The influence of the magnetic field on the quasistationary electric field penetration from the ground to the ionosphere

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Abstract. A quantitative model of the penetration of a quasistationary electric field from the Earth’s surface into the ionosphere with an inclined magnetic field has been developed. The results of the three-dimensional model are simpler regarding interpretation and explanation than those dimensional ones. The electric conductivity problem is solved using Fourier transform at each height. The space distributions of the electric field strength and the current density are calculated. The current lines are plotted. The electric field strength in the ionosphere is in the microvolt per meter range when it is about 100 V/m in the air near ground. The known approximate estimates of decrease in the electric field penetrating into the ionosphere with decrease in the magnetic latitude are confirmed and detailed. The penetration of the electric field and currents from the ground to the ionosphere through the atmospheric conductor cannot be a physical process which creates the observed ionospheric precursors of the earthquakes.

1. Introduction

The interest to the lithosphere-ionosphere relationship is mainly caused by the need to forecast earthquakes, since many satellite observations show the ionospheric electric field perturbations above seismic regions. Nowadays, the most popular models consider the lithosphere as a generator of an electric current or an electric field in the atmosphere. These models take into account the atmosphere and the ionosphere as a united conductor and are based on a stationary model of electric conductivity. In other words, the penetration of a quasistationary electric field from the Earth’s surface into the ionosphere is the basic physical process in such models. Such models are associated with numerous observations of disturbance in the vertical component of the atmospheric electric field before and after earthquakes have occurred. At the present time, three- and two-dimensional models of a conductor that includes the atmosphere and the ionosphere are known [1, 2]. However, such models are created for a vertical magnetic field, and therefore they are applicable only at high latitudes. An exception is the model [3], but it contains incorrect simplifications which significantly distort the results, as shown in [1]. Some new models are analyzed in [4]. In [5] is considered the role of the magnetic field slope but a 2-D model is adequate only for the preparation zones earthquakes elongated in magnetic longitude.

The purpose of this study is to build a quantitative model of the penetration of a quasistationary electric field from the Earth’s surface into the ionosphere with an inclined magnetic field. A three-dimensional model is constructed, which can be regarded as an improved two-dimensional model [5] that is applicable when an earthquake preparation zone is extended normal to the magnetic meridian.

2. Basic equations
In general, the physical model is formulated in the monograph [6]. It is adequate to use a steady state model for a conductor with the conductivity tensor $\sigma$ if the typical time of the process is much larger than the charge relaxation time $\tau = \varepsilon_0 / \sigma$. A minimum value is observed near the ground and it is $\sigma > 10^{14}$ S/m. So, the charge relaxation time in the Earth’s atmosphere does not exceed a quarter of an hour.

The basic equations for the steady state electric field $\mathbf{E}$ and the current density $\mathbf{j}$ are Faraday’s law, the charge conservation law, and Ohm’s law

$$\nabla \times \mathbf{E} = 0,$$

$$\nabla \cdot \mathbf{j} = 0,$$

$$\mathbf{j} = \sigma \mathbf{E}.$$  

Because of equation (1) the electric potential $V$ can be introduced so that

$$\mathbf{E} = -\nabla V.$$  

Then the system of equations (1) – (3) is reduced to the electric conductivity equation

$$-\nabla \cdot (\sigma \nabla V) = 0.$$  

3. Conductivity

Here we study a three-dimensional model. The origin of the Cartesian coordinates is located in the epicenter of the earthquake, the axis $x$ is directed to the North, the axis $y$ is directed to the West, and the axis $z$ is vertically directed from the ground to the ionosphere.

In the main part of the atmosphere up to a height of 50 km we use the empirical conductivity model by Rycroft and Odzimek [7]. Above 50 km, the air becomes sufficiently rarefied so that the influence of the magnetic field is manifested, and the conductivity becomes a gyrotropic tensor. Since we are considering local phenomena, we neglect the Earth’s surface curvature.

The magnetic induction vector $\mathbf{B}$ is located in the plane $x, z$. The magnetic inclination $I$ is the angle between the direction to the magnetic Northern pole and the magnetic induction such that at the Northern and Southern magnetic poles $I = 90^\circ$ and $-90^\circ$, respectively. Ohm’s law (3) in the detailed form can be written down

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \hat{T} \begin{pmatrix} \sigma_p(z) & -\sigma_h(z) & 0 \\ \sigma_h(z) & \sigma_p(z) & 0 \\ 0 & 0 & \sigma_i(z) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix},$$

where $\sigma_p, \sigma_h, \sigma_i$ are the Pedersen, The Hall and field-aligned conductivities and the matrix $\hat{T}$ defines the rotation coordinate system:

$$\hat{T} = \begin{pmatrix} -\sin I & 0 & -\cos I \\ 0 & 1 & 0 \\ \cos I & 0 & -\sin I \end{pmatrix},$$
and the conjugate matrix $\mathbf{T}^*$ means the opposite rotation.

Figure. 1. Profiles of components of the electric conductivity tensor for a mid-latitude night-time ionosphere and a low solar activity.

Above 90 km, we use the model [8], which is based on the empirical models IRI, MSISE and IGRF. Since we consider local phenomena, conductivity is regarded to be dependent only on the vertical coordinate $z$. In the layer $50<z<90$ km, the values for $\sigma_p$ and $\sigma_\parallel$ are smoothly interpolated by cubic functions, and we use the formula $\sigma_\parallel = \left[ \sigma_p \left( \sigma_\parallel - \sigma_p \right) \right]^{1/2}$, which is typical of plasma with one predominated kind of charged particles [9]. At altitudes of 90-100 km, this ratio is valid and considered below 90 km.

Thus, the electric conductivity equation (5) takes the form:

$$
-\frac{\partial}{\partial x} \left[ \sigma_{xx}(z) \frac{\partial V(x, y, z)}{\partial x} + \sigma_{xy}(z) \frac{\partial V(x, y, z)}{\partial y} + \sigma_{xz}(z) \frac{\partial V(x, y, z)}{\partial z} \right] - \frac{\partial}{\partial y} \left[ \sigma_{yx}(z) \frac{\partial V(x, y, z)}{\partial x} + \sigma_{yy}(z) \frac{\partial V(x, y, z)}{\partial y} + \sigma_{yz}(z) \frac{\partial V(x, y, z)}{\partial z} \right] +$

$$+
\frac{\partial}{\partial z} \left[ \sigma_{zx}(z) \frac{\partial V(x, y, z)}{\partial x} + \sigma_{zy}(z) \frac{\partial V(x, y, z)}{\partial y} + \sigma_{zz}(z) \frac{\partial V(x, y, z)}{\partial z} \right] = 0.
$$

(8)

According to (6) $\sigma_{xx} = \sigma_{yy} = \sigma_\parallel \cos^2 I + \sigma_p \sin^2 I$, $\sigma_{xy} = \sigma_\parallel \sin I \cos I$, $\sigma_{xz} = (\sigma_\parallel - \sigma_p) \sin I \cos I$, $\sigma_{yx} = -\sigma_{xy}$, $\sigma_{yy} = \sigma_{xx} = \sigma_\parallel \cos^2 I$, $\sigma_{yy} = \sigma_{xx} = \sigma_\parallel \cos I$, $\sigma_{yz} = -\sigma_\parallel \cos I$, $\sigma_{zy} = -\sigma_\parallel \cos I$, $\sigma_{zz} = \sigma_p \cos^2 I + \sigma_\parallel \sin^2 I$.

4. The boundary conditions
At the lower boundary of the atmosphere, where \( z = 0 \), the vertical component of the electric field above the Earth’s surface may be given by measurements or according to some certain assumptions

\[
- \frac{\partial V(x, y, 0)}{\partial z} = E_0(x, y),
\]

where \( E_0(x, y) \) is a given function. We use a bipolar sin-distribution in the direction perpendicular to the fault and a uni-polar cos-distribution along the fault, which has a maximum value of \( \overline{E}_0 = 100 \) V/m, and equals zero outside of some quadrangle \( |x| < x_0, |y| < y_0 \). Such a model distribution can be written down

\[
E_0(x, y) = -\overline{E}_0 E_1(x) \cdot E_2(y),
\]

where the \( x \)-dependent function \( E_1(x) \) and the \( y \)-dependent function \( E_2(y) \) are given by

\[
E_1(x) = \gamma \left( 1 + \cos \left( \pi x / x_0 \right) \right) \sin \left( \pi x / x_0 \right),
\]

\[
E_2(y) = \left( 1 + \cos \left( \pi y / y_0 \right) \right) / 2,
\]

where \( \gamma = 4 / \left( 3\sqrt{3} \right) \approx 0.77 \) is a normalization factor \( |E_1(x)| \leq 1 \) and \( x_0 \) and \( y_0 \) are the characteristic source areas for each direction, respectively. The function \( E_0(x, y) \) reaches its maximal value \( \overline{E}_0 \) at \( x = x_0 / 3 \) and \( y = 0 \). Figure 2 shows the two-dimensional distribution \( E_0(x, y) \) for an earthquake preparation area with \( x_0 = 400 \) km and \( y_0 = 1000 \) km, respectively.

We use Fourier series to solve the electric conductivity equation (5). This is possible for periodical functions. So, we continue functions (10) over the whole axes with periods of \( 2L_x \) and \( 2L_y \), respectively, and present them with Fourier series. To ascertain that the results of the calculations are not disturbed in the domain of interest, \( |x| \leq 2x_0 \) and \( |y| \leq 2y_0 \), the lengths of the periods chosen by \( L_x \gg x_0 \), \( L_y \gg y_0 \). The Fourier series of function (10) is given by

\[
E_0(x, y) = -\overline{E}_0 \left( \sum_{n=1}^{\infty} g_n \sin \left( k_n x \right) \right) \left( \sum_{m=1}^{\infty} f_m \cos \left( p_m x \right) \right),
\]

where \( k_n = (2n-1)\pi / L_x \), \( p_m = (2m-1)\pi / L_y \), and the Fourier coefficients \( g_n \) and \( f_m \) are given by

\[
g_n = \frac{12x_0 \sin \left( k_n x_0 \right)}{L_x \pi \left( 4 - 5 \left( k_n x_0 / \pi \right)^2 + \left( k_n x_0 / \pi \right)^4 \right)},
\]

\[
f_m = \frac{2 \sin \left( p_m y_0 \right)}{L_y \pi \left( \left( p_m y_0 / \pi \right)^2 - 1 \right)}.
\]

To avoid singularities in the last formulas we use \( L_x \neq (2n-1)x_0 \), \( L_x \neq (2n-1)x_0 / 4 \) for all \( n \) and \( L_y \neq (2m-1)y_0 \) for all \( m \). In this model, we take \( L_x = 10x_0 \), \( L_y = 10y_0 \). A detailed description of the Fourier transform for these functions can be found in [10].

The ionosphere as a conductor is not bounded from above since there is a magnetosphere, which is rather a complicated object. In this model, we do not take into consideration the polar caps and the
auroral zones. The values of the potential at the conjugate points at the upper boundary of the ionosphere \( z = z_\infty \) are assumed to be equal due to a high conductivity along magnetic field lines in the magnetosphere.

In the conjugate ionosphere, the same equation (8) is valid with the boundary condition

\[ E_z(\hat{x}, \hat{y}, \hat{z})|_{z=0} = 0, \]

which corresponds to the highest conductivity of the ground and sea water.

The solution to the boundary value problem (8), (9), (11) and (12) exists and it is unique. We numerically solve this problem using Fourier transform in the manner similar to the one used in [10].

5. Numerical method

Since now \( E_0(x, y) \) is a periodic function and the coefficients of the equation are independent of \( x, y \), the solution \( V(x, y, z) \) to problem (8), (9), (11) and (12) is also a periodic function in \( x \)- and \( y \)-directions. Thus, it can be presented by a Fourier series at each plane \( z = \text{const} \), where the coefficients depend only on \( z \)

\[
V(x, y, z) = \sum_{n=1}^{\infty} \left( a_{n,m}(z) \cos(k_n x) \cos(p_m y) + b_{n,m}(z) \cos(k_n x) \sin(p_m y) + c_{n,m}(z) \sin(k_n x) \cos(p_m y) + d_{n,m}(z) \sin(k_n x) \sin(p_m y) \right),
\]

(13)
where \( k_n \) and \( p_m \) are the same as in (11). We also use one more independent presentation for \( j_z(x, y, z) \), since it is possible to do for any periodical function:

\[
j_z(x, y, z) = \sum_{n,m=1}^{\infty} \left( e_{n,m}(z) \cos(k_n x) \cos(p_m y) + r_{n,m}(z) \cos(k_n x) \sin(p_m y) + u_{n,m}(z) \sin(k_n x) \cos(p_m y) + v_{n,m}(z) \sin(k_n x) \sin(p_m y) \right).
\]

It is better to solve a first order system instead of equation (8) that is equivalent to the basic equations (1) – (3):

\[
\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = 0,
\]

\[
\rho_{xx} j_x + \rho_{xy} j_y + \rho_{xz} j_z + \frac{\partial V}{\partial x} = 0,
\]

\[
\rho_{yx} j_x + \rho_{yy} j_y + \rho_{yz} j_z + \frac{\partial V}{\partial y} = 0,
\]

\[
\rho_{zx} j_x + \rho_{zy} j_y + \rho_{zz} j_z + \frac{\partial V}{\partial z} = 0,
\]

where \( \rho_{xx}, \rho_{xy}, \rho_{xz}, \rho_{yx}, \rho_{yy}, \rho_{yz}, \rho_{zx}, \rho_{zy}, \rho_{zz} \) are the entire inverse matrix of the conductivity tensor in (6), \( \rho = (\hat{\sigma})^{-1} \).

From (13) we can derive

\[
\frac{\partial V}{\partial x} = \sum_{n,m=1}^{\infty} k_n \left( -a_{n,m}(z) \sin(k_n x) \cos(p_m y) - b_{n,m}(z) \sin(k_n x) \sin(p_m y) +
\right.
\]

\[
\left. + c_{n,m}(z) \cos(k_n x) \cos(p_m y) + d_{n,m}(z) \cos(k_n x) \sin(p_m y) \right),
\]

\[
\frac{\partial V}{\partial y} = \sum_{n,m=1}^{\infty} p_m \left( -a_{n,m}(z) \sin(k_n x) \sin(p_m y) + b_{n,m}(z) \cos(k_n x) \cos(p_m y) -
\right.
\]

\[
\left. - c_{n,m}(z) \sin(k_n x) \sin(p_m y) + d_{n,m}(z) \sin(k_n x) \cos(p_m y) \right).
\]

If we express \( j_x \) from the second equation and \( j_y \) – from the third equation of system (14) and substitute it to the fourth equation, we will obtain the equation that can be decomposed to separate equations for each Fourier harmonic in view of the orthogonality of the functions \( \cos(k_n x) \cos(p_m y) \), \( \cos(k_n x) \sin(p_m y) \), \( \sin(k_n x) \cos(p_m y) \), \( \sin(k_n x) \sin(p_m y) \). Then we derive the following system of the ordinary differential equations

\[
a'(z) - A \cdot b(z) - B \cdot c(z) + C \cdot e(z) = 0,
\]

\[
b'(z) + A \cdot a(z) - B \cdot d(z) + C \cdot r(z) = 0,
\]

\[
c'(z) - A \cdot d(z) + B \cdot a(z) + C \cdot u(z) = 0,
\]

\[
d'(z) + A \cdot c(z) + B \cdot b(z) + C \cdot v(z) = 0,
\]

\[
e'(z) - D \cdot r(z) + F \cdot u(z) + R \cdot d(z) + S \cdot a(z) = 0,
\]

\[
r'(z) + D \cdot e(z) + F \cdot v(z) - R \cdot c(z) + S \cdot b(z) = 0,
\]

\[
u'(z) - D \cdot v(z) - F \cdot e(z) - R \cdot b(z) + S \cdot c(z) = 0,
\]
where \( A = \frac{\rho_{xy} \rho_{zy} - \rho_{zy} \rho_{xy}}{\rho_{yy} \rho_{zz} - \rho_{zy} \rho_{zy}} \), \( B = \frac{\rho_{yy} \rho_{zz} - \rho_{zy} \rho_{zy}}{\rho_{xx} \rho_{yy} - \rho_{zy} \rho_{zy}} \), \( C = \frac{\rho_{yy} \rho_{zz} - \rho_{zy} \rho_{zy}}{\rho_{xx} \rho_{yy} - \rho_{zy} \rho_{zy}} \), \( D = \frac{\rho_{xx} \rho_{yy} - \rho_{zy} \rho_{zy}}{\rho_{xx} \rho_{yy} - \rho_{zy} \rho_{zy}} \), \( F = k \left( \frac{\rho_{xx} \rho_{yy} - \rho_{zy} \rho_{zy}}{\rho_{xx} \rho_{yy} - \rho_{zy} \rho_{zy}} \right) \), \( R = \frac{\rho_{yy} \rho_{zz} - \rho_{zy} \rho_{zy}}{\rho_{xx} \rho_{yy} - \rho_{zy} \rho_{zy}} \), \( S = \frac{\rho_{xx} \rho_{yy} - \rho_{zy} \rho_{zy}}{\rho_{xx} \rho_{yy} - \rho_{zy} \rho_{zy}} \).

For the sake of simplicity in equations (15), the common indices \( n \) and \( m \) are omitted.

The boundary conditions (9) and (12) are replaced by
\[
\begin{align*}
    a(z)|_{z=2z_0} &= 0, & b(z)|_{z=2z_0} &= 0, & c(z)|_{z=2z_0} &= 0, & d(z)|_{z=2z_0} &= 0, \\
    e(z)|_{z=0} &= 0, & r(z)|_{z=0} &= 0, & u(z)|_{z=0} &= j^0, & v(z)|_{z=0} &= 0,
\end{align*}
\]
where we omit the indices \( n \) and \( m \) also for the Fourier coefficient of the given function \( j^0(x, y) \).

Figure 3. Electric potential \( V(x, y, z = 150 \text{ km}) \) for \( I = 60^\circ \). It is about the same in the ionosphere above \( z = 90 \text{ km} \). Solid lines correspond to positive, dashed lines – to negative values.

First using the Euler method we numerically solve the two initial four initial boundary value problems starting from \( z = 2z_0 \) with \( a=b=c=d=0, e=1, r=u=v=0 \) or \( a=b=c=d=0, e=0, r=1, u=v=0 \) or \( a=b=c=d=0, e=r=0, u=1, v=0 \) or \( a=b=c=d=e= r=u=0, v=1 \). Then we calculate the multipliers...
\( \alpha, \beta, \gamma, \delta \) which are used to satisfy the boundary condition at the ground. The linear combination of these four solutions is the solution to the boundary value problem (15), (16).

\[ \text{Figure. 4. Components of the electric field strength: the field-aligned component } E_\parallel (\text{dashed line}), \text{ and two components normal to the magnetic field, namely, } E_\perp^X \text{ in the plane } XZ (\text{solid line}) \text{ and } E_\perp^Y \text{ in the plane } YZ (\text{dashed-dotted line}) \text{ when the magnetic inclination } I = 60^\circ. \]

When we find the functions \( a(z), b(z), c(z), d(z), e(z), r(z), u(z), v(z) \) we use them to find all the needed functions \( V, E_x, E_y, j_x, j_y, j_z \) without differentiation with respect to \( z \), while differentiation with respect to \( x, y \) means only multiplication by \( k_x \) and \( p_m \), respectively. As we have mentioned above, we use the Euler method, so we have the first order convergence over \( h \) for all the functions.

Many test calculations are done to choose the necessary parameters of the numerical method which are the grid size \( h \), dimensions of the domain \( L_x, L_y \) \text{ and the number of the Fourier harmonics } N, M, \text{ which are sufficient to give a good approximation for the solution of problem (8), (9), (11) and (12). In this study, we use } h = 100 \, \text{m}, N = 300 \text{ and } M = 50. \text{ It should be noted that this numerical solution was compared with the analytical one when } \sigma_p, \sigma_n, \sigma_\parallel \text{ are piecewise exponential functions.} \]

6. The main results

The actual three-dimensional numerical model includes the whole ionospheric conductivity up to \( z_\infty = 500 \, \text{km} \), as well as the finiteness of the earthquake preparation zone in both directions. The resulting electric potential \( V(x, y, z = 150 \, \text{km}) \) for \( I = 60^\circ \) is shown in Figure 3. It is about the same at any height in the ionosphere above 90 km.

The altitude distributions of the electric field components are shown in Figure 4 with maximum values estimated at each height for the three components, i.e. the field-aligned and the two normal, \( E_\perp^X \) in the plane \( XZ \) and \( E_\perp^Y \) in the plane \( YZ \). It must be mentioned that these maxima are attained in three different points. For example, near the ground \( \max E_x \) at \( x = x_0 / 3, y = 0 \), \( \max E_y \) at \( x = y = 0 \),
max $E_y$ at $x = x_0/3 \text{ km}$, $y = -y_0/2$. At $z = 150 \text{ km}$, it is shown in Figure 3, that max $E_{||}$ at $x = 0.18x_0$, $y = -y_0/100$, max $E_x^\perp$ at $x = 0.75x_0$, $y = -y_0/100$, max $E_y^\perp$ at $x = 0.65x_0$, $y = y_0/2$.

Figure 5 shows the dependence of an ionospherical maximum of the $|E_\perp|$ on the magnetic inclination $I$.

In our study we find the decrease in the strength of the electric field, penetrating into the ionosphere, with a decrease in the module of the magnetic field inclination (from high to low latitudes). This is in agreement with the previous investigations.

![Graph](image)

**Figure. 5.** Dependence of a maximum of the ionospheric electric field $E_\perp$ on the magnetic inclination $I$.

In the model constructed, the obtained electric field is three orders of magnitude smaller than the discussed in [11] field variations, observed in the ionosphere before strong earthquakes. Therefore, the model confirms the conclusion about the need to explore other physical mechanisms of the lithosphere-atmosphere-ionosphere coupling.

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