On the Complexity of Breaking Symmetry

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Abstract. We can break symmetry by eliminating solutions within a symmetry class that are not least in the lexicographical ordering. This is often referred to as the lex-leader method. Unfortunately, as symmetry groups can be large, the lex-leader method is not tractable in general. We prove that using other total orderings besides the usual lexicographical ordering will not reduce the computational complexity of breaking symmetry in general. It follows that breaking symmetry with other orderings like the Gray code ordering or the Snake-Lex ordering is intractable in general.

1 Introduction

Symmetry occurs in many combinatorial problems (e.g. if trucks in a routing problem are located at the same depot and have the same capacity, they may be interchangeable in any solution). Symmetry can also be introduced by modelling decisions (e.g. using a set of finite domain variables to model a set of objects can introduce symmetries that permute these variables). A common method to deal with symmetry is to add constraints which eliminate symmetric solutions [1,2,3,4,5,6,7,8,9,10]. Crawford et al. have proved that breaking symmetry by adding constraints to eliminate symmetric solutions is intractable in general. More specifically, they prove that, for a matrix model with row and column symmetries, deciding if an assignment is the smallest in its symmetry class is NP-hard where we append rows together and compare them lexicographically. There is, however, nothing special about appending rows together or comparing assignments lexicographically. We could use any total ordering over assignments.

For example, we could break symmetry with the Gray code ordering. That is, we add constraints that eliminate symmetric solutions within each symmetry class that are not smallest in the Gray code ordering. The Gray code ordering is a total ordering over assignments used in error correcting codes. For instance, the 4-bit Gray code orders assignments as follows:

\[
0000, 0001, 0010, 0011, 0111, 0110, 0101, 0100, 1100, 1101, 1111, 1110, 1010, 1011, 1001, 1000
\]

Such an ordering will pick out different solutions in each symmetry class. We consider here the binary Gray code but note that it can be generalized to deal with non-binary domains. The Gray code ordering has some properties that may make it useful for symmetry breaking. In particular, neighbouring assignments in the ordering only differ at one position, and flipping just one bit reverses the ordering of the subsequent bits. As a second example, we could break row and column symmetry in a matrix model with the
Snake-Lex ordering [12]. This orders assignments within a symmetry class by lexicographically comparing vectors constructed by appending the entries in the matrix in a “snake like” manner in which the first column is appended to the reverse of the second column, and this is then appended to the third column, and then the reverse of the fourth column and so on. Again, this picks out different solutions in each symmetry class.

Can we be sure that different orderings like the Gray code and Snake-Lex ordering do not change the computational complexity of breaking symmetry? In this paper, we argue that breaking symmetry with a different ordering over assignments is unlikely to improve the complexity. Our argument breaks into two parts. First, we argue that using a different ordering can increase the computational complexity of breaking symmetry. Second, we argue that, under modest assumptions (which are satisfied by the Gray code and Snake-Lex orderings), we cannot reduce the computational complexity from that of breaking symmetry with a lexicographical ordering. Many dynamic methods for dealing with symmetry are equivalent to posting symmetry breaking constraints “on the fly” (e.g. [13,14,15,16,17,18,19]. Hence, our results are likely to have implications for such dynamic methods too.

2 Background

A symmetry of a set of constraints $S$ is a bijection $\sigma$ on variable-value pairs that maps solutions onto solutions [20]. We lift $\sigma$ from variable-value pairs to complete assignments (and hence solutions) in the natural way: $\sigma(A) = \{ \sigma(X = v) \mid X = v \in A \}$. Thus, a bijection $\sigma$ on variable-value pairs is a symmetry of a set of constraints $S$ iff given any solution $A$ of $S$, $\sigma(A)$ is also a solution of $S$. A variable symmetry is a bijection that just acts on the variable indices, whilst a value symmetry is a bijection that just acts on the values. The set of symmetries form a group under composition. Given a symmetry group $\Sigma$, a subset $\Pi$ generates $\Sigma$ iff any $\sigma \in \Sigma$ is a composition of elements from $\Pi$. A symmetry group $\Sigma$ partitions the solutions into symmetry classes (or orbits). We write $[A]_\Sigma$ for the symmetry class of solutions symmetric to the solution $A$. Where $\Sigma$ is clear from the context, we simply write $[A]$. Note that symmetry classes are equivalence classes. A set of symmetry breaking constraints is sound iff it leaves at least one solution in each symmetry class, and complete iff it leaves at most one solution in each symmetry class.

We will study what happens to symmetries when problems are reformulated onto equivalent problems. For example, we might consider the Boolean form of a problem in which we map $X_i = j$ onto $Z_{ij} = 1$. Two sets of constraints, $S$ and $T$ over possibly different variables are equivalent iff there is a bijection $\pi$ between solutions of $S$ and of $T$. Suppose $U_i$ and $V_i$ for $i \in [1,k]$ are partitions of the sets $U$ and $V$ into $k$ subsets. Then the two partitions are isomorphic iff there are bijections $\pi : U \mapsto V$ and $\tau : [1,k] \mapsto [1,k]$ such that $\pi(U_i) = V_{\tau(i)}$ for $i \in [1,k]$ where $\pi(U_i) = \{ \pi(u) \mid u \in U_i \}$. Two groups of symmetries $\Sigma$ and $\Pi$ of constraints $S$ and $T$ respectively are isomorphic iff $S$ and $T$ are equivalent, and their symmetry classes of solutions are isomorphic. When two groups of symmetries are isomorphic, the number and sizes of their symmetry classes are identical.
3 Using other orderings

Crawford et al. proposed the lex-leader method, a general way to break symmetry statically using lexicographical ordering constraints [11]. This method picks out the lexicographically smallest solution in each symmetry class. To do this, we post a lexicographical ordering constraint for every symmetry \( \sigma \):

\[
(X_1, \ldots, X_n) \leq_{\text{lex}} \sigma((X_1, \ldots, X_n))
\]

Where \( X_1 \) to \( X_n \) is some ordering on the variables in the problem. Many static symmetry breaking constraints can be derived from such lex-leader constraints. For example, \( \text{DOUBLELEX} \) constraints to break row and column symmetry can be derived from them [21]. Efficient algorithms have been developed to propagate such static symmetry breaking constraints (e.g. [22][23][24]).

We first argue that other orderings besides the lexicographical ordering can increase the computational complexity of symmetry breaking even when we have just a single symmetry to eliminate. Our argument has two parts. We first argue that, even with a single symmetry, we can introduce computational complexity through the complexity of deciding the ordering. We then argue that, when the ordering is polynomial to decide, we can introduce computational complexity by having many symmetries to break.

**Proposition 1** There exists a total ordering \( \preceq \) on assignments, and a class of problems \( P \) that is polynomial to decide and has a single symmetry \( \sigma \) such that finding a solution of \( \{ S \cup \{ (X_1, \ldots, X_n) \leq \sigma((X_1, \ldots, X_n)) \} \mid S \in P \} \) is NP-hard, whilst finding a solution of \( \{ S \cup \{ (X_1, \ldots, X_n) \leq_{\text{lex}} \sigma((X_1, \ldots, X_n)) \} \mid S \in P \} \) is polynomial.

**Proof:** Reduction from 1-in-3-SAT on \( m \) positive clauses. In 1-in-3-SAT, we wish to decide if \( 3 \)-cnf formula can be satisfied by a truth assignment that sets exactly 1 out of the 3 literals in each clause to true. Let \( n = 3m + 1 \), and \( S \) be a set of unary constraints that ensure \( X_{3p-2} = i, X_{3p-1} = j, X_{3p} = k \) where \( p \in [1, m] \) and the \( p \)th clause is \( i \lor j \lor k \). We also have \( X_{3m+1} \in \{ 0, 1 \} \). Consider the symmetry \( \sigma \) that interchanges the values 0 and 1 for \( X_{3m+1} \). We define an ordering \( \prec \) as follows:

\[
(X_1, \ldots, X_{3m+1}) \prec (Y_1, \ldots, Y_{3m+1}) \text{ iff one of 3 conditions holds:}
\]

1. \( (X_1, \ldots, X_{3m}) \preceq_{\text{lex}} (Y_1, \ldots, Y_{3m}) \), or
2. \( (X_1, \ldots, X_{3m}) = (Y_1, \ldots, Y_{3m}), X_{3m+1} = 0, Y_{3m+1} = 1 \), and the 1 in 3-SAT problem defined by \( X_1 \) to \( X_{3m} \) is satisfiable, or
3. \( (X_1, \ldots, X_{3m}) = (Y_1, \ldots, Y_{3m}), X_{3m+1} = 1, Y_{3m+1} = 0 \), and the 1 in 3-SAT problem defined by \( X_1 \) to \( X_{3m} \) is unsatisfiable.

Finally, we define \( \preceq \) by \( (X_1, \ldots, X_{3m+1}) \preceq (Y_1, \ldots, Y_{3m+1}) \text{ iff } (X_1, \ldots, X_{3m+1}) \prec (Y_1, \ldots, Y_{3m+1}) \) or \( (X_1, \ldots, X_{3m+1}) = (Y_1, \ldots, Y_{3m+1}) \). Now, the (only) solution of \( S \cup \{ (X_1, \ldots, X_n) \preceq (\sigma(X_1), \ldots, \sigma(X_n)) \} \) has \( X_{3m+1} = 0 \) if the corresponding 1 in 3-SAT problem is satisfiable and \( X_{3m+1} = 1 \) otherwise. Hence, finding the solution of these constraints is NP-hard. By comparison, finding a solution of \( S \cup \{ (X_1, \ldots, X_n) \leq_{\text{lex}} (\sigma(X_1), \ldots, \sigma(X_n)) \} \) is polynomial as it always has the solution in which \( X_{3m+1} = 0 \). \( \square \)
In the example in this proof, the ordering used to break symmetry was computationally intractable to decide. More precisely, deciding if \( \langle X_1, \ldots, X_n \rangle \preceq \langle Y_1, \ldots, Y_n \rangle \) was NP-hard. If we insist that the ordering used to break symmetry is polynomial to decide, then breaking a single symmetry will also be polynomial. Indeed, breaking even a polynomial number of symmetries must also be polynomial in this case. However, if we have an exponential number of symmetries to break then changing the ordering used to break symmetry from the lexicographical ordering to some other ordering can increase the computational complexity of finding a solution.

**Proposition 2** There exists a total ordering \( \succeq \) on assignments where deciding \( \succeq \) is polynomial, and a class of problems \( P \) that is also polynomial to decide, and for each \( S \in P \), a symmetry group \( \Sigma \) of \( S \) such that finding a solution of \( \{ S \cup \{ \langle X_1, \ldots, X_n \rangle \preceq \sigma(\langle X_1, \ldots, X_n \rangle) \mid \sigma \in \Sigma \} \mid S \in P \} \) is NP-hard, but finding a solution of \( \{ S \cup \{ \langle X_1, \ldots, X_n \rangle \leq \text{lex} \sigma(\langle X_1, \ldots, X_n \rangle) \mid \sigma \in \Sigma \} \mid S \in P \} \) is polynomial.

**Proof:** Reduction from SAT. Let \( \preceq \) be the reverse lexicographical ordering, \( \succeq \text{lex} \). This is clearly polynomial to decide. Consider any SAT formula \( \varphi \) on 0/1 variables \( X_1 \text{ to } X_n \). Let \( S \) be \( \varphi \lor (X_1 = \ldots = X_n = 0) \), and the symmetry group \( \Sigma \) be such that all solutions of \( S \) are in the same symmetry class. Consider any solution of \( S \cup \{ \langle X_1, \ldots, X_n \rangle \preceq \sigma(\langle X_1, \ldots, X_n \rangle) \mid \sigma \in \Sigma \} \). There are three cases. In the first case, the solution is \( X_1 = \ldots = X_n = 0 \) and this is not a solution of \( \varphi \). Then \( \varphi \) is unsatisfiable. In the second case, the solution is \( X_1 = \ldots = X_n = 0 \) and this is the only solution of \( \varphi \). Then \( \varphi \) is satisfiable. In the third case, the solution is lexicographical larger than \( X_1 = \ldots = X_n = 0 \). Then \( \varphi \) is again satisfiable. Hence finding a solution of \( S \cup \{ \langle X_1, \ldots, X_n \rangle \preceq \sigma(\langle X_1, \ldots, X_n \rangle) \mid \sigma \in \Sigma \} \) decides the satisfiability of \( \varphi \). Thus finding a solution is NP-hard. By comparison, finding a solution of \( S \cup \{ \langle X_1, \ldots, X_n \rangle \leq \text{lex} \sigma(\langle X_1, \ldots, X_n \rangle) \mid \sigma \in \Sigma \} \) is polynomial as it always has the solution in which \( X_1 = \ldots = X_n = 0 \). \( \Box \)

A criticism that can be made of the examples in the last two proofs is that both are rather “artificial”. In the first proof, we introduced complexity into symmetry breaking by making the ordering NP-hard to decide. In the second proof, we introduced complexity into symmetry breaking by making the symmetry group NP-hard to decide. In the rest of this paper, we give a more natural example to show that symmetry breaking with other orderings is intractable. Our argument uses a symmetry group whose elements are easy to generate and which is isomorphic to the symmetry group that interchanges rows and columns in a matrix model. In addition, we suppose that the ordering is polynomial to decide so cannot itself be a source of computational complexity.

### 4 Symmetry under reformulation

We will show that, under some modest assumptions, we cannot pick an ordering with which to break symmetry that will make it computationally easier than using the simple lexicographical ordering. Our argument breaks into two parts. First, we show how the symmetry of a problem changes when we reformulate onto an equivalent problem. Second, we argue that we can map symmetry breaking with any other ordering onto symmetry breaking using the lexicographical ordering on an equivalent problem.
Therefore, breaking symmetry with a different ordering cannot have a lesser computational complexity. We begin by proving that reformulation maps the symmetry group of a problem onto an isomorphic symmetry group.

**Proposition 3** If a set of constraints $S$ has a symmetry group $\Sigma$, $S$ and $T$ are equivalent sets of constraints, $\pi$ is any bijection between solutions of $S$ and $T$, and $\Pi \subseteq \Sigma$ then:

(a) $\pi \Sigma \pi^{-1}$ is a symmetry group of $T$;
(b) $\Sigma$ and $\pi \Sigma \pi^{-1}$ are isomorphic symmetry groups;
(c) if $\Pi$ generates $\Sigma$ then $\pi \Pi \pi^{-1}$ generates $\pi \Sigma \pi^{-1}$.

**Proof:** (a) Consider any solution $A$ of $T$ and any $\sigma \in \Sigma$. Then $\pi^{-1}(A)$ is a solution of $S$. As $\sigma$ is a symmetry of $S$, $\sigma(\pi^{-1}(A))$ is a solution of $S$. Hence, $\pi(\sigma(\pi^{-1}(A)))$ is a solution of $T$. Thus, $\pi \sigma \pi^{-1}$ is a symmetry of $T$. Hence $\pi \Sigma \pi^{-1}$ is a symmetry group of $T$.

(b) $S$ and $T$ are equivalent sets of constraints. Consider any solution $A$ of $S$. Then the bijection $\pi$ maps the symmetry class $[A]_\Sigma$ onto the isomorphic symmetry class $[\pi(A)]_{\pi \Sigma \pi^{-1}}$. Consider two symmetric solutions $B$ and $C$ from $[A]_\Sigma$ where $B \neq C$. As they are in the same symmetry class, there exists $\sigma \in \Sigma$ with $\sigma(B) = C$. The bijection $\pi$ maps $B$ and $C$ onto $\pi(B)$ and $\pi(C)$ respectively. Consider the symmetry $\pi \sigma \pi^{-1}$ in $\pi \Sigma \pi^{-1}$. Now $\pi(\sigma(\pi^{-1}(\pi(B)))) = \pi(\sigma(B)) = \pi(C)$. Thus $\pi(B)$ and $\pi(C)$ are in the same symmetry class. As $\pi$ is a bijection and $B \neq C$, it follows that $\pi(B) \neq \pi(C)$. Hence, this symmetry class has the same size as $[A]_\Sigma$. Thus $\Sigma$ and $\pi \Sigma \pi^{-1}$ are isomorphic symmetry groups.

(c) Consider any $\sigma \in \Sigma$. There exist $\sigma_1, \ldots, \sigma_n \in \Pi$ such that $\sigma$ is generated from the product $\sigma_1 \cdots \sigma_n$. Consider $\pi \sigma_1 \pi^{-1}, \ldots, \pi \sigma_n \pi^{-1} \in \pi \Pi \pi^{-1}$. Their product is $\pi \sigma_1 \pi^{-1} \pi \sigma_2 \pi^{-1} \cdots \pi \sigma_n \pi^{-1}$ which simplifies to $\pi \sigma_1 \cdots \sigma_n \pi^{-1}$ and thus to $\pi \sigma \pi^{-1}$. Hence $\Pi$ generates $\pi \Sigma \pi^{-1}$. $\square$

We next show that a sound (complete) set of symmetry breaking constraints will be mapped by the reformulation onto a sound (complete) set of symmetry breaking constraints for the reformulated problem.

**Proposition 4** If a set of constraints $S$ has a symmetry group $\Sigma$, $B$ is a sound (complete) set of symmetry breaking constraints for $S$, and $S$ and $T$ are equivalent sets of constraints, then $\pi(B)$ is a sound (complete) set of symmetry breaking constraints for $T$ where $\pi$ is a bijection between solutions of $S$ and $T$.

**Proof:** (Soundness) Suppose $B$ is a sound set of symmetry breaking constraints for $S$. Consider any $A \in \text{sol}(S \cup B)$. Now $A \in \text{sol}(S)$ and $A \in \text{sol}(B)$. But as $\pi$ is a bijection between solutions of $S$ and $T$, $\pi(A) \in \text{sol}(T)$. Since $A \in \text{sol}(B)$, it follows that $\pi(A) \in \text{sol}(\pi(B))$ $\square$. Thus, $\pi(A) \in \text{sol}(T \cup \pi(B))$. Hence, there is at least one solution left by $\pi(B)$ in every symmetry class of $T$. That is, $\pi(B)$ is a sound set of symmetry breaking constraints.

(Completeness) Suppose $B$ is a complete set of symmetry breaking constraints for $S$. Consider any $A \in \text{sol}(T \cup \pi(B))$. Now $A \in \text{sol}(T)$ and $A \in \text{sol}(\pi(B))$. But as $\pi$ is a bijection between solutions of $S$ and $T$, $\pi^{-1}$ is a bijection between solutions of $T$ and $S$. Hence $\pi^{-1}(A) \in \text{sol}(S)$. Since $A \in \text{sol}(\pi(B))$, it follows that $\pi^{-1}(A) \in \text{sol}(B)$.
Thus $\pi^{-1}(A) \in sol(S \cup B)$. Hence, there is at most one solution left by $\pi(B)$ in every symmetry class of $S$. That is, $\pi(B)$ is a complete set of symmetry breaking constraints.

We have shown that reformulating onto an equivalent problem just maps the symmetries onto isomorphic symmetries and simply requires the same reformulation of any symmetry breaking constraints. We will use these results to argue that symmetry breaking with any ordering besides the lexicographical ordering is intractable for a symmetry group isomorphic to the symmetry group that permutes rows and columns in a matrix model.

### 5 Breaking symmetry is intractable

Suppose we break symmetry using some other ordering than the usual lexicographical ordering on assignments. For example, suppose we break symmetry by insisting that any solution is the smallest symmetric solution in each symmetry class under the Gray code ordering. Under modest assumptions, we can lower bound the complexity of symmetry breaking. We consider orderings which are simple. In such an ordering we can compute the position of any assignment in the ordering in polynomial time, and given any position in the ordering we can compute the assignment at this position. For example, for 0/1 variables and a lexicographical ordering, $(0, \ldots, 0, 0, 0)$ is in first position in the lexicographical ordering, $(0, \ldots, 0, 0, 1)$ is in second position, $(0, \ldots, 0, 1, 0)$ is in third position, $(1, \ldots, 1, 1)$ is in $2^n$th (or last) position. Given any assignment, we can compute its position in the lexicographical ordering in polynomial time. Similarly, we can compute the $k$th assignment in the lexicographical ordering in polynomial time.

We now give our main result which generalizes the result in [11] that computing the lex-leader assignment is NP-hard. We prove that computing the smallest symmetry of an assignment according to any simple ordering is NP-hard.

**Proposition 5** Given any simple ordering $\preceq$, there exists a symmetry group such that deciding if an assignment is smallest in its symmetry class according to $\preceq$ is NP-hard.

**Proof:** For a $n$ by $n$ 0/1 matrix with row and column symmetry, deciding if an assignment is smallest in its symmetry class according to $\preceq_{\text{lex}}$ is NP-hard [11]. Since $\preceq$ and the lexicographical order are both simple orderings, there exists a polynomial function $f$ to map assignments onto their position in the lexicographical ordering, and a polynomial function $g$ to map position in the $\preceq$ ordering onto the corresponding assignment. Consider the mapping $\pi$ defined by $\pi(A) = g(f(A))$ for any complete assignment $A$. Now $\pi$ is a permutation that is polynomial to compute which maps the total ordering of assignments of $\preceq_{\text{lex}}$ onto that for $\preceq$. Similarly, $\pi^{-1}$ is a permutation that is polynomial to compute which maps the total ordering of assignments of $\preceq$ onto that for $\preceq_{\text{lex}}$. Let $\Sigma_{rc}$ be the row and column symmetry group. By Proposition 3, the problem of finding
the lexicographical least element of each symmetry class for $\Sigma_{rc}$ is equivalent to problem of finding the least element of each symmetry class for $\pi\Sigma_{rc}\pi^{-1}$. Thus, for the symmetry group $\pi\Sigma_{rc}\pi^{-1}$ deciding if an assignment is smallest in its symmetry class according to $\preceq$ is NP-hard. $\square$

If follows that even checking a constraint which decides if an assignment is the smallest member of its symmetry class according to $\preceq$ is NP-hard. Note that the Gray code ordering is simple. Hence, a corollary of Proposition 5 is that breaking symmetry with the Gray code ordering is NP-hard in general. It also follows from Proposition 5 that breaking symmetry with the Snake-Lex ordering is NP-hard in general.

6 Conclusions

We have argued that breaking symmetry with a different ordering over assignments than the usual lexicographical ordering used by the lex-leader method does not improve the complexity of dealing with symmetry. Our argument had two parts. First, we argued that using a different ordering can increase the computational complexity of breaking symmetry. Second, we argued that, under modest assumptions, we cannot reduce the computational complexity from that of breaking symmetry with a lexicographical ordering. These assumptions are satisfied by the Gray code and Snake-Lex orderings. Hence, it follows that these methods of breaking symmetry are also intractable in general.

Appendix: Gray code constraint

To demonstrate that we could break symmetry efficiently with a Gray code ordering, we give an encoding for the constraint $\text{Gray}([X_1, \ldots, X_n], [Y_1, \ldots, Y_n])$ that ensures $\langle X_1, \ldots, X_n \rangle$ is before $\langle Y_1, \ldots, Y_n \rangle$ in the Gray code ordering where $X_i$ and $Y_j$ are 0/1 finite domain variables. For a variable symmetry, we need just then set $Y_i = X_{\sigma(i)}$ where $\sigma$ is an appropriate bijection on variable indices, whilst for a value symmetry, we need just set $Y_i = \theta(X_i)$ where $\theta$ is an appropriate bijection on values.

We suppose the existence of a sequence of 0/1/-1 state variables, $Q_1$ to $Q_{n+1}$. We encode the Gray code constraint by means of the following decomposition where $1 \leq i \leq n$:

\[
\begin{align*}
Q_1 &= 1 \\
Q_{n+1} &= 0 \\
Q_i \neq 1 \lor X_i &\leq Y_i \\
Q_i \neq -1 \lor X_i &\geq Y_i \\
X_i = Y_i &\lor Q_{i+1} = 0 \\
X_i = 1 &\lor Y_i = 1 \lor Q_{i+1} = Q_i \\
X_i = 0 &\lor Y_i = 0 \lor Q_{i+1} = -Q_i
\end{align*}
\]

We can show that this decomposition does not hinder propagation.
**Proposition 6** Unit propagation on this decomposition achieves domain consistency on \( \text{Gray}(\{X_1, \ldots, X_n\}, \{Y_1, \ldots, Y_n\}) \) in \( O(n) \) time.

**Proof:** (Correctness) \( Q_i \) is set to 0 as soon as the two vectors are ordered correctly. \( Q_i \) is set to 1 iff the \( i \)th bits, \( X_i \) and \( Y_i \) are ordered in the Gray code ordering with 0 before 1. \( Q_i \) is set to -1 iff the \( i \)th bits, \( X_i \) and \( Y_i \) are ordered in the Gray code ordering with 1 before 0. \( Q_{i+1} \) stays the same polarity iff \( X_i = Y_i = 0 \) and flips polarity iff \( X_1 = Y_i = 1 \).

(Completeness) The decomposition is Berge acyclic. Thus unit propagation is enough to guarantee the existence of a support for every value.

(Complexity) There are \( O(n) \) disjuncts in the decomposition. Hence unit propagation takes \( O(n) \) time. \( \square \)

In fact, it is possible to show that the total time to enforce domain consistency at each branching decision down a branch of the search tree is \( O(n) \).

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