AN ANALYSIS OF TUBERCULOSIS MODEL WITH
EXPONENTIAL DECAY LAW OPERATOR

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Abstract. In this paper, we explore the dynamics of tuberculosis (TB) epidemic model that includes the recruitment rate in both susceptible and infected population. Stability and sensitivity analysis of the classical TB model is carried out. Caputo-Fabrizio (CF) operator is then used to explain the dynamics of the TB model. The concept of fixed point theory is employed to obtain the existence and uniqueness of the solution of the TB model in the light of CF operator. Numerical simulations based on Homotopy Analysis Transform Method (HATM) and padé approximations are performed to obtain qualitative information on the model. Numerical solutions depict that the order of the fractional derivative has great dynamics of the TB model.

1. Introduction. Tuberculosis (TB) is one of the dangerous diseases which has become a global burden in the 21st century. It is an airborne disease usually transmitted through the contact of an infected person via such as coughing, sneezing, spitting, etc [28]. The World Health Organization (WHO) observed that currently 10.4 million people are affected by TB and 1.7 million people approximately died. The high mortality rate of the disease can be found in the most developed country such as Nigeria, Pakistan, Ghana, etc [28, 11]. These developing countries constitute about 60% of the global burden associated with TB [16]. However, United Nation (UN) Sustainable development goal envisages that by 2035 the global TB related issue would reduce to 90% [28, 11].

Obtaining qualitative information on TB dynamics would help health practitioners to take a decision that would minimize the spread of the epidemics. The mathematical model plays a crucial role in enhancing the qualitative information about the disease which leads to minimizing the spread of the disease [30]. Modeling in all forms has played a major in controlling many epidemics on the globe. Because in the absence of real data models provide both qualitative and quantitative information that helps in minimizing the spread of many diseases [5].

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There have been countless number integer models on TB in the world [20, 21, 4, 2, 27, 29, 10]. However, due to setback of integer models which do not have hereditary memory effect in order to predict accurately. Fractional-order derivative has because of a powerful tool in modeling in recent times because of its characterization. These operators possess memory effect crossover property and have statistical interpretation which makes the operators efficient [12, 13, 14, 15].

There are different fractional-order derivative and the most common one is that of Caputo derivative which is just a power law. Because its kernel is not non-local singular but many researchers believe that this operator is just a filter of parameters, not a fractional derivative [24, 19, 3, 22]. A fractional-order derivative which non-singular Kernel proposed by Caputo and Fabrizio [7]. In recent times, further properties of the CF operators has been developed by Losada and Nieto [18]. The effectiveness of the CF operators has been illustrated by many researchers [23, 6, 17, 25]. Singh et al., [23] employed CF operator to explore the dynamics of giving up the smoking model and concluded that the new operator is effective in modeling real-world problems. Bushnaq et al., [6] utilized CF fractional derivative to get insight into the dynamics of HIV/AIDS. Kumar et al., [17] examine a new fractional SIRS − SI malaria model based on Caputo-Fabrizio operator. Ullah et al., [25] explored a CF fractional model for the dynamics of TB infection and observed that the operator is capable to model complex phenomena.

The aim of this work is to make use of a generalized function that is based on exponential law in order to obtain qualitative information about TB dynamics.

1.1. Mathematical preliminaries.

**Definition 1.1.** Let \( \Omega \in H'(l_1, l_2), l_2 > l_1, \sigma \in (0, 1] \). The Caputo-Fabrizio (CF) operator [7, 8] is given by

\[
D^\sigma_{t} \Omega (t) = \frac{M(\sigma)}{1-\sigma} \int_{l_1}^{t} \Omega^1 (\theta) \exp \left[ -\sigma \frac{t-\theta}{1-\sigma} \right] d\theta, \quad 0 < \sigma < 1
\]  

(1)

In this instance, \( M(\sigma) \) meets the condition \( M(0) = M(1) = 1 \).

**Definition 1.2.** The integral operator of the CF fractional derivative is express as [7, 8]

\[
I^{\sigma}_{t} \Omega (t) = \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \Omega (t) + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_{0}^{t} \Omega (\varepsilon) d\varepsilon, \quad t \geq 0
\]  

(2)

**Definition 1.3.** The Laplace transform of the operator CF is given by

\[
L \left[ \frac{CF}{0} D^\sigma_{t} \Omega (t) \right] = M(\sigma) \frac{sL \left[ \Omega (t) \right] - \Omega (0)}{s + \sigma (1-s)}
\]  

(3)

2. Mathematical model formulation. In this model, we modified the TB model constructed by Adebiyi [1] and included the inflow into infectious compartments. The total human population \( N(t) \) is partitioned into susceptible individuals, latent individual \( L \), infectious individuals \( I \), and treatment individual \( T \). Thus the total population is \( N(t) = S(t) + L(t) + I(t) + T(t) \). The human recruitment rates into susceptible and infectious class are \( \Lambda_0 \) and \( \Lambda_1 \) respectively.

The disease-induced death rate and natural mortality rate are denoted by \( d \) and \( \mu \) in that order. \( \beta \) is the chance that a susceptible individual becomes infected after effective interaction with an infected individual per contact-per unit of time.
α is the chance that treated individual (T) turns to become infected after effective contact with an infectious individual per contact-per unit time. The per-capita contact rate for the treated individual is c. The rate an individual move from the Latent (L) compartment to the infectious compartment is R and the treatment rate the infectious individual is r.

Based on Figure 1, the TB model is given by the following differential equations

\[
\begin{align*}
\frac{dS}{dt} &= \Lambda_0 - \beta cSI - \mu S \\
\frac{dL}{dt} &= \beta cSI + \alpha cIT - (\mu + k)L \\
\frac{dI}{dt} &= \Lambda_1 + kL - (d + r + \mu)I \\
\frac{dT}{dt} &= rI - \alpha cIT - \mu T
\end{align*}
\]  

(4)

3. Model stability and equilibrium. The steady states of the system model (4) are obtained by equating the model to zero and solving for each system variable. Model (4) does not have a disease-free equilibrium due to the inflow into infectious compartments. Hence, we consider the special case where there is no inflow of infectious compartments or mathematically \( \Lambda_1 = 0 \). For this case, the disease-free equilibrium (DFE) of system (4) is given by \( E^0 = (S^0, L^0, I^0, T^0) = \left( \frac{\Lambda_0}{\mu}, 0, 0, 0 \right) \).

Using the approach in [26], the threshold number \( R_0 \) that measures the spread of TB in a virgin population is given by:

\[
R_0 = \frac{\beta ck \Lambda_0}{\mu(k + \mu)(d + \mu + r)}. 
\]  

(5)

3.1. Local stability of DFE model. The local stability of DFE of the TB model (4) will be explored.

**Theorem 3.1.** The DFE \( E^0 \) of the TB model (4) is locally asymptotically stable (LAS) if \( R_0 < 1 \).
Proof 3.1. The Jacobian matrix of the model (4) calculated at $E^0$ is

$$
J_{E^0} = \begin{pmatrix}
-\mu & 0 & -\frac{\beta c A_0}{\mu} & 0 \\
0 & -k - \mu & \frac{\beta c A_0}{\mu} & 0 \\
0 & k & -d - \gamma - u & 0 \\
0 & 0 & \gamma & -\mu \\
\end{pmatrix},
$$

(6)

with characteristic equation

$$(\lambda + \mu)^2 (\lambda^2 + A_1 \lambda + A_0) = 0$$

(7)

where

$$A_1 = d + k + u + \gamma + \mu,$$

$$A_0 = (d + u + \gamma)(k + \mu)(1 - R_0).$$

(8)

Since, the eigenvalue $-\mu$ (twice) is negative and $A_1, A_0$ are positive if $R_0 < 1$. So, according to the Routh-Hurtwiz criteria, DFE of the model (4) is LAS if $R_0 < 1$.

3.2. Global stability analysis of DFE model. Here, we use the Castillo-Chavez [9] technique. We rewrite the model (4):

$$
\frac{dX(t)}{dt} = K(X,Z),
$$

$$
\frac{dZ(t)}{dt} = M(X,Z), M(X,0) = 0,
$$

(9)

where $X = S \in R_+^+$ and $Z = (L,I,T) \in R_+^3$ show the uninfected and infected classes respectively. The DFE $E^0 = (X^0, Z^0) = (X^0,0)$ and the existence of its GAS depends on conditions $(C_1)$ and $(C_2)$, which need to be satisfied, where,

$$(C_1) : \text{For } \frac{dX(t)}{dt} = K(X,0), X^0 \text{ is GAS},$$

$$(C_2) : \text{For } M(X,Z) = -M_0 Z - M_1(X,Z),$$

where $M_1(X,Z) \geq 0$ for all $(X,Z) \in \Omega$ (biological feasible region) and $M_0 = D \rho M(X^0,0)$ is an M-matrix.

Lemma 3.1. The DFE $E^0 = (X^0, Z^0) = (X^0,0)$ of the TB model (4) is globally asymptotically stable (GAS) if $R_0 < 1$ and $(C_1), (C_2)$ satisfied.

Theorem 3.2. If $R_0 < 1$, then $E^0$ is GAS.

Proof 3.2. Let $X = S \in R_+^+$ and $Z = (L,I,T) \in R_+^3$ with DFE $E^0$. We get

$$
\frac{dX(t)}{dt} = K(X,0),
$$

$$
\frac{dX(t)}{dt} = \Lambda_0 - \mu X^0.
$$

(10)

As $t \to \infty$, $X \to X^0$. So $X = X^0 = S^0$ is GAS.

Now,

$$
M_0 = \begin{pmatrix}
k + \mu & -\beta c S^0 & 0 \\
-k & d + \mu + u & 0 \\
0 & -\gamma & \mu \\
\end{pmatrix},
$$

(11)
and $Z = (L, I, T)^t$. Clearly $\mathcal{M}_0$ is an M-matrix and $\mathcal{M}_1(X, Z) \geq 0$ as $S^0$ is an upper bound. Hence, $(C_1), (C_2)$ are satisfied and this completes the proof.

3.3. Local stability of endemic equilibrium. The endemic equilibrium of TB system (4) is given by $E^* = (S^*, L^*, I^*, T^*)$ with

$$S^* = \frac{\Lambda_0}{\beta c I^* + \mu},$$

$$L^* = \frac{(\beta c S^* + \alpha c T^*) I^*}{\mu + k},$$

$$I^* = \frac{k L^* + \Lambda_1}{d + \mu + r},$$

$$T^* = \frac{\gamma I^*}{\alpha c I^* + \mu}.$$

The following theorem provides the stability of the endemic equilibrium.

**Theorem 3.3.** The equilibrium $E^*$ is asymptotically stable if all of the eigenvalues $\phi_i$ of Jacobian matrix $J(E^*)$ satisfy: $\phi_i < 0, i = 1, 2, 3, 4$.

**Proof.** The Jacobian matrix $J(E^*)$ TB model system equation (4) computed at the TB endemic equilibrium $E^*$ is expressed as the following

$$J(E^*) = \begin{pmatrix} -n_1 & 0 & -\beta c S^* & 0 \\ \beta c I^* & -n_2 & n_4 & \alpha c I^* \\ 0 & k & -n_3 & 0 \\ 0 & 0 & r - \alpha c T^* & -\mu \end{pmatrix}$$

with $n_1 = \beta c I^* + \mu, n_2 = \mu + k, n_3 = d + r + \mu$, and $n_4 = \beta c S^* + \alpha c T^*$.

It is clearly noticed that one of the negative eigenvalues are $-\mu$ and the rest of other three eigenvalues can be obtained from the roots of the characteristic equation

$$\lambda^3 + b_1 \lambda^2 + b_2 \lambda + b_3 = 0,$$

where

$$b_1 = n_1 + n_2 + n_3,$$

$$b_2 = n_1 n_2 + n_1 n_3 + n_2 n_3,$$

$$b_3 = k n_1 n_4 + \beta_2 c^2 S^* I^*.$$

Using the Routh-Hurwitz condition, the characteristic equation (14) has three eigenvalues with negative real part if only if $b_1, b_2, b_3 > 0$ and $b_1 b_2 > b_3$. $\square$
4. Sensitivity analysis. In this section, we examine the parameters behavior based on the reproduction number $R_0$ as given by

$$\frac{\partial R_0}{\partial c} = \frac{\beta k \Lambda_0}{\mu (k+\mu)(d+\mu+r)} > 0$$

$$\frac{\partial R_0}{\partial S} = \frac{c k \Lambda_0}{\mu (k+\mu)(d+\mu+r)} > 0$$

$$\frac{\partial R_0}{\partial k} = \frac{\beta c k \Lambda_0}{(k+\mu)^2 (d+\mu+r)} > 0$$

$$\frac{\partial R_0}{\partial \Lambda_0} = \frac{\beta c k \Lambda_0}{\mu (k+\mu)(d+\mu+r)} > 0$$

$$\frac{\partial R_0}{\partial d} = -\frac{\beta c k \Lambda_0}{\mu (k+\mu)(d+\mu+r)^2} < 0$$

$$\frac{\partial R_0}{\partial r} = -\frac{\beta c k \Lambda_0}{\mu^2 (k+\mu)^2 (d+\mu+r)^2} < 0$$

$$\frac{\partial R_0}{\partial \mu} = -\frac{\beta c k \Lambda_0 (2\mu (d+k+r)+k(d+r)+3\mu^2)}{\mu^2 (k+\mu)^2 (d+\mu+r)^2} < 0$$

Thus $R_0$ is rising up with the following parameters $c, k, \beta, \Lambda_0$, and reducing with the following parameters $r, \mu, d$.

5. Fractional mathematics model of Tuberculosis. Integer models have been identified as a medium that cannot capture complex phenomena. Non-integer order models have the capacity to incorporate memory effects in predicting outcomes. In recent times, Caputo-Fabrizio used exponential decay law to develop a new operator which has no singularity. Most natural phenomena obey the law and therefore, we explore CF operator derivative in studying TB and the following equations are obtained.

$$^{CF}D_0^\sigma S = \Lambda_0 - \beta c S I - \mu S$$

$$^{CF}D_0^\sigma L = \beta c S I + \alpha c I T - (\mu + k) L$$

$$^{CF}D_0^\sigma I = \Lambda_1 + k L - (d + r + \mu) I$$

$$^{CF}D_0^\sigma T = r I - \alpha c I T - \mu T$$

Suppose that $G$ is the Banach space which is of a continuous real-valued function and

$$(S, L, I, T) = \|S\| + \|L\| + \|I\| + \|T\|$$

expressed on the interval path corresponding norm.

Based on equation (17) we obtain

$$\|S\| = \sup \{ S(t) : t \in P \}, \|L\| = \sup \{ L(t) : t \in P \}, \|I\| = \sup \{ I(t) : t \in P \},$$

$$\|T\| = \sup \{ T(t) : t \in P \}$$

Simplify, $P = E(p) \times E(p) \times E(p) \times E(p)$, where $E(p)$ is the Banach space of continuous real-valued function on $P$ with its corresponding norm.
5.1. **Existence and uniqueness of the model.** It is essential to explore the existence and uniqueness of the TB model in the light of exponential decay law. This will assure the existence of a solution of the TB model and fixed point theory [18] is employed to carry out this aspect of the work.

The fractional integral (2) is applied to equation (16) which leads to

\[ S(t) - S(0) = 0_0^{C} \int_{t}^{T} \{ \Lambda_0 - \beta cSI - \mu S \} \]

\[ L(t) - L(0) = 0_0^{C} \int_{t}^{T} \{ \beta cSI + \alpha cI(T - \mu + k) L \} \]  

(18)

\[ I(t) - I(0) = 0_0^{C} \int_{t}^{T} \{ \Lambda_1 + kL - (d + r + \mu) I \} \]

\[ T(t) - T(0) = 0_0^{C} \int_{t}^{T} \{ rI - \alpha cIT - \mu T \} \]

Making use similar notations as in [18], we get

\[ S(t) - S(0) = \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \{ \Lambda_0 - \beta cS(t) I(t) - \mu S(t) \} \]

\[ + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_{0}^{t} \{ \Lambda_0 - \beta S(\varepsilon) I(\varepsilon) - \mu S(\varepsilon) \} d\varepsilon \]

\[ L(t) - L(0) = \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \{ \beta cS(t) + I(t) + \alpha cI(T(t) - \mu + k) L(t) \} \]

\[ + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_{0}^{t} \{ \beta cS(\varepsilon) I(\varepsilon) \} + \alpha cI(\varepsilon) T(\varepsilon) - (\mu + k) L(t) d\varepsilon \]  

(19)

\[ I(t) - I(0) = \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \{ \Lambda_1 + kL(t) - (d + r + \mu) I(t) \} \]

\[ + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_{0}^{t} \{ \Lambda_1 + kL(\varepsilon) - (d + r + \mu) I(\varepsilon) \} d\varepsilon \]

For simplicity purpose we express as

\[ \Psi_1(t, s) = \Lambda_0 - \beta cS(t) I(t) - \mu S, \]

\[ \Psi_2(t, L) = \beta cS(t) I(t) + \alpha cI(T(t) - (\mu + k) L(t), \]

\[ \Psi_3(t, I) = \Lambda_1 + kL(t) - (d + r + \mu) I(t), \]

\[ \Psi_4(t, T) = rI(t) - \alpha cIT(t) I(t) - \mu T(t). \]  

(20)

**Theorem 5.1.** The kernels \( \Psi_1, \Psi_2, \Psi_3 \) and \( \Psi_4 \) meet the condition of Lipschitz and condition if

\[ 0 \leq v \beta g_3 + \mu < 1 \]

**Proof.** Initially, we begin with \( \Psi_1 \). For two functions \( S \) and \( S_1 \) are arrived at

\[ \| \Psi_1(t, s) - \Psi_1(t, s) \| = \| \{ S(t) - S_1(t) \} (\beta cI(t) - \{ S(t) - S_1(t) \} \mu) \| \]  

(21)
Employing the properties of the norm on equation (21) produces
\[
\|\Psi_1(t, s) - \Psi_1(t, s)\| \leq \|\{S(t) - S_1(t)\} (\beta c I(t))\| + \{\{S(t) - S_1(t)\} \mu\|
\]
\[
\leq \{\beta c \|I(t)\| + \mu\} \|S(t) - S_1(t)\|
\]
\[
\leq \{\beta c + \mu\} \|S(t) - S_1(t)\|
\]
(22)

Considering \(q_1 = \beta cg_3 + \mu\) where \(\|S(t)\| \leq g_1, \|L(t)\| \leq g_2, \|I(t)\| \leq g_3,\) and \(\|T(t)\| \leq g_4\) are bounded functions, we obtain
\[
\|\Psi_1(t, s) - \Psi_1(t, s)\| \leq q_1 \|S(t) - S_1(t)\|
\]
(23)

In effect, the Lipschitz condition is fulfilled in \(\Psi_1\). In addition, if \(0 \leq vBcg_4 + \mu < 1\), can concludes that it is a contraction.

Using same approach one can show that the kernels \(\Psi_2(t, L), \Psi_3(t, I)\) and \(\Psi_4(t, T)\) fulfills the Lipschitz conditions
\[
\|\Psi_2(t, L) - \Psi_2(t, L_1)\| \leq q_2 \|L(t) - L_1(t)\|
\]
\[
\|\Psi_3(t, I) - \Psi_3(t, I_1)\| \leq q_3 \|I(t) - I_1(t)\|
\]
\[
\|\Psi_4(t, T) - \Psi_4(t, T_1)\| \leq q_4 \|T(t) - T_1(t)\|
\]
(24)

Making utilization of the notations of the previously stated kernels, equation (19) transforms into the system.
\[
S(t) = S(0) + \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \Psi_1(t, S) + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_0^t \Psi_1(\varepsilon, S) d\varepsilon,
\]
\[
L(t) = L(0) + \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \Psi_2(t, L) + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_0^t \Psi_2(\varepsilon, L) d\varepsilon,
\]
\[
I(t) = I(0) + \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \Psi_3(t, I) + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_0^t \Psi_3(\varepsilon, I) d\varepsilon,
\]
\[
T(t) = T(0) + \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \Psi_4(t, T) + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_0^t \Psi_4(\varepsilon, T) d\varepsilon,
\]
(25)

Following equation (25), the following recursive formulas are constructed
\[
S_n(t) = \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \Psi_1(t, S_{n-1}) + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_0^t \Psi_1(\varepsilon, S_{n-1}) d\varepsilon,
\]
\[
L_n(t) = \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \Psi_2(t, L_{n-1}) + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_0^t \Psi_2(\varepsilon, L_{n-1}) d\varepsilon,
\]
\[
I_n(t) = \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \Psi_3(t, I_{n-1}) + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_0^t \Psi_3(\varepsilon, I_{n-1}) d\varepsilon,
\]
\[
T_n(t) = \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \Psi_4(t, T_{n-1}) + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_0^t \Psi_4(\varepsilon, T_{n-1}) d\varepsilon
\]
(26)

with the associated initial conditions \(S_0 = S(0), L_0 = L(0), I_0 = I(0)\) and \(T_0 = T(0)\).
The difference between the successive terms are arranged as

\[
\begin{align*}
    w_{1n} (t) &= S_n (t) - S_{n-1} (t) \\
    &= \frac{2(1-\sigma)}{(2-\sigma) M^{(\sigma)}} \Psi_1 (t, S_{n-1}) - \Psi_1 (t, S_{n-2}) \\
    &+ \frac{2\sigma}{(2-\sigma) M^{(\sigma)}} \int_0^t (\Psi_1 (\varepsilon, S_{n-1}) - \Psi_1 (\varepsilon, S_{n-2})) d\varepsilon \\
    w_{2n} (t) &= L_n (t) - L_{n-1} (t) \\
    &= \frac{2(1-\sigma)}{(2-\sigma) M^{(\sigma)}} \Psi_2 (t, L_{n-1}) - \Psi_2 (t, L_{n-2}) \\
    &+ \frac{2\sigma}{(2-\sigma) M^{(\sigma)}} \int_0^t (\Psi_2 (\varepsilon, L_{n-1}) - \Psi_2 (\varepsilon, L_{n-2})) d\varepsilon \\
    w_{3n} (t) &= I_n (t) - I_{n-1} (t) \\
    &= \frac{2(1-\sigma)}{(2-\sigma) M^{(\sigma)}} \Psi_3 (t, I_{n-1}) - \Psi_3 (t, I_{n-2}) \\
    &+ \frac{2\sigma}{(2-\sigma) M^{(\sigma)}} \int_0^t (\Psi_3 (\varepsilon, I_{n-1}) - \Psi_3 (\varepsilon, I_{n-2})) d\varepsilon \\
    w_{4n} (t) &= T_n (t) - T_{n-1} (t) \\
    &= \frac{2(1-\sigma)}{(2-\sigma) M^{(\sigma)}} \Psi_4 (t, T_{n-1}) - \Psi_4 (t, T_{n-2}) \\
    &+ \frac{2\sigma}{(2-\sigma) M^{(\sigma)}} \int_0^t (\Psi_4 (\varepsilon, T_{n-1}) - \Psi_4 (\varepsilon, T_{n-2})) d\varepsilon.
\end{align*}
\]

(27)

It is interesting to note that

\[
S_n (t) = \sum_{i=0}^{n} w_{1i} (t), L_n (t) = \sum_{i=0}^{n} w_{2i} (t), I_n (t) = \sum_{i=0}^{n} w_{3i} (t), T_n (t) = \sum_{i=0}^{n} w_{4i} (t).
\]

(28)

It is easy to get the following results

\[
\begin{align*}
    \|w_{1n} (t)\| &= \|S_n (t) - S_{n-1} (t)\| \\
    &= \left\| \frac{2(1-\sigma)}{(2-\sigma) M^{(\sigma)}} (\Psi_1 (t, S_{n-1}) - \Psi_1 (t, S_{n-2})) \\
    &+ \frac{2\sigma}{(2-\sigma) M^{(\sigma)}} \int_0^t (\Psi_1 (\varepsilon, S_{n-1}) - \Psi_1 (\varepsilon, S_{n-2})) d\varepsilon \right\| \\
    &= \left\| \frac{2(1-\sigma)}{(2-\sigma) M^{(\sigma)}} (\Psi_1 (t, S_{n-1}) - \Psi_1 (t, S_{n-2})) \\
    &+ \frac{2\sigma}{(2-\sigma) M^{(\sigma)}} \int_0^t (\Psi_1 (\varepsilon, S_{n-1}) - \Psi_1 (\varepsilon, S_{n-2})) d\varepsilon \right\| \\
    &= \left\| \frac{2(1-\sigma)}{(2-\sigma) M^{(\sigma)}} (\Psi_1 (t, S_{n-1}) - \Psi_1 (t, S_{n-2})) \\
    &+ \frac{2\sigma}{(2-\sigma) M^{(\sigma)}} \int_0^t (\Psi_1 (\varepsilon, S_{n-1}) - \Psi_1 (\varepsilon, S_{n-2})) d\varepsilon \right\|
\end{align*}
\]

(29)

Utilizing triangle inequality principle in equation (29), we have

\[
\begin{align*}
    \|S_n (t) - S_{n-1} (t)\| &\leq \frac{2(1-\sigma)}{(2-\sigma) M^{(\sigma)}} (\Psi_1 (t, S_{n-1}) - \Psi_1 (t, S_{n-2})) \\
    &+ \frac{2\sigma}{(2-\sigma) M^{(\sigma)}} \int_0^t (\Psi_1 (\varepsilon, S_{n-1}) - \Psi_1 (\varepsilon, S_{n-2})) d\varepsilon
\end{align*}
\]

(30)

It has been shown that the kernels fulfill the Lipschitz conditions. Therefore equation (30) produces
Theorem 5.2. The TB model with \( \frac{1}{2-\sigma} \) fractional derivative presented in equation (25) possesses a solution if there exist to such that

\[
\frac{2 (1 - \sigma)}{(2 - \sigma) M(\sigma)} q_1 + \frac{2\sigma}{(2 - \sigma) M(\sigma)} q_1 t_0 < 1
\]

Proof. It is shown above that function \( S(t), L(t), I(t) \text{ and } T(t) \) are bounded. Utilizing the results given in equations (32)-(33) and applying the recursive algorithm, we have,

\[
\|w_{1n}(t)\| \leq \|S_n(0)\| \left[ \frac{2 (1 - \sigma)}{(2 - \sigma) M(\sigma)} q_1 + \frac{2\sigma}{(2 - \sigma) M(\sigma)} q_1 t \right]^n.
\]

\[
\|w_{2n}(t)\| \leq \|L_n(0)\| \left[ \frac{2 (1 - \sigma)}{(2 - \sigma) M(\sigma)} q_2 + \frac{2\sigma}{(2 - \sigma) M(\sigma)} q_2 t \right]^n.
\]

\[
\|w_{3n}(t)\| \leq \|I_n(0)\| \left[ \frac{2 (1 - \sigma)}{(2 - \sigma) M(\sigma)} q_3 + \frac{2\sigma}{(2 - \sigma) M(\sigma)} q_3 t \right]^n.
\]

\[
\|w_{4n}(t)\| \leq \|T_n(0)\| \left[ \frac{2 (1 - \sigma)}{(2 - \sigma) M(\sigma)} q_4 + \frac{2\sigma}{(2 - \sigma) M(\sigma)} q_4 t \right]^n.
\]

Thus the TB model solution exist and at the same time continuous. In order to establish that equation (25) depicts the solution of the model (16) we consider the following

\[
S(t) - S(0) = S_n(t) - E_n(t),
\]

\[
L(t) - L(0) = L_n(t) - F_n(t),
\]

\[
I(t) - I(0) = I_n(t) - G_n(t),
\]

\[
T(t) - T(0) = G_n(t) - H_n(t).
\]
Simply, we have
\[ \left\| E_n (t) \right\| = \left\| \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \left( \Psi_1 (t, S) - \Psi_1 (t, S_{n-1}) \right) \right. \]
\[ \left. + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_0^t \left( \Psi_1 (\varepsilon, S) - \Psi_1 (\varepsilon, S_{n-1}) \right) d\varepsilon \right\| \]
\[ \leq \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \left\| \Psi_1 (t, S) - \Psi_1 (t, S_{n-1}) \right\| \]
\[ + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_0^t \left\| \left( \Psi_1 (\varepsilon, S) - \Psi_1 (\varepsilon, S_{n-1}) \right) \right\| d\varepsilon \]
\[ \leq \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} q_1 \left\| S - S_{n-1} \right\| + \frac{2\sigma}{(2-\sigma)M(\sigma)} q_1 \left\| S - S_{n-1} \right\| t. \]

Employing same technique we have the following recursive solution
\[ \left\| E_n (t) \right\| \leq \left( \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} + \frac{2\sigma}{(2-\sigma)M(\sigma)} t \right) \left( \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} q_1 \left\| S - S_{n-1} \right\| + \frac{2\sigma}{(2-\sigma)M(\sigma)} q_1 \left\| S - S_{n-1} \right\| t \right)^{n+1} q_1 \]
\[ \leq \left( \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} q_1 \left\| S - S_{n-1} \right\| + \frac{2\sigma}{(2-\sigma)M(\sigma)} q_1 \left\| S - S_{n-1} \right\| t \right)^{n+1} q_1 \]
\[ \leq \left( \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} q_1 \left\| S - S_{n-1} \right\| + \frac{2\sigma}{(2-\sigma)M(\sigma)} q_1 \left\| S - S_{n-1} \right\| t \right)^{n+1} q_1 \]

Taking the limits with respect to equation (38) as \( n \) gets to infinity leads to
\[ \left\| E_n (t) \right\| \to 0 \]
In similar terms we have
\[ \left\| F_n (t) \right\| \to 0, \left\| G_n (t) \right\| \to 0 \]
and
\[ \left\| H_n (t) \right\| \to 0. \]
These results thus complete the existence of the theorem.

The uniqueness of the solution of the TB model (16) is examined. This is done by assuming that there exists another system solution of the TB model (16) \( S^* (t), L^* (t), I^* (t) \) and \( T^* (t) \). Therefore
\[ S (t) - S^* (t) = \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \left( \Psi_1 (t, S) - \Psi_1 (t, S^*) \right) \]
\[ + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_0^t \left( \Psi_1 (\varepsilon, S) - \Psi_1 (\varepsilon, S^*) \right) d\varepsilon \]

Considering the norms produces
\[ \left\| S (t) - S^* (t) \right\| \leq \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} \left\| \Psi_1 (t, S) - \Psi_1 (t, S^*) \right\| \]
\[ + \frac{2\sigma}{(2-\sigma)M(\sigma)} \int_0^t \left\| \left( \Psi_1 (\varepsilon, S) - \Psi_1 (\varepsilon, S^*) \right) \right\| d\varepsilon \]
Making use of the results obtained in equation (23) and (24) respectively we have
\[ \left\| S (t) - S^* (t) \right\| \leq \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)} q_1 \left\| S (t) - S^* (t) \right\| \]
\[ + \frac{2\sigma}{(2-\sigma)M(\sigma)} q_1 t \left\| S (t) - S^* (t) \right\| \]
Theorem 5.3. The fractional of the TB model (16) possesses a unique solution if
\[
\left( 1 - \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)}q_1 - \frac{2\sigma}{(2-\sigma)M(\sigma)}q_1t \right) < 0.
\] (43)

Proof. Based on equation (42) we get
\[
\| S(t) - S^*(t) \| \left( 1 - \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)}q_1 - \frac{2\sigma}{(2-\sigma)M(\sigma)}q_1t \right) \leq 0.
\] (44)

Employing equation (43) and the properties of a norm associated with equation (44) leads to
\[
\| S(t) - S^*(t) \| = 0.
\] (45)

It can be observed that
\[
S(t) = S^*(t).
\] (46)

In similar way it can be establish that
\[
L = L^*, I = I^*, T = T^*.
\] (47)

Therefore the TB model (16) possesses a unique solution.

6. HATM for fractional of TB model. In this section, Homotopy Analysis Transform Method (HATM) is used to study the dynamics of the TB model. Initially, we make use of Laplace transform to fractional of the TB model (16) which gives;
\[
\frac{sL[S] - S(0)}{s + \sigma(1-s)} = L[A_0 - \beta cSI - \mu S],
\]
\[
\frac{sL[L] - L(0)}{s + \sigma(1-s)} = L[\beta SI + \alpha cIT - (\mu + k) L],
\]
\[
\frac{sL[I] - I(0)}{s + \sigma(1-s)} = L[A_1 + kL - (d + r + \mu)],
\]
\[
\frac{sL[T] - T(0)}{s + \sigma(1-s)} = L[rI - \alpha cIT - \mu T].
\] (48)

Making further simplification in equation (47) we have the following:
\[
L[S] - \frac{b_1}{s} - \frac{[s+\sigma(1-s)]A_0}{s^2} - \frac{s+\sigma(1-s)}{s}L[-\beta cSI - \mu S] = 0,
\]
\[
L[L] - \frac{b_2}{s} - \frac{[s+\sigma(1-s)]A_1}{s^2} - \frac{s+\sigma(1-s)}{s}L[\beta SI + \alpha cIT - (\mu + k) L] = 0,
\]
\[
L[I] - \frac{b_3}{s} - \frac{[s+\sigma(1-s)]A_2}{s^2} - \frac{s+\sigma(1-s)}{s}L[kL - (d + r + \mu)] = 0,
\]
\[
L[T] - \frac{b_4}{s} - \frac{[s+\sigma(1-s)]A_3}{s^2} - \frac{s+\sigma(1-s)}{s}L[rI - \alpha cIT - \mu T] = 0.
\] (49)

The nonlinear operators are given by
The deformation equation of kth order is presented as

\[ N_1 (\Omega_1 (t; \nu)) = \mathbf{L} [\Omega_1 (t; \nu)] - \frac{b_1}{s} - \frac{[s + \sigma (1-s)]\Lambda_0}{s^2} \]

\[ - \frac{s + \sigma (1-s)}{s} L \left [ - \beta c \Omega_3 (t; z) \Omega_1 (t; z) - \mu \Omega_1 (t; \nu) \right ] = 0, \]

\[ N_2 (\Omega_2 (t; \nu)) = \mathbf{L} [\Omega_2 (t; \nu)] - \frac{b_2}{s} \]

\[ - \frac{s + \sigma (1-s)}{s} L \left [ - \beta c \Omega_2 (t; \nu) \Omega_3 (t; \nu) + \alpha c \Omega_2 (t; \nu) \Omega_4 (t; \nu) - (\mu + k) \Omega_2 (t; \nu) \right ] = 0, \]

\[ N_3 (\Omega_3 (t; \nu)) = \mathbf{L} [\Omega_3 (t; \nu)] - \frac{b_3}{s} - \frac{[s + \sigma (1-s)]\Lambda_1}{s^2} \]

\[ - \frac{s + \sigma (1-s)}{s} L \left [ k \Omega_2 (t; z) - (d + r + \mu) \Omega (t; \nu) \right ] = 0, \]

\[ N_4 (\Omega_4 (t; \nu)) = \mathbf{L} [\Omega_4 (t; \nu)] \]

\[ - \frac{b_4}{s} - \frac{s + \sigma (1-s)}{s} L \left [ r \Omega_3 (t; \nu) - \alpha c \Omega_3 (t; \nu) \Omega_4 (t; \nu) - \mu \Omega_4 (t; \nu) \right ] = 0. \]  

In effect we obtain

\[ R_{1,k} \left ( \mathbf{T} S_{k-1} \right ) = \mathbf{L} [S_{k-1}] - \left ( \frac{b_n}{s} + \frac{[s + \sigma (1-s)]\Lambda_0}{s^2} \right ) (1 - X_k) \]

\[ - \frac{s + \sigma (1-s)}{s} \mathbf{L} \left [ - \beta c \left ( \sum_{r=0}^{k-1} I_r S_{k-1-r} \right ) - \mu S_{k-1} \right ], \]

\[ R_{2,k} \left ( \mathbf{T} L_{k-1} \right ) = \mathbf{L} [L_{k-1}] - \frac{b_2}{s} (1 - X_k) \]

\[ - \frac{s + \sigma (1-s)}{s} \mathbf{L} \left [ \beta c \left ( \sum_{r=0}^{k-1} I_r S_{k-1-r} \right ) + \alpha c I_{k-1} T_{k-1} - (\mu + k) L_{k-1} \right ], \]

\[ R_{3,k} \left ( \mathbf{T} I_{k-1} \right ) = \mathbf{L} [I_{k-1}] - \left ( \frac{b_3}{s} + \frac{[s + \sigma (1-s)]\Lambda_1}{s^2} \right ) (1 - X_k) \]

\[ - \frac{s + \sigma (1-s)}{s} \mathbf{L} \left [ k L_{k-1} - (d + r + \mu) I_{k-1} \right ], \]

\[ R_{4,k} \left ( \mathbf{T} T_{k-1} \right ) = \mathbf{L} [T_{k-1}] - \frac{b_4}{s} (1 - X_k) \]

\[ - \frac{s + \sigma (1-s)}{s} \mathbf{L} \left [ r I_{k-1} - \alpha c T_{k-1} I_{k-1} - \mu T_{k-1} \right ]. \]

The deformation equation of kth order is presented as

\[ \mathbf{L} [S_k (t) - X_k S_{k-1} (t)] = h R_{1,k} \left ( \mathbf{T} S_{k-1} \right ), \]

\[ \mathbf{L} [L_k (t) - X_k L_{k-1} (t)] = h R_{2,k} \left ( \mathbf{T} L_{k-1} \right ), \]

\[ \mathbf{L} [I_k (t) - X_k I_{k-1} (t)] = h R_{3,k} \left ( \mathbf{T} I_{k-1} \right ), \]

\[ \mathbf{L} [T_k (t) - X_k T_{k-1} (t)] = h R_{4,k} \left ( \mathbf{T} T_{k-1} \right ), \]
Using the inverse Laplace transform technique on equation (51) gives

\[ S_k(t) = X_k S_{x-1}(t) + hL^{-1} \left[ R_{1,k} \left( \overrightarrow{S}_{k-1} \right) \right], \]

\[ L_k(t) = X_k L_{x-1}(t) + hL^{-1} \left[ R_{2,k} \left( \overrightarrow{L}_{k-1} \right) \right], \]

\[ I_k(t) = X_k I_{x-1}(t) + hL^{-1} \left[ R_{3,k} \left( \overrightarrow{I}_{k-1} \right) \right], \]

\[ T_k(t) = X_k T_{x-1}(t) + hL^{-1} \left[ R_{1,k} \left( \overrightarrow{T}_{k-1} \right) \right]. \] (52)

By considering the initial guess \( S_0(t) = b_1 + \{1 + \sigma (t + 1)\Lambda_0 \}, L_0(t) = b_2, I_0(t) = b_3 + \sigma (t + 1)\Lambda_1 \) and \( T_0(t) = b_4 \), solution to equation (51) for \( k = 1, 2, 3, \ldots \), we obtain the values of \( S_k(t), L_k(t), I_k(t), T_k(t) \) for \( k \geq 1 \). Thus, the solution of TB model (16) is expressed as

\[ S(t) = S_0 + S_1 + S_2 + \ldots \]

\[ L(t) = L_0 + L_1 + L_2 + \ldots \] (53)

\[ I(t) = I_0 + I_1 + I_2 + \ldots \]

\[ T(t) = T_0 + T_1 + T_2 + \ldots \]

7. Numerical results and discussions. This aspect of work examines the numerical simulation for the TB model with Caputo-Fabrizio fractional-order derivative. The numerical method employed on equation (16) hinge on HATM and padé approximation. The numerical simulation is carried out using the following parameter values \( \Lambda_0 = 800, \beta = 0.5, \gamma = 0.02, \alpha = 0.08, \Lambda_1 = 100, k = 0.1, \delta = 0.02, r = 0.03, \mu = 0.1 \). The initial condition for the study is given by \( S(0) = 0.273, L(0) = 0.0421, I(0) = 0.721 \) and \( T(0) = 0.416 \).

Figure 2 represents the numerical simulation based on HATM. In Figure 2(a), as the rate of \( \sigma \) decreases the population of the susceptible population increases. In Figure 2(b) the latent population decrease as \( \sigma \) value decrease. In Figure 2(c) as \( \sigma \) values increase the number of the infected human increase and similarly, in Figure 2(d) the number of treated humans increases as the fractional order \( \sigma \) value increase. Figure 3 depicts the number of simulation based on padé numerical technique. In Figure 3(a) it can be obtained that as \( \sigma \) values decrease, the number of susceptible increases. In Figure 3(b) the number of latent increases as the \( \sigma \) values are increased. Similarly, in Figure 3(c)-(3d) as the \( \sigma \) values increase, the respective populations are also increasing. The operator CF is effective as can be observed in Figure 3 as well as Figure 2. These findings provide insight that HATM and padé methods present a good result in predicting TB transmission in a population.

8. Conclusion. In this work, the TB model is examined included recruitment rate into infected humans. The stability and sensitivity analysis of the classical TB model was undertaken. The non-integer order operator known as Caputo-Fabrizio (CF) which is based in characterized by generalized function (exponential law) was employed. The concept of fixed point theory was utilized to establish the existence and uniqueness of the solution of the CF TB model. The HATM and padé
approximation technique was made use to obtain a numerical simulation results. The effect of fractional order derivative on CF was established in the numerical results. It can be concluded that the numerical simulation of the CF TB model using both HATM and padé approximation methods give relatively the same results. It has been found in the numerical simulation that the fractional-order has an effect on prediction. Fractional order derivatives generally hinged on previous state and incorporate the present condition for prediction (graph out). The Caputo-Fabrizio which is based on exponential law possesses crossover properties that guaranteed accurate predictions. Caputo-Fabrizio order derivative follows natural exponential law phenomena that are common in real life certain and also non- singular. Because of this property, the CF presents an accurate prediction as compared with the HATM which is based in integer derivative. In this study, we recommend that in future study real-life date ought to be incorporated with different operators and compared with an existence integer TB model.

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**REFERENCES**

[1] A. O. Adebiyi, *Mathematical Modeling of the Population Dynamics of Tuberculosis*, An Unpublished Thesis, University of Cape Town, South Africa, 2016.
Figure 3. Numerical simulation for TB model (16) with CF fractional using pade approximations at $\sigma = 1, 0.95, 0.90, 0.85$

[2] A. Athithan and M. Ghosh, Optimal control of tuberculosis with case detection and treatment, *World Journal of Modelling and Simulation*, 11 (2015), 111–122.

[3] I. A. Baba and B. Ghanbari, Existence and uniqueness of solution of a fractional order tuberculosis model, *Eur. Phys. J. Plus.*, 134 (2019), 489.

[4] C. P. Bhunu, Mathematical analysis of a three-strain tuberculosis transmission model, *Appl. Math. Model.*, 35 (2011), 4647–4660.

[5] E. Bonyah, A. Atangana and M. Chand, Analysis of 3D IS-LM macroeconomic system model within the scope of fractional calculus, *Chao. Solit. Frac. X*, 2 (2019), 100007.

[6] S. Bushnaq, S. A. Khan, K. Shah and G. Zaman, Mathematical analysis of HIV/AIDS infection model with Caputo-Fabrizio fractional derivative, *Cogent Math. Stat.*, 5 (2018), 1432521.

[7] M. Caputo and M. Fabrizio, A new definition of fractional derivative with-out singular kernel, *Progr Fract Differ Appl.*, 1 (2015), 73–85.

[8] M. Caputo and M. Fabrizio, Applications of new time and spatial fractional derivatives with exponential kernels, *Progr. Fract. Differ. Appl.*, 2 (2016), 1–11.

[9] C. Castillo-Chavez, S. Blower, P. van den Driessche, D. Kirschner and A.-A. Yakubu, *Mathematical Approaches for Emerging and Reemerging Infectious Diseases: An Introduction*, Springer-Verlag, New York, 2002.

[10] Fatmawati, U. D. Purwati, F. Riyudha and H. Tasman, Optimal control of a discrete age-structured model for tuberculosis transmission, *Heliyon*, 6 (2020), e03030.

[11] M. A. Khan, S. Ullah and M. Farooq, A new fractional model for tuberculosis with relapse via Atangana-Baleanu derivative, *Chao. Solit. Frac.*, 116 (2018), 227–238.

[12] A. Khan, T. Abdeljawad, J. F. Gómez-Aguilar and H. Khan, Dynamical study of fractional order mutualism parasitism food web module, *Chao. Solit. Frac.*, 134 (2020), 109685.
[13] H. Khan, J. F. Gómez-Aguilar, A. Alkhazzan and A. Khan, A fractional order HIV-TB coinfection model with nonsingular Mittag-Leffler Law, Math Method Appl Sci., 43 (2020), 3780–3806.

[14] A. Khan, H. Khan, J. F. Gómez-Aguilar and T. Abdeljawad, Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel, Chao. Solit. Frac., 127 (2019), 422–427.

[15] A. Khan, J. F. Gómez-Aguilar, T. S. Khan and H. Khan, Stability analysis and numerical solutions of fractional order HIV/AIDS model, Chao. Solit. Frac., 122 (2019), 119–128.

[16] S. Kim, A. A. de los ReyesV and E. Jung, Mathematical model and intervention strategies for mitigating tuberculosis in the Philippines, J. Theo. Bio., 443 (2018), 100–112.

[17] D. Kumar, J. Singh, M. Al Qurashi and D. Baleanu, A new fractional SIIRS-SI malaria disease model with application of vaccines, antimalarial drugs, and spraying, Adv. Differ. Equ., 2019 (2019), 278.

[18] J. Losada and J. J. Nieto, Properties of a new fractional derivative without singular kernel, Progr. Fract. Differ. Appl., 2 (2015), 87–92.

[19] H. Yépez-Martínez and J. F. Gómez-Aguilar, A new modified definition of Caputo–Fabrizio fractional order derivative and their applications to the Multi Step Homotopy Analysis Method (MHAM), J. Comp. Appl. Math., 346 (2019), 247–260.

[20] R. Naresh and A. Tripath, Modelling and analysis of HIV–TB coinfection in a variable size population, Math. Model. Anal., 10 (2005), 275–286.

[21] D. Okuonghae and S. E. Omosigho, Analysis of a mathematical model for tuberculosis: What could be done to increase case detection, J. Theor. Biol., 269 (2011), 31–45.

[22] S. Qureshi, E. Bonyah and A. A. Shaikh, Classical and contemporary fractional operators for modeling diarrhea transmission dynamics under real statistical data, Physica A., 535 (2019), 122496.

[23] J. Singh, D. Kumar, M. Al Qurashi and D. Baleanu, A new fractional model for giving up smoking dynamics, Adv. Differ. Equ., 2017 (2017), 88.

[24] N. H. Sweilam and S. M. Al-Mekhlafi, Numerical study for multi-strain tuberculosis (TB) model of variable-order fractional derivatives, J. Adv. Res., 7 (2016), 271–283.

[25] S. Ullah, M. A. Khan, M. Farooq, Z. Hammouch and D. Baleanu, A fractional model for the dynamics of Tuberculosis infection using Caputo-Fabrizio derivative, Discrete Contin. Dyn. Syst. Ser. S, 13 (2020), 975–993.

[26] P. van den Driessche and J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci., 180 (2002), 29–48.

[27] R. S. Wallis, Mathematical models of tuberculosis reactivation and relapse, Frontiers in Microbiology, 17 (2016), 669.

[28] World Health Organization, Global tuberculosis report, 2017. Available from: https://www.who.int/tb/publications/global_report/gtbr2017_main_text.pdf. Accessed on Feb 24, 2018.

[29] Y Yang, J. Wu, J. Li and X. Xu, Tuberculosis with relapse: A model, Math. Popul. Stud., 24 (2017), 3–20.

[30] A. Yusuf, S. Qureshi, M. Inc, A. I. Aliyu, D. Baleanu, and A. A. Shaikh, Two strain epidemic model involving fractional derivative with Mittag- Leffler kernel, Chaos, 28 (2018), 123121.

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