On geometry of the Lagrangian description of ideal fluids

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Abstract The Euler equation for an inviscid, incompressible fluid in a three-dimensional domain $M$ in $R^3$ implies that the vorticity is a frozen-in field. This can be used to construct a symplectic structure on $R \times M$. The normalized vorticity and the suspended velocity fields are Hamiltonian with the function $t$ and the Bernoulli function, respectively. The symplectic structure incorporates the helicity conservation law as an identity. The infinitesimal dilation for the symplectic two-form can be interpreted as a current vector for the helicity. The symplectic dilation implies the existence of contact hypersurfaces. In particular, these include contact structures on the space of streamlines and the Bernoulli surfaces. The symplectic structure on $R \times M$ can be realized as symplectisations of these through the Euler equation.

1 Introduction

The traditional theories for the problems of fluid flow are statistical and probabilistic in nature [1],[2]. The geometric descriptions of the Eulerian equations of hydrodynamics in the framework of infinite dimensional Lie groups and Hamiltonian (Lie-Poisson) formulations [3]-[7] has been attracted the attention of scientists working on different aspects of fluid motions. On the other hand, the need for some geometry, more specifically, the necessity of the symplectic geometry for tackling some open problems of the Euler equation was argued. The extension of methods of symplectic geometry to viscous flows which is described, in Eulerian coordinates, by the Navier-Stokes equation, was also questioned [2]. The bottom line of these arguments may be the question of incorporating those geometric structures appropriate to and available from the Eulerian equations into the traditional statistical descriptions of fluid flows [8]-[11],[1],[2].

Meanwhile, the relevance of geometric structures obtainable from the Eulerian equations to the statistical description of turbulence was discussed in Ref. [1]. The Lagrangian description, that is, the description by the trajectories of the velocity field of steady fluid motion in two and four dimensions was shown to have the structure of a completely integrable Hamiltonian system [10],[11].

In [12], starting solely from the Euler equation of an ideal fluid we obtained an infinitesimal symmetry, a symplectic structure on $R \times M$ for the suspended
velocity field and the conservation law for the helicity density. In [13] and [14], we extended these intrinsic geometric structures to the study of kinematical symmetries and generalized helicity conservation laws of three dimensional incompressible flows. In this work, we shall further exploit the Euler equation of ideal fluids to obtain geometric structures relevant to a qualitative study of the Lagrangian description of motion.

We shall introduce a current vector field governing the dynamics of the helicity density. This helicity current generates the dilation for the symplectic structure. With reference to the Navier-Stokes equation we shall demonstrate that the information content of the dynamical equations can be represented by the helicity current. We shall also discuss some algebraic consequences of the symplectic dilation. We shall then construct contact structures on the space of streamlines and on the Bernoulli surfaces. We shall show that the symplectic structure on the space of trajectories is symplectisations of these contact structures. For fluid dynamical content of this work we shall refer to Refs. [2], [15] and the necessary mathematical background can be found in Refs. [16]-[23].

2 An infinitesimal symmetry

We shall begin with the Euler equation of ideal fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p$$

(1)

for the divergence-free velocity field \( \mathbf{v} \) tangent to the boundary of a connected region \( M \subset \mathbb{R}^3 \) and the pressure function \( p \). The identity \( \mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{2} \nabla |\mathbf{v}|^2 - \mathbf{v} \times (\nabla \times \mathbf{v}) \) can be used to bring the Euler equation (1) into the Bernoulli’s form

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla \alpha$$

(2)

where the function \( -\alpha \equiv p + \frac{v^2}{2} \) is the Bernoulli function [11], also called to be the total (or stagnation) pressure [24]. In terms of the divergence-free vorticity field \( \mathbf{w} \equiv \nabla \times \mathbf{v} \) Eq. (2) gives

$$\frac{\partial \mathbf{w}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{w}) = 0 \ .$$

(3)

It follows from the identity

$$\nabla \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{w} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{w} + (\nabla \cdot \mathbf{w})\mathbf{v} - (\nabla \cdot \mathbf{v})\mathbf{w}$$

(4)

together with \( \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{w} = 0 \) that Eq. (3) is equivalent to

$$\frac{\partial \mathbf{w}}{\partial t} + [\mathbf{v}, \mathbf{w}] = 0 \ , \ [\mathbf{v}, \mathbf{w}] \equiv (\mathbf{v} \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{v}$$

(5)

which means that \( \mathbf{w} \) is an infinitesimal time-dependent symmetry of the velocity field \( \mathbf{v} \). That is, the time-dependent transformations generated by \( \mathbf{w} \) on \( M \) leaves the trajectories of the velocity field invariant.
3 A symplectic structure

In $R^3$ a vector field $w$ corresponds to a unique two-form

$$\omega \equiv w \cdot (dx \wedge dx) = d_M(v \cdot dx)$$

where we let $d_M \equiv dx \cdot \nabla$ be the exterior derivative on $R^3$. If we replace $d_M$ by the exterior derivative $d \equiv dt \partial_t + d_M$ on $R \times M$ and use the Euler equation to solve the time derivative $v_t$ of the velocity field we obtain the two-form

$$\Omega_e = w \cdot (dx \wedge dx) - (v \times w + \nabla \alpha) \cdot dx \wedge dt,$$

on $R \times M$. By construction, $\Omega_e$ is closed on the space of solutions of the Euler equation. In other words, the three-form

$$d\Omega_e = (\nabla \cdot w) \cdot (dx \cdot dx \wedge dx) + [w_t - \nabla \times (v \times w)] \cdot dx \wedge dx \wedge dt \tag{8}$$

vanishes for divergence-free vector field $w$ satisfying the Euler equation in the vorticity form. Thus, we have the closed two-form $\Omega_e$ on $R \times M$ as an extension

$$\omega = d_M(v \cdot dx) \rightarrow \Omega_e = d(v \cdot dx) \mod Eq. \{2\}$$

of $\omega$ on $M$ by the Euler equation and this is induced by the invariant differential operators

$$d_M \rightarrow d = dt \partial_t + d_M.$$

Under the condition that the Bernoulli function $\alpha$ is not the constant function the skew-symmetric matrix of the two-form $\Omega_e$ has a non-vanishing determinant. Equivalently, this can be expressed as

$$\frac{1}{2} \Omega_e \wedge \Omega_e = -w \cdot \nabla \alpha \cdot dx \cdot dx \wedge dx \wedge dt \neq 0 \tag{9}$$

whenever $\alpha \neq \text{constant}$. It seems to be necessary to assume in addition that $w \neq 0$. However, for a realistic fluid the validity of this condition was discussed in Ref. [25]. We thus obtained from the Euler equation a non-degenerate closed two-form $\Omega_e$ on $R \times M$, that is, a symplectic structure [19]-[21], [23], [16]. The non-zero four-form in Eq. (9) is the symplectic or the Liouville volume element on $R \times M$. The symplectic two-form is exact

$$\Omega_e = d\theta_e \mod Eq. \{2\}, \quad \theta_e = v \cdot dx + \psi(t) dt \tag{10}$$

where $\psi(t)$ is an arbitrary function.

4 Hamiltonian vector fields

The non-degeneracy of $\Omega_e$ means that given a one-form $\beta \equiv \beta_a dx^a$ on $R \times M$ with the local coordinates $(x^a) = (x^0 = t, x)$ the equation

$$i(X)(\Omega_e) = \beta, \quad (\Omega_e)_{ab} X^b = \beta_a \tag{11}$$
has a unique solution for the vector field \( X = X^a \partial_a = X^0 \partial_t + \mathbf{X} \cdot \nabla \) and vise versa. Here, \( i(X) \) denotes the interior product or the contraction with the vector field \( X \), \( (\Omega_e)_{ab} \) are the components of the skew-symmetric matrix of the symplectic two-form \( \Omega_e \) in the given coordinates and we employ the summation over repeated indices.

For \( \beta \) being an exact one-form, namely, \( \beta = df \) for an arbitrary smooth function \( f \) on \( R \times M \) the solution
\[
X_f = \frac{1}{\mathbf{w} \cdot \nabla \alpha} \left[ \mathbf{w} \cdot \nabla f \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) - \frac{df}{dt} \mathbf{w} \cdot \nabla + (\nabla \alpha \times \nabla f) \cdot \nabla \right]
\]
(12)
of Eq. (11) is said to be the Hamiltonian vector field for \( \Omega_e \) with the Hamiltonian function \( f \). From the skew-symmetry of \( (\Omega_e)_{ab} \) we have the conservation law
\[
0 \equiv i(X_f)(\Omega_e) = (\Omega_e)_{ab} X^a f \partial_b = X^0 f \partial_t + \mathbf{X} \cdot \nabla f = 0
\]
(13)
for the Hamiltonian function. We observe that \( X_f \) reduces to \( -(\mathbf{w} \cdot \nabla \alpha)^{-1} \mathbf{w} \cdot \nabla \) for the function \( f = t \) which obviously satisfies Eq. (13). In other words, the normalized vorticity field is Hamiltonian
\[
i(X_t)(\Omega_e) = dt , \quad X_t \equiv -(\mathbf{w} \cdot \alpha)^{-1} \mathbf{w} \cdot \nabla
\]
(14)
with the Hamiltonian function \( t \).

A time-dependent conserved function for the velocity field can be found again from the Euler equation. We recall that an energy consequence of the Euler equation follows by taking dot product of its Bernoulli form with the velocity field. The result is known as the Bernoulli equation [2,24]
\[
\frac{\partial}{\partial t} \left( \frac{1}{2} v^2 \right) - \mathbf{v} \cdot \nabla \alpha = 0
\]
(15)
which implies that if the pressure \( p \) does not depend explicitly on time, the Bernoulli function \( \alpha \) is a time-dependent conserved function
\[
\frac{\partial \alpha}{\partial t} + \mathbf{v} \cdot \nabla \alpha = 0 , \quad -\alpha(t, \mathbf{x}) = p(\mathbf{x}) + \frac{1}{2} v^2(t, \mathbf{x})
\]
(16)
along the trajectories of the velocity field. Using this as the Hamiltonian function we can write the suspended velocity field \( \partial_t + \mathbf{v} \) on \( R \times M \) as a Hamiltonian vector field
\[
i(\partial_t + \mathbf{v})(\Omega_e) = d\alpha .
\]
(17)
The symplectic structure \( \Omega_e \) on \( R \times M \) induces a Lie algebraic structure on the space of smooth functions on \( R \times M \) with the Poisson bracket
\[
\{ f, g \}_e = \frac{\partial f}{\partial x^a} (\Omega_e^{-1})^{ab} \frac{\partial g}{\partial x^b} = X_f(g) = \Omega_e(X_f, X_g)
\]
(18)
\[
= \frac{1}{\mathbf{w} \cdot \nabla \alpha} \left[ \frac{dg}{dt} \mathbf{w} \cdot \nabla f - \frac{df}{dt} \mathbf{w} \cdot \nabla g + (\nabla f \times \nabla g) \cdot \nabla \alpha \right]
\]
(19)
where \( (\Omega_e^{-1})^{ab} \) are the components of the inverse of the matrix of the symplectic two-form and \( d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla \) is the convective derivative.
5 Conservation laws

We shall now discuss the differential invariants of the velocity field and obtain the conservation law for the helicity as an identity involving the invariant forms. To this end, we recall that a differential \( p \)-form \( \xi \) is said to be a relative invariant if there exist a \((p-1)\)-form \( \zeta \) such that

\[
\mathcal{L}_X(\xi) \equiv i(X)d\xi + di(X)\xi = d\zeta
\]

(20)

where \( \mathcal{L}_X(\cdot) \) is the Lie derivative. If \( \zeta = 0 \), \( \xi \) is said to be an absolute invariant. It immediately follows from this definition, Eq. (17) and \( d\Omega_e = 0 \) that the symplectic two-form is an absolute invariant of the suspended velocity field. In fact, this is true for any Hamiltonian vector field, so is for the normalized vorticity field \( X_t \). For the canonical one-form we compute the Lie derivative

\[
\mathcal{L}_{\partial_t + v}(\theta_e) = d\alpha + d(v^2 + \psi(t)) = d(\psi(t) + \frac{1}{2}v^2 - p)
\]

(21)

using the Hamilton’s equation \( \text{(17)} \) for the first term in the definition \( \text{(20)} \). Thus, \( \theta_e \) is a relative invariant. Moreover, the derivation property of the Lie derivative implies the relative invariance

\[
\mathcal{L}_{\partial_t + v}(\theta_e \wedge \Omega_e) = \mathcal{L}_{\partial_t + v}(\theta_e) \wedge \Omega_e + \theta_e \wedge \mathcal{L}_{\partial_t + v}(\Omega_e) = d((\psi(t) + \frac{1}{2}v^2 - p)\Omega_e)
\]

(22)

of the three-form

\[
\theta_e \wedge \Omega_e = 2\mathcal{H} \, dx \cdot dx \wedge dx +
((v^2 + \psi)w - 2\mathcal{H}v + v \times \nabla \alpha) \cdot dx \wedge dx \wedge dt
\]

(23)

where the scalar component, namely, the coefficient of the term \( dx \cdot dx \wedge dx \), is the helicity density

\[
\mathcal{H} \equiv \frac{1}{2}v \cdot \nabla \times v = \frac{1}{2}v \cdot w.
\]

(24)

The absolute invariance of the four-form \( \Omega_e \wedge \Omega_e \) follows from the same argument. Moreover, from the right hand side of Eq. \( \text{(19)} \) we conclude that the absolute invariance of \( \Omega_e \wedge \Omega_e \) is a statement for the conservation of the Liouville density

\[
\frac{\partial}{\partial t}(v \cdot \nabla \alpha) + v \cdot \nabla (v \cdot \alpha) = 0
\]

(25)

along the trajectories of the velocity field.

Since \( \Omega_e \) is closed and \( d\theta_e = \Omega_e \) these differential invariants of the velocity field satisfy the relation

\[
d(\theta_e \wedge \Omega_e) - \Omega_e \wedge \Omega_e = 0
\]

(26)

identically. Using the expression \( \text{(9)} \) for the second term and computing the derivative of the last term in the three-form \( \text{(23)} \) as

\[
d([v \times \nabla \alpha] \cdot dx \wedge dx \wedge dt) = w \cdot \nabla \alpha \, dx \cdot dx \wedge dx \wedge dt
\]

(27)
we obtain the divergence expression
\[
\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot (\mathcal{H} \mathbf{v} - \frac{1}{2} \frac{1}{2} \mathbf{v}^2 - p) \mathbf{w} = 0
\] (28)
for the conservation law of the helicity density \( \mathcal{H} \). Note that the conserved flux, which is the vector field under the spatial divergence, is independent of the function \( \psi \) which we have introduced arbitrarily. We can take it to be zero. Then, it immediately follows from Eqs. (21) and (22) that for \( \frac{v^2}{2} - p = \text{constant} \), \( \theta_e \) and \( \theta_e \wedge \Omega_e \) become absolutely invariant. Moreover, from Eq. (28) we see that on the level surfaces of the function \( \frac{v^2}{2} - p \) the density \( \mathcal{H} \) turns into a conserved function of the velocity field.

We thus obtained the conservation law for the helicity density \( \mathcal{H} = \mathbf{v} \cdot \mathbf{w} / 2 \) associated with the vorticity field \( \mathbf{w} \) which was shown to be an infinitesimal kinematical symmetry. This means that, although, the conservation law is given as a divergence expression the density is a conserved quantity of the Lagrangian motion. The symplectic structure \( \Omega_e \) has a prominent role in relating the vorticity field \( \mathbf{w} \) to the helicity invariant.

We conclude that, apart from the assumption on the pressure function, the Euler equation itself intrinsically contains a geometric structure in which we can relate symmetries, invariant differential forms and conservation laws. In the next section we shall examine the situation for the Navier-Stokes equation. We shall then continue our discussion of geometric structures available from the Euler equation.

6 Viscous flow

The Navier-Stokes equation for a viscous incompressible fluid in a bounded domain \( M \subset \mathbb{R}^3 \) is
\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v}
\] (29)
where \( \nu \) is the kinematic viscosity \cite{2}. As in the case of the Euler equation, Eq. (29) can be brought into the form
\[
\frac{\partial \mathbf{w}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{w}) = \nu \nabla^2 \mathbf{w}
\] (30)
by which the two-form
\[
\Omega_v = \mathbf{w} \cdot (d\mathbf{x} \wedge d\mathbf{x}) - (\mathbf{v} \times \mathbf{w} - \nu \nabla \times \mathbf{w}) \cdot d\mathbf{x} \wedge dt
\] (31)
can be shown to be closed. For non-degeneracy we compute
\[
\frac{1}{2} \Omega_v \wedge \Omega_v = 2 \nu \mathcal{H}_w \ dx \cdot dx \wedge dx \wedge dt \neq 0
\] (32)
where the density
\[
\mathcal{H}_w \equiv \frac{1}{2} \mathbf{w} \cdot \nabla \times \mathbf{w} = -\frac{1}{2} \mathbf{w} \cdot \nabla^2 \mathbf{v}
\] (33)
is known as the vortical helicity \cite{2}. For viscous flows $\nabla^2 v \neq 0$ and, for a realistic fluid we have $w \neq 0$. Hence $\mathcal{H}_w \neq 0$. This makes $\Omega_w$ non-degenerate. $\Omega_w$ is also exact with the same canonical one-form $\theta_e$:

$$\Omega_w = d\theta_e \mod \text{Eq. (20)}$$

which shows that the symplectic structure on the space of trajectories manifests the dynamical properties of the fluid.

For an arbitrary smooth function $f$ on $R \times M$ the Hamiltonian vector field $X_f$ defined by the symplectic two-form $\Omega_w$ is

$$X_f = \frac{1}{2\nu \mathcal{H}_w} [-w \cdot \nabla f (\partial_t + v) + \frac{df}{dt} w + \nu \nabla f \times \nabla \times w] \cdot \nabla$$

which gives $(1/2\nu \mathcal{H}_w) w \cdot \nabla$ for $f = t$. The suspended velocity field is, however, not even locally Hamiltonian. These are vector fields obtained from Eq. (11) for closed non-exact one-forms $\beta$. The symplectic two-form is invariant under the flows of locally Hamiltonian vector fields because $\mathcal{L}_X (\Omega_w) = d(i(X) (\Omega_w)) = d\beta \equiv 0$ where we used the identity $\mathcal{L}_X = i(X) \circ d + d \circ i(X)$ for the Lie derivative and $d\Omega_w = 0$. We find, however, that the one-form

$$i(\partial_t + v)(\Omega_w) = \nu \nabla^2 v \cdot (dx - v dt)$$

obtained by contraction with the suspended velocity field is not even closed due to diffusive term. That means the symplectic two-form is not invariant under the Lagrangian motion. A consequence of this non-invariance or, equivalently, of the viscous diffusion is the non-conservation of the helicity density. Nevertheless, using the symplectic form (31) and $\theta_e$ in the identity (20) we get

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot (\mathcal{H} v + \nu v \times \nabla^2 v - \frac{1}{2} w^2) = -2\nu \mathcal{H}_w$$

as the evolution equation for $\mathcal{H}$.

These are properties of the Lagrangian motion reflected by the dynamical (or Eulerian) equations. We shall continue our discussion of the geometry of the Lagrangian motion in the framework of the Euler equation. We shall occasionally present the results for the Navier-Stokes equation in order to see how the dynamics is encoded into geometric objects.

7 A symplectic dilation

For our discussion of the geometric structure of the Lagrangian motion of the Euler flow a particularly interesting solution of Eq. (11) is obtained when we let the one-form $\beta$ be the canonical one-form $\theta_e$. The vector field $J$ satisfying the equation

$$i(J)(\Omega_e) = \theta_e$$

(38)
can be uniquely determined to be
\[ J = \frac{1}{w \cdot \nabla \alpha} [2\mathcal{H}(\partial_t + \mathbf{v} \cdot \nabla) + (\psi(t) - \frac{1}{2}v^2)w \cdot \nabla + \mathbf{v} \times \nabla \alpha \cdot \nabla] \] (39)
for the Euler equation and, for the Navier-Stokes equation we find
\[ J_\nu = \frac{-1}{2\nu \mathcal{H}_w} [2\mathcal{H}(\partial_t + \mathbf{v}) + (p - \frac{1}{2}v^2)w - \nu \mathbf{v} \times \nabla \times \mathbf{w} \cdot \nabla] . \] (40)
It follows from Eq. (38) and \( d\Omega_e = 0 \) that \( J \) fulfills the condition
\[ \mathcal{L}_J(\Omega_e) = di(J)(\Omega_e) = d\theta_e = \Omega_e \] (41)
of being an infinitesimal symplectic dilation for \( \Omega_e \) [26]. As a consequence of Eq. (41) and the derivation property of the Lie derivative we see that \( J \) expands the Liouville volume in Eq. (9). \( J \) is also said to be the Liouville vector field of \( \Omega_e \) [21].

We shall now give an interpretation of the symplectic dilation \( J \) in connection with the helicity conservation. We first compute its divergence with respect to the symplectic volume [18], [19], [21], [23]
\[ (\text{div}_{\Omega_e} J)(\frac{1}{2} \Omega_e \wedge \Omega_e) = \mathcal{L}_J(\frac{1}{2} \Omega_e \wedge \Omega_e) = \Omega_e \wedge \Omega_e \] (42)
where we used Eq. (41). By the same equation we see that the second equality is the identity in Eq. (26) resulting in the helicity conservation. Thus, the conservation law (28) can be written as the equation
\[ \text{div}_{\Omega_e} J - 2 \equiv 0 \] (43)
involving the symplectic-divergence of the symplectic dilation. With this interpretation we intend to call \( J \) to be the current associated with the helicity. The dynamical content of the helicity current can be revealed from a comparison of the symplectic structures obtained from the Navier-Stokes and the Euler equations. The canonical one-forms are the same. So, the dynamics is encoded into the symplectic two-forms. They define the current vectors by Eq. (38) for the same canonical one-form. With this definition, the dynamical properties of the fluid, such as viscosity, become implicit in the helicity current. Thus, we can think of the pair \((\theta_e, J)\) as a geometric representative of the dynamics of fluid motion on the space of trajectories.

8 Algebraic consequences

The helicity current is not a Hamiltonian vector field. However, it takes a Hamiltonian vector field into a Hamiltonian vector field by its action via Lie derivative. To see this, we compute
\[ i([J, X_f])(\Omega_e) = \mathcal{L}_J(i(X_f)(\Omega_e)) - i(X_f)(\mathcal{L}_J(\Omega_e)) = -d(J(f) - f) \] (44)
where Eq. (44) is an identity [23], [18] and we used Eq. (41). Thus, \([J, X_f]\) is Hamiltonian with the function \(J(f) - f\). Replacing \(X_f\) with \([J, X_f]\) in Eq. (44) and using Eq. (45) we get

\[
\iota([J, [J, X_f]])(\Omega_e) = \mathcal{L}_J(\iota([J, X_f])(\Omega_e)) - \iota([J, X_f])(\mathcal{L}_J(\Omega_e)) \equiv \mathcal{L}_J(\iota([J, X_f])(\Omega_e)) - \iota([J, X_f])(\mathcal{L}_J(\Omega_e)) - d(J(f) - 2J(f) + f)
\]

which is also Hamiltonian. Thus, by repeated applications of the Lie derivative with respect to the symplectic dilation \(J\) one can generate an infinite hierarchy of Hamiltonian vector fields \((\mathcal{L}_J)^k(X_f)\) anchored to \(X_f\).

In particular, we let \(f = t\) so that \(X_t \equiv - (w \cdot \nabla \alpha)^{-1} w \cdot \nabla\) and consider the Hamiltonian vector fields \((\mathcal{L}_J)^k(X_t)\) for \(k \geq 0\). The identity [23], [18]

\[
\mathcal{L}_{[J, X_t]}(\partial_t + v) = -\mathcal{L}_{X_t}(\mathcal{L}_J(\partial_t + v))
\]

evaluated on the suspended velocity field \(\partial_t + v\) gives

\[
\mathcal{L}_{[J, X_t]}(\partial_t + v) = -\mathcal{L}_{X_t}(\mathcal{L}_J(\partial_t + v))
\]

where we used \([\partial_t + v, X_t] = 0\). The vector field \([J, \partial_t + v]\) is Hamiltonian with the function \(J(\alpha) - \alpha = \psi - v^2\). So, we can express the right hand side of Eq. (49) as a Hamiltonian vector field. To find the Hamiltonian function we shall use the Lie algebra isomorphism

\[
[X_f, X_g] = -X_{[f,g]}
\]

defined by the symplectic structure \(\Omega_e\) [23], [21]. Taking \(f = t\) and \(g = \psi - v^2\) we compute the Poisson bracket

\[
\{t, \psi - v^2\} = X_t(\psi - v^2) = \frac{w \cdot \nabla v^2}{w \cdot \nabla \alpha}
\]

which gives the Hamiltonian function up to a constant. It follows from Eq. (49) that on the level surfaces defined by the constant values of the Hamiltonian function \(\psi - v^2\) of \(X_t\) we have \([[J, X_t], \partial_t + v] = 0\). Since \(w \neq 0\), this means that \(w\) lies on the two-parameter family \(p + av^2/2 = b, a, b = \text{constants}\), of surfaces. In this case, the hierarchy of Hamiltonian vector fields anchored to \(X_t\) becomes infinitesimal symmetries of the velocity field. In particular, this family of surfaces includes the Bernoulli surfaces for \(a = 1\). In other words, on the level sets \(-\alpha = b\) of the Hamiltonian function, \(\partial_t + v\) admits the infinite hierarchy \((\mathcal{L}_J)^k(X_t)\) of time-dependent infinitesimal symmetries.

9 Contact structures

A contact structure on a three dimensional manifold is a field of non-integrable, two-dimensional hyperplanes in its tangent spaces. Locally, this may be described as the kernel of a one-form \(\omega\) satisfying \(\omega \wedge d\omega \neq 0\) everywhere. The contact form \(\omega\) determines a unique vector field \(E\) by the conditions

\[
i(E)(\omega) = 1, \quad i(E)(d\omega) = 0
\]
which is called the Reeb vector field. \[20\]-[22].

A geometric consequence of the existence of a symplectic dilation is that the canonical one-form \(\theta_e\) can be related to a contact structure on some three-dimensional domain in \(R \times M\) because from Eq. (41) we have

\[
\theta_e \wedge d\theta_e = \theta_e \wedge \Omega_e = i(J)(\Omega_e) \wedge \Omega_e = i(J)(\frac{1}{2} \Omega_e \wedge \Omega_e) \neq 0.
\]

(53)

According to the definition of Ref. [28] a hypersurface in a symplectic manifold admits a contact one-form if and only if there exists a symplectic dilation which is defined on its neighborhood and is transversal to the hypersurface. It is also remarked that such hypersurfaces arise as level sets of Hamiltonian functions of Hamiltonian vector fields [26]-[28],[22]. In this section we shall describe two examples of contact hypersurfaces in \(R \times M\). In the next section we shall consider their relation with the symplectic structure \(\Omega_e\) and discuss some physical significance.

9.1 Spatial hypersurfaces

First, we consider spatial hypersurfaces \(M_c = \{(t, x) \in R \times M \mid t = c = \text{constant}\}\) as level sets of the function \(t\). Recall that \(t\) is the Hamiltonian function for the normalized vorticity field \(X_t\). Then, the transversality condition

\[
i(J)(i(X_t)(\Omega_e)) = i(J)(dt) = \frac{2H}{w \cdot \nabla \alpha} \neq 0
\]

(54)

for \(M_c\) holds for non-vanishing values of the helicity density. In other words, for \(H \neq 0\) the helicity current is not contained in the tangent spaces to \(M_c\). In this case, the contact form on \(M_c\) is obtained as follows. Let

\[
i : M_c \rightarrow R \times M : (t = c, x) \mapsto (t, x)
\]

(55)

be the inclusion of time slices \(M_c\) into space-time. A function on \(R \times M\) gives a function on \(M_c\) when composed by \(i\). This operation can then be extended to differential forms. If \(\sigma = f_a(x)dx^a\) is a one-form on \(R \times M\) its pull-back \(i^*\sigma\) to \(M_c\) by the inclusion map \(i\) is defined to be

\[
i^*\sigma = i^*(f_a(x)dx^a) = i^*(f_a(x))i^*(dx^a) = (f_a \circ i)(x)di(x^a)
\]

(56)

where we used the commutativity of the operators \(d\) and \(i^*\). In particular, we compute

\[
\omega = i^*\theta_e = v(t = c, x) \cdot dx
\]

(57)

\[
d\omega = di^*\theta_e = i^*d\theta_e = i^*\Omega_e = w(t = c, x) \cdot dx \wedge dx
\]

(58)

and it follows that \(\omega\) is a contact form on \(M_c\)

\[
\omega \wedge d_M\omega = i^*\theta_e \wedge i^*\Omega_e = i^*(\theta_e \wedge \Omega_e) = 2H dx \cdot dx \wedge dx \neq 0
\]

(59)
provided the helicity density is non-zero. Note that as a result of the transversality condition we have \( i(J)(\omega) = 0 \). In fact, \( i(J)(\theta_c) = 0 \) follows from the very definition of the symplectic dilation \( J \).

We shall now show that the investigation of the normalized vorticity field \( X_t \) as a Hamiltonian system on \( R \times M \) is equivalent to the study of the Reeb vector field on the level surfaces \( M_c \) of the Hamiltonian function \( t \). Since \( t \circ i \) is the constant function on time slices \( M_c \) we have

\[
i^*(i(X_t)(\Omega_c)) = i^*(dt) = d(i^*t) = d(t \circ i) = 0 \quad (60)
\]
as the pull-back of the Hamilton’s equations for \( X_t \) to the spatial hypersurfaces. On the other hand, for any vector field \( X \) on \( R \times M \) we have the identity

\[
i^*(i(X)(\Omega_c)) = i(i_*(X))(i^*\Omega_c) = i(i_*(X))(d_M\omega) \quad (61)
\]
where \( i_*X \) denotes the push-forward of \( X \) to \( M_c \). That means, \( i_*X \) is the pull-back of \( X \) by \( i^{-1} \) and hence is a vector field on \( M_c \) [18]. Since the one-dimensional kernel of \( d_M\omega \) (in the tangent spaces of \( M_c \)) is spanned by the Reeb vector field, Eq. (60) for \( X_t \) is possible only if the push-forward \( i_*X_t \) is proportional to the Reeb vector field of the contact structure on \( M_c \). In fact, it is easy to check that the vector field

\[
E(c, x) = (-\frac{2}{H}w)(t = c, x) = (\frac{w \cdot \nabla}{2H}X_t)(t = c, x) \quad (62)
\]
satisfies the criteria in Eq. (62).

To this end, we want to remark that the contact structure on spatial hypersurfaces \( M_c \) must be distinguished from similar geometric constructions on \( M \) obtained from the Euler equation

\[
v(x) \times w(x) = -\nabla \alpha(x) \quad (63)
\]
for the steady flow of incompressible fluid. In this latter case there is also a contact structure on \( M \) provided the (time-independent) helicity density is non-zero. However, the difference between the two cases is not merely the time-dependence of fields. They imply qualitatively different descriptions of the flows. For example, from Eq. (63) of the steady flow we obtain

\[
v \cdot \nabla \alpha = 0, \quad w \cdot \nabla \alpha = 0, \quad [v, w] = 0 \quad (64)
\]
which means that the fields \( v \) and \( w \) span the tangent spaces of the (two-dimensional) Bernoulli surfaces \( \alpha = constant \) and their flow lines commute on these surfaces [20]. On the other hand, the pull-back of the unsteady Euler equation to the hypersurfaces \( M_c \) gives

\[
v_t(t = c, x) - v(t = c, x) \times w(t = c, x) = \nabla \alpha(t = c, x). \quad (65)
\]
The qualitative analysis of this equation implies quite different and complicated results for the surfaces \( \alpha(t = c, x) = constant \) as well as the expression for the
push-forward of the suspended velocity field to the contact hypersurfaces $M_c$. The behaviour of the flows lines on $M_c$ are also different. For example, if we take the curl of Eq. (65)

$$w,_{t}(t = c, x) + [v(t = c, x), w(t = c, x)] = 0\tag{66}$$

we see that the flows of the velocity and the vorticity fields do not necessarily commute on $M_c$.

### 9.2 Bernoulli surfaces

Let $B \subset R \times M$ be the level sets of the Bernoulli function $\alpha$ which is the Hamiltonian function for the suspended velocity field. Let $i : B \rightarrow R \times M$ be the inclusion. The transversality condition reads

$$i(J)(d\alpha) = \psi(t) - v^2 \neq 0\tag{67}$$

where we used the fact that $\alpha$ is conserved under the Lagrangian motion. Since $\psi$ is arbitrary, the transversality of the Bernoulli surfaces to the helicity current is the same as the non-vanishing of the kinetic energy of the fluid. On $B$ the Euler equation becomes

$$\frac{\partial v}{\partial t} - v \times w = 0, \quad \alpha = \text{constant}.\tag{68}$$

Obviously, the pull-back of the symplectic two-form

$$i^*\Omega_e = w \cdot (dx \wedge dx) - v \times w \cdot dx \wedge dt\tag{69}$$

to Bernoulli surfaces is degenerate. Its one-dimensional kernel in the tangent spaces of $B$ is the span of the vorticity field. Since $\partial_t + v$ is also tangent to $B$ the two-dimensional tangent hypersurfaces (in the three dimensional tangent spaces of $B$) on which $i^*\Omega_e$ is non-degenerate can be defined to be the complement of $\text{span}\{\partial_t + v, w \cdot \nabla\}$ in the tangent spaces of $R \times M$. $i^*\Omega_e$ is also exact on $B$

$$i^*\Omega_e = d(v \cdot dx)\mod Eq. (68)\tag{70}$$

by the pulled-back Euler equation. So, the contact structure on the Bernoulli surfaces is defined by the time-dependent one-form

$$\omega_t = v(t, x) \cdot dx\tag{71}$$

whose derivative is the two-form in Eq. (69). The non-integrability condition

$$\omega_t \wedge d\omega_t = 2\mathcal{H} \cdot dx \cdot dx \wedge dx + (v^2w - 2\mathcal{H}v) \cdot dx \cdot dx \wedge dt\tag{72}$$

of the tangent hyperplanes defined as above requires either the helicity density or the kinetic energy to be non-zero. Recall that the non-vanishing of the kinetic energy is also required by the transversality condition. For contact one-form $\omega_t$
on the Bernoulli surfaces we can find the Reeb vector field up to an arbitrary function

\[ E = m(t, x)(\partial_t + v) + n(t, x)w \cdot \nabla, \quad mv^2 + 2n\mathcal{H} = 1. \]  

(73)

This arbitrariness is a manifestation of the fact that contrary to the case of spatial hypersurfaces the inclusion of the Bernoulli hypersurfaces into \( R \times M \) are defined only implicitly.

10 Symplectisation

If \( \omega \) is a contact form on \( M \) with local coordinates \( x \) then the two-form

\[ \Omega(t, x) = e^t(d_M \omega(x) + dt \wedge \omega(x)) = d(e^t \omega(x)) \]  

(74)

defines a symplectic structure on \( R \times M \) and is called to be the symplectisation of \( \omega \). Here, \( t \) is the coordinate on \( R \). Note that \( \Omega \) has \( \partial_t \) as its symplectic dilation and that the Reeb vector field \( E \) on \( M \) defined by \( \omega \) is a Hamiltonian vector field for the symplectisation with the Hamiltonian function \( e^t \) [20, 22].

We can think of the above symplectisation to be induced by the map \( \omega \mapsto e^t\omega \) in which the time variable is introduced artificially. The geometric fluid dynamics provides an unusual but nevertheless a natural example of symplectisation. Recall that we obtain the contact one-form on time slices by pulling the canonical one-form \( \theta_e \) back to \( M_c \) by the inclusion. Conversely, the symplectisation of the contact structure on time slices follows from the inclusion map

\[ i : M_c \to R \times M : (t = c, x) \mapsto (t, x). \]  

(75)

In this case, the time variable \( t \) is introduced naturally by the action of the invariant differential operators.

From a physical point of view the symplectisation of the time slices \( M_c \) corresponds to the construction of trajectories of the velocity field from streamlines. These are solutions of the non-autonomous and autonomous equations

\[ \frac{dx(t)}{dt} = v(t, x), \quad \frac{dx(\tau)}{d\tau} = v(t = c, x(\tau)), \]  

(76)

respectively. The solutions to the first equation on \( R \times M \) can be constructed by solving the autonomous system on \( M \) at each time \( t \) and then joining them by the inclusion

\[ i : (\text{streamlines}) \mapsto (\text{trajectories}). \]  

(77)

The symplectisation to \( R \times M \) of the contact structures on \( M_c \) means that the inclusion in Eq. (77) for solutions of the differential equations extends to

\[ i : \left( \text{contact structure on the space of streamlines} \right) \mapsto \left( \text{symplectic structure on the space of trajectories} \right) \]

which covers the corresponding geometric structures.
The symplectisation of the contact structure on the Bernoulli surfaces may be given a similar interpretation in the language of the solutions of differential equations. In this case, it will be appropriate to consider the solutions of the Euler equation. The inclusion

\[ i : \left( \text{Bernoulli surfaces} \right) \longrightarrow \left( \text{space-time} \right) \]

implies the construction

\[ i : \left( \begin{array}{c} \text{solution of} \\ v_t - \mathbf{v} \times \mathbf{w} = 0 \\ \text{on} \ \alpha = \text{constant} \end{array} \right) \longrightarrow \left( \text{solution of} \right. \]

\[ \left. \text{the Euler equation} \right) \]

of the solutions of the Euler equation from the solutions of a homogeneous equation on each Bernoulli surface. This may be interpreted as a nonlinear analog of the result in the theory of linear differential equations that the general solution of an inhomogeneous equation is the sum of the solution of its homogeneous part and the particular solution.

### 11 Conclusions

We obtained geometric and algebraic structures for the Lagrangian motion of an incompressible fluid implied by the Euler equation. The point of the paper is that the dynamical Eulerian equations of ideal fluid determine the geometry of the space of trajectories and can be used for qualitative investigations of the Lagrangian motion.

The most intriguing result in this direction may be the introduction of the symplectic dilation. We showed that the dynamical equations can be represented geometrically by a pair consisting of the canonical one-form and the symplectic dilation. As an immediate consequence of the existence of a symplectic dilation we the evolution equation for the helicity density and studied the contact structures on the spatial hypersurfaces and on the Bernoulli surfaces.

Since, the transformation generated by the symplectic dilation expands the Liouville volume we can argue that it generates the scaling transformation on the space of Eulerian fields. In spite of the general belief that a precise connection exists between the scaling transformations and helicity conservation law, we have not been able to find any source in the literature.

We obtained the helicity conservation law as an identity among invariant differential forms associated with the vorticity field. This manifests a relationship between symmetries and conservation laws which is different from the one familiar, for example, from the Noether theorems. Being realized as a symmetry of the flow of an ideal fluid the vorticity flow turns out to be the underlying symmetry of the Lagrangian motion connected with the helicity conservation.
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