Evaluation of the limit properties of a solid body system

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Abstract. This article discusses the design of modern wheel-transport vehicles, in which the problem of reducing the level of vibrations and shocks arises, since the quality and performance, reliability and durability of the operation of machines, devices and equipment depend mainly on the resulting fluctuations during their operation. Also, solutions of problems concerning the limiting properties of solid-state systems for cases of harmonic and shock effects are discussed.

The design model of many technical objects is a solid body, which is onboard radio electronic equipment (REA). For the selected model of a solid body, a system is designed for the vibration isolation of a block, at s points of which there are electrical radio-sensitive elements that can withstand a given limiting vibration acceleration. The synthesized vibration-proof system is implemented with the help of linear passive elastic-damping suspensions and active elements, which represent an ideal servo-mechanism.

1. Introduction

In the design of modern wheel transport vehicles, the problem of reducing the level of vibrations and shocks arises, since the quality and performance, reliability and durability of the operation of machines, devices and equipment mainly depend on the resulting fluctuations during their operation. In particular, this problem is particularly evident in the operation of transport systems for various purposes, including wheeled. The creation of means of protecting technical systems of machines for various purposes from vibrations and shocks led to the development of the theory of vibration protection systems, which used the mathematical apparatus of applied disciplines such as, for example, the theory of oscillations, the theory of automatic control, and sets for researchers problems whose solution requires new mathematical approaches and methods. On the other hand, the creation of effective means of protection against vibrations and shocks is closely linked to the need to improve the quality of the design of protection systems for wheeled transport vehicles. In this regard, the development of the theory of vibration protection systems in matters related to the development of methods and algorithms for solving problems of designing vibration protection systems is of great importance.

The dynamics and strength of machines, as an area of science and technology, studying the behavior of technical objects of various purposes, the laws of mechanical phenomena and related processes of a different nature, using methods of mechanics and computational mathematics, serve as the basis for
searching and developing new ways and means building new machines, and how the direction of scientific research in this regard becomes a tool for finding, substantiating and calculating new technical solutions.

The relevance of the study is determined by the tasks solved within its framework, the main one being the development of methods for solving problems on the limiting capabilities of a spatial vibration-proof system; methods for solving the problem of the synthesis of a vibration-protective system (VZS).

2. Typical design tasks and types of restrictions when designing a vibration-protective system

At present, the solution of the problem of designing a gas station involves a complex set of studies that are performed using theoretical calculations and constitute an interconnected system of procedures, the implementation of which in a certain sequence leads to the construction of a vibration protection system. In this case, it becomes necessary to single out typical design tasks, the solution of which forms a complex of interrelated procedures that provide preliminary data for design developments. Automation of these procedures leads to the creation of a computer-aided design of VZS [2, 3, 6, 7].

At all stages of design, developers operate with a description of the object - a mathematical model. Analysis and experience in solving research problems in the field of vibration protection shows that most of the objects of protection allow their presentation in the form of design schemes: a solid body or a system of solids. And this allows within the framework of these design schemes to identify the main typical design tasks: determining the need for protection; assessment of limiting properties; synthesis; analysis; the definition of recommendations for the selection of standard means of technical implementation of vibration isolation [2].

When designing a VZS, as a rule, a set of requirements to the vibration protection system is specified, which is reflected in the technical specifications. These requirements can be formulated as constraints [4, 10]. The most significant classes of constraints that have to be considered when designing a VZS are the following.

a) Restrictions on absolute accelerations. For example, if a VZS is intended to protect against kinematic effects, then its main task is to reduce the absolute accelerations of certain points of the depreciable body. Corresponding constraints can generally be written in the form:

$$J_{w_i}(\tau_i(t)) \leq l_i, \quad (i = 1, s), \quad (1)$$

where, $w_i(t)$ – the projection of the absolute accelerations of certain points of the depreciable body to given directions (for example, to the directions of the coordinate axes); $J_{w_i}$ – a functional whose form usually depends on the nature of dynamic actions.

In the case of a harmonic, polyharmonic, or stationary random effect, the constraints are usually imposed on some average acceleration values; most often on rms:

$$J_{w_i}(\tau_i(t)) = m \left| w_i^2(t) \right|, \quad (2)$$

Moreover, in the case of a deterministic process, averaging is performed over time, and for a random effect, over the implementations of the process $w_i(t)$. For non-stationary (shock) actions, it is natural to choose the maximum of the acceleration modulus as the $J_{w_i}$ functional:

$$J_{w_i}(\tau_i(t)) = \max_t \left| w_i(t) \right|, \quad (3)$$

limiting its value to some $l_i$.

b) Restrictions on relative movements. It is known that, by creating sufficiently soft shock absorbers, it is possible, in principle, to ensure the fulfillment of any of the most stringent requirements for VZS. However, the dynamic effects will cause large displacements of the object relative to the base. In order that at these displacements there would be no strikes against the travel stops, the latter should be located far enough from the depreciable body. In other words, the system of vibration protection acquires large overall dimensions and therefore may become practically unrealizable.

In this regard, when creating a VZS, the requirements for relative displacements should be formulated.
Relevant restrictions may be presented as:

\[ J_q \{ q_i(t) \} \leq L_i, \quad (i = 1, k) \]  

(4)

where \( q_i(t) \) – relative displacements of individual points of the object. The form of the functional \( J_q \) depends on the nature of the effects:

\[ J_q \{ q_i(t) \} = m \left| q_i^2(t) \right|, \]  

(5)

with stationary and:

\[ J_q \{ q_i(t) \} = \max \left| q_i^2(t) \right|, \]  

(6)

with non-stationary effects.

c) Restrictions on effort. For base depreciation, the main quality criterion is the amount of force transferred to the base:

\[ J_p \{ P_r(t) \} \leq J_{pr}, (r = 1, N) \]  

(7)

where \( P_r \) is the force in the \( r \)-th shock absorber, and the shape of the functional is determined in the same way as in the previous case.

d) Limitations on control actions. When using active devices, it is advisable to impose restrictions on control actions. They may limit the maximum value of the control module:

\[ J_U = \max_t \| U_i(t) \| \leq J_{U_i}, \]  

(8)

or its rms value:

\[ J_U = m \left| U_i^2(t) \right| \leq J_{U_i}. \]  

(9)

You can limit the average power of the control device.

e) Design constraints. For various reasons, it may be necessary to impose restrictions on the method of implementing a vibration-proof device. So, you may need to use devices that have a well-defined structure (for example, only linear passive shock absorbers, or a hydraulic actuator of a given type). It may be necessary to limit the values of system parameters. For example, for structural considerations, the stiffness and damping coefficients of passive elements must lie within the specified limits. All these requirements, in the final analysis, can be formulated in the form of some inequalities that must be satisfied by the parameters of the VZS.

Under the task of designing VZS can understand the task of determining the depreciation system, ensuring the fulfillment of all requirements. The design of the optimal VZS means the task of choosing a system of vibration isolation, satisfying all the specified constraints and delivering an extremum to some quality criterion. In this case, the optimality criterion is selected based on the purpose of depreciation. The quality criterion can be any of the functionals (1)-(9) or their linear combination with some weight coefficients.

3. The problem of choosing a vibration isolation system

The most important problem in practical terms is that of the synthesis of VZS, providing a given reduction in the level of dynamic effects with minimum overall dimensions. In this case, constraints of the form (1), (4) are imposed on the system, and at the same time one of the functionals characterizing the value of the relative displacement is minimized.

The design of the VZS implies the problem of choosing a system of vibration isolation w (Figure 1), which satisfies all specified constraints.
Figure 1. Various design implementations of vibration isolation systems $w$.

At the preliminary design stage, when the structure of the VZS has not yet been chosen, the practical interest is most likely to be the possibility of providing constraints on acceleration (1) when the constraints on relative offsets (4) are fulfilled.

For a quantitative expression of the fulfillment of these requirements, we introduce the following functional on the set $W$ of all possible constructive implementations of the VZS:

$$J(w) = \max_{z \in \mathbb{Z}} \max_{k \in \mathbb{Z}} \left\{ L_j \right\} \leq \max_{q \in \mathbb{Q}} \left\{ J_q \right\}$$

Functional (10) can be considered as a criterion for the feasibility of the requirements for the quality of VZS. Note that if the value of the functional (10) with some implementation $VZS \in W$ does not exceed unity, then one can conclude that for a given structure $VZS$ requirements for the designed system are performed. The task of designing the optimal VZS can be set as follows:

$$J(w^*) = \min_{w \in W_1} J(w), \quad W_1 = \{w \in W | J(w) \leq 1\}$$

The numerical value $J^* = J(w^*)$, which obviously should be sought in the design, characterizes the limiting capabilities of the VZS in its optimal implementation. It is not possible to determine the exact value of $J^*$. In this regard, the task is to find estimates of the value of the criterion of feasibility, such that $J < J(w^*) < J$. If the lower bound $J$ exceeds one, it means that the requirements for the VZS are not feasible for any constructive implementation. The condition $J < 1$ for the upper bound of criterion (10) is sufficient to satisfy the requirements (1) and (4). Thus, the value of the functional (10) characterizes the fulfillment of the requirements for VZS at the stages preceding the design development.

4. Methods

Methods for solving problems on the evaluation of limit properties of VZS are based on the solution of minimax optimal control problems. The paper [5] considers the problem of assessing the limiting capabilities of the VZS, which provides a reduction in the level of dynamic effects with limited dimensions, offers a general approach to the problem of assessing the limiting properties of the spatial VZS, which does not require the solution of the corresponding problems of optimal control. The approach described in [5] is used to develop problem - oriented algorithms for specific types of disturbances. This takes into account the specific effects.

We evaluate the limiting properties of the vibration protection system of a solid body. The computational model of many technical objects is a solid. For example, consider a system design of vibration isolation unit on-board radio-electronic equipment (REA), s points which are sensitive to overloads, the electronic components to withstand the specified vibrational acceleration limit. The vibration isolating device shall have the minimum overall dimensions. The movement of the block at the $n$ mounting points of the shock absorbers must not exceed the specified values.
The block of REA, as well as many other mechanisms connected to the basis by means of some shock-absorbing device, represent the oscillatory system possessing six degrees of freedom (figure 2). Fix the three Cartesian axes that coincide in equilibrium with each other: \( o \xi \eta \zeta \) – stationary; \( o \xi_1 y_1 z_1 \) – moving, rigidly associated with the object and coincident with the central axis; \( oxyz \) – the mobile rigidly attached to the base.

Figure 2. Dynamic schema VZS solids.

We demand that the absolute accelerations at \( s \) given points of the object satisfy the constraints:

\[
|w_i| \leq l_i, \quad (i = \overline{1, s}),
\]

where \( w_i \) is the absolute acceleration at the \( i \)-th point; \( l_i \) – given positive numbers.

The projection of the absolute acceleration of the \( i \)-th point on the axis of the coordinate system \( o \xi \eta \zeta \) is determined by the expressions:

\[
\begin{align*}
\dot{w}_x' &= \ddot{x} + \ddot{z}_x + (\ddot{\Theta}_x + \ddot{\Theta}_\eta) y_i - (\ddot{\Theta}_z + \ddot{\Theta}_\zeta) z_i, \\
\dot{w}_y' &= \ddot{y} + \ddot{z}_y + (\ddot{\Theta}_y + \ddot{\Theta}_\zeta) x_i - (\ddot{\Theta}_x + \ddot{\Theta}_\xi) z_i, \\
\dot{w}_z' &= \ddot{z} + \ddot{z}_z + (\ddot{\Theta}_z + \ddot{\Theta}_\xi) y_i - (\ddot{\Theta}_y + \ddot{\Theta}_\eta) x_i,
\end{align*}
\]

(12)

where \((s_x, s_y, s_z)\)’ is the displacement vector of the object center of inertia in the coordinate system \( oxyz \); \((s_\xi, s_\eta, s_\zeta)\)’ – displacement vector in the coordinate system \( o \xi \eta \zeta \) of the base point, which in the equilibrium position coincides with the center of inertia of the object; \((\Theta_x, \Theta_y, \Theta_z)\)’ – vector of small relative rotation angle of the body; \((\Theta_\xi, \Theta_\eta, \Theta_\zeta)\)’ is the vector of small rotation angle of the base; \(x_i, y_i, z_i\) – coordinates of the \( i \)-th point of the object in the coordinate system \( o \xi_1 y_1 z_1 \).

By entering symbols:

\[
\begin{align*}
\ddot{x} + \ddot{z}_x &= u_1, \\
\ddot{y} + \ddot{z}_y &= u_2, \\
\ddot{z} + \ddot{z}_z &= u_3,
\end{align*}
\]

(13)

and with the ratio (12), we write an expression that defines the absolute acceleration modulus at the \( i \)-th point of the body:

\[
|w_i| = \left(w_{11}^2 + w_{12}^2 + w_{13}^2\right)^{\frac{1}{2}},
\]

\[
\begin{align*}
w_{11} &= u_1 + u_5 z_i - u_6 y_i, \\
w_{12} &= u_2 + u_6 x_i - u_4 z_i, \\
w_{13} &= u_3 + u_4 y_i - u_5 x_i.
\end{align*}
\]

(14)

Let us suppose that the size constraints of the damping system are equivalent to the constraints imposed on the relative displacements of \( n \) given points of the body in the given directions:

\[
|g_j| \leq L_j, \quad (j = \overline{1, n}),
\]

(15)
where \( g_i \) is the relative displacement of the \( j \)-th points of the body in the direction in the direction \( n^j \); \( L_j \) – given positive numbers.

The relative displacement of \( g_j \) can be written using generalized coordinates in the form:

\[
 g_j = (s_x + \Theta_y \ddot{z}_j - \Theta_z \ddot{y}_j) \alpha_j + (s_y + \Theta_z \ddot{x}_j - \Theta_x \ddot{z}_j) \beta_j + (s_z + \Theta_x \ddot{y}_j - \Theta_y \ddot{x}_j) \gamma_j, \tag{16}
\]

where \( \ddot{x}_j, \ddot{y}_j, \ddot{z}_j \) – coordinates of the \( j \)-th point of the body in the coordinate system \( o_x(y)z \); \( a_j, \beta_j, \gamma_j \) – guides of the cosines, which determine the direction \( n^j \).

We give the ratio (13) to the form:

\[
\ddot{s}_x = u_1(t) - \ddot{z}_x(t), \quad \ddot{s}_y = u_2(t) - \ddot{y}_y(t), \quad \ddot{s}_z = u_3(t) - \ddot{z}_z(t), \tag{17}
\]

where \( \ddot{s}_x(t), \ddot{s}_y(t), \ddot{s}_z(t), \ddot{\Theta}_x(t), \ddot{\Theta}_y(t), \ddot{\Theta}_z(t) \) – given functions that determine the law of change of generalized accelerations of the base.

Let at the initial moment of time the object of protection is in the equilibrium position, that is:

\[
\dot{s}_x(0) = s_y(0) = s_z(0) = \ddot{s}_x(0) = \ddot{s}_y(0) = \ddot{s}_z(0) = 0,
\]

\[
\dot{\Theta}_x(0) = \dot{\Theta}_y(0) = \dot{\Theta}_z(0) = \ddot{\Theta}_x(0) = \ddot{\Theta}_y(0) = \ddot{\Theta}_z(0) = 0. \tag{18}
\]

Let’s set the following task. Dimensional constraints are specified (15). It is required to estimate from below the numerical value of the quality criterion:

\[
J(u(\cdot)) = \max_{1 \leq i \leq n} \max_{t \geq 0} l_i^{-2} w_i^2, \tag{19}
\]

which can be achieved if the system (17)-(18) under the constraints (15) is managed optimally in the sense of the functional minimum (19).

Let us note that if \( \bar{J} \) it is the lower estimate of the numerical value of the criterion (19), which can be achieved if the system (17)-(18) will be controlled optimally under restrictions (15), then for the feasibility of the quality requirements of the VZS (11) at the specified dimensions (15), it is necessary to fulfill the condition:

\[
\bar{J} \leq 1. \tag{20}
\]

The implementation of inequality (20) means the possibility of the existence of a technically feasible VZS that meets the specified requirements.

Now consider the algorithmic implementation of the above technique.

The expression defining the acceleration modulus at the \( j \)-th point (14) can be written in matrix form:

\[
|w_j| = (u'W_iu)^{\frac{1}{2}},
\]

where

\[
u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}, \quad W_j = \begin{pmatrix} 1 & 0 & 0 & 0 & z_i & -y_i \\ 0 & 1 & 0 & -z_i & 0 & x_i \\ 0 & 0 & 1 & y_i & -x_i & 0 \\ 0 & -z_i & y_i & z_i^2 + y_i^2 & -x_i y_i & -x_i z_i \\ z_i & 0 & -x_i & -x_i y_i & z_i^2 + x_i^2 & -y_i z_i \\ -y_i & x_i & 0 & -x_i z_i & -y_i z_i & y_i^2 + x_i^2 \end{pmatrix}
\]

Thus, the criterion of quality (19) will take the form:

\[
J(u(\cdot)) = \max_{1 \leq i \leq n} \max_{t \geq 0} l_i^{-2} u'W_iu. \tag{21}
\]

We also write in matrix form the dimensional constraints (15), taking into account the expression that determines the relative displacement (16):
\[ \begin{align*}
|d'_jq| & \leq L_j, (j = 1, n),
\end{align*} \] (22)

where

\[
q = \begin{pmatrix}
s_x \\
s_y \\
s_z \\
\Theta_x \\
\Theta_y \\
\Theta_z
\end{pmatrix}, \quad d_j = \begin{pmatrix}
\alpha_j \\
\beta_j \\
\gamma_j
\end{pmatrix}.
\]

If you enter a vector-function:

\[
f(t) = (\ddot{s}_x(t), \ddot{s}_y(t), \ddot{s}_z(t), \ddot{\Theta}_x(t), \ddot{\Theta}_y(t), \ddot{\Theta}_z(t)),
\]

system (17)-(18) will take the form:

\[
\ddot{q} = u(t) - f(t), \quad q(0) = \ddot{q}(0) = 0.
\] (23)

Expressions (21)-(23) define the problem of the limit possibilities of the spatial VZS of a rigid body in matrix notation. We multiply the system (23) by \(d'_jq\) and enter the notation:

\[
v_j = d'_jju, \quad (j = 1, n),
\]

\[
\ddot{\sigma}_j(t) = d'_jf(t).
\] (24)

The system (23) will be rewritten as follows:

\[
\ddot{g}_j(t) = v_j - \sigma_j(t)
\] (25)

with initial conditions:

\[
g_j(0) = 0, \quad \dot{g}_j(0) = 0.
\] (26)

Let us consider \(n\) problems of determining the lower bound of the numerical value of the functionals:

\[
J_j(v_j(\cdot)) = \max_{t \geq 0} v_j^2(t),
\] (27)

which is achieved by implementing optimal on a class of piecewise continuous control functions \(v_j^*(t)\), under constraints (15).

Let us consider the auxiliary problem. Let the control process be described by a system of equations:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \quad x_1(0) = 0, \\
\dot{x}_2(t) &= v(t) - \sigma(t), \quad x_2(0) = 0,
\end{align*}
\] (28)

where \(\sigma(t)\) is some given function (the second derivative of the function \(\sigma(t)\)); \(v(t)\) is a piecewise continuous function playing the role of control.

The phase variable \(x_1(t)\) is constrained:

\[
|x_1(t)| \leq L, \quad t > 0.
\] (29)

It is required to estimate from below the numerical value of the quality criterion:

\[
J(v(\cdot)) = \max_{t \geq 0} v^2(t),
\] (30)

which can be achieved if the system (28) under the conditions (29) is managed optimally in the sense of minimum functionality (30).

We note that the limit value of the criterion (30) could be determined in principle through the complete solution of the corresponding problem of the optimal equation. However, such a solution is difficult to solve due to the nonadditivity of the functional and the presence of the phase constraint (29) for sufficiently complex given functions \(\sigma(t)\).
The principal difference of the proposed formulation of the optimal control problem is that there is no need to determine the optimal control, and it is required to estimate (find an estimate from below) the limit value of the quality criterion (30) when performing phase constraints (29).

It is known [8] that the minimum value of the functional in the problem:

$$J_1(v) = \min_{v} \max_{t>0} x_1^2(t), \quad |v(t)| \leq l$$

(31)

for the system (28) monotonically depends on the parameter/ describing the control restriction. Let \(l^*\) be the limit value defined as the square root of the limit value of the criterion (30) when the phase constraints (29) are satisfied (the criterion value (30) in the problem (28) - (30) under optimal control \(v^*(t)\):

$$l^* = \left( J(v^*) \right)^{1/2} = \left( \min_{v} J(v) \right)^{1/2}.$$  

By virtue of the above at the limit value \(l^*\), the value of the quality criterion in the problem (31) for optimal control is determined by the expression:

$$J_1(v^*) = L^2.$$  

Optimal control \(v^*(t)\) in the problem (28) - (30) will be optimal in the problem with functional (21) at \(l = l^*\).

Let us consider the problem for a system (28) with a terminal criterion:

$$J_2(v) = x_1^2(t_1), \quad |v(t)| \leq l, \quad 0 \leq t \leq t_1.$$  

In this problem, the optimal control \(v^*(t)\), as follows from the maximum principle [8], takes a constant value on the interval \([0, t_1]\):

$$\bar{v}^* (t) = -l \text{sign} x_1(t_1) = -\bar{v} = \text{const}.$$  

(32)

We integrate (28):

$$x_2(t) = \bar{v}^* t - \dot{\sigma}(t) + \dot{\sigma}(0),$$

$$x_1(t) = \frac{\bar{v}^* t}{2} - \sigma(t) + \dot{\sigma}(0)t + \sigma(0).$$

Let's define \(t_1\) as the moment of time when the maximum value \(x_1^2(t)\) is reached during control \(v(t) = \bar{v}^*\). In this case, it should be executed:

$$x_2(t_1) = \bar{v}^* t_1 - \dot{\sigma}(t_1) + \dot{\sigma}(0) = 0.$$  

Here you will find:

$$\bar{v}^* (t_1) = \frac{\dot{\sigma}(t_1) - \dot{\sigma}(0)}{t_1}.$$  

(33)

Hence, \(x_1(t)\) at this moment is determined by the ratio:

$$x_1(t_1) = \frac{\dot{\sigma}(t_1) + \dot{\sigma}(0)}{2}t_1 - \sigma(t_1) + \sigma(0).$$

That is:

$$|x_1(t_1)| = \left| \frac{\dot{\sigma}(t_1) + \dot{\sigma}(0)}{2}t_1 - \sigma(t_1) + \sigma(0) \right| = L.$$  

(34)

At time \(t_1\), it follows from (33) and (34) that:

$$\text{sign} \left[ \frac{\dot{\sigma}(t_1) - \dot{\sigma}(0)}{t_1} \right] = -\text{sign} \left[ \frac{\dot{\sigma}(t_1) + \dot{\sigma}(0)}{2}t_1 - \sigma(t_1) + \sigma(0) \right].$$  

(35)

Additionally, we require that for any \(t \leq t_1\) the inequality is satisfied:
The following theorems are proved in [5].
Theorem 1. If \( I \) is determined by the ratio:
\[
I = |\hat{\sigma}(t_l) - \hat{\sigma}(0)|_1^{-1},
\]
where \( t_l \) satisfies the conditions (34) - (35) and for any \( t \leq t_1 \) is true (36), then \( I \) is the lower bound of the criterion value (30) under optimal control \( v^* (t) \):
\[
J^2 \leq J(v^*(i)), \quad (\ell \leq \ell^*).
\] (37)

It follows from this theorem that if we find a moment \( t_1 \) satisfying the conditions (34) - (36), then the lower bound of the limit value of the functional (30) can be calculated by the formula:
\[
J = I^2 = (\hat{\sigma}(t_1) - \hat{\sigma}(0))^{-2} t_1^{-2}.
\] (38)

Theorem 2. If there are moments \( t_1 \) and \( t_2 \) such that \( t_1 < t_2 \) satisfying (34) - (36), then:
\[
|\hat{\sigma}(t_1) - \hat{\sigma}(0)|_1^{-1} \leq |\hat{\sigma}(t_2) - \hat{\sigma}(0)|_2^{-1}.
\] (39)

Thus, if there are several \( t_1, t_2, \ldots, t_k \) satisfying the conditions (34) - (36), to obtain a more accurate lower estimate of the limit value (30), we select the maximum element \( t_1 \) from the set \( \{ t_1, t_2, \ldots, t_k \} \) and calculate the estimate by the formula (38).

If in the problem (15) - (17) we enter the notation \( x^1_j = g_j \), \( x^2_j = \bar{g}_j \), we get \( n \) simple problems of the form (18) - (20).

If:
\[
v_j^* = (\hat{\sigma}_j(t_{1j}) - \hat{\sigma}_j(0)) t_{1j}^{-1}, \quad (j = 1, n).
\] (40)

where \( t_{1j} \) is the maximum moment of time in the \( j \)-th problem satisfying the conditions corresponding to (34) - (36).

Without limiting generality, it can be assumed that:
\[
v_p^* > 0, \quad p = 1, k,
\]
\[
v_p^* < 0, \quad p = k + 1, n.
\]

We consider the following discrete minimax problem:
\[
U = \{ u : a_p^i u \geq v_p^*, \quad p = 1, k, a_q^i u \leq v_q^*, q = k + 1, n \}.
\] (41)

We find \( u^* \) delivering the minimum of the maximum function:
\[
\Phi(u) = \max_{1 \leq i \leq s} I_i^{-2} u^* W_i u.
\] (42)

In the work [5] it was shown that if \( u^* \) is a solution of discrete minimax problem (41)-(42), then \( \Phi(u^*) \) can be considered as an approximate lower bound of the limit value of the quality criterion (21). We note that the set \( U \) defined by the relation (41) is not empty. Indeed, otherwise the restrictions (5) could not be satisfied under any controls, which contradicts the fact that by choosing the control \( u(t) = f(t) \), we can in principle satisfy any restrictions of type (5).

5. Summary and Conclusions
Thus, the problem of evaluation of limit values of criterion (11), after solving \( n \) problems of evaluation of limit properties of the simplest system, is reduced to solving a discrete minimax problem of the set determined by linear functions. The solution of the discrete minimax problem can be obtained using the methods described in [1]. At the same time, as the experience of the solution shows, to reduce the number of iterations, it is recommended to choose solutions of the nonlinear programming problem with the objective function as an initial approximation:
\[ I(u) = \sum_{i=1}^{s} I_{i}^{2} u W_{i} u \]  

(43)
on the set U. The solution of this nonlinear programming problem is not difficult, since the objective function is convex and the constraints are linear.

From the above, we propose the following algorithm for finding the functional estimate (11).

1.1. We solve n simplest problems of finding the lower estimate of the numerical value of the quality criterion (17), which can be achieved if the system (15) with initial conditions (16) under condition (5) is optimally controlled in terms of the minimum functional (17).

1.2. The initial approximation is chosen as a solution of the nonlinear programming problem with the objective function (43) on the set (41).

1.3. The discrete problem of finding the minimum of the maximum function (42) on the set (41) is solved.

If \( \tilde{u}^{*} \) it is a solution of the problem (41) - (43), it \( \Phi(\tilde{u}^{*}) \) can be considered as an approximate assessment of the quality criterion (42).

The main difficulty of solving the problem of finding an estimate of the limit value of the quality criterion in the simplest problem (15) - (17) in the performance of dimensional constraints (5) is reduced to the search for time points \( t_{1} \) satisfying the conditions (24) - (26). To find the moments of time \( t_{1} \), we first find the moments of time satisfying the conditions (34), (35), and then choose from them the moments at which (36) is fulfilled.

By virtue of (35) the condition (34) can be written as follows:

if:

\[ (\dot{\sigma}(t_{1}) - \dot{\sigma}(0)) t_{1}^{-1} < 0, \]  

(44)

that is in the form:

\[ 0.5(\dot{\sigma}(t_{1}) - \dot{\sigma}(0)) t_{1} - \sigma(t_{1}) + \sigma(0) = L, \]  

(45)

and if:

\[ (\dot{\sigma}(t_{1}) - \dot{\sigma}(0)) t_{1}^{-1} > 0, \]  

(46)

then:

\[ 0.5(\dot{\sigma}(t_{1}) - \dot{\sigma}(0)) t_{1} - \sigma(t_{1}) + \sigma(0) = -L. \]  

(47)

We consider the auxiliary problem of finding all the roots of the equation:

\[ y = \Phi(t) = 0, \]  

(48)
belonging to the segment \([0, T]\).

To find the roots of the equation (48) we use the piecewise linear approximation method. According to this method, at each step of the algorithm, a piecewise linear model of the function (48) passes through the points \( t_{i} \Phi (t_{i}) \), \( (i = \frac{1}{N}) \). The total number of points by which the piecewise linear function is constructed at the \( i \)-th step is determined by the iterative formula:

\[ N_{i} = 2N_{i-1} - 1, \quad (i = 2, 3, \ldots). \]

The initial value of \( N \) points can be set based on the properties of the function (48) or, in general, accepted \( N_{1} = 5 \). In this case, the initial test points are evenly spaced on the interval \([0, T]\). The current \( t_{i} \) test points required to build the model are selected as follows. On partial intervals \((t_{i-1}, t_{i})\), in which the approximating curve crosses the \( Ot \) axis, the point of intersection of the \( Ot \) axis is taken, and on other intervals the midpoint of the segment is taken as the test point[9].

The main characteristic of the model obtained at each step is the number of partial intervals at which the approximating lines intersect the \( Ot \) axis. The process of sequential construction of piecewise linear functions ends as soon as the correspondence between the model obtained at the \( i \)-th step and the function (48) is achieved. This conformity is the \( P_{0} \) parameter indicating how many consecutive times is repeated a number of partial intervals which intersects the approximating straight axe \( Ot \) in the refinement of the model. When repeating the structure of a piecewise linear model \( P_{0} \) times in a row, it
is assumed that the function (48) has the same number of points at which it vanishes as its piecewise linear model constructed at the i-step. The definition of roots in this case is reduced to finding the function (48) at each of the obtained partial intervals at which the approximating line crosses the Ot axis. In this case, various methods of finding the roots can be used (dividing the segment in half, the golden section).

To find the moments $t_1$ satisfying the conditions (34) and (35), we find the roots of the equation (45) according to the above. From the roots found, choose those that satisfy the condition (44). When solving a specific problem when finding these moments, you should set the search interval depending on the type of disturbances. So, for example, in the case of polyharmonic disturbances can be taken:

$$T = 2\pi(\omega_{\text{min}})^{-1}$$

($\omega_{\text{min}}$ is the minimum of harmonic frequencies).

Similarly, we find the roots of the equation (47) satisfying (46). Note that the found moments $t_1$ must satisfy the condition (36). In this regard, you should choose from the found only those moments $t_1$, at which the condition (36) will be satisfied. This condition is satisfied if the derivative of the function:

$$x_1(t) = 0.5(\dot{\sigma}(t_1) - \dot{\sigma}(0))t_1^2 t_1^{-1} - \sigma(t_1) + \sigma(0) = 0,$$

on the interval $(0, t_1)$ keeps the sign. Otherwise, we find the roots $\tilde{t}$ of the equation:

$$x_2(\tilde{t}) = (\ddot{\sigma}(t_1) - \ddot{\sigma}(0))\tilde{t}^{1-1} - \ddot{\sigma}(\tilde{t}) + \dot{\sigma}(0) = 0,$$

lying on the interval $(0, t_1)$.

After calculating the value of the function (49) in points $\tilde{t}$ and comparing the module of the obtained value $|x_1(\tilde{t})|$ with $L$, we can verify the fulfillment or non-fulfillment of the condition (36).

From the moments of time found in this way, according to theorem 2, we choose the maximum, which will be the sought moment $t_1$.

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