Electromagnetic energy and momentum in moving media

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The problem of the electromagnetic energy-momentum tensor is among the oldest and the most controver-
sial in macroscopic electrodynamics. In the center of the issue is a dispute about the Minkowski and the
Abraham tensors for moving media. An overview of the current situation is presented. After putting the
discussion into a general Lagrange-Noether framework, the Minkowski tensor is recovered as a canonical
energy-momentum. It is shown that the balance equations of energy, momentum, and angular momentum
are always satisfied for an open electromagnetic system despite the lack of the symmetry of the canonical
tensor. On the other hand, although the Abraham tensor is not defined from first principles, one can formu-
late a general symmetrization prescription provided a timelike vector is available. We analyze in detail the
variational model of a relativistic ideal fluid with isotropic electric and magnetic properties interacting with
the electromagnetic field. The relation between the Minkowski energy-momentum tensor, the canonical
energy-momentum of the medium and the Abraham tensor is clarified. It is demonstrated that the Abraham
energy-momentum is relevant when the 4-velocity of matter is the only covariant variable that enters the
constitutive tensor.

1 Introduction

One century ago, in 1908, Hermann Minkowski [1] gave a complete relativistically covariant formulation
of classical Maxwell-Lorentzian electrodynamics. In particular, he demonstrated how the macroscopic
physical processes in moving bodies and media can be derived from the knowledge of the physics of the
matter at rest. An important issue in phenomenological macroscopic electrodynamics of moving media
is the definition of the energy and momentum of the electromagnetic field in matter. Rather surprisingly,
this topic has demonstrated a remarkable longevity, and the question of the electromagnetic energy and
momentum in matter was not settled until now, despite some theoretical and experimental achievements.

It is not our aim to give a complete history of this issue. After Minkowski proposed [1] his version of
the energy-momentum tensor, Abraham [2, 3, 4] entered the dispute with a different expression. The main
formal difference between the Minkowski and the Abraham energy-momentum tensors is that the latter is
symmetric while the former is not. Since then, the discussion was going on which of these two tensors is
correct (along with some other attempts, among which it is worthwhile to mention the results of Einstein
and Laub [5], of de Groot and Suttorp [6] and of Peierls [7, 8]).

Of the early studies, the work of Dallenbach and Ishiwara [9, 10, 11] deserves to be mentioned. The
later contributions of von Laue, Beck, Haus [12, 13, 14, 15] and many others [16, 17, 18, 19, 20, 21, 22, 23,
24, 25, 26, 27, 28] and [29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41] brought more arguments and more
confusion into the discussion. Eventually, it became clear that the study of the electromagnetic field alone
is insufficient, and the idea was put forward that all possible tensors are correct, depending on the way how
field and matter are separated into two subsystems, see for example [42, 43, 44, 45] and [46, 47, 48]. More recent discussions were devoted to the analysis of this separation question [49, 50, 51, 52, 53, 54, 55], along with [56, 57, 58, 59, 60, 61, 62] and [63, 64, 65].

From a theoretical point of view, the most important developments should perhaps be attributed to Schmutzer and Schröder [66, 67, 68, 69] who stressed the relation of the Minkowski and the canonical energy-momentum tensor, to Poincelot [70, 71, 72, 73] for underlining the universality of the Lorentz force, and to Penfield, Haus, and Mikura [74, 75, 76, 77] for coming up with the first working models for polarizable and magnetizable matter in the framework of the Lagrange variational approach. In our paper, these points will be put into the center of the discussion, following our earlier studies [78, 79]. Some relevant results were also presented in the recent interesting papers of Dereli et al. [80, 81].

It is clearly impossible to mention even briefly all the names and to discuss all the contributions to the issue of the electromagnetic energy and momentum in a short paper. Moreover, since we will mainly pay attention to the theoretical questions here, the experimental results in this area will be not be properly discussed. In order to get more information, the readers should consult the thorough reviews [6, 82, 83, 84, 85, 86, 87, 88], see also the most recent one in [89].

Our basic notations and conventions follow the book [90]. In particular, \( \epsilon_0, \mu_0 \) are the electric and the magnetic constant (earlier called vacuum permittivity and vacuum permeability). It is worthwhile to stress that we do not choose any special system of physical units. In other words, the presence of \( \epsilon_0 \) and \( \mu_0 \) in our formulas does not mean that the international system SI is used, their values are different for SI, Lorentz-Heaviside, and rational systems, and they are fixed only after the specific unit system is chosen. More on physical dimensions and units can be found in [91]. We do not discuss gravitational effects, the spacetime is flat. The Minkowski metric is \( g_{ij} = \text{diag}(c^2, -1, -1, -1) \). The totally antisymmetric Levi-Civita tensor \( \eta_{ijkl} \) is defined by \( \eta_{0123} = \sqrt{-g} \), with \( g = \det(g_{ij}) \). Latin indices from the middle of the alphabet label the spacetime components, \( i, j, k, \cdots = 0, 1, 2, 3 \), whereas those from the beginning of the alphabet refer to 3-space: \( a, b, c, \cdots = 1, 2, 3 \).

### 2 Preliminaries: macroscopic electrodynamics

Quite generally, the Lagrangian of the electromagnetic field in matter can be written as

\[
L^e = -\frac{1}{8} \chi^{ijkl} F_{ij} F_{kl}.
\]  

(1)

Here \( F_{ij} = \partial_i A_j - \partial_j A_i \) is the electromagnetic field strength constructed from the potential \( A_i \), and the tensor density \( \chi^{ijkl} \) describes the electric and magnetic properties of matter. This quantity may also depend on \( F_{ij} \), thereby taking into account possible nonlinear electromagnetic effects, but in the first approximation \( \chi^{ijkl} \) is a function only of the matter variables that describe the state of the medium. We will restrict ourselves to linear electrodynamics.

As usual, we define the electromagnetic excitation tensor density by

\[
H^{ij} = -2 \frac{\partial L^e}{\partial F_{ij}}.
\]

(2)

For the theory (1) we then find the linear constitutive relation [92, 93, 94, 95]

\[
H^{ij} = \frac{1}{2} \chi^{ijkl} F_{kl}.
\]  

(3)

Accordingly, we then have \( L^e = -\frac{1}{2} H^{kl} F_{kl} \).

In the conventional way, the components of the tensors \( F_{ij} \) and \( H^{ij} \) describe the electric and magnetic fields \( (E, B) \) and the electric and magnetic excitations \( (D, H) \) (other names of which are “electric...
displacement” and “magnetic field intensity”):

\[
F_{ij} = \begin{pmatrix}
0 & -E_1 & -E_2 & -E_3 \\
E_1 & 0 & B_3 & -B_2 \\
E_2 & -B_3 & 0 & B_1 \\
E_3 & B_2 & -B_1 & 0
\end{pmatrix}, \quad H^{ij} = \begin{pmatrix}
0 & D^1 & D^2 & D^3 \\
-D^1 & 0 & H_3 & -H_2 \\
-D^2 & -H_3 & 0 & H_1 \\
-D^3 & H_2 & -H_1 & 0
\end{pmatrix}.
\]

(4)

It is worthwhile to mention that it was Minkowski \[1\] who introduced the notion of a bivector (which he called “Raum-Zeit-Vektor IIter Art”, i.e., a second rank tensor in the modern terminology) and wrote for the first time the Maxwell electrodynamics in the explicitly covariant four-dimensional tensor form.

In vacuum, the electromagnetic action reads

\[
I = -\frac{1}{4} \int \lambda_0 g^{ik} g^{jl} F_{ij} F_{kl} \sqrt{-g} d^4 x, \quad \text{where} \quad \lambda_0 = \sqrt{\varepsilon_0/\mu_0}.
\]

As a result, in vacuum we find for the constitutive tensor density

\[
0^{ijkl} = \lambda_0 \sqrt{-g} \left( g^{ik} g^{jl} - g^{ij} g^{kl} \right).
\]

(5)

Consequently, the linear constitutive relation \[4\] in vacuum in Cartesian coordinates reads (note that \(\sqrt{-g} = c\), \(\lambda_0 c = \mu_0^{-1}\), and \(\lambda_0 / c = \varepsilon_0\))

\[
H^{ij} = \frac{1}{2} 0^{ijkl} F_{kl} = F^{ij} / \mu_0,
\]

(6)

or, equivalently, in 3-dimensional form

\[
D = \varepsilon_0 E, \quad H = \frac{1}{\mu_0} B.
\]

(7)

In macroscopic electrodynamics, a polarizable and magnetizable medium is characterized by the non-trivial polarization \(P\) and magnetization \(M\). They arise from the bound charge and current densities \((\rho, \mathbf{J})\) via the standard relations

\[
\rho = -\nabla \cdot P, \quad \mathbf{J} = \dot{P} + \nabla \times M.
\]

(8)

The vacuum constitutive law \[7\] in a medium is changed \[9\] into

\[
D = \varepsilon_0 E + P, \quad H = \frac{1}{\mu_0} B - M.
\]

(9)

Analogously to the electric and magnetic fields, the polarization and magnetization are not 3-vectors, but constitute the components of the 4-dimensional polarization tensor of second rank,

\[
M^{ij} = \begin{pmatrix}
0 & P^1 & P^2 & P^3 \\
-P^1 & 0 & -M_3 & M_2 \\
-P^2 & M_3 & 0 & -M_1 \\
-P^3 & -M_2 & M_1 & 0
\end{pmatrix}.
\]

(10)

Correspondingly, the constitutive relation \[9\] can be recast into

\[
H^{ij} = \frac{1}{\mu_0} F^{ij} + M^{ij},
\]

(11)

and, introducing the 4-vector of the bound current \(J^i = (\rho, \mathbf{j})\), equation \[8\] can be rewritten as

\[
J^i = -\partial_j M^{ij}.
\]

(12)

\[1\] See Hirst \[95\] for the definition and a discussion of magnetization and polarization in the microscopic theory.
Combining (3) and (6), we can put the constitutive relation in an alternative equivalent form:

\[ M^{ij} = \frac{1}{2} \xi^{ijkl} F_{kl}, \quad \xi^{ijkl} := \chi^{ijkl} - \frac{\delta^{ijkl}}{c^4}. \tag{13} \]

In general, the susceptibility tensor density \( \xi^{ijkl} \) describes also the magnetoelectric effects when the medium is magnetized in an electric field and/or polarized in a magnetic field.

### 3 Abrahamization of an energy-momentum tensor

Contrary to a general belief (and perhaps somewhat surprisingly) there does not exist any derivation of the Abraham energy-momentum tensor from first principles. In this section, we give a formal definition of what can be called the “abrahamization” prescription that can be applied to an arbitrary energy-momentum tensor.

More precisely, let us assume that we can associate with a given physical system a second rank tensor \( t^a_i \), the components of which are interpreted as follows: \( t_0^0 = w \) is a density of energy of the system, \( t_0^a = s^a \) (with \( a = 1, 2, 3 \)) is the energy flux density ("Poynting vector"), \( t_a^0 = -p_a \) is the momentum density, and \( t_{ab}^b \) is the stress tensor. We do not assume that this tensor is conserved, and the divergence \( \partial_i t_{ki}^i = f_k \) describes, in general, the balance equation of the system. For \( k = 0 \), this yields the familiar energy balance equation \( \dot{w} + \nabla \cdot s = f_0 \). Integrating over a three-dimensional volume \( V \), we find in a usual way that the change in time of the total energy \( \int_V w \) combined with the energy flux through the boundary, \( \int_{\partial V} s \), is equal to the total power production of the system \( \int_V f_0 \). Similarly, for \( k = a = 1, 2, 3 \) we recover the balance equation \( \dot{p}_a + \partial_b t_{ab}^b = f_a \) of the momentum of the system under the action of the force \( f_a \). The 4-vector force density vanishes, \( f_k = 0 \), for a closed physical system that does not interact with other systems.

Let us raise the first index, \( t^{ij} = g^{ik} t_{ki}^j \). The resulting covariant tensor is not symmetric in general: \( t^{ij} \neq t^{ji} \). In particular, this means that the momentum density is not equal to the energy flux density, \( p \neq s/c^2 \). There is nothing unphysical about this fact which may be explained, for example, by the nontrivial intrinsic angular momentum (spin) of the elements of the system \([98]\). A well known symmetrization prescription of Belinfante and Rosenfeld \([99,100,101]\) allows one to construct from the original intrinsic angular momentum (spin) of the elements of the system \([98]\). A well known symmetrization procedure is possible for the case when a timelike vector field \( u^i \) is available, with \( u^2 = u_i u^i > 0 \). Physically, \( u^i \) can be interpreted as a 4-velocity of the system with respect to the inertial reference frame. With the help of this field, we can introduce the transversal and longitudinal projectors \( \bar{P}^i_j = \delta^i_j - u^i u_j / u^2 \) and \( \underline{P}^i_j = u^i u_j / u^2 \). Let us denote the antisymmetric part of the original energy-momentum tensor by \( a^{ij} := t^{[ij]} = (t^{ij} - t^{ji})/2 \). Then we define

\[ a^{ij} := t^{ij} - \bar{P}^i_j a^j_i - \underline{P}^i_j a_k^i. \]

By construction, this object is symmetric,

\[ a^{ij} = t^{(ij)} = \frac{2}{u^2} u^{(i} u_k a^{j)} k, \tag{15} \]

and has the following crucial property: For the system at rest, \( u^i = \delta^i_0 \), the time-time and space-space components coincide with the original symmetrized ones, \( a^{00} = t_0^0 = w \) and \( a^{ab} = t^{(ab)} \), whereas the off-diagonal (time-space and space-time) components read

\[ a^{a0} = t^{a0} - 2u^a u^0 = t^{0a}, \quad a^{0a} = t^{0a}. \]
In other words, the energy flux density (“Poynting vector”) remains the same as before, \( A^a = 0^a = t_0^a = s^a \), but the momentum density \( p^a \) is replaced with \( A^a = 0^a = t_0^a = s^a / c^2 \).

The Belinfante-Rosenfeld procedure and the \textit{abrahamization} \((14)\) of the energy-momentum tensor both produce symmetric tensors from a given original asymmetric energy-momentum \( t^i_k \). However, these two symmetrization schemes are distinct in the following significant point. For the Belinfante-Rosenfeld approach, the balance equation remains untouched because the divergence \( \partial_i B_k^i = \partial_i t_k^i \) is preserved. In contrast, the divergence \( \partial_i A_k^i \neq \partial_i t_k^i \). Let us find the difference of the two force densities, \( f_k = \partial_i t_k^i \) and \( f_k = \partial_i A_k^i \). From \((14)\),

\[
A^A f_k = f_k - \partial_i (\tilde{P}^i_k a_j^i) - \partial_i (\tilde{P}^i_j a_k^i).
\]  

(17)

In the rest frame, \( u^i = \delta^i_0 \), we find \( A^A f_0 = f_0 \) and

\[
A^A f = f + \frac{\partial}{\partial t} \left( p - \frac{s}{c^2} \right) + \nabla \times a.
\]  

(18)

We use the boldface notation for the spatial 3-dimensional objects. In particular, \( f = \{ f_a \}, p = \{ p_a \}, s = \{ s_a \} \), whereas the antisymmetric part of the stress tensor gives rise to \( a = \{ \frac{1}{2} \varepsilon_{abc} a^a a^b \} \).

As such, the definition \((14)\) looks rather artificial. To the best of my knowledge, there does not exist any derivation of \((14)\) from first principles, such as from the Lagrange-Noether machinery, for example. One merely demands the symmetry of the energy-momentum tensor under the condition that the energy density and the energy flux density in the rest frame remain the same. The form of the resulting \( t_k^i \) is then fixed by \((14)\) which leads to the additional “Abraham force” terms \((18)\) in the balance equations.

4 Canonical energy-momentum for open systems

It is sometimes claimed that the symmetry is a fundamental property of the energy-momentum tensor that is related to the conservation of the angular momentum of the system. In order to clarify this point, let us recall the relevant facts of the Lagrange and Noether theory. Suppose we have a system of the fields denoted collectively by \( \Phi^A \). Its dynamics is described by the action integral \( I = \int \mathcal{L} d^4 x \), where the Lagrangian (density) is a function of the fields and their derivatives, \( \mathcal{L} = \mathcal{L}(\Phi^A, \partial_i \Phi^A) \), and \( x^i = (t, x^a) \) are the (local) spacetime coordinates.

When the action \( I \) is invariant under the \textit{spacetime translations}, one finds the conservation law\(^2\)

\[
\partial_i \Sigma_k^i = - \frac{\delta \mathcal{L}}{\delta \Phi^A} \partial_k \Phi^A.
\]  

(19)

Here the \textit{canonical energy-momentum tensor} is defined by

\[
\Sigma_k^i = \frac{\partial \mathcal{L}}{\partial (\partial_i \Phi^A)} \partial_k \Phi^A - L \delta_k^i,
\]  

(20)

and we use standard notation for the the variational derivative,

\[
\frac{\delta \mathcal{L}}{\delta \Phi^A} := \frac{\partial \mathcal{L}}{\partial \Phi^A} - \partial_i \left( \frac{\partial \mathcal{L}}{\partial (\partial_i \Phi^A)} \right).
\]  

(21)

\(^2\) We omit the derivations of these results which are well known since the work of Noether \cite{102}; for the detailed discussion see Sec. 19 of \cite{103}, for example.
Similarly, assuming that action is invariant under the Lorentz group, that acts on the coordinates and fields, we find

\[ \Sigma_{(jk)} + \partial_i S_{jk}^i = -\frac{\delta L}{\delta \Phi^A} (\rho_{jk})^A_B \Phi^B. \]  \hspace{1cm} (22)

Here \((\rho^i)^A_B\) are the Lorentz generators for the fields \(\Phi^A\), and the spin current density is introduced by

\[ S_{jk}^i = \frac{\partial L}{\partial (\partial_i \Phi^A)} (\rho_{jk})^A_B \Phi^B. \]  \hspace{1cm} (23)

Recall that the Lorentz generators are skew-symmetric, \((\rho_{jk})^A_B = - (\rho_{kj})^A_B\). Hence the spin is skew-symmetric too, \(S_{jk}^i = - S_{kj}^i\).

When the physical system is closed in the sense that it does not interact with other systems, then its state is completely described by the variables \(\Phi^A\) and their dynamics in time and space. The latter is determined by the field equations, \(\delta L/\delta \Phi^A = 0\). Using this in the conservation laws (19) and (22), we obtain \(\partial_i \Sigma_{k}^i = 0\) and \(\Sigma_{(jk)} + \partial_i S_{jk}^i = 0\). Accordingly, we find that the canonical energy-momentum tensor is symmetric when the spin is trivial, \(S_{jk}^i = 0\). Otherwise, we can use the Belinfante-Rosenfeld [99, 100, 101] procedure to derive the symmetric tensor

\[ \sigma_{k}^i = \Sigma_{k}^i - \partial_j (S_{ij}^k + S_{kj}^i - S_{kij}). \]

By construction, we have \(\partial_i \sigma_{k}^i = \partial_i \Sigma_{k}^i\), and \(\sigma_{ik} = 0\).

However, the above is true for closed systems only. For open systems that interact with other systems, we cannot put \(\delta L/\delta \Phi^A = 0\) and we have to keep the corresponding terms on the right hand sides of (19) and (22). These terms describe forces and torques which result from the interaction of the systems. Even when the spin is absent, the canonical energy-momentum of an open physical system is, in general, necessarily asymmetric in order to maintain the balance of the nontrivial torque present on the right-hand side of (22). We will see below how this works for the electromagnetic field interacting with polarizable and magnetizable matter.

## 5 Energy-momentum tensors in electrodynamics

After the preparations done in the previous three sections, we are now in a position to start the main discussion. Here we consider the construction of the canonical and symmetric energy-momentum tensors in the macroscopic electrodynamics of media. The corresponding Lagrangian approach was formulated in Sec. 2. Our starting point is the action \(I^e = \int L^e d^4 x\).

Without losing generality, we can choose the collective field of the system as \(\Phi^A = (A_i, \chi_{ijkl})\). The set of all fields is thus naturally divided into the two sectors, an electromagnetic and a material one. It is not necessary to specify how the tensor density \(\chi_{ijkl}\) depends on the more fundamental material variables (we can always do this at a later stage), but instead it is convenient to treat \(\chi_{ijkl}\) itself as a generalized material variable.

### 5.1 Canonical energy-momentum tensor

The Lagrangian (1) contains only derivatives of the electromagnetic potential \(A_i\) but not of \(\chi_{ijkl}\), and thus we easily derive from (20) and (1) the canonical energy-momentum tensor of the system:

\[ \Sigma_{k}^i = -H^{ij} F_{kj} - L^e \delta_{k}^i = -H^{ij} F_{kj} + \frac{1}{4} H^{jl} F_{jl} \delta_{k}^i. \]  \hspace{1cm} (24)

This is the well known Minkowski energy-momentum. Since the system is obviously open, the canonical tensor is not conserved. This becomes clear when we calculate the variational derivatives that enter the right-hand side of (19). Indeed, we find explicitly

\[ \Lambda^i := \frac{\delta L^e}{\delta A_i} = -\partial_j H^{ij}, \quad \frac{\delta L^e}{\delta \chi_{ijkl}} = -\frac{1}{8} F_{ij} F_{kl}. \]  \hspace{1cm} (25)
Both variational derivatives are clearly nontrivial. We should take into account that this (electromagnetic field) system interacts with the matter, and the total Lagrangian is thus the sum $L^{\text{tot}} = L^e + L^m$. After introducing the electric current $J^i = \delta L^m / \delta A_i$, one then derives the Maxwell field equation

$$\frac{\delta L^{\text{tot}}}{\delta A_i} = \Lambda^i + J^i = -\partial_j H^{ij} + J^i = 0. \quad (26)$$

This shows that $\Lambda^i$ vanishes only in the absence of electric charge and current densities of matter, but in general $\Lambda^i = -J^i \neq 0$. Substituting (25) into (19), we derive the energy-momentum balance equation for linear Maxwellian electrodynamics as

$$\partial_i \Sigma^e_{ki} = F_{ki} J^i + X_k, \quad X_k = \frac{1}{8} F_{ij} F_{mn} \partial_k \chi^{ijmn}. \quad (27)$$

The first term on the right-hand side is the familiar Lorentz force, whereas the second term requires some analysis. At first, we notice that

$$X_k = \frac{1}{4} \left( F_{ij} \partial_k H^{ij} - H^{ij} \partial_k F_{ij} \right) = \frac{1}{4} \left( F_{ij} \partial_k M^{ij} - M^{ij} \partial_k F_{ij} \right), \quad (28)$$

where we used (11). Now, after some straightforward algebra, we can bring this expression into the form

$$X_k = -\partial_i \Sigma^p_{ki} - F_{ki} \partial_j M^{ij}, \quad (29)$$

after introducing the polarizational energy-momentum tensor

$$\Sigma^p_{ki} := M^{ij} F_{kj} - \frac{1}{4} M^{jl} F_{jl} \delta^i_k. \quad (30)$$

The final step is to recall (12) and to substitute (29) into (27). After rearranging the terms, the result reads:

$$\partial_i \left( \Sigma^e_{ki} + \Sigma^p_{ki} \right) = F_{ki} \left( J^i + \Sigma^p_{ki} \right). \quad (31)$$

The form of this equation suggests to define the total electromagnetic energy-momentum tensor and the total electric current as the sums

$$\Sigma_{ki}^{\text{tot}} := \Sigma^e_{ki} + \Sigma^p_{ki}, \quad J^i := J^i + \Sigma^p_{ki}. \quad (32)$$

The physical interpretation is clear: $J^i$ is the current density of the free charges, and $\Sigma^p_{ki}$ is the polarizational current density of the bound charges. The corresponding energy and momentum, associated with the free and bound charges, are described by $\Sigma^e_{ki}$ and $\Sigma^p_{ki}$, respectively. As a result, the balance equation (31) is recast into

$$\partial_i \Sigma_{ki}^{\text{tot}} = F_{ki} J^i, \quad (33)$$

with the Lorentz force on the right-hand side that acts on all types of charges (free and bound) present in the medium.

Combining (24) and (30), and using the constitutive relation (11), we derive the explicit form of the total energy-momentum:

$$\Sigma_{ki}^{\text{tot}} = (-H^{ij} + M^{ij}) F_{kj} + \frac{1}{4} (H^{jl} - M^{jl}) F_{ji} \delta^i_k = \frac{1}{\mu_0} \left( -F^{ij} F_{kj} + \frac{1}{4} F^{jl} F_{jl} \delta^i_k \right). \quad (34)$$

This energy-momentum tensor was discussed by Poincelot [70, 71, 72, 73] and more recently by us [78].
Quite remarkably, we discover that the form of the total electromagnetic energy-momentum tensor in media is precisely the same as in vacuum. It is worthwhile to stress that the final result (33) is a direct consequence of the fact that the electromagnetic field is an open system (1). The energy and momentum of the electromagnetic field are described by the canonical (=Minkowski) tensor (24), but since the system is open, the balance equation (27) contains nontrivial force terms that arise from the interaction of the field with matter. A careful evaluation of the force \( X_k \) then eventually brings the balance equation into the final form (33). This general result is the best what can be done without entering into the question about the structure and dynamics of the medium. In the next section we will discuss a specific model of matter and demonstrate how the additional knowledge of the physical nature of a medium can provide a different computation of the force term \( X_k \).

\[ \Delta \Sigma = \Sigma - \Sigma_\text{Minkowski}. \]

5.2 Balance of the angular momentum

The canonical energy-momentum (24) is not symmetric. However, this does not mean that there is a problem with the angular momentum conservation. We simply have to recall once again that the system is open, and hence the right-hand side of the angular momentum balance equation (22) does not vanish, because the variational derivatives (25) are nontrivial.

The Lorentz generators for the material variable \( \chi^{ijkl} \) read

\[
\frac{\delta L}{\delta \chi_{mnlpq}} \rho^{jkn}_{m' n' p' q'} = \frac{1}{8} F_{[k|l} F_{p|q]} R_{j]}^{n} \equiv F_{[k|l} H_{j]}^{n}. \tag{36}
\]

We used (3) here. On the other hand, the antisymmetric part of the canonical energy-momentum (24) is straightforwardly found to be

\[
\Sigma_{[jk]} = - H_{[k} \chi_{j]}^{n} = F_{[k|l} H_{j]}^{n}. \tag{37}
\]

We thus demonstrated that the balance equation (22) of the angular momentum,

\[
\Sigma_{[jk]} \equiv - \frac{\delta L}{\delta \chi_{mnlpq}} \rho^{jkn}_{m' n' p' q'} \chi^{m' n' p' q'}, \tag{38}
\]

is satisfied identically for the Minkowski canonical energy-momentum. In physical terms, the skew part of the energy-momentum tensor is perfectly balanced by the torque which is present on the right-hand side because of the interaction of the field with matter.

There is a somewhat subtle point which we silently avoided in our previous discussion. However, for completeness we have to mention it now. Namely, it is well known that there exist two possible choices because of the interaction of the field with matter.

\[
\delta L = \delta_{m' n' p' q'}^{[jk]} \rho^{jkn}_{m' n' p' q'} \chi^{m' n' p' q'}, \tag{35}
\]

Using (25) we then immediately find

\[
\Sigma_{[jk]} = - H_{[k} \chi_{j]}^{n} = F_{[k|l} H_{j]}^{n}. \tag{37}
\]

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There is a somewhat subtle point which we silently avoided in our previous discussion. However, for completeness we have to mention it now. Namely, it is well known that there exist two possible choices because of the interaction of the field with matter.

\[
\delta L = \delta_{m' n' p' q'}^{[jk]} \rho^{jkn}_{m' n' p' q'} \chi^{m' n' p' q'}, \tag{35}
\]

Using (25) we then immediately find

\[
\Sigma_{[jk]} = - H_{[k} \chi_{j]}^{n} = F_{[k|l} H_{j]}^{n}. \tag{37}
\]

We thus demonstrated that the balance equation (22) of the angular momentum,
But at the same time, a new term appears on the right-hand side of (22):

$$- \delta L^e \delta A^e_n (\rho_{jk})^m_n A_m = \Lambda[j]A[k] = (\partial_i H^i_{[j]} A_k).$$

(40)

Accordingly, we find identically

$$\Delta \Sigma_{[jk]} + \partial_i S_{jk i} = - \delta L^e \delta A^e_n (\rho_{jk})^m_n A_m.$$

(41)

Thus we come to the same conclusion that the angular momentum balance equation is perfectly satisfied despite the lack of the symmetry of the canonical energy-momentum tensor.

5.3 Abraham energy-momentum tensor

In Sec. 3, we described a general procedure for constructing a symmetric tensor from an arbitrary energy-momentum. Now we can apply this scheme to the electromagnetic field, taking the canonical energy-momentum (24) as an input. Although we have demonstrated above that the argument of the “violation of the angular momentum conservation” is unsubstantiated for open systems, it seems reasonable to perform a detailed comparison of the energy-momentum tensors available on the market.

In this approach, we assume the existence of the timelike vector field $u^i$, with $u^2 = c^2$. Then substituting (24) and (37) into the definition (14), we construct the Abraham tensor of the electromagnetic energy-momentum:

$$\Lambda_k^i = - \frac{1}{2} (H^{ij} F_{kj} + F^{ij} H_{kj}) + \frac{1}{4} H^{jl} F_{jl} \delta^i_k + \frac{1}{2c^2} \left[ u^i u_l (F_{jk} H^{jl} - H_{jk} F^{jl}) + u_k u^l \left( F^{ji} H_{jl} - H^{ji} F_{jl} \right) \right].$$

(42)

In the rest frame, $u^i = \delta_0^i$, we find the Abraham energy flux density $\Lambda^a_s = \Lambda^0_0 a$ and the field momentum $\Lambda^a_p = - \Lambda^0_a$.

$$\Lambda^a = \frac{\Lambda^a}{c^2} = \frac{E \times H}{c^2}.$$

(43)

The Abraham force (18) then finally reduces to its well known rest-frame expression

$$\Lambda f = f + \frac{\partial}{\partial t} \left( D \times B - \frac{E \times H}{c^2} \right) + \frac{1}{2} \nabla \times \left( D \times E + H \times B \right).$$

(44)

The last word in deciding which energy-momentum tensor is correct (and in which sense) belongs certainly to experiment. However, the theoretical foundation of the Minkowski tensor obviously appears to be far more solid than that of the Abraham tensor. The former arises as a canonical energy-momentum tensor in the Lagrange-Noether framework, whereas the latter one is not derived from first principles. Nevertheless, as we will demonstrate in the next section, the Abraham tensor does resurface after we specify the structure and the dynamics of matter.

6 Variational model of matter

Let us now consider the dynamics of the medium. We will model the matter as an ideal fluid, the elements of which are structureless particles (i.e., no spin or other internal degrees of freedom are present). Such a continuous medium (see [104] [105] [106], for example, for the relevant earlier work on the relativistic ideal fluids) is characterized in the Eulerian approach by the fluid 4-velocity $u^i$, the internal energy density $\rho$, the particle density $\nu$, the entropy density $s$, and the identity (Lin) coordinate $X$. Furthermore, we assume that
the motion of a fluid is such that the number of particles is constant and that the entropy and the identity of the elements is conserved. In mathematical terms this means that the following constraints are imposed on the variables:

\[
\frac{\partial}{\partial i} (\nu u^i) = 0, \quad \frac{\partial u^i}{\partial s} = 0, \quad \frac{\partial u^i}{\partial X} = 0.
\]

Due to the conservation of the entropy only reversible processes are allowed. In a variational approach, these constraints are taken into account by means of Lagrange multipliers. The classical action of the fluid reads

\[
I^m = \int L^m d^4x,
\]

with the Lagrangian density

\[
L^m = -\rho (\nu, s) + \Lambda_0 (u^i u_i - c^2) - \nu u^i \partial_i \Lambda_1 + \Lambda_2 u^i \partial_i s + \Lambda_3 u^i \partial_i X.
\]

The Lagrange multipliers \(\Lambda_1, \Lambda_2, \Lambda_3\) impose the constraints \((45)-(47)\) on the dynamics of the fluid, whereas \(\Lambda_0\) accounts for the normalization condition for the 4-velocity

\[
g_{ij} u^i u^j = c^2.
\]

For the description of the thermodynamical properties of the fluid, the usual thermodynamical law (“Gibbs relation”) is used,

\[
T ds = d(\rho/\nu) + p d(1/\nu),
\]

where \(T\) is the temperature and \(p\) the pressure. From this we have

\[
\frac{\partial \rho}{\partial s} = \nu T, \quad \frac{\partial \rho}{\partial \nu} = \rho + p \nu.
\]

One can in fact treat the above equations as the definition the temperature and the pressure.

### 6.1 Equations of motion

Recalling the general formalism of Sec. 4, we describe the material system (ideal fluid) by a collective field as \(\Phi^A = (u^i, \nu, s, X, \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)\). The last six variables characterize only matter, i.e., they enter the matter Lagrangian \(L^m\) but not the Lagrangian of the electromagnetic field \(L^e\). The latter depends, in general, on the velocity of the medium \(u^i\) and on the particle density \(\nu\). This is a manifestation of the fact that both systems (electromagnetic and material) are open.

The variational derivatives \(\delta L^m / \delta \Lambda_K = 0, K = 0, 1, 2, 3\), with respect to the Lagrange multipliers yield the constraints \((45)-(47), (49)\), whereas variation of \(L^m\) with respect to \(s\) and \(X\) yields, respectively,

\[
\frac{\delta L^m}{\delta s} = \frac{\partial}{\partial i} (2\Lambda_2 u^i) + \frac{T \nu}{c} = 0, \quad \frac{\delta L^m}{\delta X} = \frac{\partial}{\partial i} (\Lambda_3 u^i) = 0.
\]

In order to derive \((53),(54)\), we used \((51)\).

It remains to find the variations with respect to \(\nu\) and \(u^i\). The direct calculation (use \((52)\)) yields

\[
\frac{\delta L^m}{\delta \nu} = -\frac{\rho + p}{\nu} - u^i \partial_i \Lambda_1, \quad \frac{\delta L^m}{\delta u^i} = 2\Lambda_0 u_i - \nu \partial_i \Lambda_1 + \Lambda_2 \partial_i s + \Lambda_3 \partial_i X.
\]
We cannot put these equations equal to zero because the material system is open. Contracting (56) with $u^i$, after using (55) and the constraints (46) and (47), we find the Lagrange multiplier:

$$2\Lambda_0 = \frac{1}{c^2} \left( -\rho - p - \nu \frac{\delta L^m}{\delta \nu} + u^i \frac{\delta L^m}{\delta u^i} \right).$$

(57)

Substituting this back into (56), we derive the following useful relation

$$-\nu \partial_i \Lambda_1 + \Lambda_2 \partial_i s + \Lambda_3 \partial_i X = \frac{u_i}{c^2} \left( \rho + p + \nu \frac{\delta L^m}{\delta \nu} \right) + \hat{P}_i \frac{\delta L^m}{\delta u^i}.$$

(58)

Here, as usual, $\hat{P}_i = \delta_i - u^i u_j / c^2$ is the transversal projector. Substituting this into (48), we see that “on-shell” (i.e., if the equations of motion are fulfilled) the Lagrangian of the medium satisfies

$$L^m = p + \nu \frac{\delta L^m}{\delta \nu}.$$

(59)

6.2 Canonical energy-momentum of matter

Since $L^m$ depends only on the derivatives of $s, X, \text{ and } \Lambda_1$, we have

$$\frac{\partial L^m}{\partial (\partial_i \Phi^A)} \partial_k \Phi^A = u^i \left( -\nu \partial_i \Lambda_1 + \Lambda_2 \partial_i s + \Lambda_3 \partial_i X \right).$$

(60)

Using (58) and (59), we then straightforwardly construct, from the definition (20), the canonical energy-momentum tensor of matter:

$$\Sigma_k^i = u^i P_k + \left( -\delta_i^k + \frac{1}{c^2} u_k u^i \right) p^{\text{eff}}.$$

(61)

Here we denoted

$$P_k = \frac{\rho}{c^2} u_k + \frac{\delta L^m}{\delta u^k} - \frac{u_k u^j}{c^2} \frac{\delta L^m}{\delta u^j},$$

(62)

$$p^{\text{eff}} = p + \nu \frac{\delta L^m}{\delta \nu}.$$

(63)

The physical interpretation of these quantities is clear. The 4-vector (62) is the relativistic 4-momentum density effectively carried by the elements of matter. The first term on the right-hand side is the usual kinetic momentum determined by the mass (energy density) of the particles, whereas the two next terms arise from the interaction of the medium with the electromagnetic field. The same applies to the second term of the effective pressure (63) which “corrects” the usual hydrodynamical pressure by the term arising due to the fact that the material system is open. For the closed system we would have to put $\delta L^m / \delta u^k = 0$ and $\delta L^m / \delta \nu = 0$, and then the energy-momentum (61) would reduce to the standard expression of the ideal fluid.

Since the material system is open, the divergence of the energy-momentum tensor is nontrivial. The energy-momentum balance equation (19) now reads

$$\partial_i \Sigma_k^i = -\frac{\delta L^m}{\delta \nu} \partial_k \nu - \frac{\delta L^m}{\delta u^i} \partial_k u^i.$$

(64)

As usual, the right-hand side describes the forces that the electromagnetic field exerts on the matter.

Let us now inspect the angular momentum balance equation. As a first step, we notice that only the 4-velocity vector field $u^i$ of the material variable $\Phi^A = (u^i, \nu, s, X, \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$ has nontrivial transformation properties under the action of the Lorentz group. The corresponding generators read

$$\frac{\partial}{\partial x^j} \delta m^i = \partial_j \delta m^i.$$ However, since $\partial L^m / \partial (\partial_j u^i) = 0$, the spin density of the medium vanishes, $S_{jk}^i = 0$. Copyright line will be provided by the publisher
The canonical energy-momentum of matter is not symmetric,
\[ m \Sigma_{[jk]} = P_{[jk]} = \frac{\delta L}{\delta u^j} u_k. \]  
(65)

However, the right-hand side of (22) is now
\[- \frac{\delta L}{\delta u^m} (\rho_{jk})^m u^n = - \frac{\delta L}{\delta u^m} u_j. \]  
(66)

We thus verify that the angular momentum balance equation (22) is satisfied
\[ m \Sigma_{[jk]} \equiv - \frac{\delta L}{\delta u^m} (\rho_{jk})^m u^n \]  
(67)
identically for the asymmetric canonical energy-momentum of the medium.

7 Coupled system of the electromagnetic field and matter

We finally can complete the picture by combining the two pieces which we studied separately above: the electromagnetic system of Sec. 2 and the material system of Sec. 6. In order to do this, we need one important additional input. Namely, we need to specify how exactly the two systems interact. In the most general form, the information about this interaction is encoded in the constitutive tensor density \( \chi^{ijkl} \) in the electromagnetic Lagrangian (1).

The model of the fluid, which we studied in the previous section, obviously describes the polarizable/magnetizable medium with isotropic electric and magnetic properties. Such matter is characterized by the two scalar quantities: the permittivity \( \varepsilon \) and the permeability \( \mu \). In order to take into account the possible electrostriction and magnetostriction effects, we allow for these quantities to be the functions of the particle density,
\[ \varepsilon = \varepsilon(\nu), \quad \mu = \mu(\nu). \]  
(68)

We thus assume that the medium is intrinsically isotropic, and the possible anisotropic effects may arise only from the nontrivial motion of the medium, i.e., they are induced by the velocity of the fluid.

7.1 Constitutive relation and the electromagnetic Lagrangian

The constitutive relation for an arbitrarily moving isotropic medium is well known. It is given by the famous Minkowski equations, see [90], Sec.E.4.2, eqs.(E.4.28)-(E.4.29) and (E.4.25)-(E.4.26). The constitutive tensor for this case is given by the formula that is very close to (5), namely,
\[ \chi^{ijkl} = \lambda \sqrt{-g_{opt}} \left( g^{ik}_{opt} g^{jl}_{opt} - g^{il}_{opt} g^{jk}_{opt} \right), \]  
(69)

where \( \lambda = \sqrt{\varepsilon \varepsilon_0 / \mu \mu_0} \) and the so-called optical metric was first introduced by Gordon [107]:
\[ g^{ij}_{opt} = g^{ij} - \frac{1 - n^2}{c^2} u^i u^j. \]  
(70)

Here \( n = \sqrt{\varepsilon \mu} \) is the refraction coefficient of the medium.

We straightforwardly find \( \sqrt{-g_{opt}} = c/n \), and thus the Lagrangian (1) of the electromagnetic field finally reads
\[ L^e = - \frac{1}{4\mu_0 \mu} g^{ij}_{opt} g^{kl}_{opt} F_{ik} F_{jl} = - \frac{1}{4\mu_0 \mu} \left[ F_{ij} F^{ij} + 2\frac{n^2 - 1}{c^2} F_{ik} u^k F^{il} u^l \right]. \]  
(71)
The components of the electromagnetic field $F_{ij}$ are given with respect to the laboratory system. In order to get some further insight into the formulas above, we notice that the electric and magnetic fields in the comoving system (in which the medium is momentarily at rest) can be described by the 4-vectors

$$E^i := F^i u^k, \quad B^i := \frac{1}{2c} \eta^{ijkl} F_{jk} u_l.$$ (72)

The field strength is uniquely reconstructed from these vectors as

$$F_{ij} = \frac{1}{c^2} \left( E_i u_j - E_j u_i + c \eta_{ijkl} u^k B^l \right).$$ (73)

The electromagnetic Lagrangian (71) looks remarkably simple in these variables:

$$L^e = -\frac{1}{2} \left( \varepsilon \varepsilon_0 E^2 - \frac{B^2}{\mu_0} \right),$$ (74)

with the obvious abbreviations $E^2 = E_i E^i$ and $B^2 = B_i B^i$. Both vectors are by construction orthogonal to the 4-velocity, $E_i u^i = 0$ and $B_i u^i = 0$. Hence, $E^2 \leq 0$ and $B^2 \leq 0$.

In Sec. 5 we treated the whole constitutive tensor density $\chi_{ijkl}$ as the material variable. Now, after specifying the model of the medium, we have something better. The fluid under consideration is described by the variables $u^i$ and $\nu$, and the constitutive tensor now becomes a known function $\chi_{ijkl}(u^m, \nu)$, given by the equation (69). The corresponding variational derivatives of the electromagnetic Lagrangian with respect to the material variables are easily computed:

$$\frac{\delta L^e}{\delta u^i} = -\frac{n^2 - 1}{\mu_0 c^2} F_{ki} F^{kl} u_l,$$ (75)

$$\frac{\delta L^e}{\delta \nu} = -\frac{1}{2} \left( \varepsilon_0 \frac{\partial \varepsilon}{\partial \nu} E^2 + \frac{1}{\mu_0 c^2} \frac{\partial \mu}{\partial \nu} B^2 \right).$$ (76)

We used (71) to derive (75), however, it is much simpler to use a different (equivalent) form of the Lagrangian (74) to obtain (76).

### 7.2 Canonical energy-momentum tensor

After all these preliminaries, we are now in a position to find the explicit form of the energy-momentum tensors and to analyse the corresponding balance equations.

The electromagnetic excitation (2) for the Lagrangian (71) is

$$H_{ij} = \frac{1}{\mu_0 c} g_{ik}^{\text{opt}} g_{jl}^{\text{opt}} F_{kl} = \frac{1}{\mu_0 c} \left[ F_{ij} + \frac{n^2 - 1}{c^2} \left( F_{ik} u_k u^j - F_{jk} u^k u^i \right) \right].$$ (77)

Substituting this into (24), we find the explicit form of the canonical (Minkowski) energy-momentum of the electromagnetic field:

$$\Sigma_k^i = \frac{1}{\mu_0} \left[ - F_{ij} F_{kj} + \frac{1}{4} F_{ij} F_{kl} \delta^i_k + \frac{n^2 - 1}{c^2} \left( F_{kn} F^{nl} u_l u^i - F_{kl} u^l F^{in} u_n + \frac{1}{2} F_{nl} u^l F^{nj} u_j \delta^i_k \right) \right].$$ (78)

This tensor is not symmetric, and its antisymmetric part (37) now reads explicitly

$$\Sigma_{[jk]} = -\frac{n^2 - 1}{\mu_0 c^2} u_{[j} F_{k]} F^{ij} u^i.$$ (79)
7.3 Balance equations of the energy-momentum and angular momentum

As we know, the Minkowski tensor (78) is not conserved, and the energy-momentum balance equation (27) contains a nontrivial force on the right-hand side, \( X_k = \frac{1}{8} F_{ij} F_{mn} \partial_k \chi^{ijmn} \). Since \( \chi^{ijkl} = \chi^{ijkl}(u^m, \nu) \), we have

\[
\partial_k \chi^{ijmn} = \frac{\partial \chi^{ijmn}}{\partial \nu} \partial_k \nu + \frac{\partial \chi^{ijmn}}{\partial u^m} \partial_k u^m.
\] (80)

Then the force density can be easily calculated,

\[
X_k = \frac{\partial}{\partial \nu} \left( \frac{1}{8} \chi^{ijkl} F_{ij} F_{kl} \right) \partial_k \nu + \frac{n^2 - 1}{\mu_0 c^2} F^{ij} u_j F_{im} \partial_k u^m
\]

\[
- \delta L_e \frac{\delta \nu}{\delta \nu} \partial_k \nu - \delta L_e \frac{\delta L^e_\nu}{\delta u^i} \partial_k u^i,
\] (81)

where we used (1) and (75).

Furthermore, for the explicit dependence of the constitutive tensor (69) on \( u^i \) and for the Lorentz generators \((\rho^i)^n_j = \delta^i_j \delta^m_n\) of the 4-velocity vector field, we prove the identity

\[
- \delta L_e \frac{\delta \nu}{\delta \nu} (\rho^i)^m_n u^m = - \frac{n^2 - 1}{\mu_0 c^2} u_j F_{ij} F_{n}^{\nu} u^{\nu}.
\] (82)

Making use of the variational derivative (75), we have

\[
- \delta L_e \frac{\delta \nu}{\delta \nu} (\rho^i)^m_n u^m = \frac{n^2 - 1}{\mu_0 c^2} (\rho^i)^m_n u^m = (\rho^i)^m_n u^m.
\] (83)

Comparing with (79), we thus verify that the balance equation of the angular momentum

\[
\Sigma^{\nu}_{ijk} = - \delta L_e \frac{\delta \nu}{\delta u^i} (\rho^i)^m_n u^m
\] (84)

is identically satisfied in this case, as before.

7.4 Total energy-momentum tensor of the coupled system

Now, the crucial step is to recall that the total system \( L^{\text{tot}} = L^e + L^m \) of the coupled electromagnetic field and medium is closed. Thus,

\[
\frac{\delta L^e_\nu}{\delta \nu} + \frac{\delta L^m_\nu}{\delta \nu} = 0, \quad \frac{\delta L^e_\nu}{\delta u^i} + \frac{\delta L^m_\nu}{\delta u^i} = 0.
\] (85)

By combining (81) with (64), we then finally express the force that acts on the electromagnetic field in terms of the material variables,

\[
X_k = - \partial_i \Sigma^m_k.
\] (86)

Substituting this into (27), and rearranging the terms, we find the true conservation law of the total energy-momentum of the closed system (field + medium):

\[
\partial_i \left( \Sigma^i_k + \Sigma^m_k \right) = 0.
\] (87)

Note that we assumed that the medium is neutral, hence \( J^i = 0 \).
Using the equations of motion \(^{(85)}\), we can find the explicit form of the canonical energy-momentum of medium, too. In particular, combining \(^{(85)}\) with \(^{(75)}\) and \(^{(76)}\), we have

\[
\mathcal{P}_k = \frac{\rho}{c^2} u_k u^i + \frac{n^2 - 1}{\mu \mu_0 c^2} \left( F_{ik} F^{il} u_l - \frac{u_k u^i}{c^2} F_{jn} u^ju^n F^{jl} u_l \right),
\]

\[
 p_{\text{eff}} = p + \frac{1}{2} \left( \varepsilon_0 \frac{\partial \varepsilon}{\partial \nu} \varepsilon^2 + \frac{1}{\mu \mu_0} \frac{\partial \mu}{\partial \nu} B^2 \right).
\]

Remarkably, the effective pressure describes the electro- and magnetostriction effects. Substituting \(^{(88)}\) into \(^{(61)}\), we obtain the final expression

\[
\sum_k^m u_k u^i + \left( -\delta^i_k + \frac{1}{c^2} a_k u^i \right) p_{\text{eff}} + \frac{n^2 - 1}{\mu \mu_0 c^2} \left( u_i F_{nk} F^{nl} u_l - \frac{u_k u^i}{c^2} F_{jn} u^n F^{jl} u_l \right).
\]

The total canonical energy-momentum tensor is the sum of the electromagnetic (Minkowski) tensor \(^{(78)}\) and of the energy-momentum of the fluid \(^{(90)}\). After some simple algebra we find

\[
\sum_k^e u_k u^i + \sum_k^m u_k u^i = \frac{\rho}{c^2} u_k u^i + \left( -\delta^i_k + \frac{1}{c^2} a_k u^i \right) p_{\text{eff}} + \frac{1}{\mu \mu_0} \left[ -F^{ij} F_{kj} + \frac{\varepsilon_0}{\mu \mu_0} \left( \varepsilon^{ij} F_{nk} F^{nl} u_l - \frac{u_k u^i}{c^2} F_{jn} u^n F^{jl} u_l + \frac{1}{2} F_{nl} u^l F^{nj} u_j \delta^i_k \right) \right].
\]

It is worthwhile to note that whereas the Minkowski electromagnetic tensor and the energy-momentum of the fluid are both asymmetric, the total canonical energy-momentum is explicitly symmetric. This is in agreement with the fact that the total system is closed. Indeed, combining the balance equations \(^{(67)}\) and \(^{(84)}\) of the angular momenta of the electromagnetic and material systems, we find

\[
\sum_{\text{em}} + \sum_{\text{m}} = - \left( \frac{\delta L_e}{\delta u^m} + \frac{\delta L_m}{\delta u^m} \right) (\rho_{jk})^n u^n = 0.
\]

To derive the last equality, we used the field equations \(^{(85)}\).

### 7.5 Abraham energy-momentum tensor

For completeness, let us compute the Abraham energy-momentum. In accordance with the definition \(^{(14)}\), we find the projections of the antisymmetric part \(^{\mathcal{A}}\) of the Minkowski tensor \(^{(79)}\):

\[
\frac{\mathcal{A}}{\mathcal{P}} \eta_{aj} = \frac{1}{2} \frac{\partial \mu}{\partial \nu} u^i \left( F_{kn} - \frac{u_k u^i}{c^2} F_{jn} \right) F^{nl} u_l.
\]

The resulting electromagnetic Abraham energy-momentum tensor reads

\[
\sum_k^e u_k u^i + \sum_k^m u_k u^i = \frac{\rho}{c^2} u_k u^i + \left( -\delta^i_k + \frac{1}{c^2} a_k u^i \right) p_{\text{eff}} + \frac{n^2 - 1}{\mu \mu_0 c^2} \left( u_i F_{nk} F^{nl} u_l - \frac{u_k u^i}{c^2} F_{jn} u^n F^{jl} u_l + \frac{1}{2} F_{nl} u^l F^{nj} u_j \right).
\]
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So to say, the Abraham tensor “absorbed” all the terms which explicitly contained the electromagnetic field, except for the electrostriction and magnetostriction terms, both from the Minkowski canonical tensor of the electromagnetic field and from the canonical energy-momentum of the matter.

8 Why Abraham?

The fact that the Abraham tensor resurfaced at the end of our derivations requires an additional analysis. Whereas the Minkowski tensor has a solid standing as a canonical energy-momentum deeply rooted in the Lagrange-Noether formalism, the Abraham tensor appears a rather artificial construct, as we saw in Sec. 3. So, why is it recovered in the equation (95)? Is this a coincidence, a specific feature of the models that we used for the description of the electromagnetic and material systems, or something more fundamental? Here we demonstrate that the latter is true, in a certain sense.

8.1 A useful mathematical fact

At first, we prove the following technical point. Let \( \alpha_{ij} \) be an antisymmetric tensor and \( \beta_k \) a 4-vector. Then \( \alpha_{ij} \) satisfies the algebraic equation

\[
\frac{1}{2} P^i_j \alpha_{ij} + \frac{1}{2} P^j_i \alpha_{kj} = u^i \frac{1}{2} P^j_k \beta_j
\]

if and only if this tensor is constructed from \( \beta_k \) and the velocity \( u^i \) as

\[
\alpha_{ij} = \beta_i [u_j].
\]

The proof is straightforward. Let us assume that (97) is true. Then

\[
\frac{1}{2} P^i_j \alpha_{ij} = \left( \delta^i_j - \frac{u_k u^j}{c^2} \right) \frac{1}{2} (\beta_k u^i - \beta^i u_k) = \frac{u^i}{2} P^j_k \beta_j,
\]

\[
\frac{1}{2} P^j_i \alpha_{kj} = \frac{u^i u^j}{c^2} \frac{1}{2} (\beta_k u^j - \beta^j u_k) = \frac{u^i}{2} P^j_k \beta_j.
\]

Hence, (96) is fulfilled.

Conversely, suppose (96) is true. Contracting this equation with \( u^i \), we find

\[
\frac{1}{2} P^i_k u_i \alpha_{ij} + u_i \frac{1}{2} P^j_i \alpha_{kj} = 2 u_i \alpha_{ki} + c^2 \frac{1}{2} P^j_k \beta_j \implies u^i \alpha_{ij} = -\frac{c^2}{2} P^j_k \beta^j.
\]

On the other hand, from (96) we have, making use of (100):

\[
\frac{1}{2} P^i_k \alpha_{ij} = -\frac{1}{2} P^j_i \alpha_{kj} + u^i \frac{1}{2} P^j_k \beta_j = \frac{1}{2} u^i P^j_k \beta_j.
\]

This finally yields, again using (100),

\[
\alpha_k^i = \left( \frac{1}{2} P^j_k + \frac{1}{2} P^i_k \right) \alpha_{ij} = \frac{1}{2} \left( u^i P^j_k \beta_j - u_k P^j_i \beta^j \right) = \frac{1}{2} \left( u^i (\beta_k - u_k \beta^i) \right).
\]

Thus, the statement is proved.

8.2 Crucial role of velocity

We now extend the electrodynamical model of Sec. 7.1 by generalizing the constitutive law (69). Namely, we assume that the constitutive tensor \( \chi^{ijkl} \) is not a particular function \( 69 \) of the 4-velocity \( u^i \) and of the particle density \( \nu \), but it rather is some general function

\[
\chi^{ijkl} = \chi^{ijkl}(u^m, \nu, \Psi^{kl}).
\]
Here the collective variable $\psi^{\Omega}$ denotes all possible parameters that describe the state of the polarizable/magnetizable medium. In the macroscopic approach, $\psi^{\Omega}$ includes the most general matrices of dielectric, magnetic, and magnetoelectric susceptibilities. An important assumption is that these parameters are not changed under the Lorentz transformations relating different reference systems.

More precisely, we thus assume that the 4-velocity of the medium $u^i$ is the only (Lorentz) covariant object that enters the electrodynamical constitutive relation \( \rho^{ij}_k \) and the other material variables \( (\nu, \rho, \psi^{\Omega}) \) are all scalars under the Lorentz transformations. Technically, this is realized by assuming that the parameters $\psi^{\Omega}$ take their genuine (or intrinsic) values in the comoving reference system with respect to which the medium is momentarily at rest. In other reference systems, these intrinsic values of $\psi^{\Omega}$ remain the same, whereas in every reference system the form of the constitutive matrices is determined only by the motion of the medium, i.e., in technical terms, by the velocity $u^i$.

This assumption has the far-reaching consequences. Since the velocity $u^i$ is the only covariant argument (with the Lorentz generators $\rho^{ij}_k = \delta_{\alpha}^{ij} \rho_{\alpha k}$) of the constitutive tensor $\chi^{ijkl}(u^m, \nu, \psi^{\Omega})$, we have

$$
\frac{\delta L^e}{\delta \chi^{ijkl}(\rho^{jk})^{mnpq} \partial \chi^{mn} (\rho^{jk})_m u^n} = \frac{\delta L^e}{\delta u^m (\rho^{jk})_m u^n} = \frac{\delta L^e}{\delta u^{jk} u^m}.
$$

(Hint: denoting $\omega^{ij} = -\omega^{ji}$ the parameters of the Lorentz group, $\delta_{\omega} \chi^{mnpq} = \omega^{ij} \chi^{mn} \chi^{pq}$ gives the infinitesimal Lorentz transformation; for differentiation the chain rule completes the proof.)

As a result, we notice that the angular momentum balance equation \( (58) \) is of the form \( (97) \) where

$$
\alpha_{ij} = \Sigma_{[ij]}, \quad \beta_k = \frac{\delta L^e}{\delta u^k}.
$$

Accordingly, combining \( (96) \) with the definition of the Abraham tensor \( (14) \), we find the relation

$$
\Sigma_k = \frac{\delta L^e}{\delta u^k}.
$$

Adding the canonical energy-momentum \( (61), (62) \), we then finally come to the equation \( (95) \).

This analysis demonstrates that the relation \( (95) \) between the canonical energy-momentum tensors and the Abraham energy-momentum is not occasional. It is valid not only for the isotropic constitutive relation \( (69) \) of the specific model which we studied in Sec. \( 7.1 \) but for the general constitutive relation \( (103) \) as well. Sec. \( 8.1 \) provides an explanation why the abrahamization prescription \( (14) \) is successful, after all.

The key to the Abraham tensor is in the 4-velocity of the medium: when $u^i$ is the only (Lorentz) covariant variable that enters the constitutive relation (and only then), the projections of the Minkowski tensor satisfy \( (96) \) that yields \( (106) \). This explains why the Abraham tensor turns out to be relevant for the discussion of the energy and momentum of moving media, especially of the isotropic ones. However, the angular momentum balance equation no longer has the form \( (97) \) if the constitutive tensor depends on additional covariant material variables, and the final nice result \( (106) \) is invalid then. The relevance of the Abraham construction is thus limited.

### 9 Discussion and conclusion

The complete century-long history of the discussion of the electromagnetic energy and momentum in moving media is still to be written. In this short paper, we presented a consistent viewpoint on this problem within the framework of the Lagrange-Noether approach. The canonical energy-momentum tensor appears then as a fundamental structure, together with the relevant balance equations of energy, momentum, and angular momentum.

By a detailed derivation, we explicitly demonstrate that the angular momentum balance equation is always perfectly satisfied for the canonical tensor of the electromagnetic system when one takes care of the fact that it is an open system. Even in absence of the intrinsic (spin) degrees of freedom, the canonical
energy-momentum tensor of an open system does not need to be symmetric, contrary to a widely spread misunderstanding.

The Minkowski energy-momentum tensor (24) arises as a canonical structure in macroscopic Maxwellian electrodynamics, which stresses its fundamental physical nature. The balance equation of an open electromagnetic system (27) contains nontrivial force terms. Without specifying a model for the medium, we demonstrate that the $X_k$ force (28) is related to the bound charges and currents, thus giving rise to the balance equation (33) of the total canonical energy-momentum tensor (34). Its validity for the analysis of the crucial experiments was demonstrated in [78].

Alternatively, the $X_k$ force (28) can be reconstructed in terms of the material variables when a model of the medium is specified. We show how this works in the framework of the variational approach to the relativistic ideal fluid. Here again the canonical energy-momentum (61) appears as a fundamental object. The interaction of matter with the electromagnetic field induces an additional field-dependent term in the material momentum (62) superimposed with the usual hydrodynamical structures. The pressure (63) picks up the electrostriction and magnetostriction contributions.

In contrast to the canonical structures, the Abraham energy-momentum tensor does not arise from first principles. We give a formal prescription that allows one to construct a symmetric tensor (14) from any given energy-momentum, provided a timelike vector field $u^i$ is available. Unlike the other symmetrization procedures (such as the Belinfante-Rosenfeld one, for example), the abrahamization does not preserve the balance equations, it rather introduces the additional forces in (17), (18). When applied to the Minkowski tensor, this prescription yields the famous Abraham electromagnetic energy-momentum (42).

Nevertheless, despite its ad hoc definition, the Abraham tensor turns out to be relevant since it resurfaces at the end from the sum of the two canonical energy-momenta of the electromagnetic and material systems in the eq. (95). It thus appears as a certain “hybrid” construct that absorbs the explicit field-dependent terms from the canonical energy-momenta of the electromagnetic and material systems. This curious fact explains why the Abraham tensor was for such a long time seriously competing with the Minkowski tensor in the analyses of the numerous experiments [88].

This result is not confined to the specific model with the constitutive relation (69), but holds in general for the class of constitutive tensor densities (103) that depend arbitrarily on the material variables, provided the 4-velocity $u^i$ of the medium is the only Lorentz-covariant object. This apparently confirms the recent observations of Dereli, Gratus and Tucker [80, 81] that the Abraham energy-momentum arises as themetrical energy-momentum tensor for media with a general constitutive relation. If we assume that the metric of spacetime is not the flat Minkowski one but is a function of coordinates, the metric energy-momentum tensor density is defined by the variational derivative

\[ \sigma_{ij} := 2c \frac{\delta (L^e + L^m)}{\delta g^{ij}}. \]  

(107)

Indeed, a (long but straightforward) computation then shows that $\sigma_k^i = e_k^i + m_k^i$ for the model (1), (69) and (48) of the isotropic moving medium above. There is a difference, strictly speaking, between our results since neither the Lagrangian of the fluid nor the hydrodynamical part of the energy and momentum is discussed in [80, 81].

It is worthwhile to stress that the model of the medium above is by no means a general one. Among its most serious limitations is that the elements of matter are assumed to be spinless. The extension of the model to the case of the nontrivial spin (along the lines similar to those of [46, 47, 48]) can bring new insight into the subject, especially with regard to the effects of magnetism.

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