FOURTH MOMENT THEOREMS FOR MARKOV DIFFUSION GENERATORS

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Abstract. Inspired by the insightful article [4], we revisit the Nualart-Peccati-criterion [13] (now known as the Fourth Moment Theorem) from the point of view of spectral theory of general Markov diffusion generators. We are not only able to drastically simplify all of its previous proofs, but also to provide new settings of diffusive generators (Laguerre, Jacobi) where such a criterion holds. Convergence towards gamma and beta distributions under moment conditions is also discussed.

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1. Introduction

In 2005, Nualart and Peccati [13] discovered the surprising fact that any sequence of random variables \( \{X_n\}_{n \geq 1} \) in a Gaussian chaos of fixed order converges in distribution towards a standard Gaussian random variable if and only if \( \mathbb{E}(X_n^2) \to 1 \) and \( \mathbb{E}(X_n^4) \to 3 \). In fact, this result contains the two following important informations of a different nature:

(i) For all non-zero \( X \) in a Wiener chaos with order \( \geq 2 \), \( \mathbb{E}(X^4) > 3\mathbb{E}(X^2)^2 \),
(ii) \( \mathbb{E}(X^4) - 3\mathbb{E}(X^2)^2 \approx 0 \) if and only if \( X \overset{law}{\approx} \mathcal{N}(0, \mathbb{E}(X^2)) \).

This striking discovery, now known as the Fourth Moment Theorem, has been the starting point of a fruitful line of research of which we shall give a quick overview. The proof of the above result given in [13] used the Dambis-Dubins-Schwartz Theorem (see e.g. [16] ch. 5) and did not provide any estimates. In [12], this phenomenon was translated in terms of Malliavin operators, whereas in [8] these operators

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were combined with the bounds arising from Stein’s method, thus yielding both a short proof and precise estimates in the total variation distance (see also [10]). The main difficulty of the proof consists of establishing the powerful inequality

\[
\text{Var}\left(\Gamma(X)\right) \leq C \left(\mathbb{E}(X^4) - 3\mathbb{E}(X^2)^2\right),
\]

where \(\Gamma\) is the carré du champ operator associated with the generator of the Ornstein-Uhlenbeck semigroup, from which one can almost immediately deduce convergence in law towards a standard Gaussian distribution \(\mathcal{N}(0, 1)\) (for instance by Stein’s lemma). Other proofs of the Fourth Moment Theorem, not necessarily relying on the inequality (1), can be found in [6], [11] and [4]. We also mention [3] for extensions to the free probability setting, [15] for the multivariate setting, [7] for Gamma approximation, as well as [14] for the discrete setting. It is important to note that virtually all the proofs (with the remarkable exception of [4]), make crucial use of the product formula for multiple integrals and thus rely on a very rigid structure of the underlying probability space. As a matter of fact, this approach does not cover other important structures like Laguerre and Jacobi, which are investigated in the present article.

In the recent article [4], M. Ledoux gave another proof of the Fourth Moment Theorem in the general framework of diffusive Markov generators, adopting a purely spectral point of view. In particular, he completely avoids the use of product formulae. Unlike the Wiener space setting, it turns out that in this more general framework it is not sufficient anymore that a random variable \(X\) is only an eigenfunction of the diffusion generator for an equality of the type (1) to hold. By imposing additional assumptions, one is thus naturally led to a general definition of chaos. We emphasize that this starting definition of chaos is the cornerstone of the whole strategy. The definition of ”general chaos” given in [4] makes use of iterated gradients and, although including the Ornstein-Uhlenbeck case, prevents one to reach the Laguerre and Jacobi structures. In this article, we keep the insightful idea from [4] of encoding the Fourth Moment Theorem in purely spectral form, but at the very beginning generalize once more the notion of ”general chaos”. As we shall see, we say that \(X \in \text{Ker}(L + \lambda \mathbb{I})\) is a chaos eigenfunction with respect to a Markov generator \(L\) with spectrum \(\{\lambda_n\}_{n \geq 0}\), if and only if

\[
X^2 \in \bigoplus_{\alpha \leq \lambda_2} \text{Ker}(L + \alpha \mathbb{I}).
\]

This definition has the following main advantages.

- Our definition of chaos covers the definition in [4], i.e. being a chaos in the sense of Ledoux implies (2). Besides (2) seems easier to check in practice (see the remark after Theorem 3.2).
- We are able to deduce almost immediately that for any \(X\) which is a chaos eigenfunction in the above sense we have:

\[
\text{Var}(\Gamma(X)) \leq C \left(\mathbb{E}(X^4) - 3\mathbb{E}(X^2)^2\right).
\]

This drastically simplifies all the known proofs of the Fourth Moment Theorem which are all based on the above inequality.

- With our definition of chaos, we can extend the Fourth Moment Theorem to eigenfunctions of the Laguerre generator with any parameters. We mention that, due to links between Hermite polynomials and Laguerre polynomial with integer parameters, the chaos of the Laguerre structure with integer parameters can be plugged into the Wiener chaos. However, for non integer parameters this case is uncovered by the existing literature and provides a new framework where the Fourth Moment Theorems holds. As an illustrative example of the efficiency of our method, we can also mix the structures of Wiener and Laguerre to obtain theorems of the following nature (a detailed proof will be given in Section 4.4):

**Theorem 1.1.** Let \(\mathcal{X} = \{\pi_{1,\nu} - \nu \mid \nu > -1\} \cup \{\mathcal{N}(0, 1)\}\), be the set containing all centered gamma and Gaussian laws. Let \(\{X_i\}_{i \geq 1}\) be a sequence of independent random variables such that for all
$i \geq 1$, the law of $X_i$ belongs to $\mathcal{X}$. Now choose $d \geq 1$ and let

$$P_n(x) = \sum_{i_1 < i_2 < \cdots < i_d} a_n(i_1, \cdots, i_d)x_{i_1} \cdots x_{i_d}$$

be a sequence of multivariate homogeneous polynomials of degree $d$. Then the two following statements are equivalent:

(i) As $n$ tends to infinity, the sequence $\{P_n(X)\}_{n \geq 1}$ converges in distribution towards $N(0, 1)$.

(ii) As $n$ tends to infinity, it holds that $\mathbb{E}(P_n(X)^2) \to 1$ and $\mathbb{E}(P_n(X)^4) \to 3$.

We stress that due to the rather complicated structure of the variables (with no symmetry), such a result seems hardly reachable by using product formulae.

- We are also able to deal with beta approximation, i.e. to provide conditions on the moments of a sequence $X_n$ of chaotic eigenfunctions under which the latter converges in distribution towards a Beta distribution. As we shall see, the only restriction is that the parameters $\alpha$ and $\beta$ have to satisfy the inequality $\alpha + \beta \leq 1$. However, many important special cases like the Arcsine law are covered. Moreover our results provide new differential inequalities for the beta distributions (in the spirit of inequality (1)) which are of independent interest.

- Finally, we would like to mention that our strategy enables us to extend the Nualart-Peccati criterion to other couples of moments than 2 and 4. For instance, one can prove that if $\mathbb{E}(X_n^4) \to 3$ and $\mathbb{E}(X_n^6) \to 15$, then the Central Limit Theorem holds. Up to now, this has remained an open question. Nevertheless, this topic is under current research and will be published in a forthcoming article.

- Due to the simplicity of our proofs which are only of spectral nature, we believe that our strategy can also be applied in the free and discrete settings (see [3] for Wigner and [14] for Poisson chaos).

2. Main Results

2.1. General principle. Throughout the whole paper, we adopt the setting introduced in [4]. Thus, we fix a probability space $(E, \mathcal{F}, \mu)$ and a symmetric Markov generator $-L$ with state space $E$ and probability measure $\mu$ as its invariant measure. We assume that $-L$ has discrete spectrum $S = \{\lambda_k\}_{k \geq 0}$ and order its eigenvalues by magnitude, i.e. $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$. In other words, $-L$ is a self-adjoint, linear operator on $L^2(E, \mu)$ with the property that $-L(1) = 0$. By our assumption on the spectrum, $L$ is diagonalizable and we have that

$$L^2(E, \mu) = \bigoplus_{k=0}^{\infty} \text{Ker}(L + \lambda_k \text{Id}).$$

The orthogonal projection of $X \in L^2(E, \mu)$ on the eigenspace $\text{Ker}(L + \lambda_k \text{Id})$ will be denoted by $J_k(X)$. Furthermore, we define the associated bilinear carré du champ operator $\Gamma$ by

$$\Gamma(X, Y) = \frac{1}{2} (L(XY) - XLY - Y LX).$$

If both arguments coincide, we write $\Gamma(X)$ instead of $\Gamma(X, X)$. It follows from the definition of $\Gamma$, that for any $X, Y \in L^2(E, \mu)$ the integration by parts formula

$$\int_E \Gamma(X, Y) \, d\mu = -\int_E XLY \, d\mu = -\int_E Y LX \, d\mu$$

holds. For further details on this setting, we refer to [1] and the forthcoming book [2].

The following Theorem is the starting point of our investigations.
**Theorem 2.1.** In the above setting, let \( \{X_n\} \) be a sequence in \( L^2(E, \mu) \) such that each \( X_n \) lies in a common finite sum of eigenspaces of \( L \), i.e. there exists \( p \geq 0 \) such that

\[
X_n \in \bigoplus_{k=0}^{p} \operatorname{Ker}(L + \lambda_k \text{Id})
\]

for all \( n \in \mathbb{N} \). Then it holds for any \( \eta \geq \lambda_p \) that

\[
\int_E X_n (L + \eta \text{Id})^2 X_n \, d\mu \leq \eta \int_E X_n (L + \eta \text{Id}) X_n \, d\mu \leq c \int_E X_n (L + \eta \text{Id})^2 X_n \, d\mu,
\]

where \( 1/c \) is the minimum of the set \( \{\eta - \lambda_k \mid 0 \leq k \leq p\} \setminus \{0\} \). In particular, the following two conditions are equivalent.

(i) As \( n \) tends to infinity, it holds that \( \int_E X_n (L + \eta \text{Id})^2 X_n \, d\mu \to 0 \).

(ii) As \( n \) tends to infinity, it holds that \( \int_E X_n (L + \eta \text{Id}) X_n \, d\mu \to 0 \).

**Proof.** It holds that

\[
\int_E X_n (L + \eta \text{Id})^2 X_n \, d\mu = \int_E X_n L (L + \eta \text{Id}) X_n \, d\mu + \eta \int_E X_n (L + \eta \text{Id}) X_n \, d\mu
\]

\[
= \sum_{k=0}^{p} (-\lambda_k)(\eta - \lambda_k) \int_E J_k(X_n)^2 \, d\mu + \eta \int_E X_n (L + \eta \text{Id}) X_n \, d\mu
\]

\[
\leq \eta \int_E X_n (L + \eta \text{Id}) X_n \, d\mu
\]

and

\[
\int_E X_n (L + \eta \text{Id}) X_n \, d\mu = \sum_{k=0}^{p} (\eta - \lambda_k) \int_E J_k(X_n)^2 \, d\mu
\]

\[
\leq c \sum_{k=0}^{p} (\eta - \lambda_k)^2 \int_E J_k(X_n)^2 \, d\mu
\]

\[
= c \int_E X_n (L + \eta \text{Id})^2 X_n \, d\mu.
\]

\[\square\]

**Remark.** We will see later that being able to remove the square from the operator \( (L + \lambda_p \text{Id})^2 \) will prove itself very useful in both abstract and concrete frameworks. To see the latter, note that if for example \( E = \mathbb{R}^d \) and \( L \) is a diffusion generator, then \( (L + \lambda_p \text{Id})^2 \) is a differential operator of order four while the non-squared version only has order two. For example, we are able to give a drastically simplified proof of the classical Fourth Moment Theorem of [13].

2.2. **Chaos of a Markov generator.** As indicated in the introduction and further detailed below, for a sequence \( (X_n)_{n \geq 1} \) of eigenfunctions of \( L \), convergence in law towards many target measures is implied by \( L^2(E, \mu) \)-convergence of an expression of the form \( \Gamma(X_n) - P(X_n) \) towards zero, where \( P(x) \) is some polynomial with degree at most two. As by definition \( 2\Gamma(X_n) = (L + 2\lambda_p \text{Id}) X_n^2 \), Theorem 2.1 suggests the following definition of chaos.

**Definition 2.2.** An eigenfunction \( X \) of the generator \( -L \) with eigenvalue \( \lambda_p \) is called a chaos eigenfunction of order \( p \), if and only if

\[
X^2 \in \bigoplus_{k=0}^{2p} \operatorname{Ker}(L + \lambda_k \text{Id}).
\]
Remark. It is not clear if the $p$th chaos, i.e. the set of all chaos eigenfunctions of order $p$, is always a linear subspace of $\text{Ker} (L + \lambda_p \text{Id})$. Indeed, there is no reason for the product $XY$ of two chaos eigenfunctions of order $p$ to have an expansion of the form (5). However, in many important examples all eigenfunctions are chaotic. For example, this phenomenon occurs if the eigenfunctions can be represented in terms of multivariate polynomials, as is always the case in the three most important diffusion structures, namely Wiener, Laguerre and Jacobi (see section 4) and also in discrete settings like the Poisson space.

Remark (Connection with Ledoux’s definition and product fromulae). By writing (5) in the equivalent form

$$X^2 = \sum_{k=0}^{2p} J_k (X^2),$$

one recovers an abstract version of the product formula. As indicated in the introduction, explicit versions of this formula have been a crucial tool in virtually all classical proofs of the Fourth Moment Theorem but are not needed for the method presented here. Note also that the projection $J_k (X^2)$ on $\text{Ker} (L + \lambda_k \text{Id})$ is explicitly given by

$$J_k (X^2) = \left( \prod_{1 \leq i \leq 2p, i \neq k} (\lambda_i - \lambda_k)^{-1} (L + \lambda_i \text{Id}) \right) X^2.$$

In the same spirit, we could state (5) in yet another equivalent way, namely

$$\left( \prod_{i=0}^{2p} (L + \lambda_i \text{Id}) \right) X^2 = 0.$$

In this form, through the identity $2\Gamma(X) = (L + 2\lambda_p \text{Id}) X^2$, valid for any eigenfunction of $-L$ with eigenvalue $\lambda_p$, we see that Ledoux’s definition of chaos in [4] is a special case of ours. Indeed, his condition $Q_p(\Gamma)(X) = 0$ (see the original article for a definition of the polynomial $Q_p$ and the operator $Q_p(\Gamma)$) is equivalent to

$$\left( \prod_{i=0}^{p} (L + 2(\lambda_p - \lambda_i)) \right) X^2 = 0.$$

The restriction that only even eigenspaces are allowed in the expansion of $X^2$ and that $2(\lambda_p - \lambda_i)$ does not neccessarily have to lie in the spectrum of $-L$ are lifted by Definition 2.2. In particular, eigenfunctions of the Laguerre and Jacobi generator, which (except for trivial cases) do not satisfy Ledoux’s definition of chaos, are always chaotic in our sense (see Section 4).

3. Fourth Moment Theorems for Diffusion Generators

Still retaining the setting introduced in the previous section, we now and until the end of this article assume that the generator $L$ is diffusive, i.e. that for any test function $\phi \in C^\infty (\mathbb{R})$ and any $X \in L^2 (E, \mu)$ it holds that

$$L\phi(X) = \phi'(X)LX + \phi''(X)\Gamma(X).$$

Equivalently, $\Gamma$ is a derivation, in the sense that $\Gamma (\phi(X), X) = \phi'(X)\Gamma(X)$. We also need the technical assumption that the eigenspaces are hypercontractive (see [1] for sufficient conditions). We will give Fourth Moment Theorems for convergence of a sequence $\{X_n\}_{n \geq 1}$ of eigenfunctions of $L$ towards a Gaussian, Gamma or Beta distribution. When seen as invariant measures of another diffusive and symmetric Markov generator $\mathcal{L}$ on $\mathbb{R}$ with discrete spectrum, these three measures are the only ones that can arise if one assumes that the eigenfunctions of $\mathcal{L}$ are orthogonal polynomials (see [5]). Convergence in law towards each of these distributions is implied by $L^2 (E, \mu)$-convergence of $\Gamma(X_n) - P(X_n)$ towards

$$P(X_n) = \frac{1}{Z} \int_{\mathbb{R}} \exp \left( \frac{X_n(e) - \mathbb{E} X_n(e)}{\sqrt{\text{Var} X_n}} \right) \text{d}e,$$

where $Z$ is the normalizing constant.
towards zero. As a last step, we notice that due to the derivation property of a integrate and then, imposing some conditions on order orthogonal polynomial with respect to the target measure). Secondly, we square both sides of (6), \( a/f_i \) expressed as a linear combination of the

Let Lemma 3.1.

\[ \exp(-x^2/2) \]

zero is equivalent to the convergence of \( P(X_0) \) towards zero, where \( P \) is a polynomial of degree 0 (Gaussian distribution), 1 (Gamma distribution) or 2 (Beta distribution). See Table 1 for the respective polynomials and 2 for details on how to obtain them in the Gaussian and Gamma case (the Beta case can be obtained analogously).

Our method now proceeds along the following route. As a first step, we exploit the identity 2\( \Gamma(X_n) = (L + \lambda_p \text{Id}) X_n, \lambda_p \) being the eigenvalue of \( X_n \), to obtain an identity of the form

\[ \Gamma(X_n) - P(X_n) = (L + a\lambda_p \text{Id}) Q(X_n), \]

where \( a \) is positive real number and \( Q \) is a polynomial of degree two (it will turn out that \( Q \) is the second order orthogonal polynomial with respect to the target measure). Secondly, we square both sides of (6), integrate and then, imposing some conditions on \( a \), use Theorem 2.1 to remove the square of the integrand on the right hand side. This allows us to reason that the \( L^2(E, \mu) \) convergence of \( \Gamma(X_n) - P(X_n) \) towards zero is equivalent to the convergence of

\[ \int_X (X_n) (L + a\lambda_p \text{Id}) Q(X_n) \, d\mu \]

towards zero. As a last step, we notice that due to the derivation property of \( \Gamma \), the integral (7) can be expressed as a linear combination of the first four moments of \( X_n \), denoted by \( m_k(X_n) \) in the sequel (1 \( \leq k \leq 4 \)). This is the content of the next Lemma.

**Lemma 3.1.** Let \( X \) be an eigenfunction of \( L \) with eigenvalue \( \lambda_p, a \in \mathbb{R} \) and \( Q \) be a polynomial of degree two in one variable. Then it holds that

\[ \int_X Q(X) (L + a\lambda_p \text{Id}) Q(X) \, d\mu = \lambda_p \int_X \left( aQ^2(x) - \frac{(Q'(x))^3 X}{3Q''(X)} \right) \, d\mu. \]

**Remark.** Note that as the degree of \( Q \) is two, \( Q'(x)/Q''(x) \) is a polynomial and so the integral on the right hand side of (8) is always well defined.

**Proof.** Using the integration by parts formula and the derivation property of \( \Gamma \), we obtain that

\[ \int_X Q(X)LQ(X) \, d\mu = -\int_X \Gamma(Q(X)) \, d\mu = -\int_X (Q'(X))^2 \Gamma(X) \, d\mu = -\int_X \Gamma \left( \frac{(Q'(X))^3}{3Q''(X)}, X \right) \, d\mu = \int_X \frac{(Q'(X))^3}{3Q''(X)} L\, d\mu = -\lambda_p \int_X \frac{(Q'(X))^3}{3Q''(X)} X \, d\mu, \]

which yields the stated result.

Using this method, we now proceed to proof Fourth Moment Theorems for the caseses where the target measure is Gaussian, Gamma or Beta. Note that our approach in principle also works for other
target measures \( \mu \), as long as it admits moments of all orders (i.e. \( \int x^n \mu(dx) \) is finite for all \( n \geq 1 \)). Indeed, if this is the case, one can obtain the corresponding sequence \((P_n)_{n \geq 0}\) of orthogonal polynomials, calculate the (unique) constant \( a \) such that

\[
aP_n^2(x) - \frac{(P'_n(x))^2}{3P_n^2(x)} x = \sum_{i=1}^{2n} a_i P_n(x)
\]

and then use Lemma 3.1 to obtain the \( L \)-expression (and by integration by parts also the \( \Gamma \)-expression). For clarity of exposition we present our results in finite dimension, but everything remains valid in the infinite dimensional setting by a simple limit procedure. It is also remarkable to note that the constant appearing in the bounds of the \( \Gamma \)-expressions by the corresponding moments are independent of the dimension of the state space.

### 3.1. Gaussian approximation

In order to converge towards a standard Gaussian distribution, we have to control the quantity \( \Gamma(X) - \lambda_p \). The next Theorem gives a precise bound in terms of moments.

**Theorem 3.2.** Let \( X \) be a chaos eigenfunction of order \( p \) with respect to \( -L \) and assume that \( \lambda_{2p} \leq 2\lambda_p \). Then it holds that

\[
\int_E (\Gamma(X) - \lambda_p)^2 \, d\mu \leq \frac{\lambda_p^2}{3} (m_4(X) - 6m_2(X) + 3).
\]

**Proof.** First note that \( \Gamma(X) - \lambda_p = (L + 2\lambda_p \text{Id}) \frac{1}{2} H_2(X) \), where \( H_2(x) = x^2 - 1 \) is the second Hermite polynomial. Thus, by symmetry of the operator \( L + 2\lambda_p \text{Id} \), we get that

\[
\int_E (\Gamma(X) - \lambda_p)^2 \, d\mu = \frac{1}{4} \int_E ((L + 2\lambda_p \text{Id}) H_2(X))^2 \, d\mu = \frac{1}{4} \int_E H_2(X) (L + 2\lambda_p \text{Id})^2 H_2(X) \, d\mu.
\]

As \( X^2 \) and therefore \( H_2(X) \) has an expansion on the first \( 2p \) eigenspaces and we assume that \( \lambda_{2p} \leq 2\lambda_p \), we can apply Theorem 2.1 to obtain that

\[
\int_E (\Gamma(X) - \lambda_p)^2 \, d\mu \leq \frac{\lambda_p}{2} \int_E H_2(X) (L + 2\lambda_p \text{Id}) H_2(X) \, d\mu.
\]

Finally we apply Lemma 3.1 and obtain that

\[
\int_E H_2(X) L_{2\lambda_p} H_2(X) \, d\mu = \lambda_p \int_E \left( 2H_2(X)^2 - \frac{(H'_2(X))^2}{3H''_2(X)} X \right) \, d\mu = \lambda_p \int_E \left( 2 (X^2 - 1)^2 - \frac{4}{3} X^4 \right) \, d\mu = \frac{2\lambda_p}{3} (m_4(X) - 6m_2(X) + 3).
\]

\( \square \)

**Remark.** Note that the condition \( \lambda_{2p} \leq 2\lambda_p \) is always satisfied if \( X \) is chaotic in the sense of Ledoux. As a matter of fact, Theorem 3.2 shows that the assumed spectral condition (17) of Corollary 7 in [4] always holds in this case.

**Remark.** By exploiting the fact that \( X^2 - m_2(X) \) is centered, we have

\[
\int_E (X^2 - m_2(X)) (L + \lambda_1 \text{Id}) (L + 2\lambda_p \text{Id}) (X^2 - m_2(X)) \, d\mu \leq 0
\]

and proceed as in the proof of Theorem 2.1 to get

\[
4 \int_E (\Gamma(X) - \lambda_p m_2(X))^2 \, d\mu \leq (2\lambda_p - \lambda_1 \text{Id}) \int_E (X^2 - m_2(X)) (L + 2\lambda_p \text{Id}) (X^2 - m_2(X)) \, d\mu.
\]
This yields the better estimate
\[ \int_E \left( \Gamma(X) - \lambda_p m_2(X) \right)^2 \, d\mu \leq \left( \frac{\lambda_p^2}{3} - \frac{\lambda_1 \lambda_p}{6} \right) (m_4(X) - 3m_2^2(X)). \]

**Corollary 3.3** (Fourth Moment Theorem for Gaussian Approximation). Let \( \{X_n\}_{n \geq 1} \) be a sequence of chaos eigenfunctions of order \( p \) with respect to the operator \(-L\), bounded in \( L^2(E, \mu) \). Then, if \( \lambda_{2p} \leq 2\lambda_p \), the following two assertions are equivalent.

(i) As \( n \) tends to infinity, the sequence \( \{X_n\}_{n \geq 1} \) converges in distribution to a standard Gaussian distribution.

(ii) As \( n \) tends to infinity, it holds that \( m_4(X_n) - 6m_2(X_n) + 3 \to 0 \).

**Proof.** (i) \( \rightarrow \) (ii): By hypercontractivity, the fact that the sequence \( \{X_n\}_{n \geq 1} \) is bounded in \( L^2(E, \mu) \) implies that it is also bounded in \( L^r(E, \mu) \) for any \( r \geq 1 \). Consequently, we obtain that \( m_4(X_n) - 6m_2(X_n) + 3 \to 0 \) by the continuous mapping theorem.

(ii) \( \rightarrow \) (i): By the integration by parts formula for \( \Gamma \), we obtain that
\[
\int_E e^{i\xi X_n} \Gamma(X_n) \, d\mu = \frac{\lambda_p}{i\xi} \int_E X_n e^{i\xi X_n} \, d\mu.
\]

By (ii), we have \( \Gamma(X_n) \approx \lambda_p \), implying that \( \int_E e^{i\xi X_n} \, d\mu \approx \frac{1}{i\xi} \int E X_n e^{i\xi X_n} \, d\mu \). For any limit \( \rho \) of any subsequence of \( X_n \) we get
\[
\hat{\rho}(\xi) = -\frac{1}{\xi} \frac{d\hat{\rho}(\xi)}{d\xi},
\]
where \( \hat{\rho} \) denotes the Fourier transform of \( \rho \), and we conclude the proof by noting that the only solution of the above differential equation satisfying \( \hat{\rho}(0) = 1 \) is given by \( \hat{\rho}(\xi) = e^{-\frac{\xi^2}{2}} \).

**Remark.**

(i) If we additionally assume that \( m_2(X_n) = 1 \), we can replace condition (ii) by \( m_4(X_n) - 3 \to 0 \).

(ii) To obtain convergence in stronger distances with precise estimates using Stein’s method, we refer to [4] page 8 and [9] page 63.

3.2. **Gamma Approximation.**

**Theorem 3.4.** Let \( X \) be a chaos eigenfunction with eigenvalue \( \lambda_p \) with respect to the operator \(-L\) such that \( 2\lambda_p \leq \lambda_{2p} \) and set \( Y = X + \nu \) for some \( \nu > 0 \). Then it holds that
\[
\int_E (\Gamma(Y) - \lambda_p Y)^2 \, d\mu \leq \frac{\lambda_p^2}{3} (m_4(X) - 6m_3(X) + 6(1 - \nu) m_2(X) + 3\nu^2).
\]

**Proof.** Using the identities \( LY = -\lambda_p (Y - \nu) \) and \( 2\Gamma(Y) = (L + 2\lambda_p) (Y - \nu)^2 = (L + 2\lambda_p \text{Id}) Y^2 - 2\lambda_p \nu Y \) it is straightforward to verify that
\[
\Gamma(Y) - \lambda_p Y = (L + 2\lambda_p \text{Id}) L_2^{(\nu-1)}(Y),
\]
where \( L_2^{(\nu-1)}(x) = \frac{2}{2} - (\nu + 1)x + \frac{\nu(\nu + 1)}{2} \) is the second Laguerre polynomial with parameter \( \nu - 1 \). Due to the chaos property of \( X \) and the assumption that \( \lambda_{2p} \leq 2\lambda_p \), we can apply Theorem 2.1 to get that
\[
\int_E \left( (L + 2\lambda_p \text{Id}) L_2^{(\nu-1)}(Y) \right)^2 \, d\mu \leq 2\lambda_p \int_E L_2^{(\nu-1)}(Y) (L + 2\lambda_p \text{Id}) L_2^{(\nu-1)}(Y) \, d\mu.
\]
Simple calculations after an application of Lemma 3.1 now give
\[
\int_E L_2^{(\nu-1)}(Y) \left( L + 2\lambda_p \text{Id} \right) L_2^{(\nu-1)}(Y) \, d\mu \\
= \lambda_p \int_E \left( 2L_2^{(\nu-1)}(Y)^2 - \frac{(L_2^{(\nu-1)}(Y))^3}{3L_2^{(\nu-1)\prime}(Y)} \right) \, d\mu \\
= \lambda_p \left( \frac{\nu^2}{2} + (1-\nu) m_2(X) - m_3(X) + \frac{m_4(X)}{6} \right),
\]
concluding the proof. \(\square\)

**Corollary 3.5** (Fourth Moment Theorem for Gamma Approximation). Let \(\{X_n\}_{n \geq 1}\) be a sequence bounded in \(L^2(E, \mu)\) of chaos eigenfunctions of order \(p\) with respect to the operator \(-L\) such that \(\lambda_{2p} \leq 2\lambda_p\), and set \(Y_n = X_n + \nu\). Then, the following two assertions are equivalent:

(i) As \(n\) tends to infinity, the sequence \(\{Y_n\}_{n \geq 1}\) converges in distribution to a Gamma distributed random variable with parameter \(\nu\).

(ii) As \(n\) tends to infinity, it holds that
\[
m_4(X_n) - 6m_3(X_n) + 6(1-\nu)m_2(X_n) + 3\nu^2 \to 0.
\]

**Proof.** The proof can be given analogously to Corollary 3.3 \(\square\)

**Remark.**

(i) If we additionally assume that \(m_2(X_n) = \nu\), the moment condition \(10\) can be replaced by \(m_4(X_n) - 6m_3(X_n) - 3\nu^2 + 6\nu \to 0\).

(ii) Using Stein’s method, it is possible to show that the \(L^2(E, \mu)\)-convergence of \(\Gamma(X_n+\nu) - \lambda_p(X_n+\nu)\) implies convergence in stronger distances. We refer to [4, page 9] and [7] for details.

### 3.3. Beta Approximation.

**Theorem 3.6.** Let \(X\) be a chaos eigenfunction of order \(p\) with respect to \(-L\) and set \(Y = X + \frac{\alpha + \beta}{\alpha + \beta}\), where \(\alpha, \beta > 0\). Then, if
\[
\lambda_{2p} \leq 2\lambda_p \frac{\alpha + \beta + 1}{\alpha + \beta},
\]
it holds that
\[
\int_E \left( \Gamma(Y) - \frac{\lambda_p}{\alpha + \beta} Y \right) (1-Y) \, d\mu \\
\leq \frac{2(\alpha + \beta + 1)^2}{3(\alpha + \beta)} \left( m_4(X) + 3(\alpha + 1)\frac{m_3(X) + 3(\alpha + 1)^2 m_2(X) - (\alpha + \beta) \left( \frac{\alpha + 1}{\alpha + \beta + 2} \right)^3} {\alpha + \beta + 2} \right) \left( \frac{\alpha + 1}{\alpha + \beta + 2} \right)^3 (1-2Y),
\]
where \(P_2^{\alpha,\beta}(x)\) denotes the second Jacobi-polynomial with parameters \(\alpha\) and \(\beta\). Under assumption \(11\), we can apply Theorem 3.1 to infer that
\[
\int_E \left( \Gamma(Y) - \frac{\lambda_p}{\alpha + \beta} Y \right) (1-Y) \, d\mu \\
\leq 2\lambda_p c_{\alpha, \beta} \int_E P_2^{(\alpha-1, \beta-1)}(1-2Y) (L + 2\lambda_p \frac{\alpha + \beta + 1}{\alpha + \beta} \text{Id}) P_2^{(\alpha-1, \beta-1)}(1-2Y) \, d\mu,
\]
where \(c_{\alpha, \beta}\) is a constant depending only on \(\alpha\) and \(\beta\).
where
\[ c_{\alpha, \beta} = \frac{1}{(\alpha + \beta)(\alpha + \beta + 1)(\alpha + \beta + 2)^2}. \]

The asserted moment expression on the right hand side of (12) are obtained after an application of Lemma 3.1 and some tedious calculations.

**Corollary 3.7 (Fourth Moment Theorem for Beta Approximation).** Let \( \{X_n\}_{n \geq 1} \) be a sequence of chaos eigenfunctions of order \( p \) with respect to the operator \( -L \), bounded in \( L^2(E, \mu) \) and set \( Y_n = X_n + \frac{\alpha}{\alpha + \beta} \) where \( \alpha, \beta > 0 \). Then, if \( \lambda_{2p} \leq 2\lambda_p \frac{\alpha + \beta + 1}{\alpha + \beta} \), the following two assertions are equivalent.

(i) As \( n \) tends to infinity, the sequence \( \{Y_n\}_{n \geq 1} \) converges in distribution to a Beta distribution with parameters \( \alpha \) and \( \beta \).

(ii) As \( n \) tends to infinity, it holds that
\[
m_4(X_n) + \frac{3(\alpha + 1)}{\alpha + \beta + 2} m_3(X_n) + 3(\alpha + 1)^2 m_2(X_n) - (\alpha + \beta) \left( \frac{\alpha + 1}{\alpha + \beta + 2} \right)^3 \rightarrow 0.
\]

**Proof.** The proof can be done as in Corollary 3.3. What is slightly more involved is deriving the characteristic function \( \hat{\rho} \) of the weak limit of \( \{Y_n\}_{n \geq 1} \): Using that \( \Gamma(Y_n) \approx \frac{\lambda_p}{\alpha + \beta} \lambda_p Y_n(1 - Y_n) \) and thus
\[
\int_E i\xi e^{i\xi Y_n} \Gamma(Y_n) \, d\mu \approx \int_E \frac{\lambda_p}{\alpha + \beta} i\xi e^{i\xi Y_n} Y_n(1 - Y_n) \, d\mu,
\]
we can infer by integration by parts that \( \hat{\rho} \) solves the differential equation
\[
i\xi \frac{d^2}{d\xi^2} \phi''(i\xi) + (\alpha + \beta - i\xi) \frac{d}{d\xi} \phi(i\xi) - \alpha \phi(i\xi) = 0,
\]
which is a version of Kummer’s equation. The only solution satisfying \( \phi(0) = \hat{\rho}(0) = 1 \) and \( \frac{d}{d\xi} \phi(0) = \frac{\alpha}{\alpha + \beta} \) is
\[
\phi(\xi) = M(\alpha, \alpha + \beta, i\xi).
\]
Here, \( M \) denotes Kummer’s confluent hypergeometric function, which is well-known to be the characteristic function of a Beta distribution with parameters \( \alpha \) and \( \beta \). \( \square \)

**Remark.** As is the case in the Gaussian and Gamma approximation, one can apply Stein’s method in the spirit of [4] to prove convergence in Wasserstein distance and derive precise bounds.

4. **Applications**

In this section, we give concrete examples to how our main results can be applied to the Ornstein-Uhlenbeck, Laguerre and Jacobi generators. To be more precise, we prove that in the Wiener/Laguerre/Jacobi diffusion structures, our definition of chaos is always satisfied in the eigenspaces and that the assumptions of the Fourth Moment Theorems of the previous sections are valid (in the Jacobi case under some parameter condition).

4.1. **Wiener Structure.** For \( d \geq 1 \), denote by \( \mu_d \) the \( d \)-dimensional standard Gaussian measure on \( \mathbb{R}^d \). It is well known (see for example [24]), that \( \mu_d \) is the invariant measure of the Ornstein-Uhlenbeck generator, defined for any test function \( \phi \) by
\[
L\phi(x) = \Delta \phi - \sum_{i=1}^d x_i \partial_i \phi(x).
\]
Its spectrum is given by \( -\mathbb{N} \) and the eigenspaces are of the form
\[
\text{Ker}(L + k\text{Id}) = \left\{ \sum_{i_1+i_2+\cdots+i_d=k} \alpha(i_1, \cdots, i_d) \prod_{j=1}^d H_{i_j}(x_j) \right\},
\]
where $H_n$ denotes the Hermite polynomial of order $n$. Any eigenfunction $X$ is thus chaotic in the sense of Definition 2.2. Assume now that $X$ is an eigenfunction of $L$ with eigenvalue $-\lambda_p = -p$. In particular, $X$ is a multivariate polynomial of degree $p$. Hence $X^2$ is a multivariate polynomial of degree $2p$. Note that, by expanding $X^2$ over the basis of multivariate Hermite polynomials $\prod_{j=1}^{d} H_{i_j}(x_j), i_j \geq 0$, we obtain that $X^2$ has a finite expansion over the first eigenspaces of the generator $L$, i.e.

$$X^2 \in \bigoplus_{k=0}^{M} \text{Ker}(L + k\text{Id}).$$

For degree reasons, we infer that $M = 2p$. As a result one can see that Theorem 2.1 is applicable and thus the finite-dimensional version of the celebrated Fourth Moment Theorem from [13] is a consequence of Corollary 3.3.

4.2. Laguerre Structure. Let $\nu \geq -1$, and $\pi_{1,\nu}(dx) = x^{\nu-1}e^{-x}1_{(0,\infty)}(x)dx$ be the Gamma distribution with parameter $\nu$ on $\mathbb{R}_+$. The associated Laguerre generator is defined for any test function $\phi$ (in dimension one) by:

$$L_{1,\nu}\phi(x) = x\phi''(x) + (\nu + 1 - x)\phi'(x).$$

By a classical tensorization procedure, we obtain the Laguerre generator in dimension $d$ associated to the measure $\pi_{d,\nu}(dx) = \pi_{1,\nu}(dx_1)\pi_{1,\nu}(dx_2) \cdots \pi_{1,\nu}(dx_d)$, where $x = (x_1, x_2, \cdots, x_d)$.

$$L_{d,\nu}\phi(x) = \sum_{i=1}^{d} \left( x_i \partial_i \phi + (\nu + 1 - x_i) \partial_i \phi \right)$$

It is well known that (see for example [2]) that the spectrum of $L_{d,\nu}$ is given by $-\mathbb{N}$ and moreover that

$$\text{Ker}(L_{d,\nu} + p\text{Id}) = \left\{ \sum_{i_1 + i_2 + \cdots + i_d = p} \alpha(i_1, \cdots, i_d) \prod_{j=1}^{d} L_{i_j}^{(\nu)}(x_j) \right\},$$

where $L_{i_j}^{(\nu)}$ stands for the Laguerre polynomial of order $n$ with parameter $\nu$ which is defined by

$$L_{i_j}^{(\nu)}(x) = \frac{x^{-\nu}e^x}{n!} \frac{d^n}{dx^n} \left( e^{-x}x^{n+\nu} \right).$$

Again, we have the following decomposition:

$$L^2(\mathbb{R}^d, \pi_{d,\nu}) = \bigoplus_{p=0}^{\infty} \text{Ker}(L_{d,\nu} + p\text{Id})$$

Similarly in this framework, we see that any eigenfunction $X$ is a chaotic in the sense of Definition 2.2. Assume now that $X$ is an eigenfunction of $L_{d,\nu}$ with eigenvalue $-\lambda_p = -p$. In particular, $X$ is a multivariate polynomial of degree $p$. Therefore, $X^2$ is a multivariate polynomial of degree $2p$. Note that, by expanding $X^2$ over the basis of multivariate Laguerre polynomials $\prod_{j=1}^{d} L_{i_j}^{(\nu)}(x_j), i_j \geq 0$, we get that $X^2$ has a finite expansion over the first eigenspaces of the generator $L_{d,\nu}$, i.e.

$$X^2 \in \bigoplus_{p=0}^{M} \text{Ker}(L_{d,\nu} + p\text{Id}).$$

Again, for degree reasons we infer that $M = 2p$ and thus Theorem 2.1 is applicable.
4.3. Beta Structure. For $\alpha, \beta > -1$, let $\gamma_{\alpha, \beta}(dx) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}1_{[0,1]} dx$ be the beta distribution and choose a dimension $d \geq 1$. Then, the generator $L_{\alpha, \beta}$ associated to the measure $\gamma_{d, \alpha, \beta} := \gamma_{\alpha, \beta}(dx_1)\cdots\gamma_{\alpha, \beta}(dx_d)$ is given by

$$L_{\alpha, \beta}\phi(x) = L_{\alpha, \beta}\phi(x_1, \ldots, x_d) = \left(\sum_{i=1}^{d} (x_i(1-x_i)\partial_{x_i}^2 + (\alpha - (\alpha + \beta) x_i) \partial_i)\right)\phi(x)$$

and its spectrum $S$ is of the form

$$(18) \quad S = \{\lambda_{i_1} + \cdots + \lambda_{i_d} \mid i_j \geq 0, j = 1, \ldots, d\},$$

where, here and throughout the rest of this section, $\lambda_k = k(k+\alpha+\beta-1)$. Again, it holds that $L^2(\gamma_{d, \alpha, \beta}) = \bigoplus_{\lambda \in S} \ker(L_{\alpha, \beta} + \lambda \Id)$ and the kernels are given by

$$(19) \quad \ker(L_{\alpha, \beta} + \lambda \Id) = \left\{ \sum_{\substack{i_1, \ldots, i_d \geq 0 \\ \lambda_{i_1} + \cdots + \lambda_{i_d} = \lambda}} a(i_1, \ldots, i_d) P_{i_1}^{(\alpha-1, \beta-1)}(1-2x_1) \cdots P_{i_d}^{(\alpha-1, \beta-1)}(1-2x_d) \right\},$$

where $P_{\alpha, \beta}^{(\alpha, \beta)}(x)$ denotes the $n$th Jacobi polynomial, given by

$$P_{\alpha, \beta}^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n} ((1-x)^{\alpha}(1+x)^{\beta}(1-x^2)^n).$$

We need the following technical Lemma.

Lemma 4.1. Let $\lambda \in S$ and

$$(20) \quad X = \sum_{\substack{i_1, \ldots, i_d \geq 0 \\ \lambda_{i_1} + \cdots + \lambda_{i_d} = \lambda}} a(i_1, \ldots, i_d) P_{i_1}^{(\alpha-1, \beta-1)}(1-2x_1) \cdots P_{i_d}^{(\alpha-1, \beta-1)}(1-2x_d) \in \ker(L_{\alpha, \beta} + \lambda \Id)$$

Then it holds that

$$X^2 \in \bigoplus_{\eta \leq M} \ker(L_{\alpha, \beta} + \eta \Id)$$

where

$$(21) \quad M = \max_{\substack{i_1, \ldots, i_d \geq 0 \\ \lambda_{i_1} + \cdots + \lambda_{i_d} = \lambda}} \lambda_{2i_1} + \lambda_{2i_2} + \cdots + \lambda_{2i_d}$$

Proof. First note that for $a, b \in \mathbb{N}$ we have $\lambda_a + \lambda_b = \lambda_{a+b} - 2ab$. By induction, we thus get for $p \geq 2$ and integers $a_1, \ldots, a_p \in \mathbb{N}$ that

$$(22) \quad \lambda_{a_1} + \cdots + \lambda_{a_p} = \lambda_{a_1+a_2+\cdots+a_p} - \sum_{1 \leq k \leq p \atop k \neq l} a_k a_l.$$

Consequently, it holds for any index $(i_1, \ldots, i_d)$ occurring in the sum on the right hand side of (20) that $\lambda_{i_1} + \cdots + \lambda_{i_d} = \lambda_m - \sum_{1 \leq k, l \leq d \atop k \neq l} i_k i_l$, where $m$ is the degree of $X$. If $(j_1, \ldots, j_d)$ is another index from this sum, we thus have

$$(23) \quad \sum_{1 \leq k, l \leq d \atop k \neq l} i_k i_l = \sum_{1 \leq j, k \leq d \atop k \neq l} j_k j_l$$
and, as \( \left( \sum_{k=1}^{d} i_k \right)^2 - \left( \sum_{k=1}^{d} j_k \right)^2 = m^2 - m^2 = 0 \),

\[
\sum_{k=1}^{d} i_k^2 = \sum_{k=1}^{d} j_k^2. \tag{24}
\]

Now observe that the polynomials in the expansion of \( X \) with maximum degree \( 2m \) are of the form

\[
\prod_{k=1}^{d} P_{x_k+1}^{(\alpha-1, \beta-1)}(1 - 2x_k),
\]

corresponding to the eigenvalue

\[
\sum_{k=1}^{d} \lambda_{i_k+j_k} = \lambda_{2m} - \sum_{k \neq l} (i_k + j_k)(i_l + j_l), \tag{25}
\]

where we have used the identity (22). Due to (23), it holds that

\[
\sum_{k \neq l} (i_k + j_k)(i_l + j_l) = 2 \left( \sum_{k \neq l} i_k i_l + \sum_{k \neq l} i_k j_l \right)
\]

and a straightforward application of the Cauchy-Schwarz inequality together with (24) shows that

\[
\sum_{1 \leq k \leq d} i_k j_l \geq \sum_{1 \leq k \leq d} i_k i_l.
\]

Plugging this into (25) yields that \( \sum_{k=1}^{d} \lambda_{i_k+j_k} \leq \sum_{k=1}^{d} \lambda_{2i_k} \) and concludes the proof. \( \square \)

This yields the following corollary.

**Corollary 4.2.** Let \( \lambda \in S \) and \( X \in \text{Ker}(L_{\alpha, \beta} + \lambda \text{Id}) \). Then, condition (11) of Theorem 3.6 is satisfied if and only if \( \alpha + \beta \leq 1 \).

**Proof.** A straightforward calculation shows that for \( p \geq 0 \) it holds that

\[
2\lambda p^{\frac{\alpha + \beta + 1}{\alpha + \beta}} \geq \lambda^{2p} \tag{26}
\]

if and only if \( \alpha + \beta \leq 1 \), with equality only for \( \alpha + \beta = 1 \). Now write \( \lambda = \lambda_{i_1} + \cdots + \lambda_{i_p} \) and apply Lemma 4.1. \( \square \)

**Remark.** Contrarily to the Gamma and Gaussian distribution, the Beta distribution is not stable under summation. However, if \( F_n \) is in the first eigenspace of the Jacobi diffusion generator, we know from above that

\[
F_n = \sum_{k=0}^{\infty} a_k(n) X_k,
\]

where \( X_k \sim \text{Beta}(\alpha, \beta) \). Thus we can use corollary to give moment conditions for convergence of a linear combination of Beta random variables towards another Beta random variable. This might be of independent interest for statisticians. We stress that of course this scenario is not empty, for instance, one can trivially take \( a_1(n) \to 1 \) and \( a_k(n) \to 0 \) for \( k \geq 2 \).
4.4. Mixed case. In this section we prove Theorem 1.1 from the introduction, which we restate here for convenience.

**Theorem 1.1.** Let $\mathcal{X} = \{\pi_{1,\nu} - \nu \mid \nu > -1\} \cup \{N(0,1)\}$, be the set containing all centered gamma and Gaussian laws. Let $\{X_i\}_{i \geq 1}$ be a sequence of independent random variables such that for all $i \geq 1$, the law of $X_i$ belongs to $\mathcal{X}$. Then we choose $d \geq 1$ and let

$$P_n(x) = \sum_{i_1 < i_2 < \cdots < i_d} a_n(i_1, \cdots, i_d)x_{i_1} \cdots x_{i_d}$$

be a sequence of multivariate homogeneous polynomials of degree $d$. Then the two following statements are equivalent:

(i) As $n$ tends to infinity, the sequence $\{P_n(X)\}_{n \geq 1}$ converges in distribution towards $N(0,1)$.

(ii) As $n$ tends to infinity, it holds that $\mathbb{E}(P_n(X)^2) \to 1$ and $\mathbb{E}(P_n(X)^4) \to 3$.

**Proof.** By assumption, for each $i \geq 1$, the law of $X_i$ belongs to $\mathcal{X}$. Then we set $\mathcal{L}_i$ the univariate diffusion generator associated to $X_i$. Then we define $\mathbf{L}_n$, by the usual tensorization procedure. For instance if

$$(X_1, X_2, X_3) \sim N(0,1) \otimes N(0,1) \otimes (\gamma(\nu) - \nu),$$

then

$$\mathbf{L}\phi(x, y, z) = \partial_{1,1}\phi + \partial_{2,2}\phi + z\partial_{3,3}\phi - x\partial_1\phi - y\partial_2\phi + (\nu + 1 - z)\partial_3\phi.$$ 

One can check that for all $n \geq 1$ the spectrum of $\mathbf{L}_n$ is $\mathbb{N}$ and

$$\text{Ker}\left(\mathbf{L}_n + d\mathbf{I}d\right) = \bigoplus_{d_1 + \cdots + d_d = d} \text{Ker}\left(\mathcal{L}_{i_1} + d_{i_1}\mathbf{I}d\right) \otimes \cdots \otimes \text{Ker}\left(\mathcal{L}_{i_d} + d_{i_d}\mathbf{I}d\right)$$

First we claim that

$$P_n(x) = \sum_{i_1 < i_2 < \cdots < i_d} a_n(i_1, \cdots, i_d)x_{i_1} \cdots x_{i_d} \in \text{Ker}\left(\mathbf{L}_n + d\mathbf{I}d\right).$$

And the degree of $P_n^2$ is $2d$ so that $P_n^2$ may be expanded over $\bigoplus_{d \leq 2d} \text{Ker}\left(\mathbf{L}_n + d\mathbf{I}d\right)$. Consequently, the eigenspaces of the mixed structures of Wiener and Laguerre are always chaotic in the sense of Definition 2.2. Moreover as one easily verifies that $\lambda_{2p} = 2\lambda_p$, we can apply corollaries 3.4 and 3.3 and obtain Theorem 1.1 as a particular instance. \(\square\)

**Remark.** We could replace the homogeneous sums by a general eigenfunction (i.e. sums of products of Hermite and Laguerre polynomials) and also mix Wiener, Laguerre and/or Jacobi generators. In the latter case, one has to impose some additional technical conditions involving the parameters of the Jacobi generators, as $\lambda_{2p} \leq 2\lambda_p$ doesn’t hold in general.

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