Invisibility in billiards

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Abstract
The problem of invisibility for bodies with a mirror surface is studied in the framework of geometrical optics. A closely related problem concerning the existence of bodies that have zero aerodynamical resistance is also studied here. We construct bodies that are invisible/have zero resistance in two directions, and prove that bodies which are invisible/have zero resistance do not exist in all possible directions of incidence.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Problems related to constructing invisible bodies are the focus of much recent attention. Apart from having the potential for various applications, such as constructing invisible submarines, creating improved lenses for DVD readers that would allow us to read denser information, the topic attracts the attention of the general public, mostly due to the concept of the invisible cloak, which is a popular idea in fiction and movies (see [4, 7, 8]). Apart from the various implementations of such a cloak, which use cameras to project the image from behind on a specially designed surface (e.g., see [11]), the bulk of studies on invisible bodies focus on constructing materials with special refractive properties. The research in this direction was pioneered by Veselago in 1960s, who published a theoretical study of materials that allowed for a negative refractive index [21]. Such materials do not exist in nature; however, they can be engineered. They are called metamaterials and have been successfully constructed for some limited settings. Following [19], researchers at Duke University have demonstrated a body invisible for microwaves (see [4]); in [20] construction of a 3D optical metamaterial with a negative refraction index was reported; in [9] metamaterials are used to hide a bump in a metallic mirror from angles up to 60° and a large bandwidth of unpolarized light.
In this paper we are concerned with invisibility in billiards. We consider bodies with a mirror surface and light rays falling on it. Invisibility in a direction \(v \in S^2\) means that any incident light ray which initially moves along a straight line in this direction, after several reflections from the body surface, will eventually move along the same straight line. Invisibility in a set of directions means that the above is true for any direction from this set. In [1] the notion of billiard invisibility was introduced and some examples of bodies invisible in one direction were provided. In this paper we continue the study of this topic; our results are twofold. First, we show that bodies invisible in two directions exist. Second, we prove that bodies invisible in all directions in \(S^2\) do not exist.

Note that somewhat similar results were obtained in wave scattering. It was shown, in the first Born approximation, that bodies invisible for any finite number of directions exist [10], and there are no bodies invisible for all directions of incidence [23].

There is a closely related sort of problem. Consider a parallel flow of point particles at a velocity \(v \in S^2\) falling on a body \(B\) at rest. The flow is so rarefied that the particles do not mutually interact. Particles reflect elastically when colliding with the body surface and move freely between consecutive collisions. The problem of minimal resistance going back to Newton [15] consists in finding a body, from a given class of bodies, that experiences the smallest possible force of flow pressure, or resistance force. Since the 1990s, many interesting results from this problem have been obtained by various authors (see, e.g. [2, 3, 6, 12, 13, 16, 17]). A body of zero resistance was provided in [1]; this means that the final velocity of any particle incident on the body coincides with the velocity of incidence \(v\).

In this paper a body having zero resistance in two directions is constructed, and it is proved that bodies having zero resistance in all directions do not exist. Note that invisibility implies zero resistance, therefore any body invisible in two directions will have zero resistance in these directions, and the impossibility of zero resistance in all directions implies that invisibility in all directions is also impossible.

There are many questions still open. Do bodies exist that are invisible/have zero resistance in three or more directions, or even in a set of directions of positive measure? We suppose that the answer to the last part of the question is negative, but cannot prove it.

We start with exact definitions.

**Definition 1.** A body is a bounded set with a piecewise smooth boundary in \(\mathbb{R}^3\).

Consider the billiard in \(\mathbb{R}^3 \setminus B\), and take a convex body \(C\) containing \(B\). For a regular point of the boundary \(\xi \in \partial C\), denote by \(n(\xi)\) the unit outer normal to \(\partial C\) at \(\xi\). Introduce the measurable spaces \((\partial C \times S^2)_{\pm} := \{(\xi, v) \in \partial C \times S^2 : \pm n(\xi) \cdot v \geq 0\}\) equipped with the measures \(d\mu_{\pm}(\xi, v) = \pm (n(\xi) \cdot v) \, d\xi \, dv\), correspondingly, where the dot means a scalar product. Note that the set of singular points of any convex set has zero Lebesgue measure, therefore the union \((\partial C \times S^2)_- \cup (\partial C \times S^2)_+\) is a full measure set in \(\partial C \times S^2\).

The motion of a billiard particle interacting with the body \(B\) can be generally described as follows. First the particle moves freely with a velocity \(v\), then intersects \(\partial C\) at a point \(\xi\) and moves in \(C\) making reflections from \(\partial B\), and finally, leaves \(C\) at a point \(\xi^+\) and moves freely with a velocity \(v^+\) afterwards (see figure 1). According to this description, a mapping \((\xi, v) \mapsto (\xi^+ = \xi^+_{B,C}(\xi, v), v^+ = v^+_{B,C}(\xi, v))\) is defined, which is a measure preserving one-to-one correspondence between full measure subsets of \((\partial C \times S^2)_-\) and \((\partial C \times S^2)_+\).

Note that for a zero measure set of values \((\xi, v) \in (\partial C \times S^2)_-\), the corresponding particle hits \(\partial B\) at a singular point, or gets trapped in \(C\), or makes infinitely many reflections in a finite time. For these values the mapping is not defined.
Figure 1. The broken line with the endpoints $\xi$ and $\xi^+$ is a billiard trajectory in the complement of $B$. The straight line with the endpoints $\xi$ and $\xi^+$ is a trajectory corresponding to the case $B = \emptyset$.

Figure 2. A typical billiard path in the cases of (a) a body having zero resistance in the direction $v$; (b) a body invisible in the direction $v$. The body is not shown in both cases.

For future use we introduce the notation $\xi^+_{0,C} := \xi^+_{\emptyset,C}$, corresponding to the case $B = \emptyset$ where all particles move freely inside $C$; see figure 1.

Definition 2.

(a) We say that the body $B$ has zero resistance in the direction $v$, if $v^+_{B,C}(\xi, v) = v$ for all $\xi$ (see figure 2(a)).

(b) We say that the body $B$ is invisible in the direction $v$, if it has zero resistance in this direction and, additionally, $\xi^+_{B,C}(\xi, v) - \xi$ is parallel to $v$ (see figure 2(b)).

(c) Let $A \subset S^2$. The body $B$ is said to be invisible/have zero resistance in the set of directions $A$, if it is invisible/has zero resistance in any direction $v \in A$.

One easily sees that this definition does not depend on the choice of the ambient body $C$.

Remark 1. This definition readily generalizes to any dimension.

The plan of the paper is as follows. In section 2 we construct bodies of zero resistance in two mutually orthogonal directions and bodies invisible in two directions. In section 3 we prove that bodies invisible in all directions and bodies having zero resistance in all directions do not exist.
2. Bodies invisible in two directions

**Theorem 1.** For any two mutually perpendicular directions \( v_1 \) and \( v_2 \) ∈ \( S^2 \),

(a) there exists a body having zero resistance in both directions;

(b) there exists a body invisible in these directions.

**Proof.** We first construct a basic two-dimensional body that has zero resistance in one direction, and then extend the construction to a three-dimensional body having zero resistance in two directions.

Take a plane \( \Pi \) containing \( v_1 \) and perpendicular to \( v_2 \), and consider two parabolas in this plane that have a common focus and common axis parallel to \( v_1 \), and are centrally symmetric to each other with respect to the focus. Take two straight lines in the same plane parallel to the common axis of the parabolas and situated at the same distance on both sides of it. Next, consider two curvilinear triangles formed by segments of these straight lines and by arcs of the parabolas, see figure 3(a). The union of these triangles is a (disconnected) two-dimensional figure having zero resistance to a parallel flow on the plane falling at the velocity \( v_1 \). Indeed, taking into account the focal property of parabola, we see that any incident particle of the flow, after reflecting from a parabola, passes through the focus, then reflects from the other parabola, and moves afterwards with the velocity \( v_1 \). That is, a parallel flow with velocity \( v_1 \) is transformed into a parallel flow with the same velocity.

Note also that the union of two trapezoids bounded by arcs of the parabolas and by two pairs of straight lines (where two lines in each pair are parallel to the axis and symmetric to each other with respect to it) is also a figure of zero resistance in the direction \( v_1 \) (see figure 3(b)).

Then we obtain a three-dimensional body \( B_1 \) invisible in the same direction \( v_1 \) by parallel translation of the two-dimensional figure of figure 3(a) in the direction \( v_2 \) orthogonal to the plane of the figure (see figure 4(a)). The length \( h \) of this translation is equal to the height of the figure (that is, to the length of the rectilinear side of a triangle), \( h = 2(\alpha \beta^2 - (1/4 \alpha)) \). Then we construct another body \( B_2 \) by rotating \( B_1 \) by \( \pi/2 \) around its symmetry axis perpendicular to \( v_1 \) and \( v_2 \) (see figure 4(b)). The resulting body \( B_2 \) has zero resistance in the direction \( v_2 \). Finally, we show that the body \( B = B_1 \cap B_2 \) (see figure 4(c)) has zero resistance in both directions \( v_1 \) and \( v_2 \).

Indeed, the intersection of \( B \) with any plane parallel to \( \Pi \) is a union of two curvilinear trapezoids, besides the outer normal vector to \( \partial B \) at any point of a curvilinear side of the trapezoids is parallel to \( \Pi \). Therefore any incident particle that initially moves in this plane
with the velocity $v_1$, after two reflections from curvilinear sides of the trapezoids will eventually move in the same plane and with the same velocity $v_1$. Therefore $B$ has zero resistance in the direction $v_1$. For $v_2$ the argument is the same.

To obtain a body invisible in the directions $v_1$ and $v_2$, it suffices to take a union of four identical bodies obtained from $B$ by shifts by 0, $hv_1$, $hv_2$ and $hv_1 + hv_2$ (see figure 5). □

3. Non-existence of bodies invisible in all directions

**Theorem 2.** Bodies do not exist that
(a) are invisible in all directions;
(b) have zero resistance in all directions.

**Proof.** Let us first outline the idea of the proof. Note that statement (b) of the theorem implies statement (a), but for methodological reasons we first prove (a), and then (b).
The phase space of the billiard in $C \setminus B$ is $(C \setminus B) \times S^2$, with the coordinate $(x, v)$ and the element of Liouville phase volume $dx \, dv$. Taking into account that the area of the unit sphere is $|S^2| = 4\pi$, we find that the volume of phase space equals $4\pi |C \setminus B|$.

The phase volume can be estimated in a different way. Summing up the lengths of all billiard trajectories (of course summation amounts to integration over the initial data), we obtain the volume of the reachable part of the phase space. Comparing the case of an invisible body $B$ (assuming that such a body exists) with the case $B = \emptyset$ (where the body is absent) and comparing the lengths of trajectories with identical initial data, we see that the length of a trajectory in the first case is always greater or equal than the length in the second one, therefore the phase volume is also greater in the first case, $4\pi |C \setminus B| \geq 4\pi |C|$. This inequality contradicts the fact that any body occupies a positive volume.

The case of a hypothetical zero resistance body $B$ is a little bit more complicated. We compare it with the case where $B = \emptyset$ and show that the sum of lengths of the billiard trajectories with fixed initial velocity in the first case is greater than in the second one. Then, summing up over all initial velocities, again we come to the conclusion that the phase volume in the first case is greater or equal than in the second one.

Let us pass to a more precise exposition. Suppose a billiard particle starts the motion at a point $\xi \in \partial C$ and with the initial velocity $v \in S^2$ turned inside $C$ (which means that $n(\xi) \cdot v \leq 0$), and let $t \geq 0$; then assign the new coordinate $(\xi, v, t)$ to the point of the phase space reached by the particle in the time $t$. The element of phase volume then takes the form $(-n(\xi) \cdot v) \, d\xi \, dv \, dt = d\mu_-(\xi, v) \, dt$. Further, denote by $\tau(\xi, v)$ the length of the particle’s trajectory inside $C$, from the starting point $\xi$ until the point $\xi^+ = \xi_{B,C}^+(\xi, v)$ where it leaves $C$. Recall that $(\partial C \times S^2)_+ = \{ (\xi, v) \in \partial C \times S^2 : \pm n(\xi) \cdot v \geq 0 \}$. Then the volume of the reachable part of the phase space equals

$$\int_{(\partial C \times S^2)_+} \tau(\xi, v) \, d\mu_-(\xi, v) = \int_{(\partial C \times S^2)_+} \tau(\xi, v) \, d\mu_-(\xi, v) \leq 4\pi |C \setminus B|. \tag{1}$$

Recall that $\xi^+ = \xi_{B,C}^+(\xi, v)$. Taking into account that the distance between the initial and final points of the trajectory does not exceed its length,

$$|\xi^+ - \xi| \leq \tau(\xi, v), \tag{2}$$

and at some point $(\xi, v)$ (and therefore in its neighbourhood) the inequality in (2) is strict, we obtain

$$\int_{(\partial C \times S^2)_+} |\xi^+ - \xi| \, d\mu_-(\xi, v) < 4\pi |C \setminus B|. \tag{3}$$

Now let $\xi_0^+ = \xi_{B,C}^+(\xi, v)$ be the point where the particle leaves $C$ in the case $B = \emptyset$. In other words, $\xi_0^+$ is the point of intersection of the ray $\xi + vt, \ t > 0$ with $\partial C$. In this case all the phase space is reachable, besides one has equality in (2), therefore in place of (3) one obtains the equality

$$\int_{(\partial C \times S^2)_+} |\xi_0^+ - \xi| \, d\mu_-(\xi, v) = 4\pi |C|. \tag{4}$$

If $B$ is invisible in all directions then $\xi_0^+ = \xi^+$, therefore from (3) and (4) one obtains

$$4\pi |C| < 4\pi |C \setminus B|,$$

which is a contradiction.

**Remark 2.** Formulae (1) and (4) are known in integral geometry, geometric probability and billiards [14, 18, 22]; see also a brief review in the paper by Chernov [5].
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Here, again, for an open set of values of $v$ over $\Omega_1$ and thus, the inequality is strict. Integrating both parts in (6) over $v$, we obtain the phase volume $4\pi|C|$ in the left-hand side, and the reachable phase
volume (which is less or equal than $4\pi |C \setminus B|$) in the right hand side. Thus, we obtain

$$4\pi |C| < 4\pi |C \setminus B|,$$

which is a contradiction. □

**Remark 3.** Literally repeating this proof for piecewise smooth surfaces (which, in contrast to bodies, have zero volume), one concludes that there are no surfaces which are invisible (or have zero resistance) in all directions.

**Remark 4.** Theorem 2, along with its proof, readily generalizes to arbitrary dimension.

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