THE LEVI-CIVITA SPACETIME

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Abstract

We consider two exact solutions of Einstein’s field equations corresponding to a cylinder of dust with net zero angular momentum. In one of the cases, the dust distribution is homogeneous, whereas in the other, the angular velocity of dust particles is constant [1]. For both solutions we studied the junction conditions to the exterior static vacuum Levi-Civita spacetime. From this study we find an upper limit for the energy density per unit length \( \sigma \) of the source equal \( \frac{1}{2} \) for the first case and \( \frac{1}{4} \) for the second one. Thus the homogeneous cluster provides another example [2] where the range of \( \sigma \) is extended beyond the limit value \( \frac{1}{4} \) previously found in the literature [3,4]. Using the Cartan Scalars technics we show that the Levi-Civita spacetime gets an extra symmetry for \( \sigma = \frac{1}{2} \) or \( \frac{1}{4} \). We also find that the cluster of homogeneous dust has a superior limit for its radius, depending on the constant volumetric energy density \( \rho_0 \).

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1. Introduction

The Levi-Civita metric [5] is the most general cylindrical static vacuum metric. It will be used as the exterior spacetime of static cylindrical sources.

Some non-vacuum exact solutions with cylindrical symmetry may be found in the literature. One very simple solution is the cluster of particles. This source is constituted by a great number of small gravitational particles which move freely under the influence of the field produced by all of them together. The first model of cluster was presented by Einstein [6] in spherical symmetry. Raychaudhuri and Som [7], based on the Einstein’s ideas, obtained
the first cylindrical analog case. They considered the source filled with dust with an equal number of particles moving in clockwise and anticlockwise directions and found some special cylindrical solutions. They also deduced a relation between the gravitational mass and the sum of the free particles masses of their clusters. Another cluster solution was obtained by Teixeira and Som [1]. They suposed a constant angular velocity for the dust particles. From the Teixeira and Som solution, Lathrop and Orsene [4] found an upper limit for the linear mass density of the

In 1969 Gautreau and Hoffman [9] demonstrated that there is no timelike circular geodesic in the Levi-Civita metric if \( \sigma \geq \frac{1}{4} \). Based on this, Bonnor and Martins [3] conjectured that the Levi-Civita metric does not represent an infinite line mass if \( \sigma \geq \frac{1}{4} \). Later, Bonnor and Davidson [2] presented a cylindrical source, filled with perfect fluid, and showed that the matching of this source with the Levi-Civita metric permits values of \( \sigma > \frac{1}{4} \), but \( < \frac{1}{2} \). In another work, Stela and Kramer [10] found a source with \( \sigma \approx 0.35 \) using a numerical interior solution.

Inspired by the Teixeira and Som solution, and imposing the constance of the energy density of the source, instead of the constance of the angular velocity, we found an exact solution for an homogeneous cylindrical cluster. For both these clusters we studied the junction conditions to the exterior static vacuum Levi-Civita spacetime. We found a superior limit for the linear energy density \( \sigma \) of the source equal \( \frac{1}{4} \) for the first case and \( \frac{1}{2} \) for the second one. The limit obtained for the Teixeira and Som cluster is in accordance with the limit found by Lathrop and Orsene [4]. The range of \( \sigma \) for the homogeneous cluster solution, extends the range found by Bonnor and Davidson [2], for a perfect fluid. Our solution allows \( \sigma = \frac{1}{2} \).
The paper is organized as follows. In the next section we present the static cylindrical spacetime in general. In the section three we give the notation and the spacetime at the exterior of the boundary of the source. In the fourth section we describe the interior spacetime. This section is subdivided into two subsections, where we present a new cylindrical cluster solution for homogeneous dust, found by us, and the Teixeira and Som [1] cluster solution. The junction conditions of these clusters to the Levi-Civita spacetime are also included in this section. In the conclusion we sum up our main results. In the appendix we analyse the symmetries of the Levi-Civita spacetime using the Cartan scalars approach.

2. Spacetime

The spacetime is described by the general cylindrically symmetric static metric

$$ds^2 = -f dt^2 + e^{\mu}(dr^2 + dz^2) + ld\varphi^2,$$

(2.1)

where $f, \mu$ and $l$ are functions only of $r$. The ranges of the coordinates $t, z$ and $\varphi$ are

$$-\infty < t < \infty, \quad -\infty < z < \infty, \quad 0 \leq \varphi \leq 2\pi,$$

(2.2)

and the hypersurface $\varphi = 0$ and $\varphi = 2\pi$ being identified. The coordinates are numbered

$$x^0 = t, \quad x^1 = r, \quad x^2 = z, \quad x^3 = \varphi.$$

(2.3)

We shall impose the Einstein’s field equations

$$R_{\mu \nu} = k \left( T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \right),$$

(2.4)
The non zero components of $R_{\mu \nu}$ for the metric (2.1) are

$$2e^\mu D R^0_0 = \left( \frac{lf'}{D} \right)',$$

(2.5)

$$2e^\mu D R^3_3 = \left( \frac{fl'}{D} \right)',$$

(2.6)

$$2R_{11} = -\mu'' + \mu \frac{D'}{D} - 2 \frac{D''}{D} + \frac{f'l'}{D^2},$$

(2.7)

$$2R_{22} = -\mu'' - \mu \frac{D'}{D},$$

(2.8)

where the primes stand for differentiation with respect to $r$ and $D^2 = fl.$

(2.9)

For the line element (2.1) we found that the circular geodesics equations are

$$\dot{r} = \dot{z} = 0,$$

(2.10)

$$l' \dot{\varphi}^2 - f' \dot{t}^2 = 0,$$

(2.11)

$$\ddot{t} = 0,$$

(2.12)

$$\ddot{\varphi} = 0,$$

(2.13)

$$\left( \frac{ds}{dt} \right)^2 = l \omega^2 - f,$$

(2.14)

where the dot stands for differentiation with respect to $s$, and the angular velocity of the particle $\omega = (dx^3/ds) (dx^0/ds)^{-1}$, is

$$\omega^2 = \left( \frac{d\varphi}{dt} \right)^2 = \frac{f'}{l}.$$

(2.15)
For a stationary spacetime the normal velocity of the particle defined as the change in the displacement normal to $\tau^{\mu} = \{1, 0, 0, 0\}$ relative to its displacement parallel to $\tau^{\mu}$, where $\tau^{\mu}$ is a timelike Killing vector, is [11]

$$W^\mu \overset{\text{def}}{=} \left[ \sqrt{-g_{00}} \left( dx^0 + \frac{g_{0a}}{g_{00}} dx^a \right) \right]^{-1} V^\mu, \quad (2.16)$$

where

$$V^{\mu} \overset{\text{def}}{=} \left(-\frac{g_{0a}}{g_{00}} dx^a, dx^1, dx^2, dx^3 \right)$$

Latin indexes range from 1 to 3. So, for the static metric (2.1), where $g_{0a} = 0$, the three velocity associated to the particle, defined by $W^2 = W^{\mu} W_\mu$, is

$$W^2 = \frac{l}{f} \omega^2 = \frac{lf'}{f^2}. \quad (2.17)$$

Considering equations (2.15) and (2.17) the geodesic equation (2.14) can be written as

$$\left( \frac{ds}{dt} \right)^2 = (W^2 - 1)f. \quad (2.18)$$

The above equation shows that circular geodesics are timelike, null or spacelike for respectively $W < 1$, $W = 1$ and $W > 1$.

The spacetime is divided into two regions: the interior, with $0 \leq r \leq R$, to a cylindrical surface of radius $R$ centered along $z$; and the exterior, with $R \leq r < \infty$. On the boundary surface $r = R$ the first and second fundamental forms have to be continuous [12]. Choosing the same coordinates for the exterior and interior spacetimes these conditions become

$$[g_{\mu\nu} - g^+_{\mu\nu}]_\Sigma = 0, \quad (2.19)$$

$$[g^\cdot_{\mu\nu} - g^+_{\mu\nu}]_\Sigma = 0, \quad (2.20)$$

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where the indexes $-$ and $+$ stand for the interior and exterior spacetimes respectively.

3. Exterior spacetime

In this section we present the exterior spacetime and discuss the symmetries.

The exterior spacetime is filled with vacuum, hence Einstein’s equations (2.4) reduce to $R_{\mu\nu} = 0$. The general solution for (2.5)-(2.8) is the static Levi-Civita metric,

$$f = ar^{4\sigma}, \quad e^{\mu} = \frac{1}{a}r^{4\sigma(2\sigma-1)}, \quad l = \frac{1}{a}r^{2(1-2\sigma)},$$

(3.1)

where $a$ and $\sigma$ are constants. The parameter $a$ is associated with the angular defect [13] while the parameter $\sigma$ can be interpreted as the linear energy density of the source [13,14].

We shall now study some properties of the circular geodesics in this spacetime. A test particle in circular geodesics has angular velocity (2.15) and three velocity (2.17) given by

$$\omega^2 = \frac{2\sigma}{1 - 2\sigma}ar^{2(4\sigma - 1)},$$

(3.2)

$$W^2 = \frac{2\sigma}{1 - 2\sigma}.$$  

(3.3)

From equation (2.14) we see that

$$\left(\frac{ds}{dt}\right)^2 = -\frac{1 - 4\sigma}{1 - 2\sigma}ar^{4\sigma}.$$  

(3.4)

Thus, circular geodesics are timelike, null or spacelike for respectively $0 \leq \sigma < \frac{1}{4}$, $\sigma = \frac{1}{4}$ and $\frac{1}{4} < \sigma \leq \frac{1}{2}$. The limit $\sigma = \frac{1}{2}$ implies $W \to \infty$.

4. Interior spacetime
The interior spacetime is described by a cylinder filled with a rotationally symmetric cluster of dust with zero net angular momentum. The energy momentum tensor is

\[ T_{\mu \nu} = \frac{1}{2} \rho (u^\mu u_\nu + v^\mu v_\nu), \quad (4.1) \]

where \( \rho \) is the energy density and \( u^\mu \) and \( v^\mu \) are the four velocities

\[ u^\mu = (u^0, 0, 0, \omega), \quad v^\mu = (u^0, 0, 0, -\omega), \quad (4.2) \]

satisfying

\[ u^\mu u_\mu = v^\mu v_\mu = -1. \quad (4.3) \]

From Einstein’s equations (2.4) and from (2.5)-(2.8) and (4.1) we obtain

\[ f l = r^2, \quad (4.4) \]

\[ \mu' = -\frac{f'}{f} \left( 1 - \frac{rf'}{2f} \right), \quad (4.5) \]

\[ k \rho e^{\mu'} = -(r \mu')', \quad (4.6) \]

\[ W^2 = \left( \frac{\omega r}{f} \right)^2 = \frac{rf'}{2f} \frac{1}{1 - \frac{rf'}{2f}}, \quad (4.7) \]

where we have computed in (4.7) the three velocity (2.17) of a particle in the cluster. Note that, although the particles of the cluster are rotating, this source, with null net angular momentum, generates a static spacetime.

**4.1. Cluster of homogeneous dust**

Considering a homogeneous distribution of dust inside the cylinder \( 0 \leq r \leq R \) we have

\[ \rho = \rho_0 = \text{constant}. \quad (4.1.1) \]
The solution of (4.5)-(4.6) with (4.1.1) suitable for a matching to the Levi-Civita metric is

\[ f = \frac{1}{\sqrt{2}} \left( 1 - 3br^2 + \sqrt{(1 + br^2)(1 - 7br^2)} \right)^{\frac{1}{2}} \times \exp \left[ \frac{7}{2\sqrt{7}} \left( \arcsin \left[ -\frac{(3 + 7br^2)}{4} \right] - \arcsin \left[ -\frac{3}{4} \right] \right) \right], \quad (4.1.2) \]

\[ e^\mu = (1 + br^2)^{-2}, \quad (4.1.3) \]

where we imposed that the geometry is euclidian on the rotation axis and \( b \stackrel{\text{def}}{=} k\rho_0/8 \). From (4.7) and (4.1.2) we have for the three velocity of the dust particles

\[ W^2 = \frac{\sqrt{1 + br^2} - \sqrt{1 - 7br^2}}{\sqrt{1 + br^2} + \sqrt{1 - 7br^2}}. \quad (4.1.4) \]

From equation (4.1.2) we can see that there is a restriction on the radius of the orbit of the homogeneous cluster particles, for a given gravitational mass per unit length of the cylinder. More specifically, we should have that

\[ r^2 \leq \frac{8}{7k\rho_0}, \quad (4.1.5) \]

in order that the metric potential \( f \) could be real. Equation (4.1.4) shows that in this limit the three velocity of the dust particles is equal to the unit. A similar behaviour is also found in the van Stockum solution [15,16].

Considering the matching between the interior and exterior spacetimes, given by (2.19) and (2.20), we obtain

\[ \sigma = \frac{1}{4} \left( 1 \pm \sqrt{\frac{1 - 7bR^2}{1 + bR^2}} \right), \quad (4.1.6) \]

\[ a = \frac{b^2 R^{4(2\sigma^2 - \sigma + 1)}}{\sigma^2(2\sigma - 1)^2} = (1 + bR^2)^2 R^{-\frac{4bR^2}{1 + bR^2}}. \quad (4.1.7) \]

Equation (4.1.6) imposes a superior limit on the radius of the source’s boundary, independently of the choice of the sign before the square root. This limit is identical to equation
Furthermore, (4.1.5) and (4.1.6) implies that there are two possible ranges for \( \sigma \), depending on the choice of the sign before the square root. If we choose the negative sign we have \( 0 \leq \sigma \leq \frac{1}{4} \), with \( \rho_0 = 0 \) corresponding to \( \sigma = 0 \), while if we choose the positive sign, then \( \frac{1}{4} \leq \sigma \leq \frac{1}{2} \), with \( \rho_0 = 0 \) corresponding to \( \sigma = \frac{1}{2} \). The value \( \sigma = \frac{1}{4} \) is included in both solutions and corresponds to \( W = 1 \) from equation (4.1.4). Observe that the limits imposed by equations (3.3) and (3.4) are related with the existence of circular geodesics for test particles in the Levi-Civita spacetime, while the above limits arise from the junction conditions. Although the junction conditions (4.1.6) and (4.1.7) do not impose any restriction on the value \( \sigma = \frac{1}{4} \), equation (4.1.4), as we had seen, shows that when \( \sigma = \frac{1}{4} \), which means \( R^2 = 8/(7k\rho_0) \), the three velocity of the particles is equal to the unit. So, the partic

\textbf{4.2. Cluster of constant rotating dust}

We consider now the Teixeira and Som solution where

\[ \omega = \omega_0 = \text{constant}, \tag{4.2.1} \]

given by

\[ f = \frac{1}{2} \left[ 1 + (1 + 4\omega_0^2 r^2)^{\frac{1}{2}} \right], \tag{4.2.2} \]

\[ e^\mu = (1 + 4\omega_0^2 r^2)^{-\frac{1}{2}}, \tag{4.2.3} \]

\[ 2\pi \rho = \frac{\omega_0^2}{(1 + 4\omega_0^2 r^2)^{\frac{3}{2}}}, \tag{4.2.4} \]

\[ W^2 = \frac{4\omega_0^2 r^2}{(1 + \sqrt{1 + 4\omega_0^2 r^2})^2}. \tag{4.2.5} \]

Now considering the matching (2.19)-(2.20) we obtain

\[ \sigma = \frac{1}{4} \left( 1 - \frac{1}{\sqrt{1 + 4\omega_0^2 R^2}} \right), \tag{4.2.6} \]
$$a = R^{4\sigma(2\sigma-1)}(1 + 4\omega_0^2 R^2)^{1/4}. \quad (4.2.7)$$

We shall now state some properties of this matching. We can see that if $\sigma = 0$ equations (4.2.6)-(4.2.7) give $R\omega_0 = 0$ and $a = 1$ producing the Minkowski spacetime as expected. Note that the solution is well behaved for $R = 0$ since this implies $\sigma = 0$ (i.e., finite density energy per unit length of the infinite line mass). On the other hand, even when $R = 0$, from equation (4.2.4), the density energy $\rho$ of the source vanishes only if we have $\omega_0 = 0$.

Equation (4.2.6) shows that

$$\lim_{R \to \infty} \sigma = \frac{1}{4}, \quad (4.2.8)$$

and

$$\lim_{R \to \infty} \omega_0 = 1. \quad (4.2.9)$$

So, $\frac{1}{4}$ represents a superior limit for $\sigma$ in order that the interior solution generates an exterior Levi-Civita spacetime. Lathrop and Orsene [4] found the same limit for the linear mass density of the Teixeira and Som cluster solution. They used a definition for this parameter given by Vishveshwara and Winicour [8]. This source, nevertheless, do not present a limitation on the radius of its boundary. As in the case of the homogeneous cluster, here the value $\frac{1}{4}$ for the parameter $\sigma$ implies that the particles of the cluster are travelling with the speed of light, as can be seen from the equation (4.2.5). So, we again need to avoid the value $\sigma = \frac{1}{4}$ or otherwise consider the possibility of counter-rotating photons.

5. Conclusion

We found the exact solutions of Einstein’s field equations for an homogeneous cylinder constituted by an equal number of particles of dust moving in clockwise and anticlockwise directions.
The matching of this source with the static vacuum of Levi-Civita is allowed only for a specific range of the linear energy density parameter, that is $0 \leq \sigma \leq \frac{1}{4}$. This extends the range of $\sigma$ found by Bonnor and Davidson [2] for a perfect fluid source. We also found that, for a given volumetric energy density, this source presents an upper limit for its radius. In the literature there is, at least, the van Stockum solution as another cylindrical example in which there is a limit for the radius of the source depending on its volumetric density [15,16]. In this case, the limitation on the radius comes out in order to avoid a change in the signature of the metric.

Using the solution for a cluster constituted by dust particles with constant angular velocity and zero net angular momentum, obtained firstly by Teixeira and Som [1], and matching it with the Levi-Civita metric, we found that the parameter $\sigma$ in this case should be smaller or equal than $\frac{1}{4}$. Considering $\sigma$ as the gravitational mass per unit length this limit is in agreement with the result of Lathrop and Orsene [4]. While for the Teixeira and Som solution the matching does not present a limitation to the radius of the source, for our homogeneous cluster solution the matching imposes a superior limit for its radius, depending on the volumetric energy density $\rho_0$.

We note from equation (3.4) that circular geodesics in the Levi-Civita spacetime become null when $\sigma$ is equal $\frac{1}{4}$. Some authors [Gautreau & Hoffmann [3,9] use this fact as a restriction for the linear density of the source. Nevertheless, as pointed out by [7], this result is similar to the Newtonian cylindrical analog case, i.e., for a higher density cylinder, all particles (with speed less than the light) should fall. However this argument is not without problems, since, as showed by Bonnor and Martins [3], in the interval $\frac{1}{4} \leq \sigma \leq \frac{1}{2}$ the gravitational field seems to get weaker as $\sigma$ increases [17]. Note that $\sigma = \frac{1}{2}$ means that there is no any matter inside the cylinder ($\rho_0 = 0$ and the spacetime is locally flat, in accordance to the Cartan scalars).
From equation (4.1.6) we can see that when we choose the positive sign, the parameter $\sigma$ grows up while the volumetric density $\rho_0$ decreases.

The analysis of the Cartan scalars, in the appendix, summarizes the symmetry properties of the Levi-Civita metric. It is in general a Petrov type I metric and becomes a Petrov type D metric when $\sigma = -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}$ and $1$. For $\sigma = 0$ or $\frac{1}{2}$ the metric becomes flat, which is in accordance with our cluster solution since when $\sigma = 0$ or $\frac{1}{2}$ the volumetric energy density $\rho_0$ of the cluster vanishes. For $\sigma = -\frac{1}{2}, \frac{1}{4}, 1$ the metric gets one extra symmetry.

Appendix

It is known [18] that the so called 14 algebraic invariants (and even all the polinomial invariants of any order) are not sufficient for locally characterizing a spacetime, in the sense that two metrics may have the same set of invariants and be not equivalent. As an example, all these invariants vanish for both Minkowski and plane-wave [18,19] spacetimes and they are not the same. A complete local characterization of spacetimes may be done by the Cartan scalars. Briefly, the Cartan scalars are the components of the Riemann tensor and its covariant derivatives, up to possibly the $10^{th}$ order, calculated in a constant frame. Nevertheless, for a complete characterization of a spacetime, up to the $3^{rd}$ order is usually sufficient. For a review, see [13] and references therein. In practice, the Cartan scalars are calculated using the spinorial formalism. For the purpose here, the relevant quantities are the Weyl spinor $\Psi_A$, and its first and second covariant symmetrized derivatives $\nabla \Psi_{AW'}$ and $\nabla^2 \Psi_{AW'}$, which represent the Weyl tensor and its covariant derivatives. It should be stressed that, although the Cartan scalars provide a local characterization of the spacetime, global properties such as topological defects do not probably appear in them.
The Cartan scalars for the Levi-Civita metric coincide with those for the Weyl class of the Lewis metric and were given in [13]. Using the null frame

\[
\omega^0 = \frac{1}{\sqrt{2}}(\theta^0 + \theta^1), \quad \omega^1 = \frac{1}{\sqrt{2}}(\theta^0 - \theta^1) \\
\omega^2 = \frac{1}{\sqrt{2}}(\theta^2 + i\theta^3), \quad \omega^3 = \frac{1}{\sqrt{2}}(\theta^2 - i\theta^3).
\]

where \(\theta^A\) is an orthonormal frame given by:

\[
\theta^0 = r^{2\sigma}dt, \quad \theta^1 = \frac{1}{a}r^{(1-2\sigma)}d\varphi, \quad \theta^2 = r^{(4\sigma^2-2\sigma)}dr, \quad \theta^3 = r^{(4\sigma^2-2\sigma)}dz,
\]

(A.1)

the non-vanishing Cartan scalars become

\[
\Psi_2 = -(2\sigma - 1)\sigma r^{4\sigma-8\sigma^2-2} \\
\Psi_4 = \Psi_0 = (4\sigma - 1)\Psi_2 \\
\nabla\Psi_{01'} = \nabla\Psi_{50'} = \sqrt{2}(8\sigma^2 - 4\sigma + 1)(4\sigma - 1)(2\sigma - 1)\sigma r^{6\sigma-12\sigma^2-3} \\
\nabla\Psi_{10'} = \nabla\Psi_{41'} = \sqrt{2}(4\sigma - 1)(2\sigma - 1)\sigma r^{6\sigma-12\sigma^2-3} \\
\nabla\Psi_{21'} = \nabla\Psi_{30'} = \sqrt{2}(4\sigma^2 - 2\sigma + 1)(2\sigma - 1)\sigma r^{6\sigma-12\sigma^2-3}.
\]

(A.3)

In order to discuss the symmetries, we list some quantities which are important in the Petrov classification [20]

\[
D = 64(4\sigma - 1)^2(2\sigma + 1)^2(\sigma - 1)^2\Psi_2^6 \\
I = ((4\sigma - 1)^2 + 3)\Psi_2^2, \\
J = ((4\sigma - 1)^2 - 1)\Psi_2^3, \\
G = 0, \\
N \equiv \Psi_0 I - 12H^2 = (4\sigma - 1)^2((4\sigma - 1)^2 - 9)\Psi_2^4.
\]

(A.4) (A.5) (A.6) (A.7) (A.8)

The metric is Petrov type I unless \(D = 0\). This can happen if and only if \(\sigma = -1/2, 0, 1/4, 1/2\) or 1. For \(\sigma = 0\) or \(1/2\), we have from (A.3) that \(\Psi_2 = 0\) and therefore the metric is flat.
For $\sigma = 1/4$, we have from (A.3) that $\Psi_0 = \Psi_4 = 0$ and therefore the metric is Petrov type D. For $\sigma = -1/2$ or 1, we see that $I$ and $J$ are different from zero and that $G = N = 0$, which also characterize Petrov type D metrics.

As it is well known, the Minkowski spacetime has the Poincaré group as its isometry group and the Lorentz group as its isotropy subgroup. For a general spacetime, the isometry and isotropy groups are subgroups of the Poincaré and Lorentz groups respectively. The Cartan scalars (A.3) do not depend on $t$, $\varphi$ and $z$, showing that the isometry group is at least 3 dimensional. Concerning the isotropies, it is known [21] that Petrov type I, II or III metrics have none, Petrov type D and N may have up to 2 and Petrov type 0 metrics may have all 6 isotropies. Therefore, the Levi-Civita metric has in general no isotropies. Isotropies may arise for $\sigma = -1/2, 0, 1/4, 1/2$ or 1. In fact, for $\sigma = 0$ or $1/2$, there are 6 isotropies, since the metric becomes flat and for $\sigma = -1/2, 1/4$ or 1, i.e., the Petrov type D cases, we shall show now that the metric gets an extra isotropy and therefore the spacetime has more symmetries than the original cylindrical symmetry.

The case $\sigma = 1/4$ can be easily analysed, since the Cartan scalars will be still in a canonical basis [22]. In fact, from (A.3) we get that the non-zero Cartan scalars become

$$\Psi_2 = 1/8r^{-3/2},$$
$$\nabla^2_2\Psi_2 = \nabla^2\Psi_3 = -3\sqrt{2}/(32)r^{-9/4},$$

(A.9)

$$\nabla^2_2\Psi_4 = \nabla^2\Psi_4 = 3/(16)r^{-3}, \quad \nabla^2\Psi_3 = 27/(128)r^{-3}.$$  

Actually, $\nabla^2\Psi_{AW'}$ does not come from (A.3) since in that case the second derivative was not necessary. Therefore, the second derivative was, here, calculated directly from the tetrad frame (using SHEEP and CLASSI [23,24]). The isotropy group corresponding to this set of Cartan scalars is a one-parameter group of boosts in the $\omega^0 - \omega^1$ plane which, from (A.1) and (A.2), involves the $t$ and $\varphi$ coordinate axis.
The two other cases are more complicated since the Cartan scalars will not be in a canonical basis for Petrov type D. The new canonical basis and the Cartan scalars in these cases are, for $\sigma = 1$,

$$
\begin{align*}
\omega^0 &= \frac{\sqrt{2}}{4} r^2 dt + \frac{\sqrt{2}}{4} r^2 dz, \\
\omega^1 &= \frac{\sqrt{2}}{4} r^2 dt - \sqrt{2} r^2 dz, \\
\omega^2 &= \frac{1}{\sqrt{2}} i a^{-1/2}r^{-1} d\varphi - \frac{1}{\sqrt{2}} r^2 dr, \\
\omega^3 &= -\frac{1}{\sqrt{2}} i a^{-1/2}r^{-1} d\varphi - \frac{1}{\sqrt{2}} r^2 dr,
\end{align*}
$$

$$
\Psi_2 = 2r^{-6}, \quad \nabla \Psi_{21'} = \nabla \Psi_{30'} = 6\sqrt{2} r^{-9},
$$

$$
\nabla^2 \Psi_{22'} = \nabla^2 \Psi_{40'} = 48r^{-12}, \quad \nabla^2 \Psi_{31'} = 54r^{-12},
$$

and, for $\sigma = -1/2$,

$$
\begin{align*}
\omega^0 &= \frac{1}{\sqrt{2}} r^{-1} dt + \frac{1}{\sqrt{2}} r^2 dr, \\
\omega^1 &= \frac{1}{\sqrt{2}} r^{-1} dt - \frac{1}{\sqrt{2}} r^2 dr, \\
\omega^2 &= -\frac{1}{\sqrt{2}} a^{-1/2}r^2 d\varphi + \frac{1}{\sqrt{2}} ir^2 dz, \\
\omega^3 &= -\frac{1}{\sqrt{2}} a^{-1/2}r^2 d\varphi - \frac{1}{\sqrt{2}} ir^2 dz,
\end{align*}
$$

$$
\Psi_2 = 2r^{-6}, \quad \nabla \Psi_{20'} = -\nabla \Psi_{31'} = -6\sqrt{2} r^{-9},
$$

$$
\nabla^2 \Psi_{20'} = \nabla^2 \Psi_{42'} = 48r^{-12}, \quad \nabla^2 \Psi_{31'} = -54r^{-12}.
$$

The isotropy group corresponding to the set of Cartan scalars for $\sigma = 1$ is the one-parameter group of boosts in the $\omega^0 - \omega^1$ plane which, from (A.10), involves the $t$ and $z$ coordinate axis and for $\sigma = -1/2$ it is the one-parameter group of spatial rotations in the $\omega^2 - \omega^3$ plane which, from (A.11), involves the $\varphi$ and $z$ coordinate axis (this last case is also discussed by [25]).

The non zero algebraic invariants of the Riemann tensor for (3.1) are

$$
R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 64\sigma^2 (2\sigma - 1)^2 (4\sigma^2 - 2\sigma + 1)r^{-4(4\sigma^2 - 2\sigma + 1)}, \quad (A.12)
$$
\[ R_{\alpha\beta\gamma\delta} R^{\gamma\delta\mu\nu} R_{\mu\nu}^{\alpha\beta} = 768\sigma^4(2\sigma - 1)^4 r^{-6(4\sigma^2-2\sigma+1)}. \]  \hspace{1cm} (A.13)

According to the Cartan scalars and to (A.12) and (A.13) the metric (3.1) has infinite curvature only at \( r = 0 \) for all \( \sigma \) except \( \sigma = 0 \) and \( \frac{1}{2} \) where the spacetime is flat.

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