Cosmic Variance of the
Three-Point Correlation Function
of the Cosmic Microwave Background

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ABSTRACT

The fluctuations in the cosmic microwave background radiation may contain deviations from gaussian statistics which would be reflected in a nonzero value of three-point correlation function of $\Delta T$. However, any potential observation of the three-point function is limited by cosmic variance, even if a whole-sky map of $\Delta T$ is available. Here I derive a general formula for the cosmic variance of the three-point function in terms of integrals over the two-point function. This formula can be applied to any cosmological model and to any experimental measurement strategy. It gives a fundamental lower limit on the magnitude of a measurable three-point correlation function, and hence on the measurable amount of skewness in the distribution of $\Delta T$.

Subject headings: cosmic microwave background — cosmology: theory

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Now that anisotropies in the cosmic microwave background radiation have been discovered on large angular scales (Smoot et al. 1992), attention can be focused on the detailed properties of these fluctuations. A question of particular importance is whether or not their distribution is gaussian. A gaussian distribution has a vanishing three-point correlation function. An apparently simple thing to check is whether or not this is true of the observed anisotropies (to within the experimental error bars).

However, theoretical predictions of properties of the CMBR are always probabilistic in nature, and fundamentally limited by the “cosmic variance” which arises from our inability to make measurements in more than one universe (e.g., Abbott & Wise 1984b; Scaramella & Vittorio 1990, 1991; Cayon et al. 1991; White et al. 1993). The prediction of a vanishing three-point function is no exception, and so comes with a purely theoretical error bar.

In this letter I address this problem. Assuming that the fluctuations are in fact gaussian, I compute the expected deviation from zero of the three-point correlation function (which includes, as a special case, the skewness), due solely to the effect of making measurements in a single universe. The answer is expressed in terms of the theoretical two-point correlation function, which depends on both the window function of the experiment and the cosmological model which is adopted. I give detailed results for the COBE window function, assuming a scale invariant \( n = 1 \) power spectrum for the fluctuations. Any measured three-point function must have a magnitude exceeding the values given here in order to be construed as evidence for non-gaussian fluctuations. I compare my results for the cosmic variance of the skewness with previous Monte Carlo simulations (Scaramella & Vittorio 1991).

To begin, let \( \tilde{T}(\hat{n}) \) denote the temperature difference which the experiment assigns to a point on the sky specified by the unit direction vector \( \hat{n} \), including the effects due to finite beam width and any “chopping” strategy which the experiment uses. We can make a multipole expansion of the form

\[
\tilde{T}(\hat{n}) = \sum_{lm} a_{lm} W_{lm} Y_{lm}(\hat{n}) ,
\]

where the \( Y_{lm}(\hat{n}) \) are spherical harmonics, the \( W_{lm} \) represent the window function of the experiment, and the \( a_{lm} \) are random variables whose distribution must be specified by a specific cosmological model. In general, rotation invariance implies that

\[
\langle a_{lm} a_{lm'}^{*} \rangle = C_l \delta_{ll'} \delta_{mm'} ,
\]

where the angle brackets denote an ensemble average over the probability distribution for the \( a_{lm} \). Assuming an \( n = 1 \) power spectrum and ignoring the possible contribution of tensor modes, the Sachs–Wolfe effect results in (Peebles 1982; Abbott & Wise 1984a; Bond & Efstathiou 1987)

\[
\frac{C_l}{4\pi} = \frac{6}{5} \frac{Q^2}{l(l+1)}
\]
for \( l \geq 2 \), where \( Q \) is the r.m.s. ensemble average of the quadrupole moment (which may differ significantly from the actual, measured quadrupole moment). Since the monopole term must be removed from \( \tilde{T}(\hat{n}) \), we have \( W_{00} = 0 \) and

\[
\int \frac{d\Omega}{4\pi} \tilde{T}(\hat{n}) = 0 ,
\]

where \( d\Omega \) denotes integration over the unit vector \( \hat{n} \). The theoretical two-point correlation function is given by

\[
C_2(\hat{n}_1, \hat{n}_2) = \left\langle \tilde{T}(\hat{n}_1)\tilde{T}(\hat{n}_2) \right\rangle = \sum_{lm} C_l |W_{lm}|^2 Y_{lm}(\hat{n}_1) Y_{lm}^*(\hat{n}_2) .
\]

(5)

If \( W_{lm} \) is independent of \( m \), we can simplify this to

\[
C_2(\hat{n}_1, \hat{n}_2) = C(x) \equiv \frac{1}{4\pi} \sum_l (2l + 1) C_l |W_l|^2 P_l(x) ,
\]

(6)

where \( x = \hat{n}_1 \cdot \hat{n}_2 \) is the cosine of the beam-separation angle and \( P_l(x) \) is a Legendre polynomial.

Given the observed values of \( \tilde{T}(\hat{n}) \) for each \( \hat{n} \), the experimental value of the skewness \( S \) is given by the sky average of \( \tilde{T}^3 \):

\[
S = \int \frac{d\Omega}{4\pi} \tilde{T}^3(\hat{n}) .
\]

(7)

(The effect of partial sky coverage will be discussed later.) The corresponding dimensionless skewness parameter is \( J = S/\left[ \int d\Omega \tilde{T}(\hat{n})/4\pi \right]^{3/2} \), but a discussion of this quantity is complicated by the need to treat both its numerator and denominator as random variables. I therefore concentrate on \( S \) as defined in equation (7).

If the distribution of the \( a_{lm} \) is gaussian, then the ensemble average of the skewness is zero: \( \langle S \rangle = 0 \). Of course, for our particular sky the experimental value of \( S \) is unlikely to be exactly zero, even if we completely neglect experimental noise (Scaramella & Vittorio 1991). In order to tell whether or not a particular measured value of \( S \) is significant evidence of a departure from gaussian statistics, we need to know the cosmic variance of \( S \), assuming that the underlying statistics are indeed gaussian. Since the mean of \( S \) vanishes, we can write the variance of \( S \) as

\[
\langle S^2 \rangle = \int \frac{d\Omega_1}{4\pi} \int \frac{d\Omega_2}{4\pi} \left\langle \tilde{T}^3(\hat{n}_1)\tilde{T}^3(\hat{n}_2) \right\rangle .
\]

(8)

Using standard combinatoric properties of gaussian distributions, we have

\[
\left\langle \tilde{T}^3(\hat{n}_1)\tilde{T}^3(\hat{n}_2) \right\rangle = 9 \left\langle \tilde{T}^2(\hat{n}_1) \right\rangle \left\langle \tilde{T}^2(\hat{n}_2) \right\rangle \left\langle \tilde{T}(\hat{n}_1)\tilde{T}(\hat{n}_2) \right\rangle + 6 \left\langle \tilde{T}(\hat{n}_1)\tilde{T}(\hat{n}_2) \right\rangle^3 .
\]

(9)
If $W_{lm}$ is independent of $m$, then ensemble averages are always rotationally invariant, and \( \langle \tilde{T}^2(\hat{n}_1) \rangle \) is independent of $\hat{n}_1$. From now on, for simplicity, we will assume that this is the case. Equation (4) then implies that the first term in equation (9) will vanish after integration over $\hat{n}_1$. Thus we have

$$\langle S^2 \rangle = 6 \int \frac{d\Omega_1}{4\pi} \frac{d\Omega_2}{4\pi} C^3_2(\hat{n}_1, \hat{n}_2).$$  \hspace{1cm} (10)$$

Making use of equation (6), we can simplify this to

$$\langle S^2 \rangle = 3 \int_{-1}^{+1} dx C^3(x).$$  \hspace{1cm} (11)$$

This formula assumes full sky coverage. In the case of partial sky coverage, the analysis of Scott et al. (1993) can be applied. Specifically, the integrals in equation (10) should range only over the solid angle $A$ which is covered, and each factor of $4\pi$ should be replaced by $A$. Since $C_2(\hat{n}_1, \hat{n}_2)$ is always sharply peaked near $\hat{n}_1 = \hat{n}_2$, the net change is an enhancement of $\langle S^2 \rangle$ by a factor of $4\pi/A$, provided that $A$ is large enough to encompass the entire peak, and provided that equation (4) still holds, at least approximately. For more details in the context of a different but similar calculation, see Scott et al. (1993).

As a specific example, the COBE group (Smoot et al. 1992) has reported results for the two-point correlation function with the monopole, dipole, and quadrupole terms removed, and the higher moments weighted with a 7.5 FWHM beam, which results in $W_{lm} = W_l = \exp[-\frac{1}{2}(l + \frac{1}{2})^2\sigma^2]$ with $\sigma = 3.2$ for $l > 2$, and $W_l = 0$ for $l \leq 2$. Assuming equation (3) for the moments of the temperature distribution gives $C(1) = 3.65 Q^2$, and performing the integral in equation (11) numerically yields $\langle S^2 \rangle = 1.10 Q^6$ for full sky coverage. Including the $l = 2$ term gives instead $C(1) = 4.63 Q^2$ and $\langle S^2 \rangle = 3.44 Q^6$. In either case, a galactic latitude cut of $|b| > 20^\circ$ enhances $\langle S^2 \rangle$ by a factor of $4\pi/A = 1/(1 - \cos 70^\circ) = 1.52$.

Thus, the skewness of the temperature distribution, as determined from the COBE data with $|b| > 20^\circ$ and the quadrupole removed, would have to be significantly larger than $\langle S^2 \rangle^{1/2} = 1.3 Q^3$ in order to constitute evidence for non-gaussian fluctuations. Here, again, $Q$ is the r.m.s. ensemble average of the quadrupole moment, which must be determined from the normalization of the full two-point correlation function. $Q$ is a convenient measure of the overall magnitude of the fluctuations; for example, in inflationary models, the amplitude of the perturbations at the time of horizon crossing is given by $\varepsilon_H = (12/5\pi)^{1/2}/Q/T_0$ (Abbott & Wise 1984a,b). If we assume an $n = 1$ spectrum, then the COBE data indicates that $Q = 16.7 \pm 4 \mu K$ (Smoot et al. 1992).

Although the probability distribution for $S$ must be symmetric about $S = 0$, it will not be gaussian. (It is easy to check, for example, that $\langle S^4 \rangle$ does not equal $3\langle S^2 \rangle^2$, as it would for a gaussian distribution.) Thus, $\pm \langle S^2 \rangle^{1/2}$ does not correspond to a true 68% confidence interval. The probability distribution for the dimensionless skewness parameter $J$ was computed via Monte Carlo methods by Scaramella & Vittorio (1991). For an $n = 1$
spectrum, beam width $\sigma = 3\circ 0$, and including the $l = 2$ term, they found the $\mathcal{J}$ distribution to be well approximated by a gaussian with $\langle \mathcal{J}^2 \rangle^{1/2} = 0.2$. For comparison, we can compute $\langle S^2 \rangle^{1/2}/C(1)^{3/2}$. This is a ratio of averages rather than the average of a ratio, so exact agreement is not expected. Nevertheless, in this case I find $C(1) = 4.78 Q^2$ and $\langle S^2 \rangle = 3.56 Q^6$, so that $\langle S^2 \rangle^{1/2}/C(1)^{3/2} = 0.18$, in good agreement with Scaramella & Vittorio’s value of $\langle \mathcal{J}^2 \rangle^{1/2}$.

Experiments on small angular scales typically result in a theoretical two-point correlation function which can be adequately approximated by $C(x) = C_0 \exp[(x - 1)/\theta_c^2]$, where $C_0$ and $\theta_c$ are computable functions of the cosmological model and the experimental parameters (e.g., Bond & Efstathiou 1987). In this case, combining equation (11) with the correction factor of Scott et al. (1993) for partial sky coverage yields

$$\langle S^2 \rangle = (4\pi/A)\theta_c^2 C_0^3. \tag{12}$$

Equation (12) applies when $A \gg \theta_c^2$, the upper limit on $\langle S^2 \rangle$ is $6C_0^3$.

These results for the cosmic variance of the skewness can be extended to the full three-point correlation function. Experimentally, the three-point correlation function is determined by first choosing a fixed configuration of three direction vectors; the simplest choice is an equilateral triangle,

$$\hat{n}_1 \cdot \hat{n}_2 = \hat{n}_2 \cdot \hat{n}_3 = \hat{n}_3 \cdot \hat{n}_1 = \cos \alpha, \tag{13}$$

and we specialize to this case from here on. The experimental three-point correlation function is then

$$\zeta(\alpha) = \int dR \ \tilde{T}(R\hat{n}_1)\tilde{T}(R\hat{n}_2)\tilde{T}(R\hat{n}_3). \tag{14}$$

Here the $\hat{n}_i$ are to be held fixed in a configuration obeying equation (13), and $R$ is a rotation matrix; appropriately integrating over $R$ results in an average over all possible triangles obeying equation (13). Note that $\zeta(0)$ is simply the skewness $S$. A specific realization of the $\hat{n}_i$ is

$$\hat{n}_1 = (s_\alpha, 0, c_\alpha),$$

$$\hat{n}_2 = (-\frac{1}{2} s_\alpha, +\frac{\sqrt{3}}{2} s_\alpha, c_\alpha),$$

$$\hat{n}_3 = (-\frac{1}{2} s_\alpha, -\frac{\sqrt{3}}{2} s_\alpha, c_\alpha), \tag{15}$$

where $c_\alpha = [(1 + 2 \cos \alpha)/3]^{1/2}$ and $s_\alpha = [(2 - 2 \cos \alpha)/3]^{1/2}$. This is an equilateral triangle centered on the $\hat{z}$ axis. A specific realization of $R$ is

$$R = \varepsilon \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

This is a product of a sign ($\varepsilon = \pm 1$) and three rotations, the first about the $\hat{z}$ axis by $\psi$, the second about the $\hat{x}$ axis by $\theta$, and the third about the $\hat{z}$ axis by $\phi$. The two possible values...
of ε must be summed over in order to get triangles of both orientations. The integration measure in equation (14) for this realization of \( R \) is

\[
\int dR = \frac{1}{16\pi^2} \sum_{\varepsilon = \pm 1} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi .
\]

Assuming a gaussian distribution, the ensemble average of the three-point correlation function vanishes: \( \langle \zeta(\alpha) \rangle = 0 \). The cosmic variance of \( \zeta(\alpha) \) is then

\[
\langle \zeta^2(\alpha) \rangle = \int dR \; dR' \; \langle \tilde{T}(R\hat{n}_1)\tilde{T}(R\hat{n}_2)\tilde{T}(R\hat{n}_3)\tilde{T}(R'\hat{n}_1)\tilde{T}(R'\hat{n}_2)\tilde{T}(R'\hat{n}_3) \rangle.
\]

Using the rotation invariance of the ensemble average, the integral over \( R' \) can be removed, leaving

\[
\langle \zeta^2(\alpha) \rangle = \int dR \; \langle \tilde{T}(R\hat{n}_1)\tilde{T}(R\hat{n}_2)\tilde{T}(R\hat{n}_3)\tilde{T}(\hat{n}_1)\tilde{T}(\hat{n}_2)\tilde{T}(\hat{n}_3) \rangle.
\]

Using the appropriate generalization of equation (9), symmetries of the equilateral triangle, and \( \int dR \; \tilde{T}(R\hat{n}) = 0 \) [which follows from equation (4)], we ultimately get

\[
\langle \zeta^2(\alpha) \rangle = 6 \int dR \; C_2(R\hat{n}_1, \hat{n}_1)C_2(R\hat{n}_2, \hat{n}_2)C_2(R\hat{n}_3, \hat{n}_3).
\]

When \( \alpha = 0 \), equation (20) reduces to equation (11).

The value of \( \langle \zeta^2(\alpha) \rangle^{1/2} \) can be computed by doing the integral in equation (20) numerically. The result is plotted in Figure 1, assuming an \( n = 1 \) power spectrum, the COBE window function (7°5 FWHM beam, \( l = 0, 1, 2 \) terms removed), and full sky coverage. This indicates the magnitude of \( \zeta(\alpha) \) which is to be expected in the COBE data due to cosmic variance, even though the ensemble average of \( \zeta(\alpha) \) is (assumed to be) zero. The experimental value of \( \zeta(\alpha) \) would have to rise significantly above the curve in Figure 1 in order to indicate a departure from gaussian statistics. The restriction to \( |b| > 20^\circ \) in the COBE data would enhance \( \langle \zeta^2(\alpha) \rangle^{1/2} \) by a factor of 1.23 for small values of \( \alpha \), and by a somewhat larger factor (which is difficult to estimate reliably) for \( \alpha \gtrsim 20^\circ \). Note again that while the probability distribution for \( \zeta(\alpha) \) must be symmetric about \( \zeta(\alpha) = 0 \), it will not be gaussian, and so \( \pm\langle \zeta^2(\alpha) \rangle^{1/2} \) does not represent a true 68% confidence interval.

As an example of the utility of this result, let us examine the effect of cosmic variance on the prediction of inflationary models for the three-point correlation function. For an \( n = 1 \) power spectrum, the angular dependence of the three-point function is determined, but the overall amplitude is model dependent (Falk et al. 1993). In single-field inflation models, the amplitude of the three-point function is proportional to \( \mu/H \), where \( \mu \) is the coefficient of the cubic term in the potential of the inflaton field, and \( H \) is the Hubble constant during inflation. In two-field models (e.g., Kofman et al. 1991), this amplitude is proportional to the strength of the coupling between the two fields. Figure 2 shows the ensemble average of the three-point correlation function, assuming the COBE window.
function (Falk et al. 1993). The overall amplitude is that of a single-field model with \( \mu/H = 0.1 \). The shaded area indicates a band of \( \pm \langle \zeta^2(\alpha) \rangle^{1/2} \) about the mean, assuming full sky coverage. Of course, there will be corrections to \( \langle \zeta^2(\alpha) \rangle \) coming from the non-gaussian aspect of the distribution, but these will be suppressed by a factor of \( \mu^2/H^2 \), and cannot significantly change the result. We see that unless \( \mu/H \) is larger by at least a factor of two, the wiggles at large \( \alpha \) drop below the uncertainty due to cosmic variance. This is unfortunate; Luo & Schramm (1993) have pointed out that the height of the \( \alpha = 0 \) peak depends sensitively on the power-spectrum index \( n \), but to determine \( n \) from the three-point function requires normalizing it at large \( \alpha \). This is impossible unless the three-point function is large enough to raise its \( \alpha \gtrsim 20^\circ \) features well above the uncertainty due to cosmic variance (to say nothing of the uncertainty due to experimental noise, which is likely to be at least comparable). Furthermore we see that we must have \( \mu/H \gtrsim 0.015 \) in order to keep even the \( \alpha = 0 \) peak out of the cosmic noise.

To summarize, the fluctuations in the cosmic microwave background radiation may contain deviations from gaussian statistics which would be reflected in a nonzero value of three-point correlation function of \( \Delta T \). Any potential observation of the three-point correlation function is limited by cosmic variance, even if a whole-sky map of \( \Delta T \) is available. The cosmic variance of the three-point correlation function depends on both the underlying cosmological model and the experimental measurement strategy. For COBE, the skewness, defined as the sky average of \( (\Delta T)^3 \) with the dipole and quadrupole terms removed from \( \Delta T \), must exceed 1.3 \( Q_3 \) in order to be indicative of non-gaussian fluctuations. For small-scale experiments, the skewness must exceed \( (4\pi/A)^{1/2}\theta_c C_0^{3/2} \).

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FIGURE CAPTIONS

Fig. 1.—The square root of the cosmic variance of the three-point correlation function \( \langle \zeta^2(\alpha) \rangle^{1/2} \) as a function of the beam-separation angle \( \alpha \), in units of \( Q^3 \) (where \( Q \) is the r.m.s. ensemble average of the quadrupole moment), assuming an \( n = 1 \) spectrum of fluctuations, the COBE window function (\( 7^\circ 5 \) FWHM beam, \( l = 0, 1, 2 \) terms removed), and full sky coverage.

Fig. 2.—The prediction of inflation for the three-point correlation function \( \langle \zeta(\alpha) \rangle \) as a function of the beam-separation angle \( \alpha \), in units of \( Q^3 \), assuming an \( n = 1 \) spectrum of fluctuations and the COBE window function. The overall amplitude is that of a single-field inflation model specified by \( \mu/H = 0.1 \). The gray band indicates a range of \( \pm \langle \zeta^2(\alpha) \rangle^{1/2} \) about the central value, assuming full sky coverage.
