DEFORMING SPACES OF M-JETS OF HYPERSURFACES SINGULARITIES

MAXIMILIANO LEYTON-ÁLVAREZ

Abstract. Deformation of Spaces of m-jets. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and $V$ a hypersurface defined by an irreducible polynomial $f$ with coefficients in $\mathbb{K}$.

In this article we prove that an Embedded Deformation of $V$ which admits a Simultaneous Embedded Resolution induces, under certain mild conditions, a deformation of the reduced scheme associated to the space of $m$-jets $V_m$, $m \geq 0$. An example of an Embedded Deformation of $V$ which admits a Simultaneous Embedded Resolution is a $\Gamma(f)$-deformation of $V$, where $V$ has at most one isolated singularity, and $f$ is non degenerate with respect to the Newton Boundary $\Gamma(f)$.

1. Introduction

Let $V$ be an algebraic variety over a field $\mathbb{K}$, and $m$ a positive integer. Intuitively the Space of $m$-jets, $V_m$, (resp. Space of arcs, $V_{\infty}$,) is the set of morphisms

$\text{Spec } \mathbb{K}[t]/(t^{m+1}) \rightarrow V$ (resp. $\text{Spec } \mathbb{K}[[t]] \rightarrow V$)

equipped with a “natural” structure of $\mathbb{K}$-schema. During the late 60s Nash studied these spaces as a tool to understand the local geometry of the singular locus of $V$. He was specifically interested in understanding the Essential Divisors over $V$ (see Section 2.2.2). He constructed an injective application, known as the Nash Application, from the set of Nash Components of $V$ (see Section 2.2.1) to the set of Essential Divisors. Then the question raised was: Is the Nash application bijective?

Obtaining a complete answer to this question took many decades. In the year 2003 Ishii and Kollar (see [15]) showed the first example of a variety such that its Nash application is not surjective. It is worth mentioning that this example is a hypersurface of dimension 4. In the year 2012 Fernández de Bobadilla and Pe-Pereira (see [10]) gave an affirmative answer to the question for the case of surfaces over $\mathbb{C}$. Finally between the years 2012 and 2013 Johnson, Kollar, and de Fernex constructed examples of hypersurfaces of dimension 3 where the Nash application is not surjective (see [3], [18] and [19]). It is important to mention that many mathematicians worked hard on this problem, making valuable progress with respect to the problem. For example: [1], [3], [11], [13], [15], [16], [17], [22], [23], [25], [26], [29], [33], [36], [37], [38], [39], [40] etc. Unfortunately it is not possible to comment on and
cite all the existing works.

However, despite all the progress made, giving an exact description of the image of the Nash application remains an open problem.

Informally speaking, hidden within the Spaces of $m$-jets and the Space of arcs lies much information on geometry of the subjacent variety. For example:

In the year 1995, Kontsevich, using these spaces, introduced the motivic integration to resolve the Batyrev conjecture on the Calabi-Yau varieties (see [20]). For more references on motivic integration see [1], [5], [6] and [27].

Other examples are found in the articles [8], [31] written by Ein and Mustaţă where, amongst other things, it is demonstrated that if $V$ is locally a complete intersection, then $V$ only has rational singularities (resp. log-canonical singularities) if and only if $V_m$ is irreducible (resp. equidimensional) for all $m \geq 0$.

Another interesting application for the Spaces of $m$-jets and the Space of arcs is obtaining identities and invariants associated to $V$, for example see the articles [2], and [5].

One of the main subjects of [23] was the study of the following problem: When a “deformation” of the variety $V$ induces a “natural deformation” of the Spaces of $m$-jets $V_m$ (for greater precision see Section 3.1). In general a deformation of $V$ does not induce a deformation of the spaces of $V_m$, see Example 1, however there exist important families of examples in which this property is satisfied. For example if $V$ is locally a complete intersection, and only has an isolated singularity of log-canonical type (see Proposition 3.1).

The idea to study the Spaces of $m$-jets in families is very natural, for example Mourtada in [30] studies families of complex plane branches with constant topological type, and obtains formulas for the calculation of the number and dimension of the irreducible components of the Spaces of $m$-jets.

In [23] it was obtained that if $V$ is a Pham-Briekson hypersurface defined by an irreducible polynomial $f$, and $W \to \text{Spec} \mathbb{K}[[s]]$ is a $\Gamma(f)$-deformation (see Section 3.3) then $W$ induces a deformation of $(V_m)_\text{red}$ (for more details see Theorem 3.2). In this article we will generalize this result.

Let $V \subset \mathbb{A}_{\mathbb{K}}^{n+1}$ be a hypersurface, and $S$ is a finite type $\mathbb{K}$-scheme. In this article we will consider an Embedded deformation of $V \subset \mathbb{A}_{\mathbb{K}}^{n+1}$ over $S$ which admits a Simultaneous Embedded Resolution (see Section 3.2), and we prove that this deformation induces a deformation of the scheme $(V_m)_\text{red}$ (see Theorem 3.8).

An example of an Embedded Deformation which admits a Simultaneous Embedded Resolution is the following: Let $V \subset \mathbb{A}_{\mathbb{K}}^{n+1}$ be the hypersurface defined by an irreducible polynomial $f$ non degenerate with respect to the Newton boundary $\Gamma(f)$. In addition we will suppose that $V$ has at most
one isolated singularity in the ‘origin’ of $\mathbb{A}^{n+1}_K$, and that $V$ does not contain orbits of the torus $T := (\mathbb{K}^*)^{n+1}$ of dimension greater than or equal to 1. So a $\Gamma(f)$-deformation of $V$ admits a Simultaneous Embedded Resolution.

It is worth mentioning that under certain conditions the deformations that we will consider have a property to preserve the topological type of $V$ (see [21], [34] and [42]).

The article is organized as follows:

In the section Preliminaries and Reminders we will briefly present the known definitions and results of the Hasse-Schmidt derivations and the Space of $m$-jets and the Space of arcs relatives to a given scheme. We will finish this section with a brief introduction to the Nash problem. The purpose of this section is to make reading the article easier.

The main results of this article will be developed in the section Deformation of Spaces of $m$-jets. In Section 3.1 we will briefly explain the problem we will study. Then, in Section 3.2 we will give the general hypothesis of the principal result of this article and we will show that the $\Gamma(f)$-deformations satisfy them. Finally, in Section 3.4 we will prove the principal result of this article. We will finish the section with an application to the Nash problem.

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2. Preliminaries and Reminders

2.1. Hasse-Schmidt Derivations. In this section we give some results and notions based on the Hasse-Schmidt derivations, the Relative Space of $m$-jets, and the Relative Space of arcs. Here our goal is not to give an exhaustive overview of the theory, but to give an overview of the latter in the context which interests us. For more details, see [15].

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and $P$ a $\mathbb{K}$-algebra and $R,Q$ two $P$-algebras.

A Hasse-Schmidt derivation of order $m \in \mathbb{Z}_{\geq 0}$ from $R$ to $Q$ is an $m + 1$-tuple $(D_0, \cdots, D_m)$ where $D_0 : R \to Q$ is a homomorphism of $P$-algebra, and $D_i : R \to Q$, $1 \leq i \leq m$ is a homomorphism of abelian groups, which satisfies the following properties:

(i) $D_i(p) = 0$ for all $p \in P$ and for all $1 \leq i \leq m$;
(ii) For all the elements $x$ and $y$ of $R$ and for all integers $1 \leq k \leq m$, we have:
\[ D_k(xy) = \sum_{i+j=k} D_i(x)D_j(y). \]

We remark that if \( \phi : Q \to Q' \) is a homomorphism of \( P \)-algebra, and \((D_0, \cdots, D_m)\) a Hasse-Schmidt derivation from \( R \) to \( Q \), then \((\phi \circ D_0, \cdots, \phi \circ D_m)\) is a Hasse Schmidt derivation from \( R \) to \( Q' \). Consequently we can define the following functor:

\[ \text{Der}^m_P(R, \cdot) : P-\text{Alg} \to \text{Ens}; Q \mapsto \text{Der}^m_P(R, Q) \]

where \( P-\text{Alg} \) (resp. \( \text{Ens} \)) is the category of \( P \)-algebras (resp. of sets) and \( \text{Der}^m_P(R, Q) \) is the the set of Hasse-Schmidt derivations of the order \( m \) of \( R \) to \( Q \).

In the same way, we can define the Hasse-Schmidt derivation of order \( \infty \), and the functor \( \text{Der}^\infty_P(R, \cdot) \) by using, in place of \( m+1 \)-tuple \((D_0, \cdots, D_m)\), an infinite sequence \( D_0, D_1, \cdots \).

Let \( \text{HS}^m_{R/P} \) be the \( R \)-algebra quotient of the algebra \( \text{Sym}\left( \bigoplus_{i=1}^{m} Rd^i x \right) \) by the ideal \( I \) generated by the union of the following sets:

- \( \{d^i(x + y) - d^i(x) - d^i(y) \mid x, y \in R; 1 \leq i \leq m \} \),
- \( \{d^i(s) \mid s \in P; 1 \leq i \leq m \} \),
- \( \{d^k(xy) - \sum_{i+j=k} d^i(x)d^j(y) \mid x, y \in R; 1 \leq k \leq m \} \).

We also remark that \( \text{HS}^m_{R/P} \) is a graduated algebra by the graduation \( d^i x \mapsto i \). According to the definition of the algebra \( \text{HS}^m_{R/P} \), we naturally obtain a sequence of homomorphisms of graduated algebras:

\[ R := \text{HS}^0_{R/P} \to \text{HS}^1_{R/P} \to \cdots \to \text{HS}^m_{R/P} \to \text{HS}^{m+1}_{R/P} \to \cdots \]

We note \( \text{HS}^\infty_{R/P} \) the inductive limit of the inductive system \( \text{HS}^m_{R/P} \), which is to say:

\[ \text{HS}^\infty_{R/P} := \lim_{m \to \infty} \text{HS}^m_{R/P}. \]

**Proposition 2.1** \([13]\). The Hasse-Schmidt algebra \( \text{HS}^m_{R/P}, 0 \leq m \leq \infty \), represents the functor of Hasse-Schmidt derivations \( \text{Der}^m_P(R, \cdot) \), which is to say:

\[ \text{Der}^m_P(R, Q) \cong \text{Hom}_P(\text{HS}^m_{R/P}, Q), \]

where \( Q \) is any \( P \)-algebra.

The Hasse-Schmidt algebra \( \text{HS}^m_{R/P}, 0 \leq m \leq \infty \) satisfies the following properties of localization:

**Proposition 2.2** \([13]\). Let \( 0 \leq m \leq \infty \) and consider a multiplicative subset \( L \) (resp. \( L_1 \)) of the algebra \( R \) (resp. \( P \)). Then we have:

(1) \( \text{HS}^m_{R/P}[L^{-1}] \cong \text{HS}^m_{R[L^{-1}]/P} \).
(2) If the morphism \( P \to R \) factors through the canonical morphism \( P \to P[L_{1}^{-1}] \), then \( \text{HS}_{R/P}^{m} \cong \text{HS}_{R/P[L_{1}^{-1}]}^{m} \).

Let \( f : X \to Y \) be a morphism of schemes. Taking a covering by affine open subsets of \( X \) compatible with a covering by affine open subsets of \( Y \), and by using the above localization properties, we can define a quasi-coherent sheaf \( \text{HS}_{\mathcal{O}_{X}/\mathcal{O}_{Y}}, 1 \leq m \leq \infty \), of \( \mathcal{O}_{X} \)-algebras that does not depend on the choice of the open coverings.

**Definition 2.3.** For each \( 0 \leq m \leq \infty \), we place:

\[
X(Y)_{m} := \text{Spec} \text{HS}_{\mathcal{O}_{X}/\mathcal{O}_{Y}}^{m},
\]

where \( \text{Spec} \text{HS}_{\mathcal{O}_{X}/\mathcal{O}_{Y}}^{m} \) is the relative spectrum of the quasi-coherent sheaf of \( \mathcal{O}_{X} \)-algebras \( \text{HS}_{\mathcal{O}_{X}/\mathcal{O}_{Y}}^{m} \). If \( Y := \text{Spec} \mathbb{K} \), we place \( X_{m} := X(\text{Spec} \mathbb{K})_{m} \).

For \( 0 \leq m < \infty \) (resp. \( m = \infty \)), the scheme \( X(Y)_{m} \) (resp. \( X(Y)_{\infty} \)) is called the Space of \( m \)-jets (resp. Space of arcs) of the morphisms \( f : X \to Y \).

The following results justify the terminology “Space of \( m \)-jets and Space of arcs of the morphism \( f : X \to Y \)”

We remark that we can define the following applications:

\[
\begin{align*}
\text{Der}_{P}^{m}(R, Q) & \to \text{Hom}_{P}(R, Q[t]/(t^{m+1})); \\
(D_{0}, \ldots, D_{m}) & \to \left\{ R \to Q[t]/(t^{m+1}); x \mapsto D_{0}(x) + \cdots + D_{m}(x)t^{m} \right\} \\
\text{Der}_{P}^{\infty}(R, Q) & \to \text{Hom}_{P}(R, Q[[t]]); \\
D_{0}, D_{1}, \cdots & \to \left\{ R \to Q[[t]]; x \mapsto D_{0}(x) + D_{1}(x)t + \cdots \right\}.
\end{align*}
\]

Using the above applications we can demonstrate the following result:

**Proposition 2.4.** The functor \( Q \mapsto \text{Der}_{P}^{m}(R, Q) \), \( 0 \leq m < \infty \), (resp. \( Q \mapsto \text{Der}_{P}^{\infty}(R, Q) \)) is isomorphic to the functor \( Q \mapsto \text{Hom}_{P}(R, Q[t]/(t^{m+1})) \) (resp. \( Q \mapsto \text{Hom}_{P}(R, Q[[t]]) \)).

The following theorems are a consequence of Propositions 2.1 and 2.3.

**Theorem 2.5.** Let \( f : X \to Y \) be a morphism of schemes. Then the Space of \( m \)-jets of \( f : X \to Y \) represents the functor of \( m \)-jets, which is to say:

\[
\text{Hom}_{Y}(Z \times_{\mathbb{K}} \text{Spec} \mathbb{K}[t]/(t^{m+1}), X) \cong \text{Hom}_{Y}(Z, X(Y)_{m}),
\]

where \( Z \) is any \( Y \)-scheme.

**Theorem 2.6.** Let \( f : X \to Y \) be a morphism of schemes. Then the Space of arcs of \( f : X \to Y \) represents the functor of arcs, which is to say:

\[
\text{Hom}_{Y}(Z \times_{\mathbb{K}} \text{Spec} \mathbb{K}[t], X) \cong \text{Hom}_{Y}(Z, X(Y)_{\infty}),
\]

where \( Z \) is any \( Y \)-scheme, and \( Z \times_{\mathbb{K}} \text{Spec} \mathbb{K}[t] \) is the formal completion of the scheme \( Z \times_{\mathbb{K}} \text{Spec} \mathbb{K}[t] \) along the subscheme \( Z \times_{\mathbb{K}} \text{Spec} \mathbb{K} \).

The Spaces of \( m \)-jets and the Space of arcs of a morphism satisfy the following base extension property:
Proposition 2.7. Let $f : X \to Y$ and $Y' \to Y$ be two morphisms of schemes, and we consider $X' := X \times_Y Y'$. Then $X'(Y')_m \cong X(Y)_m \times_Y Y'$, over $Y'$ for all $0 \leq m \leq \infty$. In particular there exists a commutative diagram:

\[
\begin{array}{ccc}
X'(Y')_m & \longrightarrow & X(Y)_m \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
\]

The symbol $\square$ of the diagram is in order to explicitly indicate that the commutative diagram is a Cartesian square.

2.2. The Space of arcs and the Nash Problem. In this section we will give the basic definitions of the Nash problem.

Let $K$ be an algebraically closed field of characteristic zero, $V$ a normal algebraic variety over $K$, and $\pi : X \to V$ a divisorial desingularization of $V$.

2.2.1. The Nash Components. We recall that $V_\infty$ is the Arcs Space over $V$, and that the $K$-points of $V_\infty$ are in bijective correspondence with the $K$-arcs over $V$ where $K$ is a field extension of $K$. Let $K_\alpha$ be the residual field of the point $\alpha \in V_\infty$. By abuse of notation we also note $\alpha$ the $K_\alpha$-arc which corresponds to the point $\alpha$. Let $p : V_\infty \to V$ be the canonical projection $\alpha \mapsto \alpha(0)$ where $0$ is the closed point of $\text{Spec } K_\alpha[[t]]$. We remark that $p$ is a morphism. The irreducible components of $V_\infty^\alpha := p^{-1}(\text{Sing } V)$ are called the Nash Components of $V$. We note $CN(V)$ the set of Nash Components of $V$.

2.2.2. The Essential Divisors. Given any desingularization $\pi' : X' \to V$, the birational application $(\pi')^{-1} \circ \pi : X \dashrightarrow X'$ is well defined in codimension 1 ($X$ is a normal variety). If $E$ is an irreducible component of the exceptional fiber of $\pi$, then there exists an open subset $E^0$ of $E$ over which the application $(\pi')^{-1} \circ \pi$ is well defined.

The divisor $E$ is called Essential Divisor over $V$ if for all desingularization $\pi'$ the adherence $(\pi')^{-1} \circ p(E^0)$ is an irreducible component of $(\pi')^{-1}(\text{Sing } V)$, where Sing $V$ is the singular locus of $V$. We note that $\text{Ess}(V)$ the set of essential divisors over $V$.

2.2.3. The Nash Application. Given $C \in CN(V)$, let $\alpha_C$ be the generic point of $C$. Nash demonstrated that the application $N_V : CN(V) \to \text{Ess}(V)$ which associates to $C \in CN(V)$ the adherence $\{\hat{\alpha}_C(0)\}$ is a well defined injective application (see [32]), where $\hat{\alpha}_C$ is the lifting to $X$ of the generic point $\alpha_C$, which is to say $\pi \circ \hat{\alpha}_C = \alpha_C$. The Nash problem consists of the study of the surjectivity of $N_V$.

In the case of surfaces, the problem remained open until the year 2012, when the authors of the article [10] demonstrated that the Nash application is bijective for all singularities of surfaces over $\mathbb{C}$. In the 2003 article [12], the authors discovered the first example of a $V$ variety such that the Nash Application $N_V$ is not bijective; this variety is a hypersurface of $\mathbb{A}_K^5$ having a unique isolated singularity. The examples of varieties of dimension three where the Nash application is not bijective appeared during the years 2012
and 2013 (see the articles [19], [3]); these examples are hypersurfaces of \( \mathbb{A}^4_\mathbb{K} \) having a unique isolated singularity. We can find more examples, for dimensions greater than or equal to three, in the article [18].

3. DEFORMATION OF SPACES OF M-JETS

3.1. Definition of the main problem. Let \( \mathbb{K} \) be an algebraically closed field of characteristic zero, \( V \) a variety over \( \mathbb{K} \), \( S \) a \( \mathbb{K} \)-scheme, and \( 0 \in S \) a closed point. Let us consider a deformation \( W \) of \( V \) over \( S \), which is to say a commutative diagram (Cartesian square):

\[
\begin{array}{ccc}
V & \longrightarrow & W \\
\downarrow & & \downarrow \\
0 & \longrightarrow & S
\end{array}
\]

where \( W \to S \) is flat, and \( V \cong W \times_S \{0\} \). Using Proposition 2.7 we obtain the following commutative diagram:

\[
\begin{array}{ccc}
V_m & \longrightarrow & W(S)_m \\
\downarrow & & \downarrow \\
0 & \longrightarrow & S
\end{array}
\]

The question we ask ourselves is: when is this diagram a deformation? In other words, When is the morphism \( W(S)_m \to S \) flat?

In general the morphism \( W(S)_m \to S \) is not flat.

Example 1. Let \( V \subset \mathbb{A}^3_\mathbb{C} \) be the hypersurface given by the equation \( x^4+y^4+z^4 = 0 \), and \( W \) the hypersurface of \( \mathbb{A}^3_\mathbb{C} \times \mathbb{A}^0_\mathbb{C} \) given by the equation \( x^4+y^4+z^4+s = 0 \). It is not difficult to verify that the morphism \( W \to \mathbb{A}^0_\mathbb{C}; (x, y, z) \mapsto s \) is a deformation of \( V \).

Let \( p \in \mathbb{A}^0_\mathbb{C} - \{0\} \), and \( W_p := W \times_S \{p\} \). The fiber \( W_p \) is a smooth variety, then the Space of 3-jets \( (W_p)_3 \) is irreducible. But the space of 3-jets \( V_3 \) is not equidimensional, which implies that \( W(S)_3 \to S \) is not flat.

The following result is a direct consequence of the results of Ein and Mustaţă stated in the introduction, and the local criterion for flatness. For more details see [23]:

**Proposition 3.1.** Let \( V \) be a locally complete intersection variety. We suppose that \( V \) has at most one isolated singularity of log-canonical type. Then the morphism \( W(S)_m \to S \) is a deformation of \( V_m \) for all \( 0 \leq m \leq \infty \).

In this article we will generalize the following result.

Let \( V \subset \mathbb{A}^n_\mathbb{K} \) be the hypersurface given by polynomial \( f \in \mathbb{K}[x_1, ..., x_n] \) and \( W \) the hypersurface given by formal power series \( F = f + \sum s^j g_j, g_j \in \mathbb{K}[x_1, ..., x_n] \). It is not difficult to verify that the morphism \( W \to S := \text{Spec} \mathbb{K}[s]; (x_1, ..., x_n, s) \mapsto s \) is flat. We proved in [23] the following result:

**Theorem 3.2 ([23]).** Let \( f := x_1^{a_1} + x_2^{a_2} + \cdots + x_k^{a_k} + \cdots + x_n^{a_n}, a_k > 1 \), and we suppose that the polynomials \( g_i \), \( i \geq 1 \), belong to the integral closure of the ideal generated by \( x_1^{a_1}, x_2^{a_2}, \cdots, x_n^{a_n} \). Then the morphism \( p_m : (W(S)_m)_\text{red} \to \)
whose corresponding toric variety is the normalized blowing up of the ideal \( \Gamma(\star) \) over \( V(\Gamma(f)) \). We suppose that \( W \) is an Embedded Deformation of \( V \) over \( S := \text{Spec} A \), that is we have the following commutative diagram:

\[
\begin{array}{ccc}
V & \xrightarrow{\varphi} & W'
z & \mapsto & t \\
\downarrow & & \downarrow \\
0 := \text{Spec} K & \xrightarrow{\theta} & S\\
\end{array}
\]

where the morphism \( \theta \) is flat.

In the following we will define what we mean by Simultaneous Embedded Resolution of an Embedded Deformation.

We consider a proper birational morphism \( \varphi : \tilde{A}_{S}^{n+1} \rightarrow A_{S}^{n+1} \) such that \( \tilde{A}_{S}^{n+1} \) is formally smooth over \( S \), and we note for \( \tilde{W}^{s} \) and \( \tilde{W}^{t} \) the strict and total transform of \( W \) in \( \tilde{A}_{S}^{n+1} \) respectively. We will say that the Embedded Deformation of \( V \) over \( S \) admits a Simultaneous Embedded Resolution if the morphisms \( \tilde{W}^{s} \rightarrow W \) is a very weak simultaneous resolution and \( \tilde{W}^{t} \) is a Normal Crossing Divisor Relative to \( S \), that is to say, the induced morphism \( \tilde{W}^{t} \rightarrow S \) is flat and for each \( p \in \tilde{W}^{t} \) there exists a Zariski open affine neighborhood \( U \subset \tilde{A}_{S}^{n+1} \) of \( p \), and an étale morphism \( \phi \).

\[
U \xrightarrow{\phi} \tilde{A}_{S}^{n+1} := \text{Spec} A[y_{1}, \ldots, y_{n+1}] \\
\downarrow \\
S
\]

such that the \( S \)-scheme \( \tilde{W}^{t} \cap U \) is defined by the ideal \( \phi^{*}\mathcal{I} \), where \( \mathcal{I} = (y_{a_{1}}^{1} \cdots y_{n+1}^{a_{n+1}}) \), \( a_{i} \geq 0 \). If \( p \in \tilde{W}^{s} \), we assume that \( a_{n+1} = 1 \), and that \( \tilde{W}^{s} \cap U \) is defined by the ideal \( \phi^{*}\mathcal{I}' \), where \( \mathcal{I}' = (y_{n+1}) \).

3.3. \( \Gamma(f) \)-Deformations. Now we will show an important class of examples which satisfy the hypothesis imposed above.

By abuse of notation we will note for 0 the point of \( A_{K}^{n+1} := \text{Spec} K[x_{1}, \ldots, x_{n+1}] \) corresponding to the maximal ideal \( (x_{1}, \ldots, x_{n+1}) \). Sometimes we will call this point the origin of \( A_{K}^{n+1} \).

Let \( V \) be a hypersurface of \( A_{K}^{n+1} \) defined by an irreducible polynomial \( f = \sum c_{e}x^{e} \in K[x_{1}, \ldots, x_{n+1}] \), where \( x^{e} := x_{1}^{e_{1}} \cdots x_{n+1}^{e_{n+1}} \) and \( c_{e} \in K \), and \( \mathcal{E}(f) := \{e \in \mathbb{Z}_{\geq 0}^{n+1} | c_{e} \neq 0\} \). The Newton polyhedron \( \Gamma_{+}(f) \) is the convex hull of the set \( \{e \in \mathbb{R}_{\geq 0}^{n+1} | e \in \mathcal{E}(f)\} \), and the Newton boundary \( \Gamma(f) \) of \( f \) is the union of compact faces of \( \Gamma_{+}(f) \). The Newton fan \( \Gamma^{*}(f) \) associated with \( f \) is the subdivision of the standard cone \( \Delta := \mathbb{R}_{\geq 0}^{n+1} \), whose corresponding toric variety is the normalized blowing up of the ideal.
The polynomial \( f \) is non-degenerate with respect to the Newton boundary if for each compact face \( \gamma \) of \( \Gamma_+(f) \), the polynomial \( f_\gamma := \sum_{e \in \gamma} c_e x^e \) is non-singular in the torus \( T := (\mathbb{R}^*)^{n+1} \).

We define the support function associated with \( \Gamma_+(f) \) as follows:

\[
\mathcal{I}(f) := \{ x^e \mid e \in \Gamma(f) \cap \mathbb{Z}^{N+1} \}.
\]

It is known that the Newton fan \( \Gamma^+(f) \) satisfies the following property (see [43] or [28]):

\[
(\ast) \text{ Let } J \subset \{1, \ldots, n+1\} \text{ and } \sigma_J := \{ (p_1, \ldots, p_{n+1}) \in \Delta \mid p_i = 0 \text{ iff } i \neq J \}. \text{ If there exists } p \in \sigma_J \text{ such that } \mathcal{I}(p) = 0, \text{ then the adherence of } \sigma_J \text{ is a cone of } \Gamma^+(f).
\]

A regular subdivision \( \Sigma \) of \( \Gamma^+(f) \) is called \textit{admissible} if it satisfies the property \((\ast)\), in other words, if there exists \( p \in \sigma_J \) such that \( \mathcal{I}(p) = 0 \), then \( \sigma_J \in \Sigma \). The existence of the regular admissible subdivisions, under the assumption that \( f \) is non-degenerate with respect to the Newton Boundary, was proved in [33].

For the rest of this section we will assume that \( f \) is non-degenerate with respect to the Newton Boundary, and that \( \Sigma_\ell \) is a regular admissible subdivision. We suppose that the origin 0 of \( \mathbb{A}^{n+1}_K \) is the unique singular point of \( V \), and that \( V \) does not contain \( T \)-orbits of a strictly positive dimension. Then the toric morphism \( \pi : X(\Sigma_\ell) \to \mathbb{A}^{n+1}_K \) is an embedded resolution of the singularity of the hypersurface \( V \), which is to say \( V_{\text{red}}^\ell \) is a simple normal crossing divisor. Observe that the irreducible components of \( V_{\text{red}}^\ell \) are smooth varieties, in particular \( V^\ell \) is a smooth variety. For more details, see [43].

It is worth noting that the property mentioned above only depends on the Newton polyhedron \( \Gamma_+(f) \), and on the fact that the polynomial \( f \) is non-degenerate with respect to \( \Gamma(f) \), which is to say if \( V' \) is a hypersurface, with a unique singular point in 0, given by the polynomial \( g \), non degenerate with respect to \( \Gamma(g) \) such that \( \Gamma_+(f) = \Gamma_+(g) \), then \( \pi : X(\Sigma_\ell) \to \mathbb{A}^{n+1}_K \) is an embedded resolution of the singularity \( V' \).

Let \( A \) be the localization of the ring of polynomials \( \mathbb{K}[s_1, \ldots, s_l] \) at the maximal ideal \( (s_1, \ldots, s_l) \). Consider the following polynomial.

\[
F := f(x_1, \ldots, x_{n+1}) + \sum_{j=1}^l s_j g_j(x_1, \ldots, x_{n+1})
\]

where \( \Gamma_+(g_j) \subseteq \Gamma_+(f) \) for all \( j \in \{1, \ldots, l\} \). Let \( W \) be the hypersurface \( \mathbb{A}^{n+1}_S, S := \text{Spec } A \), defined by the polynomial \( F \). Clearly \( W \) is an embedded deformation of \( V \) over \( S \). Let \( \text{Id} : S \to S \) be the identity morphism and let us consider the following proper birational morphism:
This morphism is a Simultaneous embedded Resolution of $W$.

3.4. **Deformation of Spaces of $m$-jets.** Let $V$ be a hypersurface of $\mathbb{A}_S^{n+1}$ and we suppose that $W$ is an Embedded Deformation of $V$ over $S := \text{Spec} A$.

We consider a proper birational morphism $\varphi : \tilde{\mathbb{A}}_{S}^{n+1} \rightarrow \mathbb{A}_{S}^{n+1}$ such that $\tilde{\mathbb{A}}_{S}^{n+1}$ is formally smooth over $S$, and we note for $\tilde{W}$ and $W$ the strict and total transform of $W$ in $\tilde{\mathbb{A}}_{S}^{n+1}$ respectively.

Initially we do not suppose that $\varphi$ is a simultaneous embedded resolution of $W$

**Proposition 3.3.** The morphism $\varphi_m : (\tilde{W}(S)_m)_{\text{red}} \rightarrow (W(S)_m)_{\text{red}}$ is dominant for all $0 \leq m \leq \infty$.

**Remark 1.** If the scheme $W(S)_m$ did not have Embedded Components, the same method of demonstration that we will use in the proof of the proposition could be used to prove that the morphism $\varphi_m : W(S)_m \rightarrow W(S)_m$ is schematically dominant.

The following corollary follows directly from Proposition 3.3.

**Corollary 3.4.** For each $0 \leq m \leq \infty$, we have $\sharp(V_m) \leq \sharp(V^t_m)$.

**Remark 2.** If we explicitly give a hypersurface $V$ and an embedded resolution of $V$, we can use the results of [12] to bound the value of $\sharp(V_m)$. In general this corollary does not give optimal bounds, for example if the singularity of $V$ is rational, $\sharp(V_m) = 1$, while $\sharp(V^t_m)$ can be large.

**Proof of Proposition 3.3.** Let us consider the following commutative diagram of $S$-morphisms:

\[
\begin{array}{ccc}
\tilde{W}^t & \rightarrow & \tilde{\mathbb{A}}_{S}^{n+1} \\
\downarrow & & \downarrow \\
W^t & \rightarrow & \mathbb{A}_{S}^{n+1}
\end{array}
\]

This diagram naturally induces the following commutative diagram:

\[
\begin{array}{ccc}
(\tilde{W}^t(S)_m)_{\text{red}} & \rightarrow & \tilde{\mathbb{A}}_{S}^{n+1}(S)_m \\
\downarrow & & \downarrow \\
(W(S)_m)_{\text{red}} & \rightarrow & \mathbb{A}_{S}^{n+1}(S)_m
\end{array}
\]

Let $\alpha$ be an associated point of $(W(S)_m)_{\text{red}}$, and let $\kappa(\alpha)$ be the residual field of the point $\alpha$. The functorial property of the Spaces of $m$-jets (see Theorem 2.5) tells us that an $m$-jet corresponds to the point $\alpha$. By abuse of notation we will note this $m$-jet for $\alpha$.

\[\alpha : \text{Spec } \kappa(\alpha)[t]/(t^{m+1}) \rightarrow W.\]
By abuse of notation, we note for $\alpha$ the $m$-jet $i \circ \alpha : \text{Spec} \kappa(\alpha)[t]/(t^{m+1}) \to \mathbb{A}^{n+1}_S$. As $\mathbb{A}^{n+1}_S$ is formally smooth over $S$, the $S$-morphism

$$p_m : \mathbb{A}^{n+1}_S(S) \to \mathbb{A}^{n+1}_S(S)_m$$

is surjective. Let $\mathcal{I}(W)$ be the ideal of $W$. As $W$ is a hypersurface, there exists a polynomial $F \in A[x_1, \ldots, x_{n+1}]$ such that $\mathcal{I}(W) = (F)$. The morphism $p_m$ is surjective, then there exists an arc $\beta : \text{Spec} \kappa(\alpha)[[t]] \to \mathbb{A}^{n+1}_S$, such that $p_m(\beta) = \alpha$ and the ideal $\mathcal{I}(W)$ satisfies that

$$\text{Ord}_t \beta^* \mathcal{I}(W) := \min \{ \text{Ord}_t r \mid r \in \beta^* \mathcal{I}(W) \} \geq m + 1.$$

observe that $\text{Ord}_t \beta^* \mathcal{I}(W) = \text{Ord}_t F(\beta^*(t))$, where $\beta^*(t)$ is the comorphism of $\beta$.

Let $Z \subset \mathbb{A}^{n+1}_S$ be an $S$-scheme such that $\varphi : \varphi^{-1}(U) \to U$ is an isomorphism, where $U := \mathbb{A}^{n+1}_S - Z$. Without loss of generality we can suppose that $\beta \notin Z(S)_\infty$. As the morphism $\varphi : \tilde{\mathbb{A}}^{n+1}_S \to \mathbb{A}^{n+1}_S$ is proper, there exists $\tilde{\beta} \in \tilde{\mathbb{A}}^{n+1}_S(S)_\infty$ such that $\varphi \circ \tilde{\beta} = \beta$. Let $U' \subset \tilde{\mathbb{A}}^{n+1}_S$ be an affine open such that $\tilde{\beta}(0) \in U'$, then:

$$\tilde{\beta}^* \mathcal{I}(U' \cap \tilde{W}^t) \geq m + 1.$$

Let us consider the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{\mathbb{A}}^{n+1}_S(S)_\infty & \xleftarrow{p_m} & \tilde{\mathbb{A}}^{n+1}_S(S)_m \\
\downarrow & & \downarrow \\
\mathbb{A}^{n+1}_S(S)_\infty & \xleftarrow{p_m} & \mathbb{A}^{n+1}_S(S)_m
\end{array}
$$

We define $\tilde{\alpha} := p_m(\tilde{\beta})$. By construction we obtain that $\varphi_m(\tilde{\alpha}) = \alpha$.

As $\tilde{\beta}^* \mathcal{I}(U' \cap \tilde{W}^t) \geq m + 1$, we obtain that $\tilde{\alpha} \in \tilde{W}^t(S)_m$. This ends the proof. \qed

The following proposition is one of the key points of the principal resolution of this article.

**Proposition 3.5.** Let us fix $m \geq 0$, if the morphism $W^1(S)_m \to S$ is flat, then the morphism $\varrho_m : (W(S)_m)_\text{red} \to S$ is a deformation of the Space of $m$-jets $(V_m)_\text{red}$.

**Remark 3.** If the scheme $V_m$ does not have embedded components the method of proof of the theorem is valid to demonstrate that $\varrho_m : W(S)_m \to S$ is a deformation of $V_m$.

**Proof.** For the proposition it suffices to prove that the morphism $(W(S)_m)_\text{red} \to S$ is flat.

To simplify the proof we will use the following notations:

$$M(S) := (W(S)_m)_\text{red} \text{ and } L(S) := (\tilde{W}^1(S)_m)_\text{red}.$$
The flatness is a local property (see, for example, Proposition 2.13 from page 10 of [24], or Proposition 9.1A from page 253 of [14]), then it is enough to consider \( p \in (V_m)_{\text{red}} \) arbitrary, and to prove that the germ of a neighborhood of \( p \), \((M(S), p)\), is flat over \( S \).

By virtue of Proposition 3.3 we can choose \( q \in \varphi_m(p)^{-1} \) such that the morphisms of germs induced by \( \varphi_m \)

\[ \varphi_m : (L(S), q) \rightarrow (M(S), p) \]

is dominant.

The following lemma will be very useful in this proof. Consider the commutative diagram

\[
\begin{array}{ccc}
(L(S), q) & \xrightarrow{\varphi_m} & (M(S), p) \\
\downarrow & & \downarrow \\
S & & S
\end{array}
\]

**Lemma 3.6.** Assume that \( S \) is irreducible. Then each irreducible component of \((M(S), p)\) dominates \( S \).

**Proof.** For the Theorem 3.8 the morphism \((L(S), q) \rightarrow S\) is flat, which implies that each component irreducible of \((L(S), q)\) dominates \( S \) (see Lemma 3.7 from page 136 of [24]). Using the fact that the morphism is dominant (Proposition 3.7), and the commutative diagram \((\star)\), we obtain that each irreducible component of \((M(S), p)\) dominates \( S \). \( \square \)

We will begin with the case \( S := (\mathbb{A}_K, 0) \). As the scheme \((M(S), p)\) is reduced, the condition of flatness over \((\mathbb{A}_K, 0)\) is equivalent to each irreducible components of \((M(S), p)\) dominating \((\mathbb{A}_K, 0)\) (see Proposition 9.7 from page 257 of [14]). Which is obtained directly from the previous lemma.

Now we will suppose that the theorem is true for all schemes \((\mathbb{A}_K^{l_0}, 0)\) with \( l_0 \leq l - 1 \).

Let \( S := (\mathbb{A}_K^{l_0}, 0) \). To demonstrate this case we will use the Corollary 6.9 on page 170 of [2]. This is to say that we will prove that exists \( s \in \Gamma(\mathcal{O}_S) \) such that \( s \) is not a divisor of the zero in \( \Gamma(\mathcal{O}_{M(S), p}) \), and that \( \Gamma(\mathcal{O}_{M(S), p})/s\Gamma(\mathcal{O}_{M(S), p}) \) is flat over \( \Gamma(\mathcal{O}_S)/(s) \).

Let us consider the injective morphism \( K[s_1, ..., s_l]/m \rightarrow \Gamma(\mathcal{O}_{M(S), p}) \), where \( m := (s_1, ..., s_l) \). Let us suppose that there exists \( h \in \Gamma(\mathcal{O}_{M(S), p}) - \{0\} \) such that \( s_1 h = 0 \). This means that the open set \((M(S), p)_h\) is contained in the closed set \( g^{-1}_m(V(s_1)) \), contradicting Lemma 3.0. Then \( s_1 \) is not a divisor of the zero in \( \Gamma(\mathcal{O}_{M(S), p}) \).

Now let us consider the following commutative diagram:
Let us consider \( S \) a local \( \mathbb{K} \)-scheme of the finite type. Then there exists an integer \( l \) and a morphism \( S \to (A^l, 0) \). Using base extension, the Proposition 9.1.A of [14], and the Proposition 2.7, we obtain that the morphism \( W(S)_{\text{red}} \to S \) is flat. \( \square \)

The followign result is a direct consequence of of propositions 3.7 and 3.5.
Theorem 3.8. For each \( m \geq 0 \), the morphism \( \varrho_m : (W(S)_m)_{\text{red}} \to S \) is a deformation of the Space of \( m \)-jets \( (V_m)_{\text{red}} \).

The following result is an application of this theorem.

Let \( v := (v_1, \ldots, v_n) \in \mathbb{Z}_+^n \), \( d \in \mathbb{Z}_{>0} \) et let \( f \in \mathbb{K}[x_1, \ldots, x_n] \) be a quasi-homogeneous polynomial of the type \( (d, v) \), which is to say that \( f \) is homogeneous of degree \( d \) with respect to the graduation \( \nu_v x_i = v_i \). Let \( h \) belong to \( \mathbb{K}[x_1, \ldots, x_n] \) such that \( \nu_v h > d \). We note \( V \) (resp. \( V' \)) the hypersurface given by the equation \( f(x_1, \ldots, x_n) = 0 \) (resp. \( f(x_1, \ldots, x_n) + h(x_1, \ldots, x_n) = 0 \)). We suppose that \( V \) (resp. \( V' \)) has a unique isolated singularity at the origin of \( \mathbb{A}_K^n \), and that \( V \) (resp. \( V' \)) does not contain any of \( T \)-orbit of \( \mathbb{A}_K^n \). In addition we suppose that \( f \) and \( f_1 := f + h \) are non degenerate with respect to the Newton boundary \( \Gamma(f) \). We remark that the morphism \( \pi : X(\Sigma_t) \to \mathbb{A}_K^n \), where \( \Sigma_t \) is an admissible regular fan, is an embedded resolution of varieties \( V \) and \( V' \), and that the exceptional fibers of the desingularizations \( \pi_0 : X \to V \), \( \pi_1 : X' \to V' \) induced by the embedded resolution \( \pi : X(\Sigma_t) \to \mathbb{A}_K^n \) are trivially homeomorphic. In the following theorem, by abuse of notation, we use the same notation to designate the irreducible components of the exceptional fibers of \( \pi_0 \) and \( \pi_1 \). We proved in [23] the following result:

Theorem 3.9 ([23]). If the divisor \( E \) belongs to the image of the Nash Application \( \mathcal{N}_V \), then \( E \) belongs to the Nash Application \( \mathcal{N}_{V'} \).

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Instituto Matemática y Física, Universidad de Talca, Camino Lircay S\N, Campus Norte, Talca, Chile.

E-mail address: leyton@inst-mat.utalca.cl