ON QUASICONFORMAL SELFMAPPINGS OF THE UNIT DISK AND ELLIPTIC PDE IN THE PLANE

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ABSTRACT. We prove the following theorem: if \( w \) is a quasiconformal mapping of the unit disk onto itself satisfying elliptic partial differential inequality
\[
|L[w]| \leq B|\nabla w|^2 + \Gamma,
\]
then \( w \) is Lipschitz continuous. This result extends some recent results, where instead of an elliptic differential operator is only considered the Laplace operator.

1. INTRODUCTION AND NOTATION

1.1. Quasiconformal mappings. Let \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \). We will consider the matrix norm:
\[
|A| = \max\{|Az| : z \in \mathbb{R}^2, |z| = 1\}
\]
and the matrix function
\[
l(A) = \min\{|Az| : z \in \mathbb{R}^2, |z| = 1\}.
\]
Let \( D \) and \( \Omega \) be subdomains of the complex plane \( \mathbb{C} \), and \( w = u + iv : D \to \Omega \) be a function that has both partial derivatives at a point \( z \in D \). By \( \nabla w(z) \) we denote the matrix \( \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \). For the matrix \( \nabla w \) we have
\[
|\nabla w| = |\partial w| + |\bar{\partial} w|
\]
and
\[
l(\nabla w) = ||\partial w| - |\bar{\partial} w||,
\]
where
\[
\partial w = w_z := \frac{1}{2} \left( \begin{pmatrix} w_x + \frac{1}{i} w_y \end{pmatrix} \right) \text{ and } \bar{\partial} w = w_{\bar{z}} := \frac{1}{2} \left( \begin{pmatrix} w_x - \frac{1}{i} w_y \end{pmatrix} \right).
\]
We say that a function \( u : D \to \mathbb{R} \) is ACL (absolutely continuous on lines) in the region \( D \), if for every closed rectangle \( R \subset D \) with sides parallel to the \( x \) and \( y \)-axes, \( u \) is absolutely continuous on a.e. horizontal and a.e. vertical line in \( R \). Such a function has of course, partial derivatives \( u_x, u_y \) a.e. in \( D \).

A sense-preserving homeomorphism \( w : D \to \Omega \), where \( D \) and \( \Omega \) are subdomains of the complex plane \( \mathbb{C} \), is said to be \( K \)-quasiconformal (\( K \)-q.c), with

\[2000 \text{ Mathematics Subject Classification.} \text{ Primary 30C55, Secondary 31C05.}
\]

\[\text{Key words and phrases. Quasiconformal maps, Beltrami equation, Elliptic PDE, Lipschitz condition.}\]
$K \geq 1$, if $w$ is ACL in $D$ in the sense that the real and imaginary part are ACL in $D$, and

(1.3) $|\nabla w| \leq Kl(\nabla w)$ a.e. on $D$,

(cf. [1], pp. 23–24). Notice that the condition (1.3) can be written as

$$|w| \leq k|w|$$ a.e. on $D$ where $k = \frac{K - 1}{K + 1}$, i.e. $K = \frac{1 + k}{1 - k}$.

If in the previous definition we replace the condition ”$w$ is a sense-preserving homeomorphism” by the condition ”$w$ is continuous”, then we obtain the definition of a quasiregular mapping.

1.2. Elliptic operator. Let $A(z) = \{a^{ij}(z)\}_{i,j=1}^2$ be a symmetric matrix function defined in a domain $D \subset \mathbb{C}$ ($a^{ij} = a^{ji}$). Assume that

(1.4) $\Lambda^{-1} \leq \langle A(z)h, h \rangle \leq \Lambda$ for $|h| = 1$,

where $\Lambda$ is a constant $\geq 1$ or written in coordinates

(1.5) $\Lambda^{-1} \leq \sum_{i,j=1}^2 a^{ij}(z)h_ih_j \leq \Lambda$ for $\sum_{i=1}^2 h_i^2 = 1$.

In addition for a certain $L \geq 0$, we suppose that

(1.6) $|A(z) - A(\zeta)| \leq L|z - \zeta|$ for any $z, \zeta \in D$.

For

(1.7) $L[u] := \sum_{i,j=1}^2 a^{ij}(z)D_{ij}u(z),$

subjected to conditions (1.5) and (1.6) we consider the following differential inequality

(1.8) $|L[u]| \leq B|\nabla u|^2 + \Gamma$,

with given $B, \Gamma \geq 0$, or, by using Einstein convention

(1.9) $|a^{ij}(z)D_{ij}u| \leq B|\nabla u|^2 + \Gamma$,

and call it elliptic partial differential inequality. Observe that, if $A$ is the identity matrix, then $L$ is the Laplace operator $\Delta$. A $C^2$ solutions $u : D \to \mathbb{R}(\mathbb{C})$ of the equation $\Delta u = 0$ is called a harmonic function (mapping) and the corresponding inequality (1.7) is called Poisson differential inequality. The class of harmonic quasiconformal mappings (HQC) has been one of recent main topics of investigation of some authors. See the subsection below. For the connection between quasiconformal mappings and PDE we refer to the book [2]. See also [8, Chapter 12], [5], [34] and [40].
1.3. **Background and statement of the main result.** Let $γ$ be a Jordan curve. By the Riemann mapping theorem, there exists a Riemann conformal mapping of the unit disk onto a Jordan domain $Ω = \text{int} \ γ$. By Caratheodory’s theorem, it has a continuous extension to the boundary. Moreover, if $γ \in C^{1,α}$, $0 < α < 1$, then the Riemann conformal mapping has $C^{1,α}$ extension to the boundary (this result is known as Kellogg’s theorem). We refer to [10] for the proof of the previous result and [35, 36, 24, 26] for related results. In particular a conformal mapping $w$ of the unit disk onto a Jordan domain $Ω$ with $C^{1,α}$ boundary is Lipschitz continuous, i.e. it satisfies the inequality $|w(z) - w(z')| \leq C|z - z'|$, $z, z' \in U := \{z \in \mathbb{C} : |z| < 1\}$.

On the other hand $K$ quasiconformal mappings between smooth domains are Hölder continuous and the best Hölder constant is $1/K$. So they are not in general Lipschitz mappings, except if $K = 1$. In this paper we are concerned with an additional condition of a quasiconformal mapping in order to guaranty its global Lipschitz character.

One of “additional condition” is to assume harmonicity of the mapping. This condition is natural since conformal mappings are quasiconformal and harmonic. Hence, quasiconformal harmonic (shortly HQC) mappings are natural generalization of conformal mappings. O. Martio [29] was the first who considered harmonic quasiconformal mappings on the complex plane.

Recently, there has been a number of authors who are working on this topic. We list below some of related results:

1) If $w$ is harmonic quasiconformal mapping of the unit disk onto itself, then $w$ is Lipschitz (Pavlovic theorem proved in [38]). See also some refinements of Partyka and Sakan [37].

2) If $w$ is a harmonic quasiconformal mapping between two $C^{1,α}$ Jordan domains, then $w$ is Lipschitz (the result of the author proved in [15]).

3) If $w$ is a quasiconformal mapping between two $C^{2,α}$ Jordan domains satisfying the partial differential inequality $|Δw| \leq C|fzf_\bar{z}|$, then $w$ is Lipschitz (the author & Mateljević result proved in [20]).

4) If $w$ is a quasiconformal mapping of the unit disk onto itself satisfying the PDE $Δw = g$ then this mapping is Lipschitz (the author & Pavlović result proved in [18]).

5) If $w$ is a quasiconformal mapping between two $C^{2,α}$ Jordan domains satisfying the partial differential inequality $|Δw| \leq B|∇w|^2 + Γ$, then $w$ is Lipschitz (the author & Mateljević result proved in [21]).

Notice that the proofs of 3)–5) depend on a Heinz theorem, see [11].

Concerning the bi-Lipschitz character of the class HQC we refer to the papers [14], [16], [27], [25] and [4]. See also [22] and [31] for some results concerning higher dimensional case.

For related result about quasiconformal harmonic mappings with respect to the hyperbolic metric we refer to the paper of Wan [41] and of Marković [28].

More recently, Iwaniec, Kovalev and Onninen in [12] have shown that the class of quasiconformal harmonic mappings is also interesting concerning the modulus of annuli in complex plane.
In this paper we study Lipschitz continuity of the class of $K$-q.c. self-mappings of the unit disk satisfying elliptic differential inequality $|Lw| \leq B|\nabla w|^2 + \Gamma$. This class contains conformal mappings and quasiconformal harmonic mappings.

The main result of this paper is the following theorem which is an extension of results 1)–5) mentioned above.

**Theorem 1.1.** If $a \in U$, and $w : U \to U$, $w(a) = 0$ and $w(U) = U$ is a $K$ q.c. solution of the elliptic partial differential inequality

$$|L[w]| \leq B|\nabla w|^2 + \Gamma,$$

then $\nabla w$ is bounded by a constant $C(K, B, \Gamma, \Lambda, \mathcal{L}, a)$ and $w$ is Lipschitz continuous.

**Remark 1.2.** The condition (1.10) is in [9, p. 179-180] called as natural growth condition. The result is new even for $B = \Gamma = 0$ i.e. for q.c. solution to elliptic PDE with Lipschitz coefficients.

The proof of Theorem 1.1 is given in Section 3. The methods of the proof differ from the methods of the proof of corresponding results for the class HQC. In Section 2 we make some estimates concerning the Green function of the disk, and some estimates concerning the gradient of a solution to elliptic partial differential inequality, satisfying certain boundary condition similar to those in the paper of Nagumo [33]. We first prove interior estimates for the gradient of a solution $u$ of elliptic PDE in terms of constants of the elliptic operator, and modulus of continuity of $u$ (Theorem 2.5). After that we recall a theorem of Nagumo ([33]), which shows that if $u$ is a solution of elliptic PDE, with vanishing boundary condition defined in a domain $D$ whose boundary has a bounded curvature from above by a constant $\kappa$, then $|\nabla u(z)| \leq \gamma$, $z \in D$, where $\gamma$ is a constant not depending on $u$ providing that $64B|u|_\infty < \pi$ (Theorem 2.8). In order to prove Theorem 1.1 we previously show that the function $u = |w|$ satisfies a certain elliptic differential inequality near the boundary of the unit disk. In order to show a priori bound, we make use of Mori’s theorem which implies that the modulus of continuity of a $K$-q.c. self-mapping of the unit disk depends only on $K$. By using Theorem 2.5 we show that the gradient is a priori bounded on compacts of the unit disk, while Theorem 2.8 serves to obtain the a priori bound of the gradient of $u$ in some “neighborhood” of the boundary of the unit disk. By using the quasiconformality, we prove that $\nabla w$ is a priori bounded as well.

2. Auxiliary results

2.1. Green function. If $h(z, w)$ is a real function, then by $\nabla_z h$ we denote the gradient $(h_x, h_y)$.

**Lemma 2.1.** If

$$h(z, w) = \log \left| \frac{1 - z\bar{w}}{|z - w|^2} \right|,$$
then
\begin{equation}
\nabla z h(z, w) = \frac{1 - |w|^2}{(\bar{z} - \bar{w})(w\bar{z} - 1)}
\end{equation}

and
\begin{equation}
\partial_w \nabla z h(z, w) = -\frac{1}{(1 - w\bar{z})^2}, \quad \partial_{\bar{w}} \nabla z h(z, w) = -\frac{1}{(w - \bar{z})^2}.
\end{equation}

**Proof.** First of all
\[ \nabla z h = (h_x, h_y) = h_x + ih_y. \]
Since
\[ h_{\bar{z}} = \frac{1}{2}(h_x + ih_y), \]
it follows that
\[ \nabla z h = 2h_{\bar{z}}. \]
Since
\[ 2h(z) = \log \left( \frac{1 - z\bar{w}}{z - w} \right), \]
by differentiating we obtain
\[ 2h_{\bar{z}}(z) = \log \left( \frac{1 - \bar{z}w}{\bar{z} - \bar{w}} \right) = \frac{|w|^2 - 1}{(z - \bar{w})^2 (1 - \bar{z}w)}. \]
This implies (2.1). From
\[ \frac{1 - |w|^2}{(\bar{z} - \bar{w})(w\bar{z} - 1)} = \frac{w}{w\bar{z} - 1} + \frac{1}{\bar{w} - \bar{z}}, \]
it follows (2.2). \hfill \Box

**Corollary 2.2.** Let \( G(\zeta, \omega) \) be the Green function of the disk \( \{ \zeta : |\zeta - \zeta_0| \leq R \} \)
defined by
\[ G(\zeta, \omega) := \log \frac{|\varphi(\zeta) - \varphi(\omega)|}{|1 - \varphi(\zeta)\varphi(\omega)|}, \]
where
\[ \varphi(\zeta) = \frac{1}{R} (\zeta - \zeta_0). \]
Then
\begin{equation}
|\nabla_\zeta G(\zeta, \omega)| \leq \frac{2}{|\zeta - \omega|}
\end{equation}
and
\begin{equation}
|\partial_{\omega_j} \nabla_\zeta G(\zeta, \omega)| \leq \frac{2}{|\zeta - \omega|^2}, \quad j = 1, 2,
\end{equation}
where \( \omega = \omega_1 + i\omega_2, \omega_1, \omega_2 \in \mathbb{R} \).
Proof. Let 
\[ \varphi(\zeta) = \frac{1}{R}(\zeta - z_0). \]
Then
\[ \varphi'(\zeta) = \frac{1}{R}. \]
Take \( z = \varphi(\zeta) \) and \( w = \varphi(\omega) \) and define \( h(z, w) = G(\zeta, \omega) \). It follows that
\[ (2.5) \quad \nabla_\zeta G(\zeta, \omega) = \nabla_z h(z, w) \cdot \varphi'(\zeta) = \frac{1}{R} \nabla_z h(z, w). \]
Thus
\[ (2.6) \quad |\nabla_\zeta G(\zeta, \omega)| = \frac{1}{R} |\nabla_z h(z, w)|. \]
Further
\[ (2.7) \quad \frac{1 - |w|^2}{|1 - \bar{w}z|} \leq \frac{1 - |w|^2}{1 - |w|} \leq 2. \]
Combining (2.7), (2.6) with (2.1), we obtain (2.3). To get (2.4), observe first that for \( \omega = \omega_1 + i\omega_2 \)
\[ (2.8) \quad \partial_{\omega_1} = \partial_\omega + \partial_{\bar{\omega}} \]
and
\[ (2.9) \quad \partial_{\omega_2} = i(\partial_\omega - \partial_{\bar{\omega}}). \]
On the other hand, for \( |z| \leq 1 \) and \( |w| \leq 1 \) we have
\[ \left| \frac{1}{1 - wz} \right| \leq \left| \frac{1}{(w - z)^2} \right|. \]
From (2.8), (2.9), (2.2), (2.5) we deduce (2.4). \( \square \)

2.2. Interior estimates of gradient.

Lemma 2.3. Let \( u : \overline{U} \to \mathbb{C} \) be a continuous mapping. Then there exists a positive function \( \varpi = \varpi_u(t), t \in (0, 2) \), such that \( \lim_{t \to 0} \varpi_u(t) = 0 \) and
\[ |u(z) - u(w)| \leq \varpi(|z - w|), \quad z, w \in \overline{U}. \]
The function \( \varpi \) is called the modulus of continuity of \( u \).

Lemma 2.4. Let \( Y : D \to \mathbb{U} \) be a \( C^2 \) mapping of a domain \( D \subset \mathbb{U} \). Define \( \mathbb{U}(z_0, \rho) := \{ z \in \mathbb{C} : |z - z_0| < \rho \} \) and assume that the closure of \( \mathbb{U}(z_0, \rho) \) is contained in \( D \), and let \( Z \in \mathbb{C} \) be any complex number. Then we have the estimate:
\[ (2.10) \quad |\nabla h(z_0)| \leq \frac{2}{\rho^2} \int_{|y - z_0| = \rho} |Y(y) - Z| dH^1(y) \]
where \( h(z), z \in \overline{\mathbb{U}(z_0, \rho)} \) is the Poisson integral of \( Y|_{z_0 + \rho T} \) and \( T \) is the unit circle. Moreover \( dH^1 \) is the Hausdorff probability measure (i.e. normalized arc length measure).
Proof. Assume that $v \in C^2(\overline{U})$ and define

\begin{equation}
H(z) = \int_T P(z, \eta) v(\eta) dH^1(\eta),
\end{equation}

where

\begin{equation}
P(z, \eta) = \frac{1 - |z|^2}{|z - \eta|^2}, \quad |\eta| = 1, \quad |z| < 1.
\end{equation}

Then $H$ is a harmonic function. It follows that

\begin{equation}
\langle \nabla H(z), e \rangle = \int_T \langle \nabla_z P(z, \eta), e \rangle v(\eta) dH^1(\eta), \quad e \in \mathbb{R}^2.
\end{equation}

By differentiating (2.12), we obtain

\(\nabla_z P(z, \eta) = \frac{-2z}{|z - \eta|^2} - \frac{2(1 - |z|^2)(z - \eta)}{|z - \eta|^{2+2}}.\)

Hence

\(\nabla_z P(0, \eta) = \frac{2\eta}{|\eta|^4} = 2\eta.\)

Therefore

\begin{equation}
|\langle \nabla_z P(0, \eta), e \rangle| \leq |\nabla_z P(0, \eta)||e| = 2|e|.
\end{equation}

Using (2.13), (2.14), we obtain

\begin{equation}
|\langle \nabla H(0), e \rangle| \leq \int_T |\nabla_z P(0, \eta)||e||v(\eta)||dH^1(\eta) = |e| \int_T |\nabla_z v(\eta)||dH^1(\eta).
\end{equation}

Hence, we have

\begin{equation}
|\nabla H(0)| \leq 2 \int_T |v(\eta)||dH^1(\eta)
\end{equation}

Let $v(z) = Y(z_0 + \rho z) - Z$ and $H(z) = P[v|\Omega](z)$. Then $H(z) = h(z_0 + \rho z) - Z$ and $\nabla H(0) = \rho \nabla h(z_0)$. Inserting this into (2.15), we obtain

\begin{equation}
\rho|\nabla h(z_0)| = |\nabla H(0)| \leq 2 \int_T |Y(z_0 + \rho \eta) - Z||dH^1(\eta).
\end{equation}

Introducing the change of variables $\zeta = z_0 + \rho \eta$ in the integral (2.16), we obtain

\begin{equation}
|\nabla h(z_0)| \leq \frac{2}{\rho^2} \int_{|\zeta - z_0| = \rho} |Y(\zeta) - Z||dH^1(\zeta)
\end{equation}

which is identical with (2.10). \qed

**Theorem 2.5.** Let $D$ be a bounded domain, whose diameter is $d$. Let $A(z) = \{a^{ij}(z)\}_{i,j=1}^n$ be a symmetric matrix function defined in a domain $\Omega \subset \mathbb{C}$ ($a^{ij} = a^{ji}$) satisfying the condition (1.5) and (1.6). Let $u(z)$ be any $C^2$ solution of elliptic partial differential inequality (1.8) such that

\begin{equation}
|u(z)| \leq M \text{ in } D.
\end{equation}
Then there exist constants $C^{(0)}$ and $C^{(1)}$, depending on modulus of continuity of $u$, $\Lambda$, $\mathcal{L}$, $B$, $\Gamma$, $M$ and $d$ such that

\begin{equation}
|\nabla u(z)| < C^{(0)} \rho(z)^{-1} \max_{|\zeta - z| \leq \rho(z)} \{|u(\zeta)|\} + C^{(1)}
\end{equation}

where $\rho(z) = \text{dist}(z, \partial D)$.

\textbf{Proof.} Fix a point $a \in D$ and let $B_p, 0 < p < 1$, be a closed disk defined by

\[ B_p = \{z; |z - a| \leq p \text{dist}(a, \partial D)\} \]

Its radius is $R_p = p \text{dist}(a, \partial D)$.

Define the function $\mu_p$ as

\begin{equation}
\mu_p = \max_{z \in B_p} |\nabla u| r_p(z)
\end{equation}

where $r_p(z) = \text{dist}(z, \partial B_p) = R_p - |z - a|$. Then there exists a point $z_p \in B_p$ such that

\begin{equation}
|\nabla u(z_p)| r_p(z_p) = \mu_p \quad (z_p \in B_p).
\end{equation}

We need the following result in the sequel.

\textbf{Lemma 2.6.} The function $\mu_p$ is continuous on $(0, 1)$ and has a continuous extension at 0: $\mu_0 = 0$.

\textbf{Proof of Lemma 2.6.} Let $p_n$ be a sequence converging to a number $p$, let $\mu_{p_n} = |\nabla u(z_n)| r_{p_n}(z_n)$ and assume it converges to $\mu'_p$. Prove that $\mu'_p = \mu_p$. Passing to a subsequence, we can assume that $z_n \to z'_p$. Then $z'_p \in B_p$. Thus, $\mu'_p \leq \mu_p$. On the other hand, $\mu_{p_n} \geq |\nabla u((1 - \varepsilon_n)z_p)| r_{p_n}((1 - \varepsilon_n)z_p)$, where $\varepsilon_n$ is a positive sequence converging to zero. It follows that $\mu'_p \geq \lim_{n \to \infty} |\nabla u((1 - \varepsilon_n)z_p)| r_{p_n}((1 - \varepsilon_n)z_p) = \mu_p$. Furthermore, since $r_p \leq R_p = p \text{dist}(a, \partial D)$, we obtain

\[ \lim_{p \to 0^+} \mu_p \leq |\nabla u(0)| \lim_{p \to 0^+} R_p = 0. \]

Now let $Tz = \zeta$ be a linear transformation of coordinates such that

\begin{equation}
\sum_{i,j=1}^{2} a^{ij}(z_p) D_{ij} u = \Delta v,
\end{equation}

where $v(\zeta) = u(z)$. By [23] Lemma 11.2.1 the transformation $T$ can be chosen so that

\begin{equation}
T = \begin{pmatrix}
\lambda_1^{-\frac{1}{2}} & 0 \\
0 & \lambda_2^{-\frac{1}{2}}
\end{pmatrix} \cdot R,
\end{equation}

where $\lambda_1$ and $\lambda_2$ are eigenvalues of the matrix $A(z_p)$ and $R$ is some orthogonal matrix. Then

\[ \frac{1}{\Lambda} \leq \lambda_1, \lambda_2 \leq \Lambda. \]
Let $\nabla^2 u$ denote the Hessian matrix of $u$:

$$
\nabla^2 u = \begin{pmatrix} D_{11} u & D_{12} u \\ D_{21} u & D_{22} u \end{pmatrix}.
$$

Since

$$
\nabla^2 u = T^t \nabla^2 v T,
$$
we obtain:

$$
\text{Trace}(A^t \nabla^2 u) = \text{Trace}((TA)^t \nabla^2 v T) = \text{Trace}(\nabla^2 v T (TA)^t) = \text{Trace}(\nabla^2 v T A^t T^t) = \text{Trace}(B^t \nabla^2 v),
$$

where

$$
(2.24) \quad B(\zeta) = TA(z) T^t.
$$

Then

$$
(2.25) \quad b^{ij}(\zeta) D_{ij} v(\zeta) = a^{ij}(z) D_{ij} u(z),
$$

where $B(\zeta) = \{b^{ij}\}_{i,j=1}^2$ and

$$
(2.26) \quad \Delta v = (\delta_{ij} - b^{ij}(\zeta)) D_{ij} v + b^{ij}(\zeta) D_{ij} v.
$$

Further, $T(U(z_p, r_p)) \subset T(B_p) \subset T(D) =: D'$. From (2.23) we see that $T(D(z_p, r_p))$ is an ellipse with axes equal to $\lambda_1^{-1/2} \cdot r_p$ and $\lambda_2^{-1/2} \cdot r_p$ and with the center at $\zeta_p = T(z_p)$. Then $D_\lambda := \{\zeta : |\zeta - \zeta_p| \leq \lambda r_p\}$ is a closed disk in $T(B_p)$ provided that

$$
(2.27) \quad 0 < \lambda < \frac{1}{2\sqrt{\Lambda}}.
$$

Let $G(\zeta, \omega)$ be the Green function of the disk $D_\lambda$. So that, from (2.26)

$$
\begin{align*}
\quad v &= -\frac{1}{\pi} \int_{D_\lambda} G(\zeta, \omega)(\delta_{ij} - b^{ij}(\omega)) D_{ij} v(\omega) d\mathcal{L}^2(\omega) \\
&\quad - \frac{1}{\pi} \int_{D_\lambda} G(\zeta, \omega)b^{ij}(\omega) D_{ij} v(\omega) d\mathcal{L}^2(\omega) + h(\zeta),
\end{align*}
$$

where $d\mathcal{L}^2(\omega) = dx dy$ is the Lebesgue two-dimensional measure in the complex plane and $h(\zeta)$ is the harmonic function which takes the same values as $v(\zeta)$ for $\zeta \in \partial D_\lambda$. Then

$$
(2.28) \quad |\nabla v(\zeta_p)| \leq \mathcal{P} + \mathcal{Q} + \mathcal{R},
$$

where...
where
\[
P = \frac{1}{\pi} \int_{D_\lambda} \nabla \zeta G(\zeta_p, \omega) b^{ij}(\omega) D_{ij} v(\omega) d\mathcal{L}^2(\omega),
\]
\[
Q = \frac{1}{\pi} \int_{D_\lambda} \nabla \zeta G(\zeta_p, \omega) (\delta_{ij} - b^{ij}(\omega)) D_{ij} v(\omega) d\mathcal{L}^2(\omega),
\]
\[
R = |\nabla \zeta h(\zeta_p)|.
\]

Further, it follows by (1.6) that \( A \) is differentiable almost everywhere. From (2.24) we obtain
\[
DB(\zeta) \cdot T = T \cdot DA(z) \cdot T^t, \quad \text{for a.e. } z.
\]
Here \( DA(z) \) is the differential operator defined by
\[
A(z + h) = A(z) + DA(z)h + o(|h|).
\]
Notice that \( DA(z)h \) is a matrix. Since \( \Lambda^{-1/2}|z| \leq |Tz| \leq \Lambda^{1/2}|z| \), having in mind (1.6), we obtain
\[
(2.29) \quad \|DB(\zeta)\| \leq |T|^3\|DA(z)\| \leq \Lambda^{3/2}\Sigma.
\]
In the previous formula we mean the following norms: the norm of a matrix \( L \) is defined by \( |L| = \max\{|Lh| : |h| = 1\} \), and the norm of an operator \( DX(z) \) by \( \|DX(z)\| = \max\{|DA(z)h| : |h| = 1\} \), \( X = A, B \). Thus
\[
(2.30) \quad |B(\zeta) - B(\zeta_p)| = |B(\zeta) - I| \leq \Lambda^{3/2}\Sigma |\zeta - \zeta_p|
\]
As
\[
|T(z) - T(z_p)| \leq \lambda r_p(z_p),
\]
by using the inequalities
\[
r_p(z_p) \leq d(z, z_p) + r_p(z),
\]
\[
d(z, z_p) \leq \Lambda^{1/2}|T(z) - T(z_p)|
\]
and by (2.20),
\[
|\nabla u(z)| r_p(z) \leq \mu_p,
\]
we obtain
\[
|\nabla u(z)| \leq (1 - \Lambda^{1/2})^{-1} r_p(z_p)^{-1} \mu_p \quad \text{for } z \in T^{-1}(D_\lambda)(\subset B_p).
\]
From (2.27) we obtain that
\[
(2.31) \quad (1 - \Lambda^{1/2})^{-2} < 4.
\]
Having in mind the formula, \( \nabla u(z) = \nabla v(\zeta) \cdot T \) we obtain
\[
(2.32) \quad |\nabla v(\zeta)| \leq 2\Lambda^{1/2} r_p(z_p)^{-1} \mu_p
\]
for \( \zeta \in D_\lambda \). Since
\[
|a^{ij}(z) D_{ij} u| \leq \mathcal{B}|\nabla u|^2 + \Gamma,
\]
\[
|b^{ij}(\zeta) D_{ij} v(\zeta)| = |a^{ij}(z) D_{ij} u(z)|,
\]
it follows that
\[
(2.33) \quad |b^{ij}(\zeta) D_{ij} v(\zeta)| \leq \mathcal{B}|T|^2|\nabla v|^2 + \Gamma = \mathcal{B} \Lambda|\nabla v|^2 + \Gamma
\]
and therefore, from (2.32) we find that

\begin{equation}
|b^{ij}(\zeta) D_{ij}v(\zeta)| \leq 4\Lambda^2 B r_p(z_p)^{-2} \mu_p^2 + \Gamma.
\end{equation}

Now we divide the proof into four steps:

\textbf{Step 1: Estimation of } P. \text{ From (2.3) and (2.34) we first have}

\begin{equation}
\left| \frac{1}{\pi} \int_{D_\lambda} \nabla \zeta G(\zeta, \omega) b^{ij}(\omega) D_{ij}v(\omega) \, d\mathcal{L}^2(\omega) \right|
\leq \frac{2}{\pi} \int_{|\omega - \zeta_p| \leq \lambda r_p(z_p)} \frac{1}{|\omega - \zeta_p|} |b^{ij}(\omega) D_{ij}v(\omega)| \, d\mathcal{L}^2(\omega)
\leq \frac{2}{\pi} \int_{|\omega - \zeta_p| \leq \lambda r_p(z_p)} \frac{1}{|\omega - \zeta_p|} (4\Lambda^2 B r_p(z_p)^{-2} \mu_p^2 + \Gamma) \, d\mathcal{L}^2(\omega)
\end{equation}

Therefore

\begin{equation}
P \leq \frac{16\Lambda^2 B \lambda \mu_p^2}{r_p} + 4\Gamma r_p \lambda.
\end{equation}

\textbf{Step 2: Estimation of } Q. \text{ Let } n_\omega = (\cos \alpha_1, \cos \alpha_2) \text{ be the unit inner vector of } \partial D_\lambda \text{ at } \omega. \text{ Then from Green’s formula}

\begin{equation}
\int_{\partial D_\lambda} \sum_{i=1}^2 u_i(\omega) \cos \alpha_i d\mathcal{H}^1(\omega) = \int_{D_\lambda} (\partial_{\omega_1} u_1 + \partial_{\omega_2} u_2) d\mathcal{L}^2(\omega),
\end{equation}

proceeding as in \cite{3} Theorem 2], we obtain

\begin{equation}
Q \leq \frac{1}{\pi} \int_{|\omega - \zeta_p| = \lambda r_p(z_p)} \nabla \zeta G(\zeta, \omega) (\delta_{ij} - b^{ij}(\omega)) \partial_i v(\omega) \cos \alpha_j d\mathcal{H}^1(\omega)
\end{equation}

\begin{equation}
+ \frac{1}{\pi} \int_{|\omega - \zeta_p| \leq \lambda r_p(z_p)} \nabla \zeta G(\zeta, \omega) \partial_{\omega_j} b^{ij}(\omega) \partial_i v(\omega) d\mathcal{L}^2(\omega)
\end{equation}

\begin{equation}
+ \frac{1}{\pi} \int_{|\omega - \zeta_p| \leq \lambda r_p(z_p)} \partial_{\omega_j} \nabla \zeta G(\zeta, \omega) (\delta_{ij} - b^{ij}(\omega)) \partial_i v(\omega) d\mathcal{L}^2(\omega).
\end{equation}

By using the Cauchy-Schwarz inequality, (2.3), (2.4), (2.29), (2.30), (2.32), we obtain

\begin{equation}
Q \leq 8\Lambda^2 \mathcal{L} \lambda \mu_p + 4\Lambda^2 \mathcal{L} \lambda \mu_p + 4\Lambda^2 \mathcal{L} \lambda \mu_p,
\end{equation}

i.e.

\begin{equation}
Q \leq 16\Lambda^2 \mathcal{L} \lambda \mu_p.
\end{equation}

\textbf{Step 3: Estimation of } R.

Let \( \varpi(t) = \varpi_p(t) \) be the modulus of continuity of \( v \) as in Lemma \cite{2}. \text{ From (2.10), for } Z = v(\zeta_p) (Z = 0), \ Y(\zeta) = v(\zeta) \text{ and } \rho = \lambda r_p(z_p), \text{ by using Lemma 2.4 and
we obtain
\[
\mathcal{R} \leq |\nabla h(z_p)| \leq \frac{2}{\lambda r_p(z_p)^2} \int_{|\omega - \zeta_p| = \lambda r_p(z_p)} |v(\omega) - Z| d\mathcal{H}^1(\omega)
\]
\[
\leq \frac{2}{\lambda r_p(z_p)} \max \{ |v(\zeta) - Z| : |\zeta - \zeta_p| = \lambda r_p(z_p) \}
\]
\[
\leq \frac{\min \{ 2\varpi(\lambda r_p(z_p)), 2K \} \lambda r_p(z_p)}{\lambda r_p(z_p)},
\]
where
\[
K = \sup_{|z - a| \leq \rho(a)} |u(z)|.
\]

Step 4: The finish of the proof. As
\[
|\nabla v(\zeta_p)| \geq \Lambda^{-1/2}|\nabla u(z_p)| = \Lambda^{-1/2}r_p(z_p)^{-1}\mu_p
\]
and \(r_p(z_p) < 2\rho(a) \leq d\), from (2.28), (2.35), (2.37) and (2.38), we get
\[
A_0\mu_p^2 + B_0\mu_p + C_0 \geq 0,
\]
where
\[
A_0 = 16B\Lambda^2\lambda,
\]
\[
B_0 = 16\Lambda^2\varpi\lambda r_p(z_p) - \Lambda^{-1/2}
\]
and
\[
C_0 = 4\Gamma r_p^2(z_p)\lambda + \frac{2\min \{ \varpi(\lambda r_p(z_p)), K \} \lambda}{\lambda}.
\]
We can take \(\lambda > 0\) depending on \(\varpi, \Lambda, \varpi, B, \Gamma\) and \(d\) so small that
\[
B_0^2 > 4A_0C_0
\]
and
\[
16\Lambda^2\varpi\lambda r_p(z_p)\lambda \leq 1/2\Lambda^{-1/2}.
\]
Let \(\mu_1\) and \(\mu_2\) \((\mu_1 < \mu_2)\) be the distinct real roots of the equation
\[
A_0\mu^2 + B_0\mu + C_0 = 0.
\]
Then from (2.40) we have
\[
\mu_p \leq \mu_1 \text{ or } \mu_p \geq \mu_2.
\]
Lemma 2.6 asserts that \(\mu_p\) depends on \(p\) continuously for \(0 < p < 1\) and \(\lim_{p \to 0} \mu_p = 0\). Then we have only \(\mu_p \leq \mu_1\). And, letting \(p\) tend to 1, by the definition of \(\mu_p\)
\[
|\nabla u(a)| \leq \mu_1\rho(a)^{-1}.
\]
As $\mu_1$ is the smaller root of (2.43),
\[
\mu_1 = \frac{-B_0 - \sqrt{B_0^2 - 4A_0C_0}}{2A_0} \leq \frac{-2C_0}{B_0}.
\]
From (2.44) and (2.39) we get
(2.45) \[|\nabla u(a)| \leq C(0)\rho(a)^{-1} \sup_{|z-a| \leq \rho(a)} |u(z)| + C(1)\]
where $C(0)$ and $C(1)$ depend on $\Lambda, \mathcal{L}, B, M, \Gamma, d$ and on modulus of continuity of $u$. 

2.3. Boundedness of gradient.

**Definition 2.7.** We say that a domain $D$ satisfies the exterior sphere condition for some $\kappa > 0$ if to any point $p$ of $\partial D$ there corresponds a ball $B_p \subset \mathbb{C}$ with radius $\kappa$ such that $D \cap B_p = \{p\}$.

**Theorem 2.8** (A priori bound). [33, Lemma 2] Let $D$ be a complex domain with diameter $d$ satisfying exterior sphere condition for some $\kappa > 0$. Let $u(z)$ be a twice differentiable mapping satisfying the elliptic differential inequality (1.8) in $D$ satisfying the boundary condition $u = 0$ ($z \in G$). Assume in addition that $|u(z)| \leq M, z \in D$.
(2.46) \[\frac{4}{\pi} \cdot 16B_\Gamma M < 1\]
and $u \in C(\overline{D})$. Then
(2.47) \[|\nabla u| \leq \gamma, \ z \in D,\]
where $\gamma$ is a constant depending only on $\kappa, M, B, \Gamma, \mathcal{L}, \Lambda$ and $d$.

**Remark 2.9.** See [8, Theorem 15.9] for a related result. In the statement of [33, Lemma 2] instead of condition (2.46) appears
\[16B_\Gamma M < 1\]
However, a related proof lays on [33, Theorem 2], which it seems that works only under the condition (2.46). Indeed, the right hand side of the inequality in the first line on [33, p. 214] should be multiplied by
\[\frac{2\Gamma(1 + m/2)}{\sqrt{\pi\Gamma((m + 1)/2)}}\]
where $m$ is the dimension of the space (in our case $m = 2$) and
\[\frac{2\Gamma(1 + 2/2)}{\sqrt{\pi\Gamma((2 + 1)/2)}} = \frac{4}{\pi}.\]
3. Proof of the main theorem

We need the following lemmas.

Lemma 3.1. \[17\] Every\ $K$-q.r.\ mapping\ $w(z) = \rho(z)S(z) : D \to \Omega$, $\rho = |w|$, $S(z) = e^{is(z)}$, $s(z) \in [0, 2\pi)$, satisfies the inequalities
\begin{equation}
\rho|\nabla S| \leq K|\nabla \rho|
\end{equation}
and
\begin{equation}
|\nabla \rho| \leq K\rho|\nabla S|
\end{equation}
almost everywhere on $D$. Inequalities (3.1) and (3.2) are sharp; the equality
\begin{equation}
\rho|\nabla S| = |\nabla \rho|
\end{equation}
holds if $w$ is a 1-quasiregular mapping. We also have
\begin{equation}
K^{-1}|\nabla w| \leq |\nabla \rho| \leq |\nabla w|.
\end{equation}

Lemma 3.2. If $w = \rho S : U \to U$, $\rho = |w|$, is twice differentiable, then
\begin{equation}
L[\rho] = \rho(a^{11}|p|^2 + 2a^{12} \langle p, q \rangle + a^{22}|q|^2) + \langle L[w], S \rangle,
\end{equation}
where $p = D_1 S$ and $q = D_2 S$.

If in addition $w$ is $K$-q.c. and satisfies
\begin{equation}
|L[w]| = \sum_{i,j=1}^2 a^{ij}(z) D_{ij} w| \leq B|\nabla w|^2 + \Gamma,
\end{equation}
then there exists a constant $\Theta$ depending on $K$, $B$ and $\Gamma$ such that
\begin{equation}
|L[\rho]| \leq \frac{\Theta}{\rho}|\nabla \rho|^2 + \Gamma.
\end{equation}

Proof: Let $w = (w_1, w_2)$ (here $w_i$ are real), $S = (S_1, S_2)$ and let $f = (f_1, f_2)$.

For real differentiable functions $a$ and $b$ define the bi-linear operator
\begin{equation}
D[a, b] = \sum_{k,l=1}^2 a^{kl}(z) D_k a(z) D_l b(z).
\end{equation}

Since $w_i = \rho S_i$, $i \in \{1, 2\}$ and
\[\rho = \sum_{i=1}^2 S_i w_i,\]
we obtain
\begin{equation}
L[w_i] = S_i L[\rho] + \rho L[S_i] + 2D[\rho, S_i], \ i \in \{1, 2\}
\end{equation}
and
\begin{equation}
L[\rho] = \sum_{i=1}^2 w_i L[S_i] + \sum_{i=1}^2 S_i L[w_i] + 2 \sum_{i=1}^2 D[S_i, w_i].
\end{equation}
From (3.8) we obtain

\[ L[\rho] = L[\rho]|S|^2 = 2 \sum_{i=1}^{2} S_i \cdot S_i L[\rho] \]

(3.10)

\[ = 2 \sum_{i=1}^{2} S_i L[w_i] - \rho \sum_{i=1}^{2} S_i L[S_i] - 2 \sum_{i=1}^{2} S_i D[\rho, S_i]. \]

By adding (3.9) and (3.10) we obtain

\[ L[\rho] = \sum_{i=1}^{2} (D[S_i, w_i] - S_i D[\rho, S_i]) + \langle L[w], S \rangle. \]

On the other hand

\[ D[S_i, w_i] - S_i D[S_i, \rho] = 2 \sum_{k,l=1}^{2} a_{kl}^i (z) D_k S_i D_l w_i - S_i \sum_{k,l=1}^{2} a_{kl}^i (z) D_k S_i D_l \rho \]

\[ = \sum_{k,l=1}^{2} a_{kl}^i (z) D_k S_i (\rho D_l S_i + S_i D_l \rho) - S_i \sum_{k,l=1}^{2} a_{kl}^i (z) D_k S_i D_l \rho \]

\[ = \rho \sum_{k,l=1}^{2} a_{kl}^i (z) D_k S_i D_l S_i, \quad i = 1, 2. \]

Thus

\[ L[\rho] = \rho \sum_{i,k,l=1}^{2} a_{kl}^i (z) D_k S_i D_l S_i + \langle L[w], S \rangle \]

\[ = \rho (a_{11}^2 |p|^2 + 2a_{12}^2 (p, q) + a_{22}^2 |q|^2) + \langle L[w], S \rangle, \]

where \( p = (D_1 S_1, D_1 S_2) \) and \( q = (D_2 S_1, D_2 S_2) \). Therefore

\[ |L[\rho]| \leq \Lambda \rho (|p|^2 + |q|^2) + (B|\nabla w|^2 + \Gamma) \]

\[ = \Lambda \rho \|\nabla S\|^2 + (B|\nabla w|^2 + \Gamma), \]

provided (3.6) holds. Here \( \| \cdot \| \) is the Hilbert-Schmidt norm which satisfies the inequality \( \|P\| \leq \sqrt{2}\|P\| \). If \( w \) is \( K \)-q.c., then according to (3.1) and (3.3) we have

\[ |L[\rho]| \leq 2K \Lambda \sqrt{\rho \rho^2 - 1} + (B K \sqrt{\rho^2 + \Gamma}). \]

Taking \( \Theta = 2K \Lambda + B K \) we obtain (3.7).

**Lemma 3.3.** If \( f = u + iv \) is a \( K \) q.c. mapping satisfying elliptic differential inequality, then \( u \) and \( v \) satisfy the elliptic differential inequality.

**Proof.** Let

\[ A := |\nabla u|^2 = 2(|u_z|^2 + |u_{\bar{z}}|^2) = \frac{1}{2}(|f_z + \bar{f}_{\bar{z}}|^2 + |f_{\bar{z}} + \bar{f}_z|^2) \]
and
\[ B := |\nabla v|^2 = 2(|v_z|^2 + |v_{\bar{z}}|^2) = \frac{1}{2}(|f_z - \overline{f_{\bar{z}}}|^2 + |f_{\bar{z}} - \overline{f_z}|^2). \]

Then
\[ \frac{A}{B} = \frac{|1 + \mu|^2}{|1 - \mu|^2}, \]
where \( \mu = \overline{f_z}/f_{\bar{z}}. \) Since \( |\mu| \leq k = \frac{K - 1}{K + 1} \)
(3.11)
\[ \frac{(1 - k)^2}{(1 + k)^2} \leq \frac{A}{B} \leq \frac{(1 + k)^2}{(1 - k)^2}. \]

As
\[ |L[f]| = |L[u] + iL[v]| \leq B|\nabla f|^2 + \Gamma \leq B(|\nabla u|^2 + |\nabla v|^2) + \Gamma, \]
the relation (3.11) yields
\[ |L[u]| \leq B \left( 1 + \frac{(1 + k)^2}{(1 - k)^2} \right) |\nabla u|^2 + \Gamma \]
and
\[ |L[v]| \leq B \left( 1 + \frac{(1 + k)^2}{(1 - k)^2} \right) |\nabla v|^2 + \Gamma. \]

\[ \square \]

Before proving the main results of this paper let us recall one of the most fundamental results concerning quasiconformal mappings.

**Proposition 3.4 (Mori).** If \( w : U \to U, w(0) = 0, \) is a \( K \) quasiconformal harmonic mapping of the unit disk onto itself, then
\[ |w(z_1) - w(z_2)| \leq 16|z_1 - z_2|^{1/K}, \ z_1, z_2 \in U. \]

Mori’s theorem for q.c. selfmappings of the unit disk has been generalized in various directions in the plane and in the space. See for example, the papers [13], [7] and [6].

**Proof of Theorem 1.1.** The idea of the proof is to estimate the gradient of \( w \) in some ”neighborhood” of the boundary together with some interior estimate in the rest of the unit disk. Put \( \alpha, \beta \in \mathbb{R} \) such that \( \frac{1+|a|}{2} \leq \alpha < 1 \) and \( \beta = \frac{\alpha + 1}{2}. \) Define \( D_\alpha = \{ z : |z| \leq \beta \} \) and \( A_\alpha = \{ z : \alpha \leq |z| < 1 \}. \)

Let \( w = (w_1, w_2). \) According to Theorem 2.5 and Lemma 3.3 there exist a constant \( C_i \) depending only on modulus of continuity of \( w_i, B, \Gamma, K, \Lambda, \mathcal{L} \) and \( \alpha \) such that
\[ |\nabla w_i(z)| \leq C_i, \ z \in D_\alpha, i = 1, 2. \] (3.12)

By Mori’s theorem, the modulus of continuity of \( w_i \) depends only on \( K \) and \( \alpha. \) Thus
\[ |\nabla w(z)| \leq |\nabla w_1| + |\nabla w_2| \leq C_1 + C_2 = C_3(K, B, \Gamma, \Lambda, \mathcal{L}, \alpha), \ z \in D_\alpha. \] (3.13)
As \( w \) is \( K \) quasiconformal selfmapping of the unit disk, by Mori’s theorem (42) it satisfies the inequality:

\[
4^{1-K} \left| \frac{a - z}{1 - za} \right|^K \leq |w(z)|, \quad |z| < 1,
\]

where \( a = w^{-1}(0) \). Let \( u = |w| \). From Lemma 3.2 and (3.14) we find that

\[
\|L[u]\| \leq 2^{3K-2} \left( \frac{1 + |a|}{1 - |a|} \right)^K \Theta \|\nabla u\|^2 + \Gamma, \quad (1 + |a|)/2 < |z| < 1.
\]

Let \( g \) be a function \( g : A_\alpha \rightarrow \mathbb{R} \) defined as

\[
g(z) = \begin{cases} 
1, & \text{if } \beta < |z| \leq 1; \\
1 + (u(z) - 1) \exp \frac{|z|^2 - \beta^2}{\alpha^2 - \beta^2}, & \text{if } \alpha \leq |z| \leq \beta.
\end{cases}
\]

Define

\[
\phi(z) := \exp \frac{1}{\exp \frac{|z|^2 - \beta^2}{\alpha^2 - \beta^2}}.
\]

Then

\[
L[g] = \begin{cases} 
0, & \text{if } \beta < |z| \leq 1; \\
(u(z) - 1)L[\phi] + \phi L[u] + D[u, \phi], & \text{if } \alpha \leq |z| \leq \beta.
\end{cases}
\]

Therefore

\[
\|L[g]\| \leq \begin{cases} 
0, & \text{if } \beta < |z| \leq 1; \\
B_1 \|\nabla u\|^2 + \Gamma_1, & \text{if } \alpha \leq |z| \leq \beta,
\end{cases}
\]

where

\[
B_1 = 2^{3K-2} \left( \frac{1 + |a|}{1 - |a|} \right)^K (2K\Lambda + B\Lambda)
\]

and \( \Gamma_1 \) is a constant depending only on \( K, B, \Gamma, \Lambda, \Theta \) and \( \alpha \). By (3.4), (3.13) and (3.16) we have

\[
\|L[g]\| \leq C_4(K, B, \Gamma, \Lambda, \Theta, \alpha), \quad z \in A_\alpha
\]

and

\[
\|\nabla g\| \leq C_5(K, B, \Gamma, \Lambda, \Theta, \alpha), \quad z \in A_\alpha.
\]

Furthermore, by using the inequalities (3.15), (3.17), and \( |a + b|^2 \leq 2(|a|^2 + |b|^2) \), we have

\[
\|L[u - g]\| \leq \|L[u]\| + \|L[g]\|
\]

\[
\leq B_1 \|\nabla u\|^2 + C_7(K, B, \Gamma, \Lambda, \Theta, \alpha)
\]

\[
\leq 2B_1 \|\nabla u - \nabla g\|^2 + C_8(K, B, \Gamma, \Lambda, \Theta, \alpha), \quad z \in A_\alpha.
\]

By Mori’s theorem, there exists a constant \( \alpha = \alpha(K, a) < 1 \) such that

\[
M = \max\{|u(z) - g(z)| : z \in A_\alpha\}
\]
is small enough, satisfying the inequality

\[(3.19) \quad \frac{64}{\pi} \cdot 2B \Lambda \Lambda < 1.\]

Thus \(\tilde{u} = u - g\) satisfies the conditions of Theorem 2.8 in the domain \(D = A_{\alpha}\). The conclusion is that \(\nabla u\) is bounded in \(\beta < |z| < 1\) by a constant depending only on \(K, B, \Gamma, \Lambda, \mathcal{L}\) and \(a\) and on the modulus of continuity of \(\tilde{u}\). From Mori’s theorem, the modulus of continuity of \(u\) depends only on \(K\) and \(a\). Combining (3.18) with (3.4), we obtain

\[(3.20) \quad |\nabla w| \leq C_0(K, B, \Gamma, \Lambda, \mathcal{L}, a), \quad \beta < |z| < 1.\]

From (3.13) and (3.20) we obtain the desired conclusion. \(\square\)

**Acknowledgment.** I am thankful to the referee for providing constructive comments and help in improving the contents of this paper.

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