Reduction Redux of Adinkras

S. James Gates, Jr[1], Stephen Randall[2][, and Kory Stiffler[3]

†Center for String and Particle Theory
Department of Physics, University of Maryland
College Park, MD 20742-4111 USA

and

*Department of Chemistry, Physics, & Astronomy
College of Arts and Sciences
Indiana University Northwest
Gary, Indiana 46408 USA

ABSTRACT

We show performing general “0-brane reduction” along an arbitrary fixed direction in spacetime and applied to the starting point of minimal, off-shell 4D, \( \mathcal{N} = 1 \) irreducible supermultiplets, yields adinkras whose adjacency matrices are among some of the special cases proposed by Kuznetsova, Rojas, and Toppan. However, these more general reductions also can lead to ‘Garden Algebra’ structures beyond those described in their work. It is also shown that for light-like directions, reduction to the 0-brane breaks the equality in the number of fermions and bosons for dynamical theories. This implies that light-like reductions should be done instead to the space of 1-branes or equivalently to the worldsheet.

[1]gatess@wam.umd.edu
[2]stephenlrandall@gmail.com
[3]kmstiff@iun.edu
1 Introduction

A number of years ago [1], we began the study of a ubiquitous mathematical substructure that appears in all off-shell linear representations of supersymmetry in one and two dimensions. The topic was developed in a series of papers which uncovered increasingly complex intricacies such as a series of mapping operations acting on these structures. In the works of [2], a procedure, “0-brane projection”, was described whereby these same structures could be derived from supersymmetric theories in all higher dimensions. Eventually, these structures were named “adinkras” [3].

It is rather simple to prove that when any standard field-theoretic formulation of a supersymmetric field theory is examined under such a projection, forcing all dynamical fields to only depend on a single temporal coordinate, this results in the revealing of adinkras. So these networks appear to be universal to all supersymmetric field theories. Shortly after the christening, there was assembled an interdisciplinary collaboration of mathematicians and physicists (the ‘DFGHILM group’) which worked diligently to define more rigorously the properties of adinkras.

By now there has been established a substantial literature that treats aspects of this approach to understanding the fundamental structure of off-shell spacetime supersymmetric representations. Along the way many unexpected developments have occurred. The most recent such surprise [4] is the discovery that adinkras are directly related to super Riemann surfaces with no reference to Superstring/M-Theory! Adinkras are equivalent to very special super Riemann surfaces with divisors.

We have long advocated [2] that adinkras may be regarded as the equivalent of genes for biological systems. That is, we have suggested it is possible to begin solely with adinkras, which are intrinsically one dimensional networks, and then reconstruct higher dimensional supersymmetric representations from this starting point. We have named this process “SUSY holography” as should it succeed, this would imply that the information to reconstruct the higher dimensional supersymmetric systems must in some way be holographically encoded within the starting point of one dimensional adinkra networks.

Within the last year, two presentations have advanced the possibility of creating a proof of the “SUSY holography conjecture” at least regarding the relation between 1D, \( \mathcal{N} = 4 \) adinkras and 4D, \( \mathcal{N} = 1 \) classes of theories.

First, it was established [5] that given some specific 1D, \( \mathcal{N} = 4 \) adinkras, their

\[\text{http://math.bard.edu/DFGHIL/index.php?n=Main.Publications}\] on-line.
associated adjacency matrices necessarily carry a representation of an SU(2) \otimes SU(2) algebra\footnote{The presence of this SU(2) \otimes SU(2) algebra was first observed in the work of \cite{6} where it was noted that while off-shell representations are representation of this algebra, on-shell representations only carry representations of an SU(2) \otimes SO(2) algebra.}. There is a group theoretical understanding of this result that leads to the conclusion that for a general adinkra with \( N \) colors, the associated adjacency matrices will carry some representation of SO\((N)\). Since locally the algebra of SO\((4)\) is isomorphic to an SU\((2) \otimes SU(2)\) algebra, the demonstrated results consequently follow.

Though it appears not generally appreciated, the covering algebra of the SO\((1, 3)\) Clifford algebra also carries a representation of an SU\((2) \otimes SU(2)\) algebra. It is possible to identify these two algebras as one and the same structure and re-construct, solely from the 1D, \( N = 4 \) adinkras, matrices in the covering algebra of the SO\((1, 3)\) \( \gamma \)-matrices. Thus, information about the SO\((1, 3)\) spin bundle associated with a 4D, \( \mathcal{N} = 1 \) supermultiplet can indeed be encoded within the network of adinkras!

Second, it was established \cite{7} that the sets of all possible quartets of \( 4 \times 4 \) matrices that can be used to describe the ‘L-matrices’ and related ‘R-matrices’ associated with minimal off-shell 4D, \( \mathcal{N} = 1 \) supermultiplets (the chiral, vector, and tensor supermultiplets) can be identified with the 384 elements of the Coexter Group \( BC_4 \). Taking absolute values of the elements of these matrices then leads to a further identification with \( S_4 \), the permutation group of four elements.

Next, it was shown there exists a discrete transformation (acting on these adinkras identified with the elements of the Coexter Group \( BC_4 \)) with the property of defining a class structure of three distinct parts. It was then argued that these three sub-classes should be identified with the respective three distinct minimal off-shell representations of 4D, \( \mathcal{N} = 1 \) supersymmetry and that the operation provides (in the space of 1D, \( N = 4 \) adinkras) a realization of a ‘one dimensional Hodge star operation’ acting on forms in the covering algebra of the SO\((1, 3)\) Clifford algebra!

These two results together establish a fairly well defined path for the reconstruction of 4D, \( \mathcal{N} = 1 \) supermultiplets from one dimensional four-color adinkras:

(a.) using the putative and proposed ‘one dimensional Hodge star operation’ any such adinkra can be associated with one of the minimal 4D, \( \mathcal{N} = 1 \) supersymmetric representations, and

(b.) the SU\((2) \otimes SU(2)\) content associated with the given adinkra determines the SO\((1, 3)\) spin bundle representation carried by the nodes of the adinkra.
Of course, more work understanding the details of these steps is required in the general case as well as a complete understanding of the filters that allow completely consistent dimensional enhancement. But the results in hand provide the essence of an existence proof that we believe can be successfully constructed.

All these results are within the context of adinkras associated with our traditional “0-brane reduction” of 4D, $\mathcal{N} = 1$ supermultiplets. However, in this work, we turn to a different question. Up to this point previously, whenever we have constructed adinkras from higher dimensional supersymmetric field theories, the single bosonic coordinate in the adinkra was associated with a purely time-like direction. It is a very natural question to ask, “What might occur if instead the reduction was done with respect to an arbitrary constant 4-vector direction among the spacetime coordinates?” This is the question probed in this current work.

We shall show in the following work, using the familiar off-shell minimal 4D, $\mathcal{N} = 1$ SUSY representations, that “0-brane reduction” utilizing a non-purely time-like direction yields adinkras whose associate L-matrices and associated R-matrices form representations that are generalizations of the “Garden Algebras” introduced in work of [1]. The authors Kuznetsova, Rojas, and Toppan have previously introduced generalizations [8] of the “Garden Algebras” some time ago. We shall indeed see that “0-brane reduction” utilizing a space-like direction can yield structures first suggested by these authors. However, we shall also see that the “0-brane reduction” utilizing a general space-like direction produces generalizations even beyond those considered in their work.

2 General 0-Brane Reduction

The starting point to revealing the adinkra sub-structure of higher dimensional supersymmetric theories in Minkowski space begins with the usual spacetime four-gradient operator $\partial_\mu$

$$\partial_\mu = \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = T_\mu \frac{\partial}{\partial t} + X_\mu \frac{\partial}{\partial x} + Y_\mu \frac{\partial}{\partial y} + Z_\mu \frac{\partial}{\partial z},$$

where the constant four-vectors $T_\mu$, $X_\mu$, $Y_\mu$, and $Z_\mu$ respectively denote

$$T_\mu = (1, 0, 0, 0), \quad X_\mu = (0, 1, 0, 0), \quad Y_\mu = (0, 0, 1, 0), \quad Z_\mu = (0, 0, 0, 1).$$

(2.2)
Next a single real parameter $\tau$ may be introduced via the equations below

$$\frac{\partial}{\partial t} = \cos \alpha \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x} = \sin \alpha \sin \beta \cos \gamma \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial y} = \sin \alpha \sin \beta \sin \gamma \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial z} = \sin \alpha \cos \beta \frac{\partial}{\partial \tau},$$

(2.3)

whereupon the four-gradient operator $\partial_\mu$ takes the restricted form $\partial_\mu = \ell_\mu \partial_{\tau}$ or more explicitly

$$\partial_\mu = [\cos \alpha T_\mu + \sin \alpha \sin \beta \cos \gamma X_\mu + \sin \alpha \sin \beta \sin \gamma Y_\mu + \sin \alpha \cos \beta Z_\mu] \frac{\partial}{\partial \tau}.$$  (2.4)

This restricted form of the spacetime four-gradient operator clearly generates motion in the Minkowski space along a straight line described by the four-vector $\ell_\mu$. We have named this process ‘0-brane reduction.’ It has long been our assertion that the group theoretic structures obtained from 0-brane reduction in the context of spacetime supersymmetric representation theory plays the same role as the Wigner ‘little group’ for non-supersymmetric representation theory.

In a four-dimensional $\mathcal{N} = 1$ theory, the anti-commutator algebra for the supersymmetry generator can be written as

$$\{ Q_a, Q_b \} = i 2 (\gamma_\mu)_{ab} \partial_\mu,$$  (2.5)

which under 0-brane reduction becomes

$$\{ Q_a, Q_b \} = i 2 (\gamma^\mu)_{ab} \ell_\mu \partial_\tau = i 2 (\gamma \cdot \ell)_{ab} \partial_\tau.$$  (2.6)

At this stage, it does not appear that any particular 0-brane projection (for arbitrary values of the parameters $\alpha$, $\beta$, and $\gamma$) possesses any distinguished behavior from any other projection. However, in the following we will see that though the change above is mild, when the questions of dynamics are engaged, depending on whether $\ell$ is time-like or space-like versus light-like, subtle differences do emerge.

As discussed in [6], our gamma matrices are chosen to be real and explicitly given by

$$\begin{align*}
(\gamma^0) &= i(\sigma^3 \otimes \sigma^2), \\
(\gamma^1) &= (I_2 \otimes \sigma^1), \\
(\gamma^2) &= (\sigma^2 \otimes \sigma^2), \\
(\gamma^3) &= (I_2 \otimes \sigma^3),
\end{align*}$$

(2.7)

thus describing a mostly plus Minkowski spacetime metric $\eta^{\mu \nu}$ that appears in

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu \nu} I_4.$$  (2.8)

so that the purely imaginary ‘gamma-five’ matrix takes the form

$$\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -(\sigma^1 \otimes \sigma^2).$$  (2.9)
From the definitions above, it follows that the generators of spatial rotations $\Sigma^{ij} = i/4[\gamma^i, \gamma^j]$ take the explicit forms

$$
\Sigma^{12} = \frac{1}{2} (\sigma^2 \otimes \sigma^3), \quad \Sigma^{23} = \frac{1}{2} (\sigma^2 \otimes \sigma^1), \quad \Sigma^{31} = \frac{1}{2} (\mathbf{I}_2 \otimes \sigma^2),
$$

and it is easily seen that these form an SU(2) algebra. In our discussions, we often refer to this as the SU$_\alpha$(2) algebra.

However, using the gamma matrices described above, it is possible to construct a second SU(2) algebra that we denote as the SU$_\beta$(2) algebra. The generators of SU$_\beta$(2) algebra take the explicit forms

$$
\Sigma^{12} = \frac{1}{2} (\sigma^2 \otimes \sigma^3) = -\frac{i}{2} \gamma^0, \quad \Sigma^{23} = \frac{1}{2} (\sigma^1 \otimes \sigma^2) = -\frac{i}{2} \gamma^5, \quad \Sigma^{31} = \frac{1}{2} (\mathbf{I}_2 \otimes \sigma^2) = \frac{1}{2} \gamma^0 \gamma^5.
$$

Due to the defining properties of the gamma matrices, these two SU(2) algebras commute. This implies that the complete set of sixteen elements in the covering algebra of the gamma matrices carry a representation of SU$_\alpha$(2) $\otimes$ SU$_\beta$(2).

On the other hand, as noted in [6] among other places, a four-color adinkra in some representation $\mathcal{R}$ has a set of L-matrices and R-matrices that satisfy the “Garden Algebra” conditions

$$
(R^{(\mathcal{R})}_i)_{ij} (L^{(\mathcal{R})}_j)_{ij} \hat{k} + (R^{(\mathcal{R})}_j)_{ij} (L^{(\mathcal{R})}_i)_{ij} \hat{k} = 2 \delta_{ij} (\mathbf{I}_d)_{ij} \hat{k},
$$

$$
L^{(\mathcal{R})} = [R^{(\mathcal{R})}]^t = [R^{(\mathcal{R})}]^{-1}.
$$

From these, one can construct a set of objects called “adjacency matrices” using a standard definition from graph theory that we can denote by $\tilde{\gamma}_i$ where

$$
\tilde{\gamma}_i = \begin{bmatrix} 0 & L_i \\ R_i & 0 \end{bmatrix}.
$$

Due to the defining property of the L-matrices and R-matrices it follows that the quantities $\tilde{\gamma}_i$ defined this way form the generators of a four-dimensional Euclidean-signatured Clifford algebra. Thus, the quantities defined by $\tilde{\Sigma}_{ij} = i/4[\tilde{\gamma}_i, \tilde{\gamma}_j]$ correspond to a set of hermitian generators of SO(4).

Owing to the fact that SO(4) is locally isomorphic to SU$_\alpha$(2) $\otimes$ SU$_\beta$(2), the results of (2.7) - (2.13) imply for every adinkra in any representation $\mathcal{R}$ with four colors, the $N = 4$ version of the “Adinkra/$\gamma$-matrix Holography Equation” [5]

$$
(R^{(\mathcal{R})}_i)_{ij} (L^{(\mathcal{R})}_j)_{ij} \hat{k} - (R^{(\mathcal{R})}_j)_{ij} (L^{(\mathcal{R})}_i)_{ij} \hat{k} = 2 \left[ \ell^{(\mathcal{R})1}_i (\gamma^2 \gamma^3)_{ij} \hat{k} + \ell^{(\mathcal{R})2}_i (\gamma^3 \gamma^1)_{ij} \hat{k} + \ell^{(\mathcal{R})3}_i (\gamma^1 \gamma^2)_{ij} \hat{k} + i \ell^{(\mathcal{R})4}_i (\gamma^0)_{ij} \hat{k} \right].
$$
must be valid for some set of constant coefficients \( \ell_{ij}^{(R)} \) and \( \ell_{ij}^{(\bar{R})} \). Given that the Adinkra/\( \gamma \)-matrix Holography Equation, for every four-color adinkra, generates a set of matrices \( (\gamma^0 \gamma^5, \Sigma^{ij}) \) solely from the data in the adinkra, we simply note the equation

\[
\gamma_i = \epsilon_{ijk} \gamma^0 \gamma^5 \Sigma^{jk},
\]

yields the remaining spatial gamma matrices. After these are constructed we multiply by \( \gamma^0 \), and then \( \gamma^5 \) respectively. Finally \( I_4 \) must be included the to construct all sixteen elements of the covering algebra.

### 2.1 The Generalized Garden Algebra

We can generalize the original notion of a Garden algebra seen in \( \mathbb{G} \) to the following form. Denoted by \( GR(d, N, \ell_\mu) \) this generalized algebra is defined by

\[
L_{(I} R_{J)} = R_{(I} L_{J)} = 2[E(\ell_\mu)]_{IJ} I_d
\]

for \( I, J = 1, 2, \ldots, N \),

\[
(2.16)
\]

where \( \ell_\mu \) is the “direction” of the reduction, a four-vector parameter of the algebra on equal footing with \( d \) and \( N \), and the \( E(\ell_\mu) \) determine the ‘metric’ on the space of SUSY parameters.

In the following, we shall show that _reduction along any spacetime axis satisfies the generalized Garden algebra_. Finally, some explicit examples of the values of this SUSY parameter space matrix for \( GR(4, 4, \ell_\mu) \) are

\[
[E(\tau_\mu)]_{IJ} = [I_4]_{IJ}, \quad [E(\chi_\mu)]_{IJ} = [\sigma^3 \otimes \sigma^3]_{IJ},
\]

\[
[E(\varphi_\mu)]_{IJ} = [\sigma^1 \otimes I_2]_{IJ}, \quad [E(\psi_\mu)]_{IJ} = -[\sigma^3 \otimes \sigma^1]_{IJ},
\]

\[
(2.17)
\]

with the eigenvalues of \( E(\tau_\mu) \) all being +1 and the eigenvalues of \( E(\chi_\mu), E(\varphi_\mu), \) and \( E(\psi_\mu) \) being ±1 (each doubly degenerate). We also see the trace of \( E(\tau_\mu) \) is equal to four, and the traces of \( E(\chi_\mu), E(\varphi_\mu), \) and \( E(\psi_\mu) \) are all equal to zero.

For the case of \( E(\chi_\mu) \), this is precisely an example of the structure presented by Kuznetsova, Rojas, and Toppan [8] who advocated considering this type of metric in the parameter space of the 1D, \( N \)-extended SUSY QM systems. However, in the case of \( E(\varphi_\mu), \) and \( E(\psi_\mu) \), we see the metric in the SUSY parameter space can be even more general than they proposed. For general values of the angles \( \alpha, \beta, \) and \( \gamma \), one finds:

\[
[E(\ell_\mu)]_{IJ} = \cos \alpha [E(\tau_\mu)]_{IJ} + \sin \alpha \sin \beta \cos \gamma [E(\chi_\mu)]_{IJ} + \sin \alpha \sin \beta \sin \gamma [E(\varphi_\mu)]_{IJ} + \sin \alpha \cos \beta [E(\psi_\mu)]_{IJ}.
\]

\[
(2.18)
\]
However, we also find a surprising result. If we impose the condition
\[(L_\ell)^I = [\mathcal{E}(\ell_\mu)]_{IJ}(R_J)\quad (2.19)\]
as the natural generalization from the temporal 0-brane reduction, this forces the angle \(\alpha\) to either take the value of \(\alpha = 0, \pi/2,\) or \(\alpha = \pi\!\!\!\!\!\

Let us introduce one other notation in order to illustrate another result. We can write
\[\mathcal{E}(n^1) = \mathcal{E}(\mathcal{X}_\mu), \quad \mathcal{E}(n^2) = \mathcal{E}(\mathcal{Y}_\mu), \quad \mathcal{E}(n^3) = \mathcal{E}(\mathcal{Z}_\mu).\quad (2.20)\]
It now follows from (2.10) that we have the nice algebraic relations
\[[\mathcal{E}(n^i), \mathcal{E}(n^j)]_{IJ} = i\,4\,\langle\Sigma^{ij}\rangle_{IJ}.\quad (2.21)\]

Let us note that we can define
\[V_{IJ} = \frac{1}{2} [L_I R_J - L_J R_I], \quad \tilde{V}_{IJ} = \frac{1}{2} [R_I L_J - R_J L_I].\quad (2.22)\]
and from these it follows that
\[[V_{IJ}, V_{KL}] = [\mathcal{E}(\ell_\mu)]_{IK} V_{JL} - [\mathcal{E}(\ell_\mu)]_{JK} V_{IL} - [\mathcal{E}(\ell_\mu)]_{IL} V_{JK} + [\mathcal{E}(\ell_\mu)]_{JL} V_{IK},\]
\[[\tilde{V}_{IJ}, \tilde{V}_{KL}] = [\mathcal{E}(\ell_\mu)]_{IK} \tilde{V}_{JL} - [\mathcal{E}(\ell_\mu)]_{JK} \tilde{V}_{IL} - [\mathcal{E}(\ell_\mu)]_{IL} \tilde{V}_{JK} + [\mathcal{E}(\ell_\mu)]_{JL} \tilde{V}_{IK}.\quad (2.23)\]
where \(\ell\) is picked along each of the coordinate axes. These commutator algebras clearly do not describe SO(4) for \(\alpha \neq 0\). This is turn implies that the adjacency matrices also do not form a representation of SO(4) for \(\alpha \neq 0\).

In closing, we note that each supermultiplet when subjected to 0-brane reduction is associated with its own adjacency matrices. For distinct multiplets, these adjacency matrices are different. In the following we will prove that the results in (2.19) and (2.21) hold independent of which multiplet on which the evaluation is made.

### 3 Reduction of the Chiral Multiplet

In order to investigate in concrete detail the procedure of general 0-brane reduction and any subtleties that arise along the way, the standard 4D, \(\mathcal{N} = 1\) chiral multiplet is considered. Using the conventions of [6], the Lagrangian that is invariant with respect to supersymmetry variations takes the form
\[\mathcal{L} = -\frac{1}{2} \partial_\mu A \partial^\mu A - \frac{1}{2} \partial_\mu B \partial^\mu B + \frac{1}{2} i (\gamma^\mu)_{bc} \psi_b \partial_\mu \psi_c + \frac{1}{2} F^2 + \frac{1}{2} G^2\quad (3.1)\]
with the SUSY transformation laws in component form realized by the supercovariant derivative $D_a$ operators acting on the fields as

$$D_a A = \psi_a, \quad D_a B = i (\gamma^5)_{ab}^b \psi_b,$$

$$D_a \psi_b = i (\gamma^\mu)_{ab} \partial_\mu A - (\gamma^5 \gamma^\mu)_{ab} \partial_\mu B - i C_{ab} F + (\gamma^5)_{ab} G,$$

$$D_a F = (\gamma^\mu)_{ab}^b \partial_\mu \psi_b, \quad D_a G = i (\gamma^5 \gamma^\mu)_{ab} \partial_\mu \psi_b,$$

in four dimensions. Under the 0-brane reduction where $\partial_\mu = \ell_\mu \partial_\tau$ these become

$$D_a A = \psi_a, \quad D_a B = i (\gamma^5)_{ab}^b \psi_b,$$

$$D_a \psi_b = i (\gamma \cdot \ell)_{ab} \partial_\tau A - (\gamma^5 \gamma \cdot \ell)_{ab} \partial_\tau B - i C_{ab} \partial_\tau F + (\gamma^5)_{ab} \partial_\tau G,$$

$$D_a F = (\gamma \cdot \ell)_{ab}^b \partial_\tau \psi_b, \quad D_a G = i (\gamma^5 \gamma \cdot \ell)_{ab} \partial_\mu \psi_b,$$

and the Lagrangian becomes

$$\ell = -\frac{1}{2} \ell_\mu \ell_\mu \left[ (\partial_\tau A)^2 + (\partial_\tau B)^2 \right] + \frac{i}{2} (\gamma \cdot \ell)^{bc} \psi_b \partial_\tau \psi_c + \frac{1}{2} F^2 + \frac{1}{2} G^2 \quad (3.4)$$

Furthermore, utilizing the explicit form of the $\gamma^0$ matrix to find $\eta^{00} = -1$ and the form of $\ell_\mu$, this becomes

$$\ell = \frac{1}{2} \cos(2\alpha) \left[ (\partial_\tau A)^2 + (\partial_\tau B)^2 \right] + \frac{i}{2} (\gamma \cdot \ell)^{bc} \psi_b \partial_\tau \psi_c + \frac{1}{2} F^2 + \frac{1}{2} G^2 \quad (3.5)$$

A next step involves defining the valise related to these equations. This is done by making the re-definitions $F \rightarrow \partial_\tau F$ and $G \rightarrow \partial_\tau G$ so that we find

$$D_a A = \psi_a, \quad D_a B = i (\gamma^5)_{ab}^b \psi_b,$$

$$D_a \psi_b = i (\gamma \cdot \ell)_{ab} \partial_\tau A - (\gamma^5 \gamma \cdot \ell)_{ab} \partial_\tau B - i C_{ab} \partial_\tau F + (\gamma^5)_{ab} \partial_\tau G,$$

$$D_a F = (\gamma \cdot \ell)_{ab}^b \psi_b, \quad D_a G = i (\gamma^5 \gamma \cdot \ell)_{ab} \psi_b,$$

and the Lagrangian becomes

$$\ell = -\frac{1}{2} \ell_\mu \ell_\mu \left[ (\partial_\tau A)^2 + (\partial_\tau B)^2 \right] + i \frac{1}{2} (\gamma \cdot \ell)^{bc} \psi_b \partial_\tau \psi_c$$

$$+ \frac{1}{2} \left[ (\partial_\tau F)^2 + (\partial_\tau G)^2 \right] \quad (3.7)$$

Implementing the further definitions

$$\Phi_i = \begin{bmatrix} A \\ B \\ F \\ G \end{bmatrix}, \quad \Psi_k = -i \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}, \quad (3.8)$$
makes it clear that (3.6) can be written in the concise forms
\[ D_I \Phi_i = i (L_I)_{ik} \Psi_k \quad \text{and} \quad D_I \Psi_k = (R_I)_{ik} \partial_\tau \Phi_i, \tag{3.9} \]
for some matrices \((L_I)\) and \((R_I)\). The Lagrangian then can be written as:
\[ L = \frac{1}{2} \delta_{ij} (\partial_\tau \Phi_i) (\partial_\tau \Phi_j) - \frac{i}{2} \delta^{k\ell} \Psi_k \partial_\tau \Psi_\ell, \tag{3.10} \]
where numerically we have
\[ \delta^{k\ell} = - (\gamma \cdot \ell)^{k\ell} \tag{3.11} \]
for purely time-like reduction \(\ell_\mu \ell^\mu = -1\). The condition in (3.11) is characteristic of Majorana representations and we have assumed its use in all our previous discussions of adinkras.

In this section, we have given a detailed discussion of the steps required to obtain the valise formulation of the chiral supermultiplet. Similar discussions could be undertaken for the vector multiplet starting from its equations
\[ D_a A_\mu = (\gamma_\mu)_{a}^{b} \lambda_b, \]
\[ D_a \lambda_b = -\frac{i}{4} [\gamma_\mu, \gamma_\nu]_{ab} F_{\mu\nu} + (\gamma^5)_{ab} d, \]
\[ D_a d = i (\gamma^5 \gamma_\mu)_{a}^{b} \partial_\mu \lambda_b, \tag{3.12} \]
or the tensor multiplet starting from its equations
\[ D_a \varphi = \chi_a, \]
\[ D_a B_{\mu\nu} = -\frac{i}{4} [\gamma_\mu, \gamma_\nu]_{a}^{b} \chi_b, \]
\[ D_a \chi_b = i (\gamma_\mu)_{ab} \partial_\mu \varphi - (\gamma^5 \gamma_\mu)_{ab} \epsilon_{\mu\rho\sigma\tau} \partial_\rho B_{\sigma\tau}. \tag{3.13} \]
However, the end point of such discussions is precisely the same as in (3.9) and (3.10) but with different L-matrices and R-matrices describing the distinct supermultiplets.

4 Coordinate Axis Reductions

Explicit reduction of the chiral, vector and tensor supermultiplet in a purely time-like direction has been presented before [6]. Once the valise form of each supermultiplet is obtained, the supersymmetry variations are completely described by the results in (3.9). Thus, all that remains is to give the explicit form of the L-matrices and the R-matrices. Below these are written in our compact matrix notation as explained in the appendix. A straightforward calculation with the matrices below
reveals that our assertion in section 2 is correct: that Eqs. (2.19) and (2.21) hold independent of multiplet or reduction coordinate.

**Chiral Multiplet**
Reduction for $\ell = T$ gives the following L and R matrices

\[
\begin{align*}
L_1 &= (1 \ 4 \ 2 \ 3), \quad L_2 = (2 \ 3 \ T \ 4), \quad L_3 = (3 \ 2 \ 4 \ 1), \quad L_4 = (4 \ 1 \ 3 \ 2), \\
R_1 &= (1 \ 3 \ 4 \ 2), \quad R_2 = (3 \ 1 \ 2 \ 4), \quad R_3 = (4 \ 2 \ 1 \ 3), \quad R_4 = (2 \ 4 \ 3 \ 1).
\end{align*}
\]

Reduction for $\ell = X$ gives the following L and R matrices

\[
\begin{align*}
L_1 &= (1 \ 4 \ 2 \ 3), \quad L_2 = (2 \ 3 \ 1 \ 4), \quad L_3 = (3 \ 2 \ 4 \ 1), \quad L_4 = (4 \ 1 \ 3 \ 2), \\
R_1 &= (1 \ 3 \ 2 \ 4), \quad R_2 = (3 \ 1 \ 2 \ 4), \quad R_3 = (4 \ 2 \ 1 \ 3), \quad R_4 = (2 \ 4 \ 3 \ 1).
\end{align*}
\]

Reduction for $\ell = Y$ gives the following L and R matrices

\[
\begin{align*}
L_1 &= (1 \ 4 \ 2 \ 3), \quad L_2 = (2 \ 3 \ 3 \ 2), \quad L_3 = (3 \ 2 \ 2 \ 3), \quad L_4 = (4 \ 1 \ 3 \ 2), \\
R_1 &= (3 \ 2 \ 2 \ 3)^t, \quad R_2 = (4 \ 1 \ 3 \ 1)^t, \quad R_3 = (1 \ 4 \ 2 \ 1)^t, \quad R_4 = (2 \ 3 \ 3)^t.
\end{align*}
\]

**Vector Multiplet**
Reduction for $\ell = T$ gives the following L and R matrices

\[
\begin{align*}
L_1 &= (2 \ 4 \ 1 \ 3), \quad L_2 = (1 \ 3 \ 2 \ 4), \quad L_3 = (4 \ 2 \ 3 \ 1), \quad L_4 = (3 \ 1 \ 4 \ 2), \\
R_1 &= (3 \ 1 \ 4 \ 2), \quad R_2 = (1 \ 3 \ 2 \ 4), \quad R_3 = (4 \ 2 \ 3 \ 1), \quad R_4 = (2 \ 4 \ 1 \ 3).
\end{align*}
\]

Reduction for $\ell = X$ gives the following L and R matrices

\[
\begin{align*}
L_1 &= (2 \ 4 \ 1 \ 3), \quad L_2 = (1 \ 3 \ 2 \ 4), \quad L_3 = (4 \ 2 \ 3 \ 1), \quad L_4 = (3 \ 1 \ 4 \ 2), \\
R_1 &= (1 \ 3 \ 2 \ 4), \quad R_2 = (4 \ 2 \ 3 \ 1), \quad R_3 = (4 \ 1 \ 3 \ 1)^t, \quad R_4 = (3 \ 2 \ 3)^t.
\end{align*}
\]

Reduction for $\ell = Y$ gives the following L and R matrices

\[
\begin{align*}
L_1 &= (2 \ 2 \ 1 \ 1), \quad L_2 = (1 \ 1 \ 2 \ 2), \quad L_3 = (4 \ 4 \ 3 \ 3), \quad L_4 = (3 \ 3 \ 4 \ 4), \\
R_1 &= (4 \ 4 \ 3 \ 3)^t, \quad R_2 = (3 \ 3 \ 4 \ 4)^t, \quad R_3 = (2 \ 2 \ 1 \ 1)^t, \quad R_4 = (1 \ 1 \ 2 \ 2)^t.
\end{align*}
\]

Reduction for $\ell = Z$ gives the following L and R matrices

\[
\begin{align*}
L_1 &= (2 \ 2 \ 4 \ 4), \quad L_2 = (1 \ 1 \ 3 \ 3), \quad L_3 = (4 \ 4 \ 2 \ 2), \quad L_4 = (3 \ 3 \ 1 \ 1), \\
R_1 &= (3 \ 3 \ 4 \ 4)^t, \quad R_2 = (2 \ 2 \ 4 \ 4)^t, \quad R_3 = (3 \ 3 \ 1 \ 1)^t, \quad R_4 = (4 \ 4 \ 2 \ 2)^t.
\end{align*}
\]
Tensor Multiplet
Reduction for $\ell = T$ gives the following L and R matrices
\begin{align*}
L_1 &= (1 \bar{3} 4 2) , & L_2 &= (2 4 \bar{3} 1) , & L_3 &= (3 1 2 \bar{4}) , & L_4 &= (4 \bar{2} 1 3) , \\
R_1 &= (1 \bar{4} 2 \bar{3}) , & R_2 &= (4 1 \bar{3} 2) , & R_3 &= (2 3 1 \bar{4}) , & R_4 &= (3 \bar{2} 4 1) .
\end{align*}
(4.9)

Reduction for $\ell = X$ gives the following L and R matrices
\begin{align*}
L_1 &= (1 3 \bar{4} 2) , & L_2 &= (2 4 \bar{3} 1) , & L_3 &= (3 1 2 \bar{4}) , & L_4 &= (4 2 \bar{1} \bar{3}) , \\
R_1 &= (1 \bar{4} 2 \bar{3}) , & R_2 &= (4 \bar{1} 3 \bar{2}) , & R_3 &= (2 \bar{3} 4 \bar{4}) , & R_4 &= (3 \bar{2} 4 1) .
\end{align*}
(4.10)

Reduction for $\ell = Y$ gives the following L and R matrices
\begin{align*}
L_1 &= (1 1 2 2) , & L_2 &= (2 \bar{2} \bar{1} 1) , & L_3 &= (3 \bar{3} 4 \bar{4}) , & L_4 &= (4 4 \bar{3} \bar{3}) , \\
R_1 &= (3 \bar{3} 4 \bar{4})^t , & R_2 &= (4 4 \bar{3} \bar{3})^t , & R_3 &= (1 1 2 2)^t , & R_4 &= (2 2 \bar{1} \bar{1})^t .
\end{align*}
(4.11)

Reduction for $\ell = Z$ gives the following L and R matrices
\begin{align*}
L_1 &= (1 \bar{1} \bar{3} 3) , & L_2 &= (2 2 4 \bar{4}) , & L_3 &= (3 \bar{3} 1 \bar{1}) , & L_4 &= (4 4 \bar{2} \bar{2}) , \\
R_1 &= (\bar{2} 2 \bar{4} 4)^t , & R_2 &= (\bar{1} \bar{1} \bar{3} 3)^t , & R_3 &= (4 4 \bar{2} \bar{2})^t , & R_4 &= (3 \bar{3} 1 \bar{1})^t .
\end{align*}
(4.12)

5 Complications of Light-like 0-Brane Projection

There is one other subtlety that we need to address shortly. We start by noting that a simple calculation using the parametrization in terms of $\alpha$, $\beta$, and $\gamma$ shows
\[ \ell \cdot \ell = \ell_\mu \eta^{\mu\nu} \ell_\nu = -\cos(2\alpha) . \]
(5.1)

Clearly the constant four-vector $\ell_\mu$ defines three distinct regimes for 0-brane reduction.
\[ -\cos(2\alpha) \propto \begin{cases} 
< 0 , & \text{for time-like } \ell_\mu \\
= 0 , & \text{for light-like } \ell_\mu \\
> 0 , & \text{for space-like } \ell_\mu 
\end{cases} \]
(5.2)

It turns out that light-like reduction has subtle differences from the other two cases. We will briefly discuss this here.

Given any fixed four-vector of the form that appears in (2.4), it is always possible to construct a second such four-vector from this one. The second four-vector is linearly independent of the first and obtained from it by reversing the signs of all the spatial components of the first such four-vector. For light-like 0-brane reduction this is important for reasons having to do with dynamics.
Thus, even for a four-vector, denoted by $\ell^{(4)}_\mu$, that satisfies $\ell^{(4)} \cdot \ell^{(4)} = 0$, there exists a second four-vector, denoted by $\ell^{(z)}_\mu$, constructed as described immediately above. These two satisfy the conditions

$$
\ell^{(4)}_\mu \eta^{\mu \nu} \ell^{(4)}_\nu = 0, \quad \ell^{(z)}_\mu \eta^{\mu \nu} \ell^{(z)}_\nu = 0, \quad \ell^{(4)}_\mu \eta^{\mu \nu} \ell^{(z)}_\nu = -1, \quad (5.3)
$$

using the parametrization provided by (2.4).

Under light-like 0-brane reduction we use $\partial_\mu = \ell^{(4)}_\mu \partial_\tau + \ell^{(z)}_\mu \partial_z$ where $\partial_\tau$ and $\partial_z$ are independent derivative operators. Since they are independent, this is no longer a 0-brane reduction because one is actually reducing to a 1-brane where the two vector fields $\partial_\tau$ and $\partial_z$ (in the mathematical sense) generate motion on a world-sheet. In this circumstance the algebra of SUSY generators take the form

$$
\{ Q_a, Q_b \} = i 2 (\gamma \cdot \ell^{(4)})_{ab} \partial_\tau + i 2 (\gamma \cdot \ell^{(z)})_{ab} \partial_z. \quad (5.4)
$$

The form of the SUSY variations in (3.6) do not exhibit any pathological behavior for any values of the parameters $\alpha$, $\beta$, and $\gamma$. The Lagrangian is a very different story in this regard and picks out a special value in the parameter space that requires a more careful analysis.

As long as $\ell_\mu \ell^\mu \neq 0$ (or alternately $\alpha \neq \pi/4$ or $\alpha \neq 3\pi/4$), the Lagrangian in (3.5) shows that by rescaling the bosons and possibly reversing the sign of the $\tau$-derivative all such reduced action can be brought to the same form. However, if $\ell_\mu \ell^\mu = 0$, then the form of (3.5) shows there is a complication — the $A$ and $B$ terms are absent from the Lagrangian and the off-shell equality in the number of bosonic versus fermionic degrees of freedom is lost.

In the case of a light-like $\ell$-parameter, the reduction must be done to a 1-brane (not a 0-brane) or equivalently to a world-sheet. But it should be clear that it is the requirement of being able to write an appropriate Lagrangian, i.e. the dynamics, that forces these changes in analysis.

For these we note $\partial_\mu = \ell^{(4)}_\mu \partial_\tau + \ell^{(z)}_\mu \partial_z$ so the Lagrangian we then find is

$$
\mathcal{L} = (\partial_\tau A)(\partial_\tau A) + (\partial_\tau B)(\partial_\tau B) \\
+ \frac{1}{2} i (\gamma \cdot \ell^{(4)})^{bc} \bar{\psi}_b \partial_\tau \psi_c + \frac{1}{2} i (\gamma \cdot \ell^{(z)})^{bc} \bar{\psi}_b \partial_z \psi_c \\
+ \frac{1}{2} F^2 + \frac{1}{2} G^2 \quad (5.5)
$$

and the SUSY transformation laws in component form realized via the supercovariant
derivative $D_a$ operator acting on the fields now become

\[
D_a A = \psi_a, \quad D_a B = i (\gamma^5)_a^b \psi_b, \\
D_a \psi_b = i (\gamma \cdot \ell^{(\pm)})_{ab} \partial_a A + i (\gamma \cdot \ell^{(\pm)})_{ab} \partial_b A \\
- (\gamma^5 \gamma \cdot \ell^{(\pm)})_{ab} \partial_a B - (\gamma^5 \gamma \cdot \ell^{(\pm)})_{ab} \partial_b B \\
- i C_{ab} F + (\gamma^5)_{ab} G, \\
D_a F = (\gamma \cdot \ell^{(\pm)}) a^b \partial_a \psi_b + (\gamma \cdot \ell^{(\pm)}) a^b \partial_a \psi_b \\
D_a G = i (\gamma^5 \gamma \cdot \ell^{(\pm)}) a^b \partial_a \psi_b + i (\gamma^5 \gamma \cdot \ell^{(\pm)}) a^b \partial_a \psi_b.
\]

So one can see the distinctive features that light-like 0-brane reduction possesses in comparison to the other cases. In fact, light-like 0-brane reduction is inconsistent with maintaining the off-shell equality of bosons and fermions.

6 Conclusions

Though we have investigated the 0-brane reduction only of the minimal off-shell 4D, $\mathcal{N} = 1$ supermultiplets, we expect our results to hold more generally. One of the most fascinating revelations of this work, is how a purely time-like 0-brane reduction is distinguished from all others. In the following, we give a detailed discussion of this point.

The results for the various on-axis reductions of the chiral supermultiplet (4.1 - 4.4), vector supermultiplet (4.5 - 4.8), and tensor supermultiplet (4.9 - 4.12), can all be substituted into equations of section 2 to show that these latter equations are valid for each of the supermultiplets, independent of which set of L-matrices and R-matrices are used for the various supermultiplets.

It is useful to reflect on analogous results for the group SU(3). The standard Gell-Mann matrices $\lambda_A$ are known to satisfy a commutator algebra

\[
[\lambda_A, \lambda_B] = i f_{ABC} \lambda_C
\]

for a very well-known set of structure constants $f_{ABC}$. Upon taking the complex conjugate of this equation, one learns that $-(\lambda_A)^*$ satisfies the same equation. The quantities $\frac{1}{2} \lambda_A$ are the SU(3) generators acting on the quark states while $-\frac{1}{2} (\lambda_A)^*$ are the SU(3) generators acting on the anti-quark states. This is similar to fact the the distinct sets of L-matrices and R-matrices satisfy the same algebra.
We know from the results in [5] that for temporal reduction, the Adinkra/\(\gamma\)-matrix Holographic Equation (2.14) holds. The discussion surrounding (2.22) and (2.23) shows this not the case if \(\alpha \neq 0\). As well, from [7], using temporal reductions, we have found a possible realization of the Hodge star operator of 4D, \(\mathcal{N} = 1\) supermultiplets which appears to have a ‘shadow’ realization acting on 1D, \(N = 4\) adinkras. This last point is critical for the definition of a class structure to exist that allows a purely one dimensional distinction between adinkras associated with the chiral, vector, and tensor supermultiplets, independent of any reference to higher dimensions. It is not at all clear whether these results can be extended to reductions that possess values of the angle \(\alpha \neq 0\).

The 0-brane reduction along the temporal direction for supermultiplets and adinkras appears to be distinguished as being different from other directions.

“There is no difference between time and the three dimensions of space except that our consciousness moves along with it.” - The Time Traveller
from H. G. Wells’ *The Time Machine*

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Added Note In Proof

T. Hübsch has suggested an alternative prescription for 0-brane projection that when utilized modifies our results. In the discussion presented in section 3, the 0-brane projection for the superspace covariant derivative takes the form

\[
D_a \rightarrow D_I .
\]  

(6.2)

The alternate suggestion is to use a projection of the form

\[
D_a \rightarrow [\mathcal{F}(\ell)]_I^J D_J .
\]  

(6.3)
where the factor $[\mathcal{F}(\ell)]_I^J$ has the form of a particular Lorentz transformation. If $\ell$ is in the forward lightcone, then this Lorentz transformation may be chosen so as to transform $\ell_\mu$ to the purely time-like axis. As there exists no Lorentz transformation that can transform a light-like $\ell_\mu$ to a time-like $\ell_\mu$, it is clear that light-like 0-brane reduction would remain distinct. Similarly, there is also no Lorentz transformation that can transform a space-like $\ell_\mu$ to a time-like $\ell_\mu$, so space-like 0-brane reduction would remain distinct as well.

Under this alternate prescription for 0-brane reduction, the only effectively distinct cases correspond to the angular parameter $\alpha$ in (2.4) being restricted to the values of 0, $\pi/4$, $\pi/2$, $3\pi/4$, or $\pi$. The first and last of these are time-like, the second and fourth are light-like, and the third is space-like.

The result of (2.14) can only occur for time-like 0-brane reduction. That ‘bridge’ between four-color adinkras and the $\gamma$-matrices of SO(1, 3) only exists for time-like 0-brane reductions. So all time-like 0-brane reductions are equivalent to a purely temporal reduction and this case remains distinct from the other cases.

A Defining A Compact Matrix Notation

In order to present our result most compactly, we have used a notation that we initiated in the work of [7]. In this work, we showed that since the L-matrices and R-matrices associated with purely time-light 0-brane reduction may be thought of as being constructed by multiplying a set of unimodal matrices of order two by elements of $S_4$, the permutation group of four elements, one of the standard notations for permutations may be adapted to our discussions.

This notation is simply a nice way to represent our matrices, most easily defined through example:

$$
\begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix} \equiv \begin{pmatrix} 2 & 4 & 1 & 3 \end{pmatrix}.
$$

(A.1)

In the expression $(ijkl)$, $i$ represents the column in which the non-zero entry of the first row sits; $j$ represents the same but for the second row, and so on. A bar over the index signifies that the element in that spot should be $-1$ instead of $+1$. Finally, we use the notation $(2 4 1 3)^t$ to refer to the transpose of the matrix in (A.1).
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