Existence and non-existence of area-minimizing hypersurfaces in manifolds of non-negative Ricci curvature

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EXISTENCE AND NON-EXISTENCE OF AREA-MINIMIZING HYPERSURFACES IN MANIFOLDS OF NON-NEGATIVE RICCI CURVATURE

By Qi Ding, J. Jost, and Y. L. Xin

Abstract. We study minimal hypersurfaces in manifolds of non-negative Ricci curvature, Euclidean volume growth and quadratic curvature decay at infinity. By comparison with capped spherical cones, we identify a precise borderline for the Ricci curvature decay. Above this value, no complete area-minimizing hypersurfaces exist. Below this value, in contrast, we construct examples.

1. Introduction. Bernstein’s theorem says that an entire minimal graph in $\mathbb{R}^3$ has to be a plane. This is a classical theorem, and several proofs have been found for it. The original proofs were strictly two-dimensional, making essential use of conformal coordinates, but the statement itself is certainly meaningful in any dimension. Therefore, it was asked whether it also holds in higher dimensions. By using and developing tools from geometric measure theory, higher dimensional generalizations of the Bernstein theorem were achieved by successive efforts of W. Fleming [13], E. De Giorgi [9], F. J. Almgren [1] and J. Simons [29] up to dimension seven within the framework of geometric measure theory. In 1969, Bombieri-De Giorgi-Giusti [4] then provided a counterexample by constructing a non-trivial entire minimal graph in $\mathbb{R}^{n+1}$ with $n > 7$ whose tangent cone at infinity had been described earlier by Simons.

Clearly, the Bernstein problem can be further generalized. Not only can we increase the dimension of the ambient space, but we can also allow for more general Riemannian geometries than the Euclidean one. In order to see what might happen then, we observe that minimal graphs in Euclidean space are automatically area minimizing. Thus, the Bernstein problem is essentially about the (non-)existence of a particular class of complete area-minimizing hypersurfaces. Therefore, the challenge of the Bernstein problem consists in finding sharp conditions for the existence or non-existence of complete area-minimizing hypersurfaces in curved ambient manifolds.

Let us therefore review the previous results in this direction. Schoen-Simon-Yau [26] obtained $L^p$-estimates for the squared norm of the second fundamental form for stable minimal hypersurfaces in certain curved ambient manifolds. As a
consequence, they showed that any stable minimal hypersurface with Euclidean volume growth in a flat $N^{n+1}$ with $n \leq 5$ has to be totally geodesic. Later, Fischer-Colbrie and Schoen [12] proved that there are no stable minimal surfaces in 3-dimensional manifolds with positive Ricci curvature. Shen-Zhu [27] proved certain rigidity results for stable minimal hypersurfaces in $N^4$ or $N^5$. On the other hand, P. Nabonnand [23] constructed a complete manifold $N^{n+1}$ with positive Ricci curvature which admits area-minimizing hypersurfaces. M. Anderson [3] proved a non-existence result for area-minimizing hypersurfaces in complete non-compact simply connected manifolds $N^{n+1}$ of non-negative sectional curvature with diameter growth conditions. For rotationally symmetric spaces with conical singularities, some explicit results were obtained by F. Morgan in [22]. These results will provide us with important model spaces for the general theory.

In the present paper we will study minimal hypersurfaces in complete Riemannian manifolds that satisfy three conditions:

(C1) non-negative Ricci curvature;

(C2) Euclidean volume growth;

(C3) quadratic decay of the curvature tensor.

Such manifolds can be much more complicated than Euclidean space, but on the other hand, this class of manifolds possesses certain topological and analytical properties [8, 24] that constrain their geometry. They admit tangent cones at infinity over a smooth compact manifold in the Gromov-Hausdorff sense. These cones may be not unique, but they have certain nice properties, proved by Cheeger-Colding [5]. Another important fact is that their Green functions have a well controlled asymptotic behavior. In particular, the Hessian of such a Green function converges to the metric tensor (up to a constant factor 2) pointwise at infinity, as shown by Colding-Minicozzi [8]. The precise results will be described in Section 4.

While our non-existence results are quite general, the existence results that we develop here, mainly for the purpose of showing that our non-existence results are sharp, are more explicit and depend on special constructions. Essentially, for these constructions, we consider ambient manifolds of the form $\Sigma \times \mathbb{R}$ where $\Sigma$ is an $n$-dimensional Riemannian manifold with a conformally flat metric whose conformal factor depends only on the radius. This class will include a capped spherical cone with opening angle $2\pi \kappa$, denoted by $MCS_{\kappa}$. Its tangent cone at infinity is the uncapped spherical cone $CS_{\kappa}$, or equivalently, the Euclidean cone over a sphere of radius $\kappa$. These cones will be on one hand our main examples for existence results and on the other hand our model spaces for the non-existence results. The border between those two phenomena, existence vs. non-existence, will be sharp. Existence takes place for $\kappa \geq \frac{2}{n} \sqrt{n-1}$, non-existence else. The intuitive geometric reason is simply that for larger values of $\kappa$, in order to minimize area, it is most efficient to go through the vertex of the cone, whereas for smaller values of $\kappa$, it is better to avoid the vertex and go around the cone. This had already been observed
by F. Morgan in [22]. As a by-product we can answer some questions raised by M. Anderson in [3].

Whereas the existence examples are specific, our non-existence results will be general. Essentially, the idea consists in reducing them to the model cases by taking cones at infinity. For this, we need some heavier machinery, including the theory of Gromov-Hausdorff limits [17, 18, 25, 16] and the theory of currents in metric spaces developed by Ambrosio-Kirchheim [2]. In order to apply those tools, we shall analyze the Green function at infinity of the ambient space and minimal hypersurfaces with Euclidean volume growth, in order to carry the stability inequality for minimal hypersurfaces over to the asymptotic limit. The corresponding results may be of interest in themselves, see Theorem 5.1.

Our main results thus are general non-existence results for stable minimal hypersurfaces in \((n + 1)\)-manifolds \(N\) with conditions (C1), (C2), and (C3) under an additional growth condition on the non-radial Ricci curvature involving a constant \(\kappa'\). For the capped spherical cones \(MCS_{\kappa}\), this constant \(\kappa'\) can be expressed in terms of the constant \(\kappa\). More precisely, we show that \(N\) admits no complete stable minimal hypersurface with at most Euclidean volume growth if the above constant \(\kappa' > \frac{(n-2)^2}{4}\), see Theorem 5.5. The existence result of Theorem 3.4 then tells us that our condition on the asymptotic non-radial Ricci curvature is optimal.

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2. Preliminaries. Let \(\Sigma\) be an \(n\)-dimensional Riemannian manifold with metric \(ds^2 = \sigma_{ij}dx_idx_j\) in local coordinates. Let \(D\) be the corresponding Levi-Civita connection on \(\Sigma\). For a subset \(\Omega \subset \Sigma\) let \(M\) be a graph in the product manifold \(\Omega \times \mathbb{R}\) with smooth defining function \(u\) on \(\Sigma\), i.e.,

\[
M = \{(x, u(x)) \in \Omega \times \mathbb{R} \mid x \in \Omega\}.
\]

Since \(N = \Sigma \times \mathbb{R}\) has the product metric \(ds^2 = \sigma_{ij}dx_idx_j + dt^2\), then the induced metric on \(M\) is

\[
ds^2 = g_{ij}dx_idx_j = (\sigma_{ij} + u_iu_j)dx_idx_j,
\]

where \(u_i = \frac{\partial u}{\partial x_i}\) and \(u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}\) in the sequel. Let \((\sigma^{ij})\) be the inverse metric tensor on \(\Sigma\). Let \(E_i\) and \(E_{n+1}\) be the dual vectors of \(dx_i\) and \(dt\), respectively. Let \(\Gamma^k_{ij}\) be the Christoffel symbols of \(\Sigma\) with respect to the frame \(E_i\), i.e., \(DE_iE_j = \sum_k \Gamma^k_{ij}E_k\).

Set \(u^i = \sigma^{ij}u_j, |Du|^2 = \sigma^{ij}u_iu_j, D_iD_ju = u_{ij} - \Gamma^k_{ij}u_k\) and \(v = \sqrt{1 + |Du|^2}\). If \(f\) stands for the immersion (2.1) of \(\Sigma\) in \(M \subset N\), then \(X_i = f*, E_i = E_i + u_iE_{n+1}\), \(i = 1, \ldots, n\), are tangent vectors of \(M\) in \(N\). Let \(\nu_M\) and \(H\) be the unit normal
vector field and the mean curvature of $M$ in $N$. Then, direct computation yields

$$
\nu_M = \frac{1}{v}(-\sigma^{ij} u_j E_i + E_{n+1}),
$$

$$
H = \text{div}_\Sigma \left( \frac{Du}{v} \right) = -\frac{1}{\sqrt{\det \sigma_{kl}}} \partial_j \left( \sqrt{\det \sigma_{kl}} \frac{\sigma^{ij} u_i}{v} \right).
$$

$M$ is a minimal graph in $\Omega \times \mathbb{R}$ if and only if $H \equiv 0$ and $u$ satisfies

$$
\text{div}_\Sigma \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = \frac{1}{\sqrt{\det \sigma_{kl}}} \partial_j \left( \sqrt{\det \sigma_{kl}} \frac{\sigma^{ij} u_i}{\sqrt{1 + |Du|^2}} \right) = 0.
$$

This is the Euler-Lagrangian equation of the volume functional of $M$ in $N$. Moreover, similar to the Euclidean case [31], any minimal graph on $\Omega$ is also an area-minimizing hypersurface in $\Omega \times \mathbb{R}$, see Lemma 2.1 below.

We introduce an operator $\mathcal{L}$ on a domain $\Omega \subset \Sigma$ by

$$
\mathcal{L} F = (1 + |DF|^2)^{\frac{3}{2}} \text{div}_\Sigma \left( \frac{DF}{\sqrt{1 + |DF|^2}} \right)
$$

$$
= (1 + |DF|^2) \Delta_\Sigma F - F_{i,j} F^i F^j,
$$

where $F^i = \sigma^{ik} F_k$, and $F_{i,j} = F_{ij} - \Gamma^k_{ij} F_k$ is the covariant derivative. Clearly, \{(x, F(x)) \mid x \in \Omega\} is a minimal graph on $\Sigma$ if and only if $\mathcal{L} F = 0$ on $\Omega$. We call $F$ $\mathcal{L}$-subharmonic ($\mathcal{L}$-superharmonic) if $\mathcal{L} F \geq 0$ ($\mathcal{L} F \leq 0$).

**Lemma 2.1.** Let $\Omega$ be a bounded domain in $\Sigma$ and $M$ be a minimal graph on $\overline{\Omega}$ as in (2.1) with volume element $d\mu_M$. For any hypersurface $W \subset \overline{\Omega} \times \mathbb{R}$ with $\partial M = \partial W$, one has

$$
\int_M d\mu_M \leq \int_W d\mu_W,
$$

with equality if and only if $W = M$.

**Proof.** Let $U$ be the domain in $N$ enclosed by $M$ and $W$. Recall that $\nu_M$ is a unit normal vector field on $M$. Viewing $u_i$ and $v$ as functions on $\Sigma$, we define a vector field $Y$ such that for every $(x,t) \in U$, $Y$ is just $\nu_M$ at $M \cap (\{x\} \times \mathbb{R})$ up to a translation along the $E_{n+1}$ axis. Namely,

$$
Y(x,t) = -\sum_{i=1}^{n} \frac{\sigma^{ij}(x) u_j(x)}{v(x)} E_i(x) + \frac{1}{v(x)} E_{n+1}.
$$
From the minimal surface equation (2.2) we have

$$\overline{\text{div}}(Y) = -\sum_i \frac{1}{\sqrt{\det \sigma_{kl}}} \partial_{x_i} \left( \frac{\sqrt{\det \sigma_{kl}} \sigma^{ij} u_j}{v} \right) = 0,$$

where \(\overline{\text{div}}\) stands for the divergence operator on \(N\). Let \(\nu_M, \nu_W\) be the unit outside normal vectors of \(M, W\) respectively. Observe that \(Y|_M = \nu_M\). Then by Green’s formula,

$$0 = \int_U \overline{\text{div}}(Y) = \int_M \langle Y, \nu_M \rangle d\mu_M - \int_W \langle Y, \nu_W \rangle d\mu_W \geq \int_M d\mu_M - \int_W d\mu_W.$$

Obviously, equality holds if and only if \(M = W\). \(\square\)

The index form from the second variational formula for the volume functional for a two-sided minimal hypersurface \(M\) in \(N\) is (see Chapter 6 of [31])

$$I(\phi, \phi) = \int_M (|\nabla \phi|^2 - |\bar{A}|^2 \phi^2 - \text{Ric}_N(\nu_M, \nu_M) \phi^2) d\mu_M,$$

for any \(\phi \in C^2_c(N)\), where \(\nabla\) and \(\bar{A}\) are the Levi-Civita connection and the second fundamental form of \(M\), respectively.

Let \(S_\kappa\) be an \(n\)-sphere in \(\mathbb{R}^{n+1}\) with radius \(0 < \kappa \leq 1\), namely,

$$S_\kappa = \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = \kappa^2 \}.$$

If \(\{\theta_i\}_{i=1}^n\) is an orthonormal basis of \(S_\kappa\), then the sectional curvature of \(S_\kappa\) is

$$K_{S}(\theta_i, \theta_j) = \frac{1}{\kappa^2} \quad \text{for} \ i \neq j.$$

Let \(CS_\kappa = \mathbb{R}^+ \times _\rho S_\kappa\) be the cone over \(S_\kappa\) with vertex \(o\), which has the metric

$$\sigma_C = d\rho^2 + \kappa^2 \rho^2 d\theta^2,$$

where \(d\theta^2\) is the standard metric on \(S^n(1)\).

Let \(\{e_\alpha\}_{\alpha=1}^n \cup \{\frac{\partial}{\partial \rho}\}\) be an orthonormal basis at the considered point of \(CS_\kappa\) away from the vertex, then the sectional curvature and Ricci curvature of \(CS_\kappa\) are

$$K_{CS_\kappa}(\frac{\partial}{\partial \rho}, e_\alpha) = 0, \quad K_{CS_\kappa}(e_\alpha, e_\beta) = \frac{1}{\rho^2} \left( \frac{1}{\kappa^2} - 1 \right),$$

$$\text{Ric}_{CS_\kappa}\left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho}\right) = \text{Ric}_{CS_\kappa}(\frac{\partial}{\partial \rho}, e_\alpha) = 0, \quad \text{Ric}_{CS_\kappa}(e_\alpha, e_\beta) = \frac{n-1}{\rho^2} \left( \frac{1}{\kappa^2} - 1 \right) \delta_{\alpha\beta}.$$
Set $\rho = r^\kappa$, then $\sigma_C$ can be rewritten as a conformally flat metric

\[
\sigma_C = \kappa^2 r^{2\kappa - 2} dr^2 + \kappa^2 r^{2\kappa} d\theta^2
\]

\[
= \kappa^2 r^{2\kappa - 2} \sum_{i=1}^{n+1} dx_i^2 = e^{2\log \kappa - 2(1-\kappa) \log r} \sum_{i=1}^{n+1} dx_i^2,
\]

where $r^2 = \sum_i x_i^2$.

Let $Y$ be an $(n-1)$-dimensional minimal hypersurface in $S_\kappa$ with the second fundamental form $A$ and $CY$ be the cone over $Y$ in $CS_\kappa$ with vertex $o$. For any $0 < \epsilon < 1$ denote

\[
CY_\epsilon = \{ tx \in S_\kappa \times \mathbb{R} | x \in Y, t \in [\epsilon, 1] \}.
\]

Clearly, $Y$ is a minimal hypersurface in $S_\kappa$ if and only if $CY_\epsilon$ is minimal in $CS_\kappa$. Moreover, let $\bar{A}$ be the second fundamental form of $CY_\epsilon$ in $CS_\kappa$, then

\[
|\bar{A}|^2 = \frac{1}{\rho^2} |A|^2.
\]

At any considered point, we can suppose that $\theta_n$ is the unit normal vector of $Y \subset S_\kappa$ and $\{\theta_i\}_{i=1}^{n-1}$ is the orthonormal basis of $TY$. Let $\nu = \frac{1}{\rho} \theta_n$ be the unit normal vector of $CY_\epsilon$. Let $d\mu$ and $d\mu_Y$ be the volume element of $CY_\epsilon$ and $Y$, respectively.

Now, from (2.5), the index form of $CY_\epsilon$ in $CS_\kappa$ becomes

\[
I(\phi, \phi) = \int_{CY_\epsilon} \left( -\phi \Delta_{CY} \phi - |\bar{A}|^2 \phi^2 - \text{Ric}_{CS_\kappa \times \mathbb{R}}(\nu, \nu) \phi^2 \right) d\mu
\]

for any $\phi \in C^2_c(CY \setminus \{o\})$. Note $\text{Ric}_{S_\kappa}(\theta_i, \theta_j) = \frac{n-1}{\kappa^2} \delta_{ij}$ and

\[
\text{Ric}_{CS_\kappa}(\nu, \nu) = \frac{1}{\rho^2} \text{Ric}_{S_\kappa}(\theta_n, \theta_n) - \frac{n-1}{\rho^2} = \frac{n-1}{\rho^2} \left( \frac{1}{\kappa^2} - 1 \right).
\]

When $\phi$ is written as $\phi(x, \rho) \in C^2(Y \times \mathbb{R})$, a simple calculation implies

\[
\Delta_{CY} \phi = \frac{1}{\rho^2} \Delta_Y \phi + \frac{n-1}{\rho} \frac{\partial \phi}{\partial \rho} + \frac{\partial^2 \phi}{\partial \rho^2},
\]

then

\[
I(\phi, \phi) = \int_{\epsilon}^1 \left( \int_Y \left( -\Delta_Y \phi - |A|^2 \phi - \frac{n-1}{\kappa^2} \phi + (n-1) \phi \right.ight.
\]

\[
- (n-1) \rho \frac{\partial \phi}{\partial \rho} - \rho^2 \frac{\partial^2 \phi}{\partial \rho^2} \right) \phi d\mu_Y \left. \right) \rho^{n-3} d\rho.
\]
When \( \kappa = 1 \) and \( Y \) is the Clifford minimal hypersurface in the unit 7-sphere

\[
Y = S^3 \left( \frac{\sqrt{2}}{2} \right) \times S^3 \left( \frac{\sqrt{2}}{2} \right),
\]

then, \( CY \) is Simons’ cone, proved to be stable in [29] (see also Chapter 6 of [31]).

3. Constructions of area-minimizing hypersurfaces. Let \( \Sigma \) be the Euclidean space \( \mathbb{R}^{n+1} \) with a conformally flat metric

\[
ds^2 = e^{\phi(r)} \sum_{i=1}^{n+1} dx_i^2,
\]

where \( r = |x| = \sqrt{x_1^2 + \cdots + dx_{n+1}^2} \) and \( \phi(|x|) \) is smooth in \( \mathbb{R}^{n+1} \). Let \( F \) be a function on \( \mathbb{R}^{n+1} \). Let \( E_i = \{ \frac{\partial}{\partial x_i} \} \) be a standard basis of \( \mathbb{R}^{n+1} \) and \( F_i = \frac{\partial}{\partial x_i} F \) be the ordinary derivative in \( \mathbb{R}^{n+1} \). Moreover,

\[
\Gamma_{ij}^k = \frac{\phi'}{2} \left( \delta_{ik} x_j \frac{x_j}{r} + \delta_{jk} x_i \frac{x_i}{r} - \delta_{ij} x_k \frac{x_k}{r} \right),
\]

Denote \( |\partial F|^2 = \sum_i F_i^2 \). Let \( \Delta \) be the standard Laplacian of \( \mathbb{R}^{n+1} \), then

\[
\Delta_\Sigma F = \sigma^{ij} F_{i,j} = e^{-\phi} \delta_{ij} \left( F_{ij} - \frac{\phi'}{2} \left( \delta_{ik} x_j \frac{x_j}{r} + \delta_{jk} x_i \frac{x_i}{r} - \delta_{ij} x_k \frac{x_k}{r} \right) F_k \right)
\]

\[
= e^{-\phi} \left( \Delta F + \frac{n-1}{2} \phi' F_i \frac{x_i}{r} \right).
\]

(3.1)

By (2.3) we can compute \( \mathfrak{L} F \) in the conformal flat metric as follows.

\[
\mathfrak{L} F = e^{-\phi} (1 + e^{-\phi} |\partial F|)^2 \left( \Delta F + \frac{n-1}{2} \phi' F_i \frac{x_i}{r} \right)
\]

\[
- e^{-2\phi} \left( F_{ij} F_i F_j - \frac{|\partial F|^2}{2} \phi' F_i \frac{x_i}{r} \right)
\]

\[
= e^{-\phi} \left( (1 + e^{-\phi} |\partial F|^2) \Delta F - e^{-\phi} F_{ij} F_i F_j \right)
\]

\[
+ e^{-\phi} \left( \frac{n-1}{2} + \frac{n}{2} e^{-\phi} |\partial F|^2 \right) \phi' F_i \frac{x_i}{r}
\]

\[
= e^{-2\phi} \left( |\partial F|^2 \left( \Delta F + \frac{n}{2} \phi' F_i \frac{x_i}{r} \right) - F_{ij} F_i F_j \right)
\]

\[
+ e^{-\phi} \left( \Delta F + \frac{n-1}{2} \phi' F_i \frac{x_i}{r} \right).
\]

(3.2)
LEMMA 3.1. Let $F = F(\theta, r)$ be a function with

\begin{equation}
\theta = \frac{x_{n+1}}{\sqrt{x_1^2 + \cdots + x_{n+1}^2}}, \quad r = \sqrt{x_1^2 + \cdots + x_{n+1}^2},
\end{equation}

on $[-1, 1] \times (0, \infty)$. Then we have

\begin{equation}
\mathcal{L}F = e^{-2\phi} \left( n \left( 1 - \theta^2 \right) \frac{F_\theta^2}{r^2} + F_r^2 \right) \left( \frac{F_r}{r} + \frac{\phi'}{2} F_r - \theta F_\theta \right) + (1 - \theta^2) \frac{F_\theta^2}{r^2} \left( \frac{\theta F_\theta + F_r}{r} + \frac{1-\theta^2}{r^2} \left( F_\theta^2 F_{rr} + F_r^2 F_{\theta\theta} - 2F_\theta F_r F_{\theta r} \right) \right) + e^{-\phi} \left( F_{rr} + \frac{1 - \theta^2}{r^2} F_{\theta\theta} + \frac{n}{r} F_r - \frac{n\theta}{r^2} F_\theta + \frac{n - 1}{2} \phi' F_r \right).
\end{equation}

Proof. For $1 \leq \alpha \leq n$ we have

\begin{equation}
F_\alpha = \partial_{x_\alpha} F = F_\theta \left( -\frac{x_\alpha x_{n+1}}{r^3} \right) + F_r \frac{x_\alpha}{r},
\end{equation}

\begin{equation}
F_{n+1} = \partial_{x_{n+1}} F = F_\theta \left( \frac{1}{r} - \frac{x_{n+1}}{r^3} \right) + F_r \frac{x_{n+1}}{r} = F_\theta \frac{\sum x_\alpha^2}{r^3} + F_r \frac{x_{n+1}}{r}.
\end{equation}

Hence

\begin{equation}
|\partial F|^2 = \sum_{\alpha} F_\alpha^2 + F_{n+1}^2 = F_\theta^2 \sum_{\alpha} \frac{x_\alpha^2}{r^4} + F_r^2 = \left( 1 - \theta^2 \right) \frac{F_\theta^2}{r^2} + F_r^2,
\end{equation}

and

\begin{equation}
\sum_{i=1}^{n+1} x_i F_i = \sum_{\alpha} x_\alpha F_\alpha + x_{n+1} F_{n+1} = r F_r.
\end{equation}

In polar coordinates,

\begin{equation}
\sum_{i=1}^{n+1} dx_i^2 = dr^2 + r^2 \left( d\beta^2 + \cos^2 \beta dS^{n-1} \right),
\end{equation}

where $\sin \beta = \theta \in [-1, 1]$ and $dS^{n-1}$ is the standard metric in the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Hence

\begin{equation}
\sum_{i=1}^{n+1} dx_i^2 = dr^2 + \frac{r^2}{1 - \theta^2} d\theta^2 + r^2 (1 - \theta^2) dS^{n-1},
\end{equation}

\begin{equation}
\sum_{i=1}^{n+1} dx_i^2 = dr^2 + \frac{r^2}{1 - \theta^2} d\theta^2 + r^2 (1 - \theta^2) dS^{n-1},
\end{equation}
Moreover,
\[
\sum_{1 \leq i,j \leq n+1} F_{ij} F_i F_j = \frac{1}{2} \sum_i F_i \partial_i |\partial F|^2
\]
\[
= \frac{1}{2} \sum_\alpha \left( -\frac{x_\alpha x_{n+1}}{r^3} F_\theta + \frac{x_\alpha}{r} F_r \right) \left( -\frac{x_\alpha x_{n+1}}{r^3} \partial_\theta |\partial F|^2 + \frac{x_\alpha}{r} \partial_r |\partial F|^2 \right)
\]
\[
+ \frac{1}{2} \left( \sum \frac{x_\alpha^2}{r^3} F_\theta + \frac{x_{n+1}}{r} F_r \right) \left( \sum \frac{x_\alpha^2}{r^3} \partial_\theta |\partial F|^2 + \frac{x_{n+1}}{r} \partial_r |\partial F|^2 \right)
\]
\[
= \frac{1}{2} \sum \frac{x_\alpha^2}{r^4} F_\theta \partial_\theta |\partial F|^2 + \frac{1}{2} F_r \partial_r |\partial F|^2
\]
\[
= \frac{1-\theta^2}{2r^2} F_\theta \partial_\theta \left( (1-\theta^2) \frac{F_\theta^2}{r^2} + F_r^2 \right) + \frac{1}{2} F_r \partial_r \left( (1-\theta^2) \frac{F_\theta^2}{r^2} + F_r^2 \right)
\]
\[
= -\tilde{\Omega} \left( (1-\theta^2) \frac{F_\theta^2}{r^2} + (1-\theta^2)^2 \frac{F_\theta F_\theta}{r^4} + 2(1-\theta^2) \frac{F_\theta F_r F_r}{r^2} \right)
\]
\[
- (1-\theta^2) \frac{F_\theta^2 F_r}{r^2} + F_r^2 F_{rr}.
\]
Hence by (3.2) we have
\[
\mathcal{L} F = e^{-\phi} \left( \left( (1-\theta^2) \frac{F_\theta^2}{r^2} + F_r^2 \right) \left( F_{rr} + \frac{n}{r} F_r + \frac{1-\theta^2}{r^2} F_\theta - \frac{n\theta}{r^2} F_{\theta} + \frac{n-1}{2} \phi \frac{F_r}{r} \right) \right.
\]
\[
- \left( -\theta (1-\theta^2) \frac{F_\theta^2}{r^4} +(1-\theta^2)^2 \frac{F_\theta F_\theta}{r^4} + 2(1-\theta^2) \frac{F_\theta F_r F_r}{r^2} - (1-\theta^2) \frac{F_\theta^2 F_r}{r^3} 
\]
\[
+ F_r^2 F_{rr} \right) \bigg) e^{-\phi} \left( F_{rr} + \frac{n}{r} F_r + \frac{1-\theta^2}{r^2} F_\theta - \frac{n\theta}{r^2} F_{\theta} + \frac{n-1}{2} \phi \frac{F_r}{r} \right)
\]
\[
= e^{-2\phi} \left( n \left( (1-\theta^2) \frac{F_\theta^2}{r^2} + F_r^2 \right) \left( F_r + \phi \frac{F_r}{r} \right) \right.
\]
\[
+ (1-\theta^2) \frac{F_\theta^2}{r^2} \left( \frac{F_\theta F_r}{r^2} + F_r \right) + \frac{1-\theta^2}{r^2} \left( F_\theta^2 F_{rr} + F_r^2 F_\theta - 2F_\theta F_r F_{\theta} \right) \bigg) 
\]
\[
+ e^{-\phi} \left( F_{rr} + \frac{1-\theta^2}{r^2} F_\theta + \frac{n}{r} F_r - \frac{n\theta}{r^2} F_{\theta} + \frac{n-1}{2} \phi \frac{F_r}{r} \right). \]
\]
THEOREM 3.2. Let $\Sigma$ be an $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, $n \geq 2$, endowed with a smooth conformally flat metric $ds^2 = e^\phi \sum dx_i^2$, where $\phi'(r) \geq -2(1-\kappa)r^{-1}$ and $\frac{2}{n}\sqrt{n-1} \leq \kappa \leq 1$. If

$$F(\theta, r) = C\theta^p = Cx_{n+1}r^{p-1} \triangleq F(x_{n+1}, r)$$

with any constant $C > 0$ and $p = \frac{n}{2}\kappa - \sqrt{\frac{n^2\kappa^2}{4} - (n-1)}$, then except at the origin we have

$$\Delta F(x_{n+1}, r) \left\{ \begin{array}{ll} \geq 0 & \text{if } (x_1, \ldots, x_n) \in \mathbb{R}^n, x_{n+1} \geq 0 \\ \leq 0 & \text{if } (x_1, \ldots, x_n) \in \mathbb{R}^n, x_{n+1} \leq 0. \end{array} \right.$$  \hspace{1cm} (3.10)

Proof. Since $\phi' \geq -2(1-\kappa)r^{-1}$ for $0 < \kappa \leq 1$ and $F_r = C\theta^p r^{-1}$. By (3.4) except at the origin we have

$$\theta \Delta F \geq \theta e^{-2\phi} \left[ n \left( (1-\theta^2) \frac{F_{\theta}^2}{r^2} + F_r^2 \right) \left( \frac{\kappa F_r}{r} - \frac{\theta F_{\theta}}{r^2} \right) \right. \hspace{1cm} (1-\theta^2) \frac{F_{\theta}^2}{r^2} \left( \frac{\theta F_{\theta}}{r^2} + F_r^2 \right) + \frac{1-\theta^2}{r^2} \left( F_{\theta r}^2 + F_{\theta}^2 - 2F_r F_{\theta} F_{r\theta} \right) + \theta e^{-\phi} \left( F_{rr} + \frac{1}{r^2} F_{\theta\theta} + ((n-1)\kappa + 1) \frac{F_r}{r} - \frac{n}{r^2} \theta F_{\theta} \right). \hspace{1cm} (3.11)$$

Furthermore, we take the derivatives of $F$ and get

$$\theta \Delta F \geq C^2 \theta e^{-2\phi} \left[ n \left( (1-\theta^2) r^{2p-2} + \theta^2 p^2 r^{2p-2} \right) \left( \kappa \theta pr^{p-2} - \theta r^{p-2} \right) \right. \hspace{1cm} + (1-\theta^2) r^{2p-2} \left( \theta r^{p-2} + \theta pr^{p-2} \right) + \frac{1-\theta^2}{r^2} \left( p(p-1)\theta r^{3p-2} - 2p^2\theta r^{3p-2} \right) \left. \right) + C \theta e^{-\phi} \left( p(p-1)\theta r^{p-2} + ((n-1)\kappa + 1) p \theta r^{p-2} - n \theta r^{p-2} \right) = C^2 \theta e^{-2\phi} \left( (n )p - (n-1)\kappa + 1 \right) \theta r^{p-2} + n \theta r^{p-2} \right) + C \theta e^{-\phi} \left( p^2 + (n-1)\kappa p - n \right) \theta r^{p-2}. \hspace{1cm} (3.12)$$

Note

$$n(\kappa p - 1) + 1 - p^2 = - \left( p - \frac{n\kappa}{2} \right)^2 + \frac{n^2\kappa^2}{4} - (n-1) = 0.$$
By the definition of $p$, we obtain

$$p = \frac{n\kappa}{2} \left( 1 - \sqrt{1 - \frac{4(n-1)}{n^2\kappa^2}} \right) = \frac{n\kappa}{2} \left( 1 - \frac{n-2}{n} \sqrt{1 - \frac{4(n-1)}{(n-2)^2 \left( \frac{1}{\kappa^2} - 1 \right)}} \right) \geq \frac{n\kappa}{2} \left( 1 - \frac{2(n-1)}{(n-2)^2 \left( \frac{1}{\kappa^2} - 1 \right)} \right) = \frac{1}{\kappa} \left( 1 + \frac{1 - \kappa^2}{n-2} \right) \geq \frac{1}{\kappa}.$$

Hence

$$\theta \mathcal{L} F \geq C^3 e^{-2\phi} n^2 (\kappa p - 1) \theta^4 r^{3p-4} + C e^{-\phi} \left( p^2 + (n-1)\kappa p - n \right) \theta^2 r^{p-2} \geq C e^{-\phi} (p^2 - 1) \theta^2 r^{p-2} \geq 0.$$

We complete the proof. \hfill \Box

**Remark 3.3.** There are other $\mathcal{L}$-sub(super)harmonic functions on $\Sigma$. For instance, for all $j > 0$, $\mathcal{L}(j x_{n+1} w^{p-1}) \geq 0$ on $x_{n+1} \geq 0$ and $\mathcal{L}(j x_{n+1} w^{p-1}) \leq 0$ on $x_{n+1} \leq 0$, where $w = \sqrt{x_1^2 + \cdots + x_n^2}$.

Denote $B_R = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 \leq R^2\}$.

**Theorem 3.4.** If $n \geq 3$ and

$$\frac{2}{n} \sqrt{n-1} \leq \kappa < 1,$$

then any hyperplane through the origin in $\Sigma$ as described in Theorem 3.2, that is, $\mathbb{R}^{n+1}$ equipped with a particular conformally flat metric, is area-minimizing.

**Proof.** We shall show that the hyperplane $T = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\}$ in $\Sigma$ with the induced metric is area-minimizing.

Set $\tilde{\phi}(r) = \int_0^r e^{\frac{\phi(t)}{r}} \, dr$. Let us define $\rho = \tilde{\phi}(r)$ and $\lambda(\rho) = r \tilde{\phi}'(r)$, then the Riemannian metric in $\Sigma$ can be written in polar coordinates as $ds^2 = d\rho^2 + \lambda^2(\rho) d\theta^2$, where $d\theta^2$ is the standard metric on $\mathbb{S}^n(1)$. Moreover,

$$\frac{d\lambda}{d\rho} = \frac{d\lambda}{dr} \frac{dr}{d\rho} = \left( \tilde{\phi}' + r \tilde{\phi}'' \right) \frac{1}{\tilde{\phi}'} = 1 + r (\log \tilde{\phi}') = 1 + \frac{1}{2} r \phi' \geq 1 - (1 - \kappa) = \kappa.$$

When $n \geq 3$ and $q = p - 1$ for $p$ as in the statement of Theorem 3.2, let $F_j(x_{n+1}, r) = j x_{n+1} r^q$ for $j > 0$ with $r = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$. By Theorem 3.2 we obtain

$$\mathcal{L} F_j(x_{n+1}, r) \begin{cases} \geq 0 & \text{in } \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} \geq 0\} \setminus \{0\} \\ \leq 0 & \text{in } \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} \leq 0\} \setminus \{0\}. \end{cases}$$
Combining (3.15) and formula (2.9) in [10], we know that any geodesic sphere \( \partial D_{\rho} \subset \Sigma \) centered at the origin has positive inward mean curvature \( H = (n - 1) \frac{\lambda'_{\rho}}{\lambda_{\rho}} > 0 \) for any \( \rho > 0 \). By the existence theorem for the Dirichlet problem for minimal hypersurfaces in \( \Sigma \times \mathbb{R} \), see Theorem 1.5 in [30], for any constant \( R > 0 \) and \( j = 1, 2, \ldots, \infty \), there is a solution \( u_{j} \in C^{\infty}(B_{jR}) \) to the Dirichlet problem

\[
\begin{aligned}
\mathcal{L}u_{j} &= 0 \quad \text{in } B_{jR} \\
u_{j} &= \mathcal{F}_{j} \quad \text{on } \partial B_{jR}.
\end{aligned}
\]

By symmetry, \( u_{j} = 0 \) on \( B_{R^{*}} \cap T \) for any fixed \( R^{*} > 0 \). In fact, by the definition (3.2) of \( \mathcal{L} \) in a conformally flat metric, \( -u_{j}(x_{1}, \ldots, x_{n}, -x_{n+1}) \) is also a smooth solution to the above Dirichlet problem. Since Lemma 2.1 implies the uniqueness of the minimal graph, we get \( u_{j}(x_{1}, \ldots, x_{n}, x_{n+1}) = -u_{j}(x_{1}, \ldots, x_{n}, -x_{n+1}) \), from which it follows that \( u_{j} = 0 \) on \( B_{R^{*}} \cap T \). Let \( U = \{(x_{1}, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_{n+1} > 0\} \), then the comparison theorem on \( B_{R^{*}} \setminus \{0\} \) implies

\[
\lim_{j \to \infty} u_{j} \geq \lim_{j \to \infty} \mathcal{F}_{j} = +\infty \quad \text{in } B_{R^{*}} \cap U
\]
and

\[
\lim_{j \to \infty} u_{j} \leq \lim_{j \to \infty} \mathcal{F}_{j} = -\infty \quad \text{in } B_{R^{*}} \cap (\mathbb{R}^{n+1} \setminus U).
\]
Let \( U_{j} \) denote the subgraph of \( u_{j} \) in \( B_{R^{*}} \times \mathbb{R} \), namely,

\[
U_{j} = \{(x, t) \in B_{R^{*}} \times \mathbb{R} | t < u_{j}(x)\}.
\]
Clearly, its characteristic function \( \chi_{U_{j}} \) converges in \( L^{1}_{\text{loc}}(B_{R^{*}} \times \mathbb{R}) \) to \( \chi_{U \times \mathbb{R}} \). By an analogous argument as in Lemma 9.1 in [15] for the Euclidean case, for any compact set \( E \subset B_{R^{*}} \times \mathbb{R} \), that \( \text{Graph}(u_{j}) \triangleq \{(x, u_{j}(x)) | x \in \mathbb{R}^{n+1}\} \) is an area-minimizing hypersurface implies that \( (U \times \mathbb{R}) \cap E \) is a minimizing set in \( E \). Hence \( U \times \mathbb{R} \) is a minimizing set in \( B_{R^{*}} \times \mathbb{R} \subset \Sigma \times \mathbb{R} \). By an analogous argument as in Proposition 9.9 in [15] for the Euclidean case, \( U \) is a minimizing set in \( B_{R^{*}} \), namely, the hyperplane \( T \) minimizes the perimeter in \( B_{R^{*}} \). Since \( R^{*} \) is arbitrary, we complete the proof. \( \square \)

As we showed in the previous section, on the cone \( CS_{\kappa} \) the usual metric can be rewritten as a conformally flat one. We shall therefore modify the constructions from the cone \( CS_{\kappa} \).
LEMMA 3.5. Let \( \Lambda \) be the rotationally symmetric function on \( \mathbb{R}^{n+1} \) defined by

\[
\Lambda(x) = \begin{cases} 
\frac{\sqrt{1 - \kappa^2}}{\kappa} \sqrt{x_1^2 + \cdots + x_n^2} & \text{on } \mathbb{R}^{n+1} \setminus B_1 \\
\frac{\sqrt{1 - \kappa^2}}{\kappa} \left( 1 - \frac{2}{\pi} \int_0^1 \left( \arctan \xi(s) \right) ds \right) & \text{on } B_1,
\end{cases}
\]

where \( \xi(s) = s \left( e^{1-x^2} - e \right) \). It is a smooth convex function on \( \mathbb{R}^{n+1} \).

Proof. In fact, \( \xi'(0) = 0 \), \( \xi^{(2k)}(0) = 0 \) for \( k \geq 0 \) and \( \xi^{(j)}(1) = +\infty \) for \( j \geq 0 \). Then on \( B_1 \)

\[
\partial_i \Lambda(x) = \frac{2\sqrt{1 - \kappa^2}}{\kappa \pi} \frac{x_i}{|x|} \arctan \xi(|x|),
\]

\[
\partial_{ij} \Lambda(x) = \frac{2\sqrt{1 - \kappa^2}}{\kappa \pi} \left( \delta_{ij} - \frac{x_i x_j}{|x|^2} \right) \frac{\arctan \xi}{|x|} + \frac{2\sqrt{1 - \kappa^2}}{\kappa \pi} \frac{\xi'}{1 + \xi^2} \frac{x_i x_j}{|x|^2}.
\]

Since

\[
\frac{\arctan \xi(\sqrt{t})}{\sqrt{t}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} t^k \left( e^{\frac{1}{\sqrt{t}}} - e \right)^{2k+1}
\]

in \([0, \epsilon)\) for small \( \epsilon > 0 \), \( t^{-\frac{1}{2}} \arctan \xi(\sqrt{t}) \) is a smooth function for \( t \in [0, 1) \) and

\[
\Lambda(x) = \frac{\sqrt{1 - \kappa^2}}{\kappa} \left( 1 - \frac{1}{\pi} \int_0^1 \frac{\arctan \xi(\sqrt{t})}{\sqrt{t}} dt \right)
\]

is a smooth convex function on \( B_1 \). Denote \( \Lambda(r) = \Lambda(|x|) \), then the radial derivative of \( \Lambda \) at 1 is

\[
\lim_{r \to 1} \partial_r \Lambda(r) = \frac{2\sqrt{1 - \kappa^2}}{\kappa \pi} \arctan \xi(1) = \frac{\sqrt{1 - \kappa^2}}{\kappa},
\]

and the higher order radial derivative of \( \Lambda \) at 1 is

\[
\lim_{r \to 1} (\partial_r)^{j+1} \Lambda(r) = \frac{2\sqrt{1 - \kappa^2}}{\kappa \pi} (\partial_r)^j \arctan \xi(r) \bigg|_{r=1} = \frac{2\sqrt{1 - \kappa^2}}{\kappa \pi} (\partial_r)^j \left( \frac{\xi'}{1 + \xi^2} \right) \bigg|_{r=1} = 0 \quad \text{for } j \geq 1.
\]

Hence \( \Lambda \) is a smooth convex function on \( \mathbb{R}^{n+1} \). \( \square \)

Now we suppose that \( MCS_{\kappa} \) is an \((n+1)\)-dimensional smooth entire graphic hypersurface in \( \mathbb{R}^{n+2} \) with the defining function \( \Lambda \). We see that it has non-negative sectional curvature everywhere. In fact, \( MCS_{\kappa} \) is a \( \kappa \)-sphere cone \( CS_{\kappa} \) with a smooth cap, which we shall call the modified \( \kappa \)-sphere cone.
We already showed that the metric of the $\kappa$-sphere cone is conformally flat, and we shall now also derive this for $MCS_\kappa$.

**Lemma 3.6.** The $(n + 1)$-dimensional $MCS_\kappa$ has a smooth conformally flat metric

$$ds^2 = e^{\Phi(r)} \sum_{1 \leq i \leq n+1} dx_i^2$$

on $\mathbb{R}^{n+1}$ with $-\frac{2}{\kappa}(1 - \kappa) \leq \Phi' \leq 0$.

**Proof.** $MCS_\kappa$ is defined as an entire graph on $\mathbb{R}^{n+2}$. Its induced metric can also be written in polar coordinates as

$$ds^2 = d\rho^2 + \lambda^2(\rho)d\theta^2,$$

where $d\theta^2$ is a standard metric on $\mathbb{S}^n(1)$, and

$$\lambda(\rho) = \begin{cases} \kappa \left( \rho + \frac{1}{\kappa} - \rho_0 \right) & \text{for } \rho \geq \rho_0 \\ \zeta(\rho) & \text{for } 0 \leq \rho \leq \rho_0. \end{cases}$$

Here

$$1 < \rho_0 = \int_0^1 \sqrt{1 + (\partial_r \Lambda)^2} dr < \frac{1}{\kappa},$$

and the inverse function of $\zeta$ satisfies

$$\zeta^{-1}(s) = \int_0^s \sqrt{1 + (\partial_r \Lambda)^2} dr,$$

where $\Lambda$ is defined in the last lemma. Moreover, $\kappa \leq \zeta' \leq 1$.

Let $\psi(r)$ be a function on $\left[0, \left(\frac{1}{\kappa}\right)^\frac{1}{n}\right]$ with $\psi\left(\left(\frac{1}{\kappa}\right)^\frac{1}{n}\right) = \rho_0$ and

$$\psi'(r) = \frac{1}{r} \zeta(\psi(r)) \quad \text{on } \left[0, \left(\frac{1}{\kappa}\right)^\frac{1}{n}\right].$$

In fact, let $\tilde{\zeta}(\rho) = \int_1^\rho \frac{1}{\zeta(t)} dt$ for $\rho \in (0, \rho_0]$, then we integrate the above ordinary differential equation and obtain

$$\tilde{\zeta}(\psi(r)) - \tilde{\zeta}(\rho_0) = \log r + \frac{1}{\kappa} \log \kappa.$$
Since $\tilde{\zeta}$ is a monotonic function, we can solve this equation for $\psi$. Since $\kappa \rho \leq \zeta(\rho) \leq \rho$ on $[0, \rho_0]$, a standard comparison argument implies that
\[
\left( \frac{1}{\kappa} \right)^{-\frac{1}{\kappa}} \rho_0 r \leq \psi(r) \leq \kappa \rho_0 r^{\kappa} \quad \text{on} \quad \left[ 0, \left( \frac{1}{\kappa} \right)^{\frac{1}{\kappa}} \right].
\]
In particular, $\psi(0) = 0$. Since
\[
\psi''(r) = \frac{\zeta'}{r} \psi' - \frac{\zeta}{r^2} = \frac{\zeta}{r^2} (\zeta' - 1),
\]
then
\[
\frac{\kappa - 1}{r} \leq \frac{\psi''(r)}{\psi'(r)} = \frac{\zeta' - 1}{r} \leq 0.
\]
Let
\[
\rho = \tilde{\psi}(r) = \begin{cases} 
  r^{\kappa} - \frac{1}{\kappa} + \rho_0 & \text{for } r \geq \left( \frac{1}{\kappa} \right)^{\frac{1}{\kappa}} \\
  \psi(r) & \text{for } 0 \leq r \leq \left( \frac{1}{\kappa} \right)^{\frac{1}{\kappa}},
\end{cases}
\]
then $\tilde{\psi}$ also satisfies (3.24) and hence $\tilde{\psi}$ is smooth on $[0, \infty)$. Set
\[
e^{\Phi(r)} = (\tilde{\psi}'(r))^2 = \begin{cases} 
  \kappa^2 r^{2\kappa - 2} & \text{for } r \geq \left( \frac{1}{\kappa} \right)^{\frac{1}{\kappa}} \\
  (\psi')^2(r) & \text{for } 0 \leq r \leq \left( \frac{1}{\kappa} \right)^{\frac{1}{\kappa}},
\end{cases}
\]
then
\[
ds^2 = e^{\Phi(r)} dr^2 + e^{\Phi(r)} r^2 d\theta^2 = e^{\Phi(r)} \sum_{1 \leq i \leq n + 1} dx_i^2,
\]
where $r^2 = \sum_i x_i^2$. By (2.7) and (3.25) we have
\[
-2 \frac{1}{r} (1 - \kappa) \leq \Phi' \leq 0.
\]
Now, Lemma 3.6 and Theorem 3.4 yield the following conclusion.

**Theorem 3.7.** Let $n \geq 3$. If
\[
\frac{2}{\sqrt{n - 1}} \leq \kappa < 1,
\]
then any hyperplane through the origin in $MCS_\kappa$ is area-minimizing.
Remark 3.8. Let $\{e_\alpha\}_{\alpha=1}^n \cup \{\frac{\partial}{\partial \rho}\}$ be an orthonormal basis at the considered point of $M_{CK}$. Compared with (2.6) we calculate the sectional curvature and Ricci curvature of $M_{CK}$ as follows (see Appendix A in [20] for instance).

$$K_{M_{CK}}\left(\frac{\partial}{\partial \rho}, e_\alpha\right) = -\frac{\lambda''}{\lambda} \geq 0, \quad K_{M_{CK}}(e_\alpha, e_\beta) = \frac{1 - (\lambda')^2}{\lambda^2} \geq 0,$$

$$\text{Ric}_{M_{CK}}\left(\frac{\partial}{\partial \rho}, e_\alpha\right) = 0, \quad \text{Ric}_{M_{CK}}\left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho}\right) = -n \frac{\lambda''}{\lambda} \geq 0,$$

$$\text{Ric}_{M_{CK}}(e_\alpha, e_\beta) = \left( (n-1) \frac{1 - (\lambda')^2}{\lambda^2} - \frac{\lambda''}{\lambda} \right) \delta_{\alpha\beta} \geq 0.$$

In particular, for $\rho \geq \rho_0$ with $1 < \rho_0 < \frac{1}{\kappa}$ we have

$$K_{M_{CK}}\left(\frac{\partial}{\partial \rho}, e_\alpha\right) = 0, \quad K_{M_{CK}}(e_\alpha, e_\beta) = \frac{1 - \kappa^2}{\kappa^2 (\rho + \frac{1}{\kappa} - \rho_0)^2},$$

$$\text{Ric}_{M_{CK}}\left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho}\right) = \text{Ric}_{M_{CK}}\left(\frac{\partial}{\partial \rho}, e_\alpha\right) = 0,$$

$$\text{Ric}_{M_{CK}}(e_\alpha, e_\beta) = (n-1) \frac{1 - \kappa^2}{\kappa^2 (\rho + \frac{1}{\kappa} - \rho_0)^2} \delta_{\alpha\beta}.$$

From the construction above we see that $M_{CK}$ is a complete simply connected manifold with non-negative sectional curvature.

Remark 3.9. Since $M_{CK}$ in Theorem 3.4 cannot split off a Euclidean factor $\mathbb{R}$ isometrically, the Cheeger-Gromoll splitting theorem [6] implies that it does not contain a line. Consequently, this gives a negative answer to the question (1) in [3], which is

If $M$ is a complete area-minimizing hypersurface in a complete simply connected manifold $N$ of non-negative curvature, does it follow that $N$ contains a line, that is a complete length-minimizing geodesic?

If we define for each $x \in \mathbb{R}^n$

$$\bar{\Lambda}(x) = \frac{2\sqrt{1 - \kappa^2}}{\pi \kappa} \int_0^{\sqrt{1 - \kappa^2}} \arctan s ds,$$

then $\bar{\Lambda}$ is a smooth strictly convex function on $\mathbb{R}^n$ and the hypersurface $\bar{\Sigma} = \{(x, \bar{\Lambda}(x)) \mid x \in \mathbb{R}^n\}$ is a smooth manifold with positive sectional curvature everywhere. In fact, $\bar{\Sigma}$ can be seen as a Riemannian manifold $(\mathbb{R}^n, \bar{\sigma})$ with

$$\bar{\sigma} = d\rho^2 + \bar{\lambda}^2(\rho)d\theta^2$$
in polar coordinates, where the inverse function of $\tilde{\lambda}$ satisfies

$$\tilde{\lambda}^{-1}(s) = \int_0^s \sqrt{1 + (\partial_r \tilde{\Lambda})^2} dr = \int_0^s \sqrt{1 + \frac{4(1 - \kappa^2)}{\pi^2 \kappa^2} \left(\arctan r\right)^2} dr.$$  

Hence

$$1 \geq \tilde{\lambda}'(s) = \left(1 + \frac{4(1 - \kappa^2)}{\pi^2 \kappa^2} \left(\arctan \tilde{\lambda}(s)\right)^2\right)^{-\frac{1}{2}} \kappa,$$

and

$$\tilde{\lambda}''(s) = -\left(1 + \frac{4(1 - \kappa^2)}{\pi^2 \kappa^2} \left(\arctan \tilde{\lambda}(s)\right)^2\right)^{-\frac{3}{2}} \frac{4(1 - \kappa^2)}{\pi^2 \kappa^2} \arctan \tilde{\lambda}(s) \frac{\tilde{\lambda}'(s)}{1 + \tilde{\lambda}^2(s)}.$$

Clearly,

$$\lim_{s \to \infty} \frac{\tilde{\lambda}(s)}{s} = \lim_{s \to \infty} \tilde{\lambda}'(s) = \kappa, \quad \text{and} \quad \lim_{s \to \infty} (s^2 \tilde{\lambda}'')(s) = -\frac{2}{\pi} (1 - \kappa^2).$$

If $\{\partial_\rho\}$ and $\{e_\alpha\}_{\alpha=1}^{n-1}$ are an orthonormal basis of $\tilde{\Sigma}$, then the sectional curvature of $\tilde{\Sigma}$ is

$$0 < K(\partial_\rho, e_\alpha) = \frac{\tilde{\lambda}''}{\tilde{\lambda}} \sim 2(1 - \kappa^2) \frac{1}{\pi \kappa s^3}, \quad K(e_\alpha, e_\beta) = \frac{1 - \tilde{\lambda}''}{\tilde{\lambda}^2} \sim \frac{1 - \kappa^2}{\kappa^2 s^2}.$$

Clearly,

$$\lim_{s \to 0} \frac{1 - \tilde{\lambda}''(s)}{\tilde{\lambda}^2(s)} = \frac{4(1 - \kappa^2)}{\pi^2 \kappa^2} > 0.$$

Hence $\tilde{\Sigma} = \{(x, \tilde{\Lambda}(x)) | x \in \mathbb{R}^n \}$ has positive sectional curvature everywhere.

**Theorem 3.10.** Let $n \geq 4$ and $\tilde{\Sigma} = (\mathbb{R}^n, \tilde{\sigma})$ be a complete manifold with positive sectional curvature as above. If

$$\frac{2}{n-1} \sqrt{n-2} \leq \kappa < 1,$$

then any hyperplane through the origin in $\tilde{\Sigma} = (\mathbb{R}^n, \tilde{\sigma})$ is area-minimizing.

**Proof.** Note $\kappa < \tilde{\lambda}' \leq 1$, then we can rewrite the metric $\tilde{\sigma}$ similar to (3.26) (3.27) (3.28). Apply Theorem 3.4 to complete the proof. \hfill \Box

**Remark 3.11.** Our theorem above gives an example for the question (2) in [3], which is

If $N$ is a complete manifold of positive sectional curvature, does $N$ ever admit an area-minimizing hypersurface?
Now scaling the manifold $MCS_\kappa$ yields $\epsilon^2 MCS_\kappa$ for $\epsilon > 0$, which is $\mathbb{R}^{n+1}$ endowed with the metric

\begin{equation}
\sigma_\epsilon = d\rho^2 + \epsilon^2 \lambda^2 \left( \frac{\rho}{\epsilon} \right) \, d\theta^2
\end{equation}

in polar coordinates, where $\lambda$ and $d\theta^2$ as in (3.22) and (3.23). Obviously $\epsilon \lambda \left( \frac{\rho}{\epsilon} \right) \geq \kappa \rho$ and $\epsilon \lambda \left( \frac{\rho}{\epsilon} \right)$ converges to $\kappa \rho$ uniformly as $\epsilon \to 0$. Hence $\sigma_\epsilon$ converges to $\sigma_C$ as $\epsilon \to 0$ (uniformly away from the origin), where $\sigma_C$ is the metric of $CS_\kappa$ defined in (2.7).

Now we can derive the result of F. Morgan in [22], obtained there by a different method due to G. R. Lawlor [19].

**Proposition 3.12.** Let $n \geq 3$ and $\kappa \geq \frac{2}{n} \sqrt{n-1}$. Then any hyperplane in $(n+1)$-dimensional $CS_\kappa$ through the origin is area-minimizing.

**Proof.** Let $T_\epsilon$ denote the hyperplane in $\epsilon^2 MCS_\kappa$ corresponding to $T \subset MCS_\kappa$ during the re-scaling procedure. Denote $T_0 = \lim_{\epsilon \to 0} T_\epsilon \subset \lim_{\epsilon \to 0} \epsilon^2 MCS_\kappa = CS_\kappa$ in the sense of Hausdorff distance.

Now we consider a bounded domain $\Omega_0 \subset T_0$ and a subset $W_0 \subset CS_\kappa$ with $\partial \Omega_0 = \partial W_0$. View $\Omega_0$ as a set $\Omega \subset \mathbb{R}^n$ with the induced metric from $T_0$, and $W_0$ as a set $W$ in $\mathbb{R}^{n+1}$ with the induced metric from $CS_\kappa$. Let $\Omega_\epsilon$ be the set $\Omega \subset \mathbb{R}^n$ with the induced metric from $T_\epsilon$, and $W_\epsilon$ be the set $W$ in $\mathbb{R}^{n+1}$ with the induced metric from $\epsilon^2 MCS_\kappa$. Clearly, $\partial \Omega_\epsilon = \partial W_\epsilon$, $\Omega_0 = \lim_{\epsilon \to 0} \Omega_\epsilon$ and $W_0 = \lim_{\epsilon \to 0} W_\epsilon$ in the sense of Hausdorff distance.

Let $\mathcal{H}^n(K)$ denote $n$-dimensional Hausdorff measure of $K$ for any set $K \subset CS_\kappa$, and $\mathcal{H}^n_\epsilon(K')$ denote $n$-dimensional Hausdorff measure of $K'$ for any set $K' \subset \epsilon^2 MCS_\kappa$. Note that $\epsilon \lambda \left( \frac{\rho}{\epsilon} \right) \to \kappa \rho$ uniformly as $\epsilon \to 0$, we have

\begin{equation}
\mathcal{H}^n(\Omega_0) = \lim_{\epsilon \to 0} \mathcal{H}^n_\epsilon(\Omega_\epsilon), \quad \limsup_{\epsilon \to 0} \mathcal{H}^n_\epsilon(W_\epsilon) \leq \mathcal{H}^n(W_0).
\end{equation}

Since $T_\epsilon$ is area-minimizing in $\epsilon^2 MCS_\kappa$, then

\begin{equation}
\mathcal{H}^n_\epsilon(\Omega_\epsilon) \leq \mathcal{H}^n_\epsilon(W_\epsilon).
\end{equation}

Combining (3.32) and (3.33), we obtain

\begin{equation}
\mathcal{H}^n(\Omega_0) = \lim_{\epsilon \to 0} \mathcal{H}^n_\epsilon(\Omega_\epsilon) \leq \limsup_{\epsilon \to 0} \mathcal{H}^n_\epsilon(W_\epsilon) \leq \mathcal{H}^n(W_0).
\end{equation}

Hence $T_0$ is an area-minimizing hypersurface in $CS_\kappa$. □

Actually, here the number $\frac{2}{n} \sqrt{n-1}$ is optimal. Namely, if $\kappa < \frac{2}{n} \sqrt{n-1}$ then every hyperplane in $CS_\kappa$ is no more area-minimizing and even not stable. This also has been proved in [22]. Let us show this fact by using the second variation formula for the volume functional.
THEOREM 3.13. Let \( \kappa \in (0,1] \) and \( n \geq 3 \). Any hyperplane in \( (n+1) \)-dimensional \( CS_\kappa \) through the origin is area-minimizing if and only if

\[
(3.34) \quad \kappa \geq \frac{2}{n} \sqrt{n-1}.
\]

Proof. By Proposition 3.12 we only need to prove that if (3.34) fails to hold, any hyperplane in \( CS_\kappa \) through the origin is not area-minimizing. Let \( X \) be a totally geodesic sphere in \( S_\kappa \), then \( X \) is minimal in \( S_\kappa \) and \( P \triangleq CX \) is a hyperplane in \( CS_\kappa \) through the origin. Clearly, \( P \) is a minimal hypersurface in \( CS_\kappa \) because it is totally geodesic. The second variation formula is (see (2.10))

\[
I(\phi,\phi) = \int_1^1 \left( \int_X \left( -\Delta_X \phi - \frac{n-1}{\kappa^2} \phi + (n-1)\phi \right) - (n-1)\rho \frac{\partial \phi}{\partial \rho} - \rho^2 \frac{\partial^2 \phi}{\partial \rho^2} \right) \phi \, d\mu_X \rho^{n-3} d\rho,
\]

where \( \phi(x,t) \in C^2(X \times \rho \mathbb{R}) \). Define a second order differential operator \( L \) by

\[
L = \rho^2 \frac{\partial^2}{\partial \rho^2} + (n-1)\rho \frac{\partial}{\partial \rho}.
\]

If \( s = \log \rho \), then

\[
L = \frac{\partial^2}{\partial s^2} + (n-2)\frac{\partial}{\partial s} = e^{-\frac{n-2}{2}s} \frac{\partial^2}{\partial s^2} \left( e^{\frac{n-2}{2}s} \right) - \frac{(n-2)^2}{4}.
\]

So the \( k(k \geq 1) \)-th eigenvalue of \( L \) on \([\epsilon,1]\) is

\[
(3.36) \quad \frac{(n-2)^2}{4} + \left( \frac{k\pi}{\log \epsilon} \right)^2
\]

with the \( k \)-th eigenfunction (see [29] or [31] for instance)

\[
\rho^{\frac{2-n}{2}} \sin \left( \frac{k\pi}{\log \epsilon} \log \rho \right).
\]

By the second variation formula (3.35), \( P \) is stable if and only if

\[
-\frac{n-1}{\kappa^2} + n - 1 + \frac{(n-2)^2}{4} \geq 0,
\]

i.e.,

\[
\kappa \geq \frac{2}{n} \sqrt{n-1}. \quad \square
\]
4. A class of manifolds with non-negative Ricci curvature. Let $N$ be an $(n+1)$-dimensional complete non-compact Riemannian manifold satisfying the following three conditions:

(C1) non-negative Ricci curvature: $\text{Ric} \geq 0$;

(C2) Euclidean volume growth:

$$V_N \triangleq \lim_{r \to \infty} \frac{\text{Vol}(B_r(x))}{r^{n+1}} > 0;$$

(C3) quadratic decay of the curvature tensor: for sufficiently large $\rho = d(x,p)$, the distance from a fixed point in $N$,

$$|R(x)| \leq \frac{c}{\rho^2(x)}.$$

By Gromov’s compactness theorem [17], for any sequence $\epsilon_i \to 0$ there is a subsequence $\{\epsilon_i\}$ converging to zero such that $\epsilon_i N = (N,\epsilon_i \bar{g},p)$ converges to a metric space $(N_{\infty},d_{\infty})$ with vertex $o$ in the pointed Gromov-Hausdorff sense. It is called the tangent cone at infinity. $N_{\infty} \setminus \{o\}$ is a smooth manifold with $C^{1,\alpha}$ Riemannian metric $\bar{g}_{\infty}(0 < \alpha < 1)$ which is compatible with the distance $d_{\infty}$. The precise statements were derived in [16] and [25] on the basis of the harmonic coordinate constructions of [18]. In fact, $N_{\infty} \setminus \{o\}$ is a $D^{1,1}$-Riemannian manifold (see [16, 25]). For any compact domain $K \subset N_{\infty} \setminus \{o\}$, there exists a diffeomorphism $\Phi_i : K \to \Phi_i(K) \subset \epsilon_i N$ such that $\Phi_i^*(\epsilon_i \bar{g})$ converges as $i \to \infty$ to $\bar{g}_{\infty}$ in the $C^{1,\alpha}$-topology on $K$.

Cheeger-Colding (see Theorem 7.6 in [5]) proved that under the conditions (C1) and (C2) the cone $N_{\infty}$ is a metric cone, namely $N_{\infty} = CX = \mathbb{R}^+ \times \rho X$ for some $n$ dimensional smooth compact manifold $X$ with $\text{Diam} X \leq \pi$ and the metric

$$\bar{g}_{\infty} = d\rho^2 + \rho^2 s_{ij}d\theta_i d\theta_j$$

where $s_{ij}d\theta_i d\theta_j$ is the metric of $X$ and $s_{ij} \in C^{1,\alpha}(X)$. Let $p_i$ be the distance function from $p$ to the considered point in $\epsilon_i N$. Set $B_i^r(x)$ be the geodesic ball with radius $r$ and centered at $x$ in $(N, \epsilon_i \bar{g})$, and $B_r(x)$ be the geodesic ball with radius $r$ and centered at $x$ in $N_{\infty}$. In particular, $X = \partial B_1(o)$.

Mok-Siu-Yau [21] showed that if (C1) and (C2) hold, then there exists the Green function $G(p, \cdot)$ on $N^{n+1}$ with $\lim_{r \to \infty} \sup_{\partial B_r(p)} |G r^{n-1} - 1| = 0$ and

$$r^{1-n} \leq G(p,x) \leq Cr^{1-n} \tag{4.1}$$

for any $n \geq 2$, $x \in \partial B_r(p)$ and some constant $C$. Set $\mathcal{R} = G^{\frac{1}{1-n}}$, then

$$\Delta_N \mathcal{R}^2 = 2(n+1)|\nabla \mathcal{R}|^2 \tag{4.2}.$$
Under the additional condition (C3), Colding-Minicozzi (see Corollary 4.11 in [8]) showed that

\[
\limsup_{r \to \infty} \left( \sup_{\partial B_r} \left| \frac{\mathcal{R}}{r} - 1 \right| + \sup_{\partial B_r} \left| \nabla \mathcal{R} \right| - 1 \right) = 0,
\]

and

\[
\limsup_{r \to \infty} \left( \sup_{\partial B_r} |\text{Hess}_{\mathcal{R}^2} - 2\bar{g}| \right) = 0,
\]

where \(\text{Hess}_{\mathcal{R}^2}\) is the Hessian matrix of \(\mathcal{R}^2\) in \(N\). In particular, \(\left| \nabla \mathcal{R} \right| \leq C(n, V_N)\) which is a constant depending only on \(n, V_N\).

For any \(f \in C^1(\partial B_1)\), we can extend \(f\) to \(N_{\infty} \setminus \{o\}\) by defining

\[
f(\rho_{\infty}\theta) = f(\theta)
\]

for any \(\rho_{\infty} > 0\) and \(\theta \in \partial B_1\). Let \(\bar{\nabla}\) be the Levi-Civita connection of \(N_{\infty}\), then

\[
\left\langle \bar{\nabla} f, \frac{\partial}{\partial \rho_{\infty}} \right\rangle = 0.
\]

For any \(K_2 > K_1 > 0\) and \(\epsilon > 0\), let \(\Phi_i : \overline{B_{2K_2} \setminus B_{\frac{1}{2}K_1}} \to \Phi_i(\overline{B_{2K_2} \setminus B_{\frac{1}{2}K_1}}) \subset \epsilon_i N\) be a diffeomorphism such that \(\Phi_i^*(\epsilon_i \bar{g})\) converges as \(i \to \infty\) to \(\bar{g}_{\infty}\) in the \(C^{1,\alpha}\)-topology on \(\overline{B_{2K_2} \setminus B_{\frac{1}{2}K_1}}\). Moreover, \(\Phi_i\) is \(C^{2,\alpha}\)-bounded relative to harmonic coordinates with a bound independent of \(i\) (see [18]).

Let \(\bar{\nabla}, \Delta^i_N, \text{Hess}^i, |\cdot|_i, \text{Ric}_{\epsilon_i N}\) and \(R_{\epsilon_i N}\) be the Levi-Civita connection, Laplacian operator, Hessian matrix, the norm, Ricci curvature and curvature tensor of \(\epsilon_i N\), respectively, then on \(\epsilon_i N\) we have the relations

\[
\rho_i = \epsilon_i^{1/2} \rho, \quad \bar{\nabla} = \nabla, \quad \Delta^i_N = \epsilon_i^{-1} \Delta_N, \quad \text{Hess}^i = \text{Hess},
\]

\[
\text{Ric}_{\epsilon_i N} = \epsilon_i^{-1} \text{Ric}, \quad |R_{\epsilon_i N}|_i = \epsilon_i^{-1} |R|_i, \quad d\mu_{\epsilon_i N} = \epsilon_i^{n+1} d\mu_N,
\]

where \(\rho_i\) and \(d\mu_{\epsilon_i N}\) are the distance function and volume element on \(\epsilon_i N\), respectively, and \(d\mu_N\) is the volume element on \(N\). Let \(\langle \cdot, \cdot \rangle_i\) be the inner product corresponding to \(|\cdot|_i\). We see that conditions (C1), (C2), and (C3) are all scaling invariant. Let

\[
\tilde{\mathcal{R}}_i = \sqrt{\epsilon_i} \mathcal{R} \quad \text{on } \epsilon_i N,
\]

then

\[
\Delta^i_N \tilde{\mathcal{R}}^2_i = \Delta_N \mathcal{R}^2 = 2(n + 1) |\nabla \mathcal{R}|^2 = 2(n + 1) |\nabla^i \tilde{\mathcal{R}}_i|^2
\]
and so $\tilde{R}^{1-n}_i$ is the Green function on $\epsilon_i N$. By (4.4) we have

$$
\limsup_{i \to \infty} \left( \sup_{B^i_{K_2} \setminus B^i_{\epsilon K_1}} \left| \text{Hess}^i_{\tilde{R}^2_i} - 2 \epsilon_i \tilde{g} \right| \right) = 0.
$$

(4.6)

The distance function $\rho_i$ is a Lipschitz function with Lipschitz constant 1. If $\rho_i$ is $C^1$-function at $x \in \epsilon_i N$, then $\nabla^i \rho_i(x) = \gamma^i_x (\rho_i(x))$, where $\gamma^i_x$ is a minimal normal geodesic from $p$ to $x$. For any $y \in \gamma^i_x$ except $p, x$, the minimal normal geodesic joining $p$ and $y$ must have a portion of $\gamma^i_x$. Because if there is a minimal normal geodesic $l^i_y$ joining $p$ and $y$, $l^i_y \cup \gamma^i_{x,y}$ is also a minimal normal geodesic, where $\gamma^i_{x,y}$ is the portion of $\gamma^i_x$ with boundary $\{x, y\}$. Then $l^i_y \cup \gamma^i_{x,y}$ is smooth and $l^i_y = \gamma^i_x (\rho_i(y))$. So $l^i_y \subset \gamma^i_x$ by the uniqueness of geodesic equation, which is a differential equation of order 2. Therefore, we have $\nabla^i \rho_i(y) = \gamma^i_x (\rho_i(y))$. When $\epsilon_i = 1$, $\nabla \rho(z)$ corresponds to the normal geodesic $\gamma_z$ if $\rho$ is $C^1$ at $z \in N$.

Now if $x \in B^i_{K_2} \setminus B^i_{\epsilon K_1}$ and $\rho_i$ is $C^1$-function at $x$, let $x = \gamma^i_x (t), \ x_\epsilon = \gamma^i_x (t_\epsilon) \in \partial B^i_{\epsilon K_1} \cap \gamma^i_x$, then for any parallel vector field $\xi$ along $\gamma^i_x$, we have

$$
\nabla^i_{\xi} \tilde{R}^2_i(x) - \nabla^i_{\xi} \tilde{R}^2_i(x_\epsilon) = \int_{t_\epsilon}^t \nabla^i_{\gamma^i_x(s)} \tilde{R}^2_i (\gamma^i_x(s)) ds
$$

(4.7)

$$
= \int_{t_\epsilon}^t \text{Hess}^i_{\tilde{R}^2_i} \left( \nabla^i \rho_i, \xi \right) (\gamma^i_x(s)) ds.
$$

Hence

$$
\left| \nabla^i_{\xi} \tilde{R}^2_i(x) - \nabla^i_{\xi} \rho^2_i(x) \right|_i \\
\leq \left| \nabla^i_{\xi} \tilde{R}^2_i(x_\epsilon) - \nabla^i_{\xi} \rho^2_i(x_\epsilon) \right|_i \\
+ \int_{t_\epsilon}^t \left| \text{Hess}^i_{\tilde{R}^2_i} \left( \nabla^i \rho_i, \xi \right) (\gamma^i_x(s)) - \text{Hess}^i_{\tilde{R}^2_i} \left( \nabla^i \rho_i, \xi \right) (\gamma^i_x(s)) \right| ds
$$

(4.8)

$$
\leq C \epsilon + \int_{t_\epsilon}^t \left| \text{Hess}^i_{\tilde{R}^2_i} \left( \nabla^i \rho_i, \xi \right) (\gamma^i_x(s)) - 2 \left( \nabla^i \rho_i (\gamma^i_x(s)) , \xi \right) \right| ds
$$

$$
\leq C \epsilon + K_2 \sup_{B^i_{K_2} \setminus B^i_{\epsilon K_1}} \left| \text{Hess}^i_{\tilde{R}^2_i} \left( \nabla^i \rho_i, \xi \right) - 2 \left( \nabla^i \rho_i, \xi \right) \right|,
$$

where $C$ depends only on $K_1, K_2$ and the manifold $N$. With (4.6) we obtain

$$
\limsup_{i \to \infty} \left( \sup_{Q_i \cap B^i_{K_2} \setminus B^i_{\epsilon K_1}} \left| \nabla^i \tilde{R}^2_i(x) - \nabla^i \rho^2_i(x) \right|_i \right) \leq C \epsilon,
$$

(4.9)

where $Q_i$ is the set including all the regular points of $\rho_i$ in $\epsilon_i N$ ($\rho_i$ is $C^1$ at such points). Moreover, the $n$-dimensional Hausdorff measure $\mathcal{H}^n(\epsilon_i N \setminus Q_i) = 0$. 
Since the geodesics $\gamma_i^x$ in $\epsilon_i N$ converge to a geodesic in $N_\infty$, with (4.5) and the continuity of $\nabla^i (f \circ \Phi_i^{-1})$, we have

$$\limsup_{i \to \infty} \sup_{B_{K_2 \setminus B_{\epsilon K_1}}^i} \left| \left< \nabla^i (f \circ \Phi_i^{-1}), \nabla^i \tilde{R}_i^2 \right> \right| \leq C_1 \epsilon, \quad (4.10)$$

and

$$\limsup_{i \to \infty} \sup_{B_{K_2 \setminus B_{\epsilon K_1}}^i} \left( \nabla_i \left| \nabla^i (f \circ \Phi_i^{-1}) \right| \right) < \infty. \quad (4.11)$$

Let $\Pi_i$ be the identity map from $N$ to itself, the two copies of the manifold being regarded with respect to different metrics: $\Pi_i : (N, \bar{g}) \to \epsilon_i N = (N, \epsilon_i \bar{g})$. Now (4.10) and (4.11) are equivalent to

$$\limsup_{i \to \infty} \sup_{\sqrt{\epsilon_i} B_{K_2 \setminus B_{\epsilon K_1}}^i} \left( \tilde{R}_i \left| \nabla (f \circ \Phi_i^{-1} \circ \Pi_i) \right| \right) \leq C_1 \epsilon, \quad (4.12)$$

and

$$\limsup_{i \to \infty} \sup_{\sqrt{\epsilon_i} B_{K_2 \setminus B_{\epsilon K_1}}^i} \left( R \left| \nabla (f \circ \Phi_i^{-1} \circ \Pi_i) \right| \right) < \infty. \quad (4.13)$$

The theory of integral currents in metric spaces was developed by Ambrosio and Kirchheim in [2]. It provides a suitable notion of generalized surfaces in metric spaces, which extends the classical Federer-Fleming theory [11]. We shall need the compactness Theorem (see Theorem 5.2 in [2]) and the closure Theorem (see Theorem 8.5 in [2]) for normal currents in a metric space $E$.

**Theorem 4.1.** Let $(T_h) \subset N_k(E)$ be a bounded sequence of normal currents, and assume that for any integer $p \geq 1$ there exists a compact set $K_p \subset E$ such that

$$\|T_h\|(E \setminus K_p) + \|\partial T_h\|(E \setminus K_p) < \frac{1}{p} \quad \text{for all } h \in \mathbb{N}. \quad (4.14)$$

Then, there exists a subsequence $(T_{h(n)})$ converging to a current $T \in N_k(E)$ satisfying

$$\|T\| \left( E \setminus \bigcup_{p=1}^\infty K_p \right) + \|\partial T\| \left( E \setminus \bigcup_{p=1}^\infty K_p \right) = 0.$$
Theorem 4.2. Let \( I_k(E) \) be the class of integer-rectifiable currents in \( E \). Let 
\((T_h) \subset N_k(E)\) be a sequence weakly converging to \( T \in N_k(E) \). Then, the conditions

\[
T_h \in I_k(E), \quad \sup_{h \in \mathbb{N}} N(T_h) < \infty
\]

imply \( T \in I_k(E) \).

Now let \( M \) denote a minimal hypersurface in \( N \) with the induced metric \( g \) from \( N \). Since \( N \) has non-negative Ricci curvature, then \( \text{Vol}(\partial B_r) \leq \omega_n r^n \), where \( \omega_n \) is the volume of the \( n \)-dimensional unit sphere in \( \mathbb{R}^{n+1} \). Suppose that \( M \) has Euclidean volume growth at most,

\[
\text{(4.14)} \quad \limsup_{r \to \infty} \left( r^{-n} \int_{M \cap B_r} 1 \, d\mu \right) < +\infty,
\]

where \( d\mu \) is the volume element of \( M \). Hence there is a smallest positive constant \( V_M \) such that

\[
\int_{M \cap B_r} 1 \, d\mu \leq V_M r^n \quad \text{for any } r > 0.
\]

Denote \( \epsilon_i M = (M, \epsilon_i g) \). For any fixed \( r > 1 \) let \( \Phi_i : \overline{B_{2r}} \setminus \overline{B_{\frac{1}{2}r}} \to \Phi_i(\overline{B_{2r}} \setminus \overline{B_{\frac{1}{2}r}}) \subset \epsilon_i N \) be a diffeomorphism such that \( \Phi_i^*(\epsilon_i \tilde{g}) \) converges as \( i \to \infty \) to \( g_\infty \) in the \( C^{1,\alpha} \)-topology on \( \overline{B_{2r}} \setminus \overline{B_{\frac{1}{2}r}} \). We see that the minimality is also scaling invariant and \( \epsilon_i M \) are also minimal hypersurfaces of \( \epsilon_i N \). Since

\[
\int_{M \cap B_{2r}} 1 \, d\mu = \int_0^{2r} \text{Vol}(M \cap \partial B_s) \, ds \leq V_M 2^n r^n
\]

which is scaling invariant, there exists a sequence \( l_i \in (r, 2r) \) such that \( \text{Vol}(\epsilon_i M \cap \partial B_{l_i}^i) + \text{Vol}(\epsilon_i M \cap \partial B_{l_i}^{i+1}) \) is uniformly bounded for every \( i \).

Let \( T_i = \epsilon_i M \cap (B_{l_i}^i \setminus B_{l_i}^{i+1}) \), then \( \Phi_i^{-1}(T_i) \) is a minimal hypersurface in \((N_\infty, \Phi_i^*(\epsilon_i \tilde{g}))\) with the unit normal vector \( \tilde{\nu}_i \). If we change the metric \( \Phi_i^*(\epsilon_i \tilde{g}) \) to \( \tilde{g}_\infty \), then we get a new metric \( \tilde{g}_i \) on the hypersurface \( \Phi_i^{-1}(T_i) \) induced from \((N_\infty, \tilde{g}_\infty)\). Set \( \tilde{T}_i = \left( \Phi_i^{-1}(T_i), \tilde{g}_i \right) \), and \( \tilde{\nu}_i \) be the unit normal vector to the smooth hypersurface \( \tilde{T}_i \) in the metric space \((N_\infty, \tilde{g}_\infty)\).

\( \Phi_i^*(\epsilon_i \tilde{g}) \to \tilde{g}_\infty \) implies \( \lim_{i \to \infty} \tilde{\nu}_i = \lim_{i \to \infty} \tilde{\nu}_i \triangleq \nu_0 \) and these two convergences are both uniform. Then obviously

\[
\mathcal{H}^n(\tilde{T}_i) + \mathcal{H}^{n-1}(\partial \tilde{T}_i)
\]
is uniformly bounded. By Theorem 4.1 and 4.2 (see also [28] for compactness of currents in the Euclidean case), there is a subsequence of $\epsilon_{ij}$ such that

$$\tilde{T}_{ij} \rightharpoonup T \quad \text{as} \quad j \to \infty,$$

(4.15)

where $T$ is an integer-rectifiable current in $N_\infty$. Denote $\tilde{T}_{ij}$ by $\tilde{T}_i$ for simplicity. Let $X(\Omega)$ be the set containing all smooth differential vector fields with compact support in $\Omega$. For any $\xi \in X\left( B_2 \setminus B_1 \right)$ we have

$$\lim_{i \to \infty} \int_{\tilde{T}_i} \langle \xi, \tilde{\nu}_i \rangle d\tilde{\mu}_i = \int_T \langle \xi, \nu_\infty \rangle d\mu_\infty,$$

(4.16)

where $d\tilde{\mu}_i$ and $d\mu_\infty$ are the volume elements of $\tilde{T}_i$ and $T$, respectively, and $\nu_\infty$ is the unit normal vector of $T$. Since $\tilde{\nu}_i \to \nu_0$ and $\tilde{\nu}_i \to \nu_0$ uniformly, then we have

$$\int_T \langle \xi, \nu_\infty \rangle d\mu_\infty = \lim_{i \to \infty} \int_{\tilde{T}_i} \langle \xi, \tilde{\nu}_i \rangle d\tilde{\mu}_i$$

(4.17)

$$= \lim_{i \to \infty} \int_{T_i} \langle \xi \circ \Phi_{i}^{-1}, \nu_i \rangle d\mu_i,$$

where $d\tilde{\mu}_i$ and $d\mu$ are the volume elements of $\Phi_{i}^{-1}(T_i)$ and $T_i$, respectively. Then we conclude that

$$T_i = \epsilon_i M \bigcap B_{i}^{j} \setminus B_{i-1}^{j} \rightharpoonup T \quad \text{as} \quad i \to \infty.$$

(4.18)

5. Non-existence of area-minimizing hypersurfaces. Before we can prove our main results, we still need volume estimates for minimal hypersurfaces. In fact, these results are interesting in their own right.

**Theorem 5.1.** Let $M$ be a complete $n$-dimensional minimal hypersurface in a complete non-compact Riemannian manifold $N$ satisfying conditions (C1), (C2), and (C3). Then

(i) every end $E$ of $M$ has infinite volume;

(ii) if $M$ is a proper immersion, then $M$ has Euclidean volume growth at least,

$$\liminf_{r \to \infty} \left( \frac{1}{r^n} \int_{M \cap B_r(p)} 1 d\mu \right) > 0, \quad \text{for any} \ p \in N;$$

(5.1)
(iii) If $M$ has at most Euclidean volume growth, i.e.,

\[
\limsup_{r \to \infty} \left( r^{-n} \int_{M \cap B_r} 1 \, d\mu \right) < \infty,
\]

then $M$ is a proper immersion.

**Proof.** For any $0 < \delta \leq 1$, set $\Omega = \left( \frac{\sqrt{c}}{\delta} + 1 \right)$ with $c$ as in condition (C3). For any fixed point $p \in N$ and arbitrary $q \in \partial B_{\Omega r}(p)$, we have

\[
d(p, x) \geq \frac{\sqrt{c}}{\delta} r, \quad \text{for any } x \in B_r(q).
\]

Then by condition (C3) the sectional curvature satisfies

\[
|K_N(x)| \leq \frac{\delta^2}{r^2}, \quad \text{for any } x \in B_r(q).
\]

Note $\Vol(B_s(q)) \geq V_N s^{n+1}$ for any $s > 0$ as implied by conditions (C1) and (C2) by virtue of the Bishop-Gromov theorem.

Now, we claim that by [7], for sufficiently small $\delta$ (depending only on $n, c, V_N$) the injectivity radius at $q$ satisfies $i(q) \geq r$ for every $r > 0$ and $q \in \partial B_{\Omega r}(p)$. Arguing by contradiction, let us assume instead the existence of sequences $\delta_j \to 0$, $r_j > 0$ and a sequence of $q_j \in \partial B_{(1 + \sqrt{\delta_j})^{r_j}}(p)$ such that the injectivity radius at $q_j$ on $N$ satisfies $i(q_j) < r_j$. Let $\bar{r}_j = (1 + \sqrt{\delta_j}) r_j$, and $Z_j = \frac{1}{\bar{r}_j} B_{\bar{r}_j}(q_j) = (B_{\bar{r}_j}(q_j), \frac{1}{\bar{r}_j} \bar{g}, q_j)$. Now $Z_j$ has non-negative Ricci curvature, and $\Vol(\hat{B}_{s}^{Z_j}(q_j)) \geq V_N s^{n+1}$ for any $0 < s < 1$, where $\hat{B}_{s}^{Z_j}(q_j)$ is the geodesic ball centered at $q_j$ with radius $s$ in $Z_j$. The sectional curvature tensor of $Z_j$ satisfies

\[
|K_{Z_j}(x)| \leq \frac{\delta_j^2}{r_j^2}, \quad \text{for any } x \in Z_j.
\]

Moreover, the injectivity radius at $q_j$ on $Z_j$ satisfies $i(q_j) < r_j = (1 + \sqrt{\delta_j})^{-1}$.

By Gromov’s compactness theorem [17], there is a subsequence $j_k$ such that $Z_{j_k}$ converges to a unit ball with the standard Euclidean metric in the pointed Gromov-Hausdorff sense. So we assume that $Z_{j_k}$ is connected without loss of generality. Suppose $y_k \in Z_{j_k}$ satisfying $d_{Z_{j_k}}(y_k, q_{j_k}) = i(q_{j_k})$, where $d_{Z_{j_k}}$ is the distance function on $Z_{j_k}$. Obviously, $y_k$ and $q_{j_k}$ are not conjugate by $|K_{Z_j}(x)| \leq \frac{\delta_j^2}{r_j^2} \to 0$ as $j \to \infty$. By the proof of Klingenberg’s lemma on the injectivity radius (see page 158 in [14] for instance), there are two normal minimal geodesics $\Gamma^1_k, \Gamma^2_k \subset Z_{j_k}$ both joining $q_{j_k}, y_k$ such that $\Gamma^1_k(y_k) = -\Gamma^2_k(y_k)$, i.e., $\Gamma^1_k \cup \Gamma^2_k$ is a geodesic loop on $q_{j_k}$.

By Theorem 4.3 in [7], there is a uniform positive lower bound independent of $k$ for the length of the loop. Namely, $d_{Z_{j_k}}(y_k, q_{j_k}) = i(q_{j_k})$ has a uniform positive lower bound, which violates $i(q_{j_k}) < (1 + \sqrt{\delta_k})^{-1} \to 0$ as $k \to \infty$. Hence we complete the proof of the above claim.
So \( \rho_q(x) \) is smooth for \( x \in B_r(q) \setminus \{ q \} \). Assume \( q \in M \). Let \( \{ e_i \} \) be a local orthonormal frame field of \( M \). Then

\[
\Delta_M \rho_q^2 = \sum_{i=1}^{n} \left( \nabla_{e_i} \nabla_{e_i} \rho_q^2 - (\nabla_{e_i} e_i) \rho_q^2 \right)
\]

\[
= \sum_{i=1}^{n} \left( \nabla_{e_i} \nabla_{e_i} \rho_q^2 - (\nabla_{e_i} e_i) \rho_q^2 \right) + \sum_{i=1}^{n} \left( \nabla_{e_i} e_i - \nabla_{e_i} e_i \right) \rho_q^2
\]

\[
= \sum_{i=1}^{n} \text{Hess}_{\rho_q}^2 (e_i, e_i).
\]

For any \( \xi \in \Gamma(TN) \) we denote \( \xi^T = \xi - \langle \xi, \partial/\partial \rho_q \rangle \partial/\partial \rho_q \). Combining \( \text{Hess}_{\rho_q}^2 (\xi^T, \partial/\partial \rho_q) = 0 \) and \( \text{Hess}_{\rho_q}^2 (\partial/\partial \rho_q, \partial/\partial \rho_q) = 2 \), we obtain

\[
\text{Hess}_{\rho_q}^2 (e_i, e_i) = \text{Hess}_{\rho_q}^2 ((e_i)^T, (e_i)^T) + 2 \left( e_i, \partial/\partial \rho_q \right)^2.
\]

The injectivity radius \( i(q) \geq r \) implies that \( \rho_q \) is a smooth function on \( B_r(q) \setminus \{ q \} \). By the Hessian comparison theorem and the sectional curvature \( K_N(x) \leq \frac{\delta^2}{r^2} \) for any \( x \in B_r(q) \), for any \( \xi \perp \partial/\partial \rho_q \) we have

\[
\text{Hess}_{\rho_q}^2 (\xi, \xi) \geq \frac{\delta}{r} \cot \left( \frac{\delta \rho_q}{r} \right) |\xi|^2.
\]

Since \( \frac{\delta \rho_q}{r} \cot \left( \frac{\delta \rho_q}{r} \right) \leq 1 \) for \( \rho_q \leq r \) with sufficiently small \( \delta \), then

\[
\Delta_M \rho_q^2 \geq 2 \rho_q \sum_{i=1}^{n} \frac{\delta \rho_q}{r} \cot \left( \frac{\delta \rho_q}{r} \right) |(e_i)^T|^2 + 2 \sum_{i=1}^{n} \left( e_i, \partial/\partial \rho_q \right)^2
\]

\[
\geq 2 \rho_q \cot \left( \frac{\delta \rho_q}{r} \right) \sum_{i=1}^{n} |(e_i)^T|^2 + 2 \rho_q \cot \left( \frac{\delta \rho_q}{r} \right) \sum_{i=1}^{n} \left( e_i, \partial/\partial \rho_q \right)^2
\]

\[
= 2n \rho_q \cot \left( \frac{\delta \rho_q}{r} \right).
\]

For any \( t \in [0, 1) \) we have \( \cos t \geq 1 - t \), then

\[
\left( \tan t - \frac{t}{1-t} \right)' = \frac{1}{\cos^2 t} - \frac{1}{(1-t)^2} \leq 0.
\]

So on \( [0, 1) \) \n
\[
\tan t \leq \frac{t}{1-t}.
\]
Denote the extrinsic ball $D_s(q) = B_s(q) \cap M$. Hence on $D_r(q)$ we have

$$\Delta_M \rho_q^2(x) \geq 2n \left( 1 - \frac{\delta}{r} \rho_q(x) \right) = 2n \frac{2n\delta \rho_q(x)}{r}. \quad (5.7)$$

Let $\rho_q^M$ and $B_s^M(q)$ be the distance function from $q$ and the geodesic ball with radius $s$ and centered at $q$ in $M$. Obviously, the intrinsic ball $B_s^M(q) \subset D_s(q)$ for any $s \in (0, r)$ and (5.7) is valid on $B_r^M(q)$.

Integrating (5.7) by parts on $B_s^M(q)$ yields

$$2n \int_{B_s^M(q)} \left( 1 - \frac{\delta \rho_q}{r} \right) \leq \int_{B_s^M(q)} \Delta_M \rho_q^2 = \int_{\partial B_s^M(q)} \nabla \rho_q^2 \cdot \nu \leq 2s \int_{\partial B_s^M(q)} |\nabla \rho_q|, \quad (5.8)$$

where $\nu$ is the normal vector to $\partial B_s^M(q)$. Then

$$\frac{\partial}{\partial s} \left( s^{-n} \int_{B_s^M(q)} 1 \right) = -ns^{-n-1} \int_{B_s^M(q)} 1 + s^{-n} \int_{\partial B_s^M(q)} |\nabla \rho_q| \geq -ns^{-n-1} \int_{B_s^M(q)} 1 + ns^{-n-1} \int_{\partial B_s^M(q)} \left( 1 - \frac{\delta \rho_q}{r} \right) \geq -\frac{n\delta}{r} s^{-n} \int_{B_s^M(q)} 1. \quad (5.9)$$

Integrating the above inequality implies for $0 < s \leq r$

$$\text{vol}(B_s^M(q)) \triangleq \int_{B_s^M(q)} 1 \geq \frac{\omega_{n-1}}{n} s^n e^{-\frac{n\delta s}{r}} \geq \frac{\omega_{n-1}}{n} s^n e^{-n\delta}. \quad (5.10)$$

Here $\omega_{n-1}$ is the measure of the standard $(n-1)$-dimensional unit sphere in Euclidean space.

(i) Let $E$ be an and of $M$. If $E$ is not contained in any bounded domain in $N$, then we choose $r$ large enough and some $q \in \partial B_{r}(p) \cap M$. By (5.10), $E$ then has infinite volume.

Now we suppose that $E \subset B_{R_0}(p)$ for some constant $R_0 > 0$. Recalling (5.10), there is a constant $r_0 > 0$ so that for any $0 < r \leq r_0$ and $z \in E$ we have a constant $C_0 > 0$ such that

$$\text{vol}(B_r^M(z)) \geq C_0 r^n. \quad (5.11)$$

Since $E$ is non-compact, then we can choose a sequence $\{z_i\}$ such that $B_{r_0}^M(z_i) \cap B_{r_0}^M(z_j) \neq \emptyset$ for $i \neq j$. Hence

$$\text{vol}(E) \geq \sum_i \text{vol}(B_{r_0}^M(z_i)) \geq C_0 \sum_i r_0^n = \infty.$$
(ii) Since \( B_s^M(q) \subset D_s(q) \) for any point \( q \in \partial B_{\Omega r}(p) \) and any \( s \in (0, r) \), then with (5.10) we obtain

\[
\int_{D_s(q)} 1 \geq \frac{\omega_{n-1}}{n} s^n e^{-n\delta} \quad \text{for every } s \in (0, r].
\]

Hence we conclude that (5.1) holds.

(iii) If \( M \) is not a proper immersion into \( N \), there exist an end \( E \subset M \) and a constant \( r_0 \), such that \( E \subset B_{r_0}(p) \). The assumption that \( M \) has at most Euclidean volume growth implies \( M \) has finite volume, which contradicts the results in (i). \( \square \)

Let \( M \) be a minimal hypersurface in \( N \) with Euclidean volume growth at most. Combining (4.1) (4.3) and the definition of \( \mathcal{R} \), the quantity

\[
r^{-n} \int_{M \cap \{ \mathcal{R} \leq r \}} \| \nabla \mathcal{R} \|^2 d\mu
\]

is uniformly bounded for any \( r \in (0, \infty) \), then there exists a sequence \( r_i \to \infty \) such that

\[
\limsup_{r \to \infty} \left( r^{-n} \int_{M \cap \{ \mathcal{R} \leq r \}} |\nabla \mathcal{R}|^2 d\mu \right) = \lim_{r_i \to \infty} \left( r_i^{-n} \int_{M \cap \{ \mathcal{R} \leq r_i \}} |\nabla \mathcal{R}|^2 d\mu \right).
\]

**Lemma 5.2.** There is a sequence \( \delta_i \to 0^+ \) such that for any constants \( K_2 > K_1 > 0 \) and \( \epsilon \in (0, 1) \) and any bounded Lipschitz function \( f \) on \( N \setminus B_1 \) we have

\[
\limsup_{i \to \infty} \left( \frac{\delta_i}{K_2 r_i} \right)^n \int_{M \cap \{ \mathcal{R} \leq \frac{K_2 r_i}{\delta_i} \}} f |\nabla \mathcal{R}|^2 - \left( \frac{\delta_i}{K_1 r_i} \right)^n \int_{M \cap \{ \mathcal{R} \leq \frac{K_1 r_i}{\delta_i} \}} f |\nabla \mathcal{R}|^2 \right) \leq C e^n \sup_{N \setminus B_1} |f| + \limsup_{i \to \infty} \int_{\frac{K_2 r_i}{\delta_i}}^{\frac{K_1 r_i}{\delta_i}} \left( s^{-n-1} \int_{M \cap \{ \mathcal{R} \leq \frac{K_1 r_i}{\delta_i} \} \leq \mathcal{R} \leq s} \mathcal{R} \nabla f \cdot \nabla \mathcal{R} \right) ds.
\]

**Proof.** Let \( \{ e_i \} \) be an orthonormal basis of \( TM \) and \( \nu \) be the unit normal vector of \( M \). Then by (4.2) we have

\[
\Delta_M \mathcal{R}^2 = \sum_{i=1}^{n} (\nabla_{e_i} \nabla_{e_i} \mathcal{R}^2 - (\nabla_{e_i} e_i) \mathcal{R}^2)
\]

\[
= \sum_{i=1}^{n} (\nabla_{e_i} \nabla_{e_i} \mathcal{R}^2 - (\nabla_{e_i} e_i) \mathcal{R}^2) + \sum_{i=1}^{n} (\nabla_{e_i} e_i - \nabla_{e_i} e_i) \mathcal{R}^2
\]

\[
= \Delta_N \mathcal{R}^2 - \text{Hess}_{\mathcal{R}^2}(\nu, \nu) = 2(n+1)|\nabla \mathcal{R}|^2 - \text{Hess}_{\mathcal{R}^2}(\nu, \nu).
\]

By (4.4) and (4.3) there exists a sequence \( \delta_i \to 0^+ \) such that on \( M \setminus B_{\sqrt{\delta_i}} \) we have

\[
|\Delta_M \mathcal{R}^2 - 2n|\nabla \mathcal{R}|^2| \leq 2 \delta_i |\nabla \mathcal{R}|^2.
\]
For any $s \geq \alpha_i r_i^1$ with $\alpha_i \geq 1$ and $f \in \text{Lip}(N \setminus B_1)$, integrating by parts yields

$$
2s \int_{M \cap \{R = s\}} f |\nabla R| - 2\alpha_i r_i^1 \int_{M \cap \{R = \alpha_i r_i^1\}} f |\nabla R| 
$$

(5.17)

$$
= \int_{M \cap \{\alpha_i r_i^1 < R \leq s\}} \text{div}_M (f \nabla R^2)
$$

$$
= \int_{M \cap \{\alpha_i r_i^1 < R \leq s\}} \nabla f \cdot \nabla R^2 + \int_{M \cap \{\alpha_i r_i^1 < R \leq s\}} f \Delta_M R^2.
$$

Hence,

$$
\frac{\partial}{\partial s} \left( s^{-n} \int_{M \cap \{R \leq s\}} f |\nabla R|^2 \right)
$$

$$
= -ns^{-n-1} \int_{M \cap \{R \leq s\}} f |\nabla R|^2 + s^{-n} \int_{M \cap \{R = s\}} \frac{\nabla R^2}{|\nabla R|}
$$

$$
= -ns^{-n-1} \int_{M \cap \{R \leq s\}} f |\nabla R|^2
$$

$$
+ s^{-n} \int_{M \cap \{R = s\}} f |\nabla R| + \int_{M \cap \{\alpha_i r_i^1 < R \leq s\}} f \Delta_M R^2
$$

$$
+ \alpha_i r_i^1 s^{-n-1} \int_{M \cap \{R = \alpha_i r_i^1\}} f |\nabla R|
$$

$$
+ s^{-n} \int_{M \cap \{\alpha_i r_i^1 < R \leq s\}} \nabla f \cdot \nabla R
$$

$$
+ s^{-n} \int_{M \cap \{R = s\}} \frac{(\nabla R, \nu)^2}{|\nabla R|}
$$

(5.18)

$$
= -ns^{-n-1} \int_{M \cap \{R \leq \alpha_i r_i^1\}} f |\nabla R|^2
$$

$$
+ \frac{1}{2} s^{-n-1} \int_{M \cap \{\alpha_i r_i^1 < R \leq s\}} f \left( \Delta_M R^2 - 2n |\nabla R|^2 \right)
$$

$$
+ \alpha_i r_i^1 s^{-n-1} \int_{M \cap \{R = \alpha_i r_i^1\}} f |\nabla R|
$$

$$
+ s^{-n} \int_{M \cap \{\alpha_i r_i^1 < R \leq s\}} \nabla f \cdot \nabla R + s^{-n} \int_{M \cap \{R = s\}} \frac{(\nabla R, \nu)^2}{|\nabla R|}.
$$
Denote
\[
V_M \triangleq \sup_{r > 0} \left( r^{-n} \int_{M \cap \{R \leq r\}} |\nabla R|^2 d\mu \right).
\]

Select \( f \equiv 1 \) and \( \alpha_i = 1 \) in (5.18) and integrate. Then for any \( r \geq \sqrt{r_i} \) there is a constant \( C \) depending only on \( N \) and \( V_M \) such that

\[
\begin{align*}
&\left( \delta_i^{-2} r \right)^{-n} \int_{M \cap \{R \leq \delta_i^{-2} r\}} |\nabla R|^2 - r^{-n} \int_{M \cap \{R \leq r\}} |\nabla R|^2 \\
&\geq -n C r_i \frac{\delta_i^{2r}}{s} \int_{\delta_i^{-2} r \leq s \leq r} \frac{1}{s} ds \\
&\quad + \int_{\delta_i^{-2} r \leq s \leq \delta_i r} \left( s^{-n} \int_{M \cap \{R = s\}} \frac{\langle \nabla R, \nu \rangle^2}{|\nabla R|} \right) ds \\
&\geq -C r_i \frac{\delta_i^{2r}}{r^{n}} + 2C \delta_i \log \delta_i + \int_{\delta_i^{-2} r \leq s \leq \delta_i r} \left( s^{-n} \int_{M \cap \{R = s\}} \frac{\langle \nabla R, \nu \rangle^2}{|\nabla R|} \right) ds.
\end{align*}
\]

Choose \( r = r_i \) in the above inequality and let \( i \) go to infinity, then we obtain

\[
\begin{align*}
&\limsup_{r \to \infty} \left( r^{-n} \int_{M \cap \{R \leq r\}} |\nabla R|^2 \right) - \lim_{i \to \infty} \left( r_i^{-n} \int_{M \cap \{R \leq r_i\}} |\nabla R|^2 \right) \\
\geq &\lim_{i \to \infty} \left( \delta_i^{-2} r_i \right)^{-n} \int_{M \cap \{R \leq \delta_i^{-2} r_i\}} |\nabla R|^2 - \lim_{i \to \infty} \left( r_i^{-n} \int_{M \cap \{R \leq r_i\}} |\nabla R|^2 \right) \\
\geq &\lim_{i \to \infty} \int_{r_i}^{\delta_i^{-2} r_i} \left( s^{-n} \int_{M \cap \{R = s\}} \frac{\langle \nabla R, \nu \rangle^2}{|\nabla R|} \right) ds.
\end{align*}
\]

Combining the above inequality and (5.13) imply

\[
\begin{align*}
&\lim_{i \to \infty} \int_{r_i \leq R \leq \delta_i^{-2} r_i} \left( s^{-n} \int_{M \cap \{R = s\}} \frac{\langle \nabla R, \nu \rangle^2}{|\nabla R|} \right) ds = 0,
\end{align*}
\]

namely, by the coarea formula

\[
\begin{align*}
&\lim_{i \to \infty} \int_{M \cap \{r_i < R \leq \delta_i^{-2} r_i\}} \frac{\langle \nabla R, \nu \rangle^2}{R^n} = 0.
\end{align*}
\]
Set \( |f|_0 \triangleq \sup_N f < \infty \) and \( \alpha_i = \varepsilon K_1 r_i^2 \delta_i^{-1} \) for any small \( \varepsilon \in (0, 1) \) in (5.18), then from (5.18), for any \( r \geq \varepsilon r_i \delta_i^{-1} \) we obtain

\[
\left| (K_2 r)^{-n} \int_{M \cap \{ R \leq K_2 r \}} f |\nabla R|^{-2} - (K_1 r)^{-n} \int_{M \cap \{ R \leq K_1 r \}} f |\nabla R|^{-2} \right|
\]
\[
\leq n C |f|_0 \left( \frac{\varepsilon K_1 r_i}{\delta_i} \right)^n \int_{K_1 r} s^{-n-1} ds + C \delta_i |f|_0 \int_{K_1 r} \frac{1}{s} ds
\]
\[
+ |f|_0 \left( \frac{\varepsilon K_1 r_i}{\delta_i} \right)^n \int_{M \cap \{ R = \frac{K_1 r_i}{\delta_i} \}} |\nabla R| \int_{K_1 r} s^{-n-1} ds
\]
\[
+ \int_{K_1 r} \left( s^{-n-1} \int_{M \cap \{ \frac{\varepsilon K_1 r_i}{\delta_i} < R \leq s \}} R \nabla f \cdot \nabla R \right) ds
\]
\[
(5.23)
\]
\[
+ |f|_0 \int_{K_1 r} \left( s^{-n} \int_{M \cap \{ R = s \}} \frac{|\nabla R, \nu|^2}{|\nabla R|} \right) ds
\]
\[
\leq C |f|_0 \frac{\varepsilon n r_i^n}{\delta_i^n r_i^n} + C \delta_i |f|_0 \log \frac{K_2}{K_1}
\]
\[
+ \int_{K_1 r} \left( s^{-n-1} \int_{M \cap \{ \frac{\varepsilon K_1 r_i}{\delta_i} < R \leq s \}} R \nabla f \cdot \nabla R \right) ds
\]
\[
+ \frac{|f|_0}{n K_1^n r_i^n} \frac{\varepsilon K_1 r_i}{\delta_i} \int_{M \cap \{ R = \frac{K_1 r_i}{\delta_i} \}} |\nabla R|
\]
\[
+ |f|_0 \int_{K_1 r} \left( s^{-n} \int_{M \cap \{ R = s \}} \frac{|\nabla R, \nu|^2}{|\nabla R|} \right) ds,
\]

where we have used (5.16) in the second inequality, and the definition of \( \mathcal{R} \) in Section 4. Since

\[
\frac{\varepsilon K_1 r_i}{\delta_i} \int_{M \cap \{ R = \frac{K_1 r_i}{\delta_i} \}} |\nabla R| = \frac{1}{2} \int_{M \cap \{ R \leq \frac{K_1 r_i}{\delta_i} \}} \Delta_M \mathcal{R}^2,
\]

we get

\[
\left| (K_2 r)^{-n} \int_{M \cap \{ R \leq K_2 r \}} f |\nabla R|^{-2} - (K_1 r)^{-n} \int_{M \cap \{ R \leq K_1 r \}} f |\nabla R|^{-2} \right|
\]
\[
\leq C |f|_0 \frac{\varepsilon n r_i^n}{\delta_i^n r_i^n} + C \delta_i |f|_0 \log \frac{K_2}{K_1}
\]
\[
+ \int_{K_1 r} \left( s^{-n-1} \int_{M \cap \{ \frac{K_1 r_i}{\delta_i} < R \leq s \}} R \nabla f \cdot \nabla R \right) ds
\]
\[
+ C_1 \frac{|f|_0}{2 n \delta_i^n r_i^n} \varepsilon n r_i^n + |f|_0 \int_{K_1 r} \left( s^{-n} \int_{M \cap \{ R = s \}} \frac{|\nabla R, \nu|^2}{|\nabla R|} \right) ds
\]
\[
(5.24)
\]
for some $C_1 > 0$. Letting $r = \frac{r_i}{\delta_i}$ and $i \to \infty$ in the above inequality, then the conclusion follows by (5.21). \hfill \square

Let $\epsilon_i = \delta_i^2 r_i^{-2}$ and suppose that $\epsilon_i N$ converges to $(N_\infty, d_\infty)$ without loss of generality. Let $\epsilon_i M = (M, \epsilon_i \tilde{g})$ and $D_i^l(x) = \epsilon_i M \cap B_i^l(x)$. Clearly, $\epsilon_i M$ is still a minimal hypersurface in $\epsilon_i N$ with $Vol(\epsilon_i M \cap B_i^l(p)) \leq V_M r^n$.

**Lemma 5.3.** There exists a subsequence $\{\epsilon_{i_j}\} \subset \{\epsilon_i\}$ such that $\epsilon_{i_j} M$ converges to a cone $CY = \mathbb{R}^+ \times \rho \ Y$ in $N_\infty$, where $Y \subset \partial B_1(o)$ is an $(n - 1)$-dimensional Hausdorff set with $\mathcal{H}^{n-1}(Y) > 0$.

**Proof.** Note (4.18). By choosing a diagonal sequence, we can assume

$$\Phi_{i_j}^{-1}(\epsilon_{i_j} M \cap \overline{B_{r_i^j} \setminus B_{\frac{r_i^j}{r}}} \to T \quad \text{as} \quad j \to \infty,$$

for any $r > 1$, where $T$ is an integer-rectifiable current in $N_\infty$. For convenience, we still write $\epsilon_i$ instead of $\epsilon_{i_j}$.

Let $f$ be a homogenous function in $C^1(N_\infty \setminus \{o\})$, that is,

$$f(\rho \theta) = f(\theta)$$

for any $\rho > 0$ and $\theta \in \partial B_1$. Let $\Pi_i$ be the map from $(N, \tilde{g})$ to $\epsilon_i N = (N, \epsilon_i \tilde{g}, p)$ defined before, then both of (4.12) and (4.13) hold. Now we can extend the function $f \circ \Phi_i^{-1} \circ \Pi_i$ to a uniformly bounded function $F_i$ in $B_{K_{\gamma_1}} = B_{K_{\gamma_3}}$ with $F_i = f \circ \Phi_i^{-1} \circ \Pi_i$ on $B_{K_{\gamma_1} \delta_i} \cap B_{\frac{K_{\gamma_1}}{\delta_i}}$. Note (4.1) and the definition of $\mathcal{R}$. Hence for sufficiently large $i$ and $s \in \left(\frac{K_{\gamma_1}}{\delta_i}, \frac{K_{\gamma_3}}{\delta_i}\right)$, we have

$$\begin{align*}
\int_{M \cap \{\frac{K_{\gamma_1}}{\delta_i} < R \leq s\}} \mathcal{R} \nabla F_i \cdot \nabla \mathcal{R} &\leq \int_{M \cap \{\frac{K_{\gamma_1}}{\delta_i} < R \leq s\}} \mathcal{R} \left(\nabla F_i \cdot \nabla \mathcal{R} + \left|\nabla F_i\right| \left|\nabla \mathcal{R}, \nu\right|\right) \\
&\leq \int_{M \cap \{\frac{K_{\gamma_1}}{\delta_i} < R \leq s\}} \left(C_2 \epsilon + C_2 \left|\nabla \mathcal{R}, \nu\right|\right) \\
&\leq C_3 \epsilon s^n + C_2 \int_{M \cap \{\frac{K_{\gamma_1}}{\delta_i} < R \leq s\}} \left|\nabla \mathcal{R}, \nu\right|
\end{align*}
$$

(5.25)
for some constants \( C_2, C_3 > 1 \), where the second inequality above has used (4.12) and (4.13). By the Cauchy inequality we get

\[
(5.26) \quad \limsup_{i \to \infty} \int_{K_n} \left( \frac{1}{s^{n+1}} \int_{M \cap \{ \frac{e^{r_i}}{\delta_i} < R \leq s \}} \nabla F_i \cdot \nabla R \right) \, ds \\
\leq \limsup_{i \to \infty} \int_{K_n} \left( \frac{C_3 \epsilon}{s} + \frac{C_2}{s^{n+1}} \left( \int_{M \cap \{ \frac{e^{r_i}}{\delta_i} < R \leq s \}} \frac{\langle \nabla R, \nu \rangle^2}{R^n} \right)^{\frac{1}{2}} \right) \, ds \\
\leq C_3 \epsilon \log \frac{K_2}{K_1} + C_4 \limsup_{i \to \infty} \left( \int_{M \cap \{ \frac{e^{r_i}}{\delta_i} < R \leq \frac{K_2 r_i}{\delta_i} \}} \frac{\langle \nabla R, \nu \rangle^2}{R^n} \right)^{\frac{1}{2}} ,
\]

where \( C_4 \) is a constant. Note \( F_i \) is uniformly bounded for all \( i \), then by Lemma 5.2 and (5.22) we obtain

\[
(5.27) \quad \limsup_{i \to \infty} \left| \left( \frac{\delta_i}{K_2 r_i} \right)^n \int_{M \cap \{ R \leq \frac{K_2 r_i}{\delta_i} \}} F_i |\nabla R|^2 - \left( \frac{\delta_i}{K_1 r_i} \right)^n \int_{M \cap \{ \frac{e^{r_i}}{\delta_i} < R \leq \frac{K_2 r_i}{\delta_i} \}} F_i |\nabla R|^2 \right| \\
\leq C_3 \epsilon \log \frac{K_2}{K_1} + C_4 \limsup_{i \to \infty} \left( \epsilon^n \sup_{B_{K_2 r_i}^{\delta_i}} |F_i| \right) \leq C_3 \epsilon \log \frac{K_2}{K_1} + C_5 \epsilon^n
\]

for some constant \( C_5 \). For any \( \delta \in (0, 1) \), together with (4.3) we have

\[
\left| \frac{1}{K_2^n} \int_{\Omega \cap (B_{K_2} \setminus B_{\delta K_1})} f - \frac{1}{K_1^n} \int_{\Omega \cap (B_{K_1} \setminus B_{\delta K_1})} f \right| \\
= \lim_{i \to \infty} \left| \left( \frac{\delta_i}{K_2 r_i} \right)^n \int_{M \cap \{ \frac{\delta K_1 r_i}{\delta_i} \leq R < \frac{K_2 r_i}{\delta_i} \}} F_i |\nabla R|^2 \right| \\
- \left( \frac{\delta_i}{K_1 r_i} \right)^n \int_{M \cap \{ R \leq \frac{K_2 r_i}{\delta_i} \}} F_i |\nabla R|^2 \right| \\
\leq \limsup_{i \to \infty} \left| \left( \frac{\delta_i}{K_2 r_i} \right)^n \int_{M \cap \{ \frac{\delta K_1 r_i}{\delta_i} \leq R < \frac{K_2 r_i}{\delta_i} \}} F_i |\nabla R|^2 \right| \\
- \left( \frac{\delta_i}{K_1 r_i} \right)^n \int_{M \cap \{ R \leq \frac{K_2 r_i}{\delta_i} \}} F_i |\nabla R|^2 \right| .
\]
\[
+ \limsup_{i \to \infty} \left| \left( \frac{\delta_i}{K_2 r_i} \right)^n \int_{M \cap \{ R \leq \frac{\delta K_1 r_i}{\delta_i} \}} F_i |\nabla R|^2 \right| - \left( \frac{\delta_i}{K_1 r_i} \right)^n \int_{M \cap \{ R \leq \frac{\delta K_1 r_i}{\delta_i} \}} F_i |\nabla R|^2 \right| 
\leq C_3 \epsilon \log \frac{K_2}{K_1} + C_5 \epsilon^n
\]

(5.28)

\[+ C_5 \left( \frac{1}{K_1^n} - \frac{1}{K_2^n} \right) \limsup_{i \to \infty} \int_{M \cap \{ R \leq \frac{\delta K_1 r_i}{\delta_i} \}} F_i |\nabla R|^2 \right| \, d\mu. \]

Letting \( \delta \to 0 \) and \( \epsilon \to 0 \) implies

(5.29)

\[\frac{1}{K_2^n} \int_{T \cap B_{K_2}} f = \frac{1}{K_1^n} \int_{T \cap B_{K_1}} f. \]

By the argument in the proof of Theorem 19.3 in [28], the above equality means that \( T \) is a cone in \( N_\infty \) up to a set of measure zero, as \( f \) is an arbitrary homogeneous function. In fact, by the coarea formula the above equality becomes

(5.30)

\[K_1^n \int_0^{K_2} \left( \int_{T \cap \partial B_s} f \right) ds = K_2^n \int_0^{K_1} \left( \int_{T \cap \partial B_s} f \right) ds. \]

Differentiating w.r.t. \( K_2 \) and \( K_1 \) implies

(5.31)

\[\frac{1}{K_2^{n-1}} \int_{T \cap \partial B_{K_2}} f = \frac{1}{K_1^{n-1}} \int_{T \cap \partial B_{K_1}} f. \]

Since \( N_\infty = CX \) is a cone and any point in it can be represented by \( (\rho, \theta) \) for some \( \theta \in X \), then we define \( \frac{1}{\rho} T \) by \( \{ (\rho, \theta) \in N_\infty | (\rho, \theta) \in T \} \). So

(5.32)

\[\int_{\frac{1}{K_2} T \cap \partial B_1} f = \int_{\frac{1}{K_1} T \cap \partial B_1} f. \]

Hence \( \frac{1}{K_2} T = \frac{1}{K_1} T \) up to a set of measure zero, namely, \( T \) is a cone, say, \( CY \), where \( Y \in \partial B_1(o) \) is an \( (n-1) \)-dimensional Hausdorff set. By (5.1), we know \( \mathcal{H}^n(CY) > 0 \), which implies \( \mathcal{H}^{n-1}(Y) > 0 \).

Remark 5.4. By a simple modification, Lemma 5.2 and Lemma 5.3 also apply to minimal submanifolds of higher codimension with Euclidean volume growth in \( N \).

Without loss of generality, we assume that \( \epsilon_i M \) converges to the cone \( CY \) in the current sense. Let \( \mathcal{X} \left( B_{2 \epsilon} \setminus B_\epsilon \right) \) be the set containing all smooth differential vector fields with compact support in \( B_{2 \epsilon} \setminus B_\epsilon \) as in Section 4. For any \( \xi \in \mathcal{X} \left( B_{2 \epsilon} \setminus B_\epsilon \right) \)
let
\[
(5.33) \quad \epsilon_i M(\omega) = \int_{\epsilon_i M} \langle \xi \circ \Phi_i^{-1}, \nu_i \rangle \, d\mu_i, \quad CY(\xi \circ \Phi_i) = \int_T \langle \xi, \nu_\infty \rangle \, d\mu_\infty,
\]
where \(d\mu_i\) and \(d\mu_\infty\) are the volume elements of \(\epsilon_i M\) and \(CY\), and \(\nu_i\) and \(\nu_\infty\) are the unit normal vectors of \(\epsilon_i M\) and \(CY\).

For any sufficiently small fixed constant \(\epsilon \in (0, 1)\), \(\epsilon_i M \cap (B^i_2 \setminus \epsilon_i^2)\) converges to \(CY \cap (B^i_2 \setminus \epsilon_i^2)\) in the varifold sense. Then
\[
(5.34) \quad \lim_{\epsilon_i \to 0} \epsilon_i M \cup (B^i_2 \setminus \epsilon_i^2) = CY \cup (B^i_2 \setminus \epsilon_i^2) \quad (\omega)
\]
for any \(\omega \in \mathcal{X}(B^i_2 \setminus \epsilon_i^2)\).

Let
\[
(5.35) \quad E_i \triangleq \left\{ x \in \epsilon_i M \cap (B^i_2 \setminus \epsilon_i^2) \mid \left| \langle \nabla\rho_i(x), \nu_i \rangle \right| \geq \epsilon, \rho_i \text{ is } C^1 \text{ at } x \right\}.
\]
Note that \(|\rho_i(x) - \rho_i(y)| \leq d_i(x, y)|\) for any \(x, y \in \epsilon_i N\), where \(d_i\) is the distance function on \(\epsilon_i N\). So \(\rho_i\) is \(C^1\)-function almost everywhere (outside a set of \(n\)-dimensional Hausdorff measure zero). This set of measure zero does not affect any of the integrals in this paper, so we can assume that \(\rho_i\) is \(C^1\) in these integrals. If \(\rho_\infty(x) = d_\infty(0, x)\) is the distance function on \(N_\infty\), then \(\lim_{\epsilon_i \to 0} \rho \circ \Phi_i = \rho_\infty\) in \(B^i_2 \setminus B^i_\epsilon\). For any compact set \(K \subset B^i_2 \setminus B^i_\epsilon\) by (5.34) we have
\[
0 = \lim_{\epsilon_i \to 0} \left( \epsilon_i M \cap \Phi_i(K) \right) \left( \frac{\partial}{\partial \rho_\infty} \circ \Phi_i^{-1} \right)
= \lim_{\epsilon_i \to 0} \int_{\epsilon_i M \cap \Phi_i(K)} \left\langle \frac{\partial}{\partial \rho_\infty} \circ \Phi_i^{-1}, \nu_i \right\rangle \, d\mu_i,
\]
and
\[
0 = \lim_{\epsilon_i \to 0} \int_{\epsilon_i M \cap \Phi_i(K)} \left| \frac{\partial}{\partial \rho_\infty} \circ \Phi_i^{-1} - \nabla\rho_i \right| \, d\mu_i.
\]
By
\[
\left| \int_{\epsilon_i M \cap \Phi_i(K)} \left\langle \nabla\rho_i, \nu_i \right\rangle \, d\mu_i \right| \leq \int_{\epsilon_i M \cap \Phi_i(K)} \left| \frac{\partial}{\partial \rho_\infty} \circ \Phi_i^{-1}, \nu_i \right\rangle \, d\mu_i \right| + \int_{\epsilon_i M \cap \Phi_i(K)} \left| \frac{\partial}{\partial \rho_\infty} \circ \Phi_i^{-1} - \nabla\rho_i \right| \, d\mu_i,
\]
we obtain
\[
0 = \lim_{\epsilon_i \to 0} \int_{\epsilon_i M \cap \Phi_i(K)} \left\langle \nabla\rho_i, \nu_i \right\rangle \, d\mu_i.
\]
We claim that for the sufficiently small $\epsilon > 0$ there exists $i_0 = i_0(\epsilon)$ such that $i > i_0$ implies

\[(5.39) \quad \mathcal{H}^n(E_i) < \epsilon^{n+1}.\]

If not, one could find a constant $\epsilon_0 > 0$ and a sequence $\mathbb{N} \ni s_i \to \infty$ such that $\mathcal{H}^n(E_{s_i}) \geq \epsilon_0^{n+1}$. Then without loss of generality, there is a subsequence $s_{ij} \to \infty$ of $s_i$ such that $\widetilde{E}_{s_{ij}} \subset E_{s_{ij}}$ with $\mathcal{H}^n(\widetilde{E}_{s_{ij}}) \geq \frac{1}{2} \epsilon_0^{n+1}$ and $\langle \nabla^i \rho_s(x), \nu_s \rangle_{ij} \geq \epsilon_0$ on $\widetilde{E}_{s_{ij}}$.

So we get $\mathcal{H}^n(\Phi^{-1}_{s_{ij}}(\widetilde{E}_{s_{ij}})) \geq \epsilon_0^{n+2}$ if $\epsilon_0$ is sufficiently small and $j$ is sufficiently large. Note that $\epsilon_{s_{ij}} M \to CY$. Then there are a set $K_0 \subset CY \cap (B_{\frac{1}{2}} \setminus B_k)$ and a subsequence $s_{ij} \to \infty$ of $s_i$ such that $K_0 \subset \Phi^{-1}_{s_{ij}} (\widetilde{E}_{s_{ij}})$ and $\mathcal{H}^n(K_0) > 0$. Denote the sequence $s_{ij}$ by $s_k$ for convenience. By (5.38), we obtain

\[(5.40) \quad 0 = \lim_{k \to \infty} \int_{\epsilon_{s_k} M \cap \Phi_{s_k}(K_0)} \langle \nabla_{s_k} \rho_{s_k}, \nu_{s_k} \rangle_{s_k} \, d\mu_{s_k} \geq \lim_{k \to \infty} \int_{\epsilon_{s_k} M \cap \Phi_{s_k}(K_0)} \epsilon_0 d\mu_{s_k} = \int_{CY \cap K_0} \epsilon_0 d\mu_{\infty} = \epsilon_0 \mathcal{H}^n(K_0) > 0.\]

This is a contradiction, and we get the inequality (5.39).

Now we assume that $M$ is a stable minimal hypersurface in $N$. Then $\epsilon_i M$ is still a stable minimal hypersurface in $\epsilon_i N$. Let $A^i$ be the second fundamental form of $\epsilon_i M$ in $\epsilon_i N$, and $\text{Ric}_{\epsilon_i N}$ the Ricci curvature of $\epsilon_i N$. For any Lipschitz function $\phi$ with compact support in $\epsilon_i M$ we have from (2.5)

\[(5.41) \quad \int_{\epsilon_i M} \left( |A^1|^2 + \text{Ric}_{\epsilon_i N}(\nu_i, \nu_i) \right) \phi^2 \leq \int_{\epsilon_i M} |\nabla^i \phi|^2,\]

where $\nabla^i$ is the Levi-Civita connection of $\epsilon_i M$. Now we suppose that there exists some sufficiently large $r_0 > 0$ such that the non-radial Ricci curvature of $N$ satisfies

\[(5.42) \quad \inf_{\partial B_r} \text{Ric} (\xi^T, \xi^T) \geq \frac{\kappa'}{r^2} |\xi^T|^2\]

almost everywhere for all $r \geq r_0$ and $n \geq 2$, where $\kappa'$ is a positive constant, and $\xi^T$ stands for the part that is tangential to the geodesic sphere $\partial B_r$ (at least away from the cut locus of the center), of a tangent vector $\xi$ of $N$ at the considered point. Then

\[
\inf_{\partial B_r} \text{Ric}_{\epsilon_i N} (\eta^T, \eta^T) \geq \frac{\kappa'}{r^2} |\eta^T|_i^2 > 0
\]

for all $s \geq \sqrt{\epsilon_i r_0}$ and $n \geq 2$, where $\eta$ is a local vector field on $\epsilon_i N$, $\eta^T = \eta - \langle \eta, \nabla^i \rho_i \rangle_{\epsilon_i} \nabla^i \rho_i$ if $\nabla^i \rho_i$ is well-defined. Using conditions (C1) and (C3) which are
both scaling invariant, we obtain

\[
\text{Ric}_{\epsilon_iN}(\nu_i, \nu_i) \geq \text{Ric}_{\epsilon_iN}(\nu_i^T, \nu_i^T) + 2 \left\langle \nu_i, \nabla^i \rho_i \right\rangle_i \text{Ric}_{\epsilon_iN}(\nu_i^T, \nabla^i \rho_i) \\
\geq \text{Ric}_{\epsilon_iN}(\nu_i^T, \nu_i^T) - c' \left\langle \nu_i, \nabla^i \rho_i \right\rangle_i \rho_i^{-2} 
\]

(5.43)

for some absolute constant \(c' > 0\). Let \(\phi\) be the Lipschitz function on \(\epsilon_iN\) defined by

\[
\phi(x) = (\rho_i(x))^{\frac{2-n}{2}} \sin \left( \pi \frac{\log \rho_i(x)}{\log \epsilon} \right)
\]

in \(B_i^i \setminus B_i^j\) and \(\phi = 0\) in other places. Here \(\epsilon\) is a small positive constant less than \(\min\{\frac{1}{2}, \frac{\epsilon_i}{2}\}\), which implies \(\kappa'(1 - \epsilon^2) - c' \epsilon \geq \frac{\epsilon'}{4}\). So from (5.35), (5.39), and (5.43)

(5.44)

\[
\int_{\epsilon_iM} \text{Ric}_{\epsilon_iN}(\nu_i, \nu_i) \phi^2 d\mu_i \\
\geq \int_{\epsilon_iM \cap (B_i^i \setminus B_i^j)} \left( \frac{\kappa'}{\rho_i} |\nu_i^T|^2 - \frac{c'}{\rho_i^2} \left\langle \nu_i, \nabla^i \rho_i \right\rangle_i \right) \sin^2 \left( \pi \frac{\log \rho_i}{\log \epsilon} \right) \rho_i^{-n} d\mu_i \\
\geq (\kappa'(1 - \epsilon^2) - c' \epsilon) \int_{\epsilon_iM \cap (B_i^i \setminus B_i^j)} \sin^2 \left( \pi \frac{\log \rho_i}{\log \epsilon} \right) \rho_i^{-n} d\mu_i \\
\geq (\kappa'(1 - \epsilon^2) - c' \epsilon) \int_{\epsilon_iM \cap (B_i^i \setminus B_i^j)} \sin^2 \left( \pi \frac{\log \rho_i}{\log \epsilon} \right) \rho_i^{-n} d\mu_i - \epsilon^{-n} \mathcal{H}^n(E_i) \\
\geq (\kappa'(1 - \epsilon^2) - c' \epsilon) \int_{\epsilon_iM \cap (B_i^i \setminus B_i^j)} \sin^2 \left( \pi \frac{\log \rho_i}{\log \epsilon} \right) \rho_i^{-n} d\mu_i - \kappa' \epsilon(1 - \epsilon^2)
\]

for sufficiently large \(i\). Substituting this into (5.41) yields

(5.45)

\[
(\kappa'(1 - \epsilon^2) - c' \epsilon) \int_{\epsilon_iM \cap (B_i^i \setminus B_i^j)} \sin^2 \left( \pi \frac{\log \rho_i}{\log \epsilon} \right) \rho_i^{-n} d\mu_i - \kappa' \epsilon(1 - \epsilon^2)
\]

\[
\leq \int_{\epsilon_iM} \text{Ric}_{\epsilon_iN}(\nu_i, \nu_i) \phi^2 \leq \int_{\epsilon_iM} |\nabla \phi|^2_i \\
\leq \int_{\epsilon_iM \cap (B_i^i \setminus B_i^j)} \left( \frac{2 - n}{2} \sin \left( \pi \frac{\log \rho_i}{\log \epsilon} \right) + \pi \frac{\log \rho_i}{\log \epsilon} \cos \left( \pi \frac{\log \rho_i}{\log \epsilon} \right) \right) \rho_i^{-n} d\mu_i.
\]

Due to Lemma 5.3, we let \(i \to \infty\), and get

(5.46)

\[
(\kappa'(1 - \epsilon^2) - c' \epsilon) \int_{CY \cap (B_i \setminus B_\epsilon)} \sin^2 \left( \pi \frac{\log \rho_\infty}{\log \epsilon} \right) \rho_\infty^{-n} d\mu_\infty - \kappa' \epsilon(1 - \epsilon^2)
\]

\[
\leq \int_{CY \cap (B_i \setminus B_\epsilon)} \left( \frac{2 - n}{2} \sin \left( \pi \frac{\log \rho_\infty}{\log \epsilon} \right) + \pi \frac{\log \rho_\infty}{\log \epsilon} \cos \left( \pi \frac{\log \rho_\infty}{\log \epsilon} \right) \right) \rho_\infty^{-n} d\mu_\infty.
\]
Since
\[
\int_{C Y \cap (B_1 \setminus B_\epsilon)} \sin^2 \left( \frac{\pi}{\log \rho_\infty} \right) \rho_\infty^{-n} d\mu_\infty = \mathcal{H}^{n-1}(Y) \int_0^1 \sin^2 \left( \frac{\pi}{\log \epsilon} \right) \frac{1}{s} ds
\]
\[
= \left( \log \frac{1}{\epsilon} \right) \mathcal{H}^{n-1}(Y) \int_0^1 \sin^2(\pi t) dt,
\]
and \(\mathcal{H}^{n-1}(Y) > 0\), then
\[
(\kappa' (1 - \epsilon^2) - \epsilon') \left( \log \frac{1}{\epsilon} \right) \mathcal{H}^{n-1}(Y) \int_0^1 \sin^2(\pi t) dt - \kappa' (1 - \epsilon^2)
\]
\[
\leq \mathcal{H}^{n-1}(Y) \int_0^1 \left( \frac{2 - n}{2} \sin \left( \frac{\pi}{\log \epsilon} \right) + \frac{\pi}{\log \epsilon} \cos \left( \frac{\pi}{\log \epsilon} \right) \right)^2 \frac{1}{s} ds
\]
\[
= \left( \log \frac{1}{\epsilon} \right) \mathcal{H}^{n-1}(Y) \left( \frac{(n-2)^2}{4} + \frac{\pi^2}{(\log \epsilon)^2} \right) \int_0^1 \sin^2(\pi t) dt,
\]
which implies
\[
\kappa' \leq \frac{(n-2)^2}{4}.
\]

Finally, we obtain the following results.

**Theorem 5.5.** Let \(N\) be an \((n+1)\)-dimensional complete Riemannian manifold satisfying conditions (C1), (C2) and (C3), and with non-radial Ricci curvature \(\inf_{\partial B_r} \text{Ric} (\xi^T, \xi^T) \geq \kappa' r^{-2}\) almost everywhere for a constant \(\kappa'\) and sufficiently large \(r > 0\), where \(\xi^T\) stands for the part that is tangential to the geodesic sphere \(\partial B_r\) (at least away from the cut locus of the center), of a tangent vector \(\xi\) of \(N\) at the considered point. If \(\kappa' > \frac{(n-2)^2}{4}\), then \(N\) admits no complete stable minimal hypersurface with at most Euclidean volume growth.

It is well known that area-minimizing hypersurfaces have Euclidean volume growth automatically. Let \(M\) be an \(n\)-dimensional area-minimizing hypersurface in \(N\). Then the \(s\)-dimensional Hausdorff measure of the singular set of \(S\) is \(H^s(\text{Sing} M) = 0\) for all \(s > n - 7\) (see [28] for example). We readily check that Lemmas 5.2 and 5.3 also hold for \(M\). Namely, there is a sequence \(\{\epsilon_i\}\) converging to zero such that \(\epsilon_i N = (N_i, \epsilon_i \bar{g}, p)\) converges to a metric cone \((N_\infty, d_\infty)\), and \(\epsilon_i M\) converges to the cone \(C Y = \mathbb{R}^+ \times_\rho Y\) in \(N_\infty\), where \(Y \in \partial B_1(o)\) is an \((n-1)\)-dimensional Hausdorff set.

**Corollary 5.6.** Let \(N\) be an \((n+1)\)-dimensional complete Riemannian manifold satisfying conditions (C1), (C2) and (C3), and with non-radial Ricci curvature \(\inf_{\partial B_r} \text{Ric} (\xi^T, \xi^T) \geq \kappa' r^{-2}\) for a constant \(\kappa'\) and sufficiently large \(r > 0\),
where $\xi$ is a local vector field on $N$ with $|\xi^T| = 1$ defined in (5.42). If $\kappa' > \frac{(n-2)^2}{4}$, then $N$ admits no complete area-minimizing hypersurface.

**Remark 5.7.** $\kappa = \frac{2}{n} \sqrt{n-1}$ in Remark 3.8 is equivalent to

$$\text{Ric}_{\text{MCS}_n}(\xi^T, \xi^T) = \frac{(n-2)^2}{4(\rho + \frac{1}{\kappa} - \rho_0)^2}$$

for all $\rho \geq \rho_0$, where $\xi^T = \xi - \langle \xi, \frac{\partial}{\partial \rho} \rangle \frac{\partial}{\partial \rho}$, $|\xi^T| = 1$ and $\rho_0 \in (1, \frac{1}{\kappa})$ is a constant. Hence the constant $\kappa'$ in Theorem 5.5 and Corollary 5.6 is optimal.
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