Directed Domination in Oriented Graphs

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Abstract

A directed dominating set in a directed graph $D$ is a set $S$ of vertices of $V$ such that every vertex $u \in V(D) \setminus S$ has an adjacent vertex $v$ in $S$ with $v$ directed to $u$. The directed domination number of $D$, denoted by $\gamma(D)$, is the minimum cardinality of a directed dominating set in $D$. The directed domination number of a graph $G$, denoted $\Gamma_d(G)$, which is the maximum directed domination number $\gamma(D)$ over all orientations $D$ of $G$. The directed domination number of a complete graph was first studied by Erdős [Math. Gaz. 47 (1963), 220–222], albeit in disguised form. We extend this notion to directed domination of all graphs. If $\alpha$ denotes the independence number of a graph $G$, we show that if $G$ is a bipartite graph, we show that $\Gamma_d(G) = \alpha$. We present several lower and upper bounds on the directed domination number.

Keywords: directed domination; oriented graph; independence number.

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1 Introduction

An asymmetric digraph or oriented graph $D$ is a digraph that can be obtained from a graph $G$ by assigning a direction to (that is, orienting) each edge of $G$. The resulting digraph $D$ is called an orientation of $G$. Thus if $D$ is an oriented graph, then for every pair $u$ and $v$ of distinct vertices of $D$, at most one of $(u,v)$ and $(v,u)$ is an arc of $D$. A directed dominating set, abbreviated DDS, in a directed graph $D = (V,A)$ is a set $S$ of vertices of $V$ such that every vertex in $V \setminus S$ is dominated by some vertex of $S$; that is, every vertex $u \in V \setminus S$ has an adjacent vertex $v$ in $S$ with $v$ directed to $u$. Every digraph has a DDS since the entire vertex set of the digraph is such a set. The directed domination number of a directed graph $D$, denoted by $\gamma(D)$, is the minimum cardinality of a DDS in $D$. A DDS of $D$ of cardinality $\gamma(D)$ is called a $\gamma(D)$-set. Directed domination in digraphs is well studied (cf. [2, 3, 6, 7, 8, 12, 15, 19, 22, 23]).

We define the lower directed domination number of a graph $G$, denote $\gamma_d(G)$, to be the minimum directed domination number $\gamma(D)$ over all orientations $D$ of $G$; that is,

$$\gamma_d(G) = \min \{ \gamma(D) \mid \text{over all orientations } D \text{ of } G \}.$$ 

The upper directed domination number, or simply the directed domination number, of a graph $G$, denoted $\Gamma_d(G)$, is defined as the maximum directed domination number $\gamma(D)$ over all orientations $D$ of $G$; that is,

$$\Gamma_d(G) = \max \{ \gamma(D) \mid \text{over all orientations } D \text{ of } G \}.$$ 

1.1 Motivation

The directed domination number of a complete graph was first studied by Erdős [11] albeit in disguised form. In 1962, Schütte [11] raised the question of given any positive integer $k > 0$, does there exist a tournament $T_{n(k)}$ on $n(k)$ vertices in which for any set $S$ of $k$ vertices, there is a vertex $u$ which dominates all vertices in $S$. Erdős [11] showed, by probabilistic arguments, that such a tournament $T_{n(k)}$ does exist, for every positive integer $k$. The proof of the following bounds on the directed domination number of a complete graph are along identical lines to that presented by Erdős [11]. This result can also be found in [23]. Throughout this paper, log is to the base 2 while ln denotes the logarithm in the natural base $e$.

**Theorem 1** (Erdős [11]) For every integer $n \geq 2$, $\log n - 2\log(\log n) \leq \Gamma_d(K_n) \leq \log(n+1)$.

In this paper, we extend this notion of directed domination in a complete graph to directed domination of all graphs.
1.2 Notation

For notation and graph theory terminology we in general follow [18]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ of order $n = |V|$ and edge set $E$ of size $m = |E|$, and let $v$ be a vertex in $V$. The open neighborhood of $v$ is $N_G(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is $N_G[v] = \{v\} \cup N_G(v)$. If the graph $G$ is clear from context, we simply write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$, respectively. For a set $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. If $A$ and $B$ are subsets of $V(G)$, we let $[A, B]$ denote the set of all edges between $A$ and $B$ in $G$. We denote the diameter of $G$ by $\text{diam}(G)$.

We denote the degree of $v$ in $G$ by $d_G(v)$, or simply by $d(v)$ if the graph $G$ is clear from context. The minimum degree among the vertices of $G$ is denoted by $\delta(G)$, and the maximum degree by $\Delta(G)$. The maximum average degree in $G$, denoted by $\text{mad}(G)$, is defined as the maximum of the average degrees $\text{ad}(H) = |E(H)|/|V(H)|$ taken over all subgraphs $H$ of $G$.

The parameter $\gamma(G)$ denotes the domination number of $G$. The parameters $\alpha(G)$ and $\alpha'(G)$ denote the (vertex) independence number and the matching number, respectively, of $G$, while $\chi(G)$ and $\chi'(G)$ denote the chromatic number and edge chromatic number, respectively, of $G$. The hitting number of $G$, denoted by $\beta(G)$, is the minimum number vertices that covers all the edges of $G$. The clique number of $G$, denoted by $\omega(G)$, is the maximum cardinality of a clique in $G$.

A vertex $v$ in a digraph $D$ out-dominates, or simply dominates, itself as well as all vertices $u$ such that $(v, u)$ is an arc of $D$. The out-neighborhood of $v$, denoted $N^+(v)$, is the set of all vertices $u$ adjacent from $v$ in $D$; that is, $N^+(v) = \{u \mid (v, u) \in A(D)\}$. The out-degree of $v$ is given by $d^+(v) = |N^+(v)|$, and the maximum out-degree among the vertices of $D$ is denoted by $\Delta^+(D)$. The in-neighborhood of $v$, denoted $N^-(v)$, is the set of all vertices $u$ adjacent to $v$ in $D$; that is, $N^-(v) = \{u \mid (u, v) \in A(D)\}$. The in-degree of $v$ is given by $d^-(v) = |N^-(v)|$. The closed in-neighborhood of $v$ is the set $N^-\{v\} = N^-(v) \cup \{v\}$. The maximum in-degree among the vertices of $D$ is denoted by $\Delta^-(D)$.

A hypergraph $H = (V, E)$ is a finite set $V$ of elements, called vertices, together with a finite multiset $E$ of subsets of $V$, called edges. A $k$-edge in $H$ is an edge of size $k$. The hypergraph $H$ is said to be $k$-uniform if every edge of $H$ is a $k$-edge. A subset $T$ of vertices in a hypergraph $H$ is a transversal (also called vertex cover or hitting set in many papers) if $T$ has a nonempty intersection with every edge of $H$. The transversal number $\tau(H)$ of $H$ is the minimum size of a transversal in $H$. For a digraph $D = (V, E)$, we denote by $H_D$ the closed in-neighborhood hypergraph, abbreviated CINH, of $D$; that is, $H_D = (V, C)$ is the hypergraph with vertex set $V$ and with edge set $C$ consisting of the closed in-neighborhoods of vertices of $V$ in $D$.
2 Observations

We show first that the lower directed domination number of a graph is precisely its domination number.

Observation 1 For every graph $G$, $\gamma_d(G) = \gamma(G)$.

Proof. Let $S$ be a $\gamma(G)$-set and let $D$ be an orientation obtained from $G$ by directing all edges in $[S, V \setminus S]$ from $S$ to $V \setminus S$ and directing all other edges arbitrarily. Then, $S$ is a DDS of $D$, and so $\gamma_d(G) \leq \gamma(D) \leq |S| = \gamma(G)$. However if $D$ is an orientation of a graph $G$ such that $\gamma_d(G) = \gamma(D)$, and if $S$ is a $\gamma(D)$-set, then $S$ is also a dominating set of $G$, and so $\gamma(G) \leq |S| = \gamma_d(G)$. Consequently, $\gamma_d(G) = \gamma(G)$. $\square$

In view of Observation 1 it is not interesting to ask about the lower directed domination number, $\gamma_d(G)$, of a graph $G$ since this is precisely its domination number, $\gamma(G)$, which is very well studied. We therefore focus our attention on the (upper) directed domination number of a graph. As a consequence of Theorem 1 we establish a lower bound on the directed domination number of an arbitrary graph.

Observation 2 For every graph $G$ on $n$ vertices, $\Gamma_d(G) \geq \log n - 2 \log(\log n)$.

Proof. Let $D$ be an orientation of the edges of a complete graph $K_n$ on the same vertex set as $G$ such that $\Gamma_d(K_n) = \gamma(D)$. Let $D_G$ be the orientation of $D$ induced by arcs of $D$ corresponding to edges of $G$. Then, $\Gamma_d(G) \geq \gamma(D_G) \geq \gamma(D) = \Gamma_d(K_n)$. The desired lower bound now follows from Theorem 1. $\square$

Observation 3 If $H$ is an induced subgraph of a graph $G$, then $\Gamma_d(G) \geq \Gamma_d(H)$.

Proof. Let $G = (V, E)$ and let $U = V(H)$. Let $D_H$ be an orientation of $H$ such that $\Gamma_d(H) = \gamma(D_H)$. We now extend the orientation $D_H$ of $H$ to an orientation $D$ of $G$ by directing all edges in $[U, V \setminus U]$ from $U$ to $V \setminus U$ and directing all edges with both ends in $V \setminus U$ arbitrarily. Then, $\Gamma_d(G) \geq \gamma(D) \geq \gamma(D_H) = \Gamma_d(H)$. $\square$

Observation 4 If $H$ is a spanning subgraph of a graph $G$, then $\Gamma_d(G) \leq \Gamma_d(H)$.

Proof. Let $D$ be an arbitrary orientation of $G$, and let $D_H$ be the orientation of $H$ induced by $D$. Since adding arcs cannot increase the directed domination number, we have that $\gamma(D) \leq \gamma(D_H)$. This is true for every orientation of $G$. Hence, $\Gamma_d(G) \leq \Gamma_d(H)$. $\square$

Hakimi [17] proved that a graph $G$ has an orientation $D$ such that $\Delta^+(D) \leq k$ if and only if $\text{mad}(G) \leq 2k$. This implies the following result.

Observation 5 ([17]) Every graph $G$ has an orientation $D$ such that $\Delta^+(D) \leq \lceil \text{mad}(G)/2 \rceil$. 
3 Bounds

In this section, we establish bounds on the directed domination number of a graph. We first present lower bounds on the directed domination number of a graph.

**Theorem 2** Let $G$ be a graph of order $n$. Then the following holds.
(a) $\Gamma_d(G) \geq \alpha(G) \geq \gamma(G)$.
(b) $\Gamma_d(G) \geq n/\chi(G)$.
(c) $\Gamma_d(G) \geq \lceil(diam(G) + 1)/2\rceil$.
(d) $\Gamma_d(G) \geq n/(\lceil\text{mad}(G)/2\rceil + 1)$.

**Proof.** Since every maximal independent set in a graph is a dominating set in the graph, we recall that $\gamma(G) \leq \alpha(G)$ holds for every graph $G$. To prove that $\alpha(G) \leq \Gamma_d(G)$, let $A$ be a maximum independent set in $G$ and let $D$ be the digraph obtained from $G$ by orienting all arcs from $A$ to $V \setminus A$ and orienting all arcs in $G[V \setminus A]$, if any, arbitrarily. Since every DDS of $D$ contains $A$, we have $\gamma(D) \geq |A|$. However the set $A$ itself is a DDS of $D$, and so $\gamma(D) \leq |A|$. Consequently, $\alpha(G) \geq \gamma(D) = |A| = \alpha(G)$. This establishes Part (a).

Parts (b) and (c) follow readily from Part (a) and the observations that $\alpha(G) \geq n/\chi(G)$ and $\alpha(G) \geq \lceil(diam(G) + 1)/2\rceil$. By Observations 5, there is an orientation $D$ of $G$ such that $\Delta^+(D) \leq \lceil\text{mad}(G)/2\rceil$. Let $S$ be a $\gamma(D)$-set. Then, $V \setminus S \subseteq \cup_{v \in S} N^+(v)$, and so $n - |S| = |V \setminus S| \leq \sum_{v \in S} d^+(v) \leq |S| \cdot \Delta^+(D)$, whence $\gamma(D) = |S| \geq n/(\Delta^+(D) + 1) \geq n/(\lceil\text{mad}(G)/2\rceil + 1)$. This establishes Part (d). □

We remark that since $\text{mad}(G) \leq \Delta(G)$ for every graph $G$, as an immediate consequence of Theorem 2(d) we have that $\Gamma_d(G) \geq n/(\lceil\Delta(G)/2\rceil + 1)$.

Next we consider upper bounds on the directed domination number of a graph. The following lemma will prove to be useful.

**Lemma 3** Let $G = (V, E)$ be a graph and let $V_1, V_2, \ldots, V_k$ be subsets of $V$, not necessarily disjoint, such that $\cup_{i=1}^k V_i = V(G)$. For $i = 1, 2, \ldots, k$, let $G_i = G[V_i]$. Then,
$$\Gamma_d(G) \leq \sum_{i=1}^k \Gamma_d(G_i).$$

**Proof.** Consider an arbitrary orientation $D$ of $G$. For each $i = 1, 2, \ldots, k$, let $D_i$ be the orientation of the edges of $G_i$ induced by $D$ and let $S_i$ be a $\gamma(D_i)$-set. Then, $\Gamma_d(G_i) \geq \gamma(D_i) = |S_i|$ for each $i$. Since the set $S = \cup_{i=1}^k S_i$ is a DDS of $D$, we have that $\gamma(D) \leq |S| \leq \sum_{i=1}^k |S_i| \leq \sum_{i=1}^k \Gamma_d(G_i)$. Since this is true for every orientation $D$ of $G$, the desired upper bound on $\Gamma_d(G)$ follows. □
As a consequence of Lemma 3, we have the following upper bounds on the directed domination number of a graph.

**Theorem 4** Let $G$ be a graph of order $n$. Then the following holds.
(a) $\Gamma_d(G) \leq n - \alpha'(G)$.
(b) If $G$ has a perfect matching, then $\Gamma_d(G) \leq n/2$.
(c) $\Gamma_d(G) \leq n$ with equality if and only if $G = K_n$.
(d) If $G$ has minimum degree $\delta$ and $n \geq 2\delta$, then $\Gamma_d(G) \leq n - \delta$.
(e) $\Gamma_d(G) = n - 1$ if and only if every component of $G$ is a $K_1$-component, except for one component which is either a star or a complete graph $K_3$.

**Proof.** (a) Let $M = \{u_1v_1, u_2v_2, \ldots, u_tv_t\}$ be a maximum matching in $G$, and so $t = \alpha'(G)$. For $i = 1, 2, \ldots, t$, let $V_i = \{u_i, v_i\}$. If $n > 2t$, let $(V_{t+1}, \ldots, V_{n-2t})$ be a partition of the remaining vertices of $G$ into $n - 2t$ subsets each consisting of a single vertex. By Lemma 3, $\Gamma_d(G) \leq \sum_{i=1}^{n} \Gamma_d(G_i) = t + (n - 2t) = n - t = n - \alpha'(G)$. Part (b) is an immediate consequence of Part (a) and the observation that $\alpha'(G) = 0$ if and only if $G = K_n$.

(d) It is well known (see, for example, Bollobás [4], pp. 87) that if $G$ has $n$ vertices and minimum degree $\delta$ with $n \geq 2\delta$, then $\alpha'(G) \geq \delta$. Hence by Part (a) above, $\Gamma_d(G) \leq n - \delta$.

(e) Suppose that $\Gamma_d(G) = n - 1$. Then by Part (a) above, $\alpha'(G) = 1$. However every connected graph $F$ with $\alpha'(F) = 1$ is either a star or a complete graph $K_3$. Hence, either $G$ is the vertex disjoint union of a star and isolated vertices or of a complete graph $K_3$ and isolated vertices. □

We establish next that the directed domination number of a bipartite graph is precisely its independence number. For this purpose, recall that König [21] and Egerváry [10] showed that if $G$ is a bipartite graph, then $\alpha'(G) = \beta(G)$. Hence by Gallai’s Theorem [13], if $G$ is a bipartite graph of order $n$, then $\alpha(G) + \alpha'(G) = n$.

**Theorem 5** If $G$ is a bipartite graph, then $\Gamma_d(G) = \alpha(G)$.

**Proof.** Since $G$ is a bipartite graph, we have that $n - \alpha'(G) = \alpha(G)$. Thus by Theorem 2(a) and Theorem 3(b), we have that $\alpha(G) \leq \Gamma_d(G) \leq n - \alpha'(G) = \alpha(G)$. Consequently, we must have equality throughout this inequality chain. In particular, $\Gamma_d(G) = \alpha(G)$. □

### 4 Relation to other Parameters

The following result establishes an upper bound on the directed domination of a graph in terms of its independence number and chromatic number.
Theorem 6 For every graph $G$, we have $\Gamma_d(G) \leq \alpha(G) \cdot \lceil \chi(G)/2 \rceil$.

Proof. Let $G$ have order $n$. If $\chi(G) = 1$, then $G$ is the empty graph, $\overline{K}_n$, and so $\Gamma_d(G) = n = \alpha(G)$, while if $\chi(G) = 2$, then $G$ is a bipartite graph, and so by Theorem 5, $\Gamma_d(G) = \alpha(G)$. In both cases, $\alpha(G) = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$, and so $\Gamma_d(G) = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$. Hence we may assume that $\chi(G) \geq 3$. If $\chi(G) = 2k$ for some integer $k \geq 2$, then let $V_1, V_2, \ldots, V_{2k}$ be the color classes of $G$. For $i = 1, 2, \ldots, k$, let $G_i$ be the subgraph $G[V_{2i-1} \cup V_{2i}]$ of $G$ induced by $V_{2i-1}$ and $V_{2i}$, and note that $G_i$ is a bipartite graph. By Theorem 5, $\Gamma_d(G_i) = \alpha(G_i) \leq \alpha(G)$ for all $i$. Hence, by Lemma 8, $\Gamma_d(G) \leq \sum_{i=1}^k \Gamma_d(G_i) \leq k\alpha(G) = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$, as desired. If $\chi(G) = 2k + 1$ for some integer $k \geq 1$, then let $V_1, V_2, \ldots, V_{2k+1}$ be the color classes of $G$. For $i = 1, 2, \ldots, k$, let $H_i$ be the subgraph of $G$ induced by $V_{2i-1}$ and $V_{2i}$, and note that $H_i$ is a bipartite graph. Further let $H_{k+1} = G[V_{2k+1}]$, and so $H_{k+1}$ is an empty graph on $|V_{2k+1}| \leq \alpha(G)$ vertices. By Lemma 8, $\Gamma_d(G) \leq \sum_{i=1}^{k+1} \Gamma_d(H_i) \leq (k+1)\alpha(G) = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$. $\Box$

As shown in the proof of Theorem 6, the upper bound of Theorem 6 is always attained if $\chi(G) \leq 2$. We remark that if $\chi(G) = 3$ or $\chi(G) = 4$, then the upper bound of Theorem 6 is achievable by taking, for example, $G = rK_t$ where $t \in \{3, 4\}$ and $r$ is some positive integer. In this case, $\chi(G) = t$ and $\Gamma_d(G) = 2r = \alpha(G) \cdot \lceil \chi(G)/2 \rceil$.

Theorem 7 If $G$ is a graph of order $n$, then $\Gamma_d(G) \leq n - \lfloor \chi(G)/2 \rfloor$.

Proof. If $\chi(G) = 1$, then the bound is immediate since $\Gamma_d(G) \leq n$ by Theorem 4(c). Hence we may assume that $\chi(G) = k \geq 2$. Let $V_1, V_2, \ldots, V_k$ be the color classes of $G$. By the minimality of the coloring, there is an edge between every two color classes. In particular for $i = 1, 2, \ldots, \lfloor k/2 \rfloor$, there is an edge between $V_{2i-1}$ and $V_{2i}$, and so $\alpha'(G) \geq \lfloor k/2 \rfloor$. Hence by Theorem 4(a), $\Gamma_d(G) \leq n - \alpha'(G) \leq n - \lfloor k/2 \rfloor$. $\Box$

We remark that the bound of Theorem 7 is achievable for graphs with small chromatic number as may be seen by considering the graph $G = \overline{K}_{n-k} \cup K_k$ where $1 \leq k \leq 4$ and $n > k$. We show next that the directed domination of a graph is at most the average of its order and independence number. For this purpose, we recall the Gallai-Milgram Theorem 14 for oriented graphs which states that in every oriented graph $G = (V, E)$, there is a partition of $V$ into at most $\alpha(G)$ vertex disjoint directed paths.

Theorem 8 If $G$ is a graph of order $n$, then $\Gamma_d(G) \leq (n + \alpha(G))/2$.

Proof. Let $D$ be an orientation of $G$. By the Gallai-Milgram Theorem for oriented graphs, there is a partition $P = \{P_1, P_2, \ldots, P_t\}$ of $V(D)$ into $t$ vertex disjoint directed paths where $t \leq \alpha(G)$. For $i = 1, 2, \ldots, t$, let $|P_i| = p_i$, and so $\sum_{i=1}^t p_i = n$. By Lemma 8, $\Gamma_d(G) \leq \sum_{i=1}^t \Gamma_d(P_i) = \sum_{i=1}^t [p_i/2] \leq \sum_{i=1}^t (p_i + 1)/2 = (\sum_{i=1}^t p_i/2) + t/2 = (n + \alpha(G))/2$. $\Box$

That the bound of Theorem 8 is best possible, may be seen by considering, for example, the graph $G = rK_3 \cup sK_1$ of order $n = 3r + s$ with $\alpha(G) = r + s$ and $\Gamma_d(G) = 2r + s = (n + \alpha(G))/2$. 

7
The following result establishes an upper bound on the directed domination of a graph in terms of the chromatic number of its complement.

**Theorem 9** If $G$ is a graph of order $n$, then $\Gamma_d(G) \leq \chi(G) \cdot \log \left( \left\lceil \frac{n}{\chi(G)} \right\rceil + 1 \right)$.

**Proof.** Let $t = \chi(G)$ and consider a $\chi(G)$-coloring of the complement $\overline{G}$ of $G$ into $t$ color classes $Q_1, Q_2, \ldots, Q_t$, where $|Q_i| = q_i$ for $i = 1, 2, \ldots, t$. For each $i = 1, 2, \ldots, t$, the subgraph $G[Q_i]$ of $G$ induced by $Q_i$ is a clique. We now consider an arbitrary orientation $D$ of $G$, and we let $D_i = D[Q_i]$ denote the orientation of the edges of the clique $G[Q_i]$ induced by $D$. Then,

$$\gamma(D) \leq \sum_{i=1}^{t} \gamma(D_i) \leq \sum_{i=1}^{t} \Gamma_d(Q_i) = \sum_{i=1}^{t} \Gamma_d(K_{q_i}).$$

This is true for every orientation $D$ of $G$, and so, by Theorem 1, we have that $\Gamma_d(G) \leq \sum_{i=1}^{t} \log(q_i + 1)$, where $\sum_{i=1}^{t} q_i = n$. By convexity the right hand side attains its maximum when all summands are as equal as possible; that is, some of the summands are $\lceil n/t \rceil$ and some are $\lfloor n/t \rfloor$. Hence, $\Gamma_d(G) \leq t \log(\lceil n/t \rceil + 1)$. \hfill $\Box$

As a consequence of Theorem 9, we have the following result on the directed domination number of a dense graph with large minimum degree.

**Theorem 10** If $G$ is a graph on $n$ vertices with minimum degree $\delta(G) \geq (k - 1)n/k$ where $k$ divides $n$, then $\Gamma_d(G) \leq n \log(k + 1) / k$.

**Proof.** Since $k | n$, we note that $n = kt$ and $\delta(G) \geq (k - 1)t$ for some integer $t$. By the well-known Hajnal-Szemerédi Theorem [16], the graph $G$ contains $t$ vertex disjoint copies of $K_k$. Further, $\chi(G) \leq t$. Thus applying Theorem 9 we have that $\Gamma_d(G) \leq t \log(k + 1) = n \log(k + 1) / k$. \hfill $\Box$

## 5 Special Families of Graphs

In this section, we consider the (upper) directed domination number of special families of graph. As remarked earlier, the directed domination number of a complete graph $K_n$ is determined by Erdős [11] in Theorem 1 while the directed domination number of a bipartite graph is precisely its independence number (see Theorem 5).

### 5.1 Regular Graphs

For each given $\delta \geq 1$, applying Theorem 2(a) to the graph $G = K_{k,n-\delta}$ yields $\Gamma_d(G) \geq n - \delta$. Hence without regularity, we observe that for each fixed $\delta \geq 1$, there exists a graph $G$ of
order \( n \) and minimum degree \( \delta \) satisfying \( \Gamma_d(G) \geq n - \delta \). With regularity, the directed domination number of a graph may be much smaller. For a given \( r \), let \( n = k(r + 1) \) for some integer \( k \) and let \( G \) consist of the disjoint union of \( k \) copies of \( K_{r+1} \). Let \( G_1, G_2, \ldots, G_k \) denote the components of \( G \). Each component of \( G \) is \( r \)-regular, and by Theorem 4(c), \( \Gamma_d(G) = \sum_{i=1}^{k} \Gamma_d(G_i) = \sum_{i=1}^{k} \Gamma_d(K_{r+1}) \leq k \log(r + 2) = n \log(r + 2)/(r + 1) \). Hence there exist \( r \)-regular graphs of order \( n \) with \( \Gamma_d(G) \leq n \log(r + 2)/(r + 1) \). In view of these observations it is of interest to investigate the directed domination number of regular graphs.

In 1964, Vizing proved his important edge-coloring result which states that every graph \( G \) satisfies \( \Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 \). As a consequence of Vizing’s Theorem, we have the following upper bound on the directed domination number of a regular graph.

**Theorem 11** For \( r \geq 2 \), if \( G \) is an \( r \)-regular graph of order \( n \), then

\[
\Gamma_d(G) \leq n(r + 2)/(r + 1).
\]

**Proof.** By Vizing’s Theorem, \( \chi'(G) \leq r + 1 \). Consider an edge coloring of \( G \) using \( \chi'(G) \)-colors. The edges in each color class form a matching in \( G \), and so the matching number of \( G \) is at least the size of a largest color class in \( G \). Hence if \( G \) has size \( m \), we have \( \alpha'(G) \geq m/\chi'(G) \geq m/(r + 1) = nr/2(r + 1) \). Hence by Theorem 4(a), \( \Gamma_d(G) \leq n - \alpha'(G) \leq n/nr/2(r + 1) = n(r + 2)/(r + 1) \). \( \Box \)

As a special case of Theorem 11 we have that \( \Gamma_d(G) \leq 2n/3 \) if \( G \) is a 2-regular graph. We next characterize when equality is achieved in this bound.

**Proposition 1** Let \( G \) be a 2-regular graph on \( n \geq 3 \) vertices. Then the following holds.

(a) If \( G \) is connected, then \( \Gamma_d(G) = \lceil n/2 \rceil \).

(b) \( \Gamma_d(G) \leq 2n/3 \) with equality if and only if \( G \) consists of disjoint copies of \( K_3 \).

**Proof.** (a) Suppose that \( G \) is a cycle \( C_n \). If \( n \) is even, \( G \) has a perfect matching, and so, by Theorem 4(c), \( \Gamma_d(G) \leq n/2 \). If \( n \) is odd, then \( \alpha'(G) = (n - 1)/2 \). By Theorem 4(b), \( \Gamma_d(G) \leq n - \alpha'(G) = n - (n - 1)/2 = (n + 1)/2 \). In both cases, \( \Gamma_d(G) \leq \lceil n/2 \rceil \). To show that \( \Gamma_d(G) \geq \lceil n/2 \rceil \), we note that if \( D \) is a directed cycle \( C_n \), then every vertex out-dominates itself and exactly one other vertex, and so \( \Gamma_d(G) \geq \gamma(D) = \lceil n/2 \rceil \). This proves part (a).

(b) To prove part (b), let \( G_1, G_2, \ldots, G_k \) be the components of \( G \), where \( k \geq 1 \). For \( i = 1, 2, \ldots, k \), let \( G_i \) have order \( n_i \). Since each component of a cycle, \( n \geq 3k \). Applying the result of part (a) to each component of \( G \), we have

\[
\Gamma_d(G) = \sum_{i=1}^{k} \Gamma_d(G_i) \leq \sum_{i=1}^{k} \left( \frac{n_i + 1}{2} \right) = \frac{n + k}{2} \leq \frac{2n}{3},
\]

with equality if and only if \( n = 3k \), i.e., if and only if \( G_i = C_3 \) for each \( i = 1, 2, \ldots, k \). \( \Box \)
We remark that the upper bound of Theorem 11 can be improved using tight lower bounds on the size of a maximum matching in a regular graph established in [20]. Applying Theorem 4(a) to these matching results in [20], we have the following result. We remark that the \((n + 1)/2\) bound in the statement of Theorem 12 is only included as it is necessary when \(n\) is very small or \(r = 2\).

**Theorem 12** For \(r \geq 2\), if \(G\) is a connected \(r\)-regular graph of order \(n\), then

\[
\Gamma_d(G) \leq \begin{cases} 
    \max \left\{ \left( \frac{r^2 + 2r}{r^2 + r + 2} \right) \times \frac{n}{2}, \frac{n+1}{2} \right\} & \text{if } r \text{ is even} \\
    \frac{(r^3 + r^2 - 6r + 2)n + 2r - 2}{2(r^3 - 3r)} & \text{if } r \text{ is odd}
\end{cases}
\]

We close this section with the following observation. Graphs \(G\) satisfying \(\chi'(G) = \Delta(G)\) are called class 1 and those with \(\chi'(G) = \Delta(G) + 1\) are class 2.

**Observation 6** Let \(G\) be an \(r\)-regular graph of order \(n\). Then the following holds.
(a) If \(G\) is of class 1, then \(\Gamma_d(G) \leq n/2\).
(b) If \(r \geq n/2\), then \(\Gamma_d(G) \leq \lceil n/2 \rceil\).

**Proof.** (a) Consider a \(r\)-edge coloring of \(G\). The edges in each color class form a perfect matching in \(G\), and so, by Theorem 4(c), \(\Gamma_d(G) \leq n/2\).

(b) If \(n = 2\), then the result is immediate. Hence we may assume that \(n \geq 3\). By Dirac’s theorem, \(G\) is hamiltonian, and so \(\alpha'(G) \geq \lceil n/2 \rceil\). By Theorem 4(b), \(\Gamma_d(G) \leq n - \alpha'(G) \leq n - \lceil n/2 \rceil = \lceil n/2 \rceil\). \(\square\)

### 5.2 Outerplanar Graphs

Let \(\mathcal{OP}_n\) denote the family of all maximal outerplanar graphs of order \(n\). We define \(\text{Mop}(n) = \max\{\Gamma_d(G)\}\) where the maximum is taken over all graphs \(G \in \mathcal{OP}_n\).

**Theorem 13** \(\text{Mop}(n) = \lceil n/2 \rceil\).

**Proof.** Let \(G \in \mathcal{OP}_n\). Since every maximal outerplanar graph is hamiltonian, we observe by Observation 4 and Proposition 11(a), that \(\Gamma_d(G) \leq \Gamma_d(C_n) = \lceil n/2 \rceil\). Since this is true for an arbitrary graph \(G\) in \(\mathcal{OP}_n\), we have \(\text{Mop}(n) \leq \lceil n/2 \rceil\). Hence it suffices for us to prove that \(\text{Mop}(n) \geq \lceil n/2 \rceil\). If \(n = 3\), then by Observation 3, \(\Gamma_d(G) \geq \Gamma_d(C_n) = \lceil n/2 \rceil\), as desired. Hence we may assume that \(n \geq 4\), for otherwise the desired result follows.
For $n \geq 4$ even, we take a directed cycle $\overrightarrow{C}_n$ on $n \geq 4$ vertices and a selected vertex $v$ on the cycle, and we add arcs from every vertex $u$, where $u$ is neither the in-neighbor nor the out-neighbor of $v$ on $\overrightarrow{C}_n$, to the vertex $v$. The resulting orientation $D$ of the underlying maximal outerplanar graph has $\gamma_d(D) = n/2$. Hence for $n \geq 4$ even, we have $\text{Mop}(n) = n/2$.

It remains for us to show that for $n \geq 5$ odd, $\text{Mop}(n) = (n + 1)/2$. For $n \geq 5$ odd, we take a directed cycle $\overrightarrow{C}_n$: $v_1v_2 \ldots v_nv_1$ on $n$ vertices. We now add the arcs from $v_i$ to $v_1$ for all odd $i$, where $3 \leq i \leq n - 2$, and we add the arcs from $v_1$ to $v_i$ for all even $i$, where $4 \leq i \leq n - 1$. Let $G$ denote the resulting underlying maximal outerplanar graph and let $D$ denote the resulting orientation of $D$. We now consider an arbitrary DDS $S$ in $D$.

Suppose first that $v_1 \in S$. In order to dominate the $(n - 1)/2$ vertices $v_{2i+1}$, where $1 \leq i \leq (n - 1)/2$, in $D$ we must have that $|S \cap \{v_{2i}, v_{2i+1}\}| \geq 1$ for all $i = 1, 2, \ldots, (n - 1)/2$. Hence in this case when $v_1 \in S$, we have $|S| \geq (n + 1)/2$.

Suppose next that $v_1 \notin S$. Then, $v_2 \in S$. In order to dominate the $(n - 3)/2$ vertices $v_{2i}$, where $2 \leq i \leq (n - 1)/2$, in $D$ we must have that $|S \cap \{v_{2i}, v_{2i-1}\}| \geq 1$ for all $i = 2, \ldots, (n - 1)/2$. In order to dominate $v_1$, there is a vertex $v_j \in S$ for some odd $j$, where $3 \leq j \leq n$. Let $j$ be the largest such odd subscript for which $v_j \in S$. If $j = n$, then $v_n \in S$ and $|S| \geq (n + 1)/2$, as desired. Hence we may assume that $j < n$. In order to dominate the vertex $v_i$ for $i$ odd with $j < i \leq n$, we must have $v_{i-1} \in S$. In particular, we have that $v_{j+1} \in S$ to dominate $v_{j+2}$, implying that $|S \cap \{v_j, v_{j+1}\}| = 2$ while for $i$ odd where $i \neq j$ and $3 \leq i \leq n - 2$, we have $|S \cap \{v_i, v_{i+1}\}| \geq 1$, implying that $|S| \geq (n + 1)/2$.

In both cases, $|S| \geq (n + 1)/2$. Since $S$ is an arbitrary DDS in $D$, we have $\gamma(D) \geq (n + 1)/2$. Hence, $\Gamma_d(G) \geq (n + 1)/2$, implying that $\text{Mop}(n) = (n + 1)/2$. □

5.3 Perfect Graphs

Recall that a perfect graph is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. Characterization of perfect graphs was a longstanding open problem. The first breakthrough was due to Lovsz in 1972 who proved the Perfect Graph Theorem.

**Perfect Graph Theorem** A graph is perfect if and only if its complement is perfect.

Let $\alpha \geq 1$ be an integer and let $\mathcal{G}_\alpha$ be the class of all graphs $G$ with $\alpha \geq \alpha(G)$. We are now in a position to present an upper bound on the directed domination number of a perfect graph in terms of its independence number.

**Theorem 14** If $G \in \mathcal{G}_\alpha$ is a perfect graph of order $n \geq \alpha$, then

$$\Gamma_d(G) \leq \alpha \log \left(\lceil n/\alpha \rceil + 1\right).$$
Proof. By the Perfect Graph Theorem, the complement $\overline{G}$ of $G$ is perfect. Hence, $\chi(\overline{G}) = \omega(\overline{G}) = \alpha(G)$. The desired result now follows from Theorem 9. $\square$

6 Interplay between Transversals and Directed Domination

In this section, we present upper bounds on the directed domination number of a graph by demonstrating an interplay between the directed domination number of a graph and the transversal number of a hypergraph. We shall need the following upper bounds on the transversal number of a uniform hypergraph established by Alon [1] and Chvátal and McDiarmid [9]. Applying probabilistic arguments, Alon [1] showed the following result.

Theorem 15 (Alon [1]) For $k \geq 2$, if $H$ is a $k$-uniform hypergraph with $n$ vertices and $m$ edges, then $\tau(H) \leq (m + n)(\ln k)/k$.

Theorem 16 (Chvátal, McDiarmid [9]) For $k \geq 2$, if $H$ is a $k$-uniform hypergraphs with $n$ vertices and $m$ edges, then $\tau(H) \leq (n + \lfloor k/2 \rfloor m)/\lfloor 3k/2 \rfloor$. bound is sharp.

We proceed further with two lemmas. For this purpose, we shall need the Szekeres-Wilf Theorem.

Theorem 17 (Szekeres-Wilf [24]) If $G$ is a $k$-degenerate graph, then $\chi(G) \leq k + 1$.

Lemma 18 If $G$ is a graph and $D$ is an orientation of $G$ such that $\Delta^-(D) \leq k$ for some fixed integer $k \geq 0$, then $\chi(G) \leq 2k + 1$.

Proof. It suffices to show that $G$ is $2k$-degenerate, since then the desired result follows from the Szekeres-Wilf Theorem. Assume, to the contrary, that $G$ is not $2k$-degenerate. Then there is a subset $S$ of $V(G)$ such that the subgraph $G_S = G[S]$ induced by $S$ has minimum degree at least $2k + 1$ and hence contains at least $(2k + 1)|S|/2$ edges. Let $D_S = D[S]$ be the orientation of $D$ induced by $S$. Since $\Delta^-(D) \leq k$, we have that $\Delta^-(D_S) \leq k$ and

$$k|S| \geq \sum_{v \in V(D_S)} d^-(v) = |E(G_S)| \geq (2k + 1)|S|/2 > k|S|,$$

a contradiction. $\square$

Lemma 19 Let $D$ be an orientation of a graph $G$. If $G$ contains $n_k$ vertices with in-degree at most $k$ in $D$ for some fixed integer $k \geq 0$, then $n_k \leq (2k + 1)\alpha(G)$.

Proof. Let $V_k$ denote the set of all vertices of $G$ with in-degree at most $k$ in $D$, and so $n_k = |V_k|$. Let $G_k = G[V_k]$ and let $D_k = D|V_k|$. Then, $D_k$ is an orientation of $G_k$.
such that $\Delta^-(D_k) \leq k$, and so by Lemma 18 $\chi(G_k) \leq 2k + 1$. Since every color class of $G_k$ is an independent set, and therefore has cardinality at most $\alpha(G)$, we have that

$$n_k = |V_k| \leq \chi(G_k)\alpha(G) \leq (2k + 1)\alpha(G). \square$$

Let $f(n, k)$, $g(n, k)$, and $h(n, k)$ be the functions of $n$ and $k$ defined as follows.

$$f(n, k) = 2n \ln(k + 2)/(k + 2) + (2k + 1)\alpha(G)$$
$$g(n, k) = n(k + 2)/3k + 2(2k + 1)\alpha(G)/3$$
$$h(n, k) = n(k + 1)/(3k - 1) + 2k(2k + 1)\alpha(G)/(3k - 1)$$

**Theorem 20** If $G$ is a graph on $n$ vertices, then

$$\Gamma_d(G) \leq \begin{cases} 
\min\{f(n, k), g(n, k)\} & \text{if } k \text{ is even} \\
\min\{f(n, k), h(n, k)\} & \text{if } k \text{ is odd}
\end{cases}$$

**Proof.** Let $D$ be an arbitrary orientation of the graph $G$ and let $k \geq 0$ be an arbitrary integer. Let $V_k$ denote the set of all vertices of $G$ with in-degree at most $k$ in $D$ and let $n_k = |V_k|$. Let $V_{>k} = V(G) \setminus V_k$, and so all vertices in $V_{>k}$ have in-degree at least $k + 1$ in $D$. Let $H_{>k}$ be the hypergraph obtained from the CINH $H_D$ of $D$ by deleting the $n_k$ edges corresponding to closed in-neighborhoods of vertices in $V_k$. Each edge in $H_{>k}$ has size at least $k + 2$.

We now define the hypergraph $H$ as follows. For each edge $e_v$ in $H_{>k}$ corresponding to the closed in-neighborhood of a vertex $v$ in $V_{>k}$, let $e'_v$ consist of $v$ and exactly $k + 1$ vertices from $N^-(v)$. Thus, $e'_v \subseteq e_v$ and $e'_v$ has size $k + 2$. Let $H$ be the hypergraph obtained from $H_{>k}$ by shrinking all edges $e_v$ of $H_{>k}$ to the edges $e'_v$. Then, $H$ is a $(k + 2)$-uniform hypergraph with $n$ vertices and $n - n_k$ edges.

Every transversal $T$ in $H$ contains a vertex from the closed in-neighborhood of each vertex from the set $V_{>k}$ in $D$, and therefore $T \cup V_k$ is a DDS in $D$. In particular, taking $T$ to be a minimum transversal in $H$, we have that $\gamma(D) \leq \tau(H) + n_k$. By Lemma 19 $n_k \leq (2k + 1)\alpha(G)$. Applying Theorem 15 to the hypergraph $H$, we have that

$$\tau(H) \leq (n + n - n_k)\ln(k + 2)/(k + 2) \leq 2n\ln(k + 2)/(k + 2),$$

and so $\gamma(D) \leq \tau(H) + n_k \leq 2n\ln(k + 2)/(k + 2) + \alpha(G)(2k + 1) = f(n, k)$. Applying Theorem 16 to the hypergraph $H$ for $k$ even, we have that

$$\tau(H) \leq (2n + k(n - n_k))/3k = n(k + 2)/3k - n_k/3,$$

and so $\gamma(D) \leq \tau(H) + n_k \leq n(k + 2)/3k + 2n_k/3 \leq n(k + 2)/3k + 2(2k + 1)\alpha(G)/3 = g(n, k)$. Thus for $k$ even, we have that $\Gamma_d(G) \leq \min\{f(n, k), g(n, k)\}$. Applying Theorem 16 to the hypergraph $H$ for $k$ odd, we have that

$$\tau(H) \leq (2n + (k - 1)(n - n_k))/(3k - 1) = n(k + 1)/(3k - 1) - (k - 1)n_k/(3k - 1),$$

13
and so \( \gamma(D) \leq \tau(H) + n_k \leq n + k \leq n(k+1)/(3k-1) + 2kn_k/(3k-1) \leq n(k+1)/(3k-1) + 2k(2k+1)\alpha(G)/(3k-1) = h(n,k) \). Thus for \( k \) odd, we have that \( \Gamma_d(G) \leq \min\{f(n,k), h(n,k)\} \). \( \square \)

Let \( f_n(\alpha), g_n(\alpha), \) and \( h_n(\alpha) \) be the functions of \( n \) and \( \alpha \) defined as follows.

\[
\begin{align*}
  f_n(\alpha) &= \sqrt{2n\alpha} \left( \ln\left( \sqrt{2n/\alpha} \right) + 2 \right) - 2\alpha \\
  g_n(\alpha) &= \frac{1}{3} \left( n + 2\alpha + 4\sqrt{2n\alpha} \right) \\
  h_n(\alpha) &= \frac{1}{3} \left( n + \frac{14}{3}\alpha + \frac{2\alpha(27n + 20\alpha)}{3\sqrt{3\alpha + 6n}} \right)
\end{align*}
\]

As a consequence of Theorem 20 we have the following upper bound on the directed domination of a graph.

**Theorem 21** If \( G \) is a graph on \( n \) vertices with independence number \( \alpha \), then

\[ \Gamma_d(G) \leq \min\{f_n(\alpha), g_n(\alpha), h_n(\alpha)\}. \]

**Proof.** By Theorem 20 we need to optimize the functions \( f(n,k), g(n,k) \) and \( h(n,k) \) over \( k \) to obtain an upper bound on \( \Gamma_d(G) \). To simplify the notation, let \( \alpha = \alpha(G) \). Optimizing the function \( g(n,k) \) over \( k \) (treating \( n \) as fixed), we get \( g(n,k) \leq g_n(\alpha) \), while optimizing the function \( h(n,k) \) over \( k \) (treating \( n \) as fixed), we get \( h(n,k) \leq h_n(\alpha) \). Optimization of the function \( f(n,k) \) is complicated. Hence to simplify the computations, we choose a value \( k^* \) for \( k \) and show that \( f(n,k^*) \leq f_n(\alpha) \). Suppose \( \alpha \geq n/2 \). Then, \( \alpha = cn \) with \( 1 \leq c \leq 1/2 \). Substituting this into \( f_n(\alpha) \) we get

\[
\begin{align*}
  f_n(\alpha) &= n\sqrt{2c}\left( \ln(2/c) + 2 \right) - 2cn = n\left( \sqrt{2c}\left( \ln(2/c) + 2 \right) - 2c \right) \geq n ,
\end{align*}
\]

and so the inequality \( \Gamma_d(G) \leq f_n(\alpha) \) holds trivially. Hence we may assume that \( \alpha \leq n/2 \). We now take \( k = \sqrt{2n/\alpha} - 2 \geq 0 \). Substituting into

\[
\begin{align*}
  f(n,k) &= 2n \ln(\sqrt{2n/\alpha})/\sqrt{2n/\alpha} + (2\sqrt{2n/\alpha} - 3)\alpha \\
       &= \sqrt{2n\alpha} \ln(\sqrt{2n/\alpha}) + 2\alpha\sqrt{2n/\alpha} - 3\alpha \\
       &= \sqrt{2n\alpha} \left( \ln(\sqrt{2n/\alpha}) + 2 \right) - 3\alpha \\
       &< f_n(\alpha),
\end{align*}
\]

as desired. \( \square \)

If every edge of a hypergraph \( H \) has size at least \( r \), we define an \( r \)-transversal of \( H \) to be a transversal \( T \) such that \( |T \cap e| \geq r \) for every edge \( e \) in \( H \). The \( r \)-transversal number \( \tau_r(H) \) of \( H \) is the minimum size of an \( r \)-transversal in \( H \). In particular, we note that \( \tau_1(H) = \tau(H) \). For integers \( k \geq r \) where \( k \geq 2 \) and \( r \geq 1 \), we first establish general upper bounds on the \( r \)-transversal number of a \( k \)-uniform hypergraph. Our next result generalizes that of Theorem 15 due to Alon [1, as well as generalizes results due to Caro [5].
**Theorem 22** For integers \( k \geq r \) where \( k \geq 2 \) and \( r \geq 1 \), let \( H \) be a \( k \)-uniform hypergraph with \( n \) vertices and \( m \) edges. Then, \( \tau_r(H) \leq n \ln k/k + rm(2 \ln k)^r/k \).

**Proof.** Pick every vertex of \( V(H) \) randomly with probability \( p \) to be determined later but such that \( (1 - p) > 1/2 \). Let \( X \) be the set of randomly picked vertices and let \( E_X \) be the set of edges of \( E(H) \) whose intersection with \( X \) is at most \( r - 1 \). For every fixed edge \( e \in E(H) \), the probability that \( e \) is in \( E_X \) is exactly

\[
\Pr(e \in E_X) = \sum_{i=0}^{r-1} \binom{k}{i} p^i (1 - p)^{k-i} = (1 - p)^k \sum_{i=0}^{r-1} \binom{k}{i} \left( \frac{p}{1 - p} \right)^i.
\]

We now choose \( p = \ln k/k \). With this choice of \( p \), we have that \( (1 - p) > 1/2 \). Hence, \( 1/(1 - p)^i < 2^i \) for all \( i \geq 1 \). Since \( 1 - x \leq e^{-x} \) for all \( x \in R \), we note that \( (1 - p)^k \leq e^{-pk} = e^{-\ln k} = 1/k \). Substituting \( p = \ln k/k \) into Equation (1) we therefore get

\[
\Pr(e \in E_X) \leq \frac{1}{k} \sum_{i=0}^{r-1} \binom{k}{i} \cdot \frac{p^i}{(1 - p)^i} \leq \frac{1}{k} \sum_{i=0}^{r-1} \frac{(2kp)^i}{i!} \leq \frac{1}{k} \sum_{i=0}^{r-1} (2 \ln k)^i \leq \frac{1}{k} (2 \ln k)^r,
\]

since \( 1 + q + q^2 + \cdots + q^{r-1} = (q^r - 1)/(q - 1) \leq q^r \) for \( q > 1 \) and \( r \geq 1 \). For each edge \( e \in E_X \), we add \( r - |e \cap X| \) (which is at most \( r \) vertices from \( e \setminus X \) to a set \( Y \). Then, \( T = X \cup Y \) is a \( r \)-transversal in \( H \) and \( |Y| \leq r|E_X| \). By the linearity of expectation, \( E(T) = E(X) + E(Y) \leq E(X) + r E(E_X) = n \ln k/k + rm(2 \ln k)^r/k \). \( \square \)

Using \( r \)-transversals in hypergraphs, we obtain the following bound on the directed \( r \)-domination number of a graph.

**Theorem 23** For \( r \geq 1 \) an integer, if \( G \) is a graph on \( n \) vertices, then

\[
\Gamma_d(G, r) \leq \min_{k \geq r} \left\{ (2k - 1)\alpha(G) + n \ln (k + 1)/(k + 1) + rn(2 \ln (k + 1))^r/(k + 1) \right\}.
\]

**Proof.** Let \( D \) be an arbitrary orientation of the graph \( G \) and let \( k \geq r \) be an arbitrary integer. Let \( V_{<k} \) denote the set of all vertices of \( G \) with in-degree at most \( k - 1 \) in \( D \) and let \( n_{<k} = |V_{<k}| \). Let \( G_{<k} \) be the subgraph of \( G \) induced by the set \( V_{<k} \) and let \( D_{<k} \) be the orientation of \( G_{<k} \) induced by \( D \). Then, \( \Delta^-(D_{<k}) \leq k - 1 \), and so, by Lemma [13]

\( \chi(G_{<k}) \leq 2k - 1 \), implying that \( n_{<k} \leq (2k - 1)\alpha(G) \).

Let \( V_k = V(G) \setminus V_{<k} \), and so all vertices in \( V_k \) have in-degree at least \( k \) in \( D \). Let \( H_k \) be the hypergraph obtained from the CINH \( H_D \) of \( D \) by deleting the \( n_{<k} \) edges corresponding to closed in-neighborhoods of vertices in \( V_{<k} \). Each edge in \( H_k \) has size at least \( k + 1 \). We now define the hypergraph \( H \) as follows. For each edge \( e_v \) in \( H_k \) corresponding to the closed in-neighborhood of a vertex \( v \) in \( V_k \), let \( e'_v \) consist of \( v \) and exactly \( k \) vertices from
Problem 2. Find good lower and upper bounds on $\min_{G}$

Thus, $\ell'_v \subseteq e_v$ and $e'_v$ has size $k + 1$. Let $H$ be the hypergraph obtained from $H_k$ by shrinking all edges $e_v$ of $H_k$ to the edges $e'_v$. Then, $H$ is a $(k + 1)$-uniform hypergraph with $n$ vertices and $n - n_{<k}$ edges.

Every $r$-transversal $T$ in $H$ contains at least $r$ vertices from the closed in-neighborhood of each vertex from the set $V_k$ in $D$, and therefore $T \cup V_{<k}$ is a DrDS in $D$. In particular, taking $T$ to be a minimum $r$-transversal in $H$, we have that $\gamma_r(D) \leq \tau_r(H) + n_{<k}$. By Lemma 19, $n_{<k} \leq (2k - 1)\alpha(G)$. Noting that $k + 1 \geq r + 1 \geq 2$, we can apply Theorem 22 to the hypergraph $H$ yielding $\tau_r(H) \leq n \ln(k + 1)/(k + 1) + r(n - n_{<k})(2 \ln(k + 1))r/(k + 1)$, and so $\gamma_r(D) \leq \tau_r(H) + n_{<k} \leq (2k - 1)\alpha(G) + n \ln(k + 1)/(k + 1) + rn(2 \ln(k + 1))r/(k + 1)$. Since this is true for every integer $k \geq r$, the desired upper bound on $\Gamma_d(G, r)$ follows. □

7 Open Questions

We close with a list of open questions and conjectures that we have yet to settle. Let $\mathcal{R}_n$ denote the family of all $r$-regular graphs of order $n$. We define $m(n, r) = \min\{\Gamma_d(G)\}$ and $M(n, r) = \max\{\Gamma_d(G)\}$, where the minimum and maximum are taken over all graphs $G \in \mathcal{R}_n$. Then, $m(n, 1) = M(n, 1) = n/2$. By Proposition 1, $m(n, 2) = n/2$ while $M(n, 2) = 2n/3$. We remark that by Theorem 11 for $r \geq 2$, we know that

$$\frac{n}{2} \leq M(n, r) \leq \left(\frac{r + 2}{r + 1}\right) \cdot \frac{n}{2}$$

(and this upper bound on $M(n, r)$ can be improved slightly by Theorem 12).

**Conjecture 1.** For $r \geq 3$, $M(n, r) = n/2$.

By Theorem 2(a), we know that if $G \in \mathcal{R}_n$, then $\Gamma_d(G) \geq \alpha(G) \geq n/(r + 1)$, and so $n/(r + 1) \leq m(n, r)$. Moreover taking $n/(r + 1)$ copies of $K_{r+1}$, we have by Theorem 1 that $m(n, r) \leq n \log(r + 2)/(r + 1)$. We pose the following question.

**Question 1.** For $r \geq 3$, does there exists a constant $c$ such that $m(n, r) \leq cn/(r + 1)$?

Let $\mathcal{OP}_n$ denote the family of all maximal outerplanar graphs of order $n$ and define $\text{mop}(n) = \min\{\Gamma_d(G)\}$, where the minimum is taken over all graphs $G \in \mathcal{OP}_n$. Since outerplanar graphs are 3-colorable, we note by Theorem 2(b) that for every graph $G \in \mathcal{OP}_n$, $\Gamma_d(G) \geq n/3$, implying that $\text{mop}(n) \geq n/3$. By Theorem 13, we know that $\text{mop}(n) \leq \lfloor n/2 \rfloor$. Thus, $n/3 \leq \text{mop}(n) \leq \lfloor n/2 \rfloor$.

**Problem 1.** Find good lower and upper bounds on $\text{mop}(n)$.

Let $\mathcal{P}_n$ denote the family of all maximum planar graphs of order $n$. We define $\text{mp}(n) = \min\{\Gamma_d(G)\}$ and $\text{Mp}(n) = \max\{\Gamma_d(G)\}$, where the minimum and maximum are taken over all graphs $G \in \mathcal{P}_n$.

**Problem 2.** Find good lower and upper bounds on $\text{mp}(n)$ and $\text{Mp}(n)$.
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