Visualizing and Understanding Large-Scale Assessments in Mathematics through Dimensionality Reduction

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Abstract

In this paper, we apply the Logistic PCA (LPCA) as a dimensionality reduction tool for visualizing patterns and characterizing the relevance of mathematics abilities from a given population measured by a large-scale assessment. Particularly, we analyse the data collected from SPAECE, a large-scale assessment in mathematics that has been applied yearly in the public educational system of the state of Ceará, Brazil. As the main result, we show that the the poor performance of examinees in the end of middle school is primarily caused by their disabilities in number sense.

1 Introduction

We are witnessing the birth of the fourth industrial revolution, being a technological trend that is transforming the way we live, work and interact to each other. This new era is driven by a set of technological innovations such as robotics, artificial intelligence, big data (massive volume data analysis), 3D printing, augmented reality, synthetic biology, nanotechnology and the commonly known technology of things whereb, increasingly the devices are connected to others through the Internet. Some of which said innovations are still in their embryonic stage, however being ready to develop quickly.

The World Economic Forum held in Davos, Switzerland in 2016, published a report \cite{WEF} which places the ability of complex problem solving as the most important skill required for students to achieve a successful career demanded by this technological revolution. It has become a consensus that the major mean of developing problem-solving skills is through mathematics (see \cite{NCTM} p.52). Indeed, mathematics helps us to analyze and think logically about new situations, devise unspecificed solution procedures, and communicate their solution clearly and convincingly to others. Moreover, one of the main goals of learning mathematics is to deal with abstractions that sometimes model and solve concrete problems which are apparently disconnected.

Public Educational Systems need to be aware of the technological advances in the world and properly prepare students for the job market. However, it has been a great challenge for (municipal, state, and national) governments to recycle teachers and improve the infrastructure and administration of schools. To assist policy makers in their decisions, large-scale assessments of mathematics are used to monitor the abilities and knowledge of students \cite{OECD}.

SPAECE (Permanent Evaluation System of Basic Education in the State of Ceará) is the local large-scale assessment in portuguese and mathematics in the state of Ceará. Using the Item Response Theory (IRT), this assessment collects yearly the proficiency in these areas from all students attending the public

\footnote{Here we will keep the acronym spelling in portuguese.}
Figure 1: Logistic PCA results of 5K individuals: two-dimensional scatter plot colored by proficiency (top left) and colored by descriptor $D_{16}$ (top right); relative loadings of the first principal component (bottom).

Schools of the state. This database has been the main reference source to diagnose school results and accountability to society by providing the big picture of the quality of public education of Ceará.

SPAECE is taken at three levels: at the end of elementary school (L1), at the end of middle school (L2) and at the end of high school (L3). SPAECE uses a Reference Matrix (RM) which is composed of a set of descriptors that explain the level of mental operation required to perform certain tasks. These descriptors are selected considering what can be evaluated by means of a multiple choice test, whose items imply the selection of a response in a given set of possible answers. There are three RMs, one for each level, and they may share a few descriptors. In Appendix C we show the RM corresponding to L3.

Our goal is to provide mathematical and visual insights of large-scale assessments in mathematics through the LPCA package, a Principal Component Analysis (PCA) tool for binary data which has become a popular alternative to dimensionality reduction of binary data.

The variables we investigate correspond to the descriptors of the RM. By applying LPCA to them, we obtain a set of 1–2 principal components which carry precious quantitative/qualitative information of the students' proficiency along with the local educational system. Next, we explain the intuition behind dimensionality reduction, drawing a parallel with an example in genetics.

Intuition behind dimensionality reduction

Based only on the genetic information, how precisely we can assign an individual to a geographic location? To investigate this question, characterizes genetic variation in a sample of 3K European individuals at over 500K variable DNA sites in human genome. PCA produces a two-dimensional visual summary of the observed genetic variation and the resulting figure bears a notable resemblance to a geographic map.
of Europe. The first two principal components (PCs 1 and 2) are good predictors (after a rotation) to the geographic latitude and longitude coordinates.

Now, based only on the scores of the students from an assessment in mathematics, how accurately we can assign each student to a common ruler within a proficiency scale? Let us consider the SPAECE assessing \( \sim 100K \) students at the ending of high school (L3). The test is composed by a set of 24 descriptors from the RM. As similar to the example above, we may think of each descriptor as a gene. When we apply LPCA to generate a two-dimensional map, we obtain a picture that carries similarities with the IRT scores of SPAECE. In Figure 1 we summarize the observations. The scatter plot at top left exhibits the correlation between the IRT scores and PC1. PC2 might correspond to a possible unknown latent trait. The bar plot at the bottom presents the loadings (coordinates) of PC1 by descriptor. They are co-related with the discrimination property of an item. At the top right, we show the profile of correct/wrong answers corresponding to descriptor with highest loading in PC1, that is, \( D_{16} \). Throughout the paper we clarify these findings and discuss how to interpret them.

This paper is organized as follows. In Section 2 we discuss related work. In Section 3 we set up the mathematical background. More specifically, we establish a connection between LPCA, IRT and the Inner Product Representation (IPR). In Section 4 we describe SPAECE data set. Then, we apply the LPCA and provide visualization tools. From the patterns we find in the visualizations, we extract useful information that can guide improvements to the educational system of Ceará. Finally, in Section 5 we summarize our contributions and discuss limitations.

2 Related work

According to Colin Ware [23], the term visualization means something more like a graphical representation of data or concepts. Visualization has also been combined with human factors and data analysis to gain insight in the problem at hand [20]. We develop a contribution that goes in this direction, more specifically, we do visual analytics [20, 8]. From the results generated by the LPCA, we generate visualization tools that make easier human perception of patterns. New patterns are translated to useful information that captures the actual status of educational systems.

The Graphical Item Analysis (GIA) [22] has been used to assess the quality of items for large-scale assessments in Brazil. A similar approach was also proposed in [10]. In both, the main goal is to obtain a visual understanding of IRT and its item’s statistics. In other words, their focus is to improve the quality of the assessment per se. On the flip side, our work aims at providing new visual ways to understand the entire proficiency of the population.

Regarding to dimensionality reduction for binary data, [1] uses ideas from generalized linear models (GLM) [12] to describe an optimization method. In [2], a similar generalization is proposed, in particular, using the IPR. The LPCA [9] generalizes PCA in such a way that the loadings of the principal component are linear functions of the data. This is done by interpreting Pearson (1901)’s formulation [4] in a slightly different manner.

We describe the connection between LPCA, IRT and IPR through the equivalence of parameters as shown in Table 1. This equivalence may help to better understand the proposed visualization tools in Section 4. Other comparisons betweeen IRT and structural equation modeling (SEM) can be found in [19] and [7].

3 Background

3.1 Classic PCA

Assume that our data consists of a matrix \( X_{n \times d} \) where every row is a vector \( x_i \in \mathbb{R}^d \), \( i = 1, 2, ..., n \), representing a set of observations and each column corresponds to an observable variable. For \( k < d \), we wish to find a translation vector \( \mu \in \mathbb{R}^d \) and a set vectors (also called principal components) \( u_1, u_2, ..., u_k \in \mathbb{R}^d \), forming a matrix \( U_{d \times k} = [u_1 \ u_2 \ ... \ u_k] \), such that \( \Theta_i = \mu + U U^T (x_i - \mu) \) is “close”
to $x_i'$ as possible and which belong to a lower parameter affine space. We denote each coordinate of each $u_i$ as loading. Let $\Theta$ be a matrix where each row is a vector $\theta_i$. The log likelihood expressed as a function of the parameter $\Theta$ is just,

$$l(\Theta; X) = \sum_{i,j} \log f(x_{ij}; \theta_{ij})$$

(1)

where $f(\cdot; \theta_{ij})$ is a probability density function of $x_{ij}$ from an exponential family with parameter $\theta_{ij}$. Therefore, to obtain the desired parameters, we need to minimize the scaled deviance criterion which is given by:

$$D(X; \Theta) = 2l(X, X) - 2l(\Theta, X)$$

(2)

If each $x_{ij}$ is an observation from Gaussian probability density $x_{ij} \sim N(\theta_{ij}, 1)$, the natural parameter is $\theta_{ij}$ and the natural parameter from the saturated model is $x_{ij}$ itself. Substituting (1) in (2) for a Gaussian probability density we obtain the Pearson’s formulation [4] which is to minimize the square error (see Appendix A to check derivation):

$$E = \sum_{i=1}^{n} ||x_i - \theta_i||^2 = \sum_{i=1}^{n} ||(x_i - \mu) - UU^T(x_i - \mu)||^2$$

(4)

over $\mu$ and $d \times k$ orthogonal matrix $U$.

### 3.2 Logistic PCA

Now let us suppose that $X$ is a binary data matrix where $x_{ij}$ is equal to zero or one, representing a correct/incorrect item responded by a student. Each line correspond to a student and each column correspond to an item.

Let $P = \{p_{ij}\}$ be a matrix where $p_{ij}$ is the probability of student $i$ answer correctly an item $j$, which means that $p_{ij} = f(x_{ij}; p_{ij})$, where $f(\cdot; p_{ij})$ is a Bernoulli mass function with parameter $p_{ij}$. The canonical link function for the Bernoulli distribution provides the natural parameter given by

$$\gamma_{ij} = \logit p_{ij}$$

(3)

and each element of matrix $\Gamma = \{\gamma_{ij}\}$ varies from $-\infty$ to $+\infty$. Let $\hat{\Gamma} = \{\hat{\gamma}_{ij}\}$ represent the natural parameter for the saturated model. Notice that $\hat{\gamma}_{ij}$ is $\infty$ if $x_{ij} = 1$ and $-\infty$ if $x_{ij} = 0$, which is unfeasible for numerical computations.

To extend principal component analysis to binary data and make it computationally realizable, it is necessary to limit this domain. For convenience, first define $q_{ij} = 2x_{ij} - 1$, which converts the binary variable from taking values in $\{0, 1\}$ to $\{-1, 1\}$. Let $Q$ be the matrix with elements $q_{ij}$. Then, we approximate $\hat{\gamma}_{ij}$ by $\hat{\theta}_{ij} = m \cdot q_{ij}$ for a large number $m$. Therefore, $\hat{\Theta} = m \cdot Q$ approximates the matrix $\hat{\Gamma}$ of natural parameters for the saturated model. As in standard PCA, for $k < d$ we seek for vector $\mu \in \mathbb{R}^d$ and a matrix $U_{d \times k}$ such that

$$\theta'_i = \mu + UU^T(\hat{\theta}'_i - \mu)$$

(4)

is close to $\hat{\gamma}'_i$. The objective function to minimize is the scaled deviance (2), subject to $U^TU = I_k$ (see Appendix B to check this formula and computations).

The numerical optimization has been implemented through an R [15] public package available in the web. For the web address and technical details we refer the reader to [9].
3.3 Logistic PCA and UIRT/MIRT

In Unidimensional Item Response Theory (UIRT), the Item Characteristic Curve (ICC) (see Figure 3.3) represents the probability that an individual \(i\), with a single ability level \(\alpha_i\), can solve correctly the item \(j\).

The ICC is a two-parameter logistic model (2PL) given by

\[
P(\alpha_i) = \frac{\exp[\delta_j(\alpha_i - \beta_j)]}{1 + \exp[\delta_j(\alpha_i - \beta_j)]},
\]

where the parameters \(\delta_j\) and \(\beta_j\) represent the discrimination and difficulty of the item, respectively. Notice that if we substitute \(-\delta_j\beta_j\) with \(d_j\), the linear expression turns into the slope-intercept form, \(\delta_j\alpha_i + d_j\). A more general model is the multidimensional two-parameter logistic model (M2PL) which is given by the probability function

\[
P_j(\alpha_i) = \frac{\exp(\delta'_j\alpha_i + d_j)}{1 + \exp(\delta'_j\alpha_i + d_j)}
\]

which takes multiple abilities in a \(k \times 1\) vector \(\alpha_i\) and mimics the slope-intercept form with the expression \(\delta'_j\alpha_i + d_j\). Here, \(\delta_j\) is a \(k \times 1\) vector that represents the relative discrimination parameters of the item \(j\). The scalar \(d_j\) parameter is not the difficult parameter in the usual sense of a 2PL model because it does not give a unique indicator of the difficulty of the item. Instead, the quotient \(-d_j/\delta_{jl}\) gives the relative difficulty of the item related to the ability axis \(l\).

By setting \(p_{ij} = P_j(\alpha_i)\), from Equation (3), we have that:

\[
\gamma_{ij} = \logit p_{ij} = \log \{P_j(\alpha_i)\} = \log \left\{ \frac{P_j(\alpha_i)}{1 - P_j(\alpha_i)} \right\} = \log \left\{ \frac{\exp(\delta'_j\alpha_i + d_j)}{1 + \exp(\delta'_j\alpha_i + d_j)} \right\} = \log \left\{ \frac{\exp(\delta'_j\alpha'_i + d_j)}{1 + \exp(\delta'_j\alpha'_i + d_j)} \right\} = \delta'_j\alpha'_i + d_j
\]

Now, recall from Equation (4), the LPCA provides a vector \(\mu \in \mathbb{R}^d\) and an orthogonal matrix \(U_{d \times k}\) such that \(\theta_i\) is close to \(\tilde{\gamma}_i\). Therefore, it is easy to see that
\[ \theta_{ij} = u_j \psi_i + \mu_j \]  \hspace{1cm} \text{(8)}

where \( \psi_i = U^T(\tilde{\theta}_i' - \mu) \) and \( u_j \) is the \( j \)th row of matrix \( U \).

Since \( \theta_{ij} \) is an approximation of \( \gamma_{ij} \), from Equations (7) and (8) we obtain a connection between UIRT/MIRT and LPCA. The relative discrimination vector \( \delta_j \) in Equation (7) expresses how well item \( j \) can differentiate among examinees with different abilities. It corresponds to the principal component \( u_j \) in Equation (8). Next, the scalar \( d_j \) represents the \( j \)-th coordinate of the translation vector \( \mu \). Finally, the vector of multiple abilities \( \alpha_i \) correspond to the resulting vector \( \psi_i \) in the LPCA model.

In general, UIRT/MIRT is used if the focus is on the scale and the item characteristics whereas LPCA is used if the focus is on structural relations among either observed or latent traits, with or without exogenous covariates [19, 7].

3.4 A Geometric Representation

Another equivalent model to LPCA is the following geometric problem: represent the rows of the data matrix as points and the columns as hyperplanes in low-dimensional Euclidean space \( \mathbb{R}^k \). Rows \( i \) correspond to points \( a_i \) and the columns are represented as hyperplanes \( (b_j, c_j) \) where \( b_j \) is a vector of slopes and \( c_j \) is a scalar intercept. As usual, the parameter \( k \) is the dimensionality.

The solution of the problem consists in construct a drawing in such a way that points \( a_i \) for which \( x_{ij} = 1 \) (here \( X = \{x_{ij}\} \) is a binary data) should be in one side of the plane and the points for which \( x_{ij} = 0 \) should be on the other side. In algebraic terms, we want to find a solution to the system of strict inequalities

\[
\begin{align*}
    b'_j a_i &> c_j, \text{ for } x_{ij} = 1 & (9a) \\
    b'_j a_i &< c_j, \text{ for } x_{ij} = 0 & (9b)
\end{align*}
\]

We will call such geometric model as \textit{Inner Product Representation} (IPR).

In general, the system of inequalities above will not have an exact solution. Then, it is necessary to find an approximate solution in the sense that it minimizes the scaled deviance as defined in Equation (2). We refer the reader to [2] for more details on the approximating solution. Figure 3 (left) shows an example of solution with three lines that split the plane into seven regions. Figure 3 (right) shows a theoretical configuration for IRT: every individual will get 1 for items with difficulty below than the proficiency of the individual and 0 otherwise. Notice that the solution for this configuration has vertical parallel lines and

![Figure 3: Examples of IPR. Three lines splitting the plane into seven regions (left) and an ideal IRT configuration (right).](image-url)
the proficiency of the individuals increases as the regions goes from left to right. From this example, we can infer that the vector of slopes \( b_j \), somehow, encodes the discrimination of the item. More precisely, let us say that there is a solution satisfying inequalities (9a) and (9b). Then, by setting \( \delta_j := b_j, \alpha_i = a_i \) and \( d_j := -c_j \), we have that:

\[
\begin{align*}
P_j(\alpha_i) &> 0.5 \text{ if } b_j' a_i > c_j \\
P_j(\alpha_i) &= 0.5 \text{ if } b_j' a_i = c_j \\
P_j(\alpha_i) &< 0.5 \text{ if } b_j' a_i < c_j
\end{align*}
\] (10a) (10b) (10c)

One can easily check that the level sets of \( P_j \) are hyperplanes. Equation (10b) represents the hyperplane \((b_j, c_j)\) and it corresponds exactly to the level set 0.5.

So far, in this section, we synthesized the theory of three different models (LPCA, UIRT/MIRT and IPR) for representing multidimensional data in a lower dimensional space. More importantly, we have established a connection among them. In Table 1, we summarize the equivalence of the parameters through the models which provides different insights that will help us to explore and understand data in the applications.

| LPCA | UIRT/MIRT | IPR |
|------|-----------|-----|
| vector \( u_j \) | vector of relative discriminations \( \delta_j \) | vector of slopes \( b_j \) |
| coordinate \( \mu_j \) | intercept \( d_j \) | intercept \( c_j \) |
| vector \( \psi_i \) | vector of multiple abilities \( \alpha_i \) | point position \( a_i \) |

Table 1: Equivalence table of parameters through the models LPCA, UIRT/MIRT and IPR.

4 Applications

Our applications are focused on representing data in two dimensions. We will show applications on visualizing bi-dimensional colored plots which highlight properties that users wish to see from the population. We will also perform visual analysis on SPAECE which shows how these tools can be used to perform data exploration.

4.1 Pre-processing Data Set

Since 2008, with a three-parameters model, SPAECE has been adopting an IRT based methodology for tests elaboration and data analysis. In the end of the school year every student at levels L1, L2 and L3 are submitted to the mathematics and portuguese language assessment. Our analysis will focus on the mathematics assessment applied from 2016 to 2018 at the level L3.

The test that is given to each examinee contains two distinct blocks with 13 items, totaling 26 items. These blocks are assembled through the balanced incomplete block design (BIBD). In each test, there are at most two items assigned to the same descriptor. We feed the input of the LPCA algorithm with a table where the rows correspond to the examinee-test and the columns correspond to the math descriptors (see appendix C). Depending on the number of items per descriptor in the test, the columns will be filled by \( NA \) (not available) or the probability of getting a correct answer. So, if the examinee gets one/two correct answer out of one/two items the probability is 1; one correct answer out of two items results in 1/2; no correct answers results in 0. In Table 2 we show the first samples filled of a typical input.

In summary, the variables under analysis are the descriptors as listed in the RM. For each examinee-test, the value of each descriptor corresponds to the rate of correct items assigned to it.

We emphasize that, although the previous section has assumed a binary matrix as input, one can...
easily check that the theory is still valid for values from 0 to 1. Moreover, LPCA can also manage missing data, that is, descriptors with values assigned as not available (NA).

4.2 Visualizing LPCA results

As we run the LPCA algorithm with two principal components we get the bi-dimensional scatter plot. We can use it to create color maps that highlight interesting distributions of the population. Below, we give examples of color maps that may be useful for analyses. For a better visualization of the figures we reduced the number of individuals from \( \sim 100K \) to 7K individuals randomly selected. We classify the maps in to groups: ability maps and social maps.

Ability Maps

Ability maps are primarily concerned with showing the efficiency behavior of the entire population. We exemplify here two types of maps.

Proficiency Map: SPAECE provides the IRT proficiency scores of the individuals in a scale that ranges from 0 to 500. This interval is subdivided into four performance standards: [0 – 250] is very critical; [251 – 300] is critical; [301 – 350] is intermediate and [351 – 500] is adequate. In Figure 1 (top left) we show the color map of the proficiency scores of the assessment in 2018 subdivided in these groups. As predicted in Section 3.3, we can see a strong correlation between the proficiency scores and PC1 (i.e. the unidimensionality property of the IRT model), which is confirmed by a Pearson’s correlation coefficient value of 0.92. It is interesting to see that the graphic also shows the density decreasing as we go over the PC1 from left to right. Since the first SPAECE assessment was applied, the higher concentration of individuals persists on the left side, that is, most of the population are located among the very critical and critical performance standards. This indicates a poor quality of the public education system delivered by state of Ceará.

Descriptor Map: The descriptor map is a binary color map that brings insights of the population regarding their performance per descriptor. For each descriptor \( j \) we split the population into two groups: those with probability higher than 0.5 (i.e. \( p_{ij} > 0.5 \)) of getting descriptor \( j \) correctly and those with probability lower than 0.5 (i.e. \( p_{ij} < 0.5 \)) of getting the descriptor \( j \) wrong. In Figure 4 (left) we show descriptor \( D_{76} \) in 2016. In this descriptor, most of examinees get correct answers. This means that the descriptor is composed by items with very low difficulty. In Figure 4 (middle) we show descriptor \( D_{56} \) of 2017 which has an unexpected behavior. Although a minority of the examinees get correct answers, the slope of the level set line at 0.5 is nearly horizontal which means that the group of items composing this descriptor have poor discrimination. The low discrimination value indicates the irrelevance of the descriptor to the trait being measured by the test \( 5 \). Even if examinee “B” can identify correctly circle’s equations and examinee “B” does it incorrectly, we might not feel confident concluding that examinee “A” has a higher proficiency than does examinee “B”. In Figure 4 (top right) we show the descriptor map of \( D_{16} \). We will discuss more about its importance in Section 4.3.

Social Maps

A social map provides visual distributions of social characteristics of the population (e.g. gender, age, family income etc.). Here present the shift map.

| examinee | \( D_{16} \) | \( D_{19} \) | \( D_{20} \) | \( D_{24} \) | \( D_{28} \) | \( D_{40} \) | \( D_{76} \) | \( D_{78} \) |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| \( S_1 \) | 0        | NA       | 1/2      | NA       | NA       | 0        | ...      | 0        | 1/2      |
| \( S_2 \) | 0        | NA       | 0        | 0        | 0        | 0        | ...      | 1        | 1        |
| \( S_3 \) | 0        | NA       | 0        | NA       | NA       | 0        | ...      | 1        | 0        |
| \( S_4 \) | 1/2      | NA       | 0        | 0        | 0        | 0        | ...      | 1        | 0        |

Table 2: First rows of the LPCA algorithm input.
Figure 4: Examples of color maps. Descriptor map of $D_{76}$ and $D_{56}$ (left and middle respectively) and shift map (right).

**Shift Map:** In Figure 4(right) we show the shift map of the population. Observe that the concentration of the evening group in the left side is higher than the others, that is, this group has a lower proficiency average. There are economic and social reasons involved for this problem. We refer the reader to [21, 17] for more details.

### 4.3 Example of Visual Analytics

Despite the poor results of the proficiency in mathematics in state of Ceará, there are small islands of excellence among schools in the public education system. Over the period from 2016 to 2018, having a growth of 26.5%, the best average proficiency growth was achieved by School $A$. In this period, the school moved from the very critical group to the intermediate performance group. This growth corresponds to an impressive jump of two levels. Using LPCA and the visualization tools described in this paper, we investigate this outstanding result in more details.

In Figure 5 (top row), we plot the three highest loadings of PC1. Notice that all plots are headed by descriptor $D_{16}$. Moreover, its values represent at least three times the mean of all 24 descriptors. Recall that, from the equivalence Table 1, $D_{16}$ corresponds to the best discrimination’s parameter in IRT.

In Figure 5 (middle row), we show the descriptor maps of $D_{16}$ where the examinees of School $A$ are represented by dark dots. In these maps we observe that all level set lines at 0.5 are nearly vertical. Again, as shown in Table 1, this confirms the equivalence between the IRT discrimination’s parameter and the IPR slope’s parameter. We observe that there is a strong migration of dark dots from the “left side” to the “right side” of the level set 0.5. If we compare the descriptor maps with the proficiency maps Figure 5 (bottom row), we find that the level set lines at 0.5 are located right between the critical and intermediate groups in the performance standards.

More important than visualizing graphs, is understand why $D_{16}$ has all these peculiar properties as described above. This descriptor is directly connected with a large research area called number sense, a topic in mathematics education which has been developed in the last few decades [3, 6]. Roughly speaking, number sense refers to an individual general comprehension with regard to numbers and flexibility in using the operations for making mathematical discernment. It is the result of mathematical experience where students could employ their sense in understanding circumstances involving numbers [11].

Number sense is extremely necessary for individuals to be successful in the other descriptors listed in the Reference Matrix. For example, to solve a problem corresponding to descriptor $D_{55}$, the examinee needs to understand that the inclination of a straight line is the fraction $\Delta y/\Delta x$, i.e., the ratio between the variations of $y$- and $x$- axes in the corresponding linear function. As a matter of fact, a great many people find fractions very difficult to learn because their cortical machinery resists such a counterintuitive concept [3] and an examinee tends to fail in this example if he did not understand well how to manipulate

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3Due to confidentiality reasons, we are not allowed to identify the school.
ratios and proportions. In Figure 5, we show the plots of the number of correct answers by School A where most of the other descriptors follows the growth trend of $D_{16}$.

In interviewing the School A’s math teachers, we checked that they made an special effort in recovering their students with disabilities in number sense. The discussion of the teaching methodology employed by them is out of the scope of this paper, however, the idea of focusing in number sense could be successfully reproduced by schools with similar conditions in the public education system of Ceará. By similar conditions, we mean schools with very critical performance. As a result, the students may surpass the “barrier” line of descriptor $D_{16}$ and migrate from an average performance in the very critical group to the intermediate group.

5 Discussion

In this paper we presented the LPCA as a visualization tool for the analysis of math assessments. We applied the LPCA with the SPAECE math assessments from 2016 to 2018, in order to understand possible intrinsic relation among descriptors that have not been noiced so far. Then, using the proposed color maps, we investigated the distribution of the math abilities of the population in more details.

Our findings have been used to guide new educational policies of the Secretariat of Education for the
State of Ceará (SEDUC-CE). In particular, the formulation of a new curriculum taking into account the deficiencies in number sense of the students in Ceará is underway in the educational system.

Although the examples of this paper are limited to large-scale assessment in mathematics, it is still possible to reproduce them for other areas. However, we alert that, depending on the context of the area and the distribution of the abilities of the population, the interpretation of the visualizations may not follow similar conclusions as those found in Section 4.

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A Normal deviance

The Gaussian probability density function with parameter \( \mu \) is

\[
f(x, \mu) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{(x - \mu)^2}{2} \right\}.
\]

Now, observe that

\[
l(\Theta, X) = \sum_{i,j} \left\{ -\frac{1}{2} \log(2\pi) - \frac{(x_{ij} - \theta_{ij})^2}{2} \right\}
\]
and,

\[
l(X, X) = \sum_{i,j} \frac{1}{2} \log(2\pi).
\]

Therefore, the scaled deviance is

\[
D(X, \Theta) = 2l(X, X) - 2l(\Theta, X) = \sum_{i,j} (x_{ij} - \theta_{ij})^2 = \sum_i ||x_i - \theta_i||^2.
\]

B  Bernoulli deviance

The Bernoulli probability mass function with parameter \( p \) is given by

\[
f(x;p) = \begin{cases} p & \text{if } x = 1, \\ 1-p & \text{if } x = 0, \end{cases}
\]

It is also convenient to express such function as \( f(x;p) = p^x(1-p)^{1-x} \). Now, using that

\[
\gamma_{ij} = \text{logit } p_{ij} = \frac{p_{ij}}{1-p_{ij}} \iff p_{ij} = \frac{\exp \gamma_{ij}}{1 + \exp \gamma_{ij}}
\]

we have,

\[
\log f(x_{ij};p_{ij}) = \log \{p_{ij}^x(1-p_{ij})^{1-x}\} = x_{ij} \log p_{ij} + (1 - x_{ij}) \log (1 - p_{ij}) =
\]

\[
x_{ij} \log \left( \frac{p_{ij}}{1-p_{ij}} \right) + \log (1 - p_{ij}) = x_{ij} \gamma_{ij} + \log \left( 1 - \frac{\exp \gamma_{ij}}{1 + \exp \gamma_{ij}} \right) =
\]

\[
x_{ij} \gamma_{ij} + \log \left( \frac{\exp \gamma_{ij} + 1 - \exp \gamma_{ij}}{1 + \exp \gamma_{ij}} \right) = x_{ij} \gamma_{ij} + \log (1 + \exp \gamma_{ij})^{-1} =
\]

\[
x_{ij} \gamma_{ij} - \log (1 + \exp \gamma_{ij})
\]

and it is easy to see that \( \log f(x_{ij}, x_{ij}) = 0 \). Thus, the Bernoulli scaled deviance is

\[
D(X, \Theta) = 2l(X, X) - 2l(\Theta, X) = 2 \sum_{i,j} \log f(x_{ij};x_{ij}) - 2 \sum_{ij} f(x_{ij} \theta_{ij}) =
\]

\[
= -\langle X, \Theta \rangle + 2 \sum_{ij} \log \left( 1 + \exp(\mu_j + UU^T(\theta_i - \mu)) \right).
\]

C  SPAECE’s Reference Matrix

The SPAECE’s reference matrix is formed by a set of minimum expected skills (descriptors) in their various levels of complexity, in each area of knowledge and each stage of schooling. The matrices are based on studies of the curricular proposals of teaching in the current curricula of the Brazil, besides researches in didactic books and debates with active educators and specialists in education. The reference matrices are elaborated without the pretension of exhausting the repertoire of the necessary skills to the full development of the student. Therefore, they should not be understood as unique skills to be worked on in the classroom. Its purpose is to mark the creation of test items, which distinguishes them from curricular proposals, teaching strategies and pedagogical guidelines. Below, we describe the Reference Matrix of the third year of high school, which is the main focus of this paper.
| THEME I: INTERACTING WITH NUMBERS AND FUNCTIONS |
|------------------------------------------------|
| $D_{16}$ Establish relations between fractional and decimal representations of rational numbers. |
| $D_{19}$ Solve problems involving simple interests. |
| $D_{20}$ Solve problems involving compound interests. |
| $D_{24}$ Factor and simplify algebraic expressions. |
| $D_{28}$ Identify the algebraic representation or graph of a polynomial function of 1st degree. |
| $D_{40}$ Relate the roots of a polynomial with its decomposition in factors of 1st degree. |
| $D_{42}$ Recognize algebraic or graphical representation polynomial function of the 1st degree. |

| THEME II: LIVING WITH GEOMETRY |
|--------------------------------|
| $D_{49}$ Solve problems involving similarities of planar figures. |
| $D_{50}$ Solve situation problem involving Pythagorean Theorem or other metric relations in the right triangle. |
| $D_{51}$ Solve a problem using polygon properties (sum of internal angles, number of diagonals, computing the interior angle regular polygons). |
| $D_{52}$ Identify flattening of some polyhedral and/or round objects. |
| $D_{53}$ Solve situation problem involving trigonometric ratios in right triangles (sine, cosine and tangent). |
| $D_{54}$ Calculate the area of a triangle given the coordinates of the vertices. |
| $D_{55}$ Determine the equation of a straight line given two points or a point and its inclination. |
| $D_{56}$ Among the equations of 2nd degree with two unknowns identify, those that represent a circumference. |
| $D_{57}$ Find the location of points in the Cartesian plane. |
| $D_{58}$ Interpret geometrically the coefficients of a straight line equation. |

| THEME III: LIVING THE MEASURES |
|--------------------------------|
| $D_{64}$ Solve a problem using the relations between different measure unities of capacity and volume. |
| $D_{65}$ Calculate the perimeter of planar figures in a situation problem. |
| $D_{67}$ Solve problem involving calculation of areas of planar figures. |
| $D_{71}$ Calculate the total surface area of prisms, pyramids, cones, cylinders and spheres. |
| $D_{72}$ Calculate the volume of of prisms, pyramids, cones, cylinders in a situation problem. |

| THEME IV: TREATMENT OF INFORMATION |
|-----------------------------------|
| $D_{76}$ Assign information presented in lists and/or tables to the graphs that represent them, and vice versa. |
| $D_{78}$ Solve problem involving central tendency measures: mean, mode and median. |

Table 3: test