Nonexistence of invariant distributions supported on the limit set

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1 Statement of the result

Let $X$ be a symmetric space of negative curvature. Then $X$ either belongs to one of the three families of real, complex, or quaternionic hyperbolic spaces, or it is the Cayley hyperbolic plane.

Let $G$ be a connected linear Lie group which finitely covers the isometry group of $X$. Furthermore, let $\Gamma \subset G$ be a discrete subgroup. We assume that $\Gamma$ is geometrically finite. We refer to Definition 2.1 for a precise explanation of this notion. If $X$ is a real hyperbolic space, then $\Gamma$ is geometrically finite iff it admits a fundamental domain with finitely many totally geodesic faces. In the other cases the definition is more complicated. Essentially, $\Gamma$ is geometrically finite if the corresponding locally symmetric space $\Gamma \backslash X$ has finitely many cusps and can be compactified by adding a geodesic boundary and closing the cusps. In
particular, if $\Gamma$ is cocompact, or convex-cocompact, or the locally symmetric space $\Gamma \backslash X$ has finite volume, then $\Gamma$ is geometrically finite.

We adjoin the geodesic boundary $\partial X$ to $X$ and obtain a compact manifold with boundary $\bar{X} := X \cup \partial X$. A point of $\partial X$ is an equivalence class of geodesic rays where two rays are in the same class if they run in bounded distance to each other. The action of $G$ extends naturally to $\bar{X}$. Let $\Lambda_{\Gamma} \subset \partial X$ denote the limit set of $\Gamma$. It is defined as the set of accumulation points of any orbit $\Gamma \cdot o$ in $\bar{X}$ for $o \subset X$.

We consider a $G$-equivariant irreducible complex vector bundle $V \to \partial X$ and a finite-dimensional representation $(\varphi, V_{\varphi})$ of $\Gamma$. Furthermore, by $\Lambda \to \partial X$ we denote the $G$-equivariant bundle of densities on $\partial X$. To $V$ we associate the $G$-equivariant bundle $\tilde{V} := \text{Hom}(V, \Lambda)$.

The space $C^{-\infty}(\partial X, V)$ of distribution sections of $V$ is then, by definition, the topological dual of $C^{\infty}(\partial X, \tilde{V})$. We define the space of invariant distribution sections of $V$ with twist $\varphi$ by

**Definition 1.1**

$$I(\Gamma, V, \varphi) := (C^{\infty}(\partial X, V) \otimes V_{\varphi})^\Gamma.$$  

Next we introduce some real quantities which represent growth properties of the geometric objects introduced so far. We first define the number $\rho \in \mathbb{R}$ which is a measure of the volume growth of the symmetric space $X$. We use this number in order to normalize the critical exponents below. Let $o$ be any point of $X$, and let $B(r, o)$ denote the ball of radius $r$ centered at $o$.

**Definition 1.2**

$$\rho := \frac{1}{2} \lim_{r \to \infty} \log \frac{\text{vol } B(r, o)}{r}.$$  

The growth of the action of $G$ on the bundle $V$ is measured by the quantity $s(V) \in \mathbb{R}$. Note that $\Lambda$ is the complexification of a real orientable line bundle. It is therefore trivial if
considered merely as a vector bundle, but it is not trivial as a $G$-equivariant bundle. The bundle $\Lambda$ can be represented by a cocycle of positive transition functions. If $\alpha \in \mathbb{C}$, then raising the transition functions to the power $\alpha$, we obtain a new cocycle which represents the $G$-equivariant bundle $\Lambda^\alpha$.

**Definition 1.3** $s(V)$ is defined as the unique number such that $V \otimes \Lambda^{s(V)} \cong V^\sharp$ as $G$-equivariant bundles, where $V^\sharp$ denotes the complex conjugate bundle of $\tilde{V}$.

For example, if $V = \partial X \times \mathbb{C}$ is the trivial bundle, then $s(V) = 1$. More generally, $s(\Lambda^\alpha) = 1 - 2\text{Re}(\alpha)$.

The normalized growth of $\Gamma$ is expressed by the critical exponent

**Definition 1.4**

$$d_\Gamma := \frac{1}{\rho} \inf \{ \nu \mid \sum_{g \in \Gamma} \text{dist}(g\mathbf{o}, \mathbf{o})^{-\rho-\nu} < \infty \} .$$

This definition is independent of the choice of $\mathbf{o} \in X$. Since $\Gamma$ is discrete and infinite we have $d_\Gamma \in (-1, 1]$.

The exponent $d_\varphi$ is a measure for the growth of $\varphi$. It is defined by

**Definition 1.5**

$$d_\varphi := \frac{1}{\rho} \inf \{ \nu \mid \sup_{g \in \Gamma} \| \varphi(g) \| \text{dist}(g\mathbf{o}, \mathbf{o})^{-\nu} < \infty \} ,$$

where we have fixed any norm $\| . \|$ on $\text{End}(V_\varphi)$ and any point $\mathbf{o} \in X$. Since $\Gamma$ is finitely generated, we have $d_\varphi < \infty$.

A cusp of $\Gamma$ is, by definition, a $\Gamma$-conjugacy class $[P]_\Gamma$ of proper parabolic subgroups $P \subset G$ such that $\Gamma \cap P$ is infinite and $\pi(\Gamma \cap P) \subset L$ is precompact, where $\pi$ is the projection onto the semisimple quotient $L$ given by the sequence

$$0 \rightarrow N \rightarrow P \xrightarrow{\pi} L \rightarrow 0$$
with $N \subset P$ denoting the unipotent radical of $P$. Note that if $[P]_\Gamma$ is a cusp of $\Gamma$, then $\Gamma_P := \Gamma \cap P$ again satisfies our assumptions. The limit set of $\Gamma_P$ consists of the unique fixed point $\infty_P \subset \partial X$ of $P$. Since $\Gamma_P$ acts properly on $\Omega_{\Gamma_P} := \partial X \setminus \{\infty_P\}$ and $\Gamma_P \setminus (\Lambda_\Gamma \setminus \{\infty_P\}) \subset \Gamma_P \setminus \Omega_{\Gamma_P}$ is compact (see Lemma 2.2) we can choose a smooth function $\chi^{\Gamma_P}$ on $\Omega_{\Gamma_P}$ such that $\text{supp}(\chi^{\Gamma_P})$ is a compact subset of $\Omega_{\Gamma_P}$, and $\sum_{g \in \Gamma_P} g^* \chi^{\Gamma_P} \equiv 1$. Assume that $s(V) > d_\Gamma + d_\varphi$.

**Definition 1.6** We say that $f \in I(\Gamma, V, \varphi)$ is strongly supported on the limit set if

1. $f$ is supported on the limit set as a distribution.
2. For any $h \in V^\infty(\partial X, \tilde{V}) \otimes \tilde{V}_\varphi$ and cusp $[P]_\Gamma$ of $\Gamma$ we have

$$\langle f, h \rangle = \sum_{g \in \Gamma_P} \langle \chi^{\Gamma_P} f|_{\Omega_{\Gamma_P}}, \tilde{\varphi}(g)^{-1} g^* h \rangle .$$

In order to see that the second condition is well-defined note that $\text{supp}(\chi^{\Gamma_P} f|_{\Omega_{\Gamma_P}}) \subset \text{supp}(\chi^{\Gamma_P}) \cap \Lambda_\Gamma$ is a compact subset of $\Omega_{\Gamma_P}$. Therefore the pairing $\langle \chi^{\Gamma_P} f|_{\Omega_{\Gamma_P}}, \tilde{\varphi}(g)^{-1} g^* h \rangle$ is defined. The sum converges because of our assumption $s(V) > d_\Gamma + d_\varphi \geq d_{\Gamma_P} + d_\varphi$, which implies that $\sum_{g \in \Gamma_P} \tilde{\varphi}(g)^{-1} g^* h|_{\Omega_{\Gamma_P}}$ converges in the space of smooth functions. In fact, the argument proving [4], Lemma 4.2, applies in the more general case when $\Gamma$ is merely geometrically finite. In Lemma 2.4 we will verify that this definition is independent of the choice of $\chi^{\Gamma_P}$.

In [3] and [2] we have expressed the condition ”strongly supported on the limit set” in the form $\text{res}^F(f) = 0$. While this definition works for all values of $s(V)$ there we must assume that $f$ is ”deformable”. Because in the present paper we are in the ”domain of convergence” we can use the simpler and more general definition above.

**Definition 1.7** By $I_{\Lambda_\varphi}(\Gamma, V, \varphi)$ we denote the subspace of all $f \in I(\Gamma, V, \varphi)$ which are strongly supported on the limit set.

The main result of the present paper can now be formulated as follows.
Theorem 1.8 If \( s(V) > d_\Gamma + d_\varphi + \max_{\mathcal{P}} (0, d_\Gamma - d_\Gamma + 1) \) (where the maximum is taken over all cusps of \( \Gamma \)), then \( I_{\Lambda_\Gamma}(\Gamma, V, \varphi) = 0 \).

Let us note the following special case which was already shown in [1], Thm 4.7. The group \( \Gamma \) is called convex cocompact if it acts freely and cocompactly on \( \tilde{X} \setminus \Lambda_\Gamma \). In this case \( \Gamma \) has no cusps and \( I_{\Lambda_\Gamma}(\Gamma, V, \varphi) \) is just the space invariant distribution sections of \( V \) with twist \( \varphi \) which are supported on \( \Lambda_\Gamma \).

Corollary 1.9 If \( \Gamma \) is convex cocompact and \( s(V) > d_\Gamma + d_\varphi \), then \( I_{\Lambda_\Gamma}(\Gamma, V, \varphi) = 0 \).

Back to the general case of a geometrically finite discrete group let 1 be the trivial representation of \( \Gamma \). Then we have \( d_1 = 0 \). In the place of \( V \) we consider \( \Lambda^{\frac{1-d_\pi}{2}} \). Note that \( s(\Lambda^{\frac{1-d_\pi}{2}}) = d_\Gamma = d_\Gamma + d_\varphi \). The space \( I_{\Lambda_\Gamma}(\Gamma, \Lambda^{\frac{1-d_\pi}{2}}, 1) \) is spanned by the Patterson-Sullivan measure [4], [8], [4], [5], hence \( \dim I_{\Lambda_\Gamma}(\Gamma, \Lambda^{\frac{1-d_\pi}{2}}, 1) = 1 \). Here we must use the definition of the condition ”strongly supported on the limit set” in terms of \( res^\Gamma \) given in [3].

In order to construct some twisted examples we consider a finite-dimensional \( M \)-spherical representation \((\pi, V_\pi)\) of \( G \). Here \((\pi, V_\pi)\) is called \( M \)-spherical, if for any parabolic subgroup \( P \subset G \) there exists a vector \( 0 \neq v \in V_\pi \) and a character \( \chi : P \to \mathbb{R} \) such that \( \pi(p)v = \chi(p)v \) for all \( p \in P \). There is a natural inclusion

\[
I_{\Lambda_\Gamma}(\Gamma, \Lambda^{\frac{1-d_\pi}{2}}, 1) \hookrightarrow I_{\Lambda_\Gamma}(\Gamma, \Lambda^{\frac{1-d_\pi-d_\varphi}{2}}, \pi)
\]

showing that \( I_{\Lambda_\Gamma}(\Gamma, \Lambda^{\frac{1-d_\pi-d_\varphi}{2}}, \pi) \neq 0 \). On the other hand, \( s(\Lambda^{\frac{1-d_\pi-d_\varphi}{2}}) = d_\Gamma + d_\pi \).

These examples show that our estimate can not be improved in general for convex cocompact \( \Gamma \). On the other hand, even for geometrically finite \( \Gamma \) we do not know any counterexample to the assertion that already \( s(V) > d_\Gamma + d_\varphi \) implies that \( I_{\Lambda_\Gamma}(\Gamma, V, \varphi) = 0 \).
2 Geometry of geometrically finite discrete subgroups

If \( \Gamma \subset G \) is a discrete subgroup and \( \Lambda_{\Gamma} \) is its limit set, then \( \Gamma \) acts on \( \bar{X} \setminus \Lambda_{\Gamma} \) properly discontinuously. Let \( \bar{Y}_{\Gamma} \) denote the manifold with boundary \( \bar{Y}_{\Gamma} := \Gamma \setminus (\bar{X} \setminus \Lambda_{\Gamma}) \). If \([P]_{\Gamma}\) is a cusp of \( \Gamma \), then we form the manifold with boundary \( \bar{Y}_{\Gamma P} := \Gamma_{P} \setminus (\bar{X} \setminus \{\infty_{P}\}) \).

**Definition 2.1** The group \( \Gamma \) is called geometrically finite if the following conditions hold:

1. \( \Gamma \) has finitely many cusps.
2. There is a bijection between the set of ends of \( \bar{Y}_{\Gamma} \) and and the set of cusps of \( \Gamma \).
3. If \([P]_{\Gamma}\) is a cusp of \( \Gamma \), then there exists a representative \( \bar{Y}_{P} \) of the corresponding end of \( \bar{Y}_{\Gamma} \) and embedding \( e_{P} : \bar{Y}_{P} \rightarrow \bar{Y}_{\Gamma P} \) which is isometric in the interior such that its image \( e_{P}(\bar{Y}_{P}) \) represents the end of \( \bar{Y}_{\Gamma P} \).

**Lemma 2.2** If \([P]_{\Gamma}\) is a cusp of \( \Gamma \), then \( \Gamma_{P} \setminus (\Lambda_{\Gamma} \setminus \{\infty_{P}\}) \) is a compact subset of \( \Gamma_{P} \setminus \Omega_{\Gamma P} \).

**Proof.** It suffices to show that \( \Gamma_{P} \setminus (\Lambda_{\Gamma} \setminus \{\infty_{P}\}) \) is compact in \( \bar{Y}_{\Gamma P} \). Note that \( (\Lambda_{\Gamma} \setminus \{\infty_{P}\}) \) is closed in \( \bar{X} \setminus \{\infty_{P}\} \). Therefore, \( \Gamma_{P} \setminus (\Lambda_{\Gamma} \setminus \{\infty_{P}\}) \) is closed in \( \bar{Y}_{\Gamma P} \). Furthermore, it is contained in the compact set \( \bar{Y}_{\Gamma P} \setminus e_{P}(\bar{Y}_{P}) \) (note that \( \bar{Y}_{P} \) is open). The assertion now follows. \( \square \)

Let \( o \in X \) be any point. We consider the Dirichlet domain \( F \subset X \) of \( \Gamma \) with respect to \( o \). It is a fundamental domain given by

\[
F := \{ x \in X | \text{dist}(x, o) \leq \text{dist}(hx, o) \ \forall h \in \Gamma \}.
\]

If \([P]_{\Gamma}\) is a cusp of \( \Gamma \), then let \( \chi_{\Gamma P} \) be the cut-off function introduced before Definition 1.6.
Lemma 2.3 We can decompose $F$ as $F_0 \cup F_1 \cup \ldots F_r$, where $r$ is the number of cusps $[P_i]_\Gamma$, $i = 1, \ldots, r$, of $\Gamma$, and the subsets $F_i$ satisfy

1. The closure of $F_0$ in $\bar{X} \setminus \Lambda_\Gamma$ is compact.
2. $\Gamma P_1 F_i \cap (\Lambda_\Gamma \setminus \{\infty_{P_i}\}) \cap \text{supp}(\chi_{\Gamma P_i}) = \emptyset$ for $i = 1, \ldots, r$.

Proof. By $\bar{Y}_0$ we denote the compact subset $\bar{Y}_\Gamma \setminus \bigcup_{i=1}^{r} \bar{Y}_{P_i}$ of $\bar{Y}_\Gamma$. Then we define $F_0 := \Gamma \bar{Y}_0 \cap F$, where $\Gamma \bar{Y}_0$ denotes the preimage of $\bar{Y}_0$ under the projection $\bar{X} \setminus \Lambda_\Gamma \to \bar{Y}_\Gamma$. By definition, $F_0 \subset (\bar{X} \setminus \Lambda_\Gamma)$.

For $i = 1, \ldots, r$ we define $F_i := \Gamma P_i \cap F$. We then have $\Gamma P_1 F_i \cap (\Lambda_\Gamma \setminus \{\infty_{P_i}\}) \cap \text{supp}(\chi_{\Gamma P_i}) = \emptyset$ since the contrary this would imply $e(\bar{Y}_{P_i}) \cap \Gamma P_i \setminus (\Lambda_\Gamma \setminus \{\infty_{P_i}\}) \neq \emptyset$. \hfill \Box

Lemma 2.4 Let $\chi_1, \chi_2$ be two choices of the cut-off function $\chi_{\Gamma P}$ in Definition 1.6. Then

$$\sum_{g \in \Gamma P} \langle \chi_1 f_{\Omega P}, \bar{\varphi}(g)^{-1} g^* h \rangle = \sum_{g \in \Gamma P} \langle \chi_2 f_{\Omega P}, \bar{\varphi}(g)^{-1} g^* h \rangle,$$

where $[P]_\Gamma$, $f$, and $h$ are as 1.6.

Proof. The estimates given in the proof of [1], Lemma 4.2, show that all sums below converge absolutely. This justifies the resummations in the following computation. In the first and the last equality we use the $\Gamma$-invariance of $f$.

$$\sum_{g \in \Gamma P} \langle \chi_1 f_{\Omega P}, \bar{\varphi}(g)^{-1} g^* h \rangle = \sum_{g \in \Gamma P} \langle g^* \chi_1 f_{\Omega P}, h \rangle$$

$$= \sum_{g \in \Gamma P} \sum_{l \in \Gamma P} \langle g^* \chi_1 l^* \chi_2 f_{\Omega P}, h \rangle$$

$$= \sum_{l \in \Gamma P} \sum_{g \in \Gamma P} \langle g^* \chi_1 l^* \chi_2 f_{\Omega P}, h \rangle$$

$$= \sum_{l \in \Gamma P} \langle l^* \chi_2 f_{\Omega P}, h \rangle$$

$$= \sum_{g \in \Gamma P} \langle \chi_2 f_{\Omega P}, \bar{\varphi}(g)^{-1} g^* h \rangle.$$
3 Proof of Theorem 1.8

We adapt the argument of the proof of [1], Thm.4.7 given there in the special case of a convex-cocompact group $\Gamma$ to the present situation where $\Gamma$ is geometrically finite.

**Definition 3.1** We call the bundle $V$ spherical, if $V = \Lambda^{1-t(V)}$ for some $t(V) \in \mathbb{C}$. Note that $\text{Re } t(V) = s(V)$. We first show the following special case.

**Proposition 3.2** Theorem 1.8 is true if $V$ is spherical.

*Proof.* Let $f \in I_{\Lambda}(\Gamma, V, \varphi)$. Then we must show that $\langle f, h \rangle = 0$ for any $h \in C^\infty(\partial X, \tilde{V}) \otimes \tilde{V}_\varphi$.

**Lemma 3.3** If $\Gamma$ does not contain any hyperbolic element, then $\langle f, h \rangle = 0$.

*Proof.* If $\Gamma$ does not contain any hyperbolic element, then $\Gamma = \Gamma_P$ for the unique cusp $[P]_{\Gamma}$ of $\Gamma$. Since $f$ is supported on $\Lambda_{\Gamma} = \{\infty_P\}$ as a distribution we have we have $f|_{\Omega_{\Gamma_P}} = 0$. This implies $\langle f, h \rangle = \sum_{g \in \Gamma_P} \langle \chi^{\Gamma_P} f|_{\Omega_{\Gamma_P}}, \tilde{\varphi}(g)^{-1} g^* h \rangle = 0$. \hfill $\Box$

It remains to consider the case that $\Gamma$ contains a hyperbolic element which we will denote by $g_0$.

**Lemma 3.4** If $\Gamma$ does contain a hyperbolic element, say $g_0$, then $\langle f, h \rangle = 0$. 

Proof. Let \( b_\pm \in \partial X \) denote the attracting and repelling fixed points of \( g_0 \). We can write \( h = h_+ + h_- \) such that \( h_\pm \) vanishes in a neighbourhood of \( b_\pm \). It suffices to show that \( \langle f, h \rangle = 0 \) for any \( h \) which vanishes in a neighbourhood of say \( b_+ \).

We fix the origin \( o \in X \) such that \( o \) is on the unique geodesic connecting \( b_- \) with \( b_+ \). Let \( \tilde{F} \subset X \) be the Dirichlet domain of \( \Gamma \) with respect to this choice of the origin. Furthermore, let \( \bar{F} \) be the closure of \( \tilde{F} \) in \( X \setminus \Lambda_\Gamma \). The Dirichlet domain \( D_{<g_0>} \) with respect to \( o \) of the group \( <g_0> \) generated by \( g_0 \) separates \( X \setminus D_{<g_0>} \) into two connected components \( X_+ \) and \( X_- \). Let \( \partial X_\pm := \bar{X}_\pm \cap \partial X \). We can assume that \( b_\pm \in \partial X_\pm \).

Replacing \( o \), if necessary, by \( g_j^0 o \), \( j \in \mathbb{N}_0 \) sufficiently large, we can assume that \( \text{supp}(h) \subset \partial X_- \). Then we define \( F := g_i^0 \tilde{F}, \bar{F} := g_i^0 \bar{F} \), where we choose \( i \in \mathbb{N}_0 \) sufficiently large such that \( F \subset X_+ \).

We use the polar coordinates \((a, k) \in \mathbb{R}_+ \times \partial X \) in order to parametrize points \( x \in X \setminus \{o\} \) such that \( a(x) = \exp(\text{dist}(o, x)) \) and \( k(x) \in \partial X \) is represented by the geodesic ray through \( x \) starting in \( o \). Using these coordinates we extend \( h \) to the interior of \( X \) setting \( \tilde{h}(x) = \chi(a(x))h(k(x)) \), where \( \chi \in C^\infty(R_+) \) is some cut-off function which is equal to one near infinity and vanishes for \( a < 1 \). Note that \( \text{supp}(\tilde{h}) \subset X \setminus X_+ \).

Note that by our assumption \( V \) is spherical and \( s(V) > 0 \). Therefore, the Poisson transformation

\[
P : C^{-\infty}(\partial X, V) \rightarrow C^\infty(X)
\]

is injective (we refer to [4] and the literature cited therein (e.g. [4]) for a definition of the Poisson transformation and its properties). We use the same symbol \( P \) in order to denote the extension of the Poisson transform to the tensor product by \( V_\varphi \).

Using the polar coordinates we pull-back the volume form of the unit sphere in \( T_o X \) to \( \partial X \) and thus obtain a volume form \( dk \) on \( \partial X \). Then the inverse of the Poisson transformation is given by the following limit formula

\[
\langle f, h \rangle = c_1 \lim_{a \to \infty} a^{\rho(1-t(V))} \int_{\partial X} \langle Pf(a(x), k), h(k) \rangle dk
\]
for some constant $c_1$. Using the fact that for large $a$ the volume form $dx$ can be written as $dx = c_2 a^{2\rho} da dk + O(a^{2\rho-1})$ we deduce

$$
\langle f, h \rangle = c \lim_{a \to \infty} a^{-\rho(1+\ell(V))} \int_{\{x \in F \mid a \leq a(x) \leq a_0 a\}} \langle Pf(x), \tilde{h}(x) \rangle dx .
$$

where $c$ depends on $c_1$, $a_0 > 1$, and $c_2$. We now employ the covering of $X$ by translates of the fundamental domain $gF$, $g \in \Gamma$, and the $\Gamma$-invariance of $Pf$

$$Pf(gx) = \varphi(g) Pf(x) .$$

We get

$$
\langle f, h \rangle = c \lim_{a \to \infty} a^{-\rho(1+\ell(V))} \sum_{g \in \Gamma} \int_{\{x \in F \mid a \leq a(gx) \leq a_0 a\}} \langle \varphi(g) Pf(x), \tilde{h}(gx) \rangle dx .
$$

Since $\text{supp}(\tilde{h}) \subset X \setminus X_+$ we have $gF \cap X \setminus X_+ \neq \emptyset$ if $g \in \Gamma$ contributes to the sum above. The triangle inequality for $X$ gives $a(x)a(g) \geq a(gx)$, where we write $a(g)$ for $a(g0)$. We will also need the following converse version of the triangle inequality.

**Lemma 3.5** There exists $a_1 \in \mathbb{R}_+$ such that for all $g \in \Gamma$ with $gF \cap X \setminus X_+ \neq \emptyset$ and $x \in F$ we have $a(g)a(x) \leq a_1 a(gx)$.

We postpone the proof of this lemma and continue the argument for Lemma 3.4. Using 3.5 we obtain

$$
\{x \in F \mid a \leq a(gx) \leq a_0 a\} \subseteq \{x \in F \mid a \leq a(x)a(g) \leq a_1 a\}
$$

$$
= \{x \in F \mid aa(g)^{-1} \leq a(x) \leq a_1 aa(g)^{-1}\}
$$

for all $g \in \Gamma$ with $gF \cap X \setminus X_+ \neq \emptyset$. Taking into account that $\tilde{h}$ is bounded and that for given $\epsilon > 0$ there exists a constant $C_0$ such that for all $g \in \Gamma$ we have $\|\varphi(g)\| \leq C_0 a(g)^{\rho(d_x+\epsilon)}$ we obtain

$$
|\int_{\{x \in F \mid a \leq a(gx) \leq a_0 a\}} \langle \varphi(g) Pf(x), \tilde{h}(gx) \rangle dx| \leq C_1 a(g)^{\rho(d_x+\epsilon)} \int_{\{x \in F \mid aa(g)^{-1} \leq a(x) \leq a_1 aa(g)^{-1}\}} |Pf(x)| dx ,
$$

where $C$ is independent of $g \in \Gamma$ and $a \in \mathbb{R}_+$. In order to proceed further we employ the following crucial estimate.
Lemma 3.6 There is a constant $C_1$ such that
\[ \int_{\{x \in F| b \leq a(x) \leq a_1b\}} |P_f(x)| \, dx \leq C_1 b^{\rho(s(V)+1-\mu)} \]
for all sufficiently small $\mu > 0$ and all $b \geq 1$.

We again postpone the proof of Lemma 3.6 and continue with the proof of Lemma 3.4. If we insert the estimate claimed in Lemma 3.6 into (1) and sum over $\Gamma$, then we obtain
\[ \sum_{g \in \Gamma} \left| \int_{\{x \in F| a \leq a(gx) \leq a_0a\}} \langle \varphi(g)P_f(x), \tilde{h}(gx) \rangle \, dx \right| \leq C_2 \sum_{g \in \Gamma} a(g)^{\rho(d_\varphi+\epsilon+\mu-s(V)-1)} a^{\rho(s(V)+1-\mu)}, \]
where $C_3$ is independent of $a \geq 1$. If we choose $\mu, \epsilon > 0$ so small such that $d_\varphi + \epsilon + \mu - s(V) < -d_\Gamma$, then the sum converges and the right-hand side can be estimated by $C_3 a^{\rho(s(V)+1-\mu)}$ with $C_3$ independent of $a \geq 1$. We conclude that
\[ \lim_{a \to \infty} a^{-\rho(l(V)+1)} \sum_{g \in \Gamma} \int_{\{x \in F| a \leq a(gx) \leq a_0a\}} \langle \varphi(g)P_f(x), \tilde{h}(gx) \rangle \, dx = 0, \]
and thus $\langle f, h \rangle = 0$.

It remains to prove Lemma 3.5 and Lemma 3.6.

Proof. [of Lemma 3.5] Note that for all $g \in \Gamma$ one of the following two conditions fails:
\[ gg_0^i \in X_+ \]
\[ gF \cap (X \setminus X_+) \neq \emptyset. \]
Indeed, if the first condition holds, then $gF \cap X_+ \neq \emptyset$. We conclude that $gF \subset X_+$ and hence $gF \cap (X \setminus X_+) = \emptyset$. Further note that
\[ \{gg_0^i| g \in \Gamma \text{ and } gF \cap (X \setminus X_+) \neq \emptyset\} \cap \partial X = \{g_0| g \in \Gamma \text{ and } gF \cap (X \setminus X_+) \neq \emptyset\} \cap \partial X. \]
We see that $\bar{F} \cap \partial X \subset \text{int} \partial X_+$ and $\{g_0| g \in \Gamma \text{ and } gF \cap (X \setminus X_+) \neq \emptyset\} \cap \partial X \subset \partial X \setminus \partial X_+$ are disjoint. We now obtain the desired inequality from Corollary 2.5 of [1].
Proof. [of Lemma 3.6]

It is at this point where we use that \( \Gamma \) is geometrically finite. Namely, let \( F_0 \cup F_1 \cup \ldots F_r \) be the decomposition of \( F \) given in Lemma 2.3.

Since \( \Lambda_\Gamma \) and \( \overline{F}_0 \) are separated we can use [1], Lemma 6.2 (2), in order to get the estimate
\[
\int_{\{x \in \partial X | b \leq a(x) \leq a_1 \}} |P f(x)| \leq C |b|^{\rho(1-s(V))},
\]
where \( C \) is independent of \( b \geq 1 \). This is the required estimate for the contribution of \( F_0 \).

It remains to consider the contributions of the cusps, i.e of \( F_i, i > 0 \). Let now \( [P]_\Gamma, P = P_i \) for one \( i > 0 \), be a cusp of \( \Gamma \). Then for \( v \in V_{\tilde{\varphi}} \) and \( x \in X \) we have
\[
\langle P f(x), v \rangle = \sum_{g \in \Gamma_p} \langle \varphi(g) P(\chi^{1/p} f_{|\Omega_p})(g^{-1}x), v \rangle.
\]
Indeed, let \( p_{x,v} \in C^\infty(\partial X, \tilde{V}) \otimes V_{\tilde{\varphi}} \) denote the integral kernel of the map \( f \mapsto \langle P(f)(x), v \rangle \). Then using the invariance properties of the kernel \( p_{x,v} \) and that \( f \) is strongly supported on the limit set we get
\[
\langle P f(x), v \rangle = \langle p_{x,v}, f \rangle = \sum_{g \in \Gamma_p} \langle \chi^{1/p} f_{|\Omega_p}, \tilde{\varphi}(g)^{-1} g^* p_{x,v} \rangle = \sum_{g \in \Gamma_p} \langle \chi^{1/p} f_{|\Omega_p}, p_{g^{-1}x, \tilde{\varphi}(g)} v \rangle = \sum_{g \in \Gamma_p} \langle \varphi(g)^{-1} P(\chi^{1/p} f_{|\Omega_p})(gx), v \rangle.
\]

Since \( \Gamma_p F_i \cap \text{supp}(\chi^{1/p} f_{|\Omega_p}) = \emptyset \) we again apply [1], Lemma 6.2 (2), in order to get the estimate
\[
|P(\chi^{1/p} f_{|\Omega_p})(gx)| \leq C a(gx)^{-\rho(s(V)+1)}, \tag{2}
\]
where \( C \) is independent of \( x \in F_i \) and \( g \in \Gamma_p \). In order to estimate the sum over \( g \in \Gamma_p \) we need the following geometric lemma.

Lemma 3.7 There is a constant \( a_3 \in \mathbb{R}_+ \) such that for all \( x \in F \) and \( g \in \Gamma_p \) we have \( a(gx) \geq a_3 \max(a(g)a(x)^{-1}, a(x)) \).
Let us postpone the proof of the lemma and continue with the estimates. We choose
\( \nu > 0 \) sufficiently small such that
\[ d_{\Gamma_P} + d_\varphi - s(V) + \nu < 0. \]
For those \( \nu \) using Lemma 3.7 and (2) we obtain for all \( x \in F_i \)
\[
|P(f)(x)| \leq C \sum_{g \in \Gamma_P} \| \varphi(g)^{-1} a(gx)^{-\rho(1+s(V))} \\
\leq C_1 \sum_{g \in \Gamma_P} a(g)^{\rho d_\varphi} a(gx)^{-\rho(1+s(V))} \\
\leq C_2 \sum_{g \in \Gamma_P} a(g)^{-\rho(1+d_{\Gamma_P} + \nu)} a(x)^{\rho(-s(V) + 1 + 2d_{\Gamma_P} + 2d_\varphi + 2\nu)} \\
\leq C_3 a(x)^{\rho(-s(V) + 1 + 2d_{\Gamma_P} + 2d_\varphi + 2\nu)}
\]
Since \( s(V) > d_\varphi + d_{\Gamma_P} + 1 \) we can choose \( \kappa, \nu, \mu > 0 \) such that
\[
- s(V) + 1 + 2d_{\Gamma_P} + 2d_\varphi + 2\nu + 2 + \kappa < s(V) + 1 - \mu.
\]
Then
\[
|P(f)(x)| \leq C_3 a(x)^{\rho(1+s(V) - \mu)} a(x)^{-\rho(2+\kappa)}.
\]
Lemma 3.6 now follows from the fact that the function \( X \ni x \mapsto a(x)^{-\rho(2+\kappa)} \in \mathbb{R} \) is integrable. In fact,
\[
\int_{\{x \in F_i | b \leq a(x) \leq a_1 b\}} |Pf(x)| dx \leq C_3 b^{\rho(1+s(V) - \mu)} \int_X a(x)^{-\rho(2+\kappa)} dx \leq C_4 b^{\rho(1+s(V) - \mu)},
\]
where \( C_4 \) is independent of \( b \).

\begin{proof} [of Lemma 3.7]
We consider the triangle inequality for the triangle \( (o, gx, go) \) in \( X \) and obtain \( \text{dist}(o, gx) + \text{dist}(go, gx) \geq \text{dist}(o, go) \). Since \( \text{dist}(go, gx) = \text{dist}(o, x) \) we conclude that \( a(x)a(gx) \geq a(g) \).

Recall that \( F \) is a Dirichlet domain of \( \Gamma \) with respect to \( g_0^i o \). We conclude that \( \text{dist}(g_0^i o, x) \leq \text{dist}(g_0^i o, gx) \) for all \( g \in \Gamma_P \) and \( x \in F \). Using this and the triangle inequality for the triangle \( (o, x, g_0^i o) \) we obtain
\[
\text{dist}(o, x) \leq \text{dist}(g_0^i o, x) + \text{dist}(o, g_0^i o) \\
\leq \text{dist}(g_0^i o, gx) + \text{dist}(o, g_0^i o) \\
\leq 2 \text{dist}(g_0^i o, o) + \text{dist}(o, gx)
\]
\end{proof}
and therefore $a(gx) \geq a(g_0^i)^{-2} a(x)$ for all $g \in \Gamma_P$ and $x \in F$. The assertion of the lemma holds true if we set $a_3 := \min(1, a(g_0^i)^{-2})$.

We now have finished the proof of Lemma 3.4 and therefore of Proposition 3.2.

\[ \square \]

**Proof.** [end of the proof of Theorem 1.8] By Proposition 3.2 we know that Theorem 1.8 holds true under the additional assumption that $V$ is spherical. We twist by finite-dimensional representations of $G$ in order to conclude the general case. Fix any parabolic subgroup $P \subset G$. Then we can write $\partial X = G/P$. If $(\pi, V_\pi)$ is a finite-dimensional representation of $G$, then we can form the $G$-equivariant bundle $V(\pi) = G \times_P V_\pi$ on $\partial X$.

The idea of twisting is based on the fact that there is an $G$-equivariant isomorphism

$$ C^\infty(\partial X, V \otimes V(\pi)) \otimes V_\varphi \cong C^\infty(\partial X, V \otimes V_\pi \otimes V_\varphi). $$

In particular, there is an isomorphism

$$ j : I_{\Lambda^t}(\Gamma, V \otimes V(\pi), \varphi) \cong I_{\Lambda^t}(\Gamma, V, \varphi). $$

If $V$ is an irreducible $G$-equivariant bundle on $\partial X$, then there exists an irreducible representation $(\pi, V_\pi)$ of $G$ and a $G$-equivariant embedding

$$ i : V \hookrightarrow \Lambda^{1-t(V) - d_G} \otimes V(\pi), $$

where $t(V) \in \mathbb{C}$ is defined such that $V \otimes \Lambda^{t(V)} = \tilde{V}$. In particular, $\text{Re}(t(V)) = s(V)$.

For these facts we refer to [1], p. 108 (in particular, to the formulas (33), (34)). The embedding $i$ composed with the isomorphism $j$ gives an embedding

$$ j \circ i : I_{\Lambda^t}(\Gamma, V, \varphi) \hookrightarrow I_{\Lambda^t}(\Gamma, \Lambda^{1-t(V) - d_G} / 2, \varphi \otimes \pi). $$

We can apply Prop. 3.2 to the right-hand side. Indeed,

$$ s(\Lambda^{1-t(V) - d_G} / 2) = s(V) + d_\pi + d_\varphi + d_\pi + \max(0, d_{\Gamma_P} - d_\Gamma + 1) = d_{\Gamma} + d_{\varphi} + \max(0, d_{\Gamma_P} - d_\Gamma + 1), $$

and hence $I_{\Lambda^t}(\Gamma, \Lambda^{1-t(V) - d_G} / 2, \varphi \otimes \pi) = 0$. This implies $I_{\Lambda^t}(\Gamma, V, \varphi) = 0$.  

\[ \square \]


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