Alternative potentials for the electromagnetic field

Shaun N. Mosley, Alumni, University of Nottingham NG7 2RD , England

e-mail: shaun.mosley@alumni.nottingham.ac.uk

Abstract

The electromagnetic field can be expressed in terms of two complex potentials \( \alpha, \beta \), which are related to the Debye potentials. The evolution equations for \( \alpha, \beta \) are derived, which are separable either in parabolic coordinates (leading to the radiation fields) or in radial coordinates (multipole fields). Potentials corresponding to focused wave fields as well as plane waves are discussed. A conserved radiation density can be constructed in terms of these potentials, which is positive (negative) for positive (negative) helicity radiation.

PACS nos. 4110 classical electromagnetism

4110H electromagnetic waves - theory

I. INTRODUCTION

The source-free electromagnetic field has only two degrees of freedom per space-time point. Some economy is achieved by reducing the six components of the electric and magnetic fields \( (E, B) \) to the usual 4 potentials \( (\Phi, A) \) which satisfy the covariant Lorentz condition \( \partial_t \Phi + \nabla \cdot A = 0 \), or the \( \Phi \) component may be dispensed with provided that the three potentials \( A \) satisfy the non-covariant Coulomb gauge \( \nabla \cdot A = 0 \). The most economical way to express the two degrees of freedom of the free electromagnetic field is in terms of the two real Debye potentials \( \psi_1, \psi_2 \). Here we introduce a pair of complex potentials \( \{\alpha, \beta\} \) from which the electromagnetic field can be derived via (8) below. Some advantages of using these potentials are: a conserved radiation density representing the difference of positive and negative helicity radiation can be constructed, they have accommodate singularities in the field (charged particles), and interesting focused wave solutions arise naturally when solving the evolution equations (equation (A7) below).

We first review the Debye potentials to which \( \{\alpha, \beta\} \) are related. The electric and magnetic fields are expressed in terms of the Debye potentials by

\[
\begin{bmatrix} E \\ B \end{bmatrix} = \begin{bmatrix} -(x \times \nabla) \partial_t \\ \nabla \times (x \times \nabla) \end{bmatrix} \psi_1 + \begin{bmatrix} -\nabla \times (x \times \nabla) \\ -(x \times \nabla) \partial_t \end{bmatrix} \psi_2
\]

where \( \psi \) are solutions of the wave equation

\[
(\partial_t^2 - \nabla^2) \psi = 0.
\]

The standard text-books may not refer to the Debye potentials by that name\(^3\), but use either \( \psi \) or \( L^2 \psi \) in their analysis of multipole radiation fields, where \( \psi \) are spherical solutions of the wave equation - see for example pp.432–433 of Reitz\(^4\), or pp.745-746 of Jackson\(^5\). The paper by Boukamp and Casimir\(^6\) discusses the various essentially equivalent approaches to multipole radiation, while Refs. 2,7 show that any field \( (E, B) \) outside the source region can be expressed in terms of the potentials \( \psi_1, \psi_2 \).
We express (1) more concisely by defining the complex field \( \mathbf{F} \equiv \mathbf{E} + i\mathbf{B} \), then
\[
\mathbf{F} = [-(\mathbf{x} \times \nabla)\partial_t + i\nabla \times (\mathbf{x} \times \nabla)] (\psi_1 + i\psi_2)
\]
where \( \mathbf{L} \equiv -i\mathbf{r} \times \nabla \) is the angular momentum operator and \( \psi \equiv -(\psi_1 + i\psi_2) \). Conversely
\[
\mathbf{r} \cdot \mathbf{F} = \mathbf{r} \cdot \nabla \times \mathbf{L}\psi = iL^2\psi
\]
and the operator \( L^2 \) is, in principle, invertible.\(^8\) The source-free Maxwell’s equations are
\[
[i\partial_t - \nabla \times ] \mathbf{F} = 0 \quad \nabla \cdot \mathbf{F} = 0 .
\]
(5\(a,b\))
It is readily verified that substituting (3) into the above yields an identity, given (2).

Despite the fact that the two Debye potentials (or the single complex Debye potential \( \psi \)) precisely contain the two degrees of freedom of the electromagnetic field, they are seldom used outside magnetostatics and multipole field analysis. Some disadvantages spring to mind:
(a) the field \( \mathbf{F} \equiv \mathbf{E} + i\mathbf{B} \) is expressed by second order differential operators acting on the potentials, so then for example the usual energy density of the field \( (\mathbf{F}^* \mathbf{F}) \) is a cumbersome expression in terms of \( \psi \),
(b) there is no regular closed form potential \( \psi \) representing a plane wave field. The standard texts\(^9\) express the plane wave potential as an infinite sum over \( l \) of the spherical harmonics \( Y_{l,\pm 1} \), although this actually represents plane waves outgoing in each direction from the \( (\theta = \pi/2) \) plane,
(c) the Debye potential formalism is ill suited to coping with point singularities representing charged particles in the field, i.e. there is no regular potential \( \psi \) representing the monopole field at the origin, as can be seen immediately from (4). See Ref.1 for irregular potentials \( \psi \) representing the monopole field.

We will now show that, given (3), the field \( \mathbf{F} \) can be expressed in terms of two complex potentials \( \alpha, \beta \) acted on by first order differential operators. These potentials \( \alpha, \beta \) admit charged particle solutions, for static and moving charges, and the potentials representing the plane wave fields are fairly simple.

From (3) we calculate the field \( (\mathbf{F} + i\mathbf{\hat{x}} \times \mathbf{F}) \) in terms of \( \psi \), where \( \mathbf{\hat{x}} \equiv \frac{\mathbf{x}}{r} \). We use the operator identities\(^2\)
\[
\nabla \times \mathbf{L} = (-i\mathbf{x} \nabla^2 + i\nabla(\partial_t \mathbf{r})) , \quad \mathbf{\hat{x}} \times \mathbf{L} = i(\nabla \mathbf{r} - \partial_t \mathbf{x}) , \quad \mathbf{\hat{x}} \times (\nabla \times \mathbf{L}) = -\mathbf{L}(\partial_t \mathbf{r}) .
\]
Then
\[
\mathbf{F} + i\mathbf{\hat{x}} \times \mathbf{F} = [i\partial_t \mathbf{L} - i\mathbf{x} \nabla^2 + i\nabla(\partial_t \mathbf{r})] \psi + i[-(\nabla \mathbf{r} - \partial_t \mathbf{x})\partial_t - \mathbf{L}(\partial_t \mathbf{r})] \psi
\]
\[
\mathbf{F} + i\mathbf{\hat{x}} \times \mathbf{F} = [\nabla + \mathbf{\hat{x}} \partial_t - \frac{1}{r} \mathbf{L}] \alpha
\]
(6\(a\))
where
\[
\alpha = i(\partial_t \mathbf{r} \psi - \partial_t \psi \mathbf{r}) .
\]
(7\(a\))
To derive (6\(a\)) we have used (2). Similarly
\[
\mathbf{F} - i\mathbf{\hat{x}} \times \mathbf{F} = [\nabla - \mathbf{\hat{x}} \partial_t + \frac{1}{r} \mathbf{L}] \beta
\]
(6\(b\))
with
\[ \beta = i(\partial_r r \psi + \partial_t r \psi) . \]  

Note that given \((F + i\hat{x} \times F)\) there is no straightforward algebraic manipulation to derive \(F\) due to the fact that the matrix \((I + i\hat{x} \times )\) is singular. We add \((6a,b)\) obtaining

\[ 2F = [\nabla + \hat{x} \partial_t - \frac{1}{r} L] \alpha + [\nabla - \hat{x} \partial_t + \frac{1}{r} L] \beta , \]  

which is the basic formula expressing \(F\) in terms of \(\{\alpha, \beta\}\). Dotting \((6a,b)\) with \(\hat{x}\) yields the constraint

\[ (\partial_r + \partial_t) \alpha = F_r = (\partial_r - \partial_t) \beta \]  

where \(F_r \equiv \hat{x} \cdot F\). The operators \((\partial_r + \partial_t), (\partial_r - \partial_t)\) are the radial differential operators on the future, past lightcones respectively. Given the radial field \(F_r(t, x)\), then \((9)\) enables us to calculate \(\alpha, \beta\) as follows. To carry out the integration on the future (past) lightcone we introduce the advanced (retarded) times \(S\) and \(T\) defined by

\[ S = t - r , \quad T = t + r . \]  

The hypersurfaces \(S = S_0, S_1 \ldots\) are future lightcones centered at the origin, while the hypersurfaces \(T = T_0, T_1 \ldots\) are past lightcones. Then \(F_r(S)\) or \(F_r(T)\) are obtained by substituting \(t = S + r\) or \(t = T - r\) into the expression for \(F_r(t)\). A solution of \((9)\) is then

\[ \alpha(S) = \int_0^r d r F_r(S) , \quad \beta(T) = \int_0^r d r F_r(T) . \]

Maxwell’s equations can alternatively be formulated from the start using advanced or retarded time,\(^9\) then \(\alpha\) or \(\beta\) is found from \(F_r\) by a straightforward radial integration.

We now find the evolution equations for \(\alpha, \beta\), noting that \(\alpha = i(\partial_r r - \partial_t r) \psi\) does not satisfy the wave equation like \(\psi\), because the operator \((\partial_r r - \partial_t r)\) does not commute with the d’Alembertian operator \(\Box \equiv (\partial_r^2 - \nabla^2)\). The d’Alembertian may be expressed as follows,

\[ (\partial_t^2 - \nabla^2) \psi = \left[- \frac{1}{r^2}(\partial_r + \partial_t)(\partial_r - \partial_t)r + \frac{1}{r^2} L^2\right] \psi = 0 . \]  

Now multiply \((11)\) by \(\frac{1}{r^2}(\partial_r - \partial_t)r^3\) obtaining

\[ \left[- \frac{1}{r^2}(\partial_r - \partial_t)r^2(\partial_r + \partial_t) + \frac{1}{r^2} L^2\right] i(\partial_r - \partial_t)r \psi \]

\[ = \left[\partial_t^2 - \nabla^2 - \frac{2}{r} \partial_t\right] \alpha = 0 . \]  

so that the evolution operator for \(\alpha\) is a modified d’Alembertian, having the extra term \(-\frac{2}{r} \partial_t\). Similarly

\[ \left[\partial_t^2 - \nabla^2 + \frac{2}{r} \partial_t\right] \beta = 0 . \]  

We see from \((8)\) that adding any function \(f(t - r)\) to \(\alpha\), or adding any function \(g(t + r)\) to \(\beta\) leaves \(F\) unchanged. Also the evolution operators \((13)\) for \(\alpha, \beta\), annihilate \(f(t - r), g(t + r)\) respectively. So the gauge transformations for \(\alpha, \beta\) are

\[ \alpha \rightarrow \alpha' , \quad \alpha' = \alpha + f(t - r) \]

\[ \beta \rightarrow \beta' , \quad \beta' = \alpha + g(t + r) . \]
II. THE POTENTIALS \( \{\alpha, \beta\} \) CORRESPONDING TO RADIATION FIELDS

We first establish some symmetries between \( \alpha, \beta \). Consider the parity transformation \( \mathcal{P} : x \to -x \) and the time reversal transformation \( \mathcal{T} : t \to -t \). Under the combined transformation \( \mathcal{PT} \) the field transforms as

\[
\mathcal{PT} F(t, x) = -F(-t, -x) .
\]

For brevity we will write a bar over any variable signifying its \( \mathcal{PT} \) transform, e.g. \( \bar{F} = \mathcal{PT} F \), \( \bar{\alpha} = \mathcal{PT} \alpha \) etc. Both \( \alpha, \beta \) are scalars under \( \mathcal{PT} \), so that \( \bar{\alpha}(t, x) = \alpha(-t, -x) \). Maxwell’s equations are invariant under the \( \mathcal{PT} \) transformation, so that if \( F \) satisfies Maxwell’s equations then so does \( \bar{F} \). In some cases of interest the field is even or odd under \( \mathcal{PT} \), by which we mean \( \bar{F}^{\text{ev}} = F^{\text{ev}} \) and \( \bar{F}^{\text{odd}} = -F^{\text{odd}} \) (from now on we will use the terms even and odd meaning even or odd under \( \mathcal{PT} \)).

Recall the fields \( (F + i\hat{x} \times F), (F - i\hat{x} \times F) \) which are derived from \( \alpha, \beta \) respectively. Then if the field is even/odd,

\[
(F_{\text{ev}}^{\text{odd}} - i\hat{x} \times F_{\text{ev}}^{\text{odd}}) = \pm \mathcal{PT} (F_{\text{ev}}^{\text{ev}} + i\hat{x} \times F_{\text{ev}}^{\text{ev}}) .
\]

(15)

In either of these cases the field is effectively given by one potential, \( \alpha \) say, because after calculating \( (F + i\hat{x} \times F) \) from \( \alpha \), one can write down \( (F - i\hat{x} \times F) \) using (15). Applying the \( \mathcal{PT} \) transformation to equations (6):

\[
\bar{\mathcal{F}} - i\hat{x} \times \mathcal{F} = -[\nabla - \hat{x} \partial_t + \frac{1}{r} \mathbf{L}] \bar{\alpha} \quad \bar{\mathcal{F}} + i\hat{x} \times \mathcal{F} = -[\nabla + \hat{x} \partial_t - \frac{1}{r} \mathbf{L}] \bar{\beta} ,
\]

so that

\[
F \to \bar{F} \quad \text{is equivalent to} \quad \{\alpha, \beta\} \to \{-\bar{\beta}, -\bar{\alpha}\} .
\]

(16)

From (16) we see that

\[
\{F_{\text{ev}} : \beta = -\bar{\alpha}\} \quad \{F_{\text{odd}} : \beta = +\bar{\alpha}\} .
\]

(17)

We now find the potentials \( \{\alpha, \beta\} \) corresponding to the radiation fields. Equations (11) enable us to calculate the potentials for a known field - which approach we follow in this section. Alternatively we can look for solutions of the evolution equations for the potentials (13), satisfying the constraint (9) - which approach we take in appendix A. It turns out that these evolution equations (13) are separable in either spherical or parabolic coordinates - the spherical solutions yield the potentials corresponding to the multipole fields (which are singular at the origin), while the parabolic solutions are potentials for non-singular fields, including the plane wave fields.

Consider the plane wave field \( F \) propagating in the \( x_3 \) direction, the left circularly polarized (positive helicity) field \( F_L \) is

\[
F_L = \{1, i, 0\} e^{ikx_3 - ikt} \quad F_{rL} = (\hat{x}_1 + i\hat{x}_2) e^{ikx_3 - ikt}
\]

and its \( \mathcal{PT} \) transform the right circularly polarized (negative helicity) field \( F_R \) is

\[
F_R = \bar{F}_L = -\{1, i, 0\} e^{-ikx_3 - ikt} \quad F_{rR} = -(\hat{x}_1 + i\hat{x}_2) e^{-ikx_3 - ikt} .
\]

(18)
Then recalling (10) we integrate \( F_{rL}(S) = (\hat{x}_1 + i\hat{x}_2) e^{-ikS} e^{-ikr(1-\hat{z}_3)} \) to find \( \alpha \), and follow the equivalent procedure for \( \beta \), obtaining

\[
\frac{\hat{x}_1 + i\hat{x}_2}{1-\hat{z}_3} [e^{ikx_3} - e^{-ikr}] \frac{e^{-ikt}}{ik} \]

\[
\beta_L = -\left( \frac{\hat{x}_1 + i\hat{x}_2}{1+\hat{z}_3} \right) [e^{ikx_3} - e^{-ikr}] \frac{e^{-ikt}}{ik} .
\]

The second term in the square brackets does not contribute to the field but its absence would make the potentials singular due to the factors \( 1/(1 + \hat{z}_3) \) in \( \alpha, \beta \). Direct calculation verifies that

\[
[\nabla + \hat{\mathbf{s}} \partial_t - \frac{1}{r} \mathbf{L}] \alpha_L = \left\{ 1 + \hat{x}_3, i + i\hat{x}_3, -\hat{x}_1 - i\hat{x}_2 \right\} e^{ikx_3 - ikt}
\]

which is \( \mathbf{F}_L + i\hat{\mathbf{x}} \times \mathbf{F}_L \), noting that the operator \( [\nabla + \hat{\mathbf{s}} \partial_t - \frac{1}{r} \mathbf{L}] \) commutes with the \( ((\hat{x}_1 + i\hat{x}_2)/r - \hat{z}_3) \) factor in \( \alpha_L \), and annihilates the term \( e^{ik(r-t)} \). Similarly it can be verified that \( [\nabla - \hat{\mathbf{s}} \partial_t + \frac{1}{r} \mathbf{L}] \beta_L = \mathbf{F}_L - i\hat{\mathbf{x}} \times \mathbf{F}_L \).

As \( \mathbf{F}_R = \bar{\mathbf{F}}_L \), then recalling (16)

\[
\frac{\hat{x}_1 + i\hat{x}_2}{1-\hat{z}_3} [e^{ikx_3} - e^{-ikr}] \frac{e^{ikt}}{ik} \]

\[
\beta_R = -\alpha_L = \left( \frac{\hat{x}_1 + i\hat{x}_2}{1+\hat{z}_3} \right) [e^{ikx_3} - e^{-ikr}] \frac{e^{ikt}}{ik} .
\]

The potentials for any plane wave propagating in the \( x_3 \) direction are a linear superposition of \( \{\alpha_L, \beta_L\}, \{\alpha_R, \beta_R\} \). Both \( \{\alpha_L, \beta_L\}, \{\alpha_R, \beta_R\} \) are eigenstates of \( L_3 \), the angular momentum operator in the direction of propagation, with eigenvalue +1.

We obtain the potential \( \alpha_L(k) \) for a left circularly polarized wave propagating in the \( \hat{k} = k/|k| \) direction with wavelength \( 2\pi/|k| \), by first defining the null complex vector \( \mathbf{h} \) satisfying

\[
\mathbf{h} \times \hat{k} = i\mathbf{h} \quad \mathbf{h}^* \cdot \mathbf{h} = 2 ,
\]

which so defined is unique up to a phase factor. Then

\[
\{\alpha_L(k), \beta_L(k)\} = \left\{ -\left( \frac{\mathbf{h} \cdot \mathbf{x}}{r - \hat{k} \cdot \mathbf{x}} \right) [e^{ik\mathbf{x}} - e^{-i|k|\mathbf{r}}] , \left( \frac{\mathbf{h} \cdot \mathbf{x}}{r + \hat{k} \cdot \mathbf{x}} \right) [e^{ik\mathbf{x}} - e^{-i|k|\mathbf{r}}] \right\} \frac{e^{-i|k|r}}{ik} ,
\]

and \( \{\alpha_R(k), \beta_R(k)\} = \{-\beta_L(k), -\alpha_L(k)\} \). We check that \( \alpha_L(k), \beta_L(k) \) are eigenstates of \( (\hat{k} \cdot \mathbf{L}) \) the angular momentum operator in the direction of propagation, with eigenvalue +1:

\[
\hat{k} \cdot \mathbf{L} (\mathbf{h} \cdot \mathbf{x}) = \hat{k} \cdot (-i\mathbf{x} \times \mathbf{h}) = -i\mathbf{x} \cdot (\mathbf{h} \times \hat{k}) = -i\mathbf{x} \cdot (i\mathbf{h}) = \mathbf{h} \cdot \mathbf{x} .
\]

In the next section we investigate the orthonormality of the basis potentials \( \{\alpha(k), \beta(k)\} \) and \( \{\alpha(k'), \beta(k')\} \).

We next briefly consider the multipole radiation fields and their corresponding potentials, noting some symmetry relations under the \( \mathcal{PT} \) transformation. The Debye potential \( \psi \) in this case is a spherical solution of the wave equation, then the potentials \( \{\alpha, \beta\} \) follow from (7). Consider

\[
\psi_{lm} = j_l(kr) Y_{lm}(\theta, \phi) \cos(kt) \]

where \( j_l \) is the spherical Bessel function of order \( l \), and \( Y_{lm} \) are the spherical harmonics. This \( \psi_{lm} \) when substituted into (2) yields \( \mathbf{F}_{lm} \), the magnetic multipole field of order \((l, m)\). (Multiplying the potential by \( i \) yields the corresponding electric multipole field \( i\mathbf{F}_{lm} \)). Then from (7) the potentials \( \{\alpha_{lm}, \beta_{lm}\} \) for the magnetic multipole field of order \((l, m)\) are

\[
\alpha_{lm} = i(\partial_r - \partial_t)[r j_l(kr) Y_{lm}(\theta, \phi) \cos(kt)]
\]

\[
\beta_{lm} = i(\partial_r + \partial_t)[r j_l(kr) Y_{lm}(\theta, \phi) \cos(kt)].
\]
We see that
\[ \beta_{lm} = (-1)^l \bar{\alpha}_{lm} \]
following the parity of \( Y_{lm} \). This means, recalling (17), that \( \mathbf{F}_{lm} \) is odd or even, depending on whether \( l \) is an even or odd number. (If we had considered the potential \( \psi \) of (21) with a \( \sin(kt) \) instead of a \( \cos(kt) \) factor, then
\[ \beta'_{lm} = (-1)^{l+1} \bar{\alpha}_{lm} \] Then, as discussed previously, \( \mathbf{F}_{lm} \) can be calculated from either one of the potentials \( \alpha, \beta \).

III. THE CONSERVED DENSITY \( \rho \) AND THE ENERGY \( E \)

A. The radiation density \( \rho \).
The electromagnetic field is the classical “first-quantized” version of the massless spin-1 photon field, and as such should have a conserved density corresponding to the well-known densities for the Klein-Gordon (spin-zero) or Dirac (spin-\( \frac{1}{2} \)) fields. The only text on classical (non-quantized) electromagnetism which discusses a radiation density, or an inner product, for the electromagnetic field that I have come across is the book by Good and Nelson,\(^{12}\) otherwise one has to turn to texts on QED such as that by Schweber.\(^{13}\) For an excellent recent review on this subject, see Ref. 14. This inner product for the electromagnetic field involves the non-local (integral) operator \( (\nabla^2)^{-1/2} \).

The evolution equations (7) for \( \alpha, \beta \) can be expressed in the form
\[ (\nabla - \mathbf{x} \partial_t) \cdot (\nabla + \mathbf{x} \partial_t) \alpha = 0 \]
\[ (\nabla + \mathbf{x} \partial_t) \cdot (\nabla - \mathbf{x} \partial_t) \beta = 0. \]

It follows that
\[ (\nabla + \mathbf{x} \partial_t) \cdot [\alpha (\nabla - \mathbf{x} \partial_t) \beta^* - \alpha^* (\nabla - \mathbf{x} \partial_t) \beta] - (\nabla - \mathbf{x} \partial_t) \cdot [\beta^* (\nabla + \mathbf{x} \partial_t) \alpha - \beta (\nabla + \mathbf{x} \partial_t) \alpha^*] = 0. \]

Collecting terms and multiplying by \((-i)\) we can write (4.3) in the form of a continuity equation
\[ \partial_t \rho + \nabla \cdot \mathbf{J} = 0, \]
where
\[ \rho = -\frac{i}{4} [\alpha (\partial_r - \partial_t) \beta^* - \alpha^* (\partial_r - \partial_t) \beta + \beta^* (\partial_r + \partial_t) \alpha - \beta (\partial_r + \partial_t) \alpha^*] \]
\[ \mathbf{J} = -\frac{i}{4} [\alpha (\nabla - \mathbf{x} \partial_t) \beta^* - \alpha^* (\nabla - \mathbf{x} \partial_t) \beta - \beta^* (\nabla + \mathbf{x} \partial_t) \alpha + \beta (\nabla + \mathbf{x} \partial_t) \alpha^*] \]

where \( \alpha_L, \beta_L \) and \( \mathbf{J}_L \) stand for the complex conjugate terms. Let us substitute the \( \{\alpha_L, \beta_L\} \) of (19) into (27):
\[ \rho_L = -\frac{1}{4k} \left[ -\left( \frac{x_1 + i x_2}{1 - x_3} \right) e^{ik x_3} - e^{-ik r} \right] - \left( \frac{x_1 + i x_2}{1 + x_3} \right) \left[ e^{ik x_3} - e^{-ik r} \right] \]
\[ = \frac{1}{2k} \left[ (1 + x_3) \left| 1 - \cos(kr - k x_3) \right| + (1 - x_3) \left| 1 - \cos(kr + k x_3) \right| \right] \geq 0. \]

Inserting \( \{\alpha_R, \beta_R\} = -\{\bar{\beta}_R, \bar{\alpha}_R\} \) into (27), we obtain \( \rho_R = -\rho_L \). Hence \( \rho_L \) is non-negative and \( \rho_R \) is non-positive.
The density \( \rho \) has close similarity with the Klein-Gordon density for a spin-zero particle: 
\[ \rho_{KG} = -\frac{i}{2}(\phi^* \partial_t \phi - \phi \partial_t \phi^*) . \]
In this case solutions \( \phi \) with time dependence \( e^{-i\omega t} \) with \( \omega \) positive are regarded as particle solutions, those with time dependence \( e^{+i\omega t} \) are regarded as anti-particle solutions. Analogously we can regard the right circularly polarized (negative helicity) field as the ‘anti-photons’ of the left circularly polarized (positive helicity) photon field. There appears to be no comparable radiation density in the literature, although of course one can project out the different polarizations when the field is expressed as a Fourier integral.\(^{15,16}\)

We now investigate the orthogonality of the potentials for two different fields \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \). It will be convenient to represent the potentials for the field \( \mathbf{F}_1 \) by the \( 2 \times 1 \) matrix \( U_1 \equiv \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \), then we construct the following indefinite scalar product space

\[ \langle U_1|U_2 \rangle \equiv \int d^3x \rho_{12} \tag{29} \]

where \( \rho_{12} \) is the conserved density

\[ \rho_{12} = -\frac{i}{4} \left[ U_1^T \left( I \partial_t + \sigma^3 \partial_r \right) U_2^* - U_2^T \left( I \partial_t + \sigma^3 \partial_r \right) U_1 \right] \]

\[ \rho_{12} = -\frac{i}{4} \left[ (\alpha_1 - \beta_1) F_{r2}^* - (\alpha_2^* - \beta_2^*) F_{r1} \right] . \tag{30} \]

Here \( U^T = [\alpha \ \beta] \) is the transpose of \( U \), and \( \sigma^3 \) is the Pauli matrix \( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \). Note that when \( U_1 = U_2 = U \), then \( \rho_{12} = \rho \). In appendix B we show that

\[ \langle U_L(k)|U_L(k') \rangle = (2\pi)^3 k \delta(k-k') \]

\[ \langle U_R(k)|U_R(k') \rangle = -(2\pi)^3 k \delta(k-k') \]

\[ \langle U_L(k)|U_R(k') \rangle = 0 . \tag{31} \]

The relations (31) enable one to project out the positive/negative helicity states of momentum \( k \), and show that \( \rho \) integrated over all space is the amount of left minus right circularly polarized radiation.

**B. The energy density \( E \)**

As a consequence of Maxwell’s equations (5)

\[ \partial_t \left( \frac{1}{2} \mathbf{F} \cdot \mathbf{F}^* \right) + \nabla \cdot \left( \frac{i}{2} \mathbf{F} \times \mathbf{F}^* \right) = 0 \]

where \( E = \frac{1}{2} \mathbf{F} \cdot \mathbf{F}^* \equiv \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \) is the usual energy density, and \( \frac{i}{2} \mathbf{F} \times \mathbf{F}^* \equiv \mathbf{E} \times \mathbf{B} \) is the momentum density. To simplify \( \mathbf{F} \cdot \mathbf{F}^* \) in terms of \( \alpha, \beta \), we note the following: that from (8) we can express \( \mathbf{F} \) as\(^{17}\)

\[ \mathbf{F} = \nabla_\uparrow \alpha + \nabla_\downarrow \beta + \hat{x} F_r \tag{32} \]

with \( F_r \) expressed in terms of \( \alpha, \beta \) by (9), and the operators \( \nabla_\uparrow, \nabla_\downarrow \) are defined as

\[ \nabla_\uparrow \equiv \frac{1}{2} \left( \nabla - \hat{x} \partial_r + i\hat{x} \times \nabla \right), \quad \nabla_\downarrow = (\nabla_\uparrow)^* \equiv \frac{1}{2} \left( \nabla - \hat{x} \partial_r - i\hat{x} \times \nabla \right) . \tag{33} \]

These operators have the property that for any scalars \( \chi, \zeta \)

\[ \nabla_\uparrow \chi \cdot \nabla_\downarrow \zeta = \nabla_\downarrow \chi \cdot \nabla_\uparrow \zeta = \hat{x} \cdot \nabla_\uparrow \chi = \hat{x} \cdot \nabla_\downarrow \chi = 0 \]

\[ \nabla_\uparrow \chi \cdot \nabla_\downarrow \zeta = \frac{1}{2} \left[ (\hat{x} \times \nabla \chi) \cdot (\hat{x} \times \nabla \zeta) + i\hat{x} \cdot (\nabla_\chi \times \nabla_\zeta) \right] . \tag{34} \]
Then substituting in (32) into \( (\mathbf{F} \cdot \mathbf{F}^*) \), and with the aid of (34), we find
\[
2E = \mathbf{F} \cdot \mathbf{F}^* = |\nabla \alpha + \nabla \beta + \hat{x}F_r| \cdot |\nabla \alpha^* + \nabla \beta^* + \hat{x} F_r^*|
\]
\[
2E = |\nabla \alpha|^2 + |\nabla \beta|^2 + |F_r|^2
\] (35)
with \( F_r \) given by (9). We will further expand (35) as we shall see that \( E \) contains a divergence term, which can be removed such that the resulting energy \( E' \) is still conserved. Again using (34),
\[
4E = |\hat{x} \times \nabla \alpha|^2 + |\hat{x} \times \nabla \beta|^2 + 2 |F_r|^2 + i\hat{x} \cdot [(\nabla \alpha \times \nabla \alpha^*) - (\nabla \beta \times \nabla \beta^*)],
\] (36)
and the last terms of (36) can be expressed as 
\[
i\nabla \cdot [- \alpha (\hat{x} \times \nabla \alpha^*) + \beta (\hat{x} \times \nabla \beta^*)],
\]
so that the modified energy density \( E' \)
\[
4E' = |\hat{x} \times \nabla \alpha|^2 + |\hat{x} \times \nabla \beta|^2 + 2 |F_r|^2
\] (37)
is also a conserved non-negative density, so is also a plausible candidate for the energy density.

IV. THE POTENTIALS CORRESPONDING TO THE FIELD OF A CHARGED PARTICLE

First consider the static case, then the constraint \( (\partial_r + \partial_t)\alpha = (\partial_r - \partial_t)\beta \) implies \( \beta = \alpha \), then from (8) we obtain
\[
\mathbf{F}_{sc} = \nabla \alpha_{sc}
\]
so that
\[
Re[\alpha] = -\Phi \quad Im[\alpha] = \Phi_m
\]
where \( \Phi \) is the usual scalar potential, and \( \Phi_m \) is the magnetic scalar potential. The potentials \( \Phi, \Phi_m \) for the electrostatic, magnetostatic fields are well known. The energy \( E' \) for the static field is \( E' = \frac{1}{2} |\nabla \alpha|^2 \).

Magnetic monopole fields are also accommodated: the potential for a magnetic mono-pole at position \( \mathbf{a} \) is just \( \alpha = \beta = i q_m/|\mathbf{x} - \mathbf{a}| \). Note that if we use (11) to derive \( \alpha \) from the radial field
\[
F_r = q(r - \hat{x} \cdot \mathbf{a})/|\mathbf{x} - \mathbf{a}|^3
\]
of a charge at position \( \mathbf{a} \), we obtain \( \alpha = \int_0^r dr F_r = -(q/|\mathbf{x} - \mathbf{a}|) + (q/|\mathbf{a}|) \). For non-radiation fields, we can lose the constant of integration by instead obtaining \( \alpha \) from \( F_r \), so that (9) is satisfied, as follows
\[
\alpha(S) = \int_0^S dr F_r(S).
\] (38)
We will use (38) to determine the potentials \( \{\alpha, \beta\} \) for a uniformly moving charge. The field of a charge passing through the origin in the \( x_1 \) direction is
\[
F = \{x_1 - vt, x_2 - iv x_3, x_3 + iv x_2\} \frac{q \gamma}{r'^{3/2}} \quad F_r = \frac{q \gamma (r - v \dot{x}_1 t)}{r'^{3/2}} \]
where \( r' = [\gamma^2 (x_1 - vt)^2 + x_2^2 + x_3^2]^{1/2} \). Substituting \( t = S + r \) into the expression for \( F_r \),
\[
F_r(S) = \frac{q \gamma (r - v \dot{x}_1 S - v \dot{x}_1 r)}{[\gamma^2 (\dot{x}_1 r - v S - v r)^2 + (\dot{x}_2^2 + \dot{x}_3^2) r^2]^{3/2}}
\] (39)
and
and integrating over $r$ we obtain

$$\alpha = \left[ r - \frac{q\gamma (1 - \hat{x}_1 v) + S}{S[\gamma^2(x_1 r - vS - vr)^2 + (\hat{x}_1^2 + \hat{x}_2^2) r^2]^{1/2}} \right]_{\infty}$$

$$\alpha = -q\left( \frac{t' - r'}{t - r} \right) \frac{1}{r'}$$

(41)

where $t' = \gamma(t - vx_1)$. This potential is non-singular except at the charge position, because when $t = r$, $r' = \gamma(r - vx_1)$, and $(\frac{t' - r'}{t - r}) = \gamma$. The field $F$ of (39) is an even field (which is only the case because the particle is passing through the origin), and so from (17)

$$\beta = -\bar{\alpha} = -q\left( \frac{t' + r'}{t + r} \right) \frac{1}{r''}.$$ 

V. OUTLOOK

We have calculated the potentials $\{\alpha, \beta\}$ corresponding to various fields: the simple relation (9) between the potentials and the radial field $F_r$ makes the calculation of the potential from a given field quite straightforward, or one can solve the equations for the potentials from which one then calculates the field, as in appendix A. Although $\{\alpha, \beta\}$ are scalars under rotations, they have complicated transformation properties under the Lorentz transformations - we will discuss these transformation properties elsewhere. (For the Lorentz transformation of the Debye potential $\psi$, also complicated, see the paper by Monroe.19) The stationary solutions for the potentials with time dependence $e^{-ikt}$ ($e^{+ikt}$) correspond to the left (right) circularly polarized waves: the orthogonality properties (31) suggest an alternative approach to second quantizing the electromagnetic field, which we hope to address in the future.

Under the duality transformation ($E \rightarrow B$, $B \rightarrow -E$), or $F \rightarrow iF$, the transformation of the potentials is just $\{\alpha, \beta\} \rightarrow i\{\alpha, \beta\}$. The absence of any duality transformation for the usual potentials $A^\mu \equiv (\Phi, A)$ has been commented on recently by Witten.20 On the other hand the interaction of a charged particle with the field is naturally described via the potentials $A^\mu$, by replacing the free momentum $p^\mu$ with the gauge invariant $(p^\mu - eA^\mu)$, whereas the role of the $\{\alpha, \beta\}$ potentials in gauge theory is not clear.

Appendix A - Solutions of (13) in parabolic coordinates

We solve the equations for the potentials (13). The spherical solutions yield the multipole fields, here we discuss a few of the parabolic solutions. Defining $\lambda, \mu, \phi$ such that $\lambda, \mu$ have the dimension of length:21,22

$$x_1 + ix_2 = 2\sqrt{\lambda \mu} e^{i\phi} \quad x_3 = \lambda - \mu \quad r = \lambda + \mu,$$

then21,22

$$\nabla^2 \equiv \frac{1}{\lambda + \mu} (\partial_\lambda \lambda \partial_\lambda + \partial_\mu \mu \partial_\mu) + \frac{1}{4\lambda \mu} \partial_\phi^2$$

$$\nabla^2 - \partial_\lambda^2 + \frac{2}{r} \partial_1 \equiv \frac{1}{\lambda + \mu} (\partial_\lambda \lambda \partial_\lambda + \partial_\mu \mu \partial_\mu + 2\partial_1) + \frac{1}{4\lambda \mu} \partial_\phi^2 - \partial_1^2$$

and with $\alpha = f(\lambda)g(\mu)e^{im\phi}e^{-ikt}$ then

$$[(\partial_\lambda \lambda \partial_\lambda + \partial_\mu \mu \partial_\mu - 2ik) - \frac{m^2}{4\lambda} - \frac{m^2}{4\mu} + k^2 \lambda + k^2 \mu] f(\lambda)g(\mu) = 0.$$
Inserting a separation constant $2ikc$, then

$$[(\partial_\lambda \partial_\lambda - ik(1 + 2c) - \frac{m^2}{4\lambda} + k^2\lambda)] f(\lambda) = 0 \quad (A1)$$

$$[\partial_\mu \partial_\mu - ik(1 - 2c) - \frac{m^2}{4\mu} + k^2\mu] g(\mu) = 0. \quad (A2)$$

The solution of (A1) is $f(\lambda) = \lambda^{m/2} e^{-ik\lambda} F_1(1 + \frac{m}{2} + c, m + 1, 2ik\lambda)$, where $F_1$ is the confluent hypergeometric function. So the solution for $\alpha$ is

$$\alpha = (\lambda\mu)^{m/2} e^{i\mu \phi} e^{-ik(\lambda + \mu)} F_1(1 + \frac{m}{2} + c, m + 1, 2ik\lambda) F_1(1 + \frac{m}{2} - c, m + 1, 2ik\mu) e^{-ikt}. \quad (A3)$$

The confluent hypergeometric function reduces to a simpler expression for integer or half-integer values of $c$.

**The case** $m = 0, c = 0$.

As $F_1(1, 1, i\zeta) \equiv e^{i\zeta}$ we have

$$\alpha = e^{-2ik\zeta} e^{-ik(\lambda + \mu)} e^{2ik\mu} = e^{-ikt} e^{ik(\lambda + \mu)} \equiv e^{ik\tau - ik t}$$

which potential substituted into (6a) yields a zero field.

**The case** $m = 1, c = 1/2$.

Inserting these values of $m, c$ into (A1), and using the identities $F_1(2, 2, 2i\zeta) \equiv e^{2i\zeta}$, $F_1(1, 2, 2i\zeta) \equiv \frac{\sin \zeta}{\zeta} e^{i\zeta}$, then

$$\alpha = \sqrt{\lambda\mu} e^{i\phi} e^{-ik(\lambda + \mu)} F_1(2, 2, 2ik\lambda) F_1(1, 2, 2ik\mu) e^{-ikt}$$

$$\alpha = \sqrt{\lambda\mu} e^{i\phi} e^{-ik(\lambda + \mu)} e^{2ik\lambda} \frac{\sin(k\mu)}{k\mu} e^{ik\mu} e^{-ikt}$$

$$\alpha = \sqrt{\lambda\mu} e^{i\phi} e^{ik\lambda} \frac{\sin(k\mu)}{k\mu} e^{-ikt}$$

which is the $\alpha_L$ of (19).

**The case** $m = 1, c = 0$.

Substituting these values for $m, c$, into (3.2) we have

$$\alpha_Z = \sqrt{\lambda\mu} e^{i\phi} e^{-ik(\lambda + \mu)} F_1(\frac{\zeta}{2}, 2, 2ik\lambda) F_1(\frac{\zeta}{2}, 2, 2ik\mu) e^{-ikt}.$$
The field $\mathbf{F}_Z$ may be calculated by substituting (A4), (A5) into (8), or (A6) into (3). After some labour one obtains the components of $\mathbf{F}_Z$

$$
2\begin{bmatrix}
  F_1 \\
  F_2 \\
  F_3
\end{bmatrix}
_{Z} = \frac{e^{-ikt}}{(\lambda + \mu)} \begin{bmatrix}
  \lambda[(J_0 + iJ_1)(k\lambda)J_0(k\mu)] + \mu[J_0(k\lambda)(J_0 - iJ_1)(k\mu)] \\
  i\lambda[(J_0 + iJ_1)(k\lambda)J_0(k\mu)] + i\mu[J_0(k\lambda)(J_0 - iJ_1)(k\mu)] \\
  -2\sqrt{\lambda \mu} e^{i\phi}[J_0(k\lambda)J_1(k\mu) + J_1(k\lambda)J_0(k\mu)] \\
  + e^{2i\phi} \left( -i\mu[J_1(k\lambda)(J_0 + iJ_1)(k\mu)] + i\lambda[(J_0 - iJ_1)(k\lambda)J_1(k\mu)] \\
  -i\mu[J_1(k\lambda)(J_0 + iJ_1)(k\mu)] + i\lambda[(J_0 - iJ_1)(k\lambda)J_1(k\mu)] \right)
\end{bmatrix}.
$$

(A7)

It is interesting to see the behaviour of this field $\mathbf{F}_Z$ along the $x_3$ axis. For the positive $x_3$ axis $\mu = 0$ and $\lambda = x_3$, and the field is just

$$
2\langle \mathbf{F}_Z \rangle_{\mu=0} = e^{-ikt} \begin{bmatrix}
  (J_0 + iJ_1)(kx_3) \\
  i(J_0 + iJ_1)(kx_3)
\end{bmatrix}
_{x_3 \geq 1} \rightarrow e^{-ix/4} \frac{2}{\pi|x_3|} \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{ikx_3-ikt}.
$$

(A8)

For the negative $x_3$ axis $\lambda = 0$ and $\mu = -x_3$, and we obtain the same expression (A8) for the field along the negative $x_3$ axis. Along the $x_3$ axis the field $\mathbf{F}_Z$ appears as a right-circularly polarized plane wave propagating in the $+x_3$ direction, with amplitude decaying by $|x_3|^{-1/2}$ from the origin. Away from the $x_3$ axis the field decays more rapidly: the $x_1, x_2$ plane through the origin is parametrized by $\lambda = \mu = \sqrt{x_1^2 + x_2^2}/2$, then substituting $\lambda = \mu = \rho \rightarrow (3.7)$ one finds that the field decays by $\rho^{-1}$ from the origin. Thus the field $\mathbf{F}_Z$ is highly directional along the axis of propagation.

When $m = 1, c > 1/2$, the resulting field becomes infinite at spatial infinity.

**Appendix B. Orthonormality of the basis functions $U(k)$**

First we will consider the orthogonality of two left-circularly polarized waves, $U_L(k), U_L(k')$. Due to the fact that the $U_L(k), U_L(k')$ are eigenstates of the angular momentum in the direction of propagation $k$ or $k'$, and the $L$ operator is Hermitian, then

$$
\langle U_L(k)|U_L(k')\rangle = 0 \quad \text{when} \quad \hat{k} \neq \hat{k}'.
$$

So we need only consider the case when the waves are propagating in the same direction but with different frequencies, i.e. when $k = \{0,0,k\}$ and $k' = \{0,0,k'\}$. Then recalling the $U_L$ of (19) we have the inner product

$$
\langle U_L(k)|U_L(k')\rangle
= -\frac{1}{4} \int d^3x \left[ -\left( \frac{\hat{x}_1 + i\hat{x}_2}{1 - \hat{x}_3} \right)[e^{ikx_3} - e^{-ikr}] - \left( \frac{\hat{x}_1 + i\hat{x}_2}{1 + \hat{x}_3} \right)[e^{ikx_3} - e^{-ikr}] \right] (k'(\hat{x}_1 - i\hat{x}_2)e^{-ikx_3})
\right.
\left. + \left[ -\left( \frac{\hat{x}_1 - i\hat{x}_2}{1 - \hat{x}_3} \right)[e^{-ik'x_3} - e^{-ik'\prime r}] - \left( \frac{\hat{x}_1 - i\hat{x}_2}{1 + \hat{x}_3} \right)[e^{-ik'x_3} - e^{-ik'\prime r}] \right] (k(\hat{x}_1 + i\hat{x}_2)e^{ikx_3})
\right]
\right]
$$

$$
= \frac{1}{4} \int d^3x \left[ k'(1 + \hat{x}_3)[e^{i(k-k')x_3} - e^{i(k-k'-k''x_3)}] + (1 - \hat{x}_3)[e^{i(k-k')x_3} - e^{-ikr-ik'x_3}] \right]
\right]
\left. + k(1 + \hat{x}_3)[e^{i(k-k')x_3} - e^{-i(k'-k''x_3)}] + (1 - \hat{x}_3)[e^{i(k-k')x_3} - e^{i(k'\prime r+ikx_3)}] \right]
\right]
\right]
$$

$$
= \frac{1}{4} \int d^3x \left[ 2(k + k') e^{i(k-k')x_3}
\right.
\left. - (1 + \hat{x}_3)[k' e^{i(kr-k'x_3)} + ke^{-i(k'r-kx_3)}] - (1 - \hat{x}_3)[k' e^{-ikr-ik'x_3} + ke^{i(k' r+ikx_3)}] \right].
$$
The first term yields the delta function \((2\pi)^3 k \delta(k - k')\). We go over to spherical coordinates to evaluate the remaining terms, putting \(\hat{x}_3 = \cos \theta = \nu\), the rest of the integral is

\[
- \frac{2\pi}{4} \int_0^\infty r^2 dr \int_{-1}^1 d\nu \left( (1 + \nu)[k' e^{i r(k - k')} + k e^{-i r(k' - k')}] + (1 - \nu)[k' e^{-i r(k + k')} + k e^{i r(k' + k')} \right)
\]

\[
= - \frac{4\pi}{4} \int_0^\infty r^2 dr \int_{-1}^1 d\nu \left( (1 + \nu)[k' \cos(r(k - k')) + k \cos(r(k' - k'))] \right)
\]

\[
= - \frac{4\pi}{4} \int_0^\infty r^2 dr \int_{-1}^1 d\nu \left( (1 + \nu)[k' \sin(r(k - k')) - k \sin(r(k' - k'))] \right)
\]

so that finally

\[
\langle U_L(k)|U_L(k') \rangle = (2\pi)^3 k \delta(k - k')
\]

with \(k \equiv |k|\). The other identities of (31) follow similarly.

References

1. A.C.T. Wu, Debye scalar potentials for the electromagnetic fields Phys.Rev.D 34, 3109-3110 (1986)
2. C.G. Gray, Multipole expansions of electromagnetic fields using Debye potentials Am. J. Phys. 46, 169-179 (1978)
3. See the notes of Ref. 2 for the original work on the Debye potentials
4. J.R. Reitz, F.J. Milford and R.W. Christy, Foundations of Electromagnetic Theory (Addison - Wesley, Reading MA, 1993)
5. J.D. Jackson, Classical Electrodynamics (John Wiley & Sons, New York, 1975) 2nd ed.
6. C.J. Bouwkamp and H.G.B. Casimir, On multipole expansions in the theory of electromagnetic radiation Physica 20, 539-554 (1954)
7. C.H. Wilcox, Debye potentials J.Math.Mech. 20, 167-201 (1957)
8. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol 1 (Interscience Publishers Inc, New York, 1953) pp.378
9. See for example pp. 767-769 of Ref. 5
10. S.N. Mosley, Electromagnetics in retarded time and photon localization Am.J.Phys. 65, 1094-1097 (1997)
11. See pp. 245-251 of Ref. 5
12. R.H. Good and T.J. Nelson, Classical Theory of Electric and Magnetic Fields (Academic Press, New York, 1971) pp.609
13. S.S. Schweber, An Introduction to Relativistic Quantum Field Theory (Harper and Row, New York, 1961) pp.117
Any complex vector $C$ can be split into three components with the following set of projection operators:
\[
P_{\uparrow} \equiv \frac{1}{2}(1 - \mathbf{x} \cdot \mathbf{x} + i \mathbf{x} \times), \quad P_0 \equiv \mathbf{x} \cdot \mathbf{x}, \quad P_{\downarrow} \equiv \frac{1}{2}(1 - \mathbf{x} \cdot \mathbf{x} - i \mathbf{x} \times),
\]
satisfying $P_{\epsilon}P_{\epsilon'} = \delta_{\epsilon\epsilon'}$. The transverse components $(P_{\uparrow}C)$, $(P_{\downarrow}C)$ can then each be expressed in terms of a complex potential by $(P_{\uparrow}C) = \nabla_{\uparrow}\chi \equiv P_{\uparrow}\nabla\chi$ and $(P_{\downarrow}C) = \nabla_{\downarrow}\zeta \equiv P_{\downarrow}\nabla\zeta$. See the paper by Wilcox (Ref.7) for equivalent methods of expressing a transverse vector $(P_{\uparrow}C + P_{\downarrow}C)$ in terms of two potentials.