Two–Connection Renormalization and Nonholonomic Gauge Models of Einstein Gravity

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Abstract

A new framework to perturbative quantum gravity is proposed following the geometry of nonholonomic distributions on (pseudo) Riemannian manifolds. There are considered such distributions and adapted connections, also completely defined by a metric structure, when gravitational models with infinite many couplings reduce to two–loop renormalizable effective actions. We use a key result from our partner work arXiv: 0902.0911 that the classical Einstein gravity theory can be reformulated equivalently as a nonholonomic gauge model in the bundle of affine/de Sitter frames on pseudo–Riemannian spacetime. It is proven that (for a class of nonholonomic constraints and splitting of the Levi–Civita connection into a ”renormalizable” distinguished connection, on a base background manifold, and a gauge like distortion tensor, in total space) a nonholonomic differential renormalization procedure for quantum gravitational fields can be elaborated. Calculation labor is reduced to one– and two–loop levels and renormalization group equations for nonholonomic configurations.

Keywords: perturbative quantum gravity, nonholonomic manifolds, nonlinear connections, Einstein gravity, gauge gravity

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1 Introduction

There were elaborated different perturbative approaches and applications to quantum gravity of the standard formalism developed in the 1970s with the aim to quantize arbitrary gauge theories. In the bulk, all those
results where derived using (which proves technically very convenient) the background field method from the very beginning; see reviews [1] and, for a short discussion of more recent results, [2]. That period can be characterized by some final results like that the general relativity is not renormalizable even at one-loop order, when the coupling to matter is considered, and neither is pure gravity finite to two loops [3, 4, 5].

The general conclusion that the Einstein’s gravity is perturbatively non-renormalizable was long time considered a failure as a quantum field theory and, as a result, different strategies have been pursued. Here, one should be mentioned supergravity and string gravity and loop quantum gravity (see comprehensive summaries of results and reviews, respectively, in Refs. [6, 7, 8], which is related in the bulk to the background field method, and [9, 10, 11], advocating background independent and non-perturbative approaches). We also note that some time ago S. Weinberg suggested that a quantum theory in terms of the metric field may very well exist, and be renormalizable on a non-perturbative level [12]. That scenario known as "asymptotic safety" necessitates an interacting ultraviolet fixing point for gravity under the renormalization group (see [13, 14, 15, 16, 17], for reviews). It is also similar in spirit to effective field theory approaches to quantum gravity [18, 19]. But unlike a truly fundamental theory, an effective model cannot be valid up to arbitrary scales. Even substantial evidence was found for the non-perturbative renormalizability of the so-called Quantum Einstein Gravity: this emerging quantum model is not a quantization of classical general relativity, see details in [20, 21].

However, until today none of the above mentioned approaches has been accepted to be fully successful: see, for example, important discussions and critical reviews of results on loop quantum gravity and spin networks [22, 23]. Not entering into details of those debates, we note that a number of researches consider that, for instance, the existence of a semi-classical limit, in which classical Einstein field equations are supposed to emerge, is still an open problem to be solved in the loop quantum gravity approach. This is also related to the problem of the nonrenormalizable ultra-violet divergences that arise in the conventional perturbative treatment. Finally, there are another questions like it is possible to succeed, or not, in achieving a 'true' quantum version of full spacetime covariance, and how to formulate a systematic treatment of interactions with matter fields etc.

In a series of our recent works [24, 25, 26, 27], we proved that the Einstein gravity theory, redefined in so-called almost Kähler variables, can be formally quantized following methods of Fedosov (deformation) quantization [28, 29, 30]. The approach was derived from the formalism of non-
linear connections and correspondingly adapted Lagrange–Finsler variables on (pseudo) Riemannian manifolds, as it was considered in Refs. [31, 32] (see there details how Finsler like distributions can be defined on Einstein manifolds which is very important for constructing generic off–diagonal solutions in general relativity and elaborating certain new schemes of quantization). We emphasize that in this article we shall not work with more general classes of Lagrange–Finsler geometries elaborated in original form on tangent bundles [33] but apply the nonholonomic manifold geometric formalism for our purposes in quantum gravity (with classical and quantum versions of Einstein manifolds, see also a summary of alternative geometric results on nonholonomic manifolds in Ref. [34]).

Having introduced nonholonomic almost Kähler variables in Einstein gravity, the problem of quantization of gravity can be approached [35] following certain constructions for nonholonomic branes and quantization of a corresponding A–model complexification for gravity as it was proposed for gauge and topological theories in a recent work by Gukov and Witten [36]. The nonholonomic canonical symplectic variables also provide a bridge to nonholonomic versions of Ashtekar–Barbero variables (and non–perturbative constructions in loop gravity) [37]. They can be applied to more general cases of noncommutative theories of (gauge and Einstein) gravity [38, 39] and connected to the theory of nonholonomic/ noncommutative Ricci flows, Perelman functionals and Dirac operators [40, 41]. The next step in developing the nonholonomic geometric formalism for classical and quantum gravity theory consists in a study of general relativity along the lines of ”conventional” quantum field theory.

Our key idea is to work with an alternative class of metric compatible linear and nonlinear connections which are completely defined by a metric tensor and adapted to necessary types of nonholonomic constraints. We shall use two basic results from the first partner work [42]; 1) The Einstein gravity theory can be equivalently reformulated in terms of new variables defined by nonholonomic frames and nonholonomically deformed connections possessing constant coefficient curvatures. 2) The contributions of distortion tensor (considering deformations from an auxiliary linear connection to a Levi–Civita one) can be encoded into formal gauge gravity models.

In this paper, we consider some new perspectives to quantum gravity following certain methods from the geometry of nonholonomic distributions

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1 but equivalently to the well known approaches with Levi–Civita, Ashtekar and various gauge like gravitational connections

2 in this work any distinguished connection and relevant distortion tensors will be completely defined by a metric structure, similarly to the Levi–Civita connection
on (pseudo) Riemannian manifolds. Our purpose is twofold:

1. We show that for distinguished connections with constant curvature coefficients the higher–derivative quadratic terms can be removed by means of nonholonomic deformations and covariant field redefinitions and vertex renormalization. This way the theories with infinitely many couplings (like Einstein gravity, see a detailed discussion in [43]) can be studied in a perturbative sense also at high energies, despite their notorious perturbative non–renormalizability.

2. We prove that quantization of nonholonomic distortions of connections in certain affine/de Sitter frame bundles is possible following standard perturbative methods for the Yang–Mills theory; for our approach, we shall use the quantization techniques summarized in Ref. [44].

Let us now outline the content of this work:

In section 2, we provide some basic formulas, denotations and necessary results on two–connection variables and nonholonomic gauge models of Einstein gravity considered in details in Ref. [42].

In section 3, we propose a new approach to the problem of renormalization of gravity theories following geometric constructions with nonholonomic distributions and alternative connections (to the Levi–Cevita one) also defined by the same metric structure.

We prove that pure gravity may be two–loop nondivergent, even on shell, but for an alternative ”distinguished” connection, from which various connections in general relativity theory can be generated by using corresponding distortion tensors also completely defined by metric tensor. The method of differential renormalization is generalized on nonholonomic spaces, for one– and two–loop calculus, in section 4.

Then, in section 5, we apply this method of quantization to a nonholonomic/ nonlinear gauge gravity theory (classical formulation being equivalent to Einstein gravity). The one– and two–loop computations on bundles spaces enabled with nonholonomic distributions are provided for certain estimations or running constants and renormalization group equations for nonholonomic gravitational configurations.

Section 6 is devoted to a summary and conclusions. In Appendix, there are given some formulas for overlapping divergences in nonholonomic spaces.
2 Nonholonomic Gauge Models of Einstein Gravity and Two-Connection Variables

We consider a four dimensional (pseudo) Riemannian manifold \( \mathbf{V} \) with the metric structure parametrized in the form

\[
\begin{align*}
\mathbf{g} &= g_{\alpha\beta} \mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_{ij} \mathbf{e}^i \otimes \mathbf{e}^j + h_{ab} \mathbf{e}^a \otimes \mathbf{e}^b = (1) \\
\mathbf{\tilde{g}} &= \tilde{g}_{\alpha\beta} \tilde{\mathbf{e}}^\alpha \otimes \tilde{\mathbf{e}}^\beta = \tilde{g}_{ij} \tilde{\mathbf{e}}^i \otimes \tilde{\mathbf{e}}^j + \tilde{h}_{ab} \tilde{\mathbf{e}}^a \otimes \tilde{\mathbf{e}}^b, \\
\tilde{\mathbf{e}}^\alpha' &= (\mathbf{e}'^\alpha = dx^\alpha, \tilde{\mathbf{e}}'^\alpha = dy^\alpha + \tilde{N}_i^a dx^i), \\
\mathbf{e}^\alpha &= (\mathbf{e}^i = dx^i, \mathbf{e}^a = dy^a + N_i^a dx^i), \\
\text{for } g_{ij} \mathbf{e}^i \mathbf{e}^j &= = \tilde{g}_{ij} \tilde{\mathbf{e}}^i \tilde{\mathbf{e}}^j, \quad \text{h}_{ab} \mathbf{e}^a \mathbf{e}^b = \tilde{h}_{ab} \tilde{\mathbf{e}}^a \tilde{\mathbf{e}}^b.
\end{align*}
\]

with respect to dual bases \( \mathbf{e}^\alpha \) and \( \tilde{\mathbf{e}}'^\alpha \), for \( \mathbf{e}^i = \mathbf{e}^i \mathbf{e}^i \) and \( \mathbf{e}^a = \mathbf{e}^a \mathbf{e}^a \), where vierbein coefficients \( e^\alpha_\alpha, e^\alpha_\alpha' \) are defined by for any given/prescribed values \( g_{i\beta} = \{g_{ij}, h_{ab}\}, \tilde{g}_{i\beta} = \{\tilde{g}_{ij}, \tilde{h}_{ab}\} \) and \( \tilde{N}_i^a \). For convenience, we can consider constant coefficients \( \tilde{g}_{ij}^{'} \) and \( \tilde{h}_{ab}^{'} \) and take \( [\mathbf{e}', \tilde{\mathbf{e}}'] = 0 \). The nonholonomic structure of \( \mathbf{V} \) is determined by a nonlinear connection (N-connection) \( \mathbf{N} = N_i^a(u)dx^i \otimes \partial_u \) such a manifold is called N-anholonomic. On adopted system of notations and details on nonholonomic manifolds and N-connection geometry and applications in modern gravity, we cite our first partner work \[32\] and Refs. \[31, 32\].

On a N-anholonomic \( \mathbf{V} \), we can construct an infinite number of (linear) distinguished connections, \( d \)-connections, \( \mathbf{D} = \{\Gamma^\gamma_{\alpha\beta}\} \) which are adapted to a chosen N-connection structure \( \mathbf{N} \) (i.e. the N-connection h- and \( v \)-splitting is preserved under parallelism) and metric compatible, \( \mathbf{Dg} = 0 \). There is a subclass of such \( d \)-connections \( \mathbf{8D} \) when their coefficients \( \mathbf{8}\Gamma^\gamma_{\alpha\beta} \)

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\[3\] Local coordinates on \( \mathbf{V} \) are denoted in the form \( u^\alpha = (x^i, y^a) \) (or, in brief, \( u = (x, y) \)) where indices of type \( i, j \ldots = 1, 2 \) are formal horizontal/ holonomic ones (h–indices), labeling h–coordinates, and indices of type \( a, b \ldots = 3, 4 \) are formal vertical/nonholonomic ones (v–indices), labeling v–coordinates. We may use ‘underlined’ indices \( \underline{\alpha} = (\underline{i}, \underline{j}), \beta = (\underline{i}, \underline{j}) \), for local coordinate bases \( e_{\underline{\alpha}} = \partial_{\underline{\alpha}} = (\partial_{\underline{i}}, \partial_{\underline{j}}) \), equivalently \( \partial/\partial u^\underline{\alpha} = (\partial/\partial u^i, \partial/\partial u^j \), for dual coordinate bases we shall write \( \underline{\mathbf{e}}^\underline{\alpha} = du^\underline{\alpha} = \underline{\mathbf{e}}^\underline{\alpha} = (dx^i, dx^j) \). There are also considered primed indices \( (\alpha' = (i', j'), \beta' = (j', j'), \ldots) \), with double primes etc, for other local abstract/coordinate bases, for instance, \( e_{\alpha''} = (e_{\alpha'}, e_{\alpha''}) \), \( e_{\alpha''}' = (e_{\alpha''}, e_{\alpha''}) \), \( e_{\alpha''} = (e_{\alpha''}, e_{\alpha''}) \), where \( i', j', = 1, 2 \ldots \) and \( a', a'' = 3, 4 \).

\[4\] On a manifold \( \mathbf{V} \), we can fix any type of coordinate and frame (equivalently, vierbein/ tetradic) and nonholonomic, in our case, N-connection, structures; this will result in different types of coefficients \( N_i^a \), \( N_i^a \) and \( e^i_\alpha, e^a_\alpha \), respectively, in formulas \[2\], \[3\] and \[1\], for any given metric \( \mathbf{g} = \{g_{\alpha\beta}\} \) and fixed values (for instance, constant) \( g_{ij} \), \( h_{ab} \) and \( h_{ab}^{'} \).
are uniquely determined by the coefficients of \( g = \hat{g} \) following a geometric principle. We can work equivalently with the Levi–Civita connection \( \hat{\nabla} = \{ \hat{\Gamma}_{\alpha \beta}^{\gamma} \} \) and any \( \hat{\Gamma}_{\alpha \beta}^{\gamma} \) related by a distortion relation \( \hat{\Gamma}_{\alpha \beta}^{\gamma} = g_{\alpha \beta} \hat{\Gamma}_{\alpha \beta}^{\gamma} + g_{\alpha \beta} \hat{\Gamma}_{\alpha \beta}^{\gamma} \) because the distortion/torsion tensor \( \hat{Z}_{\alpha \beta}^{\gamma} \) is also completely defined by the coefficients \( g_{\alpha \beta} \) for any prescribed values \( N^{a}_{i} \).

For our nonholonomic constructions in classical and quantum gravity, a crucial role is played by the Miron’s procedure (on applications in modern gravity and generalizations, see discussions in Refs. [42, 55, 33] and the original results, for Lagrange–Finsler spaces, [33]). This procedure allows us to compute the set of d–connections \( \{ \hat{D} \} \) satisfying the conditions \( \hat{D} x g = 0 \) for a given \( g \). The components of any such \( \hat{D} = \left( \hat{L}_{jk}^{i}, \hat{L}_{bk}^{a}, \hat{C}_{jc}^{i}, \hat{C}_{bc}^{a} \right) \) are given by formulas

\[
\begin{align*}
\hat{L}_{jk}^{i} &= \tilde{L}_{jk}^{i} + \frac{1}{2} \hat{O}_{jk}^{ih} (g_{j}^{i} g_{k}^{h} + g_{jk} g^{ih}), \\
\hat{L}_{bk}^{a} &= \tilde{L}_{bk}^{a} + \frac{1}{2} \hat{O}_{bk}^{ac} (g_{ab} g_{ck}^{a} - g_{bc} g_{aj}^{a}), \\
\hat{C}_{jc}^{i} &= \tilde{C}_{jc}^{i} + \frac{1}{2} \hat{O}_{jc}^{ad} (g_{ac} g_{jd}^{a} + g_{bd} g_{ec} g_{jd}^{a}),
\end{align*}
\]

(5)

where \( \hat{O}_{jk}^{ih} = \frac{1}{2} (\delta_{jk}^{i} g_{k}^{h} \pm g_{jk} g^{ih}) \), \( \hat{O}_{bk}^{ac} = \frac{1}{2} (\delta_{bk}^{a} g_{ck}^{a} \pm g_{ab} g_{ck}^{a}) \), are the so–called the Obata operators and \( \hat{\Gamma}_{\alpha \beta}^{\gamma} = \left( \hat{L}_{jk}^{i}, \hat{L}_{bk}^{a}, \hat{C}_{jc}^{i}, \hat{C}_{bc}^{a} \right) \), with

\[
\begin{align*}
\tilde{L}_{jk}^{i} &= \frac{1}{2} \hat{g}^{ir} (e_{kj} g_{jr} + e_{j} g_{kr} - e_{r} g_{jk}), \\
\tilde{L}_{bk}^{a} &= e_{b} (N_{k}^{a} + \frac{1}{2} \hat{g}^{ac} (e_{k} g_{bc} - g_{dc} e_{b} N_{k}^{d} - g_{db} e_{c} N_{k}^{d})), \\
\tilde{C}_{jc}^{i} &= \frac{1}{2} \hat{g}^{ik} e_{c} \tilde{C}_{jk}^{i} + \hat{C}_{bc}^{a} = \frac{1}{2} \hat{g}^{ad} (e_{c} g_{bd} + e_{d} g_{bc} - e_{d} g_{bc}),
\end{align*}
\]

(6)

is the canonical d–connection uniquely defined by the coefficients of d–metric \( g = [g_{ij}, g_{ab}] \) and N–connection \( N = \{ N^{a}_{i} \} \) in order to satisfy the conditions \( \hat{D} x g = 0 \) and \( \hat{C}_{jk}^{i} = 0 \) and \( \hat{C}_{bc}^{a} = 0 \) but with general nonzero values for \( \hat{L}_{ja}^{i}, \hat{L}_{ji}^{a}, \) and \( \hat{L}_{bk}^{a} \), see component formulas in [42, 33]. In formulas (5), the

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\footnote{We used the left label “\( g \)” in order to emphasize that certain values are defined by the metric structure. Such constructions do not depend explicitly on the type of nonholonomic distribution, for instance, we can consider any type of \( 2 + 2 \) distributions (i.e. we work with well defined geometric objects, not depending on a particular choice of coordinate/frame systems, even the constructions are adapted to a fixed nonholonomic structure). A general d–connection \( \Gamma_{\alpha \beta}^{\gamma} \) is not defined by a metric tensor. For simplicity, in this work, we shall work only with metric compatible d–connections.}
d–tensors $Y^m_{e_j}$, $Y^k_{mc}$, $Y^d_{ek}$ and $Y^d_{ec}$ parametrize the set of metric compatible d–connections, with a metric $g$, on a N–anholonomic manifold $V$. Prescribing any values of such d–tensors (following certain geometric/ physical arguments; in particular, we can take some zero, or non–zero, constants), we get a metric compatible d–connection $gD$ (5) completely defined by a (pseudo) Riemannian metric $g$ (1). Such d–connections can be chosen in different forms for different quantization/ renormalization procedures in modern quantum gravity (on the two–connection perturbative method, see next sections in this work). For simplicity, we shall consider that we fix a nonholonomic configuration of frames and linear connections on a spacetime manifold $V$, and lifts of fundamental geometric objects (metrics, connections, tensors, physical fields etc) on total spaces of some bundles on $V$, if prescribe certain constant (for simplicity), or tensor fields for –fields in formulas (5).

It is possible to construct nonholonomic lifts of any connections $\mathcal{g}\Gamma_{\alpha\beta\gamma}$ and $\mathcal{g}\Gamma^\gamma_{\alpha\beta}$, and related distortion tensors $\mathcal{g}Z_{\beta\gamma}$, into the bundle of affine/de Sitter frames on a N–anholonomic spacetime $V$ (see details in Section 3 of Ref. [42]). In our approach, the de Sitter nonlinear gauge gravitational theory is constructed from the coefficients of a d–metric $g$ and N–connection $N$ in a form when the Einstein equations on the base nonholonomic spacetime are equivalent to the Yang–Mills equations in a total space enabled with induced nonholonomic structure. We choose in the total de Sitter nonholonomic bundle a d–connection $\mathcal{g}\Gamma$ which with respect respect to nonholonomic frames of type (3), and their duals, is determined by a d–connection $\mathcal{g}\Gamma^\alpha_{\beta\gamma}$,

$$\mathcal{g}\Gamma = \left( \begin{array}{ccc} \mathcal{g}\Gamma^\alpha_{\beta'} & l_{0}^{-1}e^{\alpha'} \\ l_{1}^{-1}e^\beta & 0 \end{array} \right) ,$$

(7)

where

$$\mathcal{g}\Gamma^\alpha_{\beta'} = \mathcal{g}\Gamma_{\alpha'}_{\beta'}^\mu e^\mu ,$$

(8)

for

$$\mathcal{g}\Gamma_{\alpha'}_{\beta'}^\mu = e_{\alpha'}^\alpha e_{\beta'}^\beta \mathcal{g}\Gamma^\alpha_{\beta\mu} + e_{\alpha'}^\alpha e_{\mu}(e_{\alpha'}^\beta),$$

(9)

with $e_{\alpha'}^\alpha = e_{\mu}^\alpha e^\mu$ and $l_0$ and $l_1$ being dimensional constants. The indices $\alpha', \beta'$ take values in the typical fiber/ de Sitter space. We emphasize that

6see, for instance, applications of the geometry of nonholonomic distributions and nonlinear connections in Refs. [24, 25, 26, 27, 35, 37]

7In a similar form we can elaborate certain geometric constructions for nonholonomic affine frame bundles if we chose $\mathcal{g}\Gamma = \left( \begin{array}{ccc} \mathcal{g}\Gamma^\alpha_{\beta'} & l_{0}^{-1}e^{\alpha'} \\ 0 & 0 \end{array} \right)$ but this results in formal ‘non-variational’ gauge models because of degenerated Killing forms. Geometrically, this is
because of nonholonomic structure on the base and total spaces, the coefficients of $g\Gamma$ are subjected to nonlinear nonholonomic transformations laws under group/ frame/ coordinate transforms and nonholonomic deformations, see explicit formulas in [42]. If we take the limit $l_1^{-1} \to 0$, we get a d–connection for the affine frame bundle (with degenerated fiber metric) which allows to project the connection 1–form just in a d–connection on the base, when the constructions can be performed to be equivalent to the Einstein gravity. For $l_1 = l_0$, we shall develop a de Sitter model with nondegenerate fiber metric. For simplicity, we shall work only with a nonholonomic de Sitter frame model of nonholonomic gauge gravity.

The matrix components of the curvature of the d–connection (7),

$$g\mathcal{R} = d g\Gamma - g\Gamma \wedge g\Gamma,$$

can be parametrized in an invariant 4+1 form

$$g\mathcal{R} = \begin{pmatrix} \mathcal{R}^{\alpha'}_{\beta'} + l_0^{-1} \pi^{\alpha'}_{\beta'} & l_0^{-1} T^{\alpha'}_0 \\ l_0^{-1} T^{\beta'}_0 & 0 \end{pmatrix},$$

(10)

where

$$\pi^{\alpha'}_{\beta'} = e^{\alpha'} \wedge e^{\beta'}, \quad T^{\beta'}_0 = \frac{1}{2} g T^{\beta'}_{\mu \nu} \delta u^\mu \wedge \delta u^\nu$$

$$\mathcal{R}^{\alpha'}_{\beta'} = \frac{1}{2} \mathcal{R}^{\alpha'}_{\beta' \mu \nu} \delta u^\mu \wedge \delta u^\nu, \quad \mathcal{R}^{\alpha'}_{\beta' \mu \nu} = e^{\beta'} e^{\alpha'} g R^{\alpha}_{\beta' \mu \nu}.$$

when the torsion, $g T^{\beta'}_{\mu \nu}$, and curvature, $g R^{\alpha}_{\beta' \mu \nu}$, tensors are computed for the connection 1–form [8]. The constant $l_0$ in (7) and (10) and constants $l^2 = 2[l_0^2 \lambda, \lambda_1 = -3/l_0]$ considered in Ref. [42] do not characterize certain additional gravitational high curvature and/or torsion interactions like in former gauge like gravity theories [45, 46, 47, 48, 49, 50, 51, 52, 39], but define the type of nonholonomic constraints on de Sitter/affine bundles which are used for an equivalent lift in a total bundle space of the Einstein equations on a base spacetime manifold. Prescribing certain values of such constant is equivalent to a particular choice of $Y$–fields in formulas (5) in order to fix a nonholonomic configuration. Different values of such nonholonomy constants and additional tensor fields parametrize various type of

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*not a problem and, in both cases (for instance, for the affine and de Sitter frame bundles) we can work with well-defined nonholonomic structures and geometric objects in total bundle, considering necessary auxiliar tensor fields and constants defining a class of N–connections, when the projections of nonholonomic Yang–Mills equations on a spacetime base will be equivalent to the Einstein equations.*

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*see discussion of Miron’s procedure in Section 2 of Ref. [42]*
N–adapted metric compatible linear connections and effective gauge gravitational models into which the geometric and physical information of classical Einstein gravity can be encoded. For our purposes in developing a method of perturbative quantization of gravity, it is enough to fix a convenient set of constant Y–fields, which under quantization will run following certain renormalization group equations, see below the end of Section 5.

We can fix such a nonholonomic distribution (and nonholonomic frames) when \( g = \hat{g} \) induces a canonical d–connection \( \hat{\Gamma}^{a'}_{\alpha'\beta'} = (0, \hat{L}_{y'k'}, 0, 0) = \text{const} \), with constant curvature coefficients

\[
\hat{L}^{a'}_{b'c'} = (0, \hat{L}^{a'}_{b'c'}, 0, 0, 0, 0),
\]

(11)

with respect to a class of N–adapted frames. The corresponding distortion of the Levi–Civita connection with respect to \( \hat{\Gamma}^{a'}_{\alpha'\beta'} \) is written in the form \( \Gamma^{\alpha'}_{\beta'\gamma'} = \hat{\Gamma}^{\alpha'}_{\beta'\gamma'} + \hat{Z}^{\alpha'}_{\beta'\gamma'} \). The related distortions in the total space of nonholonomic fiber bundles are \( \hat{\Gamma} = \Gamma + \hat{Z} \) and \( \hat{R} = R + \hat{Z} \).

For a four dimensional (pseudo) Riemannian base \( V \), one could be maximum eight nontrivial components \( \hat{L}_{y'k'} \). We can prescribe such a nonholonomic distribution with some nontrivial values \( \hat{L}_{y'k'} \) when

\[
\hat{L}^{a'}_{b'c'} = \hat{L}^{a'}_{b'c'} - \hat{L}_{y'k'} = 0.
\]

(12)

Here it should be emphasized that in order to perform a so–called two–connection geometric renormalization of two–loop Einstein gravity, it is possible to consider any metric compatible d–connection \( \hat{\Gamma}^{\alpha}_{\beta\mu} \) with corresponding curvature d–tensor \( \hat{R}^{\alpha}_{\beta\mu} \), satisfying the condition

\[
\hat{R}^{\alpha}_{\beta\mu} \hat{R}^{\gamma}_{\tau\mu} \hat{R}^{\gamma}_{\tau\nu} = 0,
\]

(13)

see below Section 3. Such a condition is satisfied by any \( \hat{\Gamma}^{a'}_{\alpha'\beta'} \) with prescribed constant coefficients of type \( \hat{L}_{y'k'} \) with vanishing \( \hat{L}_{y'j', k'} \) (12), or

\[\text{d–connections with constant curvature matrix coefficients were introduced with the aim to encode classical Einstein equations into nonholonomic solitonic hierarchies [53], see also [54] and, on the procedure of metrization and parametrization of metric compatible d–connections on general holonomic manifolds/bundle spaces enabled with symmetric, or nonsymmetric, metrics, [55].}\]
with such prescribed constants when certain contractions of this d–tensor are constant/zero.\footnote{The geometric properties of curvature and Weyl d–tensors for a d–connection are very different from those of usual tensors and linear connections. Even the coefficients of a d–tensor may vanish with respect to a particular class of nonholonomic distributions, the real spacetime may be a general (pseudo) Riemannian one with nontrivial curvature of the Levi–Civita connection and nonzero associated/induced nonholonomically d–torsions, nonholonomy coefficients and curvature of N–connection.}

Choosing $\mathcal{R} = 0$, we can write the gauge like gravitational equations, the equivalent of the Einstein equations on $\mathbb{R}$, in a simplified form,

$$d\left(\star, \mathcal{Z}\right) + \mathcal{Z} \wedge (\star, \mathcal{Z}) - (\star, \mathcal{Z}) \wedge \mathcal{Z} = -\mathcal{J},$$

(14)

where the nonholonomically deformed source is

$$\mathcal{J} = \varepsilon \mathcal{J} + \Gamma \wedge (\star, \mathcal{Z}) - (\star, \mathcal{Z}) \wedge \Gamma,$$

\varepsilon \mathcal{J} determined by the energy–momentum tensor in general relativity and $\mathcal{Z}$ contains the same geometric/physical information as the curvature and Ricci tensor of the Levi–Civita connection $\Gamma_{\alpha}{}^{\beta\gamma}$. Such formulas were derived in Section 4.3 of Ref. \cite{42}, see there explicit component formulas for $\Gamma$, $\mathcal{Z}$, $\mathcal{Z}$ and $\mathcal{J}$.

Finally, we note that from formulas (11) and (12) one follows that

$$\hat{\mathcal{R}}_{\beta\gamma} = \hat{\mathcal{R}}_{\alpha'} = 0.$$

(15)

This is similar to the vacuum Einstein equations, but because the Ricci d–tensor $\hat{\mathcal{R}}_{\beta\gamma}$ is constructed for a "nonholonomic" d–connection $\hat{\Gamma}$ this structure is not trivial even its curvature d–tensor may vanish for certain parametrizations. In such cases, a part of "gravitational" degrees of freedom are encoded into the nonholonomy coefficients and associated N–connection structure. How to construct nontrivial exact solutions of (15) was considered in a series of our works, see reviews \cite{31,32,39}. In a more general case, we can work with nonholonomic configurations when $\hat{\mathcal{R}}_{\beta\gamma} = const$, for some nonzero values.

3 Models of Nonholonomic Gravity and Renormalization

It has been shown by explicit computations that in terms of the Levi–Civita connection the gravity with the Einstein–Hilber action gives rise to a
finite one–loop model only in the absence of both matter fields and a cosmo-
logical constant [3]. Similar computations result in a more negative result
that perturbative quantum gravity, with the same connection, diverges in
two–loop order [4], see further results and review in [5, 1]. The goal of this
section is to prove that working with nonholonomic distributions and alter-
tnative d–connections, the problem of renormalization of gravity theories
can be approached in a different form when certain formal renormalization
schemes can be elaborated.

### 3.1 Two–loop quantum divergences for nonholonomic models of gravity

We shall analyze the one– and two–loop divergences of a gravitational
model when the Levi–Civita connection \( \mathfrak{g}\nabla = \{ \mathfrak{g}\Gamma_{\alpha\beta\gamma} \} \) is substituted by an
alternative metric compatible d–connection \( \mathfrak{g}\mathcal{D} = \{ \mathfrak{g}\Gamma_{\alpha\beta\gamma} \} \) also completely
defined by the same metric structure \( \mathfrak{g} \) in such a form that the divergences
are eliminated by imposing nonholonomic constraints.

#### 3.1.1 One–loop computations

Let us consider a Lagrange density

\[
0\mathcal{L} = -\frac{1}{2\kappa^2} \sqrt{\mathfrak{g}} \left( \mathfrak{g}\mathcal{R} - 2\Lambda \right)
\]  

(16)

where \( \kappa^2 \) and \( \Lambda \) are defined respectively by gravitational and cosmological
constants and the scalar curvature of \( \mathfrak{g}\mathcal{D} \) is

\[
\mathfrak{g}\mathcal{R} = \mathfrak{g}^{\alpha\beta} \mathfrak{g}R_{\alpha\beta} = g^{ij} \mathfrak{g}R_{ij} + h^{ab} \mathfrak{g}S_{ab} = \mathfrak{g}\mathcal{R} + \mathfrak{g}\mathcal{S},
\]

see formula (A.5) in Appendix to [42]. For simplicity, we restrict our one–
and two–loop analysis only to nonholonomic vacuum configurations with
\( \Lambda = 0 \).

A one–loop computation similar to that of ’t Hooft and Veltman [3], see
also details in review [1], but for a background field method with metric \( \mathfrak{g} \)
and d–connection \( \mathfrak{g}\mathcal{D} \), results in this divergent part of the one–loop effective
action for pure nonholonomic gravity model,

\[
\Gamma^{(1)}_\infty = \int \delta^4 u \sqrt{\mathfrak{g}} \left( a_1 \mathfrak{g}\mathcal{R}^2 + a_2 \mathfrak{g}R_{\alpha\beta} \mathfrak{g}R^{\alpha\beta} + a_3 \mathfrak{g}R_{\alpha\beta\gamma\tau} \mathfrak{g}R^{\alpha\beta\gamma\tau} \right),
\]

where \( a_1, a_2 \) and \( a_3 \) are constant. For a given metric structure \( \mathfrak{g} \), we can al-
ways define a \( N \)–connection splitting and construct a d–connection \( \mathfrak{g}\Gamma_{\alpha\beta\gamma} = \)
\( \hat{\Gamma}^\alpha_{\beta\gamma} \) with constant curvature coefficients when \( \hat{\mathbf{R}}_{\alpha'\beta'\gamma'\delta'} = 0 \) (11) and \( \hat{\mathbf{R}}_{\beta'\gamma'\alpha'\delta'} = 0 \) (15). In such a case \( \Gamma^{(1)}_{\infty} [\hat{\Gamma}^\alpha_{\beta\gamma}] = 0 \) which holds true for a corresponding class of nonholonomic transform and d–connections even the metric \( g \) is a solution of certain non–vacuum Einstein equations for the Levi–Civita connection \( \nabla \) and nontrivial source of matter fields.

For distortions \( \Gamma^\alpha_{\beta\gamma} = \hat{\Gamma}^\alpha_{\beta\gamma} + \hat{\mathbf{Z}}^\alpha_{\beta\gamma} \), we construct a one–loop finite \( \Gamma^\alpha_{\beta\gamma} \) if \( \hat{\Gamma}^\alpha_{\beta\gamma} \) is made finite by certain nonholonomic transforms and \( \hat{\mathbf{Z}}^\alpha_{\beta\gamma} \) is renormalized following some standard methods in gauge theory (they will include also possible contributions of matter fields). So, we can eliminate the one–loop divergent part for a corresponding class of metric compatible d–connections, by corresponding nonholonomic frame deformations.

### 3.1.2 Two–loop computations

The geometry of d–connections adapted to a N–connection structure is more rich than that of linear connections on manifolds. There are different conservation laws for d–connections and the derived Ricci and Riemannian d–tensors contain various types of h– and v–components inducing different types of invariants etc (for instance, even in the holonomic case, the extension of ’t Hooft’s theorems to two–loop order, for renormalizable interactions, request an analysis of some 50 invariants, see review [1]), see details in Refs. [31, 32] and, for bundle spaces and Lagrange–Finsler geometry, [33].

Nevertheless, for nonholonomic geometries induced on (pseudo) Riemannian manifolds and lifted equivalently on bundle spaces, the background field method works in a similar case both for the Levi–Civita and any metric compatible d–connection all induces by the same metric structure. From formal point of view, we have to take \( \Gamma^\alpha_{\beta\gamma} \rightarrow g \Gamma^\alpha_{\beta\gamma} \) and follow the same formalism but taking into account such properties that, for instance, the Ricci d–tensor is nonsymmetric, in general, and that there are additional h– and v–components with different transformation laws and invariant properties. We provided such details for locally anisotropic gravity models, Lagrange–Finsler like and more general ones, obtained in certain limits of (super) string theory [56, 57].

In abstract form, the result of a N–adapted background field calculus for

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\[ ^{11} \]Those results can be redefined equivalently for certain limits to the Einstein gravity theory and string generalizations considering that the h– and v–components are not for tangent or vector bundles, but some respective holonomic and nonholonomic variables on Einstein manifolds.

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the only gauge–independent coefficient is

\[ \Gamma_\infty^{(2)} = \frac{209}{2880(4\pi)^4} \epsilon \int \delta^4 u \sqrt{g} \ g R_{\alpha\beta}^{\mu\nu} g R_{\mu\nu}^{\gamma\tau} g R_{\gamma\tau}^{\alpha\beta}, \]

which may written in equivalent form in terms of the Weyl tensor as it was obtained by certain simplified computations for the Levi–Civita connection in \([5]\). Choosing a d–connection \( g \Gamma_{\alpha\beta}^{\gamma} = \widehat{\Gamma}_{\alpha\beta}^{\gamma} \) we impose the nonholonomic constraints \([13]\), i.e. vanishing of sub–integral coefficients \([\text{with respect to } N–\text{adapted frames, we get constant curvature coefficients with } \widehat{\Gamma}_{\alpha\beta}^{\gamma} = 0]\) resulting in \( \Gamma_\infty^{(2)} = 0 \). This shows that pure gravity may be two–loop nondivergent, even on shell, but for an alternative d–connection which is also uniquely constructed from the metric coefficients (we can not obtain such a result if we work directly only with the Levi–Civita connection). So, there are hidden symmetries operating on the gravitational sector and they are related to the possibility that for a metric tensor we can construct an infinite number of metric compatible d–connections, all determined by this metric tensor with respect to a prescribed nonholonomic structure of frames. As we shall see in the next subsection, this fact carries new possibilities to avoid constructions with infinitely many couplings and higher order curvature terms.

### 3.2 Renormalization of gravity with infinitely many couplings

The final goal of a well–defined perturbation theory is the resummation of the series expansion which has to be performed at least in suitable correlation functions and physical quantities. For such constructions, the terms that cannot be reabsorbed by means of field redefinitions have to be reabsorbed by means of redefinitions of the coupling constants. When the classical action does not contain the necessary coupling constants \( \lambda \), new coupling constants have to be introduced. The Einstein gravity theory is not renormalizable, which means that divergences can be removed only at the price of introducing infinitely many coupling constants. In Ref. \([43]\), it was shown that the problem of renormalization of theories of gravity with infinitely many couplings can be solved when the spacetime manifold admits a metric of constant curvature. It was also proven that is possible to screen the terms of a generalized gravitational Lagrangian when, for instance, a whole class of terms is not turned on by renormalization, if it is absent at the tree level.
Working with $d$–connections, we can generalize the Anselmi’s constructions for arbitrary (pseudo) Riemannian metrics because we can always define such nonholonomic distributions when a given metric and the curvature of certain $d$–connections are characterized by constant matrix coefficients. We can give a physical sense to such quantum gravity models with infinitely many parameters and nonholonomic distributions at arbitrary energies if we show that the formalism does not drive an unitary propagator into a non–unitary (i.e. higher–derivative) propagator\[^{12}\]

A generalized gravitational action (supposed to be more convenient for purposes of renormalization of gravitational interactions) contains infinitely many couplings, but not all of the ones that might have been expected, and in the nonholonomic formalism one can be considered such constraints when only a finite number of terms are nonzero. In quantum gravity based on the Levi–Civita connection, the metric of constant curvature is an extremal, but not a minimum, of the complete action, which results in the problem how to fix a ”right” perturbative vacuum. For a correspondingly defined $d$–connection, it appears to be possible to introduce a good perturbative vacuum, choosing such nonholonomic distributions the curvature is negative and stating the conditions when such a nonholonomic quantum vacuum has a negative asymptotically constant curvature. Such properties may not be true for the Levi–Civita connection, but we can always extract it from a right perturbative nonholonomic vacuum and N–adapted quantum perturbations of a suitable metric compatible $d$–connection.

For a $d$–connection $\mathcal{D}$ with curvature $d$–tensor $\mathcal{R}_{\alpha\beta\mu\nu}$, we introduce the values

$$\mathcal{R}_{\alpha\beta\mu\nu}^\Lambda = \mathcal{R}_{\alpha\beta\mu\nu} - \frac{\Lambda}{6} (\mathcal{g}_{\alpha\mu} \mathcal{g}_{\beta\nu} - \mathcal{g}_{\alpha\nu} \mathcal{g}_{\beta\mu})$$

and

$$\mathcal{G} = \mathcal{G}_{\alpha\beta\mu\nu}^\mathcal{R} \mathcal{R}_{\alpha\beta\mu\nu} - 4 \mathcal{G}_{\alpha\beta} \mathcal{R}_{\alpha\beta} + \mathcal{G}_{\mathcal{R}}^2,$$

which are convenient to study $\rho$–expansions if we choose an appropriate gravitational vacuum $\mathcal{g}_{\alpha\nu}$ to define quantum fluctuations $\rho_{\alpha\nu}$, where $\mathcal{g}_{\alpha\nu} = \mathcal{g}_{\alpha\nu} + \rho_{\alpha\nu}$. By inductive hypothesis, assuming that the $O(\rho)$– and $O(\rho^2)$– contributions come only from Lagrange density $\mathcal{L}$\[^{16}\], we introduce a generalized Lagrange density

$$\mathcal{L} = \frac{1}{\kappa^2} \sqrt{\mathcal{g}} \left[ - \mathcal{G}^\mathcal{R} + \Lambda + \lambda \kappa^2 \mathcal{G}^\mathcal{G} + \sum_{s=1}^{\infty} \lambda_s \kappa^{2s+2} \mathcal{F} \left[ \mathcal{D}, \mathcal{R}^\mathcal{G}, \Lambda \right] \right], \quad (17)$$

\[^{12}\]One might happen that the non-renormalizability of the Einstein gravity theory and its nonholonomic deformations can potentially generate all sorts of counterterms, including those that can affect the propagator with undesirable higher derivatives.
where \( \lambda \) and \( \lambda_s \) label the set of infinite many couplings. A term \( F_s \) in (17) is a collective denotation for gauge–invariant terms of dimension \( 2s + 4 \) which can be constructed from three or more curvature d–tensors, \( g^R_{\alpha\beta\mu\nu} \), d–connection, \( g^D \), and powers of \( \Lambda \), in our approach, up to total derivatives adapted to \( N \)–connection structure. It should be noted that index contracted values \( g^{\hat{\mathcal{R}}}_{\alpha\beta} \) and \( g^{\hat{\mathcal{R}}}_{\alpha} \) can be removed from \( F_s \) by means of field redefinitions. In the first approximation, \( F_1 \) contains a linear combination for three multiples \( g^{\hat{\mathcal{R}}} \) with all possible contractions of indices, but does not contain terms like \( g^{\hat{\mathcal{R}}} g^D g^D g^{\hat{\mathcal{R}}} \) which would affect the \( \rho \)–propagator with higher derivatives.

In Ref. [43], for \( g^D = g^\nabla \), it is proven that the gravity theory derived for Lagrangian density \( \mathcal{L} \) (17) is renormalizable in the sense that such a Lagrangian preserves its form under renormalization in arbitrary spacetime dimension greater than two. Nevertheless, the gravitational field equations obtained from (17) are not just the Einstein equations and the quantum vacuum for this theory must have a negative asymptotically constant curvature. It was also concluded that if the theory has no cosmological constant or the space–time manifold admits a metric of constant curvature, the propagators of the fields are not affected by higher derivatives.

Working with d–connections, we can always impose nonholonomic constraints when conditions of type (13) are satisfied which results in an effective nonholonomic gravity model with a finite number of couplings. As a matter of principle, we can limit our computations only to two–loop constructions, because terms \( F_s \) can be transformed in zero for a corresponding nonholonomic distribution, with a formal renormalization of such a model. Of course, being well defined as a perturbative quantum model such a classical theory for \( g^D \) is not equivalent to Einstein gravity. Nevertheless, we can always add the contributions of distortion tensor quantized as a nonholonomic gauge gauge theory and reconstruct the classical Einstein theory and its perturbative quantum corrections renormalized both by nonholonomic geometric methods combined with standard methods elaborated for Yang–Mills fields.

4 Nonholonomic Differential Renormalization

In this section we shall extend the method of Differential Renormalization (DiffR) [58] on \( N \)–anholonomic backgrounds, which in our approach is to be elaborated as a renormalization method in real space when too singular coordinate–space expressions are replaced by \( N \)–elongated partial
derivatives of some corresponding less singular values. In brief, we shall denote this method \( \text{Diff}_N \). It should be noted here that differential renormalization keep all constructions in four dimension which is not the case for dimensional regularization or dimensional reductions.

Our goal is to formulate a renormalization procedure for a nonholonomic de Sitter frame gauge gravity theory with field equations (14) when gravitational distortion \( \tilde{Z}^{\a}_{\b\g} \) is encoded into geometric structures on total space\(^{13}\). The base spacetime nonholonomic manifold \( V \) is considered to be endowed with a d–connection structure \( \tilde{\nabla}^\a = \{ \tilde{\Gamma}^\a_{\b\g} \} \) determined completely by a metric, \( g = \tilde{g} \), and N–connection, \( \tilde{N}^a_i \), structures. The linear connection \( \tilde{\nabla} \) is with constant curvature coefficients subjected to the conditions \( \tilde{R}^\b_{\g\alpha'} = \tilde{R}^\b_{\g'\alpha} = 0 \) (15) and (13) allowing us to perform formal one– and two–loop renormalization of the nonholonomic background \( V \) as we discussed in previous section 3.

4.1 Two–loop diagrams on nonholonomic backgrounds and \( \text{Diff}_N \)

We sketch some key constructions how differential renormalization formalism and loop diagrams can be generalized on nonholonomic spaces.

4.1.1 Propagators on N–anholonomic manifolds

Let us consider the massless propagator

\[
\Delta(1u - 2u) \equiv 12 \Delta = \frac{1}{x^u} \Delta = \frac{1}{(2\pi)^2} \frac{1}{(1u - 2u)^2},
\]

for two points \( 1,2u = (1,2x^i,1,2y^a) \in V \). On a pseudo–Euclidean spacetime, this propagator defines the one–loop contribution of so–called scalar \( \lambda \phi^4(u) \) theory

\[
\Gamma(1u, 2u, 3u, 4u) = \frac{\lambda^2}{2} \left[ (4)\delta(1u - 2u) (4)\delta(3u - 4u) [\Delta(1u - 4u)]^2 + (2 \text{ points permutations}) \right],
\]

where \( (4)\delta = \delta \) is the four dimensional delta function. The usual \( \text{Diff}_R \) method proposes to replace the function \( \frac{1}{x^u} \) (which does not have a well

\(^{13}\)for simplicity, we shall quantize a model with zero matter field source
defined Fourier transform), for \( u \neq 0 \), with the Green function \( G(u^2) \), i.e.

\[
\frac{1}{u^4} = \Box G(u^2),
\]

for d’Alambertian \( \Box = \partial^\mu \partial_\mu \) determined by partial derivatives \( \partial_\mu \) and metric on Minkowski space. This solution (renormalized, with left label \( R \)) is

\[
\frac{1}{u^4} \rightarrow R \left[ \frac{1}{u^4} \right] = -\frac{1}{4} \frac{\Box \ln u^2 - 2}{u^2}
\]

where the constant \( \varpi \) of mass dimension is introduced for dimensional reasons. This constant parametrizes a local ambiguity

\[
\Box \ln \frac{u^2 \varpi^2}{u^2} = \Box \ln \frac{u^2 \varpi^2}{u^2} + 2 \ln \frac{\varpi^'}{\varpi} \delta(u),
\]

when the shift \( \varpi \rightarrow \varpi^' \) can be absorbed by rescaling the constant \( \lambda \), see details in Ref. [58]; we can related this property with the fact that renormalized amplitudes are constrained to satisfy certain renormalization group equations with \( \varpi \) being the renormalization group scale. It should be noted here that both non–renormalized and renormalized expressions coincide for \( u \neq 0 \), but that with label ”R” has a well defined fourier transform (if we neglect the divergent surface terms that appears upon integrating by parts).

For instance, we have

\[
\int R \left[ \frac{1}{u^4} \right] e^{ip \cdot u} d^4u = -\frac{1}{4} \int \Box \ln \frac{u^2 \varpi^2}{u^2} e^{ip \cdot u} d^4u = p^2 \frac{4}{u^2} \ln \frac{p^2}{\varpi^2}.
\]

On a N–anholonomic background \( \mathbf{V} \) enabled with constant coefficients \( \hat{g}_{\alpha \beta} \) and \( \hat{N}_\nu^a \), see formulas (2) and (4), the d’Alambert operator \( \Box^{\nu} = \Box \hat{D}_\mu^\nu \), where \( \Box \hat{D}_\mu^\nu = \hat{e}_\mu^\nu \pm \hat{\Gamma}_\mu^\nu \). contains the N–elongated partial derivative \( \hat{e}_\mu^\nu = \hat{e}_\nu^\nu = \partial_\nu - \hat{N}_\nu^a (u) \partial_\nu, \partial_\nu \) as the dual to \( \hat{e}^\nu_\mu \). We suppose that the nonholonomic structure on \( \mathbf{V} \) is such way prescribed that we have a well defined background operator \( \Box \) constructed as a quasi–linear combination (with coefficients depending on \( u^\alpha \)) of partial derivatives \( \partial_\mu \). In the infinitesimal vicinity of a point \( 0^\alpha \), we can always consider \( \Box \) to be a linear transform (depending on values \( \hat{g}_{\alpha \beta} \) and \( \hat{N}_\nu^a \) in this point) of the flat operator \( \Box \). Symbolically, we shall write

\[
\int \hat{R} \left[ \frac{1}{u^4} \right] e^{ip \cdot u} \sqrt{|\hat{g}_{\alpha \beta}|} d^4u = -\pi^2 \ln \frac{p^2}{\varpi^2}.
\]
for the formal solution of 
\[ \frac{1}{u^4} = \Box G(u^2) \]
with
\[ \frac{1}{u^4} \to R \left[ \frac{1}{u^4} \right] = -\frac{1}{4} \frac{\hat{\Box} \hat{\ln} \left( u^2 \omega^2 \right)}{u^2}. \]

Such functions can be computed in explicit form, as certain series, for a prescribed nonholonomic background, for instance, beginning with a classical exact solution of the Einstein equations and a fixed 2+2 splitting. For such computations, we use formal integration by parts and have to consider a "locally anisotropic" ball \( B_\varepsilon \) of radius \( \varepsilon \) around a point \( ^0u \in V \) and keep surface terms (we denote such an infinitesimal term by \( \hat{\delta} \sigma^{\mu'} \) and a closed region \( S_\varepsilon \) in \( V \)) like in formulas related to integration of a function \( A(u) \), with volume element \( dV(u) = \sqrt{|g_{\alpha\beta}|} d^4u \),

\[
\int V \setminus B_\varepsilon A(u) \left( \hat{\Box} \hat{\ln} \left( u^2 \omega^2 \right) \right) dV(u) = \int S_\varepsilon A(u) \hat{D}_\mu \hat{\ln} \left( u^2 \omega^2 \right) \hat{\delta} \sigma^{\mu'}
\]

and

\[
\int V \setminus B_\varepsilon A(u) \left( \hat{\Box} \hat{\ln} \left( u^2 \omega^2 \right) \right) dV(u) = \int S_\varepsilon A(u) \hat{D}_\mu \hat{\ln} \left( u^2 \omega^2 \right) \hat{\delta} \sigma^{\mu'}
\]

We can approximate the first integral in the last formula as

\[
\int S_\varepsilon A(u) \left( \hat{D}_\mu \hat{\ln} \left( u^2 \omega^2 \right) \right) \hat{\delta} \sigma^{\mu'} \to 4\pi^2 A(\hat{0}u)(1 - \ln \varepsilon^2 \omega^2) + O(\varepsilon)
\]

which is divergent for \( \varepsilon \to 0 \). So, we have a formal integration rule by parts because we use counterterms. Nevertheless, this method of regularization does not require an explicit use of counterterms in calculations even the background space may be subjected to nonholonomic constraints.
4.1.2 Higher loops

The method Diff\textsubscript{NR} can be applied also to multi-loop expressions. As an example, we consider a two loop diagram from Figure 1,\[ \Delta(1u - 2u)1I(1u - 2u), \]
for
\[ 1I(1u - 2u) = \int \Delta(1u - u) (\Delta(u - 2u))^2 dV(u). \quad (18) \]

We get divergences whenever two points come together. Proceeding recursively (starting from the most inner divergence), we can renormalize them,
\[ \hat{R} \left[ \left. \frac{1}{4(2\pi)^6} \frac{1}{(1u - 2u)^2} \int \frac{1}{(1u - u)^2} \hat{ln} \left( \frac{(2u - u)^2 1\varpi^2}{(2u - u)^2} \right) dV(u) \right] = -\frac{1}{32(2\pi)^6} \hat{\square} \left[ \left. \frac{1}{4(2\pi)^6} \frac{1}{(2u - u)^2} \hat{ln} \left( \frac{(2u - u)^2 1\varpi^2}{(2u - u)^2} \right) \right] \right. \]
where there are considered two constants \(1\varpi^2\) and \(2\varpi^2\) and integrating by parts the "anisotropic" d’Alambertian we used the local limit \(\hat{\square} \to \hat{\square}\), when \(\hat{\square}(1u - 2u)^{-2} = \delta(1u - 2u)\).

![Figure 1: A two-loop diagram with nested divergences](image)

Using nonholonomic versions of d’Alambertian and logarithm function, i.e. \(\hat{\square}\) and \(\hat{ln}\), corresponding N-adapted partial derivative operators and differentials, \(\hat{\epsilon}_{\nu'}\) and \(\hat{\epsilon}^{\nu'}\), and their covariant generalizations with \(\hat{\epsilon}^{\nu'}\), we can elaborate a systematic N-adapted differential renormalization procedure to all orders in perturbations theory, extending the constructions.
from Ref. [59]. This procedure maintains unitarity, fulfills locality and Lorentz invariance to all orders and allows to renormalize massive fields (in our nonholonomic gauge like approach to Einstein gravity, masses have to be considered for quantum systems of gravitational and matter field equations; for simplicity, we omit such constructions in this paper which are similar to holonomic ones for matter and usual Yang–Mills fields in [58]). Such nonholonomic implementations of Bogoliubov’s $R$–operator (this operation yields directly renormalized correlation functions satisfying renormalization group equations) in momentum spaces can be also applied to expression with IR divergences, when $p^\mu \to 0$ and UV divergences, when $p^\mu \to \infty$. The corresponding recursion formulas from Refs. [60, 61, 62] can be easily re–defined for nonholonomic backgrounds with constant $\hat{g}_{\alpha\beta}$ and correspondingly defined connections $\hat{\nabla}^\nu_{\mu}$ and $\hat{\tilde{D}}^\nu_{\mu}$.

4.2 Constrained differential renormalization on nonholonomic spaces

We can apply the method of constrained differential renormalization, see a review and basic references in [63] in order to avoid the necessity of imposing Ward identities in each calculation scheme. The constructions (in brief, we shall write for this method CDR$_N$) can be adapted to the $N$–anholonomic structure as we have done in the previous section. One should follow the rules:

1. $N$–adapted differential reduction;

One reduces to covariant $d$–derivatives of logarithmically divergent (at most), without introducing extra dimensional constants, all functions with singularities worse than logarithmic ones.

One introduces a constant $\varpi$ (it has dimension of mass and plays the role of renormalization group scale) for any logarithmically divergent expression which allows us to rewrite such an expression as derivatives of regular functions.

In infinitesimal limits, the ”anisotropic” logarithm $\ln$ and operator $\Box$ can transform into usual ones in flat spacetimes.

2. Integration by parts using $N$–elongated differentials $\tilde{e}^\nu(\mathbf{2})$. It is possible to omit consideration of divergent surface terms that appear under
integration by parts. For N–adapted differentiation and renormalization of an arbitrary function \( A(u) \), we have \( \hat{\mathcal{R}} \left[ \mathcal{D} A \right] = \mathcal{D} \hat{\mathcal{R}} \left[ A \right] \) and \( \hat{\mathcal{R}} \left[ \mathcal{e} A \right] = \mathcal{e} \hat{\mathcal{R}} \left[ A \right] \).

3. Renormalization of delta function and propagator equation,

\[
\begin{align*}
\hat{\mathcal{R}} \left[ A(u, \ 1_u, \ 2_u, \ ..., \ k-1_u) \ \delta \left( k_u - u \right) \right] &= \hat{\mathcal{R}} \left[ A(u, \ 1_u, \ 2_u, \ ..., \ k-1_u) \ \delta \left( k_u - u \right) \right], \\
\hat{\mathcal{R}} \left[ A(u, \ 1_u, \ 2_u, \ ..., \ k_u) \ \left( \hat{\Box} - \mu^2 \right) \mu \ \Delta \left( k_u - u \right) \right] &= - \hat{\mathcal{R}} \left[ A(u, \ 1_u, \ 2_u, \ ..., \ k-1_u) \ \delta(u) \right],
\end{align*}
\]

where \( \mu \Delta \) is the propagator of a particle of mass \( \mu \), where \( \mu = 0 \) for gravitational fields, and \( A \) is an arbitrary function.

The method CDR\(_N\) contains two steps:

- The Feynman diagrams are expressed in terms of basic functions performing all index contractions (this method does not commute with contractions of indices); using the Leibniz rule, we move all N–adapted derivatives to act on one of the propagators.
- Finally, we replace the basic functions with their renormalized versions.

In order to understand how the above mentioned method should be applied in explicit computations, we present a series of important examples.

The one–loop correction to the two–point function in \( \lambda \phi^4(u) \) theory is defined by renormalization of \( \Delta(u) \delta(u) \), which is constrained by above mentioned rules to result in \( \hat{\mathcal{R}} \left[ \Delta(u) \delta(u) \right] = 0 \). This way we get that all massless one–point functions in CDR\(_N\) are zero, i.e. a nonholonomic structure does not change similar holonomic values.

A nonholonomic configuration can be included into an operator containing a N–adapted covariant derivative, but also results in a zero contribution if the operator \( \hat{\Box} \) is introduced into consideration, i.e. \( \hat{\mathcal{R}} \left[ \Delta \hat{\Box} \Delta \right] (u) = 0 \).
One hold true the important formulas:

\[
\hat{R} \left[ \triangle^2 \right] (u) = -\frac{1}{4(2\pi)^4} \hat{\Delta} \left[ \frac{\hat{\ln} \left[ \frac{u^2}{\omega^2} \right]}{u^2} \right],
\]

\[
\hat{R} \left[ \triangle \hat{\mathcal{D}}_{\mu' \nu} \right] (u) = -\frac{1}{8(2\pi)^4} \hat{\mathcal{D}}_{\mu'} \left( \hat{\Delta} \left[ \frac{\hat{\ln} \left[ \frac{u^2}{\omega^2} \right]}{u^2} \right] \right),
\]

\[
\hat{R} \left[ \triangle \hat{\mathcal{O}}_{\mu' \nu} \right] (u) = -\frac{1}{12(2\pi)^4} \left( \hat{\mathcal{O}}_{\mu'} \hat{\mathcal{O}}_{\nu'} - \frac{\delta_{\mu' \nu'}}{4} \hat{\Delta} \right) \left( \hat{\Delta} \left[ \frac{\hat{\ln} \left[ \frac{u^2}{\omega^2} \right]}{u^2} \right] \right)
+
\frac{1}{288\pi^2} \left( \hat{\mathcal{O}}_{\mu'} \hat{\mathcal{O}}_{\nu'} - \delta_{\mu' \nu'} \hat{\Delta} \right) \delta (u).
\]

The method $\text{CDR}_N$ can be applied to more than two propagators. For instance, we can write $\hat{T} \left[ \triangle \mathcal{O} \right] = \triangle \triangle \mathcal{O} \triangle$, for three propagators, and compute

\[
\hat{R} T \left[ \hat{\mathcal{O}}_{\mu' \nu} \hat{\mathcal{O}}_{\nu'} \right] = \hat{R} T \left[ \hat{\mathcal{O}}_{\mu'} \hat{\mathcal{O}}_{\nu'} - \frac{\delta_{\mu' \nu'}}{4} \hat{\Delta} \right] + \frac{\delta_{\mu' \nu'}}{4} \hat{\Delta}
+
\frac{1}{128\pi^2} \delta_{\mu' \nu'} \delta (1u) \delta (2u),
\]

for two points $1u$ and $2u$.

### 4.3 Using one–loop results for $\text{CDR}_N$ in two–loop calculus

The CDR$_N$ can be easily developed at loop–order higher than one, which is enough to define the renormalization group (RG) equations. We restrict our geometric analysis only for such constructions and do not analyze, for instance, scattering amplitudes.

For the simplest, so–called nested divergences, we can compute in N–adapted form (applying formulas from previous section) the value [18], when
according the CDR\(_N\) rules, we get the renormalized values

\[
\hat{R} \left[ \Delta^1 I \right](u) = -\frac{1}{32(2\pi)^6} \left[ \left( \hat{\ln}[u^2 w^2] \right)^2 + \frac{11}{3} \hat{\ln}[u^2 w^2] \right] + \ldots,
\]

where "..." stand for the two–loop local terms that are not taken into account.

In order to compute overlapping divergences, we define for any differential \(i\)-operator \(i\mathcal{O}\), and (for instance) \(i_{1u} \mathcal{O}\) taken in the point \(1u\), the value \(H(1u - 2u) \equiv H(u)\), for \(\hat{\partial}_{\mu'}\) taken in the point \(1u\),

\[
\hat{R} \left[ \Delta^2 \mathcal{D}_{\mu'} \right] \left[ 1 I \right](u) = \frac{1}{32(2\pi)^4} \hat{\partial}_2 \left[ \left( \hat{\ln}[u^2 w^2] \right)^2 + \frac{11}{3} \hat{\ln}[u^2 w^2] \right] + \ldots,
\]

This value is very useful because it can be used as a basis for expressing the renormalized overlapping contributions to two–point functions in theories with derivative couplings at two loops. They are necessary if we need to obtain the beta function following the background field method. The typical expressions for such renormalized overlapping divergences are presented in Appendix A.

5 Quantization of Distortion Gauge Fields

This section focuses on quantization of the nonholonomic gauge model of gravity constructed as a lift of the Einstein theory in the total space of de Sitter frame bundle. The details of geometric formulation and classical field equations are given in sections 4.2 and 4.3 of Ref. [42].
5.1 Nonholonomic gauge gravity theory

Let $S = SO(5)$ be the continuous symmetry/gauge group (in this model, the isometry group of a de Sitter space $\Sigma$) with generators $I_1, ..., I_S$ and structure constants $f_{LP}^S$ defining a Lie algebra $\mathfrak{A} = so(5)$ through commutation relation

$$[I_S, I_P] = if_{SP}^TI_L,$$

with summation on repeating indices. The space $\Sigma$ can be defined as a hypersurface $\eta_{AB}u^Au^B = -1$ in a four-dimensional flat space endowed with a diagonal metric $\eta_{AB} = \text{diag}[±1, ..., ±1]$, where $\{u^A\}$ are global Cartesian coordinates in $R^5$, indices $A, B, C$ run values $1, 2, ..., 5$ and $l > 0$ is the constant curvature of de Sitter space. In quantum models, it is convenient to choose the so-called adjoint representation when the representation matrices are given by the structure constants $(\tau^S)_{TP}^{LR} = if_{SP}^T$. Such matrices satisfy the conditions

$$\sum_S (\tau^S)^2 = \sum_S \tau^S \tau^S = 1 C \mathbb{I} \text{ and } tr[\tau^S \tau^P] = 2 C \delta^{SP},$$

where the quadratic Casimir operator is defined by constant $1 C$, $2 C = \text{const}$, $tr$ denotes trace of matrices and $\mathbb{I}$ is the unity matrix. In the adjoint representation, we can write $f_{SP}^T f_{LP}^{TS} = 1 C \delta^{IL}$. With respect to a $N$–adapted dual basis $(\epsilon^\nu)'$ on $V$, we consider a d–connection (nonholonomic gauge potential) $\mathcal{Z}_\nu' = \mathcal{Z}_\nu^S \tau^S$ defining the covariant derivation in the total space $\mathcal{D}_\nu' = \hat{\mathcal{D}}_\nu' - i\mathcal{Z}_\nu'$, where the constant $\kappa$ is an arbitrary one, similar to a particular fixing of d–tensors $Y^m_{\epsilon j}$, $Y^k_{mc}$, $Y^d_{ck}$ and $Y^{di}_{ec}$ in (5) in order to state in explicit form a nonholonomic configuration. If we chose the adjoint representation, we get a covariant derivation

$$\mathcal{D}_{\nu}'^{TS} = \hat{\mathcal{D}}_{\nu}' \delta^{TS} + \kappa f_{TS}^{LP} \mathcal{Z}_{\nu}'^P.$$ 

14 A canonical 4 + 1 splitting is parametrized by $A = (\alpha, 5)$, $B = (\beta, 5), ..., \eta_{AB} = (\eta_{\alpha\beta}, \eta_{55})$ and $P_{\alpha} = l^{-1}M_{\alpha\beta}$ for $\alpha, \beta, ..., = 1, 2, 3, 4$ when the commutation relations are written

$$[M_{\alpha\beta}, M_{\gamma\delta}] = \eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\alpha\delta}M_{\beta\gamma} + \eta_{\beta\delta}M_{\alpha\gamma} - \eta_{\beta\gamma}M_{\alpha\delta},$$

$$[P_{\alpha}, P_{\beta}] = -l^{-2}M_{\alpha\beta}, [P_{\alpha}, M_{\beta\gamma}] = \eta_{\alpha\gamma}P_{\beta} - \eta_{\alpha\beta}P_{\gamma}.$$ 

This defines a direct sum $so(5) = so(4) \oplus 4 V$, where $4 V$ is the four dimensional vector space stretched on vectors $P_{\alpha}$. We remark that $4 \Sigma = S/ S$, where $S = SO(4)$. Choosing signature $\eta_{AC} = \text{diag}[±1, 1, 1, 1]$ and $S = SO(1, 4)$, we get the group of Lorentz rotations $S = SO(1, 3)$. 25
The d–field $\hat{\mathcal{Z}}_P^\mu$ is parametrized in matrix form as a d–connection of type d–connection (7) subjected to certain nonholonomic nonlinear gauge and frame transforms. The curvature is defined by commutator of $\hat{\mathcal{D}}_\nu^\mu \hat{\mathcal{Z}}_P^\mu$, equivalently to $\hat{\mathcal{Z}}$ from gauge gravity equations (14). In N–adapted components and adjoint representation, we get the field strength

$$\hat{\mathcal{Z}}_\mu^\nu = \hat{\mathcal{D}}_\mu^\nu \hat{\mathcal{Z}}_P^\mu - \hat{\mathcal{D}}_\nu^\mu \hat{\mathcal{Z}}_P^\nu + \kappa f^{PST} \hat{\mathcal{Z}}_S^P \hat{\mathcal{Z}}_T^\nu.$$

For zero source of matter fields, when $\hat{\mathcal{J}} = 0$ (this condition can be satisfied in explicit form for nonholonomic configurations with $\hat{\Gamma} = 0$), we can consider a nonholonomic gauge gravity field action (for distortions of the Levi–Civita connection lifted on total space)

$$\hat{\mathcal{S}}(\hat{\mathcal{Z}}) = \frac{1}{4} \int \hat{\mathcal{Z}}_{\mu^\nu} \hat{\mathcal{Z}}_{\mu^\nu} \ dV(u). \quad (19)$$

In order to quantize the action following the path integral method, we have to fix the "gauge" in order to suppress all equivalent field and nonholonomic configurations related by nonholonomic/nonlinear gauge transform, i.e. to introduce a gauge–fixing function $G^S(\hat{\mathcal{Z}}_P^\mu)$, and consider a generating source $J_P^\mu$. The partition function is taken

$$Z[J] = \int [d \hat{\mathcal{Z}}] \det \left[ \frac{\delta \hat{\mathcal{S}}(w \hat{\mathcal{Z}})}{\delta \hat{\mathcal{Z}}_P^\mu} \right]_{w=0} \times \exp \left[ - \hat{\mathcal{S}}(\hat{\mathcal{Z}}) - \frac{1}{2\alpha} \int G^S G^S \ dV(u) + J_P^\mu \hat{\mathcal{Z}}_P^\mu \right],$$

for $\alpha = \text{const}$. Choosing the value $G^S = \hat{\mathcal{D}}_\nu^\mu \hat{\mathcal{Z}}_S^\nu$, when the N–adapted infinitesimal nonlinear gauge transform are approximated

$$\hat{\mathcal{Z}}_P^\mu \rightarrow \hat{\mathcal{Z}}_P^\mu + \kappa^{-1} \hat{\mathcal{D}}_\mu^\nu w^\nu f^{PST} \hat{\mathcal{Z}}_S^P \hat{\mathcal{Z}}_T^\nu + O(w^2)$$

$$= \hat{\mathcal{Z}}_P^\mu + \kappa^{-1} (\hat{\mathcal{D}}_\nu^\mu w)^\nu + O(w^2),$$

and writing the determinant in terms of ghost (anticommutative variables) fields $\hat{\eta}_P$, and their complex conjugated values $\hat{\eta}_P^\dagger$, we get the gravitational gauge Lagrangian

$$\mathcal{L} = \frac{1}{4} \hat{\mathcal{Z}}_{\mu^\nu} \hat{\mathcal{Z}}_{\mu^\nu}^\dagger + \frac{1}{2\alpha} \left( \hat{\mathcal{D}}_\nu^\mu \hat{\mathcal{Z}}_S^\nu \right) \left( \hat{\mathcal{D}}_\mu^\nu \hat{\mathcal{Z}}_S^\nu \right) + \left( \hat{\mathcal{D}}_\mu^\nu \hat{\eta}_P \right) \left( \hat{\mathcal{D}}_\nu^\mu \hat{\eta}_P^\dagger \right) \quad (20)$$
resulting respectively in propagators for the distortion and ghost fields,
\[
\langle \hat{Z}^{P}_{\mu'}(1u)\hat{Z}^{S}_{\nu'}(2u) \rangle = \delta_{\mu'\nu'}\delta^{PS} \Delta (1u - 2u)
\]
and
\[
\langle \eta^{P}_{\mu'}(1u)\eta^{P}_{\nu'}(2u) \rangle = \delta^{PS} \Delta (1u - 2u).
\]

We conclude that in our approach, the distortion gravitational field (parametrized by de Sitter valued potentials) can be quantized similarly to usual Yang–Mills fields but with nonholonomic/nonlinear gauge transforms, all defined on a N–anholonomic base spacetime enabled with fundamental geometric structures $\hat{g}_{\alpha\beta}$, $\hat{D}_{\mu'}$ and $\hat{e}^\alpha$.

5.2 N–adapted background field method

The first examples of N–adapted background calculus were presented in Refs. [56, 57] when locally anisotropic (super) gravity configurations were derived in low energy limits, with nonholonomic backgrounds, of (super) string theory. In this section, we apply that formalism for quantization of nonholonomic gauge gravity models.

5.2.1 Splitting of nonholonomic gauge distortion fields

We begin with a splitting if the nonholonomic gauge field into two parts $\hat{Z}^{P}_{\mu'} \rightarrow \hat{Z}^{P}_{\mu'} + B^{P}_{\mu'}$, where $\hat{Z}^{P}_{\mu'}$ and $B^{P}_{\mu'}$ are called respectively the quantum and background fields. The action $\mathcal{S}(\hat{Z} + B)$ is invariant under 1) quantum transforms
\[
\delta \hat{Z}^{P}_{\mu'} = \chi^{-1} \left( \hat{D}_{\mu'} w + \chi f^{PST}_{\mu'} \hat{Z}^{P}_{\mu'} \right) + f^{PST}_{\mu'} \hat{Z}^{P}_{\mu'}
\]
and 2) background transforms
\[
\delta B^{P}_{\mu'} = 0,
\]
and
\[
\delta \hat{B}^{P}_{\mu'} = f^{PST}_{\mu'} \hat{Z}^{P}_{\mu'},
\]
\[
\delta B^{P}_{\mu'} = \chi^{-1} \hat{D}_{\mu'} w + f^{PST}_{\mu'} \hat{Z}^{P}_{\mu'}.
\]

Following standard methods of quantization of gauge fields (see, for instance, [44]), we derive from $Z[J]$ the partition function
\[
Z[B] = \int e^{-\hat{S}(\hat{Z} + B)[d\hat{Z}d\eta]} = \int \exp \{ - \hat{S}(\hat{Z} + B) + \}
\]
\[
tr \int \left( -\frac{1}{2\alpha} \left[ B\hat{D}_{\mu'} \hat{Z}_{\mu'} \right]^2 + \tau \left[ B\hat{D}_{\mu'}, \hat{D}_{\nu'} \right] c \right) dV(u) \}
\]
and
\[
Z = \int e^{-\hat{S}(\hat{Z} + B)[d\hat{Z}d\eta]} = \int \exp \{ - \hat{S}(\hat{Z} + B) + \}
\]
\[
\int \left( -\frac{1}{2\alpha} \left[ B\hat{D}_{\mu'} \hat{Z}_{\mu'} \right]^2 + \tau \left[ B\hat{D}_{\mu'}, \hat{D}_{\nu'} \right] c \right) dV(u) \}
\]
and
\[
Z = \int e^{-\hat{S}(\hat{Z} + B)[d\hat{Z}d\eta]} = \int \exp \{ - \hat{S}(\hat{Z} + B) + \}
\]
\[
\int \left( -\frac{1}{2\alpha} \left[ B\hat{D}_{\mu'} \hat{Z}_{\mu'} \right]^2 + \tau \left[ B\hat{D}_{\mu'}, \hat{D}_{\nu'} \right] c \right) dV(u) \}
\]
and
\[
Z = \int e^{-\hat{S}(\hat{Z} + B)[d\hat{Z}d\eta]} = \int \exp \{ - \hat{S}(\hat{Z} + B) + \}
\]
\[
\int \left( -\frac{1}{2\alpha} \left[ B\hat{D}_{\mu'} \hat{Z}_{\mu'} \right]^2 + \tau \left[ B\hat{D}_{\mu'}, \hat{D}_{\nu'} \right] c \right) dV(u) \}
\]
and
\[
Z = \int e^{-\hat{S}(\hat{Z} + B)[d\hat{Z}d\eta]} = \int \exp \{ - \hat{S}(\hat{Z} + B) + \}
\]
\[
\int \left( -\frac{1}{2\alpha} \left[ B\hat{D}_{\mu'} \hat{Z}_{\mu'} \right]^2 + \tau \left[ B\hat{D}_{\mu'}, \hat{D}_{\nu'} \right] c \right) dV(u) \}
\]
and
\[
Z = \int e^{-\hat{S}(\hat{Z} + B)[d\hat{Z}d\eta]} = \int \exp \{ - \hat{S}(\hat{Z} + B) + \}
\]
\[
\int \left( -\frac{1}{2\alpha} \left[ B\hat{D}_{\mu'} \hat{Z}_{\mu'} \right]^2 + \tau \left[ B\hat{D}_{\mu'}, \hat{D}_{\nu'} \right] c \right) dV(u) \}
\]
and
\[
Z = \int e^{-\hat{S}(\hat{Z} + B)[d\hat{Z}d\eta]} = \int \exp \{ - \hat{S}(\hat{Z} + B) + \}
\]
\[
\int \left( -\frac{1}{2\alpha} \left[ B\hat{D}_{\mu'} \hat{Z}_{\mu'} \right]^2 + \tau \left[ B\hat{D}_{\mu'}, \hat{D}_{\nu'} \right] c \right) dV(u) \}
\]
and
\[
Z = \int e^{-\hat{S}(\hat{Z} + B)[d\hat{Z}d\eta]} = \int \exp \{ - \hat{S}(\hat{Z} + B) + \}
\]
\[
\int \left( -\frac{1}{2\alpha} \left[ B\hat{D}_{\mu'} \hat{Z}_{\mu'} \right]^2 + \tau \left[ B\hat{D}_{\mu'}, \hat{D}_{\nu'} \right] c \right) dV(u) \}
\]
and
\[
Z = \int e^{-\hat{S}(\hat{Z} + B)[d\hat{Z}d\eta]} = \int \exp \{ - \hat{S}(\hat{Z} + B) + \}
\]
\[
\int \left( -\frac{1}{2\alpha} \left[ B\hat{D}_{\mu'} \hat{Z}_{\mu'} \right]^2 + \tau \left[ B\hat{D}_{\mu'}, \hat{D}_{\nu'} \right] c \right) dV(u) \}
\]
with

$$(D_{\mu'} w)^P = \hat{\nabla}_{\mu'} w^P + \kappa f^{PST}_{\mu'} (\tilde{Z}^S_{\mu'} + B^S_{\mu'})$$

and $(c, \bar{c})$ being the Faddeev–Popov ghost fields.

One of the most important consequences of the background field method is that the renormalization of the gauge constant, $0\kappa = c z \kappa$, and the background field, $0B = b z^{1/2}B$ are related,

$$c z = b z^{-1/2},$$

with renormalization of strength field,

$$0F^S_{\mu\nu} = b z^{1/2} \left[ \hat{\nabla}_{\mu'} B^S_{\nu'} - \hat{\nabla}_{\nu'} B^S_{\mu'} + \kappa c z b z^{1/2} f^{PST}_{\mu'} B^P_{\mu'} B^T_{\nu'} \right].$$

This value defines the N–adapted background renormalization of $\tilde{Z}^P_{\mu'\nu'}$. One should be emphasized the substantial physical difference between relations of type (23) in usual Yang–Mills theory and nonholonomic models of gravity. In the first case, they relate renormalization of the fundamental constant (in particular, electric charge) to renormalization of the background potential. In the second case, we have certain constants fixing a d–connection, and a lift in the total bundle, which must be renormalized in a compatible form with renormalization of distortion gravitational fields. We can impose any type of nonholonomic constraints on such nonlinear gauge fields but quantum fluctuations re–define them and correlate to renormalization of nonholonomic background configuration.

5.2.2 Nonholonomic background effective action

Using splitting $\tilde{Z}^P_{\mu'} \rightarrow \tilde{Z}^P_{\mu'} + B^P_{\mu'}$ and covariant derivatives $B_{\mu'}$ and $D_{\mu'}$, see respective formulas (21) and (22), we write the Lagrangian (20) in the form

$$\mathcal{L} = \frac{1}{4} \tilde{Z}^P_{\mu'\nu'} \tilde{Z}^{P\mu'\nu'} - \frac{1}{2\alpha} \left( B_{\mu'} \tilde{Z}^S_{\nu'} \right) \left( B_{\nu'} \tilde{Z}^S_{\mu'} \right) + \left( B_{\mu'} \tilde{Z}^P_{\mu'} (D_{\mu'} \eta)^P \right),$$

where the total space curvature (field strength) depends both on quantum and background fields, $\tilde{Z}^P_{\mu'\nu'} = \tilde{Z}^P_{\mu'\nu'} (\tilde{Z}, B)$. This Lagrangian is similar to that used of standard Yang–Mills fields with chosen (for convenience) Feynman gauge $(\alpha = 1)$. There are formal geometric and substantial physical differences because we use the N–adapted covariant operators $\hat{\nabla}_{\mu'}$ instead of partial derivatives $\partial_{\mu}$, our structure group is the de Sitter group.
and constants in the theory do not characterize certain fundamental gauge interactions but corresponding classes of nonholonomic configurations. Nevertheless, a very similar Feynman diagram techniques can be applied and a formal renormalization can be performed following standard methods from quantum gauge fields theory (see, for instance, [44, 58, 59, 60, 62, 63, 64]). For simplicity, we shall omit details on explicit computations of diagrams but present the main formulas and results and discuss the most important features of quantized distortion fields in one– and two–loop approximations.

The effective action for nonholonomic background derived for Lagrangian (24) can be written:

$$\text{eff} \Gamma[B] = \frac{1}{2} \int \{ \mathcal{B}_{\mu'}^S(1u) \, BB \, \Pi_{\mu'\nu'}^{ST} \, (1u - 2u) \times$$

$$\mathcal{B}_{\nu'}^T(2u) \, dV(1u) dV(2u) + \ldots \} \tag{25}$$

$$+ \frac{1}{2} \int \mathcal{B}_{\mu'}^S(1u) \{ \delta^{ST} \, (\circ \mathbf{D}_{\mu'} \circ \mathbf{D}_{\nu'} - \delta_{\mu'\nu'} \Box) \} \delta(4)(1u - 2u)$$

$$- \frac{BB \, \Pi_{\mu'\nu'}^{ST}}{\xi} \, (1u - 2u) \} \mathcal{B}_{\nu'}^T(2u) \, dV(1u) dV(2u) + \ldots$$

$$= 0S[B] + \frac{1}{\xi} \Gamma - \frac{1}{2} \int \mathcal{B}_{\mu'}^S(1u) \, BB \, \Pi_{\mu'\nu'}^{ST} \, (1u - 2u) \times$$

$$\mathcal{B}_{\nu'}^T(2u) \, dV(1u) dV(2u) + \ldots ,$$

where $0S[B]$ is called the three–level background two–point function and $\xi = \alpha - 1$ (in the Feynman gauge, we have $\xi = 0$). In the next sections, we will compute the one–loop contribution to the background self–energy in this gauge. Using functional methods, we shall also compute the term $\frac{1}{\xi} \Gamma$ after expanding the complete effective action at one loop at second order in the background fields and collecting only components being linear in $\xi$. Applying this procedure to the renormalization group equation, we will take derivatives with respect to parameter $\xi$ and impose, finally, the gauge $\xi = 0$.

### 5.2.3 One– and two–loop computations

To find the one–loop beta function we need to compute the background self–energy. At the first step, we present the result for a renormalized correction to the $\mathcal{B}_{\mu'}^S$ propagator (a similar calculus, for holonomic gauge fields, is provided in details in section 3.1.2 of [64]):

$$\hat{R} \left[ BB_{1\text{-loop}} \, \Pi_{\mu'\nu'}^{ST}(u) \right] = x^2 1 C \delta^{ST}(\circ \mathbf{D}_{\mu'} \circ \mathbf{D}_{\nu'} - \delta_{\mu'\nu'} \Box) \times$$

$$\left[ -\frac{11}{12(4\pi^4)} \ln \frac{u^2}{\omega^2} - \frac{1}{72\pi^2} \delta(u) \right] .$$

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This value is contained in effective action \((25)\).

In order to obtain an exact one–loop effective action we have to consider the part of Lagrangian \((24)\) which is quadratic on quantum distortion fields \(\hat{Z}_{\mu}^{P}\). Such an effective action can be represented

\[
2\mathcal{L} = x f^{PST} B_{\mu \nu}^{P} \hat{Z}_{\mu}^{S} \hat{Z}_{\nu}^{T} + \frac{1}{2} \left( B D_{\mu}^{P} \hat{Z}_{\mu}^{P} \right) S \left( B D_{\mu}^{P} \hat{Z}_{\mu}^{P} \right) S
\]

\[
+ \frac{\xi}{2} \left( B D_{\mu}^{P} \hat{Z}_{\mu}^{P} \right) S \left( B D_{\mu}^{P} \hat{Z}_{\mu}^{P} \right) S
\]

\[
= - \frac{1}{2} \hat{Z}_{\mu}^{P} \left[ \delta_{\mu \nu} \hat{\Sigma}^{ST} - 2 x f^{PST} B_{\mu \nu}^{P} + \xi \left( B D_{\mu}^{P} B D_{\nu}^{P} \right)^{ST} \right] \hat{Z}_{\mu}^{P},
\]

where \(\hat{\Sigma}^{ST} = ( B D_{\mu}^{P} B D_{\nu}^{P})^{ST} \) and

\[
B_{\mu \nu}^{P} = \hat{\Theta}_{\mu}^{P} \hat{B}_{\nu}^{P} - \hat{\Theta}_{\nu}^{P} \hat{B}_{\mu}^{P} + x f^{PST} B_{\mu}^{P} B_{\nu}^{P}.
\]

The terms from [...] define the generated functional \(\mathcal{G}\) for connected Green functions,

\[
\mathcal{G} = - tr \ln \left[ \delta_{\mu \nu} \hat{\Sigma}^{ST} - 2 x f^{PST} B_{\mu \nu}^{P} + \xi \left( B D_{\mu}^{P} B D_{\nu}^{P} \right)^{ST} \right]
\]

\[
\simeq G_{0} + \xi \frac{1}{2} x \mu^{2} tr \left[ \hat{\Delta} B_{\mu \nu}^{P} \hat{B}_{\nu \lambda}^{P} \Delta \left( \hat{\Theta}_{\mu}^{P} \hat{\Theta}_{\nu}^{P} \right) \right] + O(\xi^{2}, B^{2}),
\]

where \(\hat{\Delta} = \hat{\Theta}^{-1}\), for \(\hat{\Theta} = \hat{\Theta}^{P} \hat{\Theta}^{P} \hat{\Theta}^{P} \).

The renormalized version of \(\mathcal{G}\) \((27)\) gives the term used in effective action \((25)\),

\[
\chi^{1} \Gamma = - \frac{\xi}{16 \pi^{2}} \int B_{\mu}^{S} (1 u) B_{\nu}^{T} (2 u) \left( \hat{D}_{\mu}^{P} \hat{D}_{\nu}^{P} \delta_{\mu \nu} - \delta_{\mu \nu} \hat{\Theta} \right)
\]

\[
\left( \hat{\Theta} \left( 1 u - 2 u \right) \right) dV(1 u) dV(2 u),
\]

were, for instance, \(\hat{D}_{\mu}^{P}\) denotes that operator \(\hat{D}_{\mu}^{P}\) is computed in point \(1 u\).

Now we present the formula for the two–loop renormalized contribution to the background field self–energy (it is a result of cumbersome nonholonomic background computations using \(H\)–components from Appendix \(A\); see similar computations with Feynman diagrams in section 3.1.3 of \([64]\)).

\[
BB_{(2\text{–loop})}^{\Pi^{ST}}(u) = - \frac{1}{128 \pi^{2}} \frac{1}{128 \pi^{2}} \left( \hat{\Theta}_{\mu}^{P} \hat{\Theta}_{\nu}^{P} - \delta_{\mu \nu} \hat{\Theta} \right) \hat{\Theta} \left( \frac{1}{u^{2}} \frac{u^{2}}{2} \right) + ...
\]
We emphasize that applying the CDR method for one–loop formulas we fix a priori a renormalization scheme resulting in total two–loop renormalized contributions to the background self–energy for distortion gravitational field.

5.2.4 Renormalization group equations for nonholonomic configurations

Let us parametrize

\[ BB \Gamma_{\mu'\nu'}^{ST} (u) = \delta_{\mu'\nu'} \left( \circ \hat{D}_{\mu'} \circ \hat{D}_{\nu'} - \delta_{\mu'\nu'} \Box \right) \Gamma^{(2)} (u), \]

where \( \Gamma^{(2)} (u) \) is computed as the sum of terms (26), (28) and (29), i.e.

\[ \Gamma^{(2)} = \left[ \frac{1}{\pi^2} + \left( \frac{1}{9} + \xi \right) \frac{1}{8\pi^2} \right] \delta (u) + \frac{1}{2(2\pi)^4} \left( \frac{11}{6} + \frac{1}{(2\pi)^2} \right) \ln \left[ \frac{u^2}{\omega^2} \right] \]

For this function, we consider the renormalization group (RG) equation

\[ \left[ \omega \frac{d}{d\omega} + \beta(\omega) \frac{d}{d\omega} + \xi \gamma \frac{d}{d\xi} - 2 \beta \gamma \right] \Gamma^{(2)} (u) = 0 \]

when \( \xi \gamma = -5 \frac{1}{2} \frac{C}{24\pi^2} \). We can transform \( \beta \gamma = 0 \) if the background gauge gravity field is redefined \( B' = \omega B \) for the charge and background field renormalizations being related by formula (23).

We can evaluate the first two coefficients, \( 1 \beta \) and \( 2 \beta \), of the expansion of the beta function

\[ \beta (\omega) = 1 \beta \omega^3 + 2 \beta \omega^5 + O (\omega^7), \]

with

\[ 1 \beta = \frac{11}{48\pi^2}, \quad 2 \beta = \frac{17 (\frac{1}{2} C)^2}{24(2\pi)^4}. \]

These formulas are similar to those in (super) Yang–Mills theory (see for instance [65, 66]). For the nonholonomic gauge model of the Einstein gravity, such coefficients are not related to renormalization of a fundamental gauge constant but to quantum redefinition of certain constants stating a nonholonomic configuration for gravitational distortion fields.
6 Summary and Conclusions

In the present paper, we have elaborated a (one– and two–loop) perturbative quantization approach to Einstein gravity using a two–connection formalism and nonholonomic gauge gravity models. The essential technique is the use of the geometry of nonholonomic distributions and adapted frames and (non) linear connections which are completely defined by a given (pseudo) Riemannian metric tensor.

Let us outline the key steps of the quantization algorithm we developed:

1. On a four dimensional pseudo–Riemannian spacetime manifold \( V \), we can consider any distribution of geometric objects, frames and local coordinates. For our purposes, there were involved nonholonomic distributions inducing \((2 + 2)\)–dimensional spacetime splitting characterized by corresponding classes of nonholonomic frames and associated nonlinear connection (N–connection) structures.

2. We applied the formalism of N–connections and distinguished connections (d–connections) completely determined by metric structure. This allows us to rewrite equivalently the Einstein equations in terms of nonholonomic variables (vierbein fields and generalized connections). Such geometric formulations of Einstein gravity are more suitable for quantization following the background field method and techniques elaborated in Yang–Mills theory.

3. The background field method can be modified for d–connections. It is convenient to construct such a canonical d–connection which is characterized, for instance, by certain constant matrix coefficients of the Riemannian and Ricci tensors. For an auxiliary model of gravity with d–connections and infinitely many couplings, we impose such nonholonomic constraints when the Ricci and Riemannian/Weyl tensors vanish (nevertheless, similar tensors corresponding to the Levi–Civita connection are not trivial). Such a model can be quantized and renormalized as a gravity theory with two constants, for instance, with the same gravitational and cosmological constants as in Einstein gravity.

4. Any background d–connection completely defined by a metric structure can be distorted in a unique form to the corresponding Levi–Civita connection. The distortion field can be encoded into a class of Yang–Mills like equations for nonholonomic gauge gravity models. The constants in such theories do not characterize any additional
gauge like gravitational interactions but certain constraints imposed on the nonholonomic structure which give us the possibility to establish equivalence with the Einstein equations on the base spaceime.

5. The nonholonomic gauge gravity model can be quantized following methods elaborated for Yang–Mills fields. There are also some important differences: We work on background spaces enabled with d–connections with constant curvature coefficients (and quantized as in point 3); gravitational gauge group transforms are generic nonlinear and nonholonomically deformed, the constants in the theory, and their renorm group flows, are not related to certain additional gravitational–gauge interactions but determined by self–consistent quantum flows of nonholonomic configurations.

6. Above outlined geometric and quantum constructions can be redefined in terms of the Levi–Civita connection and corresponding Einstein equations. Renormalized d–connection and distortion fields and redefined gravitational, cosmological and nonholonomic configuration constants will be regrouped correspondingly for the metric/ tetradic components.

The proposed quantization algorithm is based on three important geometric ideas:

The first one is that for a metric tensor we can construct an infinite number of metric compatible linear connections (in N–connection adapted form, d–connections). Even the torsion of such a d–connection is not zero, it can be considered as a nonholonomic frame effect with coefficients induced completely by certain off–diagonal coefficients of the metric tensor. Using two linear connections completely defined by the same metric structure, we get more possibilities in approaching the problem of renormalizability of gravity. We can invert equivalently all construction in terms of the Levi–Civita connection and work in certain ”standard” variables of general relativity theory. It is not obligatory to generalize the Einstein gravity theory to some models of Einstein–Cartan/ string/ gauge gravity, were the torsion fields are subjected to satisfy certain additional field (dynamical, or algebraic) equations.

The second geometric idea is to consider such nonholonomic distributions when the so–called canonical d–connection is characterized by constant matrix coefficients of Riemannian and Ricci tensors. We can prescribe such sets of nonholonomic constants when the one– and two–loop quantum divergent terms vanish. The nonrenormalizability of Einstein’s theory (in a standard
meaning of gauge theories with certain gauge symmetry and mass dimensionality of couplings) is related to the fact that the gravitational coupling is characterized by Newton’s constant, which for a four dimensional space-time has the dimension of a negative power of mass.\footnote{This resulted in conclusion that the removal of divergences of quantum gravity is possible only in the presence of infinitely many independent coupling constants \cite{34}.}

The background field method can be redefined for $d$–connections and nonholonomic configurations when certain models of quantum nonholonomic gravity with infinite coupling constants transform into a theory with usual gravitational and cosmological constants.

The third geometric idea is to construct models of gauge theories generated from the Einstein gravity theory imposing a corresponding class of nonholonomic constraints. Certain additional constants (prescribing a nonholonomic distribution) characterize a nonholonomic field/ geometric configuration and do not involve any additional (to gravity) interactions. This type of nonholonomically constrained gauge gravitational interactions are completely defined by the components of a distortion tensor (also uniquely defined by a metric tensor) from a chosen $d$–connection to the Levi–Civita connection. For such models, we can perform quantization and apply formal renormalization schemes following standard methods of gauge theories. In this work we developed the method of differential renormalization for nonholonomic backgrounds and gauge like distortion (gravitational) fields.

The above presented geometric ideas and quantization procedure for the Einstein gravity theory in nonholonomic variables could be viewed as starting point for a perturbative approach relating our former results on nonholonomic (and nonperturbative) loop constructions, Fedosov–Lagrange–Hamilton quantization, nonholonomic string–brane quantum models of gravity and (non) commutative models of gauge gravity, see \cite{24,25,26,27,35,37,38,39,50,51,52} and references therein.

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A Overlapping Divergences on Nonholonomic Spaces

In this appendix, present some explicit formulas for overlapping divergences.

\[ \hat{R}H[1,1;1,1] = a \triangle, \]
\[ \hat{R}H[\hat{\mathbf{D}}_{\mu'},1;1,1] = \frac{a}{2} \hat{\mathbf{D}}_{\mu'} \triangle, \text{ for } a = \text{const}, \]
\[ R_{H}[1,\hat{\mathbf{D}}_{\mu'},1,1] = -\frac{1}{16(2\pi)^6} \hat{\mathbf{D}}_{\mu'} \frac{\hat{h}n [u^2 \omega^2]}{u^2} + \ldots, \]
\[ \hat{\mathbf{D}}_{\lambda'} \hat{R}H[1,\hat{\mathbf{D}}_{\mu'};1,1] = -\frac{1}{64(2\pi)^6} \hat{\mathbf{D}}_{\mu'} \frac{\hat{h}n [u^2 \omega^2]}{u^2} + \ldots, \]
\[ \hat{\mathbf{D}}_{\lambda'} \hat{R}H[1,\hat{\mathbf{D}}_{\mu'};1,\hat{\mathbf{D}}_{\lambda'}] = -\frac{1}{64(2\pi)^6} \hat{\mathbf{D}}_{\mu'} \frac{\hat{h}n [u^2 \omega^2]}{u^2} + \ldots, \]
\[ \hat{R}H[1,\hat{\mathbf{D}}_{\mu'};\hat{\mathbf{D}}_{\nu'};1,1] = -\frac{1}{64(2\pi)^6} \delta_{\mu'}^{\nu'} \frac{\hat{h}n [u^2 \omega^2]}{u^2} + \ldots, \]
\[ \hat{\mathbf{D}}_{\lambda'} \hat{R}H[1,\hat{\mathbf{D}}_{\nu'};\hat{\mathbf{D}}_{\mu'};1,1] = \frac{1}{256(2\pi)^6} \hat{\mathbf{D}}_{\mu'} \frac{\hat{h}n [u^2 \omega^2]}{u^2} \]
For simplicity, we omit the rest of formulas which are similar to those presented in Ref. [64] (we have to substitute formally those holonomic operators into nonholonomic covariant ones, for a corresponding d–connection with constant coefficient curvature).

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