Universal $R$–matrices for non-standard (1+1) quantum groups

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Abstract

A universal quasitriangular $R$–matrix for the non-standard quantum (1+1) Poincaré algebra $U_z iso(1,1)$ is deduced by imposing analyticity in the deformation parameter $z$. A family $g_\mu$ of “quantum graded contractions” of the algebra $U_z iso(1,1) \oplus U_{-z} iso(1,1)$ is obtained; this set of quantum algebras contains as Hopf subalgebras with two primitive translations quantum analogues of the two dimensional Euclidean, Poincaré and Galilei algebras enlarged with dilations. Universal $R$–matrices for these quantum Weyl algebras and their associated quantum groups are constructed.
1 Introduction

Two types of quantum deformations for the \( so(2,2) \) algebra and for its most relevant graded contractions have been recently studied in [1]. They are called standard and non-standard quantum algebras according to the fact that their corresponding coboundary Lie bialgebras come from a classical \( r \)-matrix which is a skew solution either of the modified classical Yang–Baxter equation (YBE) or of the classical YBE, respectively. In contradistinction with the standard case, the family of non-standard quantum algebras contains as quantum Hopf subalgebras two dimensional Euclidean and (1+1) Poincaré and Galilei algebras enlarged with a dilation: the so-called “Weyl” or similarity subalgebras. These quantum subalgebras share the property of including the two translation generators as primitive. This fact could be relevant in relation with the problem of discretizing two dimensional spaces in some symmetric way.

Let us also recall that a quasitriangular Hopf algebra [2] is a pair \((A, R)\) where \(A\) is a Hopf algebra and \(R \in A \otimes A\) is invertible and verifies

\[
\sigma \circ \Delta X = R(\Delta X)R^{-1}, \quad \forall X \in A, \tag{1.1}
\]

\[
(\Delta \otimes id)R = R_{13}R_{23}, \quad (id \otimes \Delta)R = R_{13}R_{12}, \tag{1.2}
\]

where, if \(R = \sum_i a_i \otimes b_i\), we denote \(R_{12} \equiv \sum_i a_i \otimes b_i \otimes 1\), \(R_{13} \equiv \sum_i a_i \otimes 1 \otimes b_i\), \(R_{23} \equiv \sum_i 1 \otimes a_i \otimes b_i\) and \(\sigma\) is the flip operator \(\sigma(x \otimes y) = (y \otimes x)\). If \(A\) is a quasitriangular Hopf algebra, then \(R\) is called a universal \(R\)-matrix and satisfies the quantum YBE:

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1.3}
\]

In this paper, the construction of universal \(R\)-matrices and quantum groups for the above mentioned family of non-standard quantum algebras is discussed. In particular, we obtain the universal \(R\)-matrices for the Weyl subalgebras.

A straightforward approach to this problem would consist on starting from a universal \(R\)-matrix for the non-standard Hopf algebra \(U_z sl(2, \mathbb{R}) \simeq U_z so(2,1)\), since the prescription \(U_z so(2,2) \simeq U_z so(2,1) \oplus U_z so(2,1)\) applied to \(R\)-matrices would lead to a universal \(R\)-matrix for \(U_z so(2,2)\). Afterwards, by introducing “quantum graded contractions” a set of \(R\)-matrices for all the family of non-standard algebras would be obtained. A similar procedure was developed in [3, 4] for getting universal \(R\)-matrices for some standard quantum algebras. Unfortunately, to our knowledge, no universal \(R\)-matrix for the non-standard \(U_z sl(2, \mathbb{R})\) has appeared in the literature, since in fact the one given in [5] neither verifies (1.1) nor satisfies (1.3).

Therefore, we have to focus the problem from a different point of view. Following the method developed in [6] for the standard (1+1) groups and in [7] for the Heisenberg group, we impose analyticity in the deformation parameter \(z\) and relation (1.1) in order to get an \(R\)-matrix for the non-standard quantum (1+1) Poincaré algebra \(U_z iso(1,1)\). The \(R\)-matrix so obtained coincides in turn with a universal \(R\)-matrix.
for the positive Borel subalgebra of the non-standard $sl(2, \mathbb{R})$ given in [8]; this fact proves its universality. This explicit construction together with a brief overview of both the quantum Poincaré algebra and group studied in [9] is presented in Section 2. The fact that

$$U_zt_4(so(1, 1) \oplus so(1, 1)) \simeq U_ziso(1, 1) \oplus U_ziso(1, 1)$$  \hspace{1cm} (1.4)$$
can be also used at the group and $R$–matrix levels and leads to the whole quantum structure for this group as it is shown in Section 3; Lie algebras with structure $t_n(so(p, q) \oplus so(r, s))$ are described in [10]. A “quantum graded contraction” is introduced providing quantum structures for two more non-standard quantum real algebras: $U_ziso(1, 1)$ and $U_zt_4(so(2) \oplus so(1, 1))$ (the latter is isomorphic to the (2+1) expanding Newton–Hooke algebra). Each of these quantum algebras contains a Hopf Weyl subalgebra. Section 4 is devoted to the obtention of the universal $R$–matrices and quantum groups corresponding to these Weyl subalgebras.

### 2 Universal $R$–matrix for the Poincaré group

The (1+1) Poincaré algebra $iso(1, 1)$ is generated by one boost generator $K$ and the translation generators along the light-cone $P_{\pm}$. The Lie brackets are:

$$[K, P_{\pm}] = \pm 2P_{\pm}, \quad [P_{+}, P_{-}] = 0. \hspace{1cm} (2.1)$$

A non-standard coboundary bialgebra of $iso(1, 1)$ is generated by the classical $r$–matrix $r = zK \wedge P_{+}$ which verifies the classical YBE.

The quantum deformations for the universal enveloping algebra $Uiso(1, 1)$ and for the algebra of smooth functions on the group $Fun(ISO(1, 1))$, denoted respectively by $U_ziso(1, 1)$ and $Fun_z(ISO(1, 1))$, are given by the following statements (see [1] for a more detailed exposition and proofs and also [11]):

**Proposition 1** The Hopf structure of $U_ziso(1, 1)$ is given by the coproduct, counit, antipode

$$\Delta P_+ = 1 \otimes P_+ + P_+ \otimes 1,$$

$$\Delta P_- = e^{-zP_+} \otimes P_- + P_- \otimes e^{zP_+},$$

$$\Delta K = e^{-zP_+} \otimes K + K \otimes e^{zP_+};$$

$$\epsilon(X) = 0; \quad \gamma(X) = -e^{zP_+} X e^{-zP_+}, \quad \text{for } X \in \{K, P_{\pm}\};$$

and the commutation relations

$$[K, P_{\pm}] = 2 \frac{\sinh zP_{\pm}}{z}, \quad [K, P_{\mp}] = -2P_{\pm} \cosh zP_{\mp}, \quad [P_{\pm}, P_{\mp}] = 0. \hspace{1cm} (2.3)$$

$$[K, P_{\pm}] = 2 \frac{\sinh zP_{\pm}}{z}, \quad [K, P_{\mp}] = -2P_{\pm} \cosh zP_{\mp}, \quad [P_{\pm}, P_{\mp}] = 0. \hspace{1cm} (2.4)$$
Proposition 2 The Hopf algebra $\text{Fun}_z(\text{ISO}(1,1))$ has multiplication given by

\[
[\hat{\chi}, \hat{a}_+] = z(e^{2\hat{\chi}} - 1), \quad [\hat{\chi}, \hat{a}_-] = 0, \quad [\hat{a}_+, \hat{a}_-] = -2z\hat{a}_-;
\]  

(2.5)

coproduct

\[
\Delta(\hat{\chi}) = \hat{\chi} \otimes 1 + 1 \otimes \hat{\chi}, \quad \Delta(\hat{a}_+) = \hat{a}_+ \otimes 1 + e^{2\hat{\chi}} \otimes \hat{a}_+;
\]  

(2.6)

counit and antipode

\[
\epsilon(X) = 0, \quad X \in \{\hat{a}_+, \hat{a}_-, \hat{\chi}\};
\]

(2.7)

\[
\gamma(\hat{\chi}) = -\hat{\chi}, \quad \gamma(\hat{a}_+) = -e^{2\hat{\chi}}\hat{a}_+.
\]  

(2.8)

The quantum coordinates $\hat{\chi}, \hat{a}_-$ and $\hat{a}_+$ of $\text{Fun}_z(\text{ISO}(1,1))$ are, respectively, the dual basis of the $U_z\text{iso}(1,1)$ generators $H = e^{zP}K, A_+ = e^{-zP}P$ and $A_+ = P_+$. Now we proceed to deduce a universal $R$–matrix for $U_z\text{iso}(1,1)$. We assume that the $R$–matrix is analytical in the quantum parameter $z (= \log q)$ and that $R = 1 \otimes 1 + zK \wedge P_+ + o(z^2)$. Hence we consider an $R$–matrix as a formal power series in $z$ with coefficients in $U\text{iso}(1,1) \otimes U\text{iso}(1,1)$. We start from the following Ansatz:

\[
R = \exp\{zf(K, P_+, z) g(P_+, P_-, z)\}.
\]  

(2.9)

Firstly, we impose $R$ to verify relation (1.1). Starting with the primitive generator $P_+$, it is implied that

\[
[R, \Delta P_+] = 0.
\]  

(2.10)

This requirement is fulfilled if

\[
[f(K, P_+, z) g(P_+, P_-, z), \Delta P_+] = [f(K, P_+, z), \Delta P_+] g(P_+, P_-, z) = 0.
\]  

(2.11)

Therefore, by taking into account commutation rules (2.4) a solution for $f(K, P_+, z)$ is

\[
f(K, P_+, z) = K \wedge \sinh zP_+.
\]  

(2.12)

Now we should apply condition (1.1) for the two remaining generators $P_-$ and $K$. Omitting the arguments of the functions $f$ and $g$ we have:

\[
R \Delta X R^{-1} = \exp(zfg) \Delta X \exp(-zfg) = \Delta X + z [fg, \Delta X] + \frac{z^2}{2!} [fg, [fg, \Delta X]] + \ldots
\]

\[
+ \frac{z^n}{n!} [fg, [fg \ldots [fg, \Delta X]^n \ldots]] + \ldots
\]  

(2.13)

Since $P_-$ commutes with $P_+$, relation (2.13) with $X \equiv P_-$ becomes into

\[
\exp(zfg) \Delta P_- \exp(-zfg) = \Delta P_- + z [f, \Delta P_-] g + \frac{z^2}{2!} [f, [f, \Delta P_-]] g^2 + \ldots
\]

\[
+ \frac{z^n}{n!} [f, [f \ldots [f, \Delta P_-]^n \ldots]] g^n + \ldots
\]  

(2.14)
We need to obtain the brackets \([f, \Delta P_ -], [f, [f, \Delta P_ -]], \ldots \) in (2.14). The first and the second ones are

\[
[f, \Delta P_ -] = A, \quad [f, [f, \Delta P_ -]] = B \, 2 \sinh z \Delta P_ +, \quad (2.15)
\]

where

\[
A \equiv 2 \exp(-z \Delta P_ +) \sinh z P_+ \otimes P_- - 2 \exp(z \Delta P_ +) P_- \otimes \sinh z P_+, \quad (2.16)
\]

\[
B \equiv 2 \exp(-z \Delta P_ +) \sinh z P_+ \otimes P_- + 2 \exp(z \Delta P_ +) P_- \otimes \sinh z P_. \quad (2.17)
\]

Due to expression (2.12) \(f\) commutes with any arbitrary function of \(P_+\), so we get

\[
[f, \sinh z \Delta P_+ ] = 0, \quad [f, B] = A \, 2 \sinh z \Delta P_+. \quad (2.18)
\]

By a recurrence method we obtain the \(2^n\) and \(2^n + 1\) iterates:

\[
[f, [\ldots [f, \Delta P_ -^{2n} \ldots ]]] = B \, (2 \sinh z \Delta P_+)^{2n-1}, \quad (2.19)
\]

\[
[f, [\ldots [f, \Delta P_ -^{2n+1} \ldots ]]] = A \, (2 \sinh z \Delta P_+)^{2n}, \quad (2.20)
\]

so that expression (2.14) can be written as follows

\[
\exp(zfg) \Delta P_ - \exp(-zfg) = \Delta P_ - + A \sum_{l=0}^{\infty} \frac{z^{2l+1}}{(2l+1)!} (2 \sinh z \Delta P_+)^{2l+1} + B \sum_{l=1}^{\infty} \frac{z^{2l}}{(2l)!} (2 \sinh z \Delta P_+)^{2l-1} g^{2l}
\]

\[
= \Delta P_ - + \frac{A}{2 \sinh z \Delta P_+} \sinh (2z \sinh (z \Delta P_+) g)
\]

\[
+ \frac{B}{2 \sinh z \Delta P_+} [\cosh (2z \sinh (z \Delta P_+) g) - 1 \otimes 1]. \quad (2.21)
\]

By introducing

\[
C = \frac{1}{2 \sinh z \Delta P_+} \sinh (2z \sinh (z \Delta P_+) g), \quad (2.22)
\]

\[
D = \frac{1}{2 \sinh z \Delta P_+} [\cosh (2z \sinh (z \Delta P_+) g) - 1 \otimes 1], \quad (2.23)
\]

expression (2.21) can be written as \(\Delta P_ - + AC + BD\), which must be equal to

\[
\sigma \circ \Delta P_ - = e^{zP_+} \otimes P_- + P_- \otimes e^{-zP_+}. \quad (2.24)
\]

Thus, if we impose that (2.21) coincides with (2.24) we obtain the following system of equations for the function \(g\):

\[
(1 \otimes P_-)(e^{-zP_+} \otimes 1 - e^{zP_+} \otimes 1 + 2e^{-zP_+} \sinh z P_+ \otimes e^{-zP_+} (D + C)) = 0, \quad (2.25)
\]

\[
(P_- \otimes 1)(1 \otimes e^{zP_+} - 1 \otimes e^{-zP_+} + 2e^{zP_+} \sinh z P_+ \otimes e^{zP_+} (D - C)) = 0. \quad (2.26)
\]

Both equations can be summarized by the expression

\[
\exp(\pm 2z \sinh (z \Delta P_+) g) = \exp(\pm 2z \Delta P_+) \quad (2.27)
\]
leading to the same result for the function $g$:

$$g = \frac{\Delta P_+}{\sinh z\Delta P_+}. \quad (2.28)$$

Finally, an explicit check shows that the $R$–matrix

$$R = \exp \left\{ K \wedge \sinh z P_+ \frac{z\Delta P_+}{\sinh z\Delta P_+} \right\} \quad (2.29)$$

also verifies property (1.1) for the last generator $K$. Note that the functions $f$ (2.12) and $g$ (2.28) commute.

The result (2.29) is in fact similar to a universal $R$–matrix given in [8] for a Hopf algebra $\{v, h\}$ which is isomorphic to the Hopf subalgebra $\{K, P_+\}$ of $U_ziso(1, 1)$. Therefore, since the $R$–matrix (2.29) does not depend on $P_-$ the universality holds and it satisfies the quantum YBE (1.3). On the other hand, it is worth remarking that expression (2.29) has been also obtained in [12] following a different procedure.

### 3 Construction of $U_ziso(1, 1) \oplus U_{-z}iso(1, 1)$

Let us consider two copies of $iso(1, 1)$ with generators $\{K^l, P^l_\pm\}$ ($l = 1, 2$). The set of generators defined by

$$J_3 = K^1 + K^2, \quad J_\pm = P^1_\pm + P^2_\pm, \quad N_3 = K^1 - K^2, \quad N_\pm = P^1_\pm - P^2_\pm, \quad (3.1)$$
closes the algebra $t_4(so(1, 1) \oplus so(1, 1)) \simeq iso(1, 1) \oplus iso(1, 1)$. The formal transformation (equivalent to a graded contraction [4]) defined by

$$(J, P_1, P_2, C_1, C_2, D) := (\sqrt{\mu}N_3/2, J_+, \sqrt{\mu}N_+, -J_-, \sqrt{\mu}N_-, J_3/2), \quad (3.2)$$
gives rise to the following non-vanishing commutation relations

$$[J, P_1] = P_2, \quad [J, P_2] = \mu P_1, \quad [D, P_1] = P_3, \quad [J, C_1] = C_2, \quad [J, C_2] = \mu C_1, \quad [D, C_1] = -C_3. \quad (3.3)$$

For $\mu$ equal to $+1$, $0$ and $-1$ we obtain, in this order, the commutators of $t_4(so(1, 1) \oplus so(1, 1))$, $iiiso(1, 1)$ and $t_4(so(2) \oplus so(1, 1))$. We denote these three algebras by $g_\mu$.

In the following, we will show how the results presented in the previous section for the quantum non-standard $(1+1)$ Poincaré algebra provide a quantum structure for the algebras $g_\mu$ and for the groups $G_\mu$ as well as their universal $R$–matrices.

The invariance of $U_ziso(1, 1)$ under the transformation $z \to -z$ allows us to write $U_z t_4(so(1, 1) \oplus so(1, 1)) = U_ziso(1, 1) \oplus U_{-z}iso(1, 1)$. The contraction (3.3) is implemented to the quantum case by considering the following definition of the contracted generators and deformation parameter

$$(J, P_1, P_2, C_1, C_2, D; w) := (\sqrt{\mu}N_3/2, J_+, \sqrt{\mu}N_+, -J_-, \sqrt{\mu}N_-, J_3/2; z/\sqrt{\mu}), \quad (3.4)$$

where $w$ is the new (contracted) quantum parameter. In this way we obtain the new (contracted) quantum parameter. In this way we obtain the Hopf structure of $U_wg_\mu$. We omit the explicit expressions so obtained since they are exactly the quantum algebras $U^{(n)}_w g_{(\mu_1, 0, +)}$ (with $\mu \equiv \mu_1$) given in [4].
3.1 Poisson–Hopf structure of Fun($G_\mu$)

Before getting the quantum groups associated to $U_\mu g_\mu$, we first study the algebra $Fun(G_\mu)$ of smooth functions on the group $G_\mu$.

A matrix realization of $g_\mu$ in terms of $4 \times 4$ real matrices is:

\[
J = \begin{pmatrix}
0 & -\mu & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad
P_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad
P_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad C_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad C_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

Hence, a real $4 \times 4$ representation of the element $g = e^{c_1C_1}e^{c_2C_2}e^{p_1P_1}e^{p_2P_2}e^{dD}e^{\theta J} \in G_\mu$ is given by

\[
g = \begin{pmatrix}
C_{-\mu}(\theta) & -\mu S_{-\mu}(\theta) & 0 & 0 \\
S_{-\mu}(\theta) & C_{-\mu}(\theta) & 0 & 0 \\
t_{31} & t_{32} & \cosh d & \sinh d \\
t_{41} & t_{42} & \sinh d & \cosh d
\end{pmatrix},
\]

with

\[
t_{31} = (p_2 - c_2) C_{-\mu}(\theta) - (p_1 - c_1) S_{-\mu}(\theta),
\]

\[
t_{32} = (p_1 - c_1) C_{-\mu}(\theta) - \mu(p_2 - c_2) S_{-\mu}(\theta),
\]

\[
t_{41} = (p_2 + c_2) C_{-\mu}(\theta) - (p_1 + c_1) S_{-\mu}(\theta),
\]

\[
t_{42} = (p_1 + c_1) C_{-\mu}(\theta) - \mu(p_2 + c_2) S_{-\mu}(\theta).
\]

The generalized sine and cosine functions are defined by

\[
C_{-\mu}(\theta) = \frac{e^{\sqrt{\mu} \theta} + e^{-\sqrt{\mu} \theta}}{2}, \quad S_{-\mu}(\theta) = \frac{e^{\sqrt{\mu} \theta} - e^{-\sqrt{\mu} \theta}}{2\sqrt{\mu}}.
\]

Note that for $\mu$ equal to $+1$ and $-1$ we recover the hyperbolic and elliptic trigonometric functions. The case $\mu = 0$ corresponds to a contraction of the group representation (3.6): $C_0(\theta) = 1$ and $S_0(\theta) = \theta$.

**Proposition 3** The fundamental Poisson brackets

\[
\{d, p_1\} = w \mu e^d S_{-\mu}(\theta), \quad \{p_1, c_1\} = w \mu c_2,
\]

\[
\{d, p_2\} = w(e^d C_{-\mu}(\theta) - 1), \quad \{p_1, c_2\} = w c_1,
\]

\[
\{\theta, p_1\} = w(e^d C_{-\mu}(\theta) - 1), \quad \{p_2, c_1\} = -w c_1,
\]

\[
\{\theta, p_2\} = w e^d S_{-\mu}(\theta), \quad \{p_2, c_2\} = -w c_2,
\]

endow $Fun(G_\mu)$ with a Poisson–Hopf algebra structure.
The Poisson brackets (3.9) comes from the Sklyanin bracket induced from a classical $r$–matrix

$$\{\Psi, \Phi\} = r^{\alpha\beta} \left( X^L_\alpha \Psi X^L_\beta \Phi - X^R_\alpha \Psi X^R_\beta \Phi \right) \quad \Psi, \Phi \in \text{Fun}(G_\mu).$$

In our case the $r$–matrix which satisfies the classical YBE is given by

$$r = w \left( J \wedge P_1 + D \wedge P_2 \right),$$

while left and right invariant vector fields are deduced from (3.6)

$$X^L_j = \partial_j, \quad X^L_D = \partial_d,$n
$$X^L_P = e^d C_{-\mu}(\theta) \partial_{p_1} + e^d S_{-\mu}(\theta) \partial_{p_2},$$
$$X^L_{C_1} = e^{-d} C_{-\mu}(\theta) \partial_{c_1} + e^{-d} S_{-\mu}(\theta) \partial_{c_2},$$
$$X^L_{C_2} = e^{-d} C_{-\mu}(\theta) \partial_{c_2} + \mu e^{d} S_{-\mu}(\theta) \partial_{c_1},$$

$$X^R_j = \partial_j + \mu p_2 \partial_{p_1} + p_1 \partial_{p_2} + \mu c_2 \partial_{c_1} + c_1 \partial_{c_2},$$
$$X^R_D = \partial_d + p_1 \partial_{p_1} + p_2 \partial_{p_2} - c_1 \partial_{c_1} - c_2 \partial_{c_2},$$
$$X^R_P = \partial_{p_1}, \quad X^R_{P_2} = \partial_{p_2}, \quad X^R_{C_1} = \partial_{c_1}, \quad X^R_{C_2} = \partial_{c_2}.$$  \hspace{1cm} (3.12)

### 3.2 Hopf structure of $\text{Fun}_w(G_\mu)$

Now we proceed to quantize the Poisson–Hopf algebra $\text{Fun}(G_\mu)$. First we consider two sets of quantum coordinates $\{\hat{\chi}^l, \hat{a}^l, \hat{a}^l_\pm\}$ ($l = 1, 2$) of $\text{Fun}_\pm(\text{ISO}(1,1))$ for $l = 1$ and of $\text{Fun}_{-\pm}(\text{ISO}(1,1))$ for $l = 2$. Then we construct the Hopf algebra $\text{Fun}_\pm(\text{ISO}(1,1)) \oplus \text{Fun}_{-\pm}(\text{ISO}(1,1))$ with the results of Prop. 2 and by using the new coordinates defined by

$$\hat{a} = \frac{1}{2} (\hat{\chi}^1 + \hat{\chi}^2), \quad \hat{a}_\pm = \frac{1}{2} (\hat{a}^1_\pm + \hat{a}^2_\pm), \quad \hat{b} = \frac{1}{2} (\hat{\chi}^1 - \hat{\chi}^2), \quad \hat{b}_\pm = \frac{1}{2} (\hat{a}^1_\pm - \hat{a}^2_\pm).$$  \hspace{1cm} (3.14)

Next we apply the quantum contraction induced at the group level from (3.4):

$$(\hat{\theta}, \hat{p}_1, \hat{p}_2, \hat{c}_1, \hat{c}_2, \hat{d}; w) := \left( 2\hat{b}/\sqrt{\mu}, \hat{a}_+ / \sqrt{\mu}, -\hat{a}_-, \hat{b}_+ / \sqrt{\mu}, 2\hat{a}; z / \sqrt{\mu} \right),$$

obtaining in this way the quantization of $\text{Fun}(G_\mu)$. The final result is summarized as follows:

**Proposition 4** The Hopf algebra $\text{Fun}_w(G_\mu)$ is given by non-vanishing commutators

$$[\hat{d}, \hat{p}_1] = w \mu e^d S_{-\mu}(\hat{\theta}), \quad [\hat{p}_1, \hat{c}_1] = w \mu \hat{c}_2,$$
$$[\hat{d}, \hat{p}_2] = w (e^d C_{-\mu}(\hat{\theta}) - 1), \quad [\hat{p}_1, \hat{c}_2] = w \hat{c}_1,$$
$$[\hat{\theta}, \hat{p}_1] = w (e^d C_{-\mu}(\hat{\theta}) - 1), \quad [\hat{p}_2, \hat{c}_1] = -w \hat{c}_1,$$
$$[\hat{\theta}, \hat{p}_2] = w e^d S_{-\mu}(\hat{\theta}), \quad [\hat{p}_2, \hat{c}_2] = -w \hat{c}_2;$$  \hspace{1cm} (3.16)
where \( R \) contraction (3.4). The final expression for the universal coproduct, counit and antipode

\[
\Delta(\hat{\theta}) = \hat{\theta} \otimes 1 + 1 \otimes \hat{\theta}, \quad \Delta(\hat{d}) = \hat{d} \otimes 1 + 1 \otimes \hat{d},
\]

\[
\Delta(\hat{p}_1) = \hat{p}_1 \otimes 1 + e^d C_{-\mu}(\hat{\theta}) \otimes \hat{p}_1 + \mu e^d S_{-\mu}(\hat{\theta}) \otimes \hat{p}_2,
\]

\[
\Delta(\hat{p}_2) = \hat{p}_2 \otimes 1 + e^d C_{-\mu}(\hat{\theta}) \otimes \hat{p}_2 + e^d S_{-\mu}(\hat{\theta}) \otimes \hat{p}_1,
\]

and

\[
\Delta(\hat{c}_1) = \hat{c}_1 \otimes 1 + e^{-d} C_{-\mu}(\hat{\theta}) \otimes \hat{c}_1 + \mu e^{-d} S_{-\mu}(\hat{\theta}) \otimes \hat{c}_2,
\]

\[
\Delta(\hat{c}_2) = \hat{c}_2 \otimes 1 + e^{-d} C_{-\mu}(\hat{\theta}) \otimes \hat{c}_2 + e^{-d} S_{-\mu}(\hat{\theta}) \otimes \hat{c}_1;
\]

\[
\epsilon(X) = 0, \quad X \in \{\hat{\theta}, \hat{p}_i, \hat{c}_i, \hat{d}\};
\]

The final step in this quantization process consists on deducing the universal \( R \)-matrix for \( U_{w}g_\mu \). We write two \( R \)-matrices (2.29) \( R^1_z \) and \( R^2_z \) with generators \( \{K^l, P^l_\pm\} (l = 1, 2) \) for the two copies \( U_{\pm z}iso(1,1) \) and compute the product \( R = R^1_z R^2_z \):

\[
R = \exp \left\{ K^1 \wedge \sinh zP^1_+ \frac{z\Delta P^1_+}{\sinh z\Delta P^1_+} \right\} \exp \left\{ -K^2 \wedge \sinh zP^2_+ \frac{z\Delta P^2_+}{\sinh z\Delta P^2_+} \right\}
\]

\[
= \exp \left\{ (K^1 \wedge \sinh zP^1_+ \Delta P^1_+ \sinh z\Delta P^2_+) \frac{z}{\sinh z\Delta P^1_+ \sinh z\Delta P^2_+} \right\}.
\]

We introduce the change of generators (3.1) and, afterwards, we apply the quantum contraction (3.4). The final expression for the universal \( R \)-matrix of \( U_{w}g_\mu \) (denoted by \( R_w \)) is

\[
R_w = \exp\{ (M_1 N_1 + M_2 N_2) L \},
\]

where

\[
M_1 = D \wedge C_{-\mu}(wP_1/2) \sinh(wP_2/2) + J \wedge S_{-\mu}(wP_1/2) \cosh(wP_2/2),
\]

\[
M_2 = \mu D \wedge S_{-\mu}(wP_1/2) \cosh(wP_2/2) + J \wedge C_{-\mu}(wP_1/2) \sinh(wP_2/2),
\]

\[
N_1 = \mu \Delta P_1 S_{-\mu}(w\Delta P_1/2) \cosh(w\Delta P_2/2) - \Delta P_2 C_{-\mu}(w\Delta P_1/2) \sinh(w\Delta P_2/2),
\]

\[
N_2 = \Delta P_2 S_{-\mu}(w\Delta P_1/2) \cosh(w\Delta P_2/2) - \Delta P_1 C_{-\mu}(w\Delta P_1/2) \sinh(w\Delta P_2/2),
\]

\[
L = \frac{2w}{C_{-\mu}(w\Delta P_1) - \cosh(w\Delta P_2)}.
\]

An interesting idea naturally arising from this result would be the use of the FRT construction [13] to quantize \( Fun(G_\mu) \). In fact, the matrix representation \( \{3.5\} \) substituted in (3.21) gives rise to a particular representation of \( R_w \):

\[
R_w = \exp\{ wr \} = \exp\{ w(J \wedge P_1 + D \wedge P_2) \}.
\]
\[ I \otimes I + w(J \wedge P_1 + D \wedge P_2) + \mu w^2 P_1 \otimes P_1, \quad (3.23) \]

where \( I \) is the four dimensional identity matrix. In this representation the commutation rules of the group coordinates \((\hat{d}, \hat{\theta}, \hat{p}_i, \hat{c}_i)\) would be deduced from equation

\[ R_w T_1 T_2 = T_2 T_1 R_w, \quad (3.24) \]

\( T \) is the generic element of the group \( G_\mu \), \( T_1 = T \otimes I \) and \( T_2 = I \otimes T \). Lengthy computations show that commutators so obtained are exactly those given in (3.16) up to a global change of sign in the deformation parameter \( w \). Furthermore, coproduct (3.17), counit (3.18) and antipode (3.19) can be got from relations \( \Delta(T) = T \otimes T, \epsilon(T) = I \) and \( \gamma(T) = T^{-1} \).

4 Universal quantizations of Weyl subalgebras

The Lie brackets

\[ [J, P_1] = P_2, \quad [J, P_2] = \mu P_1, \quad [P_1, P_2] = 0, \quad (4.1) \]

correspond for \( \mu \) negative, positive and zero to the two dimensional Euclidean, Poincaré and Galilei algebras, respectively. We can enlarge these algebras by means of a dilation generator \( D \):

\[ [D, P_i] = P_i, \quad [D, J] = 0. \quad (4.2) \]

These enlarged algebras are the algebras of similarities of the Euclidean, Minkowskian or Galilean planes, and will be denoted by \( s_\mu \). They are naturally the “Weyl” subalgebras of the corresponding conformal algebras in two dimensions. Although the conformal algebras of the family \((4.1)\) are indeed \( so(3,1), iso(2,1) \) and \( so(2,2) \) for \( \mu <, =, > 0 \) \([1]\), the crucial point is that each of the algebras in the family \( g_\mu \) also contains a subalgebra isomorphic to the Weyl subalgebra of the full conformal algebras of the Euclidean, Poincaré and Galilei spaces. Moreover, the Hopf algebra \( U_w g_\mu \) preserves this property, that is, \( U_w g_\mu \) includes quantum Weyl subalgebras that deform \((4.1,4.2)\).

Proposition 5 The algebras \( U_w s_\mu \) given by \([1]\):

\[
\begin{align*}
\Delta P_1 &= 1 \otimes P_1 + P_1 \otimes 1, \quad \Delta P_2 = 1 \otimes P_2 + P_2 \otimes 1, \\
\Delta J &= e^{-\frac{w^2}{2} P_2} C_{-\mu_1} (wP_1/2) \otimes J + J \otimes C_{-\mu_1} (wP_1/2) e^{\frac{w^2}{2} P_2} - e^{-\frac{w^2}{2} P_2} S_{-\mu_1} (wP_1/2) \otimes \mu_1 D + \mu_1 D \otimes S_{-\mu_1} (wP_1/2) e^{\frac{w^2}{2} P_2}, \\
\Delta D &= e^{-\frac{w^2}{2} P_2} C_{-\mu_1} (wP_1/2) \otimes D + D \otimes C_{-\mu_1} (wP_1/2) e^{\frac{w^2}{2} P_2} - e^{-\frac{w^2}{2} P_2} S_{-\mu_1} (wP_1/2) \otimes J + J \otimes S_{-\mu_1} (wP_1/2) e^{\frac{w^2}{2} P_2}, \\
\epsilon(X) &= 0; \quad \gamma(X) = -e^{wP_2} X e^{-wP_2}, \quad X \in \{J, P_i, D\};
\end{align*}
\]

\[ [J, P_1] = \frac{2}{w} \sinh(wP_2/2) C_{-\mu_1} (wP_1/2), \]
\[ [J, P_2] = \frac{2}{w} \mu_1 S_{-\mu_1} (wP_1/2) \cosh(wP_2/2), \]
\[ [D, P_1] = \frac{2}{w} S_{-\mu_1} (wP_1/2) \cosh(wP_2/2), \]
\[ [D, P_2] = \frac{2}{w} \sinh(wP_2/2) C_{-\mu_1} (wP_1/2), \]
\[ [P_1, P_2] = 0, \quad [D, J] = 0; \]

are quasitriangular Hopf algebras with universal \( R \)-matrix (3.21, 3.22).

Furthermore, it is clear that by taking the generators \( C_i \equiv 0 \) and the group parameters \( \hat{c}_i \equiv 0 \) in Props. 3 and 4 we find a Poisson–Hopf algebra structure for \( \text{Fun}(S_\mu) \) and a quantum Hopf algebra \( \text{Fun}_w(S_\mu) \).

\section{5 Concluding remarks}

A combined approach of the construction \( U_z A \oplus U_{-z} A \) (\( A \) being either a Lie algebra or the algebra of functions on the Lie group) together with a quantum contraction provide a simultaneous universal quantization for the algebras \( g_\mu \) in the family (3.3).

One of the groups in the family \( G_\mu \) can be realized as a kinematical group: the group \( G_{-1} \equiv T_4 \left( SO(2) \otimes SO(1, 1) \right) \) is isomorphic to the (2+1) expanding Newton–Hooke group \( \mathbb{L} \), the motion group of a non-relativistic space-time with constant negative curvature. Time is absolute in such a universe and a space-time contraction leads from it to the Galilean case. An adapted basis for \( G_{-1} \) is formed by a time translation \( \check{H} \), two spatial translations \( \check{P}_i \), two boosts \( \check{K}_i \) and one spatial rotation \( \check{J} \), with corresponding group coordinates \( \{ t, x_i, v_i, \psi \} \) \( (i = 1, 2) \). All expressions obtained for \( G_{-1} \) in section 3 can be written in terms of these new generators and group coordinates by means of the following isomorphisms:

\[ \check{J} \equiv J, \quad \check{P}_i \equiv \frac{1}{2} (P_i + C_i), \quad \check{K}_i \equiv \frac{1}{2} (P_i - C_i), \quad \check{H} \equiv -D, \quad (5.1) \]
\[ \psi \equiv \theta, \quad x_i \equiv 2(p_i + c_i), \quad v_i \equiv 2(p_i - c_i), \quad t \equiv -d. \quad (5.2) \]

An open problem to be solved is the construction of a universal \( R \)-matrix for the non-standard quantum deformation of \( sl(2, \mathbb{R}) \) which would provide a set of universal \( R \)-matrices for the whole set of (2+1) non-standard quantum algebras, following the method just described. We recall that among them there are some rather interesting cases from a physical point of view: the conformal algebras of the (1+1) Poincaré \( (so(2, 2)) \) and two dimensional Euclidean spaces \( (so(3, 1)) \), besides a “null-plane” (2+1) Poincaré algebra.

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