MONOPOLE CLASSES AND EINSTEIN METRICS

D. KOTSCHICK

ABSTRACT. We introduce the notion of a special monopole class on a four-manifold. This is used to prove restrictions on the smooth structures of Einstein manifolds. As an application we prove that there are Einstein four-manifolds which are simply connected, spin, and satisfy the strict Hitchin–Thorpe inequality, and which are homeomorphic to manifolds without Einstein metrics.

1. INTRODUCTION

Seiberg–Witten theory has been developed as a theory of moduli spaces of solutions to the monopole equations by imitating the development for the anti-self-dual Yang–Mills equations. Therefore, in the applications the use of the whole moduli space has been emphasized, following the ideas of Donaldson in Yang–Mills theory. Nevertheless, unlike Donaldson theory, Seiberg–Witten theory has incorporated geometric arguments exploiting a single solution to the monopole equations for explicit constructions, without necessarily studying all its deformations. Rather than imitating Donaldson theory, these arguments are more reminiscent of Yau’s work on the Calabi conjectures, where a single Einstein metric is constructed and used as input in geometric arguments, or of the use of minimal surfaces in three-dimensional topology or in the theory of manifolds of positive scalar curvature. Going back even further, there is a clear parallel with the use of harmonic forms and harmonic spinors in geometry. Of course, those are solutions of linear partial differential equations, rather than non-linear ones, and one might speculate that the mildness of the non-linearity of the Seiberg–Witten equations manifests itself in the way the solutions are used.

For arguments using the existence of a solution to the monopole equations rather than the non-triviality of a topological invariant, Kronheimer [25].

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1With the hindsight of Seiberg–Witten theory, it is worth rereading early appraisals of Donaldson theory, such as [2].
introduced the notion of a monopole class. This is the characteristic class of a Spin$^c$-structure for which there are solutions of the monopole equations for arbitrary metrics. The usefulness of this notion stems from the fact that one can make geometric arguments with a solution, without ever studying its deformation theory. The purpose of this paper is to derive certain constraints on the smooth structures underlying Einstein manifolds from the existence of monopole classes. In some of our arguments we need two additional properties of monopole classes, which are always satisfied by the basic classes coming from non-trivial Seiberg–Witten invariants, but which are not part of Kronheimer’s definition of monopole classes. Thus, in Section 2 we discuss variations on the definition, and the relations between these different definitions. This leads to issues of genericity and transversality for the monopole equations.

In Section 3 we derive restrictions on Einstein four-manifolds with monopole classes, and use these to give various examples of simply connected Einstein manifolds homeomorphic to smooth manifolds which cannot admit any Einstein metric. The existence of such examples is not new. I gave the first examples in [23], showing that an obstruction to the existence of Einstein metrics discovered by LeBrun [27] depends on the smooth structure, rather than being homotopy-invariant. Recently, Ishida–LeBrun [21] asked if there are such examples which are spin. They found an obstruction that is applicable to spin manifolds, but were unable to show that it is not homotopy-invariant. We shall resolve their conundrum by exhibiting a simply connected Einstein manifold which is spin and satisfies the strict Hitchin–Thorpe inequality, and is homeomorphic to manifolds not admitting any Einstein metric because of the obstruction discussed in [21].

Of course, the spin condition is a red herring. It is clear from the known examples that the obstructions to Einstein metrics derived from the Seiberg–Witten monopole equations have to do with the reducibility of the underlying manifolds, spin or non-spin. (Compare [27, 23, 28, 21, 7] and Section 3 below.) In fact, it is quite tempting to conjecture that, except for blowups of the complex projective plane, Einstein four-manifolds should be irreducible. In Section 4 we prove some partial results in this direction.

In the Appendix, written jointly with J. Wehrheim, we prove some results about Ricci-flat four-manifolds motivated by the conjecture that all closed Ricci-flat four-manifolds should be covered by $K3$ or $T^4$. From these results we conclude that the existence of strongly scalar-flat four-manifolds which are not covered by $T^4$ or $K3$ would imply the existence of some new obstruction to positive scalar curvature.

\footnote{See Remark 14 below for an instance where it does turn out to be homotopy-invariant.}
2. Monopole classes

Consider a closed smooth oriented four-manifold $X$ with a Spin$^c$-structure $s$. For every choice of Riemannian metric $g$, the Seiberg–Witten monopole equations for $(X, s)$ with respect to $g$ are a system of coupled equations for a pair $(A, \Phi)$, where $A$ is a Spin$^c$-connection in the spin bundle for $s$ and $\Phi$ is a section of the positive spin bundle $V_+$. The equations are:

(1) \[ D_A^+ \Phi = 0 , \]

(2) \[ F_A^+ = \sigma(\Phi, \Phi) , \]

with $D_A^+$ the half-Dirac operator defined on spinors of positive chirality, and $\hat{A}$ the connection induced by $A$ on the determinant of the spin bundle. The right-hand side of the curvature equation (2) is the 2-form which, under the Clifford module structure determined by $s$, corresponds to the trace-free part of $\Phi \otimes \Phi^\ast$.

Solutions $(A, \Phi)$ with $\Phi \equiv 0$ are called reducible. If there is a reducible solution, then $c = c_1(s)$ is represented by an anti-self-dual harmonic form because of (2). This implies $c^2 \leq 0$, with equality if and only if $c$ is a torsion class.

The following definition is due to Kronheimer [25].

**Definition 1.** A class $c \in H^2(X, \mathbb{Z})$ is called a monopole class, if there is a Spin$^c$-structure $s$ on $X$ with $c = c_1(s)$ for which the monopole equations (1) and (2) admit a solution $(A, \Phi)$ for every choice of metric $g$.

Of course, on manifolds for which the Seiberg–Witten invariants are well-defined, every basic class is a monopole class. The rationale for considering the concept of a monopole class is that the existence of solutions to the monopole equations has immediate consequences, even when the corresponding invariants vanish. For example, Kronheimer [25] showed that the adjunction inequality $2g(\Sigma) - 2 \geq c \cdot \Sigma$ holds for any monopole class $c$ and any smoothly embedded surface $\Sigma \subset X$ of positive genus $g(\Sigma)$ which has trivial normal bundle.

Another immediate consequence, going back to Witten [37], is that a non-torsion monopole class $c$ rules out the existence of a metric of positive scalar curvature as soon as $b_2^+(X) > 0$, i. e. the intersection form of $X$ is not negative definite. This is seen by combining the Weitzenböck formula for the Dirac operator with the curvature equation. A short calculation then shows that either $\Phi \equiv 0$, or one has the estimate $|\hat{\Phi}(p)| \leq -s_g(p)$ at points $p \in X$ where $|\Phi|$ has a local maximum. Therefore, if the scalar curvature $s_g$ is non-negative, every solution of the monopole equations is reducible. If $c^2 > 0$, this is a contradiction. If $c^2 \leq 0$, one has to perturb the metric,
using that positivity of the scalar curvature is an open condition in the space of metrics. The following lemma then concludes the argument. A proof can be found in [9].

**Lemma 2** (Donaldson). Fix a non-torsion class $c \in H^2(X, \mathbb{Z})$. If $b^+_2(X) > 0$, then for a generic metric $g$, there is no anti-self-dual harmonic form representing the image of $c$ in $H^2(X, \mathbb{R})$.

Note that the above argument rules out metrics of zero scalar curvature if the monopole class $c$ satisfies $c^2 > 0$, but not otherwise, because scalar-flat metrics do not form an open set in the space of metrics.

There are two fundamental properties of basic classes, the use of which has already become second nature to workers in the field, and which it would be desirable to have at hand for monopole classes. One is the non-negativity of the formal dimension of the associated moduli space of solutions to the monopole equations, the other is finiteness of the set of basic, respectively monopole classes. Therefore, we make the following definition, modifying Kronheimer’s [25], to ensure these extra properties.

**Definition 3.** A special monopole class $c$ is a monopole class for which $c^2 \geq 2e(X) + 3\sigma(X)$.

Here $e$ and $\sigma$ stand for the Euler characteristic and the signature. The inequality $c^2 \geq 2e(X) + 3\sigma(X)$ is equivalent to the non-negativity of the formal or expected dimension

$$d = \frac{1}{4} \left( c^2 - (2e(X) + 3\sigma(X)) \right)$$

of the solution space up to gauge equivalence. As an immediate consequence of the definition we have finiteness of special monopole classes:

**Lemma 4** (Witten [37]). On any smooth closed oriented four-manifold $X$ there are at most finitely many special monopole classes.

**Proof.** Suppose $c$ is a special monopole class, and $(A, \Phi)$ is any solution for the corresponding Spin$^c$-structure $s$ and some Riemannian metric $g$. The estimate on solutions we discussed before implies

$$\frac{1}{8} \| s_g \|_{L^2}^2 \leq \frac{1}{8} \| s_g \|_{L^2}^2 .$$

The Chern–Weil formula

$$\| F^+_A \|_{L^2}^2 = \| F^-_A \|_{L^2}^2 - 4\pi c^2$$

shows that an $L^2$ bound on $F^+_A$, such as (3), together with the assumption that $c^2$ is bounded below, implies an $L^2$ bound for $F^-_A$.
Thus the projections of $F_\hat{A}$ to the harmonic subspaces $\mathcal{H}_+^2$ and $\mathcal{H}_-^2$ are both bounded, so that the image of $c$ in $H^2(X, \mathbb{R}) = \mathcal{H}_+^2 \oplus \mathcal{H}_-^2$ is contained in a bounded set. Thus there are only finitely many possibilities for $c$. \qed

Note that the proof works with any lower bound on $c^2$, equivalently any lower bound on the formal dimension $d$.

Remark 5. Ishida–LeBrun [21] (p. 232) wrongly assert that Witten proved finiteness for all monopole classes, not necessarily special ones. Witten’s argument, see [37] (p. 782/783), is exactly the one in the proof of Lemma 4 above, with the additional assumption that $d = 0$, cf. (3.6) at the top of page 783, because he was working with the basic classes for Seiberg–Witten invariants coming from zero-dimensional moduli spaces.

Here is an example with an infinite number of monopole classes, showing that not all monopole classes are special.

Remark 6. On $X = \mathbb{C}P^2$, every odd multiple of the generator of $H^2(X, \mathbb{Z})$ is a monopole class. For metrics of positive scalar curvature on $X$, like the Fubini–Study metric, all solutions are reducible.

In fact, this remark applies to any manifold with negative definite intersection form. Using the solution spaces of the monopole equations, one can prove Donaldson’s theorem concluding that these intersection forms are diagonal over the integers, as soon as one has special monopole classes on these manifolds. That special monopole classes would have to exist for every negative definite four-manifold with non-standard intersection form is the content of Elkies’s theorem [10].

One should keep in mind that on any spin four-manifold $c = 0$ is a monopole class, because for every metric one has the trivial reducible solution for a $\text{Spin}^c$-structure induced by a spin structure. Considering connected sums of $S^2 \times S^2$ and $S^3 \times S^1$, one sees that the zero class does not obstruct the existence of positive scalar curvature metrics, and that the expected dimension of the corresponding moduli space can be positive, zero, or negative. Monopole classes which are torsion play a different role from non-torsion classes.

As soon as $b_+^3 (X) > 0$, a non-torsion monopole class guarantees the existence of only irreducible solutions for generic metrics because of Lemma 4. However, it is not known whether these are cut out transversely by the monopole equations. This is the issue of the missing “generic metrics theorem”, which would be the analog of the Freed–Uhlenbeck theorem [14] for the anti–self–dual Yang–Mills equations. If such a result were true, saying that for irreducible solutions of (1) and (2) for a generic metric $g$ and a non-torsion $c = c_1(s)$ the linearized equation has no cokernel, then it would
follow that on manifolds with $b_2^+(X) > 0$ all non-torsion monopole classes are special monopole classes.

Another useful variation of the definition is the following:

**Definition 7.** A *generic monopole class* $c$ is a monopole class for which there are solutions of the equations

\[ D_A^+ \Phi = 0, \]

\[ F_A^+ = \sigma(\Phi, \Phi) + \omega^+ \]

for all metrics $g$ and all two-forms $\omega$.

**Lemma 8.** If $b_2^+(X) > 0$, then generic monopole classes are special.

**Proof.** The usual transversality proof \[26, 29\] shows that for a generic pair $(g, \omega)$, all the irreducible solutions of the perturbed equations (4) and (5) are cut out transversely. Thus $d \geq 0$ as soon as there are irreducible solutions for generic parameters. In view of $b_2^+(X) > 0$, irreducible solutions exist for any non-torsion monopole class by Lemma 2. But even if $c$ is a torsion class, the assumption $b_2^+(X) > 0$ allows us to choose generic parameters $(g, \omega)$ for which there are no reducible solutions. □

If one does not have an invariant constructed from the moduli space which is unchanged under the perturbation by $\omega$, it seems quite hard to determine whether a given special monopole class is a generic one. Thus the monopole classes detected by three-dimensional topology that Kronheimer \[25\] discussed are all special, but whether they are generic is not clear to me. Recall that in many cases, the zero class is a monopole class, but is not a generic monopole class, because the trivial solution disappears when we perturb the curvature equation with a non-zero $\omega$. However, if a torsion class is a generic monopole class, so that not all solutions disappear under perturbation of the equations with a small non-zero $\omega$, then this monopole class does obstruct the existence of positive scalar curvature metrics if $b_2^+ > 0$, cf. Lemma 2. Thus, rephrasing Witten’s discussion in \[37\] in the language of monopole classes, we have:

**Proposition 9** (Witten \[37\]). *Let $X$ be a closed oriented four-manifold with $b_2^+(X) > 0$. If $X$ admits a monopole class which is either non-torsion or generic, then $X$ does not support any metric of positive scalar curvature.*

Obviously basic classes detected by numerical Seiberg–Witten invariants are generic monopole classes. A larger class of monopole classes, many of which are not basic, is detected by the stable cohomotopy invariant of Bauer–Furuta \[5\]. These are generic, and therefore special, and often have
$c^2 > 2e(X) + 3\sigma(X)$, equivalently the formal dimensions $d$ of the moduli spaces are strictly positive. The following theorem summarizes and paraphrases some consequences of the connected sum formula for the stable cohomotopy invariant.

**Theorem 10** (Bauer [4]). Let $X_i$ be smooth closed oriented four-manifolds with $b_1 = 0$ and $b_2^+ \equiv 3 \pmod{4}$. Assume that $c_i$ is a basic class on $X_i$ giving rise to a zero-dimensional moduli space with odd numerical Seiberg–Witten invariant. Then $c_1+c_2$, respectively $c_1+c_2+c_3$, is a generic monopole class on $X_1\#X_2$, respectively on $X_1\#X_2\#X_3$.

Similarly, $c_1+\ldots+c_4$ is a generic monopole class on $X = X_1\#\ldots\#X_4$ if $b_2^+(X) \equiv 4 \pmod{8}$.

The formal dimensions of the relevant moduli spaces are 1, 2 and 3 respectively.

3. THE EXOTIC SMOOTH STRUCTURES OF EINSTEIN MANIFOLDS

Recall that on simply connected four-manifolds the only classical obstruction to the existence of Einstein metrics is the Hitchin–Thorpe inequality

$$e(X) \geq \frac{3}{2} |\sigma(X)|.$$  

Hitchin [20] showed that Einstein manifolds for which (6) is an equality are either flat or are isometric quotients of a $K3$ surface with a Calabi–Yau metric. Thus, the existence of smooth manifolds homeomorphic but not diffeomorphic to the $K3$ surface provides examples of homeomorphic manifolds such that one admits an Einstein metric and the other one does not. I gave the first such examples satisfying the strict Hitchin–Thorpe inequality in [23]. We shall paraphrase the argument to exclude Einstein metrics in certain cases in the next section. Suffice it to say here that it uses Witten’s estimate (3) which implies that for any monopole class $c$ on $X$ and any metric $g$ the projection $c^+$ of $c$ to the self-dual summand in $H^2(X, \mathbb{R}) = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$ satisfies

$$(c^+)^2 \leq \frac{1}{32\pi^2} ||s_g||^2_{L^2}.$$  

Sometimes the following can be used in the same way to obtain stronger results.

**Theorem 11** (LeBrun [28]). Let $c$ be a monopole class on $X$. Then for any metric $g$ the projection $c^+$ of $c$ to the self-dual summand in $H^2(X, \mathbb{R}) = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$ satisfies

$$\frac{2}{3} (c^+)^2 \leq \frac{1}{4\pi^2} \left( \frac{1}{24} ||s_g||^2_{L^2} + 2 ||W_+||^2_{L^2} \right).$$
In the case of equality the metric is almost Kähler.

As an immediate consequence, we have:

**Theorem 12.** Let \( c \) be a monopole class on \( X \), and \( d \) the expected dimension of the corresponding moduli space. If \( X \) admits an Einstein metric, then \( 2e(X) + 3\sigma(X) \geq 8d \). In the case of equality \( X \) admits a symplectic structure.

**Proof.** If \( g \) is an Einstein metric on \( X \), the Gauss–Bonnet/Chern–Weil formulae for the Euler characteristic and the signature combine to give:

\[
2e(X) + 3\sigma(X) = \frac{1}{4\pi^2} \left( \frac{1}{24}||s_g||_{L^2}^2 + 2||W^+||_{L^2}^2 \right).
\]

Combining this with (7) and \((c^+)^2 \geq c^2 = 4d + 2e(X) + 3\sigma(X)\) gives the result.

The statement about the case of equality follows from the corresponding statement in Theorem 11.

Note that whenever \( d \leq 0 \), this is weaker than the Hitchin–Thorpe inequality (6). In particular, the result is empty for monopole classes which are not special.

Theorem 12 clarifies the arguments of Ishida–LeBrun [21]. It could have appeared in their paper, but did not, because they mixed up the consequences of the existence of a monopole class with the device used to find such classes. One of their statements is the following:

**Corollary 13 (Ishida–LeBrun [21]).** Let \( X, Y, \) and \( Z \) be simply connected symplectic 4-manifolds with \( b_2^+ \equiv 3 \pmod{4} \). If \( c_2^1(X) + c_2^1(Y) \leq 12 \), then \( X \# Y \) does not admit Einstein metrics. Similarly, if \( c_2^2(X) + c_2^1(Y) + c_2^1(Z) \leq 24 \), then \( X \# Y \# Z \) does not admit Einstein metrics.

**Proof.** By Taubes’s result [34][22], the Chern classes \( c_1(X) \) and \( c_1(Y) \) are basic classes with numerical Seiberg–Witten invariant = ±1 on \( X \) and \( Y \) respectively. Applying Theorem 10 above, we see that \( c_1(X) + c_1(Y) \) is a monopole class on \( X \# Y \), for which the expected dimension of the monopole moduli space is \( d = 1 \). Therefore, if \( X \# Y \) admits an Einstein metric, Theorem 12 implies \( c_1(X) + c_1(Y) \geq 12 \), with equality only if \( X \# Y \) is symplectic, which is not possible by [34][22].

The proof of the second statement is entirely similar, but now the expected dimension \( d = 2 \).

While this Corollary applies to spin manifolds, it is sometimes empty for non-obvious reasons.
Remark 14. In the spin case the first part of Corollary 13 is a consequence of the Hitchin–Thorpe inequality (6), and is thus homotopy-invariant. A spin symplectic 4-manifold with \( b^+_2 \equiv 3 \pmod{4} \) must have \( c^2_1 \equiv 0 \pmod{16} \) by Rochlin’s theorem. On the other hand, it is minimal, so that \( c^2_1 \geq 0 \) by the result of Taubes [35, 22]. Therefore, the assumption \( c^2_1(X) + c^2_1(Y) \leq 12 \) implies that \( c^2_1 \) vanishes for both \( X \) and \( Y \), and then their connected sum violates the Hitchin–Thorpe inequality (6).

The second part of Corollary 13 can be used for spin manifolds in exactly one way: one of the summands has \( c^2_1 = 16 \), and the other two have \( c^2_1 = 0 \).

Example 15. Let \( X \) be a symplectic spin manifold with \( c^2_1(X) = 16 \) and \( \chi(X) = 4 \), where \( \chi = \frac{1}{4}(e + \sigma) \) denotes the holomorphic Euler characteristic. Such manifolds exist by the results of Park and Szabó [31]. By Freedman’s classification [15], such an \( X \) is homeomorphic to \( K3#4(S^2 \times S^2) \). Take \( Y = E(2n) \) a spin elliptic surface, and \( Z \) the symplectic spin manifold obtained from \( E(2m) \) by performing a logarithmic transformation of odd multiplicity \( p \). Then Corollary 13 applies to show that the connected sum \( X \# Y \# Z \) does not support an Einstein metric. Note that \( X \# Y \# Z \) is homeomorphic to \( (n + m + 1)K3#(n + m + 2)(S^2 \times S^2) \). As we increase \( p \), the multiplicity of the logarithmic transformation, we find that there are more and more basic classes on \( Z \) whose numerical Seiberg–Witten invariants are \( \pm 1 \), see Fintushel–Stern [13], Theorem 8.7. By Theorem 10 these basic classes give rise to special monopole classes on \( X \# Y \# Z \), so we have a sequence of smooth structures for which the number of special monopole classes is unbounded. Thus Lemma 4 shows that we have infinitely many smooth structures.

Varying \( n \) and \( m \), we see that \( kK3#(k+1)(S^2 \times S^2) \) has infinitely many smooth structures not supporting Einstein metrics for every \( k \geq 3 \).

For \( k \geq 4 \) such examples were previously given by Ishida–LeBrun [21] using building blocks due to Gompf. Whenever \( k \geq 4 \), we have no way of finding a smooth structure on \( kK3#(k+1)(S^2 \times S^2) \) supporting an Einstein metric because this manifold violates the Noether inequality (or the Miyaoka–Yau inequality, if one reverses the orientation). The case \( k = 3 \) allows us to prove the following:

**Theorem 16.** There is a simply connected topological spin manifold \( M \) which satisfies the strict Hitchin–Thorpe inequality \( e(M) > \frac{3}{2} |\sigma(M)| \) and

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\(^3\)The argument of Ishida–LeBrun [21] for distinguishing infinitely many smooth structures is not correct as written, because the “bandwidth” is ill-defined taking the maximum over an infinite set, see Remark 5 above. The argument can be corrected by considering special monopole classes only.
which admits a smooth structure supporting an Einstein metric and also admits infinitely many smooth structures not supporting Einstein metrics.

Proof. We consider the manifold $M = 3K3\#4(S^2 \times S^2)$. We showed in the above example that $M$ admits infinitely many smooth structures without Einstein metrics. On the other hand, the algebraic surface $S$ obtained as the double cover of the projective plane branched in a smooth holomorphic curve of degree 10 has $c_1^2(S) = 8$ and $\chi(S) = 7$, and is spin because its canonical bundle is the pullback of $O(2)$. Thus $S$ is homeomorphic to $M$ by [15]. As it has ample canonical bundle, it admits an Einstein metric by the results of Aubin [3] and Yau [38]. □

Remark 17. The manifold $M$ has another infinite sequence of smooth structures, which are distinct from the ones discussed above. Fintushel–Stern [11] have shown that one can perform cusp surgery on a torus in $S$ to construct infinitely many distinct smooth structures which are irreducible and non-complex, and are therefore distinct from the smooth structures we consider. Whether they admit Einstein metrics is not known.

Example 18. Continuing Example [15] we can prove that the following topological spin manifolds admit infinitely many smooth structures not supporting Einstein metrics: $kK3\#k(S^2 \times S^2)$ for all odd $k \geq 5$, and $kK3\#(k + 4)(S^2 \times S^2)$ for all even $k \geq 4$.

We take the connected sum of one or two copies of the Park–Szabó manifold $X$ used in Example [15] and form the connected sum with three respectively two simply connected spin elliptic surfaces. Using the last part of Theorem [10] and Theorem [12] with $d = 3$ shows that such a connected sum has no Einstein metric if it has $b_2^{+} \equiv 4 \pmod{8}$. This covers all the cases mentioned above. As before, we can obtain infinitely many distinct smooth structures detected by monopole classes by performing logarithmic transformations on the elliptic surfaces.

These manifolds do not admit almost complex structures, so there is certainly no chance to find Kähler–Einstein metrics for some other smooth structure.

Theorem [12] or Corollary [13] can be applied to non-spin manifolds to produce some examples not admitting Einstein metrics and which have small homology.

Example 19. For every $q \geq 26$, the manifold $6\mathbb{C}P^2\#q\mathbb{C}P^2$ has smooth structures not admitting Einstein metrics. To see this we can use the symplectic manifolds with $b_2^{+} = 3$ constructed by D. Park [30]. For example, he constructs a minimal symplectic 4-manifold $X$ homeomorphic to $3\mathbb{C}P^2\#13\mathbb{C}P^2$. This has $c_1^2(X) = 6$. Applying the first part of Corollary [13] to this $X$ and blowups $Y$ of $X$ gives the result.
Note that these manifolds have rather smaller homology than similar examples in [21, 7].

More interesting than the above are examples which can be shown to be homeomorphic to Einstein manifolds. The smallest example where the second part of Corollary 13 applies is the following.

**Example 20.** The manifold $9\mathbb{C}P^2 \# 44\mathbb{C}P^2$ has a smooth structure with an Einstein metric, and infinitely many without. The smooth structure admitting an Einstein metric is that underlying a smooth quintic in $\mathbb{C}P^3$. To obtain smooth structures without Einstein metrics we can form the connected sum of two copies of D. Park’s examples [30], with $b_2 = 12$ and 13, and one copy of a $K3$ surface. Then performing logarithmic transformations on the $K3$ summand gives infinitely many distinct symplectic structures not admitting Einstein metrics as in Example 15 above.

This example is smaller than the ones in [23, 28, 21]. It is interesting to note that one does not need any of the sophisticated ingredients we have used. It is easy to see that there are infinitely many distinct smooth structures on $9\mathbb{C}P^2 \# 34\mathbb{C}P^2$ which support symplectic forms and are distinguished by the numerical Seiberg–Witten invariants, see for example [32]. The 10-fold blowups of those smooth structures are still distinct, and do not admit Einstein metrics by [27, 23]. Thus, neither Theorem 11 nor any input from the stable cohomotopy refinement of the Seiberg–Witten invariants is needed for Example 20.

4. **Towards Irreducibility of Einstein Manifolds**

Except for the connected sums $\mathbb{C}P^2 \# k\mathbb{C}P^2$ with $k \in \{1, 3, \ldots, 8\}$, all known Einstein four-manifolds are essentially irreducible. In fact, until recently the only other examples were either Kähler–Einstein, or of constant sectional curvature. Then, after the first version of this paper had been completed, Anderson [1] constructed some new examples of Einstein four-manifolds. The manifolds in his examples also admit Riemannian metrics of non-positive sectional curvature. Thus, by the Cartan–Hadamard theorem, the following Lemma applies to them, as it does to the space forms of non-positive sectional curvature. The only orientable space form of positive curvature is $S^4$.

**Lemma 21.** Let $M$ be any manifold with universal covering homeomorphic to $\mathbb{R}^4$. Then $M$ is irreducible.

**Proof.** Suppose $M = M_1 \# M_2$. The fundamental group of $M$ has only one end, and is therefore indecomposable as a free product. Thus one of the $M_i$, say $M_1$, is simply connected. Therefore $\pi_1(M_2) = \pi_1(M)$, and
so $M_2$ carries the homology of $\pi_1(M)$. It follows that $M_1$ is a homotopy sphere.

Concerning the Kähler–Einstein case, I proved in [22] that if a minimal closed symplectic four-manifold $X$ with $b_2^+(X) > 1$ decomposes as a connected sum, then one of the summands is an integral homology sphere whose fundamental group has no non-trivial finite quotients. In particular, if $\pi_1(X)$ is residually finite, then $X$ is irreducible. This applies to Kähler–Einstein surfaces of non-positive scalar curvature, because such manifolds are always minimal. Moreover, Gromov [17] proved that the fundamental group of a compact Kähler manifold never splits as a non-trivial free product. Thus, if one were to split off a non-trivial homology sphere from a Kähler surface, then the other summand would have to be simply connected.

As further, admittedly rather weak, evidence for the possible irreducibility of Einstein manifolds we have the following result.

**Theorem 22.** Let $X$ be a smooth four-manifold with a monopole class $c$. If $X$ admits an Einstein metric, then the maximal number $k$ of copies of $\mathbb{C}P^2$ that can be split off smoothly is bounded by

$$k \leq \frac{1}{2} (2e(X) + 3\sigma(X) - 8d).$$

**Proof.** Suppose that $X \cong Y \# k\mathbb{C}P^2$, and write $c = c_Y + \sum_{i=1}^k a_i e_i$, with respect to the obvious direct sum decomposition of $H^2(X, \mathbb{Z})$. Here $e_i$ are the generators for the cohomology of the $\mathbb{C}P^2$ summands. Note that the $a_i$ are odd integers because $c$ must be characteristic. Now the reflections in the $e_i$ are realised by self-diffeomorphisms of $X$, and the images of our monopole class under these diffeomorphisms are again monopole classes. Thus, moving $c$ by a diffeomorphism, we can arbitrarily change $e_i$ to its negative.

Given an Einstein metric $g$ on $X$, we choose the signs in such a way that $a_i e_i^+ \cdot c_Y^+ \geq 0$. Then we find

$$(c^+)^2 = \left(c_Y^+ + \sum_{i=1}^k a_i e_i^+ight)^2 \geq (c_Y^+)^2$$

$$\geq c_Y^2 = c^2 + \sum_{i=1}^k a_i^2 = 4d + 2e(X) + 3\sigma(X) + \sum_{i=1}^k a_i^2.$$

On the other hand, applying (7) to $c$ and $g$ gives

$$(c^+)^2 \leq \frac{3}{2} (2e(X) + 3\sigma(X)).$$
Combining the two inequalities and noting that $a_i^2 \geq 1$ because all the $a_i$ are odd integers proves the result.

Parts of this argument are reminiscent of the proof of irreducibility of minimal symplectic manifolds with residually finite fundamental groups in [22], which however uses some much deeper ingredients [35]. Note that one obtains a stronger inequality whenever not all the $a_i$ are $\pm 1$.

A precursor of the argument in the proof of Theorem [22] was first applied by LeBrun [27] in the case where $X$ is the blowup of a Kähler or symplectic manifold $Y$, and $c$ is the basic class of the symplectic structure. I then observed in [23] that the argument works for blowups of arbitrary manifolds $Y$ with a basic class $c_Y$, because the blowup formula [12, 24] shows that $c_Y + \sum_{i=1}^{k} \pm e_i$ is a basic class on $X$. The formulation above is such that we circumvent the absence of a blowup formula for monopole classes.

APPENDIX: RICCI-FLAT FOUR-MANIFOLDS

by D. Kotschick and J. Wehrheim

A manifold of dimension at least three that admits metrics of positive scalar curvature also admits metrics of vanishing scalar curvature. However, there are also manifolds which admit scalar-flat metrics although they do not have any of positive scalar curvature. Such manifolds are called strongly scalar-flat. The simplest examples are tori and other flat manifolds.

Bourguignon proved that scalar-flat metrics on strongly scalar-flat manifolds are in fact Ricci-flat, and therefore Einstein, compare [6]. In dimension three this implies that they are flat. In higher dimensions not much is known about strongly scalar-flat manifolds, although Futaki [16] and Des-sai [8] have proved some partial classification results in dimensions $> 4$.

In dimension 4, the only known Ricci-flat manifolds are strongly scalar-flat. They are flat manifolds and finite quotients of $K3$ surfaces with Calabi–Yau metrics. The isometric quotients of $K3$ surfaces were classified by Hitchin [20] in his discussion of the borderline case of the Hitchin–Thorpe inequality [6]. He showed that the possible covering groups are $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, both of which do actually occur. After earlier, unpublished, work of Calabi, Charlap–Sah and Levine, the closed orientable flat four-manifolds were classified by Hillman [19] and by Wagner [36], who showed that there are 27 distinct ones. By Bieberbach, all these manifolds are finite quotients of $T^4$.

To investigate Ricci-flat four-manifolds, we first show that if there is a special monopole class on such a manifold, then it is one of the standard examples. The argument is a generalization of Witten’s vanishing theorem for non-torsion monopole classes in the case of positive scalar curvature.
Proposition 23. Let $X$ be a closed oriented 4-manifold with a special monopole class $c$. If $X$ admits a Ricci-flat metric $g$, then $(X, g)$ is isometric to a finite quotient of $T^4$ or $K3$ with a standard metric, and $c$ is a torsion class.

Proof. As every solution of the monopole equations (1) and (2) on $(X, g)$ must be reducible, we conclude $c^2 \leq 0$.

On the other hand, $(X, g)$ is Einstein. Thus $X$ satisfies the Hitchin–Thorpe inequality $2e(X) + 3\sigma(X) \geq 0$, and the assumption that $c$ is special gives $c^2 \geq 2e(X) + 3\sigma(X) \geq 0$.

Combining the two inequalities, we conclude $c^2 = 0$. As the curvature form $F_A$ is anti-self-dual for every reducible solution $(A, \Phi)$ of the monopole equations, we conclude that $F_A$ vanishes and $c$ is a torsion class.

Moreover, we have $2e(X) + 3\sigma(X) = 0$, so we are in the borderline case of the Hitchin–Thorpe inequality (6), and Hitchin’s classification [20] shows that every Einstein metric on $X$ is either flat or an isometric quotient of a Calabi–Yau metric on $K3$. □

This implies the following restrictions on any other Ricci-flat four-manifolds.

Theorem 24. If a closed orientable four-manifold $X$ admits a Ricci-flat metric and is not a finite quotient of $T^4$ or $K3$, then:

1. it satisfies the strict Hitchin–Thorpe inequality $2e(X) > 3|\sigma(X)|$,
2. it has finite fundamental group,
3. $X$ and all its coverings have no special monopole classes for either choice of orientation, and
4. if the universal covering $\tilde{X}$ is spin, then $\sigma(X) = 0$.

Proof. We assume that $X$ admits an Einstein metric. As we have excluded the manifolds occurring in the borderline case of the Hitchin–Thorpe inequality (6), the inequality must be strict for $X$. In particular, the Euler characteristic of $X$ is positive. This, together with the splitting theorem of Cheeger–Gromoll for Ricci-flat manifolds, implies that $X$ has finite fundamental group.

By Proposition 23, the assumptions imply that, for either orientation, $X$ has no special monopole classes. The argument also applies to any finite covering.

Finally, suppose that $\tilde{X}$ is spin with non-zero signature. Then, for any metric, there must be a not identically zero harmonic spinor of pure chirality. For a scalar-flat metric, the Weitzenböck formula shows that such a spinor $\Phi$ is parallel. Now choose the orientation such that $\Phi$ has positive chirality. Then $\sigma(\Phi, \Phi)$ is a non-zero parallel self-dual two-form, i.e. a Kähler form. The Kähler structure gives rise to a basic class $c_1(\tilde{X})$. 
Now \( b_1(\tilde{X}) = 0 \), and \( |\sigma(\tilde{X})| \geq 16 \) by Rochlin’s theorem, so that

\[
b_+^2(\tilde{X}) = \frac{1}{2}(e(\tilde{X}) + \sigma(\tilde{X})) - 1 > \frac{1}{4}|\sigma(\tilde{X})| - 1 \geq 3
\]

using the strict Hitchin–Thorpe inequality for \( \tilde{X} \). Thus the basic class \( c_1(\tilde{X}) \) is a special monopole class, contradicting what we proved above.

The second conclusion of this theorem shows that a non-standard Ricci-flat four-manifold is not prevented from having positive scalar curvature by either enlargeability in the sense of Gromov–Lawson \([18]\), or by the minimal hypersurface method of Schoen–Yau \([33]\), because those only apply to manifolds with infinite fundamental groups. The last part shows that the Lichnerowicz obstruction vanishes as well. Note that the refined index-theoretic obstructions to positive scalar curvature due to Rosenberg reduce to the Lichnerowicz obstruction the case at hand, because the fundamental group is finite. As there are no special monopole classes for either orientation and all covering spaces, the only known obstruction to positive scalar curvature that could apply is the existence of a non-special monopole class. Here such a monopole class \( c \) would actually have to satisfy \( c^2 < 0 \), which is stronger than \( c^2 < 2e(X) + 3\sigma(X) \) because \( X \) satisfies the strict Hitchin–Thorpe inequality.

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Mathematisches Institut, Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 München, Germany

E-mail address: dieter@member.ams.org
E-mail address: Jan.Wehrheim@mathematik.uni-muenchen.de