State Complexity Bounds for the Commutative Closure of Group Languages

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Abstract. In this work we construct an automaton for the commutative closure of some given regular group language, i.e., a language accepted by some permutation automaton. The number of states of the resulting automaton is bounded by the number of states of the original automaton, raised to the power of the alphabet size, times the product of the order of the letters, viewed as permutations of the state set. This gives the asymptotic state bound $O((n \exp(\sqrt{n \ln n}))^{|\Sigma|})$, if the original regular language is accepted by some $n$ state automaton. Depending on the automaton in question, we label points of $\mathbb{N}^{|\Sigma|}$ by subsets of states and introduce unary automata which decompose the thus labelled grid. Based on these constructions, we give a general regularity condition, which is fulfilled for group languages. We also give trade-offs for the relative descriptional complexity of jumping finite automata given by permutational letters.

Keywords: state complexity · commutative closure · group language · permutation automaton · jumping finite automata

1 Introduction

The area of state complexity asks for sharp bounds on the size of resulting automata for regularity-preserving operations. This questions goes back at least to work by Maslov [16], but, starting with the work [18], has revived at the end of the last millennium. The class of deterministic and complete automata is the most natural, or prototypical, class. But state complexity questions have also been explored for non-deterministic automata, or other automata models, see for example the surveys [8, 14, 15]. Nowadays, it is an active and important area of research under the broader theme of descriptional complexity of systems. We refer again to the survey [8] for more information. It was shown in [9] that the commutative closure is regularity preserving on regular group languages. But the method of proof was algebraic and used Ramsey-type arguments. The general form of an accepting automaton was still open. Here we indicate methods to obtain such an automaton, and derive state bounds for the commutative closure of regular group languages. Our methods are somehow motivated by previous results of the author, namely on the state complexity of the shuffle on commutative languages [12, 13]. In this sense, the present study is a natural continuation of the investigation of state complexity questions for commutative languages. A guiding theme in this past work was to view commutative languages as composed out of unary languages, which directly leads to product automata models over unary automata in [12, 13]. Here, we also use unary automata in a non-trivial way. The constructions are a little bit more involved and interdependent.
than in the previous work, but still somehow show the close connection of commutative languages to unary languages. Furthermore, jumping finite automata where introduced in [17], and further work has investigated their descriptional and computational power, see [16]. They can be seen as an automaton model for the commutative closure of regular language. Our result directly yield trade-offs for the relative descriptional complexity of jumping finite automata to ordinary automata, for the case that the jumping finite automaton is a permutation automaton, if viewed as an ordinary automaton.

2 Prerequisites

Let $\Sigma = \{a_1, \ldots, a_k\}$ be a finite set of symbols\(^1\) called an alphabet. The set $\Sigma^*$ denotes the set of all finite sequences, i.e., of all words. The finite sequence of length zero, or the empty word, is denoted by $\varepsilon$. For a given word $w$ we denote by $|w|$ its length, and, for $a \in \Sigma$, by $|w|_a$ the number of occurrences of the symbol $a$ in $w$. Subsets of $\Sigma^*$ are called languages. With $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers, including zero. Let $L \subseteq \Sigma^*$, by $L^* = \{\varepsilon\} \cup L \cup L^2 \cup \ldots$ we denote its Kleene closure. For words $u \in \Sigma^*$, we also write $u^*$ as a shorthand for $\{u\}^*$. A finite deterministic and complete automaton will be denoted by $A = (\Sigma, S, \delta, s_0, F)$, with $\delta : S \times \Sigma \rightarrow S$ the state transition function, $S$ a finite set of states, $s_0 \in S$ the start state, and $F \subseteq S$ the set of final states. The properties of being deterministic and complete are implied by the definition of $\delta$ as a total function. The transition function $\delta : S \times \Sigma \rightarrow S$ could be extended to a transition function on words $\delta^* : S \times \Sigma^* \rightarrow S$, by setting $\delta^*(s, \varepsilon) := s$ and $\delta^*(s, wa) := \delta(\delta^*(s, w), a)$ for $s \in S$, $a \in \Sigma$ and $w \in \Sigma^*$.

In the remainder we drop the distinction between both functions and will also denote this extension by $\delta$. The language accepted by some automaton $A = (\Sigma, S, \delta, s_0, F)$ is $L(A) = \{w \in \Sigma^* \mid \delta(s_0, w) \in F\}$. A language $L \subseteq \Sigma^*$ is called regular if $L = L(A)$ for some finite automaton. An automaton is called a permutation automaton if the transformation of the states induced by a letter is a permutation, i.e., a bijective function. A regular language is called a group language if it is accepted by some permutation automaton. The map $\psi : \Sigma^* \rightarrow \mathbb{N}_0^k$ given by $\psi(w) = (|w|_{a_1}, \ldots, |w|_{a_k})$ is called the Parikh morphism. For $w \in \Sigma^*$ it fulfills the recursive equations

$$\begin{align*}
\psi(w) &= (0, \ldots, 0) \quad \text{if } w = \varepsilon \\
\psi(wa_k) &= (j_1, \ldots, j_k + 1, \ldots, j_k) \quad \text{if } w \neq \varepsilon, \text{ with } \psi(w) = (j_1, \ldots, j_k).
\end{align*}$$

(1)

For a given word $w \in \Sigma^*$ we define the commutative closure as $\text{perm}(w) := \{u \in \Sigma^* : \psi(u) = \psi(w)\}$. For languages $L \subseteq \Sigma^*$ we set $\text{perm}(L) := \bigcup_{w \in L} \text{perm}(w)$. A language is called commutative if $\text{perm}(L) = L$, i.e., with every word each permutation of this word is also in the language. Every function $f : X \rightarrow Y$ could be extended to subsets $S \subseteq X$ by setting $f(S) := \{f(x) : x \in S\}$, we will do this frequently without special mentioning. For $Z \subseteq X$ we denote by $f|_Z : Z \rightarrow Y$ the function obtained by restriction of the arguments to elements of $Z$. For a set $X$, we denote by $\mathcal{P}(X) = \{Y : Y \subseteq X\}$ the power set of $X$. If $X, Y$ are sets, by $X \times Y$ we denote their cartesian product. By $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ we denote the projections.
and $\pi_2 : X \times Y \to Y$ we denote the projection maps onto the first and second component, $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. If $a, b \in \mathbb{N}_0$ with $b > 0$, we denote by $a \mod b$ the unique number $0 \leq r < b$ such that $a = bn + r$ for some $n \geq 0$. For $n \in \mathbb{N}_0$ we set $[n] := \{k \in \mathbb{N}_0 : 0 \leq k < n\}$. Let $M \subseteq \mathbb{N}_0$ be some finite set. By max $M$ we denote the maximal element in $M$ with respect to the usual order, and we set $\max \emptyset = 0$. Also for finite $M \subseteq \mathbb{N}_0 \setminus \{0\}$, i.e., $M$ is finite without zero in it, by lcm $M$ we denote the least common multiple of the numbers in $M$, and set lcm $\emptyset = 0$.

2.1 Unary Languages

Let $\Sigma = \{a\}$ be a unary alphabet. In this section we collect some results about unary languages. Suppose $L \subseteq \Sigma^*$ is regular with an accepting complete deterministic automaton $A = (\Sigma, S, \delta, s_0, F)$. Then by considering the sequence of states $\delta(s_0, a^i), \delta(s_0, a^2), \delta(s_0, a^3), \ldots$ we find numbers $i \geq 0, p > 0$ with $i$ and $p$ minimal such that $\delta(s_0, a^i) = \delta(s_0, a^{i+p})$. We call these numbers the index $i$ and the period $p$ of the automaton $A$, note that $i + p = |S|$. Suppose $A$ is initially connected, i.e., $\delta(s_0, \Sigma^*) = Q$. Then the states from $\{s_0, \delta(s_0, a), \ldots, \delta(s_0, a^{-1})\}$ constitute the tail, and the states from $\{\delta(s_0, a^1), \delta(s_0, a^{i+1}), \ldots, \delta(s_0, a^{i+p-1})\}$ constitute the unique cycle of the automaton. If $A$ is not initially connected, when we speak of the cycle or tail of that automaton, we nevertheless mean the above sets, despite the automaton graph might have more than one cycle, or more than one straight path.

Lemma 1. Let $A = (\Sigma, Q, \delta, s_0, F)$ be some unary automaton. If $\delta(s, a^k) = s$ for some state $s \in Q$ and number $k > 0$, then $k$ is divided by the period of $A$.

3 Results

In Section 3.1 we first give a labelling of the grid $\mathbb{N}_0^k$ by states of some given automaton. This labelling is in some sense an abstract description of the commutative closure, which is more precisely stated in Corollary 1. We then construct unary automata for each letter. Very roughly, and intuitively, they read in letters parallel to the direction of this letter in $\mathbb{N}_0^k$, given by the Parikh map. We have one such automaton for each point on the hyperplane orthogonal to this direction. These unary automata are then used to describe the mentioned state labelling. In this sense, the state labelling is decomposed into these automata. This is made more precise in Proposition 2. If all the automata in this decomposition, for each letter, only have a bounded number of states, then the commutative closure is a regular language. By using the indices and periods, we give a state bound for the resulting automaton in Theorem 1. In Section 3.2 these results are applied to the case that the given automaton is a permutation automaton. It turns out, stated in Proposition 3 and Proposition 4, that the index and the period are always bounded, for a bound dependent on the input automaton, which is also stated in these Propositions. Intuitively, the main observations why this works is that 1) for permutations, the state labels cannot decrease as the unary automata read in symbols, and 2) we know when the state labels must become periodic. Finally, applying our general result, then gives that the commutative closure is regular, and also yields a state complexity bound. We then point out a connection to the class of jumping finite automata.
3.1 A Regularity Condition Based on a Decomposition into Unary Automata

First, we introduce the state label function.

**Definition 1. (state label function)** Let \( \Sigma = \{a_1, \ldots, a_k\} \) be the alphabet. Suppose \( \mathcal{A} = (\Sigma, Q, \delta, s_0, F) \) is a finite automaton. The state label function, associated to the automaton, is the function \( \sigma_\mathcal{A} : \mathbb{N}_0^{[\Sigma]} \rightarrow \mathcal{P}(Q) \) given by

\[
\sigma_\mathcal{A}(p) = \{ \delta(s_0, u) : \psi(u) = p \}.
\]

The value of the function \( \sigma_\mathcal{A} \), for some fixed automaton \( \mathcal{A} = (\Sigma, Q, \delta, s_0, F) \), will also be called the *state (set) label* for that point, or the *state set* corresponding to that point. Please see Example 1 for some specific automaton.

**Example 1.** Consider the minimal automaton for the language \((a_1a_2)^*\), its commutative closure is not regular, as it is precisely the language of words with an equal number of both symbols.

![Minimal Automaton](image)

**Fig. 1.** The minimal automaton of \((a_1a_2)^*\) and a resulting state labelling. Compare this to Example 3 where the labelling is given by a permutation automaton. The final state set is marked by a double circle. See Example 1 for an explanation.

The image of the Parikh morphism could be described by the state label function. In this sense, for a fixed regular language, it is a more fine notion of the Parikh image.

**Proposition 1.** (Connection with Parikh morphism) Assume \( \Sigma = \{a_1, \ldots, a_k\} \).

Let \( \psi : \Sigma^* \rightarrow \mathbb{N}_0^k \) be the Parikh morphism. Suppose \( \mathcal{A} = (\Sigma, Q, \delta, s_0, F) \) is a finite automaton. Let \( \sigma_\mathcal{A} : \mathbb{N}_0^k \rightarrow \mathcal{P}(Q) \) be the state label function, then

\[
\psi(L(\mathcal{A})) = \sigma_\mathcal{A}^{-1}(\{S \subseteq Q \mid S \cap F \neq \emptyset\}).
\]

\(^2\) Here the minimal automaton has the property that no word induces a non-trivial permutation on some subset of states. Languages which admit such automata are called aperiodic in the literature. In some sense these are contrary to group languages, the class considered in this paper.
As \( \text{perm}(L) = \psi^{-1}(\psi(L)) \) for every language \( L \subseteq \Sigma^* \), the next is implied.

**Corollary 1.** Let \( \Sigma = \{a_1, \ldots, a_k\} \) be our alphabet. Denote by \( \psi: \Sigma^* \to \mathbb{N}_0^k \) the Parikh morphism. Suppose \( A = (\Sigma, Q, \delta, s_0, F) \) is a finite automaton. Let \( \sigma_A: \mathbb{N}_0^k \to \mathcal{P}(Q) \) be the state label function, then

\[
\text{perm}(L(A)) = \psi^{-1}(\sigma^{-1}_A(\{S \subseteq Q \mid S \cap F \neq \emptyset\})).
\]

Next, we introduce a notion for the hyperplanes that we will use in Definition 3.

**Definition 2.** (Hyperplane aligned with letter) Let \( \Sigma = \{a_1, \ldots, a_k\} \) and \( j \in \{1, \ldots, k\} \). We set

\[
H_j = \{(p_1, \ldots, p_k) \in \mathbb{N}_0^k | p_j = 0\}.
\]

Suppose \( \Sigma = \{a_1, \ldots, a_k\} \) and \( j \in \{1, \ldots, k\} \). We will decompose the state label map into unary automata. For each letter \( a_j \) and point \( p \in H_j \), we construct unary automata \( A_p^{(j)} \). They are meant to read in inputs in the direction \( \psi(a_j) \), which is orthogonal to \( H_j \). This will be stated more precisely in Proposition 2.

**Definition 3.** (Unary automata along letter \( a_j \)) Let \( \Sigma = \{a_1, \ldots, a_k\} \). Suppose \( A = (\Sigma, Q, \delta, s_0, F) \) is a finite automaton. Fix \( j \in \{1, \ldots, k\} \) and \( p \in H_j \). We define a unary automaton \( A_p^{(j)} = (\{a_j\}, Q_p^{(j)}, \delta_p^{(j)}, s_p^{(0,j)}, F_p^{(j)}) \). But suppose for points \( q \in \mathbb{N}_0^k \) with \( p = q + \psi(b) \) for some \( b \in \Sigma \) the unary automata \( A_q^{(j)} = (\{a_j\}, Q_q^{(j)}, \delta_q^{(j)}, s_q^{(0,j)}, F_q^{(j)}) \) are already defined. Set

\[
\mathcal{P} = \{A_q^{(j)} | p = q + \psi(b) \text{ for some } b \in \Sigma\}.
\]

Let \( I \) and \( P \) be the maximal index and the least common multiple\(^3\) of the periods of the unary automata in \( \mathcal{P} \). Then set

\[
Q_p^{(j)} = \mathcal{P}(Q) \times [I + P]
\]

\[
s_p^{(0,j)} = (\sigma_A(p), 0)
\]

\[
\delta_p^{(j)}((S, i), a_j) = \begin{cases} (T, i + 1) & \text{if } i + 1 < I + P \\ (T, I) & \text{if } i + 1 = I + P. \end{cases}
\]

where

\[
T = \delta(S, a_j) \cup \bigcup_{(q,b) \in \mathbb{N}_0^k \times \Sigma} \delta(\pi_1(\delta_q^{(j)}(s_q^{(0,j)}, a_j^{i+1})), b)
\]

and \( F_p^{(j)} = \{(S, i) \mid S \cap F \neq \emptyset\} \). For some state \( (S, i) \in Q_p^{(j)} \), the set \( S \) will be called the state (set) label, or the state set associated with it.

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\(^3\) Note that in the definition of \( \mathcal{P} \), as \( p \in H_j \), we have \( b \neq a_j \) and \( q \in H_j \). In general, points \( q \in \mathbb{N}_0^k \) with \( p = q + \psi(b) \), for some \( b \in \Sigma \), are predecessor points in the grid \( \mathbb{N}_0^k \).

\(^4\) Note \( \max \emptyset = 0 \) and \( \text{lcm} \emptyset = 1 \).
See Example 2 for concrete constructions of the automata from Definition 3.

Example 2. In Figure 2 we list the unary automata $A^{(2)}_{(0,0)}$, $A^{(2)}_{(1,0)}$, $A^{(2)}_{(2,0)}$ and $A^{(2)}_{(3,0)}$ corresponding to the automaton from Example 1 in order. Some given automaton is constructed from previous ones, in terms of its index and period, according to Definition 3. Note that for example for $A^{(2)}_{(1,0)}$, the state label of the second state is the union of the action of $a_2$ on $\{s_1\}$, i.e., the set $\delta(\{s_1\},a_2)$, but also of $a_1$ on the state label $\{s_2\}$ of the second state of the previous automaton $A^{(2)}_{(0,0)}$. Note also that the second "counter" component is not enough to determine all states, as at the end some automata have equal values in this entry (this is essentially how these automata grow in size).

Start

- $a_2: \{s_0\} \rightarrow \{s_1\}$
- $a_2: \{(s_1,0)\} \rightarrow \{(s_0, s_2), 1\} \rightarrow \{(s_2), 1\}$
- $a_2: \{(s_2), 0\} \rightarrow \{(s_2), 1\} \rightarrow \{(s_0, s_2), 2\} \rightarrow \{(s_2), 2\}$
- $a_2: \{(s_2), 0\} \rightarrow \{(s_2), 1\} \rightarrow \{(s_1, s_2), 2\} \rightarrow \{(s_2), 3\}$

Fig. 2. The unary automata $A^{(2)}_{(0,0)}$, $A^{(2)}_{(1,0)}$, $A^{(2)}_{(2,0)}$ and $A^{(2)}_{(3,0)}$ from Definition 3 derived from the automaton from Example 1. In Example 1 these automata read in inputs in the up direction, but are drawn here horizontally to save space. See Example 2 for more explanation.

Suppose $p \in H_j$ and $j \in \{1, \ldots, k\}$. The next statement makes precise what we mean by decomposing the state label map along the hyperplanes into the automata $A^{(j)}_p = (\{a_j\}, Q^{(j)}_p, \delta^{(j)}_p, s^{(0,j)}_p, F^{(j)}_p)$. Also, it justifies calling the first component of any state $(S,i) \in Q^{(j)}_p$ also the state set label.

Proposition 2. (state label map decomposition) Suppose $\Sigma = \{a_1, \ldots, a_k\}$. Let $1 \leq j \leq k$ and $p = (p_1, \ldots, p_k) \in \mathbb{N}_0^k$. Assume $p \in H_j$ is the projection of $p$ onto $H_j$, i.e., $\overline{p} = (p_1, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_k)$. Then

$$\sigma_A(p) = \pi_1(\delta^{(j)}_p ((\overline{p}, 0^j), a_j^p))$$

for the automata $A^{(j)}_\overline{p} = (\{a_j\}, Q^{(j)}_\overline{p}, \delta^{(j)}_\overline{p}, s^{(0,j)}_\overline{p}, F^{(j)}_\overline{p})$ from Definition 3

With this observation, in Theorem 1 we derive a sufficient condition when the commutative image of some regular language is itself regular. It also gives us a general bound on the size of a minimal automaton, in case the commutative language is regular.

Theorem 1. Let $A = (\Sigma, Q, \delta, s_0, F)$ be some finite automaton. Suppose, for every $j \in \{1, \ldots, k\}$ and $p \in H_j$, with $H_j$ the hyperplane from Definition 3 the
automata \( \mathcal{A}_p^{(j)} = (\{a_j\}, Q_p^{(j)}, \delta_p, s_p^{(0,j)}, F_p^{(j)}) \) from Definition 3 have a bounded number of states, i.e., \( |Q_p^{(j)}| \leq N \) for some \( N \geq 0 \) independent of \( p \) and \( j \). Then the commutative closure \( \operatorname{perm}(L(A)) \) is regular and could be accepted by an automaton of size

\[ \prod_{j=1}^{k} (I_j + P_j), \]

where \( I_j \) denotes the largest index among the unary automata \( \{\mathcal{A}_p^{(j)} \mid p \in H_j\} \), and \( P_j \) the least common multiple of all the periods of these automata. In particular, by the relations of the index and period to the states from Section 2.1, the automaton size is bounded by \( N^k \).

This gives us a general bound in case the commutative closure is regular. We will apply this to the case of group languages and permutation automata in Section 3.2. Theorem 1 has a close relation to Theorem 6.5 from [5], namely case (iii), as we could link the periodic languages introduced in this paper to unary automata, as was done in [12, 13]. This linkage, in general, allows us to give more concrete bounds and constructions. For example, we can list all periodic languages inside the commutative closure, or we can even give concrete bounds on resulting automata. The proof in [5] used more abstract well-quasi order arguments that do not yield concrete automata, nor do they allow the arguments we employ in Section 3.2.

### 3.2 The Special Case of Group Languages

Here we apply Theorem 1 to derive state bounds for group languages. We need some basic observations about permutations, see for example [3]. Every permutation could be written in terms of disjoint cycles. For some element of the permutation domain, by the cycle length of that element with respect to some given permutation, we mean the length of the cycle in which this element appears. The order of some permutation is the smallest power such that the identity permutation results, it is the least common multiple of all cycle lengths for all elements. The next simple observation will be used.

**Lemma 2.** Let \( A \subseteq [n] \). Then if \( m \) is the least common multiple of all the different cycle lengths of elements from \( A \) for some permutation \( \pi : [n] \to [n] \), then \( \pi^m|_A = \text{id}|_A \), in particular \( \pi^m(A) = A \).

Before stating our results, let us make some general assumptions and fix some notions, to make the statements more concise.

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5. Equivalently, the index and period is bounded, which is equivalent with just a finite number of distinct automata, up to (semi-automaton-)isomorphism. We call two automata (semi-automaton-)isomorphic if one automaton can be obtained from the other one by renaming states and alphabet symbols.

6. In this context, the elements are also called points in the literature, but we will stick to the term elements or states.

7. For a given element \( m \in [n] \) and a permutation \( \pi : [n] \to [n] \), this is the number \(|\{\pi^i(m) \mid i \geq 0\}|\), in the literature also called the orbit length of \( m \) under the subgroup generated by \( \pi \).
Assumption 1. (general assumptions for this section) Let \( \Sigma = \{a_1, \ldots, a_k\} \).
Assume a permutation automaton \( A = (\Sigma, Q, \delta, s_0, F) \) is given with \( j \in \{1, \ldots, k\} \) and a point \( p \in H_j \), where \( H_j \) denotes the hyperplane from Definition 2. We denote by \( A_p^{(j)} = (\{a_j\}, Q_p^{(j)}, \delta_p^{(j)}, s_p^{(j)}, F_p^{(j)}) \) the automata from Definition 3. By \( L_j \) we will denote the order of the letter \( a_j \), viewed as a permutation of the states \( Q_j \), i.e., the least common multiple of the cycle lengths of all states.

A crucial ingredient of our arguments will be the following observation.

Lemma 3. Choose the notation from Assumption 4. Then the state set labels of states from \( A_p^{(j)} \) will not decrease in cardinality as we read in symbols, and their cardinality will stay constant on the cycle of \( A_p^{(j)} \). More precisely, let \( (S, x), (T, y) \in Q_p^{(j)} \) be any states. If \( (T, y) = \delta_p^{(j)}((S, x), a_i^r) \) for some \( r \geq 0 \), then \( |T| \geq |S| \). And if \( (S, x) \) and \( (T, y) \) are both on the cycle, i.e., \( (S, x) = \delta_p^{(j)}((S, x), a_i^r) \) and \( (T, y) = \delta_p^{(j)}((S, x), a_i^s) \) for some \( r, s \geq 0 \), then \( |S| = |T| \).

To give state bounds on some resulting automaton, using Theorem 1, we need bounds on the indices and periods of the unary automata from Definition 3. The following result gives us a criterion when we have reached the cycle in these automata, and will be used in deriving the mentioned bounds.

Lemma 4. Choose the notation from Assumption 4. Set

\[ P = \{A_p^{(j)} \mid p = q + \psi(b) \text{ for some } b \in \Sigma\}. \]

Denote by \( I \) the maximal index and by \( P \) the least common multiple of the periods of the unary automata in \( P \). Suppose \( S \subseteq Q \) and let \( L_S = \lcm\{|\delta(s, a_j^i) : i \geq 0| : s \in S\} \) be the least common multiple of the cycle lengths of the elements in \( S \) with respect to the letter \( a_j \), seen as a permutation of the states. Then for \( m \geq I \) and the states from \( Q_p^{(j)} \) which fulfill

\[ (S, x) = \delta_p^{(j)}(s_p^{(0, j)}, a_j^m) \quad \text{and} \quad (T, y) = \delta_p^{(j)}(s_p^{(0, j)}, a_j^{m+\lcm(P, L_S)}) \]

we have that if \( |S| = |T| \), then \( S = T \) and \( x = y \). This also implies that the period of \( A_p^{(j)} \) divides \( \lcm(P, L_S) \).

The next result gives us a bound for the periods of the automata from Definition 3.

Proposition 3. Choose the notation from Assumption 4. Let \( p \in H_j \). Then the periods of all automata \( A_p^{(j)} \) divide \( L_j \).

The criterion for the cycle detection from Lemma 4 could be a little bit relaxed by the next result, which will be more useful for proving a bound on the index of the automata from Definition 4. Intuitively, it bounds the way in which the indices of the automata from Definition 4 can grow.

8 For \( p \in H_j \), the condition \( p = q + \psi(b) \), for some \( b \in \Sigma \), implies \( q \in H_j \) and \( b \neq a_j \).

9 As we assume \( m \geq I \), by Equation 4 from Definition 3, we have \( x \geq I \).
Corollary 2. Choose the notation from Assumption 1. Set
\[ \mathcal{P} = \{ A_p^{(j)} \mid p = q + \psi(b) \text{ for some } b \in \Sigma \}. \]
Denote by \( I \) the maximal index and by \( P \) the least common multiple of the periods of the unary automata in \( \mathcal{P} \). Then for states \((S, x), (T, y) \in Q_p^{(j)}\) with \( x \geq I \) and
\[ (T, y) = \delta_p^{(j)}((S, x), a_i^L_j) \]
we have that \(|T| = |S|\) implies \( T = S \) and \( x = y \).

Finally, we state a bound for the indices of the automata from Definition 3.

Proposition 4. Choose the notation from Assumption 1. Then the index of any automaton \( A_p^{(j)} \) is bounded by \(|Q| - 1\cdot L_j\), where \( T \) is any state set label from a state on the cycle of \( A_p^{(j)} \).

Combining everything gives our state complexity bound.

Theorem 2. Choose the notation from Assumption 1. Then the commutative closure \( \text{perm}(L(A)) \) is regular and could be accepted by some automaton with at most
\[ \prod_{j=1}^{k} ((|Q| - 1)L_j + L_j) = |Q|^k \left( \prod_{j=1}^{k} L_j \right) \]
states.

Proof. First note that Proposition 4 gives in particular that the indices of all automata are at most \((|Q| - 1)L_i\). Also Proposition 3 gives the bound \( L_i \) for the periods. So Theorem 1 gives the result. \( \square \)

Example 3. Let \( \Sigma = \{ a_1, a_2 \} \) and consider the permutation automaton from Figure 3. It is the same automaton as given in \([9]\). As an example for the group language case, we give its state labelling on \( \mathbb{N}_0 \) and an automaton for the commutative closure, constructed from the unary automaton \( A_p^{(j)} \). Note that this is not the minimal automaton, which could be found in \([9]\). Also, note that, with the notational convention from Assumption 1, we have \( L_1 = 3 \) and \( L_2 = 2 \). Hence Theorem 2 gives the bound \( 3^2 \cdot 6 = 54 \). The automaton constructed from the unary automaton \( A_p^{(j)} \) is much smaller here, as the indices stabilize much faster than given by the theoretical bound.

Example 4. Let \( \Sigma = \{ a_1, a_2 \} \) and consider \( A = (\Sigma, Q, \delta, s_0, F) \) with \( Q = [n] \) for some \( n \geq 1 \), \( s_0 \) and \( F \) arbitrary, and \( \delta(0, a_1) = 1, \delta(1, a_1) = 0, \delta(x, a_1) = x \) for \( x \in \{ 2, \ldots, n-1 \}, \delta(x, a_2) = (x+1) \mod n \) for \( x \in [n] \). Then \( \text{perm}(L(A)) \) could be accepted by an automaton of size \( 2n^3 \).

As stated in \([8]\) the maximal order of any permutation on a set of size \( n \) is given by Landau’s function, which is asymptotically like \( e^{\Theta(\sqrt{n \ln n})} \). Hence the next is implied\(^{10}\) by Theorem 2.

\(^{10}\) The reader might notice a close resemblance between our asymptotic bound and the bound given in \([4]\) for the determinization of unary non-deterministic automata.
Corollary 3. For a regular group language whose state complexity is \( n \), the state complexity of the commutative closure is in \( O(n^{\sqrt{n\ln n}}) \).

Also we note a connection with the class of jumping finite automata (JFAs) considered in \([1, 6, 17]\). Without giving a formal definition, let us mention here that a JFA looks syntactically indistinguishable from a classical (nondeterministic) non-deterministic unary automaton we find us in a situation close to the one we face here with the state labels. And for the states in any cycle of some non-deterministic unary automaton we can use Lemma 2 in the power set automaton. Also note that, intuitively, our method of state labelling could somehow be interpreted as a “refined subset construction” for commutative closures.
istic) finite automaton, only the processing of an input word is different: In one step, not necessarily the next symbol of the input is digested, i.e., the first symbol of the (remaining) input word, but just any symbol from the input. Pictorially speaking, the read head of the automaton might jump anywhere in the input before digesting the input symbol that it scans then; after digestion, this input symbol is cut out from the input. A JFA could be seen as a descriptional device for the commutative closure of some regular language. As noted in [7], a regular language is accepted by some jumping finite automaton if and only if it is commutative. For a commutative regular language, we can operate a given deterministic automaton for the language as a jumping automaton, and hence the the size of a minimal jumping automaton is always smaller than the size of a minimal ordinary automaton. For the reverse direction, in the case of group languages, we can derive the next relative descriptional complexity result.

Corollary 4. Let \( L \) be a regular language accepted by some jumping finite automaton with \( n \) states, which, if seen as an ordinary automaton, is a permutation automaton\(^{11}\). Then \( L \) could be accepted by some finite deterministic automaton of size at most \( O(ne^{\sqrt{\ln n}})\).\(^{12}\)

4 Conclusion

We have shown that the commutative closure of regular group languages is regular, and have derived a bound on the size of the resulting automaton. The size is related to the least common multiples of the cycle lengths of the letters, viewed as permutations on the states, see Equation (6). I do not know if the bound is sharp. I have not found a single example that has the property that the index of the constructed automata \( A_p^j \), for the letter \( a_j \), has length \((|Q| - 1) \cdot L_i\), as would be necessary to reach the bound stated in Theorem 2. In fact, I believe that the cycles on individual elements of the state label are never "traversed" in its entirety before another element is added to the state label, or we reach the final cycle of the unary automata \( A_p^j \). So I conjecture that for larger alphabets we can improve this bound, as the state labels grow faster in the index part of the automata \( A_p^j \), as more predecessor automata\(^{12}\) add states of the original automaton to the state labels of \( A_p^j \) as inputs are read. This is somehow contrary to what usually happens in other existing state complexity results, namely that we need larger alphabets to reach the state bounds, see for example [2, 10, 11].

In our situation, I somehow conjecture that for larger alphabets (where surely, distinct letters have to be distinct permutations), indices of the unary automata \( A_p^j \) get smaller and smaller. Hence the overall state complexity bound reaches the product of the least common multiples of the cycle lengths for all letters, i.e., we have \( \prod_{j=1}^k L_j \) as a bound in the limit for \( k \to \infty \), with an alphabet of size \( k \).

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\(^{11}\) Note that by definition, a permutation automaton is deterministic.

\(^{12}\) For some automaton \( A_p^j \), with \( p \in H_j \) and \( j \in \{1, \ldots, k\} \), all automata \( A_q^j \) with \( p = q + \psi(b) \), for some \( b \in \Sigma \), are called predecessor automata of \( A_p^j \).
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5 Appendix

Here, we collect some proofs not given in the main text. Our first Lemma 5 is not stated in the main text, as it is essentially only used in the proofs of this appendix.

**Lemma 5.** (inductive form of state label function) Let $A = (\Sigma, Q, \delta, s_0, F)$ be some finite automaton and $\sigma_A : N_0^k \rightarrow \mathcal{P}(Q)$ the state label function from Definition 7. Then we have $\sigma_A(0, \ldots, 0) = \{s_0\}$, and

$$
\sigma_A(p) = \bigcup_{(q, b) \in \mathcal{P}(Q)} \delta(\sigma_A(q), b)
$$

for $p \neq (0, \ldots, 0)$.

**Proof.** If $p = (0, \ldots, 0)$, then $\delta(s_0, w) : \psi(w) = p = \{\delta(s_0, \varepsilon)\} = \{s_0\}$. Suppose $p \neq (0, \ldots, 0)$. Then

$$
\begin{align*}
\bigcup_{(q, b) \in \mathcal{P}(Q)} \delta(\sigma_A(q), b) & = \bigcup_{(q, b) \in \mathcal{P}(Q)} \delta(\{\delta(s_0, w) \mid \exists w \in \Sigma^* : \psi(w) = q\}, b) \\
& = \bigcup_{(q, b) \in \mathcal{P}(Q)} \{\delta(s_0, wb) \mid \exists w \in \Sigma^* : \psi(w) = q\} \\
& = \bigcup_{(q, b) \in \mathcal{P}(Q)} \{\delta(s_0, wb) \mid \exists w \in \Sigma^* : \psi(w) + \psi(b) = p, \psi(w) = q\} \\
& = \{\delta(s_0, wb) \mid \exists w \in \Sigma^* \exists b \in \Sigma : \psi(wb) = p\} \\
& = \{\delta(s_0, u) : \exists u \in \Sigma^* : \psi(u) = p\}. \quad \Box
\end{align*}
$$

**Remark 1.** (induction scheme used) In certain proofs, namely of Proposition 2, Proposition 3 and Proposition 4, we argue in an inductive fashion. Also, the formulation of Lemma 5 is inductive. This comes from the inductive form that the automata from Definition 3 are defined, or the recursive way that they are created from previous automata. Just in case you are wondering why, in the inductive proofs of Proposition 3 and Proposition 4 no base case is explicitly stated, I will give some justification for that in the next paragraph. But in case you are not wondering, you might well skip this explanation. Suppose we have some property $A$ that we want to show is true for all automata $A^{(j)}_p = (\{a_j\}, Q^{(j)}_p, \delta^{(j)}_p, s^{(0, j)}_p, F^{(j)}_p)$ from Definition 7 where $j \in \{1, \ldots, k\}$ and $p \in H_j$, the hyperplane from Definition 2. Then, our induction scheme is the following.

Fix some $j \in \{1, \ldots, k\}$. If we can show property $A$ for $A^{(j)}_p$ under the assumption that it is true for all automata from the set

$$
\mathcal{P} = \{A^{(j)}_q \mid p = q + \psi(b) \text{ for some } b \in \Sigma\},
$$
then it is true for all automata $A_p^{(j)}$, for $p \in H_j$ arbitrary.

In all our cases, the base case is when $P = \emptyset$, and our arguments will work in that case too. Hence, there is no need to treat that as a special (induction base) case. More specifically, we will use the maximal index and the least common multiple of the automata from $P$. As $\max \emptyset = 0$ and $\lcm \emptyset = 1$, by definition, the arguments, given below in the proofs, will work with these values. Even the original Definition 3 has no explicit base case, but relies on these definitions.

This is related to the fact that $N_0^k$ is well-quasi order (or a well partial order to be more specific). So also the points from $H_j$ are well-quasi ordered. As $\psi(a_j) = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the one appears precisely at the $j$-th position, the condition $p = q + \psi(a_j)$ says that $q$ is an immediate predecessor point. The induction scheme we use hence reduces to an induction scheme over this well partial order. A justification of this induction principle for well-quasi orders could be found, for example, in the thesis On Well-Quasi-Orderings by Forrest B. Thurman.

5.1 Proof of Lemma 1 (See page 3)

Lemma 1. Let $A = (\Sigma, Q, \delta, s, F)$ be some unary automaton. If $\delta(s, a^k) = s$ for some state $s \in Q$ and number $k > 0$, then $k$ is divided by the period of $A$.

Proof. Let $i$ be the index, and $p$ the period of $A$. We write $k = np + r$ with $0 \leq r < p$. First note that $s$ is on the cycle of $A$, i.e.,

$$s \in \{\delta(s, a^i), \delta(s, a^{i+1}), \ldots, \delta(s, a^{i+p-1})\}$$

as otherwise $i$ would not be minimal. Then if $s = \delta(s, a^{i+j})$ for some $0 \leq j < p$ we have $\delta(s, a^{i+k}) = \delta(s, a^{i+p+k}) = \delta(s, a^{i+j+k+(p-j)}) = \delta(s, a^{i+j+(p-j)}) = \delta(s, a^i)$. So $\delta(s, a^i) = \delta(s, a^{i+k}) = \delta(s, a^{i+np+r}) = \delta(s, a^{i+r})$ which gives $r = 0$ by minimality of $p$. □

5.2 Proof of Proposition 1 (See page 4)

Proposition 1. (Connection with Parikh morphism) Assume $\Sigma = \{a_1, \ldots, a_k\}$. Let $\psi : \Sigma^* \rightarrow N_0^k$ be the Parikh morphism. Suppose $A = (\Sigma, Q, \delta, s, F)$ is a finite automaton. Let $\sigma_A : N_0^k \rightarrow \mathcal{P}(Q)$ be the state label function, then

$$\psi(L(A)) = \sigma_A^{-1}(\{S \subseteq Q \mid S \cap F \neq \emptyset\}).$$

Proof. Notation as in the statement of the Proposition. For $p \in N_0^k$, we have

$$\sigma_A(p) \cap F \neq \emptyset \iff \{\delta(s, w) \mid \exists w \in \Sigma^* : \psi(w) = p\} \cap F \neq \emptyset$$

$$\iff \exists w \in \Sigma^* : \delta(s, w) \in F \text{ and } \psi(w) = p$$

$$\iff \exists w \in \Sigma^* : w \in L(A) \text{ and } \psi(w) = p$$

$$\iff p \in \psi(L(A)) \quad \square$$

Note that $\mathcal{P} = \emptyset$ if and only $p = (0, \ldots, 0)$, in which case $A_p^{(j)}$ is isomorphic to the starting automaton $A$.

Thurman, F. B. On Well-Quasi-Orderings. Thesis, University of Central Florida, Orlando, Florida (2013),
http://etd.fcla.edu/CF/CFH0004455/Thurman_Forrest_B_201304_BS.pdf
5.3 Proof of Proposition 2 (See page 6)

Proposition 2. (state label map decomposition) Suppose \( \Sigma = \{a_1, \ldots, a_k\} \). Let \( 1 \leq j \leq k \) and \( p = (p_1, \ldots, p_k) \in \mathbb{N}_0^k \). Assume \( \overline{p} \in H_j \), i.e., \( \overline{p} = (p_1, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_k) \). Then

\[
\sigma_A(p) = \pi_1(\delta^{(j)}_{\overline{p}}(s^{(0,j)}_{\overline{p}}, a^{p_j}_j))
\]

for the automata \( A^{(j)}_{\overline{p}} = (\{a_j\}, q^{(j)}_{\overline{p}}, \delta^{(j)}_{\overline{p}}, s^{(0,j)}_{\overline{p}}, f^{(j)}_{\overline{p}}) \) from Definition 3.

Proof. Notation as in the statement. For \( p = (0, \ldots, 0) \) this is clear. If \( p_j = 0 \), then \( p = \overline{p} \), and, by Equation (3),

\[
\pi_1(\delta^{(j)}_{\overline{p}}(s^{(0,j)}_{\overline{p}}, \varepsilon)) = \pi_1(s^{(0,j)}_{\overline{p}}) = \sigma_A(\overline{p}).
\]

Suppose \( p_j > 0 \) from now on. Then, the set \( \{ (q, b) \in \mathbb{N}_0^k \times \Sigma \mid p = q + \psi(b) \} \) is non-empty, and we can use Equation (7), and proceed inductively

\[
\sigma_A(p) = \bigcup_{q, b \in \mathbb{N}_0^k \times \Sigma} \delta(\sigma_A(q), b)
\]

where \( q = (q_1, \ldots, q_k) \), and \( \overline{p} = (q_1, \ldots, q_{j-1}, 0, q_{j+1}, \ldots, q_k) \in H_j \). As \( p_j > 0 \) we have \( p = q + \psi(a_j) \) for some unique point \( q = (p_1, \ldots, p_{j-1}, p_j - 1, p_{j+1}, \ldots, p_k) \).

For all other points \( r = (r_1, \ldots, r_k) \) with \( p = r + \psi(b) \) for some \( b \in \Sigma \), the condition \( r \neq q \) implies \( b \neq a_j \) and \( r_j = p_j \) for \( r = (r_1, \ldots, r_k) \). Also, if \( \overline{q} \in H_j \) denotes projection to \( H_j \), we have \( \overline{q} = \overline{p} \) for our chosen \( q \) with \( p = q + \psi(a_j) \).

Hence, taken all this together, we can write Equation (3) in the form

\[
\sigma_A(p) = \left( \bigcup_{(r, b), b \neq a_j} \delta(\pi_1(\delta^{(j)}_{\overline{p}}(s^{(0,j)}_{\overline{p}}, a^{p_j}_j)), b) \right) \cup \delta(\pi_1(\delta^{(j)}_{\overline{p}}(s^{(0,j)}_{\overline{p}}, a^{p_j-1}_j)), a_j).
\]

Let \( b \in \Sigma \). As for \( a_j \neq b \), we have that \( p = r + \psi(b) \) if and only if \( \overline{p} = \overline{r} + \psi(b) \), with the notation as above for \( p, r, \overline{p} \) and \( \overline{r} = (r_1, \ldots, r_{j-1}, 0, r_{j+1}, \ldots, r_k) \), we can simplify further and write

\[
\sigma_A(p) = \left( \bigcup_{(r, b), \overline{r} \in H_j, \overline{p} = \overline{r} + \psi(b)} \delta(\pi_1(\delta^{(j)}_{\overline{p}}(s^{(0,j)}_{\overline{p}}, a^{p_j}_j)), b) \right) \cup \delta(\pi_1(\delta^{(j)}_{\overline{p}}(s^{(0,j)}_{\overline{p}}, a^{p_j-1}_j)), a_j).
\]

Set \( S = \pi_1(\delta^{(j)}_{\overline{p}}(s^{(0,j)}_{\overline{p}}, a^{p_j-1}_j)), T = \sigma_A(p) \) and \( P = \{ A^{(j)}_{\overline{p}} \mid \overline{p} = r + \psi(b) \text{ for some } b \in \Sigma \}. \)

\(^{15}\) Note that for \( \overline{p} \in H_j \), the condition \( \overline{p} = q + \psi(b) \), for some \( b \in \Sigma \), implies \( q \in H_j \) and \( b \neq a_j \).
Let $I$ be the maximal index, and $P$ the least common multiple of all the periods, of the unary automata in $\mathcal{P}$. We distinguish two cases for the value of $p_j > 0$.

(i) $0 < p_j \leq I$.

By Equation (4), $\delta^{(j)}(s^{(0,j)}_y, a_{p_j}^{p_j-1}) = (S, p_j - 1)$. In this case Equation (3) equals Equation (2), if the state $(S, p_j - 1)$ is used in Equation (4). This gives

$$\delta_i^{(j)}((S, p_j - 1), a_j) = (T, p_j).$$

Hence $\pi_1(\delta_i^{(j)}((S, p_j - 1), a_j)) = T = \sigma_A(p)$.

(ii) $I < p_j$.

Set $y = I + ((p_j - 1 - I) \mod P)$. Then $I \leq y < I + P$. By Equation (4),

$$\delta_i^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j-1}) = (S, y).$$

So, also by Equation (4),

$$\delta_i^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j}) = \delta_i^{(j)}(S, y, a_j) = \begin{cases} (R, y + 1) & \text{if } I \leq y < I + P - 1 \\ (R, I) & \text{if } y = I + P - 1, \end{cases}$$

where, by Equation (5),

$$R = \delta(S, a_j) \cup \bigcup_{\bar{p} = \bar{r} + \psi(b)} \delta(\pi_1(\delta_i^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j+1})), b). \quad (10)$$

Let $\bar{p} \in H_j$ with $\bar{p} = \bar{r} + \psi(b)$ for some $b \in \Sigma$, and $\bar{p} \in H_j$ the point from the statement of this Proposition. Then, as the period of $A^{(j)}_\bar{p}$ divides $P$, and $y$ is greater than or equal to the index of $A^{(j)}_\bar{p}$, we have

$$\delta^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j-1}) = \delta^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j}).$$

So $\delta^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j}) = \delta^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j+1})$. Hence, comparing Equation (10) with Equation (9), we find $R = T$. \(\square\)

Alternative argument for case (ii): Set $x = I + ((p_j - I) \mod P)$. Then $I \leq x < I + P$. By Equation (4),

$$\delta^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j-1}) = \begin{cases} (S, x - 1) & \text{if } x > I \\ (S, I + P - 1) & \text{otherwise, if } x = I. \end{cases}$$

We have $\delta^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j}) = \delta^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j-1}, a_j)$. Hence, if $I < x < I + P$ we get

$$\delta^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j}) = \delta^{(j)}((S, x - 1), a_j) = (R, x),$$

with, by Equation (5),

$$R = \delta(S, a_j) \cup \bigcup_{\bar{p} = \bar{r} + \psi(b)} \delta(\pi_1(\delta_i^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j})), b),$$

where $\bar{p} \in H_j$ with $\bar{p} = \bar{r} + \psi(b)$ for some $b \in \Sigma$, and $\bar{p} \in H_j$ the point from the statement of this Proposition. Then, as the period of $A^{(j)}_\bar{p}$ divides $P$, and $y$ is greater than or equal to the index of $A^{(j)}_\bar{p}$, we have

$$\delta^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j-1}) = \delta^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j}).$$

So $\delta^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j}) = \delta^{(j)}(s_y^{(0,j)}, a_{p_j}^{p_j+1})$. Hence, comparing Equation (10) with Equation (9), we find $R = T$. \(\square\)
and if $x = I$, we get
\[ \delta_P^{(j)}(s_0^{(j)}, a_j^P) = \delta_P^{(j)}((S, I + P - 1), a_j) = (R, x), \]
with, by Equation (5),
\[ R = \delta(S, a_j) \cup \bigcup_{(\overline{\sigma}, b) \in \overline{\mathcal{P}}} \delta(\pi_1(\delta_P^{(j)}(s_0^{(j)}, a_j^{I+P})), b). \]

Let $\tau \in H_j$ with $\overline{\sigma} = \tau + \psi(b)$ for some $b \in \Sigma$, and $\overline{\sigma} \in H_j$ the point from the statement of this Proposition. Then, as $P$ divides the period of $A_P^{(j)}$, and $x$ is greater than or equal to the index of $A_P^{(j)}$, we have
\[ \delta_P^{(j)}(s_0^{(j)}, a_j^P) = \delta_P^{(j)}((s_0^{(j)}, a_j^P)). \]

Similar, $\delta_P^{(j)}(s_0^{(j)}, a_j^{I+P}) = \delta_P^{(j)}((s_0^{(j)}, a_j^{I+P})).$ So, combining everything so far, we have
\[ R = \delta(S, a_j) \cup \bigcup_{(\overline{\sigma}, b) \in \overline{\mathcal{P}}} \delta(\pi_1(\delta_P^{(j)}(s_0^{(j)}, a_j^P)), b) = T. \]

Hence $\sigma_A(p) = T = R = \pi_1(\delta_P^{(j)}(s_0^{(j)}, a_j^P)).$ \( \square \)

5.4 Proof of Theorem 1 (See page 6)

**Theorem 1.** Let $A = (\Sigma, Q, \delta, s_0, F)$ be some finite automaton. Suppose, for every $j \in \{1, \ldots, k\}$ and $p \in H_j$, with $H_j$ the hyperplane from Definition 3, the automata $A_P^{(j)} = (\{a_j\}, Q^{(j)}_P, \delta^{(j)}_P, s_0^{(j)}_P, F^{(j)}_P)$ from Definition 3 have a bounded number of states, i.e., $|Q^{(j)}_P| \leq N$ for some $N \geq 0$ independent of $p$ and $j$. Then the commutative closure $\text{perm}(L(A))$ is regular and could be accepted by an automaton of size
\[ \prod_{j=1}^k (I_j + P_j), \]
where $I_j$ denotes the largest index among the unary automata $\{A_P^{(j)} \mid p \in H_j\}$, and $P_j$ the least common multiple of all the periods of these automata. In particular, by the relations of the index and period to the states from Section 2.1, the automaton size is bounded by $N^k$.\( \text{\footnote{Equivalently, the index and period is bounded, which is equivalent with just a finite number of distinct automata, up to (semi-automaton-)isomorphism. We call two automata (semi-automaton-)isomorphic if one automaton can be obtained from the other one by renaming states and alphabet symbols.}} \)
Proof. We use the same notation as introduced in the statement of the theorem. Let \( p = (p_1, \ldots, p_k) \in \mathbb{N}_0^k \) and \( j \in \{1, \ldots, k\} \). Denote by \( \sigma_A : \mathbb{N}_0^k \to \mathcal{P}(Q) \) the state label function from Definition 1. Then, with Proposition 2 if \( p_j \geq I_j \), we have
\[
\sigma_A(p_1, \ldots, p_{j-1}, p_j + p_j, p_{j+1}, \ldots, p_k) = \sigma_A(p_1, \ldots, p_k).
\] (11)

Construct the unary semi-automaton \( A_j = (\{a_j\}, Q_j, \delta_j) \) with
\[
Q_j = \{s_0^{(j)}, s_1^{(j)}, \ldots, s_{I_j+p_j-1}^{(j)}\},
\]
\[
\delta_j(s_i^{(j)}, a_j) = \begin{cases} 
  s_i^{(j)} & \text{if } i < I_j \\
  s_{i-1}^{(j)} & \text{if } i \geq I_j
\end{cases}
\]

Then build \( \mathcal{C} = (\Sigma, Q_1 \times \cdots \times Q_k, \mu, s_0, E) \) with
\[
s_0 = (s_0^{(1)}, \ldots, s_0^{(k)}),
\]
\[
\mu((t_1, \ldots, t_k), a_i) = (t_1, \ldots, t_{j-1}, \delta_j(t_j, a_j), t_{j+1}, \ldots, t_k)
\]
for all \( 1 \leq j \leq k \),
\[
E = \{\mu(s_0, u) : u \in L(A)\}
\]

By construction, for words \( u, v \in \Sigma \) with \( u \in \text{perm}(v) \) we have \( \mu((t_1, \ldots, t_k), u) = \mu((t_1, \ldots, t_k), v) \) for any state \( (t_1, \ldots, t_k) \in Q_1 \times \cdots \times Q_k \). Hence, the language accepted by \( \mathcal{C} \) is commutative. We will show that \( L(\mathcal{C}) = \text{perm}(L(A)) \). By choice of \( E \) we have \( L(A) \subseteq L(\mathcal{C}) \), this gives \( \text{perm}(L(A)) \subseteq L(\mathcal{C}) \). Conversely, suppose \( w \in L(\mathcal{C}) \). Then \( \mu(s_0, w) = \mu(s_0, u) \) for some \( u \in L(A) \). Next, we will argue that we can find \( w' \in L(\mathcal{C}) \) and \( u' \in L(A) \) with \( \mu(s_0, w') = \mu(s_0, w) \) and \( \mu(s_0, u') = \mu(s_0, u) \) and max\(|w'|_a_i, |v'|_a_j\} < I_j + P_j \) for all \( j \in \{1, \ldots, k\} \).

(i) By construction of \( \mathcal{C} \), if \( |w'|_a_j \geq I_j + P_j \), we can find \( u' \) with \( |w'|_a_j = |w'|_a_j - P_j \) such that \( \mu(s_0, w') = \mu(s_0, w) \). So, inductively, suppose we have \( u' \in \Sigma^* \) with \( |w'|_a_j < I_j + P_j \) for all \( j \in \{1, \ldots, k\} \) and \( \mu(s_0, u) = \mu(s_0, u') \).

(ii) By Corollary 1, \( \sigma_A(\psi(u)) \cap F \neq \emptyset \). So, if \( |u|_{a_j} \geq I_j + P_j \), by Equation 11, we can find \( u' \) with \( |u'|_{a_j} = |u|_{a_j} - P_j \) and \( \sigma_A(\psi(u')) \cap F \neq \emptyset \). By definition of the state label function, as some word must induce the final state in \( \sigma_A(\psi(u')) \), we can choose \( u' \in L(A) \). Also, by construction of \( \mathcal{C} \), we have \( \mu(s_0, u') = \mu(s_0, u) \). So, after repeatedly applying the above steps, suppose we have \( u' \in L(A) \) with \( \mu(s_0, u) = \mu(s_0, u') \) and \( |u'|_{a_j} < I_j + P_j \) for all \( j \in \{1, \ldots, k\} \).

By construction of \( \mathcal{C} \), for words \( u, v \in \Sigma^* \) with \( \max\{|u|_{a_j}, |v|_{a_j}\} < I_j + P_j \) for all \( j \in \{1, \ldots, k\} \) we have
\[
\mu(s_0, u) = \mu(s_0, v) \iff u \in \text{perm}(v) \iff \psi(u) = \psi(v).
\] (12)

Hence, using Equation 12 for the words \( u' \) and \( u' \) from (i) and (ii) above, as \( \mu(s_0, u') = \mu(s_0, u') \), we find \( \psi(u') = \psi(u') \). So \( \sigma_A(\psi(u')) \cap F \neq \emptyset \) as \( u' \in L(A) \). Now, again using Equation 11, this gives \( \sigma_A(\psi(u)) \cap F \neq \emptyset \), which, by Corollary 1 yields \( w \in \text{perm}(L(A)) \). \( \square \)

The term semi-automaton is used for automata without a designated initial state, nor a set of final states.
5.5 Proof of Lemma 3 (See page 8)

**Lemma 3.** Choose the notation from Assumption 1 Then the state set labels of states from $A_p^{(i)}$ will not decrease in cardinality as we read in symbols, and their cardinality will stay constant on the cycle of $A_p^{(i)}$. More precisely, let $(S, x), (T, y) \in Q_p^{(i)}$ be any states. If $(T, y) = \delta_p^{(i)}((S, x), a_{i}^r)$ for some $r \geq 0$, then $|T| \geq |S|$. And if $(S, x)$ and $(T, y)$ are both on the cycle, i.e., $(S, x) = \delta_p^{(i)}((S, x), a_{i}^r)$ and $(T, y) = \delta_p^{(i)}((S, x), a_{i}^s)$ for some $r, s \geq 0$, then $|S| = |T|$.

**Proof.** Notation as in the statement of the Lemma. By Equation (4) and Equation (5) from Definition 3, intuitively, as we read in symbols in the automaton $A_p^{(i)}$, the state set label of the next state is composed by the transition from the previous state label, and by adding states from nearby automata. And as we only apply permutations, those sets cannot get smaller. More formally, from Equation (5) of Definition 3 we have that $\delta(S, a_{i}^r) \subseteq \pi_1(\delta_p^{(i)}((S, x), a_{i}^r))$ for all $i \geq 0$. As $a_i$ induces a permutation on the state set, we have $|S| = |\delta(S, a_i)|$, which gives the first claim. If $(S, x)$ and $(T, y)$ are both on the cycle, we can map them both onto each other by appropriate inputs, which implies $|S| = |T|$ by the aforementioned fact. □

5.6 Proof of Lemma 4 (See page 8)

**Lemma 4.** Choose the notation from Assumption 1. Set

$$P = \{A_p^{(j)} \mid p = q + \psi(b) \text{ for some } b \in \Sigma\}.$$  

Denote by $I$ the maximal index and by $P$ the least common multiple of the periods of the unary automata in $P$. Suppose $S \subseteq Q$ and let $L_S = \text{lcm}\{|\delta(s, a_{i}^j) : i \geq 0\} : s \in S\}$ be the least common multiple of the cycle lengths of the elements in $S$ with respect to the letter $a_i$, seen as a permutation of the states. Then for $m \geq I$ and the states from $Q_p^{(j)}$ which fulfill

$$(S, x) = \delta_p^{(j)}(s_p^{(0,j)}, a_j^m) \text{ and } (T, y) = \delta_p^{(j)}(s_p^{(0,j)}, a_j^{m+\text{lcm}(P, L_S)})$$

we have that if $|S| = |T|$, then $S = T$ and $x = y$. This also implies that the period of $A_p^{(j)}$ divides lcm$(P, L_S)$.

**Proof.** Notation as in the statement of the Lemma. From Equation (5) of Definition 3 we have $\delta(S, a_{i}^j) \subseteq \pi_1(\delta_p^{(j)}((S, x), a_{i}^r))$ for all $i \geq 0$. So as $\delta(S, a_{i}^j) = \text{id}|_S$, by Lemma 2 this gives $S \subseteq T$. Hence, as $|S| = |T|$, we get $S = T$. Further as $m \geq I$, by Equation (4) of Definition 3 as $P$ divides lcm$(P, L_S)$, we have $x = y$.

By Lemma 1 this implies that the period of $A_p^{(j)}$ divides lcm$(P, L_S)$. □

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18 For $p \in H_i$, the condition $p = q + \psi(b)$, for some $b \in \Sigma$, implies $q \in H_i$ and $b \neq a_j$.
19 As we assume $m \geq I$, by Equation (4) from Definition 3 we have $x \geq I$. 

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5.7 Proof of Proposition 3 (See page 8)

Proposition 3. Choose the notation from Assumption 1. Let \( p \in H_j \). Then the periods of all automata \( A_p^{(j)} \) divide \( L_j \).

Proof. Notation as in the statement of the Proposition. Set
\[
P = \{ A_q^{(j)} | p = q + \psi(b) \text{ for some } b \in \Sigma \}.
\]
Denote by \( I \) the maximal index, and by \( P \) the least common multiple of the periods, of the unary automata from \( P \). Let \( (S, x) \in Q_p^{(j)} \) be any state from the cycle of \( A_p^{(j)} \). From Equation (11) of Definition 3 by inspecting the second "counting" component of the states, we see that the index of \( A_p^{(j)} \) must be greater or equal than \( I \). Hence \( x \geq I \). By Equation (11) we have \( (S, x) = \delta_p^{(j)}((s_p^{0,(j)}, a_j^{(j)}) \).

Denote by \( L_S \) the least common multiple of the cycle lengths of states from \( S \) with respect to the letter \( a_j \), seen as a permutation of the states. Consider \( (T, y) = \delta_p^{(j)}((S, x), a_j^{lcm(P,L_S)}) \). By Lemma 3 as they are both on the cycle, we have \( |T| = |S| \). Then, using Lemma 4 we find \( T = S \) and \( x = y \), and that the period of \( A_p^{(j)} \) divides \( lcm(P,L_S) \). Obviously, \( L_S \) divides \( L_j \). Inductively, the periods of all automata in \( P \) divide \( L_j \), and so \( P \) divides \( L_j \). Hence, \( lcm(P,L_S) \) divides \( L_j \). So the period of \( A_p^{(j)} \) divides \( L_j \). □

5.8 Proof of Corollary 2 (See page 9)

Corollary 2. Choose the notation from Assumption 1. Set
\[
P = \{ A_q^{(j)} | p = q + \psi(b) \text{ for some } b \in \Sigma \}.
\]
Denote by \( I \) the maximal index and by \( P \) the least common multiple of the periods of the unary automata in \( P \). Then for states \( (S, x), (T, y) \in Q_p^{(j)} \) with \( x \geq I \) and
\[
(T, y) = \delta_p^{(j)}((S, x), a_j^{L_j})
\]
we have that \( |T| = |S| \) implies \( T = S \) and \( x = y \).

Proof. We choose the same notation as in the statement of the Corollary. By Proposition 3 the number \( P \) divides \( L_j \). For any subset \( R \subseteq Q \), if \( L_R \) is the least common multiple of the cycle lengths of elements from \( R \), it is also divided by \( L_j \). Hence \( lcm(L_R, P) \) divides \( L_j \). If \( R, S \subseteq Q \) and \( 0 \leq l \leq L_j \), by Lemma 4 if \( (R, z) = \delta_p^{(j)}((S, x), a_j^l) \), then \( |R| = |S| \), as \( |S| \leq |R| \leq |T| \), and \( |S| = |T| \) by assumption. Let \( L_S \) be the least common multiple of the cycle lengths of elements in \( S \). As \( lcm(L_S, P) \) divides \( L_j \), we have, for
\[
(R, z) = \delta_p^{(j)}((S, x), a_j^{lcm(L_S,P)}) \tag{13}
\]
by the previous reasoning that \( |R| = |S| \). As, by assumption, \( x \geq I \), and as \( (S, x) = \delta_p^{(j)}((s_p^{0,(j)}, a_j^{(j)}) \), we can use Lemma 4 which gives \( R = S \) and \( x = z \).

But, as \( lcm(P,L_S) \) is a divisor of \( L_j \), this gives \( T = S \) and \( x = y \) by repeatedly applying Equation (13) □

20 See Remark 11 for an explanation of the induction scheme used.
5.9 Proof of Proposition 4 (See page 9)

**Proposition 4.** Choose the notation from Assumption 4. Then the index of any automaton $A_p^j$ is bounded by $(|T| - 1) \cdot L_j$, where $T$ is any state set label from a state on the cycle of $A_p^j$.

**Proof.** Let $j \in \{1, \ldots, k\}$, $p \in H_j$ and $A_p^j = (\{a_j\}, Q_p^j, \delta_p^j, \sigma_p^j, F_p^j)$ the automaton from Definition 3. Set $P$ and denote by $\mathcal{P}$ the periods of the unary automata in $P$. By construction, and Equation (4) from Definition 3, we have $\delta_p^j (s_p^j (0, 0), a_j) (which implies $y_0 = I$) and 

$$ (T_n, y_n) = \delta_p^j ((T_{n-1}, y_{n-1}), a_j) $$

for $n > 0$.

(i) **Claim:** Let $(T, x) \in Q_p^j$ be some state from the cycle of $A_p^j$. Then the state $\delta_p^j (s_p^j (0, 0), a_j)$ is also from the cycle of $A_p^j$.

By construction, and Equation (4) from Definition 3 we have $y_n \geq I$ for all $n$. If $T_{n+1} \neq T_n$, then, by Corollary 2 and Lemma 3 we have $|T_{n+1}| > |T_n|$. Hence, by finiteness, we must have a smallest $m$ such that $T_{m+1} = T_m$. By Corollary 2 this also implies $y_{m+1} = y_m$. Hence, we are on the cycle of $A_p^j$, and the period of this automaton divides $L_j$ by Proposition 3. This yields $(T_n, y_n) = (T_m, y_m)$ for all $n \geq m$. By Lemma 3 the size of the state label sets on the cycle stays constant, and just grows before we enter the cycle. As we could add at most $|T_m| - |T_0|$ elements, and for $T_0, T_1, \ldots, T_m$ each time at least one element is added, we have, as $m$ was chosen minimal, that $m \leq |T| - |T_0|$, where $T$ is any state label on the cycle, which all have the same cardinality $|T| = |T_h|$ by Lemma 3. This means we could read at most $|T| - |T_0|$ times the sequence $a_j^{L_j}$, starting from $(T_0, I)$, before we enter the cycle of $A_p^j$.

(ii) **Claim:** We have $I \leq (|T_0| - 1)L_j$.

Let $A_p^j \in \mathcal{P}$ and suppose $p = q + \psi(b)$ for $b \in \Sigma$. By Equation (5) from Definition 3 for any $(S, x) = \delta_q^j (s_q^j (0, 0), a_j)$ we have $\delta(S, b) \subseteq \pi_1(\delta_p^j (s_p^j (0, 0), a_j))$. In particular for $n = I$ we get $\delta(S, b) \subseteq T_0$, which gives $|S| \leq |T_0|$, as $b$ induces a permutation on the states. Also for $n \geq I$ we are on the cycle of $A_p^j$.

More specifically, in that case $p = (0, \ldots, 0)$ and the reachable part from the start state of $A_p^j$ is essentially the reachable part of $A$, by restricting to inputs from $\{a_j\}^*$.  

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Inductively\footnote{See Remark 1 for an explanation of the induction scheme used. The arguments in Claim (i) and (ii) would also work for $P = \emptyset$ with a little adaptation, but we preferred to state the case $P = \emptyset$ here explicitly.} then the index of $A^{(j)}_p$ is at most $(|S| - 1)L_j \leq (|T_0| - 1)L_j$. As $A^{(j)}_p \in \mathcal{P}$ was chosen arbitrary, we get $I \leq (|T_0| - 1)L_j$.

With Claim (ii) we can derive the upper bound $(|T| - 1)L_j$ for the length of the word $a^{I + (|T| - |T_0|)L_j}_j$ used in Claim (i), as

$$I + (|T| - |T_0|)L_j \leq (|T_0| - 1)L_j + (|T| - |T_0|)L_j = (|T| - 1)L_j.$$ 

And because (i) essentially says that the index of $A^{(j)}_p$ equals at most $I + (|T| + |T_0|)L_j$, this gives our bound for the index of $A^{(j)}_p$. \qed

### 5.10 Proof of Corollary 4 (See page 11)

**Corollary 4.** Let $L$ be a regular language accepted by some jumping finite automaton with $n$ states, which, if seen as an ordinary automaton, is a permutation automaton\footnote{Note that by definition, a permutation automaton is deterministic.}. Then $L$ could be accepted by some finite deterministic automaton of size at most $O(n e^{\sqrt{n \ln n}} |L|)$.

**Proof.** By assumption, the jumping automaton, viewed as an ordinary automaton, is a permutation automaton. Hence, it is deterministic. As the language accepted by any jumping automaton is the permutation closure of the language accepted by it in the ordinary, reading left to right, fashion, the claim follows from Corollary 3.

### 5.11 Some Additional Remarks and Alternative Proofs

Here, I collect some remarks, or alternative proofs, that might be of additional interest. As far as the paper is concerned, it is self-contained without the content of this section.

**Definition 4.** The shuffle operation, denoted by $\shuffle$, is defined as

$$u \shuffle v := \left\{ x_1 y_1 x_2 y_2 \cdots x_n y_n \mid u = x_1 x_2 \cdots x_n, v = y_1 y_2 \cdots y_n, \quad x_1, y_i \in \Sigma^*, 1 \leq i \leq n, n \geq 1 \right\},$$

for $u, v \in \Sigma^*$ and $L_1 \shuffle L_2 := \bigcup_{x \in L_1, y \in L_2} (x \shuffle y)$ for $L_1, L_2 \subseteq \Sigma^*$.

The languages accepted by the automata $A^{(j)}_p$ do not seem to play any role in the main part of the paper. But actually, some interesting relationships nevertheless hold true.

**Lemma 6.** Suppose $\Sigma = \{a_1, \ldots, a_k\}$. Let $A = (\Sigma, Q, \delta, s_0, F)$ be a finite automaton. Choose $j \in \{1, \ldots, k\}$ and $p \in H_j$, then

$$L(A^{(j)}_p) = \pi_{a_j} \left\{ u \mid \exists v \in \Sigma \setminus \{a_j\}^* : u \in v \shuffle a^*_j, \psi(w) = p, u \in \text{perm}(L(A)) \right\}$$

where $\pi_{a_j} : \Sigma^* \rightarrow \{a_j\}^*$ is given by $\pi_{a_j}(a_j) = a_j$ and $\pi_{a_j}(a_i) = \varepsilon$ for $i \neq j$.
Proof. By Proposition 2, Proposition 1 and Definition 3 we have
\[ a_j^n \in L(A_p^{(j)}) \iff \pi_1(\delta_p^{(j)}(s_p^{(0,j)}, a_j^n)) \cap F \neq \emptyset, p \in H_j \]
\[ \iff \sigma_A(p + \psi(a_j^n)) \cap F \neq \emptyset, p \in H_j \]
\[ \iff p + \psi(a_j^n) \in \psi(L(A)), p \in H_j \]
\[ \iff a_j^n \in \pi_a_j(\{u \mid \psi(u) = \psi(a_j^n) = p, u \in L(A)\}) \]

And \( \{u \mid \psi(u) = \psi(a_j^{[u]|a_j}) = p, u \in L(A)\} = \{u \mid \exists w \in \Sigma^* : u \in w \cup a_j^*, \psi(w) = p \in H_j, u \in L(A)\} = \{u \mid \exists w \in \Sigma \setminus \{a_j\}^* : u \in w \cup a_j^*, \psi(w) = p, u \in \text{perm}(L(A))\} \).

**Proposition 5.** Suppose \( \Sigma = \{a_1, \ldots, a_k\} \). Let \( A = (\Sigma, Q, \delta, s_0, F) \) be a finite automaton. Choose \( j \in \{1, \ldots, k\} \), then
\[ \text{perm}(L(A)) = \bigcup_{p \in H_j} \bigcup_{\psi(w) = p} w \cup L(A_p^{(j)}). \]

**Proof.** Let \( w \in \Sigma^* \) with \( \psi(w) = (p_1, \ldots, p_k) \), then \( w \in u \cup a_j^{p_j} \) for some unique \( u \in (\Sigma\setminus\{a_j\})^* \). Set \( \overline{p} = (p_1, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_k) \in H_j \). By Corollary 1, Proposition 2 and Definition 3 we have the equivalences:
\[ w \in \text{perm}(L(A)) \iff \sigma_A(\psi(w)) \cap F \neq \emptyset \]
\[ \iff \pi_1(\delta_{\overline{p}}^{(j)}(s_{\overline{p}}^{(0,j)}, a_j^{p_j})) \cap F \neq \emptyset \text{ and } \psi(w) = \overline{p} + p_j \]
\[ \iff a_j^{p_j} \in L(A_{\overline{p}}^{(j)}) \text{ and } \psi(w) = \overline{p} + p_j \text{ and } \overline{p} \in H_j \]
\[ \iff w \in u \cup L(A_{\overline{p}}^{(j)}) \text{ and } \psi(u) = \overline{p} \in H_j. \quad \square \]

\[ \] Note that \( L \cup \emptyset = \emptyset \) for any language \( L \subseteq \Sigma^*. \)