Cosmic Strings in Compactified Gauge Theory

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Abstract

A solution of the vortex type is given in a six-dimensional \( SU(2) \times U(1) \) pure gauge theory coupled to Einstein gravity in a compactified background geometry. We construct the solution of an effective abelian Higgs model in terms of dimensional reduction. The solution, however, has a peculiarity in its physically relevant quantity, a deficit angle, which is given as a function of the ratio of the gauge couplings of \( SU(2) \) and \( U(1) \). The size of the extra space (sphere) is shown to vary with the distance from the axis of the “string”.

In the course of universe evolution, phase transitions are a very important subject in cosmology and particle theory [1]. The most attractive features are the generation of topological defects [2] such as domain walls, cosmic strings, and so on. Especially cosmic strings provide a mechanism to generate seeds of galaxies, and thus their cosmological roles are of great significance [2, 3].

There are two types of cosmic strings, namely global and local strings, which are distinguished by the type of symmetry broken via the associated phase transition. The global strings are formed when global symmetries break down such as in \( U(1) \) scalar theory, whereas the local strings are subject to gauge symmetry breaking.

The simplest approach to the study of local cosmic strings in ordinary three-dimensional space is to examine the abelian Higgs vortex, i.e. the Nielsen-Olesen type solution [4, 5] to equations of motion for the scalar and \( U(1) \) gauge fields. This solution represents an infinitely long isolated string in flat space-time. If one wants to consider a vortex or a string in a cosmological situation more precisely, however, we must couple the system with gravity since the core of the string has finite energy density.

In this context, Garfinkle [6] found consistent solutions of the Einstein-scalar-\( U(1) \) gauge system with an exact energy-momentum of the matter fields in the
situation that the vacuum expectation value of the scalar field is much smaller than the Planck scale. The characteristic feature of these string solutions is that at radial infinity, the space-time manifold becomes Minkowski space-time minus a wedge, i.e. a deficit angle appears around the cosmic string. Moreover, Laguna-Castillo and Matzner [7] solved the same system numerically without any assumption except for boundary conditions. They showed that when the symmetry breaking scale, or the vacuum expectation value of the scalar field, is greater than $10^2 \text{ GeV}$, the curvature of the space-time manifold is not negligible.

In more realistic models, the abelian gauge symmetry might come from some non-abelian symmetry. If we believe in a grand unified gauge symmetry, the $U(1)$ gauge symmetry of the abelian Higgs model is contained in the large group.

On the other hand, from the perspective of unified theories including gravity, higher-dimensional theories have very attractive possibilities [8]. For instance the string theory [9] is naturally formulated in more than four dimensions and is regarded as a candidate for a finite theory of gravity. In these theories we must assume that the extra space has become a very tiny internal space which is sufficiently small never to be seen by present experiments. Nevertheless, it is very interesting to speculate on the effects such extra dimensions might have on the four-dimensional world we live in. For instance, it is permissible that the vector fields on the internal space have non-vanishing vacuum expectation values, which do not break Poincaré invariance in the four-dimensional world. Therefore, gauge symmetry breaking can be caused by the vacuum gauge fields [10].

The effects of the vacuum configuration on the extra space may also play an essential role as the source of spontaneous compactification, together with the cosmological constant. In such a case the gauge field has non-zero field strength on the compact space with curvature [8, 11].

If “elementary” non-abelian gauge fields exist in such higher-dimensional theories, another interesting possibility comes about. It was shown that gauge boson-Higgs scalar systems are derived from dimensional reduction of higher-dimensional Yang-Mills theory [12]. Many authors considered dimensional reduction as a device to obtain various breaking patterns of the Higgs mechanism. We wish to make use of this scheme to investigate the multidimensional universe.

In this paper, we consider the full coupled equations of motion of effective four space-time dimensional Einstein-scalar-$U(1)$ gauge system induced from higher-dimensional Einstein-Yang-Mills-Maxwell theory. Here we take $S_z$, a sphere, as the extra space and $SU(2)$ as the non-abelian gauge group. Effective abelian Higgs-like equations are obtained from the pure Yang-Mills theory. We solve the solution of the vortex-type numerically. The conclusion is a very natural one in the sense that the properties of the “cosmic string” are almost the same as those of the ordinary string in three-dimensional space except that the size of the extra $S^2$ space varies with the radial coordinate. Moreover in our string, cosmologically relevant quantities such as the deficit angle and the scale of the extra space at the core are determined by the ratio between the $U(1)$ and $SU(2)$ couplings.
We begin with the following action:

\[ S = \int d^6x \sqrt{-g} \left( -\frac{1}{2\kappa^2} R + \frac{1}{4e^2} \text{tr}(F_{MN}F^{MN}) + \frac{1}{4g^2} G_{MN}G^{MN} + \Lambda \right). \] (1)

Here \( F_{MN} \) and \( G_{MN} \) are the field strengths of \( SU(2) \) and \( U(1) \) gauge fields, respectively. \( e \) is a \( SU(2) \) gauge coupling constant while \( g \) is a \( U(1) \) gauge coupling; \( \Lambda \) is a cosmological constant. The scalar curvature of \( S^2 \) with unit radius is defined as \( R = +N(N-1) \). The indices \( M \) and \( N \) take values in six dimensions. Although these gauge groups can be seen as subgroups of some larger gauge group, we do not pursue this line in this paper.

The field equations are directly derived from this action,

\[ R_{MN} = \frac{1}{2} \kappa^2 \Lambda g_{MN} + \kappa^2 \left( T_{MN} - \frac{1}{4} T g_{MN} \right), \] (2)

\[ D_M F^{MN} = \nabla_M F^{MN} + i [A_M, F^{MN}] = 0, \] (3)

\[ \nabla_M G^{MN} = 0, \] (4)

where the field strengths are given by

\[ F_{MN} = \partial_M A_N - \partial_N A_M + i [A_M, A_N] \quad \text{for} \quad SU(2), \] (5)

\[ G_{MN} = \partial_M B_N - \partial_N B_M \quad \text{for} \quad U(1). \] (6)

\( A_M \) and \( B_M \) are the \( SU(2) \) and \( U(1) \) gauge field respectively, and \( \nabla_M \) is the covariant derivative. \( T_{MN} \) is the energy-momentum tensor of the gauge fields given as follows:

\[ T_{MN} = \frac{1}{e^2} \text{tr} \left( F_{MP}F_N^P - \frac{1}{4} F_{PQ}F^{PQ} g_{MN} \right) + \frac{1}{g^2} \left( G_{MP}G_N^P - \frac{1}{4} G_{PQ}G^{PQ} g_{MN} \right), \] (7)

\[ T = T_{MN} g^{MN}. \] (8)

For simplicity, the background space-time is assumed to be locally \( M^4 \times S^2 \), namely the direct product of a flat “universe” in which the cosmic string is embedded and the internal two-dimensional sphere. Furthermore, we look for static solutions in the present work. Accordingly, the square of the line element can be set to be

\[ ds^2 = e^A(-dt^2 + dz^2) + dr^2 + e^C d\psi^2 + e^B d\Omega^2(S^2), \] (9)

where \( d\Omega^2(S^2) \) represents the line element of the sphere with unit radius, i.e. \( d\theta^2 + \sin^2 \theta d\phi^2 \) and \( 0 \leq \psi < 2\pi \), \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi < 2\pi \). Here \( e^{A/2} \) and \( e^{B/2} \) are the scale factors of two-dimensional “homogeneous” space-time and the internal sphere, respectively. The axis of the string is located at \( r = 0 \). The functions \( A, B, \) and \( C \) are assumed to depend only on the \( r \) coordinate, i.e. on the distance from the axis of the string.
First we assume the $U(1)$ gauge field being a monopole configuration (with minimal magnetic charge) \[11\]:

$$B_\phi = \frac{1}{2}(\cos \theta \pm 1), \quad (10)$$

and the other components of the $U(1)$ vector field are assumed to vanish. This $U(1)$ monopole is required only to support the large dimensions by cooperating with the fine-tuned cosmological constant. Thus we require a relation among the couplings \[11\]:

$$\Lambda = 2g^2/\kappa^4. \quad (11)$$

The vacuum configurations of $SU(2)$ gauge fields are crucial to generate the cosmic string. In our model, we set the components of $SU(2)$ gauge field on the internal space as

$$A_\theta = \Phi_1 \frac{1}{2} \begin{pmatrix} 0 & -i e^{-i\phi} \\ i e^{i\phi} & 0 \end{pmatrix} + \Phi_2 \frac{1}{2} \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}, \quad (12)$$

$$A_\phi = -\Phi_1 \frac{1}{2} \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} \sin \theta + \Phi_2 \frac{1}{2} \begin{pmatrix} 0 & -i e^{-i\phi} \\ i e^{i\phi} & 0 \end{pmatrix} \sin \theta,$$

$$+ \frac{1}{2} \begin{pmatrix} 1 - \cos \theta & 0 \\ 0 & -(1 - \cos \theta) \end{pmatrix}. \quad (13)$$

Here $\Phi_1$ and $\Phi_2$ are the functions independent of the $S^2$ coordinates $\theta$ and $\phi$. From the four-dimensional point of view, they can be seen effectively as components of a complex scalar field (see eqs. (14a, b)). Note that when $\Phi_1 = \Phi_2 = 0$ this gauge configuration is a magnetic monopole on $S^2$; therefore, it has a finite energy density. Further, the large-dimensional components of the $SU(2)$ gauge field are assumed to have the following form:

$$A_\mu = A_\mu(x^\mu)\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (14)$$

where $A_\mu(x^\mu)$ depends only on the coordinates of the four space-time dimensions but not on the extra dimensions. $\mu$ runs over the four coordinates. Note that here we use a coordinate basis associated with the metric and not an orthonormal one.

We emphasize that these components of the gauge field play the roles of the “$U(1)$" gauge field in four dimensions in our scheme as we will show in the following. Note that, of course, this effective “$U(1)$" fields has no relation to the six-dimensional $U(1)$ field introduced earlier.

With respect to these classical gauge configurations, the equations of motion (3) are reduced as follows:

$$D^\mu D_\mu \hat{\Phi} + \frac{1}{e^2}(1 - |\hat{\Phi}|^2) \hat{\Phi} = 0, \quad (15)$$

$$\nabla_\mu(e^B F^{\mu\nu}) + i(\hat{\Phi}^* D^\nu \hat{\Phi} - \hat{\Phi} D^\nu \hat{\Phi}^*) = 0, \quad (16)$$
where \( \Phi = \Phi_1 + 1 \Phi_2 \) and \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \). The covariant derivative is defined as \( D_\mu = \nabla_\mu + iA_\mu \) with \( \nabla_\mu \) being the covariant derivative associated with the four-dimensional metric \( g_{\mu\nu} \).

These equations closely resemble those of the abelian Higgs model, which has in general two couplings between “matter” and gauge fields. In our model, the “scalar” self-interaction and “gauge” coupling constants, which are implicitly contained in eq. (16) in our notation, originate from the same Yang-Mills coupling. Another different point is that the effective equations contain the extra dependence on the position through the factor \( e^B \).

Before solving these equations coupled with Einstein equations, we must give an ansatz for the gauge field in order to generate an isolated string. This requires

\[
\Phi = X(r)e^{i\psi}, \\
A_\psi(r) = P(r) - 1, \quad \text{otherwise } A_\mu(r) = 0.
\]

As is well known, the solution of this type has unit winding number around the axis \([4, 5]\).

With all these assumptions, the Yang-Mills equation (3), which is equivalent to eq. (16), is reduced to

\[
(KX')' = KB'X' + X[K(X^2 - 1)e^{-2B} + e^{2A+2B}P^2/K],
\]

\[
(e^{2A}P'/K)' = e^{2A}[-2B'P' + 2X^2e^{-B}P]/K,
\]

where \( K = \exp(A + B + C/2) \), and the prime means derivative with respect to \( r \).

For convenience, we adjust the unit of the length scale such that \( \kappa^2/(4g^2) = 1 \).

In this unit, we find that the asymptotic value of the radius of the extra sphere at large distance from the center of the string is equal to one.

With the relation (11), the Einstein equations for the variables \( A, B, \) and \( K \) are given in our unit by

\[
A'' = -(K'/K)A' - \frac{1}{2}(1 - e^{-2B}) + q^{-2}\{e^{-B}X^2 + e^{2A+2B}P^2X^2/K^2
+ e^{2A+2B}P'^2/(2K^2) + e^{-2B}(X^2 - 1)^2/2\},
\]

\[
B'' = -(K'/K)B' + 2e^{-B} - \frac{1}{2}(1 + 3e^{-2B}) - q^{-2}\{e^{-B}X^2 + e^{2A+2B}P^2X^2/K^2
- e^{2A+2B}P'^2/(2K^2) + 3e^{-2B}(X^2 - 1)^2/2\},
\]

\[
K'' = K(2e^{-B} - \frac{5}{4} - \frac{3}{4}e^{-2B}) - q^{-2}\frac{1}{2}K\{-e^{-B}X^2 + 3e^{2A+2B}P^2X^2/K^2
- e^{2A+2B}P'^2/(2K^2) + 3e^{-2B}(X^2 - 1)^2/2\},
\]

where \( q^2 = \epsilon^2/(2g^2) \).

To find solutions of these differential equations, we use a numerical method with the physically acceptable boundary conditions of ref. [7] and the conditions on \( B \), i.e. the scale of the internal space.
First, at the boundary the behaviors of the “scalar” $\Phi$ and the “$U(1)$ gauge field” are given by

\begin{align}
X(0) &= 0, \quad P(0) = 1, \\
X(\infty) &= 1, \quad P(\infty) = 0.
\end{align}

These come from the requirement that there are no singularities of gauge fields introduced originally at the core of the string and that integrating the energy-momentum tensor must yield a finite value.

Second, we give the condition for the metric of the four-dimensional part such that the space-time is locally Minkowski up to a constant, which can be scaled at the center of the string,

\begin{align}
A(0) &= K(0) = 0, \quad A'(0) = B'(0) = 0, \\
B(\infty) &= 0, \quad K'(0) = e^{B(0)}.
\end{align}

Eq. (27) is due to the choice of the unit of the scale, $\kappa^2/(4g^2) = \kappa^2q^2/(2e^2) = 1$.

After fixing all of these boundary conditions, we have solved the full set of coupled equations (19), (20) and (21-23) by using a computational program named COLSYS [13]. We determine $B(0)$ by a guess to converge the solution iteratively.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Solution for the effective scalar field $X$ for $q = 5$. The curve is hardly sensitive to the value of $q \geq 2$.}
\end{figure}

The numerical results are shown in figures. The behavior of the effective abelian Higgs fields indicates the characteristic configuration (figs. 1 and 2), which is very similar to the ordinary model (in four-dimensional space-time). The width of the string core in our model is $O(1)$ in our unit, i.e. the same order as the size of the extra space at radial infinity.
Figure 2: Solution for the effective gauge field $P$ for $q = 5$. The curve is hardly sensitive to the value of $q \geq 2$.

From fig. 3 we can see a new feature of the string in our model; there exists an attractive gravitational force along the radial direction in our solution with any value of $q^2$, in contrast to the case of the ordinary abelian Higgs vortex where a repulsive force is also allowed depending on the ratio of gauge and scalar self-couplings. This fact suggests that a single vortex with higher windings of the “scalar” and “gauge” configuration is stable in a static circumference as in the case of the vortex in a “type I superconductor” [5].

Next we consider the deficit angle at radial infinity. A definition of such a quantity is given in refs. [6, 7],

$$\Delta \psi \equiv 2\pi \left\{ 1 - \lim_{r \to \infty} \frac{d}{dr}(e^{C/2}) \right\}.$$  \hspace{1cm} (28)

Our numerical evaluation is shown in fig. 7, and we find that the deficit angle monotonically decreases with $q^2$. This result is quite natural. When $q^2$ is large, the energy-momentum of the $SU(2)$ gauge field is nearly decoupled from the other part in the equations (21-23), and thus the four-dimensional spacetime becomes flat everywhere. The stability of the structure of the solution is guaranteed also by the mild behavior of the metric at large $q^2$ (figs. 3, 4, and 5). The deficit angle turns out to be well approximated by $\Delta \psi = 2\pi q^{-2}$ in the region of the parameter $q^2$ where we investigate the solution.

The property of the internal space are shown in figs. 5 and 6. We should pay attention to the fact that the value of $\exp(B(0)/2)$, that is, the radius of the internal sphere at the axis of the string, grows as $q^2$ gets smaller. This radius is always larger than the value at radial infinity. We find a linear relation in a plot of $\log B(0)$ versus $\log q^2$. $B(0)$ is expressed approximately as $B(0) \sim q^{-2} \times \text{(constant of } O(1))$. The effect of the vortex-like configuration of $SU(2)$ gauge field might open a “window” to the extra dimensions in the core of the cosmic string, when $q^2 \sim O(1)$.  

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Figure 3: Solution for the metric field $1-e^A$ as a function of the radial coordinate $r$. The lines (———), (— - —), (— - - —), and (- - - - - -) correspond to the parameter $q = 2, 5, 10, 100$, respectively.

In ref. [7], the parameter $\eta$ indicates the symmetry-breaking scale in the Planck unit. In our model, the coupling between energy-momentum of matter fields and gravity is proportional to $q^{-2}$, instead of $\eta^2$ in the ordinary string model. Roughly speaking, $q^2$ gives the ratio of the radius of the extra space to the Planck length in the asymptotic region $r \to \infty$, provided that the Yang-Mills coupling in four dimensions is of order 1.

To summarize, we have obtained a vortex-type solution in a higher-dimensional gauge theory by a numerical method. The physical properties of the solution, such as the deficit angle around the string axis and variation of the size of the extra space has been shown to be dependent on the ratio of the two couplings in the theory.

There are many subjects for future investigations. One could investigate the following topics: existence (or non-existence) of the fermionic zero mode and construction of a superconducting string (possibly from a large gauge group), phase transition and string formation. In particular the investigation of the energy spectrum of fermions coupled with the string background is of great interest from the viewpoint of higher-dimensional theory. Probably different aspects from ordinary strings will be found in the study of these problems. Moreover, through this study the cosmological significance of cosmic strings in higher-dimensional theory can be argued in detail.

The energy-momentum tensor in our model and cosmic strings in de Sitter and time-dependent (higher-dimensional) background will be analyzed in a forthcoming publication. We are also interested in the behavior of the size of the extra space when a loop of the string is made and collapses, and when two strings collide or intersect. Such dynamical processes might provide the possibility of observing the extra dimensions.
Figure 4: The metric component $e^{B/2}$, i.e. the size of the extra space at the coordinate origin as a function of the radial coordinate $r$. The line definitions are the same as in fig. 3.

Finally, we give a few comments on the case of a four-dimensional homogeneous spacetime, in which $e^{A/2}$ is the scale factor. We checked the behavior of all fields at the same value of $q^2$ and they are almost the same as in the case already mentioned, as expected. The reason why we wish to consider the exotic case is that we can investigate the scenario of dimensional reduction, which is expressed by the slogan, “we live in a topological defect in a higher-dimensional space” [14]. The possibility of this scenario in an extended version of our model will be reported elsewhere.

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Figure 5: Solution for the $e^{C/2}$ metric component as a function of the radial coordinate $r$. The line definitions are the same as in fig. 3.

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Figure 6: Plot of $B(0)$ as a function of $q^2$. This line is well approximated by $B(0) = q^{-2} \times (\text{const. of } O(1))$.

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Figure 7: Plot of $d/dr(e^{C/2})$ at radial infinity (actually evaluated at $r = 100$) as a function of $q^2$. This curve is well approximated by $d/dr(e^{C/2})|_{r=100} = 1 - q^{-2}$. 