Supersymmetric Chern-Simons Models in Harmonic Superspaces

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Abstract

We review harmonic superspaces of the $D=3, N=3$ and $4$ supersymmetries and gauge models in these superspaces. Superspaces of the $D=3, N=5$ supersymmetry use harmonic coordinates of the $SO(5)$ group. The superfield $N=5$ actions describe the off-shell infinite-dimensional Chern-Simons supermultiplet.

1 Introduction

Supersymmetric extensions of the $D=3$ Chern-Simons theory were discussed in [1]-[10]. A superfield action of the $D=3, N=1$ Chern-Simons theory can be interpreted as the superspace integral of the differential Chern-Simons superform $dA + \frac{2}{3} A^3$ in the framework of our theory of superfield integral forms [3]-[6].

The Abelian $N=2$ CS action was first constructed in the $D=3, N=1$ superspace [1]. The corresponding non-Abelian action was considered in the $D=3, N=2$ superspace with the help of the Hermitian superfield $V(x^m, \theta^\alpha, \bar{\theta}^\dot{\alpha})$, where $\theta^\alpha$ and $\bar{\theta}^\dot{\alpha}$ are the complex conjugated spinor coordinates [3]. The unusual dualized form of the $N=2$ CS Lagrangian contains the second vector field instead of the scalar field [7].

The $D=3, N=3$ CS theory was first analyzed by the harmonic-superspace method [8, 9]. Supersymmetric action of the $D=3, N=4$ Yang-Mills theory can also be constructed in the $D=3, N=3$ superspace, but the alternative formalism exists in the $N=4$ superspace [15].

The authors of [18] propose using the $SO(5)/U(2)$ harmonic superspace for the superfield description of the $D=3, N=5$ Chern-Simons theory. The detailed analysis of the superfield formalism of the Chern-Simons theory in this harmonic superspace was presented in our recent paper [19]. The alternative formalism of this theory using the $SO(5)/U(1) \times U(1)$ harmonics and additional harmonic conditions was considered in [20]. It was shown that the action of this model is invariant with respect to the $D=3, N=6$ superconformal group. The superfield action without harmonic constraints describes additional matter fields [21].

2 $N = 4$ and $N = 3$ harmonic superspaces

We consider the following coordinates of the $D = 3, N = 4$ superspace:

$$z = (x^m, \theta^\alpha_\ell),$$

(2.1)
where \( i \) and \( \hat{k} \) are two-component indices of the automorphism groups \( SU_L(2) \) and \( SU_R(2) \), respectively, \( \alpha \) is the two-component index of the \( SL(2, R) \) group and \( m = 0, 1, 2 \) is the 3D vector index. The \( N = 4 \) supersymmetry transformations are

\[
\delta x^m = -i(\gamma^m)_{\alpha\beta}(\epsilon^\alpha_{jk}\theta^{\beta j\hat{k}} - \epsilon^\beta_{jk}\theta^{\alpha j\hat{k}}),
\]

where \( \gamma^m \) are the 3D \( \gamma \) matrices.

The \( SU_L(2)/U(1) \) harmonics \( u_i^\pm \) [11] can be used to construct the left analytic superspace [15] with the \( LA \) coordinates

\[
\zeta_L = (x_L^m, \theta^{+\hat{k}\alpha}).
\]

The \( L \)-analytic prepotential \( V^{++}(\zeta_L, u) \) describes the left \( N = 4 \) vector multiplet \( A_m, \phi_{ki}, \chi^\alpha_{ik}, D^i_k \).

The \( D = 3, N = 4 \) SYM action can be constructed in terms of this prepotential by analogy with the \( D = 4, N = 2 \) SYM action [14].

Let us introduce the new notation for the left harmonics \( u_i^\pm = u_i^{(\pm,0)} \) and the analogous notation \( v_k^{(0,\pm)} \) for the right \( SL_R(2)/U(1) \) harmonics. The biharmonic \( N = 4 \) superspace uses the Grassmann coordinates [15]

\[
\theta^{(\pm,\pm)\alpha} = u_i^{(\pm,0)}v_k^{(0,\pm)}\theta^{\hat{k}\alpha}.
\]

In this representation, we have

\[
\zeta_L = (x_L^m, \theta^{(1,\pm)\alpha}), \quad V^{++} \equiv V^{(2,0)},
\]

\[
D^{(1,\pm)1}(2.5)_{\alpha}V^{(2,0)} = 0, \quad D^{(0,2)}v^{(2,0)} = 0.
\]

The right analytic \( N = 4 \) coordinates are

\[
\zeta_R = (x_R^m, \theta^{(\pm,1)\alpha}), \quad x_R^m = x_L^m - 2i(\gamma^m)_{\alpha\beta}\theta^{(-1,1)\alpha}\theta^{(1,1)\beta}.
\]

The mirror \( R \) analytic prepotential \( \hat{V}^{(0,2)} \)

\[
D^{(1,\pm)\hat{V}}(0,2) = 0, \quad D^{(2,0)}u\hat{V}^{(0,2)} = 0
\]

describes the right \( N = 4 \) vector multiplet \( \hat{A}_m, \hat{\phi}_{ij}, \hat{\chi}_{ik}^\alpha, \hat{D}_{ik} \), where \( \hat{A}_m \) is the mirror vector gauge field using the independent gauge group. The right \( N = 4 \) SYM action is similar to the analogous left action. These multiplets can be formally connected by the map \( SU_L(2) \leftrightarrow SU_R(2) \).

The \( N = 4 \) superfield Chern-Simons type (or \( BF \)-type) action for the gauge group \( U(1) \times U(1) \) connects two mirror vector multiplets

\[
\int du^2 d\zeta_L d\theta^{(-4,0)}V^{(2,0)}(\zeta_L, u)D^{(1,1)\alpha}D^{(1,1)\alpha}\hat{V}^{(0,2)},
\]

where the right connection satisfies the equation

\[
D^{(0,2)}\hat{V}^{(0,-2)} = D^{(0,2)}\hat{V}^{(0,2)}, \quad D^{(2,0)}\hat{V}^{(0,-2)} = 0.
\]

The component form of this action was considered in [16, 17].
We can identify the left and right isospinor indices in the $N = 4$ spinor coordinates
\[ \theta^\alpha_j \rightarrow \theta^\alpha_{jk} = \tilde{\theta}^\alpha_{j(k)} + \frac{1}{2}\delta_{jk}\theta^\alpha, \] (2.11)
where the Grassmann coordinates $\theta^\alpha_{jk}$ describe the $N = 3$ superspace. The harmonic superspace of the $D = 3, N = 3$ supersymmetry uses the standard harmonics $u_i^\pm [8, 9]$
\[ x_3^{\alpha\beta} = x^\alpha + i(\theta^{++}\theta^{--} + \theta^{+-} + \theta^{--}) , \]
(2.12)
\[ \theta^{++} = u^+_i u^+_k \theta^{(ik)\alpha}, \quad \theta^{--} = u^-_i u^-_k \theta^{(ik)\alpha}, \quad \theta^{\alpha\alpha} = u^+_i u^-_k \theta^{(ik)\alpha}, \]
(2.13)
\[ \delta x_3^{\alpha\beta} = -2i\epsilon^{-\alpha} \theta^{+-} - 2i\epsilon^{-\beta} \theta^{++} + 2i\epsilon^{0\alpha} \theta^{0\beta}. \]

The vector $N = 3$ supermultiplet is described by the analytic superfield $V^{++}(x_3, \theta^{++}, \theta^0, u)$ and the corresponding analytic $N = 3$ superfield strength is
\[ W^{++}(x_3, \theta^{++}, \theta^0, u) = -\frac{1}{2}D^{++\alpha}D^{++\alpha}V^{--}. \] (2.14)
The action of the corresponding $CS$-theory can be constructed in the full or analytic $N = 3$ superspaces [8, 9].

3 Harmonic superspaces for the group $SO(5)$

The homogeneous space $SO(5)/U(2)$ is parametrized by elements of the harmonic 5×5 matrix
\[ U_a^K = (U_a^{+1}, u_a^0, U^-_a) = (U_a^{+1}, U_a^{+2}, u_a^-), \] (3.1)
where $a = 1, \ldots, 5$ is the vector index of the group $SO(5)$, $i = 1, 2$ is the spinor index of the group $SU(2)$, and $U(1)$-charges are denoted by symbols $+, -, 0$. The basic relations for these harmonics are
\[ U^{+i}_a U^{+k}_a = u^0_a, \quad U^-_a U^-_a = u^0_a, \quad U^{+i}_a U^-_a = \delta^i_k, \quad U^0_a U^0_a = 1, \]
\[ U^{+i}_a U^-_b + U^-_a U^{+i}_b + U^0_a U^0_b = \delta_{ab}. \] (3.2)

We consider the $SO(5)$ invariant harmonic derivatives with nonzero $U(1)$ charges
\[ \partial^{+i} = U^{-i}_a \frac{\partial}{\partial U^{+a}} - U^0_a \frac{\partial}{\partial U^-_a}, \quad \partial^{+i} U^0_a = U^{+i}_a, \quad \partial^{+i} U^-_a = -\delta^i_k U^0_a, \]
\[ \partial^{++} = U^{-i}_a \frac{\partial}{\partial U^{+a}}, \quad [\partial^{+i}, \partial^{+k}] = \varepsilon^{ki} \partial^{++}, \quad \partial^{+i} \partial^{+i} = \partial^{++}, \]
\[ \partial^{-i} = U^{-i}_a \frac{\partial}{\partial U^0_a} - U^0_a \frac{\partial}{\partial U^{+a}} , \quad \partial^{-i} U^0_a = U^{-i}_a, \quad \partial^{-i} U^{+k}_a = -\delta^i_k U^0_a, \]
\[ \partial^{--} = U^{-i}_a \frac{\partial}{\partial U^{+a}}, \quad [\partial^{-i}, \partial^{-i}] = \varepsilon_{ki} \partial^{--}, \quad \partial^{-k} \partial{-k} = -\partial^{--}, \] (3.3)
where some relations between these harmonic derivatives are defined. The $U(1)$ neutral harmonic derivatives form the Lie algebra $U(2)$
\[ \partial^i_k = U^{+i}_a \frac{\partial}{\partial U^{+a}} - U^-_a \frac{\partial}{\partial U^{-a}}, \quad [\partial^{+i}, \partial^{-i}] = -\delta^i_k, \]
\[ \partial^0 = \partial^i_k U^{+i}_a \frac{\partial}{\partial U^{+a}} - U^-_a \frac{\partial}{\partial U^{-a}}, \quad [\partial^{++}, \partial^{--}] = \partial^0, \]
\[ \partial^0 U^{+i}_a \partial^i_k = \delta^i_k U^{+i}_a, \quad \partial^0 U^{-i}_a = -\delta^i_k U^-_a. \] (3.5)
The operators \( \partial^+ \), \( \partial^{++} \), \( \partial^- \), \( \partial^{--} \) and \( \partial^i \) satisfy the commutation relations of the Lie algebra \( SO(5) \).

One defines an ordinary complex conjugation on these harmonics

\[
\overline{U_a^{++}} = U_{ia}, \quad \overline{U_a^0} = U_a^0,
\]

however, it is convenient to use a special conjugation in the harmonic space

\[
(U_a^{+})^\sim = U_a^{+}, \quad (U_{ia}^-)^\sim = U_{ia}^-, \quad (U_a^0)^\sim = U_a^0.
\]

All harmonics are real with respect to this conjugation.

The full superspace of the \( D=3, N=5 \) supersymmetry has the spinor \( CB \) coordinates \( \theta_a^\alpha \), \( a = 1, 2, 3, 4, 5 \) in addition to the coordinates \( x^m \) of the three-dimensional Minkowski space. The group \( SL(2, R) \times SO(5) \) acts on the spinor coordinates.

The superconformal transformations of these coordinates are considered in Appendix.

The \( SO(5)/U(2) \) harmonics allow us to construct projections of the spinor coordinates and the partial spinor derivatives

\[
\theta^{+i\alpha} = U_a^{+i} \theta_a^\alpha, \quad \theta^{0\alpha} = U_a^0 \theta_a^\alpha, \quad \theta^{-i\alpha} = U_{ia}^- \theta_a^\alpha,
\]

\[
\partial^{+i\alpha} = \frac{\partial}{\partial \theta^{+i\alpha}}, \quad \partial^{0\alpha} = \frac{\partial}{\partial \theta^{0\alpha}}, \quad \partial^{-i\alpha} = \frac{\partial}{\partial \theta^{-i\alpha}}.
\]

The analytic coordinates (\( AB \)-representation) in the full harmonic superspace use these projections of 10 spinor coordinates \( \theta^{+i\alpha}, \theta^{0\alpha}, \theta^{-i\alpha} \) and the following representation of the vector coordinate:

\[
x_A^m \equiv y^m = x^m + i(\theta^k \gamma^m \theta_k) = x^m + i(\theta_a \gamma^m \theta_b) U_a^{+k} U_{kb}.
\]

The analytic coordinates are real with respect to the special conjugation.

The harmonic derivatives have the following form in \( AB \):

\[
D^{+k} = \partial^{+k} - i(\theta^k \gamma^m \theta_k) \partial_m + \theta^{+k\alpha} \partial_m^\alpha - \theta^{0\alpha} \partial^{+k}_\alpha,
\]

\[
D^{++} = \partial^{++} + i(\theta^k \gamma^m \theta_k^+) \partial_m + \theta^{+k\alpha} \partial^{+k}_\alpha,
\]

\[
D^k_l = \partial^k_l + \theta^{+k\alpha} \partial_{l\alpha} - \theta^{-l\alpha} \partial^k_{\alpha}.
\]

We use the commutation relations

\[
[D^{+k}, D^{+l}] = -\varepsilon^{kl} D^{++}, \quad D^{+k} D^{+}_k = D^{++}.
\]

The \( AB \) spinor derivatives are

\[
D^{+i}_\alpha = \partial^{+i}_\alpha, \quad D^{-i}_{ia} = -\partial^{-i}_{ia} - 2i \theta^{-i}_{l\beta} \partial_{\alpha\beta},
\]

\[
D^0_\alpha = \partial^0_\alpha + i \theta^{0\beta} \partial_{\alpha\beta}.
\]

The coordinates of the analytic superspace \( \zeta = (y^m, \theta^{+i\alpha}, \theta^{0\alpha}, U_a^K) \) have the Grassmann dimension 6 and dimension of the even space 3+6. The functions \( \Phi(\zeta) \) satisfy the Grassmann analyticity condition in this superspace

\[
D^{+k}_\alpha \Phi = 0.
\]
In addition to this condition, the analytic superfields in the $SO(5)/U(2)$ harmonic superspace possess also the $U(2)$-covariance. This subsidiary condition looks especially simple for the $U(2)$-scalar superfields

$$\mathcal{D}_{\lambda}^{\alpha} \Lambda(\zeta) = 0. \quad (3.14)$$

The integration measure in the analytic superspace $d\mu^{(-4)}$ has dimension zero

$$d\mu^{(-4)} = dU d^{3}x_{\alpha}(\partial_{a}^{\alpha})^{2}(\partial_{\alpha a})^{4} = dU d^{3}x d\theta^{(-4)}. \quad (3.15)$$

The $SO(5)/U(1) \times U(1)$ harmonics can be defined via the components of the real orthogonal $5 \times 5$ matrix [20, 21]

$$U_{a}^{K} = (U_{a}^{(1,1)}, U_{a}^{(1,-1)}, U_{a}^{(0,0)}, U_{a}^{(-1,1)}, U_{a}^{(-1,-1)}) \quad (3.16)$$

where $a$ is the $SO(5)$ vector index and the index $K = 1, 2, \ldots 5$ corresponds to given combinations of the $U(1) \times U(1)$ charges.

We use the following harmonic derivatives

$$\begin{align*}
\partial_{(2,0)} &= U_{b}^{(1,1)} \partial / \partial U_{b}^{(1,1)} - U_{b}^{(1,-1)} \partial / \partial U_{b}^{(1,-1)}, \\
\partial_{(1,1)} &= U_{b}^{(1,1)} \partial / \partial U_{b}^{(0,0)} - U_{b}^{(0,0)} \partial / \partial U_{b}^{(1,1)}, \\
\partial_{(1,-1)} &= U_{b}^{(1,1)} \partial / \partial U_{b}^{(1,-1)} - U_{b}^{(1,-1)} \partial / \partial U_{b}^{(1,-1)}, \\
\partial_{(0,2)} &= U_{b}^{(1,1)} \partial / \partial U_{b}^{(1,1)} - U_{b}^{(1,1)} \partial / \partial U_{b}^{(1,1)}, \\
\partial_{(0,-2)} &= U_{b}^{(1,1)} \partial / \partial U_{b}^{(1,1)} - U_{b}^{(1,1)} \partial / \partial U_{b}^{(1,1)}. \quad (3.17)
\end{align*}$$

We define the harmonic projections of the $N=5$ Grassmann coordinates

$$\theta^{K}_{\alpha} = \theta_{a\alpha} U_{a}^{K} = (\theta^{(1,1)}_{\alpha}, \theta^{(1,-1)}_{\alpha}, \theta^{(0,0)}_{\alpha}, \theta^{(-1,1)}_{\alpha}, \theta^{(-1,-1)}_{\alpha}). \quad (3.18)$$

The $SO(5)/U(1) \times U(1)$ analytic superspace contains only spinor coordinates

$$\begin{align*}
\zeta &= (x_{A}^{m}, \theta^{(1,1)}_{\alpha}, \theta^{(1,-1)}_{\alpha}, \theta^{(0,0)}_{\alpha}, \theta^{(-1,1)}_{\alpha}, \theta^{(-1,-1)}_{\alpha}), \\
x_{A}^{m} &= x_{A}^{m} + i\theta^{(1,1)}_{\alpha} \gamma_{m} \theta^{(1,-1)}_{\alpha} + i\theta^{(0,0)}_{\alpha} \gamma_{m} \theta^{(-1,1)}_{\alpha}, \\
\delta_{\alpha} x_{A}^{m} &= -i \epsilon^{(0,0)}_{\beta} \gamma_{m} \theta^{(1,1)}_{\alpha} - 2i \epsilon^{(1,1)}_{\beta} \gamma_{m} \theta^{(1,-1)}_{\alpha} - 2i \epsilon^{(-1,1)}_{\beta} \gamma_{m} \theta^{(-1,1)}_{\alpha}, \quad (3.19)
\end{align*}$$

where $\epsilon^{K\alpha} = \epsilon_{\alpha}^{a} U_{a}^{K}$ are the harmonic projections of the supersymmetry parameters.

General superfields in the analytic coordinates depend also on additional spinor coordinates $\theta^{(1,1)}_{\alpha}$ and $\theta^{(-1,1)}_{\alpha}$. The harmonized partial spinor derivatives are

$$\begin{align*}
\partial_{(1,1)}^{(-1,1)} &= \partial / \partial \theta^{(1,1)}_{\alpha}, \\
\partial_{(1,1)}^{(-1,1)} &= \partial / \partial \theta^{(1,-1)}_{\alpha}, \\
\partial_{(0,0)}^{(0,0)} &= \partial / \partial \theta^{(0,0)}_{\alpha}, \\
\partial_{(1,1)}^{(-1,1)} &= \partial / \partial \theta^{(1,1)}_{\alpha}, \\
\partial_{(1,1)}^{(-1,1)} &= \partial / \partial \theta^{(1,1)}_{\alpha}. \quad (3.21)
\end{align*}$$

We use the special conjugation $\sim$ in the harmonic superspace

$$\begin{align*}
\bar{U}_{a}^{(p,q)} &= U_{a}^{(p,-q)}, \\
\bar{\theta}_{a}^{(p,q)} &= \theta_{a}^{(p,-q)}, \\
\bar{x}_{A}^{m} &= x_{A}^{m}, \\
(\theta^{(p,q)}_{\alpha}\theta^{(s,r)}_{\beta})^{\sim} &= \theta_{\beta}^{(s,-r)}\theta^{(p,-q)}_{\alpha}, \\
f(\bar{x}_{A}) &= \bar{f}(x_{A}). \quad (3.22)
\end{align*}$$

where $\bar{f}$ is the ordinary complex conjugation. The analytic superspace is real with respect to the special conjugation.
The analytic-superspace integral measure contains partial spinor derivatives (3.21)
\[ d\mu^{(-4,0)} = -\frac{1}{64}dUd^3x_A(\partial^{(-1,-1)})^2(\partial^{(0,0)})^2 = dUd^3x_Ad\theta^{(-4,0)}, \quad (3.23) \]
\[ \int d\theta^{(-4,0)}(\theta^{(1,1)})^2(\theta^{(1,-1)})^2(\theta^{(0,0)})^2 = 1. \]

The harmonic derivatives of the analytic basis commute with the generators of the N=5 supersymmetry
\[ \mathcal{D}^{(1,1)} = \partial^{(1,1)} - i\theta^{(1,1)}\beta\partial^{\alpha\beta} - \theta^{(0,0)}\alpha\partial^{(1,1)} + \theta^{(1,1)}\alpha\partial^{(0,0)}, \]
\[ \mathcal{D}^{(1,-1)} = \partial^{(1,-1)} - i\theta^{(1,-1)}\beta\partial^{\alpha\beta} - \theta^{(0,0)}\alpha\partial^{(1,-1)} + \theta^{(1,-1)}\alpha\partial^{(0,0)} = -(\mathcal{D}^{(1,1)})^\dagger, \]
\[ \mathcal{D}^{(2,0)} = [\mathcal{D}^{(1,-1)}, \mathcal{D}^{(1,1)}] = \partial^{(2,0)} - 2i\theta^{(1,1)}\beta\partial^{\alpha\beta} - \theta^{(1,1)}\alpha\partial^{(1,1)} + \theta^{(1,-1)}\alpha\partial^{(1,-1)}, \]
\[ \mathcal{D}^{(0,2)} = \partial^{(0,2)} + \theta^{(1,1)}\alpha\partial^{(-1,1)} - \theta^{(-1,1)}\alpha\partial^{(1,1)} \]
\[ \mathcal{D}^{(0,-2)} = -(\mathcal{D}^{(0,2)})^\dagger = \partial^{(-2,0)} + \theta^{(1,-1)}\alpha\partial^{(-1,1)} - \theta^{(-1,-1)}\alpha\partial^{(1,-1)}. \]

It is useful to define the AB-representation of the U(1) charge operators
\[ \mathcal{D}^0_1 A^{(p,q)} = p A^{(p,q)}, \quad \mathcal{D}^0_2 A^{(p,q)} = q A^{(p,q)}, \quad (3.24) \]

where \( A^{(p,q)} \) is an arbitrary harmonic superfield in AB.

The spinor derivatives in the analytic basis are
\[ D^{(-1,-1)}_\alpha = \partial^{(-1,-1)} + 2i\theta^{(-1,-1)}\beta\partial^{\alpha\beta}, \quad D^{(-1,1)}_\alpha = \partial^{(-1,1)} + 2i\theta^{(-1,1)}\beta\partial^{\alpha\beta}, \]
\[ D^{(0,0)}_\alpha = \partial^{(0,0)} + i\theta^{(0,0)}\beta\partial^{\alpha\beta}, \quad D^{(1,1)}_\alpha = \partial^{(1,1)}, \quad D^{(1,-1)}_\alpha = \partial^{(1,-1)}. \quad (3.25) \]

4 \( N = 6 \) Chern-Simons theory in harmonic superspaces

The harmonic derivatives \( \mathcal{D}^{+k}, \mathcal{D}^{++} \) together with the spinor derivatives \( D^{+k}_\alpha \) determine the CR-structure of the harmonic \( SO(5)/U(2) \) superspace. The \( U(2) \)-covariant CR-structure is invariant with respect to the \( N=5 \) supersymmetry. This CR-structure should be preserved in the superfield gauge theory.

The gauge superfields (prepotentials) \( V^{+k}(\zeta) \) and \( V^{++}(\zeta) \) in the harmonic \( SO(5)/U(2) \) superspace satisfy the following conditions of the Grassmann analyticity and \( U(2) \)-covariance:
\[ D^{+k}_\alpha V^{+l} = D^{+k}_\alpha V^{++} = 0, \quad \mathcal{D}^i_j V^{+k} = \delta^i_j V^{+i}, \quad \mathcal{D}^i_j V^{++} = \delta^i_j V^{++}. \quad (4.1) \]

In the gauge group \( SU(n) \), these traceless matrix superfields are anti-Hermitian
\[ (V^{+k})^\dagger = -V^{+k}, \quad (V^{++})^\dagger = -V^{++}, \quad (4.2) \]

where operation \( \dagger \) includes the transposition and \( \sim \)-conjugation.

Analytic superfield parameters of the gauge group \( SU(n) \) satisfy the conditions of the generalized CR analyticity
\[ D^{+k}_\alpha \Lambda = D^i_j \Lambda = 0, \quad (4.3) \]

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they are traceless and anti-Hermitian \( \Lambda^1 = -\Lambda \).

We treat these prepotentials as connections in the covariant gauge derivatives

\[
\nabla^{+i} = D^{+i} + V^{+i}, \quad \nabla^{++} = D^{++} + V^{++},
\]

\[
\delta_A V^{+i} = D^{+i}A + [\Lambda, V^{+i}], \quad \delta_A V^{++} = D^{++}A + [\Lambda, V^{++}],
\]

\[
D^{+k}_\alpha \delta_A V^{+k} = D^{+k}_\alpha \delta_A V^{++} = 0, \quad D^{+k}_j \delta_A V^{+k} = \delta^k_j \delta_A V^{+i}, \quad D^{+k}_j \delta_A V^{++} = \delta^k_j \delta_A V^{++},
\]

where the infinitesimal gauge transformations of the gauge superfields are defined. These covariant derivatives commute with the spinor derivatives \( D^{+i} \) and preserve the \( CR \)-structure in the harmonic superspace.

We can construct three analytic superfield strengths off the mass shell

\[
F^{++} = \frac{i}{2} \varepsilon_{ki}[\nabla^{+i}, \nabla^{+k}] = V^{++} - D^{+k} V^{+k} + V^{+k} V_k^+, \\
F^{(+3)k} = [\nabla^{+i}, \nabla^{+k}] = D^{++} V^{+k} - D^{+k} V^{++} + [V^{++}, V^{+k}].
\]

The superfield action in the analytic \( SO(5)/U(2) \) superspace is defined on three prepotentials \( V^{+k} \) and \( V^{++} \) by analogy with the off-shell action of the \( SYM_3^5 \) theory \[12\]

\[
S_1 = \frac{ik}{12\pi} \int d\mu^{(-4)} \text{Tr} \{ V^{+j} D^{++} V_j^+ + 2 V^{++} D^{+j} V^{+j} + (V^{++})^2 + V^{++}[V^+_j, V^{+j}] \},
\]

where \( k \) is the coupling constant, and a choice of the numerical multiplier guarantees the correct normalization of the vector-field action. This action is invariant with respect to the infinitesimal gauge transformations of the prepotentials (4.4). The idea of construction of the superfield action in the harmonic \( SO(5)/U(2) \) was proposed in \[18\], although the detailed construction of the superfield Chern-Simons theory was not discussed in this work. The equivalent superfield action was considered in the framework of the alternative superfield formalism \[20\]. The superconformal \( N = 5 \) invariance of this action was proven in \[19\].

The action \( S_1 \) yields superfield equations of motion which mean triviality of the superfield strengths of the theory

\[
F^{(+3)k} = D^{++} V^+_k - D^{+k} V^{++} + [V^{++}, V^+_k] = 0, \\
F^{++} = V^{++} - D^{+k} V^{+k} + V^{+k} V_k^+ = 0.
\]

These classical superfield equations have pure gauge solutions for the prepotentials only

\[
V^{+k} = e^{-\Lambda} D^{+k} e^\Lambda, \quad V^{++} = e^{-\Lambda} D^{++} e^\Lambda,
\]

where \( \Lambda \) is an arbitrary analytic superfield.

The transformation of the sixth supersymmetry can be defined on the analytic \( N=5 \) superfields

\[
\delta_6 V^{++} = \epsilon^0_6 D_\alpha^0 V^{++}, \quad \delta_6 V^{+k} = \epsilon^0_6 D_\alpha^0 V^{+k},
\]

\[
\delta_6 D^{+k} V^{+l} = \epsilon^0_6 D_\alpha^0 D^{+k} V^{+l}, \quad \delta_6 D^{++} V^{+l} = \epsilon^0_6 D_\alpha^0 D^{++} V^{+l},
\]

where \( \epsilon^0_6 \) are the corresponding odd parameters. This transformation preserves the Grassmann analyticity and \( U(2) \)-covariance

\[
\{ D_\alpha^0, D^{+k}_\beta \} = 0, \quad [D_\alpha^0, D^{+k}_\beta] = 0, \quad [D^{+k}_\alpha, D_\beta^0] = D^{+k}_\alpha, \quad [D^{++}_\alpha, D_\beta^0] = 0.
\]
The action $S_1$ is invariant with respect to this sixth supersymmetry

$$\delta_6 S_1 = \int d\mu (-4) \epsilon^\alpha_6 D^0_\alpha L^{(+4)} = 0. \quad (4.11)$$

In the $SO(5)/U(1) \times U(1)$ harmonic superspace, we can introduce the $D=3, N=5$ analytic matrix gauge prepotentials corresponding to the five harmonic derivatives

$$V^{(p,q)}(\zeta, U) = [V^{(1,1)}, V^{(1,-1)}, V^{(2,0)}, V^{(0,\pm 2)}],$$

$$V^{(1,1)} = -V^{(1,-1)}, \quad V^{(2,0)} = V^{(2,0)}, \quad V^{(0,-2)} = [V^{(0,2)}]^\dagger, \quad (4.12)$$

where the Hermitian conjugation $\dagger$ includes $\sim$ conjugation of matrix elements and transposition.

We shall consider the restricted gauge supergroup using the supersymmetry-preserving harmonic $(H)$ analyticity constraints on the gauge superfield parameter $H_1$:

$$H_1: \quad D^{(0,\pm 2)} \Lambda = 0. \quad (4.13)$$

These constraints yield additional reality conditions for the component gauge parameters.

We use the harmonic-analyticity constraints on the gauge prepotentials

$$H_2: \quad V^{(0,\pm 2)} = 0, \quad D^{(0,-2)} V^{(1,1)} = V^{(1,-1)}, \quad D^{(0,2)} V^{(1,1)} = 0 \quad (4.14)$$

and the conjugated constraints combined with relations (4.12).

The superfield CS action can be constructed in terms of these $H$-constrained gauge superfields [21]

$$S = -\frac{2ik}{12\pi} \int d\mu (-4,0) \text{Tr} \left\{ V^{2,0} D^{(1,-1)} V^{(1,1)} + V^{1,1} D^{(2,0)} V^{(1,-1)} + V^{1,-1} D^{(1,1)} V^{(2,0)} + V^{2,0} [V^{(1,-1)}, V^{(1,1)}] - \frac{1}{2} V^{(2,0)} V^{(2,0)} \right\}. \quad (4.15)$$

Note, that the similar harmonic superspace based on the $USp(4)/U(1) \times U(1)$ harmonics was used in [22] for the harmonic interpretation of the $D = 4, N = 4$ super Yang-Mills constraints.

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References

[1] W. Siegel, Nucl. Phys. B 156 (1979) 135.
[2] J. Schonfeld, Nucl. Phys. B 185 (1981) 157.
[3] B.M. Zupnik, D.G. Pak, Teor. Mat. Fiz. 77 (1988) 97; Eng. transl.: Theor. Math. Phys. 77 (1988) 1070.
[4] B.M. Zupnik, D.G. Pak, Class. Quant. Grav. 6 (1989) 723.
[5] B.M. Zupnik, Teor. Mat. Fiz. 89 (1991) 253, Eng. transl.: Theor. Math. Phys. 89 (1991) 1191.
[6] B.M. Zupnik, Phys. Lett. B 254 (1991) 127.

[7] H. Nishino, S.J. Gates, Int. J. Mod. Phys. 8 (1993) 3371.

[8] B.M. Zupnik, D.V. Khetselius, Yad. Fiz. 47 (1988) 1147; Eng. transl.: Sov. J. Nucl. Phys. 47 (1988) 730.

[9] B.M. Zupnik, Springer Lect. Notes in Phys. 524 (1998) 116; hep-th/9804167.

[10] J.H. Schwarz, Jour. High. Ener. Phys. 0411 (2004) 078, hep-th/0411077.

[11] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky, E. Sokatchev, Class. Quant. Grav. 1 (1984) 469.

[12] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky, E. Sokatchev, Class. Quant. Grav. 2 (1985) 155.

[13] A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Harmonic superspace, Cambridge University Press, Cambridge, 2001.

[14] B.M. Zupnik, Phys. Lett. B 183 (1987) 175.

[15] B.M. Zupnik, Nucl. Phys. B 554 (1999) 365, Erratum: Nucl. Phys. B 644 (2002) 405E; hep-th/9902038.

[16] R. Brooks, S.J. Gates, Nucl. Phys. B 432 (1994) 205, hep-th/09407147.

[17] A. Kapustin, M. Strassler, JHEP 04 (1999) 021, hep-th/9902033.

[18] P.S. Howe, M.I. Leeming, Clas. Quant. Grav. 11 (1994) 2843, hep-th/9402038.

[19] B.M. Zupnik, Chern-Simons theory in $SO(5)/U(2)$ harmonic superspace, arXiv 0802.0801 (hep-th).

[20] B.M. Zupnik, Phys. Lett. B 660 (2008) 254, arXiv 0711.4680 (hep-th).

[21] B.M. Zupnik, Gauge model in D=3, N=5 harmonic superspace, arXiv 0708.3951 (hep-th).

[22] I.L. Buchbinder, O. Lechtenfeld, I.B. Samsonov, N=4 superparticle and super Yang-Mills theory in $USp(4)$ harmonic superspace, arxiv:0804.3063 (hep-th).