Umbral Calculus in Positive Characteristic

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Abstract

An umbral calculus over local fields of positive characteristic is developed on the basis of a relation of binomial type satisfied by the Carlitz polynomials. Orthonormal bases in the space of continuous $\mathbb{F}_q$-linear functions are constructed.

**Key words:** $\mathbb{F}_q$-linear function; delta operator; basic sequence; orthonormal basis
1 INTRODUCTION

Classical umbral calculus \cite{15, 14, 16} is a set of algebraic tools for obtaining, in a unified way, a rich variety of results regarding structure and properties of various polynomial sequences. There exists a lot of generalizations extending umbral methods to other classes of functions. However there is a restriction common to the whole literature on umbral calculus – the underlying field must be of zero characteristic. An attempt to mimic the characteristic zero procedures in the positive characteristic case \cite{3} revealed a number of pathological properties of the resulting structures. More importantly, these structures were not connected with the existing analysis in positive characteristic based on a completely different algebraic foundation.

It is well known that any non-discrete locally compact topological field of a positive characteristic \( p \) is isomorphic to the field \( K \) of formal Laurent series with coefficients from the Galois field \( \mathbb{F}_q \), \( q = p^\nu \), \( \nu \in \mathbb{Z}_+ \). Denote by \(| \cdot |\) the non-Archimedean absolute value on \( K \); if \( z \in K \),

\[
z = \sum_{i=m}^{\infty} \zeta_i x^i, \quad n \in \mathbb{Z}, \zeta_i \in \mathbb{F}_q, \zeta_m \neq 0,
\]

then \(|z| = q^{-m}\). This valuation can be extended onto the field \( \overline{K} \), the completion of an algebraic closure of \( K \). Let \( O = \{z \in K : |z| \leq 1\} \) be the ring of integers in \( K \). The ring \( \mathbb{F}_q[x] \) of polynomials (in the indeterminate \( x \)) with coefficients from \( \mathbb{F}_q \) is dense in \( O \) with respect to the topology defined by the metric \( d(z_1, z_2) = |z_1 - z_2| \).

It is obvious that standard notions of analysis do not make sense in the characteristic \( p \) case. For instance, \( n! = 0 \) if \( n \geq p \), so that one cannot define a usual exponential function on \( K \), and \( \frac{d}{dt}(t^n) = 0 \) if \( p \) divides \( n \). On the other hand, some well-defined functions have unusual properties. In particular, there are many functions with the \( \mathbb{F}_q \)-linearity property

\[
f(t_1 + t_2) = f(t_1) + f(t_2), \quad f(\alpha t) = \alpha f(t),
\]

for any \( t_1, t_2, t \in K, \alpha \in \mathbb{F}_q \). Such are, for example, all power series \( \sum c_k t^{q^k}, c_k \in \overline{K} \), convergent on some region in \( K \) or \( \overline{K} \).

The analysis on \( K \) taking into account the above special features was initiated in a seminal paper by Carlitz \cite{1} who introduced, for this case, the appropriate notions of a factorial, an exponential and a logarithm, a system of polynomials \( \{e_i\} \) (now called the Carlitz polynomials), and other related objects. In subsequent works by Carlitz, Goss, Thakur, and many others (see references in \cite{6}) analogs of the gamma, zeta, Bessel and hypergeometric functions were introduced and studied. A difference operator \( \Delta \) acting on functions over \( K \) or its subsets, which was mentioned briefly in \cite{1}, became (as an analog of the operator \( t \frac{d}{dt} \)) the main ingredient of the calculus and the analytic theory of differential equations on \( K \) \cite{9, 10, 11}. It appears also in a characteristic \( p \) analog of the canonical commutation relations of quantum mechanics found in \cite{8}.

The definition of the Carlitz polynomials is as follows. Let \( e_0(t) = t \),

\[
e_i(t) = \prod_{m \in \mathbb{F}_q[x], \deg m < i} (t - m), \quad i \geq 1 \quad (1)
\]
(we follow the notation in [5] used in the modern literature; the initial formulas from [1] have different signs in some places). It is known [1, 5] that

\[ e_i(t) = \sum_{j=0}^{i} (-1)^{i-j} \left[ \begin{matrix} i \\ j \end{matrix} \right] t^{q^j} \]

where

\[ \left[ \begin{matrix} i \\ j \end{matrix} \right] = \frac{D_i}{D_j L_{i-j}}. \]

\( D_i \) is the Carlitz factorial

\[ D_i = [i][i-1]^q \ldots [1]^{q^{i-1}}, \quad [i] = x^{q^i} - x \quad (i \geq 1), \quad D_0 = 1, \]

the sequence \( \{L_i\} \) is defined by

\[ L_i = [i][i-1] \ldots [1] \quad (i \geq 1); \quad L_0 = 1. \]

It follows from (3), (4) that

\[ |D_i| = q^{-\frac{q^i-1}{q-1}}, \quad |L_i| = q^{-i}. \]

The normalized polynomials \( f_i(t) = \frac{e_i(t)}{D_i} \) form an orthonormal basis in the Banach space \( C_0(O, \overline{K}_c) \) of all \( \mathbb{F}_q \)-linear continuous functions \( O \to \overline{K}_c \), with the supremum norm \( \| \cdot \| \). Thus every function \( \varphi \in C_0(O, \overline{K}_c) \) admits a unique representation as a uniformly convergent series

\[ \varphi = \sum_{i=0}^{\infty} a_i f_i, \quad a_i \in \overline{K}_c, \quad a_i \to 0, \]

satisfying the orthonormality condition

\[ \varphi = \sup_{i \geq 0} |a_i|. \]

For several different proofs of this fact see [2, 8, 20]. Note that we consider functions with values in \( \overline{K}_c \) defined on a compact subset of \( K \).

The sequences \( \{D_i\} \) and \( \{L_i\} \) are involved in the definitions of the Carlitz exponential and logarithm:

\[ e_C(t) = \sum_{n=0}^{\infty} \frac{t^{q^n}}{D_n}, \quad \log_C(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{q^n}}{L_n}, \quad |t| < 1. \]

It is seen from (2) and (5) that the above functions are \( \mathbb{F}_q \)-linear on their domains of definition.

The most important object connected with the Carlitz polynomials is the Carlitz module

\[ C_s(z) = \sum_{i=0}^{\deg s} f_i(s) z^{q^i} = \sum_{i=0}^{\deg s} \frac{e_i(s)}{D_i} z^{q^i}, \quad s \in \mathbb{F}_q[x]. \]

Note that by (1) \( e_i(s) = 0 \) if \( s \in \mathbb{F}_q[x], \deg s < i. \)
The function $C_s$ appears in the functional equation for the Carlitz exponential,

$$C_s(e_C(t)) = e_C(st).$$

Its main property is the relation

$$C_{ts}(z) = C_t(C_s(z)), \quad s, t \in \mathbb{F}_q[x],$$

which obtained a far-reaching generalization in the theory of Drinfeld modules, the principal objects of the function field arithmetic (see [6]).

Let us write the identity (7) explicitly using (6). After rearranging the sums we find that

$$C_{ts}(z) = \sum_{i=0}^{\text{deg}t + \text{deg} s} z^i \sum_{m+n=l, m,n \geq 0} \frac{1}{D_n D_m^q} e_n(t)\{e_m(s)\}q^n,$$

so that

$$e_i(st) = \sum_{n=0}^{i} \binom{i}{n}_K e_n(t)\{e_{i-n}(s)\}q^n$$

where

$$\binom{i}{n}_K = \frac{D_i}{D_n D_{i-n}^q}.$$  

In this paper we show that the “$K$-binomial” relation (8), a positive characteristic counterpart of the classical binomial formula, can be used for developing umbral calculus in the spirit of [15]. In particular, we introduce and study corresponding (nonlinear) delta operators, obtain a representation for operators invariant with respect to multiplicative shifts, construct generating functions for polynomial sequences of the $K$-binomial type. Such sequences are also used for constructing new orthonormal bases of the space $C_0(O, \overline{K}_c)$ (in particular, a sequence of the Laguerre type polynomials), in a way similar to the $p$-adic (characteristic 0) case [18, 19, 13].

## 2 DELTA OPERATORS AND K-BINOMIAL SEQUENCES

Denote by $\overline{K}_c\{t\}$ the vector space over $\overline{K}_c$ consisting of $\mathbb{F}_q$-linear polynomials $u = \sum a_k t^{q^k}$ with coefficients from $\overline{K}_c$. We will often use the operator of multiplicative shift $(\rho_{\lambda} u)(t) = u(\lambda t)$ on $\overline{K}_c\{t\}$ and the Frobenius operator $\tau u = u^q$. We call a linear operator $T$ on $\overline{K}_c\{t\}$ invariant if it commutes with $\rho_{\lambda}$ for each $\lambda \in K$.

**Lemma 1.** If $T$ is an invariant operator, then $T(t^{q^n}) = c_n t^{q^n}$, $c_n \in \overline{K}_c$, for each $n \geq 0$.

**Proof.** Suppose that

$$T(t^{q^n}) = \sum_{l=1}^{N} c_{hl} t^{q^l}$$

where $c_{hl}$ are coefficients. By invariance, $T(t^{q^n}) = t^{q^n}T(1) = c_n t^{q^n}$. Therefore, $c_{hl} = 0$ for $h \neq n$. Thus, $T(t^{q^n}) = c_n t^{q^n}$, as desired.
where \( j_i \) are different non-negative integers, \( c_{j_i} \in \mathbb{K}_c \). For any \( \lambda \in K \)

\[
\rho_\lambda T(t^{q^n}) = T \rho_\lambda (t^{q^n}) = T \left( (\lambda t)^{q^n} \right) = \lambda^{q^n} T(t^{q^n}) = \lambda^{q^n} \sum_{l=1}^{N} c_{j_l} t^{q^{j_l}}.
\]

On the other hand,

\[
\rho_\lambda T(t^{q^n}) = \sum_{l=1}^{N} c_{j_l} \lambda^{q^{j_l}} t^{q^{j_l}}.
\]

Since \( \lambda \) is arbitrary, this implies the required result. ■

If an invariant operator \( T \) is such that \( T(t) = 0 \), then by Lemma 1 the operator \( \tau^{-1} T \) on \( \mathbb{K}_c \{ t \} \) is well-defined.

Definition 1. A \( \mathbb{F}_q \)-linear operator \( \delta = \tau^{-1} \delta_0 \), where \( \delta_0 \) is a linear invariant operator on \( \mathbb{K}_c \{ t \} \), is called a delta operator if \( \delta_0(t) = 0 \) and \( \delta_0(f) \neq 0 \) for \( \deg f > 1 \), that is \( \delta_0(t^{q^n}) = c_n t^{q^n} \), \( c_n \neq 0 \), for all \( n \geq 1 \).

The most important example of a delta operator is the Carlitz derivative \( d = \tau^{-1} \Delta \) where

\[
(\Delta u)(t) = u(xt) - xu(t).
\]

Many interesting \( \mathbb{F}_q \)-linear functions satisfy equations involving the operator \( d \); for example, for the Carlitz exponential we have \( de_C = e_C \). It appears also in \( \mathbb{F}_q \)-linear representations of the canonical commutation relations [8, 9].

Definition 2. A sequence \( \{P_n\}_{0}^{\infty} \) of \( \mathbb{F}_q \)-linear polynomials is called a basic sequence corresponding to a delta operator \( \delta = \tau^{-1} \delta_0 \), if \( \deg P_n = q^n \), \( P_0(1) = 1 \), \( P_n(1) = 0 \) for \( n \geq 1 \),

\[
\delta P_0 = 0, \quad \delta P_n = [n]^{1/q} P_{n-1}, \quad n \geq 1,
\]

or, equivalently,

\[
\delta_0 P_0 = 0, \quad \delta_0 P_n = [n] P_{n-1}^q, \quad n \geq 1.
\]

It follows from well-known identities for the Carlitz polynomials \( e_i \) (see [5]) that the sequence \( \{e_i\} \) is basic with respect to the operator \( d \). For the normalized Carlitz polynomials \( f_i \) we have the relations

\[
df_0 = 0, \quad df_i = f_{i-1}, \quad i \geq 1.
\]

The next definition is a formalization of the property (8).

Definition 3. A sequence of \( \mathbb{F}_q \)-linear polynomials \( u_i \in \mathbb{K}_c \{ t \} \) is called a sequence of \( K \)-binomial type if \( \deg u_i = q^i \) and for all \( i = 0, 1, 2, \ldots \)

\[
u_i(st) = \sum_{n=0}^{i} \binom{i}{n}_K u_n(t) \{u_{i-n}(s)\}^{q^n}, \quad s, t \in K.
\]
If \( \{u_i\} \) is a sequence of \( K \)-binomial type, then \( u_i(1) = 0 \) for \( i \geq 1 \), \( u_0(1) = 1 \) (so that \( u_0(t) = t \)).

Indeed, for \( i = 0 \) the formula (12) gives \( u_0(st) = u_0(s)u_0(t) \). Setting \( s = 1 \) we have \( u_0(t) = u_0(1)u_0(t) \), and since \( \deg u_0 = 1 \), so that \( u_0(t) \neq 0 \), we get \( u_0(1) = 1 \).

If \( i > 0 \), for all \( t \)

\[
0 = u_i(t) - u_i(t) = \sum_{n=0}^{i-1} \binom{i}{n} K_{u_i-n(1)} q^n u_n(t),
\]

and the linear independence of the polynomials \( u_n \) means that \( u_l(1) = 0 \) for \( l \geq 1 \).

**Theorem 1.** For any delta operator \( \delta = \tau^{-1} \delta_0 \), there exists a unique basic sequence \( \{P_n\} \), which is a sequence of \( K \)-binomial type. Conversely, given a sequence \( \{P_n\} \) of \( K \)-binomial type, define the action of \( \delta_0 \) on \( P_n \) by the relations (11), extend it onto \( \mathbb{K}_c \{t\} \) by linearity and set \( \delta = \tau^{-1} \delta_0 \). Then \( \delta \) is a delta operator, and \( \{P_n\} \) is the corresponding basic sequence.

**Proof.** Let us construct a basic sequence corresponding to \( \delta \). Set \( P_0(t) = t \) and suppose that \( P_{n-1} \) has been constructed. It follows from Lemma 1 that \( \delta \) is surjective, and we can choose \( P_n \) satisfying (10). For any \( c \in \mathbb{K}_c \), \( P_n + ct \) also satisfies (10), and we may redefine \( P_n \) choosing \( c \) in such a way that \( P_n(1) = 0 \).

Hence, a basic sequence \( \{P_n\} \) indeed exists. If there is another basic sequence \( \{P'_n\} \) with the same delta operator, then \( \delta(P_n - P'_n) = 0 \), whence \( P'_n(t) = P_n(t) + at, a \in \mathbb{K}_c \), and setting \( t = 1 \) we find that \( a = 0 \).

In order to prove the \( K \)-binomial property, we introduce some operators having an independent interest.

Consider the linear operators \( \delta_0^{(l)} = \tau^l \delta^l \).

**Lemma 2.** (i) The identity

\[
\delta_0^{(l)} P_j = \frac{D_j}{D_{j-l}} P_{j-l} \tag{13}
\]

holds for any \( l \leq j \).

(ii) Let \( f \) be a \( \mathbb{F}_q \)-linear polynomial, \( \deg f \leq q^n \). Then a generalized Taylor formula

\[
f(st) = \sum_{l=0}^{n} \left( \frac{\delta_0^{(l)} f}{D_l} \right) (s) P_1(t) \tag{14}
\]

holds for any \( s, t \in K \).

**Proof.** By (10),

\[
\delta^l P_j = \delta^{j-l-1} \left( [j] q^{-1} P_{j-1} \right) = [j]^{q-1} \delta^l P_{j-1} = [j]^{q-1}[j-1] q^{-(l-1)} \delta^{l-2} P_{j-2} = \ldots = [j]^{q-1}[j-1] q^{-(l-1)} \ldots [j - (l-1)] q^{-1} P_{j-l},
\]
so that
\[ \delta_0^{(l)} P_j = [j][j - 1]^q \ldots [j - (l - 1)]^q P_{j-l}^{q^l} \]
which is equivalent to (13).

Since \( \text{deg } P_j = q^j \), the polynomials \( P_1, \ldots, P_n \) form a basis of the vector space of all \( \mathbb{F}_q \)-linear polynomials of degrees \( \leq n \) (because its dimension equals \( n \)). Therefore
\[ f(st) = \sum_{j=0}^{n} b_j(s) P_j(t) \quad (15) \]
where \( b_j(s) \) are, for each fixed \( s \), some elements of \( \overline{K_c} \).

Applying the operator \( \delta_0^{(l)} \), \( 0 \leq l \leq n \), in the variable \( t \) to both sides of (15) and using (14) we find that
\[ \left( \delta_0^{(l)} f \right)(st) = \sum_{j=l}^{n} b_j(s) \frac{D_j}{D_{j-l}} P_{j-l}^{q^l}(t) \]
(note also that \( \delta_0^{(l)} \) commutes with \( \rho_s \)). Setting \( t = 1 \) and taking into account that
\[ P_{j-l}(1) = \begin{cases} 0, & \text{if } j > l; \\ 1, & \text{if } j = l, \end{cases} \]
we come to the equality
\[ b_l(s) = \frac{\left( \delta_0^{(l)} f \right)(s)}{D_l}, \quad 0 \leq l \leq n, \]
which implies (14). \( \blacksquare \)

Note that the formulas (13) and (14) for the Carlitz polynomials \( e_i \) are well known; see \([5]\). It is important that, in contrast to the classical umbral calculus, the linear operators involved in (14) are not powers of a single linear operator.

Proof of Theorem 1 (continued). In order to prove that \( \{P_n\} \) is a sequence of \( K \)-binomial type, it suffices to take \( f = P_n \) in (14) and to use the identity (13).

To prove the second part of the theorem, we calculate the action in the variable \( t \) of the operator \( \delta_0 \), defined by (11), upon the function \( P_n(st) \). Using the relation \( D_{n+1} = [n+1]D_n^q \) we find that
\[
\delta_0 t P_n(st) = \sum_{j=0}^{n} \binom{n}{j} K P_{n-j}^q(s) (\delta_0 P_j)(t) = \sum_{j=1}^{n} \frac{D_n}{D_{n-j}^q D_j} P_{n-j}^q(s) [j] P_{j-l}^q(t) \\
= \sum_{i=0}^{n-1} \frac{D_n[i+1]}{D_{n-i-1}^q D_{i+1}} P_{n-i-1}^q(t) P_i^q(t) = [n] \sum_{i=0}^{n-1} \left( \frac{D_{n-i}}{D_{n-i-1} D_i} \right)^q P_{n-i-1}^q(s) P_i^q(t) \\
= [n] \left\{ \sum_{i=0}^{n-1} \binom{n-1}{i} K P_{n-i-1}^q(s) P_i(t) \right\}^q = [n] P_{n-1}^q(st) = (\delta_0 P_n)(st),
\]
that is \( \delta_0 \) commutes with multiplicative shifts.
It remains to prove that \( \delta_0(f) \neq 0 \) if \( \deg f > 1 \). Assuming that \( \delta_0(f) = 0 \) for \( f = \sum_{j=0}^{n} a_j P_j \)
we have

\[
0 = \sum_{j=0}^{n} a_j [j] P_j^{-1} = \left\{ \sum_{i=0}^{n-1} a_{i+1}^{1/q}[i+1]^{1/q} P_i \right\}^q
\]

whence \( a_1 = a_2 = \ldots = a_n = 0 \) due to the linear independence of the sequence \( \{P_i\} \). ■

3 INVARIANT OPERATORS

Let \( T \) be a linear invariant operator on \( \overline{K}_c \{t\} \). Let us find its representation via an arbitrary fixed delta operator \( \delta = \tau^{-1} \delta_0 \). By (14), for any \( f \in \overline{K}_c \{t\} \), \( \deg f = q^n \),

\[
(Tf)(st) = (\rho s T)f(t) = T(\rho s f)(t) = T_i f(st) = \sum_{l=0}^{n} (TP_l)(t) \frac{\left( \delta_0(l)^{(l)} \right)(s)}{D_l}.
\]

Setting \( s = 1 \) we find that

\[
T = \sum_{l=0}^{\infty} \sigma_l \delta_0(l) \tag{16}
\]

where \( \sigma_l = \frac{(TP_l)(1)}{D_l} \). The infinite series in (16) becomes actually a finite sum if both sides of (16) are applied to any \( \mathbb{F}_q \)-linear polynomial \( f \in \overline{K}_c \{t\} \).

Conversely, any such series defines a linear invariant operator on \( \overline{K}_c \{t\} \).

Below we will consider in detail the case where \( \delta \) is the Carlitz derivative \( d \), so that \( \delta_0 = \Delta \), and the operators \( \delta_0^{(l)} = \Delta^{(l)} \) are given recursively \[5\]:

\[
(\Delta^{(l)}u)(t) = (\Delta^{(l-1)}u)(xt) - x^{q^{l-1}}u(t) \tag{17}
\]

(the formula (16) for this case was proved by a different method in \([7]\)).

Using (17) with \( l = 0 \), we can compute for this case the coefficients \( c_n \) from Lemma 1. We have \( \Delta^{(l)}(t^{q^n}) = 0 \), if \( n < l \),

\[
\Delta(t^{q^n}) = [n]t^{q^n}, \quad n \geq 1;
\]

\[
\Delta^{(2)}(t^{q^n}) = \tau^2 d^2(t^{q^n}) = \tau \Delta t^{-1} \Delta(t^{q^n}) = \tau \Delta \left( [n]^{1/q}t^{q^n-1} \right) = [n][n-1]^{q}t^{q^n}, \quad n \geq 2,
\]

and by induction

\[
\Delta^{(l)}(t^{q^n})[n][n-1]^q \ldots [n-l+1]^{q^{l-1}} t^{q^n} = \frac{D_n}{D_{n-l}} t^{q^n}, \quad n \geq l. \tag{18}
\]

The explicit formula (18) makes it possible to find out when an operator \( \theta = \tau^{-1}\theta_0 \), with

\[
\theta_0 = \sum_{l=1}^{\infty} \sigma_l \Delta^{(l)}, \tag{19}
\]
is a delta operator. We have \( \theta_0(t) = 0 \),

\[
\theta_0(t^{q^n}) = D_n S_n t^{q^n},
\]

where \( S_n = \sum_{l=1}^{n} \frac{\sigma_l}{D^q_{n-l}} \). Thus \( \theta \) is a delta operator if and only if \( S_n \neq 0 \) for all \( n = 1, 2, \ldots \).

**Example 1.** Let \( \sigma_l = 1 \) for all \( l \geq 1 \), that is \( \theta_0 = \sum_{l=1}^{\infty} \Delta^{(l)} \).

(20)

Since \( |D_i| = q^{\frac{q^{i+1}}{q-1}} \), we have

\[
|D^q_{n-l}| = q^{\frac{q^n-q^l}{q-1}},
\]

so that \( |S_n| = q^{\frac{q^n-q^1}{q-1}}(\neq 0) \) by the ultra-metric property of the absolute value. Comparing (20) with a classical formula from [15] we may see the polynomials \( P_n \) for this case as analogs of the Laguerre polynomials.

**Example 2.** Let \( \sigma_l = \frac{(-1)^{l+1}}{L_l} \). Now

\[
S_n = \sum_{l=1}^{n} (-1)^{l+1} \frac{1}{L_l D^q_{n-l}}.
\]

Let us use the identity

\[
\sum_{j=0}^{h-1} \frac{(-1)^j}{L_j D^q_{h-j}} = \frac{(-1)^{h+1}}{L_h}
\]

proved in [4]. It follows from (21) that

\[
\sum_{j=1}^{h} \frac{(-1)^j}{L_j D^q_{h-j}} = \frac{(-1)^{h+1}}{L_h} - \frac{1}{D_h} + \frac{(-1)^h}{L_h} = -\frac{1}{D_h},
\]

so that \( S_n = D_n^{-1} (\neq 0), \) \( n = 1, 2, \ldots \). In this case \( \theta_0(t^{q^j}) = t^{q_j} \) for all \( j \geq 1 \) (of course, \( \theta_0(t) = 0 \)), and \( P_0(t) = t, \ P_n(t) = D_n \left( t^{q^n} - t^{q^{n-1}} \right) \) for \( n \geq 1 \).

### 4 ORTHONORMAL BASES

Let \( \{P_n\} \) be the basic sequence corresponding to a delta operator \( \delta = \tau^{-1} \delta_0 \),

\[
\delta_0 = \sum_{l=1}^{\infty} \sigma_l \Delta^{(l)}
\]

(the operator series converges on any polynomial from \( \mathcal{K}_c(t) \)).
Let \( Q_n = \frac{P_n}{D_n}, \) \( n = 0, 1, 2, \ldots \). Then for any \( n \geq 1 \)

\[
\delta Q_n = D_n^{-1/q} \delta P_n = \frac{[n]^{1/q}}{D_n^{1/q}} P_{n-1} = \frac{P_{n-1}}{D_{n-1}} = Q_{n-1},
\]

and the \( K \)-binomial property of \( \{P_n\} \) implies the identity

\[
Q_i(st) = \sum_{n=0}^{i} Q_n(t) \{Q_{i-n}(s)\}^{q^n}, \quad s, t \in K.
\]

The identity (22) may be seen as another form of the \( K \)-binomial property. Though it resembles its classical counterpart, the presence of the Frobenius powers is a feature specific for the case of a positive characteristic. We will call \( \{Q_n\} \) a normalized basic sequence.

**Theorem 2.** If \( |\sigma_1| = 1, |\sigma_l| \leq 1 \) for \( l \geq 2 \), then the sequence \( \{Q_n\}_0^\infty \) is an orthonormal basis of the space \( C_0(O, \mathcal{K}_c) \) – for any \( f \in C_0(O, \mathcal{K}_c) \) there is a uniformly convergent expansion

\[
f(t) = \sum_{n=0}^{\infty} \psi_n Q_n(t), \quad t \in O,
\]

where \( \psi_n = \left(\delta^{(n)}_0 f\right)(1), \) \( |\psi_n| \rightarrow 0 \) as \( n \rightarrow \infty, \)

\[
\|f\| = \sup_{n \geq 0} |\psi_n|.
\]

**Proof.** We have \( Q_0(t) = P_0(t) = t \), so that \( \|Q_0\| = 1 \). Let us prove that \( \|Q_n\| = 1 \) for all \( n \geq 1 \). Our reasoning will be based on expansions in the normalized Carlitz polynomials \( f_n \).

Let \( n = 1 \). Since \( \deg Q_n = a^n \), we have \( Q_1 = a_0 f_0 + a_1 f_1 \). We know that \( Q_1(1) = f_1(1) = 0 \), hence \( a_0 = 0 \), so that \( Q_1 = a_1 f_1 \). Next, \( \delta Q_1 = Q_0 = f_0 \). Writing this explicitly we find that

\[
f_0 = a_1^{1/q} \tau^{-1} \sum_{l=1}^{\infty} \sigma_l \Delta^{(l)} f_1 = a_1^{1/q} \sigma_1^{1/q} \tau^{-1} \Delta f_1 = a_1^{1/q} \sigma_1^{1/q} f_0,
\]

whence \( a_1 = \sigma_1^{-1} \), \( Q_1 = \sigma_1^{-1} f_1 \), and \( \|Q_1\| = 1 \).

Assume that \( \|Q_{n-1}\| = 1 \) and consider the expansion

\[
Q_n = \sum_{j=1}^{n} a_j f_j
\]

(the term containing \( f_0 \) is absent since \( Q_n(1) = 0 \)). Applying \( \delta \) we get

\[
\delta Q_n = \sum_{j=1}^{n} a_j^{1/q} \sum_{l=1}^{\infty} \sigma_l^{1/q} \tau^{-1} \Delta^{(l)} f_j.
\]
It is known \(^5\) that
\[
\Delta^{(l)} e_j = \begin{cases} \frac{D_j}{D_{j-l}} e_{j-l}, & \text{if } l \leq j, \\ 0, & \text{if } l > j, \end{cases}
\]
so that
\[
\Delta^{(l)} f_j = \begin{cases} f^q_{j-l}, & \text{if } l \leq j, \\ 0, & \text{if } l > j. \end{cases}
\]
Therefore
\[
\delta Q_n = \sum_{j=1}^n a_j^{1/q} \sum_{l=1}^j \sigma_i^{1/q} f_{j-l}^{l-1}.
\] (25)

It follows from the identity \( f_{i-1}^q = f_{i-1} + [i] f_i \) (see \(^5\) \(^8\)) that
\[
f_{j-l}^{l-1} = \sum_{k=0}^{l-1} \varphi_{j,l,k} f_{j-l+k}
\]
where \( \varphi_{j,l,0} = 1, |\varphi_{j,l,k}| < 1 \) for \( k \geq 1 \). Substituting into (25) we find that
\[
Q_{n-1} = \sum_{j=1}^n a_j^{1/q} \sum_{l=1}^j \sigma_i^{1/q} \sum_{k=0}^{l-1} \varphi_{j,l,k} f_{j-l+k} = \sum_{j=1}^n a_j^{1/q} \sum_{i=0}^{j-1} f_i \sum_{l=j-i}^j \sigma_i^{1/q} \varphi_{j,l,i-j+l} = \sum_{i=0}^{n-1} f_i \sum_{j=i+1}^n a_j^{1/q} \sum_{l=j-i}^j \sigma_i^{1/q} \varphi_{j,l,i-j+l}
\]
whence
\[
\max_{0 \leq i \leq n-1} \left| \sum_{j=i+1}^n a_j^{1/q} \sum_{l=j-i}^j \sigma_i^{1/q} \varphi_{n,l,i-l-1} \right| = 1 \tag{26}
\]
by the inductive assumption and the orthonormal basis property of the normalized Carlitz polynomials.

For \( i = n - 1 \), we obtain from (26) that
\[
\left| a_n^{1/q} \sum_{l=1}^n \sigma_l^{1/q} \varphi_{n,l,l-1} \right| \leq 1.
\]
We have \( \varphi_{n,1,0} = 1, |\sigma_1| = 1 \), and
\[
\sum_{l=2}^n \sigma_l^{1/q} \varphi_{n,l,l-1} < 1,
\]
so that
\[
\left| \sum_{l=1}^n \sigma_l^{1/q} \varphi_{n,l,l-1} \right| = 1
\]
whence $|a_n| \leq 1$.

Next, for $i = n - 2$ we find from (26) that
\[
|a_{n-1}^{1/q} \sum_{l=1}^{n-1} \sigma_l^{1/q} \varphi_{n-1,l} + a_n^{1/q} \sum_{l=2}^{n} \sigma_l^{1/q} \varphi_{n,l} | \leq 1.
\]
We have proved that the second summand on the left is in $O$; then the first summand is considered as above, so that $|a_{n-1}| \leq 1$. Repeating this reasoning we come to the conclusion that $|a_j| \leq 1$ for all $j$. Moreover, $|a_j| = 1$ for at least one value of $j$; otherwise we would come to a contradiction with (26). This means that $\|Q_n\| = 1$.

If $f$ is an arbitrary $F_q$-linear polynomial, $\deg f = q^N$, then by the generalized Taylor formula (14)
\[
f(t) = \sum_{l=0}^{N} \psi_l Q_l(t), \quad t \in O,
\]
where $\psi_l = \left( \delta_0^{(l)} f \right)(1)$.

Since $\|Q_l\| = 1$ for all $l$, we have $\|f\| \leq \sup_l |\psi_l|$. On the other hand, $\delta_0^{(l)} f = \tau^l (\tau^{-1} \delta_0)^l f$, and if we prove that $\|\delta_0 f\| \leq \|f\|$, this will imply the inequality $\|\delta_0^{(l)} f\| \leq \|f\|$. We have
\[
\|\Delta^{(l)} f\| = \max_{t \in O} \left| (\Delta^{(l-1)} f)(xt) - x^q \cdot (\Delta^{(l-1)} f)(t) \right| \leq \max_{t \in O} \left| (\Delta^{(l-1)} f)(t) \right| \leq \ldots \leq \max_{t \in O} \left| (\Delta f)(t) \right| \leq \|f\|
\]
so that
\[
\|\delta_0 f\| = \left\| \sum_{l=0}^{\infty} \sigma_l \Delta^{(l)} f \right\| \leq \sup_l |\sigma_l| \cdot \left\| \Delta^{(l)} f \right\| \leq \|f\|
\]
whence $\|\delta_0^{(l)} f\| \leq \|f\|$ and $\sup_l |\psi_l| \leq \|f\|$.

Thus, we have proved (24) for any polynomial. By a well-known result of non-Archimedean functional analysis (see Theorem 50.7 in [17]), the uniformly convergent expansion (23) and the equality (24) hold for any $f \in C_0(O, \mathcal{K}_e)$.

The relation $\psi_n = \left( \delta_0^{(n)} f \right)(1)$ also remains valid for any $f \in C_0(O, \mathcal{K}_e)$. Indeed, denote by $\varphi_n(f)$ a continuous linear functional on $C_0(O, \mathcal{K}_e)$ of the form $\left( \delta_0^{(n)} f \right)(1)$. Suppose that $\{F_N\}$ is a sequence of $F_q$-linear polynomials uniformly convergent to $f$. Then
\[
F_N = \sum_n \varphi_n(F_N) Q_n,
\]
so that
\[
F - F_N = \sum_{n=0}^{\infty} \left\{ \psi_n - \varphi_n(F_N) \right\} Q_n,
\]
and by (24),
\[
\|F - F_N\| = \sup_n |\psi_n - \varphi_n(F_N)|.
\]
For each fixed $n$ we find that $|\psi_n - \varphi_n(F_N)| \leq \|F - F_N\|$, and passing to the limit as $N \to \infty$ we get that $\psi_n = \varphi_n(f)$, as desired. ■

By Theorem 2, the Laguerre-type polynomial sequence from Example 1 is an orthonormal basis of $C_0(O, K_c)$. The sequence from Example 2 does not satisfy the conditions of Theorem 2.

Note that the conditions $|\sigma_1| = 1$, $|\sigma_l| \leq 1$, $l = 2, 3, \ldots$, imply that $S_n \neq 0$ for all $n$, so that the series (19) considered in Theorem 2 always correspond to delta operators.

Let us write a recurrence formula for the coefficients of the polynomials $Q_n$. Here we assume only that $S_n \neq 0$ for all $n$. Let

$$Q_n(t) = \sum_{j=0}^{n} \gamma_j^{(n)} t^q^j. \quad (27)$$

We know that $\gamma_0^{(0)} = 1$.

Using the relation $\delta_0 \left(t^q^n\right) = D_n S_n t^q^n$ we find that for $n \geq 1$

$$Q_{n-1} = \delta Q_n = \tau^{-1} \sum_{j=1}^{n} \gamma_j^{(n)} D_j S_j t^q^j = \sum_{i=0}^{n-1} \left(\gamma_{i+1}^{(n)}\right)^{\frac{1}{q}} D_{i+1}^{\frac{1}{q}} S_{i+1}^{\frac{1}{q}} t^q^i.$$

Comparing this with the equality (27), with $n - 1$ substituted for $n$, we get

$$\gamma_i^{(n-1)} = \left(\gamma_{i+1}^{(n)}\right)^{\frac{1}{q}} D_{i+1}^{\frac{1}{q}} S_{i+1}^{\frac{1}{q}}$$

whence

$$\gamma_i^{(n)} = \frac{\left(\gamma_i^{(n-1)}\right)^{q}}{D_{i+1} S_{i+1}}, \quad i = 0, 1, \ldots, n - 1; \quad n = 1, 2, \ldots \quad (28)$$

The recurrence formula (28) determines all the coefficients $\gamma_i^{(n)}$ (if the polynomial $Q_{n-1}$ is already known) except $\gamma_0^{(n)}$. The latter can be found from the condition $Q_n(1) = 0$:

$$\gamma_0^{(n)} = -\sum_{j=1}^{n} \gamma_j^{(n)}.$$

5 GENERATING FUNCTIONS

The definition (6) of the Carlitz module can be seen as a generating function for the normalized Carlitz polynomials $f_i$. Here we give a similar construction for the normalized basic sequence in the general case. As in Sect. 4, we consider a delta operator of the form $\delta = \tau^{-1} \delta_0$,

$$\delta_0 = \sum_{l=1}^{\infty} \sigma_l \Delta^{(l)}$$

We assume that $S_n \neq 0$ for all $n$. 14
Let us define the generalized exponential

$$e_{\delta}(t) = \sum_{j=0}^{\infty} b_j t^{q^j}$$

(29)

by the conditions \(\delta e_{\delta} = e_{\delta},\ b_0 = 1\). Substituting (29) we come to the recurrence relation

$$b_{j+1} = \frac{b_j^q}{D_{j+1}S_{j+1}}$$

(30)

which determines \(e_{\delta}\) as a formal power series.

Since \(b_0 = 1\), the composition inverse \(\log_{\delta}\) to the formal power series \(e_{\delta}\) has a similar form:

$$\log_{\delta}(t) = \sum_{n=0}^{\infty} \beta_n t^{q^n}, \quad \beta_n \in K,$$

(31)

(see Sect. 19.7 in [12] for a general treatment of formal power series of this kind). A formal substitution gives the relations

$$\beta_0 = 1, \quad \sum_{m+n=l} b_m \beta_n t^{q^m} = 0, \quad l = 1, 2, \ldots,$$

whence

$$\beta_l = -\sum_{m=1}^{l} b_m \beta_{l-m} t^{q^m}, \quad l = 1, 2, \ldots.$$  

(32)

**Theorem 3.** Suppose that \(|\sigma_1| = 1\) and \(|\sigma_l| \leq 1\) for all \(l\). Then both the series (29) and (31) converge on the disk \(D_q = \{t \in O : |t| \leq q^{-1}\}\), if \(q \neq 2\), or \(D_2 = \{t \in O : |t| \leq q^{-2}\}\), if \(q = 2\), and

$$e_{\delta}(t \log_{\delta} z) = \sum_{n=0}^{\infty} Q_n(t) z^{q^n}, \quad t \in O, \ z \in D_q.$$

(33)

**Proof.** Since

$$\left| \frac{D_n}{D_{q^{n-1}}} \right| = q^{\frac{q^{n-1}}{q-1}},$$

under our assumptions we have \(|D_n S_n| = q^{-1}\) for all \(n\). By (30), \(|b_{j+1}| = q |b_j|^q, j = 0, 1, 2, \ldots,\) and we prove easily by induction that

$$|b_j| = q^{\frac{j}{q-1}}, \quad j = 0, 1, 2, \ldots.$$  

(34)

For the sequence (32) we obtain the estimate

$$|\beta_j| \leq q^{\frac{j}{q-1}}, \quad j = 0, 1, 2, \ldots.$$  

(35)
Indeed, this is obvious for \( j = 0 \). If (35) is proved for \( j \leq l - 1 \), then

\[
|\beta_j| \leq \max_{1 \leq m \leq l} |b_m| \cdot |\beta_{l-m}| q^m \leq \max_{1 \leq m \leq l} q^{m-1} + q^m q^{l-m-1} = q^{l-1}.
\]

It follows from (34) and (35) that both the series (29) and (31) are convergent for \( t \in D_q \) (in fact they are convergent on a wider disk from \( \overline{K_c} \), but here we consider them only on \( K \)).

Note also that

\[
|\log_\delta(t)| \leq \max_{n \geq 0} q^{\frac{1}{q^n}} q^{\frac{1}{q^n} |t|} = |t|
\]

if \( t \in D_q \).

If \( \lambda \in D_q \), then the function \( t \mapsto e_\delta(\lambda t) \) is continuous on \( O \), and by Theorem 2

\[
e_\delta(\lambda t) = \sum_{n=0}^{\infty} \psi_n(\lambda) Q_n(t)
\]

where \( \psi_n(\lambda) = \left( \delta_0^{(n)} e_\delta(\lambda \cdot) \right) (1) = \left( \delta_0^{(n)} e_\delta \right) (\lambda) \) due to the invariance of the operator \( \delta_0^{(n)} \).

Since \( \delta_0^{(n)} = \tau^n \delta^n \) and \( \delta e_\delta = e_\delta \), we find that \( \delta_0^{(n)} e_\delta = e_\delta^n \). Therefore

\[
e_\delta(\lambda t) = \sum_{n=0}^{\infty} Q_n(t) \{e_\delta(\lambda)\} q^n
\]

(36)

for any \( t \in O, \lambda \in D_q \). Setting in (36) \( \lambda = \log_\delta(z) \) we come to (33). ■
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