Fractal Structures of Yang-Mills Fields and Non Extensive Statistics: applications to High Energy Physics

Airtón Deppman, Eugenio Megías and Débora P. Menezes

1 Instituto de Física, Rua do Matão 1371-Butantã, São Paulo-SP, CEP 05580-090, Brazil; deppman@if.usp.br
2 Departamento de Física Atómica, Molecular y Nuclear and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, Avenida de Fuente Nueva s/n, 18071 Granada, Spain; emegias@ugr.es
3 Departamento de Física, CFM-Universidade Federal de Santa Catarina, Florianópolis, SC-CP. 476-CEP 88.040-900, Brazil; debora.p.m@ufsc.br
* Correspondence: emegias@ugr.es; Tel.: +34 958 241000 (E.M.)
† These authors contributed equally to this work.

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Abstract: In this work we provide an overview of the recent investigations on the non extensive Tsallis statistics and its applications to high energy physics and astrophysics, including physics at the LHC, hadron physics and neutron stars. We review some recent investigations on the power-law distributions arising in HEP experiments focusing on a thermodynamic description of the system formed, behaviour. The possible connections with a fractal structure of hadrons is also discussed. The main objective of the present work is to delineate the state-of-the-art of those studies, and show some open issues that deserve more careful investigation. We propose several possibilities to test the theory through analyses of experimental data.

Keywords: Tsallis statistics; pp collisions; hadron physics; quark-gluon plasma; thermofractals

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1. Introduction

An interesting description of hadronic systems in the hot and in the dense regimes, known as Quark-Gluon Plasma (QGP), has been developed in recent years. Although motivated by the large amount of information that emerged from high energy physics (HEP) experiments, the consequences of those advances are far-reaching, since they may be present in any Yang-Mills field theory (YMF). Here we will give a short review of those developments, discuss the experimental evidences of the new theoretical approach and present some applications to HEP, hadron physics and astrophysics.

Before entering in the main subject of this work, let us summarize the three fundamental theories that form the foundations for the developments that will be discussed below. These three theories are: the YMF; the Fractal Geometry of Benoit Mandelbrot; and the Non Extensive Statistics (S_q) proposed by Constantino Tsallis.

The Yang-Mills field theory is a prototype theory for describing most of the physical phenomena. It has been proposed by Yang and Mills in the early 60’s and is present in the description of interacting fields that is known as Standard Model. Basically, it describes the propagation of undulatory fields in space and time, as well as the interaction between those fields. One of the fundamental properties of physics laws is the renormalization group invariance, and all the four known fundamental interactions show this feature as an essential aspect of its structure. At least three of them can be described by YMF, but also for the gravitational interaction, a YMF version has been proposed.
The simplest non-Abelian gauge field theory whose Lagrangian density includes bosons and fermions is given by

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + i \bar{\psi} i \gamma^\mu D_\mu ij \psi^j, \]  

(1)

where \( F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \) is the field strength of the gauge field, and \( D_\mu ij = \partial_\mu \delta_{ij} - ig A_\mu^a T_{ij}^a \) is the covariant derivative, with \( f^{abc} \) being the structure constants of the group, and \( T_{ij}^a \) the matrices of the group generators. \( \psi \) and \( A \) represent, respectively, the fermion and the vector fields. Gerard ‘t Hooft and Martinus Veltman worked out the renormalization of Yang-Mills theories [1]. The asymptotic properties of the beta function of Yang-Mills theories, and in particular of Quantum Chromodynamics (QCD), were studied later, leading to the novel phenomenon of asymptotic freedom [2]. A comprehensive description of YMF can be found in e.g. Ref. [3].

Fractal is the name given by Benoit Mandelbrot to systems that present a geometry that is very different from the Euclidean one, although it has strong connections with natural phenomena as we observe them. Therefore, Fractal Geometry has been used to describe many natural shapes that can be observed in everyday life. The main aspect of a fractal is that it presents a fine structure with an undetermined number of components that are also fractals similar to the original system, but at a different scale. This aspect is known as self-similarity. One important characteristic of fractals is the fractal dimension, which describes how the measurable aspects of the system change with scale. Contrary to the usual quantities in Euclidean geometry, where measurements with higher precision give a better evaluation of the same value for the measured quantity, in fractals an increase of the resolution of the measurement will give a different value. This aspect is described by attributing a fractional dimension to the topological dimension of the system. The classical example is the coastline of an island, which has different lengths when measured with different precision.

A direct consequence of the self-similarity and the scaling properties with a fractional dimension is the power-law behavior of distributions observed for fractals. Thus, probability distributions are described by

\[ P(x) = Ax^\beta, \]

(2)

with \( A \) being constant and \( \beta \) an exponent related to the fractal dimension. The impressive ubiquity of fractals in physical systems, but also in mathematical iterative formulas, gives to the fractal theory an importance that can be noticed in many different areas, as Physics, Biology, Sociology, Engineering. A nice introductory account of the applications of fractals can be found in Refs. [4,5] and a comprehensive description of Fractal Geometry can be found in Ref. [6].

The Tsallis Statistics is a generalization of the Boltzmann-Gibbs (BG) statistics where the Entropy form is given by

\[ S_q = -k_B \ln_q p(x), \]

(3)

where \( p(x) \) is the probability of \( x \) to be observed and \( q \) is the entropic index that quantifies how Tsallis’ entropy departs from the extensive BG statistics. In the above expression we have used the \( q \)-logarithm function, defined as

\[ \ln_q(p) = p^{q-1} - 1 \over q-1. \]

(4)

A direct consequence of the entropic form defined above is the non additivity of entropy, since for two independent systems, \( A \) and \( B \), the entropy of the combined system, \( AB \) is

\[ S_q(A + B) = S_q(A) + S_q(B) - k_B^{-1} (q - 1) S_q(A) S_q(B). \]

(5)
As \( q \to 1 \) the BG statistics is recovered, the entropic form becomes additive and \( \ln_q(p) \to \ln(p) \).

Another consequence is that often the probability distributions obtained in the non extensive thermodynamics that result from the Tsallis entropic form are described in terms of the \( q \)-exponential function

\[
e^{-q(x)} = [1 + (q - 1)x]^{1/(q-1)}, \tag{6}
\]

which has found wide applicability in the last few years, see e.g. Refs. \([7,8]\). However, the full understanding of this statistics has not been accomplished yet. A comprehensive description of Tsallis Statistics and its applications can be found in Refs. \([9–11]\).

2. Objectives

The main objective of this work is to give a short review of the recent advances in the understanding of the fractal structures present in Yang-Mills Fields, in particular QCD, and on its main implication, namely, the need of Tsallis statistics to describe the thermodynamics aspects of the fields. We will obtain the entropic index of the Tsallis statistic as a function of the field fundamental parameters.

The main features of the fractal structure, which are summarized by the fractal dimension or fractal spectrum, will be discussed. We will show how the Tsallis statistics entropic index is related to the fractal structure that is represented as a thermofractal. Then, focusing in QCD and in high energy collisions (HEC), we shall obtain the effective coupling constant in terms of the entropic index, \( q \), thereby relating the coupling to the field parameters by an analytical expression. We will show how the power-law behavior observed in momentum distributions measured at high energy collisions is naturally explained by the thermofractal structure. We will mention several experimental observations that can be easily described by the theory cited above.

The paper is organized as follows: in section 3 we give a short description of Yang-Mills field theory, emphasizing the renormalization group invariance of the theory and evidencing the fractal structures present in the fields. We also show how the thermodynamic aspects of the Yang-Mills fields reflect the fractal structure in a way similar to the thermofractals, that were introduced to show how Tsallis statistics can emerge in thermodynamic systems, and study its main thermodynamic quantities. In section 4 we describe the main experimental findings that give support to the theory presented here, and we study some applications of the theory to HEP, hadron physics and Astrophysics. Finally in section 5 we present a summary of the paper and our conclusions.

3. Theoretical Method

3.1. The fractal structures in Yang-Mills Fields

As mentioned above, one of main aspects of the YMF theory, mathematically described by Eq. (1), is that it is renormalizable. It means that vertex functions that are regularized to avoid the ultra-violet (UV) divergence are related to the renormalized vertex functions to which renormalized parameters, \( \bar{m} \) and \( \bar{g} \), are associated by \([12,13]\)

\[
\Gamma(p, m, g) = \lambda^{-D} \Gamma(p, \bar{m}, \bar{g}). \tag{7}
\]

This property is described by the renormalization group equation, also known as Callan-Symanzik (CS) equation, which is given by \([14,15]\):

\[
\left[ M \frac{\partial}{\partial M} + \beta_g \frac{\partial}{\partial g} + \gamma \right] \Gamma = 0, \tag{8}
\]
where $M$ is the scale parameter, and the beta function is defined as $\beta_\bar{g} = M \frac{\partial \bar{g}}{\partial M}$. As it is shown in the left panel of Fig. 1, renormalization group invariance means that, after proper scaling, the loop in a higher order graph in the perturbative expansion is identical to a loop in lower orders. This is a direct consequence of the CS equation, and it is of fundamental importance in what follows.

The CS equation is a specific case, for QCD, of a general equation representing renormalization group invariance, and can be applied to any system where that symmetry is present, thereby all calculations made for the hadronic case apply in a similar form to the Electro-Weak fields. Let us now analyze the possibility of fractal structures to be formed by the Yang-Mills fields.

On the other hand, the scaling symmetry is one of the key ingredients in a fractal system, but not the only one. If we want to show that YMF can form fractal structures, it is important to show that, beside the scaling symmetry it presents an internal structure. The combination between the scaling property and the internal structure will give rise to the self-similarity, a distinguishing aspect of fractals. We will use the perturbative method and the concept of effective parton, that is, a one-particle irreducible representation of the associated quantum field, to show that the effective parton can be understood as a complex system with internal structure. This internal structure will lead to the formation of fractal structures.

It is clear that the fractal structures may be more easily identified in the strongly interacting systems governed by QCD, and in fact, evidences of the fractal structure can be found in the literature already a long time ago. In the 1950’s the existing experimental data on HEC evidenced the formation of a system in thermodynamic equilibrium. Indeed, Fermi proposed a thermodynamic model to describe hadronic collisions at high energies [17]. The main issue then was to explain how a short-lived system (half-life of a few fm/c) could reach equilibrium so fast. This system was called fireball.

A few years later Hagedorn [18] proposed his Self-Consistent Thermodynamics (SCT) theory that was able to explain several features of hot hadronic systems. His starting point was a rather weird definition for fireballs, stating that:

“A fireball is (*) a statistical equilibrium (hadronic black-body radiation) of an undetermined number of all kinds of fireballs, each of which, in turn, is considered to be (goto *)”.

Notice how the definition of fireball given by Hagedorn resembles that of fractals.

Let us introduce in this section the formalism that will allow us to understand the fractal structure of gauge theories. We will consider that at any scale the system can be described as an ideal gas of particles...
with different masses, i.e. masses might change with the scale. The time evolution of an initial partonic state is given by
\[
|\Psi\rangle = e^{-iHt} |\Psi_0\rangle.
\] (9)
The state |\Psi\rangle can be written as |\Psi\rangle = \sum_{\{n\}} \langle \Psi_n |\Psi\rangle |\Psi_n\rangle, where |\Psi_n\rangle is a state with n interactions in the vertex function. Each proper vertex gives rise to a term in the Dyson series, and at time t the partonic state is given by
\[
|\Psi\rangle = \sum_{\{n\}} \frac{(-i)^n}{n!} \int dt_1 \ldots dt_n e^{-iH_0(t_n-t_{n-1})} \ldots e^{-iH_0(t_1-t_0)} |\Psi_0\rangle,
\] (10)
where g represents the coupling constant and t_n > t_{n-1} > \cdots > t_1 > t_0, while \sum_{\{n\}} runs over all possible terms with n interaction vertices. Let us introduce states of well-defined number of effective partons, |\psi_N\rangle, so that
\[
|\Psi_n\rangle = \sum_N \langle \psi_N |\Psi_n\rangle |\psi_N\rangle.
\] (11)
Therefore |\psi_N\rangle = S |\eta_1, m_1, p_1; \ldots; \eta_N, m_N, p_N\rangle, where m_i and p_i are the mass and momentum of the i partonic state, and \eta_i represents all relevant quantum numbers necessary to completely characterize the partonic state. These states can be understood as a quantum gas of particles with different masses. Let us remark at this point that the number of particles in the state |\Psi_n\rangle is not directly related to n, since high order contributions to the N particles states can be important. The rule is N \leq N_{\text{max}}(n) := n(N-1) + 1, where N is the number of particles created or annihilated at each interaction. In Yang-Mills field theory, N = 2.1

Let us study the probability to find a state with one parton with mass between m_o and m_o + dm_o, and momentum between p_o and p_o + dp_o. This is given by
\[
\langle \eta_o, m_o, p_o, \ldots |\Psi(t)\rangle = \sum_n \sum_N \langle \Psi_n |\Psi(t)\rangle \langle \psi_N |\Psi_n\rangle \langle \eta_o, m_o, p_o, \ldots |\psi_N\rangle.
\] (12)

There are three factors in the rhs of this equation. The first one, \langle \Psi_n |\Psi(t)\rangle, is related to the probability that an effective parton with energy between E and E + dE at time t = 0 will evolve in such a way that at time t it will generate an arbitrary number of secondary effective partons in a process with n interactions. This factor can be written as \langle \Psi_n |\Psi(t)\rangle = G^n P(E)dE, where P(E) is the probability distribution of the initial particle, and G^n is the probability that exactly n interactions will occur in the elapsed time. The second bracket is the probability to get the configuration with N particles after n interactions, i.e.
\[
\langle \psi_N |\Psi_n\rangle = C_N(n) \approx \left( \frac{N}{n(N-1)} \right)^4.
\] (13)

Finally, the last bracket in Eq. (12) can be calculated statistically, leading to the following result [16]
\[
\langle \eta_o, m_o, p_o, \ldots |\psi_N\rangle \approx A(N) P_N \left( \frac{E}{E} \right) d^4 \left( \frac{p_i}{E} \right),
\] (14)
with
\[
A(N) = \frac{\Gamma(4N)}{8\pi^3 \Gamma(4N-1)} \quad \text{and} \quad P_N(x) = (1-x)^{4N-5},
\] (15)

1 We remark that 3- and 4-point contributions are not considered in our approach, since they cannot be renormalized at each order individually, cf. Ref. [19].
where $\Gamma(x)$ is the Euler gamma function, $p_j^\mu = (p^0_j, \vec{p}_j)$ is the four-momentum of particle $j$ inside the system of $N$ particles, and $\epsilon_j = p^0_j$ is the energy of that particle. Note that we are not assuming a fixed value for the mass $m_j$ of particle $j$, where $m^2_j = p^\mu_j p_\mu$, so that $p^0_j$ and $\vec{p}_j$ are variables that may change independently each other. By combining all these results in Eq. (12), one finally obtains

$$
\tilde{p}(\epsilon) d^4p_0 dE \equiv \langle \eta_o, m_o, \ldots | \Psi(t) \rangle = \sum_n \sum_N G^n \left( \frac{N}{n(N-1)} \right)^4 \left( 1 + \frac{\epsilon}{E} \right)^{-(4N-5)} d^4 \left( \frac{p}{E} \right) [P(E)]^\nu dE, \tag{16}
$$

where $\nu$ is a parameter that, together with the parameter $\tilde{N}$, determine the characteristics of the thermofractal. Its meaning will be elucidated below.

As we will show below, the distributions $\tilde{p}(\epsilon)$ and $P(E)$ can be obtained from considerations about self-similarity. Let us mention that relativistic corrections don’t play the role of mass variations, as any relativistic correction should keep the rest mass invariant. Notice that going from one level to the next level of the hierarchy of subsystems does not correspond to a Lorentz transformation, but instead to a scale transformation. Then, the four-momenta $p^\mu$ do not provide themselves the self-similar properties of the system.

### 3.2. Self-similarity and fractal structure

Due to scale invariance, the energy distribution of a parton, as given by Eq. (16), must depends on the ratio $\chi = \epsilon/E$, and this ratio must be invariant when we go from one level in the fractal structure to the other. For instance, let us consider that the system with energy $E$, in which the parton with energy $\epsilon_j$ appears as one among $N$ constituents, is itself a parton inside a larger system with energy $M$. Then the scale invariance, when expressed in terms of the variable $\chi$, gives

$$
\frac{\epsilon_j}{E} = \chi = \frac{E}{M}. \tag{17}
$$

Expressing Eq. (16) in terms of this scale-free variable, we have

$$
\tilde{p}(\chi) = \sum_n \sum_N G^n \left( \frac{N}{n(N-1)} \right)^4 (1 + \chi)^{-(4N-5)} P(\chi). \tag{18}
$$

On the other hand self-similarity implies that

$$
\tilde{p}(\chi) \propto P(\chi), \tag{19}
$$

since the effective partons at any scale have the same probability distribution in terms of the scale-free variable. This relation imposes a strong condition to the probability distribution in Eq. (18), since both distributions, $\tilde{p}(\epsilon)$ in the left-hand side and $P(E)$ in the right-hand side, must have the same shape. It is straightforward to conclude that they must follow a power-law function of the form

$$
P(\chi) = (1 + \chi)^{\alpha}. \tag{20}
$$

---

2 We have used in Eq. (16) that for $N$ sufficiently large and $x \ll 1$, one can approximate $(1 - x)^{(4N-5)} \simeq (1 + x)^{-(4N-5)}$. Notice that the total energy of the system formed in high energy collisions is of the order of tens of TeV, while the momentum range considered in most experiments is of the order of tens of MeV. Therefore, we can consider $x = (q - 1)e/E \ll 1$, where this identification of $x$ is motivated in Sec. 3.2.
Substituting the ansatz above into Eq. (18) it can be shown that [16]

\[ P \left( \frac{\varepsilon}{\lambda} \right) = \left[ 1 + (q - 1) \frac{\varepsilon}{\lambda} \right]^{-\frac{1}{q-1}} , \tag{21} \]

where \( q - 1 = (1 - \nu)/(4N - 5) \), while \( \nu \) represents the fraction of total number of d.o.f. of the state \( |\psi_N\rangle \) that is involved in each interaction, and \( \lambda = (q - 1)\Lambda \) is a reduced scale. This probability distribution is a power-law function, and it will be derived as well in Sec. 3.4 for thermofractals, cf. Refs. [20,21]. An interesting interpretation of the entropic index \( q \) arises: it is related to the number of internal degrees of freedom in the fractal structure, and the distribution of Eq. (21) describes how the energy received by the initial parton flows to its internal d.o.f.. In the context of the theory developed in this section, this probability describes how the energy flows from the initial parton to the partons that appear at higher perturbative orders. Since new orders are associated to new vertices, this suggests that this distribution plays the role of an effective coupling constant in the vertex function, i.e.

\[ \Gamma = \langle \Psi_{n+1} | g e^{iH_{d+1}} | \Psi_n \rangle \quad \text{with} \quad g = \prod_{i=1}^{N} G \left[ 1 + (q - 1) \frac{\varepsilon_i}{\lambda} \right]^{-\frac{1}{q-1}} . \tag{22} \]

The situation is schematized in the right panel of Fig. 1, where the vertex functions that are responsible for the scaling properties of the theory in multiparticle production (see Sec. 3.6) are shown.

### 3.3. Effective coupling and beta function

The CS equation together with the renormalized vertex functions were used to derive the beta function of QCD, which allowed to show that QCD is asymptotically free [2,22]. The result at the one-loop approximation is

\[ \beta_{\text{QCD}} = -\frac{g^3}{16\pi^2} \left[ \frac{11}{3} c_1 - \frac{4}{3} c_2 \right] , \tag{23} \]

where \( c_1 \delta_{ab} = f_{acd} f_{bcd} \) and \( c_2 \delta_{ab} = \text{Tr} (T_a T_b) \) are directly related to the QCD field parameters, as described by the Yang-Mills Lagrangian in Eq. (1). Quantitatively, the parameters \( c_1 \) and \( c_2 \) are related to the number of colors and flavors by \( c_1 = N_c \) and \( c_2 = N_f/2 \). In this section, we will study the beta function derived with our ansatz, and compare with that in QCD.

Let us consider a vertex in two different orders, as depicted in Fig. 2. The vertex function at first order, i.e. at scale \( \lambda_o \), is

\[ \Gamma_0 = \langle \eta_2 p_2, \eta_3 p_3 | g(\lambda_o) e^{iH_{d1}} | \eta_1 p_1 \rangle . \tag{24} \]

The next order in the perturbative approximation is given by the vertex with one additional loop at scale \( \lambda \), which results in a vertex function

\[ \Gamma = \langle \eta_2 p_2, \eta_3 p_3 | g(\lambda_o) e^{iH_{d1}} | \eta_2 p_6, \eta_3 p_3, \eta_4 p_4 \rangle \times \langle \eta_2 p_6, \eta_3 p_3, \eta_4 p_4 | g(\lambda) e^{iH_{d2}} | \eta_1 p_5, \eta_4 p_4 \rangle \langle \eta_1 p_5, \eta_4 p_4 | g(\lambda_o) e^{iH_{d1}} | \eta_1 p_1 \rangle . \tag{25} \]

By comparing this expression with \( \Gamma = \langle \eta_2 p_2, \eta_3 p_3 | g e^{iH_{d1}} | \eta_1 p_1 \rangle \), one can identify the effective coupling \( \bar{g} \) as

\[ \bar{g} = g(\lambda_o) e^{iH_{d1}} | \eta_2 p_6, \eta_3 p_3, \eta_4 p_4 \rangle \Gamma_M \langle \eta_1 p_5, \eta_4 p_4 | g(\lambda_o) , \]

where

\[ \Gamma_M = \langle \eta_2 p_6, \eta_3 p_3, \eta_4 p_4 | g(\lambda) e^{iH_{d2}} | \eta_1 p_5, \eta_4 p_4 \rangle . \tag{27} \]
The scaling properties of Yang-Mills fields allow us to relate $\Gamma_M$ to $\Gamma_o$ by an appropriate scale, $\lambda$. From dimensional analysis, the scaling behavior turns out to be $\Gamma_M(\lambda) = (\lambda/\lambda_o)^4$. Using these considerations and from the CS equation, it results that

$$\beta_g \frac{\partial \Gamma}{\partial g} = -(d + \gamma_5 + \gamma_6)\Gamma,$$

where $d = 4$ and $\gamma_5, \gamma_6$ are the anomalous dimensions. In order to compare with the QCD results, we will study the behavior of $g(\lambda)$ at $\lambda = \lambda_o/\mu$, where $\mu$ is a scaling factor. From Eq. (22), one has

$$g(\mu) = \prod_{i=5}^{6} G_i \left[ 1 + (q-1) \frac{\varepsilon_i \mu}{\lambda_o} \right]^{-\frac{1}{\varepsilon_i q-1}},$$

and substituting it into Eqs. (26) and (27) one can calculate the beta function in the one loop approximation. By considering the asymptotic limit $(q-1)\mu \gg \lambda_o/\varepsilon_i$, one obtains

$$\beta_g = -\frac{1}{16\pi^2} \frac{1}{q-1} \tilde{N}^{q-1},$$

with $\tilde{N} = 2$. Finally, by comparing with the QCD result, Eq. (23), one can make the identification

$$\frac{1}{q-1} = \frac{11}{3} c_1 - \frac{4}{3} c_2 = 7,$$

where in the last equality we have used $N_c = N_f/2 = 3$. This result leads to $q = 1.14$ leading to an excellent agreement with the current experimental data analyses, as it will be discussed in Sec. 4. Notice that this analysis also predicts that there is some dependence of $q$ on the number of flavors. It would be interesting that this prediction being confronted with experimental $q$-fits studies.

We notice that the scaling properties used here are necessary conditions to allow the renormalization of the quantum field theory after regularization, and are present independently of the renormalization scale used, hence our results are valid at any value for the scale chosen. In our approach, it becomes evident that the number of flavours gives the scaling dimension of the theory, and it is an essential part of the renormalization procedure, no matter what is the scale used to fix the theory after regularization.

In Fig. 3 we display the plots showing the behaviors of the beta function $\beta_g$ as a function of $g$, and of the coupling $g$ as a function of $\mu$. The results obtained here give a strong basis for the interpretation of

**Figure 2.** Vertex functions at scale $\lambda_o$ (left) and $\lambda$ (right). Taken from Ref. [16]
previous experimental and phenomenological studies on QCD in terms of non extensive statistics and thermofractals.

3.4. Thermofractals

So far we have shown that, in principle, fractal structures are possible to be formed in YMF. What we have to consider now is the evidences that such structures are really present in the physical systems and if they play a relevant role in the evolution of those systems in time and in the interactions with other fields. It is easy to observe that evidences of such fractal structure have been noticed long time ago.

The Hagedorn’s SCT gave predictions for some quantities that could be easily accessed experimentally. In particular it predicted the transverse momentum distribution of the particles produced in the decay of the fireball, given by
\[ \frac{dN}{dp_T} \bigg|_{y=0} = g V \frac{p_T m_T}{(2\pi)^2} \exp \left( \frac{m_T}{T} \right), \] (32)
where \( g \) is a constant, \( V \) is the volume of the system, \( m_T = \sqrt{p_T^2 + m^2} \). Another prediction was the hadron mass spectrum, given by
\[ \rho(m) = \gamma m^{-5/2} \exp (\beta_0 m), \] (33)
where \( \beta_0 = 1/T_o \) and \( T_o \) is a parameter of the theory.

One interesting aspect of the theory was the prediction of a limiting temperature for the fireball, known today as Hagedorn temperature, \( T_H \), which is numerically equal to the parameter \( T_o \), i.e., \( T_H = T_o \). The comparison between theory and experiment brought a sudden recognition of the importance of Hagedorn’s theory, and its impact in HEP is notorious. Frautschi proposed a similar theory based exclusively in hadron structure, stating that “hadrons are made of hadrons”, again using self-reference [24]. With this so-called bootstrap model he obtained the same hadron mass spectrum formula shown in Eq. (33).

The success of SCT prompted the development of ideal gas models for hadronic systems, the Hadron Resonance Gas (HRG) models [25–35], that were able to explain many features of high energy collisions and hadron physics. Before that, an important paper by Dashen, Ma and Bernstein [36] had explained, through
Dyson-Schwinger expansion, how a strongly interacting system could behave under some conditions as an ideal gas of resonant particles. Another important consequence of SCT came with the advent of the quark structure of hadrons, that was used by Cabibbo and Parisi [37] to propose that Hagedorn temperature was not a limiting temperature but a transition temperature between the confined quark and the deconfined quark regimes of hadronic matter. The last phase is known as Quark-Gluon Plasma and is one of the most interesting issues in nowadays Nuclear Physics. Despite its initial success and its important consequences, with the results coming from accelerators able to deliver particles at higher energies than those from existing at that time, it was found that SCT was not able to explain correctly the outcome of the new HEP data. Indeed, Hagedorn himself proposed, in substitution to this thermodynamics theory, an empirical model based on QCD [38]. Since his theory gave so many correct information about hadronic systems, the question imposes itself is: what went wrong with Hagedorn’s theory?

A possible answer to this question is based on the special fractal structure found for YMF, and hence to QCD, which implies in the non extensive thermodynamics that hadronic systems, and supposedly hadron, may present [20]. Such a fractal structure would result in fractional dimension of the phase-space of the compound particles of the system. Preliminary analysis of such idea lead to a fractal dimension that is compatible with the results from intermittency analyses.

The self-similarity is an evident feature of fireballs and hadrons according to their definitions by Hagedorn and Frautschi, respectively, as described above. However many other evidences point to the self-similar structure of hadrons: fractal dimension has been identified through the analysis of intermittency in experimental distributions obtained in HEP experiments [39–48]; a Parton Distribution Function (PDF) that describes the proton structure based on fractal properties was shown to fit experimental data rather well [49]; direct evidences of self-similarity in experimental data has been observed [50], but the so-called z-scaling [51,52] might be related to fractal structures as well.

The emergence of the non extensive behavior has been attributed to different causes. These include long-range interactions, correlations and memory effects [53], temperature fluctuations, and finite size of the system, among others. We will show in this section that a natural derivation of non extensive statistics in terms of thermo-fractals [16,20] is possible. These are systems in thermodynamical equilibrium presenting the following properties:

1. Total energy is given by \( U = F + E \), where \( F \) is the kinetic energy, and \( E \) is the internal energy of \( N \) constituent subsystems, so that \( E = \sum_{i=1}^{N} \epsilon^{(1)}_i \).
2. The constituent particles are thermo-fractals. This means that the energy distribution \( P_{\text{TF}}(E) \) is self-similar or self-affine, i.e. at level \( n \) of the hierarchy of subsystems, \( P_{\text{TF}(n)}(\chi) \) is equal to the distribution in any other level, with \( \chi \) being a scale-free variable that can be given by \( \chi = E/F \) in the case of thermo-fractals of the type-I, or \( \chi = E/U \) for thermo-fractals of the type-II.
3. At some level of the fractal structure, the internal energy fluctuation is small enough to be disregarded. In this case, the internal energy is considered to be equal to the component mass \( m \).

It is possible to show that thermo-fractals results in energy distributions of the kind

\[
P(\epsilon) = \left[ 1 \pm (q-1) \frac{\epsilon}{\lambda} \right]^{\frac{1}{q-1}},
\]

with the negative sign in the exponent corresponding to the type-II thermo-fractal, while the positive sign corresponds to the type-I. Each type of thermo-fractal leads to a distribution without cut-off (type-I) or with cut-off (type-II). As we will see below, \( q \) is the entropic index of the Tsallis statistics, while \( \lambda \) is a scale
variable. In what follows we will consider only the thermofractals of type-I, but the type-II can be derived in the same way [16]. The energy distribution according to BG statistics is given by

\[ P_{\text{BG}}(U) dU = A \exp(-U/(k_B T)) dU, \tag{35} \]

where \( A \) is a normalization constant. In the case of thermofractals the phase space must include the momentum degrees of freedom (\( \propto f(F) \)) of free particles as well as internal degrees of freedom (\( \propto f(\varepsilon) \)). Then, the internal energy is \( dE \propto [P_{\text{BG}}(\varepsilon)]^\kappa d\varepsilon \) where \( \kappa \) is an exponent to be determined, and one has [20]

\[ P_{\text{TF}}(0)(U) dU = A' F^{\frac{3N - 1}{2}} \exp\left(-\frac{A F}{k_B T}\right) dF [P_{\text{TF}}(1)(\varepsilon)]^\kappa d\varepsilon , \tag{36} \]

with \( \alpha = 1 + \varepsilon/(kT) \) and \( \varepsilon/(kT) = E/F \). This expression relates the distributions at level 0 and 1 of the subsystem hierarchy. After integration, and by imposing self-similarity, i.e.

\[ P_{\text{TF}}(0)(U) \propto P_{\text{TF}}(1)(\varepsilon), \tag{37} \]

the simultaneous solution of Eqs. (36) and (37) is obtained with [21]

\[ P_{\text{TF}}(n)(\varepsilon) = A(n) \cdot \left[ 1 + (q - 1) \frac{\varepsilon}{k_B T} \right]^{-\frac{1}{\tau - 1}} = A(n) \cdot e_q \left( -\varepsilon/(k_B T) \right) . \tag{38} \]

We find that the distribution of thermofractals obeys Tsallis statistics with \( q - 1 = 2(1 - \kappa)/(3N) \) and \( \tau = N(q - 1)T \). Observe that \( q \) is related to the number of degrees of freedom that are relevant to the description of the system, and the fact that \( q \) is different from unit means that this number is finite. In this aspect, thermofractals can be considered as examples of small systems with finite degrees of freedom. In the rest of this manuscript, we will study the connection between thermofractals and quantum field theory, and in particular to QCD.

### 3.5. Non extensive self-consistent thermodynamics

The Non Extensive Self-Consistent Thermodynamics (NESCT) is a generalization of the SCT theory by imposing the self-consistency principle from Hagedorn in the non extensive statistics from Tsallis [54]. The basic ingredients are the two forms of partition function for fireballs proposed by Hagedorn, only this time using the Tsallis \( q \)-exponential factor in Eq. (38), i.e.

\[ P(\varepsilon) = A \cdot e_q \left( -\varepsilon/(k_B T) \right), \tag{39} \]

which leads to

\[ Z_q(V_o, T) = \int_{0}^{\infty} \sigma(E) \left[ 1 + (q - 1)\beta E \right]^{-\frac{q}{(\tau - 1)}} dE , \tag{40} \]

where \( \beta = 1/(k_B T) \), and \( \sigma(E) \) is the density of states of the fireballs as a function of its energy, and

\[ \log \left[ 1 + Z_q(V_o, T) \right] = \frac{V_o}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{\infty} dm \int_{0}^{\infty} dp \ p^2 \rho(n; m) \left[ 1 + (q - 1)\beta \sqrt{p^2 + m^2} \right]^{-\frac{q}{(\tau - 1)}} , \tag{41} \]

which is the partition function for an ideal gas of fermions and bosons with mass \( m \). The sum in \( n \) results from the expansion of the logarithm function, and \( \rho(n; m) = \rho_f(m) - (1)^n \rho_b(m) \) is related to the fermionic and bosonic mass spectra. In the case \( n = 1 \), which will be used here, \( \rho(1; m) = \rho_f(m) + \rho_b(m) \equiv \rho(m) \) is the hadron mass spectrum.
Since fireballs are thermofractals and present self-similarity, the following weak-constraint must hold
\[
\log[\rho(m)] = \log[\sigma(E)]. \tag{42}
\]
The main task now is to find functions \(\rho(m)\) and \(\sigma(E)\) that satisfy the equality above and simultaneously let the two forms of partition function identical to each other. This can be achieved \([54,55]\) by choosing
\[
\rho(m) = \frac{\gamma}{m^{5/2}} [1 + (q - 1)\beta_o m]^{\frac{1}{q-1}}, \tag{43}
\]
and
\[
\sigma(E) = bE^a [1 + (q - 1)\beta_o m]^{\frac{1}{q-1}}, \tag{44}
\]
where \(m\) is the mass of the components of the thermofractal and \(E\) is the total energy of the system. With those functions both forms of partition function result in
\[
Z_q(V_o, T) \rightarrow b\Gamma(a + 1) \left(\frac{1}{\beta - \beta_o}\right)^{a+1}, \tag{45}
\]
with
\[
a + 1 = \alpha = \frac{\gamma V_o}{2\pi^2 \beta^{3/2}}. \tag{46}
\]
From here one can see that there is a singularity at \(\beta = \beta_o\), and therefore the limiting temperature is found.

Therefore it is possible to generalize Hagedorn’s self-consistency principle within Tsallis statistics and as a result one gets a new hadron mass spectrum formula, given by Eq. (43), the limiting temperature resulting from the singularity in Eq. (45), now expressed in terms of the Tsallis temperature, but also a universal entropic index, \(q\), which is characteristics of any hadron. It is worthy to mention that both \(T\) and \(q\) are parameters in the hadron mass spectrum, therefore they are related to the hadronic structure.

As in the case of the Hagedorn’s theory, here also the way to test experimentally the predictions of the theory is through the transversal momentum distribution, given by
\[
\left.\frac{dN}{dp_T}\right|_{y=0} = gV p_T m_T (2\pi)^2 e_q \left(\frac{m_T}{T}\right). \tag{47}
\]
Once the parameters \(T\) and \(q\) are determined by fitting of Eq. (47) to data on \(p_T\) distribution, the entire thermodynamics of hot hadronic systems is determined. The extension of the Hagedorn theory to non extensive statistics, which is characterized by the probability distribution Eq. (39), allowed to reproduce the distribution of all the species produced in \(pp\) collisions with a high accuracy, leading to the result \([56,57]\)
\[
q = 1.14 \pm 0.01, \tag{48}
\]
(see left panel of Fig. 4). Other reactions like \(AA\), \(dA\) or \(pA\) collisions show the same behavior (see e.g. Ref. [58] for a recent review). Moreover, the power-law behavior for the hadron spectrum predicted by this extension, cf. Eq. (43), can be compared with the hadron spectrum from the Particle Data Group (PDG). This comparison, as displayed in the right panel of Fig. 4, leads to an important improvement with respect to the exponential distribution \(\rho(m) = \rho_o e^{m/T_H}\) proposed by Hagedorn, specially at the lowest masses, cf.
Ref. [56]. Notice that it would be possible to consider an approximation in the extension of the Hagedorn theory to non extensive statistics, by using that the product of the functions in Eqs. (39) and (43) in the partition function can be expressed as

\[ \rho(m)P(\epsilon) \approx \rho_0 A \left[ 1 + (q-1) \frac{\epsilon - m}{M} \right]^{-\frac{1}{q-1}}. \]  

(49)

This leads to a good fit of observables as long as \( m \ll M / (q-1) \).

Finally, let us mention that it is possible to obtain any thermodynamics quantity and compare it to Lattice QCD (LQCD) results. A comparison between the results from NESCT and LQCD data has shown also a good agreement between both calculations [60], in this case without any adjustable parameter. Apart from these considerations, there are many other applications in which Tsallis statistics play an important role. This includes high energy collisions [57,61,62], hadron models [63], hadron mass spectrum [56], neutron stars [64], LQCD [60], non extensive statistics [21,54,55], and many others. In the rest of the manuscript we will see that the power-law behavior of Tsallis statistics leads, in fact, to important phenomenological consequences in these fields.

The theory presented here can be extended to \( AA \) collisions, as long as collective motion of the QGP fluid is taken into account [65]. The analyses of experimental data from HEP give room to some small variation of the entropic index, \( q \), and for the temperature, \( \tau \), with collision energy and/or particle species [66], and the variation is larger for small multiplicity events. The variation can, in principle, be due to non equilibrium processes, where perturbative QCD should be applied. In fact, as the multiplicity increases, the results seem to be independent from collision energy, and remain valid for \( pp \), \( pA \) and \( AA \) collision [67]. The use of non extensive thermodynamics is important for the precise determination of the character of the phase transition between confined and deconfined states of the QCD matter [68,69].

We mentioned above that \( q \) and \( \nu \) are related to the number of degrees of freedom in the system. In the context of the fractal structure exhibited by the YMF, \( q \) is related to the number of relevant degrees of freedom involved in the transfer of energy and momentum between interacting systems. Thus, regarding the general applicability of the theory, we observe that the most complex case was considered here, i.e., the case where the full phase-space is occupied and all possible field configurations, for both fermions and bosons, are allowed. These conditions seem to be valid for QGP, but for simpler systems, where the total quantum numbers are restricted enough or when topology does not allow the complete phase space to be occupied, as it may happen in, e.g. deep inelastic scattering or high energy particles scattering, the \( q \) value may change accordingly. Recent experimental data show, indeed, that for high multiplicity events the \( q \) value tends to a universal value in good agreement with the one found here [70].

3.6. Multiparticle production

An interesting physical situation in which the scaling properties emerge is in multiparticle production in \( pp \) collisions [71]. We show in the right panel of Fig. 1 a typical diagram in which two partons are created in each vertex. When effective masses and charges are used, the line of the respective field in Feynman diagrams represents an effective particle. An irreducible graph represents an effective parton, and the vertices are related to the creation of an effective parton. Due to the complexity of the system, it is desirable a statistical description in which the summation of all the diagrams becomes equivalent to an ideal gas of particles with different masses [36,72]. Another example is given by the hadron structure: as

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3 Observe that the hadron spectrum given by Eq. (43) is valid up to \( m = 2.5 \) GeV, and we consider \( \rho(m) = 0 \) above this limit. Similarly the Regge trajectories may relax from the straight lines (see Ref. [59] and references therein).
3.7. Non extensive thermodynamics of thermofractals

As we have seen above, the properties of the YMF lead to the formation of fractal structures, and the thermofractal systems show us that the thermodynamics formalism that describes YMF fractal structure is the Tsallis statistics. In QCD, such fractal aspects appear in the generalized version of Hagedorn’s thermodynamics, which was described by the NESCT approach. In this section we will describe some of the main aspects of the thermofractal system, since most of its characteristics are important for applications in hadron models and on the description of massive objects, as neutron stars, through the Tsallis statistics. We will present here the thermodynamics of a free quantum gas of bosons and fermions at finite temperature and chemical potential within Tsallis statistics.

We will define two versions of the $q$-exponential and $q$-logarithm functions as follows

$$e_q^{(\pm)}(x) = [1 \pm (q-1)x]^{\mp \frac{1}{q-1}}, \quad \ln_q^{(\pm)}(x) = \pm \frac{x^{\pm(q-1)} - 1}{q-1},$$

(50)

where $^{(+)}$ in $e_q(x)$ ($\ln_q(x)$) stands for $x \geq 0$ ($1$), and $^{(-)}$ stands for $x < 0$ ($1$). Then the grand-canonical partition function for a non extensive ideal quantum gas is given by [55]

$$\ln Z_q(V, T, \mu) = -\bar{\xi}V \int \frac{d^3p}{(2\pi)^3} \sum_{r=\pm} \Theta(rx) \ln_q^{(-)} \left( \frac{e_q^{(r)}(x) - \bar{\xi}}{e_q^{(r)}(x)} \right),$$

(51)

where $x = (E_p - \mu)/(k_BT)$, the particle energy is $E_p = \sqrt{p^2 + m^2}$, with $m$ being the mass and $\mu$ the chemical potential, $\bar{\xi} = \pm 1$ for bosons and fermions respectively, and $\Theta$ is the step function. Note that $e_q^{(\pm)}(x) \rightarrow e^x$ and $\ln_q^{(\pm)}(x) \rightarrow \ln(x)$, so that Tsallis statistics reduces to BG statistics in the limit

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4 The two versions of the functions in Eq. (50), denoted by $^{(+)}$ and $^{(-)}$, are in general needed when considering finite chemical potential. Notice that $e_q^{(\pm)}(x) = e_q(x)$ and $\ln_q^{(\pm)}(x) = \ln_q(x)$, where $e_q$ and $\ln_q$ were defined in Eqs. (6) and (4), respectively.
4. Results and Application

In this section we will describe how the theory presented in Sec. 3 can describe many aspects of QCD as observed in different physical situations, in particular in HEP. The main result we obtained from the theory is an analytic formula for the effective coupling constant in the non perturbative regime, which follows as a consequence of the fractal structure. The effects of fractality of the YMF extend to other aspects, as the power-law behavior of the HEC distribution, which are described by Tsallis statistics and allow to access experimentally the value of the entropic index, $q$, that results to be in very good agreement with the value found by the theory. We describe some aspects of the coupling constants and beta-function.

4.1. Transverse momentum distributions

The predictions of NESCT have been compared with HEP experimental data for $p_T$ distributions (Eq. (47)), showing a good agreement between calculation and data [56,61,77–79], resulting in $q = 1.41 \pm 0.01$ and $T = 62 \pm 5$ MeV. Also, the hadron mass spectrum formula has been used to describe the known hadronic states, resulting again in a very good agreement with data and leading to values of $q$ and $T$ very similar to those obtained with $p_T$ analysis [56]. An example of fitting to $p_T$ distribution is shown in Fig. 5 (left).

Careful fittings of $p_T$ distribution power-law formulas to high statistics data have shown some oscillations in the ratio between calculated and experimental data, which has already being attributed to fractal aspects of the hadronic interaction [80], as shown in Fig. 5 (right). Notice also the wide range of experimental data that can be fitted with power-law formula. This may indicate that there are non equilibrium events that are produced without the formation of a thermodynamically equilibrated system.
but the multiplicity for such events must be smaller than 5. More investigations in this direction would be welcome.

It is interesting the fact that the distributions for all particles produced in collisions at different energies can be described by roughly the same values for the parameters $T$ and $q$ of the non extensive thermodynamics, as shown in Ref. [56], cf. left panel of Fig. 4. However, a study of the best fitted values for those parameters as a function of the multiplicity has shown that the parameters can be considered constant only for events with multiplicity larger than 5 or 6, as shown in the Fig. 6.

![Figure 6. Behavior of the parameter q as a function of multiplicity. It indicates the self-similarity between the system formed at high energy collisions and the jets produced by that system. Also, it may indicate the region of multiplicity where particle production without the formation of an equilibrated system is relevant. Figure taken from Ref. [81].](image)

Another interesting aspect of the results shown in the Figure above is that there is a clear self-similarity in the $p_T$ distributions of particles and jets in relation to the beam direction, but there is also the same distribution for particles forming a jet in relation to the jet direction [81]. This is maybe the most direct experimental evidence of self-similarity in HEP, and should be studied in more details and with better statistics.

### 4.2. Rapidity distributions

The use of the non extensive thermodynamics allows a simple description of most of the data from HEP. In addition to the $p_T$ distribution described with just two parameters that have the same value for any secondary particle and any collision energy, also the rapidity distribution can be described by a simple model which includes other two parameters to describe the movement of two fireballs going apart after the collision. The two parameters are the position and width of two Gaussian functions describing the movement of those two fireballs. The two Gaussians are supposed to present some symmetries, so they can be described by only two parameters. Such model was presented in Ref. [57] and the results show that the parameters for those Gaussian are also approximately constant for different collision energies for the range considered in that work, as shown in figures below. In the same work, predictions for particle production in $pp$ collisions at 13 MeV were made, but up to now those predictions were not confronted with experimental data.

### 4.3. Intermittency

One of the most used techniques to unveil fractal dimensions from measured distributions is the intermittency method, that we briefly describe below. Consider the collision of two systems, each of which may be a lepton, a hadron, or a nucleus. Let $N(y)$ be the rapidity ($y$) distribution of particles produced in an event at high energy. We focus our attention on a narrow window of width $\delta$. The averaging over many
similar windows is a process that can be considered later for the purpose of increasing the experimental statistics, but should be put aside for the present. Let $k$ denote the number of particles detected in $\delta$ in one event. Obviously, $k$ will fluctuate from event to event. Let $Q_k$ be the number of times that the multiplicity $k$ recurs in $N$ events. Note that $Q_k$ is not normalized, so it is not a distribution. Define [39,40]

$$P_k = Q_k / N$$  \tag{52}$$

where

$$N = \sum_{k=0}^{\infty} Q_k .$$  \tag{53}$$

The normalized moments are

$$C_q = \frac{\sum_{k=0}^{\infty} k^q P_k}{(\sum_{k=0}^{\infty} k P_k)^q}$$  \tag{54}$$

for $q > 0$. Another useful quantity is defined by [41]

$$G_q = \sum_{k=0}^{\infty} k^q Q_k / K^q$$  \tag{55}$$

where

$$K = \sum_{k=0}^{\infty} k Q_k .$$  \tag{56}$$

The probability

$$p_i = k_i / K$$  \tag{57}$$

represents the probability to have an event with $k_i$ particles in the window with width $\delta$. For $\delta$ sufficiently small, the behavior of this probability with $\delta$ is

$$p_i \propto \delta^\alpha .$$  \tag{58}$$

The moments $C_q$ or $G_q$ depends on $\delta$ as

$$G_q \propto \delta^{\alpha - f(\alpha)} ,$$  \tag{59}$$

where $f(\alpha)$ is known as fractal spectrum. Also

$$G_q \propto \delta^{\tau(q)} ,$$  \tag{60}$$

therefore $\tau(q) = q\alpha - f(\alpha)$. Note that we can make the identification

$$\tau(q) = (q - 1)D_q ,$$  \tag{61}$$

where $D_q$ is the fractal dimension at order $q$. The advantage of the second momentum ($G_q$) is that it is not calculated for each event, but to a collection of events. A nice description of the experimental approach to the relevant quantities is given in [82]. A typical result of such analysis is shown in Figs. 7 and 8.
4.4. Tsallis statistics and QCD thermodynamics

The thermodynamics of QCD in the confined phase can be studied within the HRG approach, which is based on the assumption that physical observables in this phase admit a representation in terms of hadronic states which are treated as non-interacting and point-like particles [25]. These states are taken as
the conventional hadrons listed in the review by the PDG. Within this approach the partition function is then given by [55,64]
\[
\ln Z_q(V, T, \{\mu\}) = \sum_i \ln Z_q(V, T, \mu_i),
\]
where \(Z_q(V, T, \mu_i)\) is the partition function of a non extensive ideal quantum gas given by Eq. (51), and \(\mu_i\) refers to the chemical potential for the \(i\)-th hadron. The thermodynamic functions can be obtained from Eq. (62) by using the standard thermodynamic relations. There are different proposals for the conditions determining the transition line between the confined and the deconfined regimes of QCD. Using the arguments of [27], the phase transition line in the \(T \times \mu_B\) diagram can be determined by the condition \(\langle E \rangle / \langle N \rangle = 1\) GeV. A different method based on the entropy density has been proposed in [89]. The result for the chemical freeze-out line is displayed in Fig. 9 (left). We also display in Fig. 9 (right) the Equation of State (EoS) for hadronic matter at finite baryonic chemical potential. It is remarkable that \(P(E)\) becomes larger in Tsallis statistics as compared to BG statistics (see a discussion in [55,64]). These thermodynamic studies lead to the natural value \(q = 1.14\), which is in accordance with many previous analysis, like the results from \(pp\) collisions and the beta function of QCD studied above.

The QCD thermodynamics has been studied as well in Ref. [76] by using Tsallis statistics within the MIT bag model. This allowed to investigate the underlying fractal structure of hadrons, leading to the emergence of non extensivity of the hadronic thermodynamics. The thermodynamic quantities have been studied in the approximation of fixed mass for all bag constituents, but also within the discrete mass approximation given by the PDG spectrum, as well as the continuum spectrum provided by the NESCT through Eq. (43). The conclusion is that the continuum spectrum scenario fully applied the hypothesis that the hadron bag is an ideal gas of strongly interacting particles, and this corresponds to the picture of the hadrons as thermo-fractals. It is displayed in Fig. 10 the regime in the phase space \((\mu_B, T)\) in which the proton exists as a bound particle, i.e. \(\varepsilon \cdot V_{\text{proton}} \leq m_{\text{proton}}\). The results in the three scenarios mentioned above are displayed in the left panel of this figure.

4.5. Applications

The theoretical method developed above allows us to understand on solid grounds the emergence of Tsallis statistics in HEC. The predictions of the theory find support on experimental data available, and can explain, in a single theoretical framework, many aspects of the HEP and of the hadronic systems. The effective coupling obtained from considerations about scale symmetry of the YMF shows how four-momentum imparted to one effective parton is distributed among its internal degrees of freedom, which are, themselves, associated to new effective partons created in the interactions between the fields.
The description of the results from high energy collisions by thermodynamics methods is interesting by itself because of possible applications in Astrophysics and in Cosmology. In fact, a description of baryogenesis in the early universe must necessarily take into account the properties of the phase transition between confined and deconfined regimes of the hadronic matter.

The extension of the NESCT to systems with finite chemical potential [55] has allowed the application of the thermodynamics properties of hadrons to the study of neutron stars (NS) [64]. While high energy collisions take place at high temperatures and low chemical potentials, proton-neutron and NS are characterized by high chemical potentials and low temperatures, in a completely different region of the QCD phase diagram. When applied to describe these compact objects, the Tsallis statistics results on a very small change in the main macroscopic properties, as maximum mass and radius, but we found that the internal temperature of the stars decreases with the increase of the $q$-value (taken within a reasonable range) and, around the star central densities the temperature decreases by approximately 25% in average. This aspect may have important consequences when the star evolves from a hot and lepton rich object to a cold and deleptonized compact star. Moreover, the direct Urca process is substantially affected by non extensivity, with probable consequences on the cooling rates of the stars. Besides NS constituted of hadrons, strange stars constituted of deconfined quarks were analysed with a simple model [63]. The conclusions discussed above were all corroborated and the construction of the QCD phase diagram first order transition has shown that the limiting curve is quite sensitive to non extensivity. An investigation with a model that allows the calculation of the critical end point (CEP) remains to be done.

The interpretation of the power-law distributions by the thermofractal structure offers the possibility to connect such structure to QCD properties, as shown in Ref. [20]. Finally using the blast-wave approach to the expansion of the quark-gluon plasma, one can extend the present thermodynamics theory so as to apply to the problem of high energy nucleus-nucleus collisions as well, for which there is a large number of accurate data [90].

Finally, the hadron structure can benefit from the theory presented here. In fact, some models have already used Tsallis statistics as a way to introduce the fractal specs of the QCD in the description of hadron structure [76].

**Figure 10.** Left panel: Pressure (normalized to the ideal gas limit) as a function of temperature and baryonic chemical potential in the case of a non extensive gas with particles with discrete masses. Red dots indicate the region where the gas total energy inside a volume $V_{proton}$ is equal the proton mass, i.e. $\varepsilon \cdot V_{proton} = m_{proton}$. Right panel: Temperature as a function of baryonic chemical potential $T = T(\mu_B)$ for a non extensive gas within the constraint $\varepsilon \cdot V_{proton} = m_{proton}$. For comparison, the results are displayed in three different scenarios: i) continuum masses, ii) discrete masses, and iii) fixed mass. It is considered the value $q = 1.14$ in both panels. Taken from Ref. [76].
5. Conclusions

In this work we have reviewed the applications of Tsallis statistics to HEP, hadron physics and astrophysics. We have investigated the structure of a thermodynamical system presenting fractal properties showing that it naturally leads to Tsallis non extensive statistics. Based on the scaling properties of gauge theories and thermofractal considerations, we have shown that renormalizable field theories lead to fractal structures, and these can be studied with Tsallis statistics. By using a recurrence formula that reflects the self-similar features of the fractal, we have computed the effective coupling and the corresponding beta function. The result turns out to be in excellent agreement with the one-loop beta function of QCD. Moreover, the entropic index, \( q \), has been determined completely in terms of the fundamental parameters of the field theory. Finally, the result for \( q \) is shown to be in good agreement with the value obtained by fitting Tsallis distributions to experimental data. These results give a solid basis from QCD to the use of non extensive thermodynamics to study properties of strongly interacting systems and, in particular, to use thermofractal considerations to describe hadrons. In the last part of the manuscript we have discussed the experimental results on transverse momentum and rapidity distributions in \( pp \) collisions that provide a solid basis to study the role of Tsallis in HEP, as well as a discussion of some applications including the QCD thermodynamics, the QCD phase diagram and the physics of proto(neutron) stars.

From what was discussed here, one can see that there are still many open questions regarding the description of HEP data by power-law distributions. The main problem is to verify to what extent the idea of fractal structure can describe experimental data. In this regard, not only the \( p_T \) distributions must be carefully analysed, but also other quantities, mainly the fractal dimension that can be accessed through intermittency analysis. Although these analyses are still performed for HEP collisions on emulsion, no such analysis was performed with LHC data. Aside intermittency, the study of the parameters \( T \) and \( q \) as a function of multiplicity can shed some light on the discussion about the nature of the power-law distributions: is it due to a thermodynamical system in equilibrium or is it motivated by QCD aspects? If it is shown with accuracy that for multiplicities larger than 5 or 6 the values are constant, as predicted by the thermodynamical theory, this would favor the thermodynamics interpretation. The same analysis would allow for a discussion on the self-similarity in experimental data, as already discussed above.

The rapidity distribution can be further analysed, and the predictions made for \( pp \) collisions at 13 MeV can be compared with the data already measured at that energy. In addition, the theory can be extended to account for nucleus-nucleus collision, with the inclusion of the blast-wave model in the thermodynamical description, allowing for the analysis of \( AA \) data. In summary, the interesting analysis that could be made with recent experimental data are (initially for \( pp \) but eventually also for \( AA \)):

1. Analysis of \( p_T \) distribution with high statistics, obtaining the behaviour of \( T \) and \( q \) as a function of multiplicity for different particles.
2. Investigation of self-similarity in jet and secondary particle distributions.
3. Investigation of the rapidity distribution as predicted with the theory, and also of the distributions for 13 MeV.
4. Investigation of intermittency in HEP data from LHC.

These and other issues will be addressed in a forthcoming publication.

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Abbreviations

The following abbreviations are used in this manuscript:

- **BG**: Boltzmann-Gibbs
- **CEP**: Critical end point
- **CS**: Callan-Symanzik
- **EoS**: Equation of State
- **HEC**: High Energy Collisions
- **HEP**: High Energy Physics
- **HRG**: Hadron Resonance Gas
- **KMS**: Kubo-Martin-Schwinger
- **LHC**: Large Hadron Collider
- **LQCD**: Lattice QCD
- **NESCT**: Non extensive self-consistent thermodynamics
- **NS**: Neutron stars
- **PDF**: Parton Distribution Function
- **PDG**: Particle Data Group
- **QCD**: Quantum Chromodynamics
- **QGP**: Quark-Gluon Plasma
- **SCT**: Self-consistent thermodynamics
- **UV**: Ultra-violet
- **YMF**: Yang-Mills Field Theory

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Sample Availability: Samples of the compounds ...... are available from the authors.