Abstract. This note characterizes both boundedness and compactness of a composition operator between any two analytic Campanato spaces on the unit complex disk.

1. Introduction

On the basis of the works: [15], [5], [6, 7], [8], [18], [19], [17], [1], [9, 10], [12, 3], [20, 21, 22], [13, 14] and [2], we consider an unsolved fundamental problem in the function-theoretic operator theory, i.e., the so-called composition operator question for the analytic Campanato spaces:

**Question 1.** Let \( \phi \) be an analytic self-map of \( \mathbb{D} \) and \( -\infty < p, q < \infty \). What finite (resp. vanishing) property must \( \phi \) have in order that \( C_\phi \) is bounded (resp. compact) between \( \mathcal{Cp} \) and \( \mathcal{Cq} \)?

In the above and below, \( \mathbb{D} \) and \( \mathbb{T} \) respectively represent the unit disk and the unit circle in the finite complex plane \( \mathbb{C} \), \( C_\phi f = f \circ \phi \) is the composition of an analytic function \( f \) on \( \mathbb{D} \) with \( \phi \), and for \( p \in (-\infty, \infty) \), and \( \mathcal{Cp} \) denotes the so-called Campanato space of all analytic functions \( f : \mathbb{D} \to \mathbb{C} \) with radial boundary values \( f \) on \( \mathbb{T} \) satisfying

\[
\|f\|_{\mathcal{Cp}} = \sup_{I \subset \mathbb{T}} \left( |I|^{-p} \int_I |f(\xi) - f_I|^2 |d\xi| \right)^{\frac{1}{2}} < \infty,
\]

where the supremum is taken over all sub-arcs \( I \subset \mathbb{T} \) with \( |I| \) being their arc-lengths, and

\[
|d\xi| = |de^{\theta}| = d\theta; \quad f_I = |I|^{-1} \int_I f(\xi) |d\xi|.
\]

Neededless to say, \( \| \cdot \|_{\mathcal{Cp}} \) cannot distinguish between any two \( \mathcal{Cp} \) functions differing by a constant, but \( |f(0)| + \| \cdot \|_{\mathcal{Cp}} \) defines a norm so that \( \mathcal{Cp} \)
is a Banach space. Here, it is perhaps appropriate to mention the following table which helps us get a better understanding of the structure of $CA_p$ (see, e.g. [4, pp. 67-75] and [22, p. 52]):

| Index $p$ | Analytic Campanato Space $CA_p$ |
|-----------|----------------------------------|
| $p \in (-\infty, 0]$ | Analytic Hardy space $\mathcal{H}^2$ |
| $p \in (0, 1)$ | Holomorphic Morrey space $\mathcal{H}^{2-p}$ |
| $p = 1$ | Analytic John-Nirenberg space $BMOA$ |
| $p \in (1, 3]$ | Analytic Lipschitz space $A_{p-1}$ |
| $p \in (3, \infty)$ | Complex constant space $\mathbb{C}$ |

An answer to the boundedness part of Question 1 is the following result.

**Theorem 1.** Let $\phi$ be an analytic self-map of $\mathbb{D}$ and $(p, q) \in [0, 2) \times [0, 2)$. Then $C_\phi : CA_p \mapsto CA_q$ is bounded if and only if

$$\sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|_2 < \infty,$$

where

$$\sigma_b(z) = \frac{b - z}{1 - \overline{b}z} \quad \text{and} \quad \|f\|_2 = \sqrt{\int_{\mathbb{T}} |f(\xi)|^2 |d\xi|}.$$

It should be pointed out that (1) is not always true - in fact, we have the following consequence whose (i) with $p = q \in \{0, 1\}$ and (ii) are well-known; see e.g. [9, 12, 13, 14, 15, 20].

**Corollary 1.** Let $\phi$ be an analytic self-map of $\mathbb{D}$. For $f \in \mathcal{H}^2$ and $p \in [0, 2)$ set

$$\|f\|_{CA_p,*} = \sup_{a \in \mathbb{D}} (1 - |a|^2)^{\frac{1-p}{2}} \|f \circ \sigma_a - f(a)\|_2.$$

(i) If $p \in [0, 1]$ then $C_\phi : CA_p \mapsto CA_p$ is always bounded with

$$\|C_\phi f\|_{CA_p,*} \leq \left( \frac{1 + |\phi(0)|}{1 - |\phi(0)|} \right)^{\frac{1-p}{2}} \|f\|_{CA_p,*}.$$

(ii) If $p \in (1, 2)$ then $C_\phi : CA_p \mapsto CA_p$ is bounded when and only when

$$\sup_{a \in \mathbb{D}} \left( \frac{1 - |a|^2}{1 - |\phi(a)|^2} \right)^{\frac{1-p}{2}} |\phi'(a)| < \infty.$$

Below is a partial answer to the compactness part of Question 1.

**Theorem 2.** Let $\phi$ be an analytic self-map of $\mathbb{D}$ and $(p, q) \in [0, 2) \times [0, 2)$. If $C_\phi : CA_p \mapsto CA_q$ is compact then (1) holds and

$$\lim_{|\phi(a)| \to 1} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|_2 = 0.$$
Conversely, if (7) holds and (4) is valid for \((p, q) \in [0, 2) \times [1, 1] \cup (1, 2) \times [0, 2)\) then \(C_\phi : \mathcal{CA}_p \to \mathcal{CA}_q\) is compact.

Theorem 2 covers the corresponding \(BMOA\)-results in [15, 19, 8], but also it derives the following assertion extending the known one in [12, 10, 20].

**Corollary 2.** Let \(\phi\) be an analytic self-map of \(D\) and \(p \in [0, 2)\). If \(C_\phi : \mathcal{CA}_p \to \mathcal{CA}_p\) is compact then (3) holds and

\[
\lim_{|\phi(a)| \to 1} \left( \frac{1 - |a|^2}{1 - |\phi(a)|^2} \right)^{\frac{1}{2p}} |\phi'(a)| = 0.
\]

Conversely, if (3) holds and (5) is valid for \(p \in (1, 2)\) then \(C_\phi : \mathcal{CA}_p \to \mathcal{CA}_p\) is compact.

**Conjecture 1.** The converse part of Theorem 2 still holds for \((p, q) \in [0, 2) \times [0, 2) \setminus \{(0, 0) \times [0, 1] \cup (1, 2) \times [0, 2)\}\).

**Notation:** From now on, \(X \lesssim Y, X \gtrsim Y, \text{ and } X \approx Y\) represent that there exists a constant \(\kappa > 0\) such that \(X \leq \kappa Y, X \geq \kappa Y, \text{ and } \kappa^{-1} Y \leq X \leq \kappa Y\), respectively. In addition, \(dm\) stands for two dimensional Lebesgue measure.

2. **Boundedness**

In order to prove Theorem 1 and Corollary 1, we need two lemmas.

**Lemma 1.** Let \(p \in [0, 2)\) and \(f \in \mathcal{H}^2\). Then \(f \in \mathcal{CA}_p\) if and only if \(\|f\|_{\mathcal{CA}_p} < \infty\).

**Proof.** Case 1: \(p = 0\). This is trivial.

Case 2: \(p \in (0, 1]\). This situation can be verified by [22] Theorem 3.2.1 and the well-known Hardy-Littlewood identity for \(f \in \mathcal{H}^2\):

\[
\pi^{-1} \int_D |f'(z)|^2(-\ln |z|^2) \, dm(z) = (2\pi)^{-1} \int_T |f(\xi) - f(0)|^2 \, |d\xi|.
\]

Case 3: \(p \in (1, 2)\). Let \(g = f \circ \sigma_a - f(a)\). Then

\[
(1 - |a|^2)|f'(a)| = |g'(0)| \leq (2\pi)^{-1/2}\|g\|_2 = (2\pi)^{-1/2}\|f \circ \sigma_a - f(a)\|_2.
\]

If \(f \in \mathcal{CA}_p\) then an application of (7) yields

\[
\sup_{a \in D} (1 - |a|^2)^{\frac{1}{2p}} |f'(a)| < \infty
\]

and consequently, \(f \in \mathcal{A}_{\frac{1}{p-1}}\), as desired. Conversely, if \(f \in \mathcal{A}_{\frac{1}{p-1}}\) then

\[
A = \sup_{\xi_1 \neq \xi_2 \in D \cup T} \frac{|f(\xi_1) - f(\xi_2)|}{|\xi_1 - \xi_2|^{\frac{1}{p-1}}} < \infty.
\]
This, along with \( p \in (1, 2) \) and [23] p. 63, Ex. 8], gives

\[
\|f\|_{C, A_p, *}^2 = \sup_{a \in D} (1 - |a|^2)^{1-p} \int_T |f \circ \sigma_a(\xi) - f(a)|^2 |d\xi|
\]

\[
\lesssim A^2 \sup_{a \in D} \int_T \left( \frac{1 - |a|^2}{|\sigma_a(\xi) - a|} \right)^{1-p} |d\xi|
\]

\[
\approx A^2 \sup_{a \in D} \int_T (1 - |a|^2)^{2-p} |d\eta|
\]

\[
\lesssim A^2.
\]

\[\square\]

**Lemma 2.** For \( p \in [0, 2) \) let \( f_b(z) = (1 - |b|^2)\frac{z + |b|^2/(1 - \bar{b}z)}{(1 - |b|^2)} \). Then \( f_b \) is uniformly bounded in \( C, A_p, \) i.e., \( \sup_{b \in D} \|f_b\|_{C, A_p, *} < \infty \).

**Proof.** Using [11] Lemma 2.5], we get the following estimate:

\[
B = \int_D |f'_b(z)|^2 \frac{1 - |\sigma_a(z)|^2}{1 - |z|^2} dm(z)
\]

\[
= (|b|(1 - |b|^2)^{1+p})^2 (1 - |a|^2) \int_D \frac{1 - |z|^2}{|1 - \bar{a}z|^2|1 - \bar{b}z|^4} dm(z)
\]

\[
\lesssim \frac{(|b|(1 - |b|^2)^{1+p})^2 (1 - |a|^2)}{(1 - |b|^2)|1 - \bar{a}b|^2}.
\]

Choosing \( a = \sigma_b(c) \), we utilize \( 1 - |z| \lesssim -\ln |z| \) to obtain that if \( p \in [0, 2) \) then

\[
\|f_b\|_{C, A_p, *}^2 \lesssim \sup_{a \in D} (1 - |a|^2)^{1-p} B
\]

\[
\lesssim \sup_{a \in D} (1 - |a|^2)^{2-p} (1 - |b|^2)^p
\]

\[
\leq \sup_{c \in D} \left( \frac{1 - |c|^2}{|1 - \bar{b}c|} \right)^{2-p} |1 - \bar{b}c|^p
\]

\[
\leq 2^2,
\]

as desired. \[\square\]

**Proof of Theorem 1.** Using [6] Proposition 2.3], we have that if \( g(0) = 0 = \psi(0), \) \( g \in H^2, \) and \( \psi \) is an analytic self-map of \( D, \) then

(9) \( \|g \circ \psi\|_2 \lesssim \|g\|_2 \|\psi\|_2. \)
Setting
\[ g_a = f \circ \sigma_{\phi(a)} - f \circ \phi(a) \quad \text{and} \quad \psi_a = \sigma_{\phi(a)} \circ \phi \circ \sigma_a, \]
we get
\[ g_a \circ \psi_a = f \circ \phi \circ \sigma_a - f \circ \phi(a). \]

As a consequence of Lemma 1 and (9), we find that if (1) is valid then
\[
\| C_{\phi} f \|_{CA_{q^*}}^2 = \sup_{a \in D} (1 - |a|^2)^{1-q} \| f \circ \phi \circ \sigma_a - f \circ \phi(a) \|^2_2
\leq \sup_{a \in D} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} (1 - |\phi(a)|^2)^{1-p} \| g_a \|^2_2 \| \psi_a \|^2_2
\leq \| f \|^2_{CA_{p^*}} \sup_{a \in D} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \| \psi_a \|^2_2,
\]
and consequently, \( C_{\phi} \) exists as a bounded operator from \( CA_p \) into \( CA_q \).

For the “only-if” part, recall the so-called Navanlinna counting function of \( \phi \):
\[ N(\phi, w) = \sum_{z: \phi(z) = w} \ln |z|^{-1} \quad \forall \quad w \in \mathbb{D} \setminus \{ \phi(0) \} \]
and the associated change of variable formula:
\[
(10) \quad \int_{\mathbb{D}} |(C_{\phi} f)'(z)|^2 \ln |z|^{-1} dm(z) = \int_{\mathbb{D}} |f'(w)|^2 N(\phi, w) dm(w) \quad \forall \quad f \in H^2.
\]
A combination of (10) and (6) gives that if \( b = \phi(a) \) then
\[
(11) \quad \| \sigma_b \circ \phi \circ \sigma_a \|^2_2 = 4 \int_{\mathbb{D}} N(\sigma_b \circ \phi \circ \sigma_a, z) dm(z).
\]

Now, if \( C_{\phi} : CA_p \mapsto CA_q \) is bounded, then the test function \( f_b \) in Lemma 2 is used to imply
\[ C = \sup_{a,b \in D} (1 - |a|^2)^{1-q} \int_{\mathbb{D}} |(f_b \circ \phi \circ \sigma_a)'(z)|^2 \ln |z|^{-1} dm(z) < \infty. \]
and consequently,

$$C$$

$$\geq \sup_{a,b \in \mathbb{D}} (1 - |a|^2)^{1-q} |b|^2 (1 - |b|^2)^{p-1} \int_D |\sigma_{\phi}(z)|^2 N(\phi \circ \sigma_a, z) \, dm(z)$$

$$\geq \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q} |\phi(a)|^2}{(1 - |\phi(a)|^2)^{1-p}} \int_D N(\sigma_{\phi(a)} \circ \phi \circ \sigma_a, z) \, dm(z)$$

$$\geq \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q} |\phi(a)|^2}{(1 - |\phi(a)|^2)^{1-p}} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|^2$$

$$\geq s^2 \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|^2$$

as $|\phi(a)| > s \in (0, 1)$. Note also that the identity map $f(z) = z$ is an element of $\mathcal{CA}_p$. Thus, boundedness of $C_{\phi} : \mathcal{CA}_p \rightarrow \mathcal{CA}_{\psi}$ ensures $\|\phi\|_{\mathcal{CA}_{\psi}} < \infty$, and consequently, if $|\phi(a)| \leq s < 1$ then

$$\sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|^2$$

$$\leq (1 + (1 - s)^{p-1}) \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-q} \int_T \left| \frac{\phi(a) - \phi \circ \sigma_a(\xi)}{1 - \phi(a) \phi \circ \sigma_a(\xi)} \right|^2 |d\xi|$$

$$\leq \left( \frac{1 + (1 - s)^{p-1}}{(1 - s)^2} \right) \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-q} \|\phi(a) - \phi \circ \sigma_a\|^2$$

$$\approx \left( \frac{1 + (1 - s)^{p-1}}{(1 - s)^2} \right) \|\phi\|^2_{\mathcal{CA}_{\psi}^*}.$$ 

The above estimates imply (1).

\[ \square \]

**Proof of Corollary** (i) Under $p \in [0, 1]$, we use the Schwarz lemma for $\sigma_{\phi(0)} \circ \phi$ to deduce that (1) holds for $p = q \in [0, 1]$, and so that $C_{\phi}$ is bounded on $\mathcal{CA}_p$ due to Theorem 1. To reach (2), let us begin with the case $\phi(0) = 0$. According to the setting in the argument for Theorem 1 the well-known Littlewood subordination principle and Schwarz’s lemma for $\phi$, we have

$$\|C_{\phi}f\|_{\mathcal{CA}_{\psi}^*}^2$$

$$= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-p} \|g_a \circ \psi_a\|^2$$

$$\leq \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-p} \|g_a\|^2$$

$$\leq \|f\|^2_{\mathcal{CA}_{\psi}^*} \sup_{a \in \mathbb{D}} \left( \frac{1 - |a|^2}{1 - |\phi(a)|^2} \right)^{1-p}$$

$$\leq \|f\|^2_{\mathcal{CA}_{\psi}^*}.$$
Next, for the general case let
\[
\begin{align*}
\psi &= \sigma \phi(0) \circ \phi; \\
\lambda &= \frac{ab-1}{1-ab}; \\
b &= \phi(0); \\
c &= \sigma_a(b).
\end{align*}
\]
Then \(\psi(0) = 0\) and thus
\[
\begin{align*}
\|C_{\sigma_b}f\|_{C^2_{A_{p^*}}}^2 &= \sup_{a \in D} (1 - |a|^2)^{1-p} \|f(\lambda \sigma_c) - f(\lambda c)\|_2^2 \\
&\leq \|f\|_{C^2_{A_{p^*}}}^2 \sup_{a \in D} \frac{(1 - |a|^2)^{1-p}}{(1 - |c|^2)} \\
&\leq \|f\|_{C^2_{A_{p^*}}}^2 \left(1 + \frac{|b|}{1 - |b|}\right)^{1-p}.
\end{align*}
\]
Using the previous estimates, we get
\[
\|C_{\phi}f\|_{C^2_{A_{p^*}}}^2 \leq \|f \circ \sigma_b \circ \psi\|_{C^2_{A_{p^*}}}^2 \leq \|f \circ \sigma_b\|_{C^2_{A_{p^*}}}^2 \leq \|f\|_{C^2_{A_{p^*}}}^2 \left(1 + \frac{|b|}{1 - |b|}\right)^{1-p},
\]
whence reaching (2).

(ii) Suppose \(p \in (1, 2)\). Then (7) yields
\[
\sup_{a \in D} \left(\frac{1 - |a|^2}{1 - |\phi(a)|^2}\right)^{3-p} |\phi'(a)|^2 \leq \sup_{a \in D} \left(\frac{1 - |a|^2}{1 - |\phi(a)|^2}\right)^{1-p} \|\sigma_{\phi(a)} \circ \phi \circ \sigma_a\|_{C^2_{A_{p^*}}}^2,
\]
so, if \(C_{\phi}\) is bounded on \(C A_{p}\) then (1) holds with \(p = q\) due to Theorem 1 and hence (3) holds. Conversely, if (3) is true, then \(C_{\phi}\) is bounded on \(A_{p^1}\) (cf. [9, Theorem A]) and hence bounded on \(C A_{p}\).

3. Compactness

The arguments for Theorem 2 and Corollary 2 depend on the two basic facts below.

**Lemma 3.** Let \(p \in (0, 2)\) and \(f \in C A_{p}\) with \(f(0) = 0\). Then
\[
\int_\mathbb{T} |f(\xi)|^4 \, d\xi \leq \left\{
\begin{array}{l}
\|f\|_{C^2_{A_{p^*}}}^4 \|f\|_{2}^2 \text{ for } p \in [1, 2); \\
\|f\|_{C^2_{A_{p^*}}}^2 \int_0^\infty tH_p^p(\xi \in \mathbb{T} : |f(\xi)| > t) \, dt \text{ for } p \in (0, 1],
\end{array}
\right.
\]
where \(H_p^p(E) = \inf_{E \subseteq \bigcup_j I_j} \sum_j |I_j|^p\) is \(p\)-dimensional Hausdorff capacity of \(E \subseteq \mathbb{T}\) - the infimum is taken over all arc coverings \(\bigcup_j I_j \supseteq E\).
Proof. Let $d \mu = |f'(z)|^2(1 - |z|^2)dm(z)$. From $f \in CAC_p$ it follows that $\mu$ is a $p$-Carleson measure on $D$; in other words -

$$
\|\mu\|_{CM_p} = \sup_{I \subseteq \mathbb{T}} |I|^{-p} \mu(S(I)) \lesssim \|f\|_{CM_p}^2,
$$

where $S(I) = \{z = re^{i\theta} \in D : 1 - |I|/(2\pi) \leq r < 1 \& |\theta - \theta_I| \leq |I|/2\}$ is the Carleson box based on the arc $I \subseteq \mathbb{T}$ taking $\theta_I$ as its center. In fact, if $a = (1 - |I|/(2\pi))e^{i(\theta_I + |I|/4)}$ then a simple computation, along with (6) and $- \ln |z| \approx 1 - |z|^2$ as $|z| \geq 2^{-1}$ as well as Lemma 1 gives

$$
|I|^{-p} \mu(S(I)) \lesssim (1 - |a|^2)^{1-p} \int_{S(I)} |f'(z)|^2(1 - |\sigma_a(z)|^2) dm(z) \lesssim (1 - |a|^2)^{1-p} \|f \circ \sigma_a - f(a)\|_2^2 \lesssim \|f\|_{CM_p}^2.
$$

In particular, when $p \in (1, 2)$, $\mu$ is also 1-Carleson measure with $\|\mu\|_{CM_1} \lesssim \|\mu\|_{CM_p}^2$. According to [22, p. 79, Theorem 4.1.4], we have

$$
\int_{D} |f|^2 d \mu \lesssim \|\mu\|_{CM_p} \left\{ \begin{array}{ll}
\|f\|_2^2 & \text{for } p \in [1, 2); \\
\int_0^\infty tH_p^\infty(\{\xi \in \mathbb{T} : |f(\xi)| > t\}) dt & \text{for } p \in (0, 1].
\end{array} \right.
$$

This last estimate, along with the following Hardy-Stein identity based estimate (cf. [22 p. 36])

$$
\int_{\mathbb{T}} |f(\xi)|^4 |d\xi| \approx \int_{D} |f(z)|^2|f'(z)|^2(\ln |z|^{-1}) dm(z) \lesssim \int_{D} |f(z)|^2|f'(z)|^2(1 - |z|^2) dm(z) \approx \int_{D} |f|^2 d \mu,
$$

implies the desired estimate. \hfill \Box

Lemma 4. Let $(p, q) \in [0, 2] \times [1, 1]$. If an analytic self-map $\phi$ of $D$ satisfies (4), then one has

$$
\lim_{t \to 1} \sup_{|\phi(a)| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} ||\xi \in \mathbb{T} : |\sigma_{\phi(a)} \circ \phi \circ \sigma_a(\xi)| > t|| = 0 \forall s \in (0, 1).
$$

Proof. Note that $|\phi \circ \sigma_a| \to 1 \iff |\sigma_{\phi(a)} \circ \phi \circ \sigma_a| \to 1$ under $|\phi(a)| \leq s$. **
So, it suffices to show that \((4)\) implies
\[
\lim_{t \to \infty} \sup_{|\phi(a)| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} ||\xi \in T : |\phi \circ \sigma_a(\xi)| > t|| = 0 \ \forall \ s \in (0, 1).
\]

Following [8], for \(re^{i\theta} \in \mathbb{D}\) let
\[
J(re^{i\theta}) = \{e^{i\theta} : |t - \theta| \leq \pi(1 - r)\}.
\]

Clearly, \(J(re^{i\theta})\) is the sub-arc of \(T\) centered at \(e^{i\theta}\). Importantly, [8 Lemma 3] tells us that for any measurable set \(E \subseteq T\) with 1-dimensional Lebesgue measure \(|E| > 0\) there exists a measurable set \(F \subseteq E\) such that \(|F| > 0\) and
\[
\frac{|J(r\xi) \cap E|}{|J(r\xi)|} \geq (2^{4\pi})^{-1}|E| \ \forall \ r \in [0, 1) \ \& \ \xi \in F.
\]

Suppose now \((4)\) is valid but \((12)\) is not true. On the one hand, we have that for any \(\epsilon > 0\) there is an \(s \in (0, 1)\) such that
\[
\left(\frac{2\pi}{|J(a)|}\right) \int_{\phi(a)} \rho(\phi \circ \sigma_b(\xi), \phi \circ \sigma_b(a))^2 |d\xi| \geq (1 - |\phi \circ \sigma_b(a)|^2)^{-1-p} < \epsilon \ \forall \ |\phi \circ \sigma_b(a)| > s.
\]

Here we have used the pseudo-hyperbolic distance \(\rho(z, w) = |\sigma_w(z)|\) between \(z, w \in \mathbb{D}\) and the following basic estimate
\[
\left\{
\begin{align*}
|||\sigma_{\phi(b)} \circ \phi \circ \sigma_a||^2 &= \int_{\phi(b)} \rho(\phi(\xi), \phi(a))^2 P_a(\xi) |d\xi|; \\
P_a(\xi) &= |\sigma_a'(\xi)| \geq 2^{-1} \pi |J(a)|^{-1} \ \forall \ \xi \in J(a).
\end{align*}
\right.
\]

On the other hand, we can select two constants \(s_0 \in (0, 1)\) and \(\epsilon_0 > 0\), points \(b_j \in \mathbb{D}\), and numbers \(t_j \in (0, 1)\) with \(\lim_{j \to \infty} t_j = 1\) such that for any \(j = 1, 2, \ldots\) one has \(|\phi(b_j)| \leq s_0\) and
\[
E_j = \{\xi \in T : \phi_j(\xi) = \phi \circ \sigma_{b_j}(\xi)\} \text{ exists as radial limit and } |\phi_j(\xi)| > t_j\]
obeys
\[
\left(\frac{1 - |b_j|^2}{1 - |\phi(b_j)|^2}\right)^{1-q} \left(\frac{1 - |\phi(b_j)|^2}{1 - |\phi(b_j)|^2}\right)^{1-p} |E_j| \geq \epsilon_0.
\]

This \((15)\), plus the above-stated lemma on \((13)\), ensures that one can choose sets \(F_j \subseteq E_j\) such that \(|F_j| > 0\) and
\[
\frac{|J(r\xi) \cap E_j|}{|J(r\xi)|} \geq \frac{\left(\frac{1 - |b_j|^2}{1 - |\phi(b_j)|^2}\right)^{1-q} |E_j|}{\left(\frac{1 - |b_j|^2}{1 - |\phi(b_j)|^2}\right)^{1-p}} \geq \frac{|E_j|}{2^{4\pi}} \geq 2^{-3} \epsilon_0 \ \forall \ r \in [0, 1) \ \& \ \xi \in F_j.
\]
If $\varepsilon = 2^{-4} \varepsilon_0$ in (14), then one can take such $s$ that $s_0 < s < 1$ and (14) is true for $|\phi \circ \sigma_b(a)| > s$. Assuming $t_j \geq s$ and recalling that the definition of $E_j$ ensures

$$|\phi \circ \sigma_{b_j}(r \xi_j)| \rightarrow |\phi \circ \sigma_{b_j}(\xi_j)| > t_j \quad \text{as} \quad r \rightarrow 1 \quad \text{for each} \quad \xi \in E_j.$$ 

Of course, this last property is valid for arbitrarily chosen point $\xi_j \in F_j$. Note that $|\phi \circ \sigma_{b_j}(0)| = |\phi(b_j)| \leq s_0$. Thus, by continuity of $|\phi \circ \sigma_{b_j}|$ there exists an $r_j \in (0, 1)$ such that $|\phi \circ \sigma_{b_j}(r_j \xi_j)| = s$. If $a_j = r_j \xi_j$ then

$$\rho(\phi \circ \sigma_{b_j}(\xi_j), \phi \circ \sigma_{b_j}(a_j)) \geq \rho(t_j, s) \quad \forall \quad \xi \in E_j,$$

and hence (10) and $q = 1$ are applied to deduce

$$\frac{(1 - |\sigma_{b_j}(a_j)|^2)^{1-q}}{(1 - |\phi \circ \sigma_{b_j}(a_j)|^2)^{1-p}} \int_{J(a_j)} \rho(\phi \circ \sigma_{b_j}(\xi), \phi \circ \sigma_{b_j}(a_j))^2 \left(\frac{|d\xi|}{(2\pi)^{-1}|J(a_j)|}\right) 
\geq \left(\frac{(1 - |b_j|^2)^{1-q}}{(1 - |\phi(b_j)|^2)^{1-p}} \right) \left(\frac{1 - |\phi(b_j)|^2}{1 - |\phi \circ \sigma_{b_j}(a_j)|^2}\right)^{1-p} \rho(t_j, s)^2 
\geq 2^{-3} \varepsilon_0 \left(\frac{\min\{1, (1 - s_0^2)^{1-p}\}}{1 - s_2^2}\right) \rho(t_j, s)^2.$$ 

Since $\lim_{j \to \infty} \rho(t_j, s) = 1$, it follows from (14) that

$$0 = \lim_{j \to \infty} \frac{(1 - |\sigma_{b_j}(a_j)|^2)^{1-q}}{(1 - |\phi \circ \sigma_{b_j}(a_j)|^2)^{1-p}} \int_{J(a_j)} \rho(\phi \circ \sigma_{a_j}(\xi), \phi \circ \sigma_{a_j}(b_j))^2 \left(\frac{|d\xi|}{(2\pi)^{-1}|J(a_j)|}\right) 
\geq 2^{-3} \varepsilon_0 \left(\frac{\min\{1, (1 - s_0^2)^{1-p}\}}{1 - s_2^2}\right),$$

a contradiction. In other words, (12) must be true under (4) being valid. □

**Proof of Theorem 2** Suppose that $C_{\phi} : C_{\mathcal{A}_p} \mapsto C_{\mathcal{A}_q}$ is compact. Of course, this operator is bounded, and thus (1) holds. Choosing $b = \phi(a)$, we see that $f_b$ defined in Lemma 1 tends to 0 uniformly on compact subsets of $\mathbb{D}$ whenever $|b| \to 1$. Thus, $\lim_{|b| \to 1} \|C_{\phi} f_b\|_{C_{\mathcal{A}_p}} = 0$. As an immediate by-product of the $C$-part in the proof of Theorem 1 we have

$$0 = \lim_{|b| \to 1} \|C_{\phi} f_b\|_{C_{\mathcal{A}_p}}^2 \leq \lim_{|b| \to 1} \frac{(1 - |d|^2)^{1-q}|b|^2}{(1 - |b|^2)^{1-p}} \|\sigma_b \circ \phi \circ \sigma_a\|_2,$$

whence deriving (4).

Next, we deal with the converse part of Theorem 2 according to $(p, q) \in [0, 2) \times [1, 1]$ and $(p, q) \in (1, 2) \times [0, 2)$. In order to verify that $C_{\phi} : C_{\mathcal{A}_p} \mapsto C_{\mathcal{A}_q}$ is a compact operator, it suffices to check that $\lim_{n \to \infty} \|C_{\phi} f_n\|_{C_{\mathcal{A}_p}} = 0$ holds for any sequence $(f_n)_{n=1}^{\infty}$ in $C_{\mathcal{A}_p}$ with $\|f_n\|_{C_{\mathcal{A}_p}} \leq 1$ and $f_n \to 0$ on compact subsets of $\mathbb{D}$ as $n \to \infty$.
Situation 1 - assume that (1) holds and (4) is valid for $(p, q) \in [0, 2) \times [1, 1]$. Upon writing
\[
\|C_\phi f_n\|_{C(A_1^2, s)}^2 \leq \sup_{|\phi(a)| > s} T(n, a, q) + \sup_{|\phi(a)| \leq s} T(n, a, q),
\]
where
\[
0 < s < 1 \text{ and } T(n, a, q) = (1 - |a|^2)^{-q} \|f_n \circ \phi \circ \sigma_a - f_n \circ \phi(a)\|_{L^2}^2,
\]
we have to control $\sup_{|\phi(a)| > s} T(n, a, q)$ and $\sup_{|\phi(a)| \leq s} T(n, a, q)$ from above. To do so, set
\[
\begin{align*}
&f_{n,a} = f_n \circ \phi \circ \sigma_a - f_n(\phi(a)); \\
g_{n,a} = f_n \circ \sigma_{\phi(a)} - f_n(\phi(a)); \\
&\psi_a = \sigma_{\phi(a)} \circ \phi \circ \sigma_a; \\
&E(\phi, a, t) = \{\xi \in T : |\sigma_{\phi(a)} \circ \phi \circ \sigma_a(\xi)| > t\}.
\end{align*}
\]
Using (9) we obtain
\[
\begin{align*}
\sup_{|\phi(a)| > s} T(n, a, q) &\approx \sup_{|\phi(a)| > s} (1 - |a|^2)^{-q} \|f_{n,a}\|_{L^2}^2 \\
&\leq \sup_{|\phi(a)| > s} (1 - |a|^2)^{-q} \|g_{n,a}\|_{L^2}^2 \|\psi_a\|_{L^2}^2 \\
&\leq \sup_{|\phi(a)| > s} \frac{(1 - |a|^2)^{-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\psi_a\|_{L^2}^2 \|f_n\|_{C(A_1^2, s)}^2 \\
&\leq \sup_{|\phi(a)| > s} \frac{(1 - |a|^2)^{-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\psi_a\|_{L^2}^2,
\end{align*}
\]
whence getting by (4)
\[
(17) \quad \lim_{s \to 1} \sup_{|\phi(a)| > s} T(n, a, q) = 0 \quad \forall \quad n = 1, 2, 3, ...
\]
Meanwhile,
\[
\sup_{|\phi(a)| \leq s} T(n, a, q) \leq \sup_{|\phi(a)| \leq s} T_1(n, a, q) + \sup_{|\phi(a)| \leq s} T_2(n, a, q),
\]
where
\[
\begin{align*}
&T_1(n, a, q) = (1 - |a|^2)^{-q} \int_{E(\phi, a, t)} |f_{n,a}(\xi)|^2 |d\xi|; \\
&T_2(n, a, q) = (1 - |a|^2)^{-q} \int_{E(\phi, a, t)} |f_{n,a}(\xi)|^2 |d\xi|.
\end{align*}
\]
Applying Schwarz’s lemma to $g_{n,a}$ or using (6, (3.19)) we get
\[
\sup_{|z|\leq r} |z|^{-1} |g_{n,a}(z)| \leq 2 \sup_{|w| \leq r} |g_{n,a}(w)|
\]
thereby deriving

\[
\sup_{|\phi(a)| \leq s} T_1(n, a, q) \leq \sup_{|\phi(a)| \leq s} (1 - |a|^2)^{1-q} \sup_{|w| \leq t} |g_{n,a}(w)|^2 \int_T |\psi_a(\xi)|^2 |d\xi|
\]
\[
\leq (1 + (1 - s)^{p-1}) \sup_{|\phi(a)| \leq s} (1 - |a|^2)^{1-q} \sup_{|\phi(a)| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} \|\psi_a\|_2^2
\]
\[
\to 0 \quad \text{as} \quad n \to \infty,
\]

in which \(|\phi(a)| \leq s\) and \(|w| \leq t\) have been used. Also, a combination of (9), (1) and \(q = 1\) gives that if

\[
\begin{align*}
\lambda &= \frac{(a\bar{b} - 1)}{(1 - b\bar{a})}; \\
\tau &= \phi \circ \sigma_a; \\
c &= \sigma_b(a); \\
b &\in \mathcal{D},
\end{align*}
\]

then

\[
\|f_{n,a}\|_{C, \mathcal{A}_p, s}^2 = \sup_{b \in \mathcal{D}} (1 - |b|^2)^{1-q} \|f_n \circ \tau \circ \sigma_b - f_n \circ \tau(b)\|_2^2 \\
\leq \sup_{b \in \mathcal{D}} (1 - |b|^2)^{1-q} \|f_n \circ \sigma_b - f_n \circ \tau(b)\|_2^2 \|\sigma_{\tau(b)} \circ \tau \circ \sigma_b\|_2^2 \\
\leq \|f_n\|_{C, \mathcal{A}_p, s}^2 \sup_{b \in \mathcal{D}} \frac{(1 - |b|^2)^{1-q}}{(1 - |\tau(b)|^2)^{1-p}} \|\sigma_{\tau(b)} \circ \tau \circ \sigma_b\|_2^2 \\
\leq \sup_{c \in \mathcal{D}} \frac{(1 - |\lambda c|^2)^{1-q}}{(1 - |\phi(\lambda c)|^2)^{1-p}} \|\sigma_{\phi(\lambda c)} \circ \phi \circ (\lambda \sigma_c)\|_2^2 \\
\leq \sup_{c \in \mathcal{D}} \frac{(1 - |c|^2)^{1-q}}{(1 - |\phi(c)|^2)^{1-p}} \|\psi_c\|_2^2 < \infty,
\]
and hence from the Cauchy-Schwarz inequality, Lemmas 3-4 and \( q = 1 \) it follows that

\[
\sup_{|\phi(a)| \leq s} T_2(n, a, q) 
\leq \sup_{|\phi(a)| \leq s} (1 - |a|^2)^{1-q} \left( \int_{E(\phi, a, t)} |f_{n, a}(\xi)|^4 |d\xi| \right) \frac{1}{2} |E(\phi, a, t)|^{\frac{1}{2}} 
\leq \sup_{|\phi(a)| \leq s} \left( \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{p-1}} \int_{\mathbb{T}} |f_{n, a}(\xi)|^4 |d\xi| \right) \left( \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} |E(\phi, a, t)| \right)^{\frac{1}{2}} 
\leq (1 + (1 - s^2)^{1-p}) \sup_{|\phi(a)| \leq s} \left( \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} |E(\phi, a, t)| \right)^{\frac{1}{2}} 
\leq \left( \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} |\psi_c|^2 \right) \sup_{|\phi(a)| \leq s} \left( \frac{(1 - |a|^2)^{1-q}}{(1 - |\phi(a)|^2)^{1-p}} |E(\phi, a, t)| \right)^{\frac{1}{2}} 
\rightarrow 0 \quad \text{as} \quad t \rightarrow 1.
\]

Consequently,

\[
(18) \quad \lim_{n \to \infty} \sup_{|\phi(a)| \leq s} T(n, a, q) = 0.
\]

Putting (17) and (18) together, we reach \( \lim_{n \to \infty} \|C_\phi f_n\|_{C \mathcal{A}_{q_0}} = 0 \).

**Situation 2** - assume that (11) holds and (14) is valid for \((p, q) \in (1, 2) \times [0, 2)\). Rewriting

\[
\|C_\phi f_n\|_{C \mathcal{A}_{q_0}}^2 
\leq \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-q} \int_{\mathbb{D}} |f_n^*(w)|^2 N(\phi \circ \sigma_a, w) \, dm(w) 
\leq \sup_{a \in \mathbb{D}} \mathcal{U}(n, a, q, r) + \sup_{a \in \mathbb{D}} \mathcal{V}(n, a, q, r),
\]

where \( 2^{-1} \leq r < 1 \) and

\[
\mathcal{U}(n, a, q, r) = (1 - |a|^2)^{1-q} \int_{|\sigma_a(w)| \leq r} |f_n^*(w)|^2 N(\phi \circ \sigma_a, w) \, dm(w); \\
\mathcal{V}(n, a, q, r) = (1 - |a|^2)^{1-q} \int_{|\sigma_a(w)| > r} |f_n^*(w)|^2 N(\phi \circ \sigma_a, w) \, dm(w),
\]

we have to control \( \sup_{a \in \mathbb{D}} \mathcal{U}(n, a, q, r) \) and \( \sup_{a \in \mathbb{D}} \mathcal{V}(n, a, q, r) \) for an appropriate \( r \in [2^{-1}, 1) \). In the sequel, let \( b = \phi(a) \).

**Sub-situation 1 - estimate for \( \sup_{a \in \mathbb{D}} \mathcal{U}(n, a, q, r) \).** For this, we consider two cases for any given \( s \in (0, 1) \).
Case 1: \(|b| \leq s\). Under this case, \(|\sigma_b(w)| \leq r\) ensures that \(w\) belongs to a compact subset \(K\) of \(\mathbb{D}\), and therefore, it follows from \(f_n \to 0\) on any compact subset of \(\mathbb{D}\) and (11) that \(\lim_{n \to \infty} \sup_{w \in K} |f_n'(w)| = 0\) and consequently,

\[
\lim_{n \to \infty} \sup_{|\theta| \leq s} (1 - |a|^2)^{1-q} \int_{|\theta| \leq s} |f_n'(w)|^2 N(\phi \circ \sigma_a, w) \, dm(w) = 0.
\]

\[
\lim_{n \to \infty} \sup_{|\theta| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \int_{|\theta| \leq s} |f_n'(w)|^2 N(\phi \circ \sigma_a, w) \, dm(w) 
\leq \left( \lim_{n \to \infty} \sup_{w \in K} |f_n'(w)|^2 \right) \sup_{|\theta| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \int_{|\theta| \leq s} N(\phi \circ \sigma_a, w) \, dm(w)
\]

\[
\leq \left( \lim_{n \to \infty} \sup_{w \in K} |f_n'(w)|^2 \right) \sup_{|\theta| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \int_{\mathbb{D}} N(\sigma_b \circ \phi \circ \sigma_a, z) \, dm(z)
\]

\[
\leq \left( \lim_{n \to \infty} \sup_{w \in K} |f_n'(w)|^2 \right) \sup_{|\theta| \leq s} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \|\sigma_b \circ \phi \circ \sigma_a\|_2^2
\]

\[
= 0.
\]

Case 2: \(|b| > s\). Using (3) we get

\[
\sup_{|\theta| > s} (1 - |a|^2)^{1-q} \int_{|\theta| \leq s} |f_n'(w)|^2 N(\phi \circ \sigma_a, w) \, dm(w)
\]

\[
\leq \|f_n\|_{C, \mathbb{D}_r}^2 \sup_{|\theta| > s} (1 - |a|^2)^{1-q} \int_{|\theta| \leq s} N(\sigma_b \circ \phi \circ \sigma_a, \sigma_b(w)) \frac{dm(w)}{(1 - |w|^2)^{3-p}}
\]

\[
\leq \sup_{|\theta| > s} (1 - |a|^2)^{1-q} \int_{|\theta| \leq s} (1 - |\sigma_b(z)|^2)^{p-1} N(\sigma_b \circ \phi \circ \sigma_a, z) \frac{dm(z)}{(1 - |z|^2)^2}
\]

\[
\leq \sup_{|\theta| > s} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \int_{|\theta| \leq s} N(\sigma_b \circ \phi \circ \sigma_a, z) \frac{dm(z)}{(1 - |z|^2)^2}
\]

\[
\leq \left(1 - r^2\right)^{-2} \sup_{|\theta| > s} \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \|\sigma_b \circ \phi \circ \sigma_a\|_2^2
\]

\[
\to 0 \quad \text{as} \quad s \to 1.
\]

Putting the above two cases together, we see that for any \(\epsilon \in (0, 1)\) there are two real numbers: \(r_0 \in [2^{-1}, 1)\); \(s_0 \in (0, 1)\), and a natural number \(n_0\) such that \(n \geq n_0\)

\[
(19) \quad \sup_{a \in \mathbb{D}} U(a, a, q, r_0) \leq \sup_{|\theta| \leq s_0} U(a, a, q, r_0) + \sup_{|\theta| > s_0} U(a, a, q, r_0) < \epsilon.
\]

**Sub-situation 2 - estimate for** \(\sup_{a \in \mathbb{D}} V(a, a, q, r)\). Like Sub-situation 1, two treatments are required.
Case 2: $|b| \leq s$. For this case, we need the following by-product of Lemma 2.1: if $\psi$ is an analytic self-map of $\mathbb{D}$ with $\psi(0) = 0$ then

\[(20) \quad W = \sup_{0 < |w| < 1} |w|^2 N(\psi, w) < \infty \implies \sup_{2^{-1} \leq |w| < 1} \frac{N(\psi, w)}{\ln |w|^{-1}} \leq 4(\ln 2)^{-1} W.\]

Note that (1) and (4) imply respectively

\[(21) \quad \sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-q} \sup_{0 < |w| < 1} |w|^2 N(\sigma_b \circ \phi \circ \sigma_a, w) < \infty\]

and

\[(22) \quad \lim_{|b| \to 1} (1 - |b|^2)^{1-p} \sup_{0 < |w| < 1} |w|^2 N(\sigma_b \circ \phi \circ \sigma_a, w) = 0\]

thanks to the following (10)-based mean value estimate for $N(\sigma_w \circ \sigma_b \circ \phi \circ \sigma_a, 0)$ where $0 < |w| < 1$ (cf. [6, (2.9)]):

\[
|w|^2 N(\sigma_b \circ \phi \circ \sigma_a, w) = |w|^2 N(\sigma_w \circ \sigma_b \circ \phi \circ \sigma_a, 0) \\
\leq \int_{|z| < |w|} N(\sigma_w \circ \sigma_b \circ \phi \circ \sigma_a, z) \, dm(z) \\
\leq \int_{\mathbb{D}} N(\sigma_b \circ \phi \circ \sigma_a, \sigma_w(z)) \, dm(z) \\
\approx \int_{\mathbb{D}} |\sigma_w'(z)|^2 N(\sigma_b \circ \phi \circ \sigma_a, z) \, dm(z) \\
\approx \|\sigma_w \circ \sigma_b \circ \phi \circ \sigma_a - \sigma_w \circ \sigma_b \circ \phi \circ \sigma_a(0)\|_2^2 \\
\leq \|\sigma_b \circ \phi \circ \sigma_a\|_2^2.
\]
Thus, a combination of (20)-(21)-(22) and Hölder’s inequality gives

\[
\sup_{|b| \leq s} (1 - |a|^2)^{-q} \int_{|\sigma_n(w)| > r} |f_n'(w)|^2 N(\phi \circ \sigma_a, w) \, dm(w) \\
\approx \sup_{|b| \leq s} (1 - |a|^2)^{-q} \int_{|\sigma_n(w)| > r} |f_n'(w)|^2 N(\phi \circ \sigma_a, \sigma_b(w)) \, dm(w) \\
\leq \sup_{|b| \leq s} (1 - |a|^2)^{-p} \int_{|\sigma_n(w)| > r} |f_n'(w)|^2 N(\sigma_b, w) \, dm(w) \\
\leq \left(1 + (1 - s^2)^{1-p}\right) \int_{|\sigma_n(w)| > r} |f_n'(w)|^2 N(\sigma_b, w) \, dm(w) \\
\leq \left(1 + (1 - s^2)^{1-p}\right) \int_{|\sigma_n(w)| > r} |(f_n \circ \sigma_b)'(z)|^2 (1 - |z|^2) \, dm(z) \\
\leq \left(1 + (1 - s^2)^{1-p}\right) \left( \int_{|\sigma_n(w)| > r} |f_n'(w)|^2 (1 - |z|^2)^{4-p} \, dm(z) \right)^{1/2}.
\]

Since \(|f_n|_{C,\mathcal{H}_{p,\ast}} \leq 1\) and \(|b| \leq s < 1\) ensure \(|f_n \circ \sigma_b|_{C,\mathcal{H}_{p,\ast}} \leq 1\), one concludes that \(|(f_n \circ \sigma_b)'(z)|^2 (1 - |z|^2) \, dm(z)\) is \(p\)-Carleson measure with norm relying on \(s\) and so that \(d\mu_n(z) = |(f_n \circ \sigma_b)'(z)|^2 (1 - |z|^2)^{4-p} \, dm(z)\) is \(3\)-Carleson measure with norm relying on \(s\). Now, it follows from [16, Theorem 1.2] that

\[
\int_{|z| > r} |(f_n \circ \sigma_b)'(z)|^2 (1 - |z|^2)^{4-p} \, dm(z) \\
= \int_{|z| > r} |(f_n \circ \sigma_b)'(z)|^2 \, d\mu_n(z) \\
\leq \|\mu_n\|_{CM} \int_{D} |(f_n \circ \sigma_b)'(z)|^2 (1 - |z|^2) \, dm(z) \\
\leq \|f_n\|^4_{C,\mathcal{H}_{p,\ast}} \\
\leq 1.
\]

Note that

\[
\lim_{r \to 1} \int_{|z| > r} (1 - |z|^2)^{p-2} \, dm(z) = 0.
\]

So

\[
\limsup_{r \to 1} \sup_{|b| \leq s} (1 - |a|^2)^{-q} \int_{|\sigma_n(w)| > r} |f_n'(w)|^2 N(\phi \circ \sigma_a, w) \, dm(w) = 0
\]

holds for any \(n = 1, 2, 3, \ldots\).
Case 2: $|b| > s$. Since (22) implies that for any $\epsilon \in (0, 1)$ there is an $s_0 \in (0, 1)$ such that

$$|b| > s_0 \implies \frac{(1 - |a|^2)^{1-q}}{(1 - |b|^2)^{1-p}} \sup_{0 < |w| < 1} |w|^2 N(\sigma_b \circ \phi \circ \sigma_a, w) < \epsilon.$$ 

Thus, (20) is applied once again to deduce that

$$N(\phi \circ \sigma_a, w) = N(\sigma_b \circ \phi \circ \sigma_a, \sigma_b(w)) \leq \epsilon \frac{(1 - |b|^2)^{1-p}}{(1 - |a|^2)^{1-q}} \ln |\sigma_b(w)|^{-1} \approx \epsilon \frac{(1 - |b|^2)^{1-p}}{(1 - |a|^2)^{1-q}} N(\sigma_b, w)$$

as $|\sigma_b(w)| > r > 2^{-1}$.

Consequently,

$$\sup_{|b| > s_0} (1 - |a|^2)^{1-q} \int_{|\sigma_b(w)| > r} |f_n^r(w)|^2 N(\phi \circ \sigma_a, w) \, dm(w)$$

$$\leq \epsilon \sup_{|b| > s_0} (1 - |b|^2)^{1-p} \int_{|\sigma_b(w)| > r} |f_n^r(w)|^2 N(\sigma_b, w) \, dm(w)$$

$$\leq \epsilon \|f_n^r\|_{C_{\mathcal{A}_{\ast}}}^2 \leq \epsilon.$$

The previous discussions on Case 2 and Case 2 indicate

$$\lim_{n \to \infty} \sup_{a \in \mathbb{D}} V(n, a, q, r) = 0.$$ 

Obviously, (19) and (23) give $\lim_{n \to \infty} \|C_\phi f_n^r\|_{\mathcal{A}_{\ast}} = 0$.

Proof of Corollary 2. This follows from (7), Theorem 2 and [20, Theorem 1.4 (c)].

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