Quantum groups, Yang-Baxter maps and quasi-determinants

Zengo Tsuboi

Osaka City University Advanced Mathematical Institute (additional post member)

28 June 2018

Based on
- Z.T., 1708.06323 [Nucl.Phys.B 926(2018)200-238].

See also,
- V.Bazhanov, S.Sergeev, 1501.06984 [Nucl.Phys.B926(2018) 509-543],
- V.Bazhanov, S.Khoroshkin, S.Sergeev, Z.T. to appear (?)
The Yang-Baxter map [Drinfeld 1990, Veselov 2000] is a map $\chi$ defined on a direct product of two sets

$$\mathcal{R} : \chi \times \chi \mapsto \chi \times \chi$$
The Yang-Baxter map [Drinfeld 1990, Veselov 2000] is a map $\chi$ defined on a direct product of two sets

$$R : \chi \times \chi \mapsto \chi \times \chi$$

and satisfies the set-theoretical Yang-Baxter equation (on $\chi \times \chi \times \chi$)

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12}.$$
The Yang-Baxter map [Drinfeld 1990, Veselov 2000] is a map $\chi$ defined on a direct product of two sets

$$\mathcal{R} : \chi \times \chi \mapsto \chi \times \chi$$

and satisfies the set-theoretical Yang-Baxter equation (on $\chi \times \chi \times \chi$)

$$\mathcal{R}_{12} \circ \mathcal{R}_{13} \circ \mathcal{R}_{23} = \mathcal{R}_{23} \circ \mathcal{R}_{13} \circ \mathcal{R}_{12}. $$

This is related to discrete classical integrable systems. (discrete Toda equation, etc.).
**Introduction**

**Goal**

From the point of view of quantum groups, classify all the Yang-Baxter maps and construct the maps explicitly.
Goal

From the point of view of quantum groups, classify all the Yang-Baxter maps and construct the maps explicitly.

There exists quantum group $U_q(g)$ for each Lie algebra $g$. Then the maps will be classified in terms of classification of the quantum groups.
For any quantum group $U_q(\mathfrak{g})$, there exists the universal R-matrix $R \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ satisfying the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$
For any quantum group $U_q(\mathfrak{g})$, there exists the universal R-matrix $R \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ satisfying the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$ 

The quantum Yang-Baxter map is define as an adjoint action of the universal R-matrix [Bazhanov-Sergeev 2015]:

$$\mathcal{R} : U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \mapsto U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$$

$$\xi \mapsto \xi' = R\xi R^{-1}, \quad \xi \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}).$$
For any quantum group $U_q(\mathfrak{g})$, there exists the universal R-matrix $R \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ satisfying the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$ 

The quantum Yang-Baxter map is defined as an adjoint action of the universal R-matrix [Bazhanov-Sergeev 2015]:

$$R : U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \mapsto U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$$

$$\xi \mapsto \xi' = R\xi R^{-1}, \quad \xi \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}).$$

The classical Yang-Baxter map is given by the quasi-classical limit...
As an example, we consider $\mathfrak{g} = \mathfrak{sl}(n)$ or $\mathfrak{gl}(n)$.
Generators of $U_q(gl(n))$

$$E_{i,i+1}, E_{j+1,j}, E_{k,k},$$

\((i,j \in \{1, 2, \ldots, n-1\}, k \in \{1, 2, \ldots, n\})\)
$$\left[ E_{kk}, E_{ij} \right] = (\delta_{ik} - \delta_{jk})E_{ij},$$

$$\left[ E_{i,i+1}, E_{j+1,j} \right] = \delta_{ij}(q - q^{-1})(qE_{ii} - E_{i+1,i+1} - q^{-1}E_{ii} + E_{i+1,i+1}),$$

$$\left[ E_{i,i+1}, E_{j,j+1} \right] = \left[ E_{i+1,i}, E_{j+1,j} \right] = 0 \quad \text{for} \quad |i - j| \geq 2,$$
Relations of $U_q(gl(n))$

\[
[E_{kk}, E_{ij}] = (\delta_{ik} - \delta_{jk})E_{ij},
\]

\[
[E_{i,i+1}, E_{j+1,j}] = \delta_{ij}(q - q^{-1})(q^{E_{ii}}E_{i+1,i+1} - q^{-E_{ii}}E_{i+1,i+1}),
\]

\[
[E_{i,i+1}, E_{j,j+1}] = [E_{i+1,i}, E_{j+1,j}] = 0 \text{ for } |i - j| \geq 2,
\]

and, for $i \in \{1, 2, \ldots, n - 2\}$, the Serre relations

\[
E_{i,i+1}^2 E_{i+1,i+2} - (q + q^{-1})E_{i,i+1} E_{i+1,i+2} E_{i,i+1} + E_{i+1,i+2} E_{i,i+1}^2 = 0,
\]

\[\ldots\ldots\]
For $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, we define

$$E_{ij} = (q - q^{-1})^{-1}(E_{ik}E_{kj} - qE_{kj}E_{ik}), \quad E_{ji} = \ldots,$$

$$i < k < j.$$
Additional generators

For \( i, j \in \{1, 2, \ldots, n\} \) and \( i \neq j \), we define

\[
E_{ij} = (q - q^{-1})^{-1}(E_{ik}E_{kj} - qE_{kj}E_{ik}), \quad \quad E_{ji} = \ldots ,
\]

\[i < k < j.\]

\[c = \sum_{j=1}^{n} E_{jj}\] is a central element of \( \mathcal{A} \).
Additional generators

For $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, we define

\[ E_{ij} = (q - q^{-1})^{-1}(E_{ik}E_{kj} - qE_{kj}E_{ik}), \quad E_{ji} = \ldots, \]

\[ i < k < j. \]

$c = \sum_{j=1}^{n} E_{jj}$ is a central element of $A$. The algebra $A$ is isomorphic to $U_q(sl(n))$ under the condition $c = 0$. 
Additional generators

For $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, we define

$$E_{ij} = (q - q^{-1})^{-1}(E_{ik}E_{kj} - qE_{kj}E_{ik}), \quad E_{ji} = \ldots,$$

$$i < k < j.$$

$c = \sum_{j=1}^{n} E_{jj}$ is a central element of $\mathcal{A}$. The algebra $\mathcal{A}$ is isomorphic to $U_q(sl(n))$ under the condition $c = 0$.

- Borel subalgebras
  - $\mathcal{B}_+$: generated by $\{E_{ij}\}$
  - $\mathcal{B}_-$: generated by $\{E_{ji}\}$ for $i \leq j$, $i, j \in \{1, 2, \ldots, n\}$
Co-multiplication

The co-multiplication \( \Delta \) is an algebra homomorphism from the algebra \( \mathcal{A} \) to its tensor square

\[
\Delta : \quad \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A},
\]

defined by

\[
\Delta(E_{i,i+1}) = E_{i,i+1} \otimes q^{E_{ii} - E_{i+1,i+1}} + 1 \otimes E_{i,i+1},
\]

\[
\Delta(E_{i+1,i}) = E_{i+1,i} \otimes 1 + q^{E_{ii} + E_{i+1,i+1}} \otimes E_{i+1,i}, \quad i \in \{1, 2, \ldots, n-1\},
\]

\[
\Delta(E_{kk}) = E_{kk} \otimes 1 + 1 \otimes E_{kk}, \quad k \in \{1, 2, \ldots, n\}.
\]
Co-multiplication

The co-multiplication $\Delta$ is an algebra homomorphism from the algebra $\mathcal{A}$ to its tensor square

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A},$$

defined by

$$\Delta(E_{i,i+1}) = E_{i,i+1} \otimes q^{E_{ii} - E_{i+1,i+1}} + 1 \otimes E_{i,i+1},$$

$$\Delta(E_{i+1,i}) = E_{i+1,i} \otimes 1 + q^{E_{ii} + E_{i+1,i+1}} \otimes E_{i+1,i}, \quad i \in \{1, 2, \ldots, n - 1\},$$

$$\Delta(E_{kk}) = E_{kk} \otimes 1 + 1 \otimes E_{kk}, \quad k \in \{1, 2, \ldots, n\}.$$ 

We will also use the opposite co-multiplication $\Delta'$, defined by

$$\Delta' = \sigma \circ \Delta,$$

where $\sigma(a \otimes b) = b \otimes a$ for any $a, b \in \mathcal{A}$.
The algebra $\mathcal{A}$ is a quasi-triangular Hopf algebra. Then there exists an element $R \in \mathcal{A} \otimes \mathcal{A}$, which satisfies

$$\Delta'(a) R = R \Delta(a) \quad \text{for all } a \in \mathcal{A},$$

$$(\Delta \otimes 1) R = R_{13} R_{23},$$

$$(1 \otimes \Delta) R = R_{13} R_{12},$$

where $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$ and $R_{13} = (\sigma \otimes 1)R_{23}$. 

The quantum Yang-Baxter equation follows from these.
The algebra $\mathcal{A}$ is a quasi-triangular Hopf algebra. Then there exists an element $R \in \mathcal{A} \otimes \mathcal{A}$, which satisfies

$$\Delta'(a) R = R \Delta(a) \quad \text{for all } a \in \mathcal{A},$$

$$(\Delta \otimes 1) R = R_{13} R_{23},$$

$$(1 \otimes \Delta) R = R_{13} R_{12},$$

where $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$ and $R_{13} = (\sigma \otimes 1) R_{23}$. The quantum Yang-Baxter equation follows from these.

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$
Universal R-matrix

q-exponential function

\[ \exp_q(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{(k)_q!}, \]

\[ (k)_q! = (1)_q(2)_q \cdots (k)_q, \quad (k)_q = \frac{(1 - q^k)}{(1 - q)}. \]

\[ \exp_q(x)^{-1} = \exp_{q^{-1}}(-x). \]
Universal $R$-matrix

If we assume that the universal $R$-matrix has the form

$$ R = q \sum_{i=1}^{n} E_{ii} \otimes E_{ii} \overline{R}, $$

where $\overline{R} \in \mathcal{N}_+ \otimes \mathcal{N}_-$: $\mathcal{N}_+$ and $\mathcal{N}_-$ are nilpotent sub-algebras generated by $\{E_{ij}\}$ and $\{E_{ji}\}$ for $i < j$, $i, j \in \{1, 2, \ldots, n\}$ respectively.
Universal R-matrix

If we assume that the universal R-matrix has the form

$$R = q \sum_{i=1}^{n} E_{ii} \otimes E_{ii} \bar{R},$$

where $\bar{R} \in \mathcal{N}_+ \otimes \mathcal{N}_-$: $\mathcal{N}_+$ and $\mathcal{N}_-$ are nilpotent sub-algebras generated by $\{E_{ij}\}$ and $\{E_{ji}\}$ for $i < j$, $i, j \in \{1, 2, \ldots , n\}$ respectively.

The universal is uniquely defined by [Kirillov-Reshetikhin, Rosso, Levendorskii-Soibelman,....]

$$R = q \sum_{i=1}^{n} E_{ii} \otimes E_{ii} \prod_{i<j} \exp_{q^{-2}} \left( (q - q^{-1})^{-1} E_{ij} \otimes E_{ji} \right),$$

where the product is taken over the reverse lexicographical order on $(i, j)$: $(i_1, j_1) \prec (i_2, j_2)$ if $i_1 > i_2$, or $i_1 = i_2$ and $j_1 > j_2$. 
Quantum Yang-Baxter map

Let $X = \{E_{ij}, q^{E_{kk}}\}$, $i \neq j$ be the set of generators of $\mathcal{A}$ and $X^{(a)}$ be the corresponding components in $\mathcal{A} \otimes \mathcal{A}$,

$$X^{(1)} = \{x \otimes 1 | x \in X\}, \quad X^{(2)} = \{1 \otimes x | x \in X\}, \quad X = \{E_{ij}, q^{E_{kk}}\},$$
Quantum Yang-Baxter map

Let $\mathbf{X} = \{E_{ij}, q^{E_{kk}}\}, i \neq j$ be the set of generators of $\mathcal{A}$ and $\mathbf{X}^{(a)}$ be the corresponding components in $\mathcal{A} \otimes \mathcal{A}$,

$$\mathbf{X}^{(1)} = \{x \otimes 1|x \in \mathbf{X}\}, \quad \mathbf{X}^{(2)} = \{1 \otimes x|x \in \mathbf{X}\}, \quad \mathbf{X} = \{E_{ij}, q^{E_{kk}}\},$$

Quantum Yang-Baxter map

$\mathcal{R} : (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) \mapsto (\tilde{\mathbf{X}}^{(1)}, \tilde{\mathbf{X}}^{(2)})$,

$$\tilde{\mathbf{X}}^{(a)} = \mathcal{R} \mathbf{X}^{(a)} \mathcal{R}^{-1} = \{\mathcal{R}x^{(a)} \mathcal{R}^{-1}|x^{(a)} \in \mathbf{X}^{(a)}\} , \quad a = 1, 2.$$
Quantum Yang-Baxter map

Let $X \equiv \{ E_{ij}, q^{E_{kk}} \}$, $i \neq j$ be the set of generators of $A$ and $X^{(a)}$ be the corresponding components in $A \otimes A$,

$$X^{(1)} = \{ x \otimes 1 | x \in X \}, \quad X^{(2)} = \{ 1 \otimes x | x \in X \}, \quad X = \{ E_{ij}, q^{E_{kk}} \},$$

Quantum Yang-Baxter map

$$R : (X^{(1)}, X^{(2)}) \mapsto (\tilde{X}^{(1)}, \tilde{X}^{(2)}),$$

$$\tilde{X}^{(a)} = RX^{(a)}R^{-1} = \{ Rx^{(a)}R^{-1} | x^{(a)} \in X^{(a)} \}, \quad a = 1, 2.$$  

Note that any elements of $\tilde{X}^{(1)}$ commute with those of $\tilde{X}^{(2)}$. In addition, the algebra $\tilde{A}_a$ generated by the elements of the set $\tilde{X}^{(a)}$ is isomorphic to the algebra $A$. 
Quantum Yang-Baxter map

Let $\mathbf{X} = \{E_{ij}, q^{E_{kk}}\}$, $i \neq j$ be the set of generators of $\mathcal{A}$ and $\mathbf{X}^{(a)}$ be the corresponding components in $\mathcal{A} \otimes \mathcal{A}$,

$\mathbf{X}^{(1)} = \{x \otimes 1 | x \in \mathbf{X}\}$, $\mathbf{X}^{(2)} = \{1 \otimes x | x \in \mathbf{X}\}$, $\mathbf{X} = \{E_{ij}, q^{E_{kk}}\}$.

Quantum Yang-Baxter map

$\mathcal{R} : (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) \mapsto (\tilde{\mathbf{X}}^{(1)}, \tilde{\mathbf{X}}^{(2)})$,

$\tilde{\mathbf{X}}^{(a)} = \mathcal{R} \mathbf{X}^{(a)} \mathcal{R}^{-1} = \{ \mathcal{R} x^{(a)} \mathcal{R}^{-1} | x^{(a)} \in \mathbf{X}^{(a)} \}$, $a = 1, 2$.

Note that any elements of $\tilde{\mathbf{X}}^{(1)}$ commute with those of $\tilde{\mathbf{X}}^{(2)}$. In addition, the algebra $\tilde{\mathcal{A}}_a$ generated by the elements of the set $\tilde{\mathbf{X}}^{(a)}$ is isomorphic to the algebra $\mathcal{A}$.

$\Rightarrow$ The tensor product structure is preserved under the map.
Another universal R-matrix

One can prove that if $R_{12} \in \mathcal{B}_+ \otimes \mathcal{B}_-$ satisfies definition of the universal R-matrix, then

$$R_{12}^* = R_{21}^{-1} \in \mathcal{B}_- \otimes \mathcal{B}_+,$$

also satisfies the def of the universal R-matrix.
Another universal R-matrix

One can prove that if $R_{12} \in \mathcal{B}_+ \otimes \mathcal{B}_-$ satisfies definition of the universal R-matrix, then

$$R_{12}^* = R_{21}^{-1} \in \mathcal{B}_- \otimes \mathcal{B}_+,$$

also satisfies the def of the universal R-matrix.

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

$$R_{12}^*R_{13}R_{23} = R_{23}R_{13}R_{12}^*,$$

$$R_{12}^*R_{13}^*R_{23} = R_{23}R_{13}^*R_{12}^*, \ldots$$
$n$-dimensional fundamental representation $\pi$ of $A$:

$\pi(E_{kk}) = E_{kk}$, $\pi(E_{ij}) = (q - q^{-1})E_{ij}$, for $i \neq j$.

($E_{ij}$: $n \times n$ matrix unit.)
$n$-dimensional fundamental representation $\pi$ of $A$:

$\pi(E_{kk}) = E_{kk}, \quad \pi(E_{ij}) = (q - q^{-1})E_{ij}$, for $i \neq j$.

($E_{ij}$: $n \times n$ matrix unit.)

$L^- = (\pi \otimes 1) R^* \in \text{End}(\mathbb{C}^n) \otimes B_+$

$L^+ = (\pi \otimes 1) R \in \text{End}(\mathbb{C}^n) \otimes B_-$
\( n \)-dimensional fundamental representation \( \pi \) of \( \mathcal{A} \):

\[
\pi(E_{kk}) = E_{kk}, \quad \pi(E_{ij}) = (q - q^{-1})E_{ij}, \text{ for } i \neq j.
\]

\( (E_{ij} : n \times n \text{ matrix unit. } ) \)

\[
\mathbb{L}^- = (\pi \otimes 1)\mathbb{R}^* = \sum_{k=1}^{n} E_{kk} \otimes q^{-E_{kk}} - \sum_{i<j} E_{ji} \otimes E_{ij} q^{-E_{ii}},
\]

\[
\mathbb{L}^+ = (\pi \otimes 1)\mathbb{R} = \sum_{k=1}^{n} E_{kk} \otimes q^{E_{kk}} + \sum_{i<j} E_{ij} \otimes q^{E_{ii}} E_{ji},
\]
n-dimensional fundamental representation $\pi$ of $\mathcal{A}$:

$\pi(E_{kk}) = E_{kk}$, $\pi(E_{ij}) = (q - q^{-1})E_{ij}$, for $i \neq j$.

($E_{ij}$: $n \times n$ matrix unit.)

$$\mathbb{L}^- = (\pi \otimes 1) \mathbb{R}^* = \sum_{k=1}^{n} E_{kk} \otimes q^{-E_{kk}} - \sum_{i < j} E_{ji} \otimes E_{ij} q^{-E_{ii}},$$

$$\mathbb{L}^+ = (\pi \otimes 1) \mathbb{R} = \sum_{k=1}^{n} E_{kk} \otimes q^{E_{kk}} + \sum_{i < j} E_{ij} \otimes q^{E_{ii}} E_{ji},$$

$$(\mathbb{L}^-)_{kk}(\mathbb{L}^+)_{kk} = 1$$
Evaluating further the second space (quantum space) of these L-operators in the fundamental representation $\pi$, we obtain the block R-matrices

$$R^- = (1 \otimes \pi) L^-$$

$$R^+ = (1 \otimes \pi) L^+$$
R-matrices

Evaluating further the second space (quantum space) of these L-operators in the fundamental representation $\pi$, we obtain the block R-matrices

$$R^- = (1 \otimes \pi) L^- = \sum_{i,j} q^{-\delta_{ij}} E_{ii} \otimes E_{jj} - (q - q^{-1}) \sum_{i<j} E_{ji} \otimes E_{ij},$$

$$R^+ = (1 \otimes \pi) L^+ = \sum_{i,j} q^{\delta_{ij}} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i<j} E_{ij} \otimes E_{ji}.$$
Then we define the spectral parameter dependent L-operator

\[ L(\lambda) = \lambda L^+ - \lambda^{-1} L^- \]

and the R-matrix

\[ R(\lambda) = \lambda R^+ - \lambda^{-1} R^- . \]
Zero curvature representation

\[ R_{01}^* R_{02}^* R_{12} = R_{12} R_{02}^* R_{01}^*, \]

\[ R_{01}^* R_{02} R_{12} = R_{12} R_{02} R_{01}^*, \]

\[ R_{01} R_{02}^* R_{12} = R_{12} R_{02} R_{01}, \]

\[ R_{01} R_{02} R_{12} = R_{12} R_{02} R_{01}, \]
Zero curvature representation

\[ R_{01}^* R_{02}^* R_{12} = R_{12} R_{02}^* R_{01} \quad \leftarrow R_{12}^{-1}, \]

\[ R_{01}^* R_{02} R_{12} = R_{12} R_{02} R_{01}^*, \]

\[ R_{01} R_{02} R_{12} = R_{12} R_{02} R_{01}, \]

\[ R_{01} R_{02}^* R_{12} = R_{12} R_{02}^* R_{01}. \]
Zero curvature representation

\[ R_{01}^{*} R_{02}^{*} = (R_{12} R_{02}^{*} R_{12}^{-1})(R_{12} R_{01}^{*} R_{12}^{-1}), \]
\[ R_{01}^{*} R_{02} = (R_{12} R_{02} R_{12}^{-1})(R_{12} R_{01}^{*} R_{12}^{-1}) \]
\[ R_{01} R_{02} = (R_{12} R_{02} R_{12}^{-1})(R_{12} R_{01} R_{12}^{-1}). \]
Zero curvature representation

\[ R_{01}^* R_{02}^* = (R_{12} R_{02}^* R_{12}^{-1})(R_{12} R_{01}^* R_{12}^{-1}), \]

\[ R_{01}^* R_{02} = (R_{12} R_{02} R_{12}^{-1})(R_{12} R_{01}^* R_{12}^{-1}) \]

\[ R_{01} R_{02} = (R_{12} R_{02} R_{12}^{-1})(R_{12} R_{01} R_{12}^{-1}). \]

Evaluating the first space of these (labeled by 0) in the fundamental representation \( \pi \), we obtain
\[ R_{01}^* R_{02}^* = (R_{12} R_{02} R_{12}^{-1}) (R_{12} R_{01} R_{12}^{-1}), \]
\[ R_{01}^* R_{02} = (R_{12} R_{02} R_{12}^{-1}) (R_{12} R_{01} R_{12}^{-1}) \]
\[ R_{01} R_{02} = (R_{12} R_{02} R_{12}^{-1}) (R_{12} R_{01} R_{12}^{-1}). \]

Evaluating the first space of these (labeled by 0) in the fundamental representation \( \pi \), we obtain

\[ \hat{L}_1 \hat{L}_2 = \hat{L}_1 \hat{L}_2 \]
\[ \hat{L}_1 \hat{L}_2 = \hat{L}_1 \hat{L}_2 \]
\[ \hat{L}_1 \hat{L}_2 = \hat{L}_1 \hat{L}_2, \]
where \( \hat{L}_0 = R_{12} L_{01} R_{12}^{-1}, \) \( \hat{L}_0 = R_{12} L_{02} R_{12}^{-1} \) and we omit the space index 0.
The zero curvature representation gives a rational map among generators for the case $U_q(sl(2))$ [Bazhanov-Sergeev 2015].
Zero curvature representation

The zero curvature representation gives a rational map among generators for the case $U_q(sl(2))$ [Bazhanov-Sergeev 2015].

“In this paper we present detailed considerations of the above scheme on the example of the algebra $U_q(sl(2))$ leading to discrete Liouville equations, however the approach is rather general and can be applied to any quantized Lie algebra. ” [Bazhanov-Sergeev 2015]
The zero curvature representation gives a rational map among generators for the case $U_q(sl(2))$ [Bazhanov-Sergeev 2015].

“In this paper we present detailed considerations of the above scheme on the example of the algebra $U_q(sl(2))$ leading to discrete Liouville equations, however the approach is rather general and can be applied to any quantized Lie algebra.” [Bazhanov-Sergeev 2015]

However, this optimistic idea soon run into difficulty if we consider $U_q(sl(3))$ case. Namely, square roots appear in the map for $U_q(sl(n))$, $n \geq 3$. 
The zero curvature representation gives a rational map among generators for the case $U_q(sl(2))$ [Bazhanov-Sergeev 2015].

“In this paper we present detailed considerations of the above scheme on the example of the algebra $U_q(sl(2))$ leading to discrete Liouville equations, however the approach is rather general and can be applied to any quantized Lie algebra.” [Bazhanov-Sergeev 2015]

However, this optimistic idea soon run into difficulty if we consider $U_q(sl(3))$ case. Namely, square roots appear in the map for $U_q(sl(n))$, $n \geq 3$. To overcome this difficulty, we will make a change of variables by twisting the universal R-matrix.
Twisting [Drinfeld, Reshetikhin]

If \( F \in \mathcal{A} \otimes \mathcal{A} \) satisfies

\[
(\Delta \otimes 1) F = F_{13} F_{23},
\]

\[
(1 \otimes \Delta) F = F_{13} F_{12},
\]

\[
F_{12} F_{13} F_{23} = F_{23} F_{13} F_{12},
\]
Twisting universal R-matrices

Twisting [Drinfeld, Reshetikhin]

If $F \in A \otimes A$ satisfies

\[(\Delta \otimes 1) F = F_{13} F_{23},\]
\[(1 \otimes \Delta) F = F_{13} F_{12},\]
\[F_{12} F_{13} F_{23} = F_{23} F_{13} F_{12},\]

then gauge transformed universal R-matrices

\[R = F_{21} R F_{12}^{-1} q^{c \otimes c}, \quad R^* = F_{21} R^* F_{12}^{-1} q^{c \otimes c}\]

satisfy the defining relations for the universal R-matrix for the gauge transformed co-multiplication $\Delta^F(a) = F \Delta(a) F^{-1}, a \in A$. 
Twisting universal R-matrices

Twisting [Drinfeld, Reshetikhin]

If $F \in A \otimes A$ satisfies

$$(\Delta \otimes 1) F = F_{13} F_{23},$$

$$(1 \otimes \Delta) F = F_{13} F_{12},$$

$$F_{12} F_{13} F_{23} = F_{23} F_{13} F_{12},$$

then gauge transformed universal R-matrices

$$R = F_{21} R F_{12}^{-1} q^{c \otimes c}, \quad R^* = F_{21} R^* F_{12}^{-1} q^{c \otimes c}$$

satisfy the defining relations for the universal R-matrix for the gauge transformed co-multiplication $\Delta^F(a) = F \Delta(a) F^{-1}, a \in A$.

$R$ and $R^*$ satisfy the same Yang-Baxter relations as $R$ and $R^*$. 
Twisting the universal R-matrices

\[ F_{12} = q \sum_{i=1}^{n} \omega_{i-1} \otimes E_{ii}, \]

\[ \omega_i = E_{11} + \cdots + E_{ii}, \]

\[ \omega_0 = 0, \quad \omega_n = c. \]
Twisting L-operators

The gauge transformed $L$-operators are defined by evaluating the gauge transformed universal R-matrices.

$$L^- = (\pi \otimes 1)(R^*) = \sum_{k=1}^{n} E_{kk} \otimes q^{2\omega_k-1} - \sum_{i<j} E_{ji} \otimes q^{\omega_{i-1}+\omega_{j-1}} E_{ij},$$

$$L^+ = (\pi \otimes 1)(R) = \sum_{k=1}^{n} E_{kk} \otimes q^{2\omega_k} + \sum_{i<j} E_{ij} \otimes q^{\omega_i+\omega_j} E_{ji},$$

$$\omega_i = E_{11} + \cdots + E_{ii}.$$
The zero-curvature representation for the twisted L-operators has the same form as the one for the original L-operators

\[ L_1^+ L_2^+ = \tilde{L}_2^+ \tilde{L}_1^+, \quad L_1^- L_2^+ = \tilde{L}_2^+ \tilde{L}_1^-, \quad L_1^- L_2^- = \tilde{L}_2^- \tilde{L}_1^-. \]
Zero curvature representation for twisted L-operators

The zero-curvature representation for the twisted L-operators has the same form as the one for the original L-operators

\[
L_1^+ L_2^+ = \tilde{L}_2^+ \tilde{L}_1^+ , \quad L_1^- L_2^+ = \tilde{L}_2^+ \tilde{L}_1^- , \quad L_1^- L_2^- = \tilde{L}_2^- \tilde{L}_1^- .
\]

The set of generators \(X^{(a)} = \{L_{ij}^{(a)}, L_{ji}^{(a)}\}_{i \leq j}\) for the Yang-Baxter map \(R : (X^{(1)}, X^{(2)}) \rightarrow (\tilde{X}^{(1)}, \tilde{X}^{(2)})\), \((\tilde{X}^{(a)} = R_{12} X^{(a)} R_{12}^{-1})\) is different:

\[
L_{ij}^+ = u_i^2 u_j^2 E_{ji}, \quad L_{ji}^- = -u_i^2 u_{i-1}^2 u_{j-1}^2 E_{ij} \quad \text{for} \quad i < j,
\]

\[
L_{kk}^+ = u_k, \quad L_{kk}^- = u_{k-1}.
\]

\((u_k := q^{2\omega_k} = q^{2(E_{11} + \cdots + E_{kk})})\).
The zero-curvature representation for the twisted L-operators has the same form as the one for the original L-operators

\[ L_1^+ L_2^+ = \tilde{L}_2^+ \tilde{L}_1^+ , \quad L_1^- L_2^+ = \tilde{L}_2^+ \tilde{L}_1^- , \quad L_1^- L_2^- = \tilde{L}_2^- \tilde{L}_1^- . \]

The set of generators \( X^{(a)} = \{ L_{ij}^{+(a)}, L_{ji}^{-(a)} \} \) for the Yang-Baxter map \( R : (X^{(1)}, X^{(2)}) \to (\tilde{X}^{(1)}, \tilde{X}^{(2)}) \), \( \tilde{X}^{(a)} = R_{12} X^{(a)} R_{12}^{-1} \) is different:

\[ L_{ij}^+ = u_i^2 u_j^2 E_{ji} , \quad L_{ji}^- = -u_{i-1}^2 u_{j-1}^2 E_{ij} \quad \text{for} \quad i < j , \]

\[ L_{kk}^+ = u_k \quad L_{kk}^- = u_{k-1} . \]

\( u_k := q^{2\omega_k} = q^{2(E_{11} + \cdots + E_{kk})} \), \( L_{kk}^+ L_{kk}^- \neq 1 \).
Solving the zero curvature representation

\[
\begin{align*}
L_1^+ L_2^+ &= \tilde{L}_2^+ \tilde{L}_1^+, & L_1^- L_2^+ &= \tilde{L}_2^+ \tilde{L}_1^-, & L_1^- L_2^- &= \tilde{L}_2^- \tilde{L}_1^-,
\end{align*}
\]

To do: write the matrix elements of \( \tilde{L}_a^\pm = \left( \tilde{L}_{ij}^\pm(a) \right)_{i,j=1}^n \) in terms of matrix elements of \( L_1^+, L_2^+, L_1^-, L_2^- \).
Solving the zero curvature representation

\[
L_1^+ L_2^+ = \tilde{L}_2^+ \tilde{L}_1^+, \quad L_1^- L_2^+ = \tilde{L}_2^+ \tilde{L}_1^-, \quad L_1^- L_2^- = \tilde{L}_2^- \tilde{L}_1^-,
\]

To do: write the matrix elements of \( \tilde{L}_a^\pm = (\tilde{L}_{ij}^\pm(a))_{i,j=1}^n \) in terms of matrix elements of \( L_1^+, L_2^+, L_1^-, L_2^- \).

(\[
\begin{pmatrix}
1 & 0 & 0 \\
L_2^{-1} & u_1^{(1)} & 0 \\
L_3^{-1} & L_3^{-1} & u_2^{(1)}
\end{pmatrix}
\begin{pmatrix}
u_1^{(2)} \\
L_{12}^{+2} \\
L_{13}^{+2}
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{u}_1^{(2)} & \tilde{L}_{12}^{+2} & \tilde{L}_{13}^{+2} \\
0 & \tilde{u}_2^{(2)} & \tilde{L}_{23}^{+2} \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
\tilde{L}_{21}^{-1} & \tilde{u}_1^{(1)} & 0 \\
\tilde{L}_{31}^{-1} & \tilde{u}_2^{(1)} & 1
\end{pmatrix}
\]
Solving the zero curvature representation

\[ \mathbf{L}_1^+ \mathbf{L}_2^+ = \tilde{\mathbf{L}}_2^+ \tilde{\mathbf{L}}_1^+ , \quad \mathbf{L}_1^- \mathbf{L}_2^+ = \tilde{\mathbf{L}}_2^+ \tilde{\mathbf{L}}_1^- , \quad \mathbf{L}_1^- \mathbf{L}_2^- = \tilde{\mathbf{L}}_2^- \tilde{\mathbf{L}}_1^- . \]

To do: write the matrix elements of \( \tilde{\mathbf{L}}_a^\pm = (\tilde{\mathbf{L}}_{ij}^{\pm (a)})_{i,j=1}^n \) in terms of matrix elements of \( \mathbf{L}_1^+, \mathbf{L}_2^+, \mathbf{L}_1^-, \mathbf{L}_2^- \).

\[
\begin{pmatrix}
\mathcal{E}_{11} & \mathcal{E}_{12} & \mathcal{E}_{13} \\
0 & \mathcal{E}_{22} & \mathcal{E}_{23} \\
0 & 0 & \mathcal{E}_{33}
\end{pmatrix}
\begin{pmatrix}
\mathcal{F}_{11} & 0 & 0 \\
\mathcal{F}_{21} & \mathcal{F}_{22} & 0 \\
\mathcal{F}_{31} & \mathcal{F}_{32} & \mathcal{F}_{33}
\end{pmatrix}
= \\
= \\
\begin{pmatrix}
1 & 0 & 0 \\
\mathbf{L}_{21}^{-(1)} & \mathbf{u}_1^{(1)} & 0 \\
\mathbf{L}_{31}^{-(1)} & \mathbf{L}_{32}^{-(1)} & \mathbf{u}_2^{(1)}
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}_1^{(2)} & \mathbf{L}_{12}^{+(2)} & \mathbf{L}_{13}^{+(2)} \\
0 & \mathbf{u}_2^{(2)} & \mathbf{L}_{23}^{+(2)} \\
0 & 0 & 1
\end{pmatrix}
= \\
= \\
\begin{pmatrix}
\tilde{\mathbf{u}}_1^{(2)} & \tilde{\mathbf{L}}_{12}^{+(2)} & \tilde{\mathbf{L}}_{13}^{+(2)} \\
0 & \tilde{\mathbf{u}}_2^{(2)} & \tilde{\mathbf{L}}_{23}^{+(2)} \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
\tilde{\mathbf{L}}_{21}^{-(1)} & \tilde{\mathbf{u}}_1^{(1)} & 0 \\
\tilde{\mathbf{L}}_{31}^{-(1)} & \tilde{\mathbf{L}}_{32}^{-(1)} & \tilde{\mathbf{u}}_2^{(1)}
\end{pmatrix},
\]
\[ \tilde{u}_i^{(1)} = \prod_{k=1}^{i} \frac{H_k^{-1} u_k^{(1)} u_k^{(2)}}{H_k}, \]

\[ \tilde{u}_i^{(2)} = H_i \prod_{k=1}^{i-1} \frac{u_k^{(1)} u_k^{(2)}}{H_k} \quad \text{for} \quad 1 \leq i \leq n, \]

\[ \tilde{L}_{ij}^{(-1)} = \left( \prod_{k=1}^{j-1} \frac{H_k^{-1} u_k^{(1)} u_k^{(2)}}{H_k} \right) F_{ij} \quad \text{for} \quad 1 \leq j < i \leq n, \]

\[ \tilde{L}_{ij}^{(+2)} = \prod_{k=1}^{j-1} \frac{u_k^{(1)} u_k^{(2)}}{H_k} \quad \text{for} \quad 1 \leq i < j \leq n. \]
\[\tilde{u}_i^{(1)} = \prod_{k=1}^{i} H_k^{-1} u_k^{(1)} u_k^{(2)},\]

\[\tilde{u}_i^{(2)} = H_i \prod_{k=1}^{i-1} (u_k^{(1)} u_k^{(2)})^{-1} H_k \quad \text{for} \quad 1 \leq i \leq n,\]

\[\tilde{L}_{ij}^{-1} = \left( \prod_{k=1}^{i-1} H_k^{-1} u_k^{(1)} u_k^{(2)} \right) F_{ij} \quad \text{for} \quad 1 \leq j < i \leq n,\]

\[\tilde{L}_{ij}^{+2} = E_{ij} H_j \prod_{k=1}^{j-1} (u_k^{(1)} u_k^{(2)})^{-1} H_k \quad \text{for} \quad 1 \leq i < j \leq n.\]

**quasi-determinants** for the matrix \( J = L_1^{-} L_2^{+} \),

\[H_i = |J_{i,i+1,\ldots,n}|i\bar{i}, \quad E_{ij} = |J_{j,j+1,\ldots,n}|j\bar{i} |J_{j+1,\ldots,n}|^{-1},\]

\[F_{ji} = |J_{j,j+1,\ldots,n}|^{-1} |J_{i,j+1,\ldots,n}|j\bar{i}.\]
Solution

\[ L_1^+ L_2^+ = \tilde{L}_2^+ \tilde{L}_1^+ , \quad L_1^- L_2^+ = \tilde{L}_2^+ \tilde{L}_1^- , \quad L_1^- L_2^- = \tilde{L}_2^- \tilde{L}_1^- , \]

The other solutions \( \tilde{L}_1^+ , \tilde{L}_2^- \) can be obtained by substituting \( \tilde{L}_2^+ , \tilde{L}_1^- \) into the first and the third zero curvature relations.
\[ \mathbf{L}_1^+ \mathbf{L}_2^+ = \mathbf{\tilde{L}}_2^+ \mathbf{\tilde{L}}_1^+, \quad \mathbf{L}_1^- \mathbf{L}_2^+ = \mathbf{\tilde{L}}_2^+ \mathbf{\tilde{L}}_1^-, \quad \mathbf{L}_1^- \mathbf{L}_2^- = \mathbf{\tilde{L}}_2^- \mathbf{\tilde{L}}_1^- , \]

The other solutions \( \mathbf{\tilde{L}}_1^+, \mathbf{\tilde{L}}_2^- \) can be obtained by substituting \( \mathbf{\tilde{L}}_2^+, \mathbf{\tilde{L}}_1^- \) into the first and the third zero curvature relations.

\[ \mathbf{\tilde{L}}_1^+ = (\mathbf{\tilde{L}}_2^+)^{-1} \mathbf{L}_1^+ \mathbf{L}_2^+. \quad \mathbf{\tilde{L}}_2^- = \mathbf{L}_1^- \mathbf{L}_2^- (\mathbf{\tilde{L}}_1^-)^{-1}. \]
quasi-determinants [Gelfand, Retakh]

- $N \times N$ matrix whose matrix elements $a_{ij}$ are elements of an associative algebra (not necessary commutative algebra):

$$A = A_{1,2,\ldots,N}^{1,2,\ldots,N} = \begin{pmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
a_{N,1} & a_{N,2} & \cdots & a_{N,N}
\end{pmatrix},$$
quasi-determinants [Gelfand, Retakh]

- $N \times N$ matrix whose matrix elements $a_{ij}$ are elements of an associative algebra (not necessary commutative algebra):

$$A = A_{1,2,\ldots,N}^{1,2,\ldots,N} = 
\begin{pmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
a_{N,1} & a_{N,2} & \cdots & a_{N,N}
\end{pmatrix},$$

- $m \times n$ sub matrix:

$$A_{j_1,j_1,\ldots,j_n}^{i_1,i_2,\ldots,i_m} = 
\begin{pmatrix}
a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_n} \\
a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i_m,j_1} & a_{i_m,j_2} & \cdots & a_{i_m,j_n}
\end{pmatrix},$$

where $\{i_1, i_2, \ldots, i_m\}, \{j_1, j_2, \ldots, j_n\} \subset I = \{1, 2, \ldots, N\}$. 
In general, there are $N^2$ quasi-determinants for a $N \times N$ matrix $A$. 

Quasi-determinants are non-commutative analogues of ratios of determinants (rather than non-commutative analogues of determinants).
In general, there are $N^2$ quasi-determinants for a $N \times N$ matrix $A$. 

$(i,j)$-quasi-determinant of $A$ is denoted as $|A|_{ij}$.
In general, there are $N^2$ quasi-determinants for a $N \times N$ matrix $A$. The $(i,j)$-quasi-determinant of $A$ is denoted as $|A|_{ij}$.

In case all the quasi-determinants of $A$ are not zero, the inverse matrix of $A$ can be expressed in terms of them:

$$A^{-1} = (|A|^{-1})_{1 \leq i,j \leq N}.$$
In general, there are $N^2$ quasi-determinants for a $N \times N$ matrix $A$. 

$(i, j)$-quasi-determinant of $A$ is denoted as $|A|_{ij}$.

In case all the quasi-determinants of $A$ are not zero, the inverse matrix of $A$ can be expressed in terms of them:

$$A^{-1} = (|A|_{ji}^{-1})_{1 \leq i, j \leq N}.$$ 

If all the matrix elements of $A$ are commutative, then they reduce to

$$|A|_{ij} = (-1)^{i+j} \det A / \det A_{1,\ldots,i,\ldots,N}^{1,\ldots,j,\ldots,N}.$$
In general, there are $N^2$ quasi-determinants for a $N \times N$ matrix $A$. $(i,j)$-quasi-determinant of $A$ is denoted as $|A|_{ij}$.

In case all the quasi-determinants of $A$ are not zero, the inverse matrix of $A$ can be expressed in terms of them:

$$A^{-1} = (|A|_{ji}^{-1})_{1 \leq i, j \leq N}.$$ 

If all the matrix elements of $A$ are commutative, then they reduce to

$$|A|_{ij} = (-1)^{i+j} \det A / \det A_{1,\ldots,i,\ldots,N}^{1,\ldots,j,\ldots,N}.$$ 

$\implies$ Quasi-determinants are non-commutative analogues of ratios of determinants (rather than non-commutative analogues of determinants).
For a $1 \times 1$-sub-matrix $A_j^i = (a_{ij})$ of $A = A_{1,2,\ldots,N}^{1,2,\ldots,N}$, the $(i,j)$-th quasi-determinant is defined by $|A_j^i|_{ij} = a_{ij}$. 
quasi-determinants [Gelfand, Retakh]

- For a $1 \times 1$-sub-matrix $A^i_j = (a_{ij})$ of $A = A^{1,2,\ldots,N}_{1,2,\ldots,N}$, the $(i,j)$-th quasi-determinant is defined by $|A^i_j|_{ij} = a_{ij}$.

- $(i,j)$-th quasi-determinant of the submatrix $A^{i_1,i_2,\ldots,i_m}_{j_1,j_2,\ldots,j_m}$ of $A$ is recursively defined by

$$|A^{i_1,\ldots,i_m}_{j_1,\ldots,j_m}|_{ij} = a_{ij} - \sum_{k \in \{j_1,j_2,\ldots,j_m\} \setminus \{j\}, \ l \in \{i_1,i_2,\ldots,i_m\} \setminus \{i\}} a_{ik} (|A^{i_1,\ldots,\hat{i},\ldots,i_m}_{j_1,\ldots,\hat{j},\ldots,j_m}|_{lk})^{-1} a_{lj},$$

where $\{i_1, i_2, \ldots, i_m\}, \{j_1, j_2, \ldots, j_m\} \subset \{1, \ldots, N\}; \ m \geq 2; \ i \in \{i_1, i_2, \ldots, i_m\}, j \in \{j_1, j_2, \ldots, j_m\}$. 

Zengo Tsuboi (Osaka City University AdvanQuantum groups, Yang-Baxter maps and qua: 28 June 2018 30 / 54
quasi-Plücker coordinates [Gelfand, Retakh]

- Left quasi-Plücker coordinates of $m \times N$ matrix $A_{1,2,\ldots,N}^{1,2,\ldots,m}$

$$q_{ij}^{j_1,j_2,\ldots,j_{m-1}}(A_{1,2,\ldots,N}^{1,2,\ldots,m}) = (|A_{i,j_1,\ldots,j_{m-1}}^{1,2,\ldots,m}|s_i)^{-1}|A_{j,j_1,\ldots,j_{m-1}}^{1,2,\ldots,m}|s_j,$$

$s \in \{1, 2, \ldots, m\}; \ m < N, \ i, j, j_1, j_2, \ldots, j_{m-1} \in \{1, 2, \ldots, N\}, \ i \notin \{j_1, j_2, \ldots, j_{m-1}\}$. This does not depend on $s$. 
quasi-Plücker coordinates [Gelfand, Retakh]

- **Left quasi-Plücker coordinates** of $m \times N$ matrix $A_{1,2,\ldots,N}^{1,2,\ldots,m}$

  \[
  q_{ij}^{j_1 j_2 \ldots j_{m-1}}(A_{1,2,\ldots,N}^{1,2,\ldots,m}) = (|A_{i,j_1,\ldots,j_{m-1}}^{1,2,\ldots,m}|_s)^{-1} |A_{j,j_1,\ldots,j_{m-1}}^{1,2,\ldots,m}|_{sj},
  \]

  $s \in \{1, 2, \ldots, m\}$; $m < N$, $i, j, j_1, j_2, \ldots, j_{m-1} \in \{1, 2, \ldots, N\}$, $i \notin \{j_1, j_2, \ldots, j_{m-1}\}$. This does not depend on $s$.

- **Commutative case (ratios of Plücker coordinates)**

  \[
  q_{ij}^{j_1 j_2 \ldots j_{m-1}}(A_{1,2,\ldots,N}^{1,2,\ldots,m}) = \det(A_{i,j_1,\ldots,j_{m-1}}^{1,2,\ldots,m})^{-1} \det(A_{j,j_1,\ldots,j_{m-1}}^{1,2,\ldots,m}).
  \]
quasi-Plücker coordinates [Gelfand, Retakh]

- **Left quasi-Plücker coordinates** of $m \times N$ matrix $A^{1,2,\ldots,m}_{1,2,\ldots,N}$

$$q^{j_1,j_2,\ldots,j_{m-1}}_{i,j}(A_{1,2,\ldots,N}^{1,2,\ldots,m}) = (|A^{1,2,\ldots,m}_{i,j_1,\ldots,j_{m-1}}|_s)^{-1}|A^{1,2,\ldots,m}_{j,j_1,\ldots,j_{m-1}}|_s,$$

$s \in \{1, 2, \ldots, m\};$ $m < N,$ $i, j, j_1, j_2, \ldots, j_{m-1} \in \{1, 2, \ldots, N\},$ $i \notin \{j_1, j_2, \ldots, j_{m-1}\}$. This does not depend on $s$.

- **Commutative case (ratios of Plücker coordinates)**

$$q^{j_1,j_2,\ldots,j_{m-1}}_{i,j}(A_{1,2,\ldots,N}^{1,2,\ldots,m}) = \det(A^{1,2,\ldots,m}_{i,j_1,\ldots,j_{m-1}})^{-1} \det(A^{1,2,\ldots,m}_{j,j_1,\ldots,j_{m-1}}).$$

- **Right quasi-Plücker coordinates**
Quasi-Plücker coordinates also satisfy quasi-Plücker relations, which reduce to Plücker relations in case all the matrix elements are commutative.
Quasi-Plücker coordinates also satisfy quasi-Plücker relations, which reduce to Plücker relations in case all the matrix elements are commutative.

⇒ useful in non-commutative soliton theory: non-Abelian Hirota-Miwa equation, non-Abelian Toda equation, etc.
The solution of the zero-curvature relation can be rewritten in terms of quasi-Plücker coordinates of a block matrix:

$$M = \begin{pmatrix} 0 & L^{-(1)}L^{-(2)} \\ L^{+(1)}L^{+(2)} & L^{-(1)}L^{+(2)} \end{pmatrix} = \begin{pmatrix} 0 & L^{-(1)}L^{-(2)} \\ L^{+(1)}L^{+(2)} & J \end{pmatrix},$$
The solution of the zero-curvature relation can be rewritten in terms of quasi-Plücker coordinates of a block matrix:

\[
M = \begin{pmatrix}
0 & L^{-(1)}L^{-(2)} \\
L^{+(1)}L^{+(2)} & L^{-(1)}L^{+(2)}
\end{pmatrix}
= \begin{pmatrix}
0 & L^{-(1)}L^{-(2)} \\
L^{+(1)}L^{+(2)} & J
\end{pmatrix},
\]

Define a sub-matrix

\[
M_{\tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_a, i_1, i_2, \ldots, l_1, l_2, \ldots, l_d}^{\tilde{j}_1, \tilde{j}_2, \ldots, \tilde{j}_b, \tilde{j}_1, \tilde{j}_2, \ldots, \tilde{j}_b} = \begin{pmatrix}
0 & \left(L^{-(1)}L^{-(2)}\right)_{\tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_a}^{l_1, l_2, \ldots, l_d} \\
\left(L^{+(1)}L^{+(2)}\right)_{\tilde{j}_1, \tilde{j}_2, \ldots, \tilde{j}_b}^{k_1, k_2, \ldots, k_c} & \left(L^{-(1)}L^{+(2)}\right)_{l_1, l_2, \ldots, l_d}^{k_1, k_2, \ldots, k_c}
\end{pmatrix},
\]

\[
\tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_a, i_1, i_2, \ldots, k_c, \tilde{j}_1, \tilde{j}_2, \ldots, \tilde{j}_b, l_1, l_2, \ldots, l_d \in \{1, 2, \ldots, n\}.
\]
For $1 \leq i \leq j \leq n$, 

$$
\tilde{L}_{ij}^{+(1)} = \left( \prod_{k=1}^{i-1} q_{k, k}^{k+1, k+2, \ldots, n} \left( M_{1, 2, \ldots, \bar{n}, 1, 2, \ldots, n}^{k, k+1, \ldots, n} \right) \right) q_{i, j}^{i+1, i+2, \ldots, n} \left( M_{1, 2, \ldots, \bar{n}, 1, 2, \ldots, n}^{i, i+1, \ldots, n} \right),
$$

(and similar formulas for $\tilde{L}_{ji}^{-(1)}$, $\tilde{L}_{ij}^{+(2)}$, $\tilde{L}_{ji}^{-(2)}$) solve the zero-curvature relation.
For $1 \leq i \leq j \leq n$,

$$\tilde{L}_{ij}^{+(1)} = \left( \prod_{k=1}^{i-1} q_{k,k}^{k+1,k+2,...,n} (M_{1,2,...,\bar{n},1,2,...,n}^{k,k+1,...,n}) \right) q_{i,j}^{i+1,i+2,...,n} (M_{1,2,...,\bar{n},1,2,...,n}^{i,i+1,...,n}),$$

(and similar formulas for $\tilde{L}_{ji}^{-(1)}$, $\tilde{L}_{ij}^{+(2)}$, $\tilde{L}_{ji}^{-(2)}$) solve the zero-curvature relation.

A solution of a set theoretical (quantum) Yang-Baxter equation is obtained in terms of quasi-Plücker coordinates over a matrix composed of L-operators.
Heisenberg-Weyl realization (Minimal representation)

The Heisenberg-Weyl algebra $\mathcal{W}_q$

\[
\begin{align*}
    u_i w_j &= q^{2\delta_{ij}} w_j u_i, \\
    u_i u_j &= u_j u_i, \\
    w_i w_j &= w_j w_i.
\end{align*}
\]
Heisenberg-Weyl realization (Minimal representation)

The Heisenberg-Weyl algebra $\mathcal{W}_q$

$$u_i w_j = q^{2\delta_{ij}} w_j u_i, \quad u_i u_j = u_j u_i, \quad w_i w_j = w_j w_i.$$ 

Homomorphism from $U_q(sl(n))$ to $\mathcal{W}_q$ (minimal rep.)

$$L_{i,i}^+ = u_i, \quad L_{i,j}^+ = w_i^{-1}w_{i+1}^{-1} \cdots w_{j-1}^{-1}(u_j - \kappa u_{j-1}),$$

$$L_{i,i}^- = u_{i-1}, \quad L_{j,i}^- = \kappa^{-1}w_i w_{i+1} \cdots w_{j-1}(-u_i + \kappa u_{i-1}), \quad i < j,$$

where $\kappa \in \mathbb{C}$. 

This realizes a representation which has neither a highest weight nor a lowest weight.
Heisenberg-Weyl realization (Minimal representation)

The Heisenberg-Weyl algebra $\mathcal{W}_q$

$$u_i w_j = q^{2\delta_{ij}} w_j u_i, \quad u_i u_j = u_j u_i, \quad w_i w_j = w_j w_i.$$  

Homomorphism from $U_q(sl(n))$ to $\mathcal{W}_q$ (minimal rep.)

$$L^+_{i,i} = u_i, \quad L^+_{i,j} = w_i^{-1} w_{i+1}^{-1} \cdots w_{j-1}^{-1} (u_j - \kappa u_{j-1}),$$

$$L^-_{i,i} = u_{i-1}, \quad L^-_{j,i} = \kappa^{-1} w_i w_{i+1} \cdots w_{j-1} (-u_i + \kappa u_{i-1}), \quad i < j,$$

where $\kappa \in \mathbb{C}$.

This realizes a representation which has neither a highest weight nor a lowest weight.
Asymptotic representation

For \( \xi \in \mathbb{C} \setminus \{0\} \),

\[
\tau_\xi : \quad u_i \rightarrow u_i, \quad w_i \rightarrow \xi w_i
\]

gives an automorphism of \( \mathcal{W}_q \).
Asymptotic representation

For $\xi \in \mathbb{C} \setminus \{0\}$,

$$
\tau_{\xi} : \quad u_i \rightarrow u_i, \quad w_i \rightarrow \xi w_i
$$

gives an automorphism of $\mathcal{W}_q$. Taking note on this fact, we will take the limits $\kappa \rightarrow 0, \infty$. 
For $\xi \in \mathbb{C} \setminus \{0\}$,

$$\tau_\xi : \quad u_i \rightarrow u_i, \quad w_i \rightarrow \xi w_i$$

gives an automorphism of $\mathcal{W}_q$. Taking note on this fact, we will take the limits $\kappa \rightarrow 0, \infty$.

$L_{i,j}^{+,0} = \lim_{\kappa \rightarrow 0} L_{i,j}^{+,\kappa}$:

$$L_{i,i}^{+,0} = u_i, \quad L_{i,j}^{+,0} = w_i^{-1} w_{i+1}^{-1} \cdots w_{j-1}^{-1} u_j, \quad i < j.$$
Asymptoric representation

For $\xi \in \mathbb{C} \setminus \{0\}$,

$$\tau_\xi : \ u_i \rightarrow u_i, \quad w_i \rightarrow \xi w_i$$

gives an automorphism of $\mathcal{W}_q$. Taking note on this fact, we will take the limits $\kappa \rightarrow 0, \infty$.

$L^+;0_{i,j} = \lim_{\kappa \rightarrow 0} L^+_{i,j}$:

$$L^+;0_{i,i} = u_i, \quad L^+;0_{i,j} = w_i^{-1}w_{i+1}^{-1}\cdots w_{j-1}^{-1}u_j, \quad i < j.$$  

$L^+;\infty_{i,j} = \lim_{\kappa \rightarrow \infty} \tau_\kappa(L^+_{i,j})$:

$$L^+;\infty_{i,i} = u_i, \quad L^+;\infty_{i,i+1} = -w_i^{-1}u_i, \quad \text{otherwise} \quad L^+;\infty_{i,j} = 0.$$
Factorization of $L^+$ for minimal rep.

$$L_{1,0}^+ \tau_{\kappa^{-1}}(L_{2,\infty}^+) = U^+ L^+,$$
Factorization of $\mathbf{L}^+$ for minimal rep.

$$L_{1^+}^{+,0} \tau_{\kappa^{-1}}(L_{2^+}^{+,\infty}) = U^+ L^+,$$

$$(\kappa \to 0)(\kappa \to \infty) = \text{(diagonal)}\,(\text{minimal rep.})$$
Factorization of $L$-operators

Factorization of $L^+$ for minimal rep.

$$L_{1,0}^{+,0} \tau_{\kappa^{-1}}(L_{2,\infty}^{+,\infty}) = U^+ L^+,$$

$$(\kappa \to 0)(\kappa \to \infty) = \text{(diagonal)(minimal rep.)}$$

Factorization of $L^-$ for minimal rep.

$$L_{1,\infty}^{-,\infty} \tau_{\kappa^{-1}}(L_{2,0}^{-,0}) = U^- L^-,$$

$$(\kappa \to \infty)(\kappa \to 0) = \text{(diagonal)(minimal rep.)}$$
$L_{\pm,0}$ and $L_{\pm,\infty}$ give homomorphisms from $\mathcal{B}_{\mp}$ to $\mathcal{W}_q$. 
Factorization of an $R$-operator

$\mathbf{L}^{\pm,0}$ and $\mathbf{L}^{\pm,\infty}$ give homomorphisms from $\mathcal{B}_\mp$ to $\mathcal{W}_q$.

Evaluate the universal $R$-matrix $\mathbf{R} \in \mathcal{B}_+ \otimes \mathcal{B}_-$ by these homomorphisms.
Factorization of an R-operator

\( L^{\pm,0} \) and \( L^{\pm,\infty} \) give homomorphisms from \( \mathcal{B}_{\pm} \) to \( \mathcal{W}_q \).

Evaluate the universal R-matrix \( R \in \mathcal{B}_+ \otimes \mathcal{B}_- \) by these homomorphisms.

Factorization of the universal R-matrix for minimal rep.

\[
R^{\text{min, min}}_{13} = (\text{‘trivial’ } R) R^{0,0}_{14} R^{0,\infty}_{13} R^{\infty,0}_{24} R^{\infty,\infty}_{23} (\text{‘trivial’ } R)
\]

[cf. affine case: Meneghelli-Teschner 2015]
Quantum Yang-Baxter map gives an automorphism

\[ R : \mathcal{A}_1 \otimes \mathcal{A}_2 \mapsto R(\mathcal{A}_1 \otimes \mathcal{A}_2)R^{-1} \simeq \mathcal{A}_1 \otimes \mathcal{A}_2 \quad (\mathcal{A}_i \simeq \mathcal{A}). \]

Based on this map, we define a discrete quantum evolution system for the algebra of observables

\[ \mathcal{O} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_{2N-1} \otimes \mathcal{A}_{2N}, \quad N \geq 1. \]
Quantum Yang-Baxter map gives an automorphism

$$\mathcal{R} : \mathcal{A}_1 \otimes \mathcal{A}_2 \mapsto \mathcal{R}(\mathcal{A}_1 \otimes \mathcal{A}_2)\mathcal{R}^{-1} \simeq \mathcal{A}_1 \otimes \mathcal{A}_2 \quad (\mathcal{A}_i \simeq \mathcal{A}).$$

Based on this map, we define a discrete quantum evolution system for the algebra of observables

$$\mathcal{O} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_{2N-1} \otimes \mathcal{A}_{2N}, \quad N \geq 1.$$

$$\tilde{\mathcal{R}}_{ij} = \sigma_{ij} \circ \mathcal{R}_{ij}, \quad \mathcal{S} : (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots, \mathbf{X}^{(2N)}) \mapsto (\mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \ldots, \mathbf{X}^{(1)}),$$

$$\mathbf{X}^{(i)} : \text{set of the generators of } \mathcal{A}_i.$$
Quantum Yang-Baxter map gives an automorphism

\[ \mathcal{R} : \mathcal{A}_1 \otimes \mathcal{A}_2 \mapsto R(\mathcal{A}_1 \otimes \mathcal{A}_2)R^{-1} \simeq \mathcal{A}_1 \otimes \mathcal{A}_2 \]  

\[ (\mathcal{A}_i \simeq \mathcal{A}). \]

Based on this map, we define a discrete quantum evolution system for the algebra of observables

\[ \mathcal{O} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_{2N-1} \otimes \mathcal{A}_{2N}, \quad N \geq 1. \]

\[ \tilde{\mathcal{R}}_{ij} = \sigma_{ij} \circ \mathcal{R}_{ij}, \quad \mathcal{S} : (\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots, \mathbf{X}^{(2N)}) \mapsto (\mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \ldots, \mathbf{X}^{(1)}), \]

\[ \mathbf{X}^{(i)} : \text{set of the generators of } \mathcal{A}_i. \]

The operator

\[ \mathcal{U} = \mathcal{S} \circ (\tilde{\mathcal{R}}_{12} \circ \tilde{\mathcal{R}}_{34} \circ \cdots \circ \tilde{\mathcal{R}}_{2n-1,2n}) \]

gives one step of discrete time evolution \((t \rightarrow t + 1)\), which is an automorphism of \(\mathcal{O} : \mathcal{U}(\mathcal{O}) \simeq \mathcal{O}.\)
Commuting integrals of motion

Transfer matrices are generating function of integrals of motion.

\[ T(\lambda) = \text{Tr}_0 \left( L_{01}(\lambda)L_{02}^+ \cdots L_{0,2N-1}(\lambda)L_{0,2N}^+ \right), \]
Commuting integrals of motion

Transfer matrices are generating function of integrals of motion.

\[ T(\lambda) = \text{Tr}_0 \left( L_{01}(\lambda)L_{02}^+ \cdots L_{0,2N-1}(\lambda)L_{0,2N}^+ \right), \]

\[ \mathcal{U}(T(\lambda)) = T(\lambda), \]
Commuting integrals of motion

Transfer matrices are generating function of integrals of motion.

\[
T(\lambda) = \text{Tr}_0 \left( L_{01}(\lambda) L_{02}^+ \cdots L_{0,2N-1}(\lambda) L_{0,2N}^+ \right),
\]
\[
\overline{T}(\lambda) = \text{Tr}_0 \left( L_{01}^- L_{02}(\lambda) \cdots L_{0,2N-1}^- L_{0,2N}(\lambda) \right)
\]

\[
U(T(\lambda)) = T(\lambda), \quad U(\overline{T}(\lambda)) = \overline{T}(\lambda)
\]
Commuting integrals of motion

Transfer matrices are generating function of integrals of motion.

\[ T(\lambda) = \text{Tr}_0 \left( L_{01}(\lambda)L_{02}^+\cdots L_{0,2N-1}(\lambda)L_{0,2N}^+ \right), \]
\[ \overline{T}(\lambda) = \text{Tr}_0 \left( L_{01}^-L_{02}(\lambda)\cdots L_{0,2N-1}^-L_{0,2N}(\lambda) \right) \]

\[ U(T(\lambda)) = T(\lambda), \quad U(\overline{T}(\lambda)) = \overline{T}(\lambda) \]

\[ T(\lambda) = \lambda^N \sum_{j=0}^{N} \lambda^{-2j} G_j, \quad \overline{T}(\lambda) = \lambda^{-N} \sum_{j=0}^{N} \lambda^{2j} \overline{G}_j. \]

\[ [G_i, G_j] = [G_i, \overline{G}_j] = [\overline{G}_i, G_j] = 0. \]
Quasi-classical limit

$$q = e^{\pi i b^2}, \quad b \to 0,$$
Quasi-classical limit

Quasi-classical limit

\[ q = e^{\pi ib^2}, \quad b \to 0, \]

\[ U_q(gl(n)) \to \mathcal{P}(gl(n)) \]

\[ q^{E_{ii}} \to k_i, \quad \text{and} \quad E_{ij} \to e_{ij} \quad \text{for} \quad i \neq j, \]
Quasi-classical limit

\[ q = e^{\pi i b^2}, \quad b \to 0, \]

\[ U_q(gl(n)) \to \mathcal{P}(gl(n)) \]

\[ q^{E_{ii}} \to k_i, \quad \text{and} \quad E_{ij} \to e_{ij} \quad \text{for} \quad i \neq j, \]

Poisson brackets,

\[ [\ , \ ] \to 2\pi i b^2 \{ \ , \ \}, \quad b \to 0, \]
Poisson algebra $\mathcal{P}(gl(n))$

\[
\{ k_l, e_{ij} \} = \frac{\delta_{il} - \delta_{jl}}{2} e_{ij} k_l, \quad \{ k_i, k_j \} = 0,
\]

\[
\{ e_{i,i+1}, e_{j+1,j} \} = \delta_{ij} (k_i k_{i+1}^{-1} - k_{i+1}^{-1} k_i),
\]

\[
\{ e_{i,i+1}, e_{j,j+1} \} = \{ e_{i+1,i}, e_{j+1,j} \} = 0 \quad \text{for} \quad |i - j| \geq 2,
\]

Serre relations,

\[
f e_{i,i+1}; e_{i,i+1}; f e_{i,i+1}; e_{i,i+1}; e_{i,i+1}; e_{i,i+1}; e_{i,i+1} = 0
\]

Other generators,

\[
e_{ij} = f e_{ik}; e_{kj},
\]

\[
e_{ji} = \cdots ; i < k < j:
\]
Poisson algebra $\mathcal{P}(gl(n))$

\[
\{k_l, e_{ij}\} = \frac{\delta_{il} - \delta_{jl}}{2} e_{ij} k_l, \quad \{k_i, k_j\} = 0,
\]

\[
\{e_{i,i+1}, e_{j+1,j}\} = \delta_{ij} (k_i k_{i+1}^{-1} - k_i^{-1} k_{i+1}),
\]

\[
\{e_{i,i+1}, e_{j,j} + 1\} = \{e_{i+1,i}, e_{j+1,j}\} = 0 \quad \text{for} \quad |i - j| \geq 2,
\]

Serre relations,

\[
\{e_{i,i+1}, \{e_{i,i+1}, e_{i+1,i+2}\}\} - \frac{1}{4} e_{i,i+1}^2 e_{i+1,i+2} = 0, \ldots
\]

Other generators,

\[
e_{ij} = \{e_{ik}, e_{kj}\} - \frac{1}{2} e_{kj} e_{ik}, \quad e_{ji} = \ldots,
\]

\[i < k < j.\]
The universal R-matrix is \textit{singular} in the limit $b \to 0$. 

\[
R = \prod_{i<j} \left(1 - e_{ij} \otimes e_{ji}\right)^{-\frac{1}{2}} 
\times \exp \left( \frac{1}{i\pi b^2} \left( 2 \sum_{i\geq j} \log k_i \otimes \log k_j + \frac{1}{2} \sum_{i<j} \text{Li}_2(e_{ij} \otimes e_{ji}) \right) \right) (1 + O(b^2)), 
\]

\[
\text{Li}_2(x) = - \int_0^x \frac{\log(1 - t)}{t} \, dt.
\]
Although the quasi-classical limit of the universal $R$-matrix becomes singular, its adjoint action $\xi \in \mathcal{A} \otimes \mathcal{A} \rightarrow R\xi R^{-1} \in \mathcal{A} \otimes \mathcal{A}$ is well defined. Thus the $q \to 1$ limit of the quantum Yang-Baxter map is well defined.

\[
\overline{R} = \lim_{q \to 1} R
\]
The zero-curvature representation for the classical case has the same as the quantum case.

\[ \ell_1^+ \ell_2^+ = \ell_2^+ \ell_1^+ , \quad \ell_1^- \ell_2^+ = \ell_2^+ \ell_1^- , \quad \ell_1^- \ell_2^- = \ell_2^- \ell_1^- , \]

However, the matrix elements of the L-operators \( \ell_a^\pm \) are commutative.
One can obtain the solution by taking the limit \( q \to 1 \). In particular, the solution is written in terms of ratios of product of minor determinants (Plücker coordinates) of a single matrix.

\[
\begin{pmatrix}
0 & \ell_1^- \ell_2^-
\
\ell_1^+ \ell_2^+ & \ell_1^- \ell_2^+-
\end{pmatrix}
= \begin{pmatrix}
0 & \ell_1^- \ell_2^-
\
\ell_1^+ \ell_2^+ & J
\end{pmatrix}.
\]
Example for $\mathcal{P}(sl(3))$

\[
\vec{u}^{(1)}_1 = \begin{array}{c|cc}
J_{22} & J_{23} & (u_2^{(1)} u_2^{(2)})^{-1} \\
J_{32} & J_{33} & 1
\end{array}, \quad \vec{u}^{(1)}_2 = J_{33},
\]

\[
\vec{e}^{(1)}_{12} = \begin{array}{c|ccc}
(l_1^+ l_2^+)_{12} & J_{12} & J_{13} & (u_1^{(1)} u_1^{(2)} u_2^{(1)} u_2^{(2)})^{-1} \\
(l_1^+ l_2^+)_{22} & J_{22} & J_{23} & 1 \\
0 & J_{32} & J_{33} & 1
\end{array}, \quad \vec{e}^{(1)}_{23} = \begin{array}{c|cc}
(l_1^+ l_2^+)_{23} & J_{23} & (u_2^{(1)} u_2^{(2)})^{-1} \\
1 & J_{33} & 1
\end{array}, \quad \vec{e}^{(1)}_{31} = J_{31}, \quad \vec{e}^{(1)}_{32} = J_{32},
\]

\[
\vec{e}^{(1)}_{21} = \begin{array}{c|cc}
J_{21} & J_{23} & (u_2^{(1)} u_2^{(2)})^{-1} \\
J_{31} & J_{33} & 1
\end{array}, \quad \vec{u}^{(2)}_1 = \begin{array}{c|cc}
u_1^{(1)} & u_1^{(2)} & u_2^{(1)} \\
J_{22} & J_{23} & J_{32} \\
J_{32} & J_{33} & 1
\end{array}, \quad \vec{u}^{(2)}_2 = \frac{u_2^{(1)} u_2^{(2)}}{J_{33}},
\]

\[
\vec{e}^{(2)}_{12} = \begin{array}{c|cc}
J_{12} & J_{13} & u_2^{(1)} u_2^{(2)} \\
J_{32} & J_{33} & 1
\end{array}, \quad \vec{e}^{(2)}_{13} = \frac{J_{13}}{J_{33}}, \quad \vec{e}^{(2)}_{23} = \frac{J_{23}}{J_{33}}, \quad \vec{e}^{(2)}_{21} = \begin{array}{c|ccc}
(l_1^- l_2^-)_{21} & (l_1^- l_2^-)_{22} & 0 \\
J_{21} & J_{22} & J_{23} \\
J_{32} & J_{33} & J_{33}
\end{array}, \quad \vec{e}^{(2)}_{31} = \begin{array}{c|ccc}
(l_1^- l_2^-)_{31} & (l_1^- l_2^-)_{32} & (l_1^- l_2^-)_{33} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{array}, \quad \vec{e}^{(2)}_{32} = \begin{array}{c|ccc}
(l_1^- l_2^-)_{32} & (l_1^- l_2^-)_{33} & u_2^{(1)} u_2^{(2)} \\
J_{32} & J_{33} & 1
\end{array},
\]

\[
\vec{e}^{(2)}_{32} = \begin{array}{c|ccc}
J_{33} & J_{32} & J_{33}
\end{array}.
\]
The Heisenberg-Weyl algebra $\mathcal{W}_q$ reduces to the classical Heisenberg-Weyl algebra $\mathcal{W}$ in the quasi-classical limit.

\[
\{ u_i, w_j \} = \delta_{ij} w_j u_i, \quad \{ u_i, u_j \} = \{ w_i, w_j \} = 0.
\]
The Heisenberg-Weyl algebra $\mathcal{W}_q$ reduces to the classical Heisenberg-Weyl algebra $\mathcal{W}$ in the quasi-classical limit.

$$\{u_i, w_j\} = \delta_{ij} w_j u_i, \quad \{u_i, u_j\} = \{w_i, w_j\} = 0.$$
Solution of the zero-curvature relation for classical minimal rep

For instance, for \( n = 3 \) case, we explicitly obtain

\[
\tilde{u}_1^{(1)} = \left( \kappa_1 w_2^{(2)} (\kappa_1 u_1^{(1)} u_2^{(2)} w_1^{(2)} - w_1^{(1)} (\kappa_1 - u_1^{(1)}) (\kappa_2 u_1^{(2)} - u_2^{(2)})) - \\
\kappa_2 u_1^{(2)} w_1^{(1)} w_2^{(1)} (\kappa_1 - u_1^{(1)}) (\kappa_2 u_2^{(2)} - 1) \right) \left( \kappa_1^2 u_2^{(2)} w_1^{(2)} w_2^{(2)} \right)^{-1},
\]

(and similar relations for \( \tilde{u}_2^{(1)}, \tilde{u}_1^{(2)}, \tilde{u}_2^{(2)}, \tilde{w}_1^{(1)}, \tilde{w}_2^{(1)}, \tilde{w}_1^{(2)}, \tilde{w}_2^{(2)} \))
Rewriting this type of formula, we obtain the following relations for $\mathcal{P}(sl(n))$.

\[ u^{(1)}_i = \kappa_1 \left( \prod_{k=1}^{i-1} \frac{\tilde{w}^{(2)}_k}{w^{(2)}_k} \right) \frac{w^{(1)}_i - \tilde{w}^{(2)}_i}{w^{(1)}_i - \kappa_1 w^{(2)}_i}, \]

(and similar eqs. for $u^{(2)}_i, \tilde{u}^{(1)}_i, \tilde{u}^{(2)}_i$), \quad $i \in \{1, 2, \ldots, n-1\}$.
Symplectic form

Under these relations, the following function

\[ \Phi = \sum_{i=1}^{n-1} \sum_{a=1}^{2} \left( \log u_i^{(a)} \, d \log \tilde{w}_i^{(a)} - \log u_i^{(a)} \, d \log w_i^{(a)} \right) \]

becomes a closed form:

\[ d\Phi = \sum_{i=1}^{n-1} \sum_{a=1}^{2} \left( d \log \tilde{u}_i^{(a)} \wedge d \log \tilde{w}_i^{(a)} - d \log u_i^{(a)} \wedge d \log w_i^{(a)} \right) = 0. \]
Symplectic form

Under these relations, the following function

\[ \Phi = \sum_{i=1}^{n-1} \sum_{a=1}^{2} \left( \log \tilde{u}_i^{(a)} \, d \log \tilde{w}_i^{(a)} - \log u_i^{(a)} \, d \log w_i^{(a)} \right) \]

becomes a closed form:

\[ d\Phi = \sum_{i=1}^{n-1} \sum_{a=1}^{2} \left( d \log \tilde{u}_i^{(a)} \wedge d \log \tilde{w}_i^{(a)} - d \log u_i^{(a)} \wedge d \log w_i^{(a)} \right) = 0. \]

This is also an exact form (\( \Phi = d\mathcal{L} \)):

\[ \mathcal{L} = 2 \sum_{k < i} \log \frac{\tilde{w}_i^{(1)}}{w_i^{(1)}} \log \frac{\tilde{w}_k^{(2)}}{w_k^{(2)}} + \sum_{i=1}^{n-1} \log \frac{\tilde{w}_i^{(1)}}{\kappa_2 w_i^{(1)}} \log \frac{\tilde{w}_i^{(2)}}{w_i^{(2)}} \]

\[ + \sum_{i=1}^{n-1} \left\{ -\text{Li}_2 \left( \frac{\kappa_2 \tilde{w}_i^{(1)}}{\tilde{w}_i^{(2)}} \right) + \text{Li}_2 \left( \frac{\kappa_2 \tilde{w}_i^{(1)}}{\kappa_1 w_i^{(2)}} \right) + \cdots - \text{Li}_2 \left( \frac{\kappa_1 w_i^{(2)}}{\kappa_2 \tilde{w}_i^{(1)}} \right) \right\}. \]
We consider the map on $\mathcal{O} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_{2N-1} \otimes \mathcal{A}_{2N}$ for $q \to 1$

\[
\left( u_i^{2m+1,t+1} = \mathcal{U}(u_i^{2m-1,t}), u_i^{2m,t+1} = \mathcal{U}(u_i^{2m,t}), m = 1, \ldots, N; i = 1, \ldots, n-1 \right).
\]

\[
u_i^{2m-1,t} = \kappa_1 \left( \prod_{k=1}^{i-1} \frac{w_k^{2m,t+1}}{w_k^{2m,t}} \right) \frac{w_i^{2m-1,t} - w_i^{2m,t+1}}{w_i^{2m-1,t} - \kappa_1 w_i^{2m,t}}
\]

(and similar eqs. for $u_i^{2m,t}, u_i^{2m+1,t+1}, u_i^{2m,t+1}$).
Discrete soliton equations for $\mathcal{P}(sl(n))$

The consistency condition produces the following equations

$$
\left( \prod_{k=1}^{i-1} \frac{w_k^{2m+2,t+2}}{w_{k+1}^{2m+2,t+1}} \right) \frac{w_i^{2m+1,t+1} - w_i^{2m+2,t+2}}{w_i^{2m+1,t+1} - \kappa_1 w_i^{2m+2,t+1}} = \left( \prod_{k=1}^{i-1} \frac{w_k^{2m,t+1}}{w_{k+1}^{2m,t+1}} \right) \frac{w_i^{2m+1,t+1} - \kappa_2 w_i^{2m,t+1}}{w_i^{2m+1,t+1} - \kappa_1 \kappa_2 w_i^{2m,t+1}},
$$

(and one similar eq.)

These equations reduce to a discrete Liouville equation for $\mathcal{P}(sl(2))$ [Bazhanov-Sergeev 2015].
Discrete soliton equations for $\mathcal{P}(sl(n))$

The consistency condition produces the following equations

\[
\left( \prod_{k=1}^{i-1} \frac{w_{i+k}^{2m+2,t+2}}{w_{i+k}^{2m+2,t+1}} \right) \frac{w_i^{2m+1,t+1} - w_i^{2m+2,t+2}}{w_i^{2m+1,t+1} - \kappa_1 w_i^{2m+2,t+1}} = \left( \prod_{k=1}^{i-1} \frac{w_k^{2m,t+1}}{w_i^{2m,t}} \right) \frac{w_i^{2m+1,t+1} - \kappa_2^{-1} w_i^{2m,t+1}}{w_i^{2m+1,t+1} - \kappa_1 \kappa_2^{-1} w_i^{2m,t}} ,
\]

(and one similar eq.)

These equations reduce to a discrete Liouville equation for $\mathcal{P}(sl(2))$ [Bazhanov-Sergeev 2015].

We expected that $\mathcal{P}(sl(n))$ case corresponds to discrete Toda field equations. However, the equations seem to be something more complicated.
Quantum Yang-Baxter maps are defined in terms of adjoint action of the universal R-matrix [Bazhanov-Sergeev 2015].
Quantum Yang-Baxter maps are defined in terms of adjoint action of the universal R-matrix [Bazhanov-Sergeev 2015].

Solving the zero-curvature representation, we obtained the quantum Yang-Baxter map for $U_q(sl(n))$. It is expressed as a product of quasi-Plücker coordinates over a matrix (written in terms of L-operators, which are image of the universal R-matrix). Twisting of the universal R-matrix was essential for the rationality of the map.
Quantum Yang-Baxter maps are defined in terms of adjoint action of the universal R-matrix [Bazhanov-Sergeev 2015].

Solving the zero-curvature representation, we obtained the quantum Yang-Baxter map for $U_q(sl(n))$. It is expressed as a product of quasi-Plücker coordinates over a matrix (written in terms of L-operators, which are image of the universal R-matrix). Twisting of the universal R-matrix was essential for the rationality of the map.

Classical Yang-Baxter maps are derived through the quasiclassical limit.
Quantum Yang-Baxter maps are defined in terms of adjoint action of the universal R-matrix [Bazhanov-Sergeev 2015].

Solving the zero-curvature representation, we obtained the quantum Yang-Baxter map for $U_q(sl(n))$. It is expressed as a product of quasi-Plücker coordinates over a matrix (written in terms of L-operators, which are image of the universal R-matrix). Twisting of the universal R-matrix was essential for the rationality of the map.

Classical Yang-Baxter maps are derived through the quasiclassical limit.

Discrete integrable systems (soliton equations) follow from Yang-Baxter maps.
Quantum Yang-Baxter maps are defined in terms of adjoint action of the universal R-matrix [Bazhanov-Sergeev 2015].

Solving the zero-curvature representation, we obtained the quantum Yang-Baxter map for $U_q(sl(n))$. It is expressed as a product of quasi-Plücker coordinates over a matrix (written in terms of L-operators, which are image of the universal R-matrix). Twisting of the universal R-matrix was essential for the rationality of the map.

Classical Yang-Baxter maps are derived through the quasiclassical limit.

Discrete integrable systems (soliton equations) follow from Yang-Baxter maps.

Conjecture [Bazhanov-Sergeev 2015]: all the discrete integrable equations could be derived in this way.