Exactly solvable time-dependent non-Hermitian quantum systems from point transformations

Rebecca Tenney
City, University of London
Pseudo-Hermitian Hamiltonians in Quantum Physics

October 14, 2021
Exactly solvable time-dependent non-Hermitian quantum systems from point transformations

Rebecca Tenney
City, University of London
Pseudo-Hermitian Hamiltonians in Quantum Physics

October 14, 2021

Based on: A. Fring and R. Tenney, Phys. Lett. A. 410 127548 (2021)
Introduction

Time-dependent non-Hermitian quantum systems

Point transformations

Invariant, Dyson map and metric

Conclusions

Rebecca Tenney

Exactly solvable time-dependent non-Hermitian quantum systems from point transformations
Outline

Introduction

Time-dependent non-Hermitian quantum systems

Point transformations

Invariant, Dyson map and metric

Conclusions
Outline

- Introduction
- Time-dependent non-Hermitian quantum systems
  - Key equations
  - The Dyson map, the metric
  - Different solution procedures

Rebecca Tenney

Exactly solvable time-dependent non-Hermitian quantum systems from point transformations
2/19
Outline

Introduction

Time-dependent non-Hermitian quantum systems
  - Key equations
  - The Dyson map, the metric
  - Different solution procedures

Point transformations
Introduction

Time-dependent non-Hermitian quantum systems
  - Key equations
  - The Dyson map, the metric
  - Different solution procedures

Point transformations

Application to the Swanson Model
Outline

- Introduction
- Time-dependent non-Hermitian quantum systems
  - Key equations
  - The Dyson map, the metric
  - Different solution procedures
- Point transformations
- Application to the Swanson Model
- Conclusions
Hamiltonians need not be Hermitian to have real eigenvalues.
Hamiltonians need not be Hermitian to have real eigenvalues. If a Hamiltonian is $\mathcal{PT}$-symmetric it has real eigenvalues.

$$\mathcal{PT} : \quad p \rightarrow p, \quad x \rightarrow -x, \quad i \rightarrow -i$$
Introduction

Hamiltonians need not be Hermitian to have real eigenvalues. If a Hamiltonian is $\mathcal{PT}$-symmetric it has real eigenvalues.

$$\mathcal{PT} : \quad p \rightarrow p, \quad x \rightarrow -x, \quad i \rightarrow -i$$

$$H = p^2 + x^2(ix)^\varepsilon$$

$\varepsilon > 0 \rightarrow$ real eigenvalues

Figure 1: C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998)
Two time-dependent Schrödinger equations for $h(t) = h^\dagger(t)$, $H(t) \neq H^\dagger(t)$

$$h(t)\psi(t) = i\hbar \partial_t \psi(t) \quad \text{and} \quad H(t)\phi(t) = i\hbar \partial_t \phi(t)$$
Two time-dependent Schrödinger equations for \( h(t) = h^\dagger(t), H(t) \neq H^\dagger(t) \)

\[
h(t)\Psi(t) = i\hbar \partial_t \Psi(t) \quad \text{and} \quad H(t)\phi(t) = i\hbar \partial_t \phi(t)
\]

Time-dependent Dyson map

\[
\Psi(t) = \eta(t)\phi(t)
\]

\[\implies\] Time-dependent Dyson equation (TDDE):

\[
h(t) = \eta(t)H(t)\eta(t)^{-1} + i\hbar \partial_t \eta(t)\eta(t)^{-1}
\]
Two time-dependent Schrödinger equations for $h(t) = h^\dagger(t)$, $H(t) \neq H^\dagger(t)$

\[
h(t)\Psi(t) = i\hbar\partial_t\Psi(t) \quad \text{and} \quad H(t)\phi(t) = i\hbar\partial_t\phi(t)
\]

Time-dependent Dyson map

\[
\Psi(t) = \eta(t)\phi(t)
\]

$\implies$ Time-dependent Dyson equation (TDDE):

\[
h(t) = \eta(t)H(t)\eta(t)^{-1} + i\hbar\partial_t\eta(t)\eta(t)^{-1}
\]

$\implies$ Time-dependent quasi-Hermiticity relation (TDQH):

\[
H^\dagger(t)\rho(t) - \rho(t)H(t) = i\hbar\partial_t\rho(t), \quad \text{where} \quad \rho(t) = \eta^\dagger(t)\eta(t)
\]
Observables $o(t)$ in Hermitian system are self-adjoint.
Observables $o(t)$ in Hermitian system are self-adjoint.
Observables $\mathcal{O}(t)$ in the non-Hermitian system are quasi-Hermitian.
Observables $o(t)$ in Hermitian system are self-adjoint.
Observables $\mathcal{O}(t)$ in the non-Hermitian system are quasi-Hermitian.

$$o(t) = \eta(t)\mathcal{O}(t)\eta(t)^{-1}$$
Observables $o(t)$ in Hermitian system are self-adjoint.
Observables $\mathcal{O}(t)$ in the non-Hermitian system are quasi-Hermitian.

$$o(t) = \eta(t)\mathcal{O}(t)\eta(t)^{-1}$$

Calculate observables:

$$\langle \Psi(t)|o(t)\Psi(t)\rangle = \langle \phi(t)|\rho(t)\mathcal{O}(t)\phi(t)\rangle$$
Observables $o(t)$ in Hermitian system are self-adjoint. Observables $O(t)$ in the non-Hermitian system are quasi-Hermitian.

$$o(t) = \eta(t)O(t)\eta(t)^{-1}$$

Calculate observables:

$$\langle \Psi(t) | o(t) \Psi(t) \rangle = \langle \phi(t) | \rho(t)O(t)\phi(t) \rangle$$

$H(t)$ not observable $\rightarrow \tilde{H}(t) = \eta(t)^{-1}h(t)\eta(t)$ is observable energy operator.
Observables $o(t)$ in Hermitian system are self-adjoint. Observables $O(t)$ in the non-Hermitian system are quasi-Hermitian.

$$o(t) = \eta(t)O(t)\eta(t)^{-1}$$

Calculate observables:

$$\langle \Psi(t)|o(t)\Psi(t)\rangle = \langle \phi(t)|\rho(t)O(t)\phi(t)\rangle$$

$H(t)$ not observable $\rightarrow \tilde{H}(t) = \eta(t)^{-1}h(t)\eta(t)$ is observable energy operator.

Must start by calculating $\rho(t)$ and $\eta(t)$!
How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \rightarrow \rho(t) = \eta^\dagger(t) \eta(t)$

2. Solve TDQH directly for $\rho(t)$ -> harder to determine $\eta(t)$

3. Lewis-Riesenfeld invariants

\[ \frac{dI_H(t)}{dt} = \partial_t I_H(t) - i \hbar [I_H(t), H(t)] = 0, \]

for $H = h = h^\dagger$, $H \neq H^\dagger$.

Invariants are quasi-Hermitian

\[ I_h(t) = \eta(t) I_H \eta(t)^{-1} \]

Solution to TDSE:

\[ I_H |\varphi_H(t)\rangle = \lambda |\varphi_H(t)\rangle, \]

\[ |\Psi_H(t)\rangle = e^{i \hbar \alpha(t)} |\varphi_H(t)\rangle, \]

\[ \dot{\alpha}(t) = \langle \varphi_H(t) | (i \hbar \partial_t - H) | \varphi_H(t) \rangle, \]

\[ \dot{\lambda} = 0. \]

---

1. H. Lewis and W. Riesenfeld, J. Math. Phys. 10, 1458-1473 (1969)
2. B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus 132(6), 258 (2017).
How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t)$

---

1. H. Lewis and W. Riesenfeld, J. Math. Phys. 10, 1458-1473 (1969)
2. B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus 132(6), 258 (2017).
How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \rightarrow \rho(t) = \eta^\dagger(t)\eta(t)$

---

1. H. Lewis and W. Riesenfeld, J. Math. Phys. 10, 1458-1473 (1969)

2. B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus 132(6), 258 (2017).
How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \rightarrow \rho(t) = \eta^\dagger(t)\eta(t)$
2. Solve TDQH directly for $\rho(t)$

---

1. H. Lewis and W. Riesenfeld, J. Math. Phys. 10, 1458-1473 (1969)
2. B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus 132(6), 258 (2017).
How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \rightarrow \rho(t) = \eta^\dagger(t) \eta(t)$
2. Solve TDQH directly for $\rho(t) \rightarrow$ harder to determine $\eta(t)$

---

1. H. Lewis and W. Riesenfeld, J. Math. Phys. 10, 1458-1473 (1969)
2. B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus 132(6), 258 (2017).
How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \rightarrow \rho(t) = \eta^\dagger(t)\eta(t)$
2. Solve TDQH directly for $\rho(t) \rightarrow$ harder to determine $\eta(t)$
3. Lewis-Riesenfeld invariants$^1$:

$$\frac{dl_H(t)}{dt} = \partial_t l_H(t) - i\hbar [l_H(t), \mathcal{H}(t)] = 0, \text{ for } \mathcal{H} = h = h^\dagger, H \neq H^\dagger$$

---

$^1$H. Lewis and W. Riesenfeld, J. Math. Phys. 10, 1458-1473 (1969)

$^2$B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus 132(6), 258 (2017).
How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \rightarrow \rho(t) = \eta(t)\eta(t)$
2. Solve TDQH directly for $\rho(t) \rightarrow$ harder to determine $\eta(t)$
3. Lewis-Riesenfeld invariants

$$\frac{dI_{\mathcal{H}}(t)}{dt} = \partial_t I_{\mathcal{H}}(t) - i\hbar [I_{\mathcal{H}}(t), \mathcal{H}(t)] = 0, \text{ for } \mathcal{H} = h = h^\dagger, H \neq H^\dagger$$

Invariants are quasi-Hermitian $^2$

$$I_h(t) = \eta(t)I_{\mathcal{H}}(t)^{-1}$$

---

$^1$ H. Lewis and W. Riesenfeld, J. Math. Phys. 10, 1458-1473 (1969)

$^2$ B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus 132(6), 258 (2017).
How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \rightarrow \rho(t) = \eta^\dagger(t)\eta(t)$
2. Solve TDQH directly for $\rho(t) \rightarrow$ harder to determine $\eta(t)$
3. Lewis-Riesenfeld invariants$^1$:

$$\frac{dl_{\mathcal{H}}(t)}{dt} = \partial_t l_{\mathcal{H}}(t) - i\hbar [l_{\mathcal{H}}(t), \mathcal{H}(t)] = 0, \text{ for } \mathcal{H} = h = h^\dagger, H \neq H^\dagger$$

Invariants are quasi-Hermitian$^2$

$$l_h(t) = \eta(t)l_{\mathcal{H}}\eta(t)^{-1}$$

Solution to TDSE:

$$l_{\mathcal{H}} |\phi_{\mathcal{H}}(t)\rangle = \lambda |\phi_{\mathcal{H}}(t)\rangle, \quad |\Psi_{\mathcal{H}}(t)\rangle = e^{i\hbar\alpha(t)} |\phi_{\mathcal{H}}(t)\rangle$$

$$\dot{\alpha}(t) = \langle \phi_{\mathcal{H}}(t) | (i\hbar \partial_t - \mathcal{H}) |\phi_{\mathcal{H}}(t)\rangle, \quad \dot{\lambda} = 0$$

---

$^1$H. Lewis and W. Riesenfeld, J. Math. Phys. 10, 1458-1473 (1969)

$^2$B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus 132(6), 258 (2017).
Four step approach

For information on point transformations constructed between Hermitian Hamiltonians see\(^3\)

- Two time-dependent Schrödinger equations:

\[
H_0(\chi)\psi(\chi, \tau) = i\hbar \partial_\tau \psi(\chi, \tau) \quad \text{and} \quad H(x, t)\phi(x, t) = i\hbar \partial_t \phi(x, t)
\]

Reference Hamiltonian: \(H_0(\chi)\)  
Target Hamiltonian: \(H(x, t) \neq H^\dagger(x, t)\)

\(^3\)K. Zelaya and O. Rosas-Ortiz, Physica Scripta \textbf{95}(6), 064004 (2020).
Four step approach

For information on point transformations constructed between Hermitian Hamiltonians see\(^3\)

- Two time-dependent Schrödinger equations:
  \[
  H_0(\chi)\psi(\chi, \tau) = i\hbar \partial_\tau \psi(\chi, \tau) \quad \text{and} \quad H(x, t)\phi(x, t) = i\hbar \partial_t \phi(x, t)
  \]
  
  Reference Hamiltonian: \( H_0(\chi) \)
  Target Hamiltonian: \( H(x, t) \neq H^\dagger (x, t) \)

- Point transformation \( \Gamma \):
  \[
  \Gamma : H_0 - \text{TDSE} \rightarrow H - \text{TDSE} \quad [\chi, \tau, \psi(\chi, \tau)] \rightarrow [x, t, \phi(x, t)]
  \]
  \[
  \chi = P(x, t, \phi) \quad \tau = Q(x, t, \phi) \quad \psi = R(x, t, \phi)
  \]

\(^3\)K. Zelaya and O. Rosas-Ortiz, Physica Scripta 95(6), 064004 (2020).
Framework to determine $\rho(t)$ and $\eta(t)$

The reference Hamiltonian

The target Hamiltonian

Constructing the point transformation

**Four step approach**

- Construction of invariant

$$\Gamma : H_0(x) \rightarrow I_H(x,t)$$
Four step approach

- Construction of invariant

\[ \Gamma : H_0(x) \rightarrow I_H(x, t) \]

\[ \frac{dI_H(x, t)}{dt} = i\hbar \partial_t I_H(x, t) + [I_H(x, t), H(x, t)] = 0 \]
Four step approach

- Construction of invariant

\[ \Gamma : H_0(\chi) \rightarrow I_H(x, t) \]

\[ \frac{dI_H(x, t)}{dt} = i\hbar \partial_t I_H(x, t) + [I_H(x, t), H(x, t)] = 0 \]

- Determine \( \eta(t) \) and \( \rho(t) \):

\[ I_\hbar(t) = \eta(t) I_H(x, t) \eta(t)^{-1} \]
Four step approach

- Construction of invariant

\[ \Gamma : H_0(\chi) \rightarrow I_H(x, t) \]

\[ \frac{dI_H(x, t)}{dt} = i\hbar \partial_t I_H(x, t) + [I_H(x, t), H(x, t)] = 0 \]

- Determine \( \eta(t) \) and \( \rho(t) \):

\[ I_h(t) = \eta(t) I_H(x, t) \eta(t)^{-1} \]

\[ \Rightarrow h(t) = \eta(t) H(x, t) \eta^{-1} + i\hbar \partial_t \eta(t) \eta(t)^{-1} \]

\[ \Rightarrow \rho(t) = \eta(t) \eta^\dagger(t) \eta(t) \]
Outline

Introduction

Time-dependent non-Hermitian quantum systems

Point transformations

Invariant, Dyson map and metric

Conclusions

The reference Hamiltonian

The target Hamiltonian

Constructing the point transformation

The reference Hamiltonian

Time-independent harmonic oscillator:

\[ H_0(\chi) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\chi^2, \]

\[ \chi = \chi(x, t), \quad \tau(t), \quad \psi = A(x, t)\phi, \]

Simplify the calculation.

No \phi^2 x term so require \psi \phi \phi = 0.

Rebecca Tenney

Exactly solvable time-dependent non-Hermitian quantum systems from point transformations
The reference Hamiltonian

Time-independent harmonic oscillator:

\[ H_0(\chi) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 \chi^2, \quad P = -i\hbar \partial_\chi \]
The reference Hamiltonian

Time-independent harmonic oscillator:

\[ H_0(x) = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 x^2, \quad P = -i\hbar \partial_x \]

\[ \chi = \chi(x, t), \quad \tau(t), \quad \psi = A(x, t)\phi, \]
The reference Hamiltonian

Time-independent harmonic oscillator:

\[ H_0(\chi) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2\chi^2, \quad P = -i\hbar\partial_{\chi} \]

\[ \chi = \chi(x, t), \quad \tau(t), \quad \psi = A(x, t)\phi, \]

- Simplify the calculation.
The reference Hamiltonian

Time-independent harmonic oscillator:

\[ H_0(\chi) = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 \chi^2, \quad P = -i\hbar \partial_x \]

\[ \chi = \chi(x, t), \quad \tau(t), \quad \psi = A(x, t) \phi, \]

- Simplify the calculation.
- No \( \phi_x^2 \) term so require \( \psi_{\phi\phi} = 0. \)
The Reference Hamiltonian

Compute the total derivatives:

\[
\frac{d\psi}{dx} = \psi_x \chi_x = A\phi_x + A_x \phi \\
\frac{d\psi}{dt} = \psi_x \chi_t + \psi_t \tau_t = A\phi_t + A_t \phi \\
\frac{d^2\psi}{dx^2} = \psi_{xx} \chi_x^2 + \psi_x \chi_{xx} = A\phi_{xx} x 2A_x \phi_x + \psi A_{x,x}
\]
Compute the total derivatives:

\[
\frac{d\psi}{dx} = \psi_x x_x = A\phi_x + A_x \phi
\]

\[
\frac{d\psi}{dt} = \psi_x x_t + \psi_\tau \tau_t = A\phi_t + A_t \phi
\]

\[
\frac{d^2\psi}{dx^2} = \psi_{x,x} x_x^2 + \psi_x x_{x,x} = A\phi_{x,x} 2A_x \phi_x + \psi A_{x,x}
\]

→ Solve for \(\psi_x, \psi_\tau\) and \(\psi_{x,x}\) and sub into TDSE for \(H_0(\chi)\)
Point transformed TDSE:

\[ i\hbar \phi_t + \frac{\hbar^2}{2m} \frac{\tau_t}{\chi^2_x} \phi_{xx} + B_0(x, t) \phi_x - V_0(x, t) \phi = 0 \quad (*) \]

where

\[ B_0(x, t) = -i\hbar \frac{\chi_t}{\chi_x} + \frac{\hbar^2}{2m} \frac{\tau_t}{\chi^2_x} \left( 2 \frac{A_x}{A} - \frac{\chi_{xx}}{\chi_x} \right) \]

\[ V_0(x, t) = \frac{1}{2} m \omega^2 \tau_t \chi^2 - i\hbar \left( \frac{A_t}{A} - \frac{A_x \chi_t}{A \chi_x} \right) - \frac{\hbar^2}{2m} \frac{\tau_t}{\chi^2_x} \left( \frac{A_{xx}}{A} - \frac{A_x \chi_{xx}}{A \chi_x} \right) \]
Point transformed TDSE:

\[ i\hbar \dot{\phi}_t + \frac{\hbar^2}{2m} \frac{\tau_t}{\chi_x^2} \phi_{xx} + B_0(x, t) \phi_x - V_0(x, t) \phi = 0 \quad (*) \]

where

\[ B_0(x, t) = -i\hbar \frac{\chi_t}{\chi_x} + \frac{\hbar^2}{2m} \frac{\tau_t}{\chi_x^2} \left( 2 \frac{A_x}{A} - \frac{\chi_{xx}}{\chi_x} \right) \]

\[ V_0(x, t) = \frac{1}{2} m \omega^2 \tau_t \chi^2 - i\hbar \left( \frac{A_t}{A} - \frac{A_x \chi_t}{A \chi_x} \right) - \frac{\hbar^2}{2m} \frac{\tau_t}{\chi_x^2} \left( \frac{A_{xx}}{A} - \frac{A_x \chi_{xx}}{A \chi_x} \right) \]

→ Compare (*) directly with TDSE for target Hamiltonian \( H(x, t) \) and solve for \( A, \chi \) and \( \tau \).
The reference Hamiltonian - other choices

\[ i\hbar \phi_t + \frac{\hbar^2}{2m} \frac{\tau_t}{\chi^2} \phi_{xx} + B_i(x, t) \phi_x - V_i(x, t) \phi = 0 \]
The reference Hamiltonian - other choices

\[ i\hbar \dot{\phi} + \frac{\hbar^2}{2m} \frac{\tau_t}{\chi^2} \phi_{xx} + B_i(x, t) \phi_x - V_i(x, t) \phi = 0 \]

\[ H_0^{(1)}(\chi) = \frac{P^2}{2m}, \quad B_1(x, t) = B_0(x, t), \quad V_1(x, t) = V_0(x, t) - \frac{1}{2} m \omega^2 \chi^2 \tau_t \]
The reference Hamiltonian - other choices

\[ i\hbar \phi_t + \frac{\hbar^2}{2m} \frac{\tau_t}{\chi_x^2} \phi_{xx} + B_i(x, t) \phi_x - V_i(x, t) \phi = 0 \]

\[ H_0^{(1)}(\chi) = \frac{P^2}{2m}, \quad B_1(x, t) = B_0(x, t), \quad V_1(x, t) = V_0(x, t) - \frac{1}{2} m \omega^2 \chi^2 \tau_t \]

\[ H_0^{(2)}(\chi) = H_0(\chi) + a \chi, \quad B_2(x, t) = B_0(x, t), \quad V_2(x, t) = V_0(x, t) + a \chi \tau_t \]
The reference Hamiltonian - other choices

\[ \begin{align*}
  i\hbar \phi_t + \frac{\hbar^2}{2m} \frac{\tau_t}{\chi_x^2} \phi_{xx} + B_i(x, t) \phi_x - V_i(x, t) \phi = 0 \\
  H_0^{(1)}(\chi) = \frac{P^2}{2m}, \quad B_1(x, t) = B_0(x, t), \quad V_1(x, t) = V_0(x, t) - \frac{1}{2} m \omega^2 \chi^2 \tau_t \\
  H_0^{(2)}(\chi) = H_0(\chi) + a \chi, \quad B_2(x, t) = B_0(x, t), \quad V_2(x, t) = V_0(x, t) + a \chi \tau_t \\
  H_0^{(3)}(\chi) = H_0(\chi) + a \{ \chi, P \}, \quad B_3(x, t) = B_0(x, t) + \frac{2ia\hbar\chi \tau_t}{\chi_x}, \quad V_3(x, t) = V_0(x, t) - \frac{2ia\chi_x \tau_t}{A_{\chi_x}} - ia\hbar \tau_t
\end{align*} \]
The time-dependent Swanson model

\[ \tilde{H}_S(t) = \omega(t) \left( a^\dagger a + 1/2 \right) + \bar{\alpha}(t) a^2 + \bar{\beta}(t) \left( a^\dagger \right)^2, \quad \bar{\alpha} \neq \bar{\beta}^* \]

\[ a = (x + ip)/2, \quad a^\dagger = (x - ip)/2 \]
The time-dependent Swanson model

\[
\tilde{H}_S(t) = \omega(t) \left( a^\dagger a + 1/2 \right) + \tilde{\alpha}(t) a^2 + \tilde{\beta}(t) \left( a^\dagger \right)^2, \quad \tilde{\alpha} \neq \tilde{\beta}^* \\
 a = (x + ip)/2, \quad a^\dagger = (x - ip)/2
\]

\[\tilde{\alpha} = \frac{M\Omega^2}{4} - \frac{1}{4M} + \alpha, \quad \tilde{\beta} = \frac{M\Omega^2}{4} - \frac{1}{4M} - \alpha, \quad \omega = \frac{M\Omega^2}{2} + \frac{1}{2M},\]

\[H_S(x, t) := \tilde{H}_S(t) - \frac{\omega(t)}{2} = \frac{p^2}{2M(t)} + \frac{M(t)}{2} \Omega(t)^2 x^2 + i\alpha(t)\{x, p\}, \quad M, \Omega \in \mathbb{R}, \ \alpha \in \mathbb{C}\]
The time-dependent Swanson model

\[ \tilde{H}_S(t) = \omega(t) \left( a^\dagger a + 1/2 \right) + \tilde{\alpha}(t) a^2 + \tilde{\beta}(t) \left( a^\dagger \right)^2, \quad \tilde{\alpha} \neq \tilde{\beta}^* \]

\[ a = (x + ip)/2, \quad a^\dagger = (x - ip)/2 \]

\[ \tilde{\alpha} = \frac{M\Omega^2}{4} - \frac{1}{4M} + \alpha, \quad \tilde{\beta} = \frac{M\Omega^2}{4} - \frac{1}{4M} - \alpha, \quad \omega = \frac{M\Omega^2}{2} + \frac{1}{2M}, \]

\[ H_S(x, t) := \tilde{H}_S(t) - \frac{\omega(t)}{2} = \frac{p^2}{2M(t)} + \frac{M(t)}{2} \Omega(t)^2 x^2 + i\alpha(t)\{x, p\}, \quad M, \Omega \in \mathbb{R}, \quad \alpha \in \mathbb{C} \]

\[ \mathcal{PT} : x \rightarrow -x, \quad p \rightarrow p, \quad i \rightarrow -i, \quad (M, \Omega, \alpha) \rightarrow (M, \Omega, \alpha), \quad \alpha = \alpha_r + i\alpha_i \]

\[ \alpha_r \rightarrow \alpha_r, \quad \alpha_i \rightarrow -\alpha_i \]
Point transformation $\Gamma^S_0 : H_0(\chi) \rightarrow H_s(x, t)$

Point transformed TDSE:

$$i\hbar \psi_t + \frac{\hbar^2}{2m} \frac{\tau_t}{\chi_x^2} \phi_{xx} + B_0(x, t)\phi_x - V_0(x, t)\phi = 0 \quad (*)$$

TDSE for $H_s(x, t)$:

$$i\hbar \phi_t + \frac{\hbar^2}{2M(t)} \phi_{xx} - 2\hbar\alpha(t)x \phi_x - \hbar\alpha(t)\phi - \frac{1}{2} M(t)\Omega(t)^2 x^2 \phi = 0, \quad (\circ)$$
Point transformation $\Gamma^S_0 : H_0(\chi) \rightarrow H_s(x, t)$

Point transformed TDSE:

$$i\hbar \psi_t + \frac{\hbar^2}{2m} \frac{\tau_t}{\chi_x^2} \phi_{xx} + B_0(x, t)\phi_x - V_0(x, t)\phi = 0 \quad (*)$$

TDSE for $H_s(x, t)$:

$$i\hbar \phi_t + \frac{\hbar^2}{2M(t)} \phi_{xx} - 2\hbar\alpha(t)x\phi_x - \hbar\alpha(t)\phi - \frac{1}{2} M(t)\Omega(t)^2x^2\phi = 0, \quad (\circ)$$

Compare $(*)$ and $(\circ)$:

$$\frac{\tau_t}{m\chi_x^2} = \frac{1}{M(t)},$$
Point transformation $\Gamma^S_0 : H_0(\chi) \rightarrow H_s(x, t)$

Point transformed TDSE:

$$i\hbar \psi_t + \frac{\hbar^2}{2m} \frac{\tau_t}{\chi_x^2} \phi_{xx} + B_0(x, t) \phi_x - V_0(x, t) \phi = 0 \quad (*)$$

TDSE for $H_s(x, t)$:

$$i\hbar \phi_t + \frac{\hbar^2}{2M(t)} \phi_{xx} - 2\hbar \alpha(t) x \phi_x - \hbar \alpha(t) \phi - \frac{1}{2} M(t) \Omega(t)^2 x^2 \phi = 0, \quad (\circ)$$

Compare $(*)$ and $(\circ)$:

$$\frac{\tau_t}{m\chi_x^2} = \frac{1}{M(t)}, \quad B(x, t) = -2\hbar \alpha(t) x,$$
Point transformation $\Gamma^S_0 : H_0(\chi) \to H_s(x, t)$

Point transformed TDSE:

$$i\hbar \psi_t + \frac{\hbar^2}{2m} \frac{\tau_t}{\chi_x^2} \phi_{xx} + B_0(x, t)\phi_x - V_0(x, t)\phi = 0 \quad (*)$$

TDSE for $H_s(x, t)$:

$$i\hbar \phi_t + \frac{\hbar^2}{2M(t)} \phi_{xx} - 2\hbar\alpha(t)x\phi_x - \hbar\alpha(t)\phi - \frac{1}{2} M(t)\Omega(t)^2 x^2 \phi = 0, \quad (\circ)$$

Compare $(*)$ and $(\circ)$:

$$\frac{\tau_t}{m\chi_x^2} = \frac{1}{M(t)}, \quad B(x, t) = -2\hbar\alpha(t)x, \quad V(x, t) = \frac{1}{2} M(t)\Omega(t)^2 x^2 + \hbar\alpha(t)$$

Rebecca Tenney
Solution:

\[ M(t) = m \sigma^{-r-2s}, \quad \tau(t) = \int_0^t \sigma(y)^r \, dy, \quad \chi(x, t) = \frac{x + \gamma(t)}{\sigma(t)^s} \]
Point transformation $\Gamma_0^S : H_0(\chi) \rightarrow H_s(x, t)$

Solution:

$M(t) = m\sigma^{-r-2s}$, \quad $\tau(t) = \int_{0}^{t} \sigma(y)^{r} dy$, \quad $\chi(x, t) = \frac{x + \gamma(t)}{\sigma(t)^s}$

$\gamma = 0$:

$A(x, t) = \exp \left\{ \frac{i m \sigma^{-1-r-2s}}{\hbar} \left[ \left( i \alpha \sigma - \frac{1}{2} s \sigma_t \right) x^2 + \delta(t) \right] \right\}$

$\delta(t) = \sigma^{1+r+2s} \left( c_1 - \frac{i s \hbar}{2m} \log \sigma \right)$
Point transformation $\Gamma^S_0 : H_0(\chi) \to H_s(x, t)$

Solution:

$$M(t) = m\sigma^{-r-2s}, \quad \tau(t) = \int^t \sigma(y)^r\,dy, \quad \chi(x, t) = \frac{x + \gamma(t)}{\sigma(t)^s}$$

$\gamma = 0$:

$$A(x, t) = \exp\left\{ \frac{i\sigma^{-1-r-2s}}{\hbar} \left[ \left( i\alpha\sigma - \frac{1}{2}s\sigma_t \right)x^2 + \delta(t) \right]\right\}$$

$$\delta(t) = \sigma^{1+r+2s} \left( c_1 - \frac{ish}{2m} \log \sigma \right)$$

$\mathcal{PT}$-symmetry:

$$\alpha_i = \frac{1}{4} \partial_t \ln \left( \frac{\sigma^{r+2s}}{\alpha_r} \right)$$
Point transformation $\Gamma_0^S : H_0(\chi) \rightarrow H_s(x, t)$

Solution:

$$M(t) = m\sigma^{-r-2s}, \quad \tau(t) = \int_0^t \sigma(y)^r dy, \quad \chi(x, t) = \frac{x + \gamma(t)}{\sigma(t)^s}$$

$\gamma = 0$:

$$A(x, t) = \exp \left\{ \frac{i m \sigma^{-1-r-2s}}{\hbar} \left[ \left( i\alpha \sigma - \frac{1}{2} s \sigma_t \right) x^2 + \delta(t) \right] \right\}$$

$$\delta(t) = \sigma^{1+r+2s} \left( c_1 - \frac{i \hbar}{2m} \log \sigma \right)$$

$\mathcal{PT}$-symmetry:

$$\alpha_i = \frac{1}{4} \partial_t \ln \left( \frac{\sigma^{r+2s}}{\alpha_r} \right) \rightarrow \alpha_i \propto \partial_t$$
Point transformation $\Gamma_0^S : H_0(\chi) \rightarrow H_s(x, t)$

Solution:

$$M(t) = m\sigma^{-r-2s}, \quad \tau(t) = \int_0^t \sigma(y)^r dy, \quad \chi(x, t) = \frac{x + \gamma(t)}{\sigma(t)^s}$$

$\gamma = 0$:

$$A(x, t) = \exp\left\{ \frac{i m \sigma^{-1-r-2s}}{\hbar} \left[ \left( i\alpha \sigma - \frac{1}{2} s_\sigma t \right) x^2 + \delta(t) \right] \right\}$$

$$\delta(t) = \sigma^{1+r+2s} \left( c_1 - \frac{ish}{2m} \log \sigma \right)$$

$\mathcal{PT}$-symmetry:

$$\alpha_i = \frac{1}{4} \partial_t \ln \left( \frac{\sigma^{r+2s}}{\alpha_r} \right) \quad \rightarrow \quad \alpha_i \propto \partial_t \quad \rightarrow \quad \mathcal{PT} : \alpha_i \rightarrow -\alpha_i$$
Point transformation $\Gamma^S_0 : H_0(\chi) \rightarrow H_s(x, t)$

Auxiliary equation:

$$\sigma_{tt} = \sigma \left[ \frac{2\alpha_r(2\Omega^2\alpha_r + 8\alpha_r^3 + (\alpha_r)_{tt}) - 3(\alpha_r)^2_t}{2r\alpha_r^2} \right] + \frac{(\frac{r}{2} + 1)\sigma_t^2}{\sigma} - \frac{2\omega^2\sigma^{2r+1}}{r}$$

---

4. V. Ermakov, Univ. Izv. Kiev. 20, 1-19 (1880).
5. E. Pinney, Proc. Amer. Math. Soc 1, 681(1) (1950).
Point transformation $\Gamma_0^S : H_0(\chi) \to H_s(x, t)$

Auxiliary equation:

$$\sigma_{tt} = \sigma \left[ \frac{2\alpha_r(2\Omega^2\alpha_r + 8\alpha_r^3 + (\alpha_r)_{tt}) - 3(\alpha_r)^2_t}{2r\alpha_r^2} \right] + \frac{(r/2 + 1)\sigma_t^2}{\sigma} - \frac{2\omega^2\sigma^{2r+1}}{r}$$

Many choices for $r, s$ and $\alpha_r$, for example:

**Time-independent mass:**

- $\alpha_r = c_2\sigma^{r+2s}$, $r = -2s$, $s = 1 \rightarrow \alpha_i = 0$, $\alpha$ is time-independent,
- $\sigma_{tt} = -c_2^2\sigma + \frac{\omega^2}{\sigma^3} - \sigma\Omega^2$

---

4. V. Ermakov, Univ. Izv. Kiev. **20**, 1-19 (1880).
5. E. Pinney, Proc. Amer. Math. Soc **1**, 681(1) (1950).
Point transformation $\Gamma^S_0 : H_0(\chi) \rightarrow H_s(x, t)$

Auxiliary equation:

$$\sigma_{tt} = \sigma \left[ \frac{2\alpha_r(2\Omega^2\alpha_r + 8\alpha_r^3 + (\alpha_r)_{tt}) - 3(\alpha_r)_t^2}{2r\alpha_r^2} \right] + \frac{(r + 1)\sigma^2_t}{\sigma} - \frac{2\omega^2\sigma^{2r+1}}{r}$$

Many choices for $r, s$ and $\alpha_r$, for example:

Time-independent mass:

- $\alpha_r = c_2\sigma^{r+2s}$, $r = -2s$, $s = 1 \rightarrow \alpha_i = 0$, $\alpha$ is time-independent,

- $\sigma_{tt} = -c_2^2\sigma + \frac{\omega^2}{\sigma^3} - \sigma\Omega^2$

$\alpha$ complex:

- $\alpha_r = \sigma^{-2-r}$, $r = 0$

- $\sigma_{tt} = \frac{4}{\sigma^3} + \sigma(\Omega^2 - \omega^2)$

$\sigma$ : non-linear Ermakov-Pinney equation

---

4. V. Ermakov, Univ. Izv. Kiev. 20, 1-19 (1880).

5. E. Pinney, Proc. Amer. Math. Soc 1, 681(1) (1950).
Construct non-Hermitian invariant from the point transformation:

$$\Gamma_0^S : H_0(\chi) \rightarrow I_H(x, t)$$
Construct non-Hermitian invariant from the point transformation:

\[ \Gamma^S_0 : H_0(\chi) \rightarrow \hat{l}_H(x, t) \]

\[ \hat{l}_H(x, t) = \frac{\sigma^{2s}}{2m} p^2 + \frac{\sigma^{-r-1} (4i\sigma \alpha_r^2 + r\alpha_r\sigma_t - \sigma \alpha_{rt})}{4\alpha_r} \{x, p\} \]

\[ + \frac{\sigma^{-2(r+s+1)} [4m\omega^2 \alpha_r^2 \sigma^{2r+2} - m (4\sigma \alpha_r^2 - i\alpha_r\sigma_t + i\sigma \alpha_{rt})^2]}{8\alpha_r^2} x^2 \]
Construct non-Hermitian invariant from the point transformation:

\[ \Gamma^S_0 : H_0(\chi) \rightarrow I_H(x, t) \]

\[
\hat{I}_H(x, t) = \frac{\sigma^{2s}}{2m} p^2 + \frac{\sigma^{-r-1}}{4\alpha_t} \left( 4i\sigma\alpha_t^2 + r\alpha_t\sigma_t - \sigma\alpha_{rt} \right) \{x, p\}
\]

\[
+ \frac{\sigma^{-2(r+s+1)}}{8\alpha_t^2} \left[ 4m\omega^2\alpha_t^2\sigma^{2r+2} - m \left( 4\sigma\alpha_t^2 - ir\alpha_t\sigma_t + i\sigma\alpha_{rt} \right)^2 \right] x^2
\]

Different reference Hamiltonians \( \rightarrow \) different invariants.
The Dyson map and metric

All invariants can be written as \( (\gamma \neq 0) \)

\[
l_H = a_r p^2 + b_r p + (c_r + i c_i) \{ x, p \} + (d_r + i d_i) x^2 + (e_r + i e_i) x + f_r,
\]
The Dyson map and metric

All invariants can be written as ($\gamma \neq 0$)

$$I_H = a_r p^2 + b_r p + (c_r + ic_i) \{x, p\} + (d_r + id_i) x^2 + (e_r + ie_i) x + f_r,$$

Property:

$$\frac{e_i}{2b_r} = \frac{d_i}{4c_r} = \frac{c_i}{2a_r} = m_{\alpha r} \sigma^{-r-2s}.$$
The Dyson map and metric

All invariants can be written as ($\gamma \neq 0$)

$$I_H = a_r p^2 + b_r p + (c_r + i c_i) \{x, p\} + (d_r + i d_i) x^2 + (e_r + i e_i) x + f_r,$$

Property:

$$\frac{e_i}{2b_r} = \frac{d_i}{4c_r} = \frac{c_i}{2a_r} = m \alpha_r \sigma^{-r-2s}. \rightarrow \eta(t) = \exp \left[ -m \alpha_r \sigma^{-r-2s} x^2 \right],$$
All invariants can be written as \((\gamma \neq 0)\)

\[
I_H = a_r p^2 + b_r p + (c_r + ic_i) \{x, p\} + (d_r + id_i) x^2 + (e_r + ie_i) x + f_r,
\]

Property:

\[
\frac{e_i}{2b_r} = \frac{d_i}{4c_r} = \frac{c_i}{2a_r} = m\alpha r^{-r-2s}. \rightarrow \eta(t) = \exp \left[ -m\alpha r^{-r-2s} x^2 \right],
\]

Substitute \(\eta(t)\) into TDDE

\[
h = \frac{\sigma^{r+2s}}{2m} p^2 + \left( 2m\alpha^2 \sigma^{-r-2s} + \frac{1}{2} m\sigma^{-r-2s} \Omega^2 \right) x^2 + \frac{1}{4} \partial_t \ln \left( \frac{\sigma^{r+2s}}{\alpha_r} \right) \{x, p\}.
\]
Point transformations can be used to construct non-Hermitian invariants for time-dependent non-Hermitian systems.

---

6 A. Fring and R. Tenney, arXiv:2108.06793 [quant-ph]

7 A. Fring and R. Tenney, Phys. Lett. A. 410 127548 (2021)
Conclusions

- Point transformations can be used to construct non-Hermitian invariants for time-dependent non-Hermitian systems.
- Increase number of steps to obtain Dyson map, bypass ansatz for invariant.

---

6 A. Fring and R. Tenney, arXiv:2108.06793 [quant-ph]
7 A. Fring and R. Tenney, Phys. Lett. A. 410 127548 (2021)
Conclusions

- Point transformations can be used to construct non-Hermitian invariants for time-dependent non-Hermitian systems.
- Increase number of steps to obtain Dyson map, bypass ansatz for invariant.
- Easier to determine Dyson maps $\rightarrow$ easier to determine multiple maps $\rightarrow$ construct an infinite series of Dyson maps\(^6\).

---

\(^6\) A. Fring and R. Tenney, arXiv:2108.06793 [quant-ph]

\(^7\) A. Fring and R. Tenney, Phys. Lett. A. 410 127548 (2021)
Point transformations can be used to construct non-Hermitian invariants for time-dependent non-Hermitian systems.

Increase number of steps to obtain Dyson map, bypass ansatz for invariant.

Easier to determine Dyson maps $\rightarrow$ easier to determine multiple maps $\rightarrow$ construct an infinite series of Dyson maps$^6$.

Reference Hamiltonian need not be exactly solvable to obtain invariant for target Hamiltonian. Reference Hamiltonian doesn’t need to be Hermitian$^7$.

---

$^6$ A. Fring and R. Tenney, arXiv:2108.06793 [quant-ph]

$^7$ A. Fring and R. Tenney, Phys. Lett. A. 410 127548 (2021)
Conclusions

- Point transformations can be used to construct non-Hermitian invariants for time-dependent non-Hermitian systems.
- Increase number of steps to obtain Dyson map, bypass ansatz for invariant.
- Easier to determine Dyson maps $\rightarrow$ easier to determine multiple maps $\rightarrow$ construct an infinite series of Dyson maps$^6$.
- Reference Hamiltonian need not be exactly solvable to obtain invariant for target Hamiltonian. Reference Hamiltonian doesn’t need to be Hermitian$^7$.
- Apply technique to determine Dyson maps for more complicated systems.

Thank you for your attention.

---

$^6$ A. Fring and R. Tenney, arXiv:2108.06793 [quant-ph]

$^7$ A. Fring and R. Tenney, Phys. Lett. A. 410 127548 (2021)