The density of rational points on a certain singular cubic surface

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Abstract

We show that the number of non-trivial rational points of height at most $B$, which lie on the cubic surface $x_1x_2x_3 = x_4(x_1 + x_2 + x_3)^2$, has order of magnitude $B(\log B)^6$. This agrees with Manin’s conjecture.

1 Introduction

The purpose of this paper is to investigate the distribution of rational points on the singular cubic surface $X \subset \mathbb{P}^3$, given by the equation

$$x_1x_2x_3 = x_4(x_1 + x_2 + x_3)^2.$$ 

This surface has a unique singular point $[0, 0, 0, 1]$ which is of type $D_4$, and contains precisely 6 lines [2, Lemma 4]. These lines are all defined over $\mathbb{Q}$ and are given by

$$x_i = x_4 = 0, \quad x_i = x_j + x_k = 0,$$

for $\{i, j, k\} = \{1, 2, 3\}$. We shall denote by $U \subset X$ the open subset formed by deleting the lines from $X$.

Given a rational point $x = [x_1, x_2, x_3, x_4] \in \mathbb{P}^3(\mathbb{Q})$ such that $x_1, x_2, x_3, x_4$ are relatively prime integers, let $H(x) = \max_{1 \leq i \leq 4} |x_i|$ denote its anticanonical height, metrized by the choice of norm $\max_{1 \leq i \leq 4} |x_i|$. Then for any $B \geq 1$, we shall be concerned with estimating the quantity

$$N_{U,H}(B) = \# \{ x \in U \cap \mathbb{P}^3(\mathbb{Q}) : H(x) \leq B \}.$$ 

Manin [4] has provided a very general conjecture concerning the distribution of rational points on Fano varieties. In our case it predicts that there exists a positive constant $c_{X,H}$ such that

$$N_{U,H}(B) \sim c_{X,H}B(\log B)^6,$$

as $B \to \infty$. Here the exponent of $\log B$ is one less than the rank of the Picard group $\text{Pic} \hat{X}$ of $\hat{X}$, where $\hat{X}$ denotes the minimal desingularisation of $X$. In fact this sort of asymptotic formula is conjectured to hold for any cubic surface with canonical singular locus. Although there has been increasing interest in Manin’s conjecture for cubic surfaces, it has only been completely settled in particularly simple cases such as the toric variety

$$x_1x_2x_3 = x_4^3.$$
This can be found in the work of la Bretèche \cite{1}, for example.

More recently, Heath-Brown \cite{10} has established upper and lower bounds for the density of non-trivial rational points on the Cayley cubic surface

\[
\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = 0,
\]

which agree with Manin’s conjecture. This is a cubic surface containing four $A_1$ singularities, which is the maximal number of singularities that a non-ruled cubic surface can have. The principal tool in Heath-Brown’s work is a passage to the universal torsor above the minimal desingularisation of the Cayley cubic. Originally introduced by Colliot-Thélène and Sansuc to aid in the study of the Hasse principle and weak approximation, universal torsors were first used by Peyre \cite{12} and Salberger \cite{13} in the context of counting rational points of bounded height. After establishing a bijection between the rational points on the Cayley cubic and the integer points on the universal torsor, which in this setting is given explicitly by nine equations in thirteen variables, Heath-Brown proceeds by applying methods from the geometry of numbers to count integer solutions to certain ternary linear equations.

Our present work is largely inspired by Heath-Brown’s treatment of the Cayley cubic surface. We are now in a position to state our main result.

**Theorem.** We have

\[
B(\log B)^6 \ll N_{U,H}(B) \ll B(\log B)^6. \tag{1.1}
\]

Of the two bounds in our theorem, the lower bound $N_{U,H}(B) \gg B(\log B)^6$ is routine. It will follow from relatively minor adjustments to Heath-Brown’s treatment of the Cayley cubic. Establishing the upper bound in (1.1), however, is by far the most challenging component of this paper and the proof has two fundamental ingredients. The first is a translation of the problem to the universal torsor above $\tilde{X}$, which in this setting has the affine embedding

\[
s_0 s_1 s_2 s_3 u_1 u_2 u_3 = y_1 u_1 s_1^2 + y_2 u_2 s_2^2 + y_3 u_3 s_3^2. \tag{1.2}
\]

This has been calculated by Hassett and Tschinkel \cite{6} \S 4, although we shall present our own deduction of this equation in \S 4 below. The universal torsor can be thought of as serving to encode factorisation information about the integer solutions to the original equation. In practical terms, it allows us to work with a larger number of variables, all of which are smaller in modulus than the original variables $x_1, x_2, x_3, x_4$. The second main ingredient in our proof of the upper bound involves studying the distribution of integer solutions to the equation obtained by setting $s_0 = 0$ in (1.2). This is the focus of \S 3 and relies upon lattice methods to count integer solutions to ternary linear and quadratic forms. It seems worthwhile highlighting the fact that this need to consider the contribution from quadratic equations marks a significant departure from Heath-Brown’s treatment of the Cayley cubic. In particular, we shall need to pay careful attention to the fact that almost all ternary quadratic forms don’t have an integer solution.

Our work draws upon a diverse range of techniques. In addition to the geometry of numbers used to study linear and quadratic forms in \S 2.1, we make use of the large sieve inequality and real character sum estimates due to Heath-Brown and Pólya–Vinogradov in \S 2.2.
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2 Preliminary estimates

We begin by introducing some conventions regarding our choice of notation. Throughout this paper the letters $i, j, k$ will denote generic distinct indices from the set $\{1, 2, 3\}$. We shall use $\mathbb{N}$ to denote the set of positive integers, and for any $n \geq 2$ it will be convenient to let $\mathbb{Z}^n$ denote the set of primitive vectors in $\mathbb{Z}^n$, where $v \in \mathbb{Z}^n$ is said to be primitive if $\gcd(v_1, \ldots, v_n) = 1$. Similarly, we let $\mathbb{N}^n$ denote the set of primitive vectors in $\mathbb{N}^n$. Furthermore, we let $\mathbb{Z}^n_*$ denote the subset of $v \in \mathbb{Z}^n$ for which $v_1 \cdot \cdots \cdot v_n \neq 0$. Upon writing $F(x) = x_1x_2x_3 - x_4(x_1 + x_2 + x_3)^2$, it therefore follows that

$$N_{U,H}(B) = \frac{1}{2} \# \left\{ x \in \mathbb{Z}^4_* : \max_{1 \leq i \leq 4} |x_i| \leq B, F(x) = 0 \right\}, \quad (2.1)$$

since $x$ and $-x$ represent the same point in $\mathbb{P}^3$. It will be convenient to collect together some technical results that will be useful to us.

2.1 The geometry of numbers and ternary forms

Several of our arguments will involve estimating the number of primitive integer solutions to certain ternary homogeneous polynomial equations, which lie in lopsided regions. In the case of linear equations, such an estimate is provided by the following result of Heath-Brown [7, Lemma 3].

Lemma 1. Let $h \in \mathbb{Z}^3$ and let $W_i > 0$. Then the number of $w \in \mathbb{Z}^3$ for which $\sum_{i=1}^3 h_iw_i = 0$, and $|w_i| \leq W_i$, is

$$\leq 4 + 12\pi \frac{W_1W_2W_3}{\max |h_i|W_i}.$$

We shall also need a result which handles the corresponding problem for diagonal quadratic equations. For this we turn to the following result, in which $\omega(n)$ denotes the number of distinct prime factors of $n \in \mathbb{N}$.

Lemma 2. Let $g,h \in \mathbb{Z}^3_*$, with $g_1g_2g_3$ square-free, and let $W_i > 0$. Then the number of $w \in \mathbb{Z}^3$ for which $\sum_{i=1}^3 g_i h_iw_i^2 = 0$, and $|w_i| \leq W_i$, is

$$\ll \left( 1 + \sqrt{\frac{W_1W_2W_3R^{3/2}}{|h_1h_2h_3|}} \right)^{2\omega(h_1h_2h_3)}.$$
where $D_{g,h}$ is the product of greatest common divisors

$$
gcd(h_1h_2, h_1h_3, h_2h_3)gcd(g_1, h_2h_3)gcd(g_2, h_1h_3)gcd(g_3, h_1h_2).$$

Lemma\[2\] will follow from a rather straightforward modification to the proof of Heath-Brown’s \[9,\] Theorem 2]. In fact Heath-Brown establishes a version of Lemma\[2\] with $g=(1,1,1)$ and $d_3(h_1h_2h_3)$ in place of $2^{-(h_1h_2h_3)}$, where $d_k(n)$ denotes the number of representations of $n$ as a product of $k$ positive integers, for any $k,n \in \mathbb{N}$. It is perhaps worth pointing out that whereas $d_3(n)$ has average order $\frac{1}{2} (\log n)^2$, the function $2^{-n}$ has average order $\zeta(2)^{-1} \log n$. This saving plays an important role in our work.

In order to prove Lemma\[2\] we recall that the original idea behind the proof of \[9,\] Theorem 2] is to view the equation $\sum_{i=1}^3 g_i h_i w_i^2 = 0$ as a collection of lattice conditions upon the solutions $w \in \mathbb{Z}^3$. Let $p$ be any prime divisor of $h_1h_2h_3$, and assume without loss of generality that

$$0 \leq \nu_p(h_1) \leq \nu_p(h_2) \leq \nu_p(h_3),$$

where $\nu_p(n)$ denotes the $p$-adic order of any $n \in \mathbb{N}$. In particular it follows that $\nu_p(h_1)=0$, since $h$ is primitive. We shall only consider here the case in which $p$ is an odd prime. The case $p=2$ is handled along similar lines. Since $g_1g_2g_3$ is square-free, we may write

$$g_1 = p^{\alpha_1} g'_1, \quad g_2 = p^{\alpha_2} g'_2, \quad g_3 = p^{\alpha_3} g'_3,$$

for $(\alpha_1,\alpha_2,\alpha_3) \in \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$ and $p \nmid g'_i$. Similarly, we write

$$h_2 = p^{\beta_2} h'_2, \quad h_3 = p^{\beta_3} h'_3,$$

for $p \nmid h'_2h'_3$ and $\beta_3 \geq \beta_2 \geq 1$. Then one proceeds by considering solutions to the equation

$$p^{\alpha_1} g'_1 h_1 w_1^2 + p^{\alpha_2+\beta_2} g'_2 h'_2 w_2^2 + p^{\alpha_3+\beta_3} g'_3 h'_3 w_3^2 = 0. \quad (2.2)$$

Suppose for the moment that we are examining solutions $(u,v,w) \in \mathbb{Z}^3$ to the equation

$$au^2 + p^\sigma bv^2 + p^\tau cw^2 = 0, \quad (2.3)$$

for $0 \leq \sigma \leq \tau$ and $p \nmid abc$. Then arguing along similar lines to the proof of \[9,\] Theorem 2], we sketch how this implies that $(u,v,w)$ lies on one of at most 2 sublattices of $\mathbb{Z}^3$, each of determinant $p^{\delta(\sigma,\tau)}$, where

$$\delta(\sigma,\tau) = \begin{cases} (\sigma + \tau) - 3\sigma/2, & \text{if } \sigma \text{ even}, \\ (\sigma + \tau) - [3\sigma/2] + 1, & \text{if } \sigma \text{ odd}. \end{cases} \quad (2.4)$$

Suppose first that $\sigma = 2s$ is even. Then \[23\] implies that $p^s \mid u$. By writing $u = p^su'$, and considering the corresponding congruence $au^2 + bv^2 \equiv 0 \pmod{p^{2s-\sigma}}$, we therefore deduce that $(u,v,w)$ lies on one of at most 2 integer lattices, each of determinant $p^{s+\tau-\sigma} = p^{\delta(\sigma,\tau)}$. Suppose now that $\sigma = 2s+1$ is odd. In view of \[23\] we may again write $u = p^su'$, and consider the corresponding congruence $au^2 + p^\sigma bv^2 \equiv 0 \pmod{p^{\tau-2s}}$. Since $\tau - 2s \geq 1$, we may clearly write $u' = pu''$, and so consider solutions to the equation

$$pa(u'')^2 + bv^2 + p^{\tau-\sigma} cw^2 = 0.$$

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Now either \( \tau - \sigma = 0 \), or else we may write \( v = pu^r \) and consider the equation

\[
a(u v')^2 + p b v'^2 + p^{r-4} c w^2 = 0.
\]

In the former case we conclude that \((u, v, w)\) lies on one of at most two integer sublattices of determinant \( p \). But then \((u, v, w)\) lies on one of at most 2 integer lattices, each of determinant \( p^{r-2} = p^\delta(\sigma, \tau) \). In the latter case we have \( \tau - \sigma = 1 \), and we proceed inductively. Thus either \( \tau - \sigma = 1 \), in which case we deduce that \((u, v, w)\) lies on one of 2 integer lattices of determinant \( p^{r+2} = p^\delta(\sigma, \tau) \), or else \( \tau - \sigma \geq 2 \) and we can repeat the process. Since this process clearly terminates we therefore deduce that whenever \( \sigma \) is odd, \((u, v, w)\) lies on one of at most 2 integer lattices, each of determinant \( p^\delta(\sigma, \tau) \).

Returning to (2.3), our goal is to show that \( w \) lies on one of at most 2 integer sublattices of \( \mathbb{Z}^3 \), each of determinant

\[
p^2 + \beta_3 - \left[ \frac{3(\alpha_1 + \alpha_2 + \alpha_3 + \beta_2)}{2} \right].
\]

In view of the existing proof of [3], Theorem 2, this will suffice to establish Lemma 2 since the inequalities \( \beta_3 \geq \beta_2 \geq 1 \) imply that

\[
\nu_p(h_1b_2) = \beta_2 + \beta_3, \quad \nu_p(D_{g,h}) = \alpha_1 + \alpha_2 + \alpha_3 + \beta_2.
\]

Suppose first that \((\alpha_1, \alpha_2) = (0, 0)\). Then our work above shows that \( w \) lies on one of at most 2 integer sublattices of \( \mathbb{Z}^3 \), each of determinant

\[
p^2(\beta_2, \alpha_3 + \beta_2) \geq p^2(\beta_2, \beta_3).
\]

This is plainly satisfactory for (2.2), by (2.4). Suppose now that \((\alpha_1, \alpha_2, \alpha_3) = (1, 0, 0)\). If \( \beta_2 = 0 \), then it is not hard to conclude that \( w \) lies on one of at most 2 lattices of determinant \( p^\beta_2 \), which is also satisfactory. If now \( \beta_2 \geq 1 \) we obtain an equation of the shape (2.5), with \( \sigma = \beta_2 - 1 \) and \( \tau = \beta_3 - 1 \). Thus we obtain at most 2 integer lattices, each of determinant \( p^\delta(\beta_2 - 1, \beta_3 - 1) \). It is easily checked that this quantity is bounded below by (2.6). Finally we suppose that \((\alpha_1, \alpha_2, \alpha_3) = (0, 1, 0)\). In this case we again obtain an equation of the form (2.6). Suppose first that \( \beta_2 < \beta_3 \), so that we may take \( \sigma = \beta_2 + 1 \) and \( \tau = \beta_3 \) in (2.6). But then it easily follows that \( w \) lies on one of at most 2 integer lattices, each of determinant

\[
p^\delta(\beta_2 + 1, \beta_3) \geq p^\beta_2 + \beta_3 - \left[ \frac{3(1 + \beta_2)}{2} \right].
\]

Alternatively, if \( \beta_2 = \beta_3 = \beta \) say, then we take \( \sigma = \beta \) and \( \tau = \beta + 1 \) in (2.6), thereby deducing that \( w \) lies on one of at most 2 lattices, each of determinant

\[
p^\delta(\beta, \beta + 1) \geq p^{2\beta - \left[ \frac{3(1 + \beta)}{2} \right]}.
\]

This completes the proof of Lemma 2.

### 2.2 Solubility of quadratic forms

In addition to considering the density of integer solutions to diagonal quadratic equations, as in the previous section, we shall also need to consider how often such an equation has at least one non-trivial integer solution. Let \( \mathbf{a} \in \mathbb{Z}^3 \), let \( Y_1, Y_2, Y_3 \geq 1 \), and let \( H \in \mathbb{N} \). We shall write \( T(Y; \mathbf{a}, H) \) to denote the set of pairwise coprime non-zero integers \( y_1, y_2, y_3 \) such that \( |y_i| \leq Y_i \), and

\[
\gcd(a_i y_i, a_j y_j) \mid H, \tag{2.6}
\]
and such that the equation

\[ a_1y_1x_1^2 + a_2y_2x_2^2 + a_3y_3x_3^2 = 0 \]

has a non-zero solution \( x \in \mathbb{Z}^3 \) with \( \gcd(x_i, x_j) = 1 \). If \( y \in T(Y; a, H) \) then it necessarily follows that

\[ \left( \frac{-a_ia_jy_iy_j}{p} \right) \neq -1 \quad \text{for all odd primes } p \mid a_ky_k, \tag{2.7} \]

where \( \left( \frac{a}{p} \right) \) denotes the Legendre symbol for any \( n \in \mathbb{Z} \) and odd prime \( p \), and as usual \( \{i, j, k\} \) denotes any permutation of the set \( \{1, 2, 3\} \). Define the arithmetic function

\[ \vartheta(n) = \prod_{p \mid n} \left( 1 + \frac{1}{p} \right), \tag{2.8} \]

for any \( n \in \mathbb{N} \). We shall proceed under the assumption that

\[ Y_1 \leq Y_2 \leq Y_3. \tag{2.9} \]

With this in mind the goal of this section is to establish the following estimate.

**Proposition 1.** Let \( \varepsilon > 0 \). Then we have

\[ \sum_{y \in T(Y; a, H)} 2^{\omega(y_1y_2y_3)} \ll \varepsilon \vartheta(a_1a_2)H^{\varepsilon} \left( Y_1Y_2Y_3 + (Y_1Y_2)^{1/2+\varepsilon}Y_3m_\varepsilon(a, Y) \right), \]

with

\[ m_\varepsilon(a, Y) = \min \{|a_1a_2|, Y_3\}^\varepsilon + \log Y_3, \tag{2.10} \]

Before proceeding with the proof of Proposition 1 it is interesting to place it in the context of other work in the literature. Let \( T(Y) \) denote the set \( T(Y; a, H) \) in the special case \( Y_i = Y \) and \( a_i = H = 1 \), and let \( T_*(Y) \) denote the set of \( y \in T(Y) \) for which the product \( y_1y_2y_3 \) is square-free. Then it follows from Proposition 1 that

\[ \sum_{y \in T_*(Y)} 2^{\omega(y_1y_2y_3)} \leq \sum_{y \in T(Y)} 2^{\omega(y_1y_2y_3)} \ll Y^3. \tag{2.11} \]

This should be compared with the work of Guo [5] and Serre [14], who have both made a study of the corresponding sum weighted instead by 1. By using the large sieve inequality Serre has shown that

\[ \# T(Y) \ll \frac{Y^3}{(\log Y)^{3/2}}, \]

and Guo has proved an asymptotic formula for \( \# T_*(Y) \) which agrees with this upper bound. In particular, together with (2.11), these estimates show that a random conic in \( \mathbb{P}^2 \) does not contain a rational point.

We shall need several technical results during the proof of Proposition 1 and it will be convenient to list them here. We begin by recording a basic estimate for the average orders of \( \vartheta \) and \( \vartheta^2 \), as given by (2.8), whose simple proof we include here for completeness.
Lemma 3. Let \( z \geq 1 \). Then we have

\[
\sum_{n \leq z} \vartheta(n) \leq \sum_{n \leq z} \vartheta(n)^2 \ll z.
\]

Proof. The first inequality is trivial, since \( \vartheta(n) \geq 1 \) for all \( n \in \mathbb{N} \). To see the second inequality we note that

\[
\sum_{n \leq z} \vartheta(n)^2 = \sum_{n \leq z} \left( \sum_{d|n} \frac{|\mu(d)|}{d} \right)^2 = \sum_{n \leq z} \sum_{[d_1,d_2]|n} \frac{|\mu(d_1)|\mu(d_2)}{d_1d_2},
\]

where \([d_1,d_2] = d_1d_2/\gcd(d_1,d_2)\) denotes the least common multiple of \( d_1 \) and \( d_2 \). But on writing \( n = [d_1,d_2]e \) we easily deduce that

\[
\sum_{n \leq z} \vartheta(n)^2 \leq \sum_{d_1,d_2=1}^{\infty} \frac{1}{d_1d_2} \leq \sum_{d_1,d_2=1}^{\infty} \frac{\gcd(d_1,d_2)}{d_1^2d_2^2} \ll z,
\]

since

\[
\sum_{d_1,d_2=1}^{\infty} \frac{\gcd(d_1,d_2)}{d_1^2d_2^2} \leq \sum_{d_1,d_2=1}^{\infty} \frac{k}{d_1^2d_2^2} \leq \sum_{k,d_1,d_2=1}^{\infty} \frac{1}{k^3d_1^2d_2^2} \ll 1.
\]

This completes the proof of Lemma 3. \( \square \)

We shall also need to apply the large sieve inequality in our work. The following version of the large sieve is due to Montgomery [11].

Lemma 4. Let \( z, N \geq 1 \). Suppose that \( S \subseteq \mathbb{Z} \cap [1, N] \), and that for every prime \( p \leq z \) there exists \( \sigma(p) \in [0, p) \) such that the image of \( S \) in \( \mathbb{Z}/p\mathbb{Z} \) has \( p - \sigma(p) \) elements. Then we have

\[
\#S \ll N + \frac{z^2}{G(z)},
\]

with

\[ G(z) = \sum_{n \leq z} |\mu(n)| \prod_{p | n} \frac{\sigma(p)}{p - \sigma(p)}. \]

During the course of this section we will need some standard character sum estimates. The following estimate is due to Heath-Brown [3, Corollary 4].

Lemma 5. Let \( M, N \in \mathbb{N} \), and let \( a_1, \ldots, a_M \) and \( b_1, \ldots, b_N \) be arbitrary complex numbers satisfying \( |a_m|, |b_n| \leq 1 \). Then

\[
\sum_{m \leq M \atop 2|m} \sum_{n \leq N} a_m b_n \left( \frac{n}{m} \right) \ll \varepsilon (MN)^{\varepsilon} (M^{1/2}N + MN^{1/2}),
\]

for any \( \varepsilon > 0 \).

Next we recall the Pólya–Vinogradov inequality, which can be found in the work of Davenport [3, Chapter 23], for example.
Lemma 6. Let $M, N \in \mathbb{N}$, and let $\chi$ be a non-principal character modulo $q$. Then

$$\sum_{M \leq n \leq N} \chi(n) \ll q^{1/2} \log q.$$ 

We are now ready to commence our proof of Proposition 1 in earnest. It is important to note that in the course of this proof, as throughout our work, we shall follow common practice and allow the small positive constant $\varepsilon$ to take different values at different parts of the argument. For any $a \in \mathbb{Z}^*$, and any $Y_1, Y_2, Y_3 \geq 1$, recall the definition of the set $T(Y; a, H)$ defined above. It will be convenient to write

$$T = T(Y; a, H) = \sum_{y \in T(Y; a, H)} 2^{\omega(y_1 y_2 y_3)},$$

with which notation our task is to establish the inequality

$$T \ll \varepsilon \vartheta(a_1 a_2) H^\varepsilon \left( Y_1 Y_2 Y_3 + (Y_1 Y_2)^{1/2+\varepsilon} Y_3 m_\varepsilon(a, Y) \right),$$

where $m_\varepsilon(a, Y)$ is given by (2.10). Our approach is based upon a fusion of the ideas used by Guo and Serre in their work on this problem. Recall the assumption (2.9). Whenever $Y_2$ is much smaller in size than $Y_3$ we shall be in a position to apply Lemma 4 to estimate $T$. In the alternative case, in which $Y_3$ is bounded above by a power of $Y_2$, we shall employ Guo’s approach. We begin by studying the latter case.

2.2.1 Proof of Proposition 1 \( Y_3 \leq Y_2^{10} \)

As indicated by the title, our aim in this section is to establish Proposition 1 under the assumption that

$$Y_3 \leq Y_2^{10}. \quad (2.14)$$

Given any $n \in \mathbb{Z}$, it will be convenient to extend the definition of the Legendre symbol $(\frac{n}{p})$ to all primes $p$ by setting $(\frac{n}{p}) = 0$. Our first step is the observation that

$$1 + \left( \frac{n}{p} \right) = \begin{cases} 2, & (\frac{n}{p}) = 1, \\ 0, & (\frac{n}{p}) = -1, \\ 1, & \text{otherwise}, \end{cases} \quad (2.15)$$

for any integer $n$ and prime $p$. Recalling (2.10) and the fact that $\gcd(y_i, y_j) = 1$ for any $y \in T(Y; a, H)$, we see that

$$T = \sum_{y \in T(Y; a, H)} 2^{\omega(y_1) + \omega(y_2) + \omega(y_3)}$$

$$\ll \varepsilon H^\varepsilon \sum_{\substack{|y_i| \leq Y_i \\ \gcd(y_i, y_j) = 1}} \prod_{\{i, j, k\} = \{1, 2, 3\}} \prod_{p \mid y_k} \left( 1 + \left( \frac{-a_i a_j y_i y_j}{p} \right) \right)$$

$$\ll \varepsilon H^\varepsilon \sum_{\substack{|y_i| \leq Y_i \\ \gcd(y_i, y_j) = 1}} \sum_{d_i \mid y_i} 2^{|\mu(d_1 d_2 d_3)|} \prod_{\{i, j, k\} = \{1, 2, 3\}} \left( \frac{-a_i a_j y_i y_j}{d_k} \right).$$
where $\Sigma^3$ denotes a summation over odd divisors $d_i \mid y_i$. Here we have used (2.10) to deduce that $2^\omega(\gcd(y_i,y_i)) \leq 2^\omega(H) \ll \varepsilon H^\varepsilon$. We first show that the contribution arising from case in which $d_2 = d_3 = 1$ is $O_\varepsilon(H^\varepsilon Y_1 Y_2 Y_3)$, which is satisfactory. But the contribution from this case is clearly

$$\ll \varepsilon H^\varepsilon Y_1 Y_2 Y_3 + H^\varepsilon \sum_{|y_i| \ll Y_i \atop \gcd(y_i,y_i) = 1} \sum_{d_i \neq 1} t_i |\mu(d_i)| \left( \frac{-a_2 a_3 y_i y_3}{d_i} \right).$$

Using the Möbius function to pick out the condition $\gcd(y_1, y_1 y_1) = 1$, we may therefore apply Lemma 6 to deduce that

$$\ll \varepsilon H^\varepsilon \sum_{|y_2| \ll Y_2 \atop d_i \neq 1} \sum_{d_i \neq 1} |\mu(d_i)| \sum_{k \mid d_i \mid e_i} \mu(k) \left( \frac{z_i}{d_i} \right)$$

$$\ll \varepsilon H^\varepsilon \sum_{|y_2| \ll Y_2 \atop d_i \neq 1} \sum_{d_i \neq 1} |\mu(d_i)| \sum_{k \mid d_i \mid e_i} \mu(k) \left( \frac{z_i}{d_i} \right)$$

$$\ll \varepsilon (HY_1 Y_2)^\varepsilon \sum_{|y_2| \ll Y_2 \atop d_i \neq 1} \sum_{d_i \neq 1} d_i^{1/2}$$

$$\ll \varepsilon (HY_1 Y_2)^\varepsilon Y_1^{3/2} Y_2 \ll \varepsilon H^\varepsilon Y_1 Y_2 Y_3,$$

if $\varepsilon > 0$ is sufficiently small. It therefore suffices to establish the estimate

$$\sum_{|y_i| \ll Y_i \atop \gcd(y_i, y_i) = 1} \sum_{d_i \neq 1} t_i |\mu(d_i)| \prod_{\{i,j,k\} \in \{1,2,3\}} \left( \frac{-a_i a_j b_j y_i}{d_k} \right) \ll Y_1 Y_2 Y_3, \quad (2.16)$$

in order to complete the proof of Proposition 1 under the assumption (2.14).

Our main tool in the proof of (2.16) will be Lemma 5. Let $D_1, D_2, D_3 \geq 1$. We begin by estimating the contribution to the left hand side of (2.16) from those values of $d_i$ such that

$$D_i \leq d_i < 2D_i.$$

Let us write $\mathcal{T}(D)$ for this contribution. Ultimately we shall sum over dyadic intervals for $D_i \ll Y_i$ to deduce (2.16). Now for any permutation $\{i,j,k\}$ of $\{1,2,3\}$ we see that

$$\mathcal{T}(D) \ll \sum_{|d_i e_i| \ll Y_i} \sum_{d_j e_j \ll Y_j / D_j} \sum_{D_k e_k \ll Y_k / D_k} \left| \sum_{D_k < d_k < 2D_k} \sum_{\varepsilon_j \ll Y_j / D_j} c(d_k, e_j) \left( \frac{e_j}{d_k} \right) \right|,$$

for certain coefficients $c(d_k, e_j) \in \mathbb{Z}$ having modulus at most 1. Hence it easily follows from Lemma 5 that for any $\varepsilon > 0$ we have

$$\mathcal{T}(D) \ll \varepsilon Y_3^{\varepsilon} \sum_{|d_i e_i| \ll Y_i} \sum_{d_j e_j \ll Y_j / D_j} \sum_{D_k e_k \ll Y_k / D_k} \left( D_k^{1/2} \frac{Y_i}{D_j} + D_k \left( \frac{Y_i}{D_j} \right)^{1/2} \right)$$

$$\ll \varepsilon Y_3^{\varepsilon} \left( \frac{Y_1 Y_2 Y_3}{D_k^{1/2}} + Y_i Y_{j/2} Y_k D_k^{1/2} \right). \quad (2.17)$$
This will be satisfactory when exactly one of $D_2$ or $D_3$ is large. To handle the case in which both $D_2$ and $D_3$ are large, we proceed by noting that

$$T(D) \ll \sum_{|d_1| \leq Y_1} \sum_{d_2, d_3} \left| \sum_{d_2, d_3} c'(d_2, d_3) \left( \frac{d_2}{d_3} \right) \right|,$$

for certain coefficients $c'(d_2, d_3) \in \mathbb{Z}$ having modulus at most 1. But then an application of Lemma 6 immediately yields

$$T(D) \ll \frac{Y_1 Y_2 Y_3^{1+\varepsilon}}{D_2 D_3} \left( D_2^{1/2} D_3 + D_2 D_3^{1/2} \right), \quad (2.18)$$

for any $\varepsilon > 0$. Finally we must consider the case in which both $D_2$ and $D_3$ are small. For this we recall that we have already handled the contribution from those $d_2, d_3$ for which $d_2 d_3 = 1$. Suppose first that $d_2 \neq 1$. Then Lemma 6 yields the contribution

$$\ll \sum_{d_2} \sum_{d_3 \leq Y_2 / D_2} \sum_{d_2 \neq 1} \mu(d_1 d_2) \left| \sum_{k \mid d_1, d_2, e_2} \mu(k) \sum_{f_3 \leq Y_3 / (D_3 k)} \left( \frac{k f_3}{d_1 d_2} \right) \right|$$

$$\ll Y_2^2 \sum_{d_3 \leq Y_3 / D_3} \sum_{d_2 \neq 1} \left( D_1 D_2 \right)^{1/2} \ll Y_1 Y_2 Y_3^{1/2 + \varepsilon} D_2^{1/2} D_3$$

to $T(D)$, since $D_1 \ll Y_1 \ll Y_3$. Alternatively, if $d_2 = 1$ and $d_3 \neq 1$, then we obtain the contribution

$$\ll \sum_{e_1 \leq Y_1 / D_1} \sum_{d_3 \leq Y_3 / D_3} \sum_{d_3 \neq 1} \mu(d_1 d_3) \left| \sum_{k \mid d_1, d_3, e_3} \mu(k) \sum_{f_2 \leq Y_2 / k} \left( \frac{k f_2}{d_1 d_3} \right) \right|$$

$$\ll Y_1 Y_2 Y_3^{1/2} D_3^{1/2}$$

to $T(D)$. Thus we may combine these two estimates to deduce that

$$T(D) \ll Y_1 Y_2^{1/2} Y_3^{1/2 + \varepsilon} \left( Y_3^{1/2} Y_3^{1/2} + Y_2^{1/2} D_2^{1/2} D_3 \right). \quad (2.19)$$

We are now in a position to collect together our various estimates to establish the bound

$$T(D) \ll \varepsilon \frac{Y_1 Y_2^{7/8} Y_3^{1+\varepsilon}}, \quad (2.20)$$

for any $\varepsilon > 0$. Before doing so we note that on summing over dyadic intervals for $D_i \ll Y_i$ this is enough to establish that the left hand side of (2.16) is $O_\varepsilon (Y_1 Y_2^{7/8} Y_3^{1+\varepsilon})$. It is at this point that we employ the assumption (2.14), which implies in particular that $\log Y_3 \ll \log Y_2$. This therefore establishes that

$$T \ll \varepsilon H^3 Y_1 Y_2 Y_3$$

provided that the value of $\varepsilon$ is taken to be sufficiently small, and so completes the deduction of Proposition 1 from (2.20) under the assumption that (2.14)
holds. In order to establish (2.20) we shall need to split the argument according to the size of $D_2, D_3$. On supposing first that $D_2, D_3 \ll Y_2^{1/4}$, it clearly follows from (2.19) that

$$T(D) \ll \varepsilon Y_1 Y_2^{1/2} Y_3^{1/2+\varepsilon} \left( Y_2^{1/8} Y_3^{1/2} + Y_2^{7/8} \right) \ll \varepsilon Y_1 Y_2^{7/8} Y_3^{1+\varepsilon},$$

which is satisfactory for (2.20). Similarly, if $D_2, D_3 \gg Y_2^{1/4}$, then (2.18) implies that (2.20) holds. Next we suppose that $D_2 \ll Y_2^{1/4}$ and $D_3 \gg Y_2^{1/4}$. But then we may apply (2.17) to the permutation $(i, j, k) = (1, 2, 3)$ to get

$$T(D) \ll \varepsilon Y_3^{1+\varepsilon} \left( \frac{Y_1 Y_2 Y_3}{Y_2^{1/8}} + Y_1 Y_2^{5/8} Y_3 \right) \ll \varepsilon Y_1 Y_2^{7/8} Y_3^{1+\varepsilon},$$

which is satisfactory for (2.20). Finally, if $D_2 \gg Y_2^{1/4}$ and $D_3 \ll Y_2^{1/4}$ then an application of (2.17) to the permutation $(i, j, k) = (1, 3, 2)$ also yields (2.20).

### 2.2.2 Proof of Proposition \(Y_3 > Y_2^{10}\)

We now turn to the proof of Proposition \(Y_3 > Y_2^{10}\) under the assumption that

$$Y_3 > Y_2^{10}. \tag{2.21}$$

In view of the previous section, this will suffice to complete the proof of Proposition \(Y_3 > Y_2^{10}\). Note that $2^{\omega(n)} \leq d(n)$ for any $n \in \mathbb{N}$, where $d(n)$ is the ordinary divisor function. Then on recalling the conditions (2.14) and (2.17), we see that the quantity (2.13) satisfies

$$T \ll \sum_{|y_1 | \leq Y_1, |y_2 | \leq Y_2} 2^{\omega(y_1 y_2)} \sum_{y_3 : y \in T(Y; a, H), d|y_3} \sum_{d \leq Y_3^{1/2}} 1 \ll \sum_{|y_1 | \leq Y_1, |y_2 | \leq Y_2, \gcd(y_1, y_2) = 1} 2^{\omega(y_1 y_2)} \sum_{d \leq Y_3^{1/2}} \uparrow \# \{ e \in \mathbb{Z} : (y_1, y_2, de) \in T(Y; a, H) \},$$

where the summation $\Sigma \uparrow$ is only over integers $d \leq Y_3^{1/2}$ such that

$$\gcd(d, a_1 a_2) \mid H^2, \quad \gcd(d, y_1 y_2) = 1, \tag{2.22}$$

and

$$\left( \frac{-a_1 a_2 y_1 y_2}{p} \right) = 1$$

for all odd primes $p$ dividing $d/\gcd(d, a_1 a_2)$.

On defining the set

$$S = S(Y_3/d; y_1, y_2, a, H) = \{ e \in \mathbb{Z} : (y_1, y_2, de) \in T(Y; a, H) \},$$

so that in particular

$$T \ll \sum_{|y_1 | \leq Y_1, |y_2 | \leq Y_2, \gcd(y_1, y_2) = 1} 2^{\omega(y_1 y_2)} \sum_{d \leq Y_3^{1/2}} \uparrow \# S \tag{2.23}$$
we see that our task is now to estimate the size of $S$. For this we shall use the large sieve inequality, as presented in Lemma 4. For any odd prime $p$ we must calculate the size of the image $S_p$ of $S$ in $\mathbb{Z}/p\mathbb{Z}$. If $p \nmid a_1 y_1 y_2 d$ then the congruence

$$a_1 y_1 x_1^2 + a_2 y_2 x_2^2 + a_3 d e x_3^2 \equiv 0 \pmod{p}$$

is always soluble whenever $p \nmid e$, by the Chevalley–Warning theorem. Alternatively, if $p | e$ then this congruence is soluble if and only if

$$\left( -\frac{a_1 a_2 y_1 y_2}{p} \right) = 1. \quad (2.24)$$

Hence we conclude that

$$\#S_p = \begin{cases} 
 p-1, & \text{if } p \nmid a_1 y_1 y_2 d, \\
 p, & \text{otherwise},
\end{cases}$$

whenever $p \nmid a_1 y_1 y_2 d$. Turning to the size of $S_p$ in the case $p | a_1 y_1$ and $p \nmid a_2 a_3 y_2 d$. But then either $e = 0$, or else $e \neq 0$ and $e$ belongs to $S_p$ if and only if

$$\left( -\frac{a_2 a_3 y_2 d e}{p} \right) = 1.$$

Hence there are \( \frac{1}{2} (p+1) \) possible values of $e$ over all. Similarly one finds that \( \#S_p = \frac{1}{2} (p+1) \) if $p \mid a_2 y_2$ and $p \nmid a_1 a_3 y_1 d$. If $p \mid a_3 d$ and $p \nmid a_1 a_2 y_1 y_2$ however, then \( \#S_p = p \) since we may clearly suppose that $(2.24)$ holds automatically for such primes. Finally we note that \( \#S_p = p \) in all other cases. Taking $\sigma(p)$ to be $p - \#S_p$, we have therefore shown that

$$\frac{\sigma(p)}{p - \sigma(p)} = \begin{cases} 
 1/(p-1), & p \mid a_1 a_2 y_1 y_2 d, \left( -\frac{a_1 a_2 y_1 y_2}{p} \right) = -1, \\
 (p-1)/(p+1), & p \mid a_1 y_1, p \nmid a_2 a_3 y_2 d, \\
 (p-1)/(p+1), & p \nmid a_2 y_2, p \mid a_1 a_3 y_1 d, \\
 0, & \text{otherwise}.
\end{cases}$$

Now write $g(n) = |\mu(n)| \prod_{p|n} \frac{\sigma(p)}{p - \sigma(p)}$ for any $n \in \mathbb{N}$, so that in particular

$$g(mn) = \begin{cases} 
 g(m) g(n), & \text{if } \gcd(m,n) = 1, \\
 0, & \text{otherwise},
\end{cases}$$

and $G(z) = \sum_{n \leq z} g(n)$. But then it is easy to deduce that

$$G(z) = \sum_{k \mid a_1 y_1 y_2 d} \sum_{\gcd(n, a_1 y_1 y_2 d) = 1} g(n) \geq \sum_{k \mid a_1 y_1 y_2} \sum_{\gcd(n, a_1 y_1 y_2 d) = 1} g(n),$$

for any $z \geq 1$. We may now use the coprimality condition $\gcd(y_1, y_2) = 1$, together with $(2.6)$ and $(2.22)$ to conclude that

$$\sum_{k \mid y_1 y_2} g(k) = \prod_{p \mid y_1 y_2} \left( 1 + g(p) \right) \gg \varepsilon H^{-\varepsilon/2}(y_1 y_2) \prod_{p \mid y_1 y_2} \left( 1 + \frac{1}{p} \right)^{-1},$$

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for any $\varepsilon > 0$. Moreover it is not hard to deduce that

$$\sum_{n \leq z/k, \gcd(n, a, y_1y_2d) = 1} g(n) \gg \sum_{n \leq z/k, p|n \Rightarrow (\frac{-a_1a_2y_1y_2}{p}) = -1} \frac{|\mu(n)|}{n} \gg \log(z/k) \left( \sum_{n \leq z/k, p|n \Rightarrow (\frac{-a_1a_2y_1y_2}{p}) \neq -1} \frac{|\mu(n)|}{n} \right)^{-1}.$$  

We may now complete our estimate for $\#S$ by taking $z = Y_3^{1/4}$ in Lemma 3. On noting that $Y_3/d \geq Y_3^{1/2}$, since $d \leq Y_3^{1/2}$, it therefore follows that

$$\#S \ll \frac{Y_3}{dG(Y_3^{1/4})}.$$  

It is at this point that we apply the hypothesis (2.21), from which it follows that

$$Y_1Y_2 \leq Y_2^2 < Y_3^{1/5}.$$  

In particular we see that $Y_3^{1/4}/k > Y_3^{1/20}$ for any divisor $k | y_1y_2$. Recall the definition (2.8) of the arithmetic function $\vartheta$. We therefore deduce that

$$G(Y_3^{1/4}) \gg \varepsilon \frac{2^\omega(y_1y_2)}{H^3 \vartheta(y_1y_2)} \log(Y_3) \sum_{n \leq Y_3^{1/2}, p|n \Rightarrow (\frac{-a_1a_2y_1y_2}{p}) = 1} \frac{|\mu(n)|}{n},$$  

for any $\varepsilon > 0$, whence

$$\#S \ll \varepsilon \frac{H^3 \vartheta(a_1a_2) \vartheta(y_1y_2)^2}{2^\omega(y_1y_2)} \frac{Y_3}{d \log(Y_3)} \sum_{n \leq Y_3^{1/2}, p|n \Rightarrow (\frac{-a_1a_2y_1y_2}{p}) = 1} \frac{1}{n}.$$  

On inserting this into (2.22), we therefore deduce the statement of the following result.

**Lemma 7.** Let $\varepsilon > 0$. Then we have

$$T \ll \varepsilon \vartheta(a_1a_2) \frac{Y_3}{\log Y_3} \sum_{n \leq Y_3^{1/2}, \gcd(y_1y_2)=1} \vartheta(y_1y_2)^2 \left( \sum_{n \in N} \frac{1}{n} \right)^2,$$  

with

$$N = N(Y_3; a, y_1, y_2) = \left\{ n \in \mathbb{N} : n \leq Y_3^{1/2}, \left( \frac{-a_1a_2y_1y_2}{p} \right) = 1 \text{ for all } p | n \right\}.$$  

On writing $M = N(Y_3^2; a, y_1, y_2)$, we observe that

$$\left( \sum_{n \in N} \frac{1}{n} \right)^2 \leq \sum_{n_1, n_2 \in N} \frac{1}{n_1n_2} \leq \sum_{m \in M} \frac{d(m)}{m},$$
Lemma 6 yields

\[ T \ll \epsilon \frac{\vartheta(a_1a_2)H^3Y_3}{\log Y_3} \sum_{|y_1| \leq Y_1, |y_2| \leq Y_2} \vartheta(y_1y_2)^2 \sum_{1 \leq 2} \frac{m \sum_{d|m} \left( -\frac{a_1a_2y_1y_2}{d} \right)}{d}. \quad (2.25) \]

We proceed by considering the contribution to (2.25) from the terms for which \( d = k^2 \) is a square. Writing \( m = jk^2 \), we obtain the contribution

\[ \ll \epsilon \frac{\vartheta(a_1a_2)H^3Y_3}{\log Y_3} \sum_{y_1, y_2} \frac{1}{d} \sum_{j \leq Y_3/k^2} \vartheta(y_1)^2 \vartheta(y_2)^2 \]

\[ \ll \epsilon \vartheta(a_1a_2)H^2Y_1Y_2Y_3, \]

by Lemma 8. This is plainly satisfactory for Proposition 1. In order to handle the contribution to (2.25) from the remaining divisors, we define the characteristic function

\[ \delta(n) = \begin{cases} 0, & n = k^2 \text{ for some } k \in \mathbb{N}, \\ 1, & \text{otherwise}, \end{cases} \]

for any \( n \in \mathbb{N} \). Note that in particular we have \( \delta(1) = 0 \). Writing \( m = de \), it follows that

\[ \sum_{\gcd(y_1, y_2) = 1} \vartheta(y_1y_2)^2 \sum_{m \leq Y_3} \frac{1}{m} \sum_{d|m, \delta(d) = 1} \left( -\frac{a_1a_2y_1y_2}{d} \right) = \sum_{e \leq Y_3/e} S_e, \quad (2.26) \]

with

\[ S_e = \sum_{d \leq Y_3/e} \delta(d) \sum_{\gcd(y_1, y_2) = 1} \vartheta(y_1)^2 \vartheta(y_2)^2 \left( -\frac{a_1a_2y_1y_2}{d} \right). \]

Our next task is to establish the following inequality.

**Lemma 8.** Let \( \epsilon > 0 \). Then we have

\[ S_e \ll \epsilon Y_1Y_2 + (Y_1Y_2)^{1/2+\epsilon} \{ \min \{ |a_1a_2|, Y_3 \} + \log Y_3 \}. \]

**Proof.** Let us consider the contribution \( S_e(D_1, D_2) \) to \( S_e \) from \( d \) contained in the interval \( D_1 \ll d \ll D_2 \), for various choices of

\[ 1 \ll D_1 \ll D_2 \ll Y_3/e. \quad (2.27) \]

Suppose that \( N \in \mathbb{Z} \) is not a square, and that \( D \geq 1 \). Then an application of Lemma 8 yields

\[ \sum_{d \leq D} \delta(d) \left( \frac{N}{d} \right) = \sum_{d \leq D} \left( \frac{N}{d} \right) - \sum_{d \leq D} \frac{1}{d}= \frac{N^{1/2} \log N + D^{1/2}}{2}, \]

where
Thus for any $D_1, D_2$ in the range $\{2^2, 7\}$, we may combine partial summation with Lemma 4 to deduce that the contribution to $S_\epsilon(D_1, D_2)$ from those $y_1, y_2$ for which $-a_1 a_2 y_1 y_2$ is not a square, is

$$
\ll \sum_{y_1, y_2} \delta(-a_1 a_2 y_1 y_2) \vartheta(y_1)^2 \vartheta(y_2)^2 \sum_{D_1 \ll d \ll D_2} \frac{\delta(d)}{d} \left( \frac{-a_1 a_2 y_1 y_2}{d} \right)
$$

$$
\ll \varepsilon \sum_{y_1, y_2} \vartheta(y_1)^2 \vartheta(y_2)^2 \left( D_1^{-1/2} + D_1^{-1} |a_1 a_2 y_1 y_2|^{1/2 + \varepsilon} \right)
$$

$$
\ll \varepsilon Y_1 Y_2 + D_1^{-1} |a_1 a_2|^{1/2 + \varepsilon} (Y_1 Y_2)^{3/2 + \varepsilon}
$$

$$
\ll \varepsilon Y_1 Y_2 + D_1^{-1} |a_1 a_2|^{3/4} (Y_1 Y_2)^{7/4},
$$
on taking $\varepsilon = 1/4$. Now there are clearly $O((Y_1 Y_2)^{1/2})$ values of $y_1, y_2$ for which $-a_1 a_2 y_1 y_2$ is a square. In view of the trivial inequality $\vartheta(n) \ll n^\varepsilon$ for any $\varepsilon > 0$, it follows that the total contribution to $S_\epsilon(D_1, D_2)$ from such $y_1, y_2$ is

$$
\ll \varepsilon (Y_1 Y_2)^{1/2 + \varepsilon} \log D_2
$$

for any $\varepsilon > 0$. We have therefore established the bound

$$
S_\epsilon(D_1, D_2) \ll \varepsilon Y_1 Y_2 + D_1^{-1} |a_1 a_2|^{3/4} (Y_1 Y_2)^{7/4} + (Y_1 Y_2)^{1/2 + \varepsilon} \log D_2, \quad (2.28)
$$

for any $\varepsilon > 0$ and any $D_1, D_2$ in the range $\{2^2, 7\}$.

We shall need an alternative estimate for $S_\epsilon(D_1, D_2)$ to handle the contribution from small values of $d$. Let $[d_1, d_2]$ denote the least common multiple of $d_1$ and $d_2$, as usual. Then it follows from a combination of Lemma 6 and (2.12), that for fixed values of $d, y_1$ such that $\delta(d) = 1$, we have

$$
\left| \sum_{y_2 \leq Y_2} \vartheta(y_2)^2 \left( \frac{y_2}{d} \right) \right| \leq \sum_{k | y_1} |\mu(k_1)| \vartheta(k)^2 \sum_{y_2 \leq Y_2/k} \vartheta(y_2)^2 \left( \frac{y_2}{d} \right) \left| \sum_{k | y_1} \vartheta(k)^2 \sum_{d_1, d_2 \leq Y_2/k} \frac{|\mu(d_1) \mu(d_2)|}{[d_1, d_2]^c} \left( \frac{[d_1, d_2] c}{d} \right) \right|
$$

$$
\leq \sum_{k | y_1} \vartheta(k)^2 \sum_{d_1, d_2 \leq Y_2/k} \frac{1}{[d_1, d_2]} \sum_{c | Y_2 / (k [d_1, d_2])} \left( \frac{c}{d} \right)
$$

$$
\ll \varepsilon d^{1/2 + \varepsilon} (Y_1 Y_2)^\varepsilon.
$$

Hence a further application of Lemma 5 yields

$$
S_\epsilon(1, (Y_1 Y_2)^{1/2}) \ll \sum_{|y_1| \leq Y_1} \vartheta(y_1)^2 \sum_{d \leq (Y_1 Y_2)^{1/2}} \delta(d) \sum_{y_2 \leq Y_2} \vartheta(y_2)^2 \left( \frac{y_2}{d} \right)
$$

$$
\ll \varepsilon \sum_{|y_1| \leq Y_1} \vartheta(y_1)^2 \sum_{d \leq (Y_1 Y_2)^{1/2}} d^{-1/2 + \varepsilon} (Y_1 Y_2)^\varepsilon
$$

$$
\ll \varepsilon \sum_{|y_1| \leq Y_1} \vartheta(y_1)^2 (Y_1 Y_2)^{1/4 + \varepsilon}
$$

$$
\ll Y_1 Y_2,
$$

provided that $\varepsilon > 0$ is taken to be sufficiently small.
For larger $d$, we employ Lemma 8. Suppose that $(Y_1 Y_2)^{1/2} \leq D \leq Y_3/e$, and write $a(d) = \delta(d)(-\log a_d^2)$. Then proceeding as above we see that

$$S_e(D, D) = \sum_{D \leq d \leq D} \frac{a_d}{d} \sum_{|d_1, e_1| | f_1| \leq Y_1 \atop |d_2, e_2| | f_2| \leq Y_2} b(d, e, f) \left( \frac{|d_1, e_1| |d_2, e_2| f_1 f_2}{d} \right),$$

where $b(d, e, f) = |\mu(d_1)\mu(d_2)\mu(e_1)\mu(e_2)|$ if $\gcd(|d_1, e_1| f_1, |d_2, e_2| f_2) = 1$ and $b(d, e, f) = 0$ otherwise. In particular we observe that $|a(d)b(d, e, f)| \leq 1$, and so there exists a certain constant $c(d, f_1, f_2)$ of modulus at most 1 such that

$$S_e(D, D) \ll \sum_{d_1, e_1 \leq Y_1 \atop d_2, e_2 \leq Y_2} \frac{1}{d_1 d_2 e_1 e_2} \sum_{D \leq d \leq D} \left| \sum_{|f_1| \leq Y_1 / |d_1, e_1| \atop |f_2| \leq Y_2 / |d_2, e_2|} c(d, f_1, f_2) \left( \frac{f_1 f_2}{d} \right) \right|.$$ 

On combining the fact that $Y_1 Y_2 \leq D^2$ with Lemma 8 we therefore deduce that

$$S_e(D, D) \ll_{\varepsilon} \frac{(Y_1 Y_2)^{\varepsilon}}{D} \sum_{d_1, e_1 \leq Y_1 \atop d_2, e_2 \leq Y_2} \left( \frac{D^{1/2} Y_1 Y_2}{|d_1, e_1| |d_2, e_2|} + \left( \frac{D Y_1 Y_2}{|d_1, e_1| |d_2, e_2|} \right)^{1/2} \right)$$

$$\ll_{\varepsilon} D^{-1/2} Y_1 Y_2 + D^\varepsilon (Y_1 Y_2)^{1/2},$$

for any $\varepsilon > 0$. Summing over dyadic intervals for $(Y_1 Y_2)^{1/2} \leq D \leq D_2$, for any choice of $D_2 \leq Y_3/e$, we therefore obtain the estimate

$$S_e((Y_1 Y_2)^{1/2}, D_2) \ll_{\varepsilon} Y_1 Y_2 + D_2^\varepsilon (Y_1 Y_2)^{1/2}.$$ 

Once combined with (2.29) this yields the bound

$$S_e(1, D_2) \ll_{\varepsilon} Y_1 Y_2 + D_2^\varepsilon (Y_1 Y_2)^{1/2},$$

(2.30)

for any $(Y_1 Y_2)^{1/2} \leq D_2 \leq Y_3/e$.

Taking $D_2 = Y_3/e$ in (2.30) we obtain the estimate

$$S_e \ll_{\varepsilon} Y_1 Y_2 + (Y_1 Y_2)^{1/2} Y_3^\varepsilon,$$

(2.31)

for any $\varepsilon > 0$. In order to obtain a second estimate, and so complete the proof of Lemma 8, we apply (2.30) with $D_2 = |a_1 a_2|^{3/4}(Y_1 Y_2)^{3/4}$ and (2.28) with $D_1 = |a_1 a_2|^{3/4}(Y_1 Y_2)^{3/4}$ and $D_2 = Y_3/e$. This produces the estimate

$$S_e \ll_{\varepsilon} Y_1 Y_2 + (Y_1 Y_2)^{1/2 + \varepsilon} \left( |a_1 a_2|^\varepsilon + \log Y_3 \right),$$

for any $\varepsilon > 0$. We complete the proof of Lemma 8 by taking the minimum of the bounds provided by this inequality and (2.31). 

It remains to substitute Lemma 8 into (2.27), and then insert the resulting estimate into (2.26). Thus it follows that

$$T \ll_{\varepsilon} \frac{\vartheta(a_1 a_2) H^\varepsilon Y_3}{\log Y_3} \sum_{e \leq Y_3} S_e,$$

$$\ll_{\varepsilon} \vartheta(a_1 a_2) H^\varepsilon Y_3 \left( Y_1 Y_2 + (Y_1 Y_2)^{1/2 + \varepsilon} \left( \min\{|a_1 a_2|, Y_3\}^\varepsilon + \log Y_3 \right) \right),$$

which thereby establishes Proposition 1 under the assumption that (2.21) holds.
3 The equation $a_1b_1c_1^2 + a_2b_2c_2^2 + a_3b_3c_3^2 = 0$

The purpose of this section is to bring together the results in §2.1 and §2.2 in order to make a study of the density of integer solutions to the equation

\[ a_1b_1c_1^2 + a_2b_2c_2^2 + a_3b_3c_3^2 = 0. \]

(3.1)

Thus for any $A_i, B_i, C_i \geq 1$ we let $M(A_i, B_i, C_i)$ denote the number of $a, b, c \in \mathbb{Z}_3^*$ such that (3.1) holds and

\[ |a_i| \leq A_i, \quad |b_i| \leq B_i, \quad |c_i| \leq C_i, \]

with

\[ \gcd(a_i, c_j) = \gcd(c_i, c_j) = 1 \] (3.2)

and

\[ |\mu(a_1a_2a_3)| = 1, \quad \gcd(a_i, b_j, b_k) = 1. \] (3.3)

It will be convenient to set

\[ A = A_1A_2A_3, \quad B = B_1B_2B_3, \quad C = C_1C_2C_3. \]

With this notation in mind, we proceed by establishing the following result.

**Proposition 2.** For any $\varepsilon > 0$, we have

\[ M(A_i, B_i, C_i) \ll \varepsilon A^{2/3} B^{1/3} C^{1/3} + \sigma \tau AB^{1/2} C^{1/2}, \]

where

\[ \sigma = 1 + \frac{\min\{A, B\}^{1/3}}{\log B} \quad \tau = 1 + \frac{\log B}{\min\{B, B_j\}^{1/3}}. \]

**Proof.** We shall prove Proposition 2 by making suitable applications of Lemma 1 and Lemma 2. Our starting point is to fix choices of $a, c \in \mathbb{Z}_3^*$ and count the corresponding number of $b \in \mathbb{Z}_3^*$ satisfying (3.1) and $|b_i| \leq B_i$. Applying Lemma 1 with

\[ h = (a_1c_1^2, a_2c_2^2, a_3c_3^2), \]

we easily obtain the upper bound

\[ M(A_i, B_i, C_i) \ll \sum_{a,c} \left( 1 + \frac{B}{\max|a_i|^2 B_i} \right) \]

\[ \leq \sum_{a,c} \left( 1 + B^{2/3} |a_1a_2a_3c_1^2c_2^2c_3^2|^{-1/3} \right) \]

\[ \ll AC + A^{2/3} B^{2/3} C^{1/3}. \]

We shall use this bound whenever $C \leq B$, under which hypothesis the estimate in Proposition 2 clearly holds.

It remains to handle the case in which $C > B$. For this we fix choices of $a, b \in \mathbb{Z}_3^*$ for which (3.3) holds, and count the corresponding number of $c \in \mathbb{Z}_3^*$ satisfying (3.1), (3.2) and $|c_i| \leq C_i$. Thus we are in a position to apply Lemma 2 with

\[ g = (a_1, a_2, a_3), \quad h = (b_1, b_2, b_3). \]
In particular it follows from (3.1) – (3.3) that
\[ D_{a,b} = \gcd(b_1 b_2, b_1 b_3, b_2 b_3) \gcd(a_1, b_2 b_3) \gcd(a_2, b_1 b_3) \gcd(a_3, b_1 b_2) \]
\[ = \gcd(b_1 b_2, b_1 b_3, b_2 b_3) \leq \gcd(b_1, b_2) \gcd(b_1, b_3) \gcd(b_2, b_3) = E_b, \]
say. Moreover, since \(|b_1 b_2 b_3| \leq B\) and \(C > B\), we also have
\[ \frac{C}{|b_1 b_2 b_3|} \geq \frac{C}{B} > 1. \]

Thus under the assumption \(C > B\) we may conclude from Lemma 2 that
\[ M(A_i, B_i, C_i) \ll C^{1/2} \sum_{a,b}^* \frac{E_b^{3/4}}{|b_1 b_2 b_3|^{1/2}} 2^{\omega(b_1 b_2 b_3)}, \]

where \(\sum_{a,b}^*\) indicates a summation over \(a, b \in \mathbb{Z}^3\) for which \(|a_i| \leq A_i, |b_i| \leq B_i\), (3.3) holds, and the equation (3.1) has a solution \(c \in \mathbb{Z}^3\) with (3.2) holding.

In order to handle the term \(E_b^{3/4}\) in our estimate for \(M(A_i, B_i, C_i)\), we write
\[ b_i = h_{ij} h_{ik} b_i', \]
for fixed \(h_{12}, h_{13}, h_{23} \in \mathbb{N}\) such that \(h_{ij} = h_{ji}\). Then
\[ |b_i'| \leq \frac{B_i}{h_{ij} h_{ik}} = B_i', \quad (3.4) \]
say. Since \(b\) is primitive, it follows that \(\gcd(h_{ij}, h_{ik}) = 1\). Moreover, for fixed values of \(h_{12}, h_{13}, h_{23}\), it suffices to sum over \(a, b' \in \mathbb{Z}^3\) for which
\[ \gcd(h_{ij}, a_k b'_i') = \gcd(b'_i, b'_j') = 1, \quad (3.5) \]
by (3.3) and the fact that \(b\) is primitive. With this change of variables, the equation (3.1) clearly becomes
\[ \beta_1 b_1' c_1^2 + \beta_2 b_2' c_2^2 + \beta_3 b_3' c_3^2 = 0, \quad (3.6) \]
where we have written
\[ \beta_i = a_i h_{ij} h_{ik} \]
for fixed values of \(a_i, h_{ij}\). We shall need to record the equality
\[ \gcd(\beta_1 b_1', \beta_2 b_2', \beta_3 b_3') = h_{ij}, \quad (3.7) \]
which easily follows from combining the coprimality conditions (3.2), (3.3), (3.5) and \(\gcd(h_{ij}, h_{ik}) = 1\), with the equation (3.4).

Write \(h = h_{12} h_{13} h_{23}\) and suppose without loss of generality that
\[ B_1' \leq B_2' \leq B_3'. \]

Then upon collecting our work together, we see that
\[ M(A_i, B_i, C_i) \ll C^{1/2} \sum_{h_{ij}} \sum_a \sum_{b' \in B}^{*} \frac{2^{\omega(h^2 b_1' b_2' b_3')}}{h^{1/4} |b_1' b_2' b_3'|^{1/2}}, \]
where $B = B(B_i; a_i, h_{ij})$ denotes the set of $b' \in \mathbb{Z}_3^3$ with pairwise coprime components, for which (3.4) and (3.7) hold, and (3.6) has a solution $c \in \mathbb{Z}_3^3$ with $\gcd(c_i, c_j) = 1$. Using the trivial upper bound $2^{c(n)} \ll \varepsilon n^\varepsilon$, we may therefore combine Proposition 4 with Lemma 5 and partial summation to deduce that

$$\mathcal{M}(A_i, B_i, C_i) \ll \varepsilon C^{1/2} \sum_{h_{ij}} h^{\varepsilon - 1/4} \sum_a \sum_{b' \in B} \frac{2^{\omega(b'_1 b'_2 b'_3)}}{|h'_1 b'_2 b'_3|^{1/2}} \ll \varepsilon AC^{1/2} \sum_{h_{ij}} \frac{(B'_1 B'_2 B'_3)^{1/2}}{h^{1/4 - \varepsilon}} \left(1 + \frac{\min\{A, B\}^{\varepsilon} + \log B}{(B'_1 B'_2)^{1/2 - \varepsilon}}\right).$$

On recalling the definition (3.4) of $B'_i$, we see that

$$\sum_{h_{ij}} \frac{(B'_1 B'_2 B'_3)^{1/2}}{h^{1/4 - \varepsilon}} \ll B^{1/2},$$

provided that $\varepsilon > 0$ is taken to be sufficiently small. Similarly, using the inequalities

$$h_{12} \leq (B_1 B_2)^{1/2}, \quad h_{13} \leq B_1/h_{12}, \quad h_{23} \leq B_2/h_{12},$$

we easily check that

$$\sum_{h_{ij}} h^{\varepsilon - 1/4} (B'_1 B'_2)^{\varepsilon} B_3^{1/2} \ll (B_1 B_2)^{3/8 + \varepsilon} B_3^{1/2} \ll (B_1 B_2)^{7/16} B_3^{1/2},$$

provided that $\varepsilon > 0$ is sufficiently small. It therefore follows that the estimate in Proposition 2 holds in the case $C > B$, and so holds unconditionally.

It turns out that we shall need an alternative estimate for $\mathcal{M}(A_i, B_i, C_i)$ to handle the case in which $B_1, B_2, B_3$ have particularly awkward sizes. The following result is rather easy to establish.

**Proposition 3.** We have

$$\mathcal{M}(A_i, B_i, C_i) \ll AB_i B_j (C_k + C_i C_j A_k^{-1})(\log AC)^2,$$

for any permutation $\{i, j, k\}$ of the set $\{1, 2, 3\}$.

**Proof.** Our proof of Proposition 3 is based upon Heath-Brown’s treatment [10, Lemma 4] of the equation $n_1^2 n_2 n_3 + n_4^2 n_5 n_6 = n_7 n_8$. For fixed integers $a, b, q$ we let $\rho(q; a, b)$ denote the number of solutions to the congruence $at^2 + b \equiv 0 \pmod{q}$. For any value of $q$, we then have

$$\rho(q; a, b) \leq \sum_{d|q} \mu(d) \left| \frac{-ab}{d} \right|. \tag{3.8}$$

We shall establish Proposition 3 in the case $(i, j, k) = (1, 2, 3)$, say. The other cases will follow by symmetry. Now it follows from (3.8) that for given $a_i, b_1, b_2, c_3$, and each corresponding solution $t$ of the congruence

$$a_1 b_1 t^2 + a_2 b_2 \equiv 0 \pmod{a_3 c_3^2},$$

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we must have $c_1 \equiv tc_2 \pmod{a_3c_3^2}$. This gives rise to an equation of the form $h, w = 0$, with $h = (1, -t, a_3c_3^2)$ and $w = (c_1, c_2, k)$. Upon recalling that $\gcd(c_1, c_2) = 1$ from (3.2), an application of Lemma 1 therefore yields the bound

$$
\ll \rho(a_3c_3^2; a_1b_2, a_2b_2)
\left(1 + \frac{C_1C_2}{|a_3c_3^2|}\right),
$$

for the number of possible $b_3, c_1, c_2$ given fixed choices for $a_i, b_1, b_2$ and $c_3$. On employing (3.5) we therefore have

$$
\mathcal{M}(A_i, B_i, C_i) \ll \sum_{a_i,b_1,b_2,c_3} \rho(a_3c_3^2; a_1b_2, a_2b_2)
\left(1 + \frac{C_1C_2}{|a_3c_3^2|}\right)
\ll \sum_{a_i,b_1,b_2,c_3} \sum_{d|a_3c_3} \mu(d) \left(\frac{-a_1a_2b_1b_2}{d}\right)
\left(1 + \frac{C_1C_2}{|a_3c_3^2|}\right)
\ll \sum_{a_i,b_1,b_2,c_3} d(a_3)d(c_3)
\left(1 + \frac{C_1C_2}{|a_3c_3^2|}\right),
$$

since the sum over square-free divisors of $a_3c_3^2$ is the same as the sum over square-free divisors of $a_3c_3$. But a simple application of partial summation now reveals that

$$
\mathcal{M}(A_i, B_i, C_i) \ll \sum_{a_i,b_1,b_2,c_3} d(a_3)d(c_3)
+ C_1C_2 \sum_{a_i,b_1,b_2,c_3} \frac{d(a_3)d(c_3)}{|a_3c_3^2|}
\ll \left(AB_1B_2C_3 + A_1A_2B_1B_2C_1C_2\right)(\log AC)^2,
$$

which thereby completes the proof of Proposition 3.

Although we shall not need to do so here, it is worth pointing out that with more work it is possible to remove the term $(\log AC)^2$ from the statement of Proposition 3.

### 4 Passage to the universal torsor

Our goal in this section is to equate the quantity (2.1) to the cardinality of a certain subset of integral points on the universal torsor above $\bar{X}$. In fact our approach to the universal torsor rests upon an entirely elementary analysis of the equation defining the surface $X$, and we shall not prove here that the resulting parametrisation is actually the universal torsor above $\bar{X}$. This fact will be supplied for us by the work of Hassett and Tschinkel [6].

In any solution $\mathbf{x} \in \mathbb{Z}^4$ to the equation $F(\mathbf{x}) = 0$ we see that $x_4$ divides $x_1x_2x_3$. Hence we may write $x_4 = y_1y_2y_3$ and $x_i = y_iz_i$, for some $y, z \in \mathbb{Z}^3$ with $y_iz_i \neq 0$. Suppose that $z_i = \varepsilon_i z_i'$ for $\varepsilon_i = \pm 1$ and $z_i' \in \mathbb{N}$. Then one easily employs the equation $F(\mathbf{x}) = 0$ to deduce that $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$. Hence, upon relabelling variables we may assume that

$$
x_1 = y_1z_1, \quad x_4 = y_1y_2y_3,
$$

for $(y, z) \in \mathbb{Z}^3 \times \mathbb{N}^3$ with $y_i \neq 0$. 

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Under this substitution, the equation $F(\mathbf{x}) = 0$ becomes
\[ z_1 z_2 z_3 = (y_1 z_1 + y_2 z_2 + y_3 z_3)^2. \]  
(4.1)

Since $\mathbf{x}$ is assumed to be primitive, it follows that $\mathbf{y}$ is primitive. Moreover, if $p$ is any prime divisor of $\gcd(z_i, y_j)$, then $\gcd(z_i, y_j)$ implies that $p$ divides $y_k z_k$. But this contradicts the primitivity of $\mathbf{x}$, whence
\[ \gcd(z_i, y_j) = \gcd(y_1, y_2, y_3) = 1. \]  
(4.2)

We now write $z_i = w_i t_i^2$, for square-free $w_i \in \mathbb{N}$ and non-zero $t_i \in \mathbb{Z}$. In fact we may assume that $t_i \in \mathbb{N}$, since $t_i$ and $-t_i$ produce the same value of $z_i$.

Then it follows from (4.2) that
\[ \gcd(w_i, y_j) = \gcd(t_i, y_j) = 1, \]  
(4.3)

and from (4.1) that $w_1 w_2 w_3$ is a square. Hence we can write
\[ w_1 = u_2 u_3, \quad w_2 = u_1 u_3, \quad w_3 = u_1 u_2, \]
for square-free $u_i \in \mathbb{N}$, satisfying
\[ \gcd(u_i, u_j) = \gcd(u_i, y_i) = 1. \]  
(4.4)

Indeed $w_i$ is square-free, and any prime divisor of $\gcd(u_i, y_i)$ must also divide $\gcd(w_i w_k, y_i)$, contrary to (4.3).

Substituting the quantities $w_i = u_j u_k$, therefore yields the expressions
\[ \varepsilon t_1 t_2 t_3 t_1 u_2 u_3 = y_1 u_2 u_3 t_1^2 + y_2 u_1 u_3 t_2^2 + y_3 u_1 u_2 t_3^2, \]  
(4.5)

where $\varepsilon = \pm 1$. It is clear that $u_i$ must divide $y_i u_j u_k t_i^2$. But then $u_i$ divides $t_i$, since $u_i$ is square-free and $\gcd(u_i, y_i u_j u_k) = 1$, by (4.4). We proceed by writing
\[ s_0 = \gcd(t_1/u_1, t_2/u_2, t_3/u_3), \]
and $s_i = t_i/(s_0 u_i)$. Plainly $s_0, s_i \in \mathbb{N}$, and $\mathbf{s} = (s_0, s_1, s_2, s_3)$ is primitive. Moreover, (4.3) yields
\[ \gcd(u_i, y_j) = \gcd(s_i, y_j) = 1. \]  
(4.6)

Substituting $t_i = s_0 s_i u_i$ into (4.5), we therefore obtain the equations
\[ \varepsilon s_0 s_1 s_2 s_3 s_1 u_2 u_3 = y_1 u_1 s_1^2 + y_2 u_2 s_2^2 + y_3 u_3 s_3^2, \]  
(4.7)

where $\varepsilon = \pm 1$, and $\gcd(s_0, y_i) = 1$ by (4.3).

We proceed by using this equation, together with the fact that $\mathbf{s}$ is primitive, to establish that
\[ \gcd(s_i, u_j) = 1. \]

If $p$ is any prime divisor of $\gcd(s_i, u_j)$ then it follows from (4.4), in conjunction with the coprimality conditions (4.4) and (4.6), that $p$ divides $s_k$. Considering the corresponding $p$-adic order of each of the terms in (4.7), one is easily led to the conclusion that $p$ divides $y_j s_j^2$, since $u_j$ is square-free. But $\gcd(u_j, y_j) = 1$ by (4.4), and so $p$ divides $s_j$, which is impossible. In fact we may go further and deduce that the components of $\mathbf{s}$ satisfy the relation
\[ \gcd(s_i, s_j) = 1. \]
This follows immediately from (4.6), (4.7) and the fact that $\gcd(s_i, u_k) = 1$.

Let $\mathcal{T} \subset \mathbb{A}^{10}$ denote the set of $(s_0, s, u, y) \in \mathbb{N} \times \mathbb{N}^3 \times \mathbb{N}^3 \times \mathbb{Z}^3$ such that

$$s_0s_1s_2s_3u_1u_2u_3 = y_1u_1s_1^2 + y_2u_2s_2^2 + y_3u_3s_3^2,$$

with

$$|\mu(u_1u_2u_3)| = \gcd(s_i, s_j) = \gcd(s_i, u_j) = 1,$$ (4.9)

and

$$\gcd(s_0, y_i) = \gcd(s_i, y_j) = \gcd(u_i, y_1y_2y_3) = 1.$$ (4.10)

Now let $x \in \mathbb{Z}_4^4$ be any solution to the equation $F(x) = 0$. Then tracing back through our argument, we deduce that there exists $(\pm s_0, s, u, y) \in \mathcal{T}$ such that

$$x_i = y_iu_i^2u_ju_k\delta^{s_0s_1}, \quad x_4 = y_1y_2y_3.$$ (4.11)

Conversely, given any $(\pm s_0, s, u, y) \in \mathcal{T}$, the point given by (4.11) will be a primitive integer solution of the equation $F(x) = 0$, with $x_1x_2x_3x_4 \neq 0$. Indeed if $p$ is any prime divisor of $x_1, x_2, x_3, x_4$ then we may assume that

$$p \mid y_i, \quad p \mid s_0u_1u_2u_3\gcd(y_ju_jy_j, y_ku_ku_k).$$

But then (4.10) implies that $p \mid \gcd(y_1, y_2, y_3)$, which is impossible. We have therefore established the following result.

**Lemma 9.** We have

$$N_{U;H}(B) = \frac{1}{4}\#\{(s_0, s, u, y) \in \mathcal{T} : \max\{|y_1u_1^2u_ju_k\delta^{s_0s_1}|, |y_1y_2y_3|\} \leq B\}.$$ (4.12)

The equation (4.8) is an affine embedding of the universal torsor above the minimal desingularisation $\tilde{X}$ of $X$. As already mentioned, it has been calculated by Hassett and Tschinkel [11 §4] by computing generators for the Cox ring $\text{Cox}(\tilde{X})$ of $\tilde{X}$.

### 5 The lower bound

Our method of establishing the lower bound closely follows Heath-Brown’s treatment of the Cayley cubic. Consequently we shall adopt similar notation throughout this section.

Let $P_1, P_2 \leq B^3$, for some suitably small choice of $\delta > 0$. This choice will be specified in (5.13), below. The idea is to fix choices of $s, u \in \mathbb{N}^3$ such that (4.9) holds and

$$u_1u_2u_3 = P_1, \quad s_1s_2s_3 = P_2.$$ (5.1)

In fact we shall insist upon the stronger condition that $P_1P_2$ is square-free. This is clearly permissible for the purposes of a lower bound. We then count the number of comparatively large non-zero solutions $s_0, y_1, y_2, y_3$ to the linear equation (4.8) subject to certain constraints.

Thus for $Y_0, Y_i \geq 1$, we let

$$\mathcal{N} = \mathcal{N}(s, u; Y_0, Y_1, Y_2, Y_3)$$
denote the number of \((s_0, y_1, y_2, y_3) \in \mathbb{N} \times \mathbb{Z}^3\) constrained by \((4.8)\) and
\[
gcd(s_0, y_i) = \gcd(y_i, P_1P_2) = 1, \quad (5.1)
\]
for which
\[
Y_0 \leq s_0 < 2Y_0, \quad Y_i \leq |y_i| < 2Y_i. \quad (5.2)
\]
It should be clear that whenever \((4.8)\) and \((5.1)\) both hold, we automatically have \((4.10)\) and \(\gcd(y_1, y_2, y_3) = 1\). It will be convenient to define the quantities
\[
A_0 = P_1P_2, \quad A_i = u_is_i^2,
\]
so that \((4.8)\) may be written
\[
A_0s_0 = A_1y_1 + A_2y_2 + A_3y_3. \quad (5.3)
\]
Now it follows from Lemma \([9]\) that we are only interested in values of \(s_0, y_i\) for which
\[
A_0^2A_i|s_0^2y_i| \leq BP_1P_2^2, \quad A_1A_2A_3|y_1y_2y_3| \leq BP_1P_2^2.
\]
Hence we shall choose
\[
Y_0 = \left\lfloor \frac{(BP_1P_2^2)^{1/3}}{2A_0} \right\rfloor, \quad Y_i = \left\lfloor \frac{(BP_1P_2^2)^{1/3}}{2A_i} \right\rfloor. \quad (5.4)
\]
Much as in Heath-Brown’s treatment, the main difficulty arises from having to keep track of the coprimality conditions \((5.1)\). Let
\[
Q = P_1P_2 \prod_{p \leq \sqrt{\log B}} p.
\]
Following \([10, \text{Equation (3.6)}]\), we write
\[
\mathcal{N} \geq \mathcal{N}_1 - \mathcal{N}_2, \quad (5.5)
\]
where \(\mathcal{N}_1\) is the number of solutions in which the condition \(\gcd(s_0, y_i) = 1\) is replaced by the weaker condition
\[
\gcd(s_0, y_i, Q) = 1,
\]
and \(\mathcal{N}_2\) is the number of solutions in which some \(y_i\) shares a prime factor \(p\) with \(s_0\), such that \(p \nmid Q\).

We proceed by estimating \(\mathcal{N}_1\), for which we use the Möbius function to pick out the coprimality conditions. Let \(\mathcal{N}_3(d; e) = \mathcal{N}_3(d_1, d_2, d_3; e_1, e_2, e_3)\) denote the number of solutions of the equation \((5.3)\) in the range \((5.2)\) with \(d_i \mid y_i\) and \(e_i \mid s_0, y_i\). Then
\[
\mathcal{N}_1 = \sum_{d_i \mid P_1P_2} \mu(d_1)\mu(d_2)\mu(d_3) \sum_{e_i \mid Q} \mu(e_1)\mu(e_2)\mu(e_3) \mathcal{N}_3(d; e). \quad (5.6)
\]
Our task is to estimate \(\mathcal{N}_3(d; e)\). Define the least common multiples
\[
h_0 = [e_1, e_2, e_3], \quad h_i = [d_i, e_i].
\]
and the lattice
\[ \Lambda = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : A_i n_i, A_0 n_0 \mid n_1 + n_2 + n_3 \}. \]

Then upon defining the region
\[ \mathcal{R} = \{ r \in \mathbb{R}^3 : A_i Y_i \leq |r_i| < 2A_i Y_i, A_0 Y_0 \leq |r_1 + r_2 + r_3| < 2A_0 Y_0 \}, \]
one follows the lines of Heath-Brown’s argument in order to deduce that
\[ N_3(d; e) = \frac{\text{vol}(\mathcal{R})}{\det \Lambda} + O((\det \Lambda)^2 \max\{Y_i, Y_0\}^2), \]
where
\[ \det \Lambda = \frac{A_0 h_0 \prod_i A_i h_i}{\gcd(\Lambda)} \].

Since \( d_i \mid P_1 P_2 \) and \( e_i \mid Q \), we deduce that \( A_0 h_0 \leq P_1 P_2 Q^3 \) and \( A_i h_i \leq P_1^2 P_2 Q^3 \).

Hence we have
\[ \det \Lambda \leq P_1^2 P_2^{10} Q^6 < P_1^{13} P_2^{16} \exp(O(\sqrt{\log B})) \leq B^{308}. \]

It follows that the error term in our estimate for \( N_3(d; e) \) is \( O(B^{2/3+62\delta}) \), and so (5.6) becomes
\[ N_1 = \text{vol}(\mathcal{R}) \sum_{d_i \mid P_1 P_2, \ e_i \mid Q} \mu(d_1) \cdots \mu(d_3) \frac{\gcd(A_0 h_0, A_i h_i)}{A_0 h_0 \prod_i A_i h_i} + O(B^{2/3+63\delta}), \tag{5.7} \]

since there are at most \( O(B^3) \) divisors of \( P_1 P_2 Q \).

We now investigate the sum
\[ \sum_{d_i \mid P_1 P_2, \ e_i \mid Q} \mu(d_1) \cdots \mu(d_3) \frac{\gcd(A_0 h_0, A_i h_i)}{A_0 h_0 \prod_i A_i h_i} = \prod_{p \mid Q} E_p, \tag{5.8} \]
say. When \( p \mid Q \), but \( p \nmid P_1 P_2 \), we see that
\[ E_p = \sum_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0} \mu(p^{\varepsilon_1}) \mu(p^{\varepsilon_2}) \mu(p^{\varepsilon_3}) \frac{\gcd(p^{\max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}}, p^{\varepsilon_1})}{p^{\max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3}}, \]
from which it easily follows that
\[ E_p = 1 - \frac{3}{p^2} + \frac{2}{p^3}. \tag{5.9} \]

In this calculation we have used the fact that \( p \nmid A_0 A_i \) whenever \( p \nmid P_1 P_2 \). Next, when \( p \mid P_1 \) we may assume that \( p \) divides precisely one factor, \( u_1 \) say. Since \( P_1 P_2 \) is square-free it follows that \( p \nmid u_2 u_3 P_2 \), and that \( p^2 \nmid u_1 \). Let \( A_0' = p^{-1} A_0, A_1' = p^{-1} A_1, A_2' = A_2 \) and \( A_3' = A_3 \), so that
\[ \frac{\gcd(A_0 h_0, A_i h_i)}{A_0 h_0 \prod_i A_i h_i} = \frac{\gcd(p A_0' h_0, p A_1' h_1, A_2' h_2, A_3' h_3)}{p^2 A_0' h_0 \prod_i A_i' h_i}, \]
with $p \nmid A'_0 A'_1$. Then in this setting we see that
\[
E_p = \frac{1}{p^2} \sum_{d_i, e_i \geq 0} \mu(p^{d_i}) \mu(p^{d_2}) \ldots \mu(p^{d_3}) \frac{\gcd(p, p^{\max(d_2, e_2)}, p^{\max(d_3, e_3)})}{p^{\max(e_2 + \max(d_1, e_1) + \ldots + \max(d_1, e_1)}}.
\]
whence a straightforward calculation yields
\[
E_p = \frac{1}{p^2} \left( 1 - \frac{1}{p} - \frac{1}{p^2} + \frac{1}{p^3} \right). \tag{5.10}
\]
Finally we consider the case $p \mid P_2$; so that $p$ divides precisely one factor, $s_1$ say. Since $P_1 P_2$ is square-free it follows that $p \nmid P_1 s_2 s_3$, and that $p^2 \nmid s_1$. Let $A'_0 = p^{-1} A_0, A'_1 = p^{-2} A_1, A'_2 = A_2$ and $A'_3 = A_3$. Then arguing as above we now have
\[
E_p = \frac{1}{p^3} \sum_{d_i, e_i \geq 0} \mu(p^{d_i}) \mu(p^{d_2}) \ldots \mu(p^{d_3}) \frac{\gcd(p, p^{\max(d_2, e_2)}, p^{\max(d_3, e_3)})}{p^{\max(e_2 + \max(d_1, e_1) + \ldots + \max(d_1, e_1)}}.
\]
In view of our calculation for (5.10) we immediately deduce that
\[
E_p = \frac{1}{p^3} \left( 1 - \frac{1}{p} - \frac{1}{p^2} + \frac{1}{p^3} \right). \tag{5.11}
\]
Taking (5.9 - 5.11) together in (5.8), it therefore follows that
\[
\sum_{d_i, e_i} \mu(d_i) \ldots \mu(\varepsilon_3) \frac{\gcd(A_{0h_0}, A_{1h_1})}{A_{0h_0} A_{1h_1}} \gg \frac{1}{P_2^3} \frac{\phi(P_1 P_2)}{P_1 P_2} = \frac{\phi(P_1 P_2)}{P_1 P_2},
\]
since $\phi(n) = n \prod_{p \nmid n} (1 - 1/p)$ for any $n \in \mathbb{N}$. Our choices $\mathcal{Y}_i, Y_0$ clearly imply that $\text{vol}(\mathcal{R}) \gg B P_1 P_2^2$. We claim that
\[
\mathcal{N}_1 \gg \frac{B}{P_1 P_2^3} \frac{\phi(P_1 P_2)}{P_1 P_2}, \tag{5.12}
\]
provided that we take
\[
\delta = 1/201. \tag{5.13}
\]
In order to establish the claim, it clearly suffices to check that the lower bound in (5.12) is larger than the error term in (5.7) when $\delta$ is taken to be $1/201$. But on using the trivial lower bound $\phi(n) \geq 1$ for any $n \in \mathbb{N}$, we see that
\[
\frac{B}{P_1 P_2^3} \frac{\phi(P_1 P_2)}{P_1 P_2} \gg \frac{B}{(P_1 P_2)^2} \gg B^{1 - 4\delta}.
\]
Since $B^{1 - 4\delta} \gg B^{2/3 + 63\delta}$ for $\delta = 1/201$, the claim follows.

Next we must produce an upper bound for $\mathcal{N}_2$, for which we may ignore any coprimality conditions whenever we wish to. Suppose that $p \mid s_0, y_1$, for some prime $p \nmid Q$ lying in the range $R \leq p < 2R$. In particular we may assume that $R \ll Y_1$. There are $O(R)$ such primes, and we fix one particular choice. Following Heath-Brown’s treatment, we write $s_0 = pt_0$ and $y_1 = pt_1$ and count solutions of the linear equation
\[
pA_0 t_0 = pA_1 t_1 + A_2 y_2 + A_3 y_3. \tag{5.14}
\]
In particular $t_0, t_1$ are contained in the ranges
\[ Y_0/R \ll |t_0| \ll Y_0/R, \quad Y_1/R \ll |t_1| \ll Y_1/R. \]
Since $P_1 P_2$ is square-free, it follows that $\gcd(A_i, A_0) = u_i s_i$. Hence we may deduce from (5.4) that
\[ A_2 y_2 \equiv -A_3 y_3 \pmod{u_1 s_1}. \]
We may assume by symmetry that $A_2 Y_2 \geq A_3 Y_3$. Upon noting that $p u_1 s_1$ is coprime to $A_2$, since $\gcd(y_i, P_1 P_2) = 1$, it follows that for each choice of $y_3$, there are $O(1 + Y_2/(Ru_1 s_1))$ possibilities for $y_2$. Now (5.14) implies that
\[ Ru_1 s_1 < \max\{A_2 Y_2, A_3 Y_3\} = A_2 Y_2. \tag{5.15} \]
Moreover, it follows from (5.3) that
\[ A_2 u_1 s_1 \ll Y_2, \tag{5.16} \]
provided that $\delta \leq 1/5$. Together (5.15) and (5.16) imply that
\[ 1 < \left( \frac{A_2 Y_2}{Ru_1 s_1} \right)^{1/2} \left( \frac{Y_2}{A_2 u_1 s_1} \right)^{1/2} = \frac{Y_2}{R^{1/2} u_1 s_1}, \]
whence we deduce that there are $O(Y_2 Y_3/(R^{1/2} u_1 s_1))$ choices for $y_2, y_3$. We fix such a choice and write $A_2 y_2 + A_3 y_3 = p u_1 s_1 k$. Then it remains to count values of $t_0, t_1$ for which
\[ u_2 u_3 s_2 s_3 t_0 = s_1 t_1 + k. \tag{5.17} \]
Now we have already seen that $R \ll Y_1$. Moreover, as in (5.10), we can use (5.2) to show that $u_2^3 u_3^3 s_2 s_3 \ll Y_1$ provided that $\delta \leq 1/15$. Together these inequalities imply that
\[ 1 < \left( \frac{Y_1}{R} \right)^{2/3} \left( \frac{Y_1}{u_2^3 u_3^3 s_2 s_3} \right)^{1/3} = \frac{Y_1}{R^{2/3} u_2 u_3 s_2 s_3}. \]
Viewing (5.17) as a congruence modulo $u_2 u_3 s_2 s_3$, one easily concludes that there are $O(Y_1/(R^{2/3} u_2 u_3 s_2 s_3))$ possibilities for $t_0, t_1$.

In conclusion we have therefore shown that the total number of admissible $p, y_2, y_3, t_0, t_1$, for which $R \leq p < 2R$, is
\[ \ll R \cdot \frac{Y_2 Y_3}{R^{1/2} u_1 s_1} \cdot \frac{Y_1}{R^{2/3} u_2 u_3 s_2 s_3} \ll \frac{B P_1 P_2^2}{R^{1/6} P_1^2 P_2} = \frac{B}{R^{1/6} P_1 P_2}, \]
by (5.3). Summing $R \gg \sqrt{\log B}$ over dyadic intervals, we deduce that
\[ N_2 \ll \frac{B}{P_1 P_2} (\log B)^{-1/12}, \]
provided that $\delta \leq 1/15$. It follows from (5.12) and (5.13) that $N_2 = o(N_1)$, and so (5.5) implies that
\[ N \gg \frac{B}{P_1 P_2} \frac{\phi(P_1 P_2)}{P_1 P_2}. \]
Finally, in order to complete the proof of the lower bound in (1.1), we note that any square-free value of $P$ will factorise into values $u_1, u_2, u_3, s_1, s_2, s_3$ satisfying (4.9), in precisely $d_6(P)$ ways. It therefore follows that

$$N_{U,H}(B) \gg \sum_{P \leq B^{2/201}} |\mu(P)|d_6(P) \frac{B \phi(P)}{P}.$$  \hspace{1cm} (5.18)

To handle this quantity we define the sum

$$S(x) = \sum_{n \leq x} \frac{|\mu(n)|d_6(n)\phi(n)}{n},$$

for any $x > 1$, and proceed by establishing the following simple bound.

**Lemma 10.** For any $x > 1$ we have

$$S(x) \gg x(\log x)^5.$$

**Proof.** To establish the lemma we shall apply Perron’s formula to the corresponding Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{|\mu(n)|d_6(n)\phi(n)}{n^s},$$

defined for $\Re(s) > 1$. It is a trivial matter to calculate the Euler product

$$F(s) = \prod_p \left(1 + \frac{6(1 - 1/p)}{p^s}\right) = \zeta(s)^6 G(s),$$

for some function $G(s)$ that is holomorphic and bounded on the half-plane $\Re(s) > 1/2$. Let $\varepsilon > 0$ and let $T \in [1, x]$. Then Perron’s formula yields

$$S(x) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \zeta(s)^6 G(s) \frac{x^s}{s} ds + O_{\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right).$$

We apply Cauchy’s residue theorem to the rectangular contour joining the points $2/3 - iT, 2/3 + iT, 1 + \varepsilon + iT$ and $1 + \varepsilon - iT$, which therefore leads to the conclusion that there exists a polynomial $f$ of degree 5 such that

$$S(x) - xf(\log x) \ll_{\varepsilon} x^{1+\varepsilon/T} + \left(\int_{2/3-iT}^{1+\varepsilon-iT} + \int_{2/3+iT}^{1+\varepsilon+iT}\right) |\zeta(s)|^6 x^s/s ds.$$

Here we have used the fact that $G(s)$ is bounded for $\Re(s) > 1/2$. To estimate this error term we apply the well known convexity bound $\zeta(\sigma + it) \ll_{\varepsilon} |t|^{(1-\sigma)/3+\varepsilon}$, valid for any $\sigma \in [1/2, 1]$ and $|t| \geq 1$. Thus it follows that

$$S(x) - xf(\log x) \ll_{\varepsilon} x^{1+\varepsilon/T} + x^{2/3+\varepsilon}T^{2/3}.$$  

Selecting $T = x^{1/5}$ therefore completes the proof of Lemma 10.

On combining Lemma 10 with an application of partial summation, and then inserting the resulting estimate into (5.18), we therefore deduce that

$$N_{U,H}(B) \gg B(\log B)^6.$$  

This completes the proof of the lower bound in (1.1).
6 The upper bound

Fix a choice of \(X_1, \ldots, X_4, S_0, S_i, U_i, Y_i \geq 1\). We shall write

\[
N = N(X_1, \ldots, X_4; S_0; S_1, S_2, S_3; U_1, U_2, U_3; Y_1, Y_2, Y_3)
\]

for the total contribution to \(N_{U,H}(B)\) from \(x\) contained in the intervals

\[
X_\xi \leq |x_\xi| < 2X_\xi, \quad (1 \leq \xi \leq 4),
\]

and such that the variables \(s_0, s, u, y\) appearing in Lemma 9 satisfy

\[
S_0 \leq s_0 < 2S_0, \quad S_i \leq |s_i| < 2S_i, \quad U_i \leq |u_i| < 2U_i, \quad Y_i \leq |y_i| < 2Y_i.
\]

It will be convenient to relabel the indices so that

\[
X_1 \leq X_2 \leq X_3.
\]

Suppose that \(x \in Z^4\) is a solution of \(F(x) = 0\), with \(|x_1|, \ldots, |x_4| \leq B\). Then (6.1) implies that

\[
X_1, X_2, X_3, X_4 \leq B.
\]

If \(N = 0\) there is nothing to prove, and so we assume henceforth that the dyadic ranges in (6.1) and (6.2) produce a non-zero value of \(N\).

We proceed by showing that under the assumption that \(N \neq 0\), certain choices of dyadic ranges in (6.1) and (6.2) force certain other ranges to have fixed order of magnitude. It will be convenient to write

\[
S = S_1 S_2 S_3, \quad U = U_1 U_2 U_3, \quad Y = Y_1 Y_2 Y_3.
\]

Hence it follows from (4.11) that

\[
X_i \ll Y_i U_i U S_0^2 S_i^2 \ll X_i,
\]

and that

\[
X_4 \ll Y \ll X_4.
\]

Together, (6.3) and (6.6) imply that

\[
\left(\frac{X_1 X_2 X_3}{X_4}\right)^{1/2} \ll S_0^3 U^2 \ll \left(\frac{X_1 X_2 X_3}{X_4}\right)^{1/2}.
\]

We take a moment to record two further inequalities satisfied by the quantities \(S_0, S_i, U_i, Y_i\), which will be crucial in our final analysis. First we deduce from (6.3), (6.6) and (6.7) that

\[
S_0 S_1^{1/3} U^{2/3} Y^{2/3} = (S_0^3 U^2)^{1/3} Y^{2/3}
\ll (X_1 X_2 X_3)^{1/6} X_4^{1/2}.
\]

Similarly, we may deduce that

\[
S_0 S_1^{1/2} U^{1/2} \ll (S_0^3 U^2)^{1/2} Y^{1/2}
\ll (X_1 X_2 X_3 X_4)^{1/4}.
\]
It is clear that $\mathcal{N}$ is bounded above by the number of $s_0 \in \mathbb{N}$ and $s, u, y \in Z_4^3$ contained in the ranges (6.2), for which (4.8), (4.9) and (4.10) all hold. Ultimately we shall sum over suitable dyadic intervals for $X_1, X_2, X_3, X_4$ and $S_0, S_1, U_i, Y_i$, in order to establish the upper bound in (1.1), and so complete the proof of the theorem.

For any fixed choice of $s_0 \in \mathbb{N}$ and $s, u \in Z_4^3$ in the region (6.2), with (4.9) holding, we let
\[ N(s_0, s, u) = N(Y_1, Y_2, Y_3; s_0, s, u) \]
denote the corresponding contribution to $\mathcal{N}$ from the $y \in Z_4^3$. Clearly we are only interested in values of $s_0, s, u$ for which $N(s_0, s, u)$ is non-zero. Considering $s_0, s, u$ to be fixed, we select any vector
\[ \hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3) \]
for which the Euclidean norm $|\hat{y}|$ is least. Following the convention that this vector too is fixed, for fixed values of $s_0, s, u$, we define the change of variables
\[ z_i = y_i - \hat{y}_i. \]
We shall let $N_1(s_0, s, u; \hat{y})$ denote the overall contribution to $N(s_0, s, u)$ from those $y$ for which $z_1 z_2 z_3 \neq 0$, and we let $N_2(s_0, s, u; \hat{y})$ denote the remaining contribution to $N(s_0, s, u)$ from those $y$ for which $z_1 z_2 z_3 = 0$. With this notation we therefore have
\[ \mathcal{N} = \sum_{s_0, s, u} N(s_0, s, u) = \sum_{s_0, s, u} N_1(s_0, s, u; \hat{y}) + \sum_{s_0, s, u} N_2(s_0, s, u; \hat{y}) \]
\[ = N_1 + N_2, \]
say. Here the summations are over all $(s_0, s, u) \in \mathbb{N} \times Z_4^3 \times Z_4^3$ in the region (6.2), with (4.9) holding. It will be necessary to investigate the quantities $N_1$ and $N_2$ separately. Finally we shall conclude that
\[ N_{U,H}(B) \ll \sum_{s_0, s, u, U_i, Y_i, X_i, X_4} \mathcal{N}, \]
where the summation is over all dyadic intervals for $X_1, X_2, X_3, X_4$, subject to (6.4), and also all dyadic intervals for $S_0, S_1, U_i, Y_i$, subject to (6.5) and (6.6). Our first task is to show that the overall contribution from $N_1$ to $N_{U,H}(B)$ is satisfactory.

**Proposition 4.** We have
\[ \sum_{s_0, s, U_i, X_i, X_4} N_1 \ll B(\log B)^6, \]
where the summation is over dyadic intervals subject to (6.4)–(6.6).

Proposition 4 will be established in §6.2. Next in §6.3 we shall estimate the corresponding contribution from $N_2$ to $N_{U,H}(B)$ via the following result.

**Proposition 5.** We have
\[ \sum_{s_0, s, U_i, Y_i, X_i, X_4} N_2 \ll B(\log B)^6, \]
where the summation is over dyadic intervals subject to (6.4)–(6.6).
Once taken together in (6.12) and (6.13), Propositions 4 and 5 therefore yield the upper bound
\[ N_{U,H}(B) \ll B(\log B)^6, \]
which thereby completes the proof of (1.1).

6.1 Proof of Proposition 4

Our first step in the proof of Proposition 4 is to provide a pair of upper bounds for \( N_{1} \). For any fixed choice of \( s_0 \in \mathbb{N} \) and \( s, u \in \mathbb{Z}^3 \) in the region (6.2), with (4.9) holding, we let \( \hat{y} \) be the corresponding vector (6.10) that was selected above. On recalling the change of variables (6.11), it therefore follows from (4.8) that
\[ z_1 u_1 s_1^2 + z_2 u_2 s_2^2 + z_3 u_3 s_3^2 = 0, \]
and from (6.2) that \( |z_i| < 4Y_i \). Hence we deduce that
\[ N_{1} \ll S_0 \sum_{s,u} \# \{ z \in \mathbb{Z}^3 : z_1 z_2 z_3 \neq 0, |z_i| < 4Y_i, (6.14) \text{ holds} \}. \]

Recall that \( u_1 u_2 u_3 \) is square-free, so that \( \gcd(u_i, u_j) = 1 \). It is apparent that the \( z \) appearing in the summand need not be primitive. Moreover we no longer necessarily have coprimality conditions corresponding to (4.10). In order to recover a weaker set of coprimality relations, we shall write
\[ u_i = d_{ij} u_i', \quad z_i = d_{ij} d_{ki} e z_i', \]
say, for any \( d_{ij}, e \in \mathbb{N} \) with the convention that \( d_{ij} = d_{ji} \). Let
\[ U_i' = \frac{U_i}{d_{jk}}, \quad Y_i' = \frac{Y_i}{d_{ij} d_{ki} e}, \]
and
\[ U' = U_1' U_2' U_3', \quad Y' = Y_1' Y_2' Y_3'. \]
In particular it follows from (4.3) that
\[ |\mu(u_i' u_j' u_k')| = \gcd(s_i, s_j) = \gcd(s_i, u_j') = 1, \]
for any \( s \) and \( u' \).

For fixed values of \( d_{ij}, e \in \mathbb{N} \), our task is to estimate the number of \( s, u', z' \in \mathbb{Z}^3 \) such that
\[ \gcd(u_i', z_j', z_k') = 1, \]
(6.15) holds,
\[ S_i \leq |s_i| < 2S_i, \quad U_i' \leq |u_i'| < 2U_i', \quad |z_i'| < 4Y_i', \]
and
\[ z_1' u_1' s_1^2 + z_2' u_2' s_2^2 + z_3' u_3' s_3^2 = 0. \]
But this quantity is clearly bounded above by \( \mathcal{M} = \mathcal{M}(2U_i', 4Y_i', 2S_i) \) in the notation of (8). Thus it follows that
\[ N_{1} \ll S_0 \sum_{d_{ij}, e \in Y} \mathcal{M}, \]
(6.16)
and Proposition 2 yields
\[ M \ll \varepsilon S^{1/3} U^{2/3} Y^{2/3} + \sigma \tau S^{1/2} U Y^{1/2}, \]  
(6.17)
for any \( \varepsilon > 0 \), where
\[ \sigma = 1 + \frac{\min\{U, Y\} \varepsilon}{\min\{Y_i Y_j\}^{1/16}}, \quad \tau = 1 + \frac{\log Y}{\min\{Y_i Y_j\}^{1/16}}. \]

On applying Proposition 3 we obtain the alternative estimate
\[ M \ll U' Y' Y' \left( S_k + S_i S_j S_{k_1}^{-1} - U_{-1} \right) (\log SU')^2, \]  
(6.18)
for any permutation \( \{i, j, k\} \) of the set \( \{1, 2, 3\} \).

Similarly, since (6.3) and (6.5) imply that \( (\log SU) \ll \log B \), an application of (6.18) yields the following alternative estimate.

**Lemma 12.** We have
\[ N_1 \ll S_0 S_{1/3} U^{2/3} Y^{2/3} + \sigma \tau S_0 S^{1/2} U Y^{1/2}, \]
for any \( \varepsilon > 0 \), where \( \sigma, \tau \) are given by (6.19).

We are now ready to complete the proof of Proposition 4. Now it follows from the inequalities (6.2), (6.3) and (6.5) that
\[ Y_1 U_1 S_1^2 \ll Y_2 U_2 S_2^2 \ll Y_3 U_3 S_3^2. \]  
(6.20)

In particular (6.3) implies that
\[ S_0 SU \ll Y_3 U_3 S_3^2. \]  
(6.21)

Multiplying both sides of this inequality by \( S_0^3 U \), and recalling (6.3) and (6.5), we deduce that
\[ S_0^3 SU \ll B. \]  
(6.22)
It will also be useful to deduce an inequality involving the maximum size of the \( U_i \). Suppose temporarily that \( U_1 \leq U_2 \leq U_3 \), so that \( U_1 U_2 \leq U_3^2 \). Then in view of (6.4) and (6.5) it clearly follows that \( U_1 U_2 \leq B \), whence \( U_1 U_2 \leq B^{1/2} \). Using this sort of argument it is not hard to deduce that in general
\[
U_i U_j \leq B^{1/2}. \quad (6.23)
\]
Throughout the proof of Proposition 4 we shall make the additional assumption that
\[
Y_i \leq Y_j \leq Y_\kappa,
\]
for some permutation \( \{i, j, \kappa\} = \{1, 2, 3\} \). Our plan is to use Lemma 11 whenever
\[
Y_\kappa \leq (Y_i Y_j)^9 \quad \text{or} \quad U \leq (Y_i Y_j)^3, \quad (6.24)
\]
and Lemma 12 in the alternative case \( Y_i Y_j \leq \min\{Y_\kappa^9, U^{1/3}\} \).

Let us consider the case (6.24) first. Recalling the definition (6.19) of \( \sigma \), it follows that
\[
\sigma \ll 1 + (Y_i Y_j)^{10\varepsilon - 1/16}.
\]
Hence we may take \( \sigma \ll 1 \) in Lemma 11 provided that we take \( \varepsilon > 0 \) to be sufficiently small. It therefore follows from (6.8) and Lemma 11 that
\[
N_1 \ll (X_1 X_2 X_3)^{1/6} X_4^{1/2} + S_0 S_1^{1/2} U Y^{1/2} \left( 1 + \frac{\log B}{(Y_i Y_j)^{1/16}} \right), \quad (6.26)
\]
whenever (6.24) holds. We now sum over the various dyadic intervals for \( S_0, S_1, U_i, Y_i, X_i, X_4 \) subject to (6.3)–(6.6) and (6.24). Suppose for the moment that we want to sum over all possible dyadic intervals \( X \leq |x| < 2X \), for which \( |x| \ll X \). Then there are plainly \( O(\log X) \) possible choices for \( X \). In addition to this basic estimate, we shall make frequent use of the estimates
\[
\sum_X X^\delta \ll \delta \quad \left\{ \begin{array}{ll}
1, & \delta < 0, \\
X^\delta, & \delta > 0.
\end{array} \right.
\]
Returning to (6.24), we may deduce from (6.5) and (6.7) that values of \( S_0, Y_1, Y_2, Y_3 \) are determined by the choices of \( X_1, X_2, X_3, X_4 \) and \( S_i, U_i \). Now there are clearly \( \ll (\log B)^6 \) possible sets of values for \( S_i, U_i \). In view of (6.4), we therefore obtain the estimate
\[
\sum_{S_0, S_1, U_i, Y_i, X_i, X_4} (X_1 X_2 X_3)^{1/6} X_4^{1/2} \ll B(\log B)^6. \quad (6.27)
\]
Employing (6.9), we find similarly that
\[
\sum_{S_0, S_1, U_i, Y_i, X_i, X_4} S_0 S_1^{1/2} U Y^{1/2} \ll \sum_{S_0, S_1, U_i, Y_i, X_i, X_4} (X_1 X_2 X_3 X_4)^{1/4} \ll B(\log B)^6. \quad (6.28)
\]
Finally we turn to the term $S_0 S^{1/2} U(Y_1 Y_2) Y_{1/2} \log B$ in (6.20). We shall sum over dyadic intervals subject to the two inequalities

$$Y_\kappa \ll \frac{B}{Y_1 Y_2}, \quad S_1 \ll \frac{B}{S_0 S_2 S_3 U^2}.$$  

The first of these follows from (6.4) and (6.6), whereas the second is just (6.22).

We therefore obtain the estimate

$$\sum_{S_0, S_1, U, Y_1} S_0 S^{1/2} U(Y_1 Y_2) Y_{1/2} \ll B^{1/2} \sum_{S_0, S_1, U, Y_1} \frac{S_0 S^{1/2} U}{(Y_1 Y_2)^{1/16}} \ll B \sum_{S_0, S_2, S_3, U, Y_1} S_0^{-1/2} (Y_1 Y_2)^{-1/16} \ll B (\log B)^5.$$  

Since values of $X_1, X_2, X_3, X_4$ are determined by choices of $S_0, S_1, U_1, Y_1$, we may combine this latter estimate with (6.27) and (6.28) in (6.26), in order to conclude that

$$\sum_{S_0, S_1, U, Y_1} N_1 \ll B (\log B)^6.$$  

Next we handle the case in which (6.23) holds. For this we employ the alternative estimate Lemma 12 to deduce that

$$N_1 \ll (S_0 S_\kappa U Y_1 Y_2 + S_0 S_1 S_2 U U_1 Y_1 Y_2) (\log B)^2.$$  

Again we proceed by summing over dyadic intervals for $S_0, S_1, U_1, Y_1, X_1, X_4$, this time subject to (6.4)–(6.6) and (6.23). Let us consider the term $S_0 S_\kappa U Y_1 Y_2$. But then (6.3), (6.4), (6.7) and (6.23) together imply that

$$S_0 S_\kappa U Y_1 Y_2 = S_0^3 S U^2 \frac{Y_1 Y_2}{S_0^2 S_1 S_2} \ll (X_1 X_2 X_3)^{1/2} (Y_1 Y_2)^{3/2} (U_1 U_2)^{1/2} (X_1 X_2)^{1/2} \ll X_\kappa^{1/2} Y_\kappa^{1/6} (U_1 U_2)^{1/2} \ll B^{11/12}.$$  

Since there are at most $O_\kappa (B^\kappa)$ dyadic intervals for $S_0, S_1, U_1, Y_1$, which in turn determine values of $X_1, X_2, X_3, X_4$, this therefore leads to the conclusion that

$$\sum_{S_0, S_1, U_1, Y_1} S_0 S_\kappa U Y_1 Y_2 (\log B)^2 \ll B,$$  

whenever (6.23) holds. Lastly we consider the term $S_0 S_1 S_2 U U Y_1 Y_2$. Now there are $O(\log B)$ dyadic intervals for $Y_\kappa$, and (6.25) implies that $Y_\kappa Y_1 \ll U^{1/3}$. Employing the upper bound $S_1 \ll B/(S_0^3 S_1 S_2 U^2)$, we therefore deduce that

$$\sum_{S_0, S_1, U_1, Y_1} S_0 S_1 S_2 U U Y_1 Y_2 \ll \log B \sum_{S_0, S_1, U_1} S_0 S_1 S_2 U U^{2/3} \ll B \log B \sum_{S_0, S_1, S_\kappa, U_1} S_0^{-2} S_\kappa^{-1} U^{-1/3} \ll B (\log B)^2.$$  

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whenever $6.25$ holds. Once combined with $6.30$ this yields the overall contribution
\[
\sum_{S_0, S_1, U, Y_1, N_1, X_4} N_1 \ll B(\log B)^4. \tag{6.31}
\]

Once taken together, (6.29) and (6.31) therefore complete the proof of Proposition 4.

### 6.2 Proof of Proposition 5

We begin this section by providing an upper bound for $N_2$. For any fixed choice of $s_0 \in \mathbb{N}$ and $s, u \in \mathbb{Z}_d^2$ in the region $(6.2)$, with $(4.9)$ holding, let $\hat{y}$ be the vector counted by $N(s_0, s, u)$ that was selected at the start of §1. Then $(6.12)$ implies that
\[
N_2 = \sum_{s_0, s, u} N_2(s_0, s, u; \hat{y}),
\]
where $N_2(s_0, s, u; \hat{y})$ denotes the contribution to $N(s_0, s, u)$ from those $y$ for which
\[
\prod_{1 \leq i \leq 3} (y_i - \hat{y}_i) = 0.
\]

Let $N_2^{(i)}$ denote the total contribution to $N_2(s_0, s, u; \hat{y})$ from those $y$ for which $y_i = \hat{y}_i$ is fixed. It therefore follows that
\[
N_2 \leq \sum_{s_0, s, u} \left( N_2^{(1)} + N_2^{(2)} + N_2^{(3)} \right) = N_2^{(1)} + N_2^{(2)} + N_2^{(3)}, \tag{6.32}
\]
say. In order to estimate $N_2^{(i)}$ for fixed values of $s_0 \in \mathbb{N}$ and $s, u \in \mathbb{Z}_d^2$, it suffices to count non-zero integer solutions $y_j, y_k$ to the equation
\[
y_j u_j s_j^2 + y_k u_k s_k^2 = n, \tag{6.33}
\]
where $n = s_0 s_1 s_2 u_1 u_2 u_3 - \hat{y}_i u_i s_i^2$ is fixed. Our first step is to deduce from $(6.30)$ that
\[
\gcd(u_i s_i^2, u_j s_j^2) = 1.
\]
Noting that $|y_j| < 2Y_j$ and $|y_k| < 2Y_k$, we proceed by applying Lemma 1 to $(6.30)$. Taking
\[
h = (u_j s_j^2, u_k s_k^2, n), \quad w = (y_j, y_k, 1),
\]
we therefore deduce that
\[
N_2^{(i)} \ll 1 + \frac{Y_j Y_k}{\max\{Y_j U_j S_j^2, Y_k U_k S_k^2, |n|\}}. \tag{6.34}
\]
Since $Y_i U_i S_i^2 \ll \hat{y}_i u_i s_i^2 \ll Y_i U_i S_i^2$ and $S_0 SU \ll s_0 s_1 s_2 u_1 u_2 u_3 \ll S_0 SU$, by $(6.2)$, it is easy to see that
\[
|n| = |s_0 s_1 s_2 u_1 u_2 u_3 - \hat{y}_i u_i s_i^2|,
\]
\[
\gg \left| \hat{y}_i u_i s_i^2 \right| - |s_0 s_1 s_2 s_3 u_1 u_2 u_3| \gg Y_i U_i S_i^2,
\]
if $Y_i U_i S_i^2 \gg S_0 SU$. Upon summing (6.34) over all $s_0, s, u$, and then inserting the resulting bound into (6.32), we therefore obtain the following result.
Lemma 13. We have

\[ N_2 \ll S_0 SU + \max_{\{i,j,k\}} \left\{ \frac{S_0 SUY_i Y_k}{\max\{Y_j U_j S_j^2, Y_k U_k S_k^2, \theta_i\}} \right\}, \]

where the first maximum is over all permutations \( \{i,j,k\} \) of the set \( \{1,2,3\} \), and

\[ \theta_i = \begin{cases} Y_i U_i S_i^2, & Y_i U_i S_i^2 \gg S_0 SU, \\ 1, & \text{otherwise}. \end{cases} \] (6.35)

We now complete the proof of Proposition 5. Recall the inequality (6.36). Our first task will be to establish that

\[ \sum_{S_0, S_i U_i Y_i X_i X_4} Y_3 U_3 S_3^2 \ll B (\log B)^4, \] (6.37)

where the summation is over dyadic intervals subject to (6.37). In order to do so we observe as in (6.1) that values of \( X_1, X_2, X_3, X_4 \) are determined by the choices of \( S_0, S_i, U_i, Y_i \). Recall the inequalities (6.36). We have two basic cases to consider, according to whether or not \( Y_3 U_3 S_3^2 \) is sufficiently large compared with \( Y_2 U_2 S_2^2 \).

Suppose first that \( Y_3 U_3 S_3^2 \gg Y_2 U_2 S_2^2 \). Then the ranges (6.2) imply that

\[ |y_1 u_1 s_1^2 + y_2 u_2 s_2^2 + y_3 u_3 s_3^2| \geq |y_3 u_3 s_3^2| - |y_1 u_1 s_1^2 + y_2 u_2 s_2^2| \gg Y_3 U_3 S_3^2, \]

in any solution. Since we obviously have \( |y_1 u_1 s_1^2 + y_2 u_2 s_2^2 + y_3 u_3 s_3^2| \ll Y_3 U_3 S_3^2 \), the basic equation (4.8) implies that \( Y_3 U_3 S_3^2 \ll S_0 SU \ll Y_3 U_3 S_3^2 \), whence

\[ S_0 S_1 U_1 U_2 Y_3 S_3 \ll S_0 S_1 S_2 U_1 U_2. \] (6.37)

Summing over \( Y_3 \ll S_0 S_1 S_2 S_3^{-1} U_1 U_2 \), we therefore obtain

\[ \sum_{S_0, S_i U_i Y_i X_i X_4} Y_3 U_3 S_3^2 \ll \sum_{S_0, S_i U_i Y_i Y_2} S_0 SU, \]

where the last sum is subject to the inequality (6.36). Since there are \( (\log B)^2 \) choices for \( Y_1, Y_2 \), we therefore see that this sum is at most

\[ \ll (\log B)^2 \sum_{S_0, S_1, S_2 U_i} S_0 S_1 S_2 U \sum_{S_3 \ll B/(S_0^2 S_1 S_2 U^2)} S_3 \]

\[ \ll B (\log B)^2 \sum_{S_0, S_1, S_2 U_i} S_0^{-2} U_1^{-1} U_2^{-1} U_3^{-1} \]

\[ \ll B (\log B)^4, \]

as required for (6.36).

Next, if \( Y_2 U_2 S_2^2 \ll Y_3 U_3 S_3^2 \ll Y_2 U_2 S_2^2 \), then it follows that any choice of \( Y_3, U_2, U_3, S_2, S_3 \) determines a choice of \( Y_2 \). Proceeding in a similar fashion to above, we deduce from (6.4) and (6.5) that \( Y_3 \ll B/(U_3 U S_0^2 S_2^2) \). Hence we obtain the estimate

\[ \sum_{S_0, S_i U_i Y_i X_i X_4} Y_3 U_3 S_3^2 \ll B \sum_{S_0, S_i U_i Y_i} S_0^{-2} U_1^{-1} U_2^{-1} U_3^{-1} \ll B (\log B)^4. \]
This completes the proof of \((6.36)\).

Recall the estimate in Lemma 13 for \(N_2\). Then in view of \((6.34)\), it suffices to estimate

\[
N^{(1)}(B) = \sum_{S_0, S_1, U, Y} \frac{S_0 S U Y_1 Y_k}{\max\{Y_1 U_1 S_1^2, Y_2 U_2 S_2^2, \theta_3\}}, \tag{6.38}
\]

for each permutation \(\{i, j, k\}\) of \(\{1, 2, 3\}\), and where \(\theta_i\) is given by \((6.39)\). We begin by handling the case \(i = 3\). Suppose first that \(Y_3 U_3 S_3^2 \gg Y_2 U_2 S_2^2\), so that \((6.37)\) holds and we may take \(\theta_3 = Y_3 U_3 S_3^2\). Then

\[
\frac{S_0 S U Y_1 Y_2}{\max\{Y_1 U_1 S_1^2, Y_2 U_2 S_2^2, \theta_3\}} = \frac{S_0 S_1 S_2 U_1 Y_2}{S_3 Y_3}.
\]

Moreover, we recall the inequalities

\[
Y_2 \ll \frac{B}{U_2 U_2 S_2^2}, \quad Y_1 \ll \frac{Y_3 U_3 S_3^2}{U_1 S_1^2}, \quad S_3 \ll \frac{S_0 S_1 S_2 U_1 U_2}{Y_3},
\]

which follow from \((6.33)\), \((6.20)\) and \((6.37)\), respectively. But then it follows that

\[
N^{(3)}(B) \ll B \sum_{S_0, S_1, U, Y_3} \frac{S_1 Y_1}{S_0 S_1 S_2 U_1 U_2 Y_3}
\]

\[
\ll B \sum_{S_0, S_1, U, Y_3} \frac{S_3}{S_0 S_1 S_2 U_1 U_2}
\]

\[
\ll B \sum_{S_0, S_1, U, Y_3} \frac{Y_3^{-1}}{Y_3 Y_3},
\]

whence \(N^{(3)}(B) \ll B \log B^6\) if \(Y_3 U_3 S_3^2 \gg Y_2 U_2 S_2^2\). Next we suppose that \(Y_3 U_3 S_3^2 \ll Y_2 U_2 S_2^2\), and take \(\theta_3 = 1\) in \((6.38)\). Observe that

\[
S_0 \ll \frac{Y_3 S_3}{S_1 S_2 U_1 U_2}, \quad U_3 \ll \frac{Y_2 U_2 S_2^2}{Y_3 S_3^2}, \quad Y_1 \ll \frac{B}{Y_2 Y_3},
\]

which follow from \((6.21)\), the inequality \(Y_3 U_3 S_3^2 \ll Y_2 U_2 S_2^2\) and \((6.6)\), respectively. We then argue as above to deduce that

\[
N^{(3)}(B) = \sum_{S_0, S_1, U, Y_3} S_0 S_1 S_2^{-1} S_3 U_1 U_3 Y_1
\]

\[
\ll \sum_{S, U, Y_3} \frac{Y_3 Y_3 U_3 S_3^2}{U_2 S_2^2}
\]

\[
\ll \sum_{S, U, Y_3} Y_1 Y_2
\]

\[
\ll B \sum_{S, U, Y_3} Y_3^{-1}.
\]

Hence \(N^{(3)}(B) \ll B \log B^6\) in this case also.

Finally, we must estimate \((6.38)\), whenever \(i \neq 3\). Suppose that \(i = 1\), so that

\[
N^{(1)}(B) = \sum_{S_0, S_1, U, Y_1} \frac{S_0 S U Y_2 Y_3}{Y_3 U_3 S_3^2} \sum_{S_0, S_1, U, Y_1} \frac{S_0 S_1 S_2 S_3^{-1} U_1 U_2 Y_2}{S_0 S_1 S_2 S_3^{-1} U_1 U_2 Y_2}.
\]

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Once again we separate our arguments according to the size of \( Y_3U_3S_3^2 \). Suppose that \( Y_3U_3S_3^2 \gg Y_2U_2S_2^2 \), so that (6.37) holds. Then \( Y_3 \) is fixed by the choices of \( S_0, S_1, U_1, U_2 \). Moreover, we have the inequalities

\[
Y_2 \ll \frac{B}{U_2US_0^2S_2^2}, \quad S_1 \ll \frac{S_0S_2S_3U_3}{Y_1},
\]

which follow from (6.38) and the fact that \( Y_1U_1S_1^2 \ll Y_3U_3S_3^2 \ll S_0SU \), respectively. Summing over \( Y_2 \), and then over \( S_1 \), we therefore deduce that

\[
N^{(1)}(B) \ll B \sum_{S_0, S_1, U_1, Y_1} \frac{S_1}{S_0S_2S_3U_2U_3} \ll B \sum_{S_0, S_2, S_3, U_1, Y_1} Y_1^{-1} \ll B \sum_{S_0, S_1, S_2, U_1, U_2} Y_1^{-1} \ll B(\log B)^6,
\]

in this case.

Alternatively, if \( Y_3U_3S_3^2 \ll Y_2U_2S_2^2 \), then \( U_3 \) is determined by choices of \( S_2, S_3, U_2, Y_2, Y_3 \), and it follows that

\[
N^{(1)}(B) \ll \sum_{S_0, S_1, U_1, U_2, Y_1} S_0S_1S_2S_3^{-1}U_1U_2Y_2.
\]

Upon summing over \( S_0 \ll Y_3S_3/(S_1S_2U_1U_2) \), and then over \( Y_2 \ll B/(Y_1Y_3) \), we derive the estimate

\[
N^{(1)}(B) \ll B \sum_{S_1, U_1, U_2, Y_1, Y_3} Y_1^{-1} \ll B(\log B)^6,
\]

in this case.

An entirely similar argument handles the case \( i = 2 \). Upon combining our various estimates we therefore deduce the statement of Proposition 5.

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