On isomorphism classes and invariants of low dimensional complex filiform Leibniz algebras (part 1)

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Abstract

The paper aims to investigate the classification problem of low dimensional complex none Lie filiform Leibniz algebras. There are two sources to get classification of filiform Leibniz algebras. The first of them is the naturally graded none Lie filiform Leibniz algebras and the another one is the naturally graded filiform Lie algebras [7]. Here we do consider Leibniz algebras appearing from the naturally graded none Lie filiform Leibniz algebras. According to the theorem presented in [4] this class can be split into two subclasses. However, isomorphisms within each class there were not investigated. In [6] U.D.Bekbaev and I.S.Rakhimov suggested an approach to the isomorphism problem in terms of invariants. This paper presents an implementation of the results of [6] in low dimensional cases. Here we give the complete classification of complex none Lie filiform Leibniz algebras in dimensions at most 8 from the first class of the above mentioned result of [4] and give a hypothetic formula for the number of isomorphism classes in finite dimensional case.

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1 Preliminaries

Definition 1.1 An algebra $L$ over a field $K$ is called a Leibniz algebra if it satisfies the following Leibniz identity:

$$x(yz) = (xy)z - (xz)y.$$ 

Let $\text{Leib}_n(K)$ be a subvariety of $\text{Alg}_n(K)$ consisting of all $n$-dimensional Leibniz algebras over $K$. It is invariant under the isomorphic action of $\text{GL}_n(K)$ ("transport of structure"). Two algebras are isomorphic if and only if they belong to the same orbit under this action. As a subset of $\text{Alg}_n(K)$ the set $\text{Leib}_n(K)$ is specified by the system of equations with respect to structural constants $\gamma^k_{ij}$:

$$\sum_{l=1}^{n} \left( \gamma^l_{jk} \gamma^m_{il} - \gamma^l_{ij} \gamma^m_{lk} + \gamma^l_{ik} \gamma^m_{lj} \right) = 0.$$ 

It is easy to see that if the multiplication in Leibniz algebra happens to be anticommutative then it is a Lie algebra. So Leibniz algebras are "noncommutative" generalization of Lie algebras. As for Lie algebras case they are well known and several classifications of low dimensional cases have been given. But unless simple Lie algebras the classification problem of all Lie algebras in common remains a big problem. Yu.I.Malcev [1] reduced the classification of solvable Lie algebras to the classification of nilpotent Lie algebras. Apparently the first non-trivial classification of some classes of low-dimensional nilpotent Lie algebra are due to Umlauf. In his thesis [2] he presented the redundant list of nilpotent Lie algebras of dimension less seven. He gave also the list of nilpotent Lie algebras of dimension less than ten admitting a so-called adapted basis (now, the nilpotent Lie algebras with this property are called filiform Lie algebras). It was shown by M.Vergne [3] the importance of filiform Lie algebras in the study of variety of nilpotent Lie algebras laws.

Further if it is not asserted additionally all algebras assumed to be over the field of complex numbers.

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Let $L$ be a Leibniz algebra. We put: $L^1 = L, L^{k+1} = [L^k, L], k \in N$.

**Definition 1.2** A Leibniz algebra $L$ is said to be nilpotent if there exists an integer $s \in N$, such that $L^1 \supset L^2 \supset \ldots \supset L^s = \{0\}$. The smallest integer $s$ for which $L^s = 0$ is called the nilindex of $L$.

**Definition 1.3** An $n$-dimensional Leibniz algebra $L$ is said to be filiform if $\dim L^i = n - i$, where $2 \leq i \leq n$.

**Theorem 1.4** Arbitrary complex non-Lie filiform Leibniz algebra of dimension $n + 1$ obtained from naturally graded none Lie filiform Leibniz algebra is isomorphic to one of the following filiform Leibniz algebras with the table non zero multiplications for basis vectors $\{e_0, e_1, \ldots, e_n\}$:

a) (The first class):
\[
\begin{aligned}
e_0e_0 &= e_2, \\
e_2e_0 &= e_i + 1, \\
e_0e_1 &= \alpha_3e_3 + \alpha_4e_4 + \ldots + \alpha_{n-1}e_{n-1} + \theta e_n, \\
e_1e_1 &= \alpha_3e_{j+2} + \alpha_4e_{j+3} + \ldots + \alpha_{n+1-j}e_n, \\
e_0e_2 &= e_2, \\
e_1e_0 &= e_{i+1}, \\
e_1e_2 &= \beta_3e_3 + \beta_4e_4 + \ldots + \beta_ne_n, \\
e_2e_1 &= \gamma e_n, \\
e_2e_2 &= \beta_3e_{j+2} + \beta_4e_{j+3} + \ldots + \beta_{n+1-j}e_n,
\end{aligned}
\]

b) (The second class):
\[
\begin{aligned}
e_0e_0 &= e_2, \\
e_2e_0 &= e_{i+1}, \\
e_0e_1 &= \beta_3e_3 + \beta_4e_4 + \ldots + \beta_ne_n, \\
e_1e_1 &= \gamma e_n, \\
e_2e_1 &= \beta_3e_{j+2} + \beta_4e_{j+3} + \ldots + \beta_{n+1-j}e_n.
\end{aligned}
\]

Note that the algebras from the first class and the second class never are isomorphic to each other.

In this paper we will deal with the first class of algebras of the above Theorem, they will be denoted $L(\alpha_3, \alpha_4, \ldots, \alpha_n, \theta)$, meaning that they are defined by parameters $\alpha_3, \alpha_4, \ldots, \alpha_n, \theta$. The class here will be denoted as $FLeib_{n+1}$. As for the second class it will be considered somewhere else.

Using the method of simplification of the basis transformations in the following criterion on isomorphism of two $(n + 1)$-dimensional filiform Leibniz algebras was given. Namely: let $n \geq 3$.

**Theorem 1.5** Two algebras $L(\alpha)$ and $L(\alpha')$ from $FLeib_{n+1}$, where $\alpha = (\alpha_3, \alpha_4, \ldots, \alpha_n, \theta)$ and $\alpha' = (\alpha_3', \alpha_4', \ldots, \alpha_n', \theta')$, are isomorphic if and only if there exist complex numbers $A, B$ such that $A(A + B) \neq 0$ and the following conditions hold:

\[
\begin{aligned}
\alpha_i' &= \frac{A + B}{A} \alpha_i, \\
\alpha_i' &= \frac{1}{A + B} (A + B)\alpha_i - \frac{t-l}{A + B} \sum_{k=3}^{l} (C_{k-1}^{k-2} A^{k-2} B \alpha_{t+2-k} + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^{t} \alpha_{t+3-i_1} \cdot \alpha_{i_1-1-k}) + \\
C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^{t} \sum_{i_3=k+3}^{i_2} \alpha_{t+3-i_2} \cdot \alpha_{i_2+3-i_3} \cdot \alpha_{i_3-k} + + \\
C^{1}_{k-1} AB^{k-2} \sum_{i_3=2k-2}^{t} \sum_{i_4=2k-2}^{i_3} \sum_{i_1=2k-2}^{i_2} \alpha_{t+3-i_3} \cdot \alpha_{i_3-3-i_4} \cdot \ldots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+5-2k} + + \\
B^{k-1} \sum_{i_2=2k-1}^{t} \sum_{i_3=2k-1}^{i_2} \sum_{i_1=2k-1}^{i_2} \alpha_{t+3-i_2} \cdot \alpha_{i_2+3-i_3} \cdot \ldots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+4-2k}) \cdot \alpha'_k,
\end{aligned}
\]

where $4 \leq t \leq n$.

\[
\theta' = \frac{1}{A + B} (A + B)\theta - n \sum_{k=3}^{l} (C_{k-1}^{k-2} A^{k-2} B \alpha_{n+2-k} + C_{k-1}^{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^{n} \alpha_{n+3-i_1} \cdot \alpha_{i_1-1-k}) + \\
C_{k-1}^{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^{n} \sum_{i_3=k+3}^{i_2} \alpha_{n+3-i_2} \cdot \alpha_{i_2+3-i_3} \cdot \alpha_{i_3-k} + + \\
C^{1}_{k-1} AB^{k-2} \sum_{i_3=2k-2}^{n} \sum_{i_4=2k-2}^{i_3} \sum_{i_1=2k-2}^{i_2} \alpha_{n+3-i_3} \cdot \alpha_{i_3-3-i_4} \cdot \ldots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+5-2k} + + \\
B^{k-1} \sum_{i_2=2k-1}^{n} \sum_{i_3=2k-1}^{i_2} \sum_{i_1=2k-1}^{i_2} \alpha_{n+3-i_2} \cdot \alpha_{i_2+3-i_3} \cdot \ldots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+4-2k}) \cdot \alpha'_k,
\]

It is not difficult to notice that the expressions for $\alpha_i', \theta'$ in Theorem 1.5 can be represented in the
following form: \( \alpha'_t = \frac{1}{t!} \varphi_t \left( \frac{B}{A}; \alpha \right) \), where \( \alpha = (\alpha_3, \alpha_4, ..., \alpha_n, \theta) \) and \( \varphi_t(y; z) = \varphi_t(y; z_3, z_4, ..., z_n, z_{n+1}) = (1 + y)z_t - \sum_{k=3}^{t-1} (C_{k-1}^{k-2} y z_{t-2-k} + C_{k-1}^{k-3} y^2 \sum_{i_1=k+2}^{t} z_{t+3-i_1} \cdot z_{i_1+1-k})^+$

\[ C_{k-1}^{k-4} y^3 \sum_{i_2=k+3}^{t} \sum_{i_1=k+3}^{i_2} z_{t+3-i_2} \cdot z_{i_2+3-i_1} \cdot z_{i_1-k} + ... + \]

\[ C_{k-1}^{k-1} y^{k-2} \sum_{i_k-3=2k-2}^{i_k-4=2k-2} \sum_{i_1=2k-1}^{i_k=2k-2} z_{t+3-i_k-3} \cdot z_{i_k-3+3-i_k-4} \cdot ... \cdot z_{i_2+3-i_1} \cdot z_{i_1+5-2k} \]

\[ y^{k-1} \sum_{i_k-2=2k-1}^{i_k-3=2k-1} \sum_{i_1=2k-1}^{i_k=2k-1} z_{t+3-i_k-2} \cdot z_{i_k-2+3-i_k-3} \cdot ... \cdot z_{i_2+3-i_1} \cdot z_{i_1+4-2k} \cdot \varphi_k(y; z) \]

for \( 3 \leq t \leq n. \)

\[ \theta' = \frac{1}{A - \rho n + 1 \left( \frac{B}{A}; \alpha \right) \}, \text{where } \varphi_{n+1}(y; z) = \varphi_{n+1}(y; z_3, z_4, ..., z_n, z_{n+1}) = \]

\[ (z_{n+1} + yz_n - (1 + y) \sum_{k=3}^{n-1} (C_{k-1}^{k-2} y z_{n+2-k} + C_{k-1}^{k-3} y^2 \sum_{i_1=k+2}^{n} z_{n+3-i_1} \cdot z_{i_1+1-k})^+ \]

\[ C_{k-1}^{k-4} y^3 \sum_{i_2=k+3}^{n} \sum_{i_1=k+3}^{i_2} z_{n+3-i_2} \cdot z_{i_2+3-i_1} \cdot z_{i_1-k} + ... + \]

\[ C_{k-1}^{k-1} y^{k-2} \sum_{i_k-3=2k-2}^{i_k-4=2k-2} \sum_{i_1=2k-1}^{i_k=2k-2} z_{n+3-i_k-3} \cdot z_{i_k-3+3-i_k-4} \cdot ... \cdot z_{i_2+3-i_1} \cdot z_{i_1+5-2k} \]

\[ + y^{k-1} \sum_{i_k-2=2k-1}^{i_k-3=2k-1} \sum_{i_1=2k-1}^{i_k=2k-1} z_{n+3-i_k-2} \cdot z_{i_k-2+3-i_k-3} \cdot ... \cdot z_{i_2+3-i_1} \cdot z_{i_1+4-2k} \cdot \varphi_k(y; z) \].

For transition from the \((n+1)\)-dimensional filiform Leibniz algebra \( L(\alpha) \) to the \((n+1)\)-dimensional filiform Leibniz algebra \( L(\alpha') \) we will write \( \alpha' = \rho(\frac{1}{A}, \frac{B}{A}; \alpha) \), where \( \alpha = (\alpha_3, \alpha_4, ..., \alpha_n, \theta) \),

\[ \rho(\frac{1}{A}, \frac{B}{A}; \alpha) = (\rho_1(\frac{1}{A}, \frac{B}{A}; \alpha), \rho_2(\frac{1}{A}, \frac{B}{A}; \alpha), ..., \rho_{n-1}(\frac{1}{A}, \frac{B}{A}; \alpha)) \]

and

\[ \rho_{n-1}(x, y; z) = x^{n-2} \varphi_{n+1}(y, z) \]

Here are the main properties of the operator \( \rho \), derived from the fact that \( \rho(\frac{1}{A}, \frac{B}{A}; \cdot) \) is an action of a group.

1. \( \rho(1, 0; \cdot) \) is the identity operator.

2. \( \rho(\frac{1}{A^2}, \frac{B}{A^2}; \rho(\frac{1}{A^2}, \frac{B}{A^2}; \alpha)) = \rho(\frac{1}{A^2}, \frac{A^2 B_2+B_1 A_2}{A_1 A_2}; \alpha) \).

3. If \( \alpha' = \rho(\frac{1}{A}, \frac{B}{A}; \alpha) \) then \( \alpha = \rho(A, -\frac{B}{A+B}; \alpha') \).

From here on \( n \) is a positive integer. We assume that \( n \geq 4 \) since there are complete classifications of complex nilpotent Leibniz algebras of dimension at most four \[5\].

We first present the result of \( [6] \) that underlies our classification result.
In this section we will present the list of none Lie complex filiform Leibniz algebras from $\mathfrak{gl}_n$ whenever $\alpha_3(\alpha_4 + 2\alpha_3^2) \neq 0$, $F = \{L(\alpha) : \alpha_3(\alpha_4 + 2\alpha_3^2) = 0\}$.

**Theorem 1.6** i) Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_1$ are isomorphic if and only if
\[
\rho_i\left(\frac{2\alpha_3}{\alpha_4 + 2\alpha_3^2} : \alpha\right) = \rho_i\left(\frac{2\alpha_3'}{\alpha_4' + 2\alpha_3'^2} : \alpha'\right)
\]
whenever $i = 3, n - 1$.

ii) For any $(\alpha_3, \alpha_4, ..., \alpha_{n-1}) \in \mathbb{C}^{n-3}$ there is an algebra $L(\alpha)$ from $U_1$ such that
\[
\rho_i\left(\frac{2\alpha_3}{\alpha_4 + 2\alpha_3^2} : \alpha\right) = a_i \text{ for all } i = 3, n - 1.
\]

Note that the above theorem describes the field of invariant rational functions on $\mathfrak{leib}_{n+1}$ under the action of the adapted subgroup of $GL_{n+1}(\mathbb{C})$ and the part ii) means the algebraic independence of the generators.

The procedure that we are applying works by the following way: first we present $F\mathfrak{leib}_{n+1}$ as a disjoint union of subets then formulate for each subset the analogue of the above theorem.

### 2 The list of algebras

In this section we will present the list of none Lie complex filiform Leibniz algebras from $F\mathfrak{leib}_{n+1}$ for $n = 4, 5, 6, 7$. Later on $\Delta_4 = \alpha_4 + 2\alpha_3^2$, $\Delta_5 = \alpha_5 - 5\alpha_3^3$, $\Delta_6 = \alpha_6 + 14\alpha_3^4$, $\Delta_7 = \alpha_7 - 42\alpha_3^5$, $\Theta_4 = \theta - \alpha_4$, $i = 4, 5, 6, 7$ and the same letters $\Delta$ and $\Theta$ with $i$ will denote the same expression depending on parameters $\alpha_4', \alpha_5', \alpha_6', \alpha_7', \theta'$. Notice that $\Delta_i = \alpha_i (i = 4, 5, 6, 7)$ when $\alpha_3 = 0$.

Let $N_n$ denote the number of isomorphism classes in dimension $n$ (each parametric family here will be considered as a one class).

#### 2.1 Dimension 5

The class $F\mathfrak{leib}_5$ can be represented as a disjoint union of the following subsets:

- $F\mathfrak{leib}_5 = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6$, where
- $U_1 = \{L(\alpha) \in F\mathfrak{leib}_5 : \alpha_3 \neq 0, \Delta_4 \neq 0\}$,
- $U_2 = \{L(\alpha) \in F\mathfrak{leib}_5 : \alpha_3 = 0, \Delta_4 = 0, \Theta_4 \neq 0\}$,
- $U_3 = \{L(\alpha) \in F\mathfrak{leib}_5 : \alpha_3 = 0, \Delta_4 = 0, \Theta_4 = 0\}$,
- $U_4 = \{L(\alpha) \in F\mathfrak{leib}_5 : \alpha_3 = 0, \Delta_4 = 0, \Theta_4 = 0\}$,
- $U_5 = \{L(\alpha) \in F\mathfrak{leib}_5 : \alpha_3 = 0, \Delta_4 = 0, \Theta_4 = 0\}$,
- $U_6 = \{L(\alpha) \in F\mathfrak{leib}_5 : \alpha_3 = 0, \Delta_4 = 0, \Theta_4 = 0\}$,
- $U_7 = \{L(\alpha) \in F\mathfrak{leib}_5 : \alpha_3 = 0, \Delta_4 = 0, \Theta_4 = 0\}$.

Now we will investigate the isomorphism problem for each of these sets separately.

**Proposition 2.1.1** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_1$ are isomorphic if and only if
\[
\left(\frac{\alpha_3}{\Delta_4}\right)^2 \Theta_4 = \left(\frac{\alpha_3'}{\Delta_4'}\right)^2 \Theta_4'.
\]

This means that the expression
\[
\left(\frac{\alpha_3}{\Delta_4}\right)^2 \Theta_4
\]
can be taken as a parameter $\lambda$ and then algebras from the set $U_1$ can be parameterized as $L(1, 0, \lambda)$.

**Proposition 2.1.2.**

a) All algebras from the set $U_2$ are isomorphic to $L(1, -2, 0)$;

b) All algebras from the set $U_3$ are isomorphic to $L(1, -2, -2)$;

c) All algebras from the set $U_4$ are isomorphic to $L(0, 1, 0)$;

d) All algebras from the set $U_5$ are isomorphic to $L(0, 1, 1)$;

e) All algebras from the set $U_6$ are isomorphic to $L(0, 0, 1)$;

f) All algebras from the set $U_7$ are isomorphic to $L(0, 0, 0)$. 

f) All algebras from the set $U_7$ are isomorphic to $L(0, 0, 0)$.

**Theorem 2.1.3.** Let $L$ be a none Lie complex filiform Leibniz algebra in $FLeib_5$. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras:

1) $L(1, 0, \lambda)$:
$$e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 3, \quad e_0e_1 = e_3 + \lambda e_4, \quad e_1e_1 = e_3,$$
$$e_2e_1 = e_4, \quad \lambda \in \mathbb{C}.$$

2) $L(1, -2, 0)$:
$$e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 3, \quad e_0e_1 = e_3, \quad e_1e_1 = e_3 - 2e_4,$$
$$e_2e_1 = e_4.$$

3) $L(1, -2, -2)$:
$$e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 3, \quad e_0e_1 = e_3 - 2e_4,$$
$$e_1e_1 = e_3 - 2e_4, \quad e_2e_1 = e_4.$$

4) $L(0, 1, 0)$:
$$e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 3, \quad e_1e_1 = e_4.$$

5) $L(0, 1, 1)$:
$$e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 3, \quad e_0e_1 = e_4, \quad e_1e_1 = e_4.$$

6) $L(0, 0, 1)$:
$$e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 3, \quad e_0e_1 = e_4.$$

7) $L(0, 0, 0)$:
$$e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 3.$$

The number of isomorphism classes $N_5 = 7$.

**2.2 Dimension 6**

Now we consider the six dimensional case. The set $FLeib_6$ can be represented as a disjoint union of the subsets:

$FLeib_6 = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6 \cup U_7 \cup U_8 \cup U_9 \cup U_{10} \cup U_{11}$, where

$U_1 = \{ L(\alpha) \in FLeib_6 : \alpha_3 \neq 0, \Delta_4 \neq 0 \}$,

$U_2 = \{ L(\alpha) \in FLeib_6 : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_5 \neq 0 \}$,

$U_3 = \{ L(\alpha) \in FLeib_6 : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 \neq 0 \}$,

$U_4 = \{ L(\alpha) \in FLeib_6 : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_5 = 0 \}$,

$U_5 = \{ L(\alpha) \in FLeib_6 : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_5 \neq 0 \}$,

$U_6 = \{ L(\alpha) \in FLeib_6 : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Theta_5 \neq 0 \}$,

$U_7 = \{ L(\alpha) \in FLeib_6 : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 \neq 0, \Theta_5 = 0 \}$,

$U_8 = \{ L(\alpha) \in FLeib_6 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_5 = 0 \}$,

$U_9 = \{ L(\alpha) \in FLeib_6 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_5 \neq 0 \}$,

$U_{10} = \{ L(\alpha) \in FLeib_6 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_5 = 0 \}$,

$U_{11} = \{ L(\alpha) \in FLeib_6 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_5 = 0 \}$.

**Proposition 2.2.1.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_1$ are isomorphic if and only if

$$\frac{\alpha_3(\Delta_5 + 5\alpha_3\Delta_4)}{\Delta_4^2} = \frac{\alpha'_3(\Delta'_5 + 5\alpha'_3\Delta'_4)}{\Delta'_4^2},$$

$$\frac{\alpha_3^2\Theta_5}{\Delta_4^4} = \frac{\alpha'_3^2\Theta'_5}{\Delta'_4^4}.$$
Thus the following two expressions can be taken as parameters $\lambda_1, \lambda_2$:

\[
\frac{\alpha_3(\Delta_5 + 5\alpha_3\Delta_4)}{\Delta_4^2},
\frac{\alpha_3^3\Theta_5}{\Delta_4^2}
\]

and algebras from $U_1$ can be parameterized as

$L(1, 0, \lambda_1, \lambda_2)$.

**Proposition 2.2.2.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_2$ are isomorphic if and only if

\[
\frac{\Delta_5^3}{\alpha_3^3\Theta_5^2} = \frac{\Delta_5'^3}{\alpha_3'^3\Theta_5'^2}.
\]

Thus in the set $U_2$ the expression

\[
\frac{\Delta_5^3}{\alpha_3^3\Theta_5^2}
\]

can be taken as a parameter and algebras from $U_2$ can be parameterized as

$L(1, -2, \lambda, 2\lambda - 5)$.

**Proposition 2.2.3.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_3$ are isomorphic if and only if

\[
\frac{\alpha_3^3\Theta_5}{\alpha_5^3} = \frac{\alpha_3'^3\Theta_5'}{\alpha_5'^3}.
\]

So as a parameter $\lambda$ in $U_3$ we will take the expression

\[
\frac{\alpha_3^3\Theta_5}{\alpha_5^3}
\]

and write algebras from the set $U_3$ as

$L(0, 1, 1, \lambda)$.

**Proposition 2.2.4.**

a) All algebras from the set $U_4$ are isomorphic to the algebra $L(1, -2, 0, 0)$;

b) All algebras from the set $U_5$ are isomorphic to the algebra $L(1, -2, 5, 0)$;

c) All algebras from the set $U_5$ are isomorphic to the algebra $L(0, 1, 0, 1)$;

d) All algebras from the set $U_6$ are isomorphic to the algebra $L(0, 1, 0, 0)$;

e) All algebras from the set $U_7$ are isomorphic to the algebra $L(0, 0, 1, 0)$;

f) All algebras from the set $U_8$ are isomorphic to the algebra $L(0, 0, 1, 1)$;

g) All algebras from the set $U_9$ are isomorphic to the algebra $L(0, 0, 0, 1)$;

h) All algebras from the set $U_{10}$ are isomorphic to the algebra $L(0, 0, 0, 0)$.

**Theorem 2.2.5.** Let $L$ be a none Lie complex filiform Leibniz algebra in $FLeib_6$. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras:

1) $L(1, 0, \lambda_1, \lambda_2)$:

$e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 4, \quad e_0 e_1 = e_3 + \lambda_2 e_5,
\quad e_1 e_1 = e_3 + \lambda_1 e_5, \quad e_2 e_1 = e_4, \quad e_3 e_1 = e_5, \quad \lambda_1, \lambda_2 \in \C$.

2) $L(1, -2, \lambda, 2\lambda - 5)$:

$e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 4, \quad e_0 e_1 = e_3 - 2e_4 + (2\lambda - 5)e_5,
\quad e_1 e_1 = e_3 - 2e_4 + \lambda e_5, \quad e_2 e_1 = e_4 - 2e_5, \quad e_3 e_1 = e_5, \quad \lambda \in \C$.

3) $L(0, 1, 1, \lambda)$:
\[ e_0e_0 = e_2, \ e_i e_0 = e_{i+1}, \ 1 \leq i \leq 4, \ e_0 e_1 = e_4 + \lambda e_5, \]
\[ e_1 e_1 = e_4 + e_5, \ e_2 e_1 = e_5, \ \lambda \in \mathbb{C}. \]

4) \( L(1, -2, 0, 0) : \)
\[ e_0e_0 = e_2, \ e_i e_0 = e_{i+1}, \ 1 \leq i \leq 4, \ e_0 e_1 = e_3 - 2e_4, \]
\[ e_1 e_1 = e_3 - 2e_4, \ e_2 e_1 = e_4 - 2e_5, \ e_3 e_1 = e_5. \]

5) \( L(1, -2, 0, 0) : \)
\[ e_0e_0 = e_2, \ e_i e_0 = e_{i+1}, \ 1 \leq i \leq 4, \ e_0 e_1 = e_3 - 2e_4, \]
\[ e_1 e_1 = e_3 - 2e_4 + 5e_5, \ e_2 e_1 = e_4 - 2e_5, \ e_3 e_1 = e_5. \]

6) \( L(0, 1, 0, 1) : \)
\[ e_0e_0 = e_2, \ e_i e_0 = e_{i+1}, \ 1 \leq i \leq 4, \ e_0 e_1 = e_4 + e_5, \ e_1 e_1 = e_4, \]
\[ e_2 e_1 = e_5. \]

7) \( L(0, 1, 0, 0) : \)
\[ e_0e_0 = e_2, \ e_i e_0 = e_{i+1}, \ 1 \leq i \leq 4, \ e_0 e_1 = e_4, \ e_1 e_1 = e_4, \ e_2 e_1 = e_5. \]

8) \( L(0, 0, 1, 0) : \)
\[ e_0e_0 = e_2, \ e_i e_0 = e_{i+1}, \ 1 \leq i \leq 4, \ e_0 e_1 = e_5. \]

9) \( L(0, 0, 1, 1) : \)
\[ e_0e_0 = e_2, \ e_i e_0 = e_{i+1}, \ 1 \leq i \leq 4, \ e_0 e_1 = e_5, \ e_1 e_1 = e_5. \]

10) \( L(0, 0, 0, 1) : \)
\[ e_0e_0 = e_2, \ e_i e_0 = e_{i+1}, \ 1 \leq i \leq 4, \ e_0 e_1 = e_5. \]

11) \( L(0, 0, 0, 0) : \)
\[ e_0e_0 = e_2, \ e_i e_0 = e_{i+1}, \ 1 \leq i \leq 4. \]

The number of isomorphism classes \( N_6 = 11. \)

2.3 Dimension 7
\[ FLeib_7 = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6 \cup U_7 \cup U_8 \cup U_9 \cup U_{10} \cup U_{11} \cup U_{12} \cup U_{13} \cup U_{14} \cup U_{15} \cup U_{16} \cup U_{17}, \]

where
\[ U_1 = \{ L(\alpha) \in FLeib_7 : \alpha_3 \neq 0, \Delta_4 \neq 0 \}, \]
\[ U_2 = \{ L(\alpha) \in FLeib_7 : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 + 6\alpha_3 \Delta_5 \neq 0 \}, \]
\[ U_3 = \{ L(\alpha) \in FLeib_7 : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 \neq 0, \Delta_6 + 6\alpha_3 \Delta_5 = 0 \}, \]
\[ U_4 = \{ L(\alpha) \in FLeib_7 : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 \neq 0, \Theta_6 \neq 0 \}, \]
\[ U_5 = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 \neq 0 \}, \]
\[ U_6 = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Delta_6 \neq 0, \Theta_6 \neq 0 \}, \]
\[ U_7 = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 \neq 0, \Delta_6 \neq 0 \}, \]
\[ U_8 = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 \neq 0, \Theta_6 = 0 \}, \]
\[ U_9 = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 \neq 0, \Theta_6 \neq 0 \}, \]
\[ U_{10} = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Delta_6 = 0, \Theta_6 \neq 0 \}, \]
\[ U_{11} = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 \neq 0, \Delta_6 = 0, \Theta_6 \neq 0 \}, \]
\[ U_{12} = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 = 0, \Theta_6 \neq 0 \}, \]
\[ U_{13} = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 = 0, \Theta_6 = 0 \}, \]
\[ U_{14} = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 \neq 0, \Theta_6 \neq 0 \}, \]
\[ U_{15} = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 \neq 0, \Theta_6 \neq 0 \}, \]
\[ U_{16} = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 = 0, \Theta_6 \neq 0 \}, \]
\[ U_{17} = \{ L(\alpha) \in FLeib_7 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 = 0, \Theta_6 = 0 \}. \]
Proposition 2.3.1. Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_1$ are isomorphic if and only if

$$\frac{\alpha_3(\Delta_5 + 5\alpha_3\Delta_4)}{\Delta_4^2} = \frac{\alpha_3'(\Delta_4' + 5\alpha_3'\Delta_4)}{\Delta_4'^2}$$

$$\alpha_3(\alpha_3\Delta_6 + 6\alpha_2^2\Delta_5 - 3\Delta_4\Delta_5 + 9\alpha_3^3\Delta_4 - 12\alpha_3\Delta_3^2) = \frac{\alpha_4(\alpha'_4\Delta_6' + 6\alpha_2^2\Delta_5' - 3\Delta_4'\Delta_5' + 9\alpha_3^3\Delta_4' - 12\alpha_4'\Delta_4'^2)}{\Delta_4'^3}$$

$$\frac{\alpha_3^3\Theta_6}{\Delta_4^4} = \frac{\alpha_3'^3\Theta_6'}{\Delta_4'^4}.$$ 

Thus, in this case algebras from the set $U_1$ can be parameterized as $L(1, 0, \lambda_1, \lambda_2, \lambda_3)$.

Proposition 2.3.2. Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_2$ are isomorphic if and only if

$$\frac{\Delta_5^3}{\alpha_3(\Delta_6 + 6\alpha_3\Delta_5)^2} = \frac{\Delta_5'^3}{\alpha_3'(\Delta_6' + 6\alpha_3'\Delta_5')^2}$$

$$\frac{\Delta_4^4\Theta_6}{(\Delta_6 + 6\alpha_3\Delta_5)^4} = \frac{\Delta_4'^4\Theta_6'}{(\Delta_6' + 6\alpha_3'\Delta_5')^4}.$$ 

The expressions above can be taken as parameters in $U_2$ and the set $U_2$ can be represented as $L(-2, \lambda_1, -5\lambda_1 - 14, \lambda_2)$.

Proposition 2.3.3. Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_3$ are isomorphic if and only if

$$\frac{\alpha_3^3\Theta_6^2}{\Delta_5^4} = \frac{\alpha_3'^3\Theta_6'^2}{\Delta_5'^4}.$$ 

The parameter $\lambda$ for algebras from the set $U_3$ is

$$\frac{\alpha_3^3\Theta_6}{\Delta_5^4}$$

and $U_3$ can be parameterized as $L(1, -2, 0, 16, \lambda)$.

Proposition 2.3.4. Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_4$ are isomorphic if and only if

$$\frac{\Delta_4^4}{\alpha_3^3\Theta_6^3} = \frac{\Delta_4'^4}{\alpha_3'^3\Theta_6'^3}.$$ 

$U_4$ can be parameterized as $L(1, -2, 5, \lambda, 2\lambda - 14)$.

Proposition 2.3.5. Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_5$ are isomorphic if and only if

$$\frac{\Delta_4(\Delta_6 + 3\Delta_7^2)}{\Delta_5^2} = \frac{\Delta_4'(\Delta_6' + 3\Delta_7'^2)}{\Delta_5'^2},$$

$$\left(\frac{\Delta_4}{\Delta_5}\right)^4 \Theta_6 = \left(\frac{\Delta_4'}{\Delta_5'}\right)^4 \Theta_6'.$$

The set $U_5$ can be parameterized as $L(0, 1, 1, \lambda_1, \lambda_2)$.

Proposition 2.3.6. Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_6$ are isomorphic if and only if
\[
\frac{(\Delta_6 + 3\Delta_1^2)^2}{\Delta_1^4 \Theta_6} = \frac{(\Delta_6' + 3\Delta_1'^2)^2}{\Delta_1'^4 \Theta_6'}.
\]

\(U_6\) can be represented as a parameterized family of algebras \(L(0, 1, 0, \lambda, 2\lambda - 3)\).

**Proposition 2.3.7.** Two algebras \(L(\alpha)\) and \(L(\alpha')\) from \(U_7\) are isomorphic if and only if
\[
\left(\frac{\Delta_6}{\Delta_6'}\right)^4 \Theta_6 = \left(\frac{\Delta_6'}{\Delta_6}\right)^4 \Theta_6'.
\]

We will get one parametric family of algebras for the set \(U_7 : L(0, 0, 1, 1, \lambda)\).

**Proposition 2.3.8**

a) All algebras from the set \(U_8\) are isomorphic to \(L(1, -2, 5, 0, 0)\);

b) All algebras from \(U_9\) are isomorphic to \(L(1, -2, 5, 14, 0)\);

c) Algebras from \(U_{10}\) are isomorphic to \(L(0, 1, 0, 0, 0)\);

d) All algebras from \(U_{11}\) are isomorphic to \(L(0, 1, 0, -3, 0)\);

e) All algebras from \(U_{12}\) are isomorphic to \(L(0, 0, 1, 0, 1)\);

f) All algebras from \(U_{13}\) are isomorphic to \(L(0, 0, 1, 0, 0)\);

g) All algebras from \(U_{14}\) are isomorphic to \(L(0, 0, 0, 1, 0)\);

h) Algebras from \(U_{15}\) are isomorphic to \(L(0, 0, 0, 1, 1)\);

i) Algebras from \(U_{16}\) are isomorphic to \(L(0, 0, 0, 0, 1)\);

j) Algebras from \(U_{17}\) are isomorphic to \(L(0, 0, 0, 0, 0)\).

**Theorem 2.3.9.** Let \(L\) be a none Lie complex filiform Leibniz algebra in \(FLeib_7\). Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras:

1) \(L(1, 0, \lambda_1, \lambda_2, \lambda_3):\)
   \[
e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 5, \quad e_0e_1 = e_3 + \lambda_1e_5 + \lambda_3e_6, \quad e_1e_1 = e_3 + \lambda_1e_5 + \lambda_2e_6, \quad e_2e_1 = e_4 + \lambda_1e_6, \quad e_3e_1 = e_5, \quad e_4e_1 = e_6, \\
   \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}.
\]

2) \(L(1, -2, \lambda_1, -5\lambda_1 - 14, \lambda_2):\)
   \[
e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 5, \quad e_0e_1 = e_3 - 2e_4 + \lambda_1e_5 + \lambda_2e_6, \quad e_1e_1 = e_3 - 2e_4 + \lambda_1e_5 + (-5\lambda_1 - 14)e_6, \quad e_2e_1 = e_4 - 2e_5 + \lambda_1e_6, \quad e_3e_1 = e_5 - 2e_6, \quad e_4e_1 = e_6, \quad \lambda_1, \lambda_2 \in \mathbb{C}.
\]

3) \(L(1, -2, 0, 16, \lambda):\)
   \[
e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 5, \quad e_0e_1 = e_3 - 2e_4 + \lambda_6, \quad e_1e_1 = e_3 - 2e_4 + 16e_6, \quad e_2e_1 = e_4 - 2e_5, \quad e_3e_1 = e_5 - 2e_6, \quad e_4e_1 = e_6, \quad \lambda \in \mathbb{C}.
\]

4) \(L(1, -2, 5, \lambda, 2\lambda - 14):\)
   \[
e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 5, \quad e_0e_1 = e_3 - 2e_4 + 5e_5 + (2\lambda - 14)e_6, \quad e_1e_1 = e_3 - 2e_4 + 5e_5 + \lambda_6, \quad e_2e_1 = e_4 - 2e_5 + 5e_6, \quad e_3e_1 = e_5 - 2e_6, \quad e_4e_1 = e_6, \quad \lambda, \lambda \in \mathbb{C}.
\]

5) \(L(0, 1, 0, \lambda_1, \lambda_2):\)
   \[
e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 5, \quad e_0e_1 = e_4 + e_5 + \lambda_2e_6, \quad e_1e_1 = e_4 + 5 + \lambda_1e_6, \quad e_2e_1 = e_5 + e_6, \quad e_3e_1 = e_6, \quad \lambda_1, \lambda_2 \in \mathbb{C}.
\]

6) \(L(0, 1, 0, \lambda, 2\lambda - 3):\)
   \[
e_0e_0 = e_2, \quad e_ie_0 = e_{i+1}, \quad 1 \leq i \leq 5, \quad e_0e_1 = e_4 + (2\lambda - 3)e_6, \quad e_1e_1 = e_4 + \lambda_6, \quad e_2e_1 = e_4, \quad e_3e_1 = e_6, \quad \lambda \in \mathbb{C}.
\]

7) \(L(0, 0, 1, 1, \lambda):\)
The number of isomorphism classes $N_7 = 17$.

### 2.4 Dimension 8

$F_{Leib_8} = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6 \cup U_7 \cup U_8 \cup U_9 \cup U_{10} \cup U_{11} \cup U_{12} \cup U_{13} \cup U_{14} \cup U_{15} \cup U_{16} \cup U_{17} \cup U_{18} \cup U_{19} \cup U_{20} \cup U_{21} \cup U_{22} \cup U_{23} \cup U_{24} \cup U_{25},$

where

$U_1 = \{ L(\alpha) \in F_{Leib_8} : \alpha_3 \neq 0, \Delta_4 \neq 0 \},$

$U_2 = \{ L(\alpha) \in F_{Leib_8} : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 + 6\alpha_3 \Delta_5 \neq 0 \},$

$U_3 = \{ L(\alpha) \in F_{Leib_8} : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 + 6\alpha_3 \Delta_5 = 0, \Theta_7 \neq 0 \},$

$U_4 = \{ L(\alpha) \in F_{Leib_8} : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 + 6\alpha_3 \Delta_5 = 0, \Theta_7 = 0 \},$

$U_5 = \{ L(\alpha) \in F_{Leib_8} : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 \neq 0, \Delta_7 + 7\alpha_3 \Delta_6 \neq 0 \},$

$U_6 = \{ L(\alpha) \in F_{Leib_8} : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 \neq 0, \Delta_7 + 7\alpha_3 \Delta_6 = 0 \},$

$U_7 = \{ L(\alpha) \in F_{Leib_8} : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 = 0, \Delta_7 \neq 0 \},$

$U_8 = \{ L(\alpha) \in F_{Leib_8} : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 = 0, \Delta_7 = 0 \},$

$U_9 = \{ L(\alpha) \in F_{Leib_8} : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 = 3\Delta_2^2 \neq 0, \Delta_7 \neq 0 \},$

$U_{10} = \{ L(\alpha) \in F_{Leib_8} : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Delta_6 + 3\Delta_2^2 \neq 0, \Theta_7 \neq 0 \},$

$U_{11} = \{ L(\alpha) \in F_{Leib_8} : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Delta_6 + 3\Delta_2^2 = 0, \Delta_7 \neq 0, \Theta_7 \neq 0 \}.$
Proposition 2.4.1. Two algebras \( L(\alpha) \) and \( L(\alpha') \) from \( U_1 \) are isomorphic if and only if

\[
\frac{\alpha_3(\Delta_5 + 5\alpha_3\Delta_4)}{\Delta_4^5} = \frac{\alpha'_3(\Delta'_5 + 5\alpha'_3\Delta'_4)}{\Delta'_4^5}
\]

\[
\frac{\alpha_3(\alpha_3\Delta_6 + 6\alpha_3^2\Delta_5 - 3\Delta_4\Delta_5 + 9\alpha_3^3\Delta_4 - 12\alpha_3^4\Delta_4^2)}{\Delta_4^5} = \frac{\alpha'_3(\alpha'_3\Delta'_6 + 6\alpha'_3^2\Delta'_5 - 3\Delta'_4\Delta'_5 + 9\alpha'_3^3\Delta'_4 - 12\alpha'_3^4\Delta'_4^2)}{\Delta'_4^5}
\]

\[
\frac{\alpha_3^5\Delta_7 + 28\alpha_3^4\Delta_7^2 + 7\alpha_3^5\Delta_6 + 14\alpha_3^4\Delta_6^2 + 7\alpha_3^5\Delta_5 + 7\alpha_3^4\Delta_5^2 + 7\alpha_3^5\Delta_4\Delta_5}{\Delta_4^5} = \frac{\alpha'_3^5\Delta'_7 + 28\alpha'_3^4\Delta'_7^2 + 7\alpha'_3^5\Delta'_6 + 14\alpha'_3^4\Delta'_6^2 + 7\alpha'_3^5\Delta'_5 + 7\alpha'_3^4\Delta'_5^2 + 7\alpha'_3^5\Delta'_4\Delta'_5}{\Delta'_4^5}
\]

\[
\frac{\alpha_3^5\Theta_7}{\Delta_4^5} = \frac{\alpha'_3^5\Theta'_7}{\Delta'_4^5}
\]

\( U_1 \) is parameterized as \( L(1, 0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \).

Proposition 2.4.2. Two algebras \( L(\alpha) \) and \( L(\alpha') \) from \( U_2 \) are isomorphic if and only if

\[
\frac{\Delta_6^5}{\alpha_3(\Delta_6 + 6\alpha_3\Delta_5)^2} = \frac{\Delta'_6^5}{\alpha'_3(\Delta'_6 + 6\alpha'_3\Delta'_5)^2}
\]

\[
\frac{\Delta_6^5(\Delta_7 + 7\alpha_3\Delta_6 + 14\alpha_3^2\Delta_5)}{\alpha_3(\Delta_6 + 6\alpha_3\Delta_5)^4} = \frac{\Delta'_6^5(\Delta'_7 + 7\alpha'_3\Delta'_6 + 14\alpha'_3^2\Delta'_5)}{\alpha'_3(\Delta'_6 + 6\alpha'_3\Delta'_5)^4}
\]

\[
\frac{\Delta_6^5\Theta_7}{(\Delta_6 + 6\alpha_3\Delta_5)^5} = \frac{\Delta'_6^5\Theta'_7}{(\Delta'_6 + 6\alpha'_3\Delta'_5)^5}
\]

The set \( U_2 \) can be parameterized as \( L(1, -2, \lambda_1, -5\lambda_1 - 14, \lambda_2, \lambda_3) \).

Proposition 2.4.3. Two algebras \( L(\alpha) \) and \( L(\alpha') \) from \( U_3 \) are isomorphic if and only if

\[
\left( \frac{\Delta_5}{\alpha_3} \right)^5 \frac{1}{\Theta_7^5} = \left( \frac{\Delta'_5}{\alpha_3} \right)^5 \frac{1}{\Theta'_7^5}
\]

\[
\frac{\Delta_5^8(\Delta_7 - 28\alpha_3^2\Delta_5)}{\alpha_3^4\Theta_7^5} = \frac{\Delta'_5^8(\Delta'_7 - 28\alpha'_3^2\Delta'_5)}{\alpha'_3^4\Theta'_7^5}
\]
The algebras from the set $U_3$ can be parameterized as $L(1, -2, \lambda_1, -6\lambda_1 - 14, \lambda_2, \lambda_2^2)$.

**Proposition 2.4.4.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_4$ are isomorphic if and only if
\[
\frac{\alpha_3(\Delta_7 - 28\alpha_3^2\Delta_5)}{\Delta_5^2} = \frac{\alpha'_3(\Delta'_7 - 28\alpha'_3^2\Delta'_5)}{\Delta'_5^2}.
\]
The set $U_4$ can be parameterized as $L(1, -2, 0, 16, \lambda, \lambda)$.

**Proposition 2.4.5.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_5$ are isomorphic if and only if
\[
\frac{\Delta_6^4}{\alpha_3(\Delta_7 + 7\alpha_3\Delta_5)^3} = \frac{\Delta'_6^4}{\alpha'_3(\Delta'_7 + 7\alpha'_3\Delta'_5)^3}.
\]
\[
\left(\frac{\Delta_6}{\Delta_7 + 7\alpha_3\Delta_5}\right)^5 \Theta_7 = \left(\frac{\Delta'_6}{\Delta'_7 + 7\alpha'_3\Delta'_5}\right)^5 \Theta'_7.
\]
$U_5$ is parameterized as $L(1, -2, 5, \lambda_1, -6\lambda_1 + 42, \lambda_2)$.

**Proposition 2.4.6.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_6$ are isomorphic if and only if
\[
\left(\frac{\alpha_3}{\Delta_6}\right)^5 \Theta_7^3 = \left(\frac{\alpha'_3}{\Delta'_6}\right)^5 \Theta'_7^3.
\]
The set $U_6$ can be parameterized as $L(1, -2, 5, \lambda, -7\lambda + 42, \lambda^2)$.

**Proposition 2.4.7.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_7$ are isomorphic if and only if
\[
\left(\frac{\Delta_7}{\alpha_3}\right)^5 \frac{1}{\Theta_7} = \left(\frac{\Delta'_7}{\alpha'_3}\right)^5 \frac{1}{\Theta'_7}.
\]
$U_7$ is parameterized as $L(1, -2, 5, -14, \lambda, 2(\lambda + 21))$.

**Proposition 2.4.8.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_8$ are isomorphic if and only if
\[
\frac{\Delta_4(\alpha_6 + 3\alpha_4^2)}{\Delta_4^2} = \frac{\Delta'_4(\alpha'_6 + 3\alpha'_4^2)}{\Delta'_4^2}.
\]
\[
\frac{\Delta_4^2(\Delta_7 + 7\Delta_4\Delta_5)}{\Delta_5^3} = \frac{\Delta'_4^2(\Delta'_7 + 7\Delta'_4\Delta'_5)}{\Delta'_5^3}.
\]
\[
\left(\frac{\Delta_4}{\Delta_5}\right)^5 \Theta_7 = \left(\frac{\Delta'_4}{\Delta'_5}\right)^5 \Theta'_7.
\]
$U_8$ can be parameterized as $L(0, 1, 1, \lambda_1, \lambda_2, \lambda_3)$.

**Proposition 2.4.9.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_9$ are isomorphic if and only if
\[
\frac{(\Delta_6 + 3\Delta_4^2)^3}{\Delta_4^2\Delta_7^2} = \frac{(\Delta'_6 + 3\Delta'_4^2)^3}{\Delta'_4^2\Delta'_7^2}.
\]
\[
\left(\frac{\Delta_6 + 3\Delta_4^2}{\Delta_7}\right)^5 \Theta_7 = \left(\frac{\Delta'_6 + 3\Delta'_4^2}{\Delta'_7}\right)^5 \Theta'_7.
\]
Thus the algebras from the set $U_9$ can be parameterized as $L(0, 1, 0, \lambda_1, \lambda_1 + 3, \lambda_2)$.

**Proposition 2.4.10.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_{10}$ are isomorphic if and only if
\[
\left(\frac{\Delta_6 + 3\Delta_4^2}{\Delta_4}\right)^5 \Theta_7^2 = \left(\frac{\Delta'_6 + 3\Delta'_4^2}{\Delta'_4}\right)^5 \Theta'_7^2.
\]
$U_{10}$ can be parameterized as $L(0, 1, 0, \lambda, 0, \lambda^2)$.

**Proposition 2.4.11.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_{11}$ are isomorphic if and only if

$$
\left(\frac{\Delta_7}{\Delta_6}\right)^5 \Theta_7^3 = \left(\frac{\Delta_7'}{\Delta_6'}\right)^5 \Theta_7'.
$$

$L(0, 1, 0, -3, \lambda, \lambda^2 + \lambda)$ are representatives of $U_{11}$.

**Proposition 2.4.12.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_{12}$ are isomorphic if and only if

$$
\frac{\Delta_5 \Delta_7}{\Delta_6^2} = \frac{\Delta_5' \Delta_7'}{\Delta_6'^2}
$$

$$
\left(\frac{\Delta_7}{\Delta_6}\right)^5 \Theta_7 = \left(\frac{\Delta_7'}{\Delta_6'}\right)^5 \Theta_7'.
$$

Thus, the algebras from the set $U_{12}$ can be parameterized as $L(0, 1, 1, \lambda_1, \lambda_2)$.

**Proposition 2.4.13.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_{13}$ are isomorphic if and only if

$$
\left(\frac{\Delta_7}{\Delta_5}\right)^5 \frac{1}{\Theta_7} = \left(\frac{\Delta_7'}{\Delta_5'}\right)^5 \frac{1}{\Theta_7'}.
$$

Thus, the algebras from the set $U_{13}$ can be parameterized as $L(0, 0, 1, 0, \lambda, \lambda^2 + \lambda)$.

**Proposition 2.4.14.** Two algebras $L(\alpha)$ and $L(\alpha')$ from $U_{14}$ are isomorphic if and only if

$$
\left(\frac{\Delta_9}{\Delta_7}\right)^5 \Theta_7 = \left(\frac{\Delta_9'}{\Delta_7'}\right)^5 \Theta_7'.
$$

$L(0, 0, 1, 1, \lambda)$ is a parametrization of $U_{14}$.

**Proposition 2.4.15**

a) All algebras from the set $U_{15}$ are isomorphic to $L(0, 1, 0, 0, 0, 0)$;

b) All algebras from the set $U_{16}$ are isomorphic to $L(0, 1, 0, -3, 1, 1)$;

c) All algebras from the set $U_{17}$ are isomorphic to $L(0, 1, 0, -3, 0, 1)$;

d) All algebras from the set $U_{18}$ are isomorphic to $L(0, 0, 1, 0, 1, 1)$;

e) All algebras from the set $U_{19}$ are isomorphic to $L(0, 0, 1, 0, 0, 1)$;

f) All algebras from the set $U_{20}$ are isomorphic to $L(0, 0, 0, 1, 0, 1)$;

g) All algebras from the set $U_{21}$ are isomorphic to $L(0, 0, 0, 1, 0, 0)$;

h) All algebras from the set $U_{22}$ are isomorphic to $L(0, 0, 0, 0, 1, 0)$;

i) All algebras from the set $U_{23}$ are isomorphic to $L(0, 0, 0, 0, 1, 1)$;

j) All algebras from the set $U_{24}$ are isomorphic to $L(0, 0, 0, 0, 0, 1)$;

k) All algebras from the set $U_{25}$ are isomorphic to $L(0, 0, 0, 0, 0, 0)$.

**Theorem 2.4.16.** Let $L$ be a none Lie complex filiform Leibniz algebra in $FLeib_8$. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras:

1) $L(1, 0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$:

$e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_3 + \lambda_1 e_5 + \lambda_2 e_6 + \lambda_4 e_7,$

$e_1 e_1 = e_3 + \lambda_1 e_5 + \lambda_2 e_6 + \lambda_3 e_7, \quad e_2 e_1 = e_4 + \lambda_1 e_6 + \lambda_3 e_7,$

$e_3 e_1 = e_5 + \lambda_1 e_7, \quad e_4 e_1 = e_6, \quad e_5 e_1 = e_7, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}.$

2) $L(-1, -2, \lambda_1, -(5\lambda_1 + 14), \lambda_2, \lambda_3)$:

$e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_3 - 2 e_4 + \lambda_1 e_5 + \lambda_2 e_6 + \lambda_3 e_7,$

$e_1 e_1 = e_3 - 2 e_4 + \lambda_1 e_5 - (5\lambda_1 + 14)e_6 + \lambda_2 e_7,$

$e_2 e_1 = e_4 - 2 e_5 + \lambda_1 e_6 - (5\lambda_1 + 14)e_7, \quad e_3 e_1 = e_5 - 2 e_6 + \lambda_1 e_7,$
$e_4 e_1 = e_6 - 2 e_7, \; e_5 e_1 = e_7, \; \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}.$

3) $L(1, -2, \lambda_1, -(6 \lambda_1 + 14), \lambda_2, \lambda^2_2):$
\begin{align*}
e_0 e_0 &= e_2, \; e_i e_0 = e_{i+1}, \; 1 \leq i \leq 6, \; e_0 e_1 = e_3 - 2 e_4 + \lambda_1 e_5 + \lambda_2 e_6 + \lambda_2^2 e_7, \\
e_1 e_1 &= e_3 - 2e_4 + \lambda_1 e_5 - (6 \lambda_1 + 14) e_6 + \lambda_2 e_7, \\
e_2 e_1 &= e_4 - 2e_5 + \lambda_1 e_6 - (6 \lambda_1 + 14) e_7, \; e_3 e_1 = e_5 - 2e_6 + \lambda_1 e_7, \\
e_4 e_1 &= e_6 - 2e_7, \; e_5 e_1 = e_7, \; \lambda_1, \lambda_2 \in \mathbb{C}.
\end{align*}$

4) $L(1, -2, 0, 16, \lambda, \lambda):$
\begin{align*}
e_0 e_0 &= e_2, \; e_i e_0 = e_{i+1}, \; 1 \leq i \leq 6, \; e_0 e_1 &= e_3 - 2 e_4 + 16 e_6 + \lambda e_7, \\
e_1 e_1 &= e_3 - 2 e_4 + 16 e_6 + \lambda e_7, \; e_2 e_1 = e_4 - 2 e_5 + 16 e_7, \\
e_3 e_1 &= e_5 - 2 e_6, \; e_4 e_1 = e_6 - 2 e_7, \; e_5 e_1 = e_7, \; \lambda \in \mathbb{C}.
\end{align*}$

5) $L(1, -2, 5, \lambda_1, -6(\lambda_1 - 7), \lambda_2):$
\begin{align*}
e_0 e_0 &= e_2, \; e_i e_0 = e_{i+1}, \; 1 \leq i \leq 6, \; e_0 e_1 &= e_3 - 2 e_4 + 5 e_6 + \lambda_1 e_6 + \lambda_2 e_7, \\
e_1 e_1 &= e_3 - 2 e_4 + 5 e_6 + \lambda_1 e_6 - 6(\lambda_1 - 7) e_7, \\
e_2 e_1 &= e_4 - 2 e_5 + 5 e_6 + \lambda_1 e_7, \; e_3 e_1 = e_5 - 2 e_6 + 5 e_7, \\
e_4 e_1 &= e_6 - 2 e_7, \; e_5 e_1 = e_7, \; \lambda_1, \lambda_2 \in \mathbb{C}.
\end{align*}$

6) $L(1, -2, 5, \lambda, -7(\lambda - 6), \lambda^2):$
\begin{align*}
e_0 e_0 &= e_2, \; e_i e_0 = e_{i+1}, \; 1 \leq i \leq 6, \; e_0 e_1 &= e_3 - 2 e_4 + 5 e_5 + \lambda e_6 + \lambda^2 e_7, \\
e_1 e_1 &= e_3 - 2 e_4 + 5 e_5 + \lambda e_6 - 7(\lambda - 6) e_7, \\
e_2 e_1 &= e_4 - 2 e_5 + 5 e_6 + \lambda e_7, \; e_3 e_1 = e_5 - 2 e_6 + 5 e_7, \\
e_4 e_1 &= e_6 - 2 e_7, \; e_5 e_1 = e_7, \; \lambda \in \mathbb{C}.
\end{align*}$

7) $L(1, -2, 5, -14, \lambda, 2(\lambda + 21)):$
\begin{align*}
e_0 e_0 &= e_2, \; e_i e_0 = e_{i+1}, \; 1 \leq i \leq 6, \; e_0 e_1 &= e_3 - 2 e_4 + 5 e_5 + 14 e_6 + 2(\lambda + 21) e_7, \\
e_1 e_1 &= e_3 - 2 e_4 + 5 e_5 - 14 e_6 + 2(\lambda + 21) e_7, \\
e_2 e_1 &= e_4 - 2 e_5 + 5 e_6 + \lambda e_7, \; e_3 e_1 = e_5 - 2 e_6 + 14 e_7, \\
e_4 e_1 &= e_6 - 2 e_7, \; e_5 e_1 = e_7, \; \lambda \in \mathbb{C}.
\end{align*}$

8) $L(0, 1, 1, \lambda_1, \lambda_2, \lambda_3):$
\begin{align*}
e_0 e_0 &= e_2, \; e_i e_0 = e_{i+1}, \; 1 \leq i \leq 6, \; e_0 e_1 &= e_4 + e_5 + \lambda_1 e_6 + \lambda_3 e_7, \\
e_1 e_1 &= e_4 + e_5 + \lambda_1 e_6 + \lambda_2 e_7, \; e_2 e_1 = e_5 + e_6 + \lambda_1 e_7, \\
e_3 e_1 &= e_6 + e_7, \; e_4 e_1 = e_7, \; e_5 e_1 = e_7, \; \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}.
\end{align*}$

9) $L(0, 1, 0, \lambda_1, \lambda_1 + 3, \lambda_2):$
\begin{align*}
e_0 e_0 &= e_2, \; e_i e_0 = e_{i+1}, \; 1 \leq i \leq 6, \; e_0 e_1 &= e_4 + \lambda_1 e_6 + \lambda_2 e_7, \\
e_1 e_1 &= e_4 + \lambda_1 e_6 + (\lambda_1 + 3) e_7, \; e_2 e_1 = e_5 + \lambda_1 e_7, \; e_3 e_1 = e_6, \\
e_4 e_1 &= e_7, \; e_5 e_1 = \lambda_1, \lambda_2 \in \mathbb{C}.
\end{align*}$

10) $L(0, 1, 0, \lambda, \lambda^2, \lambda):$
\begin{align*}
e_0 e_0 &= e_2, \; e_i e_0 = e_{i+1}, \; 1 \leq i \leq 6, \; e_0 e_1 &= e_4 + \lambda e_6 + \lambda^2 e_7, \\
e_1 e_1 &= e_4 + \lambda e_6, \; e_2 e_1 = e_5 + \lambda e_7, \; e_3 e_1 = e_6, \; e_4 e_1 = e_7, \; \lambda \in \mathbb{C}.
\end{align*}$

11) $L(0, 1, 0, -3, \lambda, \lambda^2 + \lambda):$
\begin{align*}
e_0 e_0 &= e_2, \; e_i e_0 = e_{i+1}, \; 1 \leq i \leq 6, \; e_0 e_1 &= e_4 - 3 e_6 + (\lambda^2 + \lambda) e_7, \\
e_1 e_1 &= e_4 - 3 e_6 + \lambda e_7, \; e_2 e_1 = e_5 - 3 e_7, \; e_3 e_1 = e_6, \; e_4 e_1 = e_7, \; \lambda \in \mathbb{C}.
\end{align*}$

12) $L(0, 0, 1, 1, \lambda_1, \lambda_2):$
\begin{align*}
e_0 e_0 &= e_2, \; e_i e_0 = e_{i+1}, \; 1 \leq i \leq 6, \; e_0 e_1 &= e_5 + e_6 + \lambda_2 e_7, \\
e_1 e_1 &= e_5 + e_6 + \lambda_1 e_7, \; e_2 e_1 = e_6 + e_7, \; e_3 e_1 = e_7, \; \lambda_1, \lambda_2 \in \mathbb{C}.
\end{align*}$
13) \( L(0, 0, 1, 0, \lambda, \lambda^2 + \lambda) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_5 + (\lambda^2 + \lambda) e_7, \]
\[ e_1 e_1 = e_5 + \lambda e_7, \quad e_2 e_1 = e_6, e_3 e_1 = e_7, \quad \lambda \in \mathbb{C}. \]

14) \( L(0, 0, 0, 1, 1, \lambda) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_6 + \lambda e_7, \quad e_1 e_1 = e_6 + e_7, \]
\[ e_2 e_1 = e_7, \quad \lambda \in \mathbb{C}. \]

15) \( L(0, 1, 0, 0, 0, 0) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_4, \quad e_1 e_1 = e_4, \]
\[ e_2 e_1 = e_5, \quad e_3 e_1 = e_6, \quad e_4 e_1 = e_7. \]

16) \( L(0, 1, 0, \lambda, 1, 1) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_4 - 3e_6 + e_7, \]
\[ e_1 e_1 = e_4 - 3e_6 + e_7, \quad e_2 e_1 = e_5 - 3e_7, \quad e_3 e_1 = e_6, \quad e_4 e_1 = e_7. \]

17) \( L(0, 1, 0, \lambda, 3, 1) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_4 - 3e_6 + e_7, \]
\[ e_1 e_1 = e_4 - 3e_6, \quad e_2 e_1 = e_5 - 3e_7, \quad e_3 e_1 = e_6, \quad e_4 e_1 = e_7. \]

18) \( L(0, 0, 1, 0, 1, 1) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_5 + e_7, \]
\[ e_1 e_1 = e_5 + e_7, \quad e_2 e_1 = e_6, \quad e_3 e_1 = e_7. \]

19) \( L(0, 0, 1, 0, 0, 1) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_5 + e_7, \quad e_1 e_1 = e_5, \]
\[ e_2 e_1 = e_6, \quad e_3 e_1 = e_7. \]

20) \( L(0, 0, 0, 1, 0, 1) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_6 + e_7, \quad e_1 e_1 = e_6, \]
\[ e_2 e_1 = e_7. \]

21) \( L(0, 0, 0, 1, 0, 0) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_6, \quad e_1 e_1 = e_6, \quad e_2 e_1 = e_7. \]

22) \( L(0, 0, 0, 0, 1, 0) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_1 e_1 = e_7. \]

23) \( L(0, 0, 0, 0, 1, 1) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_7, \quad e_1 e_1 = e_7. \]

24) \( L(0, 0, 0, 0, 0, 1) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6, \quad e_0 e_1 = e_7. \]

25) \( L(0, 0, 0, 0, 0, 0) : \)
\[ e_0 e_0 = e_2, \quad e_i e_0 = e_{i+1}, \quad 1 \leq i \leq 6. \]

The number of isomorphism classes \( N_8 = 25. \)

**Conjecture.** The number of isomorphism classes \( N_n \) of \( n \)-dimensional none Lie complex filiform Leibniz algebras in \( FLeib_n \) can be found by the formula:

\[ N_n = n^2 - 7n + 17. \]

Note that the validity of the above formula is confirmed in dimension 9 as well.
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