MODELING TECHNICAL ANALYSIS

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Abstract. We model the behaviour of a stock price process under the assumption of the existence of a support/resistance level, which is one of the most popular approaches in the field of technical analysis. We obtain optimal dynamic trading strategies under the setup.

1. Introduction

Many traders base their trading strategies on technical analysis (TA). The analysis uses heavily the visual shape of historical price graphs (which traders call 'charts') to determine whether the asset is a good buy or not. One of the basic analyses in the field is that of a support and resistance line. In this method, the traders obtain a horizontal line called a support (resistance) line that they believe is a local support (roof) of the asset price. The analysis is that if the stock price crosses a support line from above and goes lower than the level by 'a lot', then it is considered that the stock has moved into a recession regime in which case traders should sell, or at least, not be long of the stock. On the other hand, if the asset price spikes up crossing a resistance line from below, the asset is considered to have shifted to a boom regime and the method asks the traders to buy the asset or to cover the short.

Remark 1. The method of support/resistance level can be applied to any assets as long as their historical prices are available. In the paper, we focus on the case when the asset is a stock.

We note here that the support/resistance level is not a hard limit. Therefore, the stock can go lower (higher) than the support (resistance) level, but it is expected to correct in a short period of time if the regime has not changed. We also note that there may be several support/resistance levels in one chart.

A level could, in theory, be a support level but not a resistance level and vice versa. However, the level is where the stock-price regime changes and it is natural to consider it to be both a support and resistance level in the following way. From one regime, the other regime is relatively ‘better’ or ‘worse’. Hence, if we are in the ‘better’ regime, the level which lies around the lower end of the regime is considered to be a support line; if we are in the ‘worse’ regime, the same level which now lies around the upper end of the regime is considered to be a resistance line. When a support level becomes a resistance level or vice versa, we say that the stock has a regime transition. This is in line with how traders think of the level.

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Figure 1. Example of (A) the support and (B) the resistance levels. Note that the levels are not hard limits, and the price can fluctuate around the levels.

Figure 2. An example of a line being both the support and resistance level.

Methods in TA are based on historical behaviour of stocks. They are not currently supported by any theory, though they may be partially explained using behavioural science. Nevertheless,
many traders believe they are useful and powerful. One reason is that the methods in TA are free from human emotions. Traders are consistently affected by the present performance of their portfolios and psychological stresses. Even if their trading instinct is sharp, the performance of their portfolios may deteriorate due to other non-trading factors. The decisions that TA makes are believed not to be affected by these factors.

Another reason why many traders support TA is that they believe in the strong form of the efficient market hypothesis (EMH). They believe that the stock price reflects not only the information publicly available, but even the information that is not disclosed in public. For example, if an investor has some insider information that potentially pushes the stock price lower, he might want to sell the stock before other people do to take advantage of possessing the information. He can only extract benefit for himself by selling the stock in the market, which pushes the stock price lower. Even though the information is not publicly available, it is thus reflected in the price chart of the stock.

Remark 2. Instead of selling the stock directly in the market, the investor with insider information can seek other methods of benefiting himself from the expected stock performance. For example, he can buy naked puts on the stock. Then, the counterparty who sold the option to the investor has to sell the stock to hedge the position (unless the counterparty is happy holding it without any hedges). In either case, the investor with insider information will make the market sell the stock.

Some studies on TA have been performed, but they mainly focus on how to detect the sign of the regime transition as quickly as possible and checking by comparing what the performance would have been if a trader adopted TA in his trading strategies. Some examples of research that focus on these points are [2] and [12]. We know of no literature attempting to model and justify TA methods mathematically.

In order to model the method of support/resistance level, we initially considered several approaches. One is to use stochastic delay differential equations (SDDEs; [3], [13], [14], [24]). This makes sense as TA is the method we use to forecast dynamics of the future stock price from analysing historical prices, and SDDEs are stochastic differential equations (SDEs) with coefficients that depend on the historical levels. However, this method requires many parameters and does not imply the optimal trading strategy traders should adopt under the setup.

The other method we considered was using a skew Brownian motion to model the price process. This has different probabilities of positive and negative excursions from the support/resistance level. Skew Brownian motion is the process in which the negative excursions from the origin of the standard Brownian motion are flipped with the probability $1 - \alpha$. It is described in [6]. Using this process to describe the underlying stock price process under our setup requires a lot less parameters than using SDDEs. However, as [17] and [22] show, the model with skew Brownian motion has arbitrage opportunities. It is discussed in [17] that we can get an arbitrage-free and complete market within the class of simple strategies, but not in a more general setup.
Remark 3. We think it is still possible to approach using skew Brownian motion in modeling the method of supply/resistance level by using approximated skew Brownian motion. From [5], skew Brownian motion satisfies the SDE

\begin{equation}
    dX_t = dW_t + (2\alpha - 1)dL_0^X(t),
\end{equation}

where \( L_0^X(\cdot) \) is the local time at zero defined by

\begin{equation}
    L_0^X(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{[-\epsilon,\epsilon]}(X_s) ds.
\end{equation}

We can approximate the process \( X \) by \( Y_\epsilon \) defined as

\begin{equation}
    dY_\epsilon(t) = dW_t + (2\alpha - 1)d\ell_0^{Y_\epsilon}(t),
\end{equation}

with some \( \epsilon > 0 \) with

\begin{equation}
    \ell_0^{Y_\epsilon}(t) = \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{[-\epsilon,\epsilon]}(Y_s) ds.
\end{equation}

From now on we adopt a different model: we assume that there are only two regimes in the stock price which correspond to different log normal diffusion processes. We then define criteria for deciding on buying/selling the stock via optimal control theory.

One of the things that makes our setup special is that these two regimes are not distinguishable based on the current stock price, i.e. there is a region where the stock price can have dynamics corresponding to either of the two SDEs. This feature provides some “room” for the process in each regime to move around the support/resistance level without switching to the other regime.

The rest of the paper is organized as follows: Section 2 presents the setup we use for the model with a support/resistance level. We first solve for the optimal selling problem given that we already hold the stock at time \( t = 0 \) in Section 3. Using the results from the optimal selling problem, we then solve the optimal purchasing problem in Section 4. We derive our conclusions and refer to possible future research topics in Section 5.

2. Setup

We assume that there are levels \( L \) and \( H \) (\( 0 < L < H \)) at which the regimes change. We define the positive region as the domain \([L, \infty)\) and the negative region as the domain \((0, H]\). Note that the two regions have non-empty intersection \([L, H]\).

We assume that there are only two regimes in the price process; the positive regime and the negative regime.
Under the positive regime, the process lies in the positive region and has dynamics

\begin{equation}
    dS_t = \mu_+ S_t dt + \sigma_+ S_t dW_t,
\end{equation}

where \(\mu_+\) and \(\sigma_+ > 0\) are constants and \(W_t\) is a one dimensional Brownian motion. The transition from the positive to the negative regime occurs when the positive regime is in place and \(S\) exits the positive region.

On the other hand, under the negative regime, the process lies in the negative region and has dynamics

\begin{equation}
    dS_t = \mu_- S_t dt + \sigma_- S_t dW_t,
\end{equation}

where \(\mu_-\) and \(\sigma_- > 0\) are constants. The transition from the negative to the positive regime occurs when the negative regime is in place and \(S\) exits the negative region.

Let \(r > 0\) denote the interest rate and we assume

\begin{equation}
    \mu_- < r < \mu_+.
\end{equation}

The condition (2.3) implies the discounted price process is a supermartingale under the negative regime and a submartingale in the positive regime up to the time of the first regime transition.

To keep track of which regime currently holds, we define the flag process \(F_t\) which takes values in \([-1, +1]\) as

\begin{equation}
    F_t = \begin{cases} 
    +1 & \text{if the dynamics correspond to the positive regime} \\
    -1 & \text{if the dynamics correspond to the negative regime}
    \end{cases}
\end{equation}

The flag process \(F_t\) indicates under which regime the price process \(S_t\) is at time \(t\). From the definition of the regime transition, \(F_t\) jumps from one value to the other only in the following cases:

\begin{equation}
    \begin{cases} 
    F_{t^-} = +1 \text{ and } S_t = L, \text{ then } F_t = -1 \\
    F_{t^-} = -1 \text{ and } S_t = H, \text{ then } F_t = +1
    \end{cases}
\end{equation}

We set \(M\) as the level of the asset price at which the trader is happy to take profit. In other words, the asset that the trader held at the price below \(M\) will be sold upon breaching the level \(M\). We therefore assume that the initial price is below \(M\). For each \(a \leq M\), we define the time \(T_a\) as
(2.6) \[ T_a := \inf \{ t | S_t = a \}. \]

We set \( T_M \) as the set of all stopping times that are not greater than \( T_M \). We set \( X_t \) as \( S_t \) stopped at \( T_M \).

3. Selling Problem

First, we assume that we already hold the asset and think of the optimal selling strategy. We find the selling strategy that enables us to sell at the best value among the expectations of all the future prices discounted to today. The problem is mathematically equivalent to solving the following optimal stopping problem:

\[
(3.1) \quad V(x, f) = \sup_{\tau \in T_M} \mathbb{E}_{x,f}[e^{-r\tau} X_{\tau}].
\]

We want to characterize the optimal stopping time \( \tau^* \) that leads to

\[
(3.2) \quad V(x, f) = \mathbb{E}_{x,f}[e^{-r\tau^*} X_{\tau^*}].
\]

In order to find the candidates for \( \tau^* \), we define the continuation region \( C \) and the stopping region \( D \) as

\[
(3.3) \quad C = \{ V(x, f) > x \}, \quad D = \{ V(x, f) = x \}
\]

and set \( \tau_m \) as

\[
(3.4) \quad \tau_m = \inf \{ t | X_t = m; t \leq T_M \}.
\]

We hypothesise that the optimal policy is to sell when \( X \) reaches \( m \) or \( M \) for a suitable value of \( m \) to be determined. We think of the set of \( \tau_m \) as the candidate for the solution to the optimal stopping problem [3.1]. We define \( V_m = \mathbb{E}[e^{-r\tau_m} X_{\tau_m}] \), then \( V_m \) is the solution to the following ODEs:

\[
(3.5) \quad \begin{cases}
\frac{1}{2} \sigma^2 x^2 V_m''(x, +1) + \mu_+ x V_m'(x, +1) - r V_m(x, +1) = 0, & x \in [L, M] \\
\frac{1}{2} \sigma^2 x^2 V_m''(x, -1) + \mu_- x V_m'(x, -1) - r V_m(x, -1) = 0, & x \in [m, H].
\end{cases}
\]

In order to solve the optimal selling problem [3.1], we use the following theorem.
Theorem 3.1. (13)

Consider the optimal stopping problem

\[ V_t^T = \sup_{t \leq \tau \leq T} \mathbb{E} G_\tau \]

upon assuming that the condition \( \mathbb{E}(\sup_{0 \leq t \leq T}|G_t|) < \infty \) holds. Furthermore, consider the process

\[ S_t = \text{ess sup}_{\tau \geq t} \mathbb{E}(G_\tau | F_t) \]

and the stopping time

\[ \tau_t = \inf \{ s \geq t | S_s = G_s \}. \]

Then for all \( t \geq 0 \) we have:

\[ S_t \geq \mathbb{E}(G_\tau | F_t) \quad \text{for each } \tau \in \mathcal{M}_t, \]

where \( \mathcal{M}_t \) denotes the family of all stopping times \( \tau \) satisfying \( \tau \geq t \). Moreover, if \( t \geq 0 \) is given and fixed, then we have:

- The stopping time \( \tau_t \) is optimal in (3.6).
- If \( \tau^* \) is optimal stopping time in (3.6), then \( \tau_t \leq \tau^* \) \( \mathbb{P} \)-a.s.
- The process \( (S_t)_{s \geq t} \) is the smallest right-continuous supermartingale which dominates \( (G_s)_{s \geq t} \).
- The stopped process \( (S_s\wedge \tau_t)_{s \geq t} \) is a right-continuous martingale.

The process \( S_t \) is called the Snell envelope of the process \( G_t \).

The plan of solving the optimal selling problem is as follows:

1. Find the maximizer \( \hat{m} \) of \( V_m(x, f) \).
2. Show that the process \( e^{-rt}V_{\hat{m}}(X_t, F_t) \) is a supermartingale that dominates the gains process \( e^{-rt}X_t \).

The fact that \( e^{-rt}V_{\hat{m}}(X_t, F_t) \) is minimal comes from the Optional Sampling Theorem. Then Theorem 3.1 proves that the optimal stopping time of the problem (3.1) is the first time when \( e^{-rt}V_{\hat{m}}(X_t, F_t) = e^{-rt}X_t \), hence when \( V_{\hat{m}}(X_t, F_t) = X_t \). In other words, the optimal stopping time is the first time the process \( X_t \) enters the domain \( D \). Finally, this stopping time is equivalent to the first time \( X_t \) hits the level \( \hat{m} \) when \( F_{t-} = -1 \) and when the process hits \( M \) when \( F_{t-} = +1 \).
We let \( \alpha_1 < \alpha_2 \) be the two solutions to the characteristic equation

\[
(3.10) \quad \frac{1}{2} \sigma_+ \alpha^2 + \mu_+ \alpha - r = 0
\]

and \( \beta_1 < \beta_2 \) the solutions to

\[
(3.11) \quad \frac{1}{2} \sigma_- \beta^2 + \mu_- \beta - r = 0.
\]

What we know right away from the equations (2.3), (3.10), and (3.11) is that

\[
(3.12) \quad \alpha_1, \beta_1 < 0
\]

and

\[
(3.13) \quad \alpha_2 < 1 < \beta_2.
\]

With some constants \( A, B, C, \) and \( D \) which are determined from boundary conditions, \( V_m(x, +1) \) and \( V_m(x, -1) \) can be written as

\[
(3.14) \quad \begin{cases}
V_m(x, +1) = Ax^{\alpha_1} + Bx^{\alpha_2} \\
V_m(x, -1) = Cx^{\beta_1} + Dx^{\beta_2}
\end{cases}
\]

We now solve for \( A, B, C, \) and \( D \) in (3.14) in the following two cases: the case when \( m \leq L \) and when \( m > L \).

3.1. The Case Where \( m \leq L \). In the case when \( m \leq L \), \( V_m(x, f) \) satisfies (3.5) with the boundary conditions

\[
(3.15) \quad \begin{cases}
V_m(m, -1) = m \leq L \\
V_m(L, -1) = V_m(L, +1) \\
V_m(H, +1) = V_m(H, -1) \\
V_m(M, +1) = M
\end{cases}
\]

and set \( V_m(x, -1) = x \) for \( x \in (0, m) \).

Define \( P(x) = x^{\alpha_2} - M^{\alpha_2-\alpha_1}x^{\alpha_1} \), \( Q(x) = x^{\beta_2} - m^{\beta_2-\beta_1}x^{\beta_1} \), and \( R(x) = m^{1-\beta_1}x^{\beta_1} - M^{1-\alpha_1}x^{\alpha_1} \).

We solve (3.15) for \( A, B, C, \) and \( D \) in (3.14) and obtain:
\[ A = M^{1-\alpha_1} - BM^{\alpha_2-\alpha_1} \]
\[ B = \frac{R(L)Q(H) - R(H)Q(L)}{P(L)Q(H) - P(H)Q(L)} \]
\[ C = m^{1-\beta_1} - Dm^{\beta_2-\beta_1} \]
\[ D = \frac{R(L)P(H) - R(H)P(L)}{Q(H)P(L) - Q(L)P(H)} \]

(3.16)

3.2. \( m \geq L \) Case. In the case when \( m \geq L \), \( V_m(x, f) \) solves (3.5) with the boundary conditions

\[ \begin{align*}
V_m(m, -1) &= m \geq L \\
V_m(H, -1) &= V_m(H, +1) \\
V_m(L, +1) &= L \\
V_m(M, +1) &= M
\end{align*} \]

(3.17)

We don’t consider \( m > H \) here as then the problem will be an optimal stopping problem under one regime (i.e. the positive regime).

\( V_m(L, -1) = V_m(L, +1) \) condition is replaced with \( V_m(L, +1) = L \) since the process is stopped when it goes below the level \( m \). We solve for the coefficients in (3.14) and obtain

\[ \begin{align*}
A &= \frac{L^{1-\alpha_1}M^{1-\alpha_1}(M^{\alpha_2-1} - L^{\alpha_2-1})}{M^{\alpha_2-1} - L^{\alpha_2-1}} \\
B &= \frac{M^{1-\alpha_2} - L^{1-\alpha_2}}{M^{\alpha_2-1} - L^{\alpha_2-1}} \\
C &= \frac{m^{1-\beta_1} - Dm^{\beta_2-\beta_1}}{H^{\beta_2-\beta_1} - m^{\beta_2-\beta_1}} \\
D &= \frac{H^{1-\beta_1} - m^{1-\beta_1}}{H^{\beta_2-\beta_1} - m^{\beta_2-\beta_1}}
\end{align*} \]

(3.18)

Note that \( V_m(x, +1) \) does not depend on \( m \).

3.3. Solving the Optimal Stopping Problem. In Subsection 3.1 and Subsection 3.2, we solved for \( V_m(x, f) \). We now have the candidates for the solution of the optimal stopping problem (3.1) with the stopping rule ”stop the asset price process \( X_t \) when it first hits the level \( m \) or \( M \) if this is earlier (since we are required to sell at level \( M \)”). In other words, we have candidates of \( m \) that satisfy

\[ \tau^* = \tau_m. \]

(3.19)

What we want to do next is to verify which choice of \( m \) actually works and enables us to solve the optimal stopping problem (3.1) with (3.19). If we guessed the right form of the optimal policy, the optimal \( m \) should be the one that maximizes \( V_m(x, f) \). We now solve for the maximizer of \( V_m(x, f) \).
3.4. Maximizer in the Case of $m \leq L$. We calculate the maximizer in the case when $m \leq L$.

For that, it is sufficient to maximize $V_m(L, -1)$.

Defining

\begin{equation}
\begin{aligned}
G_1 &= (L^\beta H^\alpha - L^\alpha H^\beta) + M^{\alpha_2-\alpha_1}(L^\alpha H^\beta - L^\beta H^\alpha) \\
G_2 &= M^{1-\alpha_1}(L^\alpha H^\beta - L^\beta H^\alpha) < 0 \\
G_3 &= (H^\beta_2 L^\alpha - L^\beta_2 H^\alpha) + M^{\alpha_2-\alpha_1}(H^\alpha L^\beta - L^\alpha H^\beta) \\
G_4 &= (L^\alpha - M^{\alpha_2-\alpha_1}L^\alpha_2)(L^\beta H^\beta_2 - L^\beta_2 H^\beta) < 0 \\
G_5 &= M^{1-\alpha_1}L^\beta_1(L^\alpha H^\beta_2 - L^\beta_2 H^\alpha) > 0 \\
G_6 &= M^{1-\alpha_1}L^\beta_2(L^\alpha H^\beta_2 - L^\beta_2 H^\alpha) = L^\beta_2 G_2 < 0
\end{aligned}
\end{equation}

$V_m(L, -1)$ is expressed as

\begin{equation}
V_m(L, -1) = \frac{G_4 m^{1-\beta_1} + G_5 m^{\beta_2-\beta_1} + G_6}{G_1 m^{\beta_2-\beta_1} + G_3}.
\end{equation}

We further define

\begin{equation}
f_M(m) = (1 - \beta_2)G_1 m^{\beta_2-\beta_1} - (\beta_2 - \beta_1)G_2 m^{\beta_2-1} + (1 - \beta_1)G_3.
\end{equation}

We calculate the derivative of $V_m(L, -1)$ with respect to $m$ and obtain

\begin{equation}
\begin{aligned}
(G_1 m^{\beta_2-\beta_1} + G_3)^2 \frac{dV_m(L, -1)}{dm} &= G_4 \left[(1 - \beta_2)\left(G_1 m^{\beta_2-2\beta_1} + G_2 m^{\beta_2-\beta_1-1}\right)ight] \\
&+ (1 - \beta_1)\left[G_3 m^{-\beta_1} - G_2 m^{\beta_2-\beta_1-1}\right]
\end{aligned}
\end{equation}

Let us define $\hat{m}$ as $m$ such that $f_M(m) = 0$ and $\hat{M}$ as $M$ such that $f_M(L) = 0$.

In order to find the maximizer of $V_m(L, -1)$ with respect to $m \in (0, L]$, we show that $V_m(L, -1)$ is strictly concave in $m \in (0, L]$ and show that there is a unique maximizer $\hat{m}$ that is characterized by $dV_m(L, -1)/dm = 0$. Then we can find the maximizer of $V_m(L, -1)$ when $m \in (0, L]$ by checking whether $\hat{m} \in (0, L]$.

Thanks to \[3.23\], this is equivalent in showing the following lemma on $f_M(m)$:

**Lemma 3.2.** $f_M(m)$ satisfies the following 3 conditions:

1. $f_M(0) < 0$. 
(2) $f'_M(m) > 0$.

(3) $f_M(L) \geq 0$ when $M \geq \hat{M}$

(3.24) $f_M(L) < 0$ when $M < \hat{M}$.

Proof. First, we check $f_M(0) = (1 - \beta_1)G_3 < 0$. Since $\beta_1 < 1$ from (3.12), we only need to check if $G_3 < 0$. Indeed,

$$G_3 = (H^{\beta_2}L^\alpha - L^{\beta_2}H^\alpha) + M^{\alpha_2 - \alpha_1}(H^{\alpha_1}L^{\beta_2} - L^{\alpha_1}H^{\beta_2})$$

(3.25) $< (H^{\beta_2}L^\alpha - L^{\beta_2}H^\alpha) + H^{\alpha_2 - \alpha_1}(H^{\alpha_1}L^{\beta_2} - L^{\alpha_1}H^{\beta_2})$

$= L^{\alpha_1}H^{\beta_2}(L^{\alpha_2 - \alpha_1} - H^{\alpha_2 - \alpha_1}) < 0.$

Secondly, we check if $f'_M(m) > 0$ in $m \in [0, L]$.

(3.26) $f'_M(m) = - (\beta_2 - \beta_1)(\beta_2 - 1)m^{\beta_2 - 2}\left\{G_1 m^{1 - \beta_1} + G_2\right\}$.

We define $P(m, M) := G_1 m^{1 - \beta_1} + G_2$. Since $P(0, M) < 0$ and $P(m, M)$ is monotone in $m$, if we show that $P(L, M) < 0$, then we obtain the conclusion $f'_M(m) > 0$.

$$P(L, M) = (L^{\beta_1}H^\alpha - H^{\beta_1}L^\alpha)L^{1 - \beta_1}$$

(3.27) $+ L^{1 - \beta_1}M^{\alpha_2 - \alpha_1}(L^{\alpha_1}H^{\beta_1} - L^{\beta_1}H^{\alpha_1}) + M^{1 - \alpha_1}(L^{\alpha_2}H^{\alpha_1} - L^{\alpha_1}H^{\alpha_2})$.

Denoting the derivative with respect to $M$ by $d_M$, we calculate $d_M P(L, M)$:

$$d_M P(L, M) = (\alpha_2 - \alpha_1)L^{1 - \beta_1}M^{\alpha_2 - \alpha_1 - 1}(L^{\alpha_1}H^{\beta_1} - L^{\beta_1}H^{\alpha_1})$$

(3.28) $+ (1 - \alpha_1)M^{-\alpha_1}(L^{\alpha_2}H^{\alpha_1} - L^{\alpha_1}H^{\alpha_2})$

$$= M^{\alpha_2 - \alpha_1 - 1}\left\{(\alpha_2 - \alpha_1)L^{1 - \beta_1}(L^{\alpha_1}H^{\beta_1} - L^{\beta_1}H^{\alpha_1})

+ (1 - \alpha_1)M^{1 - \alpha_2}(L^{\alpha_2}H^{\alpha_1} - L^{\alpha_1}H^{\alpha_2})\right\}.$$

If $\beta_1 \leq \alpha_1$, then it is obvious from (3.28) that

(3.29) $d_M P(L, M) < 0$.

We now assume $\beta_1 > \alpha_1$. Define
\[
Q(M) = (\alpha_2 - \alpha_1) L^{1-\beta_1} (L^{\alpha_1} H^{\beta_1} - L^{\beta_1} H^{\alpha_1}) \\
+ (1 - \alpha_1) M^{1-\alpha_2} (L^{\alpha_2} H^{\alpha_1} - L^{\alpha_1} H^{\alpha_2}).
\]

(3.30)

We see from (3.30) that \(Q(M)\) is monotonically decreasing with respect to \(M\). Therefore, we calculate \(d_M P(L, M)\) at \(M = H\) and obtain

\[
d_M P(L, M) = M^{\alpha_2-\alpha_1-1} Q(M) \\
< M^{\alpha_2-\alpha_1-1} Q(H) \\
= M^{\alpha_2-\alpha_1-1} \left\{ (\alpha_2 - \alpha_1) L^{1-\beta_1} (L^{\alpha_1} H^{\beta_1} - L^{\beta_1} H^{\alpha_1}) \\
+ (1 - \alpha_1) H^{1-\alpha_2} (L^{\alpha_2} H^{\alpha_1} - L^{\alpha_1} H^{\alpha_2}) \right\} \\
= M^{\alpha_2-\alpha_1-1} \left\{ (\alpha_2 - \alpha_1) L H^{\alpha_1} \left\{ \frac{H}{L} \right\}^{\beta_1-\alpha_1} - 1 \right\} \\
- (1 - \alpha_1) L^{\alpha_2} H^{1+\alpha_1-\alpha_2} \left\{ \left( \frac{H}{L} \right)^{\alpha_2-\alpha_1} - 1 \right\} \\
< 0.
\]

(3.31)

We again proved

\[
d_M P(L, M) < 0.
\]

(3.32)

As a consequence of (3.29) and (3.32),

\[
P(L, M) \leq P(L, M) \big|_{M=H} \\
= -L^{\alpha_1} H (H^{\alpha_2-\alpha_1} - L^{\alpha_2-\alpha_1}) \left\{ 1 - \left( \frac{L}{H} \right)^{1-\beta_1} \right\} \\
< 0,
\]

(3.33)

hence we have

\[
f_M'(m) > 0, \quad m \in [0, L].
\]

(3.34)

Finally, we check \(f_M(L)\).

Let us define

\[
q_\alpha(x) = (1 - \beta_1) x^{\beta_2-\beta_1} - (\beta_2 - \beta_1) x^{\alpha-\beta_1} + (\beta_2 - 1) \quad (\alpha < 1).
\]

(3.35)
Then, we can calculate \( f_M(L) \) as

\[
f_M(L) = (1 - \beta_2)(G_1 L^{\beta_2 - \beta_1} + G_3) + (\beta_2 - \beta_1)(G_3 - G_6 L^{-1})
\]

\[
= (\beta_2 - \beta_1) L^{\beta_2 - 1}(L^{\alpha_1} H^{\alpha_2} - L^{\alpha_2} H^{\alpha_1}) M^{1-\alpha_1}
\]

\[
- L^{\alpha_1 + \beta_2 - \beta_1} H^{\beta_1} q_{\alpha_1} \left( \frac{H}{L} \right) M^{\alpha_2 - \alpha_1} + L^{\beta_2 - \beta_1 + \alpha_2} H^{\beta_1} q_{\alpha_2} \left( \frac{H}{L} \right).
\]

We have

\[
q'_a(x) = (1 - \beta_1)(\beta_2 - \beta_1)x^{\alpha_1 - \beta_1 -1} \left( x^{\beta_2 - \alpha} - \frac{\alpha - \beta_1}{1 - \beta_1} \right) > 0 \quad (x \geq 1),
\]

hence

\[
q_a(x) \geq q_a(1) = 0.
\]

The derivative of \( f_M(L) \) with respect to \( M \) is calculated as

\[
d_M f_M(L) = (\beta_2 - \beta_1)(1 - \alpha_1) L^{\beta_2 - 1}(L^{\alpha_1} H^{\alpha_2} - L^{\alpha_2} H^{\alpha_1}) M^{-\alpha_1}
\]

\[
- (\alpha_2 - \alpha_1) L^{\alpha_1 - \beta_1 + \beta_2} H^{\beta_1} q_{\alpha_1} \left( \frac{H}{L} \right) M^{\alpha_2 - \alpha_1 -1}.
\]

We note here that the coefficient of \( M^{-\alpha_1} \) in (3.39), which is the higher order term in \( M \), is positive.

Calculating the value \( f_M(L) \) when \( M = H \),

\[
f_M(L)|_{M=H} = L^{\alpha_1} (H^{\alpha_2 - \alpha_1} - L^{\alpha_2 - \alpha_1})
\]

\[
\times \{ (\beta_2 - \beta_1) L^{\beta_2 - 1} H + (\beta_1 - 1) H^{\beta_2} + (1 - \beta_2) L^{\beta_2 - \beta_1} H^{\beta_1} \}.
\]

However, we have

\[
(\beta_2 - \beta_1) L^{\beta_2 - 1} H + (\beta_1 - 1) H^{\beta_2} + (1 - \beta_2) L^{\beta_2 - \beta_1} H^{\beta_1} < 0,
\]

hence

\[
f_M(L)|_{M=H} < 0.
\]

Therefore, \( f_M(L) \geq 0 \) when \( M \geq \hat{M} \) and \( f_M(L) < 0 \) when \( M < \hat{M} \). ♦

We obtain the following proposition directly from Lemma 3.2.
Proposition 3.3. When \( m \leq L \), the value of \( m \) that maximizes \( V_m(x, f) \) is

\[
\begin{cases}
\hat{m} & \text{when } M \geq \hat{M} \\
L & \text{when } M < \hat{M}
\end{cases}
\]

(3.43)

Remark 4. Proposition 3.3 says that if \( M \) is not as large as \( \hat{M} \), the point where \( V_m(x, f) \) takes its maximum is when \( m = L \), which is the largest \( m \) possible in the range of \( m \) considered. However, if \( M \) is large enough, \( V_m(x, f) \) takes its maximum at \( \hat{m} \in (0, L) \).

This is in line with the intuition that if \( M \) is too low, we cannot expect much profit by holding on to the stock in the negative regime, hence it is optimal to sell the position right away. However, if \( M \) is large enough, even if the stock price is currently in the negative regime, there is a hope that the stock enters the positive regime in the near future and generates a large profit. Therefore, it is optimal to hold on to the position in this case until the process breaches the level \( \hat{m} \).

3.5. Maximizer in the Case where \( m \geq L \). In the case when \( m \geq L \), the solution \( V_m(x, f) \) is calculated (3.14) with \( A, B, C, \) and \( D \) in (3.18). We solve for the maximizer of \( V_m(x, f) \) over \( m \geq L \).

Defining

\[
E = m^{-\beta_1}H^{\beta_2-\beta_1}\left\{ (1 - \beta_1)H^{\beta_2-\beta_1} + (\beta_2 - 1)m^{\beta_2-\beta_1} \\
- (\beta_2 - \beta_1)m^{\beta_2-1}H^{-\beta_1}V_m(H, +1) \right\},
\]

(3.44)

we can calculate the derivative of \( V_m(x, -1) \) with respect to \( m \) as

\[
(H^{\beta_2-\beta_1} - m^{\beta_2-\beta_1}) \frac{dV_m(x, -1)}{dm} = E x^{\beta_1} - \frac{E}{H^{\beta_2-\beta_1}} x^{\beta_2} \\
= E x^{\beta_1} \left\{ 1 - \left( \frac{x}{H} \right)^{\beta_2-\beta_1} \right\} \quad x \in [m, H].
\]

(3.45)

The sign of the derivative \( dV_m/dm \) matches with that of \( E \), so we focus on the sign of \( E \). The sign is the same as that of \( g \) defined by

\[
g(m) := (1 - \beta_1)H^{\beta_2-\beta_1} + (\beta_2 - 1)m^{\beta_2-\beta_1} \\
- (\beta_2 - \beta_1)m^{\beta_2-1}H^{-\beta_1}V_m(H, +1).
\]

(3.46)

Remark 5. From (3.45), we see that the \( m \) that maximizes \( V_m(x, -1) \) maximizes \( V_m(H, -1) \) and vice versa. For that, it is sufficient to maximize \( V_m(H, -1) \) over \( m \).

We show a few lemmas we need for later use.
Lemma 3.4. \( V_m(H, +1) \geq H \).

Proof. \( e^{-r t}(V_m(X_t, +1) - X_t) \) is a supermartingale. This is because before we stop \( X_t \) upon reaching \( L \) or \( M \), \( e^{-r t}V_m(X_t, +1) \) is a martingale (as it satisfies the ODE (3.5)) and \( e^{-r t}X_t \) is a submartingale in the positive regime, hence \(-e^{-r t}X_t\) is a supermartingale. Upon reaching the level \( L \) or \( M \), \( e^{-r t}(V_m(X_t, +1) - X_t) = 0 \) due to the boundary conditions (3.17) and it will be zero thereafter as we stop the process \( X_t \) upon reaching the level \( L \) or \( M \). Then, it follows from the Optional Sampling Theorem,

\[
V_m(X_0, +1) - X_0 \geq 0.
\]

Hence, considering the price process starting at \( H \), we have the desired result. \( \diamond \)

Lemma 3.5. \( V_m(H, +1) \) is continuous in \( M \). Furthermore, it is strictly and monotonically increasing in \( M \).

Proof. The continuity of \( V_m(H, +1) \) with respect to \( M \) is obvious from the expression in (3.14) and (3.18).

The second half of the lemma can also be verified from the expression in (3.14) and (3.18), but we can also verify it as follows. Let \( \tau_{LM} \) be the first exit time of the process from the domain \([L, M]\). \( V_m(H, +1) \) is the expected value of \( e^{-r t}X_t \) at the first exit time \( \tau_{LM} \). If we make \( M \) larger, \( \tau_{LM} \) gets larger. In the positive regime, since \( e^{-r t}X_t \) is a submartingale, this shows that \( V_m(H, +1) \) is monotonically increasing in \( M \). \( \diamond \)

If we take the derivative of \( g \) with respect to \( m \), we obtain

\[
g'(m) = (\beta_2 - \beta_1)(\beta_2 - 1)m^{\beta_2 - \beta_1 - 1} \left\{ 1 - \left( \frac{m}{H} \right)^{\beta_1} \frac{V_m(H, +1)}{m} \right\} < 0,
\]

where we used Lemma 3.4 in the last inequality.

Therefore,

\[
g(L) > g(m) > g(H).
\]

Again from Lemma 3.4 we have

\[
g(H) = (\beta_2 - \beta_1)H^{\beta_2 - \beta_1} \left( 1 - \frac{V_m(H, +1)}{H} \right) < 0.
\]

We now calculate \( g(L) \).
\[ g(L) = (1 - \beta_1)H^{\beta_2 - \beta_1} \left\{ 1 - \left( \frac{L}{H} \right)^{\beta_2} \frac{V_m(H, +1)}{L} \right\} \]

\[(3.51)\]

\[ + (\beta_2 - 1)L^{\beta_2 - \beta_1} \left\{ 1 - \left( \frac{L}{H} \right)^{\beta_1} \frac{V_m(H, +1)}{L} \right\}. \]

From (3.51), \( g(L) \) is a decreasing function of \( V_m(H, +1) \). In case \( V_m(H, +1) = H \), we have

\[ g(L) \bigg|_{V_m(H, +1)=H} = (1 - \beta_1)H^{\beta_2 - \beta_1} \left\{ 1 - \left( \frac{L}{H} \right)^{\beta_2 - 1} \right\} \]

\[ + (\beta_2 - 1)L^{\beta_2 - \beta_1} \left\{ 1 - \left( \frac{L}{H} \right)^{\beta_1 - 1} \right\}. \]

\[(3.52)\]

We define

\[ p(x) := 2 - x^{\beta_2 - 1} - x^{\beta_1 - 1}, \quad x \leq 1. \]

\[(3.53)\]

We first note that \( p(1) = 0 \). We take the derivative of \( p \) with respect to \( x \) and get

\[ p'(x) = - (\beta_2 - 1)x^{\beta_2 - 2} - (\beta_1 - 1)x^{\beta_1 - 2} \]

\[(3.54)\]

\[ = (\beta_2 - 1)x^{\beta_2 - 2} \left( 1 - \beta_1 \frac{1}{\beta_2 - 1} - x^{\beta_2 - \beta_1} \right) > 0. \]

Therefore,

\[ p(x) \leq p(1) = 0. \]

\[(3.55)\]

From (3.55), we have

\[ 1 - x^{\beta_2 - 1} \leq x^{\beta_1 - 1} - 1 \quad (x \leq 1). \]

\[(3.56)\]

Substituting \( x = L/H \) in (3.56), we have

\[ 1 - \left( \frac{L}{H} \right)^{\beta_2 - 1} \leq \left( \frac{L}{H} \right)^{\beta_1 - 1} - 1. \]

\[(3.57)\]

Let us define

\[ q(y) = (1 - \beta_1)y^{\beta_2 - \beta_1} - (\beta_2 - \beta_1)y^{1 - \beta_1} + (\beta_2 - 1). \]

\[(3.58)\]
Coming back to (3.52), reordering the terms, we equivalently have

\[
\begin{aligned}
g(L)|_{V_m(H, +1) = H} &> L^{\beta_2 - \beta_1} q(y) \\
K := H/L &> 1
\end{aligned}
\]  

We note that \( q(1) = 0 \) and

\[
q'(y) = y^{-\beta_1} (\beta_2 - \beta_1)(1 - \beta_1)(y^{\beta_2 - 1} - 1) \geq 0 \quad y \geq 1,
\]

so we have

\[
q(K) \geq q(1) = 0, \quad y \geq 1.
\]

From (3.59) and (3.61), we have

\[
g(L)|_{V_m(H, +1) = H} > 0.
\]

As a conclusion, from (3.49), (3.50), and (3.62), there exists some value of \( V_m(H, +1) \) which makes \( g(L) = 0 \). Since \( g(L) \) is monotonically decreasing with respect to \( V_m(H, +1) \) and (from Lemma 3.5) \( V_m(H, +1) \) is monotonically increasing with respect to \( M \), there exists a unique \( M \) that satisfies \( g(L) = 0 \).

We define \( \hat{m} \) as \( m \) that satisfies \( g(m) = 0 \) and \( \tilde{M} \) as \( M \) that satisfies \( g(L) = 0 \). Then, we have the following proposition:

**Proposition 3.6.** When \( m \geq L \), the value of \( m \) that maximizes \( V_m(x, f) \) is

\[
\begin{aligned}
\tilde{m} & \quad \text{when } M \leq \tilde{M} \\
L & \quad \text{when } M > \tilde{M}
\end{aligned}
\]

**Remark 6.** We have \( \hat{M} \) in Proposition 3.3 and \( \tilde{M} \) in Proposition 3.6. We note that although we introduced it in different ways, \( \hat{M} = \tilde{M} \). We can verify this easily by checking the boundary conditions when we have \( M = \hat{M} \) and \( M = \tilde{M} \) in each case. Therefore, we use \( \hat{M} = \tilde{M} = \hat{M} \).

3.6. **Optimal Stopping Problem.** We define \( \hat{m} \) as

\[
\hat{m} = \begin{cases} 
\tilde{m}, & M \geq \hat{M} \\
\hat{m}, & M \leq \hat{M}
\end{cases}
\]

We show the following theorem:
Theorem 3.7. The solution to the optimal stopping problem (3.1) \( \tau^* \) is equal to \( \tau_{\hat{m}} \).

**Proof.** We first show the theorem in the case when \( M \geq \bar{M} \).

We start by showing the following:

1. \( e^{-rt}V_{\hat{m}}(X_t, -1) \) is a supermartingale;
2. \( e^{-rt}V_{\hat{m}}(X_t, -1) \) dominates the gains process \( e^{-rt}X_t \).

For the first point, \( V_{\hat{m}}(\cdot, -1) \) satisfies the ODE which enables us to show that its discounted process \( e^{-rt}V_{\hat{m}}(t, -1) \) is a martingale up to the time when the price process breaches the level \( \hat{m} \). After it breaches the level, the process \( e^{-rt}V_{\hat{m}}(X_t, -1) \) will just be \( e^{-rt}X_t \) thereafter, which is a supermartingale in the regime.

We now focus on the second part, to show that \( e^{-rt}V_{\hat{m}}(X_t, -1) \geq e^{-rt}X_t \), hence to show \( V_{\hat{m}}(x, -1) \geq x \). We define \( \zeta \) as

\[
(3.65) \quad \zeta(x) = V_{\hat{m}}(x, -1) - x = Cx^{\beta_1} + Dx^{\beta_2} - x.
\]

We investigate this function in the domain \([\hat{m}, L]\).

First, note that \( \zeta(\hat{m}) = 0 \).

We calculate first and second derivatives of \( \zeta(x) \) with respect to \( x \):

\[
(3.66) \quad \zeta'(x) = \beta_1 Cx^{\beta_1-1} + \beta_2 Dx^{\beta_2-1} - 1
\]

and

\[
(3.67) \quad \zeta''(x) = \beta_1(\beta_1 - 1)Cx^{\beta_1-2} + \beta_2(\beta_2 - 1)Dx^{\beta_2-2}.
\]

Substituting \( C \) and \( D \), and using \( f_M(\hat{m}) = 0 \), we can further calculate

\[
(3.68) \quad \zeta''(x) = \frac{(\beta_2 - 1)(1 - \beta_1)(\beta_2 x^{\beta_2-\beta_1} - \beta_1 \hat{m}^{\beta_2-\beta_1})x^{\beta_1-2}}{(\beta_2 - \beta_1)\hat{m}^{\beta_2-1}} > 0.
\]

We calculate \( \zeta'(\hat{m}) \).

\[
(3.69) \quad \zeta'(\hat{m}) = \beta_1 C\hat{m}^{\beta_1-1} + \beta_2 D\hat{m}^{\beta_2-1} - 1
\]

\[
= \beta_1 C\hat{m}^{\beta_1-1} + \beta_2 D\hat{m}^{\beta_2-1} - (C\hat{m}^{\beta_1-1} + D\hat{m}^{\beta_2-1})
\]

\[
= C\hat{m}^{\beta_1-1}(\beta_1 - 1) + (\beta_2 - 1)D\hat{m}^{\beta_2-1},
\]
the Snell envelope of $e$ values of $C$ fact that since (3.70)
Theorem to deduce that $e$ is a supermartingale.

function in the domain $\tilde{m}$

stopping time is the first time when the process $X$

(3.71) and (3.72) respectively.

The fact that $\zeta(\tilde{m}) = 0, (3.68)$, and (3.70), we have $\zeta(x) = V_{\tilde{m}}(x, -1) - x \geq 0$ in $x \in [\tilde{m}, L]$.

In order to solve the optimal stopping problem (3.1), we want to show that $e^{-\eta t}V_{\tilde{m}}(X_t, F_t)$ is the Snell envelope of $e^{-\eta t}X_t$.

The fact that $e^{-\eta t}V_{\tilde{m}}(X_t, F_t)$ is a supermartingale is shown similarly as we showed that $e^{-\eta t}V_{\hat{m}}(X_t, -1)$ is a supermartingale.

We further have to show that $e^{-\eta t}V_{\hat{m}}(X_t, +1) - e^{-\eta t}X_t \geq 0$ in $x \in [L, M]$. For this, we use the fact that since $e^{-\eta t}V_{\hat{m}}(X_t, +1) - e^{-\eta t}X_t$ is a supermartingale, we can use the Optional Sampling Theorem to deduce that

$e^{-\eta t}\{V_{\hat{m}}(X_t, +1) - X_t\} \geq e^{-\eta \tau_+} \min\{V_{\hat{m}}(L, +1) - L, 0\}$

where $\tau_+$ is the first stopping time the process goes out of the region $[L, M]$. Note that we’ve replaced $V_{\hat{m}}(L, +1)$ with $V_{\hat{m}}(L, -1)$ thanks to the boundary condition (3.15).

Therefore, in the case $M \geq \hat{M}$, the Snell envelope of $e^{-\eta t}X_t$ is $e^{-\eta t}V_{\hat{m}}(X_t, F_t)$ and the optimal stopping time is the first time when the process $X_t$ hits the level $M$ or $\hat{m}$.

The argument in the case when $M \leq \hat{M}$ will be similar, and we show that $e^{-\eta t}V_{\hat{m}}(X_t, -1) \geq e^{-\eta t}X_t$, hence to show $V_{\hat{m}}(x, -1) \geq x$. We define the function $\zeta(x)$ as (3.65). We evaluate this function in the domain $[\hat{m}, H]$.

First, note that $\zeta(\hat{m}) = 0$.

We calculate first and second derivatives of $\zeta(x)$ with respect to $x$ and they are the same as in (3.66) and (3.67) respectively.

Since $g(\hat{m}) = 0$, we have

$(1 - \beta_1)H^{-\beta_1}\{H^{\beta_2 - \hat{m}\beta_2}V_+(H, M)\}$

$= (\beta_2 - 1)H^{-\beta_1}\{\hat{m}^{\beta_2 - \hat{m}\beta_2}V_+(H, M) - H^{\beta_1}\hat{m}^{\beta_2}V_+(H, M)\}.$

Substituting $C$ and $D$, and using (3.72), we can further calculate
\( \zeta''(x) = (\beta_2 - 1)H^{-\beta_1}(V(H, M) - \tilde{m}^{1-\beta_1}H^{\beta_1}) \frac{(\beta_2, \tilde{m}^{\beta_2-\beta_1} - \tilde{m}^{\beta_2-\beta_1})}{H^{\beta_2-\beta_1} - \tilde{m}^{\beta_2-\beta_1}} \) > 0.

We calculate \( \zeta'(\tilde{m}). \)

\( \zeta'(\tilde{m}) = \beta_1 C\tilde{m}^{\beta_1-1} + \beta_2 D\tilde{m}^{\beta_2-1} - 1 \)

\( = \beta_1 C\tilde{m}^{\beta_1-1} + \beta_2 D\tilde{m}^{\beta_2-1} - (C\tilde{m}^{\beta_1-1} + D\tilde{m}^{\beta_2-1}) \)

\( = C\tilde{m}^{\beta_1-1}(\beta_1 - 1) + (\beta_2 - 1)D\tilde{m}^{\beta_2-1}, \)

where the second equality comes from the boundary condition at \( x = \tilde{m}. \) We substitute the values of \( C \) and \( D \) in (3.74) and obtain

\( (H^{\beta_2-\beta_1} - \tilde{m}^{\beta_2-\beta_1})\zeta'(\tilde{m}) \)

\( = (\beta_1 - 1)H^{-\beta_1}(H^{\beta} - \tilde{m}^{\beta_2-1}V\tilde{m}(H, +1)) \)

\( + (\beta_2 - 1)\tilde{m}^{\beta_2-1}(H^{-\beta_1}V\tilde{m}(H, +1) - \tilde{m}^{1-\beta_1}) \)

\( = 0, \)

where we used (3.72) in the last equality.

From \( \zeta(\tilde{m}) = 0, \ (3.73), \) and (3.75), we have \( \zeta(x) = V\tilde{m}(x, -1) - x \geq 0 \) in \( x \in [\tilde{m}, H]. \)

4. Optimal Timing of Buying

Up to the previous section, we were dealing with the optimal selling problem. We now think of the optimal timing to purchase the shares. We assume that we introduce the capital to purchase the asset only at the time of purchase, and seek to maximise our discounted profit.

We solve the following optimal stopping problem:

\( U(x, f) = \sup_{\tau \in \mathcal{T}} E_{x,f} [e^{-r\tau} \{ V(X_\tau, F_\tau) - X_\tau \}] \).

We show the following theorem:

**Theorem 4.1.** It is optimal to purchase the shares when and only when the underlying process is in the positive regime.

**Proof.** We define the gains process

\( G(X_t, F_t) = e^{-rt} \{ V(X_t, F_t) - X_t \}. \)
We further define $U(X_t, F_t)$ as

\begin{align}
U(X_t, F_t) = \begin{cases}
  e^{-r(t,\tau_H)} \{ V(X_{\min(t,\tau_H)}, +1) - X_{\min(t,\tau_H)} \}, & F_t = +1 \\
  \mathbb{E}_x[e^{-r\tau_H} \{ V(H, +1) - H \}], & F_t = -1.
\end{cases}
\end{align}

In the positive regime, $e^{-rt}V(X_t, +1)$ is a martingale while $e^{-rt}X_t$ is a submartingale. Hence, $G(X_t, +1)$ is a local supermartingale. Therefore, $G(X_t, +1)$ is itself the Snell envelope that dominates $G(X_t, +1)$ and so it is optimal to stop the process right away.

In the negative regime, $e^{-rt}V(X_t, -1)$ is a martingale and $e^{-rt}X_t$ is a supermartingale. Hence, $G(X_t, -1)$ is a local submartingale. Since the process $X_t$ will hit the level $H$ almost surely, from the Optional Sampling Theorem, it is therefore optimal to run the process as long as possible, which corresponds to running the process until it leaves the negative regime.

We also see that

\begin{align}
U(X_t, -1) = \mathbb{E}_x[e^{-r\tau_H} \{ V(H, +1) - H \}] = \mathbb{E}_x[e^{-r\tau_H} \{ V(H, -1) - H \}] \\
\geq e^{-rt}(V(X_t, -1) - X_t) = G(X_t, -1).
\end{align}

The first line in (4.4) uses the boundary condition and the second line uses the fact that $G(X_t, -1)$ is a submartingale and the Optional Sampling Theorem.

Therefore, $U(X_t, F_t)$ is a supermartingale that dominates the gains process $G(X_t, F_t)$. We then use Theorem 3.1 and the solution to the optimal stopping problem (4.1) is the first time the process $(X_t, F_t)$ enters the stopping region, i.e. the region where $U(x, \pm 1) = G(x, \pm 1)$. This corresponds to the first time when $F_t = +1$ or when $F_t = -1$ and $(X_t, F_t) = (H, +1)$.

5. Conclusions

We started with a simple setup where we only have one support/resistance level and fully showed the optimal level to sell the shares. We also considered the optimal timing to purchase the shares and found out that it is only optimal to do so in the positive regime given that the investors borrow money upon buying the shares.

A possible extension of the problem considered in this section is to have the price process follow more general SDEs where $\mu$’s and $\sigma$’s are functions of the price. We believe we can follow the same calculation as we did in this paper using a general theory of ODEs.

Another possible problem to consider is to use different cost functions for the gains processes. For example, in considering the optimal purchasing problem, we assumed that the investors borrow money at the time they decided to purchase the shares. Instead, we can consider the case where

\footnote{Here, the terminology "local" means 'locally in space'.}
the investors already have cash in hand at time $t = 0$ and for this, we need to consider different gains process.

REFERENCES

[1] Arriojas, M., Hu, Y., Mohammed, S-E., and Pap, G. 2007. A Delayed Black and Scholes Formula, Stochastic Analysis and Applications, vol. 25, pp. 471-492.

[2] Blanchet-Scalliet, C., Diop, A., Gibson, R., Talay, D., and Tanré, E. 2007. Technical Analysis Compared to Mathematical Models Based Methods Under Parameter Mis-specification, Journal of Banking & Finance, vol. 31, pp. 1351 - 1373.

[3] Bao, J. and Yuan, C. 2011. Comparison Theorem for Stochastic Differential Delay Equations with Jumps, arXiv: 1102.2165v1

[4] Corns, T.R.A. and Satchell, S.E. 2007. Skew Brownian Motion and Pricing European Options, The European Journal of Finance, vol. 13, no. 6, pp. 523-544.

[5] Harrison, J.M. and Shepp, L.A. 1981. On Skew Brownian Motion, The Annals of Probability, vol. 9, no. 2, pp. 309 - 313.

[6] Itô, K. and McKean, H.P. 1965. Diffusion Processes and Their Sample Paths, Springer-Verlag.

[7] Karatzas, I. and Shreve, S.E. 1998. Brownian Motion and Stochastic Calculus, Springer Science+Business Media, Inc., second edition.

[8] Karatzas, I. and Shreve, S.E. 1998. Methods of Mathematical Finance, Springer.

[9] Karlin, S. and Taylor, H.M. 1981. A Second Course in Stochastic Processes, Academic Press Inc.

[10] Le Gall, J - F. 2016. Brownian Motion, Martingales, and Stochastic Calculus, Springer.

[11] Lejay, A. 2006. On the Constructions of the Skew Brownian Motion, Probability Surveys, vol. 3, pp. 413-466.

[12] Lo, A., Mamaysky, H., and Wang, J. 2000. Foundations of Technical Analysis: Computational Algorithms, Statistical Inference, and Empirical Implementation, Journal of Finance, vol. 55 (4), pp. 1705 - 1765.

[13] Longtin, A. 2010. Stochastic Delay-Differential Equations in Complex Time-Delay Systems (Atay, F.M. ed), Springer.

[14] Mao, X. 1997. Stochastic Differential Equations & Applications, Horwood Publishing Ltd.

[15] Mao, X. and Yuan, C. 2006. Stochastic Differential Equations with Markovian Switching, Imperial College Press.

[16] Mohammed, S-E. A. 1984. Stochastic Functional Differential Equations, Pitman Publishing Inc.

[17] Nilsen, W. and Sayit, H. 2011. No Arbitrage in Markets with Bounces and Sinks, International Review of Applied Financial Issues and Economics, vol. 3, no. 4, pp. 696-699.

[18] Peskir, G. and Shiryaev, A. 2006. Optimal Stopping and Free-Boundary Problems, Birkhäuser Verlag.

[19] Protter, P. 1990. Stochastic Integration and Differential Equations: A New Approach, Springer-Verlag.

[20] Pucci, P. and Serrin, James. 2007. The Maximum Principle, Birkhäuser.

[21] Rogers, L.C.G. and Williams, D. 2000. Diffusions, Markov Processes and Martingales Volume 2: Itô Calculus, Cambridge University Press, 2nd edition.

[22] Rossello, D. 2012. Arbitrage in Skew Brownian Motion Models, Insurance: Mathematics and Economics, vol. 50, pp50-56.

[23] Shreve, S. E. 2004. Stochastic Calculus for Finance II: Continuous-Time Models, Springer.

[24] Yang, Z., Mao, X., and Yuan, C. 2008. Comparison Theorem of One-dimensional Stochastic Hybrid Delay Systems, Systems & Control Letters, vol. 57, pp. 56-63.
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