Quantum Noether Method

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Abstract. - We present a general method for constructing perturbative quantum field theories with global symmetries. We start from a free non-interacting quantum field theory with given global symmetries and we determine all perturbative quantum deformations assuming the construction is not obstructed by anomalies. The method is established within the causal Bogoliubov-Shirkov-Epstein-Glaser approach to perturbative quantum field theory (which leads directly to a finite perturbative series and does not rely on an intermediate regularization). Our construction can be regarded as a direct implementation of Noether’s method at the quantum level. We illustrate the method by constructing the pure Yang-Mills theory (where the relevant global symmetry is BRST symmetry), and the \( N = 1 \) supersymmetric model of Wess and Zumino. The whole construction is done before the so-called adiabatic limit is taken. Thus, all considerations regarding symmetry, unitarity and anomalies are well-defined even for massless theories.

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1 Introduction

Symmetries have always played an important and fundamental rôle in our quest of understanding nature. In classical physics they lead to conserved quantities (integrals of motion). The latter constrain the classical evolution of the system and some times they even uniquely determine it (for example, in the case of two-dimensional exactly solvable models). The rôle of symmetries in quantum physics is equally important. In quantum field theory symmetries lead to relations among the Green functions of the theory (Ward-Takahashi identities). The latter are instrumental in the proof of renormalizability and unitarity of the theory under question. Furthermore, recent developments in supersymmetric theories\[1\] show that global symmetries themselves are sometimes sufficient to determine the structure of the theory. Thus, it seems desirable to carefully understand the inter-relations between symmetries and quantum theory in a manner which is free of the technicalities inherent in the conventional Lagrangian approach (regularization-renormalization), and also in a way which is model independent as much as possible. In this article we shall undertake a first step towards this goal. We shall analyze this question within perturbative quantum field theory(QFT).

One may argue that, to a large extent, the relation between symmetries and perturbative QFT is by now well-understood. However, one would like to have an understanding at a more fundamental level. Namely, to separate the generic properties that symmetries impose from the specifics of a given model that realizes this symmetry. In addition, it would desirable to have a formulation which is mathematically as sound as possible.

A framework that encompasses most of the desired properties for this kind of questions is the causal approach to perturbative quantum field theory introduced by Bogoliubov and Shirkov\[2\] and developed by Epstein and Glaser\[3, 4, 5, 6\]. The explicit construction method of Epstein and Glaser rests directly on the axioms of relativistic quantum field theory. On the one hand, it clarifies how the fundamental axioms guide the perturbative construction of the $S$ matrix, and how well-defined time-ordered products are directly constructed without the need of an intermediate regularization of the theory. On the other hand, it is an explicit construction method for the most general perturbation series compatible with causality and Poincaré invariance. The purely technical details which are essential for explicit calculations are separated from the simple physical structure of the theory. With the help of the causality condition, the well-known problem of ultraviolet (UV) divergences is reduced to a mathematically well-defined problem, namely the splitting of an operator-valued distribution with causal support into a distribution with retarded and a distribution with advanced support or, alternatively \[3, 7\], to the continuation of time-ordered products to coincident points. Implicitly, every consistent renormalization scheme solves this problem. In this sense the explicit Epstein-Glaser (EG) construction should not be regarded as a special renormalization scheme but as
a general framework in which the conditions posed by the fundamental axioms of QFT on any renormalization scheme are built in by construction. In the EG approach the $S$-matrix is directly constructed in the well-defined Fock space of free asymptotic fields in the form of a formal power series. Thus, one does not need the Haag-Ruelle (LSZ-) formalism. Interacting field operators can still be perturbatively constructed in an additional step as certain functional derivatives of the $S$-matrix ([4] section 8, [8], see also appendix B).

In classical physics, Noether’s theorem states that there is a conserved current for every invariance of the classical action under a continuous, internal or spacetime symmetry transformation of the fields. This theorem also allows for an iterative method to construct invariant actions, called the Noether method [3]. Noether’s method has been used in the construction of theories with local symmetries starting from ones with only rigid symmetries. For example, this method was extensively used in the construction of supergravity theories [10]. In a slightly different setting, starting from a free Lagrangian one can iteratively construct interactions by adding extra terms to the action and to transformation rules in such a way that the final action is invariant under the modified transformations. One may try to elevate these results to the quantum regime by quantizing the system. To this end, one should investigate the compatibility of the classical symmetry with the quantization. The latter is reflected in the absence or presence of anomalies.

In this paper we propose a general quantum method which, as we shall see, is a direct implementation of Noether’s method at the quantum level. For this reason we shall call it “Quantum Noether Method”. Starting from a free quantum field theory, well-defined in the Fock space of free asymptotic fields, the method allows for a construction of all perturbative quantum theories. Such a direct implementation of the Noether method in the quantum theory is established in the causal Epstein-Glaser approach to perturbative quantum field theory. In the Quantum Noether Method the conditions for constructing a classical action and the conditions for absence of anomalies are associated with obstructions in the construction of the $S$-matrix. The classical action emerges from the cancellation of tree-level obstructions, whereas anomalies are associated with loop obstructions. An algebraic consistency condition for possible obstructions can be derived without using the quantum action principle[11].

The strength of the EG construction lies in the operator formalism. The proof of general properties of a given quantum field theory can be simply and also rigorously reduced to the discussion of the local normalization ambiguity which is restricted by power counting. It is also the operator formalism which circumvents the classical problem of overlapping divergences - in the usual framework the latter problem is solved by the famous forest formula. Moreover, the EG formalism provides a natural framework to
discuss symmetries in perturbative quantum field theories in which the regularization and scheme independence of anomalies as well as the reduction of their discussion to local normalization ambiguities is manifest. From this it is clear that the Epstein-Glaser approach provides an ideal framework for the discussion of symmetries in quantum field theory.

In the original article [4] Epstein and Glaser applied their construction to scalar field theory. The extension of these results to abelian gauge theory in the four-dimensional Minkowski space has been worked out some years ago in [12, 13, 14, 15]. The causal Epstein-Glaser construction of (3+1)-dimensional non-abelian gauge theory in the Feynman gauge coupled to fermionic matter fields was performed in [16, 17]. There, a definition of non abelian gauge invariance was given as an operator condition in every order of perturbation theory separately. This condition involved only the linear (abelian) BRST-transformations of the free asymptotic field operators. It was claimed that this operator condition expresses the whole content of non-abelian gauge structure in perturbation theory [16, 17]. In fact, it was proven that the operator condition directly implies the unitarity of the $S$-matrix in the physical subspace, i.e. decoupling of the unphysical degrees of freedom. Furthermore, it was shown that from the operator condition one can derive the Slavnov-Taylor identities for the connected Green functions.

Up until now, however, only the Yang-Mills (YM) theory was fully constructed as a quantum theory in this framework. Moreover, a deep understanding of how the asymptotic operator condition develops the full BRST symmetry [18] was missing. With this ingredient missing, it was not a priori clear whether the quantum theory constructed using the EG procedure actually coincides with the usual YM theory or it is some kind of “Yang-Mills-like” theory which cannot be reached from the conventional Lagrangian approach. Since the condition of asymptotic BRST invariance seems to be a weaker condition than the full BRST symmetry it was not ruled out that there are new theories compatible with this symmetry - an interesting possibility because the asymptotic symmetry condition was shown to be sufficient for decoupling of the unphysical degrees of freedom. We shall argue, however, that the full BRST transformations were already present in the analysis of [16, 17], thereby establishing that the theory constructed by the EG procedure coincides with the usual YM theory.

Having understood fully the case of YM theories, we are in the position to generalize the construction to any theory with any local and global symmetry. Notice that local symmetries manifest themselves through the rigid BRST symmetry. Thus, one can treat both cases in parallel. Starting with the condition of asymptotic symmetry we show explicitly how the formalism automatically develops the anomaly-free full quantum symmetry of the interacting system, provided such an anomaly-free deformation of the given free theory exists. In this paper we exclude any possible obstructions of the symmetry at
tree and loop level. A systematic cohomological analysis of possible obstructions will be presented in a separate paper \cite{11}. The Epstein-Glaser framework allows for a simplified derivation of algebraic consistency conditions for obstructions. Moreover, the cohomological analysis of such obstructions is established before the adiabatic limit is taken. So the discussion is also applicable to massless theories such as Yang-Mills theories \cite{11}.

The idea behind the construction is very simple. Given a set of free fields and a symmetry (such as supersymmetry or BRST symmetry) generated by a Noether current (which at this point only captures the linear part of the transformation rules), one demands that the current is conserved at the quantum level, i.e. inside correlation functions (Quantum Noether Condition). We shall show that this condition at tree-level automatically produces the most general non-linear completion of the transformation rules and also the corresponding Lagrangian which is invariant under these transformation rules. We shall then examine the Quantum Noether Condition at loop level. We shall show that if the anomaly consistency condition has only trivial solutions then the theory is stable, i.e. all local terms that are produced by loops are already present at tree-level. This we shall call generalized renormalizability (this corresponds to the notion of “renormalizability in the modern sense” introduced for gauge theories in \cite{19}). If in addition the theory is also power counting renormalizable then generalized renormalizability coincides with the usual renormalizability. The only restriction needed for our construction to work is that the power counting index (singular order in EG, see section 3) is bounded in every order in perturbation theory. So, in particular, our consideration also apply to effective field theories that are not power counting renormalizable.

It is rather remarkable that the only information one needs in order to construct a perturbative quantum field theory with a given global symmetry is a set of free fields linearly realizing this symmetry (which is assumed to be generated by a Noether current, see footnote \footnote{4}). Even the first term in the $S$-matrix, which is usually regarded as an input in the EG formalism is now derived using the Quantum Noether Condition.

We have organized this paper as follows. In section 2 we shortly recall the Noether method. In section 3 we provide a self-contained summary of the basic ingredients of the EG construction. Section 4 is the main section where we establish the Quantum Noether Method. We illustrate the method in section 5 with two examples; the case of pure Yang-Mills theory and the $\mathcal{N} = 1$ supersymmetric Wess-Zumino model. Special care was taken in order to illustrate every step of the general construction explicitly. In Appendix A we explain our conventions in detail. In Appendix B we discuss several issues such as the infrared problem, the construction of interacting fields, the problem of overlapping divergences that further motivate the use of the EG formalism.
2 The Noether Method
in Classical Field Theory

In this section we shortly recall the Noether method. Let us start with a classical Lagrangian density $L(\phi^A, \partial_\mu \phi^A)$ that depends on a number of fields (both bosons and fermions) $\phi^A$ and their first derivative $\partial_\mu \phi^A$, where $A$ is an index that distinguishes different types of fields. Suppose now that the action $S = \int \mathcal{L}$ is invariant under the symmetry transformation $s\phi^A$. This means that the Lagrangian density transforms into a total derivative, $s\mathcal{L} = \partial_\mu k^\mu$. A standard way to derive Noether’s current is to let the parameter of the symmetry transformation $\epsilon$ become local. Then the Noether current is the expression multiplying the derivative of the local parameter.

$$\delta S = \int j^\mu (\partial_\mu \epsilon)$$ (2.1)

Taking the parameter $\epsilon$ rigid one sees that the variation is indeed a symmetry of the action. On the other hand, if the field equation are satisfied, $\delta S = 0$ for any $\epsilon$ and, therefore, $\partial_\mu j^\mu = 0$.

The Noether current is given by

$$j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} s\phi^A - k^\mu.$$ (2.2)

We shall always include the parameter of the transformation in the current. In this way the current is always bosonic. Direct calculation (using $\partial_\mu k^\mu = s\mathcal{L}$) yields,

$$\partial_\mu j^\mu = [\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} - \frac{\partial \mathcal{L}}{\partial \phi^A}] s\phi^A.$$ (2.3)

Clearly, $\partial_\mu j^\mu = 0$ when the field equations are satisfied. There is a natural arbitrariness in the definition of the current. One may always add terms of the form $\partial_\mu b^{\mu\nu}$, where $b^{\mu\nu}$ is antisymmetric in $\mu, \nu$.

The conserved charge $Q$ is equal to

$$Q = \int d^3x (p_A s\phi^A - k^0),$$ (2.4)

where $p_A = \partial \mathcal{L} / \partial_0 \phi^A$ is the conjugate momentum of $\phi^A$. One may check that $Q$ generates the corresponding variation when acting (by the Poisson bracket) to the fields,

$$\{Q, \phi^A\} = s\phi^A.$$ (2.5)

As mentioned previously, Noether’s theorem allows for an iterative method to construct invariant actions, called the Noether method [9]. Starting from a free Lagrangian one can iteratively construct interactions in classical field theory by adding extra terms
to the action and to the transformation rules such that the final action is invariant. The way the Noether method works is as follows. Start from an action \( S_0 = \int L_0 \) invariant under transformations \( s_0 \phi^A \). This set is assumed to be closed on-shell. The goal is then to find a new action,

\[
S = \int d^4x (L_0 + gL_1 + g^2L_2 + \cdots),
\]

(2.6)

and new transformation rules,

\[
s\phi^A = s_0 \phi^A + gs_1 \phi^A + g^2 s_2 \phi^A + \cdots,
\]

(2.7)

where \( g \) is a new coupling constant (or deformation parameter), such that the new action is invariant under the new transformation rules. To first order in \( g \) the relevant equation reads

\[
\frac{\delta L_0}{\delta \phi^A}(s_1 \phi^A) + \frac{\delta L_1}{\delta \phi^A}(s_0 \phi^A) - \partial_{\mu}k_1^\mu = 0
\]

(2.8)

Note that this is an equation for \( s_1, L_1 \) and \( k_1^\mu \) which may be not solvable (which means there is no possible deformation). Starting with an ansatz for \( s_1 \phi^A \) one tries to determine a \( L_1 \) such that the above equation holds or vice versa. If a solution \((s_1, L_1, k_1^\mu)\) of equation (2.8) is found, one tries to solve the equation that appears at order \( g^2 \), and so on. The corresponding current is given by

\[
j^\mu = [\left( \frac{\partial L_0}{\partial (\partial_\mu \phi^A)} \right) s_0 \phi^A - k_0^\mu] + (g \frac{\partial L_1}{\partial (\partial_\mu \phi^A)} + g^2 \frac{\partial L_2}{\partial (\partial_\mu \phi^A)} + \cdots) s_0 \phi^A]
\]

(2.9)

\[
+ g[\left( \frac{\partial L_0}{\partial (\partial_\mu \phi^A)} \right) s_1 \phi^A - k_1^\mu] + (g \frac{\partial L_1}{\partial (\partial_\mu \phi^A)} + \cdots) s_1 \phi^A] \cdots
\]

where we have organized the terms in a way that it will be useful in later sections.

A systematic way to organize this procedure is to use the Batalin-Vilkovisky (or anti-field) formalism. Although we will not use this formulation in the present article we provide a short description of the method as it provides a nice reformulation of the problem. We refer to the literature for a more detailed description.

In the anti-field formalism (for a detailed exposition see [22]) one first replaces the parameter of the symmetry variation by a ghost field (in the case of global symmetries the latter is a constant field [23]) and introduces a new field, the anti-field \( \phi^*_A \), for each field \( \phi^A \) (the fields \( \phi^A \) include the ghost field). The anti-fields act as sources for symmetry variation of the corresponding field, namely one adds in the Lagrangian a term \( \phi^*_A s_0 \phi^A \) (but the solution of (2.10) may contain higher powers of anti-fields). The defining equation of the theory is the master equation

\[
(S, S) = 0,
\]

(2.10)

\[3\]In this article we do not discuss the precise conditions under which the classical Noether method and the reformulation using anti-fields are equivalent.
where \((A, B)\) denotes the anti-bracket,

\[
(A, B) = \hat{\partial}_A \hat{\partial}_B - \hat{\partial}_B \hat{\partial}_A,
\]

(2.11)

where the arrow indicates from where the derivative acts. The action in this formalism generates the transformation of the fields,

\[
s\phi^A = (\phi^A, S)\).
\]

(2.12)

The problem is now formulated as follows. Starting from an \(S_0\) that solves the master equation \((S_0, S_0) = 0\) (for a theory with a closed gauge algebra \(S_0 = \int (L_0 + \phi^A s_0 \phi^A)\)) one seeks for a new action \(S = S_0 + gS_1 + g^2S_2 + \cdots\), where \(S_1, S_2, \ldots\) are local functionals, that solve the new master equation \((S, S) = 0\). This equation yields a tower of equations once it is expanded in \(g\). The first few equations are the following,

\[
(S_0, S_1) = 0,
\]

(2.13)

\[
2(S_0, S_2) + (S_1, S_1) = 0,
\]

(2.14)

and so on. Solving these equations one obtains both the new terms in the action and the new transformation rules. The latter are obtained by using (2.12). Let us define the derivations

\[
s_i = (\ , \ S_i)\).
\]

(2.15)

These derivations correspond to the ones in (2.7). Notice, however, that in this formulation there may be constant ghosts present, so one might have to first eliminate them in order to compare (2.15) and (2.7).

A nice feature of this approach is that it allows for a systematic cohomological approach to the problem. For instance, equation (2.13) tells us that \(S_1\) is an element of \(H^0(s_0, d)\), where \(H^k(s_0, d)\) denotes the cohomology group of the differential \(s_0\) relative to the differential \(d\) in ghost number \(k\) in the space of local functionals. In addition, the obstruction to the solvability of this equation lies in \(H^1(s_0, d)\). Similar remarks apply for the rest of the equations.

### 3 The Causal Method of Epstein-Glaser

We shall give a short but self-contained introduction to the causal Epstein-Glaser construction. This section may serve as a glossary for all quantities we are about to use in the next section. For some of the technical details we refer to the literature. We note that we follow the original Epstein-Glaser article in our presentation. So we differ slightly from reference regarding the causality condition, and the role of the Wick submonomials in the construction.
3.1 Inductive Construction

We recall the basic steps of the Epstein-Glaser construction in the case of a massless scalar field. For concreteness, we consider the case of four dimensional spacetime. The formalism, however, is valid in any dimension. The very starting point is the Fock space $\mathcal{F}$ of the massless scalar field (based on a representation space $H_s^{m=0}$ of the Poincaré group) with the defining equations

$$\Box \varphi = 0 \quad (a), \quad [\varphi(x), \varphi(y)] = i\hbar D_{m=0}(x-y) \quad (b),$$

where $D_{m=0}(x-y) = \frac{i}{(2\pi)^4} \int dk^4 \delta(k^2) \text{sgn}(k^0) \exp(-ikx)$ is the zero-mass Pauli-Jordan distribution (see appendix A). In contrast to the Lagrangian approach, the $S$-matrix is directly constructed in this Fock space in the form of a formal power series

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1^4 \cdots dx_n^4 \; T_n(x_1, \cdots, x_n; \hbar) \; g(x_1) \cdots g(x_n).$$

(We do not include explicit $i$ factors in (3.2) in order to reduce the number of $i$-factors in our equations.) In this approach the coupling constant $g$ is replaced by a tempered test function $g(x) \in \mathcal{S}$ (i.e. a smooth function rapidly decreasing at infinity) which switches on the interaction.

The central objects are the $n$-point operator-valued distributions $T_n$. They should be viewed as mathematically well-defined (renormalized) time-ordered products,

$$T_n(x_1, \cdots, x_n; \hbar) = T[ T_1(x_1) \cdots T_1(x_n) ],$$

of a given specific coupling, say $T_1 = \frac{i}{\hbar} : \Phi^4 :$ (c), which is the third defining equation in order to specify the theory in this formalism.

Notice that the expansion in (3.2) is not a loop expansion. Each $T_n$ in (3.2) can receive tree-graph and loop-contributions. One can distinguish the various contributions from the power of $\hbar$ that multiplies them.

Epstein and Glaser present an explicit inductive construction of the most general perturbation series in the sense of (3.2) which is compatible with the fundamental axioms of relativistic quantum field theory, causality and Poincaré invariance, which can be stated as follows:

- Let $g_1$ and $g_2$ be two tempered test functions. Then causal factorization means that

$$S(g_1 + g_2) = S(g_2)S(g_1) \quad \text{if} \quad \text{supp}g_1 \preceq \text{supp}g_2 \quad (3.4)$$

\footnote{\text{\(T_n \in \mathcal{S}',\) where \(\mathcal{S}'\) denotes the space of functionals on \(\mathcal{S}\).}}

\footnote{The fact that the distributions $T_n$ are formal Laurent series in $\hbar$ follows from the way $\hbar$ appears in defining equations (b) and (c) and the explicit construction that we describe below. Furthermore, one may deduce that $\hbar$ is a loop counting parameter (for connected graphs) by using similar arguments as in the Lagrangian formulation. In particular, connected tree-level graphs come with a factor of $1/\hbar$, 1-loop graphs with $\hbar^0$, etc.}
the latter notion means that the support of $g_1$ and the support of $g_2$, two closed subsets of $\mathbb{R}^4$, can be separated by a space like surface; more precisely $\text{supp}g_2$ does not intersect the past causal shadow of $\text{supp}g_1$:

$$\text{supp}g_2 \cap (\text{supp}g_1 + \vec{V}^-) = 0,$$

(3.5)

$$(\vec{V})^- = \{ x \in \mathbb{R}^4 | x^0 \leq |\vec{x}| \}$$

(3.6)

- Let $U(a, \Lambda)$ be the usual representation of the Poincaré group $P^+_4$ in the given Fock space $\mathcal{F}$. Then the condition of Poincaré invariance of the $S$-matrix says that

$$U(a, \Lambda) S(g) U(a, \Lambda)^{-1} = S(g^a_\Lambda) \quad \forall a \in \mathbb{R}^4, \forall \Lambda \in L^4_+, g^a_\Lambda(x) = g(\Lambda^{-1}(x - a))$$

(3.7)

Actually, in order to establish the general construction only translational invariance is needed. Lorentz invariance can be imposed in addition in a subsequent step.

It is well-known that the heuristic solution for (3.4), namely

$$T_n(x_1, \ldots, x_n; \hbar) = \sum_{\pi} T_1(x_{\pi(1)}) \ldots T_1(x_{\pi(n)}) \Theta(x^0_{\pi(1)} - x^0_{\pi(2)}) \ldots \Theta(x^0_{\pi(n-1)} - x^0_{\pi(n)}),$$

(3.8)

is, in general, affected by ultra-violet divergences ($\pi$ runs over all permutations of $1, \ldots, n$). The reason for this is that the product of the discontinuous $\Theta$-step function with Wick monomials like $T_1$ which are operator-valued distributions is ill-defined. One can handle this problem by using the usual regularization and renormalization procedures and finally end up with the renormalized time-ordered products of the couplings $T_1$.

Epstein and Glaser suggest another path which leads directly to well-defined $T$-products without any intermediate modification of the theory using the fundamental property of causality (3.4) as a guide. They translate the condition (3.4) into an induction hypothesis, $H_m, m < n$, for the $T_m$-distribution which reads

$$H_m : \begin{cases} T_m(X \cup Y) = T_{m_1}(X) T_{m-m_1}(Y) & \text{if } X \supseteq Y, \ X, Y \neq \emptyset, \ 0 < m_1 < m \\ \ [T_{m_1}(X), T_{m_2}(Y)] = 0 & \text{if } X \sim Y (\Leftrightarrow X \supseteq Y \land X \preceq Y) \ \forall m_1, m_2 \leq n \end{cases}$$

(3.9)

Here we use the short-hand notation $T_m(x_1, \ldots, x_m; \hbar) = T(X); |X| = m$.

Besides other properties they also include the Wick formula for the $T_m$ distributions into the induction hypothesis. This is most easily done by including the so-called Wick submonomials of the specific coupling $T_1 = (i/\hbar) : \Phi^4 :$ as additional couplings in the construction $T^{j}_i := (i/\hbar)(4!/(4-j)!) : \Phi^{4-j} :, 0 < j < 4$. Then the Wick formula for the $T_n$ products can be written as

$$T_m[T^{j_1}_i(x_1) \cdots T^{j_m}_i(x_m)] = \sum_{s_1, \ldots, s_m} \langle 0 | T[T^{j_1+s_1}_i(x_1) \cdots T^{j_m+s_m}_i(x_m)] | 0 \rangle : \prod_{i=1}^{m} \frac{\Phi^{s_i}(x_i)}{s_i!} :$$

(3.10)
In short-hand notation the formula reads

\[ T_m^\gamma(X) = \sum_{\vec{s}} \langle 0 \mid T_m^{\vec{s}+\vec{\gamma}}(X) \mid 0 \rangle \frac{\Phi_{\vec{s}}(X)}{\vec{s}} \]  

(3.11)

That such a quantity is a well-defined operator-valued distribution in Fock space is assured by distribution theory (see Theorem O in [4], p. 229). Note also that the coefficients in the Wick expansion are now represented as vacuum expectation values of operators.

Now let us assume that \( T_m \) distributions with all required properties are successfully constructed for all \( m < n \). Epstein and Glaser introduce then the retarded and the advanced \( n \)-point distributions (from now on, in this section, we suppress the \( \hbar \) factor in our notation):

\[ R_n(x_1, \ldots, x_n) = T_n(x_1, \ldots, x_n) + R'_n, \quad R'_n = \sum_{P_2} T_{n-1}(Y, x_n)\bar{T}_{n1}(X) \]  

(3.12)

\[ A_n(x_1, \ldots, x_n) = T_n(x_1, \ldots, x_n) + A'_n, \quad A'_n = \sum_{P_2} \bar{T}_{n1}(X)T_{n-1}(Y, x_n). \]  

(3.13)

The sum runs over all partitions \( P_2 : \{ x_1, \ldots, x_{n-1} \} = X \cup Y, \quad X \neq \emptyset \) into disjoint subsets with \( |X| = n_1 \geq 1, \quad |Y| \leq n - 2 \). The \( \bar{T} \) are the operator-valued distributions of the inverse S-matrix:

\[ S(g)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \tilde{T}_n(x_1, \ldots, x_n)g(x_1) \ldots g(x_n) \]  

(3.14)

The distributions \( \tilde{T} \) can be computed by formal inversion of \( S(g) \):

\[ S(g)^{-1} = (1 + T)^{-1} = 1 + \sum_{n=1}^{\infty} (-T)^r \]  

(3.15)

\[ \tilde{T}_n(X) = \sum_{r=1}^{n} (-)^r \sum_{P_r} T_{n1}(X_1) \ldots T_{n_r}(X_r), \]  

(3.16)

where the second sum runs over all partitions \( P_r \) of \( X \) into \( r \) disjoint subsets \( X = X_1 \cup \ldots \cup X_r, \quad X_j \neq \emptyset, \quad |X_j| = n_j \).

We stress the fact that all products of distributions are well-defined because the arguments of distributions are disjoint sets of points so that the products are tensor products of distributions. We also remark that both sums, \( R'_n \) and \( A'_n \), in contrast to \( T_n \), contain \( T_j \)'s with \( j \leq n - 1 \) only and are therefore known quantities in the inductive step from \( n - 1 \) to \( n \). Note that the last argument \( x_n \) is marked as the reference point for the support of \( R_n \) and \( A_n \). The following crucial support property is a consequence of the causality conditions (3.9):

\[ \text{supp} R_m(x_1, \ldots, x_m) \subseteq \Gamma^+_m(x_m), \quad m < n \]  

(3.17)

where \( \Gamma^+_m \) is the \( (m - 1) \)-dimensional closed forward cone,

\[ \Gamma^+_m(x_m) = \{(x_1, \ldots, x_{m-1}) \mid (x_j - x_m)^2 \geq 0, x_j^0 \geq x_m^0, \forall j\}. \]  

(3.18)
In the difference
\[
D_n(x_1, \ldots, x_n) \overset{\text{def}}{=} R'_n - A'_n
\] (3.19)
the unknown \(n\)-point distribution \(T_n\) cancels. Hence this quantity is also known in the inductive step. With the help of the causality conditions (3.9) again, one shows that \(D_n\) has causal support
\[
\text{supp}D_n \subseteq \Gamma_{n-1}^+(x_n) \cup \Gamma_{n-1}^-(x_n)
\] (3.20)
Thus, this crucial support property is preserved in the inductive step from \(n-1\) to \(n\).

Given this fact, the following inductive construction of the \(n\)-point distribution \(T_n\) becomes possible: Starting off with the known \(T_m(x_1, \ldots, x_n), m \leq n-1\), one computes \(A'_n, R'_n\) and \(D_n = R'_n - A'_n\). With regard to the supports, one can decompose \(D_n\) in the following way:
\[
D_n(x_1, \ldots, x_n) = R_n(x_1, \ldots, x_n) - A_n(x_1, \ldots, x_n)
\] (3.21)
\[
\text{supp}R_n \subseteq \Gamma_{n-1}^+(x_n), \quad \text{supp}A_n \subseteq \Gamma_{n-1}^-(x_n)
\] (3.22)
Having obtained these quantities we define \(T'_n\) as
\[
T'_n = R_n - R'_n = A_n - A'_n
\] (3.23)
Symmetrizing over the marked variable \(x_n\), we finally obtain the desired \(T_n\),
\[
T_n(x_1, \ldots x_n) = \sum_{\pi} \frac{1}{n!} T'_n(x_{\pi(1)}, \ldots x_{\pi(n)})
\] (3.24)
One can verify that the \(T_n\) satisfy the conditions (3.9) and all other further properties of the induction hypothesis [4].

3.2 Distribution Splitting

Let us now discuss the splitting the operator-valued distribution \(D_n\). As follows from our discussion this is the only nontrivial step in the construction.

Let there be an operator-valued tempered distribution \(D_n \in \mathcal{S}'(\mathbb{R}^{4n})\) with causal support,
\[
\text{supp}D_n \subseteq \Gamma_{n-1}^+(x_n) \cup \Gamma_{n-1}^-(x_n).
\] (3.25)
then the question is whether it is possible to find a pair \((R, A)\) of tempered distributions on \(\mathbb{R}^{4n}\) with the following characteristics:
\[
\begin{align*}
& \bullet \quad R, A \in \mathcal{S}'(\mathbb{R}^{4n}) \quad (A) \\
& \bullet \quad \text{supp}R \subset \Gamma^+(x_n), \quad \text{supp}A \subset \Gamma^-(x_n) \quad (B) \\
& \bullet \quad R - A = D \quad (C)
\end{align*}
\] (3.26) (3.27) (3.28)
The EG formalism reduces the usual renormalization program to this mathematically well-defined problem. Every renormalization scheme solves this problem implicitly. For example, the well-known BPHZ renormalization scheme which is often regarded as the most solidly founded explicit renormalization scheme also defines a splitting solution. As mentioned already in the introduction, there is a complication, namely the well-known problem of overlapping divergences which is solved by the famous forest formula in the BPHZ framework. The EG formalism provides a natural solution to this problem by implementing the causality condition directly on the operator level (see Appendix B).

The problem of distribution splitting has been solved in a general framework by the mathematician Malgrange in 1960 [24]. Epstein and Glaser used his general result for the special case of quantum field theory. A new solution of the splitting problem was given recently [15].

We mention that the Wick formula for $T$-products, (3.11) $\forall m < n$, directly implies the corresponding Wick formula for the causal operator-valued distributions at the level $n$. This is easily shown by the usual Wick theorem for ordinary products of Wick monomials,

$$D_n^j(X, x_n) = \sum_s \langle 0 \mid D_n^{j + \tilde{s}}(X, x_n) \mid 0 \rangle \frac{\Phi_s(X \cup \{x_n\})}{s}. \quad (3.29)$$

This formula reduces the splitting problem of operator-valued distributions to the splitting of the numerical $C$-number distributions

$$d_n^r(x_1 - x_n, \ldots, x_{n-1} - x_n) = \langle 0 \mid D_n^r(X, x_n) \mid 0 \rangle \quad (3.30)$$

The latter only depends on the relative coordinates because of translational invariance of $D_n$. Note that the causal support of all numerical distributions is assured by the fact that they are vacuum expectation values of operators with causal support. We can construct well-defined $T_n$ distributions as operators by first splitting the numerical distributions $d_n^r$ and then by defining the $T_n$’s as operators using the Wick formula (3.11).

The singular behaviour of the distribution $d_n^r$ for $x \to 0$ is crucial for the splitting problem because $\Gamma_{n-1}^+(0) \cap \Gamma_{n-1}^- (0) = \{0\}$. One therefore has to classify the singularities of distributions in this region. This can be characterized in terms of the singular order $\omega$ of the distribution under consideration which turns out to be identical with the usual power-counting degree [25, 7]. For further details of the theory of distribution splitting we refer to the literature [4, 15] and only make the following remarks:

- We exclude so-called ‘oversubtractions’, which correspond to an increase of the singular behaviour of the distribution in the splitting process. Thus, we further specify the splitting problem by requiring in addition

$$\omega(r) \leq \omega(d) \land \omega(a) \leq \omega(d). \quad (D) \quad (3.31)$$
Moreover, we have to ask whether the splitting solution of a given numerical distribution $d$ with singular order $\omega(d)$ is unique. Let $r_1 \in S'$ and $r_2 \in S'$ be two splitting solutions of the given distribution $d \in S'$. By construction $r_1$ and $r_2$ have their support in $\Gamma^+$ and agree with $d$ on $\Gamma^+ \setminus \{0\}$, from which follows that $(r_1 - r_2)$ is a tempered distribution with point support and with singular order $\omega \leq \omega(d)$:

$$\text{supp}(r_1 - r_2) \subset \{0\}, \quad \omega(r_1 - r_2) = \omega(d), \quad (r_1 - r_2) \in S'$$ (3.32)

According to a well-known theorem in the theory of distributions, we have

$$r_1 - r_2 = \sum_{|a|=0}^{\omega_0} C_a \partial^a \delta(x).$$

In the case $\omega(d) < 0$ which means that $d^n_\omega$ is regular at the zero point, the splitting solution is thus unique. In the case $\omega(d) \geq 0$ the splitting solution is only determined up to a local distribution with a fixed maximal singular degree $\omega_0 = \omega(d)$. The demands of causality (3.4) and translational invariance (3.7) leave the constants $C_a$ in (3.33) undetermined. They have to be fixed by additional normalization conditions.

We want to stress that a normalization ambiguity can already occur in tree graphs. For example the causal Pauli-Jordan distribution $d_1 := D(x - y)$ has singular order $\omega(d_1) = -2$, hence $d_2 := \partial^\mu \partial^\nu D(x - y)$ has $\omega(d_2) = 0$ (since each derivative increases the singular order by one). This implies that the splitting of $d_2$ is not unique according to (3.33). Because the normalization ambiguity in tree graphs will become important in our discussion in the next section let us discuss this point in more detail. Note that $[\phi(x), \phi(y)] = i\hbar D(x - y) = i\hbar(D^+ + D^-)$, where $D^+$ and $D^-$ are the positive, respectively negative frequency parts of the causal Pauli-Jordan distribution $D$. In $R'$ and $A'$ of equation (3.13) the $D^-$ and $-D^+$ occur in the case of $d_1$. The so-called natural splitting of the Pauli-Jordan distribution is given by (see also Appendix A)

$$D = D_{\text{ret}} - D_{\text{adv}}.$$ (3.34)

Here $D_{\text{ret}}$ has retarded support and $D_{\text{adv}}$ has advanced support. So $r = D_{\text{ret}}$ and $a = D_{\text{adv}}$ and then $t$ is defined as $t = r - r' = a - a'$ according to equation (3.23). This means in the case under consideration $t_1 = D_{\text{ret}} - D^- = D_{\text{adv}} + D^+ = D_F$, so finally we end up with the Feynman propagator. Analogously, the graph with the numerical distribution $d_2$ leads to a $t_2$-distribution $t_2 = \partial^\mu \partial^\nu D_F(x - y) + C g_{\mu\nu} \delta(x - y)$ with the Feynman propagator $D_F$ and a free normalization constant $C$ which has to be fixed by a further condition.

It is important to note that only in tree graphs the Feynman propagator $D_F$ occurs. In loop graphs one gets in $r'$ and $a'$ products of $D^+$ (or $D^-$) distributions which are well-defined as the direct product of distributions whose Fourier transform have retarded support ($r'$) and advanced support ($a'$). This does not lead to products of Feynman propagators in the $t$ distribution, as it would be the case if one would use the usual Feynman
rules that follow from the formal solution (3.8). For details of explicit splitting solutions in loop graphs we refer to [4, 15].

- We can now discuss the ambiguities on the operator level using the defining Wick formula (3.11). The field itself is included in the Wick submonomials one starts with, so we have \([T_1^j(x), \Phi(y)] = 0\) if \(x \sim y\) (i.e. if \(x\) and \(y\) are spacelike separated) according to condition (3.9). This implies that \(T_1^j\) must be in the Borchers class of the free field \(\Phi(x)\) ([28]). It is well-known that the set of Wick monomials exhausts the Borchers class of a free field. This leads to the most general solution of (3.2).

- The Wick formula for time-ordered products (3.11) was used to define the \(T_n^j\) distributions (including the Wick submonomials) as operator-valued distributions. This formula makes transparent that the normalization ambiguities in \(T_n^j\) for different \(j\) are not independent. Note that the normalization ambiguities of the \(T_n^j\)'s are introduced in this formula through the numerical distributions in (3.11) only. Thus, normalization conditions on different \(T_n^j\) might lead to a compatibility problem. We already mention here that the symmetry conditions we analyze in the following only include the physical \(T_n^{j=0}\) distributions, so, no such compatibility analysis has to be made.

- The question of renormalizability naturally arises. In the EG approach power-counting renormalizable quantum field theories are the ones where the number of the constants \(C_a\) to be fixed by physical conditions stays the same to all orders in perturbation theory. This means that finitely many normalization conditions are sufficient to determine the \(S\)-matrix completely. The latter property, however, does not necessarily mean that this is also possible when all the symmetry properties of the classical Lagrangian are maintained, a far more reaching quality generally referred to as renormalizability. As a consequence, power-counting renormalizability is a quality solely determined by the scaling properties of the theory. In a second step one tries to prove that there is also a symmetric normalization of the theory. In particular this holds if the loop normalization ambiguity can be fixed in the same way as the tree-level normalization ambiguity, i.e. if the theory is stable under quantum corrections.

If the number of the normalization constants increases with the order \(n\) of perturbation theory, then the theory is usually called non-renormalizable. However, if the singular order is bounded in every order in perturbation theory then, although the total number of physical conditions needed to fix the \(S\)-matrix completely is infinite, this number is finite at each order in perturbation theory and therefore the theory still has predictive power. Effective field theories belong to this class of theories. One may call these theories ‘generalized power-counting renormalizable’. If in addition the theory is stable under quantum correction then we are dealing with ‘generalized renormalizable’ theory.

Several other properties of the EG formalism are discussed in appendix B.
4 Basic Construction

4.1 The Quantum Noether Condition

We shall now present the basic construction of theories with global (= rigid) and/or local symmetries in the EG formalism.

As explained in detail in the last section, one starts with a set of free fields in the asymptotic Fock space. These fields satisfy their (free) field equations and certain commutation relations. To define the theory one still needs to specify $T_1$, the first term in the $S$-matrix. (Actually, as we shall see, even $T_1$ is not free in our construction method but is also constrained by the Quantum Noether Condition). Given $T_1$ one can, in a well defined manner, construct iteratively the perturbative $S$ matrix. In this construction, a finite number of constants (in the case of a power-counting renormalizable theory (see last section)) remains unspecified by the requirements of causality and Poincaré invariance.

We are interested in constructing theories where the $S$ matrix is invariant under a certain symmetry operation generated by a well-defined operator $Q$ in the asymptotic Fock space,

$$[Q, S] = 0. \quad (4.1)$$

The operator $Q$ acting on asymptotic fields generates their asymptotic transformation rules

$$[Q, \phi^A] = -i\hbar s_0 \phi^A, \quad (4.2)$$

where $[A, B]$ denotes a graded commutator. The latter are necessarily linear in the asymptotic fields. We want to carry out the construction before the adiabatic limit. Thus, instead of working with (4.1), we shall require

$$[Q, T_n(x_1, \ldots, x_n; \hbar)] = \sum_{l=1}^{n} \frac{\partial}{\partial x_{l}^\mu} T_{n/l}^\mu(x_1, \ldots, x_n; \hbar) \quad (4.3)$$

for $n \geq 1$ and for some $T_{n/l}^\mu$. We shall often suppress the spacetime arguments in the $n$-point functions. We shall also use the abbreviation $\partial/\partial x_i^\mu = \partial_{i}^\mu$. The meaning of the $T_{n/l}^\mu$ will be discussed in detail below. Equation (4.3) for $n=1$

$$[Q, T_1] = \partial_{1} T_{1/1}^\mu, \quad (4.4)$$

imposes restrictions on the starting point of the EG procedure, namely on the coupling $T_1$. Once the coupling $T_1$ has been determined the rest of the equations (4.3) impose relations among the constants left unspecified by the requirement of causality and Poincaré invariance. This is analogous to the situation in the conventional Lagrangian approach.

---

6 The $(-i\hbar)$ in the right hand side of (4.2) as compared to (2.5) is because in (2.7) we have Poisson brackets whereas in (4.2) quantum commutators.
where symmetry considerations restrict the possible terms in the Lagrangian and then
the corresponding symmetries at the quantum level impose certain relations among the
$Z$ factors.

Our considerations apply to the construction of theories with any global or local sym-
metry. In the case of linear symmetries, such as global internal symmetries or discrete $C,$
$P,$ $T$ symmetries, things are much simpler and one does not need the full machinery de-
veloped in this article. This is so because linear symmetries can be directly implemented
in the asymptotic Fock space by means of (anti-)unitary transformations. To achieve the
invariance of the $S$-matrix one only needs to start from a coupling $T_1$ invariant under the
corresponding linear symmetry. There is, of course, still the issue of compatibility of the
various symmetries imposed. This question will not be analyzed in this article.

The cases of interest here are non-linear symmetries. In this case, the asymptotic
transformations differ from those of the interacting fields. Such cases are, for example,
the BRST symmetry of gauge theories and rigid spacetime symmetries such as super-
symmetry. In the latter case, the transformation rules would be linear in the presence of
auxiliary fields. However, apart from the fact that supersymmetry auxiliary fields are not
always known, in the EG formalism the fields are on-shell and, therefore, these auxiliary
fields are necessarily absent.

Since different non-linear transformations may have the same linear limit it is not $a$
$priori$ obvious whether a theory constructed by EG satisfying (4.1) has any underlying
non-linear structure at all. To address this issue one can work out the precise conse-
quences of the operator equation (4.3) and try to reproduce the Ward identities derived
in the Lagrangian approach using the full non-linear transformation. This approach has
been followed in [16, 17] for the case of $SU(n)$ gauge theory in the Feynman gauge cou-
ped to fermions where it was shown that (4.3) implies the Slavnov-Taylor identities for
connected Green functions. An alternative and complementary approach is to try to find
a direct correspondence between the Lagrangian approach and the EG formalism.

In the conventional Lagrangian approach the theory is defined by giving the La-
grangian and specifying a meaningful way to compute (i.e. regularization/renormalization).
Our strategy is to identify the Lagrangian within the EG approach. If both approaches
describe the same theory, then the perturbative $S$ matrix should be identical in both. The
Lagrangian always appears in the $S$-matrix at the tree-level. We shall, therefore, iden-
tify the Lagrangian with the sum of $T_1$ and the local terms that arise through tree-level
normalization conditions (notice that in the EG approach one performs a perturbative
expansion around the free action and not around a classical solution of the full theory,
so one expects to recover the classical Lagrangian through tree-level graphs). If this
correspondence is correct then, for instance, one should be able to understand from the
EG point of view why adding a BRST exact term in the Lagrangian does not change
the physics of the theory. We will indeed see that this can be entirely understood using the EG formalism. If, in addition, one deals with a renormalizable theory then loops do not produce any further local terms besides the ones already present in the Lagrangian. Therefore, the question of renormalizability in the Lagrangian approach translates, in the EG formalism, to the question of whether the local normalization ambiguity to all orders reproduces the tree-graph normalizations. (The precise definition of these normalization terms well be given in the next section.)

Let us further remark that these considerations also explain why the Lagrangian is such a central object in quantum field theory: according to Epstein-Glaser the perturbative $S$-matrix is uniquely fixed once one fixes the local ambiguity. In a renormalizable theory, the Lagrangian precisely fixes this local ambiguity.

Our proposal for the construction of theories with global and/or local symmetries in the EG formalism is rather simple. One introduces in addition to $T_1$ the coupling $g_{\mu j_0}$ in the theory where $j_0$ is the Noether current that generates the asymptotic (linear) symmetry transformations. Actually, as we shall see, the coupling $T_1$ itself is determined by the construction. In addition, one imposes the condition that “the Noether current is conserved at the quantum level” (see also (4.8)),

$$\partial_{\mu} J_{\mu}^{n}(x_1, \ldots, x_n; \bar{h}) = 0$$

(4.5)

where we introduce the notation

$$\partial_{\mu} J_{\mu}^{n}(x_1, \ldots, x_n; \bar{h}) = \sum_{l=1}^{n} \partial_{\mu} J_{\mu/l}^{n/l},$$

(4.6)

and

$$J_{\mu/l}^{n/l} = T[T_1(x_1) \cdots j_{0}^{\mu}(x_l) \cdots T_1(x_n)].$$

(4.7)

(for $n = 1$, $J_{\mu}^{1}(x_1) = j_{0}^{\mu}(x_1)$). In other words we consider an $n$-point function with one insertion of the current $j_{0}^{\mu}$ at the point $x_l$. Notice that since the left hand side of (4.5) is a formal Laurent series in $\bar{h}$, this condition is actually a set of conditions. This construction is so natural that one hardly has to motivate it.

We shall show in the remaining of this section that one can construct using the symmetry condition (4.3) and the free Noether current $J_{\mu}^{1}(x_1) = j_{0}^{\mu}(x_1)$ as a starting point, any theory with global/local symmetry that can be viewed as deformation of a free theory (up to restrictions discussed in footnote [4]). This class includes all perturbative QFT’s. In addition, we shall establish the equivalence of any theory consistently constructed in the EG formalism with a Lagrangian theory (again up to restriction discussed in footnote [7]).

The present considerations do not immediately apply to theories that possess asymptotic symmetries but no associated asymptotic Noether currents. In these cases one still has a charge $Q$ that generates the symmetry, so one may still construct these theories using condition (4.3) (see subsection 4.5).
In the case of interest, namely theories with non-linear symmetry transformations, the corresponding Lagrangian can be assumed to be obtained by the classical Noether method. I.e. the Lagrangian $\mathcal{L}$ and the transformation rules $s\phi^A$ under which it is invariant, are both power series in the coupling constant, and are obtained by solving the equations arising in Noether’s method. We shall explicitly show that

1. the sum of $T_1$ and the tree-level normalizations (see next subsection) that arise from the requirement (4.5) coincides with the Lagrangian that is invariant under the non-linear transformations. This shows that the the full non-linear structure is present in the theory,

2. the free Noether’s current $j_0^\mu$ is renormalized by the condition (4.4) is such a way that it finally generates the full non-linear transformations,

3. the loop normalization ambiguity is fixed in the same way as at the tree-level one provided the anomaly consistency condition has only trivial solutions. This means that the theory is then stable under quantum correction,

4. condition (4.4) is equivalent to condition (4.3). The latter guarantees the invariance of the $S$-matrix under the corresponding asymptotic symmetry.

The way $T_1$ and $j_0^\mu$ get promoted to the full Lagrangian (point 1) and the full Noether current (point 2), respectively, completely parallels the classical Noether method. However, the EG methods generates the full quantum theory on the way (point 3), not just a classical Lagrangian invariant under certain classical symmetry. This motivates the title of this article. In particular, condition (1.3) (or the equivalent one (1.3)) also contains the symmetry constraints at the loop level. Points 1 and 2 deal with condition (1.3) at tree level and point 3 covers the loop analysis. Breaking of (1.3) by loop corrections corresponds to anomalies.

There is yet another condition equivalent to (1.3). Let $j_{0,\text{int}}^\mu$ and $\mathcal{L}^\mu_{1,\text{int}}$ be the interacting currents corresponding to $j_0$ and $\mathcal{L}_1^\mu$, respectively, constructed according to (B.3) ($\mathcal{L}_1^\mu$ is defined in (4.39)). As it shall be presented in detail [27], the Ward identity

$$\partial_\mu j_{0,\text{int}}^\mu + \partial_\mu g \mathcal{L}_{1,\text{int}}^\mu = 0$$

(4.8)

yields the same conditions as (1.5) on the normalization ambiguity of the physical correlation functions. In the adiabatic limit condition (4.8) becomes the conservation of the interacting current. This is the usual form of the Ward identity that follows from a symmetry.
4.2 Off-shell Formulation of the Inductive Hypothesis

Our goal is to find which are the restrictions on the $T$-products implied by equation (4.5). Since causality and Poincaré invariance uniquely fix the $T$-products up to local terms, as explained in section 3, we only need to discuss the conditions imposed on the local normalization ambiguity in (4.3). As already mentioned in the introduction, we assume in this paper that a consistent and anomaly-free deformation of the asymptotic symmetry exists, i.e. we assume that the Quantum Noether method works successfully in the cases we consider. This assumption excludes any true obstructions to the symmetry condition (4.5) at the tree and loop level. A cohomological analysis of possible true obstructions of condition (4.5) without using the quantum action principle will be presented in a separate paper [11].

We shall follow an iterative approach following the inductive EG construction. Namely, we shall assume that (4.5) is satisfied for all $m < n$, and then we shall examine the conditions implied by (4.5) at $n$th order. According to section 3 this involves three steps: We first construct the corresponding causal distribution $D_n[j_0 T_1 \cdots T_1]$, then we have to split $D_n$ to obtain $T_{c,n}[j_0 T_1 \cdots T_1]$, and finally we impose (4.5) that leads to conditions on the normalization ambiguity of $T_n$. The notation $T_c$ indicates that we use the natural splitting solution (i.e. the Feynman propagator is used in tree-graphs, see (3.34)) in tree graph contributions. The latter is our reference solution. When we refer to local normalization terms in tree graph contributions in EG they are always defined with respect to $T_c$. Points 1 and 2 crucially depend on this choice. There is a good reason, however, why this is what one should do: Only with natural splitting in tree-level graphs the contraction between two fields becomes equal to the Feynman propagator. As we already argued, we shall identify the Lagrangian with local terms in tree-level graphs in the $S$-matrix. In the Lagrangian approach these graphs have been constructed using Feynman propagators. So, in order to compare the two approaches one has to use the natural splitting solution. It is only for the sake of comparison that the natural splitting solution becomes distinguished. In the analysis of (4.5) at the loop level one may likewise choose a reference splitting solution. In this case, however, there is no ‘preferred’ reference solution, but also no need to explicitly specify one. In the following the subscript $c$ will denote natural splitting in tree-graphs and some fixed reference splitting in loops.

Let us start by noting that having satisfied our fundamental Quantum Noether condition (4.5) for all $m < n$, namely

$$\partial_\mu J_\mu^m(x_1, \cdots, x_m; h) = \sum_{l=1}^m \partial_\mu J_\mu^{m/l} = 0, \quad \forall m < n, \quad (4.9)$$

then equation (4.5) at the $n$th order can be violated by a local distribution $A_n(h)$ (which
we shall call anomaly term) only:

$$\partial_\mu J_\mu^n(x_1, \ldots, x_n; \bar{h}) = \sum_{l=1}^n \partial_\mu J_{n/l}^\mu = A_n(h), \quad (4.10)$$

Let us give a short proof of the latter statement: From (4.9) one can derive the analogous condition for the causal distribution $\mathcal{D}$ at the $n$th order:

$$\sum_{l=1}^n \partial_\mu J_{n/l}^\mu = 0, \quad (4.11)$$

where $\mathcal{D}_{n/l}$ denotes $J_{n/l}$ at the $D$-level. The latter step is somehow trivial using the basic formulae (3.12)-(3.13) which involve only tensor products of the known $T_m$ products with $m < n$ which fulfill (4.9). Knowing (4.11), we have to split the causal distributions $\mathcal{D}_{n/l}$:

Since the splitting solution $\mathcal{R}_{n/l}$ of $\mathcal{D}_{n/l}$ fulfills $\mathcal{R}_{n/l} = \mathcal{D}_{n/l}$ on $\Gamma^+ \setminus \{(x_n, \ldots, x_n)\}$ and $\mathcal{R}_{n/l} = 0$ on $(\Gamma^+)^c$, the symmetry condition can be violated in this process only in the single point $(x_n, \ldots, x_n)$, i.e. by local terms. Therefore, condition (4.5) at $n$th order can only be violated by local terms, denoted by $A_n(h)$ in (4.10). The anomaly terms $A(h)$ are a formal Laurent series in $\bar{h}$ since the left hand side in (4.10) is. In addition, they are restricted by the power counting condition (3.31). Notice that we allow for theories with different (but finite) maximal singular order $\omega$ at every order in perturbation theory (i.e. we consider the class of ‘generalized renormalizable’ theories, see section 3).

We shall now present an off-shell version\(^8\) of the inductive hypothesis. The assumption that the Quantum Noether method works successfully means that there exist local normalizations such that (4.9) is satisfied when the field equations are satisfied. This does not mean, however, that (4.9) is satisfied when any splitting is used. Actually, generically after natural splitting (this refers to tree-level graphs, for loop graphs one uses some reference splitting solution) one ends up with

$$\partial_\mu J_{c,m}^\mu = A_{c,m}, \quad (4.12)$$

where the subscript $c$ indicates that the natural splitting has been used at tree-graphs.

Our assumption only means that the anomaly $A_{c,m}$ is a divergence up to terms $B_m$ that vanish when the free field equations are used, i.e.

$$A_{c,m} = \partial_\mu A_{c,m}^\mu + B_m \quad (4.13)$$

where $A_{c,m}^\mu$ and $B_m$ are some local distributions (since $A_{c,m}$ is local). This decomposition is not unique since one can move derivatives of field equation terms from $B_m$ to $A_{c,m}^\mu$.

We fix this freedom by demanding that $B_m$ does not contain any derivatives of field equations. Let us show this explicitly. To derive the general form of $B_m$ we first note

\[^8\] By off-shell we mean that we relax the field equations of the fields $\phi^A$.\]
that it should have the general form of a local distribution:

\[ B_m = O_{1,m}^{(m)} + \partial_{\mu}O_{2,m}^{\mu}, \quad O_{2,m}^{\mu} = \sum O_{2,m}^{\mu_{\alpha_1...\alpha_n}} \partial_{\alpha_1} \delta(x_1 - x_n) \ldots \partial_{\alpha_n} \delta(x_{n-1} - x_n), \]

which is easy to prove by recursion on the number of derivatives and integration by parts. Notice that because the power counting degree is bounded at each order in perturbation theory the series terminates after a finite number of terms. The operators \( O_{i,m}, i = 1, 2, \ldots \), are in general unrestricted, but in our case they should be such that \( B_m \) vanishes when the free field equations are satisfied. This means in particular that \( O_{1,m} \) has the form

\[ O_{1,m} = S_{\mu_1...\mu_p}^A \partial_{\mu_1} \ldots \partial_{\mu_p} K_{AB} \phi^B \]

where

\[ K_{AB} \phi^B = \frac{\partial L_0}{\partial (\partial \phi^A)} - \frac{\partial L_0}{\partial \phi^A} \]

are the free field equations. The fact that \( O_{1,m} \) is linear in the field equations follows from the fact that the field equations are created because we act with a derivative on a T-product with one current insertion. One derivative can only create one field equation.

Remembering that (4.14) is a distributional relation, one may integrate by parts the derivatives from the field equations to obtain

\[ O'_{1,m} = (S_{\mu_1...\mu_p}^A + \sum (-1)^{p+1} \partial_{\mu_1} \ldots \partial_{\mu_p} S_{\mu_1...\mu_p}^A) K_{AB} \phi^B, \]

and appropriate modifications of the \( O_{i,m}, i > 0 \). Notice now that one may always factor out a derivative from the terms involving the \( O_{i,m}, i > 0 \). This means that these terms can be moved into \( A^\mu_{c,n} \). The latter is finally removed by appropriately fixing the local normalization freedom of the left hand side. Let us also define

\[ R^A_{m} = S_{\mu_1...\mu_p}^A + \sum (-1)^{p+1} \partial_{\mu_1} \ldots \partial_{\mu_p} S_{\mu_1...\mu_p}^A \]

An additional ambiguity is related to the global symmetries of the free action. If one makes the transformation

\[ A^\mu_{c,m} \rightarrow A^\mu_{c,m} + \tilde{j}_{\mu}; \quad R^A_{m} \rightarrow R^A_{m} - \tilde{s} \phi^A \]

where \( \tilde{j}_{\mu} \) is a Noether current that generates the symmetry transformations \( \tilde{s} \phi^A \), then the right hand side of (4.13) remains unchanged. To fix this ambiguity we demand that \( R^A_{m} \) do not contain any summand which is itself a symmetry transformation of the free action. (In practice, one would never have to deal with this problem unless one does by hand the substitutions (4.19)).

---

9 We use the following abbreviations for the delta function distributions \( \delta^{(m)} = \delta(x_1, \ldots, x_m) = \delta(x_1 - x_2) \ldots \delta(x_{m-1} - x_m) \).
In this manner we are lead to the following off-shell representation of the inductive hypothesis: for $m < n$,

$$\sum_{l=1}^{m} \partial_{\mu}^{l} J_{m/l}^{\mu} = \sum_{A} R^{A;m}(h) K_{AB} \phi_{B}(x_{1}, \ldots, x_{m}).$$  \hfill (4.20)

The coefficients $R^{A;m}(h)$ may, in general, receive tree and loop contributions. We shall show below that this off-shell representation provides an alternative and simplified way of obtaining local terms arising from tree-level graphs.

We first concentrate on analyzing the condition (4.15) at tree-level. We shall consider the loop case afterwards. We therefore only need the $\bar{h}^{0}$ part of (4.20). Let us define

$$s_{(m-1)}^{A} \phi^{A} = \frac{1}{m!} R^{A;m}(\bar{h}^{0}); \quad m > 1,$$  \hfill (4.21)

(we shall see below that this formula also holds for $m = 1$). Depending on the theory under consideration the quantities $R^{A;m}(\bar{h}^{0})$ may be zero after some value of $m$. Without loss of generality we assume that they are zero for $m > k + 1$, for some integer $k$ (which may be infinity; the same applies for $k'$ below.).

We shall show below that

$$s \phi^{A} = \sum_{m=0}^{k} g^{m} s_{m} \phi^{A}$$  \hfill (4.22)

are symmetry transformation rules that leave the Lagrangian invariant (up to total derivatives)

$$\mathcal{L} = \sum_{m=0}^{k'} g^{m} \mathcal{L}_{m},$$  \hfill (4.23)

where $k'$ is also an integer (generically not equal to $k$). The Lagrangian $\mathcal{L}$ will be determined from the tree-level normalization conditions as follows,

$$\mathcal{L}_{m} = \frac{\hbar N_{m}}{i \cdot m!}, \quad \text{for} \quad m > 1,$$  \hfill (4.24)

where $N_{m}$ denotes the local normalization ambiguity of $T_{m}[T_{1}(x_{1}) \ldots T_{1}(x_{m})]$ in tree graphs defined with respect to the naturally split solution. For $m = 1$, $\mathcal{L}_{1} = (\hbar/i) T_{1}$. The factor $m!$ reflects the fact that $T_{m} [...]$ appears in (3.2) with a combinatorial factors $m!$ while the factor $\hbar/i$ is there to cancel the overall factor $i/\hbar$ that multiplies the action in the tree-level $S$-matrix. Notice that we regard (4.24) as definition of $\mathcal{L}_{m}$.

To understand how the off-shell formulation simplifies the calculation of local terms arising from tree-level graphs we start by first describing the traditional way to do such a calculation. In order to obtain the local terms, one first constructs $T_{c,n}[j_{0}^{\mu}(x_{1})T_{1}(x_{2}) \ldots T_{1}(x_{n})]$, differentiates with respect to the variable of the current and symmetrizes in all variables. $T_{c,n}[j_{0}^{\mu}(x_{1})T_{1}(x_{2}) \ldots T_{1}(x_{n})]$ involves many terms and there will be a large number of cancellations after differentiating and symmetrizing. In particular, we already know from equation (4.10) that all non-local terms will cancel among
themselves. So the idea (which gets implemented with the help of the off-shell formulation) is to only concentrate on possible local terms anticipating the cancellation of all non-local terms.

From (3.12), (3.13) we know that the causal distribution \( D_n[j_0^{\mu}(x_1)T_1(x_2)\ldots T_1(x_n)] \) is equal to the sum of terms which are products of \( T \)-products constructed at lower orders one of which contains \( j_0^{\mu} \) as a vertex. To calculate \( T_{c,n}[j_0^{\mu}(x_1)T_1(x_2)\ldots T_1(x_n)] \) one first does all contractions in \( D_n \) and then splits the solution. Each tree-level contraction between two fields \( \phi^A \) and \( \phi^B \) yields after natural splitting a factor \( D^{AB} \), where \( D^{AB} \) is the inverse of the corresponding kinetic operator \( K_{AB} \),

\[
\frac{i}{\hbar} K_{AB} D^{BC} = \delta^C_A
\]

where \( \delta^C_A \) contains a delta function, and we have included also the \( i \) and \( \hbar \) factors (see appendix A). Let us also denote by \( O_{AB}^\mu \) the operator associated with the fields \( \phi^A, \phi^B \) that satisfies \( \partial_{\mu} O_{AB}^\mu = K_{AB} \). The expression for \( T_{c,n}[j_0^{\mu}(x_1)T_1(x_2)\ldots T_1(x_n)] \) will contain, among other terms, terms of the form \(^{10}\)[[4.25]]

\[
S^1(x)S^2(y)O_{AB}^\mu(x)D^{BC}(x-y),
\]

where \( x \) is the variable of the current and \( S^1, S^2 \) are local terms at order \( m \) (\( m < n \)), respectively, \( n-m \). Upon differentiating with \( \partial_{\mu} \) this term will bring among other terms the local term

\[
S^1 S^2 \delta^C_A \delta(x-y).
\]

(we have now explicitly written the delta function to emphasize that this is a local term). The mechanism we just described is the only one that creates local terms out of tree-level graphs. Diagrammatically, after we push the derivative in we get an inverse propagator between graphs. Diagrammatically, after we push the derivative in we get an inverse propagator between graphs. The fact that \( S^1 \) and \( S^2 \) are local is, of course, essential. Local terms proportional to derivatives of the \( \delta \) distribution are constructed in an analogous way.

The term [4.26] originated from the following \( T \)-product

\[
T_{c,n}[(S^1 O_{AB}^\mu \phi^B \delta^{(m)})(x)(S^2 \phi^C)\delta^{(n-m)}(y)],
\]

upon contraction between \( \phi^B \) and \( \phi^C \). We are ultimately interested in computing \( \partial_{\mu} T_{c,n}[j_0^{\mu}(x_1)T_1(x_2)\ldots T_1(x_n)] \), so we need to compute \( \partial_{\mu} T_{c,n}[(S^1 O_{AB}^\mu \phi^B \delta^{(m)})(x)(S^2 \phi^C)\delta^{(n-m)}(y)] \). Moving the derivative inside the correlation function we get (among other terms)

\[
T_{c,n}[(S^1 K_{AB} \phi^B)(x)(S^2 \phi^C)(y)]
\]

\(^{10}\)In [4.26], as well as in later formulae, Wick-ordering is always understood. We also suppress two delta distributions that set \( n \) of the variables equal to \( x \) and the remaining \( n - m \) equal to \( y \). In order to keep the notation as simple as possible we shall often suppress such delta distributions. In all cases one may insert these delta distribution by simple inspection of the formulae.
Upon contracting the $\phi^B$ with a $\phi$ in $(S^2\phi^C)$ one obtains a local term. In particular, the local term in (4.27) is simply obtained from the contraction between the explicit $\phi$’s (i.e. the $\phi^B$ and $\phi^C$) in (4.29). We shall call this kind of contractions, namely the ones that involve the field $\phi^B$ in the field equation $K_{AB}\phi^B$, “relevant contractions”. These yield the local terms that one obtains by doing the calculation in the “traditional way”. All other contractions (“irrelevant contractions”) generically yield non-local terms proportional to field equations. These are not relevant in our case and they will be discarded.

In this way we are led to an alternative and more systematic way of obtaining all tree-level local terms. One first differentiates and then does the “relevant contractions”. However, in doing the calculation in this way one should not use the field equations before the end of the calculation. The terms that are proportional to the field equations are the source of the local terms.

Let us now make contact with the off-shell formulation of the induction hypothesis. After differentiation, the causal distribution $\sum_{l=1}^{n} \partial^\mu_{l}D_{n/l}^\mu = 0$ at the $n$’th order consists of a sum of terms each of these being a tensor product of $T_m[T_1...T_lj_0T_1...T_1]$ ($m < n$) with $T$-products that involve only $T_1$ vertices according to the general formulae (3.12,3.13,3.19). By the off-shell induction hypothesis, we have for all $m < n$

$\sum_{l=1}^{n} \partial^\mu_{l}J_{m/l}^\mu = \sum_{A}(m!s_{m-1}\phi^A)K_{AB}\phi^B \delta^{(m)}$. (4.30)

At order $n$ the “relevant contractions”, namely the contractions between the $\phi^B$ in the right hand side of (4.30) and $\phi$ in local terms, yield the sought-for local terms. This implies, in particular, that no local term arises from terms in $D_n$ that are products of more than two $T$ products. This is in accordance with the diagrammatic picture of creation of local terms that we mentioned above. In this manner we get the following general formula for the local term $A_{c,n}$ arising through tree-level contractions at level $n$,

$A_{c,n}(\text{tree}) = \sum_{\pi \in \Pi^n} \sum_{m=1}^{n-1} \partial^\mu_{\mu}J_{m}^\mu(x_{\pi(1)}, \ldots, x_{\pi(m)})N_{n-m}\delta(x_{\pi(k+1)}, \ldots, x_{\pi(n)})$ (4.31)

where it is understood that in the right hand side only “relevant contractions” are made. The factors $N_{n-m}$ are tree-level normalization terms of the $T$-products that contain $n-m$ $T_1$ vertices.

4.3 Analysis of the Quantum Noether Condition at Tree-level

In this section we analyze the formula (4.31) for all $n$.

For $n = 1$, we have $J_1^\mu = j_0^\mu$. Then from (2.3) it follows that

$\partial^\mu_{\mu}J_1^\mu = s_0\phi^A K_{AB}\phi^B$. (4.32)
Therefore, $R^{4,1} = s_0\phi^A$ as we promised.

Let us now move to the $n = 2$ case. From our previous discussion follows immediately that

$$\partial_\mu^{\nu} T_{c,2}^{(j_0^{(\nu)}(x_1)\phi^A(x_2))} = \frac{\hbar}{i} s_0\phi^A \delta(x_1 - x_2)$$

(4.33)

where the factor $(\hbar/i)$ originate from the contraction (see (4.25)). Using the derivation property of single Wick contractions we immediately get that for any local function $f(\phi^A)$ of the fields that do not contain derivatives of the fields a similar relation holds,

$$\partial_\mu^{\nu} T_{c,2}^{(j_0^{(\nu)}(x_1) f(\phi^A)(x_2))} = \frac{\hbar}{i} s_0 f(\phi^A) \delta(x_1 - x_2).$$

(4.34)

Let us know consider derivative terms. In this case,

$$\partial_\mu^{\nu} T_{c,2}^{(j_0^{(\nu)}(\partial_k \phi^A)(x_2))} = \frac{\hbar}{i} (s_0\phi^A)(x_1) \partial_\mu^{\nu} \delta(x_1 - x_2)$$

(4.35)

Symmetrizing this expression with respect to $x_1$ and $x_2$ and using the distributional identity

$$a(x_1)\partial_\mu \delta(x_1 - x_2) + a(x_2)\partial_\nu \delta(x_1 - x_2) = (\partial a) \delta(x_1 - x_2)$$

we get

$$\partial_\mu^{\nu} T_{c}^{(j_0^{(\nu)}(\partial_k \phi^A)(x_2))} + \partial_\mu^{\nu} T_{c}^{[(\partial_k \phi^A)(x_1) j_0^{(\nu)}(x_2)]} = \frac{\hbar}{i} \partial_\mu \delta(x_1 - x_2).$$

(4.37)

Notice that the corresponding relation with $(\partial_k \phi^A)$ replaced by $\phi^A$ has an extra factor of 2 in the right hand side. Combining these results we obtain

$$\partial_\mu^{\nu} J_{c,2}^{(\nu)}(x_1, x_2) = \frac{\hbar}{i} (2 s_0 T_1 - \partial_\mu^{(\nu)} \partial_\nu^{(\nu)} s_0\phi^A) \delta(x_1 - x_2) = A_{c,2}$$

(4.38)

Inserting the definition of $L_1 = (\hbar/i) T_1$ the right hand side of (4.38) becomes real and independent of $\hbar$.

Our objective is to fix the tree-level normalization freedom with respect to $J_{c,2}^{(\nu)}$ such that (4.15) holds (when the free field equations are satisfied). This is possible if and only if

$$s_0 L_1 = \partial_\mu L_{1}^{\mu} + \frac{1}{2} B_2$$

(4.39)

---

11 The current $j_0^{(\nu)}$ contains the parameter of the transformation rule and is therefore bosonic. If one were to use the current without the parameter then one would need to graded-symmetrize in $x_1$ and $x_2$.

12 In this article we consider only theories that depend arbitrarily on $\phi^A$ and its first derivative $\partial_\mu \phi^A$. One could also study theories that depend on higher derivatives of $\phi^A$, $\partial_{m_1} \ldots \partial_{m_p} \phi^A$, for some integer $p$. However, it is not clear whether such theories have any relevance to the physical world as classically they have no phase space and therefore it is unclear how to canonically quantize them. Notice however that one may construct them in the EG approach. We will not consider these theories in this article.
for some $L_1^\mu$ and $B_2$, where $B_2$ vanishes when the free field equations are satisfied (the factor $1/2$ has been inserted such that we agree with (4.13)). By our assumption—the Quantum Noether Method works successfully—there is a pair $(L_1, L_1^\mu)$ which solves (4.39). We emphasize that also $L_1^\mu = (\bar{h}/i)T_1$ is not an input in our construction method but also is determined by the Quantum Noether condition (4.5). So only the free Noether current $j_0^\mu$ is used in the defining equation of the Quantum Noether method. Going back to (4.38) we obtain

$$
\partial_\mu J_{c,2}^\mu(x_1, x_2) = 2!(s_1\phi^A)K_{AB}\phi^B\delta(x_1 - x_2) - \partial_\mu(-2L_1^\mu + \frac{\partial L_1}{\partial(\partial_\mu \phi^A)}s_0\phi^A)\delta(x_1 - x_2) \tag{4.40}
$$

According to (4.21) the latter equation defines $s_1$.

Now we consider the normalization ambiguity of $T_2[j_0^\mu T_1]$,

$$
T_2[j_0^\mu(x_1)T_1(x_2)] = T_{c,2}[j_0^\mu(x_1)T_1(x_2)] + j_1^\mu \delta(x_1 - x_2) \tag{4.41}
$$

Demanding that

$$
\partial_\mu J_2^\mu(x_1, x_2) = 2!(s_1\phi^A)K_{AB}\phi^B\delta(x_1 - x_2) \tag{4.42}
$$

we obtain

$$
j_1^\mu = \frac{\partial L_1}{\partial(\partial_\mu \phi^A)}s_0\phi^A - 2L_1^\mu. \tag{4.43}
$$

The discussion for $3 \leq n \leq k + 1$ goes the same way as the $n = 2$ case. However, it is instructive to also directly work out the $n = 3$ case as it is still relatively easier than the general case but sufficiently more complicated than the $n = 2$ case. Using (3.12) and (3.13) for $n = 3$ and afterwards naturally splitting one gets

$$
\partial_\mu J_{c,3}^\mu(x_1, x_2, x_3) = \partial_\mu J_2^\mu(x_1, x_2)T_1(x_3) + \partial_\mu J_2^\mu(x_1)N_2(x_2, x_3) + \text{cyclic in } x_1, x_2, x_3, \tag{4.44}
$$

(this is equation (4.31) for $n = 3$). In writing this expression we have discarded all terms on the right hand side that do not contribute any local terms. In particular, it is understood that only “relevant contractions” are made. $N_2$ denotes the tree-normalization term of $T_2$ which is uniquely defined with respect to $T_{c,2}$.

Using our previous results (4.32),(4.42) and remembering that the derivative terms should be treated with care we obtain

$$
\partial_\mu J_{c,3}^\mu(x_1, x_2, x_3) = [3!(s_1L_1 + s_0L_2) - 2!\partial_\mu(2\frac{\partial L_1}{\partial(\partial_\mu \phi^A)}s_1\phi^A + \frac{\partial L_2}{\partial(\partial_\mu \phi^A)}s_0\phi^A)]\delta(x_1, x_2, x_3) \tag{4.45}
$$

where we have used the definition in (4.24). The Quantum Noether condition (4.3) is satisfied if and only if

$$
s_1L_1 + s_0L_2 = \partial_\mu L_2^\mu + s_2\phi^A K_{AB}\phi^B \tag{4.46}
$$

27
for some $\mathcal{L}_2^n$ and $s_2\phi^A$. Substituting back in \((4.35)\) we obtain
\[
\partial_\mu J^\mu_{\text{c},3}(x_1, x_2, x_3) = 3!(s_2\phi^A)\mathcal{K}_{AB}\phi^B\delta(x_1, x_2, x_3)
- \partial_\mu[-3!L_2^\mu + 2(\frac{\partial L_2}{\partial(\partial_\mu\phi^A)}s_0\phi^A + 2\frac{\partial L_1}{\partial(\partial_\mu\phi^A)}s_1\phi^A)]\delta(x_1, x_2, x_3)
\]
\[(4.47)\]

In a similar way as before we define
\[
T_3[j_0^\nu(x_1)T_1(x_2)T_1(x_3)] = T_{\text{c},3}[j_0^\nu(x_1)T_1(x_2)T_1(x_3)] + j_2^\nu\delta(x_1, x_2, x_3)
\]
\[(4.48)\]
Then
\[
\partial_\mu J^\mu_{\text{c},n}(x_1, x_2, x_3) = 3!s_2\phi^A\mathcal{K}_{AB}\phi^B\delta(x_1, x_2, x_3)
\]
\[(4.49)\]
provided
\[
j_2^\mu = -3!L_2^\mu + 2(\frac{\partial L_2}{\partial(\partial_\mu\phi^A)}s_0\phi^A + 2\frac{\partial L_1}{\partial(\partial_\mu\phi^A)}s_1\phi^A)
\]
\[(4.50)\]
The general case for $2 \leq n \leq k + 1$ can be worked out in a completely analogous way. The result for the current is
\[
j^\mu_{n-1} = -n!L_{n-1}^\mu + (n - 1)! \sum_{l=0}^{n-2} (l + 1) \frac{\partial L_{n-l-1}}{\partial(\partial_\mu\phi^A)}s_l\phi^A
\]
\[(4.51)\]
where we have used the definition in \((1.24)\). To derive this result one may use following distributional identity
\[
\sum_{\pi\in\Pi^n} \delta(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)})A\partial^\mu_{x_{\pi(k)}}(B\delta(x_{\pi(k)}, x_{\pi(k+1)}, \ldots, x_{\pi(n)})) =
\]
\[
\left[\begin{array}{c} n \\ k \end{array}\right] A\partial^\mu B - \left(\begin{array}{c} n - 1 \\ n - k - 1 \end{array}\right) \partial^\mu(AB)]\delta(x_1, \ldots, x_n)
\]
\[(4.52)\]
where \(\left(\begin{array}{c} n \\ k \end{array}\right)\) is the binomial coefficient. In addition,
\[
\partial_\mu J^\mu_{n} = n!s_{n-1}\phi^A\mathcal{K}_{AB}\phi^B\delta(x_1, \ldots, x_n)
\]
\[(4.53)\]

Let us now move to the $n > k + 1$ case. (In the case of “generalized renormalizable” theories only the analysis of the case $n \leq k + 1$ is present since $k$ tends to infinity). One gets
\[
\partial_\mu J^\mu_{\text{c},n} = n!\left[s_0\mathcal{L}_{n-1} + s_1\mathcal{L}_{n-2} + \cdots + s_k\mathcal{L}_{n-1-k}\right] - (n - 1)!\partial^\mu\sum_{l=1}^{k} l \frac{\partial \mathcal{L}_{n-l}}{\partial(\partial_\mu\phi^A)}s_{l-1}\phi^A
\]
\[(4.54)\]
where again use of \((4.52)\) has been made. This equation now implies that
\[
s_0\mathcal{L}_{n-1} + s_1\mathcal{L}_{n-2} + \cdots + s_k\mathcal{L}_{n-1-k} = \partial_\mu J^\mu_{n-1}.
\]
\[(4.55)\]
One achieves
\[ \partial_\mu J_\mu^n = 0 \] (4.56)
by renormalizing the current
\[ j_{n-1}^\mu = -n!L_{n-1}^\mu + (n - 1)! \sum_{l=1}^k l \frac{\partial L_{n-l}}{\partial (\partial_\mu \phi^A)} s_{l-1} \phi^A \] (4.57)
and without the need to use the free field equations. Depending on the theory under consideration the \( L_n \)'s will be zero for \( n > k' \), for some integer \( k' \). Given the integers \( k \) and \( k' \), there is also an integer \( k'' \) (determined from the other two) such that \( L_n^\mu = 0 \), for \( n > k'' \).

Let us recapitulate. We have calculated all local terms that arise from the tree level diagrams. Summing up the necessary and sufficient conditions (4.39), (4.46), (4.55) for the Quantum Noether method to hold at tree level we obtain,
\[ s \sum_{l=1}^{k'} g^l L_l = \sum_{l=1}^{k''} \partial_\mu L_l^\mu + (\sum_{l=1}^k g^l s_l \phi^A) K_{AB} \phi^B \] (4.58)
Using \( s_0 L_0 = \partial_\mu k_0^\mu \) and for \( l \leq k \)
\[ s_l \phi^A K_{AB} \phi^B = \partial_\mu \left( \frac{\partial L_0}{\partial (\partial_\mu \phi^A)} s_l \phi^A \right) - s_l L_0 \] (4.59)
we obtain,
\[ s L = \partial_\mu \left( \sum_{l=0}^{k'} g^l k_l^\mu \right) \] (4.60)
where, for \( 1 < l \leq k \),
\[ k_l^\mu = L_l^\mu + \frac{\partial L_0}{\partial (\partial_\mu \phi^A)} s_l \phi^A \] (4.61)
and for \( l > k \), \( k_l^\mu = L_l^\mu \). We therefore find that \( L \) is invariant under the symmetry transformation,
\[ s \phi^A = \sum_{l=0}^k g^l s_l \phi^A. \] (4.62)
According to Noether’s theorem there is an associated Noether current given by (2.9). Using (4.61) one may check that the current normalization terms \( j_m^\mu \) ([4.43]), (4.50), (4.51), (4.57)) are in one-to-one correspondence with the terms in the Noether current (compare with (2.9)); they only differ by combinatorial factors related to the perturbative expansion. Therefore the current \( j_0 \) indeed renormalizes to the full non-linear current. This finishes the proof of points 1 and 2.
4.4 Analysis of the Quantum Noether Condition at Loop-level

We now move to point 3 and consider what happens in loops. If the local normalization ambiguity to all orders in $\hbar$ reproduces the tree-graph normalizations then we are dealing with a (generalized) renormalizable theory. We have seen that the tree level analysis leads to formula (4.31) for the local terms at order $n$. At higher loop level this formula is not correct any longer. In addition to the tree level terms present in (4.31) one also has loop graphs between the correlation function that contains the $j_0^\mu$ vertex and the rest. Furthermore, the transformation rules may receive quantum correction, i.e. $R^{A;m}$ is a series in $\hbar$. Symbolically one has

$$A_{c,n}(\hbar^M) = \sum_{m_1+m_2=n} \{ (R^{A;m_1}(\hbar^0)K_{AB}\phi^B \delta^{(m_1)}N_{m_2}(\hbar^M)\delta^{(m_2)})$$

$$+ \sum_{M_1+M_2+M_3=M} [R^{A;m_1}(\hbar^{M_1})N_{m_2}(\hbar^{M_2})](M_3 \text{ loops}) \} \quad (4.63)$$

where $N_n(\hbar^M)$ denotes the local normalization freedom of $T_n[T_1(x_1)...T_1(x_n)]$ in the $M$-th loop level. In the first line in the right hand side of (4.63) tree-level “relevant contractions” are understood. These terms correspond to the right hand side of (4.31). The second line in (4.63) contains the new terms on top of the ones present in (4.31). We shall collectively denote these terms $L_n$.

The analysis of the first line in the right hand side of (4.63) is exactly the same as the tree-level analysis. The loop terms $L_n$ in (4.63) are also local terms. Their general form is therefore of the form (4.14),

$$L_n = n!L_{1,n}\delta^{(n)} + \partial_\mu L_\mu_{2,n}, \quad L_\mu_{2,m} = \sum_{\alpha_1} L_{\mu_1}^{\mu_1-\alpha_\alpha_1} \partial_\alpha_1 \delta(x_1-x_n) \cdots \partial_\alpha_\alpha_n \delta(x_{n-1}-x_n) \quad (4.64)$$

(the $n!$ in the first term was added for later convenience). The term $L_{2,n}$ is a total derivative term. We remove it by appropriately fixing the local normalization freedom of the current correlation function. Hence, we obtain

$$A_{n,c}(\hbar^M) = n![(s_0\mathcal{L}_{n-1}(\hbar^M) + \cdots + s_k\mathcal{L}_{n-1-k}(\hbar^M)) + L_{1,n}(\hbar^M)]\delta^{(n)} \quad (4.65)$$

where we have extended the definition (4.24) to cover also the loop case. From our assumption that the Quantum Noether method works successfully there follows

$$s_0\mathcal{L}_{n-1}(\hbar^M) + \cdots + s_k\mathcal{L}_{n-1-k}(\hbar^M) + L_{1,n}(\hbar^M) = \partial_\mu C_\mu_n(\hbar^M) \quad (4.66)$$

for some $C_\mu_n$. Summing up these relations we obtain,

$$s\mathcal{L}(\hbar^M) + L_1(\hbar^M) = \partial_\mu C_\mu_n(\hbar^M) \quad (4.67)$$

where we have defined $L_1 = \sum_n L_{1,n}$, $C = \sum_n C_n$ and we have extended the definition (4.23) at the loop level. Equation (4.67) constrains the local terms $N(\hbar^M)(= \ldots$
\[ \sum_n N_n(h^M) \]. Notice that \( L_1 \) only depends on \( N(h^K) \) for \( K < N \) (this follows from simple \( \hbar \) counting, see (4.63)). A sufficient and necessary condition for loop terms to satisfy the same condition as the tree-level terms is

\[ L_1(h^M) = sM(h^M) + \partial_\mu M^\mu(h^M) \quad (4.68) \]

If this relation is satisfied then the renormalized local terms \( \mathcal{L}'(h^M) = \mathcal{L}(h^M) + M(h^M) \) satisfy

\[ s\mathcal{L}'(h^M) = \partial_\mu C'_\mu \quad (4.69) \]

where \( C'_\mu = C_\mu - M_\mu \), i.e. the same equation as (4.60) satisfied by the tree-level normalizations, and the theory is stable.

If we are considering a BRST-like symmetry then (4.68) will always be satisfied if \( H_1(s, d) = 0 \). We can rephrase this condition by saying that (4.68) will always be true if the anomaly consistency condition \( sA = dB \) has only trivial solutions \( A = sA_1 + dB_1 \).

The general case goes along similar lines. Consider the case where the algebra of the symmetry transformation is given by

\[ [s(\epsilon), s(\eta)] \phi^A = s(\epsilon \times \eta) \phi^A \quad (4.70) \]

(for simplicity, we consider closed symmetry algebra) where \( s(\epsilon)\phi^A = (s\phi^A)^a\epsilon_a , \epsilon_a \) and \( \eta_a \) are the parameters of the symmetry transformations, \( a \) is an algebra index, \( (\epsilon \times \eta)^a = f^{a}{^b}{^c}\epsilon_b\eta_c \), and \( f^{a}{^b}{^c} \) are the structure constants of the algebra.

The Wess-Zumino consistency condition for the integrated anomaly reads\[28\]

\[ A_{\text{int}}(\epsilon \times \eta) = s(\epsilon)A_{\text{int}}(\eta) - s(\eta)A_{\text{int}}(\epsilon) \quad (4.71) \]

where \( A_{\text{int}}(\epsilon) = \int A(\epsilon) = \int A^a\epsilon_a \). One may easily check that

\[ A(\epsilon) = s(\epsilon)K + \partial_\mu A^\mu \quad (4.72) \]

is a solution of (4.71) for any \( K \). In the case of nilpotent symmetries, \( f^{a}{^b}{^c} = 0 \), equation (4.71) implies that the anomaly is \( s \)-closed. Furthermore, the trivial solution (4.72) correspond to an exact solution.

Assuming that (4.71) has only the (trivial) solution (4.72) we now show that the theory is stable. Consider (4.67). Let us first make explicit in our notation the parameter of the transformation and suppress the \( h^M \) as our considerations hold at any order in \( \hbar \),

\[ s(\eta)\mathcal{L} + L_1(\eta) = \partial_\mu C^\mu(\eta) \quad (4.73) \]

Act in this equation with \( s(\epsilon) \) and antisymmetrize in \( \epsilon \) and \( \eta \). Using (4.71) we get

\[ s(\epsilon \times \eta)\mathcal{L} + s(\epsilon)L_1(\eta) - s(\eta)L_1(\epsilon) = \partial_\mu(s(\epsilon)C^\mu(\eta) - s(\eta)C^\mu(\epsilon)) \quad (4.74) \]
Using again (4.73) to eliminate $s(\epsilon \times \eta)\mathcal{L}$ we obtain a local form of (4.71). Therefore, since by assumption this equation has only the trivial solution (4.72) we obtain

$$L_1 = sM + \partial_\mu C^\mu$$

which just (4.68) with $M^\mu \to C^\mu$. This finishes the proof that the theory is stable if the anomaly consistency condition has only trivial solutions.

### 4.5 Invariance of the $S$-matrix

In this subsection we analyze point 4, namely the question of equivalence between conditions (4.5) and (4.3). Having established this equivalence the invariance of the $S$-matrix follows as discussed in section 4.1.

At order $g$ the equivalence has been already established at (4.39). There we have explicitly shown that both general symmetry conditions, (4.5) and (4.3), pose the same condition on the physical coupling $T_1$, namely

$$[Q, T_1] = \partial_\mu T^\mu_{1/1},$$

where $T^\mu_{1/1} = \mathcal{L}^\mu_{1}$. We now define

$$T^\mu_{n/l} = T_n[T_1(x_1) \cdots T^\mu_{1/1}(x_l) \cdots T_1(x_n)].$$

Let us also use the notation

$$\partial_\mu T^\mu_{n/l}(x_1, \ldots, x_n) = \sum_{l=1}^n \partial_\mu T^\mu_{n/l}.$$

We shall now show that the condition (4.3), namely $[Q, T_n] = \partial_\mu T^\mu_n$, with this definition implies the same conditions on the the time-ordered products $T_n[T_1\ldots T_1]$ as the Quantum Noether condition (4.5), given by $\partial_\mu J^\mu_n = \sum_{l=1}^n \partial_\mu J^\mu_{n/l} = 0$ where $J^\mu_{n/l} = T[T_1(x_1) \cdots j^\mu_0(x_l) \cdots T_1(x_n)]$. In this sense the two general symmetry conditions are called equivalent.

Because Poincaré invariance and causality already fix the time-ordered products $T_n[T_1\ldots T_1]$ up to the local normalization ambiguity $N_n$, we only have to show that these local normalization terms $N_n$ are constrained in the same way by both conditions, (4.3) and (4.5).

The inductive proof of condition (4.3) proceeds along the same lines as the one of (4.5) (as explained around formula (4.9)): Assuming that the condition (4.3) is satisfied for all $m < n$ we can directly derive the fact that the condition at the $n$th order can only be violated by a local distribution (for a detailed proof see [17], section 2b):

$$[Q, T_n] = \partial_\mu T^\mu_n + A_n$$

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We shall now discuss in some detail the local terms $A_n$ arising from tree level graphs: To find the local tree-level obstruction terms $A_n$ that arise in the right hand side of (4.79) we use the same off-shell procedure as before. Namely, we first differentiate, keep only the field equation terms and then do the contractions. Thus, to get the local terms at $n = 2$ we first rewrite (4.76) including also the off-shell terms:

$$\partial_\mu T_{1/1}^\mu, -[Q, T_1] = -s_1\phi^A K_{AB}\phi^B$$

(4.80)

Then we get at the next order - using the natural splitting solution (denoted by the subscript ‘c’)-:

$$\partial_\mu T_{c,2}^\mu(x_1, x_2) - [Q, T_{c,2}(x_1, x_2)] = -(\hbar/i)(2s_1T_1 - \partial_\mu(\frac{\partial T_1}{\partial(\partial_\mu\phi^A)}s_1\phi^A))\delta(x_1, x_2)$$

(4.81)

Now we add the local normalization ambiguity $N_2$, which is uniquely defined with respect to the natural splitting solution $T_{c,2}$, to the equation,

$$T_2[T_1(x_1)T_1(x_2)] = T_{c,2}[T_1(x_1)T_1(x_2)] + N_2\delta(x_1, x_2)$$

(4.82)

We get

$$\partial_\mu T_{c,2}^\mu(x_1, x_2) - [Q, T_2(x_1, x_2)] = -(\hbar/i)[2!(s_1T_1 + s_0\frac{1}{2}N_2)]$$

$$-\partial_\mu(\frac{\partial T_1}{\partial(\partial_\mu\phi^A)}s_1\phi^A)]\delta(x_1, x_2)$$

(4.83)

Using the same identification $T_1 = (i/\hbar)L_1$, $N_2/2! = (i/\hbar)L_2$ as before and comparing (4.83) with the corresponding formula of the Quantum Noether condition (4.45), we see that we “miss” the term $(\partial L_2/\partial(\partial_\mu\phi^A))s_0\phi^A$. This term “covariantizes” the current $j_0^\mu$ that generates the linear transformation $s_0\phi^A$. In the present case, we have started with the $T_{1/1}^\mu$ coupling instead of the $j_0^\mu$ coupling, so we do not expect to find these “covariantization” terms. In addition, what matters is how the normalization freedom of the correlation functions without a current insertion are fixed. Also, now the combinatorial factor is 2! instead of 3!. This is due to the fact that we are in 2nd order instead of 3rd.

From (4.83) we see that condition (4.3) at order $n = 2$ is satisfied if and only if

$$s_1L_1 + s_0L_2 = \partial_\mu L_2^\mu + s_2\phi^A K_{AB}\phi^B$$

(4.84)

This coincides with the condition on the normalization terms, $L_1, L_2$, we derived from the Quantum Noether condition (see 4.46).

We now fix the tree-level normalization freedom of $T_2[T_{1/1}^\mu T_1]$

$$T_2[T_{1/1}^\mu(x_1)T_1(x_2)] = T_{c,2}[T_{1/1}^\mu(x_1)T_1(x_2)] - j_2^\mu\delta(x_1, x_2)$$

(4.85)
by condition (4.3). Then we get
\[ j_2^\mu = 2! (-\mathcal{L}_2^\mu) + \frac{\partial \mathcal{L}_1}{\partial (\partial_\mu \phi^A)} \phi^A. \] (4.86)

So we end up with
\[ \partial_\mu T_2^\mu(x_1, x_2) - [Q, T_2(x_1, x_2)] = -2!(s_2 \phi^A)\mathcal{K}_{AB}\phi^B \] (4.87)

After imposing the free field equations, we arrive at condition (4.3).

Following exactly the same techniques as the ones presented in subsection 4.3 one shows that condition (4.3) fixes all tree-normalization terms in the same way as our Quantum Noether condition (4.5). The derivation of the Noether consistency equations is totally analogous up to different combinatorial factors and up to covariantization terms.

For the purpose of completeness, we state here the formulae analogous to the equations we got from the Quantum Noether condition. In the general case, for \(1 \leq n \leq k\) (\(k\) is defined as in section 4.3, namely by the condition \(s_m = 0\) for all \(m > k\)), the result for the tree-level normalization freedom of \(T_n[T_{1/1}^\mu T_1 \ldots T_1]\) (with respect to the natural splitting solution) is
\[ j_n^\mu = -n!\mathcal{L}_n^\mu + (n-1)! \sum_{l=1}^{n-1} l \frac{\partial \mathcal{L}_{n-l}}{\partial (\partial_\mu \phi^A)} s_l \phi^A \] (4.88)

where we have used the definition in (4.24). To derive this result one may use again the distributional identity (4.52). In addition, we get
\[ \partial_\mu T_n^\mu - [Q, T_n] = -n!(s_n \phi^A)\mathcal{K}_{AB}\phi^B \] (4.89)

For the general case \(n > k\) one gets
\[ \partial_\mu T_n^\mu - [Q, T_n] = -n![s_0 \mathcal{L}_n + s_1 \mathcal{L}_{n-1} + \cdots + s_k \mathcal{L}_{n-k}] \]
\[ + (n-1)! \sum_{l=1}^{k} l \frac{\partial \mathcal{L}_{n-l}}{\partial (\partial_\mu \phi^A)} s_l \phi^A \] (4.90)

where again use of (4.52) has been made. This equation implies that
\[ s_0 \mathcal{L}_n + s_1 \mathcal{L}_{n-1} + \cdots + s_k \mathcal{L}_{n-k} = \partial_\mu \mathcal{L}_n^\mu. \] (4.91)

which coincides exactly with the constraint on the local normalization terms, \(\mathcal{L}_n\), we got from the Quantum Noether condition (see (4.53)). In full analogy with the analysis there, one achieves
\[ \partial_\mu T_n^\mu - [Q, T_n] = 0 \] (4.92)

by renormalizing the current as
\[ j_n^\mu = -n!\mathcal{L}_n^\mu + (n-1)! \sum_{l=1}^{k} l \frac{\partial \mathcal{L}_{n-l}}{\partial (\partial_\mu \phi^A)} s_l \phi^A \] (4.93)
and without the need to use the free field equations, which corresponds to the fact that there is no field-equation term in (4.92).

Comparing formulae (4.88), (4.93) with formulae (4.51), (4.57), we conclude that the (renormalized) Noether currents derived from the two different symmetry conditions, (4.3) and (4.5), also coincide up to different combinatorial factors and up to covariantization terms. In addition, the Lagrangian $\mathcal{L}$ constructed out of the tree-level normalization terms (that follow from condition (4.3)) is invariant (up to total derivatives) under the symmetry transformation,

$$s\phi^A = \sum_{l=0}^k q' s_l \phi^A.$$  \hspace{1cm} (4.94)

The issue of stability can be analyzed in exactly the same way as in section 4.4. One shows that condition (4.3) at loop level implies that the normalization ambiguity at the loop level, $N_n(h)$, is constrained in the same way as the tree-level normalizations, $N_n(h^0)$. Once the stability has been established the equivalence of (4.3) and (4.5) at loop level follows. Condition (4.3) guarantees the invariance of the $S$-matrix under the corresponding asymptotic symmetry in the adiabatic limit. In the case of BRST symmetries the asymptotic linear part of the symmetry directly implies the unitarity of the physical $S$-matrix, i.e. the crucial decoupling of the unphysical degrees of freedom ([29], see also [17], chapter 7). So the Quantum Noether condition (4.3) also directly implies this crucial property in the case of BRST symmetries.

5 Examples

5.1 Yang-Mills Theory

In this section we present the construction of the Yang-Mills theory with gauge group $G$. According to the discussion in section 3, we first need to specify a set of free field and to give their commutation relations. For the YM theory, one has the YM field $A_{\mu}^{a}$, the ghost $c^{a}$ and the anti-ghost $b^{a}$, where $a$ is an gauge group index. We shall discuss the theory in the Feynman gauge. For other gauge choices we refer to [30, 31]. The fields satisfy the free-field equations

$$\Box A_{\mu}^{a} = 0; \quad \Box c^{a} = 0; \quad \Box b^{a} = 0,$$  \hspace{1cm} (5.1)

and the (anti)-commutation relations,

$$[A_{\mu}^{(-)a}(x), A_{\nu}^{(+)}b(y)] = i\hbar \delta^{ab} g_{\mu\nu} D^{+}(x - y)$$

$$\{b_{a}^{(-)}(x), c^{(+)}b(y)\} = -i\hbar \delta_{a}^{b} D^{+}(x - y).$$  \hspace{1cm} (5.2)
where the super-index $\pm$ designates the emission and absorption parts of the corresponding field and $D^\pm$ is the (zero mass) Pauli-Jordan distribution (see appendix A).

The field equations in (5.1) may be derived from the following free Lagrangian

$$L_0 = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + b \Box c - \frac{1}{2} (\partial \cdot A)^2$$

(5.3)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. With these conventions, the kinetic operator $K_{AB} \phi^B = \partial_\mu (\frac{\partial L_0}{\partial (\partial_\mu \phi^A)}) - \frac{\partial L_0}{\partial \phi^A}$, and the corresponding propagator $D$ are equal to $\delta_{AB} \phi^B$, where $\delta_{AB}$ is the (zero mass) Pauli-Jordan distribution (see appendix A).

The field equations in (5.1) may be derived from the following free Lagrangian

$$L_0 = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + b \Box c - \frac{1}{2} (\partial \cdot A)^2$$

(5.3)

The free Lagrangian is invariant under the following BRST transformations

$$s_0 A^a_\mu = \partial_\mu c^a A; \quad s_0 c^a = 0; \quad s_0 b^a = -\partial \cdot A^a A$$

(5.4)

where we have introduced an anticommuting constant $\Lambda$. With this parameter present $s_0$ is a derivation\(^\text{13}\). One may derive the Noether current by making $\Lambda$ local, varying the Lagrangian and collecting all terms that are proportional to $\partial_\mu A$ as explained in section 2. The result is

$$j^\mu_0 = -\partial_\nu c F^{\nu\mu} - (\partial \cdot A) \partial^\mu c.$$  

(5.5)

(the overall sign is fixed by our convention to always remove the anticommuting variable from the right side of the equations). The current is conserved when the free field equations are satisfied. In particular,

$$\partial_\mu j^\mu_0 \equiv \partial_\mu J^\mu_1 = \partial_\mu c (-\Box A^\mu) + (\partial \cdot A)(-\Box c).$$  

(5.6)

This corresponds to the general formula (4.32) in section 4:

$$\partial_\mu j^\mu_0 \equiv \partial_\mu J^\mu_1 = (s_0 \phi^A) K_{AB} \phi^B. $$

(5.7)

The corresponding BRST charge is given by

$$Q = \int j^\mu_0 d^3 x = - \int ((\partial \cdot A^a) \partial^\mu c^a d^3 x$$

(5.8)

(with the above stated conventions, $s_0 \phi^A = (i/\hbar) \{Q, \phi^A \}$).

\(^{13}\)One may prefer to use an anti-derivation $s_0$ instead of the derivation $s_0$, since the current is fermionic. In this case the parameter $\Lambda$ will not be present in (5.4). To get the same signs in both cases one should use the Leibniz rule $s_0 (AB) = A s_0 B + (-1)^B (s_0 A) B$, where $(-1)^B$ denotes the grading of $B$. (We use the convention that $\Lambda$ is removed from the right side of the equations. This choice is co-related with the convention chosen in (3.2) to have the test functions to the right of the $T$-products. See the discussion in the paragraph after (5.9)). We choose $s_0$ in the following in order to show how the general analysis presented in section 4 (which involves a bosonic current) is also applicable to examples where the Noether current is fermionic.

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One may easily check that $s_0$ is nilpotent ($s_0(Λ_1)s_0(Λ_2) = 0$) when the free-field equation are satisfied. This is particular to the case of BRST symmetry\textsuperscript{14}. In addition, one has the ghost charge (which can also be obtained as a Noether current),

$$Q_c = \frac{i}{\hbar} \int d^3 x b_a \partial_0 \, c^a.$$ (5.9)

The ghost charge introduces a grading, the ghost number, in the algebra generated by the fundamental field operators. The ghost number of the gauge field $A_\mu^a$ is zero, of ghost field $c^a$ is $+1$, and of the antighost field $b_a$ is $-1$. It follows that the BRST charge has ghost number $+1$.

To obtain the theory in the Epstein-Glaser approach one starts with the coupling $g_\mu j_0^\mu$ and a yet unknown coupling $gT_1$. Since $j_0^\mu$ is fermionic, $g_\mu$ is fermionic too, and one has to be careful with signs. A practical trick that helps keeping track of them is to write $g_\mu = \Lambda g'_\mu$, where $\Lambda$ is an anticommuting constant (as in (5.4)). Now, let us define $j_0^\mu = j_0^\mu \Lambda$. Since $j_0^\mu g_\mu = j_0^\mu g'_\mu$ the $S$-matrix constructed with a coupling involving the bosonic current $j_0^\mu$ smeared out by the bosonic test function $g'_\mu$ is the same with the one constructed with a coupling involving the anticommuting current $j_0^\mu$ smeared out by the fermionic test function $g_\mu$. However, since all vertices are now bosonic one need not worry about signs. At the end we are interested in the $T$-products that involve the fermionic current. To obtain those, one simply pushes $\Lambda$ to the right where it recombines with $g'_\mu$ to give $g_\mu$. This automatically produces all correct signs. If one considers multi-current correlation functions one introduces as many anti-commuting constants as the number of current insertions, so one may need infinite number of anti-commuting constants.

As we have seen the Quantum Noether condition (4.5) at second order, $\partial_\mu J_2^\mu(x_1, x_2) = 0$, is equivalent to the condition

$$s_0 L_1 = \partial_\mu L_1^\mu$$ (5.10)

where $L_1 = (\hbar/i)T_1$. Observe that both sides of this equation involve a nilpotent differential; the left hand side the (abelian) BRST differential and the right hand side the co-differential $\delta = *d*$, where $*$ is the Hodge operator and $d$ is the exterior derivative. (Of course, the construction does not depend on the metric). Therefore, $L_1 \in H^\text{weak}_0(s_0, d)$ (the sub-index 0 denotes ghost number and denomination “weak” that we are working modulo field equations). Notice, however, that $d$ is not acyclic since we are working on a space where the free field equations are satisfied. The latter introduce non-trivial cycles. An example of the latter at ghost number 2 is $C_\mu = c^a \partial_\mu c^a$. One may check that $\partial_\mu C_\mu = 0$, but $C_\mu \neq \partial_\mu B$ for any $B$.

\textsuperscript{14}Let us mention that in the Lagrangian approach one may incorporate global symmetries into the BRST operator by introducing constant ghost fields (see, for instance, \textsuperscript{(3)}). In the EG formalism, where one starts with free fields in the asymptotic Fock space, there is no natural way to incorporate constant ghost fields. We shall, therefore, not discuss further this possibility.
The most general solution of (5.10) is equal to
\[ \mathcal{L}_1 = gf^{abc}(c^T \partial_\mu A_\nu F_{\mu\nu}^c + A_\mu a \phi_b c) + s_0 C + dD. \] (5.11)
where \( C \) and \( D \) are known local terms. We shall show in [11] that the exact terms do not
change the physics. In the following we concentrate on the non-trivial terms. We briefly
discuss the exact terms afterwards.

The Quantum Noether condition at order \( g \) implies that the structure constants \( f_{abc} \)
are totally antisymmetric, (see for instance [21], [32]), but still not further restricted by
a Jacobi identity. This contraint follows from the symmetry condition at order \( g^2 \) (see
below). Let us analyze further (4.5) at second order. We shall present this calculation in
some detail in order to illustrate how one deals with the various subtle points discussed
in section 4. We are interested in computing
\[ \partial_\mu j_{c,2}^\mu(x_1, x_2) = \partial_\mu x_1 T_{c,2}[j_\mu^0(x_1)T_1(x_2)] + \partial_\mu x_2 T_{c,2}[T_1(x_1)j_\mu^0(x_2)] \] (5.12)
at tree level. Let us start by first computing \( \partial_\mu x_1 T_{c,2}[j_\mu^0(x_1)T_1(x_2)] \). Since the tree-level
Wick contractions satisfy the Leibniz rule, and \( T_1 \) only depends only on the fields and their
first derivative, one can first compute \( \partial_\mu x_1 T_{c,2}[j_\mu^0(x_1)\phi^A(x_2)] \) and \( \partial_\mu x_2 T_{c,2}[j_\mu^0(x_1)\partial_\mu \phi^A(x_2)] \), where \( \phi^A = A_\mu^a, c^a, b^a \), and then use the Leibniz rule. To correctly take care of the various signs we insert an anti-commuting constant \( \Lambda \) next to \( j_\mu^0 \). As has been argued in detail
in section 4, one can first move the derivative inside the \( T \)-product and then do the
contractions. Let us compute \( \partial_\mu x_1 T_{c,2}[j_\mu^0(x_1)\Lambda b^q(x_2)] \) After the first step and before the
contractions one has (using (5.3))
\[ [\partial_\mu c(-\Box A^\mu) + (\partial \cdot A)(-\Box c)](x_1)\Lambda b^q(x_2) \] (5.13)
Clearly, a local term is produced when there is a contraction between the second term in
the square brackets and \( b^q(x_2) \). It is equal to \( (\hbar/i)(-\partial_\mu A_\mu^a \Lambda) \), which up to \( (\hbar/i) \) (which is
there to cancel the overall \( i/\hbar \)), is equal to \( s_0 b^q_{\lambda} \) as it should. One can also contract the
first term with \( b^q(x_2) \). This yields a non-local term proportional to the \( A \) field equation.
This is an example of “irrelevant contraction”. As explained in section 4 these terms are
irrelevant and they will be discarded. We shall, from now on, only concentrate on the

\[ f \]
"relevant" contractions, namely the ones resulting from contractions involving the field \(\phi^B\) in \(K_{AB}\phi^B\).

Following the procedure we have just outlined one obtains,

\[
\partial_{\mu}J_2^{\mu}(x, y) = g f_{abc}[2\partial_{\mu}c^a A^b_{\mu} F^{\mu \nu \rho} + 2\partial_{\mu}c^a c^b \partial^\mu b^c - 2A^a_{\mu} c^b \partial^\mu A^c + \partial^\mu (A^a_{\mu} c^b \partial^\nu A^c + A^c_{\nu} c^b \partial^\mu A^c)] \Lambda \delta(x_1, x_2)
\]

\[
= \{ \partial^\mu[2(s_1 A^a_{\mu} F^{\mu \nu} - s_1 c^a \partial_{\mu} b^a) + g f_{abc}(A^a_{\mu} A^b_{\mu} \partial^\mu c^c + A^a_{\mu} c^b \partial^\nu A^c)] \Lambda
+ 2(s_1 A^a_{\mu} (-\Box A^a_{\mu}) + s_1 c^a \Box b^a) \} \delta(x - y)
\]

(5.14)

where

\[
s_1 A^a_{\mu} = g f_{abc} A^b_{\mu} c^c \Lambda; \quad s_1 c^a = \frac{g}{2} f_{abc} c^b \Lambda
\]

Notice that the field equation terms in (5.14) were created in the process of factoring out a total derivative from the rest of the terms. Let us now fix the tree-level ambiguity such that (5.15) at second order holds when the free-field equations are satisfied. To this end, we let

\[
T[j_0^a(x_1)T_1(x_2)] = T[e[j_0^a(x_1)T_1(x_2)] + j_1^a \delta(x_1 - x_2)
\]

(5.16)

Then (5.3) implies

\[
j_1^a = -2 g f_{abc}(A^b c^c F^{\mu \nu} + \frac{1}{2} c^b c^c \partial_{\mu} b^a) - g f_{abc}(A^a_{\mu} A^b_{\mu} \partial^\nu c^c + A^a_{\mu} c^b \partial^\nu A^c)
\]

(5.17)

The first two terms in the Noether current are the ones that generate the \(s_1\) transformation. The last two "covariantize" \(j_0^a\). All of them are part of the Noether current of the non-linear theory up to combinatorial factors which take care of the additional factors in the perturbative expansion,

\[
j_\mu^{\text{non-abelian}} = -D_\nu c^a F^{\nu \mu} - (\partial \cdot A^a) D_\mu c^a - \frac{1}{2} g f_{abc} c^b \partial_{\mu} b^a,
\]

(5.18)

where

\[
F^{\mu \nu} = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu} + g f_{abc} A^b_{\mu} A^c_{\nu}; \quad D_\mu c^a = \partial_\mu c^a + g f_{abc} A^b_{\mu} c^c.
\]

(5.19)

So, finally at \(n = 2\) we have off-shell,

\[
\partial_{\mu}J_2^{\mu}(x, y) = 2[s_1 A^{\mu a} (-\Box A^a_{\mu}) + s_1 c^a \Box b^a] \delta(x_1, x_2)
\]

(5.20)

This corresponds to the general formula

\[
\partial_{\mu}J_2^{\mu}(x, x_2) = 2! s_1 A^{\mu a} K_{AB} \phi^B.
\]

(5.21)

One may also check that \(\mathcal{L}_0 + \mathcal{L}_1\) is equal to the YM action,

\[
\mathcal{L}_{YM} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + b \partial^\mu D_\mu c - \frac{1}{2} (\partial \cdot A)^2,
\]

(5.22)
but the four-gluon term. The latter is of order $g^2$ and will be recovered at next order.

To examine (4.5) at third order we need the second-order result off-shell (5.20). Using
\[ \partial_\mu J^\mu_{c,3}(x_1, x_2, x_3) = \partial_\mu J^\mu_{c,2}(x_1, x_2) T_1(x_3) + \partial_\mu J^\mu_{c,1}(x_1) N_2(x_2, x_3) + \text{cyclic in } x_1, x_2, x_3, \] (5.23)
one gets at $n = 3$,
\[ \partial_\mu J^\mu_{c,3}(x_1, x_2, x_3) = [3!(g^2 f_{abc} f_{cde} A_\mu^a A_\nu^b A_\rho^c \partial_\sigma^d + s_0 L_2)] \delta(x_1, x_2, x_3) \] (5.24)
where $L_2 = (h/i) N_2/2!$ and $N_2$ denotes the unique local normalization term of $T_2$ with respect to $T_{c,2}$. From here we determine $L_2$, which is just the missing four-gluon coupling in (5.22), and also the Noether current renormalizations,
\[ L_2 = -\frac{1}{4} (g f_{abc} A_\mu^a A_\nu^b)^2, \]
\[ j_2^\mu = -4g^2 f_{abc} A_\mu^a A_\nu^b A_\rho^c \delta(x_1, x_2, x_3) \] (5.25)
where $j_2$ is defined as in (1.48). Notice that this is precisely the “covariantization” term missing from (5.18). We note that one also finds the Jacobi identities for $f_{abc}$ as a consequence of the Quantum Noether Condition at order $n = 2$. At $n = 3$ we finally have even off-shell
\[ \partial_\mu J^\mu_{3}(x_1, x_2, x_3) = 0 \] (5.26)
Notice the absence of field equation terms in the right hand side of (5.26). This means that no new tree-level local terms will emerge at higher orders.

One may easily check that $s_0$ and $s_1$ as defined above satisfy $\{s_i, s_j\} = 0$, where $i = 0, 1$. It follows that $s = s_0 + s_1$ squares to zero. It has been argued that only non-trivial solutions of (5.10) are physically relevant. This question will be analyzed in detail in (11). Let us already briefly discuss this issue here. Suppose that instead of $L_1$ in (5.11) one considers $L_1 + s_0 C$. This is still a solution of (5.10). At next order, however, one obtains the equation
\[ s_1(s_0 C) + s_0 L_2' = 0 \] (5.27)
where $N_2 = (i/h)(L_2 + L_2')$ and $L_2$ is as in (5.23). It follows that $L_2' = s_1 C$. Thus, adding an exact $s_0$ term in $L_1$ results in the addition of an $s$-exact term in the Lagrangian. The statement that such $s$-exact term are physically irrelevant will be worked out in the EG formalism in (11). In our specific example one may check that adding the exact terms, $L_1' = \beta_1 s_0 (\partial A^a c^b c^c)$ and $L_1'' = \beta_2 \partial^\mu (f_{abc} c^b A_\mu^c)$, results in additional terms in the transformation rules at order $g$ and in the well-known four-ghost coupling at order $g^2$. 
Let us now move to loop level. It is well known\cite{38} that in order to have a candidate anomaly (i.e. non-trivial element of $H^1(s,d)$), one needs a non-vanishing $d_{abc}$ and an epsilon symbol $\epsilon_{\mu\nu\rho\sigma}$. However, in a theory without chiral fermions one does not have an epsilon symbol. So, in pure Yang-Mills the non-trivial elements of $H^1(s,d) = 0$ will not occur. According to the analysis presented in section 4.4 this is sufficient to guarantee that the loop normalization ambiguity is constrained the same way as the tree-level one. We therefore conclude that the Yang-Mills theory is renormalizable.

5.2 The $N = 1$ Wess-Zumino Model

We now turn to our supersymmetric example. The field in Wess-Zumino model\cite{39} are a complex scalar field $\phi$ and its fermionic partner $\psi^\alpha$. We use the two component spinor notation of\cite{40} (see appendix A). The fields satisfy the following field equations

$$\Box \phi = 0; \quad \partial_{\alpha\dot{\alpha}} \psi^\alpha = 0.$$  \hspace{1cm} (5.28)

The commutations relations are

$$[\phi^\alpha(x_1), \bar{\psi}^{\dot{\beta}}(x_2)] = i\hbar 2 D^+(x_1, x_2)$$

$$\{\psi_\alpha^\alpha(x_1), \psi^\dot{\beta}_{\dot{\beta}}(x_2)\} = i\hbar S^+_{\alpha\dot{\alpha}}(x_1, x_2)$$  \hspace{1cm} (5.29)

where $S^+_{\alpha\dot{\alpha}} = 2i\partial^{\alpha\dot{\alpha}} D^+$.

The field equations can be derived from the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2} \Box \phi + \bar{\psi}^\dot{\alpha} i\partial^\alpha_\alpha \psi_\alpha - \frac{1}{2} \bar{\phi} \phi + \frac{1}{2} \bar{\phi} \phi D^+(x_1, x_2) + \frac{1}{2} \bar{S}^{\alpha\dot{\alpha}} D^+_F(x_1, x_2)$$  \hspace{1cm} (5.30)

With these conventions $K_{\phi\phi} = -\Box \phi / 2$, $K_{\bar{\phi}\bar{\phi}} = -\Box \bar{\phi} / 2$, $K_{\alpha\alpha} \psi^\alpha = -i\partial^\alpha_\alpha \psi_\alpha$ and $K_{\alpha\dot{\alpha}} \psi^{\dot{\alpha}} = -i\partial_{\alpha\dot{\alpha}} \psi^{\dot{\alpha}}$. The corresponding Feynman propagators are given by $D^{\phi\phi} = 2i\hbar \Delta_F(x-y)$, $D^{\alpha\dot{\alpha}} = i\hbar S^{\alpha\dot{\alpha}}_F$, where $S^{\alpha\dot{\alpha}}_F = 2i\partial^{\alpha\dot{\alpha}}_x D_F(x-y)$. For a derivation of these formulae see appendix A.

This action is invariant under the linear supersymmetry transformations,

$$s_0 \phi = -\epsilon^\alpha \psi_\alpha, \quad s_0 \bar{\phi} = -\bar{\epsilon}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}, \quad s_0 \psi^\alpha = -\epsilon^\alpha i\partial^{\alpha\dot{\alpha}}_\alpha \bar{\phi}, \quad s_0 \bar{\psi}^{\dot{\alpha}} = -\bar{\epsilon}^{\dot{\alpha}} i\partial^{\alpha\dot{\alpha}}_{\dot{\alpha}} \phi$$  \hspace{1cm} (5.31)

The associated Noether current is equal to

$$j^{\alpha\dot{\alpha}}_0 = \epsilon^\beta \psi^\alpha \partial_{\beta} \bar{\phi} + \bar{\epsilon}^{\dot{\beta}} \bar{\psi}^{\dot{\alpha}} \partial^\alpha_\alpha \dot{\phi}$$  \hspace{1cm} (5.32)

An easy calculation yields

$$\partial_{\alpha\dot{\alpha}} j^{\alpha\dot{\alpha}}_0 = (-\epsilon^\alpha \psi_\alpha)(-\frac{1}{2} \Box \phi) + (-\epsilon^\alpha \psi_\alpha)(-\frac{1}{2} \Box \phi)$$

$$+ (-\bar{\epsilon}^{\dot{\beta}} i\partial^{\alpha\dot{\alpha}}_\beta \bar{\phi})(-i\partial_{\alpha\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}) + (-\bar{\epsilon}^{\dot{\beta}} i\partial^{\alpha\dot{\alpha}}_\beta \bar{\phi})(-i\partial_{\alpha\dot{\alpha}} \bar{\psi}_{\dot{\alpha}})$$  \hspace{1cm} (5.33)
In a similar way as in the Yang-Mills example, one may check that this current correctly produces the supersymmetry variation inside correlation functions.

Invariance at first order requires that we find a $T^1 = (i/h) L_1$ such that

$$s_0 L_1 = \partial_\mu L_1^\mu$$

holds, up to free field equations, for some $L_1$ and $L_1^\mu$. The latter are constraint by power counting. The most general solution (for simplicity we restrict ourselves to a massless theory) is

$$L_1 = (1/2)(\phi \psi^2 + \bar{\psi}^2)$$

where $\psi^2 = \psi^\alpha \psi_\alpha$ and $\bar{\psi}^2 = \bar{\psi}^{\dagger} \bar{\psi}_\dagger$. Then,

$$\partial_\alpha \bar{\psi}^{\dagger} \alpha \psi^{\dagger} \alpha = [\partial_\alpha \bar{\psi}^{\dagger} \alpha (e^\alpha \phi^2 + e^\alpha \bar{\phi}^2) + 2(-\frac{1}{2} e^\alpha \bar{\phi}^2)(-i\partial_\alpha \bar{\psi}^{\dagger} \alpha) + 2(-\frac{1}{2} e^\alpha \phi^2)(-i\partial_\alpha \bar{\psi}^{\dagger} \alpha)] \delta(x_1, x_2)$$

The first term in the right hand side is removed by fixing the tree-level ambiguity as

$$T[j_0^{\alpha \dagger} (x_1) T_1(x_2)] = T[e_0^{\alpha \dagger} (x_1) T_1(x_2)] + j_1^{\alpha \dagger} \delta(x_1 - x_2)$$

where

$$j_1^{\alpha \dagger} = -i(e^\alpha \phi^2 + e^\alpha \bar{\phi}^2)$$

So, we end up with

$$\partial_\alpha \bar{\psi}^{\dagger} \alpha \psi^{\dagger} \alpha = 2!(s_1 \psi^\alpha)K_{\alpha \dagger} \psi^{\dagger} + 2!(s_1 \psi^{\dagger} \alpha)K_{\alpha \dagger} \bar{\psi}^\alpha$$

where the new symmetry variations are given by

$$s_1 \phi = 0, \quad s_1 \psi^\alpha = -\frac{1}{2} e^\alpha \phi^2, \quad s_1 \bar{\phi} = 0, \quad s_1 \bar{\psi}^{\dagger} \alpha = -\frac{1}{2} e^{\dagger} \alpha \bar{\phi}^2$$

We now move to the next order. We have

$$\partial_\mu J^\mu_{c,3}(x_1, x_2, x_3) = \partial_\mu J^\mu_{c,2}(x_1, x_2)T_1(x_3) + \partial_\mu J^\mu_1(x_1)N_2(x_2, x_3) + \text{cyclic in } x_1, x_2, x_3$$

A straightforward calculation yields

$$\partial_\mu J^\mu_{c,3}(x_1, x_2, x_3) = 3!(1/2)(e^\alpha \psi_\alpha)\bar{\phi}^2 \phi + N_2(-e^{\dagger} \psi_\alpha)\bar{\phi}^2 + s_0 \Delta_2 \delta(x_1, x_2, x_3)$$

where $\Delta_2 = (h/i) N_2/2$. This implies that

$$\Delta_2 = -\frac{1}{4} \phi^2 \bar{\phi}^2$$

Hence,

$$\partial_\mu J^\mu_3(x_1, x_2, x_3) = 0$$
Therefore, no new local terms will arise in higher orders.

The sum \( \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \) is equal to
\[
\mathcal{L} = \frac{1}{2} \bar{\phi} \Box \phi + \bar{\psi} \gamma^\alpha \partial^\alpha \psi_\alpha + \frac{1}{2} g (\bar{\phi} \psi^2 + \bar{\psi} \bar{\psi}^2) - \frac{1}{4} g^2 \phi^2 \bar{\phi}^2
\]
which is indeed the Wess-Zumino Lagrangian. The transformation rules which are generated by \( s = s_0 + s_1 \) are given by,
\[
\begin{align*}
    s_0 \phi &= -\epsilon^\alpha \psi_\alpha, & \quad s_0 \bar{\phi} &= -\epsilon^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}, \\
    s_1 \psi_\alpha &= -\epsilon^{\dot{\alpha}} i \partial^\alpha \bar{\phi} - \frac{1}{2} g e^\alpha \bar{\phi}^2, & \quad s_0 \bar{\psi}_{\dot{\alpha}} &= -\epsilon^\alpha i \partial_{\dot{\alpha}} \bar{\phi} - \frac{1}{2} g \epsilon^\alpha \phi^2.
\end{align*}
\]
These are also the correct supersymmetry transformation rules.

Finally, we discuss the issue of stability under quantum corrections (renormalizability). It has been shown in [41] that there are no anomaly candidates in the \( N = 1 \) Wess-Zumino model. From our discussion in section 4.4 immediately follows that the loop normalization ambiguity is constrained in the same way as the tree-level one, i.e. the theory is renormalizable.

**Acknowledgements**

We thank Raymond Stora for discussions and comments regarding the Epstein-Glaser framework and for a critical reading of the manuscript, Peter van Nieuwenhuizen for discussions and comments regarding the Noether method, and Friedemann Brandt, Walter Troost, and Toine Van Proeyen for discussions. TH would like to thank the Institute for Theoretical Physics in Stony Brook and in Leuven and KS the Theory Division at CERN and the MPI in Munich where part of this work was completed for their kind hospitality. TH gratefully acknowledges financial support during his stay in Stanford by Schweizerischer Nationalfonds and by Department of Energy under contract number DE-AC03-73SF00515. KS is supported by the European Commission HCM program CHBG-CT94-0734 and by the European Commission TMR programme ERBFMRX-CT96-0045.

**A Conventions**

In this appendix we list in detail our conventions and our various sign choices. Since one of the aims of this article is to make contact between the EG and the Lagrangian formalism, it is important to have compatible conventions on both sides.

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16 Strictly speaking one would have to extend the considerations of section 4.4 to theories with open symmetry algebra in order to be applicable to the present example.
We start from a free massless scalar in four dimensions. The Lagrangian is given by
\[ L_0 = \frac{1}{2} \phi \square \phi \]  
(A.1)
The overall sign in the Lagrangian is such that the Hamiltonian is positive definite. (Our convention for metric is \( \eta_{\mu \nu} = \text{diag}(-1, 1, 1, 1) \) and \( x^0 \) is the time variable \( t \). For the space coordinates we use either \( x^i \) or \( x \)). The canonical momentum is equal to \( p = \dot{\phi} \). Then the equal-time commutation relations (ECR) read,
\[ [\phi(t, x), \phi(t, y)] = i \hbar \delta(x - y) \]  
(A.2)
The field equation is
\[ \square \phi = 0 \]  
(A.3)
The commutator of two fields in arbitrary spacetime points is
\[ [\phi^{(\pm)}(x), \phi^{(\pm)}(y)] = i \hbar D^{\pm}(x - y) \]  
(A.4)
where \( \phi^{(\pm)}(x) \) are the absorption and emission parts of \( \phi(x) \), and
\[ D^{\pm}(x - y) = \pm (-i) \int \frac{d^3 k}{(2\pi)^3 2\omega} e^{\pm i k(x-y)} \]  
(A.5)
The Pauli-Jordan distribution is then equal to
\[ D(x - y) = D^+ + D^- = (-i) \int \frac{d^3 k}{(2\pi)^3 \omega} \sin k(x - y) \]
\[ = (-i) \int \frac{d^4 k}{(2\pi)^4} \delta(k^2) \text{sgn}(k^0) e^{ik(x-y)} \]  
(A.6)
The Pauli-Jordan distribution has a causal decomposition into retarded and advanced part as follows,
\[ D_{\text{ret}}(x - y) = \theta(x^0 - y^0)D(x - y); \quad D_{\text{adv}}(x - y) = -\theta(y^0 - x^0)D(x - y). \]  
(A.7)
The Feynman propagator
\[ \mathcal{D}^{\phi\phi} \equiv \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle = i \hbar D_F(x - y) \]  
(A.8)
where \( \square x D_F(x - y) = \delta(x - y) \). From the latter equation one obtains
\[ D_F(x - y) = \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ikx}}{-k^2 + i\epsilon} \]  
(A.9)
It follows that
\[ D_F(x - y) = \theta(x^0 - y^0)D^+(x - y) - \theta(y^0 - x^0)D^-(x - y). \]  
(A.10)
It is now easy to verify (as described in section 3) that after natural splitting the commutator of two fields is replaced by the Feynman propagator exactly.

We move to the case of gauge fields. The Lagrangian in the Lorentz gauge is given by

\[ \mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial \cdot A)^2 \]  

(A.11)

The overall sign in the action is fixed by requiring positivity of the energy. The result for the commutator and the propagator are,

\[ [A_{\mu}^{(-)a}(x), A_{\nu}^{(+)}b(y)] = i\hbar\eta_{\mu\nu}\delta^{ab}D^+(x - y) \]

\[ \mathcal{D}^{AA} \equiv \langle 0|T(A_{\mu}^{a}(x)A_{\nu}^{b}(y))|0\rangle = i\hbar\eta_{\mu\nu}\delta^{ab}D_F(x - y) \]  

(A.12)

The indices \( a, b \) indices are gauge group indices.

We now move to the ghost sector. We take for Lagrangian

\[ \mathcal{L}_0 = \int b_a \Box c^a. \]  

(A.13)

The field equation are \( \Box c^a = 0 \) and \( \Box b_a = 0 \). \( b_a \) is antihermitian whereas and \( c^a \) is hermitian, so that the action is hermitian\(^1\). The anti-commutator of a ghost field with an antighost is

\[ \{b_{a}^{(-)}(x), c^{(+)}b(y)\} = -i\hbar\delta_a^b D^+(x - y) \]  

(A.14)

The Feynman propagator is

\[ \mathcal{D}^{bc} \equiv \langle 0|T(b_a(x)c^b(y))|0\rangle = -i\hbar\delta_a^b D_F(x - y) \]  

(A.15)

For the case of fermions we use the two component notation of\(^4\). With these conventions one avoids using gamma matrices, and the Fierz identities become a matter of symmetrizing and antisymmetrizing spinor indices.

The universal cover of the Lorentz group in four dimension is isomorphic to \( SL(2, C) \). The simplest non-trivial representation of the latter is the two component complex Weyl spinor \( \psi^\alpha, \alpha = +, - \), (the \((1/2, 0)\) representation). Its complex conjugate representation (the \((0, 1/2)\)) is denoted by \( \bar{\psi}^\dot{\alpha} \). Greek letters are reserved for spinor two components indices and Roman ones for vector indices. Each vector index is equivalent to one undotted and one dotted index \( (\phi^\alpha = \phi^{\dot{\alpha}}) \). Indices are raised and lowered using the \( sl_2 \) invariant antisymmetric two dimensional matrix \( C_{\alpha\beta} \). Since \( C_{\alpha\beta} \) is antisymmetric, we have to specify how exactly we use it to raise and lower indices, and our convention is the

\(^1\)Strictly speaking the ghosts and the action are pseudo-(anti-)hermitian which indicates that the actual hermiticity properties are defined in respect to a sesquilinear form (indefinite metric) and not in respect of the (positive definite) scalar product of the one-particle Hilbert space of the ghosts.
so-called ‘down-hill’ rule from left to right for both the undotted and the dotted sector. For example,

\[ \psi^\alpha C_{\alpha\beta} = \psi_\beta; \quad C^{\alpha\beta} \psi_\beta = \psi^\alpha; \quad \psi^\dot{\alpha} C_{\dot{\alpha}\dot{\beta}} = \psi_\dot{\beta}; \quad C^{\dot{\alpha}\dot{\beta}} \psi_\dot{\beta} = \psi^\dot{\alpha}. \]  

(A.16)

In addition, we have the following identity

\[ C_{\alpha\beta} C^{\gamma\delta} = \delta_\alpha^\gamma \delta_\beta^\delta - \delta_\beta^\gamma \delta_\alpha^\delta, \]  

(A.17)

From (A.17) we get

\[ C^{\alpha\beta} C_{\alpha\beta} = \delta_\alpha^\alpha = 2 \]  

(A.18)

\[ C_{\alpha\beta} \chi_\gamma - C_{\alpha\gamma} \chi_\beta = -C_{\beta\gamma} \chi_\alpha \]  

(A.19)

\[ \chi_\alpha \chi_\beta = -\frac{1}{2} C_{\alpha\beta} \chi_\gamma \chi_\gamma \]  

(A.20)

The last identity is an example of a Fierz identity.

The Lagrangian for a massless spinor field is given by

\[ L_0 = \psi^{\dot{\alpha}} i \partial_{\dot{\alpha}} \psi_\alpha \]  

(A.21)

The propagator is

\[ D^{\alpha\dot{\alpha}} \equiv \langle 0 | \psi^{\alpha}(x) \psi^{\dot{\alpha}}(y) | 0 \rangle = i \hbar S_F^{\alpha\dot{\alpha}} \]  

(A.22)

where

\[ S_F^{\alpha\dot{\alpha}} = 2 i \partial_{\alpha}^{\alpha\dot{\alpha}} D_F(x - y) \]  

(A.23)

Finally, the commutation relations of two Fermi fields at arbitrary spacetime separation are given by

\[ \{ \psi^{(-)}_\alpha(x), \psi^{(+)}_\alpha(y) \} = i \hbar S^{+}_{\alpha\alpha}(x - y) \]  

(A.24)

where

\[ S^{+}_{\alpha\alpha}(x - y) = 2 i \partial_{\alpha\alpha}^{+} D^+(x - y). \]  

(A.25)

\[ B \quad \text{Further Properties of the EG Formalism} \]

In this appendix we highlight some further properties of the EG formalism. In particular we discuss the infrared problem, the construction of interacting fields, the problem of overlapping divergences and a few other issues.

- The distributions \( T_n \) in (3.2) are smeared out by tempered test functions \( g \in S \). This provides a natural regularization of the physical infrared problem which arises in massless theories because the tempered test function cuts off the long-distance part of the distributions. In the construction of quantum electrodynamics, for example, the infrared
problem is fully separated in the causal formalism. Ultimately, one is interested in the physical (so-called adiabatic) limit \( g(x) \to g \equiv \text{const.} \). Epstein and Glaser have proven that this limit exists for the vacuum expectation values of the \( T_n \) distributions (in the sense of tempered distributions) in massive theories if a suitable normalization is chosen (section 8.2 in [4]). In this limit the Green functions possess all the expected linear properties such as causality, Lorentz covariance and the spectral condition. The existence of Green functions in the adiabatic limit for the case of quantum electrodynamics was shown by Blanchard and Seneor [12]. We implement our method before the adiabatic limit. All equations are understood as distributional ones. So our analysis is also well-defined in massless theories like pure Yang-Mills theories where the adiabatic limit is related to the confinement problem which is not expected to be solved in the framework of perturbation theory.

- We have argued in the introduction that the strength of the EG construction lies in the operator formalism. However, this strength turns into a weakness of the formalism when one is interested in non-local details of the theory, for example, when one tries to translate a simple operator condition into relations of \( \mathbb{C} \)-number distributions or if one discusses properties of a subgroup of contributions or even a single type of diagram.

- The EG formalism naturally leads to amputated connected Green functions and not to one-particle irreducible ones. This complicates the discussion of the renormalization group in this framework.

- Having constructed the most general \( S \)-matrix one can construct interacting field operators (compatible with causality and Poincaré invariance) ([4] section 8, [8]).

One starts with an extended first order \( S \)-matrix

\[
S(g, g_1, g_2, \ldots) = \int d^4x \{ T_1(x)g(x) + \Phi_1(x)g_1(x) + \Phi_2(x)g_2(x) + \ldots \} \tag{B.1}
\]

where \( \Phi_i \) represent certain Wick monomials like \((i/\hbar)\varphi\) or \((i/\hbar) : \varphi^3 :\). Following Bogoliubov and Shirkov ([2]), Epstein and Glaser defined the corresponding interacting fields \( \Phi_i^{\text{int}} \) as functional derivatives of the extended \( S \)-matrix:

\[
\Phi_i^{\text{int}}(g, x) = S^{-1}(g, g_1, \ldots) \frac{\delta S(g, g_1, \ldots)}{\delta g_i} \bigg|_{g_i = 0} \tag{B.2}
\]

One shows that the perturbation series for the interacting fields is given by the advanced distributions of the corresponding expansion of the \( S \)-matrix, namely

\[
\Phi_i^{\text{int}}(g, x) = \Phi_i(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \frac{\hbar}{i} A_{n+1/n+1}(x_1, \ldots, x_n; x) \tag{B.3}
\]

where \( A_{n+1/n+1} \) denotes the advanced distributions with \( n \) original vertices \( T_1 \) and one vertex \( \Phi_i \) at the \((n + 1)\)th position; symbolically we may write:

\[
A_{n+1/n+1}(x_1, \ldots, x_n; x) = Ad[T_1(x_1) \ldots T_1(x_n); \Phi_i(x)] \tag{B.4}
\]
One shows that the perturbative defined object $\Phi^i_{\text{int}}$ fulfils the properties like locality and field equations in the sense of formal power series. The definition can be regarded as a direct construction of renormalized composite operators. Epstein and Glaser showed that the adiabatic limit $g \to 1$ exists only in the weak sense of expectation values in massive theories. The limit possesses all the expected properties of a Green’s function such as causality, Lorentz covariance and the spectral condition.

- If the specific coupling $T_1$ is fermionic then the causality condition implies that the corresponding tempered test function has to be an anticommuting Grassmann variable. Also an ordering of fermionic couplings in the time-ordered products is introduced (for further details see [10, 17]).
- The polynomial character of the interactions $T_1$ is forced by the formalism allowing $g$ to be any element of $\mathcal{S}(\mathbb{R}^4)$. If one restricts the choice of the test functions to the Jaffe class one can also construct theories with non-polynomial specific couplings (see [12]).
- Steinmann presented an approach to perturbative quantum field theory which is related to the Epstein-Glaser method [25]. He works with retarded products and his construction is done after the adiabatic limit is taken.
- A variant of the Epstein-Glaser formalism emerges from the results of [3]: The problem of cutting a causal distribution into a retarded and advanced piece is equivalent to the problem of continuation of time-ordered products to coincident points. Actually, the renormalization scheme of differential renormalization proposed by Freedman, Johnson and Latorre [43] is an operative way to perform such a continuation in configuration space (x-space); at least at the one and two-loop level. This has been illustrated in [44]. Stora’s variant of the EG method allows for an extension of the EG method to theories on curved space-time. One of the main problems one encounters in trying to achieve such extension is the absence of translational invariance (which plays a crucial role in the original version of the formalism as presented by Epstein and Glaser (see section 3)) in curved space-time. For recent work see [7].
- Finally, let us shortly discuss how the problem of overlapping divergences is automatically solved by the EG formalism [13]. All $T$-products are defined as operators that fulfil Wick’s theorem (3.11) and both conditions of causality (3.3). From this, we can derive a general necessary condition for all renormalization schemes which is normally established at the level of $C$-number valued Green functions and not at the level of operators. Note that in the EG formalism a Green function always has a representation as a vacuum expectation value of an operator-valued distribution $\langle T(V) \rangle$.

A renormalization of a Green function $\langle T(V) \rangle$ has to be such that for all partition of the set of vertices $V = X \cup Y$, $X \neq \emptyset$, $Y \neq \emptyset$, it coincides in the region where $X \geq Y$ with

$$
\langle T(V) \rangle = \langle T(X) \rangle \prod_{x_i \in X, y_i \in Y} D^+(x_i - y_i) \langle T(Y) \rangle
$$

(B.5)
where $D^+$ represents a fundamental commutation distribution and $\langle T(X) \rangle$ ($\langle T(Y) \rangle$) is a subgraph of order $|X|$ ($|Y|$). In the EG formalism the latter Green functions are vacuum expectation values of operator-valued distributions including Wick submonomials (see (3.11)). These are well-defined and fulfil all required conditions by the induction hypothesis.

Usual renormalization schemes are directly implemented on the level of $C$-number Green functions via a regularization-substraction scheme. So it becomes a nontrivial task to show that all sub-diagrams of a given diagram can be renormalized in a consistent way, in particular such that the condition (3.3) is fulfilled. For example, this was shown for Pauli-Villars type regularizations (see i.e. [4]). In the context of the BPHZ method this is achieved by the forest formula found by Zimmermann, which solves the recursion formula for the R-operation as defined in [2], an algorithm that disentangles all divergences in sub-diagrams. In the EG formalism it is the inductive operator formalism which disentangles the problem of the renormalization of sub-diagrams.

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