An Iterative OLA Method for Inversion of Solar Spectropolarimetric Data. I. Single- and Multiple-variable Inversions of Thermodynamic Quantities

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Abstract

This paper describes an adaptation of the Optimally Localized Averaging (OLA) inversion technique, originally developed for geo- and helioseismological applications, to the interpretation of solar spectroscopic data. It focuses on inverting the thermodynamical properties of the solar atmosphere, assuming that the atmosphere and radiation field are in local thermodynamic equilibrium (LTE). We leave inversions of magnetic field and non-LTE inversions for future work. The advantage with the OLA method is that it computes solutions that are optimally depth resolved with minimal crosstalk error between variables. Additionally, the method allows for direct assessment of the vertical resolution of the inverted solutions. The primary challenges faced when adapting the method to spectroscopic inversions originate with the possible large-amplitude differences between the atmospheric model used to initiate the inversion and the underlying atmosphere it aims to recover, necessitating the development of an iterative scheme. Here, we describe the iterative OLA method we have developed for both single and multivariable inversions and demonstrate its performance on simulated data and synthesized spectra. We note that, when carrying out multivariable inversions, employing response function amplification factors can address the inherent spectral sensitivity bias that makes it hard to invert for less spectrally sensitive variables. The OLA method can, in most cases, reliably invert as well as or better than the frequently employed Stokes Inversion based on Response functions (SIR) scheme, but some difficulties remain. In particular, the method struggles to recover large-scale offsets in the atmospheric stratification. We propose future strategies to improve this aspect.

Unified Astronomy Thesaurus concepts: Radiative transfer (1335); Spectropolarimetry (1973); Helioseismology (1476)

1. Introduction

Solar physicists often rely on inversion methods to infer the physical properties of the solar atmosphere from spectroscopic (or spectropolarimetric) measurements. Photon interactions with the plasma alter their intensity and polarization state, allowing recovery of the properties of the atmosphere through which they have traveled. Depth-dependent inference of the physical properties is possible because the opacity changes with wavelength, and does so dramatically across spectral lines, providing varying sensitivity to different regions of the atmosphere.

Over the years, robust methods have been developed to model the spectral output of the solar plasma (the forward problem) or invert for the atmospheric properties from observed spectrum (the inverse problem). These employ a range of atmospheric conditions (Mihalas 1978; Del Toro Iniesta & Ruiz Cobo 1996; Socas-Navarro 2001; del Toro Iniesta 2003; Bellot Rubio 2006; Ramos et al. 2012; del Toro Iniesta & Ruiz Cobo 2016; de la Cruz Rodríguez & van Noort 2017); Milne–Eddington (e.g., Unno 1956; Harvey et al. 1972; Auer et al. 1977; Landolfi et al. 1984; Skumanich & Lites 1985, 1987; Orozco Suárez & Del Toro Iniesta 2007; Borrero & Kobel 2011; Centeno et al. 2014; Yadav et al. 2017), local thermodynamic equilibrium (LTE; e.g., Mihalas 1978; Ruiz Cobo & del Toro Iniesta 1992; Solanki et al. 1992; Frutiger & Solanki 1998; Frutiger et al. 2000; Socas-Navarro 2001; Gray 2005), and non-LTE (e.g., Mihalas 1978; Socas-Navarro et al. 2000; Asensio Ramos et al. 2008; Socas-Navarro et al. 2015; de la Cruz Rodríguez et al. 2016; Milic & van Noort 2017, 2018; de la Cruz Rodríguez et al. 2019; Ruiz Cobo et al. 2022). In general, inversion problems are underdetermined and one must carefully consider uniqueness. By underdetermined, we mean that, even if the number of equations in the system being solved is equal to or greater than the number of unknowns, the effective number of linearly independent equations is much smaller. Further, the observed intensity is a weighted integral of the physical properties of the atmosphere over a range of depths, and thus multiple inversion solutions are consistent with a given observed spectrum. Nonetheless, within the model assumptions, solar inversions have yielded valuable information about the morphology of sunspots (e.g., Collados et al. 1994; Solanki 2003; Orozco Suárez et al. 2005; Borrero & Ichimoto 2011; Borrero et al. 2011, 2019, 2021), properties of the quiet-Sun magnetism (e.g., Stenflo 1982; Trujillo Bueno et al. 2004; Orozco Suárez et al. 2007; Martínez González et al. 2008; Lites et al. 2008; Stenflo 2010; Bellot Rubio & Orozco Suárez 2019; Trelles Arjona et al. 2021), and the multithermal structure of the solar chromosphere (e.g., Asensio Ramos et al. 2008; Casini et al. 2009; Bueno 2010; Priest et al. 2018; Yadav et al. 2020; Anan et al. 2021; Morosin et al. 2022), as examples.
In this paper, we examine an inversion approach that has not previously been applied to the spectroscopic data. We invert synthetic spectra for the physical properties of a simulated solar atmosphere using the optimal localized averaging (OLA) technique. OLA was originally developed in the geoseismological context to invert for the Earth’s interior structure (Backus & Gilbert 1967, 1968, 1970) and has been used with great success in helioseismology to infer the interior structure and dynamics of the Sun based on its acoustic oscillations (Gough 1982, 1985; Jeffrey 1988; Christensen-Dalsgaard et al. 1990; Pijpers & Thompson 1992; Christensen-Dalsgaard & Thompson 1993; Pijpers & Thompson 1994; Basu 2016). The primary advantage of the OLA method lies in its ability to carefully constrain, given an observed spectrum, the depth dependence of the inverted solution and selectively invert where optimally localized, depth-averaged solutions can be obtained. Along with the inverted solution, the method provides a clear measure of the resolution achievable using a given spectrum and an assessment of the minimal smoothing required for a reliable solution at a given depth.

As a first step, this paper focuses on the use of OLA for the inversion of spectroscopic data (Stokes $I$) to determine the thermodynamic properties of the solar photosphere, temperature $T$, electronic pressure $P_e$, and the line-of-sight velocity $V_{los}$ as functions of depth, assuming that the spectrum is formed under conditions for which the LTE approximation is valid. Note that, throughout this paper, bold letters correspond to vectors (or matrices), so that $I$ indicates a vector of the intensity at wavelengths $\lambda$, while $T$, $P_e$, and $V_{los}$ represent vectors of the atmospheric properties as a function of optical depth $\tau$. Note also that, throughout this article, optical depth is evaluated in the continuum at 500 nm.

For spectroscopic inversions, adaptations of the OLA method are required to allow inversion for nonlinear perturbations (the initial guess atmospheric model on which the inversion is based may not be close to the underlying atmosphere it aims to recover). We discuss those adaptations and compare the inversion results we achieve to those obtained using the Stokes Inversion based on Response functions (SIR) inversion method (Ruiz Cobo & del Toro Iniesta 1992; Del Toro Iniesta & Ruiz Cobo 1996; del Toro Iniesta 2003; del Toro Iniesta & Ruiz Cobo 2016).

In Section 2, we provide a brief introduction to spectroscopic inversions, discuss the problems faced, and some limitations of current inversion approaches. In Sections 3 and 4, we describe the OLA method we have developed and its application to single and multivariable inversions. We then compare the OLA inversion output directly to the simulated atmospheres used to construct the synthetic spectrum and assess the method’s reliability. Finally, we summarize and discuss future prospects in Section 5.

2. Brief Introduction to Spectroscopic Inversions

To begin, we briefly review spectroscopic inversion strategy, defining the notational scheme we employ and focusing on those aspects of inversion that are critical to the development of both the single and multivariable OLA inversions we develop in Sections 3 and 4, respectively.

As an illustrative example, consider the single-variable inversion of spectroscopic data (Stokes $I$) for the plasma temperature $T$. In general, spectroscopic inversions are carried out by starting with an initial guess for $T$ and then solving for the difference $\Delta T$ between it and the actual (or underlying, observed) $T$ that accounts for the spectral difference $\Delta I$ between the observed spectra and that of the guess model. When the inversion is complete, if the difference between the spectra derived from the inverted model and that observed falls within some specified tolerance, the inverted model is taken to approximate the underlying atmosphere. However, due to the underdetermined nature of the problem, a good spectral fit does not guarantee recovery of the underlying atmosphere, and thus different atmospheric stratifications are consistent with a given observed spectrum (and spectral differences $\Delta I$).

A schematic of the canonical spectral inversion problem is shown in Figure 1. In the top left panel, the underlying $T$ profile is shown with the gray dashed curve, and the guess $T$ is shown with the red dashed curve. The difference $\Delta T$ is overplotted (with a different scaling) as red solid curve. Neither the underlying $T$ nor the difference $\Delta T$ are known ahead of the inversion, and it is the $\Delta T$ profile that the inversion aims to capture. The corresponding spectra and spectral differences (for the spectral lines used in this work, see Table 1), as synthesized using the SIR forward solver, are shown in the top right plot of the figure. The gray dashed spectrum is what would be observed if the true underlying atmosphere were to be observed, and the red dashed spectrum is the spectrum of the guess model atmosphere. The inversion itself aims to account for their difference (red solid curve).

One way to proceed is to linearize the relationship between $\Delta I$ and $\Delta T$ (e.g., del Toro Iniesta 2003). This relationship can be written as a first-order linear system of equations:

$$\Delta I = R_I^T \cdot \Delta T + \epsilon.$$  

When discretized in wavelength and optical depth, $R_I$ is a matrix containing the linear temperature response function at all optical depths and wavelengths under consideration (Landi Degl’Innocenti & Landi Degl’Innocenti 1977; Ruiz Cobo & del Toro Iniesta 1992). It is computed for the guess model, and $R_I^T$ in Equation (1) denotes its transpose. The temperature response function captures the spectral sensitivity $\delta I / \delta T$ at a given wavelength, to an infinitesimal change in temperature $\delta T$, at each optical depth, $R_I(r, \lambda) = \delta I / \delta T$. In Equation (1), $\epsilon$ captures both observational and instrumental errors and contributions from higher-order terms in the relationship between $\Delta I$ and $\Delta T$ (those not included in the linear response function $R_I(r, \lambda)$). When the guess model is far from the observed atmosphere, $\Delta T$ is not small, and the relationship between $\Delta I$ and $\Delta T$ is no longer linear. Under these conditions, the missing higher-order terms (order $(\Delta T)^2$ and higher) contribute significantly to $\epsilon$. The contribution of higher-order terms is indicated, for our example problem, by the yellow curve in the top right plot in Figure 1. In this work, we do not include any observational or instrumental noise, but their effect on the inversion solution is discussed in Section 2.2.7 of Agrawal (2021). We also ignore error due to discretization or numerical roundoff, as these are typically smaller than observational/instrumental noise.

We note that, in this paper, unlike in most previous applications, we employ fractional response functions, $R_T$, equal to the ratio of change in intensity at a given wavelength to the fractional change in guess model temperature at a given
depth. Thus, Equation (1) becomes

\[ \Delta I = R_T^\top \cdot \frac{\Delta T}{T} + \varepsilon. \]  

(2)

This removes the dimensional dependency of the spectral sensitivity to change in a given variable, facilitating comparison between the magnitudes of the response to different variables (see multivariable inversions in Section 4). For the test case shown, the fractional temperature response functions are plotted in the bottom left panel of Figure 1. Each curve plots the temperature response function for a different \( \lambda \) as a function of \( \log \tau \), as computed using the SIR solver.

Solving Equation (2) for \( \Delta T/T \) requires computing the inverse of the \( R_T \) matrix. This is challenging given that the system is ill-posed, as evident from the sharp decay of the singular values (Press et al. 2007, page 73) of the \( R_T \) matrix (bottom right plot in Figure 1). This reflects the limited linear independence of the response functions and thus the limited number of orthogonal modes available to represent \( R_T \). A full-rank matrix inverse for an ill-posed system has large-amplitude components that can amplify nonzero error \( \varepsilon \) and result in an error-dominated solution (Hansen 1990, 1994, 1998; Press et al. 2007). It is critical to minimize \( \varepsilon \) contributions so that the inverse solution largely reflects the underlying \( \Delta T/T \) difference rather than \( \varepsilon \) in Equation (2). To prevent \( \varepsilon \)-dominated solutions, we rely on a lower-rank pseudoinverse (Press et al. 2007, page 70) \( (R_T)^{-1} \), for which \( [R_T \cdot (R_T)^{-1}]^\top \) is not strictly an identity matrix. This yields an inversion solution that, at each depth, has contributions from other depths, and from other variables in multivariable inversions (see Section 4). In this paper, we refer to both of these as crosstalk errors.

We note that these errors are universal to all inversions, and is not particular to the OLA method we discuss in this paper. In general, it is not possible to minimize crosstalk error without amplifying the error due to nonlinearity or the observations, as eliminating crosstalk requires employing a larger-rank pseudoinverse matrix, while minimizing the contribution of \( \varepsilon \) requires a smaller-rank pseudoinverse. The goal is to regularize the solution so that it “optimally balances” these error contributions. Regularization, irrespective of the inversion scheme employed, effectively removes the smaller magnitude singular-value contributions to the pseudoinverse matrix (Hansen 1990, 1994; Golub & Van Loan 1996; Hansen 1998; Press et al. 2007). A strongly regularized solution (smaller rank) is typically one that is a smoother version of the underlying \( \Delta T/T \), i.e., one that is crosstalk dominated and can

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**Table 1**

List of Spectral Lines Used in This Work

| \( \lambda_0 \) (\( \AA \)) | Blends (if Any) |
|-----------------------------|----------------|
| Fe II (6147.743)            | Fe I (6147.835) |
| Fe II (6149.241)            | ...            |
| Ti I (6149.725)             | ...            |
| Fe II (6150.113)            | V I (6150.167)  |
| Fe I (6301.499)             | Fe I (6302.493) |
| Fe I (15648.515)            | Fe I (15647.423) |
| Fe I (15652.882)            | ...            |

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...
thus fail to recover sharp gradients that may be present. On the other hand, a more weakly regularized solution, obtained by keeping smaller singular values, is able to recover sharper gradients present, but is also more likely to be $\epsilon$ dominated and highly oscillatory with depth. Determining the appropriate degree of regularization (what rank inverse to employ) to optimally balance resolution (minimize crosstalk) and $\epsilon$ amplification is one of the most challenging aspects of inversions. This is because the spectral difference metric by which we assess how close the inverted solution is to the actual perturbation is degenerate. Different degrees of regularization can lead to significantly different solutions, all with similarly good fits to the observed spectrum.

Current state-of-the-art spectropolarimetric inversion codes, such as SIR, obtain a smooth solution by solving for the underlying perturbations at a limited number of user-defined depth locations called “nodes.” The solution for other depth points are interpolated using the inverted nodal values. Inverting at limited depths indirectly reduces the pseudoinverse matrix rank when solving Equation (2), and thus minimizes $\epsilon$ contributions. The resulting solutions are therefore typically dominated by crosstalk, especially in the shallower and deeper regions where response function sensitivities are less linearly independent. Importantly, the solutions do not typically correspond to the “most vertically resolved solution possible” given the set of spectral response functions available. Further, it is very hard, using this approach, to know a priori what gradients are recoverable given the observed spectrum.

The OLA method was developed in the geo- and helioseismic communities to address these difficulties. Over the remainder of this paper, we adapt and apply the method to spectropolarimetric inversions, “optimally” minimizing crosstalk and $\epsilon$ error contributions to the inverted solution at each depth. This allows solutions at the vertical-resolution limit, given observations of specific spectral content.

### 3. OLA Single-variable Inversion

In this section, we describe the OLA methodology for single-variable $\Delta T/T$ inversions using an artificially constructed test case. The test case adds a depth-dependent temperature perturbation $\Delta T$ to the horizontal mean state of a MURaM simulation (Vögler et al. 2005; Rempel 2014) to construct the underlying $T$ model. Starting with the mean MURaM model as the initial guess, we conduct an OLA inversion for $\Delta T/T$ and compare the deduced $\Delta T$ with that imposed. This allows an assessment of the capabilities and limitations of the OLA method, and it illustrates the implementation changes necessary in the spectropolarimetric context. Since we focus on inverting the spectral data for temperature perturbations in the absence of other variables, no perturbations are added to $P_e$ of the underlying mean MURaM model and $V_{loS}$ is taken to be zero.

Development of the OLA spectral inversion technique proceeds as follows: using the forward spectral synthesis capabilities of SIR, we compute the synthetic spectrum of the observed and the initial guess models (Figure 1, top right plot) and the fractional spectral response function matrix $R_T$ for the guess model (Figure 1, bottom left plot). The computed temperature response functions, along with the spectral difference $\Delta I$ between observed and guess spectra, define the first-order linear system of Equations (2), which must be solved for $\Delta T/T$. Once obtained, this inverted $\Delta T/T$ (or $\Delta T$) profile can be compared to the underlying perturbation for accuracy and reliability.

#### 3.1. Averaging Kernels

The OLA method is fundamentally different from the SIR inversion technique, which aims to find an overall smooth solution that best fits the observed spectrum by simultaneously solving for the underlying perturbation at all nodal depths. With OLA, inversion for the underlying perturbation proceeds one depth at a time, with the goal of finding the “best possible” solution at that depth, i.e., the solution that optimally minimizes contributions from both $\epsilon$ and from other depths (crosstalk). This is the optimally localized solution.

To obtain an OLA inversion at a target depth $\tau_i$, we first determine the coefficients $C = [c_{\lambda_1}, c_{\lambda_2}, c_{\lambda_3}, ...]$, with which we linearly combine response functions to construct an averaging kernel that best mimics a user-defined target function. A limited version of this idea has been applied previously, combining two response function to localize the Doppler-velocity response in the solar chromosphere (Wunnenberg et al. 2002). In the OLA method, all available response functions are linearly combined to produce the averaging kernel. Explicitly, the averaging kernel $A_T(\tau_i) = R_T \cdot C$, is obtained by solving the linear system of equations:

$$R_T \cdot C = G(\tau_i, \sigma),$$

for the coefficient vector $C$, which at any given depth $\tau_i$ is a function of wavelength only, with one $c_{\lambda_i}$ for each response function. In Equation (3), $G$ is a user-defined target function, usually taken to be a normalized Gaussian centered (and localized) at $\tau_i$. An example is shown in Figure 2. Plotted in pink is a Gaussian target function with peak at $\log \tau = -0.5$ and width $\sigma = 0.4$ (in $\Delta \log \tau$ units), and in blue the recovered $A_T(\tau_i)$ when solving Equation (3) with pseudoinverse matrix rank $k = 5$. The rank was determined based on dominant singular values of $R_T$, those that add up to 95% of the total sum.

Once computed, the coefficients $C$ can be used to obtain the inverted solution at depth $\tau_i$. Taking the dot product of Equation (2) with $C^\top$ and recognizing $R_T \cdot C$ as the averaging kernel $A_T(\tau_i)$ yields

$$C^\top \cdot \Delta I = A_T^\top(\tau_i) \cdot \frac{\Delta T}{T} + C^\top \cdot \epsilon.$$
Here, $C^\top \cdot \Delta I$ is the inversion solution at $\tau_i$, which, if the error contribution $C^\top \cdot \varepsilon$ is small, corresponds to an optical-depth-averaged value of the underlying $\Delta T/T$, defined as $(\Delta T/T)(\tau_i)$. The inverse solution for $\Delta T(\tau_i)$ is the product, guess $T(\tau_i) \times (C^\top \cdot \Delta I) \approx T(\tau_i) \times (\Delta T/T)(\tau_i)$. The narrower the averaging kernel, the closer $(\Delta T/T)(\tau_i)$ lies to the local $\Delta T/T$; if a $\delta$-function averaging kernel could be constructed, $(\Delta T/T)(\tau_i) \equiv \Delta T/T(\tau_i)$.

The minimum kernel width is, however, determined by the response function set available and the pseudoinverse matrix rank employed. Equations (3), like Equations (2), represent an ill-posed system and can only be solved approximately. Constructing a narrower $A_T$ requires employing larger-rank $k$, which typically results in larger-amplitude terms in $C$. These larger-amplitude terms can then amplify $\varepsilon$ through $C^\top \cdot \varepsilon$ in Equation (4) and lead to an inverted solution $C^\top \cdot \Delta I$ that is $\varepsilon$ dominated. The limited rank choice limits the number of orthogonal modes available for kernel construction. Thus, with limited orthogonal sensitivity, it is not possible to construct $\delta$-function kernels at any particular depth, and there will likely be crosstalk error from other depths and from other variables in multivariable inversions. Note that, while there is an upper limit to the magnitude of $A_T(\tau_i)^\top \cdot \Delta T$, which depends on the magnitude of underlying $\Delta T/T$, no such limit exists for $C^\top \cdot \varepsilon$. Further, in real-world applications, the error contribution $C^\top \cdot \varepsilon$ cannot be evaluated, as it depends on the underlying $\Delta T/T$, which is unknown.

In summary, when inverting at depth $\tau_i$, the goal is to construct as narrow an averaging kernel as possible while preventing the error due to nonlinearity and observation error ($\varepsilon$ in Equation (4)) from dominating the solution. The kernel width is limited by the linear independence of the response functions included in $R_T$ and the pseudoinverse matrix rank $k$ employed. While the orthogonal sensitivities available depend on both the guess model and the spectrum observed, rank is user-defined and determines the amount of regularization. As previously mentioned, a larger $k$ allows for narrower $A_T$ construction but typically results in $\varepsilon$-dominated solutions. A smaller rank only allows wider $A_T$ construction, which then produces a solution with contributions from a range of depths. Thus, the challenge is to optimally balances these error regimes.

In Section 3.3, we discuss an iterative scheme for OLA recovery of nonlinear perturbations that includes a method for determination of rank for each iteration. In the next section, we discuss how the OLA inversions are carried out over multiple optical depths once rank $k$ has been chosen.

### 3.2. OLA Inversion at Multiple Depth Locations

To obtain the optimally vertically resolved inversion at $\tau_i$, we need to determine the narrowest averaging kernel $A_T$ that can be constructed at that depth. For this, we define a range of target function widths spanning from 0.05 to 0.5, in steps of 0.05 (model grid spacing in log $\tau$ units). Starting with the minimum-width target function $G$, normalized to have area one and centered on the target depth $\tau_i$, we solve for coefficients $C$ given the pseudoinverse rank. We then compare the averaging kernel $(R_T \cdot C)$ achieved to the target function for goodness of fit. We chose a fit measure based on the L1 norm of the difference between $G$ and $A_T$, the sum of absolute-difference between the two, as it approximately corresponds to a % difference. A goodness-of-fit criterion that is stringent (smaller tolerance to the difference) makes it harder to construct averaging kernels (for a given rank $k$), while a relaxed criterion (larger tolerance) can result in averaging kernels that only very poorly resemble the target function. Empirically, we find that an L1 norm goodness-of-fit value of 0.2 (20% difference) works well. If the difference lies within this specified upper limit, we conclude that the averaging kernel approximately represents the target function, and we use the coefficients to compute the inverted solution $(\Delta T/T)(\tau_i) = C^\top \cdot \Delta I$. Taking the inverted $\Delta T(\tau_i)$ to be guess $T(\tau_i) \times (\Delta T/T)(\tau_i)$, the inversion is complete at location $\tau_i$, and the width of the target function approximates the vertical resolution of the inverted solution. The process is then repeated to invert at “all possible” depths using the same-rank pseudoinverse matrix. When iterating (Section 3.3), we call this a single “inversion cycle.” We note that it is the averaging kernel width achieved at each depth, not the target function goal, that determines the depth sensitivity of the solution.

For some depths, it is possible that an averaging kernel cannot be constructed to fit even the widest target function considered with the specified fit tolerance. In such cases, we set the inverted $\Delta T/T(\tau_i)$ to zero and conclude that an OLA inversion cannot be achieved at that location, with the rank $k$ employed and response function set available. This failure often occurs above and below a limit range of depths, as the orthogonal sensitivities of the response functions are non-uniformly distributed with depth. The depth range over which kernels can be constructed, which we termed the “OLA inversion window,” is depicted by the blue shaded region in Figure 3 for $k = 5$ over the first inversion cycle. The narrowest kernels that can be constructed at the either end of the window are shown with blue solid and blue dashed curves, and the averaging kernel widths achieved at all depths, which approximate the resolution achieved, are shown in pink. Note that the width of the inversion window depends on the pseudoinverse rank chosen and the response function set, which further depends on the guess model and the observed wavelengths. Using a broader selection of spectral lines (or wavelengths) with more linear independence between response functions and/or employing a larger rank should help increase the inversion depth range.

![Figure 3](image-url)
As with all inversions based on a limited spectrum, the inverted values at each depth are not independent (the averaging kernels are not \( \delta \) functions). With OLA, that dependence is made explicit via the depth-dependent averaging kernel profiles. While it is possible to compute the effective averaging kernels of an SIR inversion post hoc (see the Appendix), the key difference between OLA and SIR is its ability to determine the minimum averaging kernel width possible at each depth—and thus to invert only at depths where a localized kernel can be constructed. The latter is both a strength and a weakness. While providing a clear indication of where a solution is possible, the noninvertible regions pose a particular challenge when unknown perturbations exist outside the inversion window and iteration is required for the final solution (Section 3.3). We discuss this “edge-effect issue” in detail in Section 3.4.

Nonetheless, we note that the advantage of the OLA averaging kernel approach extends beyond the inversion itself. Even before performing an inversion, kernel construction allows direct quantitative assessment of spectral line combinations and their potential utility in inversions for a given variable at a given depth (reference in preparation). Note that, in detail, this assessment also depends on the starting guess model (via the response functions) and on the pseudoinverse rank \( k \) employed.

### 3.3. Iterative Scheme

When inverting spectropolarimetric data, the atmospheric profiles deduced after a single inversion cycle likely do not match the underlying solar profiles, and some spectral differences between the observed and model spectra persist. This is because the underlying perturbation magnitudes are usually large; the actual solar atmosphere is often quite far from the initial guess. The neglected higher-order terms captured by \( \varepsilon \) in Equation (2) are not insignificant, as the relationship between \( \Delta I \) and \( \Delta T/T \) is not linear. Using linear system of equations to recover nonlinear perturbations in a single inversion cycle is typically not possible. Employing a larger rank in the inversion results in a solution dominated by \( \varepsilon \), while a smaller rank is unable to recover steep gradients (if present). This is illustrated by Figure 4, in which OLA inversion results are shown for the test case (Figure 1, top left plot) after one inversion cycle and for two different rank choices: \( k = 5 \) and \( k = 10 \).

Multiple inversion cycles are needed to arrive at an optimal solution. After each inversion cycle, the inverted model from the previous cycle is used as the starting guess for the next. Because the guess model changes with each cycle, the response functions must be recalculated at the start of each. If the iteration is convergent (the underlying \( \Delta T/T \) gets smaller with successive cycles), the nonlinear contributions to \( \varepsilon \) should decrease, and a higher-rank pseudoinverse matrix can be employed to refine the solution. This is similar to what is done with SIR when the number of nodes is increased as the inversion proceeds. Here, we employ a smaller-rank pseudoinverse matrix during early cycles, to obtain a lower-resolution solution that is less likely to be \( \varepsilon \) dominated, and we then increase the rank as the inversion proceeds. The resolution of the iterative inversion increases as the system approaches the linear approximation, with consequently lower total error and thus increased tolerance for larger-magnitude coefficients in \( C^\top \cdot \varepsilon \). While using a lower-rank pseudoinverse matrix during early cycles likely results in solutions dominated by crosstalk error, these errors are made at depths where more localized kernels can be constructed, finer-resolution solutions can be achieved with subsequent iterations. The OLA scheme preserves, to some degree, the underlying \( \Delta T/T \) during early iterations and recovers it at higher resolution, where that is attainable in subsequent iterations, rather than losing it in a global best spectral fit solution.

The goal is thus to determine the maximum-rank \( k \) to be used at the end of the full inversion-cycle series, which ultimately sets the best resolution achieved, and to formulate an algorithm that facilitates scaling up to it from the lower-rank solutions earlier in the iteration cycle. Note that the relative error content in the inverted solutions, both from crosstalk and \( C^\top \cdot \varepsilon \), compared to the underlying perturbation magnitude remaining at each depth (in each inversion cycle), determines both whether iterative updates are convergent (underlying \( \Delta T/T \) gets smaller with cycle) and the maximum rank beyond which inversions are likely to be \( \varepsilon \) dominated. While there is an upper limit to the crosstalk error in \( A^\top_j \cdot \frac{\Delta T}{T} \), which depends on the magnitude of underlying \( \Delta T/T \), error from \( C^\top \cdot \varepsilon \) is unbounded because it additionally depends on the magnitude of the coefficients \( C \), which gets larger with increasing rank. Further, if the magnitude of the coefficients increases faster than the rate at which nonlinear contributions to \( \varepsilon \) decrease (and these depend on the magnitude of underlying unknown \( \Delta T/T \)), the solution will become \( C^\top \cdot \varepsilon \) error dominated.

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**Figure 4.** OLA inversion results for a temperature perturbation \( \Delta T \) (red, replotted from the top left panel in Figure 1) after the first inversion cycle when employing rank \( k = 5 \) (left plot), and \( k = 10 \) (right plot). Green dashed curves correspond to the unsmoothed results, while green solid curves correspond to a smoothed version of the inverted \( \Delta T \). The yellow curves plot the \( \varepsilon \) error contributions (from \( C^\top \cdot \varepsilon \times \text{guess } T(\tau_j) \)) to inversions, and pink curves (plotted with a different scaling) correspond to the minimum kernel widths achievable at each depth.
The aim is to develop an iterative algorithm that itself determines the maximum rank that can be tolerated, without knowing the actual solution to be achieved. Ideally, as the system gets closer to the largest tolerable rank and the inversions become error dominated, this will be reflected in the spectral difference measure (the fit between observed \( T \) and that obtained from the inverted model gets worse). If that is the case, one can conclude that iteration has converged on the optimal solution and that any further inversion cycles would lead to a poorer result. The iteration could then be terminated without difficulty. But, due to the undetermined nature of the inverse problems (multiple stratifications are consistent with a given observed spectrum), it is possible that there is a lag between when the inverted model fit gets worse and when the error amplification is reflected in the spectral difference.

In practice, we have developed the following scheme to minimize this possibility. We initiate the inversion by employing mean MURaM model atmosphere as the starting guess model; we compute guess \( I \), spectral difference \( \Delta I = \text{observed} - \text{guess} I \), and the temperature response function matrix \( R_T \) using it. A smaller rank helps prevent \( \varepsilon \) amplification, and we have empirically determined that a safe starting value can be based on dominant singular values that add to 95% of the total singular-value sum. For our set of initial response functions, this yields a starting rank \( k = 5 \). Using this rank, and the method described in Section 3.2 above, we obtain an OLA inversion estimate of \( \Delta T / T \). This first iteration cycle solution, for the test case perturbation we are examining (red curves in Figures 1 (top left) and 4), is shown with a green dashed curve in the left panel of Figure 4.

While this first iteration cycle solution approximates a running average of the underlying perturbation, it is irregular in shape and has sharp edges at the OLA inversion window boundary. Unlike SIR, OLA does not aim to obtain a globally smooth solution, and because the inversions are computed independently, at each depth they feel different \( C^T \cdot \varepsilon \) and crosstalk error contributions. Using the outcome of the iteration directly in the next iteration cycle iteration proves problematic. Because the starting model for the next iteration is obtained by updating the guess \( T \) profile with the result from the current iteration cycle, it will reflect the irregularities of the solution—and importantly, because the response functions must be recalculated with each new iteration, they too will not be smooth. This makes it hard to construct, in subsequent cycles, averaging kernels that match smooth target function profiles. Thus, we employ a smoothed version of the inverted \( \Delta T \) in the next iteration. The smoothing is done via convolution with a Gaussian kernel of width equal to that of the narrowest averaging kernel achieved in the previous inversion cycle. Note that smoothing extends the inverted solution outside of the inversion window, which helps with the “edge-effect” issue (see Section 3.4). The smoothed result from the first iteration cycle is plotted as the green solid curve in the left panel of Figure 5. It is this solution that is added to initial guess to obtain the inverted \( T \).

Before moving on to the next iteration cycle, a decision is required on whether to continue inverting using the same pseudoinverse rank, increase the rank, or stop the iteration altogether. This decision is based on a comparison between the L1 norm of \( \Delta I \) calculated for the inverted atmosphere (observed \( I - \text{inverted model} I \) and that of \( \Delta I \) obtained with the guess model. If the spectral fit improved, the next inversion cycle employs the same pseudoinverse rank. When using the same rank in successive iterations, the magnitudes of iterative updates get smaller, along with the improvements in the spectral fits. This is because, with constant \( k \) iteration, the magnitude of underlying \( \Delta T / T \) decreases and becomes more oscillatory, as each successive cycle refines the previous approximation. Moreover, while the response functions are recalculated in each cycle, the kernel widths that are achievable at each depth do not change significantly when the same rank pseudoinverse is employed. They are unable to resolve the oscillatory residual, and a larger rank is required in order to construct narrower kernels and make further improvements. We determine that the iterative updates at fixed rank have “stagnated” if the difference between the L1 norm of guess \( \Delta I \) and that of inverted model \( \Delta I \) (in a cycle) are within 0.5% of one another. When this is the case, we increase rank by two (an iteration acceleration arrived at by trial and error) and proceed with the next cycle. Inversion iteration thus proceeds in successive stages of rank increase and stagnation, until the rank cannot be further increased without error amplification, as indicated by worsening of the spectral fit with higher rank.

In Figure 5 (left plot), the green curve shows the inversion result after employing iterative OLA method (barely distinguishable from the underlying perturbation shown in red, which it overfits). A SIR inversion for the same perturbation is plotted in blue. While iterative OLA recovers the underlying

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Figure 5. Left-hand plot: Temperature perturbation to be inverted, replotted from the top left panel in Figure 1 in red. Iterative OLA inversion result is shown in green, that obtained using SIR is shown in blue (see Table 2 for node parameters). The minimum kernel widths achievable in the final iteration cycle, which also approximate the vertical resolution of OLA solutions, are shown in pink (plotted with a different scaling). Right-hand plot: The corresponding spectral differences \( \Delta f \) for the starting guess model spectra (red), OLA-inverted model spectra (green), and SIR-inverted model spectra (blue). Note that the L1 norm of observed \( I \) is 57.4, while those for the iterative OLA-inverted model and the SIR-inverted model are 0.03 and 0.947, respectively.
perturbation well, SIR struggles to recover it. This is largely because SIR, by design, inverts at all depths. This includes regions well outside of the OLA inversion window, where the solution is poorly constrained by the response functions and can be dominated by contributions from other depths. While poorly constrained, these regions nonetheless contribute to the global spectrum, and as illustrated by the right-hand panel of Figure 5, both the SIR and iterative OLA inversions yield comparably good fits to the observed spectrum.

To ensure that the final inverted solution is close to the underlying atmosphere, it is critical to minimize both the crosstalk error and contributions from \( \varepsilon \), especially early in the iterative process when \( \Delta I \) is the largest. With successive updates, \( \Delta I \) gets smaller and it gets increasingly harder to make significant changes to solution already at hand. The SIR solution allows more crosstalk error from regions that are poorly constrained. The iterative OLA approach, on the other hand, can optimally minimize crosstalk contributions, but struggles when nonzero perturbations exist outside the reliably invertible region, i.e., in regions that are not updated during the inversion iteration. When iteration is required, the limiting inversion window introduces an “edge effect.” This is the subject of the next section.

3.4. Iterative OLA Method: the “Edge-effect” Issue

The Gaussian-shaped perturbation employed in the last section to illustrate the application of the OLA method (left plot in Figure 5), while of sufficiently large amplitude to require an iterative solution, was placed in a region of the domain where averaging kernels could be readily constructed. This favored a successful OLA inversion. Other perturbations present greater challenge. Somewhat surprisingly, a simple large-scale offset between the underlying and guess models is particularly difficult for OLA to uncover. An example is shown in red in the left panel in Figure 6, where in addition to the localized Gaussian of last section, a constant offset of 300 K is added to the atmosphere. In this case, the iterative OLA method struggles to recover the perturbation (inversion result shown with a green dashed curve).

The failure of the OLA method in this case has two underlying and intertwined causes: iterative updates are confined to a limited depth range, outside of which updates are not made (localized kernels cannot be constructed in those regions), and the averaging kernels have finite widths, i.e., they are not \( \delta - \) functions. With successive updates, the underlying \( \Delta T/T \) within the inversion window gets smaller, but outside of that window it does not change beyond the smoothing at the end of an iteration cycle. Thus, with iteration, the averaging kernels on which the inversion solutions are based, if they admit contributions from outside of the inversion window, increasingly weigh the uncorrected perturbations. There is a consequent leakage of information from the outside of the inversion window into it (the solution becomes dominated by crosstalk error), which degrades the overall inversion quality. While smoothing inversion solutions, increasing the rank, and allowing the inversion window to grow with iteration all help, and were used to obtain the green dashed solution in Figure 6, updates cannot typically be made at all depths and the problem cannot be entirely avoided.

One solution to the problem is to extend the inversion update to regions outside of the OLA inversion window, and thus to minimize its negative influence. This can be done by including an averaging kernel that captures the large-scale perturbations that extend into those regions. For example, to capture a constant offset, one can define flat (constant) normalized target functions (pink curve in the right panel in Figure 6) and use the OLA framework to construct flat averaging kernels that are equally sensitive at all depth locations; one can then interleaved inversions based on these kernels with those using localized Gaussian kernels during iteration. If such a flat kernel can be constructed, an inversion solution based on \( C^T \cdot \Delta I \) would correspond to a large-scale average fractional contribution,

| Inverting | \( T \) Nodes | \( V_{in} \) Nodes | \( P_e \) Nodes | Number of Cycles |
|-----------|---------------|----------------|--------------|----------------|
| \( T \) | 1,2,3,4,5,6,8,11 \(^*\) | 0 | 1 \(^*\) | 8 |
| \( T, V_{in} \) and \( P_e \) | 1,2,3,4,5,6,8,11 | 1,2,3,4,5,6,8,11 | 1,2,3,4,5\(^*\) | 8 |

Notes.

\(^*\) Each comma-separated value corresponds to the number of nodes used in a given cycle.
\(^*\) If the number of nodes for \( P_e \) inversion is set to zero, then SIR inverts for \( T \), assuming hydrostatic equilibrium. Further, to be consistent with single-variable \( T \) inversion, the SIR-inverted \( \Delta P_e \) is manually set to zero.
\(^*\) SIR-inverted \( P_e \) solutions (generally) deviate drastically from the underlying \( P_e \) model for larger node values.
response functions, as terms in spectrum. Here, adiabatic sound speed $c_s$ is used to nondimensionalize the response functions, as terms in \( V_{\text{los}} \) can be zero. (Bottom panel) Fractional response functions for electronic pressure. Both sets of response functions were computed using SIR for mean MURaM model.

$$\langle \frac{\Delta T}{T} \rangle = A_T^T \cdot \frac{\Delta T}{T} \equiv \frac{1}{N} \sum_{\tau} \frac{\Delta T}{T}(\tau),$$

where \( N \) is the total number of optical depth points. In practice, the flat averaging kernel only approximates a constant with optical depth (blue curve in the right panel in Figure 6). The average is also an approximation.

The final inversion solution from this combined iterative effort is shown with the green solid curve in the left panel of Figure 6. While the solution is improved, some difficulties remain. Because the large-scale inversion is for the fractional average \( \langle \Delta T/T \rangle \) (from which \( \Delta T(\tau) \) at each depth is computed via \( \Delta T(\tau) = \langle \Delta T/T \rangle \times \text{guess } T(\tau) \) and the values of \( T(\tau) \) in the deeper regions are about 3–5 times larger than those in the shallower region, the method tends to overcorrect in the deeper regions and undercorrect in the shallower regions. Employing nonfractional response functions and directly computing constant inverted \( \Delta T \) (and not \( \Delta T/T \)) yields, in this case, a more robust solution, but induces similar issues for cases in which underlying perturbation is not a constant offset. Given that we do not know a priori the behavior of the underlying large-scale perturbation, it is hard to determine the target function that best captures it.

Another possible solution to the edge-effect problem, is to employ nonsymmetrical target functions that have low amplitudes outside of the inversion window. Unfortunately, just as the response functions typically available do not allow \( \delta - \) function averaging kernel construction, they also do not allow construction of an infinitely sharp cutoff. Because the magnitude of the perturbation inside the inversion window decreases with iteration, the importance of any residual kernel amplitude outside the inversion window increases with iteration if the underlying targeted perturbation has any magnitude in that region.

Finally, because the nodal approach of SIR is adept at recovering large-scale constant or linear trends, we developed a hybrid SIR/iterative OLA scheme. This approach met with some success (see Section 4 of Agrawal 2021), and is the subject of ongoing research efforts.

### 4. Multivariable Inversion Using Iterative OLA Method

The OLA method can be extended to include spectral sensitivity to more than one variable. Here, we examine multivariable inversions for the thermodynamic parameters temperature, electronic pressure, and line-of-sight velocity, \( T, P_e, \) and \( V_{\text{los}}, \) under the simplifying assumption that these completely define the atmosphere.

The first-order equation for a multivariable system can be written as

$$\Delta I = R_T^T \cdot \frac{\Delta T}{T} + R_{V_{\text{los}}}^T \cdot \frac{\Delta V_{\text{los}}}{c_s} + R_{P_e}^T \cdot \frac{\Delta P_e}{P_e} + \varepsilon,$$

where \( R_{V_{\text{los}}} \) and \( R_T \) represent the response function matrices for \( V_{\text{los}} \) and \( P_e \) (Figure 7), and \( R_T \) is that for \( T, \) as previously. One should note the use of adiabatic sound speed \( c_s \) when computing the fractional response function \( R_{V_{\text{los}}} \), to avoid difficulties where \( V_{\text{los}} \) is close to zero. It should also be noted that \( \varepsilon \) now collectively represents the missing higher-order error terms from all variables, plus observational and instrumental noise (if present).

#### 4.1. Averaging and Crosstalk Kernels

The underlying strategy for multivariate OLA inversions is similar to that for the single-variable counterpart; individually invert for each variable at each depth location, with the goal of computing solutions that optimally balance \( \varepsilon \) and crosstalk errors. In multivariable inversions, there are crosstalk error contributions from other depths—as was the case previously in single-variable inversions—and from other variables. The optimally error-balanced solution is achieved by computing a set of coefficients \( C \) that can be used to linearly combine the response functions of the target variable to form a narrow Gaussian averaging kernel, which at the same time (using the same coefficients) combines the response functions for the crosstalk variables so that their sum is nearly zero at all depths. For example, if the target is to invert \( T \) at depth \( \tau \), and the crosstalk variables are \( V_{\text{los}} \) and \( P_e \), then the coefficients \( C \) are determined by solving the linear system of equations:

$$\begin{pmatrix} R_T \\ R_{V_{\text{los}}} \\ R_{P_e} \end{pmatrix} \cdot C = \begin{pmatrix} G(\tau, \sigma) \\ 0 \\ 0 \end{pmatrix}.$$

The coefficients \( C \) yield the averaging kernel \( A_T(\tau) = R_T \cdot C \), which approximates \( G(\tau, \sigma) \), while minimizing cross-term sensitivities, with crosstalk kernels \( A_{V_{\text{los}}} = R_{V_{\text{los}}} \cdot C \) and \( A_{P_e} = R_{P_e} \cdot C \), approximately zero. The kernels \( A_{V_{\text{los}}} \) and \( A_{P_e} \) thus capture the \( \tau \) dependence of the residual crosstalk between variables. Once coefficients are computed, the dot product of \( C^T \) with Equation (5) yields the inverse solution:

$$C^T \cdot \Delta I = A_T^T(\tau) \cdot \frac{\Delta T}{T} + A_{V_{\text{los}}}^T \cdot \frac{\Delta V_{\text{los}}}{c_s} + A_{P_e}^T \cdot \frac{\Delta P_e}{P_e} + C^T \cdot \varepsilon.$$
temperature, in a multivariable system, with the total sum. This corresponds to \( k \) computed using rank sight velocity and electronic pressure. In all cases, the pseudoinverse matrix is preserved for crosstalk variable inversion.

A \( \Delta \) in be constructed for the target variable and the suppression of tradeoff between the minimum averaging kernel width that can be dominated. Note that, because of this bias, solutions based on it are then more likely to become dominated solutions, the \( R_T \) and \( R_{\text{Vlos}} \) sensitivities are truncated more severely than \( R_T \), and the computed coefficients are biased toward satisfying the temperature-related portions of the linear system.

The biased sensitivity of the reduced-rank pseudoinverse matrix is particularly critical when constructing \( A_P \) for a \( \Delta P_T / P_T \) inversion, which requires the simultaneous suppression of the \( \Delta T / T \) and \( \Delta \text{Vlos}/e_s \) contributions. In the solution of

\[
\begin{bmatrix}
R_T \\
R_{\text{Vlos}} \\
R_P
\end{bmatrix} \cdot C =
\begin{bmatrix}
0 \\
0 \\
G(\tau, \sigma)
\end{bmatrix},
\]

the smaller \( R_P \) sensitivity prevents construction of even a very wide averaging kernel at any depth, when employing the 95\% criterion on the pseudoinverse rank (no red curve in the lower panel of Figure 8). The coefficients computed largely ensure that \( R_T \cdot C = 0 \), with little sensitivity to \( R_P \cdot C = G \). The constructed \( A_P \) do not fit any allowed width target function (at any depth) with at least 80\% accuracy (see Section 3.2). While it is possible to construct \( A_P \) by employing larger rank, the solutions based on it are then more likely to become \( e \) dominated. Note that, because of this bias, \( P_s \) is typically not directly inverted in spectropolarimetric inversions. Instead, it is determined from the equation of state assuming the atmosphere is hydrostatically stratified (del Toro Iniesta & Ruiz Cobo 2016). An inversion for \( T \) and \( P_s \) independently would allow solutions that depart from the hydrostatic assumption, with importance for our understanding of compressible dynamics such as the generation of acoustic waves.

One way to address the differential spectral sensitivity is to artificially amplify/deamplify the spectral response functions so that sensitivities to all variables are similar. For example, in place of Equation (8), a modified system of equations

\[
\begin{bmatrix}
\alpha_1 \cdot R_T \\
\alpha_2 \cdot R_{\text{Vlos}} \\
R_P
\end{bmatrix} \cdot C =
\begin{bmatrix}
0 \\
0 \\
G(\tau, \sigma)
\end{bmatrix},
\]
is solved, where \( \alpha_1 \) and \( \alpha_2 \) are the response function amplification factors used to amplify/deamplify crosstalk variable sensitivities. While this mathematical operation is equivalent to multiplying both sides of the equation by a constant factor and thus does not change the meaning of the original equation, it does change the singular values and pseudoinverse matrix employed, manually forcing the underlying eigenvectors to carry similar sensitivities across all variables.

It is critical to choose the amplification factor judiciously. If too large, the lower-rank pseudoinverse matrix carries enhanced crosstalk variable sensitivity, resulting in a solution that primarily minimizes crosstalk at the expense of target variable kernel construction, and narrowing the averaging kernel width achievable remains difficult. If too small, the lower-rank pseudoinverse matrix carries insufficient crosstalk sensitivity, and crosstalk is not actively suppressed. Empirically, we have found that the ratio of the maximum of the absolute value of response functions works well, e.g., in Equation (9), \( \alpha_1 = \frac{\max |R_{Vlos}|}{\max |R_{T}|} \) and \( \alpha_2 = \frac{\max |R_{Vlos}|}{\max |R_{Pe}|} \). For the response function matrices obtained from the mean MURaM model, these ratios are \( \alpha_1 \approx 0.03 \) and \( \alpha_2 \approx 0.08 \), values less than unity because the peak magnitudes of \( R_T \) and \( R_{Vlos} \) are larger than \( R_P \). We have chosen, independently of the target variable, to apply the amplification factors to the crosstalk variable response functions. This avoids the need to factor out the amplification factors from the coefficients before computing the inverse solution \( C^{-1} \). \( \Delta I \).

When the amplification factors are employed, the dominant modes of the pseudoinverse matrices retain sensitivity to the variables with intrinsically lower-amplitude spectral response functions. This is shown with the green curves in Figure 8. In the particular case of \( \Delta P_e/P_e \) inversions, averaging kernels \( A_{P_e} \) can be constructed in the presence of both \( T \) and \( V_{los} \), when using the 95% criterion to determine the pseudoinverse rank. Notably, a broader range of depths become accessible to \( V_{los} \) inversions, and \( A_{Pe} \) construction and \( \Delta P_e/P_e \) inversion becomes possible (bottom panel).

4.3. Multivariable Iterative OLA Inversion: Example Implementation

We have carried out multivariable iterative OLA inversions (without large-scale offset correction) for idealized perturbations of the mean MURaM atmosphere. Equations like that of Equation (9) were used to invert for each variable at each depth, employing a minimum-width Gaussian target function for the target variable while simultaneously preserving the spectral contributions of the crosstalk variable via null target functions, as described above. OLA, by design, inverts for only one variable at a time, and thus each inversion cycle includes successive iteration over each variable. The order we have chosen to invert for the variables (at all depths for which a kernel can be constructed) is in decreasing order of their spectral sensitivity, first for \( \Delta T/T \), then for \( \Delta V_{los}/V_{los} \), and subsequently for \( \Delta P_e/P_e \). This order was chosen because it is easier to construct narrow kernels for the spectrally more influential variables and suppress sensitivities of the less dominant ones than vice versa.

In implementation, within a given cycle, we update the guess \( T \) with inverted \( T \), if inverted \( \Delta T/T \) results in a better spectral fit. We then repeat the inversion process for the next dominant variable \( \Delta V_{los}/V_{los} \), and finally \( \Delta P_e/P_e \), each with the same spectral fit criterion. Thus, the final inverted model (after any full cycle) is updated only with those variables that help improve the spectral fit measure and only at those depths at which a kernel can be constructed (combined with smoothing that smears inversion updates outside the inversion window). This approach helps reduce the overall perturbation magnitudes, which lessens the overall \( \varepsilon \) magnitude. A smaller \( \varepsilon \) implies a larger-rank \( k \) can be employed in subsequent cycles, improving the OLA inversion resolution and depth range, and thus the inversion quality achievable. Similar to its single-variable counterpart, we compare the guess spectral fit with that obtained from all variable updated inverted models in order to determine what rank to use for the next cycle—or whether to stop iteration altogether.

In Figure 9, we show a test case perturbation (red) obtained by adding Gaussian perturbations to the mean MURaM profile of each variable (peak location at \( \log \tau = -0.5, \) width = 0.40 \( \Delta \log \tau \)), with perturbation amplitudes equal to 10% of the variable value at the peak location depth. The final inverted solution obtained using iterative OLA and SIR are shown in green and blue. OLA and SIR are both able to recover the Gaussian perturbations. Iterative OLA localizes it quite robustly, while the SIR solution shows strong oscillatory wings in deeper and shallower regions. It is important to note, however, that success via iterative OLA solver is favored by this particular case. The underlying perturbation we have chosen for illustration does not have a large-scale offset, and thus the edge-effect issue discussed in Section 3.4 has been avoided. SIR inversions are much more successful at recovering such large-scale offsets between the model and observed atmosphere, and we are pursuing, among other possibilities, a hybrid approach that can robustly capture both large-scale and localized perturbations and yield quantitative assessment of the depth resolution achieved.

5. Summary and Future Work

In this work, we applied the OLA method to synthetic spectra to invert for the thermodynamic properties \( (T, P_e, V_{los}) \) of a simulated solar atmosphere, assuming that the atmosphere and radiation field are in LTE. These inversions, like those discussed previously, solve a first-order linear system of equations for the difference between the guess model and that for which the spectrum is observed. The spectral sensitivities are captured by the spectral response functions, and an iterative scheme is employed to relax the model to a state in which the observed spectrum and that from inverted model agree. In our case, as the observed atmosphere is synthetic, the final atmospheric properties can be compared for agreement.

Inverted solutions have two intrinsic error contributions: error \( \varepsilon \) from the higher-order terms truncated by the linear response function formalism plus observational/instrumental noise, if present; and error from crosstalk, the erroneous contributions from other depths and other variables in a multidepth, multivariable solution. It is generally not possible to minimize crosstalk error without amplifying \( \varepsilon \), and vice versa. The goal of inversions is to regularize and balance these error contributions by employing a truncated-rank pseudoinverse matrix. Employing a smaller rank results in solutions dominated by crosstalk error, while larger-rank solutions are dominated by \( \varepsilon \). Additionally, due to the nonlocal nature of the radiative transfer, different atmospheric solutions...
are consistent with a given spectral difference. This makes it hard to determine the rank that optimally balances the two errors. Current state-of-the-art spectral inversion methods, such as SIR, compute globally smooth solutions that minimizes $\varepsilon$, but these solutions are likely dominated by crosstalk error. It is hard for these approaches to recover steep gradients, or a priori know what gradients are recoverable, given the spectral response function set on hand. Addressing these issues is the key motive behind OLA.

OLA independently inverts for each variable at each depth location, to obtain solutions that optimally balance $\varepsilon$ and crosstalk contributions. This is achieved by solving for the coefficients that can be used to construct, as a linear combination of the response functions, a localized averaging kernel that mimics prescribed target functions, namely those that both localize depth sensitivity to the target variable and zero out sensitivity to other variables. The inverted solution at a given depth, given by the inner product of the coefficients with the spectral difference, approximates the kernel-averaged underlying field. The width of the kernel thus represents the spatial resolution of the inverted solution. The process is repeated to invert at all possible depths. Given the limited orthogonal sensitivity of the response functions, localized kernels cannot be constructed at all depths. Failure usually occurs above and below a limit range of optical depths, defining the “OLA inversion window.” The rank of the pseudoinverse matrix dictates the size of this window, the kernel width achievable at each depth, and the error regime the inverted solution falls into. We developed an iterative OLA that relaxes the solution, with increasing rank with iteration, from the likely nonlinear differences between the initial model and the observed atmosphere to the final model that matches the spectral observations.

We have applied the iterative OLA method we have developed to inversions for large-amplitude perturbations using linear response functions that are iteratively updated, significantly extending the inversion capability of traditional OLA methods used in helioseismology. The inversion results we achieve are promising and competitive with other state-of-the-art spectral inversion techniques. Nonetheless, some issues remain. Iterative OLA struggles to recover large-scale offsets due to nonzero perturbations outside the OLA inversion window. This is a direct consequence of the inability to construct strictly localized $\delta$-function kernels. Iteratively updating regions inside the inversion window eventually causes a leakage of the uncorrected perturbation information from the outside of the inversion window to the inside of it. In the worst cases, this can corrupt the entire solution, and in doing so defeats the very strength of the OLA method, which fundamentally aims to minimize leakage by minimizing crosstalk and thus preserve the locality of the underlying perturbations.

We have shown that the edge-effect issue can be minimized by making updates outside of the OLA inversion window. For this, we developed a scheme within the OLA framework where we construct “flat” averaging kernels (instead of Gaussian-shaped localized kernels) to obtain a large-scale averaged solution. We interleaved large-scale averaged solution with the high-resolution localized OLA inversion during iteration. While the inversion quality is significantly improved, some difficulties remain, and finding a more robust solution to this problem is a future endeavor.

We have also shown that the inherent spectral sensitivity bias in multivariable inversions can be overcome using response function amplification, which allows dominant modes of the pseudoinverse matrix to be equally sensitive to all variables. We expect that this idea would benefit other inversion methods.

Figure 9. Multivariable inversion of localized perturbations are shown here. Red curve in each panel (except bottom right) corresponds to the Gaussian perturbation (peak location at $\log \tau = -0.5$, width $0.40 \Delta \log \tau$, and perturbation amplitude = 10%) added to the corresponding variable in the mean MURaM model, to construct underlying model. The final solution obtained using iterative OLA method (without large-scale flat kernel updates) and using SIR (see Table 2 for node parameters) are shown in green and blue, respectively. Pink curve, plotted with a different scaling, corresponds to the minimum kernel widths achievable for each variable in the final OLA iteration cycle. Bottom right panel shows corresponding spectral differences $\Delta f$ for the OLA-inverted model spectra (green), SIR-inverted model spectra (blue), and starting guess model spectra (red, plotted after downscaling by a factor of 100).
as well, and have shown that its implementation can allow for inversions of electronic pressure, which is typically determined assuming that the atmosphere is in hydrostatic equilibrium.

It should be noted that the goal of this work is to test and validate the applicability of the OLA technique to invert spectroscopic data and demonstrate its strengths and weaknesses. No effort was made to optimize the numerical scheme, and no quantitative speed comparisons were made with other codes.

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Appendix

SIR Averaging Kernels

When solving, say Equation (2), SIR inverts an over-determined version of $R_T$, to prevent $\varepsilon$ amplification. The goal is first to compute $\Delta T/T$ solutions at a limited $p$ node locations, and finally interpolate nodal values to compute solutions at all $q$ depth points. Overdetermined $R_T$, computed with the help of interpolation coefficient matrix $F$, is given by $F \cdot R_T$, and contains “equivalent” response function sensitivity to node locations (del Toro Iniesta 2003, page 209).

Nodal $\Delta T/T$ solutions, using the inverse of equivalent response functions, are given by

$$[(F \cdot R_T)^{-1}] \cdot \Delta T = [(F \cdot R_T)^{-1}] \cdot [R_T \cdot \Delta T ^{-1} + \varepsilon]. \quad (B1)$$

From these $p$ nodal values, the SIR inversion solution at all $q$ depths is given by

$$F^T \cdot [(F \cdot R_T)^{-1}] \cdot \Delta T = F^T \cdot [(F \cdot R_T)^{-1}] \cdot [R_T \cdot \Delta T ^{-1} + \varepsilon]. \quad (B2)$$

Here, individual rows of matrix $F^T \cdot [(F \cdot R_T)^{-1}] \cdot R_T^{-1}$ correspond to SIR averaging kernels at each depth locations.

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