A Concentration Phenomenon for $p$-Laplacian Equation

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It is proved that if the bounded function of coefficient $Q_n$ in the following equation $-\text{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = Q_n(x)|u|^{q-2}u, u(x) = 0$ as $x \in \partial \Omega, u(x) \rightharpoonup 0$ as $|x| \to \infty$ is positive in a region contained in $\Omega$ and negative outside the region, the sets $\{Q_n > 0\}$ shrink to a point $x_0 \in \Omega$ as $n \to \infty$, and then the sequence $u_n$ generated by the nontrivial solution of the same equation, corresponding to $Q_n$, will concentrate at $x_0$ with respect to $W^{1,p}_0(\Omega)$ and certain $L^q(\Omega)$-norms. In addition, if the sets $\{Q_n > 0\}$ shrink to finite points, the corresponding ground states $\{u_n\}$ only concentrate at one of these points. These conclusions extend the results proved in the work of Ackermann and Szulkin (2013) for case $p = 2$.

1. Introduction

We study a new concentration phenomenon for the following $p$-Laplacian equations:

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = Q_n(x)|u|^{q-2}u,$$

as $x \in \partial \Omega,$

$$u(x) = 0 \quad \text{as} \quad x \in \partial \Omega,$$

$$u(x) \rightharpoonup 0 \quad \text{as} \quad |x| \to \infty,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain and $V \geq 0 \in L^{\infty}(\Omega)$, and $p < q < p^*$, where $p^* := Np/(N - p)$ if $N \geq p$ and $p^* := \infty$ if $N < p$. If $\Omega$ is unbounded, we assume additionally that $\sigma(-\text{div}(|\nabla \cdot |^{p-2}\nabla \cdot V|^{1/(p-2)} V) > 0, \mathbb{R}^N)$. And an assumption of $Q_n$ is as follows.

(*) The set $\{x \in \Omega : Q_n(x) > 0\}$ contained in the neighborhood of zero has positive measure, and $|Q_n|_{L^{\infty}(\Omega)} \leq C$ with the constant $C$ independent of $n$. Moreover, for each $\epsilon > 0$ there exist constants $\delta_\epsilon$ and $N_\epsilon$ such that $Q_n \leq -\delta_\epsilon$ whenever $x \notin B_{\epsilon}(0)$ and $n \geq N_\epsilon$.

As it is known, $u \equiv 0$ is the only solution to (1) if $Q_n(x) \leq 0$ for all $x \in \Omega$. In addition, if $Q_n(x) > 0$ is based on a bounded set of positive measures, it is clear that there exists a solution $u \neq 0$ (see Theorem 1). Hence, without loss of generality, we assume that $0 \in \Omega$ and let $Q = Q_n$ be such that $Q_n > 0$ on the ball $B_{1/n}(0)$ and $Q_n < 0$ on $\Omega \setminus B_{2/n}(0)$ and $u_n \neq 0$ are the solutions to (1) associated with $Q_n(x)$.

Accordingly, the question is what happens to $u_n$ as $n \to \infty$. Furthermore, this phenomenon can be found in physics. For instance, considering the materials separately from $Q$ positive or negative (see [1]), it corresponds to investigating the existence of bright ($Q > 0$) or dark ($Q < 0$) solitons.

Equations of these types have been studied extensively in many monographs and lectures (e.g., [2–10] for $p = 2$, [11–18] for general $p$). In [2], Byeon and Wang considered the standing wave solutions $\psi(x,t) \equiv \exp(-iEt/\hbar)\varphi(x)$ for the nonlinear Schrödinger equation:

$$ih\frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2}\Delta \psi - V(x)\psi + |\psi|^{p-1}\psi = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$  

(2)

Thus, they needed only to discuss the function $v$ which satisfies

$$\frac{\hbar^2}{2}\Delta v - (V(x) - E)v + |v|^{p-1}v = 0, \quad x \in \mathbb{R}^N,$$  

(3)

and rewrote it in the following form:

$$\epsilon^2 \Delta v - V(x)v + v^p = 0, \quad v > 0, \quad x \in \mathbb{R}^N,$$  

$$\lim_{|x| \to 0} v(x) = 0.$$  

(4)
By a rescaling, it is transformed to
\[ \Delta u - V(\varepsilon x) u + u^p = 0, \quad u > 0, \; x \in \mathbb{R}^N. \]  
\[ \lim_{|x| \to 0} u(x) = 0. \]  
(5)

Let the zero set \( \mathcal{Z} \triangleq \{ x \in \mathbb{R}^N \mid V(x) = 0 \} \) and \( A \) be an isolated component of \( \mathcal{Z} \), and they distinguished three cases of \( A \) to prove the concentration as \( \varepsilon \to 0 \). And then, in [3] by replacing \( v^\ast \) with a fairly general class nonlinearity \( f(v) \), they also obtained the concentration. Furthermore, in [4], Byeon and Jeanjean gave the almost optimal condition on \( f \) for the concentration. Recently, in [19], different from above with the linearity term \( V(\varepsilon x)u \), Ackermann and Szulkin considered the concentration phenomenon in the nonlinearity; that is, \( \Delta u + V(x)u = Q \) in \( \mathbb{R}^N \).

2. Concentration in the \( W_0^{1,p}(\Omega) \) and \( L^q(\Omega) \)

We begin with some notations.

Let \( E := W_0^{1,p}(\Omega) \) and
\[ \|u\| := \left( \int_{\Omega} (|\nabla u|^p + V|u|^p) dx \right)^{1/p} \]  
(6)
is an equivalent norm in \( E \) (due to \( \sigma \) (– div(|\nabla |^2 V)) + \( V \cdot |\nabla|^2 V \)) in \( (0, \infty) \)). Set
\[ |u|_{1,p,A} := \left( \int_A |u|^q dx \right)^{1/q}, \]  
(7)
\[ |u|_{1,q,A} = \text{esssup}_A |u|, \]  
and we abbreviate \( |u|_{1,q,A} \) to \( |u|_q \) sometimes. Moreover,
\[ B_r(a) := \{ x \in \mathbb{R}^n : |x - a| < r \} \]  
(8)
denotes a ball.

Here we offer the existence result for (1).

Theorem 1. Suppose that \( Q_n \) satisfies the assumption (*) above and \( q \in (p, p^\ast) \); then for all sufficiently large \( n \), there is a positive ground state solution \( u_n \in E \) to problem (1). Moreover, there exists a constant \( \alpha > 0 \) independent of \( n \), such that \( \|u_n\| \geq \alpha \).

Proof. As in [19], let \( J_n(v) = \int_\Omega Q_n |v|^q dx \) and
\[ s_n := \inf_{I_n(v) > 0} \frac{\|v\|^p}{J_n(v)^{p/q}} = \inf_{I_n(v) > 0} \frac{\int_\Omega (|\nabla v|^p + V|v|^p) dx}{\left( \int_\Omega Q_n |v|^q dx \right)^{p/q}}. \]  
(9)

Suppose that \( (v_k) \) is a minimizing sequence for \( s_n \), normalized by \( J_n(v_k) = 1 \); then \( \|v_k\| \) is bounded. Hence, \( v_k \to v \) in \( E \) and \( v_k(x) \to v(x) \) a.e. in \( \Omega \) (by choosing a subsequence). Note that \( Q_n < 0 \) on \( |x| > 1 \) for \( n \) large. The Rellich-Kondrachov Theorem and Fatou's Lemma say that
\[ s_n = \lim_{k \to \infty} \|v_k\|^p \]  
\[ = \lim_{k \to \infty} \left( \frac{\int_\Omega Q_n |v_k|^q dx + \int_{|x| > 1} Q_n |v_k|^q dx} {\int_{|x| < 1} Q_n |v_k|^q dx} \right)^{p/q} \]  
(10)
\[ \geq \frac{\int_\Omega |v|^p dx}{\int_{|x| < 1} Q_n |v|^q dx} \geq s_n. \]  
(11)

Thus \( v \) is a minimizer.

And then, the lagrange multiple rule implies that \( u_n = c_n v_n \) is a solution to (1) for some appropriate constant \( c_n \). Moreover, since \( v_n \) may be replaced by \( |v_n|, v_n \geq 0 \) (and hence \( u_n \geq 0 \)). To show that \( u_n > 0 \), we note that \( u_n \) satisfies
\[ - \text{div}(\nabla v^p \nabla v) + (V(x)u_n^{p-2} + Q_n(x)u_n(x)^{p-2})v = Q_n(x)u_n(x)^{p-1} \geq 0, \]  
where \( Q_n^+ := \max\{0, Q_n(x)\} \). Since \( V(x)u_n^{p-2} + Q_n(x)u_n(x)^{p-2} \geq 0 \), it follows from the strong maximum principle (see [20, 21]) that \( u_n > 0 \).

If \( u_n \neq 0 \) is a solution to (1), then, via multiplying the equation by \( u_n \), integrating by parts, and using the Sobolev inequality, one deduces that
\[ \|u_n\|^p = \int_\Omega Q_n |u_n|^q dx \leq c_1 |u_n|_q \leq c_2 \|u_n\|^q; \]  
(12)
hence, \( \|u_n\| \geq \alpha \) for some \( \alpha > 0 \) and all large \( n \).

The next step is to consider the property of the nontrivial solution \( \{u_n\} \) to (1) and \( u_n := u_n/\|u_n\| \).

Lemma 2. Consider
\[ \|u_n\| \to \infty \text{ as } n \to \infty. \]  
(13)

Proof. We present an abridged version of the proof highlighting the main differences to that in [19]. It will be proved by contradiction. Assume \( u_n \to u \) in \( E \) and \( u_n \to u \) in \( L^q_{\text{loc}}(\Omega) \) after passing to a subsequence. Multiplying (1) (with \( u = u_n \)) by \( u_n \), integrating by parts, and recalling that \( Q_n < 0 \) for each \( \varepsilon > 0 \) and \( n \geq N_\varepsilon \), it holds that
\[ \limsup_{n \to \infty} \|u_n\|^p = \limsup_{n \to \infty} \int_\Omega Q_n |u_n|^q dx \leq \limsup_{n \to \infty} \int_{|x| < r} Q_n |u_n|^q dx \leq c \int_{|x| < r} |u|^q dx. \]  
(14)

\[ \square \]
If \( \epsilon \to 0, u_n \to 0 \) in \( E \). It is a contradiction to \( \|u_n\| \geq \alpha > 0 \) given in Theorem 1.

**Lemma 3.** Consider

\[ w_n \to 0 \text{ in } E \quad \text{as } n \to \infty \]  
(15)

**Proof.** We prove it by contradiction as well. We may assume that \( w_n \to w(\neq 0) \) in \( E \). Multiplying (1) with \( u_n/\|u_n\| \) by \( u_n/\|u_n\| \) yields that

\[ 1 = \|w_n\|^p = \|u_n\|^{p-\epsilon} \int_Q |\nabla w_n|^p \, dx. \]  
(16)

Due to Lemma 2 with \( q > p \), \( \int \Omega Q_n|w_n|^q \to 0 \).

On the other hand, we have for \( 0 < \epsilon < \epsilon_1 \)

\[ 0 = \lim_{n \to \infty} \int \Omega Q_n|w_n|^q \, dx \]

\[ = \lim_{n \to \infty} \left( \int_{|x|<\epsilon} Q_n|w_n|^q \, dx + \int_{|x|>\epsilon} Q_n|w_n|^q \, dx \right) \]

\[ \leq \lim_{n \to \infty} \left( \int_{|x|<\epsilon} Q_n|w_n|^q \, dx + \int_{|x|>\epsilon} Q_n|w_n|^q \, dx \right) \]

\[ \leq \epsilon \int_{|x|<\epsilon} |w_n|^q \, dx - \delta \epsilon \int_{|x|>\epsilon} |w_n|^q \, dx. \]  
(17)

We may choose small \( \epsilon_1 \) such that the second integral on the right-hand side above is positive as \( w \neq 0 \). Then we get the contradiction as \( \epsilon \to 0 \).

In the sequel, we study concentration of \( \{u_n\} \) as \( n \to \infty \). Let \( \epsilon > 0 \) be given and \( \chi \in C^\infty_0(\Omega, [0, 1]) \) be such that \( \chi(x) = 0 \) for \( x \in B_{\epsilon/2}(0) \) and \( \chi(x) = 1 \) for \( x \notin B_\epsilon(0) \).

Multiplying (1) with \( u_n \) by \( \chi u_n \) we obtain

\[ \int \Omega (|\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (\chi u_n) + \chi \nabla u_n^p) \, dx = \int \Omega \chi Q_n|u_n|^q \, dx, \]  
(18)

namely,

\[ \int \Omega \chi (|\nabla u_n|^p + \nabla u_n^p) \, dx - \int \Omega \chi Q_n|u_n|^q \, dx \]

\[ = - \int \Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \chi \cdot u_n \, dx. \]  
(19)

Given \( \epsilon > 0 \), we have \( Q_n \leq -\delta \), on supp \( \chi \), provided that \( n \) is large enough. Hence for all such \( n \),

\[ 0 \leq \int \Omega (|\nabla u_n|^p + \nabla u_n^p) \, dx + \delta \int \Omega Q_n|u_n|^q \, dx \]

\[ \leq \int \Omega (|\nabla u_n|^p + \nabla u_n^p) \, dx - \int \Omega \chi Q_n|u_n|^q \, dx \]

\[ = - \int \Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \chi \cdot u_n \, dx \]

\[ \leq \delta \epsilon \int_{B_\epsilon(0)} |u_n|^q \, dx. \]  
(20)

where \( d_\epsilon \) is a constant independent of \( n \). Since \( w_n = u_n/\|u_n\| \to 0 \) in \( L^p(\Omega) \) according to Lemma 3, it follows from Hölder inequality that

\[ \int_{B_\epsilon(0)} |w_n|^p \, dx \to 0. \]  
(21)

So (20) implies

\[ \int \Omega (|\nabla w_n|^p + \nabla w_n^p) \, dx + \delta \int \Omega |w_n|^q \, dx = 0. \]  
(22)

**Theorem 4.** Suppose that \( Q_n \) satisfies the assumption (\( \ast \)) and \( q \in (p, p^*) \). Let \( u_n \) be a nontrivial solution to (1) and put \( w_n = u_n/\|u_n\| \). Then for every \( \epsilon > 0 \) they hold that

\[ \lim_{n \to \infty} \int \Omega (|\nabla w_n|^p + \nabla w_n^p) \, dx = 0, \]  
(23)

\[ \lim_{n \to \infty} \|u_n\|^{q-p} \int \Omega |w_n|^q \, dx = 0. \]  
(24)

Moreover,

\[ \lim_{n \to \infty} \int \Omega (|\nabla w_n|^p + \nabla w_n^p) \, dx = 0, \]  
(25)

\[ \lim_{n \to \infty} \int \Omega |w_n|^q \, dx = 0. \]  
(26)

**Proof.** (23) and (24) can be easily obtained by (22). Note that

\[ \int \Omega (|\nabla w_n|^p + \nabla w_n^p) \, dx = \|w_n\|^p = 1. \]  
(27)

From (23), one concludes that

\[ \lim_{n \to \infty} \int \Omega (|\nabla w_n|^p + \nabla w_n^p) \, dx \]

\[ \lim_{n \to \infty} \int \Omega (|\nabla w_n|^p + \nabla w_n^p) \, dx. \]  
(28)

This and (24) imply

\[ \lim_{n \to \infty} \int \Omega |w_n|^q \, dx = \lim_{n \to \infty} \frac{\|u_n\|^{q-p} \int \Omega Q_n|u_n|^q \, dx}{\|u_n\|^{q-p} \int \Omega |u_n|^q \, dx} = 0. \]  
(29)
3. Concentration in the $L^s$-Norm

The next is to consider the concentration in other norms.

**Theorem 5.** Let $u_n$ denote a nontrivial solution to (1) for each $n \in \mathbb{N}$. Suppose that the assumption $(\ast)$ holds and there exists $R, \lambda > 0$ such that $V \geq \lambda$ whenever $x \in \Omega \setminus B_R(0)$, and there exists $e > 0$ such that $\bar{B}_e(0) \subset \Omega$; then one can get that

(a) $\exists C$, for all $s \in [1, \infty], n \in \mathbb{N}, |u_n|_{L^s(\Omega \setminus B_R(0))} \leq C$;

(b) if $\delta = \delta_e > 0$ in $(\ast)$ can be chosen independently of $e (> 0)$, then $\lim_{n \to \infty} |u_n|_{L^s(\Omega \setminus B_R(0))} = 0$, for every $s \in [1, \infty]$;

(c) for all $s \geq 1 \in (N(q - p)/p, \infty], \Omega \setminus B_R(0)$, one has $\lim_{n \to \infty} |u_n|_{L^s} = \infty$ and

$$\lim_{n \to \infty} \frac{|u_n|_{L^s(\Omega \setminus B_R(0))}}{|u_n|_{L^s}} = 0; \quad (30)$$

(d) if $N(q - p)/p \geq 1$, then for $s = N(q - p)/p$ it holds that

$$\lim_{n \to \infty} \inf |u_n|_{L^s} > 0. \quad (31)$$

If the hypotheses in (b) are satisfied, then (30) also holds for this $s$.

**Proof.** There is clearly a positive classical solution $w$ to the equation

$$-\text{div} \left( |\nabla u|^{p-2} \nabla u \right) = -\delta_{e/2} |u|^{q-2} u, \quad x \in \mathbb{R}^n \setminus \overline{B}_{e/2}(0)$$

$$\lim_{|x| \to e/2} w(x) = \infty, \quad \lim_{|x| \to \infty} w(x) = 0. \quad (32)$$

In fact, by [22, 23], the radial solution $u_p(x) = u_p(|x|)$ satisfies the ordinary differential equation

$$\left( r^{n-1} |u|^{q-2} u \right)' = -\delta_{e/2} r^{n-1} u^q$$

$$u(r) = \infty \quad \text{as} \quad r \to e/2, \quad u(r) \to 0 \quad \text{as} \quad r \to \infty. \quad (33)$$

Set $z_n = w - u_n$ and

$$\varphi_n(x) := (q - 1) \int_0^1 |sw(x) + (1 - s) u_n(x)|^{q-2} (w - u_n) \, ds \geq 0,$$

$$\phi_n(x) := (p - 1) \int_0^1 |sw(x) + (1 - s) u_n(x)|^{p-2} \, ds \geq 0,$$

$$\varphi_n(x) z_n = (q - 1) \int_0^1 |sw(x) + (1 - s) u_n(x)|^{q-2} (w - u_n) \, ds$$

$$= \int_0^1 \frac{d}{ds} \left( |sw + (1 - s) u_n|^{p-2} (sw + (1 - s) u_n) \right) ds$$

$$= w^{q-1} - |u_n|^{q-2} u_n,$$

$$\phi_n(x) z_n = (q - 1) \int_0^1 |sw(x) + (1 - s) u_n(x)|^{p-2} (w - u_n) \, ds$$

$$= \int_0^1 \frac{d}{ds} \left( |sw + (1 - s) u_n|^{p-2} (sw + (1 - s) u_n) \right) ds$$

$$= w^{p-1} - |u_n|^{p-2} u_n.$$  \hfill (34)

and hence from $(\ast)$

$$-\text{div} \left( |\nabla u|^{p-2} \nabla u \right) - \div (|\nabla u|^{p-2} \nabla u_n)$$

$$+ \langle V \phi_n(x) - Q_n \varphi_n \rangle z_n$$

$$= -\text{div} |\nabla u|^{p-2} + V|w|^{p-2} w - Q_n w^{q-1}$$

$$- \left[ -\text{div} |\nabla u|^{p-2} u + V |\nabla u|^{p-2} u - Q_n |u_n|^{q-2} u_n \right]$$

$$= -\text{div} |\nabla u|^{p-2} + V|w|^{p-2} w - Q_n w^{q-1}$$

$$\geq -\text{div} |\nabla u|^{p-2} + \delta_{e/2} w^{q-1} = 0. \quad (35)$$

Note that $V \phi_n(x) - Q_n \varphi_n \geq 0$ in $\Omega \setminus B_{e/2}(0)$ when $n \geq N_{e/2}$. Due to the continuity of $u_n$ and the fact that $w_n(x) \to \infty$ as $x \to \partial B_{e/2}(0)$, there is $r \in (e/2, e)$ such that $z_n \geq 0$ on $\partial B_r(0)$. Moreover, $z_n \geq 0$ on $\partial \Omega$. If $\Omega$ is bounded, the maximum principle says that $z_n \geq 0$ in $\Omega \setminus B_{e/2}(0)$ (see [20, 21]). If $\Omega$ is unbounded, by virtue of $w(x)$ tending to 0 as $|x| \to \infty$ by construction, thus for any $y > 0$, we may pick $\tilde{R} > 0$ such that $z_n \geq -y$ in $\Omega \setminus B_{\tilde{R}}(0)$. Moreover, applying regularity theory to $u_n \in W_{0}^{1,p}(\Omega)$, we can get $u_n(x) \to 0$ as $|x| \to \infty$. Now the same maximum principle is applied on $\Omega \cap (B_{\tilde{R}} \setminus \overline{B}_e(0))$, which implies that $z_n \geq -y$ in all of $\Omega \setminus B_{\tilde{R}}(0)$. Letting $y \to 0$, we obtain $z_n \geq 0$ again. By analogy we obtain $u_n \geq -w$ (take $z_n := w + u_n$); hence

$$|z_n| \geq w \quad \text{in} \quad \Omega \setminus B_{\tilde{R}}(0), \quad \forall n \geq N_{e/2}. \quad (36)$$

Hence (a) follows from above arguments with the fact that $w$ is continuous in $\Omega \setminus B_{\tilde{R}}(0)$.

Next, the hypotheses in (b) imply that there is $\delta > 0$ such that $Q_n \leq -\delta$ on $\Omega \setminus B_{1/n}(0)$ for each $n$ large enough. Let $w_n$ be a positive solution to

$$-\text{div} \left( |\nabla u|^{p-2} \nabla u \right) - \div (|\nabla u|^{p-2} \nabla u_n)$$

$$\lim_{|x| \to 1/n} w_n(x) = +\infty, \quad \lim_{|x| \to \infty} w_n(x) = 0. \quad (37)$$
Then the sequence \( w_n \) is monotone decreasing, by using the maximum principle to \( w_n \geq w_{n+1} \) on \( \partial B_{n,0}(0) \) for every \( n \in \mathbb{N} \). Therefore, \( w_n \) converges locally and uniformly to a nonnegative solution \( w \) to (37) on \( \mathbb{R}^N \setminus \{0\} \). It follows from our hypotheses on \( N \) and \( p \) that \( w \) is an entire solution to (37) by applying the argument as in [24]. And then, due to [25], \( w \equiv 0 \) for all \( n \). For another, the function \( w_n \) dominates the solution \( u_n \) on \( \overline{\Omega} \setminus B_{r,0}(0) \) for some \( r \in (e/2,e) \), as seen in the proof of (a). Thus, \( u_n \) also converges to 0 locally and uniformly in \( \Omega \setminus B_{r,0}(0) \); that is, \( \lim_{n \to \infty} \|u_n\|_{L^\infty(\Omega \setminus B_{r,0})(0)} = 0 \).

For (c), we first consider the case \( s \geq 1 \) \( (N−p)/p \subseteq q \). By interpolation inequality, we have the following estimate for solution \( u_n \):

\[
\|u_n\|_p = \left( \int_\Omega Q_n |u_n|^p dx \right)^{1/p} \leq c_1 |u_n|_q \|u_n\|_{L^p}\[1-	heta\] \leq c_2 |u_n|_q \|u_n\|_{L^p}\[1-	heta\].
\]

(38)

Here \( c_1, c_2 \) are independent of \( n \), and \( \theta \) satisfies that

\[
\frac{1}{q} = \frac{\theta}{s} + \frac{1-\theta}{p}. \tag{39}
\]

According to Lemma 2, it suffices to impose that \( q(1-\theta) < p \) or equivalent \( s > N(q−p)/p \). This and (a) prove the case \( s \in (N(q−p)/p, \infty) \). And then, (38) and (a) yield \( \|u_n\|_{L^p}(\Omega \setminus B_{r,0}(0)) \to \infty \); hence \( \|u_n\|_{L^p}(\Omega \setminus B_{r,0}(0)) \to \infty \) for every \( s \in (q, \infty) \) as \( n \to \infty \). Using (a) again we get (30).

Note that (38) implies (30) for \( s = (N−p)/p \), so case (d) is easily followed.

\[\square\]

4. Concentration at Several Points

Now we assume that the function \( Q_n \) is positive in a neighbourhood of two distinct points \( x_1, x_2 \subseteq \Omega \) (indeed, the following argument is also valid for any finite number of points in \( \Omega \)). More precisely, we assume:

(∗∗) \( Q_n \geq 0 \) in a neighbourhood of \( \{x_1\} \cup \{x_2\} \), and there exists a constant \( C \) such that \( Q_n(\Omega \setminus C) \subseteq C \) for all \( n \). Moreover, for each \( \epsilon > 0 \) there exist constants \( \delta_\epsilon > 0 \) and \( N_\epsilon \) such that \( Q_n \leq \delta_\epsilon \) for all \( x \not\in B_{\epsilon}(x_1) \cup B_{\epsilon}(x_2) \) and \( n \geq N_\epsilon \).

As in Section 2, we put \( J_n(u) = \int_\Omega Q_n |u|^p dx \):

\[
s_n = \inf_{J_n(u) \leq s} \|u\|_p = \inf_{J_n(u) \leq s} \left( \int_\Omega |\nabla u|^p + V |u|^p \right)^{1/p}.
\]

(40)

Theorem 6. Suppose \( Q_n \) satisfies (∗∗) and \( q \in (p, p^*) \), and \( u_n \) is a ground state solution to (1). Then, for \( n \) large, \( u_n \) concentrates at \( x_1 \) or \( x_2 \). More precisely, for each \( \epsilon > 0 \) we have by passing to a subsequence

\[
\lim_{n \to \infty} \int_{\Omega \setminus B_{\epsilon}(x_1)} (|\nabla u|^p + V |u|^p) dx = 0,
\]

\[
\lim_{n \to \infty} \int_{\Omega \setminus B_{\epsilon}(x_2)} Q_n u_n^p dx = 0
\]

for \( j = 1 \) or 2 (but not for \( j = 1 \) and 2).

Remark 7. Note that, in view of the obvious modification of Theorem 4, the limits in (41) are 0 if \( \Omega \setminus B_{\epsilon}(x_j) \) is replaced by \( \Omega \setminus B_{\epsilon}(x_2) \cup B_{\epsilon}(x_2) \). So if \( j = 1 \) in (41), then concentration occurs at \( x_1 \) and if \( j = 2 \), it occurs at \( x_2 \).

Proof. As in [19], we may assume that \( J_n(u_n) = \int_\Omega Q_n |u_n|^p dx = 1 \) by renormalizing \( u_n \) (may not be a solution to (1), but we still have \( s_n := \|u_n\|^p / J_n(u_n)^{p/q} \). Let \( \xi_j \in C_0^{\infty}(\Omega, [0,1]) \) be a function such that \( \xi_j = 1 \) on \( B_{\epsilon/2}(x_j) \) and \( \xi_j = 0 \) on \( \Omega \setminus B_{\epsilon/2}(x_j) \), \( j = 1, 2 \), where \( e < 1 \) is so small that \( B_{\epsilon/2}(x_j) \subset \Omega \) and \( B_{\epsilon/2}(x_j) \cap B_{\epsilon/2}(x_i) = \emptyset \). Set \( \nu_n := \xi_1 u_n, \omega_n := \xi_1^2 u_n \), and \( z_n := u_n - \nu_n - \omega_n \). Since \( \supp z_n \subset \Omega \setminus (B_{\epsilon/2}(x_1) \cup B_{\epsilon/2}(x_2)) \) and the conclusion of Theorem 4 remains valid after a modification, we have

\[
\|u_n\|^p = \left( \int_\Omega (|\nabla u_n|^p + V |u_n|^p) dx \right)^{1/p} = \left( \int_\Omega (|\nabla v_n|^p + V |u_n|^p) dx \right)^{1/p} + \left( \int_\Omega (|\nabla w_n|^p + V |u_n|^p) dx \right)^{1/p} \left(1 + o(1)\right),
\]

(41)

\[
J_n(u_n) = \int_\Omega Q_n |u_n|^p dx \leq \int_\Omega Q_n |v_n|^p dx + \int_\Omega \left( Q_n |z_n|^p + \int_\Omega Q_n |u_n|^p dx \right) + o(1)
\]

(42)

First, we assume that \( \limsup_{n \to \infty} J_n(\nu_n) \geq 0 \) and \( \limsup_{n \to \infty} J_n(\omega_n) \geq 0 \). By passing to a subsequence, we may assume that \( J_n(\nu_n) \to c_0 \in [0,1] \) and \( J_n(\omega_n) \to c_0 \in [0,1] \). If \( c_0 \in (0,1) \), recalling that \( q > p \), we get a contradiction from the following inequality:

\[
s_n = \frac{\|u_n\|^p}{J_n(\nu_n)} = \left( \frac{\|v_n\|^p + \|w_n\|^p}{J_n(\nu_n) + J_n(\omega_n) + o(1)} \right)^{p/q} \geq \frac{\|v_n\|^p + \|w_n\|^p}{J_n(\omega_n)} \geq s_n.
\]

(43)

So \( c_0 = 0 \) or 1. If \( c_0 = 1 \); then the second limit in (41) is 0 for \( j = 1 \) because \( \supp w_n \subset B_{\epsilon/2}(x_1) \). The first limit is 0 as well, since \( \|w_n\|^p / \|v_n\|^p \) is otherwise bounded away from 0 for large \( n \), and we obtain a contradiction again from

\[
s_n = \frac{\|v_n\|^p + \|w_n\|^p}{J_n(\nu_n) + J_n(\omega_n) + o(1)} \geq \frac{\|v_n\|^p}{J_n(\nu_n)} \geq s_n.
\]

(44)

Finally, suppose \( \limsup_{n \to \infty} J_n(\nu_n) < 0 \) (the case \( \limsup_{n \to \infty} J_n(\nu_n) < 0 \) is of course analogous); it passes to
a subsequence \( J_n(\omega_n) \leq -\eta \) for some \( \eta > 0 \) when \( n \) is large enough. Then a contradiction (44) holds for such \( n \) because
\[
J_n(\nu_n) > J_n(\omega_n) + o(1).
\]

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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**References**

[1] A. V. Buryak, P. di Trapani, D. V. Skryabin, and S. Trillo, “Optical solitons due to quadratic nonlinearities: from basic physics to futuristic applications,” *Physics Reports*, vol. 370, no. 2, pp. 63–235, 2002.

[2] J. Byeon and Z. Q. Wang, “Standing waves with a critical frequency for nonlinear Schrödinger equations,” *Archive for Rational Mechanics and Analysis*, vol. 165, no. 4, pp. 295–316, 2002.

[3] J. Byeon and Z. Q. Wang, “Standing waves with a critical frequency for nonlinear Schrödinger equations,” *Calculus of Variations and Partial Differential Equations*, vol. 18, no. 2, pp. 207–219, 2003.

[4] J. Byeon and L. Jeanjean, “Standing waves for nonlinear Schrödinger equations with a general nonlinearity,” *Archive for Rational Mechanics and Analysis*, vol. 185, no. 2, pp. 185–200, 2007.

[5] A. Ambrosetti, D. Arcaya, and J. L. Gámez, “Asymmetric bound states of differential equations in nonlinear optics,” *Rendiconti del Seminario Matematico della Università di Padova*, vol. 100, pp. 231–247, 1998.

[6] D. Bonheure, J. M. Gomes, and P. Habets, “Multiple positive solutions of superlinear elliptic problems with sign-changing weight,” *Journal of Differential Equations*, vol. 214, no. 1, pp. 36–64, 2005.

[7] P. M. Girão and J. M. Gomes, “Multibump nodal solutions for an indefinite superlinear elliptic problem,” *Journal of Differential Equations*, vol. 247, no. 4, pp. 1001–1012, 2009.

[8] W. Y. Ding and W. M. Ni, “On the existence of positive entire solutions of a semilinear elliptic equation,” *Archive for Rational Mechanics and Analysis*, vol. 91, no. 4, pp. 283–308, 1986.

[9] S. Terracini, “On positive entire solutions to a class of equations with a singular coefficient and critical exponent,” *Advances in Differential Equations*, vol. 1, no. 2, pp. 241–264, 1996.

[10] A. Ambrosetti and Z. Q. Wang, “Nonlinear Schrödinger equations with vanishing and decaying potentials,” *Differential and Integral Equations*, vol. 18, no. 12, pp. 1321–1332, 2005.

[11] A. Lakmeche and A. Hammoudi, “Multiple positive solutions of the one-dimensional \( p \)-Laplacian,” *Journal of Mathematical Analysis and Applications*, vol. 317, no. 1, pp. 43–49, 2006.

[12] J. B. Su, Z. Q. Wang, and M. Willem, “Weighted Sobolev embedding with unbounded and decaying radial potentials,” *Journal of Differential Equations*, vol. 238, no. 1, pp. 201–219, 2007.

[13] D. Motreanu, V. V. Motreanu, and N. S. Papageorgiou, “A multiplicity theorem for problems with the \( p \)-Laplacian,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 4, pp. 1016–1027, 2008.

[14] J. H. Zhao and P. H. Zhao, “Existence of infinitely many weak solutions for the \( p \)-Laplacian with nonlinear boundary conditions,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 4, pp. 1343–1355, 2008.

[15] D. G. de Figueiredo, J. P. Gossez, and P. Ubilla, “Local “superlinearity” and “sublinearity” for the \( p \)-Laplacian,” *Journal of Functional Analysis*, vol. 257, no. 3, pp. 721–752, 2009.

[16] F. Torre and B. Ruf, “Multiplicity of solutions for a superlinear \( p \)-Laplacian equation,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 7, pp. 2132–2147, 2010.

[17] L. Iturriaga, S. Lorca, and E. Massa, “Positive solutions for the \( p \)-Laplacian involving critical and supercritical nonlinearities with zeros,” *Annales de l’Institut Henri Poincare C: Non Linear Analysis*, vol. 27, no. 2, pp. 763–771, 2010.

[18] S. B. Liu, “On ground states of superlinear \( p \)-Laplacian equations in \( \mathbb{R}^N \),” *Journal of Mathematical Analysis and Applications*, vol. 361, no. 1, pp. 48–58, 2010.

[19] N. Ackermann and A. Szulkin, “A concentration phenomenon for semilinear elliptic equations,” *Archive for Rational Mechanics and Analysis*, vol. 207, no. 3, pp. 1075–1089, 2013.

[20] J. García-Melián and J. Sabina de Lis, “Maximum and comparison principles for operators involving the \( p \)-Laplacian,” *Journal of Mathematical Analysis and Applications*, vol. 218, no. 1, pp. 49–65, 1998.

[21] M. Montenegro, “Strong maximum principles for supersolutions of quasilinear elliptic equations,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 37, no. 4, pp. 431–448, 1999.

[22] G. Díaz and R. Letelier, “Explosive solutions of quasilinear elliptic equations: existence and uniqueness,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 20, no. 2, pp. 97–125, 1994.

[23] J. García-Melián, “Nondegeneracy and uniqueness for boundary blow-up elliptic problems,” *Journal of Differential Equations*, vol. 223, no. 1, pp. 208–227, 2006.

[24] H. Brézis and L. Véron, “Removable singularities for some nonlinear elliptic equations,” *Archive for Rational Mechanics and Analysis*, vol. 75, no. 1, pp. 1–6, 1980.

[25] C. Bandle and M. Marcus, “Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour,” *Journal d’Analyse Mathématique*, vol. 58, no. 1, pp. 9–24, 1992.