Characterizations and properties of dual matrix star orders

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Abstract

In this paper, we introduce D-star order, T-star order and P-star order on the class of dual matrices. By applying matrix decomposition and dual generalized inverses, we discuss properties, characterizations and relations among these orders, and illustrate their relations with examples.

Keywords: Dual generalized inverse; D-star partial order; P-star partial order; Moore-Penrose dual generalized inverse; Dual Moore-Penrose generalized inverse

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1. Introduction

In this paper, we adopt the following notations. The symbol $\mathbb{R}^{m \times n}$ denotes the set of all $m \times n$ real matrices. $A^T$ and $\text{rk}(A)$ denote the transpose and rank of $A \in \mathbb{R}^{m \times n}$, respectively. The Moore-Penrose inverse of $A \in \mathbb{R}^{m \times n}$ is defined as the unique matrix $X \in \mathbb{R}^{n \times m}$ satisfying the Penrose equations: $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$, and is usually denoted by $X = A^\dagger$. Denote an $m \times n$ dual matrix by $\hat{A} = A + \varepsilon A_0$, in which $A$ and $A_0$ are all $m \times n$ real matrices, and $\varepsilon$ is the dual unit satisfying $\varepsilon \neq 0$, $0\varepsilon = \varepsilon 0$, $1\varepsilon = \varepsilon 1 = \varepsilon$, $\varepsilon^2 = 0$. Furthermore, $\hat{A}^T$ denotes the transpose of $\hat{A}$, that is, $\hat{A}^T = A^T + \varepsilon A_0^T$. $\mathbb{D}^{m \times n}$ denotes the set of all $m \times n$ dual matrices.

Dual matrices have been commonly used in various fields of science and engineering, such as the kinematic analysis synthesis of machines and mechanisms, robotics and machine vision[15]. Recently, dual generalized inverses attracted much attention. Many researchers have acquired fruitful findings [4, 5, 11, 12]. Let $\hat{A} = A + \varepsilon A_0 \in \mathbb{D}^{m \times n}$, then the Moore-Penrose dual generalized

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inverse (MPDGI for short) of $\hat{A}$ is denoted by $\hat{A}^p$ and displayed in the form

$$\hat{A}^p = A^\dagger - \varepsilon A^\dagger A_0 A^\dagger.$$  \hfill (1.1)

Obviously, every dual matrix has MPDGI. Pennestrì et al. propose novel and computationally efficient algorithms (formulas) for the computation of the MPDGI.

If there exists a unique matrix $\hat{X} \in \mathbb{D}^{n \times m}$ satisfying the Penrose equations:

$$(\hat{1}) \quad \hat{A} \hat{X} \hat{A} = \hat{A}, \quad (\hat{2}) \quad \hat{X} \hat{A} \hat{X} = \hat{X}, \quad (\hat{3}) \quad \hat{A} \hat{X} = (\hat{A} \hat{X})^T, \quad (\hat{4}) \quad \hat{X} \hat{A} = (\hat{X} \hat{A})^T \quad (1.2)$$

then $\hat{X}$ is the dual Moore-Penrose generalized inverse (DMPGI for short) of $\hat{A}$, and denoted by $\hat{X} = \hat{A}^\dagger$. Udwadia shows that not all dual matrices have DMPGIs, and gets some interesting properties of DMPGI. Wang gives a compact formula for DMPGI. He also puts forward some necessary and sufficient conditions for a dual matrix to have DMPGI. These theories should be main tools to carry out studies on dual matrix partial order in this paper.

Dual generalized inverses is a powerful tool to study the least-squares solutions to systems of linear dual equations. For example, Belzile uses dual generalized inverses, the characteristic length and Householder reflections over the dual ring to investigate problems of both translation and rotation in the realm of kinematic synthesis. These applications provide the impetus for the in-depth study on dual generalized inverse theory.

It is well known that an important application of generalized inverse is to study matrix partial order theory, such as characterizations and representations of star, sharp, core and minus partial orders. Matrix partial order theory can be applied to solve optimization problems like the minimization of production costs in statistics. The theory is also used to study autonomous linear systems and control system problems.

Abundant theories of dual generalized inverses provide a sufficient basis for carrying out researchs on dual matrix order theory and practice. Because the dual matrix structure is special, it makes DMPGI and MPDGI are closely related but they are different in essence. These differences provide a basis for conducting studies on dual matrix partial order to obtain rich and interesting results. The expectant research results of dual matrix order will be more diversified. For example, we can use the transpose of real matrices or Moore-Penrose inverse to characterize star partial order, but we cannot get similar results in the dual matrix partial order. Since the existence of DMPGI has strict conditions, but MPDGI always exists. Therefore, both of the dual binary relations are not
equivalent. Next, we will investigate star order of dual matrices. The theoretical results will also provide a theoretical basis for linear systems of dual equations.

The outline of this paper is as follows. In Section 2 we briefly review some preliminaries. In Section 3 when DMPGIs of dual matrices exist, we introduce the D-star order of dual matrices, and give some necessary and sufficient conditions for the existence of D-star order. Furthermore, we prove that it is a partial order and derive characterizations and properties of the partial order by applying matrix decomposition. In Section 4 we present a new binary relation (P-order) by applying MPDGIs. When DMPGIs of dual matrices exist, it is shown that the new binary relation is partial order is called P-star partial order. In Section 5 we consider relations between D-star partial order and P-star partial order, and give examples to illustrate their differences and connections.

2. Preliminaries

In this section, we give some basic theories for further research, such as the singular value decomposition (SVD for short) of real matrix, characterizations of star partial order and DMPGI and so on.

**Theorem 2.1 (SVD).** Let $A \in \mathbb{R}^{m \times n}$ and $\text{rk}(A) = a$. Then there exist orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that

$$A = U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T,$$

where $T_1 \in \mathbb{R}^{a \times a}$ is a diagonal positive definite matrix.

**Theorem 2.2 ([2, 10]).** Let $A, B \in \mathbb{R}^{m \times n}$, $\text{rk}(A) = a$, $\text{rk}(B) = b$ and $b > a$. Then the following four statements are equivalent:

1. $A \preceq B$;
2. $A^\dagger A = A^\dagger B$ and $AA^\dagger = BA^\dagger$;
3. $A^T A = A^T B$ and $AA^T = BA^T$;
4. There exist orthogonal matrices $U$ and $V$ such that

$$A = U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T, \quad B = U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \end{pmatrix} V^T,$$

(2.1)
where \( T_1 \in \mathbb{R}^{a \times a} \) and \( T_2 \in \mathbb{R}^{(b-a) \times (b-a)} \) are diagonal positive definite matrices.

**Theorem 2.3** ([17] Theorem 2.1). Let \( \widetilde{A} = A + \varepsilon A_0 \in \mathbb{D}^{m \times n} \). Then the following conditions are equivalent:

1. The DMPGI \( \widetilde{A}^\dagger \) of \( \widetilde{A} \) exists;
2. \( (I_m - AA^\dagger) A_0 (I_n - A^\dagger A) = 0 \);
3. \( \text{rk} \begin{pmatrix} A_0 & A \\ A & 0 \end{pmatrix} = 2 \text{rk} (A) \).

If the DMPGI \( \widetilde{A}^\dagger \) of \( \widetilde{A} \) exists, then

\[
\widetilde{A}^\dagger = A^\dagger + \varepsilon R, \tag{2.2}
\]

where \( R = -A^\dagger A_0 A^\dagger + (A^T A)^\dagger A_0^T (I_m - AA^\dagger) + (I_n - A^\dagger A) A_0^T (AA^T)^\dagger \).

Furthermore, let the SVD of \( A \) be as shown in Theorem 2.1, then

\[
\widetilde{A} = U \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 \\ A_3 & 0 \end{pmatrix} V^T, \tag{2.3}
\]

\[
\widetilde{A}^\dagger = V \begin{pmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^T + \varepsilon V \begin{pmatrix} -T_1^{-1} A_1 T_1^{-1} & T_1^{-2} A_3^T \\ A_2 T_1^{-2} & 0 \end{pmatrix} U^T, \tag{2.4}
\]

where \( T_1 \) is a diagonal positive definite matrix.

**Theorem 2.4** ([17]). Let \( \widetilde{A} = A + \varepsilon A_0 \). Then MPDGI of \( \widetilde{A} \), i.e. \( \widetilde{A}^p \) always exists, and there exist orthogonal matrices \( U \) and \( V \) such that

\[
\widetilde{A} = U \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} V^T, \tag{2.5}
\]

\[
\widetilde{A}^p = V \begin{pmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^T + \varepsilon V \begin{pmatrix} -T_1^{-1} A_1 T_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^T, \tag{2.6}
\]

where \( T_1 \) is a diagonal positive definite matrix.

3. D-Star Partial Order

The DMPGI satisfies Penrose equations and is closely related to Moore-Penrose generalized inverse of real matrix. Therefore, we firstly introduce the D-star order by applying DMPGI.
**Definition 3.1.** Let DMPGIs of \( \hat{A} \) and \( \hat{B} \) exist. If \( \hat{A}, \hat{B} \) satisfy
\[
\hat{A}^\dagger \hat{A} = \hat{A}^\dagger \hat{B} \quad \text{and} \quad \hat{A} \hat{A}^\dagger = \hat{B} \hat{A}^\dagger ,
\]
we say that \( \hat{A} \) is below \( \hat{B} \) under the D-star order, and denote it by \( \hat{A} \leq^* \hat{B} \).

**Theorem 3.1.** Let \( \hat{A} = A + \varepsilon A_0 \) and \( \hat{B} = B + \varepsilon B_0 \), where \( A, A_0, B, B_0 \in \mathbb{R}^{m \times n} \). And let DMPGIs of \( \hat{A} \) and \( \hat{B} \) exist, then \( \hat{A} \leq^* \hat{B} \) if and only if
\[
\begin{align*}
A \preceq B \\
A^\dagger A_0 + RA &= A^\dagger B_0 + RB \\
AR + A_0 A^\dagger &= BR + B_0 A^\dagger ,
\end{align*}
\]
where \( R = -A^\dagger A_0 A^\dagger + (A^T A)^\dagger A_0^T (I_m - AA^\dagger) + (I_n - A^\dagger A) A_0^T (AA^T)^\dagger \).

**Proof.** Let \( \hat{A} = A + \varepsilon A_0, \hat{B} = B + \varepsilon B_0, \) and DMPGI \( \hat{A}^\dagger \) of \( \hat{A} \) exist. Denote \( \hat{A}^\dagger = A^\dagger + \varepsilon R, \) where \( R \) is as in Theorem 2.3. Then
\[
\begin{align*}
\hat{A}^\dagger \hat{A} &= (A^\dagger + \varepsilon R) (A + \varepsilon A_0) = A^\dagger A + \varepsilon (A^\dagger A_0 + RA) \\
\hat{A}^\dagger \hat{B} &= (A^\dagger + \varepsilon R) (B + \varepsilon B_0) = A^\dagger B + \varepsilon (A^\dagger B_0 + RB)
\end{align*}
\]
and
\[
\begin{align*}
\hat{A} \hat{A}^\dagger &= (A + \varepsilon A_0) (A^\dagger + \varepsilon R) = AA^\dagger + \varepsilon (AR + A_0 A^\dagger) \\
\hat{B} \hat{A}^\dagger &= (B + \varepsilon B_0) (A^\dagger + \varepsilon R) = BA^\dagger + \varepsilon (BR + B_0 A^\dagger) .
\end{align*}
\]
Since \( \hat{A} \leq^* \hat{B} \), it follows from Definition 3.1 that \( \hat{A} \leq \hat{B} \) if and only if
\[
\begin{align*}
A^\dagger A &= A^\dagger B, \quad AA^\dagger = BA^\dagger \\
A^\dagger A_0 + RA &= A^\dagger B_0 + RB \\
AR + A_0 A^\dagger &= BR + B_0 A^\dagger .
\end{align*}
\]
Since \( A^\dagger A = A^\dagger B \) and \( AA^\dagger = BA^\dagger \), we get \( A \preceq B \). Therefore, \( \hat{A} \leq \hat{B} \) is equivalent to (3.2).

**Theorem 3.2.** Let \( \hat{A} = A + \varepsilon A_0 \in \mathbb{D}^{m \times n}, \hat{B} = B + \varepsilon B_0 \in \mathbb{D}^{m \times n}, \) and DMPGIs of \( \hat{A} \) and \( \hat{B} \) exist.
Then \( \hat{A} \leq \hat{B} \) if and only if there exist orthogonal matrices \( U \) and \( V \)

\[
\begin{align*}
\hat{A} &= U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & 0 & 0 \\ A_7 & 0 & 0 \end{pmatrix} V^T, \\
\hat{B} &= U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 - T_{11}^{-1} A_2^T T_2 & A_3 \\ A_4 - T_2 A_2^T T_1^{-1} & B_5 & B_6 \\ A_7 & B_8 & 0 \end{pmatrix} V^T,
\end{align*}
\]

where \( T_1 \) and \( T_2 \) are diagonal positive definite matrices.

**Proof.** \( \Rightarrow \) Denote \( \text{rk}(A) = a \) and \( \text{rk}(B) = b \). Since \( \hat{A} \leq \hat{B} \), by applying Theorem 3.1, we get \( A \leq B \). Then \( A \) and \( B \) are of the forms as in (2.1). Since the DMPGI of \( \hat{A} \) exists, we write

\[
A_0 = U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & 0 & 0 \\ A_7 & 0 & 0 \end{pmatrix} V^T,
\]

where \( A_1 \in \mathbb{R}^{a \times a} \), \( A_2 \in \mathbb{R}^{a \times (b-a)} \) and \( A_4 \in \mathbb{R}^{(b-a) \times a} \). Applying Theorem 2.3, we have

\[
\hat{A}^T = V \begin{pmatrix} T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon V \begin{pmatrix} -T_1^{-1} A_1 T_1^{-1} & T_1^{-2} A_1^2 T_1^{-1} & T_1^{-2} A_1^3 \\ A_2^2 T_1^{-2} & 0 & 0 \\ A_3^2 T_1^{-2} & 0 & 0 \end{pmatrix} U^T.
\]

Since the DMPGI of \( \hat{B} \) exists, we write

\[
B_0 = U \begin{pmatrix} B_1 & B_2 & B_3 \\ B_4 & B_5 & B_6 \\ B_7 & B_8 & 0 \end{pmatrix} V^T,
\]

where \( B_1 \in \mathbb{R}^{a \times a} \) and \( B_2 \in \mathbb{R}^{a \times (b-a)} \). Applying (2.1), (3.4), (3.5) and (3.6) gives

\[
\begin{align*}
\hat{A}^T \hat{A} &= V \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon V \begin{pmatrix} 0 & T_1^{-1} A_1 T_1^{-1} & T_1^{-1} A_2 \\ A_2^2 T_1^{-1} & 0 & 0 \\ A_3^2 T_1^{-1} & 0 & 0 \end{pmatrix} V^T, \\
\hat{A}^T \hat{B} &= V \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon V \begin{pmatrix} T_1^{-1} B_1 - T_1^{-1} A_1 & T_1^{-1} B_2 + T_1^{-2} A_1^2 T_2 & T_1^{-1} B_3 \\ A_2^2 T_1^{-1} & 0 & 0 \\ A_3^2 T_1^{-1} & 0 & 0 \end{pmatrix} V^T.
\end{align*}
\]
Since $\hat{A} \leq \hat{B}$, we have $\hat{A}^\dagger \hat{A} = \hat{A}^\dagger \hat{B}$. Applying (3.7) gives

$$
\begin{align*}
&\begin{pmatrix}
0 = T_1^{-1}B_1 - T_1^{-1}A_1 \\
T_1^{-1}A_2 = T_1^{-1}B_2 + T_1^{-2}A_1^T T_2 \\
T_1^{-1}B_3 = T_1^{-1}A_3.
\end{pmatrix}
\end{align*}
$$

Therefore, $B_1 = A_1$, $B_2 = A_2 - T_1^{-1}A_1^T T_2$ and $B_3 = A_3$. It follows from (3.6) that

$$
B_0 = U \begin{pmatrix} A_1 & A_2 - T_1^{-1}A_1^T T_2 & A_3 \\ B_4 & B_5 & B_6 \\ B_7 & B_8 & 0 \end{pmatrix} V^T. \tag{3.8}
$$

Applying (2.1), (3.4), (3.5) and (3.8), we obtain

$$
\begin{align*}
&\begin{pmatrix}
\hat{A}^\dagger = U \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon U \\ \hat{B}^\dagger = U \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon U
\end{pmatrix}
\end{align*}
$$

Applying (2.1), (3.4), (3.5) and (3.8), we get

$$
\begin{align*}
&\begin{pmatrix}
0 = T_1^{-1}B_1 - T_1^{-1}A_1 \\
T_1^{-1}A_2 = T_1^{-1}B_2 + T_1^{-2}A_1^T T_2 \\
T_1^{-1}B_3 = T_1^{-1}A_3.
\end{pmatrix}
\end{align*}
$$

Since $\hat{A} \leq \hat{B}$, we get $\hat{A}^\dagger \hat{B} = \hat{B}^\dagger \hat{B}$. It follows from (3.9) that $A_4 T_1^{-1} = B_4 T_1^{-1} + T_2 A_2^T T_1^{-2}$ and $A_7 T_1^{-1} = B_7 T_1^{-1}$, that is,

$$
B_4 = A_4 - T_2 A_2^T T_1^{-1} \quad \text{and} \quad B_7 = A_7. \tag{3.10}
$$

Therefore, applying (2.1), (3.5) and (3.10), we get

$$
\hat{B} = U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 - T_1^{-1}A_1^T T_2 & A_3 \\ A_4 - T_2 A_2^T T_1^{-1} & B_5 & B_6 \\ A_7 & B_8 & 0 \end{pmatrix} V^T.
$$

"\Rightarrow" \quad \text{Let there exist orthogonal matrices } U \text{ and } V \text{ such that } \hat{A} \text{ and } \hat{B} \text{ can be represented as}
Then the form of \( \hat{A}^\dagger \) is as in (3.3). It is easy to check that

\[
\hat{A}^\dagger \hat{A} = V_1 \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_1^T + \varepsilon V_1 \begin{pmatrix} 0 & T_1^{-1} A_2 & T_1^{-1} A_3 \\ A_2^T T_1^{-1} & 0 & 0 \\ A_3^T T_1^{-1} & 0 & 0 \end{pmatrix} V_1^T = \hat{A}^\dagger \hat{B}
\]

Therefore, applying Definition 3.1, we get \( \hat{A} \leq \hat{B} \).

**Theorem 3.3.** The D-star order is a partial order.

**Proof.** Let \( \hat{A} = A + \varepsilon A_0; \hat{B} = B + \varepsilon B_0; \) DMPGIs of \( \hat{A} \) and \( \hat{B} \) exist; and \( \hat{A} \leq \hat{B} \); i.e.; \( \hat{A}^\dagger \hat{A} = \hat{A}^\dagger \hat{B} \) and \( \hat{A} \hat{A}^\dagger = \hat{B} \hat{A}^\dagger \).

Next, we show that the D-star order satisfies reflexivity, anti-symmetry, and transitivity.

(i) Reflexivity is self-evident.

(ii) Let \( \hat{A} \leq \hat{B} \). Applying Theorem 3.1, it follows from \( \hat{A} \leq \hat{B} \) that we have \( A \leq B \) and \( B \leq A \). From the anti-symmetry of star partial order on real matrices, we have \( A = B \).

Since \( \hat{A} \leq \hat{B} \), \( \hat{A} \) and \( \hat{B} \) can be represented in the forms as in (3.3). Applying Theorem 3.2, we have \( T_2 = 0 \). Therefore,

\[
\hat{A} = U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & 0 & 0 \\ A_7 & 0 & 0 \end{pmatrix} V^T
\]

(3.11)

Since the DMPGI of \( \hat{B} \) exists, by applying Theorem 2.3 and (3.11), we get \( B_5 = 0, B_6 = 0 \) and \( B_8 = 0 \). Therefore, \( \hat{A} = \hat{B} \). So, the anti-symmetry holds.

(iii) Let \( \hat{C} = C + \varepsilon C_0; \) the DMPGI of \( \hat{C} \) exist; \( \hat{B} \leq \hat{C} \). Since \( \hat{A} \leq \hat{B} \) and \( \hat{B} \leq \hat{C} \), we get \( A \leq C \). Denote \( \text{rk} (A) = a, \text{rk} (B) = b \) and \( \text{rk} (C) = c \). Then there exist orthogonal matrices \( U \) and
where $T_1 \in \mathbb{R}^{a \times a}$, $T_2 \in \mathbb{R}^{(b-a) \times (b-a)}$ and $T_3 \in \mathbb{R}^{(c-b) \times (c-b)}$ are diagonal positive definite matrices.

Since $\hat{A} \leq \hat{B}$, applying Theorem 3.2, we get

\[
\begin{align*}
A_0 &= U \begin{pmatrix}
A_1 & A_2 & A_{31} & A_{32} \\
A_4 & 0 & 0 & 0 \\
A_{71} & 0 & 0 & 0 \\
A_{72} & 0 & 0 & 0
\end{pmatrix} V^T, \\
B_0 &= U \begin{pmatrix}
A_1 & A_2 - T_1^{-1} A_4^T T_2 & A_{31} & A_{32} \\
A_4 - T_2 A_2^T T_1^{-1} & B_5 & B_{61} & B_{62} \\
A_{71} & B_{61} & 0 & 0 \\
A_{72} & B_{62} & 0 & 0
\end{pmatrix} V^T. 
\end{align*}
\]

Since the DMPGI of $\hat{C}$ exists, we denote

\[
C_0 = U \begin{pmatrix}
C_1 & C_2 & C_3 & C_4 \\
C_5 & C_6 & C_7 & C_8 \\
C_9 & C_{10} & C_{11} & C_{12} \\
C_{13} & C_{14} & C_{15} & 0
\end{pmatrix} V^T.
\]

Applying Theorem 2.3 and 3.2 and 3.13, we get

\[
\hat{A}_1^T = V \begin{pmatrix}
T_1^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} U^T + \varepsilon V \begin{pmatrix}
-T_1^{-1} A_1 T_1^{-1} & T_1^{-2} A_4^T & T_1^{-2} A_{17}^T & T_1^{-2} A_{17}^T \\
A_2^T T_1^{-2} & 0 & 0 & 0 \\
A_{31}^T T_1^{-2} & 0 & 0 & 0 \\
A_{31}^T T_1^{-2} & 0 & 0 & 0
\end{pmatrix} U^T.
\]
Since $\hat{B} \leq \hat{C}$, by applying (3.12), (3.13), (3.14) and Theorem 3.2 we obtain

$$\hat{C} = U \begin{pmatrix} T_1 & 0 & 0 & 0 \\ 0 & T_2 & 0 & 0 \\ 0 & 0 & T_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V^T$$

$$+ \varepsilon U \begin{pmatrix} A_1 & A_2 - T_1^{-1} A_3^T T_2 & A_31 - T_1^{-1} A_3^T T_3 & A_32 \\ A_4 - T_2^T A_3^{-1} & B_5 & B_61 - T_2^{-1} B_6 T_3 & B_62 \\ A_71 - T_3 A_3^T A_3^T & B_61 - T_3 B_6 T_2^{-1} & C_{11} & C_{12} \\ A_{72} & B_{82} & C_{15} & 0 \end{pmatrix} V^T.$$  (3.16)

Then applying (3.11), (3.10) gives

$$\hat{A}^\dagger \hat{A} = V \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon V \begin{pmatrix} 0 & T_1^{-1} A_2 & T_1^{-1} A_31 & T_1^{-1} A_32 \\ A_2^T T_1^{-1} & 0 & 0 & 0 \\ A_3^T T_1^{-1} & 0 & 0 & 0 \\ A_{32}^T T_1^{-1} & 0 & 0 & 0 \end{pmatrix} V^T$$

$$\hat{A}^\dagger \hat{C} = V \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon V \begin{pmatrix} 0 & T_1^{-1} A_2 & T_1^{-1} A_31 & T_1^{-1} A_32 \\ A_2^T T_1^{-1} & 0 & 0 & 0 \\ A_3^T T_1^{-1} & 0 & 0 & 0 \\ A_{32}^T T_1^{-1} & 0 & 0 & 0 \end{pmatrix} V^T$$

and

$$\hat{A} \hat{A}^\dagger = U \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon U \begin{pmatrix} 0 & T_1^{-1} A_4 & T_1^{-1} A_4 & T_1^{-1} A_{72} \\ A_4 T_1^{-1} & 0 & 0 & 0 \\ A_{71} T_1^{-1} & 0 & 0 & 0 \\ A_{72} T_1^{-1} & 0 & 0 & 0 \end{pmatrix} U^T$$

$$\hat{C} \hat{A}^\dagger = U \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon U \begin{pmatrix} 0 & T_1^{-1} A_4 & T_1^{-1} A_4 & T_1^{-1} A_{72} \\ A_4 T_1^{-1} & 0 & 0 & 0 \\ A_{71} T_1^{-1} & 0 & 0 & 0 \\ A_{72} T_1^{-1} & 0 & 0 & 0 \end{pmatrix} U^T.$$  (3.16)

It follows that $\hat{A}^\dagger \hat{A} = \hat{A}^\dagger \hat{C}$ and $\hat{A} \hat{A}^\dagger = \hat{C} \hat{A}^\dagger$, that is, $\hat{A} \hat{B} \leq \hat{C}$. Therefore, the transitivity of D-star order holds.

**Theorem 3.4.** Let DMPGIs of $\hat{A}$ and $\hat{B}$ exist. Then $\hat{A} \leq \hat{B}$ if and only if $\hat{A}^\dagger \leq \hat{B}^\dagger$. 

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Proof. Let \( \hat{A} = A + \varepsilon A_0 \); \( \hat{B} = B + \varepsilon B_0 \); DMPGIs of \( \hat{A} \) and \( \hat{B} \) exist; \( \hat{A} \leq \hat{B} \). Then \( \hat{A} \) and \( \hat{B} \) are of the forms as in (3.3). So \( \hat{A} \) can be represented in the form as in (3.5), and
\[
\hat{B} = V \begin{pmatrix} T_{1}^{-1} & 0 & 0 \\ 0 & T_{2}^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} U^{T} + \varepsilon V \begin{pmatrix} \frac{-T_{1}^{-1} A_{1} T_{1}^{-1}}{T_{1}^{-1} A_{1} T_{1}^{-1} - T_{2}^{-1} A_{2} T_{2}^{-1} + T_{1}^{-2} A_{1}^{2} T_{1}^{-1}} & \frac{T_{1}^{-2} A_{1}^{2} T_{1}^{-1} - T_{2}^{-1} A_{2} T_{2}^{-1} + T_{1}^{-2} A_{1}^{2} T_{1}^{-1}}{T_{1}^{-2} A_{1}^{2} T_{1}^{-1} - T_{2}^{-1} A_{2} T_{2}^{-1} + T_{1}^{-2} A_{1}^{2} T_{1}^{-1}} \\ A_{2}^{2} T_{1}^{-2} - T_{2}^{-1} A_{4} T_{1}^{-1} & -T_{2}^{-1} B_{5} T_{2}^{-1} + T_{1}^{-2} B_{6} T_{2}^{-1} \end{pmatrix} U^{T}.
\]
(3.17)

Since \( (\hat{A})^{\dagger} = \hat{A} \), by applying (3.3), (3.5) and (3.17), we get that
\[
(\hat{A})^{\dagger} \hat{A} = (\hat{A})^{\dagger} \hat{B}^{\dagger}, \quad \hat{A}^{\dagger}(\hat{A})^{\dagger} = \hat{B}^{\dagger}(\hat{A})^{\dagger}.
\]
It follows from Definition 3.1 that \( \hat{A}^{\dagger} \leq \hat{B}^{\dagger} \).

\[\leq \quad \text{When } \hat{A} \leq \hat{B}, \text{ it is obvious that } (\hat{A})^{\dagger} \leq (\hat{B})^{\dagger}. \text{ Since } (\hat{A})^{\dagger} = \hat{A} \text{ and } (\hat{B})^{\dagger} = \hat{B}, \text{ we get } \hat{A}^{\dagger} \leq \hat{B}. \]

If \( A \leq B \), we have \((B - A)^{\dagger} = B^{\dagger} - A^{\dagger}\) and \((B + A)^{\dagger} = B^{\dagger} - \frac{1}{2} A^{\dagger}\). But in \( \mathbb{D}^{m \times n} \), not all dual matrices have DMPGIs. Therefore, in the following theorem, we consider properties of \( \hat{B} + \hat{A} \) and \( \hat{B} - \hat{A} \) under the D-star partial order.

**Theorem 3.5.** Let DMPGIs of \( \hat{A} \) and \( \hat{B} \) exist; \( \hat{A} \leq \hat{B} \). Then the DMPGIs of \( \hat{B} + \hat{A} \) and \( \hat{B} - \hat{A} \) exist, and
\[
(\hat{B} + \hat{A})^{\dagger} = \hat{B}^{\dagger} - \frac{1}{2} \hat{A}^{\dagger}, \quad (\hat{B} - \hat{A})^{\dagger} = \hat{B}^{\dagger} - \hat{A}^{\dagger}.
\]
(3.18)

Proof. Since \( \hat{A} \leq \hat{B} \), the DMPGIs of \( \hat{A} \) and \( \hat{B} \) exist. Applying Theorem 3.2 we obtain
\[
\begin{align*}
\hat{B} + \hat{A} &= U \begin{pmatrix} 2T_{1} & 0 & 0 \\ 0 & T_{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} V^{T} + \varepsilon U \begin{pmatrix} 2A_{1} & A_{2} - T_{1}^{-1} A_{1}^{2} & 2A_{3} \\ 2A_{4} - T_{2} A_{2} T_{1}^{-1} & B_{5} & B_{6} \\ 2A_{7} & B_{8} & 0 \end{pmatrix} V^{T} \\
\hat{B} - \hat{A} &= U \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} V^{T} + \varepsilon U \begin{pmatrix} 0 & -T_{1}^{-1} A_{1}^{2} & 0 \\ -T_{2} A_{2} T_{1}^{-1} & B_{5} & B_{6} \\ 0 & B_{8} & 0 \end{pmatrix} V^{T}.
\end{align*}
\]
It follows from Theorem 2.3 that DMPGIs of $\hat{B} + \hat{A}$ and $\hat{B} + \hat{A}$ exist, and
\[
\left(\hat{B} + \hat{A}\right)^T = V \begin{pmatrix}
\frac{1}{2}T_1^{-1} & 0 & 0 \\
0 & T_2^{-1} & 0 \\
0 & 0 & 0
\end{pmatrix} U^T
\]
\[
+ \varepsilon V \begin{pmatrix}
-\frac{1}{2}A_2^T T_1^{-1} A_1 T_1^{-1} & \frac{1}{2}T_1^{-1} A_4^T T_1^{-1} & \frac{1}{2}T_1^{-2} A_4^T \\
\frac{1}{2}A_2^T T_1^{-2} & -T_2^{-1} B_4 T_2^{-1} & T_2^{-2} B_6^T \\
T_2^{-1} & B_6^T & 0
\end{pmatrix} U^T
\]
\[
\left(\hat{B} - \hat{A}\right)^T = V \begin{pmatrix}
0 & 0 & 0 \\
T_2^{-1} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} U^T + \varepsilon V \begin{pmatrix}
0 & 0 & -T_1^{-1} A_2 T_2^{-1} \\
0 & -T_2^{-1} B_4 T_2^{-1} & T_2^{-1} B_6^T \\
0 & 0 & B_6^T
\end{pmatrix} U^T.
\]
Furthermore, applying (3.19) and (3.20), we get (3.18).

It is well known that if $A^T A = A^T B$ and $A A^T = B A^T$, then $A$ is below $B$ under the star partial order. Now, by using the method that is similar to the dual star partial order, we introduce the T-star order. Let $\hat{A}, \hat{B} \in \mathbb{D}^{m \times n}$, if $\hat{A}, \hat{B}$ satisfy
\[
\hat{A}^T \hat{A} = \hat{A}^T \hat{B}, \quad \hat{A} \hat{A}^T = \hat{B} \hat{B}^T,
\]
we say that $\hat{A}$ is below $\hat{B}$ under the T-star order, and denote it by $\hat{A} \leq T \hat{B}$.

Since any dual matrices can do transpose operation, we suppose $\hat{A} = \varepsilon$ and $\hat{B} = 2\varepsilon$. It is easy to check that $\hat{A}^T \hat{A} = 0 = \hat{A}^T \hat{B}$ and $\hat{A} \hat{A}^T = 0 = \hat{B} \hat{A}^T$, i.e., $\hat{A} \leq T \hat{B}$. Since $\hat{B}^T \hat{B} = 0 = \hat{B}^T \hat{A}$ and $\hat{B} \hat{B}^T = \hat{A} \hat{B}^T$, we have $\hat{B} \leq T \hat{A}$. Because $\hat{A} \neq \hat{B}$, T-star order is not anti-symmetric. Therefore, T-star order is not a partial order.

Next, in the following theorem, we suppose that DMPGIs of dual matrices exist, and consider the relations between D-star partial order and T-star order.

**Theorem 3.6.** Let $\hat{A}, \hat{B} \in \mathbb{D}^{m \times n}$, and DMPGIs of $\hat{A}$ and $\hat{B}$ exist. We get $\hat{A} \leq T \hat{B}$ if and only if $\hat{A} \leq T \hat{B}$.

**Proof.** \(\leq T\) Let $\hat{A} = A + \varepsilon A_0$ and $\hat{B} = B + \varepsilon B_0$. It is easy to check that $\hat{A} \leq T \hat{B}$ if and only if
\[
\begin{cases}
A \leq B \\
A^T A_0 + A_0^T A = A^T B_0 + A_0^T B, \quad A A_0^T + A_0 A^T = B A_0^T + B_0 A^T.
\end{cases}
\]
(3.20a)

Since $A \leq B$ and DMPGIs of $\hat{A}, \hat{B}$ exist, then $A, B, A_0$ and $B_0$ can be represented in the forms as in (2.14) and (3.11), respectively.
Applying $\hat{A}^T \hat{A} = \hat{A}^T \hat{B}$ gives
\[
V \begin{pmatrix}
T_1^2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} V^T + \varepsilon V \begin{pmatrix}
T_1 A_1 + A_1^T T_1 & T_1 A_2 & T_1 A_3 \\
A_2^T T_1 & 0 & 0 \\
A_3^T T_1 & 0 & 0
\end{pmatrix} V^T
\]

Then
\[
= V \begin{pmatrix}
T_1^2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} V^T + \varepsilon V \begin{pmatrix}
T_1 B_1 + A_1^T T_1 & T_1 B_2 + A_1^T T_2 & T_1 B_3 \\
A_2^T T_1 & 0 & 0 \\
A_3^T T_1 & 0 & 0
\end{pmatrix} V^T.
\]

It follows from (3.6) that
\[
\hat{B} = U \begin{pmatrix}
T_1 & 0 & 0 \\
0 & T_2 & 0 \\
0 & 0 & 0
\end{pmatrix} V^T + \varepsilon U \begin{pmatrix}
A_1 & A_2 - T_1^{-1} A_1^T T_2 & A_3 \\
A_4 T_1 & 0 & 0 \\
A_7 T_1 & 0 & 0
\end{pmatrix} V^T. \tag{3.21}
\]

Since $\hat{A} \hat{A}^T = \hat{B} \hat{A}^T$ and (3.24), we get
\[
U \begin{pmatrix}
T_1^2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} U^T + \varepsilon U \begin{pmatrix}
T_1 A_1^T + A_1 T_1 & T_1 A_2^T & T_1 A_3^T \\
A_4 T_1 & 0 & 0 \\
A_7 T_1 & 0 & 0
\end{pmatrix} U^T
\]

Then
\[
\begin{align*}
A_4 T_1 &= T_1 A_2^T + B_4 T_1 \\
A_7 T_1 &= B_7 T_1
\end{align*}
\]

It follows from (3.21) that
\[
\hat{B} = U \begin{pmatrix}
T_1 & 0 & 0 \\
0 & T_2 & 0 \\
0 & 0 & 0
\end{pmatrix} V^T + \varepsilon U \begin{pmatrix}
A_1 & A_2 - T_1^{-1} A_1^T T_2 & A_3 \\
A_4 - T_2 A_2^T T_1^{-1} & B_5 & B_6 \\
A_7 & B_8 & 0
\end{pmatrix} V^T. \tag{3.22}
\]
Applying (3.22) and Theorem 3.2, we obtain $\hat{A} \leq \hat{B}$.

$\Rightarrow$ Let $\hat{A} \leq \hat{B}$, by applying Theorem 3.2, and it is easy to check that $\hat{A}^T \hat{A} = \hat{A}^T \hat{B}$ and $\hat{A} \hat{A}^T = \hat{B} \hat{A}^T$, that is, $\hat{A}^{\ast} \leq \hat{B}$. $\square$

4. P-Star Partial Order

In Section 4, we introduce the D-star order and show that it is a partial order. By using the method similar to the D-star order as in (3.1), we introduce the P-order by using MPDGI in this section.

Let $\hat{A} = A + \varepsilon A_0$ and $\hat{B} = B + \varepsilon B_0$. If

$$\hat{A}^p \hat{A} = \hat{A}^p \hat{B} \quad \text{and} \quad \hat{A} \hat{A}^p = \hat{B} \hat{A}^p,$$  \tag{4.1}

we say that $\hat{A}$ is below $\hat{B}$ under the P-order, and if so, we write $\hat{A} \leq \hat{B}$.

It is well known that the existence of DMPGI of dual matrix needs strict conditions, but the MPDGI that is closely related to DMPGI always exists for arbitrary dual matrix. Therefore, the new binary relation is different from the D-star order. It is meaningful to introduce P-order and discuss its properties and characterizations.

**Theorem 4.1.** Let $\hat{A} = A + \varepsilon A_0$; $\hat{B} = B + \varepsilon B_0$; DMPGIs of $\hat{A}$ and $\hat{B}$ exist, then $\hat{A} \leq \hat{B}$ if and only if

$$\begin{cases} A \preceq B \\ A^\dagger A_0 - A^\dagger A_0 A^\dagger A = A^\dagger B_0 - A^\dagger A_0 A^\dagger B \\ -A A^\dagger A_0 A^\dagger + A_0 A^\dagger = -B A^\dagger A_0 A^\dagger + B_0 A^\dagger. \end{cases}$$  \tag{4.2}

**Proof.** Denote $\hat{A}^p = A^\dagger + \varepsilon R_p$, where $R_p = -A^\dagger A_0 A^\dagger$, then we have

$$\begin{cases} \hat{A}^p \hat{A} = (A^\dagger + \varepsilon R_p) (A + \varepsilon A_0) = A^\dagger A + \varepsilon \left( A^\dagger A_0 + R_p A \right) \\ \hat{A}^p \hat{B} = (A^\dagger + \varepsilon R_p) (B + \varepsilon B_0) = A^\dagger B + \varepsilon \left( A^\dagger B_0 + R_p B \right) \end{cases}$$  \tag{4.3}

and

$$\begin{cases} \hat{A} \hat{A}^p = (A + \varepsilon A_0) (A^\dagger + \varepsilon R_p) = A A^\dagger + \varepsilon \left( A R_p + A_0 A^\dagger \right) \\ \hat{B} \hat{A}^p = (B + \varepsilon B_0) (A^\dagger + \varepsilon R_p) = B A^\dagger + \varepsilon \left( B R_p + B_0 A^\dagger \right). \end{cases}$$  \tag{4.4}
Applying (4.3) and (4.4), we get that (4.1) is equivalent to

\[
\begin{align*}
A^\dagger A &= A^\dagger B, \quad AA^\dagger = BA^\dagger \\
A^\dagger A_0 + R_p A &= A^\dagger B_0 + R_p B, \quad AR_p + A_0 A^\dagger = BR_p + B_0 A^\dagger.
\end{align*}
\]

(4.5)

So, we have (4.2).

\[\Box\]

**Theorem 4.2.** Let \( \hat{A} \leq \hat{B} \). Then there exist orthogonal matrices \( U \) and \( V \) such that

\[
\begin{align*}
\hat{A} &= U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{pmatrix} V^T \\
\hat{B} &= U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & B_5 & B_6 \\ A_7 & B_8 & B_9 \end{pmatrix} V^T,
\end{align*}
\]

(4.6)

where \( T_1 \) and \( T_2 \) are diagonal positive definite matrices.

**Proof.** \( \Rightarrow \) Let \( \hat{A} \leq \hat{B} \). Then applying Theorem 2.2, we get \( A \) and \( B \) are of forms as in (2.1). Denote

\[
\hat{A} = U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{pmatrix} V^T.
\]

(4.7)

We have

\[
\hat{A}^p = V \begin{pmatrix} T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon V \begin{pmatrix} -T_1^{-1} A_1 T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T.
\]

(4.8)

Correspondingly, partition matrix \( U^T B_0 V \) as follows

\[
U^T B_0 V = \begin{pmatrix} B_1 & B_2 & B_3 \\ B_4 & B_5 & B_6 \\ B_7 & B_8 & B_9 \end{pmatrix}.
\]

(4.9)
Applying Theorem 4.2, we have

\[
\hat{A}^p \hat{A} = V \begin{pmatrix}
I & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} V^T + \varepsilon V \begin{pmatrix}
0 & T_1^{-1} A_2 & T_1^{-1} A_3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} V^T,
\]

(4.10)

\[
\hat{A}^p \hat{B} = V \begin{pmatrix}
I & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} V^T + \varepsilon V \begin{pmatrix}
T_1^{-1} B_1 - T_1^{-1} A_1 & T_1^{-1} B_2 & T_1^{-1} B_3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} V^T.
\]

Since \( \hat{A}^p \hat{A} = \hat{A}^p \hat{B} \), we obtain

\[
B_1 = A_1, \quad B_2 = A_2 \quad \text{and} \quad B_3 = A_3. \tag{4.11}
\]

Similarly, by applying \( \hat{A} \hat{A}^p = B \hat{A}^p \), \( B_1 = A_1 \) and

\[
\hat{A} \hat{A}^p = U \begin{pmatrix}
I & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} U^T + \varepsilon U \begin{pmatrix}
0 & 0 & 0 \\
A_4 T_1^{-1} & 0 & 0 \\
A_7 T_1^{-1} & 0 & 0
\end{pmatrix} U^T,
\]

(4.12)

\[
\hat{B} \hat{A}^p = U \begin{pmatrix}
I & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} U^T + \varepsilon U \begin{pmatrix}
0 & 0 & 0 \\
B_4 T_1^{-1} & 0 & 0 \\
B_7 T_1^{-1} & 0 & 0
\end{pmatrix} U^T,
\]

we obtain

\[
B_4 = A_4 \quad \text{and} \quad B_7 = A_7. \tag{4.13}
\]

Therefore, applying \( \hat{A} \), \( \hat{B} \) and \( \hat{A} \hat{B} \) gives \( \hat{A} \neq \hat{B} \). \( \square \)

**Example 4.1.** Let

\[
\hat{A} = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} + \varepsilon \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \quad \hat{B} = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} + \varepsilon \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

then

\[
\hat{A}^p = \hat{B}^p = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} + \varepsilon \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}.
\]

Applying Theorem 4.2, we have \( \hat{A} \leq \hat{B} \) and \( \hat{B} \leq \hat{A} \). Since \( \hat{A} \neq \hat{B} \), we get that the binary relation is not anti-symmetric. Therefore, the P-order is not a partial order.
**Theorem 4.3.** Let DMPGIs of \( \hat{A} \in \mathbb{D}^{m \times n} \) and \( \hat{B} \in \mathbb{D}^{m \times n} \) exist. Denote

\[
\hat{A} \leq P \hat{B} : \hat{A}^p \hat{A} = \hat{A}^p \hat{B} \quad \text{and} \quad \hat{A} \hat{A}^p = \hat{B} \hat{A}^p.
\]

We call it the P-star order. It is a partial order.

Furthermore, there exist \( U \) and \( V \) that are orthogonal matrices such that

\[
\begin{align*}
\hat{A} &= U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & 0 & 0 \\ A_7 & 0 & 0 \end{pmatrix} V^T, \\
\hat{B} &= U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & B_5 & B_6 \\ A_7 & B_8 & 0 \end{pmatrix} V^T,
\end{align*}
\]

where \( T_1 \) and \( T_2 \) are diagonal positive definite matrices.

**Proof.** Let \( \hat{A} = A + \varepsilon A_0, \hat{B} = B + \varepsilon B_0 \in \mathbb{D}^{m \times n} \). Since \( \hat{A}^p \hat{A} = \hat{A}^p \hat{B} \), \( \hat{A} \hat{A}^p = \hat{B} \hat{A}^p \) and DMPGIs of \( \hat{A} \) and \( \hat{B} \) exist, by applying Theorem 2.3 and Theorem 4.2, we get that \( \hat{A} \) and \( \hat{B} \) are of the forms as in (4.14).

Next, we show that P-star order satisfies reflexivity, the anti-symmetry, transitivity, respectively.

1) Since \( \hat{A} \leq \hat{A} \), reflexivity is self-evident.

2) Let \( \hat{A} \leq \hat{B} \) and \( \hat{B} \leq \hat{A} \). Hence, we get \( A \leq B \) and \( B \leq A \). Since \( \leq \) is a partial order, i.e., \( A = B \). Let \( \hat{A} \) and \( \hat{B} \) be of the form as in (4.14). Then \( T_2 = 0 \) and

\[
\hat{B} = U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & B_5 & B_6 \\ A_7 & B_8 & 0 \end{pmatrix} V^T.
\]

Since the DMPGI of \( \hat{B} \) exists, by using Theorem 2.3, we have \( (I_n - BB^\dagger) B_0 (I_n - B^\dagger B) = 0 \). It follows that \( B_5 = 0, B_6 = 0 \) and \( B_8 = 0 \). Therefore, \( \hat{A} = \hat{B} \). So, the anti-symmetry holds.

3) Next, we check the transitivity.
Since $\hat{A} \leq \hat{B}$, then $\hat{A}$ and $\hat{B}$ are of the form as in (4.14). Therefore, we obtain

$$
\hat{A}^p \hat{A} = V \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon V \begin{pmatrix} 0 & T_1^{-1}A_2 & T_1^{-1}A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T
$$

and

$$
\hat{B}^p \hat{B} = V \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon V \begin{pmatrix} 0 & 0 & T_1^{-1}A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T,
$$

(4.16)

Applying (4.16) and (4.17) gives

$$
\hat{A}^p \hat{A} \hat{B}^p \hat{B} = \hat{A}^p \hat{A} \quad \text{and} \quad \hat{A}^p \hat{B} \hat{B}^p = \hat{A}^p.
$$

(4.18)

In the same way, we get

$$
\hat{A}^p \hat{A} = V \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon U \begin{pmatrix} 0 & 0 & 0 \\ A_4T_1^{-1} & 0 & 0 \\ A_7T_1^{-1} & 0 & 0 \end{pmatrix} U^T
$$

and

$$
\hat{B}^p \hat{B} = U \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon U \begin{pmatrix} A_4T_1^{-1} & 0 & 0 \\ A_7T_1^{-1} & 0 & 0 \end{pmatrix} U^T,
$$

(4.19)
Proof. Applying (4.19) and (4.20) gives

\[
\begin{align*}
\hat{B}^p \hat{A}^p &= U \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon U \begin{pmatrix} 0 & 0 & 0 \\ A_4 T_1^{-1} & 0 & 0 \\ A_7 T_1^{-1} & 0 & 0 \end{pmatrix} U^T \\
\hat{B}^p \hat{A}^p &= V \begin{pmatrix} T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon V \begin{pmatrix} -T_1^{-1} A_1 T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T.
\end{align*}
\]

(4.20)

Applying (4.19) and (4.20) gives

\[
\hat{B}^p \hat{A}^p = \hat{A}^p \hat{A}^p \quad \text{and} \quad \hat{B}^p \hat{A}^p = \hat{A}^p.
\]

(4.21)

Let \( \hat{B} \leq \hat{C} \), that is, \( \hat{B}^p \hat{B} = \hat{B}^p \hat{C} \) and \( \hat{B}^p \hat{B} = \hat{C} \hat{B}^p \). Then

\[
\hat{A}^p \hat{A}^p \hat{B} = \hat{A}^p \hat{B}^p \hat{C} \quad \text{and} \quad \hat{B}^p \hat{A}^p = \hat{C} \hat{B}^p \hat{A}^p,
\]

It follows from (4.18) and (4.21) that

\[
\hat{A}^p \hat{A} = \hat{A}^p \hat{C} \quad \text{and} \quad \hat{A}^p \hat{A} = \hat{C} \hat{A}^p,
\]

(4.22)

that is, \( \hat{A} \leq \hat{C} \). Therefore, the transitivity holds.

\[
\square
\]

**Theorem 4.4.** Let \( \hat{A} = A + \varepsilon A_0; \hat{B} = B + \varepsilon B_0 \) be DMPGIs of \( \tilde{A} \) and \( \tilde{B} \) exist. Then \( \hat{A} \leq \hat{B} \) if and only if

\[
\begin{align*}
A^T A &= A^T B, \quad A A^T = B A^T \\
A^T A_0 &= A^T B_0, \quad A_0 A^T = B_0 A^T
\end{align*}
\]

(4.23)

**Proof.** \( \Rightarrow \) Since DMPGIs of \( \tilde{A} \) and \( \tilde{B} \) exist and \( \hat{A} \leq \hat{B} \), by using Theorem 4.3, we have

\[
\begin{align*}
A^T A = A^T B &= V \begin{pmatrix} T_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T, \quad A A^T = B A^T = U \begin{pmatrix} T_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T \\
A^T A_0 = A^T B_0 &= V \begin{pmatrix} T_1 A_1 & T_1 A_2 & T_1 A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T, \quad A_0 A^T = B_0 A^T = U \begin{pmatrix} A_1 T_1 & 0 & 0 \\ A_4 T_1 & 0 & 0 \\ A_7 T_1 & 0 & 0 \end{pmatrix} U^T.
\end{align*}
\]
It follows that $A^T A_0 = A^T B_0$ and $A_0 A^T = B_0 A^T$. Therefore, we get (4.23).

$\iff$ Since $A^T A = A^T B$ and $A A^T = B A^T$, we obtain $A \preceq B$. Then applying Theorem 2.2, we get the forms of $A$ and $B$ as in (2.1).

Since DMPGIs of $\hat{A}$ and $\hat{B}$ exist, we write

$$A_0 = U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & 0 & 0 \\ A_7 & 0 & 0 \end{pmatrix} V^T, \quad B_0 = U \begin{pmatrix} B_1 & B_3 \\ B_4 & B_5 \\ B_7 & B_8 \end{pmatrix} V^T.$$

Furthermore, applying $A^T A_0 = A^T B_0$ and $A_0 A^T = B_0 A^T$, we get $B_0 = U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & B_5 & B_6 \\ A_7 & B_8 & 0 \end{pmatrix} V^T$.

By applying Theorem 4.3, it follows that $\hat{A} \preceq \hat{B}$. \qed

5. Relations among dual matrix partial orders

From Theorem 2.3 and Theorem 2.4 we see that DMPGI and MPDGI are closely related in form. Therefore, D-star order induced by DMPGI and P-star order induced by MPDGI are also highly similar in form. This is what the partial order in the real field does not have. In this section, we consider relationships among various types of partial orders of dual matrices. These relations will provide motivation for our follow-up research on matrix partial order theory in the real field.

From the discussion in the above sections, we can see that the discussion on P-star partial order and D-star partial order is under the condition of the existence of DMPGI. Therefore, we suppose that the DMPGI of dual matrix discussed in this section exists. Since the DMPGI of $\hat{A}$ exists, the DMPGI of $\hat{A}^T$ and $\hat{A}^p$ exists. Therefore, we will not explain the existence of DMPGI one by one later.

When $\hat{A}^T = \hat{A}^p$, the D-star partial order is equivalent to the P-star partial order. However, in general, these two kinds of partial orders are not equivalent. Here are some examples:

**Example 5.1.** Let

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$
Applying Theorem 3.2 and Theorem 4.3, we have

(1). \( \hat{A} \leq \hat{B} \), and \( \hat{A} \) is not below \( \hat{B} \) under the partial order \( \leq \);

(2). \( \hat{A} \leq \hat{C} \), and \( \hat{A} \) is not below \( \hat{C} \) under the partial order \( \leq \).

In the following theorems, we further discuss relationship between D-star partial order and the P-star partial order. First, we consider characterizations of \( \hat{A} \leq \hat{B} \) when \( \hat{A} \leq \hat{B} \) holds.

**Theorem 5.1.** Let DMPGi of \( \hat{A} = A + \varepsilon A_0 \) and \( \hat{B} = B + \varepsilon B_0 \) exist; \( \hat{A} \leq \hat{B} \). Then the following conditions are equivalent:

1. \( \hat{A} \leq \hat{B} \);
2. There exist orthogonal matrices \( U \) and \( V \) such that
   \[
   \begin{align*}
   \hat{A} &= U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & 0 & A_3 \\ 0 & 0 & 0 \\ A_7 & 0 & 0 \end{pmatrix} V^T, \\
   \hat{B} &= U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & 0 & A_3 \\ 0 & B_5 & B_6 \\ A_7 & B_8 & 0 \end{pmatrix} V^T,
   \end{align*}
   \]
   \( (5.1) \)

where \( T_1 \) and \( T_2 \) are diagonal positive definite matrices;

3. \( A_0^T A = A_0^T B \) and \( AA_0^T = BA_0^T \);

4. \( \hat{A}! \leq \hat{B}! \).

**Proof.** Since DMPGi of \( \hat{A}, \hat{B} \) exist, and \( \hat{A} \leq \hat{B} \), by applying Theorem 3.2, we get the decomposition of \( \hat{A} \) and \( \hat{B} \) as in (3.3). Since DMPGi of \( \hat{A}, \hat{B} \) exist and \( \hat{A} \leq \hat{B} \), \( \hat{A} \) and \( \hat{B} \) have the forms as in (4.14).

(1) \( \Rightarrow \) (2): When \( \hat{A} \leq \hat{B} \) and \( \hat{A} \leq \hat{B} \), applying (3.3) and (4.14), we get

\[
A_2 - T_1^{-1} A_4^T T_2 = A_2 \quad \text{and} \quad A_4 - T_2 A_2^T T_1^{-1} = A_4.
\]

Since \( T_1 \) and \( T_2 \) are invertible, \( A_4 = 0 \) and \( A_2 = 0 \). It follows from (4.14) that we get (2).

(2) \( \Rightarrow \) (1): When \( \hat{A} \leq \hat{B} \) and \( A_4 = 0 \) and \( A_2 = 0 \), applying (3.3) and (4.14), it is easy to check that \( \hat{A} \leq \hat{B} \).
It follows from (4.14) that we get (2).

(2) ⇒ (3): Applying \(5.41\) gives

\[
\begin{align*}
A_0 \left( B^T - A^T \right) &= U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & 0 & 0 \\ A_7 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T = 0 \\
(B^T - A^T) A_0 &= U \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 & 0 & A_3 \\ 0 & 0 & 0 \\ A_7 & 0 & 0 \end{pmatrix} U^T = 0,
\end{align*}
\]

that is, \(A_0^T A = A_0^T B\) and \(AA_0^T = BA_0^T\).

(3) ⇒ (2): Since the decompositions of \(\hat{\mathbf{A}}\) and \(\hat{\mathbf{B}}\) are as in \(5.23\), \(A_0^T A = A_0^T B\) and \(AA_0^T = BA_0^T\), we have

\[
\begin{align*}
A_0 \left( B^T - A^T \right) &= U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & 0 & 0 \\ A_7 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T = U \begin{pmatrix} 0 & A_2T_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \\
(B^T - A^T) A_0 &= U \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & 0 \\ A_7 & 0 & 0 \end{pmatrix} U^T = U \begin{pmatrix} T_2A_4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T = 0.
\end{align*}
\]

Therefore, \(A_2 = 0\) and \(A_4 = 0\). It follows from \(5.29\) that we get (2).

(4) ⇒ (2) Since DMPGIs of \(\hat{A}\) and \(\hat{B}\) exist and \(\hat{A} \leq \hat{B}\), \(\hat{\mathbf{A}}\) and \(\hat{\mathbf{B}}\) have the forms as in \(6.19\). Then the forms of \(\hat{\mathbf{A}}^t\) and \(\hat{\mathbf{B}}^t\) are as in \(6.23\) and \(6.27\), respectively. Applying Theorem \(6.38\) we have

\[
\begin{align*}
T_1^{-2}A_1^t - T_2^{-2}A_2^t &= T_1^{-2}A_4^t \\
A_2^tT_1^{-2} - A_4^{-2}T_1^{-1} &= A_2^{-2}T_1^{-2}
\end{align*}
\]

(2) ⇒ (4) Applying \(\hat{\mathbf{A}} \leq \hat{\mathbf{B}}\), \(A_2 = 0\), \(A_4 = 0\), \(6.23\) and \(6.27\), we get

\[
\begin{align*}
\hat{\mathbf{A}}^t &= V \begin{pmatrix} T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon V \begin{pmatrix} -T_1^{-1}A_1T_1^{-1} & 0 & T_1^{-2}A_4^t \\ 0 & 0 & 0 \\ A_2^tT_1^{-2} & 0 & 0 \end{pmatrix} U^T \\
\hat{\mathbf{B}}^t &= V \begin{pmatrix} T_1^{-1} & 0 & 0 \\ 0 & T_2^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon V \begin{pmatrix} -T_1^{-1}A_1T_1^{-1} & 0 & T_1^{-2}A_4^t \\ 0 & 0 & 0 \\ A_2^tT_1^{-2} & 0 & 0 \end{pmatrix} U^T.
\end{align*}
\]
It follows from Theorem 4.3 that $\hat{A}^R \leq \hat{B}^R$. \hfill \qed

In Theorem 5.1 we see that $\hat{A}^D \leq \hat{B}$ if and only if $\hat{A}^R \leq \hat{B}^R$. Conversely, the same is not true for the relationship between $\hat{A}^R \leq \hat{B}$ and $\hat{A}^P \leq \hat{B}^P$. Let

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 7 & 0 & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 7 & -3 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & 2 \\ 7 & -3 & 0 \end{pmatrix}.$$  

Then

$$\hat{A}^P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{B}^P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} -1 & -1 & 0 \\ -2 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

Applying Theorem 4.3 we have $\hat{A}^P \leq \hat{B}$.

Since

$$(\hat{A}^P)^P \hat{A}^P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\hat{A}^P)^P \hat{B}^P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we get that $(\hat{A}^P)^P \hat{A}^P \neq (\hat{A}^P)^P \hat{B}^P$. Therefore, $\hat{A}^P$ is not below $\hat{B}^P$ under the P-star partial order.

In the following theorem we consider characterizations of $\hat{A}^P \leq \hat{B}^P$, when $\hat{A} \leq \hat{B}$ holds.

**Theorem 5.2.** Let DMPGIs of $\hat{A} = A + \varepsilon A_0$ and $\hat{B} = B + \varepsilon B_0$ exist, and $\hat{A}^D \leq \hat{B}$. Then the following conditions are equivalent:

1. $\hat{A}^D \leq \hat{B}$;
2. $\hat{A}^P \leq \hat{B}^P$;
3. $\hat{A}$ and $\hat{B}$ can be represented in the forms as in (5.1);
4. $A^T_0 A = A^T_0 B$ and $A A^T_0 = B A^T_0$.

**Proof.** Since the DMPGIs of $\hat{A}$ and $\hat{B}$ exist, and $\hat{A}^D \leq \hat{B}$, by applying Theorem 4.3, we get that $\hat{A}$ and $\hat{B}$ have the forms as in (4.14).

(1) $\Rightarrow$ (3): Since $\hat{A} \leq \hat{B}$, applying Theorem 3.2 and $\hat{A}^D \leq \hat{B}$, we get $A_2 - T_1^{-1} A_2^T T_2 = A_2$ and $A_4 - T_2 A_2^T T_1^{-1} = A_4$. Therefore, we get (3).

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Thereom 4.3 gives (3).

Applying Theorem 4.3, we obtain Theorem 5.3.

Applying Theorem 4.4, Theorem 5.1 and Theorem 5.2, we obtain Theorem 5.3.

Let

\[
\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & -6 & 3 \\ 0 & -2 & -1 \\ 7 & -3 & 0 \end{pmatrix}.
\]

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Applying Theorem 3.2, it is easy to check that \( \hat{A}^{D*} \leq \hat{B} \). Since

\[
\hat{A}^p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{B}^p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

we get \( (\hat{A}^p)^{\dagger} \hat{A}^p \neq (\hat{A}^p)^{\dagger} \hat{B}^p \). It follows from Theorem 5.2 that \( \hat{A}^p \) is not below \( \hat{B}^p \) under the D-star partial order. Therefore, \( \hat{A}^{D*} \neq \hat{B}^{D*} \).

**Theorem 5.4.** Let DMPGIs of \( \hat{A} = A + \varepsilon A_0 \) and \( \hat{B} = B + \varepsilon B_0 \) exist. If \( \hat{A}^{D*} \leq \hat{B} \), then the following conditions are equivalent:

1. \( \hat{A}^{D*} \leq \hat{B}^p \);
2. There exist orthogonal matrices \( U \) and \( V \) such that
   \[
   \hat{A} = U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & 0 & 0 \\ A_7 & 0 & 0 \end{pmatrix} V^T,
   \]
   \[
   \hat{B} = U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & 0 & A_3 \\ 0 & B_5 & B_6 \\ A_7 & B_8 & 0 \end{pmatrix} V^T,
   \]
   where \( T_1 \) and \( T_2 \) are diagonal positive definite matrices.
3. \( BB_0^T A = AB_0^T B = AB_0^T A \);

**Proof.** Since \( \hat{A}^{D*} \leq \hat{B} \), by using Theorem 3.2, we get the \( \hat{A} \) and \( \hat{B} \) are of forms as in (3.3), and

\[
BB_0^T A = U \begin{pmatrix} T_1 A_1^T T_1 & 0 & 0 \\ 0 & T_2 A_4 - T_2 A_2^T T_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T, \quad AB_0^T A = U \begin{pmatrix} T_1 A_1^T T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T
\]

\[
AB_0^T B = U \begin{pmatrix} T_1 A_1^T T_1 & A_2 T_2^2 - T_1 A_1^T T_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T, \quad AB_0^T A = U \begin{pmatrix} T_1 A_1^T T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T.
\]
\[(1) \Rightarrow (2)\] Applying (5.3) and Theorem 2.3, we get
\[
\begin{align*}
\hat{A}p &= V \begin{pmatrix} T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon V \begin{pmatrix} -T_1^{-1}A_1T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T, \\
\hat{B}p &= V \begin{pmatrix} T_1^{-1} & 0 & 0 \\ 0 & T_2^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon V \begin{pmatrix} -T_1^{-1}A_1T_1^{-1} & T_1^{-2}A_1^T - T_1^{-1}A_2T_2^{-1} & 0 \\ T_2^{-1}A_1T_2^{-1} & -T_2^{-1}B_5T_2^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T.
\end{align*}
\]

Applying Theorem 3.2, we obtain \(T_1^{-2}A_1^T - T_1^{-1}A_2T_2^{-1} = 0\) and \(A_2^TT_1^{-2} - T_2^{-1}A_4T_1^{-1} = 0\), that is, \(A_2 - T_1^{-1}A_1^TA_2T_2 = 0\) and \(A_4 - T_2A_2^TT_1^{-1} = 0\). It froms (5.4) that (5.3) holds.

\[(2) \Rightarrow (1):\] Applying (5.3) gives
\[
\begin{align*}
\hat{A}p &= V \begin{pmatrix} T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon V \begin{pmatrix} -T_1^{-1}A_1T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T, \\
\hat{B}p &= V \begin{pmatrix} T_1^{-1} & 0 & 0 \\ 0 & T_2^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T + \varepsilon V \begin{pmatrix} -T_1^{-1}A_1T_1^{-1} & 0 & 0 \\ 0 & -T_2^{-1}B_5T_2^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T.
\end{align*}
\]

Then applying Theorem 3.2 gives \(A^p \leq \hat{B}p\).

\[(2) \Leftrightarrow (3):\] Applying (5.4), we obtain that \(BB_0^TA = AB_0^TA\) and \(AB_0^TB = AB_0^TA\) if and only if \(T_2A_4 - T_2A_2^TA_4 = 0\) and \(A_2T_2^2 - T_1A_1^TT_2 = 0\) if and only if \(A_2 - T_1^{-1}A_1^T = 0\) and \(A_4 - T_2A_2^TT_1^{-1} = 0\). Therefore, applying Theorem 3.2, we get (2) \(\Leftrightarrow\) (3).

Let
\[
\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \\ 7 & 0 & 0 \end{pmatrix},
\hat{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & -2 \\ 7 & -3 & 0 \end{pmatrix}.
\]

Applying Theorem 2.8, we can easily check that \(\hat{A} \leq \hat{B}\).

Applying Theorem 2.2, we have
\[
\begin{align*}
\hat{A}^l &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} -1 & 4 & 7 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix},
\hat{B}^l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} -1 & 3 & 7 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}.
\end{align*}
\]
Since 
\[
\left(\hat{A}^\dagger\right)^p \hat{A}^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 4 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \left(\hat{A}^\dagger\right)^p \hat{B}^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 3 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
we obtain \(\left(\hat{A}^\dagger\right)^p \hat{A}^\dagger \neq \left(\hat{A}^\dagger\right)^p \hat{B}^\dagger\). Therefore, \(\hat{A}^\dagger\) is not below \(\hat{B}^\dagger\) under the P-star partial order. Therefore, \(\hat{A} \leq \hat{B} \Rightarrow \hat{A}^\dagger \leq \hat{B}^\dagger\).

**Theorem 5.5.** Let DMPGIs of \(\hat{A} = A + \varepsilon A_0\) and \(\hat{B} = B + \varepsilon B_0\) exist. If \(\hat{A} \leq \hat{B}\), then the following conditions are equivalent:

1. \(\hat{A}^\dagger \leq \hat{B}^\dagger\);
2. There exist orthogonal matrices \(U\) and \(V\) such that

\[
\begin{cases}
\hat{A} = U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & 0 & 0 \\ A_7 & 0 & 0 \end{pmatrix} V^T \\
\hat{B} = U \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & B_5 & B_6 \\ A_7 & B_8 & 0 \end{pmatrix} V^T,
\end{cases}
\]

where \(T_1\) and \(T_2\) are diagonal positive definite matrices, \(A_4 T_1 + T_2 A_2^T = 0\) and \(A_4^T T_2 + T_1 A_2 = 0\).

**Proof.** Since the DMPGIs of \(\hat{A}\) and \(\hat{B}\) exist, and \(\hat{A} \leq \hat{B}\), by using Theorem 4.3, we get that \(\hat{A}\) and \(\hat{B}\) are as in (4.14), \(\hat{A}^\dagger\) is of the form as in (5.5), and

\[
\hat{B}^\dagger = U \begin{pmatrix} T_1^{-1} & 0 & 0 \\ 0 & T_2^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T + \varepsilon U \begin{pmatrix} -T_1^{-1} A_1 T_1^{-1} & -T_1^{-1} A_2 T_2^{-1} & T_1^{-2} A_1 \end{pmatrix} V^T. \tag{5.6}
\]

1. \(\Rightarrow\) (2) If \(\hat{A}^\dagger \leq \hat{B}^\dagger\), then applying Theorem 4.3, we get \(T_1^{-2} A_1^T + T_1^{-1} A_2 T_2^{-1} = 0\) and \(A_2^T T_1^{-2} + T_2^{-1} A_4 T_1^{-1} = 0\). Therefore, we get (2).

2. \(\Rightarrow\) (1) If \(\hat{A}\) and \(\hat{B}\) are given as in (5.5), it is easy to check that (1) holds.

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