Abstract. This paper describes an enumeration of all words having a combinatoric property called “rhythmic oddity property” named rop-words. This property was introduced by Simha Aron in the 1990s. The set of rop-words is not a subset of the set of Lyndon words, but is very closed. We show that there is a bijection between some necklaces and rop-words. This leads to a formula for counting the rop-words of a given length. Keywords: Combinatoric on words, Lyndon words, Rhythmic oddity, Music formalization

1. The rhythmic oddity property

Patterns with rhythmic oddity property are combinations of durations equal to 2 or 3 units, such as the famous Aka pygmies rhythm 3222322222, and such that when placing the sequence on a circle, “one cannot break the circle into two parts of equal length whatever the chosen breaking point.” In the language of combinatorics of words, this property in terms of words \( \omega \) over the alphabet \( A = \{2, 3\} \) is defined as follows. The height \( h(\omega) \) of a word \( \omega = \omega_0\omega_1 \ldots \omega_{n-1} \) of length \( n \) is by definition the sum of its letters \( h(\omega) = \sum_{j=0}^{n-1} \omega_j \). A word \( \omega \) satisfies the rhythmic oddity property (rop) if \( h(\omega) \) is even and no cyclic shift of \( \omega \) can be factorized into two words \( uv \) such that \( h(u) = h(v) \). For short, we call rop-word a word over the alphabet \( \{2, 3\} \) satisfying the rhythmic oddity property. For instance, the word 32322 of height 12 is a rop-word, as well as all words of the form \( 32^n32^{n+1} \) for all non-negative integers \( n \), where the notation \( 2^n \) means the letter 2 is repeated \( n \) times. The properties of rop-words have been outlined in [6]. But contrary to what is sometimes written, the set of rop-words is not a subset of the set of Lyndon words. The words 222 and 23323233 are rop-words, but not Lyndon words. A Lyndon word is a string that is strictly smaller in lexicographic order than all of its rotations. Conversely, the set of Lyndon words is not included in the set of rop-words, since 2233 is a Lyndon word, but not a rop-word (the words 23 and 32 have the same height and 2332 is a rotation of 2233). We call a Lyndon rop-word a word
of the monoid \( \{2,3\}^* \) that is both a Lyndon word and a rop-word. The aim of this paper is to count the number of Lyndon rop-words and the number of rop-words of length \( n \).

2. A BIJECTION BETWEEN SOME NECKLACES AND ROP-WORDS

André Bouchet gave some characterizations of rop-words in \cite{Bouchet}. We present the main results of his paper. Let \( \varepsilon \) be the empty word and \( \omega \) a word over \( \{2,3\} \). The cycle \( \delta \) of \( \omega \) is defined by \( \delta(\varepsilon) = \varepsilon \) and \( \delta(aw) = \omega a \), for \( a \in \{2,3\} \). The rotations of \( \omega \) are the words \( \delta^k(\omega) \), for all positive integer \( k > 0 \). In his paper, Bouchet shows the following lemma.

**Lemma 1.** Let \( \omega = \omega_0 \omega_1 \ldots \omega_{n-1} \) be a word over the alphabet \( A = \{2,3\} \) of height \( 2h \). The word \( \omega \) is a rop-word if and only if the two conditions are satisfied:

(i) The length of \( \omega \) is odd, say \( 2\ell + 1 \).

(ii) The height of the prefixes of length \( \ell \) of the rotations of \( \omega \) are equal to \( h - 2 \) or \( h - 1 \).

From this lemma, André Bouchet shows the following theorem. Let \( \omega \) be a word of length \( n \) and \( d \) be an integer such that \( 0 < d \leq n/2 \). A \( d \)-pairing of \( \omega \) is a partition of the subset of indices \( \{i : 0 \leq i < n, \omega_i = 3\} \) into pairs of indices \( \{j, j + d\} \). Arithmetic operations on indices are to be understood mod \( n \).

**Theorem 2.** Let \( \omega \) be a word of even height. \( \omega \) is a rop-word if and only if the two conditions are satisfied:

(i) The length of \( \omega \) is odd, say \( 2\ell + 1 \).

(ii) \( \omega \) admits a \( \ell \)-pairing.

Let \( n_2 \) and \( n_3 \) be the number of symbols 2 and 3 in \( \omega \) and \( n = n_2 + n_3 \) be the length of \( \omega \). For a given \( n_2 \), we will use the \( d \)-pairing to show that there is a one to one correspondance between aperiodic necklaces of length \( n \) with \( n_2 \) black beads (represented by letter 2) and \( n - n_2 \) white beads (represented by letter 3) and Lyndon rop-words of length \( n' = 2n - n_2 \) with \( n'_2 = n_2 \) letters 2 and \( n'_3 = 2(n - n_2) \) letters 3. And also a one-to-one correspondance between necklaces (eventually periodic) of length \( n \) with \( n_2 \) black beads (represented by letter 2) and \( n - n_2 \) white beads (represented by letter 3) and rop-words. The correspondance is obtained by adding or removing the letters 3 coming from the pairing. Let us examine an example. (See fig. [1])

Fix \( n_2 \), for instance \( n_2 = 3 \), and let \( n \) be \( n = 5 \). The word 2233233 is a (Lyndon) rop-word with odd length 7 (\( n'_2 = 3 \), \( n'_3 = 4 \)) since it has a 3-pairing. Put the word on a circle, starting from the bottom and turn counterclockwise as shown on the figure [1]. Now discard the second 3 of each pairing \( (3,3) \) turning counterclockwise. Reading the remaining word clockwise starting from the bottom gives the word 22332, one of the two necklaces of length 5 with 3 letters 2. Conversely, starting from the word 22233, it is easy to add a 3-pairing by doubling each letter 3, with respect to the counterclockwise tour.

Let \( \omega = \omega_0 \omega_1 \ldots \omega_{n-1} \) be a word of \( \{2,3\}^* \) and \( p \) coprime with \( n \). Denote by \( \omega^{(p)} = x_0x_1 \ldots x_{n-1} \) the word obtained by reading all letters of \( \omega \) by step \( p \), starting from \( \omega_0 \). Namely, each letter of \( \omega^{(p)} \) is \( x_j = \omega_k \) with \( k = jp \mod p, 0 \leq j < p \). For instance, the word \( \omega = 2233233 \) depicted on fig. [1] with \( n = 7 \) and \( \ell = 3 \) becomes \( \omega^{(2)} = 233233 \). A. Bouchet shows
Theorem 3. Let $\omega$ be a word of even height. $\omega$ is a rop-word if and only if the two conditions are satisfied:

(i) The length of $\omega$ is odd, say $2\ell + 1$.

(ii) $\omega^{(\ell)}$ admits a 1-pairing.

In other words, we can always transform a (resp. Lyndon) rop-word $\omega$ of length $2\ell + 1$ by a one-to-one map $\phi$ such that the letters 3 in $\omega^{(\ell)}$ are always coupled by subwords 33. The bijection $\psi$ sending $2 \to 0$ and $33 \to 1$ maps $\omega^{(\ell)}$ to a word $\omega' \in \{0, 1\}^*$ corresponding to a (resp. aperiodic) necklace.

$$\omega \xrightarrow{\phi} \omega^{(\ell)} \xrightarrow{\psi} \omega'$$

The table 1 shows the first Lyndon rop-words for $n_2 = 3$ and the corresponding aperiodic necklaces

| Aperiodic Necklaces | n  | Lyndon Rop-words | n'  |
|---------------------|----|------------------|-----|
| 0001                | 4  | 22323            | 5   |
| 00011               | 5  | 2233233          | 7   |
| 00101               | 5  | 2323233          | 7   |
| 00111               | 6  | 223332333        | 9   |
| 001101              | 6  | 232332333        | 9   |
| 0011101             | 6  | 232333233        | 9   |

Table 1. Correspondance for $n_2 = 3$

Conversely, starting from the representing Lyndon word of a aperiodic necklace $\omega'$, we construct the word $\omega^{(\ell)}$ by the bijection $\psi^{-1}$ sending $0 \to 2$ and $1 \to 33$, and the word $\omega$ by applying $\phi^{-1}$. By construction, the height $h(\omega^{(\ell)})$ is even and also $h(\omega)$. Moreover, $\omega$ has a $\ell$-pairing and then is a rop-word.
3. Enumeration of the rop-words

The number of necklaces (see [1] for details, and also [5] for applications to music theory) with $n_2$ black beads and $n_3/2$ white beads derives from the generating function of the action of the cyclic group

$$Z(C_{n_2}, x) = \frac{1}{n_2} \sum_{d|n_2} \varphi(d) x^{n_2/d}$$

where the sum is over all divisors $d$ of $n_2$ and $\varphi$ is the Euler totient function, according to the substitution of $x_j$ by $\frac{1}{1-x_j}$. The development gives the coefficients of $x_j$ which are precisely the number of necklaces with $n_2$ black beads and $j$ white beads. For example, if $n_2 = p$ is prime, the development leads to the following equations:

$$Z(C_p, x) = \frac{1}{p} \left( \varphi(1)x_p^p + \varphi(p)x_p \right)$$

$$= \frac{1}{p} \left( \frac{1}{(1-x)^p} + \frac{p-1}{p} \frac{1}{1-x^p} \right)$$

$$= \frac{1}{p} \left( 1 + \sum_{n=1}^\infty p(p+1)\ldots\left(p+n-1\right) \frac{x^n}{n!} \right) + \frac{p-1}{p} \left( \sum_{n=0}^\infty x^{np} \right)$$

$$= 1 + \frac{1}{p} \sum_{n=1}^\infty \left( p+n-1 \right) x^n + \frac{p-1}{p} \left( x^p + x^{2p} + x^{3p} + \ldots \right)$$

$$= 1 + \sum_{n=1}^\infty a_n x^n$$

with

$$a_n = \begin{cases} \frac{(p+n-1)}{p+n} & \text{if } n \not\equiv 0 \mod p \\ \frac{(p+n-1)}{p+n} + p-1 & \text{if } n \equiv 0 \mod p \end{cases}$$

The table with $n_2$ on the horizontal axis and $n_3$ on the vertical axis shows the number of rop-words for $n_2$ and $n_3$ fixed. The number of rop-words of length $n$ is given by summing along the diagonal $n_2 + n_3 = n$. Each column of the table is obtained from the development of the generating function $Z(C_{n_2}, x)$. For $n_2$ prime, the coefficients agree with the formula of $a_n$ given above.

|   | 1  | 3  | 5  | 7  | 9  | 11 | 13 | 15 | 17 |
|---|----|----|----|----|----|----|----|----|----|
| 2 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |
| 4 | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
| 6 | 1  | 4  | 7  | 12 | 19 | 26 | 35 | 46 | 57 |
| 8 | 1  | 5  | 14 | 30 | 55 | 91 | 140| 204| 285|
| 10| 1  | 7  | 26 | 66 | 143| 273| 476| 776| 1197|
| 12| 1  | 10 | 42 | 132| 335| 728| 1428|2586|4389|

**Table 2. Number of rop-words $(n_2, n_3)$**

In each column, we recover the number of binary necklaces with length $n_2 + q$ and density $q = n_3/2$ given by the rhs of the next formula. From the bijection of the previous section, it follows that the number $R(n_2, n_3)$ of rop-words with $n_2$
symbols 2 and $n_3$ symbols 3 is the number of binary necklaces of length $n_2 + n_3/2$ and density $n_3/2$,

\[(4) \quad R(n_2, 2q) = \frac{1}{n_2 + q} \sum_{d \mid \text{gcd}(n_2 + q, q)} \varphi(d) \left( \frac{(n_2 + q)/d}{q/d} \right), \quad q = 1, 2, 3, \ldots\]

By computing Lyndon words on alphabet $\{2, 3\}$ and deleting those which are not rop-words, we get the table of the number of Lyndon rop-words for $n_2$ and $n_3$ fixed, with $n_2$ on the horizontal axis and $n_3$ on the vertical axis. The total number of Lyndon rop-words of length $n$ is obtained by summing along $n_2 + n_3 = n$. The differences between the tables 2 and 3 are in italics.

| $n$ | $1$ | $3$ | $5$ | $7$ | $9$ | $11$ | $13$ | $15$ | $17$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $n_2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $n_3$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $R(n_2, 2q)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $L(n_2, 2q)$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 |

Table 3. Number of Lyndon rop-words $(n_2, n_3)$

In each column of table 3, we recover the number of fixed density Lyndon words given by the following formula, with $n_3 = 2q$. It follows from the previous section, that the number $L(n_2, n_3)$ of Lyndon rop-words with $n_2$ symbols 2 and $n_3$ symbols 3 is

\[(5) \quad L(n_2, 2q) = \frac{1}{n_2 + q} \sum_{d \mid \text{gcd}(n_2 + q, q)} \mu(d) \left( \frac{(n_2 + q)/d}{q/d} \right), \quad q = 1, 2, 3, \ldots\]

where $\mu$ is the Mobius function.

By summing these formulas along a diagonal $n = n_2 + n_3$, we get the number $L_n$ of Lyndon rop-words of length $n$ and the number $R_n$ of rop-words of the length $n$:

\[(6) \quad L_n = \sum_{n_2 + n_3 = n} L(n_2, n_3) = \sum_{p=0}^{(n-3)/2} L(2p+1, n-2q-1)\]

and

\[(7) \quad R_n = \sum_{n_2 + n_3 = n} R(n_2, n_3) = 1 + \sum_{p=0}^{(n-3)/2} R(2p+1, n-2p-1)\]

These numbers are tabulated as follows: If $n$ is prime, the difference between the

| $n$ | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 | 27 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $R_n$ | 2 | 3 | 5 | 10 | 19 | 41 | 94 | 211 | 493 | 1170 | 2787 | 6713 | 16274 |
| $L_n$ | 1 | 2 | 4 | 8 | 18 | 40 | 90 | 210 | 492 | 1164 | 2786 | 6710 | 16264 |

Table 4. Numbers of Lyndon rop-words and rop-words of length $n$
cardinal of the two sets is 1, since the word $2^n$ (where the letter 2 is repeated $n$ times) is a rop-word but not a Lyndon word. If $n$ is a product or a power of primes, some periodic words appear that are rop-words but not Lyndon words. This explains the differences between the set of rop-words and the set of Lyndon rop-words. For instance, if $n = 9$, $(233)^3$ is a rop-word but not a Lyndon word. The same is true for the words $(22323)^3$, $(233)^5$ and $(233333)^3$ of length 15. For $n_2 = 9$ and $n_3 = 12$, there are 333 Lyndon rop-words and 335 rop-words. The two non Lyndon rop-words are: $(2233233)^3$ and $(2323233)^3$.

4. Generalization: from rop to rap words

4.1. First generalization. Let $s$ be an integer $\geq 2$. The rhythmic oddity property could be generalized in two ways. The first way is as follows.

Definition 4. A word $\omega \in \{2,3\}^*$ is a $s$-rop word if
(i) $h(\omega) \equiv 0 \mod s$
(ii) No cyclic shift of $\omega$ can be factorized into $s$ words $u_1, u_2, \ldots, u_s$ such that $h(u_1) = h(u_2) = \ldots = h(u_s)$

A Lyndon $s$-rop word is both a $s$-rop word and a Lyndon word. For instance, if $s = 3$, the first 3-rop words are: 2223, (333), 22323, (3333), 222323, 222223, 233233, 2233233, 2332323, 2233233, (333333). Non Lyndon words are given in parenthesis. A computation of the number of the first 3-rop words of length $n$ is given in table 5.

| $n$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|-----|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| $R_{(3)}^n$ | 2 | 3 | 1 | 6 | 11 | 6 | 25 | 46 | 41 | 117 | 232 | 278 | 631 | 1237 |
| $L_{(3)}^n$ | 1 | 2 | 1 | 5 | 9 | 6 | 22 | 45 | 40 | 116 | 226 | 278 | 620 | 1236 |

Table 5. Number of 3-rop words

For $s = 4$, the first 4-rop words are: 22233, 22323*, (222222), 223333, 232333, (233333), 2222233, 2222323, 22223323, 22232233, 22233233, 23333333. Non Lyndon words are in parenthesis. The star indicates 2-rop words. A computation of the number of the first 4-rop words of length $n$ is given in table 6.

| $n$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|-----|---|---|---|---|---|----|----|----|----|----|----|----|----|
| $R_{(4)}^n$ | 2 | 4 | 4 | 5 | 13 | 27 | 47 | 50 | 131 | 284 | 479 | 685 | 1450 |
| $L_{(4)}^n$ | 2 | 2 | 4 | 5 | 12 | 24 | 47 | 50 | 131 | 279 | 473 | 683 | 1440 |

Table 6. Number of 4-rop words

4.2. Second generalization. The second way of generalization is to change the alphabet and to consider words over $A = \{1,2,\ldots,s\}$.

Definition 5. A word $\omega \in \{1,2,\ldots,s\}^*$ has the rhythmic arity property (rap) of order $s$ if
(i) $h(\omega) \equiv 0 \mod s$
(ii) No cyclic shift of $\omega$ can be factorized into $s$ non-empty words $u_1, u_2, \ldots, u_s$ such that

$$h(u_1) = h(u_2) = \ldots = h(u_s)$$

For short, we call $s$-rap-word a word with the rhythmic arity property of order $s$. For example, on the alphabet $\{1,2,3\}$, the words 111 and 333 are not 3-rap-words, but 123 and 132 are. The word 11133 is not a 3-rap-word since the subwords $u_1 = 111$, $u_2 = 3$ and $u_3 = 3$ have the same height, but the word 11313 is a rap-word. A Lyndon $s$-rap-word is both a $s$-rap-word and a Lyndon word. For instance, 11133, 11313, 11322 are Lyndon 3-rap-words.

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