Polynomial maps of complex plane with branched valued set isomorphic to complex line

Nguyen Van Chau∗
Hanoi Institute of Mathematics, P.O. Box 631, Boho 10000, Hanoi, Vietnam
E-mail: nvchau@thevinh.nest.ac.vn

May 10, 2001

Abstract

We present a completed list of polynomial dominanting maps of $\mathbb{C}^2$ with branched value curve isomorphic to the complex line $\mathbb{C}$, up to polynomial automorphisms.

AMS Classification : 14 E20, 14 B25.

1. Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial dominanting map, $\text{Close}(f(\mathbb{C}^n)) = \mathbb{C}^n$, and denote by $\deg f$, the geometric degree of $f$ - the number of solutions of the equation $f = a$ for generic points $a \in \mathbb{C}^n$. The branched value set $E_f$ of $f$ is defined to be the smallest subset of $\mathbb{C}^n$ such that the map

$$f : \mathbb{C}^n \setminus f^{-1}(E_f) \longrightarrow \mathbb{C}^n \setminus E_f$$

(∗)

gives a unbranched $\deg f$-sheeted covering. It is well-known (see [M]) that the branched value set $E_f$ is either empty set or an algebraic hypersurface and $E_f = \{a \in \mathbb{C}^n : \# f^{-1}(a) \neq \deg f\}$. If $E_f = \emptyset$, then $f$ is injective, and hence, $f$ is an automorphism of $\mathbb{C}^n$ by the well-known fact that injective polynomial maps of $\mathbb{C}^n$ are automorphisms (see [R]). The famous Jacobian conjecture ([BCW]) asserts that $f$ must have a singularity if $E_f \neq \emptyset$. The question naturally raises as what can be said about polynomial dominanting

∗Supported in part by the National Basic program on Natural Science, Vietnam
maps of \( C^n \) if their branched value sets are isomorphic to a given algebraic hypersurface \( E \).

In this article we consider polynomial dominanting maps of \( C^2 \) with branched value set isomorphic to the complex line \( C \). We are interested in finding a list of such maps, up to polynomial automorphisms. We say that two map \( f, g : C^2 \rightarrow C^2 \) are equivalent if there are polynomial automorphisms \( \alpha \) and \( \beta \) of \( C^2 \) such that \( \alpha \circ f \circ \beta = g \).

**Theorem 1.** A polynomial dominanting map \( f \) of \( C^2 \) with finite fibres and branched value set isomorphic to \( C \) is equivalent to the map \((x, y) \mapsto (x^{\deg f}, y)\).

In view of this theorem the equivalence classes of polynomial dominanting maps of \( C^2 \) with finite fibres and branched value set isomorphic to \( C \) are completely determined by the geometric degree of maps. Theorem 1 is an immediate consequence of the following

**Theorem 2.** A polynomial dominanting map of \( C^2 \) with branched value set isomorphic to \( C \) is equivalent to one of the following maps

1) \((x, y) \mapsto (x^{\deg f}, y)\);  
2) \((x, y) \mapsto (x^{\deg f}, x^m y), m \geq 1\).  
3) \((x, y) \mapsto (x^{\deg f}, x^m (x^n y + \sum_{i=0}^{n-1} a_i x^i)), m \geq 1, n \geq 1, a_0 \neq 0 \) and \( a_i = 0 \) for \( i + m = 0 (\mod \deg f) \).

In fact, in the above list only maps of Type (i) have finite fibres. The fiber at \((0, 0)\) of a map of the types (ii-iii) is the line \( x = 0 \). Further, as shown in § 4, topology of maps of Type (ii) and Type (iii) are quite different. The proof of Theorem 2 presented in § 3 is an application of the famous theorem of Abhyankar, Moh and Suzuki ([AM],[S]) on the embedding of the complex line into the complex plane. § 4 is devoted to some remarks and open questions.

2. Let us recall some elementary facts on topology of polynomial in two-variables. Let \( h(x, y) \in C[x, y] \). By the exceptional value set \( E_h \) of \( h \) we mean a minimal set \( E_h \subset C \) such that the map

\[
h : C^2 \setminus h^{-1}(E_h) \rightarrow C \setminus E_h
\]
gives a smooth locally trivial fibration. The set \( E_h \) is at most a finite set (see [V]). The fiber of this fibration, denoted by \( \Gamma_h \), is called the generic
fiber of $h$. A polynomial $h$ is primitive if its generic fiber is connected. By the Stein factorization a polynomial $h(x, y)$ can be represented in the form $h(x, y) = \phi(r(x, y))$ for a primitive polynomial $r(x, y)$ and an one-variable polynomial $\phi(t)$ (see [F]).

The following lemma is an immediate consequence of Abhyankar-Moh-Suzuki on the embedding of the complex line into the complex plane, which asserts that regular embeddings of $C$ in $C^2$ are equivalent to the natural embedding, or equivalence, that if $p(x, y)$ is irreducible polynomial and if the curve $p = 0$ is a smooth contractible algebraic curve, then $p \circ \alpha(x, y) = x$ for some polynomial automorphism $\alpha$ of $C^2$.

**Lemma 1.** Let $h \in C[x, y]$. Suppose that the generic fiber $\Gamma_h$ has $d$ connected components and each of them is diffeomorphic to $C$. Then, there exists a polynomial automorphism $\alpha$ in $C^2$ such that

$$h \circ \alpha(x, y) = x^d + a_1x^{d-1} + \ldots + a_d.$$

We will use this lemma in the situation when all of fibres of $h$, excepted at most one, are diffeomorphic to a distinct union of $d$ lines $C$. Then, the lemma shows that $h(\alpha(x, y)) = x^d + a_d$ for an automorphism $\alpha$.

**Proof of Lemma 1.** By the Stein factorization we can represent $h(x, y) = \phi(r(x, y))$ for a primitive polynomial $r \in C[x, y]$ and $\phi \in C[t]$. Further, one can choose $r$ and $\phi$ so that $\phi(t) = t^{\deg \phi} + \text{lower terms}$.

Observe that for each $c \in C$ the fiber $h^{-1}(c)$ consists of the curves $r(x, y) = c_i$, $i = 1, \ldots, \deg \phi$, where $c_i$ are zero points of $\phi(t) - c = 0$. Since the generic fiber $\Gamma_h$ has $d$ connected components and each of them is diffeomorphic to $C$, the generic fiber $\Gamma_r$ of $r$ is diffeomorphic to $C$, $\deg \phi = d$ and

$$\phi(t) = t^d + a_1t^{d-1} + \ldots + a_d.$$

Now, let $\gamma \in C$ be a fixed generic value of $r$. Then, the polynomial $r(x, y) - \gamma$ is irreducible and the curve $r(x, y) - \gamma = 0$ is diffeomorphic to $C$. So, in view of Abhyankar-Moh-Suzuki Theorem there exists a polynomial automorphism $\alpha$ of $C^2$ such that $r(\alpha(x, y)) - \gamma = x$. Then, we get

$$h(\alpha(x, y)) = \phi(r(\alpha(x, y))) = \phi(x + \gamma) = x^d + a_1x^{d-1} + \ldots + a_d.$$

Q.E.D
3. **Proof of Theorem 2:** Let $f : C^2(x, y) \rightarrow C^2(u, v)$ be a given polynomial dominanting map with branched value set $E_f$ isomorphic to $C$, where $(x, y)$ and $(u, v)$ stand for coordinates in $C^2$. In view of Abhyankar-Moh-Suszuki Theorem we can choose a polynomial automorphism $\alpha$ of $C^2$ so that the image $\alpha(E_f)$ is the line $u = 0$.

Let $\bar{f} := \alpha \circ f$, $\bar{f} = (f_1, f_2)$. Then, $E_f = \{u = 0\}$, $\bar{f}^{-1}(E_f) = \bar{f}_1^{-1}(0)$ and the map

$$\bar{f} : C^2 \setminus \bar{f}_1^{-1}(0) \rightarrow C^2 \setminus \{u = 0\}.$$ 

gives a unbranched $\deg f$–sheeted covering. This covering induces unbranched $\deg f$–sheeted coverings

$$\bar{f} : \bar{f}_1^{-1}(c) \rightarrow \{u = c\} \cong C, \ c \neq 0.$$ 

Since $C$ is simply connected, for every $0 \neq c \in C$ the fiber $\bar{f}_1^{-1}(c)$ consists of exactly $\deg f$ connected components and each of these components is diffeomorphic to $C$. So, by applying Lemma 1 one can see that there exists an automorphism $\beta$ of $C^2$ such that

$$\bar{f}_1(\beta(x, y)) = x^{\deg f} + c.$$ 

Let $\bar{f} := \bar{f} \circ \beta - (c, 0)$. Then, $\bar{f}(x, y) = (x^{\deg f}, \bar{f}_2(x, y))$. Note that $E_{\bar{f}} = \{u = 0\}$ and $\bar{f}^{-1}(\{u = 0\}) = \{x = 0\}$. So, by definition for each $(a, b) \in C^2$, $a \neq 0$, the equation $\bar{f}(x, y) = (a, b)$ has exactly $\deg f$ distinct solutions. This implies that for each $(a, b) \in C^2$, $a \neq 0$, the equation $\bar{f}_2(\epsilon, y) = b$ has a unique solution for each $\deg f$ radical $\epsilon$ of $a$. Such a polynomial $\bar{f}_2(x, y)$ must be of the form

$$\bar{f}_2(x, y) = ax^k y + x^l g(x), 0 \neq a \in C, \ k \geq 0 \ l \geq 0, \ g \in C[x].$$ 

For $k = 0$, we have $\bar{f} \circ \gamma(x, y) = (x^{\deg f}, y)$ for the automorphism $\gamma(x, y) := (x, a^{-1} y - ax^l g(x))$.

Consider the case $k > 0$. Put $m = \min\{k; l\}$ and $n = k - m$.

For $n = 0$ we can represent $\bar{f}_2(x, y)$ in the form

$$\bar{f}_2(x, y) = ax^m(y + h(x)) + c(x^{\deg f}),$$

where $h(x), c(x) \in C[x]$ and

$$h(x) = \sum_{i + m \neq 0(\text{mod } \deg f)} a_i x^i.$$
Put
\[ \gamma_1(u, v) := (u, a^{-1}(v - c(u))); \]
\[ \gamma_2(x, y) := (x, y - h(x)). \]

Then, we get
\[ \gamma_1 \circ \tilde{f} \circ \gamma_2(x, y) = (x^{\deg f}, x^m y). \]

For the case \( n > 0 \) we can represent
\[ \tilde{f}_2(x, y) = ax^m(x^n y + h(x) + x^n b(x)) + c(x^{\deg f}), \]
where \( h(x), b(x), c(x) \in C[x] \) and
\[ h(x) = \sum_{i=0,...,n-1, \ i+m \neq 0(\mod \deg f)} a_i x^i. \]

Put
\[ \gamma_1(u, v) := (u, a^{-1}(v - c(u))); \]
\[ \gamma_2(x, y) := (x, y - b(x)). \]

Then,
\[ \gamma_1 \circ \tilde{f} \circ \gamma_2(x, y) = (x^{\deg f}, x^m(x^n y + h(x))). \]

Thus, the considered map \( f \) is alway equivalent to one of maps of the types (i- iii). Q.E.D

4. Let us to conclude the paper by some remarks and open questions.

i) Topologically, the behavior of maps of Type (i), Type (ii) and Type (iii) are quite different. The maps of Type (i) have finite fibres, while the fiber at \((0,0)\) of a map of the types (ii) or (iii) is the line \( \{x = 0\} \). Furthermore, for an irreducible germ curve \( \gamma \subset C^2 \) located at \((0,0)\) and intersecting with the line \( \{u = 0\} \) at \( \{(0,0)\} \), the inverse image of \( \gamma \) by a map of Type (ii) is connected and consists of the line \( x = 0 \) and a branch located at \((0,0)\). But, the inverse image of \( \gamma \) by a map of Type (iii) consists of the line \( x = 0 \) and a branch at infinity.

ii) As shown in Theorem 2 and its proof, the polynomial dominating maps of \( C^2 \) with branched value curve isomorphic to \( C \) form a narrow classes among polynomial dominating maps of \( C^2 \). The structure of covering (\(^*\)) associated to such a map is very simple. Up to automorphisms of \( C^2 \), this covering looks like a unbranched covering from \( C^2 \setminus \{x = 0\} \rightarrow C^2 \setminus \{u = 0\} \). The singular set of such a map is isomorphic to \( C \). In particular, a polynomial
map of $C^2$ with branched value curve isomorphic to $C$ must has a singularity. This is true even when the branched value curve is only homeomorphic to $C$ (see in [C1]). It is worth to present here the following observation, which is reduced from the results in [C2]. By a $J$-curve we mean a curve $E$ in $C^2$ of the form $E = \cup_i \phi_i(C)$ for some polynomial maps $\phi_i : C \rightarrow C^2$, $i = 1, \ldots k$, such that

a) Each component $\phi_i(C)$ is not isomorphic to $C$, and

b) For every polynomial automorphism $\alpha$ of $C^2$

$$\frac{\deg p_1}{\deg q_1} = \ldots = \frac{\deg p_k}{\deg q_k}$$

where $\alpha \circ \phi_i := (p_i, q_i)$.

**Theorem 3.** (See [Thm. 4.4, C2].) A polynomial dominanting map $f$ of $C^2$ with $E_f \neq \emptyset$ must has a singularity if $E_f$ is not a $J$-curve.

In fact, it is shown in [C2] that if $f$ has non-zero constant jacobian and if $E_f \neq \emptyset$, then $E_f$ is a $J$-curve in above and for every polynomial automorphism $\alpha$ of $C^2$

$$\frac{\deg p_i}{\deg q_i} = \frac{\deg (\alpha \circ f)_1}{\deg (\alpha \circ f)_2}, \ i = 1, \ldots k,$$

where $\alpha \circ \phi_i := (p_i, q_i)$ and $\alpha \circ f := ((\alpha \circ f)_1, (\alpha \circ f)_2)$. In view of this theorem, the Jacobian conjecture in $C^2$ can be reduced to the question whether a $J$-curve could not be the branched value set of a nonsingular polynomial map of $C^2$.

Note that the situation is quite different in the case of holomorphic maps. Orevkov in [O] had constructed a nonsingular holomorphic map from a Stein manifold homeomorphic to $R^4$ onto an open ball in the complex plane $C^2$ which gives a three-sheeted branching covering with the branched value set diffeomorphic to $R^2$.

iii) It is worth to find analogous statements of Theorem 1 for high-dimensional cases. Let $F = (F_1, F_2, \ldots F_n) : C^n_{(x_1, \ldots x_n)} \rightarrow C^n_{(u_1, \ldots u_n)}$ be a polynomial dominanting map with finite fibres.

**Problem:** Suppose $E_F = \{u_1 = 0\}$.

i) Does there exists an automorphism $\alpha$ of $C^n$ such that

$$F_1 \circ \alpha(x_1, \ldots x_n) = x_1^{\deg F} ?$$
ii) Is $F$ equivalent to the map

$$(x_1, \ldots, x_n) \mapsto (x_1^{\deg F}, x_2, \ldots, x_n) ?$$

Such a map $F$ gives a locally trivial fibration $F_1 : C^n \setminus F^{-1}(0) \to C \setminus \{0\}$ with fiber diffeomorphic to a distinct union of $\deg F$ space $C^{n-1}$. Further, every connected component of the fibers $F_1^{-1}(c)$, $c \neq 0$, is isomorphic to $C^{n-1}$. The situation here seems to be more simple than those in the problem of embeddings $C^{n-1}$ into $C^n$ - a generalization of the Abhyankar-Moh-Suzuki Theorem, which asks whether a regular embedding of $C^{n-1}$ in $C^n$ is equivalent to the natural embedding, which is still open for $n > 2$.

**Acknowledgments:** The author wishes to thank very much Prof. V.H. Ha for his valuable comments and suggestions.

**References**

[AM] S. S. Abhyankar and T. T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. 276 (1975), 148-166.

[BCW] H. Bass, E. Connell and D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (N. S.) 7 (1982), 287-330.

[C1] Nguyen Van Chau, *Remark on the Vituskin’s covering*, Act. Math. Vietnam, 24, No 1, 1999, 109-115.

[C2] Nguyen Van Chau, *Non-zero constant Jacobian polynomial maps of $C^2$*, J. Ann. Pol. Math. 71, No.3, 287-310 (1999).

[F] M. Furushima, *Finite groups of polynomial automorphisms in the complex affine plane*, 1, Mem. Fac. Sci., Kyushu Univ., Ser. A, 36 (1982), 83-105.

[M] D. Mumford, *Algebraic geometry, I. Complex projective varieties*, Springer-Verlag Berlin Heidelberg New York, 1976.

[O] S. Yu. Orevkov, *Rudolph diagram and analytical realization of Vitushkin’s covering*, Math. Zametki, 60 (1996), no 2, 206-224, 319.

[R] W. Rudin, *Injective polynomial maps are automorphisms*, Amer. Math. Monthly 102 (1995), No 6, 540-543.

[S] M. Suzuki, *Proprietes topologiques des polynomes de deux variables complexes at automorphismes algebriques de lespace $C^2$*, J. Math. Soc. Japan 26, 2 (1976), 241-257.

[V] J. L Verdier, *Statifications de Whitney et theoreme de Bertini-Sard*, Invent. Math. 36 (1976), 295-312.