A note on an ergodic theorem in weakly uniformly convex geodesic spaces

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Abstract

Karlsson and Margulis [5] proved in the setting of uniformly convex geodesic spaces, which additionally satisfy a nonpositive curvature condition, an ergodic theorem that focuses on the asymptotic behavior of integrable cocycles of nonexpansive mappings over an ergodic measure-preserving transformation. In this note we show that this result holds true when assuming a weaker notion of uniform convexity on the space.

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1 Introduction

Let $Y$ be a geodesic space and $D \subseteq Y$. Consider $S$ a semigroup of nonexpansive (i.e. 1-Lipschitz) self-mappings defined on $D$ and endow $S$ with the Borel $\sigma$-algebra induced by the compact-open topology on $S$. Assume that $(X, \mu)$ is a probability measure space, $T : X \to X$ is an ergodic measure-preserving transformation and $w : X \to S$ is a measurable map. Define the cocycles

$$a_n(x) = w(x)w(Tx) \cdots w(T^{n-1}x).$$
Fix \( y \in D \) and suppose that
\[
\int_X d(y, a_1(x)y) d\mu(x) < \infty.
\]

One can easily see that the sequence \( \left( \int_X d(y, a_n(x)y) d\mu(x) \right) \) is subadditive and so, by Fekete’s subadditive lemma \([1]\), the following limit exists
\[
0 \leq A = \lim_{n \to \infty} \frac{1}{n} \int_X d(y, a_n(x)y) d\mu(x) = \inf \frac{1}{n} \int_X d(y, a_n(x)y) d\mu(x) < \infty.
\]

As an immediate application of Kingman’s subadditive ergodic theorem \([8]\) one gets that
\[
\lim_{n \to \infty} \frac{1}{n} d(y, a_n(x)y) = A, \quad \text{for almost every } x \in X.
\]

Karlsson and Margulis \([5]\) proved that if \( Y \) is a complete Busemann convex geodesic space satisfying a uniform convexity condition (see Section 2, (ii) for the precise definition), the following holds: if \( A > 0 \), then for almost every \( x \in X \), there exists a unique geodesic ray \( \gamma \) in \( Y \) starting at \( y \) and depending on \( x \) such that
\[
\lim_{n \to \infty} \frac{1}{n} d(\gamma(A_n), a_n(x)y) = 0.
\]

Thus, in this case, instead of the convergence of averages as in classical ergodic results, one basically obtains for almost every \( x \in X \) the existence of a geodesic ray that issues at \( y \) such that, as \( n \to \infty \), the values of the nonexpansive mappings \( a_n(x) \) at \( y \) are ‘close’ to this geodesic ray, a property which is referred to in \([4]\) as ray approximation. This result generalizes the multiplicative ergodic theorem of Oseledec \([13]\) (see \([5, 4]\)). A discussion on the asymptotic behavior of ergodic products of nonexpansive mappings and isometries of a (proper) metric space and applications thereof can be found in \([4]\).

In this paper we show that the ergodic theorem given in \([5]\) holds true for a more general class of geodesic spaces. More precisely, we prove that we can relax the uniform convexity assumption on \( Y \) used in \([5]\) as follows: \( Y \) is said to be weakly uniformly convex if for any \( a \in Y \), \( r > 0 \) and \( \varepsilon \in (0, 2] \),
\[
\delta(a, r, \varepsilon) = \inf \left\{ 1 - \frac{1}{r} d \left( a, \frac{1}{2} x + \frac{1}{2} y \right) : d(a, x), d(a, y) \leq r, d(x, y) \geq \varepsilon r \right\} > 0.
\]

The mapping \( \delta \) is called the modulus of convexity of \( Y \). This notion was used by Reich and Shafrir \([14]\) in the setting of hyperbolic spaces. In addition, we assume that for every \( a \in Y \) and \( \varepsilon > 0 \) there exists \( s > 0 \) such that
\[
\inf_{r \geq s} \delta(a, r, \varepsilon) > 0. \tag{1}
\]

Thus, the main result of the paper is the following.
Theorem 1.1. Assume that $Y$ is a complete Busemann convex geodesic space that is weakly uniformly convex and satisfies (1). If $A > 0$, then for almost every $x \in X$, there exists a unique geodesic ray $\gamma$ in $Y$ that issues at $y$ and depends on $x$ such that
\[
\lim_{n \to \infty} \frac{1}{n}d(\gamma(An), a_n(x)y) = 0.
\] (2)

In Section 2 we recall some notions of uniform convexity used in nonlinear settings by various authors. All these concepts fit within the class of weakly uniform convex geodesic spaces defined as above and satisfy (1). Section 3 contains the proof of our result.

2 Weakly uniformly convex geodesic spaces

A metric space $(Y, d)$ admits midpoints if for every $x, y \in Y$ there exists a point $m(x, y) \in Y$ (called midpoint) such that $d(x, m(x, y)) = d(y, m(x, y)) = \frac{1}{2}d(x, y)$. If $Y$ is also complete, then it is a geodesic space.

Let $Y$ be a geodesic space. A point $z \in Y$ belonging to a geodesic segment joining $x, y \in Y$ will be denoted by $z = (1 - t)x + ty$, where $t = d(z, x)/d(x, y)$. If $Y$ is uniquely geodesic, we use the notation $[x, y]$ for the unique geodesic segment that joins $x, y \in Y$.

Any weakly uniformly convex geodesic space is strictly convex (that is, for all $a, x, y \in Y$ with $x \neq y$, $d(a, \frac{1}{2}x + \frac{1}{2}y) < \max\{d(a, x), d(a, y)\}$), and hence uniquely geodesic. A stronger notion that was first considered in a nonlinear setting in [3] is the one of uniform convexity which means that additionally the modulus of convexity $\delta$ does not depend on the points $a \in Y$, so $\delta = \delta(r, \varepsilon)$. Uniform convexity was also defined in [10] in the following way: a geodesic space $Y$ is uniformly convex if there exists a mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ such that for any $r > 0$, $\varepsilon \in (0, 2]$ and all $a, x, y \in Y$,
\[
\begin{align*}
    d(a, x) &\leq r \\
    d(a, y) &\leq r \\
    d(x, y) &\geq \varepsilon r
\end{align*}
\] \Rightarrow \ d \left( a, \frac{1}{2}x + \frac{1}{2}y \right) \leq (1 - \eta(r, \varepsilon))r. \quad (3)

Such a mapping $\eta$ is referred to in [10] as a modulus of uniform convexity. These two definitions of uniform convexity are equivalent, the difference being that the modulus of convexity $\delta(r, \varepsilon)$ gives the largest possible $\eta(r, \varepsilon)$ for $r > 0$ and $\varepsilon \in (0, 2]$. One can, of course, define a notion of a modulus of weak uniform convexity. However, in this paper we will use the modulus of convexity $\delta$.

Note that the above definitions can be given in the same way in the setting of metric spaces that admit midpoints.

We include next some more particular notions of (weak) uniform convexity:

(i) In [7], a geodesic space $Y$ with a convex bicombing that is uniformly convex in the above sense is also assumed to satisfy the condition that
\[
\forall s \geq 0, \forall \varepsilon > 0, \exists \alpha(s, \varepsilon) > 0 \text{ such that } \forall r > s, \delta(r, \varepsilon) > \alpha(s, \varepsilon) > 0.
\]
(ii) Karlsson and Margulis [5] defined uniform convexity in a metric space $Y$ that admits midpoints in the following way: $Y$ is said to be uniformly convex if there exists a strictly decreasing and continuous function $g : [0,1] \rightarrow [0,1]$ with $g(0) = 1$, so that for any $a, x, y \in Y$ and midpoint $m(x,y)$ of $x$ and $y$,

$$\frac{d(a,m(x,y))}{r} \leq g \left( \frac{d(x,y)}{2r} \right), \quad \text{where } r = \max\{d(x,a), d(y,a)\}. \quad (4)$$

Any such space is uniformly convex in the sense used in the present paper. Indeed, given $r > 0$ and $\epsilon \in (0,2]$, let $a, x, y \in Y$ satisfy $d(x,a) \leq r, d(y,a) \leq r$ and $d(x,y) \geq \epsilon r$. Set $R = \max\{d(x,a), d(y,a)\} \leq r$. Then,

$$d \left( a, \frac{1}{2}x + \frac{1}{2}y \right) \leq g \left( \frac{d(x,y)}{2R} \right) R \leq g \left( \frac{\epsilon}{2} \right) r.$$

It follows that

$$1 - \frac{1}{r} d \left( a, \frac{1}{2}x + \frac{1}{2}y \right) \geq 1 - g \left( \frac{\epsilon}{2} \right),$$

from where $\delta(r, \epsilon) \geq 1 - g \left( \frac{\epsilon}{2} \right) > 0$.

(iii) Gledsner, Karlsson and Margulis considered in [2] a strictly convex geodesic space $Y$ to be uniformly convex if it is weakly uniformly convex and

$$\forall \epsilon > 0, \exists \delta(\epsilon) > 0 \text{ such that } \forall r > 0, \forall a \in Y, \delta(a, r, \epsilon) \geq \delta(\epsilon).$$

(iv) In [6], a metric space $Y$ that admits midpoints is called uniformly $p$-convex (where $p \in [1,\infty]$) if for every $\epsilon > 0$ there exists $\rho_p(\epsilon) \in (0,1)$ such that for all $a, x, y \in Y$ with $d(x,y) > \epsilon M^p(d(a,x), d(a,y))$ for $p > 1$ and $d(x,y) > |d(a,x) - d(a,y)| + \epsilon M^1(d(a,x), d(a,y))$ for $p = 1$ we have that

$$d(a,m(x,y)) \leq (1 - \rho_p(\epsilon)) M^p(d(a,x), d(a,y)),$$

where for $\alpha, \beta \geq 0$, $M^p(\alpha, \beta) = (\alpha^p + \beta^p)^{1/p}$ and $M^\infty(\alpha, \beta) = \max\{\alpha, \beta\}$. Note that, by [6, Lemma 4], any uniformly $p$-convex space is uniformly $\infty$-convex which in turn is uniformly convex in the sense used in the present paper. Indeed, let $r > 0$ and $\epsilon \in (0,2]$ and consider $a, x, y \in Y$ with $d(x,a) \leq r, d(y,a) \leq r$ and $d(x,y) \geq \epsilon r$. Then $\max\{d(x,a), d(y,a)\} \leq r$ and $d(x,y) \geq \frac{\epsilon}{2} \max\{d(x,a), d(y,a)\}$. This yields

$$d \left( a, \frac{1}{2}x + \frac{1}{2}y \right) \leq \left( 1 - \rho_{\infty} \left( \frac{\epsilon}{2} \right) \right) \max\{d(x,a), d(y,a)\} \leq \left( 1 - \rho_{\infty} \left( \frac{\epsilon}{2} \right) \right) r.$$

It follows that

$$1 - \frac{1}{r} d \left( a, \frac{1}{2}x + \frac{1}{2}y \right) \geq \rho_{\infty} \left( \frac{\epsilon}{2} \right),$$

and so $\delta(r, \epsilon) \geq \rho_{\infty} \left( \frac{\epsilon}{2} \right) > 0$. 4
Geodesic spaces which are $p$-uniformly convex in the sense of Naor and Silberman [11] (see also [9, 12]) satisfy all four definitions (i)-(iv). These spaces are defined as follows: for a fixed $1 < p < \infty$, a geodesic space $(Y, d)$ is called $p$-uniformly convex with parameter $k > 0$ if for every $a, x, y \in Y$ and $t \in [0, 1]$, 

$$d(a, (1 - t)x + ty)^p \leq (1 - t)d(a, x)^p + td(a, y)^p - \frac{k}{2} t(1 - t)d(x, y)^p.$$

We see below that $p$-uniformly convex spaces indeed satisfy (i)-(iv). Remark first that, by [9, Proposition 2.5, (1)], $k \leq c_p$ with

$$c_p = \begin{cases} 
2(p - 1) & \text{if } p \in (1, 2) \\
8/2^p & \text{if } p \geq 2.
\end{cases}$$

Let $a \in Y$, $r > 0$ and $\varepsilon \in (0, 2]$. Take $x, y \in Y$ such that $d(a, x) \leq r$, $d(a, y) \leq r$ and $d(x, y) \geq \varepsilon r$. Then,

$$1 - \frac{1}{r} d\left(a, \frac{1}{2}x + \frac{1}{2}y\right) \geq 1 - \left(1 - \frac{k}{8}\varepsilon^p\right)^{1/p}.$$

This clearly implies (i) and (iii). To see that (ii) is satisfied, consider $a, x, y \in Y$ and let $r = \max\{d(a, x), d(a, y)\}$. Then,

$$\frac{1}{r} d\left(a, \frac{1}{2}x + \frac{1}{2}y\right) \leq \left(1 - \frac{k}{8}\left(\frac{d(x, y)}{r}\right)^p\right)^{1/p}.$$

For $t \in [0, 1]$, take $g(t) = \left(1 - \frac{k}{8}(2t)^p\right)^{1/p}$. Then, $g([0, 1]) \subseteq [0, 1]$, $g(0) = 1$, $g$ is continuous and strictly decreasing. Hence, (ii) holds.

Let $\varepsilon > 0$ and take $\rho_p(\varepsilon) = 1 - \left(\frac{1}{2} - \frac{k}{8}\varepsilon^p\right)^{1/p} > 0$. It follows immediately that for all $a, x, y \in Y$ with $d(x, y) > \varepsilon M_p(d(a, x), d(a, y))$,

$$d\left(a, \frac{1}{2}x + \frac{1}{2}y\right) \leq (1 - \rho_p(\varepsilon)) M_p(d(a, x), d(a, y)).$$

This means that (iv) is satisfied (see also [6, Example, page 3]).

For the rest of this paper we assume that any weakly uniformly convex geodesic space also satisfies condition (1). From the above arguments it is easy to see that all uniformly convex spaces described in (i)-(iv) satisfy (1).

Although we state the main result of this paper in the setting of weakly uniformly convex geodesic spaces that additionally satisfy a nonpositive curvature assumption, we remark that what we actually use in the proof is the following convexity condition which holds in any weakly uniformly convex geodesic space.

### 2.1 A convexity assumption

Let $(Y, d)$ be a geodesic space. We say that $Y$ satisfies property (C) if

$$\forall y \in Y, \forall r > 0, \forall \varepsilon \in (0, 2], \exists \Psi(y, r, \varepsilon) \in (0, 1]$$

such that

$$d\left(a, \frac{1}{2}x + \frac{1}{2}y\right) \leq (1 - \rho_p(\varepsilon)) M_p(d(a, x), d(a, y)).$$
such that \( \forall x, z \in Y \) with \( d(x, y) = r, \) \( d(y, z) \geq r, \) if \( w \) belongs to a geodesic segment joining \( y \) and \( z \) such that \( d(y, w) = r \) then

\[
r + d(x, z) \leq d(y, z) + \Psi(y, r, \varepsilon)r \implies \quad d(w, x) \leq \varepsilon r.
\]

In addition, we suppose that for each \( y \in Y \) and \( \varepsilon \in (0, 2], \) there exists \( s > 0, \) \( \inf \Psi(y, r, \varepsilon) > 0. \)

If \( Y \) is a normed space, \( \Psi \) does not depend on \( y \) and \( r, \) so one can take \( \Psi(y, r, \varepsilon) = \Psi(0, 1, \varepsilon). \) To see this, let \( y \in Y, \) \( r > 0, \) \( \varepsilon > 0 \) and \( x, z \in Y \) with \( \|x - y\| = r, \) \( \|y - z\| \geq r. \) Let \( w \) be on a geodesic segment joining \( y \) and \( z \) such that \( \|y - w\| = r \) and

\[
r + \|x - z\| \leq \|y - z\| + \Psi_0(1, \varepsilon)r.
\]

Dividing this inequality by \( r \) we obtain

\[
1 + \frac{1}{r}\|x - z\| \leq \frac{1}{r}\|y - z\| + \Psi_0(1, \varepsilon).
\]

Let

\[
x' = \frac{1}{r}(x - y), \quad z' = \frac{1}{r}(z - y), \quad w' = \frac{1}{r}(w - y).
\]

Then,

\[
\|x' - z'\| = \frac{1}{r}\|x - z\|, \quad \|z'\| = \frac{1}{r}\|z - y\|, \quad \|w'\| + \|w' - z'\| = \|z'\|.
\]

Thus, \( 1 + \|x' - z'\| \leq \|z'\| + \Psi_0(1, \varepsilon). \) By property (C) we have that \( \|w' - x'\| \leq \varepsilon, \) that is, \( \|w - x\| \leq \varepsilon r. \)

**Lemma 2.1.** Suppose \( Y \) is a weakly uniformly convex geodesic space. Then \( Y \) satisfies property (C) with \( \Psi(y, r, \varepsilon) = \delta(y, r, \varepsilon) \) for all \( y \in Y, \) \( r > 0 \) and \( \varepsilon \in (0, 2]. \)

**Proof.** Let \( y \in Y, \) \( r > 0, \) \( \varepsilon \in (0, 2]. \) Consider \( x, z \in Y \) with \( d(x, y) = r, \) \( d(y, z) \geq r \) and take \( w \in [y, z] \) such that \( d(y, w) = r. \) Suppose that \( r + d(x, z) \leq d(y, z) + \delta(y, r, \varepsilon)r. \) We may assume that \( w \neq x \) (otherwise the conclusion is immediate). Let \( m = \frac{1}{2}w + \frac{1}{2}x. \) Since \( d(m, z) < \max\{d(z, x), d(z, w)\} \) and \( d(x, z) \leq d(y, z) - r + \delta(y, r, \varepsilon)r = d(z, w) + \delta(y, r, \varepsilon)r, \) it follows that \( d(m, z) < d(z, w) + \delta(y, r, \varepsilon)r. \) Thus, \( d(y, z) - d(m, z) \leq d(m, z) < d(z, w) + \delta(y, r, \varepsilon)r, \) from where

\[
(d(y, m) > (1 - \delta(y, r, \varepsilon))r).
\]

By weak uniform convexity,

\[
d(y, m) \leq \left(1 - \delta\left(y, r, \frac{d(x, w)}{r}\right)\right)r.
\]

Hence,

\[
\delta\left(y, r, \frac{d(x, w)}{r}\right) < \delta(y, r, \varepsilon).
\]
Because $\delta$ is increasing with respect to the separation distance $\varepsilon$ for fixed $y \in Y$ and $r > 0$ this implies that $d(x, w) \leq \varepsilon r$.
Moreover, by (1), for every $y \in Y$ and $\varepsilon > 0$ there exists $s > 0$ such that
\[ \inf_{r \geq s} d(y, r, \varepsilon) > 0. \]
\[ \square \]

3 Proof of the main theorem

Let $E$ be the set of points $x \in X$ for which taking any $\varepsilon > 0$ there exist $M \in \mathbb{N}$ and infinitely many $n \in \mathbb{N}$ such that for every $k \in [M, n]$,
\[ d(y, a_n(x)y) - d(y, a_{n-k}(T^k x)y) \geq (A - \varepsilon)k. \]

By [5, Proposition 4.2] we know that $\mu(E) = 1$.

**Lemma 3.1.** Let $x \in E$ such that $\lim_{n \to \infty} \frac{1}{n} d(y, a_n(x)y) = A > 0$. Then $\forall (\alpha_i) \subseteq (0, 1), \forall (p_i) \subseteq \mathbb{N}, \exists (K_i) \subseteq \mathbb{N}, \exists (n_i) \subseteq \mathbb{N}$ satisfying

(i) $\forall i \geq 1, p_i \leq K_i$;

(ii) $\forall i \geq 2, K_i < n_{i-1} < n_i$ and $d(y, a_{n_i}(x)y) \geq \max\{d(y, a_{n_{i-1}}(x)y), A n_{i-1}\}$;

(iii) $\forall k \in [K_i, n_i], |d(y, a_k(x)y) - A k| \leq \frac{A k}{2^i}$ and

\[ (1 - \min\{2^{-i}, \alpha_i\}) d(y, a_k(x)y) + d(a_k(x)y, a_n(x)y) \leq d(y, a_n(x)y). \] (5)

**Proof.** For $i \geq 1$, take $\varepsilon_i = \min\left\{ \frac{A}{1 + 2^{i+1}}, \frac{A \alpha_i}{2 + \alpha_i} \right\}$. Then
\[ \frac{2 \varepsilon_i}{A - \varepsilon_i} \leq \min\{2^{-i}, \alpha_i\}. \] (6)

Since $x \in E_{\varepsilon_i}$, there exist $M_i$ and infinitely many $n$ such that for every $k \in [M_i, n]$,
\[ d(y, a_n(x)y) - d(y, a_{n-k}(T^k x)y) \geq (A - \varepsilon_i)k. \] (7)

Moreover, since $\lim_{n \to \infty} \frac{1}{n} d(y, a_n(x)y) = A$ it follows that there exists $J_i$ such that for every $k \geq J_i$,
\[ (A - \varepsilon_i)k \leq d(y, a_k(x)y) \leq (A + \varepsilon_i)k. \] (8)

Thus, there exist $K_i = \max\{M_i, J_i, p_i\}$ and infinitely many $n$ such that for every $k \in [K_i, n]$, (7) and (8) are satisfied.

Let $i = 1$ and $n_1 > K_1 + K_2$ such that (7) and (8) hold for $k \in [K_1, n_1]$. For each $i \geq 2$, pick $n_i > \max\{n_{i-1}, K_{i+1}\}$ such that
\[ d(y, a_n(x)y) \geq \max\{d(y, a_{n_{i-1}}(x)y), A n_{i-1}\} \]
and (7) and (8) hold for $k \in [K_i, n_i]$. Then for all $i \geq 1$ and $k \in [K_i, n_i]$, 

$$|d(y, a_k(x)y) - Ak| \leq \varepsilon_i k \leq \frac{Ak}{2^i}$$

and

$$d(y, a_{n_i}(x)y) - d(y, a_{n_i-k}(T^kx)y) + (A + \varepsilon_i)k \geq (A - \varepsilon_i)k + d(y, a_k(x)y).$$

This implies that

$$d(y, a_k(x)y) + d(y, a_{n_i-k}(T^kx)y) \leq d(y, a_{n_i}(x)y) + 2\varepsilon_i k$$

$$\leq d(y, a_{n_i}(x)y) + 2\varepsilon_i \frac{d(y, a_k(x)y)}{A - \varepsilon_i}$$

by (8)

$$\leq d(y, a_{n_i}(x)y) + \min\{2^{-i}, \alpha_i\}d(y, a_k(x)y)$$

by (6).

But

$$d(a_k(x)y, a_{n_i}(x)y) = d(a_k(x)y, a_k(x)a_{n_i-k}(T^kx)y) \leq d(y, a_{n_i-k}(T^kx)y),$$

so,

$$(1 - \min\{2^{-i}, \alpha_i\})d(y, a_k(x)y) + d(a_k(x)y, a_{n_i}(x)y) \leq d(y, a_{n_i}(x)y).$$

Recall that a geodesic space $Y$ is Busemann convex if given any pair of geodesic paths $\gamma_1 : [0, l_1] \to Y$ and $\gamma_2 : [0, l_2] \to Y$ with $\gamma_1(0) = \gamma_2(0)$ one has

$$d(\gamma_1(tl_1), \gamma_2(tl_2)) \leq td(\gamma_1(l_1), \gamma_2(l_2)), \text{ for every } t \in [0, 1].$$

### 3.1 Proof of Theorem 1.1

Let $x \in E$ such that $\lim_{n \to \infty} \frac{1}{n}d(y, a_n(x)y) = A$. To simplify the writing, we denote

$$r_n = d(y, a_n(x)y) \to \infty.$$ 

By property (C), for each $i \in \mathbb{N}$ there exists $s_i > 0$ such that

$$\alpha_i = \inf_{r \geq s_i} \Psi \left( y, r, \frac{1}{2^i} \right) > 0.$$ 

Also, for each $i \in \mathbb{N}$ there exists $P_i \in \mathbb{N}$ such that $r_k \geq s_i$ for $k \geq P_i$. Apply Lemma 3.1 to obtain the sequences $(K_i)$ and $(n_i)$ satisfying properties (i)-(iii) thereof. For each $i$, let $\gamma_i$ be the geodesic segment joining $y$ and $a_{n_i}(x)y$. 

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Let $k \in [K_i, n_i]$. If $r_k > r_{n_i}$, then, by (5),

$$d(a_k(x), \gamma_i(r_{n_i})) = d(a_k(x), a_{n_i}(x)) \leq \frac{1}{2^i} r_k. \quad (9)$$

Otherwise, since $k \geq p_i$ it follows that $r_k \geq s_i$ and so $\alpha_i \leq \Psi \left(y, r_k, \frac{1}{2^i}\right)$. Then (5) yields that

$$r_k + d(a_k(x), a_{n_i}(x)) \leq d(y, a_{n_i}(x)) + \Psi \left(y, r_k, \frac{1}{2^i}\right) r_k.$$

By property (C), we have that

$$d(a_k(x), \gamma_i(r_k)) \leq \frac{1}{2^i} r_k. \quad (10)$$

Thus, since $K_{i+1} < n_i < n_{i+1}$ and $r_{n_i} \leq r_{n_{i+1}}$, we have that

$$d(\gamma_i(r_{n_i}), \gamma_i(r_{n_{i+1}})) = d(\gamma_i(r_{n_i}), a_{n_i}(x)) \leq \frac{1}{2^{i+1}} r_{n_i}. \quad (11)$$

Fix $R > 0$. Let $I(R)$ be the smallest natural number for which $r_{n_{I(R)}} \geq R$. Since $(r_{n_i})_i$ is an increasing sequence, it follows that $r_{n_i} \geq R$ for all $i \geq I(R)$. Let $i \geq I(R)$, then, by Busemann convexity,

$$d(\gamma_i(r_{n_i}), \gamma_i(r_{n_{i+1}})) = d(\gamma_i(r_{n_i}), a_{n_i}(x)) \leq \frac{R}{r_{n_i}} \frac{1}{2^{i+1}} r_{n_i} = \frac{1}{2^{i+1}} R. \quad (12)$$

Now, by the triangle inequality, for all $m > 0$,

$$d(\gamma_i(r_{n_i}), \gamma_i(r_{n_{i+m}})) \leq \sum_{j=1}^{m} \frac{1}{2^{i+j}} R \leq \frac{R}{2^i}. \quad (13)$$

This means that $(\gamma_i(R))_{i \geq I(R)}$ is Cauchy. Define $\gamma(R) = \lim_{i \to \infty} \gamma_i(R)$. Letting $m \to \infty$ in (12) we obtain that for all $i \geq I(R)$,

$$d(\gamma_i(r_{n_i}), \gamma_i(R)) \leq \frac{R}{2^i}. \quad (13)$$

It is easy to see that $\gamma$ is a geodesic ray starting at $y$.

For each $k$ there exists $i$ such that $n_{i-1} \leq k < n_i$, which yields $K_i < k < n_i$.

Note that $A k < A n_i \leq r_{n_{i+1}}$. Also, if $r_k > r_{n_i}$, because $|r_n - A n_i| \leq \frac{A n_i}{2^{i+1}}$ and $|r_k - A k| \leq \frac{A k}{2^i}$, we have that

$$A \left(1 - \frac{1}{2^i}\right) k \leq A \left(1 - \frac{1}{2^{i+1}}\right) n_i \leq r_{n_i} < r_k \leq A \left(1 + \frac{1}{2^i}\right) k.$$
Thus, $|r_{n_i} - A_k| \leq \frac{A_k}{2i}$. Let $m_k = \min\{r_k, r_{n_i}\}$. It follows that $|m_k - A_k| \leq \frac{A_k}{2i}$ and, by (9) and (10), $d(a_k(x)y, \gamma_i(m_k)) \leq \frac{1}{2^i}r_k$. Note that since $r_{n_{i+1}} > A_k$ and $r_{n_i} \geq m_k$ we have that $i + 1 \geq I(A_k)$ and $i \geq I(m_k)$, respectively. Then, using (13) and (11),

$$d(\gamma(A_k), a_k(x)y) \leq d(\gamma(A_k), \gamma_{i+1}(A_k)) + d(\gamma_{i+1}(A_k), \gamma_{i+1}(m_k))$$
$$+ d(\gamma_{i+1}(m_k), \gamma_i(m_k)) + d(\gamma_i(m_k), a_k(x)y)$$
$$\leq \frac{A_k}{2^{i+1}} + |A_k - m_k| + \frac{1}{2^{i+1}}m_k + \frac{1}{2^i}r_k$$
$$\leq \frac{3A_k}{2^{i+1}} + \frac{6A_k}{2^{i+1}} = \frac{9A_k}{2^{i+1}}.$$

This implies that

$$\lim_{k \to \infty} \frac{1}{k} d(\gamma(A_k), a_k(x)y) = 0.$$  

By Busemann convexity it follows easily that the obtained geodesic ray is unique.

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