanned by \( \{ E_{ij} \mid i, j = 1, \ldots, m \} \). Then we have the following chain
\[ \mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n, \]
which induces the chain \( U_1 \subset U_2 \subset \ldots \subset U_n \) of the universal enveloping algebras \( U_m = U(\mathfrak{gl}_m), 1 \leq m \leq n \). Let \( Z_m \) be the center of \( U_m \). Then \( Z_m \) is the polynomial algebra in the \( m \) variables \( \{ c_{mk} \mid k = 1, \ldots, m \} \),

(4.1) \[ c_{mk} = \sum_{(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k} E_{i_1i_2}E_{i_2i_3} \ldots E_{i_{k-1}i_k}. \]

The subalgebra \( \Gamma \) of \( U \) generated by \( \{ Z_m \mid m = 1, \ldots, n \} \) is the Gelfand-Tsetlin subalgebra of \( U \) associated with the chain of subalgebras above. It is a polynomial algebra in the \( \frac{n(n+1)}{2} \) variables \( \{ c_{ij} \mid 1 \leq j < i \leq n \} \).

A finitely generated \( U \)-module \( M \) is called a Gelfand-Tsetlin module (with respect to \( \Gamma \)) if as a \( \Gamma \)-module
\[ M = \bigoplus_{m \in \max(\Gamma)} M(m), \]
where
\[ M(m) = \{ v \in M \mid m^kv = 0 \text{ for some } k \geq 0 \}, \]
and \( \max(\Gamma) \) is the set of maximal ideals of \( \Gamma \).
Since the Gelfand-Tsetlin subalgebra contains the Cartan subalgebra \( \mathfrak{h} \) spanned by \( \{ E_{ii} \mid i = 1, \ldots, n \} \) then any irreducible Gelfand-Tsetlin module is a weight module. On the other hand, every weight module with finite dimensional weight subspaces is a Gelfand-Tsetlin module. Theory of Gelfand-Tsetlin modules was developed in [Ov], [FO]. In fact, there are several maximal commutative subalgebras of Gelfand-Tsetlin type for which we can define a corresponding category of Gelfand-Tsetlin modules. These subcategories correspond to different (finitely many) chains of different embedding as above. In the case of \( \mathfrak{sl}_3 \) we will have 3 possible Gelfand-Tsetlin subalgebras. Since different chains are conjugated by the Weyl group, respective categories of Gelfand-Tsetlin modules are equivalent.

Denote by \( \mathcal{GT} \) the category of modules which are Gelfand-Tsetlin modules with respect to some Gelfand-Tsetlin subalgebra.

**Proposition 4.1.** If \( F_{26}^a(V) \) is a generic Witt module, then the \( \mathfrak{sl}_3 \)-module \( F_{26}^a(V) \) is not a Gelfand-Tsetlin module.

*Proof.* Suppose \( F_{26}^a(V) \) is irreducible and \( F_{26}^a(V) \in \mathcal{GT} \). Then \( F_{26}^a(V) \) contains a nonzero element \( v \) which is a common eigenvector of one of three Gelfand-Tsetlin subalgebras. This is possible if and only if \( v \) is an eigenvector for one of the following operators: \( \bar{E}_{12} \bar{E}_{21}, \bar{E}_{23} \bar{E}_{32}, \bar{E}_{13} \bar{E}_{31} \). From (3.2) we have

\[
\bar{E}_{12} \bar{E}_{21} v_i(r_1, r_2) = (c - i'') \bar{E}_{12} v_{i-1}(r_1 - 1, r_2 + 1) + (-i'' - b + r'_1) \bar{E}_{12} v_i(r_1 - 1, r_2 + 1) \\
= \ldots + (-i'' - b + r'_1)(c + i'') v_{i+1}(r_1, r_2),
\]

where \( \ldots \) corresponds to the terms with smaller indices. Since \( (-i'' - b + r'_1)(c + i'') \neq 0 \) for all integer \( i \), then \( v_i(r_1, r_2) \) is not an eigenvector of \( \bar{E}_{12} \bar{E}_{21} \). We immediately conclude that \( \bar{E}_{12} \bar{E}_{21} \) has no eigenvectors in \( F_{26}^a(V) \). Similarly,

\[
\bar{E}_{23} \bar{E}_{32} v_i(r_1, r_2) = -(r'_2 - b + i'')(c - i'') v_{i-1}(r_1, r_2) + \ldots,
\]

where \( \ldots \) corresponds to the terms with larger indices. Then \( \bar{E}_{23} \bar{E}_{32} \) has no eigenvectors in \( F_{26}^a(V) \) since \( (r'_2 - b + i'')(c - i'') \neq 0 \) for all integer \( i \).

Consider now

\[
\bar{E}_{13} \bar{E}_{31} v_i(r_1, r_2) = -(r'_1 - b - i'')(c + i'') v_{i+1}(r_1, r_2) + \ldots,
\]

where \( \ldots \) corresponds to the terms with smaller indices. Hence, \( \bar{E}_{23} \bar{E}_{32} \) has no eigenvectors in \( F_{26}^a(V) \) since \( (r'_1 - b - i'')(c + \lambda + i) \neq 0 \) for all integer \( i \). Thus any generic Witt module \( F_{26}^a(V) \) is not a Gelfand-Tsetlin module. \(\square\)

In [FOS] torsion theories for \( \mathfrak{gl}_n \) were studied and a stratification of the category of all weight modules by the heights of prime ideals was obtained. In this stratification the category of Gelfand-Tsetlin modules
is a starting point (when prime ideals are maximal). We believe that Witt modules will provide examples of irreducible modules in other strata as in the case of $\mathfrak{sl}_3$.

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V.F.: Institute of Mathematics and Statistics, University of Sao Paulo, Brazil, 05315-970. Email: futorny@ime.usp.br

G.L.: Department of Mathematics, Henan University, Kaifeng 475004, China. Email: liugenqiang@amss.ac.cn

R.L.: Department of Mathematics, Soochow University, Suzhou, P. R. China, Email: rencail@amss.ac.cn

K.Z.: Department of Mathematics, Wilfrid Laurier University, Waterloo, ON, Canada N2L 3C5, and College of Mathematics and Information Science, Hebei Normal (Teachers) University, Shijiazhuang, Hebei, 050016 P. R. China. Email: kzhao@wlu.ca