HARMONIC MORPHISMS BETWEEN WEYL SPACES AND TWISTORIAL MAPS II

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This paper is dedicated to the memory of James Eells.

ABSTRACT

We define, on smooth manifolds, the notions of almost twistorial structure and twistorial map, thus providing a unified framework for all known examples of twistor spaces. The condition of being harmonic morphisms naturally appears among the geometric properties of submersive twistorial maps between low-dimensional Weyl spaces endowed with a nonintegrable almost twistorial structure due to Eells and Salamon. This leads to the twistorial characterisation of harmonic morphisms between Weyl spaces of dimensions four and three. Also, we give a thorough description of the twistorial maps with one-dimensional fibres from four-dimensional Weyl spaces endowed with the almost twistorial structure of Eells and Salamon.

INTRODUCTION

A predominant theme in the theory of harmonic maps is their relationship to holomorphicity. Right from their inception, harmonic maps have been recognised by Eells and Sampson to include holomorphic maps between Kähler manifolds, and a few years later, Lichnerowicz distinguished two sub-classes of almost Hermitian manifolds for which this also holds.

A far harder task is to find conditions forcing harmonic maps to be holomorphic. The clearest situation is between two-spheres since harmonicity then implies conformality, but, in general, additional conditions are required, on the map or on curvature.

From a Riemann surface, or a complex projective space, into a compact irreducible Hermitian symmetric space harmonic stability ensures holomorphicity

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In the broader context of compact Kähler manifolds, Siu [22] used a $\overline{\partial}\partial$-Bochner argument to prove, under strong negativity of the target curvature, the holomorphicity of rank four harmonic maps, from which he deduced the biholomorphicity of compact Kähler manifolds of the same homotopy type.

Whilst these results are of great importance in understanding harmonic maps, this scheme bears its own natural limits, if only because it requires the presence of complex structures, with their topological consequences.

To overcome this hurdle, one can replace the codomain with an adequate bundle admitting a natural complex structure, such that harmonicity of any map is given by holomorphicity of its lift to this bundle.

This strategy, which could be traced to Calabi and even Weierstrass, was put into effect by Eells and Salamon [8], who modified a well-known twistor construction of Atiyah, Hitchin and Singer (see Example 3.8, below) to define, in our terminology, an almost twistorial structure on any oriented four-dimensional Riemannian manifold (see Example 3.9). This yields a bijective correspondence between conformal harmonic maps and holomorphic curves (see Proposition 5.2). These ideas were extensively pursued by Bryant [3], and Burstall and Rawnsley [5] for (even-dimensional) Riemannian symmetric spaces.

The success of this strategy leads naturally to considering lifts on both the domain and the codomain, hence removing any need of pre-existing almost complex structures. The objective is two-fold: firstly, show that the existence of a holomorphic lift implies harmonicity and, secondly, find natural conditions under which harmonic maps admit holomorphic lifts.

As holomorphic maps are closed under composition, it seems that harmonic morphisms will have an important role in this programme.

Besides, the very conformal nature of their characterization makes Weyl geometry an ideal framework for their study, but it also turns out to provide examples of twistor spaces in dimension two, three and four.

In the complex category, a twistor, that is, a point of the twistor space of a manifold $M$, determines a submanifold, possibly, with self-intersections, of $M$. For example, the twistor space of an anti-self-dual complex-conformal four-dimensional manifold $(M^4, c)$ is the space of self-dual surfaces of $(M^4, c)$; also, the twistor space of a three-dimensional Einstein–Weyl space $(M^3, c, D)$ is the space of coisotropic surfaces of $(M^3, c, D)$ which are totally-geodesic with respect to $D$ (see [20]). Similar considerations apply to the flat complex-conformal even-dimensional manifolds and to the complex-Riemannian manifolds of constant curvature (see [18]). Also, the space of isotropic geodesics of a complex-conformal
A complex analytic map $\varphi : M \rightarrow N$ between manifolds endowed with twistorial structures is twistorial if it maps (some of the) twistors on $M$ to twistors on $N$ (see [20]).

In this paper we extend the notions of almost twistorial structure and twistorial map, to the smooth category. We show that, in the smooth category, a twistor on a manifold $M$ (endowed with a twistorial structure) is a pair $(N, J)$ where $N$ is a submanifold of $M$, possibly, with self-intersections, and $J$ is a linear CR-structure on the normal bundle of $N$ in $M$ (Remark 3.2). These submanifolds may well just be points. For example, the twistor space of a four-dimensional anti-self-dual conformal manifold $(M^4, c)$ is formed of pairs $(x, J_x)$ where $x \in M$ and $J_x$ is a positive orthogonal complex structure on $(T_x M, c_x)$. Similarly, the twistor space $Z$ of a three-dimensional conformal manifold $(M^3, c)$ is a five-dimensional CR-manifold consisting of the orthogonal nontrivial CR-structures on $(M^3, c)$ [13] (see Example 3.4 and note that, if we assume $(M^3, c)$ real-analytic, then $Z$ is a real hypersurface, endowed with the induced CR-structure, in the space of isotropic geodesics of the germ-unique complexification of $(M^3, c)$).

On the other hand, the twistor space of a three-dimensional Einstein–Weyl space (see Example 3.5 and Theorem 3.10) is formed of pairs $(\gamma, J)$ where $\gamma$ is a geodesic and $J$ is an orthogonal complex structure on the normal bundle of $\gamma$. The natural generalizations of this example, to Riemannian manifolds of constant curvature, are given in [18].

As in the complex category, a (smooth) map $\varphi : M \rightarrow N$ between manifolds endowed with twistorial structures is twistorial if it maps twistors on $M$ to twistors on $N$ (Definition 4.1). It follows that twistorial maps naturally generalize holomorphic maps. Examples of twistorial maps have been previously used to obtain constructions of Einstein and anti-self-dual manifolds (see [6], [19], [20] and the references therein).

In Section 1, we recall a few basic facts on harmonic morphisms between Weyl spaces. Section 2 is preparatory for Sections 3 and 4, where we give the definitions of almost twistorial structures and twistorial maps, on smooth manifolds. In Sections 3 and 4, we also give examples of almost twistorial structures and twistorial maps.

In the first part [14] of this work, we introduced the notion of harmonic morphism between (complex-)Weyl spaces and we continued the study (initiated in [20]) of the relations between harmonic morphisms and twistorial maps. As all of the known examples of complex analytic almost twistorial structures have smooth versions, all of the main results of [14] have ‘real’ versions. But not all smooth
almost twistorial structures come from complex analytic almost twistorial structures. The first example of such an almost twistorial structure is the almost twistorial structure of Eells and Salamon, mentioned above. We introduce similar almost twistorial structures (Examples 3.7 and 6.2) with respect to which a map (between Weyl spaces of dimensions four and three) is twistorial if and only if it is a harmonic morphism (Theorem 6.4(i)); this implies that there exists a bijective correspondence between one-dimensional foliations on \((M^4, c, D)\), which are locally defined by harmonic morphisms, and certain almost CR-structures on the bundle of positive orthogonal complex structures on \((M^4, c)\). Other similar results are given in Section 5.

In Section 6, we also give the necessary and sufficient conditions for a map with one-dimensional fibres from a four-dimensional Weyl space endowed with the almost twistorial structure of Eells and Salamon to be twistorial (Theorem 6.6). It follows that, all such twistorial maps are harmonic morphisms. Also, a generalized monopole equation (Definition 4.10(1)) is naturally involved.

1. Harmonic morphisms

In this section all the manifolds and maps are assumed to be smooth.

Let \(M^m\) be a manifold of dimension \(m\). If \(m\) is even then we denote by \(L\) the line bundle associated to the frame bundle of \(M^m\) through the morphism of Lie groups \(\rho_m : \text{GL}(m, \mathbb{R}) \to (0, \infty), \rho_m(a) = |\det a|^{1/m}, (a \in \text{GL}(m, \mathbb{R}))\); obviously, \(L\) is oriented. If \(m\) is odd then we denote by \(L\) the line bundle associated to the frame bundle of \(M^m\) through the morphism of Lie groups \(\rho_m : \text{GL}(m, \mathbb{R}) \to \mathbb{R}^*, \rho_m(a) = (\det a)^{1/m}, (a \in \text{GL}(m, \mathbb{R}))\); obviously, \(L^* \otimes TM\) is an oriented vector bundle. We say that \(L\) is the line bundle of \(M^m\) (cf. [6]).

Similar considerations apply to any vector bundle.

Let \(\varphi : M \to N\) be a submersion and let \(\mathcal{H}\) be a distribution on \(M\), complementary to the fibres of \(\varphi\). Let \(L_{\mathcal{H}}\) and \(L_N\) be the line bundles of \(\mathcal{H}\) and \(N\), respectively. As \(L_{\mathcal{H}}^2 = (\Lambda^n(\mathcal{H}))^2\) and \(L_N^2 = (\Lambda^n(TN))^2\), where \(n = \text{dim } N\), the differential of \(\varphi\) induces a bundle map \(\Lambda\) from \(L_{\mathcal{H}}^2\) to \(L_N^2\). If \(n\) is odd then we also have a bundle map \(\lambda\) from \(L_{\mathcal{H}}\) to \(L_N\); obviously, \(\Lambda = \lambda^2\). Furthermore, if \(n\) is odd, \(d\varphi\) and \(\lambda\) induce a bundle map from \(L_{\mathcal{H}}^* \otimes \mathcal{H}\) to \(L_N^* \otimes TN\), which will also be denoted by \(d\varphi\); note that, \(d\varphi : L_{\mathcal{H}}^* \otimes \mathcal{H} \to L_N^* \otimes TN\) is orientation preserving, on each fibre.

Let \(c\) be a conformal structure on \(M\); that is, \(c\) is a section of \(L^2 \otimes (\otimes^2 T^* M)\) which is ‘positive-definite’; that is, for any positive section \(s^2\) of \(L^2\) we have \(c = s^2 \otimes g_s\), where \(g_s\) is a Riemannian metric on \(M\); then \(g_s\) is a representative of \(c\). Therefore, \(c\) corresponds to a Riemannian metric on the vector bundle
is called the square dilation of $2^L$ and $M$ be a horizontally conformal submersion; denote by $L^2 \to \mathcal{H}$ such that, at each $x \in M$, the positive bases of $L^2_x$ are mapped into the cone of positive definite symmetric bilinear forms on $T^*_x M$. Obviously, $c$ also corresponds to a reduction of the frame bundle of $M$ to $\text{CO}(m, \mathbb{R})$, where $m = \dim M$; the total space of the reduction corresponding to $c$ is formed of the conformal frames on $(M, c)$.

If $\dim M$ is odd then local sections of $\H$ correspond to oriented local representatives of $c$; that is, (local) representatives of $c$, on some oriented open set of $M$. Let $\mathcal{H}'$ be a distribution on $M$. Then $c$ induces a conformal structure $c|_{\mathcal{H}'}$ on $\mathcal{H}'$ and, it follows that, we have an isomorphism, which depends of $c$, between $L^2$ and $L^2_{\mathcal{H}'}$.

**Definition 1.1 (cf. [2]).** Let $(M, c_M)$ and $(N, c_N)$ be conformal manifolds. A map $\varphi : (M, c_M) \to (N, c_N)$ is horizontally weakly conformal if at each point $x \in M$, either $d\varphi_x = 0$ or $d\varphi_x(\ker d\varphi_x)^\perp$ is a conformal linear isomorphism from $((\ker d\varphi_x)^\perp, (c_M)_x(\ker d\varphi_x)^\perp)$ onto $(T^*_x N, (c_N)_{\varphi(x)})$.

Let $(M, c_M)$ and $(N, c_N)$ be conformal manifolds and let $\varphi : (M, c_M) \to (N, c_N)$ be a horizontally conformal submersion; denote by $L_M$ and $L_N$ the line bundles of $M$ and $N$, respectively. Let $\mathcal{H}' = (\ker d\varphi)^\perp$ and denote by $L_{\mathcal{H}'}$ its line bundle. As $L^2_M = L^2_{\mathcal{H}'}$, the bundle map $\Lambda$ defines an isomorphism from $L^2_M$ to $\varphi^*(L^2_N)$ which is called the square dilation of $\varphi$. We have $\Lambda = \frac{1}{n} c(d\varphi, d\varphi)$, where $n = \dim N$ and $c$ is the conformal structure on the bundle $T^*M \otimes \varphi^*(TN)$ induced by $c_M$, $c_N$ and $\varphi$. Therefore, if $\varphi$ is horizontally weakly conformal then $\Lambda$ extends to a (smooth) section of $\text{Hom}(L^2_M, \varphi^*(L^2_N))$ which is zero over the set of critical points of $\varphi$.

If $\dim N$ is odd, the bundle map $\lambda$ from $L_{\mathcal{H}'}$ to $L_N$ is called the oriented dilation of $\varphi$. Note that, $d\varphi : (L^2_{\mathcal{H}'} \otimes \mathcal{H}', c|_{\mathcal{H}'}) \to (L^2_N \otimes TN, c_N)$ is an orientation preserving isometry, on each fibre.

Let $(M, c)$ be a conformal manifold. A connection $D$ on $M$ is conformal if $Dc = 0$; equivalently, $D$ is the covariant derivation of a principal connection on the bundle of conformal frames on $(M, c)$. If $D$ is torsion-free then it is called a Weyl connection and $(M, c, D)$ is a Weyl space.

**Definition 1.2 (cf. [2]).** (i) Let $(M, c, D)$ be a Weyl space. A harmonic function, on $(M, c, D)$, is a function $f$, (locally) defined on $M$, such that $\text{tr}_c(D^2 f) = 0$.

(ii) A map $\varphi : (M, c_M, D^M) \to (N, c_N, D^N)$ between Weyl spaces is a harmonic map if $\text{tr}_c(D^2 \varphi) = 0$, where $D$ is the connection on $\varphi^*(TN) \otimes T^*M$ induced by $D^M$, $D^N$ and $\varphi$.

(iii) A map $\varphi : (M, c_M, D^M) \to (N, c_N, D^N)$ between Weyl spaces is a harmonic
morphism if for any harmonic function \( f : V \to \mathbb{R} \), on \((N, c_N, D^N)\), with \( V \) an open set of \( N \) such that \( \varphi^{-1}(V) \) is nonempty, \( f \circ \varphi : \varphi^{-1}(V) \to \mathbb{R} \) is a harmonic function, on \((M, c_M, D^M)\).

Obviously, any harmonic function is a harmonic map and a harmonic morphism, if \( \mathbb{R} \) is endowed with its conformal structure and canonical connection.

The following result is basic for the theory of harmonic morphisms (see [14], [2]).

**Theorem 1.3.** A map between Weyl spaces is a harmonic morphism if and only if it is a harmonic map which is horizontally weakly conformal.

We end this section by recalling the fundamental equation for horizontally conformal submersions between Weyl spaces ([14]; cf. [2]).

**Proposition 1.4.** Let \( \varphi : (M^m, c_M, D^M) \to (N^n, c_N, D^N) \) be a horizontally conformal submersion between Weyl spaces; \( m = \dim M, n = \dim N \).

Let \( L \) be the line bundle of \( M^m \) and let \( \mathcal{V} = \ker d\varphi \), \( \mathcal{H} = \mathcal{V}^\perp \); denote by \( D \) the Weyl connection of \((M^m, c_M, \mathcal{V})\).

Then we have the following equality of partial connections on \( L^{m-n} \), over \( \mathcal{H} \),

\[
(\mathcal{H} D)^{m-n} + \text{trace}_{c_M}(Dd\varphi)^\flat = (\mathcal{H} D^M)^{m-2} \otimes (D^N)^{-(n-2)},
\]

where we have denoted by the same symbol \( D^N \) and its pull-back by \( \varphi \).

2. Complex distributions

Unless otherwise stated, all the manifolds and maps are assumed to smooth.

**Definition 2.1.** A complex distribution on a manifold \( M \) is a complex subbundle \( \mathcal{F} \) of \( T^C M \) such that \( \dim(\mathcal{F}_x \cap \bar{\mathcal{F}}_x) \), \( (x \in M) \), is constant. If \( \mathcal{F} \cap \bar{\mathcal{F}} \) is the zero bundle then \( \mathcal{F} \) is an almost CR-structure on \( M \), and \((M, \mathcal{F})\) an almost CR-manifold.

**Example 2.2.** 1) Let \( F \) be an almost \( f \)-structure on a manifold \( M \); that is, \( F \) is a section of \( \text{End}(TM) \) such that \( F^3 + F = 0 \) [25]. We denote by \( T^0M, T^{1,0}M, T^{0,1}M \) the eigendistributions of \( F^C \in \Gamma(\text{End}(T^C M)) \) corresponding to the eigenvalues \( 0, i, -i \), respectively. Then \( T^0M \oplus T^{1,0}M \) is a complex distribution on \( M \) and \( T^{0,1}M \) is an almost CR-structure on \( M \). We say that \( T^0M \oplus T^{0,1}M \) is the complex distribution associated to \( F \); similarly, \( T^{0,1}M \) is the almost CR-structure associated to \( F \).

2) Let \( M \) be endowed with a complex distribution \( \mathcal{F} \) and let \( N \subseteq M \) be a submanifold. Suppose that \( \dim(T^C_x N \cap \mathcal{F}_x) \) and \( \dim(T^C_x N \cap \mathcal{F}_x \cap \bar{\mathcal{F}}_x) \), \( (x \in N) \),
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are constant. Then \( T^C N \cap \mathcal{F} \) is a complex distribution on \( N \) which we call the 
complex distribution induced by \( \mathcal{F} \) on \( N \).

In particular, if \( N \) is a real hypersurface in the complex manifold \( M \) then 
\( T^C N \cap T^{0,1} M \) is an almost CR-structure on \( N \).

Next, we define the notion of holomorphic map between manifolds endowed 
with complex distributions.

**Definition 2.3.** Let \( \mathcal{F}^M \) and \( \mathcal{F}^N \) be complex distributions on \( M \) and \( N \), respectively. A map \( \varphi : (M, \mathcal{F}^M) \to (N, \mathcal{F}^N) \) is holomorphic if 
\( d\varphi(\mathcal{F}^M) \subseteq \mathcal{F}^N \).

A map \( \varphi : (M, F^M) \to (N, F^N) \) between manifolds endowed with \( f \)-structures 
is holomorphic if \( \varphi : (M, \mathcal{F}^M) \to (N, \mathcal{F}^N) \) is holomorphic where \( \mathcal{F}^M \) and \( \mathcal{F}^N \) 
are the complex distributions, on \( M \) and \( N \), associated to \( F^M \) and \( F^N \), respectively.

Let \( F \) be an almost \( f \)-structure on \( M \) and \( J \) an almost complex structure on \( N \). Then, a map \( \varphi : (M, F) \to (N, J) \) is holomorphic if and only if 
\( d\varphi \circ F = J \circ d\varphi \).

**Definition 2.4.** Let \( M \) be a manifold. A subbundle \( E \) of \( T^C M \) is integrable if 
for any sections \( X, Y \) of \( E \) we have that \( [X, Y] \) is a section of \( E \).

A CR-structure is an integrable almost CR-structure; a CR-manifold is a manifold 
endowed with a CR-structure.

An almost \( f \)-structure is integrable if its associated complex distribution is 
integrable; an \( f \)-structure is an integrable almost \( f \)-structure.

Note that a complex distribution \( \mathcal{F}^M \) on \( M \) is integrable if and only if for any 
point \( x \in M \), there exists a holomorphic submersion \( \varphi : (U, \mathcal{F}^M|_U) \to (N, \mathcal{F}^N) \), 
from some open neighbourhood \( U \) of \( x \), onto some CR-manifold \( (N, \mathcal{F}^N) \), such 
that \( \ker d\varphi = (\mathcal{F}^M \cap \mathcal{F}^M)|_U \). If \( U = M \) and \( \varphi \) has connected fibres then \( \mathcal{F}^M \) 
is called simple; then, obviously, up to a CR-diffeomorphism, \( \varphi \) is unique; we call 
it the holomorphic submersion corresponding to \( \mathcal{F}^M \).

The following definition will, also, be useful in the next section.

**Definition 2.5.** Let \( (P, J) \) be a complex manifold and let \( Q \subseteq P \) be a complex 
submanifold. A holomorphic distribution (on \( P \)) along \( Q \) is a holomorphic sub-
bundle of \( (T^{1,0} P)|_Q \).

Let \( (M, \mathcal{F}) \) be a manifold endowed with an integrable complex distribution 
such that \( TM = \mathcal{F} + \mathcal{F} \). Let \( N \subseteq M \) be a submanifold on which \( \mathcal{F} \) induces 
a complex distribution. A complex subbundle \( E \) of \( \mathcal{F}|_N \) is called a holomorphic 
distribution (on \( M \)) along \( N \) if, for any \( x \in N \), there exists an open set
Let $U \subseteq M$, containing $x$, and a holomorphic submersion $\varphi$ from $(U, \mathcal{F}|_U)$ onto some complex manifold $(P, J)$, with $\ker d\varphi = (\mathcal{F} \cap \mathcal{F})|_U$, such that $\varphi|_{U \cap N}$ is a holomorphic submersion from $U \cap N$ onto a complex submanifold of $(P, J)$ and $d\varphi|_{U \cap N}$ maps $E|_{U \cap N}$ isomorphically on each fibre, onto a holomorphic distribution along $\varphi(U \cap N)$ (that is, onto a holomorphic subbundle of $T^{1,0}P|_{\varphi(U \cap N)}$); if $N = M$ then we say that $E$ is a holomorphic distribution on $M$.

Let $F$ be a $f$-structure on a manifold $M$. Obviously, $T^{1,0}M$ is a holomorphic distribution on $M$.

Let $E$ be a complex vector bundle over $M$ endowed with a complex linear connection $\nabla$ (that is, $\nabla$ is a complex-linear map $\Gamma(E) \to \Gamma(\text{Hom}_C(T_C^*M, E))$ such that $\nabla(sf) = (\nabla s)f + s \otimes df$ for any section $s$ of $E$ and any complex-valued function $f$ on $M$; equivalently, we have $\nabla^C = \nabla \oplus \overline{\nabla}$, with respect to the decomposition $E^C = E \oplus \overline{E}$).

Let $\pi : \text{Gr}_k(E) \to M$ be the bundle of complex vector subspaces of complex dimension $k$ of $E$; we shall denote by $\mathcal{H} (\subseteq T(\text{Gr}_k(E)))$ the connection induced by $\nabla$ on $\text{Gr}_k(E)$. Note that, as $\text{Gr}_k(E)$ is a bundle whose typical fibre is a complex manifold and its structural group a complex Lie group acting holomorphically on the fibre, we have $(\ker d\pi)^C = (\ker d\pi)^{1,0} \oplus (\ker d\pi)^{0,1}$.

The following result, which we do not imagine to be new, will be useful later on. We omit the proof.

**Proposition 2.6.** Let $p$ be a section of $\text{Gr}_k(E)$ (equivalently, $p$ is a complex vector subbundle of complex rank $k$ of $E$). Let $X \in T^C_{x_0}M$ for some $x_0 \in M$; denote by $p_0 = p(x_0)$.

(a) The following assertions are equivalent:
   (i) $(dp)^C(X) \in \mathcal{H}^C_{p_0}$.
   (ii) $\nabla^C_Xs \in p_0^C$ for any section $s$ of $p^C$.

(b) The following assertions are equivalent:
   (iii) $(dp)^C(X) \in \mathcal{H}^C_{p_0} \oplus (\ker d\pi)^{0,1}_{p_0}$.
   (iv) $\nabla_Xs \in p_0$ for any section $s$ of $p$.

Note that, if $X$ is real (that is, $X \in TM$) then the assertions (i), . . . , (iv), of Proposition 2.6, are equivalent.

### 3. ALMOST TWISTORIAL STRUCTURES

In this section, we define, in the smooth category, the notion of (almost) twistorial structure.
Definition 3.1 (cf. [21], [20]). An almost twistorial structure, on a manifold \( M \), is a quadruple \( \tau = (P, M, \pi, \mathcal{F}) \) where \( \pi : P \to M \) is a locally trivial fibre space and \( \mathcal{F} \) is a complex distribution on \( P \) which induces almost complex structures on each fibre of \( \pi \); if \( \mathcal{F} \) is induced by an almost \( f \)-structure \( F \) then, also, \( (P, M, \pi, F) \) is called an almost twistorial structure.

The almost twistorial structure \( \tau = (P, M, \pi, \mathcal{F}) \) is integrable if \( \mathcal{F} \) is integrable. A twistorial structure is an integrable almost twistorial structure; the leaf space of \( \mathcal{F} \cap \mathcal{F} \) is called the twistor space of \( \tau \).

A twistorial structure \( \tau = (P, M, \pi, \mathcal{F}) \) is called simple if \( \mathcal{F} \) is simple; if \( \tau \) is simple with \( \varphi : (P, \mathcal{F}) \to Z \) the corresponding holomorphic submersion then \( d\varphi(\mathcal{F}) \) is a CR-structure on \( Z \).

Remark 3.2. Let \( \tau = (P, M, \pi, \mathcal{F}) \) be a twistorial structure and let \( Z \) be its twistor space. Each \( z \in Z \) determines a pair \((N_z, J_z)\) where \( N_z \) is a submanifold of \( M \), possibly, with self-intersections, and \( J_z \) is a linear CR-structure on the normal bundle of \( N_z \).

Indeed, let \( N_z = \varphi^{-1}(z) \), where \( \varphi : P \to Z \) is the (continuous) projection whose fibres are the leaves of \( \mathcal{F} \cap \mathcal{F} \). Then \( \pi|_{N_z} : N_z \to M \) is an immersion. Also, let \((\ker d\pi)^{0,1} = \mathcal{F} \cap (\ker d\pi)^{1} \). Then the restriction to \( N_z \) of the quotient of \( \mathcal{F} \) through \( T N_z \oplus (\ker d\pi)^{0,1} \) defines a linear CR-structure \( J_z \) on the normal bundle \((\pi|_{N_z})^*(TM)/TN_z \) of \( N_z \) in \( M \).

Now, we formulate the examples of almost twistorial structures with which we shall work.

Example 3.3. If \( \mathcal{F} \) is a complex distribution on \( M \) then \((M, M, \text{Id}_M, \mathcal{F})\) is an almost twistorial structure.

In particular, let \((M^2, c)\) be a two-dimensional oriented conformal manifold. Then, as the identity component of \( \text{CO}(2, \mathbb{R}) \) is isomorphic to \( \mathbb{C}^\ast \), there exists a unique (almost) Hermitian structure \( J \) on \((M^2, c)\) such that if \( X \in TM \) then \((X, JX)\) is a positively oriented frame on \( M^2 \); then \((M, M, \text{Id}, J)\) is a twistorial structure. Similarly, any oriented vector bundle of (real) rank two, endowed with a conformal structure, is a complex line bundle.

Example 3.4 ([12]). Let \((M^m, c)\) be a conformal manifold of dimension \( m \geq 2 \). Let \( \pi : P \to M \) be the bundle of null direction on \( M^m \). Obviously, \( P \) is a bundle with typical fibre \( Q_{m-2} \) and structural group \( \text{PCO}(m, \mathbb{C}) \) where \( Q_{m-2} \) is the nondegenerate hyperquadric in \( \mathbb{C}P^{m-1} \); in particular, \( \dim P = 3m - 4 \).

Endow \((M^m, c)\) with a Weyl connection \( D \) and let \( \mathcal{H} \subseteq TP \) be the connection induced by \( D \) on \( P \). We denote by \( \mathcal{H}^{1,0} \) the subbundle of \( \mathcal{H}^\mathbb{C} \) such that, at
each \( p \in P \), the subspace \( \mathcal{H}^{1,0}_p \subseteq \mathcal{H}^0_p \) is the horizontal lift of \( p \subseteq T_{\pi(p)}M \). Then \( \mathcal{F} = (\ker d\pi)^{1,0} \oplus \mathcal{H}^{1,0} \) is a CR-structure on \( P \). Moreover, \( \mathcal{F} \) does not depend of \( D \).

Then \( \tau = (P, M, \pi, \mathcal{F}) \) is a twistorial structure on \( M^m \). If \( (M^m, c) \) is real analytic then, locally, \( P \) is a (real) submanifold of the space of null geodesics \( \mathcal{N} \) of the complexification of \( (M^m, c) \); moreover, the CR-structure \( \mathcal{F} \) of \( P \) is induced by the complex structure of \( \mathcal{N} \).

Note that, if \( m = 2 \) then \( P \) is a \( \mathbb{Z}_2 \)-covering space of \( M^2 \) which is trivial if and only if \( M^2 \) is oriented. Assuming \( M^2 \) oriented we obtain that \( T^{1,0}P \) restricted to one of the components of \( P \) defines the Hermitian structure of Example 3.3.

**Example 3.5** ([10], [21]). Let \( (M^3, c, D) \) be a three-dimensional Weyl space. Let \( \pi : P \to M \) be the bundle of nonzero skew-adjoint \( f \)-structures on \( (M^3, c) \). Obviously, \( P \) is also the bundle of nonzero skew-adjoint \( f \)-structures on the oriented Riemannian bundle \( (L^* \otimes TM, c) \). Therefore, \( P \) is isomorphic to the sphere bundle of \( (L^* \otimes TM, c) \). In particular, the typical fibre and the structural group of \( P \) are \( \mathbb{C}P^1 \) and \( \text{PGL}(2, \mathbb{C}) \), respectively.

We could also define \( P \) as follows: firstly, note that there exists a unique oriented Riemannian structure, on the vector bundle \( E \) of skew-adjoint endomorphisms on \( (M, c) \), with respect to which \([A, B] = A \times B\), for any \( A, B \in E \); then \( P \) is the sphere bundle of \( E \).

The bundle \( P \) is also isomorphic with the bundle of oriented two-dimensional subspaces on \( M^3 \). Therefore, there exists a bijective correspondence between one-dimensional foliations on \( M^3 \), with oriented orthogonal complement, and almost \( f \)-structures on \( (M^3, c) \). Furthermore, under this bijection, conformal one-dimensional foliations correspond to (integrable) \( f \)-structures.

Let \( k \) be a section of \( L^* \). We define a conformal connection connection \( \nabla \) on \( (M, c) \) by \( \nabla_XY = D_XY + \frac{1}{2} kX \times Y \) for any vector fields \( X \) and \( Y \) on \( M \). We say that \( \nabla \) is the connection associated to \( D \) and \( k \).

Let \( \mathcal{H} \subseteq TP \) be the connection induced by \( \nabla \) on \( P \). We denote by \( \mathcal{H}^0, \mathcal{H}^{1,0} \) the subbundles of \( \mathcal{H}^C \) such that, at each \( p \in P \), the subspaces \( \mathcal{H}^0_p, \mathcal{H}^{1,0}_p \subseteq \mathcal{H}_p^C \) are the horizontal lifts of the eigenspaces of \( p^C \in \text{End}(T_{\pi(p)}^CM) \) corresponding to the eigenvalues 0, i., respectively.

We define the almost \( f \)-structure \( \mathcal{F} \) on \( P \) with respect to which \( T^0 P = \mathcal{H}^0 \) and \( T^{1,0} P = (\ker d\pi)^{1,0} \oplus \mathcal{H}^{1,0} \). Then \( \tau = (P, M, \pi, \mathcal{F}) \) is an almost twistorial structure on \( M \).

**Remark 3.6.** Let \( (M^3, c) \) be a three-dimensional conformal manifold. Let \( \pi : P \to M \) be the bundle of nonzero skew-adjoint \( f \)-structures on \( (M^3, c) \).
Let $D$ be a Weyl connection on $(M^3, c)$ and let $\nabla$ be a conformal connection on $(M^3, c)$. The following assertions are equivalent:

(i) $\nabla$ and $D$ induce, by applying the construction of Example 3.5 (with $H$ the connection on $P$ induced by $\nabla$ and $D$, respectively), the same almost $f$-structure on $P$.

(ii) $\nabla$ and $D$ are projectively equivalent.

(iii) There exists a section $k$ of $L^*$ such that $\nabla$ is the connection associated to $D$ and $k$.

Example 3.7 (cf. [8]). Under the same hypotheses as in Example 3.5, we define the almost $f$-structure $F'$ on $P$ with respect to which we have $T^0P = H^0$ and $T^1P = (\ker d\pi)^{0,1} \oplus H^{1,0}$. Then $\tau' = (P, M, \pi, F')$ is an almost twistorial structure on $M$.

Example 3.8 ([1]). Let $(M^4, c, D)$ be a four-dimensional oriented Weyl space. Let $\pi : P \to M$ be the bundle of positive orthogonal complex structures on $(M^4, c)$. Obviously, $P$ is also the bundle of positive orthogonal complex structures on the oriented Riemannian bundle $(L^* \otimes TM, c)$. Let $E$ be the adjoint bundle of $(L^* \otimes TM, c)$ and let $*_c$ be the involution of $E$ induced by the Hodge star-operator of $(L^* \otimes M^4, c)$, under the isomorphism $E = \Lambda^2(L \otimes T^* M)$. Then $E = E_+ \oplus E_-$ where $E_\pm$ is the vector bundle, of rank three, formed of the eigenvectors of $*_c$ corresponding to the eigenvalue $\pm 1$. There exists a unique oriented Riemannian structure $<\cdot, \cdot>$ on $E_\pm$ with respect to which $AB = -<A, B> \text{Id}_{TM} \pm A \times B$ for any $A, B \in E_\pm$. It follows that $P$ is the sphere bundle of $E_\pm$.

Similarly to Example 3.5, there exists a bijective correspondence between two-dimensional distributions $\mathcal{F}$ on $M^4$, with oriented orthogonal complement, and pairs $(J, K)$ of almost Hermitian structures on $(M^4, c)$, with $J$ positive and $K$ negative, such that $J|_{\mathcal{F}^\perp} = K|_{\mathcal{F}^\perp}$.

Let $\mathcal{H} \subseteq TP$ be the connection induced by $D$ on $P$. We denote by $\mathcal{H}^{1,0}$ the subbundle of $\mathcal{H}^C$ such that, at each $p \in P$, the subspace $\mathcal{H}^{1,0}_p \subseteq \mathcal{H}^C_p$ is the horizontal lift of the eigenspace of $p^C \in \text{End}(T^C_{\pi(p)}M)$ corresponding to the eigenvalue $i$. We define the almost complex structure $J$ on $P$ with respect to which $T^{1,0}P = (\ker d\pi)^{1,0} \oplus \mathcal{H}^{1,0}$. Then $(P, M, \pi, J)$ is an almost twistorial structure on $M$.

Example 3.9 ([8]). Let $(M^4, c, D)$ be a four-dimensional oriented Weyl space. With the same notations as in Example 3.8, let $J'$ be the almost complex structure on $P$ with respect to which $T^{1,0}P = (\ker d\pi)^{0,1} \oplus \mathcal{H}^{1,0}$. Then $\tau' = (P, M, \pi, J')$ is an almost twistorial structure on $M$. 
Next, we recall the necessary and sufficient conditions for the integrability of the almost twistorial structures of Examples 3.5 and 3.8.

**Theorem 3.10** ([10], [9]). Let \((M^3, c, D)\) be a three-dimensional Weyl space and let \(k\) be a section of the dual of the line bundle of \(N^3\). Let \(\tau = (P, M, \pi, F)\) be the almost twistorial structure of Example 3.5. Let \(\mathcal{H} (\subseteq TP)\) be the connection induced by \(\nabla\) on \(P\).

Then \(\tau\) depends only of \((M^3, c, D)\) whilst \(T^1,0 P\) depends only of \((M^3, c)\). Moreover, \(\tau\) is integrable if and only if \((M^3, c, D)\) is Einstein–Weyl.

Furthermore, if \((M^3, c, D)\) is Einstein–Weyl then we have the following.

(i) \(\mathcal{H}^{1,0}\) restricted to any fibre \(P_x = \mathbb{CP}^1\) of \(P\) is a holomorphic distribution, along \(P_x\), isomorphic to \(\mathcal{O}(2), (x \in M)\).

(ii) Locally, the following four assertions are equivalent:

   (ii1) \(\mathcal{H}^{1,0}\) is a holomorphic distribution on \(P\).

   (ii2) \(\mathcal{H}\) is integrable.

   (ii3) The connection induced by \(\nabla\) on \(L^* \otimes TM\) is flat.

   (ii4) The scalar curvature \(s^D\) of \(D\) satisfies \(s^D = \frac{3}{2} k^2\), and \(*_c Dk = F^D\) where \(F^D\) is the curvature form of the connection induced by \(D\) on \(L\).

**Theorem 3.11** ([1], see [7]). Let \((M^4, c, D)\) be a four-dimensional oriented Weyl space and let \(\tau = (P, M, \pi, J)\) be the almost twistorial structure of Example 3.8. Let \(\mathcal{H} (\subseteq TP)\) be the connection induced by \(D\) on \(P\).

Then \(\tau\) depends only of \((M^4, c)\). Moreover, \(\tau\) is integrable if and only if \((M^4, c)\) is anti-self-dual.

Furthermore, if \((M^4, c)\) is anti-self-dual then we have the following.

(i) \(\mathcal{H}^{1,0}\) restricted to any fibre \(P_x = \mathbb{CP}^1\) of \(P\) is a holomorphic distribution, along \(P_x\), isomorphic to \(\mathcal{O}(1) \oplus \mathcal{O}(1), (x \in M)\).

(ii) \(\mathcal{H}^{1,0}\) is a holomorphic distribution on \(P\) if and only if, locally, either \(D\) is the Levi-Civita connection of an Einstein representative of \(c\) or \(D\) is the Obata connection of a hyper-Hermitian structure on \((M^4, c)\).

**Remark 3.12.** 1) Let \((M^3, c, D)\) be a three-dimensional Weyl space and let \(k\) be a section of \(L^*\), where \(L\) is the line bundle of \(N^3\). Let \(T^1,0 P\) be defined as in Example 3.5.

Then by associating to any \(p \in P\) the eigenspace corresponding to \(-i\) we can identify \(P\) with the bundle of null directions on \((N^3, c)\). Furthermore, under this isomorphism, \(T^1,0 P\) corresponds to the CR-structure of Example 3.4, associated to \((N^3, c)\). Conversely, any conformal connection on \((N^3, c)\) having this property is the connection associated to a Weyl connection on \((N^3, c)\) and a section of \(L^*\).

2) Let \((M^3, c)\) be a three-dimensional conformal manifold and let \(D\) be a Weyl
connection on \((M^3, c)\). Also, let \(k\) be a section of the dual of the line bundle \(L\) of \(M^3\). Then the almost twistorial structure \(\tau'\) of Example 3.7, associated to \((M^3, c, D, k)\) is nonintegrable; that is, always not integrable (cf. [8]; apply the integrability result proved in [18] to show that \(\tau'\) is nonintegrable). Furthermore, if \(D_1\) is another Weyl connection on \((M^3, c)\) and \(k_1\) a section of \(L^*\) such that the almost twistorial of Example 3.7, associated to \((M^3, c, D_1, k_1)\), is equal to \(\tau'\) then \(D = D_1\) and \(k = k_1\).

3) Let \((M^4, c)\) be a four-dimensional oriented conformal manifold and let \(D\) be a Weyl connection on \((M^4, c)\). Then the almost twistorial structure \(\tau'\) of Example 3.9, associated to \((M^4, c, D)\) is nonintegrable (cf. [8]; apply the integrability result proved in [18] to show that \(\tau'\) is nonintegrable). Furthermore, if \(D_1\) is another Weyl connection on \((M^4, c)\) such that the almost twistorial structure of Example 3.9, associated to \((M^4, c, D_1)\), is equal to \(\tau'\) then \(D = D_1\).

4. Twistorial maps

We start this section with the definition of twistorial maps (cf. [21], [20]).

**Definition 4.1.** Let \(\tau_M = (P_M, M, \pi_M, \mathcal{F}^M)\) and \(\tau_N = (P_N, N, \pi_N, \mathcal{F}^N)\) be almost twistorial structures and let \(\varphi : M \to N\) be a map. Suppose that there exists a locally trivial fibre subspace \(\pi_{M,\varphi} : P_{M,\varphi} \to M\) of \(\pi_M : P_M \to M\) and a map \(\Phi : P_{M,\varphi} \to P_N\) with the properties:

1) \(\mathcal{F}^M\) induces a complex distribution \(\mathcal{F}_{M,\varphi}\) on \(P_{M,\varphi}\) and almost complex structures on each fibre of \(\pi_{M,\varphi}\) such that \(\pi_M(\mathcal{F}_p^M) = \pi_{M,\varphi}(\mathcal{F}_{M,\varphi}^p)\), for any \(p \in P_{M,\varphi}\).

2) \(\varphi \circ \pi_{M,\varphi} = \pi_N \circ \Phi\).

Then \(\varphi : (M, \tau_M) \to (N, \tau_N)\) is a twistorial map (with respect to \(\Phi\)) if the map \(\Phi : (P_{M,\varphi}, \mathcal{F}_{M,\varphi}) \to (P_N, \mathcal{F}^N)\) is holomorphic. If \(\mathcal{F}^M_{M,\varphi}\) and \(\mathcal{F}^N\) are simple complex distributions, with \((P_{M,\varphi}, \mathcal{F}_{M,\varphi}) \to Z_{M,\varphi}\) and \((P_N, \mathcal{F}^N) \to Z_N\), respectively, the corresponding holomorphic submersions onto CR-manifolds, then \(\Phi\) induces a holomorphic map \(Z_{\varphi} : Z_{M,\varphi} \to Z_N\) which is called the twistorial representation of \(\varphi\).

**Remark 4.2.** With the same notations as in Definition 3.1, we have that \(\tau_{M,\varphi} = (P_{M,\varphi}, M, \pi_{M,\varphi}, \mathcal{F}_{M,\varphi})\) is an almost twistorial structure on \(M\). Obviously, \(\tau_{M,\varphi}\) is integrable if \(\tau_M\) is integrable. Then from (1) it follows that the twistor space \(Z_{M,\varphi}\) of \(\tau_{M,\varphi}\) is a topological subspace of the twistor space \(Z_M\) of \(\tau_M\). Moreover, for any \(z \in Z_{M,\varphi}\) the pair \((N_z, J_z)\) of Remark 3.2 applied to \(\tau_{M,\varphi}\) is equal to the pair determined by \(z\) as a point of \(Z_M\).

If \(\tau_M\) is simple then \(\tau_{M,\varphi}\) is also simple and \(Z_{M,\varphi}\) is a submanifold of \(Z_M\).
Moreover, if we denote by $C_M$ and $C_{M,\varphi}$ the CR-structures of $Z_M$ and $Z_{M,\varphi}$, respectively, then $C_{M,\varphi} = C_M \cap T^C Z_{M,\varphi}$.

Next, we give examples of twistorial maps between manifolds endowed with the almost twistorial structures of Examples 3.3, 3.5 or 3.8.

**Example 4.3.** Let $(M^2, c_M)$ and $(N^2, c_N)$ be two-dimensional oriented conformal manifolds. Let $\tau_M = (M, M, \text{Id}_M, J_M)$ and $\tau_N = (N, N, \text{Id}_N, J_N)$ be the twistorial structures of Example 3.3 associated to $(M^2, c_M)$ and $(N^2, c_N)$, respectively.

Let $\varphi : M^2 \to N^2$ be a map. Obviously, $\varphi : (M^2, \tau_M) \to (N^2, \tau_N)$ is twistorial (with respect to $\varphi$) if and only if $\varphi : (M^2, J_M) \to (N^2, J_N)$ is holomorphic.

The following three examples are, essentially, due to [10], [21].

**Example 4.4.** Let $(M^3, c, D)$ be a three-dimensional Weyl space and let $N^1$ be a (connected) one-dimensional submanifold of $M^3$ such that $TN^1$ is orientable.

Let $\tau_M = (P, M, \pi, F)$ be the almost twistorial structure of Example 3.5, associated to $(M^3, c, D)$. Also, endow $N^1$ with the trivial $f$-structure and let $\tau_N = (N, N, \text{Id}_N, 0)$ be the twistorial structure given by $(N^1, 0)$, as in Example 3.3.

Let $p_N$ be the section of $P^5$ over $N^1$ such that, $(p_N)_x|_{T_xN} = 0$, for any $x \in N$.

The following assertions are equivalent:

(i) $(N^1, \tau_N) \hookrightarrow (M^3, \tau_M)$ is twistorial (with respect to $p_N$).

(ii) $N^1$ is a geodesic of $D$.

**Example 4.5.** Let $(M^3, c, D)$ be a three-dimensional Weyl space and let $N^2$ be an oriented surface in $M^3$. Let $\tau_M = (P, M, \pi, F)$ be the almost twistorial structure of Example 3.5, associated to $(M^3, c, D)$, and let $\tau_N = (N, N, \text{Id}_N, J)$ be the twistorial structure of Example 3.3, associated to $(N^2, c|_N)$.

Let $p_N$ be the section of $P^5$ over $N^2$ such that, $(p_N)_x|_{T_xN} = J_x$, for any $x \in N$.

The following assertions are equivalent:

(i) $(N^2, \tau_N) \hookrightarrow (M^3, \tau_M)$ is twistorial (with respect to $p_N$).

(ii) $p_N : (N^2, J) \to (P^5, F)$ is holomorphic.

(iii) $N^2$ is a totally umbilical submanifold of $(M^3, c)$.

It follows that the equivalence (i) $\iff$ (iii) also holds for a surface in a conformal manifold endowed with the twistorial structure of Example 3.4.

**Example 4.6.** Let $(M^3, c_M, D)$ be a three-dimensional Weyl space and let $(N^2, c_N)$ be a two-dimensional oriented conformal manifold. Let $\tau_M = (P, M, \pi, F)$ be the almost twistorial structure of Example 3.5, associated to $(M^3, c_M, D)$, and let $\tau_N = (N, N, \text{Id}_N, J)$ be the twistorial structure of Example 3.3, associated to
(\(N^2, c_N\)).

Let \(\varphi : M^3 \to N^2\) be a submersion and let \(F^\varphi\) be the almost \(f\)-structure on \((M^3, c)\) determined by \(\ker d\varphi\) and the orientation of \(N^2\) (see Example 3.5). Define \(\Phi = \varphi \circ p_{\varphi}^{-1} : p_{\varphi}(M) \to N\) where \(p_{\varphi}\) is the section of \(P\) corresponding to \(F^\varphi\). Note that, \(p_{\varphi}\) restricted to any fibre \(\varphi^{-1}(y)\) is equal to \(p_{\varphi^{-1}(y)}\) of Example 4.5, \((y \in \varphi(M))\).

The following assertions are equivalent:

(i) \(\varphi : (M^3, \tau_M) \to (N^2, \tau_N)\) is twistorial (with respect to \(\Phi\)).

(ii) \(p_{\varphi} : (M^3, F^\varphi) \to (P^5, \mathcal{F})\) is holomorphic and \(\varphi : (M^3, c_M) \to (N^2, c_N)\) is horizontally conformal.

(iii) The fibres of \(\Phi\) are tangent to the connection induced by \(D\) on \(P^5\) and \(\varphi : (M^3, c_M) \to (N^2, c_N)\) is horizontally conformal.

(iv) \(\varphi : (M^3, c_M, D) \to (N^2, c_N)\) is a horizontally conformal submersion with geodesic fibres.

**Example 4.7** (\([8]\)). Let \((M^4, c)\) be a four-dimensional oriented conformal manifold and let \(N^2\) be an oriented surface in \(M^4\). Let \(\tau_M = (P, M, \pi, \mathcal{J})\) be the almost twistorial structure of Example 3.8, associated to \((M^4, c)\), and let \(\tau_N = (N, N, \text{Id}_N, J)\) be the twistorial structure of Example 3.3, associated to \((N^2, c_N)\).

Let \(p_N\) be the section of \(P^6\) over \(N^2\) such that \((p_N)_x|_{T_x N} = J_x\), for any \(x \in N\).

The following assertions are equivalent:

(i) \((N^2, \tau_N) \hookrightarrow (M^4, \tau_M)\) is twistorial (with respect to \(p_N\)).

(ii) \(p_N : (N^2, J) \to (P^6, \mathcal{J})\) is holomorphic.

If we endow \(M^4\) with the opposite orientation then Example 3.8 gives another almost twistorial structure \(\tilde{\tau}_M = (\tilde{\mathcal{P}}, \tilde{M}, \tilde{\pi}, \tilde{\mathcal{J}})\).

The following assertions are equivalent:

(a) \((N^2, \tau_N) \hookrightarrow (M^4, \tau_M)\) and \((N^2, \tau_N) \hookrightarrow (M^4, \tilde{\tau}_M)\) are twistorial.

(b) \(N^2\) is a totally umbilical submanifold of \((M^4, c)\).

**Example 4.8** (\([23]\)). Let \((M^4, c_M)\) and \((N^2, c_N)\) be oriented conformal manifolds of dimensions four and two, respectively. Let \(\tau_M = (P, M, \pi, \mathcal{J})\) be the almost twistorial structure of Example 3.8, associated to \((M^4, c_M)\), and let \(\tau_N = (N, N, \text{Id}_N, J)\) be the twistorial structure of Example 3.3, associated to \((N^2, c_N)\).

Let \(\varphi : M^4 \to N^2\) be a submersion and let \(J^\varphi\) be the positive almost Hermitian structure on \((M^4, c_M)\) determined by \(\ker d\varphi\) and the orientation of \(N^2\) (see Example 3.8). Define \(\Phi = \varphi \circ p_{\varphi}^{-1} : p_{\varphi}(M) \to N\) where \(p_{\varphi}\) is the section of \(P\) corresponding to \(J^\varphi\). Note that, \(p_{\varphi}\) restricted to any fibre \(\varphi^{-1}(y)\) is equal to
vector bundle. We define Example 3.5, associated to $(N, c_N)$ to $(M, \pi_M, \mathcal{J})$.

The following assertions are equivalent:

(i) $\varphi : (M^4, \tau_M) \to (N^2, \tau_N)$ is twistorial (with respect to $\Phi$).

(ii) $p_{\varphi} : (M^4, J^e) \to (P^6, \mathcal{J})$ is holomorphic and $\varphi : (M^4, c_M) \to (N^2, c_N)$ is horizontally conformal.

(iii) $\varphi : (M^4, c_M) \to (N^2, c_N)$ is horizontally conformal and its fibres are twistorial, in the sense of Example 4.7.

(iv) $J^e$ is integrable and $\varphi : (M^4, c_M) \to (N^2, c_N)$ is horizontally conformal.

Example 4.9 ([10], [6]). Let $(M^4, c_M)$ be a four-dimensional oriented conformal manifold and let $(N^3, c_N, D^N)$ be a three-dimensional Weyl space. Let $\tau_M = (P_M, M, \pi_M, \mathcal{J})$ be the almost twistorial structure of Example 3.8, associated to $(M^4, c_M)$, and let $\tau_N = (P_N, N, \pi_N, \mathcal{F})$ be the almost twistorial structure of Example 3.5, associated to $(N^3, c_N, D^N)$.

Let $\varphi : M^4 \to N^3$ be a submersion. Let $\mathcal{V} = \ker d\varphi$ and $\mathcal{H} = \mathcal{V}^\perp$. Then the orientation of $M^4$ corresponds to an isomorphism, which depends of $c_M$, between $\mathcal{V}$ and the line bundle of $\mathcal{H}$. Therefore $(\mathcal{V}^* \otimes \mathcal{H}, c|_{\mathcal{H}})$ is an oriented Riemannian vector bundle. We define $\Phi : P_M \to P_N$ by

$$\Phi(p) = \frac{1}{||d\varphi(V^* \otimes p(V))||} d\varphi(V^* \otimes p(V)),$$

where $\{V\}$ is any basis of $\mathcal{V}_{\pi_M(p)}$ and $\{V^*\}$ its dual basis, $(p \in P_M)$.

Let $I^\mathcal{H}$ be the $\mathcal{V}$-valued two-form on $\mathcal{H}$ defined by $I^\mathcal{H}(X, Y) = -\mathcal{V}[X, Y]$, for any sections $X$ and $Y$ of $\mathcal{H}$. Then $*^\mathcal{H} I^\mathcal{H}$ is a horizontal one-form on $M^4$, where $*^\mathcal{H}$ is the Hodge star-operator of $(\mathcal{V}^* \otimes \mathcal{H}, c|_{\mathcal{H}})$. Denote by $D_+$ the Weyl connection on $(M^4, c_M)$ defined by $D_+ = D + *^\mathcal{H} I^\mathcal{H}$ where $D$ is the Weyl connection of $(M^4, c_M, \mathcal{V})$ (see [14]).

The following assertions are equivalent:

(i) $\varphi : (M^4, \tau_M) \to (N^3, \tau_N)$ is twistorial (with respect to $\Phi$).

(ii) $\varphi : (M^4, c_M) \to (N^3, c_N)$ is horizontally-conformal and $\varphi^*(D^N) = \mathcal{H} D_+$ as partial connections on $\mathcal{H}$, over $\mathcal{H}$.

Let $\varphi : (M^4, c_M) \to N^3$ be a submersion from a four-dimensional oriented conformal manifold to a three-dimensional manifold. Let $L$ be the line bundle of $N^3$. As the orientation of $M^4$ corresponds to an isomorphism between $\mathcal{V}$ and $\varphi^*(L)$, the pull-back by $\varphi$ of any section of $L^*$ is a (vertical) one-form on $M^4$. Similarly, if $E$ is a vector bundle over $N^3$, the pull-back by $\varphi$ of any section of $L^* \otimes E$ is a $\varphi^*(E)$-valued one-form on $M^4$.

Next, we recall the following definitions (see [20] and the references therein).
Definition 4.10. Let $P$ be a principal bundle on $N^3$ endowed with a (principal) connection $\Gamma$ and let $A$ be a section of $L^* \otimes \text{Ad} P$.

1) Let $N^3$ be endowed with a conformal structure $c_N$ and a Weyl connection $D^N$. The pair $(A, \Gamma)$ is called a monopole on $(N^3, c_N, D^N)$ if

$$R = *_N (D^N \otimes \nabla)(A)$$

where $R$ is the curvature form of $\Gamma$ and $\nabla$ is (the covariant derivative of) the connection induced by $\Gamma$ on $\text{Ad} P$.

2) Let $(M^4, c_M)$ be a four-dimensional oriented conformal manifold and let $\varphi : M^4 \to N^3$ be a submersion. The connection $\tilde{\Gamma}$ on $\varphi^*(P)$ defined by

$$\tilde{\Gamma} = \varphi^*(\Gamma) + \varphi^*(A)$$

is called the pull-back by $\varphi$ of $(A, \Gamma)$.

Next, we prove the following (cf. [20] and the references therein):

Proposition 4.11. Let $(M^4, c_M)$ be a four-dimensional oriented conformal manifold and let $(N^3, c_N, D^N)$ be a three-dimensional Weyl space; denote by $\tau_M$ and $\tau_N$ the almost twistorial structures of Examples 3.8 and 3.5, associated to $(M^4, c_M)$ and $(N^3, c_N, D^N)$, respectively.

Let $P$ be a principal bundle over $N^3$ endowed with a connection $\Gamma$ and let $A$ be a nowhere zero section of $L^* \otimes \text{Ad} P$. Also, let $\varphi : (M^4, c_M) \to (N^3, c_N)$ be a surjective horizontally conformal submersion with connected fibres.

Then any two of the following assertions imply the third:

(i) $\varphi : (M^4, \tau_M) \to (N^3, \tau_N)$ is twistorial.

(ii) $(A, \Gamma)$ is a monopole on $(N^3, c_N, D^N)$.

(iii) $\tilde{\Gamma}$ is anti-self-dual.

Proof. Let $\tilde{R}$ be the curvature form of $\tilde{\Gamma}$. A straightforward calculation shows that, up to an anti-self-dual term, the following equality holds

$$(4.1) \quad \tilde{R} = \varphi^*(R) + \varphi^*((D^N \otimes \nabla)(A)) + (\varphi^*(D^N) - H D_+) \wedge \varphi^*(A)$$

where we have used the isomorphism $\Lambda^2(T^*M) = \Lambda^2(\varphi^*(T^*N)) \oplus (\varphi^*(TN) \otimes \mathcal{V}^*)$, induced by $c_M$. The proof follows. \qed

Remark 4.12. The fact that (4.1) holds, up to an anti-self-dual term, does not require $\varphi$ be horizontally conformal; moreover, this relation characterizes $\mathcal{H} D_+$ among the partial connections on $\mathcal{V}$, over $\mathcal{H}$.

It follows that a submersion from a four-dimensional oriented conformal manifold to a three-dimensional Weyl space is twistorial, as in Example 4.9, if and only if it pulls-back any (local) monopole to an anti-self-dual connection; the only
5. Twistorial maps and harmonic morphisms

We start this section by recalling, from [14], the relations between harmonic morphisms and the twistorial maps of Examples 4.3, 4.6, 4.8 and 4.9. Firstly, any map between two-dimensional orientable conformal manifolds is a harmonic morphism if and only if, with respect to suitable orientations, it is a twistorial map.

Proposition 5.1 ([14]). Let \((M^m, c, D^M)\) and \((N^n, c, D^N)\) be Weyl spaces of dimensions \(m\) and \(n\), respectively, where \((m, n) \in \{(3, 2), (4, 2), (4, 3)\}\). If \(m\) (\(n\)) is even then \(M^m\) (\(N^n\)) is assumed to be oriented. Endow \(M^m\) and \(N^n\), according to their dimensions, with the almost twistorial structures \(\tau_M\) and \(\tau_N\), respectively, of Examples 3.3, 3.5 or 3.8. Let \(\phi: M^m \to N^n\) be a submersion.

(i) If \((m, n) = (3, 2)\) then the following assertions are equivalent:

(i1) \(\phi: (M^3, c, D^M) \to (N^2, c, D^N)\) is a harmonic morphism.

(ii) \(\phi: (M^3, \tau_M) \to (N^2, \tau_N)\) is a twistorial map.

(ii) If \((m, n) = (4, 2)\) then the following assertions are equivalent:

(iii) \(\phi: (M^4, c, D^N) \to (N^2, c, D^N)\) is a harmonic morphism and \(\phi: (M^4, \tau_M) \to (N^2, \tau_N)\) is a twistorial map.

(iii2) \(\phi: (M^4, c, M) \to (N^2, c, N)\) is horizontally conformal and the fibres of \(\Phi\) are tangent to the connection induced by \(D^M\) on \(P_M\).

(iii3) If \((m, n) = (4, 3)\) then any two of the following assertions imply the third:

(iii3) \(\phi: (M^4, c, D^N) \to (N^3, c, D^N)\) is a harmonic morphism.

(iii) \(\phi: (M^4, \tau_M) \to (N^3, \tau_N)\) is a twistorial map.

(iii3) The fibres of \(\Phi\) are tangent to the connection induced by \(D^M\) on \(P_M\).

Next we discuss twistorial maps between manifolds endowed with the almost twistorial structures of Examples 3.3, 3.7 or 3.9. The notations are as in Section 4.

The following proposition is essentially due to [8]; its proof is similar to the proofs of Propositions 5.3 and 5.4.

Proposition 5.2. Let \((M^m, c, D)\) be a Weyl space, \(m = 3, 4\), and let \(k\) be a section of the dual of the line bundle of \(M^m\); if \(m = 4\) assume \(M^4\) oriented and \(k = 0\). Let \(\tau_M\) be the almost twistorial structure on \(M^m\) given by Examples 3.7 or 3.9 according to \(m = 3\) or \(m = 4\), respectively.

Let \(N^2\) be an oriented surface in \(M^m\). Let \(\tau_N\) be the twistorial structure of
Example 3.8. associated to $(N^2, c_N)$.

The following assertions are equivalent:

(i) $(N^2, \tau_N) \to (M^m, \tau_M')$ is twistorial (with respect to $p_N$).

(ii) $N^2$ is a minimal surface in $(M^m, c, D)$ and $k|_N = 0$.

Let $\varphi : (M^3, c) \to N^2$ be a submersion from a three-dimensional conformal manifold to an oriented two-dimensional manifold. Let $\mathcal{V} = \ker d\varphi$ and $\mathcal{H} = \mathcal{V}^\perp$.

From the fact that $N^2$ is oriented it follows that $c$ induces an isomorphism between $\mathcal{V}$ and the line bundle of $M^3$.

Let $B^\mathcal{H}, D$ be the second fundamental form of $\mathcal{H}$, with respect to $D$, and let $I^\mathcal{H}$ be the integrability tensor of $\mathcal{H}$. We shall denote by $*^\mathcal{H}$ the Hodge star operator of $(\mathcal{H}, c_M|_\mathcal{H})$.

Proposition 5.3. Let $(M^3, c_M, D)$ be a three-dimensional Weyl space and let $k$ be a section of the dual of the line bundle of $M^3$. Let $\tau_M'$ be the almost twistorial structure of Example 3.7, associated to $(M^3, c_M, D, k)$; denote by $\nabla$ the connection associated to $D$ and $k$.

Let $(N^2, c_N)$ be a two-dimensional oriented conformal manifold. Let $\tau_N$ be the twistorial structure of Example 3.3, associated to $(N^2, c_N)$.

Let $\varphi : M^3 \to N^2$ be a submersion. The following assertions are equivalent:

(i) $\varphi : (M^3, \tau_M') \to (N^2, \tau_N)$ is a twistorial map.

(ii) $\varphi : (M^3, c_M) \to (N^2, c_N)$ is horizontally conformal, $D$ is the Weyl connection of $(M^3, c_M, \mathcal{V})$ and $*^\mathcal{H} I^\mathcal{H} = k$.

(iii) $\varphi : (M^3, c_M, D) \to (N^2, c_N)$ is a harmonic morphism, $*^\mathcal{H} I^\mathcal{H} = k$ and $\text{trace}_{c_M}(B^\mathcal{H}, D) = 0$.

(iv) $\varphi : (M^3, c_M) \to (N^2, c_N)$ is a horizontally conformal submersion and $\text{trace}_{c_M}(\nabla F^\varphi) = 0$, $\text{trace}_{c_M}(B^\mathcal{H}, D) = 0$.

Proof. Let $p_\varphi$ be the section of $P$ corresponding to $F^\varphi$ where $\tau_M' = (P, M, \pi, \mathcal{F}')$ (see Example 4.6). Then condition (1) of Definition 4.1 is satisfied by $\varphi$, with respect to $\Phi = \varphi \circ p_\varphi^{-1}$, if and only if $\dim \mathbb{C}(\mathcal{F}'_{p_\varphi(x)} \cap d\varphi(T^c_\mathcal{F} M)) = 2$, for all $x \in M$, where $\mathcal{F}'$ is the direct sum of the eigenbundles of $\mathcal{F}'$ corresponding to 0 and $-i$. Equivalently, condition (1) of Definition 4.1 is satisfied by $\varphi$, with respect to $\Phi$, if and only if $p_\varphi : (M^3, F^\varphi) \to (P^5, \mathcal{F}')$ is holomorphic.

It follows that assertion (i) holds if and only if $p_\varphi : (M^3, F^\varphi) \to (P^5, \mathcal{F}')$ and $\varphi : (M^3, F^\varphi) \to (N^2, J^N)$ are holomorphic where $J^N$ is the positive Hermitian structure on $(N^2, c_N)$. The latter condition is equivalent to $\varphi : (M^3, c_M) \to (N^2, c_N)$ horizontally conformal.

We can find local representatives $g = U \odot U + 2Y \odot Y$ of $c_M$ such that $U$ and $Y$ (locally) generate the eigenbundles of $F^\varphi$ corresponding to 0 and $-i$, respectively;
in particular, $d\varphi(U) = 0$. Hence, under the identification of $P$ with the bundle of two-dimensional degenerate spaces on $(M^3, c_M)$, the section $p_\varphi$ corresponds to the distribution (locally) generated by $U$ and $Y$.

From Proposition 2.6 it follows that $p_\varphi : (M^3, F^\varphi) \rightarrow (P^5, F')$ is holomorphic if and only if $g(D_U U, Y) = 0$ and $g(D_Y U, Y) = 0$. Thus, after a straightforward calculation we obtain that $p_\varphi : (M, F^\varphi) \rightarrow (P, F')$ is holomorphic if and only if $*_{\varphi^*} I^{\varphi^*} = k$, trace$_{c_M}(B^{\varphi^*, D}) = 0$ and the fibres of $\varphi$ are geodesics of $D$.

Thus, we have proved that (i) $\iff$ (ii) $\iff$ (iii).

Also, if $\varphi : (M^3, c_M) \rightarrow (N^2, c_N)$ is horizontally conformal then we have trace$_{c_M}(\nabla F^\varphi) = 0$ if and only if $*_{\varphi^*} I^{\varphi^*} = k$ and the fibres of $\varphi$ are geodesics of $D$. Hence, (iii) $\iff$ (iv). \hfill $\Box$

Also, we have the following result.

**Proposition 5.4.** Let $(M^4, c_M, D)$ be a four-dimensional oriented Weyl space. Let $\tau'_M$ be the almost twistorial structure of Example 3.9, associated to $(M^4, c_M, D)$.

Let $(N^2, c_N)$ be a two-dimensional oriented conformal manifold. Let $\tau_N$ be the twistorial structure of Example 3.3, associated to $(N^2, c_N)$.

Let $\varphi : M^4 \rightarrow N^2$ be a submersion. The following assertions are equivalent:

(i) $\varphi : (M^4, \tau'_M) \rightarrow (N^2, \tau_N)$ is a twistorial map.

(ii) $\varphi : (M^4, c_M) \rightarrow (N^2, c_N)$ is horizontally conformal and $D$ is the Weyl connection of $(M^4, c_M, J^\varphi)$.

(iii) $\varphi : (M^4, c_M, D) \rightarrow (N^2, c_N)$ is a harmonic morphism and we have trace$_{c_M}(B^{\varphi^*, D}) = J^\varphi(*_{\varphi^*} I^{\varphi^*})$.

**Proof.** Let $p_\varphi$ be the section of $P$ corresponding to $J^\varphi$ where $\tau'_M = (P, M, \pi, J')$ (see Example 3.8). Similarly to the proof of Proposition 5.4, assertion (i) holds if and only if $p_\varphi : (M^4, J^\varphi) \rightarrow (P^6, J')$ is holomorphic and $\varphi : (M^4, c_M) \rightarrow (N^2, c_N)$ is horizontally conformal. It follows that (i) $\iff$ (ii).

We can find local representatives $g = 2(U \circ U + Y \circ Y)$ of $c_M$ such that $U$ and $Y$ (locally) generate the eigenbundle of $J^\varphi$ corresponding $-i$ and such that $U$ and $U$ are vertical. Hence, under the identification of $P$ with the bundle of self-dual spaces on $(M^4, c_M)$, the section $p_\varphi$ corresponds to the distribution (locally) generated by $U$ and $Y$.

From Proposition 2.6 it follows that $p_\varphi : (M^4, J^\varphi) \rightarrow (P^6, J')$ is holomorphic if and only if $g(D_{\varphi Y} U, Y) = 0$ and $g(D_{\varphi U} U, Y) = 0$.

Now, $g(D_{\varphi Y} U, Y) = 0$ if and only if the fibres of $\varphi$ are minimal with respect to $D$ whilst $g(D_{\varphi U} U, Y) = 0$ if and only if trace$_{c_M}(B^{\varphi^*, D}) = J^\varphi(*_{\varphi^*} I^{\varphi^*})$. Hence, (i) $\iff$ (iii). \hfill $\Box$
6. HARMONIC MORPHISMS AND TWISTORIAL MAPS BETWEEN WEYL SPACES OF DIMENSIONS FOUR AND THREE

We start this section by introducing a generalization of the almost twistorial structure of Example 3.7.

**Example 6.1.** Let \((M^3, c)\) be a three-dimensional conformal manifold endowed with two Weyl connections \(D'\) and \(D''\). Let \(k\) be a section of \(L^*\) and let \(\nabla\) be the connection associated to \(D''\) and \(k\).

Let \(\pi : P \rightarrow M\) be the bundle of nonzero skew-adjoint \(f\)-structures on \((M^3, c)\). We denote by \(H_0\) the subbundle of \(T^C P\) such that, at each \(p \in P\), the subspace \(H_0_p \subset T^C p\) is the horizontal lift, with respect to \(D'\), of the eigenspace of \(p \in \text{End}(T^C_{\pi(p)} M)\) corresponding to the eigenvalue 0. Also, we denote by \(H_{1,0}\) the subbundle of \(T^C P\) such that, at each \(p \in P\), the subspace \(H_{1,0}^p \subset T^C p\) is the horizontal lift, with respect to \(\nabla\), of the eigenspace of \(p \in \text{End}(T^C_{\pi(p)} M)\) corresponding to the eigenvalue 1.

We define the almost \(f\)-structure \(F''\) on \(P\) with respect to which \(T^0 P = H_0\) and \(T^1_{1,0} P = (\ker d\pi)^{0,1} \oplus \mathcal{H}^{1,0}\). Then \(\tau'' = (P, M, \pi, F'')\) is a nonintegrable almost twistorial structure on \(M\) (the nonintegrability of \(\tau''\) follows easily from the proof of the integrability result presented in [18]).

The results of Section 5, which involve the almost twistorial structure of Example 3.7, can be easily generalized by working, instead, with the almost twistorial structure of Example 6.1. In Theorem 6.6, below, we shall see that this generalization is significant when working with maps between Weyl spaces of dimensions four and three.

We shall also need the following.

**Example 6.2.** Let \((M^4, c, D)\) be a four-dimensional oriented Weyl space. Let \(\pi : P \rightarrow M\) be the bundle of positive orthogonal complex structures on \((M^4, c)\).

Let \(\varphi : M^4 \rightarrow N^3\) be a submersion; denote by \(\mathcal{V} = \ker d\varphi\) and \(\mathcal{H} = \mathcal{V}^\perp\). For any \(p \in P\) let \(l'_p\) and \(l''_p\) be the lines in \(T^C_{\pi(p)} M\) spanned by \(V - ip(V)\) and \(X - ip(X)\), respectively, for any \(V \in \mathcal{V}_{\pi(p)}\) and \(X \in p(\mathcal{V}_{\pi(p)})^\perp \cap \mathcal{H}_{\pi(p)}\).

We define the almost CR-structure \(C'_\varphi\) on \(P\) which, at each \(p \in P\), is the direct sum of \((\ker d\pi)^{0,1}\) and the horizontal lift, with respect to \(D\), of \(l'_p\).

Similarly, we define the almost CR-structure \(C''_\varphi\) on \(P\) which, at each \(p \in P\), is the direct sum of \((\ker d\pi)^{0,1}\) and the horizontal lift, with respect to \(D\), of \(l''_p\).

Then \(\tau'_\varphi = (P, M, \pi, C'_\varphi)\) and \(\tau''_\varphi = (P, M, \pi, C''_\varphi)\) are nonintegrable almost twistorial structures on \(M^4\) (the nonintegrability of \(\tau'_\varphi\) and \(\tau''_\varphi\) follows easily from the proof of the integrability result presented in [18]).
Remark 6.3. Let \((N^3, c_N)\) be a three-dimensional conformal manifold. At least locally, we may assume that \((L^* \otimes T N, c)\) is the adjoint bundle of a rank two complex vector bundle \(E\) with group \(SU(2)\). Furthermore, under this identification, \(P_N\) is isomorphic to \(P E\). As \(SO(3, \mathbb{R}) = SU(2)/\mathbb{Z}_2 = PSU(2)\), any connection on \((L^* \otimes T N, c)\) corresponds to a connection on the \(PSU(2)\)-bundle \(P_N\). Also, \((L^* \otimes T N, c)\) is the adjoint bundle of the \(PSU(2)\)-bundle \(P_N\).

Similarly, if \((M^4, c_M)\) is a four-dimensional oriented conformal manifold then the bundle \(P_M\) of positive orthogonal complex structures on \((M^4, c_M)\) is a \(PSU(2)\)-bundle. Furthermore, a submersion \(\varphi : (M^4, c_M) \to (N^3, c_N)\) is horizontally conformal if and only if \(\Phi : P_M \to P_N\) is a morphism of \(PSU(2)\)-bundles (in general, \(\Phi\) is a morphism of \(SL(3, \mathbb{R})\)-bundles). Also, note that, if \(\varphi\) is horizontally conformal then \(P_M = \varphi^*(P_N)\) as \(PSU(2)\)-bundles.

The following result shows the importance of the almost twistorial structures of Example 6.2.

Theorem 6.4. Let \((M^4, c_M, D^M)\) be a four-dimensional oriented Weyl space and let \(P_M\) be the bundle of positive orthogonal complex structures on \((M^4, c_M)\).

Let \((N^3, c_N, D^N)\) be a three-dimensional Weyl space and let \(k\) be a section of the dual of the line bundle of \(N^3\); denote by \(\nabla\) the connection associated to \(D^N\) and \(k\). Let \(\tau'_N\) be the almost twistorial structure of Example 3.7, associated to \((N^3, c_N, D^N, k)\).

Let \(\varphi : M^4 \to N^3\) be a submersion; denote by \(\mathcal{V} = \ker d\varphi\) and \(\mathcal{H} = \mathcal{V}^\perp\). Let \(\tau'_\varphi\) and \(\tau''_\varphi\) be the almost twistorial structures of Example 6.2, associated to \((M^4, c_M, D^M)\) and \(\varphi\).

(i) The following assertions are equivalent:

(i1) \(\varphi : (M^4, \tau'_\varphi) \to (N^3, \tau'_N)\) is twistorial.

(ii) \(\varphi : (M^4, c_M, D^M) \to (N^3, c_N, D^N)\) is a harmonic morphism.

(ii) The following assertions are equivalent:

(iii) \(\varphi : (M^4, \tau''_\varphi) \to (N^3, \tau'_N)\) is twistorial.

iv) \(\varphi : (M^4, c_M) \to (N^3, c_N)\) is horizontally conformal, \(\mathcal{V}(D^M - D) = \frac{1}{2} k\) and \(\varphi^*(D^N) = \mathcal{H} D^M + \frac{1}{2} \ast_{\mathcal{H}} \mathcal{H}\) as partial connections, over \(\mathcal{H}\), where \(D\) is the Weyl connection of \((M^4, c_M, \mathcal{V})\).

(iii) \(\varphi : (M^4, c_M) \to (N^3, c_N)\) is horizontally conformal and the partial connections on \(P_M\), over \(\mathcal{H}\), induced by \(D^M\) and \(\varphi^*(\nabla)\) are equal.

Proof. (i) By Remark 6.3, if (i1) holds then \(\varphi : (M^4, c_M) \to (N^3, c_N)\) is horizontally conformal.

Let \(\tau'_\varphi = (P_M, M, \pi_M, \mathcal{E}'_\varphi)\) and let \(\tau'_N = (P_N, N, \pi_N, \mathcal{F}')\). By associating to
each orthogonal complex structure on \((M^4, c_M)\) its eigenspace corresponding to 

\(-i\), we identify \(\pi_M : P_M \to M\) with the bundle of self-dual spaces on \((M^4, c_M)\).

Similarly, by associating to each skew-adjoint \(f\)-structure on \((N^3, c_N)\) the sum of its eigenspaces corresponding to 0 and 

\(-i\), we identify \(\pi_N : P_N \to N\) with the bundle of (complex) two-dimensional degenerate spaces on \((N^3, c_N)\).

Under this identifications the natural lift \(\Phi : P_M \to P_N\) of \(\varphi\) is given by 

\[\Phi(p) = d\varphi(p),\] 

for any \(p \in P_M\).

Let \(q\) be a local section of \(P_N\), over some open set \(V \subseteq N\), and let \(p\) be the local section of \(P_M\), over \(\varphi^{-1}(V)\), such that \(\Phi \circ p = q \circ \varphi\). At least locally, we may assume that \(p\) is generated by \(Y = U + iX\) and \(Z\), with \(U\) vertical, and \(X\) and \(Z\) basic. Thus, \(q\) is generated by \(d\varphi(X)\) and \(d\varphi(Z)\).

We shall assume that, for some \(x_0 \in \varphi^{-1}(V)\), we have \(dq(d\varphi(X_{x_0}))\) horizontal, with respect to the connection induced by \(\nabla\) on \(P_N\). Then Proposition \([2,0]\), the fundamental equation and a straightforward calculation show that 

\[dp(Y_{x_0}) \in (\mathcal{C}_\varphi')_{p(x_0)}\] 

if and only if \(c_M(\mathrm{trace}_{c_M}(\tilde{D}d\varphi), Z) = 0\), where \(\tilde{D}\) is the connection induced by \(D^M\) and \(D^N\) on \(\varphi^*(TN) \otimes T^*M\).

This proves that \((i2)\implies(i1)\).

To prove the converse, assume that \((i1)\) holds. Then there exists a unique \(A \in (\mathcal{C}_\varphi')_{p(x_0)}\) such that \(d\Phi(A) = dq(d\varphi(X_{x_0}))\). From the fact that \(d\pi_M((\mathcal{C}_\varphi')_{p(x_0)})\) is the line spanned by \(\nabla_{x_0}\), it follows quickly that \(d\pi_M(A) = -\nabla_{x_0}\). Together with 

\[d\Phi(A) = d\Phi(dp(-Y_{x_0}))\] 

this gives \(A = dp(-Y_{x_0})\). Hence, \(dp(Y_{x_0}) \in (\mathcal{C}_\varphi')_{p(x_0)}\) and, therefore, \(c_M(\mathrm{trace}_{c_M}(\tilde{D}d\varphi), Z) = 0\).

(ii) By Remark \([6,3]\), if \((ii1)\) holds then \(\varphi : (M^4, c_M) \to (N^3, c_N)\) is horizontally conformal.

Similarly to above, let \(p\) be a basic local section of \(P_M\); that is, there exists a local section \(q\) of \(P_N\) such that \(\Phi \circ p = q \circ \varphi\). At least locally, we may assume that \(p\) is generated by \(Y = U + iX\) and \(Z\), with \(U\) vertical, and \(X\) and \(Z\) basic. Thus, \(q\) is generated by \(d\varphi(X)\) and \(d\varphi(Z)\).

To prove \((ii1) \iff (ii2)\), we shall assume that \(p\) is horizontal, at some point \(x_0 \in M\), with respect to the connection induced by \(D^M\) on \(P_M\) (this could be done as follows: firstly, define \(p\) over some hypersurface, containing \(x_0\) and which is transversal to the fibres of \(\varphi\), such that \(p\) to be horizontal at \(x_0\); then, extend \(p\), to an open neighbourhood of \(x_0\), so that to be basic); in particular, 

\[dp(\nabla_{x_0}) \in (\mathcal{C}_\varphi'')_{p(x_0)}\]

Assertion \((ii1)\) is equivalent to the fact that, for any such \(p\), we have \(dq(d\varphi(X_{x_0}))\) is contained in the eigenbundle of \(F'\) corresponding to the eigenvalue \(i\). By using Proposition \([2,0]\), it quickly follows that \((ii1) \iff (ii2)\).
Let $h$ be an oriented representative of $c_N$ and let $g$ be a representative of $c_M$ such that $\varphi: (M^4, g) \to (N^3, h)$ is a Riemannian submersion. Let $\alpha^M$ and $\alpha^N$ be the Lee forms of $D^M$ and $D^N$ with respect to $g$ and $h$, respectively.

The equivalence (ii2) $\iff$ (ii3) follows from the fact that any two of the following assertions imply the third:

(a) $d_p(\mathcal{H}_{x_0})$ is horizontal, with respect to the connection induced by $D^M$ on $P_M$.

(b) $q$ is horizontal at $\varphi(x_0)$, with respect to the connection induced by $\nabla$ on $P_N$.

(c) At $x_0$ we have $\alpha^M|_{\mathcal{H}} = \frac{1}{2} k$ and $\alpha^N = \alpha^M|_{\mathcal{H}} + \frac{1}{2} *_{\mathcal{H}} I_{\mathcal{H}}$, where we have identified $k$ and $\alpha^N$ with their pull-backs by $\varphi$ and, in the first equality, we have used the isomorphism between $\mathcal{H}$ and the pull-back by $\varphi$ of the line bundle of $N^3$ induced by $\varphi$ and the orientation of $M^4$.

The proof is complete. □

Remark 6.5. Let $(M^4, c, D)$ be a four-dimensional oriented Weyl space and let $P$ be the bundle of positive orthogonal complex structures on $(M^4, c)$.

With the same notations as in Example 6.2, by Theorem 6.4(i), the relation $\varphi \leftrightarrow \mathcal{C}_\varphi$ induces a bijective correspondence between one-dimensional foliations on $(M^4, c, D)$, which are locally defined by harmonic morphisms, and certain almost CR-structures on $P$.

Next, we prove the following.

Theorem 6.6. Let $(M^4, c_M, D^M)$ be a four-dimensional oriented Weyl space and let $\tau'_M$ be the almost twistorial structure of Example 3.9, associated to $(M^4, c_M, D^M)$.

Let $(N^3, c_N)$ be a three-dimensional conformal manifold endowed with two Weyl connections $D'$ and $D''$. Let $k$ be a section of the dual of the line bundle of $N^3$ and let $\tau''_N$ be the almost twistorial structure of Example 6.1, associated to $(N^3, c_N, D', D'', k)$.

Let $\varphi: M^4 \to N^3$ be a submersion. The following assertions are equivalent:

(i) $\varphi: (M^4, \tau'_M) \to (N^3, \tau''_N)$ is twistorial.

(ii) $\varphi: (M^4, c_M) \to (N^3, c_N)$ is horizontally conformal and the connection induced by $D^M$ on the bundle of positive orthogonal complex structures on $(M^4, c_M)$ is the pull-back by $\varphi$ of $(A, \nabla)$, where $A = (D'' - D')^{\sharp \times}$ and $\nabla$ is the connection associated to $D''$ and $k$.

(iii) The following assertions hold:

(iii1) $\varphi: (M^4, c_M, D^M) \to (N^3, c_N, D')$ is a harmonic morphism;

(iii2) $\varphi: (M^4, \tau_M) \to (N^3, \tau_N)$ is twistorial, where $\tau_M$ and $\tau_N$ are the almost twistorial structures of Examples 3.8 and 3.7, associated to $(M^4, c_M)$ and
be defined a bundle map \( \tau \) which is the bundle of oriented lines on \( N \) and denote by \( \mathcal{H} \) the fundamental equation, this quickly gives (i) \( \iff \) (iii) of Theorem 6.4, it follows that assertion (i) holds if and only if \( \varphi : (M^4, c_M, D^M) \to (N^3, c_N, D') \) is a harmonic morphism, \( \nabla = \ker d\varphi \) and \( D \) is the Weyl connection

\[(N^3, c_N, 2D'' - D'), \text{ respectively.}\]

(iii3) \( \varphi^*(D^M - D) = \frac{1}{2} k \) where \( \nabla = \ker d\varphi \) and \( D \) is the Weyl connection of \((M^4, c_M, \nabla)\).  

**Proof.** From Theorem 6.4, it follows that assertion (i) holds if and only if \( \varphi : (M^4, c_M, D^M) \to (N^3, c_N, D') \) is a harmonic morphism, \( \varphi^*(D^M - D) = \frac{1}{2} k \) and \( \varphi^*(D^N) = \mathcal{H} D^M + \frac{1}{2} \ast_{\mathcal{H}} I_{\mathcal{H}} \) as partial connections, over \( \mathcal{H} \). Together with the fundamental equation, this quickly gives (i) \( \iff \) (iii).

Let \( \tau'_{M} = (P_M, M, \pi_M, J') \). From Theorem 6.4, it follows that assertion (i) holds if and only if \( \varphi : (M^4, c_M, D^M) \to (N^3, c_N) \) is horizontally conformal, the partial connections on \( P_M \), over \( \mathcal{H} \), induced by \( D^M \) and \( \varphi^*(\nabla) \) are equal, and we have \( \mathcal{H}(D^M - D) = D' - D'' + \frac{1}{2} \ast_{\mathcal{H}} I_{\mathcal{H}} \), as partial connections, over \( \mathcal{H} \).

Let \((X_1, \ldots, X_4)\) be a positive conformal local frame on \((M^4, c_M)\) such that \( X_1 \) is vertical and \( X_2, X_3, X_4 \) are basic. Let \( g \) be the local representative of \( c_M \) induced by \((X_1, \ldots, X_4)\). Let \( \alpha^M \) and \( \alpha \) be the Lee forms, with respect to \( g \), of \( D^M \) and \( D \), respectively. Denote by \( \Gamma_{jk}^i \), \((i, j, k = 1, \ldots, 4)\), the Christoffel symbols of \( D^M \) with respect to \((X_1, \ldots, X_4)\). Then a straightforward calculation gives the following relations:

\[
\begin{align*}
\Gamma_{21}^3 + \Gamma_{41}^3 &= (\alpha^M - \alpha - \frac{1}{2} \ast_{\mathcal{H}} I_{\mathcal{H}})(X_2), \\
\Gamma_{31}^4 - \Gamma_{41}^4 &= (\alpha^M - \alpha - \frac{1}{2} \ast_{\mathcal{H}} I_{\mathcal{H}})(X_3), \\
\Gamma_{41}^2 + \Gamma_{31}^2 &= (\alpha^M - \alpha - \frac{1}{2} \ast_{\mathcal{H}} I_{\mathcal{H}})(X_4).
\end{align*}
\]

Thus, we have proved that (i) holds if and only if \( \varphi : (M^4, c_M) \to (N^3, c_N) \) is horizontally conformal, the partial connections on \( P_M \), over \( \mathcal{H} \), induced by \( D^M \) and \( \varphi^*(\nabla) \) are equal, and the following relations hold:

\[
\begin{align*}
\Gamma_{21}^3 + \Gamma_{41}^3 &= (D' - D'')(X_2), \\
\Gamma_{31}^4 - \Gamma_{41}^4 &= (D' - D'')(X_3), \\
\Gamma_{41}^2 + \Gamma_{31}^2 &= (D' - D'')(X_4).
\end{align*}
\]

It follows that (i) \( \iff \) (ii). \( \square \)

**Remark 6.7.** 1) Let \((M^4, c_M, D^M)\) be a four-dimensional oriented Weyl space. Let \( \tau'_{M} \) be the almost twistorial structure of Example 3.9, associated to \((M^4, c_M, D^M)\); denote by \( P_M \) be the bundle of positive orthogonal complex structures on \((M^4, c_M)\). Let \( \varphi : M^4 \to N^3 \) be a submersion onto a three-dimensional manifold. Let \( P_N \) be the bundle of oriented lines on \( N^3 \). Then, similar to Example 4.9, there can be defined a bundle map \( \Phi : P_M \to P_N \).

Suppose that there exists an almost twistorial structure \( \tau \) on \( N^3 \) such that
\[\varphi : (M^4, \tau'_M) \rightarrow (N^3, \tau)\] is a twistorial map, with respect to \(\Phi\). Then there exists a section \(k\) of the dual of the line bundle of \(N^3\), a conformal structure \(c_N\) on \(N^3\) and Weyl connections \(D'\) and \(D''\) on \((N^3, c_N)\) such that \(\tau\) is the almost twistorial structure of Example 6.1, associated to \((N^3, c_N, D', D'', k)\).

Similar comments apply to all of the previous examples of submersive twistorial maps.

2) Let \((M^4, c_M)\) be an oriented four-dimensional conformal manifold and let \((N^3, c_N, D)\) be a three-dimensional Weyl space. Denote by \(\tau_M\) and \(\tau_N\) the almost twistorial structures, of Examples 3.8 and 3.5, associated to \((M^4, c_M)\) and \((N^3, c_N, D)\), respectively. Let \(\varphi : (M^4, \tau_M) \rightarrow (N^3, \tau_N)\) be a twistorial map (see \([20]\) and \([6]\) for examples of such maps).

Let \(D'\) be a Weyl connection on \((N^3, c_N)\) and let \(k\) be a section of the dual of the line bundle of \(N^3\); denote by \(D'' = \frac{1}{2}(D + D')\).

From Theorem 6.6, it follows that there exists a unique Weyl connection \(D^M\) on \((M^4, c_M)\) such that \(\varphi : (M^4, \tau'_M) \rightarrow (N^3, \tau'_N)\) is twistorial, where \(\tau'_M\) and \(\tau'_N\) are the almost twistorial structures, of Examples 3.9 and 6.1, associated to \((M^4, c_M, D^M)\) and \((N^3, c_N, D', D'', k)\), respectively.

3) Let \((M^4, c_M)\) be a four-dimensional oriented conformal manifold and let \(L\) be the line bundle of \(M^4\). Denote by \(P_\pm\) the bundles of positive/negative orthogonal complex structures on \((M^4, c_M)\). As \(P_\pm\) are, locally, the projectivisations of the bundles of positive/negative spinors on \((L^* \otimes TM, c_M)\), any conformal connection on \((M^4, c_M)\) corresponds to a pair \((\Gamma_+, \Gamma_-)\), where \(\Gamma_\pm\) are connections on \(P_\pm\).

Let \(\varphi : (M^4, c_M) \rightarrow (N^3, c_N)\) be a horizontally conformal submersion onto a three-dimensional conformal manifold. Endow \((N^3, c_N)\) with two Weyl connections \(D'\) and \(D''\) and let \(k\) be a section of the dual of the line bundle of \(N^3\). Let \(\Gamma_+\) be the connection on \(P_+\) which is the pull-back by \(\varphi\) of \((A, \nabla)\), where \(A = (D'' - D')^{\pm N}\) and \(\nabla\) is the connection associated to \(D''\) and \(k\).

Let \(\Gamma_-\) be a connection on \(P_-\) and suppose that the connection \(D^M\) corresponding to \((\Gamma_+, \Gamma_-)\) is torsion-free. Then, obviously, \(D^M\) is a Weyl connection on \((M^4, c_M)\). Moreover, by Theorem 6.6, the map \(\varphi : (M^4, \tau'_M) \rightarrow (N^3, \tau'_N)\) is twistorial, where \(\tau'_M\) and \(\tau'_N\) are the almost twistorial structures, of Examples 3.9 and 6.1, associated to \((M^4, c_M, D^M)\) and \((N^3, c_N, D', D'', k)\), respectively.

4) With the same notations as in Theorem 6.6, it can be proved that \((A, \nabla)\) is a monopole on \((N^3, c_N, 2D'' - D')\) if and only if \(D' = D''\) and \(\nabla\) is flat (cf. \([9]\); see \([14]\) for details about the resulting maps).

The following result is an immediate consequence of Theorem 6.6; note that, the equivalence (ii) \(\iff\) (iii) appears in \([14]\).
Corollary 6.8. Let \((M^4, c_M, D^M)\) be a four-dimensional oriented Weyl space and let \(\tau'_M\) be the almost twistorial structure of Example \(3.8\), associated to \((M^4, c_M, D^M)\).

Let \((N^3, c_N, D^N)\) be a three-dimensional Weyl space and let \(k\) be a section of the dual of the line bundle of \(N^3\). Let \(\tau'_N\) be the almost twistorial structure of Example \(3.7\), associated to \((N^3, c_N, D^N, k)\).

Let \(\varphi : M^4 \to N^3\) be a submersion. The following assertions are equivalent:

(i) \(\varphi : (M^4, c_M) \to (N^3, c_N)\) is horizontally conformal and the connection induced by \(D^M\) on the bundle of positive orthogonal complex structures on \((M^4, c_M)\) is the pull-back by \(\varphi\) of \(\nabla\), where \(\nabla\) is the connection associated to \(D^N\) and \(k\).

(ii) The following assertions hold:

(iii1) \(\varphi : (M^4, c_M, D^M) \to (N^3, c_N, D^N)\) is a harmonic morphism;

(iii2) \(\varphi : (M^4, \tau'_M) \to (N^3, \tau'_N)\) is twistorial, where \(\tau'_M\) and \(\tau'_N\) are the almost twistorial structure of Examples \(3.8\) and \(3.7\), associated to \((M^4, c_M)\) and \((N^3, c_N, D^N)\), respectively.

(iii3) \(\gamma(D^M - D) = \frac{1}{2} k\) where \(\gamma = \ker d\varphi\) and \(D\) is the Weyl connection of \((M^4, c_M, \gamma')\).

Example 6.9. 1) Let \(\varphi : (M^4, g) \to (N^3, h)\) be a harmonic morphism given by the Gibbons-Hawking or the Beltrami fields construction (see \[19\]). Then \(\varphi\) satisfies (ii) and (iii) of Corollary 6.8, with \(D^M\) and \(D^N\) the Levi-Civita connections of \(g\) and \(h\), respectively, and a suitable choice of \(k\) (\[14\], \[20\]). Hence, \(\varphi\) satisfies assertion (i) of Corollary 6.8 (and, also, (i) of Theorem 6.6).

2) Let \(\varphi : (M^4, g) \to (N^3, h)\) be a harmonic morphism of Killing type between Riemannian manifolds of dimensions four and three. Then \(\varphi\) satisfies (iii1), (iii2) of Theorem 6.6, with \(D^M\) and \(D'\) the Levi-Civita connections of \(g\) and \(h\), respectively, and a suitable choice of \(D'\) \[20\]; also, \(\varphi\) satisfies (iii3), with \(k = 0\). Hence, \(\varphi\) satisfies assertion (i) of Theorem 6.6.

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