CERTAIN SUBCLASSES OF BI-UNIVALENT
FUNCTIONS ASSOCIATED WITH HORADAM POLYNOMIALS

K. Dhanalakshmi\textsuperscript{1}, D. Kavitha\textsuperscript{2}§, A. Anbukkarasi\textsuperscript{3}

\textsuperscript{1}K. Dhanalakshmi PG
and Research Department of Mathematics
Theivanai Ammal College for Women (Autonomous)
Villupuram 605602, Tamilnadu, INDIA

\textsuperscript{2}Department of Mathematics
Audisankara College of Engineering and Technology (Autonomous)
Gudur-524101, Nellore District
Andra Pradesh, INDIA

\textsuperscript{3}A. Anbukkarasi Department of Mathematics
IFET College of Engineering (Autonomous Institution)
Villupuram 605 108, Tamilnadu, INDIA

Abstract: In this present paper, our goal is to introduce two new subclasses of analytic bi-univalent functions defined by means of Horadam polynomials in the open unit disc $U$. Also we find initial estimates on Taylor-Maclaurin coefficients and provided the relevant Fekete-Szegö theorem using coefficient estimates for the defined new subclasses.

AMS Subject Classification: 30C45, 30C50, 11B39

Key Words: univalent function; starlike and convex functions; subordination; Horadam polynomials; Fekete-Szegö problem

Received: July 18, 2020

\textsuperscript{§}Correspondence author
1. Introduction and preliminaries

Let $\mathcal{A}$ denote the family of normalized analytic functions $f$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U})$$

in the open disc $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are also univalent in $\mathbb{U}$.

The well-known Koebe one-quarter theorem [4] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disc of radius $1/4$. Hence every univalent function $f$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z, (z \in \mathbb{U})$ and

$$f^{-1}(f(w)) = w, (|w| < r_0(f), r_0(f) \geq 1/4),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). For example, functions in the class $\Sigma$ are given below [16]:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2} \log \left( \frac{1+z}{1-z} \right).$$

In 1967, Lewin [12] introduced the class $\Sigma$ of bi-univalent functions and shown that $|a_2| < 1.51$. In 1969, Netanyahu [14] showed that $\max_{f \in \Sigma} |a_2| = 4/3$ and Suffridge [18] have given an example of $f \in \Sigma$ for which $|a_2| = 4/3$. Later, in 1980, Brannan and Clunie [1] improved the result as $|a_2| \leq \sqrt{2}$. In 1985, Kedzierawski [11] proved this conjecture for a special case when the function $f$ and $f^{-1}$ are starlike. In 1984, Tan [19] proved that $|a_2| \leq 1.485$ which is the best estimate for the function in the class of bi-univalent functions.

Recently, many authors have introduced and studied various subclasses of analytic and bi-univalent functions. For some of the recent analysis in this topic, see e.g. [5, 6, 20, 17]. Brannan and Taha [3] introduced certain subclasses of the bi-univalent function class $\Sigma$ for the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$. Ali et al. [2] widen the result of Brannan and Taha using subordination.

Let the functions $f$ and $g$ be analytic in $\mathbb{U}$. Then we say that $f$ is subordinate to $g$, if there exits a Schwarz function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$, such that $f(z) = g(\omega(z)), z \in \mathbb{U}$. We denote this subordination by $f \prec g$.

The recurrence relation for the Horadam polynomials $h_n(x)$ was studied by Horzum and Koçer [9], as
\[ h_n(x) = px h_{n-1}(x) + q h_{n-2}(x); \quad (n \in \mathbb{N} \geq 2) \]  

(3)

with

\[ h_1(x) = a, \quad h_2(x) = bx, \quad h_3(x) = pbx^2 + aq, \]  

(4)

where \( a, b, p \) and \( q \) are some real constants. Taking various values of \( a, b, p \) and \( q \) leads to various polynomials:

- When \( a = b = p = q = 1 \), we obtain the Fibonacci polynomials,

  \[ F_n(x) = x F_{n-1}(x) + F_{n-2}(x), F_1(x) = 1, F_2(x) = x. \]

- When \( a = 2, b = p = q = 1 \), we have the Lucas polynomials,

  \[ L_{n-1}(x) = x L_{n-2}(x) + L_{n-3}(x), L_0 = 2, L_1 = x. \]

- When \( a = q = 1, b = p = 2 \), we attain the Pell polynomials,

  \[ P_n(x) = 2x P_{n-1}(x) + P_{n-2}(x), p_1 = 1, p_2 = 2x. \]

- When \( a = b = p = 2, q = 1 \), we get the Pell-Lucas polynomials,

  \[ Q_{n-1}(x) = 2x Q_{n-2}(x) + Q_{n-3}(x), Q_0 = 2, Q_1 = 2x. \]

- When \( a = 1, b = p = 2, q = 1 \), we obtain the Chebyshev polynomials of second kind sequence,

  \[ U_{n-1}(x) = 2x U_{n-2}(x) + U_{n-3}(x), U_0 = 1, U_1 = 2x. \]

- If \( a = 1, b = p = 2, q = 1 \), we have the Chebyshev polynomials of First kind sequence,

  \[ T_{n-1}(x) = 2x T_{n-2}(x) + T_{n-3}(x), T_0 = 1, T_1 = x. \]

One can refer to [7],[8] and [13] for more details related to these polynomials’ succession. Also, we can refer to [10] and [15] for the Horadam polynomials connected with bi-univalent functions.

**Remark 1.** ([9]) Let \( \Omega(x,z) \) be the generating function of the Horadam polynomials \( h_n(x) \). Then

\[ \Omega(x, z) = \frac{a + (b - ap)xz}{1 - pxz - qz^2} = \sum_{n=1}^{\infty} h_n(x) z^{n-1}. \]  

(5)
We use the Horadam polynomials $h_n(x)$ and the generating function $\Omega(x,z)$, given by the recurrence relation (3) and (5), respectively. Motivated by this result, we now introduce the new subclasses of bi-univalent function class.

**Definition 2.** A function $f \in \Sigma$ of the form (1) is said to be in the class $\mathcal{N}_\Sigma(\lambda, \delta; x)$, for $\lambda \geq 1$ and $\delta \geq 0$, if the following conditions are satisfied:

$$
\left(1 - \lambda \right) \frac{f(z)}{z} + \lambda f'(z) + \delta zf''(z) \prec \Pi(x, z) + 1 - a \quad (z \in \mathbb{U}) \quad (6)
$$

and for $g(\omega) = f^{-1}(\omega)$,

$$
\left(1 - \lambda \right) \frac{f(\omega)}{\omega} + \lambda f'(\omega) + \delta zf''(\omega) \prec \Pi(x, \omega) + 1 - a \quad (\omega \in \mathbb{U}), \quad (7)
$$

where the real constants $a$ and $b$ are as in (4).

Now, we can define how the subclass of the above defined bi-univalent analytic function $\mathcal{N}_\Sigma(\lambda, \delta; x)$ are lead to the new subclasses of analytic bi-univalent functions with suitable choice of parameters $\lambda$ and $\delta$.

**Remark 3.** Let $\delta = 0$ and $\lambda \geq 1$, one can easily see that the function $f \in \Sigma$ is in $\mathcal{N}_\Sigma(1, 0; x)$ if the following conditions are satisfied:

$$
\left(1 - \lambda \right) \frac{f(z)}{z} + \lambda f'(z) \prec \Pi(x, z) + 1 - a \quad (z \in \mathbb{U})
$$

and

$$
\left(1 - \lambda \right) \frac{f(\omega)}{\omega} + \lambda f'(\omega) \prec \Pi(x, \omega) + 1 - a \quad (\omega \in \mathbb{U}),
$$

where $g(\omega) = f^{-1}(\omega)$ defined by (2).

**Remark 4.** For $\lambda = 1$ and $\delta \geq 0$, we get the bi-univalent function class $f \in \Sigma$ is in $\mathcal{N}_\Sigma(1, \delta; x)$ if the following conditions are satisfied:

$$
\left( f'(z) + \delta zf''(z) \right) \prec \Pi(x, z) + 1 - a \quad (z \in \mathbb{U})
$$

and

$$
\left( f'(\omega) + \delta zf''(\omega) \right) \prec \Pi(x, \omega) + 1 - a \quad (\omega \in \mathbb{U}),
$$

where $g(\omega) = f^{-1}(\omega)$ defined by (2).
Remark 5. Let $\delta = 0$ and $\lambda = 1$, we have the bi-univalent function class $f \in \Sigma$ is in $\mathcal{N}_\Sigma(1, 0; x)$ if the following conditions are satisfied:

$$f'(z) \prec \Pi(x, z) + 1 - a \quad (z \in \mathbb{U})$$

and

$$f'(\omega) \prec \Pi(x, \omega) + 1 - a \quad (\omega \in \mathbb{U}),$$

where $g(\omega) = f^{-1}(\omega)$ defined by (2).

Definition 6. Let $\alpha \geq 0$, a function $f \in \Sigma$ is said to be in the class $\mathcal{F}_\Sigma(\alpha; x)$ if the following conditions hold true:

$$\frac{(zf'(z) + \alpha z^2 f''(z))'}{f'(z)} \prec \Pi(x, z) + 1 - a \quad (z \in \mathbb{U}) \quad (8)$$

and for $g(\omega) = f^{-1}(\omega)$

$$\frac{(\omega f'(\omega) + \alpha \omega^2 f''(\omega))'}{f'(\omega)} \prec \Pi(x, \omega) + 1 - a \quad (\omega \in \mathbb{U}), \quad (9)$$

where the real constants $a$ and $b$ are as in (4).

It can be clearly seen that the parameter $\alpha = 0$, the above subclass of bi-univalent function reduced to the subclass of bi-convex function $f \in \Sigma$ and defined as follows:

Remark 7. Let $\alpha = 0$, then the class $\mathcal{F}_\Sigma(\alpha; x)$ is reduced to $\mathcal{C}_\Sigma(x)$ and defined by

$$\frac{(zf'(z))'}{f'(z)} \prec \Pi(x, z) + 1 - a \quad (z \in \mathbb{U})$$

and for $g(\omega) = f^{-1}(\omega)$

$$\frac{(\omega f'(\omega))'}{f'(\omega)} \prec \Pi(x, \omega) + 1 - a \quad (\omega \in \mathbb{U}).$$
2. A set of main results

We begin this section by establishing the coefficient estimates $|a_2|$ and $|a_3|$ followed by the classical result of Fekete-Szegö problem for the function classes $N_{\Sigma}(\lambda, \delta; x)$ and $\mathfrak{F}_{\Sigma}(\alpha; x)$, respectively.

**Theorem 8.** Let the function $f \in \Sigma$ given by (1) be in the class $N_{\Sigma}(\lambda, \delta; x)$, then

$$
|a_2| \leq \frac{|bx| \sqrt{|bx|}}{\sqrt{|[(1 + 2\lambda + 6\delta)b - p(1 + \lambda + 2\delta)^2]bx^2 - qa(1 + \lambda + 2\delta)^2|}} \quad (10)
$$

and

$$
|a_3| \leq \frac{|bx|}{(1 + 2\lambda + 6\delta)} + \frac{b^2x^2}{(1 + \lambda + 2\delta)^2} \quad (11)
$$

For $\nu \in \mathbb{R}$,

$$
|a_3 - \nu a_2^2| \leq \begin{cases} 
\frac{|bx|}{(1 + 2\lambda + 6\delta)}; & |\nu - 1| \leq \mathcal{B}, \\
\frac{|bx|^3|\nu - 1|}{|[(1 + 2\lambda + 6\delta)b - p(1 + \lambda + 2\delta)^2]bx^2 - qa(1 + \lambda + 2\delta)^2|^2}; & |\nu - 1| \geq \mathcal{B},
\end{cases}
$$

where

$$
\mathcal{B} = \sqrt{\frac{|[(1 + 2\lambda + 6\delta)b - p(1 + \lambda + 2\delta)^2]bx^2 - qa(1 + \lambda + 2\delta)^2|}{b^2x^2(1 + 2\lambda + 6\delta)}}.
$$

**Proof.** Let $f \in N_{\Sigma}(\lambda, \delta; x)$. It is well known that there are two analytic functions $u, v : U \to U$ given by

$$
U(z) = |u_1z + u_2z^2 + u_3z^3 + \ldots| < 1 \quad (12)
$$

and

$$
V(\omega) = |v_1\omega + v_2\omega^2 + v_3\omega^3 + \ldots| < 1, \quad (13)
$$

for all $z, \omega \in \mathbb{U}$ with $U(0) = V(0) = 0, |U(z)| < 1, |V(z)| < 1$ such that

$$
\left( (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) + \delta zf''(z) \right) = \Pi(x, U(z)) + 1 - a
$$
and
\[
\left( (1 - \lambda) \frac{f(\omega)}{\omega} + \lambda f'(\omega) + \delta z f''(\omega) \right) = \Pi(x, V(\omega)) + 1 - a.
\]
Equivalently the above equation can be written as
\[
\left( (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) \right) = 1 + h_1(x) - a + h_2(x)U(z) + h_3(x)[U(z)]^2 + \ldots.
\]
and
\[
\left( (1 - \lambda) \frac{f(\omega)}{\omega} + \lambda f'(\omega) + \delta z f''(\omega) \right) = 1 + h_1(x) - a + h_2(x)V(\omega) + h_3(x)[V(\omega)]^2 + \ldots.
\]
Making use of (12) and (13), we have
\[
\left( (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) \right) = 1 + h_2(x)u_1z + [h_2(x)u_2 + h_3(x)u_1^2]z^2 + \ldots.
\]
and
\[
\left( (1 - \lambda) \frac{f(\omega)}{\omega} + \lambda f'(\omega) + \delta z f''(\omega) \right) = 1 + h_2(x)v_1\omega + [h_2(x)v_2 + h_3(x)v_1^2]\omega^2 + \ldots.
\]
It is fairly known that
\[
|u_i| \leq 1, \quad |v_i| \leq 1, \quad (i \in \mathbb{N}).
\]
Now comparing the corresponding coefficients of (14) and (15), we have
\[
(1 + \lambda + 2\delta)a_2 = h_2(x)u_1
\]
(17)\[
(1 + 2\lambda + 6\delta)a_3 = h_2(x)u_2 + h_3(x)u_1^2
\]
(18)\[
-(1 + \lambda + 2\delta)a_2 = h_2(x)v_1
\]
(19)\[
(1 + 2\lambda + 6\delta)(2a_2^2 - a_3) = h_2(x)v_2 + h_3(x)v_1^2.
\]
From (16) and (18), we can observe that
\[
u_1 = -v_1
\]
and

$$a_2^2 = \frac{[h_2(x)]^2(u_2^2 + v_1^2)}{2(1 + \lambda + 2\delta)^2}.$$  (21)

Adding (17) and (19), we get

$$2(1 + 2\lambda + 6\delta)a_2^2 = h_2(x)(u_2 + v_2) + h_3(x)(u_2^2 + v_2^2).$$  (22)

Substituting (21) in (22), we have

$$a_2^2 = \frac{h_2(x)^3(u_2 + v_2)}{2(1 + 2\lambda + 6\delta)[h_2(x)]^2 - 2h_3(x)(1 + \lambda + 2\delta)^2}.$$  (23)

Using (4), the above equation yields,

$$|a_2| \leq \frac{|bx|\sqrt{bx}}{\sqrt{[(1 + 2\lambda + 6\delta)b - p(1 + \lambda + 2\delta)^2]bx^2 - qa(1 + \lambda + 2\delta)^2}}.$$  (24)

Similarly, upon subtracting (19) from (17) and in view of (20), we obtain

$$2(1 + 2\lambda + 6\delta)a_3 - 2(1 + 2\lambda + 6\delta)a_2^2 = h_2(x)(u_2 - v_2) + h_3(x)(u_1^2 - v_1^2),$$

$$a_3 = \frac{h_2(x)(u_2 - v_2)}{2(1 + 2\lambda + 6\delta)} + a_2^2.$$  (25)

In view of (21), equation (25) becomes

$$a_3 = \frac{h_2(x)(u_2 - v_2)}{2(1 + 2\lambda + 6\delta)} + \frac{[h_2(x)]^2(u_1^2 + v_1^2)}{2(1 + \lambda + 2\delta)^2}.$$  (26)

Applying (4), we deduce that

$$|a_3| \leq \frac{|bx|}{(1 + 2\lambda + 6\delta)} + \frac{b^2x^2}{(1 + \lambda + 2\delta)^2}.$$  (27)

For any $\nu \in \mathbb{R}$,

$$a_3 - \nu a_2^2 = \frac{h_2(x)(u_2 - v_2)}{2(1 + 2\lambda + 6\delta)} + (1 - \nu)a_2^2.$$  (28)

Substituting (23) in (27), we have

$$a_3 - \nu a_2^2 = h_2(x) \left\{ \left( \Omega(\nu, x) + \frac{1}{2(1 + 2\lambda + 6\delta)} \right) u_2 ight.$$  

$$+ \left( \Omega(\nu, x) - \frac{1}{2(1 - 2\lambda + 6\delta)} \right) v_2 \right\},$$  (29)
CERTAIN SUBCLASSES OF BI-UNIVALENT...

where
\[ \Omega(\nu, x) = \frac{(1 - \nu)[h_2(x)]^2}{2[(1 + 2\lambda + 6\delta)h_2(x)^2 - (1 + \lambda + 2\delta)^2h_3(x)]}. \]

Hence, in view of (4), we conclude that
\[
|a_3 - \nu a_2^2| \leq \begin{cases} 
\frac{|h_2(x)|}{(1 + 2\lambda + 6\delta)}; & 0 \leq |\Omega(\nu, x)| \leq \frac{1}{2(1 + 2\lambda + 6\delta)} \\
2|h_2(x)|\Omega(\nu, x); & |\Omega(\nu, x)| \geq \frac{1}{2(1 + 2\lambda + 6\delta)} 
\end{cases}
\]

which completes the proof of Theorem 8.

\[ \square \]

**Corollary 9.** Let \( f \) be given by (1) in the class \( N_\Sigma(\lambda, 0; x) \). Then,
\[
|a_2| \leq \frac{|bx|\sqrt{bx}}{\sqrt{|(1 + 2\lambda)b - p(1 + \lambda)^2|bx^2 - qa(1 + \lambda)^2|}}
\]
and
\[
|a_3| \leq \frac{|bx|}{(1 + 2\lambda)} + \frac{b^2x^2}{(1 + \lambda)^2}.
\]

For any \( \nu \in \mathbb{R} \),
\[
|a_3 - \nu a_2^2| \leq \begin{cases} 
\frac{|bx|}{(1 + 2\lambda)}; & |\nu - 1| \leq \frac{[(1 + 2\lambda)h_2(x)^2 - (1 + \lambda)^2h_3(x)]}{b^2x^2(1 + 2\lambda)} \\
\frac{|bx|}{3(1 + 2\delta)}|\nu - 1|; & |\nu - 1| \geq \frac{[(1 + 2\lambda)h_2(x)^2 - (1 + \lambda)^2h_3(x)]}{b^2x^2(1 + 2\lambda)}
\end{cases}
\]

**Corollary 10.** Let \( f \) given by (1) be in the class \( N_\Sigma(1, \delta; x) \). Then,
\[
|a_2| \leq \frac{|bx|\sqrt{bx}}{\sqrt{|3(1 + 2\delta)b - 4p(1 + \delta)^2|bx^2 - 4qa(1 + \delta)^2|}}
\]
and
\[
|a_3| \leq \frac{|bx|}{3(1 + 2\delta)} + \frac{b^2x^2}{4(1 + \delta)^2}.
\]
For any $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2|^2 \leq \begin{cases} 
\frac{|bx|}{3(1 + 2\delta)}; \\
|\nu - 1| \leq \frac{[3(1 + 2\delta)h_2(x)^2 - 4(1 + \delta)^2h_3(x)]}{3b^2x^2(1 + 2\delta)} \\
|bx|^3|\nu - 1| \\
\frac{|3(1 + 2\delta)b - 4p(1 + \delta)^2|}{|3b - 4p|bx^2 - 4qa(1 + \delta)^2}; \\
|\nu - 1| \geq \frac{[3(1 + 2\delta)h_2(x)^2 - 4(1 + \delta)^2h_3(x)]}{3b^2x^2(1 + 2\delta)}.
\end{cases}$$

Corollary 11. Let $f$ given by (1) be in the class $N_\Sigma(1, 0; x)$. Then,

$$|a_2| \leq \frac{|bx| \sqrt{bx}}{\sqrt{|3b - 4p|bx^2 - 4qa}}$$

and

$$|a_3| \leq \frac{|bx|}{3} + \frac{b^2x^2}{4}.$$ 

For any $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2|^2 \leq \begin{cases} 
\frac{|bx|}{3(1 + 2\delta)}; \\
|\nu - 1| \leq \frac{[3h_2(x)^2 - 4h_3(x)]}{b^2x^23} \\
\frac{|bx|^3|\nu - 1|}{|3b - 4p|bx^2 - 4qa}; \\
|\nu - 1| \geq \frac{[3h_2(x)^2 - 4h_3(x)]}{b^2x^2}.
\end{cases}$$

Theorem 12. Let the function $f \in \Sigma$ given by (1) be in the class $\Sigma_\Sigma(\alpha; x)$, then

$$|a_2| \leq \frac{|bx| \sqrt{bx}}{\sqrt{2}|[(1 + 5\alpha)b - 2p(1 + \alpha)^2|bx^2 - 2qa(1 + 2\alpha)^2|}}$$

and

$$|a_3| \leq \frac{|bx|}{6(1 + 3\alpha)} + \frac{b^2x^2}{4(1 + 2\alpha)^2},$$

and for $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2|^2 \leq \begin{cases} 
\frac{|bx|}{6(1 + 3\alpha)}; \\
|\nu - 1| \leq B_1, \\
\frac{|bx|^3|\nu - 1|}{2|[(1 + 5\alpha)b - 2p(1 + 2\alpha)^2|bx^2 - 2qa(1 + 2\alpha)^2|}, \\
|\nu - 1| \geq B_1,
\end{cases}$$
where
\[ B_1 = \frac{[(1 + 5\alpha)b - 2p(1 + 2\alpha)^2]bx^2 - 2aq(1 + 2\alpha)^2}{3b^2x^2(1 + 3\alpha)}. \]

**Proof.** Let \( f \in \mathfrak{F}_\Sigma(\alpha; x) \) be given by Taylor-Maclaurin expansion (1). Then for all \( z, \omega \in U \) with \( U(0) = V(0) = 0, |U(z)| < 1, |V(z)| < 1 \) such that
\[
\frac{\left(zf'(z) + \alpha z^2f''(z)\right)}{f'(z)} = \Pi(x, U(z)) + 1 - a
\]
and
\[
\frac{\left(\omega f'(\omega) + \alpha \omega^2 f''(\omega)\right)}{f'(\omega)} = \Pi(x, V(\omega)) + 1 - a.
\]
Equivalently the above equation can be written as
\[
\frac{\left(zf'(z) + \alpha z^2f''(z)\right)}{f'(z)} = 1 + h_1(x) - a + h_2(x)U(z) + h_3(x)[U(z)]^2 + \ldots
\]
and
\[
\frac{\left(\omega f'(\omega) + \alpha \omega^2 f''(\omega)\right)}{f'(\omega)} = 1 + h_1(x) - a + h_2(x)V(\omega) + h_3(x)[V(\omega)]^2 + \ldots.
\]
Using (12) and (13), we have
\[
\frac{\left(zf'(z) + \alpha z^2f''(z)\right)}{f'(z)} = 1 + h_2(x)u_1z + [h_2(x)u_2 + h_3(x)u_1^2]z^2 + \ldots \quad (31)
\]
and
\[
\frac{\left(\omega f'(\omega) + \alpha \omega^2 f''(\omega)\right)}{f'(\omega)} = 1 + h_2(x)v_1\omega + [h_2(x)v_2 + h_3(x)v_1^2]\omega^2 + \ldots, \quad (32)
\]
it is clear that
\[
|u_i| \leq 1, \quad |v_i| \leq 1, \quad (i \in \mathbb{N}).
\]
Now comparing the corresponding coefficients of (31) and (32), we have
\[
2(1 + 2\alpha)a_2 = h_2(x)u_1 \quad (34)
\]
\[6(1 + 3\alpha)a_3 - 4(1 + 2\alpha)a_2^2 = h_2(x)u_2 + h_3(x)u_1^2 \quad (35)\]
\[-2(1 + 2\alpha)a_2 = h_2(x)v_1 \quad (36)\]
\[4(2 + 7\alpha)a_2^2 - 6(1 + 3\alpha)a_3 = h_2(x)v_2 + h_3(x)v_1^2. \quad (37)\]

From (34) and (36), we can observe that
\[u_1 = -v_1 \quad (38)\]
and
\[a_2^2 = \frac{[h_2(x)]^2(u_2^2 + v_1^2)}{8(1 + 2\alpha)^2}. \quad (39)\]

Adding (35) and (37), we get
\[4(1 + 5\alpha)a_2^2 = h_2(x)(u_2 + v_2) + h_3(x)(u_2^2 + v_2^2). \quad (40)\]

Substituting (39) in (40), we have
\[a_2^2 = \frac{h_2(x)^3(u_2 + v_2)}{4(1 + 5\alpha)[h_2(x)]^2 - 8(1 + 2\alpha)^2h_3(x)}. \quad (41)\]

Using (4) the above equation yields,
\[|a_2| \leq \frac{|bx|\sqrt{bx}}{\sqrt{2} \left[ (1 + 5\alpha)b - 2p(1 + \alpha)^2 \right] b^2x^2 - 2qa(1 + 2\alpha)^2}. \quad (42)\]

Similarly, upon subtracting (37) from (35) and in view of (38), we obtain
\[2(1 + 2\lambda + 6\delta)a_3 - 2(1 + 2\lambda + 6\delta)a_2^2 = h_2(x)(u_2 - v_2) + h_3(x)(u_1^2 - v_1^2), \quad a_3 = \frac{h_2(x)(u_2 - v_2)}{12(1 + 3\alpha)} + a_2^2. \quad (43)\]

In view of (39), equation (43) becomes
\[a_3 = \frac{h_2(x)(u_2 - v_2)}{12(1 + 3\alpha)} + \frac{[h_2(x)]^2(u_1^2 + v_1^2)}{8(1 + 2\alpha)^2}. \quad (44)\]

Applying (4), we deduce that
\[|a_3| \leq \frac{|bx|}{6(1 + 3\alpha)} + \frac{b^2x^2}{4(1 + 2\alpha)^2}. \quad (44)\]

For any \(\nu \in \mathbb{R}\),
\[a_3 - \nu a_2^2 = \frac{h_2(x)(u_2 - v_2)}{12(1 + 3\alpha)} + (1 - \nu)a_2^2. \quad (45)\]
Substituting (41) in (45), we have
\[
a_3 - \nu a_2^2 = h_2(x) \left\{ \left( \rho(\nu, x) + \frac{1}{12(1+3\alpha)} \right) u_2 + \left( \rho(\nu, x) - \frac{1}{12(1+3\alpha)} \right) v_2 \right\},
\]
(46)
where
\[
\rho(\nu, x) = \frac{(1 - \nu)[h_2(x)]^2}{4[(1 + 5\alpha)h_2(x)^2 - 2(1 + 2\alpha)^2h_3(x)]}.
\]
Hence in view of (4), we conclude that
\[
|a_3 - \nu a_2^2| \leq \begin{cases} 
\frac{|h_2(x)|}{6(1 + 3\alpha)}; & 0 \leq |\rho(\nu, x)| \leq \frac{1}{12(1 + 3\alpha)} \\
2|h_2(x)|\rho(\nu, x)|; & |\rho(\nu, x)| \geq \frac{1}{12(1 + 3\alpha)}
\end{cases}
\]
which completes the proof of Theorem 12.

**Corollary 13.** Let \( \alpha = 0 \) and \( f \) be in the class \( C_\Sigma(x) \). Then,
\[
|a_2| \leq \frac{|bx|\sqrt{bx}}{\sqrt{2|(b - 2p)bx^2 - 2qa|}}
\]
and
\[
|a_3| \leq \frac{|bx|}{6} + \frac{b^2x^2}{4}.
\]
For \( \nu \in \mathbb{R} \),
\[
|a_3 - \nu a_2^2| \leq \begin{cases} 
\frac{|bx|}{6}; & |\nu - 1| \leq \frac{(b - 2p)bx^2 - 2aq}{3b^2x^2} \\
|bx|^3|\nu - 1|; & |\nu - 1| \geq \frac{(b - 2p)bx^2 - 2aq}{3b^2x^2}
\end{cases}
\]

**References**

[1] D.A. Brannan, J. Clunie, *Aspects of Contemporary Complex Analysis*, Academic Press, New York-London (1980).

[2] R.M. Ali, L.S. Keong, V. Ravichandran, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, *Appl. Math. Lett.*, 25, No 3 (2012), 344-351.
[3] D.A. Brannan, T.S. Taha, On some classes of bi-univalent functions, *Studia Univ. Babes, Bolyai Math.*, **31**, No 2 (1986), 70-77.

[4] P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer-Verlag, New York (1983).

[5] B.A. Frasin, Coefficient bounds for certain classes of bi-univalent functions, *Hacet. J. Math. Stat.*, **43**, No 3 (2014), 383-389.

[6] T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, *Pan Amer. Math. J.*, **22**, No 4 (2012), 15-26.

[7] A.F. Horadam and J.M. Mahon, Pell and Pell-Lucas polynomials, *Fibonacci Quart.*, **23**, No 1 (1985), 7-20.

[8] A.F. Horadam, Jacobsthal representation polynomials, *The Fibonacci Quart.*, **35**, No 2 (1997), 137-148.

[9] T. Horzum and E.G. Kocer, On some properties of Horadam polynomials, *Int. Math. Forum*, **4**, No 25-28 (2009), 1243-1252.

[10] D. Kavitha, K. Dhanalakshmi and N. ArulMozhi, Coefficient estimates for certain subclasses of analytic functions associated with Horadam polynomial, *Adv. Math. Sci. J.*, **9**, No 12 (2020), 1-12.

[11] A. Kedzierawski, J. Waniurski, Bi-univalent polynomials of small degree, *Complex Variables Theory Appl.*, **10**, No 2-3 (1988), 97-100.

[12] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.*, **18** (1967), 63-68.

[13] A. Lupas, A guide of Fibonacci and Lucas polynomials, *Octogon Math. Mag.*, No 7 (1999), 2-12.

[14] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $z < 1$, *Arch. Rational Mech. Anal.*, No 32 (1969), 100-112.

[15] H.M Srivastava, S. Altunkaya and S. Yalçın, Certain subclasses of bi-Univalent functions associated with the Horadam polynomials, *Iran J. Sci. Technol Trans. Sci.*, **43**, No 4 (2019), 1873–1879.

[16] H.M. Srivastava, D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, *J. Egypt. Math. Soc.*, **23**, No 2 (2015), 242-246.
[17] H.M. Srivastava, S. Bulut, M. Caglar and N. Yagmur, Coefficient estimates for general subclass of analytic and bi-univalent functions, *Filomat*, **27**, No 5 (2013), 831-842.

[18] T.J. Suffridge, A coefficient problem for a class of univalent functions, *Michigan Math. J.*, **16**, (1969), 33-42.

[19] D.L. Tan, Coefficient estimates for bi-univalent functions, An English summary appears in: *Chinese Ann. Math. Ser. B*, **5**, No 5 (1984), 559-568.

[20] Q.-H. Xu, Y.-C. Gui and H.M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.*, **25**, No 6 (2012), 990-994.
