Intrabeam scattering growth rates for a bi-gaussian distribution.

George Parzen

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Abstract

This note finds results for the intrabeam scattering growth rates for a bi-gaussian distribution. The bi-gaussian distribution is interesting for studying the possibility of using electron cooling in RHIC. Experiments and computer studies indicate that in the presence of electron cooling, the beam distribution changes so that it develops a strong core and a long tail which is not described well by a gaussian, but may be better described by a bi-gaussian. Being able to compute the effects of intrabeam scattering for a bi-gaussian distribution would be useful in computing the effects of electron cooling, which depend critically on the details of the intrabeam scattering. The calculation is done using the reformulation of intrabeam scattering theory given in [1] based on the treatments given by A. Piwinski [2] and J. Bjorken and S.K. Mtingwa [3]. The bi-gaussian distribution is defined below as the sum of two gaussians in the particle coordinates $x, y, s, p_x, p_y, p_s$. The gaussian with the smaller dimensions produces most of the core of the beam, and the gaussian with the larger dimensions largely produces the long tail of the beam. The final result for the growth rates are expressed as the sum of three terms which can be interpreted respectively as the contribution to the growth rates due to the scattering of the particles in the first gaussian from themselves, the scattering of the particles in the second gaussian from themselves, and the scattering of the particles in the first gaussian from the particles in the second gaussian.
1 Introduction

This note finds results for the intrabeam scattering growth rates for a bi-gaussian distribution.

The bi-gaussian distribution is interesting for studying the possibility of using electron cooling in RHIC. Experiments and computer studies indicate that in the presence of electron cooling, the beam distribution changes so that it develops a strong core and a long tail which is not described well by a gaussian, but may be better described by a bi-gaussian. Being able to compute the effects of intrabeam scattering for a bi-gaussian distribution would be useful in computing the effects of electron cooling, which depend critically on the details of the intrabeam scattering. The calculation is done using the reformulation of intrabeam scattering theory given in [1] based on the treatments given by A. Piwinski [2] and by J. Bjorken and S. Mtingwa [3]. The bi-gaussian distribution is defined below as the sum of two gaussians in the particle coordinates \( x, y, s, p_x, p_y, p_z \). The gaussian with the smaller dimensions produces most of the core of the beam, and the gaussian with the larger dimensions largely produces the long tail of the beam. The final result for the growth rates are expressed as the sum of three terms which can be interpreted respectively as the contribution to the growth rates due to the scattering of the particles in the first gaussian from themselves, the scattering of the particles in the second gaussian from themselves, and the scattering of the particles in the first gaussian from the particles in the second gaussian.

2 Basic results for intrabeam scattering

This section lists some general results which can be used to find growth rates for a beam with any particle distribution \( f(x,p) \). Following [3], growth rates will be computed for \( <p_ip_j> \), where the \( <> \) indicates an average over all the particles in the bunch. From these one can compute the growth rates for the emittances, \( <\epsilon_i> \). A result that holds in any coordinate system and for any particle distribution \( f(x,p) \) is given in [1] as

\[
\delta < (p_ip_j) > = N \int d^3x \frac{d^3p_1}{\gamma_1} \frac{d^3p_2}{\gamma_2} f(x,p_1)f(x,p_2)F(p_1,p_2)C_{ij} dt
\]

\[
C_{ij} = \pi \int_0^\pi d\theta \sigma(\theta) \sin^3 \theta \Delta^2 [\delta_{ij} - 3 \frac{\Delta_i \Delta_j}{\Delta^2} + \frac{W_i W_j}{W^2}] \quad i,j = 1,3
\]
\[ \Delta_i = \frac{1}{2}(p_{1i} - p_{2i}) \]
\[ W_i = p_{1i} + p_{2i} \]  \hspace{1cm} (1)

\[ Nf(x,p) \] gives the number of particles in \( d^3xd^3p \), where \( N \) is the number of particles in a bunch. \( \delta < (p_ip_j) > \) is the change in \( < (p_ip_j) > \) due to all particle collisions in the time interval \( dt \). The invariants \( F(p_1, p_2), \Delta^2, W^2 \) are given by

\[ F(p_1, p_2) = c\frac{[(p_1p_2)^2 - m_1^2m_2^2c^4]^{1/2}}{m_1m_2c^2} \]
\[ F(p_1, p_2) = \gamma_1\gamma_2c[(\vec{\beta}_1 - \vec{\beta}_2)^2 - (\vec{\beta}_1 \times \vec{\beta}_2)^2]^{1/2} \]
\[ \Delta^2 = \vec{\Delta}^2 - \Delta_0^2, \quad \Delta_0 = (E_1 - E_2)/(2c) \]
\[ W^2 = \vec{W}^2 - W_0^2, \quad W_0 = (E_1 + E_2)/c \]

Eq.(1) is considerably simplified by going to the rest CS, which is the CS moving along with the bunch and the particle motion is non-relativistic, and putting \( \sigma \) equal to the Coulomb cross section. One gets

\[ \frac{1}{p_0^2} < \delta(p_{1i}p_{1j}) > = N \int d^3xd^3p_1d^3p_2 f(x,p_1)f(x,p_2)[\vec{\beta}c C_{ij}] dt \]
\[ \Delta_i = \frac{1}{2}(p_{1i} - p_{2i}) \]
\[ \vec{\beta}c = |\vec{\Delta}|/m \]
\[ C_{ij} = \frac{2\pi}{p_0^2}(r_0/2\vec{\beta}^2)^2 \ln(1 + (2\vec{\beta}^2b_{max}/r_0)^2) \]
\[ |\vec{\Delta}|^2\delta_{ij} - 3\Delta_i\Delta_j \] \( i,j = 1,3 \)
\[ r_0 = Z^2e^2/mc^2 \]
\[ \sigma(\theta) = \left[ \frac{r_0}{2\vec{\beta}^2} \right]^2 \frac{1}{(1 - \cos \theta)^2} \]
\[ \cot(\theta_{min}/2) = 2\vec{\beta}^2b_{max}/r_0 \]  \hspace{1cm} (2)

\( b_{max} \) is the largest allowed impact parameter in the center of mass CS. It has been assumed that one can replace \( \ln(1 + (2\vec{\beta}^2b_{max}/r_0)) - 1 \) by \( \ln(1 + (2\vec{\beta}^2b_{max}/r_0)) \).

In Eq.(1), the original 11-dimensional integral which arises from intra-beam scattering theory has been reduced in [1] to a 9-dimensional integral.
by integrating over all possible scattering angles. In [1] this reduction was
done for any particle distribution, \( f(x, p) \). In [3], Bjorken and Mtingwa first
do the integration over \( x, p_1, p_2 \) using a simple gaussian distribution before
doing the integration over the scattering angles and no general result for
doing this reduction for any \( f(x, p) \) is given. In [2] Piwinski computes the
growth rates for the emittances \(< \epsilon_i >\) instead of for \(< p_ip_j >\). A general
result for reducing the integral by integrating over all possible scattering
angles, for any \( f(x, p) \), for the growth rates of \(< \epsilon_i >\) is given. However,
using this result for a complicated distribution like the bi-gaussian would be
difficult.

3 Gaussian distribution

We will first consider the case of a gaussian particle distribution. This will
provide a more simple example of using the results in the reformulation given
in [1] and of the methods used to evaluate the integrals. Afterwards, the same
procedures will be applied to the case of the bi-gaussian distribution.

Let \( Nf(x, p) \) gives the number of particles in \( d^3xd^3p \), where \( N \) is the
number of particles in a bunch. For a gaussian distribution, \( f(x, p) \) is given by

\[
f(x, p) = \frac{1}{\Gamma} \exp[-S(x, p)]
\]

\[
\Gamma = \int d^3xd^3p \exp[-S(x, p)]
\]

\[
S = S_x + S_y + S_s
\]

\[
S_x = \frac{1}{\epsilon_x} \epsilon_x(x_\beta, x'_\beta)
\]

\[
x_\beta = x - D(p - p_0)/p_0
\]

\[
x'_\beta = x' - D'(p - p_0)/p_0 \quad x' = p_x/p_0
\]

\[
\epsilon_x(x, x') = \frac{[x^2 + (\beta_x x' + \alpha_x x)^2]/\beta_x}{2}
\]

\[
S_y = \frac{1}{\epsilon_y} \epsilon_y(y, y') \quad y' = p_y/p_0
\]

\[
\epsilon_y(y, y') = \frac{[y^2 + (\beta_y y' + \alpha_y y)^2]/\beta_y}{2}
\]
\[ S_s = \frac{1}{\bar{\epsilon}_s} \epsilon_s(s - s_c, (p - p_0)/p_0) \]

\[ \epsilon_s(s - s_c, (p - p_0)/p_0) = \frac{(s - s_c)^2 + ((p - p_0)/p_0)^2}{2\sigma_s^2} \]

\[ \epsilon_s(s - s_c, (p - p_0)/p_0) = \frac{1}{\beta_s} (s - s_c)^2 + \beta_s ((p - p_0)/p_0)^2 \]

\[ \epsilon_s(s - s_c, (p - p_0)/p_0) = \left[ (s - s_c)^2 + (\beta_s ((p - p_0)/p_0))^2 \right] / \beta_s \]

\[ \beta_s = \frac{\sigma_s}{\sigma_p} \]

\[ \bar{\epsilon}_s = 2\sigma_s \sigma_p \] \hspace{1cm} (4)

\( D \) is the horizontal dispersion. \( D' = dD/ds \). A longitudinal emittance has been introduced so that the longitudinal motion and the transverse motions can be treated in a similar manner. \( s_c \) locates the center of the bunch.

\( \Gamma \) can now be computed using Eq.(1). This will provide an example how the integrals are done in this paper. The integration methods used here are somewhat more complicated than those used in [3] but they will also work for the more complicated bi-gaussian distribution.

\[ \Gamma = \int d^3xd^3p \exp[-S_x - S_y - S_s] \]

Writing \( \Gamma = \Gamma_y \Gamma_{xs} \) and computing \( \Gamma_y \) first because this part is simpler,

\[ \Gamma_y = \int dydp_y \exp[-S_y] \]

\[ S_y = \frac{1}{\epsilon_y} \epsilon_y(y, y') \quad y' = p_y/p_0 \]

\[ \epsilon_y(y, y') = \left[ y^2 + 2(\beta_y y' + \alpha_y y) \right] / \beta_y \]

\[ \eta_y = y/\sqrt{\beta_y}, \quad p_{ny} = (\beta_y y' + \alpha_y y)/\sqrt{\beta_y} \]

\[ dydp_y = p_0 d\eta_y dp_{ny} \]

\[ \Gamma_y = p_0 \int d\eta_y dp_{ny} \exp[-(\eta_y^2 + p_{ny}^2)/\bar{\epsilon}_y] \]

\[ \Gamma_y = \pi \bar{\epsilon}_y p_0 \] \hspace{1cm} (5)

Now for the remaining integral we have

\[ \Gamma_{xs} = \int dxdp_x dsdp_s \exp[-S_x - S_s] \]
\[ \Gamma_{xs} = \int d\sigma_p \exp[-S_s] \int d\sigma_p \exp[-S_x] \]

Make the transformation

\[
\begin{align*}
x_\beta &= x - D(p - p_0) / p_0 \\
x'_\beta &= x' - D'(p - p_0) / p_0 \\
x' &= p_x / p_0, \quad x'_\beta = p_{\beta x} / p_0 \\
dx dp_x &= p_0 dx dx' \beta \\
\end{align*}
\]

\[ \int d\sigma_p \exp[-S_x] = p_0 \int dx dx' exp[-S_x] \]

\[ S_x = \frac{1}{\epsilon_x} (x_\beta, x'_\beta) \]

\[ \int d\sigma_p \exp[-S_x] = \pi \epsilon_x p_0 \text{ as in evaluating } \Gamma_y \]

\[ p \sim p_s \text{ in the Lab. CS and } \]

\[ \Gamma_{xs} = \pi^2 \epsilon_s \epsilon_x p_0^2 \]

\[ \Gamma = \pi^3 \epsilon_s \epsilon_x \epsilon_y p_0^3 \] \hspace{1cm} (6)

4 Growth rates for a Gaussian distribution

In the following, the growth rates are given in the Rest Coordinate System, which is the coordinate system moving along with the bunch. Growth rates are given for \( < p_i p_j > \). From these one can compute the growth rates for \( < \epsilon_i > \). Using the general result, Eq.(2), one gets

\[ \frac{1}{p_0^2} < \delta(p_i p_j) > = \frac{N}{\Gamma^2} \int d^3 x d^3 p_1 d^3 p_2 \exp[-S(x, p_1) - S(x, p_2)] 2 \bar{\beta} c C_{ij} \ dt \]

\[ \bar{\Delta} = \frac{1}{2} (\bar{p}_1 - \bar{p}_2) \]

\[ \bar{\beta} c = |\bar{\Delta}| / m \]

\[ C_{ij} = \frac{2\pi}{p_0^2} (r_0 / 2 \bar{\beta}^2)^2 \ln(1 + (2 \bar{\beta}^2 b_{\text{max}} / r_0)^2) \left[ |\bar{\Delta}|^2 \delta_{ij} - 3 \Delta_i \Delta_j \right] i, j = 1, 3 \]

\[ r_0 = Z^2 e^2 / mc^2 \]
\[ \Gamma = \pi^3 \epsilon_x \epsilon_y \epsilon_p p_0^3 \]  \hspace{1cm} (7)

Transform to \( W, \Delta \)

\[
P_1 = \frac{W}{2} + \Delta, \quad P_2 = \frac{W}{2} - \Delta \]
\[
W = P_1 + P_2, \quad \Delta = \frac{P_1 - P_2}{2} \]
\[
d^3P_1 d^3P_2 = d^3W d^3\Delta \]  \hspace{1cm} (8)

We will first do the integral over \( d^3x \) and over \( d^3W \). For the \( y \) part of the integral

\[
S_y(y, p_{1y}) = \frac{1}{\epsilon_y} \epsilon_y(y, y'), \quad y' = p_{1y}/p_0 \\
\epsilon_y(y, y') = [y'^2 + (\beta_y y' + \alpha_y y)^2]/\beta_y \\
S_y(y, p_{1y}) = [y'^2 + (\beta_y(y - \Delta_y)/p_0 + \alpha_y y)^2]/(\beta_y \epsilon_y) \\
S_y(y, p_{1y}) + S_y(y, p_{2y}) = (2y^2/\beta_y + 2(\beta_y(W_y/p_0)/2 + \alpha_y y)^2/\beta_y \\
+ 2\beta_y^2(\Delta_y/p_0)^2/\beta_y)/\epsilon_y \\
\]

Make the transformation

\[
\eta_y = \sqrt{2y}/\sqrt{\beta_y}, \quad p_{ny} = \sqrt{2}(\beta_y(W_y/p_0)/2 + \alpha_y y)/\sqrt{\beta_y} \\
dy dW_y = p_0 d\eta_y dp_{ny} \]  \hspace{1cm} (9)

Integrate over \( dy, dW_y \)

\[
\int dy dW_y \exp[-S_y(y, p_{1y}) - S_y(y, p_{2y})] = p_0 \int d\eta_y dp_{ny} \\
exp[-\eta_y^2 + p_{ny}^2 + 2\beta_y^2(\Delta_y/p_0)^2/\beta_y] \\
= p_0 \pi \epsilon_y \exp[-2\beta_y(\Delta_y/p_0)^2] \\
= p_0 \pi \epsilon_y \exp[-R_y] \\
R_y = \frac{2\beta_y}{\epsilon_y}(\Delta_y/p_0)^2 \]  \hspace{1cm} (10)

7
In doing the remainder of the integral, the integral over \(dxdW_xdsdW_s\) we will do the integral over \(dxdW_x\) first and then the integral over \(dsdW_s\). Note that the integral is being done in the Rest CS and in the expression for \(S_x\) one has to replace \(p - p_0 \sim p_s - p_0\) in the Lab CS by \(\gamma p_s\) in the Rest CS. Remember also that \(f(x,p)\) is an invariant (see [1]). One finds for \(S_x(x, p_{1x})\)

\[
S_x(x, p_{1x}) = \{(x - \gamma D\tilde{W}_s/2 - \gamma D\tilde{\Delta}_s)^2 + \beta_x(\tilde{W}_x/2 + \Delta_x - \gamma D'\tilde{W}_s/2 - \gamma D'\tilde{\Delta}_s) + \alpha_x(x - \gamma D\tilde{W}_s/2 - \gamma D\tilde{\Delta}_s)^2\}/(\beta_x\epsilon_x)
\]

\[
\tilde{W}_i = W_i/p_0 \quad \tilde{\Delta}_i = \Delta/p_0
\]

we then find for \(S_x(x, p_{1x}) + S_x(x, p_{2x})\)

\[
S_x(x, p_{1x}) + S_x(x, p_{2x}) = \{2[x - \gamma D\tilde{W}_s/2]^2 + 2\gamma^2 D^2\tilde{\Delta}_s^2 + 2[\beta_x(\tilde{W}_x/2 - \gamma D'\tilde{W}_s/2) + \alpha_x(x - \gamma D\tilde{W}_s/2)]^2 + 2[\beta_x\Delta_x - \gamma D\tilde{\Delta}_s]^2\}/(\beta_x\epsilon_x)
\]

Now make the transformations

\[
x^* = \sqrt{2}x - \gamma D\tilde{W}_s/\sqrt{2} \quad p_x^* = \tilde{W}_x/\sqrt{2} - \gamma D'\tilde{W}_s/\sqrt{2}
\]

\[
\eta_x = x^*/\sqrt{\beta_x} \quad p_{\eta_x} = (\beta_x p_x^* + \alpha_x x^*)/\sqrt{\beta_x}
\]

\[
dxdW_x = p_0dx^*dp_{x^*} = p_0dn_xdp_{\eta_x}
\]

Doing the integral over \(dxdW_x\) one finds

\[
\int dxdW_x\exp[-S_x(x, p_{1x}) - S_x(x, p_{2x})] = p_0\int dn_xdp_{\eta_x}
\]

\[
\exp[-\eta_x^2 + p_{\eta_x}^2 + 2[\gamma^2 D^2\tilde{\Delta}_s^2 + (\beta_x\Delta_x - \gamma D\tilde{\Delta}_s)^2]/(\beta_x\epsilon_x)]
\]

\[
R_x = 2[\gamma^2 D^2\tilde{\Delta}_s^2 + (\beta_x\Delta_x - \gamma D\tilde{\Delta}_s)^2]/(\beta_x\epsilon_x)
\]
Now do the integral over $dsdW$. One may note that the form of the integral here is similar to the integral done over $dydW_y$. The result is then the same with the proper substitutions of $s$ for $y$.

$$\int dx dW_s \exp[-S_s(s,p_1s) - S_s(s,p_2s)] = p_0 \pi \bar{\epsilon}_s \exp[-R_s]$$

$$R_s = \frac{2\gamma^2 \beta_s}{\bar{\epsilon}_s} \left(\frac{\Delta_s}{p_0}\right)^2$$

Note that the term $\beta_s((p - p_0)/p_0)^2$ in $S_s$ in the Lab. CS has to be replaced by $\gamma^2 \beta_s(p_s/p_0)^2$ in the Rest CS.

Using Eq.(7), one gets the result for the growth rates in the Rest CS for a gaussian distribution.

$$\frac{1}{p_0^2} < p_i p_j > = \frac{N}{\Gamma} \int d^3 \Delta \exp[-R] C_{ij}$$

$$C_{ij} = \frac{2\pi}{p_0^2} (r_0/2\bar{\beta}^2)^2 (|\Delta|^2 \delta_{ij} - 3\Delta_i \Delta_j) 2\bar{\beta} c \ln[1 + (2\bar{\beta}^2 b_{max}/r_0)^2]$$

$$\bar{\beta} = \beta_0 \gamma_0 |\Delta/p_0|$$

$$r_0 = \frac{Z^2 e^2}{Mc^2}$$

$$\Gamma = \pi^3 \bar{\epsilon}_s \bar{\epsilon}_x \bar{\epsilon}_y p_0^3$$

$$R = R_x + R_y + R_s$$

$$R_x = \frac{2}{\bar{\beta}_x \bar{\epsilon}_x} \left[ \gamma^2 \Delta_s^2 + (\beta_x \Delta_x - \gamma \bar{D} \Delta_s)^2 \right] / p_0^2$$

$$\bar{D} = \beta_x D' + \alpha_x D$$

$$R_y = \frac{2\bar{\beta}_y \Delta_y^2}{\bar{\epsilon}_y} / p_0^2$$

$$R_s = \frac{2\bar{\beta}_s \gamma^2 \Delta_s^2}{\bar{\epsilon}_s} / p_0^2$$

The integral over $d^3 \Delta$ is an integral over all possible values of the relative momentum for any two particles in a bunch. $\beta_0, \gamma_0$ are the beta and gamma corresponding to $p_0$, the central momentum of the bunch in the Laboratory Coordinate System. $\gamma = \gamma_0$
The above 3-dimensional integral can be reduced to a 2-dimensional integral by integrating over $|\Delta|$ and using $d^3\Delta = |\Delta|^2 d|\Delta|\sin\theta d\theta d\phi$. This gives

$$\frac{1}{p_0^2} \frac{d}{dt} <p_ip_j> = \frac{N}{\Gamma} 2\pi p_0^3 \left( \frac{r_0}{2\gamma_0\beta_0^2} \right)^2 2\beta_0\gamma_0 c \int \sin\theta d\theta d\phi \left( \delta_{ij} - 3g_ig_j \right)$$

$$\frac{1}{F} \ln \left[ \hat{C} \right]$$

$$g_3 = \cos \theta = g_s$$

$$g_1 = \sin \theta \cos \phi = g_x$$

$$g_2 = \sin \theta \sin \phi = g_y$$

$$\hat{C} = 2\gamma_0^2 \beta_0^2 b_{max}/r_0$$

$$F = R/(|\Delta|/p_0)^2$$

$$F = F_x + F_y + F_s$$

$$F_x = \frac{2}{\beta_x \epsilon_x} \left[ \gamma^2 D^2 g_s^2 + (\beta_x g_x - \gamma \tilde{D} g_s)^2 \right]$$

$$F_y = \frac{2}{\epsilon_y} \beta_y g_y^2$$

$$F_s = \frac{2}{\epsilon_s} \beta_s \gamma^2 g_s^2$$

(17)

In obtaining the above, one uses $z = |\Delta|^2, dz = 2|\Delta|d|\Delta|$ and

$$\int_0^\infty dz \ exp[-Fz] \ln[\hat{C}z] = \frac{1}{F} \ln \left[ \frac{\hat{C}}{F} \right] - .5772$$

For $Z = 80, A = 200, \gamma = 100, b_{max} = 1cm, \log_{10} \hat{C} = 18.6$

5 Bi-Gaussian distribution

The bi-gaussian distribution will be assumed to have the form given by the following.

$Nf(x, p)$ gives the number of particles in $d^3xd^3p$, where $N$ is the number of particles in a bunch. For a bi-gaussian distribution, $f(x, p)$ is given by

$$f(x, p) = \frac{N_a}{N} \frac{1}{\Gamma_a} \exp[-S_a(x, p)] + \frac{N_b}{N} \frac{1}{\Gamma_b} \exp[-S_b(x, p)]$$
\[ \Gamma_a = \pi^3 \epsilon_a \epsilon_x \epsilon_y P_0^3 \]
\[ \Gamma_b = \pi^3 \epsilon_b \epsilon_x \epsilon_y P_0^3 \]  \hspace{1cm} (18)

In the first gaussian, to find \( \Gamma_a, S_a \) then in the expressions for \( \Gamma, S \), given above for the gaussian distribution, replace \( \bar{\epsilon}_x, \bar{\epsilon}_y, \bar{\epsilon}_s \) by \( \bar{\epsilon}_xa, \bar{\epsilon}_ya, \bar{\epsilon}_sa \). In the second gaussian, in the expressions for \( \Gamma, S \), replace \( \bar{\epsilon}_x, \bar{\epsilon}_y, \bar{\epsilon}_s \) by \( \bar{\epsilon}_xb, \bar{\epsilon}_yb, \bar{\epsilon}_sb \). In addition, \( N_a + N_b = N \). This bi-gaussian has 7 parameters instead of the three parameters of a gaussian.

6 Growth rates for a Bi-Gaussian distribution

In the following, the growth rates are given in the Rest Coordinate System, which is the coordinate system moving along with the bunch. Growth rates are given for \( < p_ip_j > \). From these one can compute the growth rates for \( < \epsilon_i > \). Starting with Eq.2 and using the \( f(x, p) \) from Eq.18, one gets

\[ \frac{1}{p_0^2} < \delta(p, p) > = \int d^3xd^3p_1d^3p_2 \left[ \frac{N_a}{N} \frac{1}{\Gamma_a} \exp[-S_a(x, p_1)] + \frac{N_b}{N} \frac{1}{\Gamma_b} \exp[-S_b(x, p_1)] \right] \]
\[ \times \left[ \frac{N_a}{N} \frac{1}{\Gamma_a} \exp[-S_a(x, p_2)] + \frac{N_b}{N} \frac{1}{\Gamma_b} \exp[-S_b(x, p_2)] \right] \]
\[ 2 \bar{\beta} c C_{ij} dt \]
\[ \bar{\Delta} = \frac{1}{2} (\vec{p}_1 - \vec{p}_2) \]
\[ \bar{\beta} c = |\bar{\Delta}|/m \]
\[ C_{ij} = \frac{2\pi}{p_0^2} (r_0/2\bar{\beta}^2)^2 \ln(1 + (2\bar{\beta}^2 b_{max}/r_0)^2) \frac{|\bar{\Delta}|^2 \delta_{ij} - 3\Delta_i \Delta_j}{i, j = 1, 3} \]
\[ r_0 = Z^2 \epsilon^2/mc^2 \]  \hspace{1cm} (19)

The term in the integrand which contains \( \exp[-S_a(x, p_1) - S_a(x, p_2)] \) is similar to the integrand for the gaussian distribution except that \( \bar{\epsilon}_i \) are replaced by \( \bar{\epsilon}_{ia} \) and leads to the same result as that given by Eq.(16) for the gaussian beam except that \( R \) has to be replaced by \( R_a \) where \( R_a \) is obtained from \( R \) by replacing \( \bar{\epsilon}_i \) by \( \bar{\epsilon}_{ia} \). The term containing \( \exp[-S_b(x, p_1) - S_b(x, p_2)] \) can be evaluated in the same way leading to the same result as that given by Eq.(16) for the gaussian beam except that \( R \) has to be replaced by \( R_b \) where \( R_b \) is
obtained from $R$ by replacing $\bar{\epsilon}_i$ by $\bar{\epsilon}_{ib}$. The only terms that need further evaluation are the two cross product terms. The two cross product terms are equal because of the symmetry of $p_1$ and $p_2$ in the rest of the integrand. This leads to the remaining integral to be evaluated

$$
\int d^3x d^3p_1 d^3p_2 \frac{2N_a N_b}{N^2} \frac{1}{\Gamma_a \Gamma_b} \exp[-S_a(x, p_1) - S_b(x, p_2)] \, 2\beta c C_{ij}
$$

In evaluating this integral, we will use the same procedure as was used for the gaussian distribution. We will first transform to $W, \Delta$ from $p_1, p_2$ (see Eq.(8). We will then do the integral over $d^3x$ and over $d^3W$. For the $y$ part of the integral one finds,

$$
S_{ya}(y, p_{1y}) = \left\{ y^2 + [\beta_y (\bar{W}_y/2 + \bar{\Delta}_y) + \alpha_y y]^2 \right\}/(\beta_y \bar{\epsilon}_{ya})
$$

$$
W_y = W_y/p_0 \quad \bar{\Delta}_y = \Delta_y/p_0
$$

(20)

One then finds that

$$
S_{ya}(y, p_{1y}) + S_{yb}(y, p_{2y}) = \left\{ 2y^2/\beta_y + 2[\beta_y (\bar{W}_y/2 + \alpha_y y)]^2/\beta_y
\right\}/(\beta_y \bar{\epsilon}_{yc} + 2\beta_y^2 \bar{\Delta}_y^2/\beta_y) \right\}/(\beta_y \bar{\epsilon}_{yd}) +

\frac{1}{\bar{\epsilon}_{yc}} = \frac{1}{2} \left( \frac{1}{\bar{\epsilon}_{ya}} + \frac{1}{\bar{\epsilon}_{yb}} \right)
$$

$$
\frac{1}{\bar{\epsilon}_{yd}} = \frac{1}{2} \left( \frac{1}{\bar{\epsilon}_{ya}} - \frac{1}{\bar{\epsilon}_{yb}} \right)
$$

(21)

Make the transformation

$$
\eta_y = \sqrt{2} y/\sqrt{\beta_y}, \quad p_{\eta y} = \sqrt{2}(\beta_y \bar{W}_y/2 + \alpha_y y)/\sqrt{\beta_y}
$$

$$
dy dW_y = p_0 d\eta_y dp_{\eta y}
$$

(22)

Integrate over $dy, dW_y$

$$
\int dy dW_y \exp[-S_{ya}(y, p_{1y}) - S_{yb}(y, p_{2y})] = p_0 \int d\eta_y dp_{\eta y}
$$

$$
\exp\left[ -\frac{\eta_y^2 + p_{\eta y}^2 + 2\beta_y^2 (\bar{\Delta}_y/p_0)^2 / \beta_y}{\bar{\epsilon}_{yc}} \right] +
$$

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\[
4\beta_y \Delta_y \frac{p_{ny}/(\sqrt{2}\sqrt{\beta_y})}{\epsilon_{yd}} = p_0 \pi \bar{\epsilon}_{yc} \exp \left[\frac{-2\beta_y \Delta_y^2 + 2\beta_y \bar{\Delta}_y^2}{\bar{\epsilon}_{yc}^2} \right] = p_0 \pi \bar{\epsilon}_{yc} \exp \left[-R_{yc} + R_{yd} \right]
\]

\[
R_{yc} = \frac{2\beta_y \Delta_y^2}{\bar{\epsilon}_{yc}}
\]

\[
R_{yd} = \frac{2\beta_y \Delta_y^2}{\bar{\epsilon}_{yd}/\bar{\epsilon}_{yc}}
\]

(23)

The exponent \( R_{yc} - R_{yd} \) has to be positive. This can be made more obvious by noting that

\[
\frac{1}{\bar{\epsilon}_{yc}^2} - \frac{1}{\bar{\epsilon}_{yd}^2} = \frac{1}{\bar{\epsilon}_{yc} \bar{\epsilon}_{yd}}
\]

In doing the remainder of the integral, the integral over \( dxdW_x dsdW_s \) we will do the integral over \( dxdW_x \) first and then the integral over \( dsdW_s \). Note that the integral is being done in the Rest CS and in the expression for \( S_x \) one has to replace \( p - p_0 \sim p_s - p_0 \) in the Lab. CS by \( \gamma p_s \) in the Rest CS. Remember also that \( f(x, p) \) is an invariant (see [1]) One finds for \( S_{xa}(x, p_{1x}) \)

\[
S_{xa}(x, p_{1x}) = \left\{ [x - \gamma D\bar{W}_s/2 - \gamma D\bar{\Delta}_s]^2 + [\beta_x (\bar{W}_x/2 + \bar{\Delta}_x - \gamma D'\bar{W}_s/2 - \gamma D'\bar{\Delta}_s)] + \alpha_x (x - \gamma D\bar{W}_s/2 - \gamma D\bar{\Delta}_s)^2 \right\}/(\beta_x \bar{\epsilon}_{xa})
\]

\[
\bar{W}_i = W_i/p_0 \quad \bar{\Delta}_i = \Delta/p_0
\]

\[
S_{xa}(y, p_{1x}) = \left\{ [x - \gamma D\bar{W}_s/2 - \gamma D\bar{\Delta}_s]^2 + [\beta_x (\bar{W}_x/2 - \gamma D'\bar{W}_s/2) + \alpha_x (x - \gamma D\bar{W}_s/2) + (\beta_x \bar{\Delta}_x - \gamma D\bar{\Delta}_s)^2 \right\}/(\beta_x \bar{\epsilon}_{xa})
\]

\[
\bar{D} = \beta_x D' + \alpha_x D
\]

(24)

we then find for \( S_{xa}(x, p_{1x}) + S_{xb}(x, p_{2x}) \)

\[
S_{xa}(x, p_{1x}) + S_{xb}(x, p_{2x}) = \left\{ 2[x - \gamma D\bar{W}_s/2]^2 + 2\gamma^2 D^2 \bar{\Delta}_s^2 + 2[\beta_x (\bar{W}_x/2 - \gamma D'\bar{W}_s/2) + \alpha_x (x - \gamma D\bar{W}_s/2)^2 + 2[\beta_x \bar{\Delta}_x - \gamma D\bar{\Delta}_s]^2 \right\}/(\beta_x \bar{\epsilon}_{xa}) +
\]

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\(-4\gamma D\Delta_s [x - \gamma D\bar{W}_s/2] / \beta_x + 4(\beta_x \bar{\Delta}_x - \gamma \bar{D}\bar{\Delta}_s) [\beta_x (\bar{W}_x/2 - \gamma D\bar{W}_s/2) + \alpha_x (x - \gamma D\bar{W}_s/2)] / \epsilon_{xd}\) (25)

Now make the transformations

\[
\begin{align*}
x^* &= \sqrt{2} x - \gamma D\bar{W}_s / \sqrt{2} \quad p_x^* = \bar{W}_x / \sqrt{2} - \gamma D\bar{W}_s / \sqrt{2} \\
\eta_x &= x^* / \beta_x \quad p_{\eta_x} = (\beta_x p_x^* + \alpha_x x^*) / \beta_x \\
dx dW_x &= p_0 dx^* dp_x^* = p_0 d\eta_x dp_{\eta_x}
\end{align*}
\]

Doing the integral over \(dx dW_x\) one finds

\[
\int dx dW_x \exp[-S_{xa}(x, p_1) - S_{xb}(x, p_2)] = p_0 \int d\eta_x dp_{\eta_x} \exp[-\{\eta_x^2 + p_{\eta_x}^2
+2[\gamma^2 D^2 \Delta_s^2 + (\beta_x \bar{\Delta}_x - \gamma \bar{D}\bar{\Delta}_s)^2] / \beta_x\} / \epsilon_{xc}
+\{-4\gamma D\bar{\Delta}_s \eta_x / \sqrt{2} \beta_x
+4(\beta_x \bar{\Delta}_x - \gamma \bar{D}\bar{\Delta}_s) p_{\eta_x} / \sqrt{2} \beta_x\} / \epsilon_{xd}\]
= p_0 \pi \epsilon_{xc} \exp[-R_{xc} + R_{xd}]
\]

\[
\begin{align*}
R_{xc} &= 2[\gamma^2 D^2 \Delta_s^2 + (\beta_x \bar{\Delta}_x - \gamma \bar{D}\bar{\Delta}_s)^2] / (\beta_x \epsilon_{xc}) \\
R_{xd} &= 2\{-\gamma D\bar{\Delta}_s\}^2 \\
&\quad +[(\beta_x \bar{\Delta}_x - \gamma \bar{D}\bar{\Delta}_s)]^2\} / (\beta_x \epsilon_{xd}^2 / \epsilon_{xc})
\end{align*}
\]

(27)

Now do the integral over \(ds dW_s\). One may note that the form of the integral here is similar to the integral done over \(dy dW_y\). The result is then the same with the proper substitutions of \(s\) for \(y\).

\[
\int dx dW_s \exp[-S_{sa}(s, p_1) - S_{sb}(s, p_2)] = p_0 \pi \epsilon_{sc} \exp[-R_{sc} + R_{sd}]
\]

14
\[ R_{sc} = \frac{2\beta_s \Delta^2_s}{\bar{\epsilon}_{sc}} \]
\[ R_{sd} = \frac{2\beta_s \Delta^2_s}{\bar{\epsilon}_{sd}/\bar{\epsilon}_{sc}} \]  

(28)

Note that the term \( \beta_s ((p - p_0)/p_0)^2 \) in \( S_s \) in the Lab. CS has to be replaced by \( \gamma^2 \beta_s (p_s/p_0)^2 \) in the Rest CS.

Putting all the above results, for the bi-gaussian distribution, together one gets the final result

\[
\frac{1}{p_0^2} \frac{d}{dt} < p_i p_j > = N \int d^3 \Delta C_{ij} \left[ \left( \frac{N_a}{N} \right)^2 \frac{\exp(-R_a)}{\Gamma_a} + \left( \frac{N_b}{N} \right)^2 \frac{\exp(-R_b)}{\Gamma_b} \right] \\
+ 2 \frac{N_a N_b}{N^2} \frac{\Gamma_c}{\Gamma_a \Gamma_b} \exp(-T) \]

\[ C_{ij} = \frac{2\pi}{p_0^2} (r_0/2\bar{\beta}^2)^2 (|\Delta|^2 \delta_{ij} - 3\Delta_i \Delta_j) 2\bar{\beta} c \ln[1 + (2\bar{\beta}^2 b_{\text{max}}/r_0)^2] \]

\[ \bar{\beta} = \beta_0 \gamma_0 |\Delta/p_0| \]

\[ r_0 = Z^2 e^2 / M c^2 \]

\[
\frac{1}{\epsilon_{ic}} = \frac{1}{2} \left( \frac{1}{\epsilon_{ia}} + \frac{1}{\epsilon_{ib}} \right) \quad i = x, y, s \\
\frac{1}{\epsilon_{id}} = \frac{1}{2} \left( \frac{1}{\epsilon_{ia}} - \frac{1}{\epsilon_{ib}} \right) \\
\]

\[ r_0 = Z^2 e^2 / M c^2 \]

\[ \Gamma_a = \pi^3 \epsilon_{sa} \epsilon_{xa} \epsilon_{ya} p_0^3 \]

\[ R_a = R_{xa} + R_{ya} + R_{sa} \]

\[ R_{xa} = \frac{2}{\beta_x \epsilon_{xa}} \left[ \gamma^2 D^2 \Delta^2_x + (\beta_x \Delta_x - \gamma \hat{D} \Delta_s)^2 \right] / p_0^2 \]

\[ \hat{D} = \beta_x D' + \alpha_x D \]

\[ R_{ya} = \frac{2}{\beta_y \epsilon_{ya}} \beta_y^2 \Delta^2_y / p_0^2 \]

\[ R_{sa} = \frac{2}{\beta_s \epsilon_{sa}} \beta_s^2 \gamma^2 \Delta^2_s / p_0^2 \]
\[ \begin{align*}
T &= T_x + T_y + T_s \\
T_x &= R_{xc} - R_{xd} \\
T_y &= R_{yc} - R_{yd} \\
T_s &= R_{sc} - R_{sd}
\end{align*} \]

\[ \begin{align*}
R_{xd} &= 2\left\{ \left[ -\gamma D \hat{\Delta}_x \right]^2 \\
&\quad + \left[ (\beta_x \hat{\Delta}_x - \gamma D \Delta)_s \right]^2 \right\} \\
&\quad / (\beta_x \bar{\epsilon}_{xd}/\bar{\epsilon}_{xc}) \\
R_{yd} &= \frac{2\beta_y}{\bar{\epsilon}_{yd}/\bar{\epsilon}_{yc}} \hat{\Delta}_y^2 \\
R_{sd} &= \frac{2\beta_s}{\bar{\epsilon}_{sd}/\bar{\epsilon}_{sc}} \hat{\Delta}_s^2 \\
\hat{\Delta}_i &= \Delta_i/p_0 \quad (29)
\end{align*} \]

\( R_a, R_b, R_c \) are each the same as \( R_a \) given above except that \( \bar{\epsilon}_{ia} \) are replaced by \( \bar{\epsilon}_{ia}, \bar{\epsilon}_{ib}, \bar{\epsilon}_{ic} \) respectively. The same remarks apply to \( \Gamma_a, \Gamma_b, \Gamma_c \).

The above 3-dimensional integral can be reduced to a 2-dimensional integral by integrating over \(|\Delta|\) and using \( d^3 \Delta = |\Delta|^2 d|\Delta| \sin \theta d\theta d\phi \). This gives

\[ \begin{align*}
\frac{1}{p_0^2} \int \frac{d}{dt} < p_ip_j > &= 2\pi p_0^3 \left( \frac{r_0}{2\gamma_0^2 \beta_0^2} \right)^2 2\beta_0\gamma_0 c \int \sin \theta d\theta d\phi \left( \delta_{ij} - 3g_ig_j \right) \\
&\quad \times N\left[ \left( \frac{N_a}{N} \right)^2 \frac{1}{\Gamma_a F_a} \ln \left( \frac{\hat{C}}{F_a} \right) + \left( \frac{N_b}{N} \right)^2 \frac{1}{\Gamma_b F_b} \ln \left( \frac{\hat{C}}{F_b} \right) \right] \\
&\quad + 2 \frac{N_aN_b}{N^2} \frac{\Gamma_c}{\Gamma_a \Gamma_b} \frac{1}{\Gamma_c \Gamma_b} \ln \left( \frac{\hat{C}}{G} \right) \\
&\quad + 2 N_aN_b \frac{\Gamma_c}{N^2} \frac{1}{\Gamma_a \Gamma_b} \ln \left( \frac{\hat{C}}{G} \right)
\end{align*} \]

\[ \begin{align*}
g_3 &= \cos \theta = g_s \\
g_1 &= \sin \theta \cos \phi = g_x \\
g_2 &= \sin \theta \sin \phi = g_y \\
\hat{C} &= 2\gamma_0^2 \beta_0^2 b_{max}/r_0 \\
F_i &= R_i/(|\Delta|/p_0)^2 \quad i = a, b, c
\end{align*} \]
\[ G = T/(|\Delta|/p_0)^2 \]  

(30)

\( F_a, F_b, F_c \) are each the same \( F \) that was defined for the Gaussian distribution except that the \( \bar{\epsilon}_i \) are replaced by \( \bar{\epsilon}_{ia}, \bar{\epsilon}_{ib}, \bar{\epsilon}_{ic} \) respectively.

The above results for the growth rates for a bi-gaussian distribution are expressed as an integral which contains 3 terms, each of which is similar to the one term in the results for the gaussian distribution. These three terms may be given a simple interpretation. The first term represents the contribution to the growth rates due to the scattering of the \( N_a \) particles of the first gaussian from themselves, the second term the contribution due to the scattering of the \( N_b \) particles of the second gaussian from themselves, and the third term the contribution due to the scattering of the \( N_a \) particles of the first gaussian from the \( N_b \) particles of the second gaussian.

7 Emittance growth rates

One can compute growth rates for the average emittances, \(< \epsilon_i \rangle \) in the Laboratory Coordinate System, from the growth rates for \(< p_ip_j \rangle \) in the Rest Coordinate System. In the following, \( dt \) is the time interval in the Laboratory System and \( \tilde{t} \) is the time interval in the Rest System. \( dt = \gamma d\tilde{t} \)

\[ \frac{d}{dt} < \epsilon_x > = \beta_x \frac{d}{\gamma dt} < p_x^2/p_0^2 > + \frac{D^2 + \tilde{D}^2}{\beta_x} \frac{d}{dt} < p_s^2/p_0^2 > - 2\tilde{D} \frac{d}{dt} < p_x p_s/p_0^2 > \]

\[ \frac{d}{dt} < \epsilon_y > = \beta_y \frac{d}{\gamma dt} < p_y^2/p_0^2 > \]

\[ \frac{d}{dt} < \epsilon_s > = \beta_s \frac{d}{\gamma dt} < p_s^2/p_0^2 > \]

(31)

To derive the above results, the simplest case to treat is that of the vertical emittance. The vertical emittance is given by

\[ \epsilon_y(y, y') = [y^2 + (\beta_y y' + \alpha_y y)^2]/\beta_y \]

\[ \delta \epsilon_y = \beta_y \delta (y'^2) \]

\[ \frac{d}{dt} < \epsilon_y > = \beta_y \frac{d}{\gamma dt} < p_y^2/p_0^2 > \]

(32)
In Eq.(32), \( y' = p_y/p_0 \), \( \delta \epsilon_y \) is the change in \( \epsilon_y \) in a scattering event.

For the longitudinal emittance one finds

\[
\epsilon_s = \frac{[s^2/\gamma^2 + (\beta_s \gamma p_s/p_0)^2]}{\beta_s} \\
\delta \epsilon_s = \beta_s \delta (\gamma p_s/p_0)^2 \\
\frac{d}{dt} \langle \epsilon_s \rangle = \beta_s \gamma \frac{d}{dt} \langle p_s^2/p_0^2 \rangle
\]  

(33)

In Eq.(33), \( s, p_s \) are the coordinates in the rest system and I have used the relationship \((p - p_0)_{LAB} = (\gamma p_s)_{REST}\)

For the horizontal emittance one finds

\[
\epsilon_x = \{[x - \gamma D p_s/p_0]^2 + [\beta_x (p_x/p_0 - \gamma D' p_s/p_0) + \alpha_x (x - \gamma D p_s/p_0)]^2\} / \beta_x \\
= \{[x - \gamma D p_s/p_0]^2 + [\beta_x p_x/p_0 + \alpha_x x - \bar{D} \gamma p_s/p_0]^2\} / \beta_x \\
= \{x^2 + (\gamma D p_s/p_0)^2 - 2x \gamma D p_s/p_0 + (\beta_x p_x/p_0 + \alpha_x x)^2 + (\bar{D} \gamma p_s/p_0)^2 - 2(\beta_x p_x/p_0 + \alpha_x x)(\bar{D} \gamma p_s/p_0)\} / \beta_x \\
\delta \epsilon_x = \delta \{\beta_x^2 (p_x/p_0)^2 + \gamma^2 (D^2 + \bar{D}^2) (p_s/p_0)^2 - 2 \beta_x \bar{D} \gamma (p_x/p_0)(p_s/p_0)\} / \beta_x \\
\frac{d}{dt} \langle \epsilon_x \rangle = \frac{\beta_x}{\gamma} \frac{d}{dt} \langle p_x^2/p_0^2 \rangle + \frac{D^2 + \bar{D}^2}{\beta_x} \frac{d}{dt} \langle p_s^2/p_0^2 \rangle + \frac{-2 \bar{D} \frac{d}{dt} \langle p_s p_x/p_0 \rangle}{\beta_x}
\]  

(34)

In the result for \( \delta \epsilon_x \), the terms that are linear in \( p_x \) or \( p_s \) have been dropped as they do not contribute to \( \langle \delta \epsilon_x \rangle \). In a scattering event involving two particles, the \( \delta p_x \) of one particle is equal and opposite to the \( \delta p_x \) of the other particle. This is also true for \( p_s \).

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