Solution Operator for Non-Autonomous Perturbation of Gibbs Semigroup

To the memory of Hagen Neidhardt

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Abstract
The paper is devoted to a linear dynamics for non-autonomous perturbation of the Gibbs semigroup on a separable Hilbert space. It is shown that evolution family \( \{U(t,s)\}_{0 \leq s \leq t} \) solving the non-autonomous Cauchy problem can be approximated in the trace-norm topology by product formulae. The rate of convergence of product formulae approximants \( \{U_n(t,s)\}_{0 \leq s \leq t, n \geq 1} \) to the solution operator \( \{U(t,s)\}_{0 \leq s \leq t} \) is also established.

1 Introduction and main result

The aim of the paper is two-fold. Firstly, we study a linear dynamics, which is a non-autonomous perturbation of Gibbs semigroup. Secondly, we prove product formulae approximations of the corresponding to this dynamics solution operator \( \{U(t,s)\}_{0 \leq s \leq t} \), known also as evolution family, fundamental solution, or propagator, see [1] Ch.VI, Sec.9.

To this end we consider on separable Hilbert space \( \mathcal{H} \) a linear non-autonomous dynamics given by evolution equation of the type:

\[
\frac{\partial u(t)}{\partial t} = -C(t)u(t), \quad u(s) = u_s, \quad s \in [0, T) \subset \mathbb{R}_0^+,
\]

\[
C(t) := A + B(t), \quad u_s \in \mathcal{H},
\]

where \( \mathbb{R}_0^+ = \{0\} \cup \mathbb{R}^+ \) and linear operator \( A \) is generator of a Gibbs semigroup. Note that for the autonomous Cauchy problem (ACP) [1], when \( B(t) = 0 \), the

programme corresponds to the Trotter product formula approximation of the Gibbs semigroup generated by a closure of operator $A + B$, [20] Ch.5.

The main result of the present paper concerns the non-autonomous Cauchy problem (nACP) (1.1) under the following

**Assumptions:**

(A1) The operator $A \geq 1$ in a separable Hilbert space $\mathcal{H}$ is self-adjoint. The family $\{B(t)\}_{t \in \mathcal{I}}$ of non-negative self-adjoint operators in $\mathcal{H}$ is such that the bounded operator-valued function $(1 + B(\cdot))^{-1} : \mathcal{I} \rightarrow \mathcal{L}(\mathcal{H})$ is strongly measurable.

(A2) There exists $\alpha \in (0, 1)$ such that inclusion: $\text{dom}(A^\alpha) \subseteq \text{dom}(B(t))$, holds for a.e. $t \in \mathcal{I}$. Moreover, the function $B(\cdot)A^{-\alpha} : \mathcal{I} \rightarrow \mathcal{L}(\mathcal{H})$ is strongly measurable and essentially bounded in the operator norm:

$$C_\alpha := \text{ess sup}_{t \in \mathcal{I}} \|B(t)A^{-\alpha}\| < \infty. \quad (1.2)$$

(A3) The map $A^{-\alpha}B(\cdot)A^{-\alpha} : \mathcal{I} \rightarrow \mathcal{L}(\mathcal{H})$ is H"older continuous in the operator norm: for some $\beta \in (0, 1]$ there is a constant $L_{\alpha, \beta} > 0$ such that one has estimate

$$\|A^{-\alpha}(B(t) - B(s))A^{-\alpha}\| \leq L_{\alpha, \beta}|t - s|^\beta, \quad (t, s) \in \mathcal{I} \times \mathcal{I}. \quad (1.3)$$

(A4) The operator $A$ is generator of the Gibbs semigroup $\{G(t) = e^{-tA}\}_{t \geq 0}$, that is, a strongly continuous semigroup such that $G(t)|_{t \geq 0} \in \mathcal{C}_1(\mathcal{H})$. Here $\mathcal{C}_1(\mathcal{H})$ denotes the $\ast$-ideal of trace-class operators in $C^\ast$-algebra $\mathcal{L}(\mathcal{H})$ of bounded operators on $\mathcal{H}$.

**Remark 1.1** Assumptions (A1)-(A3) are introduced in [4] to prove the operator-norm convergence of product formula approximants $\{U_n(t, s)\}_{0 \leq s \leq T}$ to solution operator $\{U(t, s)\}_{0 \leq s \leq T}$. Then they were widely used for product formula approximations in [10] - [15] in the context of the evolution semigroup approach to the nACP, see [6] - [9].

**Remark 1.2** The following main facts were established (see, e.g., [4, 7, 18, 19]) about the nACP for perturbed evolution equation of the type (1.1):

(a) By assumptions (A1)-(A2) the operators $\{C(t) = A + B(t)\}_{t \in \mathcal{I}}$ have a common $\text{dom}(C(t)) = \text{dom}(A)$ and they are generators of contraction holomorphic semigroups. Hence, the nACP (1.1) is of parabolic type [5] [10].

(b) Since domains $\text{dom}(C(t)) = \text{dom}(A)$, $t \geq 0$, are dense, the nACP is well-posed with time-independent regularity subspace $\text{dom}(A)$.

(c) Assumptions (A1)-(A3) provide the existence of evolution family solving nACP (1.1) which we call the solution operator. It is a strongly continuous, uniformly bounded family of operators $\{U(t, s)\}_{(t, s) \in \mathcal{A}}$. $\mathcal{A} := \{(t, s) \in \mathcal{I} \times \mathcal{I} : 0 \leq s \leq t \leq T\}$, such that the conditions

$$U(t, t) = 1 \quad \text{for} \quad t \in \mathcal{I},$$

$$U(t, r)U(r, s) = U(t, s), \quad \text{for} \quad t, r, s \in \mathcal{I} \quad \text{for} \quad s \leq r \leq t, \quad (1.4)$$

are satisfied and $u(t) = U(t, s)u_s$ for any $u_s \in \mathcal{H}$, is in a certain sense (e.g., classical, strict, mild) solution of the nACP (1.1).
approximants may be defined as follows:

Proposition 1.3. The technique used for (1.9) in [4], but it is the same as that employed in [10].

In the present paper we essentially focus on convergence of the product approximants \( \{ U_n(t, s) \} \) to solution operator \( \{ U(t, s) \} \). Let

\[
  s = t_1 < t_2 < \ldots < t_{n-1} < t_n < t, \quad t_k := s + (k - 1) \frac{t-s}{n},
\]

for \( k \in \{1, 2, \ldots, n\}, n \in \mathbb{N} \), be partition of the interval \([s, t] \). Then corresponding approximants may be defined as follows:

\[
  W_k^{(n)}(t, s) := e^{-\frac{t-s}{n} A} e^{-\frac{t-s}{n} B(t_k)}, \quad k = 1, 2, \ldots, n,
\]

\[
  U_n(t, s) := W_n^{(n)}(t, s) W_{n-1}^{(n)}(t, s) \times \cdots \times W_2^{(n)}(t, s) W_1^{(n)}(t, s).
\]

It turns out that if the assumptions (A1)-(A3), adapted to a Banach space \( \mathcal{X} \), are satisfied for \( \alpha \in (0, 1) \), \( \beta \in (0, 1) \) and in addition the condition \( \alpha < \beta \) holds, then solution operator \( \{ U(t, s) \} \) admits the operator-norm approximation

\[
  \text{ess sup}_{(t, s) \in \mathcal{A}} \| U_n(t, s) - U(t, s) \| \leq \frac{R_{\beta, \alpha}}{n^{\beta - \alpha}}, \quad n \in \mathbb{N},
\]

for some constant \( R_{\beta, \alpha} > 0 \). This result shows that convergence of the approximants \( \{ U_n(t, s) \} \) is determined by the smoothness of the perturbation \( B(\cdot) \) in (A3) and by the parameter of inclusion in (A2), see [13].

The Lipschitz case \( \beta = 1 \) was considered for \( \mathcal{X} \) in [10]. There it was shown that if \( \alpha \in (1/2, 1) \), then one gets estimate

\[
  \text{ess sup}_{t \in \mathcal{I}} \| U_n(t, s) - U(t, s) \| \leq \frac{R_{1, \alpha}}{n^{1 - \alpha}}, \quad n = 2, 3, \ldots.
\]

For the Lipschitz case in a Hilbert space \( \mathcal{H} \) the assumptions (A1)-(A3) yield a stronger result [4]:

\[
  \text{ess sup}_{(t, s) \in \mathcal{A}} \| U_n(t, s) - U(t, s) \| \leq R \frac{\log(n)}{n}, \quad n = 2, 3, \ldots.
\]

Note that actually it is the best of known estimates for operator-norm rates of convergence under conditions (A1)-(A3).

The estimate (1.7) was improved in [12] for \( \alpha \in (1/2, 1) \) in a Hilbert space using the evolution semigroup approach [2, 3, 9]. This approach is quite different from technique used for (1.9) in [4], but it is the same as that employed in [10].

**Proposition 1.3** [12] Let assumptions (A1)-(A3) be satisfied for \( \beta \in (0, 1) \). If \( \beta > 2\alpha - 1 > 0 \), then estimate

\[
  \text{ess sup}_{(t, s) \in \mathcal{A}} \| U_n(t, s) - U(t, s) \| \leq \frac{R_{\beta}}{n^{\beta}},
\]

(d) Here \( \mathcal{A} \) is an appropriate regularity subspace of initial data. Assumptions (A1)-(A3) provide \( \mathcal{A} = \text{dom}(A) \) and \( U(t, s) \mathcal{A} \subseteq \text{dom}(A) \) for \( t > s \).
Then solution operator

Besides Remark 1.2(a)-(d) we also recall the following assertion, see, e.g., [16],

Preliminaries

Proposition 2.1

Theorem 1.4

Let assumptions (A1)-(A4) be satisfied. Then the estimate

holds for $n \in \mathbb{N}$ and for some constant $R_\beta > 0$.

Note that the condition $\beta > 2\alpha - 1$ is weaker than $\beta > \alpha$ [17], but it does not cover the Lipschitz case [18] because of condition $\beta < 1$.

The main result of the present paper is the lifting of any known operator-norm bounds [17]-[19] to estimate in the trace-norm topology $\| \cdot \|_1$. This is a subtle matter even for ACP, see [20] Ch.5.4.

- The first step is the construction for nACP (1.1) a trace-norm continuous solution operator $\{U(t,s)\}_{t,s \in \Delta}$, see Theorem 2.2 and Corollary 2.3.
- Then in Section 3 for assumptions (A1)-(A4) we prove (Theorem 1.4) the corresponding trace-norm estimate $R_{\alpha,\beta}(t,s)e_{\alpha,\beta}(n)$ for difference $\|U_n(t,s) - U(t,s)\|_1$.

**Theorem 1.4** Let assumptions (A1)-(A4) be satisfied. Then the estimate

\[
\|U_n(t,s) - U(t,s)\|_1 \leq R_{\alpha,\beta}(t,s)e_{\alpha,\beta}(n),
\]

holds for $n \in \mathbb{N}$ and $0 \leq s < t \leq T$ for some $R_{\alpha,\beta}(t,s) > 0$.

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Besides Remark 1.2(a)-(d) we also recall the following assertion, see, e.g., [16], Theorem 1, [17], Theorem 5.2.1.

**Proposition 2.1** Let assumptions (A1)-(A3) be satisfied.

(a) Then solution operator $\{U(t,s)\}_{t,s \in \Delta}$ is strongly continuously differentiable for $0 \leq s < t \leq T$ and

\[
\partial_t U(t,s) = -(A + B(t))U(t,s).
\]

(b) Moreover, the unique function $t \mapsto u(t) = U(t,s)u_s$ is a classical solution of (1.1) for initial data $\tilde{\delta}_s = \text{dom}(A)$.

Note that solution of (1.1) is called classical if $u(t) \in C([0,T],\tilde{\delta}) \cap C^1([0,T],\tilde{\delta})$, $u(t) \in \text{dom}(C(t))$, $u(s) = u_s$, and $C(t)u(t) \in C([0,T],\tilde{\delta})$ for all $t \geq s$, with convention that $(\partial_t u)(s)$ is the right-derivative, see, e.g., [16], Theorem 1, or [17], Ch.VI.9.

Since the involved into (A1), (A2) operators are non-negative and self-adjoint, equation (2.1) implies that the solution operator consists of contractions:

\[
|\partial_t |U(t,s)u|^2 = -2(C(t)U(t,s)u, U(t,s)u) \leq 0, \text{ for } u \in \tilde{\delta}.
\]

By (A1) $G(t) = e^{-tA} : \tilde{\delta} \rightarrow \text{dom}(A)$. Applying to (2.1) the variation of constants argument we obtain for $U(t,s)$ the integral equation:

\[
U(t,s) = G(t-s) - \int_s^t d\tau G(t-\tau)B(\tau)U(\tau,s), \quad U(s,s) = 1.
\]
Hence evolution family \( \{ U(t,s) \}_{(t,s) \in \Delta} \), which is defined by equation (2.3), can be considered as a mild solution of nACP (2.1) for \( 0 \leq s \leq t \leq T \) in the Banach space \( L(\mathcal{H}) \) of bounded operators, cf. \([1]\), Ch.VI.7.

Note that assumptions (A1)-(A4) yield for \( 0 \leq s < t \leq T \), \( \tau \in (s,t) \) and for the closure \( A^{-\alpha}B(\tau) \):

\[
\| G(t-s)A^\alpha \|_1 \leq \frac{M_\alpha}{(t-\tau)^\alpha} \quad \text{and} \quad \| A^{-\alpha}B(\tau) \| \leq C_\alpha .
\]  

(2.4)

Then (2.2), (2.4) give the trace-norm estimate

\[
\left\| \int_s^t d\tau \ G(t-\tau) \ B(\tau) \ U(\tau,s) \right\|_1 \leq \frac{M_\alpha C_\alpha}{1-\alpha} (t-s)^{1-\alpha},
\]  

(2.5)

and by (2.3) we ascertain that \( \{ U(t,s) \}_{(t,s) \in \Delta} \subset C_1(\mathcal{H}) \) for \( t > s \).

Therefore, we can construct solution operator \( \{ U(t,s) \}_{(t,s) \in \Delta} \) as a trace-norm convergent Dyson-Phillips series \( \sum_{n=0}^\infty S_n(t,s) \) by iteration of the integral formula (2.3) for \( t > s \). To this aim we define the recurrence relation

\[ S_0(t,s) = U_A(t-s), \]

\[ S_n(t,s) = - \int_s^t ds \ G(t-\tau) B(\tau) S_{n-1}(\tau,s), \quad n \geq 1. \]

(2.6)

Since in (2.6) the operators \( S_{n \geq 1}(t,s) \) are the \( n \)-fold trace-norm convergent Bochner integrals

\[
S_n(t,s) = \int_s^t d\tau_1 \int_s^{\tau_1} d\tau_2 \cdots \int_s^{\tau_{n-1}} d\tau_n \ G(t-\tau_1)(-B(\tau_1)) G(\tau_1-\tau_2) \cdots G(\tau_{n-1}-\tau_n)(-B(\tau_n)) G(\tau_n-s),
\]

(2.7)

by contraction property (2.2) and by estimate (2.5) there exist \( 0 \leq s \leq t \) such that \( M_\alpha C_\alpha (t-s)^{1-\alpha}/(1-\alpha) =: \xi < 1 \) and

\[
\| S_n(t,s) \|_1 \leq \xi^n, \quad n \geq 1.
\]

(2.8)

Consequently \( \sum_{n=0}^\infty S_n(t,s) \) converges for \( t > s \) in the trace-norm and satisfies the integral equation (2.3). Thus we get for solution operator of nACP the representation

\[
U(t,s) = \sum_{n=0}^\infty S_n(t,s).
\]

(2.9)

This result can be extended to any \( 0 \leq s < t \leq T \) using (1.4).

We note that for \( s < t \) the above arguments yield the proof of assertions in the next Theorem 2.2 and Corollary 2.3 but only in the strong ([19] Proposition 3.1, main Theorem in [18]) and in the operator-norm topology, [4] Lemma 2.1. While for \( t > s \) these arguments prove a generalisation of Theorem 2.2 and Corollary 2.3 to the trace-norm topology in Banach space \( C_1(\mathcal{H}) \):
Theorem 2.2 Let assumptions (A1)-(A4) be satisfied. Then evolution family 
\{U(t,s)\}_{(t,s)\in \Delta} (2.9) gives for \( t > s \) a mild trace-norm continuous solution of nACP (2.1) in Banach space \( C_1(\delta) \).

Corollary 2.3 For \( t > s \) the evolution family \( \{U(t,s)\}_{(t,s)\in \Delta} (2.9) \) is a strict solution of the nACP 
\[
\partial_t U(t,s) = -C(t)U(t,s), \quad t \in (s,T) \quad \text{and} \quad U(s,s) = \mathbb{1}, 
\]
\[
C(t) := A + B(t), 
\]
in Banach space \( C_1(\delta) \).

Proof. Since by Remark 1.2(c),(d) the function \( t \mapsto U(t,s) \) for \( t \geq s \) is strongly continuous and since \( U(t,s) \in C_1(\delta) \) for \( t > s \), the product \( U(t+\delta,t)U(t,s) \) is continuous in the trace-norm topology for \( |\delta| < t - s \). Moreover, since \( \{u(t)\}_{\delta \leq t \leq T} \) is a classical solution of nACP (1.1), equation (2.1) implies that \( U(t,s) \) has strong derivative for any \( t > s \). Then again by Remark 1.2(d) the trace-norm continuity of \( \delta \mapsto U(t+\delta,t)U(t,s) \) and by inclusion of ranges: \( \text{ran}(U(t,s)) \subseteq \text{dom}(A) \) for \( t > s \), the trace-norm derivative \( \partial_t U(t,s) \) at \( t > s \) exists and belongs to \( C_1(\delta) \).

Therefore, \( U(t,s) \in C((s,T],C_1(\delta)) \cap C^1((s,T],C_1(\delta)) \) with \( U(s,s) = \mathbb{1} \) and \( U(t,s) \in C_1(\delta) \), \( C(t)U(t,s) \in C_1(\delta) \) for \( t > s \), which means that solution \( U(t,s) \) of (2.10) is strict, cf. [19] Definition 1.1. \( \square \)

We note that these results for ACP in Banach space \( C_1(\delta) \) are well-known for Gibbs semigroups, see [20], Chapter 4.

Now, to proceed with the proof of Theorem 1.4 about trace-norm convergence of the solution operator approximants (1.6) we need the following preparatory lemma.

Lemma 2.4 Let self-adjoint positive operator \( A \) be such that \( e^{-tA} \in C_1(\delta) \) for \( t > 0 \), and let \( V_1, V_2, \ldots, V_n \) be bounded operators \( L(\delta) \). Then

\[
\left\| \prod_{j=1}^{n} V_j e^{-t_j A} \right\|_1 \leq \prod_{j=1}^{n} \| V_j \| \| e^{-(t_1 + t_2 + \ldots + t_n)A/4} \|_1, 
\]

for any set \( \{t_1, t_2, \ldots, t_n\} \) of positive numbers.

Proof. At first we prove this assertion for compact operators: \( V_j \in C_0(\delta) \), \( j = 1, 2, \ldots, n \). Let \( t_m := \min\{t_j\}_{j=1}^{n} > 0 \) and \( T := \sum_{j=1}^{n} t_j > 0 \). For any \( 1 \leq j \leq n \), we define an integer \( \ell_j \in \mathbb{N} \) by

\[
2^{\ell_j} t_m \leq t_j \leq 2^{\ell_j+1} t_m. 
\]

Then we get \( \sum_{j=1}^{n} 2^{\ell_j} t_m > T/2 \) and

\[
\prod_{j=1}^{n} V_j e^{-t_j A} = \prod_{j=1}^{n} V_j e^{-(t_j - 2^{\ell_j} t_m)A} (e^{-t_m A})^{\ell_j}, 
\]

(2.12)
By the definition of the \( \| \cdot \|_1 \)-norm and by inequalities for singular values \( \{ s_k(\cdot) \}_{k \geq 1} \) of compact operators
\[
\left\| \prod_{j=1}^{n} V_j e^{-t_j A} \right\|_1 \leq \sum_{k=1}^{\infty} \prod_{j=1}^{n} s_k \left( \prod_{j=1}^{n} V_j e^{-(t_j - 2^j t_m) A} (e^{-t_m A})^{2^j} \right) \\
\leq \sum_{k=1}^{\infty} \prod_{j=1}^{n} s_k \left( e^{-(t_j - 2^j t_m) A} \right) \left[ s_k(\cdot)^{2^j} \right] s_k(V_j) \\
\leq \sum_{k=1}^{\infty} s_k(e^{-t_m A})^{\Sigma_{j=1}^{n} 2^j} \prod_{j=1}^{n} \|V_j\| . \quad (2.13)
\]

Here we used that \( s_k(e^{-(t_j - 2^j t_m) A}) \leq \| e^{-(t_j - 2^j t_m) A} \| \leq 1 \) and that \( s_k(V_j) \leq \| V_j \|. \) Let \( N := \sum_{j=1}^{n} 2^j \) and \( T_m := N t_m > T/2. \) Since
\[
\sum_{k=1}^{\infty} s_k(e^{-t A / q}) = (\| e^{-t A / q} \|_q)_q ,
\]
the inequality (2.13) yields for \( q = N: \)
\[
\left\| \prod_{j=1}^{n} V_j e^{-t_j A} \right\|_1 \leq \left( \| e^{-T_m A / N} \|_N \right)^N \prod_{j=1}^{n} \|V_j\|. \quad (2.14)
\]

Now we consider an integer \( p \in \mathbb{N} \) such that \( 2^p \leq N < 2^{(p+1)} \). It then follows that \( T/4 < T_m/2 < 2^p T_m / N \), and hence we obtain
\[
\left( \| e^{-T_m A / N} \|_{q=N} \right)^N = \sum_{k=1}^{\infty} s_k^N(e^{-T_m A / N}) \\
\leq \sum_{k=1}^{\infty} s_k^{2^p} (e^{-2^{p} T_m A / 2^p N}) \leq \sum_{k=1}^{\infty} s_k^{2^p} (e^{-T A / 2^{p+1}}) = \| e^{-T A / 2^p} \|_1 , \quad (2.15)
\]

where we used that \( s_k(e^{-T_m A / N}) = s_k(e^{-2^{p} T_m A / 2^p N}) \leq \| e^{-T_m A / N} \| \leq 1 \), and that \( s_k(e^{-T A / q}) \leq \| e^{-t A / q} \|_q \leq s_k(e^{-t A}) \) for any \( t, \tau > 0 \). Therefore, the estimates (2.14, 2.15) give the bound (2.11).

Now, let \( V_j \in \mathcal{L}(\mathcal{F}) \), \( j = 1, 2, \ldots, n \), and set \( \tilde{V}_j := V_j e^{-\varepsilon A} \) for \( 0 < \varepsilon < t_m \). Hence, \( \tilde{V}_j \in C(\mathcal{F}) \subset C_0(\mathcal{F}) \) and \( s_k(\tilde{V}_j) \leq \| \tilde{V}_j \| \leq \| V_j \| \). If we set \( \tilde{t}_j := t_j - \varepsilon \), then
\[
\left\| \prod_{j=1}^{n} V_j e^{-\tilde{t}_j A} \right\|_1 \leq \prod_{j=1}^{n} \| V_j \| \| e^{-(\tilde{t}_j + \tilde{t}_j + \cdots + \tilde{t}_j) A / 4} \|_1 . \quad (2.16)
\]

Since the semigroup \( \{ e^{-t A} \}_{t \geq 0} \) is \( \| \cdot \|_1 \)-continuous for \( t > 0 \), we can take in (2.16) the limit \( \varepsilon \downarrow 0 \). This gives the result (2.11) in general case. \( \square \)
3 Proof of Theorem 1.4

We follow the line of reasoning of the lifting lemma developed in [20], Ch.5.4.1.

1. By virtue of (1.4) and (1.6) we obtain for difference in (1.11) formula:

$$U_n(t,s) - U(t,s) = \prod_{k=n}^{k+1} W_k^{(n)}(t,s) - \prod_{l=n}^{l+1} U(t_{l+1}, t_l).$$

Let integer $k_n \in (1, n)$. Then (3.1) yields the representation:

$$U_n(t,s) - U(t,s) = \left( \prod_{k=n}^{k+1} \right) W_k^{(n)}(t,s) - \left( \prod_{l=n}^{l+1} U(t_{l+1}, t_l) \right) + \prod_{l=n}^{l+1} U(t_{l+1}, t_l),$$

which implies the trace-norm estimate

$$\left\| U_n(t,s) - U(t,s) \right\|_1 \leq \left\| \prod_{k=n}^{k+1} W_k^{(n)}(t,s) - \prod_{l=n}^{l+1} U(t_{l+1}, t_l) \right\|_1 \left\| \prod_{k=n}^{k+1} W_k^{(n)}(t,s) \right\|_1$$

$$+ \left\| \prod_{l=n}^{l+1} U(t_{l+1}, t_l) \right\|_1 \left\| \prod_{k=n}^{k+1} W_k^{(n)}(t,s) - \prod_{l=n}^{l+1} U(t_{l+1}, t_l) \right\|_1.$$  (3.2)

2. Now we assume that $\lim_{n \to \infty} k_n/n = 1/2$. Then (1.5) yields $\lim_{n \to \infty} t_{k_n} = (t + s)/2$, $\lim_{n \to \infty} t_n = t$ and uniform estimates (1.7)-(1.10) with the bound $R_{\alpha, \beta} \epsilon_{\alpha, \beta}(n)$ imply

$$\text{ess sup}_{(t,s) \in \Delta} \left\| \prod_{k=n}^{k+1} W_k^{(n)}(t,s) - U(t, (t + s)/2) \right\| \leq R_{\alpha, \beta}^{(1)} \epsilon_{\alpha, \beta}(n),$$  (3.3)

$$\text{ess sup}_{(t,s) \in \Delta} \left\| \prod_{k=k_n}^{k+1} W_k^{(n)}(t,s) - U((t + s)/2, s) \right\| \leq R_{\alpha, \beta}^{(2)} \epsilon_{\alpha, \beta}(n).$$  (3.4)

for $n \in \mathbb{N}$ and for some constants $R_{\alpha, \beta}^{(1,2)} > 0$.

3. Since $\lim_{n \to \infty} k_n/n = 1/2$ and $t > s$, by definition (1.6) and by Lemma 2.4 for contractions $\{V_k = e^{-\frac{t_k}{\pi}B(t_k)}\}_{k=1}^{n}$ there exists $a_1 > 0$ such that

$$\left\| \prod_{k=k_n}^{k+1} W_k^{(n)}(t,s) \right\| = \left\| \prod_{k=k_n}^{k+1} e^{-\frac{t_k}{\pi}A} e^{-\frac{t_{k+1}}{\pi}B(t)} \right\|_1 \leq a_1 \left\| e^{-\frac{t_k}{2}A} \right\|_1.$$  (3.5)

Similarly there is $a_2 > 0$ such that
4. Since for \( t > s \) the trace-norm \( c(t-s) := \| e^{-tA} \|_1 \) is finite by (3.2)-(3.6) we obtain the proof of the estimate (1.11) for \( R_{\alpha,\beta}(t,s) := (a_1 R^{(1)}_{\alpha,\beta} + a_2 R^{(2)}_{\alpha,\beta}) c(t-s) \), (3.7)
and \( 0 \leq s < t \leq T \).

\[ \text{Corollary 3.1} \] By virtue of Lemma 2.4, the proof of Theorem 1.4 can be carried over almost verbatim for approximants \( \{ \hat{U}_n(t,s) \} \) for self-adjoint approximants \( \{ \tilde{U}_n(t,s) \} \) as well as for self-adjoint approximants \( \{ \hat{U}_n(t,s) \} \) for approximants (3.8), (3.9) is the same as in (1.11).

Note that the extension of Theorem 1.4 to Gibbs semigroups generated by a family of non-negative self-adjoint operators \( \{ A(t) \} \) can be done along the arguments outlined in Section 2 of [18].

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