Duality picture between antiferromagnetism and d-wave superconductivity in $t - J$ model at two dimensions

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Abstract

We show in this paper an interesting relation between elementary and topological excitations in the antiferromagnetic and d-wave superconducting phases of the $t - J$ model at two dimensions. The topological spin and charge excitations in one phase have the same dynamics as elementary excitations in the other phase, except the appearance of energy gaps. Moreover, the transition from one phase to another can be described as a quantum disordering transition associated with the topological excitations. Based on the above picture, a plausible phase diagram of $t - J$ model is constructed.

74.20.Mn, 74.72.-h, 75.10.Jm
I. INTRODUCTION

It is now commonly believed that a large part of the phase diagram of high-$T_c$ superconductors and corresponding complicated phenomenologies are results of subtle interplay between antiferromagnetism and (d-wave) superconductivity in these materials. There exists now many theoretical attempts to understand the relationship between the antiferromagnetic and superconducting phases in the cuprates. For example, Pines et al. have tried to interpret the d-wave superconducting phase as an almost antiferromagnetic Fermi liquid [1], with superconductivity being driven by antiferromagnetic spinwave fluctuations. A somewhat related approach was also employed by Sarker [2] using the slave-fermion approach to the $t-J$ model [4], where he proposed that d-wave superconductivity arises in the hole-driven quantum disordered phase of antiferromagnets as a result of formation of spin resonant-valence-bonds (RVB) and spin-charge recombination. Another completely different approach was proposed by Zhang et al. [3] where they proposed that antiferromagnetism and d-wave superconductivity are related by an hidden SO(5) symmetry in the system [3]. All these proposals have met criticisms of one form or other and there is no universally accepted theory at present.

In this paper we shall propose a duality relation between antiferromagnetism and d-wave superconductivity in the $t-J$ model. In existing treatments of $t-J$ model, antiferromagnetism and (d-wave) superconductivity appear separately in very different ways of treating the problem. The antiferromagnetic (or spiral) state is best described by the slave-fermion mean-field theory (SFMFT) [4] where the spins are represented by bosons and holes by fermions, whereas d-wave superconductivity is described by the slave-boson mean-field theory (SBMFT) [5] where spins are represented by fermions and holes by bosons. The properties of the ground states and the corresponding elementary excitations are very different in the two mean-field theories. In SFMFT, the elementary excitations are bosonic spins (spinwaves) and fermionic holes. d-wave superconductivity and Fermi surface satisfying the Luttinger Theorem are absent in the mean-field treatment. On the other hand, SBMFT
describes the d-wave superconducting state, spin-gap phase, and Fermi-surfaces fairly satisfactory in the underdoped to optimally-doped regime of high-$T_c$ cuprates, but fails to describe correctly the antiferromagnetic (or spiral) state. The elementary excitations in SBMFT are fermionic spins and bosonic holes. In this paper, we shall propose that despite the huge differences in the structure of the mean-field theories, an intimate relation exists between the two phases of the $t - J$ model. The elementary excitations in one mean-field theory can be viewed as topological excitations in the other mean-field theory and the mean-field ground state of one theory can be viewed as a quantum disordered state formed by “condensation” of topological excitations obtained in the other mean-field theory. Based on this physical picture and with some self-consistency requirements, we shall construct a plausible $\delta - T$ phase diagram for the $t - J$ model. The organization of our paper is as follows. In section II we shall discuss the construction and dynamics of the topological spin and charge excitations in SFMFT where we shall show that the topological excitations in SFMFT have same statistics and dynamics as elementary excitations in the SBMFT except that they are gapped in the antiferromagnetic phase. In section III We shall discuss the corresponding case of topological excitations in SBMFT where we shall show that similar relations exist between the topological excitations in SBMFT and elementary excitations in SFMFT. With these results we shall argue in section IV that the ground state described by one mean-field theory can be viewed as a quantum disordered state of topological excitations obtained in the other mean-field theory. Based on this physical picture and with some self-consistency requirements, we shall propose a plausible $\delta - T$ phase diagram of the $t - J$ model. Our results are summarized in section V where some experimental consequences of our theory will be discussed and a physical picture of the duality relation based on Resonant-Valence-Bond (RVB) picture will be presented.

Before we proceed further we first clarify a notation. In the following we shall call the topological spin and charge excitations 'spinons' and 'holons', respectively to distinguish them from corresponding elementary excitations in mean-field theories. Notice that there are two different kinds of spinons and holons, corresponding to two different mean-field
theories.

II. TOPOLOGICAL EXCITATIONS IN SLAVE-FERMION MEAN-FIELD THEORY

In SFMFT, the spins are represented by Schwinger boson operators $\bar{Z}_{i\sigma}, Z_{i\sigma}$ whereas holes are represented by slave fermion operators $f_i^+, f_i$. In terms of these operators the mean-field Hamiltonian has the following form, $H_{MF} = H^s_{mf} + H^h_{mf}$, where

$$
H^s_{mf} = -\frac{J}{2} \sum_{<i,j>} (\Delta^*(Z_{i\uparrow}Z_{j\downarrow} - Z_{i\downarrow}Z_{j\uparrow}) + H.C.) + \sum_{<i,j>,\sigma} \left(\frac{J}{2} \chi^*_{\nu} + tF^*_{\nu}\right) Z_{i\sigma} Z_{j\sigma} + H.C.) + \sum_i \lambda \left(\sum_\sigma (Z_{i\sigma}Z_{i\sigma} - (2S - \delta)\right),
$$

and

$$
H^h_{mf} = \sum_{<i,j>} \left(t\chi_{\nu} f_j^+ f_i + H.C.\right) - \sum_i \mu f_i^+ f_i,
$$

where $<i,j>$ are nearest neighbor pairs of sites on a square lattice, $j = i + \nu$ where $\nu = \pm \hat{x}, \pm \hat{y}$. We shall put site $i$’s on A-sublattice and site $j$’s on B-sublattice in all our following discussions. $S = 1/2$ is the spin magnitude in $t-J$ model. $\Delta, \chi_{\nu}, F_{\nu}$ and $\lambda$ are mean-field parameters determined by the mean-field equations $\Delta = <Z_{i\uparrow}Z_{j\downarrow} - Z_{i\downarrow}Z_{j\uparrow}>$, $\chi_{\nu} = \sum_{\sigma} <\bar{Z}_{i\sigma}Z_{j\sigma}>$, $F_{\nu} = <f_{j\uparrow}^+ f_i>$, $<\bar{Z}_{i\uparrow}Z_{j\uparrow}> + <\bar{Z}_{i\downarrow}Z_{j\downarrow}> = 2S - \delta$ and $<f_{i\downarrow}^+ f_i> = \delta$.

We shall choose a gauge where the mean-field parameters $\Delta$ have $s$-symmetry, and $\chi_{\nu}$ and $F_{\nu}$ have $p$-symmetry $\chi(F)_{-\nu} = -\chi(F)_{\nu}$ in the spiral states [4]. Notice that at two dimensions, long-ranged (spiral) antiferromagnetic order exists at zero temperature in SFMFT at small doping, corresponding to Bose-condensation $<Z> \neq 0$ in mean-field theory. We shall consider finite temperature $T \neq 0$ and $<Z> = 0$ in the following. The effect of Bose condensation will be addressed at the end of the section.

To look for topological excitations in SFMFT, we notice that the structure of the mean-field theory resembles very much the BCS theory for superconductivity, except that spin-pairs of bosons replace the electron (fermion) Cooper pairs in BCS theory. The resemblance
of the two theories leads us to study vortex excitations in SFMFT, since vortices are stable topological excitations in BCS theory at two dimensions. In BCS theory, a vortex located at $\vec{r} = 0$ is a solution of the BCS mean-field equation, where the order parameter $\Delta_{BCS}(\vec{r})$ has a form

$$\Delta_{BCS}(\vec{r}) = f(r)e^{i\theta},$$

in a polar coordinate, where $f(r)$ is real and positive. To minimize energy, a magnetix flux of $\pi$-flux quanta is trapped in the vortex core. The vortex solution in SFMFT has the same structure, except that the BCS order parameter $\Delta_{BCS}$ is replaced by the Schwinger boson order parameter $\Delta_{ij}$ and the vector potential $\vec{A}$ does not represent the physical magnetic field, but is a fictitious gauge field arising from phase fluctuations of order parameter $\Delta_{ij}$’s. The existence and stability of the vortex solution in Heisenberg model was demonstrated in Ref. [7,8] in an effective Ginsburg-Landau theory. These vortex solutions are bosonic, $S = 0$ topological excitations in SFMFT [7] (see fig.1a).

To construct topological spin excitations (spinons) we note, that like vortices in superconductors where electronic bound states often exist inside the vortex core, bosonic (spin) bound states may exist inside vortices in SFMFT. In particular, we have argued that in the zero doping limit, for a vortex centered at a lattice site (fig.1b), a bound state with one boson must be formed at the vortex center because of constraint that there is always one spin per site in the Heisenberg model [7,8]. In particular, because of statistics transmutation associated with binding quantum particle to flux-tubes of $\pi$-flux quantum in two dimensions [9], the resulting excitation is a spin-1/2 fermion [7,8,10].

In the presence of holes, a new type of vortex solution with vortex center located at a lattice site may form. In this case, one may remove a spin from the center of vortex, leaving a hole (empty site) there (fig.1c). The resulting object has charge one, spin zero and is a topological hole excitation (holon). Because of statistics transmutation effect associated with binding quantum particles to $\pi$-flux quantum, the resulting holon is a boson [7].

It is helpful to view the vortex excitations from a ”wavefunction” picture [10]. First
we consider the zero-doping limit of Heisenberg model. In this limit, the slave-fermion mean-field theory with $< Z > = 0$ can be understood as the mean-field description of a short-ranged RVB (resonant valence bond) state where the wavefunction is made up of short-ranged spin-singlet pairs with the two spins of the singlet resting on opposite sublattices. The relative phases between the spin singlets are fixed by the Marshall sign rule, and the total wavefunction represents a Bose-condensate of spin singlet pairs, just like a BCS superconducting state can be viewed as a Bose-condensed state of Cooper pairs. The coherent phase structure of the ground state wavefunction allows us to construct topological excitations (vortices) in this condensate where the phases of the spin singlets change by $2\pi$ when going around the center of a vortex. Notice that although SFMFT does not enforce the requirement of no double occupancy in the $t-J$ model rigorously, the correct phase structure of the wavefunction (Marshall sign rule) is kept. Therefore we expect that the qualitative properties of vortex excitations are correct in SFMFT. In the spiral states the consideration is similar, except that the two spins in a spin-singlet pair may occupy the same sublattice in the spiral states and the phases are not given by Marshall sign rule but are nevertheless fixed. Similar consideration can also be applied to construct vortex excitations in SBMFT.

To study the dynamics of holons and spinons, we shall consider the continuum limit of SFMFT and derive an effective action for vortices in the limit of small hole concentration $\delta$. An important difference between SFMFT and BCS theory is that the mean-field Hamiltonian breaks translational symmetry of the Heisenberg model by one lattice site. Correspondingly, to describe the dynamics in SFMFT correctly we must keep two lattice sites per unit cell in SFMFT and the fluctuations of the order parameter $\Delta_{ij}$'s are described in general by two amplitude and two phase (uniform and staggered) fields in the continuum limit, i.e.

$$\Delta_{ij\pm\nu} = \frac{1}{2} \left( \phi(i \pm \nu/2) + q_{\pm\nu}(i \pm \nu/2) \right) e^{i \int \frac{d^2 \mathbf{x}}{2} \mathbf{A}_s - A_s^{\pm\nu}(i \pm \nu/2)},$$

where $q_{-\nu} = -q_{\nu}$, $A_{-,\nu} = -A_{\nu}$ are 'staggered' components of the amplitude and phase.
fluctuations of $\Delta$, respectively, $\phi$ and $\int \vec{A}.d\vec{x}'$ are the corresponding 'uniform' components. Correspondingly, the constraint field $\lambda$ is also separated into 'uniform' and 'staggered' components $\lambda^u$ and $\lambda^s$. In momentum space ($\vec{k} \neq 0$),

$$\lambda^{u(s)}(\vec{k}) = \frac{1}{2} [\lambda^A(\vec{k}) + (-)\lambda^B(\vec{k})],$$

(3)

where $A(B)$ are sublattice indices. The effective Ginsburg-Landau action for the continuum field variables is derived in Refs. [6] and Ref. [7] in the zero hole concentration $\delta = 0$ limit. We obtain to order $O(m^0)$ ($m$ is the mass gap for spinwave excitations in SFMFT which is very small at low temperature ($m \sim J e^{-JS/T}$)), $\lambda^u \sim 2\phi$, and $S_{\text{eff}} = S_u(0) + S_s$, where

$$S_u(0) \sim \frac{J}{2} \int d\tau \int d^2x \left( (2a_1 - 4(2S + 1))\phi + 2\phi^2 + (2S + 1)\phi(2\vec{A})^2 \right),$$

(4)

describes the fluctuations of the 'uniform' fields. $\vec{A} = (A_x, A_y)$, $S = 1/2$ and $a_1 < 4$ is a numerical constant. Notice that in general a term $\sim F^{2}_{\mu\nu}$ where $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ also exists in $S_u(0)$ which we have not included in Eq. (4). We find that the 'uniform' gauge field $\vec{A}$ acquires a gap $\sim 4(2S + 1)\phi$ (Meissner effect) as a result of nonzero $<\Delta_{ij}>$ in SFMFT. The existence of stable vortex solution in SFMFT is tied with the existence of Meissner effect as in usual BCS superconductors [4,5]. Notice that we have chosen the London gauge in deriving the action $S_u$ which breaks the gauge symmetry $\Delta_{i,i\pm\nu} \rightarrow \Delta_{i,i\pm\nu} e^{i\theta}$, $Z_{i\sigma} \rightarrow Z_{i\sigma} e^{i\theta/2}$, and $Z_{i+\nu\sigma} \rightarrow Z_{i+\nu\sigma} e^{i\theta/2}$. The correct gauge transformation property can be recovered by replacing $2\vec{A} \rightarrow 2\vec{A} - \nabla \theta$ in $S_u$. Correspondingly, we also have

$$S_s \sim \frac{J}{2} \int d\tau \int d^2x \left( (1 - \frac{2b_1}{\phi})(q_\mu)^2 + \frac{1}{e^2} F^{(s)2}_{\mu\nu} + 2ic_1 F^{(s)}_{\mu\tau} q_\mu \right),$$

(5)

where $e^2 \sim m$, $b_1$ and $c_1$ are constants of order $O(1)$. $F^{(s)}_{\mu\nu} = \partial_\mu A^s_\nu - \partial_\nu A^s_\mu$ where $\mu = \hat{x}, \hat{y}, \tau$ is a space-time index. The time-component of staggered gauge field is given by $A^s_\tau = \lambda^s$. It turns out that the staggered gauge field is not crucial to formation of vortices but affects the vortex dynamics [8]. Since we shall treat the dynamics of vortices only at a phenomenological level in this paper, we shall concentrate at $S_u$ and ignore $S_s$ in the following.
In the presence of holes additional terms appear in the action. In the continuum limit and for small hole concentration \(\delta\), we have \(S_u = S_u(\delta) + S_h\), where \(S_u(\delta)\) is obtained from \(S_u(0)\) by replacing \(2\delta\rightarrow 2\delta - \delta\) in Eq. (I), and

\[
S_h = -\frac{t^2}{J} \int d\tau \int d^2x \left( d_\alpha \frac{n_h^2}{\phi} \right),
\]

where \(n_h(\vec{x},t)\) is the concentration of holes at space-time position \(\vec{x},t\), and \(d_\alpha\) is a constant of order one which depends on the type of spiral state being formed [4]. Gradient terms in \(n_h\) are being neglected.

To derive the dynamics for vortices (spinons and holons), we first minimize \(S_u\) with respect to the \(\phi\) field, obtaining for the ground state

\[
\phi_0 \sim \frac{1}{2} \left( 2(2S + 1 - \delta) - a_1 - \frac{4d_\alpha t^2\delta^2}{(2(2S + 1) - a_1)^2J^2} \right) + O(\delta^3),
\]

in the \(\delta t << J\) limit. The dynamics of the ‘uniform’ phase fluctuations can be obtained by expanding \(S_u\) around \(\phi_0\). We obtain

\[
S_{\text{phase}} = \frac{J}{2} \int d\tau \int d^2x \left( \xi_x(\nabla \theta - 2\vec{A})^2 + \xi_\tau(\partial_\tau \theta - 2A_\tau)^2 \right),
\]

where \(A_\tau = \lambda_u \sim 2(\phi - \phi_0)\), and

\[
\xi_x = (2 - \delta)\phi_0, \quad \xi_\tau = \frac{1}{2}(1 - \frac{d_\alpha t^2\delta^2}{\phi_0^2 J^2}).
\]

We have set \(S = 1/2\) in writing down \(\xi_x\) and have inserted the \(\partial_\mu \theta\) factors back to recover gauge invariance in \(S_{\text{phase}}\).

Next we introduce vortex 3-current \(\vec{j}^v\) in the boson phase field, \(\partial_\mu \theta \rightarrow \partial_\mu \theta + \partial_\mu \theta'\), where \(\theta'\) is multivalued and \(j^v_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu \partial_\lambda \theta' \neq 0\). In the London limit where we treat \(\phi_0\) as constant a duality transformation can be performed where we can integrate out the \(\theta\) field to obtain an effective action \(S_v\) for vortices [12],

\[
S_v = \frac{J}{2} \int d\tau \int d^2x \left( \frac{1}{\xi_x}|(\nabla \times \vec{a})_x|^2 + \frac{1}{\xi_\tau}|(\nabla \times \vec{a})_\tau|^2 + 2i\vec{a} \cdot (\vec{j}^v - 2\nabla \times \vec{A}) \right) + S_v^{(0)}
\]

where \(\nabla \times \vec{a} \sim (\nabla \theta' - \vec{A})\), \((\nabla \times \vec{a})_x\) and \((\nabla \times \vec{a})_\tau\) are the spacial and temporal part of \(\nabla \times \vec{a}\), respectively. \(S_v^{(0)}\) denotes an additional contribution to the vortex action from vortex
core, $S_v^{(0)} \sim \{(\text{energy needed to create vortex core, } \epsilon_v) \times (\text{length of vortex trajectory in space-time})\}$. For $N$ vortices,

$$S_v^{(0)} \sim \epsilon_v \sum_{i=1}^{N} \int dl_i = \epsilon_v \sum_{i=1}^{N} \int d\tau \sqrt{1 + \frac{1}{c_v^2}(\frac{d\vec{x}_i}{d\tau})^2},$$

(8b)

where $\vec{x}_i(\tau)$ represents the trajectory of the $i$th vortex in Euclidean space-time, $\epsilon_v \sim \xi_x J$ and $c_v \sim \sqrt{\xi_x / \xi_\tau}$. We shall first consider $S_v^{(0)}$ in the following.

Minimizing $S_v^{(0)}$ at real time it is easy to see that $S_v^{(0)}$ describes relativistic particles with energy $E = \gamma \epsilon_v$, where $\gamma = 1/\sqrt{1 - \frac{\bar{v}^2}{c_v^2}}$ and $\bar{v}$ is the vortex velocity. In the absence of $\nabla \times \vec{A}$ term the particles carry 'charge' (vorticity) and interact with each other through an effective U(1) gauge field $\vec{a}$. For bosonic vortices the corresponding quantum field theory is a relativistic theory of scalar electrodynamics with charged bosons (vortices) [12]. In the presence of trapped magnetic flux inside vortex core ($\nabla \times (2\vec{A}) - \vec{j}_v \sim 0$), the electric 3-current $\vec{j}_v$ is screened and the bosons decoupled from the gauge field $\vec{a}$. The resulting theory is a relativistic theory of bosons with short-ranged interactions, as is in the case of vortices in usual superconductors.

To derive dynamics for the spinons we assume that once the bosonic spins are bound to the vortex core, their spacial degree of freedom is quenched and the only modifications to the pure vortex action (8) are: (1) the vortices now carry spin indices $m = -1/2, 1/2$, and there are two spin component of vortices, and (2) vortices become fermions because of statistics transmutation. In particular, since the vortex action (8) is Lorentz invariant, the dynamics of spinons must be described by $(2 + 1)d$-relativistic field theories of fermions. To proceed further we examine the symmetry constraints imposed by the SFMFT. Since there are two lattice sites per unit cell in SFMFT, there are also two quantum fields $\psi^A_\sigma$ and $\psi^B_\sigma$ in the continuum theory, representing spin-$\sigma$ vortices centered at $A$- and $B$- sublattice sites, respectively [7]. Under reflection or rotation by $\pi/2$ around center of a square plaquette, the $A$- and $B$- sublattices are interchanged and correspondingly also the $\psi^A_\sigma$ and $\psi^B_\sigma$ fields. Notice that we have considered finite temperature where $< Z > = 0$ in our discussion and correspondingly parity (space-time reflection) symmetry is unbroken. The spinons are also
coupled to gauge fields $A_\mu$ and $A_\mu^s$ through the bound spins at the center of vortices. To describe these couplings, the quantum fields $\psi^A_\sigma$ and $\psi^B_\sigma$ must also carry 'charge' and are complex \[8\].

The simplest relativistic quantum field theory for fermions which satisfies these kinematic constraints and respects parity is a theory of Dirac fermions in $(2 + 1)d$ \[8,13\]. In the presence of spin degrees of freedom $\sigma$ there are more than one possible representation of Dirac fermions. For later convenience we shall consider a representation which corresponds to the quasi-particle excitations around the four nodes of a d-wave superconductor in the following. Notice that without a complete microscopic theory for spinons, we cannot determine with certainty their correct representation.

We introduce two four-component spinor field $\Psi_i$, $i = 1, 2$, where

\[
\Psi_i(x) = \begin{pmatrix}
\psi^A_{i\uparrow}(x) \\
\psi^{B+}_{i\uparrow}(x) \\
\psi^B_{i\downarrow}(x) \\
\psi^{A+}_{i\downarrow}(x)
\end{pmatrix}.
\]

(9)

In terms of $\Psi_i$ the effective Lagrangian is

\[
L_{eff}^s = \sum_{i=1,2} \Psi_i^+(x) \frac{\partial}{\partial \tau} \Psi_i(x) - H_{spinon},
\]

(10a)

where

\[
H_{spinon} = \Psi_1^+ \left[ \tau^z i c^{(sf)}_s \partial_x + (\tau^+ + \tau^-) i c^{(sf)}_s \partial_y + m^{(sf)}_s \hat{I} \right] \Psi_1 + (1 \leftrightarrow 2; x \leftrightarrow y),
\]

(10b)

where $m^{(sf)}_s \sim \epsilon_v$ and $c^{(sf)}_s \sim c_v$. For d-wave superconductors, $m^{(sf)}_s = 0$ and $i = 1, 2$ represents quasi-particle excitations around the Dirac pockets located at $\vec{k} = \pm(k_o, k_o)$ and $\vec{k} = \pm(k_o, -k_o)$, respectively \[3\[3\].

The dynamics of holons can be considered similarly. Assuming that the spacial degrees of freedom are quenched once a hole is bound to the vortex core, the holons are described by relativistic quantum field theory of charged bosons. As in the case of spinons there are two quantum fields $\pi^{(A)}$ and $\pi^{(B)}$ in the continuum theory, representing holons centered at $A$--
and $B-$ sublattices. To respect parity, we construct quantum fields $\pi^{(\pm)} = \pi^{(A)} \pm \pi^{(B)}$ which are eigenstates of $\pi/2$ rotation and reflection with eigenvalues $\pm 1$. We shall assume in the following that the low energy dynamics of the holons are described by the $\pi^{(+)}$ field, and with usual relativistic scalar dynamics

$$L_{\text{eff}}^h = |\partial_\tau \pi^{(+)}|^2 + |c^{(sf)}_h \nabla \pi^{(+)}|^2 + (m^{(sf)}_h)^2 |\pi^{(+)}|^2. \quad (11)$$

where $c^{(sf)}_h \sim c_v$ and $m^{(sf)}_h \sim \epsilon_v$. Notice that in general $c^{(sf)}_h \neq c^{(sf)}_s$ and $m^{(sf)}_h \neq m^{(sf)}_s$. In particular we expect that $m^{(sf)}_h < m^{(sf)}_s$ in SFMFT because antiferromagnetic correlation is weakened at the center of a vortex. As a result holes can hop more easily at vortex core and gain excess kinetic energy. Notice also that as in the case of spinons, the holons are also coupled to gauge fields $\vec{A}$ and $\vec{A}^s$. The precise way they are coupled to gauge fields depends on detail microscopic dynamics which cannot be obtained unambiguously in our phenomenological analysis. We have ignored these couplings in writing down $L_{\text{eff}}^h$.

It is interesting to compare the excitations described by the effective actions $L_{\text{eff}}^s$ and $L_{\text{eff}}^h$ with the corresponding elementary excitations in d-wave superconducting state in SBMFT of $t - J$ model. First we consider the spin excitations. In both cases the spin excitations are described by pockets of Dirac fermions. The number of species of Dirac fermions are the same in both cases - there are four 'half-pockets' of Dirac-fermions centered around $(k_x, k_y) \sim (\pm k_o, \pm k_o)$ in the d-wave superconducting state, and there are two 'full-pockets' of Dirac fermions in $L_{\text{eff}}^s$ in order to respect parity. The position of the 'Dirac-pockets' in $\vec{k}$-space cannot be determined with certainty in $L_{\text{eff}}^s$. Nevertheless, if we assume that the spinons are represented correctly by $L_{\text{eff}}^s$ and the position of the fermion pockets are also centered around $(k_x, k_y) \sim (\pm k_o, \pm k_o)$, then the only difference between spin excitations described by $L_{\text{eff}}^s$ and by SBMFT is that there is a gap $m^{(sf)}_s > 0$ in the spinon spectrum of SFMFT.

The holon spectrum described by $L_{\text{eff}}^h$ is also similar to the elementary hole spectrum in SBMFT at momenta $\vec{k} \sim (0, 0)$. The hole spectrum in SBMFT has dispersion

$$\epsilon(\vec{k}) = -2t_{\text{eff}}(\cos(k_x) + \cos(k_y)), \quad (11)$$
where $t_{eff} \sim t$ and has energy minimum at $\vec{k} = (0, 0)$. Around these regions the effective Lagrangians for holes have exactly the same form as (11) in the non-relativistic limit with finite chemical potential $\mu = m_h^{(sf)}$ at $T = 0$ and $(c_h^{(sf)})^2/m_h^{(sf)} \sim t_{eff}$. The main differences between holons in SFMFT and holes in SBMFT are (1) the presence of nonzero chemical potential $\mu = m_h^{(sf)}$ in SBMFT which allows for finite concentration of holes in ground state and (2) $(c_h^{(sf)})^2/m_h^{(sf)}$ is expected to be of order $J$ in SFMFT. Notice that it has been argued that the effective hole band-width in SBMFT should be of order $\sim J$ after renormalization. In this case, the only qualitative difference between holon spectrum in SFMFT and hole spectrum in SBMFT is the appearance of nonzero mass gap and absence of Bose-condensation of holons in SFMFT.

Lastly we discuss the effects of Bose-condensation of bosonic spins ($< Z > \neq 0$) in the $T \to 0$ limit. In this case the elementary 'bosons' which Bose-condense are single Schwinger bosons with (fictitious) charge one and vortices carrying $\pi$-flux quantum become unstable. The stable vortices carry $2\pi$-flux quantum. As a result the spinons and holons are confined in pairs. At finite temperatures the confining potential is effective up to length scale $\sim$ antiferromagnetic correlation length $\xi(T)$ and the effect of confinement is expected to be strong at low temperature when the system is at the 'renormalized classical' [15] regime. As a result the identity of spinons and holons as independent excitations are lost at this regime of the phase diagram. A more detailed discussion of their experimental consequences is presented in section V and also in ref. [8].

III. TOPOLOGICAL EXCITATIONS IN SLAVE-BOSON MEAN-FIELD THEORY

In SBMFT the spins are represented by fermion operators $c^+_{i\sigma}, c_{i\sigma}$ and holes by boson operators $b^+_i, b_i$. The mean-field Hamiltonian has the same form as the mean-field Hamiltonian in SFMFT, $H_{MF} = H_{mf}^s + H_{mf}^h$, where [3]

$$H_{mf}^s = -\frac{J}{2} \sum_{<i,\nu>} (\Delta_{i\nu}^* (c_{i\uparrow} c_{i+\nu\downarrow} - c_{i\downarrow} c_{i+\nu\uparrow}) + H.C.)$$

(12)
\[
- \sum_{\langle i,j \rangle, \sigma} \left( \frac{J}{2} \chi^* + t\delta c_{i\sigma}^+ c_{j\sigma} + H.C. \right) + \sum_i \lambda \left( \sum_{\sigma} (c_{i\sigma}^+ c_{i\sigma} - (1 - \delta) \right),
\]

and

\[
H_{mf}^h = - \sum_{\langle i,j \rangle} \left( t\chi b_i^+ b_j + H.C. \right) - \sum_i \mu b_i^+ b_i. \tag{13}
\]

The mean-field parameters are determined by the mean-field equations \( \Delta_{\nu} = \langle c_{i\uparrow} c_{i\downarrow}^+ - c_{i\downarrow} c_{i\uparrow}^+ \rangle, \chi = \sum_{\sigma} \langle c_{i\sigma}^+ c_{j\sigma} \rangle, \) and \( \sum_{\sigma} \langle c_{i\sigma}^+ c_{i\sigma} \rangle = 1 - \delta \) and \( \langle b_i^+ b_i \rangle = \delta. \) The spin-pairing parameter \( \Delta_{\nu} \) has d-symmetry in d-wave superconducting state of SBMFT [5]. We shall follow the \( SU(2) \) description of Wen et.al. [16] in describing the 'uniform' fluctuations of the order parameters \( \Delta \) and \( \chi \) in SBMFT. In this description, the low energy fluctuations of the system is described by an \( O(3) \) order parameter \( \vec{n} \), where \( (n_1, n_2) = (Re\Delta_{\nu}, Im\Delta_{\nu}) \) describes the usual phase fluctuations of the d-wave spin-pairing order parameter, and \( n_3 \) corresponds to staggered flux fluctuations (\( \chi \)). At half-filling \( (\delta = 0) \), the system has a \( SU(2) \) pseudo-spin symmetry and the fluctuations described by \( \vec{n} \) is \( O(3) \) symmetric. The topological excitations are Skyrmions in this case. The \( O(3) \) symmetry is broken down to \( O(2) \) upon introduction of holes [16] and the fluctuations are described by an easy-plane anisotropic \( O(3) \) model at finite doping with anisotropy energy of order \( \delta^2 J \) [17]. The topological excitations become vortex-like half-Skyrmions with configuration \( n_1 \sim n\cos \theta, n_2 \sim n\sin \theta, n_3 \sim 0 \) (in a polar coordinate) away from center of vortex, and \( n_3 \sim n \) at center, i.e. instead of destroying the spin-correlations completely, the vortex core is in the staggered-flux phase. The vortices carry \( \pi \)-flux quantum as in usual superconductors.

Notice however that because holes exist as an independent quantum liquid and can Bose-condense independently in SBMFT, two different kinds of vortices exist in SBMFT, corresponding to vortices associated with phase singularity in the d-wave spin-pairing order parameter \( \Delta_{\nu} \), and vortices associated with the phase singularity of the hole condensate [18,19]. The two kinds of vortices carry different fluxes \( \pi \) and \( 2\pi \), respectively [18,19] because of the different (fictitious) gauge charge associated with the spin pairs and holes. In the
presence of hole condensate \(< b > \neq 0\), vortex excitations carrying fictitious flux \(2\pi\) is the only stable topological excitation [20] and vortices carrying flux \(\pi\) are confined, similar to the situation in SFMFT when \(< Z > \neq 0\). We shall consider a disordered state of holes where \(< b > = 0\) in the following. In this case, vortices with \(\pi\) flux quantization are stable topological excitations. The effect of Bose-condensation will be addressed in next section when we discuss the \(\delta - T\) phase diagram. Topological spin and charge excitations (spinons and holons) can be constructed as in the case of SFMFT by considering bound states of spins and holes at the center of vortices. In particular, statistics transmutation occurs for particles binding to vortices with \(\pi\) flux quantization, as in the case of SFMFT, i.e. we have bosonic spinons and fermionic holons as topological excitations in SBMFT when \(< b > = 0\).

To study the dynamic of vortices we start with an effective \(O(2)\) GL functional describing the phase fluctuations of the d-wave spin-ordering parameter [17],

\[
S_{\text{phase}} = \frac{J}{2} \int d\tau \int d^2 x \left[ \kappa_x (\nabla \theta - 2 A)^2 + \frac{2\delta}{J} (\partial_\tau \theta - 2 A_\tau) + \kappa_\tau (\partial_\tau \theta - 2 A_\tau)^2 \right],
\]

(14)

where \(\kappa_x \sim \delta\), \(\kappa_\tau J \sim J\) is the compressibility of the spin-condensates, \(\theta\) is the phase of the spin-pairing order parameter \(\Delta\) and \(A_\mu\) is the fictitious U(1) gauge field associated with phase fluctuations of the order parameter \(\chi\). Notice the existence of linear \(\partial_\tau\) term in \(S_{\text{phase}}\) because of broken particle-hole symmetry [17]. The effective action describing dynamics of vortices can be constructed by following the same procedure as in last section. The resulting action \(S_v\) has the same form as Eq.(8), except that the parameters \(\xi_x\) and \(\xi_\tau\) are replaced by \(\kappa_x\) and \(\kappa_\tau\), respectively and the presence of additional linear \(\partial_\tau\) term in SBMFT. Correspondingly, the parameters \(\epsilon_v\) and \(c_v\) are modified to \(\epsilon_v \sim \kappa_x J\) and \(c_v \sim \sqrt{\kappa_x / \kappa_\tau}\). Notice that the linear \(\partial_\tau\) term does not introduce any strong effect on vortex dynamics because of screening of the vortex current \(\vec{j}_v\) by \(\vec{A}\) field, as in the case of SFMFT.

The dynamics for the topological spin and charge excitations can be obtained by making the same assumption as in last section that the spacial degrees of freedom of the spins and holes are quenched once they are bound to the center of vortices. In particular, the dynamics of the spinons are described by a relativistic quantum field theory of charged bosons and
the dynamics of the holons by a theory of Dirac fermions. We first consider the spinon excitations. Following the same argument as in previous section, we construct two quantum fields \( \psi^{(+)} \) and \( \psi^{(-)} \) with parity \( \pm 1 \). Assuming that the low energy physics of spinons are described by the \( \psi^{(+)} \) field, we obtain for the spinons

\[
L_{s}^{s\text{eff}} = |\partial_{\tau} \psi^{(+)}|^{2} + |c_{s}^{(sb)} \nabla \psi^{(+)}|^{2} + (m_{s}^{(sb)})^{2} |\psi^{(+)}|^{2}.
\]

(15)

where \( c_{s}^{(sb)} \sim c_{v} \) and \( m_{s}^{(sb)} \sim \epsilon_{v} \). Notice that \( L_{s}^{s\text{eff}} \) has the same form as the effective Lagrangian for spinwaves in the disordered phase of the non-linear-\( \sigma \)-model or SFMFT [4,21].

To construct dynamics for the holons we first construct a four-component Dirac spinor field \( \pi \), with

\[
\pi(x) = \begin{pmatrix} \pi^{A}(x) \\ \pi^{B}(x) \end{pmatrix}, \quad \pi^{A(B)}(x) = \begin{pmatrix} \pi_{1}^{A(B)}(x) \\ \pi_{2}^{A(B)}(x) \end{pmatrix}
\]

(16)

where \( \pi^{A(B)} \)'s are two component fermion fields introduced to describe positive and negative energy solutions of the Dirac equation. In terms of \( \pi \) the effective Lagrangian which transforms correctly under parity is [8]

\[
L_{e\text{ff}}^{s} = \frac{i}{2} \left[ \pi \gamma^{0} (\partial_{\tau} \pi) - (\partial_{\tau} \pi) \gamma^{0} \pi + \pi \gamma_{\mu} (c_{h}^{(sb)} \nabla \pi) - (c_{h}^{(sb)} \nabla \pi) \gamma_{\mu} \pi - m_{h}^{(sb)} \pi \pi \right].
\]

(17)

where \( \gamma^{\mu} \)'s are usual \( 4 \times 4 \) Dirac matrices in \((2+1)d\) with \( \mu = 0, 1, 2 \), \( c_{h}^{(sb)} \sim c_{v} \) and \( m_{h}^{(sb)} \sim \epsilon_{v} \). Notice that as in the case of spinons in SFMFT there must be two “Dirac pockets” of holes in order to respect parity. Notice also that \( c_{h}^{(sb)} \neq c_{s}^{(sb)} \) and \( m_{h}^{(sb)} \neq m_{s}^{(sb)} \) in general and we expect \( m_{h}^{(sb)} > m_{s}^{(sb)} \) in the present case because vortex core is in the flux-phase where antiferromagnetic correlation is enhanced [16,17]. As a result it costs kinetic energy for hole to stay at the vortex core.

Let us now compare the dynamics of spinons and holons in SBMFT with elementary spin and charge excitations in SFMFT. First we consider spin excitations. In both cases the dynamics of spin excitations are described by relativistic complex scalar fields (spinwaves). In the case of SFMFT, the dispersion minimum of the spinwave is located around \( \vec{q} = \)
\( \pm (\pi + q_\delta, \pi + q'_\delta) \) in the spiral states, where \( q_\delta \) and \( q'_\delta \) depends on the types of spiral state being formed. The location of dispersion minimum for spinons in SBMFT cannot be determined with certainty without further understanding of their microscopic dynamics. Nevertheless, if we assume that the dispersion minimum is located at the same \( \vec{q} \)-point as spinwaves in SFMFT, then the only difference between spinons in SBMFT and spinwaves in SFMFT is that there is a finite energy gap in the spinon spectrum at \( T = 0 \). Similar correspondence also exists between holons in SBMFT and elementary hole excitations in SFMFT. In the spiral states of SFMFT the holes form fermi sea pockets around \( \vec{k} \sim (\pm \pi/2, \pi/2) \) in our chosen gauge, which can be interpreted phenomenologically as two Dirac fermion pockets centered at the same momenta points and with finite chemical potential \( \mu > m^{(sb)}_h \) such that particle-hole symmetry is broken and there exists nonzero density of (fermionic) holes in the ground state. The interpretation is valid at low energy \( \epsilon \ll m^{(sb)}_h \). On the other hand, \( \mu = 0 \) for holons in SBMFT, implying that the ground state holon density is zero and there exists a nonzero energy gap for holon excitations in SBMFT.

**IV. DUALITY PICTURE AND PHASE DIAGRAM OF \( T - J \) MODEL**

In the previous two sections we demonstrate that topological spin and charge excitations exist as well defined excitations in the disordered phases \( < Z > = 0, < b > = 0 \) of slave-fermion and slave-boson mean-field theories of the \( t - J \) model, respectively. Moreover, the topological spin and charge excitations in one mean-field ground state behave as ”gapped” elementary excitations of the other mean-field state. In this section we shall explore this relation further by showing that a consistent phase diagram of the \( t - J \) model can be obtained if we assume that the ground state in one mean-field theory is a quantum-disordered state of the other mean-field theory. The disordering effect is coming from ”condensation” of topological excitations. We shall divide our discussions in two parts. In part I we shall study the effective actions of the topological excitations in the two mean-field theories and argue that quantum phase transitions to disordered states are natural outcomes of the effective
actions when concentration of hole $\delta$ changes. Based on this result and the requirement that
the phase diagram derived from the quantum-disordering picture of SFMFT and SBMFT
must be consistent with each other we shall construct in part II a plausible phase diagram
of $t - J$ model.

A. effective actions for topological excitations and quantum-disordered phases

First we consider the case of SFMFT. To show that quantum-disordered state of topo-
logical excitations is a natural outcome of our theory we start with the phase action (7a)
which describes the (uniform) phase fluctuations of the order parameter $\Delta$ in SFMFT. A
quantum-disordered phase is found if the phase $\theta$ is disordered in the ground state. The
action (7) is in fact a 3D $x - y$ model (coupled to gauge field) and is known to go through a
transition to a disordered phase when the coupling constant $g(\delta) = \xi_x\xi_\tau$ is less than certain
critical value $g_c$ [22]. The mass gap of vortices go to zero as the system approaches the
transition point from the ordered phase [12,14], showing that the quantum phase transition
is driven by condensation of vortex loops. (Notice that we assume here that the gauge field
$A$ is not in the confining phase [22]). It is easy to see from Eq. (7b) that $g_\delta$ is a decreasing
function of $\delta$ in SFMFT, as least for small $\delta$. It is therefore reasonable to assume that
there exists a critical value of hole concentration $\delta_c$ at which the ground state described by
SFMFT goes through a quantum phase transition to a disordered phase when $\delta > \delta_c$.

The main difference between the quantum-disordered state in SFMFT and usual 3D $x - y$
model is that we assume a bound state of spin or hole exists in the vortex core in SFMFT.
Thus there exists two kinds of vortices (spinons and holons) in SFMFT which possess differ-
ent statistics and dynamics. In particular, since we assume spin-charge separation and the
mass gap for holons $m_h^{(sf)}$ should be smaller than the mass gap for spinons $m_s^{(sf)}$ in general,
the condensation of (bosonic) holons and (fermionic) spinons should occur at two different
critical hole concentrations $\delta_{ch}^{(sf)}$ and $\delta_{cs}^{(sf)}$, respectively with $\delta_{ch}^{(sf)} < \delta_{cs}^{(sf)}$. Therefore two
different quantum-disordered states corresponding to condensation of holons and holons +
spinons are expected to exist in our theory. Unfortunately we cannot deduce the properties of the quantum-disordered states unambiguously based on our simple phenomenological treatment of spinon and holon dynamics. In particular, the nature of the quantum-disordered state coming from condensation of fermionic vortex loops has never been investigated theoretically. The nature of the quantum disordered states will be deduced phenomenologically in the next subsection. Another special feature of SFMFT which is absent in usual 3D $x - y$ model is the presence of long-ranged spiral magnetic order $< Z > \neq 0$ at zero temperature, at least when hole concentration is small. We shall assume that the long-ranged order exists at small value of $\delta$ and vanishes ($< Z > \to 0$) at exactly $\delta_{cs}^{(sf)}$, for reasons we shall see in the following. To summarize we show in fig.2a schematically the expected behavior of $m_h^{(sf)}$, $m_s^{(sf)}$ and $< Z >$ as functions of $\delta$ in SFMFT.

Next we consider the case of SBMFT. We start with the phase action describing the phase dynamics of the d-wave spin-pairing order parameter $\Delta$ in SBMFT. The phase action is also a 3D $x - y$ model as in the case of SFMFT and the system is expected to go through a transition to a disordered phase when the coupling constant $g(\delta) = \kappa_x \kappa_r$ is less than certain critical value $g_c$. It is easy to see that $g(\delta)$ is an increasing function of $\delta$ in SBMFT, at least for small $\delta$ and goes to zero as $\delta \to 0$. Therefore, it seems reasonable to assume that there exists a critical value of hole concentration $\delta_c$ below which the system is in a quantum-disordered phase of SBMFT.

As in SFMFT, the main difference between the quantum-disordered state of SBMFT and usual 3D $x - y$ model (in the context of a d-wave superconductor) is that we assume bound state of spin or hole exists in the vortex core and there are two kinds of vortices (spinons and holons) in SBMFT. Unlike SFMFT, we expect that the mass gap for holons $m_h^{(sb)}$ is larger than mass gap for spinons $m_s^{(sb)}$ in SBMFT and therefore, the condensation of (fermionic) holons and (bosonic) spinons should occur at different critical hole concentrations $\delta_{ch}^{(sb)}$ and $\delta_{cs}^{(sb)}$, respectively with $\delta_{ch}^{(sb)} < \delta_{cs}^{(sb)}$. Notice that we are not able to deduce the properties of the quantum-disordered states of SBMFT with certainty as in the case of SFMFT.
Note that because of spin-charge separation, the holes in SBMFT may exist in a quantum disordered (insulator) phase with $<b>=0$ by themselves, independent of the disordered effect on spin-pairing order parameter $\Delta$ [23]. In particular, it is expected that a zero temperature disordered phase of bosonic holes can be formed at small enough $\delta < \delta^{(sb)}_b$ because of disordered effect associated with $(2\pi$ flux) vortex-loop condensation in the boson field [23]. The different disordering effects in SBMFT are summarized in fig.2b, where we show schematically the expected behaviours of mass gap $m^{(sb)}_h$, $m^{(sb)}_s$ and hole-condensation amplitude $<b>$ as functions of $\delta$. Notice that we have assumed that $\delta^{(sb)}_b > \delta^{(sb)}_{cs}$ in drawing the figure. We shall explain why we make this assumption in the following subsection.

B. a plausible phase diagram of $t-J$ model

In this subsection we shall construct a phase-diagram of $t-J$ model, based on what we have obtained in previous sections. Our main assumption is that the quantum-disordering effect associated with condensation of topological excitations in SFMFT (SBMFT) of $t-J$ model gives rise to the ground state described by SBMFT (SFMFT), with the roles of topological excitations and elementary excitations being interchanged at the quantum-disordering transition. This assumption is consistent with the ”dual” relations we obtained in sections II and III between the elementary excitations in one mean-field phase and topological excitations in the other. Notice that this statement has to be understood with care because the quantum phase transitions associated with condensation of spinons and holons occur in general at different doping concentration $\delta$’s(see figure 2). As a result there exists at least three different regions in the zero-temperature phase diagram of $t-J$ model in the duality picture.

For our assumption to be valid, the quantum-disordering transition of holons and spinons from SFMFT to SBMFT and from SBMFT to SFMFT must occur at consistent value of doping, i.e. we must have $\delta^{(sf)}_{cs} = \delta^{(sb)}_{cs} = \delta_{cs}$ and $\delta^{(sf)}_{ch} = \delta^{(sb)}_{ch} = \delta_{ch}$. The zero-temperature phase diagram has therefore at least three different regions: (i)$\delta < \delta_{ch}$, in this region the low
energy physics is described by SFMFT, fermionic spinons and bosonic holons are high-energy topological excitations, (ii) $\delta_{ch} < \delta < \delta_{cs}$, in this region we expect that both low energy spin and charge excitations are bosonic. The precise nature of this state will be discussed below, and (iii) $\delta_{cs} < \delta$, in this region the low energy physics is described by SBMFT, bosonic spinons and fermionic holons are high-energy topological excitations.

Next we consider the transition regions $\delta \sim \delta_{ch}$ and $\delta \sim \delta_{cs}$ more carefully. First we consider $\delta \sim \delta_{ch}$. The change in the hole spectrum around this region can be described phenomenologically by a band picture with two branches of hole spectrum. The first one is the Dirac fermion spectrum with Dirac pockets centered at $\vec{k} \sim (\pm \pi/2, \pi/2)$ and the second one is the boson spectrum with dispersion minimum around $\vec{k} \sim (0, 0)$. In the regime $\delta < \delta_{ch}$, we have $m_h^{(sf)} > \mu > m_h^{(sb)}$ such that at zero temperature, the Dirac fermion pockets are occupied and the bosonic hole band is empty. The ground state of the system is described by SFMFT. As $\delta \rightarrow \delta_{ch}$, $m_h^{(sf)} \rightarrow \mu$ and the bosonic hole excitations become gapless. As $\delta$ increases, $m_h^{(sf)} = \mu$ and decreases further, until all the fermionic holes become bosonic and the Dirac pocket becomes empty. Notice that this band picture describes only the transformation of fermionic holes to bosonic ones at $\delta \sim \delta_{ch}$, but the nature of the ground state of the bosonic hole system is not addressed. We shall argue below that the bosonic holes must exist in an insulating phase and $\delta_{ch}$ is a critical point where the system undergoes a metal-insulator transition in the absence of potential disorder (Anderson localization). Notice that this is consistent with the existence of a nonzero critical hole density $\delta_{bc}^{(sb)}$ below which the bosonic holes are in a quantum-disordered phase [23].

Next we consider the region $\delta \sim \delta_{cs}$. The transition can be described by a picture with two branches of spin excitations. The first one is the Dirac fermion spectrum and the second one is the spinwave like spectrum. In the regime $\delta < \delta_{cs}$, $m_s^{(sb)}$ is zero whereas $m_s^{(sf)}$ is finite, the presence of gapless spinwave excitation indicates that the system is magnetically ordered and the Dirac fermions are high energy spin excitations. Therefore we expect $<Z> \neq 0$ at $\delta < \delta_{cs}$, as we have assumed in previous subsection (see fig.2a). The opposite is true for $\delta > \delta_{cs}$, where $m_s^{(sb)}$ becomes nonzero and $m_s^{(sf)}$ is zero. In this region the
elementary spin excitations are described by gapless Dirac fermion pockets. The spinwave excitations are 'gapped' and the system has no long-ranged magnetic order. Notice that the system may be in a spin-gap’ state or d-wave superconducting state, depending on whether \( \langle b \rangle \) is equal to zero \[5,24\].

To complete our discussion we shall now consider the regime \( \delta_{ch} < \delta < \delta_{cs} \) and ask what is the ground state of the system at this regime. Following our previous discussions this is a regime where the spin dynamics are described by SFMFT and hole dynamics by SBMFT with both \( \langle Z \rangle \neq 0 \) and \( \langle b \rangle \neq 0 \) in a naive mean-field picture. However, as pointed out in last section, this state cannot exists because \( \langle b \rangle \neq 0 \) implies that the (bosonic) topological spin excitations are confined and as a result we cannot have both \( \langle Z \rangle \neq 0 \) and \( \langle b \rangle = 0 \) in the ground state. Therefore, we must have \( \langle b \rangle = 0 \) in order to have a consistent phase diagram, i.e. we expect that the bosonic holes are in a disordered (insulator) state at this regime of the phase diagram. As a result we expect \( \delta_{b}^{th} \geq \delta_{cs} \), as is assumed in the previous subsection (fig.(2b)).

The general \( \delta - T \) phase diagram based on our duality picture is shown in fig.3. We note that there are four different low temperature regions in the phase diagram. Region I is the 'renormalized classical’ regime of SFMFT. In this regime the (topological) fermionic spin and bosonic charge excitations are gapped and confined. In the absence of potential disorder, long-ranged (spiral) magnetic ordering is expected to exist in this regime when interlayer coupling is included. Region II is a new regime coming from the duality picture. In this regime the holes are in the Mott insulator state but long-ranged magnetic ordering still exists. Notice that the nature of the magnetic ordering is unclear in this regime. In SFMFT, spiral ordering comes as a result of formation of fermionic hole pockets \[4\] at momentum points \( \vec{k} \sim (\pi/2, \pi/2) \). However, the hole pockets are gradually destroyed at this region of the phase diagram and we expect that long-range spiral ordering should also be destroyed. It is known experimentally that the effect of potential disorder is very important in the antiferromagnetic and spin-glass regime of high-\( T_c \) cuprates \[25\]. In particular, it seems that the holes in the antiferromagnetic phase are always localized by disorder and long-range spiral ordering never
appears [25]. It is expected that the potential disorder will also affect the magnetic behaviour at region II strongly and the precise nature of the ground state is unclear. In particular, a spin-glass like phase may appear if the localized holes are mobile enough at short distance scale so that random long-ranged spin-distortions are set up in the system [26]. Region III is the spin-gap phase described by SBMFT where the low energy spin excitations are described by pockets of Dirac fermions and the antiferromagnetic (spiral) spinwaves are gapped. The holes are still in the Mott insulator state. As a result the (topological) bosonic spin and fermionic charge excitations in this regime are gapped but deconfined. Region IV is the $\langle b \rangle \neq 0$ state. The ground state is a d-wave superconductor described by SBMFT. The topological excitations are confined since $\langle b \rangle \neq 0$. Corresponding to the zero-temperature quantum phase transitions between these four phases are three different quantum critical regimes (a-c) indicated on the phase-diagram. Note that region III and the corresponding quantum critical regime may vanish if $\delta_{cs} = \delta_{sb}^b$ which is allowed in the duality picture. When comparing with the experiments on monolayer copper oxides [25] we find that the gross features of the phase diagrams can be produced in the duality picture. Moreover, new branches of excitations not present in mean-field theories are predicted in the duality picture. A few experimental consequences from the duality picture will be discussed in the next section where our theory will be critically examined.

V. SUMMARY AND DISCUSSIONS

There are several new results obtained in this paper. In sections II and III, we showed that topological spin and hole excitations exist in general in the disorder states of SFMFT and SBMFT, and the dynamics of these excitations can be derived under some very general assumptions. We found that the topological spin and charge excitations in one mean-field theory have same dynamics as elementary excitations in the other mean-field theory, except the appearance of energy gaps. Based on this result, we propose in section IV a duality relation between the ground states described by the two mean-field theories, namely that the
ground state described by one mean-field theory can be viewed as a quantum-disordered state of the other mean-field theory. A phase diagram of $t - J$ model is constructed phenomenologically based on this assumption. In particular we find that the self-consistent requirement between the quantum-disordering effects from SFMFT and SBMFT implies that an intermediate insulator phase between the antiferromagnetic (spiral) and d-wave superconducting phases has to exist. In the following subsections we shall discuss some theoretical questions and experimental consequences of our theory. We first consider the theoretical questions.

A. theory

To have some qualitative understanding on the meaning of the duality relation we shall now give a wavefunction interpretation of the relation. First we consider the spin part of the ground state wavefunction described by SFMFT. As pointed out in section II, the wavefunction corresponding to SFMFT is a RVB state, where the wavefunction is a superposition of spin-singlet pairs. The relative phases between the singlet pairs are fixed in SFMFT. The spin part of the ground state wavefunction described by SBMFT has a similar interpretation. The wavefunction is also made up of superposition of spin-singlet pairs, and the relative phases between the singlet pairs are fixed by SBMFT. The difference between SFMFT and SBMFT is that because of the different particle statistics used in representing the spins, the phase relations between singlet pairs are very different in the two mean-field theories [10]. The differences in phase relations between spin-singlet pairs result in the very different physical properties represented by the two RVB wavefunctions. Our duality relation suggests a connection between the two very different RVB wavefunctions, namely, the phase relation between spin singlets in one wavefunction can be obtained from quantum-disordering the phases of the other wavefunction through condensation of spinons and holons. Notice that a natural consequence of this picture is that there should be no clear distinction between a high temperature short-ranged antiferromagnetic state and preformed Cooper pair state in $t - J$ model when the phases between the RVB singlets pairs are randomized. Differ-
ences between the two phases show up only as temperature is lowered and phase coherences between RVB pairs are built up gradually.

Notice that the effects of holes is not mentioned in the above picture. In our derivation of the duality relation, we have assumed a picture of spin-charge separation where spins and holes can be considered as independent quantum liquids. This is also a major assumption we made in deriving our phase diagram. Although it is believed that the picture of spin-charge separation should be correct at an intermediate temperature region above $T_c$, it is also believed that a proper treatment of the effect of (fictitious) gauge fields may lead to confinement of spin and charge at low enough temperature in $t - J$ model \cite{27,28} and may modify strongly our theoretical predictions. In particular, region III of our phase diagram may be washed away completely if spin and charge are confined by fictitious gauge fields at low energy. The effects of gauge fields are ignored in our present theory which treats the dynamics of topological spin and charge excitations only in a semi-phenomenological level. To include the effects of gauge fields properly a more microscopic understanding of topological excitations is needed and will be a direction of our future work \cite{29}.

B. experimental consequences

Next we consider a few experimental consequences of the duality relations. First we consider regions I and IV where the ground states are described by SFMFT and SBMFT, respectively. The new features we predict in these two regimes are the existence of topological spin and charge excitations which are absent in mean-field theories. Notice that as discussed in sections II and III, the topological excitations are confined in these regimes and are difficult to be observed in correlation functions with momentum transfer $\vec{q} \sim (0, 0)$. Nevertheless, the behavior of spin correlation functions at momentum $\vec{q} \sim (\pi, \pi)$ is determined by the short-distance behaviours of spin pairs and should not be affected strongly by confinement. As a result, we expect that topological spin excitations may be observable in the dynamic structure factor $S(\vec{q}, \omega)$ at momenta $\vec{q} \sim (\pi, \pi)$ \cite{38}. A more detailed analysis of the effects of
topological spin excitations in the $\delta = 0$ limit (Heisenberg model) was given in Ref. [8] (see also Ref. [29]). Notice that gapped incommensurate spinwave excitations with momentum $\vec{q} \sim (\pi \pm 2\pi \delta, \pi)$ and $(\pi, \pi \pm 2\pi \delta)$ have been observed experimentally in the underdoped regime of high-$T_c$ cuprates [30] which we believe, can be identified with the bosonic topological spin excitations in the d-wave superconductor phase. Notice that the topological excitations can be observed more easily in the underdoped regime at $T > T_c$ or at the spin-gap phase (region III) where confinement effect is absent. The spin-gap phase has low-energy spin dynamics of d-wave superconductors but is insulating otherwise. It would be interesting to see whether this phase can be observed in cuprates.

Lastly we discuss region II. This is a regime where our understanding is poorest. Based on our analysis we expect that the system is in an insulator phase where charge excitations are gapped, and the spins are in some kind of long-ranged order state. However, the nature of the spin-ordering is unclear, especially in the presence of impurities. We expect that this phase may corresponds to the spin-glass phase [25] observed experimentally. Theoretical understanding of spin dynamics at this regime is still very limited [26]. Notice that it is found experimentally that the in-plane resistance of the cuprates at the spin-glass regime goes as $\rho_{ab}(T) \sim k_B T$ at high enough temperature, before the systems become insulating [31]. Within our duality picture, a plausible explanation is that the low energy hole excitations are already slave-boson-like in the spin-glass phase. The transport behaviours of the systems at high enough temperature should be described by SBMFT, which predicts $\rho_{ab}(T) \sim k_B T$ at high enough temperature because of gauge field fluctuations.

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REFERENCES

[1] J. Schmalian, D. Pines and B. Stojkovic, Phys. Rev. Lett. 80, 3839(1998) and references therein.

[2] S.K. Sarker, preprint cond-mat/9904253.

[3] S.C. Zhang, Science, 275, 1089(1997).

[4] C. Jayaprakash, H.R. Krishnamurthy and S. Sarker, Phys. Rev. B 40, 2610(1989); C.L. Kane, P.A. Lee, T.K. Ng, B. Chakraborty and N. Read, ibid. 41, 2653(1990).

[5] G. Baskaran, Z. Zou and P.W. Anderson, Solid State Commun. 63, 973(1987); C. Gros, R. Joynt and T.M. Rice, Phys. Rev. B 36, 381(1987); G. Kotliar and J. Liu, ibid. 38, 5142(1988).

[6] N. Read and S. Sachdev, Phys. Rev. Lett. 62, 1694(1989).

[7] T.K. Ng, Phys. Rev. B 52, 9491(1995).

[8] T.K. Ng, Phys. Rev. Lett. 82, 3504(1999).

[9] F. Wilczek, Phys. Rev. Lett. 48, 1144(1982).

[10] N. Read and B. Chakraborty, Phys. Rev. B 40, 7133(1989).

[11] S. Liang, B. Doucot and P.W. Anderson, Phys. Rev. Lett. 61, 365(1988).

[12] C. Dasgupta and B.I. Halperin, Phys. Rev. Lett. 47, 1556(1981); M.P.A. Fisher and D.H. Lee, Phys. Rev. B 39, 2756(1989).

[13] see for example, The Quantum Theory of Fields I, by S. Weinberg (Cambridge Press 1995).

[14] L. Balents, M.P.A. fisher and C. Nayak, Int. J. Mod. Phys. B12, 1033(1998); preprint, cond-mat/9811236.

[15] S. Chakravarty, B.I. Halperin and D.R. Nelson, Phys. Rev. B 39, 2344(1989).
[16] C.L. Wu, C.Y. Mou, X.G. Wen and D. Chang, preprint cond-mat/9811146.

[17] P.A. Lee, N. Nagaosa, T.K. Ng and X.G. Wen. Phys. Rev. B 57, 6003(1998).

[18] S. Sachdev, Phys. Rev. B 45, 389(1992).

[19] N. Nagaosa and P.A. Lee, Phys. Rev. B 45, 966(1992).

[20] Notice that this is correct only for vortices binding to fictitious gauge flux. See also refs.[17,18].

[21] see for example, *Gauge Fields and Strings*, by A.M. Polyakov (harwood academic publishers 1987).

[22] E. Fradkin and S. Shenker, Phys. Rev. D 19, 3682(1979).

[23] Yong Baek Kim and Ziqiang Wang, preprint cond-mat/9901003.

[24] M.U. Ubbens and P.A. Lee, Phys. Rev. B 49, 6853(1994).

[25] see for example, M.A. Kastner and R.J. Birgeneau, Rev. Mod. Phys. 70, 897(1998).

[26] R.J. Gooding, N.M. Salem, R.J. Birgeneau and F.C. Chou, Phys. Rev. B 55, 6360(1997).

[27] X.G. Wen and P.A. Lee, Phys. Rev. Lett. 80, 2193(1998).

[28] P.A. Lee and N. Nagaosa, Phys. Rev. B 45, 5621(1992).

[29] T.K. Ng and C.H. Cheng, Phys. Rev. B 59, R6616 (1999); C.H. Cheng and T.K. Ng, preprint cond-mat/9908161.

[30] K.S. Yamada *et.al.*, Phys. Rev. Lett. 75, 1626(1995).

[31] H. Takagi *et.al.*, Phys. Rev. Lett. 69, 2975(1992).
FIGURES

FIG. 1. vortex excitations in SFMFT of $t-J$ model; (1a) $S=0$, chargeless vortex excitation. The center of vortex is located at the center of a square plaquette; (1b) spinon; (1c) holon. Notice that the center of vortex is located at a lattice site for both spinon and holon. Similar constructions can also be applied to the case of SBMFT.

FIG. 2. (2a) Schematic behavior of $m^{(sf)}_h$, $m^{(sf)}_s$ and $< Z >$ as functions of hole concentration $\delta$ in SFMFT; line (a): $m^{(sf)}_s$, line (b): $m^{(sf)}_h$, blacked region: $< Z > \neq 0$; (2b) Schematic behavior of $m^{(sb)}_h$, $m^{(sb)}_s$ and $< b >$ as functions of hole concentration $\delta$ in SBMFT; line (a): $m^{(sb)}_s$, line (b): $m^{(sb)}_h$, blacked region: $< b > \neq 0$.

FIG. 3. phase diagram of $t-J$ model. Region I: renormalized classical regime of quantum antiferromagnet described by SFMFT. Region II: spin-glass regime with spin dynamics described by SFMFT and hole dynamics by SBMFT. Region III: spin-gap regime described by SBMFT with $< b > = 0$. Region IV: d-wave superconductor phase described by SBMFT. Regions (a)-(c) are the corresponding quantum critical regimes of the zero temperature phase transitions.