Counting invariant of perverse coherent sheaves and its wall-crossing

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October 5, 2010

Abstract
We introduce moduli spaces of stable perverse coherent systems on small crepant resolutions of Calabi-Yau 3-folds and consider their Donaldson-Thomas type counting invariants. The stability depends on the choice of a component (= a chamber) in the complement of finitely many lines (= walls) in the plane. We determine all walls and compute generating functions of the invariants for all choices of chambers when the Calabi-Yau is the resolved conifold. For suitable choices of chambers, our invariants are specialized to Donaldson-Thomas, Pandharipande-Thomas and Szendroi invariants.

Introduction
In this paper we study variants of Donaldson-Thomas (DT in short) invariants [Tho00] for the crepant resolution $f: Y \to X = \{xy - zw = 0\}$ of the conifold where $Y$ is the total space of the vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The ordinary DT invariants are defined by the virtually counting of moduli spaces of ideal sheaves of curves. A variant has been introduced by Pandharipande-Thomas (PT in short) recently [PT09]. PT invariants are defined by the virtual counting of moduli spaces of stable coherent systems, i.e., pairs of 1-dimensional sheaves $F$ and homomorphisms $s: \mathcal{O}_Y \to F$. Both DT and PT invariants are defined for arbitrary Calabi-Yau 3-folds. For the resolved conifold $Y$, yet another variant was introduced by Szendroi [Sze08] (see also [You]). His invariants are defined by the virtual counting of moduli spaces of representations of a certain noncommutative algebra. This noncommutative algebra has its origin in the celebrated works of Bridgeland [Bri02] and Van den Bergh [VdB04]. In particular, we can interpret the invariants as virtual counting of moduli spaces of perverse ideal sheaves, originally introduced in order to describe the flop $f^+: Y^+ \to X$ of $Y$ as a moduli space. Those three classes of invariants have been computed for the resolved conifold and their generating functions are given by infinite products.

In this paper we introduce more variants by using moduli spaces of stable perverse coherent systems, i.e., pairs of 1-dimensional perverse coherent sheaves $F$ and homomorphisms $s: \mathcal{O}_Y \to F$. The stability condition is determined by a choice of parameter $\zeta = (\zeta_0, \zeta_1)$ in the complement of finitely many lines in $\mathbb{R}^2$. (The number of lines increases when the Hilbert polynomial of $F$ becomes large.)
The invariants depend only on the chamber containing $\zeta$. When we choose the chamber appropriately, our new invariants recover DT, PT invariants for $Y$ and the flop $Y^+$, as well as Szendroi’s invariants. See Figure 1. Our main results are

(a) the determination of all walls, and
(b) the computation of invariants for any choice of a chamber.

The generating function of invariants is given by an infinite product, dropping certain factors from the generating function of Szendroi’s invariants. The stability parameter determines which factors we should drop in a very simple way (see Theorems 3.12, 3.15). Finally in the chamber $\zeta_0, \zeta_1 > 0$ the generating function is simply 1.

\[ \mathbb{Z}_\zeta(q) = 1 \]

Besides Szendroi’s work [Sze08] our definition is motivated also by the second named author’s work with Kota Yoshioka [NYa, NYb], where very similar moduli spaces and chamber structures have been studied for the case when $Y \to X$ is the blow-up of a nonsingular complex surface. A similarity is natural as $Y$ is locally the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)$, instead of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. In [NYb] the
virtual Betti numbers of moduli spaces are computed. When the rank of sheaves is 1, the generating function is again an infinite product, dropping factors from the generating function of Betti numbers of Hilbert schemes of points on the blow-up $Y$, given by the famous Göttsche formula \cite{Got90}.

We compute invariants by proving the wall-crossing formula which describes how the generating function of invariants changes when the stability parameter crosses a wall except for the wall $\{ (\zeta_0, \zeta_1) \mid \zeta_0 + \zeta_1 = 0 \}$ (Theorem 3.12). In the companion paper \cite{Nag}, the first named author will generalize the main result to more general small crepant resolutions of toric Calabi-Yau 3-folds. The wall-crossing formula is an example of Joyce’s wall-crossing formulas \cite{Joy08}, and ones in more recent work by Kontsevich and Soibelman \cite{KS} (see \cite{Nag}, §4). Note that there is no substantial difference between virtual counting and actual Euler numbers in our setting (Theorem 4.23), so our computation can be examples of both works. In this paper we give an alternative elementary proof independent of \cite{Joy08, KS}. The wall $\{ (\zeta_0, \zeta_1) \mid \zeta_0 + \zeta_1 = 0 \}$, which we do not deal with, corresponds to the DT-PT conjecture \cite{PT09}. See Remark 3.16 for this wall.

The paper is organized as follows: In §1 we introduce perverse coherent systems and their stability. In §2 we show that our moduli spaces parameterize ideal sheaves or stable coherent systems in suitable choices of the stability parameters. This will be used to establish that our invariants are specialized to DT and PT invariants. In §3 we prove our main results.

After we posted a previous version of this paper to the arXiv, two physics papers \cite{JM} and \cite{CJ09} appeared on the arXiv. In \cite{JM}, Jafferis and Moore provide a wall-crossing formula, which looks quite similar to ours. In \cite{CJ09}, Chuang and Jafferis associate a new quiver $(\tilde{A}_{n}^{\pm})$ in §4.3 to each chamber and conjecture that the moduli spaces are isomorphic to the corresponding moduli spaces of $\zeta$-stable perverse coherent systems. The quiver $(\tilde{A}_{n}^{\pm})$ is obtained from our quiver $(\tilde{A})$ in Figure 4 by successive mutations in the sense of \cite{DWZ08}. We devote §4 to a proof of their conjecture and to a proof of Theorem 4.23.

Acknowledgements

The first named author (KN) is supported by JSPS Fellowships for Young Scientists (No. 19-2672). He is grateful to Y. Kimura for teaching him some references on Calabi-Yau algebras. He is also grateful to E. Macri for useful comments.

The second named author (HN) is supported by the Grant-in-aid for Scientific Research (No. 19340006), JSPS. This work was started while he was visiting the Institute for Advanced Study with supports by the Ministry of Education, Japan and the Friends of the Institute. He is grateful to K. Yoshioka for discussions on the result of this paper, in particular, the explanation how to construct moduli spaces for a projective $Y$ in §1.4. He is also grateful to Y. Soibelman for sending us a preliminary version of [KS].

While the authors were preparing this paper, they are informed that Bridgeland and Szendroi also reproduce the formula of Szendroi’s invariants \cite{You} via the wall-crossing formula in the spirit of \cite{Joy08, BTL, KS}. They thank T. Bridgeland for discussion.

They thank W. Chuang for answering questions on the paper \cite{CJ09}. They also thank the referee for the valuable comments.
1 Perverse coherent systems

1.1 Category of perverse coherent systems

Let \( f: Y \to X \) be a projective morphism between quasi-projective varieties over \( \mathbb{C} \) such that the fibers of \( f \) have dimensions less than 2 and such that 
\[
Rf_* \mathcal{O}_Y = \mathcal{O}_X.
\]

A perverse coherent sheaf \([\text{Bri02}, \text{VdB04}]\) is an object \( E \) of \( D^b(\text{Coh}(Y)) \) satisfying

- \( H^i(E) = 0 \) unless \( i = 0, -1 \),
- \( \mathbb{R}^1 f_*(H^0(E)) = 0 \) and \( \mathbb{R}^0 f_*(H^{-1}(E)) = 0 \),
- \( \text{Hom}(H^0(E), C) = 0 \) for any sheaf \( C \) on \( Y \) satisfying \( Rf_*(C) = 0 \).

Let \( \text{Per}(Y/X) \subset D^b(\text{Coh}(Y)) \) be the full subcategory consisting of all perverse coherent sheaves. This is the core of a t-structure of \( D^b(\text{Coh}(Y)) \). In particular, \( \text{Per}(Y/X) \) is an abelian category.

Remark 1.1. The abelian category \( \text{Per}(Y/X) \) we define here is \( -1\text{Per}(Y/X) \) in the notations of \([\text{Bri02}]\) and \([\text{VdB04}]\). We will also use \( 0\text{Per}(Y^+/X) \) to study the flop \( Y^+ \to X \) later (\S 2.2).

Definition 1.2. A perverse coherent system on \( Y \) is a pair \( (F, W, s) \) of a perverse coherent sheaf \( F \), a vector space \( W \) and a homomorphism \( s: W \otimes \mathcal{O}_Y \to F \). We call \( W \) the framing of a perverse coherent system.

A morphism between perverse coherent systems \( (F, W, s) \) and \( (F', W', s') \) is a pair of morphisms \( F \to F' \) and \( W \to W' \) which are compatible with \( s \) and \( s' \).

We sometimes denote a perverse coherent system \( (F, W, s) \) with \( W = \mathbb{C} \) by \( (F, s) \), and a perverse coherent system \( (F, W, s) \) with \( W = 0 \) by \( F \) for brevity.

Let \( \text{Per}(Y/X) \) denote the category of perverse coherent systems on \( Y \). It is an abelian category.

1.2 NCCR with framing

When \( X = \text{Spec}(R) \) is affine, \( P \in \text{Per}(Y/X) \) is called a projective generator if it is a projective object in \( \text{Per}(Y/X) \) and \( \text{Hom}(\text{Per}(Y/X), (P, E)) = 0 \) implies \( E = 0 \) for \( E \in \text{Per}(Y/X) \). For a general \( X \), \( P \in \text{Per}(Y/X) \) is called a local projective generator if there exists an open covering \( X = \bigcup U_i \) such that \( \mathcal{P}|_{U_i} \) is a projective generator of \( \text{Per}(f^{-1}(U_i)/U_i) \) for all \( i \).

A local projective generator \( \mathcal{P} \) exists and can be taken as a vector bundle (see \([\text{VdB04}] \) Proposition 3.3.2). When \( X = \text{Spec}(R) \) is affine, \( \mathcal{P} \) is constructed as follows: Let \( \mathcal{L} \) be an ample line bundle on \( Y \) generated by its global sections. Let \( \mathcal{P}_0 \) be a vector bundle given by an extension

\[
0 \to \mathcal{O}_Y^r \to \mathcal{P}_0 \to \mathcal{L} \to 0 \tag{1.1}
\]

associated to a set of generators of \( H^1(Y, \mathcal{L}^{-1}) \) as an \( R \)-module. Then \( \mathcal{P} := \mathcal{P}_0 \oplus \mathcal{O}_Y \) is a projective generator \([\text{VdB04}] \) Proposition 3.2.5]. The general case can be reduced to the affine case. We consider only a local projective generator of a form \( \mathcal{P} = \mathcal{P}_0 \oplus \mathcal{O}_Y \) hereafter.
Theorem 1.3 ([VdB04 Corollary 3.2.8 and Theorem A]). We denote the $\mathcal{O}_X$-algebra $f_*\End_Y(\mathcal{P})$ by $A$. Then the functors $R_* R\Hom_Y(\mathcal{P},-)$ and $-\otimes_A^L(\mathcal{P})$ give equivalences between $D^b(\text{Coh}(Y))$ and $D^b(\text{Coh}(A))$, which are inverse to each other. Here $\text{Coh}(A)$ denotes the category of coherent right $A$-modules.

Moreover, these equivalences restrict to equivalences between $\text{Per}(Y/X)$ and $\text{Coh}(A)$.

Definition 1.4. Let $\text{Coh}_c(A)$ (resp. $D^b_c(\text{Coh}(A))$) denote the full subcategory of $\text{Coh}(A)$ (resp. $D^b(\text{Coh}(A))$) consisting of objects $E$ which (resp. whose cohomology groups) have 0-dimensional supports. Let $\text{Per}_c(Y/X)$ and $D^b_c(\text{Coh}(Y))$ be the corresponding categories under the equivalences in Corollary 1.3. Let $\text{Per}_c(Y/X)$ denote the category of perverse coherent systems $(F, W, s)$ such that $F \in \text{Per}_c(Y/X)$ and such that $W$ is finite dimensional.

Giving an object $V \in \text{Coh}(A)$ (resp. $\in \text{Coh}_c(A)$) is equivalent to giving the following data:

- a coherent (resp. finite length) $\mathcal{O}_X$-module $V_0$,
- an $A' := f_* \End_Y(\mathcal{P}_0)$-module $V_1$ which is coherent (resp. finite length) as an $\mathcal{O}_X$-bimodule,
- a homomorphism $f_* \Hom_Y(\mathcal{O}_Y, \mathcal{P}_0) \to \Hom_X(V_0, V_1)$ of $(\mathcal{O}_X, A')$-bimodules,
- a homomorphisms $f_* \Hom_Y(\mathcal{P}_0, \mathcal{O}_Y) \to \Hom_X(V_1, V_0)$ of $(A', \mathcal{O}_X)$-bimodules.

Definition 1.5. A framed $A$-module is a pair $(V, V_\infty, \iota)$ of an $A$-module $V$, a vector space $V_\infty$ and a linear map $\iota : V_\infty \to H^0(X, V_0)$.

A morphism between framed $A$-modules $(V, V_\infty, \iota)$ and $(V', V'_\infty, \iota')$ is a pair of an $A$-module homomorphism $V \to V'$ and a linear map $V_\infty \to V'_\infty$ which are compatible with $\iota$ and $\iota'$.

We denote, with a slight abuse of notations, by $\text{Coh}(\tilde{A})$ the abelian category of framed $\tilde{A}$-modules and by $\text{Coh}_c(\tilde{A})$ the subcategory of framed $\tilde{A}$-modules $(V, V_\infty, \iota)$ such that $V \in \text{Coh}_c(A)$ and such that $V_\infty$ is finite dimensional.

Proposition 1.6. The category $\text{Per}(Y/X)$ (resp. $\text{Per}_c(Y/X)$) is equivalent to the category $\text{Coh}(\tilde{A})$ (resp. $\text{Coh}_c(\tilde{A})$).

Proof. First we put $W = V_\infty$. Using the adjunction, we have

$$\Hom_Y(W \otimes \mathcal{O}_Y, F) = \Hom_Y(p^*W, F) = \Hom_C(V_\infty, p_*F) = \Hom_C(V_\infty, H^0(X, V_0)),$$

where $p$ is the projection from $Y$ to a point. The equivalences follow immediately.

1.3 Stability

Let $\zeta = (\zeta_0, \zeta_1, \zeta_\infty)$ be a triple of real numbers. For a nonzero object $\tilde{F} = (F, W, s) \in \text{Per}_c(Y/X)$ we define

$$\theta_\zeta(\tilde{F}) := \frac{\zeta_0 \dim H^0(F) + \zeta_1 \dim H^0(F \otimes \mathcal{P}_0^\vee) + \zeta_\infty \dim W}{\dim H^0(F) + \dim H^0(F \otimes \mathcal{P}_0^\vee) + \dim W}.$$
Definition 1.7. A perverse coherent system \( \tilde{F} \in \text{Per}_c(Y/X) \) is \( \theta_{\tilde{\zeta}} \)-\( (\text{semi}) \)stable if we have
\[
\theta_{\tilde{\zeta}}(\tilde{F}') (\leq) \theta_{\tilde{\zeta}}(\tilde{F})
\]
for any nonzero proper subobject \( 0 \neq \tilde{F}' \subsetneq \tilde{F} \) in \( \text{Per}_c(Y/X) \).

Here we adapt the convention for the short-hand notation. The above means two assertions: semistable if we have ‘\( \leq \)’, and stable if we have ‘\( < \)’.

Remark 1.8. (1) As we shall see later, the space of the parameters \( \tilde{\zeta} \) has a chamber structure defined by integral hyperplanes so that the (semi)stability is unchanged if we stay in a chamber (see §3.3).

(2) Given a real number \( c \) let \( \tilde{\zeta}' \) be the triple of real numbers \( (\zeta_0 + c, \zeta_1 + c, \zeta_\infty + c) \). Then we have
\[
\theta_{\tilde{\zeta}'}(\tilde{F}) = \theta_{\tilde{\zeta}}(\tilde{F}) + c.
\]
Hence \( \theta_{\tilde{\zeta}'} \)-(semi)stability and \( \theta_{\tilde{\zeta}} \)-(semi)stability are equivalent. In particular, given a \( \theta_{\tilde{\zeta}} \)-(semi)stable perverse coherent system \( \tilde{F} \in \text{Per}_c(Y/X) \) we can normalize \( \tilde{\zeta} \) so that \( \theta_{\tilde{\zeta}}(\tilde{F}) = 0 \).

(3) This stability condition depends on the choice of \( \tilde{\zeta} \), as well as the choice of \( P_0 \).

(4) Given a pair of real numbers \( \zeta = (\zeta_0, \zeta_1) \), we define the \( \theta_{\zeta} \)-(semi)stability for a perverse coherent sheaf by the same conditions. In other words, a perverse coherent system \( F \in \text{Per}_c(Y/X) \) is \( \theta_{\zeta} \)-(semi)stable if and only if the perverse coherent system \( (F,0,0) \) with trivial framing is \( \theta_{\zeta} \)-(semi)stable for some (equivalently any) \( \zeta_\infty \).

Theorem 1.9 ([Rud97]). Let a stability parameter \( \tilde{\zeta} \in \mathbb{R}^3 \) be fixed.

(1) A perverse coherent system \( \tilde{F} \in \text{Per}_c(Y/X) \) has a unique Harder-Narasimhan filtration:
\[
\tilde{F} = \tilde{F}_0 \supset \tilde{F}_1 \supset \cdots \supset \tilde{F}_L \supset \tilde{F}_{L+1} = 0
\]
such that \( \tilde{F}_l/\tilde{F}_{l+1} \) is \( \theta_{\tilde{\zeta}} \)-semistable for \( l = 0, 1, \ldots, L \) and
\[
\theta_{\tilde{\zeta}}(\tilde{F}_0/\tilde{F}_1) < \theta_{\tilde{\zeta}}(\tilde{F}_1/\tilde{F}_2) < \cdots < \theta_{\tilde{\zeta}}(\tilde{F}_L/\tilde{F}_{L+1}).
\]

(2) A \( \theta_{\tilde{\zeta}} \)-semistable perverse coherent system \( \tilde{F} \in \text{Per}_c(Y/X) \) has a Jordan-Hölder filtration:
\[
\tilde{F} = \tilde{F}_0 \supset \tilde{F}_1 \supset \cdots \supset \tilde{F}_L \supset \tilde{F}_{L+1} = 0
\]
such that \( \tilde{F}_l/\tilde{F}_{l+1} \) is \( \theta_{\tilde{\zeta}} \)-stable for \( l = 0, 1, \ldots, L \) and
\[
\theta_{\tilde{\zeta}}(\tilde{F}_0/\tilde{F}_1) = \theta_{\tilde{\zeta}}(\tilde{F}_1/\tilde{F}_2) = \cdots = \theta_{\tilde{\zeta}}(\tilde{F}_L/\tilde{F}_{L+1}).
\]
### 1.4 Moduli spaces of perverse coherent systems

In this subsection, we assume that $\zeta_0, \zeta_1$ and $\zeta_\infty$ are rational numbers.

**Theorem 1.10.** There is a coarse moduli scheme parameterizing $S$-equivalence classes of $\theta^\ast$-semistable objects in $\overline{\text{Per}}_r(Y/X)$.

The theorem is deduced from a more general construction (Theorem 1.11), which was explained to the second named author by Kôta Yoshioka. We only give a sketch of the proof, as we are mainly interested in the case when $X$ is affine, and hence we can alternatively use the construction in [Kin94] (see [Yos03, §3.2], and Yoshioka wrote a paper containing the proof ([Yos Proposition 1.6.1])).

For a while, we assume that $X$ is projective. Take an ample line bundle $\mathcal{O}_X(1)$ over $X$. We define the $P$-twisted Hilbert polynomial of $F \in \text{Per}(Y/X)$ by (cf. [Yos03] $\S$4)

$$\chi(P, F(n)) = \chi(\mathbb{R}f_\ast(P^\vee \otimes F)(n)).$$

From Theorem 1.3 this is nothing but the usual Hilbert polynomial for the corresponding sheaf $\mathbb{R}f_\ast(P^\vee \otimes F)$. We expand this as

$$\chi(P, F(n)) = \sum_i a_i^P(F) \binom{n+i}{i}.$$

We say $F$ is $d$-dimensional, if $a_d^P(F) > 0$ and $a_i^P(F) = 0, i > d$. Then an $d$-dimensional object $F \in \text{Per}(Y/X)$ is $P$-twisted (semi)stable if

$$\chi(P, F'(n)) \leq \frac{a_d^P(F')}{a_d^P(F)} \chi(P, F(n)) \quad \text{for } n \gg 0$$

holds for any proper subobject $0 \neq F' \subseteq F$ in $\text{Per}(Y/X)$. From the inequality, $F$ cannot contain a nonzero subobject $F'$ with $a_d^P(F') = 0$. This condition is referred as ‘$F$ is of pure dimension $d$’ in the usual stability for coherent sheaves. Under that condition the above is equivalent to

$$\chi(P, F'(n))/a_d^P(F') \leq \chi(P, F(n))/a_d^P(F).$$

We can construct the moduli space of $P$-twisted semistable sheaves by modifying the construction of the moduli space of usual stable sheaves by Simpson [Sim94] (see [HL97]) as follows: By Theorem 1.3 we may construct it as a moduli space of semistable $\mathcal{A}$-modules $\mathcal{T} = \mathbb{R}f_\ast(P^\vee \otimes F)$, where the stability condition is defined as usual. Now $\mathcal{A}$ is an example of a sheaf of rings of differential operators on $X$ in the sense of [Sim94] $\S$2, hence the moduli space can be constructed. (See also [Yos03].) For a later purpose, we review the argument briefly. The moduli space is a GIT quotient of the scheme $Q$ parameterizing all quotients $[V \otimes_\mathcal{C} \mathcal{A}(-m) \to \mathcal{F}]$ of $\mathcal{A}$-modules by $\text{SL}(V)$ for the vector space $\mathcal{V} = \text{Hom}_\mathcal{A}(\mathcal{A}, F(m))$ for a fixed sufficiently large $m$. The scheme $Q$ is a closed subscheme of the usual quot-scheme parameterizing quotients in $\text{Coh}(X)$. The polarization of $Q$ comes from the embedding into the Grassmann variety of quotients $[H^0(V \otimes_\mathcal{C} \mathcal{A}(l-m)) \to H^0(\mathcal{T}(l))]$ for sufficiently large $l$. In $\text{Per}(Y/X)$, the Grassmann variety parameterizes quotients $[V \otimes H^0(P^\vee \otimes P(l-m)) \to H^0(P^\vee \otimes F(l))]$ under the equivalence $\mathcal{T} = \mathbb{R}f_\ast(P^\vee \otimes F)$.
We next generalize the stability condition slightly. Suppose $\mathcal{P}, \mathcal{P}'$ are local projective generators. We say $F \in \text{Per}(Y/X)$ is $(\mathcal{P}, \mathcal{P}')$-twisted (semi)stable if

$$ \chi(\mathcal{P}, F'(n)) \leq \frac{\chi(\mathcal{P}', F'(m))}{\chi(\mathcal{P}, F(m))} \chi(\mathcal{P}, F(n)) \quad \text{for } m \gg n \gg 0 $$

holds for any proper subobject $0 \neq F' \subset F$ in $\text{Per}(Y/X)$. If $\mathcal{P} = \mathcal{P}'$, $(\mathcal{P}, \mathcal{P}')$-twisted (semi)stability is equivalent to the above $\mathcal{P}$-twisted stability. The moduli space of $(\mathcal{P}, \mathcal{P}')$-twisted semistable sheaves can be constructed as above. In fact, the scheme $Q$ for which we take a GIT quotient is the same as above, but we use the different polarization from the embedding into the Grassmann variety using $\mathcal{P}'$ instead of $\mathcal{P}$, i.e., quotients $[V \otimes H^0(\mathcal{P}' \otimes \mathcal{P}(l - m)) \to H^0(\mathcal{P}' \otimes \mathcal{P}(l))]$ for sufficiently large $l$. This modification is very similar to (in fact, simpler than) the stability condition considered in [NY98, Sect. 2].

We can also construct moduli spaces of perverse coherent systems. Let $\alpha$ be a polynomial of rational coefficients such that $\alpha(n) > 0$ for $n \gg 0$. A perverse coherent system $(F, W, s)$ with a finite dimensional framing $W$ is $(\mathcal{P}, \mathcal{P}', \alpha)$-(semi)stable if

$$ \dim W' \cdot \alpha(n) + \chi(\mathcal{P}, F'(n))(\leq) \frac{\chi(\mathcal{P}', F'(m))}{\chi(\mathcal{P}', F(m))} (\dim W \cdot \alpha(n) + \chi(\mathcal{P}, F(n))) $$

for $m \gg n \gg 0$ \hspace{1cm} (1.2)

holds for any proper subobject $0 \neq (F', W', s') \subset (F, W, s)$ in $\text{Per}(Y/X)$. If the homomorphism $s: W \to \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, F)$ has nontrivial kernel, $(F', W', s') = (0, \ker s, 0)$ violates the inequality. Therefore $W$ can be considered as a subspace of $\text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, F)$. We can further consider $W$ as a subspace of $\text{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{P}(m)) \otimes \text{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}(m)) \otimes \mathcal{O}_Y$ for sufficiently large $m$ thanks to the projection $\mathcal{P} \to \mathcal{O}_Y$.

Now we can construct the moduli space of $(\mathcal{P}, \mathcal{P}', \alpha)$-semistable perverse coherent systems as a GIT quotient of a closed subscheme of the product of the quot-scheme and the Grassmann variety as in [He98, LP93]. In summary, we have the following theorem:

**Theorem 1.11.** Under the assumption that $X$ is projective, there is a coarse moduli scheme parameterizing $S$-equivalence classes of $(\mathcal{P}, \mathcal{P}', \alpha)$-semistable perverse coherent systems with a finite dimensional framing.

Now, we will explain how Theorem 1.11 is deduced from Theorem 1.10.

First, we replace $X$ by a projective scheme $\overline{X}$ containing $X$ as an open subscheme and construct a moduli space for $\overline{X}$. Then the moduli space for $X$ is an open subscheme of the moduli space for $\overline{X}$ (here we use the assumption that the objects are in $\text{Per}(\mathcal{O}_Y/X)$, not just in $\text{Per}(\mathcal{O}_Y))$. Therefore we may assume $X$ is projective from the beginning.

Note that for an object $F \in \text{Per}(\mathcal{O}_Y/X)$, the Hilbert polynomials are constant. We may assume that $\zeta_0, \zeta_1$ and $\zeta_\infty$ are integers and normalized so that $\theta^{c}(\hat{F}) = 0$ as we mentioned in Remark (2). Taking sufficiently large $c \in \mathbb{R}$ so that $\zeta_0 + c, \zeta_1 + c > 0$, we put $\mathcal{P} = \mathcal{O}_Y(\zeta_0 + c) \oplus \mathcal{P}_0(\zeta_1 + c)$, $\mathcal{P}' = \mathcal{O}_Y \oplus \mathcal{P}_0$ and $\mathcal{P} = \mathcal{O}_Y \oplus \mathcal{O}_Y(\zeta_0 + c) \oplus \mathcal{P}_0(\zeta_1 + c)$.
\[ \alpha = \zeta_{\infty}. \] Then the inequality (1.2) turns out to be

\[
\frac{\zeta_0 \dim H^0(F') + \zeta_1 \dim H^0(F' \otimes \mathcal{P}_0^\vee) + \zeta_{\infty} \dim W}{\dim H^0(F')} \leq \frac{\zeta_0 \dim H^0(F) + \zeta_1 \dim H^0(F \otimes \mathcal{P}_0^\vee) + \zeta_{\infty} \dim W}{\dim H^0(F)} = 0,
\]

where \(c\) cancels out in both hand sides. This is equivalent to the inequality in Definition 1.7. Therefore we can apply the above construction of the moduli space provided \(\zeta_{\infty} = -\zeta_0 \dim H^0(F) - \zeta_1 \dim H^0(F \otimes \mathcal{P}_0^\vee) > 0. \) Fortunately this condition is not restrictive, as there is no \(\theta_{\zeta}\)-stable object (with \(W \neq 0\)) except \(F = 0\) if \(\zeta_{\infty} \leq 0.\)

\section{Coherent systems as perverse coherent systems}

\subsection{Chambers corresponding to DT and PT}

Fix an ample line bundle \(\mathcal{L}\) and a vector bundle \(\mathcal{P}_0\) on \(Y\) as in Definition 1.2. Take \(\mathcal{L}\) as a polarization of \(Y\). For an element \(E \in D^b_{coh}(Y)\), let \(r(E)\) denote the degree one coefficient of the Hilbert polynomial \(\chi(E \otimes \mathcal{L}^\otimes k)\).

\begin{definition}
Let \((\zeta_0, \zeta_1)\) be a pair of real numbers. A perverse coherent system \((F, s) \in \text{Per}_c(Y/X)\) with a 1-dimensional framing is said to be \((\zeta_0, \zeta_1)\)-\((\text{semi})\)stable if it is \(\theta_{\zeta}\)-\((\text{semi})\)stable for

\[
\zeta = (\zeta_0, \zeta_1, -\zeta_0 \cdot \dim H^0(E) - \zeta_1 \cdot \dim H^0(E \otimes \mathcal{P}_0^\vee)).
\]

(See the normalization in Remark 1.8(2).)
\end{definition}

\begin{lemma}
A perverse coherent system \((F, s) \in \text{Per}_c(Y/X)\) is \((\zeta_0, \zeta_1)\)-\((\text{semi})\)stable if the following conditions are satisfied:

(A) for any nonzero subobject \(0 \neq E \subsetneq F\), we have

\[
\zeta_0 \cdot \dim H^0(E) + \zeta_1 \cdot \dim H^0(E \otimes \mathcal{P}_0^\vee) \leq 0.
\]

(B) for any proper subobject \(E \subsetneq F\) through which \(s\) factors, we have

\[
\zeta_0 \cdot \dim H^0(E) + \zeta_1 \cdot \dim H^0(E \otimes \mathcal{P}_0^\vee) \leq \zeta_0 \cdot \dim H^0(F) + \zeta_1 \cdot \dim H^0(F \otimes \mathcal{P}_0^\vee).
\]

Let \(r\) be the positive integer in Definition 1.1.

\begin{lemma}
For any \(E \in \text{Per}_c(Y/X)\), we have

\[
r \cdot \dim H^0(Y, E) - \dim H^0(Y, E \otimes \mathcal{P}_0^\vee) = r(E).
\]

Proof. Since the Hilbert polynomial \(\chi(E \otimes \mathcal{L}^\otimes k)\) is degree one, we have \(r(E) = \chi(E) - \chi(E \otimes \mathcal{L}^{-1})\). On the other hand, since \(E \in \text{Per}_c(Y/X)\) we have

\[
H^0(E) = \chi(E), \quad H^0(E \otimes \mathcal{P}_0^\vee) = \chi(E \otimes \mathcal{P}_0^\vee) = (r - 1)\chi(E) + \chi(E \otimes \mathcal{L}^{-1}).
\]

Hence the claim follows. \(\square\)
We set $\zeta^o = (-r, 1)$ and $\zeta^\pm = (-r \pm \ell, 1)$ for sufficiently small $\ell > 0$ specified later.

**Corollary 2.4.** A perverse coherent system $(F, s) \in \text{Per}_{c}(Y/X)$ is $\zeta^o$-(semi)stable (resp. $\zeta^\pm$-(semi)stable) if and only if the following conditions are satisfied:

(A) for any nonzero subobject $0 \neq E \subseteq F$, we have

$$-r(E) \leq 0 \quad (\text{resp. } -r(E) \pm \ell \cdot \chi(E) \leq 0),$$

(B) for any nonzero subsheaf $0 \neq E \subseteq F$ through which $s$ factors, we have

$$0 \leq -r(F/E) \quad (\text{resp. } 0 \leq -r(F/E) \pm \ell \cdot \chi(F/E)).$$

**Lemma 2.5.** Let $0 \to E \to F \to G \to 0$ be an exact sequence in $\text{Coh}(Y)$. If $F$ is perverse coherent, then so is $G$.

**Proof.** Applying the functor $f_*$ to the short exact sequence, we get the exact sequence

$$\mathbb{R}^1 f_* F \to \mathbb{R}^1 f_* G \to \mathbb{R}^2 f_* E.$$

Since $F$ is perverse coherent we have $\mathbb{R}^1 f_* F = 0$. Because dimensions of the fibers of $f: Y \to X$ are less than 2, we have $\mathbb{R}^2 f_* E = 0$. Hence we get $\mathbb{R}^1 f_* G = 0$.

**Corollary 2.6.** Let $0 \to E \to F \to G \to 0$ be an exact sequence in $\text{Coh}(Y)$. If $E$ and $F$ are perverse coherent, then this is an exact sequence in $\text{Per}(Y/X)$ as well.

**Lemma 2.7.** Let $(F, s)$ be a perverse coherent system such that $F$ is a sheaf. Then the sequence

$$0 \to \text{im}_{\text{Coh}(Y)}(s) \to F \to \text{coker}_{\text{Coh}(Y)}(s) \to 0$$

is exact in $\text{Per}(Y/X)$.

**Proof.** By Lemma 2.5 applied to $\mathcal{O}_Y \to \text{im}_{\text{Coh}(Y)}(s)$, we see that $\text{im}_{\text{Coh}(Y)}(s)$ is perverse coherent. Then the claim follows from Corollary 2.6.

**Lemma 2.8.** Let $(F, s)$ be a coherent system such that $\dim \text{coker}_{\text{Coh}(Y)}(s) = 0$. Then $F$ is perverse coherent.

**Proof.** We have the exact sequences

$$0 \to \ker(s) \to \mathcal{O}_Y \to \text{im}(s) \to 0$$

$$0 \to \text{im}(s) \to F \to \text{coker}(s) \to 0$$

in $\text{Coh}(Y)$. Applying Lemma 2.5 for the first exact sequence, we have $\text{im}(s)$ is perverse coherent. Since $\text{coker}(s)$ is 0-dimensional, it is perverse coherent. Hence $F$ is perverse coherent by the second exact sequence.
Lemma 2.9. If a perverse coherent system \((F, s) \in \Per_c(Y/X)\) is \(\zeta^\circ\)-semistable, then \(F\) is a sheaf.

Proof. We have the canonical exact sequence
\[
0 \to H^{-1}(F)[1] \to F \to H^0(F) \to 0
\]
in \(\Per(Y/X)\). By the condition (A) in Corollary 2.4, we have \(r(H^{-1}(F)) = -r(H^{-1}(F)[1]) \leq 0\). Since \(H^{-1}(F)\) is a sheaf, this inequality means that \(H^{-1}(F)\) is 0-dimensional. The defining condition of perverse coherent sheaves requires \(\mathbb{R}^0 f_\ast(H^{-1}(F)) = 0\). So we have \(H^{-1}(F) = 0\).

Fixing the numerical class of \(F \in D^b_c(\Coh(Y))\), we take sufficiently small \(\varepsilon > 0\) so that \(\varepsilon \cdot \chi(F) < 1\).

Proposition 2.10. Given a \(\zeta^-\)-stable perverse coherent system \((F, s) \in \Per_c(Y/X)\), then \(F\) is a sheaf and \(s\) is surjective in \(\Coh(Y)\). On the other hand, given a coherent sheaf \(F \in \Coh(Y)\) and a surjection \(s: \mathcal{O}_Y \to F\) in \(\Coh(Y)\), then \(F\) is perverse coherent and the perverse coherent system \((F, s)\) is \(\zeta^-\)-stable.

Proof. Assume that \((F, s) \in \Per_c(Y/X)\) is \(\zeta^-\)-stable. By Lemma 2.9 \(F\) is a sheaf. By Lemma 2.5 \(\text{coker}_{\Coh(Y)}(s)\) is perverse coherent. Assume \(s\) is not surjective in \(\Coh(Y)\). Since \(\text{coker}_{\Coh(Y)}(s)\) is a sheaf, we have \(r(\text{coker}_{\Coh(Y)}(s)) \geq 0\). Since \(\text{coker}_{\Coh(Y)}(s)\) is perverse coherent, we have \(\chi(\text{coker}_{\Coh(Y)}(s)) \geq 0\). Hence we have
\[
0 > -r(\text{coker}_{\Coh(Y)}(s)) - \varepsilon \cdot \chi(\text{coker}_{\Coh(Y)}(s)).
\]
By Lemma 2.7 this contradicts the condition (B) of \(\zeta^-\)-stability of \((F, s)\). So \(s\) is surjective in \(\Coh(Y)\).

On the other hand, assume that \(s\) is surjective in \(\Coh(Y)\). Let
\[
0 \to E \to F \to G \to 0
\]
be an exact sequence in \(\Per(Y/X)\). Since \(H^{-2}(G) = H^{-1}(F) = 0\), so we have \(H^{-1}(E) = 0\) by the long exact sequence. Thus \(E\) is a sheaf, and so we have \(r(E) \geq 0\). Since \(E\) is perverse coherent we have \(\chi(E) \geq 0\). So the condition (A) holds. Moreover, assume that \(s\) factors through \(E\). Then \(E \to F\) is surjective in \(\Coh(Y)\) and so \(G\) is a sheaf shifted by \([1]\). Suppose \(r(G) \leq -1\). Since \(F\) is perverse coherent, we have
\[
\varepsilon \cdot \chi(G) = \varepsilon(\chi(F) - \chi(E)) \leq \varepsilon \cdot \chi(F) < 1.
\]
So the condition (B) holds. Suppose \(r(G) = 0\). Then \(\chi(G) < 0\) and the condition (B) holds.

Proposition 2.11. Given a \(\zeta^+\)-stable perverse coherent system \((F, s) \in \Per_c(Y/X)\), then \(F\) is a sheaf and \((F, s)\) is a stable pair in the sense of [PT09], that is, the following conditions are satisfied:

1. \(F\) is pure of dimension 1, and
2. \(\text{coker}_{\Coh(Y)}(s)\) is 0-dimensional.
On the other hand, given a stable pair \((F, s)\), then \(F\) is perverse coherent and the perverse coherent system \((F, s)\) is \(\zeta^+\)-stable.

**Proof.** Assume that \((F, s) \in \text{Per}_c(Y/X)\) is \(\zeta^+\)-stable. Let

\[
0 \to E \to F \to G \to 0
\]

be an exact sequence in \(\text{Coh}(Y)\). Suppose that \(E\) is \(0\)-dimensional. Then \(E\) is perverse coherent and by Corollary 2.6 \(E\) is a subobject of \(F\) in \(\text{Per}(Y/X)\) as well. We have \(r(E) = 0\) and \(\chi(E) > 0\) because \(E\) is \(0\)-dimensional. This contradicts with the inequality

\[
-r(E) + \varepsilon \cdot \chi(E) < 0
\]

in the condition (A) of \(\zeta^+\)-stability. So \(F\) is pure of dimension 1. By Lemma 2.7 and the condition (B) of \(\zeta^+\)-stability, we have

\[
0 < -r(\text{coker}_{\text{Coh}(Y)}(s)) + \varepsilon \cdot \chi(\text{coker}_{\text{Coh}(Y)}(s)),
\]

unless \(\text{coker}_{\text{Coh}(Y)}(s) = 0\). Since \(\text{im}_{\text{Coh}(Y)}(s)\) is perverse coherent, we have

\[
\varepsilon \chi(\text{coker}_{\text{Coh}(Y)}(s)) = \varepsilon(\chi(F) - \chi(\text{im}_{\text{Coh}(Y)}(s))) \leq \varepsilon \cdot \chi(F) < 1.
\]

So \(r(\text{coker}_{\text{Coh}(Y)}(s))\) can not be positive, that is, \(\text{coker}_{\text{Coh}(Y)}(s)\) is \(0\)-dimensional.

Assume that \((F, s)\) is a stable pair. Let

\[
0 \to E \to F \to G \to 0
\]

be an exact sequence in \(\text{Per}(Y/X)\). As in the proof of Proposition 2.10 \(E\) is a sheaf. We have the exact sequences

\[
\begin{align*}
0 & \to H^{-1}(G) \to E \to \text{im}_{\text{Coh}(Y)}(E \to F) \to 0 \\
0 & \to \text{im}_{\text{Coh}(Y)}(E \to F) \to F \to H^0(G) \to 0
\end{align*}
\]

in \(\text{Coh}(Y)\). Since \(F\) is pure of dimension 1, \(\text{im}_{\text{Coh}(Y)}(E \to F)\) is 1-dimensional unless it is zero. As in the proof of Lemma 2.9, \(G\) is 1-dimensional unless \(G = 0\). Hence we have \(r(E) \geq 1\) unless \(E = 0\). Because \(G\) is perverse coherent, we have \(\chi(E) = \chi(F) - \chi(G) \leq \chi(F)\). So the condition (A) is satisfied.

Moreover, assume that \(s\) factors through \(E\). Then \(\text{im}_{\text{Coh}(Y)}(E \to F) \supset \text{im}_{\text{Coh}(Y)}(s)\) in \(\text{Coh}(Y)\) and so

\[
\text{coker}_{\text{Coh}(Y)}(s) \to \text{coker}_{\text{Coh}(Y)}(E \to F) = H^0(G),
\]

in \(\text{Coh}(Y)\). In particular, \(H^0(G)\) is \(0\)-dimensional. By an argument as in the proof of Proposition 2.8, \(H^{-1}(G)\) is 1-dimensional unless \(H^{-1}(G) = 0\). Because \(E\) is perverse coherent, we have \(\chi(G) = \chi(F) - \chi(E) \leq \chi(F)\). Hence we have

\[
-r(G) + \varepsilon \cdot \chi(G) = -r(H^0(G)) + r(H^{-1}(G)) + \varepsilon \cdot \chi(G) > 0.
\]
2.2 Coherent systems on the flop

Assume further $f$ is isomorphic in codimension 1. Let $f^+: Y^+ \to X$ be the flop. Let $^0\text{Per}(Y^+/X)$ be the full subcategory of $D^b(\text{Coh}(Y^+))$ consisting of objects $E$ satisfying the following conditions:

- $H^i(E) = 0$ unless $i = 0, -1$,
- $\mathbb{R}^1 f_*(H^0(E)) = 0$ and $\mathbb{R}^0 f_*(H^{-1}(E)) = 0$,
- $\text{Hom}(H^0(E), C) = 0$ for any sheaf $C$ on $Y$ satisfying $\mathbb{R}f_*(C) = 0$.

We can associate $\mathcal{L}$ and $\mathcal{P}_0$ with an ample line bundle $\mathcal{L}^+$ and a vector bundle $Q_0^+$ on $Y^+$ such that

- they are involved in an exact sequence
  
  $$0 \to (\mathcal{L}^+)^{-1} \to Q_0^+ \to \mathcal{O}_{Y^+}^{\oplus r-1} \to 0,$$

  where $r$ coincides with what appeared in the defining sequence of $\mathcal{P}_0$,

- $Q_0^+$ is a local projective generator of $^0\text{Per}(Y^+/X)$,

- $f^+ \text{End}(\mathcal{O}_{Y^+} \oplus Q_0^+) = A$,

and hence we have the following equivalences:

$$D^h(\text{Coh}(Y)) \simeq D^h(\text{Coh}(A)) \simeq D^h(\text{Coh}(Y^+)) \cup ^{-1}\text{Per}(Y/X) \simeq \text{Coh}(A) \simeq ^0\text{Per}(Y^+/X),$$

(see [VdB04, Theorem 4.4.2]). Here we denote by $^{-1}\text{Per}(Y/X)$ what we have denoted simply by $\text{Per}(Y/X)$ to emphasize the difference between $^{-1}\text{Per}(Y/X)$ and $^0\text{Per}(Y^+/X)$. By the same argument as the previous subsection, we can verify the following propositions:

**Proposition 2.12.** Given a $(-\zeta^+)$-stable perverse coherent system $(F^+, s^+) \in ^0\text{Per}_{c}(Y^+/X)$, then $F^+$ is a sheaf and $s^+$ is surjective in $\text{Coh}(Y^+)$. On the other hand, given a coherent sheaf $F^+ \in \text{Coh}_{c}(Y^+)$ and a surjection $s^+: \mathcal{O}_{Y^+} \to F^+$ in $\text{Coh}(Y^+)$, then $F^+$ is perverse coherent and the perverse coherent system $(F^+, s^+)$ is $(-\zeta^+)$-stable.

**Proposition 2.13.** Given a $(-\zeta^-)$-stable perverse coherent system $(F^+, s^+) \in ^0\text{Per}_{c}(Y^+/X)$, then $F^+$ is a sheaf and $(F^+, s^+)$ is a stable pair. On the other hand, given a stable pair $(F^+, s^+)$ on $Y^+$, then $F^+$ is perverse coherent and the perverse coherent system $(F^+, s^+)$ is $(-\zeta^-)$-stable.

3 Counting invariants on the resolved conifold

In this section, we study the counting invariants on the resolved conifold.

Let $f: Y \to X$ be the crepant resolution of the conifold, that is, $X = \{(x, y, z, w) \in \mathbb{C}^4 \mid xy - zw = 0\}$ and $Y$ is the total space of the vector bundle $\pi: \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathbb{P}^1$. This satisfies the assumptions at the beginning of §1.3.
3.1 Quivers for the resolved conifold

Let $Q$ be the quiver in Figure 3 and $A$ be the algebra defined by the following quiver with the relations:

$$A := \mathbb{C}Q / (a_1b_1a_2 = a_2b_1a_1, b_1a_1b_2 = b_2a_1b_1)_{i=1,2}.$$

Let $A$-Mod (resp. $A$-mod) denote the category of right $A$-modules (resp. finite dimensional right $A$-modules). For a finite dimensional $A$-module $V$, let $V_0$ and $V_1$ denote the vector spaces corresponding to the vertices 0 and 1 and $\dim V := (\dim V_0, \dim V_1) \in (\mathbb{Z}_{\geq 0})^2$.

Let us take $L := \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ as an ample line bundle. Since $H^1(Y, L^{-1}) = 0$, we have a projective generator $P := \mathcal{O}_Y \oplus L$. The endomorphism algebra $\text{End}_{\mathcal{Y}}(P)$ is isomorphic to $A$ and we have the following equivalence:

**Proposition 3.2.** (see Theorem 1.3)

$$\text{Per}(Y/X) \simeq A\text{-Mod}, \quad \text{Per}_{c}(Y/X) \simeq A\text{-mod}.$$  

Under these equivalences, we have $V_\infty = W$.

**Remark 3.1.** Note that this relation is derived from the superpotential $\omega = a_1b_1a_2 - a_1b_2a_1$ ([HD] for example. See also [DWZ08]).

Let $\tilde{A}$-Mod (resp. $\tilde{A}$-mod) denote the category of $\tilde{A}$-modules (resp. finite dimensional $\tilde{A}$-modules), which is equivalent to the category $\text{Coh}(\tilde{A})$ (resp. $\text{Coh}_{c}(\tilde{A})$) defined in §1.2.

For an $\tilde{A}$-module $\tilde{V}$, let $V_0$, $V_1$ and $V_\infty$ denote the vector spaces corresponding to the vertices 0, 1 and $\infty$ and $\dim \tilde{V} := (\dim V_0, \dim V_1, \dim V_\infty) \in (\mathbb{Z}_{\geq 0})^3$.

**Proposition 3.3.** (see Proposition 1.6)

$$\text{Per}(Y/X) \simeq \tilde{A}\text{-Mod}, \quad \text{Per}_{c}(Y/X) \simeq \tilde{A}\text{-mod}.$$  

Under these equivalences, we have $V_\infty = W$.  

![Figure 3: quiver Q](image-url)
3.2 Definition of the counting invariants

For $\zeta \in \mathbb{R}^2$ and $v = (v_0, v_1) \in (\mathbb{Z}_{\geq 0})^2$, let $\mathcal{M}_\zeta(v)$ (resp. $\mathcal{M}_\zeta^+(v)$) denote the moduli space of $\zeta$-semistable (resp. $\zeta$-stable) $\tilde{A}$-modules $\tilde{V}$ with $\dim \tilde{V} = (v_0, v_1, 1)$. They can be constructed by applying the result of [Kin94]. We define the generating function

$$Z'_\zeta(q) := \sum_{n \in \mathbb{Z}} \chi(\mathcal{M}_\zeta(v)) \cdot q^n$$

where $q^\nu = q_0^{\nu_0} q_1^{\nu_1}$ and $q_0$, $q_1$ are formal variables.

A 4-dimensional torus $(\mathbb{C}^*)^4$ acts on the moduli space $\mathcal{M}_\zeta(v)$ by rescaling the maps associated to the four arrows of the quiver $Q$. Since the subtorus $\mathbb{C}^* \simeq \{[(\alpha, \alpha, \alpha^{-1}, \alpha^{-1})] \in T\}$ acts trivially, we have the action of the 3-dimensional torus $T := (\mathbb{C}^*)^4 / \mathbb{C}^*$. We will show that

- the set of $T$-fixed closed points $\mathcal{M}_\zeta(v)^T$ is isolated (Proposition 4.14).

Hence we have

$$Z'_\zeta(q) = \sum_{n \in \mathbb{Z}} |\mathcal{M}_\zeta(v)^T| \cdot q^n.$$

We also define more sophisticated invariants. Let $\nu: \mathcal{M}_\zeta(v) \to \mathbb{Z}$ be the constructible function defined in [Beh09] (Behrend function). We define the counting invariants

$$D_\zeta(v) := \sum_{n \in \mathbb{Z}} n \cdot \chi(\nu^{-1}(n))$$

and encode them into the generating function

$$Z_\zeta(q) := \sum_{v \in (\mathbb{Z}_{\geq 0})^2} D_\zeta(v) \cdot q^n.$$

The Behrend function is defined for any scheme over $\mathbb{C}$. In [Beh09], Behrend showed that if an proper scheme has a symmetric obstruction theory then the virtual counting, which is defined by integrating the constant function 1 over the virtual fundamental cycle, coincides with the weighted Euler characteristic weighted by the Behrend function as above. Based on this result, he proposed to define the virtual counting for a non-proper variety with a symmetric obstruction theory as the weighted Euler characteristic.
A stability parameter $\zeta \in \mathbb{R}^2$ is said to be generic if $\zeta$-semistability and $\zeta$-stability are equivalent. Since the defining relation of $A$ is derived from the derivations of the superpotential, the moduli space $\mathcal{M}_\zeta(v)$ for a generic $\zeta$ has a symmetric obstruction theory \cite[Theorem 1.3.1]{Sze08}. We define the 2-dimensional subtorus

$$T' := \{[(\alpha_1, \alpha_2, \beta_1, \beta_2)] \in T \mid \alpha_1 \alpha_2 \beta_1 \beta_2 = 1\}$$

of $T$. The symmetric obstruction theory above lifts to a $T'$-equivariant symmetric obstruction theory. We will show the following propositions in §4 \cite[see \cite[Proposition 2.5.1 and Corollary 2.5.3]{Sze08}):

- $\mathcal{M}_\zeta(v)^{T'} = \mathcal{M}_\zeta(v)^T$ (Proposition 4.14).
- For each $T'$-fixed closed point $P \in \mathcal{M}_\zeta(v)^{T'}$, the Zariski tangent space to $\mathcal{M}_\zeta(v)$ at $P$ has no $T'$-invariant subspace (Proposition 4.20).
- For each $T'$-fixed point $P \in \mathcal{M}_\zeta(v)^{T'}$, the parity of the dimension of the Zariski tangent space to $\mathcal{M}_\zeta(v)$ at $P$ is same as the parity of $v_1$ (Corollary 4.19).

According to these propositions and Behrend-Fantechi’s result \cite[Theorem 3.4]{BF08}, we have the following formula \cite[Theorem 2.7.1]{Sze08}:

$$Z_\zeta(q) = \sum_{n \in \mathbb{Z}} (-1)^{v_1} \left| \mathcal{M}_\zeta(v)^T \right| q^n$$

(3.1)

In particular, we have

$$Z'_\zeta(q) = Z_\zeta(q_0, -q_1).$$

### 3.3 Classification of walls

In this subsection, we will classify non-generic parameters. The argument is a straightforward modification of one in \cite[§2]{NYa}.

**Lemma 3.4.** Let $W$ be a non-zero $\zeta$-stable $A$-module for some $\zeta \in \mathbb{R}^2$. Then at least one of the following holds:

1. $\dim W_0 = \dim W_1 = 1$,
2. $a_1 = a_2 = 0$,
3. $b_1 = b_2 = 0$.

**Proof.** (See \cite[Lemma 2.9]{NYa}.) Without loss of generality, we can assume $\zeta_0 \dim W_0 + \zeta_1 \dim W_1 = 0$ by Remark 1.3. If $W_0$ or $W_1$ is $0$, we trivially have (2) or (3). Therefore we may assume $W_0, W_1 \neq 0$.

We set $S_0 = \ker(a_1 b_1)$, $T_0 = \im(a_1 b_1)$, $S_1 = \ker(b_1 a_1)$ and $T_1 = \im(b_1 a_1)$. By the defining relation of $Q$ we can check $(S_0, S_1)$ and $(T_0, T_1)$ are $A$-submodules of $W$. The $\zeta$-stability of $W$ implies

$$\zeta_0 \dim S_0 + \zeta_1 \dim S_1 \leq 0, \ \zeta_0 \dim T_0 + \zeta_1 \dim T_1 \leq 0.$$ 

Since

$$\zeta_0 \dim W_0 + \zeta_1 \dim W_1 = 0, \ \dim S_i + \dim T_i = \dim W_i,$$
the above inequalities should be equalities. Again, the stability of $W$ implies $S_0 = S_1 = 0$ or $(S_0, S_1) = (W_0, W_1)$. In the previous case, $a_1$ and $b_1$ are isomorphisms and $\dim W_0 = \dim W_1$.

Taking arbitrary pairs $a_i, b_j (i, j = 1, 2)$, we may assume either (a) $a_1, b_1$ are isomorphisms and $\dim W_0 = \dim W_1$, or (b) $a_i b_j = 0, b_j a_i = 0$ for any $i$ and $j$. First we consider the case (b). Without loss of generality we also assume $\zeta_0 \geq 0$. Apply the stability conditions for an $A$-submodule $(\ker a_1 \cap \ker a_2, 0)$, we get $\ker a_1 \cap \ker a_2 = 0$. The equations $a_i b_j = 0$ mean $\im b_1, \im b_2 \subset \ker a_1 \cap \ker a_2 = 0$, that is $b_1 = b_2 = 0$. Similarly, for the case $\zeta_0 \leq 0$ we have $a_1 = a_2 = 0$.

Next consider the case (a), and hence $\zeta_0 + \zeta_1 = 0$. We first assume $\zeta_0 < 0$. From the defining equation of $Q$, four linear maps $b_1 a_1, b_2 a_1, b_1 a_2, b_2 a_2$ are pairwise commuting. We take a simultaneous eigenvector $0 \neq w_0 \in W_0$ and set

$$S'_0 := \mathbb{C} w_0, \quad S'_1 := \mathbb{C} a_1 w_0 + \mathbb{C} a_2 w_0.$$  

Then $(S'_0, S'_1)$ is an $A$-submodule of $W$, and hence

$$\zeta_0 \dim S'_0 + \zeta_1 \dim S'_1 \leq 0$$

by the $\theta_\zeta$-semistability of $W$. Therefore we have $\dim S_1' = \dim S_0' = 1$. On the other hand, $a_1 w_0 \neq 0$ as $a_1$ is an isomorphism by the assumption. Therefore we have $\dim S_1' = 1$, so the equality holds in the above inequality, so we have $(S_0', S_1') = (W_0, W_1)$ by the $\theta_\zeta$-stability of $W$. In particular, $\dim W_0 = \dim W_1 = 1$. Exchanging 0 and 1, we have the same assertion when $\zeta_1 < 0$.

Finally suppose $\zeta_0 = \zeta_1 = 0$. We define $S_0', S_1'$ as above. Since the equality $\zeta_0 \dim S_0' + \zeta_1 \dim S_1' = 0$ holds, the $\theta_\zeta$-stability of $W$ implies $(S_0', S_1') = (W_0, W_1)$. In particular, $\dim W_0 = 1$. Exchanging 0 and 1, we also get $\dim W_1 = 1$.  

In the case (1) with $\zeta_0 < 0$ (resp. $\zeta_0 > 0$), the $\zeta$-stable $A$-modules are parameterized by $Y$ (resp. $Y^\perp$). This is well-known, and can be checked easily (cf. [NYa, §2.3]).

In the cases (2) or (3), the representation can be considered as a representation of the Kronecker quiver. Then by the argument in [NYa, Lemma 2.12], we have $(W_0, W_1) = (\mathbb{C}^m, \mathbb{C}^{m+1})$ or $(W_0, W_1) = (\mathbb{C}^m, \mathbb{C}^{m-1})$ for some $m \geq 0$ or $m \geq 1$ in the latter case. Moreover, the $\zeta$-stable $A$-module is unique up to isomorphism. In the case (2), we denote the $\zeta$-stable $A$-module by $C_\zeta^+(m)$ if $(W_0, W_1) = (\mathbb{C}^m, \mathbb{C}^{m+1})$. Similarly, we denote the module by $C_\zeta^-(m)$ in the case (3).

We can visualize these modules as in Figure 3. Each dot corresponds to a basis vector of $W_0$ and $W_1$, and right-up (resp. right-down, left-up, left-down) arrows are $a_1$ (resp. $a_2, b_1, b_2$).

Now, we can check the following classification:

**Theorem 3.5.** Let $\zeta$ be a stability parameter.

1. If $\zeta_0 < \zeta_1$, then $\theta_\zeta$-stable $A$-modules $W$ are classified as follows:
   - $C_\zeta^+(m)$ $(m \geq 1)$,
   - the $\zeta$-stable $A$-modules $W$ with $\dim W = (1, 1)$ parameterized by $Y$,
   - $C_\zeta^-(m)$ $(m \geq 0)$.

2. If $\zeta_0 > \zeta_1$, then $\theta_\zeta$-stable $A$-modules $W$ are classified as follows:
The Jordan–Hölder filtration of the $\theta_\zeta$-semistable $\tilde{A}$-module $\tilde{V}$. Here $L \geq 1$ because $\tilde{V}$ is not $\theta_\zeta$-stable. Since $\dim \tilde{V}_\infty = 1$, at most one of $(\tilde{V}/\tilde{V}^{i+1})_\infty$ is non-zero. In particular, there exists a non-zero $\theta_\zeta$-stable $\tilde{A}$-module $\tilde{W}$ such that $\tilde{W}_\infty = 0$ and such that $\zeta \cdot \dim \tilde{W} = 0$. In other words, there exists a non-zero $\theta_\zeta$-stable $A$-module $W$ such that $\zeta \cdot \dim W = 0$. We define the following walls (half lines) on the set of stability parameters:

- $L_+^-(m) := \{(\zeta_0, \zeta_1) \mid \zeta_0 < \zeta_1, m \zeta_0 + (m-1)\zeta_1 = 0\} \ (m \geq 1)$,
- $L_-^-(\infty) := \{(\zeta_0, \zeta_1) \mid \zeta_0 < \zeta_1, m \zeta_0 + \zeta_1 = 0\}$,
- $L_-^-(m) := \{(\zeta_0, \zeta_1) \mid \zeta_0 < \zeta_1, m \zeta_0 + (m+1)\zeta_1 = 0\} \ (m \geq 0)$,
- $L_+^+(m) := \{(\zeta_0, \zeta_1) \mid \zeta_0 > \zeta_1, m \zeta_0 + (m-1)\zeta_1 = 0\} \ (m \geq 0)$,
- $L_+^+(\infty) := \{(\zeta_0, \zeta_1) \mid \zeta_0 > \zeta_1, m \zeta_0 + \zeta_1 = 0\}$,
- $L_+^+(m) := \{(\zeta_0, \zeta_1) \mid \zeta_0 > \zeta_1, m \zeta_0 + (m+1)\zeta_1 = 0\} \ (m \geq 0)$.

By Theorem 3.5, the set of non-generic stability parameters is the union of the origin $(0,0)$ and the walls above.

**Remark 3.6.** Recall that the derived category of finite dimensional representations of Kronecker quiver is equivalent to the derived category of coherent sheaves on $\mathbb{P}^1$.

Under the equivalence $D^b(A\text{-Mod}) \simeq D^b(\text{Coh}(Y))$, $C_+^-(m)$ and $C_-^-(m)$ correspond to $z_*\mathcal{O}_{\mathbb{P}^1}(-m+1)[1]$ and $z_*\mathcal{O}_{\mathbb{P}^1}(-m-1)[1]$, where $z: \mathbb{P}^1 \to Y$ is the zero section.

Under the equivalence $D^b(A\text{-Mod}) \simeq D^b(\text{Coh}(Y^+))$, $C_+^+(m)$ and $C_-^+(m)$ correspond to $z^*_+\mathcal{O}_{\mathbb{P}^1}(-m-1)[1]$ and $z^*_+\mathcal{O}_{\mathbb{P}^1}(m-1)$, where $z^+: \mathbb{P}^1 \to Y^+$ is the zero section.

Moreover the stable objects on the wall $L_-^-(\infty)$ correspond to skyscraper sheaves on $Y$, ones on the wall $L_+^+(\infty)$ correspond to skyscraper sheaves on $Y^+$.  

![Diagram of stable A-modules](image)
3.4 Wall-crossing formula

Let $L$ be one of the walls $L_\pm^+(m)$, $L_\pm^-(m)$, $L_{\pm}^+(m)$ or $L_{\pm}^-(m)$. Take a parameter $\zeta^0 = (\zeta_0, \zeta_1)$ on $L$ and set $\zeta^k = (\zeta_0 \pm \varepsilon, \zeta_1 \pm \varepsilon)$ for sufficiently small $0 < \varepsilon \ll 1$ such that they are in chambers adjacent to the wall $L$. Note that, by the classification in 3.3 we have the unique $\zeta^0$-stable $A$-module $C$ such that $\zeta \cdot \dim C = 0$. We fix these notations throughout this subsection.

**Lemma 3.7.** $\text{Ext}_{A}^1(C,C) = 0$.

**Proof.** By Remark 3.6, it is enough to check $\text{Ext}_{A}^1(z_*L,z_*L) = 0$ for any line bundle $L$ on $\mathbb{P}^1$, where $z : \mathbb{P}^1 \to O(-1) \oplus O(-1) = Y$ is the zero section. By the adjunction we have

$$\text{Ext}_{A}^1(z_*L,z_*L) = \text{Ext}_{\mathbb{P}^1}^1(\mathbb{L}z^*z_*L,L).$$

Since we have the Koszul resolution

$$0 \to \Lambda^2((\pi^*O(-1) \oplus \pi^*O(-1))^*) \oplus \pi^*L \to \Lambda^1((\pi^*O(-1) \oplus \pi^*O(-1))^*) \oplus \pi^*L \to \Lambda^0((\pi^*O(-1) \oplus \pi^*O(-1))^*) \oplus \pi^*L \to z_*L \to 0$$

of $z_*L$, the object $\mathbb{L}z^*z_*L$ is quasi-isomorphic to the complex

$$0 \to L(2) \to L(1) \oplus L(1) \to L \to 0.$$

We can compute $\text{Ext}^1_{\mathbb{P}^1}(\mathbb{L}z^*z_*L,L)$ by the spectral sequence of the double complex. The only non-zero in the $E_1$-terms are $\text{Hom}_{\mathbb{P}^1}(L,L) \simeq \mathbb{C}$ and $\text{Ext}^1_{\mathbb{P}^1}(L(2),L) \simeq \mathbb{C}$. Thus the spectral sequence degenerates and we have $\text{Hom}_{A}(z_*L,z_*L) = \text{Ext}^1_{A}(z_*L,z_*L) = \mathbb{C}$ and $\text{Ext}_{A}^1(z_*L,z_*L) = \text{Ext}_{\mathbb{P}^1}^1(\mathbb{L}z^*z_*L,L) = 0$. \hfill $\square$

**Proposition 3.8.**

(1) Let $\tilde{V}'$ be a $\zeta^+$-stable $A$-module. Then we have an exact sequence

$$0 \to \tilde{V} \to \tilde{V}' \to C^\otimes k \to 0,$$

where $\tilde{V}$ is a $\zeta^0$-stable $A$-module. The integer $k$ and the isomorphism class of $\tilde{V}$ are determined uniquely. Moreover, the composition of the maps

$$\mathbb{C}^k \xrightarrow{\sim} \text{Hom}_{A}(C,C^\otimes k) \xrightarrow{\iota_{C}} \text{Ext}^1_{A}(C,\tilde{V})$$

is injective, where we regard $\tilde{V}'$ as an element in $\text{Ext}^1_{A}(C^\otimes k,\tilde{V})$.

(2) Let $\tilde{V}''$ be a $\zeta^-$-stable $A$-module. Then we have an exact sequence

$$0 \to C^\otimes k \to \tilde{V}'' \to \tilde{V} \to 0,$$

where $\tilde{V}$ is a $\zeta^0$-stable $A$-module. The integer $l$ and the isomorphism class of $\tilde{V}$ are determined uniquely. Moreover, the composition of the maps

$$\mathbb{C}^l \xrightarrow{\sim} \text{Hom}_{A}(C^\otimes k,C) \xrightarrow{\phi_{\tilde{V}''}} \text{Ext}^1_{A}(\tilde{V},C)$$

is injective, where we regard $\tilde{V}''$ as an element in $\text{Ext}^1_{A}(\tilde{V},C^\otimes k)$.  

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Proof. We set
\[ \zeta_\infty = -\zeta_0 \cdot \dim V'_0 - \zeta_1 \cdot \dim V'_1. \]
Note that \( \tilde{V}' \) is \( \theta\zeta \)-semistable and \( \theta\zeta(\tilde{V}') = 0 \). Let
\[ \tilde{V}' = \tilde{V}^0 \supset \cdots \supset \tilde{V}^L \supset \tilde{V}^{L+1} = 0 \]
be a Jordan-Hölder filtration of \( \tilde{V}' \) with respect to the \( \theta\zeta \)-stability. As we have mentioned before, there is an integer \( 0 \leq l \leq L \) such that \( \dim(\tilde{V}^L/\tilde{V}^{L+1}) = 1 \) and \( \dim(\tilde{V}'^l/\tilde{V}'^{l+1}) = 0 \) for any \( l' \neq l \). Then for \( l' \neq l \) we have
\[
\zeta_0^+ \cdot \dim(\tilde{V}'^l/\tilde{V}'^{l+1})_0 + \zeta_1^+ \cdot \dim(\tilde{V}'^l/\tilde{V}'^{l+1})_1 \\
= \varepsilon \cdot (\dim(\tilde{V}'^l/\tilde{V}'^{l+1})_0 + \dim(\tilde{V}'^l/\tilde{V}'^{l+1})_1) > 0.
\]
From the \( \zeta^+ \)-stability of \( \tilde{V}' \), we have \( l = L \).
Due to the classification in \( \S 3.3 \) and Lemma 3.7, \( \tilde{V}'/\tilde{V}_L \) is isomorphic to the direct sum \( C^{\oplus k} \) for some \( k \). The uniqueness follows from the uniqueness of factors of a Jordan-Hölder filtration.

The composition of the maps is injective, since otherwise \( \tilde{V}' \) has \( C \) as a direct summand and can not be \( \zeta^+ \)-stable.

We can verify the claim of (2) similarly.

Let \( \text{Gr}(k, V) \) be the Grassmannian variety of \( k \)-dimensional vector subspaces of a vector space \( V \).

**Proposition 3.9.**  
(1) Let \( \tilde{V} \) be a \( \zeta^\circ \)-stable \( \tilde{A} \)-module. For an element \( x \in \text{Gr}(k, \dim \text{Ext}^1_A(C, \tilde{V})) \),
\[
\text{let } \tilde{V}' \text{ denote the framed } \tilde{A} \text{-module given by the universal extension} \\
0 \to \tilde{V} \to \tilde{V}' \to C^{\oplus k} \to 0 \\
corresponding to } x. \text{ Then } \tilde{V}' \text{ is } \zeta^+ \text{-stable.}
\]
(2) Let \( \tilde{V} \) be a \( \zeta^\circ \)-stable \( \tilde{A} \)-module. For an element \( y \in \text{Gr}(l, \dim \text{Ext}^1_A(\tilde{V}, C)) \),
\[
\text{let } \tilde{V}'' \text{ denote the } \tilde{A} \text{-module given by the universal extension} \\
0 \to C^{\oplus k} \to \tilde{V}' \to \tilde{V} \to 0 \\
corresponding to } y. \text{ Then } \tilde{V}'' \text{ is } \zeta^- \text{-stable.}
\]

**Proof.** We set \( \zeta_\infty \) and \( \zeta^+_\infty \) so that
\[ \tilde{\zeta} \cdot \dim(\tilde{V}) = \tilde{\zeta} \cdot \dim(\tilde{V}') = \tilde{\zeta}^+ \cdot \dim(\tilde{V}') = 0. \]
Let \( \tilde{S} \) be a nonzero proper subobject of \( \tilde{V}' \) in \( \tilde{A}\text{-Mod} \). We should check \( \tilde{\zeta}^+ \cdot \dim(\tilde{S}) < 0 \).
Suppose \( \tilde{S} \cap \tilde{V} = \emptyset \), then \( \tilde{S} \) is mapped into \( C^{\oplus k} \) injectively. Since \( \tilde{V}' \) does not have \( C \) as its direct summand, \( \tilde{S} \) is not isomorphic to a direct sum of \( C \). So
we have $\zeta \cdot \dim(\tilde{S}) < 0$ because of the $\zeta$-stability of $C$. Since $\varepsilon$ is sufficiently small we have $\zeta^+ \cdot \dim(\tilde{S}) < 0$ as well.

Suppose $\emptyset \neq \tilde{S} \cap \tilde{V} \subsetneq \tilde{V}$. Since $V$ is $\zeta$-stable and $C^{\oplus k}$ is $\zeta$-semistable we have

$$\tilde{\zeta} \cdot \dim(\tilde{S} \cap \tilde{V}) < 0, \quad \tilde{\zeta} \cdot \dim(\operatorname{im}(\tilde{S} \to C^{\oplus k})) \leq 0.$$ 

So we have

$$\tilde{\zeta} \cdot \dim(\tilde{S}) = \tilde{\zeta} \cdot \dim(\tilde{S} \cap \tilde{V}) + \tilde{\zeta} \cdot \dim(\operatorname{im}(\tilde{S} \to C^{\oplus k})) < 0.$$ 

Because $\varepsilon$ is sufficiently small we have $\tilde{\zeta} \cdot \dim(\tilde{S}) < 0$ as well.

Suppose $\tilde{S} \cap \tilde{V} = \tilde{V}$. Since $C^{\oplus k}$ is $\zeta$-semistable and $C^{\oplus k}$ is $\zeta$-semistable we have $\tilde{\zeta} \cdot \dim(\operatorname{coker}(\tilde{S} \to C^{\oplus k})) \geq 0$. Because $\operatorname{coker}(\tilde{S} \to C^{\oplus k}) \neq \emptyset$ and $\operatorname{coker}(\tilde{S} \to C^{\oplus k})_{\infty} = 0$ we have $\tilde{\zeta}^+ \cdot \dim(\operatorname{coker}(\tilde{S} \to C^{\oplus k})) > 0$. Hence we get

$$\tilde{\zeta}^+ \cdot \dim(\tilde{S}) = \tilde{\zeta}^+ \cdot \dim(\tilde{V}') - \tilde{\zeta}^+ \cdot \dim(\operatorname{coker}(\tilde{S} \to C^{\oplus k})) < 0.$$ 

We can verify the claim of (2) similarly.

\begin{proposition}
For a $\zeta$-stable $\tilde{A}$-module $\tilde{V}$ we have

$$\operatorname{ext}^1_{\tilde{A}}(C, \tilde{V}) = \operatorname{ext}^1_{\tilde{A}}(\tilde{V}, C) = \dim C_0.$$ 

\end{proposition}

\begin{proof}
Let $S_\infty$ be the simple $\tilde{A}$-module corresponding to the extended vertex $\infty$ and $V$ be the kernel of the natural map $\tilde{V} \to S_\infty$.

First, applying the functor $\operatorname{Hom}_{\tilde{A}}(C, -)$ to the short exact sequence we have the following exact sequence:

$$\operatorname{Hom}_{\tilde{A}}(C, S_\infty) \leftarrow \operatorname{Ext}^1_{\tilde{A}}(C, V) \leftarrow \operatorname{Ext}^1_{\tilde{A}}(C, \tilde{V}) \leftarrow \operatorname{Ext}^1_{\tilde{A}}(C, S_\infty).$$

Clearly $\operatorname{Hom}_{\tilde{A}}(C, S_\infty) = 0$. We can also see that any extension

$$0 \to S_\infty \to * \to C \to 0$$

of $\tilde{A}$-modules is always trivial, that is, $\operatorname{Ext}^1_{\tilde{A}}(C, S_\infty) = 0$. Hence we have

$$\operatorname{Ext}^1_{\tilde{A}}(C, \tilde{V}) = \operatorname{Ext}^1_{\tilde{A}}(C, V) = \operatorname{Ext}^1_{\tilde{A}}(C, V).$$

On the other hand, applying the functor $\operatorname{Hom}_{\tilde{A}}(-, C)$ to the short exact sequence we have the following exact sequence:

$$\operatorname{Hom}_{\tilde{A}}(\tilde{V}, C) \leftarrow \operatorname{Hom}_{\tilde{A}}(V, C) \leftarrow \operatorname{Ext}^1_{\tilde{A}}(\tilde{V}, C) \leftarrow \operatorname{Ext}^1_{\tilde{A}}(V, C).$$

Since both $\tilde{V}$ and $C$ are $\zeta$-stable and they are not isomorphic, we have $\operatorname{Hom}_{\tilde{A}}(\tilde{V}, C) = 0$.

Note that giving an extension

$$0 \to C \to * \to S_\infty \to 0$$

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is equivalent to giving a map $C \to C_0$. Hence we have $\text{ext}_{A}^1(S_{\infty}, C) = \dim C_0$.

Given an extension

$$0 \to C \to V' \to V \to 0$$

of $A$-modules, by taking any lift of $\iota: V_{\infty} \cong C \to V_0$ with respect to the surjection $V'_0 \to V_0$, we get an extension

$$0 \to C \to \tilde{V}' \to \tilde{V} \to 0$$

of framed $A$-modules. This means the map

$$\text{Ext}_{A}^1(V, C) \to \text{Ext}_{A}^1(\tilde{V}, C) \cong \text{Ext}_{A}^1(V, C)$$

is surjective.

Now we have

$$\text{ext}_{A}^1(C, \tilde{V}) - \text{ext}_{A}^1(\tilde{V}, C)$$

$$= \text{ext}_{A}^1(C, V) - (\text{hom}_{A}(V, C) + \dim C_0 + \text{ext}_{A}^1(V, C))$$

$$= \dim C_0 + (\text{ext}_{A}^1(C, V) - \text{ext}_{A}^1(\tilde{V}, C) + \text{ext}_{A}^1(C, V))$$

$$= \dim C_0 + \chi_{A}(C, V) - \text{hom}_{A}(C, V).$$

Since $f$ is relative dimension 1, the Euler form on $\text{Coh}(Y)$ vanishes by Hirzebruch-Riemann-Roch theorem, and so does the Euler form on $A$-mod. Both $\tilde{V}$ and $C$ are $\zeta^{\circ}$-stable and they are not isomorphic, so we have $\text{Hom}_{A}(C, V) = 0$. Since the induced map $\text{Hom}_{A}(C, V) \to \text{Hom}_{A}(C, \tilde{V})$ is injective, we have $\text{Hom}_{A}(C, V) = \text{Hom}_{A}(C, \tilde{V}) = 0$.

Finally the claim follows.

We define stratifications on $\mathbb{M}^{s}_{\zeta^{\circ}}(v)$ and $\mathbb{M}^{s}_{\zeta^{\ast}}(v)$ as follows:

- let $\mathbb{M}^{s}_{\zeta^{\circ}}(v)_N$ denote the subset of $\mathbb{M}^{s}_{\zeta^{\circ}}(v)$ consisting of $\tilde{A}$-modules $\tilde{V}$ such that $\dim \text{Ext}^1_{\tilde{Q}}(C, \tilde{V}) = N$, and

- let $\mathbb{M}^{s}_{\zeta^{\ast}}(v')_{N,k}$ denote the subset of $\mathbb{M}^{s}_{\zeta^{\ast}}(v')$ consisting of $\tilde{A}$-modules $\tilde{V}'$ such that there exists a $\zeta^{\circ}$-stable $\tilde{A}$-module $\tilde{V} \in \mathbb{M}^{s}_{\zeta}(v' - k \cdot \dim(C))_N$ and an exact sequence

$$0 \to \tilde{V} \to \tilde{V}' \to C^{\oplus k} \to 0.$$

- let $\mathbb{M}^{s}_{\zeta^{\ast}}(v')^N$ denote the subset of $\mathbb{M}^{s}_{\zeta^{\ast}}(v')$ consisting of $\tilde{A}$-modules $\tilde{V}$ such that $\dim \text{Ext}^1_{\tilde{Q}}(\tilde{V}, C) = N$, and

- let $\mathbb{M}^{s}_{\zeta^{\ast}}(v')_{N,k}$ denote the subset of $\mathbb{M}^{s}_{\zeta^{\ast}}(v')$ consisting of $\tilde{A}$-modules $\tilde{V}'$ such that there exists a $\zeta^{\circ}$-stable $\tilde{A}$-module $\tilde{V} \in \mathbb{M}^{s}_{\zeta}(v' - k \cdot \dim(C))^N$ and an exact sequence

$$0 \to C^{\oplus k} \to \tilde{V}' \to \tilde{V} \to 0.$$

**Lemma 3.11.** The subsets $\mathbb{M}^{s}_{\zeta^{\circ}}(v)_N \subset \mathbb{M}^{s}_{\zeta^{\ast}}(v)$ and $\mathbb{M}^{s}_{\zeta^{\ast}}(v)_{N,k} \subset \mathbb{M}^{s}_{\zeta^{\ast}}(v)$ have natural subscheme structures.
Proof. Note that, for a morphism \( f : E \to F \) of vector bundles on a scheme \( X \) and an integer \( n \), the subset \( \{ x \in X \mid \text{rank}(f) = n \} \) has a natural subscheme structure given by the minor determinants.

We denote the subalgebra \( \mathbb{C}Q_0 = \bigoplus_{i\in Q_0} \mathbb{C}e_i \) of \( A \) by \( S \). For an \( S \)-module \( T \) we define an \( A \)-bimodule \( F_T \) by

\[
F_T = A \otimes_S T \otimes_S A.
\]

For \( i, i' \in Q_0 \) let \( T_{i,i'} \) denote the 1-dimensional \( S \)-module given by

\[
e_j \cdot 1 = \delta_{j,i}, \quad 1 \cdot e_j = \delta_{j,i'},
\]

and we set

\[
F_i := F_{T_{i,i}}, \quad F_{i,i'} := F_{T_{i,i'}}.
\]

Note that \( F_{i,i'} \) has the following natural basis:

\[
\{ p \otimes 1 \otimes q \mid p \in Ae_i, \ q \in e_i'A \}.
\]

For a quiver with superpotential \( A \), the Koszul complex of \( A \) is the following complex of \( A \)-bimodules:

\[
0 \to \bigoplus_{i \in Q_0} F_i \xrightarrow{d_3} \bigoplus_{a \in Q_1} F_{\text{out}(a),\text{in}(a)} \xrightarrow{d_2} \bigoplus_{b \in Q_1} F_{\text{in}(b),\text{out}(b)} \xrightarrow{d_1} \bigoplus_{s \in Q_0} F_i \xrightarrow{m} A \to 0.
\]

Here the maps \( m, d_1, d_3 \) are given by

\[
m(p \otimes 1 \otimes q) = pq \quad (p \in A e_i, \ q \in e_i'A),
\]

\[
d_1(p \otimes 1 \otimes q) = (pb \otimes 1 \otimes q) - (p \otimes 1 \otimes bq) \quad (p \in A e_{\text{in}(b)}, \ q \in e_{\text{out}(b)}A),
\]

\[
d_3(p \otimes 1 \otimes q) = \left( \sum_{a : \text{in}(a) = i} pa \otimes 1 \otimes q \right) - \left( \sum_{a : \text{out}(a) = i} p \otimes 1 \otimes aq \right) \quad (p \in A e_i, \ q \in e_i'A).
\]

The map \( d_2 \) is defined as follows: Let \( e \) be a cycle in the quiver \( Q \). We define the map \( \partial_{e:a,b} : F_{\text{out}(a),\text{in}(a)} \to F_{\text{in}(b),\text{out}(b)} \) by

\[
\partial_{e:a,b}(p \otimes 1 \otimes q) = \sum_{r \in \text{in}(a), s \in e_{\text{in}(b)}A e_{\text{out}(a)}, arbs = e} ps \otimes 1 \otimes rq.
\]

Then \( d_2 = \partial_e \) is defined as the linear combination of \( \partial_e \)'s.

Since \( A \) is graded 3-dimensional Calabi-Yau algebra, the Koszul complex is exact ([Boc08, Theorem 4.3]).

We also consider the following Koszul type complex \( \tilde{A} \)-bimodules:

\[
0 \to \bigoplus_{i \in Q_0} \tilde{F}_i \xrightarrow{d_3} \bigoplus_{a \in Q_1} \tilde{F}_{\text{out}(a),\text{in}(a)} \xrightarrow{d_2} \bigoplus_{b \in Q_1} \tilde{F}_{\text{in}(b),\text{out}(b)} \xrightarrow{d_1} \bigoplus_{s \in Q_0} \tilde{F}_i \xrightarrow{\tilde{m}} \tilde{A} \to 0,
\]

where \( \tilde{F}_i, \tilde{F}_{i,i'}, \tilde{d}_* \) and \( \tilde{m} \) are defined in the same way. This is also exact. The exactness at the last three terms is equivalent to the definition of generators and relations of the algebra \( \tilde{A} \). The exactness at the first two terms is derived from that of the exactness of the Koszul complex of \( A \).
Let \( \tilde{V} = \bigoplus_{i \in \mathbb{Q}_0} \tilde{V}_i \) be the universal bundle on \( \mathcal{M}_{\zeta^+}(v) \). The Koszul complex of \( \tilde{A} \)-bimodules induces the following complexes of the vector bundles on \( \mathcal{M}_{\zeta}(v) \):

\[
\bigoplus_{a \in \mathbb{Q}_1} \text{Hom}(C_{\text{out}(a)}, \tilde{V}_{\text{in}(a)}) \xrightarrow{d_2} \bigoplus_{b \in \mathbb{Q}_1} \text{Hom}(C_{\text{in}(b)}, \tilde{V}_{\text{out}(b)}) \xrightarrow{d_1} \bigoplus_{i \in \mathbb{Q}_0} \text{Hom}(C_i, \tilde{V}_i) \to 0.
\]

If we restrict this complex to some closed point \( \tilde{V} \) of \( \mathcal{M}_{\zeta}(v) \), then the right and left cohomologies give \( \text{Hom}(C, \tilde{V}) \) and \( \text{Ext}^1(C, \tilde{V}) \) respectively.

Note that for \( \tilde{V} \in \mathcal{M}_{\zeta^+(v)} \) we have \( \text{hom}(C, \tilde{V}) = 0 \). This means that the morphism \( d_1 \) between vector bundles are surjective, and hence \( \ker(d_1) \) is a vector bundle. The subset \( \mathcal{M}_{\zeta^+(v)}(N) \) consists of closed points \( \tilde{V} \) such that \( \text{rank}(d_2(\tilde{V})) = \text{rank}(\ker(d_1)) - N \) and so has a natural subscheme structure.

Let \( \mathcal{M}_{\zeta^+(v')}^N \) denote the subscheme of \( \mathcal{M}_{\zeta^+(v')} \) consisting of closed points \( \tilde{V}' \) such that \( \text{rank}(d_2(\tilde{V}')) = \dim C \cdot v' - k \). We have the canonical morphism \( \mathcal{M}_{\zeta^+(v')}^N \to \mathcal{M}_{\zeta^+(v')}^N(v = v' - k \cdot \dim C) \) such that a closed point \( \tilde{V}' \) is mapped to the closed point \( \tilde{V} \in \mathcal{M}_{\zeta^+(v')}^N \) appeared in the exact sequence

\[
0 \to \tilde{V} \to \tilde{V}' \to C^{\oplus k} \to 0.
\]

The subset \( \mathcal{M}_{\zeta^+(v')}^N \) coincides the inverse image of \( \mathcal{M}_{\zeta^+(v')}^N \) with respect to the above morphism and so has a natural subscheme structure.

Similarly, we can define subschemes \( \mathcal{M}_{\zeta^+(v')}^N \), \( \mathcal{M}_{\zeta^+(v')}^N \) and \( \mathcal{M}_{\zeta^+(v')}^N \) using the exact sequence

\[
\bigoplus_{a \in \mathbb{Q}_1} \text{Hom}(\tilde{V}_{\text{in}(a)}, C_{\text{out}(a)}) \xrightarrow{d_2} \bigoplus_{b \in \mathbb{Q}_1} \text{Hom}(\tilde{V}_{\text{out}(b)}, C_{\text{in}(b)}) \xrightarrow{d_1} \bigoplus_{i \in \mathbb{Q}_0} \text{Hom}(\tilde{V}_i, C_i) \to 0,
\]

whose cohomologies give \( \text{Hom}(\tilde{V}, C) \) and \( \text{Ext}^1(\tilde{V}, C) \).

By Proposition 3.8 and Proposition 3.9 the natural map

\[
\mathcal{M}_{\zeta^+(v')}^N \to \mathcal{M}_{\zeta^+(v')}^N(\mathcal{M}_{\zeta}(v))
\]

is a \( \text{Gr}(k, N) \)-fibration. So, we have

\[
\chi(\mathcal{M}_{\zeta^+(v')}^N, k) = \chi(\text{Gr}(k, N)) \cdot \chi(\mathcal{M}_{\zeta}(v))
\]

and

\[
\sum_{v'} \chi(\mathcal{M}_{\zeta^+(v')}^N) \cdot q^v = \sum_{v', N, k} \chi(\mathcal{M}_{\zeta^+(v')}^N) \cdot q^v
\]

\[
= \sum_{v', N} \chi(\text{Gr}(k, N)) \cdot \chi(\mathcal{M}_{\zeta^+(v')}^N) \cdot q^{v + k \cdot \dim(C)}
\]

\[
= \sum_{v', N} \left( \sum_k \chi(\text{Gr}(k, N)) \cdot q^{k \cdot \dim(C)} \right) \chi(\mathcal{M}_{\zeta^+(v')}^N) \cdot q^v
\]

\[
= \sum_{v', N} \left( 1 + q^{\dim(C)} \right)^N \chi(\mathcal{M}_{\zeta^+(v')}^N) \cdot q^v.
\]
Similarly we have
\[
\sum_{v'} \chi(\mathfrak{M}_{\xi'}(v')) \cdot q^{v'} = \sum_{v,N} \left(1 + q^{\dim(C)}\right)^N \chi(\mathfrak{M}_{\xi'}(v))^N \cdot q^Y.
\]

By Proposition [3.10] we have \(\mathfrak{M}_{\xi'}(v)^N = \mathfrak{M}_{\xi'}(v)^{N+\dim C_0}\). Hence we have
\[
\sum_{v'} \chi(\mathfrak{M}_{\xi'}(v')) \cdot q^{v'} = \sum_{v,N} \left(1 + q^{\dim(C)}\right)^N \chi(\mathfrak{M}_{\xi'}(v))^N \cdot q^v
\]
\[
= \sum_{v,N} \left(1 + q^{\dim(C)}\right)^{N+\dim C_0} \chi(\mathfrak{M}_{\xi'}(v))^N \cdot q^v
\]
\[
= \left(1 + q^{\dim(C)}\right)^{\dim C_0} \cdot \sum_{v'} \chi(\mathfrak{M}_{\xi'}(v')) \cdot q^{v'}.
\]

In summary, we have the following wall-crossing formula:

**Theorem 3.12.**

\[Z_{\xi'}(q) = \left(1 + q^{\dim(C)}\right)^{\dim C_0} \cdot Z_{\xi'}(q).\]

### 3.5 DT, PT and NCDT

Let \(I_n(Y, d)\) denote the moduli space of ideal sheaves \(I_Z\) of one dimensional subschemes \(Z \subset Y\) whose Hilbert polynomials are given by

\[\chi(O_Z \otimes \mathcal{L}^\otimes K) = n + K \cdot \int_{[C]} d \cdot c_1(\mathcal{L}) = n + dK.\]

We define the **Donaldson-Thomas invariants** \(I_{n,d}\) from \(I_n(Y, d)\) using Behrend’s function as is \(\text{[3.12] (The08, Beh09)}\), and their generating function by

\[Z_{\text{DT}}(Y; q, t) := \sum_{n,d} I_{n,d} \cdot q^n t^d.\]

Let \(P_n(Y, d)\) denote the moduli space of stable pairs \((F, s)\) such that the Hilbert polynomials of \(F\)’s are given by the same equation as above. We define the **Pandharipande-Thomas invariants** \(P_{n,d}\) from \(P_n(Y, d)\) using Behrend’s function \(\text{([PT09])}\), and their generating function by

\[Z_{\text{PT}}(Y; q, t) := \sum_{n,d} P_{n,d} \cdot q^n t^d.\]

We define the Donaldson-Thomas invariants and the Pandharipande-Thomas invariants of \(Y^+\) using \([C^+] \in H_2(Y^+)\) instead of \([C] \in H_2(Y)\). Note that the natural isomorphism \(H_2(Y) \to H_2(Y^+)\) maps \([C]\) to \([-C^+]\).

For \(F \in D_+^b(Y)\) we have

\[\chi(F \otimes \mathcal{L}^\otimes K) = \chi(F) + K(\chi(F) - \chi(F \otimes \mathcal{L}^{-1})).\]

and so \(n = v_0, d = v_0 - v_1\). Put \(q = q_0 q_1\) and \(t = q_1^{-1}\), then we have \(q^n t^d = q_0^n q_1^d\). We set \(\zeta^\pm = (-1 \pm \varepsilon, 1)\) for sufficiently small \(\varepsilon > 0\). The results in \([2]\) are summarized as follows:
Proposition 3.13.

\[
Z_{\text{DT}}(Y; q_0 q_1, q_1^{-1}) = Z_{\zeta^-}(q), 
Z_{\text{DT}}(Y^+; q_0 q_1, q_1^{-1}) = Z_{-\zeta^+}(q), 
\]

\[
Z_{\text{PT}}(Y; q_0 q_1, q_1^{-1}) = Z_{\zeta^+}(q), 
Z_{\text{PT}}(Y^+; q_0 q_1, q_1^{-1}) = Z_{-\zeta^-}(q). 
\]

Remark 3.14. Here we denote, with a slight abuse of the notations, by \(Z_{\pm \zeta^\pm}(q)\) the generating functions of the virtual counting of \(\mathcal{M}_{\pm \zeta^\pm}(v)\) for sufficiently small \(\varepsilon > 0\) for each \(v\). We can not take \(\varepsilon > 0\) uniformly.

We set \(\zeta(\pm) = (\pm 1, \pm 1)\). Note that \(\mathcal{M}_{\zeta(\pm)}(v)\) is empty unless \(v = 0\) and so \(Z_{\zeta(\pm)}(q) = 1\). The invariants \(D_{\zeta(\pm)}(Y)\) are the non-commutative Donaldson-Thomas invariants defined in [Sze08]. We denote their generating function \(Z_{\zeta(\pm)}(q)\) by \(Z_{\text{NCDT}}(q)\).

Applying the wall-crossing formula in Theorem 3.12, we obtain the following relations between generating functions:

Theorem 3.15.

\[
Z_{\text{NCDT}}(q) = \left( \prod_{m \geq 1} (1 + q_0^m (-q_1)^m) \right) - Z_{\text{DT}}(Y; q_0 q_1, q_1^{-1}), 
\]

(3.2)

\[
Z_{\text{NCDT}}(q) = \left( \prod_{m \geq 1} (1 + q_0^m (-q_1)^m) \right) - Z_{\text{DT}}(Y^+; q_0 q_1, q_1^{-1}), 
\]

(3.3)

\[
Z_{\text{PT}}(Y; q_0 q_1, q_1^{-1}) = \prod_{m \geq 1} (1 + q_0^m (-q_1)^m), 
\]

(3.4)

\[
Z_{\text{PT}}(Y^+; q_0 q_1, q_1^{-1}) = \prod_{m \geq 1} (1 + q_0^m (-q_1)^m). 
\]

(3.5)

Remark 3.16. (1) The formula (3.2) was shown by a combinatorial method in [You].

(2) The generating function of Donaldson-Thomas invariants is described in terms of the topological vertex ([MNOP06]). The topological vertex for the conifold is computed in [BB07]:

\[
Z_{\text{DT}}(Y; q, t) = \left( \prod_{m \geq 1} (1 - (-q)^m t)^{-m} \right)^2 \left( \prod_{m \geq 1} (1 - (-q)^m t)^m \right). 
\]

(3.6)

(3) The \(DT-PT\) correspondence conjecture ([PT09]) asserts that

\[
Z_{\text{DT}}(Y; q, t) = Z_{\text{PT}}(Y; q, t) \cdot \left( \prod_{m \geq 1} (1 - (-q)^m t)^{-m} \right)^{v(Y)}. 
\]

The conjecture for the conifold follows from formula (3.4) and (3.6), although Theorem 3.13 does not cover the wall \(L^-(-\infty)\). The wall-crossing for the wall \(L^-(-\infty)\) requires Joyce’s general theory (see [Tod10] and [ST]).
in the final section, we provide an alternative description of the moduli spaces

\[ \text{Replacement of tilting bundles and stabilities} \]

\[ \text{Lemma 4.1.} \]

\[ \text{(see Figure 6).} \]

\[ \text{Let } \mathcal{M}_{\text{triv}} \text{ and } [CJ09]. \]

Since the contribution of the wall \( L^-(m) \) coincides with that of \( L^+(m) \)

\[ \text{after the change } (q_0 q_1, q_1^{-1}) \rightarrow (q_0 q_1, q_1) \text{ of variables, we get the flop invariance of } DT \text{ and } PT \text{ invariants:} \]

\[ Z_{\text{DT}}(Y; q_0 q_1, q_1^{-1}) = Z_{\text{DT}}(Y^+; q_0 q_1, q_1), \]

\[ Z_{\text{PT}}(Y; q_0 q_1, q_1^{-1}) = Z_{\text{PT}}(Y^+; q_0 q_1, q_1). \]

\[ \text{4 Replacement of tilting bundles and stabilities} \]

In the final section, we provide an alternative description of the moduli spaces \( \mathcal{M}_{\zeta}(v) \) for generic stability parameter \( \zeta \) in the case of the conifold. As by-products, we can see that the torus fixed point set \( \mathcal{M}_{\zeta}(v)^T \) is isolated and parameterized by the "pyramid partitions" which appeared in [Sze08, You] and [CIM09].

\[ \text{4.1 Characterization of stable objects} \]

Let \( \zeta_{\text{triv}} \) and \( \zeta_{\text{cyclic}} \) be stability parameters such that

\[ \zeta_{\text{triv}, 0}, \zeta_{\text{triv}, 1} > 0, \quad \zeta_{\text{cyclic}, 0}, \zeta_{\text{cyclic}, 1} < 0. \]

For \( m \geq 1 \), let \( \zeta_{m, \pm} = (\zeta_{0, m, \pm}, \zeta_{1, m, \pm}) \) be stability parameters such that

\[ \zeta_{0, m, +} < \zeta_{1, m, +}, \quad m \zeta_{0, m, +} + (m - 1) \zeta_{1, m, +} < 0, \quad (m + 1) \zeta_{0, m, +} + m \zeta_{1, m, +} > 0, \]

\[ \zeta_{0, m, -} < \zeta_{1, m, -}, \quad (m - 1) \zeta_{0, m, -} + m \zeta_{1, m, -} > 0, \quad m \zeta_{0, m, -} + (m + 1) \zeta_{1, m, -} < 0 \]

(see Figure 6).

\[ \text{Lemma 4.1.} \quad \text{(1) A perverse coherent system } (F, s) \in \mathcal{Perc}(Y/X) \text{ is } \zeta_{m, +}-\text{stable if and only if the following three conditions are satisfied:} \]

\[ \text{Hom}_{\mathcal{Perc}(Y/X)}((F, s), (z_*, \mathcal{O}_X (m - 1), 0, 0)) = 0, \quad (4.1) \]

\[ \text{Hom}_{\mathcal{Perc}(Y/X)}((z_*, \mathcal{O}_X (m), 0, 0), (F, s)) = 0, \quad (4.2) \]

\[ \text{Hom}_{\mathcal{Perc}(Y/X)}((\mathcal{O}_X, 0, 0), (F, s)) = 0 \quad (\forall x \in Y). \quad (4.3) \]

\[ \text{(2) A perverse coherent system } (F, s) \in \mathcal{Perc}(Y/X) \text{ is } \zeta_{m, -}-\text{stable if and only if the following three conditions are satisfied:} \]

\[ \text{Hom}_{\mathcal{Perc}(Y/X)}((z_*, \mathcal{O}_X (-m)[1], 0, 0), (F, s)) = 0, \quad (4.4) \]

\[ \text{Hom}_{\mathcal{Perc}(Y/X)}((F, s), (z_*, \mathcal{O}_X (-m - 1)[1], 0, 0)) = 0, \quad (4.5) \]

\[ \text{Hom}_{\mathcal{Perc}(Y/X)}((F, s), (\mathcal{O}_X, 0, 0)) = 0 \quad (\forall x \in Y). \quad (4.6) \]
ζ_0 + ζ_1 = 0
L_-(m + 1) L_-(m)
ζ_1 = 0
ζ_cyclic
ζ m, +
ζ triv
ζ m, −
ζ triv

Figure 6: ζ m, ±, ζ triv and ζ cyclic

Proof. Let (F, s) ∈ Per_c(Y/X) be a perverse coherent system satisfying the conditions (4.1)-(4.3). Assume that (F, s) is not ζ m, ±-stable. Let

(F, s) = F_0 ⊃ F_1 ⊃ ... ⊃ F_L ⊃ 0

be a filtration of (F, s) by θ_ζ m, +-stable subquotients, which is given by combining the Harder-Narasimhan filtration of (F, s) and Jordan-Hölder filtrations of its factors. Here θ_ζ m, + is chosen so that θ_ζ m, +(F, s) = 0. Hence we have θ_ζ m, +(F_0/F_1) < 0, θ_ζ m, +(F_k) > 0 and either (F_0/F_1)_∞ = 0 or (F_k)_∞ = 0. First suppose (F_0/F_1)_∞ = 0. By the classification in §3.3, F_0/F_1 is isomorphic to (z_*O_P^1(m_0 - 1), 0, 0) for some m_0 ≤ m. Next suppose (F_k)_∞ = 0. Then F_k is isomorphic to one of the following objects:

- (z_*O_P^1(m' - 1), 0, 0) (m' > m),
- (O_x, 0, 0) (x ∈ Y),
- (z_*O_P^1(-m' - 1)[1], 0, 0) (m' ≥ 0).

In both cases, the existence of nonzero homomorphisms

\[ \text{Hom}_{Per_c(Y/X)}(z_*O_P^1(m_0 - 1), z_*O_P^1(m - 1)) \neq 0 \quad (m ≥ m'), \]
\[ \text{Hom}_{Per_c(Y/X)}(z_*O_P^1(m), z_*O_P^1(m_0 - 1)) \neq 0 \quad (m < m'), \]
\[ \text{Hom}_{Per_c(Y/X)}(z_*O_P^1(m), z_*O_P^1(-m_0 - 1)[1]) \neq 0 \quad (1 ≤ m, 0 ≤ m'). \]

contradicts the conditions (4.1)-(4.3). Hence (F, s) is ζ m, ±-stable. The opposite direction is trivial. We can show the claim (2) in the same way.

4.2 New framed quivers

Recall that we put L = π^*O_P^1(1). For an integer m, we set L_m := L^⊗m. Let P_m (resp. P_m) be the full subcategory of D^b(Coh(Y)) consisting of objects F such that F ⊗ L_m[1] ∈ Per(Y/X) (resp. F ⊗ L_m+1 ∈ Per(Y/X)). Note that
$\mathcal{L}_m[-1] \oplus \mathcal{L}_{m+1}[-1]$ (resp. $\mathcal{L}_{-m-1} \oplus \mathcal{L}_{-m}$) is a projective generator in $\mathcal{P}_m^+$ (resp. $\mathcal{P}_m^-$) and gives the equivalence

$$\Phi_m^+: \mathcal{P}_m^+ \to A-\text{Mod}.$$ 

Note that $A$ is the algebra defined in §3.1, which is independent of $m$. We set $\mathcal{P}_m^0 := \mathcal{P}_m^+ \cap D^b(\text{Coh}(Y))$, which is equivalent to $A$-mod.

Let $A_m^+$ be the algebra defined by the following quiver with relations: the

![Quiver Q^+](image)

Figure 7: quiver $Q_m^+$

quiver $Q_m^+$ is given as in Figure 7 and the following relations are added to the usual ones:

$$b_1q_1 = 0, \quad b_1q_{i+1} = b_2q_i \quad (i = 1, \ldots, m - 1), \quad b_2q_m = 0. \tag{4.7}$$

Similarly, we define the algebra $A_m^-$ as follows: the quiver $Q_m^-$ is given as in

![Quiver Q^-](image)

Figure 8: quiver $Q_m^-$

and the following relations are added to the usual ones:

$$a_1r_{i+1} = a_2r_i \quad (i = 1, \ldots, m - 1). \tag{4.8}$$

Let $S_\infty$ and $P_\infty$ be the simple and indecomposable projective $A_m^\pm$-modules corresponding to the extended vertex $\infty$. Let $P$ denote the kernel of the canonical map $P_\infty \to S_\infty$.

**Proposition 4.2.**

$$\Phi_m^+(\mathcal{O}_Y) = P.$$

**Proof.** We will prove the claim for $\Phi_m^+$. Let $B_1, B_2 \in H^0(Y, \mathcal{L})$ be basis elements in

$$\mathbb{C}^2 \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \hookrightarrow H^0(Y, \mathcal{L})$$

and...
We consider the map

\[ \sum (B_1^{(i,i)} + B_2^{(i,i+1)}) : (L_m \oplus m) \to (L_m \oplus m+1), \]

where \( B_1^{(i,i)} = \pi^i \circ B \circ \eta^i \) and \( \pi^i \) and \( \eta^i \) are the canonical projection to inclusion of the \( i \)-th factor of the direct sum. This map is injective and the cokernel is isomorphic to the structure sheaf \( O_Y \). Applying \( \Phi_m \) we get the following map:

\[ \sum (b_1^{(i,i)} + b_2^{(i,i+1)}) : P_m \to P_{m+1}, \]

where \( P_0 \) and \( P_1 \) are the indecomposable projective \( A \)-modules and \( b^{(i,j)} \) is defined as above. We can verify that this map is injective and the cokernel is isomorphic to \( P \). Hence the claim follows.

Let \( \tilde{P}^\pm \) denote the category of pairs \((F, W, s)\), where \( F \in \tilde{P}^\pm_m \) and \( s : W \otimes O_Y \to F \). Let \( \tilde{c} \tilde{P}^\pm \) denote the full subcategory of pairs \((F, W, s)\) such that \( F \in \tilde{c}P^\pm_m \) and such that \( W \) is finite dimensional.

**Proposition 4.3.**

\[ \tilde{P}^\pm_m \simeq A^\pm_m \text{-Mod}, \quad \tilde{c} \tilde{P}^\pm_m \simeq A^\pm_m \text{-mod}. \]

**Proof.** First, giving a pair \((F, W, s)\) \( \in \tilde{P}^\pm_m \) is equivalent to giving a linear map \( W \to \text{Hom}(O_Y, F) \). Note that \( \text{Hom}(O_Y, F) \simeq \text{Hom}_{\tilde{A}^\pm_m}(P, V) \simeq \text{Ext}^1_{\tilde{A}^\pm_m}(S_\infty, V) \) where \( V := \Phi^\pm_m(P) \). The claim follows, since giving a linear map \( W \to \text{Ext}^1_{\tilde{A}^\pm_m}(S_\infty, V) \) is equivalent to giving an \( A^\pm_m \)-module, .

We can define \( \zeta \)-(semi)stability for finite dimensional \( A^\pm_m \)-modules as in Definition 2.1. In order to make it clear in what category we work, we use the notation \( (\zeta, \cdot \tilde{P}^\pm_m \text{-}(semi)stability) \). From now on, the \( \zeta \)-(semi)stability for modules of the original quiver \( \tilde{Q} \) is written \( (\zeta, \text{Per} \tilde{c}(Y/X))\text{-}(semi)stability \). We can construct the moduli spaces \( \mathcal{M}^\pm_{\tilde{Q}^\pm_m}(v_0, v_1) \) of \( (\zeta, \cdot \tilde{P}^\pm_m) \)-semistable \( A^\pm_m \)-modules \( V \) with \( \dim V = (v_0, v_1, 1) \).

### 4.3 Potentials

Let \( \tilde{Q}^+_m \) be the quiver in Figure 9 and \( \omega^+_m \) be the following potential:

\[ a_1b_1a_2b_2-a_1b_2a_3b_1+p_1b_1q_1+p_2(b_1q_2-b_2q_1)+\cdots+p_m(b_1q_m-b_2q_{m-1})-p_{m+1}b_2q_m. \]

Let \( \tilde{A}^+_m \) be the algebra defined by the quiver with the potential \( (\tilde{Q}^+_m, \omega^+_m) \) (see [DWZ08]).

**Lemma 4.4.**

\[ p_jb_{\varepsilon_j}a_2b_{\varepsilon_2}\ldots b_{\varepsilon_k}q_{\varepsilon_k} = 0 \quad (\varepsilon_l = 1, 2). \]
Proof. Assume $\varepsilon_L = 1$ and $j \geq j'$. Then we have

\[
p_j b_1 a_2 b_3 \ldots b_1 q_j' = p_j b_1 a_2 b_3 \ldots b_2 q_{j'-1}
= p_j b_2 a_2 b_3 \ldots b_{L-2} q_{j'-1}
= p_j-1 b_1 a_2 b_3 \ldots b_{L-2} q_{j'-1}
= p_j-1 b_1 a_2 b_3 \ldots b_{1} q_{j'-1}
= \ldots
= p_j-j'+1 b_1 a_2 b_3 \ldots b_1 q_1
= 0.
\]

We can show the claims for other cases in the same way.

Fix an element $(v_0, v_1) \in (\mathbb{Z}_{\geq 0})^2$. For $\zeta = (\zeta_0, \zeta_1)$, we put $\theta_\zeta = (\zeta_0, \zeta_1, -\zeta_0 v_0 - \zeta_1 v_1)$. Let $\mathfrak{M}^{A_m}_{\text{cyclic}}(v_0, v_1)$ denote the moduli space of $\theta_\zeta$-stable $A_m$-modules with dimension vectors $(v_0, v_1, 1)$.

**Lemma 4.5.** Let $V$ be an $A_m$-module with dimension vector $(v_0, v_1, 1)$. If $V$ is $\theta_\zeta$-cyclic-stable, then $p_j = 0$ for $j = 1, \ldots, m+1$.

**Proof.** By the $\theta_\zeta$-cyclic-stability, $V_0$ coincides with the union of the images of $b_1 a_2 b_3 \ldots b_{L-1} q_j$'s. Thus the claim follows from Lemma 4.4.

**Proposition 4.6.**

$\mathfrak{M}^{A_m}_{\text{cyclic}}(v_0, v_1) \cong \mathfrak{M}^{\tilde{A}_m}_{\text{cyclic}}(v_0, v_1)$.

In particular, $\mathfrak{M}^{A_m}_{\text{cyclic}}(v_0, v_1)$ has a symmetric obstruction theory.

**Proof.** The claim follows directly from Lemma 4.5 and the definition of the potential.

Similarly, we define $\tilde{A}_m = (\tilde{Q}_m, \omega_m)$ by the quiver $\tilde{Q}_m$ in Figure 10 and the following potential:

\[
\omega_m = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1 + s_1(a_1 r_2 - a_2 r_1) + \cdots + s_{m-1}(b_1 r_m - b_2 r_{m-1}).
\]
We can prove the following claim in the same way:

**Proposition 4.7.**

\[ \mathcal{M}_{\text{cyclic}}^{A_m^-(v_0, v_1)} \simeq \mathcal{M}_{\text{cyclic}}^{\tilde{A}_m^-}(v_0, v_1). \]

In particular, \( \mathcal{M}_{\text{cyclic}}^{A_m^-(v_0, v_1)} \) has a symmetric obstruction theory.

### 4.4 Moduli spaces

**Lemma 4.8.** Let \((F, s) \in \overline{\text{Per}}_c(Y/X)\) be a \((\zeta^{m, \pm}, \overline{\text{Per}}_c(Y/X))\)-stable object, then \(F \in \mathcal{P}_{r_m^c}^c\).

**Proof.** Take a sufficiently large \(c \in \mathbb{R}\) such that \(\zeta_0 + c, \zeta_1 + c > 0\) and set \(\zeta_{\text{triv}} = (\zeta_0 + c, \zeta_1 + c)\) (see Figure 10). Let

\[ (F, s) = \tilde{F}^0 \supset \tilde{F}^1 \supset \cdots \supset \tilde{F}^L \supset 0 \]

be a filtration of a \((\zeta^{m, \pm}, \overline{\text{Per}}_c(Y/X))\)-stable object \((F, s) \in \overline{\text{Per}}_c(Y/X)\) by \((\theta_{\zeta_{\text{triv}}}, \overline{\text{Per}}_c(Y/X))\)-stable subquotients, which is given by combining the Harder-Narasimhan filtration of \((F, s)\) and Jordan-Hölder filtrations of its factors. Since \(\theta_{\zeta_{\text{triv}}}((\tilde{F}^0/\tilde{F}^1)_\infty) = \mathbb{C}\). Moreover, we have \((\tilde{F}^0/\tilde{F}^1)_0 = (\tilde{F}^0/\tilde{F}^1)_1 = 0\) from the \((\theta_{\zeta_{\text{triv}}}, \overline{\text{Per}}_c(Y/X))\)-stability of \(\tilde{F}^0/\tilde{F}^1\). Note that we have

\[ \theta_{\zeta_{\text{triv}}}(v_0, v_1, 0) \leq \theta_{\zeta_{\text{triv}}}(v'_0, v'_1, 0) \iff \theta_{\zeta_{m,+}}(v_0, v_1, 0) \leq \theta_{\zeta_{m,+}}(v'_0, v'_1, 0) \]

for any \(v_0, v_1, v'_0\) and \(v'_1\). Since \(\theta_{\zeta_{\text{triv}}}((\tilde{F}^j/\tilde{F}^{j+1}) \leq \theta_{\zeta_{\text{triv}}}((\tilde{F}^L))\), we also have

\[ \theta_{\zeta_{m,+}}((\tilde{F}^j/\tilde{F}^{j+1}) \leq \theta_{\zeta_{m,+}}((\tilde{F}^L)) < 0, \]

where the last inequality is the consequence of \(\zeta^{m, \pm}\)-stability of \((F, s)\). By the classification in \(\mathbb{K}\) a \((\theta_{\zeta_{\text{triv}}}, \overline{\text{Per}}_c(Y/X))\)-stable object \(\tilde{F}\) with a 0-dimensional framing such that \(\theta_{\zeta_{m,+}}(\tilde{F}) < 0\) is isomorphic to \(z_{\mathcal{O}_{\mathbb{P}_1}(m') - 1}\) for some \(1 \leq
Thus we get a description of $F \in \text{Per}_c(Y/X)$ as successive extensions of $z_*\mathcal{O}_{\mathbb{P}}(m'-1)$'s ($1 \leq m' \leq m$). Since $z_*\mathcal{O}_{\mathbb{P}}(m'-1) \in \mathcal{P}_m^+$ for $m' \leq m$, we have $F \in \mathcal{P}_m^+$. We can show the claim for a $\zeta^{m,-}$-stable object in the same way.

Let $\zeta_{\text{cyclic}}$ be a stability parameter such that $(\zeta_{\text{cyclic}})_0, (\zeta_{\text{cyclic}})_1 < 0$.

**Lemma 4.9.** Let $(F, s) \in \overline{\text{Per}_c}(Y/X)$ be a $(\zeta^{m,\pm}, \text{Per}_c(Y/X))$-stable object, then $(F, s) \in \mathcal{P}_m^{\pm}$ is $(\zeta_{\text{cyclic}}, \mathcal{P}_m^{\pm})$-stable.

**Proof.** Note that the simple $\mathbb{A}_m^+$-modules $S_0$ and $S_1$ correspond to $\mathcal{O}_{\mathbb{P}}(m)[-1]$ and $\mathcal{O}_{\mathbb{P}}(m-1)$ in $\mathcal{P}_m$ respectively. For a $(\zeta^{m,\pm}, \text{Per}_c(Y/X))$-stable object $(F, s) \in \overline{\text{Per}_c}(Y/X)$, as in Lemma 4.11 it is enough to show that

$$\text{Hom}_{\mathcal{P}_m^{\pm}}((F, s), (\mathcal{O}_x[-1], 0, 0)) = 0,$$

$$\text{Hom}_{\mathcal{P}_m^{\pm}}((F, s), (z_*\mathcal{O}_{\mathbb{P}}(m)[-1], 0, 0)) = 0$$

for any $x \in Y$ and

$$\text{Hom}_{\mathcal{P}_m^{\pm}}((F, s), (z_*\mathcal{O}_{\mathbb{P}}(m-1), 0, 0)) = 0.$$

The first two equalities hold since $F, \mathcal{O}_x, \mathcal{O}_{\mathbb{P}}(m) \in \text{Per}_c(Y/X)$. For the third equation, we have

$$\text{Hom}_{\mathcal{P}_m^{\pm}}((F, s), (z_*\mathcal{O}_{\mathbb{P}}(m-1), 0, 0)) = 0$$

where the last equality follows from the $(\zeta^{m,\pm}, \overline{\text{Per}_c}(Y/X))$-stability of $(F, s) \in \overline{\text{Per}_c}(Y/X)$. We can show the claim for a $(\zeta^{m,\mp}, \text{Per}_c(Y/X))$-stable object in the same way.

The chamber structure in the space of stability parameters on $\mathcal{P}_m^{\pm}$ is the same as that on $\overline{\text{Per}_c}(Y/X)$. The stable objects $F \in \mathcal{P}_m^{\pm}$ on a wall with $F_{\infty} = 0$ is obtained from those for $\overline{\text{Per}_c}(Y/X)$ by applying $- \otimes \mathcal{L}_{m-1}$ (resp. $- \otimes \mathcal{L}_{m-1}$). Note that the parameter $\zeta_{\text{cyclic}}$ for $\mathcal{P}_m^{\pm}$ is in the chamber between the walls $L_{\mathcal{P}}(m)$ and $L_{\mathcal{P}}(m+1)$ with stable objects $z_*\mathcal{O}_{\mathbb{P}}$ and $z_*\mathcal{O}_{\mathbb{P}}(-1)$ on them.

**Lemma 4.10.** Let $(F, s) \in \mathcal{P}_m^{\pm}$ be a $(\zeta^{m,\mp}, \mathcal{P}_m^{\pm})$-stable object, then $F = 0$.

**Proof.** Take the Harder-Narasimhan filtration of a $(\zeta^{m,\mp}, \mathcal{P}_m^{\pm})$-stable object $(F, s) \in \mathcal{P}_m^{\pm}$ and Jordan-Hölder filtrations of its factors with respect to the $\theta_{\zeta_{\text{cyclic}}}$-stability. Then, as in the proof of Lemma 4.9, we get a description of $F \in \mathcal{P}_m^{\pm}$ as successive extensions of $z_*\mathcal{O}_{\mathbb{P}}(-m')$'s ($m' \geq 1$), $\mathcal{O}_x[-1]$'s ($x \in Y$) and $z_*\mathcal{O}_{\mathbb{P}}[m'][-1]$'s ($m' \geq m$). So we have $H^0(Y, F) = 0$ and hence $s = 0$. Then the stability requires that $F = 0$. We can show the claim for a $(\zeta^{m,\mp}, \mathcal{P}_m^{\pm})$-stable object in $\mathcal{P}_m$ similarly.
Lemma 4.11. Let \((F, s) \in c^\mp \P^\pm_m\) be a \((\varsigma, c^\pm \P^\pm_m)\)-stable object, then \(F \in \text{Per}_c(Y/X)\).

Proof. Take the Harder-Narasimhan filtration of \((F, s) \in c^\mp \P^\pm_m\) and Jordan-Hölder filtrations of its factors with respect to the \(\theta\)-stability. Then, by Lemma 4.10 we get a description of \(F \in c^\mp \P^\pm_m\) as successive extensions of \(z_*O_{P^1}(m'-1)\)’s \((1 \leq m' \leq m)\) and hence we have \(F \in \text{Per}_c(Y/X)\). We can show the claim for an object in \(c^\mp \P^\pm_m\) similarly.

Lemma 4.12. Let \((F, s) \in c^\mp \P^\pm_m\) be a \((\varsigma, c^\pm \P^\pm_m)\)-stable object, then \((F, s) \in \text{Per}_c(Y/X)\) is \(\varsigma\)-stable.

Proof. For a \((\varsigma, c^\pm \P^\pm_m)\)-stable object \((F, s) \in c^\mp \P^\pm_m\), as in the proof of Lemma 4.9 we have

\[
\text{Hom}_{\text{Per}_c(Y/X)}((F, s), (z_*O_{P^1}(m-1), 0, 0)) = 0.
\]

As we claimed in the proof of Lemma 4.11 \(F\) is described as a successive extensions of \(z_*O_{P^1}(m'-1)\)’s \((1 \leq m' \leq m)\). So we have

\[
\text{Hom}_{\text{Per}_c(Y/X)}((E, 0, 0), (F, s)) = 0.
\]

for \(E = z_*O_{P^1}(m)\) or \(E = O_x\ (x \in Y)\). By Lemma 4.11 \((F, s) \in \text{Per}_c(Y/X)\) is \((\varsigma, \text{Per}_c(Y/X))\)-stable. We can show the claim for an object in \(c^\mp \P^\pm_m\) similarly.

Theorem 4.13.

\[
\mathfrak{M}_{\varsigma, +}(v_0, v_1) \simeq \mathfrak{M}_{\text{cyclic}}^{A^+_m}(m-1)v_0 + (-m)v_1, mv_0 + (-m-1)v_1),
\]

\[
\mathfrak{M}_{\varsigma, -}(v_0, v_1) \simeq \mathfrak{M}_{\text{cyclic}}^{A^-_m}(mv_0 + (-m+1)v_1, (m+1)v_0 + (-m)v_1).
\]

4.5 Fixed points

The readers may refer [Sze08] and [You] for the definition of “pyramid partitions with length \(m^\prime\)” (see Figure 11), and [CJ09] for the definition of “finite type pyramid partitions with length \(m^\prime\)” (see Figure 12).

Proposition 4.14. (1) The set of \(T^\prime\)-fixed closed points

\[
\mathfrak{M}_{\varsigma, +}(v_0, v_1)^{T^\prime} \simeq \mathfrak{M}_{\text{cyclic}}^{A^+_m}(m-1)v_0 + (-m)v_1, mv_0 + (-m-1)v_1)^{T^\prime}
\]

is isolated and parameterized by finite type pyramid partitions with length \(m\) and with \((m-1)v_0 + (-m)v_1\) white stones and \(mv_0 + (-m-1)v_1\) black stones.

(2) The set of \(T^\prime\)-fixed closed points

\[
\mathfrak{M}_{\varsigma, -}(v_0, v_1)^{T^\prime} \simeq \mathfrak{M}_{\text{cyclic}}^{A^-_m}(mv_0 + (-m+1)v_1, (m+1)v_0 + (-m)v_1)^{T^\prime}
\]

is isolated and parameterized by pyramid partitions with length \(m\) and with \(mv_0 + (-m+1)v_1\) white stones and \((m+1)v_0 + (-m)v_1\) black stones.
Figure 11: the empty room configurations for pyramid partitions with length 3 and with length 4

Figure 12: the empty room configurations for finite type pyramid partitions with length 3 and with length 4
Proof. Recall that $P$ denotes the kernel of the canonical map $P_\infty \to S_\infty$. The $A$-module $P$ has the canonical $T$-weight decomposition such that each weight space is 1-dimensional and parameterized by the empty room configuration for finite type pyramid partitions with length $m$ (resp. for pyramids partition with length $m$).

We put $c := b_2a_2b_1a_1 + a_2b_2a_1(= xy = zw) \in Z(A) \subset A$. Here $Z(A)$ is the center of $A$, which is isomorphic to the coordinate ring $\mathbb{C}[x, y, z, w]/(xy - zw)$ of the conifold. Let $v_B \in P$ be a $T$-weight vector corresponding to a stone $B$ in the empty room configuration. Then $c \cdot v_B$ (unless $= 0$ in the case (2)) is the $T'$-weight vector corresponding to the stone just behind $B$. Any $T'$-weight space of $P$ is described as $\mathbb{C}[c] \cdot v_B$ for some stone $B$ in the empty room configuration.

Let $(F, s) \in \mathcal{P}_m^+(\text{resp. } \mathcal{P}_m^-)$ be a cyclic-stable object, then $F$ is a quotient of $P$ as an $A$-module. Any $T'$-weight space of $F$ is described as $I \cdot v_B$ for some stone $b$ and for some ideal $I \subset \mathbb{C}[c]$. Assume that $(F, s)$ is $T'$-invariant. Then $P/F$ must be supported at the singularity $0 \in \text{Spec}Z(A)$ and so $I$ must be a monomial ideal. Thus the claims follow.

Let $N^m_{\text{pyramid}}(n_0, n_1)$ (resp. $N^m_{\text{fin-pyramid}}(n_0, n_1)$) denote the number of pyramid partitions (resp. finite type pyramid partitions) with length $m$ and with $n_0$ white stones and $n_1$ black stones. We encode them into the generating functions

$$
Z^m_{\text{pyramid}}(p) := \sum_{(n_0, n_1)\in(\mathbb{Z}_{\geq 0})^2} N^m_{\text{pyramid}}(n_0, n_1) \cdot p_0^{n_0} p_1^{n_1},
$$

$$
Z^m_{\text{fin-pyramid}}(p) := \sum_{(n_0, n_1)\in(\mathbb{Z}_{\geq 0})^2} N^m_{\text{fin-pyramid}}(n_0, n_1) \cdot p_0^{n_0} p_1^{n_1},
$$

where $p_0$ and $p_1$ are formal variables.

**Theorem 4.15.**

$$
Z^m_{\text{fin-pyramid}}(p) = \prod_{m' = 1}^{m} \left(1 + p_0^{m-m'} p_1^{m-m'+1}\right)^{m'},
$$

$$
Z^m_{\text{pyramid}}(p) = \left(\prod_{m' \geq 1}^{\infty} (1 - p_0^{m'} p_1^{m'})^{m'}\right)^2 \left(\prod_{m' \geq 1}^{\infty} \left(1 + p_0^{m+m'-1} q_1^{m+m'}\right)^{m'}\right) \left(\prod_{m' \geq m}^{\infty} (1 + q_0^{m+m'+1} q_1^{m+m'})^{m}\right).
$$

**Proof.** The claim follows from Theorem 3.12 Proposition 4.14 and the Behrend-Bryan’s formula (4.6).

4.6 Zariski tangent spaces at the fixed points

**Lemma 4.16.** Let $V$ be an $A$-module such that

$$(\Phi_m^\pm)^{-1}(V) \in \mathcal{P}_m^\pm \cap \text{Per}_c(Y/X).$$

Then we have

$$
\text{Ext}^2_{A_m^\pm}(S_\infty, V) = 0. \quad (4.9)
$$
Proof. Note that we have

\[ \text{Ext}^2_{A_m^\pm}(S_\infty, V) \simeq \text{Ext}^1_{A_m^\pm}(P, V) \simeq \text{Ext}^1_A(P, V) \simeq \text{Ext}^1_Y(\mathcal{O}_Y, (\Phi_m^\pm)^{-1}(V)). \]

The last one vanishes by the assumption \((\Phi_m^\pm)^{-1}(V) \in \text{Per}_c(Y/X)\).

**Proposition 4.17.** Let \(\tilde{V}\) be a \(\zeta\text{-cyclic}-\text{stable} A_m^\pm\)-module with \(V_\infty \simeq \mathbb{C}\) and \(V\) be the kernel of the natural map \(\tilde{V} \to S_\infty\). Then we have

\[ \dim \text{Ext}^1_{A_m^\pm}(\tilde{V}, \tilde{V}) = \dim \text{Ext}^1_A(V, V) - \dim \text{Hom}_A(V, V) + \dim \text{Hom}_Y(\mathcal{O}_Y, F). \]

Proof. Applying the functor \(\text{Hom}_{A_m^\pm}(\tilde{V}, -)\) for the short exact sequence

\[ 0 \to V \to \tilde{V} \to S_\infty \to 0 \tag{4.10} \]

we have the following exact sequence:

\[ \text{Hom}_{A_m^\pm}(\tilde{V}, V) \to \text{Hom}_{A_m^\pm}(\tilde{V}, S_\infty) \]

\[ \text{Ext}^1_{A_m^\pm}(\tilde{V}, V) \to \text{Ext}^1_{A_m^\pm}(\tilde{V}, \tilde{V}) \to \text{Ext}^1_{A_m^\pm}(\tilde{V}, S_\infty). \]

Note that

\[ \text{Hom}_{A_m^\pm}(\tilde{V}, S_\infty) \simeq (V_\infty)^* \simeq \mathbb{C} \]

and this is spanned by the image of \(\text{id}_{\tilde{V}} \in \text{Hom}_{A_m^\pm}(\tilde{V}, \tilde{V})\). Thus the map in the upper line is surjective. It is clear that any exact sequence

\[ 0 \to S_\infty \to \ast \to \tilde{V} \to 0 \]

of \(A_m^\pm\)-modules splits, that is, \(\text{Ext}^1_{A_m^\pm}(\tilde{V}, S_\infty) = 0\). Then we have

\[ \text{Ext}^1_{A_m^\pm}(\tilde{V}, V) \simeq \text{Ext}^1_{A_m^\pm}(\tilde{V}, \tilde{V}). \]

On the other hand, applying the functor \(\text{Hom}_{A_m^\pm}(\ast, V)\) to the short exact sequence \((4.10)\), we have the following exact sequence:

\[ \text{Hom}_{A_m^\pm}(\tilde{V}, V) \to \text{Hom}_{A_m^\pm}(V, V) \]

\[ \text{Ext}^1_{A_m^\pm}(S_\infty, V) \to \text{Ext}^1_{A_m^\pm}(\tilde{V}, V) \to \text{Ext}^1_{A_m^\pm}(V, V) \]

\[ \text{Ext}^2_{A_m^\pm}(S_\infty, V). \]

Since \(\tilde{V}\) is \(\zeta\text{-cyclic}-\text{stable}, \text{Hom}_{A_m^\pm}(\tilde{V}, V) = 0\). Moreover, \(V\) satisfies the assumption of Lemma 4.16 by Lemma 4.11. Hence the claim follows.

**Lemma 4.18.**

\[ \dim \text{Ext}^1_A(V, V) - \dim \text{Hom}_A(V, V) \equiv \dim V_0 + \dim V_1 \pmod{2}. \]
Proof. It is shown in the proof of [MR10, Theorem 7.1] that
\[
\dim \text{Ext}^1_A(V, V) - \dim \text{Hom}_A(V, V) \\
\equiv \sum_{i \in Q_0} (\dim V_i)^2 - \sum_{h \in Q_1} (\dim V_{\text{in}(h)}) (\dim V_{\text{out}(h)}) \quad (\text{mod } 2).
\]

\[
\text{Corollary 4.19. For a } T'\text{-invariant closed point } x \in \mathcal{M}_{\zeta^{m, \pm}}(v_0, v_1)^{T'}, \text{ the parity of the dimension of the Zariski tangent space to } \mathcal{M}_{\zeta^{m, \pm}}(v_0, v_1) \text{ at } x \text{ equals to the parity of } v_1.
\]

Proof. Under the isomorphism in Theorem 4.13, \( x \) can be regarded as a closed point in the moduli space of \( A^{\pm}_m \)-modules. Let \( \tilde{V} \) denote the \( A^{\pm}_m \)-module corresponding to \( x \). Then the Zariski tangent space is isomorphic to \( \text{Ext}^1_{A^{\pm}_m}(\tilde{V}, \tilde{V}) \). Thus the claim is a consequence of Proposition 4.17 and Lemma 4.18.

\[
\text{Proposition 4.20. For each } T'\text{-fixed closed point } x \in \mathcal{M}_\zeta(v)^{T'}, \text{ the Zariski tangent space to } \mathcal{M}_\zeta(v) \text{ at } x \text{ has no non-trivial } T'\text{-invariant subspace.}
\]

Proof. Recall that we have the following exact sequence:
\[
0 \to \text{Hom}_{A^{\pm}_m}(V, V) \to \text{Ext}^1_{A^{\pm}_m}(S_\infty, V) \to \text{Ext}^1_{A^{\pm}_m}(\tilde{V}, V) \to \text{Ext}^1_{A^{\pm}_m}(V, V) \to 0.
\]
So, it is enough to show that neither
\[
\text{coker} \left( \text{Hom}_{A^{\pm}_m}(V, V) \to \text{Ext}^1_{A^{\pm}_m}(S_\infty, V) \right)
\]
nor
\[
\text{Ext}^1_{A^{\pm}_m}(V, V)
\]
have non-trivial \( T' \)-invariant subspace.
First, we have the exact sequence
\[
0 \to \text{Hom}_A(V, V) \to \text{Hom}_A(P, V) \to \text{Hom}_A(\ker(P \to V), V) \to \text{Ext}^1_A(V, V)
\]
of which the first map coincides with the map in (4.11) under the isomorphisms
\[
\text{Hom}_{A^{\pm}_m}(\ker(P \to V), V) = \text{Hom}_A(\ker(P \to V), V), \text{Ext}^1_{A^{\pm}_m}(S_\infty, V) = \text{Hom}_{A^{\pm}_m}(P, V) = \text{Hom}_A(P, V).
\]
Here \( P \) denotes the kernel of the canonical map \( P_\infty \to S_\infty \) as before.

The claim follows from Lemma 4.21 and Lemma 4.22.

\[
\text{Lemma 4.21. We put } I := \ker(P \to V).
\]

(1) In the case \( \zeta = \zeta^{m, +} \), there is no non-trivial \( T' \)-invariant subspace in \( \ker(\text{Hom}_A(I, V) \to \text{Ext}^1_A(V, V)) \).

(2) In the case \( \zeta = \zeta^{m, -} \), there is no non-trivial \( T' \)-invariant subspace in \( \text{Hom}_A(I, V) \).

Proof. In the proof of Proposition 4.20 we see that
\[
\bullet \text{ every } T\text{-weight vector in } P, V \text{ and } I \text{ is associated to a stone in the empty room configuration, and}
\]

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every \( T' \)-weight space in \( P \) is described as \( \mathbb{C}[c] \cdot v_B \) for some stone \( B \).

Suppose we have a nonzero \( T' \)-invariant element \( \phi \in \text{Hom}_A(I, V) \). We may assume that there is a positive integer \( n \) such that

\[
\phi(v_B) = \begin{cases} v_{B'} & \text{if } \exists [v_{B'}] \in V \text{ such that } c^n \cdot v_{B'} = v_B \in P; \\ 0 & \text{otherwise} \end{cases}
\]

for \( v_B \in I \).

1. In the case \( \zeta = \zeta^{m,+} \), we will show that the image of \( \phi \) under the map \( \text{Hom}_A(I, V) \to \text{Ext}^1_A(V, V) \) gives a non-trivial extension. Since \( \phi \) is \( T' \)-invariant it gives self-extensions of \( T' \)-weight spaces. Take a \( T' \)-weight space \( W \) of \( P \) such that the restriction of \( \phi \) to \( W \cap I \) is nontrivial. Recall that \( W \) admits a \( \mathbb{C}[c] \)-module structure. It is enough to show that the self extension of \([W] \in V\) is not trivial as a \( \mathbb{C}[c] \)-module. We have

\[
W \simeq \mathbb{C}[c]/(c^m), \quad W \cap I \simeq (c^k)/(c^m)
\]

for some positive integers \( m \) and \( k \) such that \( m - n \geq k \geq n \). Then the \( T' \)-weight space in the extension associated to \( \phi \) is isomorphic to \( \mathbb{C}[c]/(c^{k+n}) \oplus \mathbb{C}[c]/(c^{k+n}) \), which is not isomorphic to \( \mathbb{C}[c]/(c^k) \oplus \mathbb{C}[c]/(c^k) \).

2. In the case \( \zeta = \zeta^{m,-} \), take a stone \( B \) from the ridge of the empty configuration such that \( v_B \notin I \). Let \( \alpha \) be the positive integer such that \( (b_2a_2)^{n-1} \cdot v_B \notin I \) and \( (b_2a_2)^n \cdot v_B \in I \) and put \( v_1 := (b_2a_2)^\alpha \cdot v_B \). Note that \( \phi(v_1) = 0 \).

Let \( \beta \) be the positive integer such that \( (b_1a_1)^{3-1} \cdot (b_2a_2)^{n-1} \cdot v_B \notin I \) and \( (b_1a_1)^3 \cdot (b_2a_2)^{n-1} \cdot v_B \in I \).

We put

\[
v_2 := c^n \cdot (b_1a_1)^{3-1} \cdot (b_2a_2)^{n-1} \cdot v_B = c^n \cdot (b_1a_1)^{3} \cdot v_1 \in I.
\]

Then we have

\[
0 \neq [(b_1a_1)^{3-1} \cdot (b_2a_2)^{n-1} \cdot v_B] = \phi(v_2) = c^n \cdot (b_1a_1)^{3} \cdot \phi(v_1) = 0.
\]

This is a contradiction.

\[\square\]

**Lemma 4.22.** There is no non-trivial \( T' \)-invariant subspace in \( \text{Ext}^1_{A^n}(V, V) \).

**Proof.** Using the Koszul complex (see the proof of Lemma 3.11), \( \text{Ext}^1_{A^n}(V, V) \) is given as the second cohomology of the following complex:

\[
\bigoplus_{i \in Q_0} \text{Hom}(V_i, V_i) \to \bigoplus_{b \in Q_1} \text{Hom}(V_{\text{out}(b)}, V_{\text{in}(b)}) \\
\to \bigoplus_{a \in Q_2} \text{Hom}(V_{\text{in}(a)}, V_{\text{out}(a)}) \to \bigoplus_{i \in Q_0} \text{Hom}(V_i, V_i).
\]

For \( P \in \mathfrak{M}(\nu)^{T'} \), the second term has no non-trivial \( T' \)-invariant subspace, and hence neither does \( \text{Ext}^1_{A^n}(V, V) \). \[\square\]
Theorem 4.23.

\[ Z_\zeta(q) = \sum_{v \in (\mathbb{Z}_{\geq 0})^2} (-1)^{v_1} |M_\zeta(v)_{\mathcal{C}}| \cdot q^{v_1} |M_\zeta(v)_{\mathcal{C}}| \cdot q^{v_2}. \]

Proof. As we mentioned in §3.2, this is a consequence of Behrend-Fantechi’s result [BF08, Theorem 3.4], Proposition 4.14, Corollary 4.19 and Proposition 4.20.

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