A PROOF OF THE SECOND ROGERS-RAMANUJAN IDENTITY
VIA KLESHCHEV MULTIPARTITIONS

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Abstract. We give another proof of the second Rogers-Ramanujan identity by Kashiwara crystals.

1. Introduction

In [9], Lepowsky and Milne observed a similarity between the characters of the level 3 standard modules of the affine Lie algebra of type $A_1^{(1)}$

\[ \text{ch} V(2\Lambda_0 + \Lambda_1) = \frac{1}{(q; q^2)_\infty} \frac{1}{(q, q^4; q^5)_\infty}, \]

\[ \text{ch} V(3\Lambda_0) = \frac{1}{(q^2; q^2)_\infty} \frac{1}{(q^2, q^4, q^5)_\infty}, \] (1)

and the infinite products of the Rogers-Ramanujan identities

\[ \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_\infty}, \]

\[ \sum_{n \geq 0} \frac{q^{n(n+1)}}{(q; q)_n} = \frac{1}{(q^2, q^4, q^5)_\infty}. \] (2)

Here, the $q$-Pochhammer symbols are defined for $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ as follows.

\[ (a; q)_n = \prod_{0 \leq j < n} (1 - aq^j), \quad (a_1, \ldots, a_k; q)_n = (a_1; q)_n \cdots (a_k; q)_n. \]

Later, Lepowsky and Wilson promoted the observation to a vertex operator proof and gave a Lie theoretic interpretation of the infinite sums in the Rogers-Ramanujan identities [10]. The goal of this paper is to show that a result of Kashiwara crystals which is motivated by the representation theory of Hecke algebras [1 Corollary 9.6] promotes the equality (1) into a proof of the second Rogers-Ramanujan identity (2). Note that it is well-known that the Rogers-Ramanujan identities and the solvable lattice models from which the quantum groups originated are related (see [3 Chapter 8]). Several relationships between Rogers-Ramanujan type identities and Kashiwara crystals are also known (see [5] and the references therein). The author was inspired by a recent work of Corteel which gave a proof of (2) using the cylindric partitions and the Robinson-Schensted-Knuth correspondence [4].

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2. THE MAIN RESULT

A partition (resp. strict partition) is a weakly (resp. strictly) decreasing sequence \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of positive integers, i.e., \( \lambda_1 \geq \cdots \geq \lambda_k \) (resp. \( \lambda_1 > \cdots > \lambda_k \geq 1 \)). We denote the set of partitions (resp. strict partitions) by \( \text{Par} \) (resp. \( \text{Str} \)). We also denote the size \( \lambda_1 + \cdots + \lambda_k \) (resp. the length \( \ell \)) of \( \lambda \) by \( |\lambda| \) (resp. \( \ell(\lambda) \)). When \( \lambda \) is empty (i.e., \( \ell(\lambda) = 0 \)), we put \( \lambda_1 = 0 \).

**Theorem 2.1** ([1]) (The transposed version of Proposition 9.7). Let \( k \geq 1 \). Under the \( A_1^{(1)} \)-crystal isomorphism \( \text{Str} \cong B(\Lambda_0) \) due to Misra-Miwa [1], the canonical image \( B(k\Lambda_0) \) in the tensor product \( B(k\Lambda_0) \otimes^k \) coincides with

\[
S_k = \{ \lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)}) \in \text{Str}^k \mid \ell(\lambda^{(i)}) = (\lambda^{(i+1)}_1)\geq 1 \text{ for } 1 \leq i < k \}.
\]

This result is credited to Mathas in [1, §9]. It is also a Corollary of [2, Theorem 3.8] and [3] Theorem 10.1. An element of the connected component \( S_k \) is called a Kleshchev multipartition in the context of the representation theory of Hecke algebras. For a generalization to \( A_p^{(1)} \)-crystal, where \( p \geq 2 \), see [1 Corollary 9.6]. For a different characterization, see [6].

**Theorem 2.2.** For \( k \geq 1 \), we have

\[
\sum_{\lambda \in S_k} x^{\ell(\lambda)} q^{|\lambda|} = \sum_{i_1, \ldots, i_k \geq 0} \frac{q^{\sum_{\lambda = 1}^{n} a(n^{(i+1)}/2) + \sum_{1 \leq a < b \leq k} a^{i_a} b^{i_b} - \sum_{a=1}^{n} a^{i_a}}}{(q;q)_{i_1} \cdots (q;q)_{i_k}}.
\]

Here, for a \( k \)-tuple of strict partitions \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \text{Str}^k \), the size \( |\lambda| \) and the length \( \ell(\lambda) \) are defined as follows.

\[
|\lambda| = |\lambda_1| + \cdots + |\lambda_k|, \quad \ell(\lambda) = \ell(\lambda_1) + \cdots + \ell(\lambda_k).
\]

3. A PROOF OF THEOREM 2.2

As usual (see [2, Definition 3.1]), we define the \( q \)-binomial coefficient

\[
\left[ \begin{array}{c} n \\ m \end{array} \right]_q = \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}
\]

for \( n \geq m \geq 0 \). It is well-known (see [2, Theorem 3.1]) that we have

\[
\left[ \begin{array}{c} n \\ m \end{array} \right]_q = \sum_{\lambda \in \text{Par}} q^{\ell(\lambda) \leq m} \sum_{\lambda_1 \leq n-m} q^{\ell(\lambda) \leq m}.
\]

(3)

For \( i, j \geq 0 \), considering the staircase \( \Delta_j = (j, j-1, \ldots, 1) \in \text{Str} \), we see

\[
\sum_{\mu \in \text{Str} \atop \ell(\mu) = j} q^{\Delta_j} = q^{\Delta_j} \sum_{\lambda \in \text{Par} \atop \ell(\lambda) \leq j} q^{\ell(\lambda)}.
\]

(4)

**Proposition 3.1.** For \( k \geq 1 \) and \( j_1, \ldots, j_k \geq 0 \), there is a size preserving bijection

\[
f_{j_1, \ldots, j_k} : V_{j_1, \ldots, j_k} \to W_{j_1, \ldots, j_k},
\]

2
where

\[ A_{j_1, \ldots, j_k} = \{ \lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)}) \in \text{Str}^k : \ell(\lambda^{(i)}) = j_i + \cdots + j_k \text{ for } 1 \leq i \leq k \}, \]

\[ V_{j_1, \ldots, j_k} = S_k \cap A_{j_1, \ldots, j_k}, \]

\[ W_{j_1, \ldots, j_k} = \{ \lambda \in A_{j_1, \ldots, j_k} : (\lambda^{(i)})_{j_i+1}, \ldots, (\lambda^{(i)})_{\ell(\lambda^{(i)})} = \Delta_{\ell(\lambda^{(i)})} \text{ for } 1 \leq i < k \}. \]

Proof. We prove the claim by induction on \(k\). The case \(k = 1\) is trivial.

Similarly to (4), for \(i, j \geq 0\) we see

\[
\sum_{(\lambda, \mu) \in V_{i,j}} q^{\lambda \mid \mu} = \sum_{\mu \in \text{Str}^{\ell(\mu)} : \mu_{j+1} \leq \ell(\mu)} q^{\lambda \mid \mu} \frac{q^{\Delta_{\lambda \mu}}}{(q; q)_{i+j}},
\]

\[
\sum_{(\lambda, \mu) \in W_{i,j}} q^{\lambda \mid \mu} = \frac{q^{\Delta_{\lambda \mu}}}{(q; q)_i} \frac{q^{\Delta_\lambda}}{(q; q)_j}, \tag{5}
\]

which are equal to each other thanks to (3) and (4). This settled the case \(k = 2\).

For \(k \geq 3\), it is easily seen that the composite

\[ \lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)}) \mapsto \mu = (\mu^{(1)}, \ldots, \mu^{(k)}) := (f_{j_1, j_2, \ldots, j_k}(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \ldots, \lambda^{(k)}) \mapsto (\mu^{(1)}, f_{j_2, \ldots, j_k}(\mu^{(2)}, \ldots, \mu^{(k)})) \]

is a size preserving bijection from \(V_{j_1, \ldots, j_k}\) to \(W_{j_1, \ldots, j_k}\).

Theorem 2.2 is proved as follows. Clearly, we have

\[
\sum_{\lambda \in S_k} x^{\ell(\lambda)} q^{\lambda} = \sum_{j_1, \ldots, j_k \geq 0} x^{j_1 + 2j_2 + \cdots + k j_k} \sum_{\lambda \in V_{j_1, \ldots, j_k}} q^{\lambda}. \]

By Proposition 3.1, the right hand side is equal to

\[
\sum_{j_1, \ldots, j_k \geq 0} x^{j_1 + 2j_2 + \cdots + k j_k} \sum_{\lambda \in W_{j_1, \ldots, j_k}} q^{\lambda}. \]

Similarly to (5), we see

\[
\sum_{\lambda \in W_{j_1, \ldots, j_k}} q^{\lambda} = \prod_{a=1}^{k} \frac{q^{\Delta_{j_a + \cdots + j_k}}}{(q; q)_{j_a}}. \]

Using \(|\Delta_{s+t}| = |\Delta_s| + |\Delta_t| + st\) for \(s, t \geq 0\), we have

\[
|\Delta_{j_a + \cdots + j_k}| = \sum_{b=a}^{k} |\Delta_{j_b}| + \sum_{a \leq b < b' \leq k} j_b j_{b'}
\]

and thus we have

\[
\sum_{a=1}^{k} |\Delta_{j_a + \cdots + j_k}| = \sum_{a=1}^{k} a |\Delta_{j_a}| + \sum_{1 \leq b < b' \leq k} b j_b j_{b'}.
\]
4. A proof of the second Rogers-Ramanujan identity

In the proof, let
\[ F(x, q) = \sum_{s, t, u \geq 0} q^{\binom{s+1}{2} + 2\binom{t+1}{2} + 3\binom{u+1}{2}} + xt + su + 2tu + x + 2t + 3u \quad (q; q)_s (q; q)_t (q; q)_u, \]
\[ G(x, q) = \sum_{s \geq 0} q^{s(s+1)} x^{2s} (q; q)_s. \]

**Proposition 4.1.** We have the following \( q \)-difference equation.
\[ G(x, q) = (1 + x^2 q^2 + x^2 q^3)G(xq, q) - x^4 q^7 G(xq^2, q). \]

**Proof.** It is easy to verify that for all \( M \in \mathbb{Z} \) we have
\[ (1 - q^M)g_M - q^M (1 + q)g_{M-2} + q^{2M-1}g_{M-4} = 0, \]
where \( g_{2s} = q^{s(s+1)}/(q; q)_s \) for \( s \in \mathbb{Z}_{\geq 0} \) and \( g_M = 0 \) for \( M \in \mathbb{Z} \setminus 2\mathbb{Z}_{\geq 0} \). 

**Proposition 4.2.** We have the following \( q \)-difference equation.
\[ F(x, q) = (1 + xq)(1 + x^2 q^2 + x^2 q^3)F(xq, q) - x^4 q^7 (1 + xq)(1 + xq^2)F(xq^2, q). \]

**Proof.** Our proof is a typical application of a \( q \)-version of Wegschaider’s improvement of Sister Celine’s technique (see \[12\]). Let \( f(n) = \sum_{n \in \mathbb{Z}} f_n(q)x^n \) and put
\[ f(n, t, u) = q^{\binom{n-2t-3u+1}{2} + 2\binom{t+1}{2} + 3\binom{u+1}{2} + (n-2t-3u)(t+u)+2tu} (q; q)_{n-2t-3u}(q; q)_t(q; q)_u \]
for \( n, t, u \in \mathbb{Z} \), where we regard \( 1/0 \) as \( 0 \) if \( v < 0 \). Because \( f(n, t, u) \) is a proper hypergeometric (see \[12\] §2.1), one can automatically derive a \( q \)-holonomic recurrence for \( f_n \) thanks to \( f_n = \sum_{t, u \in \mathbb{Z}} f(n, t, u) \).

Let \( (Nq)(n, t, u) = g(n-1, t, u) \), \( (Tq)(n, t, u) = g(n, t-1, u) \), \( (Uq)(n, t, u) = g(n, t, u-1) \) be the shift operators for \( q : \mathbb{Z}^3 \to \mathbb{Q}(q) \) and let
\[ A = (1 - q^n) - q^n N - q^n (1 + q)(N^2 + N^3) + q^{2n-1}N^4 + q^{2n-2}(1 + q)N^5 + N^6, \]
\[ B = (q^n - q^{2n+u}) + q^n (1 + q^2)N + q^{n+u}(1 + q^{1+u})N^2 + q^{n+2u+1}UN^3, \]
\[ C = q^n (1 - q^3)N + q^{n+u+1}(1 - q^4)N^2 + q^n (1 + q^{1+u})N^3 - q^{2n-1}N^4 - q^{2n-2}(1 + q)N^5 + N^6. \]

One can check that
\[ (A + (1 - T)B + (1 - U)C)f(n, t, u) = 0. \]
By this certificate recurrence operator (see \[13\] §3 and \[13\] §7.1), we get
\[ (1 - q^n)f_n - q^n f_{n-1} - q^n (1 + q)(f_{n-2} + f_{n-3}) + q^{2n-1}f_{n-4} + q^{2n-2}(1 + q)f_{n-5} + f_{n-6} = 0 \]
for \( n \in \mathbb{Z} \). This is equivalent to the \( q \)-difference equation in the Proposition. 

**Corollary 4.3.** We have \( F(x, q) = (-xq; q)_\infty G(x, q) \).

**Proof.** By Proposition \[14\] and Proposition \[12\] \( F(x, q) \) and \((-xq; q)_\infty G(x, q)\) satisfy the same \( q \)-difference equation presented in Proposition \[12\]. Then, the equality follows from the fact that both the coefficients of \( x^0 \) (resp. \( x^n \) for \( n < 0 \)) in \( F(x, q) \) and \((-xq; q)_\infty G(x, q)\) are equal to 1 (resp. 0).
Remark 4.4. After submission to arXiv of the first version of this paper, we learned from Ole Warnaar that Corollary 4.3 is easily deduced by a trick to use $f_n = \sum_{t,u \in \mathbb{Z}} f(n, t-u, u)$ instead of $f_n = \sum_{t,u \in \mathbb{Z}} f(n, t, u)$ noticing
\[
f(n, t-u, u) = \frac{q^{(n+1)/2} + (t-n)}{(q; q)_{n-2t} (q; q)_t} (q^{-t}; q)_u (q^{-(n-2t)}; q)_u.
\]

Thanks to the $q$-Chu-Vandermonde identity 2\(\phi_1(a, q^{-m}; 0; q, q) = a^m \) for a nonnegative integer $m$ (see [3] (2.41)), we have
\[
f_n = \left\{ \begin{array}{l}
\frac{[n/2]}{(q; q)_{n-2t} (q; q)_t} \sum_{t=0}^{[n/2]} q^{(n+1)/2-t} (q; q)^{n-2t} (q; q)_t \\
\sum_{t=0}^{[n/2]} q^{(n-2t+1)/2+t(n+1)} (q; q)_{n-2t} (q; q)_t
\end{array} \right.
\]

This is equivalent to Corollary 4.3 by Euler’s identity $(-xq; q)_\infty = \sum_{m \geq 0} q^{m^2} (q; q)_\infty$. The second Rogers-Ramanujan identity 2\(\phi_1(1, q^{-m}; 0; q, q) = a^m \) is proved as follows. By Theorem 2.2 Lepowsky-Milne’s observation (1) is translated to
\[
F(1, q) = \frac{1}{(q; q^2)_\infty (q^2; q^5)_\infty}.
\]

By Corollary 4.3 and Euler’s identity $(q; q^2)_\infty = 1$, we have
\[
G(1, q) = \frac{1}{(q^2; q^5)_\infty}.
\]

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