The inverses of tails of the Riemann zeta function

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Abstract
We present some bounds of the inverses of tails of the Riemann zeta function on $0 < s < 1$ and compute the integer parts of the inverses of tails of the Riemann zeta function for $s = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

MSC: Riemann zeta function; Tails of Riemann zeta function; Inverses of tails of the Riemann zeta function

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1 Introduction
The Riemann zeta function $\zeta(s)$ in the real variable $s$ was introduced by Euler [2] in connection with questions about the distribution of prime numbers. Later Riemann [6] derived deeper results about a dual correspondence between the distribution of prime numbers and the complex zeros of $\zeta(s)$ in the complex variable $s$. In these developments, he asserted that all the non-trivial zeros of $\zeta(s)$ are on the line $\text{Re}(s) = \frac{1}{2}$, and this has been one of the most important unsolved problems in mathematics, called the Riemann hypothesis. A vast amount of research on calculation of $\zeta(s)$ on the line $\text{Re}(s) = \frac{1}{2}$, which is called the critical line, and on the strip $0 < \text{Re}(s) < 1$, which is called the critical strip, has been conducted using various methods [1].

The Riemann zeta function and a tail of the Riemann zeta function from $n$ for an integer $n \geq 1$ are defined, respectively, by: for $\text{Re}(s) > 1$,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad \text{and} \quad \zeta_n(s) = \sum_{k=n}^{\infty} \frac{1}{k^s}$$

and for $0 < \text{Re}(s) < 1$,

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s} \quad \text{and} \quad \zeta_n(s) = \frac{1}{1-2^{1-s}} \sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{k^s}.$$

To understand the values of $\zeta(s)$, it would be helpful to understand the values of tails of $\zeta(s)$, for example, the integer parts of their inverses $[\zeta_n(s)^{-1}]$, where $[x]$ denotes the greatest integer that is less than or equal to $x$. 

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Some values of $[\zeta_n(s)^{-1}]$ for small positive integers $s$ have become known recently. Xin [7] showed that for $s = 2$ and 3,

$$[\zeta_n(2)^{-1}] = n - 1 \quad \text{and} \quad [\zeta_n(3)^{-1}] = 2n(n - 1).$$

For $s = 4$, Xin and Xiaoxue [8] showed that

$$[\zeta_n(4)^{-1}] = 3n^3 - 5n^2 + 4n - 1 + \left[\frac{(2n + 1)(n - 1)}{4}\right]$$

for any integer $n \geq 2$, and Xu [9] showed that for $s = 5$,

$$[\zeta_n(5)^{-1}] = 4n^4 - 8n^3 + 9n^2 - 5n + \left[\frac{(n + 1)(n - 2)}{3}\right]$$

for any integer $n \geq 4$. Hwang and Song [3] provided an alternative proof of the case when $s = 5$ and a formula when $s = 6$ as follows. For an integer $n$, write $n_{48}$ for the remainder when $n$ is divided by 48, then

$$[\zeta_n(6)^{-1}] = \begin{cases} 5n^5 - 25n^4 + \frac{75}{2}n^3 - \frac{125}{8}n^2 + \frac{185}{48}n - \frac{5n_{48}}{48} - \frac{35 - 5n_{48}}{48}, & \text{if } n \text{ is even,} \\ 5n^5 - 25n^4 + \frac{75}{2}n^3 - \frac{125}{8}n^2 + \frac{185}{48}n - \frac{5n_{48} + 18}{48} - \frac{17 - 5n_{48}}{48}, & \text{if } n \text{ is odd} \end{cases}$$

for any integer $n \geq 829$. For the integer $s$ greater than 6, no such a formula is known.

There are other interesting results related to this theme such as bounds of $\zeta(3)$ in greater precision in [4] and [5].

We study the inverses of tails of the Riemann zeta function $\zeta_n(s)^{-1}$ for $s$ on the critical strip $0 < s < 1$. The following notation is needed to explain our results.

**Definition 1** For any positive integer $n$ and real number $s$ with $0 < s < 1$, we define

$$A_{n, s} = \left(\frac{1}{n^s} - \frac{1}{(n + 1)^s}\right) + \left(\frac{1}{(n + 2)^s} - \frac{1}{(n + 3)^s}\right) + \cdots$$

and

$$B_{n, s} = \left(-\frac{1}{n^s} + \frac{1}{(n + 1)^s}\right) + \left(-\frac{1}{(n + 2)^s} + \frac{1}{(n + 3)^s}\right) + \cdots.$$

Now the tail of the Riemann zeta function for $0 < s < 1$ can be written as follows:

$$\zeta_n(s) = \begin{cases} -\frac{1}{1 - 2^{-s}}A_{n, s}, & \text{if } n \text{ is even,} \\ -\frac{1}{1 - 2^{-s}}B_{n, s}, & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

In this paper, we present the bounds of $A_{n, s}^{-1}$ and $B_{n, s}^{-1}$, hence the bounds of the inverses of tails of the Riemann zeta function $\zeta_n(s)^{-1}$ for $0 < s < 1$ in Sect. 2.1, and compute the values $[A_{n, s}^{-1}]$ and $[B_{n, s}^{-1}]$, hence the values of the inverses of tails of the Riemann zeta function $\left[1 - 2^{-s}\right] \zeta_n(s)^{-1}$ for $s = \frac{1}{2}, \frac{1}{3},$ and $\frac{1}{4}$ in Sect. 2.2.
2 Main results

2.1 The bounds of the inverses of $\zeta_n(s)$ for $0 < s < 1$

In this section, we present the bounds of $A_{n,s}^{-1}$ and $B_{n,s}^{-1}$ in Definition 1, hence the bounds of the inverses of tails of the Riemann zeta function $\zeta_n(s)^{-1}$ for $0 < s < 1$.

**Proposition 1** Let $s$ be a real number with $0 < s < 1$. Then, for any positive even number $n$,

$$2(n - 1)^s < A_{n,s}^{-1} < 2n^s,$$

and for any positive odd number $n$,

$$-2n^s < B_{n,s}^{-1} < -2(n - 1)^s.$$

**Proof** Let $n$ be a positive even number. For every positive integer $k$, it is easy to see that

$$\left(\frac{1}{(n + 1 + 2k)^s} - \frac{1}{(n + 2 + 2k)^s}\right) < \left(\frac{1}{(n + 2k)^s} - \frac{1}{(n + 1 + 2k)^s}\right) < \left(\frac{1}{(n - 1 + 2k)^s} - \frac{1}{(n + 2k)^s}\right).$$

The summations of each term over $k$ give

$$A_{n+1,s} < A_{n,s} < A_{n-1,s}$$

and

$$\frac{1}{2}(A_{n+1,s} + A_{n,s}) < A_{n,s} < \frac{1}{2}(A_{n-1,s} + A_{n,s}).$$

Therefore, we have

$$\frac{1}{2n^s} < A_{n,s} < \frac{1}{2(n - 1)^s},$$

which gives the first statement.

The second statement can be shown similarly. □

Since every proof of the case when $n$ is an odd number is analogous to that of the case when $n$ is an even number, we omit all the proofs of the odd number cases in this paper.

Now we find tighter bounds for $A_{n,s}^{-1}$ and $B_{n,s}^{-1}$.

**Proposition 2** Let $s$ be a real number with $0 < s < 1$. Then, for any positive even number $n$,

$$2\left(n - \frac{1}{2}\right)^s < A_{n,s}^{-1}.$$
and for any positive odd number \( n \),

\[
B_{n,s} < -2 \left( n - \frac{1}{2} \right)^s.
\]

**Proof** Let \( n \) be a positive even number. We will show that

\[
A_{n,s} < \frac{1}{2(n - \frac{1}{2})^s}.
\]

Rewriting each of the both sides as a series

\[
A_{n,s} = \sum_{k=\frac{n}{2}}^{\infty} \left( \frac{1}{(2k)^s} - \frac{1}{(2k+1)^s} \right)
\]

and

\[
\frac{1}{2(n - \frac{1}{2})^s} = \sum_{k=\frac{n}{2}}^{\infty} \left( \frac{1}{2(2k - \frac{1}{2})^s} - \frac{1}{2(2k + \frac{1}{2})^s} \right),
\]

we will show that for any positive integer \( k \),

\[
\frac{1}{(2k)^s} - \frac{1}{(2k+1)^s} < \frac{1}{2(2k - \frac{1}{2})^s} - \frac{1}{2(2k + \frac{1}{2})^s}.
\]

For this, we let

\[
f(x) = \left( \frac{1}{2(2x - \frac{1}{2})^s} - \frac{1}{2(2x + \frac{1}{2})^s} \right) - \left( \frac{1}{(2x)^s} - \frac{1}{(2x+1)^s} \right)
\]

and will show that \( f(x) \) is positive for \( x \geq 1 \) and \( 0 < s < 1 \). With

\[
g(x) = \frac{1}{2(2x - \frac{1}{2})^s} + \frac{1}{2(2x + \frac{1}{2})^s} - \frac{1}{(2x)^s},
\]

we have \( f(x) = g(x) - g(x + \frac{1}{2}) \). Consider the derivative of \( g(x) \):

\[
g'(x) = -2s \left( \frac{1}{2(2x - \frac{1}{2})^{s+1}} + \frac{1}{2(2x + \frac{1}{2})^{s+1}} - \frac{1}{(2x)^{s+1}} \right).
\]

Since the function \( \frac{1}{x^r} \) is convex, we obtain that

\[
\frac{1}{2(2x - \frac{1}{2})^{s+1}} + \frac{1}{2(2x + \frac{1}{2})^{s+1}} - \frac{1}{(2x)^{s+1}} \geq 0,
\]

and therefore \( g'(x) \) is negative, that is, \( g(x) \) is decreasing. We conclude that \( f(x) \) is positive, which gives the statement. \( \square \)
Proposition 3 Let $s$ be a real number with $0 < s < 1$. Then, for any positive even number $n$,

$$A_{n,s}^{-1} < 2 \left(n - \frac{1}{4}\right)^s,$$

and for any positive odd number $n$,

$$-2 \left(n - \frac{1}{4}\right)^s < B_{n,s}^{-1}.$$

Proof Let $n$ be a positive even number. We will show that

$$\frac{1}{2(n - \frac{1}{4})^s} < A_{n,s}.$$

Rewriting each of the both sides as a series

$$A_{n,s} = \sum_{k=\frac{n}{2}}^{\infty} \left( \frac{1}{(2k)^s} - \frac{1}{(2k + 1)^s} \right)$$

and

$$\frac{1}{2(n - \frac{1}{4})^s} = \sum_{k=\frac{n}{2}}^{\infty} \left( \frac{1}{2(2k-%20-%20\frac{1}{4})^s} - \frac{1}{2(2k + \frac{3}{4})^s} \right),$$

we need to show that for any positive integer $k$,

$$\frac{1}{2(2k-%20-%20\frac{1}{4})^s} - \frac{1}{2(2k + \frac{3}{4})^s} < \frac{1}{(2k)^s} - \frac{1}{(2k + 1)^s}.$$

For this, we let

$$f(x) = \left( \frac{1}{(2x)^s} - \frac{1}{(2x + 1)^s} \right) - \left( \frac{1}{2(2x-%20-%20\frac{1}{4})^s} - \frac{1}{2(2x + \frac{3}{4})^s} \right).$$

We check that $f(1) > 0$ and now we will show that $f(x)$ is positive for $x \geq 2$ and $0 < s < 1$. With

$$g(x) = \frac{1}{(2x)^s} - \left( \frac{1}{2(2x-%20-%20\frac{1}{4})^s} + \frac{1}{2(2x + \frac{3}{4})^s} \right),$$

we have $f(x) = g(x) - g(x + \frac{1}{2})$, so we only need to show that $g(x)$ is decreasing. Consider the derivative of $g(x)$:

$$g'(x) = s \left( -\frac{2}{(2x)^{s+1}} + \frac{1}{(2x-%20-%20\frac{1}{4})^{s+1}} + \frac{1}{(2x + \frac{3}{4})^{s+1}} \right)$$

$$= s \left( \frac{1}{(2x-%20-%20\frac{1}{4})^{s+1}} - \frac{1}{(2x)^{s+1}} \right) \left( \frac{1}{(2x)^{s+1}} - \frac{1}{(2x + \frac{3}{4})^{s+1}} \right).$$
Since the function \(\frac{1}{x^{s+1}}\) is decreasing and convex, by comparing slopes at \((2x - \frac{1}{4})\) and \((2x + \frac{3}{4})\), we obtain
\[
\frac{1}{(2x - \frac{1}{4})^{s+1}} - \frac{1}{(2x + \frac{3}{4})^{s+1}} < \frac{1}{4}(s + 1)\left(\frac{1}{(2x - \frac{1}{4})^{s+2}} - \frac{3}{(2x + \frac{3}{4})^{s+2}}\right).
\]
and
\[
\frac{1}{(2x + \frac{3}{4})^{s+1}} - \frac{1}{(2x - \frac{1}{4})^{s+1}} > \frac{1}{4}(s + 1)\left(\frac{3}{(2x + \frac{3}{4})^{s+2}} - \frac{1}{(2x - \frac{1}{4})^{s+2}}\right).
\]
Therefore,
\[
g'(x) < \frac{1}{4}s(s + 1)\left(\frac{1}{(2x - \frac{1}{4})^{s+2}} - \frac{3}{(2x + \frac{3}{4})^{s+2}}\right).
\]
Consider \(h(x, s) := \frac{1}{3}\left(\frac{2x + 3/4}{2x - 1/4}\right)^s\), which is the ratio of two terms on the right-hand side of the above expression. We check that \(h(x, s) < 1\) for \(x \geq 2\) and \(0 < s < 1\). Since \(h(2, 1) = 6859/10,125\) and \(\lim_{s \to \infty} h(x, s) = \frac{4}{3}\) for \(0 < s < 1\), we obtain that \(g'(x)\) is negative and, therefore, \(g(x)\) is decreasing, which gives the statement.

We combine the results of Proposition 2 and Proposition 3.

**Theorem 1** Let \(s\) be a real number with \(0 < s < 1\). Then, for any positive even number \(n\),
\[
2\left(n - \frac{1}{2}\right)^s < A^{-1}_{n,s} < 2\left(n - \frac{1}{4}\right)^s,
\]
and for any positive odd number \(n\),
\[
-2\left(n - \frac{1}{2}\right)^s < B^{-1}_{n,s} < -2\left(n - \frac{1}{2}\right)^s.
\]

We express these bounds in terms of \(\zeta_n(s)\) using expression (1).

**Corollary 1** Let \(s\) be a real number with \(0 < s < 1\). Then, for any positive even number \(n\),
\[
2\left(1 - 2^{1-s}\right)\left(n - \frac{1}{4}\right)^s < \zeta_n(s)^{-1} < 2\left(1 - 2^{1-s}\right)\left(n - \frac{1}{2}\right)^s,
\]
and for any positive odd number \(n\),
\[
-2\left(1 - 2^{1-s}\right)\left(n - \frac{1}{2}\right)^s < \zeta_n(s)^{-1} < -2\left(1 - 2^{1-s}\right)\left(n - \frac{1}{4}\right)^s.
\]

Furthermore, we have tighter bounds of \(A^{-1}_{n,s}\) and \(B^{-1}_{n,s}\) for a sufficiently large number \(n\).

**Theorem 2** For any positive number \(\epsilon\) and any real number \(s\) with \(0 < s < 1\),
\[
2\left(n - \frac{1}{2}\right)^s < A^{-1}_{n,s} < 2\left(n - \frac{1}{2} + \epsilon\right)^s.
\]
for a sufficiently large even number \( n \) and
\[
-2 \left( n - \frac{1}{2} + \epsilon \right)^{s} < B_{n}^{-1} < -2 \left( n - \frac{1}{2} \right)^{s}
\]
for a sufficiently large odd number \( n \).

**Proof** From Theorem 1, it suffices to show that for a sufficiently large even number \( n \),
\[
\frac{1}{2(n - \frac{1}{2} + \epsilon)^{s}} < A_{n,s}.
\]

Rewriting each of the both sides as a series
\[
A_{n,s} = \sum_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} \left( \frac{1}{(2k)^{s}} - \frac{1}{(2k+1)^{s}} \right)
\]
and
\[
\frac{1}{2(n - \frac{1}{2} + \epsilon)^{s}} = \sum_{k=\lfloor \frac{n}{2} \rfloor}^{\infty} \left( \frac{1}{2(2k - \frac{1}{2} + \epsilon)^{s}} - \frac{1}{2(2k + \frac{1}{2} + \epsilon)^{s}} \right),
\]
we need to show that for a sufficiently large even number \( n \) and every integer \( k \geq \frac{n}{2} \),
\[
\frac{1}{2(2k - \frac{1}{2} + \epsilon)^{s}} - \frac{1}{2(2k + \frac{1}{2} + \epsilon)^{s}} < \frac{1}{(2k)^{s}} - \frac{1}{(2k+1)^{s}}.
\]

For this, let
\[
f(x) = \left( \frac{1}{(2x)^{s}} - \frac{1}{(2x+1)^{s}} \right) - \left( \frac{1}{2(2x - \frac{1}{2} + \epsilon)^{s}} - \frac{1}{2(2x + \frac{1}{2} + \epsilon)^{s}} \right),
\]
and we will show that \( f(x) \) is positive for \( x \geq x_0 \), where \( x_0 \) is a sufficiently large number.

With
\[
g(x) = \frac{1}{(2x)^{s}} - \left( \frac{1}{2(2x - \frac{1}{2} + \epsilon)^{s}} + \frac{1}{2(2x + \frac{1}{2} + \epsilon)^{s}} \right),
\]
we have that \( f(x) = g(x) - g(x + \frac{1}{2}) \), so we only need to show that \( g(x) \) is decreasing. Consider the derivative of \( g(x) \):
\[
g'(x) = \frac{2(2x)^{s+1} + \frac{1}{(2x - \frac{1}{2} + \epsilon)^{s+1}} + \frac{1}{(2x + \frac{1}{2} + \epsilon)^{s+1}}}{(2x)^{s+1}}
\]
\[
- \left( \frac{1}{(2x - \frac{1}{2} + \epsilon)^{s+1}} - \frac{1}{(2x + \frac{1}{2} + \epsilon)^{s+1}} \right).
\]

Since \( \frac{1}{(x)^{s+1}} \) is decreasing and convex, by comparing slopes at \( (2x - \frac{1}{2} + \epsilon) \) and \( (2x + \frac{1}{2} + \epsilon) \), we obtain
\[
\frac{1}{(2x - \frac{1}{2} + \epsilon)^{s+1}} - \frac{1}{(2x)^{s+1}} < (s + 1) \frac{1}{(2x - \frac{1}{2} + \epsilon)^{s+2}}.
\]
and
\[
\frac{1}{(2x)^{s+1}} - \frac{1}{(2x + \frac{1}{2} + \epsilon)^{s+1}} > (s + 1) \frac{\frac{1}{2} + \epsilon}{(2x + \frac{1}{2} + \epsilon)^{s+2}}.
\]

Therefore
\[
g'(x) < s(s + 1) \left( \frac{\frac{1}{2} - \epsilon}{(2x - \frac{1}{2} - \epsilon)^{s+2}} - \frac{\frac{1}{2} + \epsilon}{(2x + \frac{1}{2} + \epsilon)^{s+2}} \right).
\]

Consider \( h(x) := \frac{\frac{1}{2} + \epsilon}{(2x + \frac{1}{2} + \epsilon)^{s+2}} \), which is the ratio of two terms on the right-hand side of the above expression. We need to show that \( h(x) < 1 \) for every \( x > x_0 \), where \( x_0 \) is a sufficiently large number. We check that
\[
h(x) < 1 \iff \frac{2x + \frac{1}{2} + \epsilon}{2x - \frac{1}{2} - \epsilon} < \left( \frac{\frac{1}{2} + \epsilon}{\frac{1}{2} - \epsilon} \right)^{1/(s+2)}.
\]

For any \( \epsilon > 0 \) and \( 0 < s < 1 \), we have that \( 1 < \left( \frac{\frac{1}{2} + \epsilon}{\frac{1}{2} - \epsilon} \right)^{1/(s+2)} \) and \( \frac{2x + \frac{1}{2} + \epsilon}{2x - \frac{1}{2} - \epsilon} \) is larger than 1, decreasing and converges to 1 as \( x \) goes to infinity, so there is \( x_0 \) such that, for every \( x > x_0 \), \( h(x) < 1 \). Therefore the proof is complete. \( \square \)

We express these bounds in terms of \( \zeta_n(s) \) using expression (1).

**Corollary 2** For any positive number \( \epsilon \) and any real number \( s \) with \( 0 < s < 1 \), we have
\[
2(1 - 2^{-s}) \left( n - \frac{1}{2} + \epsilon \right)^s < \zeta_n(s)^{-1} < 2(1 - 2^{1-s}) \left( n - \frac{1}{2} \right)^s,
\]
for a sufficiently large even number \( n \) and
\[
-2(1 - 2^{1-s}) \left( n - \frac{1}{2} \right)^s < \zeta_n(s)^{-1} < -2(1 - 2^{1-s}) \left( n - \frac{1}{2} + \epsilon \right)^s
\]
for a sufficiently large odd number \( n \).

### 2.2 The value of the inverse of \( \zeta_n(s) \) for \( s = \frac{1}{2}, \frac{1}{3}, \) and \( \frac{1}{4} \)

We study firstly the value of the inverse of \( \zeta_n \left( \frac{1}{2} \right) \), where \( \zeta_n \left( \frac{1}{2} \right) \) is the tail of the Riemann zeta function from \( n \) at \( s = \frac{1}{2} \).

**Theorem 3** For any positive even number \( n \),
\[
\left[A_n^{-1/2}\right] = \left[2 \left( n - \frac{1}{2} \right)^{1/2} \right],
\]
and for any positive odd number \( n \),
\[
\left[B_n^{-1/2}\right] = \left[2 \left( n - \frac{1}{2} \right)^{1/2} \right].
\]
Proof Let \( n \) be a positive even number. By Theorem 1, we have that

\[
2 \left( n - \frac{1}{2} \right)^{1/2} < A_{n,1/2}^{-1} < 2 \left( n - \frac{1}{4} \right)^{1/2}.
\]

Note that \( 2(n - \frac{1}{3})^{1/3} - 2(n - \frac{1}{2})^{1/3} < 1 \) for \( n \geq 2 \), and it implies that there is at most one integer in the open interval from \( 2(n - \frac{1}{2})^{1/3} \) to \( 2(n - \frac{1}{3})^{1/3} \). Suppose that there is an integer \( h \) in the open interval, i.e.,

\[
2 \left( n - \frac{1}{2} \right)^{1/2} < h < 2 \left( n - \frac{1}{4} \right)^{1/2} \quad \text{or} \quad 4n - 2 < h^3 < 4n - 1.
\]

There is, however, no integer in the open interval from \( 4n - 2 \) to \( 4n - 1 \), therefore such an integer \( h \) does not exist. This gives the statement. \( \square \)

We express this result in terms of \( \zeta_n(s) \) using expression (1).

**Corollary 3** For any positive integer \( n \),

\[
\left[ \frac{1}{1 - 2^{1/2}} \zeta_n \left( \frac{1}{2} \right)^{-1} \right] = \left[ (-1)^{n+1}2 \left( n - \frac{1}{2} \right)^{1/2} \right].
\]

We study secondly the value of the inverse of \( \zeta_n(\frac{1}{3}) \), where \( \zeta_n(\frac{1}{3}) \) is the tail of the Riemann zeta function from \( n \) at \( s = \frac{1}{3} \).

**Theorem 4** For any positive even number \( n \),

\[
\left[ A_{n,1/3}^{-1} \right] = \left[ 2 \left( n - \frac{1}{2} \right)^{1/3} \right],
\]

and for any positive odd number \( n \),

\[
\left[ B_{n,1/3}^{-1} \right] = \left[ -2 \left( n - \frac{1}{2} \right)^{1/3} \right].
\]

Proof Let \( n \) be a positive even number. By Theorem 1, we have that

\[
2 \left( n - \frac{1}{2} \right)^{1/3} < A_{n,1/3}^{-1} < 2 \left( n - \frac{1}{4} \right)^{1/3}.
\]

Note that \( 2(n - \frac{1}{3})^{1/3} - 2(n - \frac{1}{2})^{1/3} < 1 \) for \( n \geq 2 \), and it implies that there is at most one integer in the open interval from \( 2(n - \frac{1}{2})^{1/3} \) to \( 2(n - \frac{1}{3})^{1/3} \). Suppose that there is an integer \( h \) in the open interval, i.e.,

\[
2 \left( n - \frac{1}{2} \right)^{1/3} < h < 2 \left( n - \frac{1}{4} \right)^{1/3} \quad \text{or} \quad 8n - 4 < h^3 < 8n - 2.
\]
This shows that the integer $h$ is of the form $h = 2(n - \frac{3}{8})^{1/3}$. If we show $A_{n,1/3}^{-1} < 2(n - \frac{3}{8})^{1/3}$ or, equivalently, \( \frac{1}{2(n - \frac{3}{8})^{1/3}} < A_{n,1/3} \), then our proof will be done. Let us rewrite

\[
A_{n,1/3} = \sum_{k=n}^{\infty} \left( \frac{1}{2(2k + 1)^{1/3}} - \frac{1}{(2k + 3/8)^{1/3}} \right)
\]

and

\[
\frac{1}{2(n - \frac{3}{8})^{1/3}} = \sum_{k=n}^{\infty} \left( \frac{1}{2(2k + 3/8)^{1/3}} - \frac{1}{2(2k + 13/8)^{1/3}} \right).
\]

Now it suffices to show that for any positive integer $k$,

\[
\frac{1}{2(2k + 3/8)^{1/3}} - \frac{1}{2(2k + 13/8)^{1/3}} < \frac{1}{(2k)^{1/3}} - \frac{1}{2(2k + 1)^{1/3}}.
\]

For this, we let

\[
f(x) = \left( \frac{1}{(2x)^{1/3}} - \frac{1}{(2x + 1)^{1/3}} \right) - \left( \frac{1}{2(2x - \frac{3}{8})^{1/3}} - \frac{1}{2(2x + \frac{13}{8})^{1/3}} \right),
\]

and we will show that $f(x)$ is positive for any positive integer $x$.

We check that $f(1) \approx 0.00053$ and $f(2) \approx 0.00081$, so it suffices to show $f(x) > 0$ for $x \geq 3$. With

\[
g(x) = \frac{1}{(2x)^{1/3}} - \left( \frac{1}{2(2x - \frac{3}{8})^{1/3}} + \frac{1}{2(2x + \frac{13}{8})^{1/3}} \right),
\]

we have that $f(x) = g(x) - g(x + \frac{1}{2})$, so we only need to show that $g(x)$ is decreasing for $x \geq 3$. Consider the derivative of $g(x)$:

\[
g'(x) = \frac{1}{3} \left( \frac{-2}{(2x)^{4/3}} + \frac{1}{(2x - \frac{3}{8})^{4/3}} + \frac{1}{(2x + \frac{13}{8})^{4/3}} \right)
\]

\[
= \frac{1}{3} \left( \left( \frac{1}{(2x - \frac{3}{8})^{4/3}} - \frac{1}{(2x)^{4/3}} \right) - \left( \frac{1}{(2x)^{4/3}} - \frac{1}{(2x + \frac{13}{8})^{4/3}} \right) \right).
\]

Since $\frac{1}{x^{4/3}}$ is decreasing and convex, by comparing slopes at $(2x - \frac{3}{8})$ and $(2x + \frac{13}{8})$, we obtain

\[
\frac{1}{(2x - \frac{3}{8})^{4/3}} - \frac{1}{(2x)^{4/3}} < \frac{3}{16} \cdot \frac{4}{3} \cdot \frac{1}{(2x - \frac{3}{8})^{7/3}}
\]

and

\[
\frac{1}{(2x)^{4/3}} - \frac{1}{(2x + \frac{13}{8})^{4/3}} > \frac{5}{16} \cdot \frac{4}{3} \cdot \frac{1}{(2x + \frac{13}{8})^{7/3}}.
\]

Therefore

\[
g'(x) < \frac{1}{18} \left( \frac{3}{(2x - \frac{3}{8})^{7/3}} - \frac{5}{(2x + \frac{13}{8})^{7/3}} \right).
\]
Consider $h(x) := \frac{3}{5} \left( \frac{2x+5/8}{2x-3/8} \right)^{7/3}$, which is the ratio of two terms of the right-hand side of the above expression. We check that $h(x) < 1$ for $x \geq 3$ because $h(3) = 0.87 \cdots$ and $\lim_{x \to \infty} h(x) = \frac{3}{5}$ and $h(x) < 0$ for $x \geq 3$. Hence we obtain that $g'(x)$ is negative and so $g(x)$ is decreasing for $x \geq 3$, which proves the statement. \hfill $\Box$

We express this result in terms of $\zeta_n(s)$ using expression (1).

**Corollary 4** For any positive integer $n$,

\[
\left[ \frac{1}{1 - 2^{2/3} \zeta_n \left( \frac{1}{3} \right)^{-1}} \right] = \left[ (-1)^{n+1/2} \left( n - \frac{1}{2} \right)^{1/3} \right].
\]

We study lastly the value of the inverse of $\zeta_n(\frac{1}{4})$, which is the tail of the Riemann zeta function from $n$ at $s = \frac{1}{4}$.

**Theorem 5** For any positive even number $n$,

\[
\left[ A_n^{-1}, 1/4 \right] = \left[ 2 \left( n - \frac{1}{2} \right)^{1/4} \right],
\]

and for any positive odd number $n$,

\[
\left[ B_n^{-1}, 1/4 \right] = \left[ -2 \left( n - \frac{1}{2} \right)^{1/4} \right].
\]

**Proof** Let $n$ be a positive even number. By Theorem 1, we have that

\[
2 \left( n - \frac{1}{2} \right)^{1/4} < A_n^{-1} < 2 \left( n - \frac{1}{4} \right)^{1/4}.
\]

Note that $2(n - \frac{1}{4})^{1/4} - 2(n - \frac{1}{2})^{1/4} < 1$ for $n \geq 2$, and it implies that there is at most one integer in the open interval from $2(n - \frac{1}{4})^{1/4}$ to $2(n - \frac{1}{4})^{1/4}$. Suppose that there is an integer $h$ in the open interval, i.e.,

\[
2 \left( n - \frac{1}{2} \right)^{1/4} < h < 2 \left( n - \frac{1}{4} \right)^{1/4} \text{ or } 16n - 8 < h^4 < 16n - 4.
\]

This shows that the integer $h^4$ is one of the form $16n - 7$, $16n - 6$, or $16n - 5$. For any integer $h$, however, $h^4 \equiv 0$ or 1 (mod 16), hence such an integer $h$ does not exist. Therefore this gives the statement. \hfill $\Box$

We express this result in terms of $\zeta_n(s)$ using expression (1).

**Corollary 5** For any positive integer $n$,

\[
\left[ \frac{1}{1 - 2^{3/4} \zeta_n \left( \frac{1}{4} \right)^{-1}} \right] = \left[ (-1)^{n+1/2} \left( n - \frac{1}{2} \right)^{1/4} \right].
\]

We express the results of Theorems 3, 4, and 5 in a single statement.
Theorem 6 For \( s = \frac{1}{2}, \frac{1}{3}, \text{ or } \frac{1}{4}, \) and for any positive even number \( n, \)

\[
[A_{n,s}^{-1}] = \left[ 2 \left( n - \frac{1}{2} \right)^s \right].
\]

and for any positive odd number \( n, \)

\[
[B_{n,s}^{-1}] = \left[ -2 \left( n - \frac{1}{2} \right)^s \right].
\]

We express the results of Corollaries 3, 4, and 5 in a single statement.

Corollary 6 For any positive integer \( n \) and \( s = \frac{1}{2}, \frac{1}{3}, \text{ or } \frac{1}{4}, \)

\[
\left[ \frac{1}{1 - 2^{1-s} \zeta_n(s)^{-1}} \right] = \left[ (-1)^{n+1} 2 \left( n - \frac{1}{2} \right)^s \right].
\]

3 Conclusion

In this paper, we have presented the bounds of \( A_{n,s}^{-1} \) and \( B_{n,s}^{-1}, \) hence the bounds of the inverses of tails of the Riemann zeta function \( \zeta_n(s)^{-1} \) for \( 0 < s < 1, \) and computed the values \( [A_{n,s}^{-1}] \) and \( [B_{n,s}^{-1}], \) hence the values of the inverses of tails of the Riemann zeta function \( \left[ \frac{1}{1 - 2^{1-s} \zeta_n(s)^{-1}} \right] \) for \( s = \frac{1}{2}, \frac{1}{3}, \text{ and } \frac{1}{4}. \) For other values of \( s, \) for example \( s = \frac{1}{5} \) or \( \frac{2}{3}, \) the values of \( A_{n,s} \) and \( B_{n,s} \) do not seem to have simple expressions.

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