Chaotic property of almost periodic frequency arrangement (APFA)

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Abstract

We propose an almost periodic frequency arrangement (APFA) constructed by an irrational number group created using the power root of a prime number. Based on findings, it is possible to connect more than one million channels at a base station using the super-multicarrier APFA systems with the same communication quality characteristics as the current system. In this paper, we propose a new frame Lyapunov exponent (FLE) which estimates the phase difference sensitivity depending on the frame number. We reveal the universal feature of FLE and elucidate several characteristics such as short-term radiations of phases.

Keywords chaotic property, frame Lyapunov exponent, APFA, irrational number, prime number

1. Introduction

Extensive communication and information services for connections and data processing are being developed for not only persons but also for machine to machine (M2M), accompanied by the rapid expansion of data content, transmission terminals, and the application industry [1]. However, as the number of users increases, the number of subcarriers increases, and the subcarrier interference by super-multicarrier multiplexing scheme loses the benefits by the frequency shift at the millimeter frequency bands and the narrower subcarrier frequency intervals.

Recently, chaotic spreading code generated by almost periodic functions was reported to be advantageous for super-multi access communications [2, 3]. In addition, simulation results for applicability to satellite communications are reported, and it has been observed that almost periodic frequency arrangement (APFA) has a different characteristics compared with that of existing periodic signals [4–6].

That means that the higher the number of subcarriers and the frequency division, the more it is necessary to construct communication systems such as APFA system corresponding to the frequency division multiplexing frequency.

We already reported that it has ability to connect more than one million channels at a base station by using the super-multicarrier APFA signal modulation scheme [7]. The APFA scheme (APFAS) is an asynchronous multi-carrier frequency arrangement with quasi-orthogonality. Here we introduce the new frame Lyapunov exponent which estimates the phase difference sensitivity of APFA systems [8, 9].

In the elucidation of frame Lyapunov exponent (FLE) \( \lambda(n) \) in double logarithm, \( Q_1 \) means the primary component (slope) of initial sensitivity, \( C_0 \) and \( C(n) \) refer to the stationary long-term and the short-term variation of FLE in which \( n \) is the number of frame numbers respectively.

2. APFA systems

2.1 Basic principle

While almost periodic functions (APF) was first introduced in the study of the spread spectrum scheme in 2014, APF itself was already proposed in 1924, by H. Bohr as an extension of a conventional periodic function \(|f(x + \tau) - f(x)| \leq \epsilon\), where \( f(x) \) is a complex function, \( x \) is a real parameter, and \( \tau \) is a distance from \( x \) on \( f(x) \) that belongs to \( \epsilon \) as a positive number [3].

The subcarrier frequency of the existing OFDM system or OFDMA system needs to be composed of rational number groups, while the APFA system has almost periodic frequency arrangement to the combination of disjoint irrational number groups for dispersion of coupled waves due to the nonlinear distortion of devices.

In order to select the subcarrier frequency \( M \) of APFA from the disjoint irrational number groups, there are the Dedekind cuts to put rational number partitions into irrational numbers [10]. The Dedekind cuts show that there are irrational numbers within rational number partitions using the Weyl uniform distribution [11].

The APFA scheme is using the signal of \( L_2 \) space [12].

2.2 Subcarrier frequency of APFA

We define APFF (almost periodic functions frequency) \( f_k(p_i) \) which transform by linear mapping of prime numbers as follows:

\[
f_k(p_i) \equiv \sqrt[p_i]{i} \mod 1, \quad i = 1, 2, \ldots I,
\]

where \( k \) is the power root index, \( p_i \) is the \( i \)-th prime number, and \( I \) is the number of prime numbers.
The function $f_k(p_i)$ is almost uniformly distributed over the normalized frequency $0 \leq f_m \leq 1$ and have a disjoint set of irrational numbers.

Fig. 1 is a scatter diagram of the APFF distribution, showing almost uniform distribution, where $k = 2$ and the $I = 10^5$.

In this paper, the APFA frequencies are created from APFF by using $M$-partitions with the same number of subcarriers in an OFDM system. When the number of prime numbers $I$ is sufficiently large, the APFF frequency probability distribution between $M$-partitions is uniformly distributed and is expressed by $p(I,m) = \frac{1}{M}$, $m = 1, 2, \ldots , M$.

The nearest irrational number of prime number $i$ ($1 \leq i \leq I$) to $m$-th partition can be selected as follows;

$$f_{\text{APFA}}(k,m) = f_k(p_i),$$  \hspace{1cm} (2)

where $i$ is the integer such that $|f_m - f_k(p_i)|$ is the minimal value for $1 \leq i \leq I$, and $\{f_m\} = \{m\Delta F - 0.5\Delta F\}$, $m = 1, 2, \ldots , M$.

The normalized offset frequency (NOF) $\Delta f(k,m)$ is presented by the almost periodic frequency $f_{\text{APFA}}(k,m)$ and $f_m$ ($m$-th partition) as follows;

$$\Delta f(k,m) = f_{\text{APFA}}(k,m) - f_m, \quad m = 1, 2, \ldots , M.$$  \hspace{1cm} (3)

The normalized standard deviation $\sigma_{k,M}$ of NOF are calculated by the follows;

$$\sigma_{k,M} = \sqrt{\frac{\sum_{m=1}^{M} \Delta f(k,m)^2}{M}}.$$  \hspace{1cm} (4)

### 2.3 APFA characteristics using prime group

Since the power root of a prime number is an irrational number, the APFF frequency can be configured with an irrational number.

Fig. 2 shows an example of the frequency arrangement of OFDM, where $\Delta F = 1/M$.

Fig. 3 shows an example of the frequency arrangement of APFA. The center frequency of the APFA subcarriers corresponds to the separation made by the rational number of the irrational frequency group, and it is the same as the OFDM system or the OFDMA system in Fig. 2.

The APFA subcarriers are distributed around the center frequency by the normalized standard deviation $\sigma_{k,M}$ from the center frequency $f_m$ same as the OFDM system or the OFDMA system. In the APFA system, signal processing is performed on the base frequency band. The frequency allocation $f_{\text{APFA}}$ on the transmission band is frequency-shifted to the basic frequency allocation $F_{\text{APFA}}$ for the positive side by $\Delta F/2$ as shown as follows;

$$f_{\text{APFA}}(k,m) \equiv f_{\text{APFA}}(k,m) + \frac{\Delta F}{2}, \quad m = 1, 2, \ldots , M.$$  \hspace{1cm} (5)

Eq. (6) is an approximate expression obtained by mapping a curve of Fig. 4 by a logarithmic mapping and by a first regression analysis. The standard deviation $\sigma_{k,M}$ of the NOF can be obtained from (6) given the prime numbers and the number of subcarriers [7]. Using Fig. 4 or (6), by determining $\sigma_{k,M}$ and the number of subcarriers, the maximum number of the prime numbers to be prepared for the simulation is obtained.

$$\sigma_{k,M}(\pi(N_a)) = 10^{-0.9719 \log_{10}(\pi(N_a)) - 0.64} + 1.635,$$  \hspace{1cm} (6)

where $\pi(N_a)$ is the number of prime numbers in the set of natural numbers less than or equal to $N_a$.

### 3. Chaotic property

#### 3.1 Frame Lyapunov exponent (FLE)

$f_{\text{APFA}}(k,m)$ in (2) is set within the transmission frequency band for the communication path. In the chaos analysis, the frequency of each subcarrier is handled at
the $F_{APFA}(k, m)$ within the basic frequency band in (5).

The phase of the transmit signal $T X_{\sigma_{k,M}}(t)$ of APFA is expressed by a sum of exponential functions using a real part $\alpha_{k,M}$ and an imaginary part $\beta_{k,M}$ as follows:

$$e^{\alpha_{k,M}(t)} + j e^{\beta_{k,M}(t)} = \frac{1}{\sqrt{M}} \sum_{m=1}^{M} e^{j2\pi F_{APFA}(k,m)t}, \quad (7)$$

where $k$ is the index of radical expression, and $\sigma_{k,M}$ is the normalized standard deviation of offset frequency between target frequency and APFA frequency.

Letting $t_n \equiv T_n \cdot n = 1/\Delta F \cdot n$ denote the discrete sampling time of each frame, (7) is expressed as follows;

$$e^{\alpha_{k,M}(t_n)} + j e^{\beta_{k,M}(t_n)} = \frac{1}{\sqrt{M}} \sum_{m=1}^{M} e^{j2\pi F_{APFA}(k,m)t_n}. \quad (8)$$

Here, $F_{APFA}(k, m)$ shown in (5) contains frequency component $(f_m + \Delta F/2)$ as the periodic function part and the almost periodic frequency component part $\Delta F_{APFA}(k, m)$, $(1 \leq m \leq M)$ in the basic frequency band that is same as $\Delta f(k, m)$ in the transmission frequency band,

$$F_{APFA}(k, m) = f_m + \Delta F/2 + \Delta f_{APFA}(k, m). \quad (9)$$

Since $(f_m - 0.5\Delta F) \cdot t_n = m\Delta F \cdot 1/\Delta F \cdot n = m \cdot n$ is an integer.

$$\beta_{k,M}(t_n) = \arg \left( \frac{1}{\sqrt{M}} \sum_{m=1}^{M} e^{j2\pi F_{APFA}(k,m)t_n} \right). \quad (10)$$

The difference between the phase $\beta_{k,M}^1(n)$ of signal 1 and the phase $\beta_{k,M}^2(n)$ of signal 2 at the point of $t = t_n$ is expressed by $\epsilon(n)$ as follows:

$$\epsilon(n) = \beta_{k,M}^1(n) - \beta_{k,M}^2(n), \quad n = 1, 2, \ldots . \quad (11)$$

Here, the frame Lyapunov exponent (FLE) is proposed to be defined by the following orbital sensitivity;

$$\lambda(n) = \frac{1}{n} \ln \left| \frac{\epsilon(n)}{\epsilon(0)} \right|, \quad n = 1, 2, \ldots . \quad (12)$$

where, $\epsilon(n)$ is the phase difference between signal 1 and signal 2 at frame number $n$, $\epsilon(0)$ is the phase difference between signal 1 and signal 2 at frame number 0.

The following transformations are performed to make it simple to calculate the local Lyapunov exponents in double-logarithms of (12) in case of finite interval.

$$\log_{10} \lambda(n) = -\log_{10} n + \log_{10} \ln \left| \frac{\epsilon(n)}{\epsilon(0)} \right|, \quad n = 1, 2, \ldots . \quad (13)$$

### 3.2 Characteristics of FLE

Orbital sensitivity is expressed with a positive value of the FLE. However, the FLE tends to decrease as $n$ increases; hence, short-term change is reduced to $1/n$.

In this paper, we reveal the elucidation using the primary component (slope) $Q_1$, short-term variation $C(n)$ at frame number $n$, and stationary long-term parameter $C_0$ on the initial sensitivity of FLE.

The natural logarithm part of the Lyapunov exponent $\ln |\epsilon(n)/\epsilon(0)|$ is expressed by using $Q_1$, $C(n)$, and $C_0$ as follows:

$$C(n)C_0^n Q_1 \equiv \ln \left| \frac{\epsilon(n)}{\epsilon(0)} \right|, \quad n = 1, 2, \ldots . \quad (14)$$

Thus, the exponent of $n$ is expressed by a function of $\log_{10} n$ with $Q_1$ as a parameter from (13) and (14).

$$\log_{10} \lambda(n) = (Q_1 - 1) \log_{10} n + \log_{10} C_0 C(n). \quad (15)$$

If we put $y(x) = \log_{10} \lambda(n)$, $x = \log_{10} n$ and express it as a linear function of $x$ of the form $y(x) = a_1 x + a_0$, then the short-term variation, $C(n)$, fluctuations can be neglected in the long-term variation. Thus, when $a_1$ and $a_0$ are estimated, $Q_1$ and $C_0$ are determined as follows:

$$C_0 = 10^{a_0}, \quad Q_1 = 1 + a_1. \quad (16)$$

By (15), the classification of the FLE can be as follows. If $Q_1 \geq 0$, then $\lambda(n)$ decreases slowly below the order of $1/n$.

1. $\lambda(n) > 0$ Unstable (initial value sensitive Chaos),

2. $Q_1 > 0 : \lambda(n) = O\left(\frac{1}{n}\right) (\gamma \leq 1),$

3. $Q_1 < 0 : \lambda(n) = O\left(\frac{1}{n}\right) (\gamma > 1),$

### 3.3 Short-term variation of FLE

The short-term variations $C(n)$ are obtained using (16) as shown in Fig. 9. Fig. 9 exhibits the short-term variation, $C(n)$, of the Lyapunov exponent at $\sigma_{k,M}^1 : 0.049$. Even if the number of frames $n$ is over than 100, the short-term variation is clearly expressed.

### 4. Conclusion

We discuss the APFA scheme on the premise of Hilbert space $L^2$ for a disjoint set of irrational numbers.

It is possible to cut the irrational number group using rational numbers at any time when the irrational number shows a uniform distribution. Thus, it is also possible to construct an APFA scheme using an irrational number group that guaranteed by a Weyl uniform distribution theorem.

| Items | $\sigma_{k,M}^1$ | $\pi(N_a)$ | $a_1$ | $Q_1$ | Reference |
|-------|-----------------|-------------|------|-------|-----------|
| 1     | 0.0099          | 700         | -0.921 | 0.079 | Fig. 5    |
| 2     | 0.0449          | 1230        | -0.383 | 0.165 | Fig. 6    |
| 3     | 0.01            | 8420        | -0.879 | 0.121 | Fig. 7    |
| 4     | 0.0005          | 16535       | -0.762 | 0.238 | Fig. 8    |
| 5     | 0.0033          | 28000       | -0.9022 | 0.0978 |           |
| 6     | 0.00001         | 81000       | -0.8875 | 0.1125 |           |
In this paper, we propose a new frame Lyapunov exponent (FLE) which estimates the phase difference sensitivity depending on the frame number in APFA. By using FLE, we feature the chaotic property of the APFA signals in which frequencies are allocated from the normalized irrational frequency group by linearly mapping the irrational number group, which comprises the power root of a prime number.

We reveal the universal feature of FLE that the frame Lyapunov exponent \( \lambda(n) \) has the power law such as \( \lambda(n) \sim O(n^{-\gamma}) \) and the exponent \( \gamma \) depends on the standard deviation \( \sigma_{k,M} \) of the normalized offset frequency (NOF) at APFA system.

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