Physical implementation of holonomic quantum computation in decoherence-free subspaces with trapped ions

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We propose a feasible scheme to achieve holonomic quantum computation in a decoherence-free subspace (DFS) with trapped ions. By the application of appropriate bichromatic laser fields on the designated ions, we are able to construct two noncommutable single-qubit gates and one controlled-phase gate using the holonomic scenario in the encoded DFS.

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Realization of practical quantum computers presents a tremendous challenge to scientists. One of the most difficulties is caused by the random errors during gate operations, which mainly roots in decoherence and stochastic errors in the control process. To suppress the latter, the holonomic quantum computation (HQC) as well as a more general geometric quantum computation (GQC) scenario were proposed, which are respectively based on the adiabatic non-Abelian holonomy and the nonadiabatic geometric phase. In the case of HQC, to realize the wanted gate operation, the Hamiltonian is driven adiabatically to undergo a designated cycle in the controllable parameter space, in which degenerate dark states (with zero eigenvalues) may be defined. Thus for an initial state, only the adiabatic geometric phases can be accumulated in the designated operation, and the corresponding holonomic quantum gates may be achieved.

Because of its global geometric feature, it is believed that the HQC scheme is rather robust against the stochastic errors occurring in the adiabatic cyclic evolution. On the other hand, decoherence is another main obstacle for quantum computation. To keep a quantum system away from the main decoherence source, one should attempt to isolate the system from the environment that causes mainly the decoherence effect. To deal with this matter, several scenarios were put forward, including quantum error correcting codes and quantum error avoiding codes. In particular, when the system-environment interaction has certain symmetry, a promising scheme based on the decoherence-free subspaces (DFS) was proposed. Depending on the type of interaction symmetry, one sort of DFS is usually immune to certain system-bath disturbances.

Very recently, combining the HQC with a four-qubit-encoding DFS and based on a kind of model Hamiltonian, Wu et al proposed an interesting strategy to realize universal quantum computation. Such a strategy was indicated to afford the ability against the collective decoherence induced by environment and to be rather robust to certain stochastic errors during operations. Motivated by the above idea and considering that the four-qubit DFS scenario may hardly be scalable, we here propose a feasible scheme based on a simple two-qubit-encoding DFS and the adiabatic non-Abelian holonomy, which is distinctly different from the four-qubit-encoding scheme. In particular, by the application of appropriate laser fields, we derive an effective laser-ion Hamiltonian in a certain kind of trapped ion systems and illustrate how to physically implement universal holonomic quantum computation in the present two-qubit-encoding DFS.

\[ H = H_0 + H_{\text{int}} \] (1)

FIG. 1: A schematic illustration of the encoded sub-systems (logical qubits) in trapped ions. Each sub-system consists of two ions enclosed by the dashed circle and the effective interaction between any two logical qubits is denoted by the dotted line.

To construct the holonomic quantum gates using the decoherence-free subspaces scheme in a linear Paul trap, we need first to obtain the wanted effective Hamiltonian for a sub-system consisting of two ions (e.g., ions \( j_1 \) and \( j_2 \) in Fig.1) with the appropriate laser fields on. Let us assume that each ion has the four-level energy structure, as shown in Fig.2. The Hamiltonian may be written as \[ H = H_0 + H_{\text{int}} \] (1).
FIG. 2: Schematic level structure for single ions. \(|e\), |0\), and |1\) as well as |a\) denote the exited state, the lowest ground state, and the other two stable (metastable) states (with the last one as an ancillary state), respectively. |0\) and |1\) are the basic units used to encode the logical qubits of DFS. \(\Omega_\alpha (\alpha = e0, e1, ea)\) are the corresponding Rabi-frequencies with the resonant laser fields being applied on the ions.

with

\[
H_0 = \nu(a^+a + 1/2) + \sum_{i,\alpha} E_\alpha |\alpha_i\rangle \langle \alpha_i| \tag{2}
\]

and

\[
H_{int} = \sum_{i,\alpha} \Omega_\alpha |e\rangle \langle [n_i(a+a^+)-\omega_i]|e\rangle |\alpha_i| + H.c., \tag{3}
\]

where \(\alpha \in \{0, 1, a\}\) denote the inner states of an ion, \(\nu\) is the frequency of the center-of-mass vibrational mode of ions chain, \(a^+\) and \(a\) are the creation and annihilation operators of the collective motion, \(E_\alpha\) is the energy of the \(|\alpha\rangle\) state, \(\Omega_\alpha\) is the resonant Rabi frequency between the states \(\alpha\) and \(e\) of the \(i\)th ion in the laser field with the frequency \(\omega_i\). The positions of the ions \(x_i\) are replaced by ladder operators \(\eta_i(a+a^+)\) [12], where \(\eta_i\) is the Lamb-Dicke parameter denoting the ratio of the ion oscillation and the wave length of the exciting radiation. Since we tune the lasers close to the the center-of-mass vibrational mode where all ions vibrate in the same way, the coupling to the vibration is uniform for all ions, i.e., \(\eta_i = \eta\) for all \(i\).

We now choose the so-called bichromatic laser field [12] to couple the two ions in the same sub-system (see Fig.2). For the \(j_1\)th ion, we let it be illuminated by two laser pulses, which are resonant to the transition between |1\) and |e\), with frequency \(\omega_{j_1} = \omega_{c1} + (\nu - \delta_{10})\), \(\omega_{j_1}' = \omega_{c1} - (\nu - \delta_{10})\), where \(\omega_{c1}\) denotes the energy difference between the state |e\) and |1\) and \(\delta_{10}\) represents the additional detuning. In the Lamb-Dicke limit that \(\eta^2(n+1) \ll 1\), where \(n\) is the vibrational numbers, we can approximately expand the exponential gene \(e^{i\eta(a+a^+)}\) to 1 + \(i\eta(a+a^+)\). Under the rotating-wave approximation and in the interaction picture, the laser-ion interaction of the \(j_1\)th takes the form

\[
H_{j_1 int} = i\eta J_{12}^j [ae^{-i\delta_{10}t} + a^+e^{i\delta_{10}t}]|e\rangle \langle 1| + H.c.. \tag{4}
\]

At the same time, we also apply two laser pulses on the \(j_2\)th ion with \(\omega_{j_2} = \omega_{c0} + (\nu - \delta_{10})\), \(\omega_{j_2}' = \omega_{c0} - (\nu - \delta_{10})\), where \(\omega_{c0}\) denotes the energy difference between the state |e\) and |0\). Similarly we can have

\[
H_{j_2 int} = i\eta J_{20}^j [ae^{-i\delta_{10}t} + a^+e^{i\delta_{10}t}]|e\rangle \langle j_2|0\rangle + H.c.. \tag{5}
\]

We here consider only the weak-field regime \(\eta \Omega \ll \delta_{10}\), where only a negligible population is transferred to the intermediate levels \((n \pm 1)\). In this case and under the above laser fields, the transition from \(|10n\rangle\) to \(|een\rangle\) can be anticipated, and the effective Rabi-frequency may be evaluated in second-order perturbation theory

\[
\Omega_1 = \sum_m \frac{\langle eeen|H_{int}|m\rangle \langle m|H_{int}|10n\rangle}{E_{m} - E_{10n} - \omega_m} = - \frac{2\eta^2 \Omega_{j_1}^1 \Omega_{j_2}^2}{\delta_{10}}, \tag{6}
\]

where \(H_{int} = H_{j_1 int}^1 + H_{j_2 int}^2\), and the intermediate states \(|m\rangle\) are \(|e0(n \pm 1)\rangle\) (with the corresponding \(E_{m} = E_e + E_0 + (n \pm 1)\nu\) and \(\omega_m = \omega_{c1} \pm (\nu - \delta_{10})\)) and \(|e(n \pm 1)\rangle\) (with the corresponding \(E_{m} = E_e + E_0 + (n \pm 1)\nu\) and \(\omega_m = \omega_{c0} \pm (\nu - \delta_{10})\)) (see Fig.3) [13, 14].

FIG. 3: Schematic diagram of the resonant transition from \(|10n\rangle\) to \(|een\rangle\) for the two ions in a sub-system. The two ions are illuminated by two different bichromatic laser fields. Only the transitions involving the intermediate states with \((n \pm 1)\) can occur.

Using the same scenario, we can obtain the effective Rabi-frequencies of the transitions between \(|01\rangle, |aa\rangle\) and \(|ee\rangle\) respectively as

\[
\Omega_0 = \frac{-2\eta^2 \Omega_{j_1}^0 \Omega_{j_2}^2}{\delta_{01}}, \tag{7}
\]

\[
\Omega_a = \frac{-2\eta^2 \Omega_{j_2}^0}{\delta_{aa}}, \tag{7}
\]

For simplicity, we may take \(\delta_{10} = \delta_{01} = \delta_{aa} = \delta\) hereafter. Combining Eqs.(6) and (7) and in the rotating frame, the effective Hamiltonian of the sub-system (\(j_1, j_2\)) with the lasers on can approximately be written as

\[
H_{eff} = [\Omega_1|ee\rangle\langle 10| + \Omega_0|ee\rangle\langle 01| + \Omega_a|ee\rangle\langle aa| + H.c.. \tag{8}
\]
in the designated operational subspace spanned by \{01, 10, ee, aa\}. At this stage, we consider a smaller subspace

\[ C_j := \text{span}\{01, 10\}, \]

and encode a pair-bit code to construct a logic qubit in this subspace: \( |00\rangle_L = |01\rangle \) and \( |11\rangle_L = |10\rangle \). If we choose this logic qubit as computational one, such an encoding constitutes the well-known DFS scheme that is immune from the decoherence induced by the system-environment interaction in the form of \( Z \otimes B \), where \( Z = \sigma_z^j + \sigma_z^j \) and \( B \) is a random bath operator, simply because for any state \( |\psi\rangle \in C_j \), we have \( Z|\psi\rangle = 0 \). Also note that, we can define the corresponding Pauli operators \( R_x, R_y, R_z \) in the subspace \( C_j \), which can be expressed as

\[
R_x = \frac{1}{2}(\sigma_x^j \sigma_z^j + \sigma_z^j \sigma_x^j),
\]

\[
R_y = \frac{1}{2}(\sigma_y^j \sigma_x^j - \sigma_x^j \sigma_y^j),
\]

\[
R_z = \frac{1}{2}(\sigma_z^j - \sigma_z^j),
\]

where the \( \sigma_x^j, \sigma_y^j, \sigma_z^j (j = j_1, j_2) \) are the Pauli matrices of the \( jth \) qubit.

We now address in detail how to realize the holonomic computation with a DFS scenario. As is well known, to achieve a universal set of quantum gates, we need to construct two noncommuting single qubit operations and a nontrivial two-qubit gate. Here we choose the two single logic-qubit gates as \( U_1 = e^{i\phi_1 R_x}, U_2 = e^{i\phi_2 R_y} \), and the controlled-phase gate as \( U_{jk}^{\theta} = e^{i\phi_{11}[j]_{k}} \) (11), similar to those discussed in Refs. [16, 17].

Denoting also \( |E\rangle_L = |ee\rangle, |A\rangle_L = |aa\rangle \), the effective Hamiltonian of Eq.(8) may be rewritten as

\[
H_{eff} = [\Omega_1 |E\rangle_L \langle 1| + \Omega_2 |E\rangle_L \langle 0| + \Omega_3 |E\rangle_L \langle A| + H.c.. \]

To implement the operation \( U_1 \), we choose \( \Omega_0 = 0, \Omega_1 = \Omega \sin \theta \), and \( \Omega_2 = \Omega \cos \theta e^{i\varphi} \), where \( \Omega \) is the absolute magnitude of the effective Rabi-frequency for the gate operations, \( \theta \) and \( \varphi \) are the control parameters in the parameter space \( M \). In this case the dark states of the Hamiltonian(11) take the form

\[
|D_0\rangle = |00\rangle_L,
\]

\[
|D_1\rangle = \cos \theta |11\rangle_L - \sin \theta e^{-i\varphi} |A\rangle_L.
\]

We then let \( \Omega_1 \) and \( \Omega_2 \) change adiabatically from the point \( \Theta (\theta = 0) \) along a close path \( C \) in the parameter space \( M \), i.e., \( \theta \) and \( \varphi \) take a cyclic evolution. Thus for an arbitrary initial state \( |\Psi_0\rangle \) in the computational subspace \( C_j \), the state evolves as \( |\Psi_0\rangle \rightarrow U(C)|\Psi_0\rangle \)

\[
U(C) = \exp \oint_C A^c
\]

is the non-Abelian holonomy associated with the path \( C \) and \( A^c = \sum_\mu A_\mu d\lambda_\mu \) is the U(2)-valued connection expressed as

\[
A^c_{\mu} = (D_\lambda (\lambda)) \frac{\partial}{\partial \lambda_\mu} |D_\lambda (\lambda)\rangle \]

where \( \lambda_\mu (\theta \text{ or } \varphi) \) are the coordinates in the parameter space. From Eqs.(12)-(14), we can derive the corresponding connection as

\[
A_\varphi = -\frac{i}{2} \sin^2 (\theta) (1 - R_z).
\]

In this way the holonomic operation is achieved as

\[
U(C) = e^{-i\phi_1 e^{i\phi_2 R_z}}
\]

Next we deduce the holonomy \( U_2 \). We parameterize \( \Omega_0 = \Omega \sin \theta \cos \varphi, \Omega_1 = \Omega \sin \theta \sin \varphi, \Omega_2 = \Omega \cos \theta \). Then the two dark states of Hamiltonian(11) are given by

\[
|D_0\rangle = \cos \theta \cos \varphi |00\rangle_L + \cos \theta \sin \varphi |11\rangle_L - \sin \theta |A\rangle_L,
\]

\[
|D_1\rangle = \cos \varphi |11\rangle_L - \sin \varphi |00\rangle_L.
\]

Taking into account another adiabatic cyclic evolution of the parameters \( \theta \) and \( \varphi \), we have

\[
A_\varphi = -i \cos \theta R_y.
\]

Therefore, we obtain

\[
U_2 = e^{i\phi_2 R_y}
\]

where \( \phi_2 \) is the geometric phase factor determined from the integral

\[
\phi_2 = -\oint_C \cos \theta d\varphi.
\]

So far, we have illustrated the implementation of two single-qubit rotations around the \( y \) and \( z \) axes in the DFS. The combination of \( U_1 \) and \( U_2 \) allows us to perform any single qubit operation. To accomplish a set of universal gates, we need to realize one more two-logic-qubit controlled-phase gate, which is usually more crucial and important. In our scheme, to realize the two qubit gate \( U^{jk}_{\theta} \) in the DFS, we propose to couple the \( jth \) and \( kth \) (see Fig.1) using the bichromatic laser field, and set the \( j_2 \)th and \( k_2 \)th’s energy levels be decoupled. In detail, we use two resonant laser fields \( \omega_{11} = \omega_{11} + (\nu - \delta_{11}) \) and \( \omega_{11} = \omega_{11} - (\nu - \delta_{11}) \) to illuminate the ions \( j_1 \) and \( k_1 \), where \( \nu \) still denotes the phonon frequency of ions. We denote the effective resonant Rabi-frequency as \( \Omega_{11}^{jk_1} \). The other two pulses \( \omega_{aa} = \omega_{ea} + (\nu - \delta_{aa}) \), \( \omega_{aa}' = \omega_{ea} - (\nu - \delta_{aa}) \) are used to resonate the transitions.
between \ket{a} and \ket{e} in both the jth and kth ions, with the corresponding Rabi-frequency as \( \Omega_{jk} \). As a result, denoting \( |EE\rangle_L = |e0\rangle_j |e0\rangle_k \) and \( |AA\rangle_L = |a0\rangle_j |a0\rangle_k \), the effective Hamiltonian of the two sub-systems \( j \) and \( k \) with the lasers on may be written in the coded space as

\[
H_{eff}^{jk} = \Omega_{11} |EE\rangle_L \langle 11| + \Omega_{AA} |EE\rangle_L \langle AA| + H.c.,
\]

where

\[
\Omega_{11} = \Omega_{11}^{jk} = -\frac{2\eta^2 \Omega_{a1}^{jk} \Omega_{e1}^{jk}}{\delta_{11}}, \\
\Omega_{AA} = \Omega_{aa}^{jk} = -\frac{2\eta^2 \Omega_{e2}^{jk} \Omega_{a2}^{jk}}{\delta_{aa}}.
\]

From the above Eq. (21), similar to the case in the single qubit operation, we have the two zero-eigenvalue eigenstates (dark states) as \( |00\rangle_L \) and \( \cos \theta |11\rangle_L - \sin \theta e^{i\phi} |AA\rangle_L \) under the parametrization of \( |11\rangle_L = \Omega_{11} L |11\rangle \) and \( |AA\rangle_L = \Omega_{AA} L |AA\rangle \). The component \( |AA\rangle \) will accumulate a Berry phase \( \phi_3 \). While at the same time the other computational bases \( |00\rangle_L, |01\rangle_L, |10\rangle_L \) are decoupled from the Hamiltonian (21). This process corresponds to a two-qubit controlled-phase shift gate \( U_{3} = e^{i\phi_3 |11\rangle \langle 11|} \) in the DFS.

We have proposed a holonomic quantum computation scheme in the DFS with trapped ions. We have illustrated how to achieve two noncommutable single logic-qubit gates and a two-logic-qubit controlled-phase gate using the non-Abelian holonomy. Note that in our scheme, the computational bases are all encoded in the subspace immune to the \( \sigma_z \)-type of decoherence. It is believed that the adiabatic holonomies combined with the DFS will provide additional virtues against decoherence and stochastic errors in controlling operations. It seems feasible to realize our scheme in physical systems like trapped ions, though it is very challenging experimentally.

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