Compression with wildcards: All models of a Boolean 2-CNF

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Abstract. Let $W$ be a finite set which simultaneously serves as the universe of any poset $(W, \preceq)$ and as the vertex set of any graph $G$. Our algorithm, abbreviated A-I-I, enumerates (in a compressed format using don’t-care symbols) all $G$-independent order ideals of $(W, \preceq)$. For certain instances the high-end Mathematica implementation of A-I-I compares favorably to the hardwired Mathematica commands BooleanConvert and SatisfiabilityCount. The A-I-I can be parallelized and adapts to a polynomial total time algorithm that enumerates the modelset of any Boolean 2-CNF.

Key words: Horn 2-CNF, All-Sat, compressed enumeration, Horn-Renamability, Mathematica

1 Introduction

We recommend [CH] for an introduction to Boolean functions and for reading up undefined terms in the present article. A Boolean 2-CNF is a Boolean function $F$ with vector of variables $\vec{x} = (x_1, x_2, ..., x_w)$ that is given in conjunctive normal form and that has merely clauses of length at most two. Any $\vec{y} \in \{0, 1\}^w$ with $F(\vec{y}) = 1$ is a model of $F$. For instance, letting $w = 4$ consider

$$F_1(\vec{x}) = (\overline{x_1} \lor x_3) \land (\overline{x_1} \lor x_4) \land (\overline{x_2} \lor x_1) \land (\overline{x_2} \lor x_4) \land (x_2 \lor x_3) \land (x_3 \lor x_4) \land \overline{x_2}.$$  

To warm up (and for later use), notice that the 1-clause $\overline{x_2}$ forces $y_2 = 0$ for each model $\vec{y}$ of $F_1$. But this implies that $F_1$ boils down\(^1\) to $F_2(\vec{x}) = (\overline{x_1} \lor x_3) \land (\overline{x_1} \lor x_4) \land (\overline{x_3} \lor x_4) \land x_4$. This further implies $y_1 = 1$ for each model $\vec{y}$ of $F_1$, and so $F_2$ further boils down to $F_3(\vec{x}) = (\overline{x_1} \lor x_3)$.

From this it is clear that each 2-CNF is easily reduced to an equivalent 2-CNF $F$ which is homogeneous in the sense of having no 1-clauses.

Convention: All 2-CNFs are henceforth silently assumed to be homogeneous.

So the clauses of a 2-CNF are either positive $(x_i \lor x_j)$, or negative $(\overline{x_i} \lor \overline{x_j})$, or mixed $(x_i \lor \overline{x_j})$. (These definitions extend to clauses of arbitrary length.)

\(^1\)While the domains of $F_1$ and $F_2$ are $\{0, 1\}^4$ and $\{0, 1\}^3$ respectively (and so $F_1 \neq F_2$), it holds that $\text{Mod}(F_1) = \{(y_1, 0, y_3, y_4) : (y_1, y_3, y_4) \in \text{Mod}(F_2)\}$. 

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In a nutshell, Sections 3 and 4 are dedicated to a carefully described and numerically evaluated algorithm (called A-I-I) that enumerates all models of certain natural 2-CNFs. The remaining two Sections are more light-weight and indicate how A-I-I adapts to all 2-CNFs.

Here comes the Section break-up in more detail. Section 2 reviews two previous publications. In [W2] all order ideals of a poset \((W, \preceq)\) are enumerated in compact fashion, using so-called \((a, b)\)-wildcards. Similarly, in [arXiv0901.4117v4] of 2019 all anticliques of a graph are enumerated using suitable \((a, c)\)-wildcards.

The core Section 3 combines the two scenarios by considering graph-endowed posets \((W, \preceq, G)\), i.e. the set \(W\) simultaneously carries a poset \((W, \preceq)\) and serves as vertex set of a graph \(G = (W, E)\). The task is to enumerate all \(G\)-independent order ideals of \((W, \preceq)\). Unfortunately, combining \((a, b)\)-wildcards and \((a, c)\)-wildcards does not work. To describe the way out, first observe that the order ideals of a poset are easily seen to match the models of a 2-CNF all of whose 2-clauses are mixed. Likewise the anticliques (=independent sets) of a graph match the models of a 2-CNF all of whose 2-clauses are negative. Hence graph-endowed posets match certain 2-CNFs without positive clauses.

For a minute consider a clause of arbitrary length. It is called Horn clause if it contains at most one positive literal. By definition a Horn CNF consists entirely of Horn-clauses. The Horn \(n\)-algorithm of [W1] compresses the modelset of any Horn CNF in polynomial total time. Because for 2-CNFs the avoidance of positive clauses suffices to be a Horn 2-CNF, the Horn \(n\)-algorithm can handle graph-endowed posets. However, since the clauses are so short, a thoroughly different method, called All-Independent-Ideals (A-I-I), is more efficient. In brief, A-I-I processes the \(w\) many variables (=nodes of \((W, \preceq)\)) rather than the up to \(4\binom{w}{2}\) many 2-clauses of \(F\). (Observe that A-I-I does not cater for arbitrary Horn 2-CNFs, just graph-endowed posets.)

The numerical experiments in Section 4 roughly speaking show that A-I-I, albeit implemented in high-level Mathematica code, outperforms the hardwired Mathematica commands BooleanConvert and SatisfiabilityCount whenever the number of models is small. As to larger numbers, A-I-I is easy to parallelize, and so in principle can be sped-up to any desired degree.

Since graph-endowed posets can be processed efficiently by A-I-I, this begs the question whether all 2-CNFs \(F\) can be reduced to graph-endowed posets. In Section 5 we handle the case that \(F\) is a Horn 2-CNF. Specifically, we factor out the strong components of the digraph induced by the mixed clauses of \(F\), while keeping an eye on what this entails for the graph \(G\) induced by the negative clauses of \(F\). Section 6 in turn reduces arbitrary (satisfiable) 2-CNFs to Horn 2-CNFs. Here the trick is a so-called Horn-Renaming of the variables. Furthermore in 6.4 we touch upon the so far only published algorithm to generate all 2-CNF models, i.e. Feder’s method [F1], and glimpse at real-word applications.

2 Merely negative or merely mixed 2-clauses

We say that a 2-CNF is mixed if all its clauses are mixed, and say it is negative if all its clauses are negative. Roughly speaking the two kinds correspond to posets (2.1) and graphs (2.2) respectively.
2.1 Consider the mixed 2-CNF \( f_{po} : \{0,1\}^8 \to \{0,1\} \) defined by

\[
f_{po}(\vec{x}) := (\overline{x_8} \lor x_1) \land (\overline{x_6} \lor x_2) \land (\overline{x_7} \lor x_3) \land (\overline{x_7} \lor x_2) \land (\overline{x_5} \lor x_3).
\]

Since \( x_i \lor x_j \) is logically equivalent to \( x_i \rightarrow x_j \), the models of \( f_{po} \) match the order ideals of the poset \( (W_1, \preceq) \) in Figure 1. Do not confuse the partial order relation \( \preceq \) with the ordinary total ordering of natural numbers, denoted by \( \leq \). In fact the two are interwoven in that the labeling 1, 2, ..., 8 of the elements of \( W_1 \) is a so-called linear extension of \( W_1 \), i.e.

\[
(\forall i, j \in W_1) \ i < j \Rightarrow i < j.
\]

The converse implication fails: 3 < 6 but 3 \( \not< \) 6. A crisper view of linear extensions is the following. They are obtained by "shelling" a poset, i.e. starting with any (globally) minimal element, one keeps on choosing arbitrary minimal elements of the shrinking posets. Apart from 1, 2, ..., 8, there are many other shellings of \( (W_1, \preceq) \), say 3, 5, 2, 6, 7, 4, 1, 8.

The set \( Id(P, \preceq) \) of all order ideals of a \( w \)-element poset \( (P, \preceq) \) can be enumerated in a compressed format as follows [W2]. The wildcard \( (a, b, ..., b) \) compresses the modelset of an implication \( x_i \rightarrow (x_j \land ... \land x_k) \), say \( (a, b, b, b) := (0, 2, 2, 2) \uplus (1, 1, 1, 1) \). Here '2' is the common don’t-care symbol which is sometimes written as ‘∗’. We choose '2' because it stresses that at the position occupied by 2 one has two options, i.e. 0 or 1. Upon applying the \((a,b)\)-algorithm of [W2] one obtains \( Id(W_1, \preceq) = \rho_1 \uplus \rho_2 \), where \( \rho_1, \rho_2 \) are the 012ab-rows in Table 1. Thus, two or more wildcards can appear within a 012ab-row; they are completely independent of each other and distinguished by subscripts. Generally the \((a,b)\)-algorithm outputs \( R \) many 012ab-rows in time \( O(R|W|^2) \) [W2, Thm. 5].

\[\text{Figure 1}\]

\[\text{Figure 2}\]

It remains the problem that an arbitrary conjunction of clauses \( x_i \rightarrow x_j \) (i.e. \( \overline{x_i} \lor x_j \)) generally does not model a poset, only a directed graph. One gets a poset by factoring out the so-called strong components. This issue will re-appear in the more sophisticated scenario of Section 5.

2.2 Consider the negative Boolean 2-CNF

\[
f_{gr}(\vec{x}) := (\overline{x_2} \lor x_3) \land (\overline{x_7} \lor x_3) \land (\overline{x_7} \lor x_2) \land (\overline{x_5} \lor x_3) \land (\overline{x_5} \lor x_3).
\]

\[A Boolean formula \( x_i \rightarrow (x_j \land ... \land x_k) \) is equivalent to \( \overline{x_i} \lor x_j \lor \cdots \lor x_k \), i.e. to a Horn-clause (see introduction). It is a pure Horn-clause in the sense that it has exactly one positive literal. However in Data Mining applications one rather speaks of implications (or functional dependencies, association rules, etc) and uses more handy notation such as \( \{x_i\} \rightarrow \{x_j, ..., x_k\} \).\]

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It follows that the models of $f_{gr}$ match the independent sets (=anticliques) of the graph $G_1$ in Figure 2. Each graph $G$ can be rendered this way and conversely (as opposed to 2.1) each negative 2-CNF yields a graph. With the $(a,c)$-algorithm of [arXiv0901.4417v4] the set $AC(G)$ of all anticliques of $G$ can be enumerated in a compressed format. Namely, the wildcard $(a,c,...,c)$ compresses the modelset of $x_i \rightarrow (\overline{x_j} \land ... \land \overline{x_k})$. Thus for instance $(a,c,c) := (0,2,2) \uplus (1,0,0)$. The $(a,c)$-algorithm yields $AC(G_1) = \sigma_1 \uplus \sigma_2$, where $\sigma_1, \sigma_2$ are the 012ac-rows in Table 1.

|   | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|-----|-----|-----|-----|-----|-----|-----|-----|
| $\rho_1$ | $b_3$ | $b_1$ | $b_2$ | $2$  | $a_2$ | $a_1$ | $0$  | $a_3$ |
| $\rho_2$ | $b_3$ | $1$  | $1$  | $2$  | $2$  | $1$  | $a_3$ |
| $\sigma_1$ | $2$  | $c_1$ | $a_1$ | $c_1$ | $2$  | $a_2$ | $0$  | $c_2$ |
| $\sigma_2$ | $2$  | $c_1$ | $a_1$ | $0$  | $a_2$ | $1$  | $c_2$ |

Table 1: Examples of 012ab-rows and 012ac-rows

3 \hspace{1em} Enumerating all independent order ideals

The content of Section 3 was surveyed sufficiently in the Introduction.

3.1 Let us determine those order ideals of $(W_1, \preceq)$ that happen to be $G_1$-dependent as well. Thus, putting $H_1 := f_{po} \land f_{gr}$ and referring to Table 1, we aim for

$$Mod(H_1) = Mod(f_{po}) \cap Mod(f_{gr}) = (\rho_1 \uplus \rho_2) \cap (\sigma_1 \uplus \sigma_2)$$

$$= (\rho_1 \cap \sigma_1) \uplus (\rho_1 \cap \sigma_2) \uplus (\rho_2 \cap \sigma_1) \uplus (\rho_2 \cap \sigma_2).$$

Here $\rho_1 \cap \sigma_2 = \rho_2 \cap \sigma_1 = \emptyset$ because the clashing 0 and 1 on the 7th position; see Table 1. Further $\rho_2 \cap \sigma_2 = \emptyset$ because $(1,1)$ clashes with $(c_1,a_1)$. In order to access the bitstrings in $\rho_1 \cap \sigma_1$ one of the two rows needs to be written as disjoint union of 012-rows. Because $\sigma_1$ has fewer wildcards we pick $\sigma_1$ and in Table 2 get $\sigma_1 = \sigma_{1,1} \uplus \sigma_{1,2} \uplus \sigma_{1,3} \uplus \sigma_{1,4}$ (focus on $x_3, x_6$). Consequently $\rho_1 \cap \sigma_1 = (\rho_1 \cap \sigma_{1,1}) \uplus \cdots \uplus (\rho_1 \cap \sigma_{1,4})$. Note that $(a_1,b_1)$ in $\rho_1$ clashes with $(1,0)$ in $\sigma_{1,4}$, and so $\rho_1 \cap \sigma_{1,4} = \emptyset$.

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3 As becomes clear in [arXiv0901.4417v4], and will be further evidenced in the near future, 012ac-rows can be pushed further. Roughly speaking, the enumeration of all anticliques (alternatively: all maximum anticliques) of any graph can be reduced to the enumeration of the (maximum) anticliques in suitable bipartite graphs.

4 The intersection of a 012-row with any multivalued row (be it of type 012ab or 012ac or something else) is always either empty or a single multivalued row.
\[ \rho_1 = b_3 b_1 b_2 2 a_2 a_1 0 a_3 \]
\[ \sigma_{1,1} = 2 2 0 2 0 0 2 \]
\[ \rho_1 \cap \sigma_{1,1} = b_3 2 0 2 0 0 0 a_3 12 \]
\[ \sigma_{1,2} = 2 2 0 2 2 1 0 0 \]
\[ \rho_1 \cap \sigma_{1,2} = 2 1 0 2 0 1 0 0 4 \]
\[ \sigma_{1,3} = 2 0 1 0 2 0 0 2 \]
\[ \rho_1 \cap \sigma_{1,3} = b_3 0 1 0 2 0 0 a_3 6 \]
\[ \sigma_{1,4} = 2 0 1 0 2 1 0 0 \]

Table 2: Intersecting the 012ab-row \( \rho_1 \) with 012-rows.

It follows that
\[ |\text{Mod}(H_1)| = |\rho_1 \cap \sigma_{1,1}| + |\rho_1 \cap \sigma_{1,2}| + |\rho_1 \cap \sigma_{1,3}| = 12 + 4 + 6 = 22. \]

As seen above, intersecting 012ab-rows with 012ac-rows is cumbersome\(^5\). We thus embark on a wholly different approach to retrieve all \( G \)-independent order ideals; first by toy example (3.2), then in general (3.3).

3.2 Consider

\[ H_1(\vec{x}) = f_{po}(\vec{x}) \land f_{gr}(\vec{x}) := \left[ (x_8 \lor x_1) \land (x_6 \lor x_2) \land (x_7 \lor x_2) \land (x_7 \lor x_3) \land (x_5 \lor x_3) \right] \]
\[ \land \left[ (x_2 \lor x_3) \land (x_3 \lor x_4) \land (x_7 \lor x_5) \land (x_7 \lor x_5) \land (x_5 \lor x_8) \right] \]

Thus the model set \( \text{Mod}(H_1) \) consists of all bitstrings \( \vec{y} \in \{0,1\}^8 \) which simultaneously are\(^6\) order ideals of \( (W_1, \preceq) \) and anticliques of \( G_1 \). For positive integers \( k \) put \( [k] := \{1,2,\ldots,k\} \). Let \( G_1[k] \) be the subgraph induced by \( [k] \). We will calculate \( \text{Mod}(H_1) \) by updating for \( k = 2,\ldots,8 \) the set of length \( k \) bitstrings which simultaneously are order ideals in \( ([k], \preceq) \) and anticliques in \( G_1[k] \). Notice that \( 1,2,\ldots,8 \) being a shelling of \( (W_1, \preceq) \) is crucial for \( ([k], \preceq) \) being an order ideal of \( (W_1, \preceq) \).

By inspection each subset of \( \{1,2\} \) (equivalently: bitstring \( \vec{y} \) with \( y_3 = \cdots = y_8 = 0 \)) is simultaneously an order ideal of \( ([2], \preceq) \) and an anticlique of \( G_1[2] \). This yields the 012-row \( r_1 \) in Table 3. Let us move from \( k = 2 \) to \( k = 3 \) and accordingly look at \( r_2 \) and \( r_3 \).

\(^5\)However, this line of attack may be sensible if one type of row is much rarer than the other. To some extent this state of affairs can be pushed by switching suitable variables. Details can be found in arXiv1208.2559v6 which is a previous version of the present article.

\(^6\)More precisely, the support \( \{i \in [8] : y_i = 1\} \) of \( \vec{y} \) simultaneously is an order ideal and an anticlique. For ease of notation we henceforth stick with \( \vec{y} \) and interpret it as a bitstring or as its support, according to context.
Table 3: Compressing the set of all $G_1$-independent order ideals of $(W_1,\prec)$

|    | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|----|-------|-------|-------|-------|-------|-------|-------|-------|
| $r_1$ | 2     | 2     |       |       |       |       |       |       |
| $r_2$ | 2     | 2     | 0     | 0     | 0     |       |       |       |
| $r_3$ | 2     | 0     | 1     | 0     | 0     |       |       |       |
| $r_4$ | 2     | 2     | 0     | 2     | 0     | 0     |       |       |
| $r_3$ | 2     | 0     | 1     | 0     | 0     |       |       |       |
| $r_5$ | 2     | 2     | 0     | 2     | 0     | 0     |       |       |
| $r_6$ | 2     | 1     | 0     | 2     | 0     | 1     | 0     | 0     | final |
| $r_3$ | 2     | 0     | 1     | 0     | 0     |       |       |       |
| $r_7$ | 2     | 2     | 0     | 2     | 0     | 0     | 0     | 0     | final |
| $r_8$ | 1     | 2     | 0     | 2     | 0     | 0     | 0     | 1     | final |
| $r_3$ | 2     | 0     | 1     | 0     | 0     |       |       |       |
| $r_9$ | 2     | 0     | 1     | 0     | 2     | 0     | 0     |       |
| $r_3$ | 2     | 0     | 1     | 0     | 2     | 0     | 0     | 0     | final |
| $r_{10}$ | 2 | 0 | 1 | 0 | 2 | 0 | 0 | 0 | final |
| $r_{11}$ | 1 | 0 | 1 | 0 | 2 | 0 | 0 | 1 | final |

As to $r_2$, one can always put a 0 at the new position $k$. That’s because an anticlique stays an anticlique in whatever way a graph increases by one vertex $k$. An order ideal stays an order ideal only because the new vertex $k$ is a maximal element in the larger poset (due to the shelling). So much about the fat 0 in $r_2$. The other 0’s in $r_2$ are due to the fact that all future order ideals (viewed as bitstrings) $\vec{y}$ with $y_3 = 0$ must have $y_5 = y_7 = 0$ since $3 < 5$ and $3 < 7$. As to $r_3$, one cannot always put a 1 at the new position $k$. However here it it works: Since the neighborhood of 3 in $G_1$ is $NH(3) = \{2, 4\}$, and since we are aiming for anticliques, we put $y_2 = y_4 = 0$ in $r_3$.

The *pending position* to be handled in a row is the position $k$ of its first ’blank’, thus $k = 4$ for $r_2$. As seen, we can always fill the first blank with 0. In $r_2$ we can also fill in 1 without altering anything else, because of $NH(k) = \{3, 7\} \subseteq zeros(r_2)$. Instead of replacing $r_2$ by two rows, one with 0, one with 1 on the blank, we write 2 on the blank and call the new row $r_4$. The current LIFO stack consists of $r_3$, $r_4$. Always turning to the stack’s top row we next handle the pending position $k = 6$ of $r_4$. Filling in 0 yields $r_5$. As to putting 1 on the 6th position, different from before 6 is no minimal element of $W_1$, and so instead of 6 the whole order ideal $6 \downarrow = \{2, 6\}$ needs to be considered. Hence $x_2 = x_6 = 1$ in $r_6$. One has $NH(2) = \{3\} \subseteq zeros(r_4)$, but $NH(6) = \{8\}$ forces a new 0 at position 8 in $r_6$.

All positions of the arising row $r_6$ happen to be filled. By construction $r_6 \subseteq Mod(H_1)$, and so $r_6$ is final. It is removed from the LIFO stack and stored in a safe place. One verifies that handling position 8 of the new top row $r_7$ yields rows $r_7$ and $r_8$. Both of them are final, and so only $r_3$ remains in the LIFO stack. It holds that $5 \downarrow = \{3, 5\}$ and $3 \in ones(r_3)$; further

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7LIFO stands for Last-In-First-Out stack. This basic data structure can be better visualized than the equivalent and (unfortunately) more popular point of view of depth first searching a tree. This well-known equivalence is illustrated, and hints to the literature are given, e.g., in Section 3 of our article’s previous version [arXiv:1208.2559v5].
Let \( k \) \( \subseteq \) \( \text{zeros}(r_3) \). Hence the blank on the 5th position of \( r_3 \) can be filled with 2, giving rise to row \( r_9 \). In turn \( r_9 \) gives rise to the final rows \( r_{10} \) and \( r_{11} \). It follows that there are \(|r_6| + |r_7| + |r_8| + |r_{10}| + |r_{11}| = 22 \) order ideals of \((W_1, \preceq)\) which are \( G_1 \)-independent.

### 3.3

Let \((W, \preceq)\) and \( G \) be a poset and graph respectively that share the same universe \( W = [w] \). We may assume that \( 1, 2, ..., w \) is a shelling of \((W, \preceq)\). Let us state systematically how to extend a partial row \( r \) to \( r' \) and \( r'' \) by filling its first blank (at position \( k \)) by 0 and 1 respectively. Inducting on \( k = 1, 2, ..., w \) these properties need to be maintained:

1. **(P1)** Up to position \( k - 1 \) there are no blanks\(^8\) and if \( k \leq i \leq w \) then the \( i \)th position is either 0 or a blank.
2. **(P2)** Whenever \( i \in \text{zeros}(r) \), then \((i \uparrow) \subseteq \text{zeros}(r)\).
3. **(P3)** Whenever \( i \in \text{ones}(r) \), then \((i \downarrow) \subseteq \text{ones}(r)\).
4. **(P4)** The vertex set \( \text{ones}(r) \) is \( G \)-independent.

Here comes the recipe to maintain these properties when moving from \( r \) to \( r' \) and \( r'' \):

1. **(R1)** The set \( k \uparrow \) is a subset of \( \{k, k+1, ..., w\} \) in view of the shelling order. Hence by (P1) for each \( i \in k \uparrow \) the \( i \)th position in \( r \) is either already 0 or a blank, and so one can write a 0 on it. It is clear that (P1) to (P4) are maintained by the new partial row \( r' \).
2. **(R2)** By the shelling order the position set \( k \downarrow \) is a subset of \( \{k\} \subseteq \{1, 2, ..., k\} \). By (P2) it holds that \((k \downarrow) \cap \text{zeros}(r) = \emptyset\). Hence one can write 1 on the \( i \)th position for all \( i \in k \downarrow \), provided \( Y := (k \downarrow) \cup \text{ones}(r) \) happens to be \( G \)-independent. In the latter case define \( Z \subseteq [w] \) as the set of \( j \) that are adjacent (in \( G \)) to some \( i \in Y \). Evidently \( Z \cap \text{ones}(r) = \emptyset \). By (P3) also \((Z \uparrow) \cap \text{ones}(r) = \emptyset \), and so we can write 0 on all positions \( i \in Z \uparrow \). One checks that (P1) to (P4) are maintained by the extension \( r'' \) of \( r \). (If \( Y \) is \( G \)-dependent then only \( r' \), not \( r'' \), is built.)
3. **(R3)** Let \( r' \) and \( r'' \) be the new partial rows arising in (R1) and (R2) respectively. Suppose \( \text{zeros}(r') \) is as small as it can possibly be, i.e. \( \text{zeros}(r') = \text{zeros}(r) \cup \{k\} \). Likewise assume that \( \text{ones}(r'') = \text{ones}(r) \cup \{k\} \) and \( \text{zeros}(r'') = \text{zeros}(r) \). Then, the two rows \( r', r'' \) can be replaced by a single row \( r''' \) that arises from \( r \) by writing '2' on the first blank.

**Theorem:** Let \((W, \preceq)\) be a \( w \)-element poset and \( G \) a graph with vertex set \( W \). Then the above algorithm, call it \textbf{All-Independent-Ideals} (A-I-I), represents the set of all \( G \)-independent order ideals as a disjoint union of \( R \) many 012-rows in time \( O(Rw^2) \).

**Proof.** By the workings of A-I-I layed out above it suffices to show that (correctly) filling in a blank costs \( O(w^2) \), and that this happens \( O(Rw) \) many times. As to the first claim, checking the independency of \((k \downarrow) \cup \text{ones}(r) \) in (R2) costs \( O(w^2) \), and this swallows all other costs (such as calculating \( k \downarrow \) and \( Z \)). As to the second claim, let \( A \) be the set of the \( Rw \) many components

\(^8\)It helps to distinguish (also notationally) the preliminary 2's up to position \( k - 1 \) from the other preliminary 2's. The latter we call \( \text{blanks} \). They match the (ordinary) blanks in Table 3.
occuring in the \( R \) many final rows; and let \( B \) the set of all 'blank-filling events'. Then there is an obvious (well-defined) function \( f \) from \( A \) to \( B \). Crucially, \( f \) is surjective\(^9\) since partial rows are never deleted. This proves the second claim. \( \square \)

### 3.4 Last not least, we emphasize that A-I-I is easy\(^{10}\) to parallelize (equivalent terminology: distributed computing), as is every algorithm based on depth first searching (dfs) a computation tree. Indeed, since dfs is equivalent to a LIFO stack, it suffices to note (rephrasing parts of [W3, Sec. 6.5]) that the preliminary 012-rows in the LIFO stack underlying A-I-I (recall Table 3) are completely independent of each other and can hence at any stage be distributed to distinct processors. Each calculates a couple of final 012-rows and sends them back to the 'Head-Control'.

### 4 Numerical experiments

We compare our high-end Mathematica implementation of All-Independent-Ideals (A-I-I) with the Mathematica command BooleanConvert (option 'ESOP') which converts any Boolean function \( F \) in a so-called 'exclusive sum of products'. Thus in effect \( \text{Mod}(F) \) gets written as a disjoint union of 012-rows. Since the Mathematica command SatisfiabilityCount merely calculates the cardinality \( |\text{Mod}(F)| \), it has an inherent advantage over A-I-I and BooleanConvert. Nevertheless it sometimes trails its competitors.

#### 4.1 Specifically, similarly to [W2, p.132] we generate random posets \((W, \preceq)\) of breadth \( br \), height \( ht \) and cardinality \( w := |W| = br(ht + 1) \). Specifically, the poset has \( ht + 1 \) many levels \( \text{Lev}(i) \), all of cardinality \( br \), in such a way that each \( a \in \text{Lev}(i) \) \((2 \leq i \leq ht + 1)\) is assigned \( lc \) random lower covers in \( \text{Lev}(i-1) \). (For instance, except for the non-uniform value of \( lc \), Figure 1 depicts such a poset with \( br = 4 \) and \( ht = 1 \).)

Apart from \((W, \preceq)\) a graph \( G \) with the same vertex set \( W \) and \( m \) random edges is created. Then A-I-I computes all \( G \)-independent order ideals as described in 3.2 and 3.3. (To check \( G \)-independency in (R2) we used the Mathematica command IndependentVertexSetQ.) The cardinalities of the final 012-rows are added up and the result\(^{11}\) is recorded in the column labelled '# models'. Likewise the various CPU-times are recorded in the columns labelled 'A-I-I', 'SatCount' and 'BConvert'. Finally the number of final 012-rows produced by A-I-I and BooleanConvert are recorded in the columns labelled \( R_{AI} \) and \( R_{BC} \) respectively.

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\(^9\)But unless there is just one final row, \( f \) is not injective. To spell it out, suppose the partial row \( r \) splits, and \( c \) is the entry to the left of the blank that triggers the splitting. Then the blank-filling event that produced \( c \) is the \( f \)-value of two distinct elements of \( A \).

\(^{10}\)As to BooleanConvert, only its programmer(s) can tell.

\(^{11}\)It goes without saying (almost) that the three wholly different algorithms always convened on the number of models.
| $(br, ht, lc) \rightarrow (w, m)$ | # models | $S\text{Count}$ | $B\text{Convert}$ | $R_{BC}$ | A-I-I | $R_{AII}$ |
|----------------|----------|--------------|----------------|----------|-------|--------|
| $(15, 4, 2) \rightarrow (75, 0)$ | $16 \cdot 10^9$ | 0.36 s | 75 s | 265’769 | 96 s | 655’406 |
| $(15, 4, 2) \rightarrow (75, 20)$ | $11’248’304$ | 0.30 s | 5.6 s | 12691 | 3.3 s | 32’143 |
| $(15, 4, 2) \rightarrow (75, 100)$ | $215’608$ | 0.45 s | 0.80 s | 1274 | 0.52 s | 3995 |
| $(15, 4, 2) \rightarrow (75, 200)$ | $62’224$ | 1.8 s | 1.0 s | 682 | 0.39 s | 2413 |
| $(15, 4, 2) \rightarrow (75, 1000)$ | $308$ | 0.25 s | 0.28 s | 47 | 0.03 s | 80 |
| $(30, 3, 7) \rightarrow (120, 10)$ | $10^{10}$ | 304 s | $> 12$ hrs | — | 9721 s | 44’904’971 |
| $(30, 3, 7) \rightarrow (120, 20)$ | $838’142’562$ | 138 s | 267 s | 187’250 | 99 s | 708’523 |
| $(30, 3, 7) \rightarrow (120, 300)$ | $20’957’072$ | 219 s | 116 s | 1072 | 1.1 s | 2405 |
| $(30, 3, 7) \rightarrow (120, 2500)$ | $5006$ | 175 s | 111 s | 621 | 1.3 s | 1616 |
| $(30, 3, 7) \rightarrow (120, 7000)$ | $41$ | 224 s | 118 s | 30 | 0.05 s | 32 |
| $(40, 1, 10) \rightarrow (80, 0)$ | $\approx 2473 \cdot 10^9$ | 153 s | $> 4$ hrs | — | $> 4$ hrs | — |
| $(40, 1, 10) \rightarrow (80, 10)$ | $\approx 354 \cdot 10^9$ | 173 s | 87 s | 88’552 | 15 s | 166’335 |
| $(40, 1, 10) \rightarrow (80, 50)$ | $\approx 92 \cdot 10^9$ | 186 s | 77 s | 1678 | 4.6 s | 32’139 |
| $(30, 3, 4) \rightarrow (120, 2500)$ | $4961$ | 2424 s | 1278 s | 577 | 0.75 s | 1413 |
| $(30, 3, 5) \rightarrow (120, 2500)$ | $8844$ | 700 s | 491 s | 633 | 1.1 s | 1638 |
| $(30, 3, 7) \rightarrow (120, 2500)$ | $5006$ | 175 s | 111 s | 621 | 1.3 s | 1616 |
| $(30, 3, 9) \rightarrow (120, 2500)$ | $3885$ | 72 s | 53 s | 409 | 1.1 s | 1439 |
| $(30, 3, 12) \rightarrow (120, 2500)$ | $7377$ | 32 s | 26 s | 786 | 1.4 s | 1632 |
| $(37, 2, 7) \rightarrow (111, 2500)$ | $6939$ | $> 30$ min | 508 s | 1018 | 1.1 s | 2119 |
| $(37, 4, 7) \rightarrow (185, 2500)$ | $1’724’713$ | $> 3$ hrs | $> 3$ hrs | — | 49 s | 57’686 |
| $(37, 6, 7) \rightarrow (259, 2500)$ | $18’118’642’$ | — | — | — | 74 s | 64’924 |
| $(37, 8, 7) \rightarrow (333, 2500)$ | $\approx 3 \cdot 10^9$ | — | — | — | 42 s | 33’522 |
| $(37, 14, 7) \rightarrow (555, 2500)$ | $\approx 28 \cdot 10^9$ | — | — | — | 120 s | 74’846 |

Table 4. Computational experiments with A-I-I, BooleanConvert, and SatisfiabilityCount.

4.2 These are the main findings. Concerning CPU-time, in order for SatisfiabilityCount to excel it takes a large number of models (as for $(15, 4, 2) \rightarrow (75, 0)$ or $(40, 1, 10) \rightarrow (80, 0)$), yet this is not sufficient, as e.g. seen in $(40, 1, 10) \rightarrow (80, 10)$. In fact we venture to say, no matter how large the type of 2-CNF, if there are "few" models, then A-I-I always wins out (in the scenario of Table 4 "few" means < 100’000). (As to 0 models, see 6.2.)

On the other hand, concerning compression, A-I-I always trails\footnote{Nevertheless the compression of A-I-I can be high as witnessed by $(40, 1, 10) \rightarrow (80, 10)$ where an average 012-row houses about 2 million models. We mention in passing that the trailing order switches when wildcards beyond don’t-cares are applicable and pitted against BooleanConvert, see [arXiv:1812.02570v3] and [W3]. Thirdly, the compression of A-I-I is likely high enough to prevail over any one-by-one method (cf 6.4). If not, the benefit of compactly packaging the models in 012-rows, is worth a longer wait. Also keep in mind parallelization (3.4).} BooleanConvert (so $R_{BC} < R_{AII}$). Nevertheless A-I-I is often faster; e.g. in $(40, 1, 10) \rightarrow (80, 50)$ but not in $(15, 4, 2) \rightarrow (75, 0)$.

Digging a bit deeper let us see what happens when either $m$ or $lc$ or $ht$ varies while the other parameters are fixed. Letting $m = 0, 20, 1000$ in $(15, 4, 2) \rightarrow (75, m)$ the number of models (unsurprisingly) drops drastically. So do the times of BooleanConvert, and more so A-I-I. The time of SatisfiabilityCount (by whatever reason) sticks to about 0.3 seconds. Similarly for $m = 0, 20, 300, 2500, 7000$ in $(30, 3, 7) \rightarrow (120, m)$, and for $m = 0, 10, 50$ in $(40, 1, 10) \rightarrow (80, m)$.
We let the reader draw his own conclusions about what happens (easy) and why (enigmatic) when we increase $lc = 4, 5, 7, 9, 12$ in $(30, 3, lc) \to (120, 2500)$. Similarly for $ht = 2, 4, 6, 8, 14$ in $(37, ht, 7) \to (w, 2500)$.

5 From graph-endowed posets to Horn 2-CNFs

Consider any Horn 2-CNF $H : \{0, 1\}^w \to \{0, 1\}$. Hence the clauses of type $x_i \to x_j$ (i.e. $\overline{x_i} \lor x_j$) define the arcs $i \to j$ of a directed graph $D$ (not yet a poset) with vertex set $W = [w]$. Call the ‘vertices’ $i$ and $k$ strongly connected if there is (in $D$) a directed path from $i$ to $k$ and a directed path from $k$ to $i$. As is well known, this yields an equivalence relation whose classes (the strong components) are the elements of a poset $(W_D, \preceq_D)$. By definition $[s] \preceq_D [t]$ iff there is a directed path from $t$ to $s$ (thus not $s$ to $t$). (In Figure 1 all strong components were singletons.)

5.1 To fix ideas, suppose $w = 19$ and $D$ contains (among others) the arcs

$1 \to 3 \to 5 \to 7 \to 1$, $2 \to 4 \to 2$, $6 \to 8 \to 6$, $9 \to 14 \to 9$,
$10 \to 15 \to 10$, $11 \to 17 \to 11$, $16 \to 18 \to 19 \to 16$.

Accordingly the strong components could be

$1^* = \{1, 3, 5, 7\}$, $2^* = \{2, 4\}$, $6^* = \{6, 8\}$, $9^* = \{9, 14\}$, $10^* = \{10, 15\}$,
$11^* = \{11, 17\}$, $12^* = \{12\}$, $13^* = \{13\}$, $16^* = \{16, 18, 19\}$.

Suppose the remaining arcs induce a 9-element factor poset $(W_D, \preceq)$ as shown in Figure 3(a).

In every model $\overline{y}$ of $H$ the bits $y_i$ matching the vertices $i$ in a fixed strong component must have the same value, thus (say) $y_1 = y_3 = y_5 = y_7 = 1$ or $y_1 = y_3 = y_5 = y_7 = 0$ for each $\overline{y} \in \text{Mod}(H)$.

Recall there also exists a graph $G$ on $W$ whose antichains are targeted. Suppose one of the clauses in $H$ is $\overline{x_1} \lor \overline{x_2}$ (i.e. vertices 1, 5 are adjacent in $G$); then each model $\overline{y}$ must satisfy $y_1 = y_3 = y_5 = y_7 = 0$. Suppose that by similar reasons each model $\overline{y}$ must also satisfy $y_2 = y_4 = 0$ and $y_6 = y_8 = 0$. The corresponding strong components are colored dark in Figure 3(a). The order filter generated by $1^*, 2^*, 6^*$ is $F = \{1^*, 2^*, 6^*, 12^*, 13^*\}$, and so also $y_{12} = 0$ and $y_{13} = 0$ for each $H$-model $\overline{y}$. Generally the set complement, here $W^* := W_D \setminus F = \{9^*, 10^*, 11^*, 16^*\}$, is an order ideal of $(W_D, \preceq)$. 

Figure 3 (a)  
```
2* 13* 1* 10* 11* 12* 16* 9* 6*
```

(b)  
```
10* 11*
16* 9*
```

10
5.2 Only $W^*$ (not $W_D$) plays a role in the sequel. Namely, we look at $W^*$ as a poset $(W^*, \preceq^*)$ on its own, the partial ordering $\preceq^*$ being inherited from $(W_D, \preceq)$, see Figure 3(b). Likewise $G$ induces a graph $G^*$ with vertex set $W^*$. Generally by definition vertices $\alpha, \beta$ of $G^*$ are adjacent if there are $u \in \alpha$ and $v \in \beta$ which are adjacent in $G$. (Thus if 15, 18 are adjacent in $G$ above, then $15^* (= 10^*)$ and $18^* (= 16^*)$ remain adjacent in $G^*$.)

We feed the graph-endowed poset $(W^*, \preceq^*, G^*)$ to A-I-I. The output semifinal 012-rows $r^*$ of length $w^* := |W^*|$ are extended to 012-rows $r'$ of length $w' := \bigcup W^*$ as follows. By definition $\text{ones}(r')$ is the union of all sets $\alpha$ where $\alpha \in \text{ones}(r^*)$. Similarly $\text{zeros}(r')$ arises from $\text{zeros}(r^*)$. But from $\alpha \in \text{twos}(r^*)$ does not follow $\alpha \subseteq \text{twos}(r')$. Rather on all positions $i \in \alpha$ we must have the same bit (be it 0 or be it 1). This can be indicated by wildcards \[d, d, ..., d.\] In our example $W^* = \{9^*, 10^*, 11^*, 16^*\}$, and so $r^* := (2, 1, 2, 0)$ (indexed by the shelling (cf 2.1) $9^*, 16^*, 10^*, 11^*$ of Fig. 3(b)) gives rise to

$$r' = (y_9, y_{14}, y_{16}, y_{18}, y_{19}, y_{10}, y_{15}, y_{11}, y_{17}) = (0, 0, d_1, d_1, d_1, 1, 1, d_2, d_2).$$

Recall from 5.1 that $r'$ in turn gives rise to the final length $w$ row

$$r = (0, 0, 0, 0, 0, 0, 0, 0, y_9, y_{10}, y_{11}, 0, 0, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}, y_{19}).$$

We call the described algorithm the adapted A-I-I.

**Corollary:** Given a Horn 2-CNF $H$ on $w$ variables, the adapted A-I-I writes $\text{Mod}(H)$ as disjoint union of $R$ disjoint 012-rows in time $O(Rw^3)$.

6 From Horn 2-CNFs to general 2-CNFs

What if the 2-CNF $F$ also features positive 2-clauses? By switching suitable variables $x_i$ (i.e. replacing each occurrence of $x_i$ by $\bar{x}_i$, and each occurrence of $\bar{x}_i$ by $x_i$) each satisfiable 2-CNF $F(\bar{x})$ turns into a Horn 2-CNF $H(\bar{x})$. For instance $F_2(\bar{x}) = (\bar{x}_1 \lor x_2) \land (\bar{x}_2 \lor x_3) \land (x_3 \lor x_4)$ becomes $H_2(\bar{x}) = (\bar{x}_1 \lor \bar{x}_2) \land (x_2 \lor x_3) \land (\bar{x}_3 \lor x_4)$ upon switching $x_2$ and $x_3$. What is more, for any such $F$ and $H$ any compressed enumeration of $\text{Mod}(H)$ by 012-rows (e.g. achieved by the adapted A-I-I) carries over to an equally compressed enumeration of $\text{Mod}(F)$. Namely, putting $\bar{0} := 1, \bar{1} := 0, \bar{2} := 2, \bar{d} := d$ it is clear that this kind of bit-switching gives $\text{Mod}(F) = \{(\bar{x}, \bar{\beta}, ..., \bar{\gamma}) : (\alpha, \beta, ..., \gamma) \in \text{Mod}(H)\}$.

It remains to clarify which variables must be switched (6.1), and how satisfiability should be tested (6.2). Finally 6.3 and 6.4 glimpse at two competing methods for calculating the model set of a 2-CNF.

6.1 According to [S, Thm. 5.19] the following holds. Suppose a Boolean 2-CNF $F(\bar{x})$ is satisfiable, and $\bar{y}$ is any model. If $H(\bar{x})$ is the Boolean function obtained from $F(\bar{x})$ by switching those variables $x_i$ for which $y_i = 1$, then $H(\bar{x})$ is a Horn 2-CNF. In [S] the proof of Thm. 5.19 is left as exercise. Let us do it for the sake of completeness. So why, upon switching the indicated vari-
ables, does each $F$-clause become (or stay) either negative or mixed? Starting with a negative clause $x_i \lor \overline{x}_j$, the only way for it to turn positive is by switching both literals. This happens iff $y_i = y_j = 1$. But then $\overline{x}_i \lor \overline{x}_j = 0$, and so $F(\overline{y}) = 0$, contradicting the assumption that $F(\overline{y}) = 1$. Hence each negative $F$-clause stays negative or becomes mixed. Similar reasoning shows that each positive $F$-clause becomes mixed or negative, and that each mixed $F$-clause stays mixed or becomes negative. To illustrate further, consider the Boolean 2-CNF

\begin{equation}
F_3(\overline{x}) := (\overline{x}_1 \lor x_1) \land (x_2 \lor x_3) \land (x_3 \lor x_2) \land (x_7 \lor x_2) \land (x_7 \lor x_3) \land (x_5 \lor x_3) \land (x_5 \lor x_2) \land (x_8 \lor x_1) \land (x_8 \lor x_2) \land (x_8 \lor x_3).
\end{equation}

Having five positive clauses $F_3$ thoroughly fails to be a Horn 2-CNF. However, one checks that $y_5 = y_6 = y_7 := 1$ and $y_i := 0$ otherwise, yields a $F_3$-model $\overline{y}$. Accordingly switching the variables $x_5, x_6, x_7$ in (6) must yield a Horn 2-CNF. In fact, one obtains the Horn 2-CNF $H_1$ from 3.1. Recall from Table 3 that $\text{Mod}(H_1) = r_6 \lor r_7 \lor r_8 \lor r_{10} \lor r_{11}$. Upon bit-switching the 5th, 6th, 7th bit in these 012-rows we obtain

\[ \text{Mod}(F_3) = (2, 1, 0, 2, 1, 0, 1, 0) \lor (2, 2, 0, 2, 1, 1, 0) \lor \cdots \lor (1, 0, 1, 0, 2, 1, 1, 1). \]

Following up the described Horn-Renaming\footnote{This technique has been applied to a variety of Boolean functions, see [CH]. Apparently the present article ist the first to exploit the Horn-renamability of satisfiable 2-CNFs.} of $F$ with the adapted A-I-I yields the twice adapted A-I-I.

**Corollary:** Given a satisfiable 2-CNF $F$ on $w$ variables, the twice adapted A-I-I writes $\text{Mod}(F)$ as disjoint union of $R$ disjoint 012-rows in time $O(Rw^3)$.

### 6.2 How to test satisfiability? To sketch the most natural and (up to fine-tuning) fastest way to decide the satisfiability of a 2-CNF, consider

\[ F_4(\overline{x}) = (\overline{x}_1 \lor x_3) \land (\overline{x}_1 \lor x_4) \land (\overline{x}_2 \lor x_1) \land (\overline{x}_2 \lor x_4) \land (x_2 \lor x_4) \land (x_3 \lor x_4). \]

The *Labelling Algorithm* [CH,p.231] begins the construction of a potential model $\overline{y}$ by guessing any component, say $y_2 = 1$. The clause $\overline{x}_2 \lor x_1$ then implies $y_1 = 1$. Likewise $y_4 = 0$ in view $\overline{x}_2 \lor \overline{x}_4$. Further $\overline{x}_1 \lor x_3$ forces $y_3 = 1$, and so $\overline{x}_1 \lor x_4$ forces $y_4 = 1$. This contradicts $y_4 = 0$, and so the initial guess $y_2 = 1$ must be refuted. Put another way, if there is any $F_4$-model $\overline{y}$, it must have $y_2 = 0$. Arguing as at the beginning of Section 1 (viewing that $F_1(\overline{x}) = F_4(\overline{x}) \land \overline{x}_2$), this yields $y_4 = 1$ and $\overline{x}_1 \lor y_3 = 1$. Generally speaking, if no more components can be forced, one resumes guessing. In our case guessing (say) $y_1 = 1$ forces $y_3 = 1$. All components are now assigned, and so $\overline{y} = (1, 0, 1, 1)$ is a model. One can show that the Labelling Algorithm either yields a model as sketched above, or at some stage both guesses $y_i = 0$ and $y_i = 1$ lead to contradictions, in which case the 2-CNF is insatisfiable.

### 6.3 Let us sketch another satisfiability test for 2-CNFs which is based on [APT]. Previously we only interpreted mixed clauses like $\overline{x}_1 \lor x_2$ as arcs $x_1 \rightarrow x_2$ in a directed graph $D$ with vertex set \{x_1, \ldots, x_w\} (or \{1, \ldots, w\}). In [APT] each clause gives rise to two arcs; for instance $x_1 \lor x_3$ yields $\overline{x}_1 \rightarrow x_3$ and $\overline{x}_3 \rightarrow x_1$. Accordingly each 2-CNF $F(\overline{x})$ yields a directed graph $\Delta = \Delta(F)$ (called implication graph [CH,p.212]) with double-sized vertex set $W = \{x_1, \overline{x}_1, \ldots, x_w, \overline{x}_w\}$.

If $S \subseteq W$ is any strong component, then each $F$-model $\overline{y}$ is such that all components $y_i$ that match a literal in $S$ have the same value. In particular, if $S$ contains both $x_i$ and $\overline{x}_i$, then $F$ is insatisfiable.
Conversely, and less obvious, if all strong components are clean (i.e. none contains any $x_i$ and $\overline{x_i}$ simultaneously), there exists an $F$-model $\vec{y}$.

Why bother mentioning the [APT]-method, it being slower (due to calculating strong components) than the Labelling Algorithm? Because the factor poset $(W_\Delta, \preceq)$ of the digraph $\Delta$ is the basis of the Bisection-Factory in [arXiv:1208.2559 v5] that enumerates all models of a given 2-CNF $F$. Some details. The map $\omega : W \rightarrow W$, $\omega(\overline{x_i}) := \overline{x_i}$, $\omega(x_i) := x_i$, induces an involution of $W_\Delta$, i.e. $\omega \circ \omega$ is the identity. The models of $F$ correspond bijectively to the bisections of $(W_\Delta, \preceq)$, i.e. to the filter-ideal partitions $(X,Y)$ of $W$ with the extra property that $\omega(X) = Y$. The Bisection-Factory enumerates all $N$ bisections in total time $O((N + 1)w^2)$. It also offers compression of the modelset but how this compares to the compression of A-I-I remains to be seen. That's because part of the former compression results from factoring out strong components and (different from Table 4) no competing methods were evaluated in [arXiv:1208.2559 v5]. More clear-cut benefits of the Bisection-Factory are these:

(a) Apart from strong components which occur in both (though w.r.t. different digraphs), the inner workings of the Bisection-Factory are more straightforward than the ones of the twice-adapted A-I-I (and even more so than the ones in [F1] below).

(b) The Bisection-Factory has been implemented in Mathematica (and is available on request), whereas the twice-adapted A-I-I has not yet been.

(c) A partial model of a Boolean function $F$ is any bitstring that can be extended to a model of $F$. The Bisection-Factory readily adapts to enumerate all partial models with respect to any fixed subset of literals.

(d) Related to partial models are quantified 2-CNFs (of which our 'plain' 2-CNFs constitute the special case where all quantifiers are existential), and probably the Bisection-Factory can be fine-tuned to the enumeration of all models of a quantified 2-CNF.

6.4 Last not least, it seems that so far the only algorithm dedicated to the enumeration of all models of a 2-CNF is Feder’s method [F1] which enumerates the models one after the other with delay $O(w)$. While it hence formally beats both the twice-adapted-AII and the Bisection-Factory (their 012-rows are output with delay $O(w^2)$), Feder’s enumeration seems doomed to be one-by-one. As to “seems”, the author admits to not understanding the fine details of the method, due to its network-theoretic framework. (For a gentle account of Feder’s work see [M,2.3.2].) Readers knowing of an implementation, or even evaluation, of Feder’s method, please let the author know. In [F2] Feder shows that each so-called Stable Roommates Instance (SRI) can be modelled by a 2-CNF, and conversely. As testified by the impressive monograph [M], many real-world problems are modelled by SRI’s, and so each fast method to (compactly) enumerate all SRI-instances (equivalently: 2-CNF models) is highly welcome; then one can pick the ”best” SRI-instance w.r.t. a variety of otherwise invincible extra criteria.

13Actually the main purpose of [APT] a linear time method for deciding the satisfiability of a quantified 2-CNF. This result is recast in fresh terminology in [arXiv:1208.2559 v1 of 2012]. All six versions v1 to v6 are precursors of the present article but overlap little with it. While the presentation in v1 to v6 is partly wanting, there are worthy bits to be unearthed (e.g. in a PHD project).
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