Anomaly cancellation in M-theory

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Abstract
We show the complete cancellation of gauge and gravitational anomalies in the M-theory of Horava and Witten using their boundary contribution, and a term coming from the existence of two and five-branes. A factor of three discrepancy noted in an earlier work is resolved. We end with a comment on flux quantization.

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In the M-theory of Horava and Witten [1] several consistency checks associated with anomaly cancellation were verified. However one test which involved a numerical coefficient of a purely gravitational anomaly was not carried out. In an earlier paper by the author [2] the coefficient of a certain M-theory Green-Schwarz term [3] was determined, but there appeared to be a factor of three discrepancy with the expression (3.12) of [1]. In this short note we review the anomaly cancellation argument and find that the M-theory topological terms do indeed cancel both gauge and gravitational anomalies. Finally we comment on flux quantization in M-theory in the light of a recent paper by E. Witten[4].

We work with the “downstairs” version of the theory, i.e. on an 11-D manifold \( M = M_{10} \times S^1 / \mathbb{Z}_2 \). The topological term in the low energy effective action of M-theory is

\[
-\frac{1}{\kappa^2} \frac{1}{6} \int_M C \wedge K \wedge K. \tag{0.1}
\]

In the above C is the three form gauge field of 11D supergravity and \( K = dC \).

In the Horava-Witten theory the manifold \( M \) has a boundary which consists of two disconnected components on each of which \( E_8 \) gauge fields live, so that on dimensional reduction to ten dimensions one gets the low energy effective action of the Heterotic \( E_8 \times E_8 \) theory. On checking the supersymmetry of the resulting theory it was found by

\[1\] In this paper it was also found that the dimensionless ratio of gauge and gravitational couplings determined by anomaly cancellation in [1] (see below) checks with a result obtained from string duality and D-brane methods, thus giving an additional test of M-theory.

\[2\] It should be noted that (0.1) has a factor two compared to the usual term since we are working in the “downstairs” version of the theory with \( M = M_{10} \times S^1 / \mathbb{Z}_2 \) where the integral goes over half the volume of the “upstairs” version where \( M = M_{10} \times S^1 \) and the fields are \( \mathbb{Z}_2 \) symmetric.

\[3\] Our 11 D supergravity conventions and definitions are the same as in [3]. In particular we define a p-form gauge field as \( A = \frac{1}{p!} A_{I_1...I_p} dx^{I_1} ... dx^{I_p} \). The field strength is then \( F = dA \) or in components \( F_{I_1...I_{p+1}} = (p + 1) \partial_{I_1} A_{I_2...I_{p+1}} \) with unit strength anti-symmetrization. The comparison with the notation of Horava and Witten is as follows. \( K = \sqrt{2} G, \ C = \sqrt{2} C^{HW} \) where \( C^{HW} = C^{HW}_{IJK} dx^I dx^J dx^K, \ G^{HW} = dC^{HW} = \frac{1}{4!} G_{IJKL} dx^I dx^J dx^K dx^L \) are the three form gauge field and field strength (called C and G in [3]) as defined by Horava and Witten. Our indices I,J,K,L, run from 1 to 11 whilst indices A,B,C,D run from 1 to 10.
Horava and Witten\textsuperscript{4} that one needed to have, on each component of the boundary,

\[ K|_{\partial M} = \frac{\kappa^2}{2\lambda^2} \hat{I}_4. \tag{0.2} \]

In the above \( \lambda \) is the gauge field coupling,

\[ \hat{I}_4 = \frac{1}{2} \text{tr} R^2 - \text{tr} F^2, \tag{0.3} \]

\( F \) is an \( E_8 \) gauge field strength, and \( R \) is the curvature two form. Now defining \( Q_3 = \frac{1}{2} \omega_3 - \omega_3 \), where the two omegas correspond to the Lorentz and gauge Chern-Simons forms, we have the standard descent equations,\textsuperscript{5}

\[ \hat{I}_4 = dQ_3, \quad \delta Q_3 = dQ_2^1. \tag{0.4} \]

where \( \delta \) is a gauge and local Lorentz variation and \( Q_2^1 \) is a two-form that is linear in the gauge parameter.

From (0.2), the relation \( K = dC \), and the first equation in (0.4), we have (up to an irrelevant exact form)

\[ C|_{\partial M} = \frac{\kappa^2}{2\lambda^2} Q_3. \tag{0.5} \]

Hence from the second equation in (0.4)

\[ \delta C|_{\partial M} = \frac{\kappa^2}{2\lambda^2} dQ_2^1. \tag{0.6} \]

Now clearly we may extend this variation to the bulk by writing

\[ \delta C = d\Lambda, \quad \Lambda|_{\partial M} = \frac{\kappa^2}{2\lambda^2} Q_2^1. \tag{0.7} \]

Hence we have from (0.1), and (0.7)

\[ \delta W = -\frac{1}{\kappa^2} \int_M d\Lambda \wedge K \wedge K \]

\[ = -\frac{1}{\kappa^2} \left( \frac{\kappa^2}{\lambda^2} \right)^3 \frac{1}{2} \int_{\partial M} Q_2^1 \wedge \hat{I}_4^2, \tag{0.8} \]

\textsuperscript{4}See equation (2.20) of the second paper of \cite{1}.

\textsuperscript{5}See for example reference \cite{3}, chapter 13.
where to get the second equality we have used Stokes’ theorem, \(dK = 0\), and (0.2).

Now the boundary theory is anomalous and the variation of the quantum effective action \(\Gamma\) is given by\(\delta\)\(\Gamma = -\frac{1}{48(2\pi)^5} \int_{\partial M} Q_2^1 \wedge \left(-\frac{\hat{I}_4^2}{4} + X_8\right),\) (0.9)
where \(X_8 = -\frac{1}{8}\text{tr}\mathcal{R}^4 + \frac{1}{32}(\text{tr}\mathcal{R}^2)^2\). Cancellation of the \(\hat{I}_4^2\) part of the anomaly then determines

\[\eta^{-1} \equiv \frac{\kappa^4}{\lambda^6} = \frac{1}{4(2\pi)^5}\]

as in [1].

Now as shown in [3] the existence of two and five branes in the theory implies that there is an additional topological (Green-Schwarz) term in M-theory.\(\)\(7\) This is given by,

\[W_5 = \left(\frac{(2\pi)^2}{2\kappa^2}\right)^{1/3} \frac{1}{24(2\pi)^4} \int_M C \wedge X_8.\]

(0.11)
The first factor in the equation above was obtained from the relation \(T_2 = \left[\frac{(2\pi)^2}{2\kappa^2}\right]^{1/3}\), which was originally determined using D-brane methods\(\)\(8\), but after correcting a factor of two in the quantization formula of [3] as discussed in the appendix to [2], it can also be fixed purely within M-theory.\(\)

Using equations (0.7), Stokes’ theorem and \(dX_8 = 0\) we have

\[\delta W_5 = \left(\frac{(2\pi)^2}{2\kappa^2}\right)^{1/3} \frac{1}{24(2\pi)^4} 2 \int_M d\Lambda \wedge X_8\]

\[= \frac{1}{48(2\pi)^5} \int_{\partial M} Q_2^1 \wedge X_8.\]

(0.12)

\(\)\(6\)The numerical coefficient in (0.9) is fixed by standard methods. See for example [6] equation (13.3.41), (13.4.5) the line before (13.5.6) and equations (13.5.5) and (13.5.8). The form of the anomaly is given in [1].

\(7\)The existence of this term may also be inferred from an earlier string theory calculation [5].

\(8\)There is an extra factor of 2 compared to equation (0.20) of [2] because we are in the “downstairs” theory - see footnote 2.
In the last equation we have used the value of $\eta$ given in (0.10). Thus we have the complete cancellation of the anomalies in the Horava-Witten M-theory,

$$\delta W + \delta W_5 + \delta \Gamma = 0.$$  \hspace{1cm} (0.13)

While the above was being written up, a paper by E. Witten appeared \cite{4} in which, \textit{inter alia}, some issues of normalization in M-theory were discussed using index theory.

To conclude this note we would like to make some related comments. Equation (0.2) may be rewritten (after using 0.10) as

$$G \pi \big|_{\partial M} \equiv \left[ \frac{(2\pi)^2}{2\kappa^2} \right]^{1/3} \frac{K}{2\pi} \big|_{\partial M} = w(V) - \frac{\lambda}{2},$$  \hspace{1cm} (0.14)

where $w = F \wedge F/(16\pi^2)$ has integer valued periods, being the second Chern class of the $E_8$ bundle, and $\lambda \equiv p_1/2 = \mathcal{R} \wedge \mathcal{R}/(16\pi^2)$ which is half the Pontryagin class $p_1$ of the tangent bundle, also has integer-valued periods for a spin manifold so that in general as pointed out in \cite{4}, $G/(2\pi)$ has half integer periods. By considering a 4-cycle ($C$) in the bulk that is homologous to one in the boundary ($C'$) this result was extended in \cite{4} to the statement

$$\int_C \left( \frac{G}{2\pi} - \frac{\lambda}{2} \right) \in \mathbb{Z},$$  \hspace{1cm} (0.15)

There is an alternate way to get the normalization of $G/(2\pi)$. This follows from the fact that $K = dC$ and $C$ is the three form field coupling to the 2-brane of M-theory. In earlier work this was given as $T_2 \int_C K/2\pi \in \mathbb{Z}$\footnote{see for example \cite{3} and the appendix of \cite{2}} but due to anomalies of fermionic determinants in odd dimensions, it was pointed out in \cite{4} that this gets modified to

$$\int_C \left( \frac{T_2 K}{2\pi} - \frac{\lambda}{2} \right) \in \mathbb{Z},$$  \hspace{1cm} (0.16)

The consistency of the two normalizations is then a consequence of the relation $T_2 = \left[ \frac{(2\pi)^2}{2\kappa^2} \right]^{1/3}$ derived in \cite{2}. Note that this is a check on the consistency of M-theory that is \footnote{\hspace{1cm}}
independent of the check coming from the pure gravity anomaly cancellation.\footnote{As noted in \cite{2} (equation (8), (11) and the discussion after the latter equation), the calculation of $T_2$ can be done in a fashion that is completely independent of M-theory quantization conditions.}

Acknowledgments

I wish to thank E. Witten for the suggestion to check M-theory consistency conditions, and S. Chaudhuri and P. Horava for discussions. This work is partially supported by the Department of Energy contract No. DE-FG02-91-ER-40672.

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