A scientific report of singular kernel on the rate-type fluid subject to the mixed convection flow

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Abstract

The response of inhomogeneous fluids whose material properties are strongly interconnected with mean normal stress and shear rate; such phenomenon undergo to rate constitutive-type theories within thermodynamical aspects. In this context, the thermodynamic approach for modeling a class of viscoelastic fluids so-called Oldroyd-B fluid is investigated through a singular kernel. This is because role of the singular kernel has become imperative due to the wide applications of rate-type fluid subject to the mixed convection flow. To have the systematical and molecular dynamics of Oldroyd-B fluid, the temperature distribution, velocity profile, and Nusselt numbers have been explored by invoking the Laplace transform on the governing equations. Owing to the strengthening of physical parameters, the enhancement of convective heat transfer is emphasized based on relaxation as well as retardation phenomenon. Additionally, the evaluation of thermal and flow characteristics of Oldroyd-B fluid has resulted in discovering more evolutionary mechanisms of the considered problem of rate-type fluid subject to the mixed convection flow.

Keywords Rate-type fluid · Singular kernel · Molecular dynamics of Nusselt number · Fractional derivative based on singular kernel

1 Introduction

During the last two decades, a lot of research in fractional calculus in engineering, science, and several other areas has been published. Fractional calculus now has vast applications in many fields such as signal processing, dynamics of fluid, viscoelastic material, biomedical field, and electro-chemistry. There are various ways to discuss many problems in mathematics and engineering, in which one of the convenient ways is the use of fractional derivatives. Many researchers such as Caputo, Grunwald–Letnikov, Hadamard, Riesz, and Hilfer give the solutions for many problems by using fractional derivatives; Riemann–Liouville among the ones who defined fractional derivatives problem. Mixed convection flow with MHD fluid is studied to investigate stagnation point (Sajid et al. 2015). The non-Newtonian fluid with unsteady helix flow of an Oldroyd-B fluid has also been analyzed (Tong and Zhang 2009).

Non-Newtonian fluid is categorized into rate-type integral-type, and differential-type liquid. An Oldroyd-B fluid is the subclass of rate-type fluid, and it has vast applications in engineering. Sajid et al. (2010) studied the effect of stagnation point with the help of an Oldroyd-B model. The boundary layer problem with convection flow was discussed by Hayat et al. (2011). Many fluids such as second-grade fluid, Maxwell fluid, and Burger’s fluid are being used to solve the heat transfer processes. Among all models, Oldroyd-B model is much more reasonable. The heat transfer process including the MHD flow for convective and boundary conditions was discussed in Abel et al. (2012); Jamil et al. (2013); Hayat et al. (2013); Zheng et al. (2011); Asghar et al. (2003).
The second- and third-grade fluid with unidirectional flow corresponding to the infinite plate due to pressure gradient effect was studied (Fetecau 2005; Abelman et al. 2009; Erdogan and Imrak 2005). Devakar et al. (2014) reviewed the analytic solution of slip boundary condition flow with couple stress fluid. Nowadays, experimental data is achieved when an Oldroyd-B model replaces an ordinary Oldroyd-B model with fractional calculus. To study the behavior of different types of fluids such as Second grade, Maxwell, and Oldroyd-B fluid, the fractional derivatives operator and potential vortex using cylindrical coordinates were examined (Salah et al. 2013; Fetecau 2005; Khan et al. 2016; Fetecau et al. 2007). The exact solutions for temperature effect velocity distributions with different physical parameters in second- and third-grade fluid have been analyzed (Fetecau 2011; Hayat and Kara 2006). In the past few years, several authors examined various flows in interesting geometries (Khan 2015; Saqib et al. 2018; Song et al. 2021; Lee et al. 2021; Ahmad et al. 2021; Saeed et al. 2021; Ali et al. 2021; Awan et al. 2021, 2020; Raza et al. 2017; Imran et al. 2015, 2011; Awan et al. 2010).

Hayat et al. (2007) studied the thermal radiation effect to find the semi-analytic solution with heat transfer. The heat and mass transfer in the porous channel by using Magnetohydrodynamics (MHD) oscillating flow with magnetic field effect and velocity distribution in the lower plate was studied (Hayat et al. 2009). The unsteady electro-osmotic flow in a straight pipe by using an Oldroyd-B fluid was discussed by Farhad et al. (2012). Zhao and Wang (2013) studied the hydro-magnetic flow and viscoelastic second-grade fluid to analyze heat transfer behavior in an oscillating wall. The heat and mass transfer phenomenon in different fluids with the help of fractional derivatives has also been examined (Misra et al. 2011). The researchers discussed viscoelastic fluids with and without fractional derivative operator (Abro and Atangana 2020; Abro 2019; Lohana et al. 2021; Abro and Atangana 2020; Abro et al. 2021).

The main objective of this work is to study the convective flow of Oldroyd-B fluid with the fractional model. The oscillating pressure gradient significantly impacts this flow, so the flow channel with the porous medium is examined. Many fractional models have been investigated to find the solutions for this problem. “Caputo-time fractional derivatives” is one of the reliable methods to solve our problem. The Laplace transform method determines the temperature field, velocity field, and some physical parameters. Depending on physical coefficients, our fluid describes the behavior of fractional model flow and standard flow. The results are analyzed graphically.

2 Statement of the problem

We assume the mixed convection flow of an Oldroyd-B fluid with a vertical plate of distance $d$ loaded with saturated porous medium. Both uniformly electrical and magnetic effects influence the liquid, but the magnetic effect is minimized by using Reynolds number. The electric field produced by polarization is also neglected, and an external electric field is assumed to be zero. Initially, the fluid is at rest, having temperature $T_0$. After time $t = 0^+$, the fluid between two plates begins to move with the temperature $T_0$ of the wall at $y = 0$ and $T_d$ of the wall at $y = d$. The flow is considered along the $x$-axis and $y$-axis to flow direction. The governing equations for energy and momentum are \([7,17,22]\):

\[
\rho \left( 1 + \lambda_1 \frac{\partial}{\partial t} \right) \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial x} + \mu \left( 1 + \lambda_1 \frac{\partial}{\partial t} \right) \frac{\partial^2 v}{\partial y^2} - \left( \mu \frac{\partial}{\partial x} \right)^2 v + \rho \beta_0 g (Q - Q_0),
\]

\[
\rho c_p \frac{\partial Q}{\partial x} = \frac{\partial}{\partial y} \left( \frac{\partial Q}{\partial y} \right) + 4 \alpha_0^2 (Q - Q_0).
\]

where $v(y,t)$ is the velocity parameter in $x$-direction, $Q(y,t)$, $\rho$, $\mu$, $p$, $\sigma$, $M_f$, $k_1$, $\beta_0$, $\alpha_0$, $g$, and $k$ are the temperature field, fluid density, fluid viscosity parameter, pressure, electrical conductance, externally applied magnetic parameter, porosity parameter, heat transfer parameter, thermal conductivity, acceleration due to gravity and the radiating parameter, respectively.

Defining the oscillatory pressure gradient as $-\frac{\partial p}{\partial x} = H(t) \lambda_0^* + \lambda_1 \exp(i\omega t)$ in the direction of flow with the Heaviside unit step function $H(t) = \frac{1}{2} \text{sign}(t') (1 + \text{sign}(t'))$ and initial and boundary conditions

\[
v(y, 0) = 0, \quad Q(y, 0) = Q_0, \quad v(0, t) = 0, \quad v(d, t) = 0, \quad Q(0, t) = Q_0, \quad Q(d, t) = Q_d.
\]

Using the dimensionless parameters

\[
v^* = \frac{v}{v_0}, \quad x^* = \frac{x}{d}, \quad t^* = \frac{t}{t_0}, \quad y^* = \frac{y}{d}, \quad p^* = \frac{d}{\mu V_0 p},
\]

\[
\lambda_0^* = \frac{\lambda_0}{t_0}, \quad \lambda_1^* = \frac{\lambda_1}{t_0}, \quad \lambda_{r_1}^* = \frac{\lambda_{r_1}}{t_0}, \quad \psi = \frac{Q - Q_0}{Q_d - Q_0}, \quad \omega^* = \frac{\omega_1 d}{V_0},
\]

\[
-\frac{\partial p^*}{\partial x^*} = \lambda_0^* + \lambda_1^* \exp(i\omega t^*).
\]
Applying Laplace transform to Eq. (10), we have

\[
\begin{align*}
\mathcal{L}\left[\frac{\partial v}{\partial t}\right] &= H(t)\left[\lambda_0 + \lambda_1 \exp(\omega_1 t)\right] + \frac{\partial^2 v}{\partial y^2} \\
& \quad - \left[M_0 + \frac{1}{K}\right]v + Gr\psi, \\
\mathcal{L}\left[\frac{\partial \psi}{\partial t}\right] &= \frac{\partial^2 \psi}{\partial y^2} + N^2\psi,
\end{align*}
\]

(5)

\[
\begin{align*}
v(y, 0) &= 0, \quad v(0, t) = 0, \quad v(1, t) = 0, \\
\psi(y, 0) &= 0, \quad \psi(0, t) = 0, \quad \psi(1, t) = 1,
\end{align*}
\]

(6)

where

\[
\begin{align*}
Re &= \frac{V_0d}{\nu}, \quad M_0 = \frac{\sigma M^2 \nu^2}{\mu}, \quad K = \frac{k_1}{\nu^2}, \quad N^2 = \frac{4\alpha^2 d^2}{k}, \\
Pr &= \frac{\mu c_p}{k}, \quad Pe = Pr Re, \quad Gr = \frac{g\beta_0 d^2 (Q_d - Q_0)}{\nu V_0},
\end{align*}
\]

(8)

where \(Re, M_0, K, Gr, Nu\) and \(Pe\) are Reynolds number, Magnetic parameter, dimensionless porosity parameter, Grashof, Nusselt, Prandtl and Peclet numbers, respectively. We take a fractional differential equation with Caputo-time fractional derivatives as

\[
\begin{align*}
Re(D_t^{\alpha_1} + \lambda_1 D_t^{\alpha_1+1})v(y, t) &= H(t)\left[\lambda_0' + \lambda_1 \exp(\omega_1 t)\right] \\
& \quad + \left(1 + \lambda_1 D_t^{\alpha_1}\right)\frac{\partial^2 v(y, t)}{\partial y^2} \\
& \quad - \left[M_0 + \frac{1}{K}\right]v(y, t) + Gr\psi, \\
Pe D_t^{\alpha_1} \psi(y, t) &= \frac{\partial^2 \psi}{\partial y^2} + N^2\psi(y, t),
\end{align*}
\]

(9)

where Caputo-fractional derivative operator is defined as

\[
D_t^{\alpha_1} v(y, t) = \begin{cases} 
\frac{1}{\Gamma(1-\alpha_1)} \int_0^t \frac{1}{(t-\tau)^{\alpha_1}} \frac{\partial v(\tau, t)}{\partial \tau} d\tau, & 0 \leq \alpha_1 < 1 \\
\frac{\partial v(y, t)}{\partial t}, & \alpha_1 = 1.
\end{cases}
\]

(10)

\[
\phi(y, q) = \frac{\sinh\left[q\sqrt{\frac{Pe}{\nu}}\sqrt{q^{\alpha_1} - \frac{N^2}{Pr}}\right]}{q \sinh\left[q\sqrt{\frac{Pe}{\nu}}\sqrt{q^{\alpha_1} - \frac{N^2}{Pr}}\right]}.
\]

(14)

Obtaining Laplace inverse transform of Eq. (14) and rewriting the function in equivalent form, we get

\[
\phi(y, q) = \frac{1}{q^{1-\alpha_1}} \sinh\left[q\sqrt{\frac{Pe}{\nu}}\sqrt{q^{\alpha_1} - \frac{N^2}{Pr}}\right].
\]

(15)

Now, from Appendix (A1), the convolution theorem and using

\[
L^{-1}\left\{\frac{1}{q^{1-\alpha_1}}\right\} = \begin{cases} 
t^{1-\alpha_1}/\Gamma(1-\alpha_1), & 0 < \alpha_1 < 1 \\
\delta(t), & \alpha_1 = 1.
\end{cases}
\]

(16)

we have

(a) For \(0 < \alpha_1 < 1\)

\[
\phi(y, t) = \int_0^\infty \frac{(t - \zeta)^{-\alpha_1}}{\Gamma(1-\alpha_1)} \frac{h}{\sqrt{Pe}}(y\sqrt{Pe}, \zeta, -\frac{N^2}{Pe}, \sqrt{Pe}) d\zeta
\]

\[
= \int_0^\infty \frac{(t - \zeta)^{-\alpha_1}}{\Gamma(1-\alpha_1)} \int_0^\infty \frac{f}{\sqrt{Pe}}(y\sqrt{Pe}, \nu, -\frac{N^2}{Pe}, \sqrt{Pe}) \phi(0, -\alpha_1, -\nu^{1-\alpha_1}) d\nu d\zeta
\]

\[
= \int_0^\infty \frac{f}{\sqrt{Pe}}(y\sqrt{Pe}, \nu, -\frac{N^2}{Pe}, \sqrt{Pe}) \int_0^\infty \frac{(t - \zeta)^{-\alpha_1}}{\Gamma(1-\alpha_1)} \zeta^{\alpha_1-1} \phi(0, -\alpha_1, 1-\nu^{1-\alpha_1}) d\nu (1-\nu^{1-\alpha_1}) d\zeta.
\]

(17)

where \(f, h, \phi\) are given in Appendix (A1, A2, and A3). Considering the Image function with the Laplace transform (Appendix, A2) and \(\tilde{v}(t) = \int_0^t \phi(0, -\tilde{\sigma}; -at^{-\alpha}) dt\). We find,

\[
t^{-\alpha_1} \phi(0, -\tilde{\sigma}, -at^{-\alpha_1}) = \frac{d\Phi'(1, -\tilde{\sigma}, -at^{-\alpha_1})}{dt}.
\]

(18)

Making use of Eqs. (18) and (17), we obtain

\[
\int_0^\infty \frac{(t - \zeta)^{-\alpha_1}}{\Gamma(1-\alpha_1)} \zeta^{-1} \phi(0, -\alpha_1, -\nu^{1-\alpha_1}) d\zeta = D_t^{\alpha_1} \phi'(1, -\tilde{\sigma}, -at^{-\alpha_1})
\]

and Eq. (17) becomes

\[
\phi(y, t) = \int_0^\infty f(y\sqrt{Pe}, \nu, -\frac{N^2}{Pe}, \sqrt{Pe}) D_t^{\alpha_1} \phi'(1, -\tilde{\sigma}, -at^{-\alpha_1}) d\nu; \quad 0 < \alpha_1 < 1.
\]

(19)
For $\alpha_1 = 1$

$$\bar{\phi}(y, \tau) = \int_0^\infty \frac{\phi'(0, -\alpha_1, -v \tau^{-\alpha_1}) dv}{\Gamma(1 - \alpha_1)} \frac{\Gamma(\hat{\rho} + 1)}{\Gamma(\alpha_1 \hat{\rho} + 1)} \tau^{-\alpha_1}, \quad \hat{\rho} > -1,$$

with $\hat{\rho} = 1$, Eq. (20) becomes

$$\bar{\phi}(1, \tau) = \frac{B(\alpha_1, 1 - \alpha_1)}{\Gamma(1 - \alpha_1) \Gamma(\alpha_1)} = \frac{\Gamma(1 - \alpha_1) \Gamma(\alpha_1)}{\Gamma(1 - \alpha_1 + \alpha_1)} = 1.$$

In the above equation, $B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx$, $Re(a) > 0$, $Re(b) > 0$, is the Beta function.

$$Nv_1(t) = \frac{\partial \phi(y, \tau)}{\partial y}|_{y=0} = L^{-1} \left\{ \frac{\sqrt{Pe(\alpha_1 + b_0^*)}}{q} \cosh \left( \frac{\sqrt{Pe(\alpha_1 + b_0^*)}}{q} \right) \right\}.$$

The Laplace inverse transform of the image function

$$S_k(q) = \frac{-\sqrt{Pe(2k + 1)} \sqrt{q^{\alpha_1} + b_0^*}}{q \sqrt{q^{\alpha_1} + b_0^*}}.$$

is given as

$$S_k(t) = \int_0^\infty \frac{1}{\sqrt{\pi \xi}} \exp \left( -\frac{Pe(2k + 1)^2}{4\xi} - b_0 \xi \right) t^{-1} \phi'(0, -\alpha_1, -\xi t^{-\alpha_1}) d\xi.$$

The Nusselt number $Nv_1(t)$ is

$$Nv_1(t) = c_1(t) \sum_{k=0}^\infty S_k(t) = \sum_{k=0}^\infty \int_0^t c_1(t - \xi) S_k(\xi) d\xi,$$

where $c_1(t) = \sqrt{Pe} \left( \frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)} + b_0 \right), \quad 0 < \alpha_1 < 1$. The transfer rate of heat from the wall $y = 1$ is given by

$$Nv_2(t) = \frac{\partial \phi(y, \tau)}{\partial y}|_{y=1} = L^{-1} \left\{ \frac{\sqrt{Pe(\alpha_1 + b_0^*)}}{q} \cosh \left( \frac{\sqrt{Pe(\alpha_1 + b_0^*)}}{q} \right) \right\}.$$

The Laplace inverse transform of the image function is

### 3.2 Nusselt number

The information related to heat transfer is given by the Nusselt number at the wall with in the fluid. At the wall $y = 0$, the rate of heat transfer is given by
\[ W_k(t) = \int_0^\infty \frac{1}{\sqrt{\pi} \xi} \left( e^{-\frac{-n^2 Pe}{\xi} - b_0^2} \right) \left( e^{-\frac{(k + 1)^2 Pe}{\xi} - b_0^2} \right) \left( \frac{t}{\xi} \right) d\xi. \]  

(29)

The Nusselt number is

\[ N v_2(t) = c_1(t) \sum_{k=0}^{\infty} W_k(t) = \sum_{n=0}^\infty \int_0^t c_1(t - \zeta) W_k(\zeta) d\zeta. \]  

(30)

In ordinary case, i.e., \( \alpha_k = 1 \) (Appendix, A2), \( t^{-1} \phi^\prime(0, -1, -\xi t^{-1}) = \delta(t - \zeta) \), where \( \delta \), Eqs. (23) and (28) become

\[ N v_1(t) = \sum_{k=0}^\infty \sqrt{Pe} \exp \left( \frac{-Pe(2k + 1)^2}{4t} - b_0 t \right) \exp \left( \frac{-Pe(2k + 1)^2}{4t} - b_0 t \right) \]  

\[ = \sum_{k=0}^\infty \sqrt{Pe} \exp \left( \frac{-Pe(2k + 1)^2}{4t} - b_0 t \right) \int_0^t \frac{1}{\pi^2} \]  

\[ \times \exp \left( \frac{-Pe(2k + 1)^2}{4t} - b_0 t \right) d\zeta \]  

(31)

where

\[ \eta(t) = \frac{1}{2} \sum_{k=0}^\infty \left[ e^{-\frac{q^2}{4t} Pe} \sqrt{Pe} \exp \left( \frac{(2k + 1)^2 Pe}{2\sqrt{t}} - \sqrt{b_0^2 t} \right) \right] \]  

(32)

respectively,

\[ N v_2(t) = \sqrt{Pe} \sqrt{\pi} \psi(t) + b_0^2 \sqrt{Pe} \int_0^t \psi(\tau) d\tau, \]  

(33)

where

\[ \psi(t) = \sum_{k=0}^\infty \frac{1}{2} \left[ \exp \left( \frac{-k^2 Pe}{t} - b_0^2 t \right) + \exp \left( \frac{-(k + 1)^2 Pe}{t} - b_0^2 t \right) \right]. \]  

(34)

### 3.3 Velocity field

The transformed problem in the velocity \( \bar{v}(y, q) \) is

\[ \left[ Re(q + \lambda q) + 1 \right] \bar{v}(y, q) = 0, \quad \bar{v}(1, q) = 0. \]  

Introducing the notations \( a_0^* = \frac{1}{Re} \left[ M_0 + \frac{1}{K} \right] \), \( b_0^* = \frac{N^2}{Pe} \) and rewriting Eq. (35) for finding the particular solution, we obtain

\[ \bar{v}_p(y, q) = \frac{1}{Re \left( q^{1-a_1} + \lambda q \right) - Pe(1 + \lambda q^q)} \]  

\[ + \frac{1}{q \sinh \left( \sqrt{Pe} \sqrt{q}^q + b_0^* \right)} \]  

\[ + \frac{b_0^* Re - b_0^* (1 + \lambda q^q) Pe}{Re \left( q^{1-a_1} + \lambda q \right) - Pe(1 + \lambda q^q) Pe} \]  

\[ + \frac{\lambda_0}{q} + \frac{\lambda}{q - \omega}. \]  

(37)

The solution of Eqs. (35) and (36) is

\[ \bar{v}(y, q) = \bar{v}_1(q) \]  

\[ \bar{v}_1(q) = \frac{1}{\sqrt{Re \sqrt{q}^q + \alpha_0^*}} \frac{\sqrt{Re \sqrt{q}^q + \alpha_0^*}}{\sqrt{1 + \lambda q^q}} \]  

\[ + \frac{\sqrt{Re \sqrt{q}^q + \alpha_0^*}}{\sqrt{1 + \lambda q^q}} \]  

\[ + \frac{\sqrt{Re \sqrt{q}^q + \alpha_0^*}}{\sqrt{1 + \lambda q^q}} \]  

\[ \bar{v}_2(q) = \frac{Gr}{q} \left( \frac{Re(q^{1-a_1} + \lambda q) - Pe(1 + \lambda q^q)} {Re(q^{1-a_1} + \lambda q) - Pe(1 + \lambda q^q) Pe} \right) \]  

(38)

where

\[ \bar{v}_1(q) = \frac{1}{Re \left( q^{1-a_1} + \lambda q \right) - Pe(1 + \lambda q^q)} - \frac{\lambda_0}{q} + \frac{\lambda}{q - \omega} \]  

(39)

\[ \bar{v}_2(q) = \frac{Gr}{q} \left( \frac{Re(q^{1-a_1} + \lambda q) - Pe(1 + \lambda q^q)} {Re(q^{1-a_1} + \lambda q) - Pe(1 + \lambda q^q) Pe} \right) \]  

(40)
Table 1 Comparison of velocity for the fractional parameter $\alpha$

| $y$  | $\bar{v}(y, t)\ (\text{Stehfest})$ | $\bar{v}(y, t)\ (\text{Tzou})$ |
|------|-----------------------------------|----------------------------------|
| 0    | $-3.453\times10^{-9}$             | 0                                |
| 0.05 | 0.013                             | 0.013                            |
| 0.1  | 0.018                             | 0.017                            |
| 0.15 | 0.022                             | 0.022                            |
| 0.2  | 0.026                             | 0.025                            |
| 0.25 | 0.030                             | 0.029                            |
| 0.3  | 0.033                             | 0.033                            |
| 0.35 | 0.037                             | 0.037                            |
| 0.4  | 0.041                             | 0.040                            |
| 0.45 | 0.044                             | 0.044                            |
| 0.5  | 0.048                             | 0.047                            |
| 0.55 | 0.051                             | 0.005                            |

Table 2 Comparison of velocity for the fractional parameter $\beta$

| $y$  | $\bar{v}(y, t)\ (\text{Stehfest's})$ | $\bar{v}(y, t)\ (\text{Tzou's})$ |
|------|-------------------------------------|----------------------------------|
| 0    | $-1.859\times10^{-11}$              | 0                                |
| 0.05 | 0.018                               | 0.017                            |
| 0.1  | 0.025                               | 0.025                            |
| 0.15 | 0.033                               | 0.032                            |
| 0.2  | 0.039                               | 0.045                            |
| 0.25 | 0.051                               | 0.050                            |
| 0.3  | 0.056                               | 0.056                            |
| 0.35 | 0.060                               | 0.060                            |
| 0.4  | 0.064                               | 0.064                            |
| 0.45 | 0.067                               | 0.067                            |
| 0.5  | 0.069                               | 0.068                            |
| 0.55 | 0.07                                | 0.069                            |

The above tables show the values of two different algorithms, Stehfest’s and Tzou’s. Numerically, it can be seen that the velocity markdowns taken from these two algorithms are nearly approximated to each other. From Table 1, it is clear that velocity field when the fractional parameter $\alpha$ is the same and $\alpha$ being varying and finding from both numerical methods have compatibility between them, also similar case for Table 2 when $\alpha$ is the same, and $\beta$ varies.

5 Results and discussion

In the present work, we find the problem solution of non-dimensional velocity, temperature field, and Nusselt number in the saturated porous medium using Caputo-time derivative. The behavior of velocity, temperature, and Nusselt number with the changing of different parameters are discussed to check the effect of fractional parameters $\alpha$, $\beta$, $Pe$ and $Re$.

Figure 1 shows the velocity field versus $y$, from the first three figures of Fig. 1, it is clear that the velocity field attains maximum value at $y = 1$ and then velocity profile gradually decreases for the small values of the fractional parameter $\alpha$ with dimensionless pressure gradient $\lambda_0 + \lambda \cos(\omega t)$; $\lambda_0 = 0.2; \lambda = 0.5; \lambda_r = 0.7; \omega = \frac{\pi}{2}$; $Pe = 8, 12, 16$; $Gr = 6$ and $Re = 12$. The last three figures of Fig. 1 show that the maximum value of velocity profile is attained at $y = 1$ with the same parameters as mentioned above for $Pe = 4$ and $Re = 12, 16, 24$. In the first three figures of Fig. 2, it is clear that the velocity field attains the maximum value at $y = 1$ with small fractional parameter $\beta$, then gradually decreases with $Re = 14$ and $Pe = 12, 14, 16$. The last three figures of Fig. 2 show that the temperature profile versus $y$. From these figures, it is clear that the maximum value is attained at $y = 1$ with fractional parameter $\alpha$. Figure 3 shows the Nusselt number profile versus time $t$. The Nusselt number profile shows the different variations of $N = 1, 6, 1, 7$ and $Pe = 0.3, 0.4, 0.5$ and becomes constant at $t = 0.04$.

6 Conclusion

The MHD convection flow of an Oldroyd-B fluid in a vertical channel is studied under a pressure gradient. The governing equations are converted into differential equations with the fractional model with the help of Caputo-time fractional derivatives. The Laplace transform method is applied to convert some physical parameters into dimensionless velocity, temperature, and Nusselt number. The Wright function represents the solution of temperature and Nusselt number in the form of Caputo-fractional derivative. The following points are concluded from the present work.

- The velocity field increases with the decrease in $\beta$ while it decreases with the increase in $Re$ and $Pe$. 
Fig. 1  Velocity profiles for different values of $Pe$ and $Re$ with the variation of $\alpha$
Fig. 2 Velocity and temperature profiles with the variation of $\beta$ and $\alpha$, respectively.
Fig. 3  Nusselt number profiles for different values of $N$ and $Pe$ with the variation of $\alpha$

- The temperature field $\psi$ increases by decreasing the values of $y$.
- For different values of $Pe$, the Nusselt number decreases. At $t = 0.4$, the graph becomes constant.

**Appendix**

\[ F(a_0, s, b_0, c_0) = \frac{\sinh[a_0 \sqrt{s + b_0}]}{s \sinh[c_0 \sqrt{s + b_0}]} \]  \hspace{1cm} (A1)

\[ \psi_k(a_0, t, b_0, c_0) = \frac{1}{2} [e^{-(2k+1)c_0-a_0}/\sqrt{\pi}] \text{erfc}(\frac{(2k+1)c_0-a_0}{2\sqrt{t}}) \]  \hspace{1cm} (A2)
−√b₀t) + erf((2k−1)tc₀−a₀√b₀)
× erf((2k−1)tc₀−a₀√b₀)
H(a₀0, s, b₀0, c₀)
= sinh[a₀0√s(x + b₀)]
sinh[c₀0√s(x + b₀)] = F(a₀0, s, b₀0, c₀)
h(a₀0, t, b₀0, c₀)
= L−1[H(a₀0, s, b₀0, c₀)]
= \left\{ ∫_0^∞ t^{-1} f(a₀0, x, b₀0, c₀)\phi(0, −a₀, −xt^{-α})dx, 0 < α < 1 \right\}
\left\{ f(a₀0, t, b₀0, c₀), \alpha = 1. \right\}

\begin{align}
U(s) &= \frac{1}{s}e^{-a₀s}\alpha, α₀ ≥ 0, 0 < σ < 1 \\
u(t) &= L^{-1}\{U(s)\}(t) = \phi(1, −a₀σ) \\
u(0) &= lim_{t→0^+} u(t) = lim_{σ→σ_0}sU(s) = lim_{σ→σ_0}\frac{1}{s}e^{-a₀s}\alpha = 0 \\
L\left\{ u'(t) \right\} &= sU(s) − U(0) = sU(s) = e^{-a₀s}\alpha \\
u'(t) &= t^{-1}\phi(0, −σ, −a₀t^{-α}).
\end{align}

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