SEGRE NUMBERS AND HYPERSURFACE SINGULARITIES

TERENCE GAFFNEY AND ROBERT GASSLER

ABSTRACT. We define the Segre numbers of an ideal as a generalization of the multiplicity of an ideal of finite colength. We prove generalizations of various theorems involving the multiplicity of an ideal such as a principle of specialization of integral dependence, the Rees-Böger theorem, and the formula for the multiplicity of the product of two ideals. These results are applied to the study of various equisingularity conditions, such as Verdier’s condition $W$, and conditions $A_f$ and $W_f$.

0. Introduction

If an ideal $I$ in a Noetherian local ring $A$ has finite colength, then the multiplicity of the ideal is a fundamental invariant of the ideal with many applications in geometry and algebra. Pierre Samuel [S] used it to define the intersection multiplicity of two algebraic sets. David Rees [R] linked the multiplicity of $I$ to its integral closure $\bar{I}$. Bernard Teissier [T1] used it to study the equisingularity of families of hypersurfaces with isolated singularities.

If an ideal does not have finite colength then the multiplicity is not defined. In this paper we study ideals of non-finite colength, and we define a set of invariants associated to $I$, the Segre numbers of $I$. This set of invariants then has similar properties to the multiplicity. If the multiplicity of $I$ is defined, it can be realized as the degree of the exceptional divisor in the blowup by $I$. The Segre numbers are constructed by first forming cycles by intersecting the exceptional divisor in the blowup by $I$ with generic hyperplanes, then pushing the cycles down to the base, and finally taking multiplicities of these cycles.

The Segre numbers provide a link between $I$ and $\bar{I}$ just as in the case of ideals of finite colength; see Corollary (4.9). They are lexicographically upper semi-continuous, as observed in (4.5). They are also useful in studying the equisingularity of families of hypersurfaces with non-isolated singularities.

Suppose we are given a family of hypersurfaces $X = \{X(t)\}$ parameterized by $(\mathbb{C}^p,0)$ and defined by $f : (\mathbb{C}^{n+1+p},0) \rightarrow (\mathbb{C},0)$ a function, which vanishes on $0 \times (\mathbb{C}^p,0)$. The goal is to find invariants which depend only on the members of the family $X(t) = V(f_t)$ whose constancy ensures that the given equisingularity condition being studied holds for the family. The blowup of $(\mathbb{C}^{n+1+p},0)$ along the Jacobian ideal $J(f)$ and the blowup of $X$ along the ideal $J(f)'$ induced by the

1991 Mathematics Subject Classification. Primary 32S15; Secondary 14B05, 13H15.

The first author was partially supported by N.S.F. grant 9403708-DMS.
Jacobian ideal are useful starting points for many equisingularity conditions. These spaces are the relative conormal of $f$ and the conormal of $X$. The latter consist of the closure of the pairs $(x, H)$ where $x$ is a smooth point of $X$ and $H$ is the tangent hyperplane to $X$ at such a point, while the relative conormal is the closure of the pairs $(x, H)$, where $x$ is a point $f$ is a submersion, and $H$ is the tangent hyperplane to the fiber of $f$ through $x$. The exceptional divisors of these blowups record the behavior of the limiting tangent hyperplanes to $X$, resp. the smooth fibers of $f$. The equisingularity conditions that we have in mind, Whitney conditions and the $W_f$ condition, are defined in terms of the behavior of these limiting tangent planes.

The Segre numbers of $J(f_t)$ and the ideal $J(f_t)'$ induced by it in $\mathcal{O}_{X(t),0}$ give us invariants which depend only on $X(t)$ and which describe the exceptional divisors of $\text{Bl}_{J(f_t)}(\mathbb{C}^{n+1}, 0)$ and $\text{Bl}_{J(f_t)'}X$. To show that an equisingularity condition holds, it is necessary to pass to controlling the exceptional divisors of $\text{Bl}_{J(f)}(\mathbb{C}^{n+1+p}, 0)$ and $\text{Bl}_{J(f)'}X$. More generally, for an an arbitrary family $X$ and an ideal $I$ in $\mathcal{O}_{X,0}$, we want to control the exceptional divisor of $\text{Bl}_I X$ by conditions imposed on the exceptional divisors of $\text{Bl}_{I(t)} X(t)$. This is done in Theorem (4.6) which is a generalization of the principle of specialization of integral dependence due to Teissier. This theorem has as a corollary (4.9) a generalization of the theorem of Rees mentioned earlier. Böger’s theorem then follows as an easy application (4.10).

It turns out that the Whitney conditions are controlled by the Segre numbers of $mJ(f_t)$, where $m$ is the maximal ideal in $\mathcal{O}_{\mathbb{C}^{n+1},0}$. It is useful to relate these Segre numbers to the Segre numbers of $J(f_t)$ since these are more closely linked to the geometry of $X(t)$. In fact, the Segre numbers of $J(f_t)$ are just the Lé numbers of David Massey [Ma], and their alternating sum is the Euler characteristic of the Milnor fiber of $f$. This is done in greater generality in Theorem (3.5) which relates the Segre numbers of the product of an ideal $I$ with an $m$–primary ideal $J$ to the Segre numbers of $I$ and other invariants of the pair $I$ and $J$. This extends a result of Teissier [T1]. The theorem of Teissier relates the multiplicity of the product of two ideals of finite coength with their mixed multiplicities. In our theorem the mixed multiplicities are seen to be the polar multiplicities of the ideal $I$ computed with respect to the ideal $J$. In the case where $I = J(f_t)$ and $J = m$ these polar multiplicities are just the relative polar multiplicities of $f_t$.

All of these results come together in the main results of section 6. There we show that the constancy of the relative polar multiplicities of $f_t$ and of the Lé numbers of $f_t$ implies that $f$ satisfies the $W_f$ condition and that the smooth strata of $X$, its singular set $\Sigma(X)$ and the components of the singular set $\Sigma(\Sigma(X))$ of the singular set which are of codimension 1 in $\Sigma(X)$ are all Whitney regular over the parameter stratum. We also show that if $X$ has a Whitney stratification which includes $0 \times (\mathbb{C}^p, 0)$ as a stratum, then the above numbers are constant. This gives as a corollary a theorem of Adam Parusinski [Pa] which says that the existence of such a Whitney stratification for $X$ implies the $W_f$ condition for $f$. This proof fulfills Parusinski’s prediction that his result could be proved using the Lé numbers.

These results are possible because of a surprising phenomenon. It turns out that the constancy of the relative polar multiplicities of $f_t$ and of the Lé numbers of $f_t$ imply that the Whitney conditions hold for any stratum $W$ in the critical set of $f$ over the parameter stratum whose closure is the image of a component of the exceptional divisor of the blowup of the ambient space by the Jacobian ideal of $f$. This means that it is possible to show that many strata satisfy the Whitney
conditions using only these two sets of invariants. It remains to find a good criteria for when strata correspond to components of the exceptional divisor.

As a further application of our results, we show that the constancy of the relative polar multiplicities and of the Lê numbers in a family of 2-dimensional hypersurfaces implies that the family is Whitney equisingular (Corollary (6.6)). This result is possible, because the failure of smooth points of the total space to be Whitney over the smooth points of the singular set implies the existence of a component of the exceptional divisor of the relative Nash blowup of $\mathbb{C}^3 \times \mathbb{C}$ of the type controlled by our numerical conditions. This example indicates the importance of relating the components of the exceptional divisor of this blowup to the geometry of $X$. We then apply our results to the study of the family of hypersurfaces given by hyperplane slices of a three dimensional hypersurface $X$ (Example (6.7)).

Finally, as an indication of the power of our approach, in (Proposition (6.9)) we show that if a hypersurface satisfies the same relations among its Lê numbers as a hypersurface defined by a homogeneous polynomial, then the smooth points of the deformation of the hypersurface to its tangent cone satisfy the Whitney conditions over the parameter axis, provided the tangent cone is reduced. If the original space is 2-dimensional, then by (6.6), the deformation is Whitney equisingular.

It is a pleasure to acknowledge our debt to Bernard Teissier. His Cargese paper [T1] convinced us to try to control equisingularity conditions by invariants depending only on the members of the family under consideration, and showed us the elements from the isolated case which we needed to generalize to handle non-isolated singularities. David Massey’s work on the Lê numbers was a constant guide to us as we developed the properties of the Segre numbers. Steven Kleiman and David Massey also supported us through many conversations over the years that this paper developed. We also thank Steven Kleiman for many helpful comments on earlier versions of this paper.

The paper is organized as follows. In section 1, we recall the basic results of the theory of integral closure of ideals and their application to the equisingularity theory of families of hypersurfaces with isolated singularities. In section 2, we define the Segre cycles and polar varieties of an ideal, and prove some of their basic properties. In section 3, we give a formulation of these notions using intersection theory (3.1), and use this to describe how the Segre numbers change under hyperplane section in (3.4). Theorem (3.5) is the expansion formula. In section 4, we discuss the upper semi–continuity properties of the Segre numbers, and the behavior of the polar varieties of $I(t)$ in a family. In (4.6) we give a necessary and sufficient criterion for the Segre numbers of $I(t)$ to be constant along the parameter stratum in terms of the behavior of the components of the exceptional divisor of $\text{Bl}_t X$ which project to the parameter stratum, and the limiting secant behavior of the components which do not project to the parameter stratum. Our generalization of the principle of specialization of integral dependence (4.7) follows easily from this result. We then discuss the theorems of Rees and Böger. In section 5 we use our machinery to study Thom’s $a_f$ condition, recovering a result of Massey, while in section 6 we apply our results to the Whitney conditions and the $W_f$ condition.
1. The classical theory

(1.1) (Integral dependence). Integral dependence is used in local analytic geometry to translate inequalities between analytic functions into algebra. The basic source for this is the work [LeT] of Monique Lejeune–Jalabert and Teissier. Another useful reference is Joseph Lipman’s more algebraic survey [Li].

Let \((X, 0) \subseteq (\mathbb{C}^N, 0)\) be a reduced analytic space germ. Let \(I\) be an ideal in the local ring \(\mathcal{O}_{X,0}\) of \(X\) at \(0\), and \(f\) an element in this ring. Then \(f\) is integrally dependent on \(I\) if one of the following equivalent conditions obtain:

(i) There exists a positive integer \(k\) and elements \(a_j\) in \(I^k\), so that \(f\) satisfies the relation \(f^k + a_1 f^{k-1} + \cdots + a_k = 0\) in \(\mathcal{O}_{X,0}\).

(ii) There exists a neighborhood \(U\) of \(0\) in \(\mathbb{C}^N\), a positive real number \(C\), representatives of the space germ \(X\), the function germ \(f\), and generators \(g_1, \ldots, g_m\) of \(I\) on \(U\), which we identify with the corresponding germs, so that for all \(x\) in \(X\) the following equality obtains: \(|f(x)| \leq C \max\{|g_1(x)|, \ldots, |g_m(x)|\}\).

(iii) For all analytic path germs \(\phi : (\mathbb{C}, 0) \to (X, 0)\) the pull–back \(\phi^* f\) is contained in the ideal generated by \(\phi^* (I)\) in the local ring of \(\mathbb{C}\) at \(0\).

As an example take \(I\) to be the ideal generated by the images of the coordinates \(z_1, \ldots, z_{N−1}\) in \(\mathcal{O}_{X,0}\) and \(f\) the image of \(z_N\) in the same local ring. Then, \(f\) is integrally dependent on \(I\) if, and only if, the \(z_N\)–axis is not contained in the tangent cone of \(X\) at \(0\). For if \(\phi(t) : (\mathbb{C}, 0) \to (X, 0)\) is an analytic path then for suitable \(k\) the limit \(\lim_{t \to 0} f(t)/t^k(z_1, \ldots, z_N)\) gives the direction of the limiting secant line to \(X\) determined by \(\phi\). So none of these limits are tangent to the \(z_N\)–axis if, and only if, the order in \(t\) of \(\phi^* z_N\) is greater than or equal to the smallest of the orders of \(\phi^* z_i\). This is equivalent to \(\phi^* z_N\) being contained in the ideal generated by \(\phi^* z_1, \ldots, \phi^* z_{N−1}\) in the local ring of \((\mathbb{C}, 0)\). Thus, condition (iii) shows the claim.

If we consider the normalization \(\bar{B}\) of the blowup \(B\) of \(X\) along the ideal \(I\) we get another equivalent condition for integral dependence. Denote the pull–back of the exceptional divisor \(D\) of \(B\) to \(\bar{B}\) by \(\bar{D}\).

(iv) For any component \(C\) of the underlying set of \(\bar{D}\), the order of vanishing of the pullback of \(f\) to \(\bar{B}\) along \(C\) is no smaller than the order of the divisor \(\bar{D}\) along \(C\).

The elements \(f\) in \(\mathcal{O}_{X,0}\) that are integrally dependent on \(I\) form the ideal \(\bar{I}\), the integral closure of \(I\); often we are only interested in the properties of the integral closure of an ideal \(I\); so we may replace \(I\) by an ideal \(J\) contained in \(I\) with the same integral closure as \(I\). Such an ideal \(J\) is called a reduction of \(I\). In the above example, the ideal generated by the first \(N−1\) coordinate functions in the local ring of \(X\) at \(0\) is a reduction of its maximal ideal if, and only if, the \(z_N\)–axis is not contained in the tangent cone of \(X\) at \(0\).

It is easy to see that \(J\) is a reduction of \(I\) iff there exists a finite map \(\text{Bl}_J X \to \text{Bl}_I X\). In particular, the fibres of the two blowups over \(0\) have the same dimension. Samuel proved that any ideal \(I\) has a reduction generated by at most \(n\) elements where \(n\) is the dimension of \(X\) at \(0\). In fact, \(n\) generic linear combinations of given generators of \(I\) generate a reduction.

In general, an ideal \(I\) has a reduction generated by \(m + 1\) elements where \(m\) is the dimension of the fibre of \(\text{Bl}_I X\) over \(0\).

If \(X\) is equidimensional, and \(I\) is primary to the maximal ideal, its integral closure is completely determined by its multiplicity \(e(I)\). This follows from the following
(1.2) Theorem. An ideal $J$ contained in $I$ is a reduction of $I$ if, and only if, the multiplicity $e(I)$ equals the multiplicity $e(J)$.

Note that for any ideal $J \subseteq I$ the multiplicity of $J$ is not smaller than $e(I)$.

(1.3) (Specialization of integral dependence). In his paper [T1] Teissier established the Principle of Specialization of Integral Dependence of Ideals: Consider a reduced equidimensional family $X \to Y$ of analytic spaces, and an ideal sheaf $I$ on $X$ with finite co-support over $Y$. Suppose $h$ is a section of $O_X$ so that for all $t$ in a Zariski-open dense subset of $Y$ the induced section of $O_{X(t)}$ on the fibre over $t$ is integrally dependent on the induced ideal sheaf $I_O X(t)$. If the multiplicity $e(I_O X(t))$ is independent of $t$ in $Y$, then $h$ is integrally dependent on $I$. (The multiplicity $e(I_O X(t))$ is the sum of the multiplicities of the ideals induced by $I$ in the local rings $O_{X(t),x_t}$, where the points $x_t^1, \ldots, x_t^{k(t)}$ form the fibre of the support of $O_X/I$ over $t$.)

We give a brief sketch of the proof the principle as contained in [T3, Appendix 1]. For simplicity assume that $O_X/I$ is supported on the image of a section $t \mapsto x_t$ of $X$ over $Y$. Denote the fibre dimension of $X \to Y$ by $n$. By Samuel’s theorem, we can find $n$ elements $g_1, \ldots, g_n$ in the stalk of $I$ at $x_0$ the image of which in $O_{X(0),x_0}$ generate a reduction of the ideal induced by $I$. We identify all germs with representatives in a small neighborhood of $x_0$ in $X$. Denote the ideal sheaf generated by $g_1, \ldots, g_n$ by $J$, its stalk at 0 by $J$. Now, we use the upper-semicontinuity of the multiplicity:

$$e(I_O X(0),x_0) = e(J_O X(0),x_0) \geq e(J_O X(t),x_t) \geq e(I_O X(t),x_t) = e(I_O X(0),x_0).$$

Hence, all inequalities are equalities and, by Rees’ theorem, the ideal induced by $J$ in $O_{X(t),x_t}$ is a reduction of the ideal induced by $I$ for all $t$ in $Y$ close to 0.

Now, note that the underlying set of the exceptional divisor $D$ of the blowup $B$ of $X$ along $J$ equals $Y \times \mathbb{P}^{n-1}$. Also, as the normalization $\bar{B} \to B$ is finite, all components of the pull-back $\bar{D}$ of $D$ to $B$ are equidimensional over $Y$. Thus, it is not hard to see, using (1.1)(iv) that an element induced by a section $h$ of $O_X$ in $O_{X,x_0}$ is integrally dependent on $J$ if, for all $t$ in a Zariski-open subset of $Y$, the element induced by $h$ in $O_{X(t),x_t}$ is integrally dependent on the ideal induced by $J$. In fact, the order of vanishing of the pullback of $h$ to $\bar{B}$ along a component $C$ of the underlying set of $\bar{D}$ can be computed in the fibre over some generic $t$ in $Y$. By assumption, this is no smaller than the order of vanishing of $\bar{D}$ along $C$. This proves the theorem.

We want to point out that the main part in the proof is to establish the equidimensionality of the exceptional divisor over $Y$.

(1.4) (Equisingularity conditions). The Principle of specialization of integral dependence can be used to establish criteria for equisingularity conditions. Since the procedure for applying the principle is the same in all cases, we review the procedure in the case of condition $w$ for families of hypersurfaces with isolated singularities; this case was worked out by Teissier [T1].

Let $f : (\mathbb{C}^{n+2}, 0) \to (\mathbb{C}, 0)$ be function, vanishing on $0 \times (\mathbb{C}, 0)$ and assume that its restrictions $f_t = f|_{(\mathbb{C}^{n+1}, 0) \times t}$ are reduced. Let $z_0, \ldots, z_n$ be coordinates
on \((\mathbb{C}^{n+1}, 0)\) and \(t\) the coordinate on \((\mathbb{C}, 0)\). Consider the hypersurface \(X \subset \mathbb{C}^{n+2}\) defined by \(f\). By assumption, the fibres \(X(t)\) over points \(t\) in \((\mathbb{C}, 0)\) are reduced. Then, the smooth part of \(X\) satisfies Verdier’s condition \(w\) over \(0 \times (\mathbb{C}, 0)\) at 0 if the distance between the tangent hyperplane to \(X\) at a smooth point \(x\) of \(X\) and \(0 \times (\mathbb{C}, 0)\) goes to zero no faster than the distance of \(x\) to \(0 \times (\mathbb{C}, 0)\). (Here, we use an appropriate distance function of linear subspaces of \(\mathbb{C}^{n+1}\); see e.g. [T4, Ch. 3].) The first step is to describe this condition in terms of integral closure. Bernard Teissier did this by showing that it is equivalent to the inclusion

\[
\frac{\partial f}{\partial t} \bigg|_X \in m_z J_z(f)\mathcal{O}_{X,0},
\]

(1.4.1)

where \(m_z\) is the ideal generated by the \(z\)-coordinates, and \(J_z(f)\) is the ideal generated by the partial derivatives of \(f\) with respect to these coordinates.

The next step is to check that the equisingularity condition holds generically. This can be seen by using Teissier’s Idealistic Bertini Theorem (see e.g. [T2, Chapter 2]). The principle of specialization then ensures that the constancy of the multiplicity of \(mJ(f_t) \subseteq \mathcal{O}_{X(t),0}\) implies that the equisingularity condition holds at the origin as well. At this point there are two directions in which to move. One involves looking for geometric interpretations of the multiplicities used to control the equisingularity condition. In his Cargèse paper [T1], Teissier showed that the constancy of \(e(mJ(f_t))\) is equivalent to the constancy of the Milnor numbers of the plane sections of \(X_t\). The other direction is to show that the constancy of the invariants is necessary as well as sufficient for the equisingularity condition to hold. This can be done by showing that the equisingularity condition gives enough control over the exceptional divisor of the blowup of \(X\) along \(m_z J_z(f)\) so that the constancy of the multiplicity can be deduced from the intersection theory principle “Conservation of Number”; see [F, Ch. 10].

2. Segre cycles and Polar Varieties

We define the main devices of our work, Segre cycles and polar varieties of an ideal in the local ring of a reduced analytic space germ, and discuss some properties of their multiplicities.

Our approach was inspired by work of Teissier [T4] and Massey [Ma]. Similar work was done by Leendert VanGastel [Gas], and Kleiman and Anders Thorup. In fact, our polar multiplicities and Segre numbers are special cases of polar multiplicities as defined in [KT1, (8.1)].

(2.1) Setup. Let \((X, 0) \subseteq (\mathbb{C}^N, 0)\) be a reduced closed analytic space of pure dimension \(n\) and \(I \subset \mathcal{O}_{X,0}\) an ideal which defines a nowhere dense subspace of \((X, 0)\). Consider the blowup \(B\) of \(X\) along \(I\):

\[
b : B = \text{Bl}_I X \to X
\]

with exceptional divisor \(D\). Suppose \(I\) is generated by \((f_1, \ldots, f_M)\). Then, the blowup \(B\) equals the closure in \(X \times \mathbb{P}^{M-1}\) of the graph of the map

\[
X - V(I) \to \mathbb{P}^{M-1}, x \mapsto (f_1(x) : \cdots : f_M(x)).
\]

The restriction to \(B\) of the projection onto the second factor of the product \(X \times \mathbb{P}^{M-1}\) induces a map \(p : B \to \mathbb{P}^{M-1}\). Hence, a hyperplane in the projective space
induces a Cartier divisor on $B$ via the pull–back $p^*$, provided $B$ is not contained in the product of $X$ and the hyperplane. We call it a hyperplane on $B$.

Let $V$ be a reduced subspace of $B$ of pure dimension $k$ no component of which is contained in $D$. A hyperplane $H$ of $B$ is general with respect to $V$ if $H \cap V$ is reduced of dimension $k-1$ and none of its components is contained in $D$. Then, the intersection equals the strict transform by $b$ of its image $b(H \cap V)$ in $X$. In fact, outside of $D$, the map $b$ is an isomorphism, and by assumption, the closure of $H \cap (V - D)$ equals $H \cap V$.

Using Kleiman’s Transversality Lemma, one can show that there exists a Zariski–open subset of the set of all hyperplanes of $B$ which are general w.r.t. $V$ (see e.g. [T4, Ch. 4]).

Consider a hyperplane $H$ on $B$ induced by a hyperplane $H'$ in $\mathbb{P}^{M-1}$. An equation of $H'$ corresponds to a linear combination $g$ of the generators of $I$. Suppose that $H$ is general w.r.t. $B$, then the above argument shows that the topological closure of $V(g) - V(I)$ in $X$ equals the image of $H \cap B$. This simple observation will allow us to give an alternative constructions of our main devices, polar varieties and Segre cycles, in Lemma (2.2).

Let $Y$ be a reduced subspace of $X$ of pure dimension $m$, no component of which is contained in $V(I)$. Consider an $m$–tuple $g = (g_1, \ldots, g_m)$ of linear combinations of the generators of $I$. Assume that each hyperplane $H_i$ on $B$ corresponding to $g_i$ is general w.r.t. $H_1 \cap \cdots \cap H_{i-1} \cap \text{Bl}_i Y$. Then, we also say that $g$ is general with respect to $Y$. The polar varieties and Segre cycles of $I$ on $Y$ with respect to $g$ are defined as follows:

$$
P^g_0(I, Y) := Y, \quad P^g_k(I, Y) := b(H_1 \cap \cdots \cap H_k \cap \text{Bl}_i Y),$$

$$\Lambda^g_k(I, Y) := b_*(H_1 \cdots H_{k-1} \cdot D \cdot \text{Bl}_i Y).$$

The polar varieties are reduced by the above discussion. Note, the index $k$ gives the codimension of $P^g_k(I, Y)$ and $\Lambda^g_k(I, Y)$ in $Y$ if they are not empty. For polar varieties, this follows from the definition of ‘general’. A dimension count together with the properties of the push–forward establishes the result for Segre cycles.

Let $I'$ be the ideal induced by $I$ in $\mathcal{O}_{Y, 0}$ and $g'$ its set of generators given by the restriction of the elements of $g$ to $Y$. Then, the blowup of $Y$ along $I'$ is isomorphic to the strict transform of $Y$ by $b$, and the exceptional divisor of $Y$ is the intersection of the blowup of $Y$ with the exceptional divisor of $X$. Therefore the following equalities obtain for $k = 1, \ldots, m - 1$.

$$P^g_k(I, Y) = P^g_k(I', Y),$$

$$\Lambda^g_k(I, Y) = \Lambda^g_k(I', Y).$$

If $g$ is formed by generic linear combinations of the generators of $I$, we omit the superscript $g$ in the notation for polar varieties and Segre cycles. This notation is slightly imprecise because different generic $m$–tuples will yield, in general, different polar varieties and Segre cycles. However, their multiplicity at 0 is independent of the choice of the generic linear combinations. Teissier [T4, IV, 1.3, p. 419] proved this for the special case of the Jacobian ideal. In fact, we will see in the next section that the multiplicities of both, generic polar varieties and generic Segre cycles, are
given by intersection formulas which are independent of the choice of generic \( g \).

Thus, the \( k \)th polar multiplicity of \((I, Y)\) is well–defined for \( k = 0, \ldots, m - 1 \) as

\[
m_k(I, Y) := \text{mult}_0 P_k(I, Y),
\]

and for \( k = 1, \ldots, m \) the \( k \)th Segre number of \((I, Y)\) is

\[
e_k(I, Y) := \text{mult}_0 \Lambda_k(I, Y),
\]

where the multiplicity of a cycle \( S = \sum a_i[V_i] \) at 0 is defined as

\[
\text{mult}_0 S = \sum a_i\text{mult}_0 V_i.
\]

By our definition of a general, a general \( m \)–tuple \( g \) of linear combinations of generators of \( I \) need not be generic. So, by the upper semi–continuity of the multiplicity, we have for \( k = 1, \ldots, m - 1 \),

\[
m_k(I, Y) \leq \text{mult}_0 P^g_k(I, Y) \quad (2.1.3)
\]

Let \( X = (\mathbb{C}^{n+1}, 0) \) and form the Jacobian ideal \( I \) of a function

\[
f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)
\]
generated by its partial derivatives

\[
g = \left( \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_0} \right).
\]

Suppose that the coordinates on \( X \) are sufficiently general so that \( g \) is a general \( m \)–tuple of linear combinations of generators of \( I \). Then \( \Lambda_k^g(I, X) \) is called the \( k \)–codimensional Lê cycle \( \Lambda_k^g(f) \) of \( f \) with respect to the coordinates \( z = (z_0, \ldots, z_n) \) (see [Ma]). Its multiplicity \( \lambda_k^g(f) \) at 0 is called the \( k \)–codimensional Lê number of \( f \) at 0 w.r.t. \( z \). We will return to this special case in Section 5.

The following Lemma gives a useful description of Segre cycles and polar varieties.

(2.2) Lemma. Assume that \( g = (g_1, \ldots, g_m) \) is a generic \( m \)–tuple of linear combinations of generators of \( I \). For \( k = 1, \ldots, m - 1 \), the \( k \)–codimensional polar variety of \( I \) on \( Y \) equals the closure of

\[
V(g_k|_{P^g_{k-1}(I,Y)}) - V(I)
\]
in \( Y \).

Also, the following equalities of cycles obtain:

\[
\Lambda_k^g(I, Y) = [V(g_k|_{P^g_{k-1}(I,Y)})] - [P^g_k(I, Y)],
\]

\[
\Lambda_m^g(I, Y) = [V(g_m|_{P^g_{m-1}(I,Y)})].
\]
**Proof.** The first statement follows directly from the definition of the polar varieties and the discussion that preceeded it.

For the second statement, using (2.1.1) and (2.1.2), we replace \( X \) by \( P_{k-1}(I,Y) \) and \( I \) by the ideal induced by \( I \) in the local ring of \( P_{k-1}(I,Y) \) at 0. Then, we may assume that \( k = 1 \). Note that there is nothing to prove if \( V(I) \) is of codimension bigger than one. So, we assume that \( V(I) \) has codimension one.

By definition, the one–codimensional Segre cycle of \( I \) on \( X \) equals the push–forward to \( X \) of the exceptional divisor of the blowup of \( I \) in \( X \). Its underlying set equals the set formed by the underlying components of the underlying set of \( V(I) \) of codimension one. So, we need to show that the multiplicity of such a one–codimensional component \( W \) in the cycle \([V(g_1)]\) equals the multiplicity of \( W \) in \( \Lambda_1(I,X) \).

Now, the first multiplicity is, by definition, given by the multiplicity of the ideal \((g_1)\mathcal{O}_{W,X}\). On the other hand, the multiplicity of \( W \) in \( \Lambda_1(I,X) \) equals the multiplicity of the ideal \( I\mathcal{O}_{W,X} \) (see [Fu, Ex. 4.3.4, p.81]). Finally, as the generators of \( I \) are assumed to be generic, the two multiplicities in question are equal by a theorem of Samuel (see [Fu, Ex. 4.3.5(a), p.81]). The proof is now complete.

The following equality follows immediately from the above lemma:

\[
e_k(I,Y) = e_1(I, P_{k-1}(I,Y)). \tag{2.2.1}
\]

Also, the lemma shows that generic polar varieties and Segre cycles do not change if we replace \( I \) by a reduction; for instance, the ideal generated by \( n \) generic linear combinations of \( I \).

The lemma is a useful tool for computing Segre cycles and Segre numbers. For example, let \( I \) be a principal ideal of \( \mathcal{O}_{\mathbb{C}^n,0} \) generated by \( f \). Then we have \( \Lambda_1(I,X) = [V(f)] \) and all other Segre cycles are empty. Therefore, \( e_1(I,X) \) equals the multiplicity of \( V(f) \) at 0.

**(2.3) Proposition.** Let \( g \) be an \( m \)–tuple of generic linear combinations of generators of \( I \), and \( k = 1, \ldots, m-1 \). Let \( p : \mathbb{C}^N \to \mathbb{C}^{m-k} \) be a linear map so that the \((N-m+k)\)–plane \( H = p^{-1}(0) \) intersects the tangent cone at 0 of \( \Lambda^g_k(I,Y) \) transversally. Let \( \epsilon \) be a general point of \( \mathbb{C}^{m-k} \) close to 0, and \( H^\epsilon = p^{-1}(\epsilon) \). Then,

\[
e_k^g(I,Y) = \sum_{x \in [\Lambda^g_k(I,Y)] \cap H^\epsilon} e(I\mathcal{O}_{p^g_k(I,Y)} \cap H^\epsilon, x). \]

**Proof.** By (2.2.1) and (2.1.2) we may replace \( X \) by \( P_{k-1}(I,Y) \). Thus, we may assume that \( k = 1 \), and \( X = Y \). Let \( W \) be a component of the underlying set of \( \Lambda_1(I,X) \). Then, by [Li, (3.5)] applied to the restriction of \( p \) to a neighbourhood in \( X \) of a point \( x \) in \( W \cap H^\epsilon \), we have

\[
e(I\mathcal{O}_{W,X}) = e(I\mathcal{O}_{X \cap H^\epsilon, x}).
\]

By [Fu, Ex. 4.3.3, p.81], the multiplicity of \( I\mathcal{O}_{W,X} \) equals the multiplicity of \( W \) in \( \Lambda_1(I,X) \). Also, the number of points in \( W \cap H^\epsilon \) equals the multiplicity of \( W \) at 0. This implies the claim.

**(2.4)** *(Moving and Fixed Components).* Let \( Y \) be a reduced subspace of \( X \) of pure dimension \( m \). A subset \( W \) of \((X,0)\) will be said to be *distinguished* by \((I,Y)\) if it is the image in \( Y \) of a component of \( D \cdot \text{Bl}_f Y \).
A set $W_k$ of codimension $k$ in $Y$ which is distinguished by $(I, Y)$ is then a component of $\Lambda_k(I, Y)$, as can be seen from a dimension argument. Such a component is called a fixed component of the $k$th Segre–cycle of $I$ on $Y$. A component of $\Lambda_k(I, Y)$ which is not distinguished by $(I, Y)$ is called a moving component. It is distinguished by some $(I, P)$ where $P$ is a polar variety of dimension bigger than $\Lambda_k(I, Y)$. (It is easy to see that it is distinguished by the pair $(I, P_{k-1}(I, Y))$. In general it may be distinguished by some higher–dimensional polar variety as well.) The reason for this terminology is simple. As the hyperplanes in the definition of the Lê cycles are varied, the fixed components will not change, while the moving components will.

A moving component of $\Lambda_k(I, X)$ comes from an irreducible component $C$ of the exceptional divisor $D$ of $\text{Bl}_I X$, whose image is of codimension less than $k$, and the fibre of $C$ over 0 has dimension at least $k - 1$. If the dimension of this fibre equals $k - 1$, then the component $C$ will not induce moving components of the Segre cycles of $(I, X)$ of codimension bigger than $k$. This follows directly from the definition of Segre cycles.

3. Intersection Formulas

In (3.2) we express the Segre numbers as intersection numbers. Hence, intersection theory is the main tool, although only the rudiments are needed. We review some basic definitions from Fulton’s book [Fu] in (3.1). In Lemma (3.3) we study the relation of the blowup of an ideal $I$ on a space germ $X$ and the blowup of the induced ideal on a hyperplane slice of $X$. This leads to relations of the Segre numbers of $I$ and the induced ideal on the hyperplane slice, as described in (3.4). The expansion formulas (3.5) express the Segre numbers of the product of $I$ and the maximal ideal in terms of the Segre number of $I$ and its polar multiplicities. We restate a result of Kleiman and Thorup that gives a length theoretic interpretation of the top Segre number in (3.7).

(3.1) (Intersection Theory). Let $X \subseteq (\mathbb{C}^N, 0)$ be a reduced analytic space germ of pure dimension $n$ containing 0, let $I \subseteq \mathcal{O}_{X, 0}$ be an ideal and $m \subset \mathcal{O}_{X, 0}$ be the maximal ideal. Let $\Lambda_k(m I, X)$ be a generic $k$–codimensional Segre cycle of $m I$ and $e_k(m I, X)$ its multiplicity at 0. The main tool for studying these Segre cycles is the following commutative diagram.

$$
\begin{array}{c}
D \subset B = \text{Bl}_m I X \xrightarrow{b_1} B_2 = \text{Bl}_I X \supset D_2 \\
\downarrow b_1 \quad \quad \quad \quad \quad \downarrow b_2
\end{array}
$$

$$
D_1 \subset B_1 = \text{Bl}_m X \xrightarrow{b_1} X
$$

The exceptional divisors of the blowups $B_1, B_2$ are $D_1, D_2$. The exceptional divisor of the blowup $b : \text{Bl}_m I X \to X$ is denoted $D$.

We will use the first Chern classes of the tautological line bundles on the blowups $h_1 = c_1(\mathcal{O}_{B_1}(1)), h_2 = c_1(\mathcal{O}_{B_2}(1)), h = c_1(\mathcal{O}_B(1))$. The reader should think of such a class, say $h_1$, as an operator that gives, applied to an irreducible variety $V \subseteq B_1 \subset (X \times \mathbb{P}^n)$, the rational equivalence class of $V \cap H$, where $H$ is a generic hyperplane of $B_1$, that is, $H = (X \times H') \cap B_1$ with $H'$ a generic hyperplane of $\mathbb{P}^n$. The divisor $H$ represents the tautological line bundle on $B_1$. (The correctness of this view
point follows from Kleiman’s Transversality Lemma which implies that \( H \) intersects \( V \) transversally and the intersection is reduced and of dimension \( \dim V - 1 \). This definition extends to cycles by linearity.

The line bundle on \( B_i \) associated to \( D_i \) is the dual of \( \mathcal{O}_{B_i}(1) \). Hence, intersecting with \( D_i \) equals the operation of \(-h_i\). We say that (the line bundle associated to) \( D_i \) is the dual of \( \mathcal{O}_{B_i}(1) \).

For \( i = 1, 2 \) the pull–back of \( h_i \) to \( B \) will be denoted by \( \tilde{h}_1 \) and the pull–back of \( D_i \) in \( B \) by \( \tilde{D}_i \). Then, the following equalities of Cartier divisors, resp. first Chern classes, obtain (see [KT2, (2.7)])

\[
D = \tilde{D}_1 + \tilde{D}_2, \quad h = \tilde{h}_1 + \tilde{h}_2.
\] (3.1.1)

For a cycle \( S \) on \( B \) the part of \( S \) formed by the components whose generic points map into \( X - 0 \) will be denoted \( S^{X - 0} \). In other words, the cycle \( S^{X - 0} \) is formed by those components not lying in the fibre of \( B \) over 0. Note that for a generic hyperplane \( \mathcal{H} \) on \( B \) we have

\[
\mathcal{H} \cap S^{X - 0} = (\mathcal{H} \cap S)^{X - 0}.
\] (3.1.2)

To see this we count dimensions, assuming \( \mathcal{H} \cap S^{X - 0} \) is not empty. Then, by Kleiman’s Transversality Lemma, we may assume that \( \mathcal{H} \) intersects every component of the fibre of \( S^{X - 0} \) over 0 properly. Hence,

\[
\dim \mathcal{H} \cap S^{X - 0} = \dim S^{X - 0} - 1 \geq \dim(S^{X - 0} \cap b^{-1}(0)) > \dim(\mathcal{H} \cap S^{X - 0} \cap b^{-1}(0)).
\]

Therefore, no component of the fibre of \( \mathcal{H} \cap S^{X - 0} \) over 0 is a component of \( \mathcal{H} \cap S^{X - 0} \).

Clearly, the same argument works for hyperplanes representing \( \tilde{h}_1 \) and \( \tilde{h}_2 \).

Also, denote the part of \( S \) formed by the components mapping into 0 by \( S^0 \). Then, (3.1.2) is equivalent to

\[
\mathcal{H} \cap S^0 = (\mathcal{H} \cap S)^0.
\] (3.1.3)

We will identify a Cartier divisor \( D \) with its associated Weil divisor. We also write \( D^0 \) for the part of its associated Weil divisor formed by the components mapping to 0. The cycle \( D^{X - 0} \) is defined analogously.

The degree of a cycle \( S \) is denoted \( \int S \). It is the sum of the multiplicities of the 0–dimensional components. For a cycle \( S \) in the fibre of \( B \) over 0, the degree of \( S \) depends only on the rational equivalence class of \( S \) in this fibre. For cycles outside this fibre the degree is no invariant of rational equivalence.

(3.2) (Intersection Formulas). We can express the multiplicities of the Segre cycles at 0 as intersection numbers

\[
e_k(m, I, X) = \int \tilde{h}_1^{n-k-1} h^{k-1} D^{X - 0} \cdot \tilde{D}_1, \quad k = 1, \ldots, n - 1,
\] (3.2.1)

\[
e_n(m, I, X) = \int h^{n-1} D^0,
\] (3.2.2)

\[
e_k(I, X) = \int \tilde{h}_1^{n-k-1} \tilde{h}_2^{k-1} \tilde{D}_2^{X - 0} \cdot \tilde{D}_1, \quad k = 1, \ldots, n - 1,
\] (3.2.3)

\[
e_n(I, X) = \int h_2^{n-1} D_2^0.
\] (3.2.4)
The following formula is a generalization of Lê and Teissier’s polar multiplicity formula [LT2, (5.1.1)]

\[
m_k(I, X) = \int \tilde{h}_1^{n-k-1} \tilde{h}_2^k \tilde{D}_1. \tag{3.2.5}
\]

They considered the special case of the Jacobian ideal, but their proof works also in our setup. Alternatively, the formula can be verified by using similar arguments to the ones that will imply (3.2.1)–(3.2.4); see also [KT1, (8.9)].

We extend the notion of strict transform to cycles by linearity. Then the blowup of a cycle is defined to be its strict transform by the blowup–map.

To prove the first formula consider

\[
\text{Bl}_m \Lambda_k(m I, X) = \text{Bl}_m(\tilde{b}_*(D \cap H_1 \cap \cdots \cap H_{k-1})) = \tilde{b}_2*(D^{X-0} \cap H_1 \cap \cdots \cap H_{k-1}),
\]

where the \( H_i \) are generic hyperplanes of \( B \). The first equality follows from the definition of \( \Lambda_k(m I) \). By the argument that led to (3.1.2) only the components of \( D \) mapping to a subset of codimension at least \( k \) are not annihilated by intersecting with \( k-1 \) generic hyperplanes of \( B \). The second equality follows. Now,

\[
e_k(m I, X) = \int \tilde{h}_1^{n-k-1} \text{Bl}_m \Lambda_k \cdot D_1 = \int \tilde{h}_1^{n-k-1}(D^{X-0} \cap H_1 \cap \cdots \cap H_{k-1}) \cdot \tilde{D}_1 = \int \tilde{h}_1^{n-k-1} h^{k-1} D^{X-0} \cdot \tilde{D}_1.
\]

Here, the first equality is the well–known intersection formula for the multiplicity of a cycle. (In fact, it is just a translation of Samuel’s definition of the multiplicity of an ideal into geometry.) The second equality follows from the equality above and the projection formula [F, Prop. 2.5, p.41]. Then, we pass to rational equivalence to get the formula.

The second formula follows directly from the definition of \( e_n \) and (3.1.3). The proof of the next two formulas runs analogously.

**(3.3) Lemma.** For \( k = 1, \ldots, n-1 \) let \( L_{n-k} \) be a generic \((n-k)\)-codimensional linear subspace of \( \mathbb{C}^N \). Consider the blowup \( \text{Bl}_1(X \cap L_{n-k}) \) with exceptional divisor \( D_{2,n-k} \). Then, its pull–back \( \tilde{D}_{2,n-k} \) to \( B \) satisfies the following relations up to rational equivalence:

\[
\tilde{D}_{2,n-k}^0 = \tilde{h}_1^{n-k} \tilde{D}_2^0 = \tilde{h}_1^{n-k-1} \tilde{h}_2 \tilde{D}_1 + \tilde{h}_1^{n-k-1} \tilde{D}_2^{X-0} \cdot \tilde{D}_1,
\]

\[
\tilde{D}_{2,n-k}^{X-0} = \tilde{h}_1^{n-k} \tilde{D}_2^{X-0}.
\]

**Proof.** We will only show the formulas for \( L_1 \). The general case follows by induction. Now,

\[
\text{Bl}_{m I}(X \cap L_1) = \text{Bl}_{b_1 I}(H \cap B_1) = (\tilde{b}_2^{-1} H) \cap B,
\]

where \( H \) is the hyperplane on \( B_1 \) induced by the hyperplane in \( \mathbb{P}^{N-1} \) corresponding to \( L_1 \). Indeed, the first equality follows from (3.1.3), and the second from the properties of the pull–back of a line bundle. Passing to rational equivalence implies the second relation. For the first we consider

\[
\tilde{D}_{2,1}^0 = \tilde{h}_1 \tilde{D}_2^0 = (-\tilde{D}_1) \cdot (\tilde{D}_2 - \tilde{D}_2^{X-0}) = \tilde{h}_2 \tilde{D}_1 + \tilde{D}_2^{X-0} \cdot \tilde{D}_1.
\]

Here, the first equality follows from the properties of \( \tilde{h}_1 \), the second and third use duality. \( \square \)
Let $L_{n-k}$ be a generic $(n-k)$–codimensional linear subspace of $\mathbb{C}^N$. Then there are the following relations of Segre numbers of $(I, X)$ and those of $(I, X \cap L_{n-k})$.

\begin{align*}
e_i(I, X \cap L_{n-k}) &= e_i(I, X) \quad \text{for } i = 1, \ldots, k - 1, \\
e_k(I, X \cap L_{n-k}) &= m_k(I, X) + e_k(I, X).
\end{align*}

Indeed, the same argument as above shows that the divisor $\tilde{D}_{1,n-k}$, defined by the pull–back of the maximal ideal $m$ to $\text{Bl}_{mI}(X \cap L_{n-k})$ is given by $\tilde{h}_0^{n-k} \tilde{D}_1$. Then the first relation follows immediately from the above lemma (3.3) and the intersection formula (3.2.3):

\begin{align*}
e_i(I, X \cap L_{n-k}) &= \int \tilde{h}_1^{k-i-1} \tilde{h}_2^{i-1} \tilde{D}_2^{X-0} \tilde{D}_{1,n-k} \\
&= \int \tilde{h}_1^{n-i-1} \tilde{h}_2^{i-1} \tilde{D}_2^{X-0} \tilde{D}_1 = e_i(I, X).
\end{align*}

For the second relation observe that the push–forward of $\tilde{D}_{2,n-k}^0$ to $B_2$ equals $D_{2,n-k}^0$. Hence, we can use the projection formula and (3.3) to compute $e_k(I, X \cap L_{n-k})$ on $B$:

\begin{align*}
e_k(I, X \cap L_{n-k}) &= \int \tilde{h}_2^{k-1} (\tilde{h}_1^{n-k-1} \tilde{h}_2 \tilde{D}_1 + \tilde{h}_1^{n-k-1} \tilde{D}_2^{X-0} \cdot \tilde{D}_1).
\end{align*}

The polar multiplicity formula (3.2.5) and (3.2.3) yield the desired relation.

**Theorem** (The expansion formulas). In the setup (3.1) the following formulas obtain.

\begin{align*}
e_n(mI, X) &= \sum_{i=0}^{n-1} \binom{n}{i} m_i(I, X) + \sum_{i=1}^{n} \binom{n-1}{i-1} e_i(I, X), \\
e_k(mI, X) &= \sum_{i=1}^{k} \binom{k-1}{i-1} e_i(I, X).
\end{align*}

**Proof.** Using the above formula (3.1.1), we expand $h = \tilde{h}_1 + \tilde{h}_2$ in (3.2.1), and get

\begin{align*}
e_k(mI, X) &= \sum_{i=0}^{k-1} \binom{k-1}{i} \int \tilde{h}_1^{n-i-2} \tilde{h}_2 \tilde{D}_2^{X-0} \cdot \tilde{D}_1 = \sum_{i=0}^{k-1} \binom{k-1}{i} e_{i+1}(I, X).
\end{align*}

This implies the second formula.

For the first formula we also expand $h = \tilde{h}_1 + \tilde{h}_2$ and $D^0 = \tilde{D}_1 + \tilde{D}_2^0$, and use the Polar Multiplicity formula (3.2.5):

\begin{align*}
e_n(mI, X) &= \sum_{i=0}^{n-1} \binom{n-1}{i} \int \tilde{h}_1^{n-i-1} \tilde{h}_2 (\tilde{D}_1 + \tilde{D}_2^0) \\
&= \sum_{i=0}^{n-1} \binom{n-1}{i} (m_i(I, X) + \int \tilde{h}_1^{n-i-1} \tilde{h}_2 \tilde{D}_2^0).
\end{align*}
Now, by (3.3) and (3.4.2), for $i$ less than $n - 1$ the degree of $\tilde{h}_1^{n-i-1}\tilde{h}_2^i\tilde{D}_0^i$ equals $e_{i+1}(I, X) + m_i(I, X)$. Hence,

$$e_n(m, X) = \sum_{i=0}^{n-1} \binom{n-1}{i} m_i(I, X)$$

$$+ \sum_{i=1}^{n-1} \binom{n-1}{i-1} (e_i(I, X) + m_i(I, X)) + e_n(I, X).$$

Adding the two sums yields the formula. 

(3.6) Remark. The results of this section remain valid, if the maximal ideal $m$ is replaced by an $m$–primary ideal $m'$. This should be useful in the study of weighted homogeneous polynomials: Let $m' = (z_1^{P_1}, \ldots, z_N^{P_N})$ where the $p_i$’s are positive integers. We define the weight of $z_i$ to be $1/p_i$. Let $X = V(f)$ be a hypersurface given by a weighted homogeneous polynomial $f$ of weighted degree $d$. Then we have $m_0(m', X) = dp_1 \ldots p_N$ (see [Li, p.113] and [MO, p.387]).

In fact, it is possible to replace $m$ by an ideal $m'$ that induces an ideal of finite codimension in the local ring of $V(I)$ at 0 but not on $O_{X,0}$. Still, most of the results of this section remain valid. This case seems to play an important role in the study of Thom’s $a_f$ condition on singular spaces.

In [KT2,(3.6)] Kleiman and Thorup define the generalized Samuel multiplicity of a closed subscheme on a module at a closed point. We apply their proposition (3.5) to the subscheme $Z$ defined by $I$ in $X$, the structure sheaf of $X$ and the origin. The $k$th infinitesimal neighborhood of 0 in $Z$ is denoted $Z_k$.

(3.7) Proposition. As a function of $m$, the dimension

$$p(m) = \dim(\oplus_{i=0}^{m} (I^i/I^{i+1}) \otimes O_{Z_k})$$

is eventually a polynomial of degree at most $r = \dim Z$. Moreover, for $k \gg 0$, the coefficient of $m^r/r!$ is independent of $k$ and equals $e_n(I, X)$.

Proof. Use [KT2,(4.3)] to see the connection of the statement of [KT2,(3.5)] with the Segre numbers of the ideal $I$.

This shows that the top Segre number of an ideal generalizes in a natural way Samuel’s multiplicity. No such length–theoretic interpretation is known for lower dimensional Segre numbers.

4. The Specialization of Integral Dependence

This section contains our main result, the principle of specialization of integral dependence. We consider a family of analytic spaces and a given ideal sheaf on the total space. As a start we study the first Segre numbers of the ideals induced on the fibers by the given one on the total space. They are upper semi–continuous. The first Segre number controls the components of the divisor mapping to codimension one subsets of the special member of the family; see (4.4). Proposition (4.3) shows that if the first Segre numbers are constant then the special fibre of the one–codimensional polar variety of the total space equals the one–codimensional polar variety of the special fibre. This serves as the starting point of an induction in the proof of the lexicographically upper semi–continuity of the Segre numbers (4.5).
The next proposition (4.6) links the constancy of the Segre numbers in a family to a very strong equidimensionality condition. Finally, the principle of specialization of integral dependence (4.7) follows from a classical connection of integral dependence with this exceptional divisor.

As a corollary of the principle of specialization we obtain a generalization of Rees’ theorem (1.2). For an ideal whose co-support is nowhere dense we show that its Segre numbers control its integral closure. Remark (4.11) indicates that a similar result is true for arbitrary ideals when the Segre numbers are replaced by numbers that come from the completed normal cone of the ideal.

As a first application we give a numerical characterization of limiting tangent hyperplanes of a hypersurface.

(4.1) Setup. Let $F : (X, 0) \to (Y, 0)$ be a map of germs of analytic spaces. Assume that the fibres $X(t)$ are reduced and equidimensional of the same dimension $n$ at least 1. We assume that $X$ is embedded in $(\mathbb{C}^N, 0) \times (Y, 0)$ and that $F$ is induced by the projection onto the second factor. Furthermore, assume that $0 \times (Y, 0)$ is contained in $X$; we will identify it with $Y$. Its defining ideal sheaf in $X$ will be denoted $m_Y$. Let $I \subseteq \mathcal{O}_X$ be a sheaf of ideals on $X$ and denote the ideal induced by $I$ in $\mathcal{O}_{X(t), 0}$ by $I(t)$. Assume that $Z(t) = V(I(t))$ is nowhere dense in $X(t)$.

We work in a modified version of the setup (3.1) where $m$ is replaced by $m_Y$. In addition, for $t$ in $Y$, we consider the blowup $B_{2, t} = Bl_{I(t)} X(t)$ with exceptional divisor $D_{2, t}$. It sits inside the fibre $B_2(t)$ of $B_2$ over $t$, and the tautological bundle $\mathcal{O}_{B_2}(1)$ restricts to the tautological bundle $\mathcal{O}_{B_{2, t}}(1)$. We define $B_t$ and $B_{1, t}$ in the analogous way.

The Segre numbers $e_k(I(t), X(t))$ will be denoted by $e_k(t)$, for $k = 1, \ldots, n$.

(4.2) Proposition. The map $t \mapsto e_1(t)$ is upper semi–continuous.

Proof. (i) The map $t \mapsto e_1(t)$ is ‘non–decreasing’; that is, if $A$ is an analytic subvariety of $Y$ and $t$ is a general point of $A$ close to 0, then $e_1(t) \leq e_1(0)$.

As our numbers are defined on the fibres, we may replace $A$ by $Y$. We are going to use the classical result on the upper semi–continuity of multiplicities of ideals of finite co-length; see e.g. [Li,(3.5)]. Choose a linear map $p : \mathbb{C}^N \to \mathbb{C}^{n-1}$ the kernel of which intersects the tangent cone of $\Lambda_1(I(0), X(0))$ at 0 transversally. Next, we may choose a neighborhood $U$ of 0 in $\mathbb{C}^N$ and a representative of $(Y, 0)$ which we denote again by $Y$ so that, after identifying all germs involved with their representatives on $U \times Y$, the map

$$\pi : \Lambda_1(I, X) \to (\mathbb{C}^{n-1} \times Y), (z, t) \mapsto (p(z), t)$$

is finite. Also, we may assume that the map

$$(\epsilon, t) \mapsto \sum_{x \in p^{-1}(\epsilon) \cap X(t)} e(I\mathcal{O}_{p^{-1}(\epsilon) \cap X(t), x})$$

is upper semi–continuous on the image of $\pi$.

Next, choose general points $t$ in $Y$ and $\epsilon$ in $\mathbb{C}^{n+1}$ close to 0 so that

$$e_1(0) = \sum_{x \in p^{-1}(\epsilon) \cap X(0)} e(I\mathcal{O}_{p^{-1}(\epsilon) \cap X(0), x})$$
and so that the above map (4.2.1) attains its generic value at \((\epsilon, t)\). Finally, again by upper semi–continuity, we may assume that this number is at least \(e_1(t)\). (Here, we need to consider the family of finite projections of \(\Lambda_1(I(t), X(t))\) onto \(\mathbb{C}^{n-1}\).) This proves the ‘non–decreasing’ statement.

(ii) The map \(t \mapsto e_1(t)\) is constant on an Zariski–open subset of \(Y\). Consider a component \(C\) of \(\tilde{D}_2^{X-Y}\) the fibre of which over 0 in \(Y\) maps to a codimension one subset in \(X(0)\). In general, not all components of the intersection \(C \cap \tilde{D}_1\) will map onto \(Y\). For a point \(t\) in the image of a component of this intersection that doesn’t map onto \(Y\) we don’t expect \(e_1(t)\) to be generic. Therefore, consider the Zariski–closed subset \(F\) in \(Y\) formed by the images of such ‘vertical’ components, and, in addition, by the singular locus of the underlying set of \(Y\) and the images in \(Y\) of components of \(\tilde{D}_1\) that don’t map onto \(Y\). Then, for \(t\) in \(Y - F\), a dimension argument shows that the intersection with \(\tilde{D}_1, t\) of the part of \(\tilde{D}_2, t\) formed by components that map to subsets of codimension one in \(X(t)\) equals the fibre over \(t\) of the part of \(\tilde{D}_2 \cdot \tilde{D}_1\) formed in the same way; see the proof of [GK, (2.1)]. This is the part of the intersection that is relevant to the computation of \(e_1(t)\). It follows that

\[
e_1(t) = \int \hat{h}_1^{n-2} \tilde{D}_2^{X(t)-0} \tilde{D}_1, t = \int \hat{h}_1^{n-2} (\tilde{D}_2^{X-Y} \tilde{D}_1)(t)
\]

is independent of \(t\) in \(U\) by ‘conservation of numbers’; see [F, Prop.10.2, p.180] and the discussion in the proof of (4.6).

(4.3) Proposition. Assume that the map \(t \mapsto e_1(t)\) is constant. Then, the one–codimensional polar variety of \(I\) on \(X\) specializes. That is,

\[
P_1(I, X)(0) = P_1(I(0), X(0)).
\]

Proof. By Lemma (2.2), the polar variety \(P_1(I(0), X(0))\) of the fibre \(X(0)\) is contained in the fibre of \(P_1(I, X)\) over 0. So, we have to show that no component of the fibre of \(P_1(I, X)\) over 0 is contained in the fibre of \(V(I)\) over 0.

To see this, we choose, as in the proof of (4.2), a generic linear projection \(p : \mathbb{C}^n \to \mathbb{C}^{n-1}\) Then, by a similar argument as in the above proof and the assumptions, the map (4.2.1) has constant value \(e_1(0)\). Hence, for a general \(\epsilon\) in \(\mathbb{C}^{n-1}\) close to 0, the multiplicity of the ideal \(I'\) induced by \(I\) on the fibres of the family of curves \(X' = (p^{-1}(\epsilon) \times Y) \cap X\) is constant. Also, note that \(P_1(I', X') = P_1(I, X) \cap (p^{-1}(\epsilon) \times Y)\). Hence, it is enough to show that \(P_1(I', X')\) is empty. This follows from the proof of the classical Principle of Specialization; see (1.3). In fact, the constancy of the multiplicity of \(I'\mathcal{O}_{X'}(t)\) implies that each component of the exceptional divisor of the blowup of \(X'\) along \(I'\) is mapped onto \(Y\) by \(F \circ b_2\). The desired result follows.

(4.4) Remark. The above proof shows that the map \(t \mapsto e_1(t)\) controls components of the exceptional divisor \(D_2\) the fibres of which over 0 in \(Y\) map to codimension one subsets in \(X(0)\). If it is constant then all such components are mapped onto \(Y\) by \(F \circ b_2\).

Also, the constancy of \(e_1(t)\) implies that the image of such a component in \(X\) contains \(Y\). If not, the generic value of the map (4.2.1) would be bigger than \(e_1(0)\).

(4.5) Corollary. The \(n\)–tuple \((e_1(t), \ldots, e_n(t))\) is lexicographically upper semi–continuous: If for some \(k\) the map \(t \mapsto (e_1(t), \ldots, e_k(t))\) is constant on \(Y\), then
Proof. The existence of a Zariski–open subset of $Y$ on which $t \mapsto e_{k+1}(t)$ is constant follows, as in the proof of (4.2), from conservation of numbers. (See also the first part of the proof of (4.6).)

We are going to prove by induction on $k$ that the constancy of the map $t \mapsto (e_1(t), \ldots, e_k(t))$ implies $e_{k+1}(0) \geq e_{k+1}(t)$ for all $t$, and the equality $P_k(I, X)(0) = P_k(I(0), X(0))$. The case $k = 0$ is proven in (4.2) and (4.3). So, assume that $k$ is non–zero.

Fix a generic $n$–tuple $g$ of linear combinations of generators of $I$. Then, by the induction hypothesis, we have

$$e_k(0) = e_1(I(0), P_{k-1}^g(I, X)(0)), \quad \text{and} \quad e_k(t) \leq e_1(I(t), P_{k-1}^g(I, X)(t))$$

for any $t$ in $(Y, 0)$ by (2.1.4) and (3.2.1). Now, by assumption, the map $t \mapsto e_k(t)$ is constant. Hence, the upper semi–continuous map

$$t \mapsto e_1(I(t), P_{k-1}^g(I, X)(t))$$

is also constant. Therefore, by (4.3), $P_k^g(I, X) = P_1(I, P_{k-1}^g(I, X))$ specializes. Furthermore, by (4.2), we have for $t$ in $Y$ close to 0

$$e_{k+1}(t) \leq e_1(I(t), P_{k}^g(I, X)(t)) \leq e_1(I(t), P_{k}^g(I, X)(0)) = e_{k+1}(0).$$

This finishes the proof of the corollary. \qed

(4.6) Proposition. (i) If the map $t \mapsto (e_1(t), \ldots, e_n(t))$ is constant on $Y$, then for a component $C$ of $D_2$ all components of $C(Y) = C \cap b_2^{-1}(Y)$ are equidimensional over $Y$.

(ii) If, in addition, 0 is a regular point of $Y$ then the constancy of the map in (i) is equivalent to all components of $D_2^X$ and $D_1^X \cdot \tilde{D}_1$ being equidimensional over $Y$.

Proof. (i) We do induction on $n$, the dimension of the fibres of $X$ over $Y$. The case $n = 1$ is the classical case of an ideal of finite co–support over $Y$.

Now, assume that the assertion holds for families of $(n – 1)$–dimensional spaces. Let $X \to Y$ be a family of fibre dimension $n$ as in the above setup. If for a component $C$ of $D_2$ the intersection $C \cap b_2^{-1}(0)$ is zero–dimensional, then, by Remark (4.4), its image in $X$ contains $Y$. Hence, the map $C(Y) \to Y$ is equidimensional. For other components $C$ of $D_2$ the claim holds if, and only if, it holds for the intersection of $C$ with a generic hyperplane of $B_2$; see (3.1.2). Therefore, by the induction hypothesis, it is enough to show that the numbers

$$e_1(I(t), P_1(I, X)(t)), \ldots, e_{n-1}(I(t), P_1(I, X)(t))$$

are independent of $t$ in $Y$.

We will show this by induction on the codimension of the Segre numbers. So, assume we have shown the claim for the numbers of codimension 1 to $k – 1$. Then,
we have
\[ e_{k+1}(I(t), X(t)) = e_{k+1}(I(0), X(0)) = e_k(I(0), P_1(I(0), X(0))) \]
\[ e_k(I(t), P_1(I(t), X(t))) \leq e_k(I(t), P_1(I, X)(t)) \leq e_k(I(0), P_1(I, X)(0)) \]

The vertical equality on the left hand side follows from (2.2.1), the other vertical equality follows from (4.3). In the top row, the second equality is (2.2.1), while in the bottom row the first inequality follows from (2.1.4), and the second from the upper semi–continuity (4.5). It follows that \( e_k(I(t), P_1(I, X)(t)) = e_{k+1}(t) = e_{k+1}(0) \) for all \( t \). This finishes the proof of claim (i).

(ii) Assume that the map in (i) is constant, and consider a component \( \tilde{C} \) of \( \tilde{D}_2^{X-Y} \) and its image \( C \) in \( D_2^{X-Y} \). Then, by (i), the map \( C(Y) \rightarrow Y \) is equidimensional, say of fibre dimension \( k \). Let \( C' \) be the intersection of \( C \) with \( k \) generic hyperplanes of \( \mathcal{B}_2 \). Then the map \( C'(Y) \rightarrow Y \) is finite, and the assumption implies that the multiplicity of \( (b_2, C')(t) \) at 0 is independent of \( t \). As \( Y \) is smooth at 0, the ideal induced by \( m_Y \) in \( \mathcal{O}_{X(t),0} \) equals the maximal ideal of this local ring. Hence, by a projection formula for multiplicities [F, (4.3.6)], this multiplicity equals the multiplicity of the ideal induced by \( b_2(m_Y) \) on \( C'(t) \). Now, by the classical Principle of Specialization, the equimultiplicity implies that the exceptional divisor of the blowup of \( m_Y \) in \( C' \) is equidimensional over \( Y \). Therefore, the intersection \( \tilde{C} \cap \tilde{D}_1 \) is equidimensional over \( Y \); and, therefore, so is \( \tilde{C}(Y) \rightarrow Y \).

Now, assume that the equidimensionality condition holds, and that the underlying set of \( Y \) is smooth at 0. Then the dimension argument [GK, (2.1)], together with the commutativity of refined Gysin homomorphisms, shows:
\[ D_2^Y(t) = D_2^{0,t}, \quad (\tilde{D}_2^{X-Y} \cdot \tilde{D}_1)(t) = \tilde{D}_2^{X(t)-0} \cdot \tilde{D}_{1,t}. \]

(Let \( t \hookrightarrow Y \) be the regular embedding. The equidimensionality assumptions and the compatibility properties [F, 6.2.1] of the refined Gysin homomorphism \( i^! \) imply \( i^!(\tilde{D}_2^{X-Y} \cdot \tilde{D}_1) = (\tilde{D}_2^{X-Y} \cdot \tilde{D}_1)(t) \) and \( \tilde{D}_2^{X(t)-0} \cdot \tilde{D}_{1,t} = \tilde{D}_2^{X(t)-0} \cdot \tilde{D}_1 \). Therefore, we have for \( k = 1, \ldots , n-1 \) that
\[ e_k(t) = \int \hat{h}_1^{n-k-1} \hat{h}_2^{k-1} \tilde{D}_2^{X(t)-0} \cdot \tilde{D}_{1,t} = \int \hat{h}_1^{n-k-1} \hat{h}_2^{k-1} (\tilde{D}_2^{X-Y} \cdot \tilde{D}_1)(t) \]
is independent of \( t \) by ‘conservation of numbers’. (The smoothness of \( |Y| \) is needed to apply ‘conservation of numbers’.) A similar argument shows that the map \( t \mapsto e_n(t) \) is constant.

The equidimensionality of \( \tilde{D}_2^{X-Y} \cdot \tilde{D}_1 \) over \( Y \) implies the equidimensionality of \( D_2^{X-Y} \cap b_2^{-1}(Y) \). The converse doesn’t hold in general: Consider a family of reduced plane curves over \((\mathbb{C}, 0)\) given by a function \( f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0) \). Let \( I \) be the ideal generated by \( f \). The blowup of the principal ideal \( I \) is isomorphic to \((\mathbb{C}^3, 0)\), the divisor \( D_2 \) is given by the vanishing of \( f \). By construction, the intersection of \( D_2^{X-Y} = D_2 = V(f) \) with the parameter axis is trivially equidimensional over \((\mathbb{C}, 0)\). But, the one–codimensional Segre number \( e_1(t) \) equals the multiplicity at 0 of the restriction of \( f \) to \( t \times (\mathbb{C}^2, 0) \) which need not be independent of \( t \).
(4.7) Theorem (The Principle of Specialization). In the setup (4.1) let \( h \in \mathcal{O}_{X,0} \) be a function so that \( h\mathcal{O}_{X(t),0} \) is integrally dependent on \( I(t) \) for all \( t \) in a Zariski–open subset of \( Y \). If the numbers \( e_1(t), \ldots, e_n(t) \) are independent of \( t \), then \( h \) is integrally dependent on \( I \).

Proof. The proof proceeds similarly to Teissier’s original proof [T3, Appendice I]. By the above Proposition (4.5) the assumption implies that all components of \( D_2 \) map onto \( Y \). Therefore, the same is true for the exceptional divisor \( D_2 \) of the normalized blowup of \( X \) along \( I \). Now, we use the characterization (0.1)(iv) of integral dependence. The fibre \( C(Y) \) of a component \( C \) of \( D_2 \) over \( Y \) maps onto \( Y \). Hence, we can compute the order of vanishing of the pullbacks of \( h \) and \( I \) along \( C \) at points that map to the Zariski open subset of \( Y \) in the statement of the theorem. Then, it is not hard to see that, after shrinking this open subset \( U \) of \( Y \), for a point \( t \) in \( U \) the order of vanishing along \( C(t) \) of the pullback of \( I(t) \) in the normalized blowup of \( X(t) \) along \( I(t) \) equals the order of vanishing of the pullback of \( I \) along \( C \); and similarly for \( h \) (see e.g. [T4, Ch. I, 1.3.4 and 1.3.6]). This proves the claim. 

(4.8) Remark. It is easy to see that a similar statement obtains if the embedding of \( Y \) into \( X \) is replaced by a subspace \( S \) in \( X \) that is finite over \( Y \): We replace the numbers \( e_k(t) \) by the sum of the Segre numbers of the ideal induced by \( I \) in the local rings \( \mathcal{O}_{X(t),x_i^t} \), where the points \( \{x_i^t\} \) form the underlying set of \( S(t) \). Note, however, that the analog of (4.6)(ii) is false. In fact, the defining ideal of \( S \) may not induce the maximal ideal in \( \mathcal{O}_{X(0),0} \).

As a corollary of the principle of specialization of integral dependence we obtain the following generalization of Rees’ Theorem (0.2).

(4.9) Corollary. Let \( X \subseteq (\mathbb{C}^N,0) \) be an analytic germ of pure dimension \( n \) and \( I \subset J \subset \mathcal{O}_{X,0} \). Then, \( \bar{I} = J \) if, and only if, \( e_k(I) = e_k(J) \) for \( k = 1, \ldots, n \).

Proof. Consider the family \( X' = X \times (\mathbb{C},0) \to (\mathbb{C},0) \) and the ideal \( I'(t) = (I + tJ)\mathcal{O}_{X',t} \), where \( t \) is the coordinate on \( (\mathbb{C},0) \). Then, \( I'(0) = I \) and \( I'(t) = J \) for \( t \neq 0 \). Now, if their Segre numbers are equal, the theorem implies \( \overline{I\mathcal{O}_{X'}} = \overline{J\mathcal{O}_{X'}} \), and so \( \bar{I} = J \).

The other direction follows from the existence of a generically one–to–one map \( \text{Bl}_J X \to \text{Bl}_I X \), which is compatible with the exceptional divisors of the blowups, if \( \bar{I} = J \).

(4.10) Remark. Böger’s theorem ([B], [Li]) is an easy corollary of this last result. The setting of Böger’s theorem is an ideal \( I \) in an equidimensional local ring \( R \) with the property that \( I \) has a reduction generated by the same number of generators as the height of \( I \). This implies that \( V(I) \) is equidimensional. Böger’s theorem says that if \( J \) is an ideal with the same radical as \( I \), \( I \subset J \) then \( J \) is in the integral closure of \( I \) iff the multiplicity of \( J \) in each of the local rings \( R_P \) is the same as the multiplicity of \( I \) in \( R_P \), \( P \) varying through the minimal primes of \( I \). The hypothesis on \( I \) implies that the only non-zero Segre number is \( e_j(I) \), where \( j \) is the height of \( I \); the hypothesis on \( J \) implies that \( e_i(J) = e_i(I), i \leq j \). The lexicographic upper semicontinuity of the the Segre numbers then implies that \( e_i(J) = e_i(I) = 0, i > j \). Then our extension of Rees’ theorem implies that \( \bar{I} = \bar{J} \).

(4.11) (Ideals with dense co–support). Let \( (X,0) \) be a reduced analytic space germ, and \( I \subset \mathcal{O}_{X,0} \) an ideal. We don’t require that \( V(I) \) is nowhere dense in \( X \).
Following ideas of Kleiman and Thorup’s work [KT2], we consider the embedding and projection
\[(X, 0) = (X, 0) \times 0 \hookrightarrow (X \times \mathbb{C}, 0) \rightrightarrows (X, 0).\]
Let \(y\) be the coordinate function on \((\mathbb{C}, 0)\). To an ideal \(I\) of \(O_{X,0}\) we associate the ideal \(\hat{I}\) of \(O_{X \times \mathbb{C},0}\) generated by the pull–back \(p^*I\) and \(y\) in \(O_{X \times \mathbb{C},0}\). Let \(h\) be an element of \(O_{X,0}\). Then, \(h\) is integrally dependent on \(I\) if, and only if, its pull–back \(p^*h\) is integrally dependent on \(\hat{I}\). In fact, an integral relation
\[h^k + a_1 h^{k-1} + \cdots + a_k = 0, \quad a_j \in I^j,\]
pulls back to an integral relation in \(O_{X \times \mathbb{C},0}\) with coefficients in the correct powers of \(\hat{I}\). On the other hand, an integral relation
\[(p^*h)^k + a'_1 (p^*h)^{k-1} + \cdots + a'_k = 0, \quad a'_j \in \hat{I}^j,\]
restricts to the required relation for \(h\) in \(O_{X,0}\) with coefficients in the correct powers of \(I\).

It follows that Theorem (4.3) holds in this more general setup if the Segre numbers of \(I\) on \((X, 0)\) are replaced by the Segre numbers of \(\hat{I}\) on \((X \times \mathbb{C}, 0)\). In fact, we can compute the Segre numbers of \(\hat{I}\) by using the pull–back \(p^*(m)\) of the maximal ideal of \(O_{X,0}\); see also Remark (3.6).

(4.12) (Limiting Tangent Hyperplanes). Consider a reduced hypersurface \(X = V(f) \subset (\mathbb{C}^{n+1},0)\). A hyperplane \(H\) is a limiting tangent hyperplane of \(X\) at 0 if it is the limit of tangent hyperplanes of \(X\) at smooth points converging to 0. There is a criteria for limiting tangent hyperplanes in terms of integral closure. Let \(J(f) \subset O_X\) be the Jacobian ideal. It is generated by the partial derivatives of \(f\). Consider the ideal \(J(f)_H \subset J(f)\) generated by the directional derivatives \(\frac{\partial f}{\partial v}\) with \(v\) in \(H\). Then, \(H\) is a limiting tangent hyperplane if, and only if, \(J(f)_H\) is not a reduction of \(J(f)\) (see [T1, p.321,308]). Using the above Corollary (4.9) we can give a necessary and sufficient numerical criteria for this to happen.

(4.13) Corollary. A hyperplane \(H \subset (\mathbb{C}^{n+1},0)\) is a limiting tangent hyperplane of the hypersurface \(X = V(f)\) at 0 if, and only if, the numbers \(e_i(J(f)_H)\) and \(e_i(J(f))\) differ for some \(i = 1, \ldots, n\). \(\square\)

5. Strict Dependence and Thom’s Condition \(a_f\)

We review the notion of strict dependence and characterize this condition in terms of vanishing of pullbacks along components of exceptional divisors of some normalized blowups. This allows us to prove a principle of specialization of strict dependence in (5.3).

Strict dependence is used to describe equisingularity conditions like Whitney’s condition condition \(a\) and Thom’s condition \(a_f\) in terms of integral closure; see [Ma] for a discussion of Thom’s condition \(a_f\). We apply our results to study this condition for a function \(f\) on \((\mathbb{C}^{n+1},0) \times (\mathbb{C}^p,0)\). This may be viewed as a family of functions on \((\mathbb{C}^{n+1},0)\). Proposition (5.5) recovers a result of Massey. It gives sufficient numerical conditions for \(a_f\) to hold. The same approach gives a similar result for a map \(f\) of reduced analytic spaces. It involves the relative Jacobian ideal and the relative Nash blowup the definition of which we review.
(5.1) Setup. In the Setup (4.1), we say that an element $h$ of $\mathcal{O}_{X,0}$ is strictly dependent on an ideal $I$ in $\mathcal{O}_{X,0}$ if for all holomorphic curves $\phi: (\mathbb{C},0) \to (X,x)$, the pullback $\phi^*h$ is contained in the ideal $m_1\phi^*(I)$ where $m_1$ is the maximal ideal in $\mathcal{O}_{\mathbb{C},0}$. All such elements form the ideal $\bar{I}$. We will consider the normalized blowups $B_{2,N}$ and $B_N$ of $X$ along $I$ and $m_YI$ with structure maps $b_{2,N},b_N$ and exceptional divisors $D_{2,N}$ and $D_N$. We denote the pullback to $B_N$ of the divisor $D_2$ by $\tilde{D}_{2,N}$.

(5.2) Proposition. Let $h$ be a holomorphic function on $X$. Then, the following conditions are equivalent.

(i) For every point $y$ in $0 \times Y$, the germ of $h$ at $y$ is strictly dependent on $IO_{X,Y}$.

(ii) For every point $z$ in $B_{2,N}$ over a point in $Y$ and any element $g$ of $I$ the pullback of which to $B_{2,N}$ generates a local equation of $D_{2,N}$ at $z$ the quotient $h \circ b_{2,N}/g \circ b_{2,N}$ is a holomorphic function near $z$ and vanishes on $|D_{2,N}| \cap b_{2,N}^{-1}(Y)$ near $z$.

(iii) For each component $C$ of the underlying set of $\tilde{D}_{2,N}$ the order of vanishing of $h \circ b_{2,N}$ along $C$ is greater than the order of vanishing of the pullback $b_N^{-1}I$ along $C$ if $b_N(C)$ is contained in $0 \times Y$, and, if $C$ is not mapped to $Y$, the order of vanishing of $h \circ b_{2,N}$ along $C$ is not smaller than the order of vanishing of the pullback of $I$ along $C$.

Proof. Assume that condition (i) obtains. Then, in particular, for a point $y$ in $Y$ the germ of $h$ at $y$ is in the integral closure of $IO_{X,Y}$. Hence, for each point $z$ in $|D_{2,N}|$ over $Y$ and an element $g$ as in (ii) there exists a neighborhood $U$ of $z$ and a holomorphic function $k$ on $U$ such that $h \circ b_{2,N} = k(g \circ b_{2,N})$ on $U$. We want to show that $k$ vanishes on the fibre of $|D_{2,N}| \cap U$ over $Y$. So, pick a point $z'$ in this fibre close to $z$. Then, for a curve $\phi_N: (\mathbb{C},\mathbb{C}-0,0) \to (B_{2,N},B_{2,N} - |D_{2,N}|,z')$ we have

$$h \circ b_{2,N} \circ \phi_N = (k \circ \phi_N)(g \circ b_{2,N} \circ \phi_N) \in m_1\phi^*(I).$$

Here $\phi$ is the path $b_{2,N} \circ \phi_N$. This implies that $k$ vanishes at $z'$.

On the other hand, if (ii) holds, then, clearly, the function $h$ vanishes on $V(I)$. Hence it is enough to consider paths $\phi: (\mathbb{C},\mathbb{C}-0,0) \to (X,X-V(I),y)$. Now, $\phi$ can be lifted to a path $\tilde{\phi}_N$ on the normalized blowup $B_{2,N}$. Denote the image of $0$ under the lifted path by $z$. Then, there is neighborhood $U$ of $z$ in $B_{2,N}$ and a holomorphic function $k$ on $U$, vanishing on $|D_{2,N}| \cap b_{2,N}^{-1}(Y) \cap U$, so that $h \circ b_{2,N} = k(g \circ b_{2,N})$ where $g$ is as in the statement of (ii). In particular, the pullback $k \circ \phi_N$ vanishes at $z$. Hence, $h \circ b_{2,N} \circ \phi_N = h \circ \phi \in m_1\phi^*(I)$.

Finally, condition (ii) and (iii) are equivalent by an argument similar to the ones already used. For a point $z$ in $B_{2,N}$ over a point of $0 \times Y$, we have $h \circ b_{2,N} = k(g \circ b_{2,N})$ with $k$ and $g$ as above. Then $k$ vanishes on the fibre over $0 \times Y$ of the underlying set of $D_{2,N}$ near $z$ if, and only if, the pullback of $k$ to $B_N$ vanishes on the part of $\tilde{D}_{2,N}$ formed by components that are mapped to $0 \times Y$. This finishes the proof. 

(5.3) Theorem. In the setup of (4.1), let $h \in \mathcal{O}_{X,0}$ be a function so that $h\mathcal{O}_{X(t),0}$ is strictly dependent on $I(t)$ for all $t$ in a Zariski-open subset of $Y$. If the numbers $e_1(t),\ldots,e_n(t)$ are independent of $t$, then $h$ is strictly dependent on $I$.

Proof. The constancy of the $e_i(t)$ implies that for each component of $D_2$ all components of its intersection with $b_{2,0}^{-1}(Y)$ map onto $Y$; hence the same statement
is true for the normalized blowup.

Now, we use an argument similar to the one used in the above proof. Let \( z \) be a point in \( B_{2,N} \) over 0 in \( X \). From the principle of specialization, we know that \( h \) is contained in the integral closure of \( I \). Hence, near \( z \), we can write \( h \circ b_{2,N} = k(g \circ b_{2,N}) \) with \( g \) and \( k \) as in the above proof. Let \( C \) be component of the underlying set of the exceptional divisor that contains \( z \). Then, its intersection with the inverse image of \( Y \) maps onto \( Y \). In particular, the set of points \( z' \) in this intersection that map to the Zariski open subset of \( Y \) where the strict dependence condition obtains are dense. By (5.2), the holomorphic function \( k \) vanishes at such a point \( z' \). Therefore, the function \( k \) vanishes on the intersection. Hence, again by (5.2), the claim follows.

(5.4) (The case of a smooth ambient space). Let \( M = (\mathbb{C}^{n+1}, 0) \times (\mathbb{C}^p, 0) \) and consider a function germ \( f : M \to (\mathbb{C}, 0) \). Let \( Y = 0 \times (\mathbb{C}^p, 0) \) and \( \Sigma = V(J(f)) \) be the critical locus of \( f \). We assume that \( \Sigma \) is nowhere dense and doesn’t contain \((\mathbb{C}^{n+1}, 0) \times 0\). Let \( z_0, \ldots, z_n \) be coordinates on \((\mathbb{C}^{n+1}, 0) \). For a series of points \( p_i \) in \( X - \Sigma \) converging to 0, we may view the series of tangent planes to the level hypersurfaces of \( f \) through the \( p_i \) as a series of points in \( \mathbb{P}^n \). We may assume that this series converges after passing to a subseries of \( \{p_i\} \). We call this limit a \textit{limiting tangent hyperplane} to the fibres of \( f \) at 0. If all limiting tangent hyperplanes contain \( Y \), we say that the pair \((M - \Sigma, Y)\) satisfies \( a_f \) at the origin. This condition translates into the integral dependence condition

\[
\frac{\partial f}{\partial v} \in \left( \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n} \right) \mathcal{O}_M \quad \text{for all } v \in T_0 Y.
\]

(See [G2] for this and the definition of the strict integral closure, denoted by the superscript \( \dagger \).) It is known that this condition obtains for generic points of \( Y \). Hence we can apply the principle of specialization (4.7), which also works for the strict closure. For a point \( t \) in \( Y \) we denote

\[
\lambda_i(t) := e_i \left( \left( \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n} \right), (\mathbb{C}^{n+1}, 0) \times t \right).
\]

Then we get the following result of Massey [Ma].

(5.5) Proposition. In the above setup the pair \((M - \Sigma, Y)\) satisfies \( a_f \) at the origin if the numbers \( \lambda_i(t) \) are independent of \( t \) in \( Y \).

(5.6) (Relative Jacobian ideal and relative Nash modification). We recall some definition and facts from [T4, Ch.II] and [T2, Chapter 2]. Suppose \( f : (X, 0) \to (S, 0) \) is a morphism of analytic germs so that the sheaf of relative differentials \( \Omega_f^1 \) is locally free of rank \( d = \dim X - \dim S \) outside a nowhere dense analytically closed subset \( F \) of \( X \). We will consider the relative Jacobian ideal \( J(f) \). It is defined as the \( d \)th Fitting ideal of the \( \mathcal{O}_X \)-module \( \Omega_f^1 \).

Alternatively, it can be defined via an embedding \( X \hookrightarrow S \times (\mathbb{C}^N, 0) \) so that \( f \) is the restriction to \( X \) of the projection onto the second component. If \( X \) is defined by \( f_1, \ldots, f_m \) in \( S \times (\mathbb{C}^N, 0) \) and \( z_1, \ldots, z_N \) are coordinates of \( \mathbb{C}^N \), then \( J(f) \) equals the ideal of \( \mathcal{O}_{X,0} \) generated by the minors of rank \( N - d \) of the relative Jacobian matrix

\[
\left( \frac{\partial f_i}{\partial z_j} \right), i = 1, \ldots, m; j = 1, \ldots, N.
\]
On the other hand, we may consider the Grassmannian space Grass$_d\Omega^1_f \to X$ of locally free quotients of $\Omega^1_f$ of rank $d$. There is a well-defined section over $X - F$. The closure of its image is called the relative Nash modification. Geometrically, this section maps a point $p$ to the tangent space at $p$ of the level surface of $f$ passing through $p$.

If $X$ is a relative complete intersection over $S$, the relative Nash modification equals the blowup of the relative Jacobian ideal in $X$. In the general case we have to embed $X$ into a relative complete intersection $X'$ over $S$ and then blow up the ideal $J(f)'$ in $\mathcal{O}_X$ induced by the relative Jacobian ideal of $X'$ over $S$.

**The general case.** Let $X \subseteq (\mathbb{C}^N, 0) \times (\mathbb{C}^p, 0)$ be a reduced analytic space germ of pure dimension $n + p$, so that the fibres of the map $X \to Y = (\mathbb{C}^p, 0)$ induced by the projection onto the second factor form a family of reduced analytic germs of pure dimension $n$. The fibre over $t \in Y$ will be denoted by $X(t)$. We assume that $0 \times (\mathbb{C}^p, 0)$ is contained in $X$ and will identify it with $Y$. Let $(S, 0)$ be a reduced analytic space germ and $f : X \to (S, 0)$ an analytic map germ that maps $Y$ to 0 and is a submersion off a nowhere dense analytic subset $F$ of $X$. The $a_f$ condition is defined as above. It is known that it is satisfied at generic points of $Y$, if there is no blowup in codimension 0 (see [HMS]). Then, $(X - F, Y)$ satisfies $a_f$ at 0 if each component of the exceptional divisor of the relative Nash blowup maps onto $(\mathbb{C}^p, 0)$. The principle of specialization (4.7) gives sufficient numerical conditions for this to happen.

**Theorem.** In the above situation (5.4) the pair $(X - F, Y)$ satisfies $a_f$ at 0 if the Segre numbers $e_1(J(f)', X(t)), \ldots, e_n(J(f)', X(t))$ are independent of $t$ in $Y$. 

6. Condition $W_f$ and the Whitney conditions

We will now use the Theorem of Specialization of Integral Dependence to study families of functions on $(\mathbb{C}^{n+1}, 0)$. Theorem (6.2) gives a sufficient criterion for the ambient space to satisfy the $W_f$ condition along a the parameter space. It requires the constancy of numbers that only depend on the family members. In particular, the constancy of these numbers implies that the smooth part of the corresponding family of hypersurfaces is Whitney regular along the parameter axis.

Our condition is also necessary for this family of hypersurfaces to be Whitney equisingular along the parameter space. One part of our numbers can be computed from the Milnor fibres of the restriction of the family members to generic linear subspaces.

We also apply our theory to the study of families of 2-dimensional hypersurfaces (6.6), families of hyperplane slices of a hypersurface (6.7), and the deformation of a hypersurface to its tangent cone (6.8).

**Setup.** Consider the family of smooth spaces $M = (\mathbb{C}^{n+1}, 0) \times (\mathbb{C}^p, 0) \overset{F}{\to} (\mathbb{C}^p, 0) = Y$, and identify $Y$ with the zero section $0 \times (\mathbb{C}^p, 0)$. Let $f : M \to (\mathbb{C}, 0)$ be a map germ that maps $Y$ to 0. Let $I$ be the Jacobian ideal of $f$, generated by its partial derivatives. Denote the restriction of $f$ to a fibre $M(t)$ by $f_t$. We assume that $f_t$ is reduced for any $t$. Also, assume that the critical locus $\Sigma$ of $f$ contains $Y$. Fix coordinates $z_0, \ldots, z_n, t_1, \ldots, t_p$ which respect the product structure of $M$. We
will use the relative Jacobian ideal \( J_z(f) \), generated by the partial derivatives of \( f \) with respect to \( z \)-coordinates.

Using a suitable distance function ‘dist’ for linear subspaces of \( M \), condition \( W_f \) can be expressed as an inequality; see e.g. [GK, (4.1)]. The pair \((M - \Sigma, Y)\) satisfies \( W_f \) at 0 if there exists a neighborhood \( U \) in \( M \) of 0 and a positive constant \( C \) so that for all points \( x \) in \( U - \Sigma \) we have

\[
\text{dist}(T_0Y, T_xV(f - f(x))) \leq C\text{dist}(x, Y),
\]

Using (1.1)(ii), one can show that this condition is equivalent to the following relations

\[
\frac{\partial f}{\partial t_i} \in m_z J_z(f)O_{\mathbb{C}^{n+1+p},0}, \quad \text{for } i = 1, \ldots, p,
\]  

(6.1.1)

where \( m_z = (z_0, \ldots, z_n) \) is the ideal defining \( Y \) in \( M \). The latter condition was named \((c)\)-equisingularity by Teissier; see [T2. 2.17, p.601].

Consider now the family \( X \) of hypersurfaces defined by \( f \in \mathcal{O}_{\mathbb{C}^{n+1+p},0} \). We use the above distance function to express the Whitney conditions as an inequality also. The pair \((X - \Sigma, Y)\) satisfies the Whitney conditions at 0 if there exists a neighborhood \( U \) in \( X \) of 0 and a positive constant \( C \) so that for all points \( x \) in \( U - \Sigma \) we have

\[
\text{dist}(T_0Y, T_xX) \leq C\text{dist}(x, Y).
\]

We also say that \( X - \Sigma \) is Whitney regular along \( Y \) at 0.

Again making use of 1.1 (ii), this condition is equivalent to the following integral dependence relation in the local ring \( \mathcal{O}_{X,0} \) of \( X \) at 0 (see [G1]):

\[
\left| \frac{\partial f}{\partial t_i} \right| \in m_z J_z(f)\mathcal{O}_{X,0}, \quad \text{for } i = 1, \ldots, p.
\]  

(6.1.2)

Note that the \( W_f \) condition implies the Whitney conditions as an integral dependence relation descends to quotient rings.

More generally, Whitney regularity is defined for any pair of submanifolds of \( M \) in an analogous way. A Whitney stratification of a singular space is a stratification such that for any strata \( S \) and \( S' \) with \( S' \) contained in the closure of \( S \) the bigger strata \( S \) is Whitney regular along \( S' \). We say that the hypersurface \( X \) is Whitney equisingular along \( Y \) if there exists a Whitney stratification of \( X \) with \( Y \) as a stratum.

If \( X \) is Whitney equisingular along \( Y \), then \( X \) is topologically trivial along \( Y \) by the Thom–Mather isotopy theorem; see e.g. [T4, Ch. VI, 4.3.1, p.482]. This trivialization can be obtained by lifting vector fields on \( Y \) to a corrugated (Fr. rugueux) vector field on \( X \). These vector fields are integrable and are tangent to the strata of \( X \); see [V].

If, in addition, \((M - \Sigma, Y)\) satisfies the condition \( W_f \), any vector field on \( Y \) can be lifted to a corrugated vector field on the amiant space \( M \) that is tangent to the level hypersurfaces of \( f \); see [T2, 2.17, Cor. 1, p.602]. The integration of these vector fields yields a right–trivialization of \( F \). In particular, the maps \( f_t \) are equivalent up to homeomorphism of their source \((\mathbb{C}^{n+1}, 0)\).
For \( k = 2, \ldots, n + 1 \), we use the following notations:
\[
\lambda_k(f_t) = e_k(J_z(f), M(t)) = e_k(J(f_t), M(t)),
\]
\[
m_{k-1}(f_t) = m_{k-1}(J_z(f), M(t)) = m_{k-1}(J(f_t), M(t)).
\]
The number \( \lambda_k(f_t) \) is called the \( k \)-th Lê number of \( f_t \). Note that \( e_1(J_z(f), M(t)) = e_1(J(f_t), M(t)) \) is zero as, by assumption, the critical locus of \( f_t \) has codimension at least two in \( M(t) \). The number \( m_k(f_t) \) is called the \( k \)-th relative polar multiplicity of \( f_t \).

Fix \( t \) in \( Y \). For \( L^k \subseteq (\mathbb{C}^{n+1}, 0) \) a generic \( k \)-dimensional linear subspace, we denote the Euler characteristic of the Milnor fibre of \( f_t|_{L^k} \) at 0 by \( \chi^{(k)}(t) \). Let \( s \) be the codimension of the singular locus of \( f_t \). Then, by a result of Massey [Ma, 10.6, p.96] and (3.4.2), we have
\[
\chi^{(k)}(t) = (-1)^k \sum_{i=2}^{k} (-1)^i (e_i(J_z(f), L^k \times t))
\]
\[
= m_k(f_t) + \lambda_k(f_t) - \lambda_{k-1}(f_t) + \lambda_{k-2}(f_t) - \cdots \pm \lambda_s(f_t).
\]
Hence, the number \( \chi^{(k)}(t) \) doesn’t depend on the choice of the \( L^k \). We define
\[
\chi^*(t) := (\chi^{(n+1)}(t), \ldots, \chi^{(2)}(t)).
\]

(6.2) Theorem. Suppose that the map \( t \mapsto (m_1(f_t), \ldots, m_n(f_t), \chi^*(t)) \) is constant on \( Y \). Then the pair \((M - \Sigma, Y)\) satisfies \( W_f \) at 0. In particular, the smooth part of \( X \) is Whitney–regular along \( Y \) at 0. Also, the smooth parts of \( \Sigma \) and the components of the singular locus of \( \Sigma \) of codimension one in \( \Sigma \) satisfy the Whitney–conditions along \( Y \) at 0.

Proof. We now use the strength of the machinery developed in this paper. The constancy of the map in the proposition is equivalent to the constancy of the map
\[
t \mapsto (m_1(f_t), \ldots, m_n(f_t), \lambda_2(f_t), \ldots, \lambda_{n+1}(f_t)).
\]
Hence, by the expansion formula (3.5), the numbers \( e_k((m_zJ_z(f))(t), M(t)) \) are independent of \( t \) in \( Y \). Also, we know that the integral dependence relation (6.1.1) is satisfied at generic points of \( Y \). Therefore, we can apply the Principle of specialization of integral dependence (4.7) to prove the first assertion.

The second assertion follows from the following observation:

Consider a stratum \( W \) in \( \Sigma \) whose closure is the image of a component \( C \) of the exceptional divisor \( D \) of the blowup of \( M \) along \( J(f) \). Denote its structure map by \( b \). Then the conormal of \( W \) in \( M \) equals \( C \). In fact, it is well–known that the pair \((M - \Sigma, W)\) satisfies Thom’s \( a_f \) condition at points of a Zariski–open dense subset \( U \) of \( W \). Hence, we have
\[
C \cap b^{-1}(U) \subseteq C(W) \cap b^{-1}(U).
\]
Replace \( U \) be a perhaps smaller Zariski–open dense subset of \( W \) so that for \( w \in U \), the fibre over \( w \) of \( C(W) \) is isomorphic to \( \mathbb{P}^{k-1} \) where \( k \) is the codimension of \( W \). A dimension count shows that, after shrinking \( U \) once more, we may assume that the fibre of \( C \) over a point \( w \) in \( U \) has dimension \( k - 1 \). It follows that the fibres of \( C \) and \( C(W) \) over \( w \) are equal. As \( C(W) \) is the closure of \( C(W) \cap b^{-1}(U) \)
in \( H \times \mathbb{P}(H) \), the claim follows. (The claim also follows from the Principle of Lagrangian Specialization [LT1, 1.2.6].)

We are going to show now that the smooth part of \( W \) is Whitney regular along \( Y \) if the hypothesis of the theorem obtains. By Teissier’s characterization of Whitney–conditions [T4, Ch.5], we have to show that the exceptional divisor of the blowup of \( C \) along the preimage of \( Y \) in \( C \) is equidimensional over \( Y \). As (6.1.1) holds, the ideal \( J_z(f) \) is a reduction of \( J(f) \); thus the component \( C \) is finite over a component \( C' \) of the exceptional divisor of the blowup of \( M \) along \( J_z(f) \). Hence, it is enough to show the analogous statement for \( C' \). But this is part of the result of (4.6).

Clearly, a component of \( \Sigma \) is the image of a component \( C \) of \( D \). Also, every component of the singular locus of \( \Sigma \) of codimension one in \( \Sigma \) is the image of a component \( C \) of \( D \) (see [Ma, Prop. 1.32, p.30]). This finishes the proof.

(6.3) Theorem. Suppose that \( X \) admits a Whitney stratification with \( Y \) as a stratum. Then, the map \( t \mapsto (m_1(f_t), \ldots, m_n(f_t), \chi^*(t)) \) is constant on \( Y \).

Proof. The proof uses topological methods. By the Mather–Thom isotopy theorem, the topological type of \( X(t) \) is independent of \( t \) in \( Y \) (see e.g. [T4, Ch. VI, 4.3.1]). Hence, by an observation of Lê Dũng Tráng [L, p. 261], the homotopy type of the Milnor fibre of \( f_t \) is independent of \( t \). In particular, the Euler characteristic is constant. That is \( \chi^{(n+1)}(0) = \chi^{(n+1)}(t) \).

Fix a point \( t \) in \( Y \). For \( k = 2, \ldots, n \), we can choose a \( k \)–dimensional linear subspace \( L^k \) of \( (\mathbb{C}^{n+1}, 0) \), generic w.r.t \( f_0 \) and \( f_t \), so that the restriction of \( f \) to \( L^k \times Y \) satisfies the same assumptions as \( f \). Hence, we get by the same argument as above \( \chi^{(k)}(t) = \chi^{(k)}(0) \). It follows that \( \chi^*(t) \) is independent of \( t \) in \( Y \).

It remains to show the constancy of the map \( t \mapsto (m_1(f_t), \ldots, m_n(f_t)) \). We prove this by induction over the dimension of the fibres of \( X \) over \( Y \). The case of a one–dimensional family follows from the classical theory of equisingularity for plane curves; Whitney equisingularity implies that both, \( \lambda_2(f_t) = \mu^{(2)}(X(t)) \) and \( m_1(f_t) \) are independent of \( t \) in \( Y \).

Assume now that the theorem has been proven for fibre dimension \( n - 1 \). Fix a point \( t \) in \( Y \). Then, we may choose a hyperplane \( H \) in \( M(0) \) so that \( X \cap (H \times Y) \) is again Whitney–equisingular along \( Y \), and \( H \) is generic w.r.t \( X(t) \) and \( X(0) \). In particular, we may assume that for \( k = 1, \ldots, n - 1 \)

\[
m_k(f_t) = m_k(J_z(f), H \times t) = m_k(J_z(f), H \times 0) = m_k(f_0)
\]

This implies, together with the constancy of \( \chi^{(n)}(t) \), that

\[
t \mapsto (m_1(f_t), \ldots, m_{n-1}(f_t), \lambda_2(f_t), \ldots, \lambda_{n-1}(f_t))
\]

(6.3.1) is constant on \( Y \). Therefore, using the expression of \( \chi^{(n)}(t) \) in terms of \( m_k(f_t) \) and \( \lambda_k(f_t) \), we see that the map \( t \mapsto m_n(f_t) + \lambda_n(f_t) \) is constant.

Furthermore, by (6.3.1) and (4.3), the polar variety \( P_n(J_z(f), H \times Y) \) specializes; therefore, it is empty. It follows that every component of \( P_n(J_z(f), M) \) contains \( Y \). Hence, it remains to show that the fibre over \( 0 \) of a component of this polar variety is not contained in the fibre of \( \Sigma \) over \( 0 \). Suppose there were such a component. Then, there exists a component \( C \) of the exceptional divisor of the blowup of \( J_z(f) \) in \( M \) that doesn’t map onto the parameter space \( Y \) and so that \( C(0) \) maps to a subset of \( M(0) \) of non–zero dimension. Consider a point \( p \) in this set close to
0, the stratum $S$ of the initial Whitney–stratification through $p$, and the family $X \to S$ near $p$ given by some retraction. The fibres of this family are of dimension at most $n - 1$. Hence, by the induction hypothesis and (4.6), every component of the exceptional divisor of the blowup of $J_z(f)$ in the germ of $M$ at $p$ maps onto $S$. This contradicts the existence of $C$. It follows that $P_n(J_z(f), M)$ specializes. Therefore, the map $t \mapsto m_n(f_t)$ is upper semi–continuous, but so is $t \mapsto \lambda_n(f_t)$ by the lexicographically upper semi–continuity of the Lê numbers (4.5). We know that the sum of the two maps is constant, and therefore each one is. This finishes the proof.

(6.4) Corollary. Assume that $X$ admits a Whitney stratification with $Y$ as a stratum, then $M - \Sigma$ satisfies the $W_f$ condition along each stratum.

Proof. For each stratum $S$, apply Theorem (6.3) to the family $X \to S$ given by some retraction which is compatible with $X \to Y$. Then, apply (6.2).

(6.5) Remark. (1) This corollary recovers a result of Parusinski [Pa] which was also proven by Briancon, Maisonobe and Merle [BMM] in a more general context.

In his proof Parusinski shows that the assumptions of the corollary imply that each component $C$ of the exceptional divisor of the blowup of $J(f)$ in $M$ is the conormal of the closure of some stratum $S$ of the Whitney stratification of $X$. By assumption, the smooth part of $S$ is Whitney regular along $Y$. Hence, the exceptional divisor of the blowup of $C$ along the preimage of $Y$ is equidimensional over $Y$. Thus, by (4.6), the Segre numbers of $J(f)$ on the fibres $M(t)$ are independent of $t$. Furthermore, the $W_f$ condition implies that $J_z(f)$ is a reduction of $J(f)$. So, these Segre numbers equal $\lambda_k(f_t)$. Also, it is not hard to see that the $W_f$ condition implies the constancy of $m_k(f_t)$. Therefore, we can use Parusinki’s proof to give a purely algebro–geometric proof of the above Theorem (6.3). (Note that all results of Lê and Teissier’s work [LT1] can also be proven by purely algebro–geometric means. This was worked out by Roberto Callejas–Bedregal in his thesis [Ca]).

(2) Consider the diagram of blowups in (4.1) for $X = M, I = J_z(f)$ and $m$ replaced by $m_z$. Results of Henry, Merle and Sabbah [HMS, 6.1, 3.3.1] show that $(M - \Sigma, Y)$ satisfies $W_f$ at 0 if, and only if, the exceptional divisor $\tilde{D}_1$ is equidimensional over $Y$.

On the other hand, the constancy of $t \mapsto (m_1(f_t), \ldots, m_n(f_t), \chi^*(t))$ implies that the Lê numbers $\lambda_k(f_t)$ are constant along $Y$. Hence, a necessary condition for the constancy of this map is that every component of $\tilde{D}_2^{X - Y} \cdot \tilde{D}_1$ and $D_Y^X$ is equidimensional over $Y$. As we have seen in remark (1), this condition controls some lower–dimensional strata of the Whitney–stratification of $X$.

(3) Denote the polar multiplicities, resp. Segre numbers of $J(f_t)$ on $X(t)$ by $m_k(X(t))$, resp. $\lambda_k(X(t))$. Then, the principle of specialization and the expansion formula show that the constancy of $t \mapsto (m_1(X(t)), \ldots, m_{n-1}(X(t)), \lambda_1(X(t)), \ldots, \lambda_n(X(t)))$ implies that the pair $(X - \Sigma, Y)$ satisfies the Whitney–conditions at 0. However, to get a converse statement, one has to control the components of the exceptional divisor $D$ of the blowup $B$ of $X$ along the ideal induced by $J_z(f)$. It seems that the existence of a Whitney stratification of $X$ with $Y$ as a stratum is not sufficient for the constancy of this map; for example, suppose $X$ is a 1–parameter family.
is a Whitney stratification is to establish the Whitney regularity of $\Sigma$ of non–isolated surface singularities in $(\mathbb{C}^3, 0)$. Denote the structure map of $B$ by $b$. Suppose the exceptional divisor $D$ has two components $C$ and $C'$, where the image of $C'$ is the parameter stratum $Y$, and the image of $C$ is a component $W$ of the singular set of the total space. Suppose $C'$ is equidimensional over $Y$, and contains $C \cap b^{-1}(Y)$. It is conceivable that $C \cap b^{-1}(Y)$ is not equidimensional over $Y$, even if the smooth part of $W$ is Whitney regular over $Y$, and the smooth part of $X$ is Whitney regular along $Y$. In fact, Whitney regularity of the smooth part of $X$ along $Y$ implies that the exceptional divisor $D_Y$ of the blowup of $B$ along $b^{-1}(Y)$ is equidimensional over $Y$ ([T4, Ch. 5, 2.1, p.470]). However, since $C \cap b^{-1}(Y)$ is hiding inside the hypersurface $D$, there need not be a component of $D_Y$ over this intersection; so Whitney regularity of the smooth part along the parameter stratum may not control the behavior of $C$ over $Y$. If the smooth part of $W$ is Whitney regular over $Y$, then, as we have seen in the proof of (6.3), we can control the behavior of $C(W)$, the conormal of $W$. However, $C$ is a proper subset of $C(W)$; generically it consists of smooth points of $W$ and tangent hyperplanes that are limits of tangent hyperplanes from the smooth part of $X$. So it may not be controlled by this condition either.

This hypothetical example underlines the possible geometric difference between the Whitney conditions and the constancy of the above numbers. In both conditions the components of $D$ that map onto $Y$ are equidimensional over $Y$. However, the constancy of the above Segre numbers requires that the preimage of $Y$ in each component of $D$ be equidimensional over $Y$ while the Whitney conditions imply the equidimensionality over $Y$ of the exceptional divisor $D_Y$ of the blowup of $B$ along $Y$. It is an open question whether or not the above hypothetical example exists or not.

The conclusions of Theorem (6.2) don’t give us any information on whether the smooth part of $X$ is Whitney regular along the smooth part of $\Sigma$, or the singular locus of $\Sigma$. Therefore, in general, we can’t use (6.2) to show that $X$ is Whitney equisingular along $Y$. However, in specific situations one may use auxiliary information to build a Whitney stratification with $Y$ as a stratum.

(6.6) Corollary. Assume that $X$ is a one–parameter family of reduced non–isolated surface singularities. Then, $X$ is Whitney–equisingular along the parameter axis $Y$ if and only if the map $t \mapsto (m_1(f_t), m_2(f_t), \lambda_2(f_t), \lambda_3(f_t))$ is constant on $Y$.

Proof. The ‘only if’ statement follows from Theorem (6.3). So, assume now that the map is constant. Thus, by Theorem (6.2), the smooth parts of $X$ and its singular locus $\Sigma$ are Whitney regular along $Y$. It follows that $\Sigma$ is either smooth, or its singular locus contains $Y$. Assume for the moment that $\Sigma$ is singular. Then, its singular locus is 1–dimensional, hence of codimension one in $\Sigma$. Therefore, by (6.2), the singular locus of $\Sigma$ equals $Y$. The remaining step to showing that

$$(X - \Sigma, \Sigma - Y, Y)$$

is a Whitney stratification is to establish the Whitney regularity of $X - \Sigma$ along $\Sigma - Y$. To see this, consider a general linear form $l$ on $\mathbb{C}^3$ the kernel of which intersects the fibres $X(t)$ and $\Sigma(t)$ transversally for all $t \in (Y, 0)$. Also, assume that for any any point $x \in X - Y$ the kernel of $l$ is not a limiting tangent plane of $X(t)$ at $x$. A form satisfying the transversality conditions exists because the
smooth parts of $X$ and $\Sigma$ are Whitney regular along $Y$. Also, by (4.6)(ii), the fibre of any component of $D^{X-Y}$ over 0 is of dimension at most 2. Here, $D$ denotes the exceptional divisor of the blowup of $M$ along $J_z(f)$. Hence, we can choose the form $l$ so that the ideal $J(f)_{z,l}$ generated by the partial derivatives of $f$ by vectors in the kernel of $l$ is a reduction of $J_z(f)$ outside $Y$. By (4.11), this implies the desired behavior with respect to limiting tangent planes.

Next, construct the family of reduced plane curves

$$\pi : X - Y \to Y \times \mathbb{C}, (x, t) \mapsto (t, l(x)).$$

The Whitney–equisingularity of this family will imply the desired result. But, for a family of plane curves this is equivalent to the constancy of the Milnor number of its fibres $X(t, s)$. Let $f$ be a defining equation of $X \subset (\mathbb{C}^3 \times \mathbb{C}, 0)$. Then, using (3.4.2), we have

$$\mu(X(t, s)) = \sum_{x \in \Sigma(t,s)} \lambda_2(f_{t,s}, x) = \sum_{x \in \Sigma(t,s)} \lambda_2(f_t, x) + \sum_{x \in \Sigma(t,s)} m_2(f_t, x).$$

Here we have used that $J(f)_{z,l}$ is a reduction of $J_z(f)$ outside $Y$. So, $\lambda_2(f_{t,s}, x) = e_2(J_z(f), (M(t,s), x)).$

Now, as the smooth part of $\Sigma$ is Whitney equisingular along $Y$, the first sum equals $\lambda_2(f_t)$ by (2.3). (The equisingularity ensures that the result of (2.3) remains valid for arbitrary $s$.) Furthermore, we claim that the second sum vanishes. Otherwise, there would be a fixed component of $\Lambda_3(j_z(f), \mathbb{C}^3 \times \mathbb{C})$ outside of $Y$. This is only possible if a component of the exceptional divisor of the blowup of $\mathbb{C}^3 \times \mathbb{C}$ along $J_z(f)$ maps to this component of the Lê cycle But, by (4.6), this cannot occur. This finishes the proof.

(6.7) Example. We can apply this corollary to families of hyperplane sections of hypersurfaces of dimension 3. Let $\mathcal{X} = V(f) \subset (\mathbb{C}^4, 0)$ be a reduced hypersurface with singular locus $S$ of dimension two. Let $H_0$ be a hyperplane in $(\mathbb{C}^4, 0)$ so that the intersection $\mathcal{X} \cap H_0$ is reduced. Suppose $H_0$ is a regular point of a curve $V$ in the projective space $\mathbb{P}^3$ of all hyperplanes in $\mathbb{C}^4$ through 0. Then the family of reduced surfaces

$$X = \{\mathcal{X} \cap H\}_{H \in (V,H_0)} \subset (\mathbb{C}^3 \times V, (0, H_0))$$

is Whitney equisingular along $Y = 0 \times (V, H_0)$ if, and only if, the map

$$H \mapsto (m_1(f|_H), m_2(f|_H), \lambda_2(f|_H), \lambda_3(f|_H))$$

is constant on $(V, H_0)$.

If we assume an extra condition, then we get the following stronger statement.

In the above setup, assume, in addition, that $H_0$ intersects $S$ and the singular locus of $S$ transversally. Then, the family of reduced hypersurfaces

$$X = \{\mathcal{X} \cap H\}_{H \in (\mathbb{P}^3, H_0)} \subset (\mathbb{C}^3 \times \mathbb{P}^3, (0, H_0))$$

is Whitney equisingular along $Y = 0 \times (\mathbb{P}^3, H_0)$ if, and only if, the map

$$H \mapsto (\lambda_2(f|_H), \lambda_3(f|_H))$$

is constant on $(\mathbb{P}^3, H_0)$.
The additional transversality conditions is needed to ensure that the dimension of any component of the singular locus of the singular locus $\Sigma$ of $X$ is of dimension 3. Hence, the same arguments as above yield the result, provided that the constancy of the Lê numbers implies that the map $H \mapsto (m_1(f|_H), m_2(f|_H))$ is constant.

To see this, we use the observation [G3, 2.9] of the first author. Let $g$ be a defining function of the hypersurface $X$ in $M = (\mathbb{C}^3 \times \mathbb{P}^3, (0, H_0))$. Then, $(M - \Sigma, Y)$ satisfies $w_g$ at $(0, H_0)$ if and only if the map $H \mapsto (\lambda_2(g|_H), \lambda_3(g|_H))$ is constant on $Y$. Now, fix a point $(0, H_1)$ on $Y$ and let $L$ be the line joining $H_0$ to $H_1$. Denote the restriction of $g$ to $\mathbb{C}^3 \times L$ by $g'$. Then, $(\mathbb{C}^3 \times L - \Sigma, 0 \times L)$ satisfies $w_{g'}$ at $(0, H_0)$ as the integral dependence relation (6.1.1) restricts to the desired relation on the subspace in question. In particular, if $(z_1, \ldots, z_3)$ are coordinates on $\mathbb{C}^3$, the ideal $J_z(g')$ is a reduction of $J(g')$. Hence, by [HMS, Thm. 6.1], the polar multiplicities of $J_z(g')$ on $(\mathbb{C}^3 \times L, (0, H))$ are independent of $H$ in $L$. Furthermore, by a polar transversality theorem [HMS, 4.4.6] and (4.3), these polar multiplicities equal the relative polar multiplicities of $g|_H$. Hence, the map $H \mapsto (m_1(g|_H), m_2(g|_H))$ is constant on $Y$. This finishes the proof of the claim.

\textbf{(6.8) (The deformation to the tangent cone).} Let $(X, 0) = V(f) \subset (\mathbb{C}^{n+1}, 0)$ be a hypersurface germ. Then, recall that the deformation of $X$ to its tangent cone $\mathcal{X} \to \mathbb{C}$ is defined by $g = t^{-d} f(tz_0, \ldots, tz_n)$, where $d$ is the degree of the initial term $f_0$ of $f$. The fibre of $\mathcal{X}$ over 0 equals the tangent cone $C_0X = V(f_0)$ of $X$. For non–zero $t$, the fiber $\mathcal{X}(t)$ is isomorphic to $X$. The parameter space is embedded into $\mathcal{X}$ by the map $t \mapsto 0 \in \mathcal{X}(t)$. (See e.g. [LT1, 1.6] for more about this deformation.)

\textbf{(6.9) Proposition.} Assume that the tangent cone $C_0X$ at 0 is reduced. Then, we have for $k = 1, \ldots, n$

$$m_k(f) = m_k(f_0), \quad \text{and} \quad \lambda_{k+1}(f) = \lambda_{k+1}(f),$$

if and only if for $k = 2, \ldots, n + 1$

$$m_1(f)^k - m_k(f) = \sum_{i=2}^{k} \lambda_i(f)m_1(f)^{k-i}.$$

In particular, if either condition obtains, then the smooth part of $\mathcal{X}$ satisfies the Whitney conditions along the parameter axis.

\textbf{Proof.} By a result of Massey [Ma, (4.7)], for a homogeneous polynomial of degree $d$ the following relations among the Lê numbers obtain:

$$\sum_{i=2}^{n} (d - 1)^{n-i} \lambda_i(f_0) = (d - 1)^{n+1}.$$

If we apply this to $f_0$ and its restrictions to generic linear subspaces, we obtain, using (3.4.2), that the second relations obtain for $f = f_0$. Therefore, the first set of equalities implies the set of relations.

Assume now that the relations hold. We will show by induction on $k$ that $m_k(f) = m_k(f_0)$ and $\lambda_k(f) = \lambda_k(f_0)$. First, $m_1(f) = d - 1 = m_1(f_0)$, and $\lambda_1(f) = 0 = \lambda_1(f_0)$, as both, $f$ and $f_0$, are reduced by assumption. Now, assume that we
have shown the equalities for $k-1$. Then, the relations imply
\[ \lambda_k(f) + m_k(f) = m_1(f)^k - \sum_{i=2}^{k-1} \lambda_i(f)m_1(f)^{k-i} = m_1(f_0)^k - \sum_{i=2}^{k-1} \lambda_i(f_0)m_1(f_0)^{k-i} = \lambda_k(f_0) + m_k(f_0). \]

By the induction assumption and by the upper semi-continuity of Segre numbers (4.5), we have $\lambda_k(f_0) \geq \lambda_k(f)$. So, it is enough to show that $m_k(f_0) \geq m_k(f)$. Now, the polar multiplicities $m_k(g_i)$ fail to be upper semi-continuous if and only if the global polar variety $P_k(J_z(g), M)$ doesn’t specialize, i.e. $P_k(J_z(g), M)(0) \neq P_k(J_z(g), M(0))$. This happens only if a component of the exceptional divisor of the blowup of $M$ along $J_z(g)$ maps to subset of codimension $k$ in $M(0)$.

Assume that such a component $C$ exists, and let $L^k$ be a general $k$–plane in $(\mathbb{C}^{n+1}, 0)$. Consider the exceptional divisor $D$ of the blowup of $L^k \times \mathbb{C}$ along $J_z(g)$. Then, by (3.3), the component $C$ gives rise to a component of $D$ that maps to $0$ in $L^k \times 0$. But, by induction hypothesis, the Lé numbers of $f|_{L^k}$ equal the Lé numbers of $f_0|_{L^k}$. Therefore, by (4.6)(i) no such component can exist.

(6.10) Corollary. Let $X = V(f) \subseteq (\mathbb{C}^3, 0)$ be a reduced hypersurface singularity. Assume that its tangent cone $C_0X$ is reduced. Then its the deformation to the tangent cone $X \to \mathbb{C}$ is Whitney–equisingular along $0 \times \mathbb{C}$ if and only if the following relations obtain:
\[ m_1(f)^2 = \lambda_2(f) + m_2(f), \]
\[ m_1(f)^3 = \lambda_3(f) + m_1(f)\lambda_2(f). \]

Proof. This follows from the above Proposition (6.9) and Corollary (6.6).

References
[B] E. Böger, Einige Bemerkungen zur Theorie der ganzalgebraischen Abhängigkeit von Idealen, Math. Ann. 185 (1970), 303–308.
[BMM] Joël Briançon, Phillipe Maisonobe, Michel Merle, Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom, Invent. Math. 117 (1995), 197–224.
[BS] Joël Briançon and Jean–Paul Speder, Les conditions de Whitney impliquent $\mu^*$ constant, Ann. Inst. Fourier 26 (1976), 153–163.
[Ca] Roberto Callejas–Bedregal, Algebraic Treatment of Whitney Conditions, thesis, MIT, Cambridge, 1992.
[Fu] William Fulton, Intersection theory, Ergeb. Math.,3. Folge, 2. Band, Springer-Verlag, 1984.
[G1] Terence Gaffney, Integral Closure of Modules and Whitney Equisingularity, Invent. Math. 107 (1992), 301–322.
[G2] Terence Gaffney, Aureoles and Integral Closure of Modules, preprint (1992).
[G3] Terence Gaffney, Equisingularity of Plane Sections, $t_1$ Condition, and the Integral Closure of Modules, Real and Complex Singularities (W.L.Marar, ed.), Putnam Research Series in Math. 333, Longman, 1995.
[G4] Terence Gaffney, Multiplicities and equisingularity of ICIS germs, Invent. Math. 123 (1996), 209–220.
[Gas] L. Van Gastel, Excess intersections and a correspondence principle, Invent.Math. 103(1) (1991), 197–222.
[GK] Terence Gaffney and Steven L. Kleiman, The Principle of Specialization of Modules, preprint (1995).
[HL] Jean–Pierre Georges Henry and Lê Dũng Tráng, *Limites d’espaces tangents*, Functions de plusieurs variable complexes II, Seminaire Norquet, Springer Lecture Notes in Mathematics 482, 1975, pp. 251–265.

[HMS] Jean–Pierre G. Henry, Michel Merle, Claude Sabbah, *Sur la condition de Thom stricte pour un morphisme analytique complexe*, Ann.Scient.Éc.Norm.Sup., 4. serie 131 (1984), 227–268.

[KT1] Steven L. Kleiman and Anders Thorup, *A geometric theory of the Buchsbaum–Rim multiplicity*, J. Algebra 167(1) (1994), 168–231.

[KT2] Steven L. Kleiman and Anders Thorup, *Mixed Buchsbaum–Rim multiplicities*, Copenhagen Univ. Preprint 131 (1994).

[L] Lê Dũng Tráng, *Calcul du nombre de cycles évanouissants d’une hypersurface complexe*, Ann. Inst. Fourier 23,4 (1973), 261–270.

[Li] Joseph Lipman, *Equimultiplicity, Reduction, and Blowing Up*, Commutative Algebra (R. N. Draper, ed.), Lect. Notes Pure Appl. Math. 68, Marcel Dekker, New York, 1982.

[LeT] Monique Lejeune-Jalabert and Bernard Teissier, *Clôture intégrale des idéaux et equisingularité, chapitre 1*, Publ. Inst. Fourier (1974).

[LT1] Lê Dũng Tráng and Bernard Teissier, *Limites d’espaces tangents en géométrie analytique*, Comment. Math. Helvetici 63 (1988), 540–578.

[LT2] Lê Dũng Tráng and Bernard Teissier, *Variétés polaires locales et classes de Chern des variétés singulières*, Ann. Math. 114 (1981), 457–491.

[Ma] David Massey, *Lé Cycles and Hypersurface Singularities*, Springer Lecture Notes in Mathematics 1615, 1995.

[MO] John Milnor and Peter Orlik, *Isolated singularities defined by weighted homogeneous polynomials*, Topology 9 (1970), 385–393.

[Pa] Adam Parusiński, *Limits of tangent spaces to fibres and the $W_f$ condition*, Duke Math. J. 72,1 (1993), 99–108.

[R] David Rees, *$A$–transforms of ideals, and a theorem on multiplicities of ideals*, Proc. Cambridge Phil.Soc. 57 (1961), 8–17.

[S] Pierre Samuel, *La notion de multiplicité en algèbre et en géométrie algébrique*, J. Math Pures Appl. 30 (1951), 159–274.

[T1] Bernard Teissier, *Cycles évanescents, sections planes et conditions de Whitney*, Astérisque 7-8 (1973), 285–362.

[T2] Bernard Teissier, *The hunting of invariants in the geometry of discriminants*, Real and Complex Singularities, Oslo 1976 (P. Holm, ed.), Proceedings of the Nordic Summer School/NAVF, Noordhoff & Sijthoff, 1976, pp. 565–677.

[T3] Bernard Teissier, *Cycles évanescents et resolution simultanée, I et II*, Seminaire sur les singularités des surfaces 1976–77, Springer Lecture Notes in Mathematics 777, 1980.

[T4] Bernard Teissier, *Variétés Polaires 2: Multiplicités polaires, sections planes, et conditions de Whitney*, Algebraic Geometry, La Rábida 1981 (J.M. Aroca, ed.), Proceedings of the International Conference, Springer Lecture Notes in Mathematics 961, 1982, pp. 314–491.

[V] Jean–Louis Verdier, *Stratifications de Whitney et théorème de Bertini–Sard*, Invent. Math. 36 (1976), 295–312.

Dept.of Mathematics, Northeastern University, Boston, MA 02115, USA
E-mail address: GAFF@neu.edu, Gassler@neu.edu