The lessons from the running of the tensor-to-scalar ratio

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Abstract

We derive a simple consistency relation from the running of the tensor-to-scalar ratio. This new relation is first order in the slow-roll approximation. While for single field models we can obtain what can be found by using other observables, multi-field cases in general give non-trivial contributions dependent on the geometry of the field space and the inflationary dynamics, which can be probed observationally from this relation. The running of the tensor-to-scalar ratio may be detected by direct laser interferometer experiments.

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Inflation [1] is supposed to be the most promising candidate to solve many cosmological problems such as horizon problem, and to provide the initial conditions for the subsequent standard hot big bang universe [2]. Among many successes of inflation, the nearly scale invariant density perturbations on the scales of the cosmic microwave background (CMB) are believed to be one of the most solid predictions of inflation. They were first probed by the Cosmic Background Explorer (COBE) satellite [3], and indeed in excellent agreement with most recent observations [4]. In the standard picture, the origin of these scale invariant perturbations is the vacuum fluctuations of one or more scalar fields which dominate the energy density of the universe during inflation, the inflaton fields.

There exists, however, another source of the CMB temperature anisotropies produced during inflation. The tensor perturbations, which emerge as the primordial gravitational waves [5], are believed to contribute to the CMB anisotropies at low multipoles up to \( l \lesssim 100 \). There are several reasons why the gravitational waves, and the detection of them, are potentially very important. One reason should be that from the amplitude of the gravitational waves, or the tensor-to-scalar ratio \( r \equiv \mathcal{P}_T/\mathcal{P}_S \), we can directly determine the inflationary energy scale [6]. Another reason which becomes popular recently is that, in many models of string inflation based on the de Sitter vacua constructed by flux compactification [7], the level of the gravitational waves are expected to be extremely small to be ever detected, \( r \lesssim 10^{-24} \) or so [8]. Meanwhile the current bound is only \( r \lesssim 0.2 \). In single field models, detectable gravitational waves means a field variation of \( \mathcal{O}(m_{\text{Pl}}) \) with \( m_{\text{Pl}} \equiv (8\pi G)^{-1/2} \approx 2.4 \times 10^{18} \text{GeV} [9] \). Constructing a stringy model of inflation with such a large field variation and in turn detectable gravitational waves remains an open challenge [10]. Moreover, it is expected that by near future CMB observations the sensitivity will be improved at most to a level of \( r \sim \mathcal{O}(10^{-2}) \). Thus it is very important to fully understand the implications of the primordial gravitational waves regarding the observations.

In this note we explore another aspect of the tensor perturbations. We point out that we can obtain a new consistency relation from the running of the tensor-to-scalar ratio. Using this relation, we might be enabled to observationally discriminate multi-field models from single field ones. More importantly, for multi-field cases, this consistency relation opens a new possibility to study the geometry of the field space and the dynamics during inflation.

Irrespective of inflation model driven by single or multiple number of fields, the spectrum of the primordial gravitational waves produced during inflation is given by [5]

\[
\mathcal{P}_T = \frac{8}{m_{\text{Pl}}^2} \left( \frac{H}{2\pi} \right)^2, \tag{1}
\]

and the dependence on the wavenumber \( k \) is described by

\[
\mathcal{P}_T \propto k^{n_T}, \tag{2}
\]

where the corresponding spectral index \( n_T \) is

\[
n_T = -2\epsilon, \tag{3}
\]

with \( \epsilon \equiv -\dot{H}/H^2 \) being the usual slow-roll parameter. This is because the equation of motion of each polarization state of the gravitational waves is exactly that of massless scalar field [11] and is thus independent of the detail of the inflation model under consideration.
Meanwhile, the spectrum and the spectral index of scalar perturbations do depend on the
detailed inflationary dynamics. More specifically, we have
\[ P_S = \left( \frac{H}{2\pi} \right)^2 \left( \frac{H}{\dot{\phi}} \right)^2, \]
(4)
and
\[ P_S = \left( \frac{H}{2\pi} \right)^2 h_{ij} N_i N_j, \]
(5)
for the cases of single and multiple fields, respectively. In Eq. (5), \( h_{ij} \) is the metric of the field
space, \( N \equiv \int H dt \) is the number of e-folds, and \( N_i \equiv \partial N / \partial \phi^i \). The corresponding spectral
indices which will be shown soon are different, but nevertheless we can write the dependence
of \( P_S \) on \( k \) as
\[ P_S \propto k^{n_{S}-1}. \]
(6)
Combining Eqs. (2) and (6), \( r \) depends on \( k \) as
\[ r \propto k^{1-n_{S}+n_{T}}, \]
(7)
so that the running of \( r \) is
\[ \frac{d \log r}{d \log k} = 2(2\epsilon - \eta). \]
(8)
We stress that this relation is valid for all the inflationary models, irrespective of driven by
single or multiple number of fields. Moreover, it is first order in the slow-roll approximation
and does not involve any model dependent parameter. The implications Eq. (8) suggests for
single and multi-field inflation, however, are quite different.

For single field case, it is well known that the spectral index is\[ n_{S} - 1 = -6\epsilon + 2\eta, \]
(9)
where
\[ \eta \equiv m_{\text{Pl}}^2 \frac{V''}{V}, \]
(10)
is another slow-roll parameter. Thus, from Eq. (8),
\[ \frac{d \log r}{d \log k} = 2(2\epsilon - \eta). \]
(11)
Note that the same factor of \( 2\epsilon - \eta \) can be derived from a higher order version of the consistency relation [16]. The relation here is, however, first order in the slow-roll parameters as we

\[ \text{Note that if we allow post-inflationary generation of perturbation via e.g. the curvaton mechanism [12], } P_S \text{ occupies only some or even negligible fraction of the total scalar perturbations, and in turn we always obtain smaller tensor-to-scalar ratio [13]. This jeopardizes the possibility of using the ‘standard’ consistency relation } r = 16\epsilon \text{ to distinguish between single and multi-field inflation models. In the present note, however, we do not consider any post-inflationary production of perturbation.} \]

\[ \text{It is noticeable that while one can proceed the calculation with corrections up to arbitrary power of the slow-roll parameters [14], more general slow-roll conditions [15] may give rise to, for example, large enough running of the scalar spectral index. This is another observationally interesting possibility.} \]
emphasized. This also provides some information in the light of a classification scheme of the models of inflation \[17\]. However, we can extract exactly the same conclusion by combining $\mathcal{P}_S$, $\mathcal{P}_T$ (or $r$) and $n_S$. Measuring $d \log r / d \log k$ will give a consistency check on the single field model and therefore may be worth in that sense, but there is nothing new.

The situation is, however, completely different for multi-field inflation models. When multiple number of light fields give rise to inflation, following $\delta N$ formalism \[18\], the spectral index is given by

$$n_S - 1 = -2\epsilon - \frac{r}{4} + 2\eta_{\text{multi}} - \frac{2N_iN_j}{h^{kl}N_kN_l} \frac{R^i_{ab} \dot{\phi}^a \dot{\phi}^b}{3m_{\text{Pl}}^2 H^2},$$

(12)

where $R^i_{jkl}$ is the Riemann curvature tensor of the field space,

$$R^i_{jkl} = \Gamma^i_{jk,l} - \Gamma^i_{jl,k} + \Gamma^m_{jk} \Gamma^i_{lm} - \Gamma^m_{jl} \Gamma^i_{km},$$

(13)

with

$$\Gamma^i_{jk} \equiv \frac{1}{2} h^{il} (h_{jl,k} + h_{kl,j} - h_{jk,l})$$

(14)

being the Christoffel symbol constructed by $h_{ij}$, and we have defined, à la Eq. \[10\],

$$\eta_{\text{multi}} \equiv m_{\text{Pl}}^2 \frac{N_iN_j}{h^{kl}N_kN_l} \frac{V^{ij}}{V},$$

(15)

with a semicolon denoting a covariant derivative in the field space. Therefore, we obtain

$$\frac{d \log r}{d \log k} - \frac{r}{4} = -2\eta_{\text{multi}} + \frac{2N_iN_j}{h^{kl}N_kN_l} \frac{R^i_{ab} \dot{\phi}^a \dot{\phi}^b}{3m_{\text{Pl}}^2 H^2},$$

(16)

where on the left hand side are only observable quantities. We can see that from Eqs. \[3\] and \[12\] the common term $-2\epsilon$ disappears and we are left with the ones which depend purely on the geometry of the inflaton field space and the inflationary dynamics. It is very important to note that in contrast to the single field case, we cannot obtain these terms with only $\mathcal{P}_S$, $\mathcal{P}_T$ and $n_S$. Therefore measuring the running of the tensor-to-scalar ratio can be another way of distinguishing between single and multi-field cases observationally, and further provide a potential probe of the underlying theory of inflation.

An immediate non-trivial contribution appears when the field space is curved. For example, let us consider the Kähler potential

$$K = -3m_{\text{Pl}}^2 \log (T + T^*) ,$$

(17)

which is typical for string moduli. Writing the complex modulus field $T$ as a sum of two real fields $X$ and $Y$, i.e. $T = X + iY$, from the kinetic term

$$\frac{\partial^2 K}{\partial \Phi^I \partial \Phi^* J} \partial^\mu \Phi^I \partial_\mu \Phi^* J = \frac{1}{2} h_{ij} \partial^\mu \phi^i \partial_\mu \phi^j ,$$

(18)

we find the metric $h_{ij}$ for the real scalar fields $\phi^i$ as

$$h_{ij} = \text{diag} \left( \frac{3m_{\text{Pl}}^2}{2X^2}, \frac{3m_{\text{Pl}}^2}{2X^2} \right).$$

(19)
Then we have

\[ R^X_{YXY} = R^Y_{XYY} = \frac{1}{X^2}, \]  

so the field space is indeed curved, and leads to additional contributions on the right hand side of Eq. (16). However, it is expected that this contribution due to curved field space is small: generally, in the slow-roll regime, the variation of the field \( \phi_i \) per each e-fold is

\[ \frac{\Delta \phi_i}{\epsilon_{\text{Pl}}} = \sqrt{2\epsilon_i}, \]  

with \( \epsilon_i \equiv (\dot{\phi}_i/H)^2/(2m^2_{\text{Pl}}) \) so that \( \epsilon = \sum_i \epsilon_i \). This means while the scales corresponding to the low multipoles \( l \lesssim 100 \) are leaving the horizon, the probed field space is very tiny. Therefore whatever geometry the field space has actually, it can be approximated as a flat space.

\[ \begin{align*}
\delta\phi & \mapsto \phi_c \\
\phi_c(\psi) & \mapsto
\end{align*} \]

Figure 1: A simple trajectory where the critical value of the inflaton field \( \phi \) is not a constant. Here \( \delta N \) depends not only on \( \delta\phi_\parallel \), the fluctuations along \( \dot{\phi} \), making \( \eta_{\text{multi}} \) non-trivial.

However, even if the field space is flat, i.e. \( h_{ij} = \delta_{ij} \), in general we still have non-trivial result. This is because \( V^{ij} \) is contracted with \( N_i N_j \), i.e. we have a ‘projection’ operator. From \( dN = H dt \) it directly follows that

\[ N_i \dot{\phi}^i = \nabla N \cdot \dot{\Phi} = H, \]  

which never necessarily means that the gradient of \( N \) in the field space is aligned in a specific manner with respect to the corresponding component of the velocity of the field. For example, consider a simple hybrid like situation depicted in Fig. 1. In usual hybrid inflation \[19\], \( \phi_c \) is a constant and is orthogonal to the direction of the evolution of \( \phi \), i.e. \( \dot{\phi} \). Thus \( \delta\phi_\perp \), the fluctuations of \( \phi \) orthogonal to \( \dot{\phi} \), cannot alter the number of e-folds \( N \) and hence the curvature perturbation, which is equivalent to \( \Delta N \), is independent of \( \delta\phi_\perp \). Only the fluctuations along \( \dot{\phi} \), which we write \( \delta\phi_\parallel \), can change the end of inflation by moving \( \phi \) backward or forward by that amount, and accordingly give rise to \( \Delta N \) and the curvature perturbation. Now, instead of a constant \( \phi_c \), let us consider the case where \( \phi_c \) is not orthogonal to \( \dot{\phi} \) due to for example
another field $\psi$, i.e. $\phi_c = \phi_c(\psi)$. In this case, $\delta N$ is not only dependent on $\delta \phi_\parallel$, but also on $\delta \phi_\perp$. Therefore $\nabla N$ does not have a simple relation with $\dot{\phi}$ such as
\[ \frac{\partial N}{\partial \phi} = \frac{H}{\dot{\phi}}, \] (23)
which holds for single field case. Contracted with $N, i$, even in flat field space generally we have different result for $m_{\text{Pl}}^2(N_iN_j/N_kN^k)V^{ij}/V$ from single field case where this corresponds to simply $\eta$. This suggests the potential role of the running of $r$ as an observational probe of the non-trivial inflationary dynamics. The simplest multiple chaotic inflation model,
\[ V = \sum_i V_i = \sum_i \lambda_i \frac{\phi^n_i}{m_{\text{Pl}}^{n-4}}, \] (24)
is in fact a very special case \[20\] where we can reproduce the predictions of the corresponding single field case
\[ V = \lambda \frac{\phi^n}{m_{\text{Pl}}^{n-4}}. \] (25)

Finally, let us discuss the practical feasibility of the detection of $d \log r/d \log k$. This amounts to, as can be read from Eqs. (8), (11) and (16), the individual detection of $n_S$, $n_T$ and $r$. Apart from $n_S$ which will be even further constrained, the $B$-mode CMB polarization anisotropies can reveal the primordial gravitational waves and thus $r$ and $n_T$. By the CMB polarization observations alone, e.g. even CMBPol, unfortunately, it is unlikely that we can determine both of them, or at least $n_T$, at a satisfactory level: for a realistic experiment with no foreground and no lensing subtraction which can probe $r \gtrsim 0.01$, the $1\sigma$ error of $n_T$ is as large as few $\times 0.01 - 0.1$ \[21\], but $n_T$ given by Eq. (3) is at most as large as $O(0.01)$. This situation, however, can be greatly improved by direct detection programmes with laser interferometers such as Big Bang Observer (BBO) and Deci-Hertz Interferometer Gravitational wave Observer (DECIGO). Their sensitivity peaks are at around 1 Hz with $\Omega_{\text{GW}}h^2 \sim 10^{-18}$ (BBO) $- 10^{-20}$ (DECIGO), which corresponds to $r \sim 10^{-4}$ (BBO) $- 10^{-6}$ (DECIGO). Then, the $1\sigma$ error of $n_T$ can be improved as much as $\sigma_{n_T} \sim 10^{-8}/r$ in this regime \[22\]. Thus from the laser interferometer experiments, it is not completely impossible to detect $d \log r/d \log k$ provided that once $r$ is detected. With the sharpening accuracy of the observations of the scalar perturbations on the CMB scales, it could be sufficient enough with laser interferometer alone: if $r$ is detected on the CMB scales, we will be able to fix everything unambiguously.

To conclude, we have found a very simple consistency relation from the running of the tensor-to-scalar ratio first order in the slow-roll parameters. In single field models this would merely reproduce what we can find by using other observables, meanwhile for multi-field cases we can extract very important information on the inflaton field space and the inflationary dynamics observationally. We can hope to detect this running by the future laser interferometer experiments.

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References

[1] A. H. Guth, *Phys. Rev.* D **23**, 347 (1981); A. D. Linde, *Phys. Lett.* B **108**, 389 (1982); A. Albrecht and P. J. Steinhardt, *Phys. Rev. Lett.* **48**, 1220 (1982)

[2] See, e.g. A. D. Linde, *Particle physics and inflationary cosmology*, Harwood Academic Press (1990); A. R. Liddle and D. H. Lyth, *Cosmological inflation and large scale structure*, Cambridge University Press (2000); V. F. Mukhanov, *Physical foundations of cosmology*, Cambridge University Press (2005)

[3] G. F. Smoot et al., *Astrophys. J.* **396**, L1 (1992)

[4] M. Tegmark et al., *Phys. Rev.* D **74**, 123507 (2006) astro-ph/0608632; J. K. Adelman-McCarthy et al., *Astrophys. J. Suppl.* **172**, 634 (2007) 0707.3380 [astro-ph]; E. Komatsu et al., 0803.0547 [astro-ph]

[5] A. A. Starobinsky, *JETP Lett.* **30**, 682 (1979)

[6] L. Knox and Y.-S. Song, *Phys. Rev. Lett.* **89**, 011303 (2002) astro-ph/0202286

[7] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, *Phys. Rev.* D **68**, 046005 (2003) hep-th/0301240

[8] D. Baumann and L. McAllister, *Phys. Rev.* D **75**, 123508 (2007) hep-th/0610285

[9] D. H. Lyth, *Phys. Rev. Lett.* **78**, 1861 (1997) hep-ph/9606387

[10] See, however, for example A. Krause, *J. Cosmol. Astropart. Phys.* **07**, 001 (2008) arXiv:0708.4414 [hep-th]

[11] J.-O. Gong, *Class. Quant. Grav.* **21**, 5555 (2004) gr-qc/0408039

[12] D. H. Lyth and D. Wands, *Phys. Lett.* B **524**, 5 (2002) hep-ph/0110002; T. Moroi and T. Takahashi, *Phys. Lett.* B **522**, 215 (2001) hep-ph/0110096; *Erratum*-ibid. B **539**, 303 (2002)

[13] J.-O. Gong, *Phys. Lett.* B **657**, 165 (2007) arXiv:0706.3599 [astro-ph]

[14] E. D. Stewart and J.-O. Gong, *Phys. Lett.* B **510**, 1 (2001) astro-ph/0101225

[15] S. Dodelson and E. Stewart, *Phys. Rev.* D **65**, 101301(R) (2002) astro-ph/0109354; E. D. Stewart, *Phys. Rev.* D **65**, 103508 (2002) astro-ph/0110322; J. Choe, J.-O. Gong and E. D. Stewart, *J. Cosmol. Astropart. Phys.* **07**, 012 (2004) hep-ph/0405155

[16] M. Cortèes and A. R. Liddle, *Phys. Rev.* D **73**, 083523 (2006) astro-ph/0603016
[17] S. Dodelson, W. H. Kinney and E. W. Kolb, Phys. Rev. D 56, 3207 (1997) astro-ph/9702166

[18] A. A. Starobinsky, JETP Lett. 42, 152 (1985); M. Sasaki and E. D. Stewart, Prog. Theor. Phys. 95, 71 (1996) astro-ph/9507001; M. Sasaki and T. Tanaka, Prog. Theor. Phys. 99, 763 (1998) gr-qc/9801017; J.-O. Gong and E. D. Stewart, Phys. Lett. B 538, 213 (2002) astro-ph/0202098; D. H. Lyth, K. A. Malik and M. Sasaki, J. Cosmol. Astropart. Phys. 05, 004 (2005) astro-ph/0411220

[19] A. Linde, Phys. Rev. D 49, 748 (1994) astro-ph/9307002

[20] L. Alabidi and D. H. Lyth, J. Cosmol. Astropart. Phys. 05, 016 (2006) astro-ph/0510441; J.-O. Gong, Phys. Rev. D 75, 043502 (2007) hep-th/0611293

[21] Y.-S. Song and L. Knox, Phys. Rev. D 68, 043518 (2003) astro-ph/0305411; L. Verde, H. V. Peiris and R. Jimenez, J. Cosmol. Astropart. Phys. 01, 019 (2006) astro-ph/0506036

[22] T. L. Smith, M. Kamionkowski and A. Cooray, Phys. Rev. D 73, 023504 (2006) astro-ph/0506422; H. Kudoh, A. Taruya, T. Hiramatsu and Y. Himemoto, Phys. Rev. D 73, 064006 (2006) gr-qc/0511145; T. L. Smith, H. V. Peiris and A. Cooray, Phys. Rev. D 73, 123503 (2006) astro-ph/0602137