REAL-ANALYTICITY OF HAUSDORFF DIMENSION OF JULIA SETS ALONG THE PARABOLIC ARCS OF THE MULTICORNS

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Abstract. In this article, we take a dimension-theoretic look at the connectedness loci of unicritical anti-polynomials, known as the multicorns and prove a regularity property of the Hausdorff dimension of the Julia sets on the persistently parabolic part of these parameter spaces. The boundaries of the odd period hyperbolic components of the multicorns contain real-analytic arcs consisting of quasi-conformally conjugate parabolic parameters. The principal result of this paper asserts that the Hausdorff dimension of the Julia sets is a real-analytic function of the parameter along these parabolic arcs. We also prove, along the way, that the dynamically natural parametrization of the parabolic arcs has a non-vanishing derivative at all but (possibly) finitely many points.

Our main result remains true for more general parabolic loci provided all the active critical points converge to attracting or parabolic cycles.

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1. Introduction

The multicorns are the connectedness loci of unicritical antiholomorphic polynomials. Any unicritical anti-polynomial, up to an affine change of coordinates, can be written in the form \( f_c(z) = \bar{z}^d + c \), for some \( c \in \mathbb{C} \) and \( d \geq 2 \). In analogy to the holomorphic case, the set of all points which remain bounded under all iterations of \( f_c \) is called the filled-in Julia set \( K(f_c) \). The boundary of the filled-in Julia set is defined to be the Julia set \( J(f_c) \) and the complement of the Julia set is defined to be its Fatou set \( F(f_c) \). This leads, as in the holomorphic case, to the notion of connectedness locus of degree \( d \) unicritical anti-polynomials:

Definition. The multicorn of degree \( d \) is defined as \( \mathcal{M}_d^* = \{ c \in \mathbb{C} : K(f_c) \text{ is connected} \} \).

It follows from classical works of Bowen and Ruelle [Ru] [Zi] that the Hausdorff dimension of the Julia set depends real-analytically on the parameter within every
hyperbolic component of $\mathcal{M}^*_d$. Ruelle’s proof makes essential use of the fact that hyperbolic rational maps are expanding (this allows one to use the full machinery of thermodynamic formalism) and the Julia sets of hyperbolic rational maps move holomorphically inside every hyperbolic component. The boundary of every hyperbolic component of odd period of $\mathcal{M}^*_d$ is a simple closed curve consisting of exactly $d+1$ parabolic cusp points as well as $d+1$ parabolic arcs, each connecting two parabolic cusps, and any two parameters on a given parabolic arc have quasi-conformally conjugate dynamics [MNS]. Since parabolic maps have a certain weak expansion property and since they are structurally stable in the parabolic locus, the following theorem can be naturally thought of as a version of Ruelle’s theorem on the boundaries of hyperbolic components:

**Theorem 1.1** (Real-analyticity of HD Along Parabolic Arcs). Let $C$ be a parabolic arc of $\mathcal{M}^*_d$ and let $c : \mathbb{R} \to C, h \mapsto c(h)$ be its critical Ecalle height parametrization. Then the function

$$\mathbb{R} \ni h \mapsto \text{HD}(J(f_{c(h)}))$$

is real-analytic.

![Figure 1. $\mathcal{M}_2^*$, also known as the tricorn and the parabolic arcs on the boundary of the hyperbolic component of period 1 (in blue)](image)

The proof of this theorem is carried out in two steps. At first, we embed the parabolic arcs (which are real one-dimensional curves) in a complex one-dimensional family of quasiconformally conjugate parabolic maps. This is performed in Section 3 by varying the Ecalle height over a bi-infinite strip by a q.c. deformation argument. Subsequently, in Section 4, we recall some basic facts from dimension theory and thermodynamic formalism, and apply the results on real-analyticity of Hausdorff dimension of Julia sets of analytic families of meromorphic functions, as developed in [SU], to our setting. It will transpire from the course of the proof that in good situations, the real-analyticity of Hausdorff dimension holds more generally
on certain regions of the parabolic curves \( \text{Per}_n(1) \) (see [Mi1] for the definition of the \( \text{Per} \) curves).

As a by-product of the q.c. deformation step, we prove that the Ecalle height parametrization of the parabolic arcs of the multicorns is non-singular at all but possibly finitely many points.

**Theorem 1.2.** Let \( C \) be a parabolic arc of odd period of \( \mathcal{M}_d^* \) and \( c : \mathbb{R} \rightarrow C \) be its critical Ecalle height parametrization. Then, there exists a holomorphic map \( \varphi : \{ w = u + iv \in \mathbb{C} : |v| < \frac{1}{2} \} \rightarrow \mathbb{C} \) such that:

1. The map \( \varphi \) agrees with the map \( c \) on \( \mathbb{R} \).
2. For all but possibly finitely many \( x \in \mathbb{R} \), \( c'(x) = \varphi'(x) \neq 0 \).

In particular, the critical Ecalle height parametrization of any parabolic arc of \( \mathcal{M}_d^* \) has a non-vanishing derivative at all but possibly finitely many points.

It is worth mentioning that the present paper adds one more item to the list of topological differences between the multicorns and their holomorphic counterparts, the multibrot sets (these are the connectedness loci of unicritical holomorphic polynomials \( z^d + c \)). Clearly, the parameter dependence of the Hausdorff dimension of the Julia sets is far from regular on the boundary of the Mandelbrot set. It has been recently proved [HS] that the multicorns are not locally connected, while the local connectivity of the Mandelbrot set is one of the most prominent conjectures in one dimensional complex dynamics. In another recent work [IM], we proved that rational parameter rays at odd-periodic angles of the multicorns do not land, rather they accumulate on an arc of positive length in the parameter space. This is in stark contrast with the fact that every rational parameter ray of the multibrot sets land at a unique parameter. In [IM], we also showed that the centers of the hyperbolic components of the multicorns are not equidistributed with respect to the harmonic measure. More such topological differences between the multibrot sets and the multicorns, including the bifurcation along arcs, the existence of real-analytic arcs of quasi-conformally equivalent parabolic parameters, the discontinuity of landing points of dynamical rays, can be found in [MNS].

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2. **Anti-holomorphic Fatou Coordinates, Equators and Ecalle heights**

In this section, we recall some basic facts about the parameter spaces of unicritical anti-polynomials. One of the major differences between the multicorns and the multibrot sets is that the boundaries of odd period hyperbolic components of the multicorns consist only of parabolic parameters.

**Lemma 2.1** (Indifferent Dynamics of Odd Period). *The boundary of a hyperbolic component of odd period \( k \) consists entirely of parameters having a parabolic orbit of exact period \( k \). In local conformal coordinates, the \( 2k \)-th iterate of such a map has the form \( z \mapsto z + z^{q+1} + \ldots \) with \( q \in \{1, 2\} \).*

*Proof.* See [MNS, Lemma 2.9]. \( \square \)

This leads to the following classification of odd periodic parabolic points.
Definition (Parabolic Cusps). A parameter $c$ will be called a cusp point if it has a parabolic periodic point of odd period such that $q = 2$ in the previous lemma. Otherwise, it is called a simple parabolic parameter.

In holomorphic dynamics, the local dynamics in attracting petals of parabolic periodic points is well-understood: there is a local coordinate $ζ$ which conjugates the first-return dynamics to the form $ζ \mapsto ζ + 1$ in a right half place (see Milnor [Mi, Section 10]). Such a coordinate $ζ$ is called a Fatou coordinate. Thus the quotient of the petal by the dynamics is isomorphic to a bi-infinite cylinder, called an Ecalle cylinder. Note that Fatou coordinates are uniquely determined up to addition by a complex constant.

In anti-holomorphic dynamics, the situation is at the same time restricted and richer. Indifferent dynamics of odd period is always parabolic because for an indifferent periodic point of odd period $k$, the $2k$-th iterate is holomorphic with positive real multiplier, hence parabolic as described above. On the other hand, additional structure is given by the anti-holomorphic intermediate iterate.

Lemma 2.2. Suppose $z_0$ is a parabolic periodic point of odd period $k$ of $f_c$ with only one petal (i.e. $c$ is not a cusp) and $U$ is a periodic Fatou component with $z_0 \in \partial U$. Then there is an open subset $V \subset U$ with $z_0 \in \partial V$ and $f_c^{2k}(V) \subset V$ so that for every $z \in U$, there is an $n \in \mathbb{N}$ with $f_c^{2nk}(z) \in V$. Moreover, there is a univalent map $Φ: V \to \mathbb{C}$ with $Φ(f_c^{2k}(z)) = Φ(z) + 1/2$, and $Φ(V)$ contains a right half plane. This map $Φ$ is unique up to horizontal translation.

Proof. See [HS, Lemma 2.3].

The map $Φ$ will be called an anti-holomorphic Fatou coordinate for the petal $V$. The anti-holomorphic iterate interchanges both ends of the Ecalle cylinder, so it must fix one horizontal line around this cylinder (the equator). The change of coordinate has been so chosen that the equator maps to the real axis. We will call the vertical Fatou coordinate the Ecalle height. Its origin is the equator. The existence of this distinguished real line, or equivalently an intrinsic meaning to Ecalle height, is specific to anti-holomorphic maps.

The Ecalle height of the critical value plays a special role in anti-holomorphic dynamics. The next theorem proves the existence of real-analytic arcs of non-cusp parabolic parameters on the boundaries of odd period hyperbolic components of the multicorns.

Theorem 2.3 (Parabolic arcs). Let $c_0$ be a parameter such that $f_{c_0}$ has a parabolic orbit of odd period and suppose that $c_0$ is not a cusp. Then $c_0$ is on a parabolic arc in the following sense: there exists a real-analytic arc of non-cusp parabolic parameters $c(h)$ (for $h \in \mathbb{R}$) with quasiconformally equivalent but conformally distinct dynamics of which $c_0$ is an interior point and the Ecalle height of the critical value of $f_{c(h)}$ is $h$.

Proof. See [MNS, Theorem 3.2].

The parametrization of a parabolic arc $C$ given in the previous theorem is called the critical Ecalle height parametrization of $C$.

The structure of the hyperbolic components of odd period plays an important role in the global topology of the parameter spaces.
Theorem 2.4 (Boundary Of Odd Period Hyperbolic Components). The boundary of every hyperbolic component of odd period of $M_d$ is a simple closed curve consisting of exactly $d+1$ parabolic cusp points as well as $d+1$ parabolic arcs, each connecting two parabolic cusps.

Proof. See [MNS] Theorem 1.2. □

3. Constructing an Analytic Family of Q.C. Deformations

Throughout this section, we fix a parabolic arc $C$ and the critical Ecalle height parametrization of the parabolic arc $C$ will be denoted by $c : \mathbb{R} \to C$. We will show that the polynomials on $C$ can be quasi-conformally deformed to yield an analytic family of q.c. conjugate maps; in particular, they will be structurally stable on a suitable algebraic curve.

We embed our family $f_c(z) = z^d + c$, $c \in C$ in the family of holomorphic polynomials $F_d = \{ p_{a,b}(z) = (z^2 + a)^d + b, a, b \in \mathbb{C} \}$. Since $f_c^2 = P_{c,c}$, the connectedness locus $C(P_{a,b})$ of this family intersects the slice $\{ a = \bar{b} \}$ in $M_d$.

It will be useful to have the following characterization of the elements of $F_d$ amongst all monic centered polynomials of degree $d^2$.

Lemma 3.1. Let $f$ be a monic centered polynomial of degree $d^2$. Then the following are equivalent:

(1) $f \in F_d$ with $a \neq 0$.

(2) $f$ has exactly $d+1$ distinct critical points $\{ \alpha_1, \alpha_2, \cdots, \alpha_{d+1} \}$ with $\deg_{\alpha_i}(f) = d$ for each $i$ and such that $f(\alpha_1) = f(\alpha_2) = \cdots = f(\alpha_d)$.

Proof. (1) $\implies$ (2). The critical points of any $f \in F_d$ are 0 and the $d$ roots of the equation $z^d + a = 0$. Since $a \neq 0$, these points are all distinct and $f$ has local degree $d$ at each of them. The other property is immediate.

(2) $\implies$ (1). Let, $b = f(\alpha_1) = f(\alpha_2) = \cdots = f(\alpha_d)$. Since each $\alpha_i$ ($i = 1, 2, \cdots, d$) maps in a $d$-to-1 fashion to $b$ and the degree of $f$ is $d^2$, these must be all the pre-images of $b$. Therefore, $f(z) = p(z)^d + b$, where $p(z) = (z - \alpha_1) \cdots (z - \alpha_d)$ (here we’ve used the fact that $f$ is monic). Note that the critical points of $f$ are precisely the zeroes and the critical points of $p$. Since we have used up the $d$ distinct zeroes of $p$ and are left with only one critical point of $f$, it follows that this critical point $\alpha_{d+1}$ must be the only critical point of $p$. Hence, $p$ must be unicritical. A brief computation (using the fact that $f$ is centered) now shows that $p(z) = z^d + a$, for some $a \in \mathbb{C}$. The fact that all the $\alpha_i$’s are distinct tells that $a$ must be non-zero. Thus, we have shown that $f(z) = (z^d + a)^d + b$ with $a \in \mathbb{C}^*, b \in \mathbb{C}$. □

Remark. Condition (2) of the previous lemma is preserved under topological conjugacies.

Since every anti-polynomial on the parabolic arc has a parabolic orbit of exact period $k$ and multiplier 1, the parabolic arc $C$ is contained in the algebraic curve:

$$P_k = \{ (a, b) \in \mathbb{C}^2 : \text{Disc}_z \left( P_{a,b}^k(z) - z \right) = 0 \}$$

Since an affine algebraic curve has at most finitely many singular points, all but finitely points of a parabolic arc are non-singular points of this algebraic set; i.e. the algebraic set is locally a graph near such points. The parabolic arc $C$ is naturally embedded in $\mathbb{C}^2$ by the map $z \mapsto (\bar{z}, z)$. 

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Lemma 3.2 (Extending the deformation). Let $c_0 = c(0) \in \mathcal{C}$. There exists an injective holomorphic map

$$F : \{ w = u + iv \in \mathbb{C} : |v| < \frac{1}{2} \} \to P_k,$$

$$w \mapsto (a(w), b(w))$$

with $F(0) = (c_0, c_0)$ such that for any $u \in \mathbb{R}$, $F(u) = (c(u), c(u))$ where $c : \mathbb{R} \to \mathcal{C}$ is the critical Ecalle height parametrization of the parabolic arc. Further, all the polynomials $P_{a(w), b(w)}$ have q.c.-conjugate (but not conformally conjugate) dynamics.

Proof. We construct a larger class of deformations which strictly contains the deformations constructed in [MNS, Theorem 3.2]. Choose the Fatou coordinate $\Phi_0 : z \mapsto \zeta$ of $f_{c_0}$ such that the unique geodesic invariant under the anti-holomorphic dynamics maps to the real line and the critical value $c_0$ has Ecalle coordinates $(\frac{3}{4}, 0)$ (this is possible since $c_0 = c(0)$). The polynomial $f_{c_0}^{2k} = P_{c_0, c_0}$ has two critical values $c_0$ and $\overline{c_0} + c_0$. The map $f_{c_0}^{2k-1} = P_{c_0, c_0}^{\frac{k-1}{2}}$ sends $\overline{c_0} + c_0$ to the Fatou component containing $c_0$. The Ecalle coordinates of $P_{c_0, c_0}^{\frac{k-1}{2}}(\overline{c_0} + c_0) = f_{c_0}^{2k}(c_0)$ is $(\frac{3}{4}, 0)$. We will change the Ecalle coordinates of the two representatives of the two infinite critical orbits in a fundamental domain in a controlled way, so that each perturbation gives a conformally different polynomial.

We can change the conformal structure within the Ecalle cylinder by the quasi-conformal homeomorphism of $[0,1] \times \mathbb{R}$:

$$L_w : \zeta \mapsto \begin{cases} 
\zeta + 2iwx & \text{if } 0 \leq x \leq 1/4 \\
\zeta + iw(1 - 2x) & \text{if } 1/4 \leq x \leq 1/2 \\
\zeta - iw(2x - 1) & \text{if } 1/2 \leq x \leq 3/4 \\
\zeta - 2iw(1 - x) & \text{if } 3/4 \leq x \leq 1 
\end{cases}$$

for $S = \{ w = u + iv \in \mathbb{C} : |v| < \frac{1}{2} \}$

Translating the map by positive integers, we get a q.c. homeomorphism of a right-half plane that commutes with the translation $z \mapsto z + 1$. By the coordinate change $z \mapsto \zeta$, we can transport this Beltrami form into all the attracting petals, and it is forward invariant under $f_{c_0}^{2k} = P_{c_0, c_0}^{k}$. It is easy to make it backward invariant by pulling it back along the dynamics. Extending it by the zero Beltrami form outside of the entire parabolic basin, we obtain an invariant Beltrami form. Using the Measurable Riemann Mapping Theorem with parameters, we obtain a q.c.-map $\varphi_w$ integrating this Beltrami form. Furthermore, if we normalize $\varphi_w$ such that the conjugated map $\varphi_w \circ P_{c_0, c_0} \circ \varphi_w^{-1}$ is always a monic and centered polynomial, then the coefficients of this newly obtained polynomial will depend holomorphically on $w$ (since the Beltrami form depends complex analytically on $w$).

We need to check that this new polynomial belongs to our family $\mathcal{F}_d$. But this readily follows from Lemma 3.1 and the remark thereafter. Therefore, this new polynomial must be of the form $P_{a(w), b(w)} = (z^d + a(w))^d + b(w)$. Since the coefficients of $P_{a(w), b(w)}$ depend holomorphically on $w$, the maps $w \mapsto a(w)$ and $w \mapsto b(w)$ are holomorphic. Its Fatou coordinate is given by $\Phi_w = L_w \circ \Phi_0 \circ \varphi_w^{-1}$. Note that the new Ecalle coordinates of the two critical values are $L_w(\frac{3}{4}, 0) = (\frac{3}{4} - \frac{3}{2}, \frac{3}{4})$ and $L_w(\frac{1}{4}, 0) = (\frac{1}{4} + \frac{3}{4}, -\frac{3}{4})$. 


Thus, we obtain a complex analytic map $F: S \rightarrow \mathbb{C}^2, w \mapsto (a(w), b(w))$. For real values of $w$, this coincided with deformation constructed in Theorem [MNS] Theorem 3.2] and $F(u) = (c(u), c(u)) \forall u \in \mathbb{R}$.

By construction, all the $P_{a(w), b(w)}$'s are q.c.-conjugate to $P_{c_0, c_0}$ and hence to each other. We now show that they are all conformally distinct. Define the Fatou vector of $P_{a(w), b(w)}$ to be the quantity $\Phi_w \left( P_{a(w), b(w)} \left( (a(w))^d + b(w) \right) \right) - \Phi_w(b(w))$, where $\Phi_w$ is the Fatou coordinate of $P_{a(w), b(w)}$. Since the Fatou coordinate is unique up to addition by a complex constant, the Fatou vector defined above is a conformal invariant. The Fatou vector of $P_{a(w), b(w)}$ is given by $\frac{1}{2} - iw$, which is different for different values of $w$. Hence, the $P_{a(w), b(w)}$'s are conformally nonequivalent. In particular, the map $F$ is injective.

It remains to show that for any $w \in S$, $(a(w), b(w)) \in P_k$. It is easy to see that each $P_{a(w), b(w)}$ has a parabolic orbit of exact period $k$ (both critical orbits are contained in the Fatou set and converge to a $k$-periodic orbit sitting on the Julia set). Also, the first return map $P_{a(w), b(w)}^k$ leaves the unique petal attached to every parabolic point invariant; so the multiplier of the parabolic orbit must be 1. Hence, $(a(w), b(w)) \in P_k$. □

**Figure 2.** $\pi_2 \circ F: w \mapsto b(w)$ is injective in a neighborhood of $\tilde{u}$ for all but possibly finitely many $\tilde{u} \in \mathbb{R}$.

**Proof of Theorem 1.3.** Part (i) follows from Lemma 3.2. The required map $\varphi$ is given by $b$.

We will show that if $(c(\tilde{u}), c(\tilde{u}))$ is a non-singular point of the algebraic curve $P_k$, then the map $\mathbb{R} \ni h \mapsto c(h)$ has a non-vanishing derivative at $\tilde{u}$. Since any
affine algebraic curve has at most finitely many singular points, this will complete the proof of the theorem.

By the definition of non-singularity, one of the partial derivatives \( \frac{\partial P}{\partial a} \) or \( \frac{\partial P}{\partial b} \) is non-zero at \((c(\tilde{u}), c(\bar{u}))\). Let \( \frac{\partial P}{\partial a}(c(\tilde{u}), c(\bar{u})) \neq 0 \). Then there exists \( \varepsilon_1, \varepsilon_2 > 0 \) and a holomorphic map \( g : B(c(\tilde{u}), \varepsilon_1) \to B(c(\bar{u}), \varepsilon_2) \) such that \( g(c(\tilde{u})) = c(\bar{u}) \) and \( P_k(a, b) = 0 \) for some \((a, b) \in B(c_0, \varepsilon_1) \times B(c_{10}, \varepsilon_2) \) if and only if \( a = g(b) \). Therefore, the projection map \( \pi_2 : (a, b) \mapsto b \) is an injective holomorphic map on an open neighbourhood \( U \subset P_k \) (in the subspace topology) of \((c(\tilde{u}), c(\bar{u}))\). It follows from Theorem 5.2 that the map \( \pi_2 \circ F : w \mapsto b(w) \) is an injective holomorphic map on an open neighbourhood \( B(\tilde{u}, \varepsilon_3) \subset \mathbb{C} \). Hence, it has a non-vanishing derivative at \( \tilde{u} \), i.e. \( b'(\tilde{u}) \neq 0 \).

Writing \( b(w) = b(u + iv) = b_1(u + iv) + ib_2(u + iv) \) and \( c(u) = c_1(u) + ic_2(u) \), we see that
\[
b'(\tilde{u}) = \frac{\partial b_1}{\partial u}(\tilde{u}) + i \frac{\partial b_2}{\partial u}(\tilde{u}) = (c_1)'(\tilde{u}) + ic_2)'(\tilde{u}) \quad \text{[For real } u, b(u) = c(u)]
\]

It follows that \((c_1)'(\tilde{u}), (c_2)'(\tilde{u}) \neq (0, 0)\), i.e. \( c'(\tilde{u}) \neq 0 \). \( \square \)

**Remark.** For the parabolic arcs of period 1 of \( M_2^* \) (also called the tricorn), one can easily compute that the corresponding algebraic curve is non-singular at each point of the parabolic arc. Hence, the critical Ecalle height parametrization is indeed non-singular for the period 1 arcs of the tricorn. However, we conjecture that the critical Ecalle height parametrization of any parabolic arc of \( M_2^* \) is non-singular everywhere.

4. REAL-ANALYTICITY OF HAUSDORFF DIMENSION

In this final section, we prove the real-analyticity of Hausdorff dimension of the Julia sets along the parabolic arcs making use of certain real-analyticity results from [SU] and applying them to the analytic family of parabolic maps constructed in the previous section. Recall that a rational map is called parabolic if it has at least one parabolic cycle and every critical point of the map lies in the Fatou set (i.e. is attracted to an attracting or a parabolic cycle). We begin with some known properties of Hausdorff dimensions and related objects in the realm of parabolic rational maps. The following concept of radial Julia sets was introduced by Urbański and McMullen [U] [MG].

**Definition** (Radial Julia Set). A point \( z \in J(f) \) is called a radial point if there exists \( \delta > 0 \) and an infinite sequence of positive integers \( \{n_k\} \) such that there exists a univalent inverse branch of \( f^{n_k} \) defined on \( B(f^{n_k}(z), \delta) \) sending \( f^{n_k}(z) \) to \( z \) for all \( k \). The set of all radial points of \( J(f) \) is called the radial Julia set and is denoted as \( J_r(f) \). Equivalently, the radial Julia set can be defined as the set of points in \( J(f) \) whose \( \omega \)-limit set non-trivially intersects the complement of the post-critical closure.

The dynamics at the radial points of a Julia set have a certain expansion property which makes these points easier to study. Indeed, for a radial point \( z \), there exists a sequence of iterates \( \{f^{n_k}(z)\} \) accumulating at a point \( w \) outside the post-critical closure and hence, there exists a sequence of univalent inverse branches of \( f^{n_k} \) defined on some \( B(w, \varepsilon) \) sending \( f^{n_k}(z) \) to \( z \) for all \( k \). Such a sequence necessarily
forms a normal family and any limit function must be a constant map (compare [Be, Theorem 9.2.1, Lemma 9.2.2]). This shows that the sequence of univalent inverse branches of \( f^{n_k} \) are eventually contracting; in other words, \( \lim_{n_k \to \infty} (f^{n_k})'(z) = \infty \).

Conformal measures have played a crucial role in the study of the dimension-theoretic properties of rational maps. It is worth mentioning in this regard that conformal measures satisfy a weak form of Ahlfors-regularity at the radial points of a Julia set. More precisely, by an immediate application of Koebe’s distortion theorem and the expansion property at the radial points discussed above, one obtains that for every point \( z \) in \( J_r(f) \), there exists a sequence of radii \( \{r_k(z)\}_{k=1}^{\infty} \) converging to 0 (\( r_k(z) \approx |f^{n_k}'(z)|^{-1} \)) such that if \( m \) is a \( t \)-conformal measure for \( f \), then

\[
C^{-1} < \frac{m(B(z, r_k(z)))}{r_k(z)^t} < C
\]

for some constant \( C \) depending on \( \delta \) and \( m \).

For parabolic rational maps, one has a rather simple but useful description of the radial Julia set. The following proposition was proved in [DU], we include the proof largely for the sake of completeness.

**Lemma 4.1.** Let \( f \) be a parabolic rational map and let \( \Omega \) be the set of all parabolic periodic points of \( f \). Then, \( J_r(f) = J(f) \setminus \bigcup_{i=0}^{\infty} f^{-i}(\Omega) \). In particular, \( \text{HD}(J_r(f)) = \text{HD}(J(f)) \).

**Proof.** For a parabolic rational map, the post-critical closure intersects the Julia set precisely in \( \Omega \), which is a finite set consisting of (parabolic) periodic points. Starting with any point of \( \bigcup_{i=0}^{\infty} f^{-i}(\Omega) \), one eventually lands on \( \Omega \) under the dynamics and the existence of infinitely many post-critical points in every neighbourhood of \( \Omega \) obstructs the existence of infinitely many univalent inverse branches. It follows that \( \bigcup_{i=0}^{\infty} f^{-i}(\Omega) \subset J(f) \setminus J_r(f) \).

It remains to prove the reverse inclusion. By passing to an iterate, one can assume that \( f^{o_q}(z_j) = z_j \forall z_j \in \Omega \). By the description of the local dynamics near a parabolic point ([MM, §10]), there is a neighbourhood \( B(\Omega, \theta') \) of the parabolic points that is contained in the union of the attracting and repelling petals. By continuity, there exists a \( 0 < \theta < \theta' \) such that \( f^{o_q}(B(z_i, \theta)) \cap B(z_j, \theta) = \emptyset \) for \( z_i \neq z_j \) and \( z_i, z_j \in \Omega \). We claim that for any \( z \in J(f) \setminus \bigcup_{i=0}^{\infty} f^{-i}(\Omega) \), there exists a sequence of positive integers \( \{n_k\} \) such that the sequence \( \{f^{o_q n_k}(z)\} \) lies outside \( B(\Omega, \theta) \). Otherwise, there would exist an \( n_0 \in \mathbb{N} \) such that \( f^{o_q n}(z) \in B(\Omega, \theta) \forall n > n_0 \). Then, there exists some \( z_j \in \Omega \) such that \( f^{o_q n}(z) \in B(z_j, \theta) \forall n > n_0 \). But since each \( f^{o_q n}(z) \) belongs to the repelling petal, it follows that \( f^{o_q n_0}(z) = z_j \), a contradiction to the assumption that \( z \notin \bigcup_{i=0}^{\infty} f^{-i}(\Omega) \). Finally, observe that any point in \( J(f) \setminus B(\Omega, \theta) \) has a small neighborhood disjoint from the post-critical set. Since \( J(f) \setminus B(\Omega, \theta) \) is a compact metric space, there exists a \( \delta > 0 \) such that the neighborhood \( B(w, \delta) \) of any point \( w \) of \( J(f) \setminus B(\Omega, \theta) \) is disjoint from


the post-critical set. Therefore, the balls $B(f^{\circ qn_k}(z), \delta)$ are disjoint from the post-critical closure and hence, there exists a univalent inverse branch of $f^{\circ qn_k}$ defined on $B(f^{\circ qn_k}(z), \delta)$ sending $f^{\circ qn_k}(z)$ to $z$ for all $k$. This proves that $J(f) \setminus \bigcup_{i=0}^{\infty} f^{-i}(\Omega) \subset J_r(f)$.

The final assertion directly follows as the set $\bigcup_{i=0}^{\infty} f^{-i}(\Omega)$ is countable. □

Before we state the main technical theorem from [SU], we need a couple of more definitions and facts from thermodynamic formalism. We will assume familiarity with the basic definitions and properties of topological pressure [W], [W1, §9]. The behaviour of the pressure function $t \mapsto P(t) = P(f|_{J(f)}), -t \log |f'|, t \in \mathbb{R}$ (where $P(f|_{J(f)}, -t \log |f'|), t \in \mathbb{R}$ is the topological pressure) has been extensively studied by many people in the context of rational and transcendental maps. In particular, much is known for hyperbolic and parabolic rational maps.

**Theorem 4.2.** [DU] For a parabolic rational map $f$,

1. The function $t \mapsto P(t)$ is continuous, non-increasing and non-negative.
2. $\exists s > 0$ such that,
   
   a. $P(t) > 0$ for $t \in [0, s)$,
   
   b. $P(t) = 0$ for $t \in [s, \infty)$,
   
   c. $P|_{[0,s]}$ is injective.

The following important theorem relates the Hausdorff dimension of the Julia set, the minimal zero of the pressure function and the minimum exponent of $t$-conformal measures for parabolic rational maps.

**Theorem 4.3.** [DU] For a parabolic rational map $f$, the following holds:

$$
\text{HD}(J(f)) = \inf\{t \in \mathbb{R} : \exists t - \text{conformal measure for } f|_{J(f)}\} = \inf\{t \in \mathbb{R} : P(t) = 0\}
$$

A more elaborate account of these lines of ideas can be found in the expository article of Urbański [DU, U].

**Definition.** A meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is called tame if its post-singular set does not contain its Julia set.

Clearly, parabolic polynomials are tame. For tame rational maps, there exist nice sets [JRL, ND] giving rise to conformal iterated function systems with the property that the Hausdorff dimension of the radial Julia set is equal to the common value of the Hausdorff dimensions of the limit sets of all the iterated function systems induced by all nice sets (compare [SU, §2.3]). One can define the pressure function for these iterated function systems (induced by the nice sets) and the system $S$ is called strongly $N$-regular if there is $t \geq 0$ such that $0 < P_S(t) < +\infty$ and if there is $t \geq 0$ such that $P_S(t) = 0$. The fact that the conformal iterated function systems arising from parabolic rational maps satisfy this property, can be proved as in Theorem 4.2.

We are now prepared to state the main result from [SU] which is at the technical heart of our real-analyticity result.

**Theorem 4.4.** [SU, Theorem 1.1] Assume that a tame meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ is strongly $N$-regular. Let $\Lambda \subset \mathbb{C}^d$ be an open set and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be an analytic family of meromorphic functions such that
(1) $f_{\lambda_0} = f$ for some $\lambda_0 \in \Lambda$.
(2) there exists a holomorphic motion $H : \Lambda \times J_{\lambda_0} \rightarrow \mathbb{C}$ such that each map $H_{\lambda}$ is a topological conjugacy between $f_{\lambda_0}$ and $f_{\lambda}$ on $J_{\lambda}$.

Then the map $\Lambda \ni \lambda \mapsto \text{HD}(J_r(f_{\lambda}))$ is real-analytic on some neighborhood of $\lambda_0$.

Proof of Theorem 1.1. Let $C$ be a parabolic arc and $c : \mathbb{R} \rightarrow C$ be its critical Ecalle height parametrization. It follows from Lemma 3.2 that there exists an injective holomorphic map $F : S \rightarrow \mathbb{C}^2$, $w \mapsto (a(w), b(w))$ and an analytic family of q.e. maps $(\varphi_w)_{w \in S}$ with $\varphi_w \circ P_{a(0), b(0)} \circ \varphi_w^{-1} = P_{a(w), b(w)}$. Setting $\Lambda = S$, and $H = \varphi(w, z) := \varphi_w(z)$, we see that all the conditions of Theorem 4.1 are satisfied and hence, the function $w \mapsto \text{HD}(J_r(P_{a(w), b(w)}))$ is real-analytic. Restricting the map to the reals, we conclude that the map $h \mapsto \text{HD}(J_r(f_{c(h)}))$ is real-analytic. The result now follows from Lemma 4.4. □

Remark. Parabolic curves arise naturally in the study of the parameter spaces of higher degree polynomials and the techniques used in this article can be generalized to prove corresponding statements about the real-analyticity of Hausdorff dimension of the Julia sets on these curves, under suitable conditions. In particular, the real-analyticity of $\text{HD}(J(f))$ continues to hold on regions of parameter spaces where the maps only have attracting or parabolic cycles and such that all the critical points converge to these cycles.

The parabolic arcs of the multicorns inherit the real-analyticity property from the connectedness loci of the sub-family $F_d$ of all polynomials of degree $d^2$.

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