GLUING PRINCIPLE FOR ORBIFOLD STRATIFIED SPACES

BOHUI CHEN, AN-MIN LI AND BAI-LING WANG

ABSTRACT. In this paper, we explore the theme of orbifold stratified spaces and establish a general criterion for them to be smooth orbifolds. This criterion utilizes the notion of linear stratification on the gluing bundles for the orbifold stratified spaces. We introduce a concept of good gluing structure to ensure a smooth structure on the stratified space. As an application, we provide an orbifold structure on the coarse moduli space $\overline{M}_{g,n}$ of stable genus $g$ curves with $n$-marked points. Using the gluing theory for $\overline{M}_{g,n}$ associated to horocycle structures, there is a natural orbifold gluing atlas on $\overline{M}_{g,n}$. We show this gluing atlas can be refined to provide a good orbifold gluing structure and hence a smooth orbifold structure on $\overline{M}_{g,n}$. This general gluing principle will be very useful in the study of the gluing theory for the compactified moduli spaces of stable pseudo-holomorphic curves in a symplectic manifold.

CONTENTS

1. Introduction and statements of main theorems 1
2. Smooth structures on stratified orbifolds 5
  2.1. Linearly stratified (Euclidean) spaces 5
  2.2. Gluing principle for manifold stratified spaces 8
  2.3. Gluing theorem for orbifold stratified spaces 15
3. Moduli spaces of stable curves as orbifold stratified spaces 20
  3.1. Teichmüller space and moduli space of Riemann surfaces (top stratum) 20
  3.2. Canonical construction of proper ’etale groupoids 22
  3.3. Moduli space of stable curves 24
4. Horocycle structures associated to marked or nodal points 29
5. Gluing data and good orbifold gluing structures for moduli spaces of stable curves 31
References 36

1. INTRODUCTION AND STATEMENTS OF MAIN THEOREMS

Assuming that we have the transversality for each moduli space of stable maps in a symplectic manifold $(X, \omega)$ with a domain of a fixed topological type, the compactified moduli space $\overline{M}_{g,n}(X, A, \omega, J)$ of stable maps in a symplectic manifold $(X, \omega)$ is usually a compact Hausdorff space stratified by smooth orbifolds for a compatible almost complex structure $J$ and $A \in H_2(X, \mathbb{Z})$. This moduli space is an example of the so-called orbifold stratified spaces in this paper, which is a disjoint union of locally closed smooth orbifolds indexed by a partial ordered set. A topological orbifold structure on $\overline{M}_{g,n}(X, A, \omega, J)$ can be obtained (for example, cf. [15]). Since then, there are further interests in study the smooth orbifold structure
There is a similar issue for other compactified moduli spaces arising from geometric elliptic partial differential equations. The central part in the study of these moduli spaces is the gluing analysis for the lower stratum. The motivational question is how much the gluing analysis we need in order to obtain a smooth structure on the moduli space. In this paper, we establish a general criterion for an orbifold stratified space to admit a smooth structure based on the gluing theory.

**Definition 1.1.** An \( n \)-dimensional orbifold stratified space is a topological space \( M \) which admits a stratification

\[
M = \bigsqcup_{\alpha \in S} M_{\alpha},
\]

a disjoint union of locally closed subspaces (called strata) indexed by a partially ordered set \((S, \prec)\) such that

1. The decomposition (1.1) is locally finite in the sense that each point \( x \in M \) has a neighbourhood \( U_x \) such that \( U_x \cap M_\alpha \) is empty except for finitely many \( \alpha \).
2. Denote by \( \overline{M}_\beta \) the closure of \( M_\beta \) in \( M \). Then \( M_\alpha \cap \overline{M}_\beta \neq \emptyset \iff M_\alpha \subset \overline{M}_\beta \iff \alpha \prec \beta \).
3. For each \( \alpha \in S \), the stratum \( M_\alpha \) has a smooth orbifold structure given by a proper étale Lie groupoid

\[
M_\alpha = (M^1_\alpha \rightrightarrows M^0_\alpha),
\]

that is, \( M_\alpha \) is the orbit space of \( M_\alpha \), also called the coarse space of \( M_\alpha \).
4. The top stratum of \( M \) is an \( n \)-dimensional smooth orbifold.

In this paper, we assume that \( M \) is compact and \( S \) is finite. Motivated by the gluing analysis for the compactified moduli space, such as \( \overline{M}_{g,n}(X, A, \omega, J) \), we propose a notion of linearly stratified vector spaces and linearly stratified vector bundles in Section 2. A prototype of a linearly stratified vector space is \( V = \mathbb{C}^m \) with a stratification given by

\[
V = \bigsqcup_{I \in S} V^{(I)}
\]

with respect to the naturally ordered power set

\[
S = 2^{\{1,2,\ldots,m\}} = \{ I \subset \{1,2,\ldots,m\} \}.
\]

Here \( V^{(I)} = \{(t_1, t_2, \ldots, t_m) | t_i \neq 0 \iff i \in I \} \), whose closure in \( V \) is a linear subspace of dimension \( |I| \). One could recover the topological structure or smooth structure on \( V \) by identifying each linear tubular neighbourhood of \( V^{(I)} \) in \( V \) with a normal bundle \( N(V^{(I)}) \) of \( V^{(I)} \) in \( V \). See Definitions 2.1 and 2.3 for precise definitions of linearly stratified vector spaces and linearly stratified vector bundles. This notion of linear stratifications can be generalised to an orbifold vector space (an Euclidean vector spaces with a linear action of a finite group) and an orbifold vector bundles.
In order to obtain a smooth orbifold structure on orbifold stratified space $M$ with respect to a finite partially ordered index set $\mathcal{S}$, we first assume that the disjoint union

$$M = \left( \bigsqcup_{\alpha \in \mathcal{S}} M^1_\alpha \right) \cup \left( \bigsqcup_{\alpha \in \mathcal{S}} M^0_\alpha \right)$$

is a topological groupoid for simplicity. This assumption can be dropped in actual applications. We introduce a gluing datum on $M$, see Section 2 for details, which briefly consists of

1. an orbifold linearly stratified smooth bundle $\text{Gl}^\alpha \rightarrow M_\alpha$ with respect to the index set $S^\alpha = \{ \beta \in \mathcal{S} | \beta < \alpha \}$, called a gluing bundle for each $\alpha \in \mathcal{S}$;
2. and a stratum-preserving strict groupoid homomorphism $\phi^\alpha : \text{Gl}^\alpha(\varepsilon)|_U \rightarrow M$, called a gluing map, for each open (full) subgroupoid $U$ of $M_\alpha$, here a metric $\eta$ is chosen on $\text{Gl}^\alpha(\varepsilon)|_U$, so the $\varepsilon$-neighbourhood of the zero section of $\text{Gl}^\alpha|_U$, is defined; moreover, the gluing map is required to satisfy the following conditions:
   a) $\phi^\alpha$ is a Morita equivalence of topological groupoids from $\text{Gl}^\alpha(\varepsilon)|_U$ to the full-subgroupoid generated by the image of $\phi^\alpha$ in $M$;
   b) for each $\beta \in S^\alpha$, the stratum-wise gluing map
      $$\phi^\alpha_\beta : \text{Gl}^\alpha(\varepsilon)|_U \rightarrow M_\beta$$
      is a Morita equivalence of Lie groupoids from $\text{Gl}^\alpha(\varepsilon)|_U$ to the full-subgroupoid generated by the image of $\phi^\alpha$ in $M_\beta$;
3. a collection of stratum-preserving smooth bundle isomorphism maps which preserve the induced stratifications
   $$\Phi^\alpha_\beta : N(\text{Gl}^\alpha_\beta) \rightarrow \text{Gl}^\beta$$
   that covers $\phi^\alpha_\beta$ for any $\beta \in S^\alpha$.

We remark that a Morita equivalence of proper étale Lie groupoid ensures that these two Lie groupoids are locally isomorphic, hence define the same orbifold structure on their coarse spaces.

Note that as $\text{Gl}^\alpha(\varepsilon)|_U$ is a smooth orbifold bundle over a smooth orbifold $U$, the gluing datum over $U$ defines a smooth orbifold structure on the image of the coarse gluing map

$$|\phi^\alpha| : |\text{Gl}^\alpha(\varepsilon)|_U \rightarrow |M| = M.$$  

A gluing atlas $\mathcal{F}$ of $M$ is a collection of gluing data such that the images of all the coarse gluing maps in $\mathcal{F}$ form an open cover of $M$. We show that a gluing atlas defines a canonical topological orbifold structure on $M$. In order to achieve a smoothly compatible gluing atlas, we further impose three conditions on a gluing atlas $\mathcal{F}$ in Section 2:

(i) $\mathcal{F}$ is closed under the induction for restriction maps and gluing maps;
(ii) $\mathcal{F}$ satisfies the sewing properties;
(iii) $\mathcal{F}$ satisfies an inward-extendibility condition.

The resulting atlas will be called a good orbifold gluing structure.
The main theorem of this paper is the following canonical orbifold structure on orbifold stratified space $M$ equipped with a good orbifold gluing structure.

**Theorem A** (Theorem 2.22 and Remark 2.23) Let $M$ be an orbifold stratified space with a good orbifold gluing structure $\mathcal{F}$, then $M$ admits a smooth orbifold structure such that each stratum is a smooth sub-orbifold.

In practice, one could just assume that $M$ is a disjoint union of smooth orbifolds indexed by a partially ordered set. Then the topology on $M$ can be obtained as a by-product of the underlying gluing data where we drop the condition of homeomorphisms of gluing maps and replace it with locally bijective maps on the set-theoretical level. This remark is particularly important when in applications, we often encounter that the disjoint union

$$\bigsqcup_{\alpha \in S} \bigcup \alpha \in S M_\alpha \Rightarrow \bigsqcup_{\alpha \in S} M_\alpha^0$$

does not a priori form a topological groupoid. We have to replace this by a set-theoretically Morita equivalent groupoid which admits a topological or smooth orbifold structure on $M$.

Another technical issue is how to achieve the inward-extension property in specific applications such as in the study of $\overline{M}_{g,n}(X, A, \omega, J)$. This is the main motivation to develop a general principle for an orbifold stratified space, and will be addressed in a separate paper.

In the rest of the paper, as an application of Theorem A, we revisit the orbifold structure on the coarse moduli space $\overline{M}_{g,n}$ of stable curves of genus $g$ with $n$-marked points. The orbifold structure on $\overline{M}_{g,n}$ have been constructed by various method in algebraic geometry [2] [10] [20] [18], and in differential geometry by [15] [30]. Our construction seems more like an a posteriori treatment of the known orbifold structure on $\overline{M}_{g,n}$. Nevertheless the novelty of our construction is to apply the horocycle structures at marked and nodal points to investigate the gluing datum for $\overline{M}_{g,n}$. These horocycle structures satisfy the convex property which is vital to get the inward-extendibility for a good orbifold gluing structure.

In Section 3, we give a preliminary review of some basics about the moduli space of stable curves. We employ the universal curves over Teichmüller space $\mathcal{T}_{g,n}$ to get an orbifold structure on the coarse moduli space $M_{g,n}$. Proposition 3.9 is a well-known result about the orbifold stratified structure on the compactified moduli space $\overline{M}_{g,n}$ with respect to a partially ordered index set $S_{g,n}$ of weighted dual graphs of type $(g,n)$. We include a proof of this Proposition in order to make this paper as self-contained as possible. In particular, we like to point out that each stratum $M_{[\Gamma]}$ for $[\Gamma] \in S_{g,n}$ admits a canonical proper étale groupoid $M_{[\Gamma]}$, but the disjoint union $\bigsqcup_{[\Gamma] \in S_{g,n}} M_{[\Gamma]}$ does not have any topological structure.

In Section 4, we introduce a horocycle structure on a stable nodal Riemann surface $C$ at a special point $p$ (a marked point or a nodal point). By a horocycle structure of $C$ at a smooth point $p$ is a triple $(\eta, \delta, h)$, where $\eta$ is a metric on the tangent space $T_pC$, $\delta > 0$ is a small constant and $h$ is a locally defined smooth map

$$h : T_pC(\delta) \rightarrow C$$
such that $h_p(0) = p$ and the differential of $h$ at the origin is the identity operator. A hyperbolic metric on the punctured Riemann surface obtained by removing these special points from $C$ defines a canonical horocycle structure on $C$. Horocycle structures used in this paper are small perturbations of those canonical horocycle structure. We remark that the convexity property on the collection of horocycle structures enables us to get a gluing atlas with the required inward-extension property. In this section, we show that each stratum in $\overline{M}_{g,n}$, there exists a smooth family of horocycle structures in the orbifold sense.

In Section 5, we employ the standard grafting construction to get a gluing atlas $GL$ on $\overline{M}_{g,n}$ using the horocycle structures from Section 4. The main result of Section 5 is to show that $GL$ satisfies the inward-extension property. Then by Theorem A, $\overline{M}_{g,n}$ admits a smooth orbifold structure.

Note that Fukaya and Ono in [15] outlined a gluing argument to provide $\overline{M}_{g,n}$ with a complex orbifold atlas. Therefore, we don’t claim any originality of this result in this paper. What we have done in some sense is to provide a full-fledged gluing theory for $\overline{M}_{g,n}$ which is applicable to $\overline{M}_{g,n}$ to get a smooth orbifold atlas on $\overline{M}_{g,n}$. In particular, as we commented earlier, this good gluing structure will be very useful in the study of gluing theory for the compactified moduli space of stable maps.

2. Smooth structures on stratified orbifolds

In this section, we will establish a general criterion for an orbifold stratified space $M$ to admit a smooth structure. In fact, motivated by the gluing theory on various moduli spaces arising from other geometric problems, we introduce the concept of gluing atlas on stratified space $M$. When $M$ admits a gluing atlas, it has a topological orbifold structure on $M$ automatically. In order to achieve the smooth compatibility we introduce the so-called “inward-extendibility” condition (cf. Definition 2.10) on a gluing atlas. We call a gluing atlas with such extension property a good gluing structure. The main theorem in this section is to establish Theorem A in the Introduction, that is, $M$ admits a smooth orbifold structure if it has a good gluing structure. We begin with the manifold stratified space first, as the arguments for this case can be adapted easily to orbifold stratified spaces.

2.1. Linearly stratified (Euclidean) spaces.

Let $K$ be the ground field, like $\mathbb{R}$ or $\mathbb{C}$. Denote $K^\times = K \setminus \{0\}$. Given a linear $K$-vector space $V$ of dimension $m$ with a fixed identification $V \cong K^m$ given by coordinate functions $(t_1, \ldots, t_m)$ on $V$ associated to a basis.

Let $\mathcal{N}$ be the power set $2^{\{1, \ldots, m\}}$ with a partial order given by the inclusion. For any $I \in \mathcal{N}$, we set

$$V[I] = \{x \in V| t_i(x) \neq 0 \iff i \in I\}.$$
Since $V^{|I|} \cong (\mathbb{K}^\times)^{|I|}$, we call it a $\mathbb{K}^\times$-space. Let $V^{|\emptyset|} = \{0\}$ and $V^{|I|} = V^{|I|c}$. Then $V = V^I \times V^{|I|c}$. Here $I^c$ is the compliment of $I$ in $\{1, \ldots, m\}$. The vector space $V$ has a canonical stratification

\begin{equation}
V = \bigsqcup_{I \subset \{1, 2, \ldots, m\}} V^{|I|}
\end{equation}

with respect to the power set $\mathcal{N}$ in the sense of Definition 1.1.

The normal bundle of $V^{|I|}$ in $V$, denoted by $N(V^{|I|})$, can be canonically identified with

$N(V^{|I|}) = V^{|I|} \times V^{|I|c}$.

With respect to the obvious inclusion $N(V^{|I|}) \subset V$, the canonical stratification of $V$ induces a fiber-wise stratification on the vector bundle $N(V^{|I|}) \to V^{|I|}$.

With these preparations, now we introduce a notion of a linear stratification on a vector space.

**Definition 2.1.** Let $V$ be a $\mathbb{K}$-linear vector space of dimension $m$ as above. We call it a *linearly stratified space* with respect to a partially indexed set $(\mathcal{S}, \prec)$ if $V$ admits a stratification

\begin{equation}
V = \bigsqcup_{\alpha \in \mathcal{S}} V_\alpha,
\end{equation}

with $V_\alpha = (\bigsqcup_{I \in \mathcal{J}_\alpha} V^{|I|})$ for $\mathcal{J}_\alpha \subset \mathcal{N}$, such that for any $\alpha \in \mathcal{S}$, all elements in $\mathcal{J}_\alpha$ have the same cardinality. Denote by $(V, \mathcal{S})$ the linearly stratified space $V$ with respect to $\mathcal{S}$. Given a linearly stratified $\mathbb{K}$-vector space $(V, \mathcal{S})$, the group of all invertible linear transformation which preserves the linear stratification is denoted by $GL(V, \mathcal{S}) = \{g \in GL(V) | g(V_\alpha) = V_\alpha, \text{ for any } \alpha \in \mathcal{S}\}$, called the *stratified general linear group* of $(V, \mathcal{S})$.

Note that the stratification (2.2) is completely determined by the partially index set $\mathcal{S}$. The following two conditions hold for $(\mathcal{S}, \prec)$.

1. $2\{1, 2, \ldots, m\} = \bigsqcup_{\alpha \in \mathcal{S}} \mathcal{J}_\alpha$, that is, the collection of $\{\mathcal{J}_\alpha\}_{\alpha \in \mathcal{S}}$ for a partition of the power set $\mathcal{N}$.
2. $\alpha \prec \beta \iff \text{for any } I \in \mathcal{J}_\alpha, \text{ there exists } J \in \mathcal{J}_\beta \text{ such that } I \subset J$.

In the following discussion, it might be helpful to keep one example in mind, such as a linear stratification on $V = \mathbb{R}^2$ with respect to $\mathcal{S} = \{\emptyset, \{1\}, \{2\}\}$.

Given a linearly stratified space $(V, \mathcal{S})$. Let $N(V_\alpha)$ be the component-wise normal bundle of $V_\alpha$ in $V$, that is,

$N(V_\alpha) = \bigsqcup_{I \in \mathcal{J}_\alpha} N(V^{|I|})$.

As each $N(V^{|I|})$ has an induced linear stratification given by

\begin{equation}
(N(V^{|I|}))_\beta = N(V^{|I|}) \cap V_\beta = \bigsqcup_{J \in \mathcal{J}_\beta, J \supseteq I} V^{|J|} \subset V_\beta,
\end{equation}
with respect to $S^\alpha = \{ \beta | \beta \geq \alpha \}$. This provides a fibre-wise linear stratification on vector bundle

$$N(V) = \bigsqcup_{\beta \in S^\alpha} (N(V))_\beta$$

over $V$ with respect to $S^\alpha$ with

$$\tau_{\alpha \beta} : (N(V))_\beta \to V \beta$$
given by the obvious inclusion (2.3) of each component in $V \beta$ for $I \in J_\alpha$. For simplicity, we call a map of this type a component-wise embedding.

**Lemma 2.2.** Let $(V, S)$ be a linearly stratified vector space and $\alpha \prec \beta$ in $S$. Then the normal bundle of $(N(V))_\beta$ in $N(V)$ is

$$N(N(V))_\beta = \bigsqcup_{I \in J_\alpha} \bigsqcup_{J \in J_\beta, J \supseteq I} N(V) [J]$$

with a canonical component-wise embedding

$$\tau^{\nu}_{\alpha \beta} : N(N(V))_\beta \to N(V) \beta.$$ 

**Proof.** Let $I \subset J_\alpha$ and $J \in J_\beta$, such that $J \supseteq I$, we have $V [J] \subset N(V[I])$. The key fact for this lemma is that the normal bundle of $V[J]$ in $N(V[I])$ is same as $N(V[J])$. As $(N(V[I]))_\beta$ consists of the disjoint union of $V [J]$ for $J \in J_\beta$ with $J \supseteq I$, this implies that the normal bundle of $(N(V[I]))_\beta$ in $N(V[I])$ is given by the disjoint union of the normal bundle $N(V[J])$ for $J \in J_\beta, J \supseteq I$, that is,

$$N(N(V[I]))_\beta = \bigsqcup_{J \in J_\beta, J \supseteq I} N(V[J]).$$

Then (2.6) is an easy consequence of this identity. The canonical component-wise embedding is also obvious. $\square$

**Definition 2.3.** A $\mathbb{K}$-vector bundle $E$ over a smooth manifold $X$ with fiber $V$ is linearly stratified if $V$ admits a linear stratification with respect to a partially ordered index set $(S, \prec)$ such that the structure group can be reduced to $GL(V, S)$.

A metric $\eta$ on $V$ is compatible with the stratification in the sense that the coordinate function $V \cong \mathbb{K}^m$ is with respect to an orthonormal basis. Then we can define

$$SO(V, S) = \{ g \in SO(V, \eta) | g(V_\alpha) = V_\alpha, \text{ for any } \alpha \in S \}.$$ 

We may equip a linearly stratified vector bundle $E$ with a compatible metric if the structure group can be reduced from $GL(V, S)$. 
From Definition 2.1, we know that a linearly stratified vector bundle $E$ has a fiberwise linear stratification with respect to $(S, \prec)$

$$E = \bigcup_{\alpha \in S} E_{\alpha}.$$  

Note that $E_{\alpha}$ is not a $K$-vector bundle as the fiber of $E_{\alpha}$ is only a $K^\times$-vector space. There is a canonical bundle $N(E_{\alpha})$ over $E_{\alpha}$, the normal bundle of the inclusion $E_{\alpha} \subset E$, with the induced linear stratification

$$N(E_{\alpha}) = \bigcup_{\beta \in S^{\alpha}} (N(E_{\alpha}))_{\beta}$$  

with respect to $S^{\alpha}$. By Lemma 2.2 we know there is a component-wise embedding

$$(2.8) \quad \tau_{\alpha \beta} : N(N(E_{\alpha}))_{\beta} \rightarrow N(E_{\beta}).$$

Given two linearly stratified $K$-vector bundles $E$ and $F$ over $X$ with the same index set $(S, \prec)$, a bundle map $\phi : E \rightarrow F$ is called strata-preserving if for any $\alpha \in S$,

$$\phi(E_{\alpha}) \subseteq F_{\alpha}.$$  

Then one can check that the induced map on the normal bundles

$$\phi : N(E_{\alpha}) \rightarrow N(F_{\alpha})$$

is also strata-preserving.

2.2. **Gluing principle for manifold stratified spaces.** An $m$-dimensional manifold stratified space is a topological space $M$ which admits a stratification

$$(2.9) \quad M = \bigcup_{\alpha \in S} M_{\alpha},$$

a disjoint union of locally closed smooth manifolds (called strata) indexed by a partially ordered set $(S, \prec)$ such that

1. the dimension of its top stratum is $m$;
2. the decomposition (2.9) is locally finite in the sense that each point $x \in M$ has a neighbourhood $U_x$ such that $U_x \cap M_{\alpha}$ is empty except for finitely many $\alpha$;
3. $M_{\alpha} \cap \overline{M}_\beta \neq \emptyset \iff M_{\alpha} \subseteq \overline{M}_\beta \iff \alpha \prec \beta.$

We always assume that $M$ is compact and $S$ is finite. It is easy to see that

$$\overline{M}_\alpha \setminus M_{\alpha} \subseteq \bigcup_{\beta \in S^{\alpha}} M_{\beta}$$

where $S^{\alpha} = \{ \beta | \beta \prec \alpha \}$. We set

$$M^{\alpha} = \bigcup_{\beta \in S^{\alpha}} M_{\beta}$$

where $S^{\alpha} = \{ \beta | \alpha \preceq \beta \}$. Then $M^{\alpha}$ is a subspace of $M$ that is stratified by $S^{\alpha}$.

Motivated by the gluing theory for the moduli spaces of stable maps, we impose the following conditions on the stratification (2.9).
**Condition A: (Existence of gluing bundles)** For any $\alpha \in \mathcal{S}$, there is a linearly stratified smooth vector bundle $G\ell^{\alpha}$ over $M_{\alpha}$ with respect to $S^{\alpha}$, write

$$G\ell^{\alpha} = \bigsqcup_{\beta \in S^{\alpha}} G\ell^{\alpha}_{\beta};$$

moreover the dimension of $G\ell^{\alpha}_{\beta}$ agrees with that of $M_{\beta}$. This bundle is called the gluing bundle over the strata $M_{\alpha}$.

**Remark 2.4.** To be consistent, we allow $\alpha$ to be the maximum element in $\mathcal{S}$, and then $G\ell^{\alpha} = M_{\alpha}$, the trivial bundle with zero dimensional fiber.

The linear stratification on $G\ell^{\alpha}$ induces a linearly stratification on the normal bundle of $G\ell^{\alpha}_{\beta}$ in $G\ell^{\alpha}$ with respect to $S^{\beta}$, written as,

$$N(G\ell^{\alpha}_{\beta}) = \bigsqcup_{\gamma \in S^{\beta}} (N(G\ell^{\alpha}_{\beta}))_{\gamma} \longrightarrow G\ell^{\alpha}_{\beta}.$$

By Lemma 2.2 we know that there are canonical component-wise embeddings

$$\tau_{\beta \gamma}: (N(G\ell^{\alpha}_{\beta}))_{\gamma} \rightarrow G\ell^{\alpha}_{\gamma}; \quad \tau'_{\beta \gamma}: N((N(G\ell^{\alpha}_{\beta}))_{\gamma}) \rightarrow N(G\ell^{\alpha}_{\gamma}).$$

For a linearly stratified vector bundle $E$ with a compatible metric $\eta$, we denote its $\delta$-ball bundle by $E(\delta)$ for any $\delta > 0$.

**Definition 2.5 (Gluing datum).** Let $U$ be any open subset of $M_{\alpha}$. By a gluing datum over $U$ we mean a metric $\eta^{\alpha}$ on $G\ell^{\alpha}|_{U}$, a stratified map (called a gluing map)

$$\phi^{\alpha}: G\ell^{\alpha}(\epsilon)|_{U} \rightarrow M^{\alpha}$$

for some constant $\epsilon > 0$ and a collection of stratum-preserving smooth bundle isomorphisms

$$\Phi^{\alpha}_{\beta}: N(G\ell^{\alpha}_{\beta}) \rightarrow G\ell^{\beta}$$

such that

1. the image of $\phi^{\alpha}$ is open and the map is a homeomorphism onto its image;
2. $\phi^{\alpha}$ is a stratified smooth map with respect to the induced stratifications, i.e, for any $\beta \in S^{\alpha}$, $\phi^{\alpha}$ maps $G\ell^{\alpha}(\epsilon)|_{U}$ to $M_{\beta}$, we denote this map by

$$\phi^{\alpha}_{\beta}: G\ell^{\alpha}(\epsilon)|_{U} \rightarrow M_{\beta},$$

then $\phi^{\alpha}_{\beta}$ is a diffeomorphism onto its image;
3. the bundle isomorphism $\Phi^{\alpha}_{\beta}$ covers $\phi^{\alpha}_{\beta}$ in the sense that the diagram

$$\begin{array}{ccc}
N(G\ell^{\alpha}_{\beta}) & \xrightarrow{\Phi^{\alpha}_{\beta}} & G\ell^{\beta} \\
\downarrow & & \downarrow \\
G\ell^{\alpha}(\epsilon)|_{U} & \xrightarrow{\phi^{\alpha}_{\beta}} & M_{\beta}
\end{array}$$

commutes.
We simply denote this gluing datum over \( U \) by
\[
(U, \eta^\alpha, \epsilon, \phi^\alpha, \Phi^\alpha := \{ \Phi^\alpha_{\beta} \}_{\beta \in S^\alpha})
\]
a gluing datum over \( U \).

**Definition 2.6.** Let \( GL(M) \) be the collection of all gluing data. Let \( F \) be a subset of \( GL(M) \). We call \( F \) a gluing atlas of \( M \) if the image of the gluing maps in \( F \) forms an open cover of \( M \).

**Theorem 2.7.** Suppose that a manifold stratified space \( M \) has a gluing atlas \( F \). Then \( M \) admits a canonical topological manifold structure defined by \( F \).

**Proof.** Note that given a gluing datum
\[
(U, \eta, \epsilon, \phi^\alpha, \Phi^\alpha)
\]
the map
\[
\phi^\alpha : GI^\alpha(\epsilon)|_U \to M^\alpha
\]
gives a manifold topological structure on the image of \( \phi^\alpha \), since \( GI^\alpha(\epsilon)|_U \) is a (smooth) manifold. For any point \( x_\alpha \in M_\alpha \) there exists a small neighborhood \( U_\alpha \) that is proper in \( M_\alpha \), an induced gluing datum \( (U_\alpha, \eta, \epsilon, \phi^\alpha, \Phi^\alpha) \) from a gluing datum in \( F \) by restriction to \( U_\alpha \). Moreover, \( U_\alpha \) can be chosen such that \( GI^\alpha(\epsilon)|_{U_\alpha} \) is trivial over \( U_\alpha \) and is homeomorphic to \( \mathbb{R}^m \). Then
\[
\phi^\alpha : GI^\alpha(\epsilon)|_{U_\alpha} \to M^\alpha \subset M,
\]
defines a coordinate chart for a neighborhood \( U_{x_\alpha} = \phi^\alpha(GI^\alpha(\epsilon)|_{U_\alpha}) \) of \( x_\alpha \) in \( M \), denoted by
\[
\psi_{x_\alpha} = (\phi^\alpha)^{-1} : U_{x_\alpha} \to GI^\alpha(\epsilon)|_{U_\alpha} \simeq \mathbb{R}^m.
\]
So locally we have a topological manifold structure on \( M \). The transition functions
\[
\psi_{x_\alpha} \circ \psi_{y_\beta}^{-1} : \psi_{y_\beta}(U_{x_\alpha} \cap U_{y_\beta}) \to \psi_{x_\alpha}(U_{x_\alpha} \cap U_{y_\beta})
\]
on overlaps \( U_{x_\alpha} \cap U_{y_\beta} \) for any \( y \in M_\beta \) are homeomorphisms of open subsets of \( \mathbb{R}^m \), as each gluing map is a homeomorphism onto its image. Therefore, \( M \) has a canonical topological manifold structure.

The next condition is motivated by standard gluing theory for moduli spaces.

**Condition B: (Existence of gluing data).** For any \( \alpha \in S \) and any proper open subset \( U \subset M_\alpha \) there exists a gluing datum over \( U \).

It is clear that Condition B implies trivially the existence of gluing atlas.

Now we come to the smooth structure on \( M \) which would follow if we have a \( C^\infty \)-compatible gluing atlas in the sense that all the transition functions \( \psi_{x_\alpha} \circ \psi_{y_\beta}^{-1} \) in the proof of Proposition 2.7 are diffeomorphisms of subsets of \( \mathbb{R}^m \). We begin with the following observations for \( GL(M) \).
(i) (Induction for restriction maps) Suppose that \((U, \eta^\alpha, \epsilon, \phi^\alpha, \Phi^\alpha)\) is a gluing datum, then for any open subset \(U' \subseteq U\) and \(0 < \epsilon' \leq \epsilon\), by taking the restriction of maps we have an obvious induced gluing datum \((U', \eta^\alpha, \epsilon', \phi^\alpha, \Phi^\alpha)\). Clearly, two coordinate charts are \(C^\infty\)-compatible if one of their associated gluing datum is obtained from the restriction of the other gluing datum.

(ii) (Induction for gluing maps) Suppose that \(\alpha \prec \beta\) be a pair in \(S\), let 
\[
(U, \eta^\alpha, \epsilon, \phi^\alpha, \Phi^\alpha)
\]
be a gluing datum over \(U \subset M_\alpha\). Fix any \(\beta \in S_\alpha\), and let \(D(\alpha, \beta)\) and \(R(\alpha, \beta)\) be the domain and image of \(\phi^\beta\). Then for any proper open subset \(U^\beta \subset R(\alpha, \beta)\) we have gluing data over \(U^\beta\) defined as the following. Note that 
\[
\Phi^\alpha_\beta : N(Gl^\alpha_{\beta}) \to Gl^\beta
\]
is a bundle isomorphism covering 
\[
\phi^\alpha_\beta : D(\alpha, \beta) \to R(\alpha, \beta).
\]
The metric \(\eta\) on \(Gl^\alpha\) induces a metric on \(N(Gl^\alpha_{\beta})\). Under the bundle isomorphism \(\Phi^\alpha_\beta\), we get a metric on \(Gl^\beta\)\(\mid_{R(\alpha, \beta)}\) denoted by \(\eta^\beta\). Define 
\[
\psi : Gl^\beta(\epsilon')\mid_{U^\beta} \to Gl^\alpha(\epsilon)\mid_U
\]
to be the composition of \((\Phi^\alpha_\beta)^{-1}\) and the canonical component-wise embedding \(\tau\). We can choose \(\epsilon'\) small enough such that 
\[
\psi : Gl^\beta(\epsilon')\mid_{U^\beta} \to Gl^\alpha(\epsilon)\mid_U
\]
is in fact an embedding. Set
\[
(2.11) \quad \phi^\beta = \phi^\alpha \circ \psi, \quad \Phi^\gamma_\beta = \Phi^\alpha_\beta \circ \psi.
\]
Then \((U^\beta, \eta^\beta, \epsilon', \phi^\beta, \Phi^\beta)\) is a gluing datum over \(U^\beta\). It is also easy to check that two coordinate charts are \(C^\infty\)-compatible if one of their associated gluing datum is obtained by the induction for the gluing map in the other gluing datum. This follows from the fact that, under the canonical component-wise embeddings \(\tau_{\beta\gamma}\) and \(\tau_{\beta\gamma}'\) in (2.10), we have 
\[
(2.12) \quad \phi^\alpha = \phi^\beta \circ \Phi^\alpha_\beta, \quad \Phi^\gamma_\alpha = \Phi^\beta_\gamma \circ \Phi^\alpha_\beta.
\]
for \(\alpha \prec \beta \prec \gamma\) in \(S\).

(iii) (Sewing property) Let 
\[
(U_k, \eta_k, \epsilon_k, \phi^\alpha_k, \Phi^\alpha_k), k = 1, 2
\]
be two gluing data, where \(U_1\) and \(U_2\) are both open subsets of \(M_\alpha\). We say that they coincide if, on the intersection domain \(V = U_1 \cap U_2\),

- \(\eta_1 = \eta_2\) on \(Gl^\alpha\)\(\mid_V\); and
- \(\phi^\alpha_1 = \phi^\alpha_2, \Phi^\alpha_1 = \Phi^\alpha_2\) on common domains.
Given such a pair, it is obvious that we can sew them together to get a new gluing datum over \( U = U_1 \cup U_2 \):

- \( \eta_1 \) and \( \eta_2 \) together yield a metric \( \eta \) on \( \text{Gl}^\alpha \) over \( U \),
- take \( \epsilon < \min(\epsilon_1, \epsilon_2) \),
- the gluing map \( \phi^\alpha \) and the bundle isomorphisms \( \{ \Phi^\alpha_{\beta} \} \) are defined in an obvious way.

Such a gluing datum over \( U = U_1 \cup U_2 \) is called a sewed gluing datum of the two coincided data.

Motivated by this sewing property, we introduce the inward-extendibility condition for gluing data over boundary-type open subsets.

**Definition 2.8.** For a stratum \( M_\alpha \), we say that an open subset \( U \subset M_\alpha \) is of boundary-type if \( M_\alpha \setminus U \) is closed in \( \overline{M_\alpha} \).

**Lemma 2.9.** Denote \( \overline{M_\alpha} \setminus M_\alpha \) by \( \partial M_\alpha \). Suppose \( U \) is an open subset of \( M_\alpha \). Then \( U \) is of boundary-type if and only if \( U \cup \partial M \) is open in \( \overline{M_\alpha} \).

**Proof.** Note that \( M_\alpha \setminus U = \overline{M_\alpha} \setminus (U \cup \partial M) \). Then the lemma is a consequence of this fact. \( \square \)

**Definition 2.10.** Let \( A := (U, \eta, \epsilon, \phi^\alpha, \Phi^\alpha) \) be a gluing datum over a boundary-type open subset \( U \subset M_\alpha \). We say that a gluing datum over \( M_\alpha \)

\[ \tilde{A} := (M_\alpha, \tilde{\eta}, \tilde{\epsilon}, \tilde{\phi}^\alpha, \tilde{\Phi}^\alpha) \]

is an inward-extension of \( A \) if there exists a boundary-type open subset \( U' \subset U \) such that \( A \) agrees with \( \tilde{A} \) over \( U' \).

**Definition 2.11.** A gluing atlas \( \mathcal{F} \) is called a good gluing structure of \( M \) if \( \mathcal{F} \) satisfies the following conditions:

(i) \( \mathcal{F} \) is closed under the induction for restriction maps and closed under the induction for gluing maps;
(ii) \( \mathcal{F} \) satisfies the sewing property in the sense that if \( \mathcal{F} \) has a pair gluing data over open subsets of a stratum of \( M \) which coincide over their intersection, then their sewed datum is also in \( \mathcal{F} \).
(iii) any boundary-type gluing datum in \( \mathcal{F} \) has an inward-extension in \( \mathcal{F} \).

**Condition C:** (Existence of a good gluing structure) There exists a good gluing structure \( \mathcal{F} \) in \( \mathcal{GL}(M) \) for the manifold stratified space \( M \).

The following lemma implies that the \( C^\infty \)-compatibility condition can be checked by applying the induction for gluing maps. The proof follows directly from the \( C^\infty \)-compatibility under the induction for gluing maps.

**Lemma 2.12.** Two coordinate charts associated to gluing data

\[ (U_{\alpha_k}, \eta^\alpha_k, \epsilon^\alpha_k, \phi^\alpha_k, \Phi^\alpha_k) \]
over $U_{\alpha_k}$ for $k = 1, 2$ are $C^\infty$-compatible if for any $\beta \in S^{\alpha_1} \cap S^{\alpha_2}$ their induced gluing data obtained from the induction for gluing maps agrees over the common domain.

**Definition 2.13 (Gluing-compatibility).** Two gluing data

$$(U_{\alpha_k}, \eta^{\alpha_k}, \epsilon^{\alpha_k}, \phi^{\alpha_k}, \Phi^{\alpha_k})$$

over $U_{\alpha_k}$ for $k = 1, 2$ are said to be gluing-compatible if for any $\beta \in S^{\alpha_1} \cap S^{\alpha_2}$, any their induced gluing data on the common domain in $M_\beta$ by gluing maps agree.

We finally come to the conclusion that a gluing atlas $F$ (Cf. Definition 2.6) for $M$ defines a canonical smooth structure on $M$ if any pair in $F$ is gluing-compatible.

**Theorem 2.14.** Suppose that $M$ admits a good gluing structure $F$. Then there exists a particular gluing atlas

$$\mathcal{G} := \{ A_\alpha = (M_\alpha, \eta^\alpha, \epsilon^\alpha, \phi^\alpha, \{ \Phi^{\alpha}_{\beta} \}_{\beta \in S^\alpha}) | \alpha \in \mathcal{S} \} \subset F$$

such that any pair in $\mathcal{G}$ are gluing-compatible.

**Proof.** We first introduce subsets $S_1, S_2, \ldots$ of $S$ inductively by letting $S_n$ consist of all smallest elements in $S \setminus \bigcup_{k<n} S_k$.

Here we assume that $S_0 = \emptyset$. Write

$$S = S_1 \cup \cdots \cup S_K.$$

We will construct a collection gluing datum

$$\{ A^{(k)}_\alpha = (M_\alpha, \eta^\alpha, \epsilon^{(k)}_\alpha, \phi^\alpha, \Phi^{\alpha}) | \alpha \in \bigcup_{i=1}^k S_i \}$$

by applying an induction argument to $k$ satisfying the following properties:

(I) any pair $A^{(k)}_\alpha$ and $A^{(k)}_\beta$ are gluing-compatible;

(II) for any incomparable indices $\alpha$ and $\beta$

$$R^{(k)}(\phi^\alpha) \cap R^{(k)}(\phi^\beta) \subset \bigcup_{\gamma \in S_\alpha \cap S_\beta} R(\phi^\gamma).$$

Here $R^{(k)}(\phi^\alpha)$ is the image of the gluing map $\phi^\alpha$ in $A^{(k)}_\alpha$ and $S_\alpha$ is defined to be $\{ \gamma | \gamma \preceq \alpha \}$. Though the domains of $\phi^\alpha$ in $A^{(k)}_\alpha$ depend on $k$ (due to the changes of $\epsilon^{(k)}_\alpha$), the maps are same on common domains. Hence we simply denote it by $\phi^\alpha$ without indicating $k$.

**Step 1** $(k = 1)$. Note that any $\alpha \in S_1$, $M_\alpha$ is compact and any two of $\{ M_\alpha | \alpha \in S_1 \}$ are disjoint. Choose an arbitrary gluing datum

$$A^{(1)}_\alpha := (M_\alpha, \eta^\alpha, \epsilon^{(1)}_\alpha, \phi^\alpha, \Phi^{\alpha})$$

with small enough $\epsilon^{(1)}_\alpha$ such that $R(\phi^\alpha) \cap R(\phi^\beta) = \emptyset$. Hence, $\{ A^{(1)}_\alpha | \alpha \in S_1 \}$ trivially satisfies the properties (I) and (II).
Step 2 \((k = 2)\). For any \(\alpha \in S_2\), by the induction for the gluing maps, the gluing data \(\{A^{(1)}_{\gamma} | \gamma \in S_1\}\) induce a gluing datum over

\[ U_\alpha = \bigcup_{\gamma < \alpha} R(\phi^\gamma_\alpha) \subset M_\alpha. \]

We denote it by

\[ B_{\alpha} = (U_\alpha, \eta, \epsilon, \tilde{\phi}^\alpha, \tilde{\Phi}^\alpha) \]

for some small \(\epsilon\). Clearly, \(U_\alpha\) is a boundary-type open subset of \(M_\alpha\). Then by the assumption that \(F\) is a good gluing structure, \(B_{\alpha}\) has an inward-extension, denoted by

\[ A^{(2)}_{\alpha} = (M_\alpha, \eta^\alpha, \epsilon^{(2)}_{\alpha}, \phi^\alpha, \Phi^\alpha) \]

such that it coincides with \(B_{\alpha}\) over some boundary-type open subset \(V_\alpha\) of \(U_\alpha\).

Next we need do some modifications on existing gluing data \(\{A^{(2)}_{\alpha} | \alpha \in S_1 \cup S_2\}\) as follows.

1. For any \(\gamma \in S_1\) we replace \(\epsilon^{(1)}_{\gamma}\) by smaller \(\epsilon^{(2)}_{\gamma}\) such that the image of the gluing map \(\phi^\gamma_\alpha\) is a open subset of \(V_\alpha\) for \(\alpha \in S_1 \cup S_2\);
2. we may downsize \(\epsilon^{(2)}_{\alpha}\), for example, \(\epsilon^{(2)}_{\alpha} \leq \frac{1}{2} \min_{\gamma \in S_1} \epsilon^{(1)}_{\gamma}\), such that (2.15) holds for any incomparable pair in \(S_2\).

We now verify that any two gluing data \(A^{(2)}_{\alpha}\) and \(A^{(2)}_{\beta}\) are gluing-compatible:

1. if \(\alpha\) and \(\beta\) are comparable, say \(\alpha < \beta\), then the induced gluing datum from \(A^{(2)}_{\alpha}\) is over a subset of \(V_\beta\) and it coincides with \(A^{(2)}_{\beta}\) by the construction;
2. if \(\alpha\) and \(\beta\) are incomparable and \(S_\alpha \cap S_\beta = \emptyset\), then two gluing data are trivially compatible due to (2.15);
3. if \(\alpha\) and \(\beta\) are incomparable and \(S_\alpha \cap S_\beta \neq \emptyset\), then the image of \(\phi^\alpha\) and \(\phi^\beta\) are covered by images of \(\phi^\gamma\) of \(\gamma \in S_{\alpha, \beta}\), but both \(A^{(2)}_{\alpha}\) and \(A^{(2)}_{\beta}\) are gluing-compatible with \(A^{(2)}_{\gamma}\), this implies that they are gluing-compatible with each other.

Step 3 (general case). Now suppose that \(A^{(k)}_{\alpha}\) for all \(\alpha \in S_1, \ldots, S_k\) are constructed. We proceed to construct \(A^{(k+1)}_{\alpha}\) for all \(\alpha \in S_{k+1}\) by repeating the same construction as in Step 2. That is, for any \(\alpha \in S_{k+1}\), the gluing data \(\{A^{(1)}_{\gamma} | \gamma < \alpha\}\) defines a boundary-type gluing over a subset in \(M_\alpha\). The inward-extension condition supplies us with a gluing datum

\[ A^{(k+1)}_{\alpha} = (M_\alpha, \eta^\alpha, \epsilon^{(2)}_{(k+1)}, \phi^\alpha, \Phi^\alpha). \]

Then as in Step 2, we modify \(\epsilon^{(k)}_{\alpha}\) to a suitable \(\epsilon^{(k+1)}_{\alpha}\) for \(\alpha \in S_1, \ldots, S_k\) and get an updated \(\{A^{(k+1)}_{\alpha}\}\) satisfying both (I) and (II).

Since \(S\) is finite, such a procedure will terminate in finite steps. Hence, we get a required gluing atlas as in the Theorem. \(\square\)

Remark 2.15. Let \(G\) be the collection of gluing data given in Theorem 2.14 Then all the bundle isomorphisms \(\Phi^\alpha_{\beta}\) in \(G\) are isometric.
2.3. **Gluing theorem for orbifold stratified spaces.**

In this subsection, we generalise the results in previous subsections to orbifold stratified spaces. We employ the language of proper étale groupoids to describe topological and smooth orbifold following the definition of proper étale groupoids as in [8, Definition 2.6]. For readers’ convenience, we recall the definitions of Lie groupoid, proper étale groupoids and vector bundles over Lie groupoids from [8]. Topological groupoids and vector bundles over topological groupoids can be defined in a similar way.

**Definition 2.16.** (Lie groupoids and proper étale groupoids) A Lie groupoid $G = (G^0, G^1)$ consists of two smooth manifolds $G^0$ and $G^1$, together with five smooth maps $(s, t, m, u, i)$ satisfying the following properties.

1. The source map and the target map $s, t : G^1 \rightarrow G^0$ are submersions.
2. The composition map $m : G^{[2]} := \{(g_1, g_2) \in G^1 \times G^1 : t(g_1) = s(g_2)\} \rightarrow G^1$ written as $m(g_1, g_2) = g_1 \circ g_2$ for composable elements $g_1$ and $g_2$, satisfies the obvious associative property.
3. The unit map $u : G^0 \rightarrow G^1$ is a two-sided unit for the composition.
4. The inverse map $i : G^1 \rightarrow G^1$, $i(g) = g^{-1}$, is a two-sided inverse for the composition.

In this paper, a groupoid $G$ will be denoted by $G = (G^1 \Rightarrow G^0)$ where $G^0$ will be called the space of objects or units, and $G^1$ will be called the space of arrows. A Lie groupoid $G$ is proper if $(s, t) : G^1 \rightarrow G^0 \times G^0$ is proper, and is called étale if $s$ and $t$ are local diffeomorphisms. Given a proper étale groupoid $(G^1 \Rightarrow G^0)$, for any $x \in G^0$,

$$G_x = (s, t)^{-1}(x, x) = s^{-1}(x) \cap t^{-1}(x)$$

is a finite group, called the isotropy group at $x$.

**Remark 2.17.** Let $G = (G^1 \Rightarrow G^0)$ be a proper étale Lie groupoid.

1. We remark that $G^1$ defines an equivalence relation on $G^0$: that is, any two points in $G^0$ are equivalent if they are the source and target of an arrow in $G^1$. The quotient space $G^0 / \sim$ is denoted by $|G|$ and is called the coarse space of $G$. Let $\pi : G^0 \rightarrow |G|$ be the projection map. There is a canonical orbifold structure on $|G|$ defined by $G$.

2. In this paper, when we say that $U = (U^1 \Rightarrow U^0)$ is an open full-subgroupoid of $G$ if it is of the form

$$U^0 = \pi^{-1}(V), \quad U^1 = s^{-1}(U^0),$$

for an open subset of $V$ of $|G|$. For example, given any open subset $U \subset G^0$ we can associate it an open full-subgroupoid $U$ by setting

$$U^0 = \pi^{-1}(V), \quad U^1 = s^{-1}(U^0),$$

where $V = \pi(U)$. 

(3) Let \( x \in G^0 \), there is a \( G_x \)-invariant open neighbourhood of \( x \) in \( G^0 \) such that the full-subgroupoid associated to \( U_x \) is Morita equivalent to the action groupoid

\[ U_x \rtimes G_x \cong U_x. \]

This latter action groupoid is called a local model of \( G \) at \( x \in G^0 \). Recall that for proper étale Lie groupoids, a Morita equivalence means that their coarse spaces are homeomorphic and their local models are isomorphic.

**Proposition 2.18.** Given a Lie groupoid \( G = (G^1 \rightrightarrows G^0) \), a Lie groupoid \( E = (E^1 \rightrightarrows E^0) \) is a vector bundle over \( G \) if and only if there is a strict Lie groupoid morphism \( \pi : (E^1 \rightrightarrows E^0) \to (G^1 \rightrightarrows G^0) \) given by the commutative diagram

\[
\begin{array}{ccc}
E^1 & \xrightarrow{\pi_1} & G^1 \\
\downarrow & & \downarrow \\
E^0 & \xrightarrow{\pi_0} & G^0
\end{array}
\]  

(2.16)

in the category of Lie groupoids with strict morphisms, such that

1. the diagram (2.16) is a pull-back groupoid diagram,
2. both \( \pi_1 : E^1 \to G^1 \) and \( \pi_0 : E^0 \to G^0 \) are vector bundles,
3. the pull-back arrows

\[ \{(v_x, \gamma, v_y) | \gamma \in G^1, (v_x, v_y) \in E_{s(\gamma)} \times E_{t(\gamma)}\} \]

define a linear isomorphism \( \xi(\gamma) : E_{h(\gamma)} \to E_{t(\gamma)} \) sending \( v_x \) to \( x_y \).

We say that \( (E^1 \rightrightarrows E^0) \) is linearly stratified if \( E^1 \) and \( E^0 \) are linearly stratified such that the commutative diagram (2.16) preserves the linear stratifications, and moreover, the pullback arrows in Proposition 2.18 is a stratum preserving isomorphism, or simply, the arrows in \( E^1 \) preserve the linear stratification on \( E^0 \).

Now we consider an orbifold stratified space as Definition 1.1

\[ M = \bigsqcup_{\alpha \in S} |M_\alpha|, \]

where \( M_\alpha = (M^1_\alpha \rightrightarrows M^0_\alpha) \) is a proper étale Lie groupoid. Denote

\[ M = (M^1 \rightrightarrows M^0) = \left( \bigsqcup_{\alpha \in S} M^1_\alpha \rightrightarrows \bigsqcup_{\alpha \in S} M^0_\alpha \right). \]

We assume that \( M \) is a proper étale topological groupoid and both \( M^0 \) and \( M^1 \) are manifold stratified spaces with respect to \( S \). We can adapt all the arguments for manifold stratified spaces to orbifold stratified spaces.

**Condition A’ (Existence of orbifold gluing bundles)** For any \( \alpha \in S \), there is a linearly stratified smooth orbifold vector bundle

\[ \text{Gl}^\alpha = (\text{Gl}\,^1,\alpha \rightrightarrows \text{Gl}\,^0,\alpha) \to M_\alpha = (M^1_\alpha \rightrightarrows M^0_\alpha) \]
with respect to $S^\alpha = \{ \beta | \alpha \preceq \beta \}$, such that for $i = 1, 2$, $G^{i,\alpha} \to M^{i}_\alpha$ is a gluing bundle for the manifold stratified space $M^{i}_\alpha$. This bundle is called the orbifold gluing bundle over the strata $M^{\alpha}$.

We can equip the gluing bundle $G^\alpha$ with a compatible smooth metric $\eta^\alpha$ so that for any $\epsilon > 0$, the open $\epsilon$-ball bundle of $G^\alpha$ with the induced stratification. Set $S^\alpha = \{ \beta | \beta \prec \alpha \}$. Let $M^{\alpha} = \bigcup_{\beta \in S^\alpha} M^{\beta}$

where $S^\alpha = \{ \beta | \alpha \preceq \beta \}$.

**Definition 2.19.** Let $U = (U^1 \rightrightarrows U^0)$ be any open full-subgroupoid of $M^{\alpha}$. A gluing datum over $U$ consists of a metric $\eta^{\alpha} = (\eta^{0,\alpha}, \eta^{1,\alpha})$ on $G^\alpha|_U$ and a gluing map $\phi^{\alpha} : G^\alpha|_U \to M^{\alpha}$ given by a strict morphism of topological groups

$$(\phi^{1,\alpha}, \phi^{0,\alpha}) : (G^{1,\alpha}(\epsilon)|_U \rightrightarrows G^{0,\alpha}(\epsilon)|_U) \to (M^{1,\alpha} \rightrightarrows M^{0,\alpha})$$

for some constant $\epsilon > 0$ such that $\phi^{i,\alpha}$, for $i = 0, 1$, are gluing maps for $M^{i}$, namely,

1. the image of $\phi^{i,\alpha}$ is open and the map $\phi^{i,\alpha}$ is a homeomorphism onto its image in the sense of topological groupoids;
2. the map $\phi^{\alpha}$ is a stratified smooth map with respect to the stratification, i.e., for any $\beta \in S^\alpha$, $\phi^{\alpha}$ maps $G^{\beta}_{\alpha}(\epsilon)|_U$ to $M_{\beta}$, we denote this map by

$$\phi^{\alpha}_{\beta} : G^{\alpha}_{\beta}(\epsilon)|_U \to M_{\beta};$$

then $\phi^{\alpha}_{\beta}$ is an isomorphism onto its image in the sense of Lie groupoids (cf. Remark 2.20);

and a collection of stratum-preserving smooth bundle isomorphism maps which preserve the induced stratifications

$$\Phi^{\alpha}_{\beta} : N(G^{\alpha}_{\beta}) \to G^{\beta}$$

that covers $\phi^{\alpha}_{\beta}$ for any $\beta \in S^\alpha$ in the sense that the diagram

$$\begin{array}{ccc}
N(G^{\alpha}_{\beta}) & \xrightarrow{\Phi^{\alpha}_{\beta}} & G^{\beta} \\
\downarrow & & \downarrow \\
G^{\alpha}_{\beta}(\epsilon)|_U & \xrightarrow{\phi^{\alpha}_{\beta}} & M_{\beta}
\end{array}$$

commutes in the category of Lie groupoids and strict morphisms. We denote this gluing datum over $U$ by

$$(U, \rho, \epsilon, \phi^{\alpha}, \{\Phi^{\alpha}_{\beta}\}_{\beta \in S^\alpha}).$$

**Remark 2.20.** Let $R^{i}(\alpha, \beta)$ be the image of $\phi^{i,\alpha}_{\beta}$, $i = 0, 1$ and $|R(\alpha, \beta)|$ be the image of $|\phi^{\alpha}_{\beta}|$ (the coarse map of $\phi^{\alpha}_{\beta}$). Then

$$R(\alpha, \beta) = (R^{1}(\alpha, \beta) \rightrightarrows R^{0}(\alpha, \beta))$$
is the full subgroupoid associated to an open subset \(|R(\alpha, \beta)|\) of \(M_\beta\). Then \(\phi^\alpha_\beta\), being an isomorphism onto its image in the sense of Lie groupoids, is a strict Lie groupoid isomorphism

\[
\text{Gl}^\alpha_\beta(\epsilon)|_U \cong R(\alpha, \beta).
\]

This is equivalent to say that the maps \(\phi^\alpha_{\beta,0}\) and \(\phi^\alpha_{\beta,1}\) are diffeomorphisms onto its images in \(M^0_\beta\) and \(M^1_\beta\) respectively.

Parallel to the manifold stratified case, we assume Condition B’ and Condition C’.

**Condition B’: (existence of gluing data)** For any \(\alpha \in S\) and any proper open subset \(U \subset M_\alpha\) there exists a gluing datum over \(U\).

One can also define a *good orbifold gluing structure* \(\mathcal{F}\) for an orbifold stratified space as follows.

**Definition 2.21.** A *good orbifold gluing structure* \(\mathcal{F}\) for an orbifold stratified space \(M\) is a collection of gluing data satisfying the following conditions:

(i) the image of coarse gluing maps associated to \(\mathcal{F}\) forms an open cover of \(M\);

(ii) \(\mathcal{F}\) is closed under the induction for restriction maps and closed under the induction for gluing maps;

(iii) \(\mathcal{F}\) satisfies the sewing property in the sense that if \(\mathcal{F}\) has a pair gluing data over open fullgroupoids of a stratum of \(M\) which coincide over their intersection, then their sewed datum is also in \(\mathcal{F}\).

(iv) any boundary-type gluing datum in \(\mathcal{F}\) has an inward-extension in \(\mathcal{F}\).

**Condition C’ (Existence of good orbifold gluing structure)** There exists a good orbifold gluing structure \(\mathcal{F}\) for the orbifold stratified space \(M\).

We remark that a good orbifold gluing structure for the orbifold stratified space \(M\) provides good gluing structures for manifold stratified spaces \(M^0\) and \(M^1\). The following theorem implies Theorem A in the Introduction. The proof is to apply the same arguments in the proof of Theorem 2.14 to good gluing structures to \(M^0\) and \(M^1\) such that \(M^1 \Rightarrow M^0\) is a proper étale groupoid.

**Theorem 2.22.** Suppose that the orbifold stratified space \(M\) has a good orbifold gluing structure \(\mathcal{F}\), then there exists a particular gluing data

\[
\mathcal{G} = \{A_\alpha = (M_\alpha, \eta^\alpha, \epsilon_\alpha, \phi^\alpha, \Phi^\alpha)|\alpha \in S\} \subset \mathcal{F}
\]

such that any pair in \(\mathcal{G}\) are gluing-compatible, hence, \(C^\infty\)-compatible.

**Remark 2.23.** Using the language of proper étale Lie (or topological) groupoids to describe smooth (or topological) orbifolds, the correct notion of morphisms between two groupoids should be generalised morphisms in the sense of [19] instead of strict morphisms, and generalised isomorphisms instead of strict isomorphisms. Recall a generalised morphism between
two proper étale Lie groupoids $G = (G^1 \rightrightarrows G^0)$ and $H = (H^1 \rightrightarrows H^0)$, denoted by

$$G \dashrightarrow H,$$

is given by a covering groupoid $G[\mathcal{U}]$ of $G$ associated to an open cover $\mathcal{U} = \{U_i\}$ of $G^0$, together with a strict morphism

$$G[\mathcal{U}] \rightarrow H.$$

Here the covering groupoid $G[\mathcal{U}]$ is defined to be

$$G[\mathcal{U}] = \bigsqcup G_{U_i}^{U_i} \rightrightarrows \bigsqcup U_i$$

where $G_{U_i}^{U_j} = \{g \in G^1 | s(g) \in U_i, t(g) \in U_j\}$ with the obvious source map and target map to $U_i$ and $U_j$ respectively. There is an obvious strict morphism

$$G[\mathcal{U}] \rightarrow G$$

which is a strong equivalence. A generalised isomorphism (also called a Morita equivalence) is a generalised morphism such that the associated strict morphism $G[\mathcal{U}] \rightarrow H$ is a local isomorphism which induces a homeomorphism between $|G| = |G[\mathcal{U}]|$ and $|H|$.

Note that this notion of generalised morphisms and generalised isomorphisms makes sense for proper étale topological groupoids. With this understood, then we can proceed to define a good orbifold gluing structure as the induction for restriction maps, the induction for gluing maps, the sewing property, and the inward-extendibility condition can be carried over accordingly. Moreover, the proof of Theorem 2.14 can be adapted to get a smooth structure for an orbifold stratified space with a good orbifold gluing structure in the category of proper étale groupoids with morphisms given by generalised morphisms.

As remarked in the introduction, the disjoint of proper étale Lie groupoids

$$M = \left( \bigsqcup_{\alpha \in S} M^1_\alpha \rightrightarrows \bigsqcup_{\alpha \in S} M^0_\alpha \right)$$

often does not admit a topological groupoid structure, as we shall see in the Deligne-Mumford moduli spaces of stables curves. The ultimate goal is still to construct a $C^\infty$-compatible orbifold gluing atlas. We need to resolve the issue of both

$$\bigsqcup_{\alpha \in S} M^1_\alpha \text{ and } \bigsqcup_{\alpha \in S} M^0_\alpha$$

have no topological structure. We remark that orbifold gluing bundles still make sense, but gluing maps in Definition 2.19 such as

$$\phi^\alpha : G|^{\alpha}(e)|_U \dashrightarrow M^\alpha,$$

don’t make sense as $M^\alpha$ is not a topological groupoid. We point out that the stratum-wise gluing map

$$\phi^\alpha_\beta : G|^{\alpha}(e)|_U \dashrightarrow M^\beta$$


as a generalised isomorphism onto its image is well-defined. We can just treat the gluing map
\[ \phi^\alpha : G^\alpha(\epsilon)|_U \rightarrow M^\alpha, \]
as a generalised isomorphism on the level set-theoretical groupoids. That is, a generalised morphism from a proper étale Lie groupoid \( G \) to a groupoid \( H \) (not necessarily a topological groupoid) is given by a covering groupoid \( G[U] \) and a strict morphism from \( G[U] \) to \( H \) as a set-theoretical groupoid. A generalised isomorphism from a proper étale Lie groupoid \( G \) to a groupoid \( H \) is a generalised morphism such that the associated strict morphism \( G[U] \rightarrow H \) is locally bijective and induced a bijective map from the topological space \( |G[U]| \) to a point-set \( |H| \).

Then the notion of good orbifold atlases still makes sense as inductions for restriction maps, inductions for stratum-wise gluing maps, sewing property, the inward-extendibility condition, gluing compatibility condition still make sense. This is due to fact that all these notions only involve smooth structures on the domain groupoids such as \( Gl^n \). We can then proceed to establish Theorem 2.22 even though the disjoint of proper étale Lie groupoids \( \{M_\alpha\}_{\alpha \in S} \) is not a topological groupoid. We still achieve a smooth orbifold on the orbifold stratified space \( M \) with its orbifold groupoid \( \bigsqcup_{\alpha \in S} G^\alpha(\epsilon) \) obtained from the particular gluing atlas
\[ G = \{A_\alpha = (M_\alpha, \eta^\alpha, \epsilon^\alpha, \phi^\alpha, \Phi^\alpha)|\alpha \in S\} \]
We shall explain how this can be done for the Deligne-Mumford moduli spaces of stable curves.

3. Moduli spaces of stable curves as orbifold stratified spaces

This section is mostly a review of moduli spaces of stable curves.

3.1. Teichmüller space and moduli space of Riemann surfaces (top stratum). We start with the Teichmüller space for genus \( g \) Riemann surfaces with \( n \)-marked points which play a central role in the description of moduli space of Riemann surfaces.

Denote by \( \Sigma_{g,n} \) a genus \( g \) smooth oriented compact surface \( \Sigma \) with ordered \( n \)-marked points \( \{p_1, p_2, \ldots, p_n\} \). Given a genus \( g \) compact Riemann surface with \( n \)-marked points
\[ (C, \{x_1, x_2, \ldots, x_n\}), \]
a Teichmüller structure on \( (C, \{x_1, x_2, \ldots, x_n\}) \) is the datum of the isotopy class \([f]\) of an orientation preserving diffeomorphism
\[ f : (C, \{x_1, x_2, \ldots, x_n\}) \rightarrow \Sigma_{g,n} = (\Sigma, \{p_1, p_2, \ldots, p_n\}), \]
where the allowable isotopies are those which map \( x_i \) to \( p_i \) for each \( i = 1, 2, \ldots, n \). Two genus \( g \) compact Riemann surfaces with Teichmüller structures
\[ (C, \{x_1, x_2, \ldots, x_n\}; [f]) \quad \text{and} \quad (C', \{x'_1, x'_2, \ldots, x'_n\}; [f']) \]
are called isomorphic if there is an isomorphism (a biholomorphism preserving the ordered \( n \)-marked points)
\[ \phi : (C, \{x_1, x_2, \ldots, x_n\}) \rightarrow (C', \{x'_1, x'_2, \ldots, x'_n\}) \]
such that \([f' \circ \phi] = [f]\). The Teichmüller space \(T_{g,n}\) of \(\Sigma_{g,n}\) is the set of isomorphism classes of genus \(g\), \(n\)-marked compact Riemann surface with Teichmüller structures. Any orientation preserving diffeomorphism between two genus \(g\) smooth oriented compact surfaces with \(n\)-marked points induces a canonical identification between their Teichmüller spaces. This justifies the simplified notation \(T_{g,n}\).

Let \(\text{Diff}(\Sigma_{g,n})\) be the subgroup of orientation-preserving diffeomorphism group \(\text{Diff}(\Sigma)\) that fix the \(n\) marked point and \(\text{Diff}_0(\Sigma_{g,n})\) be the identity component of \(\text{Diff}(\Sigma_{g,n})\). The mapping class group of \(\Sigma_{g,n}\), denoted by \(\text{Mod}_{g,n}\), is the group of all isotopy classes of orientation-preserving diffeomorphisms of \(\Sigma_{g,n}\), that is,

\[
\text{Mod}_{g,n} = \text{Diff}(\Sigma_{g,n})/\text{Diff}_0(\Sigma_{g,n}).
\]

The mapping class group acts naturally on \(T_{g,n}\) given by

\[
[\gamma] \cdot [C, \{x_1, x_2, \cdots, x_n\}; [f]] = [C, \{x_1, x_2, \cdots, x_n\}; [\gamma \circ f]].
\]

The quotient space of \(T_{g,n}\) by \(\text{Mod}_{g,n}\) is the moduli space \(M_{g,n}\) of genus \(g\) Riemann surfaces with \(n\)-marked points, this follows from the identifications

\[
\begin{align*}
T_{g,n} &= J(\Sigma)/\text{Diff}_0(\Sigma_{g,n}) = \left(J(\Sigma) \times (\Sigma^n \setminus \Delta)\right)/\text{Diff}_0(\Sigma), \\
M_{g,n} &= J(\Sigma)/\text{Diff}(\Sigma_{g,n}) = \left(J(\Sigma) \times (\Sigma^n \setminus \Delta)\right)/\text{Diff}(\Sigma).
\end{align*}
\]

Here \(J(\Sigma)\) is the space of complex structure on \(\Sigma\), and \(\Delta\) is the big diagonal so that \(\Sigma^n \setminus \Delta\) is the sub-manifold of \(\Sigma^n\) consisting of \(n\)-distinct points of \(\Sigma\).

In [12] and [30], it was showed that

\[
\begin{CD}
\text{Diff}_0(\Sigma) @>>> J(\Sigma) \times (\Sigma^n \setminus \Delta) \\
@VVV \\
T_{g,n}
\end{CD}
\]

is a principal fiber bundle. The associated fiber bundle

\[
\pi_{g,n} : \left(J(\Sigma) \times (\Sigma^n \setminus \Delta)\right) \times_{\text{Diff}_0(\Sigma)} \Sigma \longrightarrow T_{g,n}
\]

for the action of \(\text{Diff}_0(\Sigma)\) on \(\Sigma\) is a fiber bundle with fibers diffeomorphic to \(\Sigma\) and having \(n\) distinguished (disjoint) sections. This fiber bundle is the universal curve of genus \(g\) with \(n\) marked points, will be simply denoted by

\[
\begin{CD}
C_{g,n} \cr \pi_{g,n} \cr T_{g,n}
\end{CD}
\]

The mapping class group \(\text{Mod}_{g,n}\) acts on \(T_{g,n}\) as a properly discontinuous group of holomorphic transformations. This action defines an orbifold structure on \(M_{g,n}\). In terms of proper étale groupoids, this orbifold structure is defined by the action groupoid

\[
T_{g,n} \rtimes \text{Mod}_{g,n} \rightrightarrows T_{g,n}.
\]
In general, given a family of genus $g$, $n$-marked Riemann surfaces $\pi : C \to B$, there is a canonical construction to get a proper étale Lie groupoid whose unit space is $B$. When $B$ is the Teichmüller space $T_{g,n}$, then the resultant groupoid is exactly the above action groupoid. As this construction is very useful in practice. We devote the next subsection to this construction.

3.2. Canonical construction of proper étale groupoids. Consider a smooth family of genus $g$, $n$-marked Riemann surfaces

\[
\begin{array}{c}
\pi \\
\downarrow \\
B
\end{array}
\]

consisting of a smooth fiber bundle $\pi : C \to B$ and $n$-disjoint sections $\sigma_i, i = 1, 2, \cdots, n$. Denote by

\[
(C_b = \pi^{-1}(b), \{\sigma_1(b), \cdots, \sigma_n(b)\}, [f_b])
\]

the genus $g$ $n$-marked Riemann surface with Teichmüller structure for each $b \in B$. We can construct a proper étale Lie groupoid

\[
B = (G^1 \rightrightarrows G^0) = (G^1, G^0, s, t, m, u, i)
\]

where $G^0 = B$, and $G^1$ consists of triples

\[
(b_1, \psi, b_2),
\]

for $b_1, b_2 \in B$ and

\[
\psi \in \text{Isom}((C_{b_2}, \{\sigma_1(b_2), \cdots, \sigma_n(b_2)\}), (C_{b_1}, \{\sigma_1(b_1), \cdots, \sigma_n(b_1)\})�\).
\]

The source and tail map $(s, t)$ are given by the obvious projections

\[
s(b_1, \psi, b_2) = b_2, t(b_1, \psi, b_2) = b_1.
\]

For any composable pair

\[
(b_1, \psi, b_2) \quad \text{and} \quad (b_2, \phi, b_3),
\]

the groupoid multiplication is defined

\[
m((b_1, \psi, b_2), (b_2, \phi, b_3)) = (b_1, \psi, b_2) \cdot (b_2, \phi, b_3) = (b_1, \psi \circ \phi, b_3).
\]

The inverse map $i$ and the unit map $u$ are given by

\[
i(b_1, \psi, b_2) = (b_2, \psi^{-1}, b_1) \quad \text{and} \quad u(b) = (b, Id, b)
\]

respectively. To be consistent with our notations, we denote this groupoid by

\[
(3.1) \quad B = (\mathcal{A}_C(B) \rightrightarrows B).
\]

Here $\mathcal{A}_C(B)$ denote the space of arrows among $B$ that are generated from the family $C$.

Similarly, we can get a proper étale Lie groupoid for the family $C \to B$

\[
C = (\mathcal{A}_C(C) \rightrightarrows C).
\]
Here $\mathcal{A}_C(C)$ is a fibration over $\mathcal{A}(C)$: the fiber over $(b_1, \psi, b_2)$ consists of triple $(x_1, \psi, x_2)$ where $x_1 \in \pi^{-1}(b_1)$ and $x_2 = \psi(x_1)$. There is a groupoid fibration,

$$
\begin{array}{ccc}
\mathcal{A}_C(C) & \xrightarrow{\pi_1} & \mathcal{A}_C(B) \\
\downarrow & & \downarrow \\
C & \xrightarrow{\pi_0} & B.
\end{array}
$$

We remark that $C$ is the action groupoid associated to the canonical action of $A_C(B)$ on $C$.

Applying this construction to the universal family $C_{g,n} \to T_{g,n}$, we have the groupoid for $M_{g,n}$, i.e,

$$
M_{g,n} = (A_C_{g,n}(T_{g,n})) \rightrightarrows T_{g,n}).
$$

In fact, one can show that

$$
A_{C_{g,n}}(T_{g,n}) \cong T_{g,n} \rtimes \text{Mod}_{g,n}
$$

Then we also have a universal family

$$
C_{g,n} = (A_{C_{g,n}}(C_{g,n}) \rightrightarrows C_{g,n}),
$$

where $A_{C_{g,n}}(C_{g,n}) \cong C_{g,n} \rtimes \text{Mod}_{g,n}$.

**Remark 3.1.** Given a smooth family $\pi : C \to B$, we can certainly have a map $B \to |M_{g,n}|$ or a map $|B| \to |M_{g,n}|$. However we usually do not have a smooth map from $B$ to $T_{g,n}$, hence, the strict morphism between $B$ and $M_{g,n}$. Instead, we have a generalised morphism

$$
B \rightrightarrows M_{g,n},
$$

in the sense of Remark 2.23. We briefly review this construction. Let $b$ be any point in $B$ then there exists a small neighborhood $U_b$ of $b$ so that we have a morphism of families

$$
(3.2)
\begin{array}{ccc}
C|_{U_b} & \xrightarrow{\Phi_b} & C_{g,n} \\
\downarrow & & \downarrow \\
U_b & \xrightarrow{\phi_b} & T_{g,n}.
\end{array}
$$

Let

$$
B' = \bigsqcup_{b \in B} U_b, \quad C' = \bigsqcup_{b \in B} C|_{U_b}.
$$

Apply the above canonical groupoid construction to the family $C' \to B'$, we get a proper étale Lie groupoid

$$
B' = (A_{C'}(B') \rightrightarrows B').
$$

It is easy to see that the following diagram

$$
\begin{array}{ccc}
A_{C'}(B') & \xrightarrow{\pi_1} & A_C(B) \\
\downarrow & & \downarrow \\
B' & \xrightarrow{\pi_0} & B.
\end{array}
$$
define a strong equivalence of Lie groupoids. A strict morphism
\[ B' \longrightarrow M_{g,n} \]
can be obtained from the morphism of families in (3.2).

3.3. Moduli space of stable curves.

The moduli space \( M_{g,n} \) is not compact. It was shown in [10] [20] that \( M_{g,n} \) can be compactified by adding certain genus \( g \) curves with \( n \)-marked points and nodal points. This compactification is called the Deligne-Mumford compactification.

**Definition 3.2.** A stable curve \( C \) with \( n \)-marked points \( x_1, x_2, \ldots, x_n \) is a connected compact complex algebraic curve satisfying the following conditions.

1. The only non-smooth points are nodal points, locally modelled on the origin in \( \{(z_1, z_2) \in \mathbb{C}^2 | z_1 z_2 = 0\} \).
2. The marked points are distinct smooth points.
3. The automorphism group of \((C, \{x_1, x_2, \ldots, x_n\})\) is a finite group.

**Remark 3.3.** In order to better understand the definition of stable curves, a few remarks are needed.

1. Topologically, the neighbourhood of a nodal point is homeomorphic to a union of two discs with their centers identified. So removing a nodal point locally gives rise to two discs with their centers deleted.
2. A nodal point can be *smoothened* by replacing two discs with joint centres by a cylinder. If all nodal points in a stable curve \( C \) are smoothened, then the resulting surface is connected. The genus of the resulting surface is called the *genus* of a stable curve \( C \).
3. A nodal point is *normalized* if two discs with joint centres are replaced by disjoint discs. The *normalization* of a stable curve \( C \) is the smooth curve obtained from \( C \) by normalizing all its nodal points, equivalently, the normalization of a stable curve \( C \) with the finite set \( \tau \) of nodal points is a compact smooth Riemann surface \( \Sigma = \bigsqcup_v \Sigma_v \) together with a map
\[ f : \Sigma \rightarrow C \]
such that

(i) \( f : \Sigma \setminus f^{-1}(\tau) \longrightarrow C \setminus \tau \) is biholomorphic.

(ii) For each nodal point \( z \in \tau \), \( f^{-1}(z) \) consists of two point.

Let \( \Sigma = \bigsqcup_v \Sigma_v \) be the normalization of a stable curve \( C \). The image of \( \Sigma_v \) under \( f \) will be called an *irreducible component* of \( C \). Each component \( \Sigma_v \) is a smooth Riemann surface with special points consisting of ordered marked points
\[ \Sigma_v \cap f^{-1}(\{x_1, x_2, \ldots, x_n\}) \]
and unordered marked points (preimages of the nodes on $\Sigma_v$). The number of special points on $\Sigma_v$ is denoted by $m_v$. Then

$$|Aut(C, \{x_1, x_2, \ldots, x_n\})| < \infty$$

if and only if

$$2g_v - 2 + m_v > 0,$$

for each component $\Sigma_v$ of $\Sigma$, where $g_v$ is the genus of $\Sigma_v$. Note that there is a short exact sequence of automorphism groups

$$1 \to \prod_v Aut(\Sigma_{v,m_v}) \to Aut(C, \{x_1, x_2, \ldots, x_n\}) \to Aut(\Gamma) \to 1,$$

where $Aut(\Gamma)$ is the automorphism group of the weighted dual graph $\Gamma$ of $C$, and $Aut(\Sigma_{v,m_v})$ is the subgroup of the automorphism of $\Sigma$ fixing the marked points and the set of unordered marked points.

**Definition 3.4.** The coarse moduli space $\overline{M}_{g,n}$ is the space of isomorphism classes of genus $g$ stable curves with $n$-marked points. It is the result of Deligne-Mumford-Knudsen that this space is compact and will be called the Deligne-Mumford-Knudsen compactification of the coarse moduli space $M_{g,n}$, denoted by $\overline{M}_{g,n}$.

**Remark 3.5.** In fact, it is now well-known that $\overline{M}_{g,n}$ has a compact complex orbifold structure. We shall denote the resulting orbifold by $\overline{M}_{g,n}$. The proof of this fact requires a construction of (local) universal curves over $\overline{M}_{g,n}$, see [10] [20] and [30]. In the remaining part of this paper, we instead apply the gluing principle developed in Section 2 to provide an orbifold atlas on $\overline{M}_{g,n}$. The main analysis is to show that $\overline{M}_{g,n}$ admits a good orbifold gluing atlas as in Definition 2.21. Note that Fukaya and Ono outlined a differential geometric way to endow $\overline{M}_{g,n}$ with a complex orbifold atlas in [15]. What we have done below in some sense is to provide a complete detailed gluing theory for $\overline{M}_{g,n}$ as outline in [15].

Considering a genus $g$ stable curve $C$ with $n$-marked points, its topological types is classified by the weighted dual graph which we now review.

**Definition 3.6.** A **weighted dual graph** $\Gamma$ is a connected graph together with the assignment of a nonnegative integer weight to each vertex, denoted by

$$(V(\Gamma), E(\Gamma), T(\Gamma), g : V(\Gamma) \to \mathbb{Z}_{\geq 0}, \ell : T(\Gamma) \to \{1, 2, \ldots, n\})$$

where

- $V(\Gamma)$ is a finite nonempty set of vertices with a weighted function $g : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ assigning a nonnegative integer $g_v$ to each vertex $v$.
- $E(\Gamma)$ is a finite set of edges.
- $T(\Gamma)$ is a finite set of $n$-labelled tails with a partition indexed by $V(\Gamma)$, that is, $T(\Gamma) = \bigsqcup_{v \in V(\Gamma)} T_v$ and the labelling is given by a bijective map: $\ell : T(\Gamma) \to \{1, 2, \ldots, n\}$. 

The genus of a weighted dual graph $\Gamma$ is defined to be

$$g(\Gamma) = \sum_{v \in V(\Gamma)} g_v + b_1(\Gamma)$$

where $b_1(\Gamma)$ is the first Betti number of the graph $\Gamma$. A graph $\Gamma$ is called stable if for every $v \in V(\Gamma)$,

$$2 - 2g_v - m_v < 0$$

(3.3)

where $m_v$ denotes the valence of $\Gamma$ at $v$, the sum of the number of legs attached to $v$ (cf. Remark 3.7).

Remark 3.7. An edge consists of two half-edges. By a leg of $\Gamma$ we mean either a tail or a half-edge.

Two weighted stable genus $g$ dual graph with $n$-labelled tails $\Gamma_1$ and $\Gamma_2$ are called isomorphic if there exists a bijection between their vertices, edges and tails respecting all the relevant structure. Denote by $S_{g,n}$ the set of isomorphism classes of weighted stable genus $g$ dual graph with $n$-labelled tails. For a weighted graph $\Gamma$ we denote the class by $[\Gamma]$.

Let $\Gamma$ be such a weighted dual graph. For any edge $e \in E(\Gamma)$ we may contract the edge $e$ to get a new weighted graph $\Gamma'$: $V(\Gamma')$ and $E(\Gamma')$ are defined in an obvious way; the weight of new vertex is defined such that the genus of $\Gamma'$ is still $g$. Then it is easy to see that $\Gamma'$ is still a stable graph. We may also contract several edges simultaneously. Let $D \subset E(\Gamma)$ be a subset of edges, then the graph after contracting edges in $D$ is denoted by $\text{Ctr}_D(\Gamma)$.

Given $[\Gamma_1], [\Gamma_2] \in S_{g,n}$, we say that $[\Gamma_1] \prec [\Gamma_2]$ if and only if there exist representatives $\Gamma_1$ and $\Gamma_2$ for $[\Gamma_1]$ and $[\Gamma_2]$ respectively, such that $\Gamma_2$ is obtained from a contraction of $\Gamma_1$ along a subset of $E(\Gamma_1)$. The following lemma is a well known result and we skip the proof.

Lemma 3.8. $(S_{g,n}, \prec)$ is a partially ordered finite set with a unique top element given by a weighted dual graph $\Gamma$ with no edges and only one vertex of weight $g$ and $n$-labelled tails.

Given a genus $g$ stable curve $C$ with $n$-marked points

$$(C, \{x_1, x_2, \cdots, x_n\})$$

we can associate it a weighted dual graph $\Gamma$ as follows. There is a vertex for each irreducible component of $C$ with its weight given by the genus of the component, and its legs labelled by the marked points on the component, and there is an edge between a pair (not necessarily different) of vertices for each nodal point between their components. One can check that $\Gamma$ is a stable weighted dual graph of genus $g$ with $n$-labelled legs.

Given a weighted dual graph $\Gamma \in S_{g,n}$ with $2g - 2 + n > 0$, denote by $M_\Gamma$ be the set of isomorphism classes of genus $g$ stable curve $C$ with $n$-marked points whose weighted dual graph is $\Gamma$. If $\Gamma_1 \cong \Gamma_2$, then $M_{\Gamma_1} \cong M_{\Gamma_2}$ set theoretically. Let $M_{[\Gamma]}$ be the set of isomorphism classes of genus $g$ stable curve $C$ with $n$-marked points whose weighted dual graph $\Gamma$ belongs to the class $[\Gamma]$. 
The following proposition is well-known. We include the proof for the convenience of readers.

**Proposition 3.9.** The coarse moduli space $\overline{M}_{g,n}$ has an orbifold stratified structure

$$\overline{M}_{g,n} = \bigsqcup_{[\Gamma] \in S_{g,n}} M_{[\Gamma]}$$

with respect to $(S_{g,n}, \prec)$.

**Proof.** The coarse moduli space $\overline{M}_{g,n}$ is a compact Hausdorff topological space. This can be proved without resorting to the algebraic geometry machinery as in [2]. One can extend the Fenchel Nielsen coordinates to $\overline{M}_{g,n}$ to show that $\overline{M}_{g,n}$ is a compact Hausdorff topological space. See for example in [32] and [13]. In [30], a pure differential geometry proof of this result is provided.

Next we show that each stratum $M_{[\Gamma]}$ is a smooth orbifold. Fix a representative $\Gamma$ in the isomorphism class $[\Gamma] \in S_{g,n}$. The normalisation of $\Gamma$ is the new weighted graph

$$\tilde{\Gamma} = \bigsqcup_{v \in V(\Gamma)} \Gamma_v$$

obtained by severing all the edges in $\Gamma$, where each connected component $\Gamma_v$ has only one vertex $\{v\}$ of genus $g_v$ with $p_v$ ordered tails and $q_v$ half-edges attached to the vertex $v$. Note that $p_v + q_v = \text{val}(v)$, the valence of $\Gamma$ at $v$.

Denote by $T_{g_v,(p_v,q_v)}$ the Teichmüller space of Riemann surface of genus $g_v$ with $p_v$ ordered marked points associated to tails, and $q_v$ unordered marked points associated to half-edges. Note that

$$\sum_{v \in V(\Gamma)} p_v = n, \quad \sum_{v \in V(\Gamma)} q_v = 2\# E(\Gamma).$$

On the set of those unordered marked points for all $v \in V(\Gamma)$, there is a pairing relation defined by $E(\Gamma)$. Define

$$T_\Gamma = \prod_{v \in V(\Gamma)} T_{g_v,(p_v,q_v)},$$

then there is a universal family of curves

$$\{ s^\sharp | e \in E(\Gamma) \}$$

whose fiber at $([C_v, \{x_1, \ldots, x_{p_v}\}, \{y_1, \ldots, y_{q_v}\}, [f_v])]_{v \in V(\Gamma)}$ is the disjoint union of Riemann surfaces

$$\bigsqcup_{v \in V(\Gamma)} (C_v, \{x_1, \ldots, x_{p_v}\}, \{y_1, \ldots, y_{q_v}\}).$$

Here the set of sections $\{\sigma_i, i = 1, 2, \ldots, n\}$ is defined by ordered marked points

$$\bigsqcup_{v \in V(\Gamma)} \{x_1, \ldots, x_{p_v}\},$$
and the set of sections \( \{ s_e^\pm | e \in E(\Gamma) \} \) is defined by paired unordered marked points

\[
\bigsqcup_{v \in V(\Gamma)} \{ y_1, \ldots, y_{q_v} \}.
\]

On the universal family (3.4), an isomorphism between two fiber curves

\[
f : \bigsqcup_{v \in V(\Gamma)} (C_v, \{ x_1, \ldots, x_{p_v} \}, \{ y_1, \ldots, y_{q_v} \}) \rightarrow \bigsqcup_{v \in V(\Gamma)} (C'_v, \{ x'_1, \ldots, x'_{p_v} \}, \{ y'_1, \ldots, y'_{q_v} \})
\]

means an isomorphism \( f : \bigsqcup_{v \in V(\Gamma)} C_v \rightarrow f : \bigsqcup_{v \in V(\Gamma)} C'_v \) which preserves the ordered marked points

\[
f : \bigsqcup_{v \in V(\Gamma)} \{ x_1, \ldots, x_{p_v} \} \rightarrow \bigsqcup_{v \in V(\Gamma)} \{ x'_1, \ldots, x'_{p_v} \}
\]

and preserves the paired sets \( \bigsqcup_{v \in V(\Gamma)} \{ y_1, \ldots, y_{q_v} \} \) and \( \bigsqcup_{v \in V(\Gamma)} \{ y'_1, \ldots, y'_{q_v} \} \).

By identifying \( s_e^+ \) and \( s_e^- \) in \( \tilde{C}_\Gamma \) we have a new family of nodal curves \( C_{\tilde{\Gamma}} \). Set \( \tilde{T}_\Gamma = T_\Gamma \). Then we have a universal family

\[
(3.5)
\]

Applying the groupoid construction in \( \S 3.2 \) there is a canonical proper étale groupoid

\[
M_{[\Gamma]} = (\mathcal{A}_{C_{\tilde{\Gamma}}}(\tilde{T}_\Gamma) \rightrightarrows \tilde{T}_\Gamma)
\]

associated to the universal family (3.5). The orbit space of \( M_{[\Gamma]} \) is \( M_{[\Gamma]} \), the set of isomorphism classes of genus \( g \) stable curve \( C \) with \( n \)-marked points whose weighted dual graph is \( \Gamma \).

Further it is known that the closure of \( M_{[\Gamma]} \) in the coarse moduli space \( \overline{M}_{g,n} \) is given by

\[
\overline{M}_{g,n} = \bigsqcup_{[\Gamma] \in S_{g,n}} M_{[\Gamma]}.
\]

This says that \( \overline{M}_{g,n} \) is an orbifold stratified space. \( \square \)

**Remark 3.10.** Associated to the universal family \( C_{\tilde{\Gamma}} \rightarrow \tilde{T}_\Gamma \). We have a proper étale Lie groupoid description for the universal curve \( C_{\Gamma} \)

\[
C_{[\Gamma]} = (\mathcal{A}_{C_{\Gamma}}(\tilde{T}_\Gamma) \rightrightarrows C_{\Gamma})
\]

There is a natural submersion \( C_{[\Gamma]} \rightarrow M_{[\Gamma]} \).

**Remark 3.11.** Let \( M^1 \rightrightarrows M^0 \) be the disjoint of the proper étale Lie groupoids over \( S_{g,n} \). Then there is no sensible topology of \( M^0 \) and \( M^0 \) such that \( M^1 \rightrightarrows M^0 \) is a topological groupoid. Hence, this does not fit with our assumption in \( \S 2.3 \). However, we shall explain how the notion of good orbifold atlases can still be found following the remarks at the end of Section 2.
4. Horocycle structures associated to marked or nodal points

In this section, we introduce a notion of horocycle structures and show that each stratum in $\overline{M}_{g,n}$, there exists a smooth family of horocycle structures in the orbifold sense.

Let $(C, \{p_1, \cdots, p_n\})$ be a Riemann surface of genus $g$ and $n$-marked points. Equivalently, we may consider the punctured surface of $C$ with marked points removed.

$$C^* = C \setminus \{p_1, \ldots, p_n\}.$$ 

When $C^*$ is of negative Euler characteristic, $C^*$ has a complete hyperbolic metric $\rho$ (a complete metric of constant curvature $-1$) under which its punctures become cusps of the hyperbolic metric. Locally, the geometry of these cusps can be described by applying the uniformization theorem to $C$ at a puncture $p$ as follows (see [34]).

Let $H = \{\zeta = x + iy | y > 0\}$ be the half upper surface with the Poincare metric $\rho_0(\zeta) = \frac{1}{(\text{Im} \zeta)^2} d\zeta d\bar{\zeta}$.

Let $D = \{\zeta \in H | \text{Im} \zeta \geq 1\}$ be a cylinder, and $\rho_0$ induces a metric on $D$, which is still denoted by $\rho_0$.

An interesting result is that for any punctured point $p_i$ there exists a neighborhood $U_i$ of $p_i$ in $C$ such that $(U_i, \rho) \cong (D, \rho_0)$, moreover, all $U_i$’s are disjoint with each other. Hence, we fix an identification

$$\zeta_i : U_i \rightarrow D.$$

Let $z_i = e^{2\pi i \zeta_i}$. Then $z_i$ is a local complex coordinate on $U_i$ with $z_i(p_i) = 0$ and

$$U_i = \{z_i | \ln |z_i| < -2\pi\}.$$ 

For any $c \in (0, e^{-2\pi})$, the circle $|z_i| = c$ is called a closed horocycle at $p_i$ with hyperbolic length $\ell(c) = -2\pi / \log c$. We call $(U_i, z_i)$ a horodisc of $p_i$. We define a metric $\eta_{\text{can}}^i$ on $T_p C$ such that

(4.1)$$\eta_{\text{can}}^i \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right) = 1.$$ 

The canonical horodisc of $C$ at $p$ of radius $\delta$ is defined to be

$$D^\delta_p = \{x \in C | |z(x)| < \delta\}$$

for $\delta \leq e^{-2\pi}$. Denote by

$$T_p C(\delta) = \{z \in T_p C | |z|_{\text{can}} < \delta\}.$$ 

Under the canonical complex coordinate $z$ near $p$, $z$ induces a canonical biholomorphic map

$$z : D^\delta_p \rightarrow T_p C(\delta).$$

We denote the inverse map by the horo-map:

$$\text{hor} : T_p C(\delta) \rightarrow C.$$
We call the triple \((\eta_{\text{can}}, \delta, \text{hor})\) the canonical horocycle structure at \(p\).

**Definition 4.1.** Let \(C\) be a genus \(g\) stable curve with \(n\)-marked points \(\{p_1, \ldots, p_n\}\) and nodal points. Let \(\pi : \tilde{C} \to C\) be its normalization. If \(p\) is a marked point, a horocycle structure on \(C\) at \(p\) is a metric \(\eta\) on \(T_pC\) with a holomorphic embedding

\[ h : T_pC(\delta) \to C \]

for some small \(\delta\) such that \(h(0) = p\) and \((dh)_0\) is the identity operator. We denote the horocycle structure by the triple \((\eta, \delta, h)\). The local inverse map of \(h\) is called the horo-coordinate associated to the horocycle structure at \(p\). If \(p\) is a nodal point, a horocycle structure on \(C\) at \(p\) is given by a pair of metrics \(\eta^\pm\), a pair of small constants \(\delta^\pm\) and a pair of maps

\[ h^\pm = \pi \circ h_{p^\pm} : T_{p^\pm}C(\delta^\pm) \to \tilde{C} \to C, \]

where \((\eta^\pm, \delta^\pm, h^\pm)\) are the horocycle structures on \(\tilde{C}\) at \(p^\pm = \pi^{-1}(p)\).

**Remark 4.2.** Given a genus \(g\) stable curve \(C\) with \(n\)-marked points \(\{p_1, \ldots, p_n\}\) and nodal points, then \((\eta_{\text{can}}, e^{-2\pi}, \text{hor})\) constructed above defines a horocycle structure at each marked point and each nodal point. We refer it as a canonical horocycle structure. A general horocycle structure on \(C\) in Definition 4.1 differs from this canonical one by a small perturbation. All these horodiscs provided by a horocycle structure are mutually disjoint by taking a fixed small radius \(\delta\).

The following lemma is obvious from the definition of horocycle structure.

**Lemma 4.3.** Let \((\eta_0, \delta_0, h_0)\) and \((\eta_1, \delta_1, h_1)\) be two horocycle structures on \(C\) at point \(p\), then there exists a small constant \(\delta < \min(\delta_0, \delta_1)\) such that

\[(1 - t)\eta_0 + th_0, \delta, (1 - t)h_0 + th_1\]

defines a family of horocycle structures at \(p\) for \(t \in [0, 1]\). Here, we use the canonical horocycle coordinate at the neighborhood of \(p\).

Fix a choice function on \(S_{g,n}\) such that each \([\Gamma] \in S_{g,n}\) is represented by a weighted dual graph \(\Gamma\). Let \(\tilde{\Gamma}\) be the normalization of \(\Gamma\), obtained by severing all the edges in \(\Gamma\). The proper étale Lie groupoid \(M_{[\Gamma]}\) for \(M_{[\Gamma]}\) is obtained by the universal family (3.5) of stable curve of the weighted dual graph given by \(\Gamma\) by the canonical construction in Section 3.2. Recall that this universal family is obtained from the universal family (3.4). Note that the universal family \(C_\Gamma \to T_\Gamma\) comes with sections

\[ \{\sigma_i, s_e | i = 1, \ldots, n, e \in E(\Gamma)\} \]

defined by marked points and nodal points. Let \(C_\Gamma^*\) be the family of punctured Riemann surfaces, given by \(C_\Gamma\) with all these sections removed. Then there exists a Euclidean metric on the vertical tangent bundle \(T^vC_\Gamma^*\) of \(\hat{\pi} : C_\Gamma^* \to T_\Gamma\) such that the restriction of this metric on each fiber \(\hat{\pi}^{-1}(b)\) is the canonical hyperbolic metric. Moreover, this smooth family of hyperbolic metrics is invariant under the action of the proper étale Lie groupoid \(M_{[\Gamma]}\).
Let $L_i$ be the complex line over $T_\Gamma$ defined by the pull-back of $T^\ast C_\Gamma$ by the section $\sigma_i$ for $i = 1, \cdots, n$, and $L_\pm^e$ be the pair of complex line over $T_\Gamma$ defined by the pull-back of $T^\ast C_\Gamma$ by the section $s_\pm^e$ for $e \in E(\Gamma)$. Here $T^\ast C_\Gamma$ is the vertical tangent bundle of the universal family (3.4). The next proposition implies that there is a smooth family of horocycle structures at each of marked or nodal points of the universal family $C_\Gamma \to T_\Gamma$.

**Proposition 4.4.** The line bundles $\{L_i, L_+^e, L_-^e\}$ are complex line bundles over the proper étale Lie groupoid $M_{[\Gamma]}$, and will be denoted by $L_i$, $L_+^e$ and $L_-^e$ accordingly. Let $L$ be one of complex line bundles in $\{L_i, L_\pm^e | i = 1, \cdots, n, e \in E(\Gamma)\}$. There exist a canonical metric on $L$ defined by the smooth family of hyperbolic metrics on $T^\ast C_\Gamma$. Moreover, for any $\delta \in (0, e^{-2\pi})$, the canonical horo-coordinate associated to these hyperbolic metrics defines a smooth strict morphism

$$\text{hor}_{\text{can}} : L(\delta) \longrightarrow C_\Gamma.$$

**Proof.** The only nontrivial part is the smoothness of $\text{hor}$. This follows from Lemma 1.1 (iv) in [35] and the invariance of the hyperbolic metrics under the action of the proper étale Lie groupoid $M_{[\Gamma]}$. \qed

5. **Gluing data and good orbifold gluing structures for moduli spaces of stable curves**

With these preparations in Sections 4 and 5, we come to the construction of good orbifold structures for the orbifold stratified space

$$\mathcal{M}_{g,n} = \bigsqcup_{[\Gamma] \in S_{g,n}} M_{[\Gamma]}$$

with a choice function on $S_{g,n}$, and the canonical proper étale Lie groupoid $M_{[\Gamma]}$ for $M_{[\Gamma]}$.

We start with the orbifold gluing bundle over $GL_{[\Gamma]} \to M_{[\Gamma]}$. Define the orbifold gluing bundle to be

$$\text{GL}_{[\Gamma]} = \bigoplus_{e \in E(\Gamma)} L_+^e \otimes L_-^e \longrightarrow M_{[\Gamma]},$$

where $L_\pm^e$ is defined in Proposition 4.4. We remark, this bundle, as a proper étale Lie groupoid can be identified with the canonical groupoid associated to $M_{[\Gamma]}$-action on the complex vector bundle

$$GL_{[\Gamma]} = \bigoplus_{e \in E(\Gamma)} L_+^e \otimes L_-^e \longrightarrow T_\Gamma,$$

as described in Section 3.2.

Given any $[\Gamma'] \prec [\Gamma]$, let $\mathcal{E}([\Gamma], [\Gamma'])$ be the collection of subsets of $E(\Gamma)$ such that the contraction of $\Gamma$ along each element in $\mathcal{E}([\Gamma], [\Gamma'])$ is isomorphic to $\Gamma'$. Set

$$\text{GL}_{[\Gamma]}^{[\Gamma']} = \bigsqcup_{I \in \mathcal{E}([\Gamma], [\Gamma'])} \prod_{e \in I} (L_+^e \otimes L_-^e)^\times,$$

where $(L_+^e \otimes L_-^e)^\times$ is the $\mathbb{C}^\times$-bundle obtained from $L_+^e \otimes L_-^e$ with the zero section removed.
Lemma 5.1. Denote by $S^{[\Gamma]}$ the partially ordered set $S^{[\Gamma]} = \{ [\Gamma'] \in S_{g,n} | [\Gamma'] \succeq [\Gamma] \}$. There exists a canonical linear stratification on $GL^{[\Gamma]}$ with respect to $S^{[\Gamma]}$

$$GL^{[\Gamma]} = \bigsqcup_{[\Gamma'] \in S^{[\Gamma]}} GL^{[\Gamma']}_{[\Gamma']}$$

so that $GL^{[\Gamma]}$ is an orbifold gluing bundle with a canonical metric (Cf. Condition A').

Proof. Notice that the power set

$$2^{E(\Gamma)} = \bigsqcup_{[\Gamma'] \in S^{[\Gamma]}} E([\Gamma], [\Gamma'])$$

and all elements in $E([\Gamma], [\Gamma'])$ have the same cardinality. The proof of this lemma is straightforward. 

Next we describe the orbifold gluing datum for the orbifold gluing bundle $GL^{[\Gamma]}$ over any open full subgroupoid $U_{[\Gamma]}$ of $M_{[\Gamma]}$. We remark that we only need to show the stratum-wise gluing map $\phi_{[\Gamma'],[\Gamma]}$ being generalised isomorphism. We review how the standard grafting construction can be performed using the smooth family of horocycle structures on the universal family $C_{\Gamma} \to T_{\Gamma}$. By Proposition 4.4, we assume that the family of horocycle structures along the fiber of $C_{\Gamma}$ is invariant under the action of $M_{[\Gamma]}$.

Let $\eta$ be a point in $T_{\Gamma}$ represented by a nodal curve $C_{\eta}$ in $C_{\Gamma}$, whose normalization at a nodal point $p_e$ for $e \in E(\Gamma)$ is

$$(C^\pm, p_e \pm).$$

Using the horocycle structure of $C_{\eta}$ at the nodal point $p$, for $\delta \in (0, e^{-2\pi})$, we have two holomorphic maps

$$h^\pm : T_{p_e \pm} C^\pm(\delta) \to C^\pm$$

such that $h^\pm(0) = p_e \pm$ and the differentials $dh^\pm$ at 0 are the identity maps. These provide complex local coordinates at $p_e$ and $p_e'$ of $C^+$ and $C^-$ respectively, denoted by $z$ and $w$.

Suppose that $C_{\eta}$ is decomposed to be two parts

$$C^\eta_{out} = C \setminus \{ \{x \in C | |z(x)| \leq \delta\} \cup \{y \in C | |w(y)| \leq \delta\} \}$$

and

$$C^\eta_{in} = \{ \{x \in C | |z(x)| \leq \delta\} \cup \{y \in C | |w(y)| \leq \delta\} \}.$$

The gluing construction with respect to a horocycle structure $h^\pm$ on $C$ is given as the following. Let $t \in T_{p_e +} C^+ \otimes T_{p_e -} C^-$ such that $0 < |t| < \delta^2$. Define

$$C^\eta_{n,t} = C \setminus \{ \{x \in C | |z(x)| \leq \frac{|t|}{\delta}\} \cup \{y \in C | |w(y)| \leq \frac{|t|}{\delta}\} \}.$$

Then $C^\eta_{n,t} = C^\eta_{n,t} / \sim$ where $x \sim y$ in $C^\eta_{n,t}$ if and only if

$$|t| < |z(x)| < \delta, \quad \frac{|t|}{\delta} < |w(y)| < \delta \quad \text{and} \quad z(x)w(y) = t. \quad (5.1)$$

Then for $t \neq 0$, $C^\eta_{n,t}$ is a smooth Riemann surface with the weight dual graph given by the contraction of $\Gamma$ at $e \in E(\Gamma)$.
Given $I \in \mathcal{E}([\Gamma], [\Gamma'])$, we can perform the above grafting construction on $C_\eta$ simultaneously for $\epsilon \in I$. Suppose $I = \{e_1, \cdots, e_k\} \in \mathcal{E}_\Gamma$. Let $\eta \in T_\Gamma$ and $C_\eta$ be the representing curve. For $e_i$ let $y_i \in C_\eta$ be the corresponding nodal point and suppose that horodiscs associated to $e_i$ are $|x| > \delta$ and $|w_i| < \delta$ respectively. Denote 

$$t = (t_1, t_2, \cdots, t_k) \in \prod_{e \in I} (L_e^+ \otimes L_e^-)^\times.$$ 

Then for any $\epsilon < \delta^2$ and $|t_| < \epsilon$ we have

\begin{equation}
C_{\eta, \epsilon}^* = C_\eta \setminus \left( \bigcup_{i=1}^k \{x \in C_\eta | |z_i(x)| \leq \frac{|t_i|}{\delta} \} \cup \{y \in C_\eta | |w_i(y)| \leq \frac{|t_i|}{\delta} \} \right).
\end{equation}

Then the grafting construction defines a nodal curve $C_{\eta, t} = C_{\eta, \epsilon}/\sim$ where $x \sim y$ in $C_{\eta, \epsilon}$ if and only if for some $i$

$$\frac{|t_i|}{\delta} < |z_i(x)| < \delta, \quad \frac{|t_i|}{\delta} < |w_i(y)| < \delta \quad \text{and} \quad z_i(x)w_i(y) = t_i.$$ 

Therefore, we get a smooth family of nodal curve of type $[\Gamma']$ parametrised by $t$ in the fiber of

$$GL_{[\Gamma']}([\Gamma]) \epsilon = \prod_{I \in \mathcal{E}([\Gamma], [\Gamma'])} \prod_{e \in I} (L_e^+ \otimes L_e^-)^\times(\epsilon)$$

at $\eta$ for a sufficiently small $\epsilon$. This grafting construction can also be performed in a small neighbourhood $U_\eta$ of $\eta$ in $T_\Gamma$ with respect a smooth family of horocycle structure at the section $s_e$ of $C_\Gamma$ over $U_\eta$. Note this family of nodal curves is completely determined by the family of horocycle structures on the universal family over $U_\eta$. Denote the resulting family of nodal curves by

$$C_{U_\eta, [\Gamma'], e} \to GL_{[\Gamma']}([\Gamma])U_\eta(\epsilon)$$

which has the topological type $[\Gamma'] \in S_{g,n}$. From the grafting construction, we know that this family of nodal curves is a trivial family away from the grafting regions.

Note that $U_\eta$ can be chosen such that is invariant under the action of $M_{[\Gamma]}$ on $T_\Gamma$. By the invariance of horocycle structures under the action of $M_{[\Gamma]}$, for a sufficiently small $\epsilon$, we get a generalised morphism

\begin{equation}
\phi_{[\Gamma']} : GL_{[\Gamma']}([\Gamma])|_{U_\eta} \to M_{[\Gamma']}
\end{equation}

using the canonical construction of proper étale Lie groupoid in Section 3.2. Here $U_\eta$ is the open full subgroupoid of $M_{[\Gamma]}$ associated to $U_\eta$. By Theorem A in [33], we know that this generalised morphism is a generalised isomorphism onto its image using the real analytic coordinate functions on $T_{\Gamma'}$ from the gluing parameters in $GL_{[\Gamma']}([\Gamma])|_{U_\eta}$.

In particular, given a proper open full subgroupoid $U_{\Gamma'}$ of $M_{[\Gamma]}$, there is an orbifold gluing map

\begin{equation}
\phi_{[\Gamma']} : GL_{[\Gamma']}([\Gamma])|_{U_{\Gamma'}} \to M_{[\Gamma']},
\end{equation}

which is a generalized isomorphism onto its image (see Remark 2.23).
Proposition 5.2 (Verification of Condition B’). Given a proper open full subgroupoid $U_\Gamma$ of $M_{[\Gamma]}$, let $D = D([\Gamma], [\Gamma'], \epsilon)$ and $R = R([\Gamma], [\Gamma'], \epsilon)$ be the domain and the image of the gluing map $\phi_{[\Gamma']}^{[\Gamma]}$ in (5.4). There exists a stratified bundle map

\[ (5.5) \quad \Phi_{[\Gamma']}^{[\Gamma]} : N(GL_{[\Gamma']}^{[\Gamma]})|_D \longrightarrow GL^{[\Gamma']}|_R \]

in the sense of generalised morphism, such that the following diagram commutes

\[
\begin{array}{ccc}
N(GL_{[\Gamma']}^{[\Gamma]})|_D & \xrightarrow{\Phi_{[\Gamma']}^{[\Gamma]}} & GL^{[\Gamma']} \\
\downarrow & & \downarrow \\
D & \xrightarrow{\phi_{[\Gamma']}^{[\Gamma]}} & M_{[\Gamma']}.
\end{array}
\]

Moreover, $\Phi_{[\Gamma']}^{[\Gamma]}$ is a stratified bundle isomorphism onto its image $GL^{[\Gamma']}|_R$ in the sense of generalised isomorphism.

**Proof.** From the above construction, we know that we cover $U$ by a collection of full subgroupoids of the form $U_\eta$ for some $\eta \in \mathcal{T}_\Gamma$. Then by the definition of generalised morphisms and generalised isomorphisms, we only need to prove the proposition for $D$ and $R$ being the domain and the image of the gluing map $\phi_{[\Gamma']}^{[\Gamma]}$ in (5.4). From the grafting construction, we have a smooth family nodal curves of topological type $\Gamma'$, denoted by

\[ C(U_\eta, [\Gamma'], \epsilon) \rightarrow D(U_\eta, [\Gamma'], \epsilon) := GL_{[\Gamma']}^{[\Gamma]}|_{U_\eta}, \]

Hence, there is a strict morphism $\Psi_{[\Gamma']}^{[\Gamma]} : C(U_\eta, [\Gamma'], \epsilon) \rightarrow C_{[\Gamma']}$ (the universal family over $M_{[\Gamma']}$) such that the following diagram commutes

\[ (5.6) \quad \begin{array}{ccc}
C(U_\eta, [\Gamma'], \epsilon) & \xrightarrow{\psi_{[\Gamma']}^{[\Gamma]}} & C_{[\Gamma']} \\
\downarrow & & \downarrow \\
D(U_\eta, [\Gamma'], \epsilon) & \xrightarrow{\phi_{[\Gamma']}^{[\Gamma]}} & M_{[\Gamma']},
\end{array} \]

which preserves marked and nodal points. Note that $C(U_\eta, [\Gamma'], \epsilon) \rightarrow D$ is a trivial fibration away from the gluing region. From the definitions of the normal bundle $N(GL_{[\Gamma']}^{[\Gamma]})$ and the gluing bundle $GL^{[\Gamma']}$, we know that there is a stratified bundle map $\Phi_{[\Gamma']}^{[\Gamma]}$ such that the diagram in Proposition commutes. It is clear that $\Phi_{[\Gamma']}^{[\Gamma]}$ is a stratified bundle isomorphism onto its image. \qed

Let the gluing map

\[ \phi_{[\Gamma']}^{[\Gamma]} : GL^{[\Gamma']}(\epsilon)|_{U_{[\Gamma]}} = \bigcup_{[\Gamma'] \in S_{[\Gamma]}} GL_{[\Gamma']}^{[\Gamma]}(\epsilon)|_{U_{[\Gamma]}} \longrightarrow \bigcup_{[\Gamma'] \in S_{[\Gamma]}} M_{[\Gamma']} \]

be the union of $\phi_{[\Gamma']}^{[\Gamma]}$ for $[\Gamma'] \in S_{[\Gamma]}$. Hence, we get an orbifold gluing datum

\[ (U_{[\Gamma]}, \eta_{[\Gamma]}, \epsilon, \phi_{[\Gamma]}, \{\Phi_{[\Gamma']}^{[\Gamma]}\}) \]

for any proper subgroupoid $U_{[\Gamma]}$ in the sense of comments following Remark 2.23.
Remark 5.3. Suppose \( \Gamma' \) is obtained from \( \Gamma \) by the contraction along \( I \in \mathcal{E}(\Gamma, [\Gamma']) \). As the family \( \mathcal{C}(U_\eta, [\Gamma'], \epsilon) \to \mathcal{D}(U_\eta, [\Gamma'], \epsilon) \) is trivial away from the grafting regions. For any edge \( e \) in the compliment of \( I \) in \( E(\Gamma) \), the grafting construction carries the horocycle structure along the universal curves over \( U_\eta \) at \( s_e \) to
\[
\mathcal{C}(U_\eta, [\Gamma'], \epsilon) \to \mathcal{D}(U_\eta, [\Gamma'], \epsilon).
\]
Hence, by the commutative diagram (5.6) and local isomorphism of \( \Psi([\Gamma], [\Gamma']) \), we get a smooth family of horocycle structure at the section \( s_e \) along the that over the image of the orbifold gluing map \( \phi|_{\Gamma'} \). We emphasize that this horocycle structure is different from the canonical horocycle structure induced from \( C_{\Gamma'} \). From the convexity property of horocycle structures (Cf. Lemma 4.3), we know that any convex combination of these two horocycle structure provides a family of horocycle structure on the universal curves over the image of the orbifold gluing map \( \phi|_{\Gamma'} \). We remark that this observation is vital in obtaining the inward-extendibility condition for the existence of good orbifold gluing structures.

Notice that the only non-trivial issue for the existence of good orbifold gluing structures as specified by Definition 2.21 is the inward-extendibility condition. We proceed to this final issue.

Let \( U_1 \subset U_2 \) be a pair of boundary-type open full subgroupoids of \( M_{[\Gamma]} \). Suppose that we have an orbifold gluing datum
\[
\mathcal{A} = (U_2, \eta, \epsilon, \phi|_{\Gamma'}, \{\Phi|_{[\Gamma']}\})
\]
defined with respect to a family of horocycle structures \( (\eta, \delta, h) \) on the universal curve over \( U_2 \) associated to each \( e \in E(\Gamma) \). Let \( (\tilde{\eta}, \tilde{\delta}, \tilde{h}) \) be another horocycle structures on \( M_{[\Gamma]} \) (for example, the canonical horocycle structure induced from hyperbolic metrics or induced from the gluing map). Such horocycle structures induce a gluing datum
\[
\tilde{\mathcal{A}} = (\tilde{U}, \tilde{\eta}, \tilde{\epsilon}, \tilde{\phi}|_{[\Gamma']}, \{\tilde{\Phi}|_{[\Gamma']}\}),
\]
where \( \tilde{U} \) can be any proper open full subgroupoid of \( M_{[\Gamma]} \) such that
\[
\tilde{U} \cup U_1 = M_{[\Gamma]}.
\]
We want to sew these two gluing data together. This is equivalent to patch the horocycle structures. This can be easily done as follows using the convexity property of horocycle structures.

Let \( \beta_1 \) and \( \beta_2 \) be an orbifold partition of unity subordinated to the cover \( \{U_2, \tilde{U}\} \) such that \( \beta_1 \equiv 1 \) on \( U_1 \). Then applying Lemma 4.3, we get a new horocycle structures along the universal curve over \( M_{[\Gamma]} \), from which we have a new orbifold gluing datum
\[
\hat{\mathcal{A}} = (M_{[\Gamma]}, \tilde{\eta}, \tilde{\delta}, \tilde{\phi}|_{[\Gamma']}, \{\tilde{\Phi}|_{[\Gamma']}\}).
\]
It is easy to see this is an inward-extension of \( \mathcal{A} \).

Let \( \mathcal{GL} \) be the collection of gluing data that given by the grafting construction using horocycle structures. Then we have proved the following proposition.

**Proposition 5.4.** \( \mathcal{GL} \) is a good orbifold gluing structure.
As a corollary, we have a collection of orbifold gluing data
\begin{equation}
\mathcal{A}_\Gamma = (\mathcal{M}_\Gamma, \eta_\Gamma, \epsilon_\Gamma, \phi_\Gamma, \Phi_\Gamma) \mid [\Gamma] \in \mathcal{S}_{g,n}
\end{equation}
which are gluing-compatible.

Following the discussion after Remark 2.23, we have a smooth orbifold structure on the Deligne-Mumford moduli space $\overline{M}_{g,n}$ given by the canonical proper étale Lie groupoid
\[
\left( (\mathcal{A}_{\overline{\mathcal{T}}_{g,n}}) \rightarrow \overline{\mathcal{T}}_{g,n} \right)
\]
using the notation from Section 3.2, where $\overline{\mathcal{T}}_{g,n} = \bigsqcup_{[\Gamma] \in \mathcal{S}_{g,n}} GL_{[\Gamma]}(\epsilon)$, and $\overline{\mathcal{C}}_{\Gamma,\epsilon}$ with $\mathcal{C}_{\Gamma,\epsilon}$ being the universal curve over $GL_{[\Gamma]}(\epsilon)$ associated to the orbifold gluing datum $\mathcal{A}_\Gamma = (\mathcal{M}_\Gamma, \eta_\Gamma, \epsilon_\Gamma, \phi_\Gamma, \Phi_\Gamma)$ from (5.7).

Acknowledgments. This work is supported by the Australian Research Council Grant and the National Natural Science Foundation of China Grant.

References

[1] A. Adem, J. Leida and Y. Ruan, Orbifolds and stringy topology, Cambridge Tracts in Mathematics 171, Cambridge University Press (2007).

[2] E. Arbarello, M. Cornalba and P. Griffiths, Geometry of Algebraic Curves, Vol. II. Grundlehren der mathematischen Wissenschaften, 2011.

[3] N. Berline, E. Getzler and M. Vergne, Heat kernels and Dirac operators, Springer-Verlag, Berlin, 2004.

[4] R. Bott and L. Tu, Differential Forms in Algebraic Topology, Springer-Verlag, 1982.

[5] B. Chen Smoothness on bubble tree compactified instanton moduli spaces. Acta Math. Sin. (Engl. Ser.) 26 (2010), no. 2, 209-240.

[6] B. Chen, S. Hu, A deRham model for Chen-Ruan cohomology ring of abelian orbifolds. Math. Ann., 336, no. 1, 5171, 2006. math.SG/0408265.

[7] B. Chen and A. Li, Symplectic virtual localization of Gromov-Witten classes, arXiv:DG/0610370.

[8] B. Chen, A. Li and B. Wang, part I, preprint, 2013.

[9] B. Chen and G. Tian, Virtual manifolds and localization, Acta Math. Sinica, 26 (1), 1-24.

[10] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus. Inst. Hautes Etudes Sci. Publ. Math. Vol. 36, 75-109,1969.

[11] S. Donaldson, Riemann surfaces, Oxford Graduate Texts in Mathematics 22, 2011.

[12] C. Earle and J. Eells, A fibre bundle approach to Teichmüller theory. J. Differential Geometry, 3 (1969) 19-43.

[13] C. Earle and A. Marden, Holomorphic Plumbing Coordinates on Teichmüller and Compactified Moduli Space. Preprint, 2011.

[14] M. Gromov, Pseudo-holomorphic curves in symplectic manifolds. Invent. Math. Vol. 82 (1985), 307-347.

[15] K. Fukaya and K. Ono, Arnold conjecture and Gromov-Witten invariant. Topology 38 (1999), no. 5, 933-1048.

[16] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, Lagrangian Intersection Theory, Anomaly and Obstruction, Parts I and II. AMS/IP Studies in Advanced Mathematics, Amer. Math. Soc. and Internat. Press.

[17] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Technical details on Kuranishi structure and virtual fundamental chain, arXiv:1209.4410.

[18] J. Harris and I. Morrison, Moduli of curves. Grad. Texts in Math. 187, Springer-Verlag, New York 1998.
[19] Michel Hilsum and Georges Skandalis. Morphismes K-orientés désespaces de feuilles et fonctorialité en théorie de Kasparov (daprés une conjecture d A. Connes). Ann. Sci. École Norm. Sup. (4), 20(3):325-390, 1987.
[20] F. Knudsen, The projectivity of the moduli space of stable curves. Math. Scand. 39 (1976) 19-55,52(1983) 161-212.
[21] Y. Imayoshi and M. Taniguchi, An Introduction to Teichmüler Spaces. Springer-Verlag, Berlin, 1992.
[22] D. Mumford, The structure of the moduli spaces of curves and abelian varieties. In Proc. Internat. Congr. Math., pages 457-467, 1970.
[23] A. Li, Y. Ruan, Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds. Invent. Math. 145(2001), 151-218.
[24] J. Li and G. Tian, Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. J. Amer. Math. Soc., 11(1), 119-174, 1998.
[25] G. Liu and G. Tian, Floer homology and Arnold conjecture. J. Differential Geom. 49 (1998), no. 1, 1-74.
[26] G. Liu and G. Tian, Constructing virtual Euler cycles and classes. Int. Math. Res. Surveys (2008)
[27] E. Lupercio and B. Uribe, Gerbes over orbifolds and twisted K-theory, Comm. Math. Phys. 245 (2004), no. 3, 449-489.
[28] D. McDuff and D. Salamon, J-holomorphic curves and Symplectic Topology, AMS Colloquium Publications, Vol. 52.
[29] I. Moerdijk and D. A. Pronk, Simplicial cohomology of orbifolds, Indag. Math. (N.S.) Vol. 10, (1999), no. 2, 269-293.
[30] J. Robbin and D. Salamon, A construction of the Deligne-Mumford orbifold. J. Eur. Math. Soc. (JEMS) 8 (2006), no. 4, 611 - 699.
[31] Y. Ruan, Virtual neighborhoods and pseudo-holomorphic curves. Turkish Jour. of Math. 1(1999), 161-231.
[32] M. Seppälä and T. Sorvali, Geometry of Riemann surfaces and Teichmüller spaces, Elsevier Science Publishing Company, 1992.
[33] M. Wolf, S. Wolpert, Real Analytic Structures on the Moduli Space of Curves, Amer. J. Math. 114 , no. 5, 1079-1102, 1992.
[34] S. Wolpert, Cusps and the family hyperbolic metric, Duke Jou., vol. 138, no. 3, 423-443, 2007.
[35] S. Wolpert, Spectral limits for hyperbolic surfaces. II, Invent. Math., 108(1), 91-129, 1992.

SCHOOL OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, CHINA
E-mail address: bohui@cs.wisc.edu, math_li@yahoo.com.cn

DEPARTMENT OF MATHEMATICS, AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA ACT 0200, AUSTRALIA
E-mail address: bai-ling.wang@anu.edu.au