Lepton asymmetry rate from quantum field theory: NLO in the hierarchical limit

D. Bödeker and M. Sangel

Fakultät für Physik, Universität Bielefeld,
33501 Bielefeld, Germany

E-mail: bodeker@physik.uni-bielefeld.de, msangel@physik.uni-bielefeld.de

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Abstract. The rates for generating a matter-antimatter asymmetry in extensions of the Standard Model (SM) containing right-handed neutrinos are the most interesting and least trivial coefficients in the rate equations for baryogenesis through thermal leptogenesis. We obtain a relation of these rates to finite-temperature real-time correlation functions, similar to the Kubo formulas for transport coefficients. Then we consider the case of hierarchical masses for the sterile neutrinos. At leading order in their Yukawa couplings we find a simple master formula which relates the rates to a single finite temperature three-point spectral function. It is valid to all orders in $g$, where $g$ denotes a SM gauge or quark Yukawa coupling. We use it to compute the rate for generating a matter-antimatter asymmetry at next-to-leading order in $g$ in the non-relativistic regime. The corrections are of order $g^2$, and they amount to 4% or less.

Keywords: leptogenesis, baryon asymmetry, cosmology of theories beyond the SM

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1 Introduction

The Standard Model of particle physics has been very successfully tested at high energies. It can, however, neither explain neutrino oscillations, nor the matter-antimatter asymmetry of the Universe. These two open problems may have an elegant common solution. By extending the Standard Model by right-handed, or sterile neutrinos one can give masses to neutrinos which can cause them to oscillate. Their Yukawa couplings introduce a new source of $CP$ violation. These couplings are very small, so that sterile neutrinos easily deviate from thermal equilibrium in the early Universe. Electroweak sphalerons rapidly violate baryon plus lepton number at temperatures $T \gtrsim 130\text{GeV}$. Then all three Sakharov conditions are satisfied and the baryon asymmetry of the Universe can be generated [1]. This is referred to as baryogenesis through leptogenesis.
Many aspects and scenarios of leptogenesis have been studied, covering a vast range of sterile-neutrino masses (for reviews see e.g. [2, 3]). Originally leptogenesis was formulated using the Boltzmann equation, augmented with additional prescriptions, to overcome inconsistencies or to account for medium effects. The Boltzmann equation already contains a set of implicit assumptions and approximations. In order to eliminate ambiguities, and to assess the accuracy of the calculation of the baryon asymmetry one has to start from first principles. This has led various authors to start from Kadanoff-Baym or similar equations for Green’s functions (cf. refs. [4–6] for recent work and references). This way ambiguities have been clarified for resonant leptogenesis and a computation of flavor effects was possible [7]. In order to assess the theoretical error of leptogenesis calculations one has to identify appropriate expansion parameters and then compute corrections in this expansion. Being exact, the Kadanoff-Baym equations are difficult to handle, and it is particularly hard to compute corrections.

Here we follow another first-principle approach [8–10], in which radiative corrections have already been successfully included. Right from the start it makes use of two key features of leptogenesis: (i) Most degrees of freedom in the hot plasma are kept in thermal equilibrium by Standard Model processes with interaction rates much larger than the Hubble rate, the so called spectator processes. (ii) There is a separation of time scales; the quantities which deviate from equilibrium evolve on scales of order of the Hubble time or even larger times, i.e., much more slowly than the spectator processes. Under these conditions the non-equilibrium state is specified by the temperature, and by the values of the slow variables. In particular, these quantities determine the time evolution of the slowly changing variables. For sufficiently small deviations \( y_a \) from equilibrium the equations can be linearized. Then, in the absence of expansion, the non-equilibrium process can be described by the effective classical\(^1\) kinetic equations (cf. ref. [11])

\[
\dot{y}_a = -\gamma_{ab} y_b. \tag{1.1}
\]

The real coefficients \( \gamma_{ab} \) only depend on the temperature and encode the effect of the spectator processes.\(^2\) They have to be determined from the underlying microscopic theory. One arrives at relations which are quite similar to the Kubo formulas for transport coefficients. In these relations the \( \gamma_{ab} \) are written in terms of equal time correlation functions, so-called susceptibilities, and of unequal-time correlation functions of the slowly changing variables \( y_a \). These correlation functions are evaluated in an equilibrium system. This way one can relate the \( \gamma_{ab} \) to objects which can be computed in finite temperature field theory. Then the computation of the baryon asymmetry proceeds in two separate steps. First one computes the coefficients in the effective kinetic equations in (1.3), (1.4) using quantum field theory for equilibrium systems. The non-equilibrium problem is then treated by solving the effective kinetic equations. This is an enormous simplification compared to other approaches where the computation of radiative corrections appears to be prohibitively difficult.

What the slow variables are depends on the model parameters and on the relevant temperature range. In the limit of hierarchical sterile neutrino-masses \( M_1 \ll M_{I\neq1} \), only the lightest sterile neutrinos \( N_1 \) are present in the plasma for \( T \sim M_1 \). Then the slow variables are the (spatially homogeneous) \( N_1 \)-phase-space density \( f_k \), as well as global charges \( X_a \) such as \( L - B \) where \( L \) and \( B \) are lepton and baryon number. Both types of quantities are

\(^1\)Since the time scale on which the \( y_a \) evolve is larger than the inverse temperature, the \( y_a \) behave classically (see, e.g., ref. [11], §110).

\(^2\)They can also depend on the values of conserved or practically conserved charges but these are usually assumed to vanish.
conserved by Standard Model interactions, their conservation is violated only by the sterile-neutrino Yukawa interactions. The corresponding rates are small due to the smallness of the Yukawa couplings. In thermal leptogenesis these rates become similar in size to the Hubble rate. If we assume that during leptogenesis the deviation of the sterile neutrino phase-space density from equilibrium
\[ \delta f_k \equiv f_k - f_k^{eq}, \]
and the values of the charges \( X_a \) are sufficiently small, the system is described by linear kinetic equations of the form
\[ D_t f_k = -\gamma_{kq} \delta f_q - \gamma_{ka} X_a, \]
\[ D_t X_a = -\gamma_{ak} \delta f_k - \gamma_{ab} X_b, \]
where \( D_t \) is the time derivative which takes into account the expansion of the Universe.

Kubo-type relations for the washout rates \( \gamma_{ab} \) in (1.4), and the sterile-neutrino equilibration rate \( \gamma_{kq} \) in (1.3) were obtained in refs. [9] and [10], respectively. These are valid to leading order in the sterile-neutrino Yukawa couplings, and to all orders in Standard Model couplings.\(^4\) The next-to-leading order (NLO) Standard Model corrections to \( \gamma_{kq} \) are known in the regime \( T \ll M_1 \) [12–14], where the \( N_1 \) are non-relativistic, and in the relativistic regime \( T \sim M_1 \) [15]. In both cases the NLO is of order \( g^2 \), where \( g \) denotes some generic Standard Model coupling. In the ultrarelativistic regime \( T \gg M_1/g \) even the leading order (LO) result is quite involved, it was calculated in refs. [16, 17]. The leading corrections to the washout rates are only suppressed by a single power of \( g \); the order \( g \) and order \( g^2 \) corrections were computed in refs. [9, 18].

In this paper we consider the \( CP \) violating rates \( \gamma_{ak} \) and \( \gamma_{ka} \) in (1.3) and (1.4). We obtain the master formulas (3.36) and (3.38) which relate them to a single three-point spectral function of Standard Model fields. These are valid to LO in the sterile-neutrino Yukawa interaction, and to all orders in the Standard Model gauge couplings and the quark Yukawa couplings. The small \( CP \) violation of the Standard Model is neglected, together with the charged lepton Yukawa-interactions. We evaluate our master formulas in the regimes \( T \ll M_1 \) and \( T \sim M_1 \) at leading order in Standard Model couplings. Then we perform the first step of the order \( g^2 \) calculation of the \( CP \) violating lepton asymmetry rate \( \gamma_{ak} \) in the regime \( T \ll M_1 \) by computing the zero temperature contribution, which is the leading term in the low-temperature expansion.

This paper is organized as follows. In section 2 we slightly generalize the method of ref. [9] which allows us to obtain Kubo-type relations for \( CP \) violating rates. In section 3 we derive the master formulas for these rates, and in section 4.1 we demonstrate how they reproduce the leading-order lepton asymmetry rate \( \gamma_{ak} \) in the regime \( T \ll M_1 \) and \( T \sim M_1 \). Then in section 4.2 we compute the order \( g^2 \) corrections to the asymmetry rate at zero temperature. We summarize in section 5. Appendix A contains a derivation of a spectral representation for arbitrary thermal three-point functions of bosonic or fermionic operators. Implications of discrete symmetries for spectral functions are the subject of appendix B. The reductions to the master integrals and results for the master spectral functions are given in appendix C, and the calculation of the master spectral functions is described in D. In appendix E we...

\(^3\)We consider a finite volume \( V \) so that the momenta \( k \) are discrete. Summation over indices appearing twice is understood.

\(^4\)To systematically include higher orders in the sterile-neutrino Yukawa interactions, one would also have to consider higher derivative terms in eqs. (1.3), (1.4) which are similarly parametrically suppressed.
show that terms containing a $\gamma^5$ matrix do not contribute to the Dirac traces of the NLO diagrams.

**Notation:** the signature of the metric is $+\ldots -$, 4-vectors are written as lower-case italics, and bold-face letters refer to 3-vectors. $\omega_n$ are Matsubara frequencies $\omega_n = n\pi T$, with even (odd) integers $n$ for bosonic (fermionic) operators. We use the imaginary time formalism where $x^0 = -i\tau$ with real $\tau$. Matsubara sums over fermionic frequencies are written as $\sum_{\{k^0\}} F(k^0) \equiv \sum_{n \text{ odd}} F(i\omega_n)$.

### 2 Rates and real-time correlators

#### 2.1 Matching

Here we describe how we determine the rates in the effective kinetic equations (1.3) and (1.4) from the underlying microscopic quantum field theory by using the theory of quasi-stationary fluctuations (see, e.g., ref. [11], §118). Consider slowly varying quantities $y_a$ which vanish in thermal equilibrium. They satisfy the effective equations of motion (1.1). The thermal fluctuations of $y_a$ observe the same equations, but with an additional Gaussian white noise term on the right-hand side, which represents the rapidly fluctuating quantities. These equations can be used to compute the real-time correlation function

$$C_{ab}(t) = \langle y_a(t)y_b(0) \rangle$$

of the fluctuations by solving these equations and then averaging over the noise and over initial conditions. One obtains

$$C_{ab}(t) = \left( e^{-\gamma|t|} \right)_{ac} \chi_{cb},$$

where the average over initial conditions enters through the real and symmetric susceptibilities

$$\chi_{ab} \equiv \langle y_a y_b \rangle.$$  

These susceptibilities have to be computed in the microscopic theory. The rate-matrix $\gamma$ can be extracted from the one-sided Fourier transform

$$C_{ab}^+(\omega) \equiv \int_0^\infty dt \, e^{i\omega t} C_{ab}(t)$$

which is defined for $\text{Im} \, \omega > 0$. For real frequencies $\gamma \ll \omega \ll \omega_{UV}$, where $\omega_{UV}$ is the characteristic frequency of the spectator processes, and for real $\gamma_{ab}$ one obtains [9]

$$\gamma_{ab} = \omega^2 \, \text{Re} \, C_{ac}^+(\omega + i\epsilon) \left( \chi^{-1} \right)_{cb}$$

for $\gamma \ll \omega \ll \omega_{UV}$.

In this regime $C_{ab}^+$ has to match the one-sided Fourier transform of the microscopic correlation function

$$C_{ab}(t) \equiv \frac{1}{2} \langle \{ y_a(t), y_b(0) \} \rangle.$$  

The latter can be written as

$$C_{ab}^+(\omega) = \int \frac{d\omega'}{2\pi} \, \frac{i}{\omega - \omega'} \left[ \frac{1}{2} + f_B(\omega') \right] \rho_{ab}(\omega'),$$

where $f_B(\omega')$ is the Fermi function.
with the Bose-Einstein distribution \( f_B(\omega) \equiv [\exp(\omega/T) - 1]^{-1} \) and the spectral function (cf. appendix A)

\[
\rho_{ab}(\omega) \equiv \int dt e^{i\omega t} \langle [y_a(t), y_b(0)] \rangle.
\]

(2.8)

So far the discussion is identical to the one in \cite{9}. In \cite{9}, however, the spectral function \( \rho_{ab}(\omega) \) was real. Thus after taking the real part of \( C_{ab}(\omega + i\epsilon) \) only the delta function in

\[
\frac{1}{x + i\epsilon} = -i\pi \delta(x) + P.V. \frac{1}{x}
\]

(2.9)

ccontributed to the integral (2.7), but not the principal value. In this work we consider the spectral function of \( X_a \) and \( \delta f_k \) which have different signs under \( CPT \) transformation. Then the spectral function is imaginary (see appendix B), and we proceed as follows. Since we are interested in frequencies much smaller than the temperature \( T \), we can approximate the square bracket in (2.7) by \( T/\omega' \), which gives

\[
C_{ab}^+(\omega) = -i\frac{T}{\omega'} [\Delta_{ab}(\omega) - \Delta_{ab}(0)].
\]

(2.10)

Here

\[
\Delta_{ab}(\omega) \equiv \int \frac{d\omega'}{2\pi} \frac{\rho_{ab}(\omega')}{\omega' - \omega}
\]

(2.11)

is an analytic function off the real axis. \( \Delta_{ab}(\omega + i\epsilon) \) with real \( \omega \) equals the retarded two-point function. Matching \( C^+ \) and \( C^+ \), and using (2.5) as well as the fact that \( \Delta_{ab}(0) \) is real we obtain the Kubo-type formula

\[
\gamma_{ab} = \frac{T}{\omega} \text{Im} \Delta_{ac}^{\text{ret}}(\omega) \left( \chi^{-1} \right)_{cb} \quad (\gamma \ll \omega \ll \omega_{UV}).
\]

(2.12)

For real spectral functions this agrees with the Kubo-type relation found in \cite{9}.

Instead of using (2.12) it is a lot more convenient \cite{9} to compute the retarded correlator \( \Pi_{ab}(\omega) \) of the time derivatives \( \dot{y}_a \), because this way one keeps only the terms which violate the conservation of \( y_a \). Using \( \Pi_{ab}^{\text{ret}}(\omega) = \omega^2 \Delta_{ab}^{\text{ret}}(\omega) \) we obtain

\[
\gamma_{ab} = \frac{T}{\omega} \text{Im} \Pi_{ac}^{\text{ret}}(\omega) \left( \chi^{-1} \right)_{cb} \quad (\gamma \ll \omega \ll \omega_{UV}).
\]

(2.13)

2.2 Charge and phase-space density operators

Among the slow variables we have to consider are charges \( X_a \) which can be written as

\[
X_a = \int d^3 x \tilde{\gamma}_0 T_a \ell + \text{contributions from other fermions}.
\]

(2.14)

Here \( T_a \) is the generator of the corresponding symmetry transformation acting on the left-handed leptons \( \ell_i \), where \( i \) is a family index. We consider a temperature range in which the charged lepton Yukawa interactions are either much faster or much slower than the Hubble rate. In this regime the conservation of the \( X_a \) is violated only by the Yukawa-interaction involving the sterile neutrinos,

\[
\mathcal{L}_{\text{int}} = -\overline{N} h \tilde{\varphi}^\dagger \ell + \text{H.c.}
\]

(2.15)
Here $\bar{\varphi} \equiv i \sigma^2 \varphi^*$ with the Pauli matrix $\sigma^2$ is the isospin conjugate of the Higgs field $\varphi$. The Yukawa couplings are written as a matrix in flavor space, $(h)_{ij} = h_{ij}$. We describe the sterile neutrinos by Majorana spinors $N_i$. In (2.14) we have only written the SU(2) leptons doublets $\ell_i$ explicitly, because the other Standard Model fermions do not appear in $\mathcal{L}_{\text{int}}$ and do not enter the time derivatives of the $X_a$. In the Heisenberg picture we obtain

$$\dot{X}_a (x^0) = i \int d^3 x \left[ Q_a (x) - Q^\dagger_a (x) \right],$$

where

$$Q_a \equiv N^a_i h_{ij} J_{ia}$$

with

$$J_{ia} \equiv \bar{\varphi}^\dagger (T_a)_{ij} \ell_j.$$

The other slow variables we have to take into account are the phase-space densities of the sterile neutrinos. In this work we consider the hierarchical limit $M_1 \ll M_i$ ($i \neq 1$), and temperatures at which only the lightest sterile neutrinos $N_1$ are present in the plasma. We assume a homogeneous system, and we neglect spin asymmetries. Then one can define the phase space density operator $f_k$ similarly to $[10, 19]$. In the interaction picture with respect to the Yukawa interaction (2.15) the canonically normalized field can be written as

$$[N_1 (x)]_{\text{int}} = \sum_{k,s} \frac{1}{\sqrt{2E_k V}} \left[ e^{-i k x} u_{ks} a^\dagger_{ks} + e^{i k x} v_{ks} a_{ks} \right]_{k^0 = E_k},$$

where $E_k \equiv (k^2 + M_i^2)^{1/2}$. The creation and annihilation operators $a^\dagger_{ks}$ and $a_{ks}$ satisfy

$$\{a_{ks}, a^\dagger_{k's'}\} = \delta_{kk'} \delta_{ss'}.$$

Now we define $f$ as the spin average

$$[f_k]_{\text{int}} \equiv \frac{1}{2} \sum_s a^\dagger_{ks} a_{ks}.$$

Switching to the Heisenberg picture, the time derivative of $f$ can be obtained from the Heisenberg equation of motion. For doing perturbation theory it is convenient to re-express the creation and annihilation operators in terms of $N$ by using

$$a_{ks} = \frac{1}{\sqrt{2E_k V}} u_{ks} N_1 (0, k), \quad a^\dagger_{ks} = \frac{1}{\sqrt{2E_k V}} v^\dagger_{ks} N_1 (0, -k),$$

where $N_1 (t, k) \equiv \int d^3 x e^{-ikx} N_1 (t, x)$ are the spatial Fourier-transforms of the sterile-neutrino field $N_1$. With the definitions

$$R_k (t) \equiv h_{1i} \int \int d^3 x' d^3 x'' e^{i k (x - x'')} N_1 (t, x') \gamma^0 (k + M_1) J_i (t, x'')$$

and

$$J_i \equiv \varphi^\dagger \ell_i$$

we find $^5$

$$\dot{f}_k (t) = \frac{-i}{4V E_k} \left\{ R_k (t) - R^\dagger_k (t) \right\} + (k \to -k) \Big|_{k^0 = E_k}.$$

$^5$Note that in real time $N^\dagger (t, -k) = [N (t, k)]^\dagger$. 

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3 CP violating rates

3.1 General considerations

The discussion in section 2 applied to all coefficients in eqs. (1.3) and (1.4). Now we will determine the CP violating ones, $\gamma_{ak}$ and $\gamma_{ka}$, by using eq. (2.13). We assume that the slow interaction is the neutrino Yukawa interaction (2.15). The equal-time correlators (2.3) of $X_a$ with $f_k$ vanish due to CPT invariance, since $X_a$ and $f_k$ are odd and even under CPT, respectively, and because they commute at equal times. Thus the matrix (2.3) is block-diagonal, and only the elements

$$\chi_{kk'} \equiv \langle f_k f_{k'} \rangle,$$

$$\chi_{ab} \equiv \langle X_a X_b \rangle$$

enter (2.13). At leading order in the Yukawa couplings $\chi_{kk'}$ is determined by the free theory which gives

$$\chi_{kk'} = \delta_{kk} \chi_k \ (3.3)$$

with

$$\chi_k = f_F(E_k) [1 - f_F(E_k)] = -T f'_F(E_k). \ (3.4)$$

The susceptibility matrix for the charges $\chi_{ab}$ has been computed in [9, 18] up to order $g^2$ in the Standard Model couplings. Then according to (2.13) the CP violating rates are given by

$$\gamma_{ak} = \frac{T}{\omega} \text{Im} \Pi_{ak}(\omega) \frac{1}{\chi_k}, \quad (3.5)$$

$$\gamma_{ka} = \frac{T}{\omega} \text{Im} \Pi_{bk}^\text{ret}(\omega) \left( \chi^{-1} \right)_{ba}, \quad (3.6)$$

where $\gamma \ll |\omega| \ll \omega_{UV}$. The retarded correlators on the right-hand-side are given by analytical continuation of the imaginary-time correlators

$$\Pi_{ak}(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} \left\langle X_a(-i\tau) f_k(0) \right\rangle, \quad (3.7)$$

$$\Pi_{ka}(i\omega_n) = [\Pi_{ak}(-i\omega_n)]^*, \quad (3.8)$$

where $\omega_n$ is a bosonic Matsubara frequency, and $\beta \equiv 1/T$. Now we insert our results (2.16) and (2.25) for the time derivatives of $X_a$ and $f_k$ to write $\Pi_{ak}$ in terms of the operators $Q_a$ and $R_k$ which are defined in eqs. (2.17) and (2.23), respectively. Using periodicity in imaginary time $x^0 = -i\tau$ we find

$$\left\langle Q_a(x) R_k(0) \right\rangle = \left\langle Q^\dagger_a(-i\beta - x^0, x) R_k(0) \right\rangle^*, \quad (3.9)$$

and similarly\(^6\) for $Q^\dagger_a(x)$. With the help of these relations we can simplify $\Pi_{ak}$ such that it turns into

$$\Pi_{ak}(i\omega_n) = \frac{1}{2E_k V} \text{Re} \int_0^\beta d\tau \int d^3 x e^{i\omega_n \tau} \times \left\langle \left[ Q_a(x) - Q^\dagger_a(x) \right] R_k(0) + (k \rightarrow -k) \right\rangle_{k^0 \rightarrow E_k}, \quad (3.10)$$

\(^6\)One has to keep in mind that in imaginary time $Q^\dagger(x) = e^{iH x^0} Q^\dagger(0, x)e^{-iH x^0}$ is not the Hermitian conjugate of $Q_a(x)$. 

[Page 7]
A non-vanishing value of $X_a$ is generated by $CP$ violating interactions involving virtual sterile neutrinos $N_{I \neq 1}$ which first appear in the correlator at order $h^4$. Expanding (3.10) to order $h^4$ we need to expand only to second order in the interaction (2.15) since the operators $Q_a$ and $R_k$ are already linear in $h$. Using Wick’s theorem for the sterile neutrinos, we can express (3.10) in terms of free propagators

$$S_I(p) \equiv \int_0^\beta d\tau \int d^3x \ e^{ipx} \left< N_I(x)\bar{N}_I(0) \right> = (\vec{k} + M_I)\Delta_I(k),$$

with $\Delta_I(k) \equiv (-k^2 + M^2_I)^{-1}$, and four-point correlation functions of the operators (2.18), (2.24), which contain only Standard Model fields. These relations of the $CP$ violating rates to the four-point function are valid to all orders in the Standard Model couplings.

### 3.2 Hierarchical limit: relation to three-point functions

In the hierarchical limit $M_1 \ll M_{I \neq 1}$ one can integrate out the heavier sterile neutrinos, and work with the resulting effective theory for $N_1$ and the Standard Model fields. We include only the leading term in $1/M_I$ given by the dimension-5 Weinberg operator [20]

$$\mathcal{L}_5 = \frac{1}{2} C_{ij} (\bar{\ell}_i \tilde{\varphi}^* \tilde{\varphi} \ell_j) + H.c..$$

(3.12)

Here $\ell_c = C \ell^\top$ is the charge conjugate of $\ell$ with the antisymmetric and unitary charge-conjugation matrix $C$ which satisfies

$$C^{-1} \gamma_\mu C = -\gamma^\top_{\mu}.$$

(3.13)

It is convenient to first perform the contractions described in section 3.1, and then move on to the effective theory by approximating

$$S_I(p) \simeq \frac{1}{M_I} \text{ for } I \neq 1$$

(3.14)

and identifying

$$\sum_{I \neq 1} \frac{h_{Ii}h_{Ij}}{M_I} = c_{ij}.$$  

(3.15)

Then the four-point function of section 3.1 turns into a three-point function. Some of the terms drop out because they contain the scalars $\overline{J}J$ or their Hermitian conjugate, which vanish because the operators $J$ are left-handed. The remaining order $h^4$ terms are

$$\int_0^\beta d\tau \int d^3x \ e^{i\omega_n \tau} \left< Q_a(x)R_k \right> = \frac{V}{2} \int_0^\beta d\tau d\tau' \int d^3x d^3x' T \sum_{n'\text{odd}} e^{i(\omega_{n'} - \omega_n)\tau + i\vec{k}\cdot\vec{x}}$$

$$\times \ h_{Ii}h_{Ij}c_{lm} \left< T \overline{J}a(x')C S_1(i\omega_{n'}, k)\gamma^0 (\vec{k} + M_1) J_i(0) \bar{J}_l C \overline{J}_m^\top (x') \right>$$

(3.16)

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Note that the relation (3.15) is valid only for a tree-level matching of the four-vertex in (3.12) with the corresponding 4-point function in the full theory with the interaction (2.15). There are Standard Model corrections to this relation. In the following we only use the effective coupling $c_{ij}$ which should already contain these corrections.
and

\[ \int_0^\beta d\tau \int d^3x e^{i\omega_n \tau} \left( \langle Q_{i0}^\dagger(x) R_k \rangle \right) = V \int_0^\beta d\tau d\tau' \int d^3x \, d^3x' T \sum_{n' \text{odd}} e^{i(\omega_n \tau - \omega_{n'} \tau' + \mathbf{k}\mathbf{x}')} \times h_{1i} h_{1j} c_{lm}^* \left\langle T \mathcal{J}_{i a}^\dagger C \mathcal{J}_{m}^\top (x) J_i^+ (x') CS_1(i\omega_{n'}, \mathbf{k}) \gamma^0 (\mathbf{k} + M_1) J_j(0) \right\rangle. \]

(3.17)

Using eq. (3.11), and keeping in mind that the \( \ell_i \) are left-handed, the products of Dirac matrices in eqs. (3.16) and (3.17) can be simplified as follows,

\[ P_L S_1 (i\omega_{n'}, \mathbf{k}) \gamma^0 (\mathbf{k} + M_1) P_L = M_1 (i\omega_{n'} + k^0) \Delta_1 (i\omega_{n'}, \mathbf{k}) P_L, \]

(3.18)

with the left-chiral projector \( P_L = (1 - \gamma^5)/2 \). With the help of

\[ (i\omega_{n'} + k^0) \Delta_1 (i\omega_{n'}, \mathbf{k}) = -\frac{1}{i\omega_{n'} - k^0} \quad \text{for} \quad k^0 = \pm E_k \]

(3.19)

one can simplify eqs. (3.16) and (3.17) further. Then we plug them into eq. (3.10), thereby replacing \( k^0 \to E_k \) after which the variable \( k^0 \) no longer appears. It is then convenient to rename \( i\omega_{n'} \) as \( k^0 \), which yields the compact expression

\[ \Pi_{ak} (i\omega_n) = \frac{M_1}{2E_k} \text{Re} \left\{ T \sum_{\{k^0\}} \frac{1}{k^0 - E_k} \left( \left[ \frac{1}{2} h_{1i} (hT_a)_{1j} c_{lm}^\dagger \Gamma_{ijlm}(k - q, q - k) \right. \right. \right. \]

\[ \left. \left. \left. - h_{1i} h_{1j} (T_a c_{lm}^\dagger)_{ij} \Gamma_{ijlm}(k - q, -k) \right] - [k \to -k] \right\} \right\}_{q = (i\omega_n, 0)}, \]

(3.20)

with the three-point function

\[ \Gamma_{ijlm}(k_1, k_2) \equiv \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int d^3x_1 \int d^3x_2 e^{i(k_1 x_1 + k_2 x_2)} \times \left\langle T \mathcal{J}_i^\top (x_1) C \mathcal{J}_j^\top (x_2) \mathcal{J}_l^\dagger C_{\mathcal{J}_m}^\top (0) \right\rangle. \]

(3.21)

The time ordering \( T \) is defined in eq. (A.22), and the subscript 0 indicates that the expectation value can now be evaluated at \( h = 0 \), which is because we are only considering the leading order in \( h \).

### 3.3 Symmetries of the three-point correlator

To proceed further, we make use of certain symmetries of the imaginary-time correlation function (3.21). The two symmetries

\[ \Gamma_{ijlm}(k_1, k_2) = \Gamma_{ijlm}(k_2, k_1), \]

(3.22)

\[ \Gamma_{ijlm}(k_1, k_2) = \Gamma_{ijml}(k_1, k_2) \]

(3.23)

follow from the fact that \( J \) is fermionic and that the matrix \( C \) is antisymmetric. They will turn out to be particularly useful in section 3.4. From now on we neglect the Standard
Model CP violation. Then $\Gamma_{ijlm}(k_1, k_2)$ is real (see appendix B). If we further use that the correlation functions in (3.20) do not depend on the direction of $k$, we can write

$$\Pi_{ak}(i\omega_n) = \frac{iM_1}{2E_k} T \sum_{\{k^0\}} \frac{1}{k^0 - E_k}$$

$$\times \left\{ -\frac{1}{2} \text{Im} \left[ h_{1i}(hT_a)_{ij} c_{lm}^\dagger \right] \Gamma_{ijlm}(-k - q, k + q) 
+ \text{Im} \left[ h_{1i}h_{1j}(T_a c^\dagger)_lm \right] \Gamma_{ijlm}(-k - q, k) \right\}_{q=(i\omega_n,0)}.$$  \hfill (3.24)

Furthermore, time reversal invariance implies the relation (B.12). Since $\Gamma_{ijlm}(k_1, k_2)$ is real, we have $\Gamma_{ijlm}(k_1, k_2) = \Gamma_{ijlm}(-k_1, -k_2)$. Thus we find

$$\Pi_{ak}(i\omega_n) = \frac{iM_1}{2E_k} T \sum_{\{k^0\}} \frac{1}{k^0 - E_k}$$

$$\times \left\{ -\frac{1}{2} \text{Im} \left[ h_{1i}(hT_a)_{ij} c_{lm}^\dagger \right] \Gamma_{ijlm}(-k - q, k + q) 
+ \text{Im} \left[ h_{1i}h_{1j}(T_a c^\dagger)_lm \right] \Gamma_{ijlm}(-k - q, k) \right\}_{q=(i\omega_n,0)} - \omega_n \rightarrow -\omega_n. \hfill (3.25)$$

Now we also neglect the small Yukawa interactions of the charged leptons. Then the remaining Lagrangian is invariant under SU($N_{\text{fam}}$) lepton flavor transformations, where $N_{\text{fam}} = 3$ is the number of families. Together with the symmetry (3.23) this implies that the three-point correlator (3.21) has the form

$$\Gamma_{ijlm} = \frac{1}{2}(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) \Gamma,$$  \hfill (3.26)

where

$$\Gamma = \frac{1}{N_{\text{fam}}} \Gamma_{iill}. \hfill (3.27)$$

Then we find

$$\Pi_{ak}(i\omega_n) = M_1 \text{Im} \left[ hT_a c^\dagger h^\top \right]_{11} \mathcal{M}_k(i\omega_n), \hfill (3.28)$$

with

$$\mathcal{M}_k(i\omega_n) = \frac{iT}{4E_k} \sum_{\{k^0\}} \frac{-\Gamma(-k - q, k + q) + 2\Gamma(-k - q, k)}{k^0 - E_k} \biggl|_{q=(i\omega_n,0)}$$

$$- (\omega_n \rightarrow -\omega_n). \hfill (3.29)$$

A nice feature of the formula (3.28) is that the effects of CP violation, described by the imaginary part of the couplings, is separated from the kinematics, described by the function $\mathcal{M}_k(i\omega_n)$. One has to keep in mind that eq. (3.28) was obtained by integrating the out sterile neutrinos heavier than $N_1$ at leading order. It is not clear whether this factorization persists beyond this approximation.
3.4 Matsubara sum and analytic continuation

For the Kubo-type relation (2.13) we need the retarded correlator, which can be obtained from the imaginary-time correlator (3.28) via analytic continuation. But first the Matsubara sum over the frequency \( k_0 \) in (3.29) has to be performed. This can be done without knowing the three-point correlators explicitly, by using their spectral representation which we discuss in more detail in appendix A. The symmetries (3.22) and (3.23) imply a relation between the two spectral functions in (A.23), such that one can write \( \Gamma \) in terms of a single spectral function,

\[
\Gamma(k_1, k_2) = \int \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \frac{1}{k_1^0 + k_2^0 - \omega_1 - \omega_2} \left[ \frac{\rho(\omega_1, k_1, \omega_2, k_2) + \rho(-\omega_2, -k_2, -\omega_1, -k_1)}{k_1^0 - \omega_1} \right].
\]

(3.30)

\( \rho \equiv \rho_{ABC} \) is the spectral function (A.24) for the operators

\[
A = J_i^\alpha, \quad B = (C^{\dagger} J_i)^\alpha, \quad C = \overline{J_i} C \overline{J_i}^\dagger,
\]

(3.31)

where \( i \) and the spinor index \( \alpha \) are summed over in the product \( AB \). If we insert (3.30) in (3.29) we obtain a factor \( 1/(\omega_1 + \omega_2) \). It does, however, not give rise to a singularity because it multiplies a function which vanishes for \( \omega_1 = -\omega_2 \). We can therefore replace it by its principal value. After that we can re-arrange the integrations until we obtain

\[
\mathcal{M}_k(i\omega_n) = \frac{i}{2E_k} \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \rho(\omega_1, -k, \omega_2, k) T \sum_{\{k_0\}} \frac{1}{k_0^0 - E_k} \left( \frac{1}{k_0^0 + \omega_1 + i\omega_n} - \frac{1}{k_0^0 + \omega_1} \right) \left( \frac{1}{\omega_1 + \omega_2 + i\omega_n} - \text{P.V.} \frac{1}{\omega_1 + \omega_2} \right) - (\omega_n \rightarrow -\omega_n).
\]

(3.32)

Now the Matsubara sum is manifestly finite, and we obtain

\[
\mathcal{M}_k(i\omega_n) = \frac{i}{2E_k} \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \rho(\omega_1, -k, \omega_2, k) \left[ f_F(-\omega_1) - f_F(E_k) \right] \left( \frac{1}{\omega_1 + \omega_2 + i\omega_n} - \text{P.V.} \frac{1}{\omega_1 + \omega_2} \right) - (\omega_n \rightarrow -\omega_n).
\]

(3.33)

In this expression we can analytically continue \( i\omega_n \rightarrow \omega + i\epsilon \) with real \( \omega \).

While the relation (3.35) is only valid when \( |\omega| \gg \gamma \), where \( \gamma \) is at most of order \( h^2 \), we may now take \( \omega \) as small as we like because \( h \) no longer appears in the spectral function \( \rho \) (cf. eq. (3.21)). For small \( \omega \) we find

\[
\mathcal{M}_k(\omega + i\epsilon) = \frac{i}{E_k} \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \rho(\omega_1, -k, \omega_2, k) \left[ f_F(-\omega_1) - f_F(E_k) \right] \left( \frac{1}{\omega_1 + \omega_2 + i\epsilon} \right) (-i\pi) \delta(\omega_1 + \omega_2) + O(\omega^3).
\]

(3.34)

We compute the imaginary part of this function, using that \( \rho \) is real valued. This yields another delta function and leads to the simple relation

\[
\text{Im}\mathcal{M}_k(\omega + i\epsilon) = -\frac{\omega}{4E_k} \rho(-E_k, -k, E_k, k) f'_F(E_k) + O(\omega^3).
\]

(3.35)
Inserting (3.4), (3.28) and then (3.35) into (3.5) we finally obtain
\[ \gamma_{ak} = \frac{\rho(-E_k, -k, E_k, k)}{4E_k} M_1 \text{Im} \left( h T_a c^1 h^+ \right)_{11}. \] (3.36)

This is the master formula which relates the asymmetry rate in eq. (1.4) to the spectral function \( \rho \) in eq. (3.30), the Yukawa couplings \( h_{1i} \), and the dimension-5 couplings \( c_{ij} \) in eq. (3.12). Furthermore, from (3.8) and (3.33) we see that
\[ \Pi_{ka}(i\omega_n) = -\Pi_{ak}(i\omega_n). \] (3.37)
This implies
\[ \gamma_{ka} = -\gamma_{bk} f_F(E_k) [1 - f_F(E_k)] \left( \chi^{-1} \right)_{ba}. \] (3.38)

The master formulas (3.36) and (3.38) have two important features. First the CP violation due to the Yukawa couplings is separated from the kinematic part, which is described by the spectral function \( \rho \). Second the spectral function is exact to all orders in the Standard Model couplings, except for the very small charged lepton Yukawa couplings and the Standard Model CP violation. It can conveniently be computed in finite temperature perturbation theory.

4 Lepton asymmetry rate

4.1 Leading order at \( T \lesssim M_1 \)

We will now illustrate the use of the master formula (3.36) by computing the leading order asymmetry rate \( \gamma_{ak} \) in the regime \( T \lesssim M_1 \). First we compute the imaginary-time three-point function, which we analytically continue and then use the inverse relation (A.29) which yields the spectral function \( \rho \) in eq. (3.36).

At leading order\(^8\) the three-point function (3.21) is given by the diagram
\[ \Gamma^{(0)}(k_1, k_2) = \text{diagram}. \] (4.1)

The solid thick lines represent the operators (2.24) which couple to the sterile neutrinos with outgoing momenta \( k_1 \) and \( k_2 \). The dashed line corresponds to the third operator in (3.21) carrying the outgoing momentum \(-k_1 - k_2\) which according to (3.36) will be set to zero in the corresponding spectral function. The solid arrow-lines are Standard Model leptons and the dotted lines are Higgses.

Applying Wick’s theorem, computing traces in gauge group and flavor space and using the property (3.13) of the charge conjugation matrix we find
\[ \Gamma^{(0)}(k_1, k_2) = 2N_w [N_w + 1] \text{Tr} \left( \gamma_\mu P_L \gamma_\nu P_R \right) I^\mu(k_1) I^\nu(k_2), \] (4.2)
where \( N_w = 2 \) is the dimension of the fundamental representation of the gauge group SU(2), and
\[ I^\mu(k) \equiv \sum_p \frac{p^\mu}{p^2(p - k)^2} \] (4.3)

\(^8\)This is the leading order in the regime \( T \lesssim M_1 \). At higher temperature also gauge interactions will contribute at leading order like in the sterile neutrino production rate [16, 17].
is a 1-loop sum-integral. Since the loop integrals are UV divergent we use dimensional regularization by working with $d - 1$ spatial dimensions,

$$\oint \mathcal{J} \equiv T \sum_{\{\rho^\prime\}} \int \frac{d^{d-1}p}{(2\pi)^{d-1}}. \quad (4.4)$$

We have not yet performed the Dirac trace, because it contains $\gamma^5$ matrices which need a special treatment in dimensional regularization. We proceed similar to [12] and use the definition

$$\gamma^5 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \quad (4.5)$$

of ’t Hooft and Veltman [21] and apply the prescription of [22] which allows a naively commuting $\gamma^5$ with $(\gamma^5)^2 = 1$ in traces with more than one $\gamma^5$, except in closed fermion loops. Then only traces with one or no $\gamma^5$ remain. It has been shown in [12] that in the trace in (4.2) all terms with one $\gamma^5$ cancel exactly due to the total antisymmetry of the Levi-Civita symbol $\epsilon_{\mu\nu\rho\sigma}$. Then the trace becomes

$$\text{Tr} (\gamma_\mu P_L \gamma_\nu P_R) = \frac{1}{2} \text{Tr} (\gamma_\mu \gamma_\nu) = 2\eta_{\mu\nu}, \quad (4.6)$$

and we find the LO order result

$$\Gamma(k_1, k_2) = -24I_\mu(k_1)I_\mu(k_2) \quad (4.7)$$

for the three-point function (3.27). The inverse relation (A.29) applied to (4.7) yields the spectral function

$$\rho^{(0)}(k_1, k_2) = 96 \text{Im} I_\mu(k_1^0 + i\epsilon, k_1) \text{Im} I_\mu(k_2^0 + i\epsilon, k_2), \quad (4.8)$$

with real $k_1^0$ and $k_2^0$. Since $I_\mu(k)$ is an even function of $k$, the spectral function appearing in the master formula (3.36) becomes

$$\rho^{(0)}(-E_k, -k, E_k, k) = -96\left(\text{Im} I_\mu(E_k + i\epsilon, k)\right)^2. \quad (4.9)$$

First consider the regime $T \ll M_1$. Here the leading contribution in $T/M_1$ to the rate $\gamma_{ak}$ is given by the zero-temperature limit of (3.36). At $T = 0$ Lorentz invariance implies

$$I_\mu(k) = \frac{k^\mu}{2} I_1(k), \quad (4.10)$$

with $I_1$ defined in (D.8). Using eq. (D.11) we obtain

$$\rho(-E_k, -k, E_k, k) = \frac{24M^2_1}{(16\pi)^2}, \quad (4.11)$$

which together with the master formula (3.36) yields

$$\gamma^{(0)}_{ak} = -\frac{6M^2_1}{(16\pi)^2} E_k \text{Im} \left( h T_a c^\dagger h^\top \right)_{11}. \quad (4.12)$$

With the tree-level relation (3.15) for the dimension-5 couplings this yields

$$\gamma^{(0)}_{ak} = \varepsilon_a\gamma N_1, \quad (4.13)$$
where
\[ \gamma_{N_1} = \frac{M_1 (hh\dagger)_{11}}{8\pi} \] (4.14)
is the LO \( N_1 \) equilibration rate in the non-relativistic limit \( k^2 \ll M_1^2 \) (see e.g. [8]), and
\[ \varepsilon_a = -\frac{3}{16\pi} \sum_l \frac{M_l}{M_1} \text{Im}[\left(hT_a h\dagger\right)_{11} (hh\dagger)_{11}] \] (4.15)

Eq. (4.13) agrees with the hierarchical limit of the well known result of ref. [23].

Now consider the case \( T \sim M_1 \), usually referred to as the relativistic regime. After summing over Matsubara frequencies in eq. (4.3) one can analytically continue to real frequencies which yields
\[
\text{Im} I^\mu(E_k + i\epsilon, k) = \pi \int \frac{d^3 p}{(2\pi)^3} \delta(E_k - |p| - |p - k|) \left| \frac{p^\mu}{4|p||p - k|} \right|_{p^0 = |p|} \\
\times \left[ 1 - f_F(|p|) + f_B(|p - k|) \right].
\] (4.16)

Inserting this in (4.9), and then in (3.36) reproduces the LO asymmetry rate for the case \( T \sim M_1 \) computed in ref. [4], where analytic results for the integrals (4.16) were obtained.

### 4.2 NLO at zero temperature

In this section we calculate the NLO Standard Model corrections to the three-point spectral function at zero temperature. This is the leading term in the low temperature expansion of the \( CP \) violating rate \( \gamma_{ok} \). We take into account the U(1) and SU(2) gauge couplings \( g_1 \) and \( g_2 \), the top Yukawa coupling \( h_t \) and the Higgs self-coupling \( \lambda \).9,10

The gauge interactions give rise to factorizable and non-factorizable diagrams. At zero temperature the factorizable diagrams are
\[
\Gamma_{g,\text{fac}}(k_1, k_2) = \begin{array}{c}
\text{\begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) [circle,draw,inner sep=1pt] {}; \\
\node (v2) at (1,0) [circle,draw,inner sep=1pt] {}; \\
\node (v3) at (2,0) [circle,draw,inner sep=1pt] {}; \\
\node (v4) at (3,0) [circle,draw,inner sep=1pt] {}; \\
\end{tikzpicture}}
+ \begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) [circle,draw,inner sep=1pt] {}; \\
\node (v2) at (1,0) [circle,draw,inner sep=1pt] {}; \\
\node (v3) at (2,0) [circle,draw,inner sep=1pt] {}; \\
\node (v4) at (3,0) [circle,draw,inner sep=1pt] {}; \\
\end{tikzpicture}
+ \begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) [circle,draw,inner sep=1pt] {}; \\
\node (v2) at (1,0) [circle,draw,inner sep=1pt] {}; \\
\node (v3) at (2,0) [circle,draw,inner sep=1pt] {}; \\
\node (v4) at (3,0) [circle,draw,inner sep=1pt] {}; \\
\end{tikzpicture}
+ \begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) [circle,draw,inner sep=1pt] {}; \\
\node (v2) at (1,0) [circle,draw,inner sep=1pt] {}; \\
\node (v3) at (2,0) [circle,draw,inner sep=1pt] {}; \\
\node (v4) at (3,0) [circle,draw,inner sep=1pt] {}; \\
\end{tikzpicture}
+ \begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) [circle,draw,inner sep=1pt] {}; \\
\node (v2) at (1,0) [circle,draw,inner sep=1pt] {}; \\
\node (v3) at (2,0) [circle,draw,inner sep=1pt] {}; \\
\node (v4) at (3,0) [circle,draw,inner sep=1pt] {}; \\
\end{tikzpicture}
+ \begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) [circle,draw,inner sep=1pt] {}; \\
\node (v2) at (1,0) [circle,draw,inner sep=1pt] {}; \\
\node (v3) at (2,0) [circle,draw,inner sep=1pt] {}; \\
\node (v4) at (3,0) [circle,draw,inner sep=1pt] {}; \\
\end{tikzpicture}
+ \begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) [circle,draw,inner sep=1pt] {}; \\
\node (v2) at (1,0) [circle,draw,inner sep=1pt] {}; \\
\node (v3) at (2,0) [circle,draw,inner sep=1pt] {}; \\
\node (v4) at (3,0) [circle,draw,inner sep=1pt] {}; \\
\end{tikzpicture}
\end{array}
\] (4.17)

and the non-factorizable ones are
\[
\Gamma_{g,\text{nfac}}(k_1, k_2) = \begin{array}{c}
\text{\begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) [circle,draw,inner sep=1pt] {}; \\
\node (v2) at (1,0) [circle,draw,inner sep=1pt] {}; \\
\node (v3) at (2,0) [circle,draw,inner sep=1pt] {}; \\
\node (v4) at (3,0) [circle,draw,inner sep=1pt] {}; \\
\end{tikzpicture}}
+ \begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) [circle,draw,inner sep=1pt] {}; \\
\node (v2) at (1,0) [circle,draw,inner sep=1pt] {}; \\
\node (v3) at (2,0) [circle,draw,inner sep=1pt] {}; \\
\node (v4) at (3,0) [circle,draw,inner sep=1pt] {}; \\
\end{tikzpicture}
+ \begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) [circle,draw,inner sep=1pt] {}; \\
\node (v2) at (1,0) [circle,draw,inner sep=1pt] {}; \\
\node (v3) at (2,0) [circle,draw,inner sep=1pt] {}; \\
\node (v4) at (3,0) [circle,draw,inner sep=1pt] {}; \\
\end{tikzpicture}
+ \begin{tikzpicture}[baseline=-0.5ex]
\node (v1) at (0,0) [circle,draw,inner sep=1pt] {}; \\
\node (v2) at (1,0) [circle,draw,inner sep=1pt] {}; \\
\node (v3) at (2,0) [circle,draw,inner sep=1pt] {}; \\
\node (v4) at (3,0) [circle,draw,inner sep=1pt] {}; \\
\end{tikzpicture}
\end{array}
\] (4.18)

---

9With our convention for the hypercharge gauge coupling \( g_1 \) the covariant derivative reads \( D_\mu = \partial_\mu + ig_1 B_\mu + \cdots \), where \( g_\mu = 1/2 \) for the Higgs field and \( B_\mu \) is the hypercharge gauge field. The quartic term in the Higgs potential is \( \lambda (\phi^\dagger \phi)^2 \).

10\( O(g^2 T^2) \) and \( O(g^2 T^4) \) contributions (\( g \in \{ g_1, g_2, \lambda^{1/2}, y_t \} \)) to the \( CP \) asymmetry in the decays of the lightest sterile neutrino have been computed in ref. [25], but no connection with the kinetic equations for leptogenesis was made.
The wiggled lines represent electroweak gauge bosons. Both sets are independently gauge fixing independent. The top-quark contributions are

$$\Gamma_t(k_1, k_2) = \ldots$$

(4.19)

Here the lines in the closed fermion loop represent top-quarks. At zero temperature the only contribution containing the Higgs self-coupling is

$$\Gamma_\lambda(k_1, k_2) = 1.$$

(4.20)

Then the complete NLO correlator is the sum

$$\Gamma^{(2)}(k_1, k_2) = \Gamma_{g,\text{fac}}(k_1, k_2) + \Gamma_{g,\text{nfac}}(k_1, k_2) + \Gamma_t(k_1, k_2) + \Gamma_\lambda(k_1, k_2).$$

(4.21)

We compute the diagrams with a FORM code [24] which applies the following steps.

1. Generate the diagrams using Wick’s theorem.

2. Perform traces in flavor and gauge group space.

3. Insert the expressions for the propagators, with arbitrary gauge fixing parameter $\xi_1$ and $\xi_2$ for $B$- and $W$-bosons.

4. Use the properties of the charge conjugation matrix $C$. The Feynman-gauge results of this step are listed in appendix E for all diagrams.

5. Perform Dirac traces in naive dimensional regularization. In appendix E we show that terms with $\gamma^5$ do not contribute.

6. Express scalar products in the integrals in terms of inverse scalar propagators through the relations

$$p_i \cdot p_j = \frac{1}{2} \left( p_i^2 + p_j^2 - (p_i - p_j)^2 \right),$$

(4.22)

$$p_i \cdot k_1 = \frac{1}{2} \left( p_i^2 + k_1^2 - (p_i - k_1)^2 \right),$$

(4.23)

$$p_i \cdot k_2 = \frac{1}{2} \left( (p_i + k_2)^2 - p_i^2 - k_2^2 \right).$$

(4.24)

Then all three-point Feynman-integrals appearing in the computation of the NLO spectral functions have the generic form

$$I_{a_1,\ldots,a_{12}}(k_1, k_2) = \int \frac{1}{p_1^{2a_1} p_2^{2a_2} p_3^{2a_3} (p_1 - k_1)^{2a_4} (p_2 - k_1)^{2a_5} (p_4 - k_1)^{2a_6}}{p_1^{2a_7} (p_1 + k_2)^{2a_8} p_2^{2a_9} (p_1 + k_2)^{2a_{10}} (p_1 - p_3)^{2a_{11}} (p_2 - p_3)^{2a_{12}}}$$

(4.25)

with integer numbers $a_i$. 
7. Express all scalar integrals in terms of master-integrals. For this step we use the program Reduce [26] which uses the method of integration by parts (IBP) [27] and the Laporta algorithm [28] for IBP. The result of this step in given in appendix C.1.

We apply the relation (A.31) to the reduced three-point functions in appendix C.1 and express the results in terms of master-spectral functions listed in appendix C.2. Then the complete results for the spectral functions in terms of renormalized couplings are

\[
\rho_{g,\text{ufac}}(-k, k) = \frac{3(g_1^2 + g_2^2)M_1^2\mu^{-4\varepsilon}}{(16\pi)^2 2\pi^2} \left( \frac{3}{\varepsilon} + \frac{23}{2} + 8\ln(2) + 9\ln \left( \frac{\bar{\mu}^2}{M_1^2} \right) \right),
\]

\[
\rho_{g,\text{fac}}(-k, k) = -\frac{3(g_1^2 + 3g_2^2)M_1^2\mu^{-4\varepsilon}}{(16\pi)^2 4\pi^2} \left( \frac{3}{\varepsilon} + \frac{53}{2} + 9\ln \left( \frac{\bar{\mu}^2}{M_1^2} \right) \right),
\]

\[
\rho_{\lambda}(-k, k) = \frac{3\lambda M_1^2\mu^{-4\varepsilon}}{(16\pi)^2 \pi^2} \left( \frac{1}{\varepsilon} + \frac{13}{2} + 3\ln \left( \frac{\bar{\mu}^2}{M_1^2} \right) \right),
\]

\[
\rho_{\ell}(-k, k) = \frac{3|h_1|^2 M_1^2\mu^{-4\varepsilon}}{(16\pi)^2 \pi^2} \left( \frac{3}{\varepsilon} + \frac{45}{2} + 9\ln \left( \frac{\bar{\mu}^2}{M_1^2} \right) \right),
\]

with the \( \overline{\text{MS}} \) - scale parameter \( \bar{\mu}^2 \equiv 4\pi\mu^2 e^{-\gamma_E} \). Higher orders in \( \varepsilon \equiv (4 - d)/2 \) have been neglected.

We renormalize the \( N_1 \)-Yukawa couplings in the MS-scheme,

\[
h_{1i} = (h_{1i})_R \mu^\varepsilon Z_h
\]

with (see, e.g., ref. [29])

\[
Z_h = 1 + \frac{1}{(4\pi)^2 \varepsilon} \left( -\frac{3}{8} (g_1^2 + 3g_2^2) + \frac{N_c}{2} |h_1|^2 \right),
\]

and similarly for the dimension-5 couplings,

\[
c_{ij} = (c_{ij})_R \mu^{2\varepsilon} Z_c
\]

with (cf. refs. [30, 31])

\[
Z_c = 1 + \frac{1}{(4\pi)^2 \varepsilon} \left( -\frac{3}{2} g_2^2 + 2\lambda + N_c |h_1|^2 \right).
\]

We plug the results for the spectral function (4.26)–(4.29) into the master formula (3.36) and express the result in terms of the renormalized couplings (4.30), (4.32). Then we obtain the finite rate

\[
\gamma_{\text{ak}}^{(2)} = \gamma_{\text{ak}}^{(0)} \left\{ 1 + \frac{g_1^2 + 3g_2^2}{(8\pi)^2} \left[ 29 + 6 \ln \left( \frac{\bar{\mu}^2}{M_1^2} \right) \right] 
\right.
\]

\[
+ \frac{g_1^2 + g_2^2}{(8\pi)^2} \left( \frac{1}{2} - 8\ln(2) - 3 \ln \left( \frac{\bar{\mu}^2}{M_1^2} \right) \right) 
\]

\[
- \frac{|h_1|^2}{(8\pi)^2} \left( 84 + 24 \ln \left( \frac{\bar{\mu}^2}{M_1^2} \right) \right)
\left. \right) \left[ 20 + 8 \ln \left( \frac{\bar{\mu}^2}{M_1^2} \right) \right],
\]

where \( \gamma_{\text{ak}}^{(0)} \) is the leading-order rate (4.12) in the effective theory with the interaction (3.12). In figure 1 we show the corrections (4.34), normalized to the LO result. The renormalization scale is chosen as \( \bar{\mu} = M_1 \). The corrections are smaller than 4%. The dominant contribution is due to factorizable diagrams, and there is a cancellation between factorizable and non-factorizable ones.
Figure 1. The relative size of the radiative corrections (4.34) to the asymmetry rate in the non-relativistic regime $T \ll M_1$ versus $M_1$. The radiative corrections are smaller than 4% over the entire mass range relevant to thermal leptogenesis (cf. ref. [32]).

5 Summary and outlook

We have obtained the Kubo-type formula (2.13), by which one can relate the $CP$ violating rates in the equations for leptogenesis (1.3) and (1.4) to finite-temperature real-time correlation functions. The latter can be systematically computed in finite-temperature quantum field theory, which allows to include radiative corrections and determine the theoretical error in leptogenesis calculations. For hierarchical sterile-neutrinos masses $M_i \ll M_{i \neq 1}$ we have expressed the $CP$ violating rates in terms of the three-point function (3.21). Using the spectral representation (A.23) we found simple master formulas (3.36) and (3.38) relating them to a single three-point spectral function. These formulas are valid to leading order in the sterile-neutrino Yukawa-couplings, and to all orders in the Standard Model couplings, neglecting the small $CP$ violation and lepton Yukawa-interactions of the Standard Model. We applied them to compute the leading term in the low temperature regime $T \ll M_1$ of the asymmetry rate $\gamma_{\alpha k}$ up to NLO in Standard Model couplings. The size of the radiative corrections is smaller than 4%.

This work completes the list of Kubo-type relations for the rates in the kinetic equation for leptogenesis (1.3) and (1.4). Our low-temperature NLO result is most relevant in the so called strong washout regime where most of the asymmetry is generated at $T < M_1$. Now all rates for the non-relativistic regime $T \ll M_1$ have been computed at order $g^2$. It would be interesting to perform a leptogenesis computation including these complete corrections to see their effect on the baryon asymmetry of the Universe. We leave this for future work.

In the non-relativistic regime there are corrections suppressed by powers of $T/M_1$, which should be computable by applying the operator product expansion [40] to eq. (3.36). It would also be interesting, but also a lot more challenging, to compute the NLO in the relativistic
regime $T \sim M_1$. Furthermore, it would be interesting to compute the LO in the ultra-relativistic regime $T \gg M_1$.

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A Thermal three-point functions and their spectral representation

Before we describe the spectral representation for three-point functions we briefly recall the familiar case of the correlation function of two operators $A$ and $B$ (for a review see, e.g., [33])

$$\Delta_{AB}(t_A, t_B) \equiv \langle A(t_A)B(t_B) \rangle. \quad (A.1)$$

For complex times with

$$\text{Im} t_B \geq \text{Im} t_A \geq -\beta + \text{Im} t_B, \quad (A.2)$$

it is well defined, and its Fourier representation

$$\Delta_{AB}(t_A, t_B) = \int \frac{d\omega_A}{2\pi} \int \frac{d\omega_B}{2\pi} \exp[-i(\omega_A t_A + \omega_B t_B)]\Delta_{AB}(\omega_A, \omega_B) \quad (A.3)$$

exists. The cyclicity of the trace in the thermal average

$$\langle \cdots \rangle = Z^{-1} \text{tr}[\exp(-\beta H) \cdots] \quad (A.4)$$

implies

$$\Delta_{AB}(t_A, t_B) = \Delta_{BA}(t_B - i\beta, t_A), \quad (A.5)$$

and thus

$$\Delta_{AB}(\omega_A, \omega_B) = e^{-\beta\omega_B} \Delta_{BA}(\omega_B, \omega_A). \quad (A.6)$$

Translational invariance allows us to write

$$\Delta_{AB}(\omega_A, \omega_B) = 2\pi \delta(\omega_A + \omega_B)\tilde{\Delta}_{AB}(\omega_A). \quad (A.7)$$

In this work we pay special attention to the imaginary-time correlator$^{11}$

$$\Delta_{AB}(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} \langle A(-i\tau)B(0) \rangle. \quad (A.8)$$

Using eq. (A.3), (A.6), and (A.7) one obtains the spectral representation

$$\Delta_{AB}(i\omega_n) = \int \frac{d\omega}{2\pi} \frac{\rho_{AB}(\omega)}{\omega - i\omega_n} \quad (A.9)$$

with the spectral function

$$\rho_{AB}(\omega) \equiv \int dt e^{i\omega t} \langle [A(t), B(0)] \rangle. \quad (A.10)$$

$^{11}$To avoid a proliferation of symbols we use the same symbol as in eq. (A.1). The two can always be distinguished by their number of arguments.
Here

\[ [A, B] \equiv AB - (-1)^{\deg A \deg B} BA. \]  
(A.11)

with

\[ \deg A = \begin{cases} 
0 & \text{if } A \text{ bosonic} \\
1 & \text{if } A \text{ fermionic}
\end{cases} \]  
(A.12)

denotes the graded commutator of \( A \) and \( B \). The imaginary time correlator \( \langle A, B \rangle \) can be analytically continued into the complex frequency plane, \( i\omega_n \to \omega \). The retarded and advanced correlators are then given by

\[ \Delta_{AB}^{\text{ret}}(\omega) = \Delta_{AB}(\omega + i\epsilon), \quad \Delta_{AB}^{\text{adv}}(\omega) = \Delta_{AB}(\omega - i\epsilon), \]  
(A.13)

with real \( \omega \). Then the spectral representation (A.9) in combination with the identity (2.9) yields the inverse relation

\[ \rho_{AB}(\omega) = \frac{1}{i} \left[ \Delta_{AB}^{\text{ret}}(\omega) - \Delta_{AB}^{\text{adv}}(\omega) \right]. \]  
(A.14)

According to (A.9) it is possible to write this inverse relation in terms of the imaginary part of the retarded correlator as

\[ \rho_{AB}(\omega) = 2\text{Im} \Delta_{AB}^{\text{ret}}(\omega), \]  
(A.15)

if the spectral function is real-valued.

In the following we derive relations analogous to (A.9), (A.10) and (A.14) for three-point correlators. To our knowledge the first spectral representation of three-point functions at finite temperature has been derived [34, 35] in the real-time formalism for advanced and retarded correlators. These relations are, however, not simple integral representations like (A.9) and are therefore not useful for us. In [36, 37] three-point spectral representations have been derived, which are indeed integral representations similar to (A.9). However, these references give different spectral-representations for each retarded and advanced real-time correlator, and none for the imaginary time correlator. In [38] it has been shown how all six retarded and advanced three-point functions can be related to the imaginary-time correlator via analytical continuation. This suggests that similar to (A.9) there is a single spectral representation for the imaginary-time correlator which also covers the three spectral representations of [36, 37].

We will first derive the spectral representation of the imaginary-time three-point correlator, using the techniques of [38] and then show that the two independent spectral functions can be written in terms of (anti-)commutators similar to (A.10). Furthermore we show that there are inverse relation for the spectral functions analogous to (A.14).

For operators \( A, B, \) and \( C \) we consider the three-point correlation function

\[ \Gamma_{ABC}(t_A, t_B, t_C) \equiv \langle A(t_A)B(t_B)C(t_C) \rangle \]  
(A.16)

which is well defined for complex times with

\[ \text{Im } t_C \geq \text{Im } t_B \geq \text{Im } t_A \geq -\beta + \text{Im } t_C. \]  
(A.17)

In this region their Fourier representation

\[ \Gamma_{ABC}(t_A, t_B, t_C) = \int \frac{d\omega_A}{2\pi} \int \frac{d\omega_B}{2\pi} \int \frac{d\omega_C}{2\pi} \exp \left[ -i(\omega_At_A + \omega_Bt_B + \omega_Ct_C) \right] \times \Gamma_{ABC}(\omega_A, \omega_B, \omega_C), \]  
(A.18)
exists. The cyclicity of the trace in (A.4) implies
\[ \Gamma_{ABC}(\omega_A, \omega_B, \omega_C) = e^{-\beta \omega_C} \Gamma_{CAB}(\omega_C, \omega_A, \omega_B). \] (A.19)

Due to translational invariance in time we can write
\[ \Gamma_{ABC}(\omega_A, \omega_B, \omega_C) = 2\pi \delta(\omega_A + \omega_B + \omega_C) \tilde{\Gamma}_{ABC}(\omega_A, \omega_B). \] (A.20)

We are interested in the Fourier transform of a time ordered three-point function in imaginary time,\(^\text{12}\)
\[ \Gamma_{ABC}(i\omega_n, i\omega_{n'}) \equiv \int_0^\beta d\tau \int_0^\beta d\tau' \exp(i\omega_n \tau + i\omega_{n'} \tau') \langle [\mathcal{T}A(-i\tau)B(-i\tau')C(0)] \rangle, \] (A.21)
where the time ordering \( T \) is defined as
\[ \mathcal{T}A(-i\tau)B(-i\tau') \equiv \theta(\tau - \tau') A(-i\tau) B(-i\tau') + (-1)^{\deg A \deg B} \theta(\tau' - \tau) B(-i\tau') A(-i\tau). \] (A.22)

Following [38], we use the Fourier representation (A.18) of the correlators on the right-hand side to perform the \( \tau \) and \( \tau' \) integrals. This gives an integral representation of the imaginary-time correlator, containing the real-time correlation functions \( \gamma_{ABC} \) and \( \gamma_{BAC} \) and temperature-dependent exponential functions. Unlike [38], we rearrange the pre-factors of the real-time correlators using the cyclicity property (A.19) which allows us to cancel all exponentials. This yields the simple spectral representation
\[ \Gamma_{ABC}(i\omega_n, i\omega_{n'}) = \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \int \frac{d\omega_3}{2\pi} \sum_{n=1}^3 \frac{1}{\omega_n - \omega_n'} \left[ \frac{\rho_{ABC}(\omega_1, \omega_2)}{i\omega_n - \omega_1} + (-1)^{\deg A \deg B} \frac{\rho_{BAC}(\omega_2, \omega_1)}{i\omega_n' - \omega_2} \right]. \] (A.23)

It contains two spectral functions
\[ \rho_{ABC}(\omega, \omega') \equiv \int dt \int dt' \exp(i\omega t + i\omega' t') \left\langle \left[ A(t), [B(t'), C(0)] \right] \right\rangle \] (A.24)
which contain the graded commutators (A.11). Like in the case of two-point functions, the three-point spectral functions (A.24) can be computed from the imaginary time correlator via analytic continuation. This yields six different retarded functions which in the notation of [38] read
\[ R_1(\omega_A, \omega_B) = \Gamma_{ABC}(\omega_A + 2i\epsilon, \omega_B - i\epsilon) \] (A.25)
\[ R_2(\omega_A, \omega_B) = \Gamma_{ABC}(\omega_A - i\epsilon, \omega_B + 2i\epsilon) \] (A.26)
\[ R_3(\omega_A, \omega_B) = \Gamma_{ABC}(\omega_A - i\epsilon, \omega_B - i\epsilon) \] (A.27)
\[ \overline{R}_i(\omega_A, \omega_B) \mid_{\epsilon \to -\epsilon} = R_i(\omega_A, \omega_B) \] (A.28)

In contrast to the integral representation in [38] our spectral representation (A.23) provides simple inverse relations. We obtain them by combining (A.23) and (2.9), and we easily find
\[ \rho_{ABC} = R_2 + \overline{R}_2 - R_3 - \overline{R}_3, \] (A.29)
\[ \rho_{BAC} = (-1)^{\deg A \deg B} \left( R_1 + \overline{R}_1 - R_3 - \overline{R}_3 \right). \] (A.30)
\(^\text{12}\)Cf. the footnote on page 18.
If the spectral function is real, these relations can be further simplified to

\[
\rho_{ABC} = 2\text{Re}(R_2 - R_3) \quad \text{(A.31)}
\]
\[
\rho_{BAC} = (-1)^{\text{deg}A \cdot \text{deg}B} 2\text{Re}(R_1 - R_3). \quad \text{(A.32)}
\]

Let us summarize the results of this section. We have shown that the imaginary-time three-point correlator \((A.21)\) can be expressed in terms of a single spectral representation \((A.23)\), containing two independent spectral functions. The retarded and advanced correlators are related to the imaginary-time correlator by \((A.25)–(A.28)\) and yield the inverse relations \((A.29)\) and \((A.30)\).

### B Implications of discrete symmetries for spectral functions

First consider two-point spectral functions of Hermitian operators \(X\) which satisfy

\[
\text{CPT} X(t) (\text{CPT})^{-1} = \vartheta_X X(-t) \quad \text{(B.1)}
\]

with a phase factor \(\vartheta_X\). \(\text{CPT}\) invariance implies that

\[
\langle \mathcal{O} \rangle = \langle \text{CPT} \mathcal{O} (\text{CPT})^{-1} \rangle^* \quad \text{(B.2)}
\]

for the thermal expectation value of any operator \(\mathcal{O}\), and in our case

\[
\rho_{AB}(\omega) = \vartheta_A^* \vartheta_B^* \rho_{AB}(\omega)^* \quad \text{(B.3)}
\]

Thus the two-point spectral functions are real or imaginary if \(\vartheta_A \vartheta_B = 1\) or \(\vartheta_A \vartheta_B = -1\), respectively.\(^{13}\) For the case \(A = X_a\) and \(B = \delta f_k\), defined through \((2.14)\) and \((2.21)\), the spectral function is imaginary.

Now we turn to three-point correlators and their spectral functions. Here we assume \(\text{CP}\) and thus \(T\) invariance. We are interested in field operators, not necessarily Hermitian, which satisfy

\[
\text{CP} [A(t_1, x_1), [B(t_2, x_2), C(0)]] (\text{CP})^{-1} = \vartheta_{\text{CP}} [A(t_1, -x_1), [B(t_2, -x_2), C(0)]]^T \quad \text{(B.4)}
\]
\[
\text{T} [A(t_1, x_1), [B(t_2, x_2), C(0)]] T^{-1} = \vartheta_{\text{T}} [A(-t_1, x_1), [B(-t_2, x_2), C(0)]] \quad \text{(B.5)}
\]

with \(\vartheta_T\) and \(\vartheta_{\text{CP}}\) being \(\pm 1\). Since the operators \(A, B\) depend on spatial coordinates \(x_1\) and \(x_2\), the corresponding spectral function (cf. eq. \((3.30)\)) now also depends on the conjugate variables \(k_1\) and \(k_2\). \(\text{CP}\) invariance implies

\[
\rho_{ABC}(\omega_1, k_1, \omega_2, k_2) = \vartheta_{\text{CP}} \rho_{ABC}(-\omega_1, k_1, -\omega_2, k_2)^* \quad \text{(B.6)}
\]

and time reversal invariance gives

\[
\rho_{ABC}(\omega_1, k_1, \omega_2, k_2) = \vartheta_{\text{T}} \rho_{ABC}(\omega_1, -k_1, \omega_2, -k_2)^* \quad \text{(B.7)}
\]

Assuming that \([A, B, C]\) is a scalar, rotational invariance allows us to rewrite this as

\[
\rho_{ABC}(\omega_1, k_1, \omega_2, k_2) = \vartheta_{\text{T}} \rho_{ABC}(\omega_1, k_1, \omega_2, k_2)^* \quad \text{(B.8)}
\]

\(^{13}\)In [39] this was shown under the much stronger assumption of time-reversal invariance.
Thus, depending on the sign of $\vartheta_T$, the spectral function is either real or imaginary. Combining eqs. (B.6) and (B.7) we obtain

$$\rho_{ABC}(\omega_1, k_1, \omega_2, k_2) = \vartheta_C P \vartheta_T \rho_{ABC}(-\omega_1, k_1, -\omega_2, k_2).$$

Therefore the spectral function is either even or odd under $(\omega_1, \omega_2) \rightarrow (-\omega_1, -\omega_2)$.

Now we specialize to the three-point function (3.21), which enters the asymmetry rate. Here we can use

$$\text{CP} J_i(t, x)(\text{CP})^{-1} = \eta_{\text{CP}} C [J_i(t, -x)]^*$$

$$\mathcal{T} J_i(t, x)\mathcal{T}^{-1} = \eta_T \gamma_1 \gamma_3 J_i(-t, x),$$

where $C$ is the charge conjugation matrix, $\gamma_1, \gamma_3$ are Dirac matrices, and $\eta_{\text{CP}}$ and $\eta_T$ are phase factors. Eqs. (B.10), (B.11) imply $\vartheta_{\text{CP}} = \vartheta_T = 1$ for the phase factors in eq. (B.9).

Thus the corresponding spectral function $\rho_{ABC}$ and $\rho_{BAC}$ are real and even under $(k_1, k_2) \rightarrow (-k_1, -k_2)$. Using the spectral representation (A.23) and eq. (B.7) one finds

$$[\Gamma_{ijlm}(k_1, k_2)]^* = \Gamma_{ijlm}(-k_1, -k_2).$$

Furthermore, in imaginary time eq. (B.6) implies that

$$[\Gamma_{ijlm}(k_1, k_2)]^* = \Gamma_{ijlm}(k_1, k_2).$$

\section{C Master integrals and master spectral functions}

\subsection{C.1 Results of the reduction to master integrals}

We use the program Reduce [26] to obtain the following gauge-parameter independent contributions to the three-point correlator (3.27) in terms of master integrals, defined in (4.25). For the factorizable diagrams in eq. (4.17) we find

$$\Gamma_{g,\text{fac}}(k_1, k_2) = 2N_w(N_w + 1) \left( y_2^2 g_1^2 + C_2(r) g_2^2 \right) \times \left( \frac{(d-2)(-4-d+d^2)k_2 \cdot k_1}{(-4+d)^2 k_2^2} I_{1011001100100} \right.$$

$$\left. + \frac{(d-2)(-4-d+d^2)k_2 \cdot k_1}{(-4+d)^2 k_1^2} I_{10101000110} \right)$$

$$- \frac{(d-2)k_2 \cdot k_1}{(d-4)} I_{11101111000} - \frac{(d-2)k_2 \cdot k_1}{(d-4)} I_{111001111000},$$

and for the non-factorizable ones in eq. (4.18)

$$\Gamma_{g,\text{nfac}}(k_1, k_2) = -2N_w \left( y_2 y_6 (N_w + 1) g_1^2 + C_2(r) g_2^2 \right) \times \left( \frac{-(d-2)(2d-5)(-20 + 79d - 48d^2 + 8d^3)}{(d-3)^2(3d-10)(3d-8)k_2^2} I_{001000010110} \right.$$

$$\left. + \frac{4(d-2)(2d-5)(2d-3)}{(d-4)(3d-8)k_2^2} I_{000001010110} \right)$$

$$+ \frac{(d-2)(2(9 - 9d + 2d^2)(k_1 + k_2)^2 + (-25 + 23d - 5d^2)k_2^2)}{(d-3)(3d-8)k_2^2} I_{001001010110},$$

$$- \frac{(d-2)(-20 + 79d - 48d^2 + 8d^3)}{(d-3)(3d-8)k_2^2} I_{111001111000},$$

$$+ \frac{4(d-2)(2d-5)(2d-3)}{(d-4)(3d-8)k_2^2} I_{000001010110}.$$
\[
\Gamma_{\lambda}(k_1, k_2) = -4N_w(N_w + 1) \lambda \\
\times \left( \frac{(-2 + d)(-5 + 2d)(-20 + 7d)}{(-3 + d)(-10 + 3d)(-8 + 3d)k_1^4 I_{00100010110}} + \frac{(-2 + d)}{(-8 + 3d) I_{00100010110}} + \frac{(-5 + 2d)(-1040 + 1064d - 362d^2 + 41d^3)}{(-4 + d)(-3 + d)(-10 + 3d)(-8 + 3d)k_1^2 I_{00000010110}} \\
- \frac{(100 - 72d + 13d^2)}{(-10 + 3d)(-8 + 3d)} I_{010001010110} - \frac{(-4 + d)(k_1 + k_2)^2}{(-3 + d)(-8 + 3d) I_{01000020110}} + \frac{-2((-4 + d)(-5 + 2d)k_2^2 + (-2 + d)^2 k_1 \cdot k_2)}{(-8 + 3d)^2} I_{0110001010110} \\
+ \frac{-8(-4 + d)(-5 + 2d)k_1^2 k_2^2}{(-10 + 3d)(-8 + 3d)^2 I_{02100010110}} - \frac{(-2 + d)k_1^2 (k_1 + k_2)^2}{(-3 + d)(-8 + 3d)k_2^3 I_{00100201110}} \right). 
\]
C.2 Results for master spectral functions

The only master integrals in appendix C.1 which contribute to the spectral functions (4.26)–(4.29) are

\[ I_{BB}(k_1, k_2) \equiv I_{011001010110}(k_1, k_2), \]
\[ I_{BBdot}(k_1, k_2) \equiv I_{012001010110}(k_1, k_2), \]
\[ I_{LR}(k_1, k_2) \equiv I_{100001010110}(k_1, k_2), \]
\[ I_{2L}(k_1, k_2) \equiv I_{1100001010110}(k_1, k_2), \]
\[ I_{2R}(k_1, k_2) \equiv I_{101001010110}(k_1, k_2), \]
\[ I_{fac3L}(k_1, k_2) \equiv I_{0110100010001}(k_1, k_2), \]
\[ I_{fac3R}(k_1, k_2) \equiv I_{0110011001000}(k_1, k_2), \]
\[ I_{fac4L}(k_1, k_2) \equiv I_{1110111100000}(k_1, k_2), \]
\[ I_{fac4R}(k_1, k_2) \equiv I_{1110011110000}(k_1, k_2), \]

with \( I_{a_1, \ldots, a_{12}}(k_1, k_2) \) defined in (4.25). For all other master integrals we find that either the corresponding spectral function \( \rho(-k, k) \) vanishes, or the integrals are multiplied by \((k_1 + k_2)^2\) which is put to zero at the end of the calculation. We have checked that such spectral functions do not have a \(1/(k_1 + k_2)^2\) pole, which could cancel the factor \((k_1 + k_2)^2\).

Following the steps of appendix D we find for the master spectral functions expanded in \(\varepsilon = (4 - d)/2\)

\[ \rho_{BB}(-k, k) = \frac{\mu^{-6\varepsilon}}{(16\pi)^2 4\pi^2} \left[ \frac{1}{\varepsilon} + 7 + 3 \ln \left( \frac{\bar{\mu}^2}{k^2} \right) \right], \]
\[ \rho_{BBdot}(-k, k) = \frac{\mu^{-6\varepsilon}}{(16\pi)^2 4\pi^2} \left[ \frac{1}{\varepsilon} + 4 + 3 \ln \left( \frac{\bar{\mu}^2}{k^2} \right) \right], \]
\[ \rho_{LR}(-k, k) = \frac{k^2 \mu^{-6\varepsilon}}{(16\pi)^2 8\pi^2} \left[ 1 + \varepsilon \left[ 10 + 3 \ln \left( \frac{\bar{\mu}^2}{k^2} \right) \right] \right], \]
\[ \rho_{2L}(-k, k) = -\frac{1 + \ln(2)}{(16\pi)^2 4\pi^2} \mu^{-6\varepsilon}, \]
\[ \rho_{2R}(-k, k) = -\frac{1 + \ln(2)}{(16\pi)^2 4\pi^2} \mu^{-6\varepsilon}, \]
\[ \rho_{fac3L}(-k, k) = \frac{k^2 \mu^{-6\varepsilon}}{(16\pi)^2 8\pi^2} \left[ 1 + \varepsilon \left[ \frac{17}{2} + 3 \ln \left( \frac{\bar{\mu}^2}{k^2} \right) \right] \right], \]
\[ \rho_{fac3R}(-k, k) = \frac{k^2 \mu^{-6\varepsilon}}{(16\pi)^2 8\pi^2} \left[ 1 + \varepsilon \left[ \frac{17}{2} + 3 \ln \left( \frac{\bar{\mu}^2}{k^2} \right) \right] \right], \]
\[ \rho_{fac4L}(-k, k) = -\frac{\mu^{-6\varepsilon}}{(16\pi)^2 2\pi^2} \left[ \frac{1}{\varepsilon} + 6 + 3 \ln \left( \frac{\bar{\mu}^2}{k^2} \right) \right], \]
\[ \rho_{fac4R}(-k, k) = -\frac{\mu^{-6\varepsilon}}{(16\pi)^2 2\pi^2} \left[ \frac{1}{\varepsilon} + 6 + 3 \ln \left( \frac{\bar{\mu}^2}{k^2} \right) \right]. \]

D Computation of master spectral functions

In this section we explain the method which we used to compute the master three-point spectral functions in appendix C.2.
D.1 Factorizable integrals

If the master integrals can be written as a product of two-point integrals $I_a(k_1)$ and $I_b(k_2)$ as

$$\Gamma(k_1, k_2) = I_a(k_1) I_b(k_2),$$  \hspace{1cm} (D.1)

one can simplify the inverse relation (A.31) to

$$\rho(k_1, k_2) = \rho_a(k_1) \rho_b(k_2),$$  \hspace{1cm} (D.2)

where $\rho_a(k) = 2 \text{Im} I_a(k_0 + i \epsilon, k)$  \hspace{1cm} (D.3)

is the two-point spectral function of the integral $I_a$. In this work we have to deal with the factorizable integrals

$$I_{\text{fac}3L}(k_1, k_2) = I_3(k_1) I_1(k_2)$$  \hspace{1cm} (D.4)

$$I_{\text{fac}3R}(k_1, k_2) = I_1(k_1) I_3(k_2)$$  \hspace{1cm} (D.5)

$$I_{\text{fac}4L}(k_1, k_2) = I_2(k_1) I_1(k_2)$$  \hspace{1cm} (D.6)

$$I_{\text{fac}4R}(k_1, k_2) = I_1(k_1) I_2(k_2),$$  \hspace{1cm} (D.7)

where

$$I_1(k) = \frac{1}{(2\pi)^d} \int \frac{dp}{p^2(p-k)^2}$$  \hspace{1cm} (D.8)

$$I_2(k) = \frac{1}{(2\pi)^d} \int \frac{dp_1}{(2\pi)^d} \frac{dp_2}{p_1(p-k)^2(p_2-k)^2}$$  \hspace{1cm} (D.9)

$$I_3(k) = \frac{1}{(2\pi)^d} \int \frac{dp_1}{(2\pi)^d} \frac{dp_2}{p_1(p_1-p_2)^2(p_2-k)^2}.$$  \hspace{1cm} (D.10)

The imaginary parts of their analytic continuation to real $k^0$ have been computed in [12] and read for $k^2 > 0$

$$\text{Im}I_1(k^0 + i \epsilon, k) = \frac{\text{sgn}(k^0)}{16\pi} \mu^{-2\epsilon} \left[ 1 + \epsilon \left( \ln \frac{\mu^2}{k^2} + 2 \right) \right] + O(\epsilon^2)$$  \hspace{1cm} (D.11)

$$\text{Im}I_2(k^0 + i \epsilon, k) = \frac{\text{sgn}(k^0)}{2(4\pi)^3} \mu^{-4\epsilon} \left( \frac{1}{\epsilon} + 2 \ln \frac{\mu^2}{k^2} + 4 \right) + O(\epsilon),$$  \hspace{1cm} (D.12)

$$\text{Im}I_3(k + i \epsilon, k) = -\frac{\text{sgn}(k^0)k^2}{8(4\pi)^3} + O(\epsilon).$$  \hspace{1cm} (D.13)

This yields the results (C.19)–(C.22).

D.2 Non-factorizable integrals

In the more general case that the three-point correlator cannot be written as a product of two two-point correlators, we proceed as follows. We consider the $N_L$-loop Feynman integrals with $N_p$ propagators

$$I(k_1, k_2) = \left( \prod_{l=1}^{N_L} \int \frac{d^d p_l}{(2\pi)^d} \right) \prod_{l=1}^{N_p} \frac{1}{q_l^2}, \quad q_i = a_{in} p_n + b_{im} k_m,$$  \hspace{1cm} (D.14)

with real coefficients $a_{in}$ and $b_{im}$. In order to compute the spectral function $\rho(-k, k)$ we apply the following steps.
1. Compute the integrals over $p_i^0$. The result can be written in the form

$$I(k_1, k_2) = \left( \prod_{l=1}^{N_L} \int \frac{d^{d-1}p_l}{(2\pi)^{d-1}} \right) \frac{A(k_i, p_i)}{(B(k_i, p_i) - k_0^0)(C(k_i, p_i) - k_0^0)}$$

$$+ \text{many similar terms} \quad \text{(D.15)}$$

Here $A, B$ and $C$ are real functions which only depend on scalar products of the spatial components of the external momenta and the loop momenta.

2. Apply the inverse relation (A.31) and (2.9). The result yields two delta-functions such that

$$\rho(k_1, k_2) = 4\pi^2 \left( \prod_{l=1}^{N_L} \int \frac{d^{d-1}p_l}{(2\pi)^{d-1}} \right) A(k_i, p_i) \delta \left( B(k_i, p_i) - k_0^0 \right) \delta \left( C(k_i, p_i) - k_0^0 \right)$$

$$+ \text{many similar terms} \quad \text{(D.16)}$$

3. Set $k_2 = -k_1 = k$ with $k_2^2 = M_2^2$ and drop all terms which do not contribute due to the constraints of the delta functions.

4. Solve the remaining $(d-1)$-dimensional integrals.

5. Expand the result in $\varepsilon = (d - 4)/2$.

This yields the results (C.14), (C.16)–(C.22).

If one propagator is squared, that is,

$$I_{\text{squared}}(k_1, k_2) = \left( \prod_{l=1}^{N_L} \int \frac{d^d p_l}{(2\pi)^d} \right) \frac{1}{q_k^2} \prod_{i \neq k} \frac{1}{q_i^2}, \quad \text{(D.17)}$$

we introduce an artificial mass as

$$I_m(k_1, k_2) = \left( \prod_{l=1}^{N_L} \int \frac{d^d p_l}{(2\pi)^d} \right) \frac{1}{q_k^2 - m^2 \prod_{i \neq k} \frac{1}{q_i^2}} \quad \text{(D.18)}$$

Then we apply the steps (1.)–(5.) for $I_m$ and obtain the spectral function $\rho_{\text{squared}}$ as

$$\rho_{\text{squared}}(-k, k) = \left. \frac{d^2}{dm^2} \rho_m(-k, k) \right|_{m^2=0}, \quad \text{(D.19)}$$

which yields the result (C.15).

### E Treatment of $\gamma^5$ in Dirac traces

In this appendix we argue that terms with $\gamma^5$ do not contribute to the Dirac traces. Contracting gauge indices and using the properties of the charge conjugation matrix $C$ the NLO diagrams read in Feynman gauge

$$= 2N_w(N_w + 1) \frac{g_2^2 g_1^2}{C_2(r) g_2^2} \text{Tr} (\gamma_{\mu_1} P_L \gamma_{\mu_2} P_R)$$

$$\times \int_{p_1, p_2, p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} (p_1 + p_2 - 2k_1)^2}{p_1^2 p_3^2 (p_1 - k_1)^2 (p_2 - k_1)^2 (p_3 + k_2)^2 (p_1 - p_2)^2}, \quad \text{(E.1)}$$

$$= 2N_w(N_w + 1) \frac{g_2^2 g_1^2}{C_2(r) g_2^2} \text{Tr} (\gamma_{\mu_1} P_L \gamma_{\mu_2} P_R)$$

$$\times \int_{p_1, p_2, p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} (p_1 + p_2 - 2k_1)^2}{p_1^2 p_3^2 (p_1 - k_1)^2 (p_2 - k_1)^2 (p_3 + k_2)^2 (p_1 - p_2)^2}, \quad \text{(E.1)}$$
\[
2N_w(N_w + 1)(y_1^2 g_1^2 + C_2(r)g_2^2) \text{Tr}(\gamma_{\mu_1} P_L \gamma_{\mu_2} P_R) \\
\times \int_{p_1, p_2, p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} (p_3 + p_2 + 2k_2)^2}{p_1^2 p_2^2 (p_1 - k_1)^2 (p_2 + k_2)^2 (p_3 + k_2)^2 (p_3 - p_2)^2},
\]
(E.2)

\[
2N_w(N_w + 1)(y_1^2 g_1^2 + C_2(r)g_2^2) \\
\times \text{Tr}(\gamma_{\mu_1} P_L \gamma_{\mu_5} P_R \gamma_{\mu_2} P_L \gamma_{\mu_3} P_L \gamma_{\mu_4} P_R) \\
\times \int_{p_1, p_2, p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} p_4^{\mu_4}}{p_1^2 p_2^2 p_3^2 p_4^2 (p_1 - k_1)^2 (p_2 + k_2)^2 (p_3 + k_2)^2 (p_3 - p_2)^2},
\]
(E.3)

\[
2N_w(N_w + 1)(y_2 y_1 g_1^2 + C_2(r)g_2^2) \\
\times \text{Tr}(\gamma_{\mu_1} P_L \gamma_{\mu_5} P_R \gamma_{\mu_2} P_L \gamma_{\mu_3} P_L \gamma_{\mu_4} P_R) \\
\times \int_{p_1, p_2, p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} (p_4 + p_2 - 2k_1)^{\mu_5}}{p_1^2 p_2^2 p_3^2 p_4^2 (p_1 - k_1)^2 (p_2 - k_1)^2 (p_3 + k_2)^2 (p_3 - p_2)^2},
\]
(E.4)

\[
2N_w(N_w + 1)(y_2 y_1 g_1^2 + C_2(r)g_2^2) \\
\times \text{Tr}(\gamma_{\mu_1} P_L \gamma_{\mu_5} P_R \gamma_{\mu_2} P_L \gamma_{\mu_3} P_L \gamma_{\mu_4} P_R) \\
\times \int_{p_1, p_2, p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} (p_2 + p_3 + 2k_2)^{\mu_5}}{p_1^2 p_2^2 p_3^2 p_4^2 (p_1 - k_1)^2 (p_2 + k_2)^2 (p_3 + k_2)^2 (p_3 - p_2)^2},
\]
(E.6)

\[
2N_w(N_w + 1)(y_2 y_1 g_1^2 + C_2(r)g_2^2) \\
\times \text{Tr}(\gamma_{\mu_1} P_L \gamma_{\mu_2} P_R \gamma_{\mu_5} P_L \gamma_{\mu_3} P_L \gamma_{\mu_4} P_R) \\
\times \int_{p_1, p_2, p_3} \frac{p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} (p_4 + k_2)^{\mu_5}}{p_1^2 p_2^2 p_3^2 p_4^2 (p_1 - k_1)^2 (p_2 + k_2)^2 (p_3 + k_2)^2 (p_3 - p_2)^2},
\]
(E.8)

\[
-2N_w(N_w + 1) N_c |h| t^2
\]
(E.9)

\[
-2N_w(N_w + 1) N_c |h| t^2
\]
(E.11)
The total anti-symmetry of the Levi-Civita symbol $\epsilon$ analyzed in [E.13] so that $\epsilon$ anti-symmetric combinations of factorizable diagrams (E.13)–(E.14) contain the same trace as in the leading order case, $\epsilon$ totally anti-commuting $\gamma^5$ and $(\gamma^5)^2 = 1$ in traces with more than one $\gamma^5$, only traces with no or one $\gamma^5$ remain. With the definition (4.5) we can write these traces as

$$\text{Tr} \left( \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4} \gamma^5 \right) = -\frac{i}{4!} \epsilon^{\nu_1 \nu_2 \nu_3 \nu_4} \text{Tr} \left( \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\nu_3} \gamma_{\nu_4} \gamma_{\nu_1} \gamma_{\nu_2} \gamma_{\nu_3} \gamma_{\nu_4} \right).$$

(E.20)

The total anti-symmetry of the Levi-Civita symbol $\epsilon^{\nu_1 \nu_2 \nu_3 \nu_4}$ guarantees that only totally anti-symmetric combinations of $\eta^{\mu_1 \nu_1} \eta^{\mu_2 \nu_2} \eta^{\mu_3 \nu_3} \eta^{\mu_4 \nu_4}$ contribute to the trace, which yields

$$\text{Tr} \left( k_1 k_2 k_3 k_4 \gamma^5 \right) = -4i \epsilon_{\mu \nu \rho \sigma} k_1^\mu k_2^\nu k_3^\rho k_4^\sigma.$$

(E.21)
Therefore, all tensor integrals in the diagrams (E.15)–(E.19) coming from terms with traces containing one $\gamma^5$, have the generic form

$$I^\mu_{\nu\rho\sigma}(k_1, k_2) = \int_{p_1, p_2, p_3} \frac{T^{\mu\nu\rho\sigma}(\{p_1\}, k_1, k_2)}{p_1^2 p_2^2 p_3^2 (p_1 - k_1)^2 (p_2 + k_2)^2 (p_1 - p_3)^2 (p_2 - p_3)^2}, \tag{E.22}$$

where $T^{\mu_1\mu_2\mu_3\mu_4}(\{p_1\}, k_1, k_2)$ is one of the six total anti-symmetric rank-4-tensors of the set

$$\{p_1^\mu p_2^\nu p_3^\rho p_4^\sigma, p_1^\mu p_2^\nu k_1^\rho k_2^\sigma\}. \tag{E.23}$$

Some of the tensor integrals of the class (E.22) contain the vector-integral

$$J^\mu(p_1, p_2) = \int p_3^\mu (p_3 - p_1)^2 (p_3 - p_2)^2 \tag{E.24}$$

as a sub-integral. Due to Lorentz invariance it can be written in terms of scalar functions $f_1$ and $f_2$ as

$$J^\mu(p_1, p_2) = p_1^\mu f_1(p_1, p_2) + p_2^\mu f_2(p_1, p_2). \tag{E.25}$$

Therefore, we can express all integrals of the class (E.22) in terms of integrals containing only the tensor

$$T^{\mu\nu\rho\sigma}(p_1, p_2, k_1, k_2) = f_1^\mu p_2^\nu k_1^\rho k_2^\sigma. \tag{E.26}$$

Lorentz symmetry allows to compute the three-point correlator for $k_1 = (k_1^0, 0)$ and $k_2 = (k_2^0, 0)$. Since $T^{\mu\nu\rho\sigma}(p_1, p_2, k_1, k_2) = 0$, terms with $\gamma^5$ do not contribute. This argument holds only at zero temperature. For a finite-temperature computation it would be necessary to compute the tensor sum-integrals explicitly.

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