Coupling and Strong Feller for Jump Processes on Banach Spaces*

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Abstract

By using lower bound conditions of the Lévy measure w.r.t. a nice reference measure, the coupling and strong Feller properties are investigated for the Markov semi-group associated with a class of linear SDEs driven by (non-cylindrical) Lévy processes on a Banach space. Unlike in the finite-dimensional case where these properties have also been confirmed for Lévy processes without drift, in the infinite-dimensional setting the appearance of a drift term is essential to ensure the quasi-invariance of the process by shifting the initial data. Gradient estimates and exponential convergence are also investigated. The main results are illustrated by specific models on the Wiener space and separable Hilbert spaces.

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1 Introduction

In recent years, the coupling property, the strong Feller property, and gradient estimates have been intensively investigated for linear stochastic differential equations driven by Lévy

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processes on $\mathbb{R}^d$, see e.g. \[14, 22, 21, 17, 19, 18, 9, 8\] and references within. In these references the shift-invariance of the Lebesgue measure plays an essential role. When the state space is infinite-dimensional so that the Lebesgue measure is no longer available, we need a reference measure which is quasi-invariant under a reasonable class of shift transforms. Typical examples of the reference measure include the Wiener measure on the continuous path space and the Gaussian measure on a Hilbert space, see Section 5 for details. The purpose of this paper is to investigate regularity properties of linear SDEs driven by Lévy processes on a Banach space equipped with such a nice reference measure. To ensure the quasi-invariance of the solution, a strong enough linear drift term will be needed.

On the other hand, concerning (semi-)linear SDEs on Hilbert spaces, when the noise is a cylindrical $\alpha$-stable process, many regularity results derived in finite dimensions can be extended to the infinite-dimensional setting (see \[15, 13, 25\]); and when the noise has a non-trivial Gaussian part, the regularity properties can be derived by using the drift part and the Gaussian part (see e.g. \[26, 6, 7, 16\]). But there seems to be no results concerning the strong Feller and coupling properties for SDEs driven by purely jump non-cylindrical Lévy processes. In this paper we intend to investigate these properties for linear SDEs driven by non-cylindrical Lévy noise on Banach spaces.

Let $(\mathbb{B}, \| \cdot \|_B)$ be a Banach space and let $\mu$ be a probability measure on $\mathbb{B}$ having full support. Let $\mathbb{B}'$ be the dual space of $\mathbb{B}$ with $\langle \cdot, \cdot \rangle$ the duality between $\mathbb{B}$ and $\mathbb{B}'$. Let $(\mathbb{H}, \| \cdot \|_H)$ be another Banach space which is densely and continuously embedded into $\mathbb{B}$ such that for any $h \in \mathbb{H}$, $\mu$ is quasi-invariant under the shift $x \mapsto x + h$; that is, there exists a non-negative measurable function $\varphi_h$ on $\mathbb{B}$ such that

$$
\mu(dz - h) = \varphi_h(z) \mu(dz).
$$

Let $L_t$ be a Lévy process on $\mathbb{B}$ with Lévy measure $\nu$. Recall that a $\sigma$-finite measure $\nu$ on $\mathbb{B}$ is called a Lévy measure if $\nu(\{0\}) = 0$ and the mapping from $\mathbb{B}'$ to $\mathbb{R}$ given by

$$
\mathbb{B}' \ni a \mapsto \exp \left[ \int_{\mathbb{B}} \left( \cos \langle x, a \rangle - 1 \right) \nu(dx) \right]
$$

is the characteristic function of a random variable on $\mathbb{B}$. Note that since $\cos$ is an even function, one may replace $\nu$ by the symmetric measure $\nu + \nu^*$ as in \[2\], where $\nu^*(A) = \nu(-A)$ for any $A \in \mathcal{B}$. When $\mathbb{B}$ is a Hilbert space, $\nu$ is a Lévy measure if and only if $\nu(\{0\}) = 0$ and $\int_{\mathbb{B}} (1 \wedge \|x\|_B^2) \nu(dx) < \infty$; while in general, $\nu$ is a Lévy measure provided $\nu(\{0\}) = 0$ and $\int_{\mathbb{B}} (1 \wedge \|x\|_B) \nu(dx) < \infty$ (see \[11, 2\]).

Let $\sigma : \mathbb{B} \to \mathbb{B}$ be a bounded linear operator and let $(A, \mathcal{D}(A))$ be a linear operator on $\mathbb{B}$ generating a $C_0$ semigroup $(T_s)_{s \geq 0}$. Consider the following linear SDE on $\mathbb{B}$:

$$
dX_t = AX_t \, dt + \sigma \, dL_t.
$$

For any $x \in \mathbb{B}$, the solution with initial data $x$ is

$$
X^x_t = T_t x + \int_0^t T_{t-s} \sigma \, dL_s, \quad t \geq 0.
$$
See [11, 12, 13, 2] for the detailed construction of this solution. Let $\mathcal{B}_b(\mathbb{B})$ be the class of all bounded measurable functions on $\mathbb{B}$. We aim to investigate the coupling property and the strong Feller property for the associated Markov semigroup

$$P_t f(x) := \mathbb{E} f(X^x_t), \quad t \geq 0, x \in \mathbb{B}, f \in \mathcal{B}_b(\mathbb{B}).$$

Recall that the solution has successful coupling if and only if (cf. [10, 5])

$$\lim_{t \to \infty} \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} = 0, \quad x, y \in \mathbb{B},$$

where $P_t(x, dy)$ is the transition kernel of $P_t$ and $\| \cdot \|_{\text{var}}$ is the total variation norm. Let $\rho_0$ be a non-trivial non-negative measurable function on $\mathbb{B}$ such that

$$(1.4) \quad \nu(dz) \geq \rho_0(z) \mu(dz) =: \nu_0(dz)$$

holds. Thus, the Lévy process considered here is essentially different from the cylindrical $\alpha$-stable process used in [15, 13]. Indeed, for $\mathbb{B}$ being a Hilbert space with ONB $\{e_i\}_{i \geq 1}$, the Lévy measure (if exists) for a cylindrical Lévy process is supported on $\bigcup_{i \geq 1} \mathbb{R} e_i$ and hence, is singular w.r.t. e.g. a non-trivial Gaussian probability measure $\mu$. Assume

(A) $\text{Ker}(\sigma) = \{0\}$ and $T_s \mathbb{B} \subset \sigma \mathbb{H}$ holds for any $s > 0$.

Obviously, (A) implies that for any $s > 0$, the operator $\sigma^{-1} T_s : \mathbb{B} \to \mathbb{H}$ is well defined.

**Theorem 1.1.** Assume (A). Suppose that $\nu_0$ in (1.4) is infinite; i.e. $\nu_0(\mathbb{B}) = \infty$.

(1) If for any $h \in \mathbb{H}$

$$(1.5) \quad \sup_{\varepsilon \in (0,1)} \varphi_{\varepsilon h} (\cdot + \varepsilon h) < \infty, \quad \mu\text{-a.e.},$$

then for any $f \in \mathcal{B}_b(\mathbb{B})$ and $t > 0$, $P_t f$ is directionally continuous; i.e. $\lim_{\varepsilon \to 0} P_t f(x + \varepsilon y) = P_t f(x)$ holds for any $x, y \in \mathbb{B}$.

(2) If for any $s > 0$

$$(1.6) \quad \sup_{\|y\| \leq 1} \varphi_{\sigma^{-1} T_s y} (\cdot + \sigma^{-1} T_s y) < \infty, \quad \mu\text{-a.e.},$$

then $P_t$ is strong Feller for $t > 0$; i.e. $P_t \mathcal{B}_b(\mathbb{B}) \subset C_b(\mathbb{B})$.

A simple example for $\nu_0(\mathbb{B}) = \infty$ to hold is as follows. Let $z \to \|z\|_\mathbb{B}$ have a strictly positive distribution density function $\rho$ under the probability measure $\mu$, for instance it is the case when $\mu$ is the Wiener measure (see Subsection 5.1 below). Let $r_0 \in (0, \infty]$, and let $\alpha \in (0, 2)$ when $\mathbb{B}$ is a Hilbert space and $\alpha \in (0, 1)$ otherwise. Then

$$\nu_0(dz) := \frac{1_{(0, r_0)}(\|z\|_\mathbb{B})}{\rho(\|z\|_\mathbb{B}) \|z\|_\mathbb{B}^{1+\alpha} \mu(dz)}$$
is a Lévy measure on \( \mathbb{B} \) with \( \nu_0(\mathbb{B}) = \infty \). This measure is an infinite-dimensional version of the \( \alpha \)-stable jump measure. Modifying arguments from [22 Theorem 3.1] and [18 Theorem 1.1] where the coupling property has been investigated in the finite-dimension setting, we have the following two assertions on the coupling property with estimates on the convergence rate. For \( r > 0 \) and \( z \in \mathbb{B} \), let \( B(z, r) = \{ y \in \mathbb{B} : |z - y|_B < r \} \) be the open ball at \( z \) with radius \( r \).

**Theorem 1.2.** Assume (A). Suppose that \( \nu_0 \) in (1.4) is finite; i.e. \( \nu_0(\mathbb{B}) < \infty \), \( \sigma \) is invertible with \( \|\sigma^{-1}\|_B < \infty \), and \( \|T_s\|_B \leq c \) holds for some constant \( c > 0 \) and all \( s > 0 \).

(i) If there exist \( z_0 \in \mathbb{B} \) and \( r_0 > 0 \) such that

\[
\delta_1(\varepsilon) := \sup_{s \geq \varepsilon, \|x\|_B \leq 1} \int_{B(z_0, r_0)} \frac{\varphi_{\sigma^{-1}T_s x}(z)^2 \rho_0(z - \sigma^{-1}T_s x)^2}{\rho_0(z)} \mu(dz) < \infty, \; \varepsilon > 0,
\]

then there exists a constant \( C > 0 \) such that

\[
\|P_t(x, \cdot) - P_t(x + y, \cdot)\|_{\text{var}} \leq C(1 + \|y\|_B) \inf_{\varepsilon \in (0, 1)} \left( \varepsilon + \sqrt{\frac{\delta_1(\varepsilon)}{t}} \right), \; t > 0, \; x, y \in \mathbb{B}
\]

holds.

(ii) If there exist \( z_0 \in \mathbb{B} \) and \( r_0 > 0 \) such that

\[
\delta_2(\varepsilon) := \sup_{s \geq \varepsilon, \|x\|_B \leq 1} \int_{B(z_0, r_0)} \frac{\varphi_{\sigma^{-1}T_s x}(z)^2 \lor 1}{\rho_0(z)} \mu(dz) < \infty, \; \varepsilon > 0,
\]

then there exist two constants \( C > 0 \) such that for all \( x, y \in \mathbb{B} \) and \( t > 0 \),

\[
\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \leq C(1 + \|x - y\|_B) \inf_{\varepsilon \in (0, 1)} \left( \varepsilon + \sqrt{\frac{\delta_2(\varepsilon)}{t}} \right).
\]

Using \( \rho_0 \land 1 \) in place of \( \rho_0 \), one may replace (1.7) by

\[
\tilde{\delta}_1(\varepsilon) := \sup_{s \geq \varepsilon, \|x\|_B \leq 1} \int_{B(z_0, r_0)} \frac{\varphi_{\sigma^{-1}T_s x}(z)^2}{\rho_0(z) \land 1} \mu(dz) < \infty, \; \varepsilon > 0.
\]

If \( \inf_{z \in B(z_0, r_0)} \rho_0(z) > 0 \), then this condition and (1.9) are equivalent. But in general (1.7) and (1.9) are incomparable. Next, it is easy to see that the convergence rate implied by (1.8) or (1.10) is in general slower than \( \frac{1}{\sqrt{t}} \). Our next result shows that if \( \varphi \) and \( \rho_0 \) are regular enough, the convergence could be exponentially fast.

**Theorem 1.3.** Assume (A). Suppose that \( \nu_0 \) in (1.4) is finite with \( \lambda_0 := \nu_0(\mathbb{B}) \in (0, \infty) \), \( \|T_s\|_B \leq ce^{-\lambda s} \) and

\[
\int_{\mathbb{B}} \left( |\rho_0(z) - \rho_0(z + h)| + \rho_0(z) |\varphi_h(z) - 1| \right) \mu(dz) \leq c\|h\|_H, \; \|h\|_H \leq 1
\]

(1.11)
holds for some constants $c, \lambda > 0$ and all $s \geq 0$. If

\[
(1.12) \quad \sup_{t \geq 1} \frac{1}{1 - e^{-\lambda_0 t}} \int_0^t e^{-\lambda_0 r} \left( \sup_{\|z\| \leq 1} \sup_{s \geq r} \|\sigma^{-1} T_s z\|_{\mathbb{H}} \right) \, dr < \infty,
\]

then there exists a constant $C > 0$ such that

\[
(1.13) \quad \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \leq C(1 + \|x - y\|_{\mathbb{B}}) e^{\frac{-\lambda_0 \lambda t}{\lambda_0 + \lambda}}, \quad x, y \in \mathbb{B}, t \geq 0.
\]

Following the line of [23, Section 3], one may also naturally investigate gradient estimates and derivative formula for $P_t$. It is not difficult to present a formal result under a condition similar to [23, (3.1)], for instance:

**Proposition 1.4.** Assume that $\{h \in \mathbb{H} : \sup_{s \in [0,1]} \|\sigma^{-1} T_s h\|_{\mathbb{H}} < \infty\}$ is dense in $\mathbb{B}$. If there exists a non-negative function $g$ on $\mathbb{B}$ such that $\nu_0\{g > 0\} = \infty$, $\rho_0 g$ is bounded and Lipschitz continuous in $\| \cdot \|_{\mathbb{H}}$, and

\[
q(t) := \sup_{\|h\|_{\mathbb{H}} \in (0,1]} \left\{ \left( \frac{\lambda_0}{\lambda_0 + \lambda} \|h\|_{\mathbb{H}} + 1 \right) \int_0^\infty e^{-t \nu_0(1 - \exp[-rg])} \, dr \right. \\
+ \left. \frac{\mu \|\sigma^{-1} T_s y\|_{\mathbb{H}}}{\|h\|_{\mathbb{H}}} \right\} < \infty, \quad t > 0,
\]

then there exists a constant $C_1 > 0$ such that

\[
|\nabla_y P_t f(x)| = \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} |P_t f(x + \varepsilon y) - P_t f(x)| \\
\leq C_1 \|f\|_{\infty} q(t) \int_0^t \|\sigma^{-1} T_s y\|_{\mathbb{H}} \, ds, \quad f \in \mathcal{B}_b(\mathbb{B}), t > 0, x, y \in \mathbb{B}.
\]

Suppose moreover that $\|T_s\|_{\mathbb{H}} \leq c e^{-\lambda s}$ for some constants $c, \lambda > 0$ and all $s \geq 0$. Then

\[
\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \leq C_2(1 + \|x - y\|_{\mathbb{B}}) e^{-\lambda t}, \quad x, y \in \mathbb{B}, t \geq 0
\]

holds for some constant $C_2 > 0$.

Unfortunately, in the moment we do not have any non-trivial example in infinite dimensions to illustrate condition (1.14). Indeed, it seems that in infinite dimensions the uniform norm of the gradient of $P_t$

\[
\|\nabla P_t\|_{\infty} := \sup\{\|\nabla_y P_t f(x)\| : \|y\|_{\mathbb{B}} \leq 1, x \in \mathbb{B}, \|f\|_{\infty} \leq 1\}
\]

is most likely infinite for any $t > 0$. The intuition is that comparing with a cylindrical noise given in [13, Assumption 2.2], which is strong enough along single directions so that the noise might not take values in $\mathbb{B}$, our non-cylindrical Lévy process seems too weak to imply a bounded gradient estimate of $P_t$. Nevertheless, we are able to estimate the uniform
gradient of a modified version of \( P_t \) (cf. Proposition 1.1 below), which implies the desired exponential convergence in (1.13).

We remark that the derivative formula and gradient estimate are investigated in [20, 27] for SDEs on \( \mathbb{R}^d \) driven by Lévy noises, where in [20] the process may contain a diffusion part but extensions of the main results to infinite dimensions are not yet available, while in [27] the main result was also extended to a class of semi-linear SPDEs driven by cylindrical \( \alpha \)-stable processes. Both papers are quite different from the present one, where we aim to describe regularity properties of the semigroup merely using the Lévy measure of the noise.

We will prove Theorems 1.1 (also Proposition 1.4), 1.2 and 1.3 in the following three sections respectively. In Section 5 we present two specific examples, with \( \mu \) the Wiener measure on a Brownian path space and the Gaussian measure on an Hilbert space respectively, to illustrate these results.

2 Proofs of Theorem 1.1 and Proposition 1.4

The key technique of the study is the coupling by change of measure. For readers’ convenience, let us briefly recall the main idea of the argument. To investigate e.g. the continuity of \( P_t f \) along \( y \in \mathbb{B} \), for any \( x \in \mathbb{B} \) we construct a family of processes \( \{X_\varepsilon \}_{\varepsilon \in [0,1)} \) and the associated probability densities \( \{R_\varepsilon \}_{\varepsilon \in [0,1)} \) such that

1. \( X_\varepsilon^0 = x + \varepsilon y, \ X_\varepsilon^t = X_\varepsilon^0, \ \varepsilon \in [0,1), \ t > 0; \)

2. Under the probability \( R_\varepsilon \mathbb{P} \), the process \( X_\varepsilon^t \) is associated to the transition semigroup \( (P_s)_{s \geq 0} \);

3. \( \lim_{\varepsilon \to 0} R_\varepsilon = R_0 = 1 \) holds in \( L^1(\mathbb{P}) \).

Then, for any bounded measurable function \( f \) and \( t > 0 \),

\[
\lim_{\varepsilon \to 0} P_t f(x + \varepsilon y) = \lim_{\varepsilon \to 0} \mathbb{E}[R_\varepsilon f(X_\varepsilon^t)] = \lim_{\varepsilon \to 0} \mathbb{E}[R_\varepsilon f(X_\varepsilon^0)] = \mathbb{E}[R_0 f(X_0^t)] = P_t f(x).
\]

To realize this idea in the present setting, the following Lemma 2.1 will play a crucial role.

For fixed \( t > 0 \), let \( \Lambda \) be the distribution of \( L := (L_s)_{s \in [0,t]} \) which is a probability measure on the paths space

\[
W_t = \{ w : [0,t] \to \mathbb{B} \text{ is right-continuous having left limits} \}
\]
equipped with the Skorokhod metric. For any \( w \in W_t \), let

\[
w(\text{d}z, \text{d}s) := \sum_{s \in [0,t], \Delta w_s \neq 0} \delta_{(\Delta w_s, s)}.
\]
which records jumps of the path \( w \), where \( \Delta w_s = w_s - w_{s-} \). Let

\[
    w(g) = \int_{B \times [0,t]} g(z, s) w(\text{d}z, \text{d}s) = \sum_{s \in [0,t], \Delta w_s \neq 0} g(\Delta w_s, s), \quad g \in L^1(w).
\]

A function \( g \) on \( B \) will be also regarded as a function on \( B \times [0,t] \) by letting \( g(z, s) = g(z) \) for \( (z, s) \in B \times [0,t] \).

Moreover, write \( L = L^1 + L^0 \), where \( L^1 \) and \( L^0 \) are two independent Lévy processes with Lévy measure \( \nu - \nu_0 \) and \( \nu_0 \) respectively, and \( L^0 \) does not have a Gaussian term. Let \( \Lambda^1 \) and \( \Lambda^0 \) be the distributions of \( L^1 \) and \( L^0 \) respectively. We have \( \Lambda = \Lambda^1 \ast \Lambda^0 \).

Repeating the proof of [23, Lemma 2.1] where \( B = \mathbb{R}^d \), we have the following result.

**Lemma 2.1.** For any \( h \in L^1(W_t \times B \times [0,t]; \Lambda^0 \times \nu_0 \times ds) \),

\[
    \int_{W_t \times B \times [0,t]} h(w, z, s) \Lambda^0(\text{d}w) \nu_0(\text{d}z) \text{d}s \quad (2.1)
\]

\[
= \int_{W_t} \Lambda^0(\text{d}w) \int_{B \times [0,t]} h(w - z\mathbb{1}_{[s,t]}, z, s) w(\text{d}z, \text{d}s).
\]

To prove Theorem 1.1, we also need the following two more lemmas.

**Lemma 2.2.** Let \( y \in B \) such that \( \sigma^{-1}T_s y \in \mathbb{R} \) for any \( s > 0 \), and let \( g \) be a non-negative measurable function on \( B \) such that \( \nu_0(g) := \int_B g \text{d}\nu_0 < \infty \) and \( w(g) > 0 \) for \( \Lambda^0 \)-a.e. \( w \). Let

\[
\Phi_\varepsilon(w, z, s) = \frac{\varphi_\varepsilon \sigma^{-1}T_s y(z)(\rho_0 g)(z - \varepsilon \sigma^{-1} T_s y)}{w(g) + g(z - \varepsilon \sigma^{-1} T_s y)}, \quad \varepsilon \geq 0.
\]

If (1.5) holds for any \( h \in \mathbb{R} \), then \( \{\Phi_\varepsilon\}_{\varepsilon \in (0,1)} \) is uniformly integrable w.r.t. \( \Lambda^0 \times \mu \times ds \) on \( W_t \times B \times [0,t] \).

**Proof.** Since \( \varphi_0 \equiv 1 \), applying (2.1) to \( h(w, z, s) = \frac{g(z)}{w(g)} \) we obtain

\[
\int_{W_t \times B \times [0,t]} \Phi_\varepsilon(w, z, s) \Lambda^0(\text{d}w) \mu(\text{d}z) \text{d}s
\]

\[
= \int_{W_t \times B \times [0,t]} \frac{g(z)}{w(g) + g(z)} \Lambda^0(\text{d}w) \nu_0(\text{d}z) \text{d}s
\]

\[
= \int_{W_t} \Lambda^0(\text{d}w) \int_{B \times [0,t]} \frac{g(z)}{w(g)} w(\text{d}z, \text{d}s)
\]

\[
= 1.
\]

Next, by (1.1) and the integral transform \( z \mapsto z - \varepsilon \sigma^{-1} T_s y \), for any \( F \in \mathcal{D}_0(W_t \times B \times [0,t]) \)
we have
\[
\int_{W_t \times B \times [0,t]} F(w, z + \varepsilon^{-1} T_s y, s) \Phi_\varepsilon(w, z, s) \Lambda^0(dw) \mu(dz) \, ds
\]
(2.3)
\[
= \int_{W_t \times B \times [0,t]} \frac{F(w, z + \varepsilon^{-1} T_s y, s)(\rho_0 g)(z)}{w(g) + g(z)} \Lambda^0(dw) \mu(dz) \, ds
\]
\[
= \int_{W_t \times B \times [0,t]} F(w, z, s) \Phi_\varepsilon(w, z, s) \Lambda^0(dw) \mu(dz) \, ds.
\]
Letting \( F = 1 \) and combining this with (2.2), we conclude that \( \{ \Phi_\varepsilon \}_{\varepsilon \in (0,1)} \) are probability densities w.r.t. \( \Lambda^0 \times \mu \times ds \). Moreover, applying (2.3) to \( F(w, z, s) = 1_{\{ \varepsilon > R \}} \) for \( R > 0 \) and letting
\[
\eta(w, z, s) = \sup_{\varepsilon \in (0,1)} \frac{(\rho_0 g)(z)}{w(g) + g(z)} \varphi_{\varepsilon^{-1} T_s y}(z + \varepsilon^{-1} T_s y)
\]
which is finite \( \Lambda^0 \times \mu \times ds \)-a.e., we obtain
\[
\sup_{\varepsilon \in (0,1)} \int_{W_t \times B \times [0,t]} (\Phi_\varepsilon 1_{\{ \varepsilon > R \}})(w, z, s) \Lambda^0(dw) \mu(dz) \, ds
\]
\[
\leq \int_{W_t \times B \times [0,t]} (\Phi_0 1_{\{ \eta > R \}})(w, z, s) \Lambda^0(dw) \mu(dz) \, ds
\]
which goes to zero as \( R \to \infty \) by the dominated convergence theorem. \( \square \)

**Lemma 2.3.** Let \( E \) be a topology space and \( C_\sigma(E) \) be the class of all bounded continuous functions on \( B \). Let \( \mu_0 \) be a finite measure on the Borel \( \sigma \)-field \( \mathcal{B} \) such that \( C_\sigma(E) \) is dense in \( L^1(\mu_0) \). Let \( \{ f_n \}_{n \geq 1} \) be a sequence of uniformly integrable functions w.r.t. \( \mu_0 \) such that
\[
\lim_{n \to \infty} \int_E (F f_n) \, d\mu_0 = \int_E (F f_0) \, d\mu_0
\]
holds for some \( f_0 \in L^1(\mu_0) \) and all \( F \in C_\sigma(E) \). Then it holds also for any \( F \in \mathcal{B}_b(E) \).

**Proof.** Let \( \varepsilon(R) = \sup_{n \geq 1} \mu_0( |f_n - f_0| 1_{\{ |f_n - f_0| > R \}}) \) which goes to zero as \( R \to \infty \). For any \( F \in \mathcal{B}_b(E) \), let \( \{ F_m \}_{m \geq 1} \subset C_\sigma(E) \) such that \( \| F_m \|_\infty \leq \| F \|_\infty \) and \( \mu_0( |F_m - F|) \leq \frac{1}{m} \). Then
\[
\left| \int_E F(f_n - f_0) \, d\mu_0 \right| \leq \int_E F(f_n - f_0) 1_{\{ |f_n - f_0| \leq R \}} \, d\mu_0 + \| F \|_\infty \varepsilon(R)
\]
\[
\leq \int_E F_m(f_n - f_0) 1_{\{ |f_n - f_0| \leq R \}} \, d\mu_0 + \| F \|_\infty \varepsilon(R) + \frac{R}{m}
\]
\[
\leq \int_E F_m(f_n - f_0) \, d\mu_0 + 2\| F \|_\infty \varepsilon(R) + \frac{R}{m}.
\]
By first letting \( n \to \infty \) then \( m \to \infty \) and finally \( R \to \infty \), we complete the proof. \( \square \)
Proof of Theorem 1.1. (1) Let $f \in \mathcal{B}_0(\mathbb{B})$ and $x, y \in \mathbb{B}$ be fixed. For any $\varepsilon > 0$, let

$$F_{\varepsilon}(w) = f\left(T_t(x + \varepsilon y) + \int_0^t T_{t-s} \sigma \, dw_s\right),$$

where $\int_0^t T_{t-s} \sigma \, dw_s$ is the Itô stochastic integral which is $\Lambda$-a.e. well-defined. Let e.g. $g = \frac{1}{\rho_0 \sqrt{\varepsilon}}$. We have $\nu_0(g) < \infty$ and, since $\nu_0(\mathbb{B}) = \infty$ and $g > 0$, $w(g) > 0$ for $\Lambda^0$-a.e. $w$. Then, by (1.3) and Lemma 2.1 for

$$h(w^0, z, s) = \frac{F_0(w^1 + w^0 + (z + \varepsilon \sigma^{-1} T_s y) 1_{[s,t]} g(z))}{w^0(g) + g(z)},$$

we obtain

$$P_t f(x + \varepsilon y) = \mathbb{E} F_{\varepsilon}(L^1 + L_0)$$

$$= \int_{\mathbb{B}^2} \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{\mathbb{B}^2} \frac{F_{\varepsilon}(w^1 + w^0) g(z)}{w^0(g)} \nu_0(dz, ds)$$

$$= \int_{\mathbb{B}^2} \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{\mathbb{B}^2} \frac{F_0(w^1 + w^0 + z + \varepsilon \sigma^{-1} T_s y) 1_{[s,t]} g(z)}{w^0(g) + g(z)} \nu_0(dz) ds$$

$$= \int_{\mathbb{B}^2} \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{\mathbb{B}^2} \frac{F_0(w^1 + w^0 + z + \varepsilon \sigma^{-1} T_s y) 1_{[s,t]} (\rho_0 g)(z)}{w^0(g) + g(z)} \mu(dz) ds.$$

Since $\varepsilon \sigma^{-1} T_s y \in \mathbb{H}$ so that (1.1) implies

$$\mu(dz - \varepsilon \sigma^{-1} T_s y) = \varphi_{\varepsilon \sigma^{-1} T_s y}(z) \mu(dz),$$

by using the integral transform $z \mapsto z - \varepsilon \sigma^{-1} T_s y$ and noting that $\Lambda = \Lambda^1 \ast \Lambda^0$, we obtain

$$P_t f(x + \varepsilon y) = \int_{\mathbb{B}^2} \Lambda(dw) \int_{\mathbb{B}^2} \frac{F_0(w + z 1_{[s,t]} (\rho_0 g)(z - \varepsilon \sigma^{-1} T_s y))}{w^0(g) + g(z - \varepsilon \sigma^{-1} T_s y)} \varphi_{\varepsilon \sigma^{-1} T_s y}(z) \mu(dz) ds$$

$$= \int_{\mathbb{B}^2} \Lambda^1(dw^1) \int_{\mathbb{B}^2} F_0(w^1 + w^0 + z 1_{[s,t]} \Phi_{\varepsilon}(w^0, z, s) \Lambda^0(dw^0) \mu(dz) ds.$$

Therefore, it suffices to show that

$$\lim_{\varepsilon \to 0} \int_{W_t \times \mathbb{B}^2} (F \Phi_{\varepsilon})(w, z, s) \Lambda^0(dw) \mu(dz) ds = \int_{W_t \times \mathbb{B}^2} (F \Phi_0)(w, z, s) \Lambda^0(dw) \mu(dz) ds$$

holds for any $F \in \mathcal{B}_b(W_t \times \mathbb{B} \times [0, t])$. According to (2.3), this holds provided $F \in C_b(W_t \times \mathbb{B} \times [0, t])$. Since the Borel $\sigma$-field on the Polish space $W_t \times \mathbb{B} \times [0, t]$ is induced by bounded
continuous functions, $C_b(W_t \times \mathbb{B} \times [0,t])$ is dense in $L^1(\Lambda^0 \times \mu \times ds)$. Thus, the desired assertion follows from Lemmas \text{2.2 and 2.3}.

(2) For any sequence $\{y_n\} \subset \mathbb{B}$ converging to 0 as $n \to \infty$, define

$$\Psi_n(w, z, s) = \frac{\varphi_{\sigma^{-1}Tsy_n}(z)(\rho_0g)(z - \sigma^{-1}Tsy_n)}{w(g) + g(z - \sigma^{-1}Tsy_n)}, \quad n \geq 1.$$ 

Using $\sigma^{-1}Tsy_n$ to replace $\varepsilon h$ in the proof of Lemma \text{2.2} we see that\text{ (1.6)} implies that $\{\Psi_n\}_{n \geq 1}$ is uniformly integrable w.r.t. $\Lambda^0 \times \mu \times ds$ on $W_t \times \mathbb{B} \times [0,t]$. Therefore, using $\sigma^{-1}Tsy_n$ to replace $\varepsilon h$ in the proof of (1), we obtain $\lim_{n \to \infty} P_tf(x + y_n) = P_tf(x)$ for any $f \in \mathcal{B}^b(\mathbb{B}), t > 0$ and $x \in \mathbb{B}$. \hfill \square

Proof of Proposition \text{1.4}. Since $\{h \in \mathbb{H} : \sup_{s \in [0,1]} \|\sigma^{-1}Tsh\|_{\mathbb{H}} < \infty\}$ is dense in $\mathbb{B}$, it suffices to prove for $y \in \mathbb{H}$ such that $\|\sigma^{-1}Tsy\|_{\mathbb{H}} \leq 1$ for $s \in [0,1]$. Since the boundedness of $\rho_0g$ implies $\nu_0(g) < \infty$ and $\nu_0(\{g > 0\}) = \infty$ implies $w(g) > 0, \Lambda^0$-a.e., \text{ (2.3)} holds true. By \text{ (2.4)} and \text{ (1.14)} we have

$$\left| P_tf(x + \varepsilon y) - P_tf(x) \right|$$

$$\leq 2 \frac{\|f\|_{\infty}}{\varepsilon} \int_{W_t \times \mathbb{B} \times [0,t]} |\Phi_{E}(w, z, s) - \Phi_{E}(w, z, s)| \Lambda^0(dw) \mu(dz) ds, \quad \varepsilon > 0.$$ 

Since $\rho_0g$ is bounded and Lipschitz continuous in $\| \cdot \|_{\mathbb{H}}$, there exists a constant $c_1 > 0$ such that

$$\left| \Phi_{E}(w, z, s) - \Phi_{E}(w, z, s) \right|$$

$$\leq \frac{|\varphi_{\sigma^{-1}Tsy}(z) - 1|}{w(g)} + \frac{|(\rho_0g)(z - \varepsilon \sigma^{-1}Tsy) - (\rho_0g)(z)|}{w(g) + g(z - \varepsilon \sigma^{-1}Tsy)}$$

$$\leq \frac{|\varphi_{\sigma^{-1}Tsy}(z) - 1|}{w(g)} + c_1 \varepsilon \sigma^{-1}Tsy \|_{\mathbb{H}} + c_1 g(z - \varepsilon \sigma^{-1}Tsy) - g(z)\|_{\mathbb{H}}.$$ 

Moreover, according to \text{ [23, Lemma 2.2]} with $\mathbb{B}$ in place of $\mathbb{R}^d$, for any $\theta > 0$, we have

$$\int_{W_t} \frac{\Lambda^0(dw)}{w(g)} = \frac{1}{\Gamma(\theta)} \int_0^\infty r^{\theta - 1} e^{-t\nu_0(1-e^{-r})} dr.$$ 

Combining this with \text{ (2.5)} and \text{ (2.6)} and letting $\varepsilon \to 0$, we obtain the desired gradient estimate. According to the proof of Theorem \text{1.3} in Section 4 with $P^1_t$ replaced by $P_t$, this along with the assumption on $T_t$ implies the second assertion. \hfill \square

3 Proof of Theorem \text{1.2}

By the triangle inequality for $\| \cdot \|_{\text{var}}$, it suffices to prove both assertions for small enough $\|y\|_{\mathbb{B}}$. 

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3.1 Case (i)

Let \( \|y\|_\mathcal{B} \leq \frac{1 + \|y\|_\mathcal{B}}{1 + \|\sigma^{-1}Ty\|_\mathcal{B}} \), which implies that

\[
\|\sigma^{-1}Ty\|_\mathcal{B} + \|y\|_\mathcal{B} \leq 1 + \frac{r_0}{2}, \quad s \in [0, t].
\]

Moreover, since \( \|P_i(x, \cdot) - P_i(x + y, \cdot)\|_{var} \leq 2 \) holds for all \( x, y \in \mathcal{B} \) and \( t > 0 \), we only have to prove the desired inequality for large \( t > 0 \). From now on, let us assume \( t \geq 2 \) and (3.1).

Now, let \( t \geq 2 \) and \( x, y \in \mathcal{B} \) such that (3.1) holds. Since \( T_s^\sigma \) is bounded in \( \mathcal{B} \) uniformly in \( s \), for any \( z \in \mathcal{B} \),

\[
J^z(w) := T_iz + \int_0^t T_{t-s}^\sigma dw_s
\]
is \( \Lambda \)-a.e. (also \( \Lambda^1 \)-a.e. and \( \Lambda^0 \)-a.e.) defined. Moreover, due to (1.3) and \( L = L^1 + L^0 \),

\[
X_t^z = J^z(L) = J^z(L^1 + L^0), \quad z \in \mathcal{B}, \quad t > 0.
\]

Next, let

\[
\tau_i(w) = \inf\{s > 0 : \Delta w_s \neq 0\}, \quad \tau_{i+1}(w) = \inf\{s > \tau_i(w) : \Delta w_s \neq 0\}, \quad i \geq 1.
\]

Since \( \lambda_0 = \nu_0(\mathbb{B}) \in (0, \infty) \), we have \( \mathbb{P}(\tau_i(L^0) > s) = e^{-\lambda os} \in (0, 1) \) for \( s > 0 \), and \( \tau_i(L^0) \uparrow \infty \) as \( i \uparrow \infty \). Moreover, let

\[
N_s(w) = \#\{i \geq 1 : \tau_i(w) \leq s\}, \quad s \geq 0.
\]

Then \( \{N_s(L^0)\}_{s \in [0, t]} \) is a Poisson process with parameter \( \lambda_0 \). Similarly, let

\[
\tilde{\tau}_i(w) = \inf\{s > 1 : \Delta w_s \neq 0\}, \quad \tilde{\tau}_{i+1}(w) = \inf\{s > \tilde{\tau}_i(w) : \Delta w_s \neq 0\}, \quad i \geq 1
\]
and

\[
\tilde{N}_s(w) = N_{s+1}(w) - N_1(w) = \#\{i \geq 1 : \tilde{\tau}_i \leq s+1\} = \#\{i \geq 1 : 1 < \tau_i \leq s+1\}, \quad s \in [0, t-1].
\]

Then \( \{\tilde{N}_s(L^0)\}_{s \in [0, t-1]} \) is a Poisson process with parameter \( \lambda_0 \), which is independent of \( \{\tau_1(L^0) > \varepsilon\} = \{N_\varepsilon(L^0) = 0\} \) for \( \varepsilon \in (0, 1) \). Finally, let

\[
\xi_i(w) = 1_{B(z_0, \frac{r_0}{2})}(\Delta w_{\tilde{\tau}_i(w)}),
\]

\[
\tilde{\xi}_i(w) = \frac{\rho_0(\Delta w_{\tilde{\tau}_i(w)} + \sigma^{-1}T_{\tilde{\tau}_i(w)}y)}{\rho_0(\Delta w_{\tilde{\tau}_i(w)})}(1_{B(z_0 - \sigma^{-1}T_{\tilde{\tau}_i(w)}y, \frac{r_0}{2})}(\varphi_{-\sigma^{-1}T_{\tilde{\tau}_i(w)}y})(\Delta w_{\tilde{\tau}_i(w)})), \quad i \geq 1.
\]

We have

\[
\int_{\mathbb{B} \times [1, t]} 1_{B(z_0, \frac{r_0}{2})}(z) w(dz, ds) = \sum_{i=1}^{\tilde{N}_{t-1}(w)} \xi_i(w),
\]

\[
\int_{\mathbb{B} \times [1, t]} \frac{\rho_0(z + \sigma^{-1}Tsy)}{\rho_0(z)}(1_{B(z_0 - \sigma^{-1}Ts_y, \frac{r_0}{2})}(\varphi_{-\sigma^{-1}Ts_y})(z)) w(dz, ds) = \sum_{i=1}^{\tilde{N}_{t-1}(w)} \tilde{\xi}_i(w),
\]
where we set \( \sum_{i=1}^{0} = 0 \) by convention. From now on, we will simply denote

\[
\tau_i = \tau_i(L^0), \quad \bar{\tau}_i = \bar{\tau}_i(L^0), \quad \xi_i = \xi_i(L^0), \quad N_s = N_s(L^0), \quad \tilde{N}_s = \tilde{N}_s(L^0).
\]

To characterize the coupling property of the solution, we first prove the following relation formula for \( X_t^x \) and \( X_t^{x+y} \).

**Lemma 3.1.** For any \( f \in \mathcal{B}_t(\mathbb{B}) \) and \( \varepsilon \in (0,1) \),

\[
\mathbb{E}\left\{ f(X_t^z) 1_{\{\tau_{1} > \varepsilon\}} \sum_{i=1}^{\tilde{N}_{t-1}} \xi_i \right\} = \mathbb{E}\left\{ f(X_t^{x+y}) 1_{\{\tau_{1} > \varepsilon\}} \sum_{i=1}^{\tilde{N}_{t-1}} \xi_i \right\}.
\]

**Proof.** Since \( \varepsilon \in (0,1) \), \( \{\tau_1(w) > \varepsilon\} = \{\tau_1(w + z1_{[s,t]}) > \varepsilon\} \) holds for \( s \in [1,t] \) and \( z \in \mathbb{B} \). Moreover, by the definition of \( J^x \) we have

\[
J^x(w^1 + w^0) + T_{t-s}\sigma z = J^x(w^1 + w^0 + z1_{[s,t]}).
\]

By Lemma 2.1 for

\[
h(w^0, z, s) = f(J^x(w^1, +w^0) + T_{t-s}\sigma z) 1_{\{\tau_1 > \varepsilon\}} \times B(z, \frac{\nu_0(z)}{\tau_1}) (w^0, z, s)
\]

with fixed \( w^1 \) and using \( (3.3) \), we obtain

\[
\int_{W_t^2} \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{B(z, \frac{\nu_0(z)}{\tau_1}) \times [1,t]} f(J^x(w^1 + w^0) + T_{t-s}\sigma z) 1_{\{\tau_1 > \varepsilon\}}(w^0) \nu_0(dz) \, ds
\]

\[
= \int_{W_t^2} \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{B(z, \frac{\nu_0(z)}{\tau_1}) \times [1,t]} f(J^x(w^1 + w^0 + z1_{[s,t]}) 1_{\{\tau_1 > \varepsilon\}}(w^0 + z1_{[s,t]}) \nu_0(dz) \, ds
\]

\[
= \int_{W_t^2} 1_{\{\tau_1 > \varepsilon\}}(w^0) f(J^x(w^1 + w^0)) \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{B(z, \frac{\nu_0(z)}{\tau_1}) \times [1,t]} w^0(dz, ds).
\]

Combining this with \( (3.2) \) and the first equation in \( (3.3) \) we arrive at

\[
\int_{W_t^2} \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{B(z, \frac{\nu_0(z)}{\tau_1}) \times [1,t]} f(J^x(w^1 + w^0) + T_{t-s}\sigma z) 1_{\{\tau_1 > \varepsilon\}}(w^0) \nu_0(dz) \, ds
\]

\[
= \mathbb{E}\left\{ f(X_t^x) 1_{\{\tau_{1} > \varepsilon\}} \sum_{i=1}^{\tilde{N}_{t-1}} \xi_i \right\}.
\]

On the other hand, noting that

\[
J^x(w^1 + w^0) + T_{t-s}\sigma z = J^{x+y}(w^1 + w^0 + (z - \sigma^{-1}T_s y)1_{[s,t]}),
\]
by Lemma 2.1 and the integral transform $z \mapsto z + \sigma^{-1}T_s y$, we obtain
\[
\int_{W_t^2} \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{B(z_0, \frac{r_0}{2}) \times [1, t]} f(J^x(w^1 + w^0) + T_{t-s} \sigma z)1_{\{\tau_1 > \varepsilon\}}(w^0) \nu_0(dz) \, ds \\
= \int_{W_t^2} \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{B(z_0, \frac{r_0}{2}) \times [1, t]} f(J^x^y(w^1 + w^0 + \{z - \sigma^{-1}T_s y\})1_{[s, t]}) \\
\times 1_{\{\tau_1 > \varepsilon\}}(w^0 + \{z - \sigma^{-1}T_s y\}1_{[s, t]}) \nu_0(dz) \\
= \int_{W_t^2} \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{[1, t]} ds \int_{B(z_0 - \sigma^{-1}T_s y, \frac{r_0}{2})} f(J^x^y(w^1 + w^0 + z1_{[s, t]}) \\
\times 1_{\{\tau_1 > \varepsilon\}}(w^0 + z1_{[s, t]}) \frac{\rho_0(z + \sigma^{-1}T_s y)}{\rho_0(z)} \varphi_{-\sigma^{-1}T_s y}(z) \nu_0(dz) \\
= \int_{W_t^2} 1_{\{\tau_1 > \varepsilon\}}(w^0) f(J^x^y(w^1 + w^0)) \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{B \times [1, t]} \frac{\rho_0(z + \sigma^{-1}T_s y)}{\rho_0(z)} \\
\times (1_{B(z_0 - \sigma^{-1}T_s y, \frac{r_0}{2})} \varphi_{-\sigma^{-1}T_s y})(z) w^0(dz, ds).
\]
Combining this with (3.2) and the second equation in (3.3), we conclude that
\[
\int_{W_t^2} \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{B(z_0, \frac{r_0}{2}) \times [1, t]} f(J^x(w^1 + w^0) + T_{t-s} \sigma z)1_{\{\tau_1 > \varepsilon\}}(w^0) \nu_0(dz) \, ds \\
= \mathbb{E}\left\{ f(X_t^{x+y})1_{\{\tau_1 > \varepsilon\}} \sum_{i=1}^{\tilde{N}_{t-1}} \tilde{\xi}_i \right\}.
\]
The desired formula follows from this and (3.4). \hfill \Box

**Lemma 3.2.** Given $\tilde{N}$, $\{\xi_i\}$ and $\{\tilde{\xi}_i\}$ are two conditionally i.i.d. sequences with
\[
\mathbb{E}(\xi_i | \tilde{N}) = \mathbb{E}(\tilde{\xi}_i | \tilde{N}) = \frac{\nu_0(B(z_0, \frac{r_0}{2}))}{\lambda_0},
\]
and
\[
\mathbb{E}(\tilde{\xi}_i | \tilde{N}) = \frac{\nu_0(B(z_0, \frac{r_0}{2}))}{\lambda_0}, \quad \mathbb{E}(\tilde{\xi}_i^2 | \tilde{N}) \leq \frac{\delta_1(\tau_1)}{\lambda_0}, \quad i \geq 1.
\]

**Proof.** Since $\{\Delta L_{\tilde{\eta}_i}^0\}$ are i.i.d. and independent of $\tilde{N}$ with common distribution $\frac{1}{\lambda_0} \nu_0$, and since $\tilde{\eta}_i$ is determined by $\tilde{N}$, it is clear that both $\{\xi_i\}$ and $\{\tilde{\xi}_i\}$ are conditionally i.i.d. sequences given $\tilde{N}$. Moreover, we have
\[
\mathbb{E}(\xi_i^2 | \tilde{N}) = \mathbb{E}(\xi_i | \tilde{N}) = \mathbb{E}(\tilde{\xi}_i) = \frac{\nu_0(B(z_0, \frac{r_0}{2}))}{\lambda_0}.
\]
Noting that $\nu_0(dz) = \rho_0(z) \mu(dz)$ and $\mu(dz + h) = \varphi_h(z) \mu(dz)$, we have

$$
\mathbb{E}(\xi_t | \tilde{N}) = \frac{1}{\lambda_0} \int_{B(z_0 - \sigma^{-1}T_{\tilde{\tau}_1}, \frac{\sigma^2}{2})} \frac{\rho_0(z + \sigma^{-1}T_{\tilde{\tau}_1}y) \varphi_{-\sigma^{-1}T_{\tilde{\tau}_1}y}(z)}{\rho_0(z)} \nu_0(dz)
$$

$$
= \frac{1}{\lambda_0} \int_{B(z_0 - \sigma^{-1}T_{\tilde{\tau}_1}, \frac{\sigma^2}{2})} \rho_0(z + \sigma^{-1}T_{\tilde{\tau}_1}y) \varphi_{-\sigma^{-1}T_{\tilde{\tau}_1}y}(z) \mu(dz)
$$

$$
= \frac{1}{\lambda_0} \int_{B(z_0, \frac{\sigma^2}{2})} \rho_0(z) \mu(dz)
$$

$$
= \nu_0(B(z_0, \frac{\sigma^2}{2})).
$$

Moreover, since $\|\sigma^{-1}T_{\tilde{\tau}_1}y\|_\mathcal{B} \leq 1 \wedge \frac{\sigma}{2}$ and $\tilde{\tau}_i \geq \tilde{\tau}_1$, we obtain

$$
\mathbb{E}(\xi_t^2 | \tilde{N}) = \frac{1}{\lambda_0} \int_{B(z_0 - \sigma^{-1}T_{\tilde{\tau}_1}, \frac{\sigma^2}{2})} \frac{\rho_0(z + \sigma^{-1}T_{\tilde{\tau}_1}y)^2 \varphi_{-\sigma^{-1}T_{\tilde{\tau}_1}y}(z)^2}{\rho_0(z)^2} \nu_0(dz)
$$

$$
\leq \frac{1}{\lambda_0} \int_{B(z_0, \sigma)} \frac{\rho_0(z + \sigma^{-1}T_{\tilde{\tau}_1}y)^2 \varphi_{-\sigma^{-1}T_{\tilde{\tau}_1}y}(z)^2}{\rho_0(z)} \mu(dz)
$$

$$
\leq \frac{\delta_1(\tilde{\tau}_1)}{\lambda_0}.
$$

This completes the proof. \(\square\)

**Proof of Theorem 1.2 (i).** As explained in the beginning of this section, we assume that $t \geq 2$ and let $y$ satisfy (3.1). By Lemma 3.1 and $\tilde{\tau}_1 \geq \tau_1$, for any $f \in \mathcal{B}_0(\mathcal{B})$ with $\|f\|_\infty \leq 1$ we have

$$
\left| \mathbb{E}(f(X_t^x) - f(X_t^{x+y})) 1_{\{\tilde{\tau}_1 > \epsilon\}} \right|
$$

$$
\leq \mathbb{E} \left| 1 - \frac{\nu_0(B(z_0, \frac{\sigma^2}{2}))(t - 1)}{\nu_0(B(z_0, \frac{\sigma^2}{2}))(t - 1)} \sum_{i=1}^{\tilde{N}_{t-1}} \xi_i \right| + \mathbb{E} \left\{ 1_{\{\tilde{\tau}_1 > \epsilon\}} \left| 1 - \frac{\nu_0(B(z_0, \frac{\sigma^2}{2}))(t - 1)}{\nu_0(B(z_0, \frac{\sigma^2}{2}))(t - 1)} \sum_{i=1}^{\tilde{N}_{t-1}} \xi_i \right| \right\}.
$$

Noting that $\tilde{\tau}_1$ is determined by $\tilde{N}$, we obtain from Lemma 3.2 that

$$
\mathbb{E} \left\{ 1_{\{\tilde{\tau}_1 > \epsilon\}} \left| 1 - \frac{\nu_0(B(z_0, \frac{\sigma^2}{2}))(t - 1)}{\nu_0(B(z_0, \frac{\sigma^2}{2}))(t - 1)} \sum_{i=1}^{\tilde{N}_{t-1}} \xi_i \right| \right\}^2
$$

$$
= \mathbb{E} \left\{ 1_{\{\tilde{\tau}_1 > \epsilon\}} \left( \frac{\sum_{i,j=1}^{\tilde{N}_{t-1}} \mathbb{E}(\xi_i \xi_j | \tilde{N})}{\nu_0(B(z_0, \frac{\sigma^2}{2}))(t - 1)^2} - \frac{2 \sum_{i=1}^{\tilde{N}_{t-1}} \mathbb{E}(\xi_i | \tilde{N})}{\nu_0(B(z_0, \frac{\sigma^2}{2}))(t - 1)} + 1 \right) \right\}
$$

$$
\leq \mathbb{E} \left\{ 1_{\{\tilde{\tau}_1 > \epsilon\}} \left( \tilde{N}_t^2 - \tilde{N}_{t-1} \frac{\delta_1(\tilde{\tau}_1)}{\lambda_0 \nu_0(B(z_0, \frac{\sigma^2}{2}))(t - 1)^2} \right) - \frac{2 \tilde{N}_{t-1} \delta_1(\tilde{\tau}_1)}{\lambda_0 (t - 1)^2} + 1 \right\}
$$

$$
\leq \frac{\delta_1(\epsilon)}{\nu_0(B(z_0, \frac{\sigma^2}{2}))(t - 1)^2}.
$$
Similarly and even simpler, we have
\[ \mathbb{E} \left( 1 - \frac{1}{\nu_0(B(z_0, \frac{r}{2}))(t-1)} \sum_{i=1}^{\bar{N}_t-1} \xi_i \right)^2 \leq \frac{1}{\nu_0(B(z_0, \frac{r}{2}))(t-1)}. \]

Combining these with (3.5) and noting that \( t - 1 \geq 1 \), we arrive at
\[ \left| \mathbb{E} \left( f(X_t^x) - f(X_t^{x+y}) \right) \right|_{\tau_1 > \varepsilon} \leq \frac{C_1 \sqrt{\delta_1(\varepsilon)}}{\sqrt{t}} \]
for some constant \( C_1 > 0 \) independent of \( t, x, y \) and \( \varepsilon \in (0, 1) \). Therefore, there exists a constant \( C > 0 \) independent of \( t, x, y \) and \( \varepsilon \in (0, 1) \) such that for \( \|f\|_{\infty} \leq 1 \),
\[ |P_t f(x) - P_t f(x + y)| \leq \frac{C_1 \sqrt{\delta_1(\varepsilon)}}{\sqrt{t}} + \mathbb{E} \left| \left( f(X_t^x) - f(X_t^{x+y}) \right) \right|_{\tau_1 \leq \varepsilon} \]
\[ \leq \frac{C_1 \sqrt{\delta_1(\varepsilon)}}{\sqrt{t}} + 2\mathbb{P}(\tau_1 \leq \varepsilon) \]
\[ = \frac{C_1 \sqrt{\delta_1(\varepsilon)}}{\sqrt{t}} + 2(1 - e^{-\lambda_0 \varepsilon}) \]
\[ \leq C \left( \varepsilon + \frac{\sqrt{\delta_1(\varepsilon)}}{\sqrt{t}} \right). \]

This completes the proof.

\[ \square \]

### 3.2 Case (ii)

For every \( \eta > 0 \), define \( \nu_\eta \) on \( B \) as follows:
\[ \nu_\eta(A) = \begin{cases} 
\nu(A), & \text{if } \nu(B) < \infty; \\
\nu(A \setminus \{z : \|z\|_B < \eta\}), & \text{if } \nu(B) = \infty,
\end{cases} \]
where \( A \in \mathcal{B} \). Then \( \nu_\eta \) is a finite measure on \( (B, \mathcal{B}) \). Recall that for any two finite measures \( \pi_1 \) and \( \pi_2 \) on \( (B, \mathcal{B}) \), \( \pi_1 \wedge \pi_2 := \pi_1 - (\pi_1 - \pi_2)^+ \), where \( (\pi_1 - \pi_2)^\pm \) refers to the Jordan-Hahn decomposition of the signed measure \( \pi_1 - \pi_2 \). In particular, \( \pi_1 \wedge \pi_2 = \pi_2 \wedge \pi_1 \), and
\[ (\pi_1 \wedge \pi_2)(B) = \frac{1}{2} (\pi_1(B) + \pi_2(B) - ||\pi_1 - \pi_2||_{\text{var}}). \]

The following is an extension of the main result in \[18\] to the infinite-dimensional setting.

**Theorem 3.3.** Let \( X_t \) be the process determined by (1.2). Assume that \( \sigma \) is invertible, and that there exist \( \eta, \varrho > 0 \) such that
\[ \gamma(\eta, \varrho, \varepsilon) := \inf_{t \geq \varepsilon, \|x\| \leq \varrho} \left\{ \nu_\eta \wedge (\delta_{\sigma^{-1}T_1 x} * \nu_\eta) \right\}(B) > 0 \]
holds for any \( \varepsilon > 0 \). Then there exists a constant \( C > 0 \) such that for all \( x, y \in \mathbb{B} \) and \( t > 0 \),

\[
\|P_t(x, \cdot) - P_t(x + y, \cdot)\|_{\text{var}} \leq C \left(1 + \|y\|_{\mathbb{B}}\right) \inf_{\varepsilon \in (0,1)} \left(\varepsilon + \frac{1}{\sqrt{\gamma(\eta, \rho, \varepsilon)t}}\right).
\]

We postpone the proof to the end of this subsection and present the proof of Theorem 1.2 (ii).

Proof of Theorem 1.2 (ii). Without loss of generality, we assume that \( 0 \notin B(z_0, r_0) \). Otherwise, we may take \( z'_0 \in B(z_0, r_0) \) and \( r'_0 > 0 \) such that \( 0 \notin B(z'_0, r'_0) \subset B(z_0, r_0) \), and use \( B(z'_0, r'_0) \) to replace \( B(z_0, r_0) \). Moreover, we take \( \varrho \in (0,1) \) small enough such that \( \|\sigma^{-1}T_t x\| \leq 1 \wedge \frac{\varrho^2}{4} \) holds for all \( \|x\|_{\mathbb{B}} \leq \varrho \) and \( t > 0 \).

By (1.4), (1.11) and the Cauchy-Schwarz inequality, for any \( t \geq \varepsilon \) and \( \eta \in (0, \frac{\varrho}{4}) \),

\[
\inf_{\|t \geq \varepsilon, \|x\|_{\mathbb{B}} \leq \varrho} \left\{ \nu_{\eta} \wedge (\delta_{\sigma^{-1}T_t x} \ast \nu_{\eta}) \right\}(\mathbb{B}) \\
\geq \inf_{\|t \geq \varepsilon, \|x\|_{\mathbb{B}} \leq \varrho} \int_{B(x_0, \frac{\varrho}{4})} \left( \rho_0(z) \wedge (\rho_0(z - \sigma^{-1}T_t x) \varphi_{\sigma^{-1}T_t x}(z)) \right) \mu(\mathrm{d}z) \\
\geq \inf_{\|t \geq \varepsilon, \|x\|_{\mathbb{B}} \leq \varrho} \left( \int_{B(x_0, \frac{\varrho}{4})} \varphi_{\sigma^{-1}T_t x}(z) \mu(\mathrm{d}z) \right)^2 \\
\times \left[ \sup_{\|t \geq \varepsilon, \|x\|_{\mathbb{B}} \leq \varrho} \int_{B(x_0, \frac{\varrho}{4})} \rho_0(z) \wedge (\rho_0(z - \sigma^{-1}T_t x) \varphi_{\sigma^{-1}T_t x}(z)) \mu(\mathrm{d}z) \right]^{-1}.
\]

Since the measure \( \mu \) has full support,

\[
\inf_{\|t \geq \varepsilon, \|x\|_{\mathbb{B}} \leq \varrho} \int_{B(x_0, \frac{\varrho}{4})} \varphi_{\sigma^{-1}T_t x}(z) \mu(\mathrm{d}z) \\
= \inf_{\|t \geq \varepsilon, \|x\|_{\mathbb{B}} \leq \varrho} \int_{B(x_0, \frac{\varrho}{4})} \mu(\mathrm{d}z - \sigma^{-1}T_t x) \\
\geq \int_{B(x_0, \frac{\varrho}{4})} \mu(\mathrm{d}z) > 0.
\]

On the other hand, by (1.9), for any \( t \geq \varepsilon \),

\[
\sup_{\|t \geq \varepsilon, \|x\|_{\mathbb{B}} \leq \varrho} \int_{B(x_0, \frac{\varrho}{4})} \frac{\varphi_{\sigma^{-1}T_t x}(z)^2}{\rho_0(z) \wedge (\rho_0(z - \sigma^{-1}T_t x) \varphi_{\sigma^{-1}T_t x}(z))} \mu(\mathrm{d}z) \\
\leq \sup_{\|t \geq \varepsilon, \|x\|_{\mathbb{B}} \leq \varrho} \left[ \int_{B(x_0, \frac{\varrho}{4})} \frac{\varphi_{\sigma^{-1}T_t x}(z)^2}{\rho_0(z)} \mu(\mathrm{d}z) + \int_{B(x_0, \frac{\varrho}{4})} \frac{\varphi_{\sigma^{-1}T_t x}(z)}{\rho_0(z - \sigma^{-1}T_t x)} \mu(\mathrm{d}z) \right] \\
\leq \sup_{\|t \geq \varepsilon, \|x\|_{\mathbb{B}} \leq \varrho} \left[ \int_{B(x_0, \frac{\varrho}{4})} \frac{\varphi_{\sigma^{-1}T_t x}(z)^2}{\rho_0(z)} \mu(\mathrm{d}z) + \int_{B(x_0, \frac{\varrho}{4})} \frac{\mu(\mathrm{d}z - \sigma^{-1}T_t x)}{\rho_0(z - \sigma^{-1}T_t x)} \right] \\
\leq \sup_{\|t \geq \varepsilon, \|x\|_{\mathbb{B}} \leq \varrho} \left[ \int_{B(x_0, \frac{\varrho}{4})} \frac{\varphi_{\sigma^{-1}T_t x}(z)^2}{\rho_0(z)} \mu(\mathrm{d}z) + \int_{B(x_0, \rho_0)} \frac{1}{\rho_0(z)} \mu(\mathrm{d}z) \right] < \infty.
\]
The required assertion (1.10) follows from the conclusions above and (3.7).

Proof of Theorem 3.3. As indicated in the proof of Theorem 1.2 (i), we only have to prove the result for \( \|x - y\|_\mathbb{B} \leq \varrho \) and \( t \geq 1 \). To this end, we modify the argument from the proof of [18, Theorem 1.1]. For any \( \eta > 0 \), let \( L^n \) be a compound Poisson process on \( \mathbb{B} \) with Lévy measure \( \nu_\eta \) such that \( L^n \) and \( L - L^n \) are independent Lévy processes. Then the random variables

\[
X^n_t := T_t x + \int_0^t T_{t-s} \sigma \, dL^n_s
\]

and

\[
X^n_t - X^n_s := \int_0^t T_{t-s} \sigma \, d(L_s - L^n_s)
\]

are independent. Denote by \( \mu_{\eta,t} \) the law of random variable

\[
X^n_t := X^n_t - T_t x = \int_0^t T_{t-s} \sigma \, dL^n_s.
\]

Construct a sequence \( \{\tau_i\} \) of i.i.d. random variables which are exponentially distributed with intensity \( C_\eta = \nu_\eta(\mathbb{B}) \), and introduce a further sequence \( \{U_i\} \) of i.i.d. random variables on \( \mathbb{B} \) with law \( \tilde{\nu}_\eta = \nu_\eta/C_\eta \). We will assume that the random variables \( \{U_i\} \) are independent of the sequence \( \{\tau_i\} \). Then, according to [24 Examples, Section 2], \( L^n_t = \sum_{i=1}^{N_t} U_i \) for every \( t \geq 0 \), where \( N_t := \sup \{k : \sum_{i=1}^{k} \tau_i \leq t\} \), for \( \sum_{i \in \mathbb{N}} := 0 \) by convention, is a Poisson process of intensity \( C_\eta \). Therefore, the random variable

\[
1_{\{\tau_1 \leq t\}} \sum_{k=1}^{\infty} 1_{\{N_t = k\}} \left( T_{t-\tau_1} \sigma U_1 + \cdots + T_{t-(\tau_1 + \cdots + \tau_k)} \sigma U_k \right)
\]

has the probability distribution \( \mu_{\eta,t} \).

Let \( P_t(x, \cdot) \) and \( P_t \) be the transition kernel and the transition semigroup of the Ornstein-Uhlenbeck process \( X^n_t \). Similarly, we denote by \( P^n_t(x, \cdot) \) and \( P^n_t \) the transition kernel and the transition semigroup of \( X^n_t \), and by \( Q^n_t(x, \cdot) \) and \( Q^n_t \) the transition kernel and the transition semigroup of \( X^n_t - X^n_t \). By the independence of the processes \( X^n_t \) and \( X^n_t - X^n_t \), we get

\[
\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} = \sup_{\|f\|_{\infty} \leq 1} \left| P_t f(x) - P_t f(y) \right|
\]

\[
= \sup_{\|f\|_{\infty} \leq 1} \left| P^n_t Q^n_t f(x) - P^n_t Q^n_t f(y) \right|
\]

\[
\leq \sup_{\|h\|_{\infty} \leq 1} \left| P^n_t h(x) - P^n_t h(y) \right|
\]

\[
= \sup_{\|h\|_{\infty} \leq 1} \left| \mathbb{E}(h(X^n_t)) - \mathbb{E}(h(X^n_t)) \right|.
\]

Following the argument leading to [18, (2.11)], we may write

\[
\mathbb{E} f(X^n_t) = \int_{\mathbb{B}} f(T_t x + z) \mu_{\eta,t}(dz) = f(T_t x) e^{-C_\eta t} + H f(x), \quad f \in \mathcal{B}_0(\mathbb{B})
\]
for
\[
H f(x) = \sum_{k=1}^{\infty} \int_{I_{t,k}} C^{k+1}_{\eta} e^{-C_{\eta}(t_1 + \cdots + t_{k+1})} \, dt_1 \cdots \, dt_{k+1} \int_{B} f(T_t x + z) \, \mu_{t_1, \ldots, t_k}(dz),
\]
where
\[
I_{t,k} := \{(t_1, \ldots, t_k, t_{k+1}) \in (0, \infty)^{k+1} : \sum_{i=1}^{k} t_i \leq t < \sum_{i=1}^{k+1} t_i\},
\]
\[
\mu_{t_1, \ldots, t_k} := (\tilde{\nu}_{\eta})^{k} \circ J_{t_1, \ldots, t_k}^{-1},
\]
\[
J_{t_1, \ldots, t_k}(y_1, \ldots, y_k) := T_{t-t_1} \sigma y_1 + \cdots + T_{t-(t_1 + \cdots + t_k)} \sigma y_k, \quad y_1, \ldots, y_k \in B.
\]

Then, for any \( t \geq 1 \) and \( \varepsilon \in (0, 1) \),
\[
\begin{align*}
\sup_{\|h\|_{\infty} \leq 1} \left| \mathbb{E}(h(X_t^{\eta,x})) - \mathbb{E}(h(X_t^{\eta,y})) \right| & \leq \sup_{\|h\|_{\infty} \leq 1} \left| \mathbb{E}\left((h(X_t^{\eta,x}) - h(X_t^{\eta,y}))1_{\{t_1 \leq \varepsilon\}}\right) \right| + \sup_{\|h\|_{\infty} \leq 1} \left| \mathbb{E}\left((h(X_t^{\eta,x}) - h(X_t^{\eta,y}))1_{\{t_1 \geq \varepsilon\}}\right) \right| \\
& \leq 2\mathbb{P}(t_1 \leq \varepsilon) + 2e^{-C_{\eta}t} + \sum_{k=1}^{\infty} \int_{I_{t,k} \cap \{(0, \infty)^{k+1} : t_1 \geq \varepsilon\}} C^{k+1}_{\eta} e^{-C_{\eta}(t_1 + \cdots + t_{k+1})} \, dt_1 \cdots \, dt_{k+1} \\
& \quad \times \sup_{\|h\|_{\infty} \leq 1} \left| \int_{B} h(T_t x + z) \mu_{t_1, \ldots, t_k}(dz) - \int_{B} h(T_t y + z) \mu_{t_1, \ldots, t_k}(dz) \right| \\
& = 2(1 - e^{-C_{\eta}\varepsilon}) + 2e^{-C_{\eta}t} + \sum_{k=1}^{\infty} \int_{I_{t,k} \cap \{(0, \infty)^{k+1} : t_1 \geq \varepsilon\}} C^{k+1}_{\eta} e^{-C_{\eta}(t_1 + \cdots + t_{k+1})} \, dt_1 \cdots \, dt_{k+1} \\
& \quad \times \sup_{\|h\|_{\infty} \leq 1} \left| \int_{B} h(T_t (x-y) + z) \mu_{t_1, \ldots, t_k}(dz) - \int_{B} h(z) \mu_{t_1, \ldots, t_k}(dz) \right| \\
& \leq 2C_{\eta}\varepsilon + 2e^{-C_{\eta}t} \\
& \quad + \sum_{k=1}^{\infty} \int_{I_{t,k} \cap \{(0, \infty)^{k+1} : t_1 \geq \varepsilon\}} C^{k+1}_{\eta} e^{-C_{\eta}(t_1 + \cdots + t_{k+1})} \\
& \quad \times \|\delta T_t(x-y) \ast \mu_{t_1, \ldots, t_k} - \mu_{t_1, \ldots, t_k}\|_{\text{var}} \, dt_1 \cdots \, dt_{k+1}.
\end{align*}
\]

To estimate \(\|\delta T_t(x-y) \ast \mu_{t_1, \ldots, t_k} - \mu_{t_1, \ldots, t_k}\|_{\text{var}}\) for any \( t_1 \geq \varepsilon \) and \( t \geq t_1 + \cdots + t_k \), we will use the Mineka and Lindvall-Rogers couplings for random walks as in [17, 18]. The remainder of this part is based on steps 4 and 5 in the proof of [18 Theorem 1.1]. In order to ease notations, we set \( n := \tilde{\nu}_{\eta} \) and \( n_a := \delta_a \ast \tilde{\nu}_{\eta} \) for any \( a \in B \). For any \( i \geq 1 \), let \( (U_i, \Delta U_i) \in B \times B \) be a pair of random variables with the following distribution
\[
\mathbb{P}\{(U_i, \Delta U_i) \in C \times D\} = \begin{cases} 
\frac{1}{2}(n \wedge n^{-a_i})(C), & \text{if } D = \{a_i\}; \\
\frac{1}{2}(n \wedge n^a)(C), & \text{if } D = \{-a_i\}; \\
(n - \frac{1}{2}(n \wedge n^{-a_i} + n \wedge n^a))(C), & \text{if } D = \{0\};
\end{cases}
\]

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where $C \in \mathcal{B}$, $a_i = \sigma^{-1} T_{t_i + \ldots + t_{i-1}} (x - y)$ and $D$ is any of the following three sets: $\{-a_i\}$, $\{0\}$ or $\{a_i\}$. It follows that, cf. see [17, Lemma 3.2],

$$\mathbb{P}(\Delta U_i = -a_i) = \frac{1}{2} (n \wedge (\delta_{-a_i} \ast n))(\mathbb{B}) = \frac{1}{2} (n \wedge (\delta_{-a_i} \ast n))(\mathbb{B}) = \mathbb{P}(\Delta U_i = a_i).$$

It is clear that the distribution of $U_i$ is $n$. Let $U'_i = U_i + \Delta U_i$. We claim that the distribution of $U'_i$ is also $n$. Indeed, for any $C \in \mathcal{B}$,

$$\mathbb{P}(U'_i \in C) = \mathbb{P}(U_i - a_i \in C, \Delta U_i = -a_i) + \mathbb{P}(U_i + a_i \in C, \Delta U_i = a_i) + \mathbb{P}(U_i \in A, \Delta U_i = 0)$$

$$= \frac{1}{2} (\delta_{-a_i} \ast (n \wedge n^{a_i})) (C) + \frac{1}{2} (\delta_{a_i} \ast (n \wedge n^{-a_i})) (C) + \left( n - \frac{1}{2} (n \wedge n^{-a_i} + n \wedge n^{a_i}) \right) (C)$$

$$= n(C),$$

where we have used that

$$\delta_{a_i} \ast (n \wedge n^{-a_i}) = n \wedge n^{a_i} \quad \text{and} \quad \delta_{-a_i} \ast (n \wedge n^{a_i}) = n \wedge n^{-a_i}.$$

Without loss of generality, we can assume that the pairs $(U_i, U'_i)$ are independent for all $i \geq 1$. Now we construct the coupling

$$(S_k, S'_k)_{k \geq 1} = \left( \sum_{i=1}^{k} T_{t_i - \ldots - t_{i-1}} \sigma U_i, \sum_{i=1}^{k} T_{t_i - \ldots - t_{i-1}} \sigma U'_i \right)_{k \geq 1}$$

of

$$\left( S_k \right)_{k \geq 1} := \left( \sum_{i=1}^{k} T_{t_i - \ldots - t_{i-1}} \sigma U_i \right)_{k \geq 1}.$$

Since $U'_i - U_i = \Delta U_i$ is either $\pm a_i$ or 0, we know that

$$\left( S'_k - S_k \right)_{k \geq 1} = \left( \sum_{i=1}^{k} T_{t_i - \ldots - t_{i-1}} \sigma (U'_i - U_i) \right)_{k \geq 1} = \left( \sum_{i=1}^{k} T_{t_i - \ldots - t_{i-1}} \sigma \Delta U_i \right)_{k \geq 1}.$$

is a random walk on $\mathbb{B}$ whose steps are independent and attain the values $-T_t (x - y)$, 0 and $T_t (x - y)$ with probabilities $\frac{1}{2} (1 - p_i)$, $p_i$ and $\frac{1}{2} (1 - p_i)$, respectively; the values of the $p_i$ are given by

$$p_i = \left( n - \frac{1}{2} (n \wedge n^{-a_i} + n \wedge n^{a_i}) \right) (\mathbb{B}) = 1 - n \wedge n^{-a_i} (\mathbb{B}).$$

Note that $\mu_{t_1, \ldots, t_k}$ is the law of the random variable $\sum_{i=1}^{k} T_{t_i - \ldots - t_{i-1}} \sigma U_i$. We get

\begin{equation}
\| \delta_{T_t (x - y)} \ast \mu_{t_1, \ldots, t_k} - \mu_{t_1, \ldots, t_k} \|_{\text{var}} \leq 2 \mathbb{P}(T^S > k),
\end{equation}
where
\[ T^S = \inf \{ i \geq 1 : S_i = S'_i + T_t(x - y) \}. \]

From (3.6) we get that for all \( i \geq 1, t_1, t \geq t_1 + \cdots + t_k \) and \( x, y \in B \) with \( \|x - y\| \leq \rho \),
\[ \frac{1}{2}(1 - p_i) = \frac{1}{2}(n \wedge (\delta_{a_i} * n))(B) \]
\[ \geq \frac{1}{2} \inf_{s \geq \epsilon, \|z\| \leq \rho} n \wedge (\delta_{\sigma^{-1}T_z} * n)(B) \]
\[ = \frac{1}{2} C_{\eta} \gamma(\eta, \rho, \varepsilon) > 0. \]

We will now estimate \( P(T^S > k) \). Let \( V_i, i \geq 1 \), be independent symmetric random variables on \( B \), whose distributions are given by
\[ P(V_i = z) = \begin{cases} \frac{1}{2}(1 - p_i), & \text{if } z = -T_t(x - y); \\ \frac{1}{2}(1 - p_i), & \text{if } z = T_t(x - y); \\ p_i, & \text{if } z = 0. \end{cases} \]

Set \( Z_k := \sum_{i=1}^k V_i \). We have seen earlier that
\[ T^S = \inf \{ k \geq 1 : Z_k = T_t(x - y) \}. \]

For any \( k \geq 1 \), let
\[ \kappa = \kappa(k) := \# \{ i : i \leq k \text{ and } V_i \neq 0 \} \]
and set \( \tilde{Z}_k := \sum_{i=1}^k \tilde{V}_i \), where \( \tilde{V}_i \) denotes the ith \( V_j \) such that \( V_j \neq 0 \). Then, \( \tilde{Z}_k \) is a symmetric random walk on \( B \) with iid steps which are either \(-T_t(x - y)\) or \( T_t(x - y)\) with probability \( 1/2 \). Define
\[ T^{\tilde{Z}} := \inf \{ k \geq 1 : \tilde{Z}_k = T_t(x - y) \}. \]

By (3.12),
\[ P(T^S > k) = P(T^S > k, \kappa \geq \frac{1}{2} C_{\eta} \gamma(\eta, \rho, \varepsilon)k) + P(T^S > k, \kappa \leq \frac{1}{2} C_{\eta} \gamma(\eta, \rho, \varepsilon)k) \]
\[ \leq P(T^{\tilde{Z}} > \frac{1}{2} C_{\eta} \gamma(\eta, \rho, \varepsilon)k) + P(\kappa \leq \frac{1}{2} \sum_{i=1}^k (1 - p_i)) \]
\[ \leq P(T^{\tilde{Z}} > \frac{1}{2} C_{\eta} \gamma(\eta, \rho, \varepsilon)k) + P \left( \left| \kappa - \sum_{i=1}^k (1 - p_i) \right| \geq \frac{1}{2} \sum_{i=1}^k (1 - p_i) \right). \]

Note that \( \kappa = \kappa(k) = \sum_{i=1}^k \zeta_i \), where \( \zeta_i = 1_{\{V_i \neq 0\}}, 1 \leq i \leq k \), are independent random variables.
variables with $P(\zeta_i = 0) = p_i$ and $P(\zeta_i = 1) = 1 - p_i$. Chebyshev’s inequality shows that

$$P \left( \left| \kappa - \sum_{i=1}^{k} (1 - p_i) \right| \geq \frac{1}{2} \sum_{i=1}^{k} (1 - p_i) \right) \leq \frac{4 \text{var}(\kappa)}{\left( \sum_{i=1}^{k} (1 - p_i) \right)^2} \leq 4 \sum_{i=1}^{k} p_i (1 - p_i) \left( \sum_{i=1}^{k} (1 - p_i) \right)^2 \leq 4 \frac{(1 - C^{-1}_{\eta} \gamma(\eta, \rho, \varepsilon)) \sum_{i=1}^{k} (1 - p_i)}{C^{-1}_{\eta} \gamma(\eta, \rho, \varepsilon)}.$$

(3.14)

For the second and the last inequalities we have used (3.12). On the other hand, by [18, Lemma 2.3],

$$P \left( T^\tilde{Z} > \frac{1}{2C_{\eta}} \gamma(\eta, \rho, \varepsilon) k \right) = P \left( \max_{i \leq \left\lceil \frac{\gamma(\eta, \rho, \varepsilon) k}{2C_{\eta}} \right\rceil} \tilde{Z}_i, \theta^* \right) < \|T_i(x - y)\|_B \right) \leq 2 \left( 0 \leq \left\| \tilde{Z}_i \right\|_B \leq \|T_i(x - y)\|_B \right),$$

where in the first equality $\theta^*$ is an element in the dual space $E^*$ of the Banach space $E$ such that the duality $\langle T_i(x, y), \theta^* \rangle = \|T_i(x - y)\|_B$, and in the second equality $\|\tilde{Z}_i\|_B = \langle \tilde{Z}_i, \theta^* \rangle$ for $i \geq 1$. From the construction above, we know that $(\|\tilde{Z}_k\|_B)_{k \geq 1}$ is a symmetric random walk on $\mathbb{R}$ with iid steps with values $\pm \|T_i(x - y)\|_B$. Using the central limit theorem we find for sufficiently large values of $k \geq k_0$ and some constant $C_0 = C_0(k_0) \geq 1$

$$P \left( T^\tilde{Z} > \frac{1}{2C_{\eta}} \gamma(\eta, \rho, \varepsilon) k \right) \leq 2 \left( 0 \leq \left\| \tilde{Z}_i \right\|_B \leq \|T_i(x - y)\|_B \right),$$

(3.15)
Combining (3.13), (3.14) and (3.15) gives for all \( x, y \in \mathbb{B} \) with \( \left\| x - y \right\|_\mathbb{B} \leq \varrho, \ t \geq (t_1 + \cdots + t_k) \lor 1, \ t_1 \geq \varepsilon \) and \( k \geq k_0 \) that
\[
\mathbb{P}(T^S > k) \leq \frac{C_0 \sqrt{C_{\eta}}}{\sqrt{\pi \gamma(\eta, \varrho, \varepsilon)k}} + \frac{4(1 - C_{\eta}^{-1} \gamma(\eta, \varrho, \varepsilon))}{C_{\eta}^{-1} \gamma(\eta, \varrho, \varepsilon)k}.
\]

According to the estimate above and (3.11), we can find an integer \( k_0 \) and a constant \( C_1 > 0 \) such that
\[
\left\| \delta_{t_1(x-y)} * \mu_{t_1, \cdots, t_k} - \mu_{t_1, \cdots, t_k} \right\|_{\text{var}} \leq C_1 \left( \frac{1}{\gamma(\eta, \varrho, \varepsilon)k} + \frac{1}{\gamma(\eta, \varrho, \varepsilon)k} \right), \ \ k \geq k_0, \ \varepsilon \in (0, 1), \ t \geq 1
\]
holds for all \( x, y \in \mathbb{B} \) with \( \left\| x - y \right\|_\mathbb{B} \leq \varrho \) and \( (t_1, \cdots, t_k) \in I_{t, k} \land \{(0, \infty)^{k+1} : t_1 \geq \varepsilon\} \).

Combining this with (3.9) and (3.10), we obtain that for all \( x, y \in \mathbb{B} \) with \( \left\| x - y \right\|_\mathbb{B} \leq \varrho, \ t \geq 1 \) and \( \varepsilon > 0 \),
\[
\left\| P_t(x, \cdot) - P_t(y, \cdot) \right\|_{\text{var}} \leq 2C_\eta \varepsilon + 2e^{-C_\eta t} + 2 \sum_{k=1}^{k_0} \int_{I_{t, k}} C_{\eta}^{k+1} e^{-C_\eta (t_1 + \cdots + t_k+1)} dt_1 \cdots dt_{k+1}
\]

\[
+ \frac{C_1}{\sqrt{\gamma(\eta, \varrho, \varepsilon)}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \int_{I_{t, k}} C_{\eta}^{k+1} e^{-C_\eta (t_1 + \cdots + t_k+1)} dt_1 \cdots dt_{k+1}
\]

\[
+ \frac{C_1}{\gamma(\eta, \varrho, \varepsilon)} \sum_{k=1}^{\infty} \frac{1}{k} \int_{I_{t, k}} C_{\eta}^{k+1} e^{-C_\eta (t_1 + \cdots + t_k+1)} dt_1 \cdots dt_{k+1}
\]

\[
\leq 2C_\eta \varepsilon + 2e^{-C_\eta t} \left( 1 + C_\eta \sum_{k=1}^{k_0} \frac{C_{\eta}^{k+1}}{k!} \right)
\]

\[
+ \frac{C_1 C_\eta}{\sqrt{\gamma(\eta, \varrho, \varepsilon)}} \sum_{k=1}^{\infty} \frac{C_{\eta}^{k+1} k!}{\sqrt{k}} e^{-C_\eta t} + \frac{C_1 C_\eta}{\gamma(\eta, \varrho, \varepsilon)} \sum_{k=1}^{\infty} \frac{C_{\eta}^{k+1} k!}{k!} e^{-C_\eta t}
\]

\[
\leq C_2 \left( \varepsilon + e^{-\frac{t}{2C_\eta}} + \frac{1}{\sqrt{\gamma(\eta, \varrho, \varepsilon)t}} + \frac{1}{\gamma(\eta, \varrho, \varepsilon)t} \right)
\]
holds for some constant \( C_2 > 0 \) depending only on \( C_\eta \) and \( C_1 \). To finish the proof, let
\[
\delta_t := \inf_{\varepsilon > 0} \left( \varepsilon + \frac{1}{\sqrt{\gamma(\eta, \varrho, \varepsilon)t}} \right),
\]
\[
\varepsilon_t := \sup \left\{ \varepsilon > 0 : \varepsilon^2 \gamma(\eta, \varrho, \varepsilon) \leq \frac{1}{t} \right\}, \ \ t \geq 1.
\]

Then it is easy to see that \( \delta_t, \varepsilon_t \downarrow 0 \) as \( t \uparrow \infty \) and \( \varepsilon_t \geq \frac{1}{\sqrt{C_\eta t}} \). Moreover, since \( \gamma(\eta, \varrho, \varepsilon) \) is increasing in \( \varepsilon \), for any \( \varepsilon \in (0, \varepsilon_t) \),
\[
\varepsilon + \frac{1}{\sqrt{\gamma(\eta, \varrho, \varepsilon)t}} \geq \lim_{\varepsilon' \uparrow \varepsilon_t} \frac{1}{\sqrt{\gamma(\eta, \varrho, \varepsilon')}t} \geq \lim_{\varepsilon' \uparrow \varepsilon_t} \varepsilon' = \varepsilon_t.
\]
So, $\delta_t \geq \varepsilon_t$. Therefore, there exists a constant $C_3 > 0$ such that

$$
\inf_{\varepsilon > 0} \left( \varepsilon + \frac{1}{\gamma(\eta, \varnothing, \varepsilon) t} + \frac{1}{\gamma(\eta, \varnothing, \varepsilon) t} \right)
\leq \delta_t + e^{-\frac{1}{2}\varepsilon t} + \delta^2 + \delta_t \leq \delta_t + e^{-\frac{1}{2}\varepsilon t} + \delta^2 \leq C_3 \delta_t, \quad t \geq 1.
$$

Combining this with (3.16) we complete the proof. \qed

4 Proof of Theorem 1.3

Let $L^1, L^0, \Lambda^1, \Lambda^0$ be in Section 3.1. In particular, $L^0$ is a compound Poisson process with jump measure $\nu_0$. Then $L^0$ can be formulated as

$$
L^0_t = \sum_{i=0}^{N_t} \xi_i, \quad t > 0,
$$

where $N_t := \#\{s \in [0, t] : \Delta L^0_s \neq 0\}$, $\xi_i = \Delta L^0_{\tau_i}$ for $\tau_i$ the $i$-th jump time of $L^0$. It is well-known that $N$, $\{\xi_i\}$ are independent, $N$ is the Poisson process with parameter $\lambda_0$, and $\{\xi_i\}$ have common distribution $\frac{1}{\lambda_0} \nu_0$. To derive exponential convergence of $P_t$ in the total variational norm, we make use of the decomposition

$$
P_t f(x) = \mathbb{E}(1_{\{N_t = 0\}} f(X^x_t)) + P^1_t f(x),
\quad P^1_t f(x) = \mathbb{E}(1_{\{N_t \geq 1\}} f(X^x_t)), \quad f \in \mathcal{B}(\mathbb{B}), t \geq 0, x \in \mathbb{B}.
$$

(4.1)

Since when $t \to \infty$, $\mathbb{E}(1_{\{N_t = 0\}} f(X^x_t))$ decays exponentially fast, it suffices to prove the exponential convergence of $P^1_t$. To this end, we first consider the gradient estimate of $P^1_t$.

**Proposition 4.1.** Assume (A), (1.11) and

$$
\Gamma_t := \frac{1}{1 - e^{-\lambda_0 t}} \int_0^t e^{-\lambda_0 r} \left( \sup_{\|z\|_\mathbb{B} \leq 1} \sup_{s \geq r} \|T_s z\|_\mathbb{B} \right) dr < \infty, \quad t > 0.
$$

Then there exists a constant $c > 0$ such that

$$
\|\nabla P^1_t f\|_\infty \leq c \Gamma_t \|f\|_\infty, \quad t > 0, f \in \mathcal{B}(\mathbb{B}).
$$

**Proof.** The proof is modified from that of [21, Theorem 3.1]. Since $\cup_{s > 0} T_s \mathbb{B}$ is dense in $\mathbb{B}$, it suffices to find out a constant $c > 0$ such that for any $s_0 > 0$ and $z_0 \in T_{s_0} \mathbb{B}$ with $\|z_0\|_\mathbb{B} \leq 1$, one has

$$
|\nabla z_0 P^1_t f(x)| \leq c \Gamma_t \|f\|_\infty, \quad x \in \mathbb{B}, f \in \mathcal{B}(\mathbb{B}).
$$

(4.2)
To prove this inequality, we first establish a formula for $P_t^1$ as in [21 (3.8)] where $\sigma = I$ is considered. Recall that for a random variable $(\xi, \tau)$ on $\mathbb{B} \times [0, t]$ such that the distribution of $(L^0, \xi, \tau)$ is
\[ g(w, z, s) \Lambda^0(dw) \nu_0(dz) ds, \]
[22 Corollary 2.3] implies that
\[
\mathbb{E}\{(F1_{\{U > 0\}})(L^0)\} = \mathbb{E}\frac{F1_{\{U > 0\}}(L^0 + \xi1_{[\tau, t]})}{U}
\]
holds for positive measurable function $F$ on $W_t$, where
\[
U(w) := \sum_{s \in [0, t]: \Delta w_s \neq 0} g(w - \Delta w_s 1_{[s, t]}, \Delta w_s, s).
\]
Now, let $(\xi, \tau)$ be independent of $(L^1, L^0)$ with distribution $\frac{1}{\lambda_0}1_{[0, t]}(s)\nu_0(dz)ds$. We have $g(w, z, s) = \frac{1}{\lambda_0}1_{[0, t]}(s)$, so that
\[
U(L^0 + \xi1_{[\tau, t]}) = \frac{N_t + 1}{\lambda_0 t} > 0.
\]
Therefore, letting $Y_t = \int_0^t T_{t-s} \sigma dL^1_s$ which is independent of $(L^0, \xi, \tau)$, combining (1.3) with (4.3) we obtain
\[
P_t^1 f(x + \varepsilon z_0) = \mathbb{E}\left\{ f(Y_t + T_t(x + \varepsilon z_0) + \int_0^t T_{t-s} \sigma dL^0_s) 1\{|N_t| \geq 1\}\right\}
= \frac{1}{\lambda_0 t} \mathbb{E}\left\{ \frac{f(Y_t + T_t x + \int_0^t T_{t-s} \sigma d\{L^0 + (\xi + \varepsilon \sigma^{-1}T_{t-z_0})1_{[\tau, t]}\})}{N_t + 1} \right\}.
\]
On the other hand, it is easy to see from (1.1) that the distribution of $(L^0, \xi + \varepsilon \sigma^{-1}T_{t-z_0}, \tau)$ is
\[
\frac{\varphi_{\varepsilon^{-1}T_{t-z_0}}(z) \rho_0(z - \varepsilon^{-1}T_{t-z_0})1_{[0, t]} \Lambda^0(dw) \nu_0(dz) ds}{\lambda_0 \rho_0(z)} =: g(w, z, s) \Lambda^0(dw) \nu_0(dz) ds.
\]
According to (4.4) we have $\{U(L^0) > 0\} = \{N_t \geq 1\}$ and
\[
U(L^0) = \frac{1}{\lambda_0 t} \sum_{i=1}^{N_t} \frac{\varphi_{\varepsilon^{-1}T_{t-z_0}}(\xi_i) \rho_0(\xi_i - \varepsilon^{-1}T_{t-z_0})}{\rho_0(\xi_i)}.
\]
So, applying (4.3) to $FU$ in place of $F$, we obtain
\[
\mathbb{E}\{(FU)(L^0)1_{\{N_t \geq 1\}}\} = \mathbb{E}\{(F1_{\{U > 0\}})(L^0 + (\xi + \varepsilon \sigma^{-1}T_{t-z_0})1_{[\tau, t]}\}.
\]
Taking \( n_t(w) = \sum_{s \leq t} 1_{\{\nabla w_i \neq 0\}} \) such that \( N_t = n_t(L^0) \), and letting
\[
F(w) = \frac{f(Y_t + T_t x + \int_{\mathbb{B} \times [0,t]} T_{t-s} \sigma z w(dz, ds))}{n_t(w)} 1_{\{n_t(w) \geq 1\}},
\]
we arrive at
\[
\frac{1}{\lambda_0 t} \mathbb{E} \left\{ f(Y_t + T_t x + \int_0^t T_{t-s} \sigma dL_s^0) 1_{\{N_t \geq 1\}} \frac{\sum_{i=1}^{N_t} \phi_{\varepsilon \sigma^{-1} T_{t_i} z_0}(\xi_i) \rho_0(\xi_i - \varepsilon \sigma^{-1} T_{t_i} z_0)}{\rho_0(\xi_i)} \right\}
\]
\[
= \mathbb{E} \left\{ \frac{f((Y_t + T_t x + \int_0^t T_{t-s} \sigma d\{L^0 + (\xi + \varepsilon \sigma^{-1} T_{t_i} z_0)1_{[\tau_\varepsilon,d]}\})_s))}{N_t + 1} \right\}.
\]
Combining this with (4.5) and noting that \( X^x_t = Y_t + T_t x + \int_0^t T_{t-s} \sigma dL_s^0 \) due to (4.3), we obtain
\[
P_t^1 f(x + \varepsilon z_0) = \mathbb{E} \left\{ f(X^x_t) 1_{\{N_t \geq 1\}} \frac{\sum_{i=1}^{N_t} \phi_{\varepsilon \sigma^{-1} T_{t_i} z_0}(\xi_i) \rho_0(\xi_i - \varepsilon \sigma^{-1} T_{t_i} z_0) - \rho_0(\xi_i)}{\varepsilon \rho_0(\xi_i)} \right\}.
\]
Therefore,
\[
\frac{|P_t^1 f(x + \varepsilon z_0) - P_t^1 f(x)|}{\varepsilon} = \mathbb{E} \left\{ f(X^x_t) 1_{\{N_t \geq 1\}} \frac{1}{N_t} \sum_{i=1}^{N_t} \phi_{\varepsilon \sigma^{-1} T_{t_i} z_0}(\xi_i) \rho_0(\xi_i - \varepsilon \sigma^{-1} T_{t_i} z_0) - \rho_0(\xi_i) - \varepsilon \rho_0(\xi_i) \right\}
\]
\[
\leq \frac{\|f\|_\infty}{\varepsilon \lambda_0} \mathbb{E} \left\{ 1_{\{N_t \geq 1\}} \frac{1}{N_t} \sum_{i=1}^{N_t} \int_{\mathbb{B}} |\phi_{\varepsilon \sigma^{-1} T_{t_i} z_0}(z) \rho_0(z - \varepsilon \sigma^{-1} T_{t_i} z_0) - \rho_0(z) | \mu(dz) \right\}
\]
\[
\text{for any } \varepsilon > 0. \text{ Since } z_0 \in T_{s_0} \mathbb{B}, \text{ there exists } z_1 \in \mathbb{B} \text{ such that } z_0 = T_{s_0} z_1. \text{ So, (4.12) yields }
\]
\[
c(z_0) := \sup_{s \geq 0} \|\sigma^{-1} T_s z_0\|_{\mathbb{H}} \leq \|z_1\|_\mathbb{B} \sup_{\|z\|_\mathbb{B} \leq 1} \sup_{s \geq s_0} \|\sigma^{-1} T_s z\|_{\mathbb{H}} < \infty.
\]
This implies
\[
\|\varepsilon \sigma^{-1} T_{t_i} z_0\|_\mathbb{H} \leq \varepsilon c(z_0) \leq 1
\]
for small enough \( \varepsilon > 0 \). Therefore, it follows from (4.11) and (4.11) that
\[
\int_{\mathbb{B}} |\phi_{\varepsilon \sigma^{-1} T_{t_i} z_0}(z) \rho_0(z - \varepsilon \sigma^{-1} T_{t_i} z_0) - \rho_0(z) | \mu(dz)
\]
\[
\leq \int_{\mathbb{B}} |\rho_0(z - \varepsilon \sigma^{-1} T_{t_i} z_0) - \rho_0(z) | \phi_{\varepsilon \sigma^{-1} T_{t_i} z_0}(z) \mu(dz) + \int_{\mathbb{B}} \rho_0(z) |\phi_{\varepsilon \sigma^{-1} T_{t_i} z_0}(z) - 1 | \mu(dz)
\]
\[
= \int_{\mathbb{B}} |\rho_0(z) - \rho_0(z + \varepsilon \sigma^{-1} T_{t_i} z_0) | \mu(dz) + \int_{\mathbb{B}} \rho_0(z) |\phi_{\varepsilon \sigma^{-1} T_{t_i} z_0}(z) - 1 | \mu(dz)
\]
\[
\leq c \|\varepsilon \sigma^{-1} T_{t_i} z_0\|_\mathbb{H} \leq \varepsilon c \sup_{s \geq t_1} \|\sigma^{-1} T_s z_0\|_\mathbb{H}
\]
holds for small enough $\varepsilon > 0$ and some constant $c > 0$. Combining this with (4.6) and using the fact that the conditional distribution of $\tau_1$ under $N_t \geq 1$ is $\frac{\lambda_0 e^{-\lambda_0 t} I_{\{0,1\}}}{1-e^{-\lambda_0 t}} \, ds$, we obtain

$$\frac{|P_t^1 f(x + \varepsilon z_0) - P_t^1 f(x)|}{\varepsilon} \leq c\|f\|_\infty \int_0^t e^{-\lambda_0 r} \left( \sup_{s \geq r} \|\sigma^{-1} T_s z_0\|_H \right) \, dr \leq c \Gamma_t \|f\|_\infty$$

for small enough $\varepsilon > 0$. Then (4.2) follows by letting $\varepsilon \to 0$. \hfill \Box

**Proof of Theorem 1.3.** By (4.1) and Proposition 4.1 we have

$$|P_t f(x) - P_t f(y)| \leq 2\|f\|_\infty e^{-\lambda_0 t} + |P_t^1 f(x) - P_t^1 f(y)|$$

$$\leq 2\|f\|_\infty e^{-\lambda_0 t} + c \Gamma_t \|f\|_\infty \|x - y\|_B.$$

Since $\|T_s\|_B \leq ce^{-\lambda t}$, it follows from (1.3) that

$$\|X_t^x - X_t^y\|_B \leq ce^{-\lambda t} \|x - y\|_B, \quad x, y \in B, t \geq 0.$$

Combining this with (4.7) and using the Markov property, we arrive at

$$|P_t f(x) - P_t f(y)|$$

$$\leq \mathbb{E} |P_s f(X_{t-s}^x) - P_s f(X_{t-s}^y)|$$

$$\leq 2\|f\|_\infty e^{-\lambda_0 s} + c \Gamma_s \|f\|_\infty \|X_{t-s}^x - X_{t-s}^y\|_B$$

$$\leq c_1 \|f\|_\infty (1 + \|x - y\|_B) \{e^{-\lambda_0 s} \lor (\Gamma_s e^{-\lambda(t-s)})\}, \quad s \in (0, t)$$

for some constant $c_1 > 0$. Taking $s = \frac{\lambda t}{\lambda_0 + \lambda}$ and using (1.12), we prove the desired estimate for $t \geq \frac{\lambda_0 + \lambda}{\lambda}$. The proof is then finished since the inequality trivially holds for some constant $C > 0$ for $t \leq \frac{\lambda_0 + \lambda}{\lambda}$. \hfill \Box

## 5 Two specific models

In the following two examples we take the reference measure $\mu$ to be the Wiener measure on the Brownian path space, and the Gaussian measure on a separable Hilbert space, respectively.

### 5.1 Wiener measure

Let $B = \{x \in C([0,1]; \mathbb{R}^d) : x_0 = 0\}$, and let $\mu$ be the Wiener measure on $B$, i.e. the distribution of the $d$-dimensional Brownian motion $(B_s)_{s\in[0,1]}$. Let $H = \{h \in B : \int_0^1 |\dot{h}_s|^2 \, ds < \infty\}$ be the Cameron-Martin space. Then $(B, H, \mu)$ is known as the Wiener space (see Chapter 1).

By the Cameron-Martin theorem (or the Girsanov theorem), (1.1) holds for

$$\varphi_h(z) = \exp \left[ \int_0^1 \langle \dot{h}_s, dz_s \rangle - \frac{1}{2} \int_0^1 |\dot{h}_s|^2 \, ds \right].$$
where \( \int_0^1 \langle \dot{h}_s, dz_s \rangle \) is the Itô stochastic integral w.r.t. \((z_s)_{s \in [0,1]}\), which is the Brownian motion under \( \mu \).

Let \((\mathbb{B}, \mathbb{H}, \mu)\) be the Wiener space specified above, and let \( \Delta \) be the Laplace operator on \([0, 1]\) with Dirichlet boundary condition at 0, and with either Dirichlet or Neumann boundary condition at 1. We call \( \Delta \) the Dirichlet or the Dirichlet-Neumann Laplacian on \([0, 1]\). Let \( P_t \) be the semigroup associated with the SDE

\[
dX_t = \Delta X_t \, dt + dL_t,
\]

where \( L_t \) is a Lévy process on \( \mathbb{B} \) with Lévy measure \( \nu \), and let \( \nu_0 \) satisfy (1.4).

**Proposition 5.1.** (1) If \( \nu_0(\mathbb{B}) = \infty \), then \( P_t \) is strong Feller for any \( t > 0 \).

(2) If \( \nu_0(\mathbb{B}) < \infty \) and there exist \( z_0 \in \mathbb{B} \) and \( r_0 > 0 \) such that \( \inf_{B(z_0, r_0)} \rho_0 > 0 \), then

\[
\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\var}, \quad t > 0, x, y \in \mathbb{B}
\]

holds for some constant \( C > 0 \).

(3) If \( \rho_0 \) is Lipschitz continuous and \( \lambda_0 := \nu_0(\mathbb{B}) \in (0, \infty) \), then (1.4) holds for \( \lambda > 0 \) the first eigenvalue of \( \Delta \) on \([0, 1]\) under the underlying boundary condition.

**Proof.** By the gradient estimate for the (Dirichlet or Dirichlet-Neumann) heat semigroup \( T_s \) on the interval \([0, 1]\) (see e.g. [24, Section 2.4] and the references therein), there exists a constant \( c_1 > 0 \) such that

\[
\left| \frac{d}{dr}(T_s y)(r) \right| \leq \frac{c_1 \|y\|_{\mathbb{B}}}{\sqrt{s}}, \quad s > 0, r \in [0, 1], y \in \mathbb{B}.
\]

Then

\[
(5.2) \quad \|T_s y\|_{\mathbb{B}} \leq \frac{c_1 \|y\|_{\mathbb{B}}}{\sqrt{s}}, \quad s > 0, y \in \mathbb{B}.
\]

Therefore, (A) holds for \( \sigma = I. \) By (5.1) and (5.2), for \( \mu \)-a.e. Brownian path \( z \), we have

\[
\sup_{\|y\|_{\mathbb{B}} \leq 1} \varphi_{T_s y}(z + T_s y) \leq \sup_{\|y\|_{\mathbb{B}} \leq c_1 s^{1/2}} e^{\int_0^1 \langle \dot{h}_u, dz_u \rangle - \int_0^1 |\dot{h}_u|^2 \, du} \mu(\mathbb{B}) < \infty.
\]

Thus, (1.6) holds and Theorem 1.1 implies the first assertion.

Next, noting that

\[
\int_{\mathbb{B}} e^{2 \int_0^1 \langle \dot{h}_r, dz_r \rangle - 2 \int_0^1 |\dot{h}_r|^2 \, dr} \mu(\mathbb{B}) = 1, \quad h \in \mathbb{H},
\]

we obtain

\[
\int_{\mathbb{B}} \varphi_{T_s y}(z)^2 \mu(\mathbb{B}) = \int_{\mathbb{B}} e^{2 \langle \dot{h}_r(T_s y), dz_r \rangle - \int_0^1 |\dot{h}_r(T_s y)|^2 \, dr} \mu(\mathbb{B})
\]

\[
= e^{\int_0^1 |\dot{h}_r(T_s y)|^2 \, dr} \leq e^{c_2 \|y\|_{\mathbb{B}}^2 / s}, \quad s > 0, y \in \mathbb{B}.
\]
This implies that $\delta_2(\varepsilon) \leq c_2 e^{c_2^2/\varepsilon}$ for some constant $c_2 > 0$ and all $\varepsilon \in (0, 1)$. Thus, the second assertion follows from Theorem 1.2.

Finally, to prove (3) it suffices to verify (1.11) and (1.12) in Theorem 1.3. Since (1.12) follows from (5.2), we only have to prove (1.11). By the Lipschitz continuity of $\rho_0$, there exist constants $c_3, c_4 > 0$ such that

$$|\rho_0(z) - \rho_0(z + h)| \leq c_3 \|h\|_B \leq c_3 \|h\|_H$$

and

$$\mu(p_0^2) \leq c_4 E \sup_{s \in [0,1]} (1 + |B_s|^2) < \infty.$$ 

Moreover,

$$\mu((\varphi_h - 1)^2) = \mu(\varphi_h^2) - 1 = E e^{2 \int_0^t (\varphi_s, dB_s) - \|h\|_B^2} - 1 = e^{\|h\|_B^2} - 1 \leq e^{\|h\|_H^2}$$

holds for $\|h\|_H \leq 1$. Then (1.11) holds for some constant $c > 0$. \hfill \Box

### 5.2 Gaussian measure

Let $B$ be a separable Hilbert space with ONB $\{e_k\}_{k \geq 1}$, and $\mu$ the Gaussian measure with trace class covariance operator $Q$ such that $Q e_k = q_k^{-1} e_k, q_k > 0$ and $\sum_{k=1}^{\infty} q_k^{-1} < \infty$ (see [6], Chapter 2). Coordinating $z \in B$ by $(z_k = \langle z, e_k \rangle)_{k \geq 1}$, we have

$$\mu(dz) = \prod_{k=1}^{\infty} \mu_k(dz_k), \quad \mu_k(dz_k) = \frac{\sqrt{q_k}}{\sqrt{2\pi}} \exp\left[-\frac{q_k z_k^2}{2}\right] dz_k, \quad k \geq 1.$$ 

Next, let $A$ be the self-adjoint operator on $B$ with $A e_k = -\lambda_k e_k, \lambda_k \geq 0$ for $k \geq 1$ and

$$\beta(\varepsilon) := \sup_{k \geq 1} e^{-\varepsilon \lambda_k} q_k^2 \varepsilon < \infty, \quad \varepsilon > 0.$$ 

Let $L_t$ be a Lévy process on $B$ with Lévy measure $\nu$ satisfying (1.4). Let $P_t$ be the Markov semigroup associated to the linear SDE

$$dX_t = AX_t dt + dL_t.$$

**Proposition 5.2.** (1) If $\nu_0(B) = \infty$, then $P_t$ is strong Feller for $t > 0$.

(2) If $\nu_0(B) < \infty$ and there exist $z_0 \in B$ and $r_0 > 0$ such that $c_0 := \inf_{B(z_0, r_0)} \rho_0 > 0$, then (1.10) holds for

$$\delta_2(\varepsilon) = \frac{1}{c_0} \left[ 1 + \exp\left( \sup_{k \geq 1} q_k e^{-2\varepsilon \lambda_k} \right) \right] < \infty, \quad \varepsilon > 0.$$ 

If, in particular, $q_k \approx k^{(1+\delta)}$ and $\lambda_k \approx k^2/d$ for some constants $\delta, d > 0$ and large $k$, then there exists a constant $C > 0$ such that

$$\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \leq C(1 + \|x - y\|_B^2 / (1+t)^{1+\delta}), \quad t > 0, x, y \in B.$$

(3) Suppose $\lambda := \inf_{k \geq 1} \lambda_k > 0$. Then (1.13) holds for any Lipschitz continuous $\rho_0$ with $\lambda_0 := \nu_0(B) \in (0, \infty)$. 

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Proof. Let \( H = \{ h \in \mathbb{B} : \sum_{k=1}^{\infty} h_k^2 q_k^2 < \infty \} \). By (5.4) it is easy to check that (A) holds for \( \sigma = I \). Moreover, by (5.3), for any \( h \in H \) we have \( \mu(dz - h) = \varphi_h(z)\mu(dz) \) for

\[
(5.5) \quad \varphi_h(z) = \exp \left[ \sum_{k=1}^{\infty} \left( q_k h_k z_k - \frac{1}{2} q_k h_k^2 \right) \right], \quad h_k = \langle h, e_k \rangle, \quad k \geq 1.
\]

Then it is easy to see from (5.4) that there exists a constant \( c_1 > 0 \) such that

\[
\sup_{\|z\| \leq 1} \varphi_{T_s y}(z + T_s y) \leq \exp \left[ \|z\|_B^2 + c_1 \beta(2s) \right] < \infty, \quad z \in \mathbb{B}.
\]

Therefore, the first assertion follows from Theorem 1.1.

Next, since \( \sigma = I \), it follows from (5.5) that

\[
(5.6) \quad \int_{\mathbb{B}} \varphi_{\sigma^{-1} T_s y}(z)^2 \mu(dz) = \prod_{k=1}^{\infty} \frac{\sqrt{q_k}}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp \left[ q_k (T_s y)_k^2 - \frac{1}{2} q_k (z_k - 2(T_s y)_k)^2 \right] dz_k
\]

\[
= \exp \left[ \sum_{k=1}^{\infty} q_k (T_s y)_k^2 \right] = \exp \left[ \sum_{k=1}^{\infty} q_k e^{-2\lambda_k s} y_k^2 \right]
\]

\[
\leq \exp \left[ \|y\|_B^2 \sup_{k \geq 1} q_k e^{-2\lambda_k s} \right].
\]

Thus, due to (5.4), Theorem 1.2 holds for the claimed \( \delta_2(\varepsilon) \).

Finally, under (5.4), we have \( \sup_{k \geq 1} q_k e^{-s \lambda_k} < \infty \) for \( s > 0 \), which implies (1.12). Moreover, replacing \( T_s y \) by \( h \) in (5.6) we obtain

\[
\mu(\varphi_h^2) - 1 = \exp \left[ \sum_{k \geq 1} q_k h_k^2 \right] - 1 \leq e^{c_2 \|h\|_H^2} - 1 \leq e^{c_2 \|h\|_H^2}, \quad \|h\|_H \leq 1
\]

for some constant \( c_2 > 0 \). Then as in the proof of Proposition 5.1 we prove (1.11). \( \square \)

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