Off-critical SLE(2) and SLE(4): a field theory approach

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Abstract. Using their relationship with the free boson and the free symplectic fermion, we study the off-critical perturbations of SLE(4) and SLE(2) obtained by adding a mass term to the action. We compute the off-critical statistics of the source in the Loewner equation describing the two-dimensional interfaces. In these two cases we show that ratios of massive and massless partition functions, expressible as ratios of regularized determinants of massive and massless Laplacians, are (local) martingales for the massless interfaces. The off-critical drifts in the stochastic source of the Loewner equation are proportional to the logarithmic derivative of these ratios. We also show that massive correlation functions are (local) martingales for the massive interfaces. In the case of massive SLE(4), we use this property to prove a factorization of the free boson measure.

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1. Introduction

Schramm–Loewner evolution (SLE) has been introduced to deal with conformally invariant random curves. These curves may for instance be thought of as interfaces in two-dimensional critical statistical systems. SLE is by now a (if not ‘the’) standard tool for efficiently formulating questions concerning these curves and, in many simple but important cases, for getting the answer by routine computations. See [9, 3, 6] for detailed introductions to SLE.

Having reached this depth of insight in the critical case, it is a natural question to wonder how SLE measures are deformed when considering interfaces in statistical models not at the critical point but slightly away from it (still in the scaling regime). Interfaces
out of criticality are at the moment very poorly understood to say the least. There are several reasons for investing some efforts in this, some of them more mathematical and some more physical, though the frontier is fuzzy.

The first obvious remark is that interfaces, or domain walls, are macroscopic structures that can be (and are) observed more directly than microscopic correlations (though the average over the sample of a local magnetization is the first accessible observable). So they are interesting to study for their own sake. A second obvious remark is that some interface models are purely geometric. The canonical example is percolation, for which all non-trivial observables are non-local and deal with cluster correlations. But even if there are local observables described by a local quantum field theory, this alone does not yield a straightforward description of interfaces: local quantum field theory does not deal easily with extended objects. As an example, the fields in the Kac table for minimal conformal field theories form a closed algebra but other fields which are non-local with respect to this algebra are crucial for describing the interface. SLE and its perturbations provide a framework for deciphering properties of such non-local excitations. A situation where this should be relevant is that of the $O(n)$ model: one can see the introduction of defects on the boundary to force the existence in the statistical mechanics system of macroscopic interfaces pinned at special points as a trick to get rid of some of the difficulties involved in the direct study of a gas of loops at all scales in the continuum.

Of interest both to the physicist and the mathematician is the concept of change of measure. In quantum field theory, the starting point is often a formal measure $d\mu_S \equiv D\phi e^{-S(\phi)}$ whose rigorous construction is usually a formidable challenge. The action $S(\phi)$ may depend on parameters and changing these parameters leads to families of measures. For instance, if $S = S_0 + \int \lambda O(\phi)$ is a perturbation of $S_0$ by some operator $O(\phi)$ one gets formally that $d\mu_S = d\mu_{S_0} \exp(-\int \lambda O(\phi))$, or in a more measure-theoretic language, that $\exp(-\int \lambda O(\phi))$ is the Radon–Nikodym derivative of $\mu_S$ with respect to $\mu_{S_0}$. In general quantum field theory, it is well-known that there is something poisonous in this statement due to renormalization but a rigorous analysis is mostly out of reach. However the framework of interface growth is a playground where analogous questions can be tackled. By this we mean two things: first that for some concrete models (see below) one can prove that some interface measures have or do not have Radon–Nikodym derivatives with respect to others, and second that when the answer is no for two interface measures, it is also no for the two measures on local degrees of freedom that induce the interface measures. To elaborate on this issue, we need to introduce some background.

Consider first a finite system described by statistical mechanics. Each configuration has a Boltzmann weight, usually strictly positive, which may depend on some continuous parameters. Then two measures corresponding to different values of the parameters have a well-defined Radon–Nikodym derivative. In the thermodynamic (a fortiori in the continuum) limit, this is another matter. In general, a positive measure $d\mu$ is said to be absolutely continuous with respect to another positive measure $d\mu_0$ on the same space if for any set $B_0$ such that $\mu_0(B_0) = 0$ there is a set $B \supset B_0$ such that $\mu(B) = 0$, or more loosely, if negligible sets for $\mu_0$ are also negligible for $\mu$. Under a technical condition, this ensures that there is a $\mu_0$-measurable function $f$, called the Radon–Nikodym derivative of $\mu$ with respect to $\mu_0$, such that $d\mu = f d\mu_0$. The theorem is obvious for finite or countable spaces but delicate in general; see e.g. [7,15]. Observe that one does not look at individual configurations (which have zero measure usually) but at subsets of configurations. So the
issue is that when going to the thermodynamic limit, a subset of configurations can carry a finite weight for a first choice of parameters but a vanishing weight for a second one. Then the first measure is said to be singular with respect to the second.

Suppose now that the system has boundary conditions imposing the presence of one or several interfaces. From the point of view of statistical mechanics, one could obtain the measure on interfaces as the outcome of summing all the Boltzmann weights of configurations of local degrees of freedom leading to a given position of the interface, and the same holds in the continuum limit if we assume that interfaces still make sense in that limit. If the measure on local degrees of freedom depends on parameters, so usually does the measure on interfaces. Assume that \( \mu \) and \( \mu_0 \) are measures on local degrees of freedom for two sets of parameters, and \( \nu \) and \( \nu_0 \) are the corresponding measures on interfaces. Then if a set \( I \) of interfaces has measure 0 for \( \nu \), the set \( B \) of local configurations leading to an interface in \( I \) has measure 0 for \( \mu \). So if \( \mu \) has a Radon–Nikodym derivative with respect to \( \mu_0 \), so does \( \nu \) with respect to \( \nu_0 \). The opposite does not need to hold.

Along these lines a spectacular result has been obtained in \([12]\). For site percolation on the 2D triangular lattice (where each site is occupied with probability \( p \) and empty with probability \( 1 - p \) independently of the other sites) the measure describing interfaces in the off-critical continuum limit (if \( a \) is the lattice mesh, one lets \( p \) go to the critical value \( p_c = 1/2 \) while keeping \( (p - p_c)a^{-3/4} \) fixed) is singular with respect to the critical interface measure. By the above remark, this also entails that the local measures on hexagons are singular in the continuum limit, a fact that can be understood as follows. The typical fluctuation of the number of occupied sites is \( \sim a^{-1} \) because there are \( \sim a^{-2} \) independent sites. However if \( (p - p_c) \sim a^{3/4} \), the typical asymmetry is \( \sim a^{3/4}a^{-2} = a^{-5/4} \), which is much larger that the fluctuation, so one can assert with certainty that an individual sample is critical or not. In fact, the same counting implies that on any set containing \( \sim a^{-d} \) hexagons, the asymmetry \( \sim a^{3/4}a^{-d} \) is much larger than the fluctuation \( \sim a^{-d/2} \) if \( d > 3/2 \). The critical percolation interface is bounded by \( \sim a^{-7/4} \) hexagons so it covers enough of the sample to feel a macroscopic effect of the tiny bias out of criticality. Of course, this is cheating because the interface as a set is correlated with the hexagon configuration. But this leads us to expect that along a typical interface sample out of criticality the asymmetry between occupied and empty sites causes a systematic excess of turns in one direction with respect to the other which is larger than what could be attributed to fluctuations. This is the intuitive basis for the result in \([12]\), but the actual proof involves subtleties that are well beyond the scope of the above intuition.

The theorem in \([12]\) is proved essentially without any recourse to SLE and stochastic processes. However, SLE and more generally stochastic processes can also provide relevant tools for addressing these matters. Computing the interface measure from the local Boltzmann weights is a very hard task, even at a critical point, and the approach of SLE was to study measures on interfaces, defined directly in the continuum, under two conditions, conformal invariance (covariance under conformal transport to go from one domain to another) and the domain Markov property (which asserts that the probability distribution of the curves in a domain conditioned on an initial portion of the curves is identical to the probability distribution of the curves but in the domain minus the portion on which we condition). This analysis led Schramm to the classification—in a one-parameter family usually indexed by \( \kappa \in [0, +\infty[ \)—of conformally invariant measures on random curves drawn on simply connected planar domains. This should be contrasted
with the present status of conformal field theory, were several mathematical axiomatics have been proposed but no general classification is in view.

One of the remarkable features of Schramm’s approach and result is that the measure on critical curves can be realized in a natural way as a 1D Brownian motion measure. This goes via a construction of interfaces via a stochastic growth process: with any continuous non-self-crossing curve joining two boundary points of a domain of the complex plane one can associate, via a trick discovered by Loewner involving a refinement of the Riemann mapping theorem, a real continuous function \( \xi_t \) for \( t \in [0, +\infty) \). When the curves in the domain have the statistics of a critical interface model, there is a \( \kappa \in [0, +\infty[ \) such that \( \xi_t / \sqrt{\kappa} \) is a Brownian motion \( B_t \). In particular the \( \xi_t \) corresponding to a critical interface measure is a Markov process.

Suppose now that the curves in the domain are non-critical interfaces. For such measures, conformal invariance is broken, i.e. the measures are not transported trivially by conformal transformations in a change of domain. This is because going out of criticality introduces a scale in the system, the correlation length \( \zeta \). But the domain Markov property, which has its roots in the locality of the underlying statistical mechanics system, usually survives. The Loewner trick can still be used to associate with each realization of the interface a continuous function \( \xi_t \) for \( t \in [0, +\infty] \), and one of the ways to have a description of the interface measure would be to give the measure on \( \xi_t \).

There is little doubt that the probability that the interface has a certain topology with respect to a finite number of points in the domain should depend smoothly on the correlation length \( \zeta \). There is a small subtlety here: a collection of consistent finite-dimensional distributions fixes the law of a process, but the fact that all finite-dimensional distributions depend smoothly on \( \zeta \) does not imply that the measures of the corresponding processes depend smoothly on \( \zeta \). Some observables, like the fractal dimension, are computed by using an infinite number of points but are nevertheless expected to be local enough to remain the same out of criticality; in particular they depend smoothly on \( \zeta \). On the other hand, the construction of [12] introduces an observable that depends on infinitely many points on the interface (but on arbitrarily small segments) and uses it to prove that the measures at criticality and out of criticality are not mutually absolutely continuous.

At scales much smaller that the correlation length, i.e. in the ultraviolet regime, the deviation from criticality is small and, as a function of \( t \), the off-critical \( \xi_t \) is expected to share some local features with its critical counterpart. This raises the question of whether \( \xi_t \) can be decomposed as \( \xi_t = \sqrt{\kappa} B_t + A_t \), i.e. as the sum of a Brownian motion (scaled by \( \sqrt{\kappa} \)) plus some process \( A_t \) whose precise regularity would remain to be understood but at least tamer than \( B_t \) on small scales.

The results from [12] imply that there must be a problem with that decomposition for percolation. The authors in [12] argue that the decomposition \( \xi_t = \sqrt{6} B_t + A_t \), but that \( A_t \) is too wild to expect absolute continuity. Stochastic calculus for continuous stochastic processes deals with processes \( X_t \) that are called in the probabilistic jargon semi-martingales. This means that they are defined as functionals of a Brownian motion \( B_t \) and have the following properties. First, \( X_t \) is causal (mathematicians say adapted) in the sense that \( X_t \) can be computed knowing

\[ \text{doi:10.1088/1742-5468/2009/07/P07037} \]

\[ \text{5} \]

\[ \text{4} \] That is, interfaces in a system out of criticality but in the scaling region. See section 5.1 for an example illustrating the passage to the continuum limit in the case of loop erased random walks.
only \{B_s, s \in [0,t]\} roughly speaking. Second, there is a splitting \(X_t = M_t + A_t\) as a sum where \(M_t\) is a stochastic integral\(^5\) \(\int_0^t Y_s \, dB_s\) and \(A_t\) is of locally finite variation, i.e. \(\sup \sum_i |A_{t_{i+1}} - A_{t_i}| < \infty\) when the sup is taken over all subdivisions \(0 = t_0 < t_1 \ldots t_n = t\) for fixed \(t\). The separation of scales (very crudely, the variation of \(M_t\) is of order \(\sqrt{dt}\) while that of \(A_t\) is of order \(dt\)) between the two terms implies that such a decomposition, if it exists, is unique. In our case, we would have \(M_t = \sqrt{\kappa} B_t\). If \(A_t\) is regular enough, the measures on the processes \(\xi_s\) and \(\sqrt{\kappa} B_s\), \(s \in [0,t]\) are absolutely continuous with respect to each other. But for percolation \(A_t\) is conjectured to be too wild.

Now look at scales large compared to \(\zeta\). In this regime, the behavior is different and the interface should look like another SLE with a new \(\kappa_{ir}\). Take the Ising model as an example. At criticality \(\kappa = 3\) but if the temperature is raised above the critical point, general renormalization group arguments indicate that at large scale the interface looks like the interface at infinite temperature. From the explicit example of the hexagonal lattice (plus maybe some confidence in universality) this limit is percolation and \(\kappa_{ir} = 6\). But the infrared regime is never attained in a bounded domain.

Let us close this long introduction by stressing again that conformal covariance and the domain Markov property have rather different status. Whereas conformal invariance emerges (at best) in the continuum limit at criticality, the domain Markov property makes sense and is satisfied on the lattice without tuning parameters for many systems of interest. It can be considered as a manifestation of locality (in the physicists’ terminology). Hence the domain Markov property is still expected to hold off criticality. However the consequences of this property on \(\xi_s\) do not seem to have a simple formulation. As for conformal ‘covariance’, there is a trick for preserving it formally out of the critical point: instead of perturbing with a scaling field \(O(z, \bar{z})\) times a coupling constant \(\lambda\), one perturbs by a scaling field times a density \(\lambda(z,\bar{z})\) of appropriate weight, in such a way that \(\lambda(z,\bar{z})O(z,\bar{z})\, d\bar{z} \wedge dz\) is a 2-form. If \(\lambda(z,\bar{z})\) has compact support, one also gets rid of infrared divergences that occur in unbounded domain. We shall use this trick in some places, but beware that if perturbation theory contains divergences, problems with scale invariance will arise; hence the cautious word ‘formally’ used above.

2. Summary

We are now in position to give a summary of our approach and results.

Besides \[12\], a few works on off-critical SLE have already appeared but the study of this problem is still in its infancy. In \[4\], we exposed a possible framework for dealing with deformations of SLE adapted to off-critical perturbations of the underlying statistical models. This approach links off-critical SLE to off-critical partition functions and field theories. It was perturbatively applied, to first order in the perturbing mass only, to off-critical loop erased random walks (LERW). The aim of this paper is to develop this method for two simple off-critical SLE, namely massive SLE(2) and massive SLE(4).

\(^5\) The stochastic integral \(\int_0^t Y_s \, dB_s\) is well-defined if the process \(Y_t\) is causal and \(\int_0^t Y_s^2 \, ds\) is almost surely finite. Then the stochastic integral itself is also causal. Note that \(\int_0^t Y_s \, dB_s\) does not need to be a martingale (i.e. to be conserved in average) because it can get too large. However it becomes a martingale if it is stopped as soon as its absolute value reaches \(n\) for \(n = 1, 2, \ldots\). So it is called a local martingale, a term we sometimes use in the sequel. Thus being bounded is a sufficient condition for the process \(\int_0^t Y_s \, dB_s\) to be a martingale. A milder useful criterion for being a true martingale is that \(E \int_0^t Y_s^2 \, ds < +\infty\).
These perturbations are simple enough to be treated non-perturbatively. Apparently some unpublished related work on similar perturbations of SLE has been reported in [11]. There is no doubt that the perturbation of the Ising model by the energy operator, corresponding to a shift of the temperature, is amenable to the same techniques. These three cases (corresponding to certain perturbations of $\kappa = 2, 3, 4$, i.e. central charge $c = -2, 1/2, 1$), all correspond to free field theory and this is the crucial point for our approach because it leads to computations of (variations of) determinants.

For more general cases, the situation is less favorable. Basic rules of CFT fix unambiguously the process $A_t$ alluded to before to first order in perturbation theory but not to second order and beyond. Indeed, for the computation $\langle \exp(-\int \lambda(z, \bar{z})O(z, \bar{z}) \, d\bar{z} \wedge dz)\rangle_{bc}$ at order $n$ in $\lambda$, one needs first to evaluate $\langle O(z_1, \bar{z}_1) \cdots O(z_n, \bar{z}_n)\rangle_{bc}$, which involves two boundary changing operators but $n$ bulk fields. So if $n = 1$ the differential equation coming from the fact that the boundary fields are degenerate at level 2 is enough to fix (almost) everything. But if $n > 2$ a detailed knowledge of the operator algebra of the theory, i.e. which states are allowed as intermediate states in a correlator, is required. One could restrict to perturbations of minimal models by minimal operators. Then the value of $\langle O(z_1, \bar{z}_1) \cdots O(z_n, \bar{z}_n)\rangle_{bc}$ can be expressed in terms of more and more complicated contour integrals. The explicit perturbative computation of $\langle \exp(-\int \lambda(z, \bar{z})O(z, \bar{z}) \, d\bar{z} \wedge dz)\rangle_{bc}$ looks even more formidable as it involves renormalization to remove singularities in the $(z, \bar{z})$ integrals. Anyway, many interesting perturbations are not generically by minimal operators, as shown be the example of the operator controlling the Hausdorff dimension of SLE, which means perturbing the SLE measure using the ‘natural’ length (i.e. the continuum limit of the discrete lattice length) of interfaces.

One word on our strategy. That an interface measure is the result of tracing over the other degrees of freedom of some statistical mechanics model yields some general compatibility conditions. At criticality, this is the clue for relating SLE to conformal field theory (CFT): via the growth process construction of interfaces, CFT becomes a provider of martingales for SLE, i.e. of observables which are conserved on average under the growth process. Out of criticality, the quantum field theory that describes macroscopic correlations in the system close to criticality should for the same reasons be a martingale provider for the corresponding interface measure. This is one hopes enough to characterize this measure. This is the approach that we follow in this paper, using the ratio of partition functions as the observable.

SLE is most simply formulated in the upper half-plane $\mathbb{H}$. There, it describes curves originating from a boundary point that we choose to be the origin 0 of the real axis. The curves $\gamma_{[0,t]}$, parameterized by $t$, are coded in a conformal map $g_t$ uniformizing $\mathbb{H} \setminus \gamma_{[0,t]}$ onto $\mathbb{H}$. To make this map unique we require that its behavior at infinity is $g_t(z) = z + 2t/z + O(z^{-2})$. This is called the hydrodynamic normalization with the parameter $t$ identified with half the capacity. The SLE measures are then defined by making the maps $g_t$ random and solutions of the stochastic Schramm–Loewner evolution:

$$\frac{dg_t(z)}{g_t(z) - \xi_t} = \frac{2 \, dt}{d\xi_t}$$

with $d\xi_t = \sqrt{\kappa} \, dB_t + F_0^t \, dt$ and $B_t$ a standard one-dimensional Brownian motion. The points of the curves are reconstructed from the maps $g_t$ via $\gamma_t = \lim_{\rho \to 0^+} g_t^{-1}(\xi_t + i\epsilon)$ and the measure on the curves is that induced via this reconstruction formula from the one on
the maps $g_t$. Above $F^0_t$ is a possible drift which depends on the variants of SLE that one is considering. Different variants of SLE correspond to the different boundary conditions that one imposes on the critical statistical models. As explained in [2,3] these drifts are intimately related to the partition functions of the conformal field theories describing the continuum limit of these statistical models.

Now look at perturbations away from criticality, with a perturbing parameter $m$ which may depend on position. In the sequel we denote by $P_m$ the corresponding measure on interfaces, so that $P_0$ is the critical measure. We assume that the perturbation simply modifies the drift (see the motivating discussion above, together with its ‘caveat’) so that the Schramm–Loewner stochastic equation is

$$d\xi_t = \sqrt{\kappa} dB_t^{[m]} + F_t^{[m]} dt$$

with $B_t^{[m]}$ another standard Brownian motion and the drift $F_t^{[m]}$ depending on the perturbation driving the systems out of criticality. However, contrary to the critical case, the off-critical drift $F_t^{[m]}$ at ‘time’ $t$ depends on the full past history of the curves$^6$.

If the drift $F_t^{[m]}$ is well-defined, then under regularity conditions, the off-critical measure can be shown to be regular with respect to the critical one and so the Radon–Nikodym derivative $dP_m / dP_0$ exists and the two measures differ by a density. In that case, expectation values of events depending only on the curves up to ‘time’ $t$ differ in the off-critical $E^{[m]}[\cdots]$ and critical $E[\cdots]$ measures by the insertion of a positive martingale:

$$E^{[m]}[\cdots] = E[Z_t^{[m]} \cdots].$$

Here, $Z_t^{[m]}$ has to be a positive martingale for the critical process. Its insertion reflects the difference between the Boltzmann weights of the underlying statistical model at criticality and away from it. Again its existence is not guaranteed. But in the favorable case, by Girsanov’s theorem [13,8], it is linked to the off-critical drift by $F_t^{[m]} - F^0_t = \kappa \partial_t \log Z_t^{[m]}$. The approach of [4] relates $Z_t^{[m]}$ to the ratio of partition functions of the quantum field theories describing the off-critical models in the continuum limit.

Determining the martingale $Z_t^{[m]}$ or the drift $F_t^{[m]}$—and proving that they make sense—is a significant step towards specifying what off-critical SLE is about. Of course it is only a first step and a lot would remain to be done to determine and compute properties of the off-critical curves. One of the obstacles is that we cannot rely on the Markov property of $\xi_t$ as in critical SLE.

The aim of this paper is to determine $Z_t^{[m]}$ and $F_t^{[m]}$ in two simple cases: massive SLE(2) and massive SLE(4).

Massive SLE(4) in its chordal version in $\mathbb{H}$ describes curves from 0 to $\infty$ in the upper half-plane. Its corresponding field theory is a massive Gaussian free field$^7$ which is of course a non-scale invariant perturbation—by a mass term—of the free field conformal

$^6$ A word of caution is needed here. This phenomenon also happens for variants of SLE like SLE$_{\kappa,\rho}$... but in these cases one can introduce a finite number of auxiliary random processes in such a way as to get a usual (vector) Markov process. It is doubtful that such a trick exists for off-critical interfaces.

$^7$ We choose a position dependent mass so that all statements established here in the case of the upper half-plane can be transported to any domain by conformal covariance. Under conformal transport by a map $g$ the mass is modified covariantly as $m(z) \rightarrow |g'(z)| m(z)$.
field theory associated with SLE(4). We prove that

\[ Z_t^{[m]} = \left[ \frac{\text{Det}[\Delta + m^2(z)]}{\text{Det}[\Delta]} \right]^{-\frac{1}{2}} \exp \left[ - \int \frac{d^2z}{8\pi} m^2(z) \varphi_t(z) \Phi_t^{[m]}(z) \right] \]

is a (local) martingale for critical chordal SLE(4). Here the determinants are determinants (regularized using \( \zeta \)-functions) of the massive and massless Laplacian in the cut domain \( \mathbb{H}_t \equiv \mathbb{H} \cup \gamma_{[0,t]} \) with Dirichlet boundary conditions and \( \varphi_t(z) \) and \( \Phi_t^{[m]}(z) \) are the one-point functions of the massless and massive free fields. They satisfy \([ -\Delta ] \varphi_t = 0 \) and \([ -\Delta + m^2 ] \Phi_t^{[m]} = 0 \) with appropriate discontinuous Dirichlet boundary conditions (with a discontinuity of \( \pi \sqrt{2} \) in our normalization). The off-critical drift for massive SLE(4) is

\[ F_t^{[m]} = -\sqrt{2} \int \frac{d^2z}{2\pi} m^2(z) \Theta_t^{[m]}(z) \varphi_t(z) = -\sqrt{2} \int \frac{d^2z}{2\pi} m^2(z) \theta_t(z) \Phi_t^{[m]}(z) \]

with \( \theta_t(z) \) and \( \Theta_t^{[m]}(z) \) the massless and massive Poisson kernel. See section 4 for details.

For this case, we have a satisfactory argument that the drift is indeed of locally finite variation so we are on the safe side of standard stochastic calculus. In particular, \( F_t^{[m]} \) is always non-negative.

This drift can also be found by demanding that the one-point function \( \Phi_t^{[m]}(z) \) is a martingale [11]. Let \( X \) be a Gaussian free field with discontinuous Dirichlet boundary condition: \( X = 0 \) on \( \mathbb{R}_+ \) and \( X = \pi \lambda_c \) on \( \mathbb{R}_- \) (with \( \lambda_c = \sqrt{2} \) in our normalization). We actually prove that any correlation function of \( X \) in the cut domain \( \mathbb{H}_t \), with an arbitrary number of marked points, is a local martingale for massive SLE(4). Pushing this result in the limit \( t \to \infty \) provides arguments for the decomposition of \( X \) as the sum of two independent Gaussian fields. Namely, at infinite time the curve \( \gamma_{[0,\infty)} \) almost surely reaches the boundary point at infinity\(^9\) and it separates the domain \( \mathbb{H} \) into two sub-domains \( \mathbb{H}_+ \) and \( \mathbb{H}_- \) with \( \mathbb{R}_+ \) part of the boundary of \( \mathbb{H}_\pm \). Conditioned on \( \gamma_{[0,\infty)} \), the field \( X \) can be written as the sum

\[ X = X_+ + X_- \]

with \( X_+ |_{\partial \mathbb{H}_+} = 0 \) and \( X_- |_{\partial \mathbb{H}_-} = \pi \lambda_c \).

where the fields \( X_\pm \), respectively restricted to \( \mathbb{H}_\pm \), are massive Gaussian free fields. Consequently, conditioned on \( \gamma_{[0,\infty)} \) the Gaussian measure for \( X \) can be factored as the product of the Gaussian measures for \( X_\pm \) and so

\[
\int_{X_\pm = \pi \lambda_c}^X D X e^{-S_{m^2}[X]} \cdot \cdot \cdot = E^{[m]} \int_{X_+ |_{\partial \mathbb{H}_+} = 0} D X_+ e^{-S_{m^2}[X_+]} \int_{X_- |_{\partial \mathbb{H}_-} = \pi \lambda_c} D X_- e^{-S_{m^2}[X_-]} \cdot \cdot \cdot
\]

for any observable \( \cdot \cdot \cdot \). Here \( S_{m^2} \) are the massive free field actions and \( E^{[m]} \) is the expectation with respect to the massive SLE(4) measure. This decomposition strongly indicates that the curve \( \gamma_{[0,\infty)} \) may be seen as the discontinuity curve of \( X \), as proved in [18] in the critical case. See figure 1.

\(^8\) Here and in the following, there is an implicit normalization constant to ensure that \( x_{t=0}^{[m]} = 1 \).

\(^9\) Here, we assume that this result proved in [14] for the critical SLE remains valid for massive SLE(4).
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Figure 1. Decomposition of the Gaussian measure as the product of two Gaussian measures defined on each of the curves times the massive SLE measure on the curves.

Note that the expectation of the free field, \( \langle X(z) \rangle^m \equiv \Phi_0^m(z) \), has a simple interpretation as \( \pi \lambda_c \) times the probability that the interface passes to the right of point \( z \).

SLE(2) is the continuum limit of critical loop erased random walks (LERW) as proved in the seminal Schramm paper [16]. Massive SLE(2) describes a deformation of LERW in which the fugacity attached to the underlying random walks has been moved away from criticality. See e.g. [4] for a more detailed introduction. Its associated field theory is that of a pair of massive symplectic fermions. We prove that, for any two marked points \( a \) and \( b \) on the real axis, \( \tilde{Z}_t^m = \left[ \frac{\text{Det}[-\Delta + m^2(z)]_{\mathbb{R}^2}}{\text{Det}[-\Delta]_{\mathbb{R}^2}} \right] \times \Gamma_{t,[a,b]}^m \) is a local martingale for critical chordal SLE(2). Here \( \Gamma_{t,[a,b]}^m \) is an appropriate limit of a massive Poisson kernel; see equation (12) and appendix C. At criticality, \( \Gamma_{t,[a,b]}^0 \) is the chordal SLE(2) martingale which intertwines chordal and dipolar SLE(2) (with marked points \( a \) and \( b \)). Hence, \( \tilde{Z}_t^m \) is the martingale intertwining critical chordal SLE(2) and massive dipolar SLE(2); i.e. it describes the massive deformation of dipolar SLE(2). The corresponding drift is

\[
F_{t,[a,b]}^m = 2 \partial_{\xi_t} \log \Gamma_{t,[a,b]}^m.
\]

This drift can alternatively be determined by requiring that correlation functions of the symplectic fermions are local martingales.

The ratio \( \frac{\Gamma_{0,[x,y]}^m}{\Gamma_{0,[a,b]}^m} \) is nothing but the probability that massive LERW dipolar aiming at \( [a, b] \) exits in the sub-interval \( [x, y] \subset [a, b] \).

The paper is organized as follows. In section 3 we recall basic facts about variants of critical SLE and about the formulation of off-critical SLE `à la Girsanov’ following [4]. In section 4 we study massive SLE(4). We first compute the drift using perturbation theory. We then prove non-perturbatively that \( \tilde{Z}_t^m \), defined above, is a chordal SLE(4) local martingale and re-derive the drift this way. We also prove that any correlation functions of the massive Gaussian field in the cut domain are martingales for massive SLE(4) and use this to derive the decomposition of \( X \) mentioned above. In section 5 we use massive symplectic fermions to compute the drift and we prove that \( \tilde{Z}_t^m \) is a critical chordal SLE(2) local martingale. We also check that correlation functions of symplectic fermions are massive SLE(2) local martingales and this provides another way to derive the off-critical drift. Appendices A and B are devoted to details concerning the computation of...
the Ito derivative of the determinants of the massive and massless Laplacian regularized using $\zeta$-functions.

3. SLE basics and notation

3.1. Chordal and dipolar SLEs

Here we recall the (by now standard) definition of SLE [16, 9]. We shall use two variants of SLE: chordal and dipolar. The former describes curves in a (planar) domain $\mathbb{D}$ from a boundary point to another boundary point, the latter describes curves in $\mathbb{D}$ from a boundary point to a sub-arc of the boundary of $\mathbb{D}$. In the following we choose $\mathbb{D}$ to be the upper half-plane $\mathbb{H} = \{ z \in \mathbb{C}, y = \text{Im} z > 0 \}$, but our statements may be transported to any planar simply connected domain by conformal covariance. In SLE, random curves $\gamma_{[0,t]}$, parameterized by $t > 0$, are coded into the conformal map which uniformizes $\mathbb{H}_t \equiv \mathbb{H} \setminus \gamma_{[0,t]}$ onto $\mathbb{H}$.

- Chordal SLE in $\mathbb{H}$ from 0 to $\infty$. The Loewner equation is
  \[ \frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \xi_t}, \]
  with initial condition $g_0(z) = z$ and $\xi_t = \sqrt{\kappa} B_t$ a Brownian motion with variance $\kappa$. The solution exists up to a time $t$ for $z \in \mathbb{H} \setminus \gamma_{[0,t]}$. The points of the curves are such that $g_t(\gamma_t) = \xi_t$. Furthermore, $g_t$ is the unique conformal map from $\mathbb{H} \setminus \gamma_{[0,t]}$ to $\mathbb{H}$ with the hydrodynamic normalization $g_t(z) = z + O(z^{-1})$, so that any property of $g_t$ reflects one of the curve $\gamma_{[0,t]}$. In particular, the measure on the curves is that induced by the Brownian motion.

- Dipolar SLE in $\mathbb{H}$ from 0 to $[a, b]$. It is a particularly symmetric case of SLE $(\kappa, \rho)$. The Loewner equation is
  \[ \frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad \frac{d\xi_t}{dt} = \sqrt{\kappa} dB_t + F^0_{t, [a, b]} \, dt, \]
  \[ F^0_{t, [a, b]} = \frac{(6 - \kappa)/2}{a_t - \xi_t} + \frac{(6 - \kappa)/2}{b_t - \xi_t}, \quad \frac{da_t}{dt} = \frac{2}{a_t - \xi_t}, \quad \frac{db_t}{dt} = \frac{2}{b_t - \xi_t}, \]
  that is $a_t = g_t(a)$ and $b_t = g_t(b)$. Dipolar SLE is defined up to time $T$ where $T > 0$ is the random stopping time such that $\gamma_T \in [a, b]$, i.e. the process is stopped at the moment it touches the interval $[a, b]$.

3.2. Intertwining variants of SLEs

Girsanov’s theorem describes the way stochastic equations are modified by insertions of martingale weights in the measure [13, 8]. It provides a way to intertwine stochastic equations with different drift terms. In the physics literature, this may be coded into the Martin–Siggia–Rose path integral representation of stochastic differential equations.

More precisely, let $B_t$ be a Brownian motion and $E[\cdots]$ the corresponding expectation. Let $M_t$ be a positive martingale with respect to $E[\cdots]$. To be a martingale implies that

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the Ito derivative of $M_t$ is proportional to $dB_t$, so we can write $M^{-1}_t \, dM_t = f_t \, dB_t$. Then Girsanov’s theorem tells us that with respect to the weighted measure $\tilde{\mathbb{E}}[\cdots] = \mathbb{E}[M_T \cdots]$, the process $B_t$, $t < T$, satisfies the stochastic differential equation

$$dB_t = \hat{\mathbb{B}}_t + f_t \, dt$$

where $\hat{B}_t$ is a Brownian motion with respect to $\tilde{\mathbb{E}}[\cdots]$. In other words, inserting a martingale adds a drift in the stochastic equation and reciprocally.

As an illustration, let us apply Girsanov’s theorem to intertwine from chordal SLE from $0$ to $\infty$ martingales of chordal SLE from $0$ to $[a, b]$. From the CFT/SLE correspondence \cite{1}, martingales of chordal SLE from $0$ to $\infty$ on $\mathbb{H}$ may be constructed as CFT correlation functions $\langle \mathcal{O} \psi(\gamma_t) \rangle_{\mathbb{H}}/\langle \psi(\infty) \psi(\gamma_t) \rangle_{\mathbb{H}}$ with $\psi$ the operator (with scaling dimension $(6 - \kappa)/2\kappa$) creating the curve and $\mathcal{O}$ any spectator operator. To go from chordal to dipolar SLE we need to choose $\mathcal{O} = \psi_{0,1/2}(a)\psi_{0,1/2}(b)$ with $\psi_{0,1/2}$ a primary operator of dimension $h_{0,1/2} = (\kappa - 2)(6 - \kappa)/16\kappa$. The result is the following chordal SLE martingale:

$$\Gamma_{t,[a,b]}^{0} = |g_1(a)g_1(b)|^{h_{0,1/2}}[b_t - a_t]^{((6 - \kappa)/2\kappa)}|\xi_t - a_t|^{((6 - \kappa)/2\kappa)}|\xi_t - b_t|^{((6 - \kappa)/2\kappa)}.$$

Its Ito derivative reproduces the dipolar drift:

$$\sqrt{\kappa} \Gamma_{t,[a,b]}^{0} - 1(d\Gamma_{t,[a,b]}^{0}/dB_t) = \Gamma_{t,[a,b]}^{0} = \frac{(6 - \kappa)/2}{b_t - \xi_t} + \frac{(6 - \kappa)/2}{a_t - \xi_t}.$$

This is simply found by computing the logarithmic derivative of $\Gamma_{t,[a,b]}^{0}$ with respect to $\xi_t = \sqrt{\kappa} B_t$.

### 3.3. Off-critical SLEs

We shall formulate off-critical SLE using the approach described in \cite{4} in which off-critical SLE is viewed as SLE twisted ‘à la Girsanov’ by a martingale, which we denote by $\hat{Z}_t^{[m]}$. The off-critical measure is then $\mathbb{E}^{[m]}[\cdots] = \mathbb{E}_{\text{SLE}}[\hat{Z}_t^{[m]} \cdots]$ so the insertion of the martingale $\hat{Z}_t^{[m]}$ amounts to weighting differently SLE configurations in a way reflecting the off-critical Boltzmann weights. The off-critical martingales are ratios of partition functions\textsuperscript{10}:

$$\hat{Z}_t^{[m]} = \frac{\hat{Z}_t^{[m]}}{\hat{Z}_t^{[m=0]}}$$

where $\hat{Z}_t^{[m]} = \frac{Z_t^{[m]} / Z_t^{[m=0]}}{Z_t^{[m=0]}}$ is the partition function of the off-critical model (for $m \neq 0$ but critical for $m = 0$) in the cut domain normalized by that in the upper half-plane. See \cite{4} for a more detailed introduction and for extra (lattice) motivations.

Computing these martingales by taking the scaling limit of the off-critical lattice model is an impossible task. In the continuous field theory they may naively be presented as expectation values

$$Z_{\mathbb{H}_t}^{[m]} = \left\langle \exp \left[ - \int_{\mathbb{H}_t} d^2 z \, m^2(z)\Phi(z) \right] \left( \text{’b.c.’} \right) \right\rangle_{\mathbb{H}_t}$$

where the brackets $\langle \cdots \rangle$ refer to critical CFT expectation values and the boundary conditions (‘b.c.’) are implemented by insertions of appropriate operators including the

\textsuperscript{10} As discussed in \cite{4}, there may also be an extra term in the formula for $Z_t^{[m]}$ corresponding to a surface energy associated with the interface. But we do not need to include it at this point of the discussion.
operators generating the curves. Of course this definition is plagued with infinities and needs regularization and renormalization. As a consequence of these infinities and of the fact that the perturbing weight \(\exp[-\int_{\mathbb{H}} d^2 z \, m^2(z)\Phi(z)]\) is not a local operator, it may turn out that \(Z_t^{[m]}\) is not a SLE martingale although it is naively expected to be one since it is an appropriate ratio of expectation values of CFT operators. See the relevant discussion for self-avoiding walks in [4].

One of the main aims and results of the following sections is giving a precise meaning to \(Z_t^{[m]}\) in the case of massive SLE(4) and SLE(2), and proving that they are (local) martingales.

Assuming that \(Z_t^{[m]}\) is a martingale, Girsanov’s theorem tells us that the driving source in the Loewner equation satisfies the stochastic equation

\[
d\xi_t = \sqrt{\kappa} dB_t^{[m]} + F_t^{[m]} dt, \quad \text{with} \quad \sqrt{\kappa} Z_t^{[m]} dZ_t^{[m]} = (F_t^{[m]} - F^0_t) dB_t
\]

with \(B_t^{[m]}\) a Brownian motion with respect to \(E^{[m]}[\cdots]\) and \(F^0_t\) the critical SLE drift.

In summary, off-critical SLEs may be defined using an appropriate martingale \(Z_t^{[m]}\), provided that \(Z_t^{[m]}\) is well-defined. (This is not always the case as for instance in near critical percolation [12].) Proving that it is a (local) martingale amounts to showing that the drift term in its Ito derivative vanishes. The drift term in the off-critical stochastic Loewner equation is then given by \(\sqrt{\kappa} Z_t^{[m]} dZ_t^{[m]}\).

4. Massive SLE(4)

We look at massive SLE(4) in the chordal setting describing curves from 0 to \(\infty\) in \(\mathbb{H}\). As shown by Sheffield and Schramm [18], samples of SLE(4) may be viewed as discontinuity lines of samples of a Gaussian massless free field. The aim of this section is to describe what happens to these lines when we consider a massive Gaussian free field.

4.1. SLE(4) and free massless boson

A Gaussian massless free field is a conformal field theory with central charge \(c = 1\). Denoting by \(X\) the free field, its action is

\[
S_0[X] = \int \frac{d^2 z}{2\pi} (\partial X)(z)(\bar{\partial} X)(z)
\]

with \(d^2 z\) the Lebesgue measure. For simplicity we first consider the system in the upper half-plane \(\mathbb{H}\),\(^{11}\) but we may extend our discussion to any domain by conformal covariance. We impose Dirichlet boundary conditions on the real axis \(\mathbb{R}\) with a discontinuity at the origin, so that \(X|_{\mathbb{R}_+}\) and \(X|_{-\mathbb{R}_+}\) are constants on the positive real axis and on the negative real axis respectively. The discontinuity at 0, \(X|_{\mathbb{R}_+} - X|_{-\mathbb{R}_+}\), is written as \(\pi \lambda_c\) and the constant \(\lambda_c\) will be fixed to the critical value \(\lambda_c = \sqrt{2}\) to ensure a perfect matching between chordal SLE(4) from 0 to \(\infty\) and the Gaussian massless free field.

Let us note that in the massless (critical) theory the symmetry \(X \to X + c \xi\) implies that only the value of the discontinuity at 0 matters, not the individual constants \(X|_{\mathbb{R}_+}\).

\(^{11}\) Points in the complex plane will be identified to complex numbers \(z = x + iy, \bar{z} = x - iy\) with \((x, y)\) real, \(y > 0\). We define \(\partial = \frac{1}{2}(\partial_x - i\partial_y)\) and \(\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)\). The Laplacian is \(\Delta = 4\partial \bar{\partial}\).
and $X|_{\mathbb{R}_-}$. This is not true when the perturbation is turned on and changing $X|_{\mathbb{R}_+}$ for fixed $\lambda_c$ changes the statistics of the interface. For compactness what follows is written assuming that $X|_{\mathbb{R}_+} = 0$, but all formulas below remain correct in the general case if the appropriate one-point (massive and massless) functions are used. The connected two-point functions are not affected by a translation of the boundary conditions.

With the boundary conditions, $X|_{\mathbb{R}_+} = 0$ and $X|_{\mathbb{R}_-} = \pi \lambda_c$ the massless one-and two-point functions are

$$\varphi_{\mathbb{H}}(z) \equiv \langle X(z) \rangle_{\mathbb{H}} = \lambda_c \Im \log z,$$

$$G_{\mathbb{H}}(z, w) \equiv \langle X(z)X(w) \rangle^c_{\mathbb{H}} = -\log \left| \frac{z - w}{z - \bar{w}} \right|^2,$$

where $\langle X(z)X(w) \rangle^c$ denotes the connected two-point function, defined as $\langle X(z)X(w) \rangle^c = \langle X(z)X(w) \rangle - \langle X(z) \rangle \langle X(w) \rangle$. Here $G_{\mathbb{H}}$ is the Green function of the Laplacian with Dirichlet boundary conditions: $-\Delta G_{\mathbb{H}}(z, w) = 4\pi \delta^{(2)}(z, w)$ with $\delta^{(2)}(\cdot, \cdot)$ the Dirac point measure.

In a maybe more probabilistic verbatim, $X$ may be viewed as a Gaussian distribution valued variable with characteristic function:

$$\langle e^{i(J, X)} \rangle_{\mathbb{H}} = \exp \left[ \int d^2 z J(z) \varphi_{\mathbb{H}}(z) + \frac{i}{2} \int d^2 z d^2 w J(z) G_{\mathbb{H}}(z, w) J(w) \right]$$

for any source $J(z)$ suitably well-behaved on the upper half-plane and $(J, X) = \int d^2 z J(z)X(z)$.

To couple this Gaussian massless free field to SLE(4) we consider its correlation functions in the domain $\mathbb{H}_t$ cut along a SLE sample: $\mathbb{H}_t \equiv \mathbb{H} \setminus \gamma_{[0, t]}$. Since $X$ is a scalar field, its expectation values in $\mathbb{H}_t$ are simply computed from those in $\mathbb{H}$ by conformal transport. If $h_t(z) \equiv g_t(z) - 2B_t$ denotes the uniformizing SLE(4) map from $\mathbb{H}_t$ onto $\mathbb{H}$ mapping the tip of the curve back to the origin, $h_t(\gamma_t) = 0$, we have

$$\varphi_t(z) \equiv \langle X(z) \rangle_{\mathbb{H}_t} = \varphi_{\mathbb{H}}(h_t(z)),
\quad G_t(z, w) \equiv \langle X(z)X(w) \rangle^c_{\mathbb{H}_t} = G_{\mathbb{H}}(h_t(z), h_t(w)).$$

As is known from the SLE/CFT correspondence [1], multi-point correlation functions of the Gaussian massless free field in the cut domain are SLE(4) (local) martingales. This is true for the one-point function, as can be checked by computing its Ito derivative,

$$d\varphi_t(z) = \lambda_c \theta_t(z) \, dB_t, \quad \theta_t(z) \equiv -\Im \frac{2}{h_t(z)},$$

but also for the non-connected two-point function iff $\lambda^2 = 2$, as follows from the Hadamard formula which gives the variation of the Green function:

$$dG_t(z, w) = -2 \theta_t(z) \theta_t(w) \, dt.$$

As a consequence since the theory is Gaussian, this is also true for the characteristic function for any source $J$ but in the cut domain $\mathbb{H}_t$, so

$$\langle e^{i(J, X)} \rangle_{\mathbb{H}_t}$$

is an SLE(4) martingale.

All multi-point correlation functions of $X$ in the cut domain are discontinuous along $\gamma_{[0, t]}$ with a jump of $\lambda_c$ indicating that effectively $\gamma_{[0, t]}$ is almost surely the discontinuity lines of $X$. Notice that this requires adjusting the Dirichlet discontinuity to its critical value $\lambda_c = \sqrt{2}$. 

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4.2. Massive perturbation

We consider perturbing the massless action by a mass term:

\[ S_{m^2}[X] = \int \frac{d^2z}{8\pi} \left[ 4(\partial X)(z)(\bar{\partial} X)(z) + m^2(z)X^2(z) \right]. \]

We assume the mass to be position dependent in order to avoid possible infrared (large distance) divergences and also to make the theory conformally covariant (but at the price of modifying the mass when implementing conformal transformations). As before we consider the theory on \( \mathbb{H}_t \) and write the correlation functions in the massive theory with the mass as an upper index, e.g. \( \langle O \rangle_{m}^{[\mathbb{H}_t]} \). We impose discontinuous Dirichlet boundary conditions as in the massless case. Namely: \( X = 0 \) and \( X = \pi \lambda_c \), respectively to the right and to the left of the tip of the curve \( \gamma_t \).

With this definition, the one-point function in the massive theory is

\[ \langle X(z) \rangle_{m}^{[\mathbb{H}_t]} = \Phi_t^{[m]}(z) \]

with \( \Phi_t^{[m]}(z) \) the solution of the classical equation of motion \( [-\Delta + m^2(z)]\Phi_t^{[m]}(z) = 0 \) with discontinuous Dirichlet boundary conditions as defined above. The connected two-point function is the massive Green function:

\[ \langle X(z) X(w) \rangle_{\mathbb{H}_t}^{[m]c} = G_t^{[m]}(z, w) \]

with \( [-\Delta + m^2(z)]\Theta_t^{[m]}(z, w) = 4\pi \delta^{(2)}(z, w) \) and \( G_t^{[m]}(z, w) = 0 \) for \( z \) or \( w \) on the boundary of \( \mathbb{H}_t \).

Alternatively, the massive Gaussian free field may be defined by its generating functions:

\[ \langle e^{(J, X)} \rangle_{\mathbb{H}_t}^{[m]} = \exp \left[ \int d^2z J(z)\Phi_t^{[m]}(z) + \frac{1}{2} \int d^2z d^2w J(z)G_t^{[m]}(z, w)J(w) \right] \]

for any source \( J(z) \).

An explicit expression for the massive classical solution \( \Phi_t^{[m]}(z) \) may be written in terms of the massless solution and the massive Green function:

\[ \Phi_t^{[m]}(z) = \varphi_t(z) - \frac{1}{4\pi} G_t^{[m]}(z, \cdot) \ast m^2(\cdot) \varphi_t(\cdot) \]

where \( \ast \) denotes the convolution product\(^{12}\). For later convenience we also need to introduce the so-called massive Poisson kernel defined similarly as

\[ \Theta_t^{[m]}(z) = \theta_t(z) - \frac{1}{4\pi} G_t^{[m]}(z, \cdot) \ast m^2(\cdot) \theta_t(\cdot). \]

Of course the massive Green function satisfies a convolution formula whose iteration reproduces the perturbative series.

\(^{12}\) The convolution is defined in the usual way: \( H(z, \cdot) \ast f(\cdot) = \int d^2z' H(z, z')f(z') \).
4.3. The off-critical drift

Recall from Girsanov’s theorem that the off-critical drift $F_t^{[m]}$ at $\kappa = 4$ is given by the Ito derivative of the partition function martingale: $F_t^{[m]} dB_t = 2 Z_t^{[m]} dZ_t^{[m]}$ with $Z_t^{[m]}$ the massive partition function (normalized by the massless one so that $Z_t^{[m=0]} = 1$) in the cut upper half-plane\(^\text{13}\):

$$Z_t^{[m]} \equiv \left\langle \exp \left[ - \int \frac{d^2 z}{8\pi} m^2(z) X^2(z) \right] \right\rangle_{\mathbb{H}_t}$$

where the expectation is with respect to the massless Gaussian measure.

We have to give a meaning to $X^2$. This is done via a point splitting subtraction of the logarithmic singularity in $X(z) X(w)$ as $w$ approaches $z$:

$$X^2(z) \equiv \lim_{w \to z} X(z) X(w) + \log|z - w|^2. \quad (4)$$

It is a local definition and insertions of $X^2$ are then well-behaved in any expectation values. With this definition, $Z_t^{[m]}$ is finite (in any order in perturbation theory).

4.3.1. First-order computation. To first order in perturbation theory, the massive partition function is

$$Z_t^{[m]} = 1 - \int \frac{d^2 z}{8\pi} m^2(z) \langle X^2(z) \rangle_{\mathbb{H}_t} + \cdots.$$ 

Although $X$ is a scalar—and thus it transforms as a scalar under conformal transformations—$X^2$ is not a scalar as the logarithmic subtraction in the point splitting definition produces an anomaly in its transformation laws. As a consequence its one-point function in the cut domain $\mathbb{H}_t$ is

$$\langle X^2(z) \rangle_{\mathbb{H}_t} = \varphi_t^2(z) + 2 \log \rho_t(z)$$

where $\rho_t(z) \equiv 2 \text{ Im } h_t(z)/|h_t(z)|$ is the conformal radius at $z$ which, by Kobe’s theorem, is an estimate of the distance between $z$ and the boundary of $\mathbb{H}_t$.

The formula for $\langle X^2(z) \rangle_{\mathbb{H}_t}$ has a nice probabilistic interpretation. By construction, $\varphi_t(z)$ is a SLE(4) martingale (recall that $d\varphi_t(z) = \lambda t \theta_t(z) dB_t$), but its square is not. However, as a CFT expectation value in $\mathbb{H}_t$, $\langle X^2(z) \rangle_{\mathbb{H}_t}$ is a SLE(4) martingale. So, $2 \log \rho_t(z)$ is what is needed to be added to $\varphi_t^2(z)$ to make it a martingale, i.e. its time derivative is the quadratic variation of $\varphi_t(z)$, provided (again) that $\lambda^2 = 2$. Explicitly $d \log \rho_t(z) = -(\text{Im } \varphi_t(z))/h_t(z))^2dt$. As a consequence,

$$d \langle X^2(z) \rangle_{\mathbb{H}_t} = 2 \varphi_t(z) d\varphi_t(z) = 2 \lambda \theta_t(z) \varphi_t(z) dB_t.$$

Computing the off-critical drift to first order is now very easy. We just have to Ito differentiate the partition function and, permuting integration and Ito derivative\(^\text{14}\), we get

$$d Z_t^{[m]} = -2 \lambda \int \frac{d^2 z}{8\pi} m^2(z) \theta_t(z) \varphi_t(z) dB_t + \cdots$$

where the dots refer to higher order term in the mass perturbation.

\(^{13}\) Here we assume (and we shall prove it in the following) that $Z_t^{[m]}$ is a SLE(4) martingale.

\(^{14}\) There is no problem in doing this permutation as the integrand is regular enough.
4.3.2. All order computation. Since the theory is Gaussian the partition function $Z_t^{[m]}$ can be computed to all orders. Let us assume for a while that this partition function is an SLE(4) martingale. This will be proved in the following section. To determine the drift we need to compute $Z_t^{[m]} - 1 \ dZ_t^{[m]}$. Since we only have to extract the term proportional to $dB_t$, which is a first-order term in the Ito derivative (the higher order terms in the Ito derivative would cancel as $Z_t^{[m]}$ is a martingale), it is enough to look at the first-order $dB_t$ term in $\log Z_t^{[m]}$. In perturbative expansion, $\log Z_t^{[m]}$ is the sum of the connected diagrams:

$$
\log Z_t^{[m]} = \sum_{n \geq 0} \frac{(-1)^n}{n!} \int \prod_{j=1}^{n} \frac{d^2z_j m^2(z_j)}{8\pi} \cdot \langle X^2(z_1) \cdots X^2(z_n) \rangle_{\approx \text{connected}}.
$$

There are two kinds of connected diagrams: (i) diagrams which produce terms like $G_t(z_1, z_2) \cdots G_t(z_{n-1}, z_n) G_t(z_n, z_1)$ up to permutations—there are $2^{n-1}(n-1)!$ such diagrams—and (ii) diagrams which produce terms like $\varphi_t(z_1) G_t(z_1, z_2) \cdots G_t(z_{n-1}, z_n) \varphi_t(z_n)$ up to permutations—there are $2^{n-1}n!$ such diagrams. Only diagrams of the second kind contribute to the $dB_t$ term in the Ito derivative because the first ones only involve the Green function. Using $d\varphi_t(z) = \lambda_c \theta_t(z) dB_t$ and summing up, we find

$$
Z_t^{[m]} - 1 \ dZ_t^{[m]} = dB_t \sum_{n \geq 1} (-2)^n \lambda_c \int \prod_{j=1}^{n} \frac{d^2z_j m^2(z_j)}{8\pi} \times \theta_t(z_1) G_t(z_1, z_2) \cdots G_t(z_{n-1}, z_n) \varphi_t(z_n).
$$

The sum reproduces the perturbative expansion of the massive Green function:

$$
-2\lambda_c \int \frac{d^2z}{8\pi} m^2(z) \theta_t(z) \left[ \varphi_t(z) - \frac{1}{4\pi} G_t^{[m]}(z, \cdot) \ast m^2(\cdot) \varphi_t(\cdot) \right]
$$

where again $\ast$ denotes convolution. We here recognize the solution of the classical equation of motion $\Phi_t^{[m]}$. Thus ($\lambda_c = \sqrt{2}$),

$$
Z_t^{[m]} - 1 \ dZ_t^{[m]} = -2\lambda_c \int \frac{d^2z}{8\pi} m^2(z) \theta_t(z) \Phi_t^{[m]}(z) \ dB_t.
$$

Since $G_t^{[m]}$ is symmetric, we can also write the drift as

$$
F_t^{[m]} = -2\sqrt{2} \int \frac{d^2z}{4\pi} m^2(z) \Theta_t^{[m]}(z) \varphi_t(z). \tag{5}
$$

Recall that $\sqrt{\kappa} Z_t^{[m]} - 1 \ dZ_t^{[m]} = F_t^{[m]} dB_t$. In the following sections we will see two different ways of obtaining this result.

### 4.4. Perfect matching and decomposition

From basic rules of statistical mechanics, we expect that massive correlation functions in the cut domain are martingales for massive SLEs. This is how Makarov and Smirnov computed the off-critical drift for massive SLE(4).\(^{15}\)

\(^{15}\) We thank S Smirnov for a discussion concerning this point.
Let us first look at the one-point function \( \langle X(z) \rangle_{H_t}^{[m]} \). This correlation function is the probability that the massive SLE curve passes to the right of point \( z \), conditioned on the beginning of the curve up to time \( t \). The argument leading to this result is the same as in the massless case and it uses the fact that \( \langle X(z) \rangle_{H_t}^{[m]} \) is a martingale. It is also positive. Thus Fubini’s theorem applies and we can

\[
\begin{align*}
\langle X(z) \rangle_{H_t}^{[m]} &= \Phi_t^{[m]}(z) = \varphi_t(z) - \frac{1}{4\pi} G_t^{[m]}(z, \cdot) \ast m^2(\cdot) \varphi_t(\cdot)
\end{align*}
\]

with \( \varphi_t(z) = \varphi_{2\pi}(h_t(z)) \). Computing its Itô derivative we have \( d\varphi_t(z) = \lambda_c \theta_t(z) dB_t \).

Recall that \( 2dB_t = 2dB_t^{[m]} + F_t^{[m]} dt \). To compute \( d\Phi_t^{[m]}(z) \) we need to known the derivative of the massive Hadamard formula. This is provided by the massive Hadamard formula (which follows for instance from the massless Hadamard formula and the convolution formula satisfied by the Green function):

\[
dG_t^{[m]}(z, w) = -2 \Theta_t^{[m]}(z) \Theta_t^{[m]}(w) dt.
\]

This gives (with \( \lambda_c = \sqrt{2} \))

\[
d\Phi_t^{[m]}(z) = \lambda_c \Theta_t^{[m]}(z) [dB_t^{[m]} + \frac{1}{2} F_t^{[m]} dt] + \Theta_t^{[m]}(z) dt \cdot \int \frac{d^2w}{4\pi} m^2(w) \Theta_t^{[m]}(w) \varphi_t(w).
\]

Hence, \( \Phi_t^{[m]}(z) \) is a \( P_m \) local martingale provided the drift is

\[
F_t^{[m]} = -2\sqrt{2} \int \frac{d^2w}{4\pi} m^2(w) \Theta_t^{[m]}(w) \varphi_t(w)
\]

which coincides with what we field-theoretically computed in the previous section. Notice that then

\[
d\Phi_t^{[m]}(z) = \lambda_c \Theta_t^{[m]}(z) dB_t^{[m]}.
\]

Consider now the two-point function \( \langle X(z)X(w) \rangle_{H_t}^{[m]} \) which is the sum of the product of two one-point functions plus the massive Green function. Thanks to the massive Hadamard formula and to the formula for \( d\Phi_t^{[m]}(z) \) it is then readily checked that \( \langle X(z)X(w) \rangle_{H_t}^{[m]} \) is a martingale (i.e. the drift term vanishes) provided that \( \lambda_c^2 = 2 \).

Since the theory is Gaussian, the fact that the one-and two-point functions are martingales implies that any \( n \)-point function is a local martingale. This is also true for the generating function:

\[
\langle e^{J(X)} \rangle_{H_t}^{[m]} \quad \text{is a } P_m\text{-SLE}(4) \text{ martingale}
\]

for any source \( J \) (with compact support, say). This was expected from naive statistical mechanics arguments. Statement (6) actually needs a few justifications because it applies to the exponential of the integral of a martingale. Consider first the integrated one-point function \( I_t \equiv \int d^2z J(z) \Phi_t^{[m]}(z) \). We know that \( \Phi_t^{[m]}(z) \) is a bounded local martingale and thus a martingale. It is also positive. Thus Fubini’s theorem applies and we can permute the \( d^2z \) integration and the expectation \( E^{[m]} \) which is enough to prove that \( I_t \) is a bounded martingale. Consider now the integrated two-point functions. \( I_t^2 \) is not a martingale but \( I_t^2 - \langle \delta I_t \rangle^2 \) is a martingale [13, 8]. This quadratic variation is bilinear in the current \( J \). Considering \( J \) as equal to a sum, a weighted Dirac measure localized at arbitrary points then determines this bilinear form.
and \((\delta I_t)^2 = -\int d^2z d^2w J(z) \Delta G_t^0(z, w) J(w)\) with \(\Delta G_t^0 \equiv G_t^0 - G_0^0\). Finally, the exponential \(\langle e^{(J,X)} \rangle_t^m = e^{t(1/2)(\delta I_t)^2}\) is a bounded local martingale and thus a martingale.

We now use the property (6) to derive the decomposition of \(X\) mentioned in section 1. In the limit \(t \to \infty\), this property gives that
\[
E^m\left[\langle e^{(J,X)} \rangle_t^m \right] = \langle e^{(J,X)} \rangle_\infty^m
\]
where \(E^m\) is the massive SLE(4) measure on the complete curve \(\gamma_{[0,\infty]}\). Almost surely (this was proved in the critical case but we assumed that it is still true in the massive case), the curve \(\gamma_{[0,\infty]}\) reaches infinity and cuts the domain \(\mathbb{H}\) into two parts \(\mathbb{H}_+\) and \(\mathbb{H}_-\) whose boundaries are respectively \(\mathbb{R}_+\) (or \(\mathbb{R}_-\)) and the right \(\gamma_{[0,\infty]}^+\) (or the left \(\gamma_{[0,\infty]}^-\)) side of the curve. The expectations \(\langle e^{(J,X)} \rangle_t^m\) are fully determined by the limiting behavior as \(t \to \infty\) of the one- and two-point functions. Almost surely, we have
\[
\lim_{t \to \infty} \Phi_t^m(z) = \begin{cases} 
0, & z \in \mathbb{H}_+ \\
\pi \sqrt{2}, & z \in \mathbb{H}_-
\end{cases}
\]
and
\[
\lim_{t \to \infty} G_t^m(z, w) = \begin{cases} 
G_{\mathbb{H}_+}^m(z, w), & z, w \in \mathbb{H}_- \\
0, & z \in \mathbb{H}_-, w \in \mathbb{H}_+ \\
G_{\mathbb{H}_-}^m(z, w), & z, w \in \mathbb{H}_+
\end{cases}
\]
where \(G_{\mathbb{H}_+}^m\) are the massive Green functions in the two sub-domains \(\mathbb{H}_\pm\) with Dirichlet boundary conditions. If these limits exist their values can only be those written above because of the differential equations that they satisfy. So we only have to argue that they exist. In the massless case, convergence of the one-point function was proved in [17] on the basis of the fact that \(\varphi_t(z)\) is proportional to the harmonic measure of \(\mathbb{R}_- \cup \gamma_{[0,t]}\) viewed from \(z\). Convergence of the massless Green function is based on the fact that \(G_t^0\) and \(G_{\mathbb{H}_+}^0\) are solutions of the same differential equations with slightly different boundary conditions but whose difference converges to zero as \(t \to \infty\). Let us sketch the argument. Assume for instance that \(z, w \in \mathbb{H}_+\) and consider the differences \(G_t^0 - G_{\mathbb{H}_+}^0\) and \(G_{\mathbb{H}_+}^0 - G_{\mathbb{H}_-}^0\), say as functions of \(z\) at \(w\) fixed. The first one is harmonic on \(\mathbb{H}_+\); it reaches its maximum on the boundary \(\partial \mathbb{H}_+\) and this maximum is bounded by \(\max \gamma_{[0,\infty]} G_{\mathbb{H}_-}^0\). The second one is harmonic on \(\mathbb{H}_+\); it reaches its maximum on the boundary \(\partial \mathbb{H}_+\) which is therefore also bounded by \(\max \gamma_{[0,\infty]} G_{\mathbb{H}_-}^0\). Hence, the difference \(G_t^0 - G_{\mathbb{H}_+}^0\) is harmonic on \(\mathbb{H}_+\), with boundary condition bounded by \(2 \max \gamma_{[0,\infty]} G_{\mathbb{H}_-}^0\) and non-vanishing only a sub-arc of the boundary of the domain vanishing as \(t \to \infty\) (because almost surely the curve \(\gamma_{[0,\infty]}\) goes to infinity). Similar arguments apply for \(z \in \mathbb{H}_-\) and \(w \in \mathbb{H}_-\). The functional relations satisfied by the massive and the massless Green functions and the one-point functions imply that once the statement is proved for the massless quantities it is also true for the massive one.

As a consequence, \(\langle e^{(J,X)} \rangle_t^m\) factors into the product of expectations in the two sub-domains, as expected:
\[
\langle e^{(J,X)} \rangle_t^m = \langle e^{(J,X)} \rangle_{\mathbb{H}_+}^m \times \langle e^{(J,X)} \rangle_{\mathbb{H}_-}^m
\]
In each sub-domain, the correlation functions are those of a Gaussian free field with Dirichlet boundary conditions 0 in $\mathbb{H}_+$ and $\pi \sqrt{2}$ in $\mathbb{H}_-$. That is: conditioned on $\gamma_{[0,\infty)}$ the field $X$ can be decomposed as the sum $X = X_+ + X_-$ of two Gaussian fields $X_\pm$ respectively defined on $\mathbb{H}_\pm$ with Dirichlet boundary conditions (0 in $\mathbb{H}_+$ and $\pi \sqrt{2}$ in $\mathbb{H}_-$), as mentioned in the introduction.

### 4.5. Partition functions and the off-critical martingale

We have seen in section 4.3 that as a consequence of Girsanov’s theorem we can compute the off-critical drift by taking the Ito derivative of the ratio of massive and massless partition functions with discontinuous Dirichlet boundary conditions. This can be written as a correlation function in the massless theory:

$$Z_t^{[m]} = \exp \left[ - \int \frac{d^2z}{8\pi} m^2(z) X^2(z) \right]_{H_t}.$$  

The usual heuristic arguments from statistical mechanics tell us that this is a martingale for the critical SLE. Actually, since both the massless and the massive theories are Gaussian, one can compute their partition functions in a fully non-perturbative way. This allows us to prove rigorously that the ratio of the massive/massless partition functions is a (local) martingale for the critical SLE. Actually, since both the massless and the massive theories are martingales for the critical measure and at the same time to compute the off-critical drift. The simplest way to proceed is by first decomposing $X$ as the sum of its one-point function plus a Gaussian field $\tilde{X}$ with zero Dirichlet boundary conditions. In the cut domain $\mathbb{H}_t$ this reads $X = \varphi_t + \tilde{X}$. Notice that this decomposition is done on the massless Gaussian field as the partition function is defined via an expectation value in the massless theory. Then $X^2 = \varphi_t^2 + 2\varphi_t \tilde{X} + \tilde{X}^2$ (with $\tilde{X}^2$ defined with a similar point splitting regularization) and the expectation value can be reduced to an expectation value in the boundary zero Gaussian field. Thus,

$$Z_t^{[m]} = Z_t^{[m];\tilde{X}} \cdot \exp \left( - \int \frac{d^2z}{8\pi} m^2(z) \varphi_t^2(z) \right) \exp \left( -2 \int \frac{d^2z}{8\pi} m^2(z) \varphi_t(z) \tilde{X}(z) \right)_{H_t}.$$  

Here $Z_t^{[m];\tilde{X}}$ is the partition function (relative to the massless theory) of the massive boundary zero Gaussian field and the last expectation value is an expectation value in the massive boundary zero Gaussian field. It is thus equal to

$$\exp \left[ \frac{1}{2} \int \frac{d^2z}{4\pi} \frac{d^2w}{4\pi} m^2(z) \varphi_t(z) G_t^{[m]}(z, w) m^2(w) \varphi(w) \right].$$  

The integration over $w$ involves the convolution of $G_t^{[m]}(z, \cdot)$ with $m^2(\cdot) \varphi_t(\cdot)$ which, combined with the function $\varphi_t(z)$ in the second factor of the previous expression of the partition function, reproduces the massive classical solution $\Phi_t^{[m]}(z)$. Hence,

$$Z_t^{[m]} = Z_t^{[m];\tilde{X}} \cdot \exp \left[ - \int \frac{d^2z}{8\pi} m^2(z) \varphi_t(z) \Phi_t^{[m]}(z) \right].$$  

(7)

The partition function $Z_t^{[m];\tilde{X}}$ is the ratio of the square roots of the determinants of the massive and massless Laplacian with Dirichlet boundary conditions:

$$Z_t^{[m];\tilde{X}} = \left[ \frac{\text{Det}[\Delta + m^2(z)]}{\text{Det}[\Delta]} \right]_{H_t}^{-1/2}.$$  

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4.6. A representation of the partition function

To arrive at an alternative representation of the partition function, let us introduce a fictitious parameter $\tau$ multiplying $m^2(z)$, and consider the path integral representation of the determinant of the massive Laplacian with Dirichlet boundary conditions. Taking the derivative with respect to $\tau$ we get

$$\frac{d}{d\tau} \text{Det}[\Delta + \tau m^2(z)]\bigg|_{\|}\overset{\text{(1/2)}}{=} - \int D\tilde{X} \left( \int \frac{d^2z}{8\pi} m^2(z) \tilde{X}^2(z) \right) e^{-S_{\tau m^2}[\tilde{X}]}.$$  

Hence

$$\frac{d}{d\tau} \log Z_t^{[\tau m];\tilde{X}} = - \int \frac{d^2z}{8\pi} m^2(z) \langle \tilde{X}^2(z) \rangle_{\|^{\sqrt{\tau m}}}.$$  

(8)

Of course this result is only formal. We have given no prescription for how to regularize the composite operator $\tilde{X}^2(z)$. The proper computation, which is done in appendix A, uses the definition of the functional determinant through the $\zeta$-function regularization. It turns out that—up to an irrelevant term proportional to $\int (d^2z/4\pi)m^2(z)$—the $\zeta$-function regularization corresponds to the point splitting regularization of $\tilde{X}^2(z)$ (as done in the perturbative computation of section 4.3; see equation (4)):

$$\langle \tilde{X}^2(z) \rangle_{\|^{\sqrt{\tau m}}} = \lim_{z' \to z} \langle \tilde{X}(z') \tilde{X}(z') \rangle_{\|^{\sqrt{\tau m}}} + \log |z' - z|^2.$$  

Integrating back equation (8) and inserting the expression for $\langle \tilde{X}^2(z) \rangle_{\|^{\sqrt{\tau m}}}$ we arrive at

$$\log Z_t^{[\tau m];\tilde{X}} = - \int \frac{d^2z}{8\pi} m^2(z) \left[ \log |\rho_t(z)|^2 + \int_0^1 K_t^{|\sqrt{\tau m}|}(z) \, d\tau \right]$$  

(9)

where

$$K_t^{[\tau m]}(z) \equiv \lim_{z' \to z} G_t^{[\tau m]}(z', z) - G_t^{[0]}(z', z) = - \int \frac{d^2z'}{4\pi} G_t^{[\tau m]}(z, z') m^2(z') G_t^{[0]}(z', z),$$  

(10)

and the integrals are convergent.

4.6.1. Proof that $Z_t^{[\tau m]}$ is a martingale. To prove that $Z_t^{[\tau m]}$ is a local martingale, we use its representation in equation (9) and compute its Ito derivative. Evaluating separately the Ito derivatives of $Z_t^{[\tau m];\tilde{X}}$ and of $\exp[-\int (d^2z/8\pi)m^2(z) \varphi_t(z) \Phi_t^{[\tau m]}(z)]$ would lead to the appearance of diverging integrals. In order to avoid this problem we perform a slightly different splitting by extracting the logarithm of the conformal radius from formula (9) and putting it together with $Z_t^{[\tau m];\tilde{X}}$. We therefore write

$$Z_t^{[\tau m]} = \hat{Z}_t^{[\tau m];\tilde{X}} Y_t$$

where we have defined

$$Y_t \equiv \exp \left[ - \int \frac{d^2z}{8\pi} m^2(z) (\varphi_t(z) \Phi_t^{[\tau m]}(z) + \log |\rho_t(z)|^2) \right]$$

and

$$\hat{Z}_t^{[\tau m];\tilde{X}} \equiv \exp \left[ - \int \frac{d^2z}{8\pi} m^2(z) \left( \int_0^1 K_t^{|\sqrt{\tau m}|}(z) \, d\tau \right) \right].$$

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We first compute the Ito derivative of $Y_t$. From equation (2), we know that $d\varphi_t(z) = \lambda_t \theta_t(z) dB_t$. Using the Hadamard formula, we obtain $d\Phi_t^{[m]}(z) = \lambda_t \Theta_t^{[m]}(z) dB_t - \frac{d}{2} F_t^{[m]} dt$ with $\lambda_t F_t^{[m]} = -2 \int (d^2 z / 2 \pi) m^2(z) \varphi_t(z) \Theta_t^{[m]}(z)$. The last piece of information that we need is $d \log \rho_t(z) = -\theta_t^2(z) dt$. The result for the Ito derivative of $Y_t$ is

$$Y_t^{-1} dY_t = \frac{1}{2} F_t^{[m]} dB_t - 2N_t dt.$$  

The drift term $-2N_t dt$ comes form the second-order (crossed) term when computing the Ito derivative of $Y_t$ and reads

$$N_t = \int \frac{d^2 z}{16 \pi} m^2(z) \left[ \lambda^2 \theta_t(z) \Theta_t^{[m]}(z) - 2 \theta_t(z)^2 \right].$$

Actually the integral defining $N_t$ does not diverge at $t = 0$ for $\lambda^2 = 2$, which coincides with the value previously determined by other considerations. Hence setting $\lambda_c = \sqrt{2}$ we have

$$N_t = -2 \int \frac{d^2 z}{8 \pi} \frac{d^2 z'}{8 \pi} m^2(z) m^2(z') \theta_t(z) \theta_t(z') G_t^{[m]}(z', z).$$

Although it is the main result of this section, the computation of the derivative of $\tilde{Z}_t^{[m] : \bar{X}}$ is not particularly illuminating and we report it in appendix B. Its Ito derivative does not contain any ‘$d B_t$’ terms and there is only a drift term. The result is

$$d \log \tilde{Z}_t^{[m] : \bar{X}} = 2N_t dt.$$  

(11)

This drift compensates that of $Y_t$ and we thus find that $Z_t^{[m]}$ is a local martingale:

$$Z_t^{[m]}^{-1} dZ_t^{[m]} = \frac{1}{2} F_t^{[m]} dB_t, \quad \text{with } F_t^{[m]} = -2 \sqrt{2} \int \frac{d^2 z}{4 \pi} m^2(z) \theta_t(z) \Phi_t^{[m]}(z).$$

In summary, $Z_t^{[m]}$ is a local chordal SLE martingale and, if used as a massive perturbation, the associated massive drift is $F_t^{[m]}$, defined above. Let us note here that this drift is always non-positive if the function $m$ is non-negative. Indeed, $\theta_t(z) = -\Im(2 / h_t(z))$ is positive everywhere. As for $m^2(z) \Phi_t^{[m]}(z)$, it is non-negative on the boundary, so if it assumed some negative values, it would have some negative absolute minimum inside the domain (if $m$ has compact support, in particular if the domain is bounded). At such a minimum, $-\Delta \Phi_t^{[m]}(z)$ is non-positive and $m^2(z) \Phi_t^{[m]}(z)$ is negative, contradicting the defining equation $(-\Delta + m^2(z)) \Phi_t^{[m]}(z) = 0$. Hence $F_t^{[m]}$ is the integral of a non-positive function. In particular, the driving process $\xi_t$ in the Loewner equation is a super-martingale and we have obtained its (so-called Doob–Meyer) decomposition as a sum of a martingale and a decreasing process explicitly. A concrete interpretation of this decreasing process, even at small $m^2$ (the first order in perturbation theory), would probably be of interest.

5. Massive dipolar LERW

In [4] the massive drift for dipolar LERWs has been computed to first order in the mass perturbation. This has been done in two different ways. The first one was by looking at the sub-interval hitting probability, i.e. the probability that a LERW from $x_0$ to the

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interval \([a, b]\) ends on the sub-interval \([x, y]\). Requiring this probability to be a martingale for massive \(\text{SLE}(2)\) gives (perturbatively in the mass) the drift. The second approach goes through Girsanov’s formula, as explained in section 3 in the case of the Gaussian free field.

5.1. Discrete massive LERW

For the convenience of the reader, we recall here the basic definitions for (massive) loop erased random walks.

Let us first recall the definition of a LERW. Let us start with a lattice of mesh \(a\) embedded in a domain. Given a path \(W = (W_0, W_1, \ldots, W_n)\) on the lattice its loop erasure \(\gamma\) is defined as follows: let \(n_0 = \max\{m: W_m = W_0\}\) and set \(\gamma_0 = W_{n_0} = W_0\), next let \(n_1 = \max\{m: W_m = W_{n_0+1}\}\) and set \(\gamma_1 = W_{n_1}\), and then inductively let \(n_{j+1} = \max\{m: W_m = W_{n_j'+1}\}\) and set \(\gamma_j = W_{n_j}\). This produces a simple path \(\gamma = \mathcal{L}(W) = (\gamma_0, \gamma_1, \ldots, \gamma_l)\) from \(\gamma_0 = W_0\) to \(\gamma_l = W_n\), called the loop erasure of \(W\), but its number of steps \(l\) is in general much smaller than that of the original path \(W\). We emphasize that the starting and end points are not changed by the loop erasing.

We point out that the above definition of loop erasure is equivalent to the result of a recursive procedure of chronological loop erasing: the loop erasure of a zero-step path \((W_0)\) is itself \(\gamma = (W_0)\) and if the erasure of \(W = (W_0, \ldots, W_m)\) is the simple path \(\mathcal{L}(W) = (\gamma_0, \ldots, \gamma_l)\) then for the loop erasure of \(W' = (W_0, \ldots, W_m, W_{m+1})\) there are two cases depending on whether a loop is formed on step \(m + 1\). If \(W_{m+1} \notin \{\gamma_0, \ldots, \gamma_l\}\) then the loop erasure of \(W'\) is \(\gamma' = (\gamma_0, \ldots, \gamma_l, W_{m+1})\). But if a loop is formed, \(W_{m+1} = \gamma_k\) for some \(k \leq l\) (unique because \(\gamma\) is simple), then the loop erasure of \(W'\) is \(\gamma' = (\gamma_0, \ldots, \gamma_k)\).

In this paper we shall be interested in paths starting at a boundary point \(x_0\) and ending on a subset \(S\) of the boundary of \(\mathbb{D}\).

The statistics of LERW is defined by associating with any simple path \(\gamma\) a weight \(w_\gamma = \sum_{W: \mathcal{L}(W) = \gamma} \mu^{|W|}\), where the sum is over all nearest neighbor paths \(W\) whose erasures produce \(\gamma\), and \(|W|\) denotes the number of steps of \(W\). There is a critical value \(\mu_c\) of the fugacity at which the underlying paths \(W\) become just ordinary random walks. The partition function \(\sum_\gamma w_\gamma\) of LERWs from \(z\) to \(S\) in \(\mathbb{D}\) can be rewritten as a sum over walks in the domain \(\mathbb{D}\), starting from \(z\) and counting only those that exit the domain through \(S\):

\[
Z_{\text{RW}; S}^{\mathbb{D}; z} = \sum_{\text{\gamma simple path from } z \text{ to } S \text{ in } \mathbb{D}} w_\gamma = \sum_{\text{walk from } z \text{ to } S \text{ in } \mathbb{D}} \mu^{|W|}.
\]

Written in terms of critical random walks, the partition function thus reads \(Z_{\text{RW}}^{\mathbb{D}; z}[(\mu/\mu_c)^{|W|}] 1_{W_{\tau_{\text{RW}}}}\), where \(\tau_{\text{RW}}^\mathbb{D}\) denotes the exit time of the random walk \(W\) from \(\mathbb{D}\).

Critical LER\(\overline{\text{W}}\) corresponds to the critical fugacity and is described by \(\text{SLE}_2\); see \([16, 10, 19]\). For \(\mu < \mu_c\)—which is the case that we shall consider—paths of small lengths are more favorable and renormalization group arguments tell us that at large distances the path of smallest length dominates. The off-critical theory in the scaling regime corresponds to non-critical fugacity \(\mu\) but approaching the critical one as the mesh size tends to zero. At fixed typical macroscopic size, the number of steps of typical critical random walks (not of their loop erasures) scales as \(a^{-2}\), so the scaling limit is such that \(\rho := -a^2 \log(\mu/\mu_c)\) is finite as \(a \to 0\), i.e. \((\mu - \mu_c)/\mu_c \simeq -\rho a^2\) and \(\rho\) has scaling dimension
2 and fixes a mass scale $m^2 \simeq \rho$ and a correlation length $\zeta \simeq 1/m$. In this scaling limit the weights become $(\mu / \mu_c)^{|W|} \simeq e^{-a^2 |W|}$ and the random walks converge to two-dimensional Brownian motions $B$ with $a^2 |W| = a^2 \tau_D$ converging to the times $\tau_D$ spent in $\mathbb{D}$ by $B$ before exiting. The off-critical partition function can thus be written as a Brownian expectation value $Z^\text{off-critical} \rightarrow \mathbb{E}_B[e^{-\rho \tau} 1_{B_{\tau_D} \in S}]$ as $a \downarrow 0$. We may generalize this by letting $\rho$ vary in space: steps out of site $w \in \mathbb{D}$ are given weight factor $\mu(w) = \mu_c e^{-a^2 \rho(w)}$, in which case the partition function is a random walk expectation value

$$Z^\text{off-critical}_\rho = \mathbb{E}_\text{RW}^z \left[ \exp \left( - \sum_{0 \leq j < \tau_\text{RW}} a^2 \rho(W_j) \right) 1_{W_{\tau_\text{RW}} \in S} \right]$$

$$- \sum_{a \geq 0} \mathbb{E}_\text{BM}^z \left[ \exp \left( - \int_0^{\tau_D} \rho(B_s) \, ds \right) 1_{B_{\tau_D} \in S} \right].$$

The explicit weighting by $e^{-\rho \tau_D}$ is transparent for the random walk, but becomes less concrete for the LERW since the same path $\gamma$ can be produced by random walks of different lengths and by walks that visit different points.

### 5.2. Continuous massive LERW

As argued in [4], the field theory corresponding to (massive) LERW is that of free massive symplectic fermions $\chi^+, \chi^-$, with action

$$S_{sf}[\chi^+] = \int d^2 z \left( 4 \partial \chi^+ \partial \chi^- - m^2(\chi^+ \chi^-) \right).$$

Both in the massless and in the massive case, the partition function corresponding to dipolar SLEs can be expressed in terms of correlation functions of boundary fields creating/annihilating the curve: $\psi^\pm(x) \equiv \lim_{z \to 0} \delta^{-1} \chi^\pm(x + i \delta)$. As a consequence, Girsanov’s martingale for massive dipolar SLE from 0 to $[a, b]$ reads

$$Z_t^{[m]} = \left[ \frac{\text{Det}[-\Delta + m^2(z)]_{\mathbb{H}_t}}{\text{Det}[-\Delta]_{\mathbb{H}_t}} \right] \frac{\langle \psi^+(\gamma_t) \int_a^b dx \psi^-(x) \rangle_{\mathbb{H}_t}^{[m]}}{\langle \psi^+(\gamma_t) \int_a^b dx \psi^-(x) \rangle_{\mathbb{H}_t}^{[m=0]}}$$

where the correlation function in the numerator is computed in the massive theory, while the one in the denominator is computed in the massless theory. The determinants are $\zeta$-regularizations of determinants for the (massive) Laplacian with Dirichlet boundary conditions.

By definition of the curve-creating fields $\psi^\pm$, this ratio of correlation functions is defined by a limiting procedure:

$$\frac{\langle \psi^+(\gamma_t) \int_a^b dx \psi^-(x) \rangle_{\mathbb{H}_t}^{[m]}}{\langle \psi^+(\gamma_t) \int_a^b dx \psi^-(x) \rangle_{\mathbb{H}_t}^{[m=0]}} = \lim_{z \to 0} \frac{\Psi_t^{[m]}_{l(a, b)}(z)}{\Psi_t^{[0]}_{l(a, b)}(z)}$$

where $\Psi_t^{[m]}_{l(a, b)}(z) = \langle \chi^+(z) \int_a^b dx \psi^-(x) \rangle_{\mathbb{H}_t}^{[m]}$. By construction, $\Psi_t^{[m]}_{l(a, b)}(z)$ satisfies the massive Laplace equation $(-\Delta + m^2(z))\Psi_t^{[m]}_{l(a, b)}(z) = 0$ with specific boundary conditions. This allows us to write it in terms of the massless correlation function $\Psi_t^{[0]}_{l(a, b)}(z)$ and of the
massive Green function $G_t^{[m]}(z, w)$. We may then take the limit $z \to \gamma_t$ as the limit $g_t(z) \to \xi_t$, so this ratio becomes

$$\frac{\langle \psi^+(\gamma_t) \int_a^b \mathrm{d}x \, \psi^-(x) \rangle_{[m]}}{\langle \psi^+(\gamma_t) \int_a^b \mathrm{d}x \, \psi^-(x) \rangle_{[0]}} = \frac{\Gamma^{[m]}_{t, [a, b]}}{\Gamma^{[0]}_{t, [a, b]}}$$

with

$$\Gamma^{[m]}_{t, [a, b]} = \Gamma^{[0]}_{t, [a, b]} - \int \frac{\mathrm{d}^2 z}{4\pi} m^2(z) \Theta^{[m]}_t(z) \Psi^{[0]}_{t, [a, b]}(z) \quad (12)$$

where $\Theta^{[m]}_t(\cdot)$ is the massive Poisson kernel. From this expression, we see that $\Gamma^{[m]}_{t, [a, b]}$ depends explicitly on $\xi_t$, on $a = g_t(a)$ and $b = g_t(b)$ and on $t$. When computing its Ito derivative, only the explicit dependence on $\xi_t$ contributes to the ‘d$B_t$’ term; the rest contributes to the ‘dt’ term. See appendix C for the definitions of $\Gamma^{[0]}_{t, [a, b]}$ and $\Psi^{[0]}_{t, [a, b]}(z)$ and more details.

From Girsanov’s theorem, $\sqrt{2} \mathbb{Z}_t^{[m]} - 1 \mathrm{d} \mathbb{Z}_t^{[m]}$ gives the additional drift due to the massive perturbation. As explained in section 2, the critical drift $F_t^{[0]}_{t, [a, b]}$ derives from the critical chordal SLE martingale $\Gamma^{[0]}_{t, [a, b]}$, which intertwines dipolar and chordal SLEs. Therefore, $\mathbb{Z}_t^{[m]}$ is a dipolar martingale whenever $\mathbb{Z}_t^{[m]} \equiv \mathbb{Z}_t^{[0]} \Gamma^{[0]}_{t, [a, b]}$ is a chordal martingale. Explicitly,

$$\mathbb{Z}_t^{[m]} = \left[ \frac{\det[-\Delta + m^2(z)]}{\det[-\Delta]} \right]_{[a, b]} \Gamma^{[m]}_{t, [a, b]} \quad (13)$$

Let $\tilde{B}_t$ be the Brownian motion associated with the critical chordal LERW (not that of the dipolar LERW). The massive dipolar drift is then

$$\mathrm{d}\xi_t = \sqrt{2} \mathrm{d} \tilde{B}_t^{[m]} + F_t^{[m]}_{t, [a, b]} \, \mathrm{d}t, \quad \sqrt{2} \mathbb{Z}_t^{[m]} - 1 \, \mathrm{d} \mathbb{Z}_t^{[m]} = F_t^{[m]}_{t, [a, b]} \, \mathrm{d} \tilde{B}_t,$$

where $B_t^{[m]}$ is a Brownian motion with respect to the off-critical measure $\mathbb{E}^{[m]}[\cdot]$.

In order to avoid infinities appearing like for the Gaussian free field, we consider the Ito derivative of the product $\Gamma^{[m]}_{t, [a, b]} e^{J_t}$ where

$$J_t = \int \frac{\mathrm{d}^2 z}{4\pi} m^2(z) \log |\rho_t(z)|^2.$$

The computation of this derivative which is again based on the Hadamard formula is reported in appendix C. It reads

$$\mathrm{d} \left[ \Gamma^{[m]}_{t, [a, b]} e^{J_t} \right] = \Gamma^{[m]}_{t, [a, b]} e^{J_t} \left[ \sqrt{2} \left( \partial_{\xi_t} \log \Gamma^{[m]}_{t, [a, b]} \right) + 4N_t \, \mathrm{d}t \right] \quad (14)$$

where

$$N_t = \int \frac{\mathrm{d}^2 z}{8\pi} m^2(z) \left[ \Theta^{[m]}_t(z) \theta_t(z) - \theta_t^2(z) \right]$$

is the same quantity that we have encountered in section 4.5. The key point here is that the drift term in $\mathrm{d}[\Gamma^{[m]}_{t, [a, b]} e^{J_t}]$ is $4N_t \Gamma^{[m]}_{t, [a, b]} e^{J_t}$. The derivative of the ratio of functional determinants has already been computed in section 4.5 with the result

$$\frac{\mathrm{d} \left[ \log \left( e^{-J_t} \frac{\det[-\Delta + m^2(z)]_{\xi_t}}{\det[-\Delta]_{\xi_t}} \right) \right]}{\mathrm{d}t} = -4N_t \, \mathrm{d}t.$$
This drift cancels exactly the one coming from $d[\Gamma_{t,[a,b]}^{[m]} e^{lt}]$. In conclusion we find

$$Z_t^{[m]} dZ_t^{[m]} = \sqrt{2} (\partial_t \log \Gamma_{t,[a,b]}^{[m]}) d\tilde{B}_t,$$

which means that $Z_t^{[m]}$ is a (local) martingale for the critical chordal measure and the off-critical drift reads

$$F_{t,[a,b]}^{[m]} = 2(\partial_t \log \Gamma_{t,[a,b]}^{[m]}).$$

5.3. Massive symplectic correlation functions

We now show that, as expected from basic rules of statistical mechanics, the ratios of correlation functions of massive symplectic fermions

$$\frac{\langle \phi^+(\gamma_t) O_{\bar{H}_t}^{[m]} \rangle}{\langle \phi^+(\gamma_t) \int_a^b dx \psi^-(x) \rangle_{\bar{H}_t}^{[m]}}$$

are local martingales for massive dipolar SLE(2). These ratios are defined by a limiting procedure which can be written as

$$\lim_{z \to \gamma_t} \frac{\langle \chi^+(z) O_{\bar{H}_t}^{[m]} \rangle}{\langle \chi^+(z) \int_a^b dx \psi^-(x) \rangle_{\bar{H}_t}^{[m]}},$$

As in the previous section, this limit is taken by letting $g_t(z)$ approach $\xi_t$, which leads us to write

$$\frac{\langle \phi^+(\gamma_t) O_{\bar{H}_t}^{[m]} \rangle}{\langle \phi^+(\gamma_t) \int_a^b dx \psi^-(x) \rangle_{\bar{H}_t}^{[m]}} = \frac{\langle O \rangle_{\bar{H}_t}^{[m]}}{\Gamma_{t,[a,b]}^{[m]}}.$$

This serves as a definition for $\langle O \rangle_{\bar{H}_t}^{[m]}$.

To prove that these ratios are local martingales, we have to compute their Ito derivatives with the massive drift. These can be presented in the following form:

$$d \left[ \langle O \rangle_{\bar{H}_t}^{[m]} e^{J_t} \right] = \langle O \rangle_{\bar{H}_t}^{[m]} e^{J_t} \left[ X_t^{\partial} d\tilde{B}_t + R_t^{\partial} dt \right]$$

$$= \langle O \rangle_{\bar{H}_t}^{[m]} e^{J_t} \left[ X_t^{\partial} \left( dB_t^{[m]} + \frac{1}{\sqrt{2}} F_{t,[a,b]}^{[m]} dt \right) + R_t^{\partial} dt \right].$$

Combining this equation with the formula (14) of the Ito derivative of $\Gamma_{t,[a,b]}^{[m]} e^{J_t}$ and $d\tilde{B}_t = dB_t^{[m]} + (1/\sqrt{2}) F_{t,[a,b]}^{[m]} dt$, we obtain the Ito derivative of the ratio $\langle O \rangle_{\bar{H}_t}^{[m]} / \Gamma_{t,[a,b]}^{[m]}$:

$$d \left[ \langle O \rangle_{\bar{H}_t}^{[m]} / \Gamma_{t,[a,b]}^{[m]} \right] = \left[ \langle O \rangle_{\bar{H}_t}^{[m]} / \Gamma_{t,[a,b]}^{[m]} \right] \left[ \left( X_t^{\partial} - \frac{1}{\sqrt{2}} F_{t,[a,b]}^{[m]} \right) dB_t^{[m]} + (R_t^{\partial} - 4N_t) dt \right].$$

The condition for $\langle O \rangle_{\bar{H}_t}^{[m]} / \Gamma_{t,[a,b]}^{[m]}$ to be a martingale for massive dipolar SLE(2) is thus

$$R_t^{\partial} = 4N_t$$

independently of $O$.

Let us check this for a few examples.
Examples

- Consider $\Gamma_{t,[x,y]}^{[m]}$ for two points $x, y$ different from $a, b$. From equation (14) we know that

$$d\left[\Gamma_{t,[x,y]}^{[m]}e^{J_t}\right] = \Gamma_{t,[x,y]}^{[m]}e^{J_t} \left[ \sqrt{2} \left( \partial_t \log \Gamma_{t,[x,y]}^{[m]} \right) d\tilde{B}_t + 4N_t \, dt \right].$$

Therefore $\Gamma_{t,[x,y]}^{[m]} / \Gamma_{t,[a,b]}^{[m]}$ is a $P_m$ martingale. Actually such a martingale has a simple interpretation when the points $x$ and $y$ belong to the interval $[a, b]$. In such a case the ratio $\Gamma_{0,[x,y]}^{[m]} / \Gamma_{0,[a,b]}^{[m]}$ gives the probability that a massive LERW started at the origin and conditioned to end on the interval $[a, b]$ hits the sub-interval $[x, y]$; see [4].

- Consider $\mathcal{O} = \chi_-(z)$; then $\langle\mathcal{O}\rangle_t^{[m]} = \Theta_t^{[m]}(z)$ is the Poisson kernel. In appendix C we compute its Ito derivative and the result is

$$d \left[ \Theta_t^{[m]}(z)e^{h} \right] = e^{h} \left[ Q_t^{[m]}(z) \sqrt{2} d\tilde{B}_t + 4\Theta_t^{[m]}(z)N_t \right] \, dt.$$

Thus $\Theta_t^{[m]}(z) / \Gamma_{t,[a,b]}^{[m]}$ is a $P_m$ SLE(2) martingale.

- We generalize the previous two examples by considering an arbitrary product of fermions

$$\mathcal{O} = \prod_{j=1}^{N+1} \chi_-(z_j) \prod_{k=1}^{N} \chi_+(w_k).$$

The total charge has to be $-1$ as $\psi^+$ carries charge $+1$. Using Wick’s theorem we have

$$\langle\mathcal{O}\rangle_t^{[m]} = \det \begin{bmatrix} G_t^{[m]}(z_1, w_1) & \ldots & G_t^{[m]}(z_N, w_1) & \Theta_t^{[m]}(w_1) \\ G_t^{[m]}(z_1, w_2) & \ldots & G_t^{[m]}(z_N, w_2) & \Theta_t^{[m]}(w_2) \\ \vdots & \ddots & \vdots & \vdots \\ G_t^{[m]}(z_1, w_{N+1}) & \ldots & G_t^{[m]}(z_N, w_{N+1}) & \Theta_t^{[m]}(w_{N+1}) \end{bmatrix}.$$  

Looking at the drift term of the Ito derivative of $\langle\mathcal{O}\rangle_t^{[m]}e^{J_t}$ we notice that there are no contributions coming from the second-order term, since $dG_t^{[m]}(z, w)$ has no term proportional to $d\tilde{B}_t$. The first-order terms are of two kinds. The first one, which is of the expected form $4\langle\mathcal{O}\rangle_t^{[m]}e^{J_t}N_t$, comes from the derivative of the last column. There are other contributions coming from the derivative of each other column. Thanks to the Hadamard formula, the contribution of the derivative of the $j$th column is proportional to

$$\det \begin{bmatrix} G_t^{[m]}(z_1, w_1) & \ldots & \Theta_t^{[m]}(z_j)\Theta_t^{[m]}(w_1) & \ldots & \Theta_t^{[m]}(w_1) \\ G_t^{[m]}(z_1, w_2) & \ldots & \Theta_t^{[m]}(z_j)\Theta_t^{[m]}(w_2) & \ldots & \Theta_t^{[m]}(w_2) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ G_t^{[m]}(z_1, w_{N+1}) & \ldots & \Theta_t^{[m]}(z_j)\Theta_t^{[m]}(w_{N+1}) & \ldots & \Theta_t^{[m]}(w_{N+1}) \end{bmatrix}.$$  

This however is zero because the last and the $j$th columns are proportional.

We therefore conclude that the $\langle\mathcal{O}\rangle_t^{[m]}$ satisfy conditions (16) and thus that all correlation functions $\langle\mathcal{O}\rangle_t^{[m]} / \Gamma_{t,[a,b]}^{[m]}$ are $P_m$ (local) martingales. This is analogous to a perfect matching but between (massive) symplectic fermions and (massive) LERW.

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Appendix A. Computation of the determinant ratio

In this appendix we compute the ratio of spectral determinants that we use in sections 4.5 and 5:

$$\frac{\text{Det}[-\Delta + m^2(z)]_{\mathcal{H}_t}}{\text{Det}[-\Delta]_{\mathcal{H}_t}}.$$

We define the determinant of a self-adjoint elliptic operator $\mathcal{D}$ defined on a domain $\mathcal{M}$ through the $\zeta$-function regularization. Let $\zeta_\mathcal{D}(s)$ be defined as

$$\zeta_\mathcal{D}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{-t\mathcal{D}}) \, dt$$

where $e^{-t\mathcal{D}}$ is the heat kernel associated with the operator $\mathcal{D}$. The integral defining the $\zeta$-function is convergent only for $\text{Re}(s) > s_0 > 0$ but the $\zeta$-function itself can be analytically continued in $s = 0$ where it is holomorphic. Then the prescription for the determinant is

$$\log \text{Det}[\mathcal{D}] \equiv -\zeta'_\mathcal{D}(0).$$

In our case we are interested in getting a difference of logarithms of determinants

$$\log \frac{\text{Det}[-\Delta + m^2(z)]_{\mathcal{H}_t}}{\text{Det}[-\Delta]_{\mathcal{H}_t}} = -\zeta'_{-\Delta + m^2}(0) + \zeta'_{-\Delta}(0) = -\int_0^1 d\tau \frac{d}{d\tau} \zeta'_{-\Delta + \tau m^2}(0).$$

We are going to evaluate $(d/d\tau)\zeta_{-\Delta + \tau m^2}(s)$ for $s$ close to zero. Taking the derivative is easy because $(d/d\tau) \text{Tr}(e^{(\Delta - \tau m^2)t}) = -\text{Tr}(m^2 e^{(\Delta - \tau m^2)t})$. In order to perform the analytic continuation which gives the $\zeta$-function in 0 we separate the integral in equation (A.1) into two parts introducing a cut-off $\epsilon$:

$$\frac{d}{d\tau} \zeta_{-\Delta + \tau m^2}(s) = -\frac{1}{\Gamma(s)} \left( \int_0^\epsilon d\tau + \int_\epsilon^\infty d\tau \right) t^s \text{Tr}(m^2 e^{(\Delta - \tau m^2)t}).$$

This equation is true for any $\epsilon$ but we shall take the limit $\epsilon \to 0$ after having implemented the analytic continuation. The second integral can be directly continued to $s$ around 0 since the divergence has been cut off. The first integral of course cannot be computed for $s$ around 0 but, since we are going to send $\epsilon \to 0$, we can compute it using the small time expansion of the heat kernel [5]. So let $P_t[^{\sqrt{m}}] \equiv e^{(\Delta - \tau m^2)t}$. For small $t$ we have the expansion

$$P_t[^{\sqrt{m}}](z, w) = P_t[^{0}](z, w) \left( 1 + \sum_{j \geq 1} t^{j/2} \phi_j(z, w) \right)$$

with $P_t[^{0}]$ the massless heat kernel with Dirichlet boundary conditions. Inserting this expansion in the first integral and using the fact that along the diagonal $P_t[^{0}](z, z) = (1/4\pi t)$, up to exponentially small terms as $t \to 0$, gives

$$\int_0^\epsilon dt \, t^{s-1} \text{Tr}(m^2 e^{(\Delta - \tau m^2)t}) = \frac{\epsilon^s}{s} \int \frac{d^2z}{4\pi} m^2(z) + \cdots$$

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where the dots refer to sub-leading terms in $\epsilon$. Taking the derivative of the $\zeta$-function w.r.t. $s$ (recall that $s\Gamma(s) = \Gamma(s + 1)$) we arrive at

$$\frac{d}{d\tau} \zeta_{-\Delta + \tau m^2}(0) = \lim_{\epsilon \to 0} \left[ (\Gamma'(1) - \log \epsilon) \int \frac{dz}{4\pi} m^2(z) - \int_0^\infty \text{Tr}(m^2 e^{(\Delta - \tau m^2)\epsilon}) \, dt \right]$$

$$= \lim_{\epsilon \to 0} \left[ (\Gamma'(1) - \log \epsilon) \int \frac{dz}{4\pi} m^2(z) - \text{Tr} \left( \frac{1}{-\Delta + \tau m^2} m^2 e^{(\Delta - \tau m^2)\epsilon} \right) \right].$$

(A.3)

It is now again a matter of small time expansion of the heat kernel. We have

$$\text{Tr} \left( \frac{1}{-\Delta + \tau m^2} m^2 e^{(\Delta - \tau m^2)\epsilon} \right) = \int \frac{dz}{4\pi} \frac{d^2 z'}{4\pi} G_t^{[\sqrt{\tau m}]}(z', z) m^2(z) P_t^{[\sqrt{\tau m}]}(z, z').$$

We compute this integral by adding and subtracting $\log |z - z'|^2$ to $G_t^{[\sqrt{\tau m}]}(z', z)$ and splitting the integral into two integrals. The first one involves $G_t^{[\sqrt{\tau m}]}(z, z) + \log |z - z'|^2$. There we can directly take the limit $\epsilon \to 0$. Using the fact that $\lim_{\epsilon \to 0} P_t^{[\sqrt{\tau m}]}(z, z') = \delta(z, z')$ we get

$$\int \frac{dz}{4\pi} m^2(z) \lim_{z' \to z} (G_t^{[\sqrt{\tau m}]}(z', z) + \log |z - z'|^2).$$

By definition the last term is $\langle \tilde{X}^2(z) \rangle^{[\sqrt{\tau m}]}$. The second integral involves $\log |z - z'|^2$. In the limit $\epsilon \to 0$ of that integral we can replace $P_t^{[\sqrt{\tau m}]}(z, z')$ by $P_t^{[0]}(z, z')$. The integral over $z'$ can then be exactly evaluated to give

$$\int \frac{dz'}{4\pi} \log |z - z'|^2 P_t^{[0]}(z, z') = (\log(4\epsilon) + \Gamma'(1)) \int \frac{dz}{4\pi} m^2(z) + \cdots.$$

Putting everything together we get for $\text{Tr}((1/(-\Delta + \tau m^2)) m^2 e^{(\Delta - \tau m^2)\epsilon})$

$$= -(\log(4\epsilon) + \Gamma'(1)) \int \frac{dz}{4\pi} m^2(z) + \int \frac{dz}{4\pi} m^2(z) \langle \tilde{X}^2(z) \rangle^{[\sqrt{\tau m}]} + O(\epsilon),$$

where $\langle \tilde{X}^2(z) \rangle^{[\sqrt{\tau m}]}$ is given exactly by the point splitting regularization. Once we substitute this expression for the trace into equation (A.3), we get

$$\frac{d}{d\tau} \log \left[ \frac{\text{Det}[-\Delta + \tau m^2]}{\text{Det}[-\Delta]} \right] = - \frac{d}{d\tau} \zeta_{-\Delta + \tau m^2}(0)$$

$$= \int \frac{dz}{4\pi} m^2(z) \langle \tilde{X}^2(z) \rangle^{[\sqrt{\tau m}]} + \text{const.} \int \frac{dz}{4\pi} m^2(z).$$

(A.4)

Up to the irrelevant term proportional to $\int (dz/4\pi) m^2(z)$ that we can and shall ignore, this coincides with the naive field theory derivation, equation (8).

**Appendix B. Derivative of \( \tilde{Z}_t^{[m]}: \tilde{X} \)**

Here we compute the derivative of $\tilde{Z}_t^{[m]}: \tilde{X}$:

$$d \log \tilde{Z}_t^{[m]}: \tilde{X} = - \frac{1}{2} \int_0^1 d\tau \left( \int \frac{dz}{4\pi} m^2(z) \, dK_t^{[\sqrt{\tau m}]}(z) \right).$$
We are going to show that $dK_t^{[\sqrt{\tau_m}]}(z)$ can be written as a total derivative w.r.t. $\tau$:

$$dK_t^{[\sqrt{\tau_m}]}(z) = 2 \left( \int \frac{d^2z'}{4\pi} \frac{d}{d\tau} \left[ \tau^2 m^2(z') \theta_t(z)\theta_t(z')G_t^{[\sqrt{\tau_m}]}(z', z) \right] \right) dt. \quad (B.1)$$

Indeed, on one hand we can use the expression for $K_t^{[\sqrt{\tau_m}]}(z)$ given in equation (10) and the massless and massive Hadamard formulae to write the left-hand side of equation (B.1) as

$$dK_t^{[\sqrt{\tau_m}]}(z) = 2 \int_{\mathbb{H}_t} \frac{d^2z'}{4\pi} \Theta_t^{[\sqrt{\tau_m]}}(z)\Theta_t^{[\sqrt{\tau_m]}}(z') \tau m^2(z')G_t^{[0]}(z', z) dt$$

$$+ 2 \int_{\mathbb{H}_t} \frac{d^2z'}{4\pi} G_t^{[\sqrt{\tau_m}]}(z, z') \tau m^2(z')\theta_t(z')\theta_t(z) dt.$$

On the other hand, if we develop the derivative w.r.t. $\tau$ on the right-hand side of equation (B.1) we get

$$4 \int \frac{d^2z'}{4\pi} \tau m^2(z') \theta_t(z)\theta_t(z')G_t^{[\sqrt{\tau_m}]}(z', z) dt$$

$$- 2 \int \frac{d^2z'}{4\pi} \frac{d^2z''}{4\pi} \tau^2 m^2(z') \theta_t(z)\theta_t(z')G_t^{[\sqrt{\tau_m}]}(z', z'')m^2(z'')G_t^{[\sqrt{\tau_m}]}(z'', z)$$

$$= 2 \int \frac{d^2z'}{4\pi} \tau m^2(z') \theta_t(z)\theta_t(z')G_t^{[\sqrt{\tau_m}]}(z', z) dt$$

$$+ 2 \int \frac{d^2z'}{4\pi} \tau m^2(z') \Theta_t^{[\sqrt{\tau_m}]}(z)\theta_t(z')G_t^{[\sqrt{\tau_m}]}(z', z) dt.$$

Equation (B.1) follows from the fact that

$$\int d^2z' \Theta_t^{[\sqrt{\tau_m}]}(z') \tau m^2(z')G_t^{[0]}(z', z) = \int d^2z' \theta_t(z') \tau m^2(z')G_t^{[\sqrt{\tau_m}]}(z', z).$$

Once we have this relation we can plug it into the equation for the derivative of $d \log \bar{Z}_t^{[m];X}$ and we get

$$d \log \bar{Z}_t^{[m];X} = -4 \int \frac{d^2z \ d^2z'}{8\pi} \frac{m^2(z)m^2(z')}{8\pi} \theta_t(z)\theta_t(z')G_t^{[m]}(z', z) dt. \quad (B.2)$$

The right-hand side is nothing else than $2N_t dt$. This proves equation (11).

### Appendix C. LERW: Ito derivatives

In this appendix we present some explicit formulas which are used in carrying out the computations performed in section 5. We comment also on some apparent divergences present in the computation of the Ito derivative of $\Gamma^{[m]}_{t,[a,b]}$ and of other quantities.

Recall the definition

$$\Psi^{[m]}_{t,[a,b]}(z) = \left\langle \chi^+(z) \int_a^b dx \psi^-(x) \right\rangle_{\mathbb{H}_t}^{[m]}.$$

By construction and an appropriate choice of normalization, $\Psi^{[m]}_{t,[a,b]}(z)$ satisfies the massive Laplace equation $(-\Delta + m^2(z))\Psi^{[m]}_{t,[a,b]}(z) = 0$ with boundary conditions: $\Psi^{[m]}_{t,[a,b]}(z) = \pi$.

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when \( z \in [a, b] \), and \( \Psi_{t,[a,b]}^{[m]}(z) = 0 \) when \( z \) lies outside the interval \([a, b]\). We may write it in terms of the massive Green function:

\[
\Psi_{t,[a,b]}^{[m]}(z) = \Psi_{t,[a,b]}^{[0]}(z) - \frac{1}{4\pi}G_t^{[m]}(z, \cdot) * m^2(\cdot)\Psi_{t,[a,b]}^{[0]}(\cdot)
\]

with \( (a_t = g_t(a) \) and \( b_t = g_t(b) \)

\[
\Psi_{t,[a,b]}^{[0]}(z) = \text{Im} \log \left( \frac{g_t(z) - a_t}{g_t(z) - b_t} \right).
\]

From the relation between \( \psi^\pm \) and \( \chi^\pm \), it follows that \( \Gamma_{t,[a,b]}^{[m]} \) is defined from a limiting procedure from \( \Psi_{t,[a,b]}^{[m]}(z) \). We set

\[
\Gamma_{t,[a,b]}^{[m]} = \lim_{\delta \to 0} \frac{1}{2\delta} \left. \Psi_{t,[a,b]}^{[m]}(z) \right|_{g_t(z) = \xi_t \pm i\delta}.
\]

In the massless case we have

\[
\Gamma_{t,[a,b]}^{[0]} = \frac{(a_t - b_t)}{\xi_t - a_t(\xi_t - b_t)}.
\]

As usual we can write the massive solutions in terms of the massless ones and of the massive propagator. This gives

\[
\Gamma_{t,[a,b]}^{[m]} = \Gamma_{t,[a,b]}^{[0]} - \frac{1}{4\pi} \Theta_t^{[m]}(\cdot) * m^2(\cdot)\Psi_{t,[a,b]}^{[0]}(\cdot)
\]

(C.1)

with \( \Theta_t^{[m]}(z) = \theta_t(z) - (1/4\pi)G_t^{[m]}(z, \cdot) * m^2(\cdot)\theta_t(\cdot) \).

We can now compute the Ito derivatives. The ingredients that we need are

\[
d\Gamma_{t,[a,b]}^{[0]} = \Gamma_{t,[a,b]}^{[0]} F_{t,[a,b]}^0 \sqrt{2} dB_t,
\]

\[
d\Psi_{t,[a,b]}^{[0]}(z) = -2\theta_t(z) \Gamma_{t,[a,b]}^{[0]} dt,
\]

\[
d\theta_t(z) = Q_t^{[0]}(z) \sqrt{2} dB_t,
\]

with \( Q_t^{[0]}(z) = -2\text{Im}(1/(z_t - \xi_t)^2) \). The last equation and the Hadamard formula

\[
dG_t^{[m]}(z, w) = -2\Theta_t^{[m]}(z)\Theta_t^{[m]}(w) dt \]

imply that

\[
d\Theta_t^{[m]}(z) = Q_t^{[m]}(z) \sqrt{2} dB_t + 4\Theta_t^{[m]}(z) \hat{N}_t dt,
\]

with

\[
Q_t^{[m]}(z) = Q_t^{[0]}(z) - \frac{1}{4\pi} G_t^{[m]}(z, \cdot) * m^2(\cdot)Q_t^{[0]}(\cdot),
\]

\[
\hat{N}_t = \int \frac{d^2 z}{8\pi} m^2(z)\Theta_t^{[m]}(z) \theta_t(z).
\]

Ito differentiating equation (C.1) and putting all these pieces together we find

\[
d\left[ \Gamma_{t,[a,b]}^{[m]} \right] = \left( \Gamma_{t,[a,b]}^{[0]} F_{t,[a,b]}^0 \right) \sqrt{2} dB_t + 4\Gamma_{t,[a,b]}^{[m]} \hat{N}_t dt.
\]

By construction,

\[
\partial_t \Gamma_{t,[a,b]}^{[m]} = \Gamma_{t,[a,b]}^{[0]} F_{t,[a,b]}^0 - \frac{1}{4\pi} Q_t^{[m]}(\cdot) * m^2(\cdot)\Psi_{t,[a,b]}^{[0]}(\cdot).
\]

Again the key point is that the drift term in the previous equation is \( 4\Gamma_{t,[a,b]}^{[m]} \hat{N}_t dt \).
Here we encounter an unpleasant problem. Indeed \( \hat{N}_t \) naively diverges as \( t \to 0 \).

In order to avoid such a problem one can instead consider \( \Gamma_{t|a,b}^{[m]} e^{J_t} \), where we recall the definition of \( J_t \):

\[
J_t = \int \frac{d^2 z}{4\pi} m^2(z) \log |\rho_t(z)|^2.
\]

Recall that \( d \log |\rho_t(z)| = -\theta_t(z)^2 \, dt \). Taking now the Itô derivative of \( \Gamma_{t|a,b}^{[m]} e^{J_t} \), we get for \( t > 0 \)

\[
d \left[ \Gamma_{t|a,b}^{[m]} e^{J_t} \right] = \left( e^{J_t} \partial_t \Gamma_{t|a,b}^{[m]} \right) \sqrt{2} \, dB_t + 4 \Gamma_{t|a,b}^{[m]} e^{J_t} N_t \, dt
\]

with

\[
N_t = \int \frac{d^2 z}{8\pi} m^2(z) \left[ \Theta_t^{[m]}(z) \theta_t(z) - \theta^2(z) \right].
\]

This quantity is now finite as \( t \to 0 \). This proves equation (14).

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