Mean Absolute Deviation: Analysis and Applications

K. M.T. Elsayed

Department of Education, National Institute of Educational Research, Cairo, Egypt

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Abstract: One way analysis of mean absolute deviation (ANOMAD) about mean and median is derived where the total sum of absolute deviation is partition into exact between sum of absolute deviation and within sum of absolute deviation. Because of the presence of absolute function in mean absolute deviation, the middle term does not exist where it is included in each term. Therefore, the exact partitions are derived using the idea of creating error terms from the same type and take it away from each term to obtain the exact partitions. ANOMAD has advantages: does not square data and offers meaningful measure of dispersion. However, the variance-gamma distribution is used to fit the sampling distributions of between sum of absolute and within sum of absolute. Consequently, two tests of equal medians and means are introduced under the assumption of Laplace distribution. Moreover, two measures of effect sizes are re-defined and studied in terms of ANOMAD.

Keywords: ANOVA, Effect sizes, Laplace distribution, MAD, Variance-gamma distribution.

1. INTRODUCTION

Mean absolute deviation (MAD) is a natural scale parameter of the Laplace distribution and offers a direct and meaningful measure of the dispersion of a random variable. It has been used as an alternative to the standard deviation where the standard deviation is motivated from optimality results in independent random sampling from the normal distribution, an analysis dating back to Fisher; see, [18]. However, the mean absolute deviation may be more appropriate in case of departures from normality or in presence of outliers; see, [19] and [9]. It may also offer certain pedagogical advantages; see, [1] and [7]. For extensive discussion and comparisons; see [15] and [6] and the references therein.

The population of MAD about mean and median is defined as

\[ \Delta_\mu = E|Y - \mu| \]

and

\[ \Delta_\nu = E|Y - \nu| \]

where \( \mu = E(Y) \) and \( \nu = Med(Y) \) are the population mean and median, respectively.

The sample MAD is defined as

\[ d_x = \frac{1}{n} \sum_{i=1}^{n} |Y_i - \bar{Y}| \quad \text{and} \quad d_x = \frac{1}{n} \sum_{i=1}^{n} |Y_i - \bar{Y}| \]

where \( \bar{Y} \) and \( \bar{Y} \) are the sample mean and median, respectively.

A random variable has a Laplace distribution with location parameter \( -\infty < \mu < \infty \) and scale \( \Delta > 0 \) if its probability density function is

\[ f(y) = \frac{1}{2\Delta} e^{-\frac{|y-\mu|}{\Delta}}, \quad -\infty < y < \infty \]

The Laplace distribution has

\[ E(Y) = \mu, \quad V(Y) = 2\Delta^2 \quad \text{and} \quad MAD(Y) = \Delta \]

The probability density function of the Laplace distribution is also reminiscent of the normal distribution; whereas the normal distribution is expressed in terms of the squared difference from the mean while the Laplace density is expressed in terms of the absolute difference from the mean or median. Consequently, the Laplace distribution has flatter tails than the normal distribution and recently applied to many fields, engineering, financial, inventory management and quality control; see, [10]. Because of the presence of absolute function in MAD, the middle term does not exist where it is included inside each term.
Therefore, an exact partition of the total sum of absolute deviation (TSA) about mean and median into exact between sum of absolute (EBSA) and exact within sum of absolute (EWSA) is derived by using the idea of creating error term of the same type (between and within) and take it away from each term to obtain EBSA and EWSA.

Since the MAD is a natural parameter of the Laplace distribution the sampling distributions of the EBSA and EWSA are studied under the assumption of Laplace distribution using variance-gamma distribution (generalized Laplace distribution). Consequently, an analysis of mean absolute deviation (ANOMAD) is introduced to test for equal population means and medians. Moreover, two measures of effect sizes are re-expressed and studied in terms of TSA, EBSA and EWSA.

Representation of MAD as a covariance is presented in Section 2. The exact partitions of TSA into EBSA and EWSA are derived in Section 3. The sampling distributions of EBSA and EWSA, and tests for equal means and median are presented in Section 4. Section 5 is devoted to conclusion.

2. **REPRESENTATION OF MAD AS A COVARIANCE**

Let \( Y_1, Y_2, \ldots, Y_n \) be a random sample from a continuous distribution with density function \( f(y) \), quantile function \( y(F) = F^{-1}(y) = Q(F) \), \( 0 < F < 1 \), cumulative distribution function \( F(y) = F \). Let over indicator functions are

\[
I_{y}(Y, \mu) = I_{y>\mu} = \begin{cases} 1, & Y > \mu \\ 0, & \text{else} \end{cases}
\]

and

\[
I_{v}(Y, \nu) = I_{v>\nu} = \begin{cases} 1, & Y > \nu \\ 0, & \text{else} \end{cases}
\]

[8] and [4] used the general dispersion function that defined by [14] as

\[
\Delta_y(a) = E|Y - a| = a[2F_y(a) - 1] + E(Y)
\]

and

\[
\Delta_y(\mu) = D_\mu = E|Y - \mu| = E\{2[I_{y>\mu} - E(I_{y>\mu})]Y\}
\]

Consequently, the population MAD is re-defined as twice the covariance between the random variable \( Y \) and its indicator function as

\[
\Delta_\mu = 2\left[E(YI_{y>\mu}) - E(Y)E(I_{y>\mu})\right] = 2Cov(Y, I_{y>\mu}),
\]

and

\[
\Delta_\nu = 2E(YI_{y>\nu}) - 2E(Y)E(I_{y>\nu}) = 2Cov(Y, I_{y>\nu})
\]

Note that \( E(I_{y>\mu}) = F(\mu), \quad V(I_{y>\mu}) = F(\mu)(1 - F(\mu)) \), \( E(I_{y>\nu}) = 0.5 \) and \( V(I_{y>\nu}) = 0.25 \).

3. **EXACT MAD PARTITIONS ABOUT MEAN AND MEDIAN**

Assume there are \( G \) different groups with individuals in each group \( y_{gi}, i = 1, \ldots, n_g \), \( n = n_1 + \ldots + n_G \) and \( g = 1, \ldots, G \). Let \( y_{gi} - \bar{y} \) is the total deviation \( \bar{y} = \sum_G n_G y_{gi}/n \), \( \bar{y} \) is the deviation of grouped mean \( \bar{y}_g = \sum_{i=1}^{n_g} y_{gi}/n_g \) around total mean, and \( y_{gi} - \bar{y}_g \) is the deviation of individuals around the grouped mean. By squaring and taking the summation over both \( g \) and \( i \) then

\[
\sum_{g=1}^{G} \sum_{i=1}^{n_g} (y_{gi} - \bar{y})^2 = \sum_{g=1}^{G} \sum_{i=1}^{n_g} (\bar{y}_g - \bar{y})^2 + \sum_{g=1}^{G} \sum_{i=1}^{n_g} (y_{gi} - \bar{y}_g)^2
\]

That is known as the partitions the total sum of squares, in other words, the total sum of squares (TSS) is equal to the between sum of squares (BSS) plus the within sum of square (WSS), therefore

\[
TSS = BSS + WSS
\]

By using the absolute value instead of the square it is known that

\[
|y_{gi} - \bar{y}| \leq |\bar{y}_g - \bar{y}| + |y_{gi} - \bar{y}_g|
\]

By taking the summation over both \( g \) and \( i \) then

\[
\sum_{g=1}^{G} \sum_{i=1}^{n_g} |y_{gi} - \bar{y}_g| \leq \sum_{g=1}^{G} \sum_{i=1}^{n_g} |\bar{y}_g - \bar{y}| + \sum_{g=1}^{G} \sum_{i=1}^{n_g} |y_{gi} - \bar{y}_g|
\]

The total sum of absolute (TSA = \( \sum_{g=1}^{G} \sum_{i=1}^{n_g} |y_{gi} - \bar{y}_g| \)) is less than or equal to between sum of absolute (BSA = \( \sum_{g=1}^{G} \sum_{i=1}^{n_g} |\bar{y}_g - \bar{y}| \)) plus within sum of absolute (WSA = \( \sum_{g=1}^{G} \sum_{i=1}^{n_g} |y_{gi} - \bar{y}_g| \)), therefore.
To obtain equality this can be re-written as

\[ TSA = BSA + WSA + R \]

where \( R \leq 0 \) and \( R = TSA - BSA - WSA \) is the residuals.

These residuals could be separated to pure within-groups residuals and pure between-groups residuals and added to each term to obtain exact MAD partitions

\[ TSA = (BSA + R_0) + (WSA + R_0) = EBSA + EWSA \]

This is shown in the following theorem.

Theorem 1

The MAD partitions about mean into the exact between sum of absolute and the exact within sum of absolute are

\[
\sum_{g=1}^{G} \sum_{i=1}^{n_g} |y_{gi} - \bar{y}_g| = \sum_{g=1}^{G} \sum_{i=1}^{n_g} |\bar{y}_g - \bar{y}|_δ + \sum_{g=1}^{G} \sum_{i=1}^{n_g} |y_{gi} - \bar{y}_g|_δ \quad (5)
\]

where

\[
|\bar{y}_g - \bar{y}|_δ = \begin{cases} 
|\bar{y}_g - \bar{y}| & \text{if } y_{gi} \leq \bar{y} < \bar{y}_g \\
|\bar{y}_g - \bar{y}| & \text{else}
\end{cases}
\]

and

\[
|y_{gi} - \bar{y}_g|_δ = \begin{cases} 
|y_{gi} - \bar{y}_g| & \text{if } \bar{y}_g < y_{gi} \leq \bar{y} \text{ or } \bar{y} < y_{gi} \leq \bar{y}_g \\
|\bar{y}_g - y_{gi}| & \text{else}
\end{cases}
\]

Note that \(|\bar{y}_g - \bar{y}|_δ\) is the exact between absolute and \(|y_{gi} - \bar{y}_g|_δ\) is the exact within absolute, then for mean

\[ TSA_M = EBSA_M + EWSA_M \]

**Proof:**

Using the covariance representation of the mean absolute deviation about mean given in (4), the partitions can be written as

\[ TSA_M = \sum_{g=1}^{G} \sum_{i=1}^{n_g} |y_{gi} - \bar{y}_g| = 2Cov(y_{gi}, \bar{y}_g) = 2E[(y_{gi} - \bar{y})\bar{y}_g] = 2\sum_{g=1}^{G} \sum_{i=1}^{n_g} t_{gi} \]

\[ t_{gi} = \begin{cases} 
\bar{y}_g - \bar{y}, & y_{gi} > \bar{y} \\
0, & \text{else}
\end{cases} \]

the \( BSA_M = \sum_{g=1}^{G} \sum_{i=1}^{n_g} |\bar{y}_g - \bar{y}| = 2Cov(\bar{y}_g, \bar{y}) = 2\sum_{g=1}^{G} \sum_{i=1}^{n_g} b_{gi} \]

\[ b_{gi} = \begin{cases} 
\bar{y}_g - \bar{y}, & \bar{y}_g > \bar{y} \\
0, & \text{else}
\end{cases} \]

and the

\[ WSA_M = \sum_{g=1}^{G} \sum_{i=1}^{n_g} |y_{gi} - \bar{y}_g| = 2\sum_{g=1}^{G} \sum_{i=1}^{n_g} w_{gi} \]

\[ w_{gi} = \begin{cases} 
\bar{y}_g - \bar{y}, & y_{gi} > \bar{y}_g \\
0, & \text{else}
\end{cases} \]

Let the residuals \( R = \bar{y}_g - b_g - w_{gi} \), we have the following cases:

1. If \( w = 0, b > 0, t = 0 \), we have \( R = -b_g = -(\bar{y}_g - \bar{y}) \) then \( y_{gi} \leq (\bar{y}_g, \bar{y}) \), \( \bar{y}_g > \bar{y} \). this imply \( y_{gi} \leq \bar{y} < \bar{y}_g \). Similarly, if \( w > 0, b = 0 \), \( t > 0 \), we have \( R = t_{gi} - w_{gi} = 0 = \bar{y} - \bar{y} \), this imply \( y_{gi} > \bar{y} \geq \bar{y}_g \). These residuals must be excluded from BSA to obtain PBSA.

2. Similarly, if \( w > 0, b = 0, t = 0 \), we have \( R = -w_{gi} = -(y_{gi} - \bar{y}_g) \) this imply \( \bar{y} \geq y_{gi} > \bar{y}_g \). If \( w = 0, b > 0, t > 0 \), we have \( R = t_{gi} - b_g = (y_{gi} - \bar{y}_g) \), this imply \( \bar{y} < y_{gi} \leq \bar{y}_g \). These residuals must be excluded from WSA to obtain PWSA.

3. If \( w = 0, b = 0, t = 0 \), we have \( R = 0 \) then \( y_{gi} \leq \bar{y}_g \leq \bar{y} \), If \( w > 0, b > 0, t > 0 \), we have \( R = 0 \) then \( y_{gi} > \bar{y} \geq \bar{y}_g \).

4. Two impossible cases when \( w > 0, b > 0 \) and \( t = 0 \) and \( w = 0, b = 0 \) and \( t > 0 \).

Note that the residuals are zeros when the group mean is between the overall mean and the data.

A. **Explanation of the subtracting the error term**

In the analysis of variance the total sum of squares can be written as

\[
\sum_i \sum_g [(Y_{ig} - \hat{\theta}) + (\hat{\theta} - \theta)]^2
\]

\[
= \sum_i \sum_g (Y_{ig} - \hat{\theta})^2 + 2\sum_i \sum_g (Y_{ig} - \hat{\theta})(\hat{\theta} - \theta) + \sum_i \sum_g (\hat{\theta} - \theta)^2
\]

In the case of \( \hat{\theta} = \bar{y}_g \) and \( \theta = \bar{y} \) the middle term is zero and the exact partitions are obtained directly. On contrast, with respect to
\[ TSA = \sum_{i} \sum_{g} |(Y_{gi} - \hat{\theta}) + (\hat{\theta} - \theta)| \]

Because of the presence of absolute function the middle term does not exist where it is included inside each term. To obtain the exact partitions it is necessary to create error term from the same type and subtract it from each term.

**Illustrative example**

To have an idea on how the method work. Table 1 shows MAD partitions about mean for a hypothetical data. Note that \( TSA = 38 \), while \( BSA + WSA = 36 + 18 = 54 \) that overestimate TSA by \( 54 - 38 = 16 \). On the other hand \( EBSA_M + EWSA_M = 32 + 6 = 38 \) that is equal exactly to \( TSA \). Also, TSA is equal to EBSA plus EWBSA for each value.

| Table 1 | \( EBSA_M \) and \( EWSA_M \) for a hypothetical data |
|---------|-----------------|
| \( y \) | \( i \) | \( y_{gi} \) | \( y_i \) | \( \bar{y} \) | \( \bar{y}_i \) | \( \bar{y}_i \) | \( \bar{y}_i \) | \( \bar{y}_i \) |
| 1 | 1 | 4 | 2 | 2 | 0 | 2 | 0 |
| 2 | 10 | 9 | 5 | 9 | -5 |
| 3 | 2 | 4 | 0 | 4 | 0 |
| 4 | 3 | 5 | 4 | 1 | 4 |
| Total | 48 | 38 | 36 | 18 | 32 | 6 |

The MAD partitions about median can be shown in the same manner as MAD partitions about mean in the following theorem.

**Theorem 2**

The pure partitions of the total sum of absolute about median into the between sum of absolute and the within sum of absolute are

\[ \sum_{g=1}^{G} \sum_{i=1}^{n_g} |y_{gi} - \bar{y}| = \sum_{g=1}^{G} \sum_{i=1}^{n_g} |\bar{y}_i - \bar{y}| \]

\[ + \sum_{g=1}^{G} \sum_{i=1}^{n_g} |y_{gi} - \bar{y}_g| \]

where

\[ |\bar{y}_g - \bar{y}| \]

\[ = \begin{cases} 
-|\bar{y}_g - \bar{y}| & \text{if } y_{gi} < \bar{y}_g \text{ or } y_{gi} > \bar{y} \geq \bar{y}_g \\
|\bar{y}_g - \bar{y}| & \text{else} 
\end{cases} \]

and

\[ |y_{gi} - \bar{y}_g| \]

\[ = \begin{cases} 
-|y_{gi} - \bar{y}_g| & \text{if } y_{gi} < \bar{y}_g \leq \bar{y} \text{ or } \bar{y}_g \geq y_{gi} > \bar{y} \\
|y_{gi} - \bar{y}_g| & \text{else} 
\end{cases} \]

Note that \( |\bar{y}_g - \bar{y}| \) is the exact between absolute deviation about median and \( |y_{gi} - \bar{y}_g| \) is the exact within absolute deviation about median, i.e.

\[ TSA_{med} = EBSA_{med} + PWSA_{med} \]

**Proof:** similarly as given in the mean.

4. **Applications**

The one way analysis of mean absolute deviation (ANOMAD) is introduced and used to test for equal population means and medians under the following assumptions.

1. The observations are random and independent samples from the populations.
2. The distributions of the populations from which the samples are selected are Laplace distribution.
3. The MADs of the distributions in the populations are equal.

It is difficult to obtain the exact sampling distributions for EBSA and EWSA, therefore, a general distribution is chosen to fit these sampling distributions. One of the families that connected to Laplace distribution is the variance-gamma distribution or also known as generalized Laplace distribution; see, [10]. More recently, the variance gamma model became popular among some financial modellers, due to its simplicity, flexibility, and an excellent fit to empirical data; [12], [13] and [11]. The variance-gamma distribution will be used to fit the sampling distributions of EBSA and EWSA via method of moments where the patterns between the moments of the data and the distribution will be equated.

A. **Fitting sampling distributions**

The random variable \( Y \) is said to have Variance-Gamma (VG) with parameters \( c, \theta \in R, \sigma, \sigma > 0 \), if it has probability density function given by

\[ f(y; c, \theta; \sigma) = \frac{2e^{-\frac{(y-c)^2}{\sigma^2}}}{\sigma \sqrt{2\pi \sigma^2}} \left( \frac{1}{\sigma} \right) \right)^{\frac{1}{2}} \right] \left( \frac{1}{\sigma^2} \right) \right)^{\frac{1}{2}} \right] K_{v} \left( \sqrt{\frac{2\sigma^2 + \theta^2}{\sigma^2}} \right) \]

\[ y \in R \]

Where \( K_{v}(x) \) is a modified Bessel function of the third kind; see, for example, [17] and [5].
Note that there are other versions of this distribution available but this version is chosen because there is a software package in R called gamma-variance based on this version that be used to obtain all the simulations and graphs. The moments of this distribution are

\[
E(Y) = c + \theta, \\
V(Y) = \sigma^2 + v\theta^2, \\
\text{sk} = \frac{2\theta^2\nu^2 + 3\sigma^2\theta\nu}{\sqrt{(\theta^2\nu + \sigma^2)^3}},
\]

and

\[
k\nu = 3 + \frac{3\sigma^4\nu + 12\sigma^2\theta^2\nu^2 + 6\theta^4\nu^3}{(\theta^2\nu + \sigma^2)^2}
\]

This distribution is defined over the real line and has many distributions as special cases or limiting distributions such as Gamma distribution in the limit \(\sigma \to 0\) and \(c = 0\), Laplace distribution as \(\theta = 0\) and \(v = 2\) and normal distribution as \(\theta = 0, v = 1/r\) and \(r \to \infty\).

Note that if \(a > 0\) then

\[aY \sim VG(ac, a\sigma, a\theta, v)\]

The Gamma distribution for the random variable \(Y\) is defined as

\[
f(y; k, \omega) = \frac{1}{\Gamma(k)\omega^k} y^{k-1} e^{-\frac{y}{\omega}}, \ y > 0, k, \omega > 0
\]

where \(k\) and \(\omega\) are the shape and scale parameters and the moments are

\[
E(Y) = k\nu, \ V(Y) = k\omega^2, \ sk = \frac{2}{k} \quad \text{and} \quad k\nu = 3 + \frac{6}{k}
\]

1) **MAD about median**

The scaled exact between sums of absolute can be written as

\[
U_1 = \frac{EBSA_{med}}{\Delta} = \frac{\sum_{g=1}^{G} \sum_{i=1}^{n_g} |\bar{y}_g - \bar{y}|_g}{\Delta}
\]

Since \(EBSA_{med}\) depends on one parameter \(G\), the moments of \(EBSA_{med}\) is used to fit the sampling distribution of \(U_1\) based on VG distribution. For this purpose a simulation study is conducted to obtain the first four moments for \(U_1\) based on simulated data from Laplace distribution \((\mu, \Delta)\) with different values of \(n\) and \(G\). Table 2 gives the simulated first four moments of \(U_1\).

From Table 2 it is noted that there is a pattern between the mean and the variance for different \(n\) and \(G\) values where the mean is approximately \(G - 1\) and the variance is twice the mean \((2(G - 1))\) whatever the values of \(n\). Therefore VG distribution is used to fit the sampling distribution of \(U_1\) as

\[
U_1 = \frac{EBSA_{med}}{\beta} \approx VG\left(c = 0, \sigma \downarrow 0, \theta = (G - 1)/\nu = \frac{2}{(G - 1)}\right) \cong \Gamma\left(\frac{G - 1}{2}, \frac{2}{G - 1}\right)
\]

The moments can be computed as

\[
E(U_1) = (G - 1), \quad V(U_1) = 2(G - 1),
\]

\[
\text{sk} = \frac{4}{\sqrt{2(G - 1)}} \quad \text{and} \quad k\nu = 3 + \frac{6}{G - 1}
\]

Hence,

\[
T_1 = \frac{EBSA_{med}}{(G - 1)\beta} \approx VG\left(c = 0, \sigma \downarrow 0, \theta = 1, \nu = \frac{2}{(G - 1)}\right) \cong \Gamma\left(\frac{G - 1}{2}, \frac{2}{G - 1}\right)
\]

The moments are

\[
E(T_1) = 1, \quad V(T_1) = \frac{2}{(G - 1)},
\]

\[
\text{sk} = \frac{4}{\sqrt{2(G - 1)}} \quad \text{and} \quad k\nu = 3 + \frac{6}{G - 1}
\]
Table 2 simulated mean, variance, skewness and kurtosis for $U_1$ and $U_2$ with different values of $G$ and $n$ from Laplace distribution ($\mu$, $\Delta$) and the number of replications is 10000.

| $n$ | $G$ | $\text{mean}$ | $\text{var}$ | $sk$ | $ku$ | $\text{mean}$ | $\text{var}$ | $sk$ | $ku$ |
|-----|-----|---------------|--------------|------|-----|--------------|--------------|------|-----|
| 30  | 3   | 2.056         | 4.017        | 1.870| 8.169| 27.383       | 29.710       | 0.397| 3.279|
| 50  | 5   | 4.075         | 7.975        | 1.410| 6.389| 45.249       | 49.622       | 0.270| 3.185|
| 80  | 8   | 7.019         | 14.07        | 1.195| 6.110| 72.372       | 80.508       | 0.253| 3.134|
| 100 | 10  | 9.045         | 17.73        | 0.903| 4.086| 90.325       | 100.186      | 0.239| 3.082|
| 150 | 15  | 14.002        | 27.85        | 0.740| 3.885| 135.530      | 146.248      | 0.124| 3.045|
| 45  | 3   | 2.059         | 4.042        | 1.961| 8.451| 42.325       | 44.427       | 0.289| 3.176|
| 75  | 5   | 4.072         | 8.111        | 1.415| 6.088| 70.292       | 74.770       | 0.274| 3.114|
| 120 | 8   | 7.053         | 13.96        | 1.072| 4.671| 112.391      | 120.144      | 0.171| 3.083|
| 150 | 10  | 9.016         | 17.88        | 0.950| 4.424| 140.246      | 150.247      | 0.127| 3.075|
| 225 | 15  | 14.03         | 28.01        | 0.762| 3.748| 210.147      | 225.218      | 0.108| 3.056|
| 75  | 3   | 2.03          | 3.997        | 1.982| 8.965| 72.399       | 75.067       | 0.241| 3.095|
| 125 | 5   | 4.08          | 8.250        | 1.478| 6.431| 120.384      | 125.345      | 0.173| 3.058|
| 200 | 8   | 7.05          | 14.130       | 1.038| 4.582| 192.430      | 201.591      | 0.090| 3.048|
| 250 | 10  | 9.02          | 17.796       | 0.925| 4.264| 240.365      | 250.939      | 0.155| 3.037|
| 375 | 15  | 14.05         | 27.973       | 0.784| 3.888| 360.229      | 384.630      | 0.118| 3.021|

Moreover, Figure 1 shows the histogram of $U_1$ based on simulated data from Laplace distribution with fitting VG superimposed. The VG clearly gives a very good fit to $U_1$ for different values of $G$ and $n$.

The exact within sums of absolute can be written as

$$U_2 = \frac{\text{EWSA}_{med}}{\Delta} = \frac{\sum_{g=1}^{G} \sum_{i=1}^{n} |y_{gi} - \bar{y}_g|}{\Delta}$$

Since $\text{EWSA}_{med}$ depends on two parameters $G$ and $n$, the moments of $\text{EWSA}_{med}$ is used to fit the sampling distribution of $U_2$ based on VG distribution. For this purpose a simulation study is conducted to obtain the first four moments for $U_2$ based on simulated data from Laplace distribution ($\mu$, $\Delta$) with different values of $n$ and $G$. Table 2 gives the simulated first four moments of $U_2$ and it is noted that as expected there is a pattern between the mean and the variance for different $n$ and $G$ values where the mean is approximately $n - G$ and the variance is twice the mean plus $G$. Consequently VG distribution is used to fit the sampling distribution of $U_2$ as

$$U_2 = \frac{\text{EWSA}_{med}}{\beta} \approx \text{VG}(c = 0, \sigma = \sqrt{G}, \theta = (n - G), v = \frac{1}{(n - G)})$$

with moments

$$E(U_2) = n - G, \quad V(U_2) = n, \quad sk = \frac{2n + G}{\sqrt{n^3}} \quad \text{and} \quad ku = 3 + \frac{3g^2 + 6g + 6n}{n^2}$$

Figure 1 histogram of $U_1$ based on simulated data from Laplace distribution with VG distribution superimposed and (a) $G = 8$ and $n = 160$ (b) $G = 6$ and $n = 120$ (c) $G = 5$ and $n = 100$ and (d) $G = 3$ and $n = 60$.  

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Therefore,
\[
T_2 = \frac{PWSA_{med}}{(n - G)\beta} \approx VG\left(c = 0, \sigma = \frac{\sqrt{G}}{n - G}, \theta = 1, \nu \right) = \frac{1}{(n - G)}
\]
The moments are
\[
E(T_2) = 1, \quad V(T_2) = \frac{n}{(n - G)^2}, \quad sk = \frac{2n + G}{\sqrt{n}} \quad \text{and} \quad ku = 3 + \frac{3G^2 + 6G + 6n}{n^2}
\]
Moreover, Figure 2 shows the histogram of \(U_2\) based on simulated data from Laplace distribution with fitting VG superimposed. The VG clearly gives a very good fit to \(U_2\) for different values of \(G\) and \(n\).

Figure 2 histogram of \(U_2\) based on simulated data from Laplace distribution with VG distribution superimposed (a) \(G = 6, n = 48\), (b) \(G = 5, n = 40\), (c) \(G = 4, n = 32\), and (d) \(G = 3, n = 24\).

The ratio of mean EBSA to mean EWSA can be expressed as
\[
R_{med} = \frac{T_1}{T_2} = \frac{EBSA_{med}}{(G - 1)EWSA} = \frac{(n - G)EBSA}{(n - G)EWSA}
\]
An approximation is obtained based on gamma-variance distribution by using
\[
E(R_{med}) \approx \frac{E(T_1)}{E(T_2)}
\]
and
\[
V(R_{med}) \approx \frac{V(T_2)E^2(T_1)}{E^4(T_2)} + \frac{V(T_1)}{E^2(T_2)} = \frac{n}{(n - G)^2} + \frac{2}{(G - 1)}
\]
Therefore,
\[
R_{med} \approx VG\left(c = 0, \sigma = \frac{\sqrt{n}}{n - G}, \theta = 1, \nu = \frac{2}{(G - 1)} \right)
\]

2) MAD about mean
The scaled EBSA deviation about mean can be written as
\[
U_3 = \frac{EBSA_M}{\Delta} = \frac{\sum_{g=1}^{G} \sum_{i=1}^{n_g} (\bar{Y}_g - \bar{Y})}{\Delta}
\]
Since \(EBSA_M\) depends on one parameter \(G\), the moments of \(EBSA_M\) is a good choice to be used to fit the sampling distribution of \(U_3\) based on VG distribution. For this purpose a simulation study is conducted to obtain the first four moments for \(U_2\) using simulated data from Laplace distribution (\(\mu, \Delta\)) with different values of \(n\) and \(G\). Table 4 gives the simulated first four moments of \(U_3\). From Table 3 it is noted that there is a pattern between the mean and the variance for different \(n\) and \(G\) values where the mean is approximately \(G - 1\) and the variance is three times the mean \((3(G - 1))\) whatever the value of \(n\). Therefore VG distribution is used to fit the sampling distribution of \(U_3\) as
\[
U_3 = \frac{EBSA_M}{\Delta} \approx VG\left(c = 0, \sigma = \sqrt{(G - 1)}, \theta = (G - 1), \nu = \frac{2}{(G - 1)} \right)
\]
The moments can be computed as
\[
E(U_3) = (G - 1), \quad V(U_3) = 3(G - 1),
\]
\[
sk = \frac{14}{\sqrt{27(G - 1)}} \quad \text{and} \quad ku = 3 + \frac{102}{9(G - 1)}
\]
Hence,
\[ T_3 = \frac{EBSA_M}{\Delta(G - 1)} \approx VG\left(c = 0, \sigma = \frac{1}{\sqrt{G - 1}}, \theta = 1, \nu \right) = \frac{2}{(G - 1)} \]

Table 3 simulated mean, variance, skewness and kurtosis for \( U_3 \) and \( U_4 \) with different values of \( G \) and \( n \) from Laplace distribution and number of replications is 10000.

| \( n \) | \( G \) | mean | var | sk  | ku  | mean | var | sk  | ku  |
|-------|-------|------|-----|-----|-----|------|-----|-----|-----|
| 30    | 3     | 2.079| 5.980| 1.795| 7.953| 27.871| 31.752| 0.366| 3.199|
| 50    | 5     | 4.052| 11.606| 1.302| 5.534| 46.033| 55.172| 0.284| 3.118|
| 80    | 8     | 7.043| 20.706| 0.976| 4.339| 72.884| 86.579| 0.215| 3.084|
| 100   | 10    | 9.032| 26.759| 0.867| 4.271| 91.055| 110.502| 0.199| 3.061|
| 150   | 15    | 14.015| 41.891| 0.734| 3.815| 136.041| 163.125| 0.161| 3.047|
| 45    | 3     | 2.063| 5.972| 1.781| 7.901| 42.875| 48.324| 0.299| 3.130|
| 75    | 5     | 4.053| 11.752| 1.293| 5.431| 70.771| 80.412| 0.245| 3.071|
| 120   | 8     | 7.021| 20.811| 0.952| 4.353| 112.895| 128.012| 0.182| 3.049|
| 150   | 10    | 9.019| 26.522| 0.871| 4.301| 141.049| 160.125| 0.164| 3.038|
| 225   | 15    | 14.025| 41.902| 0.733| 3.791| 211.052| 238.752| 0.133| 3.018|
| 75    | 3     | 2.033| 5.941| 1.801| 7.899| 73.066| 77.432| 0.244| 3.076|
| 125   | 5     | 4.031| 11.842| 1.311| 5.432| 120.795| 128.840| 0.178| 3.054|
| 200   | 8     | 7.019| 20.811| 0.962| 4.360| 192.82| 208.176| 0.147| 3.046|
| 250   | 10    | 9.022| 26.522| 0.886| 4.331| 241.034| 258.552| 0.126| 3.037|
| 375   | 15    | 14.017| 41.902| 0.724| 3.801| 360.974| 390.05| 0.101| 3.022|

Moreover, Figure 4 shows the histogram of \( U_3 \) based on simulated data from Laplace distribution with fitting VG superimposed. The VG clearly gives a very good fit to \( U_3 \) for different values of \( G \) and \( n \).

The EWSA deviation about mean can be written as

\[ U_4 = \frac{EWSA_M}{\Delta} = \frac{\sum_{g=1}^{G} \sum_{i=1}^{n_g} |X_{gi} - \bar{X}_g|}{\Delta} \]

Since \( EWSA_M \) depends on two parameters \( G \) and \( n \), the moments of \( EWSA_M \) could be used to fit the sampling distribution of \( U_4 \) based on VG distribution. For this purpose a simulation study is conducted to obtain the first four moments for \( U_4 \) based on simulated data from Laplace distribution with different values of \( n \) and \( G \).

Table 3 gives the simulated first four moments of \( U_4 \). From Table 3 it is noted that there is a pattern between the mean and the variance for different \( n \) and \( G \) values where the mean is approximately \( n - G + 1 \) and the variance is equal to the mean plus 2(G - 1). Therefore VG distribution is used to fit the sampling distribution of \( U_4 \) as

\[ U_4 = \frac{PWSA_M}{\beta} \approx VG\left(c = 0, \sigma = \sqrt{2(G - 1)}, \theta = (n - G + 1), \nu = \frac{1}{(n - G + 1)} \right) \]

with moments.
\[ E(U_4) = n - G + 1, \quad V(U_4) = n + G - 1, \quad sk = \frac{2n + 4G - 4}{\sqrt{(n + G - 1)^3}} \]

and
\[ ku = 3 + \frac{6n^2 - 6g^2 + 12gn + 12g - 12n + 42}{(n + g + 1)(n + g - 1)^2} \]

Therefore,
\[ T_4 = \frac{PWSA_M}{(n - G + 1)\beta} \approx VG\left(c = 0, \sigma = \frac{\sqrt{2(G - 1)}}{(n - G + 1)}, \theta = 1, \nu = \frac{1}{(n - G + 1)}\right) \]

The moments are
\[ E(T_4) = 1, \quad V(T_4) = \frac{n + G - 1}{(n - G + 1)^2}, \quad sk = \frac{2n + 4G - 4}{(n + G - 1)^2} \]

and
\[ ku = 3 + \frac{6n^2 - 6g^2 + 12gn + 12g - 12n + 42}{(n + g + 1)(n + g - 1)^2} \]

Figure 4 shows the histogram of \( U_4 \) based on simulated data from Laplace distribution with fitting VG superimposed. The VG clearly gives a very good fit to \( U_4 \) for different values of \( G \) and \( n \). Note that more simulation results for different \( G \) and \( n \) are available from the author upon request.

The ratio of mean between absolute deviations to mean within absolute deviation can be expressed as
\[ R_M = \frac{T_3}{T_4} = \frac{PBSA_M}{PWSA_M \beta} = \frac{(n - G + 1)PBSA}{(G - 1)PWSA} = \frac{MBSA_M}{MWSA_M} \]

An approximation to \( R_M \) is obtained based on gamma-variance distribution by using
\[ E(R_M) \approx E(T_3) = \frac{E(T_4)}{E(T_4)} = 1 \]

and
\[ V(R_M) \approx \frac{V(T_3)E^2(T_3)}{E^4(T_4)} + \frac{V(T_3)}{E^2(T_4)} = \frac{n + G - 1}{(n - G + 1)^2} + \frac{3}{(G - 1)} \]

Hence, the approximation gamma-variance is
\[ R_M \approx VG\left(c = 0, \sigma = \frac{\sqrt{n + G - 1}}{(n - G + 1)^2} + \frac{1}{G - 1}, \theta = 1, \nu = \frac{2}{(G - 1)}\right) \]

Figure 5 histogram of \( U_4 \) based on simulated data from Laplace distribution with VG distribution superimposed and (a) \( G = 6 \) and \( n = 60 \) (b) \( G = 5 \) and \( n = 50 \) (c) \( G = 4 \) and \( n = 40 \) and (d) \( G = 3 \) and \( n = 30 \).

Figure 6 histogram of \( R_{med} \) based on simulated data from Laplace distribution with VG distribution superimposed (a) \( G = 8, n = 160 \), (b) \( G = 7 \) and \( n = 140 \), (c) \( G = 5 \) and \( n = 100 \) and (d) \( G = 3, n = 60 \).
Figure 6 shows the histogram of an approximate distribution of $R_m$. It is clearly that it gives a very good fit to $R_m$.

### B. Tests for equal medians and means

#### 1) Test for equal medians

The null hypothesis $H_0$ tested in one way ANOMAD is that the population medians from which the $G$ samples are selected are equal

$$H_0: \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_G$$

The alternatively hypothesis $H_a$ is that at least two of the group medians are significantly different. Table 4 gives a summary of ANOMAD for medians.

#### Table 4 summary ANOMAD for medians

| variation | Sum of absolute | Divisor | MAD estimate (mean absolute) | $R_{med}$ |
|-----------|----------------|---------|-----------------------------|-----------|
| Between   | $EBSA_{med}$   | $G - 1$ | $MEBSA_{med}$               | $MEBSA_{med}$ |
| Within    | $EWSA_{med}$   | $n - G$ | $MEWSA_{med}$               | $MEWSA_{med}$ |
| Total     | $TSA$          |         |                             |           |

To test for the assumption of Laplace distribution, the function `laplace.test()` in package `lawstat` in R-software is used where it gives five goodness of fit for the Laplace distribution based on the work of [16]. A one way ANOMAD is used to examine if students’ scores on a standardized test is a function of the teaching method. The independent variable represented the three different types of teaching method: 1) lecture only; 2) hands-on only; and 3) both. The dependent variable is the students’ score on a standardized test. See Table 5 for the sample data with means, medians and MAD reported for each of the three groups. To test the assumption of Laplace distribution, Kolmogorov-Smirnov (D) is used from package `lawstat` in R-software. The results for the three groups are given in Table 5 where $p$-values more than 0.01, 0.05 and 0.10, therefore, the assumption of Laplace cannot be rejected. Because the maximum MAD to minimum MAD is 1.14, the assumption of homogeneity of MAD’s cannot be rejected.

#### Table 5 students’ scores for three different methods of teaching and Laplace goodness of fit using Kolmogorov-Smirnov test

| Teaching methods | Laplace test |
|------------------|--------------|
| L.  | H.on | B | Kolmogrov-Smirnov |
| 65 | 82 | 87 | Group | p-value |
| 63 | 70 | 86 | Lect. | 0.67 |
| 78 | 70 | 78 | Lab. | 0.68 |
| 88 | 75 | 88 | Both | 0.73 |
| 59 | 37 | 72 | | |
| 71 | 70 | 98 | | |
| 70 | 74 | 95 | Lect. | 68 |
| 72 | 63 | 85 | Mean | 84 |
| 70 | 81 | 72 | Median | 71 |
| 81 | 70 | 90 | $MAD_M$ | 70 |
| 81 | 70 | 90 | $MAD_{med}$ | 6.12 |
| 72 | 68 | 75 | | 6.5 |
| 68 | 69 | 84 | | |
| 56 | 63 | 86 | | 6.4 |
| 65 | 64 | 92 | | |
| 71 | 68 | 74 | | |

#### Table 6 ANOMAD of testing equal medians for standardized test scores

| variation | Sum of absolute | Divisor | MAD estimate (mean absolute) | $R_{med}$ | $qV_{G_{0.95}}^2$ |
|-----------|----------------|---------|-----------------------------|-----------|------------------|
| Between   | 202            | 2       | 101                         | 24.82     | 3.02             |
| Within    | 171            | 42      | 4.07                        |           |                 |
| Total     | 373            |         |                             |           |                  |

*This value from Variance-Gamma package in R-software

where $R_{med} = 24.82 > Critical = qV_{G_{0.95}} = 3.02$. $H_0$ is rejected, i.e., this indicates that not all three groups of the teaching methods resulted in the same standardized test score.

#### 2) Test for equal means

The null hypothesis $H_0$ tested in one way ANOMAD is that the population means from which the $G$ samples are selected are equal

$$H_0: \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_G$$

The alternatively hypothesis $H_a$ is that at least two of the group means are significantly different. Table 4 gives summary of ANOMAD for means.
Table 7 summary ANOMAD for means

| Variation | Sum of absolute | Divisor | MAD estimate (mean absolute) | $R_M$ |
|-----------|-----------------|---------|-----------------------------|-------|
| Between $EBSA_M$ | $G - 1$ | $MEBSA_M$ | $MBSA_M$ | $MEWSA_M$ |
| Within $EWSA_M$ | $n - G + 1$ | $EWSA_M$ | $n - G + 1$ |
| Total | | | $TSA$ |

Table 8 gives ANOMAD to test equal means for standardized test scores.

Table 8 ANOMAD of testing equal means for standardized test scores

| variation | Sum of absolute | Divisor | MAD estimate (mean absolute) | $R_M$ | $qVG^2_{0.95}$ |
|-----------|-----------------|---------|-----------------------------|-------|----------------|
| Between | 197 | 2 | 98.5 | 22.6 | 3.45 |
| Within | 187 | 43 | 4.35 | |
| Total | 384 | |

*This value from Variance-Gamma package in R-software

The first type of standardized ES measure is

$$\eta^2 = \frac{S_B^2}{S_B^2 + S_W^2}$$

Where $S_B^2$ and $S_W^2$ are between group variance and within group variance; see, for example, [3]. Another measure of the strength of the association between the independent variable and the dependent variable in ANOVA is $\omega^2$ that indicates the proportion of the total variance in the dependent variable that is accounted for by the levels of the independent variable. This is analogous to the coefficients of determination $r^2$. The formula for $\omega^2$ is

$$\omega^2 = \frac{SS_B - (G - 1)MS_W}{SS_T + MS_W}$$

See, for example Cohen (1988).

These two measures are defined in terms of ANOMAD as

$$\eta_{mad} = \frac{EBSA}{EBSA + EWSA}$$

and

$$\omega_{mad} = \frac{EBSA - (G - 1)MEWSA}{TSA + MEWSA}$$

Therefore, $\eta_{mad}$ by using median is $202/373 = 0.54$ and mean is $197/384 = 0.51$ while $\omega_{mad}$ by using median is $(202 - 2(4))/(373 + 4) = 0.51$ and mean is $(196 - 2(4.4))/(384 + 4.4) = 0.48$. For example, when $\omega_{mad}$ is 0.51 this means that the independent variable in the ANOMAD accounts for 51% of the total variance in the dependent variable.

5. Conclusion

There are many areas in sciences where the normal distribution is not an appropriate approximation and the Laplace distribution provides a good approximation to the data. In these cases, when the tests of equal means and medians are needed, the ANOMAD will be appropriate. The ANOMAD was derived using the idea of creating error terms from the same type (BSA and WSA) and subtract it from each term to obtain the exact partitions.

The ANOMAD had important information about the shifts in means and medians of the groups and that studied by fitting variance-gamma distribution to EBSA and EWSA and test for equal means or medians. Because of no square in MAD, the ANOMAD ensured high stability of statistical inference. Finally, the effect sizes were re-expressed and studied in terms of ANOMAD.
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