Pfaffian structures and certain solutions to BKP hierarchies II.
Multiple integrals.

A. Yu. Orlov∗ T. Shiota† K. Takasaki‡

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Abstract

We introduce a useful and rather simple classes of BKP tau functions which which we shall shall
call “easy tau functions”. We consider the “large BKP hierarchy” related to $O(2\infty + 1)$ which was
introduced in [3] (which is closely related to the DKP $O(2\infty)$hierarchy introduced in [9]). Actually
“easy tau functions” of the small BKP was already considered in [29], here we are more interested
in the large BKP and also the mixed small-large BKP tau functions [3]. Tau functions under
consideration are equal to sums over partitions and to multi-integrals. In this way they may be
applicable in models of random partitions and to multi-integrals. In this way they may be
applicable in models of random partitions and to multi-integrals. Here in the part II we
consider multi-integrals and series of N-ply integrals in N. Relations to matrix models is explained.
This part of our work may be viewed as a development of the paper by J.van de Leur [4] related to
orthogonal and symplectic ensembles of random matrices.

Key words: integrable systems, Pfaffians, symmetric functions, Schur and projective Schur func-
tions, random partitions, random matrices, orthogonal ensembles, symplectic ensembles, interpolating
ensembles.

1 Introduction

This is the second part of the paper ”Pfaffian structures and certain solutions to BKP hierarchies”
devoted to the special family of tau functions which may be called “easy tau functions”; for the first part
see [2]. In the first part we consider sums over partitions originating from both small and large BKP
hierarchies introduced respectively in [9] and [3]. We shall refer the large BKP hierarchy just as the
BKP one, and tau functions of the large BKP as BKP tau function. Here we consider certain classes of
multiple integrals which depend on parameters and which may be treated as BKP tau functions where
the parameters play the role of higher times. Let us note that sums and integrals related to the small
BKP hierarchy was previously considered in [29].

In both parts of our work we use the fermionic approach to tau functions suggested in [9] and used
in [3] to study large BKP, multicomponent BKP and mixed of large and small BKP tau functions.

Sums of the previous part of our paper [2] and the integrals considered below may be related in two
different ways. (A) The first way is the straightforward specification of the integration measure which
may be chosen as the sum of Dirac delta functions. (B) The second is the presentation of an integral
in form of the asymptotic series (in this case the integral may be viewed as the Borel summation of the
series). In this second way we equate multiple integrals to multiple series and this may be viewed as a
sort of Fourier transform. In the fermionic method which we use the second way follows directly from
the formula

$$\psi(x) = \sum_{n \in \mathbb{Z}} \psi_n x^n$$

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sort of Fourier transform. In the fermionic method which we use the second way follows directly from
the formula

$$\psi(x) = \sum_{n \in \mathbb{Z}} \psi_n x^n$$ (1)
which allows to re-write integrals over \( x \) which enter expectation values in terms of sums.

Some multiple integrals considered below originate from studies of ensembles of random matrices. The link between matrix integrals and soliton theory was found in [18] for \( \beta = 2 \) ensembles, and in [5], [6] for orthogonal \( \beta = 1 \) and symplectic \( \beta = 4 \) ensembles of random matrices. Here we complete the list of multi-integrals and sums which may be recognized as (large) BKP tau functions. In particular we consider partition functions for circular \( \beta = 1, 2, 4 \) ensembles as BKP tau functions. In case of applications to matrix models BKP higher times play the role of the so-called coupling constants; then it is convenient to write the perturbation series in coupling constants in form of a sum over partitions. The perturbation series may be asymptotic one in this case the multi-integral if it is well-defined play a role of Borel summation of the series.

2 Multiple integrals

We shall consider two types of multi-integrals. The first type is a \( N \)-integral where each of these integrals is evaluated along the same contour, say \( \gamma \), namely, this is a integral over \( \gamma \times N \). Examples of such integrals applicable for the presentation of the orthogonal and symplectic ensembles are widely known. The contour may be a circle \( S^1 \) (for circular ensembles), or, it may be a real line \( \mathbb{R} \). The second type are integrals over \( \mathbb{C}^N, N = 2n \) where each integral over \( \mathbb{C} \) is actually evaluated over a domain in the complex plane, say, upper halfplane. To describe ensembles like the real Ginibre one we need the both types of integrals.

To describe integrals below we need certain data denoted by \( \bar{A} := (A, a) \) (compare with [2]) where \( A \) and \( a \) are respectively functions of two and one variables provided \( A(z, z') = -A(z', z) \) (instead of functions distribution may be also considered). The notation \( \bar{A}(z) \) is analogous to (200), denoting the Pfaffian (see Appendix A.1 for the definition) of an skew symmetric matrix \( \bar{A} \):

\[
\bar{A}(z) := \text{Pf}[\bar{A}]
\]

whose entries are defined, depending on the parity of \( N \), in terms of a skew symmetric kernel \( A(z, w) \) (possibly, a distribution) and a function (or a distribution) \( a(z) \) as follows:

For \( N = 2n \) even

\[
\bar{A}_{ij} = -\bar{A}_{ji} := A(z_i, z_j), \quad 1 \leq i < j \leq 2n
\]

For \( N = 2n - 1 \) odd

\[
\bar{A}_{ij} = -\bar{A}_{ji} := \begin{cases} A(z_i, z_j) & \text{if } 1 \leq i < j \leq 2n - 1, \\ a(z_i) & \text{if } 1 \leq i < j = 2n \end{cases}
\]

In addition we define \( \bar{A}_0 = 1 \).

2.1 Integrals along contours

Integrals along \( \gamma \times N \). First we generalize some results of [4], where nice fermionic expressions were found for the partition functions of orthogonal and symplectic ensembles in \( \gamma = \mathbb{R} \) case.

Let \( d\mu \) be a measure supported on a contour \( \gamma \) on the complex plane. We suppose that \( \varsigma \) is a parameter along the contour. Our main examples of \( \gamma \) are as follows:

(A) An interval on the real axes \( -\infty < z < \infty \). Then \( \varsigma(z) = z \in \mathbb{R} \)
(B) A segment of the unit circle: given by \( z = e^{i\varphi}, 0 \leq \varphi < \theta, \ 0 \leq \theta < 2\pi \). In this case \( \varsigma(z) = \arg(z) \)

We shall study \( N \)-fold integrals over the cone \( \Lambda_N \)

\[
z = (z_1, \ldots, z_N) \in \Lambda_N \quad \text{iff} \quad z_1, \ldots, z_N \in \gamma \quad \text{and} \quad \varsigma(z_1) > \cdots > \varsigma(z_N)
\]

defined as follows

\[
I^{(1)}(t^*, N, \bar{A}) := \int_{\Lambda_N} \Delta_N(z) \bar{A}(z) \prod_{i=1}^N d\mu(z_i, t^*)
\]
where the notation $\tilde{A}(z)$ was defined earlier in [2] and where

$$\Delta_N(z) = \prod_{i<j} (z_i - z_j)$$

and

$$d\mu(z, t^*) := e^{\sum_{n=1}^{\infty} t_n z^n + t_0 \log z - \sum_{n=1}^{\infty} t_n z^{-n}} d\mu(z)$$

(8)

Here the set $t^* = \{t_n^*, n \in \mathbb{Z}\}$ are parameters (sometimes called coupling constants). We assume that the measure $d\mu(z, t^*)$ is chosen in a way that the integral $\tilde{A}$ is convergent.

Using various specifications of $\tilde{A}$ we in particular obtain

$$I_1^{(1)}(t^*, N) := \int_{\Lambda_N} \Delta_N(z) \prod_{i=1}^{N} d\mu_1(z_i, t^*)$$

(9)

$$I_2^{(1)}(t^*, N = 2n) := \int_{\Lambda_n} \Delta_n(z^2) \prod_{i=1}^{n} d\mu_2(z_i, t^*)$$

(10)

$$I_4^{(1)}(t^*, N = 2n) := \int_{\Lambda_n} \Delta_n(z^4) \prod_{i=1}^{n} d\mu_4(z_i, t^*)$$

(11)

where

$$d\mu_1(z, t^*) = d\mu(z, t^*) = e^{\sum_{n=1}^{\infty} t_n z^n + t_0 \log z - \sum_{n=1}^{\infty} t_n z^{-n}} d\mu(z)$$

(12)

$$d\mu_2(z, t^*) = (-1)^{t_0} e^{2\sum_{n=1}^{\infty} t_n z^n + 2t_0 \log z - 2 \sum_{n=1}^{\infty} t_n z^{-2n}} d\mu(z)$$

(13)

$$d\mu_4(z, t^*) = e^{2\sum_{n=1}^{\infty} t_n z^n + 2t_0 \log z - 2 \sum_{n=1}^{\infty} t_n z^{-2n}} d\mu(z)$$

(14)

These three integrals are related to the well-known random ensembles if $\gamma = \mathbb{R}$ or $\gamma = S^1$, see [16]. In case $\gamma = \mathbb{R}$ integrals $I_1^{(1)}, I_2^{(1)}$ and $I_4^{(1)}$ describe respectively the models of random symmetric, of real symmetric and of symplectic matrices. In case $\gamma = S^1$ integrals $I_1^{(1)}, I_2^{(1)}$ and $I_4^{(1)}$ describe respectively $\beta = 1, 2, 4$ circular ensembles.

Other examples:

$$I_m^{(1)}(t^*, N) := \int_{\Lambda_N} \Delta_N^{(m)}(z) \Delta_N(z) \prod_{i=1}^{N} d\mu_m(z_i, t^*)$$

(15)

where by $\Delta^{(m)}$ we denote the following various Vandermond-like products:

$$\Delta_N^{(5)}(z) = \Delta_N^{(5)}(z, f, c) := \prod_{i<j} \frac{f(z_i) - f(z_j)}{f(z_i) + f(z_j) + cf(z_i)f(z_j)}$$

(16)

($c$ is a constant and $f$ is an arbitrary function). The case $c = 0, f(z) = z$ is related to the so-called Bures ensembles [12] and for $c = 0$ let us use the notation

$$\Delta_N(z) := \prod_{i<j} \frac{z_i - z_j}{z_i + z_j}$$

(17)

used earlier in [29]. Then

$$\Delta_N^{(6)}(z) := \prod_{i<j} \frac{f(z_i) - f(z_j)}{1 - f(z_i)f(z_j)}$$

(18)

$$\Delta_N^{(7)}(z) := \prod_{i<j} \frac{f(z_i) - f(z_j)}{f(z_i) + f(z_j)} \frac{1}{Hf} \left( \frac{1}{f(z_i) + f(z_j)} \right)$$

(19)

1 Via fermionic construction used both in [2] and in the present paper sums $S_i^{(1)}$ may be related to $I_i^{(1)}$. In our notations below we keep the numeration adopted in [2] which is rather conventional.
where $H_f$ denotes the Hafnian (see Appendix A.1, where one needs to pay attention to the case of $N$ odd where we add a variable $z_{N+1}$ to the set of $z_1, \ldots, z_N$ and then put $z_{N+1} = 0$).

$$\Delta_N^{(8)}(z) := \Delta_N(f(z)) = \prod_{i<j}^N (f(z_i) - f(z_j))$$

$$\Delta_N^{(9)}(z) := \text{Pf} \frac{1}{f(z_i) - f(z_j)} = \frac{1}{\Delta_N(f(z))} \text{Symm} \left( \prod_{i,j \in J_1} (f(z_i) - f(z_j))^2 \prod_{i,j \in J_2} (f(z_i) - f(z_j))^2 \right)$$

where the last equality is known due to the works on quantum Hall effect \cite{90} where $z_i$ is related to the particle $i$. The two subsets, $J_1$ and $J_2$, each have $N/2$ particles for $N$ even and $(N-1)/2$ and $(N+1)/2$ for odd $N$. $\text{Symm}$ indicates the symmetrization over the distributions of the particles (variables $z_i$, $i = 1, \ldots, N$) into these subsets.

At last $\Delta_n^{(10)}(z)$ is the Vandermond determinant

$$\Delta_n^{(10)}(z) := \Delta_{2n}(z_1, \frac{1}{z_1}, \ldots, z_n, \frac{1}{z_n}) = \Delta_n(z)^2 \prod_{i,j=1}^{n} (1 - z_i z_j) \prod_{i=1}^{n} z_i^{1-2n}$$

Let us note that up to a sign factor $\Delta_n^{(10)}(z)$ coincides with its absolute value in case all $|z_i| = 1$.

Now let us present the specifications of data $A$ giving rise to the integrals (9)–(15). For (26)–(31) we have used the series of papers by Ishikawa and co-authors (see \cite{89, 87} and references therein) as a source of Pfaffian relations. Eq-s. (21), (30) borrowed from \cite{90}.

$$A_1(z_i, z_j) = \frac{1}{2} \text{sgn} \left( \zeta(z_i) - \zeta(z_j) \right), \quad a_1(z) = 1$$

$$A_2(z_i, z_j) = \delta(z_i + z_j) \text{sgn} \left( \zeta(z_i) - \zeta(z_j) \right)$$

$$A_4(z_i, z_j) = \frac{1}{2} (\partial_{z_i} \delta(z_i - z_j) - (i \leftrightarrow j))$$

$$A_5(z_i, z_j) = \frac{f(z_i) - f(z_j)}{f(z_i) + f(z_j) + cf(z_i)f(z_j)}$$

$$A_6(z_i, z_j) = \frac{f(z_i) - f(z_j)}{1 - f(z_i)f(z_j)}$$

$$A_7(z_i, z_j) = \frac{f(z_i) - f(z_j)}{(f(z_i) + f(z_j))^2}$$

$$A_8(z_i, z_j) = \frac{(f(z_i))^n - (f(z_j))^n}{(f(z_i) - f(z_j))^2}$$

$$A_9(z_i, z_j) = \frac{1}{f(z_i) - f(z_j)}$$

$$A_{10}(z_i, z_j) = \frac{1}{2} \left( \delta \left( z_i - \frac{1}{z_j} \right) - (i \leftrightarrow j) \right)$$

**Proposition 1.** For each choice of contour $\gamma$, data $d\mu$ and $A$ provided that the integral $I^{(1)}(t^*, N, \tilde{A})$ exists, this integral is a tau function of the large 2-BKP hierarchy with respect to the time variables $t^*$ and the discrete variable $N$. 

Applications and remarks. Applications of the integrals \(I_1^{(1)}\) we know are as follows

(0) The series \(\sum\) for a special choice of data \(A\) may be identified with the partition function of the Mehta-Pandey interpolating ensembles, see [10], Chapter 14. This link will be explained below.

(1) In case \(\gamma = \mathbb{R}\) the integral \(I_1^{(1)}\) (up to a factor equal to the volume of \(O(N)\) group) coincides with the partition function of the orthogonal Wigner-Dyson ensemble (see [16], Chapter 7) with a generalized (not necessarily Gaussian) probability weight parametrized by \(t^*\). The link of this model with integrable systems (Pfaff lattice) was discovered in [5]. The nice expression for \(I_1^{(1)}\) as fermionic vacuum expectation value was found in [4] and in this way it was shown that \(I_1^{(1)}\) is a tau function of the large BKP hierarchy. In case \(\gamma = S^1\) the integral \(I_4^{(1)}\) is a partition function for the \(\beta = 1\) circular ensemble, see [10], Section 10.1. In the present paper we consider an arbitrary \(\gamma\) and add the dependence on \(t_n\), \(n < 0\).

(2) The integral \(I_2^{(1)}\) (up to a factor) coincides with the partition function of the ensemble of anti-symmetric Hermitian matrices, see [16], Chapter 13. The relation of this ensemble to integrable systems follows from the fact that in the variables \(x = z^2\) it coincides with the known “one-matrix model” which known to be Toda chain tau function [18].

(3) In case \(\gamma = \mathbb{R}\) the integral \(I_4^{(1)}\) (up to a factor) coincides with the partition function of the symplectic Wigner-Dyson ensemble (see [16], Chapter 8) with a generalized (not necessarily Gaussian) probability weight parametrized by \(t^*\). The nice expression for \(I_4^{(1)}\) as fermionic vacuum expectation value was found in [4] and in this way it was shown that \(I_4^{(1)}\) is a tau function of the large BKP hierarchy. In case \(\gamma = S^1\) the integral \(I_4^{(1)}\) is a partition function for the \(\beta = 4\) circular ensemble, see [10], Section 10.2 The integral \(I_6^{(1)}\) where \(f(z) = z\) and \(c = 0\) describe the so-called \(\hat{A}_0\) statistical model, see formula (30) in [66], and also the so-called Bures ensembles which appears in quantum chaos problems where random density matrix appear [12]. The integrals \(I_5^{(1)}\) where \(f(z) = e^{iz}\) and \(c = 0\) contains

\[
\Delta_N(z) \Delta_N^{(0)}(z, f, 0) \prod_{i=1}^N dz_i = \prod_{k<j}^N (z_k - z_j) \prod_{k<j}^N \tanh \pi (z_k - z_j) \prod_{i=1}^N dz_i
\]

which up to a normalization constant coincides with the Plancheral measure for the group \(SL(N, \mathbb{R})\) (and for the symmetric space \(SL(N, \mathbb{R})/SO(N)\)), see Section 17.2.8 in [10] where put \(\lambda_i = \frac{1}{2} z_i^*\).

(6)-(7) Unknown

(8) Integral \(I_8^{(1)}\) where \(f(z) = z\) may be considered as the ground state wave function for fractional quantum Hall state with filling factor 5/2 (MooreRead state), see for instance [91], re-written in moment representation (moments \(t\) were introduced in [72] to describe quantum Hall droplets in the quasiclassical limit).

(9) Unknown

(10) Unitary ensemble (under the change \(u_i = z_i + z_i^{-1}\)).

2.2 Integrals over complex plane

Here we shall consider more general case of multi-integrals over complex planes. First let us introduce the ordering of a given number of points \(z_1, \ldots, z_N\) on the complex plane. We impose

\[
\Re z_i \geq \Re z_{i+1}
\]

and if \(\Re z_i = \Re z_{i+1}\) then \(\Im z_i \geq \Im z_{i+1}\). Then we introduce

\[
z = (z_1, \ldots, z_N) \in \mathbb{M}_N \quad \text{iff} \quad \begin{cases} \Re z_i > \Re z_{i+1} \\
\Im z_i > \Im z_{i+1}
\end{cases} \quad \text{if} \quad \Re z_i = \Re z_{i+1}.
\]

By \(\bar{z}\) we shall denote complex conjugated to \(z\) (it should not be mixed with the special notation \(\bar{A} \coloneqq (A, a)\)).

\[\text{In [66] the grand partition function for } \hat{A}_0 \text{ model was considered and it was shown that it is a KdV tau function}\]
Now let us consider 2N-ply integrals similar to (37) where the integral over \( \Lambda_N \) is replaced by the integral over \( \mathbb{M}_N \):
\[
I^c(t, N) = \int_{\mathbb{M}_N} \Delta_N(z) \tilde{A}(z) \prod_{i=1}^{N} e^{\varphi(z_i,t^*)} d\mu(z_i, \bar{z}_i)
\]  
(34)
where \( d\mu(z, \bar{z}) = d\mu(\bar{z}, z) \) is an arbitrary (perhaps, complex) symmetric measure.

**Ginibre and interpolating ensembles. The models.** Details concerning the so-called Ginibre ensembles may be found in [10], Ch. 15.

Consider the Ginibre ensembles \( (i = 1, 2, 4 \text{ belows is related respectively to real, complex and real quaternionic Ginibre ensembles}) \) with a deformed measure
\[
J_i^{Gin}(t, N) = \int d\mu_i(Z, t), \quad i = 1, 2, 4
\]  
(35)
where \( Z \)
\[
d\mu_i(Z, t) = e^{\sum_{m=1}^{\infty} t_m \text{Tr}^m d_i \mu(Z)}
\]  
(36)
where the measure \( d_i \mu(Z), i = 1, 2, 4 \) is defined respectively by (295), (296) and (297).

**Ginibre and interpolating ensembles and large BKP tau functions**

**Proposition 2.** Integrals (35) are large BKP tau functions where \( t \) are higher times.

Let us restrict ourselves to the case
\[
A(z, \bar{z}, w, \bar{w}) = \frac{1}{2} \left( \mu(z, \bar{z}) \delta^{(2)}(z - \bar{w}) - \mu(w, \bar{w}) \delta^{(2)}(w - \bar{z}) \right)
\]  
(37)
that is
\[
z_{2i-1} = \bar{z}_{2i}, \quad i = 1, \ldots, N = 2n
\]  
(38)
In this way we obtain partition function for the celebrated Ginibre ensembles, see see [93] for a review. Below we re-enumerate variables \( z_i \) as follows: \( z_{2i} \rightarrow z_i, \bar{z}_{2i-1} \rightarrow \bar{z}_i, i = 1, \ldots, n \) which a natural in view of (38). Quaternionic Ginibre ensemble with a deformed measure
\[
I_{qu-in}^{Gin}(t^*, N = 2n) = \int_{\mathbb{M}_n} \Delta_{2n}(z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_n, \bar{z}_n) \prod_{i=1}^{n} |z_i - \bar{z}_i| e^{\varphi(z_i,t^*)+\varphi(\bar{z}_i,t^*)-|z_i|^2} d^2 z_i
\]  
(39)
where now in view of the re-numeration above we write that \( \mathbb{M}_n \) consists of the sets of \( z \) where \( \Re z_i > \Re z_{i+1} \) and \( 3z_i > 0 \), and
\[
\mu(z, \bar{z}) = |z - \bar{z}| e^{-|z|^2} e^{\varphi(z,t^*)+\varphi(\bar{z},t^*)}
\]  
(40)
where
\[
\varphi(z, t^*) = \sum_{n=1}^{\infty} t_n e^n z^n + t_0 \log z - \sum_{n=1}^{\infty} t_n e^n \bar{z}^{-n}
\]  
(41)
In case \( D \) is the upper half-plane and \( t^*_n = t \delta_{n,2} \) we obtain an interpolating ensemble where \( t \) is an interpolating parameter, see [91].

In the similar way we obtain the complex part of the real Ginibre ensemble. This time we take
\[
d\mu(z, \bar{z}) = \text{erfc} \left( \frac{|z - \bar{z}|}{\sqrt{2}} \right) e^{-\frac{1}{2} z^2 - \frac{1}{2} \bar{z}^2} e^{\varphi(z,t^*)+\varphi(\bar{z},t^*)} d^2 z
\]  
(42)
The whole partition function which takes into account both complex and real parts of the spectrum of this ensemble is easily obtained.
\[
I_{r}^{Gin}(t^*, N) = \sum_{m=0}^{\infty} \int_{\mathbb{M}_{2m}, N - 2m} \Delta_{2n}(z_1, \bar{z}_1, \ldots, z_m, \bar{z}_m, x_{m+1}, \ldots, x_N) d\Omega^C_{2m} d\Omega^R_{N-2m}
\]  
(43)
where the integration domain $\mathbb{M}_{2m,N-2m}$ is as follows $\Re z_1 > \cdots > \Re z_m, x_{2m+1} > \cdots > x_N, z_i \in \mathbb{C}_+$ (upper halfplane), $x_i \in \mathbb{R}$, and where

$$d\Omega_{2m}^C = \prod_{i=1}^{m} \text{erfc}\left(\frac{|z_i - \bar{z}_i|}{\sqrt{2}}\right) e^{\varphi(z_i,t^*)+\varphi(z_i,t^*)} d^2z_i, \quad d\Omega_{N-2m}^U = \prod_{i=2m+1}^{N} e^{\varphi(z_i,t^*)} dx_i$$

(to get real Ginibre ensemble itself we put $t^*_c = -\frac{1}{2} \delta_{k,2}$.)

### 2.3 Integrals as fermionic vacuum expectation values

We have

$$I^{(1)}(t^*, N, \bar{A}) = \langle N + t_0^* \rangle \Gamma(t^*_c) e^{\int_{A_2} A(z_1,z_2)\psi(z_1)\psi(z_2)dz_1dz_2 + \int c a(z)\bar{\psi}(z)\phi_0 dz} \Gamma(t^*) | t_0^* \rangle$$

where for $A_2$ see (5).

In particular

$$I^{(1)}_1(t^*, N) = \langle N + t_0^* \rangle \Gamma(t^*_c) e^{\int \frac{\beta}{2} f \bar{f} \text{sgn}(c(z_1)-c(z_2))\psi(z_1)\psi(z_2)dz_1dz_2 + \int c a(z)\bar{\psi}(z)\phi_0 dz} \Gamma(t^*) | t_0^* \rangle$$

where, say, for the $\beta = 1$ circular ensemble we take $\gamma = S^1$ and $\zeta(z) = \text{arg}(z)$. Then

$$I^{(1)}_2(t^*, N = 2n) = \langle N + t_0^* \rangle \Gamma(t^*_c) e^{\int \frac{\delta}{2} f \bar{f} \text{sgn}(c(z)-c(z))\psi(z)\psi(z)dz + \int c a(z)\bar{\psi}(z)\phi_0 dz} \Gamma(t^*) | t_0^* \rangle$$

For circular $\beta = 4$ ensemble we take $\gamma = S^1$ and $A(z_1, z_2) = \delta'(\text{arg}(z_1) - \text{arg}(z_2))$

$$I^{(1)}_4(t^*, N = 2n) = \langle N + t_0^* \rangle \Gamma(t^*_c) e^{\int \frac{\delta}{2} f \bar{f} \text{sgn}(c(z)-c(z))\psi(z)\psi(z)dz + \int c a(z)\bar{\psi}(z)\phi_0 dz} \Gamma(t^*) | t_0^* \rangle$$

Now turn to the integrals over domains in complex plane. We have

$$I^c(t^*, N) = \langle N + t_0^* \rangle \Gamma(t^*_c) e^{\int_{A_2} A(z_1,z_2,z_3,z_4)\psi(z_1)\psi(z_2)dz_1dz_2dz_3dz_4 + \int c a(z)\bar{\psi}(z)\phi_0 dz} \Gamma(t^*) | t_0^* \rangle$$

(see (33) for $\mathbb{M}_2$.)

Quaternionic Ginibre ensemble with the deformed measure

$$I^{Gin}_{qu=\tau}(t^*, N = 2n) = \langle N + t_0^* \rangle \Gamma(t^*_c) e^{\int c a(z)\bar{\psi}(z)\phi_0 dz} \Gamma(t^*) | t_0^* \rangle$$

Real Ginibre ensemble with the deformed measure

$$I^{Gin}_{r} =$$

$$\langle N + t_0^* \rangle \Gamma(t^*_c) e^{\int c e^{\text{erfc}\left(\frac{\beta}{2} \bar{f} \psi(z)dz^2 + \frac{\beta}{2} \bar{f} \text{sgn}(c(z)-c(z))\psi(z)\psi(z)dz + \int c a(z)\bar{\psi}(z)\phi_0 dz} \Gamma(t^*) | t_0^* \rangle$$

where the real Ginibre ensemble itself is related to the case $t^*_m = -\frac{1}{2} \delta_{m,2}$.

### 2.4 Integrals in Pfaffian form

$$I^{(1)}(t^*, N, \bar{A}) = b \text{Pf} \left[ \bar{M}^{(1)}(t^*) \right]_{i,j=1,\ldots,N}$$

where $\bar{M}$ is the moment $N \times N$ matrix which is defined as follows:

For $N = 2n$ even

$$\bar{M}^{(1)}_{ij} = -\bar{M}^{(1)}_{ji} = M^{(1)}_{ij}(t^*)$$

where

$$M^{(1)}_{ij}(t^*) = \int_{A_2} z_1^i z_2^j A(z_1, z_2) d\mu(z_1, t^*) d\mu(z_2, t^*)$$
For \( N = 2n - 1 \) odd
\[
M_{ij}^{(1)} = -\tilde{M}_{ji}^{(1)} \begin{cases} 
M_{ij}^{(1)}(t^*) & \text{if } 1 \leq i < j \leq 2n - 1 \\
M_i^{(1)}(t^*) & \text{if } 1 \leq i < j = 2n \end{cases}
\] (55)
where
\[
M_n^{(1)}(t^*) = \int_{\gamma} z^n a(z) d\mu(z, t^*)
\] (56)
In (62)
\[
b = b(t^*_+, t^*_-) = \sum_{m=1}^{\infty} m t^*_m t^-_m
\] (57)
For the other cases we have similar formulae.
For general complex case we have
\[
I^{(c)}(t^*, N, \tilde{A}) = b Pf \left[ \tilde{M}^{(c)}(t^*) \right]_{i,j=1,\ldots,N}
\]
where
\[
M_{nm}^{(c)}(t^*_+) = \int_{M_2} z_1^n z_2^m A(z_1, \bar{z}_1, z_2, \bar{z}_2) d\mu(z_1, \bar{z}_1, z_2, \bar{z}_2) d\mu(z_2, \bar{z}_2, t^*_+), \quad M_{n}^{(c)}(t^*_+) = \int_{C} z^n a(z, \bar{z}) d\mu(z, \bar{z}, t^*)
\] (59)
For the Ginibre cases
\[
I_{q_{qu-r}}^{Gin}(t^*, N, \tilde{A}) = b Pf \left[ M_{q_{qu-r}}^{Gin}(t^*) \right]_{i,j=1,\ldots,N}, \quad I_{r}^{Gin}(t^*, N, \tilde{A}) = b Pf \left[ M_{r}^{Gin}(t^*) \right]_{i,j=1,\ldots,N}
\]
where for the Ginibre quaternionic case we have
\[
(M_{q_{qu-r}}^{Gin})_{nm}(t^*_+) = \int_{C_+} z^n z^m (z - \bar{z}) e^{\varphi(z, t^*) + \varphi(z, t^*) - |z|^2} d^2z,
\]
while for Ginibre real case
\[
(M_{r}^{Gin})_{nm}(t^*_+) = \int_{C_+} \text{erfc} \left( \frac{|z - \bar{z}|}{\sqrt{2}} \right) z^n z^m e^{\varphi(z, t^*) + \varphi(z, t^*)} d^2z + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \text{sgn}(x_1 - x_2) x_1^n x_2^m e^{\varphi(z, t^*) + \varphi(z, t^*)} dx_1 dx_2
\]
(62)
(63)
\]
\]
\]
\]
\]

2.5 Perturbation series - series in the Schur functions

To read this section we shall need some notations used in [2], see Appendix B.

Having the fermionic expressions we can re-write our integrals as series of the Schur functions over partitions, see [2] as follows
\[
I(t^*_+, t^*_-, N, \tilde{A}^*) = \sum_{\omega \in \mathcal{P}} \tilde{A}_n^*(t^*_+) s_{\lambda}(t^*_+) = S^{(1)}(t^*_+, N, U = 0, \tilde{A}^*(t^*_-))
\]
(64)
where we use notations of [2], see formula (199) and formulae (200)–(202) in Appendix B where for \( \tilde{A} = (A, a) \) we take \( \tilde{A}^* = (A^*, a^*) \), defined as follows:

For integrals \( I^{(1)}(t^*, N, \tilde{A}^*) \) from (55), we have
\[
A_n^{nm}(t^*_+) = \int_{\Lambda_2} z_1^n z_2^m A(z_1, z_2) e^{\varphi(z_1, t^*_+) + \varphi(z_2, t^*_-)} dz_1 dz_2, \quad a_n^{nm}(t^*_+) = \int_{\gamma} z^n e^{\varphi(z, t^*_+)} a(z) dz
\]
(65)
where for \( \Lambda_2 \) see (55).
For integral (69)

\[ A_{nm}^{*}(t_{-}^{\ast}) = \int_{M_{2}} z_{1}^{n} z_{2}^{m} A(z_{1}, z_{2}, z_{1}, z_{2}) e^{\varphi(z_{1}, t_{-}^{\ast}) + \varphi(z_{2}, t_{+}^{\ast})} d^{2}z_{1} d^{2}z_{2}, \quad a_{n}^{*}(t_{-}^{\ast}) = \int_{C} z^{n} e^{\varphi(z_{1}, t_{-}^{\ast})} a(z, \bar{z}) d^{2}z \]  

(66)

(for \( M_{2} \) see (33)).

For quaternionic Ginibre ensemble (39) from (50) we get

\[ A_{nm}^{*}(t_{-}^{\ast}) = \int_{C} z_{1}^{n} z_{2}^{m} e^{\varphi(z_{1}, t_{-}^{\ast}) + \varphi(z_{2}, t_{-}^{\ast}) - |z|^{2}} d^{2}z \]

(67)

(for \( M_{2} \) see (33)).

For real Ginibre ensemble (43) from (51) we get

\[ A_{nm}^{*}(t_{-}^{\ast}) = \int_{C} e^{\text{erfc} \left( \frac{|z - \bar{z}|}{\sqrt{2}} \right)} z_{1}^{n} z_{2}^{m} e^{\varphi(z_{1}, t_{-}^{\ast}) + \varphi(z_{2}, t_{-}^{\ast})} d^{2}z + \frac{1}{2} \int_{R} \int_{R} \text{sgn}(x_{1} - x_{2}) e^{x_{1}^{n} x_{2}^{m} e^{\varphi(x_{1}, t_{-}^{\ast}) + \varphi(x_{2}, t_{-}^{\ast})}} dx_{1} dx_{2} \]

(71)

Let us consider further examples

**Circular \( \beta = 4 \) ensemble.** Fermionic representation was written down in (48). In this case

\[ A_{nm}^{*}(t_{-}^{\ast}) = \frac{n - m}{2} \int x^{n+m-1} d\mu_{4}(x, t_{-}^{\ast}) \]

(70)

and

\[ a_{n}^{*}(t_{-}^{\ast}) = \int x^{n} d\mu_{4}(x, t_{-}^{\ast}) \]

(71)

where

\[ d\mu_{4}(x, t_{-}^{\ast}) := x^{2t_{-}^{\ast}} e^{-2 \sum_{m=1}^{\infty} x^{-m} t_{-}^{\ast}} dx \]

(72)

and where for the symplectic ensemble we take \( \gamma = R \) while for the circular ensemble \( \gamma = S^{1} \). In the last case

\[ A_{nm}^{*}(0, t_{-}^{\ast}) = \frac{n - m}{2} s_{n+m-t_{-}^{\ast}}(-2t_{-}^{\ast}) \]

(73)

where \( s_{n} := s_{(n)} \) is a one-row Schur function (or, the same, complete symmetric function).

**Ginibre quaternionic ensemble.** Take \( t_{+}^{\ast} = 0 \). We obtain

\[ A_{nm} = -A_{nm} = ml \delta_{n+1, m}, \quad n < m \]

(74)

which yields a formal perturbation series over partitions as follows

\[ f_{A_{nm}}^{Gin} = \sum_{\lambda} \prod_{i=1}^{N} h_{i}^{\lambda_{i}}(h_{i}) (t_{-}^{\ast}) \]

(75)

If we put \( t_{k}^{\ast} \rightarrow t_{k}^{\ast} - \frac{i}{2} \delta_{k, 2} \), then for \( n < m \) we obtain

\[ A_{nm} = -A_{nm} = 2i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+iy)^{n}(x-iy)^{m} e^{-2\rho^{2}} y dxdy = 2\pi i \int_{0}^{\infty} d\rho \int_{0}^{2\pi} \rho^{n+m} e^{i(n-m)\phi} 2\rho^{2} \sin^{2} \phi d\rho d\phi \]

(76)

which may be expressed in terms of hypergeometric functions.

Ginibre real ensemble in details will be considered elsewhere.
2.6 Series in zonal functions

Series in zonal functions are difficult to analyze, nevertheless we write down some of them here. Let us write it in terms of Jacks polynomials.

First consider circular $\beta = \frac{2}{\alpha}$ ($\beta = 1, 4$) ensemble with $N$-ply integral. This is the integral $I^{(1)}_\beta$ of [9] and [11] where $\Lambda_N = (S^1)^N$. Here we used

$$\prod_{i \neq j} (1 - x_i x_j^{-1})^{\frac{1}{\beta}} J_\lambda^{(\alpha)}(x) J_\lambda^{(\alpha)}(x) N \prod_{i=1}^N dx_i = \prod_{1 \leq i < j \leq N} \Gamma(\xi_i - \xi_j + \frac{1}{\beta}) \Gamma(\xi_i - \xi_j - \frac{1}{\beta} + 1) =: e_\lambda^{(\alpha)}$$

where $\xi_i := \lambda_i + \frac{1}{\beta}(N - 1)$, $1 \leq i \leq N$, [17]. This formula together with

$$e^{\frac{1}{n} \sum_{m=1}^n m t_m} = \sum_\lambda d^{(\alpha)}_\lambda \times J_\lambda^{(\alpha)}(t_+) J_\lambda^{(\alpha)}(t_-)$$

$$I^{(1)}_\beta(t^*, N) = \sum_\lambda e^{(\alpha)}_\lambda \times J_\lambda^{(\frac{1}{2})}(t^*_+) J_\lambda^{(\frac{1}{2})}(t^*-).$$

Some formulas on Jack polynomials see [95]:

Let $G_s$ be the multiplication operator which increases all the pseudo-moments by $s$, i.e., $\lambda = (\lambda_1, \ldots, \lambda_r) \mapsto \lambda + (s^*) = (\lambda_1 + s, \ldots, \lambda_r + s)$:

$$G_s : J_\lambda^{(1/\beta)}(z_1, \ldots, z_r) \mapsto J_{\lambda+(s^*)}^{(1/\beta)}(z_1, \ldots, z_r) = \prod_{i=1}^r z_i^s \cdot J_\lambda^{(1/\beta)}(z_1, \ldots, z_r).$$

The second operator $N_{NM}$ changes the number of particles from $M$ to $N$:

$$N_{NM} : J_\lambda^{(1/\beta)}(t_1, \ldots, t_M) \mapsto J_\lambda^{(1/\beta)}(z_1, \ldots, z_N) =$$

$$= \oint \prod_{j=1}^M dt_j \prod_{1 \leq i < j \leq M, i \neq j} \left(1 - \frac{z_i}{t_j}\right)^{-\beta} \prod_{1 \leq i \leq M} \left(1 - \frac{t_i}{t_j}\right)^{\beta} J_\lambda^{(1/\beta)}(t_1, \ldots, t_M),$$

where the integral is over a small cycle around the origin.

Repeatedly applying these operators to the $s_1$-particle vacuum we obtain all the Jack polynomials, i.e., if $\lambda := \sum_{a=1}^{n-1} (s_a^*)$

$$\lambda := \sum_{a=1}^{n-1} (s_a^*) = \begin{array}{cccc} s_{n-1} & s_{n-2} & \ldots & s_1 \\ r_{n-1} & r_{n-2} & \ldots & r_1 \end{array}$$

the corresponding Jack polynomial has the following integral representation:

$$J_\lambda^{(1/\beta)}(z_1, \ldots, z_N) = N_{r_n r_{n-1-1}} G_{s_{n-1}} N_{r_{n-1} r_{n-2}} \cdots G_{s_2} N_{r_2 r_1} G_{s_1} \cdot 1 =$$

$$= \oint \prod_{a=1}^{n-1} \prod_{i=1}^{r_a} dt_i^{(a)} \prod_{1 \leq i \leq r_{a+1}, i \neq j} \left(1 - \frac{t_i^{(a+1)}}{t_j^{(a)}}\right)^{\beta} \prod_{1 \leq i,j \leq r_a} \left(1 - \frac{t_i^{(a)}}{t_j^{(a)}}\right)^{-\beta} \\ \prod_{i=1}^{r_a} \prod_{i,j}^{r_a} t_i^{(a)},$$

where $z_j := t_j^{(n)}$, $N = r_n$. In particular, for a rectangular $\lambda = (s^r)$ we have

$$J_\lambda^{(1/\beta)}(z_1, \ldots, z_N) = \oint \prod_{i=1}^{r} \prod_{j=1}^{N} \prod_{i \neq j} \left(1 - \frac{z_i}{t_j}\right)^{-\beta} \prod_{i=1}^{r} \prod_{i,j \neq j} t_i^r.$$

If $\beta = 2$, the latter becomes a $r$-function:
Let \( \theta := \theta(x) := \sum_{k=1}^{\infty} x^k t_k, \theta_i := \theta(x_i) \) with \( t_k := \frac{1}{k} \sum_{j=1}^{N} x_j^k \). Since \( e^{2\theta} = e^{2\sum_{k=1}^{\infty} \sum_{j=1}^{N} x_j^k t_k/k} = \prod_{j=1}^{N} e^{2\sum_{k=1}^{\infty} x_j^k t_k/k} = \prod_{j=1}^{N} (1 - x_j z_j)^{-2} \) we have
\[
\langle 2n | \Gamma(t) \exp \oint \psi(x) \psi'(x) x^a dx | 0 \rangle = \langle 2n | \exp \oint e^{2\theta} \psi(x) \psi'(x) x^a dx | 0 \rangle
= \oint \langle 2n | \prod_{i=1}^{n} \psi(x_i) \psi'(x_i) x_i^a e^{2\theta} | 0 \rangle \prod dx_i
= \oint \Delta(x)^4 \prod_{i=1}^{n} (x_i^a e^{2\theta}) \prod dx_i
= \oint \Delta(x)^4 \prod_{i=1}^{n} x_i^a \prod_{i,j} (1 - x_i z_j)^{-2} \prod dx_i.
\]

2.7 Asymmetric two-matrix ensembles and Interpolating Mehta-Pandey Ensembles

This section was written as a proof of the conjecture of E.Kanzieper and V.Osipov [20] that interpolating ensembles [22] are related to integrable systems. Interpolating ensembles have wide applications in description of quantum chaos. They were also studied in relation to non-colliding Brownian motion, see [59], [58].

Asymmetric two matrix models. Let a \( N \) by \( N \) matrix \( X \) is one of
\[
\begin{align*}
X &= R \text{ real symmetric} \quad (80) \\
X &= R^a \text{ real anti-symmetric} \quad (81) \\
X &= Q \text{ real self-dual} \quad (82) \\
X &= Q^a \text{ real anti-self-dual} \quad (83)
\end{align*}
\]

We introduce the four types of two-matrix models
\[
Z_N^{HX}(l, t, t') = \int \int dH dX \, e^{cTrHX + TrV(H,l,t) + TrV(X,l,t')}
\]
where \( H \) is a Hermitian \( N \) by \( N \) matrix while a \( N \) by \( N \) matrix \( X \) is to be specified according to [80]-[83]. In [84]
\[
V(H, l, t) = l \ln \det H + \sum_{n=1}^{\infty} (t_n H^n - \bar{t}_n H^{-n})
\]
\[
V(X, l, t') = l \ln \det X + \sum_{n=1}^{\infty} (t'_n X^n - \bar{t}'_n X^{-n})
\]
where \( t = (t_1, t_2, \ldots), t' = (t'_1, t'_2, \ldots) \) and an integer \( l \) are parameters.

Remark 1. In case \( t_n = \delta_{n,2} t_2, t'_n = \delta_{n,2} t'_2 \) these models coincide with the interpolating ensembles introduced by Mehta and Pandey, see Chapter 14 [10].

Repeating the calculation by Pandey and Mehta in Ch 14 [10] integrals [84] may be reduced to the integral [87].

We obtain

Proposition 3.
\[
Z_N^{HX}(l, t, t') = e^{\int_{\Lambda_N} \Delta_N(z) \tilde{A}(z) \prod_{i=1}^{N} d\mu(z_i, t^*)}
\]
where
For the real symmetric case

\[ A_{ij} = \text{erf} \left( \frac{(\alpha^2 - 1)}{8v^2\alpha^2} (x_i - x_j) \right), \quad a_{i,N-1} = \] (88)

For the antisymmetric case

\[ A_{ij} = \exp \left( -\frac{(\alpha^2 - 1)}{8v^2\alpha^2} (x_i^2 + x_j^2) \right) \text{erf} \left( \frac{(\alpha^2 - 1)}{8v^2\alpha^2} (x_i - x_j) \right), \quad a_{i,N-1} = \exp \frac{\alpha^2 - 1}{8v^2\alpha^2} x_i^2 \] (89)

For self-dual case

\[ A_{ij} = (x_i - x_j) \exp \left( -(x_i - x_j)^2 \frac{1 - \alpha^2}{8v^2\alpha^2} \right), \quad \alpha^2 < 1 \] (90)

For anti-self-dual case for \( t'_n = \delta_{n,2} \)

\[ A_{ij} = (x_i - x_j) \exp \left( -(x_i - x_j)^2 \frac{\alpha^2 - 1}{8v^2\alpha^2} \right), \quad \alpha^2 > 1 \] (91)

**Proposition 4.** Integrals (84) are large BKP tau functions with respect to parameters \( N, l, t \).

In case \( c = t_2 = \frac{1}{4v^2} \), \( t'_2 = -\frac{1}{4v^2} \left( 1 + \frac{1}{1 - \alpha^2} \right) \), all other \( t_n \) and \( t'_n \) vanish, we obtain

\[ \int \int dH dA e^{-\frac{1}{4v^2} \text{Tr} H A - \frac{1}{8v^2} \text{Tr} H^2 - \frac{1}{4v^2} \frac{8 - \alpha^2}{1 - \alpha^2} \text{Tr} A^2} \]

\[ \equiv \int \int d(RH)d(\Im H)e^{-\frac{1}{4v^2} \text{Tr} (RH)^2 - \frac{1}{8v^2} \text{Tr} (\Im H)^2} \] (93) (94)

The last integral was an object of intensive study (see for instance [21]) because it describes the ensemble which interpolates between Gauss unitary and Gauss orthogonal ones [22]. It is known (see Chapter 14 of [16]) that integral (84) may be written as

\[ \int_R \cdots \int_R \Delta_N(x) \text{Pf} [A(x_i, x_j)] \prod_{i=1}^N dx_i \] (95)

with some matrix \( A \) described below. This is exactly the expression [37], therefore, the interpolating ensemble is an example of BKP tau function.

For the consideration of Harish-Chandra-Itzykson-Zuber (HCIZ) integral in case where \( H \) is Hermitian and \( A \) is symmetric was done in [16]. It is clear that exactly as in the case where both \( H, A \) are Hermitian we have

\[ \int_{V \in \mathbb{U}(N)} e^{TVV^{-1}V} dV = \frac{\det \left[ e^{\text{Tr} V} \right]}{\Delta_N(x) \Delta_N(y)} \] (96)

where \( H = UXU^{-1}, A = OYO^{-1} \) and \( V = O^{-1}U \) and \( X = \text{diag}(x_i), U \in \mathbb{U}(N), O \in \mathbb{O}(N) \).

Then it follows that

\[ Z_N^H(t, \tilde{t}, t') = \]

\[ \int_{R^N} \int_{R^N} \Delta_N(x) \det \left[ e^{\text{Tr} V} \right] sgn \Delta_N(y) \prod_{i=1}^N d\nu(x_i) \, d\mu(y_i) \] (97)

where

\[ d\mu(y) = e^{\sum_{n=1}^\infty (t'_n y^n - t_n y^{-n})} dy, \quad d\nu(x) = x^t e^{\sum_{n=1}^\infty (t_n x^n - t'_n x^{-n})} dx \] (98)

Now we apply the following Lemma by Mehta [16].
Lemma 1.
\[
\int \cdots \int \prod_{i=1}^{N} dp(y_i) \det [\theta_i(y_j)] \sgn \Delta(y) = N! \text{Pf} \{a_{ij}\}_{i,j=1,\ldots,2m}
\]
(99)
where \(2m = N\) if \(N\) is even and \(2m = N + 1\) if \(N\) is odd, and
\[
a_{ij} = \int \int_{x \leq y} dp(x) dp(y) [\theta_i(x) \theta_j(y) - \theta_j(x) \theta_i(y)], \quad i, j = 1, \ldots, N
\]
(100)
When \(N\) is odd we have in addition \(a_{N+1,N+1} = 0\) and
\[
a_{i,N+1} = -a_{N+1,i} = \int \theta_i(y) dp(y), \quad i = 1, \ldots, N
\]
(101)
In our case \(dp(y)\) is given by (98) and
\[
\theta_i(y_j) := e^{c_{x_i} y_j}
\]
(102)
As a result we obtain
\[
a_{ij} = a(x_i, x_j, t') := \int \int_{x \leq y} e^{\sum_{n=1}^{\infty} (t'_n x^n y^n - t_n (x^{-n} y^{-n}))} (e^{c_{x_i} x + c_{x_j} y} - e^{c_{x_i} y + c_{x_j} x}) \, dx dy
\]
(103)
\[
a_{i,N+1} = a(x_i, t') := \int e^{\sum_{n=1}^{\infty} (t'_n y^n - t_n y^{-n})} e^{c_{x_i} y} \, dy
\]
(104)
Here it is supposed that there exists a certain domain \(D\) of parameters \(t'\) where \(a(t', x_1, x_2)\) exist for all \(x_1, x_2 \in \mathbb{R}\). At last for we obtain

**Proposition 5.** The partition function for asymmetric matrix model (100) is the following 2-BKP tau function with respect to the variables \(t = (t_1, t_2, \ldots)\):
\[
\int_{V \in U(N)} \tau^KP(I_N, H_A) \, dV = e^{\sum_{n=1}^{\infty} t'_n W_n \psi(x)} e^{-\sum_{n=1}^{\infty} t'_n W_n} \int \psi(x) \frac{d^N \psi}{dz^N} \, dz
\]
(105)
where
\[
g(t') = e^{\int a(x_1, x_2, t') \psi(x) \psi(x_2) \, dx_1 dx_2} e^{\sqrt{\tau t(t')}}
\]
(106)
where \(a(t', x_1, x_2)\) is given by (104), \(t' \in D, t\) may be also written in the following form
\[
\int_{V \in U(N)} \tau^KP(I_N, H_A) \, dV = e^{\sum_{n=1}^{\infty} (t'_n - t_n) W_n} g(t'') e^{\sum_{n=1}^{\infty} (t'_n - t_n) W_n} \int \psi(x) \, dx
\]
(107)
where \(t'' := (t''_1, t''_2, \ldots) \in D\) and
\[
W_n = \frac{1}{2\pi i e^n} \int \psi(z) \frac{d^N \psi}{dz^N} \, dz
\]
(108)

Representation (107) results from the re-writing of (104) as follows
\[
a(x_1, x_2, t') = \int \int_{x \leq y} e^{\sum_{n=1}^{\infty} t'_n c^{-n}(\theta'_n + \theta_n)} (e^{c_{x_1} x + c_{x_2} y} - e^{c_{x_1} y + c_{x_2} x}) \, dx dy
\]
and from formula
\[
e^{\sum_{n=1}^{\infty} t'_n W_n} \psi(x) e^{-\sum_{n=1}^{\infty} t'_n W_n} = e^{\sum_{n=1}^{\infty} t'_n c^{-n}(\theta'_n) \psi(x)}
\]
Remark 2. More generally for the following series
\[
\tau^KP(I_N, H_A) := \sum_{\ell(\lambda) \leq N} e^{U_0 - U_\lambda \lambda} s_\lambda(I_N) s_\lambda(H_A), \quad U_\lambda := \sum_{i=1}^{N} U_{\lambda - i + N}
\]
(109)
(which is KP tau function (105) where higher times are chosen as \(t_m = \frac{1}{m} Tr (H_A)^m\) and \(\ell_m = \frac{1}{m} Tr I_N^m\), \(I_N\) is \(N\) by \(N\) unity matrix) we have [38, 23]
\[
\int_{V \in U(N)} \tau^KP(I_N, VXV^{-1}Y) \, dV = c_N \frac{\det [\theta_i(y_j)]_{i,j=1,\ldots,N}}{\Delta N(\psi) \Delta N(y)}
\]
(109)
where \( c_N \) is some constant and where

\[
\theta_i(y_j) = \sum_{n=0}^{\infty} e^{U_{1-n} - U_{n+1-N} x_i^r y_j^r}
\]

(110)

Now, similarly to the case of two-matrix models considered in [23, 38] (where both \( H, A \) were Hermitian) we can replace the Itzykson-Zuber interaction term \( e^{cTr^{HA}} \) by \( \tau^{KP} (I_N, HA) \). The partition function of the resulting asymmetric two-matrix model is a DKP tau function \( \Omega \) where instead of \( \Omega \) one put \( \Omega \). Namely

\[
Z_N^{HA}(t, t', U) := \int \int dHdA \tau^{KP} (I_N, HA) e^{Tr \sum_{n=1}^{\infty} t_n H^n + Tr \sum_{n=1}^{\infty} t_n A^n} =
\]

(111)

\[
= c_N \int \int \Delta_N(x) \det [\theta_i(y_j)] \mathrm{sgn} \Delta_N(y) \prod_{i=1}^{N} dv(x_i, t, t') \prod_{i=1}^{N} d\mu(y_i, t')
\]

(112)

Thus \( Z_N^{HA}(t, t', U) \) is the DKP tau function defined by (105) (or, the same, by (107)) where \( a(t', x_i, x_j) = a_{ij} \) is given by (103) where \( \theta_i(y_j) \) now is given by (110).

Examples of series (108) were considered in [35]. One of the examples is the hypergeometric function of matrix argument \( pF_q \), case C, see [11], (see also [10], Section 17.4). The matrix argument is \( HA \). In this case

\[
Z_N^{HA}(a; b; t, t', U) :=
\]

(113)

\[
\int \int dHdA pF_q \left( \frac{a_1 + N, \ldots, a_p + N}{b_1 + N, \ldots, b_q + N}; HA \right) e^{Tr \sum_{n=1}^{\infty} t_n H^n + Tr \sum_{n=1}^{\infty} t_n A^n} =
\]

\[
c_N \int \int \Delta_N(x) \det \left[ pF_q \left( \frac{a_1 + 1, \ldots, a_p + 1}{b_1 + 1, \ldots, b_q + 1}; x_i, y_j \right) \right] \mathrm{sgn} \Delta_N(y) \prod_{i=1}^{N} dv(x_i, t) \prod_{i=1}^{N} d\mu(y_i, t')
\]

(114)

In the last formula \( pF_q \) is the ordinary hypergeometric tau function of one variable and \( c_N \) is some constant. Here

\[
\theta_i(y_j) = pF_q \left( \frac{a_1 + 1, \ldots, a_p + 1}{b_1 + 1, \ldots, b_q + 1}; x_i, y_j \right)
\]

For instance \( g_0(HA) = e^{cTr^{HA}} \) yields Itzykson-Zuber interaction term as in (??), while \( 1F_0(a|HA) = \det (I_N - HA)^{-a} \) results in Cauchy-type interaction for asymmetric two-matrix model (the model with Cauchy-type interaction was introduced in [23] for the "symmetric" case where both \( H \) and \( A \) are Hermitian).

The other example of series (108) is a hypergeometric function introduced by Milne [39], see [34]. In this case \( \theta_i(y_j) \) coincides with a basic hypergeometric function of argument \( x_i, y_j \).

\( Z_N^{HA}(t, t') \) as DKP-nBKP tau function. Let \( c = \sqrt{-1} \) and let \( t' \) be a set of non-vanishing times with odd subscripts \( t' = (t'_1, 0, t'_3, 0, t'_5, \ldots) \), then, we have the following

\[
\int \int dHdA e^{cTr^{HA} + Tr \sum_{n=1}^{\infty} t_n H^n + Tr \sum_{n=1,3,5,\ldots} t_n A^n}
\]

(115)

\[
= \langle N | \Gamma(t) \Gamma_B(t') \Omega_B(U) e^{f(\psi(x))} | 0 \rangle
\]

(116)

\[
\int \cdots \int \prod_{i>j} \prod_{i=1}^{N} \left( x_i - x_j \right) \prod_{i=1}^{N} e^{\sum_{n=1}^{\infty} t_n x_i^n} \prod_{i=1}^{N} \left( \sum_{n=1,3,5,\ldots} t_n A_i^n \prod_{i>j} \frac{x_i - x_j}{x_i + x_j} \right) \prod_{i>j} \left( x_i - x_j \right)^2 \prod_{i=1}^{N} e^{\sum_{n=1}^{\infty} t_n x_i^n} d\xi_i
\]

(117)

where \( U = (U_1, U_2, \ldots), U_n := \log \Gamma(n + 1) \). Thanks to the fermionic representation (110) we see that the integral (117) is a DKP-nBKP tau function where \( t \) and \( t' \) are respectively DKP and nBKP higher times.

In particular in case \( t' = 0 \) we obtain

\[
\int \int dHdA e^{cTr^{HA} + Tr \sum_{n=1}^{\infty} t_n H^n} = \int \cdots \int \prod_{i>j} \left( x_i - x_j \right)^2 \prod_{i=1}^{N} e^{\sum_{n=1}^{\infty} t_n x_i^n} d\xi_i
\]

(118)

which is the continues analog (??) of (??) and may be compare with the so-called Bures ensemble.

\(^3\)As one can see \( \theta_i(y_j) \) up to a constant coincides with \( \tau^{KP} (I_N, HA) \) where \( H = x_1, A = y_1 \).
Formula (117) follows from the equality

\[
a(x_i, x_j, 0) = \int \int_{x \leq y} \left( e^{ex_i x + cx_j y} - e^{ex_j y + cx_j x} \right) dxdy
\]

\[
= -\frac{1}{x_i x_j} \frac{x_i - x_j}{x_i + x_j} = -\frac{2}{x_i x_j} \langle 0 | \phi(x_i) \phi(x_j) | 0 \rangle
\]

and the fact that for the choice of \( U \) as above we have

\[
\Gamma_B(t') T_B(U) \cdot \phi(x) \cdot T_B(U)^{-1} \Gamma_B(t')^{-1} = e^{\sum_{n=1, 3, \ldots} m_n \beta_n^m \cdot \phi(x)}
\]

The following Lemmas were used in the Chapter 14 of [16] devoted to the interpolating ensembles:

Lemma 2.

\[
\int \cdots \int \prod_{i=1}^{n} d\mu(y_i) \det [\theta_i(y_j), \nu_i(y_j)]_{i=1, \ldots, 2n; j=1, \ldots, n} = n! \text{Pf} [b_{ij}]_{i, j=1, \ldots, 2n} \tag{119}
\]

where

\[
b_{ij} = \int \int_{x \leq y} d\mu(y) [\theta_i(y) \nu_j(y) - \nu_i(y) \theta_j(y)], \quad i, j = 1, \ldots, N
\]

Lemma 3.

\[
\int \cdots \int \prod_{i=1}^{n} d\mu(y_i) \det [\theta_i(y_j), \nu_i(y_j), \chi_i(y_n)]_{i=1, \ldots, 2n; j=1, \ldots, n-1} = (n-1)! \text{Pf} [c_{ij}]_{i, j=1, \ldots, 2n} \tag{121}
\]

where

\[
c_{ij} = \int \int_{x \leq y} d\mu(y) [\theta_i(y) \nu_j(y) - \nu_i(y) \theta_j(y)], \quad i, j = 1, \ldots, 2n-1
\]

and

\[
c_{i, 2n} = -c_{2n, i} = \int \chi_i(y) d\mu(y), \quad i = 1, \ldots, 2n-1
\]

Consider the following \( 2N \) fold integral

\[
\int \int \cdots \int e^{\Delta_N^2(x)} \prod_{i=1}^{N} \Delta_N(x) \sum_{\Delta_N(y)} \prod_{i=1}^{N} d\mu(x_i) d\mu(y_i)
\]

First of all let us note that for every reasonable choice of a function \( f \)

Consider the following \( 2N \)-fold integral

\[
\int \int \cdots \int \Delta_N(x) \det [f(x_i, y_j)]_{i, j=1, \ldots, N} sgn \Delta_N(y) \prod_{i=1}^{N} d\mu(x_i) d\nu(y_i) \tag{124}
\]

Thanks to Lemma ... it is equal to \( N \)-ply integral

\[
= N! \int \int \cdots \int \Delta_N(x) \text{Pf} [A(x_i, x_j)] \prod_{i=1}^{N} d\mu(x_i)
\]

where

\[
A(x_i, x_j) = \int \int_{y \leq y'} d\nu(y) d\nu(y') [f(x_i, y) f(x_j, y') - f(x_j, y) f(x_i, y')]
\]
3 Grand partition function for 2N-fold integrals

Let $d\mu^\pm$ be measures supported respectively on contours $\gamma^\pm$ on the complex plane. Our main examples of $\gamma^\pm$ are the same contours (A) and (B) as in subsection 12.1.

Let us adopt the following notation:

$$\Delta_N^{(1)}(x) := \prod_{i>j}^N |x_i - x_j|, \quad \Delta_N^{(2)}(x) := \prod_{i>j}^N (x_i - x_j)^2, \quad \Delta_N^{(4)}(x) := \prod_{i>j}^N (x_i - x_j)^4 \quad (125)$$

also

$$\Delta_N^{(11)}(x) := \prod_{i>j}^N \frac{(x_i - x_j)^2}{x_i + x_j} \quad (126)$$

and

$$\Delta_N^{(12)}(x) := \prod_{i>j}^N \frac{x_i - x_j}{(x_i + x_j)^2} \quad (127)$$

where in the right-hand side of the last equality we add a variable $x_{2n}$ to the set of $x_1, \ldots, x_N$, $N = 2n - 1$, and then put $x_{2n} = 0$.

Consider the following series over $N$ in 2N-fold integrals:

$$J_{\beta_-, \beta_+} := \sum_{N=0}^\infty \frac{\nu^{2N}}{N!} \frac{1}{\gamma_+} \int_{\gamma_+} \cdots \int_{\gamma_+} \Delta^{(\beta_-)}(x) \Delta^{(\beta_+)}(y) \prod_{i=1}^N f(x_i, y_i) d\mu^-(x_i) d\mu^+(y_i) \quad (128)$$

where $\beta_-$ denotes any of the index among the set $1, 2, 4, 11, 12$ which are used in (125)-(127). The same convention is chosen for $\beta_+$ which is independent of $\beta_-$ (thus formula (128) contains 15 cases.)

Series in integrals (128) may be obtained as particular cases of the series

$$J(a_-, a_+^n) := \sum_{N=0}^\infty \frac{\nu^{2N}}{N!} \frac{1}{\gamma_+} \int_{\gamma_+} \cdots \int_{\gamma_+} \Delta(x) \Delta(y) a_-(x) a_+^n(y) \prod_{i=1}^N f(x_i, y_i) d\mu^-(x_i) d\mu^+(y_i) \quad (129)$$

$$J_1 := \sum_{N=0}^\infty \frac{\nu^{2N}}{N!} \frac{1}{\gamma_+} \int_{\gamma_+} \cdots \int_{\gamma_+} |\Delta(x) \Delta(y)| \prod_{i=1}^N f(x_i, y_i) d\mu^-(x_i) d\mu^+(y_i) \quad (130)$$

$$J_{11} := \sum_{N=0}^\infty \frac{\nu^{2N}}{N!} \frac{1}{\gamma_+} \int_{\gamma_+} \cdots \int_{\gamma_+} \Delta^*(x) \Delta(x) \Delta^*(y) \Delta(y) \prod_{i=1}^N f(x_i, y_i) d\mu^-(x_i) d\mu^+(y_i) \quad (131)$$

$$J_2 := \sum_{N=0}^\infty \frac{\nu^{2N}}{N!} \frac{1}{\gamma_+} \int_{\gamma_+} \cdots \int_{\gamma_+} \Delta(x)^2 \Delta(y)^2 \prod_{i=1}^N f(x_i, y_i) d\mu^-(x_i) d\mu^+(y_i) \quad (132)$$

$$J_3 := \sum_{N=0}^\infty \frac{\nu^{2N}}{N!} \frac{1}{\gamma_+} \int_{\gamma_+} \cdots \int_{\gamma_+} \Delta(x) \Delta(y) a_-(x) a_+^n(y) \prod_{i=1}^N f(x_i, y_i) d\mu^-(x_i) d\mu^+(y_i) \quad (133)$$

$$J_4 := \sum_{N=0}^\infty \frac{\nu^{2N}}{N!} \frac{1}{\gamma_+} \int_{\gamma_+} \cdots \int_{\gamma_+} \Delta(x)^4 \Delta(y)^4 \prod_{i=1}^N f(x_i, y_i) d\mu^-(x_i) d\mu^+(y_i) \quad (134)$$

where, as before,

$$\Delta(x) = \Delta_N(x) = \prod_{i>j}^N (x_i - x_j), \quad \Delta^*(x) = \Delta_N^*(x) = \prod_{i>j}^N \frac{x_i - x_j}{x_i + x_j}$$
The first \((N = 0)\) term in the series is assumed to be 1. The notation \(a^c_\pm(x)\) is analogous to \(\text{Pf}[\tilde{a}_\pm]\):

\[
a^c_\pm(x) := \text{Pf}[\tilde{a}_\pm]
\]

where \(\tilde{a}_{ij}\) is a skew symmetric kernel \(a(x, w)\) (possibly, a distribution) and a function (or distribution) \(a(x)\) as follows:

For \(N = 2n\) even

\[
\tilde{a}_{ij} = -\tilde{a}_{ji} := a(x_i, x_j), \quad 1 \leq i < j \leq 2n
\]

For \(N = 2n - 1\) odd

\[
\tilde{a}_{ij} = -\tilde{a}_{ji} := \begin{cases} a(x_i, x_j) & \text{if } 1 \leq i < j \leq 2n - 1 \\ a(x_i) & \text{if } 1 \leq i = j \leq 2n \end{cases}
\]

In addition we define \(a_0^c = 1\).

To relate these integrals to the 2-BKP hierarchy we introduce deformations \(I_i(N) \rightarrow I_i(N; t, \bar{t})\) through the following deformation of the measure

\[
d\nu(z) \rightarrow d\nu(z|t, \bar{t}) = b(t, \{z\})b(-\bar{t}, \{z^{-1}\})d\nu(z)
\]

where

\[
b(s, t) = \exp \sum_{\text{odd } n} \frac{n}{2} s_n t_n
\]

and

\[
\{z\} = (2z, \frac{2z^3}{3}, \frac{2z^5}{5}, \cdots).
\]

Below, we show that the generating series obtained by Poissonization (the grand partition function)

\[
Z_i(\mu; t, \bar{t}) = b(t, \bar{t}) \sum_{N=0}^\infty I_i(N; t, \bar{t}) \frac{\mu^N}{N!}, \quad i = 1, 2, 3, 4,
\]

are particular 2-BKP tau functions (??).

We also consider the following \(2N\)-fold integrals:

\[
I_5(N; t^{(1)}, t^{(2)}, \bar{t}^{(1)}, \bar{t}^{(2)}) := \int \Delta_N^c(z) \Delta_N^c(y) \prod_{i=1}^N d\nu(z_i, y_i|t^{(1)}, t^{(2)}, \bar{t}^{(1)}, \bar{t}^{(2)}),
\]

where

\[
d\nu(z, y|t^{(1)}, t^{(2)}, \bar{t}^{(1)}, \bar{t}^{(2)}) = b(t^{(1)}, \{z\})b(-\bar{t}^{(1)}, \{z^{-1}\})b(t^{(2)}, \{y\})b(-\bar{t}^{(2)}, \{y^{-1}\})d\nu(z, y)
\]

(here \(d\nu(z, y)\) is an arbitrary bi-measure), and show that the generating series

\[
Z_5(\mu; t^{(1)}, t^{(2)}, \bar{t}^{(1)}, \bar{t}^{(2)}) = b(t^{(1)}, \bar{t}^{(1)})b(t^{(2)}, \bar{t}^{(2)}) \sum_{N=0}^\infty I_5(N; t^{(1)}, t^{(2)}, \bar{t}^{(1)}, \bar{t}^{(2)}) \frac{\mu^N}{N!}
\]

is a particular case of the two-component 2-BKP tau function (??). 

**Remark 3.** Note that

\[
Z_2(\mu; t, \bar{t}) = Z_5(\mu; t^{(1)}, t^{(2)}, \bar{t}^{(1)}, \bar{t}^{(2)})
\]

if

\[
d\nu(z, y) = \delta(z - y)d\nu(z)d\nu(y), \quad t = t^{(1)} + t^{(2)}, \quad \bar{t} = \bar{t}^{(1)} + \bar{t}^{(2)}.
\]

The integrals \(Z_1(\mu; t, \bar{t}), Z_2(\mu; t, \bar{t}), Z_4(\mu; t, \bar{t})\) and \(Z_5(\mu; t^{(1)}, t^{(2)}, \bar{t}^{(1)}, \bar{t}^{(2)})\) may be obtained as continuous limits of \(S_1(t_\infty, t^*)\), \(S_2(t_\infty, t_\infty, t^*)\), \(S_4(t_\infty, t_\infty, t^*)\) and \(S_5(t_\infty, t_\infty, t^*)\), respectively of [2] (see also Appendix [3].)
Consider DKP tau function

\[ \tau^A(t, \bar{t}) := \langle 0 | \Gamma(t) g^{--} g^{++} \Gamma(\bar{t}) | 0 \rangle \]  

(147)

where

\[ g^{--} = e^{\frac{a + i}{\pi i} \int \psi(z) \psi(y) \text{sgn}(\text{arg}(x) - \text{arg}(y)) \text{d}x \text{d}y} \]

\[ g^{++} = e^{\frac{a + i}{\pi i} \int \psi'(z) \psi'(y) \text{sgn}(\text{arg}(x) - \text{arg}(y)) \text{d}x \text{d}y} \]

In case \( g = 1 \) tau function (147) resembles a grand partition function for two-(unitary)-matrix models with Cauchy-kernel type interaction, see (A-41) in [23]:

\[ \tau = e^{\frac{a + i}{\pi i} \sum_{n=1}^{\infty} \frac{(-a^2)}{N} \left( \frac{1}{N!} \right)^2 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\nu(z_i) \prod_{i=1}^{N} d\nu(z_i') \prod_{i=1}^{N} \left( \int \Delta_N(z) \Delta_N(z') \right) \prod_{i=1}^{N} \left( \int \Delta_N(z) \Delta_N(z') \right) \]  

(148)

where

\[ d\nu^\pm = d\nu^\mp(z, t, \bar{t}) = \sum_{n=1}^{\infty} (z^n t_n - \bar{z}^n \bar{t}_n) d\mu^\pm(z) \]  

(149)

and where \( c = \exp \sum_{n=1}^{\infty} m t_n \bar{t}_m \).

(B) Tau function

\[ \tau^B(t, \bar{t}) := \langle 0 | \Gamma(t) e^{\frac{a + i}{\pi i} \int \psi(z) \psi(y) \text{sgn}(x - y) \text{d}x \text{d}y} \text{d}y \text{d}y \int \psi'(z) \psi'(y) \text{sgn}(x - y) \text{d}x \text{d}y \text{d}y \Gamma(\bar{t}) | 0 \rangle \]  

(150)

in case \( g = 1 \) resembles a grand partition function for two-(Hermitian)-matrix models with Cauchy-kernel type interaction, see (A-41) in [23]:

\[ \tau = e^{\frac{a + i}{\pi i} \sum_{n=1}^{\infty} \frac{(-a^2)}{N} \left( \frac{1}{N!} \right)^2 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\nu(z_i) \prod_{i=1}^{N} d\nu(z_i') \prod_{i=1}^{N} \left( \int \Delta_N(z) \Delta_N(z') \right) \prod_{i=1}^{N} \left( \int \Delta_N(z) \Delta_N(z') \right) \]  

(151)

where \( c \) and the dependence of \( d\nu^\mp \) in variables \( l, t, \bar{t} \) is the same formula (149) as in the case (A).

Remark 1. As we mentioned both expression (148) and (151) are similar to grand partition functions for two-matrix models - unitary matrices for the case (A) and Hermitian ones for the case (B). The difference is that in expressions (148) and (151) we have absolute values of Vandermonde determinants instead of their own values. I do not know are there applications for integrals (148) and (151).

Remark 2. In certain cases both integrals (148) and (151) may be identified with the following DKP tau function from [2]

\[ \tau(t, U(t^*)) = 1 + \sum_{\alpha, \beta} e^{U(1-\alpha)(t^*)-U(\alpha)(t^*)} \delta(\alpha, \beta) \]  

where \( (t, U(t^*)) \) are specified either by (??) (this leads to \( \tau(t, U(t^*)) = \tau^A(t^*) \)) or by (??) (this leads to \( \tau(t, U(t^*)) = \tau^B(t^*) \)). These identifications are achieved respectively due to (??) and to (??).

Remark 3. In case in formulae (148) and (151) we put \( g = \exp \sum_{\alpha, \beta} C_{\alpha, \beta} \psi_{\alpha} \psi_{\beta} \) the interaction term which is the Cauchy kernel \( I \) for \( g = 1 \) and the dependence of the measures \( d\nu^\mp \) on the variables will be different and expressed in terms of 2D Toda lattice Baker functions defined by the tau function \( \tau^{TL} := \langle \Pi(t) g^{\Gamma(\bar{t})} \rangle \).

(C) One can observe that

\[ g_{00}^{--} = e^{\sum_{n>m} \psi_n \psi_m} = e^{\frac{a + i}{\pi i} \int \psi(z) \psi(z^{-1})} \]

(152)

where

\[ \frac{1 + z}{1 - z} := 1 + 2z + 2z^2 + \cdots \]  

(153)

4The Cauchy type interaction for two matrix models was considered in [23], see also [57].
Then

$$g_0^{-}\langle 0 \rangle = e^{\sum_{n>0} \psi_n \psi_{n}} \langle 0 \rangle = e^{\sum_{n>0} \psi_n \psi_{m}} \langle 0 \rangle = e^{-\frac{i}{2} \oint \psi(z) \psi(z^{-1}) \frac{dz}{z}} \langle 0 \rangle$$

(154)

Also we have for (??)

$$g_0^{-0} = e^{\gamma N \sum \psi_m \phi_n} = e^{\gamma N \psi(z) \phi(z) \frac{dz}{z}}$$

Then we obtain the following N-fold integral representation for (??):

$$\sum_{\ell(\lambda) \leq N} s_{\lambda}(t) = \langle N | \Gamma(t) g_0^{-0} \langle 0 \rangle = e^{\sum_{n \neq 0} \psi_n \psi_{n}} \langle 0 \rangle = e^{\sum_{n \neq 0} \psi_n \psi_{m}} \langle 0 \rangle$$

(155)

where $\Delta_N(z)$ is the Vandermond determinant $\Delta_{2N}(z_1, \ldots, z_N, \frac{1}{z_N})$ factor $\tilde{\Delta}_N(z)$ coincides with its absolute value in case all $|z_i| = 1$.

4 Symmetries and $\beta = 2$ circular ensemble. One-matrix model.

It is well known that one matrix model both for Hermitian and unitary matrices may be expressed as KP tau functions [48]. Here we obtain partition functions of these models as DKP tau functions

$\beta = 2$ circular ensemble known also as the model of unitary matrices. Let us notice that the is a set of $\alpha(2\pi)$ elements $\{I_n^\pm, n \text{ odd integers} \}

$$I_n^- := \sum_{i \in \mathbb{Z}} (-1)^i \psi_{n-i}, \quad I_n^+ := \sum_{i \in \mathbb{Z}} (-1)^i \psi_{n-i}^\dagger,$$

(156)

commuting with currents $J_n$ as follows

$$[J_{2m-1}, I_n^\pm] = 0, \quad [J_{2m}, I_n^\pm] = \pm 2I_n^{\pm}_{n+2m}$$

(157)

It results from $[J_i, \psi(x)] = x^i \psi(x), \ [J_i, \psi^\dagger(x)] = -x^i \psi^\dagger(x)$ and from the following representation:

$$I_n^- = \frac{1}{2\pi i} \oint x^i \psi(x) \psi(-x) dx, \quad I_n^+ = \frac{1}{2\pi i} \oint x^{n+1} \psi^\dagger(x) \psi^\dagger(-x) dx$$

(158)

Now

$$[I_n^+, I_m^-] = ???$$

(159)

Operators $e^{\sum_{n \text{ odd}} s_n^i t_n^i}$ may be considered as symmetry operators which commute with odd DKP flows. Parameters $s_n^i$ play the role of group times. Let us consider a simple DKP tau function

$$\tau_{2N}(t, \bar{t}, s^-) := \langle 2N | \Gamma(t) e^{\sum_{n \text{ odd}} s_n^i t_n^i} \Gamma(\bar{t}) | 0 \rangle$$

(160)

We introduce a (deformed) measure as

$$d\mu(x, s^-) := \left( \sum_{n \in \mathbb{Z}} s_{2n-1} x^{2n} \right) \frac{dx}{x}$$

(161)

where $s^-$ are deformation parameters, then

$$\sum_{n \text{ odd}} s_n^- I_n^- = \frac{1}{2\pi i} \oint \psi(-x) \psi(x) d\mu(x, s^-)$$

Introducing $z = x^2$ and notations

$$c_N = (-\pi i)^{-N} \frac{1}{N!} \exp \sum_{n=1}^\infty n t_n I_n, \quad \Delta_N(z) := \det \left( z_i^{N-k} \right)_{i,k=1,\ldots,N}$$

19
we find that our tau function is the following $N$-ply integral
\[
\tau_{2N}(t, \tilde{t}, s) = c_N \int \ldots \int \Delta_N^2(z) \prod_{i=1}^N d\nu(z_i),
\]
which may be interpreted as the partition function for $\beta = 2$ circular ensemble.

Let us mention that
\[
\tau(t, \tilde{t}, s) := \langle 0 | \Gamma(t) e^{\frac{\alpha}{\pi t}} \sum_n e^{s_n t_n} e^{a \sum_n s_n t_n} \tilde{\Gamma}(\tilde{t}) | 0 \rangle
= c_2^2 \left( 1 + \sum_{N=1}^{\infty} \frac{(\frac{\alpha}{\pi t})^{2N}}{N!} \right)^2 \int \ldots \int \frac{\Delta_N^2(z) \Delta_N^2(z')}{\prod_{i,k=1}^N (z_i - z'_k)^2} \prod_{i=1}^N d\nu(z_i) d\nu(z'_i)
\]

For a special choice of measure $d\mu$ it may be equated to a special KP tau function (??) evaluated at special value of KP times $t = t$ given by (??), see (??)
\[
\tau^{KP} = \sum \sum e^{U(-\beta \bar{t}) - U(\bar{t})} \left( s_{(\alpha|\beta)}(t) \right)^2
\]
where variables $U_n$ are related to the even (coupled) DKP higher times $t_{2m}, t_{2m}$ as follows
\[
U_n = U_n(t^*, \tilde{t}^*, c) = \sum_{m=1}^{\infty} ((n + c)^m t_{2m}^* - (n + c)^{-m} t_{2m}^*)
\]
where $c$ is an arbitrary non-vanishing non-integer parameter introduced to get rid of divergent terms in the expression for $U_0$.

One-Matrix Model. Apart from $\beta = 2$ circular ensemble one obtains partition functions of the so-called matrix models as DKP tau functions.

Consider DKP tau function
\[
\langle 2N|\Gamma(t) e^{\int \psi(x) \psi(x) d\mu(z)} \tilde{\Gamma}(\tilde{t}) | 0 \rangle
\]
Repeating the previous calculation we obtain the $N$-ply integral
\[
c_N \int \ldots \int \Delta_N^2(z) z^N e^{\sum_{n=1}^{\infty} (n^m t_{2m}^* - n^{-m} t_{2m}^*)} d\nu(z), \quad d\nu(z) := \frac{1}{2} d\mu(z^*)
\]

5 Remarks

5.1 Replacing $|0\rangle$ by $|\Omega\rangle$: From KP and TL to BKP

There is a simple way how a TL tau function $\tau^{TL}$ (where for simplicity we fix the discrete variable $t_0 = 0$) may be transformed to a BKP tau function. This is as follows. We present $\tau^{TL}$ as the vacuum expectation value. Now, to get the BKP tau function we replace the vacuum $|0\rangle$ by the vector $|\Omega\rangle$
\[
|0\rangle \rightarrow \sum_{\lambda \in P} |\lambda\rangle := |\Omega\rangle
\]
\[
\tau^{TL}(t, \tilde{t}) = \langle 0 | \Gamma(t) g \tilde{\Gamma}(\tilde{t}) | 0 \rangle \rightarrow \tau^{BKP}(t|\tilde{t}) = \langle 0 | \Gamma(t) g \tilde{\Gamma}(\tilde{t}) | \Omega \rangle
\]
where $t$ is the BKP higher time, while $\tilde{t}$ is a hidden parameter. This replacement may be also presented as the action of a certain vertex-like operator on the TL tau function:
\[
\tau^{BKP}(t|\tilde{t}) = e^{\hat{\Omega} \cdot \tau^{TL}(t, \tilde{t} + s)}
\]
where
\[
\hat{\Omega} = \frac{1}{2} \sum_{m>0} \frac{1}{m^2 \partial_m^2} + \sum_{m>0, odd} \frac{1}{m} \partial_m
\]
is a sort of Laplacian operator.

In this subsection we write down few examples.
Example 1. A multiple integral. Here, starting from the partition function of the two matrix model known to be a tau function of the 2KP (TL) hierarchy we shall obtain new "integrable" multiple integral ("integrable" means related to an integrable hierarchy, in this particular case, related to the BKP one).

Lemma 4. We have
\[
\tau_o(t)C(x^n, y^m) = \langle n-m| \tau^{(1)} \psi(x_1) \cdots \psi(x_n) \psi(y_1) \cdots \psi(y_m) \rangle = (167)
\]
where
\[
C(x^n, y^m) = \prod_{i=1}^{n} (1 + x_i^{-1})^{-1} \prod_{i=1}^{m} (1 - y_i^{-1})^{-1} \prod_{i<j \leq n} (1 - x_i^{-1} x_j^{-1})^{-1} \prod_{i<j \leq m} (1 - y_i^{-1} y_j^{-1})^{-1}
\]
(169)

Lemma 4. We have
\[
\tau_o(t)C(x^n, y^m) = \langle n-m| \tau^{(1)} \psi(x_1) \cdots \psi(x_n) \psi(y_1) \cdots \psi(y_m) \rangle = (167)
\]
where
\[
C(x^n, y^m) = \prod_{i=1}^{n} (1 + x_i^{-1})^{-1} \prod_{i=1}^{m} (1 - y_i^{-1})^{-1} \prod_{i<j \leq n} (1 - x_i^{-1} x_j^{-1})^{-1} \prod_{i<j \leq m} (1 - y_i^{-1} y_j^{-1})^{-1}
\]
(169)

Example 1. A multiple integral. Here, starting from the partition function of the two matrix model known to be a tau function of the 2KP (TL) hierarchy we shall obtain new "integrable" multiple integral ("integrable" means related to an integrable hierarchy, in this particular case, related to the BKP one).

Proposition 7. If
\[
\sum_{\lambda, \mu \in \mathbb{P}} s_\lambda(t) g_{\lambda, \mu} s_\mu(\bar{t})
\]
is a TL tau function (the Takasaki series [27]), then,
\[
\sum_{\lambda, \mu \in \mathbb{P}} s_\lambda(t) g_{\lambda, \mu}
\]
is the BKP tau function.
Let us notice that the BKP tau function $\tau_{BKP}(t)$ may be obtained as a certain linear combination of any TL tau function $\tau_{TL}(t, t)$ by integration with respect to variables $t = (t_1, t_2, \ldots)$

$$\tau_{BKP}(t) = \int \tau_{TL}(t, t) e^{-\sum_{m=1}^{\infty} m \ell_m \ell_m^* \tau_0(t^* \tau_0^*)} \prod_{m=1}^{\infty} \frac{df_m df_m^*}{4\pi^2 m!}$$

(174)

where each $\ell_m^*$, $m = 1, 2, \ldots$, is complex conjugated to $\ell_m$, and where

$$\tau_0(t) = \sum_{\lambda \in \mathbb{P}} s_\lambda(t) = e^{\sum_{m>n} \left( \frac{1}{2} m \ell_m^2 + \frac{1}{2} m \ell_{m+1}^2 \right)}$$

(175)

Let us notice that there is a scalar product where the Schur functions are ortho-normal: $\langle s_\mu, s_\nu \rangle = \delta_{\mu,\nu}$ [17]. Then $\tau_{BKP}(t) = \langle \tau_{TL}(t, t), \sum_\mu s_\mu(t) \rangle$. More generally one BKP tau function may be obtained from the other by $\tau_{K}(t) = \langle \tau_{TL}(t, t), \tau_{K}(t) \rangle$.

**Example 3.** Let us introduce the following KP tau function

$$\tau_{K}(t, g, \lambda) := \langle N | \Gamma(t) g | \lambda, N \rangle = \sum_{\mu \in \mathbb{P}} s_\mu(t) g_{\mu \lambda}(N), \quad g_{\mu \lambda}(N) := \langle \mu, N | g | \lambda, N \rangle$$

(176)

**Remark 4.** If we choose $g = \exp \sum_{n,m>N} a_{nm} v_n v_m^*$ we obtain a tau function related to the Schubert cell of the Sato Grassmannian [10] marked by a partition $\lambda$.

One may call it the generalized Schur function since the specialization $g = 1$ yields the ordinary Schur function. Then in many formulae which express DKP and BKP tau functions as series in the Schur functions one can replace the Schur functions $s_\lambda$ by the generalized Schur functions $\tau_{K}(t, g, \lambda)$.

Instead of the Proposition ?? we obtain

**Proposition 8.**

$$\tau_{K}(t, g, \lambda) := \langle N+1 | \Gamma(t) g T(U) g^{-1} (A) g^{-1} (A)^{-1} | l \rangle$$

(177)

$$= e^{N+l} \sum_{\ell, \lambda \leq N} e^{-U_\lambda(N+l)} A_{[l+h]} \tau_{K}(t, g, \lambda)$$

(178)

where $h_i = \lambda_i - i + N$, $U_\lambda(N+l)$ and $A_{[l+h]}$ are the same as in Proposition ???. In (177) $g^{-1} (A)$ and $g^{-1} (A)^{-1}$ are given respectively by (??) and (??) and $T(U)$ is as in (??). The constant $e^{N+l}$ is defined by (??).

Tau function (177) vanishes if $N < 0$.

**Remark 5.** The series (178) may be also equated to a DKP tau function.

Examples (??)-(182) are replaced by

$$\tau_{K}(t, t', g, U) = \sum_{\ell, \lambda \leq N} e^{-U_\lambda(N+l)} Q_{l+\lambda}^{-1} \left( \frac{1}{2} t^2 \right) \tau_{K}(t, g, \lambda)$$

(179)

$$\tau_{K}(t, t', g, U) = 0$$

(180)

$$\tau_{N}(t, t', g, U) = \sum_{\ell, \lambda \leq N} e^{-U_\lambda(N+l)} \tau_{K}(t, g, \lambda)$$

(181)

$$\tau_{N}(g, t) := \sum_{\ell, \lambda \leq N} \tau_{K}(t, g, \lambda)$$

(182)

Notations are the same as in (??)-(182).
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V. Venkateswaran, “Vanishing integrals for HallLittlewood polynomials”, Transformation Groups: Volume 17, Issue 1 (2012), Page 259-302
A Appendices

A.1 Pfaffians and Hafnians

(A) Pfaffians. We need the notion of Pfaffian. If $A$ an anti-symmetric matrix of an odd order its determinant vanishes. For even order, say $k$, the following multilinear form in $A_{ij}, i < j \leq k$

$$\text{Pf}[A] := \sum_\sigma \text{sgn}(\sigma) A_{\sigma(1), \sigma(2)} A_{\sigma(3), \sigma(4)} \cdots A_{\sigma(k-1), \sigma(k)}$$

(183)

where sum runs over all permutation restricted by

$$\sigma: \sigma(2i-1) < \sigma(2i), \quad \sigma(1) < \sigma(3) < \cdots < \sigma(k-1),$$

(184)

coincides with the square root of $\det A$ and is called the Pfaffian of $A$, see, for instance [16]. As one can see the Pfaffian contains $1 \cdot 3 \cdot 5 \cdots \cdot (k-1) =: (k-1)!!$ terms.

The following equality is known as Schur identity

$$\text{Pf} \left( \frac{x_i - x_j}{x_i + x_j} \right)_{1 \leq i,j \leq 2n} = \Delta_{2n}^+(x)$$

(185)

where

$$\Delta_{k}^+(x) := \prod_{1 \leq i<j \leq k} \frac{x_i - x_j}{x_i + x_j}$$

(186)

Let us mark that a special case of this relation is obtained if $x_{2n}$ vanishes. In this case we write

$$\text{Pf}(A) = \Delta_{2n-1}^+(x)$$

(187)

where $A$ is an antisymmetric $2n \times 2n$ matrix defined by

$$A_{ij} = \begin{cases} \frac{x_i - x_j}{x_i + x_j} & \text{if } 1 < i < j < 2n \\ 1 & \text{if } i < j = 2n, \end{cases}$$

(188)

Hafnians The Hafnian of a symmetric matrix $A$ of even order $N = 2n$ is defined as

$$\text{Hf}(A) := \sum_\sigma A_{\sigma(1), \sigma(2)} A_{\sigma(3), \sigma(4)} \cdots A_{\sigma(2n-1), \sigma(2n)}$$

(189)

where sum runs over all permutation restricted by

$$\sigma: \sigma(2i-1) < \sigma(2i), \quad \sigma(1) < \sigma(3) < \cdots < \sigma(2n-1),$$

(190)

As one can see the this sum contains $1 \cdot 3 \cdot 5 \cdots \cdot (2N-1) =: (2N-1)!!$ terms.

Remark 6. Let us note that entries on the diagonal of the matrix $A$ does not contribute the sum $\text{Hf}(C) =: \Delta_{2N}^{**}(x)$

The following equality was found in [51].

$$\text{Pf} \left( \frac{x_i - x_j}{x_i + x_j} \right)_{1 \leq i,j \leq 2n} = \prod_{1 \leq i<j \leq 2n} \frac{x_i - x_j}{x_i + x_j} \text{Hf} \left( \frac{1}{x_i + x_j} \right)_{1 \leq i,j \leq 2n}$$

(191)

Another proof of this relation was presented in [50]. Let us mark that a special case of this relation is obtained if $x_{2n}$ vanishes. In this case we write

$$\text{Pf}(B) = \Delta_{2n-1}^{**}(x) \text{Hf}(C) =: \Delta_{2n}^{**}(x)$$

(192)

where $B$ and $C$ are respectively antisymmetric and symmetric $2n \times 2n$ matrices whose relevant entries (see Remark [5]) are given by

$$B_{ij} = \begin{cases} \frac{x_i - x_j}{(x_i + x_j)^2} & \text{if } 1 \leq i < j < 2n \\ \frac{1}{x_i} & \text{if } i < j = 2n, \end{cases}$$

$$C_{ij} = \begin{cases} \frac{1}{x_i + x_j} & \text{if } 1 \leq i < j < 2n \\ \frac{1}{x_i} & \text{if } i < j = 2n, \end{cases}$$

(193)
B  Sums of Schur functions [2]

In this Appendix we recall some relations from the first part of this work.

Subsets of partitions. In the following, we consider sums over partitions and strict partitions, which will be denoted by Greek letters $\alpha, \beta$. Recall [17] that a strict partition $\alpha$ is a set of integers (parts) $(\alpha_1, \ldots, \alpha_k)$ with $\alpha_1 > \cdots > \alpha_k \geq 0$. The length of a partition $\alpha$, denoted $l(\alpha)$, is the number of non-vanishing parts, thus it is either $k$ or $k - 1$.

Let $P$ be the set of all partitions. We shall need two special subsets of $P$. The first one consists of all partitions $\lambda = (\lambda_1, \ldots, \lambda_{2n})$, $0 \leq n \in \mathbb{Z}$, $\lambda_{2n} \geq 0$, which satisfy

$$\lambda_i + \lambda_{2n+1-i} \text{ is independent of } i, \quad i = 1, \ldots, 2n,$$

or equivalently

$$h_i + h_{2n+1-i} = 2c \text{ is independent of } i \text{ (hence } h_1 + h_{2n} \geq 2n - 1), \quad i = 1, \ldots, 2n, \quad (194)$$

where $h_i = \lambda_i - i + 2n$, and $2c$ is a natural number conditioned by $2c \geq 2n$. This subset consists of all partitions $\lambda$ of length $l(\lambda) \leq 2n$ whose Young diagram satisfies the property that its complement in the rectangular Young diagram $\overline{\lambda}$ corresponding to $(\lambda_1 + \lambda_{2n})^{2n}$ coincides with itself rotated 180 degrees around the center of $\overline{\lambda}$. This set of partitions will be denoted by $SCP(c)$ or simply $SCP$, for “self-complementary partitions”. If we introduce

$$y_i := h_i - c, \quad c = \frac{h_1 + h_{2n-1}}{2}, \quad (195)$$

then relation (194) may be rewritten as

$$y_i + y_{2n+1-i} = 0. \quad (196)$$

The second subset we need consists of the partitions $\lambda$ which satisfy, equivalently,

$$\lambda_{2i} = \lambda_{2i-1}, \quad i = 1, 2, \ldots, \quad (197)$$

or $\lambda = \mu \cup \mu := (\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_k, \mu_k)$ ($2\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in P$), or that the conjugate partitions of $\lambda$ are even, i.e., the ones whose parts are even numbers. This set of partitions will be denoted by $FP$, for “fat partitions”.

Following [17] we will denote by $DP$ the set of all strict partitions (partitions with distinct parts), namely, partitions $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, $1 \leq k \in \mathbb{Z}$ with the strict inequalities $\alpha_1 > \alpha_2 > \cdots > \alpha_k > 0$.

Strict partitions $\alpha$ with the property

$$\alpha_{2i} = \alpha_{2i-1} + 1 \text{ for } 2i - 1 \leq l(\alpha), \quad (198)$$

where we set $\alpha_{2i} = 0$ if $l(\alpha) = 2i - 1$, will be called fat strict partitions. The set of all fat strict partitions will be denoted by $FDP$\footnote{This subset was used in [29] where it was denoted by $DP'$.}

The set of all self-complementary strict partitions will be denoted by $SCDP$.

Let $R_{NM}$ denote the set of all partitions whose Young diagram may be placed into the rectangle $N \times M$, namely, $R_{NM}$ is the set of all partitions $\lambda$ restricted by the conditions $\lambda_1 \leq M$ and $l(\lambda) \leq N$.

Sums over partitions. Consider the following sums (for $t := (t_1, t_3, \ldots)$, $t^* := (t_1^*, t_3^*, \ldots)$, $\bar{t} := (\bar{t}_1, \bar{t}_3, \ldots)$, $\bar{t} := (\bar{t}_1, \bar{t}_3, \ldots)$, $t := (t_1, t_3, \ldots)$, $N := (t_1, t_3, \ldots)$), $N$).

$$S^{(1)}(t, N; U, \bar{A}) := \sum_{\lambda \in P, l(\lambda) \leq N} \hat{A}_h(\lambda)e^{-U(h)} s_{\lambda}(t) \quad (199)$$

where $h(\lambda) = \lambda_i - i + N$. The factors $\hat{A}_h$ on the right-hand side of (199) are determined in terms a pair $(A, a) := \bar{A}$ where $A$ is an infinite skew symmetric matrix and $a$ an infinite vector. For a strict partition
\(h = (h_1, \ldots, h_N)\), the numbers \(\tilde{A}_h\) are defined as the Pfaffian of an antisymmetric \(2n \times 2n\) matrix \(\tilde{A}\) as follows:

\[
\tilde{A}_h := \text{Pf}[\tilde{A}]
\]

where for \(N = 2n\) even

\[
\tilde{A}_{ij} = -\tilde{A}_{ji} := A_{h_i, h_j}, \quad 1 \leq i < j \leq 2n
\]

and for \(N = 2n - 1\) odd

\[
\tilde{A}_{ij} = -\tilde{A}_{ji} := \begin{cases} A_{h_i, h_j} & \text{if } 1 \leq i < j \leq 2n - 1 \\ a_{h_i} & \text{if } 1 \leq i < j = 2n. \end{cases}
\]

In addition we set \(A_0 = 1\).

Then

\[
U_h := \sum_{i=1}^{N} U_{h_i}
\]

where \(U_n, n = 0, 1, 2, \ldots\) is a set of given complex numbers. This set is denoted by \(U\).

As we see the factor \(e^{-U(h)}\) can be included into the factor \(\tilde{A}_h\) by redefinition of the data \(A\) as follows:

\[
A_{nm} \rightarrow A_{nm}e^{-U_n-U_m}, \quad a_n \rightarrow a_ne^{-U_n}
\]

However we prefer to keep \(U\) as a set of parameters.

**Example 0** We choose the following matrix \(A\) is given by

\[
A_{ik} = (A_0)_{ik} := \begin{cases} \text{sgn}(i-k) & \text{if } 1 \leq i, k \leq L \\ 0 & \text{otherwise} \end{cases}, \quad a_k = \begin{cases} 1 & \text{if } k \leq L \\ 0 & \text{otherwise} \end{cases}.
\]

Remark 7. The matrix \(A_1\) is infinite. However if in series \((199)\) we put \(U_n = +\infty\) for \(n > L\), it will be the same as if we deals with the finite \(L\) by \(L\) matrix \(A\), given by \((204)\).

**Example 1**

\[
A_{ik} = (A_1)_{ik} := 1, \quad i < k, \quad a_k = 1
\]

Then

\[
(\tilde{A}_1)_{(h)} = 1
\]

**Example 2**

The matrix \(A\) is a finite \(2n\) by \(2n\) matrix, and \(a = 0\), thus the sum \((199)\) ranges only partitions with even number of non-vashing parts. We put

\[
A_{ik} = (A_2)_{ik} := -\delta_{i,2e-i}, \quad i < k
\]

Then

\[
(\tilde{A}_2)_{(h)} = \begin{cases} 1 & \text{if } \lambda \in \text{SCP}(e) \\ 0 & \text{otherwise} \end{cases}
\]

where \(h = (h_1, \ldots, h_N)\) is related to \(\lambda = (\lambda_1, \ldots, \lambda_N)\) as \(h_i = \lambda_i - i + N, i = 1, \ldots, N = 2n\).

**Example 3** Given set of additional variables \(t' = (t'_1, t'_2, t'_3, \ldots)\) where we take

\[
A_{nm} = (A_3)_{nm} := \frac{1}{2} e^{-U_m-U_n} Q_{(n,m)}(\frac{1}{2}t'), \quad a_n = (a_3)_{n} := e^{-U_n} Q_{(n)}(\frac{1}{2}t')
\]

Here, the projective Schur functions \(Q_{\alpha}\) are weighted polynomials in the variables \(t'_m\), \(\text{deg} t'_m = m\), labeled by strict partitions (See [17] for their detailed definition.)

**Remark 8.** Let us introduce notation \(t'_\infty = (1,0,0,\ldots)\). It is known that \(Q_{h}(\frac{1}{2}t'_\infty) = \Delta^*(h) \prod_{i=1}^{N} \frac{1}{h_i!}\) where

\[
\Delta^*(h) := \prod_{i<j} h_i - h_j
\]

Thus for this choice of \(t'\) we obtain

\[
(\tilde{A}_3)_{(h)} = \Delta^*(h) \prod_{i=1}^{N} \frac{1}{h_i!}
\]

One may compare it with Example 5 where \(f(n) = n\).
Example 4

\[ A_{nm} = (A_4)_{nm} := \delta_{n+1,m} - \delta_{m+1,n}. \]  

Then

\[ (\bar{A}_4)_{\{h\}} = \begin{cases} 1 & \text{iff } \lambda = (\lambda_1, \ldots, \lambda_{2n}) \in \text{FP} \\ 0 & \text{otherwise} \end{cases} \]  

where \( h = (h_1, \ldots, h_N) \) is related to \( \lambda = (\lambda_1, \ldots, \lambda_N) \) as \( h_i = \lambda_i - i + N, i = 1, \ldots, N = 2n. \)

Remark 9. For some applications we may need further examples. In Examples 5-7 \( \bar{A} \) depends on a given function on the lattice denoted by \( f \). In particular one can choose \( f(n) = n \). Below are examples of matrices \( A \) whose Pfaffians are well-known (see [89] and references there).

Example 5

\[ A_{nm} = (A_5)_{nm} := \frac{f(n) - f(m)}{f(n) + f(m)} \]  

Then for \( h_i = \lambda_i - i + N, i = 1, \ldots, N, \) we have

\[ (\bar{A}_5)_{\{h\}} = \Delta_N^{(5)}(f(h)) \]  

where

\[ \Delta_N^{(5)}(f(h)) := \prod_{i < j \leq N} \frac{f(h_i) - f(h_j)}{f(h_i) + f(h_j)} \]  

Example 6

\[ A_{nm} = (A_6)_{nm} := \frac{f(n) - f(m)}{1 - f(n)f(m)} \]  

Then for \( h_i = \lambda_i - i + N, i = 1, \ldots, N, \) we have

\[ (\bar{A}_6)_{\{h\}} = \Delta_N^{(6)}(f(h)) \]  

where

\[ \Delta_N^{(6)}(f(h)) := \prod_{i < j \leq N} \frac{f(h_i) - f(h_j)}{1 - f(h_i)f(h_j)} \]  

Example 7

\[ A_{nm} = (A_7)_{nm} := \frac{f(n) - f(m)}{(f(n) + f(m))^2}. \]  

Then for \( h_i = \lambda_i - i + N, i = 1, \ldots, N, \) we have

\[ (\bar{A}_7)_{\{h\}} = \Delta_N^{(7)}(f(h)) \]  

where

\[ \Delta_N^{(7)}(f(h)) := \left( \prod_{i < j \leq N} \frac{f(h_i) - f(h_j)}{(f(h_i) + f(h_j))^2} \right) Hf \left( \frac{1}{f(h_i) + f(h_j)} \right) \]  

Having these examples we introduce the notation

\[ S_i^{(1)}(t, N; U) := S_i^{(1)}(t, N; U, \bar{A}_i) = \sum_{\lambda \in \mathcal{P}} (\bar{A}_i)_{\lambda(h)} e^{-U\lambda} s_\lambda(t), \quad i = 1, \ldots, 6 \]
In particular we obtain

\[ S_{0}^{(1)}(t; N, M, U) := \sum_{\lambda \in R_{N,M}} e^{-U_{\lambda}} s_{\lambda}(t) \]  

\[ S_{1}^{(1)}(t; N; U) := \sum_{\lambda \in P_{\ell(\lambda)} \leq N} e^{-U_{\lambda}} s_{\lambda}(t) \]  

\[ S_{2}^{(1)}(t, N; U, c) := \sum_{\lambda \in \text{SCP}(c) \ell(\lambda) \leq N} e^{-U_{\lambda}} s_{\lambda}(t) \]  

\[ S_{3}^{(1)}(t, N, t'; U) := \sum_{\lambda \in P_{\ell(\lambda)} \leq N} e^{-U_{\lambda}} Q_{\alpha(\lambda)} \frac{1}{2} (t') s_{\lambda}(t) \]  

\[ S_{4}^{(1)}(t, N = 2n, U) := \sum_{\lambda \in FP_{\ell(\lambda)} \leq N} e^{-U_{\lambda}} s_{\lambda}(t) \]  

\[ S_{i}^{(1)}(t, N; U, f) := \sum_{\lambda \in P_{\ell(\lambda)} \leq N} \Delta^{(i)}_{N} (f(h)) e^{-U_{\lambda}} s_{\lambda}(t), \quad i = 5, 6, 7 \]  

The coefficients \( U_{\{\alpha\}} \) are defined as

\[ U_{\{\alpha\}} := \sum_{i=1}^{k} U_{\alpha_i} \]  

The notation \( U_{\lambda} \) serves for \( U_{\lambda} := U_{\{h\}}, \quad h_{i} = \lambda_{i} - i + \ell(\lambda) \) \( \)  

**Proposition 9.** Sums (199),(224)-(229) are tau functions of the “large” BKP hierarchy introduced in [3] with respect to the time variables \( t \). Sums (227) are tau functions of the “small” BKP hierarchy introduced in [9] with respect to the time variables \( t' \).

Actually the sum (228) where we put \( U = 0 \) and \( t = (1, 0, 0, \ldots) \) was used in [75].

**Sums over pairs of strict partitions.** In the Frobenius notations [17] we write \( \lambda = (\alpha|\beta) = (\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_k) \), where \( \alpha = (\alpha_1, \ldots, \alpha_k), \alpha_1 > \cdots > \alpha_k \geq 0 \) and \( \beta = (\beta_1, \ldots, \beta_k) \) may be viewed as strict partitions. It is clear that \( \ell(\alpha) = \ell(\beta), \ell(\beta) \pm 1 \), and we imply this restriction in sums over pairs of strict partitions below.

Now we consider

\[ S^{(2)}(t; U, A, B) := 1 + \sum_{\alpha, \beta \in \text{DP}} e^{U_{(-\beta-1)}} U_{(\alpha)} A_{\alpha} s_{(\alpha|\beta)}(t) B_{\beta} \]  

where given infinite skew matrices \( A \) and \( B \) and given vectors \( a \) and \( b \), the factors \( A_{\alpha} \) and \( B_{\alpha} \) are defined in the same way as before.

\[ U_{(\alpha)} = \sum_{i=1}^{k} U_{\alpha_i}, \quad U_{(-\beta-1)} = \sum_{i=1}^{k} U_{-\beta_i-1} \]  

We introduce the following notation

\[ S^{(2)}_{i,j}(t; U) := S^{(2)}(t; U, \bar{A}_{i}, \bar{A}_{j}) \]  

where \( i, j = 1, \ldots, 7 \) and matrices \( A_{i} \) are taken from the Examples 1-7 above. In particular we obtain

\[ S^{(2)}_{i,j}(t; U) := S^{(2)}(t; U, \bar{A}_{i}, \bar{A}_{j}) \]  

where \( i, j = 1, \ldots, 7 \) and matrices \( A_{i} \) are taken from the Examples 1-7 above. In particular we obtain
For $B.1$ Pfaffian representations

\[
S_{11}^{(2)}(t; U) := \sum_{\lambda \in \mathcal{P}} e^{-U_{\lambda}} s_\lambda(t) \quad (235)
\]

\[
S_{22}^{(2)}(t; U) := 1 + \sum_{\alpha, \beta \in \text{SCDP}} e^{U_{(\alpha)} - U_{(\beta)}} s_{(\alpha|\beta)}(t) \quad (236)
\]

\[
S_{24}^{(2)}(t; U) := 1 + \sum_{\alpha \in \text{SCDP}, \beta \in \text{FDP}} e^{U_{(\alpha)} - U_{(\beta)}} s_{(\alpha|\beta)}(t) \quad (237)
\]

\[
S_{31}^{(2)}(t, t'; U) := 1 + \sum_{\alpha, \beta \in \text{SCDP}} e^{U_{(\alpha)} - U_{(\beta)}} Q_{\alpha(\alpha)}(\frac{1}{2} t') s_{(\alpha|\beta)}(t) \quad (238)
\]

\[
S_{41}^{(2)}(t; U) := 1 + \sum_{\alpha \in \text{FDP}, \beta \in \text{FDP}} e^{U_{(\alpha)} - U_{(\beta)}} s_{(\alpha|\beta)}(t) \quad (239)
\]

\[
S_{41}^{(2)}(t; U) := 1 + \sum_{\alpha \in \text{FDP}, \beta \in \text{FDP}} e^{U_{(\alpha)} - U_{(\beta)}} s_{(\alpha|\beta)}(t) \quad (240)
\]

\[
S_{33}^{(2)}(t, t', t''; U) := 1 + \sum_{\alpha, \beta \in \text{SCDP}} e^{U_{(\alpha)} - U_{(\beta)}} Q_{\alpha(\alpha)}(\frac{1}{2} t') s_{(\alpha|\beta)}(t) Q_{\beta(\beta)}(\frac{1}{2} t'') \quad (241)
\]

\[
S_{34}^{(2)}(t, t'; U) := 1 + \sum_{\alpha \in \text{FDP}, \beta \in \text{FDP}} e^{U_{(\alpha)} - U_{(\beta)}} Q_{\alpha(\alpha)}(\frac{1}{2} t') s_{(\alpha|\beta)}(t) \quad (242)
\]

\[
S_{ij}^{(2)}(t; U, f) := 1 + \sum_{\alpha \in \text{FDP}, \beta \in \text{FDP}} e^{U_{(\alpha)} - U_{(\beta)}} \Delta^{(i)}(f(\alpha)) s_{(\alpha|\beta)}(t) \Delta^{(j)}(f(\beta)), \ i, j = 5, 6, 7 \quad (243)
\]

Each $Q_{\alpha}(\frac{1}{2} t')$ is known to be a “small” BKP [7, 9] tau function. (This was a nice observation of [13, 14].) The fact that only odd subscripts appear in the BKP higher times $t_{2m-1}$ is related to the reduction from the KP hierarchy.

**Proposition 10.** Sums (232, 233) are tau functions of the “large” BKP hierarchy introduced in [3] with respect to the time variables $t$. Sums (238) are tau functions of the BKP hierarchy introduced in [9] with respect to the time variables $t'$. Sums (241) are tau functions of the two-component BKP hierarchy introduced in [9] with respect to the time variables $t'$ and $t''$.

**Remark 10.** Let us remind that for the small BKP hierarchy obtained from KP we have the following [22]

\[
S = \sum_{\alpha \in \text{DP}} \hat{A}_\alpha Q_\alpha(t') \quad (244)
\]

By specification of the data $\hat{A}$ we obtain

\[
\sum_{\alpha \in \text{DP}} e^{-U_{(\alpha)}} Q_\alpha(\frac{1}{2} t'), \quad \sum_{\alpha \in \text{DP}} e^{-U_{(\alpha)}} Q_\alpha(\frac{1}{2} t') Q_{\alpha}(\frac{1}{2} t''), \quad \sum_{\alpha \in \text{DP}} e^{-U_{(\alpha)}} Q_\alpha(\frac{1}{2} t') \quad (245)
\]

The sums (246) are particular examples (see [29]) of BKP tau functions, as introduced in [7], defining solutions to what was called the small BKP hierarchy in [3].

The coupled small BKP yields series

\[
S_5(t', t'', D) := \sum_{\alpha, \beta \in \text{FDP}} Q_{\alpha}(\frac{1}{2} t') D_{\alpha, \beta} Q_{\beta}(\frac{1}{2} t'') \quad (246)
\]

The coefficients $D_{\alpha, \beta}$ in (246) are defined as determinants:

\[
D_{\alpha, \beta} = \det (D_{\alpha, \beta}) \quad (247)
\]

where $D$ is a given infinite matrix. Taking $D_{nm} = e^{U_{m} - U_{n}} s_{(n|m)}(t)$ we reproduce (241).

**B.1 Pfaffian representations**

For

\[
t = t(x^{(M)}) =: [x_1] + \cdots + [x_M] \quad (248)
\]
we have for any \( N \geq M = 1 \)

\[
S^{(1)}(t(x_i); N, U, \bar{A}) = \sum_{n=0}^{\infty} a_n e^{-U_n t x_i^n}
\]  

(249)

and for any \( N \geq M = 2 \) we have

\[
S^{(1)}(t(x_i, x_j); N, U, \bar{A}) = \frac{1}{x_i - x_j} \sum_{m > n \geq 0} A_{nm} e^{-U_n - U_m} (x_i^m x_j^n - x_i^n x_j^m)
\]  

(250)

**Proposition 11.** For \( M = N \) we have

\[
S^{(1)}(t(x^{(M)}); N, U, \bar{A}) = \frac{1}{\Delta_N(x)} \text{Pf} [\bar{S}]
\]  

(251)

where for \( N = 2n \) even

\[
\bar{S}_{ij} = -\bar{S}_{ji} := (x_i - x_j) S^{(1)}(t(x_i, x_j), N, U, \bar{A}), \quad 1 \leq i < j \leq 2n
\]  

(252)

and for \( N = 2n - 1 \) odd

\[
\bar{S}_{ij} = -\bar{S}_{ji} := \begin{cases} 
(x_i - x_j) S^{(1)}(t(x_i, x_j), N, U, \bar{A}) & \text{if } 1 \leq i < j \leq 2n - 1 \\
S^{(1)}(t(x_i), N, U, \bar{A}) & \text{if } 1 \leq i < j = 2n
\end{cases}
\]  

(253)

and where

\[
\Delta_N(x) := \prod_{0 \leq i < j \leq N} (x_i - x_j)
\]  

(254)

We shall omit more spacious formulae for the case \( M \neq N \).

**Remark 11.** Let us write down the entries of \( \bar{S} \) to express \( S^{(1)}_i, i = 0, \ldots, 6 \)

\[
S^{(1)}_1(t(x_i), N, U) = (x_i - x_j)^{-1} \sum_{m > n \geq 0} e^{-U_n - U_m} (x_i^m x_j^n - x_i^n x_j^m)
\]  

(255)

\[
S^{(1)}_1(t(x_i), N, U) = \sum_{n=0}^{\infty} e^{-U_n} x_i^n
\]  

(256)

\[
S^{(1)}_2(t(x_i, x_j), 2n, U) = (x_i - x_j)^{-1} \sum_{n=0}^{\infty} e^{-U_n - U_{2n}} (x_i^{2n} x_j^n - x_i^n x_j^{2n})
\]  

(257)

\[
S^{(1)}_3(t(x_i, x_j), N, U) = (x_i - x_j)^{-1} \sum_{m > n \geq 0} e^{-U_n - U_m} Q_{(m, n)}(\bar{t}')(x_i^m x_j^n - x_j^n x_i^m)
\]  

(258)

\[
S^{(1)}_3(t(x_i), N, U) = \sum_{n=0}^{\infty} e^{-U_n} Q_{(n)}(\bar{t}') x_i^n
\]  

(259)

\[
S^{(1)}_4(t(x_i, x_j), N, U) = (x_i - x_j)^{-1} \sum_{n=0}^{\infty} e^{-U_n - U_{n+1}} (x_i^n x_j^{n+1} - x_j^{n+1} x_i^n)
\]  

(260)

\[
S^{(1)}_4(t(x_i), N, U) = \sum_{n=0}^{\infty} e^{-U_n} x_i^n
\]  

(261)

In particular substituting (205), (212) we obtain

\[
S^{(1)}_1(t, N, U = 0) = \frac{1}{\Delta_N(x)} \text{Pf} \left( \frac{x_j - x_i}{(1 - x_i)(1 - x_j)(1 - x_i x_j)} \right)
\]  

(262)

\[
S^{(1)}_4(t, N, U = 0) = \frac{1}{\Delta_N(x)} \text{Pf} \left( \frac{x_j - x_i}{1 - x_i x_j} \right)
\]  

(263)

Then it follows that

\[
\sum_{\lambda \in \mathcal{P}} s_{\lambda}(t(x^N)) = \prod_{i=1}^{N} (1 - x_i)^{-1} \prod_{i < j \leq N} (1 - x_i x_j)^{-1}
\]  

(264)
Proposition 14.

\[ \sum_{\lambda \in P} s_{\lambda \cup \lambda}(t(x^n)) = \prod_{i<j \leq N} (1 - x_i x_j)^{-1} \quad (265) \]

Formulæ (264) and (265) are known, see Exs 4-5 in I-5 of [17]. (265) is called Schur-Littlewood identity.

It is convenient to re-write these formulæ in a way independent of the choice of \( N \):

**Proposition 12.**

\[ \sum_{\lambda \in P} s_\lambda(t) = e^{\frac{1}{2} \sum_{m=1}^{\infty} m t_m^2 + \sum_{m=1}^{\infty} t_{2m-1}} \quad (266) \]

and

\[ \sum_{\lambda \in P} s_{\lambda \cup \lambda}(t) = e^{\frac{1}{2} \sum_{m=1}^{\infty} m t_m^2 - \sum_{m=1}^{\infty} t_{2m}} \quad (267) \]

Relations (266) and (267) will be used later in Section to solve certain combinatorial problem. From

\[ s_{\lambda^c}(t) = (-1)^{|\lambda|} s_\lambda(-t) \quad (268) \]

we obtain

\[ \sum_{\lambda \in P} (-1)^{|\lambda|} s_\lambda(t) = e^{\frac{1}{2} \sum_{m=1}^{\infty} m t_m^2 - \sum_{m=1}^{\infty} t_{2m-1}} \quad (269) \]

and

\[ \sum_{\lambda \in P \text{ even}} s_\lambda(t) = e^{\frac{1}{2} \sum_{m=1}^{\infty} m t_m^2 + \sum_{m=1}^{\infty} t_{2m}} \quad (270) \]

By the simple re-scaling \( t_m \rightarrow z^m t_m \) in equations (266)–(270) and equating factors before same powers of \( z \) we obtain

**Proposition 13.**

\[ \sum_{\lambda \in P} s_\lambda(t) = s_{(\bar{t})}(\bar{t}) \quad [\bar{t}_{2m-1} = t_{2m}, \bar{t}_{2m} = \frac{m}{2} t_m^2] \quad (271) \]

\[ \sum_{\lambda \in P \text{ even}} (-1)^{|\lambda|} s_\lambda(t) = s_{(\bar{t})}(\bar{t}) \quad [\bar{t}_{2m-1} = -t_{2m}, \bar{t}_{2m} = \frac{m}{2} t_m^2] \quad (272) \]

\[ \sum_{\lambda \in P \text{ even}} s_{\lambda \cup \lambda}(t) = s_{(\bar{t})}(\bar{t}) \quad [\bar{t}_m = \frac{m}{2} t_m^2 - t_{2m}] \quad (273) \]

\[ \sum_{\lambda \in P} s_\lambda(t) = s_{(\bar{t})}(\bar{t}) \quad [\bar{t}_m = \frac{m}{2} t_m^2 + t_{2m}] \quad (274) \]

where auxiliary sets of times \( \bar{t} = (\bar{t}_1, \bar{t}_2, \ldots) \) are specified in the brackets to the right of equalities.

For instance we get (271) from (266) using the equality

\[ \sum_{\lambda \in P} z^{|\lambda|} s_\lambda(t) = e^{\sum_{m=1}^{\infty} \frac{z^m}{m} t_m^2 + \sum_{m=1}^{\infty} z^{2m-1} t_{2m-1}} = \sum_{\bar{t} = 0}^{\infty} z^{T(\bar{t})} s_{\bar{t}}(\bar{t}) \]

where \( \bar{t} = (t_1, \frac{1}{2} t_2, t_3, \frac{3}{2} t_4, t_5, \frac{5}{2} t_6, \ldots) \).

Formula (271) in case \( t = (1, 0, 0, \ldots) \) has an interpretation in terms of total numbers of standard tableaux of weight \( (1^n) \) and numbers of involutive permutations of \( S_n \), see Ex 12.15 of [17], [17], [76].

We get from Proposition 11.

**Proposition 14.**

\[ S_1^{(1)}(t(x^{2n}), 2n, U) = \frac{1}{\Delta_{2n}(x)} \text{Pf} \left[ \sum_{m,n \geq 0} e^{-U_m - U_n} (x_j^n x_i^m - x_i^n x_j^m) \right]_{i,j = 1, \ldots, 2n} \quad (275) \]

35
Choosing $U_m = 0, m \leq L + 2n - 1$ and $U_m = +\infty, m > L + 2n - 1$ we obtain

$$\sum_{\lambda \in P, \lambda_1 \leq L} s_\lambda(t(x^{(2n)})) = \frac{1}{\Delta_{2n}(x)} \text{Pf} \left[ \frac{x_j - x_i}{(1 - x_j)(1 - x_i)} \left(1 - (x_i x_j)^{L+N} + \frac{x_j^{L+N} - x_i^{L+N}}{x_j - x_i} \right) \right]_{i,j=1,\ldots,2n} \quad (276)$$

**Proposition 15.**

$$S_4^{(1)}(t(x^{(2n)}), 2n, U) = \frac{1}{\Delta_{2n}(x)} \text{Pf} \left[ (x_j - x_i) f(x_i x_j, U) \right]_{i,j=1,\ldots,2n} \quad (277)$$

where

$$f(z, U) = \sum_{m=0}^{\infty} e^{-U_{m-1} z^m} = S_4^{(1)}(t(x_i, x_j), U)$$

Choosing $U_m = 0, m \leq L + 2n - 1$ and $U_m = +\infty, m > L + 2n - 1$ we obtain

$$\sum_{\lambda \in \text{SCP}(c)} s_\lambda(t(x^{(2n)})) = \frac{1}{\Delta_{2n}(x)} \text{Pf} \left[ \frac{1 - (x_i x_j)^{L+2n-1}}{1 - x_i x_j} \right]_{i,j=1,\ldots,2n} \quad (278)$$

Next, as a corollary of Proposition 15 we obtain

**Proposition 16.**

$$\sum_{\lambda \in \text{SCP}(c)} s_\lambda(t(x^{(2n)})) = \frac{1}{\Delta_{2n}(x)} \text{Pf} \left[ \left( \frac{|c+\frac{1}{2}| - |c+\frac{1}{2}| x_j - x_i \right)^2 \right]_{i,j=1,\ldots,2n} \quad (279)$$

$$\sum_{\lambda \in \text{SCP}(c)} (-1)^{\sum_{i=1}^{n} (\lambda_i - 1 + 2n)} s_\lambda(t(x^{(2n)})) = \frac{1}{\Delta_{2n}(x)} \text{Pf} \left[ \left( \frac{|c+\frac{1}{2}| - |c+\frac{1}{2}| x_j - x_i \right)^2 \right]_{i,j=1,\ldots,2n} \quad (280)$$

where $[a]$ is equal to the integer part of $a$. Notice that in case $c = n$ we have only one term related to $\lambda = 0$ and thus the both sides of identity (278) are equal to 1 (compare to Lemma 5.7 in [89]).

**Integral representations for the sums.**

Let us recall the fermionic expression for the sums of Schur functions (??)

$$S^{(1)}(t, N; U, \bar{A}) = \langle N|\Gamma(t)\mathbb{T}(U)g^{--}(A)|0 \rangle, \quad \mathbb{T}(U) = \exp \sum_{i \geq 0} \left( U_{i-1} \psi_{i-1}^{\dagger} \psi_{i-1} - U_i \psi_i^{\dagger} \psi_i \right)$$

where for the Examples 0-6 above we have

$$g^{--}((A_0)^c) := e^{\sum_{m \geq m_0} \psi_m \psi_n + \sum_{m \leq m_0} \psi_m \phi_n} \quad (281)$$

$$g^{--}(\bar{A}_1) := e^{\sum_{m > m_0} \psi_m \psi_n + \sum_{m \in \mathbb{Z}} \psi_m \phi_n} = e^{\int \psi(x^{-1}) \phi(x) dx} \quad (282)$$

$$g^{--}(\bar{A}_2) := e^{\sum_{m < m_0} (-1)^m \psi_{2m} \psi_n + \sum_{m \in \mathbb{Z}} \psi_m \phi_n} = e^{\int x^{-2m} \psi(x) \phi(x) dx} \quad (283)$$

$$g^{--}(\bar{A}_3(t^c)) := e^{\sum_{m > m_0} \frac{t^c}{m^2 + \sum_{m} \psi_m \psi_n} + \sum_{m \in \mathbb{Z}} \psi_m \phi_n} = (284)$$

$$g^{--}(\bar{A}_3(t^\phi)) := e^{\sum_{m > m_0} \frac{\psi_m \psi_n}{m^2 + \sum_{m} \psi_m \psi_n} + \sum_{m \in \mathbb{Z}} \psi_m \phi_n} = (285)$$

$$g^{--}(\bar{A}_4) := e^{\sum_{m \in \mathbb{Z}} \psi_m \psi_{m-1}} = e^{\int \psi(x^{-1}) \phi(x) dx} \quad (286)$$

$$g^{--}(\bar{A}_5) := e^{\sum_{m > m_0} \frac{\psi_m \psi_n}{m^2 + \sum_{m} \psi_m \psi_n} + \sum_{m \in \mathbb{Z}} \psi_m \phi_n} = (287)$$

$$g^{--}(\bar{A}_6) := e^{\sum_{m > m_0} \frac{\psi_m \psi_n}{m^2 + \sum_{m} \psi_m \psi_n} + \sum_{m \in \mathbb{Z}} \psi_m \phi_n} = (288)$$

The corollary of the right hand side expressions is the fact that sums (223) may be re-written as certain multiply integrals ($\frac{1}{2}$-N-ply integrals for $S_2^{(1)}$, $S_4^{(1)}$, and N-ply integrals in other cases).
For a sum \((199)\) it is true in case we can present \(A_{nm}\) as moments, or, the same the following inverse moment problem: given \(A_{nm} = -A_{mn}, \ m, n \geq 0\) to find such an integration domain \(D\) and an antisymmetric measure \(dA(x,y) = -dA(y,x)\) such that

\[
A_{nm} = \int_D x^n y^m dA(x,y), \quad n, m \geq 0
\] (289)

Also
\[
a_n = \int_\gamma x^n da(x)
\] (290)

If we have \((289)\) then in case \(N = 2n\) we can write \(N\)-ply integral [[to be fixed]]

\[
S^{(1)}(t, 2n, U, A_c) = e^{-\sum_{i=0}^{N-1} U_i} \int_{D^n} \left( \prod_{i=1}^{2n} e^{\xi_r(t, x_i)} \cdot \Delta_{2n}(x) \right) Pf [dA(x_i, x_j)]
\] (291)

where \(\xi_r(t, x)\) is the following \(\Psi DO\) operator
\[
\xi_r(t, x) = \sum_{m=1}^\infty t_m (x r(D))^m, \quad D = x \partial_x
\] (292)

and \(r\) is related to \(U\) as follows
\[
r(n) = e^{U_n - U_{n+1}}
\] (293)

The case \(U = 0\) causes \(\xi_r(t, x) = \sum_{m=1}^\infty t_m x^m\) and we obtain more familiar expression

\[
S^{(1)}(t, 2n, U = 0, A_c) = \int_{D^n} \prod_{i=1}^{2n} e^{\sum_{m=1}^\infty t_m x_i^m} \Delta_{2n}(x) Pf [dA(x_i, x_j)]
\] (294)

In case \(N = 2n + 1\) we have more involved expressions.

In case the solution of the inverse problem is not unique we have a set of different integral representations for the sum \((199)\).

C Matrix integrals. Integration measures

Three Ginibre ensembles The complex Ginibre ensemble of complex \(N \times N\) matrices \(Z\) is defined by the following probability measure

\[
d\mu_2(Z) = e^{-Tr ZZ^\dagger} \prod_{i,j=1,...,N} \frac{|dZ_{ij}|^2}{2\pi}
\] (295)

If we write \(Z = U(\Lambda + \Delta) U^\dagger\) where \(U \in U(N), \Lambda = \text{diag}(z_1, \ldots, z_N)\) and \(\Delta\) is strictly upper-triangular, then

\[
d\mu_2(Z) = C_N e^{-Tr (\Lambda\Lambda^\dagger - \Delta\Delta^\dagger)} \prod_{i<j} |z_i - z_j|^2 |d\Delta|^2 |d\Lambda|^2 |dU|^2
\] (296)

The real quaternionic

\[
d\mu_4(X) = e^{-Tr XX^\dagger} \prod_{i,j=1,...,N} \frac{|dX_{ij}|^2}{2\pi}
\] (297)

The real Ginibre ensemble of \(N \times N\) real matrices \(X\) is defined be the following measure

\[
d\mu_1(X) := \prod_{i,j=1,...,N} e^{-\frac{1}{2}Tr X_{ij}^2} \frac{dX_{ij}}{\sqrt{2\pi}}
\] (298)
Integrals over orthogonal and symplectic groups. Now

For $O(2n)$ the Haar measure is
\[
dµ(O) = \frac{2^{(n-1)^2}}{\pi^{n/2} n!} \prod_{i=1}^{n} (\cos \theta_i - \cos \theta_j)^2 \prod_{i=1}^{n} \sin \theta_i \prod_{i<j} (\cos \theta_i - \cos \theta_j)
\]
(299)

where $e^{\theta_1}, e^{-\theta_1}, \ldots, e^{\theta_n}, e^{-\theta_n}$ are eigenvalues of $O$.

For $O(2n+1)$ the Haar measure is
\[
dµ(O) = \frac{2^{n^2}}{\pi^{n/2} n!} \prod_{i=1}^{n} (\cos \theta_i - \cos \theta_j)^2 \prod_{i=1}^{n} \sin^2 \theta_i \prod_{i<j} (\cos \theta_i - \cos \theta_j)
\]
(300)

where $e^{\theta_1}, e^{-\theta_1}, \ldots, e^{\theta_n}, e^{-\theta_n}, 1$ are eigenvalues of $O$.

For $Sp(2n)$ the Haar measure is
\[
dµ(O) = \frac{2^{n^2}}{\pi^{n/2} n!} \prod_{i=1}^{n} (\cos \theta_i - \cos \theta_j)^2 \sin^2 \theta_i \prod_{i<j} (\cos \theta_i - \cos \theta_j)
\]
(301)

where $e^{\theta_1}, e^{-\theta_1}, \ldots, e^{\theta_n}, e^{-\theta_n}$ are eigenvalues of $S$.

D More details

D.1

Integrals $I^{(1)}_1, I^{(1)}_2$ and $I^{(1)}_4$ may be considered as $\beta = 1, 2, 4$ ensembles [16] related to a contour $\gamma$. They may be obtained as particular cases of $I^{(1)}$ as follows:

Integral $I^{(1)}_1(N)$ is a particular case of $I^{(1)}(N)$ where in the (A) case
\[
A(x_i, x_j) = \text{sgn}(x_i - x_j), \quad a(x) = 1
\]
(302)

while in case (B)
\[
A(x_k, x_j) = e^{-\frac{\pi i}{4} \text{sgn}(\varphi_k - \varphi_j)}, \quad a(x) = e^{-\frac{\pi i}{4}},
\]
(303)

with $\varphi_i = \arg x_i$. To prove this we use:

Lemma 5.

\[
Pf[\text{sgn}(x_k - x_j)] = \text{sgn} \Delta(x), \quad x_k \in \mathbb{R},
\]
(304)

\[
Pf[\text{sgn}(\varphi_k - \varphi_j)] = \text{sgn} \left( e^{-\frac{\pi i}{4}(N^2-N)} \Delta(x) \right), \quad x_k = e^{i \varphi_k}
\]
(305)

where $k, j = 1, \ldots, N$.

Integral $I^{(1)}_3(N)$ is obtained from $I^{(1)}(N)$ by setting
\[
A(x_i, x_j) = \frac{x_i - x_j}{x_i + x_j}, \quad a(x) = 1
\]
(306)

We use the fact that
\[
\Delta^*(x) = Pf \begin{bmatrix} x_i - x_j \\ x_i + x_j \end{bmatrix}
\]
(307)

Integral $I^{(1)}_4(N)$ is obtained from $I^{(1)}(N)$ as follows. In case (A) we set
\[
A(x_i, x_j) = \frac{1}{2} \left( x_j \frac{\partial}{\partial x_j} \delta(x_i - x_j) - (x_i \leftrightarrow x_j) \right)
\]
(308)

and in case (B) we set
\[
A(x_i, x_j) = \frac{\partial}{\partial \varphi_j} \delta(\varphi_i - \varphi_j).
\]
(309)
To relate these integrals to the 2-BKP hierarchy we introduce deformations $I^{(1)}(N) \to I^{(1)}(N; t, \tilde{t})$ through the following deformation of the measure
\[\text{d}u(x) \to \text{d}u(x|t, \tilde{t}) = b(t, \{x\})b(-\tilde{t}, \{x^{-1}\})\text{d}u(x)\]
where
\[b(s, t) = \exp \sum_{n \text{ odd,}} \frac{n}{2} s_n t_n\]
and
\[\{z\} = \left(2z, \frac{2z^3}{3}, \frac{2z^5}{5}, \ldots \right).\]

Below, we show that the generating series obtained by Poissonization (the grand partition function)
\[Z_i(\mu; t, \tilde{t}) = b(t, \tilde{t}) \sum_{N=0}^{\infty} I_i(N; t, \tilde{t}) \mu^N N!, \quad i = 1, 2, 3, 4,\]
are particular 2-BKP tau functions (??).

We also consider the following 2N-fold integrals:
\[I_5(N; t^{(1)}, t^{(2)}, \tilde{t}^{(1)}, \tilde{t}^{(2)}) := \int \Delta_N(z) \Delta_N(y) \prod_{i=1}^{N} dv(z_i, y_i|t^{(1)}, t^{(2)}, \tilde{t}^{(1)}, \tilde{t}^{(2)}),\]
where
\[dv(z, y|t^{(1)}, t^{(2)}, \tilde{t}^{(1)}, \tilde{t}^{(2)}) = b(t^{(1)}, \{z\})b(-\tilde{t}^{(1)}, \{z^{-1}\})b(t^{(2)}, \{y\})b(-\tilde{t}^{(2)}, \{y^{-1}\})dv(z, y)\]
(here $dv(z, y)$ is an arbitrary bi-measure), and show that the generating series
\[Z_5(\mu; t^{(1)}, t^{(2)}, \tilde{t}^{(1)}, \tilde{t}^{(2)}) = b(t^{(1)}, \tilde{t}^{(1)})b(t^{(2)}, \tilde{t}^{(2)}) \sum_{N=0}^{\infty} I_5(N; t^{(1)}, t^{(2)}, \tilde{t}^{(1)}, \tilde{t}^{(2)}) \mu^N N!,\]
is a particular case of the two-component 2-BKP tau function (??).

**Remark 12.** Note that
\[Z_2(\mu; t, \tilde{t}) = Z_5(\mu; t^{(1)}, t^{(2)}, \tilde{t}^{(1)}, \tilde{t}^{(2)})\]
if
\[dv(z, y) = \delta(z - y)dv(z)dv(y), \quad t = t^{(1)} + t^{(2)}, \quad \tilde{t} = \tilde{t}^{(1)} + \tilde{t}^{(2)}.\]

The integrals $Z_1(\mu; t, \tilde{t})$, $Z_2(\mu; t, \tilde{t})$, $Z_3(\mu; t, \tilde{t})$ and $Z_5(\mu; t^{(1)}, t^{(2)}, \tilde{t}^{(1)}, \tilde{t}^{(2)})$ may be obtained as continuous limits of $S_1(t_\infty, t^*)$, $S_2(t_\infty, t_\infty, t^*)$, $S_3(t_\infty, t^*)$ and $S_5(t_\infty, t_\infty, t^*)$, respectively.

**D.2 Partition functions of the $\beta = 1$ and $\beta = 4$ circular ensembles as tau functions**

As it well known [16] the integral over random orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) $N$ by $N$ matrices may be reduced to the $N$-ply integral over real lines
\[Z_N^\beta := \int_{\mathbb{R}^1} \cdots \int_{\mathbb{R}^1} \prod_{n<m \leq N} |x_n - x_m|^\beta \prod_{i=1}^{N} e^{x_i y_i} \text{d}u(x_i)\]
where $\text{d}u$ is an integration measure which yields the probability weight, while the exponential factor in front of it is a deformation of this measure with a given set of deformation parameters $t = (t_1, t_2, \ldots \cdots)$.

The wonderful result of [51–6] that $Z_N(t)$ is a tau function of a certain hierarchy of integrable equations which were introduced and studied in a series of papers and named as Pfaff lattice and which were found in [9] to be a tricky constraint of 2D Toda lattice hierarchy. Then it was proven by J. van de Leur in [1] that $Z_N^\beta$, $\beta = 1, 4$ is a tau function of the charged BKP hierarchy introduced in [3] which may
be viewed as a modification of the DKP hierarchy described in [4]. Here we remind and modify some of the results of [4].

We generalize [4] as follows. First we replace the integrals along the real axis in [4] by the integrals along any contour $\gamma$ on $C^1$. Second we add an additional set of deformation parameters: a set $t = (t_1, t_2, \ldots)$ and an integer $l$.

Thus we study

$$Z_N^\gamma \leq \int_\gamma \cdots \int_\gamma (\Upsilon_N(x))^\beta \prod_{i=1}^N d\mu(x_i; t, \tilde{t}, l)$$

(320)

Here

$$\Upsilon_N(x) := \prod_{n<m\leq N} (x_n - x_m)\text{sgn}(\varsigma(x_n) - \varsigma(x_m))$$

(321)

where we introduce the parameter $\varsigma$ along the curve, and where the deformed measure $d\mu(x; t, \tilde{t}, l)$ is

$$d\mu(x; t, \tilde{t}, l) = e^{\sum_{n=1}^\infty (t_n x^n - \tilde{t}_n x^{-n})} d\mu(x)$$

(322)

where $d\mu$ is any measure along $\gamma$ and $l, t, \tilde{t}$ are deformation parameters.

Our main example will be circular beta-ensembles ($\beta = 1, 4$)

$$Z_N^{C, \beta}(l, t, \tilde{t}) := \int_0^{\infty} \cdots \int_0^{\infty} \prod_{n<m\leq N} |x_n - x_m|^\beta \prod_{i=1}^N x_i^{l+i(N-1)} e^{\sum_{n=1}^\infty t_n x^n + \tilde{t}_n x^{-n}} d\mu(x_i)$$

(323)

which were not considered in [4]. We shall consider these ensembles in a way similar to [4]. Because of notational reasons we prefer to consider DKP hierarchy introduced in [9] rather than the charged BKP one introduced in [3] in spite of the fact that BKP hierarchy seems to be more natural for such problems.

Proposition 17. Let $d\mu(x)$ be any measure on the circle.

$$\tau_N(l, t, \tilde{t}) := \langle N + l \rangle \Gamma(t) e^{\frac{x}{2} \int_0^1 \psi(x)\psi(y)\text{sgn}(\arg(x) - \arg(y))d\mu(x)d\mu(y)} g^{-\frac{1}{2}} \Gamma(\tilde{t}) |l|$$

(324)

$$\tau_N(l, t, \tilde{t}) := \langle N + l \rangle \Gamma(t) e^{\frac{x}{2} \int_0^1 \psi(x)\psi(y)\text{sgn}(\arg(x) - \arg(y))d\mu(x)d\mu(y)} g^{-\frac{1}{2}} \Gamma(\tilde{t}) |l|$$

(325)

where $c = \exp\sum_{m=1}^\infty mt_m \tilde{t}_m$ and

$$g^{-\frac{1}{2}} := e^{\frac{1}{2} \int_0^1 \psi(x)d\mu(x)} e^{\frac{1}{2} \int_0^1 \psi(x)d\mu(x)}$$

(326)

Here $\tau_N(l, t, \tilde{t})$ is a tau function of the 2-BKP hierarchy. In particular it means that if we fix the variables $t = (t_1, t_2, \ldots)$ then $\tau_N(l, t, \tilde{t})$ is a 2-BKP tau function with respect to the variables $t = (t_1, t_2, \ldots)$. The complete set of Hirota equations for the 2-BKP is written down in the Appendix.

The proof basically repeats the proof of [3] of the similar statement for ensembles [5] which we write down in a little bit more unsophisticated way as follows.

Lemma 6. Let $\gamma$ be a contour on a complex plane whose points $x \in \gamma$ are parameterized by a parameter $\varsigma$: $x = x(\varsigma)$. Let $d\mu(x,y) = d\mu(y,x)$ be a symmetric bi-measure on a $\gamma \times \gamma$. Then, for any antisymmetric function $A(x,y)$ we have

$$e^{\frac{x}{2} \int_0^1 \int_0^1 \psi(x)\psi(y)A(x,y)d\mu(x)d\mu(y)} = \sum_{N=0}^\infty z^N I_N$$

(327)

where $I_N$ is the following 2N-ary integer

$$I_N = \int_\gamma \cdots \int_\gamma \psi(x_1) \cdots \psi(x_N) \text{Pf}[A(x)]$$

(328)

where $\varsigma_i := \varsigma(x_i)$ and $\varsigma_1 > \cdots > \varsigma_N$ and where

$$A_{nm}(x) = A(x_n, x_m)d\mu(x_n, x_m), \quad n, m = 1, \ldots, N$$

(329)
The proof is straightforward. We have

\[ \frac{1}{2} \int \int_{\gamma} \psi(x)\psi(y)A(x, y)d\mu(x, y) = \int \int_{\varsigma(x) > \varsigma(y)} \psi(x)\psi(y)A(x, y)d\mu(x, y) \]

Now

\[ I_N = \frac{1}{N!} \left( \int \int_{\varsigma(x) > \varsigma(y)} \psi(x)\psi(y)A(x, y)d\mu(x, y) \right)^N = \int \cdots \int \psi(x_1)\psi(x_2) \cdots \psi(x_N) \sum_{\sigma \in S^N} \text{sgn} \sigma A(x_{\sigma(1)}, x_{\sigma(2)})A(x_{\sigma(3)}, x_{\sigma(4)}) \cdots A(x_{\sigma(N-1)}, x_{\sigma(N)}) \]

where the integration domain is restricted by the cone \( \varsigma(x_1) > \varsigma(x_2) > \cdots > \varsigma(x_N) \) and where the sum runs over all possible permutations conditioned by \( \sigma(2i-1) > \sigma(2i) \), \( i = 1, \ldots, \frac{1}{2}N \) and by \( \sigma(1) > \sigma(3) > \cdots > \sigma(N-1) \) (the last restrictions gives rise to the additional factor \( N! \)).

Now we see that the following statement is valid

**Lemma 7.**

\[ \langle N + l \rangle \mathbf{F}_{\gamma, \varsigma} \psi(x(y)A(x, y)d\mu(x, y) = \frac{(N - 1)!!}{N!} \int \int \cdots \int \Delta_N(x, x_2)A(x_3, x_4) \cdots A(x_{N-1}, x_N) \]

Now we apply this Lemma. For circular ensembles we take \( \arg x \) as the parameter \( \varsigma \) used in the Lemma.

To get standard circular ensembles we imply that

\[ d\mu(x, y) = d\mu(x, \bar{t})d\mu(y, t, \bar{t}) \]

and

\[ A(x, y) = \text{sgn}(\arg x - \arg y) \]

\( N \) even. It is enough to consider the case \( t = \bar{t} = 0 \). Writing the exponential in \( (337) \) as Taylor series and keeping the \( N \)-th term we obtain the following 2\( N \)-ply integral

\[ \frac{1}{2^N} \int \cdots \int \langle N + l \rangle \psi(x_1) \cdots \psi(x_{2N}) \rangle \text{Pf} [\text{sgn}(\arg(x_i) - \arg(x_j))] d\mu(x_i, x_j) |_{i,j=1,\ldots,N} \]

\[ = \frac{1}{2^N} \int \cdots \int \prod_{i<j} (x_i - x_j) \prod_{i=1}^N x_i^j d\mu(x_i, \bar{t}) \]

(334)

where the integration domain is restricted by \( \arg x_1 > \arg x_2 > \cdots > \arg x_N \). Here we took into account that

\[ \langle N + l \rangle \psi(x_1) \cdots \psi(x_{2N}) \rangle = \prod_{i=1}^N x_i^j \prod_{i<j} (x_i - x_j) \]

(335)

and the last Lemma. Now using

\[ \prod_{n<m \leq N} \frac{|x_n - x_m|}{x_n - x_m} = \prod_{i=1}^N (-x_i)^{\frac{1-N}{2}} \prod_{n<m \leq N} \text{sgn} \left( \frac{\sin \arg x_n - \arg x_m}{2} \right) \]

we can get rid of the restriction \( \arg x_1 > \arg x_2 > \cdots > \arg x_N \) getting the factor \( (N!)^{-1} \). Thus we re-write \( I_N \) as

\[ Z_N^{C_{1\beta}}(t, \bar{t}) = (-1)^{N-\frac{1}{2}} \frac{1}{N!} \int \cdots \int \prod_{n<m \leq N} |x_n - x_m| \prod_{i=1}^N \left( x_i^{\frac{1}{2}} (N-1) \right) d\mu(x_i, t, \bar{t}) \]

in accordance with \( (338) \). In the similar way we prove \( (??) \).
\( \beta = 4 \) circular ensemble. Somehow circular ensembles were not considered in the paper [3]. The circular \( \beta = 4 \) ensemble is also related to a DKP tau function which is rather similar to the the charged BKP (cBKP) tau function (found in [3]) which gives rise to the integral over symplectic matrices:

**Proposition 18.** Let \( d\mu(x) \) be any measure on the circle. We have

\[
\tau_{2N+l,l}(t, \bar{t}) := (2N + l) \Gamma(t) e^{\int F(x,t) d\mu(x) \Gamma(\bar{t})} l \tag{337}
\]

\[
= c \int \cdots \int \prod_{n<m\leq N} |x_n - x_m|^4 \prod_{i=1}^{N} x_i^{l+2(N-1)} e^{2 \sum_{n=1}^{\infty} t_n x_i^n + \bar{t}_n x_i^{-n}} d\mu(x_i) \tag{338}
\]

where \( c = c(N) \exp \sum_{m=1}^{\infty} m t_m \bar{t}_m \).

For the proof we use (335) where we consider the limit \( x_{2l-1} \rightarrow x_l \).

Expectation values similar to (147) and (150) yields sums of integrals similar to (148) and (151) where the interaction \( \frac{\Delta_N(z) \Delta_N(z')}{\prod_{k=1}^{N} (z_k - z_k')} \) is replaced by \( \frac{\Delta_N(z) \Delta_N(z')^4}{\prod_{k=1}^{N} (z_k - z_k')^4} \). Since I do not yet know links of such series to anything else I shall not write down them explicitly.

### D.3 Perturbation series in coupling constants for \( \beta = 1, 4 \) ensembles and discrete matrix models

The fermionic language provides the simplest method to convert integrals into sums based on the equalities

\[
\int \int A(x, y) \psi(x) \psi(y) = \sum_{n>m} A_{nm} \psi_n \psi_m,
\]

\[
\int \int A(x, y) \phi(x) \phi(y) = \sum_{n>m} A_{nm} \phi_n \phi_m,
\]

\[
\int a(x) \psi(x) \phi_0 = \sum_n a_n \psi_n \phi_0
\]

where

\[
A_{nm} = \int \int A(x, y) x^n y^m dx dy, \quad a_n = \int \int a(x) x^n dx
\]

In this way each multiple integral considered in Section D.2 may be converted into a multiple sum considered either in Section ?? or in Section ??.

In this subsection we shall consider perturbation series for \( \beta = 1, 4 \) ensembles and in particular for \( \beta = 1, 4 \) circular ensembles. Perturbation series for each ensemble (323, 319, ...) may be written as

\[
Z_{N}^{\beta=1}(t, \bar{t}) = c \sum_{A(\lambda) \in \mathcal{N}} A_{\{h\}}(0, \bar{t}) s_{\lambda}(t) \tag{340}
\]

where the sum ranges over all partition whose length (that is the number of non-vanishing parts \( \lambda_i \)) does not exceed \( N \) and where \( A(\lambda) \) are defined via a matrix \( A \) and numbers \( \{a_n\} \) as follows

Let \( N = 2K \). Given ordered set \( h := (h_1, \ldots, h_N) \) we define \( A(\lambda) \) as the Pfaffian of 2K \( \times 2K \) matrix \( a = a(h) \)

\[
A(\lambda) = \text{Pf}[a], \quad a_{nm}(h) := (A)_{h_n h_m} |n, m = 1, \ldots, 2K
\]

For \( N = 2K + 1 \) we define \( A(\lambda) \) as the Pfaffian of the \((2K + 2) \times (2K + 2)\) matrix \( \tilde{a} = \tilde{a}(h) \)

\[
A(\lambda) = \text{Pf}[\tilde{a}], \quad \tilde{a}_{nm}(h) := (A)_{h_n h_m} |n, m = 1, \ldots, 2K + 1
\]

\[
\tilde{a}_{n,2K+2} = -\tilde{a}_{2K+2,n} = \int x^n d\mu(x, t, \bar{t}), \quad n = 1, \ldots, 2K + 2 \tag{343}
\]

The matrix \( \tilde{A} \) is defined in term of matrices of moments of each of ensembles separately.
For $\beta = 1$ ensembles From
\[
e^{\frac{1}{2} \int f_n f_n^* \psi(x)\psi(y) \text{sgn}(c(x) - c(y)) d\mu(x) d\mu(y)} = g^{-\gamma}(A)
\]
we obtain
\[
A_{nm} = A_{nm}(t, \bar{t}) = \frac{1}{2} \int_\gamma \int_\gamma x^n y^m \text{sgn} \left( \arg(x) - \arg(y) \right) d\mu(x, t, \bar{t}) d\mu(y, \bar{t}, t)
\]
(344)
Also
\[
a_n(t, \bar{t}) = \int_\gamma x^n d\mu(x, t, \bar{t})
\]
(345)
In particular for circular ensemble we have
\[
A_{nm} = A_{nm}(t, \bar{t}) = \frac{1}{2} \int_\gamma \int_\gamma x^n y^m \text{sgn} \left( \arg(x) - \arg(y) \right) e^{\sum_{i=1}^{\infty} (x^i + y^i) \ell_i - (x^{-i} + y^{-i}) \bar{\ell}_i} d\mu(x) d\mu(y)
\]
(346)
Also
\[
a_n(t, \bar{t}) = \int_\gamma x^n d\mu(x, t, \bar{t})
\]
(347)
There are two ways to get such series.

(I) Given fermionic representation \[337\] we have the following straightforward way. Here we put $l = 0$ for the sake of simplicity. Because
\[
e^{\frac{1}{2} \int f \psi(x)\psi(y) \text{sgn}(\arg(x) - \arg(y)) d\mu(x) d\mu(y)} = g^{-\gamma}(A)
\]
where $A$ is the matrix of moments
\[
A_{nm} = A_{nm}(t, \bar{t}) = \frac{1}{2} \int_\gamma \int_\gamma x^n y^m \text{sgn} \left( \arg(x) - \arg(y) \right) e^{\sum_{i=1}^{\infty} (x^i + y^i) \ell_i - (x^{-i} + y^{-i}) \bar{\ell}_i} d\mu(x) d\mu(y)
\]
(348)
we obtain representation (??)
\[
Z_N^{C\beta=1}(t, \bar{t}) = \sum_{\ell(\lambda) \leq N} A_{\{\bar{h}_i\}}(0, t) s_{\lambda}(t) = \sum_{\ell(\lambda) \leq N} A_{\{\bar{h}_i\}}(t, 0) s_{\lambda}(\bar{t})
\]
(349)
where the sum ranges over all partition whose length (that is the number of non-vanishing parts $\lambda_i$) does not exceed $N$ and where $A_{\{\bar{h}_i\}}$ is defined as follows as in (??):

(II) Formula \[334\] may be also obtained in a different way: via development of $Z_N$ in Taylor series in deformation parameters which is written as series in the Schur functions, usage of the Cauchy-Littlewood relation \[17\] in form
\[
e^{\sum_{n=1}^{\infty} \sum_{i=1}^{N} x^n_i} = \sum_{\lambda \in \mathcal{P}} s_{\lambda}(t) \prod_{n \leq \lambda \leq \lambda_N} (x_n - x_m^{k=1,\ldots,N}) \quad h_i = \lambda_i - i + N
\]
(350)
and the following known (see \[16\], section 14.3)

Lemma 8.
\[
\int \prod_{i=1}^{2m} \int x_i \text{sgn}(x) \Delta(x) = N! \text{Pf} \left[ a_{ij} \right]_{i, j = 1, \ldots, 2m}
\]
(351)
where $2m = N$ if $N$ is even and $2m = N + 1$ if $N$ is odd, and
\[
a_{ij} = \int x_i \text{sgn}(x) \text{sgn}(x) \left[ \theta_i(x) \theta_j(y) - \theta_j(x) \theta_i(y) \right], \quad i, j = 1, \ldots, N
\]
(352)
When $N$ is odd we have in addition $a_{N+1, N+1} = 0$ and
\[
a_{i, N+1} = -a_{N+1, i} = \int \theta_i(x) \text{sgn}(x), \quad i = 1, \ldots, N
\]
(353)
The perturbation series for the orthogonal ensemble \(A_{\beta}^{(h)}\) is the same series \((319)\) where, now, the moment matrix is

\[
A_{nm}(t, \tilde{t}) = \int_R \int_R x^n y^m \sgn(x - y) d\mu(x, t, \tilde{t}) d\mu(y, t, \tilde{t})
\]  

(354)

and

\[
a_n(t, \tilde{t}) = \int_R x^n d\mu(x, t, \tilde{t})
\]

(355)

where

\[
d\mu(x, t, \tilde{t}) := e^{\sum_{m=1}^{\infty} x^m t_m} e^{-\sum_{m=1}^{\infty} z^{-m} t_m} d\mu(x)
\]

(356)

Let \(N = 2K\). Given ordered set \(h := (h_1, \ldots, h_N)\) we define \(A_{\{h\}}\) as the Pfaffian of \(2K \times 2K\) matrix \(a = a(h)\)

\[
A_{\{h\}} = \text{Pf}[a], \quad a_{nm}(h) := (A)_{h_a h_m | n, m = 1, \ldots, 2K}
\]

(357)

For \(N = 2K + 1\) we define \(A_{\{h\}}\) as the Pfaffian of the \((2K + 2) \times (2K + 2)\) matrix \(\tilde{a} = \tilde{a}(h)\)

\[
A_{\{h\}} = \text{Pf}[\tilde{a}], \quad \tilde{a}_{nm}(h) := (A)_{h_a h_m | n, m = 1, \ldots, 2K + 1},
\]

(358)

\[
\tilde{a}_{n, 2K + 2} = -\tilde{a}_{2K + 2, n} = \int x^n d\mu(x), \quad n = 1, \ldots, 2K + 2
\]

(359)

**Perturbation series for \(\beta = 4\) ensemble.**

First, we write

\[
\int_{\gamma} \frac{\psi(x)}{dx} \psi(x) d\mu(t, \tilde{t}, x) = \sum_{n, m} A_{nm}(t, \tilde{t}) \psi_n \psi_m
\]

(360)

where

\[
A_{nm}(t, \tilde{t}) = \frac{n - m}{2} \int_{\gamma} x^{n + m - 1} d\mu(t, \tilde{t}, x)
\]

(361)

and

\[
a_n(t, \tilde{t}) = \int_R x^n d\mu(x, t, \tilde{t})
\]

(362)

where for the symplectic ensemble we take \(\gamma = R\) while for the circular ensemble \(\gamma = S^1\).

\[
d\mu(t, \tilde{t}, x) := e^{\sum_{m=1}^{\infty} x^m t_m} e^{-\sum_{m=1}^{\infty} z^{-m} t_m} d\mu(x)
\]

(363)

Then we obtain

\[
\langle N + l \vert \Gamma(t) \rangle e^{\sum_{n, m} A_{nm} \psi_n \psi_m} e^{\sum_{n=0}^{\infty} a_n \psi_n \phi_0} \bar{\Gamma}(\bar{t}) \vert l \rangle = \sum_{\lambda \in \mathcal{P}} A_{\{h\}}(0, \bar{t}) s_{\lambda}(t)
\]

**D.4 On character formulae**

**Remark** 13. There is a known relation (see [II]) between the Schur functions and the odd orthogonal character \(s_{\lambda}\) of rectangular shape as follows

\[
\sum_{\lambda \in \mathcal{P}} s_{\lambda}(x_1, \ldots, x_m) = (x_1 \ldots x_m)^{\frac{1}{2}} s_{\lambda}(y^m)(x_1^{\perp 1}, \ldots, x_m^{\perp 1}, 1)
\]

(364)

The odd orthogonal characters \(s_{\lambda}(x_1^{\perp 1}, x_2^{\perp 1}, \ldots, x_m^{\perp 1}, 1)\), where \(x_1^{\perp 1}\) is a shorthand notation for \(x_1, x_1^{-1}, \ldots, \) and where \(\lambda\) is an \(m\)-tuple \((\lambda_1, \lambda_2, \ldots, \lambda_m)\) of integers, or of half-integers, is defined by

\[
s_{\lambda}(x_1^{\perp 1}, \ldots, x_m^{\perp 1}, 1) := \frac{\det \left( x_j^{\lambda_i - i + m + \frac{1}{2}} - x_j^{-(\lambda_i - i + m + \frac{1}{2})} \right)}{\det \left( x_j^{i - m + \frac{1}{2}} - x_j^{-(i - m + \frac{1}{2})} \right)}
\]

(365)

(see, say, (3.3) in [II]).
The even orthogonal character may defined as
\[
so_\lambda(x_1^{\pm 1},\ldots,x_m^{\pm 1}) := \frac{\det (x_j^{\lambda_{-i+m}} + x_j^{-(\lambda_{-i+m}}}) + \det (x_j^{\lambda_{-i+m}} - x_j^{-(\lambda_{-i+m}}})}{\det (x_j^{-(i+m)} + x_j^{-(i+m))})}
\] (366)

(see, say, (2.12) in [12])

The even symplectic character is defined as
\[
sp_\lambda(x_1^{\pm 1},\ldots,x_m^{\pm 1}) := \frac{\det (x_j^{\lambda_{-i+m+1}} - x_j^{-(\lambda_{-i+m+1}}})}{\det (x_j^{-(i+m+1)} - x_j^{-(i+m+1))})}
\] (367)

(see, say, (4.5) in [11])

In [15] there was also defined odd symplectic characters \(sp_\lambda(x_1^{\pm 1},x_2^{\pm 1},\ldots,x_n^{\pm 1},1)\), which are for example defined by
\[
sp_\lambda(x_1^{\pm 1},x_2^{\pm 1},\ldots,x_n^{\pm 1},1) = \frac{1}{2} \det_{1\leq i,j\leq n} (h_{\lambda_{-i+j}}(x_1^{\pm 1},x_2^{\pm 1},\ldots,x_n^{\pm 1},1) + h_{\lambda_{-i+j+2}}(x_1^{\pm 1},x_2^{\pm 1},\ldots,x_n^{\pm 1},1)),
\] (368)

(see, say, (4.6) in [11]) where \(h_k(z_1,z_2,\ldots,z_r)\) denotes the \(k\)-th complete homogeneous symmetric function.

**The fermionic approach. Odd orthogonal character.** Formula (420) may be written in form
\[
so_\lambda(x_1^{\pm 1},\ldots,x_m^{\pm 1},1) := \{0 \mid \Psi(x_1) \ldots \Psi(x_m) \mid \{\lambda\}\}
\] (369)
\[
= (-\sqrt{2}x_1) \ldots (-\sqrt{2}x_m) \{0 \mid (g^{--})^{-1} \psi^\dagger(x_1) \ldots \psi^\dagger(x_m) g^{--} \mid \{\lambda,m\}\}
\] (370)

where
\[
g^{--} := e^{\frac{i}{2} \int \psi(x) \psi(-x^{-1}\d x) \equiv e^{\frac{i}{2} \sum_{n \in \mathbb{Z}} (-)^n \psi_n \psi_{n+1}}
\] (371)
\[
\Psi(x) := \sqrt{2}x \psi(-x^{-1}) - \sqrt{2}x \psi^\dagger(x)
\] (372)

and where for \(\lambda = (\lambda_1,\ldots,\lambda_r)\)
\[
\{\lambda,m\} := \phi_{\lambda_1-1+m} \ldots \phi_{\lambda_r-r+m} \{0\}
\] (373)
\[
\phi_n := \frac{1}{\sqrt{2}} \left( \psi_n + (-)^n \psi^\dagger_{-n} \right)
\] (374)

**The fermionic approach. Even symplectic case.**
\[
sp_\lambda(x_1^{\pm 1},\ldots,x_m^{\pm 1}) := (-1)^m \{0 \mid (\psi^\dagger(x_1) + \psi(-x_1^{-1})) \ldots (\psi^\dagger(x_m) + \psi(-x_m^{-1})) \mid \{\lambda,m\}\}
\] (375)

Now take
\[
\{\lambda,m\} := \tilde{\psi}_{\lambda_1-1+m} \ldots \tilde{\psi}_{\lambda_r-r+m} \{0\}
\] (376)
\[
\tilde{\psi}_n := \frac{1}{\sqrt{2}} \left( \psi_n - (-)^n \psi^\dagger_{-1-n} \right), \quad \tilde{\psi}_{n}^\dagger := \frac{1}{\sqrt{2}} \left( \psi_n^\dagger - (-)^n \psi_{-n-1} \right)
\] (377)

We have
\[
[\tilde{\psi}_{n}^\dagger,\tilde{\psi}_{k}^\dagger]_+ = 0 \quad [\tilde{\psi}_{n},\tilde{\psi}_{k}]_+ = 0 \quad [\tilde{\psi}_{n}^\dagger,\tilde{\psi}_{k}]_+ = \delta_{nk}
\] (378)
\[
\psi_n := \frac{1}{\sqrt{2}} \left( \psi_n + (-)^n \psi^\dagger_{-1-n} \right), \quad \psi_{n}^\dagger := \frac{1}{\sqrt{2}} \left( \psi_{n}^\dagger + (-)^n \psi_{-n-1} \right)
\] (379)
Bogolyubov transform. Introduce fermionic operators \( \tilde{\psi}_n \) and \( \tilde{\psi}_n^\dagger \) as the following Bogolyubov transform of the fermionic operators \( \psi_n \) and \( \psi_n^\dagger \)
\[
\tilde{\psi}_n := e^{-\frac{1}{2} J_0^-} \psi_n e^{\frac{1}{2} J_0^-} = \frac{1}{\sqrt{2}} (\psi_n - (-)^n \psi_{-1-n}^\dagger) 
\]
\[
\tilde{\psi}_n^\dagger := e^{-\frac{1}{2} J_0^-} \psi_n^\dagger e^{\frac{1}{2} J_0^-} = \frac{1}{\sqrt{2}} (\psi_n^\dagger - (-)^n \psi_{-n-1}) 
\]
where
\[
J_0^- := \sum_{n \in \mathbb{Z}} (-1)^n (\psi_{n-1} \psi_n + \psi_{n-1}^\dagger \psi_n^\dagger) 
\]
The inverse transformation is
\[
\psi_n := \frac{1}{\sqrt{2}} (\tilde{\psi}_n + (-)^n \tilde{\psi}_{-1-n}^\dagger), \quad \psi_n^\dagger := \frac{1}{\sqrt{2}} (\tilde{\psi}_n^\dagger + (-)^n \tilde{\psi}_{-n-1}) 
\]
Introduce fermionic fields
\[
\tilde{\psi}(x) := \sum \tilde{\psi}_n x^n = \frac{1}{\sqrt{2}} (\psi(x) - \psi^\dagger(-x)), \quad \tilde{\psi}^\dagger(x) := \sum \tilde{\psi}_n^\dagger x^{-n-1} = \frac{1}{\sqrt{2}} (\psi^\dagger(x) + \psi(-x)) 
\]
then
\[
\psi(x) := \frac{1}{\sqrt{2}} (\tilde{\psi}(x) + \tilde{\psi}^\dagger(-x)), \quad \psi^\dagger(x) := \frac{1}{\sqrt{2}} (\tilde{\psi}^\dagger(x) - \tilde{\psi}(-x)) 
\]
By (380), (381) we obtain
\[
\tilde{\psi}_n|m\rangle = \tilde{\psi}_{-n-1}^\dagger|m\rangle = \psi_n|m\rangle = \psi_{-n-1}^\dagger|m\rangle = 0, \quad n < m 
\]
We obviously have
\[
[\tilde{\psi}_n^\dagger, \tilde{\psi}_k]^+ = 0, \quad [\tilde{\psi}_n, \tilde{\psi}_k]^+ = 0, \quad [\tilde{\psi}_n^\dagger, \tilde{\psi}_k]^+ = \delta_{nk} 
\]
and
\[
[\tilde{\psi}^\dagger(x), \tilde{\psi}(y)]^+ = 0, \quad [\tilde{\psi}(x), \tilde{\psi}(y)]^+ = 0, \quad [\tilde{\psi}^\dagger(x), \tilde{\psi}(y)]^+ = [\tilde{\psi}^\dagger(x), \psi(y)]^+ = \delta(x/y) 
\]
As one can verify, for any \( N, n, m \)
\[
\langle N| \tilde{\psi}_n \tilde{\psi}_m^\dagger |N\rangle = \langle N| \psi_n \psi_m^\dagger |N\rangle, \quad \langle N| \tilde{\psi}_n \tilde{\psi}_m |N\rangle = \langle N| \psi_n^\dagger \psi_m^\dagger |N\rangle = 0 
\]
As one can see the normal ordering given by
\[
: \psi_n \psi_m^\dagger := \psi_n \psi_m^\dagger - (0|\psi_n \psi_m^\dagger |0\rangle, \quad : \psi_n \psi_m := \psi_n \psi_m, \quad : \psi_n^\dagger \psi_m^\dagger := \psi_n^\dagger \psi_m^\dagger 
\]
yields
\[
: \tilde{\psi}_n \tilde{\psi}_m^\dagger := \tilde{\psi}_n \tilde{\psi}_m^\dagger - (0|\tilde{\psi}_n \tilde{\psi}_m^\dagger |0\rangle, \quad : \tilde{\psi}_n \tilde{\psi}_m := \tilde{\psi}_n \tilde{\psi}_m, \quad : \tilde{\psi}_n^\dagger \tilde{\psi}_m^\dagger := \tilde{\psi}_n^\dagger \tilde{\psi}_m^\dagger 
\]
Direct computation with the use of (383), (380) yields
\[
\langle N| \tilde{\psi}(x) \tilde{\psi}^\dagger(y) |N\rangle = \sum_{n=0}^{\infty} x^{-n-N-1} y^{n+N} = \langle N| \psi(x) \psi^\dagger(y) |N\rangle 
\]
\textbf{Currents and flows.}
\[
: \psi(x) \psi^\dagger(x) := \sum_{n=-\infty}^{+\infty} J_n x^{n-1}, \quad : \tilde{\psi}(x) \tilde{\psi}^\dagger(x) := \sum_{n=-\infty}^{+\infty} \tilde{J}_n x^{n-1} 
\]
By (391) and (390) both sets \{J_n\} and \{\tilde{J}_n\} are the subjects of two Heisenberg algebras
\[
[J_n, J_m] = [\tilde{J}_n, \tilde{J}_m] = n \delta_{n+m,0} 
\]
By (380), (381) and taking into account that \( \sum (-1)^i \psi_n \psi_{n-i} = \sum (-1)^i \psi^\dagger_{n-i} \psi^\dagger_{n-i} = 0 \) for even \( n \) we obtain

\[
J_{2n-1} = \tilde{J}_{2n-1}
\]

(395)

\[
J_{2n} = \frac{1}{2} \sum_{i=-\infty}^{+\infty} (-1)^i \left( \tilde{\psi}_i \tilde{\psi}_{i-1-2n} + \tilde{\psi}_{i-1} \tilde{\psi}_{i+2n} \right)
\]

(396)

One can compare the last expression with formulae (156).

Let us introduce

\[
J_{2n}^- = \frac{1}{2} \sum_{i=-\infty}^{+\infty} (-1)^i \left( \tilde{\psi}_i \tilde{\psi}_{i-1-2n} - \tilde{\psi}_{i-1} \tilde{\psi}_{i+2n} \right)
\]

(397)

and use the notation \( J_{2n}^- := J_{2n} \). Then

\[
[J_{2n+1}, J_{2m-1}] = (2n+1) \delta_{n+m,0}, \quad [J_{2n+1}, J_{2m}] = 0
\]

(398)

\[
[J_{2n+2n}, J_{2n+2n}] = J_{2n+2n}^\dagger
\]

(399)

Similarly

\[
\tilde{J}_{2n}^- := \tilde{J}_{2n} = \frac{1}{2} \sum_{i=-\infty}^{+\infty} (-1)^i \left( \psi^\dagger_i \psi_{i-1-2n} + \psi^\dagger_{i-1} \psi^\dagger_{i+2n} \right)
\]

(400)

\[
\tilde{J}_{2n}^- := \frac{1}{2} \sum_{i=-\infty}^{+\infty} (-1)^i \left( \psi^\dagger_i \psi_{i-1-2n} - \psi^\dagger_{i-1} \psi^\dagger_{i+2n} \right)
\]

(401)

And we get

\[
[J_{2n}^\dagger, J_{2n}] = J_{2n}^\dagger
\]

(402)

\textbf{Remark 14.}

\textbf{Lemma 9.}

One can consider the transformation

\[
\tilde{\psi}_n := c \left( \psi_n + a(-1)^n \psi^\dagger_{1-n} \right), \quad \tilde{\psi}_n^\dagger := c \left( \psi^\dagger_n + a(-1)^n \psi_{-n-1} \right)
\]

(403)

\[
\psi_n := c \left( \tilde{\psi}_n - a(-1)^n \tilde{\psi}^\dagger_{1-n} \right), \quad \psi_n^\dagger := c \left( \tilde{\psi}^\dagger_n - a(-1)^n \tilde{\psi}_{-n-1} \right)
\]

(404)

where \( c = \frac{1}{\sqrt{1+a^2}} \). Instead of (383) we have

\[
\tilde{\psi}(x) := c \left( \psi(x) + a \psi^\dagger(-x) \right), \quad \tilde{\psi}^\dagger(x) := c \left( \psi^\dagger(x) - a \psi(-x) \right)
\]

(405)

\[
\psi(x) := c \left( \tilde{\psi}(x) - a \tilde{\psi}^\dagger(-x) \right), \quad \psi^\dagger(x) := c \left( \tilde{\psi}^\dagger(x) + a \tilde{\psi}(-x) \right)
\]

(406)

The previous case was related to the choice \( a = -1 \).

Then all properties (380)-(382) are still true. Then again the currents (393) satisfy the Heisenberg algebra relations (394). However, the explicit relations for currents are as follows. The relation for odd components of the currents (395) are still correct (therefore relations (398) are also correct), while for the even components we obtain

\[
J_{2n}^\dagger = \frac{1-a^2}{1+a^2} J_{2n} + \frac{a}{1+a} \sum_{i=-\infty}^{+\infty} (-1)^i \left( \tilde{\psi}_{i-1} \tilde{\psi}_{i+2n} + \tilde{\psi}_{i-1} \tilde{\psi}_{i-1-2n} \right), \quad n \in \mathbb{Z}
\]

(405)

\[
J_{2n}^\dagger(a) = \frac{1-a^2}{1+a^2} J_{2n} + \frac{a}{1+a} \sum_{i=-\infty}^{+\infty} (-1)^i \left( \psi^\dagger_{i-1} \psi^\dagger_{i+2n} + \psi^\dagger_{i-1} \psi_{i-1-2n} \right), \quad n \in \mathbb{Z}
\]

(406)

\[
J_{2n} = \frac{1}{2} [J_{2n}, J_{2n}^\dagger] = -\frac{a}{1+a^2} \sum_{i=-\infty}^{+\infty} (-1)^i \left( \tilde{\psi}_{i-1} \tilde{\psi}_i^\dagger + \tilde{\psi}_{i-1} \tilde{\psi}_{i-1} \right), \quad n \in \mathbb{Z}
\]

(407)

\[
J_{2n}^- = \frac{1}{2} [J_{2n}, J_{2n}^\dagger] = \frac{a}{1+a^2} \sum_{i=-\infty}^{+\infty} (-1)^i \left( \psi^\dagger_{i-1} \psi^\dagger_{i+2n} - \psi^\dagger_{i-1} \psi^\dagger_{i-1-2n} \right), \quad n \in \mathbb{Z}
\]

(408)

We notice that

\[
\tilde{\psi}_n = e^{a \tilde{J}_0} \psi_n e^{-a \tilde{J}_0}, \quad \tilde{\psi}^\dagger_n = e^{a \tilde{J}^\dagger_0} \psi^\dagger_n e^{-a \tilde{J}^\dagger_0}, \quad \tan \alpha = a
\]

(409)

and therefore for \( n \neq 0 \)

\[
\tilde{J}_n = e^{a \tilde{J}_0} J_n e^{-a \tilde{J}_0}
\]

(410)

It is not true for \( n = 0 \) thanks to the fact that the definition of \( J_0 \) includes the normal ordering.
**Symmetric polynomials.** Let us notice that

\[
\langle N_1| \Gamma(t) g | N_2 \rangle = \langle N_1| \tilde{\Gamma}(t) \tilde{g} | N_2 \rangle
\]

where

\[
\tilde{\Gamma}(t) := e^{\sum_{n=1}^{\infty} J_n t_n}
\]

and \(\tilde{g}\) is of form (??) where fermions \(\psi\) and \(\psi^\dagger\) are replaced by the fermions \(\tilde{\psi}\) and \(\tilde{\psi}^\dagger\). As we see

\[
s_{\lambda}(\tilde{\psi}) := \langle 0| e^{\sum_{n=1}^{\infty} J_n(\alpha) t_n} | \{\lambda, a\} \rangle
\]

Consider

\[
\tilde{s}_{\lambda}(t, a) := \langle 0| e^{\sum_{n=1}^{\infty} J_n(-a) t_n} | \lambda \rangle = \langle 0| e^{\sum_{n=1}^{\infty} J_n a t_n} | \lambda \rangle = \langle 0| e^{\sum_{n=1}^{\infty} J_n t_n} | \lambda, a \rangle
\]

This polynomial is an example of BKP tau function. \(\tilde{s}_{\lambda}(t, a)\) is a homogeneous polynomial of the same weight as \(s_{\lambda}(t)\). We have \(\tilde{s}_{\lambda}(t, 0) = s_{\lambda}(t)\).

The polynomial \(\tilde{s}_{\lambda}(t)\) may be expressed as a linear combination of the Schur functions as follows. First, we need the Frobenius notation for \(\lambda\). Let \(\lambda = (\alpha/\beta)\). Given \(\lambda\) one can construct another partition by a permutation of a certain number, say, \(n\) of variables \(\alpha_{n_i}, i = 1, \ldots, k\) and \(\beta_{m_i}, i = 1, \ldots, k\). The set of all partitions obtained in this way will be denoted by \(P_k(\alpha, \beta)\). In case \(t_2 = t_4 = t_6 = \cdots = 0\) all Schur functions in the sum are equal.

**D.5 Double series in the Schur functions**

**D.6 Matrix integrals**

Using

\[
\int_{O \in O(N)} s_{\lambda}(O) d_s = \begin{cases} 1 & \text{\(\lambda\) is even} \\
0 & \text{otherwise} \end{cases}, \quad \int_{S \in S_p(N)} s_{\lambda}(S) d_s = \begin{cases} 1 & \text{\(\lambda^r\) is even} \\
0 & \text{otherwise} \end{cases}
\]

we get

\[
J_1(t, N) := \int_{O \in O(N)} e^{\sum_{m=1}^{\infty} t_m \text{Tr} O^m} d_s = \sum_{\lambda=\lambda^{\text{even}}}^{\lambda \leq N} \sum_{\ell(\lambda) \leq N} s_{\lambda}(t)
\]

\[
J_2(t, N) := \int_{S \in S_{p}(2n)} e^{\sum_{m=1}^{\infty} t_m \text{Tr} S^m} d_s = \sum_{\lambda=\lambda^{\text{PP}}} \sum_{\ell(\lambda) \leq 2n} s_{\lambda}(t)
\]

The right hand sides were obtained in [2] (see eq. (38) and for \(N \to \infty\) see eqs.(83),(86) there) as examples of the BKP tau function.

There are three different fermionic representations of these integrals.

1. There are representations for \(S_1^{(1)}\) see eq.(162) in [2]

2. The fermionic representation for \(N = 2n\) is

\[
J_3(t, 2n) = b(t)/(2n|\Gamma(t)e^{\int_{z_1} z_2 A(z_1, z_2) w_\alpha(z_1) w_\alpha(z_2) z_1 d z_1 z_2} \Gamma(-t)|0) \}
\]

where for \(w^{(\alpha)}\) see [123] below. and where

\[
A^{(\alpha)}(z_1, z_2) = \frac{(z_1 z_2^{-1} - z_2 z_2^{-1})^n}{z_1 z_1^{-1} - z_2 z_2^{-1}}
\]

Here

\[
b(t) := e^{\sum_{m=1}^{\infty} m t_m^2}
\]
To get the fermionic representation we use that for \( \text{Pf} [A(z_i, z_j)] = \prod_{i<j}^n (z_i + z_j^{-1} - z_j - z_j^{-1})^2 \), see Lemma 5.7 in [89].

(3) The other way is to present the both matrix integrals as the following tau function of the two-component 2-KP

\[
(2n, -2n) \Gamma^{(1)}(t^{(1)}) \Gamma^{(2)}(t^{(1)}) e^{\int w_\alpha(z) w_\alpha^{(1)}(z) \Gamma(-t) \Gamma(-t) d\overline{t}}|0 \rangle
\]

for \( w_\alpha \) see (426) below, superscript shows the numerous of the component of two-component fermions.

**Character expansion: KP tau function**

\[
\chi_\lambda(z, n) := \frac{\langle n + N, \lambda | w(z_1) \cdots w(z_N) | n \rangle}{\langle n + N | w(z_1) \cdots w(z_N) | n \rangle} = \det \left[ \frac{\langle n | \psi^t_{n+\alpha} w(z_j) | n \rangle}{\langle n | \psi^t_{n+N-\alpha} w(z_j) | n \rangle} \right], \quad h_i = \lambda_i - i + N
\]

where \( w(z) \) is a linear combination of the Fermi fields. Denote

\[
\chi_\lambda^{(\alpha)}(z) := \frac{\langle n, \lambda | w_\alpha(z_1) \cdots w_\alpha(z_N) | 0 \rangle}{\langle n | w_\alpha(z_1) \cdots w_\alpha(z_N) | 0 \rangle}
\]

where

\[
w_\alpha(z) = z^{-\frac{\pi i}{2}} (Q_\alpha(z))^{-1} \psi(z) Q_\alpha(z), \quad Q_\alpha = \exp \sum_{n \in \mathbb{Z}} \psi_n \psi^t_{n+\alpha} = \exp \frac{1}{2\pi i} \int z^{\alpha} \psi(z^{-1}) \psi^t(z) dz
\]

namely

\[
w_1(z) = \psi(z) z^{-\frac{\pi i}{2}} - \psi(z^{-1}) z^{\frac{\pi i}{2}}, \quad w_2(z) = \psi(z) z^{-1} - \psi(z^{-1}) z
\]

Then the odd character of the orthogonal group is

\[
s_\lambda(z^{\pm 1}, \ldots, z^{\pm 1}, 1) = \frac{\det (z^{\lambda_i-i+m+\frac{1}{2}} - z^{-(\lambda_i-i+m+\frac{1}{2})})}{\det (z^{-i+m+\frac{1}{2}} - z^{-i+m+\frac{1}{2}})} = \chi^{(1)}_\lambda(z)
\]

(see, say, (3.3) in [11]).

The even symplectic character is defined as

\[
s_\lambda(z^{\pm 1}, \ldots, z^{\pm 1}) := \frac{\det (z^{\lambda_i-i+m+1} - z^{-(\lambda_i-i+m+1)})}{\det (z^{-i+m+1} - z^{-i+m+1})} = \chi^{(2)}_\lambda(z)
\]

(see, say, (4.5) in [11]).

Now, introduce the following KP tau function

\[
\tau^{(\alpha)}(t) = \langle n + N | \Gamma(t) w_\alpha(z_1) \cdots w_\alpha(z_N) | n \rangle = \sum_\lambda \chi^{(\alpha)}_\lambda(z) s_\lambda(t)
\]

where the characters play the role of Plucker coordinates in the Sato formula for KP tau functions.

**BKP tau functions related to 0(N) and Sp(N) characters**

\[
(2n) \Gamma(t) e^{\int_{\Lambda_2(A(z_1, z_2) w_\alpha(z_1) w_\alpha(z_2)) d\overline{t}} \Gamma(\overline{t}) |0 \rangle
\]

\[
= b(t) \sum_\lambda s_\lambda(t + \overline{t}) \int_{\Lambda_N} A(\overline{z}) \chi^{(\alpha)}_\lambda(z) \prod_{i=1}^N e^{-\sum_{m=1} \overline{t}_m (z^m_i + z_i^{-m})} d\overline{z}_i
\]

In case \( t + \overline{t} = 0 \) and \( \Lambda \) chosen as in [119] we come to \( J_\alpha(N, t) \).

**Littelwood-Hall [97]**

49
Appendix. Mehta-Pandey integration trick.

This is a short review of some facts from Chapters 2 and 14 of [16].

Consider a Hermitian matrix \( H = R + iS \) where \( R \) is real symmetric and \( S \) is real anti-symmetric. Let the matrix \( H \) and an auxiliary matrix \( X \) are \( N \times N \) matrices. Below we shall consider four cases where \( X \) is (1) symmetric, (2) anti-symmetric, (3) quaternionic self-dual and (4) quaternionic anti-self-dual.

The following simple relation will be of use below

\[
\int e^{iY_{jk}X_{kj} + tX_{kj}^2} dX_{kj} = \frac{\pi}{\sqrt{t}} e^{-\frac{t}{4} Y_{jk}^2}
\]

where \( Y_{jk}, X_{kj} \) and \( t \) are real.

Interpolation between Gauss unitary (GUE) and Gauss orthogonal ensembles GOE. The integration measure for the unitary ensemble of \( N \times N \) matrices is given by

\[
dH = \prod_{j \leq k} dR_{jk} \prod_{j < k} dS_{jk}
\]

The so-called Mehta-Pandey GUE – GOE interpolating ensemble is defined by the following Gauss probability measure

\[
P(H) = \text{const} \exp \left[ -\sum_{j,k} \left( \frac{(R_{jk})^2}{4v^2} + \frac{(S_{jk})^2}{4v^2\alpha^2} \right) \right]
\]

In case \( \alpha^2 = 1 \) yields GUE. In the limit \( \alpha^2 \to 0 \) Gauss fluctuations of the anti-symmetric part \( S \) of the Hermitian matrix \( H = R + iS \) is suppressed and statistics coincides with the statistic of Gauss symmetric matrices. Therefore, the interval \( 0 < \alpha^2 < 1 \) is related to the GOE-GUE interpolation between Gauss ensembles of Hermitian matrices and of symmetric ones, \( \alpha^2 \) being the interpolation parameter. Similarly, the interval \( 1 < \alpha^2 < +\infty \) may be related to the GOE-GUE interpolation between Gauss ensembles of Hermitian and of anti-symmetric matrices.

(1) If \( X \) is a symmetric matrix. For a given \( i,j \) we have

\[
\int e^{iR_{jk}X_{kj} + tX_{kj}^2} dX_{kj} = \frac{\pi}{\sqrt{t}} e^{-\frac{t}{4} R_{jk}^2}
\]

and thanks to \( \text{Tr}HX = \text{Tr}RX \) we obtain

\[
\int e^{\text{Tr}HX + t\text{Tr}X^2} dX = \left( \frac{\pi}{\sqrt{t}} \right)^{\frac{N(N+1)}{2}} e^{-\frac{t}{4} \text{Tr}R^2}
\]

With the help of this relation the one can write the mean value of any function \( f \) of entries of \( H \) in the Mehta-Pandey ensemble (MP1) as follows

\[
\langle f(H) \rangle_{MP1} := \int f(H) \exp \left[ -\sum_{j,k} \left( \frac{(R_{jk})^2}{4v^2} + \frac{(S_{jk})^2}{4v^2\alpha^2} \right) \right] \prod_{j \leq k} dR_{jk} \prod_{j < k} dS_{jk}
\]

\[
= \int f(\{H_{jk}\}) \int e^{t_2 \text{Tr}H^2 + c\text{Tr}HX + t_2' \text{Tr}X^2} dHdX
\]

where \( t_2 = -\frac{1}{4v^2\alpha^2}, t_2' = -\frac{1}{4v^2(1-\alpha^2)}, c = -\frac{1}{2v^2\alpha^2} \).

(2) If \( X \) is an anti-symmetric matrix then then \( \text{Tr}HX = \text{Tr}SX \). Now we obtain in a similar way

\[
\int e^{-\omega \text{Tr}HX + \omega' \text{Tr}X^2} dX = \left( \frac{\pi}{\sqrt{t}} \right)^{\frac{N(N+1)}{2}} e^{-\frac{t}{4} \text{Tr}S^2}
\]
Interpolation between Gauss unitary (GUE) and Gauss symplectic ensembles GSE. Here we use the standard notions of quaternions $e_i$, $i = 0, 1, 2, 3$ which may be viewed as two by two matrices, $e_0$ is a unity matrix and $e_i, i = 1, 2, 3$ are anti-Hermitian Pauli matrices, $e_1$ is diagonal imaginary, $e_2$ is real anti-symmetric, $e_3$ is symmetric imaginary.

A $2n$ by $2n$ Hermitian matrix $H$ may be written as $n$ by $n$ matrix with quaternionic entries as follows

$$H_{jk} = [R_{jk} + iS_{jk}]$$

where $2n$ by $2n$ matrices

$$R = R^0 \cdot e_0 + \sum_{\mu=1}^3 R^\mu \cdot e_\mu, \quad S = S^0 \cdot e_0 + \sum_{\mu=1}^3 S^\mu \cdot e_\mu$$

are written via $n$ by $n$ matrices $R^\mu$ and $S^\mu$, $\mu = 0, 1, 2, 3$, where $R^0$ and $S^0$ are real symmetric while $S^0$ and $R^0$ are real anti-symmetric.

The operation $e_0 \to e_0$, $e_1 \to -e_1$, $i = 1, 2, 3$ is called conjugation. A matrix, say $R$, with quaternionic entries is called self-dual if $R_{jk}$ is conjugated to $R_{kj}$. A matrix, say $S$, with quaternionic entries is called anti-self-dual if $S_{jk}$ is conjugated to $-S_{kj}$.

In terms of matrices $R^\mu$ and $S^\mu$ the integration measure (431) of the Hermitian matrix $H$ may be written as

$$dH = \prod_{j<k \leq n} \left( dR^\mu_{jk} \prod_{\mu=1}^3 dS^\mu_{jk} \right) \prod_{j<k \leq n} \left( dS^0_{jk} \prod_{\mu=1}^3 dR^\mu_{jk} \right)$$

The Mehta-Pandey interpolating ensemble is defined by the following probability measure

$$P(H) = \exp \left[ -\sum_{j,k} \sum_{\mu=0}^3 \left( \frac{R^\mu_{jk}}{4\mu^2} + \frac{S^\mu_{jk}}{4\mu^2\alpha^2} \right) \right]$$

(437)

In case $\alpha^2 = 1$ (432) yields GSE: the Gauss ensemble of self-dual matrices. In the limit $\alpha^2 \to 0$ Gauss fluctuations of the anti-self-dual part $S$ of the Hermitian matrix $H = R + iS$ is suppressed and statistics coincides with the statistic of Gauss self-dual matrices. Therefore, the interval $0 < \alpha^2 < 1$ is related to the GSE-GUE interpolation between Gauss ensembles of Hermitian matrices and of self-dual ones, $alpha^2$ being the interpolation parameter. Similarly, the interval $1 < \alpha^2 < +\infty$ may be related to the GSE-GUE interpolation between Gauss ensembles of Hermitian and of anti-self-dual matrices.

(3) If $N = 2n$ is even and $X$ is real quaternionic self-dual $n$ by $n$ matrix

F Appendix. Some properties of vector $|\Omega\rangle$

On ASEP and 1D Ising model

Introduce $\prod_{i<j}(\cos \theta_i - 2)$

$$n^\pm_{\lambda} := \langle \lambda | J_{\pm 1} | \Omega \rangle$$

(438)

One can interprets $n^+_{\lambda}$ and $n^-_{\lambda}$ as the numbers of respectively in- and out-incurvities of a Young diagram $\lambda$. Then

$$n^+_{\lambda} = n^-_{\lambda} + 1$$

(439)

In terms of Fock vector interpretation of the state $\lambda$ the number $n^+_{\lambda}$ ($n^-_{\lambda}$) counts the number of fermions which can hop up (resp. down) to a free neighboring site. The appearance of $1$ in the right-hand side of (439) is related to the fact that the bottom of Dirac see is fully packed.

Now introduce

$$V = \frac{1}{2} \sum_{i \in \mathbb{Z}} \left( (2\hat{n}_i - 1)(2\hat{n}_{i+1} - 1) - 1 \right), \quad \hat{n}_i = \psi_i \psi_i^\dagger$$

(440)

Then obviously

$$\langle \lambda | V = n_{\lambda} \langle \lambda |$$

(441)
where
\[ n_\lambda = n_\lambda^{(+)} + n_\lambda^{(-)} = 2n_\lambda^{(+)} - 1 = 2n_\lambda^{(-)} + 1 \] (442)
is the number of free neighboring sites for fermions in the state \( \langle \lambda \rangle \).

Introduce
\[ |\Omega_\lambda\rangle := \sum_{\lambda \in \mathcal{P}} |\lambda\rangle e^{-a|\lambda|} \] (444)
Then
\[ \langle \lambda | J_{\pm 1} | \Omega_\lambda \rangle = e^{\pm a n_\lambda^{(\pm)}} \] (445)
or, the same,
\[ J_{\pm 1} | \Omega_\lambda \rangle = \sum_{\lambda \in \mathcal{P}} |\lambda\rangle e^{-a|\lambda|} e^{\pm a n_\lambda^{(\pm)}} \] (446)

\[ (rJ_{-1} + r^{-1}J_{1} - pV) | \Omega_\lambda \rangle = \sum_{\lambda \in \mathcal{P}} |\lambda\rangle e^{-a|\lambda|} \left( \frac{1}{2}r e^{-a}(n_\lambda - 1) + \frac{1}{2}r^{-1}e^{a}(n_\lambda + 1) - pn_\lambda \right) \] (447)
when \( p = \frac{1}{2}r e^{-a} + \frac{1}{2}r^{-1}e^{a} \) then
\[ \hat{H} | \Omega_\lambda \rangle = 0, \quad \hat{H} := \left( rJ_{-1} + r^{-1}J_{1} - \left( \frac{1}{2}r e^{-a} + \frac{1}{2}r^{-1}e^{a} \right)V + \left( \frac{1}{2}r e^{-a} - \frac{1}{2}r^{-1}e^{a} \right) \right) \] (448)

At last we obtain
\[ \langle \lambda | e^{\hat{H}t} | \Omega_\lambda \rangle \equiv 1 \] (449)
Via Jordan-Wigner transform \( \hat{H} \) may be identified with the Hamiltonian of a non-Hermitian spin chain \[ \hat{H} \rightarrow \sum rS_i^- S_{i+1}^+ + r^{-1}S_i^+ S_{i+1}^- + pS_i^z S_{i+1}^z, \] which is the Hamiltonian of the well-known XXX model if \( r = 1 \), \( p = \frac{1}{2} \). This representation was used in [83, 84] for a description of the asymmetric simple exclusion model (ASEP). Then relation (449) describes the fact that the sum of the probabilities to achieve each of admissible states is equal to one. Then
\[ p_{\lambda \to \mu}(t) = \langle \lambda | e^{\hat{H}t} | \mu \rangle \]
is the probability to achieve the state \( \mu \) starting from the state \( \lambda \) after the lapse of the time \( t \).

**Determinants of infinite matrices**
\[ (\Omega | e^{\sum_{n,m \geq 0} \psi_n \psi_m^\dagger D_{nm}} | 0) = \det(1 + D) \] (450)

### G Zonal functions
\[ \int \ldots \int \Delta_N^\gamma(z) \prod_{i=1}^N e^{2\sum_{m=1}^\infty z^m t_m e^{2\sum_{m=1}^\infty z^m t_m} d z_i} = \sum_{\lambda \in \mathcal{P}} e^{-P_\lambda^\pm} J_\lambda^{(\pm)}(t)J_\lambda^{(\pm)}(t) \] (451)

is the particular case of

\[ \int \ldots \int \prod_{i \neq j}^N (1 - x_i x_j^{-1})^\alpha J_\lambda^{(\alpha)}(x) J_\lambda^{(\alpha)}(x) \prod_{i=1}^N dx_i = \prod_{1 \leq i < j \leq N} \frac{\Gamma(\xi_i - \xi_j + \frac{1}{\alpha})\Gamma(\xi_i - \xi_j - \frac{1}{\alpha} + 1)}{\Gamma(\xi_i - \xi_j)\Gamma(\xi_i - \xi_j + 1)} \] (452)

where \( \xi_i := \lambda_i + \frac{1}{\alpha}(N - 1), 1 \leq i \leq N, \) [17] we have
\[ \int \ldots \int \Delta_N^\gamma(z) \prod_{i=1}^N e^{2\sum_{m=1}^\infty z^m t_m e^{2\sum_{m=1}^\infty z^m t_m} d z_i} = \sum_{\lambda \in \mathcal{P}} e^{-P_\lambda^\pm} J_\lambda^{(\alpha)}(t)J_\lambda^{(\alpha)}(t) \]

\[ \text{**Notice that**} \]
\[ Z_t := \langle \Omega | e^V | \Omega \rangle = \sum_\lambda e^{\beta n_\lambda} \] (435)
may be interpreted as the partition function of the 1D Ising model, a parameter \( \beta \) being the inverse temperature.
where
\[ e^{-F^{(α)}_N} = \ldots \prod_{1 ≤ i < j ≤ N} \frac{Γ(ξ_i - ξ_j + \frac{1}{α})Γ(ξ_i - ξ_j - \frac{1}{α} + 1)}{Γ(ξ_i - ξ_j)Γ(ξ_i - ξ_j + 1)} \]

G.1 Appendix. On complex beta-ensembles

Below everything is wrong (kak polnyj kozel: v pyatyj raz oshibayuš' v odnom i tom zhe meste)

In this section the bar means complex conjugation: \( \bar{t} = (t_1, t_2, \ldots) \) denote the complex conjugated \( t = (t_1, t_2, \ldots) \); \( \bar{z}_i \) is the complex conjugated \( z_i \).

Consider the following complex beta-ensemble:

\[ Z_N^{(β)} := \int_C \ldots \int_C \prod_{n < m ≤ N} |z_n - z_m|^β e^{W(z_1, \ldots, z_N)} e^{\frac{1}{2} Σ_{n,m=1}^N (t_n z_n^2 + t_m z_m^2)} dz_i d\bar{z}_i \]  

(453)

where \( W(z_1, \ldots, z_N) \) is some potential, which is a symmetric function of variables \( z_i \). Such integrals appear in the theory of quantum Hall droplets.

\[ t_j = \frac{1}{j} \sqrt{β} \sum_{n=1}^N z_n^j, \quad β = \frac{2}{α} \]

(454)

As we see

\[ dt_1 ∧ \ldots ∧ dt_N = \left(\frac{β}{2}\right)^{\frac{N}{2}} Δ_N(z) dz_1 ∧ \ldots ∧ dz_N \]

In the large \( N \) limit we want to replace integrals over \( z_i, \bar{z}_i, i = 1, \ldots, N \) by integrals over \( t_i, \bar{t}_i i = 1, \ldots, N \).

\[ t_j = \frac{γ}{j} \sum_{n=1}^N z_n^j \]

(455)

As we see

\[ dt_1 ∧ \ldots ∧ dt_N = γ^N Δ_N(z) dz_1 ∧ \ldots ∧ dz_N \]

Introduce

\[ τ^{(α)}(t, 0, \bar{t}) := e^{\frac{2}{α} Σ_{n=1}^∞ n t_n t_n} = \sum_λ P^{(α)}_λ (t) Q^{(α)}_λ (\bar{t}) \]

(456)

and

\[ τ^{(α)}(t, t^*, \bar{t}) := e^{-Σ_{n=1}^∞ \overline{H^{(α)}_n}(t)n^*_n} e^{\frac{1}{2} Σ_{n=1}^∞ n t_n t_n} = e^{-Σ_{n=1}^∞ H^{(α)}(t)n^*_n} \sum_λ P^{(α)}_λ (t) Q^{(α)}_λ (\bar{t}) \]

(457)

\[ \sum_λ e^{-Σ_{n=1}^∞ ε^{(α)}_n(λ) n^*_n} P^{(α)}_λ (t) Q^{(α)}_λ (\bar{t}) \]

(458)

\[ \text{????????????? where } ε^{(α)}_λ \text{ is the "energy" of a configuration } λ = (λ_1, \ldots, λ_ℓ) \text{ defined as} \]

\[ ε^{(α)}_λ := \sum_{k=1}^{ℓ(λ)} \left( ε^{(α)}_{λ_k - \frac{1}{2} kβ} - ε^{(α)}_{-\frac{1}{2} kβ} \right) \]

(459)

\[ \text{????????????? where } (???? n) \]

\[ ε^{(α)}_λ P^{(α)}_λ = H^{(α)}_n P^{(α)}_λ \]

(460)

As one can see for any \( N \) we have the equality

\[ \int \ldots \int \prod_{n=1}^N e^{\frac{1}{2} n t_n'} t_n' e^{\frac{1}{2} n t_n} t_n e^{\frac{1}{2} n t_n} t_n e^{\frac{1}{2} n t_n'} t_n' \frac{dn n t_n d\bar{t}_n}{2π \sqrt{-1}} = e^{\frac{1}{2} Σ_{n=1}^N n t_n''} \]

(461)
This follows from
\[
\int \cdots \int p_\lambda p_\mu \prod_{n=1}^N e^{-\frac{2i}{\pi}nt_n t_n} \frac{\alpha n dt_n df_n}{2\pi\sqrt{-1}} = \alpha^\ell(\lambda) z_\lambda \delta_{\lambda,\mu}
\] (462)
where for \( \lambda = (\lambda_1, \ldots, \lambda_k) \), \( \mu = (\mu_1, \ldots, \mu_s) \) it is supposed that \( 0 \neq \lambda_k \leq N \), \( 0 \neq \mu_s \leq N \), and where \( p_\lambda := p_{\lambda_1} \cdots p_{\lambda_k} \), \( z_\lambda = \prod_i \frac{\mu_i}{m_i!} \) where \( m_i = m_i(\lambda) \) is the number of parts of \( \lambda \) equal to \( i \) and \( p_n := nt_n, n = 1, 2, \ldots \).

**Remark 15.** Changing variables via (455) we obtain
\[
N! \frac{\alpha^N}{(2\pi\sqrt{-1})^N} \int \cdots \int \prod_{i,j=1}^N (1 - z_i z_j) \frac{2\alpha^2}{|\Delta_N(z)|^2} \prod_{n=1}^N e^{\frac{\alpha}{\sqrt{-1}} \sum_{n=1}^N (t_n' z_m' + t_n z_m')} dz_m d\bar{z}_m
\] (463)
Then, relation (462) results in
\[
\int \cdots \int p_\lambda Q_\mu \prod_{n=1}^N e^{-\frac{2i}{\pi} nt_n t_n} \frac{\alpha n dt_n df_n}{2\pi\sqrt{-1}} = \delta_{\lambda,\mu}
\] (464)
which gives rise to (461) via the second equality in (460).

In the large \( N \) limit we can write
\[
\int \tau(\alpha)(t', 0, t) \tau(\alpha)(t, 0, t') \tau(\alpha)(\tilde{t}, 0, \tilde{t}') \prod_{n=1}^\infty \frac{\alpha n dt_n df_n}{2\pi\sqrt{-1}} = \tau(\alpha)(t', 0, t')
\] (465)
and thanks to the Hermitian property of Calogero Hamiltonians inside the integral we obtain
\[
\int \tau(\alpha)(t', t', t) \tau(\alpha)(t, t^*, \tilde{t}) \tau(\alpha)(\tilde{t}, t^{**}, t'') \prod_{n=1}^\infty \frac{\alpha n dt_n df_n}{2\pi\sqrt{-1}} = \tau(\alpha)(t', t'^* - t + t^{**}, t'')
\] (466)
(For \( \alpha = 1 \) it was marked in (46).)

**wrong:**

In the large \( N \) limit, \( Z_N^{(\beta)} \) of (453)
\[
\prod_{n < m \leq \infty} |z_n - z_m|^\beta e^{W(z_1 \ldots, z_N)} \prod_{i=1}^\infty d z_i d \bar{z}_i := \tau(\alpha)(t, t^* \tilde{t}) \prod_{i=1}^\infty dt_i d\bar{t}_i
\]
and where variables \( t_i, i = 1, 2, \ldots, t_N \) are related to variables \( z_i, i = 1, 2, \ldots, z_N \) via (454) where we send \( N \) to \( \infty \).

**Example.** \( \beta = 2 \). Choose \( t^* \) in a way that
\[
\tau(\alpha)(t, t^*, \tilde{t}) = \sum_\lambda \frac{1}{(N)_\lambda} s_\lambda(t) s_\lambda(\tilde{t}) = \frac{\det [e^{z_i z_j}]}{|\Delta(z)|^2}
\] (467)

******************************************************************************

\[
\int \cdots \int \prod_{i \neq j}^N (1 - x_i x_j^{-1})^{\frac{1}{\alpha}} J_\lambda^{(\alpha)}(x) J_\lambda^{(\alpha)}(x) \prod_{i=1}^N dx_i = \prod_{1 \leq i < j \leq n} \frac{\Gamma(\xi_i - \xi_j + \frac{1}{\alpha}) \Gamma(\xi_i - \xi_j - \frac{1}{\alpha} + 1)}{\Gamma(\xi_i - \xi_j) \Gamma(\xi_i - \xi_j + 1)}
\]
where \( \xi_i := \lambda_i + \frac{1}{\alpha}(N - 1), 1 \leq i \leq N \), \( \lambda_i \).