THE EXISTENCE AND SINGULARITY STRUCTURE OF LOW REGULARITY SOLUTIONS OF HIGHER-ORDER DEGENERATE HYPERBOLIC EQUATIONS

ZHUOPING RUAN, INGO WITT, AND HUICHENG YIN

ABSTRACT. This paper is a continuation of our previous work [24], where we have established that, for the second-order degenerate hyperbolic equation $(\partial^2_t - t^{m_1} \Delta_x) u = f(t, x, u)$, locally bounded, piecewise smooth solutions $u(t, x)$ exist when the initial data $(u, \partial_t u)(0, x)$ belongs to suitable conormal classes. In the present paper, we shall study low regularity solutions of the higher-order degenerate hyperbolic equations $(\partial^2_t - t^{m_1} \Delta_x) u = f(t, x, u)$ and $(\partial^2_t - t^{m_2} \Delta_x) v = f(t, x, v)$ in $\mathbb{R}_+ \times \mathbb{R}^n$ with discontinuous initial data $\partial_t^i u(0, x) = \varphi_i(x)$ $(0 \leq i \leq 2)$ and $\partial_t^j v(0, x) = \psi_j(x)$ $(0 \leq j \leq 3)$, respectively; here $m, m_1, m_2 \in \mathbb{N}, m_1 \neq m_2, x \in \mathbb{R}^n$, $n \geq 2$, and $f$ is $C^\infty$ smooth in its arguments. When the $\varphi_i$ and $\psi_j$ are piecewise smooth with respect to the hyperplane $\{x_1 = 0\}$ and are homogeneous of degree zero close to $x = 0$, then there exist local solutions $u(t, x), v(t, x) \in L^\infty((0, T) \times \mathbb{R}^n)$ which are $C^\infty$ away from $\Gamma_0 \cup \Gamma_1^k \cup \Gamma_2^k$ in $[0, T] \times \mathbb{R}^n$, respectively; here $\Gamma_0 = \{(t, x) : t \geq 0, x_1 = 0\}$ and the $\Gamma_1^k = \{(t, x) : t \geq 0, x_1 = \frac{4^{k+2}}{(k+2)^2}\} (k = m, m_1, m_2)$ is a cuspidal conic surface ("forward light cone") and $\Gamma_0 = \{(t, x) : t \geq 0, x_1 = 0\}$ is a ray.

1. Introduction

In this paper, we shall study the local existence and singularity structure of low regularity solutions of the higher-order degenerate hyperbolic equations

\begin{equation}
\begin{aligned}
&\partial_t (\partial^2_t - t^{m_1} \Delta_x) u = f(t, x, u), \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \\
&\partial_t^j u(0, x) = \varphi_j(x), \quad 0 \leq j \leq 2, \\
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
&\partial_t (\partial^2_t - t^{m_2} \Delta_x) (\partial^2_t - t^{m_2} \Delta_x) u = f(t, x, u), \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \\
&\partial_t^k u(0, x) = \psi_k(x), \quad 0 \leq k \leq 3,
\end{aligned}
\end{equation}

where $m, m_1, m_2 \in \mathbb{N}, m_1 \neq m_2, x \in \mathbb{R}^n, n \geq 2$, $f$ is $C^\infty$ in its arguments and has compact support with respect to the variable $x = (x_1, \ldots, x_n)$. The discontinuous initial data $\varphi_j(x)$ $(0 \leq j \leq 2)$ and $\psi_k(x)$ $(0 \leq k \leq 3)$ satisfy one of the following assumptions:

\begin{enumerate}
\item[(A1)] $\varphi_j(x) = \begin{cases} \varphi_{j1}(x) & \text{for } x_1 > 0, \\
\varphi_{j2}(x) & \text{for } x_1 < 0, \end{cases}$ $\psi_k(x) = \begin{cases} \psi_{k1}(x) & \text{for } x_1 > 0, \\
\psi_{k2}(x) & \text{for } x_1 < 0, \end{cases}$

where $\varphi_{j1}, \varphi_{j2}, \psi_{k1}, \psi_{k2} \in C^0(\mathbb{R}^n)$ with $\varphi_{j1}(0) \neq \varphi_{j2}(0)$ and $\psi_{k1}(0) \neq \psi_{k2}(0)$;

\item[(A2)] $\varphi_j(x) = g_j \left( x, \frac{x_1}{|x|} \right)$, $\psi_k(x) = h_k \left( x, \frac{x_1}{|x|} \right)$,

where $g_j(x, y)$ and $h_k(x, y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ have compact support in $B(0, 1) \times B(0, 2)$.
\end{enumerate}
Under assumptions $\{A_1\}$ and $\{A_2\}$, we will prove the following main results:

**Theorem 1.1.** Let assumption $\{A_1\}$ hold. Then there is a constant $T > 0$ such that the following holds true:

(i) Eq. (1.1) admits a unique solution $u \in L^\infty((0, T) \times \mathbb{R}^n) \cap C([0, T], H^{1/2-}((\mathbb{R}^n))) \cap C((0, T], H^{n/2+}((\mathbb{R}^n)))$. Moreover, $u \in C^\infty \cap C((0, T] \times \mathbb{R}^n) \setminus (\Gamma_0 \cup \Gamma_1 \cup \Gamma_2)$, where $\Gamma_0 = \{(t, x): t \geq 0, x_1 = \pm \frac{2((m+2)/2)}{m+2}\}$ and $\Gamma_1 = \{(t, x): t \geq 0, x_1 = 0\}$. $\Gamma_2 = \{(t, x): t \geq 0, x_1 = \pm \frac{2f(m+2)/2}{m+2}\}$ for $i = 1, 2$.

(ii) Eq. (1.2) admits a unique solution $u \in L^\infty((0, T) \times \mathbb{R}^n) \cap C([0, T], H^{1/2-}((\mathbb{R}^n))) \cap C((0, T], H^{n/2+}((\mathbb{R}^n)))$. Moreover, $u \in C^\infty \cap C((0, T] \times \mathbb{R}^n) \setminus (\Gamma_{m_1} \cup \Gamma_{m_2})$, where $\Gamma_{m_1} = \{(t, x): t \geq 0, x_1 = \pm \frac{2f(m+2)/2}{m+2}\}$ for $i = 1, 2$.

**Theorem 1.2.** Let assumption $\{A_2\}$ hold. Further let $f$ satisfy

\[|\partial_{t,x} f(t, x, u)| \leq C_{T_{0}, n, l} (1 + |u|)^{K - l} \]

for $\alpha \in \mathbb{N}_0^{1+n}$, $l \in \mathbb{N}_0$, $0 \leq t \leq T_0$, where $K > 0$ is fixed. Then there is a constant $0 < T \leq T_0$ such that the following holds true:

(i) Eq. (1.1) admits a unique solution $u \in L^\infty_{loc}((0, T) \times \mathbb{R}^n) \cap C([0, T], H^{1/2-}((\mathbb{R}^n))) \cap C((0, T], H^{n/2+}((\mathbb{R}^n)))$. Moreover, $u \in C^\infty \cap C((0, T] \times \mathbb{R}^n) \setminus (\Gamma_m \cup l_0)$, where $\Gamma_m = \{(t, x): t \geq 0, |x|^2 = \frac{4(t+m)/2}{m+2}\}$ and $l_0 = \{(t, x): t \geq 0, |x| = 0\}$.

(ii) Eq. (1.2) admits a unique solution $u \in L^\infty_{loc}((0, T) \times \mathbb{R}^n) \cap C([0, T], H^{1/2-}((\mathbb{R}^n))) \cap C((0, T], H^{n/2+}((\mathbb{R}^n)))$. Moreover, $u \in C^\infty \cap C((0, T] \times \mathbb{R}^n) \setminus (\Gamma_{m_1} \cup \Gamma_{m_2})$, where $\Gamma_{m_1} = \{(t, x): t \geq 0, |x|^2 = \frac{4(t+m+2)/2}{m+2}\}$ for $i = 1, 2$.

**Remark 1.3.** Consult the following figures to see the singularity structure of the solutions as described by Theorem 1.1 and Theorem 1.2 respectively.

**Figure 1.** The singularity set $\Gamma^0 \cup \Gamma^1$ of the solution $u(t, x)$ of (1.1) under assumption $\{A_1\}$.

**Remark 1.4.** For $n = m = 1$, the operator $\partial_t^2 - t \partial_x^2$ is the classical Tricomi operator that arises, e.g., in continuous transonic gas dynamics of isentropic and irrotational flow. The principal symbol of the third-order operator $\partial_t (\partial_t^2 - t \partial_x^2)$ in (1.1) resembles the one of the 2-D steady compressible isentropic Euler system in continuous transonic gas dynamics. Indeed, when introducing the flux function $\psi(x)$ and the generalized potential $\varphi(x)$ as independent variables in place of the spatial variables $(x_1, x_2)$, one arrives at a system the linearization of
which has principal symbol \( \tau \left( \tau^2 - \psi \xi^2 \right) \) for \( \psi \geq 0 \). The latter has three simple real eigenvalues for \( \psi > 0 \), where all three of them merge into one at the sonic line \( \psi = 0 \). (See [17, Chapter 2] for details.)

**Remark 1.5.** For the multi-dimensional compressible Euler system and initial data which is \( H^s \) \((s > n/2 + 5)\) conormal with respect to the origin, J.-Y. Chemin [6] has shown that the classical solution is (weakly) singular only along the set \( \Gamma \cup l \) (see Figure 5), where \( \Gamma \) is the characteristic conic surface and \( l \) is the stream curve both emanating from the origin.

For the quasilinear equation
\[
\left( \partial_t^2 - \sum_{i=1}^n c_i^2(t, x, \nabla_{t,x} u) \partial_x^2 \right) \left( \partial_t^2 - \sum_{i=1}^n d_i^2(t, x, \nabla_{t,x} u) \partial_x^2 \right) \left( \nabla_{t,x}^\alpha u \right)_{|\alpha| \leq 3} = f(t, x),
\]
which is strictly hyperbolic with respect to time \( t \), and initial data \( \partial_t^j u|_{t=0} \in H^{s-j-\infty}(0) \) \((0 \leq j \leq 3)\) conormal with respect to the origin, where \( s > (n + 1)/2 + 9 \), it has been shown in [5] that the local classical solution \( u(t, x) \in H^s_{\text{loc}}(\mathbb{R}^{n+1}) \) is (weakly) singular only along the two characteristic conic surfaces \( \Gamma_1 \) and \( \Gamma_2 \) emanating the origin (see Figure 6).
Compared to solutions of higher regularity studied in [5, 6], Theorem 1.2 deals with unbounded and discontinuous solutions.

Remark 1.6. Utilizing the technique of edge Sobolev space as in [10, 11], and microlocal analysis tools, Theorems 1.1 and 1.2 can be extended to more general third-order and fourth-order semilinear degenerate hyperbolic equations of the form

$$\left( \partial_t + t^l \sum_{j=1}^n a_j(t, x) \partial_j \right) \left( \partial_t^2 + 2t^l \sum_{j=1}^n b_{lj}(t, x) \partial_l \partial_j - t^{2l} \sum_{i,j=1}^n b_{ij}(t, x) \partial_{ij} \right) u = f(t, x, u)$$

and

$$\left( \partial_t^2 + 2t^l \sum_{j=1}^n a_{lj}(t, x) \partial_l \partial_j - t^{2l} \sum_{i,j=1}^n a_{ij}(t, x) \partial_{ij} \right) \left( \partial_t^2 + 2t^l \sum_{j=1}^n c_{0j}(t, x) \partial_l \partial_j - t^{2l} \sum_{i,j=1}^n c_{ij}(t, x) \partial_{ij} \right) u$$
respectively; here \( t \in \mathbb{N} \). We shall study even more general nonlinear degenerate hyperbolic equations in a forthcoming paper.

**Remark 1.7.** For the semilinear \( N \times N \) strictly hyperbolic system \( \partial_t U + \sum_{j=1}^{n} A_j(t,x)\partial_j U = F(t,x,U) \) with piecewise smooth initial data or initial data which is \( H^s \) conormal with respect to some \( C^\infty \) hypersurface \( \Delta_0 \subset \mathbb{R}^n \) (where \( s > n/2 \)) (in particular, this includes discontinuous Riemann initial data), the local well-posedness of piecewise smooth solutions and solutions that are \( H^s \) conormal with respect to the \( N \) pairwise transverse characteristic surfaces \( \Sigma_j \) passing through \( \Delta_0 \), respectively, has been established in \([3, 4, 18, 19]\). In the present paper, in Theorem 1.1 we establish the corresponding result for higher-order semilinear degenerate hyperbolic equations.

**Remark 1.8.** For the second-order generalized Tricomi equation

\[
\begin{align*}
\partial_t^2 u - t^m \Delta_x u &= f(t,x,u), \quad (t,x) \in (0, +\infty) \times \mathbb{R}^n, \\
\partial_t^j u(0,x) &= \varphi_j(x), \quad j = 0, 1,
\end{align*}
\]

where \( m \in \mathbb{N}, n \geq 2, f(t,x,u) \) and \( \varphi_j(x) \) (\( j = 0, 1 \)) satisfy all the assumptions of Theorems 1.1 and 1.2 by the same method one obtains conclusions analogous to those of Theorems 1.1 and 1.2. Now we have weaken the regularity assumptions of \([2]\) on the initial data inasmuch as there \( u(0,x) = 0 \) holds.

**Remark 1.9.** If we are only concerned with the local existence of solutions of (1.1) and (1.2), then it is enough to assume the nonlinearity \( f \) be of class \( C^1 \). For instance, \( f = \pm |u|^p \) or \( f = \pm |u|^{p-1}u \) with \( p > 1 \) will do. By results of \([2]\), one has that in general weak solutions of (1.1) and (1.2) blow up in finite time.

**Remark 1.10.** Because of the low regularity of initial data near the origin when assumption (A) holds, it seems to be difficult to show \( u \in L^\infty((0,T) \times \mathbb{R}^n) \) in Theorem 1.2. In fact, even for the linear equation \( \partial_t^2 w - t^m \Delta_x w = 0 \) with initial data \( (w(0,x), \partial_t w(0,x)) = (\varphi_0(x), \varphi_1(x)) \), where \( \varphi_0(x) \) and \( \varphi_1(x) \) satisfy (A), the solution \( w(t,x) \) can be shown only to satisfy \( |w(t,x)| \leq C_T(1 + |\ln t|^2) \) for \( 0 < t \leq T \) (see Lemma 2.3(ii)). In this case, the polynomial bound on \( f(t,x,u) \) with respect to the variable \( u \) (a suitable exponential bounds would do as well) is necessary to guarantee that \( f(t,x,u(t,x)) \in L^1((0,T) \times \mathbb{R}^n) \) in Theorem 1.2 and then to obtain the local existence of solutions in \( L^\infty_{loc}((0,T) \times \mathbb{R}^n) \) by a fixed-point argument.

For the semilinear Tricomi equation \( \partial_t^2 u - t \Delta_x u = f(t,x,u) \) and initial data of regularity \( H^s \) (\( s > n/2 \)), M. Beals \([2]\) has proven the local existence of a classical solution \( u \in C([0,T], H^s(\mathbb{R}^n)) \cap C^1([0,T], H^{s-5/6}(\mathbb{R}^n)) \cap C^2([0,T], H^{s-11/6}(\mathbb{R}^n)) \) for some \( T > 0 \) under the assumption that the support of \( f(t,x,u) \) with respect to the variable \( t \) lies in \( \{t \geq 0\} \). Conormal regularity of the classical \( H^s \) solutions \( u(t,x) \) with

\[
\Gamma_1 \cup \Gamma_2
\]
respect to the characteristic cusp surfaces $x_1 = \pm 3t^{5/2}$ has also been established in [3]. For more general nonlinear degenerate hyperbolic equations with data of higher regularity, the authors of [3, 9] have studied the local existence and propagation of weak singularity of classical solutions. For the Cauchy problem for linear degenerate hyperbolic equations, there are rather complete results on the well-posedness and the regularity of solutions (see [7, 13–16, 20, 22, 23] and the references therein). In [21], we have established that bounded and piecewise smooth solutions $u(t, x)$ exist locally for the second-order semilinear degenerate hyperbolic equation $(\partial_t^2 - t^m \Delta_x) u = f(t, x, u)$, where $u(0, x)$ is continuous and piecewise smooth, while $\partial_t u(0, x)$ is piecewise smooth, but might be discontinuous. In the present paper, we will focus on solutions (even of lower regularity) of higher-order degenerate hyperbolic equation in the category of piecewise smooth and possibly unbounded functions.

We now comment on the proofs of Theorems 1.1 and 1.2. In order to prove the local existence of solutions of (1.1) and (1.2) with the low regularity as given, we first will establish $L^\infty$ (or $L^\infty_{\text{loc}}$) bounds on the solutions $v(t, x)$ of the linear problem $\partial_t^2 v - t^m \Delta_x v = F(t, x)$ with discontinuous initial data $(v(0), \partial_t v(0)) = (\varphi_0(x), \varphi_1(x))$ so that nonlinear superposition $v \mapsto f(t, x, v)$ is well-defined for $t > 0$. When doing this, we shall make full use of the special structure of the piecewise smooth and single-point singular initial data, respectively, as well as some tools from harmonic analysis such as the Hilbert transformation and Fourier analysis methods. This is necessary as the energy method and Sobolev embedding theorems cannot be applied directly to obtain $v(t, x) \in L^\infty_{\text{loc}}$ because of its low $H^s$ regularity (with $s < n/2$). (For instance, initial data is in $H^{1/2}_t (\mathbb{R}^n)$ in the case of assumption (A1) and in $H^{n/2}_t (\mathbb{R}^n)$ in case of assumption (A2), respectively.) Based on these $L^\infty$ (or $L^\infty_{\text{loc}}$) estimates and invoking the theory of confluent hypergeometric functions, we construct suitable nonlinear maps related to problems (1.1) and (1.2), respectively, and further show that these maps possess fixed points in the space $L^\infty((0, T) \times \mathbb{R}^n) \cap C([0, T], H^s(\mathbb{R}^n))$ for some $T > 0$ and a suitable regularity $s_0 > 0$. This then establishes the local solvability of (1.1) and (1.2). After this, we start to deal with the singularity structure of the solutions $u(t, x)$ of (1.1) and (1.2). Note that the initial data is conformal with respect to the hypersurface $\{x_1 = 0\}$ under assumptions (A1) and (A2) and conformal with respect to the origin $\{x = 0\}$ under assumption (A3), i.e., it holds $(x, \partial_j)_{k_1} \prod_{2 \leq i \leq n} \partial_t^{k_i} \varphi_i(x) \in H^{1/2}_t (\mathbb{R}^n)$ for all $k_i \in \mathbb{N}_0$ ($1 \leq i \leq n$) in the first case and

\[
\prod_{1 \leq i,j \leq n} (x, \partial_j)^{k_i} \varphi_i(x) \in H^{n/2}_t (\mathbb{R}^n)
\]

for all $k_{ij} \in \mathbb{N}_0$ in the second case. We then intend to use commutator arguments as in [2, 3] to prove conormality of the solutions $u(t, x)$ of (1.1) and (1.2). The fact that the hypersurfaces $\Gamma_l, \Gamma_l^\pm (l = m, m_1, m_2)$ form cusp singularities, however, makes it difficult to use directly smooth vector fields $Z_1, \ldots, Z_d$ tangent to $\Gamma_l$ and to $\Gamma_l^\pm$, respectively, to define the conormal spaces and to perform the analysis of the commutators $[\partial_t (\partial_t^2 - t^m \Delta), Z_1, \ldots, Z_d]$ and $[(\partial_t^2 - t^m \Delta), \partial_t^2 - t^m \Delta, Z_1, \ldots, Z_d]$, since this leads to a violation of the Levi condition on lower-order terms which results in a loss of regularity for $Z_1, \ldots, Z_d u$. (More detailed explanations can be found in Remark 3.2.) Motivated by [1, 2, 21], to overcome these difficulties we shall work with nonsmooth vector fields instead and gain extra regularity by some specific relations provided by the operator under study itself and some parts of these vector fields (see Proposition 3.7). To obtain the full conormal regularity of the solutions $u(t, x)$, this completes the proofs of Theorems 1.1 and 1.2. We point out that although some of the statements in this paper are analogous to those of [21], here due to the lower regularity of the initial data and the higher order of the degenerate hyperbolic equations under consideration, we have to perform a more thorough analysis including a more technically involved treatment of the linear problems. This is caused by the fact that some commutator relations turn out not to be “good” in the sense that some of the resulting coefficients are not admissible (see Definition 3.2 for the notion of an admissible function). (Compare the expressions for $[F_1, V^{(m)}]$ in Lemma 3.1.) Likewise, the operators $\partial_t^2 - t^m \Delta$ and $\partial_t^2 - t^{m_1} \Delta$ ($m_1 \neq m_2$) have different “radial” vector fields $2t \partial_t + (m_1 + 2)(x_1 \partial_1 + \cdots + x_n \partial_n)$ and $2t \partial_t + (m_2 + 2)(x_1 \partial_1 + \cdots + x_n \partial_n)$ which needs a special treatment in the proof of Theorem 1.2 in Section 4.

This paper is organized as follows: In Section 2 we provide preliminary results and prove $L^\infty$ (or $L^\infty_{\text{loc}}$) bounds on the solutions of the linear problems. In Section 3 the conormal spaces related to the equations under study are introduced and corresponding commutator relations which are crucial for the following are established. Based on the results of Section 2, local solvability of Eq. (1.1) and Eq. (1.2) is shown to hold in Section 4. In Sections 5 and 6 the proofs of Theorems 1.1 and 1.2 are completed utilizing the conormal spaces and commutator relations of Section 3.
LOW REGULARITY SOLUTIONS OF DEGENERATE HYPERBOLIC EQUATIONS

2. Preliminaries

In this section, we recall some results of [3] and establish the $L^\infty$ (or $L^\infty_{\text{loc}}$) property of low regularity solutions of second-order linear degenerate hyperbolic equations using rather delicate techniques.

Lemma 2.1 ([3, Proposition 3.3]). Let $\phi_1 \in H^{s}(\mathbb{R}^n)$ and $\phi_2 \in H^{s-\frac{m+2}{2(m+2)}}(\mathbb{R}^n)$, where $s \in \mathbb{R}$. Then the homogeneous equation

$$
\begin{aligned}
\partial_t^2 u - tm^2 \Delta_x u &= 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
u(0, x) &= \phi_1(x), \quad \partial_t u(0, x) = \phi_2(x),
\end{aligned}
$$

has a unique solution $u(t, x) \in C([0, T], H^{s}(\mathbb{R}^n)) \cap C([0, T], H^{s-\frac{m+2}{2(m+2)}}(\mathbb{R}^n)) \cap C^1([0, T], H^{s-\frac{m+4}{2(m+2)}}(\mathbb{R}^n))$.

Moreover, this solution satisfies, for $0 < t \leq T$,

$$
\|u(t, \cdot)\|_{H^{s}(\mathbb{R}^n)} + t^{m/4}\|u(t, \cdot)\|_{H^{s-\frac{m+4}{2(m+2)}}(\mathbb{R}^n)} + \|\partial_t u(t, \cdot)\|_{H^{s-\frac{m+4}{2(m+2)}}(\mathbb{R}^n)} \leq C\left(\|\phi_1\|_{H^{s}(\mathbb{R}^n)} + \|\phi_2\|_{H^{s-\frac{m+4}{2(m+2)}}(\mathbb{R}^n)}\right).
$$

Let

$$
\begin{aligned}
V_1(t, |\xi|) &= e^{-z/2|\xi|^2} \Phi\left(\frac{m}{2(m+2)} \frac{m}{m+2} \frac{m^2}{m+2} z\right), \\
V_2(t, |\xi|) &= te^{-z/2|\xi|^2} \Phi\left(\frac{m+4}{2(m+2)} \frac{m+4}{m+2} \frac{m^2}{m+2} z\right),
\end{aligned}
$$

where $z = \frac{4i}{m+2} \frac{m+2}{m-2} |\xi|; \Phi(\frac{m}{2(m+2)} \frac{m}{m+2} \frac{m^2}{m+2} z)$ and $\Phi(\frac{m+4}{2(m+2)} \frac{m+4}{m+2} \frac{m^2}{m+2} z)$ are confluent hypergeometric functions. (The definition of confluent hypergeometric function can be found in [12].) These are analytic functions of $z$ that satisfy, for large $|z|$,

$$
\begin{aligned}
\left| \Phi\left(\frac{m}{2(m+2)} \frac{m}{m+2} \frac{m^2}{m+2} z\right) \right| &\leq C|z|^{-\frac{1}{(m+2)}} \left(1 + O\left(|z|^{-1}\right)\right), \\
\left| \Phi\left(\frac{m+4}{2(m+2)} \frac{m+4}{m+2} \frac{m^2}{m+2} z\right) \right| &\leq C|z|^{-\frac{1}{(m+2)}} \left(1 + O\left(|z|^{-1}\right)\right).
\end{aligned}
$$

Then:

Lemma 2.2 ([3, Lemma 3.2]). Let $0 \leq s_1 \leq \frac{m}{2(m+2)}, 0 \leq s_2 \leq \frac{m+4}{2(m+2)}$, $T > 0$, and $g(x) \in H^s(\mathbb{R}^n)$, where $s \in \mathbb{R}$. Then one has, for $0 < t \leq T$,

$$
\begin{aligned}
\left\| (V_1(t, |\xi|) \hat{g}(\xi))^\wedge \right\|_{H^{s+s_1}} &\leq Ct^{-\frac{1}{2(m+2)}} \|g\|_{H^s}, \\
\left\| (V_2(t, |\xi|) \hat{g}(\xi))^\wedge \right\|_{H^{s+s_2}} &\leq Ct^{-\frac{1}{2(m+2)}} \|g\|_{H^s},
\end{aligned}
$$

and

$$
\begin{aligned}
\left\| (\partial_t V_1(t, |\xi|) \hat{g}(\xi))^\wedge \right\|_{H^{s-s_1}} &\leq C\|g\|_{H^s}, \\
\left\| (\partial_t V_2(t, |\xi|) \hat{g}(\xi))^\wedge \right\|_{H^{s-s_2}} &\leq C\|g\|_{H^s},
\end{aligned}
$$

where $^\wedge$ and $^\vee$ denote the Fourier transform and inverse Fourier transform, respectively, with respect to $x \in \mathbb{R}^n$.

Lemma 2.3. Let $u(t, x)$ be a solution of the inhomogeneous problem

$$
\begin{aligned}
\partial_t^2 u - t^m \Delta_x u &= f(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
u(0, x) &= 0, \quad \partial_t u(0, x) = 0.
\end{aligned}
$$

(i) If $f(t, x) \in L^p((0, T), H^s(\mathbb{R}^n))$, where $s \in \mathbb{R}$ and $1 < p < \infty$, then, for any $t \in (0, T]$,

$$
\begin{aligned}
\|u(t, \cdot)\|_{H^{s+p_1}} &\leq Ct^{\frac{1}{2(m+2)} - \frac{1}{p}} \|f(t, x)\|_{L^p((0, T), H^s)}, \\
\|\partial_t u(t, \cdot)\|_{H^{s-\frac{m+2}{2(m+2)} + p_2}} &\leq Ct^{\frac{1}{2(m+2)} - \frac{1}{p}} \|f(t, x)\|_{L^p((0, T), H^s)},
\end{aligned}
$$

where $0 \leq p_1 < p_1(m) = \min\left\{\frac{p(m + 8) - 4}{2p(m + 2)}, 1\right\}$ and $0 \leq p_2 < p_2(m) = \min\left\{\frac{2(p - 1)}{p(m + 2)}, \frac{m}{2(m + 2)}\right\}$. 


(ii) (See [21, Lemma 3.4].) If \( f(t, x) \in C([0, T], H^s(\mathbb{R}^n)) \), where \( s \in \mathbb{R} \), then, for any \( t \in (0, T) \),

\[
\|u(t, \cdot)\|_{H^{s+p_3}} \leq C t^2 \| f(t, x) \|_{L^\infty((0, T), H^s)},
\]

\[
\| \partial_t u(t, \cdot) \|_{H^{s-m+2}} + p_4 \leq C p t^2 \| f(t, x) \|_{L^\infty((0, T), H^s)},
\]

where \( 0 \leq p_3 < p_3(m) = \min \left\{ \frac{m + 8}{2(m + 2)}, 1 \right\} \) and \( p_4 < p_4(m) = \min \left\{ \frac{2(m + 4)}{m + 2}, \frac{m}{2(m + 2)} \right\} \).

**Remark 2.4.** If \( f(t, x) \in L^1((0, T), H^s(\mathbb{R}^n)) \), then it follows from the proof of Lemma 2.3(i), that, for \( t \in (0, T) \),

\[
\left\{ \begin{array}{l}
\|u(t, \cdot)\|_{H^{s+p_1}} \leq C t^2 \| f(t, x) \|_{L^1((0, T), H^s)}, \\
\| \partial_t u(t, \cdot) \|_{H^{s-m+2}} \leq C \| f(t, x) \|_{L^1((0, T), H^s)},
\end{array} \right.
\]

where \( 0 \leq p_1 \leq \frac{m}{2(m + 2)} \).

**Proof.** We only need to prove (i). It is readily seen that the solution \( u \) of (2.4) can be expressed as

\[
\hat{u}(t, \xi) = \int_0^t \left( V_2(t, [\xi]) V_1(\tau, [\xi]) - V_1(t, [\xi]) V_2(\tau, [\xi]) \right) \hat{f}(\tau, \xi) d\tau,
\]

where the definition of \( V_1(t, [\xi]) \) and \( V_2(t, [\xi]) \) is given in \( [21] \). It follows from the Minkowski inequality and (2.6) that

\[
\|u(t, \cdot)\|_{H^{s+p_1}} \leq \left( \int_0^t \left( \int_{\mathbb{R}^n} \left( 1 + |\xi|^2 \right)^{s/2 + 1/p_1} \| V_2(t, [\xi]) V_1(\tau, [\xi]) - V_1(t, [\xi]) V_2(\tau, [\xi]) \|_2^2 \right) d\xi \right)^{1/2} d\tau
\]

\[
\leq \int_0^t \left( (1 + |\xi|^2)^{s/2 + 1/p_1} \| V_2(t, [\xi]) V_1(\tau, [\xi]) \|_2^2 \right) d\tau
\]

\[
+ \int_0^t \| (1 + |\xi|^2)^{s/2 + 1/p_1} V_1(t, [\xi]) V_2(\tau, [\xi]) \|_2^2 d\tau
\]

\[
\equiv I_1 + I_2.
\]

Let \( p_1 = s_1 + s_2 \) with \( 0 \leq s_1 < \min \left\{ \frac{m}{2(m + 2)}, \frac{2(p - 1)}{p(m + 2)} \right\} \) and \( 0 \leq s_2 \leq \frac{m + 4}{2(m + 2)} \). Then one has by Lemma 2.2 and the Hölder inequality that

\[
\left\{ \begin{array}{l}
I_1 \leq C t^2 \int_0^t \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s_1/2} \| V_1(\tau, [\xi]) \|_2^2 \right) d\xi \right)^{1/2} d\tau
\end{array} \right.
\]

\[
\leq C t^{1 - \frac{s_1(m + 2)}{2}} \int_0^t \| f(\tau, \cdot) \|_{H^{s_1}} d\tau
\]

\[
\leq C t^{1 - \frac{s_1(m + 2)}{2}} \| f(t, x) \|_{L^p((0, T), H^{s_1})}.
\]

If one sets \( p_1 = \tilde{s}_1 + \tilde{s}_2 \) with \( 0 \leq \tilde{s}_1 \leq \frac{m}{2(m + 2)} \) and \( 0 \leq \tilde{s}_2 < \min \left\{ \frac{m + 4}{2(m + 2)}, \frac{2(2p - 1)}{p(m + 2)} \right\} \), then by Lemma 2.2 and the Hölder inequality

\[
I_2 \leq C t^{-\frac{\tilde{s}_1(m + 2)}{2}} \int_0^t \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s_2/2} \| V_1(\tau, [\xi]) \|_2^2 \right) d\xi \right)^{1/2} d\tau
\]

\[
\leq C t^{-\frac{\tilde{s}_1(m + 2)}{2}} \| f(\tau, \cdot) \|_{H^{s_2}} d\tau
\]

\[
\leq C t^{-\frac{\tilde{s}_1(m + 2)}{2}} \| f(t, x) \|_{L^p((0, T), H^{s_2})}.
\]

Substituting (2.8)–(2.9) into (2.7) yields (2.4) for \( 0 \leq p_1 < p_1(m) \).

Next we prove (2.5). In view of

\[
\| \partial_t u(t, \cdot) \|_{H^{s-m+2}} \leq C p t^2 \| f(t, x) \|_{L^\infty((0, T), H^s)},
\]

one has by the Minkowski inequality

\[
\| \partial_t u(t, \cdot) \|_{H^{s-m+2}} \leq C p t^2 \| f(t, x) \|_{L^\infty((0, T), H^s)}.
\]
Lemma 2.5. with piecewise smooth or single-point singular initial data.

Applying for Lemma 2.2 and the Hölder inequality yields for

\[
\int_0^t \left\| (1 + |\xi|^2)^{s/2 - \frac{m+n}{2} + \frac{p}{2}} \partial_t V_2(t, |\xi|) V_2(\tau, |\xi|) \partial_t \right\|_{L^2}^2 d\tau \leq \int_0^t \left\| (1 + |\xi|^2)^{s/2 - \frac{m+n}{2} + \frac{p}{2}} \partial_t V_2(t, |\xi|) V_2(\tau, |\xi|) \partial_t \right\|_{L^2}^2 d\tau \\
+ \int_0^t \left\| (1 + |\xi|^2)^{s/2 - \frac{m+n}{2} + \frac{p}{2}} \partial_t V_2(t, |\xi|) \right\|_{L^2}^2 d\tau \\
\equiv I_3 + I_4.
\]

Applying for Lemma 2.2 and the Hölder inequality yields for $0 < t \leq T$

\[
I_3 \leq C \int_0^t \left\| (1 + |\xi|^2)^{s/2 - \frac{p}{2}} V_1(\tau, |\xi|) \right\|_{L^2}^2 d\tau \leq C \int_0^t \tau \left\| f(\tau, \cdot) \right\|_{H^s} d\tau \leq C p_2 t \left\| f \right\|_{L^p(0,T)} d\tau
\]

and

\[
I_4 \leq C \int_0^t \left\| (1 + |\xi|^2)^{\frac{p}{2} - \frac{m+n}{2}} \partial_t V_2(\tau, |\xi|) \right\|_{L^2}^2 d\tau \leq C p_2 t \left\| f \right\|_{L^p(0,T)} d\tau.
\]

The estimates of $I_3$ and $I_4$ yield (2.5).

Now we start to establish the $L^\infty$ (or $L^\infty_\text{loc}$) property of solutions $u$ of the linear problem $\partial_t^2 u - t^m \Delta_x u = 0$ with piecewise smooth or single-point singular initial data.

**Lemma 2.5.** Assume that $u(t, x) \in C([0,T], H^{1/2} - (\mathbb{R}^n))$ is a solution of the linear equation

\[
\begin{cases}
\partial_t^2 u - t^m \Delta_x u = 0, & (t, x) \in (0,T) \times \mathbb{R}^n, \\
u(0, x) = \varphi_1(x), & \partial_t u(0, x) = \varphi_2(x).
\end{cases}
\]

(i) If $\varphi_1(x)$ and $\varphi_2(x)$ satisfy assumption (A), then there is a constant $C = C(n, T) > 0$ such that, for $(t, x) \in (0,T) \times \mathbb{R}^n$,

\[
|u(t, x)| \leq C \left( 1 + |\ln t| \right).
\]

In addition, if $n = 1$, then $u(t, x) \in L^\infty((0,T) \times \mathbb{R})$.

(ii) If $\varphi_1(x)$ and $\varphi_2(x)$ satisfy assumption (A2), then there is a constant $C = C(n, T) > 0$ such that, for $(t, x) \in (0,T) \times \mathbb{R}^n$,

\[
|u(t, x)| \leq C \left( 1 + |\ln t|^2 \right).
\]

**Proof.** (i) For $j = 1, 2$, one can write

\[
\varphi_j(x) = \varphi_2(x) + \langle \varphi_1(x) - \varphi_2(x) \rangle E(x_1),
\]

where $E(x_1)$ is the Heaviside function with $E(x_1) = \begin{cases} 1, & x_1 > 0, \\
0, & x_1 < 0. \end{cases}$ Recall that the Fourier transform of $E(x_1)$ is

\[
E(\xi_1) = \frac{1}{2} \left( \delta(\xi_1) - i \frac{p}{\pi} \text{p. v.} \frac{1}{\xi_1} \right),
\]

where $\delta$ is the Dirac delta function, $i = \sqrt{-1}$, and p. v. denotes the principal value. Then it follows from (2.10) that one has

\[
\bar{u}(t, \xi) = \frac{1}{2} V_1(t, |\xi|) \langle \varphi_1(\xi) + \varphi_2(\xi) \rangle - i H \langle \varphi_{12}(\xi_1 - \xi_1, \xi') \rangle (\xi_1)
\]

\[
+ \frac{1}{2} V_2(t, |\xi|) \langle \varphi_{21}(\xi_1 + \xi_1, \xi') \rangle - i H \langle \varphi_{22}(\xi_1 - \xi_1, \xi') \rangle (\xi_1),
\]

where the functions $V_i (i = 1, 2)$ have been defined in (2.1), $\xi' = (\xi_2, \ldots, \xi_n)$, and $H$ is the Hilbert transformation. The latter means that, for $1 \leq j, k \leq 2$,

\[
H \langle \varphi_{jk}(\xi_1, \xi') \rangle (\xi_1) = \frac{1}{\pi} \left( \text{p. v.} \frac{1}{\xi_1} \right) \langle \varphi_{jk}(\xi_1, \xi') \rangle (\xi_1) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\xi_1| < \varepsilon} \frac{\hat{\varphi}_{jk}(\xi_1 - \xi_1, \xi')}{\xi_1} d\xi_1.
\]
Therefore,

$$(2.11) \quad u(t, x) = \frac{1}{2} \int_{\mathbb{R}^n} e^{2\pi i x \xi} \sum_{j,k,\ell=1}^2 V_t(t, |\xi|) \hat{\varphi}_{jk}(\xi) \, d\xi - \frac{i}{2} \int_{\mathbb{R}^n} e^{2\pi i x \xi} \sum_{j,k,\ell=1}^2 V_r(t, |\xi|) H(\hat{\varphi}_{jk}(\xi)) \, d\xi.$$

To show that $u(t, x) \in L^\infty((0, T) \times \mathbb{R}^n)$, we will show that each term in (2.11) is bounded. By (2.2), one has that, for $1 \leq j, k, \ell \leq 2$,

$$\left| \int_{\mathbb{R}^n} e^{2\pi i x \xi} V_t(t, |\xi|) \hat{\varphi}_{jk}(\xi) \, d\xi \right| \leq \|V_t(t, |\xi|)\|_{L^\infty((0, T) \times \mathbb{R}^n)} \|\hat{\varphi}_{jk}\|_{L^1(\mathbb{R}^n)} \leq C.$$

Next we treat the terms $V_t(t, |\xi|) H(\hat{\varphi}_{jk}(\xi)) (\xi) (1 \leq j, k, \ell \leq 2)$ in (2.11). Due to the Schwartz inequality and the $L^2(\mathbb{R}^n)$ boundedness of the Hilbert transformation $H$, one has that, for $(\xi_1, \xi') \in (0, M) \times \mathbb{R}^n - 1$ for some large $M > 0$,

$$\left| \int_{\mathbb{R}^n} e^{2\pi i x \xi} H(\hat{\varphi}_{jk}(\xi_1, \xi')) \xi_1 \xi' \right| \leq C_M \|V_t(t, |\xi|)\|_{L^\infty((0, T) \times \mathbb{R}^n)} \int_{\mathbb{R}^n} \left( \int_{|\xi_1| \leq M} |H(\hat{\varphi}_{jk}(\xi_1, \xi')) (\xi)|^2 \xi_1 \, d\xi_1 \right)^{1/2} \xi' \leq C_M \|\hat{\varphi}_{jk}\|_{L^1(\mathbb{R}^n - 1, L^2(\mathbb{R}^n))} \leq C_M.$$

For $(\xi_1, \xi') \in (M, \infty) \times \mathbb{R}^n$, we set

$$I \equiv \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{2\pi i x \xi} H(\hat{\varphi}_{jk}(\xi_1, \xi')) (\xi_1) \xi_1 \xi' \right) \xi_1 \xi' \, d\xi_1.$$

Denote by $\Phi_{jk}(x_1, \xi')$ the Fourier transform of $\varphi_{jk}$ with respect to the variables $x_2, \ldots, x_n$. If $n = 1$, then $\Phi_{jk} = \varphi_{jk}$. Notice that, for any fixed $\xi_1 \neq 0$,

$$\xi_1 H(\varphi_{jk}(\xi_1, \xi')) (\xi_1) = \frac{1}{\pi^2} \Phi_{jk}(0, \xi') + H(\xi_1 \varphi_{jk}(\xi_1, \xi')) (\xi_1)$$

which gives

$$I \leq C \left( \int_{\mathbb{R}^n} \left| H(\xi_1 \varphi_{jk}(\xi_1, \xi')) (\xi_1) \right| \, d\xi_1 \xi' \right)^{1/2} + C \int_{\mathbb{R}^n} \left| \Phi_{jk}(0, \xi') \right| \, d\xi' \leq I_1 + I_2 \leq I_1 + I_{2,1} + I_{2,2} + I_{2,3},$$

where $z_1 = \frac{4}{m+2} \frac{m^2}{\pi^{m+1}} \xi_1$. (If $n = 1$, then $z_1 = z$.) The $L^2$ boundedness of the Hilbert transformation further yields

$$I_1 \leq C \left( \int_{\mathbb{R}^n} \left| H(\xi_1 \varphi_{jk}(\xi_1, \xi')) (\xi_1) \right| \, d\xi_1 \xi' \right)^{1/2} \leq C_M \|\xi_1 \varphi_{jk}(\xi_1, \xi') \|_{L^1(\mathbb{R}^n - 1, L^2(\mathbb{R}^n))} \leq C_M.$$

We now estimate the term $I_2$. For $l = 1$, one has from the estimates (2.11) that

$$I_{2,1} \leq C \left( \int_{\mathbb{R}^n} \left| \Phi_{jk}(0, \xi') \right| \, d\xi' \right)^{1/2} \leq C \int_{\mathbb{R}^n} \left| \Phi_{jk}(0, \xi') \right| \, d\xi' \left( \int_{1/M}^\infty s^{-1} \frac{m}{(m+1)s^{m+1}} \, ds \right) \leq C_M,$$

and similarly for $l = 2$, where $I_{2,1}$ is also dominated by a constant $C_M$. Using the fact that $V_t(t, |\xi|) \in L^\infty((0, T) \times \mathbb{R}^n)$ once again, one has that

$$I_{2,2} \leq \left( \int_{\mathbb{R}^n} \left| \Phi_{jk}(0, \xi') \right| \, d\xi' \right)^{1/2} \leq C_M.$$
In addition, one has, for \( t \in (0, T) \),

\[
I_{2,3} \leq C \| V(t, |\xi|) \|_{L^\infty((0, T) \times \mathbb{R}^n)} \| \varphi_{jk} \|_{L^1(\mathbb{R}^n)} \left( \int_{M}^{\infty} \frac{r^{\frac{m+2}{2}}}{s} \frac{ds}{s} \right) \leq C \left( 1 + |\ln t| \right).
\]

Collecting the estimates in (2.12) – (2.14) yields \( I_2 \leq C \left( 1 + |\ln t| \right) \) for \( t \in (0, T) \). Thus,

\[
|u(t, x)| = \left| \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \check{u}(t, \xi) d\xi \right| \leq C \left( 1 + |\ln t| \right), \quad (t, x) \in (0, T) \times \mathbb{R}^n.
\]

Finally, we discuss uniformly boundedness of the solution \( u \) in case \( n = 1 \). Following the arguments above, one only needs to show that the term

\[
I_{2,3} = C |\varphi_{jk}(0)| \left| \int_{|\xi_1| > M, |z_1| > 1/M} e^{2\pi i x_1 \xi_1} \frac{V(t, |\xi_1|)}{\xi_1} d\xi_1 \right|
\]
is uniformly bounded for \((t, x) \in (0, T) \times \mathbb{R} \). For \( \ell = 1 \),

\[
I_{2,3} = C |\varphi_{jk}(0)| \left| \int_{|\xi_1| > M, |z_1| < 1/M} e^{2\pi i x_1 \xi_1} e^{i |z_1|/2} \Phi \left( \frac{m}{2(m+2)} \cdot \frac{m}{m+2} : z_1 \right) \frac{d\xi_1}{\xi_1} \right|
\]

\[
= C |\varphi_{jk}(0)| \left| \int_{|\xi_1| > M, |z_1| < 1/M} \sin \left( 2\pi x_1 \xi_1 \right) \left( \cos \left( \frac{|z_1|}{2} \right) + i \sin \left( \frac{|z_1|}{2} \right) \right) \Phi \left( \frac{m}{2(m+2)} \cdot \frac{m}{m+2} : z_1 \right) \frac{d\xi_1}{\xi_1} \right|
\]

\[
\leq C \int_{|\xi_1| > M, |z_1| < 1/M} \sin \left( 2\pi x_1 \xi_1 \right) \cos \left( \frac{|z_1|}{2} \right) \Phi \left( \frac{m}{2(m+2)} \cdot \frac{m}{m+2} : z_1 \right) \frac{d\xi_1}{\xi_1} \right|
\]

\[
\equiv I_{2,3}^{(1)} + I_{2,3}^{(2)}.
\]

Obviously, the nonnegative term \( I_{2,3}^{(1)} \) is dominated by a positive constant \( C \).

Next we show the uniform boundedness of \( I_{2,3}^{(2)} \). Denote by

\[
a_1 = \left( 2\pi x_1 + \frac{2}{m+2} t^{\frac{m+2}{2}} \right) M, \quad b_1 = \left( 2\pi x_1 + \frac{2}{m+2} t^{\frac{m+2}{2}} \right) \frac{4}{m+2} t^{-\frac{m+2}{2}} M^{-\frac{1}{2}},
\]

and

\[
a_2 = \left( 2\pi x_1 - \frac{2}{m+2} t^{\frac{m+2}{2}} \right) M, \quad b_2 = \left( 2\pi x_1 - \frac{2}{m+2} t^{\frac{m+2}{2}} \right) \frac{4}{m+2} t^{-\frac{m+2}{2}} M^{-\frac{1}{2}}.
\]

Then the term \( I_{2,3}^{(2)} \) is dominated by

\[
C \sum_{i=1}^{2} \left| \int_{a_i < s < b_i, |z_1| < 1/M} \Phi \left( \frac{m}{2(m+2)} \cdot \frac{m}{m+2} : z_1 \right) \frac{\sin s}{s} ds \right|.
\]

Since the confluent hypergeometric function \( \Phi \left( \frac{m}{2(m+2)} \cdot \frac{m}{m+2} : z_1 \right) \) is an analytic function of \( z_1 \) and \( \Phi \left( \frac{m}{2(m+2)} \cdot \frac{m}{m+2} : 0 \right) = 1 \), one has, for \( M > 1 \) large and \( |z_1| < 1/M \),

\[
\Phi \left( \frac{m}{2(m+2)} \cdot \frac{m}{m+2} : z_1 \right) = 1 + O(z_1),
\]

which yields

\[
I_{2,3}^{(2)} \leq C \sum_{i=1}^{2} \left| \int_{a_i}^{b_i} \frac{\sin s}{s} ds \right| + C \sum_{i=1}^{2} \left| \int_{a_i < s < b_i, |z_1| < 1/M} \frac{|z_1|}{s} ds \right|
\]

\[
\leq C \sum_{i=1}^{2} \left| \int_{a_i}^{b_i} \frac{\sin s}{s} ds \right| + \frac{C t^{\frac{m+2}{2}}}{2\pi x_1 + t^{\frac{m+2}{2}}} |b_1 - a_1| + \frac{C t^{\frac{m+2}{2}}}{2\pi x_1 - t^{\frac{m+2}{2}}} |b_2 - a_2| \leq C.
\]

Thus, \( I_{2,3} \leq C \) for \( \ell = 1 \). Similarly, one shows that \( I_{2,3} \leq C \) for \( \ell = 2 \). Hence, one has that \( u(t, x) \in L^\infty((0, T) \times \mathbb{R}) \) in case \( n = 1 \).
(ii) Suppose that $\varphi_1(x)$ and $\varphi_2(x)$ satisfy assumption (A). From the proof of [21], Lemma 2.1 (a), one has that $|\hat{\varphi}_i(\xi)| \leq C \frac{(1 + \ln |\xi|)}{|\xi|^m}$ for $|\xi| > M > 1$ and $\hat{\varphi}_i \in L^1(\{|\xi| < M\})$. Thus, for $\ell = 1, 2$,
\[
\int_{|\xi| < M} |V_\ell(t, |\xi|)| \, |\hat{\varphi}_i(\xi)| \, d\xi \leq \|V_\ell(t, |\xi|)\|_{L^\infty((0, T) \times \mathbb{R}^n)} \|\hat{\varphi}_i\|_{L^1(\{|\xi| < M\})} \leq C
\]
and
\[
\int_{|\xi| > M, |z| > M} |V_\ell(t, |\xi|)| \, |\hat{\varphi}_i(\xi)| \, d\xi \leq C \int_{|\xi| > M, |z| > M} \frac{1 + \ln |\xi|}{|\xi|^n} \, d\xi
\]
\[
\leq C \left( t^{-m/4} \int_{r > M} r^{-\frac{m-2}{2m+2}} \left( 1 + \ln r \right) \, dr \right)
\leq C \int_{s > M} s^{-\frac{m+2}{2m+2}} \left( 1 + \ln s + |\ln t| \right) \, ds \leq C \left( 1 + |\ln t| \right).
\]
Similarly,
\[
\int_{|\xi| > M, |z| > M} |V_\ell(t, |\xi|)| \, |\hat{\varphi}_i(\xi)| \, d\xi \leq C \left( 1 + |\ln t| \right).
\]
Further,
\[
\int_{|\xi| > M, |z| < M} |V_\ell(t, |\xi|)| \, |\hat{\varphi}_i(\xi)| \, d\xi \leq C \|V_\ell(t, |\xi|)\|_{L^\infty((0, T) \times \mathbb{R}^n)} \int_{|\xi| > M, |z| < M} \frac{1 + \ln |\xi|}{|\xi|^n} \, d\xi
\]
\[
\leq C \|V_\ell(t, |\xi|)\|_{L^\infty((0, T) \times \mathbb{R}^n)} \int_{M}^{\frac{m+2}{2m+2}} \frac{1 + \ln r}{r} \, dr \leq C \left( 1 + |\ln t| \right)^2,
\]
which completes the proof of Lemma 2.6.

Lemma 2.6. Let $u(x, t)$ be a solution to the problem
\[
\begin{align*}
\partial_t^2 u - t^m \Delta_x u &= f(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
u(0, x) &= \partial_t u(0, x) = 0,
\end{align*}
\]
(2.15)

(i) If $f(t, x) \in C([0, T], H^s(\mathbb{R}^n))$ and $\partial_{x}^\beta f(t, x) \in L^\infty((0, T), H^1(\mathbb{R}^n))$, where $s > \frac{m-2}{2(m+2)}$, $x \in (x_2, \ldots, x_n)$, and $|\alpha| \leq \lfloor n/2 \rfloor + 1$, then $u(t, x) \in L^\infty((0, T) \times \mathbb{R}^n)$.

(ii) If $f(t, x) \in L^p((0, T), H^r(\mathbb{R}^n))$ and $\partial_{x}^\beta f(t, x) \in L^p((0, T), H^s(\mathbb{R}^n))$, where $1 < p < \infty$, $r > \frac{m}{2(m+2)}$, and $|\alpha| \leq \lfloor n/2 \rfloor + 1$, then $u(t, x) \in L^\infty((0, T) \times \mathbb{R}^n)$.

Proof. Because of $|\partial_x^\alpha \partial_t^\beta - t^m \Delta_x| = 0$, one has from (2.15) that, for $|\alpha| \leq \lfloor n/2 \rfloor + 1$,
\[
\begin{align*}
(\partial_t^2 - t^m \Delta_x) \partial_x^\beta u &= (\partial_x^\beta f)(t, x), \\
(\partial_t^2 u)(0, x) &= \partial_t (\partial_x^\beta u)(0, x) = 0.
\end{align*}
\]
(i) Applying Lemma 2.3 (ii) with $p_3 = \frac{2}{m+2}$, one sees that, for any $t \in [0, T]$, there is a constant $C > 0$ independent of $t$ such that
\[
\|\partial_x^\beta u(t, \cdot)\|_{H^s(\mathbb{R}^n)} \leq C \|\partial_x^\beta f(t, x)\|_{L^\infty((0, T), H^s(\mathbb{R}^n))}.
\]
One further has that, for $s > \frac{m-2}{2(m+2)}$ and $\beta \in \mathbb{N}^n$ with $|\beta| = \lfloor n/2 \rfloor + 1 > \frac{n-1}{2}$,
\[
\|u(t, x_1, x')\|_{L^\infty((0, T) \times \mathbb{R}^n)} \leq C \|u(t, x_1, x')\|_{L^\infty((0, T), H^s(\mathbb{R}^n))}
\]
\[
\leq C \|\partial_x^2 u(t, x_1, x')\|_{L^\infty((0, T), H^s(\mathbb{R}^n))}
\leq C \|\partial_x^2 u(t, x_1, x')\|_{L^\infty((0, T), H^s(\mathbb{R}^n))}
\]
where $\otimes$ denotes the (completed) Hilbert space tensor product. Combining (2.16) and (2.17) yields $u(t, x) \in L^\infty((0, T) \times \mathbb{R}^n)$. 

\[
\frac{1}{2}
\]
(ii) Applying Lemma 2.3(i) with $p_1 = \frac{1}{m+2}$ and using an argument analogous to that in (i) one sees that, for $p > 1, u(t, x) \in L^\infty((0, T) \times \mathbb{R}^n)$. 

Finally, based on Lemmas 2.5 and 2.6 we are able to show that $u \in L^\infty((0, T) \times \mathbb{R}^n)$ for the solution of (2.10) under assumption (A1) on the initial data.

**Lemma 2.7.** Let $u(t, x)$ be a solution of Eq. (2.10). If the initial data $\varphi_1(x)$ and $\varphi_2(x)$ satisfies assumption (A1), then, for any $T > 0$, there is a constant $C = C(n, T) > 0$ such that $u(t, x) \in L^\infty((0, T) \times \mathbb{R}^n)$ and $|u(t, x)| \leq C, \quad (t, x) \in (0, T) \times \mathbb{R}^n$.

**Proof.** Write Eq. (2.10) as

\begin{equation}
(\partial_t^2 - t^m \partial_x^2) u = t^m \sum_{i=2}^n \partial_x^2 u(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
\quad u(0, x) = \varphi_1(x), \quad \partial_t u(0, x) = \varphi_2(x).
\end{equation}

The solution $u(t, x)$ of problem (2.15) decomposes as

\begin{equation}
\begin{aligned}
&u(t, x) = w(t, x) + v(t, x), \\
&\text{where } w(t, x) \text{ satisfies}
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\left\{ \begin{array}{l}
(\partial_t^2 - t^m \partial_x^2) w = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
\quad w(0, x) = \varphi_1(x), \quad \partial_t w(0, x) = \varphi_2(x). 
\end{array} \right.
\end{aligned}
\end{equation}

and $v(t, x)$ satisfies

\begin{equation}
\begin{aligned}
&\left\{ \begin{array}{l}
(\partial_t^2 - t^m \partial_x^2) v = t^m \sum_{i=2}^n \partial_x^2 u(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
\quad v(0, x) = 0, \quad \partial_t v(0, x) = 0.
\end{array} \right.
\end{aligned}
\end{equation}

The Fourier transform $\hat{w}(t, \xi_1, x')$ of $w$ with respect to the variable $x_1$ can be written as

\begin{align*}
\hat{w}(t, \xi_1, x') = & \frac{1}{2} \left( V_1(t, |\xi_1|) \left( (\hat{\varphi}_1(<\xi_1, x'>) + \hat{\varphi}_2(\xi_1, x')) - iH(\hat{\varphi}_1(<\xi_1, x'>) - \hat{\varphi}_2(\xi_1, x'))(\xi_1) \right) \\
&+ \frac{1}{2} \left( V_2(t, |\xi_1|) \left( (\hat{\varphi}_21(\xi_1, x') + \hat{\varphi}_22(\xi_1, x')) - iH(\hat{\varphi}_21(\xi_1, x') - \hat{\varphi}_22(\xi_1, x'))(\xi_1) \right) \right).
\end{align*}

Following the arguments in the proof of Lemma 2.5(i) in case $n = 1$, one deduces that (2.20)

$\begin{equation}
w(t, x) \in L^\infty((0, T) \times \mathbb{R}^n).
\end{equation}$

Moreover, for any $\alpha \in \mathbb{N}^n_{0, 1}$, from $[\partial^\alpha_x, \partial^2_t - t^m \Delta_x] = 0$ and (2.10) one obtains that $\partial^\alpha_x u$ satisfies

\begin{equation}
\begin{aligned}
&\left\{ \begin{array}{l}
(\partial_t^2 - t^m \Delta_x) (\partial^\alpha_x u) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
\quad (\partial^\alpha_x u)(0, x) = (\partial^\alpha_x \varphi_1)(x), \quad \partial_t (\partial^\alpha_x u)(0, x) = (\partial^\alpha_x \varphi_2)(x).
\end{array} \right.
\end{aligned}
\end{equation}

Note that $\partial^\alpha_x \varphi_1$ and $\partial^\alpha_x \varphi_2$ also satisfy assumption (A1). Then $\partial^\alpha_x \varphi_1, \partial^\alpha_x \varphi_2 \in H^{1/2-}(\mathbb{R}^n)$. Hence, it follows from Lemma 2.1 and (2.21) that $\partial^\alpha_x u \in C([0, T], H^{1/2-}(\mathbb{R}^n)) \cap C\left((0, T], H^{1/2+}(\mathbb{R}^n)\right) \cap C^2\left((0, T], H^{1/2+}(\mathbb{R}^n)\right)$ satisfies, for any $\delta > 0$ and all $0 < t \leq T$,

\begin{align*}
&\left\| \partial^\alpha_x u(t, \cdot) \right\|_{H^{1/2+}(\mathbb{R}^n)} + t^{m/4} \left\| (\partial^\alpha_x u)(t, \cdot) \right\|_{H^{1/2+}(\mathbb{R}^n)} \\
&\quad + \left\| \partial_t (\partial^\alpha_x u)(t, \cdot) \right\|_{H^{1/2+}(\mathbb{R}^n)} \leq C(n, T, \delta).
\end{align*}

It then follows that

$\begin{equation}
t^m (\partial^\alpha_x u)(t, x) \in L^\infty((0, T), H^{1/2+}(\mathbb{R}^n)).
\end{equation}$

Together with Lemma 2.3(i) and Eq. (2.19), this yields $v(t, x) \in L^\infty((0, T), H^{1/2+}(\mathbb{R}^n))$.

Using $[\partial^\alpha_x, \partial^2_t - t^m \Delta_x] = 0$ once again, one has that

$\begin{equation}
\begin{aligned}
&\left\{ \begin{array}{l}
(\partial_t^2 - t^m \partial_x^2) (\partial^\alpha_x v) = t^m \sum_{i=2}^n (\partial_x^2 \partial^\alpha_x u)(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
\quad (\partial^\alpha_x v)(0, x) = 0, \quad \partial_t (\partial^\alpha_x v)(0, x) = 0.
\end{array} \right.
\end{aligned}
\end{equation}$
One then obtains as in Lemma 2.6(i) that
\[ \|v(t, x)\|_{L^\infty((0, T) \times \mathbb{R}^n)} \leq C. \]
Hence, combining (2.20) with the latter yields \( u(t, x) \in L^\infty((0, T) \times \mathbb{R}^n). \)

3. CONORMAL SPACES AND COMMUTATOR RELATIONS

In order to study the singularity structure of solutions of (1.1) and (1.2), we are required to introduce spaces of conormal functions which relate to these problems.

For \( k \in \mathbb{N} \), set
\[ \Gamma_k = \{(t, x) : t \geq 0, |x|^2 = \frac{4t^{k+2}}{(k+2)^2}\}, \quad \Gamma_k^\pm = \{(t, x) : t \geq 0, x_1 = \pm \frac{2t^{(k+2)/2}}{k+2}\} \]
and
\[ l_0 = \{(t, x) : t \geq 0, |x| = 0\}, \quad \Sigma_0 = \{(t, x) : t = 0\}. \]
A basis of the \( C^\infty \) vector fields tangent to \( \Gamma_k \) is given by (see [21, Lemma 4.2])
\[
V_0^{(k)} = 2t\partial_t + (k + 2)(x_1\partial_1 + \cdots + x_n\partial_n),
\]
\[
V_\ell^{(k)} = 2t^{\ell+1}\partial_\ell + (k + 2)\frac{x_\ell}{t^{1/2}}\partial_t, \quad 1 \leq \ell \leq n,
\]
\[
L_{ij} = x_i\partial_j - x_j\partial_i, \quad 1 \leq i < j \leq n.
\]
Moreover, a basis of the \( C^\infty \) vector fields tangent to \( l_0 \) is given by
\[
t\partial_t, \quad V_0^{(m)}, \quad L_{ij}, \quad 1 \leq i < j \leq n.
\]
To prepare for the commutator argument handling the degenerate equations whose characteristic cones and characteristic surfaces exhibit cusp singularities, we shall use the following slightly altered vector fields tangent to \( \Gamma_k \):
\[
V_0^{(k)} = 2t\partial_t + (k + 2)(x_1\partial_1 + \cdots + x_n\partial_n),
\]
\[
V_\ell^{(k)} = 2t^{\ell+1}\partial_\ell + (k + 2)\frac{x_\ell}{t^{1/2}}\partial_t, \quad 1 \leq \ell \leq n,
\]
\[
L_{ij} = x_i\partial_j - x_j\partial_i, \quad 1 \leq i < j \leq n.
\]
Set
\[ P_1 = \partial_t (\partial_t^2 - t^m\Delta_x), \quad Q_k = \partial_t^2 - t^k\Delta_x. \]
Further, let \([A, B] = AB - BA\) denote the commutator of \( A \) and \( B \).

By direct verification, one then has:

**Lemma 3.1.** For \( 1 \leq i \leq n \) and \( 1 \leq i < j \leq n \),
\[
[Q_k, V_0^{(k)}] = 4Q_k, \quad [Q_k, V_\ell^{(k)}] = -k(k + 2)\frac{x_\ell}{t^{k/2+1}}Q_k + \frac{k(k + 2)}{4t^2}V_\ell^{(k)}, \quad 1 \leq \ell \leq n,
\]
\[
[Q_k, L_{ij}] = 0, \quad 1 \leq i < j \leq n,
\]
\[
[V_0^{(k)}, V_\ell^{(k)}] = 0, \quad [V_\ell^{(k)}, L_{ij}] = 0, \quad [V_\ell^{(k)}, V_\ell^{(k)}] = V_j^{(k)}V_\ell^{(k)},
\]
\[
[V_i^{(k)}, V_j^{(k)}] = 2(k + 1)(k + 2)L_{ij} + \frac{k(k + 2)}{2}\left(\frac{x_i}{t^{k/2+1}}V_j^{(k)} - \frac{x_j}{t^{k/2+1}}V_i^{(k)}\right),
\]
\[
[V_\ell^{(k)}, L_{ij}] = 0, \quad i \neq \ell \neq j,
\]
\[
[L_{ij}, L_{lk}] = 0, \quad 1 \leq k < l \leq n, \quad k \neq i, \quad l \neq j, \quad [L_{ij}, L_{ik}] = L_{kj} \quad k \neq j,
\]
\[
[P_1, V_0^{(m)}] = 6P_1, \quad [P_1, L_{ij}] = 0,
\]
\[
[P_1, V_i^{(m)}] = -\frac{3m(m+2)}{2}l^{-m/2-1}x_iP_1 + (m + 2)l^{m/2}Q_m\partial_i + \frac{3m(m+2)^2}{4}l^{-m/2-2}x_i\partial_i^2
\]
\[
- \frac{m(m+2)}{2}l^{m/2-2}x_i\Delta_x - \frac{m(m+2)}{4}l^{m/2-1}\partial_i\partial_i + \frac{m(m^2 - 4)}{4}l^{m/2-2}\partial_i,
\]
\[
- \frac{m(m+2)^2(m+4)}{8}l^{-m/2-3}x_i\partial_i.
\]
In addition,
\[ [P_i, t \partial_t] = 3P_i + (m + 2)\partial_t \Delta_x + m(m + 2)t^{m-1} \Delta_x, \quad [t \partial_t, V_0^{(m)}] = 0, \quad [t \partial_t, L_{ij}] = 0. \]

**Remark 3.2.** The commutator \([P_i, \tilde{V}_i^{(m)}]\) for \(1 \leq i \leq n\) in Lemma 3.1 contains the two singular terms \(\frac{3m(m + 2)}{4} t^{-m/2-2} x_i \partial_t^2\) and \(-\frac{m(m + 2)^2}{8} t^{-m/2-3} x_i \partial_t\), and further the term \(\frac{m(m + 2)^2}{2} t^{m/2-2} x_i \Delta_x\), which violates the Levi condition to be imposed on degenerate hyperbolic equations. Both will cause a loss of regularity for \(\tilde{V}_i^{(m)} u\) and further for \(\tilde{V}_i^{(m)} u, \tilde{V}_j^{(m)} u\), where \(k, i_1, \ldots, i_s \in \mathbb{N}\). To avoid such a loss, we shall look for relations that provide extra regularity of \(\tilde{V}_i^{(m)} u, \tilde{V}_j^{(m)} u\). Such relations are realized by the operator itself and by some parts of the vector fields. (See formulas (3.1), (3.7) in Proposition 3.7 below.)

**Lemma 3.3.** The vector fields
\[ V^{(k)} = 2t \partial_t + (k + 2)x_1 \partial_1, \quad V_i^{(k)} = 2k^{1/2+1} \partial_1 + (k + 2) \frac{x_1}{k^{1/2}} \partial_t, \quad R_\ell = \partial_\ell, \quad 2 \leq \ell \leq n, \]
are tangent to \(\Gamma^+_k\). Moreover, one has
\[
[V^{(k)}, V_i^{(k)}] = 0, \quad [V^{(k)}, R_\ell] = -(k + 2)R_\ell, \quad 2 \leq \ell \leq n, \quad [V_i^{(k)}, R_\ell] = 0, \quad 2 \leq \ell \leq n,
\]
\[
[P_i, V^{(m)}] = 6P_i + 2(m + 2)t^n \sum_{i=2}^n \partial_t R_i^2 + 2m(m + 2)t^{m-1} \sum_{i=2}^n R_i^2, \quad [P_i, R_\ell] = 0, \quad 2 \leq \ell \leq n,
\]
\[
[Q_k, V^{(k)}] = 4Q_k, \quad [Q_k, R_\ell] = 0, \quad 2 \leq \ell \leq n.
\]

Following [1, 2], we now introduce the notion of an admissible function.

**Definition 3.4** (Admissible functions). A function \(h(x) \in L^\infty(\Omega) \cap C^\infty(\Omega)\) is said to be admissible with respect to the vector fields \(N_1, \ldots, N_k\) if \(N_1^{j_1} \cdots N_k^{j_k} h \in L^\infty(\Omega) \cap C^\infty(\Omega)\) for all \((j_1, \ldots, j_k) \in \mathbb{N}_0^k\).

The module over the algebra of admissible functions with basis \(N_1, \ldots, N_k\) constitutes a Lie algebra of vector fields on \(\Omega\) provided that
\[(*) \quad \text{each commutator } [N_i, N_j] (1 \leq i < j \leq k) \text{ is a linear combination of } N_1, \ldots, N_k \text{ with admissible coefficients.}\]

Next we define admissible tangent vector fields with respect to the surface \(\Gamma_m\) and the ray \(l_0\).

**Definition 3.5** (Admissible tangent vector fields for \(\Gamma_m \cup l_0\)).

1. (Exterior to \(\Gamma_m\)) For \(\Omega_1 = \{(t, x): 0 < t < C|x| \leq \varepsilon\}, S_1\) denotes the Lie algebra of vector fields with admissible coefficients in \(\Omega_1\) generated by \(x_i \partial_t, t^{m/2} |x| \partial_t (1 \leq \ell \leq n), L_{ij} (1 \leq i \leq j < n)\).

2. (Near \(\Gamma_m\)) For \(\Omega_2 = \{(t, x): 0 < |x| < Ct \leq \varepsilon\} \cap \{(t, x): |x| - \frac{2}{m+2} t^\frac{m+2}{m} \leq C t^\frac{m+2}{m}\}, S_2\) denotes the Lie algebra of vector fields with admissible coefficients in \(\Omega_2\) generated by \(V_0^{(m)}, V_1^{(m)} (1 \leq \ell \leq n), L_{ij} (1 \leq i < j \leq n)\).

3. (Near \(l_0\)) For \(\Omega_3 = \{(t, x): |x| < Ct \leq \varepsilon\} \cap \{(t, x): t^\frac{m+2}{m} \leq C|x| - \frac{2}{m+2} t^\frac{m+2}{m}\}, S_3\) denotes the Lie algebra of vector fields with admissible coefficients in \(\Omega_3\) generated by \(t \partial_t, V_0^{(m)}, L_{ij} (1 \leq i < j \leq n)\).

4. (Between \(\Gamma_m\) and \(l_0\)) For \(\Omega_4 = \{(t, x): 0 < t < C|x| \leq \varepsilon\} \cap \{(t, x): t^{m+1} \partial_t \leq C|x| - \frac{2}{m+2} t^\frac{m+2}{m}\}, S_4\) denotes the Lie algebra of vector fields with admissible coefficients in \(\Omega_4\) generated by \(t \partial_t, t^{m+1} \partial_t (1 \leq i < j \leq n)\).

In all four cases (1)--(4), one checks that condition (*) is fulfilled.

The conormal space \(I^\infty H^s(\Gamma_m \cup l_0)\) for \(0 \leq s < n/2\) is defined as follows:

**Definition 3.6** (Conormal space \(I^\infty H^s(\Gamma_m \cup l_0)\)). Define \(u(t,x) \in I^\infty H^s(\Gamma_m \cup l_0)\) in \(\{(t,x): 0 < t < T, x \in \mathbb{R}^n\}\) if, away from the origin \(\{t = |x| = 0\}\) and near \(\Gamma_m, Z_1 \cdots Z_k u \in L^\infty(0, T), H^s(\mathbb{R}^n)\) for all smooth vector fields \(Z_1, \ldots, Z_k \in \{V_0^{(m)}, V_1^{(m)}, 1 \leq \ell \leq n, L_{ij}, 1 \leq i < j \leq n\}\), away from \(\{t = |x| = 0\}\), and near \(l_0, Z_1 \cdots Z_k u \in L^\infty(0, T), H^s(\mathbb{R}^n)\) for all smooth vector fields \(Z_1, \ldots, Z_j \in \{t \partial_t, V_0^{(m)}, L_{ij}, 1 \leq i < j \leq n\}\). Near \(\{t = |x| = 0\}\), the following properties hold:
(1) If \( h_1(t,x) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\}) \) is homogeneous of degree zero and supported in \( \Omega_1 \), then \( Z_1 \ldots Z_k h_1(t,x) u(t,x) \in L^\infty((0,T), H^s(\mathbb{R}^n)) \) for all \( Z_1, \ldots, Z_k \in S_1 \).

(2) If \( h_2(t,x) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\}) \) is homogeneous of degree zero and supported in \( \{ (t,x) : 0 < |x| < Ct \leq \varepsilon \} \) and \( \chi(\theta) \in C^\infty \) has compact support near \( \{ \theta = 1 \} \), then \( Z_1 \ldots Z_k h_2(t,x) \chi \left( \frac{(m+2)|x|}{2t} \right) u \in L^\infty((0,T), H^s(\mathbb{R}^n)) \) for all \( Z_1, \ldots, Z_k \in S_1 \).

(3) If \( h_3(t,x) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\}) \) is homogeneous of degree zero and supported in \( \{ (t,x) : |x| < Ct \leq \varepsilon \} \) and \( \chi_1(\theta) \in C^\infty \) has compact support away \( \{ \theta = 1 \} \), then \( Z_1 \ldots Z_k h_3(t,x) \chi_1 \left( \frac{(m+2)|x|}{2t} \right) u \in L^\infty((0,T), H^s(\mathbb{R}^n)) \) for all \( Z_1, \ldots, Z_k \in S_1 \).

(4) If \( h_4(t,x) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\}) \) is homogeneous of degree zero and supported in \( \{ (t,x) : 0 < t < C|x| < \varepsilon \} \) and \( \chi_2(\theta) \in C^\infty \) has compact support away \( \{ \theta = 1 \} \), then \( Z_1 \ldots Z_k h_4(t,x) \chi_2 \left( \frac{(m+2)|x|}{2t} \right) u \in L^\infty((0,T), H^s(\mathbb{R}^n)) \) for all \( Z_1, \ldots, Z_k \in S_1 \).

Note that \( h_1(t,x), h_2(t,x) \chi \left( \frac{(m+2)|x|}{2t} \right), h_3(t,x) \chi_1 \left( \frac{(m+2)|x|}{2t} \right), \) and \( h_4(t,x) \chi_2 \left( \frac{(m+2)|x|}{2t} \right) \) are admissible functions in the regions \( \Omega_1, \Omega_2, \Omega_3, \) and \( \Omega_4 \), respectively. Moreover, they belong to \( L^\infty((0,T), H^{n/2}(\mathbb{R}^n)) \).

Because some of the vector fields in Definition 3.6 (e.g., \( \tilde{V}_i^{(m)} \)), \( 1 \leq i \leq n \) do not admit good commutator relations with \( P_1 = \partial_t \left( \partial_t^2 - m^2 \Delta_x \right) \) (in the sense that not all coefficients appearing in the commutators are admissible), one has to look for additional relations among these vector fields. From now on, we will often write \( \tilde{V}_i \) (\( 1 \leq i \leq n \)) and \( \tilde{V}_0 \) instead of \( V_i^{(m)} \) and \( \tilde{V}_0^{(m)} \), respectively.

By a careful computation as in (21), one obtains the following result:

**Proposition 3.7.** Let \( \Omega_i (1 \leq i \leq 4) \) be as given in Definition 3.5. Then:

(1) With \( N_1^0 = |x| \partial_x, N_1^1 = t m^2/2 |x| \partial_t \), one has, for \( \nu = 0, \ldots, n \),

\[
(\bar{N}_1^1)^2 = h_1^{(0)} Q_m + h_1^{(1)} V_0^2 + \sum_{1 \leq i < k \leq n} a_{ik}\chi_{mil} L_{ik} \chi_{ml} + \sum_{1 \leq i < k \leq n} b_{ik} \chi_{mil} L_{ik} + \sum_{0 \leq i \leq n} r_i^{(0)} N_i^1 V_0
\]

\[
+ \sum_{0 \leq i \leq n} b_{i mil} N_i^1 L_{ml} + \sum_{0 \leq i \leq n} c_{i mil} N_i^1 + c_{i mil} V_0,
\]

where all the coefficients \( h_1^{(0)}, h_1^{(1)}, a_{ik}, b_{ik}, r_i^{(0)}, b_{i mil}, c_{i mil}, \) and \( c_{i mil} \) are admissible in \( \Omega_1 \).

(2) With \( N_2^1 = \left( |x| - \frac{m^2}{2 t} \right) \partial_t \), one has, for \( \nu = 1, \ldots, n \),

\[
\tilde{V}_i = \frac{1}{2 t} \left( (m+2) x_i V_0 - (m+2)^2 \sum_{k \neq i} x_k L_{ik} - (m+2) ((m+2) |x| + 2 t \frac{m^2}{2 t}) N_2^1 \right)
\]

and

\[
(\bar{N}_2^1)^2 = h_2^{(0)} Q_m + h_2^{(1)} V_0^2 + \sum_{1 \leq i < k \leq n} a_{ik}\chi_{mil} L_{ik} \chi_{ml} + \sum_{1 \leq i < k \leq n} b_{ik} \chi_{mil} L_{ik} + \sum_{1 \leq i \leq n} r_i^{(0)} N_i^2 V_0
\]

\[
+ \sum_{1 \leq i \leq n} b_{i mil} N_i^2 L_{ml} + \sum_{1 \leq i \leq n} c_{i mil} N_i^2 + c_{i mil} V_0,
\]

where all the coefficients \( h_2^{(0)}, h_2^{(1)}, a_{ik}, b_{ik}, r_i^{(0)}, b_{i mil}, c_{i mil}, \) and \( c_{i mil} \) are admissible in \( \Omega_2 \).

(3) With \( N_3^0 = t \partial_t \), one has

\[
\tilde{V}_i = \frac{2 t \frac{m^2}{2 t}}{(m+2)^2 |x|^2} V_0 + \frac{x_i \left((m+2)^2 |x|^2 - 4 m^2 \right)}{(m+2)^2 |x|^2 + \frac{m^2}{2 t}} \sum_{k \neq i} x_k L_{ik}, 1 \leq i \leq n,
\]

and

\[
(\bar{N}_3^0)^2 = h_3^{(0)} Q_m + h_3^{(1)} V_0^2 + \sum_{1 \leq i < k \leq n} a_{ik}\chi_{mil} L_{ik} \chi_{ml} + \sum_{1 \leq i < k \leq n} b_{ik} \chi_{mil} L_{ik} + r_0^{(0)} N_3^0 V_0.
\]
where all the coefficients $h^3_{0i}$, $h^3_{ik}$, $a^3_{ikml}$, $b^3_{ik}$, $r^3_{0i}$, $d^3_{0i}$, and $c^3_{0i}$ are admissible in $\Omega_3$.

(4) With $N^i_1 = t^{m+2} \partial_i (1 \leq i \leq n)$ in $\Omega_4$, one has, for $\nu = 1, \ldots, n$,

$$
V^\nu_\nu = \frac{(m + 2)x^\nu}{2t^{m+2}} V_0 + \frac{4t^{m+2} - (m + 2)^2|x|^2}{t^{m+2}} N^\nu_0 - \frac{(m + 2)^2}{2t^{m+2}} \sum_{k \neq \nu} x_k L_{\nu k}
$$

and

$$
(N^1_1)^2 = h^4_{0i} Q_m + b^4_{i} V_0^2 + \sum_{1 \leq i < k \leq n} \sum_{1 \leq m \leq n} a^4_{ikml} L_{lm} V_0 + \sum_{1 \leq i < k \leq n} b^4_{ik} V_0 L_{ik} + \sum_{1 \leq i \leq n} r^4_{i} N^1_1 V_0 + \sum_{1 \leq i < k \leq n} b^4_{i} N^1_1 L_{ik} + \sum_{1 \leq i \leq n} c^4_{i} N^1_1 + d^4_{i} V_0^2,
$$

where all the coefficients $h^4_{0i}$, $h^4_{i}$, $a^4_{ikml}$, $b^4_{ik}$, $r^4_{i}$, $b^4_{i}$, $c^4_{i}$, and $d^4_{i}$ are admissible in $\Omega_4$.

Remark 3.8. Admissibility of the coefficients in each of the regions $\Omega_i (1 \leq i \leq 4)$ refers to the vector fields in $\mathcal{S}_i$.

Proof. (1) It follows from a direct computation that

$$
(N^0_1)^2 = \frac{1}{4t^{m+2} - (m + 2)^2|x|^2} \left(-4|x|^2t^{m+2} Q_m - |x|^2 t^{m+2} \sum_{j=1}^n \bar{V}_j^2 + 4|x|t^{m+1} N^0_0 V_0 \right)
+ (m + 2)|x|^2 t^{m} V_0 + \left((2m+2)(n-1) - 8\right) t^{m+1} |x| - \frac{m(m+2)^2|x|^3}{2t} N^0_0,
$$

(3.9) $\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quart
+ f_2 x_i \sum_{j=1}^n x_j N_j^i V_{ij} + g_2 \sum_{k \neq i} x_k V_i L_{ik} + h_{21} V_0 + h_{22} N_j^i + \sum_{j \neq i} h_j^2 V_j,

where

\begin{align*}
a_2 &= 4t^3(x^2 - \frac{2}{m+2} t^{m+2})^2 \\
b_2 &= -t(x^2 - \frac{2}{m+2} t^{m+2})^2 ((m+2)^2 x^2 + 4t^{m+2})}
\end{align*}

\begin{align*}
c_2 &= 2m(m+2)(x^2 - \frac{2}{m+2} t^{m+2})
\end{align*}

\begin{align*}
ce_2 &= \frac{(m+2)(x^2 - \frac{2}{m+2} t^{m+2})^2 ((m+2)^2 x^2 + 4t^{m+2})}{2t^{m+2}((m+2)^2 x^2 - 4t^{m+2})^2},
\end{align*}

\begin{align*}
f_2 &= \frac{n(x^2 - \frac{2}{m+2} t^{m+2})^2 ((m+2)^2 x^2 + 4t^{m+2})}{2|x|^2 t^{m+2}((m+2)^2 x^2 - 4t^{m+2})^2},
\end{align*}

and \( h_{21}, h_{22}, h_j^2 \) are admissible in \( \Omega_2 \). Note that, for \( 1 \leq i \leq n \),

(3.12) \quad \hat{V}_i = \frac{1}{2t^{m+2}}((m+2)x_i V_0 - (m+2)^2 \sum_{k \neq i} x_k L_{ik} - (m+2)((m+2)x| + 2t^{m+2})N_j^i).

Then combining (3.11) and (3.12) yields (3.2) and (3.3).

(3) Since one has

(3.13) \quad (N_j^0)^2 = \frac{1}{4t^{m+2} - (m+2)^2 x^2} \left(-4t^{m+2} Q_m - (m+2)^2 \sum_{j=1}^n \hat{V}_{ij}^2 + 4t^{m+2} N_j^0 V_0 + (m+2)t^{m+2} V_0 \right.
\left. + 2(n - 1)(m+2)t^{m+2} - \frac{(4+m)(m+2)^2 x^2}{2} N_j^0 \right)

and

(3.14) \quad \hat{V}_i = \frac{2t^{m+2} x_i}{(m+2)|x|^2} V_0 + x_i ((m+2)^2 x^2 - 4t^{m+2}) \frac{N_j^0}{(m+2)|x|^2 t^{m+2}} \sum_{k \neq i} x_k L_{ik}, \quad i = 1, \ldots, n,

it follows from (3.13) and (3.14) that (3.4) and (3.5) hold.

(4) A direct computation yields for \( 1 \leq i \leq n \)

(3.15) \quad (N_j^i)^2 = a_4 Q_m + b_4 V_0^2 + c_4 x_i N_j^i V_0 + d_4 \sum_{j=1}^n \hat{V}_{ij}^2 + e_4 \sum_{k=1}^n x_k x_i V_0 V_k
\begin{align*}
+ f_4 x_i \sum_{j=1}^n x_j N_j^i V_{ij} + g_4 \sum_{k \neq i} x_k V_i L_{ik} + h_{41} V_0 + h_{42} N_j^i + \sum_{j \neq i} h_j^2 V_j,
\end{align*}

where

\begin{align*}
a_4 &= \frac{4t^{m+2}}{(m+2)^2 x^2 - 4t^{m+2}}, \quad b_4 = -t^{m+2} ((m+2)^2 x^2 + 4t^{m+2})
\end{align*}

\begin{align*}
c_4 &= 2n(m+2)t^{m+2}
\end{align*}

\begin{align*}
d_4 &= \frac{4}{(m+2)^2 x^2 - 4t^{m+2}},
\end{align*}

\begin{align*}
e_4 &= \frac{n(m+2)^2 x^2 + 4t^{m+2}}{2t^{m+2}((m+2)^2 x^2 - 4t^{m+2})},
\end{align*}

\begin{align*}
f_4 &= \frac{n(m+2)^2 x^2 + 4t^{m+2}}{2t^{m+2}((m+2)^2 x^2 - 4t^{m+2})},
\end{align*}

\begin{align*}
g_4 &= -\frac{nt^{m+2}}{2|x|^2},
\end{align*}

and \( h_{41}, h_{42}, h_j^i \) are admissible in \( \Omega_4 \). In addition,

(3.16) \quad \hat{V}_i = \frac{(m+2)x_i}{2t^{m+2}} V_0 + \frac{4t^{m+2} - (m+2)^2 x^2}{t^{m+2}} N_j^i - \frac{(m+2)^2 x^2}{2t^{m+2}} \sum_{k \neq i} x_k L_{ik}, \quad 1 \leq i \leq n.

Then substituting (3.16) into (3.15) yields (3.6) and (3.7).
LOW REGULARITY SOLUTIONS OF DEGENERATE HYPERBOLIC EQUATIONS

Next we define admissible tangent vector fields with respect to the hypersurfaces \(\Gamma^\pm_m \cup \Sigma_0\). As before, we denote \(V^{(m)}\) by \(V\).

**Definition 3.9** (Admissible tangent vector fields for \(\Gamma^\pm_m \cup \Sigma_0\)).

1. (Near \(\{t = 0\}\)) For \(W_1 = \{(t, x): 0 \leq t < C|x_1| \leq \varepsilon\}\), \(M_1\) denotes the Lie algebra of vector fields in \(W_1\) with admissible coefficients generated by \(x_1 \partial_1, x_i \partial_i, R_\ell\) (\(2 \leq \ell \leq n\)).

2. (Near \(\Gamma^\pm_m\)) For \(W_{2, \pm} = \{(t, x): 0 < |x_1| < C t \leq \varepsilon\} \cap \{(t, x): |x_1| \pm \frac{2 t^{m+2}}{m+2} < C t\}\), \(M_{2, \pm}\) denotes the Lie algebra of vector fields in \(W_{2, \pm}\) with admissible coefficients generated by \(V, \tilde{V}_1, R_\ell\) (\(2 \leq \ell \leq n\)).

3. (Near \(\Sigma_0\)) Let \(W_3 = \{(t, x): |x_1| < C t \leq \varepsilon\} \cap \{(t, x): t^{m+2} < C|x_1| \pm \frac{2 t^{m+2}}{m+2}\}\), \(M_3\) denotes the Lie algebra of vector fields in \(W_3\) with admissible coefficients generated by \(t \partial_t, V, \tilde{V}_1, R_\ell\) (\(2 \leq \ell \leq n\)).

4. (Between \(\Gamma^\pm_m\) and \(\Sigma_0\)) For \(W_4 = \{(t, x): \frac{C t^{m+2}}{m+2} < |x_1| < C t \leq \varepsilon\} \cap \{(t, x): t^{m+2} < C|x_1| \pm \frac{2 t^{m+2}}{m+2}\}\), \(M_4\) is the Lie algebra of vector fields in \(W_4\) with admissible coefficients generated by \(t \partial_t, t^{m+2} \partial_1, R_\ell\) (\(2 \leq \ell \leq n\)).

**Remark 3.10.** In \(W_{2, \pm}\), for computations we will also use the equivalent vector fields \(V, N_{2, \pm}, R_2, \ldots, R_n\) with \(N_{2, \pm} = \left(1 \mp \frac{2 t^{m+2}}{m+2}\right) \partial_1\) instead of \(V, \tilde{V}_1, R_2, \ldots, R_n\). This equivalence stems from the fact that

\[
N_{2, \pm} = \frac{t^{m+2}}{(m + 2)^2 \left(1 \pm \frac{2 t^{m+2}}{m+2}\right)^2 \left(1 \mp \frac{2 t^{m+2}}{m+2}\right)} \left(\frac{(m + 2)x_1}{t^{m+2}} - 1\right),
\]

\[
\tilde{V}_1 = \frac{m + 2}{2t^{m+2}} \left(\frac{1}{2} x_1 - \frac{2 t^{m+2}}{m+2}\right) N_{2, \pm},
\]

where all the coefficients are admissible in \(W_{2, \pm}\).

We similarly define the conormal space \(I^s H^s(\Gamma^\pm_m \cup \Sigma_0)\) for \(0 \leq s < 1/2\).

**Definition 3.11** (Conormal space \(I^s H^s(\Gamma^\pm_m \cup \Sigma_0)\)). Define \(u(t, x) \in I^s H^s(\Gamma^\pm_m)\) in \(t > 0\) if, away from \(\{t = x_1 = 0\}\) and near \(\Gamma^\pm_m, Z_1 \ldots Z_k u \in L^\infty((0, T), H^s(\mathbb{R}^n))\) for all smooth vector fields \(Z_1, \ldots, Z_k \in \{V, \tilde{V}_1, R_2, \ldots, R_n\}\), away from \(\{t = x_1 = 0\}\) and near \(\Sigma_0\), \(Z_1 \ldots Z_k u \in L^\infty((0, T), H^s(\mathbb{R}^n))\) for all smooth vector fields \(Z_1, \ldots, Z_k \in \{t \partial_t, V_0, L_{ij}, 1 \leq i < j \leq n\}\). Near \(\{t = x_1 = 0\}\), the following properties hold:

1. If \(h_1(t, x_1) \in C^\infty(\mathbb{R}^2 \setminus \{0\})\) is homogeneous of degree zero and supported in \(W_1 = \{(t, x_1): t < C|x_1| \leq \varepsilon\}\), then \(Z_1 \ldots Z_k h_1(t, x_1) u(t, x) \in L^\infty((0, T), H^s(\mathbb{R}^n))\) for all \(Z_1, \ldots, Z_k \in M_4\).

2. If \(h_2(t, x_1) \in C^\infty(\mathbb{R}^2 \setminus \{0\})\) is homogeneous of degree zero and supported in \(W_2 = \{t, x_1: 0 < |x_1| < C t \leq \varepsilon\}\) and \(\eta_{1, \pm}(\theta) \in C^\infty\) has compact support near \(\{\theta = \pm 1\}\), then \(Z_1 \ldots Z_k h_2(t, x_1) \eta_{1, \pm} \left(\frac{(m+2)x_1}{2t^{m+2}}\right) u(t, x) \in L^\infty((0, T), H^s(\mathbb{R}^n))\) for all \(Z_1, \ldots, Z_k \in M_{4, \pm}\).

3. If \(h_3(t, x_1) \in C^\infty(\mathbb{R}^2 \setminus \{0\})\) is homogeneous of degree zero and supported in \(W_3 = \{t, x_1: |x_1| < C t \leq \varepsilon\}\) and \(\eta_{2, \pm}(\theta) \in C^\infty\) has compact support away from \(\{\theta = \pm 1\}\), then \(Z_1 \ldots Z_k h_3(t, x_1) \eta_{2, \pm} \left(\frac{(m+2)x_1}{2t^{m+2}}\right) u(t, x) \in L^\infty((0, T), H^s(\mathbb{R}^n))\) for all \(Z_1, \ldots, Z_j \in M_3\).

4. If \(h_4(t, x_1) \in C^\infty(\mathbb{R}^2 \setminus \{0\})\) is homogeneous of degree zero and supported in \(W_4 = \{t, x_1: \frac{C t^{m+2}}{m+2} < |x_1| < C t \leq \varepsilon\}\) and \(\eta_{3, \pm}(\theta) \in C^\infty\) has compact support away from \(\{\theta = \pm 1\}\), then \(Z_1 \ldots Z_k h_4(t, x_1) \eta_{3, \pm} \left(\frac{(m+2)x_1}{2t^{m+2}}\right) u(t, x) \in L^\infty((0, T), H^s(\mathbb{R}^n))\) for all \(Z_1, \ldots, Z_j \in M_4\).

Note that the cut-off functions \(h_1, h_2 \eta_{1, \pm}, h_3 \eta_{2, \pm}, h_4 \eta_{3, \pm}\) are admissible in the regions \(W_1, W_{2, \pm}, W_3, W_4\) respectively. Moreover, they belong to the space \(L^\infty((0, T), H^{n/2-}(\mathbb{R}^n))\).

Similar to Proposition 3.7, one has:
Proposition 3.12. Let $W_1, W_2, W_3$, and $W_4$ be given as in Definition 3.9. Then one has:

1. With $N_1 = x_1 \partial_t$ in $W_1$,
\[
N_1^2 = \frac{1}{(m+2)^2 x_1^2 - 4t^{m+2}} \left( (m+2)^2 x_1^2 Q_m + x_1^2 t^m V^2 - 4x_1 t^{m+1} N_1 V \right) + (m+2)^2 x_1^2 \sum_{i=2}^n R_i^2 - (m+2)x_1^2 t^m V + 2(m+4)x_1 t^{m+1} N_1 \right) .
\]

2. With $N_{2,\pm} = (x_1 \pm \frac{2}{m+2} t^{\frac{m+2}{m+2}}) \partial_t$ in $W_{2,\pm}$,
\[
N_{2,\pm}^2 = \frac{x_1 \pm \frac{2}{m+2} t^{\frac{m+2}{m+2}}}{(m+2)^2 (x_1 \pm \frac{2}{m+2} t^{\frac{m+2}{m+2}})} \left( 4t^2 Q_m - V^2 + 4t^{m+2} \sum_{i=2}^n R_i^2 + 2V \right) + \frac{2x_1}{(m+2)(x_1 \pm \frac{2}{m+2} t^{\frac{m+2}{m+2}})} N_{2,\pm} V - \frac{2(x_1 \pm \frac{m+2}{m+2})}{(m+2)(x_1 \pm \frac{m+2}{m+2})} N_{2,\pm}.
\]

3. With $N_3 = t \partial_t$ in $W_3$,
\[
N_3^2 = \frac{1}{(m+2)^2 x_1^2 - 4t^{m+2}} \left( (m+2)^2 x_1^2 t^2 Q_m + t^{m+2} V^2 - 4t^{m+2} N_3 V + (m+2)^2 x_1^2 t^{m+2} \sum_{i=2}^n R_i^2 \right.
\]
\[
- (m+2)t^{m+2} V + ((m+2)^2 x_1^2 + 2(m+2)t^{m+2}) N_3 \right). \]

4. With $N_4 = t \frac{m+2}{m+2} \partial_t$ in $W_4$,
\[
N_4^2 = \frac{1}{(m+2)^2 x_1^2 - 4t^{m+2}} \left( 4t^{m+2} Q_m - t^{m+2} V^2 + 2(m+2)x_1 t^{\frac{m+2}{m+2}} N_4 V + 4t^{2(m+2)} \sum_{i=2}^n R_i^2 \right.
\]
\[
+ 2t^{m+2} V - 3(m+2)^2 x_1 t^{\frac{m+2}{m+2}} N_4 \right).
\]

Remark 3.13. As in Remark 3.8 one verifies that, in Proposition 3.12, all coefficients are admissible in the corresponding region with respect to the corresponding vector fields.

Without loss of generality, one can assume that $m_1 > m_2$ and $0 < t < T \leq 1$.

Definition 3.14 (Admissible tangent vector fields for $\Gamma_{m_1} \cup \Gamma_{m_2}$).

1. (Near $\{t = 0\}$) For $D_1 = \{(t, x) : t < C|x| \leq \varepsilon\}$, $X_1$ denotes the Lie algebra of vector fields in $D_1$ with admissible coefficients generated by $|x| \partial_t$, $t^{m_2/2} |x| \partial_t$ (1 \(\leq \ell \leq n\), $L_{ij}$ (1 \(\leq i < j \leq n\)).

2. (Near $\Gamma_{m_2}$) For $D_2 = \{(t, x) : 0 < |x| < Ct \leq \varepsilon\} \cap \{(t, x) : |x| + \frac{m_2}{m_2^2 + 2} |x|^{\frac{m_2^2}{m_2^2 + 2}} \leq Ct \frac{m_2^2}{m_2^2 + 2}\}$, $X_2$ denotes the Lie algebra of vector fields in $D_2$ with admissible coefficients generated by $V_0^{(m_2)}$, $\tilde{V}_k^{(m_2)}$ (1 \(\leq \ell \leq n\), $L_{ij}$ (1 \(\leq i < j \leq n\)).

3. (Between $\Gamma_{m_2}$ and $\Gamma_{m_1}$) Let $D_3 = \{(t, x) : 0 < |x| < Ct \leq \varepsilon\} \cap \{(t, x) : \frac{m_2^2}{m_2^2 + 2} \leq C|\tilde{x}| \leq \frac{2}{m_2^2 + 2} \frac{m_2^2}{m_2^2 + 2}\}$ \(\cap\) $\{(t, x) : \frac{m_2^2}{m_2^2 + 2} \leq C|\tilde{x}| \leq \frac{2}{m_2^2 + 2} \frac{m_2^2}{m_2^2 + 2}\}$, $X_3$ denotes the Lie algebra of vector fields in $D_3$ with admissible coefficients generated by $t \partial_t$, $t^{m_2 + 1} \partial_t$ (1 \(\leq \ell \leq n\), $L_{ij}$ (1 \(\leq i < j \leq n\)).

4. (Near $\Gamma_{m_1}$) For $D_4 = \{(t, x) : 0 < |x| < Ct \leq \varepsilon\} \cap \{(t, x) : \frac{m_1}{m_1^2 + 2} \leq C|\tilde{x}| \leq \frac{2}{m_1^2 + 2} \frac{m_1^2}{m_1^2 + 2}\}$ \(\cap\) $\{(t, x) : \frac{m_1}{m_1^2 + 2} \leq C|\tilde{x}| \leq \frac{2}{m_1^2 + 2} \frac{m_1^2}{m_1^2 + 2}\}$, $X_4$ is the Lie algebra of vector fields in $D_4$ with admissible coefficients generated by $V_0^{(m_1)}$, $\tilde{V}_k^{(m_1)}$ (1 \(\leq \ell \leq n\), $L_{ij}$ (1 \(\leq i < j \leq n\)).

5. (Inside $\Gamma_{m_1}$) Let $D_5$ be the region $\{(t, x) : \frac{m_1}{m_1^2 + 2} \leq C|\tilde{x}| \leq \frac{2}{m_1^2 + 2} \frac{m_1^2}{m_1^2 + 2}\}$ \(\cap\) $\{(t, x) : \frac{m_1}{m_1^2 + 2} \leq C|\tilde{x}| \leq \frac{2}{m_1^2 + 2} \frac{m_1^2}{m_1^2 + 2}\}$, $X_5$ be the Lie algebra of vector fields with admissible coefficients on $D_5$ generated by $\{t \partial_t, t^{m_1 + 1} \partial_t \leq \ell \leq n; L_{ij}, 1 \leq i < j \leq n\}$.

Then we define the conormal spaces $I^{\infty} H^s(\Gamma_{m_1} \cup \Gamma_{m_2})$ with $s < n/2$ and $m_1 > m_2$. 
Definition 3.15 (Conormal space $I^\infty H^s(\Gamma_{m_1} \cup \Gamma_{m_2})$). Define $u(t, x) \in I^\infty H^s(\Gamma_{m_1} \cup \Gamma_{m_2})$ in $t > 0$ if, away from $\{ t = |x| = 0 \}$ but near $\Gamma_m$, $(i = 1, 2)$, $Z_1 \ldots Z_k u \in L^\infty((0, T), H^s(\mathbb{R}^n))$ for all smooth vector fields $Z_1, \ldots, Z_k \in \{ V^{(m_1)}, V^{(m_2)}, 1 \leq \ell \leq n, L_{k,j}, 1 \leq k < j \leq n \}$, and near $\{ t = |x| = 0 \}$, the following properties hold:

(1) If $h_3(t, x) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ is homogeneous of degree zero and supported on $D_1 = \{ (t, x) : 0 < t < C|x| \leq \epsilon \}$, then $Z_1 \ldots Z_k h_3(t, x)(t, x) \in L^\infty((0, T), H^s(\mathbb{R}^n))$ for all $Z_1, \ldots, Z_k \in X_1$.

(2) If $h_3(t, x) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ is homogeneous of degree zero and supported on $\{ (t, x) : 0 < |x| < C \leq t \leq \epsilon \}$ and $\chi_0(\theta) \in C^\infty$ has compact support near $\{ \theta = 1 \}$, then $Z_1 \ldots Z_k h_3(t, x) \chi_0 \left( \frac{(m_2+2)|x|}{2t} \right) u(t, x) \in L^\infty((0, T), H^s(\mathbb{R}^n))$ for all $Z_1, \ldots, Z_k \in X_2$.

(3) If $h_3(t, x) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ is homogeneous of degree zero and supported on $\{ (t, x) : 0 < |x| < C \leq \epsilon \}$, and $\chi_1(\theta) \in C^\infty$ has compact support away $\{ \theta = 1 \}$, then $Z_1 \ldots Z_k h_3(t, x) \chi_1 \left( \frac{(m_2+2)|x|}{2t} \right) u(t, x) \in L^\infty((0, T), H^s(\mathbb{R}^n))$ for all $Z_1, \ldots, Z_j \in X_3$.

(4) If $h_3(t, x) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ is homogeneous of degree zero and supported on $\{ (t, x) : 0 < |x| < C \leq \epsilon \}$, and $\chi_3(\theta) \in C^\infty$ has compact support away $\{ \theta = 1 \}$, then $Z_1 \ldots Z_k h_3(t, x) \chi_3 \left( \frac{(m_2+2)|x|}{2t} \right) u(t, x) \in L^\infty((0, T), H^s(\mathbb{R}^n))$ for all $Z_1, \ldots, Z_j \in X_4$.

(5) If $h_3(t, x) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ is homogeneous of degree zero and supported on $\{ (t, x) : 0 < |x| < C \leq \epsilon \}$, and $\chi_5(\theta) \in C^\infty$ has compact support away $\{ \theta = 1 \}$, then $Z_1 \ldots Z_k h_3(t, x) \chi_5 \left( \frac{(m_2+2)|x|}{2t} \right) u(t, x) \in L^\infty((0, T), H^s(\mathbb{R}^n))$ for all $Z_1, \ldots, Z_j \in X_5$.

One similarly defines admissible tangent vector fields for $\Gamma^\pm_{m_1} \cup \Gamma^\pm_{m_2}$ and the conormal spaces $I^\infty H^s(\Gamma^\pm_{m_1} \cup \Gamma^\pm_{m_2})$ with $s < n/2$ and $m_1 > m_2$.

Definition 3.16 (Admissible tangent vector fields for $\Gamma^\pm_{m_1} \cup \Gamma^\pm_{m_2}$).

(1) (Near $\{ t = 0 \}$) For $E_1 = \{ (t, x) : t < C|x| \leq \epsilon \}$, $Y_1$ is the Lie algebra of vector fields in $E_1$ with admissible coefficients generated by $x_1 \partial_t, x_1 \partial_1, \partial_1 (2 \leq \ell \leq n)$.

(2) (Near $\Gamma^\pm_{m_2}$) For $E_2 = \{ (t, x) : 0 < |x| < C \leq \epsilon \} \cap \{ (t, x) : |x_1| + \frac{2}{m_2+2} \left( \frac{m_2+2}{2t} \right) \leq C \frac{m_2+2}{2t} \}$, $Y_2$ is the Lie algebra of vector fields in $E_2$ with admissible coefficients generated by $V^{(m_2)}, V^{(m_2)}, \partial_t (2 \leq \ell \leq n)$.

(3) (Between $\Gamma^\pm_{m_2}$ and $\Gamma^\pm_{m_1}$ or $\Gamma^\pm_{m_2}$ and $\Gamma^\pm_{m_1}$) For $E_3 = \{ (t, x) : 0 < |x| < C \leq \epsilon \} \cap \{ (t, x) : t^{m_2+2} \left( \frac{m_2+1}{m_2+2} \right) \leq C \frac{m_2+1}{m_2+2} \}$, $Y_3$ is the Lie algebra of vector fields in $E_3$ with admissible coefficients generated by $t \partial_t, t^{m_2+1} \partial_1, \partial_t (2 \leq \ell \leq n)$.

(4) (Near $\Gamma^\pm_{m_1}$) For $E_4 = \{ (t, x) : 0 < |x_1| < C \leq \epsilon \} \cap \{ (t, x) : t^{m_2+2} \left( \frac{m_2+1}{m_2+2} \right) \leq C \frac{m_2+1}{m_2+2} \}$, $Y_4$ is the Lie algebra of vector fields in $E_4$ with admissible coefficients generated by $V^{(m_1)}, V^{(m_1)}, \partial_1 (2 \leq \ell \leq n)$.

(5) (Between $\Gamma^\pm_{m_1}$ and $\Gamma^\pm_{m_2}$) For $E_5 = \{ (t, x) : 0 < |x_1| < C \leq \epsilon \} \cap \{ (t, x) : t^{m_2+2} \left( \frac{m_2+1}{m_2+2} \right) \leq C \frac{m_2+1}{m_2+2} \}$, $Y_5$ is the Lie algebra of vector fields in $E_5$ with admissible coefficients generated by $t \partial_t, t^{m_2+1} \partial_1, \partial_t (2 \leq \ell \leq n)$.

Then one defines the conormal spaces $I^\infty H^s(\Gamma^\pm_{m_1} \cup \Gamma^\pm_{m_2})$ with $-n/2 < s < n/2$ and $m_1 > m_2$.

Definition 3.17 (Conormal space $I^\infty H^s(\Gamma^\pm_{m_1} \cup \Gamma^\pm_{m_2})$). A function $u(t, x)$ defined for $t > 0$ belongs to $I^\infty H^s(\Gamma^\pm_{m_1} \cup \Gamma^\pm_{m_2})$ if, away from $\{ t = |x| = 0 \}$ but near $\Gamma^\pm_{m_1}$, $\Gamma^\pm_{m_2}$, $Z_1 \ldots Z_k u \in L^\infty((0, T), H^s(\mathbb{R}^n))$ for all smooth vector fields $Z_1, \ldots, Z_k \in \{ V^{(m_1)}, V^{(m_2)}, \partial_t, 2 \leq \ell \leq n \}$, and near $\{ t = |x| = 0 \}$, the following properties hold:

(1) If $h_3(t, x) \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ is homogeneous of degree zero and supported in $E_1 = \{ (t, x) : t < C|x_1| \leq \epsilon \}$, then $Z_1 \ldots Z_k h_3(t, x)(u(t, x)) \in L^\infty((0, T), H^s(\mathbb{R}^n))$ for all $Z_1, \ldots, Z_k \in Y_1$. 

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(2) If \( h_2(t, x_1) \in C^\infty(\mathbb{R}^2 \setminus \{0\}) \) is homogeneous of degree zero and supported in \( \{(t, x_1) : 0 < |x_1| < Ct \leq \varepsilon\} \) and \( x_{\pm}(\theta) \in C^\infty \) has compact support near \( \theta = \pm 1 \), then \( Z_1 \ldots Z_k \left( h_2(t, x_1) \chi_{\pm} \left( \frac{(m_2+2)x_1}{2t^{\frac{m_2+2}{2}}} \right) u \right) \in L^\infty((0, T), H^s(\mathbb{R}^n)) \) for all \( Z_1, \ldots, Z_k \in Y_2 \).

(3) If \( h_3(t, x_1) \in C^\infty(\mathbb{R}^2 \setminus \{0\}) \) is homogeneous of degree zero and supported in \( \{(t, x_1) : 0 < |x_1| < Ct \leq \varepsilon\}, x_{1,\pm}(\theta) \in C^\infty \) has compact support away from \( \{\theta = \pm 1\} \), then \( Z_1 \ldots Z_k \left( h_3(t, x_1) x_{1,\pm} \left( \frac{(m_2+2)x_1}{2t^{\frac{m_2+2}{2}}} \right) \chi_{2,\pm} \left( \frac{(m_2+2)x_1}{2t^{\frac{m_2+2}{2}}} \right) u \right) \in L^\infty((0, T), H^s(\mathbb{R}^n)) \) for all \( Z_1, \ldots, Z_k \in Y_3 \).

(4) If \( h_4(t, x_1) \in C^\infty(\mathbb{R}^2 \setminus \{0\}) \) is homogeneous of degree zero and supported in \( \{(t, x_1) : 0 < |x_1| < Ct \leq \varepsilon\}, x_{3,\pm}(\theta) \in C^\infty \) has compact support away from \( \{\theta = \pm 1\} \), then \( Z_1 \ldots Z_k \left( h_4(t, x_1) x_{3,\pm} \left( \frac{(m_2+2)x_1}{2t^{\frac{m_2+2}{2}}} \right) \chi_{4,\pm} \left( \frac{(m_2+2)x_1}{2t^{\frac{m_2+2}{2}}} \right) u \right) \in L^\infty((0, T), H^s(\mathbb{R}^n)) \) for all \( Z_1, \ldots, Z_k \in Y_4 \).

(5) If \( h_5(t, x_1) \in C^\infty(\mathbb{R}^2 \setminus \{0\}) \) is homogeneous of degree zero and supported in \( \{(t, x_1) : 0 < |x_1| < Ct \leq \varepsilon\}, x_{5,\pm}(\theta) \in C^\infty \) has compact support away from \( \{\theta = \pm 1\} \), then \( Z_1 \ldots Z_k \left( h_5(t, x_1) x_{5,\pm} \left( \frac{(m_2+2)x_1}{2t^{\frac{m_2+2}{2}}} \right) \chi_{6,\pm} \left( \frac{(m_2+2)x_1}{2t^{\frac{m_2+2}{2}}} \right) u \right) \in L^\infty((0, T), H^s(\mathbb{R}^n)) \) for all \( Z_1, \ldots, Z_k \in Y_5 \).

4. LOCAL EXISTENCE OF SOLUTIONS OF Eqs. (1.1) AND (1.2)

In this section, we will use the Banach fixed point theorem to obtain the local existence of low regularity solutions of (1.1) and (1.2). The method is to reduce both the third-order equation in (1.1) and the fourth-order equation in (1.2) to the corresponding Tricomi-type problem. Let us stress that the conditions on the initial data are much weaker than those in [21]. Thanks to Lemmas 2.3, 2.7, we are able to overcome the difficulties induced by the low regularity.

It is readily seen that problem (1.1) is equivalent to the second-order degenerate hyperbolic equation

\[
\begin{align*}
\partial_t^2 u - t^n \Delta_x u &= \varphi_2(x) + \int_0^t f(s, x, u) \, ds, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \\
u(0, x) &= \varphi_0(x), \quad \partial_t u(0, x) = \varphi_1(x).
\end{align*}
\]

which contains a nonlocal nonlinear term.

Let us first consider problem (4.1) under assumption (A3) which is easier to handle than assumption (A1).

**Theorem 4.1.** Let assumption (A3) hold. If \( f(t, x, u) \) satisfies the assumptions of Theorem 1.2, then there is a constant \( 0 < T \leq T_0 \) such that (4.1) has a local solution \( u \in L^\infty((0, T) \times \mathbb{R}^n) \cap C((0, T], H^{n/2+\frac{m_2}{m_2+2}}(\mathbb{R}^n)) \cap C^1([0, T], H^{n/2+\frac{m_2}{m_2+2}}(\mathbb{R}^n)) \cap C^1([0, T], H^{n/2+\frac{m_2}{m_2+2}}(\mathbb{R}^n)).

**Proof.** Observe that the \( \varphi_j(x) \) \((0 \leq j \leq 2)\) belong to \( H^{n/2}(\mathbb{R}^n) \) under assumption (A3). Let \( u_1(t, x) \) satisfy

\[
\begin{align*}
\partial_t^2 u_1 - t^n \Delta_x u_1 &= 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \\
u_1(0, x) &= \varphi_0(x), \quad \partial_t u_1(0, x) = \varphi_1(x).
\end{align*}
\]

For any small \( \delta > 0 \), it follows from Lemmas 2.3, 2.7 (choose \( s = n/2 - \delta \)) that

\[
u_1 \in L^\infty((0, T_0) \times \mathbb{R}^n)
\]

\( \cap C((0, T_0], H^{n/2-\delta}(\mathbb{R}^n)) \cap C((0, T_0], H^{n/2+\frac{m_2}{m_2+2}}(\mathbb{R}^n)) \cap C^1([0, T_0], H^{n/2+\frac{m_2}{m_2+2}}(\mathbb{R}^n)) \cap C^1([0, T_0], H^{n/2-\frac{m_2}{m_2+2}}(\mathbb{R}^n))\)

which satisfies, for any \( t \in (0, T_0] \),

\[
\begin{align*}
\|u_1(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} &\leq C(\delta) (1 + \ln t)^2, \\
\|u_1(t, \cdot)\|_{H^{n/2-\delta}(\mathbb{R}^n)} &+ t^{m/4} \|u_1(t, \cdot)\|_{H^{n/2-\frac{m_2}{m_2+2}}(\mathbb{R}^n)} \\
+ \|\partial_t u_1(t, \cdot)\|_{H^{n/2-\frac{m_2}{m_2+2}}(\mathbb{R}^n)} &\leq C(\delta).
\end{align*}
\]
Let \( u_2(t, x) \) be the solution of

\[
\frac{\partial_t^2 u_2 - t^m \Delta_x u_2}{u_2(0, x)} = \varphi_2(x) + \int_0^t f(s, x, 0) \, ds, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \\
u_2(0, x) = \partial_t u_2(0, x) = 0.
\]

From Lemmas 2.3(ii) and 2.6(i) one has that (choose \( s = n/2 - \delta \) and \( p_3 = \frac{m}{2(m+2)} \))

\[
u_2 \in L^\infty((0, T_0) \times \mathbb{R}^n) \\
cap C\left( (0, T_0], H^{n/2-\delta}(\mathbb{R}^n) \right) \cap C\left( (0, T_0], H^{n/2+\frac{m}{2(m+2)}-\delta}(\mathbb{R}^n) \right) \cap C^1\left( (0, T_0], H^{n/2-\frac{m}{2(m+2)}-\delta}(\mathbb{R}^n) \right)
\]

which satisfies, for any \( t \in (0, T_0] \),

\[
\|u_2(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} + \|u_2(t, \cdot)\|_{H^{n/2-\delta}(\mathbb{R}^n)} + t^{n/4} \|u_2(t, \cdot)\|_{H^{n/2+\frac{m}{2(m+2)}-\delta}(\mathbb{R}^n)} + \|\partial_t u_2(t, \cdot)\|_{H^{n/2-\frac{m}{2(m+2)}-\delta}(\mathbb{R}^n)} \leq C(\delta).
\]

Set \( v(t, x) = u(t, x) - u_1(t, x) - u_2(t, x) \). It follows from 4.1, 4.2, and 4.3 that \( v \) is a solution of

\[
\begin{aligned}
\frac{\partial_t^2 v - t^m \Delta_x v}{v(0, x)} &= \int_0^t \left( f(s, x, u_1 + u_2 + v) - f(s, x, 0) \right) ds, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \\
v(0, x) &= \partial_t v(0, x) = 0.
\end{aligned}
\]

For \( w \in C\left( (0, T], H^{n/2+p_0(m)-\delta}(\mathbb{R}^n) \right) \cap C^1\left( (0, T], H^{n/2-p_1(m)-\delta}(\mathbb{R}^n) \right) \cap C^1\left( (0, T], H^{n/2-\frac{m}{2(m+2)}+p_2(m)-\delta}(\mathbb{R}^n) \right) \), where \( 0 < T \leq T_0 \) and \( p > 1 \) is large, define

\[
\|w\| \equiv \sup_{0 \leq t \leq T} \|w(t, \cdot)\|_{H^{n/2+p_0(m)-\delta}(\mathbb{R}^n)} + \sup_{0 \leq t \leq T} \left( \frac{T^{1/p} (n+2+p_1(m))}{2(p+1)} \right)^{2/p} \|w(t, \cdot)\|_{H^{n/2+p_1(m)-\delta}(\mathbb{R}^n)} + \sup_{0 \leq t \leq T} \|\partial_t w(t, \cdot)\|_{H^{n/2-\frac{m}{2(m+2)}+p_2(m)-\delta}(\mathbb{R}^n)},
\]

where \( p_0(m) = \min \left\{ \frac{4p-2}{p+1}, 1 \right\}, p_1(m) = \min \left\{ \frac{2p-2}{p(m+2)}, 1 \right\}, \) and \( p_2(m) = \min \left\{ \frac{2p-1}{p(m+2)}, \frac{m}{2(m+2)} \right\} \). Let the set \( G \) be defined by

\[
G \equiv \left\{ w \in C\left( (0, T], H^{n/2+p_0(m)-\delta} \right) \cap C^1\left( (0, T], H^{n/2-p_1(m)-\delta} \right) \cap C^1\left( (0, T], H^{n/2-\frac{m}{2(m+2)}+p_2(m)-\delta} \right): \|w\| \leq 1 \right\}.
\]

For \( w \in G \), one has \( u_1 + u_2 + w \in L^\infty((0, T) \times \mathbb{R}^n) \cap L^q((0, T) \times \mathbb{R}^n) \) for all \( 1 < q < \infty \). Let

\[
(4.4) \quad E(f(t, x, u) - f(t, x, 0)) \equiv \left( \int_0^t \left( V_t(t, [\xi]) V_1(t, [\xi]) - V_1(t, [\xi]) \right) \left( V_2(t, [\xi]) \right) ds \right)^{1/q} \left( f(s, x, u(s, x)) - f(s, x, 0) \right) \, ds \right)^{1/q} (t, x)
\]

and define the nonlinear map \( F \) by

\[
F(w) = E(f(t, x, u_1 + u_2 + w) - f(t, x, 0)).
\]

We will show that \( F \) maps \( G \) into itself, and that it is a contraction for small \( T > 0 \).

By Lemma 2.3(i) (with \( p_1 = p_0(m) - \delta/2 < p_1(m) \)), the Hölder inequality, and the polynomial increase of \( f(t, x, u) \) with respect to the variable \( u \), for \( w \in G \) and \( T > 0 \) small, one has

\[
\|F(w)(t, \cdot)\|_{H^{n/2+p_0(m)-\delta}} \leq C(\delta) T^{2 - \frac{8p-4}{p(m+2)(m+2)} - \frac{m}{p+2} - \frac{1}{p}}.
\]
\[ \|F(t, u_1(t, \cdot) + u_2(t, \cdot) + w(t, \cdot)) - f(t, \cdot, 0)\|_{L^p((0,T), H^{n/2-\delta/2})} \leq C(\delta) T^{3 - \frac{(m+2)p(m+1)}{2} - \frac{1}{p} - \frac{1}{2} - \frac{1}{p}}. \]

For \( T > 0 \) small, one obtains that
\[
\|F(w)(t, \cdot)\|_{H^{n/2+p_2(m)-\delta}} \leq \frac{1}{3}. \tag{4.5}
\]

Moreover, by Lemma 2.3(i) (with \( p_1 = p_1(m) - \delta/2 \)), one has
\[
\|F(w)(t, \cdot)\|_{H^{n/2+p_2(m)-\delta}} \leq C T^{3 - \frac{(m+2)p(m+1)}{2} - \frac{1}{p} - \frac{1}{2} - \frac{1}{p}}.
\]

which yields, for \( T > 0 \) small and \( t \in (0, T] \),
\[
\frac{1}{p} + \frac{(m+2)p(m+1)}{2} - \frac{1}{p} \]

For \( p_2 = p_2(m) - \frac{\delta}{2} \) in Lemma 2.3(i) and \( T > 0 \) small, one has
\[
\|\partial_t F(w)(t, \cdot)\|_{H^{n/2-\delta}} \leq C T^{3 - \frac{(m+2)p(m+1)}{2} - \frac{1}{p} - \frac{1}{2} - \frac{1}{p}}. \tag{4.7}
\]

Collecting (4.5)–(4.7) yields, for \( T > 0 \) small,
\[
\|F(w)\| \leq 1, \tag{4.8}
\]

which shows that \( F \) maps \( G \) into \( G \).

Next we prove that the map \( F \) is strongly contractible for \( T > 0 \) small. For \( w_1, w_2 \in G \), in view of
\[
f(\tau, x, u_1 + u_2 + w_1) - f(\tau, x, u_1 + u_2 + w_2) = \int_0^1 f'(\tau, x, u_1 + u_2 + \theta w_1 + (1 - \theta) w_2) w_1 - w_2 \, d\theta,
\]

by a direct computation as for (4.5)–(4.7) one has that, for \( T > 0 \) small,
\[
\|F(w_1) - F(w_2)\| = \|E(f(t, x, u_1 + u_2 + w_1(\cdot)) - E(f(t, x, u_1 + u_2 + w_2(\cdot))\|
\leq C T^{(m+2)p(m+1)} \| w_1 - w_2 \| \leq \frac{1}{T} \| w_1 - w_2 \|. \tag{4.9}
\]

Thus, by the Banach fixed-point theorem and (4.8)–(4.9), we have completed the proof of Theorem 4.1. \( \square \)

Next, we prove the local existence of solutions of (1.1) under assumption (A2).

**Theorem 4.2.** Under assumption (A2), there is a constant \( T > 0 \) such that (1.1) has a local bounded solution
\[
u \in L^\infty((0,T) \times \mathbb{R}^n) \cap C([0, T], H^{1/2-\delta} \cap (0,T], H^{\frac{m+1}{2}-\delta}(\mathbb{R}^n)) \cap C^1([0, T], H^{\frac{m+1}{2}-\delta}(\mathbb{R}^n)).
\]

**Proof.** Let \( u_1(t, x) \) and \( u_2(t, x) \) be defined as in (4.2) and (4.3), respectively. Then, for any fixed \( \delta > 0 \) with
\[
\delta < \frac{1}{2(m+2)}, \]

one infers from \( \varphi_j \in H^{1/2-\delta}(\mathbb{R}^n)(0 \leq j \leq 2) \), Lemma 2.2 and Lemma 2.1(with \( s = 1/2 - \delta \)) that
\[
u(t, x) \in L^\infty((0,T) \times \mathbb{R}^n) \cap C([0, T], H^{1/2-\delta}(\mathbb{R}^n)) \cap C([0, T], H^{\frac{m+1}{2}-\delta}(\mathbb{R}^n)) \cap C^1([0, T], H^{\frac{m+1}{2}-\delta}(\mathbb{R}^n)).
\]
which satisfies, for $t \in (0, T]$,
\begin{equation}
\begin{aligned}
&\quad \|u_1(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C, \\
&\quad \|u_1(t, \cdot)\|_{H^{1/2-\delta}(\mathbb{R}^n)} + \|u_1(t, \cdot)\|_{H^{-s/4,4}(\mathbb{R}^n)} + \|\partial_t u_1(t, \cdot)\|_{H^{-s,4}(\mathbb{R}^n)} \leq C(\delta),
\end{aligned}
\end{equation}
and from Lemma 2.4 and Lemma 2.2 (with $s = 1/2 - \delta$ and $p_1 = \frac{n}{2(n+2)}$),
\begin{equation}
u_2(t, x) \in L^\infty((0, T) \times \mathbb{R}^n)
\end{equation}
\begin{equation}
\cap C([0, T], H^{1/2-\delta}(\mathbb{R}^n)) \cap C((0, T], H^{-s/4,4}(\mathbb{R}^n)) \cap C^1([0, T], H^{-s,4}(\mathbb{R}^n))
\end{equation}
which satisfies, for $t \in (0, T]$,
\begin{equation}
\begin{aligned}
&\quad \|u_2(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} + \|u_2(t, \cdot)\|_{H^{1/2-\delta}(\mathbb{R}^n)} + \|u_2(t, \cdot)\|_{H^{-s/4,4}(\mathbb{R}^n)} + \|\partial_t u_2(t, \cdot)\|_{H^{-s,4}(\mathbb{R}^n)} \leq C(\delta).
\end{aligned}
\end{equation}
Because of $[\partial^2_{\alpha} - t^n \Delta_x] = 0$, one obtains higher regularity of $u_1(t, x)$ and $u_2(t, x)$ in the $x'$ directions. In fact, for any $|\alpha| \geq 1$,}
\begin{equation}
\begin{aligned}
&\quad (\partial^2_{\alpha} - t^n \Delta_x)(\partial^0_{\alpha} u_1) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \\
&\quad \partial^0_{\alpha} u_1(0, x) = \partial^0_{\alpha} \varphi_0(x), \quad \partial_t \partial^0_{\alpha} u_1(0, x) = \partial^0_{\alpha} \partial^1 \varphi_1(x)
\end{aligned}
\end{equation}
which gives
\begin{equation}
\begin{aligned}
&\quad \partial^0_{\alpha} u_1(t, x) \in L^\infty((0, T) \times \mathbb{R}^n) \\
&\quad \cap C([0, T], H^{1/2-\delta}(\mathbb{R}^n)) \cap C((0, T], H^{-s/4,4}(\mathbb{R}^n)) \cap C^1([0, T], H^{-s,4}(\mathbb{R}^n)),
\end{aligned}
\end{equation}
and, for any $t \in (0, T]$,}
\begin{equation}
\begin{aligned}
&\quad \|\partial^0_{\alpha} u_1(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha \\
&\quad \|\partial^0_{\alpha} u_1(t, \cdot)\|_{H^{1/2-\delta}(\mathbb{R}^n)} + \|\partial^0_{\alpha} u_1(t, \cdot)\|_{H^{-s/4,4}(\mathbb{R}^n)} + \|\partial_t \partial^0_{\alpha} u_1(t, \cdot)\|_{H^{-s,4}(\mathbb{R}^n)} \leq C_\alpha(\delta).
\end{aligned}
\end{equation}
Furthermore, for $|\alpha| \geq 1$,}
\begin{equation}
\begin{aligned}
&\quad (\partial^2_{\alpha} - t^n \Delta_x)(\partial^0_{\alpha} u_2) = \partial^0_{\alpha} \varphi_2(x) + \int_0^t (\partial^0_{\alpha} f)(s, x, 0) ds, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \\
&\quad \partial^0_{\alpha} u_2(t, 0, x) = 0, \quad \partial_t \partial^0_{\alpha} u_2(t, 0, x) = 0
\end{aligned}
\end{equation}
which gives
\begin{equation}
\begin{aligned}
&\quad \partial^0_{\alpha} u_2(t, x) \in L^\infty((0, T) \times \mathbb{R}^n) \\
&\quad \cap C([0, T], H^{1/2-\delta}(\mathbb{R}^n)) \cap C((0, T], H^{-s/4,4}(\mathbb{R}^n)) \cap C^1([0, T], H^{-s,4}(\mathbb{R}^n))
\end{aligned}
\end{equation}
and, for any $t \in (0, T]$,}
\begin{equation}
\begin{aligned}
&\quad \|\partial^0_{\alpha} u_2(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} + \|\partial^0_{\alpha} u_2(t, \cdot)\|_{H^{1/2-\delta}(\mathbb{R}^n)} + \|\partial^0_{\alpha} u_2(t, \cdot)\|_{H^{-s/4,4}(\mathbb{R}^n)} + \|\partial_t \partial^0_{\alpha} u_2(t, \cdot)\|_{H^{-s,4}(\mathbb{R}^n)} \leq C(\delta).
\end{aligned}
\end{equation}
Set $v(t, x) = u(t, x) - u_1(t, x) - u_2(t, x)$. Then one has from (4.11) that
\begin{equation}
\begin{aligned}
&\quad \partial^2_{\alpha} v - t^n \Delta_x v = \int_0^t (f(s, x, u_1 + u_2 + v) - f(s, x, 0)) ds, \\
&\quad v(0, x) = \partial_t v(0, x) = 0.
\end{aligned}
\end{equation}
In order to solve (4.11), it suffices to solve (4.13). This requires to establish an $a priori$ $L^\infty$ bound on $\partial^0_{\alpha} v$ in (4.15) for $|\alpha| \leq [n/2] + 1$. To this end, motivated by Lemma 2.3 and Lemma 2.4 one should establish $\partial^0_{\alpha} \partial^\beta v \in L^\infty((0, T), H^s(\mathbb{R}^n))$ with some constant $s > 1/2$ and $|\beta| \leq [n/2] + 1$.}
Applying $\partial^\gamma_x (|\gamma| \leq 2 \lfloor n/2 \rfloor + 2)$ on both sides of (4.15) yields
\[
\begin{aligned}
\partial^\gamma_x v - t^m \Delta_x \partial^\gamma_x v = F_s (t, x, \partial^\gamma_x v)_{|\alpha| \leq |\gamma|} \\
\equiv \sum_{|\beta| + |t| \leq |\gamma|} C_{\beta t} \int_0^t \left( (\partial^\beta_x f)(s, x, u_1 + u_2 + v) - (\partial^\beta_x f)(s, x, 0) \right) \\
\times \partial^t_x f(s, x, u_1 + u_2 + v) \prod_{\beta_t \leq |t|, \beta_t = 1} \partial^{\beta_t} (u_1 + u_2 + v) \, ds,
\end{aligned}
\]
(4.16)
If
\[
\sum_{|\alpha| \leq \lfloor n/2 \rfloor + 1} \| \partial^\alpha_x v \|_{L^\infty ((0, T) \times \mathbb{R}^n)} + \sum_{|\gamma| \leq 2 \lfloor n/2 \rfloor + 2} \| \partial^\gamma_x v \|_{L^\infty ((0, T), H^s (\mathbb{R}^n))} \leq 2,
\]
where $s = s \left( 2\lfloor n/2 \rfloor + 1/2 - \delta > 1/2 \right)$ for $\delta > 0$ small and $T \leq 1$, then by Lemma 2.6, (4.12), and (4.14), one has from (4.16) that, for $T > 0$ small,
\[
\sum_{|\alpha| \leq \lfloor n/2 \rfloor + 1} \| \partial^\alpha_x v \|_{L^\infty ((0, T) \times \mathbb{R}^n)} \leq 1.
\]
(4.17)
Relying the preparations above, we will now use the Banach fixed-point theorem to establish Theorem 4.2.
For $w \in L^\infty ((0, T) \times \mathbb{R}^n) \cap C([0, T], H^{1/2 + p_0(\eta - \delta)}) \cap C((0, T), H^{1/2 + p_0(\eta - \delta)}) \cap C^1 ([0, T], \mathbb{H}^{1/2 + p_0(\eta - \delta)} \cap C^1 ([0, T], H^{1/2 + p_0(\eta - \delta)} \cap C^1 ([0, T], H^{1/2 + p_0(\eta - \delta)}) \cap C^1 ([0, T], H^{1/2 + p_0(\eta - \delta)}) \cap C^1 ([0, T], H^{1/2 + p_0(\eta - \delta)}) \cap C^1 ([0, T], H^{1/2 + p_0(\eta - \delta)}) \cap C^1 ([0, T], H^{1/2 + p_0(\eta - \delta)})) \cap C^1 ([0, T], H^{1/2 + p_0(\eta - \delta)})) \cap C^1 ([0, T], H^{1/2 + p_0(\eta - \delta)})) \cap C^1 ([0, T], H^{1/2 + p_0(\eta - \delta)})) \cap C^1 ([0, T], H^{1/2 + p_0(\eta - \delta)})$, define
\[
\| w \| = \sum_{|\alpha| = 0}^{\lfloor n/2 \rfloor + 1} \| \partial^\alpha_x w \|_{L^\infty ((0, T) \times \mathbb{R}^n)} + \sup_{0 \leq t \leq T} \sum_{|\gamma| = 0}^{2 \lfloor n/2 \rfloor + 2} \| \partial^\gamma_x w (t, \cdot) \|_{H^{1/2 + p_0(\eta - \delta)} (\mathbb{R}^n)}
+ \sup_{0 \leq t \leq T} \sum_{|\gamma| = 0}^{2 \lfloor n/2 \rfloor + 2} \| \partial^\gamma_x w (t, \cdot) \|_{H^{1/2 + p_0(\eta - \delta)} (\mathbb{R}^n)}
\]
(4.18)
The set $Q$ is defined by
\[
Q \equiv \left\{ w \in L^\infty ((0, T) \times \mathbb{R}^n) \cap C([0, T], H^{1/2 - \delta} (\mathbb{R}^n)) \cap C^1 ([0, T], H^{1/2 - \delta} (\mathbb{R}^n)) : \| w \| \leq 2 \right\}.
\]
Further define a nonlinear map $F$ by
\[
F(w) = E (f(t, x, u_1 + u_2 + w) - f(t, x, 0)),
\]
where the operator $E$ has been introduced in (4.4).
As in the proof of Theorem 4.1 we now show that $F$ maps $Q$ into itself and that it is strongly contractible for $T > 0$ small. Indeed, $F(w)$ for $w \in Q$ solves the problem
\[
\begin{aligned}
\left\{ \begin{array}{l}
(\partial^\gamma_x - t^m \Delta_x) F(w) = \int_0^t \left( f(s, x, u_1 + u_2 + v) - f(s, x, 0) \right) \, ds,
\\
F(w)|_{t = 0} = \partial_t F(w)|_{t = 0} = 0.
\end{array} \right.
\end{aligned}
\]
From (4.17) one concludes that, for $T > 0$ small,
\[
\sum_{|\alpha| = 0}^{\lfloor n/2 \rfloor + 1} \| \partial^\alpha_x F(w) (t, x) \|_{L^\infty ((0, T) \times \mathbb{R}^n)} \leq 1.
\]
(4.18)
Similar to the proof of Theorem 4.1 one has, for $T > 0$ small,
\[
\sup_{0 \leq t \leq T} \sum_{|\gamma| = 0}^{2 \lfloor n/2 \rfloor + 2} \| \partial^\gamma_x w(t, \cdot) \|_{H^{1/2 + p_0(\eta - \delta)} (\mathbb{R}^n)}
\]
(4.19)
\[ u(4.22) \cap g \quad \text{and} \quad (4.20) \quad (4.23) \]

where is a constant i.e.,

\[ w \quad \text{F maps } Q \text{ into itself.} \]

Invoking the Banach fixed-point theorem completes the proof of Theorem 4.2 \qed

Next we study the local existence of solutions of (1.2).

**Theorem 4.3.** (i) Under assumption \( (A_2) \), if \( f(t, x, u) \) satisfies the assumption in Theorem 4.2 then there is a constant \( T > 0 \) such that \( (1.2) \) has a local solution \( u \in L^\infty_{\text{loc}}((0, T] \times \mathbb{R}^n) \cap C([0, T], H^{n/2-}\mathcal{R}^{m_+1}\mathcal{R}^{m_+2}\mathcal{R}^{m_+3}\mathcal{R}^{m_+0}=0) \cap C^1([0, T], H^{n/2-}\mathcal{R}^{m_+1}\mathcal{R}^{m_+2}\mathcal{R}^{m_+3}\mathcal{R}^{m_+0}=0) \cap C((0, T], H^{n/2-}\mathcal{R}^{m_+1}\mathcal{R}^{m_+2}\mathcal{R}^{m_+3}\mathcal{R}^{m_+0}=0) \cap C^1((0, T], H^{n/2-}\mathcal{R}^{m_+1}\mathcal{R}^{m_+2}\mathcal{R}^{m_+3}\mathcal{R}^{m_+0}=0) \]

(ii) Under assumption \( (A_3) \), (1.2) has a unique local solution \( u \in L^\infty((0, T) \times \mathbb{R}^n) \cap C([0, T], H^{1/2-}(\mathbb{R}^n)) \cap C((0, T], H^{1/2-}(\mathbb{R}^n)) \]

\[ \quad \cap C^1([0, T], H^{1/2-}(\mathbb{R}^n)) \cap C^1((0, T], H^{1/2-}(\mathbb{R}^n)). \]

**Proof.** (i) Let \( u_1 \) satisfy

\[ (\partial_t^2 - t^{m_1}\Delta_x) (\partial_t^2 - t^{m_2}\Delta_x) u_1(t, x) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \]

\[ u_1(0, x) = \varphi_0(x), \quad \partial_t u_1(0, x) = \varphi_1(x), \quad \partial_t^2 u_1(0, x) = \partial_t^2 u_1(0, x) = 0 \]

and \( u_2(t, x) \) satisfy

\[ (\partial_t^2 - t^{m_1}\Delta_x) (\partial_t^2 - t^{m_2}\Delta_x) u_2(t, x) = f(t, x, 0), \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \]

\[ u_2(0, x) = \partial_t u_2(0, x) = 0, \quad \partial_t^2 u_2(0, x) = \varphi_2(x), \quad \partial_t^3 u_2(0, x) = \varphi_3(x). \]

Then \( u_1 \) is the solution of

\[ (\partial_t^2 - t^{m_2}\Delta_x) u_1(t, x) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \]

\[ u_1(0, x) = \varphi_0(x), \quad \partial_t u_1(0, x) = \varphi_1(x), \]

while \( u_2 \) is the solution of

\[ (\partial_t^2 - t^{m_2}\Delta_x) u_2(t, x) = g_1(t, x) + g_2(t, x), \quad (t, x) \in [0, +\infty) \times \mathbb{R}^n, \]

\[ u_2(0, x) = \partial_t u_2(0, x) = 0, \]

where \( g_1(t, x) \) satisfies

\[ (\partial_t^2 - t^{m_1}\Delta_x) g_1(t, x) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \]

\[ g_1(0, x) = \varphi_2(x), \quad \partial_t g_1(0, x) = \varphi_3(x) \]

and \( g_2(t, x) \) satisfies

\[ (\partial_t^2 - t^{m_1}\Delta_x) g_2(t, x) = f(t, x, 0), \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \]

\[ g_2(0, x) = \partial_t g_2(0, x) = 0. \]

From (4.22), one has by Lemma 4.5 and Lemma 4.6 that

\[ u_1 \in L^\infty_{\text{loc}}((0, T] \times \mathbb{R}^n) \]

\[ \cap C([0, T], H^{n/2-}(\mathbb{R}^n)) \cap C((0, T], H^{n/2-}(\mathbb{R}^n)) \cap C_1([0, T], H^{n/2-}(\mathbb{R}^n)). \]
which satisfies, for any $t \in (0, T]$, 
\[
\begin{align*}
&\left\| u_1(t, \cdot) \right\|_{L^{\infty}(\mathbb{R}^n)} \leq C(1 + |\ln t|)^2, \\
&\left\| u_1(t, \cdot) \right\|_{H^{3/2-\delta}(\mathbb{R}^n)} + t^{m_2/4} \left\| u_1(t, \cdot) \right\|_{H^{\gamma+1/2-\delta}(\mathbb{R}^n)} + \left\| \partial_t u_1(t, \cdot) \right\|_{H^{\gamma+1/2-\delta}(\mathbb{R}^n)} \leq C(\delta).
\end{align*}
\]

Further Lemma 2.6 implies that $g_1 \in C([0, T], H^{n/2-\delta}(\mathbb{R}^n))$. In addition, one has from Lemma 2.3 that $g_2 \in C([0, T], H^{n/2-\delta}(\mathbb{R}^n))$. Therefore, in view of Lemma 2.6 and Lemma 2.3 one obtains that 
\[
u_2(t, x) \in L^{\infty}((0, T) \times \mathbb{R}^n)
\]
\[\cap C([0, T], H^{1/2-\delta}(\mathbb{R}^n)) \cap C((0, T], H^{\frac{m_1+1}{m_2}+\delta}(\mathbb{R}^n)) \cap C^1([0, T], H^{\frac{m_1+1}{m_2}+\delta}(\mathbb{R}^n))
\]
which satisfies, for any $t \in [0, T]$, 
\[
\begin{align*}
&\left\| u_2(t, \cdot) \right\|_{L^{\infty}(\mathbb{R}^n)} + \left\| u_2(t, \cdot) \right\|_{H^{3/2-\delta}(\mathbb{R}^n)} + t^{m_2/4} \left\| u_2(t, \cdot) \right\|_{H^{\gamma+1/2-\delta}(\mathbb{R}^n)} \\
&+ \left\| \partial_t u_2(t, \cdot) \right\|_{H^{\gamma+1/2-\delta}(\mathbb{R}^n)} \leq C(\delta).
\end{align*}
\]

Set $w(t, x) = u(t, x) - u_1(t, x) - u_2(t, x)$. Then one has from (4.22), (4.20), and (4.21) that 
\[
\begin{align*}
&\left\{ \begin{array}{l}
(\partial_t^2 - t^m \Delta_x) w(t, x) \\
= f(t, x, u_1 + u_2 + w) - f(t, x, 0),
\end{array} \right. \\
&\text{for } t \in (0, +\infty) \times \mathbb{R}^n,
\end{align*}
\]
which implies that $w$ satisfies 
\[
\begin{align*}
&\left\{ \begin{array}{l}
(\partial_t^2 - t^{m_2} \Delta_x) w(t, x) = f(t, x, u(t, x)) - f(t, x, 0),
\end{array} \right. \\
&w(0, x) = \partial_t w(0, x) = 0,
\end{align*}
\]
where 
\[
F(s, t, u) = \mathcal{F}^{-1}_{\xi}(\{V_2(t, |\xi|) V_1(s, |\xi|) - V_1(t, |\xi|) V_2(s, |\xi|)\} \mathcal{F}_x(f(t, u(t, x)) - f(t, x, 0))(\xi))(s, t, x).
\]

Then following the arguments as in the proof of Theorem 4.1 one obtains by the Banach fixed-point theorem that there is a unique solution $w$ of (4.25) such that $w \in C([0, T], H^{n/2+\delta(\gamma+1/2-\delta)}(\mathbb{R}^n)) \cap C^1([0, T], H^{n/2+\delta(\gamma+1/2-\delta)}(\mathbb{R}^n))$. This shows that (1.2) has a local solution $u \in C([0, T], H^{n/2-\delta}(\mathbb{R}^n)) \cap C((0, T], H^{n/2+\delta(\gamma+1/2-\delta)}(\mathbb{R}^n)) \cap C^1([0, T], H^{n/2-\delta}(\mathbb{R}^n))$.

(ii) Let $u_1$ and $u_2$ be defined as in (4.20) and (4.21), respectively. Then one infers from (4.22), Lemma 2.7 and Lemma 2.3 that 
\[
u_1(t, x) \in L^{\infty}((0, T) \times \mathbb{R}^n)
\]
\[\cap C([0, T], H^{1/2-\delta}(\mathbb{R}^n)) \cap C((0, T], H^{\frac{m_1+1}{m_2}+\delta}(\mathbb{R}^n)) \cap C^1([0, T], H^{\frac{m_1+1}{m_2}+\delta}(\mathbb{R}^n))
\]
which satisfies, for any $t \in (0, T]$, 
\[
\begin{align*}
&\left\| u_1(t, \cdot) \right\|_{L^{\infty}(\mathbb{R}^n)} \leq C, \\
&\left\| u_1(t, \cdot) \right\|_{H^{3/2-\delta}(\mathbb{R}^n)} + t^{m_2} \left\| u_1(t, \cdot) \right\|_{H^{\gamma+1/2-\delta}(\mathbb{R}^n)} + \left\| \partial_t u_1(t, \cdot) \right\|_{H^{\gamma+1/2-\delta}(\mathbb{R}^n)} \leq C(\delta),
\end{align*}
\]
and from (4.23), Lemma 2.6 and Lemma 2.3 that 
\[
u_2(t, x) \in L^{\infty}((0, T) \times \mathbb{R}^n)
\]
\[\cap C([0, T], H^{1/2-\delta}(\mathbb{R}^n)) \cap C((0, T], H^{\frac{m_1+1}{m_2}+\delta}(\mathbb{R}^n)) \cap C^1([0, T], H^{\frac{m_1+1}{m_2}+\delta}(\mathbb{R}^n))
\]
which satisfies, for any $t \in [0, T]$, 
\[
\begin{align*}
&\left\| u_2(t, \cdot) \right\|_{L^{\infty}(\mathbb{R}^n)} + \left\| u_2(t, \cdot) \right\|_{H^{3/2-\delta}(\mathbb{R}^n)} + t^{m_2} \left\| u_2(t, \cdot) \right\|_{H^{\gamma+1/2-\delta}(\mathbb{R}^n)} + \left\| \partial_t u_2(t, \cdot) \right\|_{H^{\gamma+1/2-\delta}(\mathbb{R}^n)} \leq C(\delta).
\end{align*}
\]
Because of $[\partial^n_{0x}(\partial^2_t - t^{m1}\Delta_x)(\partial^2_t - t^{m2}\Delta_x)] = 0$ and $[\partial^n_{0x}, \partial^2_t - t^{m1}\Delta_x] = 0$ ($j = 1, 2$), one also has that, for any $|\alpha| \geq 1$,

$$\begin{align*}
\left\{ \begin{array}{ll}
(\partial^2_t - t^{m2}\Delta_x)(\partial^n_{0x}u_1) = 0, & (t, x) \in (0, +\infty) \times \mathbb{R}^n,
\partial^n_{0x}u_1(0, x) = \varphi_0(x),
\end{array} \right.
\end{align*}$$

and

$$\begin{align*}
\left\{ \begin{array}{ll}
(\partial^2_t - t^{m2}\Delta_x)(\partial^n_{0x}u_2)(t, x) = \partial^n_{0x}g_1(t, x) + \partial^n_{0x}g_2(t, x), & (t, x) \in (0, +\infty) \times \mathbb{R}^n,
\partial^n_{0x}u_2(0, x) = \partial_t\partial^n_{0x}u_2(0, x) = 0,
\end{array} \right.
\end{align*}$$

where $\partial^n_{0x}g_1(t, x)$ satisfies

$$\begin{align*}
\left\{ \begin{array}{ll}
(\partial^2_t - t^{m2}\Delta_x)\partial^n_{0x}g_1(t, x) = 0, & (t, x) \in (0, +\infty) \times \mathbb{R}^n,
\partial^n_{0x}g_1(0, x) = \varphi_2(x),
\partial_t\partial^n_{0x}g_1(0, x) = \varphi_3(x)
\end{array} \right.
\end{align*}$$

while $\partial^n_{0x}g_2(t, x)$ satisfies

$$\begin{align*}
\left\{ \begin{array}{ll}
(\partial^2_t - t^{m2}\Delta_x)\partial^n_{0x}g_2(t, x) = f(t, x), & (t, x) \in (0, +\infty) \times \mathbb{R}^n,
\partial^n_{0x}g_2(0, x) = \partial_t\partial^n_{0x}g_2(0, x) = 0.
\end{array} \right.
\end{align*}$$

Then

$$\begin{align*}
\partial^n_{0x}u_1(t, x) & \in L^\infty((0, T) \times \mathbb{R}^n),
\cap C([0, T], H^{1/2-\delta}(\mathbb{R}^n)) \cap C([0, T], H^{m+1/2}\leq t<1}(\mathbb{R}^n)) \cap C^1([0, T], H^{-1/2-\delta}(\mathbb{R}^n))
\end{align*}$$

which satisfies, for any $t \in [0, T]$,

$$\begin{align*}
\|\partial^n_{0x}u_1(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} & \leq C,
\|\partial^n_{0x}u_1(t, \cdot)\|_{H^{1/2-\delta}\leq t<1}(\mathbb{R}^n) & + t^{m/4}\|\partial^n_{0x}u_1(t, \cdot)\|_{H^{m+1/2}\leq t<1}(\mathbb{R}^n) & + \|\partial_t\partial^n_{0x}u_1(t, \cdot)\|_{H^{-1/2-\delta}(\mathbb{R}^n)} \leq C(\delta),
\end{align*}$$

while

$$\begin{align*}
\partial^n_{0x}u_2(t, x) & \in L^\infty((0, T) \times \mathbb{R}^n),
\cap C([0, T], H^{1/2-\delta}(\mathbb{R}^n)) \cap C([0, T], H^{m+1/2}\leq t<1}(\mathbb{R}^n)) \cap C^1([0, T], H^{-1/2-\delta}(\mathbb{R}^n))
\end{align*}$$

which satisfies, for any $t \in [0, T]$,

$$\begin{align*}
\|\partial^n_{0x}u_2(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} & + \|\partial^n_{0x}u_2(t, \cdot)\|_{H^{1/2-\delta}\leq t<1}(\mathbb{R}^n) & + t^{m/4}\|\partial^n_{0x}u_2(t, \cdot)\|_{H^{m+1/2}\leq t<1}(\mathbb{R}^n) & + \|\partial_t\partial^n_{0x}u_2(t, \cdot)\|_{H^{-1/2-\delta}(\mathbb{R}^n)} \leq C(\delta).
\end{align*}$$

Set $w(t, x) = u(t, x) - u_1(t, x) - u_2(t, x)$. Then $w$ satisfies (4.24) and also (4.25). Thus $\partial^n_{0x}w$ satisfies

$$\begin{align*}
\left\{ \begin{array}{ll}
\partial^2_t\partial^n_{0x}w - t^{m1}\Delta_x\partial^n_{0x}w = G(t, x, \partial^n_{0x}, v)|_{\alpha| \leq |\gamma|}
\end{array} \right.
\end{align*}$$

(4.26)

Then following the same argument as in the proof of Theorem 4.2 one obtains by applying the Banach fixed-point theorem that there is a unique bounded solution $w$ of (4.26) such that $w \in L^\infty((0, T) \times \mathbb{R}^n) \cap C([0, T], H^{1/2+p_0(m_3)\leq t<1}(\mathbb{R}^n)) \cap C^1([0, T], H^{-1/2+p_2(m_3)\leq t<1}(\mathbb{R}^n))$ with $\partial^n_{0x}w \in L^\infty((0, T) \times \mathbb{R}^n) (|\alpha| \leq |\gamma| \leq |\alpha| + 1)$ and $\partial^n_{0x}w \in C([0, T], H^{m+1/2+p_2(m_3)\leq t<1}(\mathbb{R}^n)) \cap C^1([0, T], H^{1/2+p_2(m_3)\leq t<1}(\mathbb{R}^n)) \cap C^1([0, T], H^{-1/2+p_2(m_3)\leq t<1}(\mathbb{R}^n))$. □
5. PROOF OF THEOREM 1.1

Based on the results of Sections 3 to 4, we now prove Theorem 1.1. To this end, we first establish conormal regularity of the local solutions \( u(t, x) \) obtained in Theorems 4.2 and 4.3(ii), respectively.

Theorem 5.1. Under assumption \( (A_1) \), one has

(i) \( u(t, x) \in I^\infty H^{-\frac{m}{m-2}} \cap (\Gamma_+^m \cup \Sigma_0) \) for the local solution \( u \) of (1.1).

(ii) \( u(t, x) \in I^\infty H^{-\frac{m}{m-2}} \cap (\Gamma_+^m \cup \Gamma_0^m) \) for the local solution \( u \) of (1.2).

Proof. (i) Note that for \( \varphi_i \) \( (i = 0, 1, 2) \) satisfying assumption \( (A_1) \), one has \( (x_1 \partial_1)^{k_1} \prod_{2 \leq i \leq n} \partial_i^{k_i} \varphi_i(x) \in L^\infty(\mathbb{R}^n) \cap H^{1/2}(\mathbb{R}^n) \) for any \( k_i \in \mathbb{N}_0 \) \( (1 \leq i \leq n) \). Thus, by the commutator relations of Lemma 3.2, one has from Eq. (4.1) that, for \( k \geq 2 \) and \( j \geq 1 \), there exists a \( \Phi(x) \in L^\infty(\mathbb{R}^n) \cap H^{1/2}(\mathbb{R}^n) \) such that

\[
\begin{aligned}
\partial^2_t U_k - t^m \Delta_x U_k &= \sum_{\beta_0 + \beta_1 \leq k_0} \mathcal{C}_{\beta_0} \int_0^t (V^{\beta_0} \prod_{2 \leq i \leq n} R_i^{\beta_1} f(s, x, u)) \, ds + \Phi(x), \\
U_k(0, x) &\in L^\infty(\mathbb{R}^n) \cap H^{1/2}(\mathbb{R}^n), \quad \partial_t U_k(0, x) \in L^\infty(\mathbb{R}^n) \cap H^{1/2}(\mathbb{R}^n),
\end{aligned}
\]

where \( U_k = \left\{ V^{k_0} \prod_{2 \leq i \leq n} R_i^{k_i} u \right\}_{k_0 + \sum k_i = k} \) for \( k \in \mathbb{N}_0 \).

By Lemma 2.4 together with Lemmas 2.2–2.3 and Lemma 2.7, one has from (5.1) that

\[
U_k(t, x) \in L^\infty((0, T) \times \mathbb{R}^n) \cap C([0, T], H^{1/2}(\mathbb{R}^n)) \cap C((0, T], H^{-\frac{m}{m-2}}(\mathbb{R}^n)) \cap C^1([0, T], H^{-\frac{m}{m-2}}(\mathbb{R}^n))
\]

which satisfies, for \( t \in (0, T] \) and any \( \delta > 0 \) with \( 0 < \delta \leq \frac{1}{2(m+2)} \),

\[
\begin{aligned}
\|U_k(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} &\leq C_k, \\
\|U_k(t, \cdot)\|_{H^{1/2}(\mathbb{R}^n)} + t^{m/4}\|U_k(t, \cdot)\|_{H^{-\frac{m}{m-2}}(\mathbb{R}^n)} + \|\partial_t U_k(t, \cdot)\|_{H^{-\frac{m}{m-2}}(\mathbb{R}^n)} &\leq C_k(\delta).
\end{aligned}
\]

The latter yields

\[
N_{2, k} \left( h_2(t, x_1) \chi_1 \left( \frac{(m+2)x_1}{2t^{\frac{m}{m-2}}} \right) \right) = N_{2, k} \left( h_2 \chi \right) u + h_2 \chi \left( \frac{(m+2)x_1}{2t^{\frac{m}{m-2}}} \right) \right) \in L^\infty((0, T), H^{-\frac{m}{m-2}}),
\]

where the functions \( h_2 \) and \( \chi_\pm \) have been defined in Definition 3.1. Furthermore, applying (2) of Proposition 3.12 together with (5.2) yields

\[
N_{2, k} \left( h_2 \chi \right) u \in L^\infty((0, T), H^{-\frac{m}{m-2}}),
\]

Analogously, by (1), (3), and (4) in Proposition 3.12 and the same proof as for (5.3), one obtains

\[
Z_{1, i} \ldots Z_k u(t, x) \in L^\infty((0, T), H^{-\frac{m}{m-2}}) \quad \text{on } W_i \text{ for } Z_1, \ldots, Z_k \in M_i, \quad i = 1, 3, 4.
\]

Therefore,

\[
u(t, x) \in I^\infty H^{-\frac{m}{m-2}} \cap (\Gamma_+^m \cup \Gamma_0).
\]

(ii) The solution \( u(t, x) \) of (1.2) satisfies

\[
\begin{aligned}
(\partial_t^2 - t^m \Delta_x) u(t, x) &= v_1(t, x) + v_2(t, x, u), \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \\
u_0(x) &= \varphi_0(x), \quad \partial_t u(0, x) = \varphi_1(x),
\end{aligned}
\]

where \( v_1(t, x) \) satisfies

\[
\begin{aligned}
(\partial_t^2 - t^m \Delta_x) v_1(t, x) &= 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \\
v_1(0, x) &= \varphi_2(x), \quad \partial_t v_1(0, x) = \varphi_3(x).
\end{aligned}
\]
and $v_2(t, x)$ satisfies
\[
\begin{cases}
(\partial_t^2 - t^{m_2} \Delta_x)v_2(t, x) = f(t, x, u), \\ v_2(0, x) = \partial_t v_2(0, x) = 0.
\end{cases}
\]

By the commutator relations of Lemma 3.3 one has from Eq. (5.4) that, for $k \geq 2$ and $j \geq 1$,
\[
\left\{ \begin{array}{l}
(\partial_t^2 - t^{m_2} \Delta_x) U_k = \sum_{\alpha \leq k_0} C_\alpha \left( (V^{(m_2)})^\alpha \prod_{1 \leq i \leq j} L_{ij}^{k_i} v_1 \right)(t, x) \\
+ \sum_{\beta \leq k_0} C_{\beta_l} \left( (V^{(m_2)})^{\beta_0} \prod_{1 \leq i \leq j} L_{ij}^{\beta_i} \partial_u v_2 \right)(t, x, u) \\
\end{array} \right.
\]
\[
U_k(0, x) \in L^\infty(\mathbb{R}^n) \cap H^{1/2-} \left( \mathbb{R}^n \right), \\
\partial_t U_k(0, x) \in L^\infty(\mathbb{R}^n) \cap H^{1/2-} \left( \mathbb{R}^n \right)
\]
which satisfies, for $t \in [0, T]$ and any fixed $\delta > 0$ with $0 < \delta \leq \frac{1}{2(m_2 + 2)}$, \begin{align}
\|U_k(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} & \leq C_k, \\
\|U_k(t, \cdot)\|_{H^{1/2-t} \left( \mathbb{R}^n \right)} + \left( \frac{m_2 + 2}{2} \right)^{1/2} \left( \frac{m_2 + 2}{2} \right)^{1/2} + \|\partial_t U_k(t, \cdot)\|_{H^{-\frac{m_2 + 2}{2}} - t \left( \mathbb{R}^n \right)} & \leq C_k(\delta).
\end{align}

This yields that for $Z_{2, \pm} \equiv \left( x_1 \mp \frac{2}{m_2 + 2} t \frac{m_2 + 2}{2} \right) \partial_1$,
\[
Z_{2, \pm} \left( h_2(t, x_1) \chi_{\pm} \left( \frac{(m_2 + 2) x_1}{2l} \right) u \right) = Z_{2, \pm} (h_2 \chi) u + \left( \frac{(m_2 + 2) x_1}{2l} \right) \partial_1 u \in L^\infty([0, T], H^{-\frac{m_2 + 2}{2}} - t \left( \mathbb{R}^n \right)),
\]
where the functions $h_2$ and $\chi_{\pm}$ have been defined in Definition 3.15 and we have used that $w_1(x)w_2(x) \in H^{s-} \left( \mathbb{R}^n \right)$ for $w_1(x) \in H^{n/2-} \left( \mathbb{R}^n \right)$ and $w_2(x) \in H^{s} \left( \mathbb{R}^n \right)$ with $-n/2 < s < n/2$. Moreover, applying (2) of Proposition 3.12 together with (5.6) yields
\[
Z_{2, \pm} (V^{(m_2)})^{k_0} \prod_{2 \leq i \leq n} R_i^{k_i} (h_2 \chi u) \in L^\infty([0, T], H^{-\frac{m_2 + 2}{2}} - t \left( \mathbb{R}^n \right)).
\]

Analogously, by (1), (3), and (4) in Proposition 3.12 and the same proof as for (5.7), one obtains
\[
Z_1 \ldots Z_k u(t, x) \in L^\infty([0, T], H^{-\frac{m_2 + 2}{2}} - t \left( \mathbb{R}^n \right)) \text{ on } E_i \text{ for } Z_1, \ldots, Z_k \in Y_i, \quad i = 1, 3, 4, 5.
\]

Therefore, $u(t, x) \in I^\infty H^{-\frac{m_2 + 2}{2}} - t \left( \mathbb{R}^n \right)$ on $\Gamma_{m_1}^\pm \cup \Gamma_{m_2}^\pm$. \hfill $\square$

Finally, we complete the proof of Theorem 1.1.

End of proof of Theorem 1.1
(i) Theorem 4.1 and Theorem 4.2(ii) show the local existence of solutions of (1.1) and (1.2), respectively, under assumption (A_1).

(ii) From Theorem 5.1 one then obtains for these local solutions that $u \in C^\infty (([0, T] \times \mathbb{R}^n) \setminus (\Gamma_{m_1}^\pm \cup \Gamma_0))$ and $u \in C^\infty (([0, T] \times \mathbb{R}^n) \setminus (\Gamma_{m_1}^\pm \cup \Gamma_{m_2}^\pm))$, respectively. \hfill $\square$
6. PROOF OF THEOREM 1.2

Under assumption (A₂), we first establish conormal regularity for local solutions \( u(t, x) \) of (1.1) and (1.2).

**Theorem 6.1.** Under assumption (A₂), one has

(a) \( u(t, x) \in I^\infty H^{n/2 - \frac{m+2}{2(m+k)}}(\Gamma_m \cup l_0) \) for the local solution \( u \) of (1.1),

(b) \( u(t, x) \in I^\infty H^{n/2 - \frac{m+4}{2(m+k)}}(\Gamma_m \cup \Gamma_m) \) for the local solution \( u \) of (1.2).

**Proof.** (a) By the commutator relations of Lemma 3.1 and a direct computation, one has from (1.1) that

\[
\partial_t \left( \partial_t^2 - t^m \Delta_x \right) U_k = \sum_{\beta + \alpha \leq k} C_{\beta \alpha} \left( L_0^{\beta_0} \prod_{1 \leq j \leq n} L_{ij}^{\beta_j} \partial_u f \right)(t, x, u)
\]

\[
U_k(0, x) \in L^\infty(\mathbb{R}^n) \cap H^{n/2 - \frac{m}{m+k}}(\mathbb{R}^n), \quad \partial_t^i U_k(0, x) \in L^\infty(\mathbb{R}^n) \cap H^{n/2 - \frac{m}{m+k}}(\mathbb{R}^n), \quad i = 1, 2,
\]

where \( U_k = \left\{ L_0^{k_0} \prod_{1 \leq j \leq n} L_{ij}^{k_j} u \right\} \) for \( k \in \mathbb{N}_0 \). Note that in the process of deriving the regularity of \( U_k(0, x) \) and \( \partial_t U_k(0, x) \) we have used that \( \prod_{1 \leq j \leq n} (x, \partial_x)_{k_j} \varphi(x) \in H^{n/2 - \frac{m}{m+k}}(\mathbb{R}^n) \) and \( w_1(x)w_2(x) \in H^{n/2 - \frac{m}{m+k}}(\mathbb{R}^n) \) for \( w_1(x), w_2(x) \in L^\infty(\mathbb{R}^n) \cap H^{n/2 - \frac{m}{m+k}}(\mathbb{R}^n) \).

We next prove by induction on \( k \) that

\[ U_k(t, x) \in L^\infty(\{0, T\} \times \mathbb{R}^n) \]

\[ \cap C([0, T], H^{n/2 - \frac{m}{2(m+k)}}(\mathbb{R}^n)) \cap C^1([0, T], H^{n/2 - \frac{m+4}{2(m+k)}}(\mathbb{R}^n)), \]

which satisfies, for any fixed small \( \delta > 0 \),

\[ \left\| U_k(t, \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \leq C_k(\delta) (1 + |\ln t|^2), \]

\[ \left\| U_k(t, \cdot) \right\|_{C([0, T], H^{n/2 - \frac{m}{2(m+k)}}(\mathbb{R}^n))} + t^{m/4} \left\| U_k(t, \cdot) \right\|_{H^{n/2 - \frac{m+4}{2(m+k)}}(\mathbb{R}^n)} \leq C_k(\delta). \]

Note that (6.2)–(6.3) has been shown in Theorem 4.1 in case of \( k = 0 \). Assume that (6.2)–(6.3) holds up to the order \( k - 1 \). Then one has by (6.1) that

\[
\left\{ \begin{array}{l}
\partial_t (\partial_t^2 - t^m \Delta_x) U_k - (\partial_u f)(t, x, u) U_k = F_k(t, x), \\
U_k(0, x) \in L^\infty(\mathbb{R}^n) \cap H^{n/2 - \frac{m}{2(m+k)}}(\mathbb{R}^n), \quad \partial_t U_k(0, x) \in L^\infty(\mathbb{R}^n) \cap H^{n/2 - \frac{m}{2(m+k)}}(\mathbb{R}^n), \quad i = 1, 2,
\end{array} \right.
\]

where \( F_k(t, x) \in C([0, T], H^{n/2 - \frac{m}{2(m+k)}}(\mathbb{R}^n)) \). From (6.4), one sees that \( U_k \) satisfies

\[
\left\{ \begin{array}{l}
(\partial_t^2 - t^m \Delta_x) U_k - \int_0^t (\partial_u f)(s, x, u) U_k ds = G_k(t, x), \\
U_k(0, x) \in L^\infty(\mathbb{R}^n) \cap H^{n/2 - \frac{m}{2(m+k)}}(\mathbb{R}^n), \quad \partial_t U_k(0, x) \in L^\infty(\mathbb{R}^n) \cap H^{n/2 - \frac{m}{2(m+k)}}(\mathbb{R}^n),
\end{array} \right.
\]

where \( G_k(t, x) \in C([0, T], H^{n/2 - \frac{m}{2(m+k)}}(\mathbb{R}^n)) \). Then Lemma 2.1 and Lemma 2.3(i) yield (6.2)–(6.3) (for \( k \)).

We now prove that \( u(t, x) \in I^\infty H^{n/2 - \frac{m+4}{2(m+k)}}(\Gamma_m \cup l_0) \). Note that, for \( 1 \leq i \leq n \), one has by (6.3) that

\[
N_x^2 \left( h_2(t, x) \left( \frac{(m+2)|x|}{2t^{m+2}} \right) u \right) = \left( N_x^2 (\varphi_2 \chi) u + \frac{2}{m+2} \left( \frac{(m+2)|x|}{2t^{m+2}} - 1 \right) h_2 \chi t^{\frac{m+4}{2(m+k)}} \partial_t u \right) \in L^\infty((0, T), H^{n/2 - \frac{m+4}{2(m+k)}}(\mathbb{R}^n)),
\]

where the definitions of \( h_2(t, x) \) and \( \chi \left( \frac{(m+2)|x|}{2t^{m+2}} \right) \) have been given in Definition 3.6. Furthermore, by Proposition 3.7 and (6.3), one obtains that, for any \( k_1, k_0, k_{ij} \in \mathbb{N}_0 \),

\[
(N_x^2)^{k_1} V_0^{k_0} \prod_{1 \leq i < j \leq n} L_{ij}^{k_{ij}} (h_2 \chi u) \in L^\infty((0, T), H^{n/2 - \frac{m+4}{2(m+k)}}(\mathbb{R}^n)).
\]
Together with Proposition 3.7, this yields

\[
\hat{V}_{k_i} V_0^{k_0} \prod_{1 \leq i < j \leq n} L_{i,j}^{k_i} (h_2 \chi u) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}}). 
\]

In order to show \( u(t, x) \in L^\infty H^{n/2 - (\Gamma_m \cup l_0)} \), we need to prove that

\[
\prod_{1 \leq i \leq n} \hat{V}_{k_i} V_0^{k_0} \prod_{1 \leq i < j \leq n} L_{i,j}^{k_i} (h_2 \chi u) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}}) 
\]

or equivalently

\[
\prod_{1 \leq i \leq n} (N_2)^{k_i} V_0^{k_0} \prod_{1 \leq i < j \leq n} L_{i,j}^{k_i} (h_2 \chi u) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}}).
\]

To this end, by the commutator relations of Lemma 3.1 and Proposition 3.7, it suffices to prove that

\[
N_2^i N_2^j \cdots N_2^{i_k} (h_2 \chi u) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}}), \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n, \quad 2 \leq k \leq n,
\]

because the proof on \( N_2^i N_2^j \cdots N_2^{i_k} V_0^{k_0} \prod_{1 \leq i < j \leq n} L_{i,j}^{k_i} (h_2 \chi u) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}}) \) is completely analogous.

In fact, by \( N_2^i \equiv a(t, x) \partial_i \) with \( a(t, x) = |x| - \frac{2}{m+4} (\frac{m+4}{m+8}) \) and (6.5), one has, for \( 1 \leq i \leq n, \)

\[
\partial_i^2 (a^2(t, x)h_2 \chi u) = (a\partial_i a)(h_2 \chi u) + \partial_i a(a\partial_i)(h_2 \chi u) + 2a(\partial_i^2 a)h_2 \chi u + 2(\partial_i a)^2 h_2 \chi u
\]

\[
= (N_2^i)^2 (h_2 \chi u) + (\partial_i a) N_2^i (h_2 \chi u) + 2a(\partial_i^2 a)h_2 \chi u + 2(\partial_i a)^2 h_2 \chi u
\]

\[
\in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}}),
\]

where we have used that \( x_i/|x| \in H^{n/2 - (\mathbb{R}^n)} \) and \( w_1(x)w_2(x) \in H^{\min(s_1, s_2) - \frac{n+2}{2}}(\mathbb{R}^n) \) for \( w_1(x) \in H^{s_1}(\mathbb{R}^n) \) and \( w_2(x) \in H^{s_2}(\mathbb{R}^n) \) when \( s_1, s_2 \geq 0 \). It follows that

\[
\Delta (a^2(t, x)h_2 \chi u) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}})
\]

which gives by the regularity theory of second-order elliptic equations

\[
\partial_{ij} (a^2(t, x)h_2 \chi u) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}}), \quad 1 \leq i < j \leq n,
\]

or equivalently

\[
N_2^i N_2^j (h_2 \chi u) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}}), \quad 1 \leq i < j \leq n.
\]

Analogously, one obtains, for any \( 1 \leq i, k \leq n, \)

\[
\partial_i^2 (a^3 \partial_k (h_2 \chi u)) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}})
\]

and

\[
\Delta (a^3 \partial_k (h_2 \chi u)) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}})
\]

which gives

\[
\partial_{ij} (a^3 \partial_k (h_2 \chi u)) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}})
\]

and further by (6.9)

\[
N_2^i N_2^j N_2^k (h_2 \chi u) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}}).
\]

By induction, we have completed the proof of (6.8). Consequently, one has

\[
V_0^{k_0} \prod_{1 \leq i \leq n} \hat{V}_{k_i} \prod_{1 \leq i < j \leq n} L_{i,j}^{k_i} (h_2 \chi u) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}}).
\]

Similarly, by (1), (3), and (4) of Proposition 3.7 (note that \( \hat{V}_i \) can be expressed as a linear combination of \( V_0 \) and \( L_{jk} \) with admissible coefficients in \( \Omega_1, \Omega_2, \) and \( \Omega_4 \), respectively), one arrives at

\[
Z_1 \cdots Z_k (h_2(t, x)u(t, x)) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}}), \quad Z_1, \ldots, Z_k \in S_1,
\]

\[
Z_1 \cdots Z_k \left( h_3(t, x) \chi_1 \left( \frac{|m+2| |x|}{2t^{(m+2)/2}} \right) u \right) \in L^\infty ((0, T), H^{n/2 - \frac{m+4}{m+8}}), \quad Z_1, \ldots, Z_k \in S_3,
\]
and
\[ Z_1 \ldots Z_k \left( h_4 (t,x) \chi_2 \left( \frac{(m + 2)|x|}{2t^{\frac{m+4}{2(m+2)}}} \right) \right) u \in L^\infty \left( (0,T), H^{n/2-\frac{m+4}{2(m+2)}} \right), \quad Z_1, \ldots, Z_k \in S_4, \]

where the functions \( h_1, h_3, h_4, \) and \( \chi_i \) (\( 1 \leq i \leq 2 \)) have been given in Definition \ref{def:6.3}. Therefore,
\[ u(t,x) \in I^\infty H^{n/2-\frac{m+4}{2(m+2)}} (\Gamma_m \cup I_0), \]
as required.

(b) By the commutator relations of Lemma \ref{lem:5.1} and a direct computation, one from (5.4) that
\[
\begin{cases}
(\partial_t^2 - t^{m2} \Delta_x) U_k = \sum_{\alpha \leq k_0} C_{\alpha} \left( (V_0^{(m2)})^\alpha \prod_{1 \leq i < j \leq n} L_{ij}^{k_{ij}} v_1 \right) (t,x) \\
\qquad + \sum_{\beta_0 + \alpha \leq k_0} \sum_{\beta_1, \beta_2, \ldots, \beta_l} C_{\beta_0} \left( (V_0^{(m2)})^{\beta_0} \prod_{1 \leq i < j \leq n} L_{ij}^{\beta_{ij}} \partial_t v_2 \right) (t,x,u) \\
\qquad \times \prod_{1 \leq s < l} \left( (V_0^{(m2)})^{\beta_s} \prod_{1 \leq i < j \leq n} L_{ij}^{\alpha_{ij}} \right),
\end{cases}
\]
\[
U_k (0,x) \in L^\infty (\mathbb{R}^n) \cap H^{n/2-} (\mathbb{R}^n), \quad \partial_t U_k (0,x) \in L^\infty (\mathbb{R}^n) \cap H^{n/2-} (\mathbb{R}^n),
\]
where \( U_k = \left\{ (V_0^{(m2)})^{k_0} \prod_{1 \leq i < j \leq n} L_{ij}^{k_{ij}} \right\}_{k_0 + \sum k_{ij} = k} \) for \( k \in \mathbb{N}_0; \) \( v_1 \) and \( v_2 \) have been defined in (5.4).

We next prove by induction on \( k \) that
\[
U_k (t,x) \in L^\infty ((0,T] \times \mathbb{R}^n)
\]
\[
\cap C \left( (0,T], H^{n/2-} \right) \cap C \left( (0,T], H^{n/2-\frac{m}{2(m+2)}} \right) \cap C^1 \left( (0,T], H^{n/2-\frac{m+4}{2(m+2)}-} \right)
\]
which satisfies, for any small \( \delta > 0 \)
\[
\begin{cases}
||U_k (t, \cdot)||_{L^\infty(\mathbb{R}^n)} \leq C_k (\delta) (1 + |t|)^2, \\
||U_k||_{C ((0,T], H^{n/2-\delta}) \cap C \left( (0,T], H^{n/2-\frac{m}{2(m+2)}}-\delta \right) \cap C^1 \left( (0,T], H^{n/2-\frac{m+4}{2(m+2)}-}\delta \right) \leq C_k (\delta).
\end{cases}
\]
Note that (6.12)–(6.13) have been shown in Theorem \ref{thm:4.3}(i) in case \( k = 0 \). Assume that (6.12)–(6.13) holds up to the order \( k - 1 \). Then one has from (6.11) that
\[
\begin{cases}
(\partial_t^2 - t^{m2} \Delta_x) U_k = F_k (t,x), \\
U_k (0,x) \in L^\infty (\mathbb{R}^n) \cap H^{n/2-} (\mathbb{R}^n), \quad \partial_t U_k (0,x) \in L^\infty (\mathbb{R}^n) \cap H^{n/2-} (\mathbb{R}^n),
\end{cases}
\]
where \( F_k (t,x) \in L^p ((0,T], H^{n/2-}) \) with any \( 1 < p < \infty \). Then Lemma \ref{lem:2.1} and Lemma \ref{lem:2.3}(i) yield (6.12)–(6.13) (for \( k \)).

We now prove that \( u(t,x) \in I^\infty H^{n/2-} (\Gamma_m, \cup \Gamma_m) \). On \( D_2 \), set \( Z_2^i = \left( |x| - \frac{2}{m+2} t^{\frac{m+2}{2}} \right) \partial_i \) for \( 1 \leq i \leq n \). Note that by (6.13), for \( 1 \leq i \leq n \),
\[
Z_2^i \left( h_2^i (t,x) \chi \left( \frac{(m+2)|x|}{2t^{\frac{m+4}{2}}} \right) \right) u = Z_2^i (h_2^i \chi) u + \frac{2}{m+2} \left( \frac{(m+2)|x|}{2t^{\frac{m+4}{2}}} - 1 \right) h_2^i \chi t^{\frac{m+4}{2}} \partial_i u \in L^\infty ((0,T), H^{n/2-\frac{m+4}{2(m+2)}}),
\]
where the definitions of \( h_2^i (t,x) \) and \( \chi \left( \frac{(m+2)|x|}{2t^{\frac{m+4}{2}}} \right) \) have been given in Definition \ref{def:3.15}. Furthermore, by Proposition \ref{prop:3.4} and (6.13), one obtains that, for any \( k_i, k_0, k_{ij} \in \mathbb{N}_0 \),
\[
(Z_2^i)^{k_i} (V_0^{(m2)})^{k_0} \prod_{1 \leq i < j \leq n} L_{ij}^{k_{ij}} (h_2^i \chi) u \in L^\infty ((0,T), H^{n/2-\frac{m+4}{2(m+2)}}).
Together with Proposition 5.7, this yields
\[
\left( V_0^{(m_2)} \right)^{k_0} \prod_{1 \leq i \leq n} L_{ij}^{k_0} (h_2 \chi u) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right).
\]

In order to show that \( u(t, x) \in L^\infty H^{n/2} (\Omega_1 \cup \Omega_{m_2}) \), we need to prove that
\[
\prod_{1 \leq i \leq n} \left( V_0^{(m_2)} \right)^{k_i} \prod_{1 \leq i < j \leq n} L_{ij}^{k_i} (h_2 \chi u) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right)
\]
or equivalently
\[
\prod_{1 \leq i \leq n} \left( Z_2^{k_i} \right)^{k_0} \prod_{1 \leq i < j \leq n} L_{ij}^{k_0} (h_2 \chi u) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right).
\]

To this end, by the commutator relations of Lemma 3.1 and Proposition 3.7, it suffices to prove that
\[
(6.16) \quad Z_2^{i_1} Z_2^{i_2} \ldots Z_2^{i_k} (h_2 \chi u) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right), \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n, \quad 2 \leq k \leq n,
\]
because the proof on \( Z_2^{i_1} Z_2^{i_2} \ldots Z_2^{i_k} (V_0^{(m_2)})^{k_0} \prod_{1 \leq i < j \leq n} L_{ij}^{k_0} (h_2 \chi u) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right) \) is completely analogous.

In fact, by \( Z_2^{i_1} \equiv b(t, x) \partial_t \) with \( b(t, x) = |x| - \frac{1}{m+2} \frac{\ln \ln \rho}{m} \) and (6.15), one has, for \( 1 \leq i \leq n \),
\[
\partial_t^2 \left( b^2(t, x) h_2 \chi u \right) = (b \partial_t)^2 (h_2 \chi u) + \partial_t (b \partial_t h_2 \chi u) + 2 b (\partial_t^2 h_2 \chi u) + 2 (\partial_t b)^2 h_2 \chi u
\]
(6.17)
\[
= (Z_2^{i_1})^2 (h_2 \chi u) + (\partial_t h)_2 (h_2 \chi u) + 2 b (\partial_t^2 h_2 \chi u) + 2 (\partial_t b)^2 h_2 \chi u
\]
\[
\in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right).
\]

From (6.17), one has
\[
\Delta \left( b^2(t, x) h_2 \chi u \right) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right)
\]
which gives by the regularity theory of second-order elliptic equations
\[
\partial_{ij} (b^2(t, x) h_2 \chi u) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right), \quad 1 \leq i < j \leq n,
\]
or equivalently
\[
(6.18) \quad Z_2^{i_1} Z_2^{i_2} (h_2 \chi u) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right) \quad \text{for any } 1 \leq i < j \leq n.
\]

Analogously, we can get, for any \( 1 \leq i, k \leq n \),
\[
\partial_t^k \left( b^2 \partial_k (h_2 \chi u) \right) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right)
\]
and
\[
\Delta (b^2 \partial_k (h_2 \chi u)) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right)
\]
which gives
\[
\partial_{ij} (b^2 \partial_k (h_2 \chi u)) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right)
\]
and further by (6.18),
\[
(6.19) \quad Z_2^{i_1} Z_2^{i_2} Z_2^{i_3} (h_2 \chi u) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right).
\]

By induction, we have completed the proof of (6.16). Consequently, one has
\[
(6.20) \quad (V_0^{(m_2)})^{k_0} \prod_{1 \leq i \leq n} \left( V_0^{(m_2)} \right)^{k_i} \prod_{1 \leq i < j \leq n} L_{ij}^{k_i} (h_2 \chi u) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right).
\]

Similarly, by (1), (3), and (4) of Proposition 3.7 (note that \( M_i \) can be expressed as a linear combination of \( M \) and \( L_{jk} \) with admissible coefficients in \( \Omega_1 \), \( \Omega_2 \), and \( \Omega_4 \), respectively), one arrives at
\[
Z_1 \ldots Z_k (h_1(t, x) u(t, x)) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right), \quad Z_1, \ldots, Z_k \in X_1,
\]
\[
Z_1 \ldots Z_k \left( \left( \frac{(m_2+2)|x|}{2t^{\frac{m+2}{2}}} \right)^{\chi_1 (\frac{(m_2+2)|x|}{2t^{\frac{m+2}{2}}})} \right) \chi_2 \left( \left( \frac{(m_1+2)|x|}{2t^{\frac{m+2}{2}}} \right)^{u} \right) \in L^\infty \left( (0, T), H^{n/2 - \frac{m+4}{m(m+2)+n}} \right),
\]
Therefore, solutions of (1.1) and (1.2) under assumption (A)
End of proof of Theorem 1.2.

one has from (6.12) and (6.20) by an argument similar to the one dealing with $V_j^{(m_2)}$ (1 $\leq j \leq n$) that, for all $Z_1, \ldots, Z_k \in X_4,

$$Z_1 \ldots Z_k \left( h_4(t,x) \chi_3 \left( \frac{(m_2 + 2)|x|}{2i \omega_{\ell} + \bar{\omega}_{\ell} + 2} \right) \chi_4 \left( \frac{(m_1 + 2)|x|}{2i \omega_{\ell} + \bar{\omega}_{\ell} + 2} \right) u(t,x) \right) \in L^\infty((0,T), H^{n/2 - \frac{m_2 + k}{2(m_2 + 2)}}(\mathbb{R}^n)).$$

Therefore, $u(t,x) \in L^\infty H^{n/2 - \frac{m_2 + 4}{2(m_2 + 2)}}(\Gamma_{m_1} \cup \Gamma_{m_2})$, as required.

Finally, we finish the proof of Theorem 1.2.

**End of proof of Theorem 1.2**

(i) From Theorem 4.1 and Theorem 4.3(ii), one obtains the local existence of solutions of (1.1) and (1.2) under assumption (X5).

(ii) Based on Theorem 6.1 one then sees that $u(t,x) \in L^\infty H^{n/2 - \frac{m_2 + 4}{2(m_2 + 2)}}(\Gamma_{m_1} \cup l_0)$ for the solution $u$ of (1.1) and $u(t,x) \in L^\infty H^{n/2 - \frac{m_2 + 4}{2(m_2 + 2)}}(\Gamma_{m_1} \cup \Gamma_{m_2})$ for the solution $u$ of (1.2).

Thus, the proof of Theorem 1.2 is finished.

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DEPARTMENT OF MATHEMATICS AND IMS, NANJING UNIVERSITY, NANJING 210093, P.R. OF CHINA

E-mail address: XXX@nju.edu.cn

MATHEMATICAL INSTITUTE, UNIVERSITY OF GÖTTINGEN, D-37073 GÖTTINGEN, GERMANY

E-mail address: iwitt@uni-math.gwdg.de

DEPARTMENT OF MATHEMATICS AND IMS, NANJING UNIVERSITY, NANJING 210093, P.R. OF CHINA

E-mail address: huicheng@nju.edu.cn