HAMILTONIAN STRUCTURES ON
FOLIATIONS

by

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ABSTRACT. We discuss hamiltonian structures of the Gelfand-Dorfman complex of projectable vector fields and differential forms on a foliated manifold. Such a structure defines a Poisson structure on the algebra of foliated functions, and embeds the given foliation into a larger, generalized foliation with presymplectic leaves. In a so-called tame case, the structure is induced by a Poisson structure of the manifold. Cohomology spaces and classes relevant to geometric quantization are also considered.

1 Preliminaries

Let $\mathcal{S}$ be a moving body with supplementary physical characteristics, expressed by scalar parameters, which have no impact on the motion but depend on the latter. For instance, the temperature of a rigid body which moves with high friction.

The mathematical model of such a system will consist of a configuration space which is an $s$-dimensional differentiable manifold $N$ endowed with a $p$-dimensional foliation $\mathcal{G}$ such that the supplementary parameters are the coordinates along the leaves of $\mathcal{G}$, and the position coordinates are constant along these leaves. Then, the phase space of $\mathcal{S}$ will be the total space $M$ of the annihilator bundle $\nu^*\mathcal{G} \subseteq T^*N$ of the tangent bundle $T\mathcal{G}$, and $M$ is

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endowed with the natural lift $\mathcal{F}$ of $\mathcal{G}$, which is such that the leaves of $\mathcal{F}$ are covering spaces of the leaves of $\mathcal{G}$ (e.g., see [7]).

Since the motion does not depend on the supplementary parameters, the hamiltonian function $H$ of the system will be a $\mathcal{F}$-foliated function on $M$ i.e., a function which is constant along the leaves of $\mathcal{F}$. On the other hand, since we want the motion to determine the time evolution of the supplementary parameters, we should be able to define the hamiltonian vector field of $H$ as a foliated vector field on the phase space of $\mathcal{S}$.

Therefore, $(M, \mathcal{F})$ should be endowed with a generalized hamiltonian structure that prescribes foliated hamiltonian vector fields to foliated functions. The aim of this paper is to initiate the study of such hamiltonian structures.

The generalized hamiltonian structures we need may be defined within the general Gelfand-Dorfman scheme of hamiltonian structures on complexes over a Lie algebra $[1, 2]$. For convenience, we refer to such complexes as Gelfand-Dorfman complexes $[3]$, and recall their definition below.

**Definition 1.1** A Gelfand-Dorfman complex consists of:

i) a real Lie algebra $(\chi, [\cdot, \cdot])$;

ii) a cochain complex of real vector spaces $C = (\bigoplus_{k=0}^{\infty} \Omega^k, d : \Omega^k \to \Omega^{k+1}, d^2 = 0)$;

iii) mappings $X \mapsto i(X) \in L_{\mathbb{R}}(\Omega^1, \chi), (\Omega^{-1} := 0; := denotes a definition), defined for all $X \in \chi$ and $k = 0, 1, 2, \ldots$, such that

a) if $\alpha \in \Omega^1$ and $i(X)\alpha = 0$ for all $X \in \chi$ then $\alpha = 0$;

b) if $L_X := di(X) + i(X)d$ then

(1) $i(X)i(Y) + i(Y)i(X) = 0, \ i([X, Y]) = L_X i(Y) - i(Y)L_X$.

Usually, one says that $C$ is a complex over $\chi$, and the mapping $X \mapsto i(X)$ encountered in Definition [1] may be seen as a representation of $\chi$ on $C$. This mapping also defines a pairing

$< \alpha, X > = < X, \alpha > := i(X)\alpha, \quad X \in \chi, \alpha \in \Omega^1$,

and, in particular, one denotes $Xf := < df, X >, f \in \Omega^0, X \in \chi$.

A linear mapping $H \in L_{\mathbb{R}}(\Omega^1, \chi)$ is said to be skew symmetric if

(2) $< \alpha, H\beta > = - < \beta, H\alpha >, \quad \forall \alpha, \beta \in \Omega^1$. 

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The hamiltonian structures of a Gelfand-Dorfman complex are defined by generalizing the notion of a Poisson bivector (e.g., [11]). For this purpose, one notices that the formula [1, 2]

\[
[H, K](\alpha, \beta, \gamma) := \sum_{\text{Cycl}(\alpha, \beta, \gamma)} \{<KL_{H\alpha}\beta, \gamma > + <HL_{K\alpha}\beta, \gamma >\},
\]

where \( H, K \in L_R(\Omega^1, \chi) \) are skew symmetric and \( \alpha, \beta, \gamma \in \Omega^1 \), may be seen as defining a bracket

\[ [H, K] \in L_{alt,R}((\Omega^1)^3, \Omega^0), \]

which is a generalization of the Schouten-Nijenhuis bracket of bivector fields on manifolds. We call the bracket (3) the Gelfand-Dorfman bracket. Then, one defines

**Definition 1.2** A skew-symmetric homomorphism \( H \in L_R(\Omega^1, \chi) \) which satisfies the Poisson condition \([H, H] = 0\) is called a hamiltonian structure on the Gelfand-Dorfman complex \((\chi, C)\).

For a hamiltonian structure one defines the following generalizations of classical notions:

i) \( \forall f \in \Omega^0 \), \( X_f := H(df) \in \chi \) is the hamiltonian vector of \( f \);

ii) \( \forall f, g \in \Omega^0 \), \( \{f, g\} := X_f g \) is the Poisson bracket; this bracket is skew-symmetric because of (2), and it satisfies the Jacobi identity because (3) yields

\[ [H, H](df, dg, dh) = 2 \sum_{\text{Cycl}(f, g, h)} \{\{f, g\}, h\}; \]

iii) \( \forall \alpha, \beta \in \Omega^1 \), one has a \( \Omega^1 \)-bracket

\[ \{\alpha, \beta\} := L_{H\alpha}\beta - L_{H\beta}\alpha - d <H\alpha, \beta>, \]

with the particular case

\[ \{df, dg\} = d\{f, g\}. \]

The \( \Omega^1 \)-bracket (4) may be defined for any skew-symmetric mapping \( H \in L_R(\Omega^1, \chi) \), and it satisfies the following fundamental identities [4, 12]

\[ <\gamma, H\{\alpha, \beta\}> = <\gamma, [H\alpha, H\beta]> + \frac{1}{2} [H, H](\alpha, \beta, \gamma), \]
\[
\sum_{Cycl(\alpha, \beta, \gamma)} <\{\alpha, \beta\}, \gamma>, X >= [H, L_X H](\alpha, \beta, \gamma)
\]
\[
+ \frac{1}{2} \sum_{Cycl(\alpha, \beta, \gamma)} [H, H](\alpha, \beta, d <\gamma, X >),
\]
where \(\alpha, \beta, \gamma \in \Omega^1\), \(X \in \chi\), and

\[
L_X H(\alpha) := [X, H\alpha] - H(L_X \alpha).
\]

In the Hamiltonian case \([H, H] = 0\), it follows from (8) that the \(\Omega^1\)-bracket is a Lie algebra bracket. Furthermore, under the supplementary \textit{regularity hypothesis}: if \(\forall \alpha \in \Omega^1 < \alpha, X >= 0\) then \(X = 0\) \((X \in \chi)\), \(H\) is a homomorphism of Lie algebras i.e.,

\[
H\{\alpha, \beta\} = [H\alpha, H\beta].
\]

On the other hand, even without the regularity hypothesis, (8) shows that if we ask \(H \in L_\mathbb{R}(\Omega^1, \chi)\) to be skew symmetric and satisfy (9), \(H\) is a Hamiltonian structure.

### 2 Hamiltonian structures of foliations

With the motivation of Section 1 in mind, let us consider an arbitrary \(n\)-dimensional differentiable manifold \(M\) (in the present paper “everything” is of differentiability class \(C^\infty\)) endowed with a \(p\)-dimensional foliation \(\mathcal{F}\). An object of \(M\) that projects to the space of the leaves of \(\mathcal{F}\) is called either \textit{projectable} or \textit{foliated}. We refer the reader to [7] for all the notions of foliation theory which we are going to use.

The Lie algebra \(\chi_\mathcal{F}\) of the \(\mathcal{F}\)-foliated vector fields and the complex of projectable differential forms \(\Omega_\mathcal{F} = \bigoplus_{k=1}^l \Omega^k_\mathcal{F}\) \((q := n - p)\), with the usual exterior differential and contraction operators \(i(X), X \in \chi_\mathcal{F}\), define a Gelfand-Dorfman complex associated with the pair \((M, \mathcal{F})\). One might consider general hamiltonian structures on this complex, but, such a structure may have a non-local character. We avoid non-locality by

**Definition 2.1** A \textit{Hamiltonian structure} on (or of) the foliation \(\mathcal{F}\) is a vector bundle morphism \(h : \nu^\ast \mathcal{F} \to TM\) \((\nu \mathcal{F} = TM/T\mathcal{F}\) is the transversal bundle of \(\mathcal{F}\)) such that the induced map of cross sections \(H : \Omega^1_{\mathcal{F}} \to \chi(M)\) \((\chi(M)\) is the space of all the tangent vector fields of \(M\)) is a Hamiltonian structure of the Gelfand-Dorfman complex of \((M, \mathcal{F})\).
In particular, Definition 2.1 implies that the morphism \( h \) is skew symmetric (i.e., it satisfies (2) pointwisely), and that the values of the mapping \( H \) are in \( \chi_F \).

**Example 2.1** Any skew symmetric \( h \in L_R(\nu^*, F, TF) \) may be seen as a trivial hamiltonian structure of the foliation \( F \). Indeed, formula (3) shows that \([H, H] = 0\) if the values of \( H \) are vector fields tangent to \( F \).

**Example 2.2** Let \( P \) be a Poisson bivector field on the foliated manifold \((M, F)\), such that for any foliated function \( f \in \Omega^0_F \) the hamiltonian vector field \( X^P_f \) is a foliated vector field. Then, \( h := \sharp_P|_{\nu^*F} \) \((\sharp_P : T^*M \to TM, \langle \sharp_P\alpha, \beta \rangle := P(\alpha, \beta))\) defines a hamiltonian structure of the foliation \( F \).

**Example 2.3** A bivector field \( P \) is called a transversal Poisson structure of \( F \) if the bracket
\[
\{f, g\} := P(df, dg) \quad (f, g \in C^\infty(M))
\]
makes \( \Omega^0_F \) a Poisson algebra \([6]\). In this case, again, \( h := \sharp_P|_{\nu^*F} \) is a hamiltonian structure of \( F \). Moreover, for any hamiltonian structure \( h \) of \( F \) and any choice of a decomposition \( TM = E \oplus TF \), the bivector field \( P \) defined by
\[
\sharp_P|_{E \approx \nu^*F} = h, \quad \sharp_P|_{T^*F} = 0
\]
is a transversal Poisson structure of \( F \).

We also show how to express hamiltonian structures of a foliation \( F \) by means of adapted local coordinates \((x^a, y^u)\), where \( a = 1, \ldots, q; u = q + 1, \ldots, n \), and \( x^a = \text{const.} \) are the local equations of \( F \). In order to get an expression by tensors, we fix a decomposition \( TM = E \oplus TF \) where \([8]\)
\[
E = \text{span}\{X_a := \frac{\partial}{\partial x^a} - t^u_a \frac{\partial}{\partial y^u}\}, \quad TF = \text{span}\{\frac{\partial}{\partial y^u}\},
\]
for some local coefficients \( t^u_a \) and with the Einstein summation convention. The local bases of \( TM \) defined by \((1)\) have the dual co-bases
\[
dx^a, \quad \theta^u := dy^u + t^u_a dx^a,
\]
and \( \nu^* \mathcal{F} = \text{span}\{dx^a\} \).

Then, a skew-symmetric morphism \( h : \nu^* \mathcal{F} \to TM \) has local equations

\[
(12) \quad h(dx^a) = h^{ab} X_b + k^{au} \frac{\partial}{\partial y^u}.
\]

The components \( h^{ab} \) define a global cross section \( W \) of \( \wedge^2 E \), therefore, a global cross section of \( \wedge^2 \nu \mathcal{F} \), which is independent on the choice of \( E \), and the components \( k^{au} \) define a global cross section of \( E \otimes T \mathcal{F} \). The following assertion is obvious

**Proposition 2.1** The morphism \( h \) defined by (12) is a hamiltonian structure of \( \mathcal{F} \) iff the cross section \( W \) with local components \( h^{ab} \) is foliated and defines a structure of Poisson algebra on \( \Omega^0_F \).

The Poisson bracket defined by \( W \) on \( \Omega^0_F \) is of the local type, and it has the following interpretation. Let \( U \) be an open neighborhood of \( M \) such that the manifold \( N \) of the slices of \( \mathcal{F} \) in \( U \) exists, and let \( p : U \to N \) be the natural projection (constant along the slices of \( \mathcal{F} \) in \( U \)). Then \( h_N(p(x)) := p_*(x) \circ (h|_U(x)) \circ p^*(p(x)), x \in U \), is the morphism \( \sharp_{P_N} \) of a well defined Poisson bivector field \( P_N \) on \( N \), which defines the same local Poisson brackets as \( W \).

\( h_N \) is well defined since the values of the mapping \( H \) defined by \( h \) are foliated vector fields.

Furthermore, any Poisson algebra structure of local type on \( \Omega^0_F \) is defined by a family of foliated hamiltonian structures on \( \mathcal{F} \). Indeed, the required structure is equivalent to a foliated section \( W \) of \( \wedge^2 \nu (\mathcal{F}) \), which satisfies the Poisson condition \( [W, W] = 0 \). Choose a decomposition \( TM = E \oplus T \mathcal{F} \), and, \( \forall \alpha \in \Omega^1_F \), define \( h(\alpha) \) to be the unique vector of \( E \) with projection \( \sharp_{W} \alpha \) on \( \nu \mathcal{F} \). Since by (3)

\[
[H, H](\alpha, \beta, \gamma) = [W, W](\alpha, \beta, \gamma) \quad (\alpha, \beta, \gamma \in \Omega^1_F),
\]

\( h \) is a hamiltonian structure of \( \mathcal{F} \), and \( h \) induces \( W \).

More exactly, if \( h_0 \) is one of the foliated hamiltonian structures which define \( W \), the whole family which defines \( W \) is \( h_0 + k \), where \( k \in L_R(\nu^* \mathcal{F}, T \mathcal{F}) \) is skew symmetric. This holds since for any hamiltonian structure \( h \) of \( \mathcal{F} \) and any skew symmetric \( k \in L_R(\nu^* \mathcal{F}, T \mathcal{F}) \), the corresponding morphisms \( H, K \) of global cross sections satisfy the relation \( [H, K] = 0 \) (see (3)).
**Proposition 2.2** For any hamiltonian structure \( h \) on a foliation \( \mathcal{F} \), the generalized distribution \( \mathcal{H} := TF + \mathcal{H}_0 \) (\( \mathcal{H}_0 := \text{im} h \)) is a projectable, completely integrable distribution, and its leaves are presymplectic manifolds with kernel \( TF \). Furthermore, \( h(\text{ann} \mathcal{H}) = \mathcal{H}_0 \cap TF \) (\( \text{ann} \) denotes the annihilator of a vector space or bundle).

**Proof.** We continue to use the previous notation. Let \( x_0 \in U \subseteq M \) where \( U \) is a neighborhood such that \( \mathcal{F}|_U \) has a \( q \)-dimensional, transversal submanifold \( N \). Since the projection \( p \) is a submersion, if \( L_p(x_0) \) is the symplectic leaf of the Poisson structure \( P_N \) through \( p(x_0) \in N \), \( \tilde{L}_{x_0} := p^{-1}(L_{x_0}) \) is an integral submanifold of \( \mathcal{H} \) through \( x_0 \). The existence of these integral submanifolds shows the complete integrability of \( \mathcal{H} \). Projectability follows from the fact that \( \mathcal{H} \) is spanned by the projectable vector fields \( H(\alpha) \), \( \alpha \in \Omega^1_F \), and \( \mathcal{H} \) projects onto the symplectic distribution of \( P_N \). The lift of the symplectic form of \( L_p(x_0) \) by \( p^* \) yields the required presymplectic form of the corresponding leaf of \( \mathcal{H} \). Finally, notice that \( \alpha \in \text{ann} \mathcal{H} \) iff \( \alpha = p^*(\lambda) \) for some \( \lambda \in \ker \sharp P_N \), and then \( p_* h(\alpha) = 0 \). This implies \( h(\text{ann} \mathcal{H}) \subseteq \mathcal{H}_0 \cap TF \). On the other hand, if \( h(\alpha) \in TF \), we must have \( \alpha = p^*(\lambda) \) where \( \lambda \in \ker \sharp P_N \), and this justifies the converse inclusion. (All these also follow immediately from the local equations (12) of \( h \).) Q.e.d.

The distribution \( \mathcal{H} \) will be called the **characteristic distribution** of the hamiltonian structure \( h \), and its leaves constitute the **presymplectic foliation**. The hamiltonian structure \( h \) of the foliation \( \mathcal{F} \) on \( M \) will be called **transitive** if the characteristic distribution is \( \mathcal{H} = TM \). In this case, Proposition 2.2 tells us that \( M \) is a presymplectic manifold with the kernel foliation \( \mathcal{F} \), and that \( TM = \mathcal{H}_0 \oplus TF \). The latter equality also shows that the corresponding local Poisson structures \( P_N \) are the symplectic reduction of the presymplectic form of \( M \). Conversely, if \( M \) is a presymplectic manifold with the presymplectic 2-form \( \sigma \), and if \( E \) is a complementary distribution of the kernel foliation \( \mathcal{F} \) of \( \sigma \), there exists a well defined, transitive, hamiltonian structure \( h \) of \( \mathcal{F} \) such that \( \mathcal{H}_0 = E \) and the local Poisson structures \( P_N \) are the symplectic reductions of \( \sigma \).

**Example 2.4** Let \( \mathcal{H} \) be a coisotropic foliation of dimension \( n + k \) \( (k \leq n) \) of a symplectic manifold \( M \) of dimension \( 2n \), with the symplectic form \( \omega \). It is well known that the \( \omega \)-orthogonal distribution of \( \mathcal{H} \) is tangent to a foliation \( \mathcal{F} \), and that, \( \forall x \in M \), there exist local coordinates \( (x^a, x^u, y^i) \) around \( x \) such that \( a = 1, \ldots, p := n - k \), \( u = p + 1, \ldots, n \), \( i = 1, \ldots, n \), \( x^a = \text{const.} \) are the
local equations of $\mathcal{H}$, and the symplectic form has the canonical expression
\begin{equation}
\omega = \sum_{a=1}^p dx^a \land dy^a + \sum_{u=p+1}^n dx^u \land dy^u.
\end{equation}

(This result is a Lie’s theorem \cite{5}.) The local equations of the foliation $\mathcal{F}$ are $x^a = \text{const.}$, $x^u = \text{const.}$, $y^u = \text{const.}$, and the computation of the hamiltonian vector field $X^\omega_f$ of an $\mathcal{F}$-foliated function (via (13)) shows that $X^\omega_f$ is an $\mathcal{F}$-foliated vector field tangent to the leaves of $\mathcal{H}$. Therefore, $h := -\flat\omega|_{\nu^*\mathcal{F}}$ is a hamiltonian structure of the foliation $\mathcal{F}$ with the presymplectic foliation $\mathcal{H}$. Moreover, in this case we have $TF \subseteq \mathcal{H}_0$.

**Example 2.5** Example 2.4 can be generalized as follows. Let $(M, \omega)$ be an almost symplectic manifold (i.e., we ask $\omega$ to be non-degenerate but not necessarily closed), and let $\mathcal{H}$ be a coisotropic foliation such that the pullback of $\omega$ to every leaf of $\mathcal{H}$ is closed on the leaf. Then formula (13) is to be replaced by
\begin{equation}
\omega = \sum_{a=1}^p dx^a \land \varpi^a + \sum_{u=p+1}^n dx^u \land dy^u,
\end{equation}
where $\varpi^a$ are linearly independent, local, 1-forms which contain only the differentials $dy^a$. Now, we obtain the foliation $\mathcal{F}$ and its hamiltonian structure $h$ in the same way as in the symplectic case.

We finish this section by a remark about the chosen definition of the notion of a hamiltonian structure on a foliation.

If we start with the physical motivation of Section 1, and do not think of Gelfand-Dorfman complexes a priori, the natural definition of a generalized hamiltonian structure (g.h.s.) that suites the problem is that of an $\mathbf{R}$-linear morphism of sheaves
\begin{equation}
\Phi : \Omega^0_F \longrightarrow \chi, \quad f \mapsto X_f,
\end{equation}
(underlining means passing to germs of the corresponding type of objects), such that the bracket defined by
\begin{equation}
\{f, g\} = X_fg, \quad f, g \in \Omega^0_F,
\end{equation}
makes $\underline{\Omega}^0_F$ a Poisson algebra sheaf.
In particular, the action of a hamiltonian vector field $X_f$ on foliated functions $g \in \Omega^0_F$ depends only on the first jet $j^1 f$. This is not enough to ensure that the g.h.s. has local type. A natural condition for the latter property is to ask $X_f = 0$ for all $f \in \Omega^0_F$ such that $j^1_x f = 0$ at each point $x \in M$. If the g.h.s. structure $\Phi$ satisfies this locality condition, $\Phi$ is completely defined by local vector fields

$$X_{x^a} = h^{ab} X_b + k^{au} \frac{\partial}{\partial y^u},$$

that satisfy the conditions of Proposition 2.1.

Therefore, the generalized hamiltonian structures of local type are exactly the hamiltonian structures of foliations which we defined earlier.

## 3 Tame hamiltonian structures

The Gelfand-Dorfman complex of a foliation does not satisfy the regularity hypothesis formulated at the end of Section 1. The equality $\langle \alpha, X \rangle = 0$, $\forall \alpha \in \Omega^1_F$, only implies $X \in \Gamma TF$ ($\Gamma$ denotes the space of global cross sections). Therefore, (1), or the equivalent property

$$X_{\{f, g\}} = [X_f, X_g], \quad \forall f, g \in \Omega^0_F,$$

obtained by taking $\alpha = df, \beta = dg, f, g \in \Omega^0_F$ in (4), may not hold, and we shall define

**Definition 3.1** A skew symmetric morphism $h : \nu^* F \to TM$ which satisfies condition (18) is a **strong hamiltonian structure** on $F$.

**Remark 3.1** If $h$ is a strong hamiltonian structure, the sheaf $\nu^* F$ has a natural structure of a sheaf of twisted Lie algebras $[3]$ over $(\mathbb{R}, \Omega^0_F)$, with the action of germs $\alpha \in \nu^*_x F$ defined as the action of $H(\alpha)$.

Formula (3) shows that a strong hamiltonian structure is hamiltonian. The hamiltonian structures indicated in Examples 2.2 and 2.4 are strong but, this is not necessarily true for Examples 2.3 and 2.5. If $h$ is a strong hamiltonian structure, the generalized distribution $\mathcal{H}_0 = \text{im} h$ is involutive. Conversely, if $\mathcal{H}_0$ is involutive and if $\mathcal{H}_0 \cap TF = 0$, $h$ is a strong hamiltonian structure (use (3)). These facts suggest
Definition 3.2 A hamiltonian structure $h$ of a foliation $\mathcal{F}$ is transversal (to $\mathcal{F}$) if there exists a differentiable complementary distribution $E$ of $T\mathcal{F}$ ($E \oplus T\mathcal{F} = TM$) such that $\mathcal{H}_0 \subseteq E$. The distribution $E$ will be called an image extension of $h$. (It is possible to have more than one image extension.) A transversal hamiltonian structure of $\mathcal{F}$ is a tame structure if all the brackets of differentiable vector fields that belong to $\mathcal{H}_0$ are contained in an image extension $E$. (In the tame case, only such image extensions will be used.)

A tame hamiltonian structure is strong (see (6)), and a transversal, strong hamiltonian structure is tame. The condition $\mathcal{H}_0 \cap T\mathcal{F} = 0$, which is implicit in the definition of transversality, is equivalent to $h(\text{ann } \mathcal{H}) = 0$ and also to the fact that the rank of the morphism $h$ is equal to the rank of the Poisson structures induced by $h$ on the manifolds of local slices of $\mathcal{F}$. (See Proposition 2.2 and formula (12). This condition is not enough for transversality. Indeed, there always exists a smallest regular distribution $\tilde{\mathcal{H}}_0$ which contains the generalized distribution $\mathcal{H}_0$ but, we may have $\tilde{\mathcal{H}}_0 \cap T\mathcal{F} \neq 0$.

Example 3.1 Let $TM = F \oplus F'$ be a locally product structure on the manifold $M$, and $\mathcal{F}$ the foliations tangent to $F$. Assume that one has a Poisson algebra structure of the local type on $\Omega^0_{\mathcal{F}}$. Then, the hamiltonian structure $h$ which induces the former and has its hamiltonian vector field in $F'$ is tame. Indeed, $F'$ is an image extension of $h$ of the kind required for tame structures. Notice also that a transitive, tame, hamiltonian structure must be of the locally product type shown in the example.

Proposition 3.1 Let $h$ be a transversal hamiltonian structure of the foliation $\mathcal{F}$ with image extension $E$. Then $h$ is tame with image extension $E$ iff the Nijenhuis tensor $N_E$ of the projection $p_E : TM \to TM$ of $TM = E \oplus T\mathcal{F}$ onto $E$ satisfies the condition

$$N_E(h\alpha, h\beta) = 0, \quad \forall \alpha, \beta \in \nu^*_x\mathcal{F}, \forall x \in M.$$

Proof. Following the general definition of a Nijenhuis tensor e.g., [4,13] and since $p_E^2 = p_E$, for $X, Y \in \Gamma TM$, one has

$$N_E(X,Y) = [p_E(X, p_EY) - p_E[p_EX, Y] - p_E[X, p_EY] + p_E[X,Y].$$

Consider the local equations (12) of $h$ using an image extension $E$, which implies that $k^{au} = 0$. Then, $h$ is tame iff

$$H(dh^{ab}) = [H(dx^a), H(dx^b)].$$
which is equivalent to

\[ h^{ac} h^{be} \tau_{ce}^u = 0, \quad \tau_{ce}^u := \frac{\partial v^e}{\partial x^c} - \frac{\partial v^e}{\partial x^c} + t^e \frac{\partial v^u}{\partial y^v} - t^e \frac{\partial v^u}{\partial y^v}. \]

The invariant meaning of (21) is exactly (19). Q.e.d.

In the case of a transversal hamiltonian structure \( h \) on a foliated manifold \((M, F)\) it is possible to extend the hamiltonian formalism in a way similar to what was done for presymplectic manifolds in [9].

Let us recall that, if \((M, F)\) is a foliated manifold and if \( E \) is a complementary distribution of \( TF \), the use of the local bases (10), (11) yields a bigrading of tensor fields and differential forms, with the convention that the first degree is the \( E \)-degree and the second is the \( TF \)-degree [8]. For instance, a differential \( k \)-form is of bidegree \((s, t)\) if its local expressions contain \( s \) forms \( dx^a \) and \( t \) forms \( \theta^u \) \((s + t = k)\). Then, one has a decomposition

\[ d = d'_{(1,0)} + d''_{(0,1)} + \partial_{(2,-1)}, \]

and \( d^2 = 0 \) is equivalent to

\[ d'' d' = 0, \quad \partial^2 = 0, \quad d'' + d'' \partial + \partial d'' = 0, \]

\[ d'd'' + d'' d' = 0, \quad \partial d'' + d' \partial = 0. \]

Now, we return to the transversal hamiltonian structure \( h \) of \( F \), and fix an image extension \( E \) of \( h \). Then the corresponding section mapping \( H \) is well defined for any differential form \( \alpha \in \Omega^{(1,0)}(M) \) of bidegree \((1, 0)\), and \( H\alpha \in \Gamma E \). For any differentiable function \( f \in C^\infty(M) \), we can define the hamiltonian vector field \( X'_f \in \Gamma E \) by

\[ X'_f = H(d' f) \]

and \( \forall f, g \in C^\infty(M) \) we get an extended Poisson bracket

\[ \{ f, g \}' := X'_f g =< H d' f, d g > = < H d' f, d' g > = -\{ g, f \}'. \]

Furthermore, if \( X \in \Gamma E \) and \( \alpha \in \Omega^{(1,0)}(M) \), (22) leads to

\[ L_X \alpha = L'_X \alpha + L''_X \alpha, \]

where

\[ L'_X = i(X) d' + d' i(X), \quad L''_X = i(X) d'' + d'' i(X). \]
Accordingly, it is possible to extend the Gelfand-Dorfman bracket (3) to arbitrary \((1, 0)\)-forms \(\alpha, \beta, \gamma\) by

\[
[H, K]'(\alpha, \beta, \gamma) := \sum_{cyc(\alpha, \beta, \gamma)} \left\{<KL'_{H\alpha}\beta, \gamma> + <HL'_{K\alpha}\beta, \gamma>\right\},
\]

(28)

where \(H, K\) are defined by skew symmetric morphisms \(h, k : \nu^*\mathcal{F} \to E\). A straightforward computation shows that the extended bracket is trilinear over \(C^\infty(M)\), and for a hamiltonian structure \(h\) we have \([H, H]'(\alpha, \beta, \gamma) = 0\) for any \(\alpha, \beta, \gamma \in \Omega^{(1, 0)}(M)\).

In particular, using (25), (27), one gets

\[
[H, H]'(d'f, d'g, d'k) = 2 \sum_{cyc(f, g, k)} \left\{\{\{f, g\}', k\}' + d^2f(X'_g, X'_k)\right\} = 0.
\]

(29)

**Proposition 3.2** If \(h\) is a tame hamiltonian structure on \((M, \mathcal{F})\) the Poisson bracket \(\{ , \}'\) defines a Poisson structure on the manifold \(M\).

**Proof.** For any foliation and any choice of a complementary distribution \(E\) one gets

\[
d^2f(X, Y) =< d'f, N_E(X, Y)>, \quad \forall f \in C^\infty(M), \forall X, Y \in \Gamma E,
\]

(30)

where \(N_E\) is the Nijenhuis tensor (20). Indeed, if \(X, Y \in \Gamma E\), (20) yields

\[
N_E(X, Y) = p_{TF}[X, Y],
\]

(31)

where \(p_{TF}\) denotes the projection onto the second term of the decomposition \(TM = E \oplus TF\). On the other hand,

\[
d^2f(X, Y) = d(d'f)(X, Y) = XYf - YXf - <d'f, [X, Y] >
\]

\[
= [X, Y]f - (p_E[X, Y])f =< df, p_{TF}[X, Y] > =< d'f, p_{TF}[X, Y] >.
\]

Thus, (30) is justified, and the conclusion follows from the characterization (19) of the tame hamiltonian structures and formula (29). Q.e.d.

Theorem 3.2 tells us that a tame hamiltonian structure \(h\) is defined by a usual Poisson structure \(P\) on the foliated manifold \((M, \mathcal{F})\). The hamiltonian vector fields of foliated functions with respect to \(h\) coincide with those with
respect to \( P, \tilde{z}_P|_{T^*F} = h \) and \( \tilde{z}_P|_{T^*F} = 0 \). Thus, the tame hamiltonian structures are included in Example 2.2. But, not all the structures of Example 2.2 are tame.

Similarly, it is possible to extend the bracket (4) of foliated 1-forms to any \( \alpha, \beta \in \Omega^{(1,0)}(M) \) by

\[
\{\alpha, \beta\}' := L'_H \alpha \beta - L'_H \beta \alpha - d' <H \alpha, \beta>.
\]

(32)

From (32), it follows that \( \forall f, g \in C^\infty(M) \) one has

\[
\{f \alpha, g \beta\}' = fg\{\alpha, \beta\}' + f(H(\alpha)g)\beta - g(H(\beta)f)\alpha.
\]

(33)

In particular, we see that the bracket (32) is skew symmetric because it is such for foliated 1-forms, where it reduces to (4).

Let us also evaluate the bracket (32) on an argument \( X \in \Gamma E \). First we define

\[
L'_X H := p_E[X, H(\alpha)] - H(L'_X \alpha).
\]

(34)

Taking the derivative of (2) in direction \( X \), and with the decomposition (26), we see that \( L'_X H \) is skew symmetric. Then, if the derivatives \( L' \) of (32) are replaced by \( L - L'' \) one gets

\[
\{\alpha, \beta\}'(X) = H(\alpha)i(X)\beta - H(\beta)i(X)\alpha - <\alpha, L'_X H(\beta) >.
\]

(35)

In particular, if \( \alpha = d'f, \beta = d'g \) (33) yields

\[
\{d'f, d'g\}' = d'\{f, g\} + L_{X'_y} d''f - L_{X'_y} d''g.
\]

(36)

The result follows by an easy computation which takes into account the fact that the space of \((1,0)\)-forms is the annihilator of \( E \).

If \( L_{X'_y} d''f = 0 \ \forall g \in C^\infty(M) \), we will say that \( f \in C^\infty(M) \) is a distinguished function, and we will denote by \( \Omega^0_d \) the space of distinguished functions. For instance, any foliated function is distinguished but, not conversely. By separating the \((1,0)\)-term and the \((0,1)\)-term in the definition of a distinguished function, we see that \( f \in \Omega^0_d \) iff: a) \( d'f \) is a foliated 1-form, and b) \( \mathcal{H} \subseteq \text{ker} d^2 f \). Formula (23) shows that the extended Poisson bracket of distinguished functions satisfies the Jacobi identity, and a) implies that \( \{f, g\}' \in \Omega^0_d \ \forall f, g \in \Omega^0_d \). Therefore, \( \Omega^0_d \) is a Poisson algebra and \( \Omega^0_d \) is an ideal of the former. Furthermore, (36) implies

\[
\{d'f, d'g\}' = d'\{f, g\}' \quad \forall f, g \in \Omega^0_d,
\]

(37)
and, if we take $f, g \in \Omega_0^d, k \in C^\infty(M)$ in (29) and use (31), we get
\begin{equation}
X'_{\{f,g\}} = p_E[X'_f, X'_g] \quad f, g \in \Omega_0^d.
\end{equation}

**Proposition 3.3** Let $h$ be a tame hamiltonian structure of the foliation $\mathcal{F}$, $E$ an image extension of $h$, and $P$ the Poisson structure defined by the brackets $\{ , \}'$. Then, the triple $(\nu^*\mathcal{F}, \{ , \}', h)$, with the bracket (32), is a Lie subalgebroid of the cotangent Lie algebroid $(T^*M, \{ , \}_P, \sharp_P)$.

**Proof.** The bracket $\{ , \}_P$ is given by (4) with $H$ replaced by $\sharp_P$, and, since $\sharp_P|_{E^*} = h$, we have $\forall \alpha, \beta \in \Omega_1^\mathcal{F}$
\begin{equation}
\{\alpha, \beta\}_P = \{\alpha, \beta\} = \{\alpha, \beta\}'.
\end{equation}
Then, (33) implies
\begin{equation}
\{f\alpha, g\beta\}_P = \{f\alpha, g\beta\}' \quad \forall f, g \in C^\infty(M), \forall \alpha, \beta \in \Omega_1^\mathcal{F}.
\end{equation}
Q.e.d.

Now, let us notice that there exist an inclusion and a splitting morphism of Lie algebroids
\begin{equation}
\iota : \nu^*\mathcal{F} \hookrightarrow T^*M, \quad \pi = p_{E^*} : T^*M \to \nu^*\mathcal{F} \quad (\pi \circ \iota = \text{id}),
\end{equation}
where $p_{E^*}$ is the projection onto $E^*$ in the decomposition $T^*M = E^* \oplus T^*\mathcal{F}$.

**Proposition 3.4** Under the hypotheses of Proposition 3.3, the projection $\pi$ induces an injection $\pi^*$ of the de Rham cohomology of the Lie subalgebroid $\nu^*\mathcal{F}$ into the Lichnerowicz-Poisson cohomology of $(M, P)$. For any complex vector bundle $S$ over $M$, the Lichnerowicz-Poisson Chern classes $c^\text{LP}_k(S)$ belong to the image of the injection $\pi^*$.

**Proof.** For the definition of the de Rham cohomology of Lie algebroids, see [4]; the Lichnerowicz-Poisson cohomology is the de Rham cohomology of the cotangent Lie algebroid $T^*M$ of the Poisson manifold $(M, P)$ (e.g., [11]). These definitions show the existence of homomorphisms
\begin{align*}
H^*_\text{deRham}(M) & \xrightarrow{\iota^*} H^*_\text{LP}(M, P) \xrightarrow{\pi^*} H^*(\nu^*\mathcal{F}), \\
H^*_\text{deRham}(M) & \xrightarrow{j^*} H^*(\nu^*\mathcal{F}) \xrightarrow{\pi^*} H^*_\text{LP}(M, P),
\end{align*}
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where the morphisms are naturally induced by \( j_1 = \sharp_P, j_2 = h, \iota, \pi \). (For instance, at the level of cochains we define

\[
(j_2^* \lambda)(\alpha_1, \ldots, \alpha_k) = \lambda(H\alpha_1, \ldots, H\alpha_k), \quad (\lambda \in \Omega^k(M), \alpha_1, \ldots, \alpha_k \in \Gamma E^*),
\]

eq \) etc.) The following relations are obvious: \( \iota^* \circ j_1^* = j_2^*, \pi^* \circ j_2^* = j_1^* \), \( \iota^* \circ \pi^* = \text{id} \).
The last one shows that \( \pi^* \) is injective; the others were mentioned for a later utilization.

Now, we remind that the Lichnerowicz-Poisson Chern classes are the \( j_1^* \)-image of the real Chern classes. Representatives of \( c_{LP}^k(S) \) are obtained by evaluating Chern-Weil polynomials on the curvature of an arbitrary contravariant derivative \( ^hD \) on \( S \) (i.e., a connection of the Lie algebroid \( T^* M \) on \( S \)) like in the usual Chern-Weil theory \([11]\). In particular, if \( ^hD \) is a connection of the Lie algebroid \( \nu^* F \) on \( S \) then

\[
P_D \alpha s = ^hD \pi \alpha s
\]
is a contravariant derivative on \( S \), and, if \( C \) denotes curvatures, one has

\[
C_{^hD} = \pi^* C_{^D},
\]
where \( \pi^* \) is used at the level of cochains. Now, the same procedure of evaluating Chern-Weil polynomials on curvature applied to \( C_{^D} \) yields Chern classes \( c_k^h(S) \in H^{2k}(\nu^* F) \), which are the \( j_2^* \)-images of the real Chern classes. Furthermore, \([11]\) shows that \( c_k^{LP}(S) = \pi^* c_k^h(S) \). Q.e.d.

**Corollary 3.1** Let \( h \) be a tame hamiltonian structure and let \( P \) be the bivector field of the Poisson brackets \( \{ , \} \). Then, there exists a prequantization bundle of the \( h \)-Poisson bracket iff \( \iota^*[P] \in j_2^* (H^2(M, Z)) \).

**Proof.** \([P] \in H^2_{LP}(M, P) \) is the cohomology class defined by the cocycle \( P \). We refer the reader to \([11]\) for the geometric quantization theory involved in the corollary. Since \( P \) defines the same Poisson brackets as \( h \), the existence of a prequantization bundle implies \([P] = j_1^*(\zeta) \) for some \( \zeta \in H^2(M, Z) \), which implies \( \iota^*[P] = j_2^*(\zeta) \). Conversely, if this condition is satisfied, and if (as a consequence of \([11]\)) we see the Kostant-Souriau prequantization formula as

\[
\hat{f}(s) = ^hD_g f s + 2\pi \sqrt{-1} f s, \quad s \in \Gamma K,
\]

where \( K \) is the required prequantization bundle, the Dirac quantization principle implies that \( c_k^h(K) = \iota^*[P] \). Since we assumed that \( \iota^*[P] \) is an integral cohomology class, \( K \) exists. Q.e.d.
Now, let us consider the case of a transversal hamiltonian structure $h$ on $(M, \mathcal{F})$, and fix an image extension $E$. In this case, we may still see the cross sections of $\wedge^k E$ as a kind of generalized cochains with a coboundary $\delta^{(k)} = \delta$ defined by

\[(\delta Q)(\alpha_0, ..., \alpha_k) = \sum_{i=0}^{k} (-1)^i H(\alpha)(Q(\alpha_0, ..., \hat{\alpha}_i, ..., \alpha_k)) + \sum_{i<j} (-1)^{i+j} Q(\{\alpha_i, \alpha_j\}', \alpha_0, ..., \hat{\alpha}_i, ..., \hat{\alpha}_j, ..., \alpha_k),\]

where $Q \in \wedge^k E$, $\alpha_i \in \Gamma E^*$ $(i = 0, ..., k)$, and the hat denotes the absence of the corresponding argument.

If we denote $\delta^2 = \delta^{(k+1)} \circ \delta^{(k)}$, a straightforward computation yields

\[(\delta^2 Q)(\alpha_0, ..., \alpha_{k+1}) = \sum_{i<j=1}^{k+1} (-1)^{i+j} \Delta_h(\alpha_i, \alpha_j)(Q(\alpha_0, ..., \hat{\alpha}_i, ..., \hat{\alpha}_j, ..., \alpha_{k+1})) + \sum_{i<j<k} \sum_{Cycl(i,j,k)} (-1)^{i+j+k} Q(\{\alpha_k, \{\alpha_i, \alpha_j\}'\}') \]

\[\alpha_0, ..., \hat{\alpha}_i, ..., \hat{\alpha}_j, ..., \hat{\alpha}_k, ..., \alpha_{k+1}),\]

where

\[\Delta_h(\alpha_i, \alpha_j) := H(\{\alpha_i, \alpha_j\}') - [H(\alpha_i), H(\alpha_j)].\]

Since $\delta^2 \neq 0$, we can only define the twisted cohomology spaces (e.g., [10])

\[H_{tw}^k(h) := \frac{\ker \delta^{(k)}}{\text{im } \delta^{(k-1)} \cap \ker \delta^{(k)}}.\]

For instance, by straightforward computations one gets

\[H_{tw}^0(h) = \{f \in C^\infty(M) / X'_f = 0\}, \quad H_{tw}^1(h) = \frac{\{Q \in \Gamma E / L_Q H = 0\}}{\{X'_f / f \in C^\infty(M), L'_{X'_f} = 0\}}.\]

But, if we define $W' \in \Gamma \wedge^2 E$ by $W'(d'f, d'g) = \{f, g\}'$, we do not get a cocycle since

\[(\delta W')(d'f, d'g, d'k) = -2 \sum_{Cycl(i,j,k)} \{\{f, g\}', k\}'.\]
and the Jacobi identity may not hold.

Several interpretations of twisted cohomology as a usual cohomology exist (e.g., \cite{10}). For instance, the subspaces \( \tilde{C}^k(h) = \ker(\delta^{k+1} \circ \delta^k) \) with the coboundary \( \delta \) constitute a usual cochain complex \( \tilde{C}(h) \), and \( H^k_{tw}(h) \) are the usual cohomology spaces of \( \tilde{C}(h) \).

On the other hand, since the Poisson bracket \( \{ \cdot, \cdot \}' \) defines a representation of the Lie-Poisson algebra \( \Omega^0_{\mathcal{F}} \) of distinguished functions on the space \( \Omega^0_{\mathcal{F}} \) of foliated functions, we get corresponding cohomology spaces \( H^*_{\mathcal{F}}(h) := H^*(\Omega^0_{\mathcal{F}}, \Omega^0_{\mathcal{F}}) \). Then, the cochains

\[
c(f_1, \ldots, f_k) = Q(d'f_1, \ldots, d'f_k), \quad Q \in \Gamma \wedge^k E, \quad f_1, \ldots, f_k \in \Omega^0_{\mathcal{F}},
\]

with values in \( \Omega^0_{\mathcal{F}} \) and the coboundary \( \delta \) define the cochain complex of projectable cross sections of \( \wedge E \) with the Lichnerowicz-like coboundary (see \cite{11}) \( \delta Q = -p_{\Gamma \wedge^{k+1} E}[W, Q] \) (\( p \) denotes the projection), where \( W \) defines the \( h \)-Poisson bracket of foliated functions, and \( [\cdot, \cdot] \) is the Schouten-Nijenhuis bracket. We may say that the cohomology spaces, say \( H^*_L(h) \), of this complex are the basic Lichnerowicz-Poisson cohomology spaces of \( h \). The restriction of the cochain \( W' \) to distinguished functions is \( W \), and we have a fundamental class \( [W] \in H^2_L(h) \).

Now, remember that a foliated manifold also has basic de Rham cohomology spaces \( H^*_b(M, \mathcal{F}) \) \cite{7}, defined as the cohomology spaces of the complex \( (\Omega^*_{\mathcal{F}}, d) \), and there exist natural homomorphisms

\[
\varphi : H^*_b(M, \mathcal{F}) \to H^*_{de Rham}(M), \quad \psi : H^*_b(M, \mathcal{F}) \to H^*_L(h),
\]

induced by inclusion and \( h \), respectively.

These facts have the following consequences for geometric quantization. Assume that \( [W] = \psi[\Phi] \) where \( \varphi[\Phi] \) is an integral de Rham cohomology class. Then \( \Phi \in \Omega^2_{\mathcal{F}} \) is a closed 2-form with integral periods, such that

\[
\{f, g\}' = \Phi(X'_f, X'_g), \quad \forall f, g \in \Omega^0_{\mathcal{F}}.
\]

Accordingly, there exists a Hermitian line bundle \( K \) over \( M \) with a connection \( \nabla \) of curvature \( 2\pi\sqrt{-1}\Phi \), and the Kostant-Souriau formula

\[
\hat{f}s = \nabla_{X_f}s + 2\pi\sqrt{-1}fs
\]

provides a prequantization such that the Dirac principle holds for distinguished functions but, generally, not for arbitrary functions (use (47)). The transitive case, i.e., presymplectic manifolds, was discussed in \cite{17}.
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