AN ELEMENTARY PROOF OF THE STRING TOPOLOGY STRUCTURE OF COMPACT ORIENTED SURFACES

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Abstract. We give a new proof of the string topology structure of a compact-oriented surface of genus $g \geq 2$, using elementary algebraic topology. This reproves the result of Vaintrob.

Introduction

Let $\Sigma_g$ be a compact oriented surface of genus $g$, then $\Sigma_g$ is an Eilenberg-Mac Lane space $K(\pi_g, 1)$ for $\pi_g$ the surface group $\pi_1(\Sigma_g, p)$, where $p \in \Sigma_g$ is any basepoint. This group has a presentation given by:

$$\pi_g = \langle A_1, \ldots, A_g, B_1, \ldots, B_g \rangle / ([A_1, B_1] \cdots [B_g, A_g])$$

We will calculate the string topology structure on $H_*(L\Sigma_g; \mathbb{Z})$ for $g \geq 2$, i.e. we will give the string product, BV-operator and string coproduct as defined in [CS99] [CV06]. These comprise all explicitly known non-zero string operations. In doing this we reprove the results of Vaintrob [Vai07]. The cases $g = 0$ and $g = 1$ are a consequence of Menichi’s work on the string topology of spheres [Men09] and, in the second case, of the fact that the string topology structure of a product is the product of the string topology structures. From now we always assume $g \geq 2$.

1. A property of $\pi_g$

We will need the following fact: the centralizer of $h \in \pi_g$ is infinite cyclic unless $h = e$, in which case it is obviously the entire group $\pi_g$.

Proposition 1.1. If $h \neq e \in \pi_g$, then the centralizer of $h$ is infinite cyclic.

Proof. We will use the theory of Riemann surfaces and in particular of $PSL(2, \mathbb{R})$ as isometries of the upper half-plane as model for hyperbolic space. We also use that all discrete subgroups of $\mathbb{R}$ are isomorphic to $\mathbb{Z}$.

A particular universal cover of $\Sigma_g$ is the upper-half plane $\mathbb{H}$ and $\pi_g$ can be made to act – of course still freely and properly discontinuous – on $\mathbb{H}$ by isometries. This means that we have an inclusion $\pi_g \hookrightarrow PSL(2, \mathbb{R})$. Because the fundamental domain has non-zero volume, the image has to be discrete. The non-identity elements of $PSL(2, \mathbb{R})$ can be classified into three types, depending on their fixed points when acting on the compactification $\overline{\mathbb{H}}$.

1. Elliptic elements have a fixed point in $\mathbb{H}$, so can’t be in the image of $\pi_g$.
2. Parabolic elements have a single fixed point on the boundary and are all conjugate to $z \mapsto z + 1$. Thus, if $h$ is sent to a parabolic element, without loss of generality we can assume it is sent to $z \mapsto z + 1$. Another element in $PSL(2, \mathbb{R})$ only commutes with this if it is of the form $z \mapsto z + t$. This means that the image of the centralizer of $h$ must lie in $\{z \mapsto z + t | t \in \mathbb{R}\} \cong \mathbb{R}$ and be discrete, hence be infinite cyclic.
3. Hyperbolic elements have two distinct fixed points on the boundary and are all conjugate to $z \mapsto \lambda z$ for some $\lambda \in \mathbb{R}_{>0}$. An element of $PSL(2, \mathbb{R})$ only commutes with this if it is of the form $z \mapsto \rho z$ for some $\rho \in \mathbb{R}_{>0}$. Hence, if $h$ is sent to a hyperbolic element without loss of generality it is sent to $z \mapsto \lambda z$. Then the centralizer of $h$ is sent to a discrete subgroup of $\{z \mapsto \rho z | \rho \in \mathbb{R}_{>0}\} \cong \mathbb{R}$ and again we conclude that it is infinite cyclic.

□
Remark 1.2. In fact, because we required \( g \geq 2 \), the element \( h \) always has to be mapped to an hyperbolic element.

2. The homotopy and integral homology groups of \( L\Sigma_g \)

We will show that that the fact that \( \Sigma_g \) is a \( K(\pi_g, 1) \) implies that \( L\Sigma_g \) is a disjoint union of \( \Sigma_g \) with a number of circles, which makes its homology easy to compute. First note that \( \pi_0(L\Sigma_g) \) consists of the conjugacy classes in \( \pi_g \), which in canonical bijection with the set \( L \) of isotopy classes of closed curves on \( \Sigma_g \). This means we get a connected component \( L[h]\Sigma_g \) for each conjugacy class of \( \pi_g \) or equivalently for each isotopy class of closed curve.

Consider the long exact sequence of homotopy groups associated to the restriction of the fibration \( \Omega\Sigma_g \to L\Sigma_g \to \Sigma_g \) to each of the connected components \( L[h]\Sigma_g \) of \( LG \). Because the connected components of \( \Omega\Sigma_g \) and \( \Sigma_g \) is a \( K(\pi_g, 1) \), we conclude that \( L[h]\Sigma_g \) is a \( K(G, 1) \) as well. But for which groups \( G \)? To find out, we fix a basepoint \( p \in \Sigma \) and a based representative \( h \) of \( [h] \) and use the following lemma:

Lemma 2.1. \( \pi_1(L[h]\Sigma_g, h) \) is given by the centralizer in \( \pi_1(\Sigma_g, p) \) of the representative \( h \) of the conjugacy class \( [h] \).

Proof. Although this fits into the more general context of Whitehead products on homotopy groups, we give an elementary proof. An element of this group is a homotopy class of maps \( S^1 \to L\Sigma_g \) which sends 1 to the based loop \( h \). This is the same as a homotopy class of maps \( S^1 \times S^1 \to \Sigma_g \) which sends \( S^1 \times 1 \) to \( h \). Consider the other restriction \( f : 1 \times S^1 \to \Sigma_g \) and hence the map \((h, f) : S^1 \vee S^1 \to \Sigma_g \). The fact that this factors over \( S^1 \times S^1 \) is equivalent to the fact that \( hfh^{-1}f^{-1} \) is homotopically trivial, which is in turn equivalent to \( f \) being a centralizer of \( h \). Conversely, any element of the centralizer gives such a map.

Let \( e \) denote the constant loop at \( p \). Using proposition 1.1 we obtain that \( \pi_1(L[e]\Sigma_g, e) = \pi_g \) and \( \pi_1(L[h]\Sigma_g, h) \equiv \mathbb{Z} \) for \( [h] \neq [e] \). The former implies \( L[e]\Sigma \simeq \Sigma \), and indeed evaluation at the basepoint and inclusion of constant loops are inverse up to homotopy. The latter tells us that \( L[h]\Sigma \) is homotopy equivalent to a circle for \( [h] \neq [e] \). This allows for a complete calculation of the homology of \( L\Sigma_g \).

Theorem 2.2. The integral homology of \( L\Sigma_g \) is given by

\[
H_*(L\Sigma_g; \mathbb{Z}) = \bigoplus_{[h] \in \text{Conj}(\pi_g)} H_*(L[h]\Sigma_g; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}[L] & \text{if } * = 0 \\
\mathbb{H}_1(\Sigma_g; \mathbb{Z}) \oplus \mathbb{Z}[L]/(\mathbb{Z} \cdot [e]) & \text{if } * = 1 \\
\mathbb{Z} & \text{if } * = 2 \\
0 & \text{otherwise}
\end{cases}
\]

Here we have to be careful: in \( H_1 \), the summand corresponding to a conjugacy class \([h]\) is not always generated by the class that one would expect.

Let \( k_h \) be a loop generating the centralizer of \( h \). Note that we can arrange that \( h = k_h^l \) for some \( l \in \mathbb{N} \), since \( h \) lies in its own centralizer. The example of \( A_2^{11} \) shows that we can have \( l \neq 1 \). It is easy to see that \( l \) is independent of the choice of representative \( h \). We call \( l = l([h]) \) the level of the conjugacy class\( ^{[1]} \). Using the proof of the lemma 2.1 we now see that a generator of \( H_1(L[h]\Sigma_g; \mathbb{Z}) \) is given by the homology class of the cycle

\[
\tilde{h} := \theta \mapsto \left( t \mapsto h \left( t + \frac{\theta}{l} \right) \right)
\]

\(^{[1]}\)It is essentially a winding number: \( l \) is the largest number such that there exists a representative \( S^1 \to \Sigma_g \) of \( h \) which factors as \( S^1 \to S^1 \to \Sigma_g \). If \( l = 1 \) then the corresponding isotopy class of closed curve is called primitive, exactly because it can not be written as a power of a smaller closed curve.
3. The BV-operator

Rotation of loops gives an action of $S^1$ on the free loop space:

$$\rho: S^1 \times L\Sigma_g \to L\Sigma_g$$

$$(\theta, \gamma) \mapsto (t \mapsto \gamma(t + \theta))$$

The BV-operator is given by $\Delta(a) = \rho_\ast([(S^1) \times a])$ and we can calculate it explicitly. Note that $\rho$ respects connected components, so it suffices to calculate the BV-operator for each connected component separately. We start with $[h] = [e]$. Then the action $\rho$ is homotopy trivial, so $\Delta$ vanishes on the homology of $L[e]\Sigma_g$.

Now let’s do $[h] \neq [e]$. Only in degree one could the BV-operator be non-zero. If we pick a representative $h$ as our generator for $H_0(L[h]\Sigma_g; \mathbb{Z})$ we see directly that $\Delta(h) = l([h])h$, where $l$ is the level of the conjugacy class of $h$ as before.

4. The string product and coproduct

We will use the actual definition of the string product as being induced by the following diagram

$$
\begin{array}{ccc}
L\Sigma_g \times L\Sigma_g & \xrightarrow{i} & \text{Map}(S^1 \lor S^1, \Sigma_g) \\
\downarrow{ev \times ev} & & \downarrow{ev} \\
\Sigma_g \times \Sigma_g & \overset{\triangledown}{\longrightarrow} & \Sigma_g
\end{array}
$$

as $H_\ast(L\Sigma_g; \mathbb{Z}) \otimes H_\ast(L\Sigma_g; \mathbb{Z}) \ni a \otimes b \mapsto j_\ast i_\ast(a \otimes b) \in H_{\ast-\delta}(L\Sigma_g; \mathbb{Z})$. Because our surfaces are even-dimensional we do not need to worry about signs.

We know that the string product drops the degree by -2 and has a unit in $H_2(L\Sigma_g; \mathbb{Z})$ given by the image $c_\ast([\Sigma_g])$ of fundamental class of $\Sigma_g$ under the induced map $c_\ast: H_\ast(\Sigma_g; \mathbb{Z}) \to H_\ast(L[1]\Sigma_g; \mathbb{Z})$ of the constant map. This unit is therefore exactly the generator of $H_2(L\Sigma_g; \mathbb{Z})$. Hence it suffices to calculate the string product on $H_1(L\Sigma_g; \mathbb{Z}) \otimes H_1(L\Sigma_g; \mathbb{Z})$, which is mapped to $H_0(L\Sigma_g; \mathbb{Z})$.

The connected components of $L\Sigma_g \times L\Sigma_g$ are given by the products $L_{[h_1]}\Sigma_g \times L_{[h_2]}\Sigma_g$. If $[h_1] = [e] = [h_2]$ then the vertical arrows become homotopy equivalences and the string product reduces to the ordinary intersection product $\langle - , - \rangle$ on the homology of $\Sigma_g$: $A_i \cdot A_j = B_i \cdot B_j = 0$ and $A_i \cdot B_j = \delta_{ij}$.

With two lemmas we calculate the other two cases. The first case is $[h_1] = [e]$ and $[h_2] = [h] \neq [e]$.

**Lemma 4.1.** Let $\tilde{h} \in H_1(L[h]\Sigma_g, \mathbb{Z})$ be the generator. We have that $A_i \cdot \tilde{h} = \frac{1}{\langle [h], [h] \rangle}(A_i, h)\tilde{h}$, where $\langle - , - \rangle$ is the ordinary intersection product. A similar formula holds for $B_i \cdot \tilde{h}$.

**Proof.** In this case the diagram reduces to

$$
\begin{array}{ccc}
\Sigma_g \times L_{[h]}\Sigma_g & \xrightarrow{i = ev \times id} & L_{[h]}\Sigma_g \\
id \times ev & \Downarrow{ev} & \downarrow{ev} \\
\Sigma_g \times \Sigma_g & \overset{\triangledown}{\longrightarrow} & \Sigma_g
\end{array}
$$

The class $i\ast(A_i \otimes \tilde{h})$ must be a multiple of the generator $h$ of $H_0(L[h]\Sigma_g; \mathbb{Z})$. It is given by $(id \times ev)^\ast(u) \cap (A_i \otimes \tilde{h})$, where $u$ is Thom class for $\triangledown$. To determine which multiple we can compose with $ev$, we get the following formula:

$$
ev_*((id \times ev)^\ast(u) \cap (A_i \otimes \tilde{h})) = u \cap (A_i \otimes ev_\ast(\tilde{h})) = u \cap (A_i \otimes k_h)
$$

where $k_h$ was the generator of the centralizer of $h$. This is the intersection product $\langle A_i, k_h \rangle$, which in turn is equal to $1/\langle [h], [h] \rangle\langle A_i, h \rangle$ because by definition $h = (k_h)^{\langle [h] \rangle}$. This proves the formula. $\square$

The second case is $[h_1], [h_2] \neq [e]$. The result uses the Goldman bracket $[Go86]$. 

Lemma 4.2. For \( i = 1, 2 \), let \( \tilde{h}_i \in H_1(L_{[h_i]} \Sigma_g, \mathbb{Z}) \) be the generator. We have that \( \tilde{h}_1 \cdot \tilde{h}_2 \) is equal to a multiple of the Goldman bracket:

\[
\tilde{h}_1 \cdot \tilde{h}_2 = \frac{1}{l([h_1])l([h_2])} [h_1, h_2]
\]

Proof. It is well-known that the Goldman bracket \([h_1, h_2]\) of \( h_1 \) and \( h_2 \) is given by \( \Delta(h_1) \cdot \Delta(h_2) \)\[CS99\] example 7.1. We just have to note that \( \Delta(h_1) = l([h_1])\tilde{h}_1 \), \( \Delta(h_2) = l([h_2])\tilde{h}_2 \) and that the string product and the Goldman bracket are bilinear. Because all homology groups are free abelian, we do not need to worry about torsion. \( \square \)

Finally, we get to the string coproduct. The string coproduct is almost trivial: it vanishes except on \( H_2(L_{[e]} \Sigma_g; \mathbb{Z}) \) and is given on the unit as follows:

\[
H_2(L_{[e]} \Sigma_g; \mathbb{Z}) \ni c_*([\Sigma_g]) \mapsto \chi([\Sigma_g]) [e] \otimes [e] = (2 - 2g) [e] \otimes [e] \in H_0(L_{[e]} \Sigma_g; \mathbb{Z}) \otimes H_0(L_{[e]} \Sigma_g; \mathbb{Z})
\]

This is a consequence of results by Tamanoi \[Tam07\].

References

[CS99] M. Chas and D. Sullivan, String topology, to appear in the Annals of Mathematics, preprint (1999), arXiv:math.GT/9911159.

[CV06] R.L. Cohen and A.A. Voronov, Notes on string topology, String topology and cyclic homology, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2006, pp. 1–95.

[Gol86] W. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface group representations, Invent. Math. (1986), no. 85, 263–302.

[Men09] L. Menichi, String topology for spheres, Comment. Math. Helv. 84 (2009), no. 1, 135–157.

[Tam07] H. Tamanoi, Loop coproducts in string topology and triviality of higher genus TQFT operations, preprint (2007), arXiv:0706.1276v3.

[Vai07] D. Vaintrob, The string topology BV algebra, Hochschild cohomology and the Goldman bracket on surfaces, preprint (2007), arXiv:math/0702859v1.