Mass-ratio condition for non-binding of three two-component particles with contact interactions

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Abstract The binding of two heavy fermions interacting with a light particle via a contact interaction is possible only for a sufficiently large heavy-light mass ratio. The two-variable inequality is derived to determine the specific mass-ratio bound providing the absence of three-body bound states for lower values of the mass ratio. By means of this inequality, the mass-ratio bound is found to be 5.26 for the total angular momentum and parity $L^P=1^-$. For other $L^P$ sectors, the mass-ratio bounds providing the absence of three-body bound states is found in a similar way. For generality, the method is extended to determine also the mass-ratio bounds for a system consisting of two identical bosons and a distinct particle for different $L^P$ ($L>0$) sectors.

1 Introduction

In recent years, few-body dynamics of multi-component ultra-cold quantum gases has attracted much attention. A particular form of short-range interaction between particles becomes insignificant in the low-energy limit, and the zero-range model provides a universal description. Particularly important is the two-component three-body system with zero-range interaction, which was investigated, e. g., in [1–10]. A single parameter of the zero-range model, e. g., a two-body scattering length $a$, can be chosen as a scale, thus there is only one essential parameter, the mass ratio of different particles. One should mention that the introduction of the zero-range model in the few-body problem could be ambiguous and needs special efforts, which were discussed, e. g., in [9–14].

A system of two identical particles (bosons or fermions) of mass $m$ and a distinct particle of mass $m_1$ in the universal low-energy limit of zero-range two-body interaction was considered in [2, 5, 9, 15–17] in different sectors of the total angular momentum $L$ and parity $P$. Three-body bound states exist for the mass ratio exceeding the critical values $\mu_B$, which were determined in numerical calculations [2, 5, 9, 15], namely, $\mu_B \approx 8.17259$ for $L^P=1^-$, $\mu_B \approx 22.6369$ for $L^P=2^+$ and $\mu_B \approx 43.3951$ for $L^P=3^-$ etc.

Besides these numerical results, it is of interest to determine the lower bound of the mass ratio $\mu^*$, below which the three-body bound states do not exist. So far, only one mass-ratio bound $\mu^* = 2.617$ was obtained for the fermionic system in the sector of $L^P=1^-$ by analyzing the momentum-space integral equation [10]. In this work, the mass-ratio bound was determined using hyper-radial equations [18] and exploiting the fact that in the two-dimensional (also one-dimensional) quantum problem a bound state exists for any attractive potential [19].

2 Formulation

Consider a particle 1 of mass $m_1$ interacting with two identical particles 2 and 3 of masses $m_2=m_3=m$. In the framework of the zero-range model, two identical fermions do not interact with each other and the same is assumed for generality if two identical particles are bosons. The zero-range interaction in pairs (1-2) and (1-3) is completely determined by a single parameter, the two-body scattering length $a$. In the center-of-mass frame, one defines the scaled Jacobi variables as $x = \sqrt{\frac{2mm_1}{m+m_1}}(r_2-r_1)$ and $y = \sqrt{\frac{2m(m+m_1)}{m_1+2m}}\left(r_3 - \frac{m_1r_1 + mr_2}{m_1+m}\right)$, where $r_i$ is the position vector of the $i$-th particle. The units are chosen by means of the condition $\hbar = |a| = \frac{2mm_1}{m+m_1} = 1$, which gives the unit two-body binding energy $\varepsilon_2 = 1$ for finite $a > 0$. The three-body

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Hamiltonian is the sum of the kinetic energy, which is the six-dimensional negative Laplace operator, and the potential energy expressed by boundary conditions imposed at zero distance between interacting particles,

$$\lim_{r \to 0} \frac{\partial \ln(r \Psi)}{\partial r} = -\text{sgn}(a)$$  \hspace{1cm} (1)

where $r$ denotes either $|\mathbf{r}_1 - \mathbf{r}_2|$ or $|\mathbf{r}_1 - \mathbf{r}_3|$. The problem formally depends on a single parameter, the mass ratio $m_1/m_2$, alternatively, the kinematic angle $\omega$ defined by $\sin \omega = 1/(1+m_1/m_2)$ will be used for convenience.

The total angular momentum $L$, its projection $M$ and parity $P$ are conserved quantum numbers that label the solutions. As the zero-range interaction acts in the $s$-wave, it is sufficient to consider only the case $P = (-)^L$. In addition, it is convenient to introduce $P_s$, the permutation operator of identical particles 2 and 3, whose eigenvalues $P_s = \pm 1$ designate whether the identical particles are fermions or bosons.

In the following analyses, the method of hyper-radial equations proposed by Macek [18] will be used. Thus, one introduces a hyper-radius $\rho = \sqrt{x^2 + y^2}$ and hyper-angles $(\alpha, \hat{x}, \hat{y})$ by $x = \rho \cos \alpha$, $y = \rho \sin \alpha$, $\hat{x} = x/x$, and $\hat{y} = y/y$. In these variables the Hamiltonian is expressed as

$$H = -\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) + \Delta_\rho$$  \hspace{1cm} (2)

supplemented with boundary conditions that follow from (1). In Eq. (2), $\Delta_\rho$ denotes the Laplace operator on the hyper-sphere whose explicit form can be found, e. g., in [2, 9, 20]. One can define an auxiliary eigenproblem on the hyper-sphere (for fixed $\rho$),

$$\left( \Delta_\rho + \gamma^2(\rho) - 4 \right) \Phi(\alpha, \hat{x}, \hat{y}; \rho) = 0,$$

$$\lim_{\alpha \to \pi/2} \frac{\partial \log[(\alpha - \pi/2)\Phi(\alpha, \hat{x}, \hat{y}; \rho)]}{\partial \alpha} = \rho \cdot \text{sgn}(a),$$  \hspace{1cm} (3)

whose eigenvalues $\gamma^2(\rho)$ and eigenfunctions $\Phi(\alpha, \hat{x}, \hat{y}; \rho)$ inherit symmetry of the total wave function and will be chosen in the form [2, 9, 15]

$$\Phi(\alpha, \hat{x}, \hat{y}; \rho) = (P_s + P_\alpha) \frac{\psi_{\gamma}^L(\alpha)}{\sin 2\alpha} Y_{LM}(\hat{y}).$$  \hspace{1cm} (5)

Here $Y_{LM}(\hat{y})$ is the spherical function, the labels $L, M$, and $P_s$ in the left-hand side are suppressed for brevity, and the function $\psi_{\gamma}^L(\alpha)$ satisfies the equations

$$\left[ \frac{d^2}{d\alpha^2} - \frac{L(L+1)}{\sin^2 \alpha} + \gamma^2 \right] \psi_{\gamma}^L(\alpha) = 0,$$  \hspace{1cm} (6a)

$$\lim_{\alpha \to \pi/2} \left( \frac{d}{d\alpha} - \rho \cdot \text{sgn}(a) \right) \psi_{\gamma}^L(\alpha) = \frac{2(-)^L P_s}{\sin 2\omega} \frac{\psi_{\gamma}^L(\omega)}{\sin \omega},$$  \hspace{1cm} (6b)

$$\psi_{\gamma}^L(0) = 0.$$  \hspace{1cm} (6c)

The solution of Eq. (6a) under the boundary condition (6c) can be written via the Legendre function of the second kind $Q_{\gamma}^L(x)$ or as a finite sum

$$\psi_{\gamma}^L(\alpha) = (\sin \alpha)^{L+1} \left( \frac{1}{\sin \alpha} \frac{d}{d\alpha} \right)^L \frac{\sin \gamma \alpha}{\sin \alpha}.\hspace{1cm} (7)$$

Substituting (7) into (6b), one arrives at the transcendental equation [9, 20]

$$\rho \cdot \text{sgn}(a) \Gamma\left( \frac{L + \gamma + 1}{2} \right) \Gamma\left( \frac{L - \gamma + 1}{2} \right) = 2 \Gamma\left( \frac{L + \gamma + 1}{2} \right) \Gamma\left( \frac{L - \gamma + 1}{2} \right) \Gamma\left( \frac{L - \gamma}{2} + 1 \right) \Gamma\left( \frac{L + \gamma}{2} + 1 \right) + P_s \frac{(-)^{L-\gamma} L \pi (\sin \omega) L}{\sin \gamma \pi \cos \omega} \left( \frac{d}{\sin \omega \ d\omega} \right)^L \frac{\sin \gamma \omega}{\sin \omega},$$

which determines an infinite set of eigenvalues $\gamma_n^2(\rho)$ and the corresponding eigenfunctions $\Phi_n(\alpha, \hat{x}, \hat{y}; \rho)$. Using the Hamiltonian (2) and the expansion of the total wave function in terms of eigenfunctions on the hyper-sphere

$$\Psi(x, y) = \rho^{-5/2} \sum_{n=1}^{\infty} f_n(\rho) \Phi_n(\alpha, \hat{x}, \hat{y}; \rho)$$  \hspace{1cm} (8)

one can obtain a system of hyper-radial equations [2, 9, 20] for the channel functions $f_n(\rho)$. 

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As it will be proven in the next section, the lower energy bound and, consequently, the mass-ratio bound \( \mu^* \) can be found by using the one-channel approximation of an infinite system of hyper-radial equations, in which the diagonal coupling term \( \int \left( \frac{\partial \Phi}{\partial \rho} \right)^2 \sin^2 \alpha \, d\alpha \, d\mathbf{k} \, d\mathbf{j} \) is omitted,\(^{(10)}\)

\[
\frac{d^2}{d\rho^2} - \frac{\gamma^2(\rho) - 1/4}{\rho^2} + E \left( f(\rho) = 0. \right)
\]

Here \( \gamma^2(\rho) \) and \( \Phi(\alpha, \mathbf{k}, \mathbf{j}; \rho) \) denote the lowest eigenvalue and the corresponding eigen-function. The reduction to Eq. (10) is known also as the "extreme" adiabatic approximation \([21, 22]\). One should notice that if the diagonal coupling term is not omitted, one can obtain the usual adiabatic approximation whose solution provides the upper energy bound.

It is known \([9, 10, 12, 14, 20]\) that the above formulation of the tree-body problem does not define the self-adjoint Hamiltonian for a sufficiently large mass ratio. In particular, it was shown \([9, 20]\) that the singular term \( \frac{\gamma^2(\rho) - 1/4}{\rho^2} \) for \( \rho \to 0 \) in the lowest channel should be considered for analysis of self-adjointness. As a matter of fact, the three-body Hamiltonian is self-adjoint if \( \gamma^2(0) \geq 1 \) and an additional parameter is needed for an unambiguous formulation of the problem if \( \gamma^2(0) < 1 \). In terms of the mass ratio, the above formulation is valid for \( m/m_1 \leq \mu_r \), where the critical value \( \mu_r \) is determined by the condition \( \gamma^2(0) = 0 \). Using Eq. (8), one can find the equation for \( \mu_r \),

\[
2^{L-1}(L+1)!^2 \left( \frac{L+1}{2} \right) \cos \omega_r + (\sin \omega_r)^L \left( \frac{1}{\sin \omega_r} \frac{d}{d\omega_r} \right)^L (\omega_r \cot \omega_r) = 0,
\]

\(^{(11)}\) where \( \sin \omega_r = \mu_r/(\mu_r + 1) \). Numerical values of \( \mu_r \) are approximately 8.6185769247 for \( L_P = 1^-, 32.947611782 \) for \( L_P = 2^+ \), and 70.070774958 for \( L_P = 3^-, 119.73121698 \) for \( L_P = 4^-, 181.86643779 \) for \( L_P = 5^- \) etc.

In the following, it is important that the mass-ratio bound is less than this critical value \( \mu_r^* < \mu_r \); therefore, the problem is completely defined by the requirement of self-integrability or, equivalently, by the boundary condition \( f(\rho) \to 0 \) as \( \rho \to 0 \). This is supported by the fact that the numerically calculated critical values \( \mu_B \), where the first bound state appears, are smaller than \( \mu_r \).

Simpler example when the singular inverse square potential determines the self-adjointness of the Hamiltonian is the relativistic Coulomb problem \([23, 24]\). Note also a similar problem that arises if the Minlos-Faddeev method is used to regularize the three-body system with zero-range interaction \([25–30]\).

As it follows from (8), \( \gamma^2(\rho) \geq 1 \) for any \( m/m_1 < \mu_r \), if either \( a > 0 \) and \( P_s = (-)^{L+1} \) or \( a < 0 \), which entails the absence of bound states. Besides, it is trivial that an infinite number of bound states exist for any \( m/m_1 \) in the case \( L_P = 0^+ \) for two identical bosons and a distinct particle \( (P_s = 1) \). Thus, it remains to determine the value \( \mu^* \) only for the positive scattering length \( (a > 0) \) and \( P_s = (-)^L \), i.e., for odd \( L \) and \( P \) (even \( L > 0 \) and \( P \)) if the identical particles are fermions (bosons).

### 3 Non-binding condition

For the determination of the specific value \( \mu^* \), it is sufficient to construct the lower bound \( E_{LB} \) of the exact three-body energy \( E \) and prove that \( E_{LB} \) exceeds the two-body threshold for \( m/m_1 \leq \mu^* \). This will be done in the following three steps.

1. The lower bound \( E_{LB} \) for the ground-state energy \( E \) will be obtained by solving Eq. (10). This follows from the general derivation of the energy lower bound for any Hamiltonian separated into two parts

\[
h = T_1 + V_1(\xi) + T_2 + V_2(\xi, \eta),
\]

\(^{(12)}\) where \( \xi \) and \( \eta \) denote the sets of “slow” and “fast” variables. The kinetic energies \( T_1 \) and \( T_2 \) depend on \( \xi \) and \( \eta \), respectively. The statement is that the lowest eigenvalue \( E_{LB} \) of \( T_1 + V_1(\xi) + E(\xi) \) is the lower bound for all eigenvalues of the initial Hamiltonian \( h = (E_{LB} \leq E) \), where \( E(\xi) \) is the lowest eigenvalue of \( T_2 + V_2(\xi, \eta) \).

To sketch a simple proof, one should notice the inequality,

\[
T_2 + V_2(\xi, \eta) \geq E(\xi),
\]

\(^{(13)}\) i.e., the self-adjoint operator bounded from below exceeds its lowest eigenvalue (lowest threshold), where \( A \geq B \) means \( \langle \phi | A | \phi \rangle \geq \langle \phi | B | \phi \rangle \) for any \( \phi \). Using the evident inequality \( A + B \geq A + C \) for \( B \geq C \), one obtains the desired result \( h \geq T_1 + V_1 + E(\xi) \) and \( E \geq E_{LB} \).

This lower bound for the ground-state energy has been discussed many times in the literature, e.g., this line of proof was carried out for the adiabatic description of molecules \([31, 32, 34, 35]\), the \( N \)-body problem within the hyper-spherical framework \([22]\), and the hydrogen atom in a magnetic field \([33]\). These arguments will be applied to the problem under consideration taking the hyper-radius \( \rho \) as a “slow” variable and the hyper-angles \( (\alpha, \mathbf{x}, \mathbf{j}) \) as “fast” variables. Furthermore, the part of the kinetic-energy operator depending on the hyper-radius
corresponds to $T_1$, the potential $V_1 = 0$, $T_2 + V_2$ is determined by Eqs. (3) and (4), and $\gamma^2 / \rho^2$ corresponds to $\varepsilon(\rho)$. As a result, the eigenvalue equation for $T_1 + V_1 + \varepsilon(\rho)$ reduces to Eq. (10), whose solution provides the energy lower bound.

(2) Let us introduce the reference Hamiltonian $h_\gamma = -d^2 / dx^2 - 1 / 4x^2$ supplemented with the restriction on the functions $\varphi(x)$ to satisfy the condition $\varphi \to x^{1/2}[1 + O(x)]$. It is well known that $h_\gamma$ is non-negative, i.e., $\langle \varphi | h_\gamma | \varphi \rangle \geq 0$ for any $\varphi$, as $h_\gamma$ is the radial part of the two-dimensional kinetic-energy operator. Thus, the spectra of $h_\gamma$ as well as of any operator $\hat{h}$ are pure continuous and there are no bound states, if $h \geq h_\gamma$. Nevertheless, a bound state of $h_\gamma + V(x)$ arises for an arbitrarily small $V(x)$ provided $\int V(x) x dx \leq 0$, and it is known that the bound-state energy is exponentially small for $V(x) \to 0$ [19].

(3) Determining whether the operator in Eq. (10) exceeds the reference Hamiltonian $h_\gamma$, one comes to the condition providing the absence of bound states. As $\gamma^2(\rho)/\rho^2$ tends to the two-body threshold $\varepsilon_2 = -1$ for $\rho \to \infty$, the bound states do not exist if $\gamma^2(\rho)/\rho^2 \geq -1$ for any $\rho$. The final result is

$$\rho(i \kappa) - \kappa \geq 0,$$

where $\gamma = i \kappa$ and the function $\rho(\gamma)$ is defined in transcendental Eq. (8). For $P = P_s = (-1)^L$ and $a > 0$ the condition (14) takes the form $B_L(\kappa, \omega) \geq 0$, where

$$B_L(\kappa, \omega) = 2\Gamma\left(1 + \frac{i \kappa}{2}\right) - \frac{1}{2\sin \omega \frac{d}{d \omega}} \left[ \frac{1}{2} \sinh \kappa \omega \sin \frac{\kappa \pi}{2} \right] \left( \frac{L - i \kappa + 1}{2} \right) \left( \frac{L + i \kappa + 1}{2} \right),$$

Finally, to determine the mass-ratio bound $\mu^*$, one needs to find $\omega^*$, for which the condition $B_L(\kappa, \omega) \geq 0$ is valid for any $0 < \omega < \omega^* < \pi / 2$ and $\kappa > 0$.

3.1 Fermionic system in the sector $L^P = 1$−

In the system of identical fermions and a distinct particle, the three-body bound state of $L^P = 1$− arises for the smallest mass ratio. Moreover, $L^P = 1$− state is most important for describing the low-energy processes for fermions. The non-binding condition on the mass-ratio, i.e., the inequality $B_1(\kappa, \omega) \geq 0$, takes the form

$$F(\kappa) - G(\kappa, \omega) \geq 0,$$

where $F(\kappa) = (\kappa^2 + 1) \sinh \frac{\kappa \pi}{2} - \kappa^2 \cosh \frac{\kappa \pi}{2}$ and $G(\kappa, \omega) = 2 \frac{\kappa \omega}{\sin 2 \omega} - \frac{\sinh \kappa \omega}{\sin \omega} \sin \frac{\kappa \pi}{2}$.

Firstly, one should prove that $B_1(\kappa, \omega)$ is a monotonically decreasing function of $\omega$ ($0 < \omega < \pi / 2$) for any $\kappa > 0$. The condition $\frac{\partial B_1(\kappa, \omega)}{\partial \omega} \leq 0$ is equivalent to $\frac{\partial G(\kappa, \omega)}{\partial \omega} \geq 0$, which essentially gives

$$\left(\kappa^2 \tan \omega + 2 \cot \omega\right) \tanh \kappa \omega + \kappa \left(\tan^2 \omega - 2\right) \geq 0.$$  

Inequality (17) is evidently fulfilled for $\tan^2 \omega \geq 2$. To proceed further, after simple transformations the inequality is written as

$$\left[4 + \kappa^2(\kappa^2 + 4)z^2 - \kappa^2z^2\right] \sinh^2 \kappa \omega - \kappa^2z(2z - 2)^2 \geq 0,$$

where $z = \tan^2 \omega$ is used for brevity. Using the inequality $\sinh \kappa \omega \geq \kappa^2 \sin^2 \omega \equiv \kappa^2z/(1 + z)$ in Eq. (18), one comes to a simple result

$$\kappa^2 + 3 - z \geq 0,$$

which is fulfilled for any $\kappa$ if $z \equiv \tan^2 \omega \leq 3$. This completes the proof that $\frac{\partial B_1}{\partial \omega} \leq 0$; therefore, the implicit condition $B_1(\kappa, \omega) = 0$ determines the single-valued function $\omega_0(\kappa)$. At last, if one finds

$$\omega^* = \min \omega_0(\kappa), \quad 0 \leq \kappa < \infty$$

and the corresponding mass ratio $\mu^*$, it provides the absence of bound states for any $\omega \leq \omega^*$ (consequently, for $m/m_1 \leq \mu^*$).

The function $\omega_0(\kappa)$, as shown in Fig. 1, has one minimum at $\kappa^*$ and its value $\omega^* = \omega_0(\kappa^*)$ determines the mass ratio $\mu^*$.

Numerical values $\kappa^*$, $\omega^*$, and $\mu^*$ are presented in Table 1. The condition $m/m_1 \leq \mu^*$ in the sector $L^P = 1$− means a complete unbinding of the three-body system. As it was claimed in Sect. 2, the condition $\mu^* < \mu_r$ is valid, which confirms the self-consistency of the procedure.
for three-body systems containing either two identical fermions or two non-interacting bosons. The non-binding conditions were

Analogously to the preceding Sect. 3.1, one can suppose that identical fermions (bosons) and a distinct particle. Using the described approach, the values of the positive two-body scattering length $a$ provides the absence of bound states for any mass ratio below as an inequality for the function of two variables. It is proven that two fermions and a distinct particle are not bound for any mass ratio below $<\omega<\pi/2$ for an angular momentum from $0$. Thus, the condition $B_L(\kappa, \omega) = 0$ again determines the single-valued function $\omega_0(\kappa)$ and finding its global minimum $\omega^*$ yields the absence of bound states for any $\omega < \omega^*$, respectively, for $m/m_1 < \mu^*$.

The expressions for $B_L(\kappa, \omega)$ become lengthy and difficult to handle for higher $L$. In particular, for the three-body system containing two identical bosons in the sector $L^P = 2^+$,

$$B_L(\kappa, \omega) = \frac{(1 + \kappa^2)\pi}{2 \sinh \kappa \pi} \left[ \frac{\kappa(\kappa^2 + 4)}{\kappa^2 + 1} \cosh \frac{\kappa \pi}{2} - \kappa \sinh \frac{\kappa \pi}{2} \right. + \left. 3 \frac{\kappa \cosh \kappa \omega - \sinh \kappa \omega \cot \omega}{(\kappa^2 + 1) \sin^2 \omega} \right] \frac{\sinh \kappa \omega}{\sin 2\omega}.$$  \hspace{1cm} (21)

Numerical calculations reveal that for all $0 < L \leq 5$ the functions $\omega_0(\kappa)$ exhibit one minimum, as shown in Fig. 1. The positions of these minima $(\kappa^*, \omega^*)$ are calculated and presented in Table 1 jointly with the corresponding values $\mu^*$. All the values $\mu^*$, $\omega^*$, and $\kappa^*$ increase with $L$, thus reflecting the general trend for the $L$-dependence of the critical mass-ratio value $\mu_B$ at which the first bound state appears. For comparison, the result of numerical calculations [2, 20] of $\mu_B$ is presented in the last column of Table 1.

| $L^P$ | $\kappa^*$ | $\omega^*$ | $\mu^*$ | $\mu_B$ |
|------|-----------|-----------|--------|--------|
| $1^-$ | 2.17701   | 0.997755  | 5.26002| 8.17259|
| $2^+$ | 3.30822   | 1.243618  | 17.85119| 22.6369|
| $3^-$ | 4.51245   | 1.340135  | 36.75782| 43.3951|
| $4^+$ | 5.74050   | 1.392347  | 61.97274| 70.457 |
| $5^-$ | 6.97890   | 1.425184  | 93.49356| 103.823|

3.2 Angular momenta $L \geq 2$

As the case $L^P = 1^-$ provides a complete unbinding of the three-body system, it is also of interest to derive the corresponding conditions in any $L^P$ sectors. As discussed previously, one should consider odd (even) $L$ and $P$ correspond to the system containing two identical fermions (bosons) and a distinct particle. Using the described approach, the values $\mu^*$ will be determined below for $L \leq 5$.

He Analogously to the preceding Sect. 3.1, one can suppose that $B_L(\kappa, \omega)$ in Eq. (15) monotonically decrease with increasing $0 < \omega < \pi/2$ for any $\kappa > 0$. Thus, the condition $B_L(\kappa, \omega) = 0$ again determines the single-valued function $\omega_0(\kappa)$ and finding its global minimum $\omega^*$ provides the absence of bound states for any $\omega < \omega^*$, respectively, for $m/m_1 < \mu^*$.

The expressions for $B_L(\kappa, \omega)$ become lengthy and difficult to handle for higher $L$. In particular, for the three-body system containing two identical bosons in the sector $L^P = 2^+$,

$$B_L(\kappa, \omega)\left[ \frac{(1 + \kappa^2)\pi}{2 \sinh \kappa \pi} \left[ \frac{\kappa(\kappa^2 + 4)}{\kappa^2 + 1} \cosh \frac{\kappa \pi}{2} - \kappa \sinh \frac{\kappa \pi}{2} \right. + \left. 3 \frac{\kappa \cosh \kappa \omega - \sinh \kappa \omega \cot \omega}{(\kappa^2 + 1) \sin^2 \omega} \right] \frac{\sinh \kappa \omega}{\sin 2\omega} \right].$$  \hspace{1cm} (21)

Numerical calculations reveal that for all $0 < L \leq 5$ the functions $\omega_0(\kappa)$ exhibit one minimum, as shown in Fig. 1. The positions of these minima $(\kappa^*, \omega^*)$ are calculated and presented in Table 1 jointly with the corresponding values $\mu^*$. All the values $\mu^*$, $\omega^*$, and $\kappa^*$ increase with $L$, thus reflecting the general trend for the $L$-dependence of the critical mass-ratio value $\mu_B$ at which the first bound state appears. For comparison, the result of numerical calculations [2, 20] of $\mu_B$ is presented in the last column of Table 1.

The relative difference $\frac{\mu^* - \mu_B}{\mu_B}$ decreases from 0.36 to 0.1 for increasing angular momentum from $L = 1$ to $L = 5$. Again, it is possible to confirm the statement in Sect. 2 that $\mu^* < \mu_r$ for the considered total angular momenta.

4 Conclusion

Using the one-channel approximation for a system of hyper-radial equations, the non-binding condition for three particles is written as an inequality for the function of two variables. It is proven that two fermions and a distinct particle are not bound for any mass ratio below $\mu^* = 5.26$. This bound is sufficiently close to the result of numerical calculations [2, 9] for the mass ratio $\mu_B \approx 8.17259$, at which the first bound state arises. This non-binding condition on the mass ratio was obtained by considering the most important case of describing the low-energy processes for fermions, namely, states of the total angular momentum and parity $L^P = 1^-$ for the positive two-body scattering length $a > 0$. The alternative lower bound $\mu^* \approx 2.617$ was obtained by the rigorous analysis of the momentum-space integral equations [10] (Theorem 2.5) and is compatible with the present results of $\mu^*$.

Furthermore, the same procedure was used to find non-binding conditions also for states of higher total angular momenta $L \leq 5$ for three-body systems containing either two identical fermions or two non-interacting bosons. The non-binding conditions were
determined for odd (even) \( L \) and \( P \) for the system containing fermions (bosons). As expected, the one-channel approximation works better for higher \( L \), which leads to better agreement between \( \mu^* \) and the numerically calculated \( \mu_R \) at which the first bound state arises in a given \( L^P \) sector. The described approach seems to be useful for derivation the lower bounds in the other few-body problems.

**Data Availability** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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