CONVERGENCE OF A SEMI-LAGRANGIAN SCHEME FOR THE ELLIPSOIDAL BGK MODEL OF THE BOLTZMANN EQUATION

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ABSTRACT. The ellipsoidal BGK model is a generalized version of the original BGK model designed to reproduce the physical Prandtl number in the Navier-Stokes limit. In this paper, we propose a new implicit semi-Lagrangian scheme for the ellipsoidal BGK model, which, by exploiting special structures of the ellipsoidal Gaussian, can be transformed into a semi-explicit form, guaranteeing the stability of the implicit methods and the efficiency of the explicit methods at the same time. We then derive an error estimate of this scheme in a weighted $L^\infty$ norm. Our convergence estimate holds uniformly in the whole range of relaxation parameter $\nu$ including $\nu = 0$, which corresponds to the original BGK model.

1. INTRODUCTION

The BGK model [8, 49] has been widely used as an efficient model for the Boltzmann equation because the BGK model is much more amenable to numerical treatment, and still maintains many of the important qualitative properties of the Boltzmann equation. But several shortcomings are also reported where this model fails to reproduce the correct physical data of the Boltzmann equation. One such example is the Prandtl number, which is defined as the ratio between the viscosity and the heat conductivity. The Prandtl number computed via the BGK model does not match with the one derived from the Boltzmann equation, resulting in the incorrect hydrodynamic limit at the Navier-Stokes level. To overcome this, Holway [28] suggested a generalized version of the BGK model by replacing the local Maxwellian with an ellipsoidal Gaussian parametrized by a free parameter $-\frac{1}{2} < \nu < 1$. This model is called the ellipsoidal BGK model (ES-BGK model), whose initial value problem reads

$$
\partial_t f + v \cdot \nabla f = \frac{1}{\kappa} A_\nu (M_\nu(f) - f),
$$

$$
f(x, v, 0) = f_0(x, v).
$$

(1)

The velocity distribution function $f(x, v, t)$ is the number density of the particle system on the phase point $(x, v) \in \mathbb{T}^{d_1} \times \mathbb{R}^{d_2}$ ($d_1 \leq d_2$) at time $t \in \mathbb{R}_+$. Here, $\mathbb{T}$ denotes the unit interval with periodic boundary condition and $\mathbb{R}$ is the whole real line. The Knudsen number $\kappa$ is a dimensionless number defined by the ratio between the mean free path and the characteristic length. For later convenience, we allowed a slight abuse of notation so that the convection term $v \cdot \nabla_x f$ is understood as

$$
v \cdot \nabla_x f = \sum_{1 \leq i \leq d_1} v^i \partial_{x_i} f.
$$

The collision frequency $A_\nu$ takes various forms depending on modeling assumptions. In this paper, we only consider the fixed collision frequency: $A_\nu = (1 - \nu)^{-1}$. The ellipsoidal Gaussian $M_\nu(f)$ reads:

$$
M_\nu(f) = \frac{\rho}{\sqrt{\det(2\pi T_\nu)}} \exp \left( -\frac{1}{2} (v - U)^\top T_\nu^{-1} (v - U) \right),
$$

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where the macroscopic density, velocity, temperature and the stress tensor are defined by
\[
\rho(x, t) = \int_{\mathbb{R}^d_2} f(x, v, t) dv,
\]
\[
\rho(x, t)U(x, t) = \int_{\mathbb{R}^d_2} f(x, v, t) v dv,
\]
\[
d_2 \rho(x, t) T(x, t) = \int_{\mathbb{R}^d_2} f(x, v, t) |v - U(x, t)|^2 dv,
\]
\[
\rho(x, t) \Theta(x, t) = \int_{\mathbb{R}^d_2} f(x, v, t) (v - U) \otimes (v - U) dv.
\]
The temperature tensor \( T_\nu \) is given by a convex combination of \( T \) and \( \Theta \):
\[
T_\nu = (1 - \nu) T Id + \nu \Theta,
\]
where \( Id \) is the \( d \times d \) identity matrix. The ellipsoidal relaxation operator satisfies the following cancellation property:
\[
\int_{\mathbb{R}^d_2} (\mathcal{M}_\nu(f) - f) (1, v, |v|^2) dv = 0,
\]
which leads to the conservation of mass, momentum and energy:
\[
\frac{d}{dt} \int_{\mathbb{R}^d_2} f dx dv = \frac{d}{dt} \int_{\mathbb{R}^d_2} v f dx dv = \frac{d}{dt} \int_{\mathbb{R}^d_2} f |v|^2 dx dv = 0.
\]

When Holway first suggested this model, \( H \)-theorem was not verified, which was the main reason why the ES-BGK model has been neglected in the literature until very recently. It was resolved in [2] (See also [10, 53]):
\[
\frac{d}{dt} \int_{\mathbb{R}^d_2} f \ln f dx dv \leq 0,
\]
and ignited the interest on this model [1, 10, 22, 25, 31, 32, 44, 51, 52, 53, 54].

It can be verified via the Chapman-Enskog expansion that the Prandtl number computed using the ES-BGK model is \( 1/(1 - \nu) \). Therefore, the correct physical Prandtl number can be recovered by choosing appropriate \( \nu \), namely, \( \nu = 1 - 1/Pr \approx -1/2 \), where \( Pr \) denotes the correct Prandtl number. When \( \nu = 0 \), the ES-BGK model reduces to the original BGK model. Hence, any results for the ES-BGK model automatically hold for the original BGK model either. We also mention that, in the range \(-1/2 < \nu < 1\), the only possible equilibrium state of the ellipsoidal relaxation operator is the usual Maxwellian, not the ellipsoidal Gaussian. That is, the only solution satisfying the relation \( M_\nu(f) = f \) is the local Maxwellian (See [51, 53] for the proof):
\[
f = M_0(f) = \frac{\rho}{\sqrt{(2\pi T)^d}} \exp \left( -\frac{|v - U|^2}{2T} \right).
\]
Therefore, the ES-BGK model correctly captures the two most important asymptotic behavior of the Boltzmann equation, namely, the time asymptotic limit \( (t \to \infty) \) and the hydrodynamic limit at the Euler level \( (\kappa \to 0) \).

In recent years, semi-Lagrangian (SL in short) methods for the numerical solutions of kinetic equations have attracted the attention of several authors, see for example, the papers [5, 11, 12, 17, 18, 20, 21, 24, 32, 33, 34, 35, 43, 44, 50] on the use of SL schemes for the Vlasov-Poisson equations and the works [20, 23, 27, 33, 37] on SL schemes for the BGK model. SL methods are very attractive, since they allow one to use large time step, with no CFL-type accuracy restriction typical of Eulerian-based schemes. In the limit of small Knudsen number, they can capture the underlying fluid dynamic limit with an implicit L-stable scheme adopted for the treatment of the collision term. It is also relatively easy to obtain high accuracy, by using high order reconstruction of the solution at the foot of the characteristics. One of the difficulties with semi-Lagrangian schemes is that they are
naturally constructed in non-conservative form. Several papers have been devoted to the construction of conservative SL schemes (see, for example, [19, 33, 36, 50]). SL schemes have also been used to solve specific problems where accurate solutions are needed. In [41], the authors adopt a SL method to study the decay of an oscillating plate in a rarefied gas described by the BGK model and compare the results with those obtained by a scheme specifically designed for the problem. For a survey on numerical schemes on BGK model, we refer to the review paper [21, 30] and references therein.

In view of the increasing interest in the subject, our aim is to introduce a new family of semi-Lagrangian scheme for the ES-BGK model and to investigate their convergence properties. Numerical results of the schemes will be presented in a separate paper.

1.1. Implicit semi-Lagrangian scheme. First, we propose an implicit semi-Lagrangian scheme. In the numerical computation of the collisional or relaxational kinetic equations, it is common to employ the so called splitting method, which amounts to computing the transport part:

\[ \frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0, \]  
and the relaxational time evolution:

\[ \frac{df}{dt} = \frac{1}{\kappa} A_v \{ \mathcal{M}_\nu(f) - f \}, \]  
separately. The most naive way for this is to use the forward in time methods and the explicit Euler type method respectively for (2) and (3). It is, however, well-known that the first procedure leads to the restriction on the temporal grid size due to the CFL condition: \( \Delta t < \Delta x / \max_j |v_j| \), while the second procedure entails a stability condition: \( \Delta t < C\kappa \) for some constant \( C > 0 \), resulting in the following two scale restriction on the size of time step:

\[ \Delta t < \min \left\{ C\kappa, \Delta x / \max_j |v_j| \right\}. \]

Since the two parameters of the right-hand side are independent of each other, the discrepancy between these two scale can get arbitrary large, and the scheme becomes severely resource-consuming accordingly. Such a stiffness problem has been one of the key difficulties in developing efficient stable schemes for kinetic equations. In this paper, we propose a new semi-Lagrangian scheme, which combines two numerical methods known to guarantee stable performances, namely the semi-Lagrangian treatment for the transport part (2), and the implicit Euler for the evolution part (3), to overcome the CFL restriction and secure the stability of the scheme over the large range of Knudsen number at the same time:

\[ f_{n+1}^{i,j} - f_n^{i,j} = \frac{1}{\kappa} A_v \{ \mathcal{M}_{\nu,j}(f_{n+1}^{i}) - f_{n+1}^{i,j} \}. \]

Here \( f_{n}^{i,j} \) denotes the linear reconstruction. (See Definition 2.1.) At first sight, the scheme seems very time consuming due to the implicit implementation of the relaxation part. In the case of the original BGK model (\( \nu = 0 \)), such difficulties can be circumvented by a clever trick using the fact that (1) the local Maxwellian depends on the distribution function only through the conservative macroscopic fields, and (2) the macroscopic fields satisfy the following identities:

\[ \rho_{i}^{n+1} = \tilde{\rho}_{i}^{n}, \quad U_{i}^{n+1} = \tilde{U}_{i}^{n}, \quad T_{i}^{n+1} = \tilde{T}_{i}^{n}, \]

with small error, enabling one to explicitly solve for the numerical solutions [23, 33]. Here, the macroscopic variables with tilde are those constructed from \( \tilde{f}_{n}^{i,j} \) (see Section 2.) In this surprising turn of events, the two seemingly contradicting properties: the stable performance of the implicit scheme and the efficiency of the explicit scheme, are reconciled. Such nice feature, of course, can never be expected for the Boltzmann type collision operators.

In the case of the ES-BGK model, however, the conservation laws are not sufficient to make this trick work, since the ellipsoidal Gaussian contains the stress tensor, which is not a conserved quantity.
Even though we cannot expect the stress tensor to satisfy similar conservation identities as \( \Theta \), we observe that the following approximation holds with small error (See \( \Theta \) in Section 2):

\[
\Theta_{n+1} \approx \frac{\Delta t}{\kappa + \Delta t} \tilde{T}_i^n T_i + \frac{\kappa}{\kappa + \Delta t} \hat{\Theta}_i^n,
\]

which enables us to rewrite the implicit ellipsoidal Gaussian in a semi-explicit manner. The resultant scheme can now be written in an explicit manner as (See Section 2.2)

\[
f_{i,j}^{n+1} = \frac{\kappa}{\kappa + \Delta t} \tilde{f}_{i,j}^n + \frac{A_v \Delta t}{\kappa + \Delta t} \mathcal{M}_{i,j}(\tilde{f}_i^n).
\]

We remark that the implementation of the scheme and its check on several numerical tests will be reported in an independent paper \([9]\).

1.2. \( L^\infty \) convergence theory. We then develop a convergence theory for this scheme that will guarantee the credibility of the method. In this regard, our main result is the following error estimate stated in Theorem 3.2 in Section 3:

\[
\|f(T_f) - f_{n}^N\|_{L^\infty} \leq C \left\{ \sum_{x} \Delta x + \Delta v + \Delta t + \frac{\Delta x}{\Delta t} \right\},
\]

for some constant \( C > 0 \). Compared to our previous result \([37]\) where the convergence of a semi-Lagrangian scheme for the original BGK model was established in \( L^1 \) space:

\[
\|f(T_f) - f_{n}^N\|_{L^1} \leq C \left\{ \sum_{x} \Delta x + \Delta v + \Delta t + \frac{\Delta x}{\Delta t} \right\},
\]

we make four improvements. The error estimate in the spatial node is improved from \( \Delta x \) to \( (\Delta x)^2 \) by assuming additional regularity on the initial data and refining the analysis of the interpolation part. This enables one to recover the first order error estimate by choosing \( \Delta = \Delta t \). Secondly, we impose size restriction only on the velocity nodes, whereas in \([37]\), we needed to restrict the size of all the node size: \( \Delta x, \Delta v \) and \( \Delta t \). Thirdly, we develop a theory to measure the error in a weighted \( L^\infty \) norm instead of the weighted \( L^1 \) norm, which provides a more clear and detailed picture on the convergence of the scheme, since the error estimate in \( L^\infty \) norm gives a node-wise convergence estimate. Finally, the proof is greatly simplified, enabling one to extend the convergence theory to the whole range of relaxation parameter \(-1/2 < \nu < 1\). As a result, we derive a convergence estimate that is uniform in \(-1/2 \nu < \nu < 1\) uniformly, it also holds for the original BGK model, which corresponds to \( \nu = 0 \).

The most important step of the proof is the derivation of the following uniform stability estimate (For notations, see the next subsection):

\[
C_{0,1} e^{-\frac{\Delta t}{\kappa} T_f} e^{-C_{0,1} |v| \alpha} \leq f_{i,j}^n \leq e^{-\frac{(C_{M}-1)\Delta t}{\kappa + A_v \Delta t}} \| f^0 \|_{L^\infty} (1 + |v_j|)^{-q},
\]

which comes from the uniform-in-\( n \) control of the discrete ellipsoidal Gaussian in a weighted \( L^\infty \) norm (See Lemma 5.1):

\[
\| M_{\rho}(f^n) \|_{L^\infty} \leq C_M \| f^n \|_{L^\infty}.
\]

Two technical issues arise in the process of obtaining the above estimates. First, we need to show that the discrete temperature tensor \( \tilde{T}_{\nu,i}^n \) is strictly positive definite uniformly in \( n \):

\[
k^\top (\tilde{T}_{\nu,i}^n) k \geq C_{\nu,q,n,T_f} > 0 \quad \text{for all} \ \kappa \in S^2.
\]

Otherwise, since the discrete ellipsoidal Gaussian involves the inverse of \( \tilde{T}_{\nu,i}^n \) and \( \det(\tilde{T}_{\nu,i}^n) \), it may blow up as \( \tilde{T}_{\nu,i}^n \) approaches arbitrarily close to zero. Second issue is more subtle. It turns out that we need to show that ratios such as

\[
\frac{\tilde{\rho}_i^n \tilde{T}_{i}^n}{\| f^n \|_{L^\infty}}, \quad \| f^n \|_{L^\infty}/\tilde{\rho}_i^n
\]
remain strictly larger than $\triangle v$, to guarantee the existence of proper decomposition of the macroscopic fields to derive necessary moment estimates. (See Lemma 5.6.) The restriction on the velocity node in Theorem 3.2 mostly comes from this subtle technical issue.

1.3. Notation. Before we finish this introduction, we summarize the notational convention kept throughout this paper:

- $C$ denotes generic constants. The exact value may change in each line, but they are explicitly computable in principle.
- We will use lower indices $n$, $i$, $j$ exclusively for time, space and velocity variable respectively. For example $x_i$, $v_j$, $t_n$.
- We use upper indices for the components of vectors as $v = (v^1, v^2, v^3)$, while the lower indices are reserved for the spatial, velocity and temporal nodes.
- $T^f$ will denote the final time, whereas $T_f$ represents for the local temperature constructed from the distribution function $f$.
- We use the following notation for weighted $L^\infty$-Sobolev norm for continuous solution:
  \[ \| f \|_{L^\infty_q}^\alpha = \sup_{x,v} |f(x,v)(1 + |v|)^\alpha|, \quad \| f \|_{W^\ell,q,\infty} = \sum_{|\alpha|+|\beta|\leq \ell} \| \partial^\alpha_x \partial^\beta_v f \|_{L^\infty_q}, \]
  where $\alpha, \beta \in \mathbb{Z}_+$ and, the differential operator $\partial^\alpha_x$ stands for $\partial^\alpha_x = \partial^\alpha_x \partial^\beta_v$.
- For any sequence $a^n_{i,j}$, we use the following notation for the weighted $L^\infty$-norm for of the sequence:
  \[ \| a^n \|_{L^\infty_q} = \sup_{i,j} |a^n_{i,j}(1 + |v_j|)^q|. \]
- In view of the above norms, $\| f(t_n) - f^n \|_{L^\infty_q}$ is understood, with a slight abuse of notation, as
  \[ \| f(t_n) - f^n \|_{L^\infty_q} = \sup_{i,j} \left| \left\{ f(x_i,v_j,t_n) - f^n_{i,j} \right\}(1 + |v_j|)^q \right|, \]
  for simplicity.

The paper is organized as follows. In the following Section 2, we derive a semi-Lagrangian scheme for the ES-BGK model. The main convergence result of this paper is presented in Section 3. In section 4, we establish some technical lemmas. Section 5 is devoted to the stability estimate of the scheme. In Section 6, we transform the ES-BGK model to a form consistent with our scheme. In Section 7, we estimate the discrepancy of discrete Gaussian and the continuous one. Finally, we prove the convergence of our scheme in Section 7.

2. Description of the numerical scheme

We fix $d_1 = 1$ and $d_2 = 3$ case with periodic boundary condition throughout this paper in order to stay in the simplest possible framework. We believe that the analysis of this paper can be extended to more general conditions such as higher dimensions in $x$ and/or different boundary conditions, although such extensions may give rise to unexpected difficulties. This will be a topic of future work. Note that, in contrast to the original BGK model, the velocity domain must be at least 2-dimensional for the ellipsoidal BGK to be meaningful. Otherwise the model reduces to the original BGK model. We choose a constant time step $\triangle t$ with final time $T^f$. The spatial domain and the velocity domain are divided into uniform grids with mesh size $\triangle x$, $\triangle v$ respectively:

\[
\begin{align*}
t_n &= n\triangle t, \quad n = 0, 1, \ldots, N_t, \\
x_i &= i\triangle x, \quad i = 0, \pm 1, \ldots, \pm N_x, \pm (N_x + 1), \ldots \\
\end{align*}
\]

where $N_t\triangle t = T^f$, $N_x\triangle x = 1$, and

\[ v_j = (v_{j1}, v_{j2}, v_{j3}) = (j_1\triangle v, j_2\triangle v, j_3\triangle v), \quad j = (j_1, j_2, j_3) \in \mathbb{Z}^3. \]

Note that the spatial node is defined on the whole line instead of unit interval, even though we are considering periodic problem. Periodicity will be imposed on the initial data $f_0$, which is defined on
we then approximate \( f \approx \) and energy at the discrete level (throughout this subsection in the semi-Lagrangian setting in [23, 27, 37]. We first impose the conservation of mass, momentum such as the stability property. This idea seems to trace back to [18, 33], and successfully implemented. We attempt to convert (6) into a semi-explicit scheme keeping beneficial features of implicit schemes.

2.1. Derivation of the scheme (13). Let \( f_j = f(x, v_j, t) \). We rewrite the ES-BGK model (1) in the characteristic formulation:

\[
\begin{align*}
\frac{df_j}{dt} &= \frac{1}{\kappa} A_{\nu}(M_{\nu,j}(f_j) - f_j), \\
\frac{dx}{dt} &= v_j.
\end{align*}
\]

Using implicit Euler scheme on (5), we obtain

\[
f_j^{n+1}(x_i) - f_j^n(x_i - v_{ji} \Delta t) = \frac{\Delta t}{\kappa} A_{\nu} \{M_{\nu,j}(f^{n+1}) - f_j^{n+1}\}(x_i).
\]

We then approximate \( f_j^n(x_i) \) by \( f_{i,j}^n \), and \( f_j^n(x_i - v_{ji} \Delta t) \) by \( \tilde{f}_{i,j}^n \) to obtain

\[
\left( f_j^{n+1} - \tilde{f}_{i,j}^n \right) = \frac{\Delta t}{\kappa} A_{\nu} \{M_{\nu,j}(f^{n+1}) - f_j^{n+1}\}.
\]

We attempt to convert (6) into a semi-explicit scheme keeping beneficial features of implicit schemes such as the stability property. This idea seems to trace back to [15, 33], and successfully implemented in the semi-Lagrangian setting in [23, 27, 37]. We first impose the conservation of mass, momentum and energy at the discrete level (throughout this subsection \( \approx \) means that they are identical up to negligible error):

\[
\sum_j f_{i,j}^{n+1}(x_i) - \sum_j \tilde{f}_{i,j}^n(x_i) = \frac{\Delta t}{\kappa} \sum_j \{M_{\nu,j}(f^{n+1}) - f_j^{n+1}\}(x_i) \approx 0,
\]

\[
\sum_j (\hat{f}_{i,j}^n(x_i) - \tilde{f}_{i,j}^n(x_i) - \hat{f}_{i,j}^n(x_i)) = \frac{\Delta t}{\kappa} \sum_j \{M_{\nu,j}(f^{n+1}) - f_j^{n+1}\}(x_i) \approx 0,
\]

\[
\frac{\Delta t}{\kappa} \sum_j \{M_{\nu,j}(f^{n+1}) - f_j^{n+1}\}(x_i) = \frac{\Delta t}{\kappa} \sum_j \{M_{\nu,j}(f^{n+1}) - f_j^{n+1}\}(x_i) \approx 0.
\]
Therefore, if we define
\[
\rho_t^n = \sum_j \tilde{f}_{t,j}^n (\Delta v)^3, \quad \rho_t^n \tilde{U}_i^n = \sum_j \tilde{f}_{t,j}^n v_j (\Delta v)^3, \quad 3\rho_t^n \tilde{T}_i^n = \sum_j \tilde{f}_{t,j}^n |v_j - \tilde{U}_i^n|^2 (\Delta v)^3,
\]
then (7) gives the following relation:
\[
\rho_t^{n+1} \approx \tilde{\rho}_i^n, \quad U_i^{n+1} \approx \tilde{U}_i^n, \quad T_i^{n+1} \approx \tilde{T}_i^n.
\]
In the case of the original BGK model ($\nu = 0$), identities in (8) are sufficient to conclude that $M_{0,j}(f_i^{n+1}) \approx M_{0,j}(f_i^n)$. But this is not the case for the ellipsoidal case, since the ellipsoidal Gaussian contains the temperature tensor $\Theta$, which is not a conserved quantity. For this, we introduce
\[
\phi_{t,j}^n \equiv (v_j - U_i^n) \otimes (v_j - U_i^n).
\]
Multiplying $\phi_{t,j}^{n+1} (\Delta v)^3$ on both sides of (6) and summing over $i$ and $j$, we get:
\[
\frac{\sum_j f_{t,j}^{n+1} \phi_{t,j}^{n+1} (\Delta v)^3 - \sum_j f_{t,j}^n \phi_{t,j}^{n+1} (\Delta v)^3}{\Delta t} = \frac{1}{\kappa A r} \sum_j \left\{M_{t,j}(f_i^{n+1})\phi_{t,j}^{n+1} - f_{t,j}^{n+1} \phi_{t,j}^{n+1}\right\} (\Delta v)^3.
\]
Let’s denote the r.h.s of (6) by $R$ and the l.h.s by $L$ for simplicity. We then recall (8) to observe
\[
\phi_{t,j}^{n+1} = (v_j - U_i^{n+1}) \otimes (v_j - U_i^{n+1}) \approx (v_j - \tilde{U}_i^n) \otimes (v_j - \tilde{U}_i^n) = \phi_{t,j}^n.
\]
Therefore, the second term of $L$ becomes
\[
\sum_j \tilde{f}_{t,j}^n \phi_{t,j}^{n+1} (\Delta v)^3 \approx \sum_j \tilde{f}_{t,j}^n \phi_{t,j}^n (\Delta v)^3 = \rho_t^n \tilde{\Theta}_t^n,
\]
where $\tilde{\Theta}_t^n$ is defined by
\[
\rho_t^n \tilde{\Theta}_t^n = \sum_j \tilde{f}_{t,j}^n (v_j - \tilde{U}_i^n) \otimes (v_j - \tilde{U}_i^n) (\Delta v)^3,
\]
so that
\[
\frac{L}{\Delta t} = \frac{\rho_t^{n+1} \Theta_t^{n+1}}{\Delta t} - \frac{\rho_t^n \tilde{\Theta}_t^n}{\Delta t}.
\]
On the other hand, we find for the right hand side,
\[
R \approx \frac{1}{\kappa} A r \left\{\rho_t^{n+1} T_{t,j}^{n+1} - \rho_t^{n+1} \Theta_t^{n+1}\right\} = \frac{1}{\kappa} A r \left\{\rho_t^{n+1} (1 - \nu) T_{t,j}^{n+1} \text{Id} + \nu \Theta_t^{n+1}\right\} = \frac{1}{\kappa} A r (1 - \nu) \left\{\rho_t^{n+1} T_{t,j}^{n+1} \text{Id} - \rho_t^{n+1} \Theta_t^{n+1}\right\} = \frac{1}{\kappa} \left\{\rho_t^{n+1} T_{t,j}^{n+1} \text{Id} - \rho_t^{n+1} \Theta_t^{n+1}\right\},
\]
and recall (8) to see that
\[
R = \frac{1}{\kappa} \left\{\rho_t^{n+1} T_{t,j}^{n+1} \text{Id} - \rho_t^{n+1} \Theta_t^{n+1}\right\} = \frac{1}{\kappa} \left\{\rho_t^n \tilde{T}_t^n \text{Id} - \rho_t^n \Theta_t^{n+1}\right\}.
\]
Now, equating (10) and (11), we rewrite (9) as
\[ \frac{\rho_i^{n+1} \Theta_i^{n+1} - \rho_i^n \Theta_i^n}{\Delta t} = \frac{1}{\kappa} \left\{ \tilde{\rho}_i^n \tilde{T}_i^n \text{Id} - \rho_i^n \Theta_i^{n+1} \right\}. \]
Dividing both sides by \( \rho_i^{n+1} \) and gathering relevant terms, we get
\[ \Theta_i^{n+1} = \frac{\Delta t}{\kappa + \Delta t} \tilde{T}_i^n \text{Id} + \frac{\kappa}{\kappa + \Delta t} \tilde{\Theta}_i^n. \]
Therefore, \( T_{\nu,i}^{n+1} \) can be expressed as
\[ T_{\nu,i}^{n+1} = (1 - \nu) T_i^n \text{Id} + \nu \Theta_i^{n+1} \]
\[ \approx (1 - \nu) T_i^n \text{Id} + \nu \left\{ \frac{\Delta t}{\kappa + \Delta t} \tilde{T}_i^n \text{Id} + \frac{\kappa}{\kappa + \Delta t} \tilde{\Theta}_i^n \right\} \]
\[ = \left( 1 - \frac{\kappa \nu}{\kappa + \Delta t} \right) \tilde{T}_i^n \text{Id} + \left( \frac{\kappa \nu}{\kappa + \Delta t} \right) \tilde{\Theta}_i^n \]
\[ \equiv (1 - \tilde{\nu}) T_i^n \text{Id} + \tilde{\nu} \tilde{\Theta}_i^n \]
\[ \equiv \tilde{T}_{\nu,i}^n, \]
where we denoted
\[ \tilde{\nu} = \frac{\kappa \nu}{\kappa + \Delta t}. \]
Using (8) and (13), the implicitly defined discrete ellipsoidal Gaussian can now be rewritten in an explicit way as:
\[ M_{\nu,j}(\tilde{f}_i^n) = M_{\nu,j}(\tilde{\rho}_i^n, \tilde{u}_i^n, \tilde{T}_{\nu,i}^n) = \frac{\tilde{\rho}_i^n}{\sqrt{\det(2\pi \tilde{T}_{\nu,i}^n)}} \exp \left( -\frac{1}{2} (U_j - \tilde{U}_i^n)^\top \{ \tilde{T}_{\nu,i}^n \}^{-1} (U_j - \tilde{U}_i^n) \right). \]

With a slight abuse of notation, we now denote the r.h.s as \( M_{\nu,j}(\tilde{f}_i^n) \). We should note carefully that (for example in Lemma 7.3)
\[ \tilde{T}_{\nu,j}^n = (1 - \tilde{\nu}) T_i^n + \tilde{\nu} \tilde{\Theta}_i^n \neq (1 - \nu) T_i^n + \nu \Theta_i^n = \tilde{T}_{\nu,i}^n \]
throughout this paper. We then use this to rewrite the implicit scheme (6) as the following explicit form:
\[ f_{i,j}^{n+1} = \frac{\kappa}{\kappa + A_{\nu} \Delta t} \tilde{f}_{i,j}^n + \frac{A_{\nu} \Delta t}{\kappa + A_{\nu} \Delta t} M_{\nu,j}(\tilde{f}_i^n). \]

2.2. Implicit semi-Lagrangian scheme. Summarizing, our semi-Lagranian scheme for the ES-BGK model (1) reads:
\[ f_{i,j}^{n+1} = \frac{\kappa}{\kappa + A_{\nu} \Delta t} \tilde{f}_{i,j}^n + \frac{A_{\nu} \Delta t}{\kappa + A_{\nu} \Delta t} M_{\nu,j}(\tilde{f}_i^n), \]
where the discrete ellipsoidal Gaussian \( M_{\nu,j}(\tilde{f}_i^n) \) is defined as follows:
\[ M_{\nu,j}(\tilde{f}_i^n) = \frac{\tilde{\rho}_i^n}{\sqrt{\det(2\pi \tilde{T}_{\nu,i}^n)}} \exp \left( -\frac{1}{2} (U_j - \tilde{U}_i^n)^\top \{ \tilde{T}_{\nu,i}^n \}^{-1} (U_j - \tilde{U}_i^n) \right), \]
and discrete macroscopic field $\tilde{\rho}_i^n, \tilde{U}_i^n, \tilde{T}_i^n, \tilde{\rho}_{p,i}^n$ and $\tilde{T}_{p,i}^n$ ($n \geq 1$) are given by
\[
\tilde{\rho}_i^n = \sum_j \tilde{f}_{i,j}^n (\Delta v)^3, \quad \tilde{\rho}_i^n \tilde{U}_i^n = \sum_j \tilde{f}_{i,j}^n v_j (\Delta v)^3, \quad 3\tilde{\rho}_i^n \tilde{T}_i^n = \sum_j \tilde{f}_{i,j}^n v_j - \tilde{U}_i^n (\Delta v)^3,
\]

(15) $\tilde{\rho}_i^n \tilde{U}_i^n = \sum_j \tilde{f}_{i,j}^n \{(v_j - \tilde{U}_i^n) \otimes (v_j - \tilde{U}_i^n)\}(\Delta v)^3, \quad \tilde{T}_{p,i}^n = (1 - \tilde{\nu}) \tilde{T}_i^n \text{Id} + \tilde{\nu} \tilde{\Theta}_i^n.$

In the last line, $\tilde{\nu}$ denotes $\tilde{\nu} = \frac{k\nu}{\kappa + \Delta t}.$

For the initial step ($n = 0$), to ignore the error arising in the discretization of the initial data and simplify the convergence proof, we sample values directly from continuous distribution function and macroscopic fields at $t = 0$:
\[
f_{i,j}^0 = f_0(x_i, v_j), \quad \tilde{f}_{i,j}^0 = \tilde{f}_0(x_i, v_j) = f_0(x_i - v_j \Delta t, v_j),
\]
and
\[
\tilde{\rho}_i^0 = \tilde{\rho}(x_i, 0) = \int_{\mathbb{R}^3} f_0(x_i - v \Delta t, v) dv,
\]
\[
\tilde{\rho}_i^0 \tilde{U}_i^0 = \tilde{\rho}(x_i, 0) \tilde{U}(x_i, 0) = \int_{\mathbb{R}^3} f_0(x_i - v \Delta t, v) v dv,
\]
\[
3\tilde{\rho}_i^0 \tilde{T}_i^0 = 3\tilde{\rho}(x_i, 0) \tilde{T}(x_i, 0) = \int_{\mathbb{R}^3} f_0(x_i - v \Delta t, v) |v - \tilde{U}_i^0|^2 dv.
\]

3. Main results

We are now ready to state our main result. We first record the existence result relevant to our convergence proof.

**Theorem 3.1.** Let $-1/2 < \nu < 1$ and $q > 5.$ Let $f_0$ satisfy $\|f_0\|_{W_{q,\infty}^2} < \infty.$ Suppose further that there exist positive constants $C_1^1, C_2^1$ and $\alpha$ such that
\[
f_0(x, v) \geq C_0^1 e^{-C_0^2|v|^\alpha}.
\]
Then, for any final time $T^f > 0$, the ES-BGK model has a unique solution $f \in C([0,T^f], \| \cdot \|_{W_{q,\infty}^2})$ such that

1. $f$ is bounded in $\| \cdot \|_{W_{q,\infty}^2}$ for $[0,T^f]$:
\[
\|f(t)\|_{W_{q,\infty}^2} \leq C_1 e^{C_2 t} \left\{ \|f_0\|_{W_{q,\infty}^2} + 1 \right\}, \quad t \in [0,T^f],
\]
for some constants $C_1$ and $C_2$.

2. The macroscopic fields satisfy the following lower and upper bounds:
\[
\rho(t) \leq C_{q} e^{C_{q} t}, \quad \rho(x, t) \geq C_{N,q} e^{-C_{N,q} t},
\]
\[
k^\top \{ T_{\nu}(x, t) \} k \geq C_{N,q} e^{-C_{N,q} t} > 0, \text{ for any } k \in \mathbb{S}^2.
\]

Now, we state our main result.

**Theorem 3.2.** Let $-1/2 < \nu < 1.$ Let $f$ be the unique smooth solution of corresponding to a nonnegative initial datum $f_0$ satisfying the hypotheses of Theorem 3.1. Let $f^n$ be the approximate solution constructed iteratively by given in Section 2. Then, there exists a positive number $r_{\Delta v}$, which is explicitly determined in Theorem 5.3 in Section 4, such that, if $\Delta v < r_{\Delta v}$, then we have
\[
\|f(T^f) - f^n\|_{L_{q}^\infty} \leq C \left\{ (\Delta x)^2 + \Delta v + \Delta t + \frac{(\Delta x)^2}{\Delta t} \right\},
\]
where $N_t$ is defined by $T^f = N_t \Delta t$ and $C = C(T^f, f_0, q, \kappa, \nu) > 0$. Here, $C$ is uniformly bounded in $\nu$.

**Remark 3.3.** (1) $\nu = 0$ corresponds to the original BGK model. Therefore, our result holds for the original BGK model too. (2) For the precise definition of $r_{\Delta t}$, see Theorem 5.5. (3) When $\kappa = 0$, this error estimate breaks down, since the coefficients of the estimate contain $\kappa^{-1}$. Currently, it is not clear whether this is of inherent nature, or can be avoided by developing finer convergence analysis. (4) The bad term $1/\Delta t$ is removed in [11]. But the argument cannot be implemented in our case since it depends heavily on the fact that the distribution function for the Vlasov-Poisson equation remains compactly supported, once it is so initially. (5) We believe the argument we develop in this work is robust, and can be extended in many directions such as semi-Lagrangian scheme for polyatomic BGK models, high order semi-Lagrangian schemes, and semi-Lagrangian BDF methods, or Runge-Kutta method. We leave them for the future.

4. Technical Lemmas

**Lemma 4.1.** Discrete solutions to (14) are periodic in the spatial nodes:

$$f_{i+1}^{n+1,x,j} = f_{i,j}^n.$$  

**Proof.** We use induction. We recall the definition of $f_{i,j}^0$ to get

$$f_{i+1}^{0,x,j} = f_0(x_i + N_x \Delta x, v_j) = f_0(x_i + 1, v_j) = f_0(x_i, v_j) = f_{i,j}^0.$$  

Similarly, we have

$$\tilde{f}_{i+1}^{0,x,j} = \tilde{f}_0(x_i + N_x \Delta x, v_j) = \tilde{f}_0(x_i + 1, v_j)$$  

$$= f_0(x_i - \Delta t v_j + 1, v_j) = f_0(x_i - \Delta t v_j, v_j)$$  

$$= \tilde{f}_0(x_i, v_j)$$  

$$= \tilde{f}_{i,j}^0.$$  

Then, the periodicity of $f_{i,j}^0$ and $\tilde{f}_{i,j}^0$ implies the periodicity of $\rho_{i,j}^0, U_{i,j}^0, T_{i,j}^0$ and $\rho_{i,j}^0, U_{i,j}^0, T_{i,j}^0$ by definition. This completes the proof of the initial step of the induction. Now, assume that $f_{i,j}^n, \tilde{f}_{i,j}^n, \rho_{i,j}^n, U_{i,j}^n, T_{i,j}^n$ and $\tilde{f}_{i,j}^n, U_{i,j}^n, T_{i,j}^n$ are all periodic in spatial variable. Then the periodicity of $f_{i,j}^{n+1}$ is immediate from (14). For the periodicity of $\tilde{f}_{i,j}^{n+1}$, we first observe

$$x(i + N_x, j) = x(i + N_x, j) - \Delta t v_j = x(i + N_x \Delta x - \Delta t v_j = x(i, j) + N_x \Delta x,$$

so that

$$s(i + N_x, j) = s(i, j) + N_x.$$  

Therefore,

$$x(i + N_x, j) - x_s(i + N_x, j) = x(i, j) - x_{s(i,j)}.$$  

Likewise,

$$x_s(i + N_x, j) + 1 - x(i + N_x, j) = x_s(i,j) + 1 - x(i, j).$$
We then use these identities together with the periodicity of $f_{i,j}^{n+1}$ to derive
\[ f_{i,N_x,j}^{n+1} = \frac{x(i + N_x,j) - x(i + N_x,j)}{\Delta x} f_{i,N_x,j+1}^{n+1} + \frac{x(i + N_x,j+1) - x(i + N_x,j)}{\Delta x} f_{i,N_x,j}^{n+1} \]
\[
= \frac{x(i,j) - x(i,j)}{\Delta x} f_{i,j}^{n+1} + \frac{x(i,j+1) - x(i,j)}{\Delta x} f_{i,j+1}^{n+1} + \frac{x(i,j) - x(i,j+1)}{\Delta x} f_{i,j}^{n+1} + \frac{x(i,j+1) - x(i,j)}{\Delta x} f_{i,j+1}^{n+1} = f_{i,j}^{n+1},
\]
which gives the periodicity of $f_{i,N_x,j}^{n+1}$. Then the macroscopic fields associated with $f_{i,j}^{n+1}$ and $f_{i,j}^{n+1}$ are periodic by construction. Therefore, the desired result follows from induction. \( \square \)

Lemma 4.2. \[ \text{The reconstruction procedure does not increase the } \| \cdot \|_{L_\infty} \text{-norm of the discrete distribution function:} \]
\[ \| \tilde{f}^n \|_{L_\infty} \leq \| f^n \|_{L_\infty}. \]

Proof. We observe from the definition of $\tilde{f}^n$ that, for $n \neq 0$
\[
\| \tilde{f}^n \|_{L_\infty} = \sup_{i,j} |\tilde{f}^n_{i,j}(1 + |v_j|)^q| \\
= \sup_{i,j} \left| \frac{x(i,j) - x_{i,j}}{\Delta x} f_{i,j}^{n+1} + \frac{x_{i,j+1} - x(i,j)}{\Delta x} f_{i,j+1}^{n+1} \right| (1 + |v_j|)^q \\
\leq \sup_{i,j} \max \{ f_{i,j}^{n+1}, f_{i,j+1}^{n+1} \} (1 + |v_j|)^q \\
\leq \sup_{i,j} |f_{i,j}^{n+1}(1 + |v_j|)^q| \\
= \| f^n \|_{L_\infty}.
\]
Here, $s$ denotes $s(i,j)$. When $n = 0$, we have
\[
\| \tilde{f}^0 \|_{L_\infty} = \sup_{i,j} \left| f_0(x_i - v_j \Delta t, v_j)(1 + |v_j|)^q \right| \leq \sup_{i,j} \left| f_0(x_i, v_j)(1 + |v_j|)^q \right| = \| f_0 \|_{L_\infty}.
\]
\( \square \)

Lemma 4.3. Let $-1/2 < \nu < 1$. Assume $\tilde{f}_{i,j}^n > 0$ and $\tilde{p}_i^n > 0$. Then the discrete temperature tensor $\tilde{T}_{\nu,i}$ and its determinant $\det \tilde{T}_{\nu,i}$ satisfy the following equivalence estimates:

1. $\min \{ 1 - \tilde{v}, 1 + 2\tilde{v} \} T^n \text{Id} \leq \tilde{T}_{\nu,i} \leq \max \{ 1 - \tilde{v}, 1 + 2\tilde{v} \} T^n \text{Id},$
2. $\min \{ 1 - \tilde{v}, 1 + 2\tilde{v} \}^3 (T^n)^3 \leq \det \tilde{T}_{\nu,i} \leq \max \{ 1 - \tilde{v}, 1 + 2\tilde{v} \}^3 (T^n)^3.$

In the first inequality, $A \geq B$ for $3 \times 3$ symmetric matrices $A$ and $B$ means $A - B$ is positive definite.

Proof. From \([13]\), we see that
\[
\tilde{p}_{i,j}^n \tilde{T}_{\nu,i} = (1 - \tilde{v}) \tilde{p}_{i,j}^n \tilde{T}_{\nu,i} \text{Id} + \tilde{v} \tilde{p}_{i,j}^n \tilde{\Theta}_{\nu}^n \\
= \frac{(1 - \tilde{v})}{3} \left\{ \sum_j f_{i,j}^n |v_j - U_i^n|^2 (\Delta v)^3 \right\} + \tilde{v} \sum_j f_{i,j}^n (v_j - U_i^n) \otimes (v_j - U_i^n)(\Delta v)^3.
\]
Then, in view of the identity: $k^T \{ U \cup U \} k = (k \cdot U)^2 \ (k, U \in \mathbb{R}^3)$, we have
\[
k^T \{ \tilde{p}_{i,j}^n \tilde{T}_{\nu,i} \} k = \frac{(1 - \tilde{v})}{3} \left\{ \sum_j f_{i,j}^n |v_j - U_i^n|^2 (\Delta v)^3 \right\} |k|^2 + \tilde{v} \sum_j f_{i,j}^n \left\{ (v_j - U_i^n) \cdot k \right\}^2 (\Delta v)^3.
\]
We note from the definition of \( \tilde{v} \) that \(-1/2 < \nu < 1\) implies \(-1/2 < \tilde{v} < 1\), and divide our estimate into the following two cases:

(a) \(0 < \tilde{v} < 1\): Since the second term is non-negative, we have

\[
k^T \{ \tilde{\nu}^n \tilde{T}_{\tilde{\nu}, i}^n \} k \lor \begin{pmatrix} 1 - \tilde{v} \\ \sum_j f_{i,j} |v_j - \tilde{U}_{i}^n| (\Delta v)^3 \end{pmatrix} |k|^2 = (1 - \tilde{v}) \tilde{\nu}^n \tilde{T}_{\tilde{\nu}}^n |k|^2.
\]

(b) \(-\frac{1}{2} < \tilde{v} < 0\): By Cauchy-Schwartz inequality, we see that

\[
k^T \{ \tilde{\nu}^n \tilde{T}_{\tilde{\nu}, i}^n \} k \lor \begin{pmatrix} 1 - \tilde{v} \\ \sum_j f_{i,j} |v_j - \tilde{U}_{i}^n| (\Delta v)^3 \end{pmatrix} |k|^2 \\
= (1 + 2\tilde{v}) \begin{pmatrix} 1 + 2\nu \\ \sum_j f_{i,j} |v_j - \tilde{U}_{i}^n| (\Delta v)^3 \end{pmatrix} |k|^2 \\
= (1 + 2\tilde{v}) \tilde{\nu}^n \tilde{T}_{\tilde{\nu}}^n |k|^2.
\]

We then combine the above estimates to get

\[
\min \{1 - \tilde{v}, 1 + 2\tilde{v} \} \tilde{T}_{\tilde{\nu}}^n \leq k^T \{ \tilde{\nu}^n \tilde{T}_{\tilde{\nu}, i}^n \} k.
\]

The r.h.s of the inequality follows in a similar manner.

(2) Let \( \lambda_i \) be the eigenvalues of \( \tilde{T}_{\tilde{\nu}} \). Then (1) implies that the values of these eigenvalues lie between

\[
\min \{1 - \tilde{v}, 1 + 2\tilde{v} \} \tilde{T}_{\tilde{\nu}}^n \text{ and } \max \{1 - \tilde{v}, 1 + 2\tilde{v} \} \tilde{T}_{\tilde{\nu}}^n.
\]

This gives

\[
\min \{ \lambda_1^n, \lambda_2^n, \lambda_3^n \} = \min \{1 - \tilde{v}, 1 + 2\tilde{v} \} \tilde{T}_{\tilde{\nu}}^n,
\]

and

\[
\max \{ \lambda_1^n, \lambda_2^n, \lambda_3^n \} = \max \{1 - \tilde{v}, 1 + 2\tilde{v} \} \tilde{T}_{\tilde{\nu}}^n,
\]

so that

\[
\min \{1 - \nu, 1 + 2\nu \}^3 \tilde{T}_{\tilde{\nu}}^n \leq \det \tilde{T}_{\tilde{\nu}, i} = \lambda_1^n \lambda_2^n \lambda_3^n \leq \max \{1 - \nu, 1 + 2\nu \}^3 \tilde{T}_{\tilde{\nu}}^n.
\]

\[\square\]

In the following lemma, the symbol \([x]\) denotes, as usual, the largest integer that does not exceed \(x\).

**Lemma 4.4.** Fix velocity grid index \(j_1\) and the grid size \(\Delta x, \Delta v, \Delta t\). We define \(s^k\) inductively as

\[
s^{(1)} = s(i, j_1), \quad s^{(2)} = s(s^{(1)}, j_1), \quad s^{(3)} = s(s^{(2)}, j_1), \cdots
\]

Using this notation, we define \(a_{j_1}^{(n)}\) by

\[
a_{j_1}^{(n)} = \frac{x_{s(n)} - \Delta vt_{j_1} - x_{s(n+1)}}{\Delta x}.
\]

Then, \(a_{j_1}^{(n)}\) is constant for all \(n > 0\), that is

\[
a_{j_1}^{(n)} = a_{j_1}^{(m)},
\]

for all positive integers \(m\) and \(n\).

**Proof.** We take a positive integer \(\ell_{s(n)}\) such that

\[
x_{s(n)} = \ell_{s(n)} \Delta x.
\]

On the other hand, we can find a positive integer \(m_j\) such that

\[
m_j \Delta x \leq v_{j_1} \Delta t \leq (m_j + 1) \Delta x.
\]
That is,
\begin{equation}
\langle 17 \rangle \quad m_{j_1} = \left[ \frac{v_{j_1} \Delta t}{\Delta x} \right].
\end{equation}

From these, we immediately see that
\[ \{ \ell_x(n) - (m_{j_1} + 1) \} \Delta x \leq x_{s(n)} - v_{j_1} \Delta t \leq \{ \ell_x(n) - m_{j_1} \} \Delta x. \]
Since \( x_{s(n+1)} \) denotes the closest spatial node that lies before \( x_{s(n)} - v_{j_1} \Delta t \), this gives
\[ x_{s(n+1)} = \{ \ell_x(n) - (m_{j_1} + 1) \} \Delta x. \]
Therefore,
\begin{equation}
\langle 18 \rangle \quad a_{j_1}^{n+1} = \frac{x_{s(n)} - \Delta t v_{j_1} - x_{s(n+1)}}{\Delta x} = (m_{j_1} + 1) - \frac{v_{j_1} \Delta t}{\Delta x}.
\end{equation}

In view of \( \langle 10 \rangle \), this can be rewritten as
\[ \left[ \frac{v_{j_1} \Delta t}{\Delta x} \right] - \frac{v_{j_1} \Delta t}{\Delta x} + 1, \]
which is dependent on \( j \) but not on \( n \). This completes the proof. \( \Box \)

5. Stability of the discrete distribution function

In this section, we derive uniform lower and upper bounds for the discrete distribution function \( \tilde{f}_{i,j}^n \) and corresponding macroscopic fields. We start with series of definitions most of which were introduced for technical reasons.

**Definition 5.1.**
1. We define \( C_\alpha, C_{q,\alpha} \) and \( C_{q-m} \) (\( q \geq m \)) by
   \[ C_\alpha = \int_{\mathbb{R}^3} e^{-C_{0,1} |v|^\alpha} dv, \quad C_{q,\alpha} = \sup_v \{(1 + |v|)^q e^{-C_{0,1} |v|^\alpha}\}, \quad C_{q-m} = \int_{\mathbb{R}^3} \frac{1}{(1 + |v|)^{q-m}} dv. \]
2. Throughout this section, we fix \( C_M, C_M' \) as is defined in Lemma 5.6 and Lemma 5.7 respectively.

**Definition 5.2.**
1. We say that \( f_{i,j}^n \) satisfies \( E_1^n \) if the following two statements hold:
   \[ (A^n) \quad \| \tilde{f}_{i,j}^n \|_{L_q^\infty} \leq \left( \frac{\kappa + C_M A_{\nu} \Delta t}{\kappa + A_\nu \Delta t} \right)^n \| f_0 \|_{L_q^\infty} \leq e^{\frac{(C_{M-1} A_{\nu} T_f)}{\kappa + A_\nu \Delta t}} \| f_0 \|_{L_q^\infty}, \]
   \[ (B^n) \quad \tilde{f}_{i,j}^n \geq C_{0,1} \left( \frac{\kappa}{\kappa + A_\nu \Delta t} \right)^n e^{-C_{0,1} |v|^\alpha} \geq C_{0,1} e^{-\frac{\Delta t}{\kappa} T_f} e^{-C_{0,1} |v|^\alpha}. \]
2. We say that \( f_{i,j}^n \) satisfies \( E_2^n \) if the following two statements hold:
   \[ (C^n) \quad \tilde{f}_{i,j}^n \geq \frac{1}{2} C_{0,1} C_\alpha e^{-\frac{\Delta t}{\kappa} T_f}, \quad \tilde{T}_{i,j}^n \geq \left( \frac{C_{0,1} C_\alpha}{2 C_M \| f_0 \|_{L_q^\infty}} \right)^{2/3} e^{-\frac{1}{2} \left( \frac{C_M - 1}{\kappa + A_\nu \Delta t} \right) A_{\nu} T_f}, \]
   \[ (D^n) \quad \| \tilde{p}^n \|_{L_q^\infty}, \| \tilde{U}^n \|_{L_q^\infty}, \| \tilde{T}^n \|_{L_q^\infty} \leq 2 C_q \left( 1 + \left( C_{0,1} C_\alpha \right)^{-1} \right) e^{-\frac{1}{2} \left( \frac{C_{M-1} A_{\nu} T_f}{\kappa + A_\nu \Delta t} \right) A_{\nu} T_f} \| f_0 \|_{L_q^\infty}. \]
3. We define \( E^n = E_1^n \land E_2^n \).

**Remark 5.3.** In fact, the first inequality in \( (A^n) \) implies the second inequality due to the elementary inequality \((1 + x)^n \leq e^{nx}\). We stated them in this seemingly redundant manner since both estimates are interchangeably used in the following proofs. \( (B^n) \) is stated in such a redundant manner for the same reason.
Lemma 5.6. We begin with the discrete moment estimates:

\begin{align}
\sum_{v} (1) \quad & \text{We split the macroscopic density into the following two parts:} \\
\text{Proof.} & \quad \text{for all} \\
\text{(19)} & \quad \text{Then we see that} \\
\end{align}

Definition 5.4. Theorem 5.5. Choose \( \ell > 0 \) sufficiently small such that \( \Delta v < \ell \) implies

\begin{align}
\frac{1}{2} C_{\alpha} \leq & \sum_{j} e^{- C_{\alpha} |v_j|^\alpha} (\Delta v)^3 \leq 2 C_{\alpha}, \\
\frac{1}{2} C_{q,\alpha} \leq & \sup_{j} \left\{ (1 + |v_j|)^q e^{- |v_j|^\alpha} \right\} \leq 2 C_{q,\alpha}, \\
\frac{1}{2} C_{q-m} \leq & \sum_{j} (\Delta v)^3 \left(1 + |v_j|\right)^{q-m} \leq 2 C_{q-m}.
\end{align}

Now, define \( r_{\Delta v} \) by

\[ r_{\Delta v} = \min\{a_1, a_2, a_3, \ell, 1/2\}, \]

and suppose \( \Delta v \) is sufficiently small in the following sense:

\[ \Delta v < r_{\Delta v}. \]

Then, \( f_{n,i,j}^{u} \) satisfies \( E^n \) for all \( n \geq 0 \).

We postpone the proof to the end of this section, after establishing several preliminary results. We begin with the discrete moment estimates:

Lemma 5.6. Let \( q > 5 \). Suppose \( f_{n,i,j}^{u} \) satisfies \( E^n \) and \( \Delta v \) satisfies the smallness condition stated in Theorem 5.5. Then there exists a positive constant \( C_M \) which depends only on \( q \), such that

\begin{align}
(1) & \quad \frac{\bar{\rho}_n^i}{(T_n^i)^2} \leq C_M \| f^n \|_{L_\infty^n}, \\
(2) & \quad \frac{\bar{\rho}_n^i}{(T_n^i)^2} \left[ T_n^i + |U_n^i|^2 \right]^{\frac{4}{3}} \leq C_M \| f^n \|_{L_\infty^n}, \\
(3) & \quad \frac{\bar{\rho}_n^i}{|(T_n^i) + |U_n^i|^2 | T_n^i|^2} \leq C_M \| f^n \|_{L_\infty^n},
\end{align}

for all \( n \).

Proof. (1) We split the macroscopic density into to the following two parts:

\[ \bar{\rho}_n^i = \sum_{|v_j - \bar{v}_n^i| < r + \Delta v} f_{n,i,j}^u (\Delta v)^3 + \sum_{|v_j - \bar{v}_n^i| \geq r + \Delta v} f_{n,i,j}^u (\Delta v)^3. \]

Then we see that

\[ \sum_{|v_j - \bar{v}_n^i| < r + \Delta v} f_{n,i,j}^u (\Delta v)^3 \leq \bar{f}^n \left( r + 2 \Delta v \right)^3 \]

\[ \leq \pi \| \bar{f}^n \|_{L_\infty^n} \left( r + 2 \Delta v \right)^3 \]

\[ \leq 8 \pi \| \bar{f}^n \|_{L_\infty^n} \left( r + \Delta v \right)^3, \]
For the second term, we compute

\[ \sum_{|v_j - U^n_1| \geq r + \Delta v} f_{i,j}^n (\Delta v)^3 = \sum_{|v_j - U^n_1| \geq r + \Delta v} f_{i,j}^n \frac{|v_j - U^n_1|^2}{|v_j - U^n_1|^2} (\Delta v)^3 \]

and

\[ = \frac{1}{(r + \Delta v)^2} \sum_j f_{i,j}^n |v_j - U^n_1|^2 (\Delta v)^3 \]

yielding

\[ \tilde{\rho}_i^n \leq 8\pi \| f^n \|_{L^\infty} (r + \Delta v)^3 + \frac{3\tilde{T}_i^n}{(r + \Delta v)^2}. \]

We then optimize \( r \) by equating the two terms on the right hand sides, to obtain

\[ (20) \quad r + \Delta v = \left( \frac{3\tilde{T}_i^n}{8\pi \| f^n \|_{L^\infty}} \right)^{1/5}. \]

Using the fact that \( f_{i,j}^n \) satisfies \( E^n \), it can be easily verified that the r.h.s is greater than \( a_1 \), so that

\[ \left( \frac{3\tilde{T}_i^n}{8\pi \| f^n \|_{L^\infty}} \right)^{1/5} \geq a_1 > \Delta v. \]

Therefore, we can always find a positive number \( r \) satisfying \( (20) \). Inserting this, one finds

\[ \tilde{\rho}_i^n \leq C \tilde{T}_i^n \frac{1}{3/2} \| f^n \|_{L^\infty}. \]

Since the positivity of \( \tilde{T}_i^n \) is guaranteed by \( (C^n) \), we can divide both sides by \( \tilde{T}_i^n \) to get the desired estimate.

(2) We split the domain into \( \{ |v_j| > r + 2\Delta v \} \) and \( \{ |v_j| \leq r + 2\Delta v \} \):

\[ \tilde{\rho}_i^n (3\tilde{T}_i^n + |U_i^n|^2) = \sum_{|v_j| > r + 2\Delta v} f_{i,j}^n |v_j|^2 (\Delta v)^3 + \sum_{|v_j| \leq r + 2\Delta v} f_{i,j}^n |v_j|^2 (\Delta v)^3. \]

The first term is bounded by

\[ \sum_{|v_j| > r + 2\Delta v} f_{i,j}^n \frac{|v_j|^q}{|v_j|^{q-2}} (\Delta v)^3 \leq \| f^n \|_{L^\infty} \sum_{|v_j| > r + 2\Delta v} \frac{(\Delta v)^3}{|v_j|^{q-2}} \int_{|v| > r + \Delta v} \frac{dv}{|v|^{q-2}} \leq 4\pi \| f^n \|_{L^\infty} \]

For the second term, we compute

\[ \sum_{|v_j| \leq r + 2\Delta v} f_{i,j}^n |v_j|^2 (\Delta v)^3 \leq (r + 2\Delta v)^2 \sum_j f_{i,j}^n (\Delta v)^3 \leq 4\tilde{\rho}_i^n (r + \Delta v)^2. \]

Consequently,

\[ \tilde{\rho}_i^n (3\tilde{T}_i^n + |U_i^n|^2) \leq \frac{4\pi}{q - 5} \| f^n \|_{L^\infty} + 4\tilde{\rho}_i^n (r + \Delta v)^2 \leq C \tilde{\rho}_i^n \left( \frac{1}{\tilde{T}_i^n} \right) \| f^n \|_{L^\infty} \left( \frac{2}{3/2} \right), \]
where the latter inequality follows from optimizing \( r \) by taking
\[
r + \Delta v = \left( \frac{\pi}{q - 5} \frac{\| \tilde{f}^n \|_{L^q}^2}{\rho_i^n} \right)^{1/(q-3)}.
\]
The r.h.s is larger than \( a_2 \) by the fact that \( f_{i,j}^n \) satisfies \( E^n \), and we can find the optimizing \( r \) by a similar argument as in the previous case.

(3) We decompose the summational index of \( \rho_i^n |\tilde{U}_i^n| \) as follows:
\[
\rho_i^n |\tilde{U}_i^n| \leq \sum_{|v_j| \leq r + \Delta v} \tilde{f}_{i,j}^n |v_j| (\Delta v)^3 + \sum_{|v_j| > r + \Delta v} \tilde{f}_{i,j}^n |v_j| (\Delta v)^3 \equiv I + II.
\]
Then, we apply the H"older inequality to bound
\[
I \leq \left\{ \sum_{|v_j| \leq r + \Delta v} \tilde{f}_{i,j}^n (\Delta v)^3 \right\}^{1-1/q} \left\{ \sum_{|v_j| \leq r + \Delta v} \tilde{f}_{i,j}^n |v_j|^{q/(\Delta v)^3} \right\}^{1/q}
\]
\[
\leq \pi^{1/q} \{ \rho_i^n \}^{1-\frac{1}{q}} \| f \|_{L^q}^{1/q} (r + 2\Delta v)^{3/q} \]
\[
\leq (8\pi)^{1/q} \{ \rho_i^n \}^{1-1/q} \| f \|_{L^q}^{1/2} (r + \Delta v)^{3/q}.
\]

For \( II \), we employ the Schwartz inequality to see that
\[
II \leq \frac{1}{r + \Delta v} \sum_{|v_j| \leq r + 2\Delta v} \tilde{f}_{i,j}^n |v_j| \tilde{U}_i^n |v_j| (\Delta v)^3
\]
\[
\leq \frac{1}{r + 2\Delta v} \left\{ \sum_{j} \tilde{f}_{i,j}^n |v_j|^2 (\Delta v)^3 \right\}^{1/2} \left\{ \sum_j \tilde{f}_{i,j}^n |v_j| - \tilde{U}_i^n |v_j| (\Delta v)^3 \right\}^{1/2}
\]
\[
\leq \frac{1}{r + \Delta v} \left\{ \tilde{f}_i^n (3\tilde{T}_i^n + |\tilde{U}_i^n|^2) \right\}^{1/2} \left\{ 3\tilde{T}_i^n \right\}^{1/2}
\]
\[
= \frac{3^{1/2} \tilde{T}_i^n}{r + \Delta v} \left\{ 3\tilde{T}_i^n + |\tilde{U}_i^n|^2 \right\}^{1/2} \left\{ \tilde{f}_i^n \right\}^{1/2}.
\]
Therefore,
\[
\rho_i^n |\tilde{U}_i^n| \leq (8\pi)^{1/q} \{ \rho_i^n \}^{1-1/q} \| f \|_{L^q}^{1/2} (r + \Delta v)^{3/q} + \frac{3^{1/2} \tilde{T}_i^n}{r + \Delta v} \left\{ 3\tilde{T}_i^n + |\tilde{U}_i^n|^2 \right\}^{1/2} \left\{ \tilde{f}_i^n \right\}^{1/2}.
\]
We then derive the desired result by optimizing the above estimate by setting \( r \) as
\[
r + \Delta v = \left( \frac{3^{1/2} \tilde{T}_i^n}{(8\pi)^{1/q} \| f \|_{L^q}^{1/2}} \right)^{q/(q+3)} \left( 3^{1/q} \rho_i^n \right)^{1/q} \left( 3\tilde{T}_i^n + |\tilde{U}_i^n|^2 \right)^{1/2} \left\{ \tilde{f}_i^n \right\}^{1/2}
\]
The fact that \( f_{i,j}^n \) satisfies \( E^n \) guarantees that the r.h.s is greater than or equal to \( a_3 \), which guarantees the existence of \( r > 0 \).

We now show that the ellipsoidal Gaussian is controlled by the discrete distribution in \( L_q^\infty \).

**Lemma 5.7.** Suppose \( f_{i,j}^n \) satisfies \( E^n \), and \( \Delta v < r_\Delta v \). Then we have
\[
\| M_{\beta} (\hat{f}^n) \|_{L^q} \leq C_M \| f^n \|_{L^q}^\infty,
\]
for some constant \( C_M \) which depends only on \( q \) and \( \nu \).
Proof. We divide the proof into the two cases: \( q = 0 \) and \( q \neq 0 \).

(1) \( q = 0 \): Since the exponential part is less than or equal to 1, we see from Lemma 4.3 that

\[
\mathcal{M}_{\beta,j}(\tilde{f}^n_1) \leq \frac{\tilde{\rho}_i^n}{\sqrt{\det(2\pi T^n_i)}} \leq C_\beta \frac{\tilde{\rho}_i^n}{(T^n_i)^{3/2}}.
\]

Then Lemma 5.6 (1) gives the desired estimate.

(2) \( q \neq 0 \): We split \( v_j \) as:

\[
|v_j|^q \mathcal{M}_{\beta,j}(\tilde{f}^n_1) \leq C_q \left( |\tilde{U}_i^n|^q + |v_j| - |\tilde{U}_i^n|^q \right) \mathcal{M}_{\beta,j}(\tilde{f}^n_1) = I_1 + I_2.
\]

(i) The estimate for \( I_1 \): We first bound the exponential part by 1 to get

\[
I_1 \leq C_\beta |\tilde{U}_i^n|^q \tilde{\rho}_i^n \left( T^n_i \right)^{3/2}.
\]

(a) \( |\tilde{U}_i^n| > \left\{ T^n_i \right\}^{\frac{1}{2}} \): Lemma 5.6 (3) gives:

\[
I_1 \leq C \frac{|\tilde{U}_i^n|^q \tilde{\rho}_i^n}{\left( T^n_i \right)^{3/2}} \leq C \frac{|\tilde{U}_i^n|^q + 3 \tilde{\rho}_i^n}{\left( T^n_i + |\tilde{U}_i^n|^2 \right)^{3/2}} \leq C_{\nu,q} \|\tilde{f}^n\|_{L_q^\infty}.
\]

(b) \( |\tilde{U}_i^n| \leq \left\{ T^n_i \right\}^{\frac{1}{2}} \): In this case, we employ Lemma 5.6 (2) as

\[
I_1 \leq \frac{\left( T^n_i \right)^{\frac{1}{2}} \tilde{\rho}_i^n}{\left( T^n_i + |\tilde{U}_i^n|^2 \right)^{3/2}} \leq \tilde{\rho}_i^n \left( T^n_i + |\tilde{U}_i^n|^2 \right)^{\frac{2}{3}} \leq C_q \|\tilde{f}^n\|_{L_q^\infty}.
\]

(ii) The estimate for \( I_2 \): By Lemma 4.3, we have

\[
I_2 \leq C_\beta |v_j - \tilde{U}_i^n|^q \tilde{\rho}_i^n \left( T^n_i \right)^{3/2} \exp \left( -C_\nu \frac{|v_j - \tilde{U}_i^n|^2}{T^n_i} \right)
\]

\[
= C_\beta \tilde{\rho}_i^n \left( T^n_i \right)^{3/2} \left( \frac{|v_j - \tilde{U}_i^n|^2}{T^n_i} \right)^{\frac{2}{3}} \exp \left( -C_\nu \frac{|v_j - \tilde{U}_i^n|^2}{T^n_i} \right)
\]

\[
\leq C_\beta \tilde{\rho}_i^n \left( T^n_i \right)^{\frac{2}{3}}.
\]

In the last line, we used the elementary inequality \( |x^a e^{-bx}| \leq C_{a,b} \) for some positive \( C_{a,b} (a, b, x > 0) \). Then, Lemma 5.6 (2) gives

\[
\tilde{\rho}_i^n \left( T^n_i \right)^{\frac{2}{3}} \leq \tilde{\rho}_i^n \left( T^n_i + |\tilde{U}_i^n|^2 \right)^{\frac{2}{3}} \leq C_q \|\tilde{f}^n\|_{L_q^\infty}.
\]

This completes the proof. \( \square \)

**Lemma 5.8.** Let \( \Delta v < r_{\Delta v} \). Assume that \( f_0 \) satisfies the assumptions of Theorem 3.1 Then \( f_0 \) satisfies \( E^0 \).

**Proof.** • \( (A^0) \) Thanks to the assumption on the initial data, we have \( \|f_0\|_{L_q^\infty} < \infty \). Therefore, in view of Lemma 4.2, we have

\[
\|\tilde{f}^0\|_{L_q^\infty} \leq \|f^0\|_{L_q^\infty} \leq \|f_0\|_{L_q^\infty} < \infty.
\]

• \( (B^0) \) We recall the lower bound assumption imposed on the initial data in Theorem 3.1 to see that

\[
\tilde{f}^0_{i,j} = f_0(x_i - v_j \Delta t, v_j) \geq C_{0,1} e^{-C_{0,2} \|v_j\|^\alpha}.
\]

• \( (C^0) \) Using the lower bound on the initial data again, one finds

\[
\tilde{\rho}_i^n = \int_{\mathbb{R}^3} f_0(x_i - v \Delta t, v) dv \geq C_{0,1} \int_{\mathbb{R}^3} e^{-C_{0,2} \|v\|^\alpha} dv = C_{0,1} C_\alpha > 0.
\]
For the estimate of $\bar{\rho}_i^0$, we decompose the integral domain as

$$
\bar{\rho}_i^0 = \int_{\mathbb{R}^3} f_0(x_i - v\Delta t, v) dv
\leq \int_{|v - \bar{v}_i^0| \leq r} f_0(x_i - v\Delta t, v) dv + \int_{|v - \bar{v}_i^0| > r} f_0(x_i - v\Delta t, v) dv
\leq \frac{4\pi}{3} \|f_0\|_{L^\infty} r^3 + 3 \frac{\bar{\rho}_i^0}{\bar{T}_i^0}
$$

and optimize $r$ with

$$
r = \left( \frac{9\bar{\rho}_i^0}{4\pi\|f_0\|_{L^\infty}} \right)^{1/5}
$$

to get

$$
\bar{T}_i^0 \geq \left( \frac{4\pi}{9} \right)^{1/3} \left( \frac{\bar{\rho}_i^0}{\|f_0\|_{L^\infty}} \right)^{2/3} \geq \left( \frac{4\pi}{9} \right)^{1/3} \left( \frac{C_{0,1}C_\alpha}{\|f_0\|_{L^\infty}} \right)^{2/3}.
$$

If necessary, we can replace $C_M$ in Lemma 4.4 by $\max \left\{ C_M, \frac{3}{4\sqrt{\pi}} \right\}$ to get the desired result.

- (D\textsuperscript{0}) Lemma 5.10 gives

$$
\bar{\rho}_i^0 = \int_{\mathbb{R}^3} f_0(x_i - v\Delta t, v) dv \leq \|f_0\|_{L^\infty} \int_{\mathbb{R}^3} \frac{1}{(1 + |v|)^q} dv = C_q \|f_0\|_{L^\infty}.
$$

The estimate for $\bar{U}_i^0$ follows from

$$
|\bar{U}_i^0| = \left| \frac{1}{\bar{\rho}_i^0} \int_{\mathbb{R}^3} f_0(x_i - v\Delta t, v) dv \right|
\leq |\{C_\alpha C_{0,1}\}^{-1} \|f_0\|_{L^\infty} \int_{\mathbb{R}^3} \frac{1}{(1 + |v|)^q} dv
= C_{q-1} \{C_\alpha C_{0,1}\}^{-1} \|f_0\|_{L^\infty}.
$$

For the estimate of $\bar{T}_i^n$, we compute

$$
3\bar{T}_i^n = \frac{1}{\bar{\rho}_i^n} \int_{\mathbb{R}^3} f_0(x_i - v\Delta t, v) |v|^2 dv - |\bar{U}_i^n|^2
\leq \frac{1}{\bar{\rho}_i^n} \int_{\mathbb{R}^3} f_0(x_i - v\Delta t, v) |v|^2 dv
\leq \frac{1}{\bar{\rho}_i^n} \|f_0\|_{L^\infty} \int_{\mathbb{R}^3} \frac{1}{(1 + |v|)^{q-2}}
= C_{q-2} \{C_{0,1} C_\alpha\}^{-1} \|f_0\|_{L^\infty}.
$$

This completes the proof for $E^0$. \hfill \Box

**Lemma 5.9.** Assume $f_{i,j}^{n-1}$ satisfies $E^{n-1}$. Then, $f_{i,j}^n$ satisfies $B^n$:

$$
\bar{f}_{i,j}^n \geq C_{0,1} \left( \frac{\kappa}{\kappa + A_\nu \Delta t} \right)^n e^{-C_{0,1} |v_j|^n} \geq C_{0,1} e^{-\frac{A_\nu}{\kappa} T' e^{-C_{0,2} |v_j|^n}},
$$

for all $i, j$. From this, we also have

$$
\|\bar{f}^n\|_{L^\infty} \geq \frac{1}{2} C_{0,1} C_{q,\alpha} e^{-\frac{A_\nu}{\kappa} T'}.\]
Proof. Since \( f_{i,j}^{-1} \) satisfies \( E^{n-1} \), \( \mathcal{M}_{i,j}^{n} \) is strictly positive. Therefore, we have from (14)
\[
 f_{i,j}^{n} \geq \left( \frac{\kappa}{\kappa + A_{\nu} \Delta t} \right) \left( \frac{\kappa}{\kappa + A_{\nu} \Delta t} \right)^{n-1} e^{-C_{0,2} |\nu_{j}|^{n}} = C_{0,1} \left( \frac{\kappa}{\kappa + A_{\nu} \Delta t} \right) e^{-C_{0,2} |\nu_{j}|^{n}}.
\]
This immediately leads to the same lower bound estimate for \( \tilde{f}_{i,j}^{n} \):
\[
 \tilde{f}_{i,j}^{n} = a_{j_{1}} f_{s_{(i,j)},j_{1}}^{n} + (1 - a_{j_{1}}) f_{s_{(i,j)+1,j}}^{n} \geq C_{0,1} \left( \frac{\kappa}{\kappa + A_{\nu} \Delta t} \right) e^{-C_{0,2} |\nu_{j}|^{n}}
\]
with \( a_{j_{1}} = a_{j_{1}}^{(n)} \). We suppressed the dependence on \( n \), which is justified by Lemma 4. Then we employ the following elementary inequality \( (1 + x)^{n} \leq e^{nx}, (x \geq 0) \) to derive
\[
 \left( \frac{\kappa}{\kappa + A_{\nu} \Delta t} \right)^{n} = \left( 1 + \frac{A_{\nu} \Delta t}{\kappa} \right)^{-n} \geq e^{-n A_{\nu} \frac{\Delta t}{\kappa}} \geq e^{-\frac{n A_{\nu}}{\kappa} T'},
\]
where we used \( n \Delta t \leq N_{i} \Delta t = T' \). The second estimate follows directly from this:
\[
 \| \tilde{f}_{i,j}^{n} \|_{L_{q}^{\infty}} \geq C_{0,1} e^{-\frac{n A_{\nu}}{\kappa} T'} \sup_{j_{1}} \left( (1 + |\nu_{j}|)^{n} e^{-C_{0,2} |\nu_{j}|^{n}} \right) \geq \frac{1}{2} C_{0,1} C_{q,\alpha} e^{-\frac{n A_{\nu}}{\kappa} T'}.
\]
This completes the proof. \( \square \)

**Lemma 5.10.** Assume \( f_{i,j}^{-1} \) satisfies \( E^{n-1} \). Then, \( f_{i,j}^{n} \) satisfies \( A' \):
\[
 \| f^{n} \|_{L_{q}^{\infty}} \leq \left( \frac{\kappa + C_{M} A_{\nu} \Delta t}{\kappa + A_{\nu} \Delta t} \right)^{n} \| f^{0} \|_{L_{q}^{\infty}} \leq e^{\left( \frac{C_{M}^{3} - 1}{\kappa + A_{\nu} \Delta t} \right) T'} \| f^{0} \|_{L_{q}^{\infty}}.
\]

**Proof.** Applying Lemma 5.7 and to (14), one finds
\[
 \| f^{n} \|_{L_{q}^{\infty}} \leq \frac{\kappa}{\kappa + A_{\nu} \Delta t} \| \tilde{f}^{n-1} \|_{L_{q}^{\infty}} + \frac{A_{\nu} \Delta t}{\kappa + A_{\nu} \Delta t} \| \mathcal{M}_{\nu} (\tilde{f}^{n-1}) \|_{L_{q}^{\infty}} \leq \frac{\kappa}{\kappa + A_{\nu} \Delta t} \| \tilde{f}^{n-1} \|_{L_{q}^{\infty}} + \frac{A_{\nu} \Delta t}{\kappa + A_{\nu} \Delta t} C_{M} \| \tilde{f}^{n-1} \|_{L_{q}^{\infty}} \leq \frac{\kappa + C_{M} A_{\nu} \Delta t}{\kappa + A_{\nu} \Delta t} \| \tilde{f}^{n-1} \|_{L_{q}^{\infty}}.
\]
We then recall \( A_{n-1} \) to bound this further by
\[
 \left( \frac{\kappa + C_{M} A_{\nu} \Delta t}{\kappa + A_{\nu} \Delta t} \right)^{n} \| f^{0} \|_{L_{q}^{\infty}} \leq \left( \frac{\kappa + C_{M} A_{\nu} \Delta t}{\kappa + A_{\nu} \Delta t} \right)^{n} \| f^{0} \|_{L_{q}^{\infty}}.
\]
The second estimate follows from
\[
 \left( \frac{\kappa + C_{M} A_{\nu} \Delta t}{\kappa + A_{\nu} \Delta t} \right)^{n} \leq \left( 1 + \frac{(C_{M} - 1) A_{\nu} \Delta t}{\kappa + A_{\nu} \Delta t} \right)^{n} \leq e^{\left( \frac{(C_{M} - 1) A_{\nu}}{\kappa + A_{\nu} \Delta t} \right) T'} \leq e^{\left( \frac{(C_{M} - 1) A_{\nu}}{\kappa + A_{\nu} \Delta t} \right) T'},
\]
where we used \( (1 + x)^{n} \leq e^{nx} \) and \( n \Delta t \leq N_{i} \Delta t = T' \). \( \square \)

Using this, we can prove the uniform lower bound of the macroscopic fields:

**Lemma 5.11.** Assume \( f_{i,j}^{n} \) satisfies \( A' \land B' \). Then, \( f_{i,j}^{n} \) satisfies \( C' \):
\[
 \bar{u}_{i}^{n} \geq \frac{1}{2} C_{0,1} C_{\alpha} e^{-\frac{A_{\nu}}{\kappa} T'},
\]
\[
 \bar{T}_{i}^{n} \geq \left( \frac{C_{0,1} C_{\alpha}}{2 C_{M} \| f^{0} \|_{L_{q}^{\infty}}} \right)^{2/3} e^{-\frac{3}{2} \left( \frac{(C_{M} - 1) T'}{\kappa + A_{\nu} \Delta t} \right) A_{\nu} T'}.
\]
Note also that Lemma 4.3 then immediately yields the lower bound for $\tilde{T}_i^n$:

$$\tilde{T}_i^n \geq C_M C_{0,1} C_\alpha \min\{1 - \nu, 1 + 2\nu\} e^{-\left(\frac{1}{2} + \frac{(C_M-1)A_{T'}}{\nu} + \frac{(C_M-1)A_{T'}}{\nu}^2\right) A_{T'} I_d}.$$

**Proof.** For lower bound control for the discrete local density, we multiply $\hat{f}_{i,j}^n$ by $(\Delta v)^3$ and sum over $j$ to get

$$\tilde{\rho}_i^n = \sum_j \hat{f}_{i,j}^n (\Delta v)^3 \geq C_{0,1} e^{-\frac{A_{T'}}{2}} \sum_j e^{-C_{0,2}(\nu)^n} (\Delta v)^3 \geq \frac{1}{2} C_{0,1} C_\alpha e^{-\frac{A_{T'}}{2}}.$$

This, together with Lemma 5.6 (1) and Lemma 5.10 gives the lower bound for $\tilde{T}_i^n$:

$$\tilde{T}_i^n \geq \left(\frac{\tilde{\rho}_i^n}{C_M \|f^n\|_{L^\infty}}\right)^{2/3} \geq \left(\frac{C_{0,1} C_\alpha}{2CM \|f_0\|_{L^\infty}} e^{-\left(\frac{1}{2} + \frac{(C_M-1)A_{T'}}{\nu} + \frac{(C_M-1)A_{T'}}{\nu}^2\right) A_{T'}}\right)^{2/3}.$$

Then the lower bound for $\tilde{T}_i^n$ follows from the equivalence estimate in Lemma 4.3.

**Lemma 5.12.** Assume $f_{i,j}^n$ satisfies $A^n \land B^n$. Then, $f_{i,j}^n$ satisfies $D^n$:

$$\|\hat{\rho}_i^n\|_{L^\infty}, \|\hat{U}^n\|_{L^n}, \|T^n\|_{L^\infty} \leq 2C_q \left(1 + (C_{0,1} C_\alpha)^{-1} e^{-\left(\frac{1}{2} + \frac{(C_M-1)A_{T'}}{\nu} + \frac{(C_M-1)A_{T'}}{\nu}^2\right) A_{T'}}\right) \|f_0\|_{L^\infty}.$$

**Proof.** Lemma 5.10 gives

$$\tilde{\rho}_i^n = \sum_j \hat{f}_{i,j}^n (\Delta v)^3 \leq \|f^n\|_{L^\infty} \sum_j \frac{(\Delta v)^3}{(1 + |v_j|)^q} \leq 2C_q e^{-\frac{(C_M-1)A_{T'}}{\nu} A_{T'} \|f_0\|_{L^\infty}}.$$

We combine this with the lower bound estimates of $\tilde{\rho}_i^n$ established in Lemma 5.11 and the upper bound of the discrete solution in Lemma 5.10 to obtain

$$|\hat{U}^n_i| = \left|\frac{1}{\tilde{\rho}_i^n} \sum_j \hat{f}_{i,j} v_j (\Delta v)^3\right| \leq \left\{C_{0,1} C_\alpha \right\}^{-1} e^{-\frac{A_{T'}}{2}} \|f^n\|_{L^\infty} \sum_j \frac{(\Delta v)^3}{(1 + |v_j|)^q-1} \leq 2C_q \left(1 + (C_{0,1} C_\alpha)^{-1} e^{-\left(\frac{1}{2} + \frac{(C_M-1)A_{T'}}{\nu} + \frac{(C_M-1)A_{T'}}{\nu}^2\right) A_{T'}}\right) \|f_0\|_{L^\infty}.$$

The estimate for $\tilde{T}_i^n$ follows from

$$3\tilde{T}_i^n = \frac{1}{\tilde{\rho}_i^n} \sum_j \hat{f}_{i,j}^n |v_j|^2 (\Delta v)^3 - |\hat{U}^n_i|^2 \leq \frac{1}{\tilde{\rho}_i^n} \sum_j \hat{f}_{i,j}^n |v_j|^2 (\Delta v)^3 \leq \frac{1}{\tilde{\rho}_i^n} \|f^n\|_{L^\infty} \sum_j \frac{(\Delta v)^3}{(1 + |v_j|)^q-2} \leq 2C_q^{-2} \left(1 + (C_{0,1} C_\alpha)^{-1} e^{-\left(\frac{1}{2} + \frac{(C_M-1)A_{T'}}{\nu} + \frac{(C_M-1)A_{T'}}{\nu}^2\right) A_{T'}}\right) \|f_0\|_{L^\infty}.$$

by a similar manner.

**5.1. Proof of Theorem 5.5** Due to Lemma 5.8 $E_0$ holds. Assume $E^n_{n-1}$ is satisfied. Then Lemma 5.9, 5.10, 5.11 and 5.12 respectively show that $f_{i,j}^n$ satisfies $A^n, B^n, C^n, D^n$, that is $E^n$. Therefore, we can conclude that $f_{i,j}^n$ satisfies $E^n$ for all $n \geq 0$ by induction.
6. Consistent form

In this section, we transform the ES-BGK model \( \text{(1)} \) into a form which is consistent to our scheme \( \text{(14)} \). We use the following notation for continuous solutions:

\[
\tilde{f}(x, v, t) = f(x - v_1 \Delta t, v, t).
\]

**Theorem 6.1.** Under the assumption of Theorem 3.1, \( \text{(1)} \) can be represented in the following form:

\[
f(x, v, t + \Delta t) = \frac{\kappa}{\kappa + A_\nu \Delta t} \tilde{f}(x, v, t) + \frac{A_\nu \Delta t}{\kappa + A_\nu \Delta t} M_\nu(\tilde{f})(x, v, t) + \frac{A_\nu}{\kappa + A_\nu \Delta t} \{ R_1 + R_2 \},
\]

where

\[
R_1 = \int_0^{t + \Delta t} \left\{ M_\nu(f)(x, v, t) - M_\nu(\tilde{f})(x, v, t) \right\} ds
\]

\[-\int_0^{t + \Delta t} \left\{ (t + \Delta t - s) \partial_x M(x_{\theta_1}, v, t_{\theta_1}) + (s - t) \partial_t M_\nu(f)(x_{\theta_1}, v, t_{\theta_1}) \right\} ds,
\]

\[
R_2 = \int_0^{t + \Delta t} (s - t - \Delta t)(M(f) - f)(x_{\theta_2}, v, t_{\theta_2}) ds
\]

for some \( (x_{\theta_i}, v, t_{\theta_i}) \) \( (i = 1, 2) \).

**Proof.** Along the characteristic line, \( \text{(1)} \) reads

\[
\frac{df}{dt}(x + v_1 t, v, t) = \frac{1}{\kappa} A_\nu (M_\nu(f) - f)(x + v_1 t, v, t).
\]

Integrating on \( [t, t + \Delta t] \), we get

\[
f(x, v, t + \Delta t) = f(x - \Delta t v_1, v, t) + \frac{1}{\kappa} A_\nu \int_t^{t + \Delta t} (M_\nu(f) - f)(x - (t + \Delta t - s)v_1, v, s) ds
\]

\[
eq f(x - \Delta t v_1, v, t) + \frac{1}{\kappa} A_\nu (I_1 - I_2).
\]

By Taylor’s theorem around \( (x - \Delta t v_1, v, t) \), we see that there exist \( x_{\theta_1} \) which lies between \( x \) and \( x - (t + \Delta t - s) \) and \( t_{\theta_1} \in [s, t] \) such that

\[
M_\nu(f)(x - (t + \Delta t - s)v_1, v, s) = M_\nu(f)(x, v, t) - (t + \Delta t - s) \partial_x M_\nu(f)(x_{\theta_1}, v, t_{\theta_1})
\]

\[
+ (s - t) \partial_t M_\nu(f)(x_{\theta_1}, v, t_{\theta_1})
\]

\[
= M_\nu(\tilde{f})(x, v, t) + \{ M_\nu(f)(x, v, t) - M_\nu(\tilde{f})(x, v, t) \}
\]

\[-(t + \Delta t - s) \partial_x M_\nu(f)(x_{\theta_1}, v, t_{\theta_1})
\]

\[
+ (s - t) \partial_t M_\nu(f)(x_{\theta_1}, v, t_{\theta_1}).
\]

Therefore, we have

\[
I_1 = \Delta t M_\nu(\tilde{f})(x - \Delta t v_1, v, t) + R_1.
\]

On the other hand, by Taylor expansion around \( (x, v, t + \Delta t) \), we get

\[
f(x - (t + \Delta t - s)v, v, s) = f(x, v, t + \Delta t) + (s - t - \Delta t)(\partial_t v \cdot \nabla_x) f(x_{\theta_2}, v, t_{\theta_2})
\]

\[
= f(x, v, t + \Delta t) + \frac{1}{\kappa}(s - t - \Delta t) \{ M_\nu(f) - f \}(x_{\theta_2}, v, t_{\theta_2})
\]
Lemma 6.3. \( R \)

Proof. \( (25) \)

Substituting \( (23) \) and \( (24) \) into \( (22) \), we get

\[
\text{We then collect relevant terms to derive the desired result.}
\]

We now estimate the remainder terms. First, we need the following estimates.

**Lemma 6.2.** [51] Let \( f \) be solution in Theorem 3.1 corresponding to the initial data \( f_0 \). Then we have for \( q \geq 5 \)

\[
\| M_\nu(f) - M_\nu(g) \|_{L_q^\infty} \leq C_q \| f - g \|_{L_q^\infty},
\]

\[
\sum_{\theta \in [0,1]} \| \partial_\nu^\theta M_\nu(f) \|_{L^\infty} \leq CT_{r,\nu} \| f \|_{W^{1,\infty}} + 1.
\]

**Lemma 6.3.** \( R_1, R_2 \) satisfy the following estimate:

\[
(25) \quad \| R_1 \|_{L_q^\infty} + \| R_2 \|_{L_q^\infty} \leq C_{T_{r,\nu}} (\Delta t)^2.
\]

**Proof.** We start with \( R_1 \). We decompose

\[
M_\nu(\tilde{f}) - M_\nu(f) = \{ M_\nu(\tilde{f}) - M_\nu(f) \} + \{ M_\nu(f) - M_\nu(f) \}
\]

\[
\equiv I + II.
\]

For \( I \), we compute

\[
I = (\tilde{v} - \nu) \frac{\partial T_{\nu}}{\partial \nu} \frac{\partial M_\nu}{\partial T_{\nu}} = -\nu \frac{A_\nu \Delta t}{\kappa + A_\nu \Delta t} (T_{Id} - \tilde{\Theta}) \frac{\partial M_\nu}{\partial T_{\nu}}.
\]

Then, since

\[
| T_{Id} - \Theta | \leq \frac{1}{\rho} \int_{R^3} f |v - U|^2 Id + (v - U) \otimes (v - U) dv
\]

\[
\leq \frac{C}{\rho} \int_{R^3} f |v - U|^2 dv
\]

\[
= \frac{C}{\rho} \left\{ \int_{R^3} f |v|^2 dv + \rho |U|^2 \right\}
\]

\[
\leq C \rho^{-1} \| f \|_{L_q^\infty} + |U|^2,
\]

the lower and upper bound estimates of the macroscopic field given in Theorem 3.1 yield \( | T_{Id} - \Theta | \leq C_{T_{r,\nu}} \). On the other hand, it was derived in [51] that

\[
\left\| \frac{\partial M_\nu}{\partial T_{\nu}} \right\|_{L_q^\infty} \leq C_{T_{r,\nu}}.
\]

Therefore we can estimate \( I \) as

\[
\| I \|_{L_q^\infty} \leq C T_{r,\nu} \Delta t.
\]

For \( II \), we first recall Lemma 6.2 to deduce

\[
\| II \|_{L_q^\infty} \leq C \| \hat{f} - f \|_{L_q^\infty}.
\]

Then we apply the mean value theorem to estimate

\[
\| f - \hat{f} \|_{L_q^\infty} \leq \| \Delta t v \cdot \nabla_x f \|_{L_q^\infty} \leq C_q \| f \|_{W_{q+1}^{1,\infty}} \Delta t \leq C_{T_{r,\nu}} \left\{ \| f_0 \|_{W_{q+1}^{1,\infty}} + 1 \right\} \Delta t
\]

to get

\[
\| II \|_{L_q^\infty} \leq C_{T_{r,\nu}} \left\{ \| f_0 \|_{W_{q+1}^{1,\infty}} + 1 \right\} \Delta t.
\]
We combine these estimates to obtain
\[ \| \mathcal{M}_\nu(\tilde{f}) - \mathcal{M}_\nu(f) \|_{L^\infty_q} \leq C_{T^t, f_0} \Delta t; \]
from which we can estimate
\[ \left\| \int_t^{t+\Delta t} \mathcal{M}_\nu(\tilde{f}) - \mathcal{M}_\nu(f) ds \right\|_{L^\infty_q} \leq \int_t^{t+\Delta t} \left\| \mathcal{M}_\nu(\tilde{f}) - \mathcal{M}_\nu(f) \right\|_{L^\infty_q} ds \leq C_{T^t, f_0}(\Delta t)^2. \]
On the other hand, again from Lemma 6.2
\[ \left\| \int_t^{t+\Delta t} (t + \Delta t - s) \partial_x \mathcal{M}_\nu(f) ds \right\|_{L^\infty_q} \leq \| \partial_x \mathcal{M}_\nu(f) \|_{L^\infty_q} \int_t^{t+\Delta t} (t + \Delta t - s) ds \leq C_{f_0,q}(\Delta t)^2, \]
and
\[ \left\| \int_t^{t+\Delta t} (s - t) \partial_t \mathcal{M}_\nu(f) ds \right\|_{L^\infty_q} \leq \| \partial_t \mathcal{M}_\nu(f) \|_{L^\infty_q} \int_t^{t+\Delta t} (s - t) ds \leq C_{f_0,q}(\Delta t)^2. \]
This gives the desired remainder estimate for $R_1$. We compute $R_2$ similarly as
\[ \left\| \int_t^{t+\Delta t} (s - t - \Delta t) (\mathcal{M}_\nu(f) - f)(x_{\theta_2}, v, t_{\theta_2}) ds \right\|_{L^\infty_q} \leq \| \mathcal{M}_\nu(f) - f \|_{L^\infty_q} \int_t^{t+\Delta t} (s - t - \Delta t) ds \leq C_{T^t} \| f^0 \|_{L^\infty_q}(\Delta t)^2. \]
together with
\[ |\partial_x^2 f(x_{\theta_i}, v_j, t_n)| \leq \frac{\|f\|_{H^{2,\infty}_q}}{(1 + |v_j|)^q} \leq C_{T_f} \frac{\|f_0\|_{H^{2,\infty}_q} + 1}{(1 + |v_j|)^q}, \]
to estimate \( R_\theta \) as
\[ |R_\theta| \leq C_{T_f} \frac{\|f_0\|_{H^{2,\infty}_q}}{(1 + |v_j|)^q} (\Delta x)^2. \]

With these estimates, we can compute the difference of \( \tilde{f}_{i,j}^{n} \) and \( f(x_i, v, t_n) \) as follows:
\[
|\tilde{f}_{i,j}^{n} - \tilde{f}(x_i, v_j, t_n)| \leq |(1 - a_{j_1})f_{s,j}^{n} + a_{j_1}f_{s+1,j}^{n} - (1 - a_{j_1})f(x_s, v_j, t_n) - a_{j_1}f(x_{s+1}, v_j, t_n)| + |R_\theta|
\leq (1 - a_{j_1})|f_{s,j}^{n} - f(x_s, v_j, t_n)| + a_{j_1}|f_{s+1,j}^{n} - f(x_{s+1}, v_j, t_n)| + |R_\theta|.
\]
We then multiply \((1 + |v_j|)^q\) on both sides and take supremum over \(i, j\). The desired result follows from (26).

**Lemma 7.2.** Assume \( q > 5 \) and \(|\Delta v| < r_{\Delta v}\). Let \( f \) and \( f^n \) denote the solution of (4) and (14) respectively. Let \( \phi(v) \) denote one of \( 1, v, |v|^2, (1 - v)|v|^2 Id + \nu v \otimes v \) for \( v \in \mathbb{R}^3 \). Then we have
\[
\left| \sum_j \tilde{f}_{i,j}^{n} \phi(v_j) (\Delta v)^3 - \int_{\mathbb{R}^3} \tilde{f}(x_i, v, t_n) \phi(v) dv \right| 
\leq C_{T_f} \|f^n - f(t_n)\|_{L^\infty_q} + C_{T_f} \|f_0\|_{H^{2,\infty}_q} \{(\Delta x)^2 + \Delta v + \Delta v \Delta t\}. \]

**Proof.** For simplicity, we define
\[
\Delta_j = [v_{j_1}, v_{j_1+1}] \times [v_{j_2}, v_{j_2+1}] \times [v_{j_3}, v_{j_3+1}],
\]
so that
\[
\int_{\mathbb{R}^3} \tilde{f}(x_i, v, t_n) \phi(v) dv = \sum_j \int_{\Delta_j} \tilde{f}(x_i, v, t_n) \phi(v) dv.
\]
Therefore,
\[
\sum_j \tilde{f}_{i,j}^{n} \phi(v_j) (\Delta v)^3 - \int_{\mathbb{R}^3} \tilde{f}(x_i, v, t_n) \phi(v) dv 
= \sum_j \tilde{f}_{i,j}^{n} \phi(v_j) (\Delta v)^3 - \sum_j \int_{\Delta_j} \tilde{f}(x_i, v, t_n) \phi(v) dv 
= \left\{ \sum_j \tilde{f}_{i,j}^{n} \phi(v_j) (\Delta v)^3 - \sum_j \int_{\Delta_j} \tilde{f}(x_i, v, t_n) \phi(v_j) dv \right\} 
+ \sum_j \int_{\Delta_j} \tilde{f}(x_i, v, t_n) \{ \phi(v_j) - \phi(v) \} dv 
= I + II.
\]

- **(The estimate of I):** Assume \( v \in \Delta_j \) and expand \( \tilde{f}(x_i, v, t_n) = f(x_i - v^1 \Delta t, v, t_n) \) in the following two ways:

\[
f(x_s, v_j, t_n) = \tilde{f}(x_i, v, t_n) + (x_s - x_i + v^1 \Delta t) \partial_x \tilde{f}(x_i, v, t_n) + \frac{1}{2} (x_s - x_i + v^1 \Delta t)^2 \partial^2_x \tilde{f}(z_{\theta,s}) 
+ (v - v_j) \cdot \nabla_v \tilde{f}(z_{\theta,s}),
\]
We first estimate \( I \) case. We rewrite \( f(x, v, t_n) \) as
\[
\begin{align*}
\tilde{f}(x, v, t_n) &= \tilde{f}(x_i, v, t_n) + (x_{i+1} - x_i + v^1 \Delta t) \tilde{\rho_x} f(x, v, t_n) + \frac{1}{2} (x_{i+1} - x_i + v^1 \Delta t)^2 \tilde{\rho_x^2} f(z_{j+1}, 1) \\
&+ (v - v_j) \cdot \tilde{\nabla_v} f(z_{j+1, 2}),
\end{align*}
\]
where \( z_{j+1, i} \) denote the mean value points defined similarly as in the previous case. We rewrite \( f(x, v, t_n) \) and \( f(x_{i+1}, v, t_n) \) as
\[
\begin{align*}
f(x, v, t_n) &= \tilde{f}(x, v, t_n) + (x - x_i + v_j \Delta t) \tilde{\rho_x} f(x, v, t_n) + \frac{1}{2} (x - x_i + v^1 \Delta t)^2 \tilde{\rho_x^2} f(z_{j+1}) \\
&+ (v^1 - v_j) \tilde{\rho_x} f(x, v, t_n) + (v - v_j) \cdot \tilde{\nabla_v} f(z_{j+1, 2}),
\end{align*}
\]
and make a linear combination and cancel out the first order derivatives in \( x \) using (27) as in the proof of the previous lemma, to get
\[
(30) \quad \tilde{f}(x, v, t_n) = (1 - a_j) f(x, v, t_n) + a_j f(x_{i+1}, v, t_n) + R_\theta
\]
where
\[
\begin{align*}
R_\theta &= -(v^1 - v_j) \Delta t \tilde{\rho_x} f(x, v, t_n) \\
&+ \frac{1}{2} (v - v_j) (x - x_i + v^1 \Delta t)^2 \tilde{\rho_x^2} f(z_{j+1}) + \frac{1}{2} a_j (x - x_i + v^1 \Delta t)^2 \tilde{\rho_x^2} f(z_{j+1, 1}) \\
&+ (1 - a_j) (v - v_j) \cdot \tilde{\nabla_v} f(z_{j+1, 2}) + a_j (v - v_j) \cdot \tilde{\nabla_v} f(z_{j+1, 2})
\end{align*}
\]
From this, we see that
\[
\begin{align*}
\sum_j \int_{\Delta_j} \tilde{f}(x, v, t_n) \phi(v_j) dv &= \sum_j \int_{\Delta_j} \{(1 - a_j) f(x, v, t_n) + a_j f(x_{i+1}, v, t_n) + R_\theta\} \phi(v_j) dv \\
&= \sum_j \{(1 - a_j) f(x, v, t_n) + a_j f(x_{i+1}, v, t_n)\} (\Delta v)^3 \phi(v_j) \\
&+ \sum_j \int_{\Delta_j} R_\theta \phi(v_j) dv,
\end{align*}
\]
so that
\[
\begin{align*}
\sum_j \tilde{f}^n_{i,j} \phi(v_j) (\Delta v)^3 - \int_{\mathbb{R}^3} \tilde{f}(x, v, t_n) \phi(v) dv \\
&\leq \sum_j \left| \tilde{f}^n_{i,j} - \{(1 - a_j) f(x, v, t_n) + a_j f(x_{i+1}, v, t_n)\}\right| |\phi(v_j)||\Delta v|^3 \\
&+ \sum_j \int_{\Delta_j} |R_\theta| |\phi(v_j)| dv \\
&= I_1 + I_2.
\end{align*}
\]
We first estimate \( I_1 \) since we have from the definition of \( \tilde{f}_{i,j} \),
\[
\begin{align*}
|\tilde{f}^n_{i,j} - (1 - a_j) f(x, v, t_n) - a_j f(x_{i+1}, v, t_n)| \\
&= |(1 - a_j) f^a_{i,j} + a_j f^a_{i+1,j} - (1 - a_j) f(x, v, t_n) - a_j f(x_{i+1}, v, t_n)| \\
&\leq (1 - a_j) |f^a_{i,j} - f(x, v, t_n)| + a_j |f^a_{i+1,j} - f(x_{i+1}, v, t_n)|,
\end{align*}
\]
\( I_1 \) can be estimated as follows:

\[
I_1 = \sum_j \left| f^n_{i,j} - \left\{ (1 - a_{j,i})f(x_s, v_j, t_n) + a_{j,i}f(x_{s+1}, v_j, t_n) \right\}|\phi(v_j)|(\Delta v)^3 \right|
\]

\[
\leq \sum_j \left\{ (1 - a_{j,i})\|f^n - f(t_n)\|_{L^\infty_{v}} + a_{j,i}\|f^n - f(t_n)\|_{L^\infty_{v}} \right\} \frac{\|\phi(v_j)|(\Delta v)^3 (1 + |v_j|)^q}{(1 + |v_j|)^q}
\]

\[
\leq \left\{ \sum_j \frac{\|\phi(v_j)|(\Delta v)^3}{(1 + |v_j|)^q} \right\} \|f^n - f(t_n)\|_{L^\infty_{v}}
\]

\[
\leq C\|f^n - f(t_n)\|_{L^\infty_{v}},
\]

where we used \( \phi(v_j) \leq C(1 + |v_j|)^p, p = 0, 1, 2 \) and \( q - p > 3 \).

For \( I_2 \), we first observe from the definition of \( x_s \) and the fact that \( v \in \Delta_j \),

\[
|x_i - v^1\Delta t - x_s|^2 \leq 2|x_i - v_j, \Delta t - x_s|^2 + 2|v^1 - v_j, \Delta t|^2 \leq 2 \{(\Delta x)^2 + (\Delta v)^2(\Delta t)^2 \}.
\]

Similarly,

\[
|x_{s+1} - (x_i - v^1\Delta t)| \leq 2 \{(\Delta x)^2 + (\Delta v)^2(\Delta t)^2 \}.
\]

On the other hand, since \( \Delta v < 1/2 \), we have \((i = 1, 2)\)

\[
\sum_{|\alpha| + |\beta| \leq 2} \left| \partial^\alpha_{x} \partial^\beta_{t} f(z_{\alpha}) \right| + \left| \partial^\alpha_{x} \partial^\beta_{t} f(z_{\alpha}) \right| \leq \frac{\|f\|_{W^{2,\infty}_v}}{(1 + |v_j + \theta|\Delta v)^q}
\]

\[
\leq \frac{\|f\|_{W^{2,\infty}_v}}{(1 - |\Delta v| + |v_j|)^q}
\]

\[
\leq C_T, \frac{\|f_0\|_{W^{2,\infty}_v} + 1}{(1 + |v_j|)^q},
\]

so that

\[
|R_\theta| \leq C_T \{(\Delta x)^2 + (\Delta v)^2(\Delta t)^2 + \Delta v + \Delta v \Delta t \} \frac{\|f_0\|_{W^{2,\infty}_v} + 1}{(1 + |v_j|)^q}
\]

\[
\leq C_T \{(\Delta x)^2 + \Delta v + \Delta v \Delta t \} \frac{\|f_0\|_{W^{2,\infty}_v} + 1}{(1 + |v_j|)^q}.
\]

Hence we have

\[
I_2 \leq C_T \{(\Delta x)^2 + \Delta v + \Delta v \Delta t \} \|f_0\|_{W^{2,\infty}_v} \sum_j \frac{|\phi(v_j)|(\Delta v)^3}{(1 + |v_j|)^q} \leq C_{T, f_0} \{(\Delta x)^2 + \Delta v + \Delta v \Delta t \},
\]

where we used \( \int_{\Delta_j} dv = (\Delta v)^3 \). Therefore, we have the following estimate for \( I \)

\[
I \leq C\|f^n - f(t_n)\|_{L^\infty_{v}} + C_{T, f_0} \{(\Delta x)^2 + \Delta v + \Delta v \Delta t \}.
\]

\( \bullet \) (The estimate of \( II \)): Since \( |v_j| = |v + \theta \Delta v| \leq (1 + |v|) \) in \( \Delta_j \), we have for \( v \in \Delta_j \)

\[
|\phi(v) - \phi(v_j)| \leq C|v - v_j| \{|v|^p + |v_j|^p \} \leq C\Delta v(1 + |v|)^p. \quad (p = 0, 1, 2)
\]
With this, we can estimate $II$ as

$$II \leq C \Delta v \sum_j \int_{\Delta_j} \tilde{f}(x_i, v, t_n)(1 + |v|)^p \, dv$$

$$\leq C \Delta v \|f\|_{L^\infty_q} \sum_j \int_{\Delta_j} \frac{dv}{(1 + |v|)^{q-p+1}}$$

$$\leq C \Delta v \|f\|_{L^\infty_q} \int_{\mathbb{R}^3} \frac{dv}{(1 + |v|)^{q-p+1}}$$

$$\leq C_{q-p+1} \|f\|_{L^\infty_q} \Delta v,$$

which, upon applying Lemma 6.2 yields

$$II \leq C_{Tl} \|f_0\|_{L^\infty_q} \Delta v. \tag{32}$$

Finally, we combine the estimates for $I$ and $II$ to get the desired result. \hfill \Box

**Lemma 7.3.** Let $\Delta v < r_\Delta$. Suppose $f_0$ satisfies the assumptions in Theorem 3.1, then we have

$$|\tilde{\rho}_i^n - \tilde{\rho}(x_i, t_n)|, |\tilde{\nu}_i^n - \tilde{\nu}(x_i, t_n)|, |\tilde{\nu}_{vi}^n - \tilde{\nu}_{vi}(x_i, t_n)|$$

$$\leq C \|f^n - f(t_n)\|_{L^\infty_q} + C_{Tl, f_0} \{(\Delta x)^2 + \Delta v + \Delta v \Delta t\}.$$

**Proof.** Note that

$$\tilde{\rho}_i^n - \tilde{\rho}(x_i, t_n) = \sum_j \tilde{f}_{i,j}^n (\Delta v)^3 - \int_{\mathbb{R}^3} \tilde{f}(x_i, v, t_n) \, dv.$$

Therefore, it’s a direct consequence of the Lemma 7.2. For the second estimate, we observe that

$$\tilde{\nu}_i^n - \tilde{\nu}(x_i, t_n) = \frac{1}{\tilde{\rho}_i^n} \{\tilde{\rho}_i^n \tilde{\nu}_i^n - \{\tilde{\rho} \tilde{\nu}\}(x_i, t_n)\} - \frac{\tilde{\nu}}{\tilde{\rho}_i^n} (\tilde{\rho}_i^n - \tilde{\rho}(x_i, t_n)) \}.$$
which, in view of Lemma 7.2, gives the desired result. For the estimate of the temperature tensor, we recall that \( \tilde{T}_{\nu,i}^n \) contains \( \tilde{\nu} \) and decompose it as

\[
\tilde{T}_{\nu,i}^n - \tilde{T}_\nu(x_i, t_n) = (1 - \nu) \tilde{T}_\nu^n I_d + \nu \tilde{\Theta}_i^n - \left\{ (1 - \nu) \tilde{T}_\nu(x_i, t_n) I_d + \nu \tilde{\Theta}(x_i, t_n) \right\}
= (\nu - \tilde{\nu}) \left\{ \tilde{T}_\nu^n I_d - \tilde{\Theta}_i^n \right\}
+ \left\{ (1 - \nu) \tilde{T}_\nu^n I_d + \nu \tilde{\Theta}_i^n \right\} - \left\{ (1 - \nu) \tilde{T}_\nu(x_i, t_n) I_d + \nu \tilde{\Theta}(x_i, t_n) \right\}
\equiv I + II.
\]

Since \( |\tilde{\nu} - \nu| = \frac{\nu^2 \Delta t}{k + \Delta t} \), \( I \) is bounded by \( C_T I \frac{\nu^2 \Delta t}{k + \Delta t} \). The estimate of \( II \) can be carried out in the exactly same manner as in the previous case, through a tedious computation, using the following identity:

\[
\tilde{T}_\nu - \tilde{T}_\nu' = \rho_f^{-1} \left\{ (\rho T + \rho U \otimes U)_f - (\rho T + \rho U \otimes U)'_f \right\} + \frac{1}{\rho_f} \{ \rho_f U_f \otimes \rho_f U_f - \rho g U_f \otimes \rho g U_f \} - \frac{\rho_f + \rho g}{\rho_f \rho g} (\rho g U_f)^2 (\rho_f - \rho g).
\]

We omit the computation. \( \square \)

**Proposition 7.1.** Assume that \( ||f^n||_{L^\infty} < \infty \) with \( q > 5 \). Then we have

\[
||\mathcal{M}_{\tilde{\nu}}(\tilde{f}(t_n)) - \mathcal{M}_{\tilde{\nu}}(\tilde{f}^n)||_{L^\infty} \leq C_T ||f^n - f(t_n)||_{L^\infty} + C_T \{ (\Delta x)^2 + \Delta v + \Delta v \Delta t \}.
\]

The constants depend only on \( q, \nu, T \) and \( f_0 \).

**Proof.** We note that

\[
\mathcal{M}_{\tilde{\nu}}(\tilde{f}(x_i, t_n))(v_j) - \mathcal{M}_{\tilde{\nu,j}}(\tilde{f}_j^n) = \mathcal{M}_{\tilde{\nu}}(\tilde{\rho}(x_i, t_n)), \tilde{U}(x_i, t_n)), \tilde{T}(x_i, t_n)))(v_j) - \mathcal{M}_{\tilde{\nu}}(\tilde{\rho}_i^n, \tilde{U}_i^n, \tilde{T}_i^n)(v_j).
\]

Therefore, applying the Taylor series, we expand this as

\[
\begin{align*}
\mathcal{M}_{\tilde{\nu}}(\tilde{f}(x_i, t_n))(v_j) - \mathcal{M}_{\tilde{\nu,j}}(\tilde{f}_j^n) &= \left\{ \tilde{\rho}(x_i, t_n) - \tilde{\rho}_i^n \right\} \int_0^1 \frac{\partial \mathcal{M}_{\tilde{\nu}}(\theta)}{\partial \rho} d\theta \\
&+ \left\{ \tilde{U}(x_i, t_n) - \tilde{U}_i^n \right\} \int_0^1 \frac{\partial \mathcal{M}_{\tilde{\nu}}(\theta)}{\partial U} d\theta \\
&+ \left\{ \tilde{T}(x_i, t_n) - \tilde{T}_i^n \right\} \int_0^1 \frac{\partial \mathcal{M}_{\tilde{\nu}}(\theta)}{\partial T} d\theta \\
&\equiv I_1 + I_2 + I_3,
\end{align*}
\]

where

\[
\frac{\partial \mathcal{M}_{\tilde{\nu}}(\theta)}{\partial X} = \frac{\partial \mathcal{M}_{\tilde{\nu}}}{\partial X} |_{(\tilde{\rho}_{i}, \tilde{U}_{i}, \tilde{T}_{i})}.
\]

For simplicity of notation, we define transitional macroscopic fields \( \tilde{\rho}_{0i}, \tilde{U}_{0i}, \tilde{T}_{0i} \) by

\[
(\tilde{\rho}_{0i}, \tilde{U}_{0i}, \tilde{T}_{0i}) = (1 - \theta) \left( \tilde{\rho}(x_i, t_n), \tilde{U}(x_i, t_n), \tilde{T}(x_i, t_n) \right) + \theta \left( \tilde{\rho}_i^n, \tilde{U}_i^n, \tilde{T}_i^n \right).
\]

Since this is a linear combination, we can derive the following estimates for the transitional macroscopic fields:

\[
\begin{align*}
||\tilde{\rho}_{0i}||_{L^\infty}, ||\tilde{U}_{0i}||_{L^\infty}, ||\tilde{T}_{0i}||_{L^\infty} &\leq C_T, \\
\tilde{\rho}_{0i} &\geq C_T e^{-CT}, \quad \tilde{T}_{0i} \geq C_T e^{-CT}, \\
k^T \{ \tilde{T}_{0i} \} &\geq C_T e^{-CT} |k|^2, \quad k \in \mathbb{R}^3.
\end{align*}
\]
from the lower and upper bounds of continuous and discrete macroscopic fields given in Theorem 5.11, Proposition 5.11 and Proposition 5.12. On the other hand, Brum-Minkowski inequality implies that

\[
\det \{ \bar{T}_{\theta i}^n \} = \det \left\{ (1 - \theta) \bar{T}_{\theta i}^n (x_i, t_n) + \theta \bar{T}_{\theta i}^n \right\} \\
\geq \det \left\{ \bar{T}_{\theta i}^n (x_i, t_n) \right\}^{1 - \theta} \det \left\{ \bar{T}_{\theta i}^n \right\}^\theta \\
= \left\{ C_{Tf} e^{-CTf} \right\}^{1 - \theta} \left\{ C_{Tf} e^{-CTf} \right\}^\theta \\
= C_{Tf} e^{-CTf}.
\]

From these observations we have

\[
\mathcal{M}_\varphi(\theta) = \frac{\tilde{\rho}_{\theta i}^n}{\sqrt{\det(2\pi \bar{T}_{\theta i}^n)}} \exp \left( -\frac{1}{2} (v_j - \tilde{U}_{\theta i}^n)^\top \{ \tilde{T}_{\theta i}^n \}^{-1} (v_j - \tilde{U}_{\theta i}^n) \right) \\
\leq C_{Tf} \exp \left( -C_{Tf} |v_j - \tilde{U}_{\theta i}^n|^2 \right) \\
\leq C_{Tf} \exp \left( C_{Tf} |\tilde{U}_{\theta i}^n|^2 \right) \exp \left( -C_{Tf} |v_j|^2 \right) \\
\leq C_{Tf,1} e^{-C_{Tf,2}|v_j|^2}.
\]

We now estimate each integral in \( I_i \) \((i = 1, 2, 3)\). The integral in \( I_1 \) comes straightforwardly from (36):

\[
\left| \int_0^1 \frac{\partial \mathcal{M}_\varphi(\theta)}{\partial \tilde{\rho}_{\theta i}^n} \right| ds = \left| \int_0^1 \frac{1}{\tilde{\rho}_{\theta i}^n} \mathcal{M}_\varphi(\theta) ds \right| \leq C_{Tf,1} e^{-C_{Tf,2}|v_j|^2}.
\]

For the integral in \( I_2 \), we compute

\[
\left| \frac{\partial \mathcal{M}_\varphi(\theta)}{\partial \tilde{U}_{\theta i}^n} \right| = \left| (v_j - \tilde{U}_{\theta i}^n)^\top \{ \tilde{T}_{\theta i}^n \}^{-1} + \{ \tilde{T}_{\theta i}^n \}^{-1} (v_j - \tilde{U}_{\theta i}^n) \right| \mathcal{M}_\varphi(\theta) \\
\leq C_{Tf,1} \left| (v_j - \tilde{U}_{\theta i}^n)^\top \{ \tilde{T}_{\theta i}^n \}^{-1} + \{ \tilde{T}_{\theta i}^n \}^{-1} (v_j - \tilde{U}_{\theta i}^n) \right| e^{-C_{Tf,2}|v_j|^2}.
\]

In the last line, we used (36). For simplicity, set \( X = v_j - \tilde{U}_{\theta i}^n \). Then,

\[
\left| X^\top \{ \tilde{T}_{\theta i}^n \}^{-1} \right| = \sup_{|Y|=1} |X^\top \{ \tilde{T}_{\theta i}^n \}^{-1} Y| \\
= \frac{1}{2} \sup_{|Y|=1} \left| (X + Y)^\top \{ \tilde{T}_{\theta i}^n \}^{-1} (X + Y) - X^\top \{ \tilde{T}_{\theta i}^n \}^{-1} X - Y^\top \{ \tilde{T}_{\theta i}^n \}^{-1} Y \right| \\
\leq C \sup_{|Y|=1} \left( \frac{|X + Y|^2 + |X|^2 + 1}{\tilde{T}_{\theta i}^n} \right) \\
\leq C \left( 1 + \frac{|v_j - \tilde{U}_{\theta i}^n|^2}{\tilde{T}_{\theta i}^n} \right),
\]

which is, by (34), bounded by \( C_{Tf} (1 + |v_j|)^2 \). Similarly, we can derive

\[
\left| (\tilde{T}_{\theta i}^n)^{-1} (v_j - \tilde{U}_{\theta i}^n) \right| \leq C_{Tf} (1 + |v_j|)^2,
\]

so that from (38)

\[
\left| \int_0^1 \frac{\partial \mathcal{M}_\varphi(\theta)}{\partial \tilde{U}_{\theta i}^n} ds \right| \leq C_{Tf,1} (1 + |v_j|)^2 e^{-C_{Tf,2}|v_j|^2}.
\]
We now turn to the integral in $I_3$. We first observe
\[
\frac{\partial M_\nu(\theta)}{\partial T_{\theta_i}^{\alpha\beta}} = \frac{1}{2} \left\{ \frac{\det T_{\theta_i}^n}{\det T_{\theta_i}^{\alpha\beta}} \right\} \frac{\partial}{\partial T_{\theta_i}^{\alpha\beta}} \frac{1}{2} \left\{ \frac{\det T_{\theta_i}^n}{\det T_{\theta_i}^{\alpha\beta}} \right\} M_\nu(\theta)
\]
\[+ \frac{1}{2} \left\{ (v_j - \tilde{U}_{\theta_i}^n)^T \{ T_{\theta_i}^n \}^{-1} \left( \frac{\partial T_{\theta_i}^n}{\partial T_{\theta_i}^{\alpha\beta}} \right) \{ T_{\theta_i}^n \}^{-1} (v_j - \tilde{U}_{\theta_i}^n) \right\} M_\nu(\theta).
\]
To estimate the first term, we observe through an explicit computation that
\[
\det T_{\theta_i}^n = \left\{ T_{\theta_i}^{n11} T_{\theta_i}^{n22} T_{\theta_i}^{n33} - 2 T_{\theta_i}^{n12} T_{\theta_i}^{n23} T_{\theta_i}^{n31} - T_{\theta_i}^{n11} \left\{ T_{\theta_i}^{n23} \right\}^2 - T_{\theta_i}^{n22} \left\{ T_{\theta_i}^{n13} \right\}^2 - T_{\theta_i}^{n33} \left\{ T_{\theta_i}^{n12} \right\}^2 \right\}.
\]
Therefore, $\frac{\partial \det T_{\theta_i}^n}{\partial T_{\theta_i}^{\alpha\beta}}$ is a homogeneous polynomial of degree 2 having $T_{\theta_i}^{\alpha\beta}$ $(1 \leq \alpha, \beta \leq 3)$ as variables.

Now, recalling $T_{\theta_i}^{\alpha\beta} \leq C e^{T \epsilon}$ from (34), together with the lower bound of $\det T_{\theta_i}^n$ in (35), we conclude
\[
\left( \frac{\partial \det T_{\theta_i}^n}{\partial T_{\theta_i}^{\alpha\beta}} \right) \leq C_{T \epsilon}^\prime.
\]
On the other hand, since all the entries of $\frac{\partial T_{\theta_i}^n}{\partial T_{\theta_i}^{\alpha\beta}}$ are 0 except for the $\alpha, \beta$ entry, we see that
\[
\left| (v - \tilde{U}_{\theta_i}^n)^T \{ T_{\theta_i}^n \}^{-1} \left( \frac{\partial T_{\theta_i}^n}{\partial T_{\theta_i}^{\alpha\beta}} \right) \{ T_{\theta_i}^n \}^{-1} (v - \tilde{U}_{\theta_i}^n) \right| \leq \left| (v - \tilde{U}_{\theta_i}^n)^T \{ T_{\theta_i}^n \}^{-1} (v - \tilde{U}_{\theta_i}^n) \right|.
\]
which can be shown to be bounded by $C_{T \epsilon} e^{1 + (|v|)^2}$ using the same argument as in the previous case. Combining these two estimates, we bound the integral in $I_3$ as
\[
\int_0^1 \left| \frac{\partial M_\nu(\theta)}{\partial \theta} \right| d\theta \leq C_{T \epsilon} e^{-C_{T \epsilon} |v|^2}.
\]
We now insert all the above estimates into (33) to get
\[
\left| M_{\bar{\nu}, \bar{j}}(\tilde{f}^n) - M_{\bar{\nu}}(\tilde{f}(x_i, t_n))(v_j) \right| \leq C_{T \epsilon} \left\{ |\tilde{\rho}_{\theta_i}^n - \bar{\rho}(x_i, t_n)| + |\tilde{U}_{\theta_i}^n - \bar{U}(x_i, t_n)| + |\tilde{T}_{\theta_i}^n - T(x_i, t_n)| e^{-C_{T \epsilon} |v|^2}.
\]
Then, Lemma [7.3] gives the desired result.}

\section{8. Proof of Theorem 5.2}

We are now ready to prove our main theorem. Subtracting (13) from (22) and taking $L_\infty$ norms, we get
\[
\| f(t_{n+1}) - f^n + 1\|_{L_\infty} \leq \frac{\kappa}{\kappa + A_\nu \Delta t} \| f(t_n) - \tilde{f}^n \|_{L_\infty} + \frac{A_\nu \Delta t}{\kappa + A_\nu \Delta t} \| M_\nu(\tilde{f}(t_n)) - M_\nu(\tilde{f}^n) \|_{L_\infty} + \frac{A_\nu}{\kappa + A_\nu \Delta t} \| R_1 \|_{L_\infty} + \frac{A_\nu}{\kappa + A_\nu \Delta t} \| R_2 \|_{L_\infty}.
\]
We then recall the estimates in Lemma 5.3, Lemma 7.1, and Proposition 7.3
\[
\| f(t_n) - \tilde{f}^n \|_{L_\infty} \leq \| f(t_n) - f^n \|_{L_\infty} + C_{T \epsilon} (\Delta x)^2,
\]
\[
\| M_\nu(\tilde{f}(t_n)) - M_\nu(\tilde{f}^n) \|_{L_\infty} \leq C_{T \epsilon} \| f(t_n) - f^n \|_{L_\infty} + C_{T \epsilon} \left\{ (\Delta x)^2 + \Delta v + \Delta v \Delta t \right\} + R_1 \|_{L_\infty} + R_2 \|_{L_\infty} \leq C_{T \epsilon} \| f_0 \|_{W_1, \infty} (\Delta t)^2,
\]
and to derive the following recurrence inequality for the numerical error:
\[
\| f(t_{n+1}) - f^n + 1\|_{L_\infty} = \left( 1 + \frac{C_{T \epsilon} \Delta t}{\kappa + A_\nu \Delta t} \right) \| f(t_n) - f^n \|_{L_\infty} + C_{T \epsilon} P(\Delta x, \Delta v, \Delta t),
\]
where
\[ P(\Delta x, \Delta v, \Delta t) = \frac{(\Delta x)^2 + \Delta t \{ (\Delta x)^2 + \Delta v + v \Delta t \}}{\kappa + A_\nu \Delta t}. \]

Put \( \Gamma = \frac{C_T f \parallel A_\nu T f}{\kappa + A_\nu \Delta t} \) for simplicity of notation and iterate (11) until the initial step is reached, to derive:

\[ f(N_i \Delta t) - f^N_i \parallel L_q^\infty \leq (1 + \Gamma)^{N_i} f_0 - f^0 \parallel L_q^\infty + C_\nu \sum_{i=1}^{N_i} (1 + \Gamma)^{i-1} P. \]

We first note from the definition of \( f^i_{i,j} = f_0(x_i, v_j) \) that
\[ \| f_0 - f^0 \parallel L_q^\infty = \sup_{i,j} \{ f_0(x_i, v_j) - f^0 \parallel (1 + \| v \|)^q \} = 0. \]

Besides, we use \((1 + x)^n \leq e^{nx}\) to see that
\[ (1 + \Gamma)^{N_i} \leq e^{N_i \Gamma} \leq e^{C_T f \parallel A_\nu T f} = e^{C_T f \parallel A_\nu T f}, \]
so that
\[ \sum_{i=1}^{N_i} (1 + \Gamma)^{i-1} = \left( \frac{(1 + \Gamma)^{N_i}}{(1 + \Gamma)} - 1 \right) \leq C_T f \left( \frac{\kappa + A_\nu \Delta t}{\Delta t} \right) \left\{ e^{C_T f \parallel A_\nu T f} - 1 \right\}, \]
which gives
\[ \sum_{i=1}^{N_i} (1 + \Gamma)^{i-1} P \leq C_T f \left( e^{C_T f \parallel A_\nu T f} - 1 \right) \left\{ \Delta x + \Delta v + \Delta t + \frac{\Delta x}{\Delta t} \right\}. \]
Substituting these estimates into (12), we find
\[ \| f(T^f) - f^N_i \parallel L_q^\infty \leq C_T f \left( e^{C_T f \parallel A_\nu T f} - 1 \right) \left\{ (\Delta x)^2 + \Delta v + \Delta t + \frac{(\Delta x)^2}{\Delta t} \right\}. \]
This completes the proof.

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