Towards the Exact Simulation Using Hyperbolic Brownian Motion

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Abstract

In the present paper, an expansion of the transition density of Hyperbolic Brownian motion with drift is given, which is potentially useful for pricing and hedging of options under stochastic volatility models. We work on a condition on the drift which dramatically simplifies the proof.

1 Introduction

As is well recognized, “local stochastic volatility models” can be reduced to Brownian motion with drift thanks to Lamperti’s transform. This is not the case when one works with stochastic volatility (henceforth SV) models where the stock price $S$ and its instantaneous volatility $V$ are modeled by a two-dimensional diffusion process. One can not transform it into a two dimensional Brownian motion with drift in general.

As is pointed in [4], however, most of existing stochastic volatility models are “conformally equivalent” to hyperbolic Brownian motion (HBM for short) instead; or in other words, many SV diffusion processes $(S, V)$ can be transformed to HBM with drift by a diffeomorphism.

In the present paper, we shall give an asymptotic expansion formula of the transition density of HBM with drift with respect to the so-called McKean kernel; density kernel. That is, the HBM without drift. We claim that this formula can be used in numerical calculations for the under SV models, although in this paper we will not go in depth in this direction.

Our formula is in fact a parametrix one, so along the line of Bally-Kohatsu [1]’s idea, we give an exact simulation interpretation of the parametrix formula.\footnote{Here the term “exact” is used because it is not an approximation, but the equality. It may be also referred to as “unbiased” since it is only simulate the expectation of a functional of $(S_t, V_t)$.}
The present paper is organized as follows. In section 2, we briefly recall some basic facts about HBM. In section 3, we introduce a drift to the HBM, and describe its transition density by using as parametrix a HBM (Theorem 2). In section 4, we give an interpretation of the formula given in Theorem 2 that it gives a description of an exact simulation.

In the present paper we restrict ourselves to 1) working on a simple situation given by (4); no drift in the volatility, and (5), which reduce the computational complexity of the proof dramatically. Further, 2) we omit the description of how SV models can be transformed to HBM in this paper. The main aim of the present paper is then to show that the condition (4) simplifies the proof quite a lot.

## 2 Hyperbolic Brownian Motions

In this section, we recall basic facts about Hyperbolic Brownian motions.

Let $n \geq 2$ and

$$ \mathbb{H}^n := \{ z = (x, y) = (x^1, \ldots, x^{n-1}, y); x \in \mathbb{R}^{n-1}, y > 0 \}, $$

the upper half space in $\mathbb{R}^n$, endowed with the Poincaré metric

$$ ds^2 = y^{-2}((dx)^2 + (dy)^2). $$

The Riemannian volume element is given by $dv = y^{-2}dx dy$ and the distance $d_{\mathbb{H}^n}(z, z')$ for $z = (x, y)$, $z' = (x', y') \in \mathbb{H}^n$ is given by

$$ \cosh(d_{\mathbb{H}^n}(z, z')) = \frac{d_{\mathbb{R}^{n-1}}(x, x')^2 + y^2 + (y')^2}{2yy'}. $$

(1)

The Laplace-Beltrami operator is

$$ \Delta_n := y^2 \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + y^2 \frac{\partial^2}{\partial y^2} - (n - 2)y \frac{\partial}{\partial y}. $$

We denote by $q_n(t, z, z')$ the heat kernel with respect to the volume element $dv$ of the semigroup generated by $\Delta_n/2$; that is to say,

$$ \partial_t q_n = \frac{1}{2} \Delta_n q_n, $$

---

2A metric, at each point, is a bi-linear form on the tangent space, or equivalently, an element of the tensor product of the cotangent space. The convention $(dx)^2$ should then be understood as $dx \otimes dx$, and so on.
and
\[
\lim_{t \to 0} \int_{\mathbb{H}^n} q_n(t, z', (x, y)) f(x, y) y^{-2} dxdy = f(z')
\]
for any bounded continuous function \( f \). In other words,
\[
\mathbb{P}(\{(X_t, Y_t) \in dxdy | (X_0, Y_0) = z\}') = q_n(t, z', (x, y)) y^{-2} dxdy,
\]
where \((X_t, Y_t)\) is the solution to the following stochastic differential equation:
\[
\begin{align*}
 dX^i_t &= Y^i_t dW^i_t, \quad i = 1, \ldots, n-1, \\
 dY^i_t &= Y^i_t dW^n_t,
\end{align*}
\]
where \(W^1, \ldots, W^n\) are mutually independent Brownian motion defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The diffusion \((X, Y)\) is the one associated with the semigroup \(\Delta_n/2\).

The following formulas for \(q_n\) are known (see e.g. [2] and [6]):

**Theorem 1.** The heat kernel with respect to the volume form has the following explicit expressions.

i) (McKean’s kernel) In the case of \(n = 2\):
\[
q_2(t, z, z') =: p_2(t, r) = \frac{\sqrt{2} e^{-t/8}}{(2\pi t)^{3/2}} \int_r^\infty \frac{be^{-b^2/2t}}{(\cosh(b) - \cosh(r))^{1/2}} db.
\]

ii) (Milson’s formula) For \(n \geq 2\), we have the following recursive relation:
\[
q_{n+2}(t, z, z') =: p_{n+2}(t, r) = -\frac{e^{-nt/2}}{2\pi \sinh(r)} \frac{\partial}{\partial r} p_n(t, r).
\]

(iii) (Gruet’s formula [3]) For every \(n \geq 2, t > 0, z, z' \in \mathbb{H}^n\), it holds that
\[
q_n(t, z, z') = p_n(t, r) = \frac{e^{-(n-1)^2t/8}}{\pi(2\pi)^{n/2}t^{1/2}} \Gamma\left(\frac{n+1}{2}\right) \int_0^\infty \frac{e^{(x^2-b^2)/2t} \sinh(b) \sin(\pi b/t)}{(\cosh(b) + \cosh(r))^{(n+1)/2}} db,
\]
where \(r = d_{\mathbb{H}^2}(z, z')\).

### 3 HBM with drift, and its parametrix

We consider the following stochastic differential equation:
\[
\begin{align*}
 dX_t &= Y_t dW^1_t + \mu(X_t, Y_t) dt \\
 dY_t &= Y_t dW^n_t, \\
(X_0, Y_0) &= (x, y) = z,
\end{align*}
\]
where \(\mu(X, Y)\) is a measurable function. The parametrix of the solution \((X_t, Y_t)\) is given by:
\[
\mathbb{P}(\{(X_t, Y_t) \in dxdy | (X_0, Y_0) = z\}') = p_n(t, r) y^{-2} dxdy,
\]
where
\[
p_n(t, r) = \frac{e^{-(n-1)^2t/8}}{\pi(2\pi)^{n/2}t^{1/2}} \Gamma\left(\frac{n+1}{2}\right) \int_0^\infty \frac{e^{(x^2-b^2)/2t} \sinh(b) \sin(\pi b/t)}{(\cosh(b) + \cosh(r))^{(n+1)/2}} db.
\]
where \((x, y) = z \in \mathbb{H}^2\), \(\mu : \mathbb{H}^2 \to \mathbb{R}\) be a Lipschitz function, bounded in \(x\) and
\[
|\mu(x, y)| \leq K_0 |y|, \quad (x, y) \in \mathbb{H}^2
\]
with some positive constant \(K_0\). The unique strong solution to (4) exists, and will be denoted by \((X^\mu, Y^\mu) =: Z^\mu\), while the 2-dimensional HBM given by (3) with \(n = 2\) will be denoted by \((X^0, Y^0) =: Z^0\).

Put
\[
\theta(t, z, z') := \mu(x, y) \frac{\partial}{\partial x} \log q_2(t, (x, y), (x', y'))
\]
\[
= \mu(x, y) \frac{\partial}{\partial x} q_2(t, (x, y), (x', y')) q_2(t, (x, y), (x', y'))^{-1},
\]
\[
t > 0, \; z, z' \in \mathbb{H}^2.
\]

For \(t > 0\) and each \(n\), let
\[
\Delta_n(t) := \{(u_1, u_2, \cdots, u_n) \in [0, t]^n : u_1 < \cdots < u_n\}.
\]

The following is the main theorem of the present paper:

**Theorem 2.** (i) We have that
\[
|\theta(t, z, z')| \leq \frac{3K_0}{2}
\]
and therefore for each \(n \geq 2\), \(t > 0\) and \((s_1, \cdots, s_{n-1}) \in \Delta_{n-1}(t)\), the random variable \(\prod_{i=1}^n \theta(s_i - s_{i-1}, Z^0_{s_{i-1}}, Z^0_{s_i})\), where \(s_0 = 0\) and \(s_n = t\), is in \(L^\infty(\mathbb{P})\) and
\[
\mathbb{E}[\prod_{i=1}^n \theta(s_i - s_{i-1}, Z^0_{s_{i-1}}, Z^0_{s_i}) | Z^0_t = z'] \in L^\infty(\Delta_{n-1}(t))
\]
for each \(t > 0\) and \(z, z' \in \mathbb{H}^2\).

(ii) Set
\[
h_1(t, z, z') = \mu(x, y) \frac{\partial}{\partial x} q_2(t, (z, z'), (y'))^{-2}.
\]
and
\[
h_n(t, z, z') := \int_{\Delta_{n-1}(t)} \mathbb{E}[\prod_{i=1}^n \theta(s_i - s_{i-1}, Z^0_{s_{i-1}}, Z^0_{s_i}) | Z^0_t = z'] q_2(t, (z, z'), (y')) ds_1 \cdots ds_{n-1}
\]
for \(n \geq 2\). Then, the series \(\sum_{n=1}^N h_n(t, z, z')\) is absolutely convergent as \(N \to \infty\) uniformly in \((t, z, z')\) on every compact set.
(iii) The transition density of \( Z \) is given by

\[
\frac{q_2(t, z, z')}{(y')^2} + \int_{y^2}^{t} \int_0^t \frac{q_2(t-s, z, z'')}{(y'')^2} \Phi(s, z'', z') ds dz',
\]

where \( \Phi(t, z, z') = \sum_{n=1}^{\infty} h_n(t, z, z') \).

Proof. Since \( q_n(t, z, z') = p_n(t, r(z, z')) \), we have that

\[
\frac{\partial}{\partial x} q_2(t, (x, y), (x', y')) = \frac{\partial}{\partial x} p_2(t, r((x, y), (x', y')))
\]

\[
= \frac{\partial r}{\partial x} \frac{\partial p_2}{\partial r}(t, r((x, y), (x', y')))
\]

\[
= \frac{x - x'}{yy'} \left(-e^t 2\pi \sinh(r) p_4(t, r((x, y), (x', y'))))
\]

\[
= \frac{x - x'}{yy'} \left(-e^t (2\pi) p_4(t, r((x, y), (x', y'))))
\]

by (ii) of Theorem 1. Also, (iii) of Theorem 1 tells us that

\[
e^t (2\pi) p_4(t, r) = e^{-t/8}\pi (2\pi)^{1/2} \frac{1}{2} \int_0^\infty \frac{e^{(\pi^2 - b^2)/2t} \sinh(b) \sin(\pi b/t)}{(\cosh(b) + \cosh(r))^{3/2}} db
\]

\[
\leq 3 \frac{1}{2} \frac{1}{1 + \cosh(r)} p_2(t, r)
\]

since \( \cosh(x) \geq 1 \) for all \( x \). Therefore, we see that

\[
|\theta(t, (x, y), (x', y'))| \leq |\mu(x, y)| \left| \frac{\partial}{\partial x} q_2(t, (x, y), (x', y')) \right| q_2(t, (x, y), (x', y'))
\]

\[
\leq 3 K_0 \frac{|y||x - x'|}{2 yy'(1 + \cosh(r(z, z')))}
\]

Here, we have used (5) in the last inequality. By (1),

\[
\frac{|y||x - x'|}{yy'(1 + \cosh(r(z, z')))} = \frac{|y||x - x'|}{yy'(1 + |x - x'|^2 + |y - y'|^2)} = \frac{2|y||x - x'|}{|x - x'|^2 + |y + y'|^2}
\]

\[
\leq \frac{|y|}{|y + y'|} \leq 1.
\]
Thus we obtained (6). Here in the last line we have used the following elementary inequality:

$$|x - x'|^2 + |y + y'|^2 \geq 2|x - x'||y + y'|.$$ 

Let us consider (ii). By (6), we have that for $n$ bigger than 2,

$$h_n(t, z, z') \leq \frac{q_2(t, z, z')}{(y')^2} \int_{\Delta_{n-1}(t)} \mathbb{E}\left[\left(\frac{3}{2}K_0\right)^n |Z^0_i = z'|ds_1 \cdots ds_{n-1}\right]$$

$$= \left(\frac{3}{2}K_0\right)^n \frac{q_2(t, z, z')}{(y')^2} \int_{\Delta_{n-1}(t)} ds_1 \cdots ds_{n-1}$$

$$= \left(\frac{3}{2}K_0\right)^n \frac{q_2(t, z, z')}{(y')^2} \frac{t^{n-1}}{(n-1)!}.$$ 

Here we have used

$$\mathbb{E}\left[1|Z^0_i = z'\right] = \frac{q_2(t, z, z')}{(y')^2}.$$ 

Hence we have

$$\sum_{n=1}^{\infty} |h_n(t, z, z')| \leq \frac{q_2(t, z, z')}{(y')^2} \sum_{n=1}^{\infty} \left(\frac{3}{2}K_0\right)^n \frac{t^{n-1}}{(n-1)!}$$

$$= \frac{3}{2}K_0 \frac{q_2(t, z, z')}{(y')^2} \sum_{n=0}^{\infty} \left(\frac{3}{2}K_0t\right)^n \frac{1}{n!}$$

$$= \frac{3}{2}K_0 \frac{q_2(t, z, z')}{(y')^2} e^{\frac{3}{2}K_0t},$$

which complete the proof of (ii).

Finally, we shall prove (iii). Since

$$h_n(t, z, z') = \int_{\mathbb{H}^2} \int_0^t h_1(t - s, z, z'')h_{n-1}(s, z'', z')dsdz'';$$

we see that the sum $\sum_{n=1}^{\infty} h_n(t, z, z') = \Phi(t, z, z')$ satisfies

$$\Phi(t, z, z') = h_1(t, z, z') + \int_{\mathbb{H}^2} \int_0^t h_1(t - s, z, z'')\Phi(s, z'', z')dsdz''.$$ 

Note that since we have, by (6),

$$|\Phi(t, z, z')| = |\sum_{n=1}^{\infty} h_n(t, z, z')|$$

$$\leq \sum_{n=1}^{\infty} |h_n(t, z, z')| \leq \frac{3}{2}K_0 \frac{q_2(t, z, z')}{(y')^2} e^{\frac{3}{2}K_0t},$$

which completes the proof of (iii).
we see that $\Phi$ is integrable:

$$
\int_0^T \int_{\mathbb{H}^2} |\Phi(t, z, z')|dz'dt \leq \frac{3}{2} K_0 \int_0^T \int_{\mathbb{H}^2} \frac{q_2(t, z, z')}{(y')^2} e^{\frac{3}{2}K_0 t} dz'dt \\
\leq \frac{3}{2} K_0 e^{\frac{3}{2}K_0 T} \int_0^T \int_{\mathbb{H}^2} \frac{q_2(t, z, z')}{(y')^2} dz'dt = \frac{3}{2} K_0 T e^{\frac{3}{2}K_0 T} < \infty.
$$

We know that

$$
(1/2 \Delta_2 - \partial_t) q_2(t, z, z') = 0,
$$

and

$$
(1/2 \Delta_2 - \partial_t) \int_{\mathbb{H}^2} \int_0^t \frac{q_2(t-s, z, z'')}{(y'')^2} \Phi(s, z'', z') dsdz'' = -\Phi(t, z, z')
$$

by Feynman-Kac formula (see e.g. [5, Theorem 7.6]). Therefore, we have that

$$
\left(1/2 \Delta_2 + \mu \frac{\partial}{\partial x_1} - \partial_t\right) p_2(t, z, z')
$$

$$
= \left(1/2 \Delta_2 + \mu \frac{\partial}{\partial x_1} - \partial_t\right) \left(\frac{q_2(t-s, z, z')}{(y')^2}\right) + \int_{\mathbb{H}^2} \int_0^t \frac{q_2(t-s, z, z'')}{(y'')^2} \Phi(s, z'', z') dsdz''
$$

$$
= \mu \frac{\partial q_2}{\partial x_1} \frac{1}{(y')^2} + \int_{\mathbb{H}^2} \int_0^t \frac{\mu q_2}{(y'')^2} \frac{\partial}{\partial x_1} (t-s, z, z'') \Phi(s, z'', z') dsdz'' - \Phi(t, z, z'),
$$

which is seen to be zero by (7) and (9).

Clearly, the property that $p_2(t, z, z')dz$ converges to $\delta_{z'}(dz)$ is inherited from $q_2$.

\[\square\]

4 Exact Simulation Interpretation

In the spirit of Bally-Kohatsu [1], we give the following “exact simulation interpretation” to Theorem 2.

**Theorem 3.** Let $S_i$, $i = 1 \cdots$, are independent copies of an exponentially distributed random variable with mean 1, which are also independent of the Brownian motion $(W^1, W^2)$. Let $T_i := S_i + \cdots + S_i$ and $N_t := \sum_i 1_{T_i \leq t}$, $t > 0$. Then, for any bounded measurable $f$, we have that

$$
\mathbb{E}[f(Z_t^i)] = e^t \mathbb{E}\left[\prod_{i=1}^{N_t} \theta(T_i - T_{i-1}, Z_{T_i}^{i-1}, Z_{T_i}^i, Z_{T_i}^0) f(Z_0^i)\right].
$$
Even though this is an almost direct corollary to Theorem 2 and Bally-Kohatsu’s general theory, we give a self-contained proof below.

**Proof.** First we claim that for a positive measurable function 
\[ G \equiv G(s_1, \cdots, s_{k+1}, z_1, \cdots, z_{k+1}), \]
we have that 
\[ \mathbb{E} \left[ 1 \{ N_t = k \} G(T_1, \cdots, T_{k+1}, Z_0, Z_1, \cdots, Z_{k+1}) \right] \]
\[ = \mathbb{E} \left[ \int_{\Delta_k(t) \times [t, \infty)} G(s_1, \cdots, s_{k+1}, Z_1^0, \cdots, Z_{k+1}^0, Z_{k+1}) ds_1 \cdots ds_k e^{-s_{k+1} ds_{k+1}} \right]. \quad (10) \]

In fact, since 
\[ \mathbb{E} \left[ 1 \{ N_t = k \} G(T_1, \cdots, T_{k+1}, Z_0, Z_1, \cdots, Z_{k+1}) \right] \]
\[ = \mathbb{E} \left[ \mathbb{E} \left[ 1 \{ T_1 \leq t, \cdots, T_{k+1} > t \} G(T_1, \cdots, T_{k+1}, Z_1^0, \cdots, Z_{k+1}^0, Z_{k+1}) \mid \mathcal{F}^Z \right] \right] \]
\[ = \mathbb{E} \left[ \int_{[0, t]^{k+1} \times [t, \infty)} G(s_1, \cdots, s_{k+1}, Z_1^0, \cdots, Z_{k+1}^0) ds_1 \cdots ds_{k+1} \mathbb{P} (T_1 \in ds_1, \cdots, T_{k+1} \in ds_{k+1}) \right], \]
and since the joint density of \( T_1, \cdots, T_k \) is given by
\[ \mathbb{P}(T_1 \in ds_1, \cdots, T_{k+1} \in ds_{k+1}) = 1_{\{s_{k+1} > s_k > s_{k-1} > \cdots > s_1 > 0\}} e^{-s_{k+1} ds_1 \cdots ds_{k+1}}, \]
we have (10).

In particular, if \( G \) is independent to \( s_{k+1} \), we have the following reduction:
\[ \mathbb{E} \left[ 1 \{ N_t = k \} G(T_1, \cdots, T_k, Z_1^0, \cdots, Z_{k+1}^0, Z_{k+1}) \right] \]
\[ = e^{-t} \mathbb{E} \left[ \int_{\Delta_k(t)} G(s_1, \cdots, s_k, Z_1^0, \cdots, Z_{k+1}^0) ds_1 \cdots ds_{k+1} \right]. \quad (11) \]

We note that we can apply (11) to
\[ G_+(s_1, \cdots, s_k, z_1, \cdots, z_{k+1}) = \left( \prod_{i=1}^{k} \theta(s_i - s_{i-1}, z_{i-1}, z_i) f(z_{k+1}) \right)_+, \]
\[ G_-(s_1, \cdots, s_k, z_1, \cdots, z_{k+1}) = \left( \prod_{i=1}^{k} \theta(s_i - s_{i-1}, z_{i-1}, z_i) f(z_{k+1}) \right)_-. \]
and so we have
\[
\mathbb{E} \left[ \prod_{i=1}^{k} 1_{\{N_i=k\}} \theta(T_i - T_{i-1}, Z_{T_{i-1}}^0, Z_t^0) f(Z_t^0) \right]
\]
\[= e^{-t} \mathbb{E} \left[ \int_{\Delta_k(t)} \prod_{i=1}^{k} \theta(s_i - s_{i-1}, Z_{s_{i-1}}^0, Z_{s_i}^0) f(Z_t^0) ds_1 \cdots ds_k \right]. \quad (12)
\]
Since we know from (i) of Theorem 2 that \( \prod_{i=1}^{k} \theta(s_i - s_{i-1}, Z_{s_{i-1}}^0, Z_{s_i}^0) \in L^\infty(\mathbb{P}) \), we see that \( \prod_{i=1}^{k} \theta(s_i - s_{i-1}, Z_{s_{i-1}}^0, Z_{s_i}^0) f(Z_t^0) \) is in \( L^1(\mathbb{P}) \) by the requirement that \( f(Z_t^0) \in L^1(\mathbb{P}) \). Therefore, the right-hand-side of (12) is equal to
\[
e^{-t} \int_{\Delta_k(t)} \mathbb{E} \left[ \prod_{i=1}^{k} \theta(s_i - s_{i-1}, Z_{s_{i-1}}^0, Z_{s_i}^0) f(Z_t^0) \right] ds_1 \cdots ds_k.
\]
Noting that
\[
\mathbb{E} \left[ \prod_{i=1}^{k} \theta(s_i - s_{i-1}, Z_{s_{i-1}}^0, Z_{s_i}^0) f(Z_t^0) \right]
\]
\[= \int_{(\mathbb{H}^2)^{k+1}} \prod_{i=1}^{k} h_1(s_i - s_{i-1}, z_{i-1}, z_i) f(z\prime) \frac{q_2(t - s_k, z_k, z\prime)}{(y\prime)^2} dz_1 \cdots dz_k dz\prime,
\]
we obtain that
\[
\int_{\Delta_k(t)} \mathbb{E} \left[ \prod_{i=1}^{k} \theta(s_i - s_{i-1}, Z_{s_{i-1}}^0, Z_{s_i}^0) f(Z_t^0) \right] ds_1 \cdots ds_k
\]
\[= \int_{\mathbb{H}^2} \left( \int_{(\mathbb{H}^2 \times [0,t])^k} \prod_{i=1}^{k} h_1(s_i - s_{i-1}, z_{i-1}, z_i) \frac{q_2(t - s_k, z_k, z\prime)}{(y\prime)^2} ds_i dz_i \right) f(z\prime) dz\prime
\]
\[= \int_{\mathbb{H}^2} \left( \int_{\mathbb{H}^2 \times [0,t]} h_k(s_k, z, z\prime, z\prime) \frac{q_2(t - s_k, z, z\prime)}{(y\prime)^2} ds_k dz\prime dz\prime \right) f(z\prime) dz\prime,
\]
which is bounded by
\[
\left( \frac{3}{2} k_0 \right)^k \frac{t^{k-1}}{(k-1)!} \int_{\mathbb{H}^2} \frac{q_2(t, z, z\prime)}{(y\prime)^2} |f(z\prime)| dz\prime
\]
\[= \left( \frac{3}{2} k_0 \right)^k \frac{t^{k-1}}{(k-1)!} \mathbb{E}[|f(Z_t^0)|],
\]
as we see from (8). Therefore, we can change the order between the summation and the expectation in

\[
\mathbb{E} \left[ \prod_{i=1}^{N_t} \theta(T_i - T_{i-1}, Z^0_{T_i}, Z^0_{T_{T_i}}) f(Z^0_i) \right]
\]

\[
= \mathbb{E} \left[ f(Z^0_i) \mathbb{1}_{\{N_t=0\}} + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \mathbb{1}_{\{N_t=k\}} \theta(T_i - T_{i-1}, Z^0_{T_i}, Z^0_{T_{T_i}}) f(Z^0_i) \right].
\]

On the other hand, by (13),

\[
\mathbb{E} \left[ f(Z^0_i) \mathbb{1}_{\{N_t=0\}} \right] + \sum_{k=1}^{\infty} \mathbb{E} \left[ \prod_{i=1}^{k} \mathbb{1}_{\{N_t=k\}} \theta(T_i - T_{i-1}, Z^0_{T_i}, Z^0_{T_{T_i}}) f(Z^0_i) \right]
\]

\[
= e^{-t} \int_{\mathbb{H}^2} \frac{q_2(t, z, z')}{(y^\prime)^2} f(z') \, dz'
+ e^{-t} \sum_{k=1}^{\infty} \int_{\mathbb{H}^2} f(z') \, dz' \int_{\mathbb{H}^2} \int_{0}^{t} \frac{q_2(t-s, z, z'')}{(y^{\prime\prime})^2} \Phi(s, z', z') \, ds \, dz''
\]

\[
= e^{-t} \int_{\mathbb{H}^2} \frac{q_2(t, z, z')}{(y^\prime)^2} f(z') \, dz'
+ e^{-t} \int_{\mathbb{H}^2} f(z') \, dz' \int_{\mathbb{H}^2} \int_{0}^{t} \frac{q_2(t-s, z, z'')}{(y^{\prime\prime})^2} \Phi(s, z', z') \, ds \, dz''
= e^{-t} \mathbb{E}[f(Z^0_t)],
\]

where the last equality is valid by (iii) of Theorem 2.

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