RESUMMED EFFECTIVE ACTION IN INHOMOGENEOUS EXTERNAL FIELD AT ZERO AND FINITE TEMPERATURE

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Abstract

The two ways of resumming the effective action for the massless test particles in inhomogeneous external field at zero and finite temperature providing the infrared finite answer are discussed. The case of the massive test particles having a mass which is parametrically small with respect to a scale set by the inhomogeneous external field is briefly considered.

1To appear in the Proc. Workhop "Quantum Field Theoretical Aspects of High Energy Physics" (Kyffhäuser near Bad Frankenhausen, Germany, September 1993)
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**Abstract**

The two ways of resumming the effective action for the massless test particles in inhomogeneous external field at zero and finite temperature providing the infrared finite answer are discussed. The case of the massive test particles having a mass which is parametrically small with respect to a scale set by the inhomogeneous external field is briefly considered.

1. Below we discuss the ways of operationally defining the contributions to the effective action induced by interaction of a massless test scalar particle with the external field. Namely, we shall be interested in dealing with a well-known problem of the appearance of the infrared-divergent contributions to the effective action when using the expansion in the powers of the external field. The presentation follows the two papers written in collaboration with A. Zelnikov [1,2]. In this contribution we shall also discuss the small-mass expansion in the case when a mass of the test particle is parametrically small with respect to a characteristic scale set by an inhomogeneous external field configuration.

Let us begin by reminding which problem we are going to deal with and consider the simplest case of a scalar field $\varphi(x)$, interacting with some background fields so that a corresponding Lagrangian has the form

$$L[\varphi, V] = \frac{1}{2} \varphi(x) \left(-\Box + m^2 + V(x)\right) \varphi(x)$$  \hspace{1cm} (1)

where $\Box$ is a free Laplacian, $V$ is a potential depending on some inhomogeneous external fields and the corresponding effective action induced by the test particles $\varphi(x)$ (or using the terminology from statistical mechanics, the induced entropy of the external field) at the one loop level is

$$W[V] = -\frac{1}{2} \int_0^\infty \frac{ds}{s} e^{-m^2 s} Tr(e^{-(\Box + V(x))s} - e^{\Box s})$$ \hspace{1cm} (2)

where we have used the proper time representation and the interesting contribution is isolated by subtracting the free propagation contribution. In a situation when the external field potential $V(x)$ is essentially inhomogeneous one is usually confined to using only a certain number of terms in the expansion of the trace of the heat kernel

$$TrK(s) = Tr e^{-(\Box + V(x))s}$$ \hspace{1cm} (3)

in the powers of the proper time $s$

$$TrK(s) = \sum_{n=0}^{\infty} b_{-\omega+n}(V) s^{-\omega+n}$$ \hspace{1cm} (4)

where $2\omega$ is a (eucledian) space-time dimension. We see that in the massless case $m = 0$ starting from some term in the heat kernel expansion the corresponding
effective action is term by term divergent at the upper limit of the integration over \( s \) and therefore the effective action is an operationally ill-defined quantity in the infrared domain. In the case when \( m \) is parametrically small, the contribution of the higher order terms to the effective action is proportional to the increasing inverse powers of \( m \), so that although these are formally infrared finite, the expansion itself has no sense. The same problem arises in the finite temperature calculations of the free energy of a system of massless particles in the inhomogeneous static external field. Namely, we have for the free energy of this system at the one-loop level

\[
- \beta F = \frac{1}{2} \int_0^\infty \frac{ds}{s} \left( 1 + 2 \sum_{n=1}^\infty e^{-\frac{4\pi^2 n^2}{\beta m^2}} T r K_{2\omega-1}(s) \right)
\]

where \( \beta \) is an inverse temperature (let us recall that here the euclidean time is compactified on a circle of a length \( \beta \)). We see that for \( n = 0 \) (entropy) term when using a usual semiclassical expansion for the spatial heat kernel \( K_{2\omega-1}(s) \) we face the same problem of a term by term infrared divergence at the upper limit of the integration over \( s \).

As it is clear that the origin of a problem is in expanding the exponential in (3), the reasonable way to solve it is to make a resummation of the proper time expansion keeping its exponential character and thus providing the infrared convergent answer. In the following we shall consider two such approximations.

The first method corresponds to a summation of some sequence of terms of all orders in the external field potential \( V(x) \) using the analogy between the trace of the heat kernel and the partition function, for which the corresponding resummation has been discussed in the literature [3]. We find that for the simple external scalar field configurations the infrared problem is cured and it is possible to get an infrared-finite answer for the effective action (entropy) already for the simplest of previously discussed resummation for the partition function. The reasonable small mass expansion is also easily obtainable.

The second method corresponds to summing all derivatives for the terms having some definite power in the external field. This corresponds to a calculation of a nonlocal effective action [4-6,2]. In this method it was proved both at zero [4-6] and finite temperature [2] that this procedure allows to get an infrared-finite answer for the effective action (free energy). Below we illustrate this situation - again using the simplest localized scalar field configurations and present some more general expressions for the case of a finite temperature [2].

Let us notice, that formally it is possible to exponentiate the nonlocal terms in the effective action too (at least for scalar [7] and electromagnetic [8] interactions). However it is not clear whether this method could be used for actual computations.

Let us also mention that in estimating the effective action one can use the method of term-by-term infrared regularization of the effective action using the series (4) and optimizing with respect to a cutoff [9], but in this case it seems to be difficult to trace the interrelations between the relevant scales to motivate the expansion parameter.

2. The first possible way of preserving the exponential character of the heat kernel expansion is to keep the interaction potential \( V \) in the exponent as if it is constant, perform the derivative expansion and then integrate over the space-time coordinates. This procedure is completely analogous to the one used in statistical physics [3], where the analogue of the trace of the heat kernel is a partition function, the inverse temperature being the analogue of the proper time. Let us now consider a simplest case of a scalar interaction potential \( V \). Then in our notations the expression for the trace of the heat kernel for the scalar particle propagating in the
external scalar potential \( V(x) \) reads

\[
Tr(K(s) - K_0(s)) = e^{-m^2s} \int d^2x (e^{-V(x)s}(1 - s^2(\frac{1}{6}\Box V(x) - \frac{1}{12}(\nabla V(x))^2 + ...)) = 1) \tag{6}
\]

Let us now consider a localized spherically symmetric static external field configuration in 3 dimensions corresponding to a scalar external field potential

\[
V(r) = \frac{\Phi_0^2}{(1 + \tau/a)^3} \tag{7}
\]

and first calculate the effective action in the zero temperature case. Neglecting for the moment the derivative terms in (6), we get

\[
W[\Phi_0, a, m, M] = -\frac{1}{64\pi^2} r (\frac{4}{3} \pi a^3)(\Phi_0^2 + m^2)^2 \log \frac{M^2}{\Phi_0^2 + m^2} (-1 + \frac{4\Phi_0^2}{\Phi_0^2 + m^2} - \frac{104}{35} \frac{\Phi_0^4}{(\Phi_0^2 + m^2)^2})
\]

\[
-\frac{1}{64\pi^2} r (\frac{4}{3} \pi a^3)m^4 \log \frac{M^2}{m^2} \tag{8}
\]

where \( r \) is an euclidean time and \( M \) is an ultraviolet regulator. Let us notice that at \( \Phi_0 = 0 \) the above expression vanishes. This happens because we have subtracted the free field contribution in (6). Taking into account the derivative terms and keeping the lowest order term in the small mass expansion, we get

\[
W[\Phi_0, a, M] = -\frac{1}{64\pi^2} r (\frac{4}{3} \pi a^3)\Phi_0^4 \frac{1}{35} \log \frac{M^2}{\Phi_0^2} \frac{1}{\Phi_0^2} \Phi_0^2 \frac{1}{3} \left( 1 + \sum_{n=1}^{\infty} (1 + \mu n^2)^2 f(\mu, n) \right) \tag{9}
\]

where \( \mu = \frac{4\pi^2 r^2}{\Phi_0^2} \) and

\[
f(\mu, n^2) = 2F_1(1; -\frac{3}{2}; -\frac{3}{4}; \frac{1}{1 + \mu n^2}) - 2F_1(1; -\frac{3}{2}; -\frac{1}{2}; 1 + \mu n^2)
\]

\[
+ \frac{1}{3} 2F_1(1; -\frac{3}{2}; -\frac{1}{4}; 1 + \mu n^2) - \frac{1}{3} (\frac{\mu n^2}{1 + \mu n^2})^2 \tag{11}
\]

We see that in the proposed approximation for the exponentiated heat kernel the basic difference from the familiar constant external field case (the first term in the above expression would just correspond to a usual effective potential) is the appearance of the effective volume \( a^3 \) of the localized configuration instead of the total spatial volume \( V^3 \). The logarithmic factor still depends only on the external field amplitude and taking into account the terms with derivatives results in the expansion in the inverse powers of \( \Phi_0^2 a^2 \).

Let us now calculate the free energy of a massless scalar field propagating in the static background field configuration Eq.7. We obtain

\[
-\frac{F}{T} = \frac{\Phi_0 a^3}{90} - \frac{2}{3} (\Phi_0 a)^3 \sum_{n=1}^{\infty} (1 + \mu n^2)^2 f(\mu, n) \tag{10}
\]

where \( \mu = \frac{4\pi^2 r^2}{\Phi_0^2} \) and

\[
f(\mu, n^2) = 2F_1(1; -\frac{3}{2}; -\frac{3}{4}; \frac{1}{1 + \mu n^2}) - 2F_1(1; -\frac{3}{2}; -\frac{1}{2}; 1 + \mu n^2)
\]

\[
+ \frac{1}{3} 2F_1(1; -\frac{3}{2}; -\frac{1}{4}; 1 + \mu n^2) - \frac{1}{3} (\frac{\mu n^2}{1 + \mu n^2})^2 \tag{11}
\]

where \( 2F_1(a; b; c; x) \) is a hypergeometric function and for simplicity we took into account only the leading exponential term (the derivative corrections can be derived
in a completely analogous way). From this expression one can work out the low-
and high-temperature expansions in a standard way.

3. Let us now turn to the second possibility of constructing an infrared-safe
approximation for the calculation of the effective action for massless test particles.
This method corresponds to a summation of all derivative terms for a given power of
the external field. The resulting effective action is therefore an essentially nonlocal
object. In a pioneering paper [4] Barvinsky and Vilkovisky have shown that this
procedure provides infrared-convergent integrals for the effective action in all orders
in the external field. Later this technique was generalized to a finite-temperature
case [2]. Below we shall analyse the explicitly nonlocal effective action for the same
case of a massless scalar field propagating in an localized external field configuration
[1] and consider the more general formal expressions for the finite temperature case
[2].

For the trace of the heat kernel we have a general expansion

\[ TrK(s) = \sum_{n=0}^{\infty} TrK_n(s) \]  \hspace{1cm} (12)

where

\[ TrK_n(s) = \frac{s^n}{n} \int_{\alpha \geq 0} d^n \alpha \delta(1 - \sum_{i=1}^{n} \alpha_i) Tr[V e^{s \alpha_1 \Box} ... V e^{s \alpha_n \Box}] \]  \hspace{1cm} (13)

and for the external field potential \( V \) we shall take an \( O(3) \)-symmetric external
field configuration

\[ V = \Phi_0^2 e^{-\frac{r^2}{a^2}} \]  \hspace{1cm} (14)

where the choice of the configuration to consider is dictated by a computational
simplicity. For the effective action at zero temperature we get in the third order in
the external field and keeping the lowest order term in the small mass expansion
perturbation:

\[ W = \frac{1}{64\pi^2} \frac{\tau}{a} (\Phi_0^2 a^2)^2 \log(\frac{1}{M^2 a^2}) [1 + m^2 a^2 + ...] - \frac{c}{192(2\pi)^2} (\frac{\tau}{a} (\Phi_0^2 a^2)^3) \]  \hspace{1cm} (15)

where the constant \( c = 1.30348 \) was obtained by numerical integration. We see
that the basic difference of this answer from that obtained in the first section is
that the charge renormalization logarithm is now saturated by the slope of the
field, and not by its amplitude and that the expansion is now in powers of \( \Phi_0 a \).

For the free energy one gets in the same limit \((m = 0)\)

\[ -\beta F = \frac{\pi}{16} (\Phi_0 a)^4 \]

\[ + \frac{\pi^2}{4} (\Phi_0 a)^4 (aT)^2 \sum_{n=1}^{\infty} n^2 \int_0^1 d\alpha \alpha^{-\frac{3}{2}} (1 - \alpha)^{-\frac{1}{2}} \Psi(\frac{3}{2}, 2, \frac{4\pi^2 a^2 T^2 n^2}{\alpha (1 - \alpha)}) \]  \hspace{1cm} (16)

where \( \Psi(a, c; x) \) is a confluent hypergeometric function. This expression can serve
as a starting point for constructing the low- and high-temperature expansions by
standard methods.

Let us now consider a more general case when we have external scalar and
Yang-Mills fields and develop an expression for the second order effective action in
the second order in the external fields at finite temperature using the method of
derivatives resummation [2]. Due to the lack of space we shall not present here the
details of the computation and just present the final formulas demonstrating the above-described derivative resummation providing the infrared-finite answer.

The formula for the trace of the heat kernel considered up to the second order in the external field has a form

\[
\text{Tr } K(s) = \frac{1}{(4\pi s)^{1/2}} \int d^{2\omega-1}x \sum_{n=-\infty}^{\infty} \text{tr} [I + s\Phi - 2s\Gamma_0^2\beta^2 \frac{\partial}{\partial \beta^2}] + s^2(\Phi f_4(-s\Box)\Phi + G_{\mu\nu}f_5(-s\Box)G_{\mu\nu} + 4G_{\mu0}f_5(-s\Box)G_{\mu\nu}\beta^2 \frac{\partial}{\partial \beta^2}) \exp(-\frac{\beta^2}{4s}n^2) (17)
\]

where \( \Phi \) is an external scalar field, \( G_{\mu\nu} \) is an external Yang-Mills field, \( \Gamma_0 = \nabla_\mu (1/\Box)G_{\mu0} \) and the formfactors \( f \) are the same as introduced in [6]:

\[
f_4 = \frac{1}{2}f, \quad f_5 = -\frac{1}{2}f - \frac{1}{2}, \quad f(-s\Box) = \int_0^1 d\alpha \exp(\alpha(1-\alpha)s\Box) (18)
\]

The corresponding \( \zeta \)-regularized expression for the effective action in four spacetime dimensions reads [2]

\[
W_2 = -\int d^{2\omega-1}x \sum_{n=-\infty}^{\infty} \int_0^1 d\alpha \text{tr} \left[ \frac{1}{4(4\pi)^{1/2}} \Phi[(n^2 + B)^{1/2} - |n|]\Phi + \frac{1}{2\beta^2}G_{\mu\nu}[(n^2 + B)^{1/2} - |n|] - \frac{B}{2|n|}G_{\mu\nu} \right.
\]

\[
+ \left. \beta^2 \frac{\partial}{\partial (\beta^2)} \frac{1}{2\beta^2}G_{\mu0}[(n^2 + B)^{1/2} - |n|] - \frac{B}{2|n|}R_{\mu0} \right] + \Delta W_2 (19)
\]

where the explicit expression for the local contribution for the effective action \( \Delta W_2 \) arising due to the necessity of regularizing the Matsubara frequency series can be found in [2]. The answer for the third order term will have the same character (with the more complicated structure of the parametric integration). The exponential formfactors in the trace of the heat kernel (13) sandwiched between the external fields provide the infrared finiteness of the effective action.

**Acknowledgments**

The author is grateful to Prof. H. Satz for kind hospitality at the University of Bielefeld and for A.v.Humboldt Foundation for financial support. His work was partially supported by the Russian Fund for Fundamental Research, Grant 93-02-3815.

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