Pati-Salam model in curved space-time from square root Lorentz manifold

De-Sheng Li

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There is a $U(4')U(4)$-bundle on four-dimensional square root Lorentz manifold. Then a Pati-Salam model in curved space-time (Lagrangian) and a gravity theory (Lagrangian) are constructed on square root Lorentz manifold based on self-parallel transportation principle. An explicit formulation of Sheaf quantization on this square root Lorentz manifold is shown. Sheaf quantization is based on superposition principle and construct a linear Sheaf space in curved space-time. The transition amplitude in path integral quantization is given which is consistent with Sheaf quantization. All particles and fields in Standard Model (SM) of particle physics and Einstein gravity are found in square root metric and the connections of bundle. The interactions between particles/fields are described by Lagrangian explicitly. There are few new physics in this model. The gravity theory is Einstein-Cartan kind with torsion. There are new particles, right handed neutrinos, dark photon, Fiona, $X^E$ and $Y^0, Y^1, Y^2, Y^1_1, Y^2_2$.

INTRODUCTION

Four-dimensional pseudo Riemann geometry with signature $(−,+,+,+)$, Lorentz manifold, is the geometry background of the general relativity, space-time is described by the metric, and the gravitational field is described as the curve of space-time. In general relativity, the geodesic equation describes the trajectories of free particles, and the Einstein equation determines how matter curves space-time. At the last life time of Einstein, he attempted to establish a new geometry unifying electromagnetic interaction and gravity. This idea was developed by Weil into the early idea of gauge invariance and by the Kaluza and Klein into the idea of extra dimensions.

Later, the Yang-Mills theory [11] was confirmed. Yang-Mills theory takes gauge invariance as its basic principle and to be the theoretical framework of electromagnetic, weak and strong interaction in Standard Model (SM) of particle physics [2–7]. The Yang-Mills theory is the theoretical framework of SM and has a good correspondence with the complex structure group G fiber bundle theory [8]. General relativity can actually be rewritten in the framework of fiber bundle theory also, except that the G structure group of general relativity is real, and the corresponding fiber bundles are the tangent and cotangent bundles. Another way to build unified field theory is introducing extra dimensions to give all fields their geometric positions. And lots of attempts in extra dimension were made. Is it possible to fuse the tangent (cotangent) bundle of general relativity with the complex structure group G-bundle of Yang-Mills theory?

Inspired by the Dirac’s way of finding his equation and spinors through making square root of the Klein-Gordon equation, we researched the square root of the four-dimensional Lorentz manifold, which similar with the papers in Clifford algebra or Clifford bundle [9][52], spin-gauge theory in Riemann-Cartan space-time [53 54], sedenion [55] and Einstein-Cartan theory [56][58] etc.

Four-dimensional square root Lorentz manifold has extra $U(4') × U(4)$ principal bundle than Lorentz manifold. Two Lagrangians based on four-dimension square root Lorentz manifold are constructed which describe a $U(4') × U(4)_L × U(4)_R$ Pati-Salam model in curved space-time and a gravity theory, respectively. In the Pati-Salam model [59], the $SU(4')$ is color group with “lepton number as the fourth color”, and the $SU(4)_L × SU(4)_R$ is chiral flavor group. We realize an explicit formulation of Sheaf quantization [60][76] scheme which consists with path integral quantization. The particles spectrum on this model is discussed.

GEOMETRY AND LAGRANGIAN

The notations are introduced here. $a,b,c,d$ represent frame indices, and $a,b,c,d = 0,1,2,3$. $\mu, \nu, \rho, \sigma$ represent coordinates indices, and $\mu, \nu, \rho, \sigma = 0,1,2,3$. $\alpha$ represent group indices, and $\alpha = 0,1,\cdots,15$. $i,j,k,l,m,n = 1,2,3,4$. $C = R,G,B = 1,2,3$ is quarks color. $\kappa$ is Sheaf space index. Repeated indices are summed by default.

The pseudo Riemann manifold is described by a metric

$$g(x) = −g_{\mu\nu}(x)dx^\mu \otimes dx^\nu,$$ (1)

$$g_{\mu\nu}(x) = g_{\nu\mu}(x), \quad g_\nu = det(g_{\mu\nu}(x)),$$ (2)

where $\{x| (x^\mu ) = (r, \vec{x})\}$ is a four-dimensional topological space. Here we discuss the four-dimensional pseudo Riemann manifold with signature (−, +, +, +), Lorentz manifold. And it can be described by orthonormal frame formalism as

$$g^{-1}(x) = −\eta^{ab}_\mu \theta_a(x)\theta_b(x),$$ (3)

where $\eta^{ab} = diag(1,−1,−1,−1)$ and $\theta_a(x) = \theta^{\mu}_a(x)\frac{\partial}{\partial x^\mu}$ are orthonormal frames and describe gravitational field.

The definition of gamma matrices is

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{ab}I_{\mu\nu}4.$$

(4)

The Hermiticity conditions for gamma matrices are

$$\gamma^\mu \theta^\dagger \gamma^\nu + \gamma^\nu \theta^\dagger \gamma^\mu = 2I^{ab}I_{\mu\nu}4,$$

(5)

where $I^{ab} = diag(1,1,1,1)$. We define

$$L(x) = i\gamma^{\mu}_a(x)\gamma^\nu_b(x)e^\nu_j \otimes e_i \theta_a(x),$$

(6)

$$\bar{L}(x) = i\gamma^\dagger_a(x)\gamma^\nu_b(x)e^\nu_j \otimes e_i \theta_a(x),$$

(7)
where $e_i$ are the orthogonal bases expanding four-dimension complex space $\mathbb{C}^4$ and

$$\text{tr}(e_j^i \otimes e_i) = e_ie_j = \delta_{ij}. \quad (8)$$

One simple choice of $e_i$ is

$$e_1 = (e^{i\theta_1},0,0,0), \quad e_2 = (0,e^{i\theta_2},0,0), \quad e_3 = (0,0,e^{i\theta_3},0), \quad e_4 = (0,0,0,e^{i\theta_4}). \quad (9)$$

After using $\gamma^\dagger = \gamma^0 \gamma^\rho \gamma^\sigma$, we find that

$$g^{-1}(x) = \frac{1}{4} \text{tr} [\tilde{l}(x)l(x)]. \quad (11)$$

Then $l(x)$ or $\tilde{l}(x)$ are the square root of metric $g$ in some sense. The $\eta_{ij}(x)$ can be shown as

$$\gamma^\mu_{\nu j}(x) = \eta^i_{\nu j}(x)\gamma^\mu_{\nu i}(x) = \tilde{\eta}_i(x) \gamma^\mu \psi_i(x), \quad (12)$$

$$\gamma^\mu_{\nu j}(x) = \eta^i_{\nu j}(x)\gamma^\mu_{\nu i}(x) = \tilde{\eta}_i(x) \gamma^\mu \psi_i(x), \quad (13)$$

where $\psi_i$ are the Dirac fermions field with flavor related index $i = 1,2,3,4$. $\psi_i(x) = \psi_i^\dagger(x)\gamma^\rho \psi_i(x) \in U(4)$ is $4 \times 4$ matrix. So, the square root metric are defined as follow

$$l(x) = i\tilde{\psi}(x) \gamma^\rho \psi(x) e_i^\dagger \otimes e_i \theta_a(x), \quad (12)$$

$$\tilde{l}(x) = i\psi(x) \gamma^\rho \psi(x) e_i^\dagger \otimes e_i \theta_a(x). \quad (13)$$

The square root Lorentz manifold is described by square root metric and gauge fields on the $U(4') \times U(4)$-bundle are defined as follows

$$\nabla_{\mu} \gamma^\rho \gamma^\sigma = i[V_{\mu}(x) \gamma^\rho \gamma^\sigma - \gamma^0 \gamma^\rho V_{\mu}(x)], \quad (16)$$

$$\nabla_{\mu} e_i^\dagger = iW_{\mu ij}(x)e_j^\dagger, \quad (17)$$

where $\Gamma^\mu_{\rho \sigma}(x)\theta^\rho_{\mu}(x) = \partial_\mu \theta^\rho_{\mu}(x) + \theta^\rho_{\nu}(x)\Gamma^\mu_{\nu \mu}(x)$ is found and $V_{\mu}(x) = V_{\mu}(x), W_{\mu ij}(x) = W_{\mu ij}(x)$. The uniqueness of definition of gauge fields is originated from restriction $\epsilon_3$ and $g$. The equation as follow can be derived from $\epsilon_3$

$$\nabla_{\mu} (\gamma^\rho \gamma^\sigma) = i[V_{\mu}(x) \gamma^\rho \gamma^\sigma - \gamma^0 \gamma^\rho V_{\mu}(x)], \quad (18)$$

$$\nabla_{\mu} e_i^\dagger = iW_{\mu ij}(x)e_j^\dagger, \quad (19)$$

where $V_{\mu}(x) = V_{\mu}(x), W_{\mu ij}(x) = W_{\mu ij}(x)$ are gauge bosons fields. The $\gamma^\alpha$ are the generators of $U(4)$ and an explicit one can be seen in appendix. A equation which satisfying the $U(4') \times U(4)$ gauge invariant, locally Lorentz invariant and generally covariant principles is constructed

$$\text{tr}[\hat{l}(x)l(x)] = 0, \quad (21)$$

this equation originated from generalized self parallel transportation principle. Eliminating index $x$, the explicit form of equation (21) is

$$[(i\partial_\mu \tilde{\psi}_i(x) + V_\mu \psi_i(x) + \theta_{\mu ij}(x)) \gamma^\alpha \psi_i(x) + \psi_i(x) \gamma^\alpha (i\partial_\mu \psi_i(x) + V_\mu \psi_i(x) - \psi_j(x))W_{\mu ij}(x)] \theta_a^\mu = 0. \quad (22)$$

We define a Lagrangian

$$\mathcal{L} = \tilde{\psi}_i(x) \gamma^\rho \psi_i(x) + \psi_i(x) \gamma^\rho \psi_i(x) \in U(4) \times U(4)$$

The last term in Lagrangian (22) is Yukawa coupling term $\psi_i(x) \gamma^\rho \psi_i(x)$ and the scalar (Higgs) field is gamma matrix valued and originatd from gravitational field

$$\phi = \frac{i}{2} \Gamma^\mu_{\nu \mu} \theta_a^\mu. \quad (23)$$

Then, the Lagrangian (22) describes $U(4') \times U(4)$ Yang-Mills theory in curved space-time. The Lagrangian (22) has relation with (21)

$$\text{tr}\hat{l}(x) = \mathcal{L} - \mathcal{L}^\dagger. \quad (24)$$

If equation (21) being satisfied, the Lagrangian (22) is Hermitian

$$\mathcal{L} = \mathcal{L}^\dagger. \quad (25)$$

So, the unitary principle of quantum field theory (25) consists with generalized self parallel transportation principle (21). The equations of motion for the Lagrangian (22) are

$$\gamma^\rho (i\partial_\mu \psi_i + V_\mu \psi_i - \psi_j W_{\mu ij}^a \theta_a^\mu + \frac{i}{2} \gamma^\rho \psi_i \Gamma^a_{\nu \mu} \theta_a^\mu) = 0. \quad (26)$$
and this equation’s conjugate transpose. We point out that (21) is density matrix version of (26). The effective equation of motion of (26) has signature $(1, -1, -1, -1)$. For example, the massless Dirac equation in curved space-time of this model is

$$iγ^μ \partial_μψ_i = 0.$$  \hspace{1cm} (27)

The square of equation (27) is massless Klein-Gordon equation in curved space-time.

$$η^{ab}θ^α_μ \theta^β_ν \partial_μ \partial_νψ_i = 0.$$  \hspace{1cm} (28)

The signature of equation (28) is $(1, -1, -1, -1)$ and consistent with special relativity.

Then, a Lagrangian (22) which describes the $U(4') \times U(4)$ Yang-Mills theory in curved space-time is constructed. Where the gravitational field $θ^c_μ$ is all other fields (except Higgs field) dynamical background which satisfies the characteristic of the gravitational field in our real world.

Lagrangian (22) is $U(4') \times U(4)$ gauge invariant, locally Lorentz invariant and generally covariant. So, Lagrangian (22) is demanded invariant under the transformations

$$ψ_i^T = Uψ_iU_{ji}, γ^µ = U^Tγ^µU \Lambda^a_μ, \theta^a_μ = U\Lambda^a_μ,$$  \hspace{1cm} (29)

where $U \in U(4'), (U_{ji}) \in U(4), \Lambda^a_μ \in O(1,3), \Lambda^a_μ \in GL(4, \mathbb{R})$. Then, the transformation rules have to be derived as follows

$$V^T_μ = UV_μU^T_μ - (∂_μ U)U^T_μ,$$  \hspace{1cm} (30)

$$W^T_μ = UW_μU_μ + U_μi∂_μU_μ,$$  \hspace{1cm} (31)

$$\Gamma^T_α_μ_β = \Lambda^a_γ \Gamma^c_γ_μ_β \Lambda^b_ε - \Lambda^a_γ \partial_μ \Lambda^c_γ \Lambda^b_ε,$$  \hspace{1cm} (32)

where $U^T_μ = I, U_μU_μ = δ_μ and \Lambda^a_μ \Lambda^b_ε = δ^c_ε \Lambda^b_ε$ are used. Now, we complete the proof of the Lagrangian (22) is $U(4') \times U(4)$ gauge invariant, locally Lorentz invariant and generally covariant.

The gauge field strength tensors and curvature tensor are defined as follows

$$H^a_μ_ν = \partial_μ V_ν - \partial_ν V_μ - iV_μV_ν + iV_νV_μ,$$  \hspace{1cm} (33)

$$F^a_μ_νij = \partial_μ W_νij - \partial_ν W_μij - iW_μikW_νkj + iW_νikW_μkj,$$

$$R^a_μ_ν_σ = \partial_μ \Gamma^a_ν_σ - \partial_ν \Gamma^a_μ_σ + \Gamma^a_ρ_μ \Gamma^c_ρ_ν_σ - \Gamma^c_ρ_ν \Gamma^a_ρ_μ_σ,$$

where $R^a_μ_ν_σ = -R_μ_ν_σ$ if $∇g = 0$ and $H^T_μ_ν = H_μ_ν, F^a_μ_νij = F^a_μ_νij$. The gauge field strength can be decomposed by the $U(4)$ generators $H^a_μ_ν = H^a_μ_ν \bar{σ}^α, F^a_μ_νij = F^a_μ_νij \bar{σ}^α$. After the torsion being defined

$$T^α_μ_ν = 2\bar{σ}^α_μ [ψ_ν],$$  \hspace{1cm} (34)

we have the Ricci identity and Bianchi identity (78) on this geometry structure as follows

$$\partial_μ H^a_ν_ρ = H^a_μ_ν_δ - V_μ H^a_ν_ρ,$$  \hspace{1cm} (35)

$$\partial_μ F^a_ν_ρ_σ = F^a_μ_ν_ρ_σ - W^a_μ_ρ_σ,$$

$$T^α_μ_ν_ρ_σ = R^α_ρ_σ_μ_ν + ∇_{[ρ} T^α_μ_ν],$$  \hspace{1cm} (36)

$$∇_{[ρ} R^α_μ_ν_σ] = R^α_μ_ν_σ T^α_ρ_σ.$$  \hspace{1cm} (37)

There is Yang-Mills Lagrangian for gauge bosons in this model

$$L^γ = -\frac{1}{2} \text{tr} (H^μ_ν H_μ_ν) - \frac{ξ}{2} F^μ_ν F_μ_ν,$$  \hspace{1cm} (38)

where $ξ \in \mathbb{R}$ is constant.

For the gravity, the Einstein-Hilbert action in Lorentz manifold be showed as follow

$$S = \int ω R,$$  \hspace{1cm} (39)

where $R$ is the Ricci scalar curvature in Lorentz manifold, $ω = \sqrt{-g} dx^0 ∧ dx^1 ∧ dx^2 ∧ dx^3$ is volume form. And in this geometry framework, the equations can be derived as follows

$$∇_{[μ} ψ_ν] i = \frac{1}{2} \left( ψ_μ γ^ν ψ_ν F_μ_ν i - F^σ_μ_ν ψ_ν γ^ν ψ_μ i + \frac{1}{2} ψ_ν γ^ν ψ_μ R^σ_μ_ν_σ \right) e^γ_ν ⊗ e^i_σ,$$

$$∇_{[μ} ψ_ν] i = \frac{1}{2} \left( ψ_μ γ^ν ψ_ν F_μ_ν i - F^σ_μ_ν ψ_ν γ^ν ψ_μ i + \frac{1}{2} ψ_ν γ^ν ψ_μ R^σ_μ_ν_σ \right) e^γ_ν ⊗ e^i_σ,$$  \hspace{1cm} (40)

$$\left( ψ_μ γ^ν ψ_ν F_μ_ν i - F^σ_μ_ν ψ_ν γ^ν ψ_μ i + \frac{1}{2} ψ_ν γ^ν ψ_μ R^σ_μ_ν_σ \right) e^γ_ν ⊗ e^i_σ,$$

where $H_μ_ν = ψ_μ γ_ν ψ_ν$. We define $∇_μ = \nabla_μ dx^μ ∧ dx^ν$, the equation of this gravity theory is constructed

$$\text{tr} ∇_μ (l(x) l(x)) = 0.$$  \hspace{1cm} (42)

This equation (42) is obviously $U(4') \times U(4)$ gauge invariant, locally Lorentz invariant and generally covariant. The explicit formula of equation (42) is

$$R = \frac{i}{4} \left( F^a_μ_ν ij \left( γ^a γ^μ - γ^μ γ^a \right) ψ_i - H_μ_ν \left( γ^μ γ^ν - γ^ν γ^μ \right) ψ_i \right),$$  \hspace{1cm} (43)

where $∇_μ dx^ν ⊗ dx^ν \partial_μ = δ_μ \partial_ν α^μ, dx^μ ⊗ dx^ν \partial_ν = δ^ν_μ α^μ$ are used and $F^a_μ_ν ij = F^a_μ_ν ij \bar{σ}^α, H_μ_ν = H_μ_ν \bar{σ}^α$. So we define a $U(4') \times U(4)$ gauge invariant, locally Lorentz invariant, generally covariant Lagrangian

$$L_γ = R^a_μ_ν_σ ψ_i - i \left( F^a_μ_ν ij \left( γ^μ γ^ν - γ^ν γ^μ \right) ψ_i - H_μ_ν \left( γ^μ γ^ν - γ^ν γ^μ \right) ψ_i \right).$$  \hspace{1cm} (44)

The Lagrangian (44) is Hermitian

$$L_γ = L_γ^\dagger.$$  \hspace{1cm} (45)

The $R^a_μ_ν_σ ψ_i$ in Lagrangian (44) gives us the Einstein-Hilbert action. The equation (43) and the Einstein tensor can be derived from the Einstein-Hilbert action.

**Sheaf Quantization and Path Integral Quantization**

The entities $l(x)$ and $i(x)$ are two sections of the two bundles, respectively, where these two bundles are dual to each
other. Further, the Sheaf valued entities \( \tilde{l}(x) \) and \( \hat{l}(x) \) can be defined

\[
\tilde{l}(x) = \sum_{\kappa} \eta_{\kappa}(x) \langle \kappa, x | l_k(x) \rangle,
\]

\[
\hat{l}(x) = \sum_{\kappa} \eta_{\kappa}(x) \langle \kappa, x | \hat{l}_k(x) \rangle,
\]

where \( \eta_{\kappa}(x) \in [0, 1] \) are probability of corresponding section \( l_k(x) \) and \( \hat{l}_k(x) \), \( \kappa \) is Sheaf space index and evaluated in an abelian group. The density matrix corresponds to \( \tilde{l}(x) \) and \( \hat{l}(x) \) is

\[
\rho(x) = \sum_{\kappa} \eta_{\kappa}(x) \langle \kappa, x | \kappa, x \rangle.
\]

We have orthogonal bases in Sheaf space and probability complete formulas

\[
\langle \kappa, x | \kappa', x' \rangle = \delta(x - x') \delta(\kappa - \kappa'),
\]

\[
\text{tr} \rho(x) = \sum_{\kappa} \eta_{\kappa}(x) = 1.
\]

In mathematic, a Sheaf is a collection of sections, the index \( \kappa \) of each section correspond to an abelian group element. In physics, the Sheaf spaces \( Sh \) of each section correspond to an abelian group element. In mathematic, a Sheaf is a collection of sections, the index \( \kappa \) of each section correspond to an abelian group element. In physics, the Sheaf spaces \( Sh \) of each section correspond to an abelian group element.

The corresponding total Lagrangian density is

\[
\mathcal{L} = \sum_{\kappa} \eta_{\kappa}(x) \mathcal{L}_k + g \mathcal{L}_g + \bar{g} \mathcal{L}_\bar{g},
\]

where \( g, \bar{g} \) are Lagrange multipliers.

For pure state

\[
\rho(x) = |\Psi(x)\rangle \langle \Psi(x)|,
\]

\[
|\Psi(x)\rangle = \sum_{\kappa} \alpha_{\kappa}(x) |\kappa\rangle,
\]

where the \( |\Psi(x)\rangle \) is the quantum state of quantum field theory. The transition amplitude can be defined through path integral formula

\[
\alpha_{\kappa}(t, \bar{x}) = \int_{t' \in (t_0, t)} D\kappa(t', \bar{x}) e^{i \alpha \mathcal{L}[\kappa(t', \bar{x})]} \alpha_{\kappa}(t_0, \bar{x}).
\]

FIG. 2. Left: The fiber bundle structure and \( l(x) \) is a section of the bundle. Right: A Sheaf is a collection of the sections. \( \tilde{l}(x) \) is Sheaf valued.

### PARTICLES SPECTRUM

\( V^\alpha_\mu \) and \( W^\alpha_\mu \) (\( \alpha = 0, 1, \ldots, 15 \)) are gauge bosons fields. The interactions related with \( W^\alpha_\mu \) always preserves the possibility of chiral symmetry breaking such that the gauge group can decomposed to \( U(4') \times U(4)_L \times U(4)_R \), where \( U(4') \) is color group and \( U(4)_L \times U(4)_R \) is chiral flavor group. The \( V^0_\mu \) is dark photon and \( W^0_\mu \) is Fiona particle. The left over part gauge group is a Pati-Salam gauge group \( SU(4') \times SU(4)_L \times SU(4)_R \) [59] [79] and the \( SU(4') \) can be decomposed as follow

\[
SU(4') = SU(3') + U(1') + U_{X+} + U_{X-}.
\]

The \( SU(3') \) is the gauge group of quantum chromodynamics (QCD) and the corresponding gauge bosons \( V^{\alpha}_\mu \) (\( \alpha = 1, 2, \ldots, 8 \)) are gluons. The \( U(1') \) is electro-magnetic interaction group and corresponding gauge boson \( V^{15}_\mu \) is photon \( \gamma \). The \( X^{\pm} \) particles transport semi-leptonic processes and

\[
X^{\pm C} = V^{8+c}_\mu \mp iV^{9+c}_\mu.
\]

The electric charge of \( X^+ \) and \( X^- \) are \( \frac{1}{3} \) and \( -\frac{1}{3} \). The chiral gauge group \( SU(4)_{L,R} \) can be decomposed as

\[
SU(4)_{L,R} = SU(3)_Y + U(1)_{1Z} + U_{W^+} + U_{W^-},
\]

and related gauge bosons \( W^\alpha_\mu \) (\( \alpha = 1, 2, \ldots, 15 \)) contain weak bosons \( W^\pm \) and \( Z \)

\[
W^\pm_\mu = W^9_\mu \pm iW^{10}_\mu = W^{11}_\mu \pm iW^{12}_\mu = W^{13}_\mu \pm iW^{14}_\mu,
\]

\[
Z_\mu = W^{15}_\mu.
\]

The left over gauge bosons are \( Y^0, Y^1, Y^2 \) and \( Y', Y'' \) with 0 electric charge. The gauge bosons \( Y^1, Y^2, Y'^1, Y'^2 \) transport non-SM flavor changing neutral currents (FCNCs) and

\[
\begin{pmatrix}
(2 - \eta) Y^0_\mu & Y^1_\mu & Y^2_\mu \\
\eta Y^0_\mu & Y^1_\mu & Y^2_\mu \\
0 & -2Y^0_\mu & Y^1_\mu
\end{pmatrix} = 2 \sum_{\alpha=1}^8 W^\alpha_\mu \mathcal{T}^\alpha.
\]

The \( X^{\pm} \) and \( Y^1, Y^2, Y'^1, Y'^2 \) must be superheavy from the restrictions of experimental data.

The fermionic fields \( \psi_i \) transform as the \( U(4') \times U(4) \) fundamental representation according to [20]. So, fermions are filled into the \( SU(4) \) fundamental representation \( 4 \otimes 6 \) naturally. The fundamental representation \( 4 \) corresponds to 3 colors and 1 lepton and leads us reobtain “Lepton number as the
The decomposition of $SU(4)$ adjoint representation is $15 = 8 \oplus 1 + 3 + 3^*$. (a) The weight diagram of $V^\alpha_\mu (\alpha = 1, 2, \ldots, 15)$ related gauge bosons. (b) The weight diagram of $W^\alpha_\mu (\alpha = 1, 2, \ldots, 15)$ related gauge bosons.

The particles spectrum in this model is discussed. Lorentz manifold are derived. A Sheaf quantization scheme might be

$$\psi_i = \begin{pmatrix} \sqrt{2} \mu_R & \sqrt{2} \tau_R & \sqrt{2} t_R & d'_R \\ \sqrt{2} \mu_G & \sqrt{2} \tau_G & \sqrt{2} t_G & d'_G \\ \sqrt{2} \mu_B & \sqrt{2} \tau_B & \sqrt{2} t_B & B \\ e & \mu & \tau & \nu' \end{pmatrix}.$$ (65)

where $u, c, t$ and $d'$ are quarks fields, $e, \mu, \tau$ and $\nu'$ are electron, mu, tau and neutrinos fields. The corresponding fermions electric charges of (65) are

$$Q = \begin{pmatrix} 2/3 & 2/3 & 2/3 & -1/3 \\ 2/3 & 2/3 & 2/3 & -1/3 \\ 2/3 & 2/3 & 2/3 & -1/3 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$ (66)

The quarks states like $|d\rangle, |s\rangle, |b\rangle$ and neutrinos states $|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle$ are eigen states of the Lagrangian (79).

**CONCLUSION AND DISCUSSION**

A Pati-Salam model and a gravity theory from square root Lorentz manifold are derived. A Sheaf quantization scheme which consists with path integral quantization is shown. The particles spectrum in this model is dicussed.

Some possible new physics on this model are listed as follows. There are exotic gauge bosons such as dark photon, Fiona, $X^\pm$ and $Y^0, Y_1^+, Y_2^+, Y_1^-, Y_2^-$. The $X^\pm$ transports semi-leptonic processes, the $Y_1^+, Y_2^+, Y_1^-, Y_2^-$ transport non-SM FC-NCs. The right handed neutrinos are existed. The Higgs field is gamma matrix valued.

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Generators of $U(4)$

\[
\mathcal{J}^1 = \frac{1}{2} \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \mathcal{J}^2 = \frac{1}{2} \begin{pmatrix}
0 & 0 & -i & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\mathcal{J}^3 = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \mathcal{J}^4 = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 1 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\mathcal{J}^5 = \frac{1}{2} \begin{pmatrix}
0 & 0 & i & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \mathcal{J}^6 = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\mathcal{J}^7 = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \mathcal{J}^8 = \frac{\sqrt{3}}{6} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\mathcal{J}^9 = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \mathcal{J}^{10} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\mathcal{J}^{11} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad \mathcal{J}^{12} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0
\end{pmatrix}, \\
\mathcal{J}^{13} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \mathcal{J}^{14} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0
\end{pmatrix}, \\
\mathcal{J}^{15} = \frac{\sqrt{5}}{12} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -3
\end{pmatrix}, \quad \mathcal{J}^0 = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

* At College of Physics, Mechanical and Electrical Engineering, Jishou University, Jishou 416000, P. R. China; Also at Interdisciplinary Center for Quantum Information, National University of Defense Technology, Changsha 410073, P. R. China; Also at Institute of High Energy Physics, Chinese Academy of Sciences, 19B Yuquan Road, Beijing 100049, P. R. China; E-mail: lideshengjiy@126.com

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