Nonlinear perturbations of higher dimensional anti-de Sitter spacetime

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Abstract

We study nonlinear gravitational perturbations of vacuum Einstein equations, with $\Lambda < 0$ in $(n + 2)$ dimensions, with $n > 2$, the $n = 2$ case already having been done before. We follow the formalism by Ishibashi, Kodama and Seto to decompose the metric perturbations into tensor, vector and scalar sectors, and simplify the Einstein equations. We render the metric perturbations asymptotically anti-de Sitter by employing a suitable gauge choice for each of the sectors. Finally, we analyze the resonant structure of the perturbed equations at second order for the five dimensional case, by starting out with tensor-type perturbations at the linear level. We find evidence for resonances at second order.

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I Introduction

The stability of the three maximally symmetric solutions to the vacuum Einstein equations (with cosmological constant) has been studied extensively. Minkowski and de Sitter spacetime have been found to be nonlinearly stable under small perturbations \[1\], \[2\]. Although Anti de Sitter (AdS) spacetime is stable under linearized perturbations, it was conjectured to be nonlinearly unstable with reflecting boundary conditions at the AdS boundary \[3\]. In \[4\], Bizon and Rostworowski did a numerical study, which involved spherically symmetric evolution of a massless scalar field in four dimensions, with \(\Lambda < 0\). The end point of the evolution was a black hole, indicating the nonlinear instability of the system. These results hold true for higher dimensions as well \[7\].

The AdS instability was also observed in the evolution of complex scalar fields \[6\]. The evolution of a massless scalar field in Gauss Bonnet gravity was numerically studied in \[7\], \[8\]. Further, renormalization group methods \[9\], \[10\] and the two-time framework (TTF) \[11\], \[12\], \[13\] were used to study the instability of AdS. An interacting scalar field in AdS was investigated in TTF in \[14\].

Certain systems like a massless scalar field enclosed in a cavity in Minkowski \[15\] and massive fields in AdS \[16\], were thought to exhibit an AdS-like instability, although the linear spectra was non-resonant. This led to the question of whether resonant spectra are required for a turbulent instability. It was later seen that there is a minimum amplitude required to trigger instability in such cases \[17\]. This minimum amplitude was too small to be observed initially in numerical studies. The reasons for these observations were discussed in \[18\].

Non collapsing solutions were studied for asymptotically AdS spacetimes in \[17\], \[19\], \[20\], \[21\], \[22\], \[23\] and \[24\].

Finally, the proof of AdS instability for the Einstein-massless Vlasov system in spherical symmetry was given by Moschidis \[25\].

Non spherically symmetric collapse was also studied in \[26\] with massless scalar field in five dimensions. It was seen that the configuration collapsed faster than the spherically symmetric case. Rotational dynamics of \(AdS_5\) was studied in \[27\] in presence of a complex doublet scalar field.

Study of pure gravitational perturbations naturally involves breaking of spherical symmetry. In this regard, numerical evolution for pure gravity was done in \(AdS_5\) using the cohomogeneity-two biaxial Bianchi IX ansatz \[28\].
A more general breakdown of spherical symmetry was done in AdS, \([29]\), where time periodic solutions called geons were constructed. Geons were also constructed in \([30], [31], [32], [33], [34], [35], [36] and [37]\). Nonlinear perturbation theory was employed to study pure gravitational perturbations in a cavity in Minkowski, in general dimensions \([38]\). Very recently, the resonant system was derived in five dimensions, within cohomogeneity-two biaxial Bianchi IX ansatz \([39]\).

In this work, we extend the work of systematically studying gravitational perturbations of AdS using nonlinear perturbation theory in four dimensions \([32]\), to general \((n + 2)\) dimensions \((n > 2)\). In section II, we give an overview of the methods used to study pure gravitational perturbations of AdS\(_{n+2}\). We use the Kodama-Ishibashi formalism \([40]\) to study nonlinear perturbations, by extending it beyond the linear level. Section III holds a brief discussion on how the metric perturbations and hence the source terms fall off, upon imposing asymptotic AdS boundary conditions. Section IV contains simplification of higher order equations. In section V-VII we systematically construct metric perturbations and render them asymptotically AdS through suitable gauge choices. Section VIII contains calculation of \(l_s = 0, 1\) scalar modes as well as \(l_v = 1\) vector modes. These modes, which are gauge at linear level, are in fact physical perturbations at higher orders. In Section IX, we look at the nature of secular resonances arising in dimension five, which corresponds to \(n = 3\). Finally, section X contains the summary and discussion of this paper.

**II \hspace{1em} Methodology**

The vacuum Einstein equation with negative cosmological constant (i.e. \(\Lambda < 0\)) in \((n+2)\) spacetime dimensions is given by

\[
R_{\mu\nu} + \frac{(n + 1)}{L^2} g_{\mu\nu} = 0 \tag{II.1}
\]

where \(L^2 = -\frac{n(n+1)}{2\Lambda}\). We are interested in the solutions of the above equation when the metric \(g_{\mu\nu}\) is expanded around the AdS metric, which we will refer to as the background metric, given by

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_n^2, \quad f(r) = 1 + \frac{r^2}{L^2} \tag{II.2}
\]
where \( d\Omega_n^2 = \gamma_{ij}(w)dw^i dw^j \) is the metric for \( n \)-sphere. We are interested in generic perturbations about the background metric to higher orders in perturbation theory. In the equations that follow, we employ the notation used by [32]. Hence the ‘bar’ quantities refer to the background AdS geometry. Next we expand the solutions of (II.1) as \( g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \), where

\[
\delta g_{\mu\nu} = \sum_{1 \leq i}^{(i)} h_{\mu\nu} e^i
\]  

(II.3)

Then the inverse metric can be represented as

\[
g_{\alpha\beta} = (\bar{g}^{-1} - \bar{g}^{-1} \delta g \bar{g}^{-1} + ...)^{\alpha\beta}
\]

(II.4)

The Christoffel symbol as:

\[
\Gamma^\alpha_{\mu\nu} = \bar{\Gamma}^\alpha_{\mu\nu} + \frac{1}{2}(\bar{g}^{-1} - \bar{g}^{-1} \delta g \bar{g}^{-1} + ...)^{\alpha\beta}(\bar{\nabla}_\mu \delta g_{\beta\nu} + \bar{\nabla}_\nu \delta g_{\beta\mu} - \bar{\nabla}_\beta \delta g_{\mu\nu})
\]

(II.5)

And the Ricci tensor as:

\[
R_{\mu\nu} = \bar{R}_{\mu\nu} + \bar{\nabla}_\alpha \delta \Gamma^\alpha_{\mu\nu} - \bar{\nabla}_\nu \delta \Gamma^\alpha_{\alpha\mu} + \delta \Gamma^\alpha_{\alpha\lambda} \delta \Gamma^\lambda_{\mu\nu} - \delta \Gamma^\lambda_{\mu\alpha} \delta \Gamma^\alpha_{\lambda\nu}
\]

(II.6)

The perturbed Einstein’s equation is given by

\[
^{(i)}R_{\mu\nu} + \frac{(n + 1)}{L^2}^{(i)}h_{\mu\nu} = 0
\]  

(II.7)

Before writing down our working equations, we define the following two quantities: The Lorentzian Lichnerowicz operator \( \triangle_L \), which is given as

\[
2 \triangle_L^{(i)} h_{\mu\nu} = - \nabla^\alpha \nabla_\alpha^{(i)} h_{\mu\nu} - \nabla_\mu \nabla_\nu^{(i)} h + \nabla_\mu \nabla_\alpha^{(i)} h^\alpha_{\nu} + \nabla_\nu \nabla_\alpha^{(i)} h^\alpha_{\mu}
\]

\[
+ \bar{R}_{\mu\nu}^{(i)} h^\alpha_{\nu} + \bar{R}_{\nu\alpha}^{(i)} h^\alpha_{\mu} - 2 \bar{R}_{\mu\nu\lambda\alpha}^{(i)} h^\alpha_{\lambda\mu}
\]  

(II.8)

and \( ^{(i)}A_{\mu\nu} \) defined as

\[
^{(i)}A_{\mu\nu} = \{ \nabla_\nu \left[ (\bar{g}^{-1} \delta g \bar{g}^{-1} + ...)^{\alpha\lambda}(\nabla_\mu \delta g_{\lambda\nu} + \nabla_\nu \delta g_{\lambda\mu} - \nabla_\lambda \delta g_{\mu\nu}) \right] 
\]

\[
+ \nabla_\mu \left[ (\bar{g}^{-1} \delta g \bar{g}^{-1} + ...)^{\alpha\lambda}(\nabla_\nu \delta g_{\lambda\mu} + \nabla_\mu \delta g_{\lambda\nu} - \nabla_\lambda \delta g_{\nu\mu}) \right]
\]

\[
- 2 \delta \Gamma^\lambda_{\alpha\lambda} \delta \Gamma^\alpha_{\mu\nu} + 2 \delta \Gamma^\lambda_{\mu\alpha} \delta \Gamma^\alpha_{\lambda\nu}
\}
\]

(II.9)
where \([\epsilon^i f]\) denotes the coefficient of \(\epsilon^i\) in the expansion of the power series \(\sum_i [\epsilon^i] f_i\). By using (II.1) and (II.7) one can easily obtain

\[
(i) G_{\mu\nu} = 2 \tilde{\Delta}_L (i) h_{\mu\nu} - (i) S_{\mu\nu} = 0
\]

(II.10)

where \(\tilde{\Delta}_L (i) h_{\mu\nu}\) is given as

\[
2 \tilde{\Delta}_L (i) h_{\mu\nu} = 2 \Delta_L (i) h_{\mu\nu} + \frac{2(n + 1)}{L^2} (i) h_{\mu\nu} - \bar{g}_{\mu\nu} (\bar{g}^{\alpha\beta} \Delta_L (i) h_{\alpha\beta} - (i) h_{\alpha\beta} R^{\alpha\beta})
\]

(II.11)

and \((i) S_{\mu\nu}\) is given in terms of \((i) A_{\mu\nu}\) is defined as

\[
(i) S_{\mu\nu} = (i) A_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{g}^{\alpha\beta} (i) A_{\alpha\beta}
\]

(II.12)

The background metric \(\bar{g}_{\mu\nu}\) is spherically symmetric and is of the form

\[
ds^2 = \bar{g}_{\mu\nu} dz^\mu dz^\nu = g_{ab} dy^a dy^b + r^2(y) d\Omega_n^2
\]

(II.13)

One can use the gauge invariant formalism given by Ishibashi, Kodama and Seto [40] to study the perturbations around such a background metric and we will be extending the same to higher orders in perturbations theory as well. Let the covariant derivative associated with \(ds^2\), \(g_{ab} dy^a dy^b\) and \(d\Omega_n^2\) be \(\nabla_M\), \(\bar{D}_a\) and \(\bar{D}_i\) respectively. The metric perturbations \((i) h_{\mu\nu}\) are decomposed according to their behaviour on the \(n\)-sphere i.e. into the scalar type, \(S\), the vector type, \(V_i\) and the tensor type, \(T_{ij}\). In the following sections, \(\Delta = \bar{D}^i \bar{D}_i\) where raising (and lowering) of the sphere indices is done with \(\gamma_{ij}\). The scalar harmonics \(S\) satisfy

\[
(\hat{\Delta} + k_s^2) S = 0
\]

(II.14)

where \(k_s^2 = l_s (l_s + n - 1)\) and \(l_s = 0, 1, ..., \) from where one can construct \(S_i\) and \(S_{ij}\)

\[
S_i = -\frac{1}{k_s} D_i S; \quad S_{ij} = \frac{1}{k_s^2} D_i D_j S + \frac{1}{n} \gamma_{ij} S
\]

(II.15)

which satisfy

\[
\bar{D}^i S_i = k_s S; \quad S_i^i = 0; \quad \bar{D}_k S^i_i = \frac{(n - 1)(k_s^2 - n)}{nk_s} S_i.
\]

(II.16)
Vector harmonics $V_i$ are defined as

$$(\hat{\Delta} + k^2 V) V_i = 0 \quad (II.17)$$

where $k^2 = l_v(l_v + n - 1) - 1$ and $l_v = 1, 2, \ldots$, such that

$$\bar{D}_i V^i = 0. \quad (II.18)$$

From $V_i$, one can construct tensors $V_{ij}$

$$V_{ij} = -\frac{1}{2k_v}(\bar{D}_i V_j + \bar{D}_j V_i) \quad (II.19)$$

which satisfy

$$V^i_i = 0; \quad D_j V^j_i = 0 \quad (II.20)$$

Tensor type harmonics, $T_{ij}$ are defined as

$$(\hat{\Delta} + k^2) T_{ij} = 0 \quad (II.21)$$

where $k^2 = l(l + n - 1) - 2$ and $l = 2, 3, \ldots$. They satisfy

$$T^i_i = 0; \quad D_j T^j_i = 0 \quad (II.22)$$

The metric perturbations can be now be expanded as

$$^{(i)}h_{ab} = \sum_{k_a} \sum_{k_b} f_{ab,ks} \delta_{ks}; \quad ^{(i)}h_{ai} = r \left( \sum_{k_a} f_{ak,ks} \delta_{ks} + \sum_{k_v} f_{av,ks} \delta_{kv} \right)$$

$$^{(i)}h_{ij} = r^2 \left( \sum_k f_{Tk,ks} \delta_{ks} + 2 \sum_k f_{Tk,sv} \delta_{sv} + 2 \sum_k f_{Tk,sv} \delta_{sv} \right) \quad (II.23)$$

The metric components are also gauge dependent. Under an infinitesimal gauge transformation $\delta z^a = \sum_i ^{(i)}\zeta^a$, metric perturbation $^{(i)}h_{\mu\nu}$ transforms as

$$^{(i)}h_{\mu\nu} \rightarrow ^{(i)}h_{\mu\nu} - \nabla_\mu ^{(i)}\zeta_\nu - \nabla_\nu ^{(i)}\zeta_\mu \quad (II.24)$$
Let \((i)\zeta_a = (i)T_a S\) and \((i)\zeta_i = r (i)M S_i + r (i)M^v \nabla_i\). Thus the gauge transformations for \((i)f_{ab}, (i)f^a, (i)f^v, (i)H_T, (i)H_T^v\) and \((i)H_T\) are

\[
(i) f_{ab} \rightarrow (i) f_{ab} - \bar{D}_a (i) \zeta_b - \bar{D}_b (i) \zeta_a \\
(i) f^a \rightarrow (i) f^a - r \bar{D}_a \left( \frac{(i)M}{r} \right) + \frac{k_a}{r} (i) T_a \\
(i) H_L \rightarrow (i) H_L - \frac{k_s}{n r} (i) M - \frac{\bar{D}^a r (i) T_a}{r} \\
(i) H_T^v \rightarrow (i) H_T^v + \frac{k_v}{r} (i) M^v \\
(i) f^v \rightarrow (i) f^v - r \bar{D}_a \left( \frac{(i)M^v}{r} \right) \\
(i) H_T \rightarrow (i) H_T
\]

For all cases except \(l_s = 0, 1\) and \(l_v = 1\) modes, one can define the following gauge invariant variables.

\[
(i) Z_a = (i)f^v + \frac{r}{k_v} \bar{D}_a (i) H_T^v
\]

\[
(i) F_{ab} = (i)f_{ab} + \frac{1}{2} \bar{D}_{(a} (i) X_{b)}; \quad (i) F = (i) H_L + \frac{(i) H_T^v}{n} + \frac{1}{r} \bar{D}^a r (i) X_a
\]

where

\[
(i) X_a = \frac{r}{k_s} \left( (i)f^s + \frac{r}{k_s} \bar{D}_a (i) H_T^s \right)
\]

It is possible to write the \(\bar{\Delta}_L\) operator in (II.10) solely in terms of these gauge invariant variables [30]. The strategy is to solve for these variables and add suitable gauge transformations to the metric perturbations to render them asymptotically AdS (aAdS).
III Asymptotic nature of source terms

In order to ensure that the metric perturbations are asymptotically AdS, the leading order behaviour of \( \delta g_{\mu\nu} \), has to be \([42],[44],[45] \):

\[
\begin{align*}
\delta g_{rr} &\sim \frac{1}{r^{n+3}}; \quad \delta g_{r\gamma} \sim \frac{1}{r^{n+2}}; \quad \delta g_{\gamma\sigma} \sim \frac{1}{r^{n-1}} \\
\end{align*}
\]

(III.36)

where \( \sigma, \gamma \neq r \). Given this, one can see that the leading order behaviour of the inverse \( \delta g^{\mu\nu} \) should be:

\[
\begin{align*}
\delta g^{rr} &\sim \frac{1}{r^{n-1}}; \quad \delta g^{r\gamma} \sim \frac{1}{r^{n+2}}; \quad \delta g^{\gamma\sigma} \sim \frac{1}{r^{n+3}} \\
\end{align*}
\]

(III.37)

The metric perturbations at higher orders depend on the sources as well. Therefore while making the relevant gauge choice it is also important to take into account the fall off of the sources \((i)S_{\mu\nu}\). Using (II.9), (III.36) and (III.37), one can deduce their leading order behaviour, which is as follows:

\[
\begin{align*}
(i)S_{rr} &\sim \frac{1}{r^{2n+4}}; \quad (i)S_{r\gamma} \sim \frac{1}{r^{2n+3}}; \quad (i)S_{\gamma\sigma} \sim \frac{1}{r^{2n}} \\
\end{align*}
\]

(III.38)

IV Linear and higher order equations

The tensor, vector and scalar sectors decouple completely at the linear level and are discussed in [43]. The higher order perturbed equations, (II.10) can be obtained in terms of the gauge invariant variables, \((i)H_T, (i)F, (i)F_{ab}\) and \((i)Z_a\). For eg., for \(\mu = a, \nu = i\), (II.10) takes the form,

\[
\begin{align*}
\sum_{k_v} \left[ -\frac{1}{r^n} \tilde{D}_b \left\{ r^{n+2} \left[ \tilde{D}_b \left( \frac{(i)Z_a}{r} \right) - \tilde{D}_a \left( \frac{(i)Z_b}{r} \right) \right] \right\} + \frac{k_v^2 - (n - 1)K}{r} (i)Z_a \right] V_{ki} \\
+ \sum_{k_s} \left[ -k_s \left( \frac{1}{r^{n-2}} \tilde{D}_b (r^{n-2}(i)F^b_a) - r \tilde{D}_a \left( \frac{1}{r} (i)F^b_b \right) - 2(n - 1) \tilde{D}_a (i)F \right] S_{ki} \\
= (i)S_{ai}. \\
\end{align*}
\]

(IV.39)

In order to decompose the various sectors we use the fact that

\[
\int T^{ij} \nabla_i d^n\Omega = \int T^{ij} S_{ij} d^n\Omega = \int V^{ij} S_{ij} d^n\Omega = \int V^{i} S_{ij} d^n\Omega = 0. 
\]

(IV.40)
After doing the necessary projections, one gets the following equations (see [38] for more details). The higher order tensor sector takes the form:

\[-r^2 \tilde{D}^a \tilde{D}_a^{(i)} H_T - nr \tilde{D}^a r \tilde{D}_a^{(i)} H_T + (k^2 + 2)^{(i)} H_T = \int \Theta^{ij}(i) S_{ij} d^n \Omega. \quad (IV.41)\]

We obtain the two equations pertaining to vector sector:

\[-\frac{1}{r^n} \tilde{D}^b \left\{ r^{n+2} \left[ \tilde{D}_b \left( \frac{(i)Z_a}{r} \right) - \tilde{D}_a \left( \frac{(i)Z_b}{r} \right) \right] \right\} + \frac{k^2_v - (n - 1)^{(i)}Z_a}{r} = \int \Psi_{k_v}^{ij}(i) S_{ij} d^n \Omega, \quad (IV.42)\]

\[-\frac{2k_v}{r^{n-2}} \tilde{D}_a (r^{n-1}(i)Z^a) = \int \chi_{k_v}^{ij}(i) S_{ij} d^n \Omega. \quad (IV.43)\]

For the scalar sector, there are three equations, (quantities with superscript \( m \) on the left side are defined for the metric \( g_{ab} \))

\[-\tilde{D}_c \tilde{D}_c^{(i)} F_{ab} + \tilde{D}_a \tilde{D}_c^{(i)} F^c_b + \tilde{D}_b \tilde{D}_c^{(i)} F^c_a + \frac{D^r}{r} (\tilde{D}_c^{(i)} F_{ab} + \tilde{D}_a^{(i)} F_{cb} + \tilde{D}_b^{(i)} F_{ca}) + m R_a^{(i)} F_{cb} + m R_c^{(i)} F_{ca} - 2m R_{a c b d}^{(i)} F^{c d} + \left( \frac{k^2_s}{r^2} + \frac{2(n + 1)}{L^2} \right)^{(i)} F_{a b} - \tilde{D}_a \tilde{D}_{b}^{(i)} F_{c}^{d} - 2n \left( \tilde{D}_a \tilde{D}_{b}^{(i)} F + \frac{1}{r} \tilde{D}_a r \tilde{D}_{b}^{(i)} F + \frac{1}{r} \tilde{D}_b r \tilde{D}_{a}^{(i)} F \right) - \left( \tilde{D}_c \tilde{D}_{d}^{(i)} F^{c d} + \frac{2n}{r} \tilde{D}_c r \tilde{D}_{d}^{(i)} F^{c d} + \left( \tilde{D}_c^{(i)} F_{a b} + \frac{n(n + 1)}{r^2} \tilde{D}_c r \tilde{D}_{a}^{(i)} F + 2(n - 1) \frac{(k^2_s - n)}{r^2} (i) F - \tilde{D}_c \tilde{D}_{c}^{(i)} F_{d}^{d} \right) \right) \]

\[-\frac{n}{r} \tilde{D}_c r \tilde{D}_{c}^{(i)} F_{d}^{d} + \frac{k^2_s}{r^2} (i) F_{d}^{d} \right) g_{a b} = \int S_{k_s}^{(i)} S_{a b} d^n \Omega, \quad (IV.44)\]

\[-k_s \left( \frac{1}{r^{n-2}} \tilde{D}_b (r^{n-2}(i) F_{a}^{b}) - r \tilde{D}_a \left( \frac{1}{r} (i) F_{b}^{a} \right) - 2(n - 1) \tilde{D}_a^{(i)} F \right) = \int S_{k_s}^{(i)} S_{a b} d^n \Omega, \quad (IV.45)\]
\[-k^2_s[2(n-2)^iF + (i)F^c_c] = \int S_{kj}^{ij}(i)S_{ij}d^n\Omega.\] (IV.46)

In the following sections we will drop the superscript (1) on metric $h^{1\mu\nu}$, while considering the leading order perturbations.

V Tensor perturbations

V.1 At linear level

Tensor perturbations are solely present in the $\delta g_{ij}$ component i.e. $h_{ij} = r^2H_Tk_Tk_{ij}$. Following [43], we let $H_T = r^{-n/2}\Phi_T$. Then the tensor perturbations at leading order are governed by:

$$\ddot{\Phi}_T - f^2\Phi''_T - f'f\Phi'_T + \left(\frac{n(n-2)}{4} f^2 + \frac{n}{2r} f' f + \frac{l(l+n-1)}{r^2}\right) \Phi_T = 0$$

(V.47)

Substituting the ansatz $\Phi_T = \cos(\omega t + b)$ in (V.47) we get:

$$\hat{L}\phi = \omega^2 \phi$$

(V.48)

where $\hat{L}$ is given by

$$\hat{L} = -f^2\partial^2_r - f'f\partial_r + \left(\frac{n(n-2)}{4} f^2 + \frac{n}{2r} f' f + \frac{l(l+n-1)}{r^2}\right)$$

(V.49)

The eigensolutions which ensure appropriate aAdS conditions at the boundary are given by

$$\phi = e_{p,l} = d_{p,l} \frac{L^{\frac{1}{2}+\nu}r^{\frac{1}{2}+\sigma}}{(r^2 + L^2)^{\frac{(\nu+\sigma+1)}{2}}} 2F_1\left(\zeta^\omega_{\nu,\sigma}, \zeta^{-\omega}_{\nu,\sigma}, 1 + \nu; \frac{L^2}{(r^2 + L^2)}\right)$$

(V.50)

where $\zeta^\omega_{\nu,\sigma} = \nu + \sigma + \omega L + 1$, $\nu = \frac{(n+1)}{2}$ and $\sigma = l + \frac{(n-1)}{2}$

The eigenfrequencies $\omega$ are determined by imposing regularity of $\phi$ at the origin, which gives us

$$\omega L = 2p + l + n + 1; \quad p = 0, 1, 2...$$

(V.51)

The eigenfunctions $e_{p,l}$ form a complete orthogonal set w.r.t the inner product

$$<e_{p,l}, e_{p',l}> = \int_0^\infty e_{p,l}e_{p',l}w(r)dr = \delta_{p'}^p$$

(V.52)
where \( w(r) \) is the appropriate weight function given by

\[
w(r) = \frac{1}{f}
\]  

(V.53)

Hence the normalization constant \( d_{p,l} \) is given by

\[
d_{p,l} = \left[ \frac{2 (2p + l + n + 1) \Gamma(p + l + n + 1)}{L^p p! \Gamma(p + l + \frac{n+1}{2}) \Gamma(p + \frac{n+3}{2})} \right]^{1/2} \left( \frac{n + 3}{2} \right)_p
\]  

(V.54)

### V.2 Higher orders

To study higher order tensor perturbations, we let \(^{(i)}H_{T_k} = r^{-\frac{n}{2}}^{(i)}\Phi_{T_k} \) in (IV.41), which leads to

\[
^{(i)}\ddot{\Phi}_T + \mathcal{L}_v^{(i)}\Phi_T = r^{\frac{n}{2} - 2} f \int T^{ij}S_{ij} d^n\Omega
\]  

(V.55)

Just like linearized perturbations, the leading order fall of \(^{(i)}\Phi_T\) at asymptotic infinity is

\[
^{(i)}\Phi_T \sim \frac{1}{r^{\frac{n}{2}+1}}
\]  

(V.56)

This automatically ensures the correct leading order asymptotic behaviour of the tensor sector of \( \delta g_{ij} \), which is \( \sim r^{-(n-1)} \). Thus the tensor sector at all orders are already in aAdS form. Further simplification of the equations can be done by using the orthonormality and completeness of eigenfunctions. The completeness of \( e_{p,l} \) allows one to write \(^{(i)}\Phi_T\) as

\[
^{(i)}\Phi_{T_k} = \sum_{p=0}^{\infty} \langle^{(i)}c_{p,k}(t) e_{p,l}(r), \rangle
\]

so that \(^{(i)}c_{p,k}\) satisfies:

\[
^{(i)}\ddot{c}_{p,k}(t) + \omega^{2(i)}c_{p,k}(t) = \langle r^{\frac{n}{2} - 2} f \int T^{ij}S_{ij} d^n\Omega, e_{p,l} \rangle
\]  

(V.57)

### VI Vector perturbations

#### VI.1 Linear level

The following two independent equations govern vector type perturbations.

\[
\dot{Z}_l - f^2 Z'_r - f' f Z_r - \frac{(n-1) f^2}{r} Z_r = 0
\]  

(VI.58)
\[
\frac{r}{f} \dddot{Z}_r - \frac{r}{f} \dddot{Z}_t + \frac{1}{f} \dddot{Z}_t + \left( \frac{k_v^2 - (n-1)}{r} \right) Z_r = 0 \quad \text{(VI.59)}
\]

From the above equations, one can obtain the following master equation in variable \(\Phi_{v k_v}\), which is defined as

\[
Z_{r k_v} = f^{-1} r^{-\frac{(n-2)}{2}} \Phi_{v k_v}.
\]

\[
\dddot{\Phi}_v - f^2 \Phi''_v - f' f \Phi'_v + \left( \frac{n(n+1)}{4} \frac{f^2}{r^2} + (l_v(l_v + n - 1) - n) \frac{f}{r^2} - \frac{n f' f'}{2} \right) \Phi_v = 0
\quad \text{(VI.60)}
\]

By letting \(\Phi_v = \cos(\omega t + b) \phi_v\), (VI.60) becomes

\[
\hat{L}_v \phi_v = \omega^2 \phi_v
\quad \text{(VI.61)}
\]

Here, \(\hat{L}_v\) is given by

\[
\hat{L}_v = -f^2 \partial_r^2 - f f' \partial_r + \left( \frac{n(n+1)}{4} \frac{f^2}{r^2} + (l_v(l_v + n - 1) - n) \frac{f}{r^2} - \frac{n f' f'}{2} \right)
\quad \text{(VI.62)}
\]

The eigensolutions which ensure appropriate aAdS conditions at the boundary are given by

\[
\phi_v = c_{p,l_v}^{(v)} = d_{p,l_v}^{(v)} \frac{L^{1/2} \nu_v r^{1/2 + \sigma_v}}{(r^2 + L^2)^{1/2 \nu_v + \sigma_v + 1/2}} F_1 \left( \zeta_{\nu_v,\sigma_v}, \zeta_{\nu_v,\sigma_v}, 1 + \nu_v; \frac{L^2}{(r^2 + L^2)} \right)
\quad \text{(VI.63)}
\]

where \(\sigma_v = l_v + \frac{(n-1)}{2}\), \(\nu_v = \frac{(n-1)}{2}\) and \(\zeta_{\nu_v,\sigma_v} = \frac{\nu_v + \sigma_v + \omega_v L^2}{2}\). Regularity of the eigensolution at origin sets the eigenfrequencies to be

\[
\omega_v L = 2p + l_v + n; \quad p = 0, 1, 2...
\quad \text{(VI.64)}
\]

The vector modes also form a complete orthogonal set with an inner product

\[
< c_{p,l_v}^{(v)}, c_{p',l_v}^{(v)} >_v = \int_0^\infty c_{p,l_v}^{(v)} c_{p',l_v}^{(v)} w_v(r) dr = \delta_{p'}
\quad \text{(VI.65)}
\]

where the weight function \(w_v(r)\) is given by

\[
w_v(r) = \frac{1}{f}
\quad \text{(VI.66)}
\]

Hence the normalization constant \(d_{p,l_v}^{(v)}\) is fixed as

\[
d_{p,l_v}^{(v)} = \left[ \frac{2 (2p + l_v + n) \Gamma(p + l_v + n)}{L p \Gamma(p + l_v + \frac{n+1}{2}) \Gamma(p + \frac{1}{2})} \right]^{1/2} \left( \frac{n+1}{2} \right)_p
\quad \text{(VI.67)}
\]
VI.2 Higher orders

The higher order vector equations will be given in terms of the following defined quantities:

\[(i) \dot{V}_{s1k_v} = \int V_{k_v}^{ij} (i)S_{ij} d^n\Omega \quad \text{(VI.68)}\]

\[(i) \dot{V}_{s2k_v} = \int V_{k_v}^i (i)S_{ir} d^n\Omega \quad \text{(VI.69)}\]

\[(i) \dot{V}_{s3k_v} = \left[ (i) \dot{V}_{s2k_v} - \frac{(i) \dot{V}_{s1k_v}}{k_v r} + \frac{1}{2k_v f} (f^{(i)} V_{s1k_v})' \right] \quad \text{(VI.70)}\]

Then the relevant equations are:

\[
(i) \dot{Z}_t = f^{2(i)} Z'_r + f' f^{(i)} Z_r + (n - 1) \frac{f^2}{r} (i) Z_r + \frac{f^{(i)} V_{s1}}{2k_v r} \quad \text{(VI.71)}
\]

\[
\frac{r}{f} (i) \ddot{Z}_v - \frac{r}{f} (i) \dot{Z}'_v + \frac{1}{f} (i) \dot{Z}_v + \frac{k_v^2 - (n - 1)}{r} (i) Z_r = (i) \dot{V}_{s2} \quad \text{(VI.72)}
\]

To get the master equation, we substitute the expression for \((i) \dot{Z}_t\) given by (VI.71) in (VI.72) gives the following:

\[
\frac{r}{f} (i) \ddot{Z}_v - r f^{(i)} Z''_r - (3r f' + (n - 2)f) (i) Z'_r + \left( -r f'' - \frac{r}{f} (f')^2 \right.

\left. - (2n - 3)f' + 2(n - 1) \frac{f}{r} + \frac{k_v^2 - (n - 1)}{r} \right) (i) Z_r = (i) \dot{V}_{s3} \quad \text{(VI.73)}
\]

Letting \((i) Z_{T_{k_v}} = f^{-1} r^{-(n-2)} (i) \Phi_{v_{k_v}}\) in (VI.73) gives us a master equation for \((i) \Phi_v:\)

\[
(i) \ddot{\Phi}_v + \hat{L}_v (i) \Phi_v = r^{\frac{n-2}{2}} f^2 (i) \dot{V}_{s3} \quad \text{(VI.74)}
\]

Further, to construct asymptotically AdS solutions to all orders, we consider the class of metric perturbations (with suitable gauge choice) where \((i) H_{p}^{(i)} = \)
0). For such a class, \( f_a \) in (II.23) is simply: \( f_a = (i)Z_a \). Hence the metric perturbations, along with their gauge transformations take the following form (summation over \( k_v \) on the R.H.S. of the following equations is implied):

\[
(i)h_{ri} = r^2 \left[ \frac{2 - \hat{f} f^{-1}(i)\Phi_v - r^2 \bar{D}_r \left( \frac{(i)M(v)}{r} \right)}{r} \right] \mathbb{V}_i \tag{VI.75}
\]

\[
(i)h_{ti} = \left[ \int^t \left( \frac{f}{r^{n-2}} \left( r^2 \frac{(i)\Phi_s}{2k_v} \right)' + \frac{f}{2k_v} (i)V_{s1} \right) dt - r(i)M(v) \right] \mathbb{V}_i \tag{VI.76}
\]

\[
(i)h_{ij} = r k_v (i)m_v \mathbb{V}_{ij} \tag{VI.77}
\]

**Gauge choice for vector perturbations:**

In order to define \( (i)M(v) \) appropriately, we notice from (VI.77) that since \( (i)h_{ij} \) needs to fall off like \( r^{-(n-1)} \), \( (i)M(v) \) should have an expansion of the following form at asymptotic infinity:

\[
(i)M(v) = \frac{(i)m_n}{r^n} + O(r^{-(n+2)}) \tag{VI.78}
\]

The solution to (VI.74) which shows the appropriate boundary behaviour has the following form as \( r \to \infty \):

\[
(i)\Phi_v = \frac{(i)c_{n/2}}{r^{n/2}} + O(r^{-(n/2+2)}) \tag{VI.79}
\]

By putting these expansions back in (VI.75) comparing the coefficients of \( r^{-(n)} \), one can relate \( (i)m_n \) and \( (i)c_{n/2} \) so as to kill off the terms contributing to \( r^{-n} \). This results in the following expression for \( (i)M(v) \).

\[
(i)M(v) = - \frac{L^2}{(n+1)} r^{-n/2} (i)\Phi_v \tag{VI.80}
\]

The above expression is similar to that given by [32] for \( n = 2 \) and is applicable for linearized perturbations (where \( (1)S_{\mu\nu} = 0 \)) as well. One can see that the source dependent term in (VI.76) falls off like \( r^{-(2n-2)} \) and hence doesn’t spoil the aAdS boundary condition for \( (i)h_{ti} \) for the given choice of \( (i)M(v) \) (even in the lowest possible \( n = 2 \) case).

Finally, we further simplify (VI.74). Because of completeness of \( e_{p,l_v}^{(v)} \), one can write, \( (i)\Phi_{v;k_v} = \sum_{p=0}^{\infty} (i)c_{p,k_v}^{(v)}(t) e_{p,l_v}^{(v)}(r) \) where \( c_{p,k_v} \) satisfies,

\[
(i)c_{p,k_v}^{(v)}(t) + \omega_v^2 (i)c_{p,k_v}^{(v)}(t) = r^{n/2 - 2} f^2 (i)V_{k_v}, e_{p,l_v}^{(v)} > v \tag{VI.81}
\]
VII Scalar perturbations

VII.1 Linear level

Following equations govern scalar perturbations \[40\]:

\[ F_c^c + 2(n - 2)F = 0 \]  
\[ \text{(VII.82)} \]

\[ \frac{nf}{r} F_{rr} + \frac{k_s^2}{r^2} F_{rt} - 2n F' + \frac{nf'}{f} \dot{F} - \frac{2n}{r} \ddot{F} = 0 \]  
\[ \text{(VII.83)} \]

\[ \frac{nf^2}{r} F_{rr}' + \left( \frac{k_s^2}{r^2} f + \frac{2n}{r} f' f + n(n - 1) \frac{f^2}{r^2} \right) F_{rr} - 2nf F'' 
- \left( nf' + 2n(n + 1) \frac{f}{r} \right) F' + \frac{2(n - 1)(k_s^2 - n)}{r^2} F = 0 \]  
\[ \text{(VII.84)} \]

\[ \frac{2n}{rf} F_{rt} - \frac{2n}{f^2} \ddot{F} - \frac{n f'}{r} F_{rr} + \frac{nf'}{f} F' + \frac{2n(n - 1)}{r} F' - \frac{n(n - 1)f}{r^2} F_{rr} + \frac{n}{r} (F_t')' 
- \frac{k_s^2}{fr^2} F_t' - \frac{2(n - 1)(k_s^2 - n)}{fr^2} F = 0 \]  
\[ \text{(VII.85)} \]

\[ \frac{1}{f} \dot{F}_{rt} + (F_t')' - \frac{1}{r} F_t' + 2(n - 1) F' - \frac{(n - 1)}{r} F_r' - \frac{f'}{2f} F_{rr} + \frac{f'}{2f} F_t' = 0 \]  
\[ \text{(VII.86)} \]

We will use the ansatz similar to \[41\] to simplify these equations, defined as

\[ F_{rt} = \frac{2r}{f} (\Phi_s + \dot{F}) \]  
\[ \text{(VII.87)} \]

Hence, we obtain a single master equation in terms of the master variable \( \Phi_s \)

\[ \ddot{\Phi}_s - f^2 \Phi_s'' - \left( f' f + \frac{nf^2}{r} \right) \Phi_s' + \left( -(n - 1) \frac{f' f}{r} + \frac{k_s^2}{r^2} f \right) \Phi_s = 0 \]  
\[ \text{(VII.88)} \]

Let \( \Phi_s = \cos(\omega_s t + b) \phi_s \), then \( \text{VII.88} \) is given by:

\[ \ddot{\phi}_s + \omega_s^2 \phi_s \]  
\[ \text{(VII.89)} \]
Here, $\hat{L}_s$ is defined as
\[
\hat{L}_s = -f^2 \partial_r^2 - \left( f' f + \frac{n f^2}{r} \right) \partial_r + \left( - (n - 1) \frac{f' f}{r} + \frac{k_s^2}{r^2} f \right)
\] (VII.90)

The eigensolutions which ensure appropriate aAds conditions at the boundary are given by
\[
\phi_s = e^{(s)}_{p,l_s} = d^{(s)}_{p,l_s} \frac{L^\frac{1}{2} + \nu_s r \frac{1}{2} + \sigma_s \frac{n}{2}}{(r^2 + L^2)^\frac{1}{2} (\nu_s + \sigma_s + 1)^2} F_1 \left( \frac{\omega_s}{\nu_s, \sigma_s}; \frac{L^2}{(r^2 + L^2)} \right)
\] (VII.91)

where $\nu_s = \frac{(n-3)}{2}$, $\sigma_s = l_s + \frac{(n-1)}{2}$ and $\frac{\omega_s}{\nu_s, \sigma_s} = \frac{\nu_s + \sigma_s + \omega_s L + 1}{2}$. The eigenfrequencies $\omega_s$ are obtained by imposing the regularity condition at the origin, which gives
\[
\omega_s L = 2p + l_s + n - 1 \quad ; p = 0, 1, 2...
\] (VII.92)

The associated eigenfunctions $e^{(s)}_{p,l_s}$ form a complete orthogonal set and the inner product is given by
\[
< e^{(s)}_{p,l_s}, e^{(s)}_{p',l_s} >_s = \int_0^\infty e^{(s)}_{p,l_s} e^{(s)}_{p',l_s} w_s(r) dr
\] (VII.93)

where the weight function $w_s(r)$ is given by
\[
w_s(r) = \frac{r^n}{f}
\] (VII.94)

Hence the normalization constant is given by
\[
d^{(s)}_{p,l_s} = \left( \frac{2}{L} \frac{(2p + l_s + n - 1) \Gamma(p + l_s + n - 1)}{p! \Gamma(p + l_s + \frac{n+1}{2}) \Gamma(p + \frac{n-1}{2})} \right)^{1/2} \left( \frac{n - 1}{2} \right)_p
\] (VII.95)

**VII.2 Higher orders**

Before considering higher order perturbations, we define the following quantities:
\[
^{(i)} S^{(i)}_{0k_s} = \int S^{ij}_{k_s} \delta \Omega^i
\] (VII.96)
\(^{(i)}S_{s1k_s} = \int S_{k_s}^{(i)}S_{r't'}d^3\Omega,\) (VII.97)

\(^{(i)}S_{s2k_s} = \int S_{k_s}^{(i)}S_{tt}d^3\Omega,\) (VII.98)

\(^{(i)}S_{s3k_s} = \int S_{k_s}^{(i)}S_{rr}d^3\Omega\) (VII.99)

\(^{(i)}S_{s4k_s} = \frac{1}{k_s} \int S_{k_s}^{(i)}S_{\psi\psi}d^3\Omega\) (VII.100)

\(^{(i)}S_{s5k_s} = \left(\frac{k_s^2}{nr} + 2f' + \frac{(n - 1)f}{r}\right) \int \limits^t \left((i)S_{s1}dt - \frac{(i)S_{s2}}{f} + \frac{f^2}{r} \left(\frac{r}{f} \int \limits^t (i)S_{s1}dt\right)'\right)\) (VII.101)

\(^{(i)}S_{s6k_s} = \frac{f^2}{2n} \left\{ (i)S_{s3} - \frac{n}{r} (i)S_{s4} - \left(1 - \frac{n}{k_s^2}\right) \frac{(i)S_{s0}}{r^2f} - \left(\frac{(k_s^2 - n)}{nr} - \frac{f'}{2f^2}\right) \int \limits^t (i)S_{s1}dt\right\} + \frac{1}{f} (i)S_{s5} - \frac{1}{k_srf} \left(nr^2 f (i)S_{s5}\right)\) (VII.102)

Then the following equations govern scalar perturbations:

\[-k_s^2[(i)F_c^c + 2(n - 2)(i)F] = (i)S_{s0}\] (VII.103)

\[\frac{n}{r} f(i)F_{rr'} + \frac{k_s^2}{r^2} (i)F_{rt} - 2n(i)\dot{F}' + n \frac{f'}{f}(i)\dot{F} - \frac{2n}{r}(i)\ddot{F} = (i)S_{s1}\] (VII.104)

\[\frac{n}{r} f^2(i)F_{rr'} + \left(\frac{k_s^2}{r^2} f + \frac{2n}{r} f' f + n(n - 1) \frac{f^2}{r^2}\right)(i)F_{rr} - 2nf(i)F''\]

\[\quad - \left(n f' + 2n(n + 1) \frac{f'}{r}\right)(i)F' + 2(n - 1) \frac{(k_s^2 - n)}{r^2}(i)F = \frac{(i)S_{s2}}{f}\] (VII.105)
\[ \frac{2n}{f} (i) \dot{F}_{rt} \left( \frac{1}{f^2} \right) - \frac{1}{r} (i) F_{rt} + \frac{n f'}{f} (i) F_{rr} + \frac{2n(n - 1)}{r} (i) F' - \frac{n(n - 1)}{r^2} f (i) F_{rr} + \frac{n}{r} \left( (i) F_{rt} \right)' - \frac{k^2}{f r^2} (i) F_t - \frac{2(n - 1)(k^2 - n)}{f r^2} (i) F = (i) S_{s3} \] (VII.106)

\[ \frac{1}{f} (i) \dot{F}_{rt} + (i) F_{tt}' - \frac{1}{r} (i) F_{t} + 2(n - 1) (i) F' - \frac{(n - 1)}{r} (i) F_{rr}' - \frac{f'}{2} (i) F_{rr}' + \frac{f'}{2f} (i) F_{t} = (i) S_{s4} \] (VII.107)

Using the ansatz:

\[ (i) F_{rt} = \frac{2r}{f} ((i) \dot{\Phi}_s + (i) \dot{F}) \] (VII.108)

it is possible to use the system of five equations to obtain a single equation in terms of the higher order master variable \((i) \Phi_s\)

\[ (i) \ddot{\Phi}_s + \hat{L}_s (i) \dot{\Phi}_s = (i) S_{s6} \] (VII.109)

The solution to the above equation can be written as

\[ (i) \Phi_s = (i) \Phi_s^H + (i) \Phi_s^P \] (VII.110)

where the behaviour of \((i) \Phi_s^H\) as \(r \to \infty\) is similar to homogeneous solution of (VII.109), i.e.

\[ (i) \Phi_s^H = \frac{1}{r^{n-1}} \left( (i) \Phi_{n-1} + \frac{(i) \Phi_{n+1}}{r^2} + \ldots \right) \] (VII.111)

The nature of the \((i) \Phi_s^P\) at infinity can be deduced by looking at the behaviour of \((i) S_{s6}\), whose leading order behaviour as \(r \to \infty\) goes like \(r^{-(2n-2)}\). Hence, asymptotically

\[ (i) \Phi_s^P = \frac{(i) \Phi_{2n}}{r^{2n}} + O(r^{-(2n+1)}) \] (VII.112)

The various gauge invariant quantities in terms of \((i) \Phi_s\) are as follows.

\[ (i) F = \frac{1}{(-k^2 + n)} \left[ n r f (i) \Phi_s' + (k^2 + n(n - 1)f) (i) \Phi_s - n r^2 f (i) S_{s5} \right] \] (VII.113)
Rest of the variables can be expressed in terms of \((i)F\) and \((i)\Phi_s\). For e.g.

\[
(i)F_{rr} = \frac{2r}{f}(i)F' + \frac{(-k_s^2 + n)}{nf^2}(i)F - \frac{2k_s^2}{nf^2}(i)\Phi_s + \frac{r}{nf} \int (i)S_{rt} dt \quad \text{(VII.114)}
\]

Similarly, \((i)F_{tt}\) is obtained from \((\text{VII.102})\). In order to construct a\(\text{AdS}\) solutions, we consider a class of perturbations where \((i)H_L = (i)f_a = 0\) at each order. For this choice, \((i)f_{ab} = (i)F_{ab}\) and \((i)H^{(s)}_L = (i)F\). Hence the metric perturbations along with the gauge transformations are given as (summation over \(k_s\) on the R.H.S. of each of the equations is implied):

\[
(i)h_{tt} = \left[(i)F_{tt} - 2(i)\dot{T}_t + f' f(i)T_r\right] S \quad \text{(VII.115)}
\]

\[
(i)h_{rr} = \left[(i)F_{rr} - 2(i)\dot{T}_r - \frac{f'}{f}(i)T_r\right] S \quad \text{(VII.116)}
\]

\[
(i)h_{rt} = \left[(i)F_{rt} - (i)\dot{T}_r - (i)\dot{T}_t + \frac{f'}{f}(i)T_t\right] S \quad \text{(VII.117)}
\]

\[
(i)h_{ti} = \left[-(i)\dot{M} + k_s (i)T_t\right] S_i \quad \text{(VII.118)}
\]

\[
(i)h_{ri} = \left[-r^2 \left(\frac{(i)M}{r}\right)' + k_s T_r\right] S_i \quad \text{(VII.119)}
\]

\[
(i)h_{ij} = 2 \left[r^2 (i)F - \frac{k_s r}{n}(i)M - rf(i)T_r\right] \gamma_{ij} S + 2k_s r (i)MS_{ij} \quad \text{(VII.120)}
\]

In order to ensure that the metric perturbations remain asymptotically AdS, we need to make suitable gauge choices.

**Gauge choice for scalar perturbations:**

In order to fix \((i)M\) and \((i)T_a\), we first note that \((i)T_r\) should be expanded as

\[
(i)T_r = \frac{(i)T_r^{(n)}}{r^n} + \mathcal{O}(r^{-(n+1)}) \quad \text{(VII.121)}
\]
Putting this expansion back in (VII.116) and expanding \((i)\Phi_s\) as in (VII.110), one can compare the coefficients of \(r^{-(n+1)}\). This means writing \((i)T_r^{(n)}\) in terms of \((i)a_{n-1}\) and \((i)a_{n+1}\) (defined in (VII.111)) so that \(r^{-(n+1)}\) terms are killed off. This way, we can construct \((i)T_t\) in terms of \((i)\Phi_s\). Similarly \((i)T_t\) and \((i)M\) can be constructed from (VII.117) and (VII.119) respectively, by assuming an expansion for \((i)T_t\) of the form

\[
(i)T_t = \frac{(i)T_t^{(n-1)}}{r^{n-1}} + \mathcal{O}(r^n)
\]  

Hence, the final form of gauge choices are as follows:

\[
(i)T_r = \frac{1}{(k_s^2 + n)} \left[ \frac{k_s^2 L^2}{r} (i)\Phi_s + \frac{n}{r^{n-3}} \partial_r (r^{n-1}(i)\Phi_s) \right]
\]  

\[
(i)T_t = -\frac{L^2}{(n+1)(k_s^2 + n)} \left[ (-k_s^2 + 2n)(i)\Phi_s + \frac{n}{L^2 r^{n-4}} \partial_r (r^{n-1}(i)\Phi_s) \right]
\]  

\[
(i)M = -\frac{k_s}{(n+1)} (i)T_r
\]  

These gauge choices are also valid for linearized perturbations \((i)S_{\mu\nu} = 0\). One important thing to note here is that there is a \(r^{-2n}\) piece in the expression for \((i)F_{rr}\) arising from the source dependent terms. In four dimensions it means the presence of non-zero \((i)b_{n+2}\), since \(n + 2 = 2n\) in four dimensions. This means that \((i)T_r\) will have a second derivative of \((i)\Phi_s\) w.r.t to radial coordinate. But in higher dimensions \(n + 2 < 2n\), which implies \((i)b_{n+2} = 0\) as well. Hence the above gauge choices are suitable. Finally, since \(e^{(s)}_{p,l_s}\) form a complete orthonormal set, one can write \((i)\Phi_s\) as \((i)\Phi_{s k_s} = \sum_{p=0}^{\infty} (i)c_{p,k_s}(t)e^{(s)}_{p,l_s}(r)\). From (VII.109), we see \((i)c_{p,k_s}(t)\) satisfies:

\[
(i)c_{p,k_s} + \omega_s^2 (i)c_{p,k_s} = (i)S_{s6k_s}, e^{(s)}_{p,l_s} > s
\]  

\[\text{VIII Special modes}\]

These modes satisfy first order equations and beyond the linearised level, are no longer gauge degrees of freedom.
VIII.1 Scalar perturbations $l_s = 0, 1$ modes

VIII.1.1 $l_s = 0$ mode

Now we consider the $l_s = 0$ mode for scalar perturbations. Let $(i)\tilde{S}_{0\mu\nu}$ be the source terms associated with these modes. For this case $S$ is just a constant and only $(i)f_{ab}$ and $(i)H_L$ exist. We will make a gauge choice so that

$$(i)H_L = (i)f_{rt} = 0 \quad \text{(VIII.127)}$$

We get the following equations for the case $(i)G_{rt} = 0$, $(i)G_{tt} = 0$ and $(i)G_{rr} = 0$ respectively.

$$nf_{r}(i)f''_r = (i)\tilde{S}_{0}\rt \quad \text{(VIII.128)}$$

$$\frac{nf}{r}(i)f'_r' + \left(\frac{n(n-1)f}{r^2} + \frac{nf'}{r}\right)(i)f'_r = \frac{1}{f}(i)\tilde{S}_{0}\tt \quad \text{(VIII.129)}$$

$$\frac{n}{r}(i)f'_t' - \left(\frac{nf'}{fr} + \frac{n(n-1)}{r^2}\right)(i)f'_t = (i)\tilde{S}_{0}\rr \quad \text{(VIII.130)}$$

Upon solving (VIII.128), one obtains

$$(i)f_{rr} = \int_{t_1}^{t} \frac{r}{nf}(i)\tilde{S}_{0}\rt dt + (i)f_{rr}(t_1, r) \quad \text{(VIII.131)}$$

where $(i)f_{rr}(t_1, r)$ can be obtained from (VIII.129):

$$(i)f_{rr}(t_1, r) = \frac{1}{f^2 r^{n-1}} \int_{0}^{r} \frac{r^n}{nf}(i)\tilde{S}_{0}\tt(t_1, r) dr \quad \text{(VIII.132)}$$

Similarly $(i)f_{tt}$ is given by

$$(i)f_{tt} = f^2(i)f_{rr} - \frac{f}{n} \int_{0}^{r} dr \left((i)\tilde{S}_{0}\rr + \frac{1}{f^2}(i)\tilde{S}_{0}\tt\right) r \quad \text{(VIII.133)}$$
VIII.2 \( l_s = 1 \) mode

Let \( \tilde{S}_1^{\mu \nu} = \int S^{(i)} S_{\mu \nu} d^3 \Omega \) be the source associated for this mode. Using gauge choice freedom, \( H_L \) and \( f_a^{(s)} \) is put to zero. So we need to solve for \( f_{a b}^{(i)} \). From the \( G_{tt} \) equation

\[
(i)^{f_{rr}'} + \left( \frac{1}{rf} + \frac{2f'}{f} + \frac{(n-1)}{r} \right) (i)^{f_{rr}} = \frac{r}{nf^3} (i)^{\tilde{S}_1}_{tt} \tag{VIII.134}
\]

Hence,

\[
(i)^{f_{rr}} = \frac{1}{rnf^{3/2}} \left( \int_0^r \frac{r^{n+1}}{n f^{3/2}} (i)^{\tilde{S}_1}_{tt} dr \right) \tag{VIII.135}
\]

From \( G_{rt} = 0 \) and \( G_{it} = 0 \), we obtain

\[
\frac{nf}{r} (i)^{f_{rr}'} + \frac{n}{r^2} (i)^{f_{rt}} = (i)^{\tilde{S}_1}_{rt} \tag{VIII.136}
\]

\[
-f \sqrt{n} \left[ (i)^{f_{rt}'} + \left( \frac{(n-2)}{r} + \frac{f'}{f} \right) (i)^{f_{rt}} - (i)^{f_{rr}} \right] = (i)^{\tilde{S}_1}_{it} \tag{VIII.137}
\]

Hence, from the two equations,

\[
(i)^{f_{rt}} = \frac{1}{f^{1/2} r^{n-1}} \left\{ \int_0^r \frac{r^{n-1}}{f^{1/2}} \left( \frac{r}{n} (i)^{\tilde{S}_1}_{rt} - \frac{1}{\sqrt{n}} (i)^{\tilde{S}_1}_{tt} \right) dr \right\} \tag{VIII.138}
\]

Similarly from \( G_{ir} = 0 \) equation we get

\[
(i)^{f_{tt}} = rf^{1/2} \left\{ \int_0^r f^{1/2} \left( \frac{(i)^{f_{tr}'} + (n-1)f}{rf} (i)^{f_{rr}} - \frac{(i)^{\tilde{S}_1}_{ir}}{\sqrt{n} r} - \frac{f'}{2r} (i)^{f_{rr}} \right) dr \right\} \tag{VIII.139}
\]

VIII.3 Vector modes \( l_v = 1 \) mode

Since \( \nabla_{ij} \) is undefined, only \( f_a^{(i)} \) exist. We will use gauge freedom to put \( f_t^{(i)} \) to zero. Let \( \tilde{S}_{1ia}^{(i)} = \int S^{(i)} S_{ia} d^3 \Omega \). Then from \( G_{iv} = 0 \), one obtains

\[
(i)^{\dot{f}_r} = \frac{f}{r} \int_{t_1}^t (i)^{\tilde{S}_{1ir}^{(i)}} dt + (i)^{\dot{f}_r}(t_1, r) \tag{VIII.140}
\]

where \( (i)^{\dot{f}_r}(t_1, r) \) can be determined from \( G_{it} = 0 \):

\[
(i)^{\dot{f}_r}(t_1, r) = \frac{1}{r^{n+1}} \int_0^r \frac{r^n}{f} (i)^{\tilde{S}_{1it}}(t_1, r) dr \tag{VIII.141}
\]
Analysis of higher order equations indicate the presence of secular resonances. In four dimensions, it was seen that for specific examples of single mode data, resonances are completely absent at second order [31], [34]. At third order, some of these lead to irremovable resonances (for a proposal to construct geons starting from any linear eigenfrequency, see [46]).

Here, we will concentrate on dimension five, which is equivalent to $n = 3$. We consider a case, where the initial single mode data, contains only tensor-type harmonics, with labels, $\mathbf{k} = \{l, l', m\}$. Let the single mode data have value,

$$\tilde{\mathbf{k}} = \{l = 2, l' = 2, m = 0\} \quad (IX.142)$$

The associated source frequency spectrum is then $\tilde{\omega} L = 2p' + 6$. (see (V.51)) Let us now inspect the R.H.S of (V.57) when $i = 2$:

$$< r^{-1/2} f \int \tau^{ij} (2) S_{ij} d^3 \Omega, e_{p,2} > = < r^{-1/2} f \int \tau^{ij} (2) A_{ij} d^3 \Omega, e_{p,2} > \quad (IX.143)$$

The above replacement of $(2) S_{ij}$ with $(2) A_{ij}$ is possible because of the traceless property, $T^i_i = 0$ satisfied by the tensor harmonics. The explicit form of $(2) A_{ij}$, in case we start out only with tensor type perturbations, at linear level is given by (see appendix C of [38])

$$(2) A_{ij} = \sum_{k_1} \sum_{k_2} H_{T_{k_1} H_{T_{k_2}}} \left( T^{kl}_{k_1} \left( -D_i D_j T_{kl}^{k_2} + D_k D_i T_{jl}^{k_2} + D_k D_j T_{lk}^{k_2} \right) 
- D_k D_l T_{ij}^{k_2} \right) 
+ \frac{D_j T^{kl}_{k_1} D_l T_{kl}^{k_2} + D_k T^{l}_{k_1} D_j T_{lk}^{k_2} - D^{k} T_{il}^{k_1} D_k T_{lj}^{k_2}}{2} 
- r D^a r D_a H_{T_{k_1} H_{T_{k_2}}} \gamma^{ij} T^{kl}_{k_1} T_{kl}^{k_2} - r^2 D^a H_{T_{k_1}} D_a H_{T_{k_2}} T_{ik}^{k_1} T_{jk}^{k_2} \quad (IX.144)$$

Let us first tackle the kind of integral, arising as a result of a term like $H_{k_1} H_{k_2}$ in $(2) A_{ij}$. We take $k_1 = k_2 = \tilde{k}$, where $\tilde{k}$ is given by (IX.142). Then, (IX.143) contains the following term

$$\int_{0}^{\infty} r^{-1/2} (H_{T_{p',l=2}})^2 e_{p,2} dr \sim \int_{-1}^{1} (1 - y)^2 (P_{p'}^{2,3}(y))^2 d^p dy \left[ (1 - y)^{p+2} + (1 + y)^{p+3} \right] dy \quad (IX.145)$$
where \( r^2 = (1+y)/(1-y) \) and we have used the fact that the hypergeometric function \( _2F_1(p+\alpha+\beta+1,-p,1+\alpha;z) \sim P_{p}^{\alpha,\beta}(1-2z) \) (the Jacobi polynomial of degree \( p \)). The only frequency excited here is \( 2\tilde{\omega} \). The resonant modes correspond to those frequencies \( \omega \), which satisfy \( \omega_{p,l=2} = 2\tilde{\omega} \). This translates to

\[
p = 2p' + 3 \tag{IX.146}
\]

Hence, upon inspecting (IX.145), and doing integration by parts, one can conclude that this particular integral vanishes (because the derivative operator in (IX.145) acts \( p = 2p'+3 \) times on \( (1-y)^p[P_{p'}^2]^2 \), which is a polynomial of degree \( 2p' + 2 \)). Since, \( T_i = 0 \), the part of the integral (IX.143), arising out of a term proportional to \( \gamma_{ij} \) in \(^{(2)}A_{ij} \) also vanishes.

The last part in integral (IX.143) arises out of term \( r^2\tilde{D}^aH_{T_{K_1}}\tilde{D}_aH_{T_{K_2}} \). We find on inspection that this integral is not trivially zero. We therefore perform this integral for the case with \( \tilde{\omega}L = 6 \), which corresponds to \( p' = 0 \). The tensor frequency \( \omega \) satisfying the resonant condition will thus have a mode number corresponding to \( p = 3 \) by (IX.146). Hence, we obtain the following forced harmonic oscillator equation at second order for this particular single mode initial data:

\[
^{(2)}c_{p=3,k}(t) + \omega_{p=3,t=2}^2c_{p=3,k}(t) = a_1 \cos \left( \frac{12}{L}t \right) + a_2 \tag{IX.147}
\]

where we checked explicitly through Mathematica and found constants \( a_1 \) and \( a_2 \) to be non zero.

Since there is no mode corresponding to frequency \( 2\tilde{\omega} \) at the linear level, we are not aware of any methods similar to the Poincare-Lindstedt technique [47] by which these resonances could be removed at this order, thus we suspect the presence of irremovable secular resonances at second order.

### X Discussion

The main objective of this work was to extend the study of nonlinear perturbations of Anti de Sitter spacetime to dimensions greater than four. In higher dimensions, apart from vector and scalar type, we also have tensor type perturbations. We see that the tensor perturbations are by default in aAdS form at all orders. The analysis of the vector type perturbations is very similar to
the axial perturbations in [32] and reduces to a similar expression for \( n = 2 \).

In case of scalar perturbations, especially at higher orders, the metric perturbations also have source dependent terms in them. It so happens that in four dimensions, the leading order behaviour at asymptotic infinity of these terms is slower compared to higher dimensions. This is because, the source dependent terms fall off like \( r^{-(2n+k)} \), where \( k \) is some integer i.e. fall off is faster in higher dimensions. Hence the gauge choice needs to have second order derivatives of \( \Phi_s \) with respect to the radial coordinate, compared to that in higher dimension.

So far, analysis of the resonant structure of perturbed equations in four dimensions [31], [34] as well as the five dimensional biaxial Bianchi IX case [39] had revealed the presence of irremovable resonances at the third order. In this paper, we also study the perturbed equations for special classes of perturbations of the vacuum Einstein equations in dimension five. We examine the resonant structure of equations at the second order, by starting out with only tensor-type perturbations at the linear level. At least for the single mode data we took, we find that the resonant terms, which arise at second order don’t trivially vanish. Moreover, we are not aware of any way to remove these resonances, similar to methods like Poincare-Lindstedt technique, thus we suspect the presence of irremovable resonances at second order.

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