A subexponential bound on the cardinality of abelian quotients in finite transitive groups

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Abstract

We show that, for every transitive permutation group $G$ of degree $n \geq 2$, the largest abelian quotient of $G$ has cardinality at most $4^n/\sqrt{\log_2 n}$. This gives a positive answer to a 1989 outstanding question of László Kovács and Cheryl Praeger.

1. Introduction

László Kovács and Cheryl Praeger [5] have investigated large abelian quotients in arbitrary permutation groups of finite degree. Their work was motivated by recent (at that time) investigations on minimal permutation representations of a finite group [2]. One of the main results in [5] (which is independently proved in [1]) shows that, for every permutation group of degree $n$, the largest abelian quotient has order at most $3^{n/3}$. Clearly, this bound is attained, whenever $n$ is a multiple of 3, by an elementary abelian 3-group of order $3^{n/3}$ having all of its orbits of cardinality 3. Furthermore, the authors conjecture that, for transitive groups of degree $n$, a subexponential bound in $n(\log_2 n)^{-1/2}$ holds. More history on this conjecture and more details can be found in the survey paper [8].

The first substantial evidence towards the conjecture goes back to the work of Aschbacher and Guralnick [1]; they proved the striking result that the largest abelian quotient of a primitive group of degree $n$ has order at most $n$. In the concluding remarks, the authors also independently ask whether one can obtain a subexponential bound on the order of abelian quotients of transitive groups in terms of their degrees. We refer to [1, 8] for an infinite family of transitive groups $G$ of degree $n$ with $|G/G'|$ asymptotic to $\exp(bn/\sqrt{\log_2 n})$, for some constant $b$.

The second substantial evidence towards the conjecture is in [4], where many of the results in Section 7 get very close to the desired upper bound. In particular, [4, Theorem 7.6] says that if $G$ is a transitive permutation group of degree $n \geq 2$ and $N < G$ is a still transitive normal subgroup of $G$, then the product of the orders of the abelian composition factors of $G/N$ is at most $4^n/\sqrt{\log_2 n}$.

In this paper, we settle in the affirmative the conjecture of Kovács and Praeger.

Theorem 1. For every positive integer $n \geq 2$ and for every transitive permutation group $G$ of degree $n$, we have

$$|G/G'| \leq 4^n/\sqrt{\log_2 n}.$$
The constant 4 in Theorem 1 should not be taken too seriously, but it seems remarkably hard to pin down the exact constant. The choice of the constant 4 in our work is a compromise: it makes the statement of Theorem 1 explicit and valid for every \( n \geq 2 \).

2. Preliminaries

Unless otherwise explicitly stated, all the logarithms are to base 2. Given a field \( \mathbb{F} \), a group \( G \), a subgroup \( H \) of \( G \) and an \( \mathbb{F}H \)-module \( W \) (or simply \( H \)-module), we denote by \( W \uparrow^G_H \) the induced \( G \)-module of \( W \) from \( H \) to \( G \), that is, \( W \uparrow^G_H := W \otimes_{\mathbb{F}H} \mathbb{F}G \). Moreover, given a \( G \)-module \( M \), we denote by \( d_G(M) \) the minimal number of generators of \( M \) as a \( G \)-module.

We are ready to report a fundamental result from [7].

**Lemma 2.1** (See [7, Lemma 4]). There is a universal constant \( b' \) such that whenever \( H \) is a subgroup of index \( n \geq 2 \) in a finite group \( G \), \( \mathbb{F} \) is a field, \( V \) is an \( H \)-module of dimension \( a \) over \( \mathbb{F} \) and \( M \) is a \( G \)-submodule of the induced module \( V \uparrow^G_H \), then

\[
\begin{align*}
d_G(M) &\leq ab' n \sqrt{\log n}.
\end{align*}
\]

**Remark 2.2.** Gareth Tracey, in his monumental work [10] on minimal sets of generators of transitive groups, has refined Lemma 2.1 in various directions. For instance, [10, Section 4] gives a more quantitative form of Lemma 2.1. Indeed, using the notation in Lemma 2.1, from [10, Corollary 4.27 (iii)], we deduce

\[
\begin{align*}
d_G(M) &\leq aE(n,p) \leq \begin{cases} an \frac{2}{\sqrt{\pi} \log n} & \text{when } 2 \leq n \leq 1260, \\
\frac{an}{\sqrt{\pi} \log n} & \text{when } n > 1261,
\end{cases}
\end{align*}
\]

where \( c' := 0.552282 \), \( p \) is the characteristic of \( M \) and \( E(n,p) \) is explicitly defined in [10, Section 4]. In particular, we immediately see that in Lemma 2.1 we may take \( b' := 2/\sqrt{\pi} \) whenever \( n > 1261 \). With the help of a computer, we have implemented the function \( E(n,p) \) and we have checked that \( E(n,p) \leq 2n/\sqrt{\pi} \log n \) also when \( n \leq 1260 \). Therefore, in Lemma 2.1 we may take \( b' := 2/\sqrt{\pi} \).

Let \( R \) be a finite group. For each prime number \( p \), let \( a_p(R) \) be the number of abelian composition factors of \( R \) of order \( p \), and let

\[
a(R) := \sum_{p \text{ prime}} a_p(R) \log p.
\]

We now report a useful result of Pyber.

**Lemma 2.3** (See [9, Theorem 2.10]). Let \( c_0 := \log_9(48 \cdot 24^{1/3}) \). The product of the orders of the abelian composition factors of a primitive permutation group of degree \( r \) is at most \( 24^{-1/3} r^{1+c_0} \).

From Lemma 2.3, we deduce the following.

**Lemma 2.4.** Let \( R \) be a primitive group of degree \( r \) and let \( c_0 \) be the constant in Lemma 2.3. Then

\[
a(R) \leq (1 + c_0) \log r - \log(24)/3.
\]
Proof. By definition, the product of the orders of the abelian composition factors of $R$ is
\[
\prod_{p \text{ prime}} p^{\omega_p(R)} = \prod_{p \text{ prime}} 2^{\omega_p(R) \log p} = 2^{a(R)}.
\]
From Lemma 2.3, this number is at most $2^{4^{-1/3}r^{1+c_0}}$. The proof follows by taking logarithms.

Notice that Lemma 2.3 is often used in order to bound the composition length of a primitive permutation groups. A more precise bound on this composition length has been recently proved by Glasby, Praeger, Rosa and Verret [3, Theorem 1.3]. However, this stronger bound is not sufficient for our application, which requires information not only on the number of the composition factors but also on their order.

Finally, given a finite group $G$, we denote by $G_{ab}$ the quotient group $G/G'$.  

3. Proof of Theorem 1

Let $R$ be a finite group, let $\Delta$ be a finite set and let $W := R \wr_\Delta \text{Sym}(\Delta)$ be the wreath product of $R$ via $\text{Sym}(\Delta)$. We denote by
\[
\pi : W \to \text{Sym}(\Delta)
\]
the projection of $W$ over the top group $\text{Sym}(\Delta)$. Let $\prod_{\delta \in \Delta} R_\delta$ be the base subgroup of $W$ and, for each $\delta \in \Delta$, consider $W_\delta := N_W(R_\delta)$. As $W_\delta = R_\delta \times R \wr \text{Sym}(\Delta \setminus \{\delta\})$, we may consider the projection $\rho_\delta : W_\delta \to R_\delta$. Using this notation, we adapt the proof of [6, Lemma 2.5] to prove the following.

**Lemma 3.1.** Let $R$ be a finite group, let $\Delta$ be a set of cardinality at least 2 and let $G$ be a subgroup of the wreath product $R \wr_\Delta \text{Sym}(\Delta)$ with the properties

1. $\pi(G)$ is transitive on $\Delta$,
2. $\rho_\delta(N_G(R_\delta)) = R_\delta$, for every $\delta \in \Delta$.

Then
\[
\log |G_{ab}| \leq \frac{a(R)b'|\Delta|}{\sqrt{\log |\Delta|}} + \log |(\pi(G))_{ab}|,
\]
where $b'$ is the absolute constant appearing in Lemma 2.1, and $a(R)$ is defined in Section 2.

**Proof.** We argue by induction on the order of $R$. When $|R| = 1$, there is nothing to prove because $\pi(G) \cong G$ and hence $\log |G_{ab}| = \log |(\pi(G))_{ab}|$. Suppose then $R \neq 1$. We write
\[
|G_{ab}| = |G : G'M||G'M : G'| = |(G/M)_{ab}| |M : M \cap G'|.
\]  \hspace{1cm} (3.1)

Let $L$ be a minimal normal subgroup of $R$. Fix $\delta_0 \in \Delta$. We identify $L$ with a normal subgroup $L_{\delta_0}$ of the direct factor $R_{\delta_0}$ of the base group $\prod_{\delta \in \Delta} R_\delta$ of $W$. Let $B_L$ be the direct product of the distinct $G$-conjugates of $L_{\delta_0}$ and consider $M := B_L \cap G$. We have $M \leq G$ and
\[
\frac{G}{M} = \frac{G}{B_L \cap G} \cong \frac{GB_L}{B_L}.
\]
Now, from (1), we deduce that $GB_L/B_L$ is isomorphic to a subgroup of the wreath product $(R/L) \wr_\Delta \text{Sym}(\Delta)$. 

Therefore, by induction,
\[
\log |(G/M)_{ab}| \leq \frac{a(R/L)b'|\Delta|}{\sqrt{\log |\Delta|}} + \log |(\pi(G))_{ab}|. \tag{3.2}
\]

We now distinguish two cases.

L IS NON-ABELIAN:
Since \( M \leq W_{\delta_0} \cap G \), we deduce \( \rho_{\delta_0}(M) \leq \rho_{\delta_0}(W_{\delta_0} \cap G) \). From (2), we have \( \rho_{\delta_0}(W_{\delta_0} \cap G) = \rho_{\delta_0}(N_G(R_{\delta_0})) = R_{\delta_0} \) and hence \( \rho_{\delta_0}(M) \leq R_{\delta_0} \). Observe that \( \rho_{\delta_0}(M) \) is contained in \( L_{\delta_0} \). As \( L_{\delta_0} \) is a minimal normal subgroup of \( R_{\delta_0} \), we get either \( \rho_{\delta}(M) = 1 \) or \( \rho_{\delta}(M) = L_{\delta_0} \). From (1), \( \pi(G) \) is transitive on \( \Delta \) and hence either \( \rho_{\delta}(M) = 1 \) for each \( \delta \in \Delta \), or \( \rho_{\delta}(M) = L_{\delta} \) for each \( \delta \in \Delta \).

Suppose \( \rho_{\delta_0}(M) = 1 \). As \( \rho_{\delta}(M) = 1 \) for each \( \delta \in \Delta \), we get \( M = 1 \). Now the proof immediately follows from (3.2) because \( G/M \cong G \).

Suppose \( \rho_{\delta_0}(M) = L_{\delta_0} \). Then \( M \) is a subdirect product of \( L_{\Delta} = \prod_{\delta \in \Delta} L_{\delta} \). As \( L \) is a non-abelian minimal normal subgroup of \( R \), we deduce that \( M \) is a direct product of non-abelian simple groups. Thus \( M \) has no abelian composition factor and hence (3.1) gives \( |G_{ab}| = |(G/M)_{ab}| \). Moreover, \( a(R/L) = a(R) \), and hence, once again, the proof immediately follows from (3.2).

L IS ABELIAN:
As \( L \) is a minimal normal subgroup of \( R \), it is an elementary abelian \( p_0 \)-group, for some prime number \( p_0 \). Let \( a_{p_0} \) be the composition length of \( L \). In particular,
\[
a(R) = a(R/L) + a_{p_0} \log p_0.
\]

The group \( B_L \) is abelian and the action of \( G \) by conjugation on \( B_L \) endows \( B_L \) with a natural structure of \( G \)-module. From its definition, as \( G \)-module, \( B_L \) is isomorphic to the induced module
\[
L_{\delta_0} \uparrow^G_K,
\]
where \( K := N_G(L_{\delta_0}) \). From (1), \( G \) acts transitively on \( \Delta \) and hence \( |\Delta| = |G : N_G(L_{\delta_0})| = |G : K| \). From Lemma 2.1, we deduce
\[
d_G(M/(M \cap G')) \leq d_G(M) \leq \frac{a_{p_0} b'|\Delta|}{\sqrt{\log |\Delta|}}.
\]

However, as \( G \) acts trivially by conjugation on \( M/(M \cap G') \), we get that \( d_G(M/(M \cap G')) \) is just the dimension of \( M/(M \cap G') \) as a vector space over the prime field \( \mathbb{Z}/p_0\mathbb{Z} \). Therefore,
\[
|M : M \cap G'| \leq p_0^{a_{p_0} b'|\Delta|/\sqrt{\log |\Delta|}}. \tag{3.3}
\]

From (3.1), (3.2) and (3.3), we get
\[
\log |G_{ab}| \leq \log |(G/M)_{ab}| + \log |M : M \cap G'|
\leq \frac{a(R/L)b'|\Delta|}{\sqrt{\log |\Delta|}} + \log |(\pi(G))_{ab}| + \log(p_0) \frac{a_{p_0} b'|\Delta|}{\sqrt{\log |\Delta|}}
= (a(R/L) + a_{p_0} \log p_0) \frac{b'|\Delta|}{\sqrt{\log |\Delta|}} + \log |(\pi(G))_{ab}|.
\]

With Lemma 3.1 in hand, we prove Theorem 1 by induction on \( n \).
Let $G$ be a transitive permutation group of degree $n \geq 2$. From the main result of [5], we have $|G_{ab}| \leq 3^{n/3}$. Now the inequality $3^{n/3} \leq 4^{n/\sqrt{\log n}}$ is satisfied for each $n \leq 20,603$. In particular, for the rest of the proof, we may suppose that $n \geq 20,604$.

Suppose first that $G$ is primitive. In this case, from [1], we have $|G_{ab}| \leq n$ and the inequality $n \leq 4^{n/\sqrt{\log n}}$ follows with an easy computation.

Suppose now that $G$ is imprimitive and let $\Omega$ be the domain of $G$. Among all non-trivial blocks of imprimitivity of $G$, choose one (say $\Lambda$) minimal with respect to the inclusion. Let $G_{(\Lambda)} := \{ g \in G \mid \Lambda^g = \Lambda \}$ be the setwise stabilizer of $\Lambda$ in $G$ and let $R \leq \text{Sym}(\Lambda)$ be the permutation group induced by $G_{(\Lambda)}$ in its action on $\Lambda$. The minimality of $\Lambda$ yields that $R$ acts primitively on $\Lambda$.

Let $\Delta := \{ \Lambda^g \mid g \in G \}$ be the system of imprimitivity determined by the block $\Lambda$. Then $G$ is a subgroup of the wreath product $R \wr \Delta \text{Sym}(\Delta)$.

We now use the notation of Lemma 3.1 for wreath products. In particular, let $\pi : R \wr \Delta \text{Sym}(\Delta) \to \text{Sym}(\Delta)$ be the projection onto the top group $\text{Sym}(\Delta)$ and for each $\delta \in \Delta$, let $R_{\delta}$ be the direct factor of the base group $\prod_{\delta \in \Delta} R_{\delta}$ corresponding to $\delta$. From the fact that $G$ acts transitively on $\Omega$ and from the definition of $R$, we get that the two hypotheses (1) and (2) are satisfied. Therefore, from Lemma 3.1 itself, we deduce

$$\log |G_{ab}| \leq \frac{a(R)b(|\Delta|)}{\sqrt{\log(|\Delta|)}} + \log |\pi(G)_{ab}|.$$  

Set $r := |\Lambda|$. Thus $|\Delta| = n/r$. From Lemma 2.4 and from induction (as $n/r < n$), we get

$$\log |G_{ab}| \leq \frac{b'(n/r)}{\sqrt{\log(n/r)}} \left( 1 + c_o \log r - \frac{\log(24)}{3} \right) + 2 \frac{(n/r)}{\sqrt{\log(n/r)}}.$$  

From Remark 2.2, we see that we may take $b' = 2/\sqrt{\pi}$. Now, for $n \geq 20,604$, a careful calculation shows that the right-hand side of (3.4) is at most $2n/\sqrt{\log n}$ for every divisor $r$ of $n$ with $4 < r < n$.

We now discuss the cases $r \in \{2, 3, 4\}$ separately. When $r = 2$, we have $a(R) = 1$ and hence

$$\log |G_{ab}| \leq \frac{b'(n/2)}{\sqrt{\log(n/2)}} + 2 \frac{(n/2)}{\sqrt{\log(n/2)}}.$$  

Now, the right-hand side of (3.5) is less than $2n/\sqrt{\log n}$ for each $n \geq 20,604$. The computation when $r \in \{3, 4\}$ is analogous using $a(R) \leq 1 + \log(3)$ when $r = 3$, and $a(R) \leq 3 + \log(3)$ when $r = 4$.

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