ALIGNING SPATIAL FRAMES THROUGH QUANTUM CHANNELS

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We review the optimal protocols for aligning spatial frames using quantum systems. The communication problem addressed here concerns a type of information that cannot be digitalized. Asher Peres referred to it as “unspeakable information”. We comment on his contribution to this subject and give a brief account of his scientific interaction with the authors.

I. INTRODUCTION

In his later years, Asher Peres became interested in a type of information that he suggestively called “indiscrete” and, later on, “unspeakable” —paraphrasing the title of J. S. Bell’s famous book [1]. In Asher’s own words [2]:

“The role of a dictionary is to define unknown words by means of known ones. However there are terms, like left or right, which cannot be explained in this way. In the absence of a formal definition, material objects must be used to illustrate these terms: for example, we may say that the human liver is on the right side. Likewise, the sign of helicity may be referred to the DNA structure, or to the properties of weak interactions.”

In this context, there are interesting questions which can be readily translated into quantum information problems. Imagine Bob went on an intergalactic journey and because of a classical computer failure, he is now completely lost in space. Alice and Bob can communicate through a classical channel, but there is no way for them to establish a common reference frame. To help Bob come back home, Alice needs to send him information about vectors in space (about the one that points home). In these circumstances sending any string of bits has no meaning whatsoever, since there is no shared reference frame to which Bob can refer them. Alice can only send Bob information “analogically”: e.g., she can send a physical object that carries intrinsic orientation. In the quantum world there are such kind of objects: particles with spin. So, the question is: how efficiently can one communicate directions, align reference frames, etc. using such particles?

About five years ago, we became interested in this kind of topic. Sandu Popescu, who at the time visited Barcelona, explained to us that a state of two antiparallel spin-1/2 particles (spins for short) codifies space directions (three dimensional unit vectors) better than a state of two parallel spins [3]. Then the obvious questions arose: what is the best quantum state for this purpose if \( N \) spins are available? what is the best measurement to retrieve this information?

Shortly after we published the answer to these questions and the general solution to the \( N \)-copy problem [4], a paper by Asher and Petra Scudo, who was Asher’s student at the time, appeared in the archive [5]. The paper addressed the same problem from a more physical point of view. They were not that interested in analytical results, and rather resorted to clever numerical analysis. Minor disagreements, which were finally overcome, triggered a very intense e-mail exchange out of which we got the chance to get acquainted with Asher.

Having raised the question of how to communicate through a quantum channel when the parties share no reference frame, the natural next question we wished to address was how to optimally align spatial frames (SF) —i.e. orthogonal trihedra— using this channel. A few months later our analysis was completed, and before we could even start writing a draft a new paper by Asher and Petra appeared in the archive [6]. As it turned out, once again we had had the same agenda, tough this time Asher and Petra were quicker than we were. Their numerics revealed some stronger disagreement with our results and ‘there we go again!’ We exchanged e-mails for weeks, trying to figure out what was wrong, until we all agreed. In the meantime, we completed our draft and got it published [7] some months later. We got a lot out of this interaction, because Asher’s physical insight was astonishingly deep and he was always eager to share, but a great deal of it goes far beyond the scientific realm.

Our approaches to the alignment of SFs were slightly different. While Asher and Petra considered superpositions of orbital angular momentum states —such as those of a hydrogen atom— we focused on systems made out of \( N \) spins. The two approaches are formally equivalent if \( N \) is even and if in the latter the multiplicity of the irreducible representations that show up in the Clebsch-Gordan decomposition of \( (1/2)^N \) is not taken into account. The two approaches boil down to a sort of rigid covariant encoding of \( SO(3) \) rotations into a quantum state of spins using that \( SO(3) \approx SU(2) \). As we see it now, covariance is inherent to the problem at hand. In the absence of a shared frame, all Alice can do is rotate (a fixed anisotropic state of) a system and send it to Bob, who can afterwards attempt to align his spacecraft (his SF) with the system.
Shortly after, yet another new paper by Asher and Petra challenged us. They pointed out that by splitting the system of spins into three sets, and using them to transmit the directions of each of the three axes of the SF separately, one could outstrip the performance of our frame-transmission protocol. Actually, by that time, we had realized that our protocol was less efficient than that Asher was then proposing, and we were already looking for improvements. Asher and Petra’s paper boosted our commitment to the search we had already undertaken, but it took us quite a while to complete. In the meantime, together with Netanel Lindner and Daniel Terno, we considered a concrete realization of the alignment problem. They proposed to use Elliptic Rydberg states, which are the quantum analogues for the hydrogen atom of the classical Keplerian orbits and are feasible in a laboratory.

All this progress has triggered other interesting developments such as e.g., the optimal protocols for communication of digital information without shared reference frame, an issue not far removed from superselection rules, or from the quantification of the information contained in the establishment of a shared reference frame.

This paper is a review in perspective of our present knowledge of optimal alignment of SFs using systems of spins. The paper is organized as follows. In the next section we formulate the problem and introduce the notation. In Sec. III we discuss the optimal protocol for a hydrogen atom. In Sec. IV we show that an efficient use of a $N$ spin system decreases the communication error drastically. In Sec. V we present the absolute optimal protocol. This protocol, which requires full-fledged entanglement, can be viewed as dense covariant coding of “unspeakable information” (using Asher’s terminology) or continuous variables. The last section contains our conclusions.

II. PRELIMINARIES AND NOTATION

To lend a dramatic touch to the presentation, let us consider the situation described in the introduction, in which Bob’s spacecraft is lost in space while Alice, on Earth, is sending him (through a quantum channel) “unspeakable” information consisting of a SF $n = \{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$. Let us assume that Alice has a quantum system that she can prepare in a superposition of states with angular momentum $J$. From the outcomes of his measurements he makes a guess for the rotation that takes an external observer’s spin system $\vec{n}$ into $n$. We will use $g$ as a shorthand for the three Euler angles, i.e., $g = (\alpha, \beta, \gamma)$. Of course, neither Alice nor Bob need to be aware of the existence of $n_0$. Following Holevo, we use the error, defined by

$$h(g, g') = \sum_{a=1}^{3} |\vec{n}_a - \vec{n}_a'|^2 = \sum_{a=1}^{3} |\vec{a}(g) - \vec{a}(g')|^2,$$

(1)

to quantify the quality of Bob’s guess. Alice encodes $n = n(g)$ into a suitable quantum state $|A(g)\rangle$ of her system. Covariance allows us to write

$$|A(g)\rangle = U(g)|A\rangle,$$

(2)

where $|A\rangle$ is a fixed fiducial state and $U(g)$ is the unitary representation on Alice’s (system) Hilbert space of the rotation that takes $n_0$ into $n$.

Since Bob is completely lost in space, he has absolutely no information about $n$. Therefore, if we wish to compute how well Bob is doing on average, we must assume that $a$ priori Alice’s SF are isotropically distributed, i.e., we must define $dg$ by the Haar measure of $SO(3)$, which in terms of the Euler angles reads $dg = \sin \beta d\beta d\alpha d\gamma/(8\pi^2)$.

To compute the average error $\langle h \rangle$, the optimal measurement —represented in full generality by a Positive Operator Valued Measure (POVM)— can be chosen to be covariant and to have a rank-one seed $O(g') = U(g')|B\rangle\langle B|U^\dagger(g')$.

(3)

The state $|B\rangle$ defines Bob’s POVM much in the same way as $|A\rangle$ defines Alice’s messenger state.

The conditional probability of Bob guessing $n(g')$ if Alice’s SF is $n(g)$ is given by quantum mechanics through the Born rule $p(g'|g) = \langle A(g)|O(g')|A(g)\rangle$. Then the average error reads

$$\langle h \rangle = \int dg \int dg' h(g, g')p(g'|g) = \int dg h(0, g)|A|U(g)|B\rangle|A|U(g)|B\rangle|^2,$$

(4)
where \(\mathbf{0}\) stands for \((\alpha, \beta, \gamma) = (0, 0, 0)\). To derive the second equality we have used the invariance of the Haar measure and the normalization condition \(\int dg = 1\). One can easily check that

\[
h(0, g) = 6 - 2\chi^{(1)}(g),
\]

(5)

where, following a widely used notation, we denote by \(\chi^{(j)}\) the character of the spin-\(j\) representation of \(SO(3)\). Namely,

\[
\chi^{(j)}(g) \equiv \sum_m D^{(j)}_{mm}(g),
\]

(6)

where the elements of the matrix \(D^{(j)}_{mm}\) are defined in the standard way as \(D^{(j)}_{mm}(g) \equiv \langle jm|U(g)|jm'\rangle\). One also has \(\chi^{(1)}(g) = \cos \beta + (1 + \cos \beta) \cos (\alpha + \gamma)\), from which it follows that the values of \(\chi^{(1)}\) lie in the real interval \([-1, 3]\). The value \(\chi^{(1)} = 3\) corresponds to perfect determination of Alice’s SF and implies that \(h = 0\). Note also that \(\langle h \rangle = 6 - 2\chi^{(1)}\). Random guessing implies \(\langle \chi^{(1)} \rangle = 0 \((\langle h \rangle = 0)\), while perfect determination of one axis and random guessing of the remaining two yield \(\langle \chi^{(1)} \rangle = 1 \((\langle h \rangle = 4)\).

There is no consensus on the loss function for this frame-transmission problem. Instead of \(h\), Asher and Petra [6] used the Mean Square Error per axis, defined as \(\text{MSE} = \frac{1}{N} \sum (1 - \bar{n}_a \cdot \bar{n}'_a)\). One can readily see that \(\text{MSE} = (3 - \langle \chi^{(1)} \rangle)/6\). In the following, we will focus on \(\langle \chi^{(1)} \rangle\). From our results, values for the reader’s favorite loss function can be obtained trivially.

In the following sections we will show that perfect SF transmission (i.e \(\langle \chi^{(1)} \rangle \rightarrow 3\) ) is possible in the asymptotic limit of \(J \rightarrow \infty\) (or equivalently \(N \rightarrow \infty\)), provided the appropriate signal state \(|A\rangle\) and POVM seed state \(|B\rangle\) are chosen. We will also compute the rate at which this limit is approached for each of the protocols discussed below.

### III. HYDROGEN ATOM

Following Asher’s approach, consider that Alice prepares a hydrogen atom in the state

\[
|A\rangle = \sum_j |A^j\rangle = \sum_{jm} A^j_m |jm\rangle; \quad \sum_{jm} |A^j_m|^2 = 1.
\]

(7)

A system of an even number \(N\) of spins can also be prepared in such a state if one neglects all but one of the equivalent spin-\(j\) representations that appear with multiplicity

\[
n_j = (N + j - 2j + 1)(N/2 + j + 1)
\]

(8)

in the Clebsch-Gordan decomposition of \((1/2)^\otimes N\). In Eq. (4), \(j\) runs from 0 to \(J\) \((N/2)\), and \(m\) runs from \(-j\) to \(j\). The state \(|A\rangle\) is a \(n'\)th energy level of a hydrogen atom (a Rydberg state) \(\text{R}\). As mentioned above, a single transmission of the rotated state \(|A(g)\rangle\) in Eq. (4) can give Bob ("unspeakable") information about Alice’s SF. The larger the value of \(J\) (equiv. \(N\)) the better Bob’s guess will be and, as we will show, perfect determination is possible in the asymptotic limit.

In full analogy with Eq. (4) we write

\[
|B\rangle = \sum_j \sqrt{d_j}|B^j\rangle; \quad |B^j\rangle = \sum_{jm} B^j_{jm} |jm\rangle,
\]

(9)

where the square root of the dimension of the spin-\(j\) representation, \(d_j = 2j + 1\), is introduced for later convenience. The condition \(I = \int dg O(g)\) requires that

\[
\sum_{jm} |B^j_{jm}|^2 = 1, \quad \forall j,
\]

(10)

as follows from Schur lemma.

From Eq. (4) we have

\[
\langle \chi^{(1)} \rangle = \int dg \langle A|U(g)|B\rangle^2 \chi^{(1)}(g),
\]

(11)

and to optimize the transmission we just have to maximize \(\langle \chi^{(1)} \rangle\) over \(A^j_m\) and \(B^j_{jm}\). Hence, we write \(\langle \chi^{(1)} \rangle_{\text{max}} = \max_{A,B} \langle \chi^{(1)} \rangle\). Rather than attempting a numerical maximization, as Asher and Petra did [6], we chose to play with
At the same time, \( |A^j \tilde{A}^l\rangle = |A^j\rangle \otimes |\tilde{A}^l\rangle \), where the state \(|\tilde{A}^l\rangle\) is the time reversed of \(|A^j\rangle\) [i.e., \(\tilde{A}^l_m = (-1)^m A^l_m\)] and similarly for \(|B^l \tilde{B}^l\rangle\) and \(|\tilde{B}^l\rangle\) [and \(P_l\) is the projector over the Hilbert space of the representation of spin \(j = 1\)]. With the help of the Schwarz inequality, we find that the maximum in Eq. (12) occurs for states \(|A\rangle\) such that

\[
A^j_m = C^j B^j_m, \quad \text{with} \quad \sum_j |C^j|^2 = 1. \tag{13}
\]

This equation just tells us that, for an optimal communication, the messenger states \(|A(g)\rangle\) must be as similar as possible to the states \(U(g)|B\rangle\) on which the measuring device projects [4]. We then have

\[
\langle \chi^{(1)} \rangle_{\text{max}} = \max_{BC} \sum_{jj'} C^j M^j_{B} C^{j'}, \tag{14}
\]

where

\[
M^j_{B} = \frac{\sqrt{d_j d_{j'}}}{3} \langle B^j \tilde{B}^{j'} | P_l | B^j \tilde{B}^{j'} \rangle, \tag{15}
\]

and the maximization is over all \(B^j_m\) and \(C^j\) subject to the normalizations (10) and (13). The maximum \(\langle \chi^{(1)} \rangle_{\text{max}}\) is thus given by the largest value, \(\lambda_{\text{op}}\), in the set \(\{\lambda_B\}\) of largest eigenvalues of matrices of the form (15). Notice that the entries of all these matrices are non-negative. In this situation, it is easy to see that if a matrix \(M_{\text{op}}\), such that \(M^j_{B} \geq M^j_{B'}\) for all \(B\), exists, then \(\lambda_{\text{op}}\) is precisely its largest eigenvalue. Notice also that all the matrices in (15) are tri-diagonal. It is easy to verify [7] that such \(M_{\text{op}}\) indeed exists and is given by

\[
M_{\text{op}} = \begin{pmatrix}
\frac{j}{J+1} & \sqrt{\frac{2j-1}{2j+1}} & \cdots & 0 \\
\sqrt{\frac{2j-1}{2j+1}} & \sqrt{\frac{2j-2}{2j+1}} & \cdots & \frac{2}{3} \sqrt{\frac{3}{5}} \\
\cdots & \cdots & \frac{2}{3} \sqrt{\frac{3}{5}} & \sqrt{\frac{3}{5}} \\
0 & \sqrt{\frac{4}{5}} & \frac{1}{2} & \sqrt{\frac{4}{5}} \\
\sqrt{\frac{4}{5}} & \frac{1}{2} & \sqrt{\frac{4}{5}} & 0
\end{pmatrix}. \tag{16}
\]

A straightforward choice of the state \(|B\rangle\) that saturates these bounds is given by

\[
|B_{\text{op}}\rangle = \sum_j \sqrt{d_j} |j,j\rangle \quad \Leftrightarrow \quad B^j_{\text{op}m} = \delta^j_m. \tag{17}\]

The form of the optimal seed state \(|B_{\text{op}}\rangle\), which in turn determines through Eq. (16) Alice’s optimal messenger state, agrees with our physical intuition. If Alice’s state had a well defined total spin along some axis (i.e., if it were an eigenstate of \(\vec{n} \cdot \vec{J}^\dagger\), for some unit vector \(\vec{n}\)), \(M_{\text{op}}\) would become diagonal and \(\langle \chi^{(1)} \rangle_{\text{max}} = J/(J+1) = N/(N+2)\) thus, at most (in the limit \(N \to \infty\)) \(\langle \chi^{(1)} \rangle = 1\). In average, Bob could not determine more than just one axis of Alice’s SF. The structure of the state \(|B_{\text{op}}\rangle\) is such that, within each irreducible representation, the determination of a single axis is optimal [21] (this is the best Alice could do if she only were allowed to use a single irreducible representation). At the same time, \(|B_{\text{op}}\rangle\) is as different from an eigenstate of \(\vec{n} \cdot \vec{J}\) as it can possibly be.

The problem is now solved; the optimal measurement is determined by Eq. (16). \(\langle \chi^{(1)} \rangle_{\text{max}}\) is given by the largest eigenvalue of \(M_{\text{op}}\), and the optimal messenger state, \(|A\rangle\), is computed through Eq. (16) substituting \(C^j\) by the corresponding eigenvector \(C^j_{\text{op}}\). For small \(N\), one can easily obtain analytic expressions for \(\langle \chi^{(1)} \rangle_{\text{max}}, |B\rangle\) and \(|A\rangle\).

In the asymptotic regime of large \(J\) (\(N\)), it is again possible to compute \(\langle \chi^{(1)} \rangle\) explicitly up to sub-leading order in \(1/J (1/N)\). Sub-leading orders are important because, among other things, they enable us to compare different acceptable protocols (those that achieve perfect determination in the strict limit \(J,N \to \infty\)) independently of \(J\).
as these orders tell us the rate at which perfect determination \( (\chi^{(1)})_{\text{max}} \to 3 \) is reached. They are also very important in quantum statistics [22]. For that purpose we give simple upper and lower bounds of \( (\chi^{(1)})_{\text{max}} \). A useful upper bound is provided by the condition

\[
\langle \chi^{(1)} \rangle_{\text{max}} \leq \max_j \sum_{j'} M_{jj'},
\]

while a lower bound can be derived (with hard work and determination) using a variational method with a judicious choice of the vector \( C \) (see Ref. [7] for details). We obtain

\[
3 - \frac{4}{N} + O(N^{-4/3}) \leq \langle \chi^{(1)} \rangle_{\text{max}} \leq 3 - \frac{4}{N} + O(N^{-2}).
\]

We see that perfect determination of Alice’s SF is attained at a rate linear in \( 1/N \).

IV. \( N \) SPINS. USE OF THE EQUIVALENT REPRESENTATIONS.

We will now show that with \( N \) spins Alice can devise a communication protocol that outperforms that of the previous section, provided she makes proper use of some of the \( n_j \) equivalent spin-\( j \) representations of \( SO(3) \).

Instead of (7), the most general state in which Alice can actually prepare her \( N \)-spin system is

\[
|A⟩ = \sum_j |A⟩^j = \sum_{j\alpha} A^j_{\alpha} |j\alpha⟩,
\]

where we have introduced the additional index \( \alpha \) to label the \( n_j \) equivalent spin-\( j \) representations. Recall that the index \( \alpha \) does not rotate under \( SO(3) \) and that the corresponding irreducible spaces are orthogonal, namely,

\[
⟨jm\alpha | U(g) | jm'\alpha'⟩ = \delta_{m\alpha} D_{mm'}^{(j)}(g).
\]

This index \( \alpha \) corresponds to a truly additional degree of freedom of the system and, based on the work by Acín et al. [23], one could argue that by entangling it with the magnetic number \( m \) [which does rotate under \( SO(3) \)] one could improve on the previous protocol (see also next section).

\[
|A⟩^j = \frac{1}{\sqrt{d_j}} \sum_m |jm\alpha_m⟩.
\]

Note that this (maximum) entanglement of degrees of freedom can be established on any of the spin-\( j \) invariant subspaces but on the \( j = J \) one (\( J \equiv N/2 \) throughout this section), which corresponds to the highest spin. This is so because from Eq. (8) one has \( n_J = 1 \), whereas \( n_j \geq d_j \) if \( j < J \). Therefore, \( |A⟩^J = \sum_m A^J_m |Jm⟩ \) has no entanglement of this type at all.

It is not difficult to convince oneself that the \( n_j - d_j \) equivalent representations that do not show up in Eq. (21) are actually sterile, i.e we cannot make them play any role in the estimation problem at hand. Recalling from Refs. [7] and [18] that \( |JJ⟩ \) is optimal when only one of the equivalent representations is allowed, we propose the following fiducial messenger state

\[
|A⟩ = a_J |JJ⟩ + \sum_{j<J} \frac{a_j}{\sqrt{d_j}} \sum_m |jm\alpha_m⟩
\]

(22)

A covariant POVM for these signal states is given by Eq. (3), where \( |B⟩ \) is now chosen to be

\[
|B⟩ = \sum_j \sqrt{d_j} \sum_m |jm\alpha_m⟩.
\]

(23)

Substituting (22) and (23) in (11) we can write \( \langle \chi^{(1)} \rangle \) as the quadratic form [which plays the role of (14)]

\[
\langle \chi^{(1)} \rangle_{\text{max}} = \max_{a^t} a^t (1 + M) a,
\]

(24)

where \( a^t = (a_J, a_{J-1}, a_{J-2}, \ldots) \) in the transposed of \( a \). Considering for simplicity \( N \) odd (i.e. \( J \) half integer or
around $\lambda P$. Hence, the smallest zero of $P$ where $\theta_1, i.e, $
abla = 0 = 2N$, the matrix $M$ is

$$M = \begin{pmatrix}
\frac{1}{\sqrt{d_j}} & 0 & 1 \\
\frac{1}{\sqrt{d_j}} & 0 & 1 \\
0 & 1 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{pmatrix},$$

(25)

The maximum value of $\langle \chi^{(1)} \rangle = 1 - 2\lambda_0$, where $-2\lambda_0$ is the largest eigenvalue of $M$. The characteristic polynomial of $M$, defined here to be $P_n(\lambda) = \det(M + 2\lambda)$, satisfies the recursive relation of the Tchebychev polynomials $\chi_n$, hence, $P_n(\lambda)$ is a linear combination of them. One can check that the explicit solution is

$$P_n(\lambda) = U_n(\lambda) - \frac{2}{2n + 1} U_{n-1}(\lambda) + \frac{2n - 1}{2n} U_{n-2}(\lambda),$$

(26)

where $P_n(\lambda) \equiv P_n^{n-1/2}(\lambda)$ and $U_n$ are the Tchebychev polynomials

$$U_n(\cos \theta) = \sin [(n + 1)\theta] / \sin \theta.$$  

(27)

Hence, the smallest zero of $P_n(\lambda)$, which we write as $\lambda_0 \equiv \cos \theta_0$, can be easily computed in the large $n$ limit expanding around $\lambda_0 = -1$, i.e, $\theta_0 = \pi (1 - n^{-1} + an^{-2} + bn^{-3} + \ldots)$. We find

$$\langle \chi^{(1)} \rangle_{\text{max}} = 3 - 4\pi^2/N^2 + 8\pi^2/N^3 + \ldots.$$  

(28)

(See Refs. 24 and 25 for alternative derivations of this equation.) By comparing with Eq. 16, we note the communication protocol presented here outperforms that of the previous section, as already stated above. The rate at which it achieves perfect determination of Alice’s SF is quadratic in $1/N$, in contrast to the former protocol which attains this limit linearly in $1/N$. Despite this success, from our derivation it should be clear to the reader that this protocol may not be optimal. For finite $N$ this is indeed the case. However, the results in the next section show that the protocol is asymptotically optimal.

V. OPTIMAL PROTOCOL. DENSE COVARIANT CODING.

The optimal protocol requires that Alice and Bob share a maximally entangled state. It bears a great similarity with dense coding.

We now assume that Alice and Bob have each of them a system of $N$ spins. Before Bob’s departure for the intergalactic journey, they prepare a suitable entangled state $|\Phi \rangle$. Also before departure, Bob locks the orientation of his quantum system to that of his spacecraft (SF and spacecraft are of course synonyms in this section), while Alice locks her spins to her laboratory on Earth. After Bob’s classical computers crashed, far away from home, the state of the 2N spins is still given by $|\Phi \rangle$ but Alice’s and Bob’s parts now refer to their respective SFs. Relative to Bob’s SF this state can be written as

$$|\Phi(g) \rangle \equiv U_A(g) \otimes I_B |\Phi \rangle,$$

(29)

where the subscripts $A$ and $B$ refer to Alice’s and Bob’s Hilbert spaces respectively and $g$ stands for the three Euler angles of the $SO(3)$ rotation that takes Bob’s SF into Alice’s. With no other resource available, Alice sends her $N$ spins to Bob, with the hope that he will retrieve from them the information he needs. To do so, he is allowed to perform generalized collective measurements on both his own spins and Alice’s (now in his possession), namely, on the state 26.
In Ref. [23] it was demonstrated that the maximally entangled state

\[ |\tilde{\Phi}\rangle = \sum_j a_j |\tilde{\Phi}^j\rangle, \]  

(30)

with

\[ |\tilde{\Phi}^j\rangle = \frac{1}{\sqrt{d_j n_j}} \sum_{m=-j}^j |j m\rangle_A |j m\rangle_B, \]  

(31)

is the optimal encoding state of an \( SU(2) \cong SO(3) \) operation (notice that this result does not preclude the existence of other optimal states). As in previous sections, the coefficients \( a_j \) have to be properly chosen to maximize \( \langle \chi^{(1)} \rangle \). It is easy to prove that with this setup the equivalent representations do not play any role\(^1\), so, without loss of generality, the optimal messenger state can be chosen as

\[ |\Phi\rangle = \sum_j a_j |\Phi^j\rangle = \sum_j \frac{a_j}{\sqrt{d_j n_j}} \sum_{m=-j}^j |j m\rangle_A |j m\rangle_B. \]  

(32)

Bob’s optimal measurement is defined by the (once again) covariant POVM

\[ O(g) = U_A(g) \otimes I_B |\Psi\rangle \langle \Psi| U_A^\dagger(g) \otimes I_B, \]  

(33)

where \( |\Psi\rangle \) can be taken to be the maximally entangled state

\[ |\Psi\rangle = \sum_{jm} \sqrt{d_j} |jm\rangle_A |jm\rangle_B. \]  

(34)

In analogy with Eq. (11), \( \langle \chi^{(1)} \rangle \) is given by

\[ \langle \chi^{(1)} \rangle = \int dg \langle \Phi|O(g)|\Phi\rangle^{(1)}(g) = \int dg \chi^{(1)}(g) \left| \sum_j a_j \chi^{(1)}(g) \right|^2. \]  

(35)

The group integral can be easily performed by recalling the Clebsch-Gordan series \( \chi^{(j)}(g)\chi^{(l)}(g) = \sum_{k=|j-l|}^{j+l} \chi^{(k)}(g) \) of \( SO(3) \) and the orthogonality of the characters [28], namely, \( \int dg \chi^{(j)}(g)\chi^{(l)}(g) = \delta_{jl} \). Again, the result can be conveniently written as in (24), where now \( M \) is the tri-diagonal matrix

\[ M = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 & 1 \\ & & & & 1 & \zeta \end{pmatrix}, \]  

(36)

with \( \zeta = -1 \) (\( \zeta = 0 \)) for \( N \) even (odd).

Proceeding as in the previous section, we obtain the maximum value of \( \langle \chi^{(1)} \rangle \) by computing the maximum eigenvalue of \( M \). The characteristic polynomials \( P_n(\lambda) = \det(M + 2\lambda I) \), where \( n \) is the dimension of \( M \) (one has \( n = N/2 + 1 \) for \( N \) even and \( n = N/2 + 1/2 \) for \( N \) odd) again satisfy the recursion relation of the Tchebychev polynomials, and

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\(^1\) One just has to realize that any state in the space spanned by the set \( \{U_A(g) \otimes I_B |\tilde{\Phi}^j\rangle\} \) is orthogonal to a space spanned by \( (n_j - 1) \times d_j \) mutually orthogonal states. Hence, \( |\tilde{\Phi}^j\rangle \) effectively lives in only one of the equivalent spin-\( j \) representations and can be chosen as in Eq. (32).
the solution for the initial conditions derived from (36) is $P_n(\lambda) = U_n(\lambda) + \zeta U_{n-1}(\lambda)$. The largest eigenvalue of $M$ is $2 \cos[2\pi/(N + 3)]$, hence,

$$\langle \chi_1 \rangle_{\text{max}} = 1 + 2 \cos \frac{2\pi}{N + 3},$$

(37)

and one can also verify that the corresponding eigenvector is

$$a_j = \frac{2}{\sqrt{N + 3}} \sin \frac{(2j + 1)\pi}{N + 3}.$$  

(38)

Remarkably this is an exact closed solution valid for any $N$. For large $N$ this expression reads

$$\langle \chi^{(1)} \rangle_{\text{max}} = 3 - \frac{4\pi^2}{N^2} + \ldots.$$  

(39)

Eq. (37) gives the minimum error one can ever attain when aligning SF through a quantum channel and proves that the protocol of the previous section, which does not use shared entanglement between Alice and Bob and requires half the number of spins, is also optimal asymptotically.

**VI. CONCLUSIONS**

We have reviewed the alignment of spatial frames using quantum states; a problem that interested Asher in his later years. We have obtained the optimal protocols assuming different setups. In all the cases, perfect alignment is possible in the asymptotic limit. We have also shown that entanglement —either of internal degrees of freedom or shared entanglement between Alice and Bob— dramatically improves the efficiency of the communication of frames.

A comment on the intrinsicallity of the protocols discussed in this paper is in order. It is clear that this communication problem requires that the encoding of the information about the frame (“unspeakable” information in Asher’s terminology) on the messenger state (as well as the decoding via measurements) has to be accomplished through spatial rotations, since no reference is assumed to be shared by sender and recipient. This is the reason why we consider spin or angular momentum. By the same reason, the whole procedure needs be covariant. The precise implementation of the communication protocols is, of course, well beyond the scope of this paper. Here we have just presented theoretical lower bounds on the communication error.

To end, we would like to point out that the problem of aligning spatial frames can also be translated into the reverse engineering problem of estimating an unknown $SU(2)$ operation on qubits. We refer the interested reader to Refs. [23] and [10] for details.

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