THREE RESULTS FOR $\tau$-RIGID MODULES

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Abstract. $\tau$-rigid modules are essential in the $\tau$-tilting theory introduced by Adachi, Iyama and Reiten. In this paper, we give equivalent conditions for Iwanaga-Gorenstein algebras with self-injective dimension at most one in terms of $\tau$-rigid modules. We show that every indecomposable module over iterated tilted algebras of Dynkin type is $\tau$-rigid. Finally, we give a $\tau$-tilting theorem on homological dimension which is an analog to that of classical tilting modules.

1. Introduction

In 2014, T. Adachi, O. Iyama and I. Reiten [AIR] introduced $\tau$-tilting theory to generalize the classical tilting theory. $\tau$-tilting theory is closely related to silting theory [AI, BZ] and cluster tilting theory [KR, IY, BMRRT] which are popular in the recent years. Therefore, $\tau$-tilting theory has attracted widespread attention. For the latest general results on $\tau$-tilting theory, we refer to [DIJ, DIRRT, EJR, IJY, IZ1, IZ2, J, K, W] and references there.

Note that $\tau$-rigid modules are important objects and tools in the $\tau$-tilting theory. It is interesting to study the properties of $\tau$-rigid modules and find the indecomposable $\tau$-rigid modules for a given algebra. For the recent development of this topic, we refer to [A1, A2, DIP, HZ, Mi, Z1, Z2, Zi1, Zi2] and so on. In this paper, we also focus on the properties of $\tau$-rigid modules.

For an algebra $A$, denote by $\text{mod}A$ the category of finitely generated right $A$-modules. Recall that an algebra $A$ is Iwanaga-Gorenstein, that is, $\text{id}_A < \infty$ and $\text{id}_{A^{op}} < \infty$. In this case, $\text{id}_A = \text{id}_{A^{op}}$. Our first main result gives some new equivalent conditions for an Iwanaga-Gorenstein algebra $A$ with $\text{id}_A \leq 1$ in terms of $\tau$-rigid modules. We remark that this result was inspired by Osamu Iyama and Yingying Zhang.

Theorem 1.1. (Theorems 2.6, 2.7) For an algebra $A$, the following are equivalent
(1) $A$ is Iwanaga-Gorenstein with $\text{id}_A \leq 1$.
(2) Every classical cotilting module in $\text{mod}A$ is a classical tilting module.
(3) $D_A$ is a $\tau$-rigid module in $\text{mod}A$.
(4) $A$ is a $\tau^{-1}$-rigid module in $\text{mod}A$.

We are also interested in algebras $A$ satisfying every indecomposable module in $\text{mod}A$ is $\tau$-rigid. Easy examples of such algebras are hereditary algebras of Dynkin type. We aim to find more examples in this paper. Recall that Assem and Happel [AsH] introduced the following notation of iterated tilted algebras of Dynkin type as a generalization of tilted algebras of Dynkin type [HR]. Let $Q$ be a finite, connected, and acyclic quiver. An algebra $A_m$ ($m \geq 1$) is called an iterated tilted algebra of type $Q$ if (1) $A_0 = KQ$, (2) $T_i$ is a splitting classical tilting module in $\text{mod}A_i$ and (3) $A_{i+1} = \text{End}_{A_i}T_i$ are satisfied, where $0 \leq i \leq m - 1$. Our second main result is the following.

Theorem 1.2. (Theorem 3.7) Let $B$ be an iterated tilted algebra of Dynkin type. Then every indecomposable module in $\text{mod}B$ is $\tau$-rigid.

For a classical tilting module $T$ in $\text{mod}A$ with $B = \text{End}_AT$, by using the tilting theorem of Brenner and Butler [BB], one gets that the homological dimension of $N \in \text{Fac}T$ gives an upper bound of the homological dimension of $\text{Hom}_A(T, N)$, where $\text{Fac}T$ (resp. $\text{Sub}T$) is the subcategory.
of mod\(A\) consisting of modules \(N\) generated (resp. cogenerated) by \(T\). It is natural to ask: Is there a similar result for \(\tau\)-tilting modules? We give a positive answer to this question and get our third main result. We should remark that Buan and Zhou have studied the global dimension of 2-term silting complexes in \([BZ]\).

**Theorem 1.3.** (Theorems 4.4, 4.5) Let \(A\) be an algebra, \(T\) be a \(\tau\)-tilting module in \(\text{mod}A\), \(B = \text{End}_A T \) and \(C = \text{End}_A T^{\text{op}}\).

1. For any \(M \in \text{Fac} T\) with \(\text{pd}_A M \leq 1\), \(\text{pd}_B \text{Hom}_A(T, M) \leq \text{pd}_A M\) holds.
2. For any \(M \in \text{Fac} T\) with \(\text{Ext}^i_A(T, M \oplus T) = 0\) for any \(i \geq 1\), \(\text{pd}_B \text{Hom}_A(T, M) \leq \text{pd}_A M\) holds.
3. For any \(N \in \text{Sub}\tau T\) with \(\text{id}_A N \leq 1\), \(\text{id}_C \text{Hom}_A(N, \tau T) \leq \text{id}_A N\) holds.
4. For any \(N \in \text{Sub}\tau T\) with \(\text{Ext}^i_A(N \oplus \tau T, \tau T) = 0\) for any \(i \geq 1\), \(\text{id}_C \text{Hom}_A(N, \tau T) \leq \text{id}_A N\) holds.

The paper is organized as follows:

In Section 2, we study \(\tau\)-rigid modules over Iwanaga-Gorenstein algebras and show Theorem 1.1.

In Section 3, we study the indecomposable \(\tau\)-rigid modules over iterated tilted algebras of Dynkin type and show Theorem 1.2. In Section 4, we give the \(\tau\)-rigid (resp. \(\tau^{-1}\)-rigid) version of Watanabe’s Lemma and then give an upper bound for some special modules over the endomorphism ring of \(\tau\)-tilting (resp. \(\tau^{-1}\)-tilting) modules.

Throughout this paper, all algebras are finite dimensional algebras over an algebraically closed field \(K\) and \(D = \text{Hom}_K(\_, K)\) is the standard duality.

2. Gorenstein algebras and \(\tau\)-rigid modules

In this section, we aim to study Iwanaga-Gorenstein algebras in terms of \(\tau\)-rigid modules.

For an algebra \(A\), denote by \(\text{gl.dim}\ A\) the global dimension of \(A\). For a right \(A\)-module \(M\), denote by \(\text{pd}_A M\) (resp. \(\text{id}_A M\)) the projective dimension (resp. injective dimension) of \(M\), denote by \(\text{add}_A M\) the subcategory of direct summands of finite direct sums of \(M\) and denote by \(|M|\) the number of pairwise nonisomorphic indecomposable summands of \(M\). Firstly, we recall the definition of tilting (resp. cotilting) modules, see \([AIR]\) for details.

**Definition 2.1.** A module \(T \in \text{mod}A\) is called a tilting module, if it satisfies

1. \(\text{pd}_A T \leq n\).
2. \(\text{Ext}^i_A(T, T) = 0\) for all \(i \geq 1\).
3. There exists an exact sequence \(0 \to A \to T_0 \to T_1 \to \cdots \to T_n \to 0\), for all \(T_i \in \text{add} T\), \(0 \leq i \leq n\).

In particular, we call \(T\) in Definition 2.1 a classical tilting module whenever \(n = 1\). In this case, Definition 2.1(3) is equivalent to \(|T| = |A|\). Dually, one can define cotilting modules and classical cotilting modules.

We also need the following definitions in \([AIR]\).

**Definition 2.2.** (1) We call \(T \in \text{mod}A\) \(\tau\)-rigid if \(\text{Hom}_A(T, \tau T) = 0\), where \(\tau\) is the Auslander-Reiten translation. Moreover, \(T\) is called \(\tau\)-tilting if \(T\) is \(\tau\)-rigid and \(|T| = |A|\).

(2) We call \(T \in \text{mod}A\) \(\tau^{-1}\)-rigid if \(\text{Hom}_A(\tau^{-1} T, T) = 0\). Moreover, \(T\) is called \(\tau^{-1}\)-tilting if \(T\) is \(\tau^{-1}\)-rigid and \(|T| = |A|\).

Clearly, \(T\) is \(\tau^{-1}\)-rigid (resp. \(\tau^{-1}\)-tilting) module in \(\text{mod}A\) if and only if \(\mathbb{D}T\) is \(\tau\)-rigid (resp. \(\tau\)-tilting) module in \(\text{mod}A^{\text{op}}\).

Recall that \(T \in \text{mod}A\) is called faithful if the right annihilator of \(T\) is zero. Now we can state the following proposition in \([AIR]\).

**Proposition 2.3.** (1) Any faithful \(\tau\)-rigid module \(T\) in \(\text{mod}A\) is a partial tilting \(A\)-module, that is, \(\text{Ext}^i_A(T, T) = 0\) and \(\text{pd}_A T \leq 1\).

(2) Any faithful \(\tau\)-tilting module in \(\text{mod}A\) is a classical tilting \(A\)-module.
Now we can state the properties of $\tau$-rigid cotilting modules, which is essential in the proof of the main result.

**Proposition 2.4.** (1) If a cotilting (resp. tilting) module $T$ in $\text{mod} A$ is $\tau$-rigid, then $T$ is a classical tilting $A$-module.

(2) If a tilting (resp. cotilting) module $T$ in $\text{mod} A$ is $\tau^{-1}$-rigid, then $T$ is a classical cotilting $A$-module.

**Proof.** We only prove (1), since the proof of (2) is similar. Because $T$ is cotilting, $T$ is faithful by [AsSS, Chapter VI, Lemma 2.2]. By Proposition 2.3 any faithful $\tau$-rigid module in $\text{mod} A$ is a partial tilting $A$-module. Note that $|T| = |A|$, we are done.

For any $X \in \text{mod} A$, denote by $\text{Fac} X = \{M|X^n \to M \text{ for some integer } n\}$. The following proposition [AS, Proposition 5.8] are useful.

**Proposition 2.5.** For $X$ and $Y$ in $\text{mod} A$, we have the following

(1) $\text{Hom}_A(X, \tau Y) = 0$ if and only if $\text{Ext}^1_A(Y, \text{Fac} X) = 0$.

(2) $X$ is $\tau$-rigid if and only if $\text{Ext}^1_A(X, \text{Fac} X) = 0$.

For an algebra $A$, denote by $\mathcal{P}(A)$ the subcategory of finitely generated projective right $A$-modules and denote by $\mathcal{I}(A)$ the subcategory of finitely generated injective right $A$-modules. Now we recall the following result due to Happel and Unger [HU, Lemma 1.3]. We provide a new proof for this result.

**Theorem 2.6.** For an algebra $A$, the following are equivalent.

(1) $A$ is Iwanaga-Gorenstein.

(2) Every cotilting module in $\text{mod} A$ is tilting.

(3) Every tilting module in $\text{mod} A$ is cotilting.

(4) There exists a tilting-cotilting module in $\text{mod} A$.

**Proof.** (1) $\Rightarrow$ (2) Assume that $A$ is Iwanaga-Gorenstein and $T$ is a cotilting module in $\text{mod} A$. Since $T$ is selforthogonal and $\text{id}_A T$ is finite, we only need to show that every module in $\mathcal{P}(A)$ has a finite exact coresolution in $\text{add} T$.

Denote by $\perp T = \{M \in \text{mod} A | \text{Ext}^i_A(M, T) = 0 \text{ for } i \geq 1\}$ and

$$X_T = \{X \mid 0 \to X \to T_0 \xrightarrow{f_0} T_1 \to \cdots \to T_n \xrightarrow{f_n} T_{n+1} \to \cdots, T_i \in \text{add} T, \text{Im} f_i \in \perp T, n \geq 0\}.$$ 

Since $T$ is a cotilting $A$-module, we have that $\mathcal{P}(A)$ is contained in $X_T = \perp T$.

Then there exists an exact sequence

$$0 \to P \to T_0 \xrightarrow{f_0} T_1 \to \cdots \to T_n \xrightarrow{f_n} T_{n+1} \to \cdots$$

for all $A$-module $P \in \mathcal{P}(A)$, where $T_i \in \text{add} T$ and $X_i = \text{Im} f_i$ is in $X_T$ for all $i \geq 0$. Let $\text{id}_A \mathcal{P}(A) \leq r$. Then $\text{Ext}^1_A(X_r, X_{r-1}) = \text{Ext}^2_A(X_r, X_{r-2}) = \cdots = \text{Ext}^{r+1}_A(X_r, P) = 0$, hence the exact sequence $0 \to X_{r-1} \to T_r \to X_r \to 0$ splits, such that $X_{r-1} \in \text{add} T$. This implies that $T$ is a tilting $A$-module.

(2) $\Rightarrow$ (1) Assume that a module $T$ in $\text{mod} A$ is a cotilting-tilting module. By the definitions of cotilting and tilting modules every module in $\mathcal{I}(A)$ has a finite exact resolution in $\text{add} T$ and every module in $\mathcal{P}(A)$ has a finite exact coresolution in $\text{add} T$. Since $\text{id}_A T$ and $\text{pd}_A T$ both are finite, it follows immediately that both $\text{pd}_A \mathcal{I}(A)$ and $\text{id}_A \mathcal{P}(A)$ are finite. Therefore $A$ is Gorenstein.

Similarly, one can prove the equivalence of (1) and (3).

In the following we show the equivalence of (1) and (4).

(1) $\Rightarrow$ (4) Assume that $A$ is Gorenstein. Then $A$ is a tilting-cotilting module.

(4) $\Rightarrow$ (1) is similar to (2) $\Rightarrow$ (1).

Now we are in a position to show the main result in this section.

**Theorem 2.7.** For an algebra $A$, the following are equivalent.

(1) $A$ is Iwanaga-Gorenstein with $\text{id}_A A \leq 1$. 

(2) \( \mathbb{D}A \) is a \( \tau \)-rigid module in \( \mod A \).

(3) \( A \) is a \( \tau^{-1} \)-rigid module in \( \mod A \).

**Proof.** We show the equivalence of (1) and (2). Similarly, one can show the equivalence of (1) and (3).

(1) \( \Rightarrow \) (2) For any \( M \in \Fac \mathbb{D}A \), there exists a short exact sequence

\[
0 \to N \to \mathbb{D}A^n \to M \to 0
\]  

(2.1)

Applying the functor \( \Hom_A(\mathbb{D}A, -) \) to the short exact sequence (2.1) yields the following long exact sequence

\[
0 \to \Hom_A(\mathbb{D}A, N) \to \Hom_A(\mathbb{D}A, \mathbb{D}A^n) \to \Hom_A(\mathbb{D}A, M) \to \Ext_A^1(\mathbb{D}A, N) \\
\to \Ext_A^1(\mathbb{D}A, \mathbb{D}A^n) \to \Ext_A^1(\mathbb{D}A, M) \to \Ext_A^2(\mathbb{D}A, N) \to \Ext_A^2(\mathbb{D}A, \mathbb{D}A^n) \to \cdots
\]

Then \( \Ext_A^1(\mathbb{D}A, M) \simeq \Ext_A^2(\mathbb{D}A, N) \) since \( \mathbb{D}A \) is an injective \( A \)-module, and \( \pdim \mathbb{D}A \leq 1 \) since \( \id_A \leq 1 \). Thus \( \Ext_A^1(\mathbb{D}A, M) \simeq \Ext_A^2(\mathbb{D}A, N) = 0 \). We have \( \Ext_A^1(\mathbb{D}A, \Fac \mathbb{D}A) = 0 \), therefore \( \mathbb{D}A \) is a \( \tau \)-rigid \( A \)-module by Proposition 2.5. Since |\( \mathbb{D}A \)| = |\( A \)|, one gets \( \mathbb{D}A \) is a \( \tau \)-tilting \( A \)-module.

(2) \( \Rightarrow \) (1) Since \( \mathbb{D}A \) is \( \tau \)-rigid and |\( \mathbb{D}A \)| = |\( A \)|, \( \mathbb{D}A \) is a \( \tau \)-tilting \( A \)-module. By [ASSS, Chapter VI, Lemma 2.2], \( \mathbb{D}A \) is faithful. Then \( \mathbb{D}A \) is a classical tilting \( A \)-module by Proposition 2.3(2), and hence \( \pdim \mathbb{D}A \leq 1 \). Thus \( \id_{\mathbb{D}A} \leq 1 \). Note that \( \id_A \leq 1 \) if and only if \( \id_{\mathbb{D}A} \leq 1 \), then \( \id_A \leq 1 \).

The following corollary is immediate.

**Corollary 2.8.** For an algebra \( A \), if one of the following conditions is satisfied,

(1) Every \( \tau \)-tilting \( A \)-module is a \( \tau^{-1} \)-tilting \( A \)-module;

(2) Every \( \tau^{-1} \)-tilting \( A \)-module is a \( \tau \)-tilting \( A \)-module,

then \( A \) is Iwanaga-Gorenstein with \( \id_A \leq 1 \).

**Proof.** We only prove (1) since the proof of (2) is similar. By assumption we have that the \( \tau \)-tilting module \( A \) is a \( \tau^{-1} \)-tilting module. Then \( A \) is a Gorenstein algebra with \( \id_A \leq 1 \) by Theorem 2.7(3).

We should remark that the converse of Corollary 2.8 is not true in general.

**Example 2.9.** Let \( A \) be the algebra given by the quiver

\[
\begin{array}{c}
1 \\
\alpha \downarrow \beta
\end{array} \quad \begin{array}{c}
2 \oplus 1
\end{array}
\]

with relations \( \alpha \beta = \beta \alpha = 0 \).

Then the support \( \tau \)-tilting quiver of \( A \) is the following:

\[
\begin{array}{c}
\frac{1}{2} \oplus \\
\frac{1}{2} \oplus \\
2 \oplus 1
\end{array} \quad \begin{array}{c}
\frac{1}{2} \oplus \\
\frac{1}{2} \oplus \\
2 \oplus 1
\end{array}
\]

One can show that \( 2 \oplus \frac{1}{2} \) is \( \tau \)-tilting but not \( \tau^{-1} \)-tilting.

At the end of this section, we give an example to show that the existence of \( \tau \)-tilting-\( \tau^{-1} \)-tilting modules (even \( \tau \)-rigid classical cotilting modules) is not equivalent to \( 1 \)-Gorensteinness in general.

**Example 2.10.** Let \( A \) be the algebra given by the quiver

\[
\begin{array}{c}
1 \\
\alpha \downarrow \beta \downarrow
\end{array} \quad \begin{array}{c}
2 \oplus 1
\end{array}
\]

with the relation \( \alpha \beta = 0 \).

Then \( T = \frac{1}{2} \oplus \frac{3}{2} \oplus 2 \) is a \( \tau \)-tilting-\( \tau^{-1} \)-tilting module in \( \mod A \) (actually a classical tilting-cotilting module) but \( \gl \dim A = 2 \).
3. ITERATED TILTED ALGEBRAS AND T- RIGID MODULES

In this section, we focus on the T-rigid modules over iterated tilted algebras and show every indecomposable module over an iterated tilted algebra of Dynkin type is T-rigid. Throughout this section, all tilting modules are classical tilting modules.

Firstly, we need the notion of torsion pairs.

**Definition 3.1.** Let $A$ be an algebra. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\text{mod}A$ is called a torsion pair if the following conditions are satisfied:

1. $\text{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}$, $N \in \mathcal{F}$.
2. $\text{Hom}_A(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$.
3. $\text{Hom}_A(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

To introduce the tilting theorem due to Brenner and Butler, we also need the following:

**Definition 3.2.** Let $A$ be an algebra. Any tilting module $T$ in $\text{mod}A$ induces torsion pairs $(\mathcal{T}(T), \mathcal{F}(T))$ in $\text{mod}A$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\text{mod}B$ with $B = \text{End}_A T$, where

\[
\mathcal{T}(T) = \{ M_A[\text{Ext}_A^1(T, M) = 0] \}, \\
\mathcal{F}(T) = \{ M_A[\text{Hom}_A(T, M) = 0] \}, \\
\mathcal{X}(T) = \{ X_B[\text{Hom}_B(X, \mathbb{D}T) = 0] = \{ X_B[X \otimes_B T = 0] \}, \\
\mathcal{Y}(T) = \{ Y_B[\text{Ext}_B^1(Y, \mathbb{D}T) = 0] = \{ Y_B[\text{Tor}^B_1(Y, T) = 0] \}.
\]

Now we can state the tilting theorem of Brenner and Butler [BB] as follows:

**Theorem 3.3.** Let $A$ be an algebra, $T$ be a tilting module in $\text{mod}A$ and $B = \text{End}_A T$. Let $(\mathcal{T}(T), \mathcal{F}(T))$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ be the induced torsion pairs in $\text{mod}A$ and $\text{mod}B$, respectively. Then $T$ has the following properties:

1. $\mu T$ is a tilting $B$-module, and the canonical $K$-algebra homomorphism $A \rightarrow \text{End}_B T^\oplus$ defined by $a \rightarrow (t \mapsto ta)$ is an isomorphism.
2. The functors $\text{Hom}_A(T, -)$ and $- \otimes_B T$ induce quasi-inverse equivalences between $\mathcal{T}(T)$ and $\mathcal{Y}(T)$.
3. The functors $\text{Ext}_A^1(T, -)$ and $\text{Tor}^B_1(-, T)$ induce quasi-inverse equivalences between $\mathcal{F}(T)$ and $\mathcal{X}(T)$.

Recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{mod}A$ is called splitting if for any indecomposable $M \in \text{mod}A$ either $M \in \mathcal{T}$ or $M \in \mathcal{F}$ holds. For a tilting module $T_A$ with $B = \text{End}_A T$, $T_A$ is said to be splitting if the induced torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\text{mod}B$ is splitting. The following propositions in [AsSS] are critical in the proof of the main result in this section.

**Proposition 3.4.** [AsSS] VI, Corollary 5.7] For an algebra $A$, if $\text{gl.dim} A \leq 1$, then every tilting module in $\text{mod}A$ is splitting.

**Proposition 3.5.** [AsSS] VI, Proposition 5.2] Let $A$ be an algebra, $T$ be a splitting tilting module in $\text{mod}A$, and $B = \text{End}_A T$. Then any almost split sequence in $\text{mod}B$ lies entirely in either $\mathcal{X}(T)$ or $\mathcal{Y}(T)$, or else it is of the form

\[
0 \rightarrow \text{Hom}_A(T, I) \rightarrow \text{Hom}_A(T, I/\text{soc} I) \oplus \text{Ext}_A^1(T, \text{rad} P) \rightarrow \text{Ext}_A^1(T, P) \rightarrow 0,
\]

where $P$ is an indecomposable projective $A$-module not lying in add$T$ and $I$ is the indecomposable injective $A$-module such that $P/\text{rad} P \cong \text{soc} I$.

Keeping the symbols as above, we can recall the following proposition.

**Proposition 3.6.** [AsSS] VI, Lemma 5.3] Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an almost split sequence in $\text{mod}B$.

1. If $L, M, N \in \mathcal{Y}(T)$, then $0 \rightarrow L \otimes_B T \rightarrow M \otimes_B T \rightarrow N \otimes_B T \rightarrow 0$ is almost split in $\mathcal{T}(T)$.
2. If $L, M, N \in \mathcal{X}(T)$, then $0 \rightarrow \text{Tor}^B_1(L, T) \rightarrow \text{Tor}^B_1(M, T) \rightarrow \text{Tor}^B_1(N, T) \rightarrow 0$ is almost split in $\mathcal{F}(T)$.
Let $Q$ be a finite, connected, and acyclic quiver. Recall that an algebra $B$ is called an itera-
ted tilted algebra of type $Q$ if there is a series of algebras $A_0 = KQ, A_1, \ldots, A_m = B$ such that $T_i$ is a splitting classical tilting module over $A_i$ and $A_{i+1} = \text{End}_{A_i} T_i$ for $0 \leq i \leq m - 1$. Now we are in a position to show the main result of this section.

**Theorem 3.7.** Let $B$ be an iterated tilted algebra of Dynkin type $Q$. Then every indecomposable module in $\text{mod} B$ is $\tau$-rigid.

**Proof.** Assume that $B = A_m$ is the iterated tilted algebra of Dynkin type $Q$ with the corresponding splitting tilting modules $T_i$ for $0 \leq i \leq m - 1$. We prove the assertion by induction on $m$.

If $m = 1$, then $B = A_1 = \text{End}_{A_0} T_0$ is a tilted algebra of Dynkin type.

Let $N$ be any indecomposable module in $\text{mod} B$. By Proposition 3.4, $N$ is either in $\mathcal{X}(T)$ or in $\mathcal{Y}(T)$.

If $N$ is projective, then there is nothing to show. Now assume that $N$ is not projective. Then there is an almost split sequence $0 \to L \to M \to N \to 0$. By Proposition 3.5, the exact sequence is either in $\mathcal{X}(T)$, $\mathcal{X}(T)$ or a connecting sequence.

1. If $0 \to L \to M \to N \to 0$ in $\mathcal{Y}(T)$, then $0 \to L \otimes_B T \to M \otimes_B T \to N \otimes_B T \to 0$ is an Auslander-Reiten sequence in $\text{mod} A_0$. By Proposition 3.6, since $A_0$ is the path algebra of a Dynkin quiver, $A_0$ is a representation-finite hereditary algebra. This implies every indecomposable module in $\text{mod} A_0$ is projective and thus $\tau$-rigid. By Theorem 3.3, $\text{Hom}_B(N, L) \simeq \text{Hom}_{A_0}(N \otimes_B T, L \otimes_B T) = 0$, hence $N$ is $\tau$-rigid.

2. If $0 \to L \to M \to N \to 0$ in $\mathcal{X}(T)$, then $0 \to \text{Tor}^B_1(L, T) \to \text{Tor}^B_1(M, T) \to \text{Tor}^B_1(N, T) \to 0$ is an Auslander-Reiten sequence in $\text{mod} A_0$. By Proposition 3.6. As we showed in (1) every indecomposable module in $\text{mod} A_0$ is $\tau$-rigid. By Theorem 3.3, $\text{Hom}_B(N, L) \simeq \text{Hom}_{A_0}(\text{Tor}^B_1(N, T), \text{Tor}^B_1(L, T)) = 0$, hence $N$ is $\tau$-rigid.

3. If $0 \to L \to M \to N \to 0$ is a connecting sequence, then $N \simeq \text{Ext}^1_{A_0}(T, P(a)) \in \mathcal{X}(T)$, $L \simeq \text{Hom}_{A_0}(T, I(a)) \in \mathcal{Y}(T)$ by Proposition 3.5. Thus, $\text{Hom}_B(N, \tau N) = \text{Hom}_B(N, L) = 0$, $N$ is $\tau$-rigid.

Now assume the assertion holds for $B = A_m$. In the following we show the assertion holds for $B = A_{m+1}$.

By induction assumption, every indecomposable module in $\text{mod} A_m$ is $\tau$-rigid. For any indecomposable module $N \in \text{mod} B$, if $N$ is projective, then there is nothing to show. We assume that $N$ is not projective. Since $T_m$ is splitting, then $N$ is either in $\mathcal{Y}(T_m)$ or in $\mathcal{X}(T_m)$. Putting $T = T_m$ in the proof of the case $m = 1$, one gets the desired result.

**Example 3.8.** Let $A_0 = KQ$ be the algebra given by the quiver $Q : 1 \to 2 \to 3 \to 4$ and let $T_0$ be the tilting module $\frac{1}{4} \oplus 1 \oplus \frac{1}{2} \oplus 4$ in $\text{mod} A_0$. Then

1. $A_1 = \text{End}_{A_0} T_0$ is given by the quiver $Q' : 1 \overset{\alpha_1}{\rightarrow} 2 \overset{\alpha_2}{\rightarrow} 3 \overset{\alpha_3}{\rightarrow} 4$ with the relation $\alpha_2 \alpha_3 = 0$ and $\text{gl.dim} A_1 = 2$.
2. $T_1 \cong \frac{1}{3} \oplus 2 \oplus \frac{3}{4} \oplus 4$ in $\text{mod} A_1$ is a classical tilting module and $A_2 = \text{End}_{A_1} T_1$ is given by the quiver $Q'' : 1 \overset{\beta_1}{\rightarrow} 2 \overset{\beta_2}{\rightarrow} 3 \overset{\beta_3}{\rightarrow} 4$ with relations $\beta_1 \beta_2 = 0$ and $\beta_2 \beta_3 = 0$.
3. $\text{gl.dim} A_2 = 3$ implies that $A_2$ is iterated tilted but not tilted.
4. The Auslander-Reiten quiver of $A_2$ is as follows:

One can show that every indecomposable module in $\text{mod} A_2$ is $\tau$-rigid.
4. \( \tau \)-Tilting Modules and Homological Dimension

In this section, we give the relationship between \( \tau \)-tilting modules and homological dimension, which is an analog of that of classical tilting modules (see \cite{AsSS} Lemma 4.1 for details).

For an \( A \)-module \( M \), denote by \( M^{\perp_0} \) (resp. \( ^{\perp_0}M \)) the subcategory consisting of \( N \) such that \( \text{Hom}_A(M,N) = 0 \) (resp. \( \text{Hom}_A(N,M) = 0 \)). Firstly, we introduce the following lemma known as Wakamatsu’s Lemma.

**Lemma 4.1.** (1) Let \( \theta : 0 \to Y \to T' \xrightarrow{\theta} X \) be an exact sequence in \( \text{mod}A \), where \( T \) is \( \tau \)-rigid, and \( g : T' \to X \) is a right \( \text{add}T \)-approximation. Then we have \( Y \in ^{\perp_0}(\tau T) \).

(2) Let \( \theta : Y \xrightarrow{f} U \to Z \to 0 \) be an exact sequence in \( \text{mod}A \), where \( T \) is \( \tau \)-rigid, \( U \in \text{add}T \), and \( f : Y \to U \) is a left \( \text{add}(\tau T) \)-approximation. Then we have \( Z \in T^{\perp_0} \).

**Proof.** (1) is given by Adachi, Iyama and Reiten in \cite{AIR}. We only prove (2).

Replacing \( Y \) by \( \text{Ker}f \), we can assume that \( f \) is an injective. We apply \( \text{Hom}_A(T, -) \) to \( \theta \) and get the exact sequence

\[
0 = \text{Hom}_A(T, U) \to \text{Hom}_A(T, Z) \to \text{Ext}_A^1(T, Y) \xrightarrow{\text{Ext}_A^1(T, f)} \text{Ext}_A^1(T, U)
\]

where we have \( \text{Hom}_A(T, U) = 0 \) because \( U \in \text{add}T \). Since \( f : Y \to U \) is a left \( \text{add}(\tau T) \)-approximation, the induced map \( (f, \tau T) : \text{Hom}_A(U, \tau T) \to \text{Hom}_A(Y, \tau T) \) is surjective. Then the induced map \( \text{Hom}_A(U, \tau T) \to \text{Hom}_A(Y, \tau T) \) of the maps modulo injectives is surjective. By the Auslander-Reiten duality, the map \( \text{Ext}_A^1(T, f) : \text{Ext}_A^1(T, Y) \to \text{Ext}_A^1(T, U) \) is injective. It follows that \( \text{Hom}_A(T, Z) = 0 \).

Dually, one can show Wakamatsu’s Lemma in terms of \( \tau^{-1} \)-rigid modules.

Recall from \cite{AsSS} Chapter VI, Lemma 4.1, for an algebra \( A \), \( T \) a classical tilting module in \( \text{mod}A \) and \( B = \text{End}_A T \), if \( M \in \text{Fac} T \), then \( \text{pd}_B \text{Hom}_A(T, M) \leq \text{pd}_A M \) holds. We prove an analog result in terms of \( \tau \)-tilting modules as follows.

**Theorem 4.2.** Let \( A \) be an algebra, \( T \) be a \( \tau \)-tilting module in \( \text{mod}A \) and \( B = \text{End}_A T \). For any \( M \in \text{Fac} T \), we have

(1) If \( \text{pd}_A M \leq 1 \) holds, then \( \text{pd}_B \text{Hom}_A(T, M) \leq \text{pd}_A M \) holds.

(2) If \( \text{Ext}_A^i(T, M \otimes T) = 0 \) holds for any \( i \geq 1 \), then \( \text{pd}_B \text{Hom}_A(T, M) \leq \text{pd}_A M \) holds.

**Proof.** (1) If \( \text{pd}_A M = 0 \), then \( M \in \text{Fac} T \) implies \( M \in \text{add}T \). One gets \( \text{Hom}_A(T, M) \) is a projective module in \( \text{mod}B \) since \( \text{Hom}_A(T, -) \) induces an equivalence between \( \text{add}T \) and \( \text{add}B \).

Now, assume \( \text{pd}_A M = 1 \). Since \( M \in \text{Fac} T \), by Lemma 4.1 we get a short exact sequence

\[
0 \to L \to T_0 \to M \to 0
\]

with \( L \in ^{\perp_0}(\tau T) = \text{Fac} T \).

Recall that \( L \in C \subseteq \text{mod}A \) is Ext-projective if \( \text{Ext}_A^2(L, C) = 0 \). In the following we show \( L \in \text{add}T \), that is, \( L \) is Ext-projective in \( \text{Fac} T \).

For any \( N \in \text{Fac} T \), applying the functor \( \text{Hom}(-, N) \) to the exact sequence (4.1), we get a long exact sequence

\[
\text{Ext}_A^1(T, M) \to \text{Ext}_A^1(T, N) \to \text{Ext}_A^2(T, L, N) = 0
\]

holds because of \( \text{pd}_A M = 1 \) and \( N \in \text{Fac} T \). We are done.

Applying the functor \( \text{Hom}_A(T, -) \) to the sequence (4.1) again, we get the assertion since \( \text{Hom}(T, -) \) is an equivalence between \( \text{add}T \) and \( \text{add}B \).

(2) If \( \text{pd}_A M = \infty \), then there is nothing to show.

Now we can assume that \( \text{pd}_A M = t < \infty \). Since \( M \in \text{Fac} T \), by Lemma 4.1 we get a short exact sequence

\[
0 \to L \to T_0 \to M \to 0
\]

with \( L \in ^{\perp_0}(\tau T) = \text{Fac} T \). Applying the functor \( \text{Hom}_A(T, -) \) to the sequence (4.1), we get a long exact sequence

\[
\text{Ext}_A^1(T, L) \to \text{Ext}_A^2(T, L, N) = 0
\]

for any \( i \geq 1 \) by assumption, and hence \( \text{Ext}_A^1(T, L) = 0 \) for any \( i \geq 1 \). Continuing the similar process, we get the following long exact sequence

\[
\cdots \to T_n \xrightarrow{f_n} T_{n-1} \to \cdots \to T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0
\]

(4.2)
with $T_i \in \text{add}T$ and $L_{i+1} = \ker f_i \in \tilde{\text{add}}(\tau T) = \text{Fac}T$ for $i \geq 0$ and $\text{Ext}^j_A(T, L_{i+1}) = 0$ for $j \geq 1$ and $i \geq 0$.

Next we show that the exact sequence $0 \to L_{t+1} \to T_t \to T_t \to 0$ splits.

Since $\text{pd}_A M = t < \infty$, then $\text{Ext}^{t+1}_A(M, L_{t+1}) = 0$. On the other hand, applying the functor $\text{Hom}_A(−, L_{t+1})$ to the sequence (4.2), one gets $0 = \text{Ext}^{t+1}_A(M, L_{t+1}) \simeq \text{Ext}^t_A(L_1, L_{t+1}) \simeq \cdots \text{Ext}^1_A(L_t, L_{t+1})$ since $\text{Ext}^0_A(T, M) = 0$ and $L_t \in \text{Fac}T$ hold for any $i \geq 1$. Hence we have a long exact sequence.

$0 \to T_t \xrightarrow{f_t} T_{t−1} \to \cdots \to T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0 \quad (4.3)$

Applying the functor $\text{Hom}_A(T, −)$ to the exact sequence (4.3), we have

$0 \to \text{Hom}_A(T, T_t) \to \text{Hom}_A(T, T_{t−1}) \to \cdots \to \text{Hom}_A(T, T_1) \to \text{Hom}_A(T, T_0) \to \text{Hom}_A(T, M) \to 0$

and hence $\text{pd}_B \text{Hom}_A(T, M) \leq t = \text{pd}_A M$. □

For a module $T$ in $\text{mod}A$, we denote by $\text{Sub}T = \{N|N \to T^n \text{ for some integer } n\}$. Then we have the following on the injective dimensions.

**Theorem 4.3.** Let $A$ be an algebra, $T$ be a $τ$-tilting module in $\text{mod}A$ and $C = \text{End}_A τT^{\text{op}}$. For any $N \in \text{Sub}τT$, we have

1. If $\text{id}_A N ≤ 1$ holds, then $\text{pd}_C \text{Hom}_A(N, τT) ≤ \text{id}_A N$ holds.
2. If $\text{Ext}^j_A(τT ⊕ N, τT) = 0$ holds for any $j ≥ 1$, then $\text{pd}_C \text{Hom}_A(N, τT) ≤ \text{id}_A N$ holds.

**Proof.** Throughout the proof, we denote by $U = τT$.

1. If $\text{id}_A N = 0$, then $N \in \text{Sub}U$ implies $N \in \text{add}U$. One gets $\text{Hom}_A(N, U)$ is a projective $C$-module since $\text{Hom}_A(−, U)$ induces a duality between $\text{add}U$ and $\text{add}C$.

Assume $\text{id}_A N = 1$. Since $N \in \text{Sub}U$, by Lemma 4.1 we get a short exact sequence

$0 \to N \to U_0 \to L \to 0 \quad (4.4)$

where $L \in T^{\tilde{\text{add}}U} = \text{Sub}U$. In the following we show $L \in \text{add}U$, that is, $\text{Ext}^1_A(N', L) = 0$ holds for any $N' \in \text{Sub}U$. Applying the functor $\text{Hom}_A(N', −)$ to the exact sequence (4.4), one gets the exact sequence $\text{Ext}^1_A(N', U) \to \text{Ext}^1_A(N', L) \to \text{Ext}^2_A(N', N)$. The assertion follows from the facts $U$ is Ext-injective and $\text{id}_A N = 1$.

2. If $\text{id}_A N = ∞$, then there is nothing to show. So we can assume that $\text{id}_A N = s < ∞$.

By Lemma 4.1 we get the following exact sequence

$0 \to N \xrightarrow{f_0} U_0 \xrightarrow{f_1} U_1 \cdots \xrightarrow{f_s} U_s \to \cdots \quad (4.5)$

with $f_i$ the minimal left $\text{add}U$-approximation. Denote by $L_i = \text{Im} f_i$, then one gets $\text{Ext}^k_A(L_i, U) = 0$ for any $k ≥ 1$ and $i ≥ 0$.

In the following we show the exact sequence $0 \to L_s \to U_s \to L_{s+1} \to 0$ splits. Applying the functor $\text{Hom}_A(L_{s+1}, −)$ to the exact sequence (4.5), one gets $0 = \text{Ext}^{s+1}_A(L_{s+1}, N) \simeq \text{Ext}^s_A(L_{s+1}, L_1) \simeq \cdots \simeq \text{Ext}^1_A(L_{s+1}, L_s)$ since $\text{id}_A N ≤ n$. Hence we have the following exact sequence

$0 \to N \xrightarrow{f_0} U_0 \xrightarrow{f_1} \cdots \xrightarrow{f_s} U_s \to 0 \quad (4.5)$

Applying the functor $\text{Hom}_A(−, U)$, one gets the assertion since $\text{Ext}^k_A(L_i, U) = 0$ holds for any $k, i ≥ 1$. □

At the end of this section, we give an example to show our main results.

**Example 4.4.** Let $A$ be the algebra given by the quiver $Q : 1 \xrightarrow{α_1} 2 \xrightarrow{α_2} 3$ with relations $α_1 β_2 = 0$ and $α_2 β_1 = β_2 α_1$. Then

1. $A$ is an Auslander algebra and $T = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} ⊕ \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} ⊕ \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ is a $τ$-tilting module in $\text{mod}A$. 


(2) \( B = \text{End}_A T \) is given by the quiver \( Q' : \begin{array}{ccc} 3 & \xrightarrow{\gamma_3} & 2 \\ \xleftarrow{\gamma_1} & & \xleftarrow{\gamma_2} 1 \end{array} \) with the relation \( \gamma_1 \gamma_2 = 0 \) and \( \text{gl.dim} B = 2 \).

(3) One can show \( M = \begin{array}{c} 2 \\ 3 \end{array} \in \text{Fac} T \) with \( \text{pd}_A M = 1 \), \( \text{Hom}_A (T, M) = S(2) \) in \( \text{mod} B \), and \( \text{pd}_B \text{Hom}_A (T, M) \leq \text{pd}_A M \).

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