ON A CLASS OF HYPOELLIPTIC OPERATORS WITH UNBOUNDED COEFFICIENTS IN $\mathbb{R}^N$

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Abstract. We consider a class of non-trivial perturbations $A$ of the degenerate Ornstein-Uhlenbeck operator in $\mathbb{R}^N$. In fact we perturb both the diffusion and the drift part of the operator (say $Q$ and $B$) allowing the diffusion part to be unbounded in $\mathbb{R}^N$. Assuming that the kernel of the matrix $Q(x)$ is invariant with respect to $x \in \mathbb{R}^N$ and the Kalman rank condition is satisfied at any $x \in \mathbb{R}^N$ by the same $m < N$, and developing a revised version of Bernstein's method we prove that we can associate a semigroup $\{T(t)\}$ of bounded operators (in the space of bounded and continuous functions) with the operator $A$. Moreover, we provide several uniform estimates for the spatial derivatives of the semigroup $\{T(t)\}$ both in isotropic and anisotropic spaces of (Hölder-) continuous functions. Finally, we prove Schauder estimates for some elliptic and parabolic problems associated with the operator $A$.

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1. Introduction

In the last decades the interest towards elliptic and parabolic operators with unbounded coefficients in unbounded domains has grown considerably due to their applications to stochastic analysis and mathematical finance.

The literature on uniformly elliptic operators with unbounded coefficients in $\mathbb{R}^N$ is nowadays rather complete (we refer the interested reader, e.g., to [3]). The picture changes drastically when one considers degenerate elliptic operators with unbounded coefficients. The prototype of such
operators is the degenerate Ornstein-Uhlenbeck operator defined on smooth functions by

$$\mathcal{A} \varphi(x) = \sum_{i,j=1}^{N} q_{ij} D_{ij} \varphi(x) + \sum_{i,j=1}^{N} b_{ij} x_j D_i \varphi(x), \quad x \in \mathbb{R}^N,$$

(1.1)

where $Q = (q_{ij})$ and $B = (b_{ij})$ are suitable square matrices such that $Q$ is singular and the condition $\det Q > 0$ is nevertheless satisfied for any $t > 0$. Here,

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} ds, \quad t > 0.$$

The condition $\det Q_t > 0$ is equivalent to the well-known Kalman rank condition which requires that

$$\text{rank}[Q^{\frac{1}{2}}, B Q^{\frac{1}{2}}, \ldots, B^m Q^{\frac{1}{2}}] = N,$$

(1.2)

for some $m < N$. In particular, $\mathcal{A}$ is hypoelliptic in Hörmander’s sense.

A suitable change of the orthonormal basis of $\mathbb{R}^N$ (see Remark 2.5) allows to rewrite the operator $\mathcal{A}$ on smooth functions $\varphi$ as

$$\mathcal{A} \varphi(x) = \sum_{i,j=1}^{N} \tilde{q}_{ij} D_{ij} \varphi(x) + \sum_{i,j=1}^{N} \tilde{b}_{ij} x_j D_i \varphi(x), \quad x \in \mathbb{R}^N,$$

(1.3)

for some positive definite and not singular $p_0 \times p_0$ matrix $\tilde{Q} = (\tilde{q}_{ij})$ and some $p_0 \in \{1, \ldots, N-1\}$.

In [17] Lunardi proves that one can associate a semigroup of bounded operators $\{T(t)\}$ in $C_b(\mathbb{R}^N)$ (the space of all bounded and continuous functions) with the operator $\mathcal{A}$ in a natural way, i.e., for any $f \in C_b(\mathbb{R}^N)$, $T(t)f$ is the value at $t > 0$ of the (unique) classical solution to the homogeneous Cauchy problem

$$\begin{cases}
D_t u(t, x) = \mathcal{A} u(t, x), & t \in [0, +\infty[, \quad x \in \mathbb{R}^N, \\
u(0, x) = f(x), & x \in \mathbb{R}^N,
\end{cases}$$

(1.4)

where by classical solution we mean a function $u$ which (i) is once continuously differentiable with respect to the time variables and twice continuously differentiable with respect to the spatial variable in $[0, +\infty[ \times \mathbb{R}^N$, (ii) is continuous in $[0, +\infty[ \times \mathbb{R}^N$ and bounded in $[0, T_0] \times \mathbb{R}^N$ for any $T_0 > 0$ and (iii) solves (1.4).

One of the main peculiarities of the Ornstein-Uhlenbeck operator is that an explicit representation formula for the associated semigroup is available. This fact allows the author of [17] to prove uniform estimates for the spatial derivatives of the function $T(t)f$ when $t$ approaches 0 and $f$ belongs to various spaces of (Hölder-) continuous functions. In fact, the behavior of the spatial derivatives of $T(t)f$ depends on the variable along which one differentiates. As a byproduct, this shows that the right (Hölder-) spaces where to study the semigroup $\{T(t)\}$ are not the usual ones but rather anisotropic spaces modelled on the degeneracy of the operator $\mathcal{A}$. Denoting, roughly speaking, by $C^\alpha(\mathbb{R}^N)$ these anisotropic spaces, Lunardi shows that

$$\|T(t)f\|_{C^\alpha(\mathbb{R}^N)} \leq C t^{-\frac{2\alpha}{N}} \|f\|_{C^\alpha(\mathbb{R}^N)}, \quad t \in [0, 1],$$

(1.5)

for any $0 < \alpha \leq \theta$ and some positive constant $C$, independent of $t$, i.e., what one can expect in the non-degenerate case when $\mathcal{C}^\alpha$ and $C^\theta$ are the usual Hölder spaces, even for unbounded coefficients; see e.g., [2, 18]. Estimate (1.5) represents the key stone to apply an abstract interpolation argument from [16] to prove optimal Schauder estimates for the solution both to the elliptic equation

$$\lambda u(x) - \mathcal{A} u(x) = h(x), \quad x \in \mathbb{R}^N, \quad \lambda > 0,$$

(1.6)

and to the non-homogeneous Cauchy problem

$$\begin{cases}
D_t u(t, x) = \mathcal{A} u(t, x) + g(t, x), & t \in [0, T_0[, \quad x \in \mathbb{R}^N, \\
u(0, x) = f(x), & x \in \mathbb{R}^N,
\end{cases}$$

(1.7)

when $f, g, h$ are suitable continuous functions such that $g(t, \cdot)$, $f$, $h$ have some additional degrees of smoothness.
Recently, the second author, in [13, 14], has extended these results to some non-trivial perturbations of the Ornstein-Uhlenbeck operator in (1.1). More precisely, in [13, 14] the operator (1.3) has been studied under the assumption that

$$p_0 \geq N/2, \quad \hat{B} = \begin{pmatrix} \hat{B}_1 & \hat{B}_2 \\ \hat{B}_3 & \hat{B}_4 \end{pmatrix},$$

the $(N - p_0) \times p_0$ matrix $\hat{B}_3$ has full rank, and assuming that the matrix $\hat{Q}$ depends on $x \in \mathbb{R}^N$ and its entries are possibly unbounded functions at infinity. These assumptions imply that the Kalman rank condition (1.2) is satisfied at any $x \in \mathbb{R}^N$, with $m = 1$.

To prove the crucial estimates (1.5) a different technique than that in [17] has been applied since in this new situation no explicit representation formulas for the associated semigroup is available. More precisely, such estimates have been obtained by developing a variant of the classical Bernstein method in [1].

Recently, the results in [17] have been generalized, both with analytic and probabilistic methods, in [22, 23, 25] to non-trivial perturbations of the operator $\mathcal{A}$ in (1.3) in which an additional unbounded drift term is added. More specifically, Saintier in [25] considers the case when the differential operator is of type $\mathcal{A} = \mathcal{A} + \sum_{j=1}^{p_0} F_j D_j$, with $\mathcal{A}$ being given by (1.3), with an even $N$ and $p_0 = N/2$ and $Q = B = I$. Here, $F$ is any smooth function with bounded derivatives up to the third-order. This operator arises e.g., in the study of the motion of a particle $y$ of mass one subject to a force field depending on $y$ and its first-order derivative, perturbed by a noise. We refer the interested reader to [8] for further details. Applying the same techniques as those in [13, 14], Saintier proves optimal Schauder estimates for both the solutions to (1.6) and (1.7). Note that in this situation, the operator $\mathcal{A}$ satisfies the Kalman rank condition with $m = 1$. The same problem is investigated with a stochastic approach in [22].

Very recently the results in [22, 25] have been generalized in [23] with both analytic and stochastic methods to the case when $\mathcal{A} = \mathcal{A} + \sum_{j=1}^{p_0} F_j D_j$ with some $p_0 < N$, $\mathcal{A}$ still being given by (1.3).

In this paper we extend a part of the results in [13, 14, 22, 23, 25] considering a class of elliptic operators that, up to a change of the coordinates, may be written in the following form

$$\mathcal{A} \varphi(x) = \sum_{i,j=1}^{p_0} q_{ij}(x) D_{ij} \varphi(x) + \sum_{i,j=1}^{N} b_{ij} x_j D_i \varphi(x) + \sum_{j=1}^{p_0} F_j(x) D_j \varphi(x), \quad x \in \mathbb{R}^N, \quad (1.8)$$

for some $p_0 < N$, where the matrices $Q_0(x) = (\hat{q}_{ij}(x))$, defined by $\hat{q}_{ij} = q_{ij}$ if $i, j \leq p_0$ and $\hat{q}_{ij} \equiv 0$ otherwise, and $B$ satisfy the Kalman rank condition (1.2) for some $m$ independent of $x$. We assume that $F : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth function with derivatives whose growth at infinity is comparable with the growth of the minimum eigenvalue of the matrix $\hat{Q}^\ast(x)$. In the particular case when $F \equiv 0$, our results apply to any elliptic operator of the type

$$\mathcal{A} \varphi(x) = \sum_{i,j=1}^{N} q_{ij}(x) D_{ij} \varphi(x) + \sum_{i,j=1}^{N} b_{ij} x_j D_i \varphi(x), \quad x \in \mathbb{R}^N, \quad (1.9)$$

when the Kalman rank condition is satisfied, by any fixed $x \in \mathbb{R}^N$, for some $m < N$, independent of $x$.

The paper is organized as follows. First, in Section 2 we introduce the function spaces we deal with, as well as some notation. Moreover, we introduce the Hypotheses that will be assumed in the whole of the paper and we recall some preliminary results mainly from [13]. Next, in Section 3, the main part of this paper, we prove uniform estimates of the spatial derivatives for the semigroups associated with the family of non-degenerate elliptic operators $\mathcal{A}_\varepsilon := \mathcal{A} + \varepsilon \Delta_\ast$ with $\Delta_\ast$ being the Laplacian containing the missing second order derivatives, i.e., $\Delta_\ast = D_{(p_0 + 1)(p_0 + 1)} + \cdots + D_{N,N}^2$. More precisely, we show that the constants appearing in the estimates can be chosen to be independent of $\varepsilon \in [0,1]$. Then, in Section 4, using these estimates, we prove that we can associate a semigroup $\{T(t)\}$ of bounded operators in $C_b(\mathbb{R}^N)$ with the operators $\mathcal{A}$ in (1.8) and (1.9) and that the uniform estimates of the preceding section may be extended to $\{T(t)\}$. We also state some remarkable continuity properties of the semigroup $\{T(t)\}$. Further, we show that we can associate a "weak" generator with the semigroup $\{T(t)\}$, a generalization of the classical concept of infinitesimal generator of a strongly continuous semigroup, and we give a characterization of its
domain. In Section 5, we prove Schauder estimates for the distributional solutions to the elliptic equation \( (1.6) \) and the non-homogeneous Cauchy problem \( (1.7) \). Finally, in Appendix A we prove some technical lemmas that are used in the proof of the uniform estimates.

2. Main assumptions and preliminaries

In this section we introduce the main assumptions on the operators we consider. We also fix the notation and the define the function spaces we use in this paper.

2.1. Hypotheses. The assumptions on the coefficients of the operator \( \mathcal{A} \) in \( (1.8) \) and \( (1.9) \), we always put throughout this paper are the following. We begin by considering the case when \( \mathcal{A} \) is given by \( (1.8) \).

**Hypotheses 2.1.**

(i) \( Q(x) = (q_{ij}(x)) \) is a \( p_0 \times p_0 \) symmetric matrix, with entries which belong to \( C^\kappa(\mathbb{R}^N) \) for some \( \kappa \in \mathbb{N}, \kappa \geq 3 \), such that

\[
\sum_{i,j=1}^{p_0} q_{ij}(x)|\xi_i\xi_j| \geq \nu(x)|\xi|^2, \quad x \in \mathbb{R}^N, \ \xi \in \mathbb{R}^r, \tag{2.1}
\]

for some positive function \( \nu \) such that \( \inf_{\mathbb{R}^N} \nu(x) = \nu_0 > 0 \). Further,

\[
|D^\alpha q_{ij}(x)| \leq C_{||\alpha||}|x|^{1-||\alpha||^+}\sqrt{\nu(x)}, \quad x \in \mathbb{R}^N, \ i,j = 1, \ldots, p_0, \ ||\alpha|| \leq \kappa, \tag{2.2}
\]

for some positive constant \( C_{||\alpha||} \).

(ii) There exist integers \( p_1, \ldots, p_r \) with \( p_0 \geq p_1 \geq \ldots \geq p_r \) such that the matrix \( B \) can be split into blocks as follows:

\[
B = \begin{pmatrix}
\ast & \ast & \cdots & \ast \\
B_1 & \ast & \cdots & \ast \\
0 & B_2 & \ast & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & B_r & \ast
\end{pmatrix}, \tag{2.3}
\]

where \( B_h \) is a \( p_h \times p_{h-1} \) matrix with full rank, i.e., \( \text{rank}(B_h) = p_h \) \( (h = 1, \ldots, r) \).

(iii) \( F \in C^\kappa(\mathbb{R}^N, \mathbb{R}^{p_0}) \) and

\[
|D^\alpha F(x)| \leq C\sqrt{\nu(x)}, \quad x \in \mathbb{R}^N, \ ||\alpha|| \leq \kappa. \tag{2.4}
\]

**Remark 2.2.**

(i) Since the coefficients \( q_{ij} \) \( (i,j = 1, \ldots, p_0) \) need to satisfy both \( (2.1) \) and \( (2.2) \), the \( q_{ij} \)'s \( (i,j = 1, \ldots, p_0) \) and \( \nu \) may grow at most as \( |x|^2 \) as \( |x| \to +\infty \).

(ii) Hypotheses 2.1 guarantee that the operator \( \mathcal{A} \) is hypoelliptic in the sense of Hörmander, at any \( x \in \mathbb{R}^N \).

The hypotheses on the coefficients of the operator \( \mathcal{A} \) in \( (1.9) \) are the following.

**Hypotheses 2.3.**

(i) \( Q = (q_{ij}) \) is a \( N \times N \) symmetric matrix with \( q_{ij} \in C^\kappa(\mathbb{R}^N) \) \( (i,j = 1, \ldots, N) \) for some \( \kappa \in \mathbb{N}, \kappa \geq 3 \), and there exists a function \( \nu : \mathbb{R}^N \to ]0, +\infty[ \) such that \( \nu_0 := \inf_{x \in \mathbb{R}^N} \nu(x) > 0 \) and

\[
\sum_{i,j=1}^{N} q_{ij}(x)|\xi_i\xi_j| \geq \nu(x)|\xi|^2, \quad \xi \in (\ker(Q(0)))^\perp, \quad x \in \mathbb{R}^N.
\]

(ii) For any \( \alpha \in \mathbb{N}^N_0 \) with length at most \( \kappa \), there exists a positive constant \( C = C_{||\alpha||} \) such that

\[
|D^\alpha q_{ij}(x)| \leq C|x|^{1-||\alpha||^+}\sqrt{\nu(x)}, \quad x \in \mathbb{R}^N, \ i,j = 1, \ldots, N, \ ||\alpha|| \leq \kappa. \tag{2.4}
\]

(iii) The kernel of the matrix \( Q(x) \) is independent of \( x \in \mathbb{R}^N \) and it is a proper subspace of \( \mathbb{R}^N \). Moreover, \( \ker(Q(0)) \) does not contain non-trivial subspaces which are invariant for \( B^* \).

**Remark 2.4.** Note that Hypothesis 2.3(iii) can be rewritten in one of the following equivalent forms:

(a) the matrix \( Q_t(x) = \int_0^t e^{sB}Q(x)e^{sB^*}ds \) is positive definite for any \( t > 0 \) and any \( x \in \mathbb{R}^N \);

(b) for every \( x \in \mathbb{R}^N \), the kernel of \( Q(x) \) is a proper subspace of \( \mathbb{R}^N \) and is invariant under the action of \( B^* \).
(b) there exists \( r < N \) such that the rank of the block matrix 
\[
[Q(x), BQ(x), B^2Q(x), \ldots, B^rQ(x)]
\]

is \( N \) for any \( x \in \mathbb{R}^N \).

To prove this claim, it suffices to adapt to our situation the proof of [12, Proposition A.1]. For the reader's convenience we give a detailed proof in the appendix (see Lemma A.1).

**Remark 2.5.** If the coefficients of the operator \( \mathcal{A} \) in (1.9) satisfy Hypotheses 2.1, then one can find a suitable change of variables which transforms \( \mathcal{A} \) in an operator of the type (1.8) (with \( F \equiv 0 \)). To check this fact, let us denote by \( \{V_k : k \in \mathbb{N}\} \) the sequence of nested vector spaces defined by
\[
V_k = (\ker(Q(0)) \cap \ker(Q(0)B^*) \cap \ldots \cap \ker(Q(0)(B^*)^k))^\perp,
\]
for any \( k \in \mathbb{N} \). In view of Lemma A.1 and Hypothesis 2.3(iii), there exists a positive integer \( p_0 < N \) such that \( V_{p_0} = \mathbb{R}^N \) and \( V_k \) is properly contained in \( V_{k+1} \) if \( k < p_0 \).

Let now \( W_0 = V_0 \) and \( W_k \) be the orthogonal of \( V_{k-1} \) in \( V_k \), for any \( k = 1, \ldots, p_0 \). Let \( p_k = \dim(W_k) \) for any \( k \leq p_0 \). Of course, \( \mathbb{R}^N = \bigoplus_{k=0}^{p_0} W_k \). Fix an orthonormal basis \( \{e'_1, \ldots, e'_N\} \) of \( \mathbb{R}^N \) consisting of vectors of the spaces \( W_k \) \((k = 0, \ldots, r)\). Adapting the proof of [12, Proposition 2.1] to our situation, we can show that in the basis \( \{e'_1, \ldots, e'_N\} \) the operator \( \mathcal{A} \) may be written as in (1.8) with the coefficients satisfying Hypotheses 2.1.

In view of Remark 2.5, without loss of generality, throughout the paper, we can limit ourselves to dealing with the case when \( \mathcal{A} \) is given by (1.8) and its coefficients satisfy Hypotheses 2.1.

2.2. General notation.

**Functions.** For any real-valued function \( u \) defined on a domain of \( \mathbb{R} \times \mathbb{R}^N \), we indiscriminately write \( u(t, \cdot) \) and \( u(t) \) when we want to stress the dependence of \( u \) on the time variable \( t \). Moreover, for any smooth real-valued function \( v \) defined on a domain of \( \mathbb{R}^N \), we denote by \( Dv \) its gradient and by \( |Dv(x)| \) the Euclidean norm of \( Dv(x) \) at \( x \). Similarly, by \( D^k v \) \((k \in \mathbb{N})\) we denote the vector consisting of all the \( k \)th order derivatives of \( v \) with no repetitions. This means that we identify \( k \)th order derivatives of type \( \frac{\partial^k v}{\partial x_{i_1} \ldots \partial x_{i_k}} \) and \( \frac{\partial^k v}{\partial x_{i_1} \ldots \partial x_{i_k}} \) when \((j_1, \ldots, j_k)\) is a permutation of \((i_1, \ldots, i_k)\).

We agree that the vector \( D^k v \) contains only derivatives \( \frac{\partial^k v}{\partial x_{i_1} \ldots \partial x_{i_k}} \) with \( i_1 \leq i_2 \leq \ldots \leq i_k \). We denote by \( |D^k v(x)| \) the Euclidean norm of the vector \( D^k v(x) \).

**Asymptotics.** Given any real-valued function \( u \) defined in some neighborhood of \(+\infty\) and \( m \in \mathbb{N} \), we use the usual notation \( u = o(s^m) \) when \( \lim_{s \to +\infty} s^{-m}u(s) = 0 \). If \( \{u_a\}_{a \in \mathcal{F}} \) is a family of functions which are defined in a right-neighborhood of \( 0 \) (independent of \( a \)), we write \( u_a = o(t^m) \) (for some \( m \in \mathbb{N} \)) when \( \lim_{t \to 0^+} t^{-m}u_a(t) = 0 \) for any of such parameters.

**Matrices.** We denote the \( k \times k \) identity matrix by \( I_k \) and the transposed of a matrix \( A \) by \( A^* \). For any matrix \( A \) we denote by \( \|A\| \) its Euclidean norm. If \( A \) is symmetric, \( \lambda_{\min}(A) \) is the minimum eigenvalue of \( A \). Finally, we use the notation \( \*^s \) to denote matrices when we are not interested in their entries.

**Miscellanea.** We agree that \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Given a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m \), we denote by \( \|\alpha\| := \sum_{i=1}^m \alpha_i \) its length. Moreover, by \( a^s \) we denote the maximum between \( a \in \mathbb{R} \) and \( 0 \). For any \( R > 0 \), we denote by \( B(R) \) the open ball in \( \mathbb{R}^N \) centered at \( x = 0 \) and with radius \( R \). \( \overline{B(R)} \) is its closure.

2.3. Ordering the derivatives of smooth functions.** Here, we introduce a splitting of the vector of all the derivatives of a function \( u : \mathbb{R}^N \to \mathbb{R} \) of a given order into sub-blocks. This splitting will be extensively used in Section 3.

Given \( k, q \in \mathbb{N} \), we introduce a (total) ordering \( \preceq_q \) in the set \( \mathcal{F}_{k,q} \) of all the multi-indices in \( \mathbb{N}_0^{q+1} \) with length \( k \). We say that \( (m_0, \ldots, m_q) \preceq_q (m'_0, \ldots, m'_q) \) if there exists \( h = 0, \ldots, q \) such that \( m_j = m'_j \) for any \( j = 0, \ldots, h-1 \) and \( m_h > m'_h \). We thus may order the elements of \( \mathcal{F}_{k,q} \) in a sequence \( i_1^{(k,q)}, \ldots, i_q^{(k,q)} \). Here, \( c_{k,q} := \binom{q+k}{q} \).

Now, to order the entries of the vector \( D^k u \) \((k \in \mathbb{N})\) we proceed as follows. Let \( \{p_0, \ldots, p_r\} \) be a given set of non-increasing integers such that \( p_0 + \cdots + p_r = N \), throughout the paper these will be
fixed as in Hypotheses 2.1 (ii). We set \( p_{-1} := 0 \) and introduce the sets \( \mathcal{I}_j = \{ i \in \mathbb{N} : r_j < i \leq r_{j+1} \} \), \((j = 0, \ldots, r)\), where \( r_j = \sum_{l=0}^{j} p_{l-1} \) for any \( l = 0, \ldots, r + 1 \). Moreover, we split \( \mathbb{R}^N \) into the direct sum \( \mathbb{R}^N = \bigotimes_{j=0}^{N} \mathbb{R}^{p_j} \). Hence, any multi-index \( \alpha = (\alpha_0, \ldots, \alpha_r) \) with \( \alpha_j \in \mathbb{N}_0^p \) \((j = 0, \ldots, r)\) and we can write \( |\alpha| := (|\alpha_0|, \ldots, |\alpha_r|) \). We can now split the vector \( D^k u \) as follows:

(i) we split \( D^k u \) into blocks according to the rule: \( D^k u = (D^k u_{1, \ldots, c_{k,r}}, \ldots, D^k u_{r, \ldots, c_{k,r}}) \), where \( D^k u_{j} (j = 1, \ldots, c_{k,r}) \) contains all the derivatives \( D^\alpha \varphi \) of order \( k \) such that \( |\alpha| = i_j (k, r) \), where (ii) we order the entries of the vectors \( D^k u_{j} \) \((j = 1, \ldots, c_{k,r})\) according to the following rule: if \( D^\alpha u \) and \( D^\beta u \) belong to the block \( D^k u_{i} \), we say that \( D^\alpha u \) precedes \( D^\beta u \) if \( \beta \leq N \) \( \alpha \).

2.4. Hölder spaces. Here, we introduce most of the isotropic function spaces we deal with in this paper.

**Definition 2.6.** For any \( k \geq 0 \), \( C^k_\theta (\mathbb{R}^N) \) denotes the subset of \( C^k (\mathbb{R}^N) \) of functions which are bounded together with their derivatives up to the \([k]^{\text{th}}\) order. We endow it with the norm

\[
||u||_{C^k_\theta (\mathbb{R}^N)} = \sum_{|\alpha| \leq [k]} ||D^\alpha f||_{\infty} + \sum_{|\alpha| = [k]} [D^\alpha f]_{C^1_\theta (\mathbb{R}^N)},
\]

where \( ||D^\alpha f||_{\infty} \) denotes the sup-norm of \( D^\alpha f \) and \( [D^\alpha f]_{C^1_\theta (\mathbb{R}^N)} \) is the \((k - [k])\)-Hölder seminorm of \( f \). We say that \( u \in C^\theta_\infty (\mathbb{R}^N) \) if it belongs to \( C^k_\theta (\mathbb{R}^N) \) for any \( k \geq 0 \). Finally, given an open set \( \Omega \) \((\text{eventually}, \Omega = \mathbb{R}^N)\), by \( C^\infty (\Omega) \) we denote the set of all infinitely many times differentiable functions with compact support.

We now define the anisotropic spaces \( \mathcal{C}^\theta (\mathbb{R}^N) \) \((\theta \in \mathbb{R}_+)\). Let \( p_0, \ldots, p_r \) be as in Hypothesis 2.1(ii). To simplify the notation, we split any \( x \in \mathbb{R}^N \) as \( x = (x_0, \ldots, x_r) \) with \( x_j \in \mathbb{R}^{p_j} \) \((j = 0, \ldots, r)\).

**Definition 2.7.** For any \( \theta > 0 \), \( \mathcal{C}^\theta (\mathbb{R}^N) \) consists of all bounded functions \( f : \mathbb{R}^N \rightarrow \mathbb{R} \) such that \( f(x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r) \) belongs to the Hölder space \( C_b^{\theta/(2j + 1)} (\mathbb{R}^{p_j}) \) for any \( x_j := (x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r) \in \mathbb{R}^{N-p_j} \), and

\[
||f||_{\mathcal{C}^\theta (\mathbb{R}^N)} := \sup_{x_j \in \mathbb{R}^{N-p_j}} \|f(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r)\|_{C_b^{\theta/(2j + 1)} (\mathbb{R}^{p_j})} < +\infty.
\]

We norm it by \( ||f||_{\mathcal{C}^\theta (\mathbb{R}^N)} = \sum_{j=0}^{r} ||f||_{\mathcal{C}^\theta (\mathbb{R}^N)} \) for any \( f \in \mathcal{C}^\theta (\mathbb{R}^N) \). When \( \theta \) is such that \( \theta / (2j + 1) \in \mathbb{N} \) for some \( j = 0, \ldots, r \), we assume that all the existing derivatives of \( f \in \mathcal{C}^\theta (\mathbb{R}^N) \) are continuous in \( \mathbb{R}^N \).

3. Uniform estimates for the approximating semigroups

To investigate the elliptic and parabolic problems associated with \( \mathcal{A} \) we approximate this operator by the uniformly elliptic operator \( \mathcal{A}_\varepsilon \) defined on smooth function \( \varphi \) by

\[
\mathcal{A}_\varepsilon \varphi (x) := \mathcal{A} \varphi (x) + \varepsilon \sum_{i=p_{-1}+1}^{N} D_i \varphi (x), \quad x \in \mathbb{R}^N,
\]

for any \( \varepsilon > 0 \). It is known that one can associate a semigroup of bounded linear operators \( \{T_t (t)\} \) on \( C_b (\mathbb{R}^N) \) with each operator \( \mathcal{A}_\varepsilon \). For any \( f \in C_0 (\mathbb{R}^N) \) and any \( t > 0 \), \( T_t (t) f \) is the value at \( t \) of the unique classical solution to the Cauchy problem

\[
\begin{aligned}
D_t u (t, x) &= \mathcal{A}_\varepsilon u (t, x), \quad t \in [0, +\infty), \quad x \in \mathbb{R}^N, \\
u(0, x) &= f (x), \quad x \in \mathbb{R}^N.
\end{aligned}
\]

The uniqueness of the classical solution to problem (3.1) follows from a corresponding maximum principle \((\text{see, e.g., Proposition 3.1(ii)})\). The existence of a solution to problem (3.1) can be proved approximating such a problem with Dirichlet Cauchy problems in balls centered at 0 and radius \( n \) and using classical Schauder estimates and a compactness argument to show that the sequence of solutions \( \{u_n\} \) to such Dirichlet Cauchy problems converges, as \( n \rightarrow +\infty \), to a function \( u_\varepsilon \) which turns out to solve problem (3.1). We refer the reader, for example, to [3, Chapter 1] and [20, Section 4] for more details.
By letting $\varepsilon$ go to 0 and applying a compactness argument we will show the existence of a semigroup “generated by” $\mathcal{A}$. For this purpose we need estimates for the spatial derivatives of $\{T_\varepsilon(t)\}$ uniformly for $\varepsilon \in [0,1]$. This section is devoted to the proof of such estimates.

We start with a maximum principle for (degenerate) elliptic and parabolic equation, which leads to uniqueness of the distributional solutions to the problems (1.6) and (1.7), but which will be also crucial in the proof of the estimates for the spatial derivatives in Theorem 3.2 and Theorem 3.3. We postpone the, more or less standard, proof to Appendix A.

**Proposition 3.1.** Let $\mathcal{L}$ be any, degenerate or non-degenerate, elliptic operator defined on smooth functions $\psi$ by

$$
\mathcal{L}\psi(x) = \sum_{i,j=1}^{m} q_{ij}(x) D_{ij}\psi(x) + \sum_{i,j=1}^{N} b_{ij} x_j D_i\psi(x) + \sum_{j=1}^{m} F_j(x) D_j\psi(x), \quad x \in \mathbb{R}^N,
$$

with the coefficients $q_{ij}$ and $F_j$ ($i,j = 1, \ldots, m$) being (possibly) unbounded functions in $\mathbb{R}^N$ which may grow, respectively, at most quadratically and linearly at infinity. Then the following assertions hold true.

(i) Let $u \in C_0^\infty(\mathbb{R}^N)$ be a distributional solution to the equation $\mathcal{L}u = f$, corresponding to some $f \in C_0^\infty(\mathbb{R}^N)$ and $\lambda > 0$. Further, suppose that $D_iu$ and $D_{ij}u$ exist in the classical sense for any $i, j = 1, \ldots, m$. Then,

$$
\lambda \|u\|_{C_0^\infty(\mathbb{R}^N)} \leq \|f\|_{C_0^\infty(\mathbb{R}^N)}.
$$

(ii) Let $u : [0,T_0] \times \mathbb{R}^N \rightarrow \mathbb{R}$ ($T_0 > 0$) be a distributional solution of the Cauchy problem

$$
\begin{cases}
D_t u(t,x) = \mathcal{L}u(t,x) + g(t,x), & t \in [0,T_0], \quad x \in \mathbb{R}^N, \\
u(0,x) = f(x), & x \in \mathbb{R}^N,
\end{cases}
$$

corresponding to some $f \in C_0^\infty(\mathbb{R}^N)$ and $g \in C([0,T_0] \times \mathbb{R}^N)$. Further, assume that $D_t u, D_{ij}u$ ($i,j = 1, \ldots, m$) exist in the classical sense. If $g \leq 0$ in $[0,T_0] \times \mathbb{R}^N$, then $\sup_{[0,T_0] \times \mathbb{R}^N} u \leq \sup_{\mathbb{R}^N} f$. Similarly, if $g \geq 0$ in $[0,T_0] \times \mathbb{R}^N$, then $\inf_{[0,T_0] \times \mathbb{R}^N} u \geq \inf_{\mathbb{R}^N} f$. In particular, if $g \equiv 0$, then

$$
\|u(t,\cdot)\|_{\infty} \leq \|f\|_{\infty}, \quad t \in [0,T_0].
$$

The following theorem will be the most crucial ingredient for the construction of the semigroup associated with $\mathcal{A}$.

**Theorem 3.2.** For any $\varepsilon > 0$, any $h \in \mathbb{N}$ and any $f \in C_0^h(\mathbb{R}^N)$, the function $T_\varepsilon(t)f$ belongs to $C_0^h(\mathbb{R}^N)$ for any $t > 0$. Moreover, for any $T_0 > 0$ and any $h,l \in \mathbb{N}$ with $h \leq l$, the function $T_\varepsilon(t)f$ is bounded and continuous in $[0,\infty[ \times \mathbb{R}^N$, and when $l > h$ it vanishes at $t = 0$.

**Proof.** We restrict ourselves to showing the assertion in the case when $h = 0$, the other cases being similar and even easier. We split the proof into two steps. In the first one, we prove that there exists a positive constant $C$, independent of $f$, such that

$$
\|D^l T_\varepsilon(t)f\|_{\infty} \leq Ct^{-\frac{l}{2}}\|f\|_{\infty}, \quad (3.3)
$$

for any $f \in C_0^\infty(\mathbb{R}^N)$, $t > 0$. Next, in Step 2, we prove that the function $(t,x) \mapsto t^{l/2}(D^l T_\varepsilon(t)f)(x)$ is continuous up to $t = 0$.

**Step 1.** Without loss of generality, we limit ourselves to proving (3.3) in the particular case when $f \in C_0^\infty(\mathbb{R}^N)$. Indeed, in the general case it suffices to approximate $f \in C_0^\infty(\mathbb{R}^N)$ by a sequence of smooth functions $f \in C_0^\infty(\mathbb{R}^N)$, bounded in $C_b(\mathbb{R}^N)$ and converging to $f$ locally uniformly in $\mathbb{R}^N$. It is well-known that $T_\varepsilon(\cdot) f_n$ converges to $T_\varepsilon(\cdot) f$ uniformly in $[0,T_0] \times B(M)$, as $n \rightarrow +\infty$, for any $M, T_0 > 0$ (see e.g., [3, Proposition 2.2.9] or [20, Proposition 4.6]). Moreover, the classical interior estimates in [11, Chapter 4, Theorem 5.1] imply that

$$
\|D^l T_\varepsilon(\cdot) f_n - D^l T_\varepsilon(\cdot) f\|_{C([T_0/2,T_0] \times B(M))} \leq \hat{C}\|T_\varepsilon(\cdot) f_n - T_\varepsilon(\cdot) f\|_{L^\infty([T_0/2,T_0] \times \overline{B(2M)})},
$$

for any $M, T_0 > 0$ and some positive constant $\hat{C}$, depending on $M, T_0$. Hence, $D^l T_\varepsilon(t)f_n$ converges to $D^l T_\varepsilon(t)f_n$ locally uniformly in $[0,\infty[ \times \mathbb{R}^N$ and this allows us to extend (3.3) to any $f \in C_0^\infty(\mathbb{R}^N)$. 


Now for the proof of (3.3) for $f \in C^\infty_c(\mathbb{R}^N)$, let $\varphi \in C^\infty_c(\mathbb{R})$ be a non-increasing function such that $\varphi(t) = 1$ for any $t \in [-1/2, 1/2]$, $\varphi(t) = 0$ for any $t \in \mathbb{R} \setminus [-1, 1]$. For $R > 1$ define the functions $\eta_R : \mathbb{R}^N \rightarrow \mathbb{R}$ by $\eta_R(x) := \varphi(|x|/R)$ and $v_R$ by

$$v_R(t, x) := \sum_{m=0}^l a^m t^m \eta_R^{2m}(x)|D^m u_R(t, x)|^2, \quad t \in [0, T_0], \ x \in B(R),$$

where $u_R$ denotes the classical solution to the Dirichlet Cauchy problem in the ball $B(R)$ with initial value $f$, and $a$ is a positive parameter to be fixed later on ($a$ will be small). To simplify the notation, we drop out the index $R$, when there is no danger of confusion.

The classical Schauder estimates of [11, Chapter 4, Theorem 5.1] imply that $v$ is continuous in $[0, T_0] \times \overline{B(R)}$. Moreover, a straightforward computation shows that $v$ solves the Cauchy problem

$$\begin{align*}
\frac{D_t v(t, x)}{\partial x} = & \mathcal{A} v(t, x) + g(t, x), \quad t \in [0, T_0], \ x \in B(R), \\
v(t, x) = & 0, \quad t \in [0, T_0], \ x \in \partial B(R), \\
v(0, x) = & (f(x))^2, \quad x \in B(R),
\end{align*}$$

where, for any $t \in [0, T_0]$ and any $x \in B(R)$, the function $g$ is given by $g(t, x) = \sum_{j=1}^4 g_j(t, x)$ with

\begin{align*}
g_1(t, \cdot) &= -2 \sum_{i,j=1}^N \sum_{m=0}^l a^m t^m \eta^{2m} q_{ij} \langle D^m D_i u(t), D^m D_j u(t) \rangle, \\
g_2(t, \cdot) &= -\langle D^2 \eta, D \eta \rangle \sum_{m=1}^l 2m(2m-1)a^m t^m \eta^{2m-2}|D^m u(t)|^2 \\
&\quad + \sum_{m=1}^l ma^m t^{m-1} \eta^{2m}|D^m u(t)|^2, \\
g_3(t, \cdot) &= -2\mathcal{A} \eta \sum_{m=1}^l ma^m t^m \eta^{2m-1}|D^m u(t)|^2 \\
&\quad - 8 \sum_{i,j=1}^N \sum_{m=1}^l ma^m t^m \eta^{2m-1} q_{ij} D_i \eta \langle D^m u(t), D^m D_j u(t) \rangle, \\
g_4(t, \cdot) &= 2 \sum_{m=1}^l a^m t^m \eta^{2m} \langle [D^m, \mathcal{A}] u(t), D^m u(t) \rangle.
\end{align*}

Here, $[D^m, \mathcal{A}]$ denotes the commutator between the operators $D^m$ and $\mathcal{A}$. Using the ellipticity assumption on $q_{ij}$ we get

$$g_1(t) \leq -2\nu \sum_{m=0}^l a^m t^m \eta^{2m}|D^{m+1} u(t)|^2 - 2\varepsilon \sum_{m=0}^l a^m t^m \eta^{2m}|D^{m+1} u(t)|^2$$

(3.4)

$$= -2\nu \sum_{m=1}^{l+1} a^m t^m \eta^{2m-2}|D^m u(t)|^2 - 2\varepsilon \sum_{m=1}^{l+1} a^m t^{m-1} \eta^{2m-2}|D^m u(t)|^2,$$

where $D^m u$ (respectively $D^{m} u$) denotes the vector whose entries are the $m^{th}$ order derivatives $\frac{D^m u}{\partial x_{i_1} \cdots \partial x_{i_m}}$ with $i_j \leq p_0$ for some $j = 1, \ldots, m$ (respectively $i_j > p_0$ for all $j = 1, \ldots, m$).

We turn to estimating the function $g_3$. From Hypotheses 2.1 it follows easily that

$$|\mathcal{A} \eta(x)| \leq C_1, \quad |(Q(x) D \eta(x))| \leq C_1 \left\{ \begin{array}{ll}
\sqrt{v(x)}, & \text{if } i \leq p_0, \\
\varepsilon, & \text{if } i > p_0,
\end{array} \right.$$

(3.5)
for any $x \in \mathbb{R}^N$ and some positive constant $C_1$. Taking this into account and using Young’s inequality we conclude

$$
g_3(t) \leq 2C_1 \sum_{m=1}^{l} ma^m t^m \eta^{2m-1} |D^m u(t)|^2$$

$$+ C_2 \sum_{m=1}^{l} ma^m t^m \eta^{2m-1} \sqrt{\nu} |D^m u(t)| \cdot |D^m u(t)|$$

$$+ C_2 \varepsilon \sum_{m=1}^{l} ma^m t^m \eta^{2m-1} |D^m u(t)| \cdot |D^m u(t)|$$

$$\leq 2C_1 \sum_{m=1}^{l} ma^m t^m \eta^{2m-1} |D^m u(t)|^2$$

$$+ C_2 \sum_{m=1}^{l} m \left(a^{m-\frac{1}{2}} t^{m-\frac{1}{2}} \eta^{2m-2} |D^m u(t)|^2 + a^{m+\frac{1}{2}} t^{m+\frac{1}{2}} \eta^{2m} |D^m u(t)|^2\right)$$

$$+ C_2 \varepsilon \sum_{m=1}^{l} m \left(a^{m-\frac{1}{2}} t^{m-\frac{1}{2}} \eta^{2m-2} |D^m u(t)|^2 + a^{m+\frac{1}{2}} t^{m+\frac{1}{2}} \eta^{2m} |D^m u(t)|^2\right),$$

for any $t \in [0,T_0]$ and some positive constant $C_2$, independent of $\varepsilon$ and $t$. The term $g_4$ can be estimated similarly, taking now (2.4) into account. We obtain

$$
g_4(t) \leq 2\|B\|_{\infty} \sum_{m=1}^{l} a^m t^m \eta^{2m} |D^m u(t)|^2$$

$$+ 2 \sum_{m=1}^{l} a^m t^m \eta^{2m} \sum_{n=1}^{m} \left(\|D^n Q\|_{\infty} |D^{m-n} u(t)| \cdot |D^n u(t)| \right.$$}

$$+ \left.\|D^n F\|_{\infty} |D^{m-n} u(t)| \cdot |D^n u(t)|\right),$$

for any $t \in [0,T_0]$, where $\|D^h Q\|_{\infty}$ (respectively $\|D^h F\|_{\infty}$) $(h=1,\ldots,l)$ denotes the maximum of the sup-norm of the functions $D^h q_j$ (respectively $D^h f_j$) $(i,j=1,\ldots,N)$. Hence, taking Hypotheses 2.1(i) and 2.1(ii) into account, we can write

$$
g_4(t) \leq C_3 \nu \sum_{m=1}^{l+1} a^{m-\frac{1}{2}} t^{m-\frac{1}{2}} \eta^{2m} |D^m u(t)|^2 + C_3 \sum_{m=1}^{l} a^m t^m \eta^{2m} |D^m u(t)|^2$$

$$+ C_3 \sum_{m=1}^{l} a^{m-\frac{1}{2}} t^{m-\frac{1}{2}} \eta^{2m} |D^m u(t)|^2,$$

(3.7)

for any $t \in [0,T_0]$ and some positive constant $C_3$, independent of $t$. Summing up, from (3.4), (3.6) and (3.7) we easily deduce that

$$
g(t) \leq \sum_{m=1}^{l+1} M^*_m(a,T_0) a^{m-1} t^{m-1} \eta^{2m-2} |D^m u(t)|^2$$

$$+ \sum_{m=1}^{l+1} M^{**}_m(a,T_0) a^{m-1} t^{m-1} \eta^{2m-2} |D^m u(t)|^2,$$

for any $t \in [0,T_0]$, where

$$M^*_m(a,T_0) := (-2 + C_2(m-1)\sqrt{a} \sqrt{T_0} + C_3 \sqrt{a} \sqrt{T_0}) \nu$$

$$+ (C_2 m + C_2 \nu m + C_3 + C_2 \varepsilon (m-1)) \sqrt{a} \sqrt{T_0}$$

$$+ (m + 2C_1 m T_0 + C_3 T_0) a,$$

$$M^{**}_m(a,T_0) := -2 \varepsilon + (C_2 m + C_2 \nu m + C_3 + C_2 \varepsilon (m-1)) \sqrt{a} \sqrt{T_0}$$

$$+ (m + 2C_1 m T_0 + C_3 T_0) a,$$
for any \( m = 1, \ldots, l + 1 \). Since in \( M_m^*(a, T_0) \) and \( M_m^{**}(a, T_0) \) apart from the first negative term everything vanishes as \( a \to 0 \) for any \( m = 1, \ldots, l + 1 \), it follows that for sufficiently small \( a > 0 \) (independent of \( R \)) the inequality \( g(t, x) \leq 0 \) holds for any \( t \in [0, T_0] \) and any \( x \in B(R) \). The classical maximum principle principle yields then

\[
|v_R(t, x)| \leq \|f\|_{C^2(R)}^2 \quad \text{and so} \quad t^m \eta_R^m(x)|D^m u_R(t, x)|^2 \leq C_m \|f\|_{C^2}^2,
\]

for any \( (t, x) \in [0, T_0] \times B(R) \).

Now, (3.3) follows by letting \( R \to +\infty \).

**Step 2.** We now conclude the proof by showing that the function \( w_t : [0, +\infty] \times \mathbb{R}^N \to \mathbb{R} \) defined by \( w_t(x) := t^{-1/2}D^t u(t, x) \) is continuous on \( [0, +\infty] \times \mathbb{R}^N \).

If \( f \in C^\infty_c(\mathbb{R}^N) \) this claim is easily checked. Indeed, in this case, if \( u_R \) denotes the solution of the Dirichlet Cauchy problem on \( B(R) \) with initial value \( f \), it is well-known (see [11, Chapter 4, Theorem 5.1]) that, for any \( T_0 > 0 \) and any \( m, M \in \mathbb{N} \), with \( m < M \) and \( \text{supp}(f) \subset B(m) \), there exists a positive constant \( C_l = C_l(m, M, T_0) \) such that

\[
\|u_R\|_{C^{l+\theta/2, 2l+\theta}(0, T_0) \times B(m)} \leq C_l \left( \|f\|_{C^{l+\theta}(\mathbb{R}^N)} + \|u_R\|_{C([0,2T_0] \times B(M))} \right)
\]

for any \( R > 0 \). Hence, by a compactness argument, we can easily show that \( u_R \) converges to \( T_\varepsilon(\cdot)f \) in \( C^{l, 2l}(0, +\infty) \times \mathbb{R}^N \). Since the function \( |D^t u_R| \) is continuous in \( [0, +\infty] \times \mathbb{R}^N \) so is the functions \( w_t \), too.

Let us now consider the general case when \( f \in C_b(\mathbb{R}^N) \). Then, there exists a sequence \( \{f_n\} \subset C^\infty_c(\mathbb{R}^N) \) which is bounded in \( C_b(\mathbb{R}^N) \) and converges to \( f \) locally uniformly in \( \mathbb{R}^N \). Let us fix \( k, m \in \mathbb{N} \). By [15, Proposition 1.1.3(iii)], we know that

\[
\|\psi\|_{C^l(B(M))} \leq P_l \|\psi\|_{C^l(B(M))} \|\psi\|_{C^{l+\theta}(B(M))},
\]

for some positive constant \( P_l \). Then, \( P_l \) is replaced by some \( P_1 \) in this case. The right-hand side of (3.9) vanishes as \( n \to +\infty \). By the arbitrariness of \( T_0 \) and \( M \), it follows immediately that the function \( w_t \) is continuous in \( [0, +\infty] \times \mathbb{R}^N \). In particular, it vanishes at \( t = 0 \) since the function \( (t, x) \to t^{1/2}(D^t u_R) f_n(x) \) does for any \( n \in \mathbb{N} \). This completes the proof.

We are now in a position to prove the main result of this section. Our ultimate aim is to show that the semigroups \( \{T_n(t)\} \) converge to a semigroup \( \{T(t)\} \) which is associated with the operator \( a \), and we also wish to establish estimates for the spatial derivatives of \( \{T(t)\} \). Contrary to the uniformly elliptic situation of \( \{T_n(t)\} \) the behavior near \( t = 0 \) of the partial derivatives of \( D^\alpha T(t) \) is expected to depend not only on the length \( \|\alpha\| \) of the multi-index \( \alpha \), but also on the directions along which we differentiate. Thus the well-known behavior \( t^{-\|\alpha\|/2} \) is replaced by some function growing faster near 0. The exact behavior is well-known, e.g., for the Ornstein-Uhlenbeck semigroup (see [17]) and the optimal exponent is actually given by the following function \( q \). We define \( q : \mathbb{N}_0^{r+1} \to \mathbb{R} \) as

\[
q(\alpha) = \sum_{k=0}^r \frac{2k + 1}{2} \alpha_k = \frac{1}{2} \|\alpha\| + \sum_{k=1}^r k \alpha_k, \quad \alpha \in \mathbb{N}_0^{r+1}.
\]

With this function the, still to be constructed, semigroup \( \{T(t)\} \) will obey the estimate

\[
\|D^\alpha T(t) f\|_{\infty} \leq C t^{-q(\|\alpha\|)}\|f\|_{\infty},
\]

for any \( m = 1, \ldots, l + 1 \).
for any $f \in C_h(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}_0^N$ (recall the notation $|\alpha| = (\alpha_0, \alpha_1, \ldots, \alpha_r)$ from Subsection 2.3). Whereas, if we have a control over certain derivatives of $f$, say $f \in C_b^h(\mathbb{R}^N)$ we expect a better behavior. Indeed, this will be the case. For the precise statement we will need the following function $q_h : \mathbb{N}_0^{r+1} \to \mathbb{R}$ ($h \in \mathbb{N}_0$). We let

$$q_h(\beta) = \frac{1}{2} \|\beta\| - \frac{1}{2} h + \sum_{k=0}^{j(\beta) - 1} k \beta_k + (j(\beta) - 1) \left( \sum_{k=j(\beta)}^{r} \beta_k - h \right),$$

for any multi-index $\beta \in \mathbb{N}_0^{r+1}$, where $j(\beta) \in \mathbb{N}_0$ is the smallest integer such that $\sum_{k=0}^{j(\beta)} \alpha_j \leq h$, and we agree that $\sum_{k=0}^{j(\beta) - 1} \beta_k := 0$ and $q_h(\beta) = 0$, if $h \geq \|\beta\|$. This function describes the expected behavior near $t = 0$ in the estimates of the derivatives, and it models the following: if we have a function in $C_b^h(\mathbb{R}^N)$, then we can drop out any $h$ partial derivatives from a multi-index $\alpha$, since these should not contribute to the power of $t$. We do this in a way that derivatives which would give the largest contribution in the derivative-estimate are dropped out. Then, we can evaluate our $q$ on this new multi-index and get the right behavior near $t = 0$.

The in $\varepsilon \in [0,1]$ uniform estimates for spatial derivatives of $\{T_\varepsilon(t)\}$ are given by the following result.

**Theorem 3.3.** For any compact interval $J \subseteq [0, +\infty[$, multi-index $\alpha = (\alpha_0, \ldots, \alpha_r)$, with $\alpha_j \in \mathbb{N}_0^p$ $(j = 0, \ldots, r)$ and $\|\alpha\| \leq \kappa$, any $h \in \mathbb{N}_0$, with $h \leq \|\alpha\|$, there exists a positive constant $\tilde{C}$, depending on $\alpha$, but being independent of $\varepsilon \in [0,1]$, such that

$$\|D^\alpha T_\varepsilon(t)f\|_\infty \leq \tilde{C} \varepsilon^{-q_h(|\alpha|)} \|f\|_{C_b^h(\mathbb{R}^N)}, \quad t \in J, \quad \varepsilon \in [0,1].$$

(3.10)

For the proof we need some preparation and auxiliary results. Define the function $\ell : \mathbb{N} \setminus \{1\} \to \mathbb{N}$ as follows. Let $i_{\ell(m)}^{(k,r)} = (0, \alpha_1, \ldots, \alpha_r)$, where $k = \|\alpha\|$, and let $j$ be the smallest integer such that $\alpha_j > 0$. Then, $\ell(m)$ is the index such that $i_{\ell(m)}^{(k,r)} = (0, \ldots, 1, \alpha_j - 1, \alpha_{j+1}, \ldots, \alpha_r)$. As it is immediately seen, $q(i_{\ell(m)}^{(k,r)}) = q(i_{\ell(m)}^{(k,r)}) - 1$. Moreover, we have $\ell(m) < m$. These will be used in the sequel without further mentioning. We also need some properties of the function $q_h$ presented in the next lemma (the proof is in Appendix A).

**Lemma 3.4.** For $\alpha \in \mathbb{N}_0^{r+1}$, the following hold.

(i) $\|\alpha\| \leq h$ if and only if $q_h(\alpha) = 0$.

(ii) We have $q_h(\alpha) \geq (\|\alpha\| - h)^+ / 2$.

(iii) If $\|\alpha\| \geq h$ and $\beta = (\alpha_0 + 1, \alpha_1, \ldots, \alpha_r)$, then $q_h(\beta) = q_h(\alpha) + 1 / 2$.

(iv) If $\beta = \alpha - e_j^{(r+1)} + e_{j'}^{(r+1)}$ for some $0 \leq j, j' \leq r$ such that $\alpha_j > 0$ and $j' \leq j + 1$, then $q_h(\beta) \geq q_h(\beta) - 1$.

(v) Suppose that $\alpha_0, \ldots, \alpha_{j_0-1} = 0$ and $\alpha_{j_0} > 0$ for some $j_0 > 0$. Set $\beta = \alpha - e_{j_0}^{(r+1)} + e_{j_0-1}^{(r+1)}$. If $h < l$, then $q_h(\alpha) > 1$ and $q_h(\beta) = q_h(\alpha) - 1$.

(vi) Suppose that $\alpha_0, \ldots, \alpha_{j_0-1} = 0$ and $\alpha_{j_0} > 0$ for some $j_0 > 0$. Set $\alpha' = \alpha - e_{j_0}^{(r+1)} + e_{j_0-1}^{(r+1)}$ and $\beta = \alpha' - e_{j_0}^{(r+1)} + e_{j_0-1}^{(r+1)}$ for some $0 \leq j', j'' \leq r$ such that $\alpha_j > 0$ and $j' \leq j + 1$. Then $q_h(\alpha') + q_h(\beta) \geq 2q_h(\beta) - 1$.

(vii) Let $\alpha$ and $\beta$ be two multi-indices such that $\alpha_j \leq \beta_j$ for all $j = 0, \ldots, r$ and $\alpha_{j_0} < \beta_{j_0}$ for some $j_0$. Further, let $\beta = 2e_{j_0}^{(r+1)} + \tilde{\alpha}$. Then, we have $q_h(\alpha) \geq q_h(\beta) - 1$.

Also the next linear algebraic lemma will be used in the proof Theorem 3.3. For a proof we refer to [13, Lemma 2.6].

**Lemma 3.5.** Suppose that $Q = (q_{ij})$ and $A$ are non-negative definite $N \times N$ square matrices. Further, assume that, for some $m \in \mathbb{N}$, the $m \times m$-submatrix $Q_0 = (q_{ij})$, obtained erasing the last $N - m$ rows and columns, is positive definite and $q_{ij} = 0$ if $\max\{i, j\} > m$. Then,

$$\text{Tr}(QA) \geq \lambda_{\min}(Q_0) \text{Tr}(A_1),$$

where $A_1$ is the submatrix obtained from $A$ by erasing the last $N - m$ rows and columns.

**Proof of Theorem 3.3.** Throughout the proof, we simply write $c_k$ and $i_m^{(k,r)}$ instead of $c_{k,r}$ and $i_m^{(k,r)}$.\]
Let $\varepsilon \in [0, 1]$, $h, k \in \mathbb{N}$ with $h \leq k \leq \kappa$, $f \in C^0_b(\mathbb{R}^N)$. Further, we introduce the function $v_\varepsilon : [0, +\infty] \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$v_\varepsilon(t, x) = \sum_{l=0}^{k}(\mathcal{H}^{(l)}(t)D^ju_\varepsilon(t, x), D^ju_\varepsilon(t, x)),$$  

where $u_\varepsilon = T_\varepsilon(\cdot)f$ and $\mathcal{H}^{(l)}(t)$ ($l = 0, \ldots, k$) are suitable symmetric matrices. Namely, $\mathcal{H}^{(0)} = 1$ and the matrices $\mathcal{H}^{(l)}(t)$ ($l = 1, \ldots, k$) are split into $c_l$ blocks $H_{m,p}^{(l)}(t)$ according to the splitting of the vector $D^j u_\varepsilon$ introduced in Subsection 2.3. We set $s_p^{(l)} = \#\{\alpha \in \mathbb{N}_0^N : |\alpha| = t_p^{(l)}\}$. Now the matrices $H_{m,p}^{(l)}(t)$ have the form

$$H_{m,p}^{(l)}(t) = \begin{cases} \alpha_{p,p}^{(l)}I_{s_p^{(l)}} & \text{if } m = p, \\
\alpha_{m,m}^{(l)}H_{m,m}(t) & \text{if } m > c_l-1 \text{ and } p = \ell(m), \\
\alpha_{m,\ell(p)}^{(l)}(H_{p,p}^{(l)})^* & \text{if } p > c_l-1 \text{ and } m = \ell(p), \\
0, & \text{otherwise}, \end{cases}$$

for any $t > 0$ and some constant $s_m^{(l)} \times s_{\ell(m)}^{(l)}$-matrices $H_{m,\ell(m)}^{(l)}$ to be determined later on just as well as the positive parameters $a > 1$, $\eta_{m,m}$ and $\eta_{m,\ell(m)}^{(l)}$. We put the following requirements on these parameters:

$$\begin{cases} (a) & \eta_{m,m}^{(l)} + \eta_{m,\ell(m)}^{(l)} > 2\eta_{m,\ell(m)}^{(l)}, \\
(b) & \eta_{m,\ell(m)}^{(l)} > \eta_{p,p}^{(l)}, \\
(c) & 2\max_{m=1, \ldots, c_l-1} \eta_{m,m}^{(l)} < \min_{m=1, \ldots, c_l-1} \eta_{m,m}^{(l-1)}=: \eta^{(l)}, \\
(d) & \eta_{m,\ell(m)}^{(l)} < \eta_{p,p}^{(l)} & p \leq c_l-1 < m, \\
(e) & \eta_{m,\ell(m)}^{(l)} < \eta_{\ell(m),\ell(m)}^{(l)} & m, \ell(m) > c_l-1, \end{cases}$$

for any $l = 1, \ldots, k$. For the moment, as it will be crucial in the following, we assume that the constants $\eta_{m,m}^{(l)}, \eta_{p,p}^{(l)}$ ($l = 1, \ldots, k$, $m = 1, \ldots, c_l$, $p = c_l-1 + 1, \ldots, c_l$), satisfying the conditions (3.11)(a) and (3.12), can be actually determined. We will return to this point at the end and show that this is actually the case.

From Theorem 3.2 it follows that the function $v_\varepsilon$ is continuous on $[0, +\infty] \times \mathbb{R}^N$. A straightforward computation shows that it satisfies the Cauchy problem

$$\begin{cases} D_tv_\varepsilon(t, x) = \mathcal{A} v_\varepsilon(t, x) + g_\varepsilon(t, x), & t \in [0, +\infty[, \ x \in \mathbb{R}^N, \\
\varepsilon(0, x) = \sum_{l=0}^{k}(\mathcal{H}^{(l)}(0)D^j f(x), D^j f(x)), & x \in \mathbb{R}^N, \end{cases}$$

where the function $g_\varepsilon$ is given by

$$g_\varepsilon = -2 \sum_{i,j=1}^{n} \sum_{l=0}^{k} q_{ij}^{(l)}(\mathcal{H}^{(l)}D^j D^j u_\varepsilon, D^j D^j u_\varepsilon) + 2 \sum_{l=1}^{k}(\mathcal{H}^{(l)}(D^l, (B^l, D^l))u_\varepsilon, D^j u_\varepsilon)$$

$$+ \sum_{l=k+1}^{k}(\mathcal{H}^{(l)}D^j u_\varepsilon, D^j u_\varepsilon) + 2 \sum_{l=1}^{k}(\mathcal{H}^{(l)}D^j (Q^l D^j u_\varepsilon) + (Q^l D^j D^j u_\varepsilon), D^j u_\varepsilon)$$

$$+ 2 \sum_{l=1}^{k}(\mathcal{H}^{(l)}(D^j, (F, D^j))u_\varepsilon, D^j u_\varepsilon) := \sum_{j=1}^{5} g_{j, \varepsilon},$$

(3.13)
the matrix $\mathcal{H}^{(l)}$ is obtained by entrywise differentiating the matrix $\mathcal{H}^{(l)}$ with respect to time, and we have $D^{\nu} D_{t} u_{\varepsilon} = D_{i} u_{\varepsilon}$. Note also that the commutators here are understood coordinatewise. When $h = k$ we agree that the first sum in the second line of (3.13) disappears.

We are going to prove that we can fix $T_{0}$ small enough, but independent of $\varepsilon$, such that $g_{\varepsilon} \leq 0$ in $[0, T_{0}] \times \mathbb{R}^{N}$. Proposition 3.1(ii) then will yield $v_{\varepsilon} \leq \sum_{l=0}^{h} (\mathcal{H}^{(l)}(0) D^{l} f, D^{l} f)$ in $[0, T_{0}] \times \mathbb{R}^{N}$. In particular, this implies that

$$\langle \mathcal{H}^{(j)}(t) D^{l} u_{\varepsilon}(t, x), D^{l} u_{\varepsilon}(t, x) \rangle \leq \hat{C} \|f\|_{C^{2}_{0}(\mathbb{R}^{N})}^{2}, \quad (t, x) \in [0, T_{0}] \times \mathbb{R}^{N}, \quad j = 1, \ldots, k,$$

for some positive constant $\hat{C}$. Since the matrices $\mathcal{H}^{(j)}(t)$ are positive definite for any $j$ and any $t$ if we assume (3.11), we obtain that (3.10) holds in the time interval $[0, T_{0}]$. The semigraf property allows then to extend this estimate to any compact time interval $J \subset [0, +\infty[$.

We now turn to the estimation of $g_{\varepsilon}$.

**Estimating the function $g_{1, \varepsilon}$.** Lemma 3.5 and the ellipticity condition (2.1) imply that

$$g_{1, \varepsilon}(t) \leq -2\nu \sum_{j=1}^{p_{0}} \sum_{l=0}^{k} \langle \mathcal{H}^{(l)}(t) D^{l} D_{j} u_{\varepsilon}(t), D^{l} D_{j} u_{\varepsilon}(t) \rangle, \quad t \in [0, +\infty[. \quad (3.14)$$

This is a term of negative type and it will help us to control (most of) the remaining terms in (3.13). More precisely, the right-hand side of (3.14) contains all the derivatives $D^{\alpha} u_{\varepsilon}$ of order less than or equal to $k + 1$ such that, if we split $\alpha = (\alpha_{0}, \ldots, \alpha_{r})$ (as explained in Subsection 2.3), then $\|\alpha_{0}\| \neq 0$. So, we miss all the derivatives of $u$ of the type $D^{\alpha} u$ with $\|\alpha\| \leq k + 1$ and $\|\alpha_{0}\| = 0$. We will recover these latter derivatives from (a part of) the term $g_{2, \varepsilon}$.

Using the very definition of the matrices $\mathcal{H}^{(l)}$ ($l = 1, \ldots, k$) we obtain

$$\sum_{j=1}^{p_{0}} \sum_{l=1}^{k} \langle \mathcal{H}^{(l)}(t) D^{l} D_{j} u_{\varepsilon}(t), D^{l} D_{j} u_{\varepsilon}(t) \rangle$$

$$\geq \sum_{j=1}^{p_{0}} \sum_{l=1}^{k} c_{l} \sum_{m=1}^{c_{l}} a_{m, m}^{(l)} t^{2q_{0}(l)_{m}} \|D^{l}_{m} D_{j} u_{\varepsilon}(t)\|^{2}$$

$$- 2 \sum_{j=1}^{p_{0}} \sum_{l=1}^{k} c_{l} \sum_{m=c_{l}+1}^{c_{l}} a_{m, l_{j}}^{(l)} t^{q_{0}(l)_{j}+q_{0}(l)_{m}} \|H^{(l)}_{l_{j}}(t), m\| \|D^{l}_{m} D_{j} u_{\varepsilon}(t)\| \|D^{l}_{m} D_{j} u_{\varepsilon}(t)\|,$$

for any $t > 0$. Thanks to (3.11)(a), we can fix $\gamma^{(l)}_{m}$ and $\gamma^{(l)}_{m, l_{j}}$ such that

$$\gamma^{(l)}_{m} < \eta_{m, m}^{(l)}, \quad \gamma^{(l)}_{m, l_{j}} < \eta_{m, l_{j}}^{(l)}, \quad 2\eta_{m, l_{j}}^{(l)} = \eta^{(l)}_{m} + \gamma^{(l)}_{m, l_{j}},$$

By Young’s inequality (we will use the same trick several times in the sequel) and Lemma 3.4(iii) we now infer that

$$2a_{m, l_{j}}^{(l)} t^{q_{0}(l)_{m}+q_{0}(l)_{j}} \|H^{(l)}_{l_{j}}(t, m)\| \cdot \|D^{l}_{m} D_{j} u_{\varepsilon}(t)\| \cdot \|D^{l}_{m} D_{j} u_{\varepsilon}(t)\|$$

$$\leq a_{m, l_{j}}^{(l)} t^{2q_{0}(l)_{m}} \|H^{(l)}_{l_{j}}(t, m)\| \cdot \|D^{l}_{m} D_{j} u_{\varepsilon}(t)\|^{2} + a_{m, l_{j}}^{(l)} t^{2q_{0}(l)_{j}} \|H^{(l)}_{l_{j}}(t, m)\| \cdot \|D^{l}_{m} D_{j} u_{\varepsilon}(t)\|^{2}$$

$$= o(a_{m, l_{j}}^{(l)} t^{2q_{0}(l)_{m}}) \|D^{l}_{m} D_{j} u_{\varepsilon}(t)\|^{2} + o(a_{m, l_{j}}^{(l)} t^{2q_{0}(l)_{j}}) \|D^{l}_{m} D_{j} u_{\varepsilon}(t)\|^{2},$$

for any $t > 0$. Since $j \leq p_{0}$, we obtain

$$g_{1, \varepsilon}(t) \leq -2\nu |D^{l}_{1} u_{\varepsilon}(t)|^{2} - 2\nu \sum_{j=1}^{p_{0}} \sum_{l=1}^{k} \sum_{m=1}^{c_{l}} \left\{ a_{m, m}^{(l)} + o(a_{m, m}^{(l)}) \right\} t^{2q_{0}(l)_{m}} |D^{l}_{m} D_{j} u_{\varepsilon}(t)|^{2}$$

$$\leq -2\nu |D^{l}_{1} u_{\varepsilon}(t)|^{2} - 2\nu \sum_{l=2}^{k+1} \sum_{m=1}^{c_{l}} \left\{ a^{(l)}_{0} + o(a^{(l)}_{0}) \right\} t^{(2q_{0}(l)_{m})-1} |D^{l}_{m} u_{\varepsilon}(t)|^{2}, \quad (3.15)$$

for any $t > 0$, where (see (3.12)(c)) we have

$$\eta^{(l)}_{m} = \min_{m=1, \ldots, c_{l-1}} \eta^{(l-1)}_{m}, \quad l = 2, \ldots, k + 1.$$
Estimating the term $g_{2,z}$. Observe that

\[
g_{2,z}(t) = 2 \sum_{l=1}^{k} \sum_{m=1}^{c_l} a^{(l)}_{m}\mathbf{t}^{2q_h(i_l^{(m)})} (t) \langle [D^l_m, \langle B, D \rangle] u_z(t), D^l_m u_z(t) \rangle \\
+ 2 \sum_{l=1}^{k} \sum_{m=c_l-1+1}^{c_l} a^{(l)}_{m,\ell(m)} t^{q_h(i_l^{(m)})} \langle H^{(l)}_{m,\ell(m)} [D^l_m, \langle B, D \rangle] u_z(t), D^l_m u_z(t) \rangle \\
+ 2 \sum_{l=1}^{k} \sum_{m=c_l-1+1}^{c_l} a^{(l)}_{m,\ell(m)} t^{q_h(i_l^{(m)})} \langle H^{(l)}_{\ell(m),m} [D^l_m, \langle B, D \rangle] u_z(t), D^l_m u_z(t) \rangle,
\]

for any $t > 0$. By virtue of Lemma A.2 and a straightforward computation, we can write

\[
g_{2,z}(t) = 2 \sum_{l=1}^{k} \sum_{m=c_l-1+1}^{c_l} a^{(l)}_{m,\ell(m)} t^{q_h(i_l^{(m)})} \langle H^{(l)}_{m,\ell(m)} \mathcal{J}^{(l)}_m D^l_m u_z(t), D^l_m u_z(t) \rangle \\
+ 2 \sum_{l=1}^{k} \sum_{m=c_l-1+1}^{c_l} \sum_{p \in A^{(l)}_m \setminus \{m\}} a^{(l)}_{m,\ell(m)} t^{q_h(i_l^{(m)})} \langle \mathcal{J}^{(l)}_m D^l_m u_z(t), D^l_m u_z(t) \rangle \\
+ 2 \sum_{l=1}^{k} \sum_{m=c_l-1+1}^{c_l} \sum_{s \in B^{(l)}_m} a^{(l)}_{m,\ell(m)} t^{2q_h(i_s^{(m)})} \langle \mathcal{A}^{(l)}_{m,s} D^l_s u_z(t), D^l_s u_z(t) \rangle \\
+ 2 \sum_{l=1}^{k} \sum_{m=c_l-1+1}^{c_l} \sum_{s \in B^{(l)}_m} a^{(l)}_{m,\ell(m)} t^{q_h(i_s^{(m)})} \langle H^{(l)}_{\ell(m),m} \mathcal{A}^{(l)}_{m,s} D^l_s u_z(t), D^l_s u_z(t) \rangle,
\]

(3.16)

for any $t > 0$, where $A^{(l)}_m$ is given by (A.2), and

\[
B^{(l)}_m = \left\{ s : i_s^{(l)} = i_m^{(l)} - e_j^{(r+1)} + e_h^{(r+1)} \text{ for some } j, \ldots, r \right\},
\]

such that $\alpha_l > 0$, $h \leq \min\{j + 1, r\}$,

if $i_l^{(m)} = (\alpha_0, \ldots, \alpha_r)$, moreover the entries of the matrices $\mathcal{K}^{(l)}_{m,p}$ $(m = c_l-1+1, \ldots, c_l)$ and $\mathcal{A}^{(l)}_{m,s}$ $(m = 1, \ldots, c_l, s \in B^{(l)}_m)$ depend (linearly) only on the entries of the matrix $B$.

Since by Lemma A.2 the matrix $\mathcal{J}^{(l)}_m$ has maximum rank (which equals the number of its columns) for any $m$ and any $l$, we can fix the matrix $H^{(l)}_{m,\ell(m)}$ such that the matrix $-H^{(l)}_{m,\ell(m)} \mathcal{J}^{(l)}_m - (H^{(l)}_{m,\ell(m)} \mathcal{J}^{(l)}_m)^*$ is positive definite. We set

\[
\lambda^{(l)} = \min_{m=c_l-1+1, \ldots, c_l} \lambda_{\min}(-H^{(l)}_{m,\ell(m)} \mathcal{J}^{(l)}_m - (H^{(l)}_{m,\ell(m)} \mathcal{J}^{(l)}_m)^*).
\]

Hence, observing that, by properties (ii) and (v) in Lemma 3.4 we have $q_h(i_l^{(m)}) = (2q_h(i_m^{(l)}) - 1)^+$, we can estimate

\[
2 \sum_{l=1}^{k} \sum_{m=c_l-1+1}^{c_l} a^{(l)}_{m,\ell(m)} t^{2q_h(i_m^{(l)})} \langle H^{(l)}_{m,\ell(m)} \mathcal{J}^{(l)}_m D^l_m u_z(t), D^l_m u_z(t) \rangle \\
\leq -\lambda^{(l)} \sum_{l=1}^{k} \sum_{m=c_l-1+1}^{c_l} a^{(l)}_{m,\ell(m)} t^{(2q_h(i_m^{(l)}) - 1)^+} |D^l_m u_z(t)|^2,
\]

(3.17)

for any $t > 0$. Now, we estimate the second and the third terms in (3.16). For this purpose, we first conclude from Lemma 3.4(iv) the following: if $p \in A^{(l)}_m$, then $q_h(i_p^{(l)}) - 1 \leq q_h(i_m^{(l)})$; and if $s \in B^{(l)}_m$, then $q_h(i_s^{(l)}) - 1 \leq q_h(i_m^{(l)})$. It follows that

\[
2(q_h(i_m^{(l)}) + q_h(i_m^{(l)})) \geq (2q_h(i_m^{(l)}) - 1)^+ + (2q_h(i_p^{(l)}) - 1)^+, \quad m > c_l-1, p \in A^{(l)}_m,
\]

\[
2q_h(i_m^{(l)}) \geq (2q_h(i_s^{(l)}) - 1)^+ + (2q_h(i_m^{(l)}) - 1)^+, \quad m \leq c_l, s \in B^{(l)}_m.
\]
or, equivalently,

\[
t^{q_h(i_m^{(l)})} + q_h(i_{\ell(m)}^{(l)}) \leq t^{\frac{(2q_h(i_m^{(l)})-1)^+}{2}} + \frac{(2q_h(i_{\ell(m)}^{(l)})-1)^+}{2}, \quad m > c_{l-1}, \quad p \in A_{m}^{(l)},
\]

\[
t^{q_h(i_m^{(l)})} \leq t^{\frac{(2q_h(i_{\ell(m)}^{(l)})-1)^+}{2}} + \frac{(2q_h(i_{\ell(m)}^{(l)})-1)^+}{2}, \quad m \leq c_{l-1}, \quad s \in B_{m}^{(l)},
\]

for any \( t \in [0, 1] \). Inequalities (3.18a) and (3.18b) will allow us to split the powers of \( t \) by Young’s inequality in the estimate of the second and third terms in (3.16). Since we are looking for a right-neighborhood of \( t = 0 \) where \( g_z \) is non-positive, without loss of generality we can assume that \( t \in [0, 1] \). We now consider several cases according to the values of \( p \) and \( s \). We handle the different cases for \( p \), respectively for \( s \), parallely. First, suppose that \( p, s \leq c_{l-1} \). Using conditions (3.12)(c) and (3.12)(d), we obtain \( 2\eta_{m,t}^{(l)} < \eta_{m,t}^{(l)} + \eta_{m,t}^{(l)} \) and \( 2\eta_{m,m}^{(l)} < \eta_{m,t}^{(l)} + \beta_{m}^{(l)} \), where \( \beta_{m}^{(l)} = \eta_{m,t}^{(l)} \) if \( m \leq c_{l-1} \) and \( \beta_{m}^{(l)} = \eta_{m,t}^{(l)} \) otherwise. From this, (3.18a), (3.18b) and Young’s inequality we can conclude

\[
\begin{align*}
|\mathcal{A}_{m,t}^{(l)} & \cdot t^{q_h(i_m^{(l)})} + q_h(i_{\ell(m)}^{(l)})\langle \mathcal{A}_{m,t}^{(l)} \cdot D_p u_z(t), D_m u_z(t) \rangle | \\
& \leq o(\mathcal{A}_{m,t}^{(l)}) t^{(2q_h(i_m^{(l)})-1)^+} |D_p u_z(t)|^2 + o(\mathcal{A}_{m,t}^{(l)}) t^{(2q_h(i_{\ell(m)}^{(l)})-1)^+} |D_m u_z(t)|^2,
\end{align*}
\]

(3.19a)

\[
\begin{align*}
|\mathcal{A}_{m,m,t}^{(l)} & \cdot t^{q_h(i_m^{(l)})} \langle \mathcal{A}_{m,m,t}^{(l)} \cdot D_s u_z(t), D_m u_z(t) \rangle | \\
& \leq o(\mathcal{A}_{m,m,t}^{(l)}) t^{(2q_h(i_m^{(l)})-1)^+} |D_s u_z(t)|^2 + o(\mathcal{A}_{m,m,t}^{(l)}) t^{(2q_h(i_m^{(l)})-1)^+} |D_m u_z(t)|^2,
\end{align*}
\]

(3.19b)

for any \( t \in [0, 1] \). The case when \( m \leq c_{l-1} \) and \( s > c_{l-1} \) can be addressed similarly, taking now (3.12)(c) into account. We thus obtain

\[
\begin{align*}
|\mathcal{A}_{m,m,t}^{(l)} & \cdot t^{2q_h(i_m^{(l)})} \langle \mathcal{A}_{m,m,t}^{(l)} \cdot D_s u_z(t), D_m u_z(t) \rangle | \\
& \leq o(\mathcal{A}_{m,m,t}^{(l)}) t^{(2q_h(i_m^{(l)})-1)^+} |D_s u_z(t)|^2 + o(\mathcal{A}_{m,m,t}^{(l)}) t^{(2q_h(i_m^{(l)})-1)^+} |D_m u_z(t)|^2,
\end{align*}
\]

(3.20)

for any \( t \in [0, 1] \). Finally, we consider the case when \( m, p, s > c_{l-1} \). Observe that \( p < m \) for \( p \in A_{m}^{(l)}, p \neq m \), and hence from condition (3.12)(b) we obtain \( 2\eta_{m,t}^{(l)} < \eta_{m,t}^{(l)} + \eta_{p,t}^{(l)} \), whereas we also have \( 2\eta_{m,m}^{(l)} < \eta_{m,t}^{(l)} + \eta_{s,t}^{(l)} \). By (3.12)(a). These yield

\[
\begin{align*}
|\mathcal{A}_{m,m,t}^{(l)} & \cdot t^{q_h(i_m^{(l)})} + q_h(i_{\ell(m)}^{(l)})\langle \mathcal{H}_{m,t}^{(l)} \cdot D_s u_z(t), D_m u_z(t) \rangle | \\
& \leq o(\mathcal{A}_{m,m,t}^{(l)}) t^{(2q_h(i_m^{(l)})-1)^+} |D_s u_z(t)|^2 + o(\mathcal{A}_{m,m,t}^{(l)}) t^{(2q_h(i_{\ell(m)}^{(l)})-1)^+} |D_m u_z(t)|^2,
\end{align*}
\]

(3.21a)

\[
\begin{align*}
|\mathcal{A}_{m,s,t}^{(l)} & \cdot t^{q_h(i_m^{(l)})} \langle \mathcal{A}_{m,s,t}^{(l)} \cdot D_s u_z(t), D_m u_z(t) \rangle | \\
& \leq o(\mathcal{A}_{m,s,t}^{(l)}) t^{(2q_h(i_m^{(l)})-1)^+} |D_s u_z(t)|^2 + o(\mathcal{A}_{m,s,t}^{(l)}) t^{(2q_h(i_m^{(l)})-1)^+} |D_m u_z(t)|^2,
\end{align*}
\]

(3.21b)

for any \( t \in [0, 1] \). We now estimate the fourth term in (3.16). First, notice that

\[
2(q_h(i_m^{(l)}) + q_h(i_{\ell(m)}^{(l)})) \geq (2q_h(i_s^{(l)}) - 1)^+ + (2q_h(i_{\ell(m)}^{(l)}) - 1)^+,
\]

by Lemma 3.4(iv), or, equivalently

\[
t^{q_h(i_m^{(l)})} + q_h(i_{\ell(m)}^{(l)}) \leq t^{\frac{(2q_h(i_m^{(l)})-1)^+}{2}} + \frac{(2q_h(i_{\ell(m)}^{(l)})-1)^+}{2}, \quad t \in [0, 1], \ m > c_{l-1}, \ s \in B_{m}^{(l)}.
\]

If \( \ell(m) > c_{l-1} \), then because of \( 2\eta_{m,t}^{(l)} < \eta_{m,t}^{(l)} \), (3.12)(e) we can write

\[
\begin{align*}
|\mathcal{A}_{m,m,t}^{(l)} & \cdot t^{q_h(i_m^{(l)})} + q_h(i_{\ell(m)}^{(l)})\langle \mathcal{H}_{m,m}^{(l)} \cdot D_s u_z(t), D_m u_z(t) \rangle | \\
& \leq o(\mathcal{A}_{m,m,t}^{(l)}) t^{(2q_h(i_m^{(l)})-1)^+} |D_s u_z(t)|^2 + o(\mathcal{A}_{m,m,t}^{(l)}) t^{(2q_h(i_{\ell(m)}^{(l)})-1)^+} |D_m u_z(t)|^2,
\end{align*}
\]

(3.22)

for all \( t \in [0, 1] \), where again \( \beta_{m}^{(l)} = \eta_{m,t}^{(l)} \) if \( s \leq c_{l-1} \) and \( \beta_{m}^{(l)} = \eta_{s,t}^{(l)} \) otherwise. On the other hand, if \( \ell(m) \leq c_{l-1} \), then we have \( 2\eta_{m,t}^{(l)} < \eta_{m,t}^{(l)} \) (see (3.12)(c) and (3.12)(d)). Thus, we may conclude

\[
\begin{align*}
|\mathcal{A}_{m,m,t}^{(l)} & \cdot t^{q_h(i_m^{(l)})} + q_h(i_{\ell(m)}^{(l)})\langle \mathcal{H}_{m,m}^{(l)} \cdot D_s u_z(t), D_m u_z(t) \rangle | \\
& \leq o(\mathcal{A}_{m,m,t}^{(l)}) t^{(2q_h(i_m^{(l)})-1)^+} |D_s u_z(t)|^2 + o(\mathcal{A}_{m,m,t}^{(l)}) t^{(2q_h(i_{\ell(m)}^{(l)})-1)^+} |D_m u_z(t)|^2,
\end{align*}
\]

(3.23)

for all \( t \in [0, 1] \), where again \( \beta_{m}^{(l)} = \eta_{m,t}^{(l)} \) if \( s \leq c_{l-1} \) and \( \beta_{m}^{(l)} = \eta_{s,t}^{(l)} \) otherwise.
that
\[
\left| a^{(i)}_{l,m,z}(t) q_h(i^{(i)}_{l,m}) + q_h(i^{(i)}_{l,m}) \left( H^l_{l,m} + M_{l,m,z} D^l_d u_z(t) \right) D^l_d u_z(t) \right|
\leq o(\beta^{(l)}_{l,m}) t^{2q_h(i^{(i)}_{l,m}) - 1} |D^l_d u_z(t)|^2 + o(\beta^{(l)}_{l,m}) t^{2q_h(i^{(i)}_{l,m}) - 1} |D^l_d u_z(t)|^2,
\]
holds for \( t \in [0, 1] \). Therefore, by summing up (3.16), (3.17), (3.20)–(3.23), we can deduce that
\[
g_{2, \varepsilon}(t) \leq - \varepsilon t^{(k)} \sum_{\ell=1}^{c_1} \sum_{m=\ell+1}^{c_1} \left( a^{(i)}_{\ell,m} + o(a^{(i)}_{\ell,m}) \right) t^{(2q_h(i^{(i)}_{\ell,m}) - 1)} |D^l_d u_z(t)|^2
+ \sum_{\ell=1}^{c_1} \sum_{m=\ell+1}^{c_1} o(\beta^{(l)}_{\ell,m}) t^{2q_h(i^{(i)}_{\ell,m}) - 1} |D^l_d u_z(t)|^2
\]
holds for any \( t \in [0, 1] \).

**Estimating the term** \( g_{3, \varepsilon} \). As it has been already remarked, this term occurs only if \( h < k \). We begin by estimating the term \( t^{q_h(i^{(i)}_{l,m}) + q_h(i^{(i)}_{l,m}) - 1} a^{(i)}_{l,m} |D^l_d u_z(t)| |D^l_d u_z(t)| \), when \( l > h \) and \( m > c_{l-1} \); note that, by Lemma 3.4(v), \( q_h(i^{(i)}_{l,m}) + q_h(i^{(i)}_{l,m}) - 1 > 0 \). By (3.12)(b), (3.12)(c) and (3.12)(d), we have that \( 2n(i^{(i)}_{l,m}) < \eta^{(l)}_{l,m} + \eta^{(l)}_{l,m} \), if \( \ell(m) \leq c_{l-1} \), and \( 2n(i^{(i)}_{l,m}) < \eta^{(l)}_{l,m} + \eta^{(l)}_{l,m} \) if \( \ell(m) > c_{l-1} \). Hence, we can estimate
\[
t^{q_h(i^{(i)}_{l,m}) + q_h(i^{(i)}_{l,m}) - 1} a^{(i)}_{l,m} |D^l_d u_z(t)| |D^l_d u_z(t)|
\leq o(\beta^{(l)}_{l,m}) t^{2q_h(i^{(i)}_{l,m}) - 1} |D^l_d u_z(t)|^2 + o(\beta^{(l)}_{l,m}) t^{2q_h(i^{(i)}_{l,m}) - 1} |D^l_d u_z(t)|^2,
\]
for any \( t > 0 \), where \( \beta^{(l)}_{l,m} \) is as above. From (3.25) and taking condition (3.12)(a) into account, we now get
\[
g_{3, \varepsilon}(t) = 2 \sum_{l=h+1}^{c_1} \sum_{m=\ell+1}^{c_1} q_h(i^{(i)}_{l,m}) a^{(i)}_{l,m} t^{2q_h(i^{(i)}_{l,m}) - 1} |D^l_d u_z(t)|^2 + 2 \sum_{l=h+1}^{c_1} \sum_{m=\ell+1}^{c_1} \left( q_h(i^{(i)}_{l,m}) + q_h(i^{(i)}_{l,m}) \right) a^{(i)}_{l,m} t^{2q_h(i^{(i)}_{l,m}) + q_h(i^{(i)}_{l,m}) - 1}
\times \left( H_{l,m} D^l_d u_z(t), D^l_d u_z(t) \right)
\leq \sum_{l=h+1}^{c_1} \sum_{m=\ell+1}^{c_1} o(\beta^{(l)}_{l,m}) t^{2q_h(i^{(i)}_{l,m}) - 1} |D^l_d u_z(t)|^2
+ \sum_{l=h+1}^{c_1} \sum_{m=\ell+1}^{c_1} o(\beta^{(l)}_{l,m}) t^{2q_h(i^{(i)}_{l,m}) - 1} |D^l_d u_z(t)|^2,
\]
for any \( t > 0 \).

**Estimating the terms** \( g_{4, \varepsilon} \) and \( g_{5, \varepsilon} \). We begin with \( g_{4, \varepsilon} \), the case of \( g_{5, \varepsilon} \) being completely analogous.

Let us observe that we have
\[
[D^l_d, \text{Tr}(Q D^2)] u_z = \sum_{i,j=1}^{N} [D^l_d, q_{ij} D_{ij}] u_z = \sum_{i,j=1}^{m_0} [D^l_d, q_{ij} D_{ij}] u_z = \sum_{z=2}^{l+1} \mathcal{P}^{(l,z)} D^z u_z,
\]
for any \( m = 1, \ldots, c_t \) and some matrices \( \mathcal{P}^{(l,z)} \) whose entries linearly depend only on the derivatives (of order at least 1 and at most \( l \)) of the functions \( q_{ij} \) (\( i, j = 1, \ldots, p_0 \)). In particular, these matrices are independent of \( \varepsilon \). Moreover, if we split the matrices \( \mathcal{P}^{(l,z)} \) into sub-blocks \( F^{(l,z)}_{m,s} \) (\( m = 1, \ldots, c_t \), \( s = 1, \ldots, c_z \)) according to the rule in Subsection 2.3, it follows that \( F^{(l,z)}_{m,s} = 0 \) if \( s > c_{l-1} \). To see the above let \( \alpha \in \mathbb{N}_0^l \) be a multi-index with \( |\alpha| = l \), \( |\alpha| = (\alpha_0, \alpha_1, \ldots, \alpha_r) \in \mathbb{N}_0^{r+1} \). From the second equality in (3.27) (which is immediate if we recall that \( q_{ij} \) is constant if at least one of \( i \) and \( j \) is greater than \( p_0 \)) it follows that the terms appearing in (3.27) and obtained from \([D^\alpha, \text{Tr}(Q D^2)] u_z \) are \( D^\beta u_z \) with coefficients in front depending on the derivatives of \( q_{ij} \), \( i, j \leq p_0 \), and with some \( \beta \in \mathbb{N}_0^r \) such that \( |\beta| = (\beta_0, \ldots, \beta_z) \) with \( 1 \leq \beta_0 \leq \alpha_0 + 2 \), \( \beta_j \leq \alpha_j \) for any
The summands in the second and the third terms in (3.28) can be estimated likewise. By (3.12)(c), here, we can use (3.12)(c) and estimate 

$$m = \|\beta\| \leq \|\alpha\| + 1.$$ 

In particular, since $\beta_0 > 1$, then $|\beta| = \gamma^{(z)}_s$ with $s \leq c_{z-1}$, where $z = \|\beta\|$. Denote by $C_m^{(l)}$ the indices $i_s^{(l)}$ obtained in this way from multi-indices $\alpha$ with $|\alpha| = \gamma^{(l)}_m$. By Lemma 3.4(vii) we have $q_{\gamma}(i_s^{(l)}) \geq q_{\gamma}(i_s^{(z)}) - 1$ for $i_s^{(z)} \in C_m^{(l)}$, and $q_{\gamma}(i_s^{(l)}) \geq q_{\gamma}(i_s^{(z)}) - 1$ for $i_s^{(z)} \in C_m^{(l)}$. These inequalities will allow us to split the powers of $t$ by Young's inequality. Hence we can write

$$g_{\nu}(t) = \sum_{l=1}^{N} \sum_{m=1}^{C_{l+1}} \sum_{z=2}^{l+1} a^{(l)}_{m,m} t^{2q_{\gamma}(i_s^{(l)})} \langle \varphi (l,z) D^z u_{\nu}(t), D^m u_{\nu}(t) \rangle$$

$$+ \sum_{l=1}^{N} \sum_{m=cl_{-1}+1}^{l} \sum_{z=2}^{l+1} a^{(l)}_{m,m} t^{q_{\gamma}(i_s^{(l)}) + q_{\gamma}(i_s^{(z)})} \langle H^{(l)}_{m,l}, \varphi (l,m) D^z u_{\nu}(t), D^m u_{\nu}(t) \rangle$$

$$+ \sum_{l=1}^{N} \sum_{m=cl_{-1}+1}^{l} \sum_{z=2}^{l+1} a^{(l)}_{m,m} t^{q_{\gamma}(i_s^{(l)}) + q_{\gamma}(i_s^{(z)})} \langle H^{(l)}_{l,m}, \varphi (l,m) D^z u_{\nu}(t), D^l u_{\nu}(t) \rangle,$$

(3.28)

for any $t > 0$. In the following, we assume again $t \in [0, 1]$ and denote by $C$ positive constants, independent of $\nu$, $t$ and $a$, which may vary from line to line. We can estimate the summands in the first term in the right-hand side of (3.28) as follows:

$$\sum_{l=2}^{l+1} a^{(l)}_{m,m} t^{2q_{\gamma}(i_s^{(l)})} \langle \varphi (l,z) D^z u_{\nu}(t), D^m u_{\nu}(t) \rangle$$

$$\leq C \sqrt{t} a^{(l)}_{m,m} t^{2q_{\gamma}(i_s^{(l)})} \sum_{l=2}^{l+1} \sum_{s, i_s^{(s)} \in C_m^{(l)}} |D^z u_{\nu}(t) \cdot |D^m u_{\nu}(t)|$$

$$+ C \sqrt{t} a^{(l)}_{m,m} t^{2q_{\gamma}(i_s^{(l)})} \sum_{s, i_s^{(s)} \in C_m^{(l)}} |D^l u_{\nu}(t) \cdot |D^m u_{\nu}(t)|,$$

(3.29)

for any $m \leq c_l$. Let us consider the first term in the right hand side of (3.29). Since we have $z \leq l$ here, we can use (3.12)(c) and estimate

$$\sqrt{t} a^{(l)}_{m,m} t^{2q_{\gamma}(i_s^{(l)})} \sum_{l=2}^{l+1} \sum_{s, i_s^{(s)} \in C_m^{(l)}} |D^z u_{\nu}(t) \cdot |D^m u_{\nu}(t)|$$

$$\leq C \left( o(a^{(l)}_{m,m}) t(2q_{\gamma}(i_s^{(s)}) - 1^+) |D^m u_{\nu}(t)| 2^l \sum_{s=1}^{c_{l-1}} o(a^{(l)}_{m,m}) t(2q_{\gamma}(i_s^{(s)}) - 1^+) \nu |D^s u_{\nu}(t)| 2^l \right),$$

where $\beta_m^{(l)} = \gamma^{(l)}_m$ if $m \leq c_{l-1}$ and $\beta_m^{(l)} = \eta^{(l)}_{m,l}$ otherwise. On the other hand, in the case when $z = l + 1$ we use conditions (3.12)(a), (3.12)(c) to obtain $2q_{\gamma}(i_s^{(l)}) < \beta_m^{(l)} + \eta^{(l+1)}$. Thus, we can estimate by Young's inequality

$$C \sqrt{t} a^{(l)}_{m,m} t^{2q_{\gamma}(i_s^{(l)})} \sum_{s, i_s^{(s)} \in C_m^{(l)}} |D^l u_{\nu}(t) \cdot |D^m u_{\nu}(t)|$$

$$\leq C \left( o(a^{(l)}_{m,m}) t(2q_{\gamma}(i_s^{(s)}) - 1^+) |D^m u_{\nu}(t)| 2^l \sum_{s=1}^{c_{l-1}} o(a^{(l)}_{m,m}) t(2q_{\gamma}(i_s^{(s)}) - 1^+) \nu |D^s u_{\nu}(t)| 2^l \right).$$

The summands in the second and the third terms in (3.28) can be estimated likewise. By (3.12)(c), (3.12)(d) and (3.12)(e) we have $\eta^{(l)}_{m,l} < \eta^{(l+1)}$, $2 \eta^{(l)}_{m,l} < \eta^{(l)}$ and $2 \eta^{(l)}_{m,l} < 2 \eta^{(l)}_{m,l}$. (this
latter for \( \ell(m) > c_{l-1} \), so by Young’s inequality we can deduce

\[
\sum_{z=2}^{l+1} a_{m,\ell(m)}^{(l)} t q_{m,\ell(m)} \langle H_{m,\ell(m)}, \mathcal{L}^{(l)} \rangle \mathcal{D}^z u(t), D_m^l u(t) \leq C \sqrt{T} \sum_{z=2}^{l+1} \sum_{s,z \in C_m^{(l)}} \frac{\mathcal{D}_u^z(t)}{|D_m^l u(t)|}
\]

\[
\leq o(a_{m,\ell(m)}^{(l)}) t^{(2q_{m,\ell(m)}(i^{(l)}_z)-1)^+} |D_m^l u(t)|^2 + \sum_{z=2}^{l+1} \sum_{s=1}^{c_{l-1}} o(a_{m,\ell(m)}^{(l)}) t^{(2q_{m,\ell(m)}(i^{(l)}_z)-1)^+} |D_m^l u(t)|^2
\]

and

\[
\sum_{z=2}^{l+1} a_{m,\ell(m)}^{(l)} t q_{m,\ell(m)} \langle H_{m,\ell(m)}, \mathcal{L}^{(l)} \rangle \mathcal{D}^z u(t), D_m^l u(t) \leq C \sqrt{T} \sum_{z=2}^{l+1} \sum_{s,z \in C_m^{(l)}} \frac{\mathcal{D}_u^z(t)}{|D_m^l u(t)|}
\]

\[
\leq o(a_{m,\ell(m)}^{(l)}) t^{(2q_{m,\ell(m)}(i^{(l)}_z)-1)^+} |D_m^l u(t)|^2 + \sum_{z=2}^{l+1} \sum_{s=1}^{c_{l-1}} o(a_{m,\ell(m)}^{(l)}) t^{(2q_{m,\ell(m)}(i^{(l)}_z)-1)^+} |D_m^l u(t)|^2,
\]

for \( t \in [0, 1] \), \( m > c_{l-1} \), where, again, \( \beta_{\ell(m)}^{(l)} = \eta_{\ell}^{(l)} \), if \( \ell(m) \leq c_{l-1} \), and \( \beta_{\ell(m)}^{(l)} = \eta_{\ell(m),\ell(l(m))}^{(l)} \), otherwise.

By putting everything together, we obtain

\[
g_{4,T}(t) \leq \nu \sum_{l=1}^{k+1} \sum_{m=1}^{c_{l-1} \cap c_{l-1}} o(a_{m,\ell(m)}^{(l)}) t^{(2q_{m,\ell(m)}(i^{(l)}_z)-1)^+} |D_m^l u(t)|^2 + \sum_{l=1}^{k} \sum_{m=1}^{c_{l-1} \cap c_{l-1} + 1} o(a_{m,\ell(m)}^{(l)}) t^{(2q_{m,\ell(m)}(i^{(l)}_z)-1)^+} |D_m^l u(t)|^2,
\]

(3.30)

Just in the same way, we can estimate the function \( g_{5,T} \) and get

\[
g_{5,T}(t) \leq \nu \sum_{l=1}^{k} \sum_{m=1}^{c_{l-1} \cap c_{l-1}} o(a_{m,\ell(m)}^{(l)}) t^{(2q_{m,\ell(m)}(i^{(l)}_z)-1)^+} |D_m^l u(t)|^2 + \sum_{l=1}^{k} \sum_{m=1}^{c_{l-1} \cap c_{l-1} + 1} o(a_{m,\ell(m)}^{(l)}) t^{(2q_{m,\ell(m)}(i^{(l)}_z)-1)^+} |D_m^l u(t)|^2,
\]

(3.31)

**Final estimate of the function \( g_T \).** Now, collecting (3.15), (3.24), (3.26), (3.30) and (3.31) together, we get

\[
g_T(t) \leq -2\nu \sum_{l=1}^{k} \sum_{m=1}^{c_{l-1} \cap c_{l-1}} \{a_{m,\ell(m)}^{(l)} + o(a_{m,\ell(m)}^{(l)})\} t^{(2q_{m,\ell(m)}(i^{(l)}_z)-1)^+} |D_m^l u(t)|^2
\]

\[
- \nu \sum_{l=1}^{k} \sum_{m=1}^{c_{l-1} \cap c_{l-1}} \{a_{m,\ell(m)}^{(l)} + o(a_{m,\ell(m)}^{(l)})\} t^{(2q_{m,\ell(m)}(i^{(l)}_z)-1)^+} |D_m^l u(t)|^2
\]

\[
- 2\nu \sum_{m=1}^{c_{l-1} \cap c_{l-1} + 1} o(a_{m,\ell(m)}^{(l)}) t^{(2q_{m,\ell(m)}(i^{(l)}_z)-1)^+} |D_m^l u(t)|^2,
\]

(3.32)

for any \( t \in [0, 1] \).

If we now fix the parameter \( a \) sufficiently large, condition (3.11)(b) is satisfied and, for an even larger constant \( a \), the terms in right-hand side of (3.32) will be negative, provided that one can choose the parameters \( \eta_{m,m}^{(l)} \) and \( \eta_{p,\ell(p)}^{(l)} \) such that conditions (3.11)(a) and (3.12), which we have
used in the previous estimates, are satisfied. Hence, the last part of the proof is devoted to address this point, and leads us to the conclusion of the proof.

Choice of the parameters \( \eta_{m,m}^{(l)} \) and \( \eta_{p,\ell(p)}^{(l)} \). We now show that we can fix all the constants \( \eta_{m,m}^{(l)} \) and \( \eta_{p,\ell(p)}^{(l)} \) such that the conditions (3.11)(a) and (3.12) are satisfied. For each \( l = 1, \ldots, k + 1 \) we will take a positive, strictly decreasing sequence \( \{a_{n}^{(l)}\} \) with \( a_{n}^{(l)} < 1 \), and set \( \eta_{m,m}^{(l)} := a_{n}^{(l)} a_{m}^{(l)} \). By this restriction, condition (3.11)(a) will be satisfied. So from now we concentrate only on (3.12). Notice that for \( c_{l} < m < p \) we have \( \ell(m) < \ell(p) \), so (3.12)(b) is automatically satisfied by monotonicity. Note also that for such \( m \) we have \( \ell(m) < m \). Hence, if we choose \( a_{n+1}^{(l)} < \frac{1}{2} a_{n}^{(l)} \) for all \( n \), also (3.12)(a) and (3.12)(c) will be satisfied. Now we turn to the actual construction keeping all the above requirements on \( a_{n}^{(l)} \). First we choose \( a_{n}^{(l)} \) for \( n = 1, \ldots, c_{1} = r + 1 \) according to the above. Then, for \( l \geq 2 \) we proceed inductively, first taking \( a_{1}^{(l)} < \frac{1}{2} \sqrt{\eta^{(l)}} \) and then, choosing \( a_{n}^{(l)} \) \( (n = 2, \ldots, c_{l}) \) satisfying \( a_{n}^{(l)} < \frac{1}{2} a_{n-1}^{(l)} \) for any \( n \) and with \( a_{c_{l}+1}^{(l)} a_{c_{l}+1}^{(l)} \). The first condition implies that (3.12)(c) is satisfied. On the other hand, since \( a_{n}^{(l)} < 1 \), the latter condition implies (3.12)(d). Indeed, for \( m > c_{l+1} \) and \( p < c_{l} \) we have, \( a_{m} a_{l}^{(m)} < a_{c_{l}+1} a_{c_{l}+1} < a_{l}^{(l)} < a_{l}^{(l)} \). \( \blacksquare \)

**Remark 3.6.** Notice that the above proof works also for other functions \( q : \mathbb{N}_{0}^{+} \rightarrow \mathbb{R}^{+} \) replacing \( q_{h} \), as long as this function \( q \) has the properties similar to that of \( q_{h} \) as listed in Lemma 3.4.

4. Construction of the semigroup

In this section we prove that, for any \( f \in C_{b}(\mathbb{R}^{N}) \), the Cauchy problem

\[
\begin{align*}
D_{t}u(t,x) &= \mathcal{A} u(t,x), & t > 0, & x \in \mathbb{R}^{N}, \\
\psi(0,x) = f(x), & x \in \mathbb{R}^{N},
\end{align*}
\]

admits a unique classical solution \( u \), and consequently we can associate a semigroup of bounded operators in \( C_{b}(\mathbb{R}^{N}) \) with the operator \( \mathcal{A} \).

**Theorem 4.1.** Suppose that Hypotheses 2.1 are satisfied. Then the following assertions hold:

(i) For any \( f \in C_{b}(\mathbb{R}^{N}) \) there exists a unique classical solution \( u \) to problem (4.1).

(ii) The family \( \{T(t)\} \), defined by \( T(t)f := u(t,\cdot) \) for any \( t > 0 \), where \( u \) is the classical solution to problem (4.1) corresponding to the initial value \( f \), is a positivity preserving semigroup of linear contractions in \( C_{b}(\mathbb{R}^{N}) \).

(iii) If \( f \in C_{c}(\mathbb{R}^{N}) \), then \( T(t)f \) converges to \( f \), as \( t \rightarrow 0^{+} \), uniformly in \( \mathbb{R}^{N} \).

(iv) For any \( f \in C_{b}(\mathbb{R}^{N}) \) and any multi-index \( \alpha \in \mathbb{N}_{0}^{N} \), with \( \|\alpha\| \leq \kappa - 1 \), the derivative \( D^{\alpha} T(t) \cdot f \) exists in the classical sense in \( [0, +\infty[ \times \mathbb{R}^{N} \) and it is a continuous function. Moreover, there exists a positive constant \( C \), depending only on \( \omega \), \( h \) and \( \|\alpha\| \) such that, for any \( f \in C_{b}(\mathbb{R}^{N}) \) and any \( \alpha \) as above, we have

\[
\|D^{\alpha} T(t)f\|_{C_{b}(\mathbb{R}^{N})} \leq C t^{-q_{h}(\|\alpha\|)} e^{\omega t} \|f\|_{C_{b}^{k}(\mathbb{R}^{N})}, \quad t \in [0, +\infty[.
\]

**Proof.** (i) First of all, notice that uniqueness follows immediately from the maximum principle, Proposition 3.1. Throughout the proof, we denote by \( C \) positive constants, independent of \( \varepsilon \in [0,1] \), which may vary from line to line. As a first step, we show that, for any \( \omega > 0 \), there exists a positive constant \( \bar{C} = \bar{C}(\omega) \), independent of \( \varepsilon \), such that

\[
\|D^{\alpha} T_{\varepsilon}(t)f\|_{\infty} \leq \bar{C} t^{-q_{h}(\|\alpha\|)} e^{\omega t} \|f\|_{C_{b}^{k}(\mathbb{R}^{N})}, \quad t \in [0, +\infty[ , \quad \varepsilon \in [0,1],
\]

for any \( f \in C_{b}^{k}(\mathbb{R}^{N}) \) and any \( \|\alpha\| \leq \kappa \). Estimate (4.3) follows from the semigroup law and from (3.10). Indeed, fix \( \omega > 0 \) and let \( C_{0} := \min\{1, \inf_{t \in [1, +\infty[ [t^{-q_{h}(\|\alpha\|)} e^{\omega t}] \}. \) Splitting \( T_{\varepsilon}(t) = T_{\varepsilon}(1) T_{\varepsilon}(t-1) \), for any \( t > 1 \), and taking (3.10) in Theorem 3.3 into account, we get

\[
\|D^{\alpha} T_{\varepsilon}(t)f\|_{\infty} \leq \bar{C} \|T_{\varepsilon}(t-1)f\|_{\infty} \leq \bar{C} C_{0}^{-1} C_{0} \|f\|_{\infty} \leq \bar{C} C_{0}^{-1} t^{-q_{h}(\|\alpha\|)} e^{\omega t} \|f\|_{\infty}.
\]

Hence, (4.3) follows with \( \bar{C} = \bar{C} C_{0}^{-1} \).

We can now prove that problem (4.1) admits a unique classical solution for any \( f \in C_{b}(\mathbb{R}^{N}) \). For this purpose, as in the proof of Theorem 3.2, we set \( u_{\varepsilon} := T_{\varepsilon}(\cdot) f \). Then, from (4.3), we easily
deduce that for any $0 < T_1 < T_2$
\[
\sup_{t \in [0,1]} \sup_{t \in [T_1,T_2]} ||u_\varepsilon(t,\cdot)||_{C^2_0(\mathbb{R}^N)} < +\infty \text{ holds.}
\]

Since the function $u_\varepsilon$ solves the Cauchy problem (3.1) and the coefficients of the operator $\mathcal{A}_\varepsilon$ are locally bounded, uniformly with respect to $\varepsilon \in [0,1]$, the function $D_t u_\varepsilon$ is bounded in $[T_1,T_2] \times B(R)$, for any $R > 0$, by a constant, independent of $\varepsilon$. Therefore we have $u_\varepsilon \in \text{Lip}([T_1,T_2]; C(B(R)))$ with norm independent of $\varepsilon \in [0,1]$. By applying [15, Propositions 1.1.2(iii) and 1.1.4(i)], we now deduce that $u_\varepsilon \in C^{\theta/2, \kappa - 1 + \theta}(\{T_1,T_2\} \times B(R))$ for any $\varepsilon$ as above and some $\theta \in [0,1]$, and with $C^{\theta/2, \kappa - 1 + \theta}$ norm being bounded by a constant independent of $\varepsilon$. As a byproduct, using that $u_\varepsilon$ solves (3.1), we deduce that $D_t u_\varepsilon \in C^{\theta/2, \kappa - 3 + \theta}(\{T_1,T_2\} \times B(R))$ and, again, its $C^{\theta/2, \kappa - 3 + \theta}$ norm is bounded by a constant independent of $\varepsilon$. Since $T_1, T_2, R$ are arbitrarily fixed, using both a compactness and a diagonal argument, we can determine an infinitesimal sequence $\{\varepsilon_n\}$ such that $\{u_\varepsilon_n\}$ converges in $C^{1, \kappa - 1}(K)$, for any compact set $K \subset \mathbb{R}^N$, to a function $u_f \in C^{1+\theta/2, \kappa - 1 + \theta}_{\text{loc}}([0,\infty[ \times \mathbb{R}^N)$. Of course, the function $u_f$ solves the differential equation in (4.1) for $t > 0$. The continuity of $u_f$ up to $t = 0$ and the condition $u_f(0, \cdot) = f$, are obtained in three steps.

\textbf{Step 1.} Suppose that $f \in C^2_0(\mathbb{R}^N)$. Then, by the proof of [20, Proposition 4.3], we know that
\[
||u_n(t) - f||_\infty \leq t \sup_{s \in [0,\infty[} ||T_{\varepsilon_n}(s) f||_\infty \leq t ||f||_\infty \leq C t, \quad \text{if } 0 < t.
\]

Hence, taking the limit, first as $n \to +\infty$ and then as $t \to 0^+$, we obtain that $u_f$ is continuous at $t = 0$. Hence $u_f$ is the unique classical solution to problem (4.1). Moreover, we infer that $u_n$ converges to $u_f$, as $\varepsilon \to 0^+$, in $C^{1, \kappa - 1}(\{T_1,T_2\} \times B(R))$ for any $T_1, T_2, R$ as above. Indeed, by uniqueness (or by the maximum principle in Proposition 3.1), any sequence $u_n$ (with $\varepsilon_n$ being positive and infinitesimal) which converges in $C^{1, \kappa}_{\text{loc}}([0,\infty[ \times \mathbb{R}^N)$, must converge to $u_f$. Again, the maximum principle implies that for a non-negative $f \in C^0(\mathbb{R}^N)$ the solution $u_f$ is also non-negative.

\textbf{Step 2.} Suppose now that $f$ vanishes at $\infty$. Then, we can approximate $f$ by a sequence of smooth and compactly supported functions $f_n$. By estimate (3.2) we know that
\[
\sup_{t \geq 0} ||u_{f_n}(t,\cdot) - u_f(t,\cdot)||_\infty \leq ||f_n - f||_\infty, \quad \text{if } f_n \to f, \quad n \in \mathbb{N}.
\]

Hence $u_{f_n}$ converges to $u_f$ uniformly in $[0,\infty[ \times \mathbb{R}^N$. Since $u_{f_n}$ is continuous in $[0,\infty[ \times \mathbb{R}^N$ and $u_{f_n}(0,\cdot) = f_n$, it follows that $u_f$ is continuous in $[0,\infty[ \times \mathbb{R}^N$ as well, and that $u_f(0,\cdot) = f$ holds.

The same argument in the last part of Step 1, shows that, also in this situation, the function $T_{\varepsilon_n}(t)f$ converges to $u_f$ in $C^{1, \kappa}_{\text{loc}}([0,\infty[ \times \mathbb{R}^N)$ as $\varepsilon \to 0^+$. Moreover for non-negative $f$ we see the solution $u_f$ to be non-negative as well.

\textbf{Step 3.} We now consider the general case when $f \in C_b(\mathbb{R})$. We fix $R > 0$ and a function $\psi \in C^\infty_c(\mathbb{R}^N)$ satisfying $\chi_{B(R)} \leq \psi \leq \chi_{B(R)+1}$. Further, we split first $f = \psi f + (1 - \psi)f$, and then we can write $T_{\varepsilon_n}(t)f = T_{\varepsilon_n}(t)(\psi f) + T_{\varepsilon_n}(t)((1 - \psi)f)$. We remark that the semigroups $\{T_e(t)\}$ preserve positivity, which is well-known but also follows immediately from the maximum principle, Proposition 3.1. This implies
\[
\{T_{\varepsilon_n}(t)((1 - \psi)f)\}(x) \leq ||f||_\infty (T_{\varepsilon_n}(t)(1 - \psi))(x), \quad \text{if } t > 0, \quad x \in \mathbb{R}^N.
\]

Recall that $T_{\varepsilon_n}(1 - \psi) = 1 - T_{\varepsilon_n}(\cdot)\psi$ and let $n \to +\infty$ in (4.4) to conclude
\[
||u_f(t,x) - u_{\psi f}(t,x)||_\infty \leq ||f||_\infty (1 - u_{\psi f}(t,x)), \quad \text{if } t \in ]0,\infty[, \quad x \in \mathbb{R}^N.
\]

Since, for any $x \in B(R)$, $(\psi f)(x) = f(x)$ and $u_{\psi f}(t,x)$ tends to 1 as $t \to 0^+$, we obtain that $u_f$ is continuous in $[0,1] \times B(R)$ and $u_f(0,\cdot) = f$ in $B(R)$. The arbitrariness of $R > 0$ allows us to complete the proof. Moreover, as in the previous cases, $T_{\varepsilon}(\cdot)f$ converges to $u_f$ in $C^{1, \kappa}_{\text{loc}}([0,\infty[ \times \mathbb{R}^N)$, as $\varepsilon \to 0^+$.

(ii) and (iii). They follow from the maximum principle in Proposition 3.1 and Steps 1 and 2 in the proof of (i).
(iv). By (i), we know that the function $T(t)f$ belongs to $C^{\kappa-1}(\mathbb{R}^N)$ for any $t > 0$ and any $f \in C_0(\mathbb{R}^N)$, and $T(t)f$ converges to $T(t)f$ in $C^{\kappa-1}_{\text{loc}}(\mathbb{R}^N)$ as $\varepsilon \to 0^+$. Since the constant in (4.3) is independent of $\varepsilon \in [0,1]$, it is immediate to conclude that (4.2) holds for any $\|\alpha\| \leq \kappa - 1$.

With respect to derivatives in the first and second block of variables we can prove more regularity.

**Theorem 4.2.** Suppose that Hypotheses 2.1 are satisfied and let $\{T(t)\}$ be the semigroup constructed in Theorem 4.1. Then for any $f \in C_0(\mathbb{R}^N)$ and any multi-index $\alpha \in \mathbb{N}_0^N$, with $\|\alpha\| = \kappa$ and $\alpha_j \neq 0$ for some $j \leq p_0 + p_1$, the derivative $D^\alpha T(\cdot)f$ exists in the classical sense in $[0, +\infty[ \times \mathbb{R}^N$ and it is a continuous function. Moreover, there exists a positive constant $C$, depending only on $\omega, h$ such that, for any $f \in C_0(\mathbb{R}^N)$ and any $\alpha$ as above, we have

$$
\|D^\alpha T(t)f\|_{C_0(\mathbb{R}^N)} \leq Ct^{-\theta_0(\|\alpha\|)}e^{\omega t}\|f\|_{C_0(\mathbb{R}^N)}, \quad t \in [0, +\infty[.
$$

**Proof.** We set $u = T(\cdot)f$ and split the proof into several steps. In the first one we show a formula that will be used in Steps 2 to 5, in the actual proof of (4.5). Until Step 5, we will assume at least $f \in C^{\kappa-1}_{\text{loc}}(\mathbb{R}^N)$, and then in Step 5 we proceed with an approximation argument. Finally, in Step 6, we show that the function $D^\alpha u$ is continuous in $[0, +\infty[ \times \mathbb{R}^N$.

**Step 1.** We fix $R > 0$, $j \in \{1, \ldots, N\}$ and $f \in C^{\kappa-1}_{\text{loc}}(\mathbb{R}^N)$, and prove that, for any $\eta = \eta_R \in C^\infty(\mathbb{R}^N)$ such that $\chi_R(\mathbb{R}) \leq \eta_R \leq \chi(2R)$, and any $\vartheta = \vartheta_R \in C^\infty([0, +\infty[)$ such that $\chi_{[R-1,R]} \leq \vartheta \leq \chi_{[2R-1,2R]}$, it holds that

$$
D_j u(t,x) = \int_0^t \langle T(t-s)g_j(s,\cdot)(x)ds, \quad t \in [R^{-1}, R], \ x \in B(R),
$$

where

$$
g_j = \vartheta \text{Tr}((D_0Q_0)D_j^2(\eta_R)) + \vartheta(B^* D_j(\eta_R)) + \vartheta((D_jF,F_j(\eta_R))
- \vartheta D_j(u \eta R) - 2\vartheta(Q_0D_ju, \vartheta D_j(\eta_R)
- 2\vartheta(Q_0D_ju, \vartheta D_j(\eta_R)
- 2\vartheta(D_jQ_0D_ju, \vartheta D_j(\eta_R) + \vartheta D_j(\eta_R), \quad \text{for } D_j u \text{ and } D_j^2 u \text{ denoting, respectively, the vector of first-order derivatives of } u \text{ with respect to indices not greater than } p_0 \text{ and the quadratic submatrix obtained erasing the last } N - p_0 \text{ rows and columns from } D^2 u.

To prove (4.6), for any $\delta \in [-1, 1]$, we introduce the operator $\tau^\delta_\psi$ defined on $C_0(\mathbb{R}^N)$ by

$$(\tau^\delta_\psi)(x) = \frac{\psi(x + \delta \epsilon_j) - \psi(x)}{\delta}, \quad x \in \mathbb{R}^N, \ \psi \in C_0(\mathbb{R}^N).$$

Moreover, we set $w^\delta_{t, \epsilon} = \vartheta \tau^\delta_\psi v_{\epsilon}$ where $v_{\epsilon} = \eta u_{\epsilon}$. In the sequel, in order to shorten the notation, if there is no danger of confusion we only stress explicitly the dependence on $\epsilon$ of the functions considered. As it is easily seen,

$$
\begin{aligned}
D_t w_{\epsilon}(t,x) &= \omega_{\epsilon} w_{\epsilon}(t,x) + g_{\epsilon}(t,x), \quad t \in [0, +\infty[, \ x \in \mathbb{R}^N, \\
 w_{\epsilon}(0,x) &= 0, \quad \text{for } x \in \mathbb{R}^N,
\end{aligned}
$$

holds with

$$
g_{j, \epsilon, \tau} = g_{\epsilon} = \vartheta \text{Tr}(\tau(Q_0)D_j^2 v_{\epsilon}) + \vartheta(B^* D_j v_{\epsilon}) + \vartheta((F, D_j v_{\epsilon})
- \vartheta D_j(u \epsilon \tau(\cdot, + \delta \epsilon_j), D_j(\eta_R)
- 2\vartheta(D_ju, D_j(\eta_R) - 2\vartheta((Q_0D_ju, \vartheta D_j(\eta_R)
- 2\vartheta(D_jQ_0D_ju, \vartheta D_j(\eta_R) - 2\vartheta(D_jQ_0D_ju, \vartheta D_j(\eta_R) + \vartheta D_j(\eta_R), \quad \text{see (4.8)}
$$

and $D_j \psi$ denotes the vector of the first-order derivatives of the function $\psi : \mathbb{R}^N \to \mathbb{R}$ with respect to the last $N - p_0$ variables. In view of the variation of constants formula (see [21, Theorem 3.5]), we obtain that $w_{\epsilon}$ satisfies

$$
w_{\epsilon}(t,x) = \int_0^t (T_s(t-s)g_{\epsilon}(s,\cdot))(x)ds, \quad t \in [0, +\infty[., \ x \in \mathbb{R}^N. \quad (4.9)
$$

We are going to show that we can take the limit as $\epsilon \to 0^+$ in (4.9) and write

$$
v_{\delta}(t,x) := \eta(x)(\tau_\delta u(t,\cdot))(x) = \int_0^t (T(t-s)g_{\delta j, \delta}(s,\cdot))(x)ds, \quad t \in [0, +\infty[., \ x \in \mathbb{R}^N, \quad (4.10)
$$
where \( g_{j,\delta} \) is obtained from \( g_{j,\delta,\varepsilon} \) by replacing \( u_\varepsilon \) by \( u \) and letting \( \varepsilon = 0 \) in (4.8). By the results in the proof of Theorem 4.1(i), it follows immediately that the continuous function \( g_{j,\delta,\varepsilon} \) converges to the function \( g_{j,\delta} \) uniformly in \([0, +\infty) \times \mathbb{R}^N\), as \( \varepsilon \to 0^+ \). This implies that, for any \( r, s > 0 \), \( T_\varepsilon(r)g_{j,\delta,\varepsilon}(s, \cdot) \) converges to \( T(r)g_{j,\delta}(s, \cdot) \) locally uniformly in \( \mathbb{R}^N \), as \( \varepsilon \to 0^+ \). Indeed, for any compact set \( K \subset \mathbb{R}^N \), we have

\[
\|T_\varepsilon(r)g_{j,\delta,\varepsilon}(s, \cdot) - T(r)g_{j,\delta}(s, \cdot)\|_{C(K)} \\
\leq \|T_\varepsilon(r)(g_{j,\delta,\varepsilon}(s, \cdot) - g_{j,\delta}(s, \cdot))\|_{C(K)} + \|T_\varepsilon(r)(g_{j,\delta}(s, \cdot) - T(r)g_{j,\delta}(s, \cdot))\|_{C(K)} \\
\leq \|g_{j,\delta,\varepsilon}(s, \cdot) - g_{j,\delta}(s, \cdot)\|_{\infty} + \|T_\varepsilon(r)g_{j,\delta}(s, \cdot) - T(r)g_{j,\delta}(s, \cdot)\|_{C(K)}.
\]

From the proof of Theorem 4.1(i) we see that the last term in the previous chain of inequalities vanishes as \( \varepsilon \to 0^+ \). Moreover, since the semigroups \( \{T_\varepsilon(t)\} \) are contractive, the function \( (r, s) \to T_\varepsilon(r)g_{j,\delta,\varepsilon}(s, \cdot) \) is bounded in \([0, +\infty) \times [0, +\infty) \times \mathbb{R}^N \), uniformly with respect to \( \varepsilon \in [0,1] \). Therefore, the dominated convergence theorem yields (4.10).

We can now prove formula (4.6). Since, by Theorem 4.1(iv), the function \( u \) is bounded in \([0, T_0]\) with values in \( C_0^{\kappa - 1}(\mathbb{R}^N) \) for any \( T_0 > 0 \) (use \( f \in C_0^{\kappa - 1}(\mathbb{R}^N) \)), it is immediate to see that \( g_{j,\delta}(s, \cdot) \) converges to \( g(s, \cdot) \) uniformly in \( \mathbb{R}^N \) for any \( s > 0 \). Formula (4.6) now follows from (4.10) via the dominated convergence theorem.

**Step 2.** Here, and in the forthcoming Steps 3 and 4, we assume that \( f \in C_0^\kappa(\mathbb{R}^N) \). Let us fix a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N \) with \( \|\alpha\| = \kappa \) and \( \|\alpha_1, \ldots, \alpha_N\| \geq 1 \). We denote by \( j \) the largest integer such that \( \alpha_j \neq 0 \) and set \( \beta := \alpha_1 - \varepsilon_j^{(N)} \). Set \( \beta' := |\beta| - \varepsilon_j^{(r + 1)} = (\beta_0 - 1, \beta_1, \ldots, \beta_r) \), and denote by \( j \) the smallest integer with \( \beta_j' > 0 \). To prove that the derivative \( D^\alpha u \) exists in the classical sense it suffices to show that we can differentiate, with respect to the multi-index \( \beta \), the function in (4.6). For this purpose, we observe that, from (4.2) with \( h = \kappa - 3 \) and \( \kappa = 2 \), we deduce that

\[
\|D^\beta T(t)\psi\|_{C_0^{\kappa - 3 + \theta}(\mathbb{R}^N)} \leq C t^{\frac{-1}{2} - (1-\theta)\frac{2\alpha_j + 1}{\kappa + 1}} e^{\omega t^2} \|\psi\|_{C_0^{\kappa - 3 + \theta}(\mathbb{R}^N)},
\]

holds for any \( t \in [0, +\infty[ \), \( \psi \in C_0^{\kappa - 3 + \theta}(\mathbb{R}^N) \) and \( \theta = 0, 1 \). By interpolation, we can extend the previous estimate to any \( \theta \in [0, 1] \). Estimate (4.3) implies that, for any multi-index \( \gamma \) with length \( \kappa - 1 \) and any \( t > 0 \), the function \( D^\alpha u(t, \cdot) \) is Lipschitz continuous in \( \mathbb{R}^N \) with Lipschitz semi-norm that can be bounded by \( C e^{\omega t^2} \) for any \( \omega > 0 \) and some \( C = C(\omega) \), where the constants are uniform in \( \varepsilon \). Since \( u_\varepsilon \) converges to \( u \) in \( C_0^{\kappa - 1}(\mathbb{R}^N) \), the function \( D^\beta u(t, \cdot) \) is Lipschitz continuous in \( \mathbb{R}^N \) and its norm can be bounded by \( C e^{\omega t^2} \). As a byproduct, we infer that, for any \( \theta \in [0, 1] \) and any \( T_0 > 0 \), the function \( \|g_j(s, \cdot)\|_{C_0^{\kappa - 3 + \theta}(\mathbb{R}^N)} \) is bounded in \([0, T_0]\). Moreover, for \( 0 < s < t \leq T_0 \) we have

\[
\|D^\beta T(t - s)g_j(s, \cdot)\|_{C_0^{\kappa - 3 + \theta}(\mathbb{R}^N)} \leq C (t - s)^{-\frac{1}{2} - (1-\theta)\frac{2\alpha_j + 1}{\kappa + 1}} e^{\omega T_0} \sup_{s \in [0, T_0]} \|g_j(s, \cdot)\|_{C_0^{\kappa - 3 + \theta}(\mathbb{R}^N)},
\]

(4.11)

Consequently, if we take \( \theta > 2t/(2t + 1) \), we get an integrable function on the right hand side of (4.11), and hence we can differentiate under the integral sign in (4.6). This proves that the derivative \( D^\alpha u \) exists in the classical sense. Moreover, it satisfies (4.2). Indeed, the sup-norm of \( D^\alpha u \) can be controlled from above by the Lipschitz seminorm of \( D^\beta u \) which, as we have shown, can be estimated from above by \( C e^{\omega t^2} \) for any \( t > 0 \), any \( \omega > 0 \) and some \( C = C(\omega) \).

**Step 3.** We now assume that \( \|\alpha\| = \kappa \) and \( \alpha_j = 0 \) for all \( i = 1, \ldots, p_0 \), whereas \( \alpha_j \neq 0 \) for some \( j \in \{p_0 + 1, \ldots, p_0 + p_1\} \); we again set \( \beta := \alpha - \varepsilon_j^{(N)} \). We are going to show that we can differentiate the formula (4.6) with respect to the multi-index \( \beta \). For this purpose, let again \( \alpha \) be the largest integer with \( \beta_j \neq 0 \), and note that it suffices to prove that, for any \( T_0 > 0 \), the function \( g_j \) is bounded in \([0, T_0]\) with values in \( C_0^{\kappa - 2 + \theta}(\mathbb{R}^N) \), for some \( \theta \in \frac{2r}{2r + 1}, 1 \). Indeed, once this property is proved, estimate (4.2) gives

\[
\|D^\beta T(t - s)g_j(s, \cdot)\|_{C_0^{\kappa - 3 + \theta}(\mathbb{R}^N)} \leq C (t - s)^{-(1-\theta)\frac{2\alpha_j + 1}{\kappa + 1}} e^{\omega T_0} \sup_{s \in [0, T_0]} \|g_j(s, \cdot)\|_{C_0^{\kappa - 3 + \theta}(\mathbb{R}^N)},
\]

for all \( 0 < s < t \leq T_0 \), for arbitrary \( T_0 > 0 \) and some \( C = C(\omega) \), and we can complete the proof applying the same arguments as in the previous step.

Due to the structure of \( g_j \), in order to prove that \( g_j \) is bounded in \([0, T_0]\) with values in \( C_0^{\kappa - 2 + \theta}(\mathbb{R}^N) \), it suffices to show that for any pair of indexes \( l \leq p_0 \) and \( l' \leq p_0 + p_1 \), with \( l \leq l' \), the function \( D_{l' - l}^\alpha u(t, \cdot) \) belongs to \( C_0^{\kappa - 2 + \theta}(\mathbb{R}^N) \) and \( \sup_{t \in [0, M]} \|D_{l' - l}^\alpha u(t, \cdot)\|_{C_0^{\kappa - 2 + \theta}(\mathbb{R}^N)} < +\infty \).
for any \( M > 0 \). Actually, only the first and the fifth, second-order terms have to be taken care of in (4.7). Indeed, applying \( D^\gamma \) with \( \|\gamma\| = \kappa - 2 \) to any of the other terms \( \tilde{g} \) from (4.7) (in which there are only first order derivatives of \( u \)), we get that \( D^\gamma \tilde{g}(t, \cdot) \) is Lipschitz continuous, uniformly in \([0, t_0] \) and \( \sup_{t \in [0, t_0]} \|g(t, \cdot)\|_{C^{\kappa - 2 + \eta}(\mathbb{R}^N)} < +\infty \) (these follow by approximating \( u \) by \( u_\varepsilon \) as we have done several times above).

We now prove the assertion about \( D^\ell u \). So let now \( \beta' \in \mathbb{N}_0^N \) with \( \|\beta'\| = \kappa - 2 \). Denote furthermore by \( i \) the largest integer such that \( \beta_i' > 0 \), and define \( \beta = \beta' + e_i^{(N)} + e_i^{(N)} - e_i^{(N)} \). From (4.2) and from the already proved assertion in Step 2 we obtain, by using interpolation as well, that

\[
\|D^\beta T(t)\psi\|_{C^\rho_{\mathbb{C}}(\mathbb{R}^N)} \leq Ct^{-\frac{\alpha - \beta}{2} - (1 - \alpha)\frac{\rho + 1}{2 \rho + 1}} e^{\omega t}\|\psi\|_{C^\rho_{\mathbb{C}}(\mathbb{R}^N)}, \quad t \in [0, +\infty],
\]

for any \( \theta, \rho \in [0, 1] \). From (4.12) it now follows that

\[
\|D^\beta T(t - s)g_t(s, \cdot)\|_{C^\rho_{\mathbb{C}}(\mathbb{R}^N)} \leq C(t - s)^{-\frac{\alpha - \beta}{2}} \|g_t(s, \cdot)\|_{C^\rho_{\mathbb{C}}(\mathbb{R}^N)} \leq C(t - s)^{-\frac{\alpha - \beta}{2}} \quad (4.13)
\]

holds for any \( 0 < s < t \leq T_0 \). Hence, if we fix \( \gamma \in [0, 1] \) and take \( \rho = \theta_1 = \frac{3}{2} \), we see that the function in the right-hand side of (4.13) is integrable in \([0, T_0] \) for any \( T_0 > 0 \). Thus we can differentiate under the integral in (4.6), and conclude that the function \( D^\beta u \) is bounded in \([R - t, R] \) with values in \( C^{\kappa - 2 + \beta_1}(B(\mathbb{R})) \). Due to the arbitrariness of \( R \), it follows that \( D^\beta u \) is bounded in \( H \) with values in \( C^{\kappa - 2 + \beta_1}(K) \) for any compact set \( H \times K \subset [0, +\infty] \times \mathbb{R}^N \).

As a second step, using (4.12), we deduce that \( D^\beta u \) is bounded in \( H \) with values in \( C^{\kappa + \beta_2}(K) \) for any \( H \) and \( K \) as above, where \( \beta_2 = \gamma \frac{1 + \theta}{3 - 2\theta} \). Iterating, this argument, we see that \( D^\beta u \) is bounded in \( H \) with values in \( C^{\kappa + \beta_t}(K) \), where the sequence \( \{\beta_t\} \) is defined by recurrence as follows:

\[
\begin{align*}
\beta_{k+1} &= \gamma \frac{1 + \beta_k}{3 - 2\beta_k}, \quad k \leq k_0, \\
\beta_0 &= 0,
\end{align*}
\]

where either \( k_0 = +\infty \) or \( k_0 \) is the largest integer such that \( \beta_t < 3/2 \).

It easy to see that \( \beta_k < \beta_{k+1} \) holds for any \( k \leq k_0 \) and choice \( \gamma \in [0, 1] \). For the choice \( \gamma = \frac{3}{2} \), the equation \( \ell = \gamma \frac{1 + \beta}{3 - 2\beta} \) has no real solutions. This fact combined with the monotonicity property implies that there exists \( k_1 \) such that \( \beta_{k_1} > 1 \). It follows that \( D^\beta u \) is bounded in \( H \) with values in \( C^{\kappa - 2 + \beta_1}(K) \) for any \( \beta \in [0, 1] \) and, consequently, \( g_t \) is locally bounded in \([0, +\infty]\) with values in \( C^{\kappa - 2 + \beta}(\mathbb{R}^N) \) for any \( \beta \in [0, 1] \). The proof of Step 3 is complete.

Step 4. We now show that \( \|D^\alpha T(t)f\|_{C^\rho_{\mathbb{C}}(\mathbb{R}^N)} \leq Ct^{-\alpha}(\|f\|_{C^\rho_{\mathbb{C}}(\mathbb{R}^N)}) \) for any \( t \in [0, +\infty[ \), any \( f \in C^\rho_{\mathbb{C}}(\mathbb{R}^N) \), any \( h \in \mathbb{N} \) with \( h < \kappa \), and any \( \alpha \in \mathbb{N}_0^N \) with length \( \kappa \) and such that \( \alpha_j \neq 0 \) for some \( j \leq p_0 + p_1 \). For this purpose, fix \( j \leq p_0 + p_1 \) such that \( \alpha_j \neq 0 \). Further, we let \( \beta = \alpha - e_j^{(N)} \). From (4.2) it follows that for any \( x \in \mathbb{R}^{N-1} \) the Lipschitz seminorm of the function \( \psi = (D^\beta T(t)f)(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n) \) does not exceed \( C e^{\omega t} t^{-\alpha(\|f\|_{C^\rho_{\mathbb{C}}(\mathbb{R}^N)})} \), with \( C \) depending only on \( \omega \). Since the Lipschitz seminorm of the function \( \psi \) equals the sup-norm of the function \( (D^\alpha T(t)f)(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n) \) (which is already known to exist by Steps 2 and 3), the desired estimate follows.

Step 5. We now prove (4.5) for a general \( f \in C_{b}^\rho(\mathbb{R}^N) \) (\( h < \kappa \)) and any multi-index \( \alpha \in \mathbb{N}_0^N \) such that \( \|\alpha\| = \kappa \) and \( \alpha_j \neq 0 \) for some \( j \leq p_0 + p_1 \). Let us notice that we can limit ourselves to proving that the derivative \( D^\alpha T(t)f \) exists in the classical sense for any \( t > 0 \) and any \( f \in C_0^\rho(\mathbb{R}^N) \). Indeed, once this property is checked, estimate (4.5) can be proved arguing as in Step 4.

We begin by considering the case when \( f \in BUC(\mathbb{R}^N) \), and we fix a sequence \( \{f_n\} \in C_b^\rho(\mathbb{R}^N) \) converging to \( f \) uniformly in \( \mathbb{R}^N \). We can write

\[
\|D^\alpha T(t)f_n - D^\alpha T(t)f_m\|_{\infty} \leq C e^{\omega t} t^{-\alpha(\|\alpha\|)} \|f_n - f_m\|_{\infty},
\]

for any \( t > 0 \) and any \( n, m \in \mathbb{N} \). If follows that \( \{D^\alpha T(t)f_n\} \) is a Cauchy sequence in \( C_b^\rho(\mathbb{R}^N) \) and, consequently, \( D^\alpha T(t)f \in C_b^\rho(\mathbb{R}^N) \).

We now assume that \( f \in C_b^\rho(\mathbb{R}^N) \). By splitting \( T(t)f = T(t/2)T(t/2)f \) and noting that \( T(t/2)f \in C_b^1(\mathbb{R}^N) \subset BUC(\mathbb{R}^N) \), from the preceding we deduce that \( D^\alpha T(t)f = D^\alpha T(t/2)(T(t/2)f) \) exists in the classical sense.

Step 6. To complete the proof, we have to show that for any multi-index \( \alpha \) with length \( \kappa \) such that \( \alpha_j \neq 0 \) for some \( j \leq p_0 + p_1 \), the function \( D^\alpha T(t)f \) is continuous in \([0, +\infty[ \times \mathbb{R}^N \). For this we will need the following lemma.
purpose, let $i$ be the largest integer such that $\alpha_i > 0$. Let us fix $y \in \mathbb{R}^{N-1}$, and introduce the function $\psi = D^\beta u(y_1 + e_1 \cdot y, \ldots, y_N)$ where, again, $\beta = \alpha - e_1^{[N]}$, and still $\beta_{ji} > 0$ for some $j' \leq p_0 + p_1$. From the results in Steps 2 to 5 we know that $\psi$ is bounded in $[a, b]$ with values in $C^{1+\theta}(B(R))$ for some $\theta \in [0, 1]$ and any $a, b, R > 0$, with $a < b$. Applying [15, Propositions 1.1.2(iii) and 1.1.4(iii)] to the function $\psi(t, \cdot) - \psi(s, \cdot)$ $(s, t \in [a, b])$, we immediately see that

$$
\|\psi(t, \cdot) - \psi(s, \cdot)\|_{C^1(B(R))} \leq C\|\psi(t, \cdot) - \psi(s, \cdot)\|_{C(B(R))} \|T(t) - T(s)\|_{C^1(B(R))},
$$

for some constant $C$, independent of $y$. Since $u \in C^{1,\kappa-1}([0, +\infty[ \times \mathbb{R}^N)$, we immediately deduce that the right-hand side of the previous chain of inequalities vanishes as $|t - s| \to 0^+$, implying that the function $D^\alpha u(y, x)$ is continuous in $[a, b]$ uniformly with respect to $x \in \mathbb{R}^N$. This is enough to conclude that $D^\alpha u$ is continuous in $[0, +\infty[ \times \mathbb{R}^N$.

**Remark 4.3.**

(i) We remark that the results proved in Theorem 4.1 are stronger than those in [24].

(ii) Some calculation yields that the bootstrap argument used in Step 3 of the proof of Theorem 4.2 cannot be applied to prove the existence of the derivative $D^\alpha T(t)f$ in the classical sense when $\|\alpha\| = \kappa$ and $\alpha_j = 0$ for all $j = 1, \ldots, p_0 + p_1$.

### 4.1. Properties of the semigroup.

In this section we first state some continuity property of the semigroup $\{T(t)\}$ that will play a fundamental role in order to prove the Schauder estimates of Section 5. Then, we characterize the domain of the weak generator of the semigroup. Since the proofs of the following proposition can be obtained arguing as in [14], we omit it.

**Proposition 4.4.** The following assertions hold.

(i) Let $\{f_n\} \subset C_b(\mathbb{R}^N)$ be a bounded sequence of continuous functions converging to $f \in C_b(\mathbb{R}^N)$, pointwise in $\mathbb{R}^N$. Then, $T(\cdot)f_n$ converges to $T(\cdot)f$ pointwise in $[0, +\infty[ \times \mathbb{R}^N$.

(ii) If $f_n$ converges to $f$, locally uniformly in $\mathbb{R}^N$ and $\|f_n\|_{\infty}$ is bounded, then $T(\cdot)f_n$ converges to $T(\cdot)f$ locally uniformly in $[0, +\infty[ \times \mathbb{R}^N$ and in $C^{1,2}(F)$ for any compact set $F \subset [0, +\infty[ \times \mathbb{R}^N$.

(iii) There exists a family of probability Borel measures $\{p(t, x, dy) : t > 0, x \in \mathbb{R}^N\}$ such that, for any $f \in C_b(\mathbb{R}^N)$,

$$
(T(t)f)(x) = \int_{\mathbb{R}^N} f(y)p(t, x, dy), \quad t > 0, \ x \in \mathbb{R}^N.
$$

Consequently, $\{T(t)\}$ can be extended to the space $B_b(\mathbb{R}^N)$ of all bounded and Borel measurable functions $f : \mathbb{R}^N \to \mathbb{R}$ with a semigroup of positive contractions.

(iv) $\{T(t)\}$ is strong Feller, i.e., $T(t)f \in C_b(\mathbb{R}^N)$ (actually $T(t) \in C_b^{\kappa-1}(\mathbb{R}^N)$) for any $f \in B_b(\mathbb{R}^N)$.

Differently from what happens in the classical case when the coefficients are bounded, in general the semigroup associated with elliptic operators with unbounded coefficients is neither analytic in $C_b(\mathbb{R}^N)$, nor strongly continuous in $BUC(\mathbb{R}^N)$. Assertion (ii) above in Proposition 4.4, however, expresses the fact that the semigroup $\{T(t)\}$ is bi-continuous for the topology of locally uniform convergence $\tau_c$ (see [9, 10] or [6]), or which is essentially the same it is a locally-locally convergent semigroup with respect to the mixed topology. The mixed topology is finest locally convex topology agreeing with $\tau_c$ on $\| \cdot \|_{\infty}$-bounded sets. (See [27] or [26] for the definition of the mixed topology; [6] for the equivalence of these two families of semigroups; and [28, Section IX.2.1] for locally-locally convergent semigroups). This allows us to associate an infinitesimal generator $(A, D(A))$ to the semigroup (see [9, 10]):

$$
D(A) := \left\{ f \in C_b(\mathbb{R}^N) : \exists \tau_c \lim_{t \to 0^+} \frac{T(t)f - f}{t} \text{ and } \sup_{t \in [0, 1]} \|T(t)f - f\|_{\infty} \right\}
$$

$$
Af := \tau_c \lim_{t \to 0^+} \frac{T(t)f - f}{t}.
$$

With this definition the infinitesimal generator $(A, D(A))$ is a Hille-Yosida operator, and the resolvent of $A$ can be calculated

$$
R(\lambda, A)f = \int_0^{+\infty} e^{-\lambda t}T(t)f dt
$$
where the integral exists in the topology $\tau_c$ and for all positive $\lambda$. In general one could replace here the $\tau_c$-convergence by pointwise convergence resulting in the so-called "weak-generator", in our case, however, this would not result in any difference.

**Remark 4.5.** We note that assertion (iii) in Proposition 4.4 follows also directly from the first part of (ii). Actually, we even have the equivalence of these two statements, for details see, e.g., [6].

The next proposition characterizes the domain $D(A)$.

**Proposition 4.6.** The following characterization holds true:

$$D(A) = \left\{ f \in C_b(\mathbb{R}^N) : \exists \{f_n\} \subset C^2_b(\mathbb{R}^N), \exists g \in C_b(\mathbb{R}^N) : 
\begin{align*}
&f_n \to f, \mathcal{A} f_n \to g \text{ loc. uniformly in } \mathbb{R}^N \\
&\text{and } \sup_{n \in \mathbb{N}} (\|f_n\|_\infty + \|\mathcal{A} f_n\|_\infty) < +\infty \right\}.
\right. \quad (4.14)$$

Moreover, $Af = \mathcal{A} f$ for any $f \in D(A)$. Here and above, $\mathcal{A} f$ is meant in the sense of distributions.

This tells us essentially that $C^2_b(\mathbb{R}^N)$ is a core for the generator $A$ with respect to the mixed topology, or which is the same is a bi-core with respect to $\tau_c$ (see [10]). For the proof we use an invariance argument and need the following preparatory lemma.

**Lemma 4.7.** For the semigroup $\{T(t)\}$ we can state the following.

(i) For any $t > 0$, $T(t)$ commutes with $\mathcal{A}$ on $D_0(\mathcal{A}) := \{ f \in C^2_b(\mathbb{R}^N) : \mathcal{A} f \in C_b(\mathbb{R}^N) \}$;

(ii) if $\{f_n\} \subset C_b(\mathbb{R}^N)$ is a bounded sequence converging locally uniformly to some function $f \in C_b(\mathbb{R}^N)$, then, for any $\lambda > 0$, $R(\lambda, A)f_n$ converges to $R(\lambda, A)f$, locally uniformly in $\mathbb{R}^N$;

(iii) for any $\lambda > 0$, $R(\lambda, A)$ is a bounded operator mapping $C^b_0(\mathbb{R}^N)$ into itself for any $h \in \mathbb{N}$ such that $h < \kappa$.

**Proof.** (i). We begin the proof by recalling that, for any $t > 0$, $T_c(t)$ and $\mathcal{A}_c$ commute on $D(A_0)$ since they commute on

$$D_{\max}(\mathcal{A}_c) := \left\{ g \in C_b(\mathbb{R}^N) \cap \bigcap_{1 < p < +\infty} W^{2,p}_{\text{loc}}(\mathbb{R}^N) : \mathcal{A}_c g \in C_b(\mathbb{R}^N) \right\}$$

(see e.g., [3, Propositions 2.3.1, 2.3.6, 4.1.1 and Lemma 2.3.3]). Hence, we only have to show that, for any $f \in D(A_0)$, $\mathcal{A}_c T_c(t)f$ and $T_c(t)\mathcal{A}_c f$ converge to $\mathcal{A} T(t)f$ and $T(t)\mathcal{A} f$, respectively, as $\varepsilon \to 0^+$. The proof of Theorem 4.1 shows that $T_c(t)f$ converges to $T(t)f$ in $C^2(K)$, as $\varepsilon \to 0^+$, for any compact set $K \subset \mathbb{R}^N$. Therefore, $\mathcal{A}_c T_c(t)f$ converges to $\mathcal{A} T(t)f$ locally uniformly in $\mathbb{R}^N$.

On the other hand, recalling that $\{T_c(t)\}$ is a contraction semigroup, we can write

$$\|T_c(t)\mathcal{A}_c f - T(t)\mathcal{A} f\|_{C(K)} \leq \|T_c(t)(\mathcal{A}_c f - \mathcal{A} f)\|_{C(K)} + \|(T_c(t) - T(t))\mathcal{A} f\|_{C(K)}$$

$$\leq \|\mathcal{A}_c f - \mathcal{A} f\|_\infty + \|(T_c(t) - T(t))\mathcal{A} f\|_{C(K)}, \quad (4.15)$$

for any $t > 0$. Since $\mathcal{A}_c f$ converges uniformly in $\mathbb{R}^N$ to $\mathcal{A} f$ as $\varepsilon \to 0^+$, estimate (4.15) implies that $T_c(t)\mathcal{A}_c f$ tends to $T(t)\mathcal{A} f$, locally uniformly in $\mathbb{R}^N$.

(ii) This is a property shared by resolvents of generators of bi-continuous semigroups, see [9, 10]. For the sake of completeness we give the straightforward proof. Let $\{f_n\}$ and $f$ be as in the statement of the lemma. Observe that for any compact set $K \subset \mathbb{R}^N$,

$$\|R(\lambda, A)(f_n - f)\|_{C(K)} \leq \int_0^{+\infty} e^{-\lambda t} \|T(t)(f_n - f)\|_{C(K)} dt, \quad \lambda > 0.$$ 

Theorem 4.1(ii) and Proposition 4.4(ii) show that $\{\|T(\cdot)(f_n - f)\|_{C(K)}\}$ is a bounded sequence converging pointwise in $[0, +\infty]$ to 0 as $n \to +\infty$. The assertion now follows from the dominated convergence theorem.

(iii) It follows immediately from the estimate (4.2) with $h = k$. 

\[\blacksquare\]
Proof of Proposition 4.6. Taking Lemma 4.7 into account, it is easy to check that, for any \( f \in D_0(\mathcal{A}) \), it holds that

\[
(R(1, A) \mathcal{A} f)(x) = \int_0^{+\infty} e^{-t} (\mathcal{A} T(t)f)(x) dt = \int_0^{+\infty} e^{-t} \left( \frac{\partial}{\partial t} T(t)f \right)(x) dt = -f(x) + (R(1, A)f)(x),
\]

(4.16)

for any \( x \in \mathbb{R}^N \). Therefore, \( f \in D(A) \) and \( Af = \mathcal{A} f \), so that \( D_0(\mathcal{A}) \subset D(A) \) and \( A|_{D_0(\mathcal{A})} = \mathcal{A} \).

We could now conclude the proof by using density and the invariance under \( \{T(t)\} \) of \( D(A_0) \) and by referring, e.g., to [10, Proposition 1.21], or to [19, Proposition 2.12] (the analogous statement for strongly-continuous semigroups is in [5, Proposition II.1.7]). We nevertheless give a complete proof.

Let us fix \( f \in \hat{D} \) (the function space defined by the right-hand side of (4.14)) and let \( \{f_n\} \subset C^2_b(\mathbb{R}^N) \) be a bounded sequence with respect to the sup-norm which converges to \( f \) locally uniformly in \( \mathbb{R}^N \) and it is such that the sequence \( \{\mathcal{A} f_n\} \subset C_b(\mathbb{R}^N) \) is bounded and converges locally uniformly in \( \mathbb{R}^N \) to some function \( g \in C_b(\mathbb{R}^N) \). By the above results we know that

\[
f_n = R(1, A)(f_n - \mathcal{A} f_n), \quad n \in \mathbb{N}.
\]

(4.17)

Lemma 4.7(ii) allows us to take the limit as \( n \to +\infty \) in (4.17), getting \( f = R(\lambda, A)(\lambda f - g) \), so that \( f \in D(A) \) and \( Af = g \). We claim that \( Af = \mathcal{A} f \) (where \( \mathcal{A} f \) is meant in the distributional sense). For this purpose, it suffices to observe that, for any \( \varphi \in C^\infty(\mathbb{R}^N) \), we have

\[
\int_{\mathbb{R}^N} \varphi \mathcal{A} f_n dx = \int_{\mathbb{R}^N} f_n \mathcal{A}^* \varphi dx, \quad n \in \mathbb{N},
\]

(4.18)

where \( \mathcal{A}^* \) is the formal adjoint of the operator \( \mathcal{A} \). Letting \( n \to +\infty \) in (4.18), the claim follows.

We have so proved that \( \hat{D} \) is contained in \( D(A) \) and \( A = \mathcal{A} \) on \( \hat{D} \).

We now prove that \( D(A) \subset \hat{D} \). For this purpose, we fix \( f \in D(A) \), and \( h \in C_b(\mathbb{R}^N) \) be such that \( f = R(1, A)h \). By convolution, we can determine a sequence of smooth functions \( \{h_n\} \subset C^2_b(\mathbb{R}^N) \), bounded in \( C_b(\mathbb{R}^N) \) and converging locally uniformly to \( h \) as \( n \to +\infty \). By Lemma 4.7(ii) and (iii), the sequence \( \{R(1, A)h_n\} \) is contained in \( C^2_b(\mathbb{R}^N) \) and it converges to \( f \) locally uniformly in \( \mathbb{R}^N \). Further, arguing as in the proof of (4.16), one can easily show that \( \mathcal{A} R(1, A)h_n = -h_n + R(1, A)h_n \) for any \( n \in \mathbb{N} \). Hence, the sequence \( \{\mathcal{A} R(1, A)h_n\} \) is bounded in \( C_b(\mathbb{R}^N) \) and it converges to \( -h + f \in C_b(\mathbb{R}^N) \), locally uniformly in \( \mathbb{R}^N \). It follows that \( f \in D(A) \).

5. Schauder estimates

In this section we prove Schauder estimates for the (distributional) solutions to the elliptic equation

\[
\lambda u - \mathcal{A} u = f, \quad \lambda > 0,
\]

(5.1)

and to the non-homogeneous Cauchy problem

\[
\begin{aligned}
D_t u(t, x) &= \mathcal{A} u(t, x) + g(t, x), \quad t \in [0, T_0], \quad x \in \mathbb{R}^N, \\
u(0, x) &= f(x), \quad x \in \mathbb{R}^N.
\end{aligned}
\]

(5.2)

Throughout the section, we assume that Hypotheses 2.1 are satisfied with \( \kappa \) equal to the least common multiple of the odd numbers between 1 and \( 2r + 1 \).

The main results of this section are collected in the following two theorems.

Theorem 5.1. Let \( \theta \in ]0, 1[ \) and \( \lambda > 0 \). Then, for any \( f \in C^\theta_b(\mathbb{R}^N) \) there exists a function \( u \in \mathcal{C}^{2+\theta}(\mathbb{R}^N) \) solving equation (5.1) in the sense of distributions. Moreover, there exists a positive constant \( C \), independent of \( u \) and \( f \), such that

\[
\|u\|_{\mathcal{C}^{2+\theta}(\mathbb{R}^N)} \leq C \|f\|_{C^\theta_b(\mathbb{R}^N)}.
\]

(5.3)

Such a function \( u \) is the unique distributional solution to the equation (5.1) which is bounded and continuous in \( \mathbb{R}^N \) and it is twice continuously differentiable in \( \mathbb{R}^N \) with respect to the first \( p_0 \) variables, with bounded derivatives.
Theorem 5.2. Let $\theta \in [0, 1]$, $T_0 > 0$ and $f \in C_b^{2+\theta}(\mathbb{R}^N)$ and $g \in C_b([0,T_0] \times \mathbb{R}^N)$ be such that $g(t, \cdot) \in C_b^\theta(\mathbb{R}^N)$ for any $t \in [0, T_0]$, and

$$
\sup_{t \in [0, T_0]} \|g(t, \cdot)\|_{C_b^\theta(\mathbb{R}^N)} < +\infty.
$$

Then, there exists a function $u \in C_b([0,T_0] \times \mathbb{R}^N)$, solution to problem (5.2) in the sense of distributions, such that $u(t, \cdot) \in C_b^{2+\theta}(\mathbb{R}^N)$ for any $t \in [0, T_0]$ and

$$
\sup_{t \in [0, T_0]} \|u(t, \cdot)\|_{C_b^{2+\theta}(\mathbb{R}^N)} \leq C\left(\|f\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \sup_{t \in [0, T_0]} \|g(t, \cdot)\|_{C_b^\theta(\mathbb{R}^N)}\right),
$$

(5.4)

for some positive constant $C$, independent of $u, f, g$. Moreover, $u$ is the unique distributional solution to problem (5.2) which is bounded and continuous in $[0,T_0] \times \mathbb{R}^N$, and there, it is twice continuously differentiable with respect to the first $p_0$ spatial variables, with bounded derivatives.

To begin with, we prove an interpolation result. We need to introduce the auxiliary spaces $\mathcal{E}^\theta(\mathbb{R}^N) (\theta \in [0, +\infty))$ that are defined analogously to the spaces $\mathcal{E}^\theta(\mathbb{R}^N)$, with the Hölder spaces $C^{\theta/(2j+1)}(\mathbb{R}^p_j)$ being replaced by the Zygmund spaces $C^{\theta/(2j+1)}(\mathbb{R}^p_j)$ ($j = 0, \ldots, r$); see Definition 2.7. It is clear that $\mathcal{E}^\theta(\mathbb{R}^N) = \mathcal{E}^\theta(\mathbb{R}^N)$ if $\theta/(2j+1) \notin \mathbb{N}$ for any $j = 0, \ldots, r$.

**Proposition 5.3.** Fix $\theta \in [0, 1]$ and $\beta \in [0, \kappa]$ such that $\beta/(2j+1) \notin \mathbb{N}$ for any $j = 0, \ldots, r$. Then,

$$
(\mathcal{E}^\beta(\mathbb{R}^N), \mathcal{E}^\kappa(\mathbb{R}^N))_{\theta, \infty} = \mathcal{E}^{(1-\theta)\beta + \kappa\theta}(\mathbb{R}^N),
$$

(5.5)

with equivalence of the corresponding norms. Here, $\mathcal{E}^0(\mathbb{R}^N) = C_b(\mathbb{R}^N)$.

**Proof.** We first prove (5.5) in the case when $\beta = 0$. For this purpose, we recall that, in [17, Theorem 2.2], the author has proved that, for any $\gamma > 0$ and any $\theta \in [0, 1]$, the topological equality ($BUC(\mathbb{R}^N), \mathcal{E}^\gamma(\mathbb{R}^N))_{\theta, \infty} = \mathcal{E}^\gamma(\mathbb{R}^N)$ holds. Since $BUC(\mathbb{R}^N)$ belongs to both the classes $J_0(C_b(\mathbb{R}^N), \mathcal{E}^\gamma(\mathbb{R}^N))$ and $K_0(C_b(\mathbb{R}^N), \mathcal{E}^\gamma(\mathbb{R}^N))$, the Reiteration Theorem (see, e.g., [15, Theorem 1.2.15]) implies that

$$
(C_b(\mathbb{R}^N), \mathcal{E}^\gamma(\mathbb{R}^N))_{\theta, \infty} = \mathcal{E}^\gamma(\mathbb{R}^N),
$$

(5.6)

with equivalence of the corresponding norms.

Let us now fix $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ such that $\gamma > \kappa$. This choice of $\gamma$ implies that $\mathcal{E}^\gamma(\mathbb{R}^N) = \mathcal{E}^\gamma(\mathbb{R}^N)$. Therefore, the formula (5.6) with $\theta = \kappa/\gamma$ yields the equality ($C_b(\mathbb{R}^N), \mathcal{E}^\gamma(\mathbb{R}^N))_{\kappa/\gamma, \infty} = \mathcal{E}^\gamma(\mathbb{R}^N)$ with equivalence of the corresponding norms. Since $\mathcal{E}^\gamma(\mathbb{R}^N) \subset \mathcal{E}^\kappa(\mathbb{R}^N)$ with a continuous embedding, we easily see that $\mathcal{E}^\kappa(\mathbb{R}^N)$ is continuously embedded in ($C_b(\mathbb{R}^N), \mathcal{E}^\gamma(\mathbb{R}^N))_{\kappa/\gamma, \infty}$, or, equivalently, $\mathcal{E}^\gamma(\mathbb{R}^N)$ belongs to the class $K_{\kappa/\gamma}(C_b(\mathbb{R}^N), \mathcal{E}^\gamma(\mathbb{R}^N))$.

Let us prove that $\mathcal{E}^\kappa(\mathbb{R}^N)$ belongs also to the class $J_{\kappa/\gamma}(C_b(\mathbb{R}^N), \mathcal{E}^\gamma(\mathbb{R}^N))$. For this purpose, we recall that, there exists a positive constant $C$ such that

$$
\|\psi\|_{C_b^{\gamma/(2j+1)}(\mathbb{R}^p_j)} \leq C \|\psi\|_{C_b^{\gamma/(2j+1)}(\mathbb{R}^p_j)},
$$

(5.7)

for any $\psi \in C_b^{\gamma/(2j+1)}(\mathbb{R}^p_j)$ and any $j = 0, \ldots, r$ (see e.g., [15, Proposition 1.1.3(ii)]). Fix $f \in \mathcal{E}^\kappa(\mathbb{R}^N)$ and $1 \leq j \leq r$. By applying (5.7) to the function

$$
\psi = f(x_0, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_r)
$$

(where we have split $x \in \mathbb{R}^N$ as $x = (x_0, \ldots, x_r)$, with $x_i \in \mathbb{R}^p_i$ ($i = 0, \ldots, r$)) and then, by taking the supremum when we let the variable $x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r$ run over $\mathbb{R}^{N-p_j}$, we conclude that $\|f\|_{J_{\kappa/\gamma}(C_b(\mathbb{R}^N), \mathcal{E}^\gamma(\mathbb{R}^N))} \leq C \|f\|_{C_b^{\kappa/\gamma}(\mathbb{R}^N)}$ (see (2.5) for the definition of these seminorms), so that, summing over $j = 0, \ldots, r$, we get

$$
\|f\|_{\mathcal{E}^\kappa(\mathbb{R}^N)} \leq C \|f\|_{\mathcal{E}^\gamma(\mathbb{R}^N)},
$$

that is, $\mathcal{E}^\kappa(\mathbb{R}^N)$ belongs to the class $J_{\kappa/\gamma}(C_b(\mathbb{R}^N), \mathcal{E}^\gamma(\mathbb{R}^N))$. Since $C_b(\mathbb{R}^N)$ belongs to both classes $J_0(C_b(\mathbb{R}^N), \mathcal{E}^\gamma(\mathbb{R}^N))$ and $K_0(C_b(\mathbb{R}^N), \mathcal{E}^\gamma(\mathbb{R}^N))$, the Reiteration Theorem yields now the equality ($C_b(\mathbb{R}^N), \mathcal{E}^\kappa(\mathbb{R}^N))_{\kappa/\gamma, \infty} = (C_b(\mathbb{R}^N), \mathcal{E}^\kappa(\mathbb{R}^N))_{\kappa/\gamma, \infty}$ (with equivalence of the corresponding norms) that, combined with (5.6), yields (5.5) with $\beta = 0$. 

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The general case when \( \beta \in [0, \kappa] \) is such that \( \beta/(2j+1) \notin \mathbb{N} \) for any \( j = 0, \ldots, r \) now follows from the interpolation theorem. Indeed,
\[
(\mathcal{C}^\beta(\mathbb{R}^N), \mathcal{C}^\kappa(\mathbb{R}^N))_{\theta, \infty} = ((\mathcal{C}^0(\mathbb{R}^N), \mathcal{C}^\kappa(\mathbb{R}^N))_{\beta/\kappa, \infty}, \mathcal{C}^\kappa(\mathbb{R}^N))_{\theta, \infty} = (\mathcal{C}^0(\mathbb{R}^N), \mathcal{C}^\kappa(\mathbb{R}^N))_{(1-\theta)\beta/\kappa + \theta, \infty}.
\]

The following is a straightforward consequence of the estimates in Theorem 4.1

**Lemma 5.4.** For any \( \omega > 0 \), there exists a positive constant \( C = C(\omega) \) such that
\[
\|T(t)f\|_{\mathcal{C}^\omega(\mathbb{R}^N)} \leq Ct^{-\frac{\beta}{2\omega}}e^{-\omega t}\|f\|_{\mathcal{C}^0(\mathbb{R}^N)}, \quad \text{holds for } t \in [0, +\infty[.
\]  

Combining Theorem 4.1 and Lemma 5.4, we can now prove the following.

**Proposition 5.5.** For any \( T_0 > 0 \), \( 0 < \beta \leq \theta < 3 \) with \( \beta, \theta \notin \mathbb{N} \), there exists a positive constant \( C = C(T_0) \) such that, for any \( f \in \mathcal{C}^0(\mathbb{R}^N) \) the following inequality holds:
\[
\|T(t)f\|_{\mathcal{C}^\omega(\mathbb{R}^N)} \leq Ct^{-\frac{\beta}{\omega}}e^{-\omega t}\|f\|_{\mathcal{C}^0(\mathbb{R}^N)}, \quad t \in [0, +\infty[.
\]  

**Proof.** The proof follows from an interpolation argument. To simplify the notation, in the sequel we denote by \( \omega \) any positive number and by \( C \) a positive constant, possibly depending on \( \omega \) but being independent of \( t \) and \( f \), which may vary from line to line. By applying [15, Proposition 1.2.6] with \( X_1 = X_2 = Y_1 = C_b(\mathbb{R}^N), \quad Y_2 = \mathcal{C}^\kappa(\mathbb{R}^N) \), and by taking estimate (5.8), Theorem 4.1(ii) (which implies that \( \{T(t)\} \) is a contractive semigroup) and Proposition 5.3 into account, we obtain
\[
\|T(t)\|_{\mathcal{C}^{\omega_3}(\mathbb{R}^N), \mathcal{C}^{\omega_3}(\mathbb{R}^N))} \leq Ct^{-\frac{\beta}{\omega}}e^{-\omega t}, \quad t \in [0, +\infty[.
\]  

for any \( \theta_1 \in [0, \kappa[ \) such that \( \theta_1 \) is not rational. Of course, (5.10) holds also with \( \theta_1 = 0 \). Using again [15, Proposition 1.1.13] now with \( X_1 = C_b(\mathbb{R}^N), \quad X_2 = C^\kappa_b(\mathbb{R}^N), \quad Y_1 = Y_2 = \mathcal{C}^\kappa(\mathbb{R}^N) \), we get
\[
\|T(t)\|_{\mathcal{C}^{\omega_3}(\mathbb{R}^N), \mathcal{C}^{\omega_3}(\mathbb{R}^N))} \leq Ct^{-\frac{\beta}{\omega}}e^{-\omega t}, \quad t \in [0, +\infty[.
\]  

for any \( \theta_2 \in [0, \kappa[ \) such that \( \theta_2 \) is not integer and even for \( \theta_2 = 0, \kappa \). Finally, interpolating the estimates (5.10) and (5.11), we get
\[
\|T(t)\|_{\mathcal{C}^{\omega_3+\omega_4}(\mathbb{R}^N), \mathcal{C}^{\omega_3+\omega_4}(\mathbb{R}^N))} \leq Ct^{-\frac{(1-\theta_3)\theta_1 + (\kappa - \theta_2)\theta_3}{\omega}}e^{-\omega t}, \quad t \in [0, +\infty[,
\]
and (5.9) follows by taking \( \theta_1, \theta_2, \theta_3 \) such that \( \theta_2 \theta_3 = \beta \) and \( (1-\theta_3)\theta_1 + \kappa\theta_3 = \theta \). \( \blacksquare \)

The estimate (5.9) is the keystone in the proof of Theorems 5.1 and 5.2. The candidate to be the solutions to the equation (5.1) and the non-homogeneous Cauchy problem (5.2) are, respectively, the functions \( R(\lambda, A)f \) and \( u \) defined by
\[
u(t, x) = (T(t)f)(x) + \int_0^t (T(t-s)g(s, \cdot))(x)ds, \quad t \in [0, T_0[, \quad x \in \mathbb{R}^N.
\]  

The results in the following proposition are now a straightforward consequence of the estimate (5.9) and the interpolation arguments in [16, Section 3]. For this reason we skip the proof, referring the reader to the quoted paper.

**Proposition 5.6.** For fixed \( \theta \in [0, 1[ \) and \( T_0 > 0 \) the following are true.

(i) For any \( f \in C^0(\mathbb{R}^N) \), the function \( R(\lambda, A)f \) belongs to \( \mathcal{C}^{2+\theta}(\mathbb{R}^N) \) and the estimate (5.3) is satisfied by some positive constant \( C \) independent of \( f \).

(ii) For any \( f \in C^{2+\theta}(\mathbb{R}^N) \) and any function \( g \in C([0, T_0] x \mathbb{R}^N) \) such that \( g(t, \cdot) \in C^\theta_b(\mathbb{R}^N) \) for any \( t \in [0, T_0] \), with \( \sup_{t \in [0, T_0]} \|g(t, \cdot)\|_{C^\theta_b(\mathbb{R}^N)} < +\infty \), the function \( u \) in (5.12) is bounded and continuous in \([0, T_0] \times \mathbb{R}^N \). Moreover, \( u(t, \cdot) \in \mathcal{C}^{2+\theta}(\mathbb{R}^N) \) for any \( t \in [0, T_0] \) and estimate (5.4) is satisfied by some positive constant \( C \) independent of \( f \) and \( g \).

We can now complete the proofs of Theorems 5.1 and 5.2.
Proof of Theorem 5.1. By Proposition 4.6, we know that $A\psi = \mathcal{A}\psi$ for any $\psi$ in $D(A)$, where $\mathcal{A}\psi$ is meant in the sense of distributions. Hence, the resolvent equality immediately implies that the function $R(\lambda, A)f$ is a distributional solution of the equation (5.1). Moreover, by Proposition 5.6(i), $R(\lambda, A)f \in \mathcal{C}^{2+\theta}(\mathbb{R}^N)$ and satisfies estimate (5.3). As a byproduct, Proposition 3.1(i) implies that $R(\lambda, A)f$ is the unique distributional solution to the equation (5.1) satisfying the properties of Theorem 5.1. The proof is now complete.

Proof of Theorem 5.2. The uniqueness part of the statement follows immediately from the maximum principle in Proposition 3.1(ii). Moreover, by virtue of Proposition 5.6, we can limit ourselves to proving that the convolution term in (5.12), that we simply denote by $\mathcal{A}f$, is positive definite for any $f \equiv 0$. Actually for smooth $f$ with compact support this is an easy and classical argument using variation of constants. For the general case we pick a sequence $\{g_n\} \subset C_b^{1,2}(\mathbb{R}^N)$, bounded in the sup-norm, and converging locally uniformly in $[0, T_0] \times \mathbb{R}^N$ to $g$. Moreover, for any $n \in \mathbb{N}$, we denote by $v_n$ the convolution function defined as $v$, but with $g$ being replaced by $g_n$. As already indicated above, a straightforward computation, based on estimate (4.2) with $|\alpha| = 2$ and $h = 2$, shows that $v_n$ is a classical solution to problem (5.2) (with $f \equiv 0$ and $g$ being replaced by $g_n$). Moreover, its sup-norm may be bounded by a positive constant, independent of $n$ and, by Proposition 4.4, $v_n$ converges to $v$ pointwise in $[0, T_0] \times \mathbb{R}^N$.

Now, we observe that, for any smooth function $\varphi \in C_\infty(\mathbb{R}^N)$, it holds that

\[
\int_{[0,T_0] \times \mathbb{R}^N} g_n \varphi \, dt \, dx = \int_{[0,T_0] \times \mathbb{R}^N} (D_t v_n - \mathcal{A} v_n) \varphi \, dt \, dx = \int_{[0,T_0] \times \mathbb{R}^N} v_n (-D_t \varphi - \mathcal{A}^* \varphi) \, dt \, dx,
\]

where $\mathcal{A}^*$ is the formal adjoint to operator $\mathcal{A}$. Letting $n \to +\infty$, we deduce that $v$ is a distributional solution of (5.2) with $f \equiv 0$.

Appendix A. Technical results

Lemma A.1. Suppose that $\text{Ker}(Q(x))$ is independent of $x \in \mathbb{R}^N$. Then, the following conditions are equivalent:

(i) for any $x \in \mathbb{R}^N$, $\text{Ker}(Q(x))$ does not contain non-trivial subspaces which are $B^*$-invariant;

(ii) for any $x \in \mathbb{R}^N$, let $W(x) = \{\xi \in \mathbb{R}^N : Q(x)(B^*)^k \xi = 0, \ k \in \mathbb{N}\}$. Then, $W(x) = \{0\}$;

(iii) for any $x \in \mathbb{R}^N$ and any $r \in \mathbb{N}$, let $W_r(x) = \{\xi \in \mathbb{R}^N : Q(x)(B^*)^k \xi = 0, \ k = 0, \ldots, r-1\}$. Then, there exists $k_0 \leq N$, independent of $x$, such that $W_{k_0}(x) = \{0\}$;

(iv) the matrix $Q_t(x) = \int_0^t e^{sB^*}Q(x)e^{sB} \, ds$ is positive definite for any $t > 0$ and any $x \in \mathbb{R}^N$;

(v) the rank of the matrix $\mathcal{F}(x)^r = [Q(x),BQ(x),B^2Q(x),\ldots,B^rQ(x)]$ is $N$, for any $x \in \mathbb{R}^N$ and some $r < N$, independent of $x$.

Proof. We will show that (i) $\iff$ (ii), (ii) $\iff$ (iii), (ii) $\iff$ (iv), (iii) $\iff$ (v). We preliminarily note that both $W(x)$ and $W_r(x)$ are independent of $x$, so that, in the rest of the proof, we simply write $W$ and $W_r$ instead of $W(x)$ and $W_r(x)$.

(i) $\iff$ (ii): To prove this equivalence, it suffices to observe that, for any $x \in \mathbb{R}^N$, the set $W(x)$ is contained in $\text{Ker}(Q(x))$ and is its largest subspace, which is invariant for $B^*$.

(ii) $\iff$ (iii): Of course, we have only to prove that (ii) $\implies$ (iii). So, let us suppose that $W = \{0\}$. Since, $W_r \supset W_{r+1}$, then $\dim(W_r) \geq \dim(W_{r+1})$ for any $r \in \mathbb{N}$. Further, $\dim(W_1) = \dim(\text{Ker}(Q(0)))$ is positive and strictly less than $N$, since $Q(0)$ is a singular and not trivial matrix. It follows easily that there exists $k_0 \leq N$ such that $W_{k_0} = W_{k_0+1}$. We claim that $W_{k_0} = \{0\}$. Let $\xi \in W_{k_0}$. Then, $Q(0)(B^*)^j \xi = 0$ for any $j = 0, \ldots, k_0 + 1$. It follows that $B^*\xi \in W_{k_0}$ and, consequently, $W_{k_0}$ is a $B^*$-invariant subspace of $\text{Ker}(Q(0))$. Therefore, $W_{k_0} \subset W = \{0\}$ and we are done.

(ii) $\iff$ (iv): Let us fix $t > 0$, $x \in \mathbb{R}^N$ and let $\xi \in \mathbb{R}^N$ be such that $\langle Q_t(x)\xi, \xi \rangle = 0$. This implies that $\langle e^{sB}Q(x)e^{sB^*}\xi, \xi \rangle = 0$ for any $s \in [0, t]$. Hence, $Q(x)e^{sB^*}\xi = 0$ for any $s$ as above. Since

\[
Q(x)e^{sB^*}\xi = \sum_{k=0}^{\infty} \frac{s^k}{k!} Q(x)(B^*)^k \xi, \quad s \in [0, t],
\]

$Q(x)e^{sB^*}\xi = 0$ if and only if $Q(x)(B^*)^k \xi = 0$ for any $k \in \mathbb{N}_0$, that is if and only if $\xi \in W$. The equivalence between (ii) and (iv) follows immediately.
when: Let us fix $x \in \mathbb{R}^N$ and denote by $\mathcal{F}_j^{(r)}(x)$ ($j = 1, \ldots, N$) the rows of the matrix $\mathcal{F}^{(r)}(x)$. Further, fix $\xi_1, \ldots, \xi_N \in \mathbb{R}$ and set $\xi := (\xi_1, \ldots, \xi_N)$. As it is immediately checked, $\sum_{j=1}^N \xi_j \mathcal{F}_j^{(r)}(x) = 0$ if and only if $\xi \in W_{r-1}$. Hence, the rows of the matrix $\mathcal{F}^{(r)}$ are linearly independent if and only if $W_{r-1} = \{0\}$. From this, the equivalence between (iii) and (v) clearly follows.

The following lemma plays a crucial role in the proofs of Theorems 3.3.

**Lemma A.2.** Fix $l \geq 1$ and $m > c_{l-1,r}$. Then, for any function $w \in C^{l+1}_b(\mathbb{R}^N)$ it holds that

$$[D^l_{(m)}, \langle B, D \rangle]w = \sum_{k \in A_{m}^{(l)}} \mathcal{F}_k^{(l)} D^l_k w,$$

where the set $A_{m}^{(l)}$ is defined as follows: if $d_{j_1}, d_{j_2}, \ldots, d_{j_k}$ ($1 \leq j_1 < \ldots < j_k \leq r$) are all the non-zero entries of the vector $i^{(l)}_m = (0, d_{j_1}, \ldots, d_{j_k})$, then

$$A_{m}^{(l)} = \{ s : i^{(l)}_s = i^{(l)}_m - e^{(r+1)}_{j_1} + e^{(r+1)}_{j_1-1} - e^{(r+1)}_{j_i} + e^{(r+1)}_h \}
$$

for some $i = 2, \ldots, k$, and $h \leq j_i + 1$.

$\cup \{ s : i^{(l)}_s = i^{(l)}_m - e^{(r+1)}_{j_1} + e^{(r+1)}_h \text{ for some } h \leq j_1 \}

\cup \{ s : i^{(l)}_s = i^{(l)}_m - 2e^{(r+1)}_{j_1} + e^{(r+1)}_{j_{i-1}} + e^{(r+1)}_h \}
$$

for some $h \leq \min\{j_1 + 1, r\}$, if $\alpha_{j_i} > 1$, where $e^{(r+1)}_h$ denotes the $h^{th}$ vector of the Euclidean basis of $\mathbb{R}^{r+1}$. The entries of the matrices $\mathcal{F}_k^{(l)}$ ($k \in A_{m}^{(l)}$) linearly depend only on the entries of the matrix $B$. In particular, the matrix $\mathcal{F}_m^{(l)}$ has full rank.

**Proof.** By using the chain rule and by taking the structure of the matrix $B$ in (2.3) into account, it is easy to see that for any multi-index $\alpha \in \mathbb{N}_0^N$ we have

$$[D^\alpha, \langle B, D \rangle]w(x) = \sum_{i,j=1}^N \sum_{\beta \leq \alpha \atop |\beta| = 1} \binom{\alpha}{\beta} b_{ij} D^\beta x_j D^{\alpha-\beta} D_i w(x)
$$

$$= \sum_{i,j=1}^N \sum_{s=0}^r \sum_{\tau \in \mathbb{S}_s} \langle \alpha, \epsilon^{(N)}_\tau \rangle b_{ij} D_\tau x_j D^{\alpha-\epsilon^{(N)}_\tau+\epsilon_i^{(N)}} w(x)
$$

$$= \sum_{s=0}^r \sum_{\tau \in \mathbb{S}_s} \sum_{h=0}^{\min\{s+1,r\}} \sum_{i \in \mathbb{S}_h} \langle \alpha, \epsilon^{(N)}_\tau \rangle b_{ir} D^{\alpha-\epsilon^{(N)}_\tau+\epsilon_i^{(N)}} w(x),$$

for any $x \in \mathbb{R}^N$. By definition we have $i^{(l)}_m = (0, \ldots, 0, 1, d_{j_1} - 1, \ldots, d_{j_2}, \ldots, d_{j_k}, 0, \ldots, 0)$. In (A.3) consider all the possible multi-indices $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = i^{(l)}_m$. We see immediately that $[D^l_{(m)}, \langle B, D \rangle]w$ is given by the right-hand side of (A.1) for some matrices $\mathcal{F}_k^{(l)}$ ($k \in A_m$) and $\mathcal{F}_m^{(l)}$. It remains to show that the matrix $\mathcal{F}_m^{(l)}$ has full rank which equals the number of its columns. We split the rest of the proof in two steps.

**Step 1.** First, we show that we can make some reduction. More precisely, we show that, without loss of generality, we can limit ourselves to prove the assertion for a generic smooth function $w$ when:

(i) the only non-trivial blocks of the matrix $B$ in (2.3) are $B_1, \ldots, B_r$;

(ii) $i^{(l)}_m = (0, \ldots, 0, 1, d_{j_1} - 1, 0, \ldots, 0)$.

As a straightforward computation shows, the entries of the matrix $\mathcal{F}_m^{(l)}$ depend only on the matrices $B_1, \ldots, B_r$. Hence, we can assume (i). This implies that formula (A.3) can be rewritten as follows:

$$[D^\alpha, \langle B, D \rangle]w = \sum_{s=0}^{r-1} \sum_{\tau \in \mathbb{S}_s} \sum_{i \in \mathbb{S}_{s+1}} \langle \alpha, \epsilon^{(N)}_\tau \rangle b_{ir} D^{\alpha-\epsilon^{(N)}_\tau+\epsilon_i^{(N)}} w.$$
But if we have here \( |\alpha| = i_{\ell(m)} \), then the only possibilities to obtain a multi-index \( \alpha - e^{(N)}_\tau + e^{(N)}_i \) having the block form \( i_m \), are exactly the choices \( \tau \in J_{j-1} \) and \( i \in J_i \). So if we split

\[
i_{\ell(m)} = (0, \ldots, 0, 1, d_1 - 1, 0, \ldots, 0) + (0, \ldots, 0, 0, d_{j+1}, \ldots),
\]

and, accordingly \( \alpha = \beta + \gamma \) with \( |\beta| = (0, \ldots, 0, 1, d_1 - 1, 0, \ldots, 0) \) and \( |\gamma| = (0, \ldots, 0, 0, d_{j+1}, \ldots) \), we see

\[
[D^\alpha, (B, D)]w = [D^\beta D^\gamma, (B, D)]w = \sum_{\tau \in J_{j-1}} \sum_{i \in J_i} (\alpha, e^{(N)}_\tau) b_{i\tau} D^{\beta - e^{(N)}_\tau + e^{(N)}_i} D^\gamma w + \cdots,
\]

where we haven’t written out the terms, which do not contribute to \( J_m^{(l)} \). This means we that we can argue for the function \( D^\gamma w \) hence assuming (ii), and the general case will follow, as well.

**Step 2.** Let us take a derivative \( D^\alpha \) in the block \( D^{i_{\ell(m)}}_m \). Then, there exists an index \( \tau \in J_{j-1} \) such that \( (\alpha, e^{(N)}_\tau) = 1 \). Taking formula (A.4) into account, it is immediate to see that

\[
[D^\alpha, (B, D)]w = \sum_{i \in J_i} b_{i\tau} D^{\alpha - e^{(N)}_\tau + e^{(N)}_i} w + \cdots = [B^*_m D^l_\tau D^{e^{(N)}_\tau}_m w]_\tau + \cdots = \mathcal{H}_\alpha(w) + \cdots, \tag{A.5}
\]

where \( [\cdot]_\tau \) and “\( \cdots \)” denote, respectively, the \( \tau \)th component of the vector in brackets and terms which depend on (some of) the \( l \)th derivatives of \( w \) that are in a block different from \( D^l_\tau \).

Finally, we recall that \( D^{i_m}_m z \) denotes the vector of the first order derivatives \( D_h z \) of the function \( z \), with \( h \in J_{j_i} \). We are interested exclusively in \( \mathcal{H}_\alpha(w) \), because only this term will contribute to \( J_m^{(l)} \). In the following, we are going to reorder the vectors \( D^l_m \) and \( D^{i_{\ell(m)}}_m \) in such a way that the assertion about the rank of \( J_m^{(l)} \) will be clear.

Order the set \( \Gamma \subset \mathbb{N}_0^P \) of multi-indices of length \( d_j - 1 \) by \( \prec \) anti-lexicographically. That is we have

\[
(d_j - 1, 0, \ldots, 0) \prec (d_j - 2, 1, \ldots, 0) \prec \cdots \prec (0, \ldots, 0, d_j - 1).
\]

Next, we introduce the set

\[
\Lambda := \left\{ \lambda_{i, \gamma} := (0, \ldots, 0, e^{(P_{j-1})}_i, \underbrace{0, \ldots, 0}_{j_i \text{th block}}, i = 1, \ldots, P_{j-1}, \gamma \in \Gamma} \right\},
\]

which we order again anti-lexicographically, still denoting the ordering by \( \prec \). The set \( \Lambda \) describes the possible multi-indices having block from \( i_{\ell(m)}^{(l)} \). Reorder the vector \( D^{i_{\ell(m)}}_m w \) according to this ordering. Now pick \( \gamma \in \Gamma \). By considering multi-indices

\[
\gamma^{+i} := (0, \ldots, 0, \underbrace{\gamma + e^{(P_{j-1})}_i, 0, \ldots, 0}_{j_i \text{th block}}, i = 1, \ldots, P_{j-1}, \gamma \in \Gamma),
\]

we recover all the multi-indices of block form \( i^{(l)}_m \), but most of them even several times. For a \( \gamma \in \mathbb{N}_0^P \) let \( n(\gamma) \) denote the smallest non-negative integer \( n \) such that for all \( n + 1 < k \leq P_j \), we have \( \gamma_k = 0 \). We have, for instance, \( n((d_1 - 1, 0, \ldots, 0)) = 0 \) (only in this case is \( n(\gamma) = 0 \)), \( n((d_1 - 1, 1, 0, \ldots, 0)) = 1 \) and \( n((0, \ldots, 0, d_{j-1} - 1)) = P_j - 1 \). Consider now a multi-index \( \gamma \in \Gamma \) and all the multi-indices \( \gamma^{+i}, i = 1, \ldots, P_{j-1} \). Precisely for \( i = 1, \ldots, n(\gamma) \) we obtain multi-indices \( \beta \) which can be written both as \( \gamma^{+i} \) and \( \gamma^{+i'} \) for some \( \gamma' \prec \gamma \) and for some \( 1 \leq i' \leq P_j \). We set

\[
D^{\gamma^{+i}}_m := \left( D^{(n(\gamma)+1)}_m w, D^{(n(\gamma)+2)}_m w, \ldots, D^{(n(\gamma)+P_j)}_m w \right)^T.
\]

If \( \gamma_1 \prec \gamma_2 \prec \cdots \) is an enumeration of \( \Gamma \), we have now that \( (D^{\gamma}_m w, D^{\gamma_2}_m w, \ldots)^T \) is a reordering of \( D^{\gamma}_m w \). Further we set \( \mathcal{H}_\gamma(w) := \sum_{\gamma \in \Gamma} \mathcal{H}_\gamma(w) \).
The function $u$ argument as in the proof of part (i), one can show that a straightforward computation, based on an integration by parts. To prove it for any $\parallel \function$, suffices to write it with the following hold:

\[
\begin{pmatrix}
\mathcal{K}_{\gamma_1}(w) \\
\mathcal{K}_{\gamma_2}(w) \\
\vdots \\
\mathcal{K}_{\gamma_k}(w)
\end{pmatrix}
= 
\begin{pmatrix}
[B_{\gamma_1}^*]_{-n(\gamma_1)} & \cdots & \cdots & 0 & \cdots & 0 \\
* & [B_{\gamma_2}^*]_{-n(\gamma_2)} & 0 & \cdots & \cdots & 0 \\
* & \cdots & \cdots & \cdots & \cdots & \cdots \\
* & \cdots & \cdots & \cdots & \cdots & \cdots \\
* & \cdots & \cdots & \cdots & \cdots & \cdots \\
* & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\begin{pmatrix}
D_{\gamma_1}^m w \\
D_{\gamma_2}^m w \\
\vdots \\
D_{\gamma_k}^m w
\end{pmatrix},
\]

where $[B_{\gamma_i}^*]_{-s}$ denotes the matrix obtained from $B_{\gamma_i}^*$ by dropping out the first $s$ columns. The block matrix above is block-lower triangular with full rank, as all its blocks on the diagonal do so, and its rank is equal to the number of its columns. Thus $\mathcal{F}_m^{(i)}$ which is similar to the above block matrix, has the asserted properties.

The following two lemmas are used in the forthcoming proof of the maximum principle of Proposition 3.1.

**Lemma A.3.** For the first-order differential operator $\mathcal{B}$, formally defined by the equality $\mathcal{B}u(x) = \langle Bx, Du(x) \rangle$ for any $x \in \mathbb{R}^N$ and any $u \in C(\mathbb{R}^N)$, where $Du$ is meant in the sense of distributions, the following hold:

(i) For any $u \in \mathcal{B}UC(\mathbb{R}^N)$ such that $\mathcal{B}u \in C(\mathbb{R}^N)$, there exists a sequence $\{u_n\}$ of smooth functions, converging to $u$ uniformly in $\mathbb{R}^N$, such that $\mathcal{B}u_n \in C_b(\mathbb{R}^N)$ for any $n \in \mathbb{N}$ and it converges to $\mathcal{B}u$ locally uniformly in $\mathbb{R}^N$. In particular, if $u$ is compactly supported in $\mathbb{R}^N$, then $u_n$ is compactly supported in $\text{supp}(u) + B(1)$, for any $n \in \mathbb{N}$.

(ii) For any $u \in \mathcal{B}UC(0, +\infty \times \mathbb{R}^N)$ such that $D_t u - \mathcal{B}u \in C([0, +\infty \times \mathbb{R}^N)$, where both $D_t u$ and $Du$ are meant in the sense of distributions, there exists a sequence $\{u_n\}$ of smooth functions, converging to $u$ uniformly in $[0, +\infty \times \mathbb{R}^N$, such that $D_t u_n - \mathcal{B}u_n \in C_b([0, +\infty \times \mathbb{R}^N)$ for any $n \in \mathbb{N}$ and it converges to $D_t u - \mathcal{B}u$ locally uniformly in $[0, +\infty \times \mathbb{R}^N$. In particular, if $u$ is compactly supported in $[0, +\infty \times \mathbb{R}^N$, then $\text{supp}(u_n)$ is compact and contained in a compact set, which is independent of $n$.

**Proof.** (i) For any $n \in \mathbb{N}$, let $u_n = u \ast g_n$, where $g_n = n^N g(n \cdot)$, $g \in C^\infty_c(B(1))$ being a positive function with $\|g\|_{L^1(\mathbb{R}^N)}$ and “$\ast$” denotes the convolution operator. As it is immediately checked, the function $u_n$ is smooth and converges to $u$ uniformly in $\mathbb{R}^N$. Moreover, if $u$ is compactly supported in $\mathbb{R}^N$, then each function $u_n$ is compactly supported in $\text{supp}(u) + B(1)$.

To prove that $\mathcal{B}u_n$ converges to $\mathcal{B}u$ locally uniformly in $\mathbb{R}^N$, we observe that

\[
\mathcal{B}u_n = \mathcal{B}u + \gamma(B)u_n + u \ast \gamma_0,
\]

(A.6)

This is enough for our aims. Indeed, as it is immediately seen, $u \ast \gamma_0$ converges to $\gamma(B)u$ uniformly in $\mathbb{R}^N$. It follows that the right-hand side of (A.6) tends to $\mathcal{B}u$ as $n \to +\infty$, locally uniformly in $\mathbb{R}^N$.

Formula (A.6) is immediately checked in the particular case when $u \in C^1_b(\mathbb{R}^N)$, by means of a straightforward computation, based on an integration by parts. To prove it for any $u \in C_b(\mathbb{R}^N)$, it suffices to write it with $u_n$ and $u$ being replaced, respectively, by $u_n = v_n \ast g_n$ and $v_n$, where $\{v_n\}$ is a sequence of smooth functions converging to $u$ uniformly, and then take the pointwise limit as $m \to +\infty$. Indeed, it is immediate to check that $u_n^m$, $\mathcal{B}u_n^m$ and $v_n \ast \gamma_0$ converge, respectively, to $u_n$, $\mathcal{B}u_n$ and $u \ast \gamma_0$, locally uniformly in $\mathbb{R}^N$, as $m \to +\infty$. Moreover, since $\mathcal{B}u_n^m$ converges to $\mathcal{B}u$ in the sense of distributions, then $\mathcal{B}u_n^m \ast \gamma_0$ converges to $\mathcal{B}u \ast \gamma_0$ pointwise in $\mathbb{R}^N$ as $m \to +\infty$.

(ii) The proof is similar to the previous one. We extend $u$ to $]-1, 0[\times \mathbb{R}^N$, by setting $\tilde{u}(t, x) = u(-t, x)$ for such $(t, x)$'s. Next, we approximate $\tilde{u}$ by the sequence $\{u_n\}$ defined by taking the convolution of $\tilde{u}$ with a standard sequence $\{\varphi_n\}$ of mollifiers in $\mathbb{R}^{N+1}$. Using the same approximation argument as in the proof of part (i), one can show that $D_t \tilde{u}_n - \mathcal{B}\tilde{u}_n = (D_t u - \mathcal{B}u) \ast \gamma_0 - \gamma(B)u_n - u \ast \gamma_0$, in $[0, +\infty) \times \mathbb{R}^N$ for any positive number $a$ such that $na > 1$. Letting $n \to +\infty$, it is easy to check that $D_t \tilde{u}_n - \mathcal{B}\tilde{u}_n$ converges to $D_t u - \mathcal{B}u$ locally uniformly in $[0, +\infty] \times \mathbb{R}^N$. 


Lemma A.4. The following hold true:

(i) Let \( u \in C(\mathbb{R}^N) \) be such that \( \mathcal{B}u \in C(\mathbb{R}^N) \), where \( \mathcal{B}u \) is meant in the sense of distributions. If \( x_0 \in \mathbb{R}^N \) is a maximum (resp. minimum) point of \( u \), then \( (\mathcal{B}u)(x_0) = 0 \).

(ii) Let \( u \in C([0,T_0] \times \mathbb{R}^N) \) be such that \( D_tu - \mathcal{B}u \in C([0,T_0] \times \mathbb{R}^N) \), where \( D_tu \) and \( \mathcal{B}u \) are meant in the sense of distributions. If \( (t_0,x_0) \in [0,T_0] \times \mathbb{R}^N \) is a maximum (resp. minimum) point of \( u \), then \( (D_tu - \mathcal{B}u)(t_0,x_0) = 0 \).

Proof. (i) We adapt the proof of [15, Proposition 3.1.10] to our situation. Without loosing in generality we can assume that \( x_0 \) is a maximum point of \( u \) and \( u(x_0) > 0 \). Indeed, if \( x_0 \) is a minimum point, it suffices to replace the function \( u \) by \(-u\). Similarly, if \( x_0 \) is a maximum point and \( u(x_0) < 0 \), then the function \( u - 2u(x_0) \) has at \( x_0 \) a positive maximum.

Let \( R > 0 \) be such that \( u(x) \leq u(x_0) \) for any \( x \in x_0 + B(R) \). Further, let \( \vartheta \in C_0^\infty (x_0 + B(R)) \) satisfy \( \vartheta(x) < \vartheta(x_0) \) for any \( x \in x_0 + B(R) \) such that \( x \neq x_0 \). As it is immediately seen, the function \( v = u\vartheta \) is compactly supported in \( x_0 + B(R) \) and assumes its maximum value only at \( x_0 \). A straightforward computation shows that \( \mathcal{B}v \in BUC(\mathbb{R}^N) \). Let now \( v_n \) be a sequence of smooth functions compactly supported in \( x_0 + B(R+1) \), converging to \( v \) uniformly in \( \mathbb{R}^N \) and such that \( \mathcal{B}v_n \) converges to \( \mathcal{B}v \) locally uniformly in \( \mathbb{R}^N \), whose existence is guaranteed by Lemma A.3(i). Without loss of generality, we can also assume that \( \sup_{\mathbb{R}^N} v_n > 0 \) for any \( n \in \mathbb{N} \). Let \( \{x_n\} \subset x_0 + B(R+1) \) be a sequence such that \( \sup_{\mathbb{R}^N} v_n = v_n(x_n) \) for any \( n \in \mathbb{N} \). Up to a subsequence, we can assume that \( x_n \) converges in \( \mathbb{R}^N \) to a maximum point of \( v \). Hence, it converges to \( x_0 \). To complete the proof, it suffices to observe that \( (\mathcal{B}v_n)(x_n) = 0 \), for any \( n \in \mathbb{N} \), and \( (\mathcal{B}v)(x_n) = (\mathcal{B}u)(x_n) = 0 \).

(ii) The proof can be obtained arguing as above, taking Lemma A.3(ii) into account, and replacing the function \( \vartheta \), defined in (i), by a cut-off function \( \psi \in C_0^\infty ([0,T_0] \times \mathbb{R}^N) \), compactly supported in \([t_0 - R^{-1}, t_0 + R^{-1}] \times x_0 + B(R)\), for some \( R > 0 \) sufficiently large, and such that \( \psi(t,x) < \psi(t_0,x_0) = 1 \) for any \((t,x) \in [0,\infty) \times x_0 + B(R)\), with \((t,x) \neq (t_0,x_0)\).

We conclude this section with the proof of the maximum principle in Proposition 3.1 and with the proof of Lemma 3.4.

Proof of Proposition 3.1. (i) Let \( \varphi(x) = 1 + |x|^2 \) for any \( x \in \mathbb{R}^N \), with \( \lambda_0 \) sufficiently large such that \( \mathcal{A} \varphi - \lambda_0 \varphi < 0 \) in \( \mathbb{R}^N \). The existence of such a \( \lambda_0 \) is guaranteed by our assumptions on the growth of the coefficients of the operator \( \mathcal{A} \) at infinity (see Remark 2.2(i)).

Set \( u_n = u - n^{-1} \varphi \). Suppose that \( \lambda \geq \lambda_0 \) and \( \lambda u - \mathcal{A} u = f \) for some \( u \) and \( \mathcal{A} \) as in the statement of the proposition. Then, \( \lambda u_n - \mathcal{A} u_n < f \). Since \( u_n \) tends to \( -\infty \) as \( |x| \to +\infty \), then, for any \( n \in \mathbb{N} \), there exists \( x_n \in \mathbb{N} \) such that \( u_n(x_n) = \sup_{\mathbb{R}^N} u_n \). Taking Lemma A.4 into account, we can easily show that \( \mathcal{A} u_n(x_n) \leq 0 \). Hence, \( \lambda u_n(x_n) \leq f(x_n) \leq \|f\|_\infty \). Since \( \sup_{\mathbb{R}^N} u = \lim_{n \to +\infty} \sup_{\mathbb{R}^N} u_n \), it follows that \( \lambda u(x_n) \leq \|f\|_\infty \) for any \( x \in \mathbb{R}^N \). Applying the same argument to \( -u \) leads us to the assertion in the case when \( \lambda \geq \lambda_0 \).

Finally, if \( \lambda \in ]0,\lambda_0[ \), we can rewrite the equation \( \lambda u - \mathcal{A} u = f \) as \( \lambda_0 u - \mathcal{A} u = g \), where \( g = f + (\lambda_0 - \lambda)u \). Applying the estimate so far obtained gives \( \lambda_0 \|u\|_\infty \leq ||g||_\infty \leq (\lambda_0 - \lambda)\|u\|_\infty + ||f||_\infty \), which leads us to the assertion also in this situation.

(ii) The proof is similar to the previous one. Suppose that \( g \geq 0 \) in \([0,T_0]\times\mathbb{R}^N\) and introduce the function \( u_n : [0,T_0] \times \mathbb{R}^N \to \mathbb{R} \) defined by \( u_n(t,x) = e^{-\lambda_0 t} u(t,x) - \sup_{\mathbb{R}^N} f \) for any \( (t,x) \in [0,T_0] \times \mathbb{R}^N \), where \( \lambda_0 \) and \( \varphi \) are as in the proof of (i). The function \( u_n \) satisfies the equation \( D_tu_n - (\mathcal{A} \varphi - \lambda_0)u_n < 0 \) in \([0,T_0]\times\mathbb{R}^N\) and \( u_n(0,\cdot) \leq 0 \). Moreover, it attains its maximum value at some point \((t_0,x_0) \in [0,T_0] \times \mathbb{R}^N \). If \( t_0 = 0 \), then \( u_n(t,x) \leq 0 \) for any \((t,x) \in [0,T_0] \times \mathbb{R}^N \). On the other hand, if \( t_0 > 0 \) by elementary analysis and Lemma A.4(ii), \( (D_tu_n - \mathcal{A} u_n)(t_0,x_0) \geq 0 \). Therefore, \( u_n \leq 0 \) in this case, as well. Taking the limit as \( n \to +\infty \) gives \( e^{-\lambda_0 t} u(t,x) - \sup_{\mathbb{R}^N} f \leq 0 \) for any \((t,x) \in [0,T_0] \times \mathbb{R}^N \), and we are done.

To prove the assertion when \( g \geq 0 \), it suffices to consider this part to \(-u \). Finally, estimate (3.2) follows straightforwardly from these results.

Proof of Lemma 3.4. (i)-(iii) Trivial from the definition.

(iv) Consider all the possible multi-indices \( \beta \) which are of the form \( \beta = \alpha - e_0^{(r+1)} + e_j^{(r+1)} \) for some \( 0 \leq j, j' \leq r \) such that \( \alpha_j > 0 \) and \( j' \leq j + 1 \). We get the largest value of \( q_{\beta}(\beta) \), if actually \( j' = j + 1 \) holds. For this choice we have \( q_{h_\beta}(\beta) = q_{h_\alpha}(\alpha) + 1 \).
(v) Let $\alpha$ and $\beta = \alpha - e_{j_0}^{(r+1)} + e_{j_0-1}^{(r+1)}$ be as in the assertions. Since $\alpha_0 = 0$ and $\|\alpha\| > h$, after dropping out $h$ “derivatives” from $\alpha$, starting from the right, there will remain at least one positive entry which is not at the 0th position. This gives $q_h(\alpha) > 1$. Now, the equality $q_h(\beta) = q_h(\alpha) - 1$ is clear from the definition.

(vi) Observe that, by (v), $q_h(\tilde{\alpha}) = q_h(\alpha) - 1$ if $\|\alpha\| > h$. Now use (iii) to conclude $q_h(\beta) \leq q_h(\tilde{\alpha}) + 1$ and finish the proof.

(vii) Let $\alpha$ and $\tilde{\alpha}$ be as in the assertion. By definition we have $q_h(\beta) = q_h(\tilde{\alpha}) + 1$ if $\|\tilde{\alpha}\| \geq h$. If $\|\tilde{\alpha}\| = h - 1$, we have $q_h(\tilde{\alpha}) = 0$, $\|\beta\| = h + 1$ and $q_h(\beta) = 1/2$. For $\|\tilde{\alpha}\| \leq h - 2$ we have $q_h(\tilde{\alpha}) = q_h(\beta) = 0$ (we have used (i)). So in all cases we conclude $q_h(\tilde{\alpha}) \geq q_h(\beta) - 1$. The inequality $q_h(\tilde{\alpha}) \leq q_h(\alpha)$ is trivial, and hence the proof is complete.

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