EHRHART SERIES OF POLYTOPES RELATED TO SYMMETRIC DOUBLY-STOCHASTIC MATRICES

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Abstract. In Ehrhart theory, the \( h^* \)-vector of a rational polytope often provide insights into properties of the polytope that may be otherwise obscured. As an example, the Birkhoff polytope, also known as the polytope of real doubly-stochastic matrices, has a unimodal \( h^* \)-vector, but when even small modifications are made to the polytope, the same property can be very difficult to prove. In this paper, we examine the \( h^* \)-vectors of a class of polytopes containing real doubly-stochastic symmetric matrices.

1. Introduction

For a rational polytope \( \mathcal{P} \subseteq \mathbb{R}^n \) of dimension \( d \), consider the counting function \( \mathcal{L}_{\mathcal{P}}(m) = |m\mathcal{P} \cap \mathbb{Z}^n| \), where \( m\mathcal{P} \) is the \( m \)-th dilate of \( \mathcal{P} \). The Ehrhart series of \( \mathcal{P} \) is

\[
E_{\mathcal{P}}(t) := 1 + \sum_{m \in \mathbb{Z}_{\geq 1}} \mathcal{L}_{\mathcal{P}}(m) t^m.
\]

Let \( \text{den} \mathcal{P} \) denote the least common multiple of the denominators appearing in the coordinates of the vertices of \( \mathcal{P} \). Combining two well-known theorems due to Ehrhart [5] and Stanley [10], there exist values \( h^*_0, \ldots, h^*_k \in \mathbb{Z}_{\geq 0} \) with \( h^*_0 = 1 \) such that

\[
E_{\mathcal{P}}(t) = \frac{\sum_{j=0}^k h^*_j t^j}{(1 - t^{\text{den} \mathcal{P}})^{d+1}}.
\]

We say the polynomial \( h^*_\mathcal{P}(t) := \sum_{j=0}^k h^*_j t^j \) is the \( h^* \)-polynomial of \( \mathcal{P} \) (sometimes referred to as the \( \delta \)-polynomial of \( \mathcal{P} \)) and the vector of coefficients \( h^*(\mathcal{P}) \) is the \( h^* \)-vector of \( \mathcal{P} \). That \( E_{\mathcal{P}}(t) \) is of this rational form is equivalent to \( |m\mathcal{P} \cap \mathbb{Z}^n| \) being a quasipolynomial function of \( m \) of degree at most \( d \); the non-negativity of the \( h^* \)-vector is an even stronger property. If \( \text{den} \mathcal{P} \neq 1 \) then the form of \( E_{\mathcal{P}}(t) \) above may not be fully reduced, yet we still refer to the coefficients of this form when discussing \( h^*(\mathcal{P}) \). Even more tools are available when \( \mathcal{P} \) is a lattice polytope, that is, when its vertices are integral.

Recent work has focused on determining when \( h^*(\mathcal{P}) \) is unimodal, that is, when there exists some \( i \) for which \( h^*_0 \leq \cdots \leq h^*_i \geq \cdots \geq h^*_k \). The specific sequence in question may not be of particular interest, but unimodal behavior often suggests an underlying structure that may not be immediately apparent. Thus, the proofs of various \( h^* \)-vectors being unimodal are often more enlightening than the sequences themselves. There are a number of approaches possible for proving unimodality, taken from fields such as Lie theory, algebraic statistics, and others [11].

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In this paper, we consider a variation of the Birkhoff polytope, which is defined as follows.

**Definition 1.1.** The *Birkhoff polytope* is the set of \( n \times n \) matrices with real nonnegative entries such that each row and column sum is 1.

We denote this polytope by \( B_n \) and note that it is also often referred to as the polytope of real \( n \times n \) doubly-stochastic matrices or the polytope of \( n \times n \) magic squares. The fact that \( B_n \) is a polytope at all is due to the Birkhoff-von Neumann theorem, which finds that \( B_n \) is the convex hull of the permutation matrices. The \( h^* \)-vector of the Birkhoff polytope is difficult to compute in general, and is known only for \( n \leq 9 \); its volume only for \( n \leq 10 \) [2]. As limited as the data is, it has still been shown that \( h^*(B_n) \) is symmetric as well as unimodal [1, 8, 9].

On the other hand, little is known about the polytope \( \Sigma_n \) obtained by intersecting \( B_n \) with the hyperplanes \( x_{ij} = x_{ji} \) for all \( i, j \), that is, by requiring the corresponding matrices to be symmetric. Nothing is new when \( n \leq 2 \), but complications arise once \( n \geq 3 \) since the vertices of \( \Sigma_n \) are no longer always integral. They are contained in the set

\[
L_n = \left\{ \frac{1}{2} (P + P^T) | P \in \mathbb{R}^{n \times n} \text{ is a permutation matrix} \right\},
\]

but \( L_n \) is not necessarily equal to the vertices of \( \Sigma_n \). A description of the vertices and a generating function for the number of them can be found in [13]. The \( h^* \)-vector of \( \Sigma_n \) is known to be symmetric [12] and \( E_{\Sigma_n}(t) \) has been computed in a reduced form for some small \( n \) [14], but it is still unknown whether the \( h^* \)-vector is unimodal in this case.

**Definition 1.2.** Denote by \( S_n \) the polytope containing all real \( n \times n \) symmetric matrices with nonnegative entries such that every row and column sum is 2. That is, \( S_n \) is the dilation of \( \Sigma_n \) by two.

Fortunately, some information about \( \Sigma_n \) is retained by \( S_n \), a polytope that is combinatorially equivalent but with integral vertices.

Our main purpose is to show the following.

**Theorem 1.3.** Let \( \Sigma_n \) and \( S_n \) be as above. If \( n = 2k \) for some \( k \in \mathbb{Z}_{>0} \), then

1. \( h^*(\Sigma_n) = (h^*_0, h^*_1, \ldots, h^*_d) \) for \( d = 2k^2 - 2k + 1 \), and
2. \( h^*(S_n) = (h^*_0, h^*_2, \ldots, h^*_{2d-2}, h^*_{2d}) \) is unimodal.

We approach these results by first showing that \( S_n \) is integrally closed for all \( n \). Often this is done by proving the existence of a unimodular triangulation, or at least a unimodular covering, of the polytope, but we will approach this result by passing through graph theory. Showing integral closure will actually help prove the existence of a regular unimodular triangulation. We then construct a reverse lexicographic monomial order for the toric ideal of \( S_n \) and examine the reduced Gröbner basis for this. We find particularly nice properties among the binomials of degree two, and these properties, when checking \( S \)-polynomials, allow us to show that each element of the reduced Gröbner basis has a squarefree initial term. The correspondence given by Theorem 8.9 of [15] tells us that this is enough to prove the existence of a regular, unimodular triangulation. Finally, the unimodality of \( h^*(S_n) \) will follow from a previously established result.
2. Basic Properties, Symmetry, and Integral Closure

Although relatively little has been established about the Ehrhart theory of $S_n$, it has still been studied and some basic information is known. For $\Sigma_n$, the degrees of the constituent polynomials of its Ehrhart quasipolynomial are known.

**Theorem 2.1** (Theorem 8.1, [7]). The Ehrhart quasipolynomial of $\Sigma_n$ is of the form $f_n(t) + (-1)^t g_n(t)$, where $\deg f(t) = \binom{n}{2}$ and

$$\deg g_n(t) = \begin{cases} \binom{n-1}{2} - 1 & \text{if } n \text{ odd} \\ \binom{n-2}{2} - 1 & \text{if } n \text{ even} \end{cases}.$$

Stanley first proved that the above degrees are upper bounds and conjectured equality [9], and the conjecture was proven using analytic methods. These degrees provide an upper bound on the degree of $h^*_{\Sigma_n}(t)$; we will provide exact degrees later. Since the Ehrhart series of $S_n$, as a formal power series, consists of the even-degree terms of the monomials appearing in $E_{\Sigma_n}(t)$, we get $L_{S_n}(t) = f_n(2t) + g_n(2t)$.

The defining inequalities of our polytopes will be helpful in some contexts. For $S_n$, these are

$$x_{ij} \geq 0 \text{ for all } 1 \leq i \leq j \leq n,$$

$$x_{ij} = x_{ji} \text{ for all } 1 \leq i < j \leq n,$$

$$\sum_{i=1}^{n} x_{ij} = 2 \text{ for each } j = 1, \ldots, n.$$

The first set of inequalities provided indicate that the facet-defining supporting hyperplanes of $S_n$ are $x_{ij} = 0$: if any of these are disregarded, the solution set strictly increases in size.

**Definition 2.2.** A lattice polytope $P \subseteq \mathbb{R}^n$ is called *integrally closed* if, for every $v \in mP \cap \mathbb{Z}^n$, there are $m$ points $v_1, \ldots, v_m \in P \cap \mathbb{Z}^n$ such that $v = v_1 + \cdots + v_m$.

This idea is not to be confused with a normal polytope, in which we instead choose $v$ from $mP \cap (my + N)$ for an appropriate choice of $y \in P \cap \mathbb{Z}^n$ and $N$ is the lattice

$$N = \sum_{z_1, z_2 \in P \cap \mathbb{Z}^n} \mathbb{Z}(z_1 - z_2) \subseteq \mathbb{Z}^n.$$

There is more discussion of this difference in [6]. It is currently an open problem to determine whether integrally closed polytopes have unimodal $h^*$-vectors. This is unknown even in highly restricted cases, such as if the polytope is reflexive, a simplex, or even both. The last case is explored more in [3].

We first would like to prove that $S_n$ is integrally closed. To do so, we must interpret the lattice points of $S_n$ as certain incidence matrices of graphs.

**Proposition 2.3.** For all $n$, $S_n$ is integrally closed.

**Proof.** The can be seen as a corollary of a theorem of Petersen’s 2-factor theorem. For any $m \in \mathbb{Z}_{\geq 0}$, each lattice point $X = (x_{ij}) \in mS_n$ can be interpreted as the incidence matrix of an undirected $m$-regular multigraph $G_X$ on distinct vertices $v_1, \ldots, v_n$, with loops having degree 1. We first observe that the total number of loops will be even: if there were an odd number of loops, consider the
graph with the loops removed. The sum of degrees of the vertices in the resulting graph would be odd, which is an impossibility.

Denote by $V_{\text{odd}}(G_X)$ the vertices of $G_X$ with an odd number of loops, and write $|V_{\text{odd}}(G_X)| = 2mt + s$, where $t, s$ are nonnegative integers and $s < 2m$. Note in particular that $s$ will be even. Construct a new graph $G_Y$ with vertex set $V(G_Y) = \{v_1, \ldots , v_n, w_0, w_1, \ldots , w_t\}$ with the same edges as in $G_X$ with the following modifications:

1. For each $v_i \notin V_{\text{odd}}(G_X)$, $v_i$ will have $\frac{1}{2}x_{ii}$ loops in $G_Y$.
2. For each $v_i \in V_{\text{odd}}(G_X)$, $v_i$ will have $\frac{1}{2}(x_{ii} - 1)$ loops and an edge between $v_i$ and the lowest-indexed $w_j$ such that $\deg w_j < 2m$.
3. Vertex $w_i$ will have $\frac{1}{2}(2m - s)$ loops.

This new graph will be $2m$-regular, now counting loops as degree 2. Thus, by Petersen’s 2-factorization theorem, $G_Y$ can be decomposed into 2-factors. Hence the matrix $Y$ corresponding to $G_Y$ will decompose as the sum of $Y_1, \ldots , Y_m$, each summand a lattice point of $mS_{n+\ell+1}$.

Now we must “undo” the changes we made to $G_X$ to obtain the desired sum. Index the rows and columns by $\{v_1, \ldots , v_n, w_0, w_1, \ldots , w_t\}$. Each edge $v_i w_j$ will appear in some $Y_k$ as a 1 in positions $(v_i, w_j)$ and $(w_j, v_i)$. Replace these entries with 0 and add 1 to entry $(v_i, v_i)$. Denote by $X_k$ the submatrix of $Y_k$ consisting of rows and columns indexed by $v_1, \ldots , v_n$ after any appropriate replacements have been made. Each replacement preserves the sum of row/column $v_i$, and applying this to each $Y_k$ leaves any entry $(v_i, w_j)$ as 0, so each $X_k$ is a lattice point of $S_n$. Thus $X = \sum X_k$, as desired.

**Definition 2.4.** For a lattice polytope $\mathcal{P} \subseteq \mathbb{R}^n$, denote by $k[\mathcal{P}]$ the semigroup algebra

$$k[\mathcal{P}] := k[x^a z^m | a \in m\mathcal{P} \cap \mathbb{Z}^{n+1}] \subseteq k[x_1^{\pm 1}, \ldots , x_n^{\pm 1}, z].$$

Then $\mathcal{P}$ is called **Gorenstein** if $k[\mathcal{P}]$ is Gorenstein. More specifically, $\mathcal{P}$ is **Gorenstein of index $r$** if there exists a monomial $x^c z^r$ for which

$$k[\mathcal{P}^g] \cong (x^c z^r) k[\mathcal{P}].$$

Having the hyperplane description of a polytope can make it easier to determine if it is Gorenstein, as evidenced by the following lemma.

**Lemma 2.5** (Lemma 2(ii), [4]). Suppose $\mathcal{P}$ has irredundant supporting hyperplanes $l_1, \ldots , l_s \geq 0$, where the coefficients of each $l_i$ are relatively prime integers. Then $\mathcal{P}$ is Gorenstein (of index $r$) if and only if there is some $c \in r\mathcal{P} \cap \mathbb{Z}^n$ for which $l_i(c) = 1$ for all $i$.

Generally, proving the unimodality of an $h^*$-vector is a challenging task. There are more techniques available, though, if we have a Gorenstein polytope, that is, if the semigroup algebra $k[\mathcal{P}]$ is Gorenstein.

**Lemma 2.6** (Corollary 7, [4]). Suppose $\mathcal{P} \subseteq \mathbb{R}^n$ is an integrally closed Gorenstein polytope with irredundant, integral supporting hyperplanes $l_1, \ldots , l_s$, and contains lattice points $v_0, \ldots , v_k$. If these points form a $k$-dimensional simplex and $l_i(v_0 + \cdots + v_k) = 1$ for each $i$, then $\mathcal{P}$ projects to an integrally closed reflexive polytope $Q$ of dimension $n - k$ with equal $h^*$-vector.
Theorem 2.7. \( S_n \) is Gorenstein if and only if \( n \) is even. When \( n = 2k \), \( S_n \) is Gorenstein of type \( k \), and \( h^*(S_n) \) is the \( h^* \)-vector of a reflexive polytope of dimension \( 2k^2 - 2k + 1 \). Hence, 
\[
\deg h_{S_n}^*(t) = 2k^2 - 2k + 1.
\]

Proof. By Lemma 2.5 and knowing the facet description of \( S_n \), we can see that the polytope is Gorenstein by choosing integer matrices of \( S_n \) whose sum is the all-ones matrix. When \( n \) is odd, this is impossible: such a matrix has an odd line sum, whereas any sum of matrices in \( S_n \) has even line sum.

When \( n = 2k \), we use a family of circulant matrices
\[
\begin{pmatrix}
 a_0 & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \\
 a_1 & a_0 & a_{n-1} & \cdots & a_3 & a_2 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_0 & a_{n-1} \\
 a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 
\end{pmatrix},
\]
arising from setting \( a_i = a_{n-i} = 1 \) and \( a_j = 0 \) for each \( i = 1, 2, \ldots, k - 1 \) and \( j \neq i \), and the additional matrix with \( a_0 = a_k = 1 \) and \( a_j = 0 \) for \( j \neq 0, k \). These matrices are symmetric and have pairwise disjoint support by construction. These are therefore vertices of a simplex of dimension \( k - 1 \), and Lemma 2.6 provides the reflexivity result.

Note that this is not the only class of simplices satisfying the conditions of Lemma 2.5 contained in \( S_n \) for even \( n \); others may be found. It may be interesting to ask how many such distinct simplices in \( S_n \) exist.

Example 2.8. For \( n = 6 \), we construct the special simplex described above. It has three vertices, which are
\[
\begin{pmatrix}
 0 & 1 & 0 & 0 & 0 & 1 \\
 1 & 0 & 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 1 \\
 1 & 0 & 0 & 0 & 1 & 0 
\end{pmatrix},
\]
\[
\begin{pmatrix}
 0 & 0 & 1 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 1 \\
 1 & 0 & 0 & 0 & 1 & 0 \\
 0 & 1 & 0 & 0 & 0 & 1 \\
 1 & 0 & 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 1 & 0 & 0 
\end{pmatrix},
\]
\[
\begin{pmatrix}
 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 1 & 0 \\
 0 & 0 & 1 & 0 & 0 & 1 \\
 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 1 & 0 \\
 0 & 0 & 1 & 0 & 0 & 1 
\end{pmatrix}.
\]

Proposition 2.9. If \( n = 2k + 1 \), then the first scaling of \( S_n \) containing interior lattice points is \((\frac{n+1}{2}) \) \( S_n \). Specifically, the number of interior lattice points in this scaling is the number of symmetric permutation matrices, i.e. the number of involutions of the set \( \{1, 2, \ldots, n\} \). Thus, 
\[
\deg h_{S_n}^*(t) = 2k^2.
\]

Proof. For an interior point, each matrix entry must be positive. However, the matrix of all 1s does not work since this results in an odd line sum. Thus there must be a 2 in each row and column as well. Thus by subtracting the all-1s matrix, each lattice point corresponds to a symmetric permutation matrix, that is, an involution. The line sum for the interior lattice points will be \( n + 1 \), and we remember that the line sums of matrices in \( S_n \) is 2.

By Theorem 1.5 of [10],
\[
E_{(S_n)\circ}(t) = (-1)^t \binom{n}{2} E_{S_n} \left( \frac{1}{t} \right).
\]
When expanded as a power series, the lowest-degree term will be \( t^{(n^2)/2+1-d} \), where \( d = \deg h^*_n(t) \).

The degree of \( h^*_n(t) \) follows. \( \Box \)

With these, we can deduce the degrees of \( h^*_n(t) \) for each \( n \).

**Corollary 2.10.** For even \( n \), \( \deg h^*_n(t) = 2(\deg h^*_n(t)) \). For odd \( n \), \( \deg h^*_n(t) = 2(\deg h^*_n(t)) + 1 \).

**Proof.** Although the first scaling of \( S_n \) with an interior point for odd \( n \) is \( \frac{n+1}{2} \), the first such scaling for \( \Sigma_n \) is \( n \). Using the same technique as in the previous proposition, the computations quickly follow. \( \Box \)

With this corollary we prove the first half of Theorem 1.3 (1). We also gain some insight into part of Theorem 1.3 (2).

**Corollary 2.11.** For all \( n \), \( h^*(S_n) \) consists of the even-indexed entries of \( h^*(\Sigma_n) \).

**Proof.** As power series, the coefficient of \( t^m \) in \( E_{S_n}(t) \) is the same as the coefficient of \( t^{2m} \) in \( E_{\Sigma_n}(t) \). Recalling Theorem 2.4 this gives

\[
E_{\Sigma_n}(t) = E_{S_n}(t^2) + t \sum_{m \geq 0} f(m)t^{2m}
\]

for some polynomial \( f \). So, as rational functions, the first summand of the above will have entirely even-degree terms in the numerator while the second summand will have entirely odd-degree terms in the numerator. Thus, \( h^*(S_n) \) consists of the even-indexed entries of \( h^*(\Sigma_n) \). \( \Box \)

### 3. Toric Ideals and Regular, Unimodular Triangulations

For a polytope \( P \subseteq \mathbb{R}^n \) let \( P \cap \mathbb{Z}^n = \{a_1, \ldots, a_s\} \). We define the toric ideal of \( P \) to be the kernel of the map

\[
\pi : T_P = k[t_1, \ldots, t_s] \to k[P],
\]

where \( \pi(t_i) = (\prod x^{a_i}) z \), using the multivariate notation. This ideal we denote \( I_P \). Because the lattice points of \( S_n \) correspond to matrices, it will sometimes be more convenient to use the indexing

\[
T_{S_n} = k[t_A|A \in S_n \cap \mathbb{Z}^{n \times n}] \quad \text{and} \quad k[S_n] = k[x^A z^m|A \in mS_n \cap \mathbb{Z}^{n \times n}],
\]

where we now use

\[
x^A z^m = \prod_{0 \leq i,j \leq n} x_{i,j}^{a_{ij}} z^m
\]

with \( A = (a_{ij}) \). Thus \( \pi : T_{S_n} \to k[S_n] \) is given by \( \pi(t_M) = x^M z \).

The toric ideal of a polytope has been widely studied, in large part for its connections to triangulations of the polytope. Various properties of the initial ideal of \( I_P \) are equivalent to corresponding properties of the triangulation, with perhaps the most well-known connection being the following result.

**Theorem 3.1** (Theorem 8.9, [13]). Given a monomial ordering \( \prec \) on \( T_P \), the initial ideal in \( \prec(I_P) \) is squarefree if and only if the corresponding regular triangulation of \( P \) is unimodular.
In general, in$_{\text{rlex}}(I_P)$ cannot be guaranteed to be squarefree. This does not rule out the existence of in$_{\text{rlex}}(I_P)$ being squarefree for some ordering of their lattice points, though this may require much more work; the generators of a toric ideal are notoriously difficult to compute in general. Fortunately, the following ordering we place on $S_n$ provides enough structure to prove the existence of a regular, unimodular triangulation.

**Definition 3.2.** We place a total order $\prec_{S_n}$ on the lattice points of $S_n$ by setting $M \prec_{S_n} N$ if $M$ contains more 2s in its entries than $N$, and by then taking a linear extension. This induces a graded reverse lexicographic monomial order $\prec_{S_n}$ on the variables of $T_{S_n}$, specifically $t_M \prec_{S_n} t_N$ if $M \prec_{S_n} N$.

**Lemma 3.3.** For any monomial ordering, $I_{S_n}$ has a reduced Gröbner basis consisting of binomials $u - v$, at least one monomial of which is squarefree.

**Proof.** Let $\mathcal{G}$ be the reduced Gröbner basis of $I_{S_n}$ with respect to any ordering. It is known to consist of binomials itself. Suppose $\mathcal{G}$ has a binomial $u - v$ with both terms containing squares, and $\pi(u) = \pi(v) = x^M z^k$. Note in particular that the indeterminates in $u$ and $v$ are distinct. Suppose $t_i$ and $t_j$ are the indeterminates in the separate terms with powers greater than 1. Then $\pi(t_it_j)$ is the average of the points corresponding to $\pi(t_i^2)$ and $\pi(t_j^2)$, thus is subtractable from $M$. By the integral closure of $\mathcal{P}$, there is some third monomial $b$ such that $\pi(t_it_jb) = x^M z^k$. So $u - t_it_jb$ is in $I_P$; however, we can factor out $t_i$ from this to get $u - t_it_jb = t_i(u_1 - u_2)$. We may similarly factor $t_j$ from $v - t_it_jb$ to get $t_j(v_1 - v_2)$. Therefore $u_1 - u_2$ and $v_1 - v_2$ must be in $I_P$ themselves, and $u - v$ can be written as

$$u - v = u - t_it_jb + t_it_jb - v = t_i(u_1 - u_2) - t_j(v_2 - v_1)$$

which contradicts $\mathcal{G}$ being reduced. Therefore no binomial in $\mathcal{G}$ can have both terms containing a square. \hfill $\square$

**Proposition 3.4.** Let $\mathcal{P}$ be integrally closed, and let $\mathcal{G}$ be a reduced Gröbner basis of $\mathcal{P}$ with respect to $\prec$ such that

1. every element of $\mathcal{G}$ consists of binomials, one monomial of which is squarefree,
2. the initial term of each degree-two binomial in $\mathcal{G}$ is squarefree, and
3. every indeterminate of $T_\mathcal{P}$ appears in a degree-two binomial in $\mathcal{G}$.

Then in$_{\prec}(I_P)$ is squarefree, that is, $\mathcal{P}$ has a regular, unimodular triangulation.

**Proof.** Let $f \in \mathcal{G}$ be degree at least 3 and of the form

$$f = \prod_{k=0}^{r} i_k^{a_k} - \prod_{l=0}^{s} t_jl$$

where each $a_k \geq 1$. We know $f$ has such a representation from Lemma 3.3. We then wish to show that the initial term of $f$ is squarefree. By way of contradiction, we will assume $a_m \geq 2$ for some $m$.

By assumption there is some $g \in \mathcal{G}$ of the form $t_{i_0}t_{x} - t_{y}t_z$. There are two cases to consider: first, assume in$_{\prec}(g) = t_{i_0}t_{x}$. We know that $x \notin \{i_0, \ldots, i_m\}$ since otherwise in$_{\prec}(g)$ divides in$_{\prec}(f)$, violating $\mathcal{G}$ being reduced. Thus we look at the $S$-polynomial $S(f, g)$, which must reduce to 0:
\[ S(f, g) = t_x \sum_{i_0}^{s} t_{i_0} \cdots t_{i_m}^{a_m} \cdot t_{j_0} \cdots t_{j_r}^{a_r} t_y t_z. \]

Since we assume \( a_m \geq 2 \), some variable in the first term must be equal to \( t_{i_m} \). If it is \( t_x \), then this contradicts what the initial term of \( g \) should be (a square cannot be an initial term among degree-two binomials). If it is some \( t_{j_l} \) then this contradicts the disjointness of the variables in \( f \). Therefore the first case fails.

For the second case we assume \( \text{in}_{\prec}(g) = t_y t_z \), so \( y \neq z \) and \( y, z \neq i_m \). This time, the term being multiplied by \( g \) will contain \( t_{i_m}^{a_m+1} \). But this cannot be cancelled out, since this would force \( y = z = j \) for some \( l \). Therefore, the initial term of \( f \) must be the squarefree term.

\[ S(f, g) = t_y t_z \sum_{r=0}^{k} t_{i_r}^{a_r} g \]

This time we get \( a_{i_m} + 1 \geq 3 \), so \( y = z = j_\alpha \) for some \( \alpha \), but this again implies \( \text{in}_{\prec}(g) \) is a square.

**Theorem 3.5.** For all \( n \), \( S_n \) has a regular, unimodular triangulation.

**Proof.** It suffices to show that the conditions of Proposition 3.4 hold. First let \( \mathcal{G} \) be the reduced Gröbner basis for \( I_{S_n} \). There exist degree-two binomials in \( \mathcal{G} \) since the boundary lattice points of \( S_n \) that are not vertices are merely midpoints of vertices themselves. Consider \( t_M t_N - t_X t_Y \in \mathcal{G} \). By Theorem 2.3 checking that the initial term is squarefree requires only checking the case where exactly one of the terms is a square, say \( M = N \). This can only occur if \( M \) is not a vertex; hence, \( M \) is the midpoint of \( X \) and \( Y \). Thus if any entries of \( M \) are 2, the corresponding entries of \( X \) and \( Y \) must also be 2. Since \( X \) and \( Y \) are distinct, though, they have distinct support. This implies that some entry of \( M \) is 1, which arises from one of the corresponding entries of \( X \) and \( Y \) being 0 and the other being 2. So, one of \( X \) or \( Y \) will contain more twos than \( M \), giving us \( \text{in}_{\prec_{S_n}}(t_M t_N - t_X t_Y) = t_X t_Y \).

For the second condition, we must show that, for any lattice point \( M \in S_n \), we can find a second lattice point \( N \in S_n \) such that \( M + N \) can be represented in a second, distinct sum. Since these are degree 2, the relation must be recorded in \( I_{S_n} \), meaning both term appear individually in \( \mathcal{G} \) (even if not as part of the same binomial). While this can be proven in terms of matrices, it will be easier to work with in terms of graph labelings: for each 2-factor of the graph, we want to find a second 2-factor so that their union can be written as a union of 2-factors, each distinct from the given 2-factor and the one we chose.

As we saw in Proposition 2.3, each lattice point \( M \in S_n \) corresponds to a \( G_M \), a covering of \( n \) vertices so that each vertex is incident to two edges. So, each covering is a disjoint union of two possible connected components: first, a path, possibly of length 0, whose endpoints also have loops;
second, a $k$-cycle for some $k \leq n$. This allows us to break the remainder of the proof into three cases.

First suppose $G_M$ contains a path $v_1, v_2, \ldots, v_k$, $k > 1$, with loops at its endpoints. Set $G_N$ to be the graph agreeing with $G_M$ except on these vertices. Here we place a single loop on each of $v_1$ and $v_k$, an edge between these two vertices, and two loops on each of $v_2, \ldots, v_{k-1}$. The union $G_M \cup G_N$ can be decomposed appropriately as a cycle $v_1, v_2, \ldots, v_k, v_1$ and as two loops on each vertex.

Next suppose that $G_M$ contains no such paths but does contain a cycle $v_1, v_2, \ldots, v_k$, $v_1, v_k$ for some $k \geq 2$. Let $G_N$ be the cover with two loops on each $v_i$. Then $G_M \cup G_N$ decomposes as the path $v_1, \ldots, v_k$ with a loop on $v_1$ and $v_k$ as one covering and the other covering as the edge $v_1, v_k$ with loops $v_1, v_1$ and $v_k, v_k$ along with two loops on all other vertices.

If $G_M$ does not fit into either of the previous cases, then it can only consist of the disjoint union of two loops on each of the $n$ vertices. If we choose two vertices $v_1$ and $v_2$ and replace their loops by two edges between them, then the $G_N$ created will be $G_M$ itself. Thus this replacement will qualify as the second summand needed to decompose its union with $G_M$ into distinct 2-factors.

This covers all cases, so the corresponding $M$ will always appear in a degree-two binomial of $I_{S_n}$. □

4. Future Directions, Questions, and Conjectures

The regular unimodular triangulations for $S_n$ provide subdivisions of $\Sigma_n$ into simplices, though not using their lattice points. The Ehrhart series of each of the simplices can be computed, and they will each be of the form

$$\frac{1}{(1-t)^a(1-t^2)^b},$$

where $a, b \in \mathbb{Z}_{\geq 0}$, $a + b = \binom{n}{2} + 1$.

An inclusion-exclusion argument on the intersections of the simplices will then produce the desired Ehrhart series. However, proving unimodality of $h^*(\Sigma_n)$ from this is not necessarily straightforward.

The results we have shown lead to some natural questions and conjectures.

**Question 4.1.** For even $n$, how can we determine the behavior of the odd-degree terms of $h^*_\Sigma (t)$?

Another modification that can be made to $\Sigma_n$ is the following. Define by $P_n$ the convex hull of the lattice points in $\Sigma_n$. In general, $P_n$ is neither Gorenstein nor integrally closed. However, based on experimental data, we conjecture the following.

**Conjecture 4.2.** For all $n$, $h^*(P_n)$ is unimodal.

Many methods for showing unimodality aim to show that the $h^*$-vector of a polytope is the same as the $h$-vector of a simplicial polytope, but another approach is necessary for $P_n$, as well as $S_n$ for odd $n$.

Instead of looking at all lattice points of $S_n$, one can form triangulations using only the vertices. These will not be unimodular triangulations, but they might lead to something interesting.

**Conjecture 4.3.** For $n \geq 2$, any reverse lexicographic initial ideal of the toric ideal $I_{S_n}$ (no new vertices) is $n$-free and is generated by monomials of degree $3(n-2)$. 

The conjecture is experimentally true for $n = 3$ by an exhaustive search. Higher dimensions result in exponentially increasing numbers of vertices, vastly increasing the computational difficulty of experimentation.

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