A NOTE ON THE A-NUMBERS AND P-RANKS OF KUMMER COVERS

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ABSTRACT. We study the a-numbers and p-ranks of Kummer covers of the projective line, and we give bounds for these numbers.

1. INTRODUCTION

For some special curves, explicit formulas exist for the p-rank in terms of p, the degree of C, and the degree of the ramification divisor. One of the most famous of these formulas is due to Deuring and Shafarevich and dates back to the 1940s (see [9]). However, as Crew pointed out much later in [2], such a formula is impossible for Kummer covers since even for elliptic curves the p-rank can vary with the other numbers fixed. The same argument works equally well for the a-number.

We will study the a-numbers and p-ranks of Kummer covers. Our method uses Čech cohomology to produce a natural basis of $H^1(C, \mathcal{O}_C)$, and we calculate the action of Frobenius on this basis. Using this action, we give bounds for the a-number and the p-rank. This extends recent results of Elkin who used a similar method for a more specialized class of Kummer cover (see [4]).

As an application, we recover Ekedahl’s bound $g(C) < \frac{p^2 + 1}{p + 1}$ for superspecial hyperelliptic curves (see [3]). We show that there are numbers less than this upper bound that do not occur as the genus of such a curve.

2. KUMMER COVERS

In this section, the main result is a decomposition theorem for the induced action of Frobenius on the first cohomology group of a Kummer cover.

Definition 1. An irreducible projective smooth curve C over a field k is a Kummer cover of degree n if there exists a finite separable morphism $\psi : C \to \mathbb{P}^1_k$ of degree n such that $K(C)/K(\mathbb{P}^1_k)$ is a Kummer extension.

This definition automatically assumes that the characteristic of k does not divide n and that k contains the nth roots of unity. For example, hyperelliptic curves over algebraically closed fields of characteristic not equal to 2 are the Kummer covers of degree 2. We will need the following algebraic fact.

Lemma 1. Let $R$ be a noetherian unique factorization domain, and let $R[\alpha]$ be the cyclic extension of $R$ defined by a root $\alpha$ of the irreducible polynomial $z^n - u \prod_{j=1}^{n-1} f_j^{f_j/k/n}$, where $f_j \in R$ is square-free and $u \in R^\ast$. The integral closure of $R[\alpha]$ is generated as an $R$-module by

$$\frac{\alpha^k}{\prod_{j=1}^{n-1} f_j^{f_j/k/n}}, \quad k = 0, 1, \ldots, n - 1.$$
Proof. See \[5\].

We use the previous result to find an affine cover of our curve.

**Lemma 2.** Let $C$ be a Kummer cover of degree $n$ over a field $k_0$. After a base extension $k/k_0$, we can find a generator $y$ for the cyclic extension $K(C)/K(\mathbb{P}^1_k)$ such that $y^n = f = u \prod_{j=1}^{n-1} f_j$ for $u \in k^*$ and $f_j \in k[x]$ separable. Then $C$ has an affine cover consisting of two parts $U' = \text{Spec } A$ and $V' = \text{Spec } B$, where

$$A = \sum_{i=0}^{n-1} \frac{y^i}{\prod_{j=1}^{n-1} f_j^{ji/n_j}} \cdot k[x], \quad B = \sum_{i=0}^{n-1} \frac{y^i}{\prod_{j=1}^{n-1} f_j^{ji/n_j}} \cdot \frac{1}{x^{m_i}} \cdot k[1/x],$$

and

$$m_i = [i \deg(f)/n] - \sum_{j=1}^{n-1} \deg(f_j) ji/n_j.$$

**Proof.** Let $K(\mathbb{P}^1_{k_0}) = k_0(x)$. Since $K(C)/k_0(x)$ is a Kummer extension, we can find a field extension $k/k_0$ such that $\alpha^n = q \in k_0[x]$ and $Z^n - q \in k_0(x)[Z]$ is irreducible. We can also find a field extension $k/k_0$ such that all square-free factors of $q$ in $k[x]$ are separable. Base extending $C$ and $\mathbb{P}^1_{k_0}$ by $k$, we get a Kummer extension $K(C)/k(x)$ with a generator $y$ such that $y^n = f$, $f$ divides $q$, and $Z^n - f \in k(x)[Z]$ is irreducible. We have the square-free factorization $f = u \prod_{j=1}^{n-1} f_j^j \in k[x]$, where each $f_j$ is separable since it divides $q$.

Write $\mathbb{P}^1_k = \text{Proj } [t_0, t_1]$ and cover it by $U = \text{Spec } k[x]$ and $V = \text{Spec } k[1/x]$ with $x = t_1/t_0$. Using the finite morphism $\psi : C \to \mathbb{P}^1_k$, we form a cover of $C$ consisting of two affine open sets $U' = \psi^{-1}(U)$ and $V' = \psi^{-1}(V)$.

We know that $A = \Gamma(U', \mathcal{O}_C)$ is the integral closure of $k[x]$ in $K(C)$ and $B = \Gamma(V', \mathcal{O}_C)$ is the integral closure of $k[1/x]$ in $K(C)$ since $C$ is isomorphic to its normalization. Lemma 1 immediately gives us the generators for $A$. To find the generators for $B$, let $\alpha = (y/x^n)$ be the root of the irreducible polynomial $Z^n - u(\prod_{j=1}^{n-1} f_j)/x^{n s} \in k[1/x, Z]$, where $s = [\deg(f)/n]$. We can use Lemma 1 to compute the integral closure of $k[1/x, \alpha]$. Since the integral closure of $k[1/x]$ in $K(C)$ is the smallest integrally closed ring in $K(C)$ that contains $k[1/x]$, this computation is all we need. Rearranging the basis elements for $B$ using elementary algebra gives us the desired form where $m_i = s - [i s - i \deg(f)/n] - \sum_{j=1}^{n-1} \deg(f_j) ji/n_j$. We get the definition of $m_i$ used above by the equality $i s - [i s - i \deg(f)/n] = [i \deg(f)/n]$.

**Lemma 3.** Using the same notation, let $C$ be a Kummer cover of degree $n$ over $k$.

1. $H^1(C, \mathcal{O}_C) = \sum_{i=1}^{n-1} \sum_{l=1}^{m_i-1} \frac{y^i}{\prod_{j=1}^{n-1} f_j^{ji/n_j}} \cdot \frac{1}{x^l} \cdot k$.

2. The genus of $C$ is $g(C) = \left( \sum_{i=1}^{n-1} m_i \right) - n + 1$. Moreover,

$$0 \leq g(C) \leq \frac{1}{2}(n - 1)(\deg(f) - 1).$$

3. Let $\text{char}(k) = p$. The induced Frobenius map $F$ on $H^1(C, \mathcal{O}_C)$ is determined by

$$\frac{y^i}{\prod_{j=1}^{n-1} f_j^{ji/n_j}} \cdot \frac{1}{x^l} \mapsto \sum_{w=1}^{m_{(pi \mod n)}-1} c_w \cdot \frac{y^{(pi \mod n)}}{\prod_{j=1}^{n-1} f_j^{(pi \mod n) j/n}} \cdot \frac{1}{x^w},$$

where $c_w$ is the coefficient of $x^{pt-w}$ in $f^{(pi/n)}/\prod_{j=1}^{n-1} f_j^{(pi \mod n) j/n}$. 

\[\square\]
Proof. (1) Let $R = \Gamma(U' \cap V', \mathcal{O}_C)$ and note that $R$ is the integral closure of $k[x, 1/x]$ in $K(C)$. Lemma 1 tells us that $R$ is generated as a $k[x, 1/x]$-module by the same set of generators that formed $A$ as a $k[x]$-module. We form the standard Čech complex

$$A \oplus B \xrightarrow{d} R \xrightarrow{coker(d)} 0$$

and the result is immediate after taking the quotient.

(2) The genus formula is obvious from part (1). The upper bound for $g(C)$ comes from considering $f$ to be square-free: it is clear we obtain the largest possible $m_i$ in this case, and hence the largest possible $g(C)$ for fixed $n$ and $\deg(f)$. An obvious lower bound for the genus of a Kummer cover with a square-free $f$ is obtained by replacing $m_i$ with $i \cdot \deg(f)/n$, which gives us $\frac{1}{2}(n-1)(\deg(f) - 2) \leq g(C)$. The upper bound comes from the basic numerical fact that $\sum_{i=1}^{n-1} (m_i - (i) \deg(f)/n) \leq (n-1)/2$, which is added to the formula for the lower bound.

(3) We can determine the action of $F$ on $H^1(C, \mathcal{O}_C)$ by the action of Frobenius on the Čech complex $A \oplus B \to R$. Since $F$ is semi-linear on $k$, it is completely determined by its action on the basis vectors of $H^1(C, \mathcal{O}_C)$. To determine the action of $F$ on a basis vector, let Frob denote the absolute Frobenius map on $C$ and look at the following commutative diagram.

$$
\begin{array}{ccc}
R & \xrightarrow{Frob} & R \\
\downarrow{\text{coker}(d)} & & \downarrow{\text{coker}(d)} \\
H^1(C, \mathcal{O}_C) & \xrightarrow{\text{Frob}} & H^1(C, \mathcal{O}_C)
\end{array}
$$

We have already computed the basis vectors of $H^1(C, \mathcal{O}_C)$ as the images of elements of $R$ under $\text{coker}(d)$ of the form

$$\frac{y^i}{\prod_{j=1}^{n-1} f^{[ji/n]}}, \quad \frac{1}{x^t}$$

To compute the action of $F$ on a basis vector of $H^1(C, \mathcal{O}_C)$, we will simply apply Frob to the above term of $R$ and then apply $\text{coker}(d)$. Applying Frob obviously gives us

$$\frac{y^i}{\prod_{j=1}^{n-1} f^{[ji/n]}} \cdot \frac{1}{x^t} \mapsto \left(\frac{y^i}{\prod_{j=1}^{n-1} f^{[ji/n]}} \cdot \frac{1}{x^t}\right)^p,$$

and we have that the image is equal to

$$\frac{f^{[pi/n]}}{\prod_{j=1}^{n-1} f^{[ji/n]}} \cdot \frac{1}{x^{pt}}$$

by elementary algebra. If we let $Q_i$ denote the leftmost term, we see that $Q_i \in k[x]$ since $j[pi/n] \geq p[ji/n] - [j(pi \mod n)/n]$. To finish the calculation, we take the image of the above expression under $\text{coker}(d)$, which is clearly $0$ if $m(pi \mod n) \leq 1$. If $m(pi \mod n) > 1$, the image is the sum of the terms

$$\left[ e_w \cdot \frac{y^{(pi \mod n)}}{\prod_{j=1}^{n-1} f^{[ji \mod n]/n}} \cdot \frac{1}{x^w} \right]$$

for $w = 1, \ldots, m(pi \mod n) - 1$ and $e_w$ the coefficient of $x^{pt-w}$ as a term of $Q_i$. \qed
Remark. The bounds given for $g(C)$ in (2) are sharp. The lower bound occurs for curves with affine equations $y^n = x^j$ for $j > 0$. The upper bound occurs for all Kummer covers with an affine equation of the form $y^n = f(x)$, where $f$ is separable and $\deg(f)$ is coprime to $n$. Also, we have seen that the computation of the $\check{\text{C}}$ech map involves the polynomial

$$Q_i = f^{[pi/n]} / \prod_{j=1}^{n-1} f_j^{p^{[ji/n]} - [j:(pi \mod n)/n]} \in k[x].$$

It is important to note that the exponents of the $1/f_j$ terms may be negative.

We now turn our attention to the $a$-number and $p$-rank of a Kummer cover. To define these numbers, we will need some facts about semi-linear maps. Recall that a semi-linear map of a $k$ vector space $L : V \to V$ is an additive map satisfying $L(\lambda x) = \theta(\lambda)L(x)$ for some $\theta \in \text{End}(k)$. For any semi-linear map $L$, the set $\ker(L)$ is a vector space over $k$ and $\text{im}(L)$ is a vector space over $\theta(k)$. Since it is more desirable to view the image of an $L$ as a vector space over $k$, we define $\text{im}_k(L) = \text{im}(L) \otimes_{\theta(k)} k$.

Many of the decomposition theorems from linear algebra carry over to semi-linear maps. Recall that Rank-Nullity holds for $L$ in the sense that $\dim_k \ker(L) = \dim_k \text{im}_k(L)$. We also have that $\ker(L^m)$ stabilizes for some $m \geq 0$, where the smallest such $m$ is denoted by $i(L)$ and called the index of $L$. Finally, the Range-Nullspace decomposition tells us that $L|_{\ker(L^m)}$ is nilpotent and $\dim_k \text{im}_k(L^m) = \dim_k \text{im}_k(L^{m+1})$. Of course, the semi-linear map we are interested in is $F$ acting on $H^1(C, \mathcal{O}_C)$, where $\theta$ is $\lambda \mapsto \lambda^p$ on $k$.

From this point on, we assume that $\text{char}(k) = p > 0$. The semi-simple rank of $F$ is $\text{rk}(F) = \dim_k \text{im}_k(F)$. The $a$-number $a(C)$ of a curve $C$ over $k$ is $a(C) = \dim_k \ker(F)$. Rank-Nullity gives us the relation $\text{rk}(F) = g(C) - a(C)$. The $p$-rank $f(C)$ of $C$ is $f(C) = \text{rk}(F^m)$ for any $m \geq i(F)$. This is well-defined because $\ker(F^m)$ stabilizes. Moreover, it is easy to see that $i(F) \leq g(C)$, so we can always take $m$ to be $g(C)$ in the definition of $f(C)$. The integers $\text{rk}(F)$, $a(C)$, and $f(C)$ are all between 0 and $g(C)$. The curve $C$ is called superspecial if $F = 0$.

The partition of $\mathbb{Z}/n\mathbb{Z}$ into subsets via the action of multiplication by $p$ plays an important role in our next result. We fix the notation for this as follows.

**Notation 1.** Let $S = \mathbb{Z}/n\mathbb{Z} - \{0\}$ and let $G$ be the cyclic group $\{p^q : q \geq 0\} \subset (\mathbb{Z}/n\mathbb{Z})^\times$. Consider the group action of $G$ on $S$ given by $p^q \cdot s = p^q s \mod n$. Let $S/G$ be the set of distinct orbits of this action.

**Theorem 1.** Using the same notation, let $C$ be a Kummer cover over $k$ of degree $n$. Set $B_i = \text{span}_k \left( \prod_{j=1}^{n-1} \frac{y^j}{f_j} \right)^{m_i-1}$.

1. $F^q(B_i) \subset B_{(ip^q \mod n)}$ for $q > 0$.
2. $\text{rk}(F) = \sum_{i=1}^{n-1} \text{rk}(F|_{B_i})$.
3. $f(C) = \sum_{\Omega \in S/G} \text{rk}(F^m|_{B_i})$, where $m \geq i(F)$ and $i \in \Omega$ is any element.

**Proof.** Since $\sum_{i=1}^{n-1} B_i = H^1(C, \mathcal{O}_C)$ by the first part of Lemma 3, (2) and (3) follow from (1), so we prove (1). Part (3) of Lemma 3 tells us that the action of $F$ takes $m_i - 1$ basis vectors and maps them to $m_i(p \mod n) - 1$ number of basis vectors. Since multiplication by $p$ defines a bijection from $\mathbb{Z}/n\mathbb{Z}$ to itself, the $m_i - 1$ number of vectors are the only terms...
to be mapped to the \( m_{(pi \mod n)} - 1 \) number of vectors. This proves \( F(B_i) \subset B_{(pi \mod n)} \). Iterating \( F \) finishes the proof. \( \square \)

**Example.** Let \( C \) be the Kummer cover defined by \( y^{11} = x^2(x + 1) \) over a field of characteristic 13 that contains the 11th roots of unity. We will show that \( a(C) = 1 \) and \( f(C) = 0 \) using the theorem. The orbit of 1 under the action of \( G \) on \( S \) is \( \{1, 2, 4, 8, 5, 10, 7, 3, 6\} \). Thus, \( S/G \) consists of the single orbit \( S \). Moreover, the set \( \{4, 5, 8, 9, 10\} \) consists of all values of \( i < 11 \) where \( m_i > 1 \); since \( m_i = 2 \) for these values, \( g(C) = 5 \). On the other hand, since \( S/G \) consists of a single orbit and there is some \( j \) for which \( m_j = 1 \), the image of \( F^q(B_i) \) passes through a zero dimensional \( B_j \) for some iteration \( q \) for any \( i \). Hence, \( f(C) = 0 \). Since \( m_7 = 1 \) and \( 9 \) maps to \( 7 \) in one iteration, \( \text{rk}(F|_{B_9}) = 0 \), so we can compute \( \text{rk}(F) \) by taking the sum of \( \text{rk}(F|_{B_i}) \) for \( i \in \{4, 5, 8, 10\} \). To determine \( \text{rk}(F|_{B_i}) \), all we need to know is if the coefficient \( a_{i,12} \) of \( x^{12} \) in \( Q_i \) is zero or not. A simple computation reveals \( a_{4,12} = 4, a_{5,12} = 5, a_{8,12} = 10, \) and \( a_{10,12} = 3 \). Thus, \( \text{rk}(F) = 4 \) and \( a(C) = 1 \). We see that \( C \) is an example of a curve of genus 5 with \( a \)-number 1 and \( p \)-rank 0.

**Bounds for the invariants**

Using Theorem 1, we can easily produce the following bounds. The group action plays an important role in the calculation of the \( p \)-rank.

**Corollary 1.** Using the notation introduced in Lemma 2 and Notation 7, let \( C \) be a Kummer cover of degree \( n \) over \( k \).

1. \( a(C) \geq 1 - n + \sum_{i=1}^{n-1} \max\{1, m_i - m_{(pi \mod n)} + 1\} \).
2. \( f(C) \leq \sum_{\Omega \in S/G} \min_{i \in \Omega} \{m_i - 1\} \).

**Proof.** (1) Using part (1) of Theorem 1 and Rank-Nullity, we have the bound \( \text{rk}(F|_{B_i}) \leq \min\{\dim_k B_i, \dim_k B_{(pi \mod n)}\} = \min\{m_i - 1, m_{(pi \mod n)} - 1\} \). We get the lower bound for \( a(C) \) by subtracting the upper bound of \( \text{rk}(F) \) from \( g(C) \) given in part (2) of Lemma 3.

(2) Taking iterations in part (1) and using the Range-Nullspace decomposition, we have \( \text{rk}(F|_{B_i}) \leq \min\{\dim_k B_i, \dim_k B_{(pi \mod n)}, \dim_k B_{p^2i \mod n}, \ldots\} = \min_{i \in \Omega} \{m_i - 1\} \) where \( \Omega \) is the action of \( G \) on \( i \). \( \square \)

**Example.** The upper bounds are sharp. For instance, take \( C \) to be the curve \( y^6 = x^3 + x^2 + 1 \) over a field \( k \) of characteristic 5 that contains the 6th roots of unity. In this case, \( G = \{1, 5\} \) and \( S/G \) consists of the orbits \( \{3\}, \{1, 5\}, \) and \( \{2, 4\} \). Only \( i \) in \( \{3, 4, 5\} \) satisfies \( m_i > 1 \), where \( m_3 = m_4 = 2 \) and \( m_5 = 3 \). From this information alone, we obtain the following: \( g(C) = 4, f(C) \leq 1, \text{rk}(F) \leq 1, a(C) \geq 3, \) and \( F = F|_{B_5} \). The action of \( F \) on \( B_3 \) is easy to determine: it is multiplication by the coefficient \( a_4 \) of \( Q_3 \), which is 1. Thus, our bounds are all met. We see that \( C \) is an example of a curve of genus 4 with \( a \)-number 3 and \( p \)-rank 0.

**Corollary 2.** Using the notation introduced in Lemma 2, let \( C \) be a Kummer cover of degree \( n \) over \( k \).

\[ a(C) \leq 1 - n + \sum_{i=1}^{n-1} \min\{m_i, \max\{m_i - q_i + v_i, 1 + v_i\}\}. \]
where
\[ q_i = \lfloor (\deg(Q_i) + m_{(pi \mod n)} - 1)/p \rfloor \quad \text{and} \quad v_i = \lfloor \deg(Q_i)/p \rfloor. \]

**Proof.** Our task is to compute a lower bound for \( \text{rk}(F|_{B_i}) \). The entries in \( F|_{B_i} \) come from the coefficients of the polynomial \( Q_i \) as described by Lemma 3. Let \( c \) denote the leading coefficient of \( Q_i \). We will exploit the following fact: when \( c \) is used in \( F|_{B_i} \), we can use row-reduction to easily see that it must contribute 1 to the rank (indeed, any coefficient of \( Q_i \) can be used at most once on any given row and all entries below those coming from \( c \) are zero). This means we get a lower bound for \( \text{rk}(F|_{B_i}) \) by counting the minimal number of rows where \( c \) must occur; we compute this number as follows. The integer \( v_i \) is the largest possible row of \( F|_{B_i} \) where \( c \) may not occur since \( pv_i \leq \deg(Q_i) \). The largest row of \( F|_{B_i} \) where \( c \) must occur is \( q_i \) since \( p(q_i + 1) - m_{(pi \mod n)} + 1 > \deg(Q_i) \). Thus,
\[
\max\{0, \min\{q_i - v_i, m_i - 1 - v_i\}\} \leq \text{rk}(F|_{B_i}).
\]
We conclude by taking the sum over \( i \) of this lower bound for \( \text{rk}(F|_{B_i}) \) and subtracting it from \( g(C) \) as we did before. □

This lower bound can be made much stronger for superelliptic curves, see [4].

**Hyperelliptic Curves**

In this section, we look at hyperelliptic curves over an algebraically closed field \( k \). Since \( a(C) \) and \( f(C) \) are invariants under separable base extension, the assumption that \( k \) is algebraically closed is no loss of generality for our purposes. Hyperelliptic curves are Kummer covers in every characteristic except 2, so we only need to extend our results to characteristic 2.

**Lemma 4.** Let \( C \) be a hyperelliptic curve of genus \( g = g(C) \) over an algebraically closed field \( k \) of characteristic 2. Assume that \( C \) is ramified at infinity.

1. \( C \) has an affine cover consisting of two parts \( U' = \text{Spec } A \) and \( V' = \text{Spec } B \), where
\[
A = k[x,y]/(y^2 + Qy - P),
\]
\[
B = k\left[\frac{1}{x}, \frac{y}{x^{g+1}}\right]/(\frac{y^2}{x^{2g+2}} + \frac{Qy}{x^{2g+2}} - \frac{P}{x^{2g+2}}),
\]
and where \( Q, P \in k[x] \) satisfy \( \deg(Q) \leq g, \deg(P) = 2g + 1, \) and \( Q \) is coprime to \((Q')^2P + (P')^2\).

2. \( H^1(C, \mathcal{O}_C) = \sum_{i=1}^g k \cdot y/x^i \). The induced action of Frobenius is given by
\[
y/x^i \mapsto \sum_{j=1}^g c_{i,j}y/x^j,
\]
where \( c_{i,j} \) is the coefficient of \( x^{2i-j} \) as a term of \( Q \).

3. If \( f(C) = 0 \), then \( a(C) = \lfloor \frac{x^{g+1}}{x^2} \rfloor \).
Proof. (1) See Proposition 7.4.24 of [8].

(2) We have that 
\[ R = \Gamma(U' \cap V', \mathcal{O}_C) = k[x, 1/x, y]/(y^2 + Qy - P). \]
The result follows by forming the standard Čech complex \( A \oplus B \to R \) and passing to the quotient. As for the action induced by Frobenius, if we square \( y/x^i \) in \( H^1(C, \mathcal{O}_C) \), we have the coset relation 
\[ [(Qy - P)/x^{2i}] = [Qy/x^{2i}] + \sum_{j=1}^{g} c_{i,j} y/x^j, \]
where \( c_{i,j} \) is the coefficient of \( Q \) as stated.

(3) View \( F \) as the \( g \times g \) matrix \( (c_{i,j}) \) and use the notation \( F[i, j] \) to denote \( c_{i,j} \). Part (3) tells us that the \( c_{i,j} \) are coefficients of the polynomial \( Q = \sum a_i x^i \) of degree at most \( g \).

Since \( f(C) = 0 \), we also know that \( F \) is nilpotent. We have that 
\[ (F^n)[g, g] = a_0^n, \]
which forces \( a_g = 0 \). Using this, we continue our elimination: we have that 
\[ a_{g-1} = 0, \]
\[ (F^n)[g - 2, g - 2] = a_{g-2} = 0, \]
and so on, until we have 
\[ (F^n)[1, 1] = a_1^n, \]
which forces \( a_1 = 0 \). Hence, the only \( Q \) that satisfies a nilpotent \( F \) is the constant \( Q = a_0 \). It must be non-zero because 0 is not coprime to \( (P^*)^2 \). If \( g \) is even, \( a_0 \) appears on \( g/2 \) rows. If \( g \) is odd, \( a_0 \) appears on \( (g - 1)/2 \) rows. \( \square \)

A much stronger version of (3) has been proved by G. van der Geer (see Lemma 11.1 of [6]).

Ekedahl’s bound \( g(C) < (p+1)/2 \) for superspecial hyperelliptic curves is an immediate consequence of part (3) and Corollary [2] when we take \( C \to \mathbb{P}^1 \) to be ramified over infinity. It is well-known that this bound is sharp. What we want to know is if all the numbers below Ekedahl’s bound occur as the genus of some superspecial hyperelliptic curve in characteristic \( p \). For \( g(C) = 2 \) and \( p > 3 \), such curves exist by a result of Ibukiyama, Katsura, and Oort in [7]. The case \( g(C) = 3 \) and \( p > 5 \) follows from a result in [4]. Despite these early successes, we will show that there are gaps for genus 4 in the next example by showing that there is no superspecial hyperelliptic curve of genus 4 in characteristic 11.

Example. Assume that \( C \) is a superspecial hyperelliptic curve of genus 4 over an algebraically closed field of characteristic 11. Use a fractional linear transformation of \( C \) to force 0 and infinity to be ramification points. Using Lemma [2], \( C \) has an affine equation of the form
\[ y^2 = f(x) = a_1 x + \cdots + a_9 x^9, \]
with \( a_1 \neq 0 \) and \( a_9 \neq 0 \). Lemma [3] tells us that \( f(x)^5 = \sum b_i x^i \) has \( b_j = 0 \) for \( j \in \{7, 8, 9, 10, 11, 18, 19, 20, 21\} \). Since \( 0 = b_7 = 10a_1^2 a_2^2 + 5a_4a_3 \), we have the relation \( a_3 = -2a_2^2/a_1 \). Likewise, \( a_4 = -3a_2^3/a_1^2 \), \( a_5 = -6a_2^4/5a_1^3 \), and \( a_6 = -8a_2^5/5a_1^4 \). For \( b_{18} \), we have
\[ \frac{7a_2^{13}}{a_1^8} + \frac{8a_2^7 a_7}{a_1^4} + 8a_1^2 a_2 a_7^2 + 4a_2^5 a_8}{a_1^4} + 9a_1^3 a_7 a_9 = 0. \]
This breaks down into three possible statements that we enumerate and eliminate below.

I. \( a_8 = 0 \) and \( -7a_2^{13} - 8a_5 a_7 a_7 - 8a_1^{10} a_2 a_7 = 0 \). This gives us
\[ 0 = b_{19} = \frac{9a_1^{14}}{a_1^2} + \frac{4a_1^8 a_7}{a_1^4} + 3a_1 a_2 a_7^2 + 4a_2^5 a_9}{a_1^2} + 9a_1^3 a_7 a_9. \]
Since \( a_9 \neq 0 \), we have two subcases from the condition above. We enumerate and eliminate them.

I.a. \( a_7 = -4a_2^7/9a_1^7 \) and \( a_2^{14} = 0 \). This case is eliminated because \( 0 = b_{21} = 10a_1^4 a_2^2 \), so either \( a_1 \) or \( a_9 \) is zero, which is not possible.

I.b. \( a_9 = -9a_1^{14} + 4a_1^8 a_7^2 + 3a_1^{10} a_2 a_7^2)}/a_1^2 = 4a_2^5 + 9a_1^3 a_7 \). Returning to \( b_{18} = 0 \), this forces \( a_2 = 0 \), which in turn forces \( a_9 = 0 \).

II. \( a_7 = -4a_2^6/9a_1^6 \) and \( a_2 = 0 \). We have that the relation \( b_{19} = 0 \) yields \( a_8 = 0 \), and then \( b_{21} = 0 \) forces either \( a_1 \) or \( a_9 \) to be zero.
III. \( a_8 = -(7a_2^{13} + 8a_5^5a_2^2a_7 + 8a_2^{10}a_2a_7^2)/(a_5^6(4a_2^6 + 9a_5^5a_7)). \) The relation \( b_{19} = 0 \), gives us the following subcases.

III.a. \( a_9 = 0 \) and \(-5a_2^{14} - 7a_1^{10}a_2^2a_7^2 = 0\). This case is impossible because \( a_9 \neq 0 \).

III.b. \( a_7 = -4a_2^5/9a_5^3 \) and \(-5a_2^{14} - 7a_1^{10}a_2^2a_7^2 = 0\). This case is impossible because this definition of \( a_7 \) conflicts with the definition of \( a_8 \) (it causes a division by 0).

III.c. \( a_9 = -(5a_2^{14} + 7a_1^{10}a_2^2a_7^2)/(a_5^1(4a_2^6 + 9a_5^5a_7)). \) We have that

\[
0 = b_{20} = \frac{7a_1^{15}a_2^5 + 8a_5^{30}a_2^3a_7 + 8a_1^{35}a_2^2a_7^3}{a_1^{35}}.
\]

On the other hand,

\[
0 = b_{21} = \frac{5a_1^{24}a_2^{16} + a_1^{20}a_2^{10}a_7 + a_1^{34}a_2^4a_7^2}{a_1^{35}}.
\]

Combining the two yields \( a_2 = 0 \), which forces \( a_9 = 0 \).

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(Extensive details and references included for academic rigor and completeness.)