Abstract

We consider off-diagonal Jacobi matrices $J$ with (faster–than–exponential) sparse perturbations. We prove (Theorem 3.1) that the Fourier transform $\widehat{|f|^2 \, d\rho(t)}$ of the spectral measure $\rho$ of $J$, whose sparse perturbations are at least separated by a distance $\exp\left(cj(\ln j)^2 \right) / \delta^j$, for some $c > 1/2, 0 < \delta < 1$ and for a dense subset of $C^\infty_0(-2,2)$–functions $f$, decays as $t^{-1/2} \Omega(t)$, uniformly in the spectrum $[-2,2]$, $\Omega(t)$ increasing less rapidly than any positive power of $t$, improving earlier results obtained by Simon (Commun. Math. Phys. 179, 713-722 (1996)) and by Krutikov–Remling (Commun. Math. Phys. 223, 509-532 (2001)) for Schrödinger operators with sparse potential that increases as fast as exponential–of–exponential. Applications to the spectrum of the Kronecker sum of two (or more) copies of the model are given.

1 Introduction

The present paper deals with the Kronecker sum ($I$ denotes the identity matrix)

$$K = J \otimes I + I \otimes J$$

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of off–diagonal Jacobi matrices

\[
J = \begin{pmatrix}
0 & p_0 & 0 & 0 & \cdots \\
p_0 & 0 & p_1 & 0 & \cdots \\
0 & p_1 & 0 & p_2 & \cdots \\
0 & 0 & p_2 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

which are sparse perturbations of the free Jacobi matrix \( J_0 \) in the sense that the sequence \( (p_n)_{n \geq 0} \) differs from the unit on a lacunary subset of natural numbers \( \mathbb{A} = \{a_j^\omega \}_{j \geq 1} \), i. e.,

\[
p_n = \begin{cases}
p & \text{if } n \in \mathbb{A}, \\
1 & \text{otherwise},
\end{cases}
\]

for some \( p \in (0, 1) \). We assume that \( \mathbb{A} \) is possibly a random set, \( a_j^\omega = a_j + \omega_j \), with

\[
a_j^\omega - a_j^{\omega-1} \geq 2
\]

and \( a_j \) satisfying the ”sparseness” condition \( \lim_{j \to \infty} a_j/a_{j-1} = \beta > 1 \) (\( \beta = \infty \) is included). A concrete example is given by

\[
a_j - a_{j-1} = \beta^j, \quad j = 1, 2, \ldots
\]

with \( a_0 = 0, \beta \geq 3 \) and \( \omega_j, j \geq 1 \), independent random variables defined on a probability space \((\Omega, \mathcal{B}, \mu)\), uniformly distributed over the sets \( \Lambda_j \equiv \{-j, \ldots, j\} \). These variables introduce uncertainty in the position of the points where the \( p_n \) differ from the unit and their support increases linearly with the index \( j \) (see Remark 1.4 of \cite{CMW1} for less restrictive examples). A disordered potential of this type was introduced by Zlatoš \cite{Z}. We shall however consider the deterministic case with (1.4) replaced by a sequence \( (\beta_j)_{j \geq 1} \) that increases faster–than–exponential.

The Jacobi matrices (1.2) when applied to a vector \( u = (u_n)_{n \geq 0} \in l_2(\mathbb{Z}_+) \) can be written as a difference equation

\[
(Ju)_n = p_n u_{n+1} + p_{n-1} u_{n-1},
\]

for \( n \geq 0 \) with \( u_{-1} = 0 \). We denote by \( J^\phi \) the Jacobi matrix \( J \) which satisfies \( \phi \)–boundary condition at \(-1\):

\[
u_{-1} \cos \phi - u_0 \sin \phi = 0.
\]

\( J^\phi \) is a (noncompact) perturbation of the free Jacobi matrix \( J_0^\phi \), where \( p_n = 1 \) for all \( n \geq -1 \):

\( J^\phi = J_0^\phi + V \), the “potential” \( V \) composed by infinitely many random barriers whose separations increase, at least, exponentially fast. The Jacobi matrix \( J \) corresponds to the matrix \( J^0 \) satisfying 0–Dirichlet boundary condition at \(-1\).

There have been a few results on models that exhibit spectral transition, supporting spectra of different types in complementary set of parameters. The Anderson model in a Bethe lattice \cite{K} is
an example. Our intention is to provide another instance of models whose spectrum contains pure point and absolutely continuous nonempty components. For the model (1.1) with random sparse potential satisfying (1.4), the essential spectrum \( \sigma_{\text{ess}}(K) \) can be decomposed into continuous \( \sigma_c(K) \) at its center and dense pure point \( \sigma_{\text{pp}}(K) \) near to the edges. Whether \( \sigma_c(K) \) has an absolutely continuous component still remains unknown.

A method of study the spectrum of sparse Jacobi matrices \( J \), exploiting the uniform distribution of the Prüfer angles with fixed energy, has been introduced in [MWGA]. With this method, the Hausdorff dimension of the spectral measure of sparse block-Jacobi matrices \( J \otimes I_L + I \otimes J_L \) can be determined with any degree of precision, provided \( \beta \) is large enough and \( J_L \) homogeneous (see [CMW]). A sharp spectral transition from singular continuous to pure point spectrum, announced by Zlatoš for a model \( J^\phi = J_0^\phi + V \) with diagonal potential \( V \) (see Theorem 6.3 of [Z]), has been proved in [CMW1] for Jacobi matrices \( J \) with random sparseness using this method. Here we address the Fourier–Stieltjes transform \( \hat{d}\rho \) of the spectral measure \( \rho \) of \( J \) and investigate the least faster–than–exponential sparse condition required for pointwise decay of \( \hat{d}\rho(t) \). While we consider off-diagonal case for reasons explained in [MWGA], there will be no difficulties of principle in applying our methods to the diagonal case considered in [Z].

The layout of the present paper is as follows. Some notions and motivations are presented in Section 2 and precise statements, Theorem 3.1 and Corollary 3.2 are formulated in Section 3. Section 3 also contains the mathematical tools used in Section 4 to prove Theorem 3.1. None of those tools depends on whether the Prüfer angles are uniformly distributed. Lemma 3.3 in Subsection 3.2 uses the idea of the proof of Theorem 10.12 in Chap. XII of [Zy] and the Gevrey type estimates of Subsection 3.3 permit integration by parts to be applied an unlimited number of times.

2 Preliminaries

According to [MWGA] (see e.g. Theorem 4.4), if the angles of Prüfer are uniformly distributed, \( \lambda \in [-2, 2] \) belongs to the essential support of the singular continuous spectrum of \( J \) provided

\[
\frac{r}{\beta} < 1
\]

where \( r = r(p, \lambda) = 1 + \vartheta(p)/(4 - \lambda^2) \) and \( \vartheta(p) = (1 - p^2)^2 / p^2 \) is monotone decreasing function of \( p \in (0, 1) \). The local Hausdorff dimension of \( \rho \) in this case reads

\[
\alpha_H = \max\left(1 - \frac{\ln r}{\ln \beta}, 0\right)
\]

(see Theorem 3.11 of [CMW]). Note that \( \alpha_H = \alpha_H(p, \beta, \lambda) \) varies from 0 to 1 as \( p \) varies from \( p^* \) to 1, where \( p^* \) is defined by \( r(p^*, \lambda) = \beta \); as \( \lambda \) varies from 0 to \( \pm 2 \), \( \alpha_H \) varies from \( \alpha_H(p, \beta, 0) < 1 \) to 0, attained at \( \lambda^* \) defined by \( r(p, \lambda^*) = \beta \).
Let \( f : [-2, 2] \to \mathbb{C} \) be a smooth function with \( \text{supp} f \ni \lambda \) compact and sufficiently localized so that \( \psi = f(J)\delta_0 \) is in \( l_2(\mathbb{Z}_+) \). The probability \( |(\psi, \exp(itJ)\psi)|^2 \) of finding at time \( t \) the system in its initial state \( \psi \) can be estimated observing that

\[
(\psi, \exp(itJ)\psi) = \int_{-2}^{2} |f(\lambda)|^2 e^{it\lambda} d\rho(\lambda) = |f|^2 d\rho(t),
\]

where \( \rho(I) = \|E(I)\delta_0\|^2 \) is the spectral measure of the state \( \delta_0 \) localized at 0, and \( E(\cdot) \) is the spectral resolution of \( J \). We shall also denote by \( \rho_\psi(I) = \|E(I)\psi\|^2 \) the spectral measure of an state \( \psi = f(J)\delta_0 \) and the Fourier–Stieltjes transform of \( \rho_\psi \) shall also be written as \( \widehat{d\rho_\psi}(t) \).

Our purpose is primary to prove that \( (2.3) \) decays to zero as \( |t| \) goes to infinity. Let \( \alpha_F \) be the supremum of \( \alpha \geq 0 \) such that

\[
|f|^2 d\rho(t) \leq C (1 + |t|)^{-\alpha/2},
\]

with \( C < \infty \), holds for any \( t \in \mathbb{R} \), uniformly in a dense (in \( L_2(-2, 2) \)) set of the \( f \)'s, sufficiently localized around a point \( \lambda \) of the spectrum. It follows by a theorem of Frostman (see e.g. [M]) that the so called Fourier dimension \( \alpha_F \) satisfies

\[
\alpha_F \leq \alpha_H
\]

(Fourier dimension and Hausdorff dimension do not agree in general). Measures for which \( \alpha_F > 0 \) and \( \alpha_F = \alpha_H \) are, respectively, known as Raychmann measures (see e.g. [Ly]) and Salem measures, after Salem’s work [S] on continuous distribution functions which are constant in each interval contiguous to a perfect set of Lebesgue measure zero.

Disorder plays a crucial role in most examples of sets and measures in which the Hausdorff dimension and the Fourier dimension are equal (see e.g. [M] Sa and Chap. 17 of [K]). The equality \( \alpha_H = \alpha_F \) of dimensions defined by \( (2.2) \) and \( (2.4) \) is thus expected to be attained for models with random sparseness condition \( (1.4) \). However, there are also singular continuous measures constructed by a deterministic method (see e.g. [Ko, P]) and for the deterministic sparse model \( J \) (or its diagonal version considered in references [Z, KR]) with faster–than–exponential \( (\beta = \infty) \) sparsity, it can actually prove that this property is satisfied with \( \alpha_H = \alpha_F = 1 \).

Pointwise decay

\[
|f|^2 d\rho(t) \leq C |t|^{-1/2} \ln |t|
\]

(with \( |t| \geq 2 \), because of the behavior of \( \ln |t| \) for \( |t| \leq 1 \)) has been obtained earlier by Simon [S] for continuous Schrödinger operators with generic and sufficiently sparse potentials and, by using a different method, Krutikov–Remling [KR] have found, for a model similar to the one considered here, a resonant set \( \mathcal{R} \subset \mathbb{R} \) in which \( |f|^2 d\rho(t) \leq C (1 + |t|)^{-1/2+\varepsilon} \) holds if \( t \in \mathcal{R} \) and

\[
|f|^2 d\rho(t) \leq C (1 + |t|)^{-m}
\]

for an arbitrary large but finite \( m \), otherwise. The work of [KR] was motivated by the “little control” of reference [S] on the rate with which the barrier separations
$a_j - a_{j-1}$ have to increase (Simon’s method requires $a_j \sim \exp(\exp(cj^{3/2}))$ and Krutikov–Remling need e.g. $a_j \sim \exp(\exp(cj))$, $c > 0$, to satisfy their condition $a_j \leq Ca_{j+1}^{-\mu}$ for some $C > 0$ and $\mu > 0$). In the present work, the sparseness condition is less restrictive than those stated in references [S] and [KR]. Thanks to the Gevrey type estimates developed in Subsection 3.3, the technique of integration by parts, employed in reference [KR], has been exploited to its limit and (2.3) (with $\Omega(|t|)$ in the place of $\ln |t|$, $\Omega(t)$ increasing less rapidly than any positive power of $t$) has been established for $a_j - a_{j-1}$ increasing slightly–faster–than factorial: $\exp(cj(\ln j)^2) / \delta^j$ for some $c > 1/2$ and $\delta < 1$ (see (3.4) for sparseness improvement: if $a_j - a_{j-1} = \exp\left(\frac{1}{\varepsilon} j \ln j \right) / \delta^j$ for some $\varepsilon > 0$, then (2.5) holds with $\ln |t|$ replaced by $|t|^{\varepsilon}$).

Our motivation in considering the Kronecker sum (1.1) is the same that led Simon to investigate the pointwise decay (2.3): it comes from the observation that the convolution of two singular continuous measures, $d\rho_* \ast d\rho_*$, may be absolutely continuous. Note that $d\rho_* \ast d\rho_*(t) = |\hat{d\rho_*}|^2(t)$ is square integrable and this, according to the well-known folklore result, implies that the corresponding measure is absolutely continuous with respect to Lebesgue measure. For model with faster–than–exponential sparseness the spectral measure is singular with respect to Lebesgue and the responding measure is absolutely continuous with respect to Lebesgue measure. For model with $\alpha_H$–singular sparseness the spectral measure is singular with respect to Lebesgue and has Hausdorff dimension 1, uniformly over the essential spectrum $[-2, 2]$ (see Theorem 1.4 of [Z], which also applies to the off–diagonal model). The spectrum of $K$ in this case is purely absolutely continuous (see Corollary 3.2). We are, however, interested in spectral transitions and to achieve the decay (2.4) depending on the local Hausdorff dimension $\alpha_H$, a more sophisticated method that exploits the randomness of \{$a_j^\varepsilon$\}$_{j \geq 1}$ is required. Investigation in this direction will be carried in a separate paper.

As in [KR], our starting point is the representation of the spectral measure as a weak–star–limit of absolutely continuous measures (see also [P]):

$$\int f(\lambda)d\rho(\lambda) = \lim_{N \to \infty} \frac{1}{\pi} \int_{-2}^{2} f(\lambda) \frac{\Im(w(\lambda))}{|y_N(\lambda) - w(\lambda)y_{N+1}(\lambda)|^2} d\lambda$$ (2.6)

for every continuous function $f : [-2, 2] \to \mathbb{C}$. Here, $y_n = y_n(z)$ denotes the solution of the eigenvalue equation

$$p_n y_{n+1} + p_{n-1} y_{n-1} = z y_n ,$$

satisfying the initial conditions $y_{-1}(z) = 0$ and $y_0(z) = 1$, and $w(z) = z/2 + i \sqrt{1 - z^2/4}$ is a Herglotz function (maps the upper half-plane $\mathbb{H}$ into itself). We shall apply this formula with $f(\lambda) = |f(\lambda)|^2 e^{it\lambda}$.

It turns out that $y_n(\lambda)$ is a subordinate solution of $Ju = \lambda u$ (see details in [CMW, CMW1]). Let $T(N, \lambda)$ denote the $2 \times 2$ transfer matrix associated with the eigenvalue equation $Ju = \lambda u$ with

\[1\]The above mentioned decay $(1 + |t|)^{-1/2+\varepsilon}$ holds in ref. [KR] for every $t \in \mathcal{R}$ if $a_{j-1} < Ca_j^{1/2}$ is satisfied (i.e. for $\mu = 1/2$) and $a_j \sim \exp(c \exp j)$ with $c > \ln 2$. In ref. [S], this decay holds provided $a_j \sim \exp(C \varepsilon j^{3/2})$ for some $C_\varepsilon > 0$. 


\( \lambda = 2 \cos \varphi \) for some \( \varphi \in (0, \pi) \) and define \( \phi_j \), for the subsequence \( (N_j)_{j \geq 1} \) with \( N_j = a_j + 1 \), by the equation

\[
|T(N_j, \lambda)|^2 v_{\phi_j} = t_j^{-2} v_{\phi_j}
\]

where \( |T(N_j, \lambda)|^2 = T^*(N_j, \lambda)T(N_j, \lambda) \) is a real symmetric unimodular matrix, \( t_j = \|T(N_j, \lambda)\| \) is the spectral norm of \( T(N_j, \lambda) \) and \( v_{\phi} = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} \). Under the assumption that the Prüfer angles are uniformly distributed (satisfied if \( J \) is random; see [CMW]), \( t_j^2 = O(r^j) \) and we have (see Proposition 3.9 of [CMW]) for an improved version of Lemma 2.1 of [Z])

\[
|y_{N_j}(\lambda) - w(\lambda)y_{N_j+1}(\lambda)|^2 = |UT(N_j, \lambda)v_{\phi^*}|^2 = O(r^{-j})
\]

for \( \lambda \) in the essential support of \( \rho \). Because the hypothesis of Theorem 8.1 of Last–Simon [LS] is verified, the limit \( \phi^* \) of the sequence \( (\phi_j)_{j \geq 1} \) exists and by the Gilbert–Pearson theory \( \phi^* = 0 \) for a.e. \( \varphi \in [0, \pi] \) in the essential support of \( \rho \) (see [GP]), establishing the decay property (2.7).

We also have (with \( \lambda = 2 \cos \varphi \) and \( U = \begin{pmatrix} 0 & \sin \varphi \\ 1 & -\cos \varphi \end{pmatrix} \))

\[
|UT(N_j, \lambda)v_0|^2 = R_j^2,
\]

where \( R_j = R_j(\varphi) \), \( j = 1, 2, \ldots \), are the radius of Prüfer associated with \( J \). Hence, our next ingredient is related with the following identity

\[
\frac{1}{R_j^2} = \frac{1}{R_{j-1}^2} + \sum_{k=j^*-1}^{j-1} \frac{1}{R_k^2} \left( \frac{R_k^2}{R_{k+1}^2} - 1 \right)
\]

for some conveniently chosen \( j^* \leq j \), together with

\[
\frac{R_k^2}{R_{k+1}^2} - 1 = \frac{p^2}{a + b \cos 2\theta_{k+1} + c \sin 2\theta_{k+1}} - 1 \equiv H(\varphi, \theta_{k+1})
\]

where \( \theta_k = \theta_k(\varphi) \) is the \( k \)-th angle of Prüfer; \( a, b \) and \( c \) are functions of \( p \) and \( \varphi \) defined in [MWGA] and \( \bar{H} = \frac{1}{\pi} \int_{0}^{\pi} H(\varphi, \theta)d\theta = 0 \). Plugging (2.9) and (2.10) into (2.9), yields

\[
|f|^2d\rho(t) = \frac{1}{\pi} \int_{0}^{\pi} |f(2 \cos \varphi)|^2 \frac{\sin^2 \varphi}{R_{j^*-1}^2} e^{2it\cos \varphi} d\varphi
\]

\[
+ \sum_{k=j^*-1}^{\infty} \frac{1}{\pi} \int_{0}^{\pi} |f(2 \cos \varphi)|^2 \frac{\sin^2 \varphi}{R_k^2} H(\varphi, \theta_{k+1}) e^{2it\cos \varphi} d\varphi.
\]

Expanding \( H(\varphi, \theta) \) in Fourier series (see [KR], for details)

\[
H(\varphi, \theta) = \sum_{n=1}^{\infty} \left( A(\varphi)^n e^{2in\theta} + \bar{A}(\varphi)^n e^{-2in\theta} \right)
\]

\[
A(\varphi) = \sqrt{1 - r^{-1}e^{i(\delta + \pi)}},
\]
with $r$ given by (2.1) and $\tan \delta = c/b$, each integral involved in the sum (2.11) is of the form
\[ I(t) = \int_{-\infty}^{\infty} e^{ith(\phi)} d(G \circ \lambda)(\phi) \]
for some $C_0^\infty$ function $dG/d\lambda$ with support $2 \cos [\phi_-, \phi_+] \subset (-2, 2)$, and $h$ of the type:
\[ h_k(t, n; \phi) = 2(\cos \phi + \frac{n}{t} \theta_k(\phi)), \quad n \in \mathbb{Z} \]
for some $k \geq j^\ast$.

The main contribution to (2.12) comes from the stationary phase
\[ h'_k(\phi) = -2 \sin \phi + \frac{2n}{t} \theta'_k(\phi) = 0 \]
and the $\phi$'s that satisfy the equation will be called resonant or critical values. Although the usual method of stationary phase does not apply, its ideas can, nevertheless, be easily traced. To each integral for which there are no resonant values inside the interval $[\phi_-, \phi_+]$, the standard stationary phase method applies integration by parts once. In order to apply integration by parts $m$ times, with $m$ a large fixed number, obtaining therefore a pointwise decay $t^{-m}$, Krutikov–Remling in [KR] have exploited the fact that $dG/d\lambda$ is analytic and the sequence $(a_j)_{j \geq 1}$ were super-exponentially sparse.

We have in the present paper extended Krutikov–Remling’s method in two directions. Firstly, for pointwise decay $t^{-1}$ fixed to each of those no resonant integrals, we use infinitely many integration by parts to obtain the least possible sparseness condition. Secondly, concerning the integrals in which there are resonant values inside the interval $[\phi_-, \phi_+]$, we apply Lemma 3.3. Following the ideas in the proof of Theorem 10.12 of Zygmund’s book on trigonometric series, we use Plancherel identity in order to obtain the main contribution of the stationary phase, up to a logarithmic correction, provided the integral with the phase $e^{ith(\phi)}$ replaced by $e^{it\lambda(\phi)}$ decays as $t^{-1}$.

### 3 Faster–than–exponential Sparse Models

#### 3.1 Statement of Results

We devote this section to show that $|\hat{f}|^2 d\rho(t)$ decays as $t^{-1/2} \Omega(t)$, uniformly with respect to any $C^\infty$ function $f$ with support contained into the essential spectrum $[-2, 2]$ of $J$, provided $\rho$ is the spectral measure of $J$ given by (1.2) and (1.3) with the sparseness increment
\[ a_j - a_{j-1} = \beta_j, \quad j = 1, 2, \ldots \]
\(^2\text{Note that } |A(\phi)| \leq a < 1 \text{ uniformly in each compact set } K \text{ of } (0, \pi) \text{ and the series is uniformly and absolutely convergent.}\)
an increasing faster-than-exponential sequence:

$$\lim_{j \to \infty} \beta_{j-1}/\beta_j = 0.$$  \hfill (3.2)

From now on, $J$ will always be such that (3.2) is satisfied. Our goal is to find the least increasing sequence that leads to this result. Without loss of generality, we assume that

$$\frac{\beta_{j-1}}{\beta_j} \leq \delta$$  \hfill (3.3)

holds uniformly in $j$ for some $0 < \delta < 1$ satisfying

$$\sup_{\phi \in \text{supp} \circ \lambda} \sup_{\theta} \frac{p^2}{a + b \cos 2\theta + c \sin 2\theta} < \frac{1}{\delta}$$  \hfill (3.4)

Note that

$$\frac{\delta^j}{R_j^2(\phi)} \to 0$$

exponentially fast, as $j$ goes to infinity, uniformly in $\text{supp} f \circ \lambda$. We now state our result.

**Theorem 3.1** Let $\rho$ the spectral measure of $J$ associated with the state $\delta_0$ localized at 0 and let $f$ be a smooth function with compact support inside $(-2, 2)$ and such that $0 \notin \text{supp} f$. Suppose that the sequence $(a_j)_{j \geq 1}$ satisfies (3.1)–(3.4) with

$$\beta_j = 1 \frac{1}{\delta^j} \exp \left( cj (\ln j)^2 \right)$$  \hfill (3.5)

for some $c > 1/2$, as $j$ tends to infinity. Then, there exist a constant $C$, depending on $f$ and $p$ (the intensity parameter of $J$), such that

$$|\widehat{|f|^2 d\rho|(t)|} \leq C |t|^{-1/2} \Omega(|t|)$$  \hfill (3.6)

holds for $|t| \geq 2$, where $\Omega(t)$ increases less rapidly than any positive power of $t$.

Moreover, if the sparseness increments $(\beta_j)_{j \geq 1}$ is chosen as

$$\beta_j = 1 \frac{1}{\delta^j} \exp \left( \varepsilon^{-1} j \ln j \right)$$  \hfill (3.7)

for some $\varepsilon > 0$ small enough, then the conclusion (3.6) holds with the upper bound replaced by $C(1 + |t|)^{-1/2+\varepsilon}$.

The proof of Theorem 3.1 uses a classical result on the decay of Fourier–Stieltjes coefficients $c_n(dG)$ of a monotone increasing singular continuous function $G$ originated from a Riesz product.

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3By equation (4.15) of [MWGA] $\delta > \min_{\phi^{-}\leq \phi \leq \phi^{+}} \left( a/p - \sqrt{(a/p)^2 - 1} \right) > 0$ if $0 < p < 1$. 

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(see Theorem 10.12 in Chap. XII of [Zy]). The Lemma stated below reduces the resonant estimate to a non–resonant one, which will be studied in Subsection 4. Our proof resembles, in this sense, the proof of Simon (compare Lemma 3.3 below with Lemma 4.2 of [S]) with the non–resonant estimate given by Krutikov–Remling’s method of integration by parts.

A well-known folklore result (see, e.g., [C], Exercise 11, Section 6.2) states that if the Fourier–Stieltjes transform of a finite Borel measure is square-integrable, the corresponding measure is absolutely continuous with respect to Lebesgue measure. We have, thus, as a direct corollary of Theorem 3.1:

**Corollary 3.2** The spectrum of $K = J \otimes I + I \otimes J$, with $J$ defined as in Theorem 3.1, is purely absolutely continuous.

An alternative proof of a theorem that includes the above mentioned folklore result (which has been used as early as 1958 to provide nontrivial examples of the statement that the convolution of two singular measures may be absolutely continuous; see [KS]) may be found in [CMW2] (see also [S], Corollary 3.2). Note that the spectral measure of $K$ associated with the tensor product of two vectors in $l_2(\mathbb{Z}_+)$ is given by the convolution of the two measures associated to each of these vectors (see [S] for details).

Of course the corollary extends to the Kronecker sum of any number of copies higher than two, and thus the nature of the spectrum changes dramatically in dimension two or higher. This will be exploited further in [CMW2].

### 3.2 Basic Lemma

The goal of this subsection is to prove the following

**Lemma 3.3** Suppose $G : \mathbb{R} \rightarrow \mathbb{R}$ is a monotone increasing continuous function, with $dG$ supported in some closed interval $[a, b] \subset (-2, 0) \cup (0, 2)$, whose Fourier–Stieltjes transform satisfies

$$\hat{dG}(t) = \int_{-\infty}^{\infty} e^{it\lambda} dG(\lambda) \leq \frac{C}{1 + |t|} \quad (3.8)$$

for some constant $C < \infty$ and every $t \in \mathbb{R}$. Let $\gamma : \mathbb{R} \rightarrow \mathbb{C}$ be defined by

$$\gamma(t) = \int_{-\infty}^{\infty} e^{itx(t, \lambda)} dG(\lambda)$$

where

$$tx(t, \lambda) = t\lambda + \frac{\kappa}{\pi} \cos^{-1} \frac{\lambda}{2} \quad (3.9)$$

is a mapping from $[-2t, 2t]$ into $[-2t + \kappa, 2t]$. If $\kappa = \kappa(t) = O(|t|)$, then

$$|\gamma(t)| \leq \frac{B}{|t|^{1/2}} \ln |t|$$

holds for some $B < \infty$ and every $t \in \mathbb{R}$ with $|t| \geq 2$. 


Proof. Denoting by \( \chi \) the characteristic function of the interval \([a, b]\): \( \chi(\lambda) = 1 \) if \( a \leq \lambda \leq b \) and \( = 0 \) otherwise, by the Plancherel theorem

\[
\gamma(t) = \int_{-\infty}^{\infty} e^{itx(\lambda)} \chi(\lambda) dG(\lambda) = \int_{-\infty}^{\infty} \Lambda(t, \tau) \hat{\delta}(\tau) d\tau
\]

where

\[
\Lambda(t, \tau) = \frac{1}{2\pi} \int_{a}^{b} e^{i(tx(\lambda)+\tau\lambda)} d\lambda.
\]

We apply van der Corput estimates (see e.g. Lemma 4.3, Chap. V of [Zy]) in order to obtain the asymptotic behavior of \( \Lambda(t, \tau) \) for large \( t \) and \( \tau \). The integral (3.11) is of the form

\[
2\pi \Lambda(t, \tau) = \int_{a}^{b} e^{2\pi i f(\lambda)} d\lambda
\]

where the second derivative \( f'' \) of \( f \) is strictly negative (under the hypothesis \( 0 \notin [a, b] \), \( \cos^{-1} \lambda/2 \) is strictly concave for every \( \lambda \in [a, b] \)) and proportional to \( \kappa = O(t) \), uniformly in \( \tau \): \( |f''(\lambda)| \geq \rho |\kappa| \) where \( 2\pi \rho = \sup_{\lambda \in [a, b]} \lambda/(4 - \lambda^2)^{3/2} \). By van der Corput’s lemma

\[
|\Lambda(t, \tau)| \leq \frac{4}{\sqrt{\rho \kappa}} \leq \frac{K}{\sqrt{1 + |t|}}
\]

holds for a constant \( K \), independent of \( t \) and \( \tau \). On the other hand, for \( |\tau| > \Delta |t| \) for \( \Delta = 1 + |\kappa/t| \sup_{\lambda \in [a, b]} 1/(\pi \sqrt{4 - \lambda^2}) \), the derivative \( f' \) of \( f \) is of order \( \tau \) and Van der Corput’s lemma gives

\[
|\Lambda(t, \tau)| \leq \frac{K'}{|\tau|}
\]

for some constant \( K' \), independent of \( t \) and \( \tau \).

Estimates (3.12) and (3.13) together with (3.8), yield

\[
|\gamma(t)| \leq \int_{|\tau| \leq \Delta |t|} |\Lambda(t, \tau)| \left| \hat{\delta}(\tau) \right| d\tau + \int_{|\tau| > \Delta |t|} |\Lambda(t, \tau)| \left| \hat{\delta}(\tau) \right| d\tau
\]

\[
\leq \frac{K}{\sqrt{1 + |t|}} \int_{|\tau| \leq \Delta |t|} \frac{C}{1 + |\tau|} d\tau + \int_{|\tau| > \Delta |t|} \frac{K'}{|\tau|} \frac{C}{1 + |\tau|} d\tau
\]

\[
\leq B' |t|^{-1/2} \ln |t| + B'' |t|^{-1}
\]

for \( |t| \geq 2 \) and some finite constants \( B' \) and \( B'' \), concluding the proof of the lemma.

□
Remark 3.4 If the pointwise behavior of (3.8) is replaced by \( C/(1 + |t|)^{1-\varepsilon} \) for some \( \varepsilon > 0 \), then the logarithmic correction \( \ln|t| \) in the conclusion of Lemma has to be replaced by \( |t|^\varepsilon \) (see the last inequality of (3.14)). We shall also apply Lemma 3.3 with \( n^*_\kappa = O(t) \) where \( n^*_\kappa = n^*(t) \) is a nonvanishing integer valued, piecewise monotone function, increasing slower than any positive power of \( t \). In this case, logarithmic correction will be replaced by \( \sqrt{n^*(t)} \) (see Section 4 for details).

Remark 3.5 The function \( x = x(\lambda) \) in Theorem 10.12 of Chap. XII of [Zy], whose proof has suggested us Lemma 3.3, is a one–to–one mapping of \([-\pi, \pi]\) onto itself (it does not depend on \( t \), as in our case; see (3.9)) and \( \gamma(t) = \hat{d}F(t) \), thereby, where \( F(x) \) is an increasing function with \( F \circ x(\lambda) = \sqrt{2\pi G(\lambda)} \). These facts are not necessary for the conclusion of Lemma 3.3. The hypothesis on \( \hat{d}G(t) \) in that theorem is, in addition, stronger than ours. Because \( \hat{d}G(t) \) decays only “on the average” it is necessary an improved estimate \( K'/\tau^2 \) instead of a simpler one (3.13).

3.3 Gevrey Type Estimates

To prove Theorem 3.1 we need to improve results of [KR] in order to bring \( \beta_j \) down from the asymptote \( \exp(\exp cj) \) to \( (j!)^{1/\varepsilon} \) for some \( \varepsilon > 0 \). The following proposition gives a Gevrey type estimate for the derivatives of the Prüfer angles.

Proposition 3.6 Let \( \theta_k = \theta_k(\varphi), k = 0,1, \ldots, \) be the sequence of Prüfer angles starting from \( \theta_0 \), consistent with the initial condition (2.7). Then, for every \( m \geq 1 \),

\[
|\theta'_m(\varphi)| \leq C_1 \delta \eta \beta_m
\]  \hspace{1cm} (3.15)

and for every \( m > 1 \) and \( n > 1 \)

\[
\frac{1}{n!} |\theta_m^{(n)}(\varphi)| \leq C_n \eta^n \beta_{m-1}^n
\]  \hspace{1cm} (3.16)

hold uniformly in compact subsets of \((0, \pi)\) with \( C_n = K/n^2, K \leq 3/(2\pi^2) = 0.151981 \ldots \) and

\[
\eta = \frac{1 + \Delta}{\delta K}
\]

where \( \Delta < 1 \) (\( \Delta = O(\delta) \)) is a constant satisfying \( \theta'_m(\varphi) \leq (1 + \Delta) \beta_m \), which is computable from Proposition 5.2 of [MWGA], and \( \delta \) is suitably small.

Remark 3.7 Proposition 3.6 replaces the unspecified constant \( C_j \) appearing in Lemma 3.1 of [KR] by \( K\eta^j j!/j^2 \). Detailed information on the growth of \( \theta_m^{(j)}(\varphi) \) in both \( j \) and \( m \) are an essential ingredient of our method.

To prove Proposition 3.6 we need the following
Lemma 3.8 Let $C \ast D$ denote the convolution product in $\mathbb{R}^{\mathbb{Z}^+}$:

$$(C \ast D)_n = \sum_{i=0}^{n} C_i D_{n-i} \quad (3.17)$$

for $n \geq 0$. If $C$ has components given by $C_i = K/i^2$, $i \geq 1$, with $C_0 = K$, then

$$\underbrace{C \ast C \ast \cdots \ast C}_{k-\text{factors}} \leq C$$

holds for every $k \geq 1$ provided $K \leq 1/(2 + 2\pi^2/3)$. If $C_0 = 0$, then same result holds with $K \leq 3/(2\pi^2)$.

Remark 3.9 The $n$–th component of the convolution of $C$ with itself $k$–times satisfies

$$\sum_{i_1, \ldots, i_k \geq 0 \atop \sum i_j = n} C_{i_1} \cdots C_{i_k} \leq C_n \quad (3.18)$$

by Lemma 3.8.

Remark 3.10 We have stated Lemma 3.8 for sequences $C = (C_0, C_1, \ldots) \in \mathbb{R}^{\mathbb{Z}^+}$ with the 0–th component $C_0 = 0$ and $C_0 = K$ since both cases will be considered in this subsection (see Proposition 3.13 below for the case $C_0 \neq 0$). Lemma 3.8 plays a key role in every estimate involving higher order chain rule.

Proof of lemma. By definition,

$$\frac{1}{C_n} \sum_{i=0}^{n} C_i C_{n-i} = K \left(2 + \sum_{i=1}^{n-1} \frac{n^2}{i^2(n-i)^2} \right). \quad (3.19)$$

Writing

$$\frac{n^2}{i^2(n-i)^2} = \frac{n^2}{(n-i)^2 + i^2} \left(\frac{1}{i^2} + \frac{1}{(n-i)^2} \right), \quad (3.20)$$

the pre–factor in the r.h.s. of (3.20) can be bounded using $0 \leq (a-b)^2 = 2(a^2 + b^2) - (a+b)^2$, which holds for any real numbers $a$ and $b$, with $a = n - i$ and $b = i$. We have

$$0 \leq (n-2i)^2 = 2 \left((n-i)^2 + i^2\right) - n^2$$

or, equivalently,

$$\frac{n^2}{(n-i)^2 + i^2} \leq 2. \quad (3.21)$$
Plugging (3.20) into (3.19) together with (3.21), gives

\[ \frac{1}{C_n} \sum_{i=0}^{n} C_i C_{n-i} \leq 2K \left( 1 + \sum_{i=1}^{n-1} \left( \frac{1}{i^2} + \frac{1}{(n-i)^2} \right) \right) \]

\[ \leq 2K \left( 1 + \frac{\pi^2}{3} \right) \leq 1 \quad (3.22) \]

provided \( K \leq 1/ (2 + 2\pi^2/3) \). The case \( C_0 = 0 \), the terms with \( i = 0 \) and \( n \) do not contribute to the sum and the inequality (3.22) holds provided \( K \leq 3/(2\pi^2) \). Once we have \( C \ast C \leq C \), Lemma 3.8 is proved by induction.

Proof of Proposition 3.6. The proof uses the recursive relation

\[ \theta_m(\varphi) = g \circ \theta_{m-1}(\varphi) + \beta_m \varphi \quad (3.23) \]

where

\[ g = g(\varphi, \theta) = \tan^{-1} \left( (\tan \theta + \cot \varphi)/p^2 - \cot \varphi \right) \quad (3.24) \]

together with the Scott’s formula for higher order chain rule (see e.g. [FLy])

\[ (g \circ f)^{[n]} = \sum_{k=1}^{n} g^{[k]} \circ f \sum_{i_1, \ldots, i_k \geq 1 \atop i_1 + \cdots + i_k = n} f^{[i_1]} \cdots f^{[i_k]} \quad (3.25) \]

where, from here on, \( h^{[n]} \) stands for \( h^{(n)}/n! \), the \( n \)-th derivative of \( h \) divided by \( n! \).

Upper and lower bounds for the first derivative has been provided in [MWGA]:

\[ (1 - \Delta) \beta_m \leq \theta'_m(\varphi) \leq (1 + \Delta) \beta_m \quad (3.26) \]

with \( \Delta < 1 \) a constant. Now, choosing \( \eta = \frac{1 + \Delta}{\delta K} \), (3.26) establishes (3.15) for every \( m \geq 1 \).

Since \( (\beta_j)_{j \geq 1} \) is a fast increasing sequence we apply the Scott’s formula to \( g \) in (3.24) as it were a function of a single variable \( \theta \). This really gives the main contribution to the derivatives. \( g \) as a function of \( z = e^{i\theta} \), continued to the complex plane, is analytic outside a disc of radius strictly less than \( 1 - e/\xi < 1 \). The derivatives of \( q(e^{i\theta}) = g(\theta) \) may be estimate by Cauchy formula:

\[ |g^{[k]}(\theta)| \leq c_1 \xi^k \quad (3.27) \]

holds for \( k \geq 1 \) with \( c_1 \) as small as one wishes, by increasing \( \xi \) accordingly ((3.27) can be bounded, e. g., by \( \varepsilon (c_1/\varepsilon)^k = \varepsilon \xi^k \), for any \( \varepsilon > 0 \)). Replacing \( f \) by \( \theta_{m-1} \) in (3.25), gives

\[ \theta_m^{[n]}(\varphi) = \sum_{k=1}^{n} g^{[k]} \circ \theta_{m-1}(\varphi) \sum_{i_1, \ldots, i_k \geq 1 \atop i_1 + \cdots + i_k = n} \theta_{m-1}^{[i_1]}(\varphi) \cdots \theta_{m-1}^{[i_k]}(\varphi) \]
We prove (3.16) by induction in \( n \). Consider the case \( n = 2 \), for any \( m > 1 \). By equation (3.23), together with (3.27) and (3.15), we have
\[
\theta_m'' = g'' \circ \theta_{m-1} \cdot (\theta_{m-1}')^2 + g' \circ \theta_{m-1} \cdot \theta_m''
\]
\[
\leq c_1 \xi^2 C_1^2 \delta^2 \eta^2 \beta_{m-1}^2 + g' \circ \theta_{m-1} \cdot \theta_m'' \cdot \theta_m'' .
\]
The iteration of this relation together with (3.27) and (3.3), yields
\[
\theta_m'' \leq c_1 \xi^2 C_1^2 \delta^2 \eta^2 \beta_{m-1}^2 \sum_{j=1}^{m-1} (c_1 \xi)^{j-1} \frac{\beta_{m-j}^2}{\beta_{m-1}^2}
\]
\[
\leq \frac{c_1 \xi^2 \delta^2}{1 - c_1 \xi \delta^2} C_1^2 \eta^2 \beta_{m-1}^2
\]
\[
\leq 2C_2 \eta^2 \beta_{m-1}^2
\]
provided \( \delta \) is chosen so small that \( c_1 \xi \delta^2 < 1 \) and
\[
2K \frac{c_1 \xi^2 \delta^2}{1 - c_1 \xi \delta^2} \leq 1
\]
are both satisfied, establishing (3.16) for \( n = 2 \).

Now, suppose
\[
\theta_{m}^{[j]}(\varphi) \leq C_j \eta^j \beta_{m-1}^j
\]
holds for \( m > 1 \) and \( j = 2, \ldots, n-1 \) and we shall establish the inequality for \( n \). By this assumption together with (3.3), we have
\[
\theta_{m-1}^{[j]}(\varphi) \leq C_j \eta^j \beta_{m-2}^j = C_j \eta^j \left( \frac{\beta_{m-2}}{\beta_{m-1}} \right)^j \beta_{m-1}^j \leq C_j (\delta \eta)^j \beta_{m-1}^j .
\]
(3.29)

Plugging (3.15), (3.27) and (3.29) into (3.28), together with (3.18), yields
\[
\theta_m^{[n]}(\varphi) \leq c_1 (\delta \eta)^n \beta_{m-1}^n \sum_{k=2}^{n} \xi^k \sum_{i_1, \ldots, i_k \geq 1 \atop i_1 + \cdots + i_k = n} C_{i_1} \cdots C_{i_k} + g' \circ \theta_{m-1} \cdot \theta_m^{[n]}
\]
\[
\leq c_1 \frac{\xi}{\xi - 1} C_n (\delta \xi \eta)^n \beta_{m-1}^n + g' \circ \theta_{m-1} \cdot \theta_m^{[n]}
\]
Here, we have separated the term with \( k = 1 \) which applies \( n \) derivatives on \( \theta_{m-1} \). Note that, for all the other terms with \( k \geq 2 \), we have \( i_1, \ldots, i_k \geq 1 \) and the derivatives applied on the \( \theta_{m-1} \) are of order strictly smaller than \( n \). The iteration of this relation, gives
\[
\theta_m^{[n]}(\varphi) \leq c_1 \frac{\xi}{\xi - 1} C_n (\delta \xi \eta)^n \beta_{m-1}^n \sum_{j=1}^{m-1} (c_1 \xi)^{j-1} \frac{\beta_{m-j}^n}{\beta_{m-1}^n}
\]
\[
\leq C_n \eta^n \beta_{m-1}^n
\]

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provided
\[
c_1 \frac{\xi}{\xi - 1} (\xi \delta)^n \frac{1}{1 - c_1 \xi \delta^n} \leq 1
\]
holds for every \( n > 2 \). We pick \( \delta \) satisfying both (3.3) and (3.30) for \( n \geq 2 \), concluding the proof of Proposition 3.6.

\[\Box\]

**Remark 3.11** The well known formula for higher derivative of composite functions, Faà di Bruno’s formula, cannot be used recursively since the constant \( \eta \) in equation (3.16) deteriorates each time it is applied (see eq. (6.10) in Subsection 6.2 of [BM]). The proof of (3.16), by induction, using Scott’s formula was based on yet unpublished manuscript “O(N) Hierarchical Model Approached by the Implicit Function Theorem” by W. R. P. Conti and D. H. U. Marchetti.

To obtain the \( t^{-1} \) decay from the summation in (2.11), it is necessary to apply an arbitrarily large number of the integration by parts for integrals of the type (2.12):
\[
\int_0^\pi f_0 e^{i\theta} d\varphi = i \int_0^\pi \left( \frac{1}{i \theta} f_0 \right)' e^{i\theta} d\varphi
\]
where \( \text{supp} f_0 = [\varphi_-, \varphi_+] \subset (0, \pi) \). The following propositions gather tools to implement the estimate.

**Proposition 3.12** Let \( f_0(\varphi) \) and \( \varrho(\varphi) \) be, respectively, \( C^\infty \) complex and real–valued functions on \([0, \pi)\) and let \( L = \frac{d}{d\varphi} \varrho(\varphi) \) be an operator defined in this space. If
\[
f_n = \frac{1}{n!} L^n f_0 = \frac{1}{n!} \frac{d}{d\varphi} \varrho \frac{d}{d\varphi} \varrho \cdots \frac{d}{d\varphi} \varrho f_0
\]
denotes the \( n \)–th application of \( L \) over \( f_0 \), divided by factorial of \( n \), \( n = 0, 1, \ldots \), then
\[
f_n = \sum_{k_1, \ldots, k_n, p_n \geq 0 \atop k_1 + \cdots + k_n + p_n = n} \varrho^{[k_1]} \cdots \varrho^{[k_n]} f_0^{[p_n]}. \tag{3.31}
\]

**Proof.** The proof is by induction. For \( n = 1 \), we have \( f_1 = \varrho f_0 + \varrho f_0' \). Assuming that (3.31) holds,
\[
f_{n+1} = \frac{1}{n+1} (\varrho' f_n + \varrho f_n')
\]
\[
= \frac{1}{n+1} \sum_{k_1, \ldots, k_n, p_n, \hat{p}_n \geq 0 \atop k_1 + \cdots + k_n + \hat{p}_n + p_n = n+1} (\hat{k}_1 + \cdots + \hat{k}_n + \hat{p}_n + p_n) \varrho^{[k_1]} \cdots \varrho^{[k_n]} f_0^{[p_n+\hat{p}_n+1]}
\]
\[
= \sum_{k_1, \ldots, k_n, p_n, \hat{p}_n \geq 0 \atop k_1 + \cdots + k_n + \hat{p}_n + p_n = n+1} \varrho^{[k_1]} \cdots \varrho^{[k_n]} f_0^{[p_n+\hat{p}_n+1]}
\]

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where we have applied the product rule
\[ f'_n = \sum_{j=1}^{n} \sum_{k_1, \ldots, k_n, p_n \geq 0} q^{[k_1]} \cdots q^{[k_j]} f'_0 + \sum_{k_1, \ldots, k_n, p_n \geq 0} q^{[k_1]} \cdots q^{[k_n]} (f_0^{[p_n]})', \]
used \((q^{[k_j]})' = (k_j + 1)q^{[k_j+1]}\) for \(j = 1, \ldots, n\) (analogously for \((f_0^{[p_n]})' = (p_n + 1)q_0^{[p_n+1]}\)), redefined variables:
\[ \hat{k}_{j+1} = k_j + 1, \quad \hat{k}_{l+1} = k_l \quad \text{for} \quad l \neq j \quad \text{and} \quad \hat{p}_{n+1} = p_n \]
(the same for \(\hat{p}_{n+1} = p_n + 1\) and \(\hat{k}_{j+1} = k_j\) for \(j = 1, \ldots, n\)) and have added a new variable \(\hat{k}_1\). Observe that \(\hat{k}_{j+1} = k_j + 1 \geq 1\) but we can start the sum over \(\hat{k}_{j+1}\) from 0 since \(\hat{k}_{j+1} q^{[\hat{k}_{j+1}]}\) is identically 0 at \(\hat{k}_{j+1} = 0\). This completes the proof of the proposition.

In our application, \(\varrho = \frac{1}{t h'_{m+1}(\varphi)}\) and \(f_0 = |f(2\cos \varphi)|^2 \frac{\sin^2 \varphi}{R_m^2(\varphi)} A^n(\varphi)\) (or its complex conjugate). The main contribution for \(m \geq j^* + 1\), where \(j^* = j^*(t)\) is such that
\[ \beta_{j^*} \leq t < \beta_{j^*+1}, \quad (3.32) \]
comes from the derivatives of the Prüfer angles \(\theta_k(\varphi)\) and in this case it is thus sufficient to consider
\[ \varrho = \frac{1}{\theta'_{m+1}(\varphi)} \equiv s \circ \theta'_{m+1}(\varphi). \]
For \(m < j^*\), we have
\[ t h'_{m+1}(\varphi) = -2t \sin \varphi + n \theta'_{m+1} = -2t \sin \varphi (1 + O(1)) \quad (3.33) \]
and the derivatives of higher order
\[ t h^{(k)}_{m+1}(\varphi) - n \theta^{(k)}_{m+1} = \begin{cases} 2t (-1)^{(k+1)/2} \sin \varphi & \text{if} \quad k \text{ is odd} \\ 2t (-1)^{k/2} \cos \varphi & \text{if} \quad k \text{ is even} \end{cases} \]
satisfies, in view of (3.16),
\[ t h^{(k)}_{m+1}(\varphi) \leq 2 |t| + nC_k \eta^k \beta_{m,k}^k. \quad (3.34) \]
It is also sufficient to consider in both cases
\[ f_0 = \frac{1}{R_m^2(\varphi)} = \frac{1}{R_0^2} \prod_{j=1}^{m} \frac{p^2}{a + b \cos 2\theta_j + c \sin 2\theta_j} \]
\[ \equiv \frac{1}{R_0^2} \prod_{j=1}^{m} F \circ \theta_j(\varphi) \]
where $F(\theta)$ satisfies, by direct computation,

$$F^{[k]} \leq \frac{1 - \delta / \zeta}{\delta} \left( \frac{\zeta}{\delta} \right)^k$$

for $k \geq 0$ and some positive number $\zeta$. We also need

$$s^{[k]}(x) = \frac{(-1)^k}{x^{k+1}}, \quad k = 1, \ldots.$$  

(3.36)

**Proposition 3.13** Let $\theta_k = \theta_k(\varphi)$, $k = 0, 1, \ldots$, be the sequence of Prüfer angles and let $\eta, \delta$ and $\{C_n\}$ be the constants that appear in Proposition 3.6. Then, there exist positive numbers $d$ and $\tilde{\eta}$, which can be expressed in terms of the previous constants, such that (with $g = g^0 \leq d/\beta_{m+1}$)

$$g^{[n]} = (s \circ \theta_{m+1})^{[n]} \leq \frac{d}{\beta_{m+1}} C_n \tilde{\eta}^n \beta_m^n$$

(3.37)

as well as ($f_0 = f_0^0 \leq R_0^{-2}\delta^{-m}$)

$$f_0^{[n]} = \frac{1}{R_0^2} \left( \prod_{j=1}^m F \circ \theta_j \right)^{[n]} \leq \frac{1}{R_0^2} C_n (\zeta \eta)^n \frac{1}{\delta m} \beta_m^n$$

(3.38)

hold for every non–negative integer $n$, with $\zeta$ as in (3.35).

**Proof.** These inequalities are established as in Proposition 3.6 by using the Scott’s formula. We begin with (3.37). If $\tilde{\eta}$ is the smallest constant such that $(i - 1)^2 \eta^i / i \leq \tilde{\eta}$ holds for every $i \geq 1$, by (3.16), (3.29) and (3.25) with $g$ and $f$ replaced by $s$ and $\theta_{m}'$, we have

$$g^{[n]} = \sum_{k=1}^n s^{[k]} \circ \theta_{m+1}' \sum_{i_1, \ldots, i_k \geq 1} (i_1 + 1) \theta_{m+1}^{[i_1+1]} \cdots (i_k + 1) \theta_{m+1}^{[i_k+1]}$$

$$\leq \tilde{\eta}^n \beta_m^n \sum_{k=1}^n \frac{1}{(\theta_{m+1})_{k+1}^n} \tilde{\eta}^k \beta_m^k \sum_{i_1, \ldots, i_k \geq 1} C_{i_1} \cdots C_{i_k}$$

$$\leq d C_n \tilde{\eta}^n \frac{\beta_m^n}{\beta_{m+1}}$$

where $\tilde{\eta} = \delta \tilde{\eta}^2 / (1 - \Delta)$ and $d = \delta \eta / (\delta \tilde{\eta} + \Delta - 1)$. In the third inequality we have used the lower bound (3.26) for $\theta_{m}'$ and (3.18). Note $\delta \tilde{\eta} > \delta \eta > 1$, by definition of $\eta$ in Proposition 3.6.
For (3.38), we start with the Scott’s formula (3.25) with \(g\) and \(f\) replaced by \(F\) and \(\theta_j\) which, together with (3.16), (3.29) and (3.35), gives

\[
(F \circ \theta_j)^[n] = \sum_{k=1}^{n} F[k] \circ \theta_j \sum_{i_1,\ldots,i_k \geq 1}^{i_1+\cdots+i_k=n} \theta_j[i_1] \cdots \theta_j[i_k]
\]

\[
\leq C_n (\delta \eta)^n \beta^n_j \sum_{k=1}^{n} \left( \frac{\zeta}{\delta} \right)^k
\]

\[
\leq \frac{1}{\delta} C_n (\zeta \eta)^n \beta^n_m.
\]

Now we take the \(n\)–th derivative of the product. For this, we use the variation of Lemma 3.8 mentioned in Remark 3.10:

\[
\left( \prod_{j=1}^{m} F \circ \theta_j(\varphi) \right)^[n] = \sum_{n_1,\ldots,n_m \geq 0}^{n_1+\cdots+n_m=n} (F \circ \theta_1)^[n_1] \cdots (F \circ \theta_m)^[n_m]
\]

\[
\leq C_n (\zeta \eta)^n \frac{1}{\delta^m} \beta^n_m
\]

(3.39)

concluding the proof of this proposition.

\[\square\]

**Remark 3.14** An estimate of (3.37) with \(\varrho = 1/\theta'_{m+1}(\varphi)\) for \(m < j^*\) is analogously given by

\[
(\varrho)^[k] = (s \circ \theta'_{m+1})^[k] \leq \frac{d_n}{|t|} C_n \eta^k \beta^k_m
\]

(4.40)

with \(d_n = 2dn/c\) where \(c = \min_{\varphi \in \text{supp} \, f \circ \lambda} 2 \sin \varphi\). Note that, for any fixed \(m\) and \(t\) satisfying (3.32), the second term of the l.h.s. of (3.34) rapidly overcomes \(t\). On the other hand, an estimate for \(k\) in which \(t\) still dominates (3.34) is, by the Scott’s formula (3.25), much better than (3.40):

\[
(\varrho)^[k] = \frac{1}{t} \sum_{l=1}^{k} s[l] \circ h'_{m+1} \sum_{i_1,\ldots,i_l \geq 1}^{i_1+\cdots+i_l=k} \frac{1}{i_1!} h_{m+1}^{(i_1+1)} \cdots \frac{1}{i_l!} h_{m+1}^{(i_l+1)}
\]

\[
\leq \frac{2^k}{|t|} \sum_{l=1}^{k} \frac{2^l}{c^{l+1}} \sum_{i_1,\ldots,i_l \geq 1}^{i_1+\cdots+i_l=k} \frac{1}{i_1!} \cdots \frac{1}{i_l!}
\]

\[
\leq \frac{2^k}{|t|} \frac{1}{k!} \sum_{l=1}^{k} \frac{2^l k^k}{c^{l+1}}.
\]

(3.41)
Let us put all together. Plugging (3.37) and (3.38) into (3.31), deduced in Proposition 3.12 by applying \( n \) times integration by parts \( n! f_n = L_n f_0 \) to the integrand \( f_0 \) of (2.12), we arrive at the following estimate:

\[
n! |f_n| \leq \sum_{k_1, \ldots, k_n, p_n \geq 0} \left| \varrho^{[k_1]} \cdots \varrho^{[k_n]} f_0^{[p_n]} \right| \leq \frac{1}{R_0^2} D^{n} \frac{1}{\delta^m} \left( \frac{\beta_m}{\beta_{m+1}} \right)^n n! \sum_{k_1, \ldots, k_n, p_n \geq 0} C_{k_1} \cdots C_{k_n} C_{p_n}
\]

where \( D = d \cdot \max (\hat{\eta}, \zeta \eta) \). Estimate (3.42) will be used to get an upper bound for all non-resonant integrals of (2.11).

**Remark 3.15** As \( C_n = K/n^2 \), \( n \geq 1 \), \( (C_0 = K) \) with \( K \leq 1/(2 + 2\pi^2/3) \) are bounded constants, (3.42) makes explicit the dependence on the number \( n \) of times that integration by parts is applied to integral of type (2.12). Explicit dependence of \( n \) was not necessary in reference [KR], since \( n \) is an arbitrarily large but fixed number. Apart this, (3.42) agrees with the estimate used on p. 522 of [KR].

### 4 Proof of Theorem 3.1

Let \( t \) be a fixed number. We assume \( t \) positive but the negative value can be dealt similarly. Let \( f \) be a \( C^\infty \) function with compact support in \((0, 2)\) and let \( I_f = [\varphi_-, \varphi_+] \) be smallest closed interval that contains \( \text{supp} f \circ \lambda, \lambda(\varphi) = 2 \cos \varphi \). Since the spectral measure is symmetric, \( d\rho(-\lambda) = d\rho(\lambda) \), we need only to consider \( f \) supported in one-half of the essential spectrum. We have excluded the origin to avoid that the curvature of \( \cos^{-1} \lambda/2 \) vanishes (see observation right before (3.12)).

For \( j^* = j^*(t) \) defined by equation (3.32), let \( n^* = n^*(t) \) be given by

\[
(n^* - 1)\beta_{j^*} \leq t < n^* \beta_{j^*}.
\]

Since \( \beta_{j^*}/t > 1/n^* \) there are at most \( n^* \) points \( \varphi_1, \ldots, \varphi_{n^*} \) in the support of \( f \circ \lambda \) satisfying

\[
- \sin \varphi_t + \frac{t \theta_{j^*}}{t} = 0.
\]

Observe that \( 1 \leq n^*(t) \leq \beta_{j^*+1}/\beta_{j^*} + 1 \), by (3.32) and (4.1). For the sparseness increment \( \beta_j \) in (3.5), we have

\[
j^*(t) = \frac{\ln t}{c \ln^2 \ln t} \left( 1 + O \left( \frac{\ln \ln \ln t}{\ln \ln t} \right) \right)
\]
and, consequently, the number of resonant values \( n^* \) is a monotone nondecreasing function of \( t \) in each interval \((\beta_{j^*}, \beta_{j^*+1})\). Let \( L(t) \) denote the continuous interpolation of \( n^*(t) \). It follows from these observations that \( L \) is a piecewise linear function with inclination \( 1/\beta_j \) satisfying

\[
1 < L(t) < E e^{\ln^2 t} \equiv \Omega^2(t), \quad \beta_j < t \leq \beta_{j+1}
\]

for some constant \( E \), independent of \( t \). The inequality (4.3) will be used at the end of this section.

By (2.11), the Fourier–Stieltjes transform of \( \rho \) can be written as

\[
|\widehat{f}|^2 d\rho(t) = I_{j^*-1,0}(t) + \sum_{j=j^*-1}^{\infty} \sum_{n=1}^{\infty} (I_{j,n}(t) + \bar{I}_{j,n}(-t))
\]

where

\[
I_{j,n}(t) = \frac{1}{\pi} \int_0^\pi |f(2 \cos \varphi)|^2 \sin^2 \varphi \frac{A^*(\varphi) e^{2it(\cos \varphi + n\theta_{j+1}/t)}}{R_j^2} d\varphi.
\]

By integration by parts to all terms of this sum not satisfying the resonant condition (4.2). Since the support of \( f \) is compact, every boundary term vanishes. Integration by parts may be repeated \( N_j \) times depending on the index \( j \) of the sum. Propositions 3.12 and 3.13, together with its combined estimate (3.42), can be used to get an upper bound for each integral (4.4) with \( j \geq j^* \).

This yields

\[
\sum_{j=j^*}^{\infty} \sum_{n=1}^{\infty} (|I_{j,n}| + |\bar{I}_{j,n}(-t)|) \leq 2 \sum_{j=j^*}^{\infty} \left( \sum_{n=1}^{\infty} \frac{1}{n^a} \right) \frac{1}{R_0^2} C_{N_j} D^{N_j} \frac{1}{\delta_j^j} \left( \frac{\beta_j}{\beta_{j+1}} \right)^{N_j} \delta_j! N_j!
\]

where \( a = \sup_{\varphi \in I_j} |A(\varphi)| < 1 \). If the sequences \( (\beta_j)_j \) and \( (N_j)_j \) are chosen so that

\[
D^{N_j} \frac{1}{\delta_j^j} \left( \frac{\beta_j}{\beta_{j+1}} \right)^{N_j} \delta_j! N_j! \leq \frac{1}{\beta_{j+1}}
\]

then the series in (4.5) converges uniformly in \( t \). By (3.3) and (3.32), we have

\[
\frac{1}{\beta_{j+1}} = \frac{1}{\beta_{j^*+1}} \frac{\beta_{j^*+1}}{\beta_{j^*+2}} \cdots \frac{\beta_j}{\beta_{j+1}} \leq \delta^{j^*-j} \frac{1}{\beta_{j^*+1}}.
\]

and \( t/\beta_{j^*+1} < 1 \). Consequently,

\[
\sum_{j=j^*}^{\infty} \sum_{n=1}^{\infty} (|I_{j,n}| + |\bar{I}_{j,n}(-t)|) \leq 2a \frac{1}{1-a} \frac{1}{R_0^2} \sum_{j=j^*}^{\infty} C_{N_j} \frac{1}{\beta_{j+1}}
\]

\[
\leq 2a \frac{1}{1-a} \frac{1}{R_0^2} \frac{t}{\beta_{j^*+1}} K \sum_{l=0}^{\infty} \delta^l \leq \frac{C}{t}
\]

holds with \( C < \infty \) independent of \( t \).
Let us now verify that the sparseness condition stated in Theorem 3.1 satisfies (4.6). Choosing \( \beta_j \) as given by (3.5) and \( N_j = j + 1 \), by the Stirling formula,

\[
\beta_{j+1} \left( \frac{\beta_j}{\beta_{j+1}} \right)^{N_j} N_j! = \frac{\delta^j}{\sqrt{2\pi(j+1)}} \left( \frac{j+1}{e} \right)^{j+1} \exp \left( -2c(j+1) \ln(j+1) \right) \left( 1 + O \left( \frac{\ln j}{j} \right) \right)
\]

and (4.6) holds for any \( c > 1/2 \) provided \( j \) is large enough.

Note that, for the sparseness increment \( \left( \frac{\beta_j}{\beta_{j+1}} \right)^{N_j} N_j! = \delta_j \sqrt{2\pi(j+1)} (j+1) e^{j+1} \exp \left( -2c(j+1) \ln(j+1) \right) \left( 1 + O \left( \frac{\ln j}{j} \right) \right) < 1 \) and (4.6) holds with \( \frac{1}{\beta_{j+1}} \) replaced by \( \frac{1}{\beta_{j+1} - \epsilon} \), provided \( 1 + \frac{1}{\epsilon} > \ln D - (1 - \epsilon) \ln \delta \) and \( j \) is large enough. Together with Remark 3.4, the proof may continue exactly as for decaying \( t^{-1} \). We shall consider only the latter case.

For \( j < j^* \), (3.33) holds and we need replace the estimate (3.37) by (3.40) and \( \tilde{t} \) occupies now the place of \( \tilde{t}_m \) in (3.42). Applying successive integration by parts to \( I_{j^*-1,0} \), gives, analogously

\[
|I_{j^*-1,0}| \leq \frac{1}{R_{j^*-1}^2 C_{N_{j^*-1},1}} D^{N_{j^*-1}} \frac{1}{\delta^{j^*-1}} \left( \frac{\beta_{j^*-1}}{\tilde{t}} \right)^{N_{j^*-1}} \frac{N_{j^*-1}!}{\tilde{t}^{N_{j^*-1} - 1}} \leq C' \tag{4.8}
\]

for some constant \( C' \). Note that, by (3.32),

\[
\left( \frac{\beta_{j^*-1}}{\tilde{t}} \right)^{N_{j^*-1}} \leq \frac{1}{\tilde{t}} \frac{\beta_{j^*-1}^{N_{j^*-1}}}{\delta^{N_{j^*-1} - 1}}
\]

and by (4.6)

\[
D_k \frac{1}{\delta^k} \frac{\beta_k^{N_k}}{\beta_{k+1}^{N_{k+1}}} \leq 1 \tag{4.9}
\]

for \( k \) large enough.

It remains to estimate the sum \( S_{j^*}(t) = \sum_{n=1}^\infty \left( I_{j^*-1,n}(t) + \tilde{I}_{j^*-1,n}(-t) \right) \) which contains the most significant terms responsible for \( t^{-1/2} \) decaying behavior. To extract this decay we write

\[
I_{j^*-1,n}(t) = \frac{1}{\pi} \int_{0}^{\pi} \left| f(2 \cos \varphi) \right|^2 \sin^2 \varphi \frac{B_n^{j^*-1}(\varphi)}{R_{j^*-1}^2} e^{2it(c \varphi + n\beta_j \varphi/t)} \, d\varphi
\]

(analogously for \( \tilde{I}_{j^*-1,n}(t) \)) where, by (3.23), \( B_k \) is a function of the Prüfer angles \( \theta_k(\varphi) \) such that \( |B_k| = |A| \) and

\[
\arg B_k = \arg A + \theta_{k+1}(\varphi) - \beta_{k+1} \varphi = \arg A + g \circ \theta_k(\varphi)
\]
with \( g \) given by (3.24). We then apply Lemma 3.3 to (4.10) with
\[
d (G_n \circ \lambda) (\varphi) = \frac{1}{\pi} |f(2 \cos \varphi)|^2 \frac{\sin^2 \varphi}{R_{j^* - 1}^2} B_{j^* - 1}^n (\varphi) d\varphi
\]
and
\[
tx(t, \lambda) = t\lambda + 2n\beta_{j^*} \cos^2 \frac{\lambda}{2}.
\]
Note that, by (3.32), \( \kappa = 2\pi n\beta_{j^*} \) and \( n^*\beta_{j^*} = O(t) \). In order to fulfill all assumptions of Lemma 3.3 it remains to show that \( \hat{dG_n}(t) \) decays as \(|t|^{-1} \) (see equation (3.8)).

We estimate the Fourier–Stieltjes transform
\[
\hat{dG_n}(t) = \frac{1}{\pi} \int_0^\pi |f(2 \cos \varphi)|^2 \frac{\sin^2 \varphi}{R_{j^* - 1}^2} B_{j^* - 1}^n (\varphi)e^{2it\cos \varphi} d\varphi
\]
as the non–resonant integrals (4.4) with \( j < j^* \) (see Remark 3.14). The estimate (3.37) is replaced by (3.41) and \( f_0 = |f(2 \cos \varphi)|^2 \frac{\sin^2 \varphi}{R_{j^* - 1}^2} A^n(\varphi) \exp (ing \circ \theta_{j^* - 1}(\varphi)) \) includes now an extra exponential term depending on \( \theta_{j^* - 1}(\varphi) \).

To deal with this new term we need some more estimates. By the Scott’s formula (3.25) together with (3.27), we have
\[
\left| \left( e^{ing} \right)^{[N]} \right| \leq \sum_{k=1}^N \frac{n^k}{k!} \sum_{i_1, \ldots, i_k \geq 1} \sum_{i_1 + \cdots + i_k = N} |g[i_1] \cdots g[i_k]|
\]
\[
\leq c_2 \xi^N
\]
with \( c_2 = e^{nc_1} - 1 \). The Scott’s formula (3.25) applied once again together with Proposition 3.6 (3.29) and (4.12), yield
\[
\left| \exp (ing \circ \theta_{j^* - 1})^{[N]} \right| \leq \sum_{k=1}^N \left( e^{ing} \right)^{[k]} \circ \theta_{j^* - 1} \sum_{i_1, \ldots, i_k \geq 1} \sum_{i_1 + \cdots + i_k = N} |\theta_{i_1}^{[i_1]} \cdots \theta_{i_k}^{[i_k]}|
\]
\[
\leq c_3 C_N (\delta \xi \eta)^N \beta_{j^* - 1}^N,
\]
with \( c_3 = c_2 \xi/(\xi - 1) \). Finally, we shall replace (3.39) by
\[
\left| \left( \prod_{j=1}^{j^* - 1} F \circ \theta_j \cdot e^{ing_0 \theta_{j^* - 1}} \right)^{[N]} \right| \leq \sum_{n_1, \ldots, n_j \geq 0} \sum_{n_1 + \cdots + n_j = N} \left| (F \circ \theta_1)^{[n_1]} \cdots (F \circ \theta_{j^* - 1})^{[n_{j^* - 1}]} (e^{ing_0 \theta_{j^* - 1}})^{[n_{j^*}]} \right|
\]
\[
\leq c_3 C_N (\bar{\xi} \eta)^N \frac{1}{\delta^{j^* - 1}} \beta_{j^* - 1}^N
\]
with $\bar{\zeta} = \max (\delta \xi, \zeta)$.

Since the required estimates didn’t change significantly, we integrate (4.11) by parts $N_{j-1}$ times and use (3.42) with (3.41) in the place of (3.37) to get, exactly as for (4.8),

$$\left| \frac{dG_n(t)}{dt} \right| \leq \frac{C''}{t} d^k \tilde{a}^n, \quad \beta_k \leq t < \beta_{k+1}$$

for some constants $C'' < \infty$, $d < 1$ and $\tilde{a} = e^{c_1} \sup_{\varphi \in I_j} |A(\varphi)| < 1$, as $c_1$ is arbitrarily small by the observation after (3.27). Here, we have used the fact that (4.9) holds with $D$ replaced by $D/d$, for any $d > 1$, provided $k$ is large enough. This immediately imply, by a slight modification of Lemma 3.3 (see Remark 3.4),

$$|I_{j^-1,n}(t)| \leq \int_{|\tau| \leq \Delta |t|} |\Lambda(t, \tau)| \left| \frac{dG(t)}{d\tau} \right| d\tau + O \left( \frac{1}{t} \right)$$

and by (4.1) and the fact that $\ln \beta_{k+1}/\beta_k = O(\ln^2 k)$, we have

$$|S_j(t)| \leq \sum_{n=1}^{\infty} \left( |I_{j^-1,n}(t)| + |I_{j^-1,n}(-t)| \right) \leq C \sqrt{|t|} \Omega(|t|),$$

where $\Omega(t)$ is defined in (4.3).

Now, we show that $\Omega(t)$ increases slower than $t^\varepsilon$, for any $\varepsilon > 0$. Suppose, by contradiction, that $\lim_{t \to \infty} \Omega(t)/t^\varepsilon = k > 0$ holds for some $\varepsilon > 0$. Then, by L'Hospital,

$$\lim_{t \to \infty} \frac{\Omega'(t)}{t^\varepsilon} = \lim_{t \to \infty} \frac{\Omega(t)}{t^{\varepsilon-1}} = c \cdot \lim_{t \to \infty} \frac{\Omega(t)}{t^{\varepsilon}} \cdot \lim_{t \to \infty} \frac{\ln \ln t}{\ln t} = 0,$$

concluding the proof of Theorem 3.1.

□

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