A DUALITY THEOREM FOR THE IC-RESURGENCE OF EDGE IDEALS

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Abstract. The aim of this work is to use linear programming and polyhedral geometry to prove a duality formula for the ic-resurgence of edge ideals. We show that the ic-resurgence of the edge ideal $I$ of a clutter $C$ and the ic-resurgence of the edge ideal $I^V$ of the blocker $C^V$ of $C$ coincide. If $C$ is the clutter of bases of certain uniform matroids, we recover a formula for the resurgence of $I$, and if $C$ is a connected non-bipartite graph with a perfect matching, we show a formula for the Waldschmidt constant of $I^V$.

1. Introduction

Let $C$ be a clutter with vertex set $V(C) = \{t_1, \ldots, t_s\}$, that is, $C$ is a family of subsets $E(C)$ of $V(C)$, called edges, none of which is contained in another [11]. We assume that all edges of $C$ have at least two vertices. For example, a graph (no multiple edges or loops) is a clutter. Regarding each vertex $t_i$ as a variable, we consider the polynomial ring $S = K[t_1, \ldots, t_s]$ over a field $K$. The monomials of $S$ are denoted by $t^a := t_1^{a_1} \cdots t_s^{a_s}$, $a = (a_1, \ldots, a_s)$ in $\mathbb{N}^s$, where $\mathbb{N} = \{0, 1, \ldots\}$. The edge ideal of $C$, denoted $I(C)$, is the ideal of $S$ given by

$$I(C) := \langle \{\prod_{t_i \in e} t_i \mid e \in E(C)\} \rangle.$$ 

The minimal set of generators of $I(C)$, denoted $G(I(C))$, is the set of all squarefree monomials $t_e = \prod_{t_i \in e} t_i$ such that $e \in E(C)$. Any squarefree monomial ideal $I$ of $S$ is the edge ideal $I(C)$ of a clutter $C$ with vertex set $V(C) = \{t_1, \ldots, t_s\}$. A set of vertices $C$ of $C$ is called a vertex cover if every edge of $C$ contains at least one vertex of $C$. A minimal vertex cover of $C$ is a vertex cover which is minimal with respect to inclusion. The clutter of minimal vertex covers of $C$, denoted $C^V$, is called the blocker of $C$, and the edge ideal $I(C^V)$ of $C^V$ is called the Alexander dual of $I(C)$ and is denoted by $I(C)^V$ [40, p. 221]. We assume that $|C| \geq 2$ for all $C \in E(C^V)$.

We denote the edge ideal $I(C)$ of $C$ by $I$ and denote the minimal set of generators of $I$ by $G(I) := \{t^{v_1}, \ldots, t^{v_q}\}$. The incidence matrix of $I$, denoted by $A$, is the $s \times q$ matrix with column vectors $v_1, \ldots, v_q$. This matrix is the incidence matrix of $C$. The covering polyhedron of $I$, denoted by $Q(I)$, is the rational polyhedron

$$Q(I) := \{x \mid x \geq 0; xA \geq 1\},$$

where $1 = (1, \ldots, 1)$. The map $E(C^V) \to \{0, 1\}^q, C \mapsto \sum_{t_i \in C} e_i$, induces a bijection between $E(C^V)$ and the set $\{u_1, \ldots, u_m\}$ of integral vertices of $Q(I)$ [40, Corollary 13.1.3]. Given an integer $n \geq 1$, the $n$-th symbolic power of $I$, denoted $I^{(n)}$, is given by [18, p. 78]:

$$I^{(n)} = \langle \{t^a \mid a/n \in Q(I^V)\} \rangle = \langle \{t^a \mid \langle a, u_i \rangle \geq n \text{ for } i = 1, \ldots, m\} \rangle,$$

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where \( \langle , \rangle \) denotes the ordinary inner product. The \textit{Newton polyhedron} of \( I \), denoted \( \text{NP}(I) \), is the integral polyhedron

\[
\text{NP}(I) = \mathbb{R}^s_+ + \text{conv}(v_1, \ldots, v_q),
\]

where \( \mathbb{R}^s_+ = \{ \lambda \in \mathbb{R} \mid \lambda \geq 0 \} \). It is well known that \( \text{NP}(I) \) is equal to

\[
Q(B) := \{ x \mid x \geq 0; x^T B \geq 1 \},
\]

where \( B \) is the rational matrix whose columns are precisely the vertices \( u_1, \ldots, u_p \) of the covering polyhedron \( Q(I) \) of \( I \) \cite{18} Proposition 3.5(b)]. Note that \( m \leq p \), that is, not all vertices of \( Q(I) \) are integral. The \textit{integral polyhedron} \( Q \) is normal. A related well studied invariant is \( \gamma(I) \), the \textit{Waldschmidt constant} of \( I \) \cite{10} \cite{13}: \( \gamma(I) := \text{min} \{ n/r \mid I^{(n)} \not\subset I^r \} \).

Our main result is a formula for the ic-resurgence \( \rho_{ic}(I) \) of \( I \) in terms of the vertices of \( Q(I) \) and \( Q(I^\vee) \) (Theorem 3.7).

As we now explain computing the ic-resurgence \( \rho_{ic}(I) \) of the ideal \( I \) is a linear-fractional programming problem \cite{7}. Let \( V(Q(I)) = \{ u_1, \ldots, u_p \} \) be the vertex set of \( Q(I) \), we can write \( u_i = \gamma_i/d_i, \gamma_i \in \mathbb{N}_s \setminus \{0\}, d_i \in \mathbb{N}_+ \) for all \( i \) (Section 3). Recall that we may assume that \( d_i = 1 \) for \( i = 1, \ldots, m \), i.e., \( u_1, \ldots, u_m \) are the characteristic vectors of the minimal vertex covers of \( C \).
and $m \leq p$. By Eqs. (1.1)-(1.2), a monomial $t^a$ is in $I^{(n)} \setminus \overline{T}$ if and only if $a/n \in Q(I')$ and $a/r \notin \text{NP}(I)$, that is, $t^a$ is in $I^{(n)} \setminus \overline{T}$ if and only if $a = (a_1, \ldots, a_s)$ satisfies
\begin{equation}
\langle a, u_i \rangle \geq n \text{ for } i = 1, \ldots, m \text{ and } \langle a, \gamma_j \rangle \leq rd_j - 1 \text{ for some } 1 \leq j \leq p. \tag{1.3}
\end{equation}

Let $x_1, \ldots, x_s$ be variables that correspond to the entries of $a$ and let $x_{s+1}, x_{s+2}$ be two extra variables that correspond to $n$ and $r$, respectively. By Eq. (1.3), $t^a$ is in $I^{(n)} \setminus \overline{T}$ if and only if there exists $1 \leq j \leq p$ such that $(a, n, r)$ is a feasible point for the linear-fractional program:

\begin{align}
\text{maximize} & \quad h_j(x) = \frac{x_{s+1}}{x_{s+2}} \\
\text{subject to} & \quad \langle (x_1, \ldots, x_s), u_i \rangle - x_{s+1} \geq 0, \quad i = 1, \ldots, m, \quad x_{s+1} \geq 1 \\
& \quad x_i \geq 0, \quad i = 1, \ldots, s \\
& \quad d_j x_{s+2} - \langle (x_1, \ldots, x_s), \gamma_j \rangle \geq 1, \quad x_{s+2} \geq 1
\end{align}

This type of program can be solved using linear programming [4, Section 4.3.2, p. 151]. The linear-fractional program of Eq. (1.4) is equivalent to the linear program of Eq. (1.5) below [21, p. 18]. The next result solves the problem of computing the ic-resurgence of $I$. An algorithm to compute $\rho_{ic}(I)$ has been implemented in Macaulay2 using its interface to Normaliz [5] (see [21, Procedure A.3, Algorithm A.4]). We use the next result to prove a duality formula for the ic-resurgence of $I$ that implies the equality $\rho_{ic}(I) = \rho_{ic}(I')$.

**Theorem 1.1.** [21, Theorem 5.3] For each $1 \leq j \leq p$, let $\rho_j$ be the optimal value of the following linear program with variables $y_1, \ldots, y_{s+3}$. Then, $\rho_{ic}(I) = \max \{ \rho_j \}^p_{j=1}$ and $\rho_j$ is attained at a rational vertex of the polyhedron $P_j$ of feasible points of Eq. (1.5).

\begin{align}
\text{maximize} & \quad g_j(y) = y_{s+1} \\
\text{subject to} & \quad \langle (y_1, \ldots, y_s), u_i \rangle - y_{s+1} \geq 0, \quad i = 1, \ldots, m, \quad y_{s+1} \geq y_{s+3} \\
& \quad y_i \geq 0, \quad i = 1, \ldots, s, \quad y_{s+3} \geq 0 \\
& \quad d_j y_{s+2} - \langle (y_1, \ldots, y_s), \gamma_j \rangle \geq y_{s+3}, \quad y_{s+3} = 1
\end{align}

First, we determine all vertices $y$ of $P_j$ with $y_{s+1} > 0$ and relate them with the vertices of $Q(I)$ and $Q(I')$. To introduce our result, let $V(Q(I')) = \{ v_1, \ldots, v_{p_1} \}$ be the vertex set of $Q(I')$, we can write $v_i = \delta_i/f_i$, $\delta_i \in \mathbb{N}^s \setminus \{0\}$, $f_i \in \mathbb{N}_+$ for all $i$ (Section 3). Recall that we may assume that $f_i = 1$ for $i = 1, \ldots, q$, i.e., $v_1, \ldots, v_q$ are the characteristic vectors of the edges of $\mathcal{C}$ and $q \leq p_1$. We introduce the Rees cones of $I$ and $I'$ as a device to compute the vertices of $Q(I)$ and $Q(I')$, and the linear constraints that define the Newton polyhedron of $I$ (Theorems 3.1-3.4). Then, we collect some basic properties of $P_j$ showing that $P_j$ could be unbounded, and proving that $\rho_j \leq d_j$ and $\langle \gamma_i, \delta \rangle \geq d_i \geq 1$ for all $i, \ell$ (Lemma 3.5).

We come to our main auxiliary result.

**Theorem 3.6** Let $y = (y_1, \ldots, y_{s+3})$ be a point in $\mathbb{R}^{s+3}$ with $y_{s+1} > 0$ and let $1 \leq j \leq p$, then $y$ is a vertex of $P_j$ if and only if $y$ has one of the following two forms:

(a) $y = (y_{s+1}(\delta_k/f_k), y_{s+1}, 1, 0)$, where $y_{s+1} = d_j f_k/\langle \gamma_j, \delta_k \rangle$ and $1 \leq k \leq p_1$, or

(b) $y = (y_{s+1}(\delta_k/f_k), y_{s+1}, 1, y_{s+1})$, where $y_{s+1} = d_j f_k/\langle \gamma_j, \delta_k \rangle + f_k$ and $1 \leq k \leq p_1$.

The first evidence that the equality $\rho_{ic}(I) = \rho_{ic}(I')$ could be true came from [13] and [21, 26] where it is shown that for the edge ideal $I(G)$ of a perfect graph $G$, one has

$$\rho_{ic}(I(G)) = \frac{2(\omega(G) - 1)}{\omega(G)} = \rho_{ic}(I(G')'),$$
respectively, where \( \omega(G) \) is the clique number of \( G \), that is, \( \omega(G) \) is the number of vertices in a maximum complete subgraph of \( G \). If \( G \) is a perfect graph, then \( \rho(I(G)^\vee) \) is equal to \( \rho_{ic}(I(G)^\vee) \) because \( I(G)^\vee \) is normal [35, Theorem 2.10].

We come to our main result.

**Theorem 3.7** (Duality formula) If \( I \) is the edge ideal of a clutter \( \mathcal{C} \), then

\[
\frac{1}{\rho_{ic}(I)} = \min \{ \langle u, v \rangle \mid u \in V(Q(I)), v \in V(Q(I^\vee)) \},
\]

where \( V(Q(I)) \) is the vertex set of \( Q(I) \). In particular, \( \rho_{ic}(I) = \rho_{ic}(I^\vee) \).

A covering polyhedron is integral if and only if it has only integral vertices [30, p. 232]. The ic-resurgence of \( I \) classifies the integrality of the covering polyhedron \( Q(I) \) because \( \rho_{ic}(I) = 1 \) if and only if \( Q(I) \) is integral (Proposition 3.8). As a consequence of the duality formula for the ic-resurgence, we recover the fact that \( Q(I) \) is integral if and only if \( Q(I^\vee) \) is integral [11, Theorem 1.17] (Corollary 3.9), and recover two results of Jayanthan, Kumar and Mukundan [26, Theorems 4.8 and 5.3] showing that when \( I(G) \) is the edge ideal of a graph \( G \) the following conditions are equivalent

(a) \( \rho(I(G)) = 1 \); (b) \( \rho_{ic}(I(G)) = 1 \); (c) \( G \) is bipartite; (d) \( \rho(I(G)^\vee) = 1 \);

see Corollary 3.10. Then, we show a formula for the ic-resurgence of the sum of two edge ideals of clutters generated by monomials in disjoint sets of variables (Corollary 3.11). A similar formula is known for the Waldschmidt constant (Remark 3.12). If \( G \) is a connected non-bipartite graph with a perfect matching, we show that \( \hat{\alpha}(I(G)^\vee) = |V(G)|/2 \) (Proposition 3.17).

To compute the vertices of the covering polyhedron \( Q(I) \) of the edge ideal \( I \) of a clutter \( \mathcal{C} \), we have used Theorems 3.1 and 3.2, see Procedure A.1. Another way to compute the vertices of \( Q(I) \) is to find the extreme rays, i.e., the 1-dimensional faces of the following polyhedral cone

\[ SC(I^\vee) := \{ x \in \mathbb{R}^{s+1} \mid x \geq 0; \langle x, (v_i, -1) \rangle \geq 0 \text{ for } i = 1, \ldots, q \}, \]

see [21, Proposition 3.15]. The cone \( SC(I^\vee) \) is called the Simis cone of \( I^\vee \) [15]. For a discussion on how to find all vertices of a general polyhedron we refer to [2 3].

The algebraic properties and invariants of the ideal \( I = I_{d,s} \), generated by all squarefree monomials of \( S \) of degree \( d \) in \( s \) variables, were studied in [1, 4, 16, 37, 39]. This ideal is normal [37] and its Alexander dual \( I^\vee \) is equal to \( I_{s-d+1,s} \). The minimal generators of \( I \) correspond to the bases of a matroid of rank \( d \). The statement about resurgence in Theorem 3.19 is due originally to Lampi-Baczynska and Malara [28, Theorem C]. In a more general form, it appears in a paper by Geramita, Harbourne, Migliore and Nagel [16, Theorem 4.8]. The Waldschmidt constant of \( I \) was computed in [4, Theorem 7.5]. Using the Rees cones of \( I \) and \( I^\vee \), we compute the vertices of \( Q(I) \) and \( Q(I^\vee) \) and use Theorem 3.7 to recover the formulas for \( \rho(I) \) and \( \hat{\alpha}(I) \).

**Theorem 3.19** [4, 28] The resurgence and ic-resurgence of \( I_{d,s} \) and \( I^\vee_{s,d} \) are given by

\[
\rho(I_{d,s}) = \rho_{ic}(I_{d,s}) = \rho_{ic}((I_{d,s})^\vee) = \frac{d(s - d + 1)}{s} = \frac{d}{\hat{\alpha}(I_{d,s})} = \frac{s - d + 1}{\hat{\alpha}((I_{d,s})^\vee)}.
\]

In Section 4 we present examples illustrating our results. Then in Appendix A we give the procedures for Macaulay2 [20], that are used in the examples.

For unexplained terminology and additional information, we refer to [12, 25, 31, 35] for the theory of integral closure, [23, 40] for the theory of edge ideals and monomial ideals, and [27, 30, 31] for combinatorial optimization and integer programming.
In this section we introduce a few results from polyhedral geometry and commutative algebra. We continue to employ the notations and definitions used in Sections 1 and 2.

Let \( I \subseteq R[\{a, c \}] \) be a squarefree monomial ideal. If \( Q(I) \) is integral, then \( \hat{\alpha}(I) = \alpha(I) \) and \( \hat{\alpha}(I^V) = \alpha(I^V) \).

Theorem 2.2. Let \( I \) be a squarefree monomial ideal. The following hold.

(a) \( I^n = I^{(n)} \) for all \( n \geq 1 \) if and only if \( Q(I) \) is integral and \( I \) is normal.
(b) \( \mathcal{P}^n = I^{(n)} \) for all \( n \geq 1 \) if and only if \( Q(I) \) is integral.

3. Duality formula for the ic-resurgence

In this part we give a duality formula for the ic-resurgence of edge ideals of clutters. To avoid repetitions, we continue to employ the notations and definitions used in Sections 1 and 2.

Let \( S = K[t_1, \ldots, t_s] \) be a polynomial ring over a field \( K \), let \( I \subseteq S \) be the edge ideal of a clutter \( C \), and let \( G(I) = \{t^{e_1}, \ldots, t^{e_s}\} \) be the minimal set of generators of \( I \). To study \( \rho_{ic}(I) \), we need a convenient way to determine the polyhedron \( \mathcal{P}_j \) of feasible points of the linear program of Theorem 1.1 for \( 1 \leq j \leq p \). Our approach is based on the computation of the supporting hyperplanes of the Rees cone of \( I \) which is the finitely generated rational cone defined as [15]:

\[
(3.1) \quad \text{RC}(I) := \mathbb{R}_+ \{e_1, \ldots, e_s, (v_1, 1), \ldots, (v_q, 1)\}.
\]

Theorem 3.1. [10] Proposition 1.1.51, Theorem 14.1.1] The Rees cone of \( I \) has a unique irreducible representation

\[
(3.2) \quad \text{RC}(I) = \left( \bigcap_{i=1}^{s+1} H_{e_i}^+ \right) \bigcap \left( \bigcap_{i=1}^{m} H_{(\gamma_i, -d_i)}^+ \right) \bigcap \left( \bigcap_{i=m+1}^{p} H_{(\gamma_i, -d_i)}^+ \right),
\]

where none of the closed halfspaces can be omitted from the intersection, \( d_i = 1 \) for \( i = 1, \ldots, m \), \( t^{e_1}, \ldots, t^{e_s} \) are the minimal generators of \( I^V \), \( \gamma_i \in \mathbb{N}^s \setminus \{0\} \) for \( i > m \), \( d_i \in \mathbb{N} \setminus \{0, 1\} \) for \( i > m \), and the non-zero entries of \( (\gamma_i, -d_i) \) are relatively prime for all \( i \).
The hyperplanes defining the closed halfspaces of Eq. (3.2) are the supporting hyperplanes of the Rees cone of $I$, and the $\gamma_i$’s and $d_i$’s can be computed using Normaliz \[8\]. Thus, for each $1 \leq j \leq p$, we can determine the polyhedron $P_j$.

The following theorem justifies the a priori naming coincidence between the constants appearing in Theorem 3.1 and the description given for the $u_i$ in the introduction before Eq. (1.3).

**Theorem 3.2.** [8] Theorem 3.1 The vertex set of $Q(I)$ is $V(Q(I)) = \{\gamma_1/d_1, \ldots, \gamma_p/d_p\}$.

The last two results say that finding the supporting hyperplanes of $RC(I)$ is equivalent to finding the vertices of $Q(I)$.

For use below we set $u_i = \gamma_i/d_i = \gamma_i$ for $i = 1, \ldots, m$. To study the vertices of $P_j$ we need the following dual versions of Theorems 3.1 and 3.2. The Rees cone of $I^\vee$ is given by

$$(3.3) \quad RC(I^\vee) = \mathbb{R}_+ \{ e_1, \ldots, e_s, (u_1, 1), \ldots, (u_m, 1) \}. $$

**Theorem 3.3.** The Rees cone of $I^\vee$ has a unique irreducible representation

$$(3.4) \quad RC(I^\vee) = \bigcap_{i=1}^{p+1} H_{e_i}^+ \bigcap \bigcap_{i=1}^{q} H_{(\delta_i, -f_i)}^+ \bigcap \bigcap_{i=q+1}^{p+1} H_{(\delta_i, f_i)}^+, $$

where $f_i = 1$ for $i = 1, \ldots, q$, $t^{\delta_1}, \ldots, t^{\delta_q}$ are the minimal generators of $I$, $\delta_i \in \mathbb{N}^s \setminus \{0\}$ for $i > q$, $f_i \in \mathbb{N} \setminus \{0, 1\}$ for $i > q$, and the non-zero entries of $(\delta_i, -f_i)$ are relatively prime for all $i$.

**Theorem 3.4.** The vertex set of $Q(I^\vee)$ is $V(Q(I^\vee)) = \{\delta_1/f_1, \ldots, \delta_p/f_p\}$.

In what follows we set $v_i = \delta_i/f_i$ for $i = 1, \ldots, p_1$ and $u_i = \gamma_i/d_i$ for $i = 1, \ldots, p$. Recall that $(v_1^s, \ldots, v_q^s)$ is the minimal generating set of the edge ideal $I = I(C)$ of $C$. The minimal generating set for $I^\vee$ is $(v_1^u, \ldots, v_m^u)$ and $I^\vee = I(C^\vee)$ is the edge ideal of the blocker $C^\vee$ of $C$.

**Lemma 3.5.** Let $I$ be the edge ideal of a clutter $C$. The following hold.

(a) Given $\gamma_i$ and $\delta_i$, there are $u \in \{u_i\}_{i=1}^m$ and $v \in \{v_i\}_{i=1}^q$ such that $\gamma_i \geq u$ and $\delta_i \geq v$ componentwise.

(b) $\langle \gamma_i, \delta_\ell \rangle \geq d_\ell \geq 1$ and $\langle \gamma_i, \delta_\ell \rangle \geq f_\ell \geq 1$ for all $i, \ell$.

(c) If $p_j$ is the optimal value of the linear program of Eq. (1.5), then $p_j \leq d_j$.

(d) If $\gamma_j = (\gamma_{j,1}, \ldots, \gamma_{j,s})$ and $\gamma_{j,k} = 0$ for some $k$, then $P_j$ is an unbounded polyhedron.

(e) If $\gamma_j = (\gamma_{j,1}, \ldots, \gamma_{j,s})$ and $\gamma_{j,k} > 0$ for $k = 1, \ldots, s$, then $P_j$ is a bounded polyhedron.

**Proof.** (a)-(b) Given $a \in \mathbb{N}^s$, $a = (a_1, \ldots, a_s)$, we set $F_a = \text{supp}(t^a) = \{ t_k \mid a_k > 0 \}$. From Eq. (3.2), $\langle v_n, \gamma_i \rangle \geq d_i$ for $n = 1, \ldots, q$. Then, $F_{v_n} \bigcap F_{\gamma_i} \neq \emptyset$ for $n = 1, \ldots, q$. As $F_{v_1}, \ldots, F_{v_q}$ are the edges of $C$, $F_{\gamma_i}$ is a vertex cover of $C$, and consequently $F_{v_n}$ contains a minimal vertex cover $C$. Similarly, from Eq. (3.4), $\langle v_n, \gamma_i \rangle \geq f_i \geq 1$ for $n = 1, \ldots, m$. Then, $F_{v_n} \bigcap F_{\delta_i} \neq \emptyset$ for $n = 1, \ldots, m$. As $F_{v_1}, \ldots, F_{v_m}$ are the edges of $C^\vee$, $F_{\delta_i}$ is a vertex cover of $C^\vee$, and consequently $F_{\delta_i}$ contains a minimal vertex cover $D$ of $C^\vee$. Note that $D$ is an edge of $C$ because $(C^\vee)^\vee = C$.

Let $u = \sum_{i \in C} e_i$ and $v = \sum_{i \in D} e_i$ be the characteristic vectors of $C$ and $D$, respectively. Then, $\delta_i \geq v$ and $\gamma_i \geq u$. Hence, using Eq. (3.2) and Eq. (3.3), we obtain

$$\langle \gamma_i, \delta_\ell \rangle \geq \langle \gamma_i, v \rangle \geq d_\ell \geq 1 \text{ and } \langle \gamma_i, \delta_\ell \rangle \geq \langle u, \delta_\ell \rangle \geq f_\ell \geq 1. $$

(c) Let $y = (y_1, \ldots, y_{s+3})$ be any point in $P_j$, that is, $y$ is feasible for Eq. (1.5). By part (a), one can pick $u \in \{u_j\}_{j=1}^m$ such that $\gamma_j \geq u$. Then, using the constraints that define $P_j$, we get

$$y_s + 1 \leq \langle (y_1, \ldots, y_s), u \rangle \leq \langle (y_1, \ldots, y_s), \gamma_j \rangle \leq d_j - y_{s+3} \leq d_j. $$
Thus, \( y_{s+1} \leq d_j \), and consequently \( \rho_j \leq d_j \).

(d) For simplicity, we may assume that \( k = 1 \). Setting \( x = (x_1, \ldots, x_{s+3}) = \ell e_1 + e_{s+2} \) for \( \ell \geq 0 \) and using that \( \gamma_{j,1} = 0 \), one obtains

\[
\langle (x_1, \ldots, x_s), u_i \rangle \geq x_{s+1} = 0, \ i = 1, \ldots, m, \ x_{s+1} \geq x_{s+3} \\
x_i \geq 0, \ i = 1, \ldots, s, \ x_{s+3} \geq 0 \\
d_j x_{s+2} - \langle (x_1, \ldots, x_s), \gamma_j \rangle = d_j - x_1 \gamma_{j,1} = d_j \geq x_{s+3} = 0, \ x_{s+2} = 1,
\]

and consequently \( x \in \mathcal{P}_j \) for all \( \ell \geq 0 \). This proves that \( \mathcal{P}_j \) is unbounded.

(e) Let \( y = (y_1, \ldots, y_{s+3}) \) be any point in \( \mathcal{P}_j \). It suffices to show that \( y_i \leq d_j \) for \( i = 1, \ldots, s+3 \).

By part (a), one can pick \( u \in \{u_i\}_{i=1}^m \) such that \( \gamma_j \geq u \). As \( \gamma_j \) is integral, \( \gamma_{j,k} \geq 1 \) for all \( k \).

Then, using the constraints that define \( \mathcal{P}_j \), we get

\[
y_i \leq \langle (y_1, \ldots, y_s), \gamma_j \rangle \leq d_j - y_{s+3} \leq d_j, \ i = 1, \ldots, s \\
y_{s+3} \leq y_{s+1} \leq \langle (y_1, \ldots, y_s), u \rangle \leq \langle (y_1, \ldots, y_s), \gamma_j \rangle \leq d_j - y_{s+3} \leq d_j,
\]

and the proof is complete. \( \square \)

**Theorem 3.6.** Let \( \mathcal{P}_j \) be the polyhedron of feasible points of Eq. (1.3), \( 1 \leq j \leq p \), and let \( y = (y_1, \ldots, y_{s+3}) \) be a point in \( \mathbb{R}^{s+3} \) with \( y_{s+1} > 0 \), then \( y \) is a vertex of \( \mathcal{P}_j \) if and only if \( y \) has one of the following two forms:

(a) \( y = (y_{s+1}(\delta_k/f_k), y_{s+1}, 1, 0) \), where \( y_{s+1} = d_j f_k / \gamma_{j,k} \) and \( 1 \leq k \leq p_1 \), or

(b) \( y = (y_{s+1}(\delta_k/f_k), y_{s+1}, 1, y_{s+1}) \), where \( y_{s+1} = d_j f_k / (\gamma_{j,k} + f_k) \) and \( 1 \leq k \leq p_1 \).

**Proof.** \( \Rightarrow \) Assume that \( y = (y_1, \ldots, y_{s+3}) \) is a vertex of \( \mathcal{P}_j \) with \( y_{s+1} > 0 \). The constraints that define \( \mathcal{P}_j \) can be written as

\[
\langle y, (-u_i, 1, 0, 0) \rangle \leq 0, \ i = 1, \ldots, m, \ \langle y, -e_{s+1} + e_{s+3} \rangle \leq 0, \\
\langle y, -e_i \rangle \leq 0, \ i = 1, \ldots, s, \ \langle y, -e_{s+3} \rangle \leq 0, \\
\langle y, (\gamma_j, 0, 0, 0) - d_j e_{s+2} + e_{s+3} \rangle \leq 0, \ \langle y, e_{s+2} \rangle \leq 1, \ \langle y, -e_{s+2} \rangle \leq -1.
\]

As \( y \) is a vertex of \( \mathcal{P}_j \), by [20] Corollary 1.1.47, there are \( s+3 \) linearly independent constraints that are satisfied with equality. The hyperplane \( H_{(y_1, \ldots, y_{s+1})} \) of \( \mathbb{R}^{s+1} \) contains at most \( s \) linearly independent vectors because \( y_{s+1} > 0 \). Therefore, there are exactly \( s \) constraints of the form

\[
\langle y, (-u_i, 1, 0, 0) \rangle \leq 0, \ i = 1, \ldots, m, \\
\langle y, -e_i \rangle \leq 0, \ i = 1, \ldots, s,
\]

that are satisfied with equality because the remaining constraints that define \( \mathcal{P}_j \) are

\[
\langle y, -e_{s+1} + e_{s+3} \rangle \leq 0, \\
\langle y, -e_{s+3} \rangle \leq 0, \\
\langle y, (\gamma_j, 0, 0, 0) - d_j e_{s+2} + e_{s+3} \rangle \leq 0, \\
\langle y, e_{s+2} \rangle \leq 1, \ \langle y, -e_{s+2} \rangle \leq -1,
\]

and the constraints of Eqs. (3.8) and (3.9) cannot hold simultaneously with equality. Then, recalling that \( y_{s+2} = 1 \) in \( \mathcal{P}_j \), one has

\[
\langle (y_1, \ldots, y_s), \gamma_j \rangle = d_j - y_{s+3}
\]

because the constraint of Eq. (3.10) must hold with equality, and either \( y_{s+1} = y_{s+3} \) if Eq. (3.8) holds with equality or \( y_{s+3} = 0 \) if Eq. (3.9) holds with equality. Let

\[
\mathcal{B} = \{((-w_i, 1))_{i=1}^{r-1} \cup \{(-w_i, 0)\}_{i=r}^s
\]
be the set of vectors in $\mathbb{R}^{s+1}$ that correspond to the $s$ linearly independent constraints of Eqs. (3.6) and (3.7) that are satisfied with equality. Note that $\mathcal{B}$ is linearly independent if and only if $\mathcal{B}' = \{(w_i, 1)\}_{i=1}^{r-1} \cup \{(w_i, 0)\}_{i=r}^{r}$ is linearly independent. Consider the hyperplane $H = H(y_1, \ldots, y_s, -y_{s+1})$.

The set $\mathcal{B}'$ is contained in $H$, and $\mathcal{B}'$ is also contained in the Rees cone $RC(I^\vee)$ of $I^\vee$ because $w_i \in \{u_1, \ldots, u_m\}$ for $1 \leq i \leq r - 1$ and $w_i \in \{e_1, \ldots, e_s\}$ for $r \leq i \leq s$. Then, $F = H \cap RC(I^\vee)$ is a facet of the Rees cone $RC(I^\vee)$ of $I^\vee$. Hence, by Theorem 3.3 and [41, Theorem 3.2.1], we obtain that $F = H(\delta_k, -f_k) \cap RC(I^\vee)$ for some $1 \leq k \leq p_1$, and consequently

$$(\delta_k, -f_k) = \lambda(y_1, \ldots, y_s, -y_{s+1})$$

for some $\lambda \in \mathbb{R}$. Thus, $(y_1, \ldots, y_s)/y_{s+1} = \delta_k/f_k$ for some $1 \leq k \leq p_1$.

Case (I) $y_{s+3} = 0$. Then, dividing Eq. (3.12) by $y_{s+1}$, we get

$$\frac{\langle \delta_k, \gamma_j \rangle}{f_k} = \frac{d_j}{y_{s+1}},$$

and solving for $y_{s+1}$ gives $y_{s+1} = d_j f_k / \langle \gamma_j, \delta_k \rangle$. Thus, $y$ is as in (a).

Case (II) $y_{s+1} = y_{s+3}$. Then, dividing Eq. (3.12) by $y_{s+1}$, we get

$$\frac{\langle \delta_k, \gamma_j \rangle + f_k}{f_k} = \frac{\langle \delta_k, \gamma_j \rangle}{f_k} + 1 = \frac{d_j}{y_{s+1}},$$

and solving for $y_{s+1}$ gives $y_{s+1} = d_j f_k / (\langle \gamma_j, \delta_k \rangle + f_k)$. Thus, $y$ is as in (b).

$\Leftarrow$ Assume that $y$ is as in (a). As $(y_1, \ldots, y_s)/y_{s+1} = \delta_k/f_k$ is a vertex of $Q(I^\vee)$, it follows that $y$ satisfies the constraints of Eqs. (3.6)–(3.7) and that $s$ of them occur with equality and are linearly independent. Let $\Gamma$ be the set of vectors in $\mathbb{R}^{s+3}$ that correspond to these independent constraints. Then, $\langle \alpha, y \rangle = 0$ for all $\alpha \in \Gamma$. The following three constraints are also satisfied with equality

$$\langle y, -e_{s+3} \rangle \leq 0,$$

$$\langle y, (\gamma_j, 0, 0, 0) - d_j e_{s+2} + e_{s+3} \rangle \leq 0,$$

$$\langle y, e_{s+2} \rangle \leq 1.$$

As $y \in \mathcal{P}_j$, to prove that $y$ is a vertex of $\mathcal{P}_j$ we need only show that the set

$$\Gamma \cup \{-e_{s+3}, (\gamma_j, 0, 0, 0) - d_j e_{s+2} + e_{s+3}, e_{s+2}\}$$

is linearly independent or equivalently that $\Gamma \cup \{e_{s+3}, (\gamma_j, 0, 0, 0), e_{s+2}\}$ is linearly independent. This follows noticing that $\Gamma \cup \{e_{s+2}, e_{s+3}\}$ is linearly independent and observing that $(\gamma_j, 0, 0, 0)$ cannot be in $\mathbb{R}(\Gamma \cup \{e_{s+2}, e_{s+3}\})$, the linear span of $\Gamma \cup \{e_{s+2}, e_{s+3}\}$, because $\langle y, (\gamma_j, 0, 0, 0) \rangle = d_j$ and $\langle \alpha, y \rangle = 0$ for all $\alpha \in \Gamma$.

Assume that $y$ is as in (b). As $(y_1, \ldots, y_s)/y_{s+1} = \delta_k/f_k$ is a vertex of $Q(I^\vee)$, it follows that $y$ satisfies the constraints of Eqs. (3.6)–(3.7) and that $s$ of them occur with equality and are linearly independent. Let $\Gamma$ be the set of vectors in $\mathbb{R}^{s+3}$ that correspond to these independent constraints. Then, $\langle \alpha, y \rangle = 0$ for all $\alpha \in \Gamma$. The following constraints are satisfied with equality

$$\langle y, -e_{s+1} + e_{s+3} \rangle \leq 0,$$

$$\langle y, (\gamma_j, 0, 0, 0) - d_j e_{s+2} + e_{s+3} \rangle \leq 0,$$

$$\langle y, e_{s+2} \rangle \leq 1.$$
As \( y \in \mathcal{P}_j \), to prove that \( y \) is a vertex of \( \mathcal{P}_j \) we need only show that the set
\[
\Gamma \cup \{-e_{s+1} + e_{s+3}, (\gamma_j, 0, 0, 0) - d_j e_{s+2} + e_{s+3}, e_{s+2}\}
\]
is linearly independent or equivalently that the set
\[
\Gamma \cup \{-e_{s+1} + e_{s+3}, (\gamma_j, 0, 0, 0) + e_{s+3}, e_{s+2}\}
\]
is linearly independent. Clearly the set \( \Gamma \cup \{-e_{s+1} + e_{s+3}, e_{s+2}\} \) is linearly independent. Hence, it suffices to notice that \((\gamma_j, 0, 0, 0)\) cannot be in \( \mathbb{R}(\Gamma \cup \{-e_{s+1} + e_{s+3}, e_{s+2}\}) \) because \( \langle y, (\gamma_j, 0, 0, 0) \rangle \) is equal to \( d_j - y_{s+1}, \langle \alpha, y \rangle = 0 \) for all \( \alpha \in \Gamma \), and \( d_j - y_{s+1} \neq 0 \) (Lemma 3.5).

**Theorem 3.7.** (Duality formula) If \( I = I(\mathcal{C}) \) is the edge ideal of a clutter \( \mathcal{C} \) and \( \rho_{ic}(I) \) is the ic-resurgence of \( I \), then
\[
1/\rho_{ic}(I) = \min \{ \langle u, v \rangle \mid u \in V(Q(I)), v \in V(Q(I^\forall)) \},
\]
where \( V(Q(I)) \) is the vertex set of \( Q(I) \). In particular, \( \rho_{ic}(I) = \rho_{ic}(I^\forall) \).

**Proof.** The sets of vertices of \( Q(I) \) and \( Q(I^\forall) \) are \( \{\gamma_i/d_i\}_{i=1}^p \) and \( \{\delta_{\ell}/f_{\ell}\}_{\ell=1}^{p_1} \), respectively. This follows from Theorems 3.2 and 3.4. Then, noticing the equalities
\[
1/\min\{\langle \gamma_i/d_i, \delta_{\ell}/f_{\ell}\rangle\}_{i,\ell} = \max\{1/\langle \gamma_i/d_i, \delta_{\ell}/f_{\ell}\rangle\}_{i,\ell} = \max\{(d_i f_{\ell})/\langle \gamma_i, \delta_{\ell}\rangle\}_{i,\ell},
\]
it suffices to prove the following equality
\[
(3.13) \quad \rho_{ic}(I) = \max\{(d_i f_{\ell})/\langle \gamma_i, \delta_{\ell}\rangle\}_{i,\ell}.
\]

Given integers \( 1 \leq i \leq p \) and \( 1 \leq \ell \leq p_1 \), by Theorem 3.6, there is a vertex \( y_{i,\ell} \) of \( \mathcal{P}_i \) such that the \( (s+1) \)-th entry \( \langle y_{i,\ell} \rangle_{s+1} \) of \( y_{i,\ell} \) is equal to \( (d_i f_{\ell})/\langle \gamma_i, \delta_{\ell}\rangle \). As \( y_{i,\ell} \) is in \( \mathcal{P}_i \), we obtain that \( \rho_i \geq (d_i f_{\ell})/\langle \gamma_i, \delta_{\ell}\rangle \), and consequently one has
\[
(3.14) \quad \rho_{ic}(I) \geq \rho_i \geq d_i f_{\ell}/\langle \gamma_i, \delta_{\ell}\rangle \quad \forall \ i, \ell.
\]

By Theorems 1.1 and 3.6, \( \rho_{ic}(I) = \rho_j \) for some \( 1 \leq j \leq p \) and there is \( y = (y_1, \ldots, y_{s+3}) \) a vertex of \( \mathcal{P}_j \) such that \( y_{s+1} = \rho_{ic}(I) \), and either
\[
y_{s+1} = d_j f_{k}/\langle \gamma_j, \delta_k \rangle \quad \text{or} \quad y_{s+1} = d_j f_{k}/(\langle \gamma_j, \delta_k \rangle + f_k)
\]
for some \( k \). The second equality cannot occur because
\[
d_j f_{k}/(\langle \gamma_j, \delta_k \rangle + f_k) < d_j f_{k}/\langle \gamma_j, \delta_k \rangle \leq \rho_{ic}(I).
\]

Thus, one has \( y_{s+1} = d_j f_{k}/\langle \gamma_j, \delta_k \rangle \) and, by Eq. (3.14), we get equality in Eq. (3.13). \( \square \)

**Proposition 3.8.** If \( I \) is a squarefree monomial ideal, then \( \rho_{ic}(I) \geq 1 \) with equality if and only if \( Q(I) \) is integral.

**Proof.** By [13] Corollaries 4.14 and 4.16, \( \rho_{ic}(I) \geq 1 \) with equality if and only if \( I^{(n)} = \overline{T^n} \) for every \( n \geq 1 \). Therefore, by Theorem 2.2, \( \rho_{ic}(I) = 1 \) if and only if \( Q(I) \) is integral. \( \square \)

**Corollary 3.9.** [11] Theorem 1.17] \( Q(I) \) is integral if and only if \( Q(I^\forall) \) is integral.

**Proof.** If \( Q(I) \) is integral, by Proposition 3.5, \( \rho_{ic}(I) = 1 \). Then, by Theorem 3.7, \( \rho_{ic}(I^\forall) = 1 \). Thus, by Proposition 3.5, \( Q(I^\forall) \) is integral. The converse follows by replacing \( I \) with \( I^\forall \), in the first part of the proof, and recalling that \( (I^\forall)^\forall \) is equal to \( I \). \( \square \)

**Corollary 3.10.** [20] Theorems 4.8 and 5.3] Let \( I = I(G) \) be the edge ideal of a graph. The following conditions are equivalent.

(a) \( \rho(I) = 1 \);  (b) \( \rho_{ic}(I) = 1 \);  (c) \( G \) is bipartite;  (d) \( \rho(I^\forall) = 1 \).
Remark 3.12. There is a similar formula for \( \hat{\gamma} \) to max
\[
\{ \rho_{ic}(I) \} = \max_{\gamma \in \mathcal{C}} \{ \rho_{ic}(I^\gamma) \}
\]
and the proof is complete.

Corollary 3.11. If \( I_1 \) and \( I_2 \) are squarefree monomial ideals of \( S \) generated by monomials in disjoint sets of variables, then
\[
\rho_{ic}(I_1 + I_2) = \max\{\rho_{ic}(I_1), \rho_{ic}(I_2)\}.
\]

Proof. We may assume that \( C_1 \) and \( C_2 \) are clutters with vertex sets \( \{t_1, \ldots, t_r\} \) and \( \{t_{r+1}, \ldots, t_s\} \) such that \( I_i \) is the edge ideal of \( C_i \) for \( i = 1, 2 \). If \( C = C_1 \cup C_2 \) and \( I = I_1 + I_2 \), then \( I \) is the edge ideal of \( C \) and \( C \) is a minimal vertex cover of \( C \) if and only if \( C = C_1 \cup C_2 \) with \( C_i \) a minimal vertex cover of \( C_i \), for \( i = 1, 2 \). Hence, \( I^\gamma = I_1^\gamma I_2^\gamma \) and, by \( \mathbb{Q} \) Proposition 3.5, \( \rho_{ic}(I^\gamma) \) is equal to
\[
\max\{\rho_{ic}(I_1^\gamma), \rho_{ic}(I_2^\gamma)\}.
\]
Thus, \( \rho_{ic}(I) = \max\{\rho_{ic}(I_1), \rho_{ic}(I_2)\} \) and the proof is complete.

Remark 3.12. There is a similar formula for \( \hat{\alpha}(I_1 + I_2) \) \( \mathbb{Q} \) Corollary 7.10:
\[
\hat{\alpha}(I_1 + I_2) = \min\{\hat{\alpha}(I_1), \hat{\alpha}(I_2)\}.
\]

Definition 3.13. \( \mathbb{Q} \) p. 541] Let \( a = (a_i) \neq 0 \) be a vector in \( \mathbb{N}^q \) and let \( b \in \mathbb{N} \). If \( a, b \) satisfy
\[
\langle a, v_i \rangle \geq b \quad \text{for} \quad i = 1, \ldots, q,
\]
we say that \( a \) is a \( b \)-cover of \( C \).

The notion of a \( b \)-cover occurs in combinatorial optimization \( \mathbb{Q} \) Chapter 77, p. 1378] and algebraic combinatorics \( \mathbb{Q} \) 15] [24].

Definition 3.14. A \( b \)-cover \( a \) of \( C \) is called reducible if there exists an \( i \)-cover \( c \) and a \( j \)-cover \( d \) of \( C \) such that \( a = c + d \) and \( b = i + j \). If \( a \) is not reducible, we call \( a \) irreducible.

Lemma 3.15. \( \mathbb{Q} \) Lemma 1.8] If \( (\gamma_k, -d_k) \) is any of the vectors of Eq. (3.2), then \( \gamma_k \) is an irreducible \( d_k \)-cover of \( C \).

Definition 3.16. Let \( G \) be a graph. A set \( P \) of pairwise disjoint edges of \( G \) is a perfect matching of \( G \) if \( V(G) = \bigcup_{e \in P} e \). A set of vertices of \( G \) is stable if no two of them are adjacent.

Proposition 3.17. Let \( G \) be a connected non-bipartite graph with a perfect matching and let \( I(G) \) be its edge ideal. Then, \( \hat{\alpha}(I(G)^\gamma) = |V(G)|/2 \).

Proof. Let \( u_i = \gamma_i/d_i \) be any vertex of \( Q(I) \). By Theorem 3.2 \( (\gamma_i, -d_i) \) occurs in Eq. (3.2). Hence, by Lemma 3.15 \( \gamma_i \) is an irreducible \( d_i \)-cover of \( G \). Therefore, using the classification of the irreducible covers of a graph \( \mathbb{Q} \) Theorem 1.7], \( \gamma_i \) and \( d_i \) have one of the following forms
(a) \( d_i = 1 \) and \( \gamma_i = \sum_{t \in C} e_j \) for some \( C \in E(G^\gamma) \), that is, \( 1 \leq i \leq m \),
(b) \( d_i = 2 \) and \( \gamma_i = (1, \ldots, 1) \),
(c) \( d_i = 2 \) and up to permutation of vertices

\[
(3.15) \quad \gamma_i = (0, \ldots, 0, 2, \ldots, 2, 1, \ldots, 1)
\]

for some stable set of vertices \( \mathfrak{A} \neq \emptyset \) of \( G \), where \( N_G(\mathfrak{A}) \) is the neighbor set of \( \mathfrak{A} \).

The incidence matrix of \( I \) has rank \( s = |V(G)| \) because \( G \) is connected and non-bipartite \([36] \), Lemma 2.1]. Hence, \( \beta = (1/2, \ldots, 1/2) \) is a vertex of \( Q(I) \). Indeed, note that \( \beta A \geq 1 \), where \( A \) is the incidence matrix of \( I \). There are \( s \) linearly independent columns of \( A \), say \( v_1, \ldots, v_s \), such that \( \langle \beta, v_i \rangle = 1 \) for \( i = 1, \ldots, s \). Thus, \( \beta \) is a basic feasible solution of the system \( xA \geq 1 \); \( x \geq 0 \) and, by \([40] \) Corollary 1.1.49], \( \beta \) is a vertex of \( Q(I) \). Then, by \([4] \) Theorem 3.2], one has

\[
\hat{\alpha}(I') = \min\{|u_k| : u_k \in V(Q(I))\} \leq |\beta| = s/2.
\]

Therefore, it suffices to show that \( |u_i| \geq s/2 \). If \( u_i = \gamma_i/d_i \) is as in (a), then \( |u_i| \geq s/2 \) because \( G \) has a perfect matching and any minimal vertex cover \( C \) of \( G \) has at least \( s/2 \) vertices of \( G \). If \( u_i \) is as in (c), by Eq. (3.15), we obtain

\[
|u_i| = |\gamma_i/2| = |N_G(\mathfrak{A})| + (1/2)(s - |\mathfrak{A}| - |N_G(\mathfrak{A})|) = (1/2)|N_G(\mathfrak{A})| + (s/2) - (1/2)|\mathfrak{A}|.
\]

As the graph \( G \) has a perfect matching and the set \( \mathfrak{A} \) contains no edges of \( G \) since \( \mathfrak{A} \) is stable, one has \( |N_G(\mathfrak{A})| \geq |\mathfrak{A}| \). Thus, \( |u_i| \geq s/2 \).

**Lemma 3.18.** Let \( d, k, \ell, s \) be integers such that \( 1 \leq d \leq \ell \leq s \) and \( s - d + 1 \leq k \leq s \). If \( A \) and \( B \) are subsets of \( \{1, \ldots, s\} \), \( k = |A| \), \( \ell = |B| \), then

\[
(3.16) \quad \frac{|A \cap B|}{(k - s + d)(\ell - d + 1)} \geq \frac{s}{d(s - d + 1)}
\]

with equality if \( A = B = \{1, \ldots, s\} \).

*Proof.* We can write \( \ell = d + i \) and \( |A \cup B| = s - \epsilon \), where \( 0 \leq \epsilon \leq s - d \) and \( \epsilon \geq 0 \). As \( |A \cap B| \) is equal to \( k + \ell - |A \cup B| \), we obtain that Eq. (3.16) is equivalent to

\[
(k + d + i - s + \epsilon)d(s - d + 1) \geq s(k - s + d)(i + 1).
\]

By factoring out \( i \), this inequality is equivalent to

\[
(3.17) \quad \epsilon d(s - d + 1) + (k + d - s)(d - 1)(s - d) \geq i[s(k - s + d) - d(s - d + 1)].
\]

We set \( f = s(k - s + d) - d(s - d + 1). \) If \( f \leq 0 \), the inequality of Eq. (3.17) holds because the left-hand side is non-negative and the right-hand side is if. Thus, we may assume that \( f \geq 1 \). Since \( s - d \geq i \), one has \( (s - d)f \geq if \), and we need only prove the inequality

\[
(3.18) \quad \epsilon d(s - d + 1) + (k + d - s)(d - 1)(s - d) \geq (s - d)[s(k - s + d) - d(s - d + 1)].
\]

This inequality is equivalent to

\[
d(s - d + 1)(\epsilon + (s - d)) \geq (s - d)(s - (d - 1))(k - s + d).
\]

By cancelling out \( s - d + 1 \), the proof reduces to showing that

\[
d(\epsilon + (s - d)) \geq (s - d)(k - s + d).
\]

As \( d(s - d) \) appears as a summand on both sides of this inequality it suffices to notice that \( d \epsilon \geq (s - d)(k - s) \) because \( k \leq s \). \(\Box\)
Theorem 3.19. \cite{25} If $I_{d,s}$ is the $d$-th squarefree Veronese ideal of $S$ generated by all squarefree monomials of $S$ of degree $d$ in $s$ variables, then

\begin{align}
(p(I_{d,s}) = \rho_{ic}(I_{d,s}) &= \rho_{ic}((I_{d,s})^\vee) = \frac{d(s - d + 1)}{s} = \frac{d}{\alpha(I_{d,s})} = \frac{s - d + 1}{\alpha((I_{d,s})^\vee)}.\end{align}

Proof. We set $I = I_{d,s}$ and $J = I_{s-d+1,s}$. The primary decomposition of $I$ is

\begin{align}
I = \bigcap_{1 \leq j_1 < \cdots < j_{s-d+1} \leq s} (t_{j_1}, \ldots, t_{j_{s-d+1}}).
\end{align}

Then, $(I_{d,s})^\vee = I_{s-d+1,s}$ and $J = (I_{d,s})^\vee$. If $d = s$, then $I = (t_1 \cdots t_s)$, $I^\vee = (t_1, \ldots, t_s)$, the vertex set of $Q(I)$ is $\{e_1, \ldots, e_s\}$, and the vertex set of $Q(I^\vee)$ is $\{e_1 + \cdots + e_s\}$. Then, by \cite[Theorem 3.2]{1} and \cite[Theorem 3.3]{2} we get $\alpha(I^\vee) = 1$, $\alpha(I) = s$, and $\rho_{ic}(I) = \rho_{ic}(I^\vee) = 1$. Then, Eq. (3.19) holds. For similar reasons Eq. (3.19) holds if $d = 1$.

Thus, we may assume that $2 \leq d < s$. We claim that the vertex sets of $Q(I)$ and $Q(I^\vee)$ are

\begin{align}
V(Q(I)) &= \left\{ \frac{e_{i_1} + \cdots + e_{i_k}}{k - s + d} \Big| s - d + 1 \leq k \leq s, 1 \leq i_1 < \cdots < i_k \leq s \right\},
\end{align}

\begin{align}
V(Q(I^\vee)) &= \left\{ \frac{e_{j_1} + \cdots + e_{j_k}}{\ell - d + 1} \Big| d \leq \ell \leq s, 1 \leq j_1 < \cdots < j_\ell \leq s \right\}.
\end{align}

To prove the first equality, we use the Rees cone $RC(I)$ of $I$ defined in Eq. (3.1). The second equality will follow by using the Rees cone $RC(J)$ of $J$ or by replacing $I_{d,s}$ with $I_{s-d+1,s}$. Next we prove that the irreducible representation of $RC(I)$ is given by

\begin{align}
RC(I) = \left( \bigcap_{i=1}^{s+1} H^+_{e_i} \right) \bigcap \left( \bigcup_{1 \leq i_1 < \cdots < i_k \leq s} \bigcup_{s-d+1 \leq k \leq s} H^+_{(e_{i_1} + \cdots + e_{i_k}) - (k-s+d)} \right).
\end{align}

First we show that the non-trivial closed halfspaces on the right-hand side of this equality occur in the irreducible representation of $RC(I)$. Consider the hyperplane

\begin{align}
H = H_{(e_{i_1} + \cdots + e_{i_k}) - (k-s+d)},
\end{align}

where $1 \leq i_1 < \cdots < i_k \leq s$ and $s - d + 1 \leq k \leq s$. To show that $H^+$ occurs in the irreducible representation of $RC(I)$ we need only show that the set $F = H \cap RC(I)$ is a facet of $RC(I)$ \cite[Theorem 3.1]{11}. Recall that $RC(I)$ is the cone $\mathbb{R}_+ \mathcal{A}'$ generated by the set

\begin{align}
\mathcal{A}' = \{e_1, \ldots, e_s\} \cup \{e_{j_1} + \cdots + e_{j_d} + e_{s+1} \mid 1 \leq j_1 < \cdots < j_d \leq s\}.
\end{align}

That $H$ is a supporting hyperplane of the Rees cone, that is, $RC(I) \subset H^+$, follows using that for any sequence of integers $1 \leq j_1 < \cdots < j_d \leq s$ one has

\begin{align}
\langle e_{i_1} + \cdots + e_{i_k}, e_{j_1} + \cdots + e_{j_d} \rangle = |\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_d\}| = k + d - |\{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_d\}| \geq k + d - s.
\end{align}

Hence, to prove that $F$ is a facet of $RC(I)$, it suffices to prove that $F$ contains $s$ linearly independent vectors in $\mathcal{A}'$. Recall that $d - s + k < k$ because $d < s$. Take any subset $A_1$ of $\{i_1, \ldots, i_k\}$ with $d - s + k$ elements and any subset $A_2$ of $\{1, \ldots, s\} \setminus \{i_1, \ldots, i_k\}$ with $s - k$ elements. Then

\begin{align}
\sum_{i \in A_1 \cup A_2} e_i = (A_1 \cup A_2) \cap \{i_1, \ldots, i_k\} = s - d + k, \text{ and}
\end{align}

\begin{align}
e_i \in H \text{ for } i \in \{1, \ldots, s\} \setminus \{i_1, \ldots, i_k\}.
\end{align}
Let $I_{d-s+k,k}$ be the $(d - s + k)$-th squarefree Veronese ideal in the $k$ variables $\{t_{i_1}, \ldots, t_{i_k}\}$. The incidence matrix of $I_{d-s+k,k}$ has rank $k$ because $d - s + k < k$ \[10\] Remarks 12.4.1, 12.4.11. Hence, using Eqs. (3.24)-(3.25), it follows that $F$ contains $s$ linearly independent vectors in $A'$, and $F$ is a facet of $RC(I)$.

Conversely, let $H^+_{(\gamma_p,-d_p)}$ be any of the non-trivial closed halfspaces that occur in the irreducible representation of $RC(I)$ in Eq. (3.2) of Theorem 3.1. Then, by \[39\] Theorem 3.5,
\[
(\gamma_p, -d_p) = (e_{i_1} + \cdots + e_{i_k}, -f_p)
\]
for some $1 \leq i_1 < \cdots < i_k \leq s$. Let $C$ be the clutter with edge ideal $I$ and let $C^\vee$ be its blocker. Consider the induced subclutter $D = C^\vee[\{t_{i_1}, \ldots, t_{i_k}\}]$ of $C^\vee$ consisting of all minimal vertex covers \{t_{j_1}, \ldots, t_{j_\ell}\} of $C$ contained in \{t_{i_1}, \ldots, t_{i_k}\}. Then, $I(D) = I_{s-d+1,k}$. Using Lemma 3.15 and \[29\] Theorem 2.6, it follows that $\gamma_p$ is an irreducible $d_p$-cover of $C$ and $\alpha_0(D)$ is equal to $d_p$, where $\alpha_0(D)$ is the covering number of $D$, i.e., the size of the smallest minimal vertex cover of $D$. In particular $D \neq \emptyset$ because $d_p \geq 1$, that is, $k \geq s - d + 1$. Using the primary decomposition of $I_{s-d+1,k}$ (cf. Eq. (8.20)), we get
\[
\alpha_0(D) = k - (s - d + 1) + 1 = k - s + d.
\]
Thus, $d_p = k - s + d$ and $H^+_{(\gamma_p,-d_p)}$ occurs in the right-hand side of Eq. (5.23).

Using Eq. (5.23) and Theorem 8.2 we obtain that Eq. (3.21) holds. Let $u$ and $v$ be any vertices of $Q(I)$ and $Q(I')$, respectively. Then, by Eqs. (3.21)-(3.22) and Lemma 3.18, we get
\[
\langle u, v \rangle = \frac{e_{i_1} + \cdots + e_{i_k}}{k - s + d}, \frac{e_{j_1} + \cdots + e_{j_\ell}}{\ell - d + 1} = \frac{|A \cap B|}{(k - s + d)(\ell - d + 1)} \geq \frac{s}{d(s - d + 1)},
\]
where $A = \{i_1, \ldots, i_k\}$ and $B = \{j_1, \ldots, j_\ell\}$, with equality if $A = B = \{1, \ldots, s\}$. Hence
\[
\min \{\langle u, v \rangle : u \in V(Q(I)), v \in V(Q(I'))\} = s/d(s - d + 1)
\]
and by Theorem 8.7 we get $\rho_{ic}(I) = \rho_{ic}(I') = d(s - d + 1)/s$. Furthermore, by Eqs. (3.21)-(3.22) and \[13\] Theorem 3.2, one has $\tilde{\alpha}(I) = \min\{|v| : v \in V(Q(I'))\} = \frac{s}{s - d + 1}$ and $\tilde{\alpha}(I') = \min\{|u| : u \in V(Q(I))\} = \frac{s}{d}$, where equality is attained at $(1/(s - d + 1))(1, \ldots, 1)$ and $(1/d)(1, \ldots, 1)$, respectively. As $I$ is normal \[37\] Proposition 2.9, one has $\rho(I) = \rho_{ic}(I)$. \[\square\]

Remark 3.20. If $I = I_{d,s}$ and $2 \leq d < s$, then the number of vertices of $Q(I)$ and $Q(I')$ are
\[
|V(Q(I))| = \sum_{k=s-d+1}^{s} \binom{s}{k} \quad \text{and} \quad |V(Q(I'))| = \sum_{\ell=d}^{s} \binom{s}{\ell}.
\]
This follows from Eqs. (3.21)-(3.22).

4. EXAMPLES

Example 4.1. Let $S = \mathbb{Q}[t_1, \ldots, t_7]$ be a polynomial ring and let
\[
I = I(G) = (t_1t_3, t_1t_4, t_2t_4, t_1t_5, t_2t_5, t_3t_5, t_1t_6, t_2t_6, t_3t_6, t_4t_6, t_2t_7, t_3t_7, t_4t_7, t_5t_7)
\]
be the edge ideal of the graph $G$ of Figure 1. This graph is the complement of a cycle of length 7 and is called an odd antihole in the theory of perfect graphs.
Figure 1. Graph $G$ is the complement of a cycle of length 7.

From the minimal generating set of $I$, we get that the incidence matrix $A$ of $I$ is given by

$$A = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.$$ 

Using Procedure A.1 Macaulay2 [20] and Theorem 3.7 we obtain the following information. The vertex set $V(\mathcal{Q}(I))$ of $\mathcal{Q}(I)$ is given by

$$V(\mathcal{Q}(I)) = \{(0, 0, 1, 1, 1, 1, 1, 1, 0), (1, 0, 0, 1, 1, 1, 1, 1, 1), \(1, 1, 0, 0, 1, 1, 1, 1, 1), (1, 1, 0, 0, 1, 1, 1, 1, 1), (1, 1, 1, 1, 0, 0, 1, 1), (1, 1, 1, 1, 0, 0, 1, 1), (1, 1, 1, 1, 0, 0, 1, 1), (1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2)\},$$

and $\tilde{\alpha}(I^\lor) = 7/2$. The incidence matrix $B$ of $I^\lor$ is

$$B = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.$$
the vertex set \( V(Q(I')) \) of \( Q(I') \) is given by

\[
V(Q(I')) = \{(0, 0, 0, 0, 1, 0, 1), (0, 0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0, 0),
(0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 1, 0, 0),
(0, 1, 0, 0, 0, 1, 0, 0), (0, 1, 0, 0, 1, 0, 0, 0), (0, 1, 0, 0, 1, 0, 0, 0),
(0, 1, 2, 0, 0, 1/2, 0, 1/2), (0, 1, 0, 0, 0, 0, 1),
(1, 0, 0, 0, 1, 0, 0, 0), (1, 0, 0, 1, 0, 0, 0, 0), (1, 0, 1, 0, 0, 0, 0, 0),
(1/2, 0, 0, 1/2, 0, 1/2, 0), (1/2, 0, 1/2, 0, 0, 1/2, 0),
(2, 0, 1, 2, 0, 1/2, 0, 0, 1/2), (1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5).\]

and \( \hat{\alpha}(I) = 7/5 \). The ic-resurgences of \( I \) and \( I' \) are equal to \( 10/7 \).

Example 4.2. Let \( S = \mathbb{Q}[t_1, \ldots, t_d] \) be a polynomial ring and let \( I = (t_1t_2t_3, t_1t_4, t_2t_4, t_3t_4) \) be the edge ideal of the clutter \( C \) (cf. [13 Example 2.26]). Using Procedure \( \text{A.1} \) we obtain the following information. The vertex set of \( Q(I) \) is

\[
V(Q(I)) = \{(0, 0, 1, 1), (0, 1, 0, 1), (1, 0, 0, 1), (1, 1, 1, 0), (1/3, 1/3, 1/3, 2/3)\},
\]

\( \hat{\alpha}(I) = 5/3, I = I', \) and \( \rho_{ic}(I) = 9/7. \)

Example 4.3. Let \( S = \mathbb{Q}[t_1, \ldots, t_7] \) be a polynomial ring and let

\[
I = (t_1t_3t_4, t_1t_3t_5, t_1t_3t_6, t_2t_4t_6, t_5t_6, t_1t_3t_7, t_2t_3t_7, t_3t_5t_7, t_2t_6t_7)
\]

be the edge ideal of the clutter \( C \). Using Procedure \( \text{A.1} \) and Theorem 3.7 we obtain

\[
I' = (t_1t_2t_5, t_2t_3t_4t_5, t_3t_6, t_4t_5t_7, t_1t_6t_7),
\]

\( \hat{\alpha}(I) = \hat{\alpha}(I') = 2, \) and \( \rho_{ic}(I) = \rho_{ic}(I') = 4/3. \)

Example 4.4. Let \( S = \mathbb{Q}[t_1, \ldots, t_7] \) be a polynomial ring and let

\[
I = (t_1t_2, t_2t_3, t_3t_4, t_1t_5, t_1t_6, t_2t_6, t_3t_6, t_4t_6, t_5t_6, t_1t_7, t_2t_7, t_3t_7, t_4t_7, t_5t_7, t_6t_7)
\]

be the edge ideal of the clutter \( C \). Using Procedure \( \text{A.1} \) and Theorem 3.7 we obtain

\[
I' = (t_1t_2t_3t_4t_5t_6, t_1t_2t_3t_4t_5t_7, t_1t_2t_4t_6t_7, t_1t_3t_4t_6t_7, t_1t_3t_5t_6t_7, t_1t_3t_5t_6t_7, t_2t_3t_5t_6t_7, t_2t_4t_5t_6t_7),
\]

\( \alpha = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2) \) is a vertex of \( Q(I) \), \( \hat{\alpha}(I') = |\alpha| = 7/2, \)

\[
\beta = (1/7, 1/7, 1/7, 1/7, 1/7, 1/7, 2/7, 2/7)
\]

is a vertex of \( Q(I') \), \( \hat{\alpha}(I) = |\beta| = 9/7, \) and \( \rho_{ic}(I) = \rho_{ic}(I') = 14/9. \)

Example 4.5. Let \( S = \mathbb{Q}[t_1, \ldots, t_6] \) be a polynomial ring and let \( I = I_{d,s} \) be the \( d \)-th squarefree Veronese ideal of \( S \) with \( d = 3 \) and \( s = 6 \). The ideal \( I' \) is \( I_{d-4+1, s} = I_{4,6} \), that is, \( I' \) is generated in degree 4, \( \alpha = (1/4)(1, 1, 1, 1, 1, 1) \) is a vertex of \( Q(I') \), \( \hat{\alpha}(I') = |\alpha| = 3/2, \)

\[
\beta = (1/3)(1, 1, 1, 1, 1, 1)
\]

is a vertex of \( Q(I') \), \( \hat{\alpha}(I') = |\beta| = 6/3 = 2, \) and \( \rho_{ic}(I) = \rho_{ic}(I') = 2. \) The number of vertices of \( Q(I) \) and \( Q(I') \) are 22 and 42, respectively.
Appendix A. Procedures

In this appendix we give procedures for Macaulay2 [20] to compute the vertices of covering polyhedra and the ic-resurgence of any squarefree monomial ideal.

Procedure A.1. Let $I$ be a squarefree monomial ideal. We implement a procedure—that uses the interface of Macaulay2 [20] to Normaliz [8]—to compute the vertices of the covering polyhedra $Q(I)$ and $Q(I^\vee)$, using the Rees cones of $I$ and $I^\vee$ defined in Eqs. (3.1) and (3.3). Then, using Theorem 3.7, we compute the ic-resurgence $\rho_{ic}(I)$ of $I$. For convenience, in this procedure we also include the function “rhoichypes” that was given in [21, Algorithm A.4]. This procedure corresponds to Example 4.1. To compute other examples, in the next procedure simply change the polynomial rings $R$ and $S$, and the generators of $I$.

```
restart
loadPackage("Normaliz", Reload=>true)
loadPackage("Polyhedra", Reload => true)
load "SymbolicPowers.m2"
R =QQ[x1,x2,x3,x4,x5,x6,x7];
--C7-antihole
I=monomialIdeal(x1*x3,x1*x4,x1*x5,x1*x6,x2*x4,x2*x5,
x2*x6,x2*x7,x3*x5,x3*x6,x3*x7,x4*x6,x4*x7,x5*x7)
waldschmidt(I)
--Alexander dual of I
J=dual(I)
waldschmidt(J)
--transpose incidence matrix of I
A=matrix flatten apply(flatten entries gens I, exponents)
--transpose incidence matrix of J
AJ=matrix flatten apply(flatten entries gens J, exponents)
--generators of the Rees cone of I
M = id_(ZZ^(numcols(A)+1))^{0..numcols(A)-1}||
(A|transpose matrix {for i to numrows A-1 list 1})
--generators of the Rees cone of J
MJ=id_(ZZ^(numcols(AJ)+1))^{0..numcols(AJ)-1}||
(AJ|transpose matrix{for i to numrows AJ-1 list 1})
--rows of M
l= entries M
--rows of MJ
lJ= entries MJ
S=QQ[x1,x2,x3,x4,x5,x6,x7,x8]
L=for i in l list S_i
LJ=for i in lJ list S_i
nmzFilename="rproj1"
--Rees algebra of I
intclToricRing L
--supporting hyperplanes of the Rees cone of I
hypes=readNmzData("sup")
```
A DUALITY THEOREM FOR THE IC-RESURGENCE

nmzFilename="rproj1"
--Rees algebra of J
intclToricRing LJ
--supporting hyperplanes of the Rees cone of J
hypesJ=readNmzData("sup")
--nontrivial supporting hyperplanes of RC(I)
choices = select (entries hypes, l->
   not isSubset({last l}, {0,1}))
--nontrivial supporting hyperplanes of RC(J)
choicesJ = select (entries hypesJ, lJ->
   not isSubset({last lJ}, {0,1}))
A1=set choices, A2=set choicesJ
V1=apply(toList A1,toList), V2=apply(toList A2,toList)
H1=apply(V1,x->{x/-last x}), H2=apply(V2,x->{x/-last x})
--vertices of Q(I)
F1=set flatten apply(H1,
x->entries submatrix'(matrix x, ,{#x#0-1}))
--Vertices of Q(J)
F2=set flatten apply(H2,
x->entries submatrix'(matrix x, ,{#x#0-1}))
--Cartesian product Q(I) x Q(J)
E=apply(toList (F1**F2),toList)
f=(x)->matrix{x#0}*transpose(matrix{x#1})
--This is the ic-resurgence of I
rhoic=1/min toList set flatten flatten apply(apply(E,f), entries)
--Now we compute the ic-resurgence of I using
--the function ‘‘rhoichypes’’
rhoichypes = hypes -> (choices = select (entries hypes, l-> not
   isSubset({last l}, {0,1})); possibilities = for i to #choices-1 list
   (l = choices_i;
    l' = apply( drop(l,-1)|{0, last(l), 1}, a-> a_ZZ); b = {0};
    s = select (entries hypes, l-> isSubset({last l},
       set {0, -1})); s' = apply(s, l -> -l |{0,0});
    s' = apply(s', a-> apply(a, c-> c_ZZ));
    b = b | for i to #s'-1 list 0;
    v = for i to #1-2 list 0;
    t = { v | {-1, 0, 1}, v | {0, 0, -1} };
    t' = { v | {0, 1, 0} };
    t = apply(t, a-> apply(a, c-> c_ZZ));
    b = b | { 0, 0 }; b = apply(b, c-> c_ZZ);
    A = matrix{{l'} | s' | t};

\begin{verbatim}
b = transpose matrix{b};
C = matrix t';
d = matrix{{1}};
P = polyhedronFromHData( A, b, C, d);
vert = vertices P;
max flatten entries vert^{numrows vert -3});
return max possibilities

--This is the ic-resurgence of I
time rhoichypes hypes
\end{verbatim}

Acknowledgments

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