C-ROBIN FUNCTIONS AND APPLICATIONS

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Abstract. We continue the study in [1] in the setting of pluripotential theory arising from polynomials associated to a convex body \(C\) in \((\mathbb{R}^+)^d\). Here we discuss \(C\)-Robin functions and their applications. In the particular case where \(C\) is a triangle in \((\mathbb{R}^+)^2\) with vertices \((0,0),(b,0),(0,a)\), \(a,b > 0\), we generalize results of T. Bloom to construct families of polynomials which recover the \(C\)-extremal function \(V_{C,K}\) of a nonpluripolar compact set \(K \subset \mathbb{C}^2\).

1. Introduction

As in [1], we fix a convex body \(C \subset (\mathbb{R}^+)^d\) and we define the logarithmic indicator function

\[
H_C(z) := \sup_{J \in C} \log |z^J| := \sup_{(j_1, \ldots, j_d) \in C} \log[|z_1|^{j_1} \cdots |z_d|^{j_d}].
\]

We assume throughout that

\[
\Sigma \subset kC \quad \text{for some } k \in \mathbb{Z}^+.
\]

where

\[
\Sigma := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1, \sum_{i=1}^d x_i \leq 1\}.
\]

Then

\[
H_C(z) \geq \frac{1}{k} \max_{j=1,\ldots,d} \log^+ |z_j| = \frac{1}{k} H_\Sigma(z)
\]
where \( \log^+ |z_j| = \max[0, \log |z_j|] \). We define

\[
L_C = L_C(\mathbb{C}^d) := \{ u \in PSH(\mathbb{C}^d) : u(z) - H_C(z) = O(1), \ |z| \to \infty \},
\]

and

\[
L_{C,+} = L_{C,+}(\mathbb{C}^d) = \{ u \in L_C(\mathbb{C}^d) : u(z) \geq H_C(z) + C_u \}
\]

where \( PSH(\mathbb{C}^d) \) denotes the class of plurisubharmonic functions on \( \mathbb{C}^d \). These are generalizations of the classical Lelong classes \( L := L_{\Sigma}, L^+ := L_{\Sigma,+} \) when \( C = \Sigma \). Let \( \mathbb{C}[z] \) denote the polynomials in \( z \) and

\[
\text{Poly}(nC) := \left\{ p \in \mathbb{C}[z] : p(z) = \sum_{\alpha \in nC} a_\alpha z^\alpha \right\}.
\]

For a nonconstant polynomial \( p \) we define

\[
\deg_C(p) = \min\{ n \in \mathbb{N} : p \in \text{Poly}(nC) \}.
\]

If \( p \in \text{Poly}(nC), n \geq 1 \) we have \( \frac{1}{n} \log |p| \in L_C \); also each \( u \in L_{C,+}(\mathbb{C}^d) \) is locally bounded in \( \mathbb{C}^d \).

The \( C \)-extremal function of a compact set \( K \subset \mathbb{C}^2 \) is defined as the uppersemicontinuous (usc) regularization \( V_{C,K}^*(z) := \limsup_{\zeta \to z} V_{C,K}(\zeta) \) of

\[
V_{C,K}(z) := \sup\{ u(z) : u \in L_C, u \leq 0 \text{ on } K \}.
\]

If \( K \) is regular (\( V_K := V_{\Sigma,K} \) is continuous), then \( V_{C,K} = V_{C,K}^* \) is continuous (cf., [9]). In particular, for \( K = T^d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d : |z_j| = 1, j = 1, \ldots, d \} \), \( V_{C,T^d} = V_{C,T^d}^* = H_C \) (cf. [1, (2.7)]). If \( K \) is not pluripolar, i.e., for any \( u \) psh with \( u = -\infty \) on \( K \) we have \( u \equiv -\infty \), the Monge–Ampère measure \( (dd^c V_{C,K}^*)^d \) is a positive measure with support in \( K \) and \( V_{C,K}^* = 0 \) quasi-everywhere (q.e.) on \( \text{supp}(dd^c V_{C,K}^*)^d \) (i.e., everywhere except perhaps a pluripolar set).

Much of the recent development of this \( C \)-pluripotential theory can be found in [9], [1] and [2]. One noticeable item lacking from these works is a constructive approach to finding natural concrete families of polynomials associated to \( K,C \) which recover \( V_{C,K} \). In order to do this, following the approach of Tom Bloom in [4] and [5], we introduce a \( C \)-Robin function \( \rho_u \) for a function \( u \in L_C \). The “usual” Robin function \( \rho_u \) associated to \( u \in L_{\Sigma} \) is defined as

\[
\rho_u(z) := \limsup_{|\lambda| \to \infty} [u(\lambda z) - \log |\lambda|]
\]

and this detects the asymptotic behavior of \( u \). This definition is natural since the “growth function” \( H_{\Sigma}(z) = \max_{j=1,\ldots,d} \log^+ |z_j| \) satisfies \( H_{\Sigma}(\lambda z) = \)
\( H_\Sigma(z) + \log |\lambda| \) if \( \max_{j=1,\ldots,d} |z_j| \geq 1 \) and \( |\lambda| \geq 1 \). Let \( C \) be the triangle in \( \mathbb{R}^2 \) with vertices \((0,0), (b,0), (0,a)\) where \( a, b \) are relatively prime positive integers. Then

(1) \( H_C(z_1, z_2) = \max[\log^+ |z_1|^b, \log^+ |z_2|^a] \) (note \( H_C = 0 \) on the closure of the unit polydisk \( P^2 := \{(z_1, z_2): |z_1|, |z_2| < 1\})

(2) defining \( \lambda \circ (z_1, z_2) := (\lambda^az_1, \lambda^bz_2) \), we have

\[ H_C(\lambda \circ (z_1, z_2)) = H_C(z_1, z_2) + ab \log |\lambda| \]
for \((z_1, z_2) \in \mathbb{C}^2 \setminus P^2\) and \( |\lambda| \geq 1 \).

Given \( u \in L_C(\mathbb{C}^2) \), we define the \( C \)-Robin function of \( u \) (Definition 4.2) as

\[ \rho_u(z_1, z_2) := \limsup_{|\lambda| \to \infty} [u(\lambda \circ (z_1, z_2)) - ab \log |\lambda|] \]
for \((z_1, z_2) \in \mathbb{C}^2 \). This agrees with (1.5) when \( a = b = 1 \); i.e., when \( C = \Sigma \subset (\mathbb{R}^+)^2 \). For general convex bodies \( C \), it is unclear how to define an analogue to recover the asymptotic behavior of \( u \in L_C \).

The next two sections give some general results in \( C \)-pluripotential theory which will be used further on but are of independent interest. Section 4 begins in earnest with the case where \( C \) is a triangle in \( \mathbb{C}^2 \). The key results utilized in our analysis are the use of an integral formula of Bedford and Taylor [3], Theorem 6.1 in Section 6, yielding the fundamental Corollary 6.4, and recent results on \( C \)-transfinite diameter in [12] and [13] of the second author in Section 7. Our arguments in Sections 5 and 8 follow closely those of Bloom in [4] and [5]. The main theorem, Theorem 8.3, is stated and proved in Section 8; then explicit examples of families of polynomials which recover \( V_{C,K} \) are provided. We mention that the results given here for triangles \( C \) in \( \mathbb{R}^2 \) with vertices \((0,0), (b,0), (0,a)\) where \( a, b \) are relatively prime positive integers should generalize to the case of a simplex

\[ C = \text{co}\{(0, \ldots, 0), (a_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, a_d)\} \]
in \( (\mathbb{R}^+)^d \), \( d > 2 \) where \( a_1, \ldots, a_d \) are pairwise relatively prime (cf., Remark 4.5). Section 9 indicates generalizations to weighted situations.

2. Rumely formula and transfinite diameter

We recall the definition of \( C \)-transfinite diameter \( \delta_C(K) \) of a compact set \( K \subset \mathbb{C}^d \) where \( C \) satisfies (1.2). Letting \( N_n \) be the dimension of \( \text{Poly}(nC) \) in (1.3), we have

\[ \text{Poly}(nC) = \text{span}\{e_1, \ldots, e_{N_n}\} \]
where \( \{ e_j(z) := z^{\alpha_j} \} \) are the standard basis monomials in Poly\((nC)\) in any order. For points \( \zeta_1, \ldots, \zeta_N \in \mathbb{C}^d \), let

\[
VDM(\zeta_1, \ldots, \zeta_N) := \det [e_i(\zeta_j)]_{i,j=1,\ldots,N},
\]

and for a compact subset \( K \subset \mathbb{C}^d \) let

\[
V_n = V_n(K) := \max_{\zeta_1, \ldots, \zeta_N \in K} |VDM(\zeta_1, \ldots, \zeta_N)|.
\]

Then

\[
\delta_C(K) := \limsup_{n \to \infty} V_n^{1/n}
\]

is the \( C \)-transfinite diameter of \( K \) where \( l_n := \sum_{j=1}^N \deg (e_j) \). The existence of the limit is not obvious but in this setting it is proved in [1]. We return to this issue in Section 7.

Next, for \( u, v \in L_{C,+} \), we define the mutual energy

\[
E(u, v) := \int_{\mathbb{C}^d} (u - v) \sum_{j=0}^d (dd^c u)^j \wedge (dd^c v)^{d-j}.
\]

Here \( dd^c = 2i \partial \bar{\partial} \) and for locally bounded psh functions, e.g., for \( u, v \in L_{C,+} \), the complex Monge–Ampère operators \((dd^c u)^j \wedge (dd^c v)^{d-j}\) are well-defined as positive measures. We have that \( E \) satisfies the cocycle property; i.e., for \( u, v, w \in L_{C,+} \), (cf., [1, Proposition 3.3])

\[
E(u, v) + E(v, w) + E(w, u) = 0.
\]

Connecting these notions, we recall the following formula from [1].

**Theorem 2.1.** Let \( K \subset \mathbb{C}^d \) be compact and nonpluripolar. Then

\[
\log \delta_C(K) = \frac{-1}{c} E(V_{C,K}^*, H_C)
\]

where \( c \) is a positive constant depending only on \( d \) and \( C \).

We will use the global domination principle for general \( L_C \) and \( L_{C,+} \) classes associated to convex bodies satisfying (1.2) (cf., [11]):

**Proposition 2.2.** For \( C \subset (\mathbb{R}^+)^d \) satisfying (1.2), let \( u \in L_C \) and \( v \in L_{C,+} \) with \( u \leq v \) a.e.-\((dd^c v)^d\). Then \( u \leq v \) in \( \mathbb{C}^d \).
We use these ingredients to prove the following.

**Proposition 2.3.** Let $E \subset F$ be compact and nonpluripolar. If $\delta_C(E) = \delta_C(F)$ then $V_{C,E}^* = V_{C,F}^*$.

**Proof.** By Theorem 2.1, the hypothesis implies that $\mathcal{E}(V_{C,E}^*, H_C) = \mathcal{E}(V_{C,F}^*, H_C)$. Using the cocycle property,

$$0 = \mathcal{E}(V_{C,E}^*, H_C) + \mathcal{E}(H_C, V_{C,F}^*) + \mathcal{E}(V_{C,F}^*, V_{C,E}^*)$$

$$= \mathcal{E}(V_{C,E}^*, H_C) - \mathcal{E}(V_{C,F}^*, H_C) + \mathcal{E}(V_{C,F}^*, V_{C,E}^*) = \mathcal{E}(V_{C,F}^*, V_{C,E}^*).$$

From the definition (2.1),

$$0 = \mathcal{E}(V_{C,F}^*, V_{C,E}^*) = \int_{\mathbb{C}^d} (V_{C,F}^* - V_{C,E}^*) \sum_{j=0}^{d} (dd^c V_{C,F}^*)^j \wedge (dd^c V_{C,E}^*)^{d-j}$$

$$= \int_{\mathbb{C}^d} (V_{C,F}^* - V_{C,E}^*) (dd^c V_{C,F}^*)^d + \int_{\mathbb{C}^d} (V_{C,F}^* - V_{C,E}^*) (dd^c V_{C,E}^*)^d$$

$$+ \int_{\mathbb{C}^d} (V_{C,F}^* - V_{C,E}^*) \sum_{j=1}^{d-1} (dd^c V_{C,F}^*)^j \wedge (dd^c V_{C,E}^*)^{d-j}.$$

Now $E \subset F$ implies $V_{C,F}^* \leq V_{C,E}^*$; i.e., $V_{C,F}^* - V_{C,E}^* \leq 0$ on $\mathbb{C}^d$. Also,

$$V_{C,F}^* = V_{C,E}^* = 0 \text{ q.e. on supp}(dd^c V_{C,E}^*)^d$$

and

$$V_{C,F}^* = 0 \text{ q.e. on supp}(dd^c V_{C,F}^*)^d.$$ 

Thus we see that

$$0 = \int_{\mathbb{C}^d} (-V_{C,E}^*) (dd^c V_{C,F}^*)^d + \int_{\mathbb{C}^d} (V_{C,F}^* - V_{C,E}^*) \sum_{j=1}^{d-1} (dd^c V_{C,F}^*)^j \wedge (dd^c V_{C,E}^*)^{d-j}$$

where each term on the right-hand side is nonpositive. Hence each term vanishes. In particular,

$$0 = \int_{\mathbb{C}^d} V_{C,E}^* (dd^c V_{C,F}^*)^d$$

implies that $V_{C,E}^* = 0$ q.e. on supp$(dd^c V_{C,F}^*)^d$ (and hence a.e.$-(dd^c V_{C,F}^*)^d$).

We finish the proof by using the domination principle (Proposition 2.2): we have $V_{C,E}^*, V_{C,F}^* \in L_{C,+}(\mathbb{C}^d)$ with

$$V_{C,E}^* \leq V_{C,F}^* \text{ a.e.-(}dd^c V_{C,F}^*)^d$$
and hence \( V_{C,E}^* \leq V_{C,F}^* \) on \( \mathbb{C}^d \); i.e., \( V_{C,E}^* = V_{C,F}^* \) on \( \mathbb{C}^d \). \( \square \)

**Remark 2.4.** For \( C = \Sigma \), Proposition 2.3 was proved for regular compact sets \( E, F \) in [5] and in general (compact and nonpluripolar sets) in [6]. Both results utilized the “usual” Robin functions \((1.5)\) of \( V_{E}^*, V_{F}^* \).

3. Other preliminary results: general

Let \( K \subset \mathbb{C}^d \) be compact and nonpluripolar and let \( \mu \) be a positive measure on \( K \) such that one can form orthonormal polynomials \( \{ p_{\alpha} \} \) using Gram-Schmidt on the monomials \( \{ z^{\alpha} \} \). We use the notion of degree given in \((1.4)\): \( \deg_C(p) = \min\{ n \in \mathbb{N} : p \in \text{Poly}(nC) \} \). Here we use an ordering \( \prec_C \) on \((\mathbb{Z}^+)^d \) which respects \( \deg_C(p) \) in the sense that \( \alpha \prec_C \beta \) whenever \( \deg_C(z^{\alpha}) < \deg_C(z^{\beta}) \) (cf., [13]). We have the Siciak–Zaharjuta type polynomial formula

\[
V_{C,K}(z) = \sup \left\{ \frac{1}{\deg_C(p)} \log |p(z)| : p \in \mathbb{C}[z], \|p\|_K \leq 1 \right\}
\]

(cf., [1, Proposition 2.3]). It follows that \( \{ z \in \mathbb{C}^d : V_{C,K}(z) = 0 \} = \hat{K} \), the polynomial hull of \( K \):

\[
\hat{K} := \{ z \in \mathbb{C}^d : |p(z)| \leq \|p\|_K, \text{ all polynomials } p \}.
\]

In this section, we follow the arguments of Zeriahi in [17].

**Proposition 3.1.** In this setting, \( \limsup_{|\alpha| \to \infty} \frac{1}{\deg_C(p_{\alpha})} \log |p_{\alpha}(z)| \geq V_{C,K}(z) \), \( z \notin \hat{K} \).

**Proof.** Let \( Q_n \in \text{Poly}(nC) \) with \( \|Q_n\|_K \leq 1 \). From the property of the ordering \( \prec_C \), we can write \( Q_n = \sum_{\alpha \in nC} c_{\alpha} p_{\alpha} \). Then

\[
|c_{\alpha}| = \left| \int_K Q_n \overline{p_{\alpha}} \, d\mu \right| \leq \int_K |\overline{p_{\alpha}}| \, d\mu \leq \sqrt{\mu(K)}
\]

by Cauchy–Schwarz. Hence

\[
|Q_n(z)| \leq N_n \sqrt{\mu(K)} \max_{\alpha \in nC} |p_{\alpha}(z)|
\]

where recall \( N_n = \text{dim}(\text{Poly}(nC)) \).

Now fix \( z_0 \in \mathbb{C}^d \setminus \hat{K} \) and let \( \alpha_n \in nC \) be the multiindex with \( \deg_C(p_{\alpha_n}) \) largest such that

\[
|p_{\alpha_n}(z_0)| = \max_{\alpha \in nC} |p_{\alpha}(z_0)|.
\]
We claim that taking any sequence \( \{Q_n\} \) with \( \|Q_n\|_K \leq 1 \) for all \( n \),
\[
\lim_{n \to \infty} \deg_C(p_{\alpha_n}) = +\infty.
\]
For if not, then by the above argument, there exists \( A < +\infty \) such that for any \( n \) and any \( Q_n \in \text{Poly}(nC) \) with \( \|Q_n\|_K \leq 1 \),
\[
|Q_n(z_0)| \leq N_n \sqrt{\mu(K)} \max_{\deg_C(p_n) \leq A} |p_{\alpha}(z_0)| = N_n M(z_0)
\]
where \( M(z_0) \) is independent of \( n \). But then
\[
V_{C,K}(z_0) = \sup \left\{ \frac{1}{\deg_C(p)} \log |p(z_0)| : p \in \mathbb{C}[z], \|p\|_K \leq 1 \right\}
\leq \limsup_{n \to \infty} \left[ \frac{1}{n} \log N_n + \frac{1}{n} \log M(z_0) \right] = 0
\]
which contradicts \( z_0 \in \mathbb{C}^d \setminus \hat{K} \). We conclude that for any \( z \in \mathbb{C}^d \setminus \hat{K} \), for any \( n \) and any \( Q_n \in \text{Poly}(nC) \) with \( \|Q_n\|_K \leq 1 \),
\[
\frac{1}{n} \log |Q_n(z)| \leq \frac{1}{n} \log N_n + \frac{1}{n} \log |p_{\alpha_n}(z)|
\]
where we can assume \( \deg_C(p_{\alpha_n}) \uparrow +\infty \). Hence, for such \( z \),
\[
V_{C,K}(z) \leq \limsup_{n \to \infty} \frac{1}{n} \log |p_{\alpha_n}(z)| \leq \limsup_{n \to \infty} \frac{1}{\deg_C(p_{\alpha_n})} \log |p_{\alpha_n}(z)|
\leq \limsup_{|\alpha| \to \infty} \frac{1}{\deg_C(p_{\alpha})} \log |p_{\alpha}(z)|
\]
where we have used \( \deg_C(p_{\alpha_n}) \leq n \). \( \square \)

Suppose \( \mu \) is any Bernstein–Markov measure for \( K \); i.e., for any \( \varepsilon > 0 \), there exists a constant \( c_\varepsilon \) so that
\[
\|p_n\|_K \leq c_\varepsilon (1 + \varepsilon)^n \|p_n\|_{L^2(\mu)}, \quad p_n \in \text{Poly}(nC), \; n = 1, 2, \ldots.
\]
From (1.2), \( \Sigma \subset kC \subset m\Sigma \) for some \( k, m \) and we can replace \( (1 + \varepsilon)^n \) by \( (1 + \varepsilon)^{\deg_C(p_n)} \). In particular, for the orthonormal polynomials \( \{p_{\alpha}\} \),
\[
\|p_{\alpha}\|_K \leq c_\varepsilon (1 + \varepsilon)^{\deg_C(p_{\alpha})}.
\]
Thus
\[
\limsup_{|\alpha| \to \infty} \frac{1}{\deg_C(p_{\alpha})} \log \|p_{\alpha}\|_K \leq 0
\]
and we obtain equality in the previous result:
Corollary 3.2. In this setting, if \( \mu \) is any Bernstein–Markov measure for \( K \),
\[
\limsup_{|\alpha| \to \infty} \frac{1}{\deg_{S_C}(p_\alpha)} \log |p_\alpha(z)| = V_{C,K}(z), \quad z \notin \hat{K}.
\]

We remark that Bernstein–Markov measures exist in abundance; cf., [8]. Our goal in subsequent sections is to generalize the results in [4] and [5] of T. Bloom to give more constructive ways of recovering \( V_{C,K} \) from special families of polynomials.

4. C-Robin function

We begin with the observation that a proof similar to that of [10, Theorem 5.3.1] yields the following result.

Theorem 4.1. Let \( C, C' \subset (\mathbb{R}^+)^d \) be convex bodies and let \( F: \mathbb{C}^d \to \mathbb{C}^d \) be a proper polynomial mapping satisfying
\[
0 < \liminf_{|z| \to \infty} \frac{\sup_{J \in C} |[F(z)]^J|}{\sup_{J' \in C'} |z^{J'}|} \leq \limsup_{|z| \to \infty} \frac{\sup_{J \in C} |[F(z)]^J|}{\sup_{J' \in C'} |z^{J'}|} < \infty.
\]

Then for \( K \subset \mathbb{C}^d \) compact,
\[
V_{C,K}(F(z)) = V_{C',F^{-1}(K)}(z).
\]

Proof. Since \( H_C(z) := \sup_{J \in C} \log |z^J| \), the hypothesis can be written
\[
0 < \liminf_{|z| \to \infty} \frac{e^{H_C(F(z))}}{e^{H_{C'}(z)}} \leq \limsup_{|z| \to \infty} \frac{e^{H_C(F(z))}}{e^{H_{C'}(z)}} < \infty.
\]

We first show that \( \liminf_{|z| \to \infty} \frac{e^{H_C(F(z))}}{e^{H_{C'}(z)}} > 0 \) implies
\[
V_{C',F^{-1}(K)}(z) \leq V_{C,K}(F(z)).
\]

Indeed, starting with \( u \in L_{C'} \) with \( u \leq 0 \) on \( F^{-1}(K) \), take
\[
v(z) := \sup u(F^{-1}(z))
\]
where the supremum is over all preimages of \( z \). Then \( v \in \text{PSH}(\mathbb{C}^d) \) and \( v \leq 0 \) on \( K \). Note that \( v(F(z)) = u(z) \). Now \( u \in L_{C'} \) implies
\[
\limsup_{|z| \to \infty} |u(z) - H_{C'}(z)| \leq M < \infty.
\]
To show \( v \in L_C \), since \( F \) is proper it suffices to show
\[
\limsup_{|z| \to \infty} [v(F(z)) - H_C(F(z))] < \infty.
\]
We have
\[
\limsup_{|z| \to \infty} [v(F(z)) - H_C(F(z))]
= \limsup_{|z| \to \infty} [v(F(z)) - H_C'(z) + H_C'(z) - H_C(F(z))]
\leq \limsup_{|z| \to \infty} [u(z) - H_C'(z)] - \liminf_{|z| \to \infty} [H_C(F(z)) - H_C'(z)]
\leq M - \liminf_{|z| \to \infty} [H_C(F(z)) - H_C'(z)] < \infty
\]
from the hypothesized condition in (4.1) so \( v \in L_C \) and (4.2) follows.

Next we show that
\[
\limsup_{|z| \to \infty} \frac{e^{H_C(F(z))}}{e^{H_C'(z)}} < \infty \implies V_{C',F^{-1}(K)}(z) \geq V_{C,K}(F(z)).
\]
Letting \( u \in L_C \) with \( u \leq 0 \) on \( K \), we have \( u(F(z)) \in PSH(\mathbb{C}^d) \) and \( u(F(z)) \leq 0 \) on \( F^{-1}(K) \) and we are left to show \( u(F(z)) \in L_{C'} \). Now
\[
\limsup_{|z| \to \infty} [u(F(z)) - H_C'(z)]
= \limsup_{|z| \to \infty} [u(F(z)) - H_C(F(z)) + H_C(F(z)) - H_C'(z)]
\leq \limsup_{|z| \to \infty} [u(F(z)) - H_C(F(z))] + \limsup_{|z| \to \infty} [H_C(F(z)) - H_C'(z)] < \infty
\]
from the hypothesized condition in (4.1) and \( u \in L_C \). □

We can apply this in \( \mathbb{C}^d \) with \( C' = c\Sigma \) where \( c \in \mathbb{Z}^+ \) and \( C \) is an arbitrary convex body in \((\mathbb{R}^+)^d\). Given \( K \subset \mathbb{C}^d \) compact, provided we can find \( F \) satisfying the hypotheses of Theorem 4.1, from the relation
\[
V_{C,K}(F(z)) = V_{c\Sigma,F^{-1}(K)}(z) = cV_{F^{-1}(K)}(z) \in cL(\mathbb{C}^d)
\]
we can form a scaling of the standard Robin function (1.5) for \( V_{F^{-1}(K)} \), i.e., \( \rho_{F^{-1}(K)} := \rho_{\Sigma,F^{-1}(K)} \), and we have
\[
c\rho_{F^{-1}(K)}(z) = \limsup_{|\lambda| \to \infty} [V_{C,K}(F(\lambda z))] - c \log |\lambda|.
\]
This gives a connection between the standard Robin function $\rho_{F^{-1}(K)}$ and something resembling a possible definition of a $C$-Robin function $\rho_{C,K}$ (the right-hand side). Given $K \subset \mathbb{C}^d$, the set $F^{-1}(K)$ can be very complicated so that, apriori, this relation has little practical value.

For the rest of this section, and for most of the subsequent sections, we work in $\mathbb{C}^2$ with variables $z = (z_1, z_2)$ and we let $C$ be the triangle with vertices $(0,0), (b,0), (0,a)$ where $a, b$ are relatively prime positive integers. We recall from the introduction:

1. $H_C(z_1, z_2) = \max\{|\log^+|z_1|^b, \log^+|z_2|^a|\}$ (note $H_C = 0$ on the closure of the unit polydisk $P^2 := \{(z_1, z_2): |z_1|, |z_2| < 1\}$);
2. defining $\lambda \circ (z_1, z_2) := (\lambda^a z_1, \lambda^b z_2)$, we have

$$H_C(\lambda \circ (z_1, z_2)) = H_C(z_1, z_2) + ab \log |\lambda|$$

for $(z_1, z_2) \in \mathbb{C}^2 \setminus P^2$ and $|\lambda| \geq 1$.

**Definition 4.2.** Given $u \in L_C$, we define the $C$-Robin function of $u$:

$$\rho_u(z_1, z_2) := \limsup_{|\lambda| \to \infty} |u(\lambda \circ (z_1, z_2)) - ab \log |\lambda||$$

for $(z_1, z_2) \in \mathbb{C}^2$.

We claim that $\rho_u \in L_C$. To see this, we lift the circle action on $\mathbb{C}^2$,

$$\lambda \circ (z_1, z_2) := (\lambda^a z_1, \lambda^b z_2),$$

to $\mathbb{C}^3$ in the following manner:

$$\lambda \circ (t, z_1, z_2) := (\lambda t, \lambda^a z_1, \lambda^b z_2).$$

Given a function $u \in L_C(\mathbb{C}^2)$, we can associate a function $h$ on $\mathbb{C}^3$ which satisfies

1. $h(1, z_1, z_2) = u(z_1, z_2)$ for all $(z_1, z_2) \in \mathbb{C}^2$;
2. $h \in L_{\tilde{C}}(\mathbb{C}^3)$ where $\tilde{C} = \text{co}\{(0,0,0), (1,0,0), (0,b,0), (0,0,a)\}$;
3. $h$ is $ab$-log-homogeneous:

$$h(\lambda \circ (t, z_1, z_2)) = h(t, z_1, z_2) + ab \log |\lambda|.$$

Indeed, we simply set

$$h(t, z_1, z_2) := \begin{cases} u\left(\frac{z_1}{t^a}, \frac{z_2}{t^b}\right) + ab \log |t| & \text{if } t \neq 0, \\ \limsup_{(t,w_1,w_2) \to (0,z_1,z_2)} h(t, w_1, w_2) & \text{if } t = 0. \end{cases}$$

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Now since $h$ is psh on $\mathbb{C}^3$, we have
\begin{equation}
(4.4) \quad h(0,z_1,z_2) := \limsup_{(t,z_1,z_2) \to (0,z_1,z_2)} h(t,z_1,z_2) = \rho_u(z_1,z_2).
\end{equation}

**Proposition 4.3.** For $u \in L_C$, we have $\rho_u \in L_C$. In particular, $\rho_u$ is plurisubharmonic.

**Proof.** The psh of $\rho_u$ follows directly from (4.4) since $h$ is psh on $\mathbb{C}^3$. To show $\rho_u \in L_C$, note that
$$
\rho_u(\lambda \circ (z_1,z_2)) = \rho_u(z_1,z_2) + ab \log |\lambda| \quad \text{for } \lambda \in \mathbb{C}.
$$
From (4.3) $H_C$ satisfies the same relation for $(z_1,z_2) \in \mathbb{C}^2 \setminus P^2$ and $|\lambda| \geq 1$ which gives the result. □

**Remark 4.4.** Since $\rho_u(\lambda \circ (z_1,z_2)) = \rho_u(z_1,z_2) + ab \log |\lambda|$, in particular,
$$
\rho_u(e^{i\theta} \circ (z_1,z_2)) = \rho_u(z_1,z_2).
$$
Moreover, any point $(z_1,z_2) \in \mathbb{C}^2$ is of the form $(\lambda^a \zeta_1, \lambda^b \zeta_2)$ for some $(\zeta_1, \zeta_2) \in \partial P^2$ and some $\lambda \in \mathbb{C}$. Indeed, if $b \geq a$ then we get all points $(z_1,z_2) \in \mathbb{C}^2$ with $|z_2|^a \leq |z_1|^b$ as $(\lambda^a \zeta_1, \lambda^b \zeta_2)$ for some $(\zeta_1, \zeta_2)$ with $|\zeta_1| = 1$ and $|\zeta_2| \leq 1$ and we get all points $(z_1,z_2) \in \mathbb{C}^2$ with $|z_2|^a \geq |z_1|^b$ as $(\lambda^a \zeta_1, \lambda^b \zeta_2)$ for some $(\zeta_1, \zeta_2)$ with $|\zeta_1| \leq 1$ and $|\zeta_2| = 1$. Thus we recover the values of $\rho_u$ on $\mathbb{C}^2$ from its values on $\partial P^2$.

**Remark 4.5.** In the general case where
$$
C = \text{co}\{(0,\ldots,0), (a_1,0,\ldots,0), \ldots, (0,\ldots,0, a_d)\} \in (\mathbb{R}^+)^d
$$
where $a_1,\ldots,a_d$ are pairwise relatively prime, we have
$$
H_C(z_1,\ldots,z_d) = \max\{a_j \log^+ |z_j| : j = 1,\ldots,d\}
$$
and we define
$$
\lambda \circ (z_1,\ldots,z_d) := (\lambda \prod_{j \neq 1} a_j z_1, \ldots, \lambda \prod_{j \neq d} a_j z_d)
$$
so that
$$
H_C(\lambda \circ (z_1,\ldots,z_d)) = H_C(z_1,\ldots,z_d) + \left(\prod_{j=1}^d a_j \right) \log |\lambda|
$$
for $(z_1,\ldots,z_d) \in \mathbb{C}^d \setminus P^d$ and $|\lambda| \geq 1$. Then given $u \in L_C$, we define the $C$-Robin function of $u$ as
$$
\rho_u(z_1,\ldots,z_d) := \limsup_{|\lambda| \to \infty} \left[u(\lambda \circ (z_1,\ldots,z_d)) - \left(\prod_{j=1}^d a_j \right) \log |\lambda|\right]
$$
for $(z_1,\ldots,z_d) \in \mathbb{C}^d$.
We recall the Siciak–Zaharjuta formula (3.1) for $K \subset \mathbb{C}^d$ compact:

$$V_{C,K}(z) = \sup\left\{ \frac{1}{\deg_C(p)} \log |p(z)| : p \in \mathbb{C}[z], \|p\|_K \leq 1 \right\}$$

For simplicity in notation, we write $\rho_{C,K} := \rho_{V^*_C,K}$. The following result will be used in Section 8.

**Theorem 4.6.** Let $K \subset \mathbb{C}^2$ be nonpluripolar and satisfy

$$e^{i\theta} \circ K = K.$$  

Then $K = \{ \rho_{C,K} \leq 0 \}$ and $V^*_C,K = \rho^+_{C,K} := \max[\rho_{C,K},0]$.

**Proof.** We first define a $C$-homogeneous extremal function $H_{C,K}$ associated to a general compact set $K$. To this end, for each $n \in \mathbb{N}$ we define the collection of $nC$-homogeneous polynomials by

$$H_n(C) := \left\{ h_n(z_1,z_2) = \sum_{(j,k):aj+bk=nab} c_{jk} z_1^j z_2^k : c_{jk} \in \mathbb{C} \right\} \subset \text{Poly}(nC).$$

Note that for $h_n \in H_n(C)$,

$$h_n(\lambda \circ (z_1,z_2)) = \lambda^{nab} \sum_{(j,k):aj+bk=nab} c_{jk} z_1^j z_2^k = \lambda^{nab} h_n(z_1,z_2)$$

and thus $u := \frac{1}{n} \log |h_n|$ satisfies

$$u(\lambda \circ (z_1,z_2)) = u(z_1,z_2) + ab \log |\lambda|.$$  

Define

$$H_{C,K}(z_1,z_2) := \sup_n \sup\left\{ \frac{1}{n} \log |h_n(z_1,z_2)| : h_n \in H_n(C), \|h_n\|_K \leq 1 \right\}.$$  

Then $H_{C,K}$ satisfies the property in (4.6). Clearly

$$H^+_{C,K} := \max[H_{C,K},0] \leq V_{C,K}$$

and hence $K \subset \{ H_{C,K} \leq 0 \}$.

For a polynomial $p \in \text{Poly}(nC)$, we write

$$p(z_1,z_2) = \sum_{aj+bk \leq nab} c_{jk} z_1^j z_2^k = \sum_{l=0}^{nab} \tilde{h}_l(z_1,z_2)$$

where $\tilde{h}_l(z_1,z_2) := \sum_{aj+bk=l} c_{jk} z_1^j z_2^k$ satisfies

$$\tilde{h}_l(\lambda \circ (z_1,z_2)) = \lambda^l \tilde{h}_l(z_1,z_2).$$

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Then for each $l = 0, 1, \ldots, nab$,

$$
\|\tilde{h}_l\|_K \leq \|p\|_K.
$$

To prove (4.9), note that

$$
p(\lambda \circ (z_1, z_2)) = \sum_{l=0}^{nab} \lambda^l \tilde{h}_l(z_1, z_2).
$$

Take $(z_1, z_2) \in K$ at which $|\tilde{h}_l(z_1, z_2)| = \|\tilde{h}_l\|_K$. Then by the Cauchy estimates for $\lambda \to F(\lambda) := p(\lambda \circ (z_1, z_2))$ on the unit circle,

$$
|\tilde{h}_l(z_1, z_2)| = \|\tilde{h}_l\|_K = |F(l)(0)|/l! \leq \max_{|\lambda|=1} |F(\lambda)| \leq \|p\|_K,
$$

proving (4.9).

We define

$$
\tilde{H}_l := \{\tilde{h}_l(z_1, z_2) := \sum_{a_j+bk=l} c_{jk} z_1^j z_2^k, c_{jk} \in \mathbb{C} \}.
$$

If $a = b = 1$, $\tilde{H}_l = H_l(\Sigma) = H_l(C)$ are the usual homogeneous polynomials of degree $l$ in $C^d$. Moreover, if $\tilde{h}_l \in \tilde{H}_l$, then $\tilde{h}_l^{ab} \in H_l(C)$. Since $\|\tilde{h}_l\|_K \leq 1$ if and only if $\|\tilde{h}_l^{ab}\|_K \leq 1$, this shows

$$
H_{C,K}(z_1, z_2) = ab \cdot \sup_l \sup \left\{ \frac{1}{l} \log |\tilde{h}_l(z_1, z_2)| : \tilde{h}_l \in \tilde{H}_l, \|\tilde{h}_l\|_K \leq 1 \right\}.
$$

We define the $C$-homogeneous polynomial hull $\hat{K}_C$ of a compact set $K$ as

$$
\hat{K}_C := \left\{(z_1, z_2) : |k(z_1, z_2)| \leq \|k\|_K, k \in \bigcup_l \tilde{H}_l \right\}.
$$

It is clear $\hat{K} \subset \hat{K}_C$ for any compact set $K$. We show the reverse inclusion, and hence equality, for $K$ satisfying (4.5). To this end, let $a \in \hat{K}_C$. For $p \in \text{Poly}(nC)$, write $p = \sum_{l=0}^{nab} \tilde{h}_l$ as in (4.8). Then

$$
|p(a)| \leq \sum_{l=0}^{nab} |\tilde{h}_l(a)| \leq \sum_{l=0}^{nab} \|\tilde{h}_l\|_K \leq (nab + 1) \|p\|_K.
$$

Thus

$$
|p(a)| \leq (nab + 1) \|p\|_K.
$$
Apply this to \( p^m \in \text{Poly}(nmC) \):

\[
|p(a)|^m \leq (nmab + 1)\|p\|_K^m
\]

so that

\[
|p(a)|^{1/n} \leq (nmab + 1)^{1/nm}\|p\|_K^{1/n}.
\]

Letting \( m \to \infty \), we obtain \( |p(a)| \leq \|p\|_K \) and hence \( a \in \hat{K} \).

We use this to show

\[
\{ V_{C,K} = 0 \} = \{ H_{C,K} \leq 0 \}
\]

for sets satisfying (4.5). To see this, we observe from (4.10) that the right-hand side of (4.11) is the \( C \)-homogeneous polynomial hull \( \hat{K}_C \) of \( K \) while the left-hand-side is the polynomial hull \( \hat{K} \) of \( K \). Thus (4.11) follows from the previous paragraph.

Now we claim that \( V_{C,K}^* = H_{C,K}^+ \). We observed that \( H_{C,K}^+ \leq V_{C,K} \); for the reverse inequality, we observe that \( H_{C,K}^+ \) is in \( L_C \) and since \( H_{C,K} \) satisfies (4.6), we have \( H_{C,K}^+ \) is maximal outside \( \hat{K} \). From (4.11) we can apply the global domination principle (Proposition 2.2) to conclude that \( H_{C,K}^+ \geq V_{C,K}^* \) and hence \( H_{C,K}^+ = V_{C,K}^* \).

Using \( H_{C,K}^+ = V_{C,K}^* \),

\[
\rho_{C,K}(z_1, z_2) := \limsup_{|\lambda| \to \infty} [H_{C,K}(\lambda \circ (z_1, z_2)) - ab \log |\lambda|] = \limsup_{|\lambda| \to \infty} H_{C,K}(z_1, z_2) = H_{C,K}(z_1, z_2)
\]

for \( (z_1, z_2) \in \mathbb{C}^2 \setminus K \) by the invariance of \( H_{C,K} \) (i.e., it satisfies (4.6)). Thus, from Proposition 4.3 (and the invariance of \( \rho_{C,K} \)) we have

\[
\rho_{C,K}^+ = H_{C,K}^+ = V_{C,K}^*.
\]

This shows \( K = \{ \rho_{C,K} \leq 0 \} \) and \( V_{C,K}^* = \rho_{C,K}^+ := \max[\rho_{C,K}, 0] \). □

**Remark 4.7.** It follows that for

\[
p = \sum_{l=0}^{nab} \tilde{h}_l = h_n + r_n \in \text{Poly}(nC)
\]

where \( h_n := \tilde{h}_{nab} \in H_n(C) \) and \( r_n = p - h_n = \sum_{aj+bk<nab} c_{jk}z_1^jz_2^k \), if \( u := \frac{1}{n} \log |p_n| \) then

\[
\rho_u = \frac{1}{n} \log |\tilde{h}_{nab}| = \frac{1}{n} \log |h_n|.
\]
We write $\hat{p}_n := h_n = \hat{h}_{nab}$; thus $\rho_u = \frac{1}{n} \log |\hat{p}_n|.$

In the case $a = b = 1$ where $C = \Sigma$, we know from [7, Corollary 4.6] that if $K$ regular then $\rho_K := \rho_{\Sigma,K}$ is continuous. We need to know that for our triangles $C$ where $a, b$ are relatively prime positive integers we also have $\rho_{C,K}$ is continuous. To this end, we begin with the observation that applying Theorem 4.1 in the special case where $d = 2$ and $C$ is our triangle with vertices $(0,0), (b,0), (0,a)$, we can take

$$F(z_1, z_2) = (z_1^a, z_2^b)$$

and $c = ab$ to obtain

$$ab\rho_{F^{-1}(K)}(z_1, z_2) = \limsup_{|\lambda| \to \infty} [V_{C,K}(\lambda^a z_1^a, \lambda^b z_2^b) - ab \log |\lambda|]$$

$$= \limsup_{|\lambda| \to \infty} [V_{C,K}(\lambda \circ (z_1^a, z_2^b)) - ab \log |\lambda|] = \rho_{C,K}(z_1^a, z_2^b) = \rho_{C,K}(F(z_1, z_2)).$$

We use this connection between $\rho_{C,K}$ and the standard Robin function $\rho_{F^{-1}(K)}$ to show that $\rho_{C,K}$ is continuous if $K$ is regular.

**Proposition 4.8.** Let $K \subset \mathbb{C}^2$ be compact and regular. Then $\rho_{C,K}$ is uniformly continuous on $\partial P^2$.

**Proof.** With $F(z_1, z_2) = (z_1^a, z_2^b)$ as above, from [10, Theorem 5.3.6], we have $F^{-1}(K)$ is regular. Thus, from [7, Corollary 4.6], $\rho_{F^{-1}(K)}$ is continuous. Hence $\rho_{C,K}(z_1^a, z_2^b) = \rho_{C,K}(F(z_1, z_2))$ is continuous. To show $\zeta \to \rho_{C,K}(\zeta)$ is continuous at $\zeta = (\zeta_1, \zeta_2) \in \partial P^2$, we use the fundamental relationship that

$$ab\rho_{F^{-1}(K)}(z_1, z_2) = \rho_{C,K}(z_1^a, z_2^b).$$

To this end, let $\zeta^n = (\zeta_1^n, \zeta_2^n) \in \partial P^2$ converge to $\zeta = (\zeta_1, \zeta_2)$. Then

$$\rho_{C,K}(\zeta_1^n, \zeta_2^n) = \rho_{C,K}((\zeta_1^n)^{1/a}, (\zeta_2^n)^{1/b})$$

for any $a$-th root $(\zeta_1^n)^{1/a}$ of $\zeta_1^n$ and any $b$-th root $(\zeta_2^n)^{1/b}$ of $\zeta_2^n$. But

$$\rho_{C,K}((\zeta_1^n)^{1/a}, (\zeta_2^n)^{1/b}) = ab\rho_{F^{-1}(K)}((\zeta_1^n)^{1/a}, (\zeta_2^n)^{1/b}).$$

By continuity of $\rho_{F^{-1}(K)}$,

$$\lim_{n \to \infty} \rho_{C,K}(\zeta_1^n, \zeta_2^n) = \lim_{n \to \infty} ab\rho_{F^{-1}(K)}((\zeta_1^n)^{1/a}, (\zeta_2^n)^{1/b})$$

$$= ab\rho_{F^{-1}(K)}((\zeta_1)^{1/a}, (\zeta_2)^{1/b})$$
for the appropriate choice of \((\zeta_1)^{1/a}\) and \((\zeta_2)^{1/b}\). But

\[
abrho_{F^{-1}(K)}((\zeta_1)^{1/a}, (\zeta_2)^{1/b}) = \rho_{C,K}([(\zeta_1)^{1/a}], [(\zeta_2)^{1/b}]) = \rho_{C,K}(\zeta_1, \zeta_2).
\]

Note that this also yields that the value of \(\rho_{F^{-1}(K)}((\zeta_1)^{1/a}, (\zeta_2)^{1/b})\) is independent of the choice of the roots \((\zeta_1)^{1/a}\) and \((\zeta_2)^{1/b}\). This can also be seen from the definitions of \(\rho_{F^{-1}(K)}\) and \(F\). □

Remark 4.9. The relationship

\[
\rho_{F^{-1}(K)}(z_1, z_2) = \rho_{C,K}(z_1^a, z_2^b)
\]

is a special case of a more general result. Let \(u \in L_C\). Then

\[
\tilde{u}(z) := u(F(z_1, z_2)) = u(z_1^a, z_2^b) \in abL = abL_\Sigma
\]

and

\[
\rho_u(F(z_1, z_2)) = \rho_u(z_1^a, z_2^b) = \limsup_{|\lambda| \to \infty} \left[ u(\lambda \circ (z_1^a, z_2^b)) - ab \log |\lambda| \right]
= \limsup_{|\lambda| \to \infty} \left[ u(\lambda^a z_1, \lambda^b z_2) - ab \log |\lambda| \right] = \limsup_{|\lambda| \to \infty} \left[ \tilde{u}(\lambda) - ab \log |\lambda| \right].
\]

Since \(\tilde{u} \in abL\), this last line is equal to the “usual” Robin function of \(\tilde{u}\) in the sense of (1.5). To be precise, it is equal to \(ab\rho_{\tilde{u}/ab}\) where \(\rho_{\tilde{u}/ab}\) is the standard Robin function (1.5) of \(\tilde{u}/ab \in L\). This observation will be crucial in Section 6.

We need an analogue of formula (18) in [17] in order to verify a calculation in the next section. We follow the arguments in [17]. Recall we may lift the circle action on \(C^2\) to \(C^3\) via

\[
\lambda \circ (t, z_1, z_2) := (\lambda t, \lambda^a z_1, \lambda^b z_2).
\]

This gave a correspondence between \(L_C(C^2)\) and a subclass of \(L_{\tilde{C}}(C^3)\) where

\[
\tilde{C} = \text{co}\{(0,0,0), (1,0,0), (0,b,0), (0,0,a)\}.
\]

In analogy with our class

\[
H_n(C) := \left\{ h_n(z_1, z_2) = \sum_{(j,k): a_j + b_k = nab} c_{jk} z_j^1 z_k^2 : c_{jk} \in \mathbb{C} \right\} \subset \text{Poly}(nC)
\]

in \(C^2\), we can consider

\[
H_n(\tilde{C}) := \left\{ h_n(t, z_1, z_2) = \sum_{(i,j,k): i + a_j + b_k = nab} c_{ijk} t^i z_j^1 z_k^2 : c_{ijk} \in \mathbb{C} \right\} \subset \text{Poly}(n\tilde{C})
\]
in \( \mathbb{C}^3 \). For \( h_n \in H_n(\tilde{C}) \), we have
\[
 u_n(t, z_1, z_2) := \frac{1}{n} \log |h_n(t, z_1, z_2)|
\]
belongs to \( L_{\tilde{C}}(\mathbb{C}^3) \) and \( u_n \) is \( ab \)-log-homogeneous. That \( u_n \in L_{\tilde{C}}(\mathbb{C}^3) \) is clear; to show the \( ab \)-log-homogeneity, note that
\[
h_n(\lambda \circ (t, z_1, z_2)) = h_n(\lambda t, \lambda^a z_1, \lambda^b z_2)
\]
\[
= \sum_{(i,j,k): i+aj+bk=nab} c_{ijk} (\lambda t)^i (\lambda^a z_1)^j (\lambda^b z_2)^k
\]
\[
= \sum_{(i,j,k): i+aj+bk=nab} c_{ijk} \lambda^i a^j b^k t^{i} z_1^{j} z_2^{k} = \lambda^{nab} h_n(t, z_1, z_2)
\]
so that
\[
u_n(\lambda \circ (t, z_1, z_2)) = u_n(t, z_1, z_2) + ab \log |\lambda|.
\]
Moreover, for \( h_n \in H_n(\tilde{C}) \), the polynomial
\[
p_n(z_1, z_2) := h_n(1, z_1, z_2) = \sum_{(j,k): aj+bk \leq nab} c_{ijk} z_1^j z_2^k \in \text{Poly}(nC);
\]
conversely, if \( p_n(z_1, z_2) = \sum_{(j,k): aj+bk \leq nab} c_{jk} z_1^j z_2^k \in \text{Poly}(nC) \), then
\[
h_n(t, z_1, z_2) := t^{nab} \cdot p_n(\frac{z_1}{ta}, \frac{z_2}{tb}) \in H_n(\tilde{C}).
\]
Next, given a compact set \( E \subset \mathbb{C}^3 \), we define the \( ab \)-log-homogeneous \( \tilde{C} \)-extremal function
\[
H_{\tilde{C},E}(t, z_1, z_2) := \sup \left\{ \frac{1}{\deg_C(p)} \log |p(t, z_1, z_2)| : p \in \bigcup_n H_n(\tilde{C}), ||p||_{E} \leq 1 \right\}
\]
and its usc regularization \( H^*_{\tilde{C},E} \). Given the one-to-one correspondence between \( \text{Poly}(nC) \) in \( \mathbb{C}^2 \) and \( H_n(\tilde{C}) \) in \( \mathbb{C}^3 \), we see that for \( K \subset \mathbb{C}^2 \) compact,
\[
(4.12) \quad V_{C,K}(z_1, z_2) = H_{\tilde{C},\{1\} \times K}(1, z_1, z_2) \quad \text{for all } (z_1, z_2) \in \mathbb{C}^2
\]
and hence a similar equality holds for the usc regularizations of both sides. Using this, we observe that for \( \zeta = (\zeta_1, \zeta_2) \neq (0,0) \), we have
\[
\rho_{C,K}(\zeta) = \limsup_{|\lambda| \to \infty} \left[ V^*_{C,K}(\lambda \circ \zeta) - ab \log |\lambda| \right]
\]
\[ \limsup_{|\lambda| \to \infty} H^*_{C,(1) \times K}(1, \lambda^a \zeta_1, \lambda^b \zeta_2) - ab \log |\lambda| \]

\[ = \limsup_{|\lambda| \to \infty} H^*_{C,(1) \times K} \left( 1/\lambda, \zeta_1, \zeta_2 \right) = H^*_{C,(1) \times K} \left( 0, \zeta_1, \zeta_2 \right). \]

Here we have used the fact that

\[ H^*_{C,(1) \times K}(\lambda \circ (1/\lambda, \zeta_1, \zeta_2)) = H^*_{C,(1) \times K}(1, \lambda^a \zeta_1, \lambda^b \zeta_2). \]

We state this as a proposition:

**Proposition 4.10.** For \( K \subset \mathbb{C}^2 \) compact,

\[ \rho_{C,K}(\zeta_1, \zeta_2) = H^*_{C,(1) \times K}(0, \zeta_1, \zeta_2) \quad \text{for all} \ (\zeta_1, \zeta_2) \neq (0,0). \]

**Remark 4.11.** Using the relation (4.12) and following the reasoning in [15, Proposition 2.3], it follows that a compact set \( K \subset \mathbb{C}^2 \) is regular; i.e., \( V_{C,K} \) is continuous in \( \mathbb{C}^2 \), if and only if \( H^*_{C,(1) \times K} \) is continuous in \( \mathbb{C}^3 \). Thus we get an alternate proof of Proposition 4.8.

5. Preliminary results: triangle case

We continue to let \( C \) be the triangle with vertices at \((0,0), (b,0), \) and \((0,a)\) where \( a, b \) are relatively prime positive integers. For \( K \subset \mathbb{C}^2 \) compact and \( \zeta := (\zeta_1, \zeta_2) \in \partial P^2 \), we define Chebyshev constants

\[ \kappa_n := \kappa_n(K, \zeta) := \inf \{ \|p_n\|_K : p_n \in \text{Poly}(nC), \ |\hat{p}_n(\zeta)| = 1 \}. \]

We note that \( \kappa_{n+m} \leq \kappa_n \kappa_m \): if we take \( t_n, t_m \) achieving \( \kappa_n, \kappa_m \), then \( t_n t_m \in \text{Poly}(n+m)C \) and \( t_n \hat{t}_m = \hat{t_n} t_m \) (see Remark 4.7) so that

\[ \kappa_{n+m} \leq \|t_n t_m\|_K \leq \kappa_n \kappa_m. \]

Thus \( \lim_{n \to \infty} \kappa_n^{1/n} \) exists (this follows from a classical lemma of Fekete) and we set

\[ \kappa(K, \zeta) = \lim_{n \to \infty} \kappa_n^{1/n} = \inf_n \kappa_n^{1/n}. \]

The following relation between \( \kappa(K, \zeta) \) and \( \rho_{C,K}(\zeta) \) is analogous to [14, Proposition 4.2].

**Proposition 5.1.** For \( \zeta \in \partial P^2 \),

\[ \kappa(K, \zeta) = e^{-\rho_{C,K}(\zeta)}. \]
Proof. We first note that
\[
\kappa_n(K, \zeta) = \inf \left\{ \frac{\|p_n\|_K}{|p_n(\zeta)|} : p_n \in \text{Poly}(nC) \right\}
\]
\[
= \inf \left\{ \frac{1}{|\hat{p}_n(\zeta)|} : p_n \in \text{Poly}(nC), \|p_n\|_K \leq 1 \right\}.
\]
Thus for any \(p_n \in \text{Poly}(nC)\) with \(\|p_n\|_K \leq 1\), \(\kappa_n(K, \zeta) \leq \frac{1}{|\hat{p}_n(\zeta)|}\). For such \(p_n\), \(\frac{1}{n} \log |p_n(z)| \leq V_{C,K}(z)\) for all \(z \in \mathbb{C}^2\) so that
\[
\frac{1}{n} \log |\hat{p}_n(\zeta)| \leq \rho_{C,K}(\zeta); \quad \text{i.e.} \quad \frac{1}{|\hat{p}_n(\zeta)|^{1/n}} \geq e^{-\rho_{C,K}(\zeta)}
\]
for all \(\zeta \in \partial P^2\). Taking the infimum over all such \(p_n\),
\[
\kappa_n(K, \zeta)^{1/n} \geq e^{-\rho_{C,K}(\zeta)}
\]
for all \(n\); taking the limit as \(n \to \infty\) gives
\[
\kappa(K, \zeta) \geq e^{-\rho_{C,K}(\zeta)}.
\]

To prepare for the reverse inequality, we let \(\{b_j\}\) be an orthonormal basis of \(\bigcup_n \text{Poly}(nC)\) in \(L^2(\mu)\) where \(\mu\) is any Bernstein–Markov measure for \(K\): thus for any \(\varepsilon > 0\), there exists a constant \(c_\varepsilon\) so that
\[
\|p_n\|_K \leq c_\varepsilon (1 + \varepsilon)^{\deg_{C}(p_n)} \|p_n\|_{L^2(\mu)}, \quad p_n \in \text{Poly}(nC), \quad n = 1, 2, \ldots.
\]
In particular,
\[
\|b_j\|_K \leq c_\varepsilon (1 + \varepsilon)^{\deg_{C}(b_j)}
\]
and from Corollary 3.2,
\[
\limsup_{j \to \infty} \frac{1}{\deg_{C}(b_j)} \log |b_j(z)| = V_{C,K}(z), \quad z \not\in \hat{K}.
\]

We next show that for \(\zeta \in \partial P^2\),
\[
\limsup_{j \to \infty} \frac{1}{\deg_{C}(b_j)} \log |\hat{b}_j(\zeta)| = \rho_{C,K}(\zeta).
\]
For one inequality, we use the fact that for a function \(u\) subharmonic on \(\mathbb{C}\) with \(u \in L\), the function \(r \to \max_{|t|=r} u(t)\) is a convex function of \(\log r\). Hence
\[
\limsup_{|t| \to \infty} \frac{u(t) - \log |t|}{r} = \inf_r \left( \max_{|t|=r} u(t) - \log r \right).
\]
Thus if \( u \in abL \); i.e., \( u(z) - ab \log |z| = 0(1) \), \( |z| \to \infty \), we have

\[
\limsup_{|t| \to \infty} [u(t) - ab \log |t|] = \inf_{r} (\max_{|t|=r} u(t) - ab \log r).
\]

Fix \( \zeta \in \partial P^2 \) and letting \( d_j := \deg_C(b_j) \) apply this to the function

\[
\lambda \to \frac{1}{d_j} \log |b_j(\lambda \circ \zeta)| = \frac{1}{d_j} \log |b_j(\lambda^a \zeta_1, \lambda^b \zeta_2)|.
\]

We obtain (using also Remark 4.7), for any \( r \),

\[
\frac{1}{d_j} \log |\hat{b}_j(\zeta)| = \limsup_{|\lambda| \to \infty} \left[ \frac{1}{d_j} \log |b_j(\lambda \circ \zeta)| - ab \log |\lambda| \right]
\]

\[
\leq \max_{|\lambda|=r} \frac{1}{d_j} \log |b_j(\lambda \circ \zeta)| - ab \log r.
\]

Thus

\[
\limsup_{j \to \infty} \frac{1}{d_j} \log |\hat{b}_j(\zeta)| \leq \limsup_{j \to \infty} \left( \max_{|\lambda|=r} \frac{1}{d_j} \log |b_j(\lambda \circ \zeta)| - ab \log r \right)
\]

\[
\leq \max_{|\lambda|=r} \left( \limsup_{j \to \infty} \frac{1}{d_j} \log |b_j(\lambda \circ \zeta)| - ab \log r \right) = \max_{|\lambda|=r} [V_{C,K}(\lambda \circ \zeta) - ab \log r]
\]

where we used Hartogs lemma and (5.1). Thus, letting \( r \to \infty \),

\[
\limsup_{j \to \infty} \frac{1}{d_j} \log |\hat{b}_j(\zeta)| \leq \rho_{C,K}(\zeta).
\]

In order to prove the reverse inequality in (5.2), we use Proposition 4.10. With the notation from the previous section, and following the proof of [17, Théorème 2], let \( h \in H_n(\tilde{C}) \) with \( \|h\|_{1 \times K} \leq 1 \). Then

\[
p(z_1, z_2) := h(1, z_1, z_2) \in \text{Poly}(nC)
\]

with \( \|p\|_K \leq 1 \). Writing \( p = \sum_{j=1}^{N_n} c_j b_j \) where \( N_n = \dim(\text{Poly}(nC)) \) as in the proof of Proposition 3.1, we have \( |c_j| \leq 1 \) and hence

\[
|h(1, z_1, z_2)| = |p(z_1, z_2)| \leq \sum_{j=1}^{N_n} |b_j(z_1, z_2)|.
\]

Then

\[
|h(1/\lambda, z_1, z_2)| = |\lambda^{-nab} \cdot p(\lambda^a z_1, \lambda^b z_2)| \leq \left| \lambda^{-nab} \sum_{j=1}^{N_n} b_j(\lambda^a z_1, \lambda^b z_2) \right|.
\]

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Fixing \((z_1, z_2) = (\zeta_1, \zeta_2)\) and letting \(|\lambda| \to \infty\), we get
\[
|h(0, \zeta_1, \zeta_2)| \leq \sum_{b_j \in \text{Poly}(nC) \setminus \text{Poly}((n-1)C)} |\hat{b}_j(\zeta_1, \zeta_2)| \leq (N_n - N_{n-1})|\hat{b}_{j_n}(\zeta_1, \zeta_2)|
\]
where \(N_{n-1} \leq j_n \leq N_n\). Using Proposition 4.10 we conclude that
\[
\rho_{C,K}(\zeta_1, \zeta_2) \leq \limsup_{j \to \infty} \frac{1}{\deg_{C}(b_j)} \log |\hat{b}_j(\zeta_1, \zeta_2)|
\]
and (5.2) is proved.

We now use (5.2) to prove that \(\kappa(K, \zeta) \leq e^{-\rho_{C,K}(\zeta)}\) for \(\zeta \in \partial P^2\) which will finish the proof of the proposition. Fixing such a \(\zeta\) and \(\varepsilon > 0\), take a subsequence \(\{b_{k_j}\}\) with \(d_j := \deg_{C}(b_{k_j})\) such that
\[
\frac{1}{d_j} \log |\hat{b}_{k_j}(\zeta)| \geq \rho_{C,K}(\zeta) - \varepsilon, \quad j \geq j_0(\varepsilon).
\]

Letting
\[
p_j(z) := \frac{b_{k_j}(z)}{c_{\varepsilon}(1 + \varepsilon)^{d_j}},
\]
we have \(\|p_j\|_K \leq 1\) and
\[
\rho_{C,K}(\zeta) - \varepsilon \leq \frac{1}{d_j} \log |\hat{b}_{k_j}(\zeta)| = \frac{1}{d_j} \log |\hat{p}_j(\zeta)| + \frac{1}{d_j} \log c_{\varepsilon} + \log(1 + \varepsilon).
\]

Thus
\[
\varepsilon - \rho_{C,K}(\zeta) \geq \frac{1}{d_j} \log \frac{1}{|\hat{p}_j(\zeta)|} - \frac{1}{d_j} \log c_{\varepsilon} - \log(1 + \varepsilon)
\]
\[
\geq \frac{1}{d_j} \log \kappa_{d_j}(K, \zeta) - \frac{1}{d_j} \log c_{\varepsilon} - \log(1 + \varepsilon).
\]

Letting \(j \to \infty\),
\[
\varepsilon - \rho_{C,K}(\zeta) \geq \log \kappa(K, \zeta) - \log(1 + \varepsilon);
\]
which holds for all \(\varepsilon > 0\). Letting \(\varepsilon \to 0\) completes the proof. \(\square\)

Using this proposition, and the observation within its proof that
\[
\kappa_n(K, \zeta) = \inf \left\{ \frac{1}{|\hat{p}_n(\zeta)|} : p_n \in \text{Poly}(nC), \|p_n\|_K \leq 1 \right\},
\]
we obtain a result which will be useful in proving Theorem 5.3.
Corollary 5.2. Let $K \subset \mathbb{C}^2$ be compact and regular. Given $\varepsilon > 0$, there exists a positive integer $m$ and a finite set of polynomials $\{W_1, \ldots, W_s\} \subset \text{Poly}(mC)$ such that $\|W_j\|_K = 1$, $j = 1, \ldots, s$ and

$$\frac{1}{m} \log \max_j |\hat{W}_j(\zeta)| \geq \rho_{C,K}(\zeta) - \varepsilon \quad \text{for all } \zeta \in \partial P^2.$$

Proof. From Proposition 5.1, given $\varepsilon > 0$, for each $\zeta \in \partial P^2$ we can find a polynomial $p \in \text{Poly}(nC)$ for $n \geq n_0(\varepsilon)$ with $\|p\|_K = 1$ and

$$\frac{1}{n} \log |\hat{p}(\zeta)| \geq \rho_{C,K}(\zeta) - \varepsilon.$$

By continuity of $\rho_{K,C}$, which follows from Proposition 4.8, such an inequality persists in a neighborhood of $\zeta$. We take a finite set $\{p_1, \ldots, p_s\}$ of such polynomials with $p_i \in \text{Poly}(n_i C)$ such that

$$\max_i \frac{1}{n_i} \log |\hat{p_i}(\zeta)| \geq \rho_{C,K}(\zeta) - \varepsilon \quad \text{for all } \zeta \in \partial P^2.$$

Raising the $p_i$’s to powers to obtain $W_i$’s of the same $C$-degree $m$, we still have $\|W_i\|_K = 1$ and

$$\frac{1}{m} \log \max_j |\hat{W}_j(\zeta)| \geq \rho_{C,K}(\zeta) - \varepsilon \quad \text{for all } \zeta \in \partial P^2. \quad \Box$$

Given $K \subset \mathbb{C}^2$ compact, and given $h_n \in H_n(C)$, we define

$$\text{Tch}_K h_n := h_n + p_{n-1} \quad \text{where } p_{n-1} \in \text{Poly}(n - 1)C$$

and

$$\| \text{Tch}_K h_n \|_K = \inf \{ \|h_n + q_{n-1}\|_K : q_{n-1} \in \text{Poly}(n - 1)C \}.$$

The polynomial $\text{Tch}_K h_n$ need not be unique but each such polynomial yields the same value of $\| \text{Tch}_K h_n \|_K$. The next result is similar to [4, Theorem 3.2].

Theorem 5.3. Let $K \subset \mathbb{C}^2$ be compact, regular and polynomially convex. If $\{Q_n\}$ is a sequence of polynomials with $Q_n \in H_n(C)$ satisfying

$$\limsup_{n \to \infty} \frac{1}{n} \log |Q_n(\zeta)| \leq \rho_{C,K}(\zeta), \quad \text{for all } \zeta \in \partial P^2,$$

then

$$\limsup_{n \to \infty} \| \text{Tch}_K Q_n \|_K^{1/n} \leq 1.$$
Proof. We follow the proof in [15]. Given $\varepsilon > 0$, we start with polynomials \( \{W_1, \ldots, W_s\} \subset \text{Poly}(mC) \) such that $\|W_j\|_K = 1$, $j = 1, \ldots, s$ and

$$\frac{1}{m} \log \max_j |\hat{W}_j(\zeta)| \geq \rho_{C,K}(\zeta) - \varepsilon$$

for all $\zeta \in \partial P^2$

(and hence on all of $\mathbb{C}^2$) from Corollary 5.2. From the hypotheses on $\{Q_n\}$ and the continuity of $\rho_{C,K}$ (Proposition 4.8), we apply Hartogs lemma to conclude

$$\frac{1}{n} \log |Q_n(\zeta)| < \rho_{C,K}(\zeta) + \varepsilon, \quad \zeta \in \partial P^2, \quad n \geq n_0(\varepsilon).$$

Thus

$$(5.4) \quad \frac{1}{n} \log |Q_n(\zeta)| < \frac{1}{m} \log \max_j |\hat{W}_j(\zeta)| + 2\varepsilon, \quad \zeta \in \partial P^2, \quad n \geq n_0(\varepsilon).$$

Note that $Q_n \in H_n(C)$ implies $Q_n(\lambda \circ \zeta) = \lambda^{nab}Q_n(\zeta)$ so that

$$\frac{1}{n} \log |Q_n(\lambda \circ \zeta)| = \frac{1}{n} \log |Q_n(\zeta)| + ab \log |\lambda|.$$ 

Similary $\hat{W}_j \in H_m(C)$ implies

$$\frac{1}{m} \log |\hat{W}_j(\lambda \circ \zeta)| = \frac{1}{m} \log |\hat{W}_j(\zeta)| + ab \log |\lambda|$$

so that (5.4) holds on all of $\mathbb{C}^2$.

We fix $R > 1$ and define

$$G := \{ z \in \mathbb{C}^2 : |\hat{W}_j(z)| < R^m, \quad j = 1, \ldots, s \}.$$ 

Since $\hat{W}_j(\lambda \circ \zeta) = \lambda^{nab}\hat{W}_j(\zeta)$, we have $e^{i\theta} \circ G = G$; since $\hat{W}_j(0) = 0$, we have $0 \in G$. We claim $G$ is bounded. To see this, choose $r > 0$ so that

$$K \subset rP^2 = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|, |z_2| < r \}.$$ 

Then $V_{C,K}(z_1, z_2) \geq H_{C}(z_1/r, z_2/r)$ and hence

$$\rho_{C,K}(z_1, z_2) \geq H_{C}(z_1/r, z_2/r), \quad (z_1, z_2) \in \mathbb{C}^2 \setminus rP^2.$$ 

Since

$$\frac{1}{m} \log \max_j |\hat{W}_j(z_1, z_2)| \geq \rho_{C,K}(z_1, z_2) - \varepsilon$$

for all $(z_1, z_2) \in \mathbb{C}^2$,

$G$ is bounded.
Next, choose $\delta > 0$ sufficiently large so that

$$K \cup G \subset \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|, |z_2| < \delta R^m\}.$$ 

Define

$$\Delta := \{(z, w) \in \mathbb{C}^2 \times \mathbb{C}^s : |z_1|, |z_2| < \delta R^m, |w_j| < \delta R^m, j = 1, \ldots, s\}.$$ 

Given $\theta > 1$, we can choose $p > 0$ sufficiently large so that

$$D := \{(z, w) \in \mathbb{C}^2 \times \mathbb{C}^s : |z_1|^p + |z_2|^p + |w_1|^p + \cdots + |w_s|^p < (\theta \alpha \beta R^m)^p\},$$

which is complete circled (in the ordinary sense) and strictly pseudoconvex, satisfies

$$\Delta \subset D \subset \theta \alpha \beta m \Delta$$

(note this is just a replacement of an $l^\infty$-norm with an $l^p$-norm).

We write $z := (z_1, z_2) \in \mathbb{C}^2$ and $\hat{W}(z) := (\hat{W}_1(z), \ldots, \hat{W}_s(z)) \in \mathbb{C}^s$ for simplicity in notation. Let

$$Y := \{(z, \delta \hat{W}(z)) \in \mathbb{C}^2 \times \mathbb{C}^s : z \in \mathbb{C}^2\}.$$ 

Then $Y$ is a closed, complex submanifold of $\mathbb{C}^2 \times \mathbb{C}^s$. Appealing to the bounded, holomorphic extension result stated as [15, Theorem 3.1], there exists a positive constant $M$ such that for every $f \in H^\infty(Y \cap D)$ there exists $F \in H^\infty(D)$ with

$$\|F\|_D \leq M\|f\|_{Y \cap D} \quad \text{and} \quad F = f \text{ on } Y \cap D.$$ 

We will apply this to the polynomials $Q_n(z)$. First, we observe that if $\pi: \mathbb{C}^2 \times \mathbb{C}^s \to \mathbb{C}^2$ is the projection $\pi(z, w) = z$, then

$$G = \pi(Y \cap \Delta) \subset \pi(Y \cap D) \subset \pi(Y \cap \theta \alpha \beta m \Delta) \subset \theta \circ G.$$ 

To see the last inclusion (note we use $\theta \circ G$, not $\theta G$) first note that

$$s = \theta \circ z \iff z = \frac{1}{\theta} \circ s$$

and thus since $\hat{W}_j(\frac{1}{\theta} \circ s) = \frac{1}{\theta \alpha \beta m} \hat{W}_j(s)$,

$$\theta \circ G = \left\{z \in \mathbb{C}^2 : |\hat{W}_j(z)| < (\theta \alpha \beta R)^m, \ j = 1, \ldots, s\right\}.$$ 

On the other hand,

$$\pi(Y \cap \theta \alpha \beta m \Delta) = \left\{z \in \mathbb{C}^2 : (z, w) \in Y \cap \theta \alpha \beta m \Delta\right\}.$$ 

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= \{ z \in \mathbb{C}^2 : |z_1|, |z_2| < \theta^{abm}\delta R^m, \delta|\hat{W}_j(z)| < \theta^{abm}\delta R^m, j = 1, \ldots, s \}.

Applying the bounded holomorphic extension theorem to \( f(z, w) := Q_n(z) \) for each \( n \), we get \( F_n(z, w) \in H^\infty(D) \) with

\[
Q_n(z) = F_n(z, \delta\hat{W}(z))
\]

for all \( z \in \pi(Y \cap D) \) and

\[
\|F_n\|_D \leq M\|Q_n\|_{\pi(Y \cap D)}.
\]

Utilizing the set inclusion \( \pi(Y \cap D) \subset \theta \circ G \), the definition of \( \theta \circ G \) and (5.4) (which recall is valid on all of \( \mathbb{C}^2 \)),

\[
(5.5) \quad \|F_n\|_D \leq M\|Q_n\|_{\theta \circ G} \leq M(e^{2\varepsilon\theta^{ab}}R)^n \quad \text{for} \ n \geq n_0(\varepsilon).
\]

Since \( D \) is complete circled, we can expand \( F_n \) into a series of homogeneous polynomials which converges locally uniformly on all of \( D \). Rearranging into a multiple power series, we write

\[
F_n(z, w) := \sum_{|I|+|J| \geq 0} a_{IJ}z^I w^J, \quad (z, w) \in D.
\]

Using \( Q_n(z) = F_n(z, \delta\hat{W}(z)) \) for \( z \in \pi(Y \cap D) \), we obtain for such \( z \),

\[
Q_n(z) = \sum' a_{IJ}z^I(\delta\hat{W}(z))^J
\]

where the prime denotes that the sum is taken over multiindices

\[
I = (i_1, i_2) \in (\mathbb{Z}^+)^2, \quad J \in (\mathbb{Z}^+)^s,
\]

where

\[
(5.6) \quad ai_1 + bi_2 =: iab \quad \text{and} \quad i + |J|m = n.
\]

This is because \( Q_n \in H_n(C) \) and \( \hat{W}_j \in H_m(C) \), \( j = 1, \ldots, s \). Precisely, each \( \hat{W}_j(z) \) is of the form \( \sum_{a\alpha + b\beta = mab} c_{\alpha\beta}z_1^{\alpha}z_2^{\beta} \) so that if \( J = (j_1, \ldots, j_s) \), a typical monomial occurring in \( Q_n(z) \) must be of the form

\[
(5.7) \quad z_1^{i_1}z_2^{i_2}(z_1^{\alpha_1}z_2^{\beta_1})^{j_1} \cdots (z_1^{\alpha_s}z_2^{\beta_s})^{j_s}
\]

where \( a\alpha_k + b\beta_k = mab \), \( k = 1, \ldots, s \); hence

\[
a(\alpha_1j_1 + \cdots + \alpha_sj_s) + b(\beta_1j_1 + \cdots + \beta_sj_s) = |J|mab.
\]
In order for (5.7) to (possibly) appear in \( Q_n(z) \), we require (5.6). The positive integers \( i \) in (5.6) are related to the lengths \(|I|\) by \(|I| = i_1 + i_2 \leq ai_1 + bi_2 = iab\); and if, say, \( a \leq b \) we have a reverse estimate

\[
iab = ai_1 + bi_2 \leq b|I| \quad \text{so that} \quad |I| \geq ia.
\]

However, all we will need to use is the fact that the number of multi-indices occurring in the sum for \( Q_n(z) \) is at most \( N_n = \dim(Poly(nC)) \) and \( \lim_{n \to \infty} N_n^{1/n} = 1 \).

Applying the Cauchy estimates on the polydisk \( \Delta \subset D \), we obtain

\[
(5.8) \quad |a_{IJ}| \leq \frac{\|F_n\|_D}{(\delta R^m)|I| + |J|}
\]

for \( n \geq n_0(\varepsilon) \).

We now define

\[
p_n(z) := \sum a_{IJ}z^I(\delta W(z))^J.
\]

From (5.6) and the previously observed fact that

if \( q_j \in Poly(n_jC), \ j = 1, 2 \) then \( \hat{q_1}q_2 = \hat{q_1}\hat{q_2} \),

we have \( \hat{p_n}(z) = Q_n(z) \). Using the estimates (5.5), (5.8), the facts that \( \|W_j\|_K = 1, \ j = 1, \ldots, s \) and

\[
K \cup G \subset \{(z_1, z_2) : |z_1|, |z_2| < \delta R^m\},
\]

we obtain

\[
\|Tch_K Q_n\|_K \leq \|p_n\|_K \leq \sum M(e^{2\varepsilon \theta ab}R)^n \cdot (\delta R^m)^{|I| + |J|} \cdot (\delta R^m)^{|I| |J|}
\]

\[
= \sum M(e^{2\varepsilon \theta ab}R)^n \cdot R^{n-m|J|} \leq \sum M(e^{2\varepsilon \theta ab}R)^n \cdot R^n
\]

\[
\leq C_n(e^{2\varepsilon \theta ab}R)^n, \quad n \geq n_0(\varepsilon),
\]

where \( C_n \) can be taken as \( M \) times the cardinality of the set of multiindices in (5.6). Clearly \( \lim_{n \to \infty} C_n^{1/n} = 1 \) so that

\[
\limsup_{n \to \infty} \|Tch_K Q_n\|^{1/n}_K \leq e^{2\varepsilon \theta ab}R.
\]

Since \( \varepsilon > 0, \ R > 1 \) and \( \theta > 1 \) were arbitrary, the result follows. \( \square \)
6. The integral formula

In the standard setting of the Robin function $\rho_u$ associated to $u \in L(C^2)$ (cf., 1.5), for $z = (z_1, z_2) \neq (0, 0)$ we can define

$$\rho_u(z) := \limsup_{|\lambda| \to \infty} [u(\lambda z) - \log |\lambda z|] = \rho_u(z) - \log |z|$$

so that $\rho_u(tz) = \rho_u(z)$ for $t \in C \setminus \{0\}$. Here $|z|^2 = |z_1|^2 + |z_2|^2$. Thus we can consider $\rho_u$ as a function on $\mathbb{P}^1 = \mathbb{P}^2 \setminus C^2$ where to $p = (p_1, p_2)$ with $|p| = 1$ we associate the point where the complex line $\lambda \to \lambda p$ hits $\mathbb{P}^1$. The integral formula [3, Theorem 5.5] in this setting is the following.

**Theorem 6.1** (Bedford–Taylor). Let $u, v, w \in L^+(C^2)$. Then

$$\int_{C^2} (udd^c v - vdd^c u) \wedge dd^c w = 2\pi \int_{\mathbb{P}^1} (\rho_u - \rho_v)(dd^c \rho_w + \Omega)$$

where $\Omega$ is the standard Kähler form on $\mathbb{P}^1$.

We use this to develop an integral formula for $u, v, w \in L_{C,+}$. Letting

$$\zeta = (\zeta_1, \zeta_2) = F(z) = F(z_1, z_2) = (z_1^a, z_2^b),$$

we recall that for $u \in L_C$, we have

(6.1) \hspace{1cm} \tilde{u}(z) := u(F(z_1, z_2)) = u(\zeta) \in abL,

(6.2) \hspace{1cm} \rho_u(\zeta) = \rho_u(F(z_1, z_2)) = ab \rho_{\tilde{u}/ab}(z)

where $\rho_{\tilde{u}/ab}$ is the standard Robin function of $\tilde{u}/ab \in L$. It follows from the calculations in Remark 4.9 that if $u \in L_{C,+}$ then $\tilde{u} \in abL^+$. From (6.1), if $u, v, w \in L_{C,+}$,

$$ab \int_{C^2} (udd^c v - vdd^c u) \wedge dd^c w = \int_{C^2} (\tilde{u}dd^c \tilde{v} - \tilde{v}dd^c \tilde{u}) \wedge dd^c \tilde{w}.$$ 

We apply Theorem 6.1 to the right-hand side, multiplying by factors of $ab$ since $\tilde{u}, \tilde{v}, \tilde{w} \in abL^+$, to obtain the desired integral formula:

(6.3) \hspace{1cm} \int_{C^2} (udd^c v - vdd^c u) \wedge dd^c w

$$= 2\pi (ab)^2 \int_{\mathbb{P}^1} (\rho_{\tilde{u}/ab} - \rho_{\tilde{v}/ab})(dd^c \rho_{\tilde{w}/ab} + \Omega).$$
Corollary 6.2. Let \( u, v \in L_{C,+} \) with \( u \geq v \). Then
\[
\int_{\mathbb{C}^2} u(ddc v)^2 \leq \int_{\mathbb{C}^2} v(ddc u)^2 + 2\pi(ab)^2 \int_{\mathbb{P}^1} (\rho\hat{\mu}/ab - \rho\hat{\nu}/ab)[(ddc \rho\hat{\mu}/ab + \Omega) + (ddc \rho\hat{\nu}/ab + \Omega)].
\]

Proof. From (6.3) we have
\[
\int_{\mathbb{C}^2} (uddc v - vddc u) \wedge ddc(u + v) = 2\pi(ab)^2 \int_{\mathbb{P}^1} (\rho\hat{\mu}/ab - \rho\hat{\nu}/ab)[(ddc \rho\hat{\mu}/ab + \Omega) + (ddc \rho\hat{\nu}/ab + \Omega)].
\]

We observe that
\[
u(ddc v)^2 - v(ddc u)^2 = (uddc v - vddc u) \wedge ddc(u + v) + (v - u)ddc u \wedge ddc v.
\]

Using this and the hypothesis \( u \geq v \) gives the result. \( \square \)

We also obtain a generalization of [3, Theorem 6.9]:

Corollary 6.3. Let \( E, F \) be nonpluripolar compact subsets of \( \mathbb{C}^2 \) with \( E \subset F \). We have \( \rho_{C,E} = \rho_{C,F} \) if and only if \( V_{C,E}^* = V_{C,F}^* \) and \( \hat{E} = \hat{F} \setminus P \) where \( P \) is pluripolar.

Proof. The “if” direction is obvious. For “only if” we may assume \( E = \hat{E} \) and \( F = \hat{F} \) since \( V_{C,K}^* = V_{C,\tilde{K}}^* \) for \( K \) compact. It suffices to show \( V_{C,E}^* \leq V_{C,F}^* \) as \( E \subset F \) gives the reverse inequality. We have
\[
0 \leq \int_{\mathbb{C}^2} V_{C,E}^*(ddc V_{C,F}^*)^2 = \int_{\mathbb{C}^2} [V_{C,E}^*(ddc V_{C,F}^*)^2 - V_{C,F}^*(ddc V_{C,E}^*)^2]
\]
since \( V_{C,F}^* = 0 \) q.e. on \( F \) (and hence a.e.-\((ddc V_{C,F}^*)^2\)). Applying Corollary 6.2 with \( u = V_{C,E}^* \) and \( v = V_{C,F}^* \), the right-hand side of the displayed inequality is nonpositive since \( \rho_{C,E} = \rho_{C,F} \) implies \( \rho\hat{V}_{C,E}/ab = \rho\hat{V}_{C,F}/ab \) on \( \mathbb{C}^2 \) by (6.2) so that \( \rho\hat{V}_{C,E}/ab = \rho\hat{V}_{C,F}/ab \) on \( \mathbb{P}^1 \). Hence \( \int_{\mathbb{C}^2} V_{C,E}^*(ddc V_{C,F}^*)^2 = 0 \). We conclude that \( V_{C,E}^* \leq V_{C,F}^* \) a.e.-\((ddc V_{C,F}^*)^2\). By Proposition 2.2, \( V_{C,E}^* \leq V_{C,F}^* \). Then \( \hat{E} = \hat{F} \setminus P \) follows since \( \{ z \in \mathbb{C}^2 : V_{C,E}^* = 0 \} \) differs from \( E = \hat{E} \) by a pluripolar set. \( \square \)

Again using Corollary 6.2 and Proposition 2.2, we get an analogue of [4, Lemma 2.1], which is the key result for all that follows.

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Corollary 6.4. Let $K \subset \mathbb{C}^2$ be compact and nonpluripolar and let $v \in L_C$ with $v \leq 0$ on $K$. Suppose that $\rho_v = \rho_{\mathbb{C},K}$ on $\partial \mathbb{P}^2$. Then $v = V_{\mathbb{C},K}^*$ on $\mathbb{C}^2 \setminus \hat{K}$.

Proof. Fix a constant $c$ so that $H_C(z) < c$ on $K$ and let

$$w := \max[v, 0, H_C - c].$$

Then $w \in L_{\mathbb{C},+}$ with $w = 0$ on $\hat{K}$ and since $H_C - c \leq V_{\mathbb{C},K}$ we have $\rho_w = \rho_{\mathbb{C},K}$ on $\partial \mathbb{P}^2$. Then $\rho_w = \rho_{\mathbb{C},K}$ on $\mathbb{C}^2$ (see Remark 4.4) and by (6.2), $\rho_{\hat{w}/ab} = \rho_{\hat{V}_{\mathbb{C},K}/ab}$ on $\mathbb{C}^2$. Thus $\rho_{\hat{w}/ab} = \rho_{\hat{V}_{\mathbb{C},K}/ab}$ on $\mathbb{P}^1$. Since $w \leq V_{\mathbb{C},K}^*$, by Corollary 6.2,

$$\int_{\mathbb{C}^2} V_{\mathbb{C},K}^*(dd^c w)^2 \leq \int_{\mathbb{C}^2} w(dd^c V_{\mathbb{C},K}^*)^2 = 0,$$

the last equality due to $w = 0$ on supp$(dd^c V_{\mathbb{C},K}^*)^2$. Thus $V_{\mathbb{C},K}^* = 0$ a.e.-$(dd^c w)^2$ and hence $V_{\mathbb{C},K}^* \leq w$ a.e.-$(dd^c w)^2$. By Proposition 2.2, $V_{\mathbb{C},K}^* \leq w$ on all of $\mathbb{C}^2$. Since $V_{\mathbb{C},K}^* \geq H_C - c$, $v = V_{\mathbb{C},K}^*$ on $\mathbb{C}^2 \setminus \hat{K}$. □

As in [4, Theorem 2.1], we get a sufficient condition for a sequence of polynomials to recover the $C$-extremal function of $K$ outside of $\hat{K}$. This will be used in Section 8.

Theorem 6.5. Let $K \subset \mathbb{C}^2$ be compact and nonpluripolar. Let $\{p_j\}$ be a sequence of polynomials, $p_j \in \text{Poly}(d_j C)$, with $\deg_C(p_j) = d_j$ such that

$$\limsup_{j \to \infty} \|p_j\|_{1/K}^{1/d_j} = 1$$

and

$$\left(\limsup_{j \to \infty} \frac{1}{d_j} \log |\hat{p}_j(\zeta)|\right)^* = \rho_{\mathbb{C},K}(\zeta), \quad \zeta \in \partial \mathbb{P}^2. \quad (6.4)$$

Then

$$\left(\limsup_{j \to \infty} \frac{1}{d_j} \log |p_j(z)|\right)^* = V_{\mathbb{C},K}^*(z), \quad z \in \mathbb{C}^2 \setminus \hat{K}.$$

Remark 6.6. Given an orthonormal basis $\{b_j\}$ of $\bigcup_n \text{Poly}(nC)$ in $L^2(\mu)$ where $\mu$ is a Bernstein–Markov measure for $K$, using

$$\limsup_{j \to \infty} \frac{1}{\deg_C(b_j)} \log |b_j(z)| = V_{\mathbb{C},K}(z), \quad z \notin \hat{K},$$

from Corollary 3.2, in the proof of Proposition 5.1 we showed

$$\limsup_{j \to \infty} \frac{1}{\deg_C(b_j)} \log |\hat{b}_j(\zeta)| = \rho_{\mathbb{C},K}(\zeta) \quad \text{for} \quad \zeta \in \partial \mathbb{P}^2.$$
Theorem 6.5 is a type of reverse implication.

**Proof.** The function

\[ v(z) := \left( \limsup_{j \to \infty} \frac{1}{d_j} \log |p_j(z)| \right)^* \]

is psh in \( \mathbb{C}^2 \). Given \( \varepsilon > 0 \),

\[ \frac{1}{d_j} \log |p_j(z)| \leq \varepsilon, \quad z \in K, \quad j \geq j_0(\varepsilon). \]

Thus

\[ \frac{1}{d_j} \log |p_j(z)| \leq V_{C,K}(z) + \varepsilon, \quad z \in \mathbb{C}^2, \quad j \geq j_0(\varepsilon). \]

We conclude that \( v \in L_C \) and \( v \leq V_{C,K} \). Hence \( \rho_v \leq \rho_{C,K} \).

From Corollary 6.4, to show \( v = V_{C,K}^* \) outside \( \hat{K} \) it suffices to show \( \rho_v \geq \rho_{C,K} \) on \( \partial P^2 \). We use the argument from Proposition 5.1. Recall from (5.3) for \( u \in abL(\mathbb{C}) \) we have

\[ \limsup_{|t| \to \infty} \left[ u(t) - ab \log |t| \right] = \inf_{r} \left( \max_{|t|=r} u(t) - ab \log r \right). \]

Fix \( \zeta \in \partial P^2 \) and apply the above to the function

\[ \lambda \to \frac{1}{d_j} \log |\hat{p}_j(\lambda \circ \zeta)| = \frac{1}{d_j} \log |p_j(\lambda^a \zeta_1, \lambda^b \zeta_2)|. \]

We see that for any \( r \)

\[ \frac{1}{d_j} \log |\hat{p}_j(\zeta)| = \limsup_{|\lambda| \to \infty} \left[ \frac{1}{d_j} \log |p_j(\lambda \circ \zeta)| - ab \log |\lambda| \right] \leq \max_{|\lambda|=r} \frac{1}{d_j} \log |p_j(\lambda \circ \zeta)| - ab \log r. \]

Thus

\[ \limsup_{j \to \infty} \frac{1}{d_j} \log |\hat{p}_j(\zeta)| \leq \limsup_{j \to \infty} \left( \max_{|\lambda|=r} \frac{1}{d_j} \log |p_j(\lambda \circ \zeta)| - ab \log r \right) \leq \max_{|\lambda|=r} \left( \limsup_{j \to \infty} \frac{1}{d_j} \log |p_j(\lambda \circ \zeta)| - ab \log r \right) = \max_{|\lambda|=r} \left[ v(\lambda \circ \zeta) - ab \log r \right] \]

where we used Hartogs lemma. Thus, letting \( r \to \infty \),

\[ \limsup_{j \to \infty} \frac{1}{d_j} \log |\hat{p}_j(\zeta)| \leq \rho_v(\zeta). \]
Since \( \rho_v \) is usc, \( \left( \limsup_{j \to \infty} \frac{1}{d_j} \log |\hat{p}_j(\zeta)| \right)^* \leq \rho_v(\zeta) \) and using the hypothesis (6.4) finishes the proof. \( \square \)

7. \( C \)-transfinite diameter and directional Chebyshev constants

From [13], we have a Zaharjuta-type proof of the existence of the limit

\[
\delta_C(K) := \limsup_{n \to \infty} V_n^{1/l_n}
\]

(see Section 2) in the definition of \( C \)-transfinite diameter \( \delta_C(K) \) of a compact set \( K \subset \mathbb{C}^d \) where \( C \) satisfies (1.2). In the classical \( (C = \Sigma) \) case, Zaharjuta [16] verified the existence of the limit in (7.1) by introducing directional Chebyshev constants \( \tau(K, \theta) \) and proving

\[
\delta_{\Sigma}(K) = \exp \left( \frac{1}{|\sigma|} \int_{\sigma^0} \log \tau(K, \theta) d|\sigma|(\theta) \right)
\]

where

\[
\sigma := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1, \sum_{i=1}^d x_i = 1 \right\}
\]

is the extreme “face” of \( \Sigma \);

\[
\sigma^0 = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : 0 < x_i < 1, \sum_{i=1}^d x_i = 1 \right\};
\]

and \( |\sigma| \) is the \((d - 1)\)-dimensional measure of \( \sigma \).

In [13], a slight difference with the classical setting is that we have

\[
\delta_C(K) = \left[ \exp \left( \frac{1}{\text{vol}(C)} \int_{C^o} \log \tau(K, \theta) \, dm(\theta) \right) \right]^{1/A}
\]

where the directional Chebyshev constants \( \tau(K, \theta) \) and the integration in the formula are over the interior \( C^o \) of the entire \( d \)-dimensional convex body \( C \) and \( A = A(C, d) \) is a positive constant depending only on \( C \) and \( d \). Moreover in the definition of \( \tau(K, \theta) \) the standard grevlex ordering \( \prec \) on \((\mathbb{Z}^+)^d \) (i.e., on the monomials in \( \mathbb{C}^d \)) was used. This was required to obtain the submultiplicativity of the “monic” polynomial classes

\[
M_k(\alpha) := \left\{ p \in \text{Poly}(kC) : p(z) = z^\alpha + \sum_{\beta \in kC \cap (\mathbb{Z}^+)^d, \ \beta \prec \alpha} c_\beta z^\beta \right\}
\]
for $\alpha \in kC \cap (\mathbb{Z}^+)^d$ and corresponding Chebyshev constants

$$T_k(K, \alpha) := \inf\{\|p\|_K : p \in M_k(\alpha)\}^{1/k}.$$ 

Then for $\theta \in C^\circ$, we have existence of the limit

$$\tau(K, \theta) := \lim_{k \to \infty, \alpha/k \to \theta} T_k(K, \alpha).$$

In our triangle setting, following [13], the grevlex ordering $\prec$ on $(\mathbb{Z}^+)^2$ is defined by $\alpha = (\alpha_1, \alpha_2) \prec \beta = (\beta_1, \beta_2)$ if

(1) $|\alpha| := \alpha_1 + \alpha_2 < |\beta| := \beta_1 + \beta_2$ or

(2) when $|\alpha| = |\beta|$ we have $\alpha_2 < \beta_2$.

Then

(1) one has submultiplicativity of the corresponding “monic” polynomial classes defined as in (7.3) and one gets the formula (7.2) with $\theta \to \tau(K, \theta)$ continuous; and

(2) if $\phi = (\phi_1, \phi_2)$ is on the open hypotenuse $C$ of $C$; i.e., $a\phi_1 + b\phi_2 = ab$ with $\phi_1\phi_2 > 0$, we can apriori define

$$\tau(K, \phi) := \limsup_{k \to \infty, \alpha/k \to \phi} T_k(K, \alpha)$$

and verify in this setting that the limit still exists. If $\phi = r\theta$ where $\theta$ lies on the interior of $C$ and $r > 1$, then $r \log \tau(K, \theta) = \log \tau(K, \phi)$ (cf., the proof of [12, Lemma 5.4]). (Note $C = \sigma^0$ if $C = \Sigma \subset (\mathbb{R}^+)^2$).

As a consequence, in our triangle case $C = \text{co}\{(0, 0), (b, 0), (0, a)\}$,

$$\delta_C(K) = \exp\left(\frac{1}{\sqrt{a^2 + b^2}} \int_C \log \tau(K, \theta) d|\sigma|(\theta)\right)$$

where the directional Chebyshev constants $\tau(K, \theta)$ in (7.4) and the integration in the formula are over $C$; and $\theta \to \tau(K, \theta)$ is continuous on $C$. In what follows, we fix our triangle $C$ and use this $\prec$ ordering to define these directional Chebyshev constants

$$\tau(K, \theta) := \lim_{k \to \infty, \alpha/k \to \theta} T_k(K, \alpha) \quad \text{for } \theta \in C.$$

Using (7.4) we have a result along the lines of [5, Proposition 3.1]:

**Proposition 7.1.** For $E \subset F$ compact subsets of $\mathbb{C}^2$,

(1) for all $\theta \in C$, $\tau(E, \theta) \leq \tau(F, \theta)$ and

(2) if $\tau(E, \theta) = \tau(F, \theta)$ for all $\theta \in C$ then $\delta_C(E) = \delta_C(F)$.
Remark 7.2. The converse of (2) in Proposition 7.1 is true in this situation, and it is this converse appearing in the version of [5, Proposition 3.1], but we won’t need this.

However, in our triangle setting, following [13] we can also define an ordering $\prec_C$ on $(\mathbb{Z}^+)^2$ which respects $\deg_C$ by

$$\alpha = (\alpha_1, \alpha_2) \prec_C \beta = (\beta_1, \beta_2) \text{ if}$$

1. $\deg_C(z^\alpha) < \deg_C(z^\beta)$ or
2. when $\deg_C(z^\alpha) = \deg_C(z^\beta)$ we have $\alpha < \beta$.

Defining

$$M^C_k(\alpha) := \left\{ p \in \text{Poly}(kC) : p(z) = z^\alpha + \sum_{\beta \in kC \cap (\mathbb{Z}^+)^d, \beta \prec_C \alpha} c_\beta z^\beta \right\}$$

for $\alpha \in kC \cap (\mathbb{Z}^+)^d$ and corresponding Chebyshev constants

$$T^C_k(K, \alpha) := \inf \{ \|p\|_K : p \in M^C_k(\alpha) \}^{1/k},$$

for $\theta \in C^\circ$ we define

$$\tau_C(K, \theta) := \limsup_{k \to \infty, \alpha / k \to \theta} T_k(K, \alpha).$$

In [13] it is proved that the limit exists for $\theta \in C^\circ$ and that formula (7.2) is still valid with $\tau(K, \theta)$ replaced by $\tau_C(K, \theta)$. Extending the definition of $\tau_C(K, \theta)$ to $\theta \in C$, as with $\tau(K, \theta)$ we have

$$r \log \tau_C(K, \theta) = \log \tau_C(K, \phi)$$

if $\phi = r\theta$ where $\theta$ lies on the interior of $C$ and $r > 1$ and hence formula (7.4) also holds with $\tau(K, \theta)$ replaced by $\tau_C(K, \theta)$. Thus Proposition 7.1 holds with $\tau(K, \theta)$ replaced by $\tau_C(K, \theta)$. This version of Proposition 7.1 will be used in the next section.

8. Polynomials approximating $V_{C,K}$

Following [5], given a nonpluripolar compact set $K \subset \mathbb{C}^2$ and $\theta \in C$, a sequence of polynomials $\{Q_n\}$ is $\theta$-asymptotically Chebyshev (we write $\theta aT$) for $K$ if

1. for each $n$ there exists $k_n \in \mathbb{Z}^+$ and $\alpha_n$ with $Q_n \in M^\prec_C(k_n \alpha_n)$;
2. $\lim_{n \to \infty} k_n = +\infty$ and $\lim_{n \to \infty} \frac{\alpha_n}{k_n} = \theta$; and
3. $\lim_{n \to \infty} \|Q_n\|_K^{1/k_n} = \tau_C(K, \theta).$

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Proposition 8.1. Let $K \subset \mathbb{C}^2$ be compact and nonpluripolar and satisfy $e^{i\theta} \circ K = K$. Let $\{Q_n\}$ be $\theta aT$ for $K$. Then $\{\hat{Q}_n\}$ is $\theta aT$ for $K$.

Proof. This follows from (4.9) giving $\|\hat{Q}_n\|_K \leq \|Q_n\|_K$ for such $K$. Here it is important that we use the $\prec_C$ order. □

Given $K \subset \mathbb{C}^2$ compact and nonpluripolar, define

$$K_\rho := \{ z \in \mathbb{C}^2 : \rho_{C,K}(z) \leq 0 \}.$$  

Note that if $e^{i\theta} \circ K = K$ then Theorem 4.6 shows that $K = K_\rho$. Moreover, from Remark 4.4,

$$\rho_{C,K}(e^{i\theta} \circ z) = \rho_{C,K}(z)$$  

so that $e^{i\theta} \circ K_\rho = K_\rho$ and $V^*_{C,K_\rho} = \rho_{C,K}^+ := \max[0, \rho_{C,K}]$.

Theorem 8.2. Let $K \subset \mathbb{C}^2$ be compact and regular and let $\{Q_n\}$ be $\theta aT$ for $K$ with $Q_n \in M^\prec_C(\alpha_n)$. Then $\{\hat{Q}_n\}$ is $\theta aT$ for $K_\rho$. Conversely, if $\{H_n\}$ is $\theta aT$ for $K_\rho$ with $H_n \in H_{k_n}(C) \cap M^\prec_C(\alpha_n)$, then the sequence $\{\text{Tch}_K H_n\}$ is $\theta aT$ for $K$. Moreover,

$$\tau_C(K_\rho, \theta) = \tau_C(K, \theta) \quad \text{for all } \theta \in \mathbb{C}.$$  

Proof. Given $\{Q_n\}$ which are $\theta aT$ for $K$ with $Q_n \in M^\prec_C(\alpha_n)$, we have

$$\frac{1}{k_n} \log \frac{|Q_n(z)|}{\|Q_n\|_K} \leq V_{C,K}(z), \quad z \in \mathbb{C}^2.$$  

Thus

$$\frac{1}{k_n} \log \frac{\hat{Q}_n(z)}{\|Q_n\|_K} \leq \rho_{C,K}(z), \quad z \in \mathbb{C}^2.$$  

Hence for $z \in K_\rho$,

$$\frac{1}{k_n} \log |\hat{Q}_n(z)| \leq \frac{1}{k_n} \log \|Q_n\|_K; \quad \text{i.e., } \|\hat{Q}_n\|_{K_\rho} \leq \|Q_n\|_K.$$  

Since

$$\lim_{n \to \infty} \|Q_n\|_K^{1/k_n} = \tau_C(K, \theta),$$  

(8.1)  

$$\tau_C(K_\rho, \theta) \leq \limsup_{n \to \infty} \|\hat{Q}_n\|_{K_\rho}^{1/k_n} \leq \tau_C(K, \theta).$$  

On the other hand, considering $\{H_n\}$ which are $\theta aT$ for $K_\rho$ (we can assume $H_n \in H_{k_n}(C)$ from Proposition 8.1 and our use of the $\prec_C$ order) we have

$$\lim_{n \to \infty} \|H_n\|_{K_\rho}^{1/k_n} = \tau_C(K_\rho, \theta).$$  

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Thus
\[ \limsup_{n \to \infty} \frac{1}{k_n} \log |H_n(z)| = \limsup_{n \to \infty} \frac{1}{k_n} \log |H_n(z)| + \log \tau_C(K_\rho, \theta) \]
\[ \leq \log \tau_C(K_\rho, \theta) + \rho_{C,K}^+(z), \quad z \in \mathbb{C}^2. \]

By rescaling $K$; i.e., replacing $K$ by $rK$ for appropriate $r \geq 1$ if need be, we can assume that
\[ K_\rho \subset \{(z_1, z_2) : |z_1|, |z_2| \leq 1\}. \]

In particular, $\rho_{C,K} \geq 0$ on $\partial P^2$. From Theorem 5.3 we conclude that
\[ \limsup_{n \to \infty} \frac{1}{k_n} \log \| Tch_K H_n \|_K \leq \log \tau_C(K_\rho, \theta). \]

We have $Tch_K H_n \in M_{k_n}^*(\alpha_n)$ since $H_n \in H_{k_n}(C) \cap M_{k_n}^\prec(\alpha_n)$ and we are using the $\prec_C$ order, hence
\[ \tau_C(K, \theta) \leq \limsup_{n \to \infty} \| Tch_K H_n \|_{K}^{1/k_n} \leq \tau_C(K_\rho, \theta). \]

Together with (8.1) we conclude that $\tau_C(K_\rho, \theta) = \tau_C(K, \theta)$ and the inequalities in (8.1) and (8.2) are equalities. □

Finally, we utilize Theorems 8.2 and 6.5 together with Proposition 2.3 to prove our main result.

**Theorem 8.3.** Let $K \subset \mathbb{C}^2$ be compact and regular. Let $\{p_n\}$ be a countable family of polynomials with $p_n \in \text{Poly}(k_n C)$ such that for every $\theta \in C$, there is a subsequence which is $\theta a T$ for $K$. Then
\[ \left( \limsup_{n \to \infty} \frac{1}{k_n} \log \left| p_n(z) \right| / \| p_n \|_K \right)^* = V_{C,K}(z), \quad z \in \mathbb{C}^2 \setminus \hat{K}. \]

**Proof.** Let
\[ v(z) := \left( \limsup_{n \to \infty} \frac{1}{k_n} \log \frac{|p_n(z)|}{\| p_n \|_K} \right)^*. \]

Clearly $v \leq V_{C,K}$ on all of $\mathbb{C}^2$. Let
\[ w(z) := \left( \limsup_{n \to \infty} \frac{1}{k_n} \log \frac{|\hat{p}_n(z)|}{\| p_n \|_K} \right)^*. \]

To finish the proof, it suffices, by Theorem 6.5, to show that
\[ w(z) = \rho_{C,K}(z), \quad z \in \partial P^2. \]
Clearly \( w \leq \rho_{C,K} \) in \( \mathbb{C}^2 \) since \( v \leq V_{C,K} \). To show the reverse inequality, we proceed as follows. Let
\[
Z := \{ z \in \mathbb{C}^2 : w(z) < 0 \}.
\]
Then \( Z \) is open since \( w \) is usc. We claim that \( \text{int}(K_{\rho}) \subset Z \). For if \( z \in \text{int}(K_{\rho}) \), we have \( \rho_{C,K}(z) = -a < 0 \). Thus \( w(z) \leq -a < 0 \) and \( z \in Z \). Moreover, both sets \( K_{\rho} \) and \( Z \) satisfy the invariance property
\[
e^{i\theta} \circ K_{\rho} = K_{\rho} \quad \text{and} \quad e^{i\theta} \circ Z = Z
\]
(for \( Z \) this follows since \( \hat{p}_n \in H_{k_n}(C) \)). Thus to show the equality \( w(z) = \rho_{C,K}(z) \) it suffices to verify the equality
\[
\text{int}(K_{\rho}) = Z.
\]
Suppose this is false. Then we take a point \( z_0 \in \partial K_{\rho} \cap Z \) and a closed ball \( B \) centered at \( z_0 \) contained in \( Z \). Since \( B \) is regular and \( K \) is assumed regular, by Proposition 4.8 together with [5, Lemma 4.1], \( B \cup K_{\rho} \) is regular.

Given \( \theta \in \mathbb{C} \), by assumption there exists a subsequence \( N_{\theta} \subset N \) such that \( \{p_n \}_{n \in N_{\theta}} \) is \( \theta aT \) for \( K \). From Theorem 8.2, \( \{\hat{p}_n \}_{n \in N_{\theta}} \) is \( \theta aT \) for \( K_{\rho} \). Since \( w \leq 0 \) on \( B \cup K_{\rho} \), for \( z \in B \cup K_{\rho} \) we have
\[
\limsup_{n \in N_{\theta}} \frac{1}{k_n} \log |\hat{p}_n(z)| \leq \limsup_{n \in N_{\theta}} \frac{1}{k_n} \log \|\hat{p}_n\|_K = \log \tau_C(K,\theta).
\]
Using Hartogs lemma, we conclude that
\[
\log \tau_C(B \cup K_{\rho},\theta) \leq \limsup_{n \in N_{\theta}} \frac{1}{k_n} \log \|\hat{p}_n\|_{B \cup K_{\rho}} \leq \log \tau_C(K,\theta).
\]
Hence
\[
\tau_C(B \cup K_{\rho},\theta) \leq \tau_C(K,\theta) = \tau_C(K_{\rho},\theta)
\]
for all \( \theta \in \mathbb{C} \). Since \( \tau_C(B \cup K_{\rho},\theta) \geq \tau_C(K_{\rho},\theta) \) we see that
\[
\tau_C(B \cup K_{\rho},\theta) = \tau_C(K_{\rho},\theta)
\]
for all \( \theta \in \mathbb{C} \). From Proposition 2.3 and the \( \tau_C \) version of Proposition 7.1 (and regularity of the sets \( K_{\rho}, B \cup K_{\rho} \)),
\[
V_{C,B \cup K_{\rho}} = V_{C,K_{\rho}}.
\]
But \( V_{C,K_{\rho}} = \rho_{C,K}^+ = \max[0,\rho_{C,K}] \) thus if \( B \setminus K_{\rho} \neq \emptyset \), since \( \rho_{C,K} > 0 \) on \( \mathbb{C}^2 \setminus K_{\rho} \) and \( V_{C,B \cup K_{\rho}} = 0 \) on \( B \cup K_{\rho} \), this is a contradiction. \( \square \)
As examples of sequences of polynomials satisfying the hypotheses of Theorem 8.3, as in [5] we have

1. the family \( \{ t_{k,\alpha} \in M_k^{<C}(\alpha) \}_{k,\alpha} \) of Chebyshev polynomials (minimal supremum norm) for \( K \) in these classes;

2. for a Bernstein–Markov measure \( \mu \) on \( K \), the corresponding polynomials \( \{ q_{k,\alpha} \in M_k^{<C}(\alpha) \}_{k,\alpha} \) of minimal \( L^2(\mu) \) norm (see Corollary 3.2);

3. any sequence \( p_{\alpha(s)} = z^{\alpha(s)} - L_{\alpha(s-1)}(z^{\alpha(s)}) \) where \( \{ z^{\alpha(s)} \} \) is an enumeration of monomials with the \( <C \) order and \( L_{\alpha(s-1)}(z^{\alpha(s)}) \) is the Lagrange interpolating polynomial for the monomial \( z^{\alpha(s)} \) at points \( \{ z_{s-1,j} \}_{j=1,...,s-1} \) in the \( (s-1) \)-st row in a triangular array \( \{ z_{jk} \}_{j=1,2,...; k=1,...,j} \subseteq K \) where the Lebesgue constants \( \Lambda_{\alpha(s)} \) associated to the array grow subexponentially. Here

\[
\Lambda_{\alpha(s)} := \max_{z \in K} \sum_{j=1}^{s} |l_{sj}(z)|,
\]

where \( l_{sj} \in \text{Poly}(\deg_C(z^{\alpha(s)})C) \) satisfies \( l_{sj}(z_{sk}) = \delta_{jk} \), \( j, k = 1, \ldots, s \) and we require

\[
\lim_{s \to \infty} \Lambda_{\alpha(s)}^{1/\deg_C(z^{\alpha(s)})} = 1.
\]

We refer to [5, Corollary 4.4] for details.

Remark 8.4. Example (3) includes the case of a sequence of \( C \)-Fekete polynomials for \( K \) (cf. [5, p. 1562]). The case of \( C \)-Leja polynomials for \( K \), defined using \( C \)-Leja points as in [13], also satisfy the hypotheses of Theorem 8.3. This can be seen by following the proof of [5, Corollary 4.5]. The proof that \( C \)-Leja polynomials satisfy the analogue of [5, (4.28)] is given in [13, Theorem 5.1].

9. Further directions

We reiterate that the arguments given in the note for triangles \( C \) in \( \mathbb{R}^2 \) with vertices \( (0,0), (b,0), (0,a) \) where \( a, b \) are relatively prime positive integers should generalize to the case of a simplex

\[
C = \text{co}\{(0,\ldots,0),(a_1,0,\ldots,0),\ldots,(0,\ldots,0,a_d)\}
\]

in \( \mathbb{R}^d \) with \( a_1, \ldots, a_d \) pairwise relatively prime using the definition of the \( C \)-Robin function in Remark 4.5. Indeed, following the arguments on [6, pp. 72–82] one should also be able to prove \textit{weighted} versions of the \( C \)-Robin results for such simplices \( C \) in \( \mathbb{R}^d \). We indicate the transition from
the $C$-weighted situation for $d = 2$ to a $\tilde{C}$-homogeneous unweighted situation for $d = 3$. As in Section 4, we lift the circle action on $\mathbb{C}^2$,

$$\lambda \circ (z_1, z_2) := (\lambda^a z_1, \lambda^b z_2),$$

which is a circle action on $\mathbb{C}^3$ via

$$\lambda \circ (t, z_1, z_2) := (\lambda t, \lambda^a z_1, \lambda^b z_2).$$

Given a compact set $K \subset \mathbb{C}^2$ and an admissible weight function $w \geq 0$ on $K$, i.e., $w$ is usc and $\{ z \in K : w(z) > 0 \}$ is not pluripolar, we associate the set

$$\tilde{K}_w := \{(t \circ (1, z_1, z_2) : (z_1, z_2) \in K, |t| = w(z_1, z_2)\}.$$

It follows readily that

$$e^{i\theta} \circ \tilde{K}_w = \tilde{K}_w.$$

Setting $\tilde{C} = \text{co}\{(0, 0, 0), (1, 0, 0), (0, b, 0), (0, 0, a)\}$, we can relate a weighted $C$-Robin function $\rho_{C,K}^w$ to the $\tilde{C}$-Robin function $\rho_{\tilde{C},\tilde{K}_w}$. Using these weighted ideas, the converse to Proposition 2.3 should follow as in [6].

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