RATIONAL GROWTH IN THE HEISENBERG GROUP

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Abstract. A group presentation is said to have rational growth if the generating series associated to its growth function represents a rational function. (In the polynomial growth range, being rational is the same as being eventually quasi-polynomial.) A long-standing open question asks whether the Heisenberg group has rational growth with respect to all finite generating sets, and we settle this question affirmatively. Previously, the only groups known to have this property were virtually abelian groups and hyperbolic groups. Our method involves a very precise description of families of geodesics (in any word metric) that suffice to represent all group elements.

1. Introduction

Growth functions of finitely-generated groups count the number of elements that can be spelled as words in a generating alphabet, as a function of spelling length. Though the functions themselves depend on a choice of generating set, they become group invariants under the standard equivalence relation that allows affine rescaling of domain—in particular, this preserves the property of having polynomial growth of a particular degree is independent of generators.

It has been known since the early 1970s that all nilpotent groups have growth functions in the polynomial range, in fact bounded above and below by polynomials of the same degree, and the degree was computed by Bass and Guivarc’h independently [1, 16, 17]. A breakthrough theorem of Gromov showed that in fact any group with growth bounded above by a polynomial is virtually nilpotent [14].

One can still wonder, however, whether the growth function is precisely polynomial. This turns out to be a bit too much to ask for nilpotent groups. Virtually abelian groups, for instance, have a slightly more general property called rational growth: no matter what finite generating set is chosen, the power series associated to the growth function represents a rational function.

Hyperbolic groups have rational growth for all generators—this is an important theorem from the early 1980s for which credit can be shared among Cannon, Thurston, and Gromov [7, 8, 12, 15]. (This has a somewhat interesting history: Cannon’s argument for fundamental groups of closed hyperbolic manifolds directly generalized to hyperbolic groups once that definition was in place. And Thurston’s definition of automatic groups was partly motivated by these ideas.) At almost the same time, Benson established the same result for virtually abelian groups [2]. Given the work at the time understanding the growth of nilpotent groups, it was a natural question to ask whether nilpotent groups also have rational growth, which was open even for the simplest non-abelian nilpotent group, the integer Heisenberg group. This question was posed or referred to by many authors, including

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By the late 1980s, Benson and Shapiro had independently established one piece of this: the Heisenberg group has rational growth in its standard generators. We settle the full question here.

**Theorem 1.** The Heisenberg group has rational growth for all generating sets.

In the process of establishing this fact, we will get quite precise information about the combinatorial geometry of Heisenberg geodesics (Theorem 26) that will be useful in the further study of the geometric group theory of $H(\mathbb{Z})$. We give remarks, applications, and open questions in the last section.

### 1.1. Literature.

We review what is known about rationality of growth in groups and classes of groups.

| For all $S$ | For at least one $S$ | For no $S$ |
|------------|---------------------|-----------|
| hyperbolic groups | some automatic groups | unsolvable word problem |
| virtually abelian groups | Coxeter groups, standard $S$ | intermediate growth |
| **Heisenberg group $H$** | $H$, standard $S$ | $H_5$, cubical $S$ |
| | $BS(1,n)$, standard $S$ | |

Automatic groups have rational growth with respect to an automatic structure that bijects onto the word geodesics; this is used in [21] to study groups that act geometrically finitely on hyperbolic space. There are more examples belonging in the middle category—known to have rational growth in a special generating set—found in work of Barré (quotients of triangular buildings), Alonso (amalgams), Brazil (other Baumslag-Solitar groups), Johnson (wreath products and torus knot groups), and others. For references and an excellent survey, see [13].

The nilpotent cases go as follows. As mentioned above, [3, 24] show that $H$ has rational growth in standard generators. In [25], Stoll proves the following remarkable result: the higher Heisenberg group $H_5$ has transcendental growth in its standard generators, but rational growth in a certain dual generating set, which we will call **cubical generators**. (See Sec 3.3 for a definition of $H_5$.) On the other hand, Stoll establishes the following theorem to use as a criterion for transcendental growth.

**Theorem 2** (Stoll [25]). If $\frac{\beta(n)}{\alpha \cdot n^2} \to 1$ and $\alpha$ is a transcendental number, then $\mathbb{B}(x) = \sum \beta(n)x^n$ is a transcendental function.

A volume computation gives $\alpha = \frac{6027 + 2 \ln 2}{65610}$, establishing that $(H_5, \text{std})$ has transcendental growth. Over fifteen years later, this (with small variations explained by Stoll) still provides the only known example of a group with both rational and irrational growth series.

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2. Outline

In this paper, we will give a way to compare geodesics in the Cayley graph of $H(\mathbb{Z})$ with geodesics in a geometrically much simpler continuous metric on $\mathbb{R}^3$. We will show that word geodesics are not too different from these simpler paths, classifying them by “shape.” Since these geodesics in $\mathbb{R}^3$ can be understood in terms of their projection into $\mathbb{R}^2$, this allows us to use planar pictures to understand geodesics in $H(\mathbb{Z})$.

By an important theorem of Pansu [22], any word metric on the Heisenberg group $H(\mathbb{Z})$ is asymptotic to a left-invariant metric on its ambient Lie group $H(\mathbb{R})$, known as a Carnot-Carathéodory (cc) Finsler metric, which admits $\mathbb{R}^3$ coordinates. (Pansu’s theorem is much more general, and was further generalized by Breuillard in [4].) Several authors have studied the geodesics in these cc metrics, including Krat, Stoll, Breuillard, and Duchin–Mooney, and we will stay close to the notation of [11]. It is to Pansu’s cc geodesics that we will compare our word geodesics.

The $xy$–plane in $\mathbb{R}^3$ will be identified with the $XY$–subspace of the Lie algebra of $H(\mathbb{R})$, and we will denote this plane by $\mathfrak{m}$. A generating set of $H$ induces a norm on $\mathfrak{m}$ in a manner described further below, and word geodesics come in two kinds. The “unstable” kind behave much like geodesics in free abelian groups, and these are modeled by finitely many patterns which are close to reorderings of free abelian geodesics.

The “regular” geodesics behave differently. Among these is a subset whose projections to $\mathfrak{m}$ fellow-travel the boundary of a characteristic polygon $\mathcal{I}$ determined by $S$. We will see that this subset contains at least one geodesic for each group element. Once we make this precise we will have described finitely many languages, which we call shapes. The words of each language are parameterized by the lengths of runs of particular letters. Thus, each shape is a map from a subset of $\mathbb{Z}^M$ to spellings in $S^*$.

We will show that every group element has a geodesic spelling produced by a pattern or a shape (Theorem 26), even though it is not true that all geodesics are so obtained, nor is it necessarily true that all spellings produced by these shapes and patterns are geodesic.

The domains of patterns and shapes in $\mathbb{Z}^M$ are determined by linear equalities, inequalities, and congruences, and so counting the spellings enumerated by the shapes amounts to solving congruences in rational polyhedra. By a marvelous theorem of Benson [3], enumeration over rational polyhedra yields a rational function.

However, this is not yet sufficient for rationality of growth. For each group element $g = (a, b, c)$ we must determine which shapes might produce a spelling for $g$ and among these shapes we must determine which one(s) win the competition to produce a shortest spelling. While the horizontal position $(a, b)$ varies linearly over the shape’s domain, the height or $z$–coordinate $c$ of the elements produced by each shape $\omega$ varies quadratically, which poses a problem for counting. However, we show that whenever two shapes compete for geodesity in spelling a group element, the difference in the heights they produce is linear on the domain of competition (Sec 9). These linear comparison lemmas then allow us to enumerate the elements of each length $n$ using only linear equations, inequalities, and congruences, which finally establishes rational growth.
2.1. **Example: Shapes in** $\mathbb{Z}^2$. To illustrate the idea of shapes of geodesics, consider the example of $\mathbb{Z}^2$, first with standard generators $a, b$. Here, we will introduce four shapes: $a^m b^n$, $a^{-m} b^n$, $a^m b^{-n}$ and $a^{-m} b^{-n}$.

![Diagram of four shapes of geodesics for $(\mathbb{Z}^2, \text{std})$.](image)

**Figure 1.** Four shapes of geodesics for $(\mathbb{Z}^2, \text{std})$. (Take $m, n \geq 0$ in each case.)

One quickly observes a few basic properties:

- There are finitely many shapes.
- Each shape is a language, and a map from a subset of some $\mathbb{Z}^M$ to $S^*$. (Here, $M = 2$ for each shape, and the domain is the first quadrant of $\mathbb{Z}^2$.)
- Every group element admits a geodesic spelling by at least one shape (even though not every geodesic is realized this way).

This case is too simple to capture some features of the situation, so consider the slightly more complicated case of $\mathbb{Z}^2$ with *chess-knight* generators $\{(\pm 2, \pm 1), (\pm 1, \pm 2)\}$. Consider the case of geodesically spelling the group element (100,100). If we let $u = (2, 1)$, $v = (1, 2)$, and $w = (2, -1)$ then $u^{33} v^{33}$ reaches an adjacent position in $\mathbb{Z}^2$, but an exact spelling (in fact a geodesic spelling) requires two more letters: $(wv)u^{33} v^{33}$. (This is because of the well-known property of chess-knights that it takes several moves to arrive at an adjacent square on the chessboard.) These correction terms never have more than three letters, so we can arrive at a finite list of shapes: every shape has the form $x \cdot a_i^m a_{i+1}^n$, where $x$ is in the ball of radius three, $a_i$ and $a_{i+1}$ are cyclically successive generators, and $m, n \geq 0$.

Now we can add to the list of properties:

- A shape may not evaluate to a geodesic for every value of its arguments.
- The set of positions reached by each shape is given by the simultaneous solution of finitely many linear inequalities and congruences in the plane.
- If a group element is reached by more than one shape, there is a linear function that compares the spelling length required by each shape.

These are the essential features that we will establish in the Heisenberg group for an appropriate finite list of shapes though the linear comparison feature will play a somewhat different role.

3. **Background**

3.1. **Growth of groups.** Suppose a group $G$ is generated by the finite symmetric generating set $S = S^{-1}$. We take $S^n$ to be the set of all (unreduced) strings of
length \( n \) in the elements of \( S \) (sometimes called spellings) and \( S^* = \cup_{n=0}^{\infty} S^n \) to be the set of all spellings of any finite length. This \( S^* \) comes equipped with two important maps, spelling length and evaluation into \( G \). Length, denoted \( \ell(\gamma) \), is defined on \( \gamma \in S^n \subset S^* \) via \( \ell(\gamma) = n \). Evaluation into \( G \) is given by the monoid homomorphism which carries concatenation in \( S^* \) to group multiplication in \( G \). We denote this by
\[
\gamma \mapsto T.
\]
An element of \( S^* \) can be thought of as a path in the Cayley graph \( \text{Cay}(G, S) \) from \( e \) to \( T \).

We define the word length of a group element \( g \in G \) by
\[
|g| = |g|_S = \min\{\ell(\gamma) \mid \gamma \in S^* \text{ and } T = g\},
\]
i.e., the shortest spelling length of any spelling.

The sphere and ball of radius \( n \) are denoted \( S_n, B_n \) respectively, and the associated growth functions are
\[
\sigma(n) := \#S_n = \#\{g \in G : |g| = n\} ;
\beta(n) := \#B_n = \#\{g \in G : |g| \leq n\},
\]
related of course by \( \sigma(n) = \beta(n) - \beta(n-1) \). Then we can form associated generating functions, called the spherical growth series and the growth series of \( (G, S) \), as follows:
\[
S(x) := \sum_{n=0}^{\infty} \sigma(n)x^n ; \quad B(x) := \sum_{n=0}^{\infty} \beta(n)x^n.
\]
Since \( \sigma(n) \leq \beta(n) \leq \sum_{i=0}^{n} |S^i| = \sum_{i=0}^{n} |S|^i \), the coefficients are bounded above by an exponential, ensuring a positive radius of convergence for both series.

We say that \( (G, S) \) has rational growth if the growth series are rational functions (i.e., each is a ratio of polynomials in \( x \)). Note that the relationship between \( \sigma \) and \( \beta \) implies that \( (1 - x)B(x) = S(x) \), so either is rational iff the other is.

It is a standard fact that rationality of a generating function \( F(x) = \sum f(n)x^n \) is equivalent to the property that the values \( f(n) \) satisfy a finite-depth linear recursion for \( n \gg 1 \), i.e., there exist \( N_0 \) and \( P \) such that for \( n > N_0 \),
\[
f(n + P) = a_0 \cdot f(n) + a_1 \cdot f(n + 1) + \cdots + a_{P-1} \cdot f(n + P - 1).
\]
(Here, the coefficients \( a_i \) come from the same base field as the polynomials in the rational function.)

The growth of a regular language is necessarily rational with integer coefficients, and therefore the values \( \sigma(n) \) satisfy an integer recursion. In fact, this recursion can be described in terms of the finite-state automaton which accepts the language, and therefore can be written with non-negative integer coefficients in the recursion. In the case of groups, if there is a generating set for which there is a regular language of geodesics which bijects to the group, then the corresponding growth function is rational. This can be used to prove rational growth for free abelian groups and for word hyperbolic groups.

In this paper we focus on the integer Heisenberg group \( H = H(\mathbb{Z}) \) and consider its growth functions with various finite generating sets. Shapiro 1989 \cite{24} shows that for the standard Heisenberg generators \( S = \text{std} \), there is no regular language of geodesics for \( (H, \text{std}) \). Nevertheless, the growth function is rational \cite{24}. In this paper, we will show the same holds for arbitrary generating sets.
3.2. Rational families. We now review material from Max Benson’s papers [2, 3], which express the principle that *counting in polyhedra is rational*. Benson uses these techniques in [2] to show that virtually abelian groups have rational growth with respect to arbitrary generating set.

Suppose we have a parameter $n$ which we will take to lie in the non-negative integers and we consider sets of points in $P(n) \subset \mathbb{Z}^d$ defined by finitely many equalities, inequalities, and congruences

$$a_i \cdot x = b_i(n) ;$$

$$a_j \cdot x \leq b_j(n) ;$$

$$a_k \cdot x \equiv b_k(n) \pmod{c_k} ,$$

where each $a_i$, $a_j$ and $a_k$ are in $\mathbb{Z}^n$, and each $b_i$, $b_j$ and $b_k$ is an affine function of $n$ with integer coefficients. The sequence of sets $\{P(n)\}$ constitutes a rational family of polyhedra. If each $P(n)$ is bounded, this is a bounded rational family of polyhedra.

**Observation.** Families of polyhedra are closed under finite intersection. That is, if $\{P(n)\}_{i=1}^m$ is a rational family of polyhedra, then $\bigcap_{i=1}^m P(n)_i$ is a rational family of polyhedra. It is bounded if any of the families $\{P(n)\}$ is bounded.

We will want to consider the closure of families of polyhedra under the operations of finite intersection, finite union and set difference, and we will call these rational families; i.e., any family of polyhedra is a rational family, and $\{P(n) \cup Q(n)\}$, $\{P(n) \setminus Q(n)\}$ are rational families whenever $\{P(n)\}$ and $\{Q(n)\}$ are.

**Theorem 3** (Counting over polytopes [2, 3]). Suppose that $\{P(n)\}$ is a bounded rational family in $\mathbb{Z}^n$. Suppose also that $f : \mathbb{Z}^d \to \mathbb{Z}$ is a polynomial with integer coefficients. Then

$$g(x) = \sum_{n=0}^{\infty} \sum_{v \in P(n)} f(v) x^n$$

is a rational function of $x$.

**Corollary 4** (Counting over rational families). Suppose that $\{P(n)\}$ is a bounded rational family in $\mathbb{Z}^d$. Then any polynomial image has rational growth. That is, suppose that $f : \mathbb{Z}^d \to \mathbb{Z}$ is a polynomial with integer coefficients. Then

$$g(x) = \sum_{n=0}^{\infty} \sum_{v \in P(n)} f(v) x^n$$

is a rational function of $x$.

To see this, notice the following.

**Lemma 5.** Suppose that $P(n)$ and $Q(n)$ are families of polyhedra. Then $P(n) \cap Q(n)$, $P(n) \cup Q(n)$ and $P(n) \setminus Q(n)$ are each the finite disjoint union of families of polyhedra.

**Proof.** Clearly $P(n) \cap Q(n)$ is a family of polyhedra.

We now consider $P(n) \setminus Q(n)$. Let $p_1, \ldots, p_m$ be the criteria—equations, inequalities and congruences—defining $P(n)$, and $q_1, \ldots, q_k$ be the criteria defining $Q(n)$. If $q_i$ is an equation, the complement of its solution set is the disjoint union of the solution sets of two inequalities. If it is an inequality, the complement of its solution
set is given by a single inequality. If it is a congruence mod \( r \), then the complement of its solution set is the disjoint union of solutions to \( r - 1 \) congruences. For each non-empty \( A \subseteq \{1, \ldots, n\} \), write \( \Sigma(A, n) \) for the set defined by the criteria \( \{p_i\} \) together with \( \{q_i \mid i \notin A\} \) and the negation of \( \{q_i \mid i \in A\} \). Clearly, the \( \Sigma(A, n) \) are disjoint, each is the finite disjoint union of families of polyhedra, and \( P(n) \setminus Q(n) \) is the union of the \( \Sigma(A, n) \). Thus \( P(n) \setminus Q(n) \) is the finite disjoint union of families of polyhedra.

To complete the proof, observe that

\[
P(n) \cup Q(n) = (P(n) \setminus Q(n)) \sqcup (P(n) \cap Q(n)) \sqcup (Q(n) \setminus P(n)).
\]

\( \square \)

**Proof of Corollary 4.** An induction now shows that every rational family is the disjoint union of finitely many families of polyhedra. Thus the series computed over any rational family is the sum of finitely many rational functions. \( \square \)

**Observation.** These sets have the useful property of being closed under affine push-forward. Suppose \( \{P(n)\} \) is a rational family in \( \mathbb{Z}^d \) and \( f : \mathbb{Z}^d \to \mathbb{Z}^n \) is an affine map. Then \( \{f(P(n))\} \) is a rational family, and it is bounded if the sets \( \{P(n)\} \) are bounded.

### 3.3. The Heisenberg groups.

Most of this paper will focus on the Heisenberg group \( H(\mathbb{Z}) \), which is also the first in the family \( H_k, k = 3, 5, 7, \ldots \) of two-step nilpotent groups realized as inside the \( N \times N \) matrices, where \( N = \frac{k+3}{2} \). (This parametrization has \( k \) as the number of integer parameters in each matrix.) For \( i = 1, 2, \ldots, N - 2 \), let \( a_i \) be the \((1, i + 1)\) elementary matrix, let \( b_i \) be the \((N, i + 1)\) elementary matrix, and write \( c \) for top-right elementary matrix. Then we have the commutator relations \( [a_i, b_i] = c \) and all other commutators are trivial. Thus for any \( k \), the commutator subgroup is \( \langle c \rangle \), so that the lower central series is

\[
1 \leq \mathbb{Z} \leq H_k.
\]

For any nilpotent group, the well-known Bass-Guivarch’s formula tells us that the degree of polynomial growth in \( H_k \) is \( \beta(n) \asymp n^d \) for \( d = (k - 1)\cdot1 + 1\cdot2 = k + 1 \).

For the Heisenberg group \( H(\mathbb{Z}) \), we will drop the subscripts and write the elementary matrices as \( e_1, e_2, e_3 \), so that \( [e_1, e_2] = e_3 \) and \( [e_1^n, e_2^m] = e_3^{mn} \). The standard generating set for \( H(\mathbb{Z}) \) is \( \{e_1, e_2\}^{\pm 1} \), and from the above formula we know that the growth function in these generators is bounded above and below by fourth-degree polynomials.
3.4. Geometric model, spelling paths, and boost. We will use the exponential coordinates on $H(\mathbb{Z}) \leq H(\mathbb{R})$ given by the following representation:

$$(a, b, c) \leftrightarrow \begin{pmatrix} 1 & a & c + \frac{1}{2}ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$ 

These coordinates have the property that $(a, b, c)^n = (na, nb, nc)$, and in this notation $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$.

For integers $a$ and $b$, define $\epsilon(a, b)$ to be $1/2$ if $a$ and $b$ are both odd, and $0$ otherwise. In these coordinates, $H(\mathbb{Z})$ looks just like the standard lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ shifted by $\epsilon$ in the $z$ direction, and the Haar measure on $H(\mathbb{R})$ is identified with Lebesgue measure in $\mathbb{R}^3$.

**Definition 6.** A *spelling path* is a string of letters from $S$, i.e., an element of $S^*$, regarded as a path in the Cayley graph that represents a group element from $H(\mathbb{Z})$. Define $\ell$, $(a, b)$, and $z$ to be the length, horizontal position, and height of $\gamma$, respectively: if the group element represented by $\gamma$ is $(a, b, c)$, then $\ell(\gamma)$ is the spelling length of the string, $(a, b) \in \mathfrak{m}$ is the projection of the endpoint, and the height of the group element and hence the path is $z(\gamma) = c$. Also let the shadow, denoted $\pi(\gamma)$, be the projection to $\mathfrak{m}$ (the path in $\mathfrak{m}$ obtained by concatenating the projections of the generators to $\mathfrak{m}$ in the order of appearance in $\gamma$).

Define the area of a spelling path $\gamma$, denoted $z_A(\gamma)$, to be the balayage area of its projection, that is, the signed area of the concatenation of $\pi(\gamma)$ with the chord between its endpoint and $0$. The *boost* of a generating letter $a_i$ is its height $z(a_i)$. Then the boost of a spelling, denoted $z_b(\gamma)$, is the sum of the boosts of the letters in the spelling.

Note that the height of a spelling path is equal to its balayage area plus its boost:

$$z(\gamma) = z_b(\gamma) + z_A(\gamma).$$

3.5. cc metrics and Pansu’s theorem. As mentioned above, Pansu’s theorem states that the large-scale structure of the Cayley graph $(H, S)$ is a metric on $H(\mathbb{R})$. It is not a Riemannian metric, but rather a sub-Finsler metric called a cc metric. See [4, 11] for some explicit descriptions of the geometry of the limit metric, and [10] for general background on sub-Riemannian geometry and the Heisenberg group. We collect a few salient features here.

The cc metrics are defined as follows. Let $\mathfrak{m}$ denote the horizontal subspace of the Lie algebra $\mathfrak{h}$ of $H(\mathbb{R})$; that is, the span of the tangent vectors $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and identify $\mathfrak{m}$ with the $xy$-plane in $\mathbb{R}^3$ in exponential coordinates.

We can regard $\mathfrak{m}$ as a copy of $\mathbb{R}^2$ and make use of the linear projection $\pi : H(\mathbb{R}) \to \mathfrak{m}$ given by $(a, b, c) \mapsto (a, b)$.

Fix a centrally symmetric convex polygon $L \subset \mathfrak{m}$; this uniquely defines a norm $\| \cdot \|_L$ on $\mathfrak{m}$ for which $L$ is the unit sphere. The push-forwards of $\mathfrak{m}$ by left multipliers give admissible planes at every point in $H(\mathbb{R})$, which are similarly normed; the plane field is a sub-bundle of the tangent bundle to $H(\mathbb{R})$. We say that a curve in $H(\mathbb{R})$ is admissible if all of its tangent vectors lie in these normed planes. The length of an admissible curve is simply the integral of the lengths of its tangent vectors, and it is easily verified that this is the same as the length in the $L$-norm of the projection $\pi(\gamma)$, and that any two points are connected by an admissible
path. Then the cc distance $d_{cc}(x, y)$ is (well-)defined as the infimal length of an admissible path between $x$ and $y$.

In exponential coordinates, all cc metrics are equipped with a dilation $\delta_t(a, b, c) = (ta, tb, t^2c)$ that is a metric similarity, scaling lengths and distances by $t$, areas in $\mathbb{m}$ by $t^2$, and volumes by $t^4$.

Pansu also tells us which polygon $L$ is induced by a generating set $S$: namely, $L$ is the boundary of the convex hull of the projection $\pi(S)$ of the generators to $\mathbb{m}$. For example, the two most basic generating sets for $H(\mathbb{Z})$ are $\{e_1, e_2\}^\pm$ and $\{e_1, e_2, e_3\}^\pm$. In either case, the cc metric is induced by the $L^1$ norm on $\mathbb{m}$. By contrast, if one took the nonstandard generators $\{e_1, e_2, e_1e_2\}^\pm$, the polygon $L$ would be a hexagon.

In this language, we can state this special case of Pansu’s theorem as follows: for any finite symmetric generating set $S$ of $H(\mathbb{Z})$, 
\[
\lim_{x \to \infty} \frac{d_{cc}(x, 0)}{|x|} \to 1.
\]

While Pansu’s result extends to a statement for all nilpotent groups, there is a substantial strengthening due to Krat [19] which was shown only in the case of $H(\mathbb{Z})$: there is a global bound (depending on $S$) in the additive difference between word and cc lengths: $\sup |d_{cc}(x, 0) - |x|| < \infty$. Below in Section 5.2 we will give a new proof of Krat’s (and therefore Pansu’s) result for $H(\mathbb{Z})$. We note that Breuillard (4) has shown that bounded difference does not hold for all 2-step nilpotent groups, though arguments from [5] can be adapted to show bounded difference for all of the higher Heisenberg groups.

3.6. Significant directions, isoperimetries, structure of cc geodesics. It is a standard fact in Heisenberg geometry that for any admissible path $\gamma$ based at the origin $0 \in H(\mathbb{R})$, the height or $z$ coordinate of $\gamma(t)$ is equal to its balayage area: the signed (Lebesgue) area enclosed by the concatenation of the curve’s shadow $\pi(\gamma)$ with a straight line segment connecting its endpoints.

As a consequence of the connection between height and balayage area, we have a criterion for geodesicity in the cc metric: a curve $\gamma$ in $\mathbb{m}$ based at $(0, 0)$ lifts to a geodesic in $H(\mathbb{R})$ iff its $L$-length is minimal among all curves with the same endpoints and enclosing the same area. As a result, to classify geodesics one uses the solution to the isoperimetric problem in the normed plane $(\mathbb{m}, \| \cdot \|_L)$. By a classical theorem of Busemann from 1947 [6], the solution is described in terms of a polygon which he called the isoperimetrix.

**Definition 7.** For a finite symmetric generating set $S$, let $Q = \text{CHull}(\pi(S))$ be the convex hull of the projection of $S$ to $\mathbb{m}$ and let $L$ be its boundary polygon, as above. The polar dual of $Q$ is defined as $Q^* = \{v \in \mathbb{m} : v \cdot x \leq 1 \ \forall x \in Q\}$ with respect to the standard dot product. Busemann’s isoperimetrix is the polygon $\partial (e^{i\pi/2}Q^*)$, obtained by rotating the polar dual of $Q$ through a right angle.

**Definition 8.** The vertices of the polygon $L$ will be labelled cyclically as $a_1, \ldots, a_{2k}$ and these vectors will be called significant directions. For the significant directions, we will extend the subscripts periodically by defining $a_m$ to equal $a_n$ if $m \equiv n \pmod{2k}$.

Each significant direction is the shadow of at least one significant generator in $S$ and we will label the generators projecting to $a_i$ as $a_i, a_i', a_i''$, etc. Elements of
S which project to the edges of $L$ are called \emph{edge letters} and those that project properly inside $L$ are called \emph{interior letters}.

**Remark.** We will maintain this font distinction as much as possible to mark the difference between group elements $a \in H$ and their corresponding projections $a \in m$, the latter thought of as vectors in the plane.

With this terminology, Busemann’s theorem can be stated as follows. Use Lebesgue measure on $\mathbb{R}^2$ for area and length in the Minkowski norm for perimeter. Then up to dilation and translation \emph{the isoperimetrix is the unique closed curve realizing the maximal value of area divided by perimeter-squared}. The following properties follow from Busemann’s construction.

- If the vertices of $Q$ have rational coordinates, then the same is true of the vertices of the isoperimetrix.
- The edges of the isoperimetrix are parallel to the significant directions.

It follows that by clearing common denominators we can find positive integers $\sigma_1, \ldots, \sigma_{2k}$ with gcd $= 1$ and an integer $\lambda$ such that the edge vectors of $\lambda \partial (e^{i\pi/2}Q^*)$ are $\sigma_i a_i$.

**Definition 9.** Define the \emph{standard isoperimetrix} to be the closed polygon $I = I(S)$ having vertices

$0, \quad \sigma_1 a_1, \quad \sigma_1 a_1 + \sigma_2 a_2, \quad \ldots$

(This is a translated and scaled copy of Busemann’s curve.)

![Figure 2](image)

**Figure 2.** This example shows an isoperimetrix which is twice the rotated polar dual of the original polygon.

CC geodesics based at 0 are classified into two kinds: \emph{regular geodesics}, which project to $m$ as an arc of an isoperimetrix, and \emph{unstable geodesics}, which project to $m$ as geodesic in the $L$-norm.

Fix a polygon $L$ in $m$, which determines a CC metric on $H(\mathbb{R})$. Then for any $(a, b) \in m$, each length $\ell \geq \|(a, b)\|_L$ uniquely determines a height $c = c(a, b, \ell) \geq 0$ so that there exists a regular geodesic connecting 0 to $(a, b, c)$ at length $\ell$. That is, for each $\ell$ there exists a scale $s$ and a translation vector $q$ so that $sI + q$ passes through 0 and $(a, b)$; the subarc between those two points has length $\ell$ with respect to $L$ and encloses area $c$, and it lifts to a CC geodesic. On the other hand if $\ell = \|(a, b)\|_L$, there are $L$-norm geodesics connecting 0 to $(a, b)$ with length $\ell$, and these can enclose any area in an interval of possibilities.
Lemma 10 (Types of \(cc\) geodesics [11]). Let \(c_0(a, b) = c(a, b, \| (a, b) \|_\rho)\) be the first height reached by a regular geodesic. Then \((a, b, c_0)\) is reached by a geodesic that is both regular and unstable, and the same is true for \((a, b, -c_0)\). Any point \((a, b, t) \in H(\mathbb{R})\) is reached by only unstable geodesics if \(|t| \leq c_0\), and by only regular geodesics if \(|t| \geq c_0\).

Furthermore, if \((a, b)\) is not in a significant direction then there are infinitely many unstable geodesics in the first case and a uniquely determined regular geodesic in the second.

Recall that “stability of geodesics” means that for each pair of endpoints, geodesics between the endpoints fellow travel with some fixed constant (as in the Morse Lemma in hyperbolic geometry). Here, \(cc\) geodesics of the second type are called “unstable” because they are highly non-unique and can fail to fellow travel arbitrarily badly.

4. Euclidean geometry lemmas

Below, we will use multiplicative vector notation for polygonal paths in the plane, so that for instance \(a_1 a_2 a_1\) denotes the concatenated path obtained by starting with the vector \(fa_1\) followed by the vector \(a_2\) followed by the vector \(a_1\), ending at the point \((t + 1)a_1 + a_2\) in the plane. Note that in this path notation, the exponents need not be integers.

Any closed polygon in the plane with a vertex at the origin is traced out by a path \(P = v_1 v_2 \ldots v_r\) with \(\sum v_i = 0\). The polygon is convex if and only if the vectors \(v_1, \ldots, v_r\) are cyclically ordered (that is, if their arguments proceed in a monotone fashion around the circle). In this notation, the standard isoperimetric described above can be written \(I = a_1^{\sigma_1} \cdots a_{2k}^{\sigma_k}\). For each \(1 \leq i \leq 2k\), define \(\tilde{a}_i = a_i^{\sigma_i}\) and \(\tilde{a}_i = a_i^{\sigma_i}\) so that \(I = \tilde{a}_1 \cdots \tilde{a}_{2k}\). We refer to these as blocks of significant letters or more simply significant blocks, relying on context to distinguish between \(\tilde{a}_i\) and \(\tilde{a}_i\). Note that \(\tilde{a}_i\) is an element of \(S^{\sigma_i}\), not a weighted generator added to the generating set \(S\).

For any closed convex polygon \(P\) in the plane, we consider the family of polygons with the same ordered set of interior angles, i.e., the family obtained by moving the sides of \(P\) parallel to themselves. Let us call this the parallel family of \(P\). If \(P = v_1 v_2 \ldots v_r\), then an element of the parallel family is of the form \(P_s = v_1^{s_1} v_2^{s_2} \ldots v_r^{s_r}\) for some \(s = (s_1, \ldots, s_r) \in \mathbb{R}^r\), and it is closed and convex if all \(s_i \geq 0\) and \(\sum s_i v_i = 0\).

Lemma 11 (Isoperimetric problem in parallel families). Let \(P = v_1 v_2 \ldots v_r\) be a closed convex polygon in the plane. For fixed arbitrary positive numbers \(\ell_1, \ldots, \ell_r\) and for any \(\lambda > 0\), let

\[
M(\lambda) := \{s \in [0, \infty)^r : \sum s_i v_i = 0, \sum s_i \ell_i = \lambda\}.
\]

Then the function \(\text{Area}(P_s)\) has a unique local maximum on \(M(\lambda)\).

Proof. \(\text{Area}(P_s)\) varies quadratically over \(\mathbb{R}^r\), and therefore also over the convex polytope \(M(\lambda)\). From the form of \(\text{Area}(P_s)\), one checks that it is a negative-definite quadratic form.

This simple observation says that for an arbitrary convex polygon in an arbitrary normed plane, the parallel family contains a unique set of best proportions to maximize area relative to perimeter, by taking the \(\ell_i\) to be the lengths of the sides.
(This is slightly more general than what is implied by Busenmann’s theorem, which for a given norm only treats the polygons in the parallel family of the isoperimetric.)

**Definition 12.** For a convex closed \( P = v_1v_2 \ldots v_{r-1}v_r \) as above, let the indices be considered cyclically. A \( P \)-arc \( \tau \) of scale \( s \) is a path \( v_1^{s^-}v_{i+1}^{s^+} \ldots v_{j-1}^{s^-}v_j^{s^+} \) where \( 0 \leq s^-, s^+ \leq s \), and its combinatorial length is the sum of the exponents, \( \ell(\tau) = s^- + (j - i - 1)s + s^+ \). Note that \( \tau \) begins at the origin and lies on a scaled and translated copy of \( P \), i.e., \( \tau \subset sP + r \). There are two possible ambiguities: first, if \( \tau \) is one- or two-sided, the scale is underdetermined, so we take \( 0 \leq s \leq 100 \). Second, if \( s^- \) or \( s^+ \) equals 0 or \( s \), then the arc is of more than one combinatorial type; for instance, \( v_1^{100}v_4^{100}v_5^{100} \) is of types (2, 5), (2, 6), (3, 5), and (3, 6).

The combinatorial type of \( \tau \) is the pair \((1, i)\) of starting and ending sides. Given \( K > 0 \), we say that the arc \( K \)-almost has combinatorial type \((1, i)\) if it can be modified to an arc of combinatorial type \((1, i)\) by modifying \( s^+ \) and \( s^- \) by at most \( K \) (possibly making them equal either 0 or \( s \) to change type). If there is \((1, i)\) so that \( \tau \) and \( \tau' \) both \( K \)-almost have combinatorial type \((1, i)\), we say that \( \tau \) and \( \tau' \) \( K \)-almost have the same combinatorial type.

**Lemma 13** (Combinatorial types of nearby arcs). Fix \( K_1 \) and \( K_2 \). Then there are \( K_3, K_4, K_5 \) with the following property. If \( \tau \) and \( \tau' \) are \( P \)-arcs whose combinatorial lengths are within \( K_1 \) and whose endpoints are within distance \( K_2 \), then their scales differ by at most \( K_3 \). Further, one of the following holds.

- \( \tau \) and \( \tau' \) are of \( K_4 \)-almost the same combinatorial type and so \( K_5 \)-fellow travel;
- \( \tau \) and \( \tau' \) are \( K_4 \)-almost of type \((1, i)\) for some \( i \), and the values of \( s^- + s^+ \) are close for \( \tau, \tau' \), so that the modified arcs \( v_1^{(s^-+s^+)}v_{i+1}v_{i-1}^{s^+} \) for each of \( \tau, \tau' \) \( K_5 \)-fellow travel.
- \( \tau \) and \( \tau' \) are \( K_4 \)-almost closed polygons. In this case \( \tau \) can be \( K_4 \)-almost of type \((1, i)\) and \( \tau' \) can be \( K_4 \)-almost of type \((1', i')\) for any \( i, i' \), and they need not fellow travel but both must end \( K_5 \)-close to the origin.

**Proof.** Suppose \( \tau \) is a \( P \)-arc with endpoint \((a, b) \in \mathbb{R}^2 \), and the length of \( \tau \) is \( \ell \). By convexity of \( P \), there are only very limited ways to find \((a, b)\) as a chord of a scaled copy of \( P \) with given arclength.

**Case 1:** If \((a, b)\) is a nonzero vector which is not parallel to any \( v_i \), then the triple \((a, b, \ell)\) uniquely determines not only \( s \) but determines \( \tau \) completely (by convexity of \( P \)), and the starting side and ending side are different \((i \neq j)\). Within a combinatorial type, the scale \( s \) is a linear function of \((a, b, \ell)\), and indeed it is piecewise linear (and continuous) across combinatorial types as \((a, b)\) varies over the sector between any successive \( v_i, v_{i+1} \). (See [11] for details and examples.)

**Case 2:** If \((a, b)\) is a nonzero multiple of \( v_1 \) and \( \ell \) sufficiently long, it can also be a chord in a \( P \)-arc of type \((1, i)\). In this case there is clearly a family of polygons with the same \((a, b, \ell)\) and type \((1, i)\) obtained by shifting weight between \( s^- \) and \( s^+ \), and these are the only solutions to the chord problem.

**Case 3:** If \((a, b) = (0, 0)\) then \( \tau \) can be any translate of \( P \) containing the origin.

This completes the proof: nearby endpoints \((a, b)\) and \((a', b')\) are either in the same sector or they are close to a dividing line, which puts them both nearly in Case 2 or both nearly in Case 3. These are the three possibilities listed in the statement. \( \Box \)
\( P \)-arcs have the proportions required to belong to \( P \) as sub-arcs. Generalizing slightly, in the family
\[
\gamma_t = c_1 \cdot v_i^1 \cdot c_{i+1} \cdot v_{i+1}^i \cdots c_J \cdot v_J^i \cdot c_{J+1}
\]
as \( t \in \mathbb{R}^r \) varies, a path with \( J = 1 \geq 2 \) will be called \textit{balanced} if \( t_1, t_J \leq t_{J+1} = t_{J+2} = \cdots = t_{J-1} \). Such a path is \( K \)-\textit{almost balanced} if all of these equalities and inequalities hold within \( K \), i.e., \( |t_i - t_j| \leq K \) for \( i < j \) and \( t_i, t_J \leq t_i + K \) for all \( i < j \).

Below, we will take the area of a not-necessarily-closed path to be its balayage area: the signed area enclosed by concatenating the path with the chord from its endpoint to its start point.

**Lemma 14** (Balancing paths). Suppose a closed convex polygon \( P = v_1 v_2 \ldots v_r \) encloses maximal area among all closed \( P \) with \( \sum_{i=1} r_i = r \). Let
\[
\gamma_t = c_1 \cdot v_i^1 \cdot c_{i+1} \cdot v_{i+1}^i \cdots c_J \cdot v_J^i \cdot c_{J+1},
\]
and consider the affine subspace
\[
M(\lambda) := \{ t \in \mathbb{R}^r : \sum_{i=1}^r t_i = \lambda \}.
\]
Then for every \( \Delta > 0 \) and every vector \((k_1, \ldots, k_r) \in \mathbb{R}^r\), there is a constant \( K \) such that, for \( \lambda \gg \Delta \) and any integer lattice \( L \leq \mathbb{Z}^r \cap M_\Delta \) with covolume \( \leq \Delta \), the maximum value of \( \text{Area}(\gamma_t) + \sum_{i=1}^r k_i t_i \) over \( t \in L \) occurs when the path is \( K \)-almost balanced with respect to \( P \).

**Proof.** There is a constant area difference between each \( \gamma \) and a modified path \( \tau \) of the form \( v_i^1 \cdot c_{i+1} \cdot v_{i+1}^i \cdots v_J^i \) (with a slightly different start and end). We know that the \( \tau \) paths enclose optimal area when they are balanced, and that iso-length deformations in the family of \( \tau \) improve area quadratically as they tend towards balanced exponents, which beats the linear contribution of the \( \sum k_i t_i \). Negative-definiteness of the quadratic form from the previous lemma finishes the proof: the balance point for \( \tau \) lies in some fundamental domain for \( L \leq M(\lambda) \), and one of the vertices of that fundamental domain supplies the needed optimum. \( \square \)

Notice that in the application of this to the isoperimetric, this produces a path with roughly the same number of significant blocks on each side.

**Remark.** This is the first of several places where something is shown to be bounded with reference to a constant \( K \). To avoid proliferating notation, we will maintain the symbol \( K \) in each successive place that a constant bound is derived, enlarging it each time as necessary. No earlier statement will be hurt by subsequent enlargement, so that in the end one value of \( K \) depending only on \( S \) will suffice for all applications.

### 5. Simple shapes and approximate geodesics

#### 5.1. Simple shapes and highest height

Suppose \( u, v \in H(\mathbb{Z}) \) project to integer vectors \( \mathbf{u}, \mathbf{v} \in \mathbb{m} \). We write \( \mathbf{u} \wedge \mathbf{v} \) to denote the determinant of the matrix with those column vectors, i.e., the area of the parallelogram they define. Then when letters \( u \) and \( v \) are exchanged, the effect on area is given by the wedge: \( z(uv) = z(vu) + u \wedge v \). For instance, \( z(e_1 e_2) = \frac{1}{2} \) \( z(e_2 e_1) = -\frac{1}{2} \); and \( e_1 \wedge e_2 = 1 \). Note that two group elements commute if and only if they project to the same direction in the plane.
To keep track of all the possible effects of rearranging letters, we once and for all define
\[ N = N(S) := \text{lcm}\{u \land v : u, v \in S\}. \]

**Definition 15.** For \((a, b) \in \mathfrak{m}\), define \(n_0(a, b) = |(a, b)|_{\pi(S)}\), so that the fiber \((a, b, \ast)\) can be reached by a spelling path of length \(n\) if and only if \(n \geq n_0\). Then for \(n \geq n_0\), we define the highest height at length \(n\) over \((a, b)\) to be the largest \(z\) coordinate reachable with at most \(n\) letters,
\[ w_n = w_n(a, b) := \max\{t : \ell(a, b, t) \leq n\}. \]
A spelling (or a group element) will be called highest-height if it realizes \((a, b, w_n)\) at length \(n\). Let \(W(a, b) := w_{n_0}(a, b)\) be first positive \(w_n\).

Note that \(w_n < w_{n+2}\), but that there may be no spellings at all of a certain parity reaching \((a, b)\), in which case \(w_n = w_{n+1}\).

**Definition 16.** Given a constant \(K\), let \(C(K) = \bigcup_{i=0}^{K} S^i\) be the strings in \(S\) whose length is at most \(K\) (so that the evaluation map sends \(C(K)\) onto the ball of radius \(K\) in the word metric). Then a break word is an element \(c \in C(K)\) and a break vector is a tuple of break words \(c = (c_1, \ldots, c_{2k})\).

A simple shape is a tuple \(\omega = (i, j, b, c)\), where \(c\) is a break vector, \(i, j\) are indices \((1 \leq i, j \leq 2k)\), and \(b = (b_1, \ldots, b_{2k})\) is a vector of integers. The simple shape domain is \(\text{Cone} := \{(s^-, s, s^+) \in \mathbb{Z}^3 : 0 \leq s^-, s^+ \leq s\}\) and the restricted domain is \(\text{Cone}_0 := \{(s^-, s, 0) \subset \text{Cone}\} \). (Compare to Lemma 13.)

Each such shape induces a map from the shape domain to spellings in the group. That is, define the evaluation of a simple shape to be
\[ \omega(s^-, s, s^+) = c_{1\ldots2k}^{s^- + b_1} \cdot \hat{c}_1 \cdot c_{1+1}^{s^+ + b_{1+1}} \cdot \hat{c}_{1+1} \cdot \cdots \cdot \hat{c}_{1-j-1} \cdot c_{J-1} \cdot \hat{c}_j \cdot c_{J+1}, \]
recalling that \(\hat{c}_i = c_i^{q_i}\) is defined so that \(I = \hat{a}_1 \cdots \hat{a}_{2k}\). Further, we take the convention that if \(i = j\), i.e., if the shape starts and ends with the same generator, then the domain is restricted to \(\text{Cone}_0\).

**Example.** Consider the nonstandard generators for \(H(\mathbb{Z})\) given by \(S = \{a, b, A, B\}^\pm\), where \(a, b\) are the standard generators and \(A, B\) are big generators \(A = a^3, B = b^3\), and a bar denotes the inverse of an element. Then the word \(A^5ab^9b\bar{a}^1b^3\bar{A}^5\bar{B}^3\) is given by evaluating the shape with \(c = (e, a, b\bar{a}, b, e), b = (0, 0, 1, 0), 1 = 1, j = 4\) at \(s = 9, s^- = 5, s^+ = 3\).

**Remark.** There are other shapes with other data that evaluate to the same path.

### 5.2. Bounded difference between word and \(cc\) metrics.

**Proposition 17** (Form for highest-height geodesics). Given a finite generating set \(S\), there is a number \(K = K(S)\) such that any highest-height spelling path is the evaluation of some simple shape with break words from \(C(K)\) separating runs of significant letters given by integer values \(s^-, s^+ \leq s\) with exponent corrections \(0 \leq b_i \leq K\).

That is, in a very strong sense, highest-height spellings track along an arc of a canonical polygon (Busemann’s isoperimetrix), which has a spelling of the form \(\hat{a}_1^{-s} \cdot \hat{a}_{1+1}^{-s_2} \cdots \hat{a}_{j-1}^{-s_{j-1}} \hat{a}_j^{s_j} \) with \(s^- \leq s \) and \(s^+ \leq s\), not necessarily integers. The highest-height spellings only differ by bounded break words appearing in the corners, and by bounded deviation in run lengths.
Proof. We suppose that $\gamma$ is a highest-height geodesic over $(a, b)$ and that its length is $n$. We claim that the letters of $\gamma$ are in cyclic order. If not, we produce $\gamma'$ by putting its letters into cyclic order. This changes neither the horizontal endpoint $(a, b)$ nor the boost $z_0(\gamma)$. If two of the letters which we move past each other in this process do not lie in the same direction in projection, then $z(\gamma') > z(\gamma)$, contradicting our assumption. Thus the letters appearing in $\gamma$ are arranged in cyclic order in projection. Also if there are multiple letters $a_i, a_i', a_i''$ projecting to the same significant $a_i$, then clearly $\gamma$ must use the one with greatest boost to achieve highest height.

We now claim that there is a bound $K$ on the total exponent of any non-significant generator. To see this, suppose that $u$ is a non-significant generator appearing with large exponent, as a subword $u^m$. Supposing $a_i$ and $a_i+1$ are the significant generators whose directions bound the sector that $u$ lies in, there must be integers $p, q, r$ so that $qu = pa_i + ra_i+1$, with $q \geq p + r$. We can then replace $u^{pq}$ by $a_i^{kp}a_i^{kr}$. The area gained by this operation is quadratic in $k$ while any boost lost is linear in $k$. Consequently, if $m$ is sufficiently large, this operation increases height. So if the total exponent of $u$ in $\gamma$ is $m$, then the reshuffling which brings all powers of $u$ together and then performs the subword replacements above will produce a path over $(a, b)$ with no greater length and with higher height, contradicting the assumption.

It follows now that $\gamma$ consists of corner words of bounded length between ordered runs of highest-boost significant letters. That is, we have

$$\gamma = \hat{a}_1 \cdot \hat{a}_2^n \cdot \hat{a}_3 \cdot \hat{a}_4^n_1 \cdots c_1 \cdot \hat{a}_j \cdot c_{j+1}.$$ 

The statement now follows from an application of the Balancing Lemma (Lemma 13).

Proof. We suppose that $\gamma$ is a highest-height geodesic over $(a, b)$ and that its length is $n$. We claim that the letters of $\gamma$ are in cyclic order. If not, we produce $\gamma'$ by putting its letters into cyclic order. This changes neither the horizontal endpoint $(a, b)$ nor the boost $z_0(\gamma)$. If two of the letters which we move past each other in this process do not lie in the same direction in projection, then $z(\gamma') > z(\gamma)$, contradicting our assumption. Thus the letters appearing in $\gamma$ are arranged in cyclic order in projection. Also if there are multiple letters $a_i, a_i', a_i''$ projecting to the same significant $a_i$, then clearly $\gamma$ must use the one with greatest boost to achieve highest height.

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The statement now follows from an application of the Balancing Lemma (Lemma 13).

Corollary 18 (Bounded difference). For each generating set $S$, there exists a constant $K = K(S)$ with the following property. If $w_n(a, b) < c \leq w_{n+1}(a, b)$ then $n < |(a, b, c)| \leq n + K$, and if $0 \leq c \leq W = w_{n0}(a, b)$, then $n_0 \leq |(a, b, c)| \leq n_0 + K$.

Consequently, there exists a constant $K = K(S)$ such that

$$d_{cc}(x, 0) - K \leq |x|_S \leq d_{cc}(x, 0) + K.$$ 

Proof. We define $(a, b, w_n)$ is the highest-height element of the fiber over $(a, b)$ which can be reached by a spelling of length less than or equal to $n$, so $n < |(a, b, c)|$.

Let $\omega(s^-, s, s^+)$ and $\omega'(t^-, t, t^+)$ be shapes evaluating to geodesic spellings for $g = (a, b, w_n)$ and $g' = (a, b, w_{n+1})$. Let $\tau$ and $\tau'$ be $I$-arcs which fellow-travel these in projection. If $\tau$ and $\tau'$ are of almost the same combinatorial type, then the polygonal paths $\beta = \pi(\omega(s^-, s, s^+))$ and $\beta' = \pi(\omega'(t^-, t, t^+))$ fellow-travel. Consider the sequence of paths $\beta' = \beta_0, \beta_1, \ldots, \beta_{n+1} = \beta$ formed as follows. For $i = 1, \ldots, n$, let $\beta_i$ be the path starting along $\beta$ until $\beta(i)$, taking a geodesic from $\beta(i)$ to $\beta'(i)$, and continuing along $\beta'$. Since $\beta$ and $\beta'$ $K_0$-fellow-travel for some $K_0$, the connecting geodesics have bounded length, so each $\beta_i$ has length at most $n + 1 + K_0$. Take $\gamma_i$ to be the lift of $\beta_i$. These $\gamma_i$ end at group elements $x(a, b, c_i)$ with $|c_i+1 - c_i| \leq 2K_0 + 2$. Thus any value $(a, b, c)$ in the range in question can be reached by taking a bounded-length path on to the end of an appropriate $\gamma_i$. It
follows that there is $K$ such that for each $c$ with $w_n < c \leq w_{n+1}$, $|(a, b, c)| \leq n + K$ as required.

If $\tau$ and $\tau'$ are not of almost the same combinatorial type, then by Lemma 13 $(a, b)$ is close to the origin and $\tau$ and $\tau'$ almost complete the entire boundary of the isoperimetrix. It follows that we can replace $\omega'(t^-, t, t^+)$ by a path which fellow-travels $\omega(s^-, s, s^+)$ in projection and is only boundedly longer than $\omega'(t^-, t, t^+)$. The proof now goes through as before.

Note that for an $I$-arc of spelling length $n$, its length the $L$-norm is $n$, so its lift has $cc$ length $n$ as well and it is geodesic. Therefore $d_{cc}((a, b, w_n), 0)$ is boundedly close to $n$ and we are done with the case $w_n < c \leq w_{n+1}$.

For heights below $W$, we begin with a highest-height spelling realizing $(a, b, w_n)$. By permuting the letters, we can lower the height in bounded increments down to below zero. Since the intermediate heights can be reached by appending a bounded-length correction word, we have $n_0 \leq |(a, b, c)| \leq n_0 + K$. On the other hand, by Lemma 10 the $cc$ distance from 0 is constant in the $(a, b)$ fiber up to the first height reached by a regular geodesic, which his boundedly close to $(a, b, W)$. We enlarge the constant $K$ from the first statement in the Lemma to be sufficient for the second statement.

Finally, observe that the map $g \mapsto g^{-1}$ is a length-preserving bijection which carries $(a, b, c)$ to $(-a, -b, -c)$, so the $c < 0$ case is similar.

This gives a new proof of Krat’s result. And in particular, since Krat’s theorem (bounded difference) has a stronger conclusion than Pansu’s theorem (ratio goes to 1), our argument also gives a direct geometric proof of Pansu’s theorem for the special case of arbitrary word metrics on $H(Z)$.

5.3. **Simplification.** We will see below that every regular element has a geodesic which is close to a simple shape. To this end, we show that we can modify paths to become simple shapes while staying in the same fiber and increasing height in a controlled manner.

**Lemma 19** (Simplifying paths). There is a constant $K = K(S)$ so that for each spelling path $\gamma$ there exists a refined path $\gamma_1$ with the following properties.

- If $(a, b, c)$ is the evaluation of $\gamma$, then $\gamma_1$ evaluates to $(a, b, c + kN)$ for some $k \geq 0$;
- the length of $\gamma_1$ is less than or equal to the length of $\gamma$;
- $\gamma_1 = \omega(s^-, s, s^+)$ for some simple shape $\omega$ with corners from $C(K)$.

**Proof.** Suppose that $u, v$ are any two letters appearing in $\gamma$ such that $u \land v > 0$ (so that $u$ comes before $v$ in the cyclic ordering of their projections, and replacing $vu$ with $uv$ increases area). Let $\Lambda_{u, v} = \Lambda_{u, v}(\gamma)$ be the sum of all of the exponents $k$ appearing in distinct subwords $vu^k w$ of $\gamma$ with $w \neq u$. Then we can make generator swaps of $u$ and $v$ letters to change the height by any multiple of $u \land v$ less than or equal to $\Lambda_{u, v} \land v$. By rounding $\Lambda_{u, v} \land v$ down to the nearest multiple of $N$, we can perform generator swaps to obtain $\gamma_1$, so that $\Lambda_{u, v}(\gamma_1) \leq N$ for all pairs $u, v$.

Notice that we may have to perform this procedure many times. A single application of this procedure reduces $\Lambda_{u, v}$ to be less than $N$, but may increase $\Lambda_{u, v'}$. We can perform this procedure whenever there is some pair $u, v$ so that $\Lambda_{u, v} \land v > N$. We claim that repeated applications of procedure must eventually terminate with a spelling where there is no such pair. To see this, consider the total number of pairs
of letters in the spelling which are out of order. This total number decreases at every application of the procedure, and hence we must terminate with $\Lambda_{u,v}, u \land v < N$ for every pair $u$ and $v$.

Next, we will cash in any big blocks of non-significant letters for significant letters. Recall that significant letters project to corner points of the polygon $L$, while edge letters project to other boundary points and interior letters project to the interior. That is, for an edge letter $u$ and an interior letter $v$ with projections in the sector between $a_i$ and $a_{i+1}$, we have $qu = pa_i + ra_{i+1}$ and $q'v = p'a_i + r'a_{i+1}$ such that $q = p + r$ while $q' > p' + r'$.

Consider the subword replacements $u^{kNq} \rightarrow a_i^{kNp} a_{i+1}^{kNq}$, or $u^{kNq'} \rightarrow a_i^{kNp'} a_{i+1}^{kNq'}$.

As above, the new paths reach the same endpoint in $m$ while either preserving or reducing the total spelling length of the path, gaining area by an amount proportional to $k^2$, and reducing boost by an amount proportional to $k$. We perform these replacements in every instance where $k$ is large enough to produce is a net height increase; and note that the height change is a multiple of $N$.

Repeat the reordering and the replacement steps one after the other until neither can be performed any further. Then $z(\gamma_1) \geq z(\gamma)$, and they differ by a linear combination of the wedges; namely, $\Delta \gamma = z(\gamma_1) - z(\gamma) = \sum_{u,v} k_{uv}(u \land v)$, for integers $k_{uv} \geq 0$ with $k_{uv} = 0$ (mod $N$). At this stage, the path $\gamma_1$ has well-defined sides with mostly $a_i$ letters and only boundedly many exceptions.

Next, we push the remaining “out of place” letters to the corners so that the $a_i$ side is mostly a single long block of the $a_i$ letter. So far we have a spelling $\gamma_1$ that contains boundedly many non-significant letters and boundedly many significant letters on the wrong side. Consider a side which consists of significant generator $a_i$ with a bounded number of letters which are not $a_i$. For each letter $u$ on the $a_i$ side, the sign of $u \land a_i$ tells us whether replacing $a_i u$ with $u a_i$ is height-increasing or height-decreasing (note that the case $u \land a_i = 0$ is the case that $u$ and $a_i$ commute). We swap each $u$ past $a_i^N$ in the height-increasing direction (or an arbitrary direction if they commute) until we create a block $w = u' a_i^m u$ with $m < N$. This $w$ itself can be commuted with $a_i^N$ to the left or right, not decreasing height. Since there are boundedly many out of place letters on each side, this process ends with all these letters within a bounded distance of the corner, so we merge them with the corner words. At each move we have increased height by a multiple of $N$.

Finally, we balance the side lengths of $\gamma_1$. To do this we apply the balancing lemma (Lemma 14) to the lattice of integer tuples which differ from the original $(t_1, t_1+1, \ldots, t_j)$ by multiples of $N$ in each coordinate. This ensures that area and boost, and therefore height, changes by a multiple of $N$.

This final step has produced a modified path, again called $\gamma_1$, which still has the same $(a,b)$ endpoint as $\gamma$ and may have higher height by a multiple of $N$. Now there are bounded-size exceptional corner words between the sides, and the exponents of significant letters differ only by a bounded amount, so this is the evaluation of a simple shape.

\[ \Box \]

6. General shapes and unsimplification

6.1. General shapes. In general, geodesic spellings of the regular type have not only break words at the corners, but between them have runs of significant generators separated by finitely many other break words.
Definition 20. Given a generating set $S$ for which the isoperimetrix has $2k$ sides, a general shape with parameter $K \geq 1$ is a tuple $\omega = (i, j, b, \chi)$, where

- $1 \leq i, j \leq 2k$ are a starting and ending side;
- $b = (b_1, \ldots, b_{2k})$ is a vector of integers $0 \leq b_i \leq K$;
- $\chi$ is a $(K - 1) \times 2k$ matrix whose entries are break words from $C(K)$.

Let $\text{Shape}_K$ be the set of all such shapes, clearly a finite set for each value $K$. We will evaluate each shape at a matrix $X \in M_{K \times 2k}$. Let $\Lambda : \text{Shape}_K \times M_{K \times 2k} \to \mathbb{Z}^{2k}$ be given by $\Lambda(\omega, X) = (\lambda_1, \ldots, \lambda_{2k})$, where $\lambda_i := \left( \sum_{j=1}^{K} x_{ij} \right) - b_j$. Then the shape domain $\text{Dom}_K(\omega)$, for $\omega \in \text{Shape}_K$, is the set of $K \times 2k$ matrices $X$ of non-negative integers satisfying a condition on the image of $\Lambda$, namely:

- $\lambda_i = \lambda_j$ for all $1 \leq i, j \leq K$;
- $\lambda_i, \lambda_j \leq \lambda_i$;
- $\lambda_i = 0$ for the $t$ that are not between 1 and $j$.

With slight abuse of notation, we will then write $\Lambda : \text{Shape}_K \times \text{Dom}_K \to \text{Cone}$ given by $\Lambda(\omega, X) = (s^-, s, s^+)$ where $s^- = \lambda_1$, $s = \lambda_{i+1} = \cdots = \lambda_{j-1}$, and $s^+ = \lambda_j$.

It is immediate from this definition that $\text{Dom}_K(\omega)$ is given by pulling back a rational family under an affine map.

The matrix $X$ is to be thought of as a matrix of length $2k$. The evaluation of a shape, $\omega(X)$, is the concatenation of the break words with the runs of significant generator blocks of length prescribed by $X$. The $b$ vector records the failure of the column sums to be equal, i.e., the failure of the shadow to be balanced in terms of its side lengths. (Since its entries are bounded, the column sums are nearly equal, which means that the spelling will track close to an isoperimetrix.) Simple shapes are a subset of general shapes for which the break words only appear at the corners.

Remark. Note that the triple $(a, b, \ell)$ associated to a spelling $\omega(X)$ factors through $\Lambda$. That is, as $X$ ranges over $\text{Dom}_K(\omega)$, the three integers $\Lambda(\omega, X) = (s^-, s, s^+)$ determine the horizontal position and the word length of the evaluation word. Thus we can regard this as a map $\omega : \text{Cone} \to \mathbb{Z}^3$ that is affine and injective.

Remark. If significant generators include several options with same projection and different boost, then we also need $Y$, a matrix specifying for each side how many of each different boost level get used, and in this case the evaluation will be $\omega(X, Y)$. This makes no meaningful difference anywhere in the argument below.

6.2. Unsimplication. We describe a 2-sided surgery and a 3-sided surgery for paths and then explain how to use them algorithmically to begin with a path described by a simple shape and produce a path ending lower in the same fiber and still described by a general shape. In both of these moves, we will suppose that $a_1, a_2, a_3$ are successive significant generators and that $p, q, r$ are the values with $\gcd = 1$ so that where $qa_2 = pa_1 + ra_3$. (In the special case that $a_1 = -a_3$ (the parallel case), we have such a surgery with $p = r = 1, q = 0$.) Here we describe the surgeries on side $a_2$.

2–sided surgery. Here, a subword of the form $a_1^{s_1} c_1 a_2^{s_2}$ is replaced by $a_1^{s_1-3Np} c_1 w a_2$, where $w$ is a permutation of the letters in $a_1^{3Np} a_2^{-1}$.

3–sided surgery. Here, a subword of the form $a_1^{s_1} c_1 a_2^{s_2} c_2 a_3^{s_3}$ is replaced by $a_1^{s_1-2Np} c_1 a_2^{s_2+2Nq} c_2 a_3^{s_3-2Nr}$.
Figure 3. Examples of 2-sided and 3-sided surgery with corners.
If the length $s_2$ of the second side is long enough, then 2-sided surgery can make a larger change to area because it is thicker: the width of the surgery is proportional to $3N$ rather than $2N$.

Lemma 21 (Unsimplification for shapes). Given a starting word $\gamma$, let $\gamma_1$ be the simplification described above and suppose the height difference $\Delta z = z_1 - z_0$ is sufficiently large. Then for any full side of $\gamma_1$, a sequence of (possibly zero) 3-sided surgeries on that side followed by at most one 2-sided surgery on that side produces a word $\gamma_2$ which evaluates to the same group element as $\gamma$.

Note that if there are fewer than three sides (so that there is no well-defined “full side”), then we can appeal to the unstable (pattern) case presented in the next section.

Proof. The change in area for each application of three-sided surgery equals

$$(3SS) := 2Np(a_1 \wedge c_1) + 2Nps_2(a_1 \wedge a_2) + 2N^2pr(a_1 \wedge a_3) + 2Nr(c_2 \wedge a_3).$$

We note that since the wedges are all integers, this is divisible by $N$ and therefore also by $a_1 \wedge a_2$.

On the other hand, the area difference from performing two-sided surgery depends on the permutation parameter; the area change equals

$$(2SS)_k := 3Np(a_1 \wedge c_1) + k(a_1 \wedge a_2),$$

where $k$ is an arbitrary integer, $0 \leq k \leq 3Np(s_2 - 1)$.

The lemma’s assumption that the height difference is large enough can be taken to precisely mean that $\Delta z = z_3 - z_0 > (2SS)_0$.

Perform $(3SS)$ repeatedly, updating $z_3$ each time, until

$$\Delta z < (2SS)_0 + (3SS).$$

Then we must show that there exists $k$ such that $\Delta z = (2SS)_k$. We know that $\Delta z$ is a multiple of $N$ and therefore of $a_1 \wedge a_2$. On the other hand, $\Delta z$ is greater than $(2SS)_0$, and $(2SS)_k$ achieves all multiples of $a_1 \wedge a_2$ past that threshold and up to its maximum. Thus it is enough to show that $\Delta z < (2SS)_{\text{max}}$. Since we saw above that
\[ \Delta z < (2SS)_0 + (3SS) \], this amounts to showing that \((2SS)_{\max} - (2SS)_0 > (3SS) \). Since all the wedges of vectors and the values \(p, q, r\) are fixed by the choice of side, it suffices to take \(s_3\) sufficiently large: since the left-hand side has a term \(3N_{ps2}(a_1 \wedge a_2)\) and the right-hand side has a term \(2N_{ps2}(a_1 \wedge a_2)\), eventually the difference between these overwhelms all the other fixed terms. \hfill \square

7. Patterns

Recall that cc geodesics are classified into two kinds (see Sec. 3.6), regular and unstable. In a particular fiber \((a, b, *)\), only unstable geodesics reach positive heights below a certain threshold height and only regular geodesics reach above that level. We will consider the corresponding situation for word geodesics.

We defined \(W = W(a, b)\) to be the highest height reached by a spelling path of length \(n_0 = |(a, b)|_{\pi(S)}\). In each fiber \(\{(a, b, *)\}\), the general shapes defined in the previous section will reach the elements \(\{(a, b, c) : c > W\}\), which may be called regular elements. In this section we turn to the growth of the unstable elements. Here we will consider the unstable elements \(\{(a, b, c) : 0 \leq c \leq W\}\) at non-negative heights. (Later, we will appeal to the map \(g \mapsto g^{-1}\) which carries \((a, b, c)\) to \((-a, -b, -c)\) to deal with the negative heights.)

**Definition 22.** A pattern is a tuple \(w = (i, c_1, c_2, c_3)\), where each \(c_i \in C(K)\) is a break word (a string of length at most \(K\)), and \(1 \leq i \leq 2k\) picks out a sector between successive significant directions \(a_i, a_{i+1}\). The (finite) set of all such patterns will be denoted \(\text{Patt}_K\). Each pattern \(w\) gives a map \(\mathbb{N}^2 \to S^*\) via \(w(n_1, n_2) = c_1a_1^{n_1}c_2a_2^{n_2}c_3\).

**Lemma 23** (Simplifying to a pattern). Let \((a, b)\) lie in the sector between \(a_i\) and \(a_{i+1}\), and let \(N\) be the lcm of all possible area swaps, as usual. Then there is \(K = K(S)\) with the following property. If \(\gamma\) is a geodesic for an unstable element \((a, b, c)\), then there is a pattern \(w \in \text{Patt}_K\) and \(n_1, n_2 \in \mathbb{N}\) such that the spelling path \(\tau = w(n_1, n_2) = c_1a_1^{n_1}c_2a_2^{n_2}c_3\) has the following properties:

- the paths \(\tau\) and \(\gamma\) have the same length;
- the \(\gamma\) evaluates to an element \((a, b, c+kN)\) with \(k \geq 0\) and \(c+kN \leq W+K\); and
- the letters \(a_1\) and \(a_{i+1}\) are the highest-boost generators projecting to \(a_i\) and \(a_{i+1}\), respectively.

**Proof.** First note that if that \(\gamma\) is a geodesic for an unstable element \((a, b, c)\) where \((a, b)\) lies in the sector between \(a_i\) and \(a_{i+1}\), then all but boundedly many letters in \(\gamma\) project to convex combinations of \(a_i\) and \(a_{i+1}\). This is because, by Bounded Difference (Corollary [15]), there is \(K\) such that if \((a, b, c)\) is unstable, then \(n_0 \leq |(a, b, c)| \leq n_0 + K\), so that the projection \(\pi(\gamma)\) must reach \((a, b)\) in at most \(n_0 + K\) letters. This means \(\pi(\gamma)\) can only use boundedly many letters that are not on the edge between those points (i.e., convex combinations of \(a_i\) and \(a_{i+1}\))—to see this, just consider orthogonal projection to the normal of that edge, so every time any other letter is used, the projection falls behind by a definite amount.

We now carry out the simplification procedure used above (Lemma [19]), making a few extra observations as we go. We note that the length of the path in this case will be maintained and not shortened, because there are only boundedly many interior letters and so we need not cash them in for significant letters.
If \( a_i \) is the highest-boost lift and \( a'_i \) is another letter projecting to \( a_i \), then we can replace any \( N \) occurrences of \( a'_i \) by \( a_i \). The remaining (boundedly many) \( a'_i \), which commute with \( a_i \), can be pushed to the corner position.

Finally, \( \pi(\tau) \) fellow-travels the \( L \)-norm geodesic \( a_1^{n_1}a_2^{n_2}, \) and \( W \) is boundedly close to the highest height achievable by unstable \( cc \) geodesics, so \( |z(\tau) - W| \leq K \).

On the other hand, by controlled rearrangement of letters, patterns can produce a range of group elements in the same fiber.

**Definition 24.** For a pattern \( w = c_1a_1^{n_1}c_2a_2^{n_2}c_3 \) evaluating to \((a, b, c)\), define a process of rearrangements as follows. Consider letters \( b_1, \ldots, b_k \) appearing in the word \( c_2 \). For \( j = 1, \ldots, k \), let \( d_j = \frac{N}{\alpha_j + \beta_j} \), so that commuting \( a_{i+1}^{d_j} \) through \( b_j \) decreases height by \( N \). We greedily perform commutations to move \( a_{i+1} \) letters past \( c_2 \), then continue if possible by commuting groups of \( a_{i+1} \) letters through \( a_i \) letters. Consider the set of \((a, b, c')\) achievable by this process for which \( 0 \leq c' \leq W \), and let the *height interval* of the pattern, denoted \( I_w(a, b) \), be the \( z \) coordinates in this set.

For example, for the generators \( \{a, b, A, B\} \) described above, if \( w = aA^*aB^*b \), then \( I_w(52, 131) = \{6, 106, 206, \ldots, 3406\} \). Here \( W(52, 131) = 3406 \) and \( N = 100 \).

**Lemma 25** (Unsimplification for patterns). Let \( w(n_1, n_2) \) evaluate to \((a, b, C_w)\), and define \( C'_w = \max I_w \) and \( C''_w = \min I_w \), so that \( C_w, C'_w, C''_w \) are functions of \( n_1, n_2 \) or of \( a, b \) representing the possible heights of rearrangements of patterns. Then there is a partition of \( \mathbb{N}^2 \) given by finitely many linear equations, inequalities, and congruences such that \( C_w, C'_w, C''_w \) are given by quadratic polynomials in \( n_1, n_2 \) on each set in the partition. Therefore there is a corresponding partition of \( m \) so that these heights are quadratic on each piece on which \( I_w \neq \emptyset \).

**Proof.** Fixing \( w \), the height \( C_w \) can be seen as a function of \((n_1, n_2)\) whose degree-two term equals \( \frac{1}{2}n_1n_2(a_1 \wedge a_{1+1}) \), because \( w(n_1, n_2) \) fellow-travels the two-sided figure \( a_1^{n_1}a_{1+1}^{n_2} \). Fellow traveling ensures that the enclosed areas differ by at most an amount proportional to the length of the shape plus the boost provided by corner words, which are terms of degree one and zero.

\( W \) is the highest height of a minimal-length spelling path reaching the shadow of \( w(n_1, n_2) \). The simplification argument above shows that the spelling path realizing height \( W \) must also be boundedly close in projection to \( a_1^{n_1}a_{1+1}^{n_2} \), so the difference \( W - C_w \) is a linear function as well. If it is positive, then \( C'_w = C_w \); if it is negative, then \( C''_w \) are given by quadratic polynomials on each residue class of \( C_w \) (mod \( N \)).

The lowering process can take the pattern all the way down below height zero as long as \( n_2 \) is sufficiently large compared to \( N \). If it is not, then the quadratic expression for \( C''_w \) in terms of \( n_1, n_2 \) is given by linear functions of \( n_1 \) for each small value of \( n_2 \).

Finally, the \((a, b)\) are linearly related to \((n_1, n_2)\) via \( (a, b) = n_1a_1 + n_2a_{1+1} + \tau \), where \( \tau \) is the sum of the corner words, so a change of basis finishes the proof.

8. Word geodesics tracking close to \( cc \) geodesics

**Theorem 26** (Realization by shapes and patterns). For every generating set \( S \), the following two equivalent conditions hold:
there is a \( K = K(S) \) such that every group element has a geodesic spelling for which the shadow is \( K \)-close to the shadow of a \( cc \) geodesic;

- there is a \( K = K(S) \) such that every group element has a geodesic spelling which is either the rearrangement of some pattern from \( \text{Patt}_K \) or the evaluation of some general shape from \( \text{Shape}_K \).

**Proof.** Suppose \( \gamma \) is a geodesic spelling in \( (H(\mathbb{Z}), S) \) evaluating to \( (a, b, c) \in H(\mathbb{Z}) \). Recall that \( N \) was defined as the least common multiple of the areas spanned by pairs of letters in the generating alphabet. Then any single neighboring generator-swap suffices, if performed enough times, to produce area changes of any multiple of \( N \). The steps will be organized to ensure that, though the height may change, it stays in the same residue class modulo \( N \). Throughout, we will be assuming \( n = \ell(\gamma) \gg N \).

First we simplify \( \gamma \) to \( \gamma_1 \) (Lemma 29) by shuffling letters, cashing in insignificant generators, and balancing lengths. We know that \( \gamma_1 \) is \( K \)-almost balanced with respect to the induced norm on \( m \). This means that it has the form \( c_{1-1}a_i^n c_1 \cdots a_j^m c_j \) and that for \( 1 < i < j < \infty \), the values \( \frac{n_i}{\sigma_i} \) and \( \frac{n_j}{\sigma_j} \) differ by at most a bounded amount and that the values \( \frac{n_i}{\sigma_i} \) and \( \frac{n_j}{\sigma_j} \) can exceed these by at most a bounded amount, though of course these values are not necessarily integral. We can rewrite such a spelling \( \gamma_1 \) as

\[
\gamma_1 = c_{1-1}^{-1} \tilde{a}_1^{-1} c_{1}^{1} \tilde{a}_{1+1}^{-1} \cdots \tilde{a}_{j-1}^{1} c_{j-1}^{-1} \tilde{a}_{j}^{-1} c_{j+1}^{-1}.
\]

In particular the projection \( \pi(\gamma_1) \) fellow-travels a \( cc \) geodesic.

Depending on the number of sides, we next apply unsimplification for shapes or patterns (Lemma 21 or 25) to obtain \( \gamma_2 \). In the pattern case, note that \( (a, b, c) \) is geodesically spelled by some rearrangement of the pattern \( w \), because the pattern was obtained in the first place by shuffling the original spelling.

To complete the proof of the Theorem for shapes, we observe that we are in one of three cases: either the height difference \( z_{2} - z_0 < (2SS)_0 \) so that we can not apply unsimplification; the unsimplification process had at least one three-sided surgery; or unsimplification had only two-sided surgery. If any three-sided surgery was performed, then our new spelling \( \gamma_2 \) evaluates to the same word as \( \gamma \) but is shorter, contradicting geodesity of \( \gamma \). If only two-sided surgery was needed, then a \( \gamma_2 \) of equal length to \( \gamma \) has been produced, but with lower eccentricity. Finally, if \( z_{2} - z_0 \) is smaller than some fixed bound, then the steps in the proof only made minor changes to \( \gamma \), and retracing the argument this implies that \( \gamma \) was boundedly close to isoperimetric at the beginning of the process.

**Example.** We will run an example to illustrate an eccentric word geodesic being improved by the shape algorithm above. Consider the standard generators, fix a value \( D \) and take \( \gamma \gg N \). Let \( \gamma \) be the closed rectangular path

\[
e_1^{N-D} e_2^{N+D} e_1^{N+D} e_2^{N-D}.
\]

This has length \( 4N \) and encloses area \( N^2 - D^2 \), so it evaluates to the group element \( (0, 0, N^2 - D^2) \). The \( cc \) geodesic reaching the same element would have length \( 4\sqrt{N^2 - D^2} \), which is strictly greater than \( 4N - 1 \) if \( N \) is large enough compared to \( D \), and this means that \( \gamma \) is a geodesic. It is already cyclically ordered and has no out-of-place letters, so \( \gamma_1 = \gamma \). Balancing the sides produces
\[ \gamma_2 = e_1^N e_2^N e_1^{-N} e_2^{-N}, \] which has area \( N^2 \). Now we perform a 2-sided surgery, replacing \( e_1^N e_2^N \) with \( e_1^{N-1} e_2^{D^2} e_1 e_2^{N^2-D^2} \). This reduces the area by \( D^2 \) while preserving length, so creates a geodesic to \((0, 0, N^2 - D^2)\) that 1-fellow-travels the cc geodesic.

\[ \gamma_2 \]
\[ \tau \]
\[ \gamma \]

**Figure 4.** Here, a word geodesic \( \gamma \) with large eccentricity is shown compared to the corresponding cc geodesic \( \tau \), which can’t be realized with integers. The algorithm *balances* \( \gamma \) and then *chips away* area to produce a geodesic \( \gamma_2 \) which evaluates to the same group element as the original \( \gamma \) but tracks close to \( \tau \).

## 9. Picking out geodesics

### 9.1. Linear comparison for shapes
We have seen that when \( \omega \) is a shape (simple or general), the map \( \text{Dom}_K \to \mathbb{Z}^3 \) induced by \( \omega \) taking \( X \mapsto (s^-, s, s^+) \mapsto (a, b, \ell) \) is injective and affine. Therefore, for a given shape \( \omega \), the inverse map \( (a, b, \ell) \mapsto s = (s^-, s, s^+) \) is an affine function on \( \omega(\text{Dom}_K) \subset \mathbb{Z}^3 \).

First, we define the domain of competition for a pair of shapes to be the inputs for which they reach the same horizontal position at nearby lengths:

\[ \text{DomComp}_K(\omega, \omega') = \{(X, X') \in \text{Dom}_K(\omega) \times \text{Dom}_K(\omega') : |\ell - \ell'| \leq K\} \]

Define a competition function \( f_{\omega\omega'} : \text{DomComp}(\omega, \omega') \to \mathbb{Z} \) to be the difference in heights, \( z(\omega(X)) - z(\omega'(X')) \). We show that if two shapes ever compete, then the domain of competition decomposes into rational families where that height difference is given by a linear function.

**Definition 27.** Given a general shape \( \omega \) of type \((i, j)\) and data \( X \) with lengths \( s = (s^-, s, s^+) \), we define the trace \( \tau = \tau(\omega(X)) \) to be the corresponding I-arc \( \tau = a_1^s a_{i+1}^s \ldots a_j^s a_{j-1}^s a_j^s \).

That is, \( \tau \) is equal to \( \omega(X) \) with the break words deleted and the exponent differentials erased. Observe that by construction,

- the dependence of \( \tau \) on \( X \) factors through \((s^-, s, s^+)\);
- \( \tau \) begins at \( 0 \in \mathfrak{m} \), and synchronously fellow-travels \( \omega(X) \) with a fellow-traveller constant which depends only on \( \omega \) and is independent of \( X \); and
for a given $\omega$ the difference between the endpoint of $\omega(X)$ and the endpoint of $\tau$ is independent of $X$, i.e., is constant on $\text{Dom}_K(\omega)$. This is because this difference depends only on the $b$ and $c$ data from $\omega$.

**Lemma 28** (Linear comparison for shapes). If $\text{DomComp}_K(\omega, \omega')$ is nonempty, then there is a finite partition such that each piece $U_\delta \subset \text{DomComp}_K(\omega, \omega')$ is defined by linear equations, linear inequalities, and congruences, and the comparison function $f_{\omega\omega'}|_{U_\delta}$ is linear.

**Proof.** We will partition the domain of competition into pieces for each $-K \leq \delta \leq K$ consisting of the subset of positions $(a,b)$ reached by $\omega$ at some length $\ell$ and by $\omega'$ at length $\ell + \delta$. Call this subset $U_\delta$. Let $(s^-,s,s^+)$ and $(t^-,t,t^+)$ denote the length data extracted from $X$ and $X'$ respectively.

Fixing this $\delta$, we first consider the case where $\omega$ and $\omega'$ have the same combinatorial type, that is, $i = i'$, $j = j'$. Further, if $i = j$, recall that we have restricted the domain so that $s^+ = t^+ = 0$. We claim that the trace $\tau'$ fellow travels $\tau$ in projection and that the distance between corresponding sides is independent of $a$, $b$ and $\ell$. This is because the affine maps

\[
(s^-,s,s^+) \mapsto (a,b,\ell) \quad (t^-,t,t^+) \mapsto (a,b,\ell + \delta)
\]

have the same linear part, hence so do their inverses. Thus, $(s^- - t^-, s - t, s^+ - t^+)$ is constant on the domain of competition, which ensures fellow-traveling.

It follows that the area between the traces is linear on $U_\delta$, as is the area between each of the shapes and its respective trace. (Area between two planar paths with different endpoints is measured by closing up with a straight chord.) Clearly the boost of each shape is also linear on $\text{DomComp}$. Thus $f_{\omega\omega'}$ is linear for each value of $\delta$ in the case where $\omega$ and $\omega'$ have the same combinatorial type. (Notice that in the case where $\tau$ and $\tau'$ might a priori differ as in the second case of Lemma 13 because of our restriction to $\text{Cone}_0$, they actually fellow-travel and the argument goes through.)

Next, consider the case where $\tau$ and $\tau'$ are almost the same combinatorial type. In this case $\omega(X)$ and $\omega'(X')$ are also of almost the same combinatorial type. For specificity let us consider the case where $i + 1 = i'$, $j = j'$, and $s^-$ and $t - t^-$ are both bounded, so that there are only finitely many possible pairs $(s^-, t - t^-)$. For any such pair, the subset of $U_\delta$ realizing that pair is defined by linear equations. If we fix those values—i.e., treat $a_i^- a_i^{i+1}$ as a break word in $\omega(X)$—we can define new traces of the same combinatorial type and appeal to the case above.

Finally, we turn to the case where $\tau$ and $\tau'$ end close to the origin and have different types. Here, $\omega$ and $\omega'$ can only compete when $\tau$ and $\tau'$ are close to being the full polygon. But this implies that there are finitely many values $(a,b)$ for which they compete. Furthermore, the set of $(X, X')$ mapping to each of these finitely many $(a,b)$ is determined by linear equalities and inequalities. For each such $(a,b)$, the areas of $\omega(X)$ and $\omega(X')$ differ from a full isoperimetrix of scale $s$ by amounts which are linear in $X$ and $X'$ respectively. Thus their areas differ from each other by amounts which are linear in $X$ and $X'$, and once again their respective boosts are also linear in $X$ and $X'$. The result now follows. \hfill $\square$

**9.2. Testing geodesity for shapes.** Consider the set

\[
\{(a,b,w_n + j) : n \geq n_0(a,b), \quad 1 \leq j \leq w_{n+1} - w_n\},
\]
containing the elements of the Heisenberg group in the (positive) regular range, i.e., the set of \((a, b, c) \in H(\mathbb{Z})\) with \(c > W = w_n(a, b)\). By Bounded Difference (Cor \[8\]), such an element \((a, b, w_n + j)\) has word length between \(n + 1\) and \(n + K\).

Since \(\text{Shape}_K\) is a finite set, we can fix an arbitrary ordering of its elements.

**Definition 29.** For a general shape \(\omega\), let \(G_{\omega}^+(n)\) be the set of \((a, b, j) \in \mathbb{Z}^3\) such that \(1 \leq j \leq w_{n+1} - w_n\) and \(\omega\) is the first shape to geodesically realize \((a, b, w_n + j)\) at length \(n + \Delta\).

**Theorem 30** (Deciding geodesity for shapes). For each shape \(\omega\) and each \(0 \leq \Delta \leq K\), the \(G_{\omega}^+(n)\) form a bounded rational family in \(\mathbb{Z}^3\).

**Proof.** We will show that membership in \(G_{\omega}^+(n)\) is tested by finitely many linear equations, linear inequalities, and congruences.

For a shape \(\omega\), consider

\[A_{\omega}(n) = \{(a, b) : \omega \text{ produces a spelling of length } n \text{ over } (a, b)\}.

To see that \(A_{\omega}(n)\) is a rational family, recall that \(X \mapsto \ell(\omega(X)) = n\) is an affine map. Thus the sets \(\{X \in \text{Dom}_{\omega}(\omega) : \ell(\omega(X)) = n\}\) constitute a rational family. The map \(X \mapsto (a, b)\) is also affine and thus the sets \(A_{\omega}(n)\) are the affine pushforwards of a rational family and hence themselves rational.

For each shape \(\omega\) consider

\[H_{\omega}(n) = \{(a, b) : \omega \text{ realizes the highest-height element } (a, b, w_n) \text{ at length } n\},

which is empty unless \(\omega\) is a simple shape. We claim that for each simple shape \(\omega\), \(H_{\omega}(n)\) is a rational family. For each \((a, b) \in A_{\omega}(n)\), \(\omega\) fails to produce the highest-height element if either there is \(\omega'\) producing a higher element over \((a, b)\) at length at most \(n\). However, since \(w_n - 2 < w_n\) this only needs to be tested for length \(n + 1\). Thus, for each potential competitor \(\omega'\) only two inequalities need to be tested. But these are tested by the linear inequality \(f_{\omega, \omega'}(X, X') \geq 0\) at \(\delta = 0\) and \(\delta = -1\). It follows that the sets \(H_{\omega}(n)\) form a rational family as claimed.

Now for each pair of shapes \(\alpha\) and \(\beta\), note that

\[H_{\alpha}(n + 1) \cap H_{\beta}(n) = \{(a, b) : \beta \text{ realizes } w_n(a, b) \text{ and } \alpha \text{ realizes } w_{n+1}(a, b)\}.

This is a rational family picking out positions at which \(\alpha\) is highest-height at length \(n + 1\) and \(\beta\) is highest-height at length \(n\). Given \((a, b)\), we can search the finite list of shapes to find such a pair, and then \((a, b, n)\) affinely determine \(s_\alpha\) and \((a, b, n+1)\) determine \(s_\beta\) so that \(\alpha(s_\alpha)\) and \(\beta(s_\beta)\) are the highest-height paths. Thus, we can test the requirement that \(j\) satisfy \(1 \leq j \leq w_{n+1}(a, b) - w_n(a, b)\) using equations which are linear in our data by seeing whether there exist shapes \(\alpha, \beta\) for which \(j \leq f_{\alpha, \beta}(s_\alpha, s_\beta)\) at \(\delta = 1\).

The requirement that \(\omega\) realizes \((a, b, w_n + j)\) at length \(n + \Delta\) is similarly tested by \(j = f_{\omega}(s, s_\beta)\) at \(\delta = \Delta\), i.e., by linear equalities and inequalities.

Finally, for any \(\omega\) which realizes \((a, b, w_n + j)\) at length \(n + \Delta\), we must test whether this is geodesic, i.e., whether this length is shortest-possible. This is accomplished by testing all potential competitors \(\omega'\) at lengths \(\Delta' < \Delta\). This is finitely many competitors \(\omega'\) and finitely many values \(\Delta'\), and therefore determined by finitely many linear equalities and inequalities.

Finally, to see that \(\omega\) is the lowest-numbered shape to produce such a geodesic, we simply check \(\omega' < \omega\) at length \(\Delta\).

\(\square\)
9.3. Linear comparison for patterns. We will establish linear competition for patterns as we did for shapes above. For patterns $w$ and $w'$, define

$$\text{DomComp}(w, w') = \{(n_1, n_2, n'_1, n'_2) : (a, b) = (a', b')\},$$

requiring that both paths end at the same horizontal position. Notice that on DomComp($w, w'$), the length difference $\ell(w(n_1, n_2)) - \ell(w'(n'_1, n'_2))$ is constant.

**Lemma 31** (Linear comparison for patterns). The comparison functions $\max I_w - \max I_{w'}$ and $\min I_w - \min I_{w'}$ are affine on a finite partition of DomComp($w, w'$).

Equivalently, these can be regarded as affine functions on $(n_1, n_2)$, or affine in $(a, b)$ on those $(a, b)$ whose fibers are reached by both $w$ and $w'$.

**Proof.** The three statements are equivalent because on the appropriate sets, each of the three quantities, $(n_1, n_2, n'_1, n'_2), (n_1, n_2)$ and $(a, b)$ determines the other two by an affine map. Linearity follows from the fact that the tops of the intervals are given piecewise by quadratic polynomials with the same leading coefficient, and the bottoms of the intervals are piecewise linear, over finitely many rational families that partition DomComp. □

9.4. Testing geodesity for patterns. Each pattern $w$ is easily seen to determine maps from $(n_1, n_2)$ to length, horizontal position, and $\ell(w(n_1, n_2)) - n_0$. Notice that for each $w$, $(a, b, \ell)$ is an affine function of $(n_1, n_2)$, and the map $(n_1, n_2) \mapsto (a, b)$ is injective.

Since Patt$_K$ is a finite set, we can fix an arbitrary ordering of patterns as we did for shapes.

**Definition 32.** For a pattern $w$, let $G^\Delta_w(n)$ be the set of $(a, b) \in \mathbb{Z}^2$ such that $n = n_0(a, b) = |(a, b)|_S$ and $w$ realizes some $(a, b, c)$ at length $n + \Delta$.

**Lemma 33** (Positions reached by patterns). For each shape $w$ and each $0 \leq \Delta \leq K$, the $G^\Delta_w(n)$ form a bounded rational family in $\mathbb{Z}^2$.

**Proof.** The set of $(a, b)$ reached by $w$ is the push-forward under an affine map of the set of non-negative pairs $(n_1, n_2)$. Now observe that in the $i$th sector of the plane, the length $n_0 = n_0(a, b)$ is a periodic linear function in which the linear coefficient is independent of $(a, b)$ and the constant term depends on the congruence class of $(a, b)$ modulo the group generated by $a_1$ and $a_{i+1}$. Note also that if $w(n_1, n_2)$ ends over $(a, b)$, then $(n_1, n_2)$ and $(a, b)$ are affine functions of each other. Of course then length of $w(n_1, n_2)$ is an affine function of $(n_1, n_2)$. Thus the difference $\ell(w(n_1, n_2)) - n$, which gives $\Delta$, is a periodic function, and the result follows. □

**Corollary 34** (Counting with patterns). For each $w$ and $0 \leq \Delta \leq K$ there are polynomials $p^\Delta_w(a, b, n)$ of degree at most two such that for $(a, b) \in G^\Delta_w(n)$ the number of group elements $(a, b, c)$ with $c \geq 0$ geodesically spelled by $w$ at length $n + \Delta$, and by no smaller-numbered pattern, is given by $p^\Delta_w(a, b, n)$.

**Proof.** Clearly, the unstable elements of length $n_0 + \Delta$ are those reached by some pattern $w$ at length $n_0 + \Delta$ but not by $w'$ with length $n_0 + \Delta'$ for any $\Delta' < \Delta$.

The $p_w$ are defined by making the comparisons of the interval $I_w$ against competing intervals $I_{w'}$, and enumerating the points over $(a, b)$ assigned to $w$ as a finite sum/difference of the appropriate quadratic polynomials. □
10. The growth series

The growth series of \((H, S)\) is now given as follows. The generators \(S\) determine a constant \(K\) so that the positive-height regular elements are enumerated by

\[ S^{\text{reg}}(x) = \sum_{\omega} \sum_{n=0}^{\infty} \sum_{\Delta=0}^{K} \sum_{G_{\omega}(n)} x^\Delta x^n, \]

where \(\omega \in \text{Shape}_K\) are the shapes described above.

The series enumerating unstable elements with \(c \geq 0\) is

\[ S^{\text{uns}}(x) = \sum_{w} \sum_{n=0}^{\infty} \sum_{\Delta=0}^{K} \sum_{G_{w}(n)} w(n) p_w(a, b, \Delta) x^\Delta x^n, \]

where \(w \in \text{Patt}_K\) are the patterns described above. The difference in appearance between the two expressions comes from the fact that regular geodesics of a certain length only hit each fiber in a single point, while unstable geodesics may hit in an interval of size that is quadratic in the length.

Both series are rational by Corollary 4, because \(\text{Shape}_K\) and \(\text{Patt}_K\) are finite sets, the \(G(n)\) are bounded rational families, and the \(p\) are polynomial. We then appeal to the height-reversing bijection \(g \mapsto g^{-1}\) to similarly count the elements of negative height. This double-counts the elements at height zero.

Lemma 35 (Zero-height elements). Let \(\sigma^0(n) = \#\{(a, b, 0) : |(a, b, 0)|_S = n\}\) be the spherical growth function of height-zero elements. Then \(S^0(x) = \sum \sigma^0(n) x^n\) is rational.

Proof. The fiber over \((a, b)\) has an element with \(c = 0\) if and only if \(ab\) is even. Thus, our problem reduces to counting the set of such \((a, b) \in \mathbb{Z}^2\) with respect to the generating set \(\pi(S)\). It is well-known that the set of lex-least geodesics in an abelian group is a regular language. Those ending at an element \((a, b)\) with \(ab\) even is a regular subset of these. The set in question therefore has rational growth. \(\square\)

Finally, we have

\[ S(x) = 2 \cdot S^{\text{reg}}(x) + 2 \cdot S^{\text{uns}}(x) - S^0(x). \]

This establishes that the spherical growth series \(S(x)\) and thus also the growth series \(B(x)\) is rational for any finite generating set of \(H(\mathbb{Z})\), finishing Theorem 1.

11. Remarks and questions

11.1. Languages. Each shape defines a language \(L(\omega)\). For \(j > 1 + 1\), these languages are not regular. For \(j > 1 + 2\), they are not context-free.

This is attributable to non-commutativity: what could be accomplished with a bounded counter if the group were abelian is a non-regular language otherwise. For instance, \(\{a^nb^n\}\) is non-regular, even though \(\{(ab)^n\}\) enumerates words with the same letters. The words represented by our shapes of geodesics need to be nearly balanced, and this breaks regularity.

It was pointed out to us by Cyril Banderier that a recursion with positive integer coefficients implies the existence of some regular language enumerated by the function, though not necessarily the language of geodesics for \((G, S)\). This holds in the special case of \((H, \text{std})\), which is extremely intriguing.
11.2. **Cone types.** We recall the definition of cone type from [8].

**Definition 36.** Consider the Cayley graph $\text{Cay}(G, S)$ of group $g$ with generating set $S$. Given $g \in G$, the cone at $G$, denoted $C(g)$, consists of all paths $\sigma$ based at $g$ with the property that word length $|\sigma(t)|$ is strictly increasing along $\sigma$. The cone type of $g$ consists of the cone of $g$ translated to the origin, i.e., $g^{-1}(C(g))$.

For $\text{Cay}(G, S)$ to have finitely many cone types is almost exactly the same thing as having the language of geodesics in $\text{Cay}(G, S)$ be a regular language. If $\text{Cay}(G, S)$ has finitely many cone types, these cone types can be used as the states of a finite state automaton which accepts the language of geodesics. This is because the cone type of $G$ tells us exactly which generators are outbound at $g$. However, the cone type of $g$ encodes additional information, namely which edges are “half outbound”: if an edge $e$ of $\text{Cay}(G, S)$ connects two elements $g$ and $g'$ with $|g| = |g'| = n$, then the midpoint of this edge is at distance $n + \frac{1}{2}$ from the origin. As far as we know, it is not known whether there is a group with a Cayley graph where the language of geodesics is regular, but which has infinitely many cone types.

From the shape theorem we easily recover the (already known) fact that $H$ has infinitely many cone types in every generating set. In particular, it has no generating set where the language of geodesics is regular.

To see this, just note that there are infinitely many possibilities for how long a geodesic continues in a particular significant direction before turning to the successive direction, depending on what shape has reached the point $g = (a, b, c)$ at what scale.

Brian Rushton has pointed out to us that the presence of infinitely many cone types implies that there is no associated subdivision rule. (See [23].)

11.3. **Open questions.**

11.3.1. **Scope of rational growth in the nilpotent class.** Our argument should carry through with small modifications for groups that are virtually $H(\mathbb{Z}) \times \mathbb{Z}^d$. We know from Stoll’s result that not all two-step groups have rational growth, even with respect to their standard generators. However it is possible (for instance) that free nilpotent groups do.

**Question 37.** Which nilpotent groups have rational growth in all generating sets?

On the other hand, one could try to mimic and extend the Stoll construction.

**Question 38.** Does every nilpotent group have rational growth with respect to at least one generating set? In the other direction, for which nilpotent groups is the fundamental volume transcendental for standard generators (which would rule out rationality)?

11.3.2. **Period and coefficients.** In the polynomial range (i.e., $f(n) \leq An^d$ for some $A, d$), rational growth is equivalent to the property that $f(n)$ is eventually quasi-polynomial, i.e., there are a finite period $N$, polynomials $f_1, \ldots, f_N$, and a threshold $T$ such that

$$n \geq T, \quad n = kN + i \implies f(n) = f_i(n).$$
For example, Shapiro’s computation of the spherical growth for the Heisenberg group with standard generators showed it to be eventually quasipolynomial of period twelve, and in fact only the constant term oscillates:

$$\sigma(n) = \frac{1}{18} (31n^3 - 57n^2 + 105n + c_n),$$

where \(c_n = 0, -7, -14, 9, -16, -23, 18, -7, 32, 9, 2, -23,\) and then repeats mod 12, for \(n \geq 1\).

It follows that the (ball) growth function \(\beta(n) = \sum_{k=0}^{n} \sigma(k)\) is also quasipolynomial of period twelve, with only its constant term oscillating. We note that this implies that the growth function for standard generators is within bounded distance of a true polynomial in \(n\).

Preliminary calculations indicate that several other generating sets also have the property that only the constant terms oscillate; in these examples, the periods relate both to the sidedness of the fundamental polygon and to the index of the sublattice of \(\mathbb{Z}^2\) generated by its extreme points.

**Question 39.** How does the generating set \(S\) determine the period of quasipolynomiality of the growth function? Which coefficients oscillate? We know that the top coefficient of \(\beta(n)\) is the volume of the \(cc\) ball; is the second coefficient well-defined, and if so is it a “surface area”? Are all growth function bounded distance from polynomials?

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