Numerical solution of a critical Sobolev exponent problem with weight on $S^3$

Adel Almarashi a, Idir Mechai a, Ahmed Msmali a and Habib Yazidi b

a Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia; b National School of Engineers of Tunis, University of Tunis, Tunis, Tunisia

ABSTRACT

In this paper, we prove the existence of a positive solution for elliptic nonlinear partial differential equation with weight involving a critical exponent of Sobolev imbedding on $S^3$. Moreover, we discuss numerically the influence of the weight on the radius of the domain for which the given PDE has a positive solution.

1. Introduction

Let $D$ be a geodesic ball with radius $\rho_1$, centred at the north pole, on $S^3$. We study the following elliptic nonlinear partial differential equation with weight involving the critical exponent of Sobolev imbedding on $S^3$

$$
\begin{aligned}
-\text{div}_{S^3}((\alpha + \beta |x|^k)\nabla u) &= u^5 + \lambda u, & \text{in } D, \\
u &= 0 & \text{on } \partial D,
\end{aligned}
$$

where the exponent $5 + 1 = 6$ is critical in the sense of Sobolev embedding, and the constants $\alpha, k, \beta$ and the parameter $\lambda$ are assumed to be positive.

The partial differential equations are one of the most celebrated tools discovered from modelling many phenomena in nature. The differential equation in (1) can be a laboratory of finding many methods to deal with similar mathematical models which arise in different branch of sciences [1–17]. The problem (1) is interesting to study and has different new features, since this model is the stationary equation of convection–diffusion models appearing frequently in connection with conservation laws. The proposed problem has also some non-stationary equations with non-stationary solutions [1–3]. More precisely, for example in [3], Md N. Alam and C. Tunç use the modified $(G'/G)$-expansion process to obtain soliton answers of the $(3 + 1)$ dimensional conformable Zakharov–Kuznetsov equation with power law nonlinearity $\frac{\partial^2 u}{\partial t^2} + a u^3 + b \frac{\partial^3 u}{\partial x^3} + c \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$. The authors apply some change of variables, namely $\xi = (x, y, z, t)$ with $\lambda = \frac{2}{1+|x|^2}$.

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In this work, we treat the general case of $\alpha$ and $\beta$ in a bounded domain of $S^3$. More precisely, we study the influence of the function $\alpha + \beta |x|^k$ on the existence of solutions of a weighted elliptic PDE. Our method combined two directions: first, we study the existence of a positive solution. This procedure is not obvious and presents many difficulties due to the presence of critical Sobolev exponent which generate a lack of compactness. We overcome this problem using minimizing technique and variational approach. We obtain the existence of a positive solution only in the case $k > 1$ and for $\lambda$ in a well-determined interval. Therefore, second, in order to obtain a complete result of our problem, we use Newton iteration method with classical fourth-order Runge–Kutta procedure and we carry out a numerical solution in the cases that we have no theoretical results.

Using the stereographic transformation, $D$ is mapped onto a ball $B(0, R) \subset \mathbb{R}^3$ and we write (1) as

$$
\begin{aligned}
-\text{div}(\rho(x) (\alpha + \beta |x|^k) \nabla u) &= \rho^3(x) u^5, & \text{in } B(0, R), \\
u &= 0 & \text{on } \partial B(0, R),
\end{aligned}
$$

where $\rho(x) = \frac{2}{1+|x|^2}$.

CONTACT Adel Almarashi a almarashia@jazanu.edu.sa

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By a result of Padilla [16] (extending the classical result of Gidas, Ni and Nirenberg [13] to domain on manifolds of constant curvature), a solution $u$ of (2) is symmetric, i.e. it only depends on the azimuthal angle, then we write (2) as the boundary problem

$$
\begin{align*}
-\text{div}(\rho(r)^2(\alpha + \beta k^2)\nabla u) &= \rho^3(r)^2 u^2 \quad \text{in } (0,R), \\
+\lambda \rho^3(r)^2 u_t, \\
u(R) &= 0 \text{ and } u'(0) = 0,
\end{align*}
$$

with $\rho(r) = \frac{2}{1+r^2}$.

For large dimensions $N \geq 5$, there is a little difference between studying the problem for a domain in $\mathbb{S}^N$ and a domain in $\mathbb{R}^N$. However, the results differ considerably for $N = 4$ and $N = 3$, see [4] for $S^3$ and [10] for $\mathbb{R}^N$.

So, in this work we will study the case $N = 3$ where we prove the existence of a positive solution for $\lambda > 0$ and $k > 1$. We have no theoretical results for $\lambda > 0$ and $0 < k \leq 1$ or $\lambda < 0$ and $k > 1$. Nevertheless, we obtain some numerical results for the existence of a positive solution. This approach is motivated by the results of [4] and [7] for $\alpha = 1$ and $\beta = 0$, where the authors found numerical solutions for $\lambda$ negative enough when the geodesic radius $\theta_1 > \frac{\pi}{2}$.

The rest of the paper is organized as follows: in Section 2, we present some theoretical results. In Section 3, we present numerical results for existence that complete the results announced in Section 2. Furthermore, in Section 4 we give some interpretations on the influence of the parameters $k, \alpha, \beta$ and $\lambda$, in a ball with radius $R$ where the problem has a positive solution. Finally, in Section 5 we summarize our results and describe future work.

2. Theoretical results

Let $\lambda_{1}^{\text{div}}$ be the first eigenvalue of $-\text{div}((\alpha + \beta |x|^k) \nabla u)$ on $D$ with zero Dirichlet boundary condition. We define

$$
\gamma(\varphi) = \frac{\int_0^R (\alpha + \beta k^2)\rho(r)(\varphi'(r))^2 \, dr + \beta \int_0^R \rho(r)(\varphi(r))^2 \, dr}{\int_0^R (\rho(r)^3(\varphi(r))^2) \, dr},
$$

and

$$
\lambda^\ast(k) = \inf \left\{ \gamma(\varphi), \varphi \in C_0^\infty(B(0,R)) \varphi \equiv 1 \text{ in } B(0,R/4), \varphi \equiv 0 \text{ on } B(0,R) \setminus B(0,R/3) \right\}.
$$

The main result is:

**Theorem 2.1:** (1) For $k > 1$ there exists a positive solution of the Equation (3) for $\lambda \in (\lambda^\ast(k), \lambda_{1}^{\text{div}})$. (2) There is no solution of (3) for $\lambda \geq \lambda_{1}^{\text{div}}$.

Let $S$ be the best Sobolev constant for the injection of $H_0^1(D)$ into $L^6(D)$. We consider the associate minimizing problem to (2)

$$
S_{\lambda,\alpha,\beta} = \inf_{u \in H_0^1(D) \setminus \{0\}} \left\{ \int_D (\alpha + \lambda |x|^k)\rho(x)|\nabla u|^2 \, dx - \lambda \int_D (\rho(x)^3|u|^2) \, dx \right\}.
$$

The proof of the first part of Theorem 2.1 is based on the two following lemmas.

**Lemma 2.1:** If $S_{\lambda,\alpha,\beta} < \alpha S$, then the problem (5) has a minimizer.

**Proof:** See Lemma 1.1 in [10] and Lemma 3.1 in [15].

**Lemma 2.2:** We have

$$
S_{\lambda,\alpha,\beta} \leq \alpha S, \quad \text{if } k > 1 \quad \text{and} \quad S_{\lambda,\alpha,\beta} \leq \alpha S, \quad \text{if } k \leq 1.
$$

**Proof:** We define

$$
Q_{\lambda,k}(u) = \frac{\int_D q(x)\rho(x)|\nabla u|^2 \, dx - \lambda \int_D (\rho(x)^3|u|^2) \, dx}{(\int_D (\rho(x)^3|u|^6) \, dx)^{\frac{1}{3}}},
$$

with $q(x) = q(r) = \alpha + \beta |x|^k$ with $r = |x|$.

Next, we estimate the energy at $u_t(r) = \frac{1}{(e+t^2)^\frac{1}{2}}$.

Using (16) and (24) in [4] and [15], we have

$$
\int_0^R u_r^2 \rho^3 r^2 \, dr = 4\pi \varepsilon r^2 \int_0^R (\rho(r))^2 \, dr + o(\varepsilon^2),
$$

and

$$
\int_0^R q(r) \frac{\partial u_r}{\partial r} \rho^2 r^2 \, dr = 24\alpha \int_0^\infty \frac{t^2}{(1+t^2)^3} \, dt
$$

for

$$
\begin{align*}
4\pi \varepsilon \left[ \int_0^R q(r)\rho(r)(\nu'(r))^2 - 2(\varphi(r))^2 \right] \, dr + \\
\beta \int_0^R \rho(r)(\varphi(r))^2 t^2 \, dr + o(\varepsilon^2), \quad \text{if } k > 1,
\end{align*}
$$

$$
\begin{align*}
8\pi \varepsilon k \int_0^R \frac{1}{1+t^2} \, dt + o(\varepsilon^2), \quad \text{if } k < 1.
\end{align*}
$$

(9)
Combining (7), (8) and (9) we get

\[
Q_{a,λ}(u_k) = αS + (2^3π)^{\frac{1}{2}} \left[ e^2 \left[ \int_0^R q(r)ρ(r)((ϕ'(r))^2 - (ϕ(r))^2)dr \right. \\
+ β \int_0^R ρ(r)(ϕ(r))^2 k^{k-2}dr \left. \right] \right. \\
- λ e^2 \int_0^R (ϕ'(r))^2 dr + o(ϵ^{\frac{1}{2}}), \text{ if } k > 1, \\
1 \int_k^R 2ρk^2 |ln(ε)| + o(ϵ^{1/2} |ln(ε)|), \text{ if } k = 1, \\
1 \int_0^∞ 2ρk^2 \left( \frac{k^{k+2}}{(1 + ϵ^2)^2} \right) dr \\
+ \left( \frac{2k^2}{(1 + ϵ^2)^3} \right) + o(ϵ^{1/2}), \text{ if } k < 1.
\]

Therefore, we can deduce a conclusion just when \( k > 1 \), more precisely we have

\[
Q_{a,λ}(u_k) = αS + (2^3π)^{\frac{1}{2}} \left[ e^2 \left[ \int_0^R q(r)ρ(r)((ϕ'(r))^2 - (ϕ(r))^2)dr \right. \\
+ β \int_0^R ρ(r)(ϕ(r))^2 k^{k-2}dr \left. \right] \right. \\
- λ e^2 \int_0^R (ϕ'(r))^2 dr + o(ϵ^{\frac{1}{2}}), \text{ if } k > 1, \\
1 \int_k^R 2ρk^2 |ln(ε)| + o(ϵ^{1/2} |ln(ε)|), \text{ if } k = 1, \\
1 \int_0^∞ 2ρk^2 \left( \frac{k^{k+2}}{(1 + ϵ^2)^2} \right) dr \\
+ \left( \frac{2k^2}{(1 + ϵ^2)^3} \right) + o(ϵ^{1/2}), \text{ if } k < 1.
\]

Finally, from the definition of \( λ^*(k) \) in (4) and the fact that \( q(r) = α + β r^k \), we conclude that \( S_{λ,α,β} < αS \) when \( λ > λ^*(k) \) and \( k > 1 \).

**Proof of Theorem 2.1:** (1) Let \((u_j)\) be a minimizing sequence of \( S_{λ,α,β} \). More precisely,

\[
\int_D (α + β |x|^k)ρ(x)|∇u|^2dx - λ \int_D (ρ(x))^3 |u|^2dx = S_{λ,α,β} + o(1),
\]

and

\[
\left( \int_D (ρ(x))^3 |u|^6 \right)^{\frac{1}{2}} = 1.
\]

From Lemma 6, we know that \( S_{λ,α,β} < αS \) for \( λ > λ^*(k) \) and \( k > 1 \). Thus we have a minimizing sequence of \( S_{λ,α,β} \) and \( S_{λ,α,β} < αS \). Consequently, from Lemma 2.1, we deduce that \( S_{λ,α,β} \) is achieved by a function \( u \) such that \( u_j \to u \) strongly in \( H_0^1(Ω) \) and

\[
\int_D (α + β |x|^k)ρ(x)|∇u|^2dx - λ \int_D (ρ(x))^3 |u|^2dx = S_{λ,α,β} \text{ and } \left( \int_D (ρ(x))^3 |u|^6 \right)^{\frac{1}{2}} = 1.
\]

We may assume that \( u \geq 0 \) (otherwise we replace \( u \) by \(|u|\)). Since \( u \) is a minimizer of (5) then there exists a Lagrange multiplier \( μ \in \mathbb{R} \) such that

\[
- \text{div}(ρ(x)(α + β |x|^k)∇u) - λ \rho^3(x)u = μ \rho^3(x)u^5 \text{ in } Ω.
\]

In fact, \( μ = S_{λ,α,β} \) and \( S_{λ,α,β} > 0 \) since \( λ > λ^*_1 \). Then \( u > 0 \) on \( Ω \) by the strong maximum principle. Finally, there exists a positive constant \( k > 0 \) (more precisely \( k = (S_{λ,α,β})^{\frac{1}{k}} \)) such that \( k u \) is a solution of problem 2.

(2) There is no solution of (2) when \( λ > λ^*_1 \). Indeed, let \( ψ_1 \) be the eigenfunction of \( -\text{div}_Ω((α + β |x|^k)∇_3ψ_1) \) corresponding to \( λ^*_1 \), with \( ψ_1 > 0 \) on \( D \). More precisely, we have

\[
- \text{div}_Ω((α + β |x|^k)∇_3ψ_1) = λ^*_1 ψ_1 \text{ on } D,
\]
equivalently, under the stereographic projection, to

\[
- \text{div}(ρ(x)(α + β |x|^k)∇ψ_1) = λ^*_1 ρ^3(x)ψ_1 \text{ on } B(0,R).
\]

Suppose that \( u \) is a solution of (2), then we have

\[
λ^*_1 \int_D ρ^3 |ψ_1|^2 = \int_D - \text{div}(ρ(α + β |x|^k)∇u)ψ_1
\]

\[
= \int_D ρ^3 u^5 ψ_1 + λ \int_D ρ^3 |ψ_1|^2
\]

\[
> λ \int_D ρ^3 |ψ_1|^2.
\]

Thus \( λ < λ^*_1 \), since the functions \( ρ, u \) and \( ψ_1 \) are positive which concludes the proof of Theorem 2.1.

3. Numerical study

From the boundary problem (3), we have

\[
\begin{cases}
- (ρ(r) p(r) r^2 u'(r))' = r^2 p^2(r) u^5(r) \\
+ λ r^2 ρ^3(r) u(r), \text{ for } r ∈ (0,R),
\end{cases}
\]

\[
u'(0) = u(R) = 0,
\]

where

\[
ρ(r) = \frac{2}{1 + r^2} p(r) = α + β r^k.
\]

Let \( u(r) = u(r; u_0) \) be a solution of the initial value problem

\[
\begin{cases}
- (ρ(r) p(r) r^2 u'(r))' = r^2 p^2(r) u^5(r) \\
+ λ r^2 ρ^3(r) u(r), \text{ for } r ∈ (0,R),
\end{cases}
\]

\[
u'(0) = 0,
\]

\[
u(0) = u_0 > 0,
\]

such that

\[
u(R; u_0) = 0.
\]
Using Newton iteration method to approximate \( u_0 \):

\[
u_0^{+1} = u_0 - \left( \frac{1}{u_0} u \left( R; u_0 \right) \right)^{-1} u \left( R; u_0 \right),
\]

\( j = 1, 2, 3, \ldots, n, \)

(14)

and differentiating the differential Equation (13) with respect to \( u_0 \), yields

\[
\begin{align}
\left( (r^2 p^2) \right) & = 5r^2 p^3 (r) u^4 (r) v (r) + \lambda r^2 p^3 (r) v (r), \quad \text{for} \ r \in (0, R), \\
v (0) & = 0,
\end{align}
\]

(15)

where

\[
v (r) = \frac{\partial}{\partial u_0} u \left( R; u_0 \right).
\]

By substituting

\[
x (r) = r^2 (r) p (r) u' (r),
\]

\[
y (r) = r^2 (r) p (r) v' (r),
\]

in Equations (13) and (15), we get the following system of first-order differential equations:

\[
\begin{align*}
\left\{ u' (r) = (r^2 p) \left. \frac{1}{u} \right| _{x}, \\
x' (r) = -r^2 p^3 (u^5 + \lambda u), \\
v' (r) = \left( (r^2 p) \right) \left. \frac{1}{u} \right| _{y}, \\
y' (r) = -r^2 p^3 (\lambda + (2^* - 1) u^8) v,
\end{align*}
\]

(16a)

subjected to the initial conditions

\[
\begin{align*}
u (0) & = u_0, \\
x (0) & = 0, \\
v (0) & = 1, \\
y (0) & = 0,
\end{align*}
\]

(16b)

which can be written in a matrix form as

\[
w' = f (r, w),
\]

with

\[
w = \begin{pmatrix} u \\ x \\ v \end{pmatrix}
\]

and

\[
f (r, w) = \begin{pmatrix} (r^2 p) \left. \frac{1}{u} \right| _{x} \\ -r^2 p^3 (u^5 + \lambda u) \\ (r^2 p) \left. \frac{1}{u} \right| _{y} \\ -r^2 p^3 (\lambda + (2^* - 1) u^8) v \end{pmatrix}.
\]

Using the classical fourth order Runge–Kutta method for the system (16a)–(16b), and for a given step-size \( h > 0 \) we define

\[
\begin{align*}
w_{i+1} & = w_i + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right), \\
k_1 & = f (r, w), \\
k_2 & = f (r, w_{i + \frac{1}{2} h}), \\
k_3 & = f (r, w_{i + \frac{1}{2} h} + \frac{1}{2} k_2 h), \\
k_4 & = f (r, w_{i + h})
\end{align*}
\]

for \( i = 0, 1, 2, 3, \ldots, m, \)

\( r_{i+1} = r_{i} + h \)

where

\[
\begin{align*}
\left\{ u_{m} \left( u_0 \right) & = u \left( R \right) = 0,
\end{align*}
\]

(17)

and we have

\[
\begin{pmatrix} u_1 (u_0) \\
u_2 (u_0) \\
u_3 (u_0) \\
\vdots \\
u_{m-1} (u_0) \\
u_m (u_0) = u \left( R \right) \end{pmatrix} = \begin{pmatrix} u_1 (u_0) \\
u_2 (u_0) \\
u_3 (u_0) \\
\vdots \\
u_{m-1} (u_0) \\
u_m (u_0) = u \left( R \right) \end{pmatrix}.
\]

(18)

The last equation is nonlinear, which will be solved for \( u_0 \) by Newton iterative method:

\[
u^{+1} = u_0 - \left( \frac{\partial}{\partial u_0} u \left( R; u_0 \right) \right)^{-1} u \left( R; u_0 \right),
\]

\( j = 1, 2, 3, \ldots, n. \)
Next, we solve the boundary value problem (12) for different values of $\alpha$, $\beta$, $k$, $\lambda$, and the corresponding $R$ with the solutions are presented in Figures 1, 2, 3.

Figure 1 presents a numerical solution of (12) in the case $\lambda > 0$ and $0 < k \leq 1$, where $\alpha = 1$, $\beta = 1$, $\lambda = 2$, and $R = 0.981918$, $k = 0.1$ (Figure 1 a), $R = 1.08245$, $k = 0.5$ (Figure 1 b), $R = 1.08481$, $k = 0.9$ (Figure 1 c), and $R = 1.07375$, $k = 1$ (Figure 1 d).

Figure 2. Numerical solutions of (12) for $\alpha = 1$, $\beta = 1$, $\lambda = -6$, and different values of $R$ and $k$: (a) $R = 26.4421$, $k = 1.1$ (Figure 2 a), $R = 43.4183$, $k = 1.2$ (Figure 2 b), $R = 114.356$, $k = 1.3$ (Figure 2 c), and $R = 316.708$, $k = 1.35$ (Figure 2 d).

Figure 2 presents a numerical solution of (12) in the case $\lambda < 0$ and $0 < k \leq 1$, where $\alpha = 1$, $\beta = 1$, $\lambda = -6$, and $R = 0.981918$, $k = 0.1$ (Figure 2 a), $R = 1.08295$, $k = 0.5$ (Figure 2 b), $R = 1.08481$, $k = 0.9$ (Figure 2 c), and $R = 1.07375$, $k = 1$ (Figure 2 d).
Figure 3. Numerical solutions of (12) for \( \lambda = 6 \) (a), (b) and \( \lambda = -6 \) (c), (d), with different values of \( R, k, \alpha, \beta \): (a) \( R = 1.89151, k = 0.9, \alpha = 2, \beta = 5 \); (b) \( R = 1.38993, k = 0.9, \alpha = 3, \beta = 4 \); (c) \( R = 29.3202, k = 1.1, \alpha = 0.5, \beta = 0.25 \); (d) \( R = 8.90831, k = 1.3, \alpha = 2, \beta = 0.2 \).

Next, Figure 2 gives a numerical solution of (12) in the case \( \lambda < 0 \) and \( k > 1 \), where \( \alpha = 1, \beta = 1, \lambda = -6 \), and \( R = 26.4421 \), \( k = 1.1 \) (Figure 2 a), \( R = 43.4183 \), \( k = 1.2 \) (Figure 2 b), \( R = 114.356 \), \( k = 1.3 \) (Figure 2 c), \( R = 316.708 \), \( k = 1.35 \) (Figure 2 d). We observe that the numerical solution start by oscillating near the origin then decreases for a large value of the radius \( R \) to \( u(R) = 0 \).

Finally, Figure 3 illustrates a numerical solution of (12) for \( \alpha = 2, \beta = 5, \lambda = 6 \), \( R = 1.89151 \), \( k = 0.9 \) (Figure 3 a), \( \alpha = 3, \beta = 4, \lambda = 6 \), \( R = 1.38993 \), \( k = 0.9 \) (Figure 3 b), \( \alpha = 0.5, \beta = 0.25, \lambda = -6 \), \( R = 29.3202 \), \( k = 1.1 \) (Figure 3 c), \( \alpha = 2, \beta = 0.2, \lambda = -6 \), \( R = 8.90831 \), \( k = 1.3 \) (Figure 3 d). We notice similar observations as in Figure 1 and Figure 2, respectively, for \( \lambda \) positive and negative.

4. Variation of the problem parameters

Next, for a given \( u(0) = u_0 > 0 \), we solve numerically the boundary value problem (12) for different values of the parameters in both cases when \( \lambda \) positive and negative and choose the minimum radius \( R \) so that the obtained numerical solution stay positive.

- **Case 1 (\( \lambda > 0 \))**:

| \( \beta \) | \( R \) | \( u(R) \) |
|---|---|---|
| 1 | 1.30469 | 5.05173e-6 |
| 2 | 1.92968 | 2.79194e-6 |
| 3 | 2.73412 | 2.83756e-6 |
| 4 | 3.7134 | 1.66654e-5 |
| 5 | 4.85963 | 1.21794e-5 |
| 10 | 12.8083 | 4.34204e-6 |

**Table 1. Minimum radius \( R \) for (12) with \( \alpha = 1, k = 0.9, \lambda = 2 \) and \( \beta \).**

| \( \alpha \) | \( R \) | \( u(R) \) |
|---|---|---|
| 1 | 1.30469 | 5.05173e-6 |
| 2 | 1.6094 | 5.85605e-7 |
| 3 | 1.8487 | 1.03335e-5 |
| 4 | 2.05247 | 1.35971e-5 |
| 5 | 2.23364 | 1.36117e-5 |
| 10 | 2.96707 | 9.28055e-6 |

**Table 2. Minimum radius \( R \) (12) with \( \beta = 1, k = 0.9, \lambda = 2 \) and \( \alpha \).**

**Example 4.1:** We solve the boundary value problem (12) for \( \alpha = 1, k = 0.9, \lambda = 2 \) with different values of \( \beta \) and the obtained minimum radius \( R \) satisfied the boundary condition \( u(R) = 0 \) are presented in Table 1.

**Example 4.2:** We solve the boundary value problem (12) for \( \beta = 1, k = 0.9, \lambda = 2 \) with different values of \( \alpha \) and the obtained minimum radius \( R \) satisfied the boundary condition \( u(R) = 0 \) are presented in Table 2.
the parameters problem (5) has a numerical solution is depending on ing the existence of a positive solution in 0 and for different positive values of λtion for the boundary value problem (3) for all In this work, we proved the existence of a positive solu-

| β  | R         | u(R)         |
|----|-----------|--------------|
| 0.1| 6.95859   | 3.4388e-5    |
| 0.2| 8.90816   | 9.9492e-6    |
| 0.3| 11.8937   | 6.6295e-6    |
| 0.4| 16.8078   | 8.8816e-6    |
| 0.5| 25.7395   | 1.7292e-6    |
| 0.6| 44.4701   | 1.41598e-7   |

Table 4. Minimum radius R (12) with β = 0.1, k = 1.3, λ = −6 and α.

| α  | R         | u(R)         |
|----|-----------|--------------|
| 2.1| 7.32417   | 1.7822e-5    |
| 2.2| 7.76992   | 2.4366e-5    |
| 2.3| 8.29987   | 2.0350e-6    |
| 2.4| 8.92551   | 2.1239e-5    |
| 2.5| 9.65183   | 1.4909e-5    |
| 2.6| 10.507    | 2.679e-5     |
| 2.7| 11.515    | 1.4483e-5    |
| 2.8| 12.7119   | 5.7846e-6    |
| 2.9| 14.1481   | 5.8943e-6    |
| 3.0| 15.8942   | 8.8491e-6    |

Table 4. Minimum radius R (12) with β = 0.1, k = 1.3, λ = −6 and α.

We remark that by increasing β the minimum radius R satisfied the boundary condition u(R) = 0 increases significantly compared when increasing the parameter α.

Case 2 (λ < 0):

Example 4.3: We solve the boundary value problem (12) for α = 1, k = 1.3, λ = −6 with different values of β and the obtained minimum radius R satisfied the boundary condition u(R) = 0 are presented in Table 3.

Example 4.4: We solve the boundary value problem (12) for β = 0.1, k = 1.3, λ = −6 with different values of α and the obtained minimum radius R satisfied the boundary condition u(R) = 0 are presented in Table 4.

Similarly as in the first case, the minimum radius R satisfied the boundary condition u(R) = 0 increases considerably by changing the parameters α and β.

5. Conclusion

In this work, we proved the existence of a positive solution for the boundary value problem (3) for all λ ≠ 0 and for different positive values of k. Theoretically, we proved the result for only λ > 0 and k > 1, and we completed the other cases numerically. More precisely, we obtained the existence of solutions using numerical methods when λ is positive and k is between 0 and 1, also when λ is negative and k is positive. We concluded that the radius of a ball in S^3 for which the problem (5) has a numerical solution is depending on the parameters α, β and k. Future work will include proving the existence of a positive solution in S^4 especially for k = 2 and λ strictly positive close to zero or λ strictly negative.

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ORCID

Adel Almarsh et al. (http://orcid.org/0000-0002-2542-3320)
Idir Mechai (http://orcid.org/0000-0003-4107-9010)
Habib Yazidi (http://orcid.org/0000-0002-5386-772X)
Ahmed Msmali (http://orcid.org/0000-0002-0186-0526)

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