ON THE RATE OF CONVERGENCE OF LOOP-ERASED RANDOM WALK TO SLE$_2$

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Abstract. We derive a rate of convergence of the Loewner driving function for planar loop-erased random walk to Brownian motion with speed 2 on the unit circle, the Loewner driving function for radial SLE$_2$. The proof uses a new estimate of the difference between the discrete and continuous Green’s functions that is an improvement over existing results for the class of domains we consider. Using the rate for the driving process convergence along with additional information about SLE$_2$, we also obtain a rate of convergence for the paths with respect to the Hausdorff distance.

1. Introduction

The Schramm-Loewner evolution (SLE) is a one-parameter family of random planar growth processes constructed by solving the Loewner equation when the driving function is a one-dimensional Brownian motion. SLE was introduced by Schramm in [17] and has been shown to describe the scaling limits of a number of two-dimensional discrete models from statistical mechanics including percolation, loop-erased random walk, uniform spanning trees, and the Ising model. This has provided a means for developing a rigorous mathematical understanding of these models. SLE has also allowed a number of long-standing open problems about planar Brownian motion to be solved, notably, Mandelbrot’s conjecture about the Hausdorff dimension of the Brownian frontier. Despite a rapid progress in the understanding of questions involving SLE, there are still several fundamental open problems. Some of these were communicated by Schramm in [18], in particular that of “obtain[ing] reasonable estimates for the speed of convergence of the discrete processes which are known to converge to SLE.” One of the motivations for this question, besides its being of independent interest, is that results of this type could lead to improved estimates for certain critical exponents; see the discussion in [18].

The loop-erased random walk is a self-avoiding random walk obtained by chronologically erasing the loops of a simple random walk. It was proved by

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Lawler, Schramm, and Werner in [14] that the scaling limit of loop-erased random walk in a simply connected domain is SLE$_2$. Arguably the most important step in this proof is to show that the Loewner driving function for the loop-erased random walk path converges to Brownian motion with speed 2, the Loewner driving function for SLE$_2$.

The primary purpose of this paper is to establish a rate for this convergence. To the best of our knowledge, this is the first instance of a formal derivation of a rate of convergence for any of the discrete processes known to converge to SLE, although Smirnov has given without proof a rate of convergence for critical percolation crossing probabilities; see Remark 3 of [20].

1.1. Statement of the main result and outline of the paper. Let $D \subseteq \mathbb{C}$ be a simply connected domain with $0 \in D$, and let $\psi_D : D \to \mathbb{D}$, where $\mathbb{D}$ is the open unit disk, be the unique conformal map with $\psi_D(0) = 0$, $\psi_D'(0) > 0$. Let $D^n$ be the $n^{-1}\mathbb{Z}^2$ grid domain approximation of $D$; that is, the connected component containing 0 in the complement of the closed faces of $n^{-1}\mathbb{Z}^2$ intersecting $\partial D$. Let $\gamma^n$ denote the time-reversal of loop-erased random walk on $n^{-1}\mathbb{Z}^2$ started at 0 and stopped when hitting $\partial D^n$. Note that $D^n$ is simply connected and let $\psi_{D^n} : D^n \to \mathbb{D}$ be the conformal map normalized as above. Let

$$W_n(t) = W_n(0)e^{i\vartheta_n(t)}, \quad t \geq 0,$$

denote the Loewner driving function for the curve $\tilde{\gamma}^n = \psi_{D^n}(\gamma^n)$ parameterized by capacity and let $\text{inrad}(D) = \inf\{|w| : w \notin D\}$. The following is our main result.

**Theorem 1.1.** Let $0 < \epsilon < 1/24$ be fixed, and let $D$ be a simply connected domain with $\text{inrad}(D) = 1$. For every $T > 0$ there exists an $n_0 < \infty$ depending only on $T$ such that whenever $n > n_0$ there is a coupling of $\gamma^n$ with Brownian motion $B(t)$, $t \geq 0$, where $e^{iB(0)}$ is uniformly distributed on the unit circle, with the property that

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} |W_n(t) - e^{iB(2t)}| > n^{-(1/24-\epsilon)}\right) < n^{-(1/24-\epsilon)}.$$

The proof of Theorem 1.1, which follows the general strategy that was outlined in [17] and implemented in detail for the proof of convergence in [14], has four main components. Each of these is covered in a separate section following Section 2, in which we introduce some notation and preliminary results.

In Section 3 we derive a rate of convergence for the distribution of the conformal image of the starting point of the time-reversed loop-erased random walk. This result, which is given in Proposition 3.1, is an application of the strong approximation of simple random walk due to Komlós, Major, and Tusnády; see [10].

Section 4 covers the second main step of our proof which is to derive a rate of convergence of the martingale observable for the loop-erased random walk.
path. We use the same observable as in [14], but our method for proving convergence is somewhat different. More precisely, we improve estimates of the discrete Green’s function from [11] which, together with geometric arguments, yield a rate of convergence for the observable; see Theorem 4.1. Certain technical details in this section are deferred to Appendix A.

Next, in Section 5 the Loewner equation is used to transfer information about the observable to information about the Loewner driving function for a piece of the loop-erased random walk path. In particular, it is shown that the driving function is, up to explicit error terms, a martingale on a certain mesoscopic scale that depends on the rate of convergence of the observable; see Proposition 5.2.

In Section 6 we use the estimates from the previous sections, a sharp martingale maximal inequality, and the Skorokhod embedding theorem to find a coupling of the driving function with Brownian motion such that (1.1) holds. This step concludes the proof of Theorem 1.1.

In Section 7 we use Theorem 1.1 and a derivative estimate for radial SLE$_2$ (which we derive from an estimate for chordal SLE$_2$) to obtain a rate of convergence for the paths with respect to Hausdorff distance; see Theorem 7.1 for a precise statement.

In Section 8 we prove Theorem 8.1 which is a particular Green’s function estimate. This uniform estimate for the expected number of visits to $x$ by two-dimensional simple random walk starting at 0 before exiting a simply connected grid domain is an improvement of Theorem 1.2 from [11] and is a result of independent interest. It is then used in Appendix A to help prove Proposition 4.2.

Finally, Section 9 is an informal summary of the main steps in our derivation of a rate of convergence, each of which contributes to the rate. We discuss along the way the optimality of the rate we obtained.

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2. Notation and preliminaries

We now introduce the notation that will be used throughout this paper.
General information about the basics of SLE and much of the necessary background material can be found in [13]. To facilitate the reading of this paper we have tried to be consistent with the notation used in [14].

2.1. Conformal maps and grid domains. Suppose that $\mathbb{C}$ denotes the complex plane, and write $D = \{ z : |z| < 1 \}$ for the unit disk in $\mathbb{C}$. We write $B(z,r) = \{ w : |w-z| < r \}$ and use the notation $A(r,R)$ to denote the annulus $\{ z : r < |z| < R \}$. For a set $D \subset \mathbb{C}$, we define the inner radius of $D$ with respect to $z \in \mathbb{C}$ to be

$$\text{inrad}_z(D) = \sup \{ r : B(z,r) \subset D \}$$

and we write $\text{inrad}(D)$ for $\text{inrad}_0(D)$. We say that a domain $D \subset \mathbb{C}$ is a grid domain (with respect to $\mathbb{Z}^2$) if the boundary of $D$ consists of edges of the lattice $\mathbb{Z}^2$, and we write $\mathcal{D}$ for the set of all simply connected grid domains $D$ such that $0 < \text{inrad}(D) < \infty$; that is, those simply connected grid domains $D \neq \mathbb{C}$ such that $0 \in D$.

If $D$ is a simply connected domain containing the origin, we denote by $\psi_D$ the unique conformal map of $D$ onto $\mathbb{D}$ with $\psi_D(0) = 0$ and $\psi_D'(0) > 0$. For simplicity, we will call the value $\log \psi_D'(0)$ the capacity (from 0) of $D$ and we denote this by $\text{cap}(D)$. In the particular case when $D = \mathbb{D} \setminus K$ for some compact set $K$, we write $\text{cap}(K)$.

We will also have occasion to consider a similar quantity for the upper half-plane $\mathbb{H}$. Suppose that $K \subset \mathbb{H}$ is bounded. If $K = \mathbb{H} \setminus \overline{K}$ and $\mathbb{H} \setminus K$ is simply connected, then there exists a unique conformal transformation $\psi_K : \mathbb{H} \setminus K \rightarrow \mathbb{H}$ such that

$$\lim_{z \rightarrow \infty} [\psi_K(z) - z] = 0.$$ 

The half-plane capacity (from infinity) of $K$ is defined by

$$\text{hcap}(K) = \lim_{z \rightarrow \infty} z[\psi_K(z) - z] = 0.$$ 

See Chapter 3 of [13] for further details.

We say that a proper subset $A \subset \mathbb{Z}^2$ is connected if every two points in $A$ can be connected by a nearest neighbor path staying in $A$, and is called simply connected if both $A$ and $\mathbb{Z}^2 \setminus A$ are connected. The boundary of $A$ is given by $\partial A = \{ y \in \mathbb{Z}^2 \setminus A : |y-x| = 1 \text{ for some } x \in A \}$.

When $D$ is a grid domain, we write $V(D) = D \cap \mathbb{Z}^2$ for the lattice points contained in $D$ and note that $\partial V(D)$ is contained in $\partial D \cap \mathbb{Z}^2$.

If $D$ is a simply connected domain with a Jordan boundary, it is well-known that $\psi_D$ can be extended continuously to the boundary so that if $u \in \partial D$, then $\psi_D(u) = e^{i\theta_D(u)} \in \partial \mathbb{D}$. For our purposes, we will be concerned with grid domains which may not have a Jordan boundary. This means that if $D \in \mathcal{D}$, then a boundary point may correspond under conformal mapping to several points on the boundary of the unit disk. To avoid using prime ends (see [15] for a full discussion), we adopt the convention from [14] of viewing the boundary of $\mathbb{Z}^2 \cap D$ as pairs $(u,e)$ of a point $u \in \partial D \cap \mathbb{Z}^2$ and an incident
edge \( e \) connecting \( u \) to a neighbor in \( D \). We write \( V_\partial(D) \) for the set of such pairs, and if \( v \in V_\partial(D) \), then the notation \( \psi_D(v) \) means \( \lim_{z \to v} \psi_D(z) \) along \( e \), and this limit always exists. If \( v = (u, e) \in V_\partial(D) \), then we write \( A_v \) for the neighbors \( w \) of \( u \) such that the edge \( (w, u) \) corresponds to the same limit on \( \partial \mathbb{D} \) as \( e \).

Furthermore, if \( B \) is planar Brownian motion, and \( T_D = \inf \{ t \geq 0 : B_t \not\in D \} \) where \( D \) is a grid domain, it is known that the limit \( \psi_D(B_{T_D}) = \lim_{t \to T_D} \psi_D(B_t) \) exists almost surely.

We state here Koebe’s well-known distortion, growth, and one-quarter inequalities as the Koebe distortion theorem. We will use these results extensively. See [15] for further discussion and proofs. We remark that we usually refer to the first two sets of inequalities as the Koebe distortion theorem.

**Lemma 2.1.** Let \( D \) be a simply connected domain and suppose \( f : D \to \mathbb{C} \) is a conformal map. Set \( d = \text{dist}(z, \partial D) \) for \( z \in D \). If \( |z - w| \leq rd \), then

\[
\frac{1-r}{1+r^2}|f'(z)| \leq |f'(w)| \leq \frac{1+r}{1-r^2}|f'(z)|,
\]

\[
\frac{|f'(z)|}{1+r^2}|z-w| \leq |f(z) - f(w)| \leq \frac{|f'(z)|}{1-r^2}|z-w|,
\]

and

\[
\mathcal{B}(f(z), d|f'(z)|/4) \subset f(D),
\]

where \( \mathcal{B}(w, \rho) \) denotes the open disk of radius \( \rho \) around \( w \).

Let \( d_f(z) = \text{dist}(f(z), \partial D') \), where \( f \) is conformal and \( D' = f(D) \). The following result, which we will refer to as Koebe’s estimate, is a consequence of the last lemma, see [6]:

\[
\frac{1}{4}d|f'(z)| \leq d_f(z) \leq 4d|f'(z)|.
\]

We will also make use of various versions and consequences of the Beurling projection theorem, in both the continuous and discrete setting. We state three versions here; see [13], [9], and [2], respectively.

**Lemma 2.2.** Let \( D \) be a simply connected domain, and let \( \varphi : D \to \mathbb{D} \) be a conformal map with \( \varphi(0) = 0 \). If \( \beta \) is a simple curve in \( D \) with one end-point on \( \partial D \), then there exists a constant \( c < \infty \) such that

\[
\text{diam} \varphi(\beta) \leq c \left[ \frac{\text{diam} \beta}{\text{inrad}(D)} \right]^{1/2}.
\]

**Lemma 2.3.** There exists a constant \( c > 0 \) such that for any \( R \geq 1 \), any \( x \in \mathbb{C} \) with \( |x| \leq R \), any \( A \subset \mathbb{C} \) with \( [0, R] \subset \{|z| : z \in A\} \),

\[
\mathbb{P}^x(\xi_R \leq T_A) \leq c (|x|/R)^{1/2},
\]

where \( \xi_R = \inf\{ t \geq 0 : |B_t| \geq R \} \) and \( T_A = \inf\{ t \geq 0 : B_t \in A \} \), where \( B \) is planar Brownian motion.
Lemma 2.4. There exists a constant $c > 0$ such that for any $n \geq 1$, any $x \in \mathbb{Z}^2$ with $|x| \leq n$, any connected set $A \subset \mathbb{Z}^2$ containing the origin and such that $\sup\{|z| : z \in A\} \geq n$,
\[
\mathbb{P}^x(\Xi_n \leq \tau_A) \leq c(|x|/n)^{1/2},
\]
where $\Xi_n = \inf\{k \geq 0 : |S_k| \geq n\}$ and $\tau_A = \inf\{k \geq 0 : S_k \in A\}$, where $S$ is simple random walk on $\mathbb{Z}^2$.

2.2. Green’s functions. If $D$ is a domain whose boundary includes a curve, let $g_D(z, w)$ denote the Green’s function for $D$. If $z \in D$, we can define $g_D(z, \cdot)$ as the unique harmonic function on $D \setminus \{z\}$, vanishing on $\partial D$ (in the sense that $g_D(z, w) \to 0$ as $w \to w_0$ for every regular $w_0 \in \partial D$), with
\[
g_D(z, w) = -\log |z - w| + O(1) \quad \text{as } |z - w| \to 0.
\]
In the case $D = \mathbb{D}$, we have
\[
g_\mathbb{D}(z, w) = \log |w - 1| - \log |w - z|.
\]
Note that $g_\mathbb{D}(0, z) = -\log |z|$ and $g_\mathbb{D}(z, w) = g_\mathbb{D}(w, z)$. An equivalent formulation of the Green’s function can be given in terms of Brownian motion, namely $g_D(z, w) = \mathbb{E}^z[|B_{T_D} - w|] - \log |z - w|$ for distinct points $z, w \in D$ where $T_D = \inf\{t : B_t \notin D\}$. The Green’s function is a well-known example of a conformal invariant; see Chapter 2 of [13] for further details. Note that the conformal map $\psi_D : D \to \mathbb{D}$ can be written as
\[
\psi_D(z) = \exp\{-g_D(z) + i\theta_D(z)\}, \quad z \in D,
\]
where $g_D(z) = g_D(0, z)$ and $\theta_D(z) = \arg(\psi_D(z))$. In particular, we can write $g_D(z) = -\log |\psi_D(z)|$.

Thus, suppose $D \in \mathcal{D}$ is a grid domain with $\operatorname{inrad}(D) = R$. If $z \in D$ with $\operatorname{dist}(z, \partial D) = 1$, then by a Beurling estimate $g_D(z) = O(R^{-1/2})$, and if $u \in V_0(D)$ and $z \in A_u$, then $|\psi_D(u) - \psi_D(z)| = O(R^{-1/2})$ so that
\[
\theta_D(u) = \theta_D(z) + O(R^{-1/2})
\]
in the sense that for each $u$ as above, we can choose a branch such that (2.2) holds.

Suppose that $S$ is a simple random walk on $\mathbb{Z}^2$ and $A$ is a proper subset of $\mathbb{Z}^2$. If $\tau_A = \min\{j \geq 0 : S_j \notin A\}$, then we let
\[
G_A(x, y) = \sum_{j=0}^{\infty} \mathbb{P}^x(S_j = y, \tau_A > j)
\]
denote the Green’s function for random walk on $A$. Note that $G_A(x, y) = G_A(y, x)$, and set $G_A(x) = G_A(x, 0) = G_A(0, x)$. In analogy with the Brownian motion case, we have
\[
G_A(x, y) = \mathbb{E}^x[a(S_{\tau_A} - y)] - a(y - x) \quad \text{for } x, y \in A
\]
where $a$ is the potential kernel for simple random walk defined by

$$a(x) = \sum_{j=0}^{\infty} \left[ \mathbb{P}^y(S_j = 0) - \mathbb{P}^x(S_j = 0) \right].$$

For details, see Proposition 1.6.3 of [12]. It is known (see [5] for details) that

$$a(x) = \frac{2}{\pi} \log |x| + k_0 + O(|x|^{-2})$$

as $|x| \to \infty$ where $k_0 = (2\zeta + 3\ln 2)/\pi$ and $\zeta$ is Euler’s constant.

Let $x \in A$, $w = (u, e) \in V_0(A)$, and recall that $A_w$ is defined to be the set of neighbors $y \in A$ of $u$ such that the edge $(y, u)$ and $e$ correspond to the same limit on $\partial \mathbb{D}$. We say that these edges correspond to $w$. We define $H_A(x, w)$ to be the probability that a simple random walk starting at $x$ exits $A$ at $u$ using one of the edges determined by the vertices in $A_w$; that is, through one of the edges corresponding to $w$. If a boundary vertex $u$ corresponds to only one limit on $\partial \mathbb{D}$ we simply write $H_A(x, u)$. We note that a so-called last-exit decomposition implies the identity

$$H_A(x, w) = \frac{1}{4} \sum_{A_w} G_A(x, y).$$

2.3. Loop-erased random walk. We now briefly review the definition of loop-erased random walk. Further details may be found in Chapter 7 of [12]. The following loop-erasing procedure, which works for any finite simple random walk path in $\mathbb{Z}^2$, assigns a self-avoiding path to each such random walk path.

Suppose that $S = S[0, m] = [S_0, S_1, \ldots, S_m]$ is a simple random walk path of length $m$. The loop-erased part of $S$, denoted $\Lambda(S)$, is constructed recursively as follows. If $S$ is already self-avoiding, set $\Lambda(S) = S$. Otherwise, let $s_0 = \max \{ j : S_j = S_0 \}$, and for $i > 0$, let $s_i = \max \{ j : S_j = S_{s_{i-1}+1} \}$. If we let $n = \min \{ i : s_i = m \}$, then $\Lambda(S) = [S_{s_0}, S_{s_1}, \ldots, S_{s_n}]$. Observe that $\Lambda(S)(0) = S_0$ and $\Lambda(S)(s_n) = S_m$; that is, the loop-erased random walk has the same starting and ending points as the original simple random walk.

Also notice that the loop-erasing algorithm depends on the order of the points. If $a = [a_0, a_1, \ldots, a_k]$ is a lattice path, write $\overline{a} = [a_k, a_{k-1}, \ldots, a_0]$ for its reversal. Thus, if we define reverse loop-erasing by $\overline{\Lambda}(S) = \Lambda(\overline{S})$, then one can construct a path $S$ such that $\Lambda(S) \neq \overline{\Lambda}(S)$. It is, however, a fact that both $\Lambda(S)$ and $\overline{\Lambda}(S)$ have the same distribution; see Lemma 3.1 of [14]. As such, we will say that $\gamma$ is the time-reversal of loop-erased random walk if $\gamma = \Lambda(\overline{S})$.

In this paper, we will consider the loop-erasure of simple random walk started at 0 and stopped when hitting the boundary of some fixed grid domain $D$. We call this loop-erased random walk in $D$.

Loop-erased random walk has the important domain Markov property, as is further discussed in Lemma 3.2 of [14]. Suppose $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_l)$ is the loop-erasure of a time-reversed simple random walk that is started from 0
and stopped when exiting $D$. Then if we condition on the first $j$ steps of $\gamma$, the distribution of the rest of the curve is the same as the time-reversal of loop-erased random walk in $D \setminus \gamma[0,j]$ conditioned to start at $\gamma_j$; that is, it is distributed as the loop-erasure of the time-reversal of a simple random walk started from 0 conditioned to exit $D \setminus \gamma[0,j]$ through an edge ending in $\gamma_j$.

2.4. The Loewner differential equation and Schramm-Loewner evolution. Suppose the unit disk $D$ is slit by a non self-intersecting curve $\gamma$ in a way such that $D \setminus \gamma$ is simply connected and contains 0. Then we may parameterize the curve by capacity; that is, we choose a parameterization $\gamma(t)$ so that the normalized conformal map $g_t : D \setminus \gamma[0,t] \to D$ satisfies

$$g_t(z) = e^t z + O(z^2),$$

around the origin for each $t \geq 0$. It is a theorem by Loewner that the Loewner chain $(g_t)$, $t \geq 0$, satisfies the Loewner differential equation

$$\partial_t g_t(z) = g_t(z) \frac{\xi(t) + g_t(z)}{\xi(t) - g_t(z)}, \quad g_0(z) = z,$$

where $\xi(t) = g_t(\gamma(t))$ is a unique continuous unimodular function. The inverse $f_t = g_t^{-1}$ satisfies the partial differential equation

$$\partial_t f_t(z) = zf_t'(z) \frac{z + \xi(t)}{z - \xi(t)}, \quad f_0(z) = z.$$

Conversely, consider a function continuous on $[0, \infty)$, taking values in $\partial D$. Then (2.6) can be solved up to time $t$ for all $z$ outside $K_t = \{w : \tau(w) \leq t\}$, where $\tau(w)$ is the blow-up time when $g_t(w)$ hits $\xi(t)$; see [13] for precise definitions. We note that $g_t$ maps $D \setminus K_t$ conformally onto $D$ for $t \geq 0$, and that $K_t$ is called the hull of the Loewner chain. The function $\xi$ is called the driving function for the Loewner chain $(g_t)$ (or $(f_t)$).

If the limit

$$\gamma(t) = \lim_{r \to 1^-} f_t(r \xi(t))$$

exists for $t > 0$ and $t \mapsto \gamma(t)$ is continuous, we say that $(g_t)$ is generated by a curve. In this case, the connected components of $D \setminus \gamma[0,t]$ and $D \setminus K_t$ that contain the origin are the same.

By taking $\xi(t) = \exp\{iB(\kappa t)\}$, where $B(t)$ is standard Brownian motion and $\kappa > 0$, we obtain radial Schramm-Loewner evolution with parameter $\kappa$, or radial SLE$_\kappa$ for short (there are several other versions of SLE). It is known that SLE$_\kappa$ is generated by a curve; see [14] and [10].

3. A rate of convergence for discrete harmonic measure

In this short section we prove a rate of convergence for the boundary hitting distribution of simple random walk in a grid domain $D$ as the inner radius increases. Our goal is to give a quantitative statement of the fact that
the image of the starting point of the time-reversed loop-erased random walk path is close to uniform on \( \partial \mathbb{D} \), when the inner radius of \( D \) is large.

As before, for a grid domain \( D \in \mathcal{D} \) we let \( \psi = \psi_D \) denote the conformal map from \( D \) onto \( \mathbb{D} \) such that \( \psi(0) = 0, \psi'(0) > 0 \). We let \( \tau_D \) and \( T_D \) denote the hitting times of \( \partial D \) for simple random walk \( S \) on \( \mathbb{Z}^2 \) and planar Brownian motion \( B \), respectively. Our goal is now to prove the following result, the proof of which is similar to that of Proposition 3.3 of [11].

**Proposition 3.1.** Let \( 0 < \epsilon < 1/4 \) be fixed. Let \( D \in \mathcal{D} \) be a grid domain. Let \( S \) denote simple random walk on \( \mathbb{Z}^2 \) and let \( B \) denote planar Brownian motion, both started from \( 0 \). There exists \( R_0 < \infty \) such that if \( R = \text{inrad}(D) \) and if \( R > R_0 \), then there is a coupling of \( S \) and \( B \) such that

\[
\mathbb{P}\left( |\psi(S_{\tau_D}) - \psi(B_{T_D})| > R^{-1/2(1/4 - \epsilon)} \right) < R^{-1/4}.
\]

Recall that \( \psi(B_{T_D}) \) is uniformly distributed on \( \partial \mathbb{D} \). Note that \( S_{\tau_D} \) is viewed as an element of \( V_\partial(D) \); see Section 2.1. Since the error term in Proposition 3.1 only depends on the inner radius, the result when applied to the \( n^{-1} \mathbb{Z}^2 \) approximation of a given simply connected domain is independent of the boundary regularity of the domain that is being approximated.

To prove Proposition 3.1 we shall use the strong approximation of Komlós, Major, and Tusnády in a form given in [11]. A technical difficulty is that the joint process \((S, B)\) does not have the Markov property in this coupling, although, of course, each of \( S \) and \( B \) separately has it. This is the reason for introducing the stopping times \( \nu_B \) and \( \nu_S \) in the proof below. In the following, \( S \) is defined by linear interpolation for non-integer \( t \).

**Lemma 3.2.** There exists \( c_0 < \infty \) and a coupling of planar Brownian motion \( B \) and simple random walk \( S \) on \( \mathbb{Z}^2 \), both started from \( 0 \), such that

\[
\mathbb{P}\left( \sup_{0 \leq t \leq \sigma_R} |S_t - B_t / \sqrt{2}| \geq c_0 \log R \right) = O(R^{-10}),
\]

where

\[
\sigma_R = \inf \left\{ t : \min \left\{ \sup_{0 \leq s \leq t} |S_s|, \sup_{0 \leq s \leq t} |B_s| \right\} \geq R^8 \right\}.
\]

Note that if \( R = \text{inrad}(D) \), then in view of Lemma 2.3

\[
\mathbb{P}(\sigma_R < T_D) = O(R^{-7/2})
\]

and similarly for \( \tau_D \).

**Proof of Proposition 3.1.** Write \( R = \text{inrad}(D) \) and let \( S \) and \( B' \) be simple random walk on \( \mathbb{Z}^2 \) and planar Brownian motion, respectively, both started from \( 0 \). By Lemma 3.2 we may couple \( S \) and \( B' \) so that (3.1) holds. Set \( B = B'/\sqrt{2} \) and define

\[
\eta = \inf \{ t \geq 0 : \min \{ \text{dist}(S_t, \partial D), \text{dist}(B_t, \partial D) \} \leq 2c_0 \log R \},
\]
where \( c_0 \) is the constant from Lemma 3.2. Let
\[
E_1 = \{ |S_\eta - B_\eta| \leq c_0 \log R \} \cap \left\{ \sup_{0 \leq t \leq \sigma_R} |S_t - B_t| < c_0 \log R \right\}.
\]
Then it follows from (3.1) and (3.2) that \( \mathbb{P}(E_1^c) = O(R^{-7/2}) \). Indeed, since \( \eta < T_D \) we have
\[
\mathbb{P}(E_1^c) \leq \mathbb{P}(E_1^c, \sigma_R < T_D) + \mathbb{P}(E_1^c, \sigma_R \geq \eta)
\leq O(R^{-7/2}) + O(R^{-10}).
\]
Define the stopping times
\[
\nu_B = \inf\{ t \geq 0 : \text{dist}(B_t, \partial D) \leq 3c_0 \log R \},
\nu_S = \min\{ j \geq 0 : \text{dist}(S_j, \partial D) \leq 3c_0 \log R \}.
\]
On \( E_1 \) we clearly have
\[
\max\{ \nu_B, \nu_S \} \leq \eta < \min\{ T_D, \tau_D \}.
\]
Let \( 0 < \alpha < 1 \) and let \( E_2^B \subset E_1 \) be the event that \( E_1 \) occurs and that \( B \) hits \( \partial D \) before exiting the ball \( B(B_{\nu_B}, 3c_0R^\alpha \log R) \). By using the strong Markov property of \( B \) together with Lemma 2.3, we see that \( \mathbb{P}(E_2^B) \geq 1 - O(R^{-\alpha/2}) \). Let \( Q_B \) be the component of \( B(B_{\nu_B}, 4c_0R^\alpha \log R) \cap \partial D \) that contains the point \( B_{\nu_B} \). On the event \( E_2^B \) we have that \( |B_\eta - S_\eta| \leq c_0 \log R \) and that \( Q_B \) contains a ball of radius \( 3c_0 \log R/2 \) around \( B_\eta \). In particular, \( Q_B \) contains \( B_\eta, S_\eta \), and \( B_{T_D} \).

We define the event \( E_2^S \) and the set \( Q_S \) by replacing \( B \) with \( S \) in the last paragraph. By the strong Markov property of \( S \), using Lemma 2.4, we have that \( \mathbb{P}(E_2^S) \geq 1 - O(R^{-\alpha/2}) \). On the event \( E_2^B \cap E_2^S \) the set \( Q_B \cap Q_S \) is non-empty and contains the points \( B_\eta \) and \( S_\eta \).

Consequently, with probability at least \( 1 - O(R^{-\alpha/2}) \), the pair of boundary hitting points (in the sense of prime ends) \( B_{T_D} \) and \( S_{T_D} \) can be separated from 0 in \( D \) by a crosscut with length at most \( 12\pi c_0 R^\alpha \log R \). Let \( \beta \) be such a crosscut and let \( F \subset \partial D \) be the part of \( \partial D \) that is separated from 0 by \( \beta \). By Lemma 2.2 if \( R \) is sufficiently large, the harmonic measure of \( F \) from 0 in \( D \) is bounded above by \( c(R^{\alpha - 1} \log R)^{1/2} \) for some constant \( c < \infty \). Hence, by conformal invariance of harmonic measure, the length of the interval \( I = \{ z \in \partial \mathbb{D} : \psi^{-1}(z) \in F \} \) satisfies the same bound (with a different constant). Since \( \psi(S_{T_D}) \) and \( \psi(B_{T_D}) \) both are contained in \( I \), the proof is completed by choosing \( \alpha \) such that \( (\alpha - 1)/2 = -\alpha/2 \); that is, choose \( \alpha = 1/2 \). \( \square \)

4. A RATE OF CONVERGENCE FOR THE MARTINGALE OBSERVABLE

The purpose of this section is to provide a rate of convergence for the martingale observable. This result is given in Theorem 4.1 and will then be used in Section 5. Recall that if \( D \in \mathcal{D} \) is a grid domain, then \( \psi_D : D \to \mathbb{D} \) is the conformal map of \( D \) onto \( \mathbb{D} \) satisfying \( \psi_D(0) = 0 \), \( \psi'_D(0) > 0 \).
Theorem 4.1. Let \( 0 < \epsilon < 1/4 \) and let \( 0 < \rho < 1 \) be fixed. There exists \( R_0 < \infty \) such that the following holds. Suppose that \( D \in \mathcal{D} \) is a grid domain with \( \text{inrad}(D) = R \), where \( R > R_0 \). Furthermore, suppose that \( x \in D \cap \mathbb{Z}^2 \) with \( |\psi_D(x)| \leq \rho \) and \( u \in V_0(D) \). If both \( x \) and \( u \) are accessible by a simple random walk starting from \( 0 \), then
\[
H_D(x,u) = H_D(0,u) = 1 - \frac{|\psi_D(x)|^2}{|\psi_D(x) - \psi_D(u)|^2} \cdot \left[ 1 + O(R^{-\left(1/4-\epsilon\right)}) \right].
\]

The proof is given in Section 4.3. It relies on both the estimate of the discrete Green’s function outlined in Section 4.1 and the domain reduction argument given in Section 4.2. The purpose of the domain reduction argument is that it reduces the proof of Theorem 4.1 to showing that (4.1) holds for a special class of grid domains.

Definition. We call a domain \( D \subset \mathbb{C} \) a union of big squares (or UBS) domain if \( D \) can be written as
\[
D = \bigcup_{z \in V} S(z),
\]
where
\[
S(z) = \{ w \in \mathbb{C} : |\text{Re}(w) - \text{Re}(z)| < 1, |\text{Im}(w) - \text{Im}(z)| < 1 \}
\]
for some connected subset \( V \subset \mathbb{Z}^2 \).

Note that \( S(z) \) is the open square with side length 2 around the vertex \( z \). Furthermore, observe that a UBS domain is a grid domain, although the converse is not true. It will be tacitly understood that UBS domains are simply connected unless otherwise stated.

The main reason for using UBS domains is that while grid domains may have parts of the boundary with positive continuous harmonic measure but zero discrete harmonic measure, this does not happen with UBS domains if the discrete harmonic measure is interpreted appropriately. At the same time we can associate a UBS domain to each grid domain in \( \mathcal{D} \) without them differing too much from the conformal mapping point of view.

4.1. Estimates of the discrete Green’s function. The first step in the proof of Theorem 4.1 requires the following estimate which is a version for UBS domains of Proposition 3.10 of [11].

Proposition 4.2. Let \( 0 < \epsilon < 1/4 \) and let \( 0 < \rho < 1 \) be fixed. There exists \( R_0 < \infty \) such that the following holds. Suppose that \( D \) is a UBS domain with \( \text{inrad}(D) = R \) and that \( R > R_0 \). Let \( V = V(D) = D \cap \mathbb{Z}^2 \). If \( x, y \in V \) with \( |\psi_D(x)| \leq \rho \) and \( |\psi_D(y)| \geq 1 - R^{-\left(1/4-\epsilon\right)} \), then
\[
(G_D(x,y)) = \frac{G_D(x,y)}{G_D(y)} = \frac{1 - |\psi_D(x)|^2}{|\psi_D(x) - e^{\theta_D(y)}|^2} \cdot \left[ 1 + O(R^{-\left(1/4-\epsilon\right)}) \right]
\]
where \( G_D \) denotes the Green’s function for simple random walk on \( V \).
In [11], the results are proved for simply connected domains with a Jordan boundary and allow both points to be close to the boundary as long as they are not too close to each other. In the present paper, we are concerned with grid domains which, although still simply connected, need not have a Jordan boundary. Furthermore, we are not concerned with any two arbitrary points, but rather with one point near the boundary and one point near the origin. Using this additional hypothesis and improving the methods of [11] allows us to find a better exponent of $1/4$.

The derivation of Proposition 4.2 in our particular setting essentially follows the same steps as in the original proof from [11]. There is, however, the matter of adapting the original proof from a simply connected domain with Jordan boundary to a UBS domain. This change of setting requires that certain technical estimates be established. For this reason we have included an appendix outlining the proof of Proposition 4.2 in this new setting.

4.2. A domain reduction. Suppose that $D \in \mathcal{D}$ is a grid domain and that $u \in \partial D \cap \mathbb{Z}^2$ is accessible by a simple random walk starting from 0. Write $R = \text{inrad}(D)$. Let $V = V(D) = D \cap \mathbb{Z}^2$ denote those vertices contained in $D$ and let $V_0$ be the component of $V$ containing the origin; note that $V_0$ is simply connected. Define $D_0 \subset D$ by setting

$$D_0 = \bigcup_{z \in V_0} S(z),$$

where $S(z) = \{w \in \mathbb{C} : |\text{Re}(w) - \text{Re}(z)| < 1, |\text{Im}(w) - \text{Im}(z)| < 1\}$ so that $D_0$ is a UBS domain. We will call $D_0$ the UBS domain associated with $D$.

In particular, notice that

(i) $D_0 \subset D$ is a simply connected domain containing the origin,
(ii) $u \in \partial D_0$, and
(iii) for some $1 \leq M \leq \infty$, we can write

$$\partial D_0 \cap D = \bigcup_{j=1}^{M} C_j,$$

where $C_j$, $j = 1, \ldots, M$, are crosscuts of $D$ with length at most 2.

For ease of notation, throughout this section, we write $\psi$ for $\psi_D$ and $\psi_0$ for $\psi_{D_0}$. Recall that we can write $\psi(z) = \exp\{-g(z) + i\theta(z)\}$ and $\psi_0(z) = \exp\{-g_0(z) + i\theta_0(z)\}$ where $g$ and $g_0$ are the Green’s functions for $D$ and $D_0$, respectively.

By Lemma 2.4, since $\text{diam}(C_j) \leq 2$, there exists a universal constant $c < \infty$ such that

$$\text{diam}(\psi(C_j)) \leq cR^{-1/2}.$$

If $\Omega = \psi(D_0) \subset \mathbb{D}$ it follows that

$$\partial \Omega \cap \mathbb{D} \subset A(1 - cR^{-1/2}, 1),$$
where $\mathcal{A}(a,b) = \{ z : a < |z| < b \}$ denotes the annulus. Finally, we write

$$\psi_0 = \varphi \circ \psi, \quad z \in D_0,$$

where $\varphi : \Omega \to \mathbb{D}$ is the conformal map of $\Omega = \psi(D_0)$ onto $\mathbb{D}$ satisfying $\varphi(0) = 0$, $\varphi'(0) > 0$. The following estimate quantifies the fact that $\varphi$ is almost the identity away from the boundary.

**Lemma 4.3.** Let $0 < \epsilon < 1/2$ be fixed. Suppose $\Omega \subset \mathbb{D}$ with $\partial \Omega \cap \mathbb{D} \subset \mathcal{A}(1 - \epsilon, 1)$, and let $\varphi : \Omega \to \mathbb{D}$ be the conformal map of $\Omega$ onto $\mathbb{D}$ with $\varphi(0) = 0$, $\varphi'(0) > 0$. If $|z| \leq 1 - 2\epsilon$, then

$$|\varphi(z) - z| \leq c_0 \epsilon \log(1/\epsilon),$$

where $c_0$ is a uniform constant.

**Proof.** In Section 3.5 of [13] it is shown that

$$|\log(\varphi(z)/z)| \leq c\epsilon[1 - \log(1 - |z|)],$$

where the branch of the logarithm is chosen so that $\log(\varphi(0)/0) = \log \varphi'(0) \geq 0$. It follows that if $|z| \leq 1 - 2\epsilon$, then

$$0 \leq |\varphi(z)| - |z| \leq c\epsilon \log(1/\epsilon),$$

and

$$|\arg(\varphi(z)/z)| \leq c\epsilon[1 + \log(1/\epsilon)],$$

completing the proof. \qed

**Lemma 4.4.** Let $0 < \epsilon < 1/4$ be fixed. There exists $R_0$ such that if $R > R_0$ and $x, y \in V_0$ with

$$g(x) \geq R^{-(1/4-\epsilon)}$$

and $g(y) < R^{-(1/4-\epsilon)}$, then

$$\psi_0(x) = \psi(x) + O(R^{-1/2} \log R)$$

and

$$e^{i\theta_0(y)} = e^{i\theta(y)} + O(R^{-1/4}).$$

**Proof.** We may assume that $x \neq 0$. Let $c_0 > 0$ be a constant such that $\partial \psi(D_0) \subset \mathcal{A}(1-c_0R^{-1/2},1)$ and recall that $\psi_0 = \varphi \circ \psi$ for $z \in D_0$ as in [13]. Note that (4.5) implies that $|\psi(x)| \leq 1 - cR^{-(1/4-\epsilon)} \leq 1 - 2c_0R^{-1/2}$, for $R$ large enough, so Lemma 4.3 applied to the point $z = \psi(x)$ implies that there exists a uniform constant $c_1$ such that

$$|\varphi(\psi(x)) - \psi(x)| = |\psi_0(x) - \psi(x)| \leq c_1 R^{-1/2} \log R,$$

yielding (4.6).

If $y$ is as in the statement of the lemma and $|\psi(y)| \leq 1 - 2c_0R^{-1/2}$ then (4.7) follows from (4.4). Hence we may assume that $|\psi(y)| > 1 - 2c_0R^{-1/2}$. Since the boundary of $\psi(D_0)$ contained in $\mathbb{D}$ is a union of images of crosscuts with diameter bounded by $c_0R^{-1/2}$ there is a curve $\beta$ in
that connects \( \psi(y) \) to the circle \( \{ |z| = 1 - 2c_0 R^{-1/2} \} \) and satisfies \( \text{diam } \beta \leq c_2 R^{-1/2} \) for some absolute constant \( c_2 < \infty \). By Lemma 2.2 we have \( \text{diam } \varphi(\beta) \leq c_3 R^{-1/4} \) and using again Lemma 4.3 we see that
\[
e^{i\theta_0(y)} = e^{i\theta(y)} + O(R^{-1/4})
\]
yielding (4.7), and the proof is complete. \( \square \)

The final result for this section uses a particular continuity estimate for the Poisson kernel. If \( z \in \mathbb{D} \) and \( w \in \partial \mathbb{D} \), let
\[
(4.8) \quad \lambda(z, w; \mathbb{D}) = \frac{1 - |z|^2}{|z - w|^2}
\]
so that \( \lambda/(2\pi) \) is the Poisson kernel for the unit disk.

It can be shown that
\[
|\lambda(z', w'; \mathbb{D}) - \lambda(z, w; \mathbb{D})| \\
\leq |z' - z| \left[ \frac{8|z - w| + 8|z' - w|}{|z - w|^2 |z' - w|^2} \\
+ \frac{(|z - w| + |z' - w| + 2)(|z - w|^2 + |z' - w|^2)}{|z - w|^2 |z' - w|^2} \\
+ |w - w'| \left[ \frac{|z - w'| + 3|z - w|}{|z - w|^2 |z - w'|^2} \right] \right].
\]

(4.9)

**Lemma 4.5.** Let \( 0 < \epsilon < 1/4 \) and let \( 0 < \rho < 1 \) be fixed. There exists \( R_0 < \infty \) such that if \( R > R_0 \) and if \( x, y \in V_0 \) with \( |\psi(x)| \leq \rho \) and \( |\psi(y)| \geq 1 - R^{-(1/4 - \epsilon)} \), then
\[
(4.10) \quad \frac{1 - |\psi_0(x)|^2}{|\psi_0(x) - e^{i\theta_0(y)}|^2} = \frac{1 - |\psi(x)|^2}{|\psi(x) - e^{i\theta(y)}|^2} + O(R^{-1/4}).
\]

**Proof.** Let \( z = \psi(x), z' = \psi_0(x), w = e^{i\theta(y)}, \) and \( w' = e^{i\theta_0(y)} \), and note that by assumption there exists some constant \( 0 < \rho' < 1 \) such that \( |z - w| \geq \rho' \).

We also know from Lemma 4.4 that there exist constants \( c_1 \) and \( c_2 \) such that
\[
|w - w'| \leq c_1 R^{-1/4} \quad \text{and} \quad |z - z'| \leq c_2 R^{-1/2} \log R.
\]

Using the crude bounds that \( |z - w| \leq 2, |z' - w| \leq 2, |z - w'| \leq 2, \) and \( |z' - w'| \leq 2, \) it follows from (4.9) that for \( R \) large enough
\[
(4.11) \quad |\lambda(z', w'; \mathbb{D}) - \lambda(z, w; \mathbb{D})| \leq c_3 R^{-1/4}.
\]

Thus, we see from (4.8) that (4.11) is equivalent to (4.10) as required, and the proof is complete. \( \square \)
4.3. Proof of Theorem 4.1 Let $D \in \mathcal{D}$ be a grid domain, write $R = \text{inrad}(D)$, and assume that $u \in V_0(D)$ is accessible by a simple random walk starting from 0. Let $V = V(D) = D \cap \mathbb{Z}^2$, let $V_0$ be the component of $V$ containing the origin, and let $D_0$ be the UBS domain associated to $D$ as in Section 4.2. Recall that $D_0 \subset D$ is a simply connected domain containing the origin and $u \in V_0(D_0)$.

As in (2.5), if $z \in V_0$ and $w \in V_0(D_0)$, then

$$H_{D_0}(z,w) = \frac{1}{4} \sum_{A_w} G_{D_0}(z,y)$$

where $A_w$ is as in Section 2.1.

Recall that we can write $\psi_{D_0} = \varphi \circ \psi_D$. Hence if $|\psi_D(x)| \leq \rho$, there is a $\rho_0 < 1$ only depending on $\rho$ such that $|\psi_{D_0}(x)| \leq \rho_0$ whenever $R$ is sufficiently large. Since $D_0$ is a UBS domain, we can apply Proposition 4.2 to $u$ and any point $x \in V_0$ with $|\psi_{D_0}(x)| \leq \rho_0$. Hence, substituting (4.2) into (4.12) gives

$$H_{D_0}(x,u) = \frac{1}{4} \sum_{A_u} G_{D_0}(y) \cdot \frac{1 - |\psi_{D_0}(x)|^2}{|\psi_{D_0}(x) - e^{i\theta_{D_0}}(y)|^2} \cdot \left[ 1 + O(R^{-(1/4-\epsilon)}) \right].$$

Since the summation in (4.13) is over $y$, we use the fact (2.2) that $\psi_{D_0}(u) = e^{i\theta_{D_0}(y)} + O(R^{-1/2})$ to conclude

$$H_{D_0}(x,u) = \frac{1}{4} \sum_{A_u} G_{D_0}(y),$$

we see that (4.14) yields

$$H_{D_0}(x,u) = \frac{H_{D_0}(x,u)}{H_{D_0}(0,u)} = \frac{1 - |\psi_{D_0}(x)|^2}{|\psi_{D_0}(x) - \psi_{D_0}(u)|^2} \cdot \left[ 1 + O(R^{-(1/4-\epsilon)}) \right].$$

If we now observe that

$$H_{D_0}(x,u) = \frac{H_{D_0}(x,u)}{H_D(0,u)} H_D(0,u)$$

since $V_0$ consists of precisely those vertices accessible by a simple random walk starting from the origin, and that Lemma 4.5 combined with (2.2) implies

$$H_{D_0}(x,u) = \frac{1 - |\psi_D(x)|^2}{|\psi_D(x) - \psi_D(u)|^2} \cdot \left[ 1 + O(R^{-(1/4-\epsilon)}) \right],$$

then combining (4.15) and (4.16) gives (4.11) and the proof of Theorem 4.1 is complete.
5. Moment estimates for increments of the driving function

The idea is now to use Theorem 4.1 to transfer the fact that a suitable version of the discrete Poisson kernel (4.1) is a martingale with respect to the growing loop-erased random walk path to information about the Loewner driving function for a mesoscopic scale piece of the path. This is the analogue of Proposition 3.4 of [14], but with a rate of decay. Suppose that $D \in \mathcal{D}$ is a grid domain, write $R = \text{inrad}(D)$, and let $\psi_D : D \to \mathbb{D}$ be the conformal map of $D$ onto $\mathbb{D}$ with $\psi_D(0) = 0$, $\psi_D'(0) > 0$. For ease of notation, we will write $\psi = \psi_D$ in what follows. For $w \in D$ and $u \in \partial D$, define

$$\lambda(w, u; D) = \Re \left( \frac{\psi(u) + \psi(w)}{\psi(u) - \psi(w)} \right) = \frac{1 - |\psi(w)|^2}{|\psi(u) - \psi(w)|^2}$$

as in (4.8).

Let $\gamma = (\gamma_0, \ldots, \gamma_t)$ denote the loop-erasure of the time-reversal of simple random walk started at 0, stopped when it hits $\partial D$, and for $j \geq 0$, define the slit domains

$$D_j = D \setminus \bigcup_{i=1}^{j} [\gamma(i-1), \gamma(i)].$$

As before, the conformal maps $\psi_j : D_j \to \mathbb{D}$ will be those satisfying $\psi_j(0) = 0$ and $\psi_j'(0) > 0$. We write $t_j$ for the capacity of the curve $\psi(\gamma[0, j])$ from 0 in $\mathbb{D}$. Denote by $W : [0, \infty) \to \partial \mathbb{D}$ the Loewner driving function for the curve $\tilde{\gamma}^R = \psi(\gamma)$ parameterized by capacity. That is, $W$ is the unique continuous function such that solving the radial Loewner equation (2.6) with driving function $W$ gives the path $\tilde{\gamma}^R$. Moreover, we denote by $(\vartheta(t), t \geq 0)$ the continuous, real-valued function such that $\vartheta(0) = 0$ and $W(t) = W(0)e^{i\vartheta(t)}$, and we define

$$\Delta_j = \vartheta(t_j).$$

Let $0 < \epsilon < 1/4$ be fixed. Set $3\alpha = 1/4 - \epsilon$ and define

$$m = m(R) = \min\{j \geq 0 : t_j \geq R^{-2\alpha} \text{ or } \Delta_j \geq R^{-\alpha}\}.$$

The following is Lemma 2.1 of [14].

**Lemma 5.1.** Suppose $K_t$ is the hull obtained by solving (2.6) with $U_t$ as driving function. If $D(t) = \sqrt{t} + \sup_{0 \leq s \leq t} |U_s - U_0|$, then there exists a constant $c$ such that

$$c^{-1} \min\{1, D(t)\} \leq \text{diam}(K_t) \leq cD(t).$$

It follows from the last lemma and Lemma 2.2 that $t_m \leq R^{-2\alpha} + O(R^{-1})$ and

$$|\Delta_m| \leq R^{-\alpha} + O(R^{-1/2}).$$
Furthermore, if \( w = \psi(v) \) where \( |v| \leq \text{inrad}(D)/5 \), then the Koebe one-quarter theorem implies \( |w| \leq 4/5 \). By the Loewner equation, we have
\[
|\psi_m(v) - w| \leq cR^{-2\alpha}
\]
so that \( |\psi_m(v)| \leq 5/6 \) for \( R \) large enough. This means that the conditions of Theorem 4.1 are satisfied by \( v \in D_j \) for each \( 1 \leq j \leq m \).

Let \( x \in D \cap \mathbb{Z}^2 \), \( w \in V_0(D) \), and recall the definition of the hitting probability \( H_D(x, w) \) from Section 2.2. We will write \( H_j(x, w) \) for \( H_{D_j}(x, w) \). Fix \( v \in V(D) \) with \( |v| \leq R/5 \). It can be shown that
\[
M_j = \frac{H_j(v, \gamma_j)}{H_j(0, \gamma_j)}
\]
is a martingale with respect to the filtration generated by \( \gamma[0, j] \), \( j \geq 0 \); see [14]. With the definition \( \lambda_j = \lambda(v, \gamma_j; D_j) \), we know from Theorem 4.1 that
\[
\left| \frac{H_j(v, \gamma_j)}{H_j(0, \gamma_j)} - \lambda_j \right| \leq cR^{-3\alpha}
\]
for \( j \leq m \) implying that
\[
\mathbb{E}[\lambda_m - \lambda_0] = \mathbb{E}[M_m - M_0] + O(R^{-3\alpha}) = O(R^{-3\alpha}).
\]
By a Taylor expansion using the Loewner equation we get
\[
\lambda_m - \lambda_0 = \text{Re} \left( \frac{ZU(U + Z)}{(U - Z)^2} \right) (2t_m - \Delta_m^2) + 2\text{Im} \left( \frac{ZU}{(U - Z)^2} \right) \Delta_m + O(R^{-3\alpha}),
\]
where \( Z = \psi(v) \) and \( U = W(0) \). (See [14] for more details.) By taking the expectation and plugging in two different \( v \), exactly as in [14], recalling that \( 3\alpha = 1/4 - \epsilon \), we arrive at the following.

**Proposition 5.2.** Let \( 0 < \epsilon < 1/4 \) be fixed. There exist constants \( c > 0 \), \( R_0 \geq 1 \) such that for all \( R \geq R_0 \) the following holds. Let \( D \in \mathcal{D} \) be a grid domain with \( \text{inrad}(D) = R \) and let \( \gamma \) be the loop erasure of the time-reversal of simple random walk from 0 in \( D \) conditioned to exit \( D \) through an edge corresponding to \( u_0 \), where \( u_0 \in V_0(D) \) is such that this event has positive probability. If \( t_j, \Delta_j \), and \( m \) are defined as above, then
\[
|\mathbb{E}[\Delta_m]| \leq cR^{-(1/4 - \epsilon)}
\]
and
\[
|\mathbb{E}[\Delta_m^2] - 2\mathbb{E}[t_m]| \leq cR^{-(1/4 - \epsilon)}.
\]

6. **Skorokhod embedding and proof of Theorem 1.1**

Assume that \( D \in \mathcal{D} \) and write \( R = \text{inrad}(D) \). Recall that the Loewner driving function for the loop-erased random walk path \( \tilde{\gamma}^R = \psi_D(\gamma) \) in \( \mathbb{D} \) is denoted \( W(t) = W_0e^{i\vartheta(t)} \). In Proposition 3.1 we quantified that \( W_0 \) is close to uniform in terms of the inner radius \( R \). Hence, to prove Theorem 1.1 it will be enough to study \( \vartheta(t) \), and show that it is close to a standard Brownian motion with speed 2. One way of proving this is to couple (a variant of) this
process with Brownian motion, using Skorokhod embedding. The standard version of this technique is a method for coupling sums of i.i.d. random variables and Brownian motion in such a way that with large probability the processes are close at any given time. In the proof, a sequence of times \( \{ t_m \}_k \geq 1 \) is constructed which correspond to roughly constant increases in capacity for the time-reversed loop-erased random walk in \( D \). Although \( \{ \vartheta(t_m) \}_k \geq 1 \) is not a random walk, it is almost a martingale, and in view of Section 5 we can use the following version of Skorokhod embedding for martingales. A proof can be found in [8].

**Lemma 6.1** (Skorokhod embedding theorem). Suppose that \((M_k)_{k \leq K}\) is an \((\mathcal{F}_k)_{k \leq K}\) martingale, with \( \| M_{k+1} - M_k \|_\infty \leq \delta \) and \( M_0 = 0 \) a.s. There are stopping times \( 0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_K \) for standard Brownian motion \( B(t), t \geq 0 \), such that \((M_0, M_1, \ldots, M_K)\) and \((B(\tau_0), B(\tau_1), \ldots, B(\tau_K))\) have the same law. Moreover, we have for \( k = 0, 1, \ldots, K - 1 \),

\[
\mathbb{E} [\tau_{k+1} - \tau_k | B[0, \tau_k]] = \mathbb{E} [(B(\tau_{k+1}) - B(\tau_k))^2 | B[0, \tau_k]],
\]

\[
\mathbb{E} [(\tau_{k+1} - \tau_k)^p | B[0, \tau_k]] \leq C_p \mathbb{E} [(B(\tau_{k+1}) - B(\tau_k))^2p | B[0, \tau_k]],
\]

for constants \( C_p < \infty \), and also

\[
\tau_{k+1} \leq \inf \{ t \geq \tau_k : |B(t) - B(\tau_k)| \geq \delta \}.
\]

We will now prove Theorem 1.1 using Proposition 5.2 and Lemma 6.1. Although the structure of the proof is similar to that of Theorem 3.7 in [14], some estimates need to be done with more care, in particular to ensure that the exponent in our rate of convergence is optimal for the method used in this paper. Rather than including the key steps and referring the reader to [14], we write the proof in detail here to allow a more fluid reading.

The following result about the modulus of continuity of Brownian motion will be needed and is part of Lemma 1.2.1 of [4].

**Lemma 6.2.** Let \( B(t), t \geq 0 \), be standard Brownian motion. For each \( \epsilon > 0 \) there exists a constant \( C = C(\epsilon) > 0 \) such that the inequality

\[
P \left( \sup_{t \in [0, T-h]} \sup_{s \in (0, h]} |B(t + s) - B(t)| \leq \nu \sqrt{h} \right) \geq 1 - \frac{CT}{h} e^{-\frac{\nu^2}{2h+\epsilon}}
\]

holds for every positive \( \nu, T, \) and \( 0 < h < T \).

The proof of convergence in [14] uses Doob’s maximal inequality. In order to obtain an better rate of convergence, we need the following sharper maximal inequality for martingales from [7].
Lemma 6.3. Let $\xi_k, k = 1, \ldots, K$, be a martingale difference sequence with respect to the filtration $\mathcal{F}_k$. If $\lambda, u, v > 0$, then it follows that

$$
P \left( \max_{1 \leq j \leq K} \left| \sum_{k=1}^{j} \xi_k \right| \geq \lambda \right) \leq \sum_{k=1}^{K} P(\xi_k > u)$$

$$+ 2P \left( \sum_{k=1}^{K} E[\xi_k^2|\mathcal{F}_{k-1}] > v \right)$$

$$+ \exp \left\{ \lambda u^{-1} (1 - \log(\lambda uv^{-1})) \right\}.$$  

The strategy of the proof of Theorem 1.1 is the following. In Section 5, we showed that $E[\Delta_m]$ is close to zero. We use the domain Markov property to iterate this estimate to construct a sequence of random variables $\Delta_{m_k}$ that almost forms a martingale. We adjust the sequence $\Delta_{m_k}$ to make it into a martingale, so that we can couple it with Brownian motion, using Skorokhod embedding.

The next step is to show that the stopping times $\tau_k$ obtained by Skorokhod embedding are likely to be close to the capacities $2t_{m_k}$ for all $k \leq K$ for some appropriate $K$. This is done by showing separately that each of these two quantities has high probability of being close to the natural time (the quadratic variation) of the martingale.

Once that we know that the two processes run on similar clocks, all that is left to do is show that they are likely to be close at all times. The key tool needed for that is Lemma 6.2.

Proof of Theorem 1.1. Choose, without loss of generality, $T \geq 1$ and assume $R \geq R_1 > 8e^{20T}R_0$, where $R_0$ is the constant from Proposition 5.2. This choice of $R_1$ implies that Proposition 5.2 can be applied to $D$ slit by the initial piece of curve $\gamma$ up to capacity $20T$. Indeed, the Koebe one-quarter theorem implies that $\text{inrad}(D \setminus \beta)/\text{inrad}(D) \geq \exp\{-\text{cap}(\beta)/4\}$ if $D$ is slit by the curve $\beta$.

In what follows, most constants, which may depend on $T$, will be denoted by $c$ even though they may change from one line to the next. Define $m_0 = 0$ and $m_1 = m$, where $m$ is defined as in (6.1). Inductively for $k = 1, 2, 3, \ldots$, define

$$m_{k+1} = \min\{j > m_k : |t_j - t_{m_k}| \geq R^{-2\alpha} \text{ or } |\Delta_j - \Delta_{m_k}| \geq R^{-\alpha}\}.$$  

Define

$$K = \lceil 10TR^{2\alpha} \rceil$$  

and note that $t_{m_K} \leq 20T$. Set $\eta(R) = R^{-\alpha}$, where $\alpha = (1/4 - \epsilon)/3$. Then, by Proposition 5.2 and the domain Markov property of loop-erased random walk, we can find a constant $c$ such that

$$|E[\Delta_{m_{k+1}} - \Delta_{m_k}|\mathcal{F}_k]| \leq c\eta^{3}$$
and
\[|E[(\Delta_{m_{k+1}} - \Delta_{m_k})^2 - 2(t_{m_{k+1}} - t_{m_k})|\mathcal{F}_k]| \leq c\eta^3,\]
for \(k = 0, \ldots, K\), where \(\mathcal{F}_k\) is the filtration generated by \(\gamma_n[0, m_k]\).

For \(j = 1, \ldots, K\), define
\[\xi_j = \Delta_{m_j} - \Delta_{m_{j-1}} - E[\Delta_{m_j} - \Delta_{m_{j-1}}|\mathcal{F}_{j-1}].\]
This is clearly a martingale difference sequence and \(M\) defined by \(M_0 = 0\) and
\[M_k = \sum_{j=1}^{k} \xi_j\]
for \(k = 1, \ldots, K\), is a martingale with respect to \(\mathcal{F}_k\). Note that
\[\|M_k - M_{k-1}\|_\infty \leq 4\eta\]
by (5.2) for \(R\) sufficiently large.

Skorokhod embedding allows us to find stopping times \(\{\tau_k\}\) for standard Brownian motion \(B\) and a coupling of \(B\) with the martingale \(M\) (and the loop-erased random walk path \(\gamma\)) such that \(M_k = B(\tau_k), k = 0, \ldots, K\).

Consider the natural time associated to \(M\), namely
\[Y_k = \sum_{j=1}^{k} \xi_j^2, \quad k = 1, \ldots, K.\]
We will show that \(2t_{m_k}\) is close to the stopping time \(\tau_k\) for every \(k \leq K\) by showing separately that each of these quantities is close to \(Y_k\). We first show that \(Y_k\) is close to \(2t_{m_k}\) for every \(k \leq K\). Set \(\sigma_k = 2t_{m_k} - 2t_{m_{k-1}}\). For \(\phi = 3\eta|\log \eta|\) we have
\[
P \left( \max_{1 \leq k \leq K} \left| \sum_{j=1}^{k} (\xi_j^2 - \sigma_j) \right| \geq \phi \right)
\leq P \left( \max_{1 \leq k \leq K} \left| \sum_{j=1}^{k} (\xi_j^2 - E[\xi_j^2|\mathcal{F}_{j-1}]) \right| \geq \phi/3 \right)
\leq P \left( \max_{1 \leq k \leq K} \left| \sum_{j=1}^{k} (E[\xi_j^2|\mathcal{F}_{j-1}] - E[\sigma_j|\mathcal{F}_{j-1}]) \right| \geq \phi/3 \right)
+ P \left( \max_{1 \leq k \leq K} \left| \sum_{j=1}^{k} (\sigma_j - E[\sigma_j|\mathcal{F}_{j-1}]) \right| \geq \phi/3 \right)
=: p_1 + p_2 + p_3.\]
We estimate $p_1$ using the maximal inequality from Lemma 6.3 with $\lambda = \eta \log \eta$, $u = \eta/2$, and $v = e^{-2}\lambda u$. This gives

\begin{align*}
p_1 \leq & \sum_{j=1}^{K} P \left( |\xi_j^2 - E[\xi_j^2]| > \eta \right) \\
& + 2P \left( \sum_{j=1}^{K} E \left[ (\xi_j^2 - E[\xi_j^2]|F_{j-1})^2 \right] > e^{-2}\eta^2 \log \eta \right) \\
& + 2\eta.
\end{align*}

Since $\max_j |\xi_j| \leq 4\eta$, the first sum is equal to zero for $R$ sufficiently large. This bound and the definition of $K$ imply that

\[ \sum_{j=1}^{K} E \left[ (\xi_j^2 - E[\xi_j^2]|F_{j-1})^2 \right] \leq 16[10T]\eta^2. \]

It follows that the second sum also is zero if $R$ is large enough. To get a bound on $p_2$ we note that

\[ |E[\xi_j^2|F_{j-1}] - E[\sigma_j|F_{j-1}]| = |E[(\Delta_{m_j} - \Delta_{m_j-1})^2|F_{j-1}] - 2E[t_{m_j} - t_{m_j-1}|F_{j-1}] + O(\eta^4)| \leq c\eta^3. \]

Using the triangle inequality and summing over $j$ we see that $p_2 = 0$ if $R$ is large enough. Finally $p_3$ is estimated in a similar fashion as $p_1$ using the inequality $\max_k \sigma_k \leq 2\eta^2$. This shows that

\[ \mathbb{P} \left( \max_{1 \leq k \leq K} |Y_k - 2t_{m_k}| \geq 3\eta |\log \eta| \right) = O(\eta) \]

for all $R$ large enough.

We now show that $Y_k$ is close to $\tau_k$ for every $k \leq K$. Set $\zeta_k = \tau_k - \tau_{k-1}$ and let $G_k$ denote the $\sigma$-algebra generated by $B[0, \tau_k]$. Then, again with
\( \phi = 3\eta |\log \eta| \), we can write
\[
\mathbb{P} \left( \max_{1 \leq k \leq K} \left| \sum_{j=1}^{k} (\xi_j^2 - \zeta_j) \right| \geq \phi \right)
\leq \mathbb{P} \left( \max_{1 \leq k \leq K} \left| \sum_{j=1}^{k} (\xi_j^2 - \mathbb{E} \xi_j^2 | G_{j-1}) \right| \geq \phi/3 \right)
+ \mathbb{P} \left( \max_{1 \leq k \leq K} \left| \sum_{j=1}^{k} (\mathbb{E} \xi_j^2 | G_{j-1}) - \mathbb{E} \zeta_j | G_{j-1}) \right| \geq \phi/3 \right)
+ \mathbb{P} \left( \max_{1 \leq k \leq K} \left| \sum_{j=1}^{k} (\zeta_j - \mathbb{E} \zeta_j | G_{j-1}) \right| \geq \phi/3 \right)
=: p_4 + p_5 + p_6.
\]
The estimate of \( p_4 \) is identical to the estimate of \( p_1 \) above, and by (6.1) we conclude \( p_5 = 0 \). (Recall that \( \xi_j^2 = (B(\tau_j) - B(\tau_{j-1}))^2 \).) It remains to estimate \( p_6 \). We use Lemma 6.3 to get
\[
p_6 \leq \sum_{j=1}^{K} \mathbb{P} (|\zeta_j - \mathbb{E} \zeta_j | G_{j-1}) > \eta)\]
\[+ 2 \mathbb{P} \left( \sum_{j=1}^{K} \mathbb{E} \left[ (\zeta_j - \mathbb{E} \zeta_j | G_{j-1})^2 | G_{j-1}) > e^{-2} \eta^2 | \log \eta| \right) \]
\[+ 2 \eta. \]
By the definition of \( K \) as in (6.4), Chebyshev’s inequality, (6.1), (6.2), and (6.3) we have
\[
\sum_{j=1}^{K} \mathbb{P} (|\zeta_j - \mathbb{E} \zeta_j | G_{j-1}) > \eta) \leq \sum_{j=1}^{K} \eta^{-3} \mathbb{E} [|\zeta_j - \mathbb{E} \zeta_j | G_{j-1})^2]
\leq C \eta.
\]
Moreover, since \( \mathbb{E} [ (\zeta_j - \mathbb{E} \zeta_j | G_{j-1})^2 | G_{j-1}] = O(\eta^4) \), the probability (6.6) equals 0 for \( R \) large enough. Hence \( p_6 = O(\eta) \). This shows that
\[
\mathbb{P} \left( \max_{1 \leq k \leq K} |Y_k - \tau_k| > 3\eta | \log \eta| \right) = O(\eta),
\]
for \( R \) large enough.
Equations (6.5) and (6.7) now imply that
\[
\mathbb{P} \left( \max_{1 \leq k \leq K} |2t_{m_k} - \tau_k| > 3\eta | \log \eta| \right) = O(\eta),
\]
for \( R \) large enough.
ON THE RATE OF CONVERGENCE OF LERW TO SLE$_2$

Notice that (6.3) implies that for $k \leq K$,

\begin{equation}
\sup\{|B(t) - B(\tau_{k-1})| : t \in [\tau_{k-1}, \tau_k]\} \leq 4\eta, \tag{6.9}
\end{equation}

and by the definition of $m_k$ and (5.2) we have for $R$ large enough

\[\sup\{|\Delta_{m_k} - \varphi(t)| : t \in [t_{m_{k-1}}, t_{m_k}]\} \leq 2\eta.\]

Summing over $k$ and using the definition of $K$ we also get from (5.2) that

\[\sup\{|\Delta_{m_k} - M_k| : k \leq K\} \leq cT\eta.\]

As in [14], by summing, we have $Y_K + t_{m_K} \geq N\eta^2 \geq 10T$. Hence, the event that $t_{m_K} < 2T$ is contained in the event that $|Y_K - 2t_{m_K}| \geq 4T$. It follows from (6.5) that

\begin{equation}
P(t_{m_K} < 2T) = O(\eta). \tag{6.10}
\end{equation}

Set $h = h(\eta) = \eta \log \eta$ and consider the event

\[\mathcal{E} = \{t_{m_K} \geq 2T\} \cap \left\{ \sup_{t \in [0,2T-h]} \sup_{s \in (0,h]} |B(t+s) - B(t)| \leq \sqrt{6h \log h}\right\} \cap \left\{ \max_{k \leq K} |\tau_k - 2t_{m_k}| \leq 2h \right\}.
\]

Then in view of the inequalities (6.8), (6.10), and Lemma 6.2 (with $v = \sqrt{6 \log h}$, $\epsilon = 1$) we have $P(\mathcal{E}^c) = O(\eta \log \eta)$. Note that on $\mathcal{E}$ we have that

\[\sup\{|\varphi(t) - B(2t)| : t \in [0,T]\} \leq \max_{1 \leq k \leq K} \left(\sup\{|\varphi(t) - \Delta_{m_k}| : t \in [t_{m_{k-1}}, t_{m_k}]\} + |\Delta_{m_k} - B(\tau_k)|
\right.

\[+ \sup\{|B(\tau_k) - B(2t)| : t \in [t_{m_{k-1}}, t_{m_k}]\}\),

and the first two terms are $O(T\eta)$ uniformly in $k$. For the last term, we can use (6.9) to see that on $\mathcal{E}$, we have

\[\sup\{|B(\tau_k) - B(2t)| : t \in [t_{m_{k-1}}, t_{m_k}]\}
\]

\[= \sup\{|B(\tau_k) - B(s)| : s \in [2t_{m_{k-1}}, 2t_{m_k}]\}
\]

\[\leq \sup\{|B(\tau_k) - B(s)| : s \in [\tau_{k-1} - 2h, \tau_k + 2h]\}
\]

\[\leq 4\eta + \sup\{|B(\tau_{k-1}) - B(s)| : s \in [\tau_{k-1} - 2h, \tau_k - 1]\}
\]

\[+ \sup\{|B(\tau_k) - B(s)| : s \in [\tau_k, \tau_k + 2h]\}
\]

\[\leq 4\eta + c(\eta \varphi(1/\eta))^{1/2},
\]

where $\varphi$ is a subpower function; that is, $\varphi(x) = o(x^\epsilon)$ for any $\epsilon > 0$. It follows that we may couple $\varphi$ and $B$ so that

\[P\left(\sup_{t \in [0,T]} |\varphi(t) - B(2t)| > c_1 T\eta^{1/2}\varphi_1(1/\eta)\right) < c_2 \eta \log \eta,
\]
where we recall that $\eta(R) = R^{-(1/12-\epsilon)}$, and $\varphi_1$ is also a subpower function. This in turn implies that there exist constants $c_1, c_2$ such that for every $\epsilon > 0$, all $R$ sufficiently large,

$$
P\left( \sup_{t \in [0,T]} |\vartheta(t) - B(2t)| > c_1 R^{-(1/24-\epsilon)} T \right) < c_2 R^{-(1/24-\epsilon)}. $$

Together with Proposition 3.1 this estimate concludes the proof of the theorem. $\square$

7. Hausdorff convergence

Recall that the Hausdorff distance between two compact sets $A, B \subset \mathbb{C}$ is defined by

$$
d_H(A, B) = \inf \left\{ \epsilon > 0 : A \subset \bigcup_{z \in B} B(z, \epsilon), B \subset \bigcup_{z \in A} B(z, \epsilon) \right\}. $$

In this section we prove a rate of convergence result for the pathwise convergence with respect to Hausdorff distance. We use the notation from Section 1 and in addition we let $\tilde{\gamma}$ denote the radial SLE$_2$ path in $\mathbb{D}$ started from 1. In this section it is convenient to parameterize $\tilde{\gamma}$ by half-plane capacity. For the rest of this section we let $\varphi(z) := i - z / i + z$.

Note that $\varphi : \mathbb{H} \to \mathbb{D}$, $\varphi(i) = 0$, and $\varphi(\infty) = -1$. We parameterize $\tilde{\gamma}(t)$, $0 \leq t \leq t_0$, so that the image in $\mathbb{H}$ satisfies

$$
\text{hcap}[\varphi^{-1}(\tilde{\gamma}[0, t])] = 2t.
$$

We assume that $t_0$ is sufficiently small so that $\text{cap}(\tilde{\gamma}[0, t_0]) < \infty$ with probability one. (One can take, e.g., $t_0 = 1/16$.) We allow all constants in this section to depend on $t_0$.

Our approach uses a uniform derivative estimate for radial SLE$_2$ that we derive from an estimate on the growth of the derivative of the chordal SLE mapping from [16]. This is the reason why it is convenient to use the half-plane capacity parameterization.

**Theorem 7.1.** Let $0 < t \leq t_0$ where $t_0$ is sufficiently small. There exists $c < \infty$ with the property that for $n$ sufficiently large there is a coupling of $\tilde{\gamma}^n$ with $\tilde{\gamma}$ such that

$$
P \left( d_H (\tilde{\gamma}^n[0, t] \cup \partial \mathbb{D}, \tilde{\gamma}[0, t] \cup \partial \mathbb{D}) > c (\log n)^{-p} \right) < c (\log n)^{-p}
$$

whenever $p < (15 - 8\sqrt{3})/66$.

**Lemma 7.2.** Let $0 < T < \infty$ be fixed. For $j = 1, 2$, let

$$
h_j(t, z) : \mathbb{D} \to \mathbb{D} \setminus \gamma_j[0, t]
$$
be the solution to the radial Loewner equation generated by the simple curve \( \gamma_j \) with \( W_j \) as driving function and write \( h_j(z) = h_j(T, z) \). Suppose that
\[
\sup_{0 \leq t \leq T} |W_1(t) - W_2(t)| < \epsilon,
\]
where \( \epsilon > 0 \) is sufficiently small. Suppose further that there exists \( 0 < \beta < 1 \) such that
\[
|h_1'(1 - \delta)\zeta| \leq \delta^{-\beta}, \quad \delta < |\log \epsilon|^{-1/2},
\]
for all \( \zeta \in \partial \mathbb{D} \). Then there is a constant \( c < \infty \) depending only on \( T \) and \( \beta \) such that
\[
d_H(\gamma_1[0, T] \cup \partial \mathbb{D}, \gamma_2[0, T] \cup \partial \mathbb{D}) \leq c |\log \epsilon|^{-(1-\beta)/2}.
\]

**Proof.** It follows by considering the reverse-time radial Loewner equation that there is a \( c_0 < \infty \) depending only on \( T \) such that
\[
|h_1(z) - h_2(z)| \leq \epsilon \left( e^{c_0/\delta^2} - 1 \right)
\]
whenever \( |z| \leq 1 - \delta \). An identical calculation for the reverse-time chordal Loewner equation may be found in Section 4.9 of [13].

Hence, by taking \( \delta = \delta_0 := 2c_0/|\log \epsilon|^{1/2} \), we see that
\[
\sup_{|z| \leq 1 - \delta_0} |h_1(z) - h_2(z)| \leq c \epsilon^{1/2}.
\]

Using this, Cauchy’s estimate implies that
\[
\sup_{|z| \leq 1 - \delta_0} |h_1'(z) - h_2'(z)| \leq c(\epsilon|\log \epsilon|)^{1/2}.
\]

For ease of notation we shall write
\[
\dot{\gamma}_j = \gamma_j[0, T] \cup \partial \mathbb{D}, \quad j = 1, 2,
\]
in the sequel. Fix \( \zeta \in \partial \mathbb{D} \). Then, using the assumption (7.1), the Koebe distortion theorem (Lemma 2.1) and an integration yield
\[
|h_1(\zeta) - h_1((1 - \delta)\zeta| \leq c\delta_0^{1-\beta}, \quad 0 \leq \delta \leq \delta_0.
\]

Hence, for some constant \( c < \infty \),
\[
h_1(|z| \geq 1 - \delta_0) \subset \bigcup_{z \in \dot{\gamma}_1} B(z, c\delta_0^{1-\beta}).
\]

It follows in view of (7.2) that, with perhaps a different \( c \), we can replace \( h_1 \) by \( h_2 \) in the last expression. Clearly \( \dot{\gamma}_2 \subset h_2(|z| \geq 1 - \delta_0) \), so it remains to show that
\[
\dot{\gamma}_1 \subset \bigcup_{z \in \dot{\gamma}_2} B(z, c\delta_0^{1-\beta}).
\]
Write \( w = \lim_{\delta \to 0} h_1((1 - \delta)\zeta) \in \hat{\gamma}_1 \). Let \( \hat{w} \in \hat{\gamma}_2 \) be a point closest to \( h_2((1 - \delta)\zeta) \). Then by Koebe’s estimate, (7.3), and (7.1) we have

\[
|\hat{w} - h_2((1 - \delta)\zeta)| \leq c\delta_0|h_2'((1 - \delta)\zeta)|
\]

\[
\leq c\delta_0 \left( |h_1'((1 - \delta)\zeta)| + c'(\epsilon|\log \epsilon|)^{1/2} \right)
\]

\[
\leq c\delta_0^{1-\beta}.
\]

Hence, using (7.2) and (7.1),

\[
|\hat{w} - w| \leq |\hat{w} - h_2((1 - \delta)\zeta)| + |h_2((1 - \delta)\zeta) - h_1((1 - \delta)\zeta)|
\]

\[
+ |h_1((1 - \delta)\zeta) - w|
\]

\[
\leq c\left( \delta_0^{1-\beta} + \epsilon^{1/2} + \delta_0^{1-\beta} \right),
\]

and this implies that (7.4) holds, which concludes the proof.

The idea is now to use the coupling whose existence follows from Theorem 11 and an estimate on the growth of the derivative of the radial SLE\(_2\) mapping to find an event with large probability on which Lemma 7.2 can be applied. We will derive the needed estimate from the corresponding chordal estimate in 16. We let \( F : \mathbb{H} \to \mathbb{H} \setminus \varphi^{-1}(\hat{\gamma}[0, t]) \) satisfy the hydrodynamical normalization, that is,

\[
F(z) = z + 2t/z + O(1/z^2), \quad z \to \infty.
\]

The mapping \( F \) has the distribution of a chordal SLE(2; -4) mapping with force point \( i \), as is further explained in 19. This also means that \( F \) can be viewed as a chordal SLE\(_2\) mapping weighted by a certain local martingale, namely, \( M_t = y_t/(x_t^2 + y_t^2) \). (Note that \( M_t \) is a local martingale with respect to the chordal SLE\(_2\) path.) Here \( x_t + iy_t = \hat{G}_t(i) - \sqrt{2}B_t \) and \( \hat{G}_t : H_t \to \mathbb{H} \) is the standard uniformizing chordal SLE\(_2\) Loewner chain. Let \( U_s \) denote the chordal driving function for \( F_s \), \( 0 \leq s \leq t_0 \), that is, \( U_s \) is the unique continuous real-valued function such that for \( 0 \leq s \leq t_0 \),

\[
\partial_s F_s = -\partial_x F_s \frac{2}{z - U_s}, \quad F_0 = z, \quad z \in \mathbb{H}.
\]

By Girsanov’s theorem, under the measure obtained by weighting by \( M_{t_0} \), a standard linear Brownian motion with speed 2 has the same distribution as the process \( U_s \), \( 0 \leq s \leq t_0 \) under the “unweighted” measure. Since \( y_0 = 1 \) the Loewner equation implies that \( y_t \geq \sqrt{1 - 4t} \) if \( t < 1/4 \). Consequently if \( 0 \leq t \leq t_0 < 1/4 \) we see that \( M_t \leq 1/\sqrt{1 - 4t_0} \) and we can write

\[
\mathbb{P}(|F'(x + iy)| \geq \eta y^{-1}) = \mathbb{E} [1\{ |\hat{F}'(x + iy)| \geq \eta y^{-1} \} M_{t_0}]
\]

\[
\leq \frac{\mathbb{P}(|\hat{F}'(x + iy)| \geq \eta y^{-1})}{\sqrt{1 - 4t_0}},
\]

where \( \hat{F} = \hat{G}_t^{-1} \) and \( 0 < \eta < 1 \). The last probability can be estimated using Corollary 3.5 of 16.
We observe now that the radial SLE$_2$ mapping
\[ f : \mathbb{D} \to \mathbb{D} \setminus \gamma[0, t], \quad f(0) = 0, \quad f'(0) > 0, \]
can be written as
\[ f = \varphi \circ F \circ \Delta, \]
where, for $G = F^{-1}$,
\[ \Delta(z) := \frac{zG(i) - \lambda G(i)}{z - \lambda} \]
is a random Moebius transformation. Note that $\Delta : \mathbb{D} \to \mathbb{H}$ and $\Delta(0) = G(i)$, $\Delta(\lambda) = \infty$, $\lambda = f^{-1}(-1)$. Indeed, if $|\lambda| = 1$ then the right hand side of (7.6) maps $\mathbb{D}$ onto $\mathbb{D} \setminus \gamma[0, t]$ and fixes the origin where it has derivative $\frac{|iF'(G(i)) \Im G(i)|}{|z - \lambda|^2}$. Hence we can choose a unique $\lambda$ with $|\lambda| = 1$ such that this derivative is real and positive. But this defines $f$ by uniqueness of Riemann mappings. We can then see that $f(\lambda) = -1$. The chain rule gives
\[ |f'(z)| = |\varphi'(F(\Delta(z)))||F'(\Delta(z))||\Delta'(z)|. \]
Note that
\[ |\varphi'(F(\Delta(z)))||\Delta'(z)| = \frac{2}{|F(\Delta(z)) + i|^2} \frac{|2\lambda \Im G(i)|}{|z - \lambda|^2} \leq c_0, \quad z \in \mathbb{D}, \]
by the hydrodynamical normalization and a simple computation.

We can now derive the needed estimate for the growth of the derivative of the radial SLE$_2$ mapping. For the purpose of stating the lemma, define $\beta_0 = (14 + 4\sqrt{6})/25$ and set
\[ \rho(\beta) = \frac{25\beta^2 - 28\beta + 4}{8\beta}. \]

**Lemma 7.3.** There exists a subpower function $\phi$ such that for $\beta_0 < \beta < 1$ and all sufficiently small $\delta_0 > 0$

\[ \mathbb{P} \left( \sup_{z \in \partial \mathbb{D}} |f'(z)| \leq \delta^{-\beta}, \delta < \delta_0 \right) \leq \phi(1/\delta_0)\delta_0^{\rho(\beta)}. \]

**Proof.** Let $c_1$ be fixed for the moment and let $\mathcal{V} = \mathcal{V}_{\delta_0}$ be the event that

\[ \sup_{0 \leq t \leq t_0} |U_t| \leq c_1 \sqrt{\log \delta_0^{-1}}. \]

Recall that $U_t$ has the same distribution as a Brownian motion weighted by the local martingale $M$. Consequently, by our assumption that $t \leq t_0$, if $c_1$ is chosen sufficiently large, we can compare with a Brownian motion using the reflection principle to see that
\[ \mathbb{P}(\mathcal{V}') = O(\exp\{-\log \delta_0^{-1}\}) = O(\delta_0). \]
We claim that there are constants $0 < c, c' < \infty$ such that if
\[
A := \left[ -c\sqrt{\log \delta_0^{-1}}, c\sqrt{\log \delta_0^{-1}} \right] \times (0, c\delta_0 \log \delta_0^{-1}],
\]
then
\[
P \left( \sup_{\zeta \in \partial D} |f'( (1 - \delta) \zeta)| \leq \delta^{-\beta}, \delta < \delta_0 \right)^c \cap V \right) \leq P \left( \exists w \in A : |F'(w)| \geq c'[\text{Im}(w \log \delta_0^{-1}]^{-\beta} \right).
\]
(7.11)

To prove (7.11), write $\hat{\gamma}(t) = \varphi^{-1}(\tilde{\gamma}(t))$ for the image of the radial SLE$_2$ path in $\mathbb{H}$ parameterized by half-plane capacity. Note that the Loewner equation implies that on $V$
\[
\hat{\gamma}[0, t_0] \subset \left[ -c_1 \sqrt{\log \delta_0^{-1}}, c_1 \sqrt{\log \delta_0^{-1}} \right] \times [0, 2\sqrt{t_0}].
\]
Consequently, a harmonic measure estimate shows that there exists $c_2 < \infty$ such that the pre-image in $\mathbb{R}$ satisfies
\[
G(\hat{\gamma}[0, t_0]) \subset \left[ -c_2 \sqrt{\log \delta_0^{-1}}, c_2 \sqrt{\log \delta_0^{-1}} \right]
\]
on $V$. We can assume that $c_2 \geq c_1$. Define
\[
L := 2c_2 \sqrt{\log \delta_0^{-1}}, \quad I := [-L, L],
\]
where $c_2$ is as in (7.12). (Note that on $V$ we have that $f^{-1}(\hat{\gamma}[0, t_0]) \subset \Delta^{-1}(I/2) \subset \partial \mathbb{D}$.) The normalization of $F$ at infinity, Schwarz’s reflection principle, and Koebe’s distortion theorem (Lemma 2.1) imply that the estimates
\[
|F'(x + iy)| \asymp 1, \quad x \in \mathbb{R} \setminus I,
\]
hold on the event $V$. Here, and in what follows, $\asymp$ means that each side is bounded by a constant times the other. Indeed, if $S(z) = (z + 1/z)/4$, then since the function $z \mapsto 4F(LS(z))/L$ belongs to the class $\Sigma$ of normalized conformal mappings of the exterior unit disk it has a uniformly bounded derivative for, e.g., $|z| > 2$. (Here we have extended $F$ to $\mathbb{C} \setminus [-L/2, L/2]$ by Schwarz reflection without changing its symbol.) Moreover, since $t \leq t_0$, the Loewner equation implies that $\text{Im} G(i) \asymp 1$ and that there is a constant $c_3 < \infty$ such that
\[
|\text{Re} G(i)| \leq c_3.
\]
Further, there is a constant $c_4 > 0$ such that
\[
|\Delta^{-1}(x) - \lambda| = \frac{2 \text{Im} G(i)}{|x - G(i)|} \geq \frac{c_4}{L}, \quad x \in I.
\]
Note also that $\text{Im} \Delta(z) = \text{Im} G(i)(2\delta - \delta^2)/|z - \lambda|^2$. Consequently, we can choose a constant for $A$ such that
\[
\Delta \left( \{(1 - \delta)\zeta : 0 \leq \delta \leq \delta_0, \zeta \in \Delta^{-1}(I) \} \right) \subset A
\]
and so (7.11) follows from (7.10) and (7.8).

By (7.10) it remains to estimate the right-hand side of (7.11). To this end, we consider a Whitney decomposition of $A$, that is, a partition using dyadic rectangles

$$S_{j,k} = \{ x + iy : 2^{-j-1} \leq y \leq 2^{-j}, (k - 1)2^{-j} \leq x \leq k2^{-j} \},$$

for integer $j \geq \lceil -\log(c_0L^2) \rceil$ and $-L2^j \leq k \leq L2^j$, where $c$ is the constant used in the definition of $A$. Let $z_{j,k}$ be the center point of $S_{j,k}$. By the Koebe distortion theorem (Lemma 2.1) $|F'(z)|/|F'(z_{j,k})| \asymp 1$ for $z \in S_{j,k}$.

By (7.5) we can now apply Corollary 3.5 of [16] to estimate

$$\mathbb{P} \left( \exists w \in A : |F'(w)| > c (L^2 \text{Im } w)^{-\beta} \right) \leq \sum_{j = \lceil -\log(c_0L^2) \rceil}^{\infty} \sum_{k = -L2^j}^{L2^j} \mathbb{P} \left( |F'(z_{j,k})| \geq c (2j/L^2)^{\beta} \right) \leq \phi(1/\delta_0) \sum_{j = \lceil -\log(c_0L^2) \rceil}^{\infty} (2^{-j})^{\rho_0(\beta)}.
$$

Here $\rho_0(\beta) := -1 - 2b + \beta b(5 - 2b)$ with $b \in [0, 3]$ (see [16] for more details) and $\phi$ is a subpower function. When $\rho_0(\beta) > 0$ the last term is finite and bounded above by $\phi_1(1/\delta_0)\delta_0^{\rho_0(\beta)}$ for a possibly different subpower function $\phi_1$. We maximize over $b$ to find $\rho(\beta) = (25\beta^2 - 28\beta + 4)/(8\beta)$, which is positive when $\beta > \beta_0 = (14 + 4\sqrt{6})/25$. $\square$

**Proof of Theorem 7.1**. Let $0 < T < \infty$ be fixed for the moment. Set $\varepsilon_n = n^{-1/(24 - \epsilon)}$ for some fixed $0 < \epsilon < 1/24$. By Theorem 1.1 if $n$ is sufficiently large, there is a coupling of $\tilde{\gamma}^n$ with Brownian motion $B$ started uniformly on $\partial \mathbb{D}$ and an event $E_n$ with $\mathbb{P}(E_n^c) < \varepsilon_n$ on which for sufficiently large $n$ the estimate

$$\sup_{0 \leq s \leq T} |W_n(s) - e^{iB(2s)}| \leq \varepsilon_n,$$

holds, where $W_n$ is the radial Loewner driving function for $\tilde{\gamma}^n$. We extend the coupling to include $\tilde{\gamma}$, the radial SLE$_2$ path, which is obtained deterministically from $B$. Let $f : \mathbb{D} \to \mathbb{D} \setminus \tilde{\gamma}[0, t]$ be the radial SLE$_2$ mapping evaluated at a fixed half-plane capacity time $t \leq t_0$ as above. We can assume that $T > \text{cap}(\tilde{\gamma}[0, t_0])$. Set

$$\delta_n = |\log \varepsilon_n|^{-1/2},$$

and let $\mathcal{F}_n(\beta)$ be the event that

$$\sup_{\zeta \in \partial \mathbb{D}} |f'((1 - \delta)\zeta)| \leq \delta^{-\beta}, \quad \delta \leq \delta_n.$$

By Lemma 7.3 there exists a subpower function $\phi$ such that

$$\mathbb{P}(\mathcal{F}_n(\beta)^c) \leq \phi(1/\delta_n)\delta_0^{\rho(\beta)},$$

where $\beta_0 < \beta < 1$ and $\rho$ is given in (7.9).
On the event \( E_n \cap F_n(\beta) \), which has probability at least \( 1 - \phi(1/\delta_n)\delta_n^\alpha(\beta) \), we can apply Lemma 7.2 to see that
\[
d_H(\tilde{\gamma}_n[0, t] \cup \partial D, \tilde{\gamma}[0, t] \cup \partial D) \leq c\delta_n^{1-\beta}.
\]
We solve \( \rho(\beta) = 1 - \beta \) for \( \beta_0 < \beta < 1 \) to find \( \rho(\beta) = (15 - 8\sqrt{3})/33 \approx 0.035 \) and this completes the proof. \( \Box \)

8. An estimate for Green’s functions

In this section, we derive a uniform bound for the difference between the discrete and continuous Green’s functions in a simply connected grid domain \( D \in \mathcal{D} \) with \( \text{inrad}(D) = R > 0 \), and let \( V = V(D) = D \cap \mathbb{Z}^2 \). If \( x, y \in V \) with \( x \neq y \) and \( |\psi_D(x)| \leq \rho \), then
\[
(8.1) \quad \left| G_D(x, y) - \frac{2}{\pi} g_D(x, y) - k_{y-x} \right| \leq cR^{-(1/2-\epsilon)}
\]
where
\[
k_z = k_0 + \frac{2}{\pi} \log |z| - a(z),
\]
a(z) is the potential kernel as in (2.4), and \( k_0 = (2\varsigma + 3 \ln 2)/\pi \) where \( \varsigma \) is Euler’s constant.

In what follows, the symbol \( \mathbb{P}^x \) will refer to the measure of random walk or Brownian motion (or both) started at \( x \). There will be no ambiguity as to which process is referred to by \( \mathbb{P}^x \). The symbol \( \mathbb{E}^x \) will represent the expected value associated with \( \mathbb{P}^x \). If \( x = 0 \), we just write \( \mathbb{P} \) and \( \mathbb{E} \). The proof of Theorem 8.1 relies on a fine estimate of \( |\mathbb{E}^x[\log |B_T|] - \mathbb{E}^x[\log |S_\tau|]| \) when \( B \) and \( S \) are linked by a strong coupling, where \( T \) and \( \tau \) are the exiting times from \( D \) by \( B \) and \( S \), respectively. The majority of this section is devoted to the derivation of this estimate. Before stating it, we introduce a symbol which we will use throughout the section to alleviate the otherwise cumbersome notation. If for two functions \( f \) and \( g \) defined on some domain \( A \) there exists a constant \( c > 0 \) such that \( f(x) \leq cg(x) \) for all \( x \in A \), we will write \( f(x) \lesssim g(x) \). It will always be clear from context what the variable and the domain are.
Lemma 8.2. For any $\epsilon > 0$ there exists a constant $c$ such that if $D \in \mathcal{D}$ is a grid domain with $\text{inrad}(D) = R > 0$, $B$ is a planar standard Brownian motion, and $S$ is a two-dimensional simple random walk, then for any $x \in \mathbb{Z}^2$ with $|x| \leq R^2$,

$$| \mathbb{E}^x[\log |B_T|] - \mathbb{E}^x[\log |S_{\tau}|] | \leq cR^{-(1/2-\epsilon)},$$

where $T = \inf\{t \geq 0 : B_t \notin D\}$ and $\tau = \inf\{k \geq 0 : S_k \notin D\}$.

Remark. We believe that the bound $R^{-(1/2-\epsilon)}$ in Lemma 8.2 can be replaced by $(|x| \vee R)^{-1/2-\epsilon}$ and give a heuristic argument for it in the proof below, but prove the weaker version, as it is all we need.

At the heart of the proof lies a coupling argument, the strong approximation of Komlós, Major, and Tusnády, which we state here and which we will refer to below as the KMT coupling. For a proof of the one-dimensional case, see [10] and for the two-dimensional case, see [1].

Theorem 8.3. There exists a coupling of planar Brownian motion $B$ and two-dimensional simple random walk $S$ with $B_0 = S_0$, and a constant $c > 0$ such that for every $\lambda > 0$, every $n \in \mathbb{R}^+$,

$$\mathbb{P}\left( \sup_{0 \leq t \leq n} |S_{2t} - B_t| > c(\lambda + 1) \log n \right) \leq cn^{-\lambda}.$$ 

In Theorem 8.3, $S$ represents random walk interpolated linearly between integer times. For the rest of this section, we use the same notation and in addition run the random walk at twice the Brownian speed. This way, on the probability space of Theorem 8.3 it is $B_t$ and $S_t$ that are close, rather than $B_t$ and $S_{2t}$.

A few other technical ingredients will be needed to cook up the proof of Lemma 8.2. Among them are Beurling estimates (see Section 2) and the following large deviations estimates giving an upper bound for the probability that in time $n$ random walk or Brownian motion travel much beyond distance $\sqrt{n}$ or remain in a disk of radius much smaller than $\sqrt{n}$. For the proofs, see [2].

Lemma 8.4. If $B$ is a planar Brownian motion and $S$ is a planar simple random walk, there exists a constant $c < \infty$ such that for every $n \geq 0$, every $r \geq 1$,

$$\mathbb{P}\left( \sup_{0 \leq t \leq n} |B_t| \geq r \sqrt{n} \right) \leq c \exp\{-r^2/2\}$$

and

$$\mathbb{P}\left( \max_{0 \leq k \leq 2n} |S_k| \geq r \sqrt{n} \right) \leq c \exp\{-r^2/4\}.$$ 

Lemma 8.5. If $B$ is a planar Brownian motion and $S$ is a planar simple random walk, there exists a constant $c > 0$ such that for every $n \geq 0$, every
\( r \geq 1, \)

\[ \mathbb{P}\left( \sup_{0 \leq t \leq n} |B_t| \leq r^{-1}\sqrt{n} \right) \leq \exp \{-cr^2\} \]

and

\[ \mathbb{P}\left( \max_{0 \leq k \leq 2n} |S_k| \leq r^{-1}\sqrt{n} \right) \leq \exp \{-cr^2\} . \]

**Proof of Lemma 8.2.** We assume throughout the proof that \( B \) and \( S \) are coupled as in Theorem 8.3. Recall that in this section, \( S \) represents random walk run at twice the usual speed. The main difficulty in deriving the optimal estimate lies in the fact that there are a number of different kinds of configurations that can make \(|\log |B_T|-\log |S_T||\) very large, for instance the rare events of Theorem 8.3 or the event that \( T \) is unusually large. To obtain a precise estimate, we fix \( \epsilon > 0 \) and define

\[ \sigma = \min\{T, \tau\}, \]

\[ A_k = \{|B_T| \in [R^{1+ke}, R^{1+(k+1)e}]\}, \quad k \geq 0, \]

\[ B_\ell = \{\sigma \in [R^{e\ell} - 1, R^{(\ell+1)e} - 1]\}, \quad \ell \geq 0, \]

\[ C_m = \{|B_T - S_T| \in [R^{me} - 1, R^{(m+1)e} - 1]\}, \quad m \geq 0, \]

\[ \mathcal{H}_r = \left\{ \sup_{0 \leq t \leq \sigma} |B_t - S_t| \in [cr\log\sigma, c(r + 1)\log\sigma) \right\}, \quad r \geq 0, \]

where \( c \) is the constant from Theorem 8.3. Throughout the proof of this theorem, multiplicative constants will be allowed to depend on \( \epsilon \).

Suppose that \( d := \text{dist}(x, \partial D) \), and assume \( s = s(\epsilon) \) is defined so that \( d = R^{s_*} \). In other words, \( s = (\log d)/((\epsilon \log R)) \). Then, clearly,

\[ |\mathbb{E}^\sigma[\log |B_T|] - \mathbb{E}^\sigma[\log |S_T|]| \]

\[ \leq \sum_{k, \ell, m, r \geq 0} \mathbb{E}^\sigma\left[ |\log |B_T| - \log |S_T|| \mathbb{1}\{A_k \cap B_\ell \cap C_m \cap \mathcal{H}_r\} \right]. \]

(8.2)

The dominant term in the sum above corresponds essentially to the terms where \( T \approx d^2, \ |B_T| \approx |x| \lor R, \) and \( |B_T - S_T| \approx |x| \lor R \). We give a rough heuristic argument for why this should be the case.

By the Beurling estimate (see Lemma 2.3 as well as Lemma 2.4 for the discrete version), the most likely place for \( B \) and \( S \) to leave \( D \) is at some point \( z \) with \( |z| = O(|x| \lor R) \). Assume that \( T < \tau \). Typically, under the KMT coupling, \( d(S_T, \partial D) = O(\log R) \). By the Beurling estimate, the distance \( S \) travels after \( T \) before leaving \( D \) will be of order \( R^a \) with probability of order \( (\log R)^{1/2} R^{-a/2} \). In that case, if \( R^a = O(|x| \lor R) \), then

\[ |\log |B_T| - \log |S_T|| = O\left(\log \left(1 + \frac{|B_T - S_T|}{|B_T|}\right)\right) = O\left(\frac{R^a}{|x| \lor R}\right). \]

On the other hand, if \( R^a \geq |x| \), then \( |\log |B_T| - \log |S_T|| = O(\log R) \) (of course the constant increases with \( a \) but this is taken care of easily). The
value of \( \alpha \) which maximizes the product of \( \log |B_T| - \log |S_T| \) and the corresponding probability is that for which \( R^\alpha \approx |x| \lor R \), which corresponds to \( |B_T - S_T| \approx |x| \lor R \). Consequently we expect that

\[
|E^x[\log |B_T|] - E^x[\log |S_T|]| \approx (\log R)^{1/2}(|x| \lor R)^{-1/2}.
\]

We now prove Lemma \textbf{8.2} formally. We begin by estimating \( \log |B_T| - \log |S_T| \), noting that this quantity can be bounded using just the information contained in \( A_k \) and \( C_m \).

If \( |B_T - S_T| < |B_T| \), we can use Taylor’s expansion to note that

\[
\log |B_T| - \log |S_T| = \log \left( 1 + \frac{S_T - B_T}{B_T} \right) \lesssim \frac{|B_T - S_T|}{|B_T|},
\]

so that if \( m \leq |k - 1 + 1/\epsilon| \), then on \( A_k \cap C_m \),

\[
(8.3) \quad |\log |B_T| - \log |S_T|| \lesssim R^{(m+1)\epsilon - 1 - \epsilon}. \tag{8.3}
\]

On the other hand, \( |B_T| \leq R^{2\epsilon}|B_T - S_T| \) when \( m \geq |k + 1/\epsilon| \), so by the triangle inequality, \( |S_T| \lesssim R^{2\epsilon}|B_T - S_T| \) and another application of the inequality gives

\[
|\log |B_T| - \log |S_T|| \lesssim |\log |B_T|| + |\log |S_T|| \lesssim \epsilon \log R + |\log |B_T - S_T||,
\]

so that on \( C_m \),

\[
(8.4) \quad |\log |B_T| - \log |S_T|| \lesssim m \epsilon \log R. \tag{8.4}
\]

We now focus on the expected value of the indicator function in \textbf{8.2}. The difficulty lies in finding a way to use the strong Markov property for the two processes which, under the KMT coupling, are not jointly Markov. (See also the proof of Proposition \textbf{8.1} where we were faced with the same problem.) We first note that on \( B_\ell \cap \mathcal{H}_r \),

\[
\sup_{0 \leq t \leq R^{\ell \epsilon}} |B_t - S_t| \leq c(r + 1)(\ell + 1) \log R.
\]

We let \( c \) be the constant of Theorem \textbf{8.3} and define

\[
\xi_{r,\ell} = \inf \left\{ t \geq 0 : \min\{d(B_t, \partial D), d(S_t, \partial D)\} \leq 2c(r + 1)(\ell + 1) \log R \right\},
\]

\[
B'_{i,r,\ell} = \left\{ \xi_{r,\ell} \in [R^{i \epsilon} - 1, R^{(i+1)\epsilon} - 1) \right\}, \quad 0 \leq i \leq \ell,
\]

\[
\mathcal{H}'_{j,r,\ell} = \left\{ \sup_{0 \leq t \leq \xi_{r,\ell}} |B_t - S_t| \in [cj \log R^{(\ell+1)\epsilon}, c(j + 1) \log R^{(\ell+1)\epsilon}) \right\}, \quad 0 \leq j \leq r,
\]

which implies

\[
(8.5) \quad \mathbb{P}^x(A_k \cap B_\ell \cap C_m \cap \mathcal{H}_r) = \sum_{i=0}^{\ell} \sum_{j=0}^{r} \mathbb{P}^x(A_k \cap B_\ell \cap C_m \cap \mathcal{H}_r \cap B'_{i,r,\ell} \cap \mathcal{H}'_{j,r,\ell}).
\]
We note that
\[
\begin{align*}
\mathbb{P}^x(A_k \cap B_{\ell} \cap C_{\ell} \cap H_{\ell} \cap B'_{i,r,\ell} \cap H'_{j,r,\ell}) \\
\leq \min \left\{ \mathbb{P}^x(B_{\ell} \cap H_{\ell}), \mathbb{P}^x(B_{\ell}), \mathbb{P}^x(A_k \cap C_{\ell} \cap B'_{i,r,\ell} \cap H'_{j,r,\ell}) \right\}
\leq \min \left\{ \mathbb{P}^x \left( \sup_{0 \leq t \leq R^{(\ell + 1)c}} |B_{\ell} - S_t| \geq cr \log R^{c\ell} \right), \mathbb{P}^x(B_{\ell}), \mathbb{P}^x \left( A_k \cap C_{\ell} \cap B'_{i,r,\ell} \cap H'_{j,r,\ell} \right) \right\}
\end{align*}
\]

for any positive integer \( \ell \), where the last inequality follows from Theorem 8.3. Recall \( d = \text{dist}(x, \partial D) \) and \( s = s(\epsilon) = (\log d)/(\epsilon \log R) \) so that \( d = R^{s\epsilon} \).

If \( \ell \geq 2s + 3 \), then by Lemmas 8.5, 2.3, and 2.4,
\[
\mathbb{P}^x(B_{\ell}) \leq \mathbb{P}^x \left( \sup_{0 \leq t \leq R^{c\ell}} |B_{\ell} - x| \leq R^{c\ell/2} \log^2 R^{c\ell/2} \right)
\]
\[
+ \mathbb{P}^x \left( \sup_{0 \leq t \leq R^{c\ell}} |B_{\ell} - x| \geq R^{c\ell/2} \log^2 R^{c\ell/2} \right)
\]
\[
+ \mathbb{P}^x \left( \sup_{0 \leq t \leq R^{c\ell}} |S_{\ell} - x| \leq R^{c\ell/2} \log^2 R^{c\ell/2} \right)
\]
\[
+ \mathbb{P}^x \left( \sup_{0 \leq t \leq R^{c\ell}} |S_{\ell} - x| \geq R^{c\ell/2} \log^2 R^{c\ell/2} \right)
\]
\[
\leq 2R^{-c(\ell/2)^2} \log R + \mathbb{P}^x(B[0, \xi] \cap D = \emptyset) + \mathbb{P}^x(S[0, \Xi] \cap D = \emptyset)
\]
\[
\lesssim R^{-c(\ell/2)^2} \log R + R^{(\ell/2 - s\epsilon)(1/2)} \log R^{c\ell}
\]
\[
\lesssim R^{s\epsilon/2} \log R^{c\ell},
\]
where \( \xi = \inf \{ t \geq 0 : |B_{\ell}| \geq R^{c\ell/2} \log^2 R^{c\ell/2} \} \) and \( \Xi = \inf \{ t \geq 0 : |S_{\ell}| \geq R^{c\ell/2} \log^2 R^{c\ell/2} \} \).

If \( \ell \leq 2s - 2 \), then by Lemma 8.4,
\[
\mathbb{P}^x(B_{\ell}) \leq \mathbb{P}^x(T \leq R^{(\ell + 1)c}) + \mathbb{P}^x(\tau \leq R^{(\ell + 1)c})
\]
\[
\lesssim \mathbb{P}^x(T \leq R^{(\ell + 1)c})
\]
\[
\leq \mathbb{P}^x \left( \sup_{0 \leq t \leq R^{(\ell + 1)c}} |B_{\ell} - x| \geq R^{c\ell} \right)
\]
\[
\leq \mathbb{P}^x \left( \sup_{0 \leq t \leq R^{(\ell + 1)c}} |B_{\ell}| \geq R^{c\ell - (\ell + 1)c/2} R^{(\ell + 1)c/2} \right)
\]
\[
\lesssim \exp \left\{ -R^{2s\epsilon - (\ell + 1)c/2} \right\},
\]
Finally, if \(|2/\epsilon| - 1 \leq \ell \leq |2/\epsilon|\), we have the trivial bound \(\mathbb{P}^x(B_\ell) \leq 1\). To summarize, we have
\[
\mathbb{P}^x(B_\ell) \lesssim \begin{cases} 
\exp\{-R^{2s-\ell /2}\epsilon /2\}, & \ell \leq 2s - 2, \\
1, & 2s - 1 \leq \ell \leq 2s + 2, \\
R^{\epsilon /4} \log R \ell^\epsilon, & \ell \geq 2s + 3.
\end{cases}
\]

To evaluate \(\mathbb{P}^x(C_m|B'_{i,r,\ell} \cap \mathcal{H}'_{j,r,\ell})\), one has to be somewhat careful, as \(S\) and \(B\) are not jointly Markov under the KMT coupling. However, they are Markov when considered separately. We define the following stopping times for \(S\) and \(B\), respectively:
\[
\tau_{r,\ell} = \inf \{t \geq 0 : d(S_t, \partial D) \leq 2c(r + 1)(\ell + 1)\epsilon \log R\}
\]
and
\[
T_{r,\ell} = \inf \{t \geq 0 : d(B_t, \partial D) \leq 2c(r + 1)(\ell + 1)\epsilon \log R\}.
\]
Then clearly, on \(B_\ell \cap H_r\),
\[
\max\{\tau_{r,\ell}, T_{r,\ell}\} \leq \xi_{r,\ell} \leq T.
\]

By the Beurling estimates (see Lemmas 2.3 and 2.4), we have
\[
\mathbb{P}^x(|S_{\tau_{r,\ell}} - S_t| \geq a \text{ for some } t \in [\tau_{r,\ell}, \tau]) \lesssim a^{-1/2}(r\ell \log R)^{1/2}
\]
and
\[
\mathbb{P}^x(|B_{T_{r,\ell}} - B_t| \geq a \text{ for some } t \in [T_{r,\ell}, T]) \lesssim a^{-1/2}(r\ell \log R)^{1/2}.
\]
In particular, applying the triangle inequality to
\[
S_T - B_T = (S_T - S_{\xi_{r,\ell}}) + (S_{\xi_{r,\ell}} - B_{\xi_{r,\ell}}) + (B_{\xi_{r,\ell}} - B_T)
\]
and noting that given \(B'_{i,r,\ell} \cap \mathcal{H}'_{j,r,\ell}\),
\[
|S_{\xi_{r,\ell}} - B_{\xi_{r,\ell}}| \leq c(r + 1)(\ell + 1)\epsilon \log R,
\]
we see that the Beurling estimates imply that
\[
\mathbb{P}^x(C_m|B'_{i,r,\ell} \cap \mathcal{H}'_{j,r,\ell}) \lesssim R^{-m\epsilon /2}(j\ell \log R)^{1/2}.
\]
Putting (8.5), (8.6), and (8.7) together, we get
(8.8) \(\mathbb{P}^x(A_k \cap B_\ell \cap C_m \cap H_r) \lesssim f(r, \ell, m)\),
where
(8.9) \(f(r, \ell, m) = \ell r \min \{cR^{(1-r/2)(\ell+1)}\epsilon, \mathbb{P}^x(B_\ell), R^{-m\epsilon /2}(r\ell \log R)^{1/2}\}\).
and

\[
\mathbb{P}^x(B_\ell) \lesssim \begin{cases} 
\exp\{-R^{2s_\epsilon - (\ell+1)/2}\}, & \ell \leq 2s - 2, \\
1, & 2s - 1 \leq \ell \leq 2s + 2, \\
R^{se/2-\ell/4} \log R^\epsilon, & \ell \geq 2s + 3.
\end{cases}
\]

We can now plug (8.8), (8.3), and (8.4) into (8.2) to see that

\[
\left| \mathbb{E}^x[\log |B_T|] - \mathbb{E}^x[\log |S_r|] \right| \lesssim \sum_{k, \ell, r \geq 0} \left[ \sum_{m=0}^{[k-1+1/\epsilon]} \left( \sum_{r=0}^{[m/2s+2]} r R^{(m+1)\epsilon - 1 - k \epsilon} R^{-(m/2)\epsilon} (r(2s + 2) \log R)^{1/2} \right. \right.
\]

\[
+ \left. \sum_{r \geq [m/2s+3]} r R^{(m+1)\epsilon - 1 - k \epsilon} R^{(2-r)\epsilon} \right]
\]

\[
+ \sum_{m \geq [k+1/\epsilon]} \left( \sum_{r=0}^{[m/2s+2]} r m \epsilon R^{-(m/2)\epsilon} (r(2s + 2) \log R)^{1/2} \log R \right)
\]

\[
+ \sum_{r \geq [m/2s+3]} r m \epsilon R^{(2-r)\epsilon} \log R \right) \right]
\]

\[
\lesssim \sum_{k \geq 0} \left[ \sum_{m=0}^{[k-1+1/\epsilon]} m^{5/2} (\log R)^{1/2} R^{(m/2+1)\epsilon - 1 - k \epsilon} + \sum_{m \geq [k+1/\epsilon]} m^{7/2} (\log R)^{3/2} R^{-me/2} \right]
\]

\[
\lesssim R^{-(1-\epsilon)/2} (\log R)^{1/2} + R^{-(1-\epsilon)/2} (\log R)^{3/2}
\]

\[
\lesssim R^{-(1/2-\epsilon)}.
\]

One can estimate the sums of the terms for \(\ell \leq 2s - 2\) and \(\ell \geq 2s + 3\) in a similar manner and find that they are both \(o(R^{-(1/2-\epsilon)})\), which proves the lemma. \(\square\)
Having established Lemma 8.2, we are now able to prove the main theorem.

Proof of Theorem 8.1. We will begin with the case that $x = 0$, $y \neq 0$. As noted in Section 2,
\[ g_D(y) = \mathbb{E}^y[\log |B_T|] - \log |y| \quad \text{and} \quad G_D(y) = \mathbb{E}^y[a(S_T)] - a(y). \]
Moreover, as $|y| \to \infty$,
\[ a(y) = \frac{2}{\pi} \log |y| + k_0 + O(|y|^{-2}), \]
so that
\[ \left| G_D(y) - \frac{2}{\pi} g_D(y) - k_y \right| = \frac{2}{\pi} \left| \mathbb{E}^y[\log |B_T|] - \mathbb{E}^y[\log |S_T|] + O(R^{-2}) \right|. \]
In order to establish (8.1) in the case that $x = 0$, $y \neq 0$, we consider $|y| \leq R^2$ and $|y| \geq R^2$ separately. If $|y| \leq R^2$, then Lemma 8.2 applies and (8.1) holds. If $|y| \geq R^2$, then (8.1) follows by virtue of the fact that there is a constant $c < \infty$ such that if $D$ is simply connected, $V = V(D) = D \cap \mathbb{Z}^2$, $R = \text{inrad}(D)$, and $|y| \geq R^2$ then
\[ G_D(y) \leq cR^{-1/2} \quad \text{and} \quad g_D(y) \leq cR^{-1/2}. \]
To establish (8.11), write $R_y = \text{inrad}_y(D)$, and note that $R_y \leq 2|y|$. There is a constant $c_0 < \infty$ such that if $R_y/8 \leq |w - y| \leq R_y/4$ then $G_D(w, y) \leq c_0$. Indeed, from the expression (2.3) for the Green’s function we have
\[
G_D(w, y) = \mathbb{E}^w[a(S_T - y)] - a(y - w)
\leq \frac{2}{\pi} \sum_{k \geq 1} \left( k \log 2 + \log R_y + O(1) \right) \mathbb{P}^w(2^{k-1} R_y \leq |S_T - y| < 2^k R_y)
- \frac{2}{\pi} \log |y - w| + O(1)
\leq \frac{2}{\pi} \log R_y - \frac{2}{\pi} \log |y - w| + O(1)
= O(1),
\]
where we used a Beurling estimate and that $\text{inrad}_w(D) \leq 5R_y/4$ to bound the sum. Set
\[ \eta = \min\{j \geq 0 : |S_j - y| \leq R_y/4 \text{ or } S_j \notin D\}. \]
As a function of $w$, $G_D(w, y)$ is discrete harmonic on $D \setminus \{y\}$, and so we see that $G_D(S_j \wedge \eta, y)$ is a bounded martingale if $|S_0 - y| > R_y/4$. Hence, if $c_0$ is as above,
\[ G_D(w, y) \leq c_0 \mathbb{P}^w(|S_\eta - y| \leq R_y/4). \]
Set $w = 0$ and note that $|S_\eta - y| \leq R_y/4$ implies that $|S_\eta| \geq R^2/2$. Consequently, since $R = \text{inrad}(D)$, a Beurling estimate shows the existence of a constant $c < \infty$ such that
\[ G_D(y) \leq cR^{-1/2} \]
and by symmetry of the Green’s function we are done. The analogue for the continuous Green’s function can be proved in a similar fashion (or by using conformal invariance). This establishes (8.11) and shows that (8.1) holds when \(|y| \geq R^2\).

To complete the proof of the theorem, we need to consider the case when \(x \in V\) with \(|\psi_D(x)| \leq \rho\) and \(y \neq 0\). In this case, if we translate \(D\) to make \(x\) the origin, then it follows directly from Koebe’s estimate that there is a constant \(c\) such that inner radius of the translated set satisfies \(\text{inrad}_x(D) \geq cR\), and so the argument for the case when \(x = 0, y \neq 0\) implies that (8.1) holds for any \(x \in V\) with \(|\psi_D(x)| \leq \rho\) as well. □

9. Some remarks on the derivation and optimality of the rate

We will now briefly review how we obtain the exponent of \(1/24\) in Theorem 1.1 and how the different scales on which we work fit together.

There are three main contributions to the exponent. The first comes from the rate of convergence of the martingale observable as given in Theorem 4.1, the second comes from Section 5, and the third comes from the Skorokhod embedding step of Section 6. There are essentially four different scales that come into play: the microscopic scale which is basically the lattice size, the MO scale corresponding to the rate of convergence of the martingale observable, the mesoscopic scale on which the discrete driving function is close to a martingale (with error terms on the MO scale), and finally the macroscopic scale which is of constant order.

Suppose that \(D \subset \mathbb{C}\) is a simply connected domain with \(0 \in D\) and \(\text{inrad}(D) = 1\), and let \(D^n\) be the \(n^{-1}Z^2\) grid domain approximation of \(D\). Let us first consider the rate of convergence of the martingale observable, which we denote here by \(\delta = \delta(n)\); this is the MO scale. The rate in this step comes essentially from the estimate comparing the continuous and discrete Green’s functions in Theorem 8.1 and is given in Theorem 4.1 to be \(\delta(n) = n^{-(1/4-\epsilon)}\). This error term, which we believe to be optimal for the chosen martingale observable, then determines a proper mesoscopic scale on which the driving function corresponding to the loop-erased random walk path is close to a martingale. Suppose that \(t_j\) denotes the capacity of \(\psi_D(\gamma^n[0,j])\), and let \(\Delta_j = \theta_n(t_j)\). In (5.1), the integer \(m\) is defined formally, but roughly it is defined so that \(t_m \approx \delta^{2/3}\) and \(\Delta_m \approx \delta^{1/3}\). Thus \(\delta^{2/3}\) is the chosen mesoscopic scale. Let us remark that the appropriate pairwise relationship between \(t_m\) and \(\Delta_m\) is essentially given by properties of the Loewner equation and the fact that the limiting driving function is expected to be a Brownian motion which is Hölder-\(h\) for any \(h < 1/2\). However, at this point, the precise relationship between the pair \((t_m, \Delta_m)\) and \(\delta\) need not necessarily be as above. Indeed, for any \(\alpha < 1\) one could define \(m\) so that \(\Delta_m \approx \delta^{\alpha/2}\) and \(t_m \approx \delta^\alpha\), and the proof of the main estimate in Section 5 would still work. Choosing some \(\alpha > 2/3\) could then in principle improve the error terms in
the Skorokhod embedding step (see below). However, our current argument uses in an essential way that \( \alpha = 2/3 \).

The penultimate goal is to show that the Loewner driving function of a macroscopic piece of the loop-erased random walk path is close to a Brownian motion. The domain Markov property of loop-erased random walk allows us to iterate the estimates for mesoscopic pieces of the curve to “build” a macroscopic piece of it. This is done in the final step of the proof, where the Skorokhod embedding scheme is employed. The convergence rate in this step is essentially determined by the maximal step size of the discretized driving function, that is, the magnitude of \( |\Delta_m| \) which we here write as \( \delta^{1/3} \). The resulting error term after Skorokhod embedding is then roughly

\[
\delta^{(1/3)(1/2)} = O(n^{-(1/4)(1/3)(1/2)+\epsilon}) = O(n^{-(1/24-\epsilon)}). 
\]

We believe this is optimal (up to subpower correction), given the magnitude of \( |\Delta_m| \). This is because it is known (see [3] for details) in the “nicer” case of (one-dimensional) simple random walk that the Skorokhod embedding scheme gives a rate of the square root of the step size of the normalized random walk, in agreement with our estimates.

As mentioned above, we believe that the exponents 1/4 and 1/2 in (9.1) are optimal. The conjectured optimality of the 1/4 exponent only pertains to our specific choice of martingale observable. Our general method may allow for some improvement of the exponent 1/3, as long as it remains smaller than 1/2. We also believe that much of our work can be used for the derivation of the rate of convergence for other processes known to converge to SLE, in the sense that it should be usable as it is or with minor modifications once one has a rate of convergence for an appropriate martingale observable. This should also be true for the derivation of the rate of convergence with respect to Hausdorff distance, at least as long as the limiting curve is simple.

### Appendix A. Proof of Proposition 4.2

In this appendix we prove Proposition 4.2 whose statement we now recall for the benefit of the reader. We also recall that a union of big squares (or UBS) domain is defined in Section 4.1.

**Proposition.** Let \( 0 < \epsilon < 1/4 \) and let \( 0 < \rho < 1 \) be fixed. There exists \( R_0 < \infty \) such that the following holds. Suppose that \( D \) is a UBS domain with \( \text{inrad}(D) = R \) and that \( R > R_0 \). Let \( V = V(D) = D \cap \mathbb{Z}^2 \). If \( x, y \in V \) with \( |\psi_D(x)| \leq \rho \) and \( |\psi_D(y)| \geq 1 - R^{-(1/4-\epsilon)} \), then

\[
\frac{G_D(x,y)}{G_D(y)} = \frac{1 - |\psi_D(x)|^2}{|\psi_D(x) - e^{i\theta_D(y)}|^2} \cdot [1 + O(R^{-(1/4-\epsilon)})]
\]

where \( G_D \) denotes the Green’s function for simple random walk on \( V \).

As noted in Section 4.1, the proof requires a modification of a result from [11] that was proved for simply connected domains with Jordan boundary. The change of setting to UBS domains requires one to establish certain
technical estimates that are not immediately obvious. In this appendix, we go over some of those results and proofs from that paper adapting them to our setting and generalizing them whenever possible. The first step in the proof of Proposition 4.2 is to establish estimates for the Green’s function for simple random walk in $D \cap \mathbb{Z}^2$ in terms of the (continuous) Green’s function for $D$. This is where we require Theorem 8.1.

The following is an estimate for comparing discrete harmonic measure with continuous harmonic measure and basically says that for a UBS domain if planar Brownian motion has a chance of exiting a domain at a particular boundary arc, then simple random walk also has a chance of exiting the domain at that arc. For UBS domains we need to define the association between boundary subsets in a different way than for the Jordan domains used in [11]. With this change, however, exactly the same proof carries through.

If $A \subset \mathbb{C}$ is any set, we define $\hat{A}$ to be the set of closed edges of $\mathbb{Z}^2$ that intersect $A$. That is, if $E$ denotes the (closed) edge set of $\mathbb{Z}^2$, then

(A.1) $\hat{A} = \{ e \in E : e \cap A \neq \emptyset \}.$

In the following proposition, as before, the exiting points should be understood in terms of prime ends and similarly for the harmonic measure.

**Proposition A.1.** Let $D$ be a UBS domain with $E \subset \partial D$, and let $z \in V(D)$. For all $\delta > 0$, there exists a $\epsilon > 0$ such that if $\omega(z, E, D) > \delta$, then $P_z(S_{T_D0} \in \hat{E}) > \epsilon$ where $\hat{E}$ is as in (A.1) and $\omega(z, E, D)$ denotes the continuous harmonic measure of $E$ in $D$ from $z$.

The next step is to establish several technical lemmas. If $E \subset \mathbb{C}$ is any set, we define the UBS covering of $E$ by

$$U(E) = \bigcup_{\{x \in \mathbb{Z}^2 : S(x) \cap E \neq \emptyset\}} S(x)$$

where

(A.2) $S(z) = \{ w \in \mathbb{C} : |\text{Re}(w) - \text{Re}(z)| < 1, |\text{Im}(w) - \text{Im}(z)| < 1 \}$

as in Section 4.1. If $E \subset D$, where $D$ is a UBS domain, we define $U_D(E)$, the UBS covering of $E$ with respect to $D$, by restricting the union in (A.2) to those squares contained in $D$.

Furthermore, for $x \in D$, where $D$ is any simply connected domain, let $d_D(x) = 1 - |\psi_D(x)| = \text{dist}(\psi_D(x), \partial D)$.

**Lemma A.2.** There exists a constant $c < \infty$ such that if $D$ is a simply connected UBS domain, $x, w \in \mathbb{Z}^2 \cap D$, and $z, z' \in S(x)$, then

(i) $d_D(z) \leq cd_D(x)$,

(ii) $|\psi_D(z) - \psi_D(w)| \leq c|\psi_D(z') - \psi_D(w)|$ for $w \neq x$. 

Proof. We write \( \psi = \psi_D \) and prove (i) first. If \( d = \text{dist}(z, \partial D) \), recall that Koebe’s estimate implies
\[
\text{(A.3)} \quad d_D(z) \asymp d|\psi'(z)|
\]
where \( \asymp \) means that each side is bounded by a constant times the other. The result follows easily from the Koebe distortion theorem and (A.3) if \( S(x) \) is away from the boundary. Hence, we may assume that \( S(x) \) is adjacent to \( \partial D \). It is enough to prove the existence of a constant \( c < \infty \) such that
\[
\frac{d|\psi'(z)|}{|\psi'(x)|} \leq c.
\]
Let \( \varphi \) map \( D_x \), the component of \( B(x, 10) \cap D \) containing \( S(x) \), onto \( \mathbb{D} \) with \( \varphi(x) = 0, \varphi'(x) > 0 \) and define \( h = \psi \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D} \). Then with \( z' = \varphi(z) \), we have
\[
\frac{d|\psi'(z)|}{|\psi'(x)|} = \frac{d|h'(z')||\varphi'(z)|}{|h'(0)||\varphi'(x)|}.
\]
Note that \( |\varphi'(z)| \leq cd^{-1/2} \) for some universal constant. Note also that \( |\varphi'(x)| \) is uniformly bounded away from 0. Using the Schwarz reflection principle and the Koebe distortion theorem we see that
\[
|h'(z')|/|h'(0)| \leq c
\]
for some uniform constant. This concludes the proof of (i). To prove (ii), note that we can assume \( w \notin D_x \) (since otherwise the estimate holds using the Koebe distortion theorem directly). We consider again \( \varphi \) and \( h \). Now, \( |\psi(z) - \psi(w)| \leq |\psi(z) - \psi(z')| + |\psi(z') - \psi(w)| \). Also there is some \( u \in D_x \) with \( |x - u| = 5 \) such that \( |\psi(z') - \psi(w)| \geq |\psi(z') - \psi(u)| \). Using Schwarz reflection and the distortion theorem (constants may change from line to line)
\[
|\psi(z) - \psi(z')| = |h(\varphi(z)) - h(\varphi(z'))|
\leq c|h'(\varphi(z'))||\varphi(z') - \varphi(z)|
\leq c|h'(\varphi(z'))||\varphi(z') - \varphi(u)|
\leq c|h(\varphi(z')) - h(\varphi(u))|
= c|\psi(z') - \psi(u)|,
\]
and (ii) follows. \( \square \)

The following lemma shows that there is a uniform lower bound on the probability that random walk leaves the pre-image of a family of polar rectangles at each of its four sides. The centers of these polar rectangles can vary but the ratio between the angular and radial lengths is constant. It is the analogue of Lemma 3.12 in [11].

Lemma A.3. Suppose \( D \) is a simply connected UBS domain and that \( x \in D \). There is a constant \( a_0 \) such that if \( d_D(x) \leq a_0 \), then there exist an \( \epsilon > 0 \)
and constants $a > 1$, $b_1 < \infty$, and $b_2 < \infty$ such that if
\[
\sigma = \min\{j \geq 0 : S_j \notin D \text{ or } d_D(S_j) \geq ad_D(x) \text{ or } |\theta_D(S_j) - \theta_D(x)| \geq b_1d_D(x)\},
\]
then
\[
\begin{align*}
(i) & \quad \mathbb{P}^x(S_\sigma \notin D) \geq \epsilon, \\
(ii) & \quad \mathbb{P}^x(d_D(S_\sigma) \geq ad_D(x)) \geq \epsilon, \text{ and} \\
(iii) & \quad \mathbb{P}^x(|\theta_D(S_\sigma) - \theta_D(x)| \leq b_2d_D(x) | S_\sigma \in D) = 1.
\end{align*}
\]

Proof. Let $x \in D \cap \mathbb{Z}^2$ satisfy $d_D(x) \leq 1/(140c_0^3)$, where $c_0$ is the maximum of the constant from Lemma A.2 and 1. In particular, if $z, w$ satisfy $\max\{d_D(z), d_D(w)\} \leq 7c_0^3d_D(x)$ we have $|\theta_D(z) - \theta_D(w)| \leq 2|\psi(z) - \psi(w)|$, where $\psi = \psi_D$.

Note first that there is a positive probability $\delta_1 > 0$ (independent of $d_D(x)$) that a planar Brownian motion started from $\psi(x)$ leaves $D$ before exiting the ball $B_1 := B(\psi(x), 2d_D(x))$. Consequently, by conformal invariance, a planar Brownian motion from $x$ exits $E_1 := \psi^{-1}(B_1 \cap \mathbb{D})$ at $\partial D$ with probability at least $\delta_1$. Let $U_1 = \mathcal{U}_D(E_1)$ be the UBS covering with respect to $D$ of $E_1$. Notice that Lemma A.2 (ii) implies that any point $z \in U_1$ satisfies
\[
|\psi(z) - \psi(x)| \leq 2c_0d_D(x).
\]

By Proposition A.1, there is an $\epsilon_1 > 0$ such that simple random walk from $x$ exits $U_1$ at $\partial D$ at a vertex $v$ at distance at most 1 from a point $w$ contained in $E_1$ with probability at least $\epsilon_1$. By Lemma A.2 (ii), for such $v$, we have
\[
|\theta_D(v) - \theta_D(x)| \leq 2|\psi(v) - \psi(x)| \leq 2c_0|\psi(w) - \psi(x)| \leq 4c_0d_D(x).
\]

Consider now the pre-image in $D$ of the ball around $\psi(x)$ with radius $6c_0^2d_D(x)$, namely
\[
E_2 := \psi^{-1}(\mathcal{B}(\psi(x), 6c_0^2d_D(x)) \cap \mathbb{D}),
\]
and let $U_2 = \mathcal{U}_D(E_2)$ be the UBS covering with respect to $D$ of $E_2$. We see that $U_1 \subset U_2$.

Let $z \in U_2$. By Lemma A.2 (ii) we have that
\[
d_D(z) \leq |\psi(z) - \psi(x)| + d_D(x) \leq (6c_0^3 + 1)d_D(x) \leq 7c_0^3d_D(x).
\]

Similarly any $z \in U_2$ satisfies $|\theta_D(z) - \theta_D(x)| \leq 12c_0^3d_D(x)$.

There is a strictly positive probability $\delta_2$ that planar Brownian motion from $\psi(x)$ exits the polar rectangle
\[
\mathcal{R} := \{z \in \mathbb{D} : |\arg(z) - \theta_D(x)| \leq d_D(x), 1 - 7c_0^3d_D(x) \leq |z| \leq 1\}
\]
at a point with $|z| = 1 - 7c_0^3d_D(x)$. Hence with probability at least $\delta_2$ planar Brownian motion from $x$ exits $\psi^{-1}(\mathcal{R}) \cap U_2$ through the “top”; that is, at a
point \( w \) contained in \( \partial U_2 \setminus \partial D \). Any such point satisfies (by the assumption \( |\theta_D(w) - \theta_D(x)| \leq d_D(x) \)),
\[
d_D(w) \geq |\psi(w) - \psi(x)| - 2d_D(x) \geq (6c_0^2 - 2)d_D(x) \geq 4c_0^2d_D(x).
\]
By Lemma A.2(i) this means that \( w \) is on the boundary of a square \( S(y) \subset U_2 \) such that \( d_D(y) \geq 4c_0d_D(x) \). By Proposition A.1 it follows that there is an \( \epsilon_2 > 0 \) such that with probability at least \( \delta_2 \) a simple random walk from \( x \) visits a point \( y \) with \( d_D(y) \geq 4c_0d_D(x) \) before exiting \( U_2 \) and thus before exiting \( \{ z : |\theta_D(z) - \theta_D(x)| \leq 12c_0^3d_D(x) \} \). Finally, using part (ii) of Lemma A.2 for the estimate on \( b_2 \), the desired conclusion follows by taking \( a_0 = 1/(140c_0^3) \), \( \epsilon = \min\{\epsilon_1, \epsilon_2\} \), \( a = 4c_0, b_1 = 12c_0^3 \), and \( b_2 = 24c_0^3 \).

The final preliminary result that is needed in order to prove Proposition 4.2 is the analogue of Corollary 3.15 of [11]. Assuming Lemma A.3 the proof in the UBS setting is identical to the proof in the original Jordan setting. We refer to Proposition 3.14 and Corollary 3.15 of [11] for more details.

For any \( a \in (0, 1/2) \) and for any \( \theta_1 < \theta_2 \), let \( \xi_D(a, \theta_1, \theta_2) \) be the first time \( t \geq 0 \) that a random walk leaves the pre-image of the polar rectangle
\[
\{ y \in V(D) : d_D(y) \leq a, \; \theta_1 \leq \theta_D(y) \leq \theta_2 \}.
\]
Consider the probability that the random walk conditioned not to exit the polar rectangle at \( \partial D \) exits at the “top”:
\[
q_d(x, a, \theta_1, \theta_2) = \mathbb{P}^x(d_D(S_{\xi_D(a, \theta_1, \theta_2)} > a | S_{\xi_D(a, \theta_1, \theta_2)} \in V(D)) ,
\]
and note that if \( \theta_1 \leq \theta'_1 \leq \theta'_2 \leq \theta_2 \), then \( q_d(x, a, \theta'_1, \theta'_2) \leq q_d(x, a, \theta_1, \theta_2) \).

**Corollary A.4.** There exist \( c \) and \( \beta \) such that if \( a \in (0, 1/2), r > 0, D \) is a UBS domain, and \( x \in V(D) \), then
\[
q_d(x, a, \theta_D(x) - ra, \theta_D(x) + ra) \geq 1 - c e^{-\beta r}.
\]

We can now complete the proof of Proposition 4.2. Given the technical results that we have just discussed, the proof essentially follows as the proof of Proposition 3.10 of [11]. (Unlike in that paper, however, we are not considering any two arbitrary points in the domains but rather one point near the boundary and one point near the origin.) Consequently, we will not give all the details in the proof below, but rather highlight the steps that affect the rate and show how the exponent of \( 1/4 \) occurs. We will begin by assuming that \( p \in (0, 1/2) \) is arbitrary, and then we will derive a number of estimates in terms of \( p \). As various steps in the proof additional restrictions on \( p \) will be added, and at the end we will optimize to find \( p = 1/4 \).

Before we give the details, let us give a heuristic argument ignoring the particular error terms. If \( D^* = \{ x \in V : g_D(x) \geq c_0 R^{-p} \} \) then \( g_D(z) \approx R^{-p} \) for \( z \in \partial D^* \). By Theorem S.1 the same is true for a constant times \( G_D(z) \). Let \( y \in V \setminus D^* \). If \( \eta \) is the first time \( S \) exits \( V \setminus D^* \), we can write \( G_D(y) \approx \text{const.} R^{-p} \mathbb{P}(\eta \in D^*) \). Also, by Corollary A.4 a simple random walk from \( y \) that hits \( D^* \) before exiting \( V \) is likely to do so without the argument of its
Proof of Proposition 4.2. Let 
for suitable error terms depending on 
universal constant 
for 
Finally, let 

By Corollary 3.19 of [13] we know \(|f'(0)| \geq \text{inrad}(D)/4 = R/4\) and so

The Koebe one-quarter theorem implies that \( B(z, cR^{-p} |f'(u)|/4) \subset D \), and so we conclude from (A.5) that \( B(z, cR^{1-2p}) \subset D \). Since \( g_D \) is a positive
bounded harmonic function in $\mathcal{B}(z, cR^{1-2p})$ we can use Exercise 2.17 of [13] to conclude

$$g_D(z) = g_D(w) + O(R^{-(1-2p)})$$

assuming that the error term is not larger than the leading term which is true as long as $p < 1 - 2p$. Thus, we have introduced a restriction on $p$, namely that $0 < p < 1/3$. Combined with (A.4) we conclude that

$$g_D(z) = c_0 R^{-p} + O(R^{-(1-2p)}).$$

Using Theorem 8.1, it now follows that

$$G_D(z) = (2/\pi) c_0 R^{-p} + O(R^{-(1-2p)}) + O(|z|^{-2}) + O(R^{-1/2} \log R).$$

Thus, since $\text{dist}(z, V \setminus D^*) = 1$, it follows that $|z| > R^{1/4}$ which implies $G_D(z) = (2/\pi) c_0 R^{-p} + O(R^{1/2} \log R) = (2/\pi) c_0 R^{-p}[1 + O(R^{p-1/2} \log R)]$ and similarly for $G_D(w)$. We also find another restriction on $p$, namely that $p - 1/2 < 0$. Therefore, using the strong Markov property, for any $y \in V \setminus D^*$,

$$G_D(y) = (2/\pi) c_0 R^{-p} \mathbb{P}^y(S_\eta \in D^*) [1 + O(R^{p-1/2} \log R)].$$

In a similar fashion, note that if $x \in V$ with $|\psi_D(x)| \leq \rho$ and $z \in D^*$, then $g_D(x, z) \geq cR^{-p}$ for some $c$, and hence by Theorem 8.1 if $|z - x| \geq R^{1/4}$, then

$$G_D(x, z) = (2/\pi) g_D(x, z) [1 + O(R^{p-1/2} \log R)].$$

If $x \in V$ with $|\psi_D(x)| \leq \rho$ and $y \in V \setminus D^*$, then there exists some $0 < \delta < 1$ such that $|\psi_D(x) - \psi_D(y)| \geq \delta$.

Using (2.1) and the fact that the Green’s function is conformally invariant, one can check that if $\zeta' = \psi_D(y) = (1 - r)e^{i\theta}$ and $\zeta = \psi_D(x) \in \mathbb{D}$ with $|\zeta - \zeta'| \geq r$, then

$$g_D(x, y) = g_D(\zeta, \zeta') = \frac{g_D(\zeta') (1 - |\zeta'|^2)}{|\zeta - e^{i\theta}|^2} \left[ 1 + O \left( \frac{r}{|\zeta - \zeta'|} \right) \right].$$

Let $y \in V \setminus D^*$ and let $z \in D^*$ be as above with one neighbor in $V \setminus D^*$. Assume also that $z \in K = K(y) := \{z \in D^*, |\psi_D(z) - \psi_D(y)| \leq c_1 R^{-p} \log R\}$. Then (A.7) and the estimate on $g_D(z)$ imply that

$$g_D(x, z) = \frac{c_0 R^{-p} (1 - |\psi_D(x)|^2)}{|\psi_D(x) - e^{i\theta D(y)}|^2} \left[ 1 + O(R^{-p} \log R) \right].$$

It now follows from Corollary A.4 that there exists a $c_1$ such that if

$$\xi = \xi(D, c_1) = \min\{j \geq 0 : S_j \not\in V \text{ or } |\psi_D(S_j) - \psi_D(y)| \geq c_1 R^{-p} \log R\},$$

then

$$\mathbb{P}^y(\xi < \eta) \leq c_1 R^{-5} \mathbb{P}^y(S_\eta \in D^*).$$

In particular

$$\mathbb{P}^y(S_\eta \in K(y)) = (1 - O(R^{-5})) \mathbb{P}^y(S_\eta \in D^*).$$
Hence, by stopping the random walk from $y$ when it hits $D^*$ it can be shown that

$$G_D(x, y) = \mathbb{P}_y(S_\eta \in D^*) \frac{(2/\pi) c_0 R^{-p} (1 - |\psi_D(x)|^2)}{|\psi_D(x) - e^{i\theta_D(y)}|^2} \cdot [1 + O(R^{p-1/2} \log R)] \cdot [1 + O(R^{-p} \log R)].$$

Combining this with (A.6) gives

$$\frac{G_D(x, y)}{G_D(y)} = \frac{1 - |\psi_D(x)|^2}{|\psi_D(x) - e^{i\theta_D(y)}|^2} \cdot \frac{[1 + O(R^{p-1/2} \log R)] \cdot [1 + O(R^{-p} \log R)]}{[1 + O(R^{p-1/2} \log R)] \cdot [1 + O(R^{-p} \log R)]}.$$

Thus, solving $p - 1/2 = -p$ gives $p = 1/4$ and so choosing $p = 1/4 - \epsilon$ for any $0 < \epsilon < 1/4$ yields (4.2) completing the proof. $\square$

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