GOING TO LORENTZ WHEN FRACTIONAL SOBOLEV, GAGLIARDO AND NIRENBERG ESTIMATES FAIL

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Abstract. In the cases where there is no Sobolev-type or Gagliardo–Nirenberg-type fractional estimate involving $|u|_{W^{s,p}}$, we establish alternative estimates where the strong $L^p$ norms are replaced by Lorentz norms.

1. Introduction

In [6, Theorem 1.1], it was shown that there exists a constant $C = C(N)$ such that

\[ \left| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + 1}} \right|_{MP(\mathbb{R}^N \times \mathbb{R}^N)} \leq C^{1/p} \| \nabla u \|_{L^p(\mathbb{R}^N)}, \quad \forall u \in C^\infty_c(\mathbb{R}^N), \forall p \geq 1. \]

(1.1)

Here $MP(\mathbb{R}^N \times \mathbb{R}^N) = L^p_w(\mathbb{R}^N \times \mathbb{R}^N) = L^{p,\infty}(\mathbb{R}^N \times \mathbb{R}^N)$, $1 \leq p < \infty$, is the Marcinkiewicz (=weak $L^p$) space modelled on $L^p(\mathbb{R}^N \times \mathbb{R}^N)$, defined by the condition

\[ [f]_{MP(\mathbb{R}^N \times \mathbb{R}^N)}^p := \sup_{\lambda > 0} \lambda^p L^{2N}(\{|(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |f(x)| \geq \lambda\}) < \infty \]

(see for example [8, Chapter 5] or [15, §1.1]).

We also know that the inequality

\[ \left| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + 1}} \right|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C(N, p) \| \nabla u \|_{L^p(\mathbb{R}^N)}, \quad \forall u \in C^\infty_c(\mathbb{R}^N), \forall p \geq 1 \]

(1.2)

does not hold. In fact the failure of (1.2) is more striking: for every $1 \leq p < \infty$ and every measurable function $u$,

\[ \left| \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + 1}} \right|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} < \infty \implies u \text{ is constant}; \]

(1.3)

See [1] and [3, 10, 20].

A natural question is whether one can improve (1.1) in the Lorentz scale, where $L^{p,q}(X, \mu)$, with $1 \leq p < \infty$ and $1 \leq q \leq \infty$, is characterized by (see for example [15, §1.4], [8, Chapter 6], [17] or [22, §1.8]), when $q < \infty$

\[ [f]^q_{L^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)} = \rho \int_0^\infty \lambda^q L^{2N}(\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |f(x, y)| \geq \lambda\}) \frac{d\lambda}{\lambda} < +\infty, \]

(1.4)

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and when $q = \infty$ by $[f]_{L^p,\infty(\mathbb{R}^N \times \mathbb{R}^N)} = [f]_{M^{p}(\mathbb{R}^N \times \mathbb{R}^N)} < +\infty$. In other words, the question is whether the estimate

$$
(1.5) \quad \frac{\|u(x) - u(y)\|}{|x - y|^\frac{N}{p} + 1}_{L^p,\varphi(\mathbb{R}^N \times \mathbb{R}^N)} \leq C (N, p, q) \|\nabla u\|_{L^p(\mathbb{R}^N)}, \quad \forall u \in C^\infty_c(\mathbb{R}^N)
$$

holds for some $q \in (p, \infty)$ (q depending on $p$ and $N$). The answer is negative, as can be seen from the following generalization of (1.3).

**Theorem 1.** Assume that $1 \leq p < \infty$, $1 \leq q < \infty$ and $u$ is measurable. Then

$$
(1.6) \quad \frac{\|u(x) - u(y)\|}{|x - y|^\frac{N}{p} + 1}_{L^p,\varphi(\mathbb{R}^N \times \mathbb{R}^N)} < \infty \implies u \text{ is constant}.
$$

The heart of the matter is the following far-reaching extension of (1.3). It was originally presented in [7, Proposition 6.3] when $p > 1$; the case $p = 1$ is essentially due to A. Poliakovsky [19, Corollary 1.1] who settled [7, Open Problem 1].

**Theorem 2.** Let $1 \leq p < \infty$ and let $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function satisfying

$$
(1.7) \quad \lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(x) - u(y)|}{|x - y|^\frac{N}{p} + 1} \geq \lambda \right\} \right) = 0.
$$

Then $u$ is constant.

The proofs of Theorems 1 and 2 are given in Section 2.

Recall the fractional Sobolev spaces $W^{s, p}$ (also called Slobodeskii spaces) is associated with the Gagliardo semi-norm, $0 < s < 1$ and $1 \leq p < \infty$ defined by

$$
(1.8) \quad |u|_{W^{s, p}}^p := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dy \, dx.
$$

In [6] we announced the following theorem, which is a substitute for a fractional Sobolev-type estimate $W^{1, 1}(\mathbb{R}^N) \hookrightarrow W^{1 - N(1 - \frac{1}{p}), p}(\mathbb{R}^N)$ that fails in dimension $N = 1$:

**Theorem 3.** [6, Corollary 4.1] There exists an absolute constant $C$ such that for every $1 \leq p < \infty$,

$$
(1.9) \quad \frac{\|u(x) - u(y)\|}{|x - y|^\frac{N}{p}}_{M^p(\mathbb{R} \times \mathbb{R})} \leq C \|u'\|_{L^1(\mathbb{R})}, \quad \forall u \in C^\infty_c(\mathbb{R}).
$$

**Remark 1.1.** When $p = 1$, estimate (1.9) is originally due to Greco and Schiattarella [16].

In [6], we also announced the following theorem, which offers an alternative when the “anticipated” fractional Gagliardo–Nirenberg-type inequality

$$
(1.10) \quad \left\| \frac{|u(x) - u(y)|}{|x - y|^\frac{N}{p}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \|u\|_{L^\infty(\mathbb{R}^N)}^{1 - 1/p} \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1/p}, \quad \forall u \in C^\infty_c(\mathbb{R}^N)
$$

fails for every $1 \leq p < \infty$:
Theorem 4. [6, Corollary 5.1] For every $N \geq 1$, there exists a constant $C = C(N)$ such that for every $1 \leq p < \infty$,
\[
(1.11) \quad \left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N+1}{p}}} \right\|_{MP(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \|u\|^{1-1/p}_{L^\infty(\mathbb{R}^N)} \|\nabla u\|^{1/p}_{L^1(\mathbb{R}^N)}, \quad \forall u \in C_c^\infty(\mathbb{R}^N).
\]

Theorem 4 clearly implies Theorem 3 since $\|u\|_{L^\infty(\mathbb{R})} \leq \|u\|_{L^1(\mathbb{R})}$, $\forall u \in C_c^\infty(\mathbb{R})$. The case $p = 1$ of (1.11) follows from (1.1) by setting $p = 1$. The proof of (1.11) in the case $p > 1$ can be reduced to the case $p = 1$ (see Section 3). Alternatively, in the same section, we show how we can establish (1.11) in the case $p > 1$ by a more elementary method, if we allow a constant $C$ that depends not only on $N$ but also on $p$. This method, which is in line with the techniques in Bourgain, Brezis and Mironescu [2], can be contrasted with the proof of (1.11) in the case $p = 1$: the latter relies on a covering argument in the one-dimensional case and the method of rotation to reach higher dimensions.

Theorems 3 and 4 can be restated equivalently as Lorentz spaces estimates. In particular, replacing $M^p$ by $L^{p,\infty}$, we get Theorems 3 and 4 at the endpoint of the Lorentz scale.

We show in Section 4 that there is no improvement of Theorems 3 and 4 in the Lorentz scale. (Recall that for any fixed $p$ the Lorentz spaces $L^{p,\infty}$ increase as $q$ increases.)

We now turn to another situation, also involving $\dot{W}^{1,1}$, where a Gagliardo–Nirenberg-type inequality fails. Let $0 < s_1 < 1$, $1 < p_1 < \infty$ and $0 < \theta < 1$. Set
\[
(1.12) \quad s = \theta s_1 + (1 - \theta) \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{p_1} + (1 - \theta).
\]

It is known that the estimate
\[
(1.13) \quad \|u\|_{W^{s,p}(\mathbb{R}^N)} = \left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N+1}{p}}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \|u\|_{W^{1,p_1}(\mathbb{R}^N)}^\theta \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1-\theta}.
\]

- holds for every $\theta \in (0, 1)$ when $s_1 p_1 < 1$ (Cohen, Dahmen, Daubechies and DeVore [9]),
- fails for every $\theta \in (0, 1)$ when $s_1 p_1 \geq 1$ (Brezis and Mironescu [5]).

We investigate here what happens in the regime $s_1 p_1 \geq 1$. Our main result in this direction is

Theorem 5. For every $N \geq 1$, $p_1 \in (1, \infty)$ and $\theta \in (0, 1)$, there exists a constant $C = C(N, p_1, \theta)$ such that for all $s_1 \in (0, 1)$ with $s_1 p_1 \geq 1$, we have
\[
(1.14) \quad \left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N+1}{p}}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \|u\|_{W^{1,p_1}(\mathbb{R}^N)}^\theta \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1-\theta}, \quad \forall u \in C_c^\infty(\mathbb{R}^N),
\]
where $0 < s < 1$ and $1 < p < \infty$ are defined by (1.12).

Note that $L^p(\mathbb{R}^N \times \mathbb{R}^N) \subset L^{p,\infty}(\mathbb{R}^N \times \mathbb{R}^N)$ since $p < \frac{N}{N+1}$; this is consistent with the fact that (1.13) fails when $s_1 p_1 \geq 1$. As an immediate consequence of (1.14) we obtain
\[
(1.15) \quad \left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N+1}{p}}} \right\|_{MP(\mathbb{R}^N \times \mathbb{R}^N)} \leq C \|u\|_{W^{1,p_1}(\mathbb{R}^N)}^\theta \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1-\theta}, \quad \forall u \in C_c^\infty(\mathbb{R}^N);
\]
a slightly more careful argument shows that the constant $C$ in (1.15) can be taken to depend only on $N$ but not on $p_1$ nor $\theta$. Estimate (1.15) was announced in [6, Corollary 5.2]. The proofs of Theorem 5 and (1.15) are given in Section 5. In the same section we establish the optimality of the exponent $\frac{p}{p_1}$ in (1.14).

To conclude this paper we mention another estimate in the spirit of Gagliardo–Nirenberg interpolation between $L^\infty$ and $W^{1,1}$. It is originally due to Figalli-Serra [13, Lemma 3.1] when $p = 2$ and $q = \infty$, with roots in Figalli–Jerison [12, Lemma 2.1] (see also [14, Lemma 2.2 and Corollary 2.3] for a simpler proof and more general version).

**Theorem 6.** Let $N \geq 1$, $1 < p < \infty$ and $N < q \leq \infty$. There exists a constant $C = C(N, p, q)$ such that

$$
\int_{B_1 \times B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+1}} \, dx \, dy
\leq C\|u\|^{p-1}_{L^\infty(B_1)}\|
abla u\|_{L^1(B_1)} \left(1 + \log \max \left\{\|\nabla u\|_{L^q(B_1)}, 1\right\}\right)
$$

for every $u \in C^1(B_1)$.

Here $B_1$ denotes the unit ball in $\mathbb{R}^N$. See Section 6 for a proof of Theorem 6.

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2. Proofs of Theorems 1 and 2

**Lemma 2.1.** Let $f$ be a measurable function on $\mathbb{R}^N \times \mathbb{R}^N$ in $L^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)$ with $1 \leq p < \infty$, $1 \leq q < \infty$. Then

$$
\lim_{\lambda \to \infty} \lambda^p L^{2N}(\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |f(x, y)| \geq \lambda\}) = 0.
$$

**Proof.** Set

$$
\varphi(\lambda) = \int_{\lambda/2}^{\lambda} t^q L^{2N}(\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |f(x, y)| \geq t\}) \frac{dt}{t}.
$$

Since $f \in L^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)$, we know that $\varphi(\lambda) \to 0$ as $\lambda \to \infty$. On the other hand,

$$
\varphi(\lambda) \geq L^{2N}(\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |f(x, y)| \geq \lambda\}) \int_{\lambda/2}^{\lambda} t^q \frac{dt}{t}
\geq L^{2N}(\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : |f(x, y)| \geq \lambda\}) \frac{\lambda^q}{q} \left(1 - \frac{1}{2q}\right),
$$

which yields (2.1). 

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Proof of Theorem 2 when \( p > 1 \). Let \( E_\lambda \subseteq \mathbb{R}^N \times \mathbb{R}^N \) denote the set in the left-hand side of (1.7). First observe that for each \( \lambda > 0 \),

\[
\iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(y)|}{|x - y|^\frac{p}{p+1}} - \lambda \right) \, dy \, dx \leq \int_0^\infty L^{2N}(E_t) \, dt \leq \frac{1}{(p-1)\lambda^{p-1}} \sup_{t \geq \lambda} t^p L^{2N}(E_t).
\]

Hence, we have

\[
\lim_{\lambda \to \infty} \lambda^{p-1} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(y)|}{|x - y|^\frac{p}{p+1}} - \lambda \right) \, dy \, dx = 0.
\]

We next use an argument similar to the one in [11, 20] and [21, Proof of Proposition 5.1]. From the triangle inequality and change of variable, we obtain

\[
\iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(y)|}{|x - y|^\frac{p}{p+1}} - \lambda \right) \, dy \, dx \leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(y)|}{|x - y|^\frac{p}{p+1}} - \lambda \right) \, dy \, dx + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(y)|}{|x - y|^\frac{p}{p+1}} - \frac{\lambda}{2} \right) \, dy \, dx
\]

\[
= 2^{\frac{N}{p}(p-1)} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(y)|}{|x - y|^\frac{p}{p+1}} - \frac{2^{\frac{N}{p}} \lambda}{2} \right) \, dy \, dx.
\]

Iterating (2.3), we have in view of (2.2),

\[
\iint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \frac{|u(x) - u(y)|}{|x - y|^\frac{p}{p+1}} - \lambda \right) \, dy \, dx = 0, \quad \forall \lambda > 0,
\]

from which it follows that \( u \) is constant. \( \square \)

Proof of Theorem 2 when \( p = 1 \). As already mentioned, in this case the conclusion of Theorem 2 is essentially due to A. Poliakovsky [19]. Indeed, if a measurable function \( u \) satisfies (1.7), then so does its truncation \( u_h \) for any \( h > 0 \) where \( u_h := \max\{\min\{u, h\}, -h\} \). Then \( u_h \in L^1(B_h) \), where \( B_h := \{x \in \mathbb{R}^N : |x| < h\} \), and [19, Cor. 1.1] (with \( q = 1 \), \( \Omega = B_h \)) shows that \( u_h \) is a constant on \( B_h \). Since this is true for every \( h > 0 \), this also shows \( u \) is a constant.

The proof of [19, Cor. 1.1] is quite intricate and we refer the reader to [19]; it would be interesting to find a simpler argument as in the case \( p > 1 \). \( \square \)

We also call the attention of the reader to

Open Problem 1. Does the conclusion of Theorem 2 still hold if “lim” is replaced by “lim inf” in (1.7)?
3. Proofs of Theorem 4

Theorem 4 can be derived as an immediate consequence of (1.1) (applied with \( p = 1 \)) and the fact that
\[
\frac{|u(x) - u(y)|}{|x - y|^{\frac{N+1}{p}}} \geq \lambda \quad \text{implies} \quad \frac{|u(x) - u(y)|}{|x - y|^{N+1}} \geq \frac{\lambda^{p}}{(2\|u\|_{L^{\infty}(\mathbb{R}^{N})})^{p-1}}.
\]
Hence
\[
\mathcal{L}^{2N}\left(\left\{(x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N} : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+1}{p}}} \geq \lambda\right\}\right) \leq \frac{2^{p-1}C}{\lambda^{p}} \|u\|_{L^{\infty}(\mathbb{R}^{N})}^{-1} \|\nabla u\|_{L^{1}(\mathbb{R}^{N})},
\]
where \( C = C(N) \) is as in (1.1); note that \( (2^{p-1})^{1/p} \) can be dominated by a constant depending only on \( N \). This proves Theorem 4.

For the enjoyment of the reader we also present an elementary qualitative argument for the case \( p > 1 \) of Theorem 4 which does not make use of (1.1). It relies on the following estimate occurring in [2]; unfortunately it yields a constant \( C \) in (1.11) which depends on \( p \) and \( N \), and which deteriorates as \( p \searrow 1 \). Note that (3.1) is a straightforward consequence of the inequality (see [4, Proposition 9.3])
\[
\int_{\mathbb{R}^{N}} |u(x + h) - u(x)| \, dx \leq |h| \int_{\mathbb{R}^{N}} |\nabla u|, \quad \forall h \in \mathbb{R}^{N}, \forall u \in C^{\infty}_{c}(\mathbb{R}^{N}).
\]

Lemma 3.1. For every \( u \in C^{\infty}_{c}(\mathbb{R}^{N}) \) and \( \rho \in L^{1}(\mathbb{R}^{N}) \),
\[
\iint_{\mathbb{R}^{N} \times \mathbb{R}^{N}} \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p}}} \rho(x - y) \, dy \, dx \leq \|\rho\|_{L^{1}(\mathbb{R}^{N})} \int_{\mathbb{R}^{N}} |\nabla u|,
\]
and in particular choosing \( \rho(z) = 1_{B_{r}(0)}(z)/|z|^{N-\delta}, \delta > 0 \), we obtain
\[
\iint_{(x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N}} \frac{|u(x) - u(y)|}{|x - y|^{N+1-\delta}} \, dy \, dx \leq C(N) \rho_{\delta} \int_{\mathbb{R}^{N}} |\nabla u|.
\]

Alternative proof of (1.11) when \( p > 1 \). Define the set
\[
E_{\lambda} := \left\{(x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N} : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+1}{p}}} \geq \lambda\right\}.
\]
Observe that
\[
E_{\lambda} \subseteq K_{\lambda} := \left\{(x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N} : |x - y| \leq (2\|u\|_{L^{\infty}(\mathbb{R}^{N})})^{\frac{p}{N+1}}\right\}.
\]
Thus
\[
1_{E_{\lambda}} \leq 1_{K_{\lambda}} \frac{1}{\lambda} \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+1}{p}}}.
\]
Hence we have
\[
\mathcal{L}^{2N}(E_{\lambda}) \leq \frac{1}{\lambda} \iint_{K_{\lambda}} \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+1}{p}}} \, dy \, dx.
\]
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It then follows by (3.2), with \[ \delta := (N + 1)^{\frac{1}{p}} > 0 \] and \( r := (2\|u\|_{L^\infty(\mathbb{R}^N)/\lambda})^{-\frac{p}{N-2}} \), that

\[ \mathcal{L}^{2N}(E_\lambda) \leq \frac{C(N) (2\|u\|_{L^\infty(\mathbb{R}^N)})^{p-1}}{(N + 1)(1 - \frac{1}{p})\lambda^p} \int_{\mathbb{R}^N} |\nabla u|. \]

4. Optimality of Theorems 3 and 4 in the Lorentz scale

Theorems 3 and 4 cannot be improved. This is a consequence of the following lemma and its proof.

**Lemma 4.1.** Assume that \( 1 \leq p < \infty \). If

\[ \left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N+1}{p}}} \right] \leq C\|u\|_{L^\infty(\mathbb{R}^N)}^{\frac{1-p}{p}} \|\nabla u\|_{L^1(\mathbb{R}^N)}^{\frac{p}{Q}}, \quad \forall u \in C_c^\infty(\mathbb{R}^N) \]

holds for some \( 1 \leq q \leq \infty \), then \( q = \infty \).

**Proof.** When \( p = 1 \), the conclusion follows from Theorem 1. Indeed, we already know that if \( q < \infty \), then for any measurable function \( u \),

\[ \left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N+1}{p}}} \right]_{L^{q,p}(\mathbb{R}^N \times \mathbb{R}^N)} = \infty, \]

unless \( u \) is a constant.

When \( p > 1 \), the argument is different since one may easily check that

\[ \left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N+1}{p}}} \right]_{L^{p,q}(\mathbb{R}^N \times \mathbb{R}^N)} < \infty, \quad \forall u \in C_c^\infty(\mathbb{R}^N), \forall q \in [1, \infty]. \]

We consider the case \( N = 1 \), the case \( N > 1 \) being similar. By an approximation argument, it follows that (4.1) holds for every \( u \in BV(\mathbb{R}) \) with compact support. However, if \( u := \chi_{[0,1]} \), we have

\[ \{ (x, y) \in (-1, 0) \times (0, 1) : |x - y| \leq \lambda^{-p/2} \} \]

\[ \subseteq E_\lambda := \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} : \frac{|u(x) - u(y)|}{|x - y|^{\frac{p}{2}}} \geq \lambda \right\}, \]

and thus, if \( \lambda \geq 1 \),

\[ \mathcal{L}^2(E_\lambda) \geq \frac{c}{\lambda^p}. \]

Hence, if \( 1 \leq q < \infty \),

\[ \left[ \frac{u(x) - u(y)}{|x - y|^{\frac{N+1}{p}}} \right]_{L^{p,q}(\mathbb{R}^N)}^q = p \int_0^\infty \lambda^q \mathcal{L}^2(E_\lambda)^{\frac{q}{p}} \frac{d\lambda}{\lambda} \geq p c^{\frac{q}{p}} \int_1^\infty \frac{d\lambda}{\lambda} = \infty, \]

which contradicts (4.1). \[ \square \]
5. Proof of Theorem 5

We deduce Theorem 5 from Theorem 1.1 in [6] and the classical product property in Lorentz spaces.

Proof of Theorem 5. By Theorem 1.1 in [6], we have
\[
\frac{\|u(x) - u(y)\|}{|x - y|^{\frac{N}{p} + s}} \leq C\|\nabla u\|_{L^1(\mathbb{R}^N)}
\]
where \(C = C(N)\). On the other hand, by definition of the Gagliardo semi-norm
\[
\frac{\|u(x) - u(y)\|}{|x - y|^{\frac{N}{p} + 1}} = |u|_{W^{s,p_1}}.
\]

We now observe that
\[
\frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + s}} = \left(\frac{|u(x) - u(y)|}{|x - y|^{N+1}}\right)^{1-\theta} \left(\frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + 1}}\right)^{\theta}.
\]
Hence, by the product property in Lorentz spaces [18, Theorem 3.4],
\[
(5.1) \quad \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + s}} \leq C \left[\left(\frac{|u(x) - u(y)|}{|x - y|^{N+1}}\right)^{1-\theta}\right]^{\frac{1}{L^p(\mathbb{R}^N)}} \left[\left(\frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + 1}}\right)^{\theta}\right]^{\frac{1}{L^p(\mathbb{R}^N)}}.
\]
Here \(C = C(N,p,\theta)\). In order to conclude, we use the fact that for \(1 \leq p < +\infty\), \(1 \leq q \leq \infty\) and \(0 < \beta < 1\), \([f^\beta]_{L^p(\mathbb{R}^N)} = [f]_{L^p(\mathbb{R}^N)}^\beta\).

If instead of (5.1), we use
\[
(5.2) \quad \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + s}} \leq 2^{1/p} \left[\left(\frac{|u(x) - u(y)|}{|x - y|^{N+1}}\right)^{1-\theta}\right]^{\frac{1}{L^p(\mathbb{R}^N)}} \left[\left(\frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + 1}}\right)^{\theta}\right]^{\frac{1}{L^p(\mathbb{R}^N)}}
\]
then we obtain (1.15) with \(C = C(N)\) independent of \(p_1\) and \(\theta\).

The optimality of Theorem 5 follows from

Lemma 5.1. Fix \(s_1 \in (0,1)\), \(p_1 \in (1,\infty)\) such that \(s_1 p_1 \geq 1\) and \(\theta \in (0,1)\). Let \(0 < s < 1\) and \(1 < p < \infty\) be defined by (1.12). If
\[
\frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + s}} \leq C\|u\|_{W^{s,p_1}}^{\theta} \|\nabla u\|_{L^1(\mathbb{R}^N)}^{1-\theta}, \quad \forall u \in C_0^\infty(\mathbb{R}^N)
\]
holds for some \(1 \leq q \leq \infty\), then \(q \geq \frac{p_1}{p}\).
**Proof of Lemma 5.1 when** $s_1 p_1 = 1$. We concentrate on the case $N = 1$, the case $N > 1$ being similar. Following [5, Proof of Lemma 4.1, Step 1], we define the function

$$u_k(x) = \varphi(k(|x| - 1/2)).$$

where $\varphi \in C^1(\mathbb{R})$, $\varphi = 1$ on $(-\infty, -1]$ and $\varphi = 0$ on $[1, \infty]$. We have as in [5]

$$\|u'_k\|_{L^1(\mathbb{R})} \leq C \quad \text{and} \quad |u_k|_{W^{s_1, p_1}(\mathbb{R})} \leq C(\log k)^{\frac{1}{p_1}}. \quad (5.3)$$

Given $\lambda > 0$, we have since $sp = 1$,

$$\left\{(x, y) \in [-1, 1] \times [-1, 1] : \frac{|u_k(x) - u_k(y)|}{|x - y|^{\frac{4}{p} + s}} \geq \lambda \right\} \supseteq \{(x, y) \in [0, \frac{1}{2} - \frac{1}{k}] \times [\frac{1}{2} + \frac{1}{k}, 1] : |x - y| \leq \lambda^{-p/2}\}. \quad \text{Hence, there is } c > 0 \text{ such that if } \lambda \leq (k/4)^{2/p},$$

$$L^2(E_\lambda) \geq \frac{c}{\lambda^p}. \quad \text{It follows from (1.4) that}$$

$$\left[\frac{u_k(x) - u_k(y)}{|x - y|^{\frac{4}{p} + s}}\right]_{L^p(\mathbb{R} \times \mathbb{R})} \geq \left(\int_1^{(k/4)^{2/p}} \frac{c d\lambda}{\lambda}\right)^{\frac{1}{2}} \geq c'(\log k)^{\frac{1}{p}}. \quad \text{By assumption and by (5.3), we have}$$

$$(\log k)^{\frac{1}{q}} \leq C(\log k)^{\frac{p}{p_1}}, \quad \square$$

**Proof of Lemma 5.1 for** $s_1 p_1 > 1$. We concentrate on the case $N = 1$, the case $N > 1$ being similar. We adapt the proof from [5, Proof of Lemma 4.1], where functions $w_j$ are constructed (in Step 2 there, with $\alpha := (s_1 - \frac{1}{p_1})/(1 - \frac{1}{p_1}) = (s - \frac{1}{p})/(1 - \frac{1}{p})$) and satisfy

$$\|(w_j^k)'\|_{L^1([0,1])} = 1, \quad \limsup_{k \to \infty} |w_j^k|_{W^{s_1, p_1}([0,1])} \leq Cj^{1/p_1} \quad (5.4)$$

and

$$\limsup_{k \to \infty} |w_j^k|_{W^{s, p}([0,1])} \geq \frac{j^{1/p}}{C}. \quad (5.5)$$

We improve (5.5) to cover the case $q \neq p$ in the Lorentz scale $L^{p,q}$.

Given $\lambda > 0$, we have

$$\left\{(x, y) \in [0, 1] \times [0, 1] : \frac{|w_j^k(x) - w_j^k(y)|}{|x - y|^{\frac{1}{p} + s}} \geq \lambda \right\} \supseteq \{(x, y) \in [0, 1] \times [0, 1] : |w_j^k(x) - w_j^k(y)| \geq \lambda\}, \quad \text{and thus if } \lambda \leq \frac{1}{q}, \text{ we have}$$

$$L^2\left(\left\{(x, y) \in [0, 1] \times [0, 1] : \frac{|w_j^k(x) - w_j^k(y)|}{|x - y|^{\frac{1}{p} + s}} \geq \lambda\right\}\right) \geq c$$
for some constant $c > 0$.

Next by the inductive definition of $w_k^j$ and by scaling, we have

$$
\mathcal{L}^2 \left( \left\{ (x, y) \in [0, 1] \times [0, 1] : \frac{|w_k^j(x) - w_k^j(y)|}{|x - y|^{\frac{1}{p} + s}} \geq \lambda \right\} \right) 
\geq \sum_{k=1}^j \mathcal{L}^2 \left( \left\{ (x, y) \in I_k^j \times I_k^j : \frac{|w_k^j(x) - w_k^j(y)|}{|x - y|^{\frac{1}{p} + s}} \geq \lambda \right\} \right) 
\geq \frac{1}{k^{\frac{1}{p} - 1}} \mathcal{L}^2 \left( \left\{ (x, y) \in [0, 1] \times [0, 1] : \frac{|w_k^j(x) - w_{k-1}^j(y)|}{|x - y|^{\frac{1}{p} + s}} \geq \frac{\lambda}{k^{\frac{1}{p} - 1}} \right\} \right),
$$

where we used $\alpha = \frac{s - \frac{1}{p}}{1 - \frac{1}{p}}$ to obtain

$$
\frac{1}{p} + s - 1 = 2 \frac{1}{p} + \frac{s - \frac{1}{p}}{\alpha} - 1 = \frac{1}{p} \left( \frac{2}{\alpha} - 1 \right).
$$

By induction, for each $i \in \{1, \ldots, j\}$ and $\lambda \leq k^{\frac{i-1}{p} \left( \frac{1}{p} - 1 \right)}/3$, we have

$$
\mathcal{L}^2 \left( \left\{ (x, y) \in [0, 1] \times [0, 1] : \frac{|w_k^j(x) - w_k^j(y)|}{|x - y|^{\frac{1}{p} + s}} \geq \lambda \right\} \right) \geq \frac{c}{k^{(i-1) \left( \frac{1}{p} - 1 \right)}}.
$$

We finally estimate in view of (1.4)

$$(6.2) \quad \left[ \frac{w_k^j(x) - w_k^j(y)}{|x - y|^{\frac{1}{p} + s}} \right]_{L^{p,q}([-1,1] \times [-1,1])} \geq C \left( \sum_{k=1}^j \int_{I_k^j} \frac{\lambda^{q-1}}{k^{\frac{i-1}{p} \left( \frac{1}{p} - 1 \right) / 3}} \frac{d\lambda}{k^{\frac{1}{p} - 1}} \right)^{1/q} \geq c^n j^{1/q}.
$$

The conclusion follows from the assumptions combined with the estimates (5.4) and (5.6).

6. PROOF OF THEOREM 6

We may always extend $u$ to $B_3$ with control of norms and assume that $\|u\|_{L^\infty(B_1)} = 1$. By the Sobolev–Morrey embedding we have (since $q > N$)

$$
\|u(x) - u(y)\| \leq C \min \{1, |x - y|^{\alpha} \|\nabla u\|_{L^q(B_1)}\}, \quad \text{for all } x, y \in B_1,
$$

where $\alpha = 1 - \frac{N}{q}$. Thus

$$
|u(x) - u(y)| \leq C \min \{1, |x - y|^{\alpha} \|\nabla u\|_{L^q(B_1)}^{-1}(\|\nabla u\|_{L^q(B_1)}^{-1}) \} |u(x) - u(y)|
$$

and therefore

$$
\left(6.2\right) \quad \int_{B_1 \times B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+1}} \, dx \, dy \leq C \int_{B_2} dh \int_{B_1} \min \{1, |h|^{\alpha} \|\nabla u\|_{L^q(B_1)}^{-1}(\|\nabla u\|_{L^q(B_1)}^{-1}) \} |u(x + h) - u(x)| \, dx.
$$
Since
\[
\int_{B_1} |u(x+h) - u(x)| \, dx \leq |h| \|\nabla u\|_{L^1(B_2)} \leq C|h| \|\nabla u\|_{L^1(B_1)}, \quad \text{for all } h \in B_2,
\]
we conclude that
\begin{equation}
\int_{B_1 \times B_1} \frac{|u(x) - u(y)|^p}{|x-y|^{N+1}} \, dx \, dy \leq C \|\nabla u\|_{L^1(B_1)} \int_{B_2} \min\{1, |h|^{\alpha(p-1)} \|\nabla u\|_{L^1(B_1)}^{p-1}\} \frac{dh}{|h|^N} 
\end{equation}
(6.3)
\[
= C \|\nabla u\|_{L^1(B_1)} \int_0^2 \min\{1, r^{\alpha(p-1)} \|\nabla u\|_{L^1(B_1)}^{p-1}\} \frac{dr}{r}
\]
and the conclusion follows from a straightforward computation. \qed

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