Twin edge coloring of infinite lattices

Qing YANG\textsuperscript{a}, Shuang Liang TIAN\textsuperscript{b}, Lang Wang Qing SUO\textsuperscript{c}

China Mathematics and computer institute, Northwest Minzu University, Lanzhou, Gansu 730030, China
\textsuperscript{a}1070329172@qq.com, \textsuperscript{b}sl-tian@163.com(corresponding author), 
\textsuperscript{c}971653292@qq.com

Abstract. Using the methods on construction of coloring and proofs by contradiction, the twin edge chromatic numbers of the infinite square, hexagonal and triangular lattices are 5, 4 and 7.

1. Introduction
Burris in [1] put forward vertex-distinguishing edge coloring of simple graphs. This coloring is related to vertex-distinguishing proper edge coloring of graphs, first examined by Burris and Schelp [2] and further discussed by many others, including Bazgan, et al. [3] and Balister, et al. [4]. The adjacent vertex-distinguishing edge coloring has been introduced by zhang et al. [5]. This type of coloring has been studied in many papers (see for example [6-10]). In 2013 Flandrin et al. [11] put forward concept of an adjacent sum-distinguishing edge coloring, and determined adjacent sum-distinguishing edge chromatic number for many basic families of graphs, including paths, cycles, tree, complete graphs and complete bipartite graphs. Flandrin et al. give an upper bound \( \lceil (7\Delta - 4)/2 \rceil \) of adjacent sum-distinguishing edge chromatic number of connected graph of the maximum degree \( \Delta \geq 2 \). Some other results about adjacent sum-distinguishing edge coloring can be found in [12-14].

In 2014 Andrews [15] introduced a concept of the twin edge coloring and determined the value of twin edge coloring for many basic families of graphs, including paths, cycles, complete graphs and complete bipartite graphs. At present, the results of the twin edge coloring of the relevant is only one, and the results obtained less.

In the existing literature on graph coloring, most of the graphs with finite graphs then the vertex set and the edge set are finite sets. Some other papers about grid graphs can be found in [16-18]. The twin edge coloring of the infinite square, hexagonal and triangular lattices are studied in this paper. The twin edge chromatic numbers of the infinite square, hexagonal and triangular lattices are determined.

Let \( G \) be a simple graph; the vertex-set of \( G \), denoted by \( V(G) \); the edge-set of \( G \), denoted by \( E(G) \); the maximum degree of \( G \), denoted by \( \Delta(G) \); denoted by \( \{k\} = \{0, 1, \ldots, k-1\} \), \( (x)_k = (x) \mod k \).

**Definition 1.1** Let \( \sigma \) is a \( k \)-distance coloring of simple graph \( G \) of order at least 3, denoted by \( C(\sigma) = \{\sigma(u) \mid uv \in E(G)\} \). For every adjacent vertex \( u, v \in V(G) \), if \( C(u) \neq C(v) \) then we say \( \sigma \) is an adjacent vertex-distinguishing edge coloring. We call the smallest \( k \) for which such a coloring of \( G \) exists the adjacent vertex-distinguishing edge chromatic number, and denote it by \( \chi'_a(G) \) [5].
Definition 1.2 Let $\sigma$ is a $k$-distance coloring of simple graph $G$ of order at least 3, the set of colors, denoted by $[k]$. Define $\sigma$ to denote coloring of $G$ that denoted by $\sigma(v) = (\sum_{u \in E_v} \sigma(uv))_k$ ($E_v$ is a set about all adjacent edge of $v$) than we say $\sigma'$ is a induced vertex coloring by $\sigma$. If $\sigma'$ is a proper vertex coloring, we say $\sigma$ is a twin edge coloring. We call the smallest $k$ for which such a coloring of $G$ exists the twin edge chromatic number, and denote it by $\chi_t(G)$ [15].

The following lemma is obtained by definition 1.

Lemma 1.1 For $G$ of order at least 3 have adjacent maximum degree vertex, then $\chi_t(G) \geq \Delta(G)+1$.

It is known by definition 1 and definition 2 that twin edge coloring must be adjacent vertex-distinguishing edge coloring then $\chi_t(G) \leq \chi(G)$.

The following definition is about the infinite square, hexagonal and triangular lattices [16].

Let $V = \{(x,y) \mid x, y \in \mathbb{Z}\}$ be a set of vertex where $\mathbb{Z}$ is a set of integer. The vertex-set of infinite square($L$), hexagonal($H$) and triangular($T_r$) lattices denoted by $V$. For any two vertex $(x,y)$ and $(x',y')$ are adjacent in $L$ if and only if $x = x'$ and $|y - y'| = 1$, or $y = y'$ and $|x - x'| = 1$. For any two vertex $(x,y)$ and $(x',y')$ are adjacent in $H$ if and only if $y = y'$ and $|x - x'| = 1$, or $x = x'$ and $(x + y)_2 = 1$, or $x - y'_2 = 1$. For any two vertex $(x,y)$ and $(x',y')$ are adjacent in $T_r$, if and only if $x = x'$ and $|y - y'| = 1$, or $y = y'$ and $|x - x'| = 1$, or $x - x'_2 = 1$ and $y - y'_2 = 1$, or $x' - x = 1$ and $y' - y = 1$.

2. Main results and proofs

The following theorem is about the twin edge coloring of infinite square lattices ($L$).

Theorem 2.1 $\chi_t(L) = 5$.

Proof If $L$ have adjacent maximum degree vertex, then $\chi_t(L) \geq \Delta(L)+1 = 5$. To prove $\chi_t(L) \leq 5$, we construct 5-twin edge coloring of $L$.

Let the set of color, be $\{0,1,2,3,4\}$. For any vertex $u = (x,y)$, we have $\sigma((x,y) + 1,y) = (x+1,y)$, $\sigma((x,y) + 1,y+1) = 3 + (x+y)$.

Clearly, this coloring $\sigma$ uses no more than 5 colors.

Now, to prove the coloring $\sigma$ is proper edge coloring, and the coloring $\sigma'$ is a proper vertex coloring.

Firstly, to prove the coloring $\sigma$ is proper edge coloring. For every vertex $u = (x,y)$, we have $u v_1 = (x,y) (x,y+1)$, $u v_2 = (x,y) (x+1,y)$, $u v_3 = (x,y-1)(x,y)$, $u v_4 = (x-1,y)(x,y)$.

By definition of the coloring $\sigma$, we have $\sigma(uv_1) = 3 + (x+y)_2$, $\sigma(uv_2) = (x+y)_3$, $\sigma(uv_3) = 3 + (x+y+1)_2$, $\sigma(uv_4) = (x+y+2)_3$.

It is easy to prove that the colors appearing on incident edge are different about coloring of the coloring $\sigma$. In fact, since $\sigma(uv_1),\sigma(uv_2) \in \{3,4\}$ and $\sigma(uv_3),\sigma(uv_4) \in \{0,1,2\}$, edge coloring of $\{uv_1,uv_3\}$ and $\{uv_2,uv_4\}$ are different. If $\sigma(uv_1) = \sigma(uv_3)$, then $(0)_2 = (1)_2$, a contradiction; if $\sigma(uv_2) = \sigma(uv_4)$, then $(0)_3 = (2)_3$, a contradiction. $\sigma$ the coloring $\sigma$ is 5-twin edge coloring of $L$.

Secondly, by the proofs by contradiction to prove $\sigma'$ is a proper vertex coloring. For any two vertex $u = (x,y)$ and $v = (x',y')$ are adjacent so that $\sigma'(u) = \sigma'(v)$, where $v = (x',y')$ have four cases then $v = (x+1,y)$, $v = (x,y+1)$, $v = (x-1,y)$ and $v = (x,y-1)$.

By definition of the coloring $\sigma'$, we have
\[ \sigma'(u) = \sigma'(x, y) = ((x + y) + (x + y + 1) + (x + y + 2) + 6) . \]

When \( v = (x + 1, y) \) or \( v = (x, y + 1) \), by definition of the coloring \( \sigma' \), we have
\[ \sigma'(x, y + 1) = ((x + y) + (x + y + 1) + (x + y + 2) + 6) . \]

Assumption \( \sigma'(u) = \sigma'(v) \), then \( (2)_3 = (1)_3 \), a contradiction.

When \( v = (x - 1, y) \) or \( v = (x, y - 1) \), by definition of the coloring \( \sigma' \), we have
\[ \sigma'(x - 1, y) = ((x + y) + (x + y + 1) + (x + y + 2) + 6) . \]

Assumption \( \sigma'(u) = \sigma'(v) \), then \( (0)_3 = (1)_3 \), a contradiction.

From the above analysis, the coloring \( \sigma' \) is a proper vertex coloring.

Thus, \( \chi'(L) = 5 \).

The following theorem is about the twin edge coloring of infinite hexagonal lattices \( (H) \).

**Theorem 2.2** \( \chi'(H) = 4 \).

**Proof** If \( H \) have adjacent maximum degree vertex, then \( \chi'(H) \geq \Delta(H) + 1 = 4 \). To prove \( \chi'(H) \leq 4 \), we construct 4-twin edge coloring of \( H \).

Let the set of colors, be \( [4] = \{0, 1, 2, 3\} \). For any vertex \( u = (x, y) \), we have
\[ \sigma(x, y)(x + 1, y) = (x + y)_3 \]
\[ \sigma(x, y)(x + 1, y) = 3, \text{when}(x + y)_2 = 0 . \]

Clearly, this coloring \( \sigma \) uses no more than 4 colors.

Now, to prove the coloring \( \sigma \) is a proper edge coloring, and \( \sigma' \) is a proper vertex coloring.

Firstly to prove the coloring \( \sigma \) is proper edge coloring. For any vertex \( u = (x, y) \), we have
\[ uv_1 = (x - 1, y)(x, y) \] and \( uv_2 = (x, y)(x + 1, y) \), when \( (x + y)_2 = 1 \), \( uv_3 = (x, y - 1)(x, y) \); when \( (x + y)_2 = 0 \), \( uv_3 = (x, y)(x + 1) \).

By definition of the coloring \( \sigma \), we have
\[ \sigma(uv_1) = (x + y - 1)_3 = (x + y + 2)_3 , \]
\[ \sigma(uv_2) = (x + y)_3 , \sigma(vu) = 3 . \]

It is easy to prove that the colors appearing on incident edge are different about coloring of the coloring \( \sigma \). In fact, since \( \sigma(uv_1), \sigma(uv_2) \in \{0, 1, 2\} \) and \( \sigma(vu) = 3 \), edge coloring of \( \{uv_1, uv_2\} \) and \( uv_3 \) are different. If \( \sigma(uv_1) = \sigma(uv_2) \), then \( (2)_3 = (0)_3 \), a contradiction. So the coloring \( \sigma \) is 4-twin edge coloring of \( H \).

Secondly, using the proofs by contradiction to prove \( \sigma' \) is a induced proper vertex coloring by \( \sigma \).

For any two adjacent \( u = (x, y) \) and \( v = (x', y') \) so that \( \sigma'(u) = \sigma'(v) \) where \( v = (x', y') \) have three cases then \( v_1 = (x - 1, y) \), \( v_2 = (x + 1, y) \) and \( v_3 = (x, y + 1) \) (where \( (x + y)_2 = 1 \), \( v_1 = (x, y - 1) \); where \( (x + y)_2 = 0 \), \( v_3 = (x, y)(x + 1) \)),

By definition of the coloring \( \sigma' \), we have
\[ \sigma'(u) = ((x + y) + (x + y + 1) + (x + y + 2) + 3)_4 , \]
\[ \sigma'(v_1) = ((x + y + 1) + (x + y + 2) + 3)_4 , \]
\[ \sigma'(v_2) = ((x + y) + (x + y + 1) + 3)_4 , \]
\[ \sigma'(v_3) = ((x + y + 1) + (x + y + 2) + 3)_4 , \] or \( \sigma'(v_3) = ((x + y) + (x + y + 1) + 3)_4 . \)

If \( v = v_1 \), because \( \sigma'(u) = \sigma'(v_1) \), then \( (0)_3 = (1)_3 \), a contradiction. If \( v = v_2 \), because \( \sigma'(u) = \sigma'(v_2) \) then \( (2)_3 = (1)_3 \), a contradiction. If \( v = v_3 \), because \( \sigma'(u) = \sigma'(v_3) \) then \( (0)_3 = (1)_3 \), a contradiction.

From the above analysis, the coloring \( \sigma' \) is a proper vertex coloring.

Thus, \( \chi'(H) = 4 \).
The following theorem is about the twin edge coloring of infinite triangular lattices \( T_r \).

**Theorem 2.3** \( \chi'_r(T_r) = 7 \).

**Proof** If \( T_r \) have adjacent maximum degree vertex, then \( \chi'_r(T_r) \geq \Delta(T_r)+1 = 7 \). To prove \( \chi'_r(T_r) \leq 7 \), we construct 7-twin edge coloring of \( T_r \).

Let the set of colors be \([7] = \{0,1,2,3,4,5,6\}\). For any vertex \( u = (x,y) \), we have
\[
\sigma((x,y)(x+1,y)) = (x+y) \in \{0,1,2,3,4,5,6\}, \\
\sigma((x,y)(x,y+1)) = (x+y+2) \in \{0,1,2,3,4,5,6\}, \\
\sigma((x,y)(x+1,y+1)) = 5+(x)_2.
\]
Clearly, this coloring \( \sigma \) uses no more than 7 colors.

Now, to prove the coloring \( \sigma \) is a proper edge coloring, and \( \sigma' \) is a proper vertex coloring.

Firstly to prove the coloring \( \sigma \) is proper edge coloring. For any vertex \( u = (x,y) \), we have
\[
\sigma(uv_1) = (x,y)(x,y+1), \\
\sigma(uv_2) = (x,y)(x+1,y+1), \\
\sigma(uv_3) = (x,y)(x,y-1), \\
\sigma(uv_4) = (x,y+1)(x+1,y), \\
\sigma(uv_5) = (x,y-1)(x,y), \\
\sigma(uv_6) = (x,y+1)(x,y).
\]
By definition of the coloring \( \sigma \), we have
\[
\sigma(uv_1) = (x+y+2)_2, \\
\sigma(uv_2) = 5+(x)_2, \\
\sigma(uv_3) = (x+y+4)_2, \\
\sigma(uv_4) = (x+y+1)_2, \\
\sigma(uv_5) = (x+y+2)_2, \\
\sigma(uv_6) = (x+y+4)_2.
\]
It is easy to prove that the colors appearing on incident edge are different about coloring of \( \sigma \). In fact, since \( \sigma(uv_1), \sigma(uv_2), \sigma(uv_4), \sigma(uv_6) \in \{0,1,2,3,4,5,6\} \), edge coloring of \( \{uv_1, uv_2, uv_4, uv_6\} \) and \( \{uv_3, uv_5\} \) are different. Since any two edges coloring are different in \( \{uv_1, uv_3, uv_4, uv_6\} \), thus \( i_2 = (j)_2 \) (where \( i \neq j \) and \( i, j \in \{0,1,2,4\} \)) a contradiction since \( i \neq j \); if \( \sigma(uv_2) = \sigma(uv_3) \), then \( (0)_2 = (1)_2 \) a contradiction since \( 0 \neq 2 \). So the coloring \( \sigma \) is 7-twin edge coloring of \( T_r \).

Secondly, using the proofs by contradiction to prove the coloring \( \sigma' \) is a induced proper vertex coloring by the coloring \( \sigma \). For any two adjacent vertex \( u = (x,y) \) and \( v = (x',y') \) so that \( \sigma'(u) = \sigma'(v) \) where \( v = (x',y') \) have six cases then \( v_1 = (x,y+1) \), \( v_2 = (x+1,y+1) \), \( v_3 = (x+1,y) \), \( v_4 = (x,y-1) \), \( v_5 = (x-1,y-1) \) and \( v_6 = (x-1,y) \).

By definition of the coloring \( \sigma' \), we have
\[
\sigma'(u) = \sigma'(x,y) = (x+y)_2 + (x+y+2)_2 + (x+y+4)_2 + (x+1)_2 + (x+1)_2 + 10)_7.
\]
When \( v = v_1 \), by definition of the coloring \( \sigma' \), we have
\[
\sigma'(v_1) = ((x+y)_2 + (x+y+1)_2 + (x+y+2)_2 + (x+y+3)_2 + (x+1)_2 + (x+1)_2) + 10)_7.
\]
Assumption \( \sigma'(u) = \sigma'(v_1) \), then \( (4)_2 = (3)_2 \), a contradiction.

When \( v = v_2 \), by definition of the coloring \( \sigma' \), we have
\[
\sigma'(v_2) = ((x+y)_2 + (x+y+2)_2 + (x+y+3)_2 + (x+1)_2 + (x+1)_2) + 10)_7.
\]
Assumption \( \sigma'(u) = \sigma'(v_2) \), then \( (0)_2 = (3)_2 \), a contradiction.

When \( v = v_3 \), by definition of the coloring \( \sigma' \), we have
\[
\sigma'(v_3) = ((x+y)_2 + (x+y+1)_2 + (x+y+2)_2 + (x+y+3)_2 + (x+1)_2 + (x+1)_2) + 10)_7.
\]
Assumption \( \sigma'(u) = \sigma'(v_3) \), then \( (4)_2 = (3)_2 \), a contradiction.

When \( v = v_4 \), by definition of the coloring \( \sigma' \), we have
\[
\sigma'(v_4) = ((x+y)_2 + (x+y+1)_2 + (x+y+2)_2 + (x+y+3)_2 + (x+1)_2 + (x+1)_2) + 10)_7.
\]
Assumption \( \sigma'(u) = \sigma'(v_4) \), then \( (2)_2 = (3)_2 \), a contradiction.

When \( v = v_5 \), by definition of the coloring \( \sigma' \), we have
\[
\sigma'(v_5) = ((x+y)_2 + (x+y+1)_2 + (x+y+2)_2 + (x+y+3)_2 + (x+1)_2 + (x+1)_2) + 10)_7.
\]
Assumption \( \sigma'(u) = \sigma'(v_5) \), then \( (1)_2 = (3)_2 \), a contradiction.

When \( v = v_6 \), by definition of the coloring \( \sigma' \), we have
\[
\sigma'(v_6) = ((x+y)_2 + (x+y+1)_2 + (x+y+2)_2 + (x+y+3)_2 + (x+1)_2 + (x+1)_2) + 10)_7.
\]
Assumption \( \sigma'(u) = \sigma'(v_6) \), then \( (1)_2 = (3)_2 \), a contradiction.
\[\sigma'(v_k) = ((x+y)_3 + (x+y+1)_3 + (x+y+3)_3 + (x+y+4)_3 + (x+1)_2 + (x+1)_2 + 10)_7.\]

Assumption \(\sigma'(u) = \sigma'(v_k)\), then \((2)_3 = (3)_3\), a contradiction.

From the above analysis, the coloring \(\sigma'\) is proper vertex coloring of \(T_r\).

Thus, \(\chi_r(T_r) = 7.\)

### 3. Conclusion

This work was financially supported by Key Laboratory of Streaming Data Computing Technologies and Applications, State Ethnic Affairs Commission of China (No.12XBZ006), Social Science Planning Project in Gansu Province (No.13YD031), State Ethnic Affairs Commission of China (No.14XBZ018) and Innovative Team Subsidize of Northwest Minzu University.

### References

[1] Burris A. Vertex-Distinguishing Edge-Coloring [D]. Ph. D. Dissertation, Memphis State University, 1993

[2] Burris A C, Schelp R H. Vertex-distinguishing proper edge colorings[J]. Journal of Graph Theory, 1997, 26: 73-82

[3] Bazgan C, Harkat-Benhamdine A, Li H, et al. On the vertex-distinguishing proper edge colorings of graphs[J]. Journal of Combinatorial Theory, 1999, 75: 288-301

[4] Balister P N, Bollobás B, Schelp R H. Vertex-distinguishing proper edge colorings of graphs with \(\Delta(G) = 2\) [J]. Discrete Mathematics, 2002, 252: 17-29

[5] Zhang Zhongfu, Liu Linzhong, Wang Jianfang. Adjacent strong edge coloring of graphs [J]. Applied Mathematical Letters, 2002, 15: 623-626

[6] Akbari S, Bidkhot H, Nosrati N. \(r\)-Strong edge colorings of graphs[J]. Discrete Math. 2006, 306: 3005-3010

[7] Balister P N, Gyori E, Lehel J, Schelp R H. Adjacent vertex distinguishing edge-colorings[J]. SIAM J. Discrete Math. 2007, 21: 237-250

[8] Baril J-L, Kheddouci H, Togni O. Adjacent vertex distinguishing edge-colorings of meshes[J]. Australasian Journal of Combinatorics, 2006, 35: 89-102

[9] Chen M, Guo X. Adjacent vertex-distinguishing edge and total chromatic numbers of hyper-cubes [J]. Information Processing Letters, 2009, 109: 599-602

[10] Hatami H. \(\Delta + 300\) is a bound on the adjacent vertex distinguishing edge chromatic number[J]. J. Combin. Theory Ser. B, 2005, 95 :246-256

[11] Flandrin E, Marczyk A, Przybylo J, et al. Neighbor sum distinguishing index[J]. Graphs Combin, 2013, 29(5):1329-1336

[12] Wang G H and Yan G Y. An improved upper bound for the neighbor sum distinguishing index of graphs[J]. Discrete Applied Mathematics, 2014, 175:126-128

[13] Bonamy M, Przybylo J. On the neighbor sum distinguishing index of planar graphs, 2014, arXiv:1408.3190

[14] Hocquard H and Przybylo J. On the neighbor sum distinguishing index of graphs with bounded maximum average degree. 2015, arXiv:1508.06112v1

[15] Andrews E, Helenius E, Johnston L, et al. On Twin Edge Colorings of Graphs [J]. Discussiones Mathematicae Graph Theory, 2014, 3(3):613-627

[16] Bondy J A, Murty U S R. Graph Theory[M], New York: Springer, 2008

[17] Diestel R. Graph Theory [M]. New York: Springer-Verlag Heidelberg, 2005

[18] Bondy J A, Murty U S R. Graph Theory with Applications [M]. New York: American Elsevier, 1976