Quantum Cosmology

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Abstract

The problems encountered in trying to quantize the various cosmological models, are brought forward by means of a concrete example. The Automorphism groups are revealed as the key element through which G.C.T.’s can be used for a general treatment of these problems. At the classical level, the time dependent automorphisms lead to significant simplifications of the line element for the generic spatially homogeneous geometry, without loss of generality. At the quantum level, the "frozen" automorphisms entail an important reduction of the configuration space –spanned by the 6 components of the scale factor matrix– on which the Wheeler-DeWitt equation, is to be based. In this spirit the canonical quantization of the most general minisuperspace actions –i.e. with all six scale factor as well as the lapse function and the shift vector present– describing the vacuum type II, I geometries, is considered. The reduction to the corresponding physical degrees of freedom is achieved through the usage of the linear constraints as well as the quantum version of the entire set of all classical integrals of motion.

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1 Introduction

Since the conception by Einstein of General Relativity Theory, a great many efforts have been devoted by many scientists to the construction of a consistent quantum theory of gravity. These efforts can be divided into two main approaches:

(a) perturbative, in which one splits the metric into a background (kinematical) part and a dynamical one: \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) and tries to quantize \( h_{\mu\nu} \). The only conclusive results existing, are that the theory thus obtained is highly nonrenormalizable [1].

(b) non-perturbative, in which one tries to keep the twofold role of the metric (kinematical and dynamical) intact. A hallmark in this direction is canonical quantization.

In trying to implement this scheme for gravity, one faces the problem of quantizing a constrained system. The main steps one has to follow are:

(i) define the basic operators \( \hat{g}_{\mu\nu} \) and \( \hat{\pi}^{\mu\nu} \) and the canonical commutation relation they satisfy.

(ii) define quantum operators \( \hat{H}_\mu \) whose classical counterparts are the constraint functions \( H_\mu \).

(iii) define the quantum states \( \Psi[g] \) as the common null eigenvector of \( \hat{H}_\mu \), i.e. these satisfying \( \hat{H}_\mu \Psi[g] = 0 \). (As a consequence, one has to check that \( \hat{H}_\mu \), form a closed algebra under the basic CCR.)

(iv) find the states and define the inner product in the space of these states.

It is fair to say that the full program has not yet been carried out, although partial steps have been made [2].

In the absence of a full solution to the problem, people have turned to what is generally known as quantum cosmology. This is an approximation to quantum gravity in which one freezes out all but a finite number of degrees of freedom, and quantizes the rest. In this way one is left with a much more manageable problem that is essentially quantum mechanics with constraints. Over the years, many models have appeared in the literature [3]. In most of them, the minisuperspace is flat and the gravitational field is represented by no more than three degrees of freedom (generically the three scale factors of some anisotropic Bianchi Type model [4]).

In order for the article to be as self-consistent as possible, we include in section 2, a short introduction to the theory of constrained systems and in section 3, the Kantowski-Sachs model is treated both at the classical and the quantum level—as an interdisciplinary example. In section 4, the importance of the Automorphism group is uncovered and the quantization of the most general Type II, I Vacuum Bianchi Cosmologies, is exhibited.
2 Elements of Constrained Dynamics

2.1 Introduction

In these short notes, we present the elements of the general methods and some techniques of the Constrained Dynamics. It is about a powerful mathematical theory (a method, more or less) –primarily developed by P. A. M. Dirac. The scope of it, is to describe singular (the definition is to be presented at the next section) physical systems, using a generalization of the Hamiltonian or the Lagrangian formalism. This theory, is applicable both for discrete (i.e. finite degrees of freedom) and continua (i.e. infinite degrees of freedom) systems.

For the sake of simplicity, the Hamiltonian point view of a physical system is adopted, and the discussion will be restricted on discrete systems. A basic bibliography, at which the interested reader is strongly suggested to consult, is quoted at the end of these notes. Also, the treatment follows reference [5].

2.2 The Hamiltonian Approach

Suppose a discrete physical system, whose action integral is:

\[ A = \int L \, dt \]  

(2.2.1)

The dynamical coordinates, are denoted by \( q^i \), with \( i \in [1, \ldots, N] \) The Lagrangian is a function of the coordinates and the velocities, i.e. \( L = L(q^i, \dot{q}^i) \).

A note is pertinent at this point. If one demands the action integral (2.2.1), to be scalar under general coordinate transformations (G.C.T.), then he can be sure that the content of the theory to be deduced, will be relativistically covariant even though the form of the deduced equations will not be manifestly covariant, on account of the appearance of one particular time in a dominant place in the theory (i.e. the time variable \( t \) occurring already, as soon as one introduces the generalized velocities, in consequently the Lagrangian, and finally the Lagrange transformation, in order to pass from the Lagrangian, to the Hamiltonian).

Variation of the action integral, gives the Euler-Lagrange equations of motion:

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = \frac{\partial L}{\partial q^i}, \quad i \in [1, \ldots, N] \]  

(2.2.2)

In order to go over to the Hamiltonian formalism, the momentum variables \( p_i \), are introduced through:

\[ p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad \forall \ i \]  

(2.2.3)

In the usual dynamical theories, a very restricting assumption is made; that all momenta are independent functions of the velocities, or –in view of the inverse map theorem for a
function of many variables— that the following (Hessian) determinant:

\[ |H_{ij}| = \left| \frac{\partial^2 L}{\partial \dot{q}^i \dot{q}^j} \right| \]  

(2.2.4)
is not zero in the whole domain of its definition. If this is the case, then the theorem guarantees the validity of the assumption, permits to use the Legendre transformation, and the corresponding physical system is called Regular. If this is not the case (i.e. some momenta, are not independent functions of the velocities), then there must exist some (say \( M \)) independent relations of the type:

\[ \phi_m(q, p) = 0, \quad m \in [1, \ldots, M] \]  

(2.2.5)
which are called Primary Constraints. The corresponding physical systems, are characterized as Singular.

Variation of the quantity \( p_i \dot{q}^i - L \) (the Einstein’s summation convention is in use), results in:

\[ \delta (p_i \dot{q}^i - L) = \ldots = (\delta p_i) \dot{q}^i - \left( \frac{\partial L}{\partial q^i} \right) \delta q^i \]  

(2.2.6)
by virtue of (2.2.2). One can see that this variation, involves variations of the \( q \)'s and the \( p \)'s. So, the quantity under discussion does not involve variation of the velocities and thus can be expressed in terms of the \( q \)'s and the \( p \)'s, only. This is the Hamiltonian. It must be laid stress on the fact that the variations, must respect the restrictions (2.2.5), i.e. to preserve them—if they are considered as conditions (see, e.g. C. Carathéodory, "Calculus of Variations and Partial Differential Equations of the First Order", AMS Chelsea (1989)).

Obviously, the Hamiltonian is not uniquely determined for, zero quantities can be added to it. This means that the following:

\[ H_T = H + u^m \phi_m \]  

(2.2.7)
where \( u^m \)'s are arbitrary coefficients in the phase space (including the time variable), is a valid Hamiltonian too. Variation of (2.2.7) results in:

\[ \dot{q}^i = \frac{\partial H}{\partial p_i} + u^m \frac{\partial \phi}{\partial p_i} + \text{term that vanishes as (2.2.5)} \]  
\[ \dot{p}_i = -\frac{\partial H}{\partial q_i} - u^m \frac{\partial \phi}{\partial q_i} + \text{term that vanishes as (2.2.5)} \]  

(2.2.8)
These are the Hamiltonian equations of motion for the system under consideration. This scheme, reflects the previous observation about variations under which, conditions must be preserved.

In order to proceed, a generalization of the Poisson Brackets must be introduced. This is done as follows:

Let \( f, g, h \) be quantities on a space, endowed with a linear map \{ , \} such that:

\[
\{ f, g \} + \{ g, f \} = 0 \quad \text{Antisymmetry} \\
\{ f + g, h \} = \{ f, h \} + \{ g, h \} \quad \text{Linearity} \\
\{ f g, h \} = f \{ g, h \} + \{ f, h \} g \quad \text{Product Law} \\
\{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0 \quad \text{Jacobi Identity}
\]  

(2.2.9)
If the space is the phase space, then these Generalized Poisson Brackets, reduce to the usual ones:

\[
\{ f, g \} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p^i}
\] (2.2.10)

otherwise are subject to the previous algebra –only.

For a dynamical variable –say \( g \), one can find –with the usage of:

\[
\dot{g} = \frac{\partial g}{\partial q^i} \dot{q}^i + \frac{\partial g}{\partial p^i} \dot{p}^i
\] (2.2.11)

and of (2.2.8), as well as the generalized Poisson Bracket Algebra (2.2.9):

\[
\dot{g} \approx \{ g, H \}
\] (2.2.12)

The symbol \( \approx \) is the Weak Equality symbol and stands for the following rule (deduced from a thorough analysis of the previous procedure):

\textit{A constraint, must not be used before all the Generalized Poisson Brackets, are calculated formally (i.e. only with the usage of the algebra (2.2.9) and the usual definition (2.2.10) –when the last is applicable).}

This rule, is encoded as:

\[
\phi_m(q, p) \approx 0, \quad m \in [1, \ldots, M]
\] (2.2.13)

In the previous procedure, the position of that rule, reflects the need to manipulate the \( u^m \)'s, which may depend on \( t \) only –since they are unknown coefficients, the definition (2.2.10) can not be used.

If the dynamical variable \( g \) is any one of the constraints, then (2.2.5) declare the preservation of zero. Thus, consistency conditions, are deduced:

\[
\{ \phi_{m'}, H \} + u^m \{ \phi_{m'}, \phi_m \} = 0
\] (2.2.14)

There are three possibilities:

\( CC_1 \) Relations (2.2.14) lead to identities –maybe, with the help of (2.2.5).

\( CC_2 \) Relations (2.2.14) lead to equations independent of the \( u \)'s. These must also be regarded as constraints. They are called \textit{Secondary}, but must be treated on the same footing as the primary ones.

\( CC_3 \) Relations (2.2.14) impose conditions on the \( u \)'s.

The above procedure must be applied to all secondary constraints. Again, the possible cases will be the previous three. The new constraints which may turn up are called secondary too. The procedure is applied for once more and so on. At the end, one will have a number of constraints (primary plus secondary) –say \( J \)– and a number of conditions on the \( u \)'s. A detailed analysis of the set of these conditions, shows that:

\[
u^m = U^m(q, p) + V^a(t)V^m_a(q, p)
\] (2.2.15)
where $V^m_a(q, p)$ are the $a$ (in number) independent solutions of the homogeneous systems:

$$V^m_a(q, p)\{\phi_m', \phi_m\} = 0$$

The functions $V^a(t)$ are related to the gauge freedom of the physical system.

Some terminology is needed at this point.

A dynamical variable $R$, is said to be **First Class**, if it has zero Poisson Bracket, with all the constraints:

$$\{R, \phi_n\} = 0, \quad n \in [1, \ldots, J]$$ (2.2.16)

where $J$ is the total number of constraints –i.e. primary plus all the secondary ones. It is sufficient for these conditions, to hold weakly –since, by definition, the $\phi$’s are the only independent quantities that vanish weakly. Otherwise, the variable $R$, is said to be **Second Class**. If $R$ is First Class, then the quantity $\{R, \phi_n\}$ is strongly equal to some linear combination of the $\phi$’s. The following relative theorem (with a trivial proof) holds:

"The Poisson Bracket of two First Class quantities, is also First Class".

Using the result (2.2.15) the Hamiltonian (2.2.7), which is called **Total Hamiltonian**, is written:

$$H_T = H + U^m \phi_m + V^a V^m_a \phi_m \equiv H' + V^a \phi_a$$ (2.2.17)

with obvious associations. It can be proved that $H'$ and $\phi_a$, are first class quantities. With this splitting and the relation (2.2.12) for a dynamical variable $g$, it can be deduced that:

*The First Class Primary Constraints $\phi_a$, are the generating functions (i.e. the quantities $\{g, \phi_a\}$) of infinitesimal Contact Transformations; i.e. of transformations which lead to changes in the $q$’s and the $p$’s which do not affect the physical state of the system.*

Successive application of two contact transformations generated by two given First Class Primary Constraints and taking into account the order, leads –for the sake of consistency– to a new generating function: $\{g, \{\phi_a', \phi_a''\}\}$. Thus one can see that First Class Secondary Constraints, which may turn up from $\{\phi_a, \phi_a'\}$, can also serve as generating functions of infinitesimal Contact Transformations. Possibly, another way to produce First Class Secondary Constraints, is the First Class quantity $\{H', \phi_a\}$. Since no one has found an example of a First Class Secondary Constraint, which affects the physical state when used as generating function, the conclusion is that all First Class quantities, are generating functions of infinitesimal Contact Transformations. Thus, the total Hamiltonian should be replaced by the **Extended Hamiltonian** $H_E$, defined as:

$$H_E = H_T + U'' \phi_a''$$ (2.2.18)

where the $\phi_a''$’s are those First Class Secondary Constraints, which are not already included in $H_T$. Finally, the equation of motion for a dynamical variable $g$ (2.2.12) is altered:

$$\dot{g} \approx \{g, H_E\}$$ (2.2.19)
2.3 Quantization of Constrained Systems

2.3.1 No Second Class Constraints are Present

The quantization of a classical physical system, whose Lagrangian, gives first class constraints only, is made in three steps:

\( S_1 \) The dynamical coordinates \( q \)'s and momenta \( p \)'s, are turned into Hermitian Operators \( \hat{q} \)'s and \( \hat{p} \)'s, satisfying the basic commutative algebra: \( [\hat{q}^i, \hat{p}_j] = i\delta^i_j \).

\( S_2 \) A kind of a Schrödinger equation, is set up.

\( S_3 \) Any dynamical function, become Hermitian Operator –provided that the ordering problem is somehow solved.

Obviously, the constraints –being functions on the phase space– are subject to the \( S_3 \) rule. Dirac, proposed that when the constraints are turned into operators, they must annihilate the wave function \( \Psi \):

\[
\hat{\phi}_i \Psi = 0, \quad \forall \quad i
\]  \hspace{1cm} (2.2.20)

Successive application of two such given conditions and taking into account the order, for sake of consistency, results in:

\[
[\hat{\phi}_i, \hat{\phi}_j] \Psi = 0 \tag{2.2.21}
\]

In order for operational conditions (2.2.21) not to give new ones on \( \Psi \), one demands:

\[
[\hat{\phi}_i, \hat{\phi}_j] = C^k_{ij} \hat{\phi}_k \tag{2.2.22}
\]

If it is possible for such an algebra to be deduced, then no new operational conditions on \( \Psi \) are found and the system is consistent. If this is not the case, the new conditions must be taken into account and along with the initial ones, must give closed algebra, otherwise the procedure must be continued until a closed algebra is found. The discussion does not end here. Consistency between the operational conditions (2.2.20) and the Schrödinger equation, is pertinent as well. This lead to:

\[
[\hat{\phi}_i, \hat{H}] \Psi = 0 \tag{2.2.23}
\]

and consistency know, reads:

\[
[\hat{\phi}_i, \hat{H}] = D^k_i \hat{\phi}_k \tag{2.2.24}
\]
2.3.2 Second Class Constraints are Present

Suppose we have a classical physical system, whose Lagrangian, gives second class constraints. Any set of constraints, can be replaced by a corresponding set of independent linear combinations of them. It is thus, in principle, possible to make arrangement such that the final set of constraints, contains as much first class constraints as possible. Using the remaining –say $S$ in number– second class constraints, the following matrix is defined:

$$
\Delta_{ij} = [\chi_i, \chi_j], \quad (i, j) \in [1, \ldots, S]
$$

(2.2.25)

where, $\chi$’s are the remaining (in classical form) second class constraints. A theorem can be proved:

"The determinant of this matrix does not vanish, not even weakly".

Since the determinant of $\Delta$ is non zero, there is the inverse of this matrix; say $\Delta^{-1}$. Dirac, proposed a new kind of Poisson Bracket, the $\{ , \}_D$:

$$
\{ , \}_D = \{ , \} - \sum_{i=1}^{S} \sum_{j=1}^{S} \{ \chi_i \} \Delta_{ij}^{-1} \{ \chi_j \}
$$

(2.2.26)

These Brackets, are antisymmetric, linear in their arguments, obey the product law and the Jacobi identity. It holds that:

$$
\{ g, H_E \}_D \approx \{ g, H_E \}
$$

(2.2.27)

because terms like $\{ \chi_i, H_E \}$, with $H_E$ being first class, vanish weakly. Thus:

$$
\dot{g} \approx \{ g, H_E \}_D
$$

(2.2.28)

But:

$$
\{ \xi, \chi_s \}_D = \ldots = 0
$$

(2.2.29)

if $\xi$ is any of the $q$’s or the $p$’s. Thus, at the classical level, one may put the second class constraints equal to zero, before calculating the new Poisson Brackets. That means that:

$M_1$ The equations $\chi = 0$ may be considered as strong equations.

$M_2$ One, must ignore the corresponding degrees of freedom and

$M_3$ quantize the rest, according to the general rules, given in the previous section.
3 A Pedagogical Example:  
The Kantowski-Sachs Model

The purpose of the present section is twofold:

- to illustrate –at the classical level– an application of Dirac’s method for constrained systems.
- to present, in an easy manner, the problems raised by the quantization of such systems.

The example chosen, is that of Kantowski-Sachs reduced Lagrangian –i.e. of a vacuum cosmological model; thus the interdisciplinary character of the section, emerges.

3.1 The Classical Case

Consider, the Kantowski-Sachs model (described in [8]), characterized by the line element:

\[ ds^2 = -N^2(t)dt^2 + a^2(t)dr^2 + b^2(t)d\theta^2 + b^2(t)sin^2(\theta)d\phi^2 \]  (3.3.1)

The corresponding Einstein’s Field Equations, are:

\[ G_{00} = -\left( \frac{N(t)}{b(t)} \right)^2 - 2\frac{a'(t)b'(t)}{a(t)b(t)} - \left( \frac{b'(t)}{b(t)} \right)^2 \]  (3.3.2)

\[ G_{11} = -\left( \frac{a(t)}{b(t)} \right)^2 + \left( \frac{a(t)b'(t)}{N(t)b(t)} \right)^2 - 2\frac{a(t)^2b'(t)N'(t)}{b(t)N(t)^3} + 2\frac{a(t)b''(t)}{b(t)N(t)^2} \]  (3.3.3)

\[ G_{22} = \frac{b(t)a'(t)b'(t)}{a(t)N(t)^2} - \frac{a'(t)b(t)^2N'(t)}{N(t)^3a(t)} - \frac{b(t)b'(t)N'(t)}{N(t)^3} + \frac{b(t)^2a''(t)}{a(t)N(t)^2} + \frac{b(t)b''(t)}{N(t)^2} \]  (3.3.4)

\[ G_{33} = sin(\theta)^2G_{22} \]  (3.3.5)

The first of these \( G_{00} \), is the quadratic constraint equation i.e. its time derivative vanishes –by virtue of the other two \( G_{11}, G_{22} \). This is a ”peculiarity” of Einstein’s system, and reflects the time reparametrization invariance \( t \rightarrow \tilde{t} = f(t) \). Under such a transformation, \( a(t) \) and \( b(t) \) change as scalars \( \left( a(t) = \tilde{a}(\tilde{t}), \text{ditto the } b(t) \right) \) while \( N(t) \), changes as density \( \left( \tilde{N}(\tilde{t})\tilde{d}\tilde{t} = N(t)dt \right) \), revealing its nature, as a Lagrange multiplier.

It must be brought to the reader’s notice that the above set of equations \( G_{\mu\nu} \), can be obtained from the following action principle:

\[ S = \int Ldt = \int \left( -\frac{a(t)b^2(t) + 2b(t)a(t)b'(t)}{2N(t)} + \frac{N(t)a(t)}{2} \right) dt \]  (3.3.6)
where \(a(t), b(t)\) and \(N(t)\), are the three degrees of freedom (the \(q\)'s) – which has the above mentioned reparametrization invariance.

The momenta are:

\[
\begin{align*}
p_a &= \frac{\partial L}{\partial \dot{a}(t)} = -\frac{b(t)\dot{b}(t)}{N(t)} \quad (3.3.7a) \\
p_b &= \frac{\partial L}{\partial \dot{b}(t)} = -\frac{\dot{a}(t)b(t) + a(t)\dot{b}(t)}{N(t)} \quad (3.3.7b) \\
p_N &= \frac{\partial L}{\partial \dot{N}(t)} = 0 \quad (3.3.7c)
\end{align*}
\]

From the third of (3.3.7a), one can see that there is one primary constraint:

\[
p_N \approx 0 \quad (3.3.8)
\]

The total Hamiltonian is:

\[
H_T = H + u(t)p_N \quad (3.3.9)
\]

where:

\[
H = p_a\dot{a}(t) + p_b\dot{b}(t) - L = N(t)\Omega(t) \quad (3.3.10)
\]

with:

\[
\Omega(t) \equiv \frac{-a(t)}{2} - \frac{p_ap_b}{b(t)} + \frac{a(t)p_a^2}{2b^2(t)} \quad (3.3.11)
\]

The consistency condition (2.2.14) applied to:

\(A_1\) the constraint \(p_N \approx 0\), gives one secondary constraint:

\[
\chi \equiv \{p_N, H\} = \{p_N, N(t)\Omega(t)\} = -\Omega(t) \approx 0 \quad (3.3.12)
\]

A straightforward calculation, results in:

\[
\{\chi, p_N\} = 0 \quad (3.3.13)
\]

\(A_2\) the previously deduced secondary constraint \(\chi \approx 0\), gives –by virtue of (3.3.11), (3.3.12) and (3.3.13)– no further constraints, since it is identically satisfied (\(CC_1\) case):

\[
\{\chi, H\} + u(t)\{\chi, p_N\} = 0 \quad (3.3.14)
\]
The Poisson Bracket (3.3.13) also declares that both $p_N$ and $\chi$, are first class quantities.

Finally, the equations of motion are:

$$\dot{a}(t) \approx \{a(t), H_T\} \quad (3.3.15)$$
$$\dot{p}_a \approx \{p_a, H_T\} \quad (3.3.16)$$
$$\dot{b}(t) \approx \{b(t), H_T\} \quad (3.3.17)$$
$$\dot{p}_b \approx \{p_b, H_T\} \quad (3.3.18)$$
$$\dot{N}(t) \approx \{N(t), H_T\} \quad (3.3.19)$$
$$\dot{p}_N \approx \{p_N, H_T\} \quad (3.3.20)$$

The first four equations constitute the usual set of the Euler-Lagrange equations for the $a(t)$ and $b(t)$, degrees of freedom. Equation (3.3.19), results in the gauge freedom related to $N(t)$ since –according to this equation– $\dot{N}(t) = u(t)$, i.e. an arbitrary function of time, while equation (3.3.20) is trivially satisfied, in view of (3.3.12).

Finally a remark concerning the existence of shift terms of the form $N_i(x^j, t)dx^i dt$ –where $N_i(x, t) \equiv N_a(t)\sigma^a(x)$: their existence entails constraint equations $(G_{0i})$ –again preserved in time, by virtue of the $(G_{ij})$ equations– which reflect the space reparametrization invariance $x^i \rightarrow \tilde{x}^i(x^j, t)$. Along with the existence of these shift terms, a change in the spatial part of the line element, is induced.

### 3.2 The Quantum Case

In trying to quantize the previously described constraint Hamiltonian system, various problems, arise [5, 6].

In the canonical approach [2] –and references therein–, the Schrödinger representation, is most frequently adopted. Applied to our example, this entails the step:

$$a \rightarrow \hat{a} = a$$
$$b \rightarrow \hat{b} = b$$
$$N \rightarrow \hat{N} = N$$
$$p_a \rightarrow \hat{p}_a = -i\frac{\partial}{\partial a}$$
$$p_b \rightarrow \hat{p}_b = -i\frac{\partial}{\partial b}$$
$$p_N \rightarrow \hat{p}_N = -i\frac{\partial}{\partial N} \quad (3.3.21)$$

When trying to implement Dirac’s proposal (steps $S_1$, $S_2$ of the section 2.3.1) we came across the factor ordering problem (see e.g. T. Christodoulakis, J. Zanelli, Nuovo
Cimento B 93 (1986) 1). Its resolution is achieved via the recipe that the kinetic term must be realized as the conformal Laplacian. This is due to the covariance in the change $\tilde{N}(\tilde{t}) = N(t)f(a(t), b(t))$—with the understanding that $f(a(t), b(t))$, is identified to an arbitrary function of time. The conformal Laplacian must be based on the metric:
\[
g^{ij} = \begin{pmatrix} a/2b^2 & -1/2b \\ -1/2b & 0 \end{pmatrix}
\] (3.3.22)
because of the correspondence principle, since $H = N(g^{ij}p_i p_j + V)$, where $p_1 \equiv p_a$, $p_2 \equiv p_b$ and $V = -a/2$. In two dimensions, the conformal Laplacian, reduces to the typical one (see next section for details). Thus, following Dirac’s quantization program, we deduce:
\[
\hat{H}_T\Psi = 0 \tag{3.3.23}
\]
or:
\[
\hat{p}_N\Psi = 0 \tag{3.3.24}
\]
(Constraint) and:
\[
\hat{H}\Psi = 0 \tag{3.3.25}
\]
(Wheeler-DeWitt equation). Under the transformation:
\[
a \rightarrow u = b \\
b \rightarrow v = a^3b
\] (3.3.26)
the Wheeler-DeWitt equation, assumes the form:
\[
4 \frac{\partial^2\Psi}{\partial u \partial v} - \Psi = 0 \tag{3.3.27}
\]
and under a second transformation:
\[
u(t) \rightarrow X = \frac{u+v}{2} \\
v(t) \rightarrow Y = \frac{a-v}{2}
\] (3.3.28)
the Wheeler-DeWitt equation, takes the form:
\[
\frac{\partial^2\Psi}{\partial X^2} - \frac{\partial^2\Psi}{\partial Y^2} - \Psi = 0 \tag{3.3.29}
\]
Now the previous equation can be solved via the method of separation of variables, e.g. $\Psi(X, Y) = A(X)B(Y)$; its general solution, consists of products of Exponentials and/or Trigonometric functions—depending on the sign and the value of the separation constant.
Of course, in order to complete the program of quantization, we need to construct the Hilbert space, i.e. to select a measure. The problem is open, because there is an
infinitude of candidates. If one invokes some sort of "naturality", one could adopt as a measure the square root of the determinant of the supermetric, i.e. $2b$ --in our case. This however, causes two unpleasant drawbacks: the first is that the wave function, is not square integrable, and the second is the violation of the conformal covariance.

In the case where shift terms and more spatial metric cross terms, are present, one would like to know, what features of the above exhibition, are generic and thus, characterize the general situation. The answer is given through the consideration of the automorphism group, which can be considered as the symmetries of the symmetry group of the 3-space. Their action entails considerable simplification, both at the classical and the quantum level. The spirit of these ideas, is exhibited in the next sections.
4 Automorphisms in Classical & Quantum Cosmology

4.1 The Simplification of Einstein’s Equations

It has long been suspected and/or known, that automorphisms, ought to play an important role in a unified treatment of this problem. The first mention, goes back to the first of [9]. More recently, Jantzen, second of [9]– has used Time Dependent Automorphism Matrices, as a convenient parametrization of a general positive definite $3 \times 3$ scale factor matrix $\gamma_{\alpha\beta}(t)$, in terms of a –desired– diagonal matrix. His approach, is based on the orthonormal frame bundle formalism, and the main conclusion is (third of [9], pp. 1138): " . . . the special automorphism matrix group $SAut(G)$, is the symmetry group of the ordinary differential equations, satisfied by the metric matrix $\gamma_{\alpha\beta}$, when no sources are present . . . " Later on, Samuel and Ashtekar in [10], have seen automorphisms, as a result of general coordinate transformations. Their spacetime point of view, has led them, to consider the –so called– ”Homogeneity Preserving Diffeomorphisms”, and link them, to topological considerations.

4.1.1 Time Dependent Automorphism Inducing Diffeomorphisms

It is well known that the vacuum Einstein field equations can be derived from an action principle:

$$ A = \frac{-1}{16\pi} \int \sqrt{-g} (4) R d^4x $$

(4.4.1)

(we use geometrized units i.e. $G = c = 1$)

The standard canonical formalism [11] makes use of the lapse and shift functions appearing in the 4-metric:

$$ ds^2 = (N^i N_i - N^2) dt^2 + 2 N_i dx^i dt + g_{ij} dx^i dx^j $$

(4.4.2)

From this line-element the following set of equations obtains, expressed in terms of the extrinsic curvature:

$$ K_{ij} = \frac{1}{2N}(N_{ij} + N_{ji} - \partial g_{ij} \partial t) $$

$$ H_0 = \sqrt{g} (K_{ij} K^{ij} - K^2 + R) = 0 $$

(4.4.3a)

$$ H_i = 2 \sqrt{g} (K^j_{ij} - K^j_i) = 0 $$

(4.4.3b)

$$ \frac{1}{\sqrt{g}} \frac{d}{dt} [\sqrt{g} (K^{ij} - K g^{ij})] = -N (R^{ij} - \frac{1}{2} R g^{ij}) - \frac{N}{2} (K_{kl} K^{kl} - K^2) g^{ij} $$

(4.4.3c)

$$ + 2N (K^{ik} K^j_k - K K^{ij}) - (N^{ij} - N^{[ij]} g^{ij}) + [(K^{ij} - K g^{ij}) N^l_{i\mid l}] $$

$$ - N^l_{i\mid l} (K^{lj} - K g^{lj}) - N^l_{j\mid l} (K^{li} - K g^{li}) $$

(4.4.3d)
This set is equivalent to the ten Einstein’s equations.

In cosmology, we are interested in the class of spatially homogeneous spacetimes, characterized by the existence of an m-dimensional isometry group of motions $G$, acting transitively on each surface of simultaneity $\Sigma_t$. When $m$ is greater than 3 and there is no proper invariant subgroup of dimension 3, the spacetime is of the Kantowski-Sachs type [8] and will not concern us further. When $m$ equals the dimension of $\Sigma_t$—which is 3—, there exist 3 basis one-forms $\sigma_i^\alpha$ satisfying:

$$d\sigma^\alpha = C^\alpha_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \Leftrightarrow \sigma^\alpha_{i,j} - \sigma^\alpha_{j,i} = 2C^\alpha_{\beta\gamma} \sigma^\gamma_i \sigma^\beta_j$$

(4.4.4a)

where $C^\alpha_{\beta\gamma}$ are the structure constants of the corresponding isometry group.

In this case there are local coordinates $t, x^i$ such that the line element in (4.4.2) assumes the form:

$$ds^2 = (N^\alpha(t)N_\alpha(t) - N^2(t))dt^2 + 2N_\alpha(t)\sigma^\alpha_i(x)dx^idt$$

$$+ \gamma_{\alpha\beta}\sigma^\alpha_i(x)\sigma^\beta_j(x)dx^i dx^j$$

(4.4.4b)

Latin indices, are spatial with range from 1 to 3. Greek indices, number the different basis 1-forms, take values in the same range, and are lowered and raised by $\gamma_{\alpha\beta}$, and $\gamma^{\alpha\beta}$ respectively.

A commitment concerning the topology of the 3-surface, is pertinent here, especially in view of the fact that we wish to consider diffeomorphisms [10]; we thus assume that $G$ is simply connected and the 3-surface $\Sigma_t$ can be identified with $G$, by singling out a point $p$ of $\Sigma_t$, as the identity $e_i$ of $G$.

If we insert relations (4.4.4) into equations (4.4.3), we get the following set of ordinary differential equations for the Bianchi-Type spatially homogeneous spacetimes:

$$E_0 \doteq K^\alpha_\beta K^\beta_\alpha - K^2 + R = 0$$

(4.4.5a)

$$E_\alpha \doteq K^\mu_\alpha C^\nu_{\mu\nu} - K^\mu_\epsilon C^\epsilon_{\alpha\mu} = 0$$

(4.4.5b)

$$E^\alpha_\beta \doteq \dot{K}^\alpha_\beta - NK^\alpha_\beta + NR^\alpha_\beta + 2N^\rho(K^\alpha_\nu C^\nu_{\beta\rho} - K^\nu_\beta C^\nu_{\alpha\rho})$$

(4.4.5c)

where $K^\alpha_\beta = \gamma^{\alpha\rho}K_{\rho\beta}$ and

$$K_{\alpha\beta} = -\frac{1}{2N}(\gamma_{\alpha\beta} + 2\gamma_{\alpha\nu}C^\nu_{\beta\rho}N^\rho + 2\gamma_{\beta\nu}C^\nu_{\alpha\rho}N^\rho)$$

(4.4.6)

$$R_{\alpha\beta} = C^\epsilon_{\sigma\tau}C^\lambda_{\mu\nu}\gamma_{\alpha\kappa}\gamma_{\beta\lambda}\gamma^{\epsilon\sigma\mu\nu} + 2C^\lambda_{\alpha\kappa}C^\epsilon_{\beta\lambda} + 2C^\epsilon_{\alpha\kappa}C^\lambda_{\beta\lambda}\gamma^{\epsilon\mu\nu\gamma^\kappa\lambda}$$

$$+ 2C^\lambda_{\beta\kappa}C^\epsilon_{\mu\nu}\gamma_{\alpha\lambda}\gamma^{\epsilon\mu\nu} + 2C^\epsilon_{\alpha\kappa}C^\lambda_{\mu\nu}\gamma_{\beta\lambda}\gamma^{\epsilon\mu\nu}$$

(4.4.7)

When $N^\alpha = 0$, equation (4.4.5c) reduces to the form of the equation given in [12]. Equation set (4.4.5), forms what is known as a --complete-- perfect ideal; that is, there are no
integrability conditions obtained from this system. So, with the help of (4.4.5c), (4.4.6), (4.4.7), it can explicitly be shown, that the time derivatives of (4.4.5a) and (4.4.5b) vanish identically. The calculation is straightforward—although somewhat lengthy. It makes use of the Jacobi identity
\[ C_{\alpha \beta} C^\alpha_{\gamma \delta} + C_{\alpha \rho} C^\rho_{\beta \gamma} + C_{\alpha \gamma} C^\gamma_{\rho \delta} = 0, \]
and its contracted form
\[ C_{\alpha \beta} C^\beta_{\gamma \delta} = 0. \]
The vanishing of the derivatives of the 4 constrained equations:
\[ E_0 = 0, \quad E_\alpha = 0, \]
implies that these equations, are first integrals of equations (4.4.5c)—moreover, with vanishing integration constants. Indeed, algebraically solving (4.4.5a), (4.4.5b) for \( N(t), N^\alpha(t) \), respectively and substituting in (4.4.5c), one finds that in all—but Type II and III—Bianchi Types, equations (4.4.5c), can be solved for only 2 of the 6 accelerations \( \gamma_{\alpha \beta} \) present. In Type II and III, the independent accelerations are 3, since \( E_\alpha \) are not independent and thus, can be solved for only 2 of the 3 \( N^\alpha \)’s. But then in both of these cases, a linear combination of the \( N^\alpha \)’s remains arbitrary, and counterbalances the extra independent acceleration. Thus, in all Bianchi Types, 4 arbitrary functions of time enter the general solution to the set of equations (4.4.5). Based on the intuition gained from the full theory, one could expect this fact to be a reflection of the only known covariance of the theory; i.e. of the freedom to make arbitrary changes of the time and space coordinates.

The rest of this section is devoted to the investigation of the existence, uniqueness, and properties of general coordinate transformations—containing 4 arbitrary functions of time—, which on the one hand, must preserve the manifest spatial homogeneity, of the line element (4.4.4b), and on the other hand, must be symmetries of equations (4.4.3). As far as time reparametrization is concerned the situation is pretty clear: If a transformation:
\[ t \to \tilde{t} = g(t) \Leftrightarrow t = f(\tilde{t}) \quad (4.4.9a) \]
is inserted in the line element (4.4.4b), it is easily inferred that:
\[ \gamma_{\alpha \beta}(t) \to \gamma_{\alpha \beta}(f(\tilde{t})) \equiv \tilde{\gamma}_{\alpha \beta}(\tilde{t}) \quad (4.4.9b) \]
\[ N(t) \to \pm N(f(\tilde{t})) \frac{df(\tilde{t})}{d\tilde{t}} \equiv \tilde{N}(\tilde{t}) \quad (4.4.9c) \]
\[ N^\alpha(t) \to N^\alpha(f(\tilde{t})) \frac{df(\tilde{t})}{d\tilde{t}} \equiv \tilde{N}^\alpha(\tilde{t}) \quad (4.4.9d) \]
Accordingly, \( K^\alpha_\beta \) transforms under (4.4.9a) as a scalar and thus (4.4.5a), (4.4.5b) are also scalar equations while (4.4.5c) gets multiplied by a factor \( df(\tilde{t})/d\tilde{t} \). Thus, given a particular solution to equations (4.4.5), one can always obtain an equivalent solution, by arbitrarily redefining time. Hence, we understand the existence of one arbitrary function of time in the general solution to Einstein’s equations (4.4.3). In order to understand the presence of the rest 3 arbitrary functions of time it is natural to turn our attention
to the transformations of the 3 spatial coordinates \( x^i \). To begin with, consider the transformation:

\[
\tilde{t} = t \iff t = \tilde{t} \tag{4.4.10}
\]

\[
\tilde{x}^i = g^i(x^j, t) \iff x^i = f^i(\tilde{x}^j, \tilde{t}) \tag{4.4.11}
\]

It is here understood, that our previous assumption concerning the topology of \( G \) and the identification of \( \Sigma \) with \( G \), is valid for all values of the parameter \( t \), for which the transformation is to be well defined.

Under these transformations, the line element (4.4.4b) becomes:

\[
ds^2 = [(N^\alpha N_\alpha - N^2) + \frac{\partial \sigma^\alpha_i}{\partial t} \frac{\partial f^j}{\partial t} \sigma^\alpha_j(f) \gamma_{\alpha\beta}(\tilde{t})
+ 2\sigma^\alpha_i(f) \frac{\partial f^j}{\partial t} N_\alpha(\tilde{t})]d\tilde{t}^2
+ 2\sigma^\alpha_i(x) \frac{\partial x^i}{\partial \tilde{x}^m} [N_\alpha(\tilde{t}) + \sigma^\beta_j(x) \frac{\partial x^j}{\partial \tilde{x}^m} \gamma_{\alpha\beta}(\tilde{t})]d\tilde{x}^m d\tilde{t}
+ \sigma^\alpha_i(x) \sigma^\beta_j(x) \gamma_{\alpha\beta}(\tilde{t}) \frac{\partial x^i}{\partial \tilde{x}^m} \frac{\partial x^j}{\partial \tilde{x}^n} d\tilde{x}^m d\tilde{x}^n
\tag{4.4.12}
\]

Since our aim, is to retain manifest spatial homogeneity of the line element (4.4.4b), we have to refer the form of the line element given in (4.4.12) to the old basis \( \sigma^\alpha_i(x) \) at the new spatial point \( \tilde{x}^i \). Since \( \sigma^\alpha_i \) –both at \( x^i \) and \( \tilde{x}^i \), as well as, \( \partial x^i/\partial \tilde{x}^j \), are invertible matrices, there always exists a non-singular matrix \( \Lambda^\mu_\alpha(\tilde{x}, \tilde{t}) \) and a triplet \( P^\alpha(\tilde{x}, \tilde{t}) \), such that:

\[
\sigma^\alpha_i(x) \frac{\partial x^i}{\partial \tilde{x}^m} = \Lambda^\mu_\alpha(\tilde{x}, \tilde{t}) \sigma^\mu_m(\tilde{x}) \tag{4.4.16}
\]

\[
\sigma^\alpha_i(x) \frac{\partial x^i}{\partial \tilde{t}} = P^\alpha(\tilde{x}, \tilde{t}) \tag{4.4.17}
\]

The above relations, must be regarded as definitions, for the matrix \( \Lambda^\mu_\alpha \) and the triplet \( P^\alpha \). With these identifications the line element (4.4.12) assumes the form:

\[
ds^2 = [(N^\alpha N_\alpha - N^2) + P^\alpha(\tilde{x}, \tilde{t}) P^\beta(\tilde{x}, \tilde{t}) \gamma_{\alpha\beta}(\tilde{t}) + 2P^\alpha(\tilde{x}, \tilde{t}) N_\alpha(\tilde{t})]d\tilde{t}^2
+ 2\Lambda^\mu_\alpha(\tilde{x}, \tilde{t}) \sigma^\mu_m(\tilde{x}) [N_\alpha(\tilde{t}) + P^\beta(\tilde{x}, \tilde{t}) \gamma_{\alpha\beta}(\tilde{t})]d\tilde{x}^m d\tilde{t}
+ \Lambda^\mu_\alpha(\tilde{x}, \tilde{t}) \Lambda^\beta_\gamma(\tilde{x}, \tilde{t}) \sigma^\mu_m(\tilde{x}) \sigma^\nu_n(\tilde{x}) d\tilde{x}^m d\tilde{x}^n \tag{4.4.18}
\]

If, following the spirit of (4.4.4), we wish the transformation (4.4.10) to be manifest homogeneity preserving i.e. to have a well defined, non-trivial action on \( \gamma_{\alpha\beta}(t) \), \( N(t) \) and \( N^\alpha(t) \), we must impose the condition that \( \Lambda^\mu_\alpha(\tilde{x}, \tilde{t}) \) and \( P^\alpha(\tilde{x}, \tilde{t}) \) do not depend on the spatial point \( \tilde{x} \), i.e. \( \Lambda^\mu_\alpha = \Lambda^\mu_\alpha(\tilde{t}) \) and \( P^\alpha = P^\alpha(\tilde{t}). \) Then (4.4.18) is written as:

\[
ds^2 = [(N^\alpha N_\alpha - N^2) + P^\alpha P^\beta \gamma_{\alpha\beta} + 2P^\alpha N_\alpha]d\tilde{t}^2
+ 2\Lambda^\mu_\alpha \sigma^\mu_m \gamma_{\alpha\beta} \sigma^\beta_n d\tilde{x}^m d\tilde{t}
+ \Lambda^\mu_\alpha \Lambda^\beta_\gamma \sigma^\mu_m \sigma^\nu_n d\tilde{x}^m d\tilde{x}^n \Rightarrow
\]

\[
ds^2 \equiv (\tilde{N}^\alpha \tilde{N}_\alpha - \tilde{N}^2) d\tilde{t}^2 + 2\tilde{N}_\alpha(\tilde{t}) \sigma^\alpha(\tilde{x}) d\tilde{x}^i d\tilde{t}
+ \tilde{\gamma}_{\alpha\beta}(\tilde{t}) \sigma^\alpha(\tilde{x}) \sigma^\beta(\tilde{x}) d\tilde{x}^i d\tilde{x}^j \tag{4.4.24}
\]
with the allocations:

\[
\tilde{\gamma}_{\alpha\beta} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \gamma_{\mu\nu} \tag{4.4.26a}
\]

\[
\tilde{N}_\alpha = \Lambda^\beta_\alpha (N_\beta + P^\rho \gamma_{\rho\beta}) \text{ and thus } \tilde{N}^\alpha = S^\alpha_\beta (N_\beta + P^\beta) \tag{4.4.26b}
\]

\[
\tilde{N} = N \tag{4.4.26c}
\]

(where \(S = \Lambda^{-1}\)).

Of course, the demand that \(\Lambda^\alpha_\beta\) and \(P^\alpha\) must not depend on the spatial point \(\tilde{x}^i\), changes the character of (4.4.16), from identities, to the following set of differential restrictions on the functions defining the transformation:

\[
\frac{\partial f^i}{\partial \tilde{x}^m} = \sigma^i_\alpha(f) \Lambda^\alpha_\beta(\tilde{t}) \sigma^\beta_m(\tilde{x}) \tag{4.4.27a}
\]

\[
\frac{\partial f^i}{\partial \tilde{t}} = \sigma^i_\alpha(f) P^\alpha(\tilde{t}) \tag{4.4.27b}
\]

Equations (4.4.27) constitute a set of first-order highly non-linear P.D.E.’s for the unknown functions \(f^i\). The existence of local solutions to these equations is guaranteed by Frobenius theorem \(\cite{13}\) as long as the necessary and sufficient conditions:

\[
\frac{\partial}{\partial \tilde{x}^j} \left( \frac{\partial f^i}{\partial \tilde{x}^m} \right) - \frac{\partial}{\partial \tilde{x}^m} \left( \frac{\partial f^i}{\partial \tilde{x}^j} \right) = 0
\]

\[
\frac{\partial}{\partial \tilde{t}} \left( \frac{\partial f^i}{\partial \tilde{x}^m} \right) - \frac{\partial}{\partial \tilde{x}^m} \left( \frac{\partial f^i}{\partial \tilde{t}} \right) = 0
\]

hold. Through (4.4.27) and repeated use of (4.4.4a), these equations reduce respectively to:

\[
\Lambda^\alpha_\beta C^\mu_\gamma = \Lambda^\beta_\gamma \Lambda^\alpha C^\mu_\sigma \tag{4.4.28}
\]

\[
P^\mu C^\alpha_\nu \Lambda_\beta^\nu = \frac{1}{2} \Lambda^\alpha_\beta \tag{4.4.29}
\]

It is noteworthy that the solutions to (4.4.28) and (4.4.29), –by virtue of (4.4.26)– form a group, with composition law:

\[
(\Lambda_3)^\alpha_\beta = (\Lambda_1)^\alpha_\epsilon (\Lambda_2)^\epsilon_\beta
\]

\[
(P_3)^\alpha = (\Lambda_1)^\alpha_\beta (P_2)^\beta + (P_1)^\alpha
\]
where $(\Lambda_1, P_1)$ and $(\Lambda_2, P_2)$, are two successive transformations of the form (4.4.26).

Note also, that a constant automorphism is always a solution of (4.4.28), (4.4.29); indeed, $\Lambda^\alpha(t) = \Lambda^\alpha_0$ and $P^a(t) = 0$ solve these equations. Thus, $\Lambda^\alpha_0$ and $P^a = 0$ can be regarded as the remaining gauge symmetry, after one has fully used the arbitrary functions of time, appearing in a solution $\Lambda^\alpha_\beta(t)$ and $P^{a}(t)$. Consequently one can, at first sight, regard all the arbitrary constants encountered when integrating (4.4.28), as absorbable in the shift, since the transformation law for the shift, is then tensorial. This is certainly true, as long as there is a non zero initial shift. However, if one has used the independent functions of time, in order to set the shift zero, then the constants remaining within $\Lambda^\alpha_\beta$, are not absorbable. It is this kind of constants that are explicitly present in “T. Christodoulakis, G. Kofinas, E. Korfiatis, G. O. Papadopoulos and A. Paschos, J. Math. Phys. 42 (2001) 3580-3608, gr-qc/0008050”. Where the solutions to (4.4.28), (4.4.29) for all Bianchi Types are given. A relevant nice discussion, distinguishing between genuine gauge symmetries (cf. arbitrary functions of time) and rigid symmetries (cf. arbitrary constants), is presented in [14]. There a different definition of manifest homogeneity preserving diffeomorphisms –stronger than the one adopted in this work– is used, and results in only the inner automorphisms being allowed to acquire $t$ dependence. In connection to this, it is interesting to observe that (4.4.28), (4.4.29) give essentially the same results: notice that $2P^\mu C^\alpha_{\mu\beta}$ is, by definition, the generator of Inner Automorphisms. Thus there is always a $\lambda^\alpha_\beta(t) \equiv \text{Exp}(2P^\mu C^\alpha_{\mu\beta}) \in \text{IAut}(G)$ satisfying (4.4.29). If we now parameterize the general solution to (4.4.28, 4.4.29) by $\Lambda^\alpha_\beta(t) = \lambda^\alpha_\beta(t) U^\beta_\gamma(t)$ and substitute in these relations, we deduce that the matrix $U$ is a constant automorphism. This analysis is verified in the explicit solutions to (4.4.28, 4.4.29), presented in references quoted above.

### 4.2 Automorphisms, Invariant Description of 3-spaces, and Quantum Cosmology

As it is well known, the quantum cosmology approximation consists in freezing out all but a finite number of degrees of freedom of the gravitational field and quantize the rest. This is done by imposing spatial homogeneity. Thus, our –in principle– dynamical variables are the scale factors $\gamma^{\alpha\beta}(t)$, of some spatially homogeneous geometry.

The basic object of the theory, is the wave function $\Psi$, which must describe the quantum evolution of the 3-geometry. Consequently, the wave function, will –in principle– depend on the 6 $\gamma^{\alpha\beta}$’s. Hence, a question naturally arises; whether all different $\gamma^{\alpha\beta}$ matrices, are characterizing different 3-geometries, or not. The answer to this question, involves the A.I.D.s of the previous section, with the difference that time does not concern us. Thus, the frozen analogue of (4.4.10) will lead us to (4.4.26) (\Lambda being now, constant) and the integrability condition (4.4.28).

So, any two matrices $\gamma^{(1)}_{\alpha\beta}$, $\gamma^{(2)}_{\alpha\beta}$, connected by an element of the automorphism group $\Lambda^a_\beta$ (for an arbitrary albeit given Bianchi Type) i.e. satisfying $\gamma^{(2)}_{\alpha\beta} = \Lambda^a_\alpha \Lambda^a_\beta \gamma^{(1)}_{\alpha\beta}$, represent the same 3-geometry.
The existence of these A.I.D.'s has very important implications for the wave functions of a given Bianchi geometry: as we have proven, points in the configuration space – spanned by the $\gamma_{\alpha\beta}$'s, named $\Delta$ – that are related through an automorphism, correspond to spatial line elements that are G.C.T. related and thus geometrically identifiable. Thus, if we want our wave-functions to depend only on the Geometry of the three-space and not on the spatial coordinate system, we must assume them to be annihilated by the generators of the entire Automorphism Group and not just by the constraint vector fields $H_\rho$, which generate only the so-called inner-automorphisms, i.e. we have to demand:

$$\hat{X}_i \Psi \equiv \lambda_{(i)\mu}^{\rho} \frac{\partial \Psi}{\partial \gamma_{\mu\nu}} = 0 \quad (4.4.30)$$

where: $\lambda_{(i)\mu}^{\rho} \equiv (C^\rho_{(i)\alpha\beta}, \varepsilon_{(i)\alpha\beta})$ are the generators of (the connected to the identity component of) the Automorphism group and $(i)$ labels the different generators. Depending on the particular Bianchi Type, the vector fields (in $\Delta$) $X_{(i)}$ may also include, except of the $H_\rho$'s, the generators of the outer-automorphisms: $E_j \equiv \varepsilon_{(j)\rho}^{\sigma\tau} \frac{\partial}{\partial \gamma_{\rho\sigma\tau}}$.

Using the method of characteristics, the solutions to the set $(4.4.30)$, can be found to have the form:

$$\Psi = \Psi(q^i)$$

where:

$$q^1(C_{\mu\nu}^\alpha, \gamma_{\alpha\beta}) = \frac{m^{a\beta} \gamma_{a\beta}}{\sqrt{\gamma}} \quad (4.4.31a)$$

$$q^2(C_{\mu\nu}^\alpha, \gamma_{\alpha\beta}) = \frac{(m^{a\beta} \gamma_{a\beta})^2}{2\gamma} - \frac{1}{4} C^a_{\mu\kappa} C^\beta_{\nu\lambda} \gamma_{a\beta} \gamma_{\mu\nu} \gamma_{\kappa\lambda} \quad (4.4.31b)$$

$$q^3(C_{\mu\nu}^\alpha, \gamma_{\alpha\beta}) = \frac{m}{\sqrt{\gamma}} \quad (4.4.31c)$$

These three quantities, serve to invariantly describe the geometry and one can prove the following relevant proposition (T. Christodoulakis, E. Korfiatis & G.O. Papadopoulos, gr-qc/0107050):

Let $\gamma_{\alpha\beta}^{(1)}$, $\gamma_{\alpha\beta}^{(2)}$, $\in \Delta$, and $C_{\mu\nu}^a$ be the structure constants of a given Bianchi Type. If $q^i(\gamma_{\alpha\beta}^{(1)}, C) = q^i(\gamma_{\alpha\beta}^{(2)}, C)$ $(i = 1, 2, 3)$, then there is $\Lambda$ such that $\gamma_{\alpha\beta}^{(2)} = \Lambda_{\alpha}^{a} \gamma_{a\beta}^{(1)}$ and $\Lambda \in Aut(G)$ i.e. $C_{\mu\nu}^a \Lambda_{\rho}^{\alpha} = \Lambda_{\mu}^{\kappa} \Lambda_{\nu}^{\lambda} C_{\kappa\lambda}^a$.

It is important to notice that the reduction from a 6-dim configuration space – spanned by the $\gamma_{\alpha\beta}$ – to a space spanned by the $q$’s, is achieved solely by kinematical considerations, i.e. the action of G.C.T.’s. Further specification of the theory, may occur only through the considerations of the dynamics i.e. the Wheeler-DeWitt equation.

We close this presentation, by giving two examples. We firstly consider the (see T. Christodoulakis, G. O. Papadopoulos, Phys. Lett. B 501 (2001) 264-8):
4.2.1 Quantization of the most general Bianchi Type II Vacuum Cosmologies

In [15], we had considered the quantization of an action corresponding to the most general Bianchi Type II cosmology, i.e. an action giving Einstein’s Field Equations, derived from the line element:

$$ds^2 = (N^2(t) - N_a(t)N^a(t))dt^2 + 2N_a(t)\sigma^a_i(x)dx^i dt + \gamma_{\alpha\beta}(t)\sigma^\alpha_i(x)\sigma^\beta_j(x)dx^idx^j$$

(4.4.32)

with:

$$\sigma^\alpha(x) = \sigma^\alpha_i(x)dx^i$$
$$\sigma^1(x) = dx^2 - x^1dx^3$$
$$\sigma^2(x) = dx^3$$
$$\sigma^3(x) = dx^1$$

$$d\sigma^\alpha(x) = \frac{1}{2}C^\alpha_{\beta\gamma}\sigma^\beta \wedge \sigma^\gamma$$

$$C^1_{23} = -C^1_{32} = 1$$

(4.4.33)

see [16].

As is well known [17], the Hamiltonian is

$$H = \tilde{N}(t)H_0 + N^a(t)H_a$$

where:

$$H_0 = \frac{1}{2}L_{\alpha\beta\mu\nu}\pi^\alpha\pi^\mu + \gamma R$$

(4.4.34)

is the quadratic constraint with:

$$L_{\alpha\beta\mu\nu} = \gamma_{\alpha\mu}\gamma_{\beta\nu} + \gamma_{\alpha\nu}\gamma_{\beta\mu} - \gamma_{\alpha\beta}\gamma_{\mu\nu}$$
$$R = C^\beta_{\lambda\mu} C^\alpha_{\theta\tau}\gamma_{\alpha\beta}\gamma^\tau_{\mu} + 2C^\alpha_{\beta\mu} C^\tau_{\nu\alpha}\gamma^\beta_{\nu} + 4C^\mu_{\nu\mu} C^\beta_{\lambda\nu}\gamma^\alpha_{\mu\nu}\gamma^{\mu\lambda}$$

(4.4.35)

$$\gamma$$ being the determinant of $$\gamma_{\alpha\beta}$$ (the last equality holding only for the Type II case), and:

$$H_a = C^\mu_{\alpha\rho} \gamma_{\beta\mu}\pi^\beta\rho$$

(4.4.36)

are the linear constraints. Note that $$\tilde{N}$$ appearing in the Hamiltonian, is to be identified with $$N/\sqrt{\gamma}$$.

The quantities $$H_0, H_a$$, are weakly vanishing [5], i.e. $$H_0 \approx 0, H_a \approx 0$$. For all class A Bianchi Types ($$C^\alpha_{\alpha\beta} = 0$$), they can be seen to obey the following first-class algebra:

$$\{H_0, H_0\} = 0$$
$$\{H_0, H_a\} = 0$$
$$\{H_a, H_\beta\} = -\frac{1}{2}C^\gamma_{\alpha\beta}H_\gamma$$

(4.4.37)

which ensures their preservation in time i.e. $$\dot{H}_0 \approx 0, \dot{H}_a \approx 0$$ and establishes the consistency of the action.

If we follow Dirac’s general proposal [5] for quantizing this action, we have to turn $$H_0, H_a$$, into operators annihilating the wave function $$\Psi$$.
In the Schrödinger representation:
\[
\gamma_{\alpha\beta} \rightarrow \hat{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta}
\]
\[
\pi^{\alpha\beta} \rightarrow \hat{\pi}^{\alpha\beta} = -i \frac{\partial}{\partial \gamma_{\alpha\beta}}
\]
(4.4.38)
satisfying the basic Canonical Commutation Relation (CCR) –corresponding to the classical ones:
\[
[\hat{\gamma}_{\alpha\beta}, \hat{\pi}^{\mu\nu}] = -i \delta^{\mu\nu}_{\alpha\beta}
\]
(4.4.39)

The quantum version of the 2 independent linear constraints has been used to reduce, via the method of characteristics \[18\], the dimension of the initial configuration space from 6 ($\gamma_{\alpha\beta}$) to 4 (combinations of $\gamma_{\alpha\beta}$), i.e. $\Psi = \Psi(q, \gamma, \gamma^{2}_{12} - \gamma^{2}_{11}\gamma^{2}_{22}, \gamma^{2}_{12}\gamma^{2}_{13} - \gamma^{2}_{11}\gamma^{2}_{23})$, where $q = C^{\alpha\mu}_{\rho\lambda} C^{\beta\nu}_{\rho\lambda} \gamma^{\mu\nu}_{\alpha\beta} \gamma^{\rho\lambda}_{\nu\lambda}$.

According to Kuchař’s and Hajicek’s \[19\] prescription, the ‘kinetic’ part of $H_{0}$ is to be realized as the conformal Laplacian, corresponding to the reduced metric:
\[
L_{\alpha\beta\mu\nu} \frac{\partial x^{i}}{\partial \gamma^{\alpha\beta}} \frac{\partial x^{j}}{\partial \gamma^{\mu\nu}} = g^{ij}
\]
(4.4.40)
where $x^{i}, i = 1, 2, 3, 4$, are the arguments of $\Psi$. The solutions had been presented in \[15\]. Note that the first-class algebra satisfied by $H_{0}, H_{a}$, ensures that indeed, all components of $g^{ij}$ are functions of the $x^{i}$’s. The signature of the $g^{ij}$, is $(+, +, -, -)$ signaling the existence of gauge degrees of freedom among the $x^{i}$’s.

Indeed, one can prove \[20\] that the only gauge invariant quantity which, uniquely and irreducibly, characterizes a 3-dimensional geometry admitting Type II symmetry group, is:
\[
q = C^{\alpha\mu}_{\rho\lambda} C^{\beta\nu}_{\rho\lambda} \gamma^{\mu\nu}_{\alpha\beta} \gamma^{\rho\lambda}_{\nu\lambda}
\]
(4.4.41)

An outline of the proof, is as follows:
Let two hexads $\gamma^{(1)}_{\alpha\beta}$ and $\gamma^{(2)}_{\alpha\beta}$ be given, such that their corresponding $q$’s, are equal. Then according to the result given at the end of the previous section \[20\], there exists an automorphism matrix $\Lambda$ (i.e. satisfying $C^{\alpha}_{\mu\nu} \Lambda_{\alpha}^{\kappa} = C^{\kappa}_{\rho\sigma} \Lambda^{\rho}_{\mu} \Lambda^{\sigma}_{\nu}$) connecting them, i.e. $\gamma^{(1)}_{\alpha\beta} = \Lambda^{\mu}_{\alpha} \gamma^{(2)}_{\beta} \Lambda^{\nu}_{\beta} \Lambda^{\rho}_{\nu}$. But as it had been shown in the appendix of \[21\], this kind of changes on $\gamma_{\alpha\beta}$, can be seen to be induced by spatial diffeomorphisms. Thus, 3-dimensional Type II geometry, is uniquely characterized by some value of $q$.

Although for full pure gravity, Kuchař \[22\] has shown that there are not other first-class functions, homogeneous and linear in $\pi^{a\beta}$, except $H_{a}$, imposing the extra symmetries (Type II), allows for such quantities to exist –as it will be shown. We are therefore, naturally led to seek the generators of these extra symmetries –which are expected to chop off $x^{2}, x^{3}, x^{4}$. Such quantities are, generally, called in the literature ‘Conditional Symmetries’.
The automorphism group for Type II, is described by the following 6 generators—in matrix notation and collective form:

\[
\lambda_{(I)\beta}^a = \begin{pmatrix}
\kappa + \mu & x & y \\
0 & \kappa & \rho \\
0 & \sigma & \mu \\
\end{pmatrix}
\]  

(4.4.42)

with the property:

\[
C_{\mu\nu}^a \lambda^\kappa_a = C_{\mu\sigma}^\kappa \lambda^\sigma_a + C_{\sigma\nu}^\kappa \lambda^\mu_a
\]  

(4.4.43)

¿From these matrices, we can construct the linear—in momenta—quantities:

\[
A_{(I)} = \lambda_{(I)\beta}^a \gamma_{\alpha\rho} \pi^{\rho\beta}
\]  

(4.4.44)

Two of these, are the \(H_a\)'s since \(C_{(\rho)\beta}^a\) correspond to the inner automorphism subgroup—designated by the x and y parameters, in \(\lambda_{(I)\beta}^a\). The rest of them, are the generators of the outer automorphisms and are described by the matrices:

\[
\varepsilon_{(I)\beta}^a = \begin{pmatrix}
\kappa + \mu & 0 & 0 \\
0 & \kappa & \rho \\
0 & \sigma & \mu \\
\end{pmatrix}
\]  

(4.4.45)

The corresponding—linear in momenta—quantities, are:

\[
E_{(I)} = \varepsilon_{(I)\beta}^a \gamma_{\alpha\rho} \pi^{\rho\beta}
\]  

(4.4.46)

The algebra of these—seen as functions on the phase space, spanned by \(\gamma_{\alpha\beta}\) and \(\pi^{\mu\nu}\)—is:

\[
\{E_I, E_J\} = \tilde{C}_{I\beta}^K E_K \\
\{E_I, H_a\} = -\frac{1}{2} \lambda_{(I)\beta}^a H_\beta \\
\{E_I, H_0\} = -2(\kappa + \mu) \gamma R
\]  

(4.4.47)

¿From the last of (4.4.47), we conclude that the subgroup of \(E_I\)'s with the property \(\kappa + \mu = 0\), i.e. the traceless generators, are first-class quantities; their time derivative vanishes. So let:

\[
\tilde{E}_I = \{E_I : \kappa + \mu = 0\}
\]  

(4.4.48)

Then, the previous statement translates into the form:

\[
\tilde{E}_I = 0 \Rightarrow \tilde{E}_I = c_I
\]  

(4.4.49)

the \(c_I\)'s being arbitrary constants.

Now, these are—in principle—integrals of motion. Since, as we have earlier seen, \(\tilde{E}_I\)'s along with \(H_a\)'s, generate automorphisms, it is natural to promote the integrals of
motion (4.4.49), to symmetries –by setting the \(c_I\)'s zero. The action of the quantum version of these \(\tilde{E}_I\)'s on \(\Psi\), is taken to be [19]:

\[
\tilde{E}_I \Psi = \varepsilon^a_{(I)} \gamma_{\alpha^I \rho} \partial_{\gamma^I \rho} \Rightarrow \Psi = \Psi(q, \gamma)
\]  

(4.4.50)

The Wheeler-DeWitt equation now, reads:

\[
5q^2 \frac{\partial^2 \Psi}{\partial q^2} - 3\gamma^2 \frac{\partial^2 \Psi}{\partial \gamma^2} + 2q \gamma \frac{\partial^2 \Psi}{\partial \gamma \partial q} + 5q \frac{\partial \Psi}{\partial q} - 3\gamma \frac{\partial \Psi}{\partial \gamma} - 2q \gamma \Psi = 0
\]  

(4.4.51)

Note that:

\[
\nabla^2_c = \nabla^2 + \frac{(d - 2)}{4(d - 1)} R = \nabla^2
\]  

(4.4.52)

since we have a 2-dimensional, flat space, with contravariant metric:

\[
g^{ij} = \begin{pmatrix}
5q^2 & q \gamma \\
q \gamma & -3\gamma^2
\end{pmatrix}
\]  

(4.4.53)

which is Lorentzian. This equation, can be easily solved by separation of variables; transforming to new coordinates \(u = q \gamma^3\) and \(v = q \gamma\), we get the 2 independent equations:

\[
16u^2 A''(u) + 16u A'(u) - cA(u) = 0
\]

\[
B''(v) + \frac{1}{v} B'(v) - \left(\frac{1}{2v} + \frac{c}{3v^2}\right) B(v) = 0
\]  

(4.4.54)

where \(c\), is the separation constant. Equation (4.4.51), is of hyperbolic type and the resulting wave function will still not be square integrable. Besides that, the tracefull generators of the outer automorphisms, are left inactive –due to the non vanishing CCR with \(H_0\).

These two facts, lead us to deduce that there must still exist a gauge symmetry, corresponding to some –would be, linear in momenta– first-class quantity. Our starting point in the pursuit of this, is the third of (4.4.47). It is clear that we need another quantity –also linear in momenta– with an analogous property; the trace of \(\pi^{\mu \nu}\), is such an object. We thus define the following quantity:

\[
T = E_I - (\kappa + \mu) \gamma_{\alpha \beta} \pi^{\alpha \beta}
\]  

(4.4.55)

in the phase space –spanned by \(\gamma_{\alpha \beta}\) and \(\pi^{\mu \nu}\). It holds that:

\[
\{T, H_0\} = 0
\]

\[
\{T, H_a\} = 0
\]

\[
\{T, E_I\} = 0
\]

(4.4.56)

because of:

\[
\{E_I, \gamma\} = -2(\kappa + \mu) \gamma
\]

\[
\{E_I, q\} = 0
\]

\[
\gamma_{\alpha \beta} \{\pi^{\alpha \beta}, q\} = q
\]

\[
\gamma_{\alpha \beta} \{\pi^{\alpha \beta}, \gamma\} = -3\gamma
\]  

(4.4.57)
Again—as for $\tilde{E}_I$'s—, we see that since $T$, is first-class, we have that:

$$\dot{T} = 0 \Rightarrow T = \text{const} = c_T \tag{4.4.58}$$

another integral of motion. We therefore see, that $T$ has all the necessary properties to be used in lieu of the tracefull generator, as a symmetry requirement on $\Psi$. In order to do that, we ought to set $c_T$ zero—exactly as we did with the $c_I$'s, corresponding to $\tilde{E}_I$'s.

The quantum version of $T$, is taken to be:

$$\hat{T} = \lambda^\alpha_\beta \gamma_{\alpha\rho} \frac{\partial}{\partial \gamma^\rho_{\beta}} - (\kappa + \mu) \gamma_{\alpha\beta} \frac{\partial}{\partial \gamma_{\alpha\beta}} \tag{4.4.59}$$

Following, Dirac’s theory, we require:

$$\hat{T}\Psi = \lambda^\alpha_\beta \gamma_{\alpha\rho} \frac{\partial\Psi}{\partial \gamma^\rho_{\beta}} - (\kappa + \mu) \gamma_{\alpha\beta} \frac{\partial\Psi}{\partial \gamma_{\alpha\beta}} = (\kappa + \mu)(q \frac{\partial\Psi}{\partial q} - \gamma \frac{\partial\Psi}{\partial \gamma}) = 0 \tag{4.4.60}$$

Equation (4.4.60), implies that $\Psi(q, \gamma) = \Psi(q\gamma)$ and thus equation (4.4.51), finally, reduces to:

$$4w^2 \Psi''(w) + 4w \Psi'(w) - 2w \Psi = 0 \tag{4.4.61}$$

where, for simplicity, $w = q\gamma$. The solution to this equation, is:

$$\Psi = c_1 I_0(\sqrt{2q\gamma}) + c_2 K_0(\sqrt{2q\gamma}) \tag{4.4.62}$$

where $I_0$ is the modified Bessel function, of the first kind, and $K_0$ is the modified Bessel function, of the second kind, both with zero argument.

At first sight, it seems that although we have apparently exhausted the symmetries of the system, we have not yet been able to obtain a wave function on the space of the 3-geometries, since $\Psi$ depends on $q\gamma$ and not on $q$ only. On the other hand, the fact that we have achieved a reduction to one degree of freedom, must somehow imply that the wave function found must be a function of the geometry. This puzzle finds its resolution as follows. Consider the quantity:

$$\Omega = -2\gamma_{\rho\sigma} \pi^{\rho\sigma} + \frac{2C^\alpha_\mu\kappa C^\beta_\nu\lambda \gamma_{\kappa\lambda \mu\nu} \gamma_{\alpha\rho} \gamma_{\beta\sigma} - 4C^\alpha_\mu\rho C^\beta_\nu\sigma \gamma_{\alpha\beta} \gamma_{\mu\nu}}{q} \pi^{\rho\sigma} \tag{4.4.63}$$

This can also be seen to be first-class, i.e.

$$\dot{\Omega} = 0 \Rightarrow \Omega = \text{const} = c_\Omega \tag{4.4.64}$$

Moreover, it is a linear combination of $T$, $\tilde{E}_I$'s, and $H_a$'s, and thus $c_\Omega = 0$. Now it can be verified that $\Omega$, is nothing but:

$$\frac{1}{N(t)} \left( \frac{\dot{\gamma}}{\gamma} + \frac{1}{3} \frac{\dot{q}}{q} \right) \tag{4.4.65}$$
So:

\[ \gamma q^{1/3} = \vartheta = constant \]  

(4.4.66)

Without any loss of generality, and since \( \vartheta \) is not an essential constant of the classical system (see [23] and reference [18] therein), we set \( \vartheta = 1 \). Therefore:

\[ \Psi = c_1 I_0(\sqrt{2}q^{1/3}) + c_2 K_0(\sqrt{2}q^{1/3}) \]  

(4.4.67)

where \( I_0 \) is the modified Bessel function, of the first kind, and \( K_0 \) is the modified Bessel function, of the second kind, both with zero argument.

As for the measure, it is commonly accepted that, there is not a unique solution. A natural choice, is to adopt the measure that makes the operator in (4.4.61), hermitian –that is:

\[ \mu(q) \propto q^{-1} \]  

(4.4.68)

It is easy to find combinations of \( c_1 \) and \( c_2 \) so that the probability \( \mu(q) |\Psi|^2 \), be defined.

Note that putting the constant associated with \( \Omega \), equal to zero, amounts in restricting to a subset of the classical solutions, since \( c_\Omega \), is one of the two essential constants of Taub’s solution. One could keep that constant, at the expense of arriving at a wave function with explicit time dependence, since then:

\[ \gamma = q^{-1/3} \exp[\int c_\Omega N(t)dt] \]  

(4.4.69)

We however, consider more appropriate to set that constant zero, thus arriving at a \( \Psi \) depending on \( q \) only, and decree its applicability to the entire space of the classical solutions. Anyway this is not such a blunder, since \( \Psi \) is to give weight to all states, –being classical ones, or not.

And the last example (see T. Christodoulakis, G. Gakis & G. O. Papadopoulos, gr-qc/0106065):

4.2.2 Conditional Symmetries and the Quantization of Bianchi Type I Vacuum Cosmologies with Cosmological Constant

Note: The original work, deals with both the cases; the models with a vanishing and those with non vanishing cosmological constant.

The case of Bianchi Type I geometries, has been repeatedly treated in the literature –both at the classical level [23] and the quantum level [26]. The main reason, is the simplicity brought by the vanishing structure constants, i.e. the high spatial symmetry of the model. Thus, the most general of these models, is described by the 6 scale factors \( \gamma_{\alpha\beta}(t) \) and the lapse function \( N(t) \) –the shift vector \( N^\alpha(t) \), being absent due to the non existence of the \( H_\alpha \)'s (linear constraints). The absence of \( H_\alpha \)'s presents –at first sight– the
complication that no reduction of the initial configuration space, is possible –in contrast to what happens in other Bianchi Types \cite{25}.

In what follows, we present a complete reduction of the initial configuration space for Bianchi Type I geometry, when the cosmological constant is present. A wave function, which depends on one degree of freedom, is found.

As is well known (first of \cite{17}) the Hamiltonian of the above system is:

\[
H = \tilde{N}(t)H_0 + N^\alpha(t)H_\alpha
\]

where:

\[
H_0 = \frac{1}{2}L_{\alpha\beta\mu\nu}\pi^{\alpha\beta}\pi^{\mu\nu} + \gamma \Lambda
\]  \hspace{1cm} (4.4.70)

Thus, the only operator which must annihilate the wave function, is \(\hat{H}_0\); and the Wheeler-DeWitt equation \(\hat{H}_0\Psi = 0\), will produce a wave function, initially residing on a 6-dimensional configuration space –spanned by \(\gamma_{\alpha\beta}\)'s. The discussion however, does not end here. If the linear constraints existed, a first reduction of the initial configuration space, would take place \cite{19}. New variables, instead of the 6 scale factors, would emerge –say \(q^i\), with \(i < 6\). Then a new ”physical” metric would be induced:

\[
g^{ij} = L_{\alpha\beta\mu\nu}\frac{\partial q^i}{\partial \gamma_{\alpha\beta}}\frac{\partial q^j}{\partial \gamma_{\mu\nu}}
\]  \hspace{1cm} (4.4.71)

According to Kuchař’s and Hajicek’s \cite{19} prescription, the ”kinetic” part of \(H_0\) would have to be realized as the conformal Laplacian (in order for the equation to respect the conformal covariance of the classical action), based on the physical metric \((4.4.71)\). In the presence of conditional symmetries, further reduction can take place, a new physical metric would then be defined similarly, and the above mentioned prescription, would have to be used after the final reduction \cite{28,22}.

The case of Bianchi Type I, is an extreme example in which all the linear constraints, vanish identically; thus no initial physical metric, exists –another peculiarity reflecting the high spatial symmetry of the model under consideration. In compensation, a lot of integrals of motion exist and the problem of reduction, finds its solution through the notion of ‘‘Conditional Symmetries’’.

The automorphism algebra of this Type, has been exhaustively treated in the literature –see e.g. \cite{27}. The relevant group, is that of the constant, real, 3 \(\times\) 3, invertible, matrices i.e. \(GL(3,\mathbb{R})\). The generators of this automorphism group, are (in a collective form and matrix notation) the following 9 –one for each parameter:

\[
\lambda^\alpha_{(I)\beta} = \begin{pmatrix}
\alpha & \beta & \delta \\
\epsilon & \zeta & \eta \\
\theta & \sigma & \rho
\end{pmatrix}, \quad I \in [1, \ldots, 9]
\]  \hspace{1cm} (4.4.72)

with the defining property:

\[
C_{\mu\nu}^{\alpha} \lambda^\kappa_\alpha = C_{\mu\alpha}^{\kappa} \lambda^\sigma_\nu + C_{\alpha\nu}^{\kappa} \lambda^\sigma_\mu.
\]  \hspace{1cm} (4.4.73)
Exponentiating all these matrices, one obtains the outer automorphism group of Type I.

For full pure gravity, Kuchař [22] has shown that there are no other first-class functions, homogeneous and linear in the momenta, except the linear constraints. If however, we impose extra symmetries (e.g. the Bianchi Type I –here considered), such quantities may emerge –as it will be shown. We are therefore –according to Dirac [5] –justified to seek the generators of these extra symmetries, whose quantum-operator analogues will be imposed as additional conditions on the wave function. Thus, these symmetries are expected to lead us to the final reduction, by revealing the true degrees of freedom. Such quantities are, generally, called in the literature ‘Conditional Symmetries’ [22]. From matrices (4.4.72), we can construct the linear –in momenta– quantities:

\[ E(I) = \lambda_\alpha^\beta \gamma_{\alpha \beta} \pi^{\alpha \beta} \]  

(4.4.74)

In order to write analytically these quantities, the following base is chosen:

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_8 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \lambda_9 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\end{align*}
\]

(4.4.75)

It is straightforward to calculate the Poisson Brackets of \( E(I) \) with \( H_0 \):

\[
\{ E(I), H_0 \} = -\gamma \Lambda \lambda_a^a
\]

(4.4.76)

But, it holds that:

\[
\dot{E}(I) = \{ E(I), H_0 \} = -\gamma \Lambda \lambda_a^a
\]

(4.4.77)

–the last equality emerging by virtue of (4.4.76). Thus:

\[
\dot{E}(I) = \{ E(I), H_0 \} = 0 \Rightarrow E(I) = K(I) = \text{constants}, \quad I \in [1, \ldots, 8]
\]

(4.4.78)

We therefore conclude that, the first eight quantities \( E(I) \), are first-class, and thus integrals of motion. Out of the eight quantities \( \dot{E}(I) \), only five are functionally independent (i.e. linearly independent, if we allow for the coefficients of the linear combination, to be functions of the \( \gamma_{\alpha \beta} \)s); numerically, they are all independent.

The algebra of \( E(I) \) can be easily seen to be:

\[
\{ E(I), E(J) \} = -\frac{1}{2} C_{IJ}^M E(M), \quad I, J, M \in [1, \ldots, 9]
\]

(4.4.79)
where:
\[
[\lambda(I), \lambda(J)] = C_{IJ}^{M} \lambda(M), \quad I, J, M \in [1, \ldots, 9]
\] (4.4.80)

the square brackets denoting matrix commutation.

The non-vanishing structure constants of the algebra (4.4.80), are found to be:

\[
\begin{align*}
C_{13}^{2} &= 1 & C_{15}^{4} &= -1 & C_{16}^{7} &= 1 & C_{18}^{1} &= -1 & C_{17}^{1} &= -2 \\
C_{24}^{1} &= 1 & C_{25}^{7} &= 1 & C_{26}^{8} &= -1 & C_{26}^{3} &= -1 & C_{27}^{2} &= -1 \\
C_{28}^{2} &= 1 & C_{34}^{8} &= -1 & C_{35}^{6} &= 1 & C_{37}^{3} &= 1 & C_{38}^{3} &= 2 \\
C_{46}^{5} &= 1 & C_{47}^{4} &= -1 & C_{48}^{4} &= -2 & C_{57}^{5} &= 1 & C_{58}^{5} &= -1 \\
C_{67}^{6} &= 2 & C_{68}^{6} &= 1
\end{align*}
\] (4.4.81)

At this point, in order to achieve the desired reduction, we propose that the quantities \(E(I)\) –with \(I \in [1, \ldots, 8]\)– must be promoted to operational conditions acting on the requested wave function \(\Psi\) –since they are first class quantities and thus integrals of motion (see (4.4.78)). In the Schrödinger representation:

\[
\hat{E}(I) \Psi = -i \lambda^{\gamma \beta} \frac{\partial \Psi}{\partial \gamma_{\alpha \beta}} = K(I) \Psi, \quad I \in [1, \ldots, 8]
\] (4.4.82)

In general, systems of equations of this type, must satisfy some consistency conditions (i.e. the Frobenius Theorem):

\[
\begin{align*}
\hat{E}(J) \Psi &= K(J) \Psi \Rightarrow \hat{E}(I) \hat{E}(J) \Psi &= K(I) K(J) \Psi \\
\hat{E}(I) \Psi &= K(I) \Psi \Rightarrow \hat{E}(J) \hat{E}(I) \Psi &= K(J) K(I) \Psi
\end{align*}
\] (4.4.83)

Subtraction of these two and usage of (4.4.79), results in:

\[
K_{IJ}^{M} \hat{E}(M) \Psi = 0 \Rightarrow C_{IJ}^{M} K_{M} = 0
\] (4.4.84)

i.e. a selection rule for the numerical values of the integrals of motion. Consistency conditions (4.4.84) and the Lie Algebra (4.4.81), impose that \(K_{1} = \ldots = K_{8} = 0\). If we also had \(E_{(9)}\) (as is the case when \(\Lambda = 0\)) then \(K_{9}\) would remain arbitrary. With this outcome, and using the method of characteristics, the system of the five functionally independent P.D.E.s (4.4.82), can be integrated. The result is:

\[
\Psi = \Psi(\gamma)
\] (4.4.85)

i.e. an arbitrary (but well behaved) function of \(\gamma\) –the determinant of the scale factor matrix.

A note is pertinent here; from basic abstract algebra, is well known that the basis of a linear vector space, is unique –modulo linear mixtures. Thus, although the form of the system (4.4.82) is base dependent, its solution (4.4.83), is base independent.

The next step, is to construct the Wheeler-DeWitt equation which is to be solved by the wave function (4.4.85). The degree of freedom, is 1; the \(q = \gamma\). According
to Kuchār’s proposal \[19\], upon quantization, the kinetic part of Hamiltonian is to be realized as the conformal Beltrami operator – based on the induced physical metric – according to (4.4.71), with \( q = \gamma \):

\[
g^{11} = L_{\alpha\beta\mu\nu} \frac{\partial\gamma}{\partial\gamma_{\alpha\beta}} \frac{\partial\gamma}{\partial\gamma_{\mu\nu}} = L_{\alpha\beta\mu\nu} \gamma^2 \gamma^{\alpha\beta} \gamma^{\mu\nu} = -3\gamma^2
\] (4.4.86)

In the Schrödinger representation:

\[
\frac{1}{2} L_{\alpha\beta\mu\nu} \pi^{\alpha\beta} \pi^{\mu\nu} \rightarrow -\frac{1}{2} c^2
\] (4.4.87)

where:

\[
c^2 = \Box = \frac{1}{\sqrt{g_{11}}} \partial_\gamma \{ \sqrt{g_{11}} g^{11} \partial_\gamma \}
\] (4.4.88)

is the 1–dimensional Laplacian based on \( g_{11} \) \((g_{11} g_{11} = 1)\). Note that in 1–dimension the conformal group is totally contained in the G.C.T. group, in the sense that any conformal transformation of the metric can not produce any change in the –trivial– geometry and is thus reachable by some G.C.T. Therefore, no extra term in needed in (4.4.88), as it can also formally be seen by taking the limit \( d = 1, R = 0 \) in the general definition:

\[
\Box_c \equiv \Box + \frac{(d-2)}{4(d-1)} R = \Box
\]

Thus:

\[
H_0 \rightarrow \hat{H}_0 = -\frac{1}{2} \left( -3\gamma^2 \partial^2 - 3\gamma \frac{\partial}{\partial\gamma} \right) + \Lambda \gamma
\] (4.4.89)

So, the Wheeler-DeWitt equation –by virtue of (4.4.85)–, reads:

\[
\hat{H}_0 \Psi = \gamma^2 \Psi'' + \gamma \Psi' + \frac{2}{3} \gamma \Lambda \Psi = 0
\] (4.4.90)

The general solution to this equation, is:

\[
\Psi(\gamma) = c_1 J_0(2\sqrt{\frac{2\gamma\Lambda}{3}}) + c_2 Y_0(2\sqrt{\frac{2\gamma\Lambda}{3}})
\] (4.4.91)

where \( J_n \) and \( Y_n \), are the Bessel Functions of the first and second kind respectively –both with zero argument– and \( c_1, c_2 \), arbitrary constants.

An important element for selecting the measure, is the conformal covariance; the supermetric \( L^{\alpha\beta\mu\nu} \) is known only up to rescalings, because instead of \( \tilde{N}(t) \) one can take any \( \tilde{N}(t) = \tilde{N}(t) e^{-2\omega} \) (with \( \omega = \omega(\gamma_{\alpha\beta}) \)) and consequently \( L^{\alpha\beta\mu\nu}(t) = L^{\alpha\beta\mu\nu}(t) e^{2\omega} \). This property, is also inherited to the physical metric (4.4.71) and is the reason for the Kuchār’s recipe, adopted in this work.
It is natural that the proposed measure \( \mu \), must be such that the probability density \( \mu | \Psi |^2 \), be invariant under these scalings. Recalling that \( \Psi = \Psi e^{(2-D)\omega/2} \), we conclude that \( \mu \) must scale as \( \mu = \mu e^{(D-2)\omega} \). The natural measure under which the Wheeler-DeWitt operator is hermitian, is \( \sqrt{\text{Det}(\text{physical metric})} \), but it scales as \( \sqrt{\text{Det}(\text{physical metric})} = \sqrt{\text{Det}(\text{physical metric})} e^{D\omega} \).

We are thus after a quantity \( \xi \) –preferably constant, (so that the hermiticity is preserved)– which scales as \( \xi = \xi e^{-2\omega} \). It is not difficult to imagine such a quantity: The inverse of any product of \( E_{(I)\alpha\beta} \) with \( E_{(J)\mu\nu} \) (where \( E_{(I)\alpha\beta} = 1/2(\lambda^\kappa_{\gamma\alpha\beta} + (\alpha \leftrightarrow \beta)) \)) has the desired property. Indeed the \( E_{(I)} \)'s do not scale at all, while the supermetric scales as mentioned before. The group metric \( \Theta_{IJ} = C^F_{IS}C^S_{JF} \) can serve to close the group indices of \( E_{(I)\alpha\beta} \). So, we arrive at the quantity:

\[
\xi = \frac{1}{L^{\alpha\beta\mu\nu} \Theta^{IJ} E_{(I)\alpha\beta} E_{(J)\mu\nu}}
\]

(where \( \Theta^{IJ} \) is the inverse of the group metric) having the desired property and being also a constant. Using the Lie algebra (4.4.81), one obtains:

\[
\Theta_{IJ} = C^F_{IS}C^S_{JF} = \begin{pmatrix}
0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 12 \\
0 & 0 & 0 & 0 & 12 & 6
\end{pmatrix}
\]

Thus:

\[
\Theta^{IJ} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/6 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/9 & -1/18 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/9 & -1/18 & 1/9
\end{pmatrix}
\]

After a straightforward calculation, one finds that:

\[
\xi = \frac{12}{5}
\]

The product of \( \xi \) with the respective natural measure, defines the final expression for the measure \( \mu \).
It is fair to say that the problem of selection of the "correct" measure, is not yet resolved; it is a reflection of the problem of time in Quantum Gravity [29].

Another issue that has not been touched upon is the problem of selecting a unique wave function. In the path integral approach, to quantum cosmology, there is the Hartle-Hawking "no boundary proposal" [30]. In the canonical approach, there are various forms of the Vilenkin proposal [31].

Finally, the problem of decoherence (i.e. of reconstruction of classical trajectories, from the knowledge of the wave function), has occupied several workers in the field [32].

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