Kink dynamics in a one-dimensional growing surface

Paolo Politi

Fachbereich Physik, Universität GH Essen, D-45117 Essen (Germany)
Dipartimento di Fisica dell’Università di Firenze e Sezione INFM, L.go E. Fermi 2, I-50125 Firenze (Italy)

I. INTRODUCTION

A high-symmetry crystal surface may undergo a kinetic instability during the growth, such that its late stage evolution resembles a phase separation process. This parallel is rigorous in one dimension, if the conserved surface current is derivable from a free energy. We study the problem in presence of a physically relevant term breaking the up-down symmetry of the surface and which can not be derived from a free energy. Following the treatment introduced by Kawasaki and Ohta [Physica 116A, 573 (1982)] for the symmetric case, we are able to translate the problem of the surface evolution into a problem of nonlinear dynamics of kinks (domain walls). Because of the break of symmetry, two different classes (A and B) of kinks appear and their analytical form is derived. The effect of the adding term is to shrink a kink A and to widen the neighbouring kink B, in such a way that the product of their widths keeps constant. Concerning the dynamics, this implies that kinks A move much faster than kinks B. Since the kink profiles approach exponentially the asymptotical values, the time dependence of the average distance \( L(t) \) between kinks does not change: \( L(t) \sim \ln t \) in absence of noise, and \( L(t) \sim t^{1/3} \) in presence of (shot) noise. However, the cross-over time between the first and the second regime may increase even of some orders of magnitude. Finally, our results show that kinks A may be so narrow that their width is comparable to the lattice constant: in this case, they indeed represent a discontinuity of the surface slope, that is an angular point, and a different approach to coarsening should be used.

II. THE SURFACE CURRENT

The study of a growth process may ideally be divided into two main steps: the first one starts from some microscopic point of view and should arrive to a contin-
uum description of the surface/interface; the second one may even assume a given evolution equation and study it. Most of the difficulties encountered in a theoretical study of MBE are related to the first step. In the present paper, we will limit ourselves to a one-dimensional high-symmetry surface, and in this section we will introduce and justify a specific Langevin-type equation.

The local surface height $z(x, t)$ is generally supposed to satisfy a local equation of the form $\partial_t z = aF + \mathcal{F}(\partial_x z, \partial_x^2 z, \ldots)$, $F$ being the incoming flux, $a$ the in-plane lattice constant, and $\mathcal{F}$ a function of the local profile of the surface (the out-of-plane lattice constant is put equal to one, i.e. $z$ is adimensional). The underlying hypotheses have been discussed in Ref. [6], where we have shown that the appearance of angular points in the surface profile may be treated correctly solely through the introduction of a nonlocal equation. We will take up this point again, at the end of the article.

The flux $F$ contains a constant part $F_0$, which is “eliminated” by redefining $z(x, t)$: $z \rightarrow z - aF_0 t$, and a fluctuating part $\delta F(x, t)$ which represents the so-called shot noise, which is supposed to follow a Gaussian-like distribution:

$$\langle \delta F(x, t) \rangle = 0$$

$$\langle \delta F(x, t) \delta F(x', t') \rangle = 2F_0 \delta(x - x') \delta(t - t')$$

In the limit of negligible desorption, and if highsymmetry surfaces grow by conserving both mass and volume: therefore, the function $\mathcal{F}$ must be derivable from a surface current $j$, and the evolution equation will be written in the form:

$$\partial_t z(x, t) = -a\partial_x j + a\delta F(x, t) ,$$

(3)

The central question is which current $j$ governs the evolution of the surface: a still debated question even for the simplified model of a one dimensional surface, as shown by the following discussion on the different terms appearing in $j$. Symmetry arguments simply tell that $j$ does not depend on $z$ \cite{10}, but on its derivatives $(m = \partial_z z, m' = \partial_z^2 z, m'' = \partial_z^3 z, \ldots)$ and that on a high-symmetry surface– it must be an odd function of $x$: so, a term proportional to $m$ or $m''$ satisfies this request, but if proportional to $m'$, it does not.

A. Ehrlich-Schwoebel current

In the Introduction, we mentioned the Ehrlich-Schwoebel effect, which gives rise to a slope-dependent current: $j_{ES}(m)$. Since it must be an odd function of $m$, its form at small $m$ will be $j_{ES} = \nu m$. The coefficient $\nu$ depends \cite{6} on the flux $F_0$, the diffusion length $\ell_D$ and the Schwoebel length $\ell_S$: $\ell_D$ measures the typical linear distance travelled by the adatom before meeting another one forming the nucleus of a growing island. It represents the “maximal” size of a terrace, because if $\ell > \ell_D$ the probability to nucleate a new island on it is very high; during the first stages of growth, when the surface is still more or less flat, $\ell_D$ is also the typical size of a terrace. It is not so when an instability develops: in this case, $\ell$ may be much smaller than $\ell_D$ and the slope $m = 1/\ell$ may be fairly large. Indeed, the slope $1/\ell_D$ discriminates between a nucleation-dominated regime ($m \ll 1/\ell_D$) and a step-flow regime ($m \gg 1/\ell_D$): the latter is generally relevant for vicinal surfaces, which grow through sticking of adatoms to preexisting steps; anyway, if a flat surface develops an instability with regions of high slope, such regime becomes important also for high-symmetry orientations.

The second relevant length, the Schwoebel length, is a measure of the asymmetry in the sticking coefficients of an adatom to a step. Its simplest form \cite{6} is: $\ell_S = a(D_+/D_- - 1)$, where $D_+$ and $D_-$ are such coefficients for an adatom approaching the step from above ($D_+$) and below ($D_-)$. The existence of an Ehrlich-Schwoebel effect means that $D_+ < D_- and therefore $\ell_S > 0$. To describe the meaning of $\ell_S$, let us consider a terrace of size $\ell (< \ell_D)$: if $\ell_S \ll \ell$, only a fraction $\ell_S/\ell$ of the fallen adatoms will contribute to the uphill current, and therefore $j_{ES} = (F_0\ell) (\ell_S/\ell) = F_0 \ell S$, since the number of atoms arriving per unit time on a terrace of size $\ell$ is nothing but $F_0 \ell$. Conversely, if $\ell_S \gg \ell$ all the adatoms will stick to the ascending step, and so: $j_{ES} = F_0 \ell$. The simplest interpolation formula, valid for any value of $\ell_S$ is:

$$j_{ES} = \frac{F_0 \ell S}{1 + \ell S |m|} \quad |m| > 1/\ell_D$$

(4)

This formula also allows to obtain a semiquantitative expression for the parameter $\nu$: in fact, when $|m| \approx 1/\ell_D$, Eq. (4) must match the expression valid at small slopes: $j_{ES} = \nu m$. The result is: $\nu = F_0 \ell S (\ell_D/|\ell_S + \ell_D|$. It is important to remark that all the previous considerations may be made more rigorous \cite{6}, but in this section we are mainly interested in justifying the expression for the current $j$, rather than in deriving it.

The main characteristic of the Ehrlich-Schwoebel current just discussed is that it has no zeros other than $m = 0$ and $m = \pm \infty$. A zero in $j_{ES}$ is extremely important \cite{10} because the other terms in $j$ will be seen to depend on higher order derivatives of $z(x, t)$. So, a constant slope $m_0$ may be a stationary slope if and only if $j_{ES}(m_0) = 0$. An extra-zero $m_0$ may have different origins: the symmetry of the crystal lattice \cite{10, 11}, non-thermal relaxation mechanisms \cite{12}, or a transient mobility of the adatom just after the deposition \cite{13}. For example, the slope at 45 degrees corresponds in a cubic lattice to the high-symmetry orientation (11): we expect that $j_{ES}$ vanishes on it, as it vanishes on the (10) ($m = 0$) and (01) ($m = \infty$) orientations. A different example is the following: if atoms falling in the vicinity of a step have a higher probability to land on the lower terrace, or to kick down the step adatom, a down-hill current $j$, proportional to the density of steps and therefore to the
slope $m$, will appear: $j_b = -\nu' m$. So, if $\nu > \nu'$, a zero will appear when $(j_{ES} + j_b) = 0$.

Whatever is the origin of extra-zero(s) in the slope-dependent current, we can introduce two different models, according to the presence (model I) or absence (model II) of zeros at finite slope. The simplest expressions of $j_{ES}$ for the two models, having the correct symmetry properties are [13]:

\[
\begin{align*}
\text{model I} & : j_{ES} = \nu m(1 - m^2/m_0^2) \\
\text{model II} & : j_{ES} = \nu m /

1 + \ell_B^2 m^2
\end{align*}
\]

Model II does not correspond to a phase separation process (see Sec. [11]); it will be discussed in Sec. [VII].

B. Mullins-like current

The most famous “equilibrium” current is perhaps the one ($j_M$) introduced by Mullins [15] forty years ago, to study the relaxation towards equilibrium of a non-singular grooved surface. A simple derivation starts from writing $j_M$ as the gradient of a chemical (surface) potential: $j_M = -\Gamma \partial_z \mu$, where $\Gamma$ is the adatom mobility, and afterwards to derive $\mu$ from a surface free-energy:

\[
\mu = \frac{\delta \mathcal{E}}{\delta z(x)}, \quad \text{with} \quad \mathcal{E} = \sigma \int dx \sqrt{1 + m^2 a^2}
\]

By combining the different equations, in the limit of small slopes we obtain:

\[
j_M = Km''(x)
\]

with $K = a^2 \Gamma \sigma$.

The usage of this expression in our problem may be questionable in at least two respects: first, it applies to a nonsingular surface, i.e. above the roughening transition $T_R$; second, it applies to a close-to-equilibrium surface. Concerning the first remark, our surface is a high-symmetry one and therefore almost necessarily below $T_R$, because for a high symmetry orientation the roughening temperature is equal or nearly equal to the melting temperature $T_M$, while ordinary temperatures for MBE are well below $T_M$. Nevertheless, our surface—which is strongly out-of-equilibrium—contains a lot of steps because the incoming flux makes the surface rough [16]: therefore, the surface current should be nonsingular at zero slope.

The latter remark is more “critical”: the Mullins current derives from thermal detachment of atoms from steps, in order to minimize the surface free-energy. It is not clear if such process is effective in presence of a flux $F$. For example, Stroscio and Pierce [17] state that thermal detachment is negligible in the homoepitaxial growth of Fe (at least at room temperature) and therefore they do not write [15] such a term in the current. Anyway, it has been shown [11,18] that the current [8] may derive also from nonequilibrium effects: nucleation noise and diffusion noise. The first one should be dominant and correspond to the value [6,16]: $K = F_0 \ell_D^2$.

C. Symmetry-breaking current

The terms in the surface current which have been introduced so far not only satisfy the $x \rightarrow -x$ symmetry (because $j(-x) = -j(x)$), but they also fulfil the up-down symmetry, corresponding to the change of sign of $z$. In fact, if $z \rightarrow -z$ both $j_{ES}$ and $j_M$ change sign. However, there is no reason to expect that surface growth proceeds by conserving such symmetry, since the flux breaks it.

A symmetry-breaking (SB) term is intrinsically nonlinear, because any current of the form $j \sim \partial_t z(x,t)$ changes sign with $z$. The lowest order expression which changes sign with $x$ but does not change sign with $z$ is:

\[
j_{SB} = \partial_z A(m^2)
\]

where $A$ is any even function of the local slope. The simplest form for $A$: $A = (\lambda/2)m^2$ has been introduced by Sun et al. [19]. It is also called “conserved Kardar-Parisi-Zhang term”, because in Eq. (4) it looks like the laplacian of $(\partial_x z)^2$, i.e. the nonlinear term of the KPZ equation [20].

The current (1) is not derivable from a free energy. As pointed out by Somfai and Sander [21] it is necessary to rise the order of $j_{SB}$ to make it derivable from some free energy (for example, $j_{SB} \sim \partial_x (m')^2 = \frac{\rho}{\mathrm{vol}} \int dx F_{SB}$ with $F_{SB} \sim (m')^3$).

Before proceeding, let us discuss the physical origin of $j_{SB}$. When there is a gradient in the density $\rho$ of adatoms, a current of the form $j = -D \partial_x \rho$ is expected, where $D$ is the diffusion constant. In the case of a growing surface, the applicability of the previous expression is not obvious, because steps are sink for diffusing atoms and —at least if thermal detachment is forbidden— interlayer diffusion is absent. In spite of this, the above expression may help in understanding: in fact, adatom density on a terrace depends on its size $\ell$, because a larger terrace collects more atoms from the flux than a smaller one. So, $\rho = \rho(\ell) = \rho(|m|)$. In other words, the function $A$ appearing in $j_{SB}$ seems to be proportional to the adatom density itself.

This interpretation can be made more rigorous for large slopes ($|m| = 1/\ell > 1/\ell_B$), where nucleation of new terraces is absent and $\rho$ can be simply determined by solving the diffusion equation $\partial_t \rho = F_0 + D \partial^2 \rho$ in the quasi-static approximation ($\partial_t \rho = 0$) and with $\rho(0) = \rho(\ell) = 0$ as boundary conditions (i.e. steps are perfect sinks). The resulting average density on the terrace is $\rho \simeq (F_0/D)\ell^2$ and the current is $j_{SB} \simeq -F_0 \partial_x (1/m^2)$.

This expression agrees with those determined, with different methods, by Politii and Villain [3] and by Krug [22]. Hunt et al. [23] suggest that $j_{SB}$ may derive from the sticking-asymmetry induced by the Ehrlich-Schwoebel effect: nevertheless, $j_{SB}$ does not vanish even if $\ell_S = 0$.
One could ask why the average value of $\rho$ is taken. The answer is that inhomogeneities in the adatom density on a given terrace give rise to $j_{ES}$! In fact, if no Ehrlich-Schoeibel effect is present, $\rho(x)$ is symmetric with respect to the center of the terrace and therefore the average value of $\partial_x \rho$ vanishes. Conversely, if $\ell_S > 0$ then $\langle \partial_x \rho \rangle_{\text{terrace}} \neq 0$ and it corresponds just to $j_{ES}$. This remark stresses the “similar” origin of $j_{ES}$ and $j_{SB}$. It is likely that a systematic derivation of the surface current should give all the terms we have introduced: $j_{ES}$ (which depends on the slope $m$), $j_{SB}$ (which depends on the curvature $m''$), and $j_M$ (which depends on a higher order derivative: $m^{(n)}$). Anyway, a rigorous derivation is still lacking at the moment, above all for a high-symmetry orientation.

**D. The current of our model**

In the following, we will study the dynamical evolution of the surface, as determined by the current:

$$j = j_{ES} + j_M + j_{SB} \quad (10)$$

where:

$$j_{ES} = \nu m \left(1 - \frac{m^2}{m_0^2}\right) \quad (11)$$

$$j_M = Km'' \quad (12)$$

$$j_{SB} = \lambda mm' \quad (13)$$

The reason of our choice is clear: we want to study the effect of the symmetry-breaking current ($j_{SB}$) on the phase separation process determined by the other two terms of the surface current ($j_{ES} + j_M$), and at this aim we choose the simplest expression for $j_{ES}$ – which must have a zero at a finite slope $m_0$ – and for $j_{SB}$ – for which we take $A(m^2) = \lambda m^2/2$. In the last section, we will discuss how the conclusions depend or not depend on the present choice.

**III. EVOLUTION IN ABSENCE OF THE SYMMETRY-BREAKING CURRENT**

In the “language” of surface growth, the evolution of the surface proceeds as follows: after a time $t^*$ an instability of the flat surface with a well-determined wavelength $L^*$ develops. In this linear regime, $L^*$ is constant and the amplitude increases exponentially. Afterwards, because of the non-linearity of $j_{ES}$ a coarsening process takes place: the wavelength $L(t)$ of the mound-like (or pyramid-like) surface profile increases in time, whilst the maximal slope tends to the constant values $\pm m_0$. So, the surface is “made up” of neighbouring regions where the slope is alternately (nearly) equal to $+m_0$ and $-m_0$.

The first stages of growth can be analyzed by linearizing Eq. (3) with the current (10):

$$a^{-1} \partial_t z(x,t) = -\nu \partial_x^2 z(x,t) - K \partial_x^4 z(x,t) \quad (14)$$

which shows [1][23] that the flat surface is unstable against deformations of wavelength larger than $L_\nu = 2\pi\sqrt{K/\nu}$. The most unstable mode corresponds to $L_\nu = \sqrt{2}L_\nu$ and its amplitude grows as $\exp \left[ (aw^2/4K)t \right]$. So, $L^* = L_\nu$ and $t^* = (4K/aw^2)$.

The nonlinear profiles of the mounds are determined as stationary solutions of Eq. (6), that is to say as solutions of the equation $j = 0$:

$$j_{ES}(m) + Km''(x) = 0 \quad (15)$$

This equation can be derived by the following Lagrangian:

$$\mathcal{L} = (K/2)m^2 - V(m) \quad \text{with } V'(m) = j_{ES}(m)$$

which corresponds to an anharmonic pendulum, once we have identified the slope $m$ as its spatial coordinate and $x$ as the time. Since the potential $V(m) = (\nu/2)m^2[1 - m^2/2m_0^2]$ has two symmetric maxima in $\pm m_0$, the period of the oscillation (i.e. the wavelength of the surface profile) diverges when its amplitude (i.e. the maximal slope of the surface profile) goes to $m_0$. If $j_{ES}$ followed model II, $V(m)$ would have no maxima and no limitation on the slope would be present.

By going on with this mechanical analogy, the existence of coarsening requires a condition on the stationary configurations: the period of the oscillation must be an increasing function of the amplitude [24]; a condition which is surely fulfilled by the potential $V(m)$, since the quartic correction has a negative sign! Clearly, coarsening also requires that these stationary solutions are not stable: more precisely, they must be unstable with respect to wavelength fluctuations, but stable with respect to amplitude fluctuations.

The previous mechanical analogy helps in understanding why the surface keeps a regular profile and also allows to determine this profile at a given time, but it is not effective in determining the time dependence of $L(t)$, i.e. the coarsening law [23]. To this end, we must observe that the evolution equation for the local slope $m$ (which represents the “order parameter” of our problem) satisfies the noisy Cahn-Hilliard equation [23]:

$$a^{-1} \partial_t m = \partial_x^2 \left( \frac{\delta F}{\delta m} \right) + \eta(x,t) \quad \text{where } \mathcal{F} = \int dx \mathcal{L}$$

This equation corresponds to a phase separation process, where the order parameter is conserved ($\partial_t \int dx m(x,t) = 0$). The system is made up of domains where $m$ equals one of the two degenerate minima of the potential energy $U(m) = -V(m)$; domains which are separated by domain walls move in order to minimize the “action” $\mathcal{F}$. Domain wall (or “kink”) movement is determined both by their (deterministic) interaction and by fluctuations induced by the conserved noise. We will see that the growing surface (even in presence of
the symmetry-breaking current \(j_{SB}\) can be mapped in
a one-dimensional system of interacting kinks which an-
nihilate, so that the average distance \(L(t)\) between kinks
increases in time.

By using this method for the symmetric case \((j_{SB} = 0)\),
Kawasaki and Ohta \[26\] have found the equation of
motion for the kinks, which has been then studied by
Kawakatsu and Munakata \[27\]. The final result is that
\(L(t)\) grows logarithmically with time if noise is absent
and grows as \(t^{1/3}\) if noise is present.

IV. KINK PROFILES

A stationary kink \(M(x)\) is defined as a monotonic so-
lution of \(j[M(x)] = 0\), with \(M(x)\) tending to (different)
minima of \(U(m)\), when \(x \to \pm \infty\). In the present case,
there are only two symmetric minima in \(\pm m_0\) and there-
fore only two kinks \(M_{\pm}(x)\) are possible, the subscript
corresponding to the sign of its first derivative, i.e. to
the curvature of the surface profile.

The surprising result is that the “shape” of the kink
does not change because of the introduction of the
symmetry-breaking term. To see it, let us replace the
expression:

\[
M_{\pm}(x) = \pm m_0 \tanh(\kappa_{\pm} x/2)
\]

in the differential equation \(j = 0\):

\[
Km''(x) + \nu m(1 - m^2/m_0^2) + \lambda mm' = 0
\]

We obtain the following second degree equation for the
parameters \(\kappa_{\pm}\):

\[
K\kappa_{\pm}^2 + \nu m_0 \kappa_{\pm} - 2\nu = 0
\]

which gives the positive solutions:

\[
\kappa_{\pm} = \left(\sqrt{\lambda^2 m_0^2 + 8\nu K} \pm \nu m_0 \right) / 2K
\]

Two limiting cases, corresponding to weak and strong
symmetry-breaking, will be frequently used:

\[
\lambda m_0 \ll \sqrt{8\nu K} \quad \kappa_+ = \kappa_- = \sqrt{2\nu / K} \equiv \kappa_0
\]

\[
\lambda m_0 \gg \sqrt{8\nu K} \quad \kappa_+ \approx \lambda m_0 / K \quad \kappa_- = 2\nu / \lambda m_0
\]

So, the effect of \(j_{SB}\) is to create two classes of kinks:
kinks “A”, given by the profile \(M_+(x)\) and characterized
by a width \((1/\kappa_+)\), and kinks “B”, given by the profile
\(M_-(x)\) and whose width is \((1/\kappa_-)\). For a strong \(j_{SB}\),
\(\kappa_+ > \kappa_-\): kinks A are much narrower than kinks B.
It must also be observed that the product \((\kappa_+ \kappa_-)\) does not
depend on \(\lambda\), since it equals (see the algebraic equation)
\((2\nu / K)\). In other terms, the effect of \(j_{SB}\) is to shrink
kinks A and to widen kinks B, in such a way that the
product of their widths keeps constant.

V. FROM SURFACE DYNAMICS TO KINK
DYNAMICS

In this section we will describe the method to solve the
growth equation for the surface-slope profile:

\[
a^{-1} \partial_t m = D_x \left[ K \nu m'' - U'(m) + \lambda mm' \right] \quad D_x = -\partial_x^2
\]

in a “multi-kink” approximation. Since our approach
follows that one introduced by Kawasaki and Ohta \[24\]
to study the above equation in absence of the \(\lambda\)-term,
we will expose the main calculations in App. \[A\] and here
we will limit ourselves to explain the general lines of the
method.

Once a kink is inserted in our problem, it moves with a
given (constant) velocity \(v_0\) and a profile \(m(x,t) = M(x - v_0 t)\),
where \(v_0\) is found by solving the eigenvalues
problem obtained by putting \(m(x,t)\) in \(\[24\]\). Our system
is made up of an ensemble of kinks \(A\) which alternate to
kinks \(B\), and we will look for an approximate solution of
\(\[24\]\) as a superposition of kinks centered in \(x_i\) and mov-
ing with velocity \(v_i\). Because of the interaction between
kinks, \(v_i\) is a constant, and depends on the position of
the other kinks. In principle, the nonlinear part of
\(U(m)\) (i.e. the quartic term \(m^4\)) gives rise to terms of
\(n\)-kinks interaction: we will adopt a “binary-interaction”
approximation, which will be further simplified by limit-
ing to nearest-neighbour interaction. This procedure is
justified by the fact that we are interested in the late
stages of growth, when the distance between kinks is
much larger than the width of their cores \((= 1/\kappa_+)\): so,
they interact only through the tails of the profiles, which
means that the interaction decays exponentially, since
\(\tanh(\kappa x/2) \approx \pm 1 \mp \exp(-\kappa |x|)\) when \(x \to \pm \infty\).
For the same reason, the velocities \(v_i\) and the accelerations
\(\dot{v}_i\) will be considered “small”, because the typical size of
the mounds grows slower than linearly: This means that the
velocity of the coarsening process goes to zero, as
time increases.

As a final result, we obtain a Langevin equation for the
discrete variables \(x_i(t)\), or equivalently— for the
kink-kink distances \(X_i(t) = x_{i+1}(t) - x_i(t)\), which will be
studied by translating it in a Fokker-Planck equation.

The treatment of Eq. \(\[22\]\) (see App. \[A\]) gives the fol-
lowing coupled equations for the kink positions:

\[
-2a^{-1} m_0^2 \sum_j (1 - j) |x_i - x_j| \dot{x}_j =
(C_1) + (C_2) \quad \text{(23)}
\]

where:

\[
(C_1) = 8\nu m_0^2 [R_{\beta}(X_i) - R_{\beta}(X_{i-1})]
\]

\[
(C_2) = \beta(4/3) m_0^3 \kappa_{\beta} \lambda [R_{-\beta}(X_i) - R_{-\beta}(X_{i-1})]
\]

\[
(C_4) = -\beta m_0^3 \kappa_{\beta} \lambda [R_{\beta}(X_i) - R_{\beta}(X_{i-1})]
\]

and:
\[
\langle \eta_i(t) \rangle = 0
\]
\[
\langle \eta_i(t)\eta_j(t') \rangle = -4m_0^2F_0(-1)^{i-j}|x_i - x_j|\delta(t-t')
\]

Let us explain the notations: The \( i \)-th kink is centered in \( x_i \), and because of the breaking of symmetry—two different classes of kinks exist. In accordance with Sec. 13, their profiles are given by: \( M_\beta(x) = \beta m_0 \tanh(\kappa_\beta x/2) \), where \( \beta = \pm 1 \). We will assume that the \( i \)-th kink is of class \( \beta \) (whatever is its value) and its nearest-neighbours of class \( -\beta \). The quantity:

\[
R_\beta(x) = \exp(-\kappa_\beta x)
\]

in the \( C_i \) expresses the interaction between kinks, when the distances \( |x_{i+1} - x_i| \) are large compared to \( 1/\kappa_\beta \).

Eq. (23) can also be written in matrix form: \( A_{ij}\dot{x}_j = I_i + \eta_i \). The matrix \( A \) takes into account the kinematical coupling between kinks, due to the conservation of the order parameter, and \( I \) contains the forces between kinks. The matrix \( A \) can be inverter \( A^{-1} \), giving a tridiagonal and symmetric \( A^{-1} \):

\[
A^{-1}_{ii} = \frac{a}{4m_0^2} \left( \frac{1}{X_i} + \frac{1}{X_{i-1}} \right)
\]

\[
A^{-1}_{i+1,i} = \frac{a}{4m_0} \frac{1}{X_i}
\]

The evaluation of \( A^{-1} I \) is trivial:

\[
(A^{-1} I)_i = \frac{a}{4m_0} \left( \frac{I_i + I_{i+1}}{X_i} + \frac{I_i + I_{i-1}}{X_{i-1}} \right)
\]

and the explicit expression of \( I_i \) is found directly from Eqs. (23):

\[
I_i = R_\beta^+(X_i) - R_\beta^-(X_{i-1})
\]

where \( R_\beta^+(X) \) is a linear combination of the two different \( R_\beta(X) \):

\[
R_\beta^+(X) \equiv c_\beta R_\beta(X) + d_\beta R_{-\beta}(X) \quad \text{with}
\]

\[
c_\beta = 8vm_0^2 - \beta 4m_0^2 \kappa_\beta \lambda \quad \text{and} \quad d_\beta = \beta(4/3)m_0^2 \kappa_\beta \lambda
\]

Concerning the noise, it is preferable to work with quantities which are not spatially correlated. To this end, the matrix \( A^{-1} \) is written as the product \( PP^T \) and new noise variables \( \tilde{\eta} = PP^{-1} \eta \) are defined. Since \( P \) is a bidiagonal matrix whose nonvanishing elements are:

\[
P_{ii} = P_{i+1,i} = \frac{a}{2m_0} \frac{1}{\sqrt{X_i}}
\]

\( \tilde{\eta}_i \) is given by \( \tilde{\eta}_i = \sqrt{a}(\eta_i + \eta_{i+1})/(2m_0 \sqrt{X_i}) \), and it results that:

\[
\langle \tilde{\eta}_i(t) \rangle = 0 \quad \langle \tilde{\eta}_i(t)\tilde{\eta}_j(t') \rangle = 2aF_0 \delta_{ij} \delta(t-t')
\]

In order to eliminate the constant factor in the correlator, we simply put: \( \tilde{\eta}_i = \sqrt{2a}F_0 \xi_i \). This way, the final equation for kink dynamics is:

\[
\dot{x}_i(t) = \frac{a}{4m_0} \left[ \frac{I_i + I_{i+1}}{X_i} + \frac{I_i + I_{i-1}}{X_{i-1}} \right] + \frac{\sqrt{2a}a}{2m_0} \frac{\xi_i}{\sqrt{X_i}} + \frac{\xi_{i-1}}{\sqrt{X_{i-1}}}
\]

\[
\langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij} \delta(t-t')
\]

VI. FORCES AND KINK VELOCITIES

In this section we want to discuss the effect of the symmetry breaking on the equations of motion for the kinks. Since we are here interested in the deterministic part of the interaction, we will not consider the noise. Therefore Eq. (23) takes the form: \( A_{ij}\dot{x}_j = I_i \). Kawasaki and Ohta \[3\] suggest to look on \( I_i \) as the force acting on the \( i \)-th kink. Let us consider the two opposite limits: \( \lambda = 0 \) and \( \lambda m_0 \gg \sqrt{8\nu K} \). For \( R_\beta^+(X) \) we obtain:

\[
R_\beta^+(x) = 8vm_0^2 \exp(-\kappa_\beta x)
\]

in the first limit (\( \lambda = 0 \), and:

\[
R^+_\beta(x) = (4\lambda^2m_0^4/3K)\exp(-\kappa_\lambda x)
\]

\[
R^-_\beta(x) = 16vm_0^2 \exp(-\kappa_\lambda x)
\]

in the second one (\( \lambda m_0 \gg \sqrt{8\nu K} \)). In the previous equations, \( \kappa_\lambda = \sqrt{2\nu K} \) and \( \kappa_\lambda = (\lambda m_0/K) \gg \kappa_0 \). They correspond to \( \kappa^- \) in the two pertinent limits.

In the case of absence of the \( \lambda \)-term, we simply get:

\[
I_i = 8vm_0^2[\exp(-\kappa_0 X_i) - \exp(-\kappa_0 X_{i-1})]
\]

This equation can be interpreted by saying that there is an attraction between kinks, proportional to \( \exp(-\kappa_0 X) \). If \( \lambda \neq 0 \) (and “strong”), than we must distinguish between positive and negative kinks:

\[
I_i = (4\lambda^2m_0^4/3K)[\exp(-\kappa_\lambda X_i) - \exp(-\kappa_\lambda X_{i-1})] \quad (\beta > 0)
\]

\[
I_i = 16vm_0^2[\exp(-\kappa_\lambda X_i) - \exp(-\kappa_\lambda X_{i-1})] \quad (\beta < 0)
\]

The first comment is that symmetry breaking implies that a positive kink is attracted (by a negative one) more strongly than a negative kink is attracted by a positive one. In other words, if we assign a mass to a kink, a negative kink weights more than a positive one, and the mass is proportional to the width of the kink itself.

This interpretation seems to be satisfactory, but if we analyse the velocities \( \dot{x}_i(t) \) rather than the “forces” \( I_i \), the picture becomes more complicated. In the limit \( \lambda = 0 \) we have:

\[
\dot{x}_i(t) = 2a \nu \left[ \frac{\exp(-\kappa_0 X_i)}{X_{i-1}} - \frac{\exp(-\kappa_0 X_{i-1})}{X_i} \right] + \frac{\exp(-\kappa_0 X_{i+1}) - \exp(-\kappa_0 X_{i-2})}{X_i}
\]
So, the effect of the conservation law (i.e., of the matrix $A$) is that $\dot{x}_i(t)$ depends not only on the positions of the nearest neighbour (nn) kinks ($x_{i\pm 1}$), but also on those of the next-nearest (nnn) ones ($x_{i\pm 2}$). Even more important, the nnn-"interaction" is of the same order of magnitude as the nn-one! Whilst the interpretation of $\exp(-\kappa_0 X_i)$ and $\exp(-\kappa_0 X_{i-1})$ in Eq. (32), as respectively the interaction with the kinks ($i + 1$) and ($i - 1$) is straightforward, in Eq. (33) the generic term $\exp(-\kappa_0 X_i)$ is divided by a different $X_j$, and therefore a similar interpretation becomes less evident. Anyway, if we don’t ascribe too much importance to the quantities $X_j$ in the denominator, Eq. (34) says that kink $i$ is attracted both by nn-kinks and nnn-kinks: the "interaction" between $i$ and ($i \pm 2$) has a kinematical origin (conservation of the order parameter) and indeed depends on ($x_{i\pm 2} - x_i$) rather than on ($x_{i\pm 2} - x_i$). A further comment is that in the evaluation of (I$_i$ + I$_{i+1}$) two terms cancel exactly, because in this case action and reaction are opposite and equal.

If now we consider the case of a strong symmetry breaking term, the velocity takes the form:

$$\dot{x}_i(t)|_{\beta > 0} = \frac{a \lambda^2 m_0^2}{3K} \left[ \left( \frac{1}{X_i} + \frac{1}{X_{i-1}} \right) \exp(-\kappa \lambda X_i) \right. - \left( \frac{1}{X_i} + \frac{1}{X_{i-1}} \right) \exp(-\kappa \lambda X_{i-1}) \right]$$

$$+ 4a \nu \left[ \exp(-\kappa \lambda X_{i+1}) - \exp(-\kappa \lambda X_{i-2}) \right]$$

and:

$$\dot{x}_i(t)|_{\beta < 0} = \frac{a \lambda^2 m_0^2}{3K} \left[ \exp(-\kappa \lambda X_i) \frac{X_i - \exp(-\kappa \lambda X_{i-1})}{X_{i-1}} \right]$$

$$+ \exp(-\kappa \lambda X_{i+1}) - \exp(-\kappa \lambda X_{i-2}) \right]$$

The surprising result is that the sign of the terms proportional to $\exp(-\kappa \lambda X_i)$ and $\exp(-\kappa \lambda X_{i-1})$ is inverted: so -because of kinematics- a negative kink is subject to a repulsive interaction with its nn-kinks! This result derives from the unbalancing of action and reaction. A closer inspection of the derivation of Eqs. (33) (34) allows to give the following interpretation: if $f_{ij}$ means the force exerted by the kink $j$ on the kink $i$ (so that $I_i = f_{i,i+1} + f_{i,i-1}$), then kinematics determines that the effective force $\tilde{f}_{i,i\pm 1}$ is a linear combination of $f_{i,i+1}$ and $f_{i,i-1}$. If $\lambda = 0$, the first term vanishes, but if $\lambda \neq 0$ it does not: furthermore, for a negative kink $f_{i,i\pm 1}$ prevails over $f_{i,i\pm 1}$ and it corresponds to a repulsive force, for kink $i$.

The conclusion we draw from the previous considerations is that negative kinks move much slower than positive kinks. This results on one side from the fact that a bigger mass can be attributed to them, and on the other side that they are subject to an effective repulsive nn-interaction.

VII. FROM KINK DYNAMICS TO COARSENING LAWS

The interesting dynamical variables are the kink-kink distances $X_i$, rather than the kink-positions $x_i$. So, from Eq. (32) we obtain:

$$\dot{X}_i(t) = \frac{a}{4m_0} \times$$

$$\left[ \frac{1}{X_{i+1}} \left( R_0^\beta (x_{i+2}) + R_0^\beta (x_{i+1}) - R_0^\beta (x_{i+1}) - R_0^\beta (x_i) \right) \right.$$ 

$$- \frac{1}{X_{i-1}} \left( R_0^\beta (x_i) + R_0^\beta (x_{i-1}) - R_0^\beta (x_{i-1}) - R_0^\beta (x_{i-2}) \right)$$

$$+ \sqrt{2\mu a \left[ \xi_{i+1} - \xi_{i-1} \right]}$$

(36)

The previous equations have the form:

$$\dot{q}_i(t) = \mathcal{U}_i(\{q\}) + \sum_j G_{ij}(\{q\}) \xi_j$$

with (37)

$$\langle \xi_j(t)\xi_j(t') \rangle = \delta_{jj'} \delta(t-t')$$

and we can therefore obtain a Fokker-Planck equation for the probability $\rho(\{q\}, t)$ of finding a given distribution $\{q\}$, at time $t$. Two different procedures exist [23], due to Ito and to Stratonovich, but as remarked by Kawakatsu and Munakata [27] the result is the same. This is true even in presence of the symmetry breaking term, because the two procedures may differ with respect of the term $G_{ij}$, which does not change if the $\lambda$-term is added.

The Fokker-Planck equation writes:

$$\frac{\partial \rho}{\partial t} = -\sum_k \frac{\partial}{\partial q_k} [\mathcal{U}_k(\{X\}) \rho] + \frac{1}{2} \sum_{kl} \frac{\partial^2}{\partial q_k \partial q_l} \sum_m G_{km} G_{lm} \rho$$

(38)

Its actual form, in our case, is [24]:

$$\frac{\partial \rho}{\partial t} = -\sum_k \frac{\partial}{\partial X_k} [\mathcal{U}_k(\{X\}) \rho]$$

$$+ \frac{F_0 a^2}{4m_0^2} \sum_k \frac{1}{X_k} \left[ \frac{\partial^2}{\partial X_k^2} + \frac{\partial^2}{\partial X_{k+1}^2} - 2 \frac{\partial^2}{\partial X_{k-1} \partial X_{k+1}} \right] \rho$$

(39)

where $\mathcal{U}_k$ is nothing but the "deterministic" velocity of the $k$-th kink.

We are interested in the time dependence of the average value of $X_i$ (which does not depend on $i$). To this end, we define the distribution functions:

$$g_1(X; t) = \int_0^{\infty} (dX_i) \rho$$

$$g_2(X, X_{i+1}; t) = \int_0^{\infty} (dX_i)_{i+1} \rho$$

$$g_3(X, X_{i+1}, X_{i+2}; t) = \int_0^{\infty} (dX_i)_{i+1,i+2} \rho$$

(40) (41) (42)
The notation \( (dX)_{i+1} \) means that the integration is performed on all the variables \( X_i \) but \( X_i, X_{i+1}, \ldots \).

The details of the calculation follow Ref. [27] and therefore they will not be given here. By using the factorization approximation:

\[
g_2(X_i, X_{i+1}; t) = g(X_i)g(X_{i+1})
\]

\[
g_3(X_i, X_{i+1}, X_{i+2}; t) = g(X_i)g(X_{i+1})g(X_{i+2})
\]

and integrating Eq. (23) over \((dX)_i\), we obtain:

\[
\frac{\partial g}{\partial t} = - \frac{\partial}{\partial X} j(X, t)
\]

with the current of probability given by:

\[
j(X, t) = \frac{a}{4m_0^2} \frac{1}{X} \times \left[ (R_+^*(X) + R_-^*(X)) - (R_+^*(X) + R_-^*(X)) \right] g - \frac{F_0 a^2}{2m_0^2} \frac{\partial g}{\partial \lambda X}
\]

In the two relevant limits, \( R_+^*(X) + R_-^*(X) \) takes the form:

\[
R_+^*(X) + R_-^*(X) = 16m_0^2 \exp(-\kappa_0 X) \lambda = 0
\]

\[
R_+^*(X) + R_-^*(X) = (4\lambda^2 m_0^2/3K) \exp(-\kappa \lambda X) \quad \lambda m_0 \gg \sqrt{\nu K}
\]

and in the limit \( \lambda = 0 \) we recover Eq. (4-4) of Ref. [27].

Important works on the solution of Eq. (47), which also go beyond the factorization approximation by taking into account correlations of consecutive domains, are given in a series of papers by Nagai and Kawasaki [30]. Here, we will follow Ref. [27] and the first of the papers cited in Ref. [30].

The time dependence of the density of kinks \( n(t) \) or alternatively of the average kink-kink distance: \( \langle X \rangle \equiv \langle X \rangle / n(t) \) is studied by assuming that at large times \( X \) represents the only relevant scale in the problem, and therefore \( g(X; t) \) satisfies the scaling expression:

\[
g(X; t) = n(t)\bar{g}(X/\overline{X})
\]

For example, for a Dirac-delta distribution (all the domains have the same size) \( \bar{g}(s) = \delta(s - 1) \), and for a Poisson distribution (randomly distributed kinks) \( \bar{g}(s) = e^{-s} \).

Secondly, we will use a steady-state approximation [27] according to which the distribution \( g(X; t) \) does not depend on time, on scales sufficiently small with respect to \( \overline{X}(t) \): more precisely, on scales \( X < \overline{X}^* \). This means that the motion of a couple of kinks at distance smaller than \( \overline{X}^* \) is essentially independent on the position of all the other kinks. Because of the scaling hypothesis, it must result: \( \overline{X}^* = \overline{X}/\alpha \), with \( \alpha \) constant.

The temporal variation of \( n(t) \) is determined by the number of kink-kink annihilations per unit time and unit length. Since each annihilation makes two kinks disappear, we have:

\[
\dot{n}(t) = 2n(t)j(X = 0; t) = 2n(t)j(X^*; t)
\]

where the second relation derives from the fact that \( \partial_t g = 0 \) implies \( \partial_X j = 0 \).

By approximating \( \langle f(X) \rangle \) with \( f(\overline{X}) \), \( f \) is a generic function, and by neglecting \( R_0^*(\overline{X}) \) with respect to \( R_0^*(X^*) \), we finally obtain the following expression for the current in \( \overline{X}^* \):

\[
j(X^*; t) = -\frac{a}{4m_0^2} \frac{1}{X} \left[ R_+^*(X^*) + R_-^*(X^*) \right] g(X^*)
\]

\[
- \frac{F_0 a^2}{2m_0^2} \frac{\partial g}{\partial X}
\]

\[
\overline{X}(t) \simeq (\alpha/\kappa) \ln(t/t_1)
\]

\[
t_1 = \left[ \frac{e\alpha^2}{8\bar{g}(1/\alpha)} \right] \frac{K}{av^2} \quad (\lambda = 0)
\]

In the opposite limit of a strong symmetry breaking \( (\lambda m_0 \gg \sqrt{\nu K}) \), a similar calculation gives:

\[
\overline{X}(t) \simeq (\alpha/\kappa \lambda) \ln(t/t_2)
\]

\[
t_2 = \left[ \frac{3e\alpha^2}{4\bar{g}(1/\alpha)} \right] \frac{K}{av^2} \quad (\lambda m_0 \gg \sqrt{\nu K})
\]

We therefore obtain that \( t_1 \simeq t_2 \simeq t^* \), where \( t^* \) was defined in Sec. [11] as the time necessary for the developing of the linear instability of the flat surface. So, the time scale for the logarithmic coarsening doesn’t depend on \( \lambda \), but the length scale does, since it depends on the width of the (largest) domain wall.

We can ask what is the meaning of the \( \alpha \)-dependence in Eqs. (47-49). As pointed out by Nagai and Kawasaki [30], since \( a \ln t = \ln t^* \) the parameter \( \alpha \) should have some “universal” value. In a mean-field calculation these authors find \( \alpha = 1 \), while in a numerical solution of the kink equations they obtain \( \alpha \simeq 3.5 \). More rigorous calculations [30] give \( \alpha = 2.27 \) if domains are completely uncorrelated, and \( \alpha = 3.56 \) if correlation effects between neighbouring domains are taken into account.
2. Noise-dominated regime

Now the current is:

\[ j(X^*; t) = \frac{F_0\alpha^2}{2m_0^2} \frac{1}{X} \frac{\partial \eta}{\partial X} \bigg|_{X^*}. \tag{56} \]

The equation for \( n(t) \) writes:

\[ \dot{n}(t) = -\left( \frac{F_0\alpha^2}{m_0^2} g'(1/\alpha) \right) n^4(t) \tag{57} \]

and the solution is:

\[ \overline{X}(t) = \overline{X}_0(t/t_0)^{1/3} \]

\[ \overline{X}_0 = \left[ \frac{3\alpha g'(1/\alpha)}{m_0^2} \right]^{1/3} t_0 = 1/F_0 a \tag{58} \]

So, we will have logarithmic coarsening at “small” times and a power-like one at later times. The cross-over time is determined by the relation \( \langle \alpha/\kappa \rangle \ln(t_c/t^*) = \overline{X}_0(t_c/\overline{X}_0)^{1/3} \). By neglecting the logarithmic dependence (also because \( t^* \gg t_0 \)), it is found that:

\[ t_c \approx t_0 \left( \frac{\alpha}{\kappa \overline{X}_0} \right)^3 \tag{59} \]

So, the ratio between the cross-over time in presence of a strong asymmetry and the cross-over time in absence of the \( \lambda \)-term is approximately given by:

\[ \frac{t_c(\lambda m_0 \gg \sqrt{\nu K})}{t_c(\lambda = 0)} \approx \left( \frac{\kappa_\lambda}{\kappa_\alpha} \right) \left( \frac{\lambda m_0}{\sqrt{\nu K}} \right)^3 \tag{60} \]

It is important to stress the cubic exponent in the previous expression: so, even a not large value of \( \langle \kappa_\alpha/\kappa_\lambda \rangle \) gives rise to a logarithmic coarsening which proceeds for a much longer time, because kink interaction is stronger and therefore a larger \( t_c \) is necessary so that noise get the better of the deterministic regime.

We want to emphasize that in the noise-dominated regime, the actual value of \( \alpha \) is much less relevant than in the deterministic regime, because of the power-law character of the coarsening.

VIII. DISCUSSION

The main result of the present paper is that “coarsening laws” don’t change if the symmetry-breaking current \( j_{SB} \) is put in the problem (at least, as far as a continuum local description is valid: see below). This is mainly due to the fact that the functional form of the kinks does not change, as shown by the exact solution we have given in Sec. \[\text{[IV]}\] for their profile.

So, a first question is how general is this result if we modify the surface current, and therefore Eq. \[\text{(10)}\]. A first obvious modification would be to replace \( \partial_x A(m^2) = \lambda m m' \) with a more complicated expression of the slope \( m \). This corresponds to have a \( \lambda \) depending on \( m \); in fact, \( \lambda = \lambda(m^2) = 2A'(m^2) \). Since in the late stages of growth the slope is almost everywhere equal to \( \pm m_0 \), \( \lambda \) is almost everywhere a constant equal to \( \lambda(m_0^2) \). Is it possible to simply replace \( \lambda \) by \( \lambda(m_0^2) \) in the final results?

This should not be a bad approximation, as suggested by the analysis of Eq. \[\text{(15)}\] when \( \lambda \) depends on \( m \). In fact, the asymptotic behaviour of \( M(x) \) [the relevant one for kink interaction] and the values of \( \kappa_\pm \) can be found by linearizing the differential equation with respect to \( (m_0 - M(x)) \) for a positive kink and to \( (-m_0 + M(x)) \) for a negative kink [in both cases, in the limit \( x \to \infty \)]. Because of the linearization, only the value \( \lambda(m_0^2) \) enters in the problem and therefore determines the profile.

In a similar way, we can take into account a possible \( m \)-dependence of the quantity \( K \). In this case, such dependence might arise from a slope-dependent mobility \( \Gamma \) —if \( K \) has an equilibrium origin— or from the dependence on the terrace length \( l \) of the probability to nucleate a new terrace \( \overline{X} \), if \( K \) derives from nucleation noise.

Let us now discuss the choice of the slope-dependent current: \( j_{ES} = \nu m(1 - m^2/m_0^2) \). The only features we require to have a phase separation process are: \( j_{ES}^{(m)}(m = 0) > 0 \) (to make the flat surface unstable) and \( j_{ES}(m_0) = 0 \) for some finite value \( m_0 \) [indeed, \( m_0 \) must be the first zero of \( j_{ES} \)]. These features define the so-called model I.

Modifications of \( j_{ES} \) inside this model do not change the given picture, as suggested by the analysis of the stationary profile of the kink (for the sake of simplicity we put \( \lambda = 0 \)). If we linearize the equation:

\[ j_{ES}(m) + Km''(x) = 0 \tag{61} \]

with respect to \( \epsilon(x) = m_0 - m(x) \), we obtain:

\[ j_{ES}^{(m_0)}(m_0)\epsilon(x) + K\epsilon''(x) = 0 \tag{62} \]

whose solution is again an exponential function. So, for \( x \to \infty \), \( m(x) = m_0 - \epsilon e^{-\kappa x} \), with \( \kappa = \sqrt{-j_{ES}(m_0)/K} \). In our expression of \( j_{ES} \) (Eq. \[\text{(10)}\]): \( j_{ES}(m_0) = -2\nu \) and \( \kappa \) reduces to \( \kappa_0 = \sqrt{2\nu/K} \).

Conversely, in model II there is no finite zero in \( j_{ES} \). This implies that the slope increases with no upper limit: for \( \lambda = 0 \), as shown by Hunt et al. \[\text{[23]}\], the maximal slope \( M_0 \) in the profile is asymptotically proportional to the size of the mounds: \( M_0(t) \sim \overline{X}(t) \). Since the potential energy \( U(m) = U''(m) = -j_{ES}(m) \) has minima, it is no more possible to define domains and domain-walls, i.e. kinks.

Concerning the time dependence of coarsening, the only existing numerical results are the ones found by Hunt et al. \[\text{[23]}\]. According to their simulations (in presence of noise), \( \overline{X}(t) \approx t^\alpha \) with \( n \approx 0.22 \): a fairly small value \[\text{[22]}\]. No (rigorous) theoretical derivation of \( n \) is available at the moment. Some scaling arguments —applicable to noiseless growth— can be found in Rost
and Krug [33] and in Golubović [34]: the former give $n \leq 1/4$ while the latter gives the equality $n = 1/4$ [35].

A final question we want to face now is how narrow kinks $A$ actually are. In the limit $\lambda m_0 \gg \sqrt{\nu K}$, from Eq. (21) we have: $\kappa_+ = \lambda m_0 / K$ and $\kappa_- = 2\nu / \lambda m_0$. A simple evaluations suggest [19]: $\lambda \approx K \approx F_{0b}^{\ell_{DF}}$. This expression for $K$ is surely wrong, if thermal detachment plays an important role. Conversely, if $\lambda$ and $K$ −or, more precisely, $\lambda(m_0^3)$ and $K(m_0^3)$− are of the same order of magnitude, we obtain $\kappa_+ \approx m_0$. This means that the width of the positive kink ($=1/\kappa_+$) is nothing but the inverse of the value of the constant slope in the surface profile: so, if $m_0$ is determined by the symmetry of the crystal lattice, $m_0 \approx 1/a$ and the positive kink is as narrow as a lattice constant! In this case, our description would break down, because the regions of positive curvature in the surface profile would correspond to a discontinuity of the slope, i.e. to angular points, which are not compatible with a local continuum equation [38].

**IX. CONCLUSIONS**

The kink picture not only has allowed to find the coarsening law in presence of the symmetry breaking term, but it has also given a qualitative description of the dynamics which allows a better comprehension of the evolution of the system: the widening or the narrowing of a kink; the consequent different velocities of kinks $A$ and $B$; the conservation of the order parameter seen as a kinematical constraint on kink movement; the difference between the “real” force acting on a kink and the “effective” force felt by the kink, because of such constraint.

In this respect, the most important consequence of the breaking of symmetry is that negative kinks feel an effective *repulsive* interaction with the nn kinks (but attractive with the nnn ones). It is important to stress this point because coarsening is the result of a global attraction between kinks: if kinks repelled each other, the configuration with the $X_1$ all equal would be stable.

Finally, the kink picture has provided the condition of applicability of the local theory:

$$\frac{1}{\kappa_+} = \frac{K(m_0^2)}{\lambda m_0 \lambda(m_0^3)} \gg a$$

(63)

If this relation is not fulfilled, a different method to study coarsening should be used. In Ref. [1] we showed that in this case the evolution of the surface is governed by a nonlocal current; alternatively, we can keep a local description, but we must add a singular term to the current $j$, and couple the Langevin equation: $\partial_t z(x,t) = -a \partial_x j$ with specific evolution equations for the angular points. It would be clearly interesting to check if a different coarsening process may arise from an “angular point” picture.

To our knowledge, the current [10] has not been formerly studied. The closest model is the one considered by Stroscio et al. [18] in two dimensions, where the Mullins term ($K m''(x)$) is replaced by a higher order one ($K m'''(x)$) and the resulting equation is studied numerically. Clearly, in two dimensions analytical treatments are much more difficult; anyway, neither a numerical solution of the model studied in the present paper is available at the moment. One reason is that in two dimensions, even the model without j$_{SB}$ is not yet fully understood, since the evolution equation for $m(x,t)$ [27] is no more equivalent to the Cahn-Hilliard equation.

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**APPENDIX A: LANGEVIN EQUATIONS FOR THE KINKS**

1. Absence of noise

The starting point is the following multi-kink expansion:

$$m(x,t) = M_i(x,t) + \sum_{j>i}[M_j(x,t) - M_j(-\infty)] + \sum_{j<i}[M_j(x,t) - M_j(\infty)]$$

(61)

$$\equiv M_i(x,t) + \delta m_i$$

(62)

which gives rise, once replaced in (23), to:

$$a^{-1} \sum_j \left[-v_j M'_j + \dot{v}_j \frac{\partial M_j}{\partial v_j}\right] = D_x j[m(x,t)]$$

(63)

$M_j$ depends on $x$ and $t$ through the combination $(x-v_j t)$ and $M'_j$ is the derivation with respect to all this argument. We will also use the notation: $d_x M$ to mean the same kind of derivation. The single kink profile is found by simply dropping the sum $\sum_j$ and the term in $\dot{v}$ in Eq. (63):

$$-a^{-1} v_0^0 M'_j = D_x j[M_j]$$

(64)

It will be useful to consider, together with $M_j$, also its spatial derivative $\tilde{M}'_j(x)$ which is localized around $x = x_j$. We define also $\tilde{M}'_j$ through the relation: $M'_j(x) = D_x \tilde{M}'_j(-x)$. They satisfy the relations:

$$-a^{-1} v_0^0 \tilde{M}'_j(x) =$$

$$D_x [K d_x^2 - U''(M_j) + \lambda M'_j + \lambda M_j d_x] M'_j(x)$$

(65)

$$a^{-1} v_0^0 \tilde{M}'_j(x) =$$

$$[K d_x^2 - U''(M_j) + \lambda M'_j + \lambda M_j d_x] D_x \tilde{M}'_j(x)$$

(66)
Now, let us multiply \( \hat{M}'_j(x) \) by \( \hat{M}'_j(x) \) and integrate on \( x \). By defining \( \delta v_j \equiv v_j - v'_j \), we can write:

\[
a^{-1} \sum_j \int dx \hat{M}'_j(x) \left\{ \delta v_j M'_j + v'_j \frac{\partial M_j}{\partial v_j} \right\} = \int dx \hat{M}' \left( D_x j[M] + a^{-1} \sum_j v'_j \int dx \hat{M}'_j M'_j \right) \tag{A7}
\]

The next step is to replace \( m(x, t) = M_i(x) + \delta m_i \) in the current \( j[m] \). The definition of the nonlinear part of the potential \( U(m) \) (or equivalently of the current \( U'(m) \)) is self-explanatory.

\[
j[m] = K m'' - U'(m) + \lambda m m' = \int dx K M''(x) + \lambda M_1 m + \delta m_i \]

\[
\approx \int dx K M''(x) + \lambda M_1 m + \delta m_i \]

\[
\equiv \int dx K M''(x) + \lambda M_1 m + \delta m_i \]

Let us consider separately some terms:

\[
\int dx \hat{M}'(x) \sum_{j \neq i} K D_x M''_j + a^{-1} v'_j M'_j = \int dx K \delta m_i + \delta m_i \]

\[
\int dx \hat{M}'(x) \sum_{j \neq i} K D_x M''_j = \int dx K \delta m_i \]

\[
\int dx \hat{M}'(x) \sum_{j \neq i} -D_x U''(M_i) \delta m_i = \int dx U''(M_i) \delta m_i \]

\[
\int dx \hat{M}'(x) \sum_{j \neq i} \lambda D_x \delta m_i M'_j + \delta m'_i M_j = \int dx \hat{M}' \left( D_x j[M] + a^{-1} \sum_j v'_j \int dx \hat{M}'_j M'_j \right) \tag{A8}
\]

This way, Eq. (A7) takes the form (LHS = left-hand-side): \( LHS|_{A7} = (A) + (B) + (C) \).

By using Eq. (A6):

\[
(A) = a^{-1} \int dx \hat{M}' \left( \sum_{j \neq i} [v'_j - v'_j M'_{j} - \delta m_i \frac{\partial M_j}{\partial v_j}] \right) \tag{A11}
\]

which can be summed to \( B \), giving:

\[
(B) = a^{-1} \int dx \hat{M}' \left( \sum_{j \neq i} [v'_j - v'_j M'_{j} - \delta m_i \frac{\partial M_j}{\partial v_j}] \right) \tag{A12}
\]

and subtracting \( LHS|_{A7} \):

\[
(C) = (A) + (B) - LHS|_{A7} \equiv (C_1) + (C_2) + (C_3) + (C_4)
\]

In the previous equation we have used the following scalar product:

\[
(R, S) = \int_{-\infty}^{+\infty} dx R(x) S(x) \tag{A14}
\]

The three terms in square brackets on the right-hand-side \([C_2] + [C_3] + [C_4]\) represent the effect of the symmetry breaking current.

By integrating by parts and by using the definition of \( \hat{M}' \):

\[
(C_1) = (M'_i(-x), U'_{NL,i}) \tag{A15}
\]

If we define the function \( G(x, y) \equiv U'(x + y) - U'(x) - yU''(x) \), then \( U'_{NL,i} = G(M_i, \delta m_i) \). In the following, we will also make use of the function: \( \hat{U}(x, y) \equiv U(x + y) - U(x) - yU'(x, y) \). It is obvious that \( G(x, y) = \partial_y \hat{U}(x, y) \).

We observe that: i) \( G(M_i, 0) = 0 \); ii) \( G \) may be written as a Taylor expansion whose generic term contains \( (\delta m_i)^n \); iii) \( G \) is not linear in \( \delta m_i \), but if we use the binary interaction approximation, it is indeed linear. This approximation corresponds to write:

\[
(\delta m_i)^n \approx \sum_{j > i} |M_j - M_j|(\infty) |M_j - M_j| \tag{A16}
\]

In this approximation, we obtain:

\[
(C_1) = \sum_{j > i} \int dx M'_i(x - x) G(M_i(x - x), M_j - M_j(\infty)) \]

\[
+ \sum_{j < i} \int dx M'_i(x - x) G(M_i(x - x), M_j - M_j(\infty)) \tag{A17}
\]

\[
\]
We must observe that $M'_i$ is not vanishing only when $x \approx x_i$; furthermore, $[M_j - M_j(\pm \infty)]$ goes to zero when $(+)$ $x > x_j$ or $(-)$ $x < x_j$. On the basis of these considerations, it is possible to write:

$$
(C_1) = \sum_{j>i} \int_{x_j}^{+\infty} dx M'_i(x_i - x) G(M_i(x - x_i), \Delta M_j) \\
+ \sum_{j<i} \int_{-\infty}^{x_i} dx M'_i(x_i - x) G(M_i(x - x_i), -\Delta M_j)
$$

where $\Delta M_j \equiv M_j(\infty) - M_j(-\infty)$.

Since $M'_i(x)$ is an even function of $x$:

$$
\int_{-\infty}^{x_i} dx M'_i(x_i - x) G(M_i(x - x_i), \text{const}) = \tilde{U}(M_i(x_i - x), \text{const}) \bigg|_{x_i}^{x_i}
$$

and $(C_1)$ can be written as:

$$
(C_1) = \sum_{j>i} [\tilde{U}(M_i(\infty), \Delta M_j) - \tilde{U}(M_i(x_j - x_i), \Delta M_j)] \\
+ \sum_{j<i} [\tilde{U}(M_i(x_j - x_i), -\Delta M_j) - \tilde{U}(M_i(-\infty), -\Delta M_j)]
$$

At the first order in the small quantities $[M_i(x_j - x_i) - M_i(\pm \infty)]$ (± resp. for $j > i$ and $j < i$), we have:

$$
(C_1) = - \sum_{j>i} [M_i(x_j - x_i) - M_i(\infty)] G(M_i(\infty), \Delta M_j) \\
+ \sum_{j<i} [M_i(x_j - x_i) - M_i(-\infty)] G(M_i(-\infty), -\Delta M_j) \quad \text{(A16)}
$$

In the following, we will restrict ourselves to nearest-neighbour kinks interaction, and therefore only the terms $j = i \pm 1$ will survive in (A16). If we also use the fact that:

$$
G(M_i(\pm \infty), \pm \Delta M_{i\pm1}) = \mp \Delta M_{i\pm1} U''(M_i(\pm \infty)) \quad \text{(A17)}
$$

we obtain the following final expression:

$$
(C_1) = [M_i(x_{i+1} - x_i) - M_i(\infty)] \Delta M_{i+1} U''(M_i(\infty)) \\
+ [M_i(x_{i-1} - x_i) - M_i(-\infty)] \Delta M_{i-1} U''(M_i(-\infty)) \quad \text{(A18)}
$$

The procedure to follow for the treatment of the other terms $(C_i)$ is similar. In poor words, if $R(x)$ and $S(x)$ are functions which are localized, resp. in $x_1$ and $x_2$, we make the approximation:

$$
R(x - x_1) S(x - x_2) \approx R(x - x_1) S(x_1 - x_2) \\
+ R(x_2 - x_1) S(x - x_2)
$$

and then we retain only the term corresponding to the function decreasing more rapidly (for example, if $R(x)$ was a Dirac-delta, only the first term would be retained, because the second one would be exactly zero). We give here only the results.

$$(C_2) = \lambda \{ [M_{i+1}(x_{i+1} - x_i) - M_{i+1}(\infty)] \\
- [M_{i-1}(x_i - x_{i-1}) - M_{i-1}(\infty)] \} \int_{-\infty}^{+\infty} dx (M'_i)^2$$

$$(C_3) = 0
$$

$$(C_4) = -\frac{\lambda}{2} (\Delta M_{i+1})^2 M'_i(x_{i+1} - x_i) \\
+ \frac{\lambda}{2} (\Delta M_{i-1})^2 M'_i(x_i - x_{i-1})
$$

The expression $(C_3)$ is 0 means that such term is always of higher order than the others.

From now on, notation must take into account explicitly the existence of two different classes of kinks. By using the following results:

$$
M_\beta(x) - M_\beta(\infty) \simeq -\beta m_0 \exp(-\kappa_\beta x) \quad \text{when } x \to \infty
$$

$$(R_\beta(x) \equiv \exp(-\kappa_\beta x))
$$

$$
\Delta M_k = \beta 2m_0 \\
U''(\pm m_0) = 2\nu \\
\int_{-\infty}^{+\infty} dx [M'_\beta(x)]^2 = \frac{2}{3} m_0^2 \kappa_\beta
$$

it is straightforward to write:

$$
(C_1) = 8\nu m_0^2 [R_\beta(X_i) - R_\beta(X_{i-1})]
$$

$$
(C_2) = \beta (4/3) m_0^3 \kappa_\beta \lambda [R_{-\beta}(X_i) - R_{-\beta}(X_{i-1})] \quad \text{(A19)}
$$

$$
(C_4) = -\beta 4m_0^3 \kappa_\beta \lambda [R_{\beta}(X_i) - R_{\beta}(X_{i-1})]
$$

We can now put together Eq. (A19) with the previous ones. A further approximation is to neglect the “deformation” of the kink-profile, due to its velocity, and to suppose that kinks are immobile in absence of interactions. This way, we obtain:

$$
a^{-1} \sum_j v_j (\tilde{M}'_i, M'_j) = (C_1) + (C_2) + (C_4) \quad \text{(A20)}
$$

where the LHS can be further developed:

$$
a^{-1} \sum_j v_j (\tilde{M}'_i, M'_j) = \int \frac{dx M'_i(x) M'_j(x)}{dx D_x \tilde{M}'_i(x) D_x^{-1} M'_j(x)} \quad \text{(A21)}
$$

We have therefore to determine the inverse of the operator $D_x$. By following Kawasaki and Ohta [20]:
The “integration constants” appearing when the operator $D_x$ is inverted, are shown to be irrelevant for the kink dynamics (Ref. [27]).

By applying Eq. (A22) to Eq. (A21):

$$\int dx M'_i(x) D_x^{-1} M'_j(x) =$$

$$-\frac{1}{2} \int dx \int dx' M'_i(x)|x - x'| M'_j(x')$$

$$\approx -\frac{1}{2} |x_i - x_j| \int dx' M'_i(x - x_i) M'_j(x - x_j)$$

$$= -\frac{1}{2} |x_i - x_j| \Delta M_i \Delta M_j$$

(A24)

So, Eq. (A13) finally writes:

$$- a^{-1} \Delta M_i \sum_j \Delta M_j |x_i - x_j| \dot{x}_j = (C_1) + (C_2) + (C_4)$$

(A25)

(A26)

2. The effect of noise

The term of noise $\delta F(x,t)$ in Eq. (B) corresponds to a term $\eta(x,t) = \partial_x \delta F(x,t)$ on the right-hand-side of Eq. (A3). To see how it affects the kink movement, it must be multiplied by $\delta \tilde{M}'(x)$ and integrated on $x$. Since the LHS of (A26) indeed corresponds to minus the LHS of (A3), if we call $\eta_i(t)$ the noise term to be added to $(C_1) + (C_2) + (C_4)$ in (A26), it will result:

$$\eta_i(t) = - \int dx \delta \tilde{M}'_i(x) \eta(x,t) = \int dx \partial_x \delta \tilde{M}'_i(x) \cdot \delta F(x,t)$$

(A27)

The following properties are found [27]:

$$\langle \eta_i(t) \rangle = 0 \quad \text{and} \quad \langle \eta_i(t) \eta_j(t') \rangle = 2F_0 \delta(t - t') \int dx \delta \tilde{M}'(x) D_x \delta \tilde{M}'(x)$$

$$= -4m_0^2 F_0 (-1)^{i-j} |x_i - x_j| \delta(t - t')$$

(A28)

To derive the spatial correlation between noise, we have used the definition of $\tilde{M}'_i$ and inverted the operator $D_x$. Finally, we have used the fact that $\Delta M_i \Delta M_j = 4m_0^2 (-1)^{i-j}$, a relation which can be used also for the LHS of Eq. (A26). So, we obtain the following system of coupled Langevin equations:

$$- 2a^{-1} m_0^2 \sum_j (-1)^{i-j} |x_i - x_j| \dot{x}_j =$$

$$(C_1) + (C_2) + (C_4) + \eta_i(t)$$

(A29)
L.M. Sander, Europhys. Lett. 27, 611 (1994); and in “Scale invariance, interfaces, and non-equilibrium dynamics”, A. McKane et al. eds. (Plenum Press, New York, 1995). In these papers authors study the model with $\lambda = 0$ and an unstable current $j_{ES}$ as given by the model II. They draw information on the dynamics from the numerical evaluation of the smallest eigenvalue of a proper operator associated to the Langevin equation: anyway, no analytical evaluation is given.

[24] In a different context [O. Pierre-Louis, C. Misbah, Y. Saito, J. Krug and P. Politi, Phys. Rev. Lett. 80, scheduled for the issue of 4 May 1998] it is indeed possible to observe no coarsening because the period is a decreasing function of the amplitude.

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[29] Eq. (4-1) of Ref. 27 contains indeed a misprinting: $\partial^2/\partial X^2 j_j$ should be replaced by $\partial^2/\partial X_{j-1} \partial X_{j+1}$.

[30] T. Nagai and K. Kawasaki, Physica 120A, 587 (1983); K. Kawasaki and T. Nagai, Physica 121A, 175 (1983); T. Nagai and K. Kawasaki, Physica 134A, 483 (1986).

[31] J. Krug, H.T. Dobbs and S. Majaniemi, Z. Phys. B 97, 281 (1995).

[32] For model I, $n = 1/3$ for a conserved order parameter and $n = 1/2$ for a nonconserved one.

[33] M. Rost and J. Krug, Phys. Rev. E 55, 3952 (1997).

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[35] This value is a bit surprising: if compared to $n \simeq 0.22$ it would lead to conclude that (in 1+1 dimensions) deterministic coarsening is not slower than the noisy one; if compared to the noiseless coarsening of model I ($L(t) \sim \ln t$), we should conclude that steepening (due to the absence of finite zeros in $j_{ES}$) favours the coarsening.

[36] The reason is simply that $j_{SB}$ (the cause of angular points) would contribute to the growth velocity with a term proportional to $\partial^2 A(m^2)$, which diverges in the angular points. More details are given in Ref. [34].

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