Resiliently evolving supply-demand networks

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The ability to design a transport network such that commodities are brought from suppliers to consumers in a steady, optimal, and stable way is of great importance for nowadays distribution systems. In this Letter, by using the circuit laws of Kirchhoff and Ohm, we provide the exact capacities of the edges that an optimal supply-demand network should have to operate stably under perturbations. The perturbations we consider are the evolution of the connecting topology, the decentralisation of hub sources or sinks, and the intermittence of suppliers/consumers characteristics. We analyse these conditions and the impact of our results, both on the current UK power-grid structure and on numerically generated evolving archetypal network topologies.

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Networks are ubiquitous in nature and man-made systems. Power and gas networks bring light and heating to our homes, telecommunication networks allow us to be entertained and to browse for information, and distribution networks allow manufacturers to supply food-stock and other products to the demand chain. In all of these cases a basic problem needs to be addressed: how to create a steady, optimal, and stable transport of commodities across such supply-demand networks.

We understand that a supply-demand network is stable when the system is not vulnerable to modifications in the network’s connectivity, switch from hub suppliers to decentralised smaller producers, or changes in the location of suppliers and/or consumers that may cause over-load failures to occur. It is optimal when the transport is done such that the cost is minimum, e.g., minimising the energy consumption.

Our problem is part of what flow network theory tries to decipher [1–4], a theory that roots back to Kirchhoff [1], answering what are the current flows in each edge of an electrical circuit as a set of voltages are applied to some nodes. The solution is then achieved by solving Kirchhoff’s equations. It is related to the probability that a random walker starts at the source and finishes at the sink [5] and to first-passage times at each node [6]. In order to model a supply-demand network, we solve the inverse problem. Namely, we calculate voltages (loads) using a conservative minimal cost transport system when input/output currents (flows) are given. This means loads are carried optimally from the source (supplier) to the sink (consumer) without losses. It is related to finding shortest-paths and community structures on weighted networks [7–8].

In particular, Kirchhoff’s flow network model is used to express electrical flow in circuits [9], but also to establish systems ecology quantities relationship [10], biologically inspired steady-state’s transport systems [11], and fractures in materials [12, 13]. Although the relationship between flows and loads in these models is restricted to be linear and conservative, the complexity in the mathematical treatment of the equations due to the topology structure is still demanding. Thus, most flow network solutions are based on optimisation schemes [3]. The results are complex and not easy to relate to other relevant network centrality measures. Moreover, if the network evolves in time (the connecting topology changes with nodes and/or edges appearing and/or disappearing), then predictions, controlling cascade of failures, and analytical solutions are scarcer [14–17].

In this Letter, we provide analytical expressions for the edge capacities that a steady optimal supply-demand network should have to operate stably under perturbations by using Kirchhoff flow network model. The perturbations we consider constitute some possible evolution factors that supply-demand networks are subjected to, such as, switching from hub suppliers to multiple smaller producers, intermittent supplying and consuming nodes, and either node and edge additions or removals.

We apply our edge capacity analytical results both to the current UK power-grid and to numerically generated evolving archetypal network topologies. We discuss the design of a modern steady-state stable power-grid system and we find that most topology modifications have a power-law behaviour of the analytically derived edge capacities as node or edges are added to the network. Our results and conclusions are general and applicable to any other system that is modelled by Kirchhoff flow network model in its steady-state. Furthermore, they are related to standard network characteristics and allow to predict cascade of failures due to over-loads.

To achieve an analytical solution of our problem, we initially assume that the network structure, the location of the supplier(s) and consumer(s), and the amount of commodities produced and consumed are known, but
loads and flows in every edge need to be calculated. Furthermore, we let loads to be linearly related to the flows by
\[ l_{ij}^{(st)} = R_{ij} f_{ij}^{(st)}. \]  
(1)

The left hand side of Eq. (1) is the load being transferred across the edge connecting nodes \( i \) and \( j \) of the network given a source located at node \( s \) and a sink located at node \( t \). The extension to many sources and sinks adds more upper indexes to the equation (see Supplementary Material). In our model, \( l_{ij}^{(st)} \) represents the voltage difference between two points in an electric circuit, where a current enters the circuit at node \( s \) and leaves at node \( t \). The right hand side of Eq. (1) is composed of a proportionality factor \( R_{ij} \), depending on the structure of the edge, and the unknown flow \( f_{ij}^{(st)} \), which the edge has for that particular location of the supply-demand nodes. Physically, it is the edge’s resistance times the electrical current. For example, in systems ecology, Eq. (1) relates the net flow at node \( i \) carried optimally from the source(s) to the sink(s). Then, the net flow at node \( i \) is

\[ \sum_{j=1}^{N} f_{ij}^{(st)} = I (\delta_{is} - \delta_{it}) = \sum_{j=1}^{N} W_{ij} \left( V_{i}^{(st)} - V_{j}^{(st)} \right), \]  
(2)

where \( V_{i}^{(st)} \) is the voltage potential at node \( i \) for the particular \( s-t \) pair and \( W_{ij} = A_{ij}/R_{ij} \) is the matrix representation of the network structure. \( A_{ij} = A_{ji} \) is the adjacency matrix entries, that are either 1 (nodes \( i \) and \( j \) are connected to each other) or 0 (\( i \) and \( j \) are not connected), and \( R_{ij} \) is the edge resistance.

We also make use of the equivalent resistance \( \rho_{ij} \), which is a network structure characteristic [18,20]. It is found analytically from the weighted Laplacian matrix \( G \) eigenvalues and eigenvectors (\( G = D - W \), with \( D_{ij} = \delta_{ij} k_{i} \), and \( k_{i} = \sum_{j=1}^{N} W_{ij} \), \( N \) being the number of nodes in the network). This quantity allows to express any connected arbitrary topology (defined with or without weighted edges) to an effective weighted complete network.

Our first analytical result is the edge capacity \( C_{ij}^{(st)} \) that a supply-demand network must have in order to operate stably avoiding over-loads is such that \( l_{ij}^{(st)} \leq C_{ij}^{(st)} \). We obtain that \( C_{ij}^{(st)} \) is a function of the equivalent resistance of the edge and proportional to the total amount of commodities per unit of time that are produced by the suppliers (the total input \( I \)). Moreover, the value of \( C_{ij}^{(st)} \) we find is independent of where the suppliers and consumers are located within the network (namely, \( C_{ij}^{(st)} = C_{ij} \)). Specifically, the exact value an edge capacity must have is

\[ C_{ij} = I \rho_{ij}, \]  
(3)

where the functional \( \rho_{ij}(W) \) is the equivalent resistance between nodes \( i \) and \( j \). The edge capacities in Eq. (3) quantify the value that each existing edge of the supply-demand network must have to secure a steady-state stable distribution, regardless of the location of the producer and consumer and regardless if, instead of a single supplier and consumer, there are many with arbitrary spatial distributions [the demonstration of Eq. (3) from Eq. (2) is given in the Supplementary Material]. To derive the exact values for node capacity \( C_{i} \), we perform a summation over all edge capacities \( C_{ij} = \sum_{j=1}^{N} C_{ij} \), which is feasible because our model of flow network is conservative. Hence, the node and edge capacities are not independent quantities.

Cascade of failures on networks are often studied by analysing how attacks and/or over-loads occur when a load surpass the node capacity. Such a node capacity is conjectured to have, with some tuning parameters, a linear relationship with the initial load distribution [16,17]. This assumption allows to draw conclusions on how the network structure should be designed to avoid failures due to over-loads. Here, we show that the capacity-load relationship is given by \( \rho_{ij} \) [Eq. (3)], and it is derived from finding the edge’s maximum loads for any of the discussed network evolution processes. In the cases that the physical edge capacity is pre-assigned, such as in a fuse network (a model that explains fractures in materials [12,13]), then Eq. (3) predicts exactly which edges will over-load due to the perturbations. This can still aid in the prevention of cascade of failures as it detects the vulnerable edges exactly.

The second analytical result we find is that all \( C_{ij} \) defined by Eq. (3) are bounded by the inverse of the largest \( (\lambda_{N-1}(G)) \) and the smallest non-zero \( (\lambda_{1}(G)) \), also known as spectral gap) weighted Laplacian matrix \( G_{ij \rightarrow} \) eigenvalues [21,22]. In particular, we find that

\[ \frac{2I}{\lambda_{N-1}} \left( 1 - \delta_{ij} \right) \leq C_{ij} \leq \frac{2I}{\lambda_{1}} \left( 1 - \delta_{ij} \right). \]  
(4)

Moreover, these eigenvalues are related to the minimal and maximal degrees of the network, thus, providing a way to modify the network topology (adding or removing nodes and edges) and keep the capacity values bounded by considering simple rules from minimal information about the structure [the derivation of Eq. (4) and the relationship of these bounds to the node degrees is provided in the Supplementary Material].

As a practical proof of concept and a way to illustrate these analytical results, we apply them both to
the UK power-grid structure \cite{23} and numerically generated random \cite{24} and small-world \cite{25} topologies. In these frameworks, we discuss the following perturbations: (i) power generator decentralisation (changing from centralised high-power generators to distributed smaller generators), (ii) source-sink intermittency (the inclusion of suppliers and consumers, such as renewable sources, storage systems, and electric cars, all possibly changing locations within the network), and (iii) connectivity modifications. These are some of the most important perturbations that modern power-grid systems are having to deal with. We address cases (i) and (ii) using the real UK power-grid structure. For the case (iii), we focus the analysis on how the connectivity modifications affect the edge’s capacity by deriving them for numerically generated networks.

Since the capacity is linearly related to $\rho_{ij}$ [Eq. (3)], to obtain the influence of the topology on the loads we first calculate the UK power-grid $\rho_{ij}$ by assuming $R_{ij} = 1$, i.e., $W_{ij} = A_{ij}$. The resultant set of $\rho_{ij}$ is represented by $\rho(A)$ in Fig. 1. We then calculate $\rho_{ij}$ considering the edge’s resistance in MVA units (mega Volt-Ampere) \cite{23}, i.e., $W_{ij} = A_{ij}/R_{ij}$. This set is represented by $\rho(W)$ in Fig. 1. The bounds in terms of the maximum and minimum eigenvalues [Eq. (4)] are shown by the vertical dashed lines in each case.

![Fig. 1: The plot shows the probability density functions of the equivalent resistance $\rho_{ij}$ (stairs-like lines) for the power-grid adjacency (green online) and weighted structure (black), and their respective bounds (vertical dashed lines are the bounds of Eq. (4)). The units of $\rho_{ij}$ are in MVA (as the power-line resistances found from \cite{23}) and include all three UK’s major transmission companies (SHETL, SPT, and NGET).](image)

From an engineering point of view, the steady-state stability of a power network is the capability of the system to maintain the power transmitted between any two nodes below the edge capacity (namely, the maximal load that the connecting power line can handle) when perturbations are applied to the network. Figure 1 shows that, if we neglect the reactance and inductance characteristics of the power lines and we model the power-grid by a conservative linear flow network model such as Eq. (1), then assigning edge capacities to the edges in the power-grid drawn from the $\rho$ distribution guarantees the steady-state stability of the system. Such a power-grid system is resilient to changing from hub generators to distributed sources [case (i)] or having intermittent sources and sinks [case (ii)].

In the case (i), a single source is substituted by multiple sources while maintaining the same inflow of power. Namely, $I$ is kept constant in the transformation while the network structure does not change. In such a case, our first analytical result for $C_{ij}$ [Eq. (3)] predicts that the new maximum loads are always less than the capacity value $|l_{ij}^{s_1,s_2,...,s_t}| < |l_{ij}^{st}| \leq I \rho_{ij}$. In the case (ii), given the intermittency property of renewable sources and electric car power stations, the supply-demand network behaves as if sources and sinks keep changing locations with time. In other words, the system explores various configurations of the many source-sink problem for a fixed structure at different times. Equation (3) describes the edge capacities for all edges that ensures no over-load will happen.

On the contrary, the modifications to the connecting topology [case (iii)] change the value of $\rho_{ij}$, which consequently, changes the edge capacities and redistributes the flows. In this case, in order to draw conclusions about edge capacities for a power-grid such as the UK, one needs a dynamic picture of the network topology as it evolves. We find that the minimum and maximum eigenvalues [Eq. (4)] must be kept fixed to design edge capacities with fixed margins as the supply-demand network topology changes. In this sense, the changes in the topology are contained within the bounds of the initial edge capacities. Thus, resilience is enforced by using the upper bound of Eq. (4) for every edge. In order to keep the minimal and maximum eigenvalues fixed, we find that it is enough to fix the minimum and maximum node degrees of the network (Supplementary Material).

In general, we find that if a supply-demand network needs to be designed with similar edge capacities, then the graph has to be set such that it resembles as much as possible a complete graph (which has all its non-null eigenvalues equal to the node degrees). This narrows the values that $\rho_{ij}$ can take. On the other hand, if the range of $C_{ij}$ values sought needs to be as broad as possible, then the network should be designed in such a way it include some nodes with higher degrees (this increases the largest eigenvalue, thus, diminishing the lower bound for the edge capacity) and, well-defined communities (which lowers the magnitude of the spectral gap, hence, increasing the upper bound for the edge capacity) or nodes with lower degrees.

We particularise now the analysis of the effect of connectivity modifications to the $\rho_{ij}$ probability distribution function (PDF) for two types of numerically generated networks: random (Fig. 2) \cite{24} and small-world (Fig. 3) \cite{25}. In both cases, two growth protocols are carried out.
The effect of these protocols on the $\rho_{ij}$ PDFs are shown in Figs. 2 and 3.

![FIG. 2: Edge [panel (a)] and node [panel (b)] addition protocols effect on the PDF of the $\rho_{ij}$ of random networks. Panel (a) simulations start from a ring graph of $N = 2^9$ nodes. Then, edges linking two disjoint nodes are added with probability $p \in [0, 1]$. The node addition in panel (b) is done by growing the ring graph from $N = 2^6$ to $2^{12}$. Then, nodes are linked with probability $p = 10^{-1}$. The colour scale corresponds to the logarithm of the PDFs values. The analytical bounds of Eq. (4) are shown with dashed lines (see insets).](image1)

For random networks (RN), the first protocol for RN keeps the number of nodes fixed but increases the number of edges (hence, increasing the density of edges). The second protocol adds nodes maintaining the average number of edges fixed (thus, decreasing the density of edges). For small-world networks (SW), the first protocol rewires the existing edges in a regular graph. This means the number of nodes, edges, and density of edges, are fixed. The second protocol increases the number of nodes and edges but keeps the average edge density constant.

![FIG. 3: Edge rewiring [panel (a)] and node addition [panel (b)] protocols effect on the PDF of the $\rho_{ij}$ of small-world networks. Panel (a) simulations are carried out for a regular network of $N = 2^9$ nodes with node degree $k = N/4$ for all nodes. Then, edges are rewired with probability $p \in [0, 1]$. The node addition in panel (b) is done by growing the regular graph from $N = 2^6$ to $2^{12}$. Then, edges are rewired with probability $p = 10^{-1}$. Colour scale and lines follow the same criteria as in Fig. 2.](image2)

RN and SW networks exhibit power-law behaviour of the $\rho_{ij}$ PDFs for both growth protocols, with the exception of the edge rewiring protocol for fixed number of nodes in SW networks [panel (a) in Fig. 3]. In other words, most growth processes lead to a power-law distribution of the edge capacities as a function of the control parameter (either $p$ or $N$), as it also happens in scale-free networks [14]. This is very advantageous when designing an invariant flow distribution for an evolving supply-demand network. Moreover, we find that at every step of the growth process the edge capacity distribution is mainly given by the behaviour of the most probable $\bar{\rho}$ value. Thus, its evolved magnitude can be predicted from the power-law exponents at any step. The evolution of $\bar{\rho}$ is derived from

$$\bar{\rho}(r) \simeq e^{-\beta/\alpha} r^{1/\alpha},$$

where $\alpha < 0$ and $\beta$ are the scaling exponents $[\log(r) \simeq \alpha \log(\bar{\rho}) + \beta]$ and the protocol control parameter $r$ is either $p$ or $N$. Furthermore, as can be seen from the insets of these figures, we find that $\alpha \sim -1$ for all the power-law cases.

We interpret the power-law behaviour in the following way. Any addition of edges results in a decrease of the equivalent resistance between nodes. This is a consequence of having shorter paths between nodes, namely, more connecting edges. This is why the edge addition protocol for RN and the node addition protocol for SW result in a power-law PDF evolution. The node addition protocol for RN keeps an average number of edges fixed, though it diminishes the edge density at every stage, thus, creating a power-law evolution of the edge’s $\rho_{ij}$ due to the existence of some short and long paths balance. When the connectivity modification protocol fixes the number of nodes, edges, and the edge density [such as the protocol in panel (a) of Fig. 3], then the $\rho_{ij}$’s PDF remains invariant.

Any real supply-demand network operates under unpredictable fluctuations, such as, the switch from hub sources to distributed smaller producer [case (i)], the change in the location of suppliers and consumers [case (ii)], topology intended modifications [case (iii)], or directed attacks. Using Eq. (3), we provide a robust edge capacity that is not surpassed in either case (i) or case (ii). In the case (iii), we find that most topology evolution factors cause the edge capacity to evolve in a power-law behaviour.

In the cases where the knowledge of the full network structure is missing, we also obtain manageable bounds for the exact capacity values in terms of minimal information of the network structure, e.g., eigenvalues of the weighted Laplacian matrix [Eq. (4)] and minimum/maximum degrees (see Supplementary Material). Our margins give simple engineering strategies for modifying the network’s topology while bounding the capacities and maintaining a stable distribution [case (iii)].

To summarise, by analytically providing exact edge capacities values (plus bounds) of conservative linear flow problems, we are able to show how to design resiliently evolving supply-demand networks.

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