On the homology of Iwasawa cohomology groups

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Abstract

A fundamental observation of Iwasawa gives a criterion for a \( \mathbb{Z}_p[\Gamma]\)-module to be torsion. In this paper, we study certain extension of this criterion. We will apply this extended result to investigate the structure of the homology of certain “Iwasawa cohomology groups”. Namely, we will answer partially a question of Hachimori on the structure of the homology of the first Iwasawa cohomology groups as modules over Iwasawa algebras and study the pseudo-nullity of the homology of the second Iwasawa cohomology groups and dual fine Selmer groups.

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1 Introduction

Let \( p \) be a fixed prime number. Denote \( \Gamma \) to be the compact abelian multiplicative group isomorphic to the additive group \( \mathbb{Z}_p \). A well known criterion of Iwasawa states that a finitely generated \( \mathbb{Z}_p[\Gamma]\)-module \( M \) with the property that \( M_\Gamma \) is finite is necessarily a torsion \( \mathbb{Z}_p[\Gamma]\)-module. This criterion is an easy consequence of the structure theory of finitely generated \( \mathbb{Z}_p[\Gamma]\)-module (see [Ho] for another approach in deriving the criterion) and has been a useful tool in the study of modules over the classical Iwasawa algebra. This criterion has been extended to solvable uniform pro-\( p \) group in [BHo, HaO]. Building on the ideas of [BHo, HaO], the author established a relative version of this generalized criterion and apply it to the study of pseudo-nullity of fine Selmer groups in [Lim]. In this paper, we show a sharper version of this result under a certain stronger assumption (see Proposition 2.3). The proof of this result makes additional use of a technical proposition of Kato [Ka2].

Following the classical situation, we will apply this criterion to study the structure of certain modules which now bears a module structure over a larger (possibly non-commutative) Iwasawa algebra. Namely, we will study an extension of the questions of Hachimori [Ha] on the structure of the homology of the first Iwasawa cohomology groups which was also studied in [Ka2]. Hachimori’s motivation of studying these questions lies in developing a criterion to determine the existence of the Euler characteristic of the fine Selmer group of an elliptic curve and he was able to provide answer to his question in the case when the Iwasawa cohomology groups are considered over a \( p \)-adic Lie extension of dimension 2. In this paper, we will formulate a refined version of his question.
and provide an answer to it for a more general Galois modules and $p$-adic extensions (see Theorem 3.2). We are also able to establish the existence of the Euler characteristic of the fine Selmer group of an abelian variety for this class of $p$-adic Lie extensions without invoking Hachimori’s criterion (see Corollary 3.5).

We will also formulate and study certain variant of the questions of Hachimori over an admissible $p$-adic extension. As we will see, the answer to these questions in this form is a consequence of various conjectures on the structure of the second Iwasawa cohomology groups (or dual fine Selmer groups) which has been studied in [CS, JiS, Lim]. We finally note that our criterion (Proposition 2.3) has a pseudo-null analog, and we will apply this to establish the pseudo-nullity of certain homology groups of the Iwasawa cohomology groups (and the dual fine Selmer groups).

We now give a brief description of the layout of the paper. In Section 2, we will discuss and prove our algebraic criterion. In Section 3, we formulate a refinement of Hachimori question and answer the question for certain class of $p$-adic Lie extensions. We will also establish the existence of the Euler characteristic of the dual fine Selmer group of an abelian variety over the same class of $p$-adic Lie extensions. We will then formulate and study a variant of Hachimori question for admissible $p$-adic Lie extension in Section 4. In Section 5, we will establish the pseudo-nullity of certain homology groups of the second Iwasawa cohomology groups and the dual fine Selmer groups.

## 2 Ranks of Iwasawa modules

Denote $p$ to be a fixed prime. Let $R$ be a complete regular local ring with a finite residue field of characteristic $p$. For a compact pro-$p$ $p$-adic Lie group $G$ without $p$-torsion, we denote $R[G]$ to be the completed group algebra which is defined by $\lim \rightarrow R[G/U]$, where the inverse limit is taken over all open normal subgroups of $G$. It is well known that $R[G]$ is an Auslander regular ring (cf. [VI] Theorems 3.26 and 3.30(ii)] or [Lim] Theorem A1(c)). In particular, the ring $R[G]$ is Noetherian local and has finite projective dimension. Therefore, it follows that every finitely generated $R[G]$-module admits a finite free resolution of finite length. In the case that either the ring $R$ has characteristic zero or $G$ is a uniform pro-$p$ group, the ring $R[G]$ has no zero divisors (cf. [Lim] Theorem A1(b)), and therefore admits a skew field $Q(G)$ which is flat over $R[G]$ (see [GW] Chapters 6 and 10] or [Lam] Chapter 4, §9 and §10]). In this case, if $M$ is a finitely generated $R[G]$-module, we define the $R[G]$-rank of $M$ to be

$$\text{rank}_{R[G]} M = \dim_{Q(G)} Q(G) \otimes_{R[G]} M.$$ 

For a general compact pro-$p$ $p$-adic Lie group $G$ without $p$-torsion, the $R[G]$-rank of a finitely generated $R[G]$-module $M$ is defined by

$$\text{rank}_{R[G]} M = \frac{\text{rank}_{R[G_0]} M}{|G : G_0|},$$
where $G_0$ is an open normal uniform pro-$p$ subgroup of $G$. This is integral and independent of the choice of $G_0$ (cf. [Lim, Section 4]). In fact, one has the following formula of Howson (cf. [Ho, Theorem 1.1] or [Lim, Lemmas 4.3 and 4.5]).

**Lemma 2.1.** Let $G$ be a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Let $M$ be a finitely generated $R[[G]]$-module. Then we have the equality

$$\text{rank}_{R[[G]]} M = \sum_{i \geq 0} (-1)^i \text{rank}_R H_i(G, M).$$

Furthermore, if $H$ is a closed normal subgroup of $G$ such that $G/H$ has no $p$-torsion, we have

$$\text{rank}_{R[[G]]} M = \sum_{i \geq 0} (-1)^i \text{rank}_{R[[G/H]]} H_i(H, M)$$

We say that $M$ is a torsion $R[[G]]$-module if $\text{rank}_{R[[G]]} M = 0$. The following result was stated in [Lim, Theorem 4.6], although the essences of the proof were essentially in [BHo, HaO, Ho].

**Proposition 2.2.** Let $G$ be a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Suppose that $H$ is a closed normal subgroup of $G$ such that $H$ is solvable uniform and $G/H$ is a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Let $M$ be a finitely generated $R[[G]]$-module. Then $M_H$ is a torsion $R[[G/H]]$-module if and only if $M$ is a torsion $R[[G]]$-module and $\sum_{n \geq 1} (-1)^i \text{rank}_{R[[G/H]]} H_n(H, M) = 0$.

The above cited proposition is a natural refinement of the results in [BHo, HaO] and, of course, the classical case when $R = H = G = \mathbb{Z}_p$. The proposition is false if one removes the “solvable” assumption (see [BHo, Ho] for discussions and counterexamples on this issue). By imposing a stronger condition on $H$, we can prove that each of the terms in the alternating sum is actually zero.

**Proposition 2.3.** Let $G$ be a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Let $H$ be a closed normal subgroup of $G$ such that $G/H$ is a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Suppose that there is a finite family of closed normal subgroups $H_i \ (0 \leq i \leq r)$ of $G$ such that $1 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_r = H$, $H_i/H_{i-1} \cong \mathbb{Z}_p$ for $1 \leq i \leq r$ and such that the action of $G$ on $H_i/H_{i-1}$ by inner automorphism is given by a homomorphism $\chi_i : G/H \rightarrow \mathbb{Z}_p^\times$. Let $M$ be a finitely generated $R[[G]]$-module. Then the following statements are equivalent.

(a) $M_H$ is a torsion $R[[G/H]]$-module.

(b) $M$ is a torsion $R[[G]]$-module, and $H_n(H, M)$ is a torsion $R[[G/H]]$-module for every $n \geq 1$.

To prove Proposition 2.3 we need a proposition of Kato [Ka2]. Before stating this proposition, we introduce some terminology. Let $\mathcal{G}$ be a compact pro-$p$ $p$-adic Lie group. For a continuous character $\kappa : \mathcal{G} \rightarrow R^\times$ and an $R[[G]]$-module $M$, we let $M(\kappa)$ denote the $R$-module $M$ with the new commuting $\mathcal{G}$-action given by the twist of the original by $\kappa$. For any continuous character
χ : G → Z_p^×, by abuse of notation, we denote χ : G → R^× to be continuous character obtained by the composition of the natural map Z_p^× → R^× with χ. Then M(χ) is the R[G]-module defined as above. We can now state the following proposition of Kato [Ka2].

**Proposition 2.4.** Let G be a compact p-adic Lie group and let H be a closed normal subgroup of G. Assume that we are given a finite family of closed normal subgroups H_i (0 ≤ i ≤ r) of G such that 1 = H_0 ⊆ H_1 ⊆ ⋯ ⊆ H_r = H, H_i/H_{i−1} ≅ Z_p for 1 ≤ i ≤ r and such that the action of G on H_i/H_{i−1} by inner automorphism is given by a homomorphism χ_i : G/H → Z_p^×.

Let M be a finitely generated R[G]-module, and let M' be a R[G]-subquotient of M. Let m ≥ 0. Then there exists a finite family (S_i)_{1 ≤ i ≤ k} = (S_{i,m})_{1 ≤ i ≤ k} of R[G/H]-submodules of H_m(H, M') satisfying the following properties.

(i) 0 = S_0 ⊆ S_1 ⊆ ⋯ ⊆ S_k = H_m(H, M').

(ii) For each j (1 ≤ j ≤ k), there is a R[G/H]-subquotient K = K_j of H_0(H, M) and a family (s_j(i))_{1 ≤ i ≤ r} = (s(i))_{1 ≤ i ≤ r} of nonnegative integers such that |{i|s(i) > 0}| ≥ m and such that S_j/S_j−1 is isomorphic to the twist K(∏_{1 ≤ i ≤ r} χ_i^{s(i)}) of K.

**Proof.** The proof of [Ka2] Proposition 4.2 basically carries over word-for-word. One only needs to make the cosmetic change by replacing Z_p in the original proof by R.

We will require another lemma. For a continuous character κ : G → R^×, we denote κ^{-1} to be the character G → R^× given by g → κ(g)^{-1}.

**Lemma 2.5.** Let G be a compact pro-p p-adic Lie group without p-torsion. Suppose that we are given a continuous character κ : G → R^× and a finitely generated R[G]-module M. If M is a torsion R[G]-module, so is M(κ).

**Proof.** By [Lim] Lemma 4.4, M is a torsion R[G]-module if and only if Hom_{R[G]}(M, R[G]) = 0. It is a straightforward exercise to see that

Hom_{R[G]}(M(κ), R[G]) = Hom_{R[G]}(M, R[G])(κ^{-1}).

The conclusion of the lemma is now immediate.

We can now prove Proposition 2.3.

**Proof of Proposition 2.3.** It follows from the above proposition of Kato that H_m(H, M) is a successive extensions of twists of R[G/H]-subquotients of H_0(H, M). Since H_0(H, M) = M_H is a torsion R[G/H]-module by hypothesis, so is every R[G/H]-subquotients of M_H. By Lemma 2.5 every twists of R[G/H]-subquotients of M_H is also a torsion R[G/H]-module. The conclusion of Proposition 2.3 is now immediate by Proposition 2.4.

We end the section giving a variant of Proposition 2.3 for pseudo-null modules. This will be used in Section 5. Recall that finitely generated torsion R[G]-module is said to be pseudo-null if Ext^1_{R[G]}(M, R[G]) = 0.
Proposition 2.6. Let $N$ and $H$ be closed normal subgroups of $G$ such that $N \subseteq H$, $G/H \cong \mathbb{Z}_p$ and $G/N$ is a pro-$p$ group of dimension $\geq 2$ without $p$-torsion. Suppose further that there is a finite family of closed normal subgroups $N_i$ $(0 \leq i \leq r)$ of $G$ such that \( 1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = N \), $N_i/N_{i-1} \cong \mathbb{Z}_p$ for $1 \leq i \leq r$ and such that the action of $H$ on $N_i/N_{i-1}$ by inner automorphism is given by a homomorphism $\chi_i : H/N_{i-1} \rightarrow \mathbb{Z}_p^\times$. Let $M$ be a finitely generated $R[[G]]$-module which is also finitely generated over $R[[H]]$. Then the following statements are equivalent.

(a) $M_N$ is a pseudo-null $R[[G/N]]$-module.

(b) $M$ is a pseudo-null $R[[G]]$-module, and $H_n(N,M)$ is a pseudo-null $R[[G/N]]$-module for every $n \geq 1$.

Proof. Since $M$ is a $R[[G]]$-module which is finitely generated over $R[[H]]$, it follows from a result of Venjakob (cf. [V2, Example 2.3 and Proposition 5.4] or [Lim, Lemma 5.1]) that $M$ is a pseudo-null $R[[G]]$-module if and only if $M$ is a torsion $R[[H]]$-module. The conclusion of the proposition is now an immediate consequence of Proposition 2.3. \qed

3 On the questions of Hachimori

In this section, we will apply the results obtained in the preceding section to study the questions of Hachimori. We begin by introducing some terminology and notation that we shall use throughout this section.

As before, $p$ will denote a fixed prime. Let $F$ be a number field. If $p = 2$, we assume further that $F$ has no real primes. Let $S$ be a finite set of primes of $F$ that contains the primes above $p$ and the infinite primes. We then denote $F_S$ to be the maximal algebraic extension of $F$ unramified outside $S$. For any algebraic (possibly infinite) extension $L$ of $F$ contained in $F_S$, we write $G_S(L) = \text{Gal}(F_S/L)$. Let $F_\infty$ be a $p$-adic extension of $F$ which is unramified outside $S$ and we will assume that $G = \text{Gal}(F_\infty/F)$ is a compact pro-$p$ Lie group with no $p$-torsion.

Let $R$ be a complete regular local ring with a finite residue field of characteristic $p$. Let $T$ denote a finitely generated free $R$-module with a continuous $R$-linear $G_S$-action. (Here the topology on $T$ is given by its filtration by powers of the maximal ideal of $R$.) We define the $i$th “Iwasawa cohomology groups” (of $T$ over $F_\infty$) to be

$$H^i_{Iw,S}(F_\infty/F,T) = \lim_{\leftarrow L} H^i(G_S(L),T),$$

where the inverse limit is taken over all the finite extensions $L$ of $F$ contained in $F_\infty$ and with respect to the corestriction maps. For ease of notation, we will drop the ‘$S$’. It is well-known that $H^i_{Iw}(F_\infty/F,T)$ is a finitely generated (compact) $R[[G]]$-module for every $i$. In fact, it is not difficult to see that $H^i_{Iw}(F_\infty/F,T) = 0$ for $i \neq 1,2$. We now record the following lemma which will be crucial in our discussion.
Lemma 3.1. Assume that $H^0(G_S(F),T) = 0$. We then have isomorphisms

$$H^2_{Iw}(F_{\infty}/F,T)_G \cong H^2(G_S(F),T)$$

and

$$H_i(G,H^1_{Iw}(F_{\infty}/F,T)) \cong H_i(G,H^2_{Iw}(F_{\infty}/F,T))$$

for $i \geq 1$.

Proof. These are immediate consequences from analyzing the homological spectral sequence

$$H^i(G,H^{-j}(F_{\infty}/F,T)) \Rightarrow H^{-i-j}(G_S(F),T)$$

which can be found, for instances, in either [FK Proposition 1.6.5(iii)] or [LSH Theorem 3.1.8].

In this section, we are interested in the following questions which were raised and studied in [Ha].

Question: Do the following statements hold?

(A) rank$_{R[G]} H^1_{Iw}(F_{\infty}/F,T) = \text{rank}_R H^1_{Iw}(F_{\infty}/F,T)_G.$

(B) $H_i(G,H^1_{Iw}(F_{\infty}/F,T))$ is a torsion $R$-module for each $i \geq 1$.

In his paper [Ha], Hachimori studied the question in the case when $T$ is the Tate module of an elliptic curve. In [Ka2], question (B) is studied in the case when $T = \mathbb{Z}_p(1)$. It follows from an application of the formula of Howson (cf. Lemma 2.1) that (B) implies (A). We can now state the main result of this section which gives an affirmative answer to the above question for certain classes of $p$-adic Lie extensions.

Theorem 3.2. Let $F_{\infty}$ be an $p$-adic Lie extension of $F$ with Galois group $G$. Assume that $H^0(G_S(F),T) = 0$. Suppose further that either one of the following statements holds.

(i) The group $G$ has dimension $\leq 2$.

(ii) There is a finite family of closed normal subgroups $G_i$ ($0 \leq i \leq r$) of $G$ such that $1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G$ and $G_i/G_{i-1} \cong \mathbb{Z}_p$ for every $i$, and $H^2(G_S(F),T)$ is a torsion $R$-module.

Then statements (A) and (B) hold.

Before proving the theorem, we make a remark.

Remark 3.3. The assumption $H^0(G_S(F),T) = 0$ is known to hold in many interesting cases (for instances, $\mathbb{Z}_p(r)$ where $r \neq 0$, and the Tate module of an abelian variety). When $T$ is the Tate module of an elliptic curve $E$, the theorem under statement (i) was proved in [Ha Theorem 1.3] under further assumptions that $E(F_{\infty})[p^{\infty}]$ is finite and $H^2(G_S(F_{\infty}),E[p^{\infty}]) = 0$. Therefore, our theorem in this case can be viewed as a refinement of that, namely both in terms of the Galois module $T$ and that we can do without the two assumptions above. We also note that if $G$ has dimension $\leq 2$, then $G$ will satisfy the group theoretical assumption of statement (ii). (This is clear if $G$ is abelian or has dimension 1. In the case when $G$ is nonabelian of dimension 2, this follows from [GSK Proposition 7.1].) The point of (i) is that if the dimension of $G$ is less than or equal to 2, one does not require the torsionness condition on $H^2(G_S(F),T)$ in (ii).
Proof of Theorem 3.2. (i) As mentioned above, it suffices to show that (B) holds. By the assumption that \( \dim G \leq 2 \) and the second isomorphism in Lemma 3.1, we have \( H_i(G, H^1_{Iw}(F_\infty/F, T)) = 0 \) for all \( i \geq 1 \). In particular, statement (B) holds.

(ii) It follows from Lemma 3.1 that \( H_0(G, H^2_{Iw}(F_\infty/F, T)) \) is isomorphic to \( H^2(G_S(F), T) \) which is \( R \)-torsion by the hypothesis. By an application of Proposition 2.3, we then have that \( H_m(G, H^2_{Iw}(F_\infty/F, T)) \) is \( R \)-torsion for all \( m \geq 1 \). By the second isomorphism in Lemma 3.1, this in turn implies that \( H_m(G, H^1_{Iw}(F_\infty/F, T)) \) is \( R \)-torsion for all \( m \geq 1 \). Thus, (B) holds.

We now define the dual Selmer group. Let \( v \) be a prime in \( S \). For every finite extension \( L \) of \( F \) contained in \( F_S \), we define

\[
K^2_v(T/L) = \bigoplus_{w | v} H^2(L_w, T),
\]

where \( w \) runs over the (finite) set of primes of \( L \) above \( v \). If \( L \) is an infinite extension of \( F \) contained in \( F_S \), we define

\[
K^2_v(T/L) = \lim_{\leftarrow L} K^2_v(T/L),
\]

where the inverse limit is taken over all finite extensions \( L \) of \( F \) contained in \( L \). For any algebraic extension \( L \) of \( F \) contained in \( F_S \), the dual fine Selmer group of \( T \) over \( L \) (with respect to \( S \)) is defined to be

\[
Y(T/L) = \ker \left( H^2_{Iw}(L/F, T) \to \bigoplus_{v \in S} K^2_v(T/L) \right).
\]

Since \( Y(T/F_\infty) \) is a \( R[G] \)-submodule of \( H^2_{Iw}(F_\infty/F, T) \), the following is an immediate consequence of Proposition 2.3 and Lemma 3.1.

**Theorem 3.4.** Let \( F_\infty \) be a \( p \)-adic Lie extension of \( F \) with Galois group \( G \). Assume that \( H^0(G_S(F), T) = 0 \). Suppose that there is a finite family of closed normal subgroups \( G_i \) (\( 0 \leq i \leq r \)) of \( G \) such that \( 1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G \) and \( G_i/G_{i-1} \cong \mathbb{Z}_p \) for \( 1 \leq i \leq r \), and that \( H^2(G_S(F), T) \) is \( R \)-torsion. Then \( H_i(G, Y(T/F_\infty)) \) is \( R \)-torsion for all \( i \geq 0 \).

For the remainder of the section, we will assume that our ring \( R \) is \( \mathbb{Z}_p \). One of the motivation of studying the questions (A) and (B) lies in developing a criterion for the existence of the Euler characteristic of the dual fine Selmer group of a \( p \)-adic representation (see [Ha, Theorem 3.1]). Following [Ha], we say that the \( G \)-Euler characteristic of \( Y(T/F_\infty) \) is defined if \( H_i(G, Y(T/F_\infty)) \) is finite for all \( i \). An interesting by-product of our discussion is that we can establish the existence of the Euler characteristic of the dual fine Selmer group in the cases considered without requiring the usage of [Ha, Theorem 3.1].

**Corollary 3.5.** Let \( A \) be an abelian variety over \( F \). Denote \( T \) to be the Tate module of the dual abelian variety \( A^* \) of \( A \). Let \( F_\infty \) be a \( p \)-adic Lie extension of \( F \) with Galois group \( G \). Suppose that there is a finite family of closed normal subgroups \( G_i \) (\( 0 \leq i \leq r \)) of \( G \) such that \( 1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G \) and \( G_i/G_{i-1} \cong \mathbb{Z}_p \) for every \( i \), and \( Y(T/F) \) is finite. Then the \( G \)-Euler characteristic of \( Y(T/F_\infty) \) is defined.
Proof. Since \( A^*(F)[p^\infty] \) is finite, we have \( H^0(G_S(F), T) = 0 \). On the other hand, for each \( v \in S \), it follows easily from Mattuck’s theorem [Mat] that \( A(F_v)[p^\infty] \) is finite. By Tate local duality, this in turn implies that \( H^2(F_v, T) \) is finite. Therefore, \( Y(T/F) \) is finite if and only if \( H^2(G_S(F), T) \) is finite. Thus, the hypotheses of Theorem 3.4 are satisfied and the required conclusion follows.

4 On a variant of the question of Hachimori

In this section, we will study certain variant of the questions of Hachimori. In particular, we show that these questions are consequences of various conjectures on the structure of the second Iwasawa cohomology groups.

As before, \( p \) will denote a fixed prime. Let \( F \) be a number field, where we assume further that it has no real primes when \( p = 2 \). Let \( S \) be a finite set of primes of \( F \) containing the primes above \( p \) and the infinite primes. Following [CS], we say that \( F_\infty \) is a \( S \)-admissible \( p \)-adic Lie extension of \( F \) if (i) Gal(\( F_\infty/F \)) is a compact pro-\( p \)-adic Lie group without \( p \)-torsion, (ii) \( F_\infty \) contains the cyclotomic \( \mathbb{Z}_p \)-extension \( F_{cyc} \) of \( F \) and (iii) \( F_\infty \) is contained in \( F_S \). Write \( G = \text{Gal}(F_\infty/F) \), \( H = \text{Gal}(F_\infty/F_{cyc}) \) and \( \Gamma = \text{Gal}(F_{cyc}/F) \). As before, we denote \( T \) to be a finitely generated free \( R \)-module with a continuous \( R \)-linear \( G \)-action, where \( R \) is a complete regular local ring with finite residue field of characteristic \( p \). We will like to study the following variant of the questions of Hachimori.

**Question:** Do the following statements hold?

\[(\alpha) \ \text{rank}_{R[G]} H^1_{\text{Iw}}(F_\infty/F, T) = \text{rank}_{R[\Gamma]} H^1_{\text{Iw}}(F_\infty/F, T)_H; \]

\[(\beta) \ \text{H}_i(H, H^1_{\text{Iw}}(F_\infty/F, T)) \text{ is a torsion } R[\Gamma]-\text{module for each } i \geq 1. \]

As before, it follows from an application of the formula of Howson (cf. Lemma 2.1) that (\( \beta \)) implies (\( \alpha \)). To facilitate further discussion, we record the following lemma (cf. [CS, Lemma 3.2] or [Lim, Lemma 5.2]).

**Lemma 4.1.** Let \( F_\infty \) be a \( S \)-admissible \( p \)-adic Lie extension of \( F \). Then the following statements are equivalent.

\[(a) \ \text{Y}(T/F_{cyc}) \text{ is a finitely generated } R\text{-module.} \]

\[(b) \ H^2_{\text{Iw}}(F_{cyc}/F, T) \text{ is a finitely generated } R\text{-module.} \]

\[(c) \ \text{Y}(T/F_\infty) \text{ is a finitely generated } R[[H]]\text{-module.} \]

\[(d) \ H^2_{\text{Iw}}(F_\infty/F, T) \text{ is a finitely generated } R[[H]]\text{-module.} \]

We now recall the following conjecture which has been studied in [CS, JhS, Lim].

**Conjecture A:** For any number field \( F \), one (and hence all) of the statements in Lemma 4.1 holds.

**Proposition 4.2.** Let \( F_\infty \) be a \( S \)-admissible \( p \)-adic Lie extension of \( F \) with Galois group \( G \). Assume that Conjecture A holds. Then \( H_n(H, H^1_{\text{Iw}}(F_\infty/F, T)) \) is a finitely generated \( R \)-module for every \( n \geq 1 \) and \( j = 1, 2 \).

In particular, statements (\( \alpha \)) and (\( \beta \)) in the Question hold.
Before proving the proposition, we record the following variant of Lemma~\ref{lem:main} which will be required in the proof of the proposition and subsequent part of this paper.

**Lemma 4.4.** Let $F'_\infty$ be an infinite $p$-adic extension of $F$ contained in $F_\infty$ which has the property that $\text{Gal}(F'_\infty/F)$ has no $p$-torsion. Write $N = \text{Gal}(F_\infty/F'_\infty)$. We then have isomorphisms

$$H^2_{\text{Iw}}(F_\infty/F,T) \cong H^2_{\text{Iw}}(F'_\infty/F,T) \quad \text{and} \quad H_i(N, H^1_{\text{Iw}}(F_\infty/F,T)) \cong H_{i+2}(N, H^2_{\text{Iw}}(F_\infty/F,T)) \quad \text{for } i \geq 1.$$

*Proof.* Since $F'_\infty$ is an infinite pro-$p$ extension of $F$, we have $H^0_{\text{Iw}}(F'_\infty/F,T) = 0$. The proof then proceeds as in Lemma~\ref{lem:main} \hfill \Box

*Proof of Proposition 4.2.* Since we are assuming that Conjecture A is valid, we have that $H^2_{\text{Iw}}(F_\infty/F,T)$ is a finitely generated $R[H]$-module. It is then an easy exercise (or see \cite[Lemma 3.2.3]{LS}) to show that $H_i(H, H^1_{\text{Iw}}(F_\infty/F,T))$ is a finitely generated $R$-module for $i \geq 1$. By the second isomorphism in Lemma 4.3 this in turn implies that $H_i(H, H^1_{\text{Iw}}(F_\infty/F,T))$ is a finitely generated $R$-module for $i \geq 1$. In particular, statement $(\beta)$ holds. Statement $(\alpha)$ then follows from an application of Howson’s formula. \hfill \Box

When $T = \mathbb{Z}_p(1)$, Conjecture A is precisely the classical conjecture of Iwasawa which asserts that the $\mu$-invariant of $\text{Gal}(K(F^{\text{cyc}})/F^{\text{cyc}})$ vanishes \cite{Iw1}. Here $K(F^{\text{cyc}})$ is the maximal unramified pro-$p$ extension of $F^{\text{cyc}}$ where every prime of $\mathcal{L}$ above $p$ splits completely. We shall call this conjecture the Iwasawa $\mu$-conjecture for $F^{\text{cyc}}$. It is interesting to note that the general statement of Conjecture A turns out to be a consequence of this classical conjecture of Iwasawa (cf. \cite[Theorem 3.1]{Lim}). However, even in this classical setting, the conjecture is only proved in the case when $F$ is abelian over $\mathbb{Q}$ (see \cite{PSim}). Building on this, we at least have the following unconditional validity of statements $(\alpha)$ and $(\beta)$.

**Corollary 4.4.** Let $F_\infty$ be a $S$-admissible $p$-adic Lie extension of $F$ with Galois group $G$. Assume further that $F(\mu_{2p}, T/mT)$ is a finite $p$-extension of an abelian extension $F'$ of $\mathbb{Q}$. Then statements $(\alpha)$ and $(\beta)$ hold.

*Proof.* It follows from an application of the main theorem of \cite[Theorem 3]{PSim} that the Iwasawa $\mu$-conjecture holds for $F(\mu_{2p}, T/mT)^{\text{cyc}}$. By \cite[Theorem 3.5]{Lim}, this implies that Conjecture A holds for $T$ over $F(\mu_{2p}, T/mT)^{\text{cyc}}$. By \cite[Lemma 3.2]{Lim}, this in turn implies that Conjecture A holds for $T$ over $F^{\text{cyc}}$. The conclusion of the corollary now follows from Proposition 4.2 \hfill \Box

In particular, Conjecture A holds for elliptic curves over $\mathbb{Q}$ that have a rational $p$-isogeny (cf. \cite[Corollary 3.6]{CS}). Therefore, we can apply Corollary 4.4 to obtain $(\alpha)$ and $(\beta)$ unconditionally. We shall give an example that is not an elliptic curve. For the rest of this paragraph, we shall assume that $p \geq 5$. For each integer $n$ such that $1 \leq n \leq p-2$, let $J_n$ be the Jacobian variety of the curve $y^p = x^n(1-x)$. Let $T = T_n$ be the Tate module of the $p$-division points of the abelian...
variety of $J_n$. Write $R = \mathbb{Z}_p[\zeta_p]$ and $\pi = 1 - \zeta_p$. Note that $\pi$ is a generator of the maximal ideal of $\mathcal{O}$. It follows from the discussion in [Gr Section 2] that $T$ is a free $R$-module of rank one and that $G_S(Q(\mu_p))$ acts trivially on $T/\pi T$. Therefore, we may apply preceding corollary to obtain validity of $(\alpha)$ and $(\beta)$ for any $p$-adic Lie extension of $Q(\mu_p)$.

We will now investigate the statements $(\alpha)$ and $(\beta)$ under the following weaker assumption on $H^2_{Iw}(F_{\infty}/F, T)$.

**Conjecture WL/F_{\infty}**: For any $S$-admissible extension $F_{\infty}$ of $F$, $H^2_{Iw}(F_{\infty}/F, T)$ is a torsion $R[[G]]$-module.

We note that Conjecture WL will follow from Conjecture A. However unlike Conjecture A, Conjecture WL/F_{\infty} does not have good descent properties and hence the dependence on $F_{\infty}$. In general, one can show that if Conjecture WL/F_{\infty} holds, then Conjecture WL/L_{\infty} holds for any solvable extension $L_{\infty}$ of $F_{\infty}$ (cf. [Lim Proposition 7.2]). Despite so, Conjecture WL/F_{\infty} is known to hold in many cases. For instance, it is well-known that Conjecture WL/F_{\infty} holds for every $F_{\infty}$ if $T = \mathbb{Z}_p(1)$ (cf. [lw1 Theorem 5] and [OcV Theorem 6.1]). In the case when $E$ is an elliptic curve defined over $Q$ with good reduction at $p$ and $F$ is an abelian extension of $Q$, a deep theorem of Kato [Ka1 Theorem 12.4] asserts that Conjecture WL/F_{cyc} is valid.

As it turns out, we are not able to establish an analogue statement as Proposition 4.2 assuming Conjecture WL only. However, we do have the following partial result in this direction whose proof, which we omit, is similar to Theorem 3.2.

**Theorem 4.5.** Let $F_{\infty}$ be a $S$-admissible $p$-adic Lie extension of $F$ with Galois group $G$. As before, denote $H = \text{Gal}(F_{\infty}/F_{\text{cyc}})$. Suppose that one of the following statements holds.

(i) The group $H$ has dimension $\leq 2$.

(ii) Suppose that there is a finite family of closed normal subgroups $H_i$ ($0 \leq i \leq r$) of $G$ such that $1 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_r = H$, $H_i/H_{i-1} \cong \mathbb{Z}_p$ for $1 \leq i \leq r$ and such that the action of $G$ on $H_i/H_{i-1}$ by inner automorphism is given by a homomorphism $\chi_i : G/H_i \longrightarrow \mathbb{Z}_p^\times$. Also, suppose that Conjecture WL/F_{cyc} holds.

Then statements $(\alpha)$ and $(\beta)$ hold.

Combining the preceding theorem with Kato’s theorem [Ka1 Theorem 12.4] on the validity of Conjecture WL/F_{cyc}, we have the following corollary.

**Corollary 4.6.** Set $F = Q(\mu_p, a_1, \ldots, a_n)$, where each $a_i$ is a nonzero element of the maximal abelian extension of $Q$ which is not a root of unity. Let $F_{\infty} = Q(\mu_p^\infty, a_1^{p^\infty}, \ldots, a_n^{p^\infty})$. Suppose that $T$ is one of the following Galois modules.

(i) The Tate module of an elliptic curve over $Q$ with good reduction at $p$. Here $T$ is a module over $\mathbb{Z}_p$. 

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(ii) A Galois invariant lattice of the Galois representation associated to a normalized cuspidal eigenform \( f \) of weight \( \geq 2 \), tame level \( N \) and character \( \psi \) that is ordinary at \( p \). Here \( T \) is a module over the ring of integers of the completion of the number field \( K_f \) at some prime above \( p \), where \( K_f \) is the number field generated by the Fourier coefficients of \( f \) and the values of \( \psi \).

(iii) The Galois representation associated to a nearly ordinary Hida deformation (cf. [Jhs, P.181(4)]) satisfying the conditions (Nor) and (Irr) as given in [Jhs, P.182]. Here \( T \) is a module over \( \mathcal{O}[X] \).

Then statements (\( \alpha \)) and (\( \beta \)) hold.

**Proof.** As is well-known (and noting that \( F \) is an abelian extension of \( \mathbb{Q} \)), the validity of Conjecture WL\( _{/F_{\text{cyc}}} \) in cases (i) and (ii) follows from [Ka1, Theorem 12.4]. The validity of Conjecture WL\( _{/F_{\text{cyc}}} \) in case (iii) follows from that in (i) and (ii) by a standard control argument (or see [Lim, Proposition 7.3]).

**Remark 4.7.** We like to mention that in a recent paper of Aribam [A], he was able to establish Conjecture A for certain elliptic curves and modular forms (which are defined over \( \mathbb{Q} \)) over the cyclotomic \( \mathbb{Z}_p \)-extension of certain number fields \( F \) that are non abelian over \( \mathbb{Q} \). His examples can therefore be applied to give (unconditional) examples of validity of statements (\( \alpha \)) and (\( \beta \)) that are not covered by our Corollary 4.4 and Corollary 4.6.

## 5 Pseudo-nullity of homology of Iwasawa cohomology groups

We retain the notation of Section 4. The aim of this section is to prove the following theorem which refines [Lim, Theorem 5.7] under a stronger assumption.

**Theorem 5.1.** Let \( F_{\infty} \) be a \( S \)-admissible \( p \)-adic Lie extension of \( F \). Assume that Conjecture A holds. Suppose that \( F'_{\infty} \) is another \( S \)-admissible \( p \)-adic Lie extension of \( F \) which satisfies the following properties.

(i) \( F'_{\infty} \) is contained in \( F_{\infty} \).

(ii) For each \( v \in S \), the decomposition group of \( \text{Gal}(F'_{\infty}/F) \) at \( v \) has dimension \( \geq 2 \).

(iii) There is a finite family of closed normal subgroups \( N_i \) (\( 0 \leq i \leq r \)) of \( \text{Gal}(F_{\infty}/F) \) such that \( 1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = \text{Gal}(F_{\infty}/F) \), \( N_i/N_{i-1} \cong \mathbb{Z}_p \) for \( 1 \leq i \leq r \) and such that the action of \( G \) on \( N_i/N_{i-1} \) by inner automorphism is given by a homomorphism \( \chi_i : G/N_i \rightarrow \mathbb{Z}_p^\times \).

Then \( Y(T/F_{\infty}) \) is a pseudo-null \( \mathcal{R}[\text{Gal}(F'_{\infty}/F)] \)-module if and only if \( Y(T/F_{\infty}) \) is a pseudo-null \( \mathcal{R}[\text{Gal}(F_{\infty}/F)] \)-module and \( H_i(N,H^w_{Iw}(F_{\infty}/F,T)) \) is a pseudo-null \( \mathcal{R}[\text{Gal}(F'_{\infty}/F)] \)-module for all \( i \geq 1 \).
Proof. By assumption (ii) and \[\text{Lemma 5.3,}\] \(Y(T/F'_\infty)\) is a pseudo-null \(R[\text{Gal}(F'_\infty/F)]\)-module if and only if \(H^2_{Iw}(F'_\infty/F, T)\) is a pseudo-null \(R[\text{Gal}(F'_\infty/F)]\)-module. We have a similar statement for \(Y(T/F_\infty)\) and \(H^2_{Iw}(F_\infty/F, T)\). The conclusion of the theorem now follows from an application of Proposition \(2.6\) and Lemma \(4.3\).

Combining the above with \[\text{Lemma 4.3,}\] we have the following corollary which will answer a pseudo-null analog of statement (β).

**Corollary 5.2.** Retain the assumptions of \[\text{Theorem 5.1,}\] Suppose that \(Y(T/F'_\infty)\) is a pseudo-null \(R[\text{Gal}(F'_\infty/F)]\)-module. Then we have that \(H_i(N, H^1_{Iw}(F_\infty/F, T))\) is a pseudo-null \(R[\text{Gal}(F'_\infty/F)]\)-module for all \(i \geq 1\).

We end by giving an example. Let \(E\) be the elliptic curve 79A1 of Cremona’s table which is given by

\[
y^2 + xy + y = x^3 + x^2 - 2x.
\]

Take \(p = 3\) and \(F = \mathbb{Q}(\mu_3)\). Let \(S\) be the set of primes of \(F\) lying above 3, 79 and \(\infty\). Let \(T\) denote either the Tate module of \(E\) or the Galois representation attached to the Hida family associated to the weight 2 newform corresponding to \(E\). It was shown in \[\text{[JH]}\] that \(Y(T/F'_\infty)\) is a pseudo-null \(R[\text{Gal}(F'_\infty/F)]\)-module when \(F'_\infty = \mathbb{Q}(\mu_{3\infty}, 79^{-3\infty})\). Since \(\mathbb{Q}(\mu_{3\infty}, 79^{-3\infty})\) satisfies statement (b) of \[\text{Theorem 5.1,}\] one can apply \[\text{Theorem 5.1,}\] to conclude that \(Y(T/F_\infty)\) is a pseudo-null \(R[\text{Gal}(F_\infty/F)]\)-module, \(H_n(\text{Gal}(F_\infty/F'), H^1_{Iw}(F_\infty/F, T_E))\) is a pseudo-null \(\mathbb{Z}_5[\text{Gal}(F'_\infty/F)]\)-module for every \(n \geq 1\) and \(i = 1, 2\), and \(H_n(\text{Gal}(F_\infty/F'), Y(T_E/F_\infty))\) is a pseudo-null \(\mathbb{Z}_5[\text{Gal}(F'_\infty/F)]\)-module for every \(n \geq 1\), where \(F_\infty\) is one of the following \(S\)-admissible 3-adic Lie extensions:

\[
\mathbb{Q}(\mu_{3\infty}, 3^{3^{-\infty}}, 79^{3^{-\infty}}), \quad L_\infty(79^{3^{-\infty}}), \quad L_\infty(3^{3^{-\infty}}, 79^{3^{-\infty}}).
\]

Here \(L_\infty\) is the unique \(\mathbb{Z}_3^\infty\)-extension of \(F\).

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