WEAK TYPE (1,1) ESTIMATES FOR INVERSES OF DISCRETE ROUGH SINGULAR INTEGRAL OPERATORS.

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Abstract. We obtain weak type (1,1) estimates for the inverses of truncated discrete rough Hilbert transform. We include an example showing that our result is sharp. One of the ingredients of the proof are regularity estimates for convolutions of singular measure associated with the sequence $[m^\alpha]$, see [13].

1. Introduction

Suppose $1 < \alpha \leq 1 + \frac{1}{1000}$, $0 < \theta < 1$ are fixed parameters. For a non-negative number $M$ we consider a family of operators on $\ell^2(\mathbb{Z})$

$$
\mathbb{H}_M f(x) = \sum_{M^\theta \leq s \leq M} \mathcal{H}_s f(x) = \sum_{M^\theta \leq s \leq M} \sum_{m > 0} \varphi_s \left( \frac{m^\alpha}{s} \right) f(x - [m^\alpha]) - f(x + [m^\alpha]), \quad x \in \mathbb{Z}
$$

for some sequence $\varphi_s$, which is uniformly in $C^\infty_c(\frac{1}{2}, 2)$. It is by now a routine fact that the operators $\mathbb{H}_M$, the truncated Hilbert transforms, are bounded on $L^p$, $1 < p < \infty$ with norm estimates uniform in $M$ and $\theta$. The analogous weak type $(1, 1)$ estimate seems to be unknown. For a fixed $\theta$, by a rather routine application of the methods of [4], [16] and [18] the operators $\mathbb{H}_M$ can be shown to be of weak type $(1, 1)$ uniformly in $M$. The subject of the current paper has been inspired by [3]. There, a theorem has been proved ([3], Theorem 3), which for our purposes can be formulated as follows:

Theorem. Suppose $K$ is a kernel in $\mathbb{R}^d$ satisfying $K(x) = \Omega(x)/|x|^d$, where $\Omega$ is homogeneous of degree 0, $\Omega \in L^q(S^{d-1})$ and has mean 0.

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Denote $Kf = K * f$. Suppose further that for some $\lambda \in \mathbb{C}$ the operator $\lambda \text{Id} + K$ is invertible in $L^2(\mathbb{R}^d)$. Then $(\lambda \text{Id} + K)^{-1}$ is of form $\Lambda \text{Id} + K'$, where the kernel $K'$ satisfies the same assumptions as $K$.

It immediately implies:

**Corollary** ([3], [4], [6], [15]). *In the setting of the above theorem, the operator $(\lambda \text{Id} + K)^{-1}$ is of weak type $(1,1)$.*

The principal object of the current work is to extend the above theorem to the case of discrete rough Hilbert transforms $H_M$. For a fixed $\theta$ we prove the uniform in $M$ estimates for $\| (\lambda \text{Id} + H_M)^{-1} \|_{\ell^1 \to \ell^1, \infty}$, provided such an estimate exists in the sense of $\ell^2$. By the previous general remark, this goal is accomplished through the following representation theorem, which is the main result of this paper

**Theorem 1.** *Suppose $1 < \alpha \leq 1 + \frac{1}{1000}$ and let $\theta$ be such, that $\alpha - 1 < \theta < 1$. Fix $\lambda \in \mathbb{C}$ and suppose that for some constant $C_I$ we have

$$\left\| (\lambda \text{Id} + H_M)^{-1} \right\|_{\ell^1 \to \ell^1, \infty} \leq C_I, \quad \text{for } M \geq M_0. \quad (2)$$

Then, there exists $M_1 = M_1(C_I, \lambda)$ such that for $M \geq M_1$ the kernel of the operator $(\lambda \text{Id} + H_M)^{-1}$ has the form

$$\lambda \text{Id} + \beta \text{Id} + K, \quad (3)$$

where $K$ is the classical discrete Calderon-Zygmund kernel, and we have a uniform in $M \geq M_1$ estimate

$$\| \lambda \| + \| \beta \| + \left\| K \right\|_{\ell^1 \to \ell^1} + \left\| K \right\|_{CZ} \leq C_1(C_I, \lambda), \quad (4)$$

where

$$\| K \|_{CZ} = \sup_{y \neq 0} \sum_{|x| \geq 2|y|} |K(x) - K(y)|.$$

Moreover, the above restriction on $\theta$ is sharp (we make this statement precise in Theorem 4 in the next section).

Applying standard Banach algebras arguments (eg. [8]), for each fixed $M$, the kernel of the operator $(\lambda \text{Id} + H_M)^{-1}$ is in $\ell^1((1 + |x|)^N)$ for any $N \geq 0$. In particular $(\lambda \text{Id} + H_M)^{-1}$ is bounded on $\ell^1$, but the weak type $(1,1)$ estimate obtained in this way becomes unbounded when $M \to \infty$. Also, by selfduality of the multiplier problem, the uniform in $M$ upper bound for $\| (\lambda \text{Id} + H_M)^{-1} \|_{\ell^1 \to \ell^1, \infty}$ requires assumption (2).
It is worthwhile to put our result in a more general context. First we note that for the convolution Calderón-Zygmund operators in the continuous setting, the invertibility theorems are by now classical. Similarly, the resolvent of the discrete Hilbert transform, if it exists as an operator on $\ell^2(\mathbb{Z})$, is a discrete Calderón-Zygmund operator. This fact seems to be folklore and can be proved by an application of Fourier transform or by the method of [3]. The discrete analogues of the classical singular integrals have been studied intensively, see some examples [1], [2], [5], [10], [11], [13]. We believe, that our results fit well within this line of research.

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2. Main Theorem

Let us recall, that we have fixed parameters $\alpha$, $\theta$ with $1 < \alpha \leq 1 + \frac{1}{1000}$, $0 < \theta < 1$. We introduce a family of algebras, which are subalgebras of the algebra of operators on $\ell^2$.

Definition 2. We consider the family of operators $T$, which are convolution operators on $\mathbb{Z}$, with kernels of the form

$$T = \lambda \text{Id} + \beta \mathbb{H}_M + \sum_{M^\theta \leq s < \infty \atop s \text{ dyadic}} K_s,$$

(we identify convolution operator with its kernel), where the operator $\mathbb{H}_M$ is the truncated Hilbert transform:

$$\mathbb{H}_M f(x) = \sum_{M^\theta \leq s \leq M \atop s \text{ dyadic}} \mathcal{H}_s f(x)$$

with

$$\mathcal{H}_s f(x) = \sum_{m > 0} \varphi_s \left( \frac{m^\alpha}{s} \right) \frac{f(x - [m^\alpha]) - f(x + [m^\alpha])}{m}$$

for some sequence $\varphi_s$ which is uniformly in $C^\infty_c(\frac{1}{2}, 2)$. We require that the kernels $K_s$ satisfy:

(i) $\sum_x K_s(x) = 0$,

(ii) $\text{supp } K_s \subset [-s, s]$,

(iii) $\sum_x |K_s(x)|^2 \leq \frac{D^2}{s}$,
(iv) $\sum_x |K_s(x + h) - K_s(x)|^2 \leq D_s^2 \left( \frac{|h|}{s} \right)^{\gamma_0}$, for some small positive $\gamma_0$ depending only on $\delta = \theta - (\alpha - 1)$.

For a fixed $M$ we put

$$\|\{K_s\}\|_{A_M} = \sup_{M^\theta \leq s < \infty} D_s,$$

and

$$\|T\|_{A_M} = \inf \{ |\lambda| + |\beta| + \|\{K_s\}\|_{A_M} \},$$

where the infimum is taken over all representations of the operator $T$ in the form $[5]$. 

In fact $A_M$ is a Banach algebra with the norm $C\|T\|_{A_M}$ for certain constant $C$ independent of $M$. Moreover, 

$$K = \sum_{M^\theta \leq s < \infty} K_s$$

is Calderón-Zygmund kernel with constant controlled by $\|T\|_{A_M}$.

We are now ready to formulate the two theorems leading immediately to Theorem 1.

**Theorem 3.** Let $\theta > \alpha - 1$. Assume that for some fixed $\lambda \in \mathbb{C}$ and a constant $C_1$ all operators $\lambda \text{Id} + \mathbb{H}_M$ are invertible for $M \geq M_0$ and $\|(\lambda \text{Id} + \beta \mathbb{H}_M)^{-1}\|_{l^2 \rightarrow l^2} \leq C_1$. Then for $M \geq M_1$ we have $\|(\lambda \text{Id} + \beta \mathbb{H}_M)^{-1}\|_{A_M} \leq C(C_1, \lambda)$.

**Theorem 4.** Let $\theta < \alpha - 1$. There exists a sequence of functions $\varphi_s$ and a compact set $\Gamma \subset \mathbb{C}$ such that the corresponding Hilbert transform $[7]$ satisfies $\|(\lambda \text{Id} + \mathbb{H}_M)^{-1}\|_{l^2 \rightarrow l^2} \leq C_1$ for all $M$ and $\lambda \in \Gamma$, and the estimate $\|(\lambda \text{Id} + \mathbb{H}_M)^{-1}\|_{l^1 \rightarrow l^1, \infty} \leq C$, does not, for any $C$, hold uniformly in $\lambda \in \Gamma$ and $M$.

**Remarks:**

(i) The range of $\alpha$’s considered in Theorem 3 is not optimal, and can be improved using the methods from [12], [18] or a variant of the argument used in this work to prove Lemma 6.

(ii) Theorem 3 is probably also true with $[m^\alpha]$ replaced by $[m^\alpha \varphi(m)]$, where $\varphi$ is a function of the Hardy class considered in [12].

(iii) For values of $\theta < 1$ close to 1 Theorem 3 could be proved using
regularising effect in $\ell^2$ of the kernel $H_M$. Known estimates for the Fourier transform $\hat{H}_M$ seem, however, to be too weak to cover the entire range of $\theta$ considered in this paper.

(iv) In the proof of Lemma 12 we could have used a weaker statement of Lemma 6 at a cost of a more sophisticated argument. We believe that Lemma 6 is of some independent interest, because of its relation to certain type of Waring problem (see [7], [17]). This is one reason we have chosen the variant of proof we present.

(v) Condition (2) is always satisfied for sufficiently large $|\lambda|$. If we only consider real valued $\varphi_s$, more can be said. Since the kernels $H_M$ are anti-symmetric, the Fourier transform $\hat{H}_M$ is purely imaginary and also anti-symmetric. Thus (2) is equivalent to $\lambda \not\in [-iN, iN]$, where $N \geq 0$. Using the estimates from [7] it can be shown that

$$N = \limsup_{M \to \infty} \sup_{\xi \in \mathbb{R}} \left| c_\alpha \sum_{M^{\theta} \leq s \leq M} \int_0^\infty \sin(\xi t^\alpha) \varphi_s(t^{1/\alpha}) \frac{dt}{t^{1-1/\alpha}} \right|$$

(where $c_\alpha$ is explicitly computable).

(vi) We refer the reader to our subsequent paper [14] for a sharper version of Theorem 4, see Remark at the end of Section 5.

Theorem 3 is an immediate consequence of the following result, which exploits the mixed-norm submultiplicity properties of algebras $A_M$.

The idea of using such estimates to solve the problem of invertibility of singular integral operators first appeared in [3].

**Theorem 5.** Let $A_M$, $M \geq M_0 \geq 1$ be a family of algebras, consisting of bounded convolution operators on $\ell^2$, with norms $\| \cdot \|_{A_M}$, satisfying

$$\|T_1 T_2\|_{A_M} \leq C_A \left( \|T_1\|_{\ell^2 \to \ell^2} \|T_2\|_{A_M} + \|T_1\|_{A_M} \|T_2\|_{\ell^2 \to \ell^2} \right) + C_A \epsilon(M) \|T_1\|_{A_M} \|T_2\|_{A_M},$$

$$\|T_1 T_2\|_{A_M} \leq C_A \|T_1\|_{A_M} \|T_2\|_{A_M},$$

where the constant $C_A$ does not depend on $M$ and $\epsilon(M) \to 0$ as $M \to \infty$. Suppose all operators from the sequence $T^{(M)}$ are invertible on $\ell^2$ and satisfy:

$$\|T^{(M)}(\cdot)^{-1}\|_{\ell^2 \to \ell^2} + \|T^{(M)}\|_{A_M} \leq K, \quad K \text{ independent of } M \geq M_0,$$

$$\|T^{(M)}\|_{\ell^2 \to \ell^2} \leq \delta < 1.$$
Then for an \( M_1 \geq M_0 \), sufficiently large and depending only on \( K \) and \( \delta \), and all \( M \geq M_1 \), \( T^{(M)} \) are invertible in \( A_M \), with
\[
\|(T^{(M)})^{-1}\|_{A_M} \leq C = C(K, \delta),
\]
with \( C(K, \delta) \) independent of \( M \geq M_1 \).

Proof. We will drop the superscript \( M \) and denote \( T^{(M)} \) by \( T \). We first prove that there exist constants \( C, N_0 \) and \( \delta_1 < 1 \), depending only on \( K, \delta, C_A \), such that
\[
\|T^n\|_{A_M} \leq C \delta_1^n, \quad n \geq N_0.
\]
A simple inductive argument shows an estimate
\[
\|T^{2^n}\|_{A_M} \leq 2^n C_A^n \|T^{2^{n-1}}\|_{\ell^2 \to \ell^2} \cdots \|T\|_{\ell^2 \to \ell^2} \|T\|_{A_M} + \epsilon G_N(\|T\|_{A_M}, \|T\|_{\ell^2 \to \ell^2}),
\]
where \( G_N \) is a polynomial of degree \( \leq 2^N \), with non-negative coefficients. Suppose an operator \( T \) satisfies (11). Then, clearly
\[
\|T^{2^n}\|_{A_M} \leq (2 C_A)^n \delta^{2^n - 1} K + \epsilon G_N(K, \delta).
\]
Choose \( N_0 \) such that
\[
(2 C_A)^{N_0} \delta^{N_0} K \leq \frac{1}{4 C_A},
\]
and \( M_1 \geq M_0 \) so that also
\[
\epsilon(M) G_{N_0}(K, \delta) \leq \frac{1}{4 C_A}, \quad M \geq M_1.
\]
We get
\[
\|T^{2^{N_0}}\|_{A_M} \leq \frac{1}{2 C_A}, \quad M \geq M_1.
\]
By (11) and a standard Banach algebras considerations we get
\[
\|T^n\|_{A_M} \leq \left( \frac{1}{2} \right)^{\frac{n}{2^{N_0}}} C_{C_A, K, \delta}.
\]
Suppose that the positive invertible on \( \ell^2 \) operator \( T \) satisfies (11). Then \( \delta \leq I - T \leq 1 - K^{-1} \) so \( I - T \) satisfies (11). Applying (13) to the Neumann series representation of \( T^{-1} \) we get an estimate \( \|T^{-1}\|_{A_M} \leq C_{K, \delta, C_A} \).

Now, if \( T \) is an arbitrary operator, invertible on \( \ell^2 \) and satisfying (11), we apply the above conclusion to \( T^* T \) and \( T T^* \) and the proof of Theorem 5 is concluded. \( \square \)
The fact that the algebra norms \( \| \cdot \|_{A_M} \) satisfy the hypotheses (9) and (10) will follow from a series of lemmas, which are gathered in the next section.

### 3. Lemmas

In this section we fix \( \theta = \alpha - 1 + \delta \), \( \delta > 0 \). Let \( \varphi \in C_c^\infty(\frac{1}{2}, 2) \), and, for convenience let us introduce an operator \( H_s \):

\[
H_s f(x) = \mathcal{H}_{s^\alpha} f(x) = \sum_{m > 0} \varphi \left( \frac{m}{s} \right) f(x - [m^\alpha]) - f(x + [m^\alpha]),
\]

where \( \mathcal{H}_s \) corresponds to the functions \( \tilde{\varphi}_s(t) = \varphi(t^{1/\alpha}) \). Let us denote by \( H_s(x) \) the kernel of this operator.

**Lemma 6.** Fix \( 1 < \alpha < 1 + \frac{1}{1000} \) and \( \delta_L > 0 \). Then there exist functions \( G_s(x) \), \( E_s(x) \) and an exponent \( \gamma(\delta_L) \) independent of \( s \), such that

\[
H_s * H_s(x) = G_s(x) + E_s(x) + C \delta_0(x)
\]

where

\[
|G_s(x)| + |E_s(x)| \leq Cs^{-\alpha}, \quad \text{supp } E_s \subset [-s^{\alpha-1+\delta_L}, s^{\alpha-1+\delta_L}]
\]

and

\[
|G_s(x + u) - G_s(x)| \leq Cs^{-\alpha} |u|^{-\gamma(\delta_L)}
\]

where the constants \( C \) depends only on \( \varphi \).

This lemma is the main technical tool we use. We postpone its proof to the next section. In this section we will apply this lemma to \( \mathcal{H}_s \), that is with \( s \) replaced by \( s^{\frac{1}{\alpha}} \).

**Lemma 7.** Let \( \psi \in C_c^\infty(\mathbb{R}) \), \( \psi \equiv 1 \) for \( |x| \leq 1 \), \( \psi \equiv 0 \) for \( |x| \geq 2 \). For a given convolution kernel \( K \) on \( \mathbb{Z} \) we define truncated kernels:

\[
K_R(x) = K(x) \cdot \psi \left( \frac{x}{R} \right).
\]

Then for \( R \geq 1 \) we have

\[
\|K_R\|_{\ell^2 \rightarrow \ell^2} \leq C \|K\|_{\ell^2 \rightarrow \ell^2},
\]

where the constant \( C \) is independent of \( R \).

**Proof.** This is immediate by Fourier transform. \( \square \)
Lemma 8. For an operator $T$ as in (5), we have

$$|\lambda| \leq \|T\|_{\ell^2 \to \ell^2} + \epsilon(M)\|T\|_{A_M}.$$ 

Proof. It suffices to observe, that

$$<H_M \delta_0, \delta_0> = 0,$$

and by $(iii)$ of definition $2$,

$$|K_s(0)|^2 \leq \frac{\|T\|_{A_M}^2}{s}.$$ 

Then, for $\epsilon(M) \leq CM^{-\theta/2}$ the conclusion follows from

$$\lambda = <T \delta_0, \delta_0> - \sum_{M^\theta \leq s < \infty \text{ s- dyadic}} K_s(0).$$

\[\square\]

Lemma 9. Let $T$ be the kernel of the form (5). Then $T$ admits a representation

$$\lambda \text{Id} + \beta \sum_{M^\theta \leq s < M \text{ s- dyadic}} \mathcal{H}_s + \sum_{M^\theta \leq s < \infty \text{ s- dyadic}} K'_s,$$

where:

$$\mathcal{H}_s(x) = \left(\psi\left(\frac{x}{2s}\right) - \psi\left(\frac{x}{2s}\right)\right) H_M(x), \quad s \geq M^\theta, \text{ dyadic},$$

the function $\psi$ is the same smooth cutoff function as in the previous lemma, the kernels $K'_s$ satisfy conditions $(i)$,$(ii)$,$(iv)$ from Definition $3$ and we have:

$$|\lambda| + |\beta| + \|\{K'_s\}\|_{A_M} \leq C \|T\|_{A_M}.$$ 

Moreover

$$\left\|\lambda \text{Id} + \sum_{M^\theta \leq s \leq s' \text{ s- dyadic}} (\beta \mathcal{H}_s + K'_s)\right\|_{\ell^2 \to \ell^2} \leq C\|T\|_{\ell^2 \to \ell^2} + \epsilon(M)\|T\|_{A_M}.$$ 

Proof. This lemma is standard and we include the proof for the reader’s convenience. Let $\psi$ be the smooth symmetric cutoff function as in the lemma $7$, and let $s'$ be the largest dyadic integer satisfying $s' \leq M^\theta/2$. We let

$$\psi^{s'}(x) = \psi\left(\frac{x}{s'}\right), \text{ and } \psi^s(x) = \psi\left(\frac{x}{s}\right) - \psi\left(\frac{2x}{s}\right) \text{ for } s > s',$$

for $s > s'$. 

and thus
\[ \sum_{s \geq s_0} \psi^s(x) = \psi \left( \frac{x}{s_0} \right) = \psi_{s_0}(x), \]
with
\[ \text{supp } \psi_{s'} \subset \{|x| \leq M^\theta\} \quad \text{supp } \psi^s \subset \{s/2 \leq |x| \leq 2s\}, \quad s > s'. \]

Given an operator \( T \) with kernel of the form (5):
\[ T = \lambda \text{Id} + \beta \mathbb{H}_M + \sum_{M^\theta \leq s < \infty} K_s, \]
we can write the decomposition of its kernel
\[ \psi_{s_0} \cdot T = \lambda \text{Id} + \beta \sum_{s \geq s_0} \psi^s \cdot \mathbb{H}_M + \sum_{s \geq s_0} \psi^s \cdot K, \]
where
\[ K = \sum_{M^\theta \leq s < \infty} K_s. \]

Now we let
\[ \mathcal{H}_s = \psi^s \cdot \mathbb{H}_M, \quad s > s', \]
\[ \tilde{K}_s = \psi^s \cdot K, \quad s \geq s'. \]

Observe, that the kernels \( \tilde{K}_s \) satisfy the requirements in the definition of the algebra \( A_M \), except, possibly, for the vanishing means. We let
\[ K'_s(x) = \tilde{K}_s(x) - \frac{c_s}{s} \psi \left( \frac{x}{s} \right) \sum_{y \in \mathbb{Z}} \tilde{K}_s(y), \]
where the constants \( c_s \) have been chosen so that
\[ \frac{c_s}{s} \sum_{x \in \mathbb{Z}} \psi \left( \frac{x}{s} \right) = 1. \]

Note, that the kernels \( K'_s \) do have vanishing means, and satisfy all the requirements of the definition of the algebra \( A_M \), with \( \|\{K'_s\}\|_{A_M} \) bounded by \( \|\{K_s\}\|_{A_M} \). Now we write the decomposition of kernel \( T(x) \)
\[ \psi_{s_0}(x) \cdot T(x) = \lambda \text{Id}(x) + \beta \sum_{s \geq s_0} \mathcal{H}_s(x) + \sum_{s \geq s_0} K'_s(x) + \]
\[ + \sum_{s \geq s_0} J_s \left( \frac{c_s}{s} \psi \left( \frac{x}{s} \right) - \frac{c_{2s}}{2s} \psi \left( \frac{x}{2s} \right) \right) + J_{s_0} \frac{c_{s_0}}{s_0} \psi \left( \frac{x}{s_0} \right), \]
where
\[ J_s = \sum_{s \geq s'} \sum_y K'_s(y) = \sum_y K(y) \psi \left( \frac{y}{s} \right) \]
and \( J_{s'/2} = 0 \). Let
\[ K''_s(x) = K'_s(x) + J_{s'/2} \left( \frac{2c_{s/2}}{s} \psi \left( \frac{2x}{s} \right) - \frac{c_s}{s} \psi \left( \frac{x}{s} \right) \right) \]
We will prove below that \( |J_s| \leq |\lambda| + C \|T\|_{\ell^2 \to \ell^2} \). This immediately imply
\[ T = \lambda \text{Id} + \beta \sum_{s \geq 2s'} \mathcal{H}_s + \sum_{s \geq 2s'} K''_s \]
in a weak sense. Moreover, by lemma 7 applied to \( \psi_{s_0} \cdot T \) and estimate on \( \lambda \) provided by lemma 8, the partial sums
\[ \lambda \text{Id} + \beta \sum_{s \geq 2s'} \mathcal{H}_s + \sum_{s \geq 2s'} K''_s \]
represents an operator with \( \ell^2 \to \ell^2 \) bounded by \( C \|T\|_{\ell^2 \to \ell^2} + \epsilon(M) \|T\|_{A_M} \),

and by the construction \( \|K''_s\|_{A_M} \leq C \|T\|_{A_M} \).

We will now show the required estimate for \( J_s \), that is
\[ \left| \sum_{y \in \mathbb{Z}} K(y) \psi \left( \frac{y}{s} \right) \right| \leq c \|T\|_{\ell^2 \to \ell^2} + |\lambda| \]
We let
\[ K^s = (K + \mathbb{H}_M) \cdot \psi_s, \quad \chi_s = \frac{1}{2s + 1} \chi_{[-s,s]} \]
and, since the kernel \( \mathbb{H}_M \) is antysymmetric
\[ \left| \sum_{y \in \mathbb{Z}} K(y) \psi_s(y) \right|^2 = \left| \sum_{y \in \mathbb{Z}} K^s(y) \sum_{y_1 \in \mathbb{Z}} \chi_s(y_1) \right|^2 
\leq 8s \sum_{y \in \mathbb{Z}} \left| K^s \ast \chi_s(y) \right|^2 
\leq 8s \|K^s\|_{\ell^2 \to \ell^2} \|\chi_s\|_{\ell^2}^2 
\leq \frac{8s}{2s + 1} \|K^s\|_{\ell^2 \to \ell^2}^2 
\leq c \|K + \mathbb{H}_M\|_{\ell^2 \to \ell^2}^2 
\leq 2c \|T\|_{\ell^2 \to \ell^2}^2 + 2|\lambda|^2, \]
where the estimate for \( \|K^s\|_{\ell^2 \to \ell^2} \) follows by lemma 7. Now we apply lemma 8. \( \square \)
Lemma 10. Let \( 0 \leq \varphi \in C^\infty_c(\mathbb{R}) \) and \( \varphi_s = \frac{s}{\varphi(\frac{s}{s})}, \) with constants \( c_s > 0 \) such that \( \| \varphi_s \|_1 = 1. \) For a given \( \delta > 0 \) and a positive dyadic integer \( s \) let \( s_1 \) be such that \( s^{\frac{\alpha - 1 + \delta}{\alpha}} \leq s_1 \leq s. \) Then for \( 0 < \gamma \leq \gamma_0(\delta) \) we have:

(i) \( \| \varphi_{s_1} \ast H_s \|_2^2 \leq C_\delta, \)

(ii) \( \| \varphi_{s_1} \ast H_s (\cdot + h) - \varphi_{s_1} \ast H_s \|_2^2 \leq C \left( \frac{\delta}{s} \right)^\gamma. \)

We can take \( \gamma_0(\delta) = \min \{ \frac{\delta}{4\alpha}, \gamma(\frac{\delta}{2}) \} \), where \( \gamma(\delta) \) is defined by (17).

Proof. It suffices to prove (ii) with \( |h| \leq C_s \) since it implies (i). For the moment, the superscript \( \cdot \) denotes the translation of a function by \( h \). We have:

\[
< (\varphi_{s_1}^h - \varphi_{s_1}) \ast H_s, (\varphi_{s_1}^h - \varphi_{s_1}) \ast H_s > = < (\varphi_{s_1}^h - \varphi_{s_1}) \ast G_s, \varphi_{s_1}^h - \varphi_{s_1} > + \\
+ \| \varphi_{s_1}^h - \varphi_{s_1} \|_2^2 \frac{1}{s^{\frac{1}{\alpha}}} + < (\varphi_{s_1}^h - \varphi_{s_1}) \ast E_s, (\varphi_{s_1}^h - \varphi_{s_1}) > \\
= I + II + III.
\]

In the above we have applied Lemma 6 with \( \delta_l = \delta/2 \) to obtain the decomposition: \( H_s \ast H_s = G_s + \frac{C_\delta}{s^{1/\alpha}} + E_s, \) satisfying estimates (16), (17). We have for \( \gamma \leq \gamma(\delta/2) \), where \( \gamma(\delta) \) is defined by (17):

\[
|I| = | < (\varphi_{s_1}^h - \varphi_{s_1}) \ast G_s, \varphi_{s_1}^h - \varphi_{s_1} > | \\
= | \varphi_{s_1} \ast (G_s^h - G_s), \varphi_{s_1}^h - \varphi_{s_1} > | \\
\leq C \frac{1}{s} \left( \frac{|h|}{s} \right)^\gamma \| \varphi_{s_1} \|_2^2.
\]

\[
|II| \leq C \frac{1}{s^{1/\alpha}} \cdot \frac{1}{s_1} \cdot \left( \frac{|h|}{s_1} \right)^\gamma \\
\leq C \frac{1}{s^{1/\alpha}} \cdot \frac{|h|}{s_1} \\
\leq C \frac{1}{s^{1/\alpha}} \cdot \frac{|h|^\gamma}{s^{1/\alpha + \delta/\alpha}} \\
\leq C \frac{1}{s} \left( \frac{|h|}{s} \right)^{\delta/2\alpha},
\]

for $\gamma \leq \delta/2\alpha$ and $s_1 \geq s^{-1/\alpha+\delta/\alpha}$. By Hölder regularity of $\varphi_{s_1}$

$$|\text{III}| \leq C \|\varphi_{s_1}^h - \varphi_{s_1}\|_{\ell^\infty} \|E_s\|_{\ell^1} \leq C \left( \frac{|h|}{s_1} \right)^{\gamma} \frac{1}{s_1} \cdot \frac{1}{s} \cdot s^{-1/\alpha+\delta/2}\alpha \leq \frac{C}{s_1^{1/\alpha}} \cdot \frac{1}{s} \cdot |h|^\gamma \|s\|^{\delta/2\alpha} \leq \frac{C}{s} \left( \frac{|h|}{s} \right)^{\delta/4\alpha} \cdot s^{-\delta/4\alpha},$$

for $\delta/4\alpha \geq \gamma$ and $s_1$ as in II.

Let

$$\tilde{T}_s = \sum_{M^\theta<s'<s} (\mathcal{H}_{s'} + \mathcal{K}_{s'}),$$

where the kernels $\mathcal{H}_{s'}, \mathcal{K}_{s'}$ comes from the representation of $\tilde{T}$ in the sense of Lemma 9.

**Lemma 11.** For $\gamma \leq \gamma_0(\delta)$ and $s_1^{-1/\alpha+\delta/\alpha} \leq s_1 \leq s$ we have

(i) $\|\varphi_{s_1} * \mathcal{H}_s * \tilde{T}_s\|_{\ell^2} \leq \frac{C}{|s|^{1+\gamma}} (\|\tilde{T}\|_{\ell^2} + \|\tilde{T}\|_{\ell^2}^2) + C \frac{|s|}{|s_1|} \|\tilde{T}\|_{\ell^2}^2$,

(ii) $\|\varphi_{s_1} * \mathcal{H}_s * \tilde{T}_s (\cdot + h) - \varphi_{s_1} * \mathcal{H}_s * \tilde{T}_s\|_{\ell^2} \leq \frac{C}{|s|} \left( \frac{|h|}{|s|} \right)^\gamma (\|\tilde{T}\|_{\ell^2} + \|\tilde{T}\|_{\ell^2}^2) + \frac{C}{|s|^{1+\gamma}} \|\tilde{T}\|_{\ell^2}^2$.

**Proof.** Immediate, from Lemmas 9 and 10.

**Lemma 12.** Let $0 \leq l \leq s^{-1/\alpha+\delta/\alpha}$, $s^\theta = s^{\alpha-1+\delta}$, $s_1 \leq s$ and $\psi_l = \varphi_l - \varphi_{2l}$, where $\varphi_l$ has been defined in Lemma 10. We have for $\gamma \leq \gamma_0(\delta)$:

(i) $\|\psi_l * \mathcal{H}_s * \mathcal{H}_{s_1}\|_{\ell^2} \leq \frac{C}{|s|^{1+\gamma/\alpha}},$

(ii) $\|\psi_l * \mathcal{H}_s * \mathcal{H}_{s_1} (\cdot + h) - \psi_l * \mathcal{H}_s * \mathcal{H}_{s_1}\|_{\ell^2} \leq \frac{C}{|s|^{1+\gamma/\alpha}} \cdot \left( \frac{|h|}{|s|} \right)^\gamma,$

(iii) $\|\psi_l * \mathcal{H}_s * \mathcal{K}_{s_1}\|_{\ell^2} \leq \frac{C}{|s|^{1+\gamma/\alpha}},$

(iv) $\|\psi_l * \mathcal{H}_s * \mathcal{K}_{s_1} (\cdot + h) - \psi_l * \mathcal{H}_s * \mathcal{K}_{s_1}\|_{\ell^2} \leq \frac{C}{|s|^{1+\gamma/\alpha}} \cdot \left( \frac{|h|}{|s|} \right)^\gamma.$
Proof. (ii) and (iv) follow from (i) and (iii), since $|h| \geq 1$. We will now prove (i). We again use Lemma 6 with $\delta_L = \delta/2$.

$$
\|\psi_l \ast H_s \ast H_{s_1}\|_2^2 =<\psi_l \ast G_s, H_{s_1} \ast H_{s_1} \ast \psi_l> + \\
+<\psi_l \ast E_s, H_{s_1} \ast H_{s_1} \ast \psi_l> + \\
+<\psi_l \cdot \frac{1}{s^{1/\alpha}}, H_{s_1} \ast H_{s_1} \ast \psi_l>
$$

$$
= I + II + III.
$$

We estimate each part:

$$
|I| \leq \|\psi_l \ast G_s\|_\infty \cdot \|H_{s_1} \ast H_{s_1} \ast \psi_l\|_1
$$

$$
= C \left( \frac{|s|^{1-1/\alpha + \delta/\alpha}}{|s|} \right)^\gamma \cdot \frac{1}{|s|}
$$

$$
\leq C \frac{1}{|s|} \cdot |s|^{\delta_1}.
$$

$$
|III| = |<\psi_l \ast H_{s_1}, H_{s_1} \ast \psi_l>| \cdot \frac{1}{s^{1/\alpha}}
$$

$$
\leq \|H_{s_1}\|_2^2 \|\psi_l\|_2^2 \cdot \frac{1}{s^{1/\alpha}}
$$

$$
\leq \frac{C}{s^{1/\alpha}} \cdot \frac{1}{s^{1/\alpha}}
$$

$$
\leq \frac{1}{s^{1/\alpha}} \cdot \frac{1}{s^{1-1/\alpha + \delta/\alpha}}
$$

$$
\leq \frac{1}{s^{1+\delta/\alpha}}.
$$

$$
|II| = |<E_s, H_{s_1} \ast H_{s_1} \ast \psi_l \ast \psi_l>| \\
\leq |<E_s \ast H_{s_1}, H_{s_1} \ast \psi_l \ast \psi_l>| \\
\leq \|E_s\|_1 \cdot \|H_{s_1}\|_2^2 \\
\leq \frac{s^{1-1/\alpha + \delta/2\alpha}}{s} \cdot \frac{1}{s^{1/\alpha}}
$$

$$
\leq \frac{s^{1-1/\alpha + \delta/2\alpha}}{s \cdot s^{1-1/\alpha + \delta/\alpha}}
$$

$$
\leq \frac{1}{s^{1+\delta/2\alpha}}.
$$

The estimates of $|II|$ is very crude but it suffices for our purposes. The proof of (iii) is identical. \qed
Lemma 13. We notice:

(18) \[ \|H_s \ast \tilde{T}_s\|_{L^2}^2 \leq \frac{C}{s} \left( \|\tilde{T}\|_{L^2}^2 + \|\tilde{T}\|_{A} \cdot \left( \frac{1}{s^{\delta/4a}} + \epsilon(s) \right) \right), \]

(19) \[ \|H_s \ast \tilde{T}_s(\cdot + h) - H_s \ast \tilde{T}_s\|_{L^2} \leq \frac{C}{|s|} \left( \frac{|h|}{s} \right)^{\gamma} \left( \|\tilde{T}\|_{L^2}^2 + \|\tilde{T}\|_{A} \cdot \left( \frac{1}{s^{\delta/4a}} + \epsilon(s) \right) \right), \]

where \( \tilde{T}_s, \tilde{T}_s \) has been defined before Lemma \([11]\).

Proof. It is a corollary of Lemmas \([11]\) and \([12]\). Let \( s_1 = s^{1-1/\alpha+\delta/\alpha}, \varphi \in C_c^\infty(-\frac{1}{2}, \frac{1}{2}) \) and \( \varphi_{s_1} \psi \) be as in Lemma \([12]\). Then \( \delta_0 = \varphi_{s_1} + \sum_{l=1}^{\frac{1}{2} s_1} \psi_l \).

The conclusion of the Lemma follows directly from the formula:

(20) \[ H_s \ast \tilde{T}_s = \varphi_{s_1} \ast H_s \ast \tilde{T}_s + \sum_{M^0 \leq k \leq s} \sum_{l=1}^{\frac{1}{2} s_1} \psi_l \ast H_s \ast (\tilde{H}_{s'} + \tilde{K}_{s'}), \]

Since the kernels \( \tilde{T}_s, \tilde{T}_s \) are supported in \([-Cs, Cs]\) for some constant \( C \), from Lemma \([11]\) we conclude that

\[ \varphi_{s_1} \ast H_s \ast \tilde{T}_s \]

satisfies \((18)\) and \((19)\), that is the \((i)_{s_2} - (iv)_{s_2}\) of the definition \([2]\) for some \( s_2 = Cs \) and with the constant \( D_{s_2} \leq C\|\tilde{T}_s\|_{L^2} \leq C\|\tilde{T}\|_{L^2} + C\epsilon(s)\|\tilde{A}_M\|. \)

Since \( s^\theta \leq s' \leq s \), each of the kernels

(21) \[ \psi_l \ast H_s \ast (\tilde{H}_{s'} + \tilde{K}_{s'}) \]

by Lemma \([12]\) satisfies \((18)\) and \((19)\), that is the \((i)_{s_2} - (iv)_{s_2}\) of the definition \([2]\) with \( s_2 = Cs \) and \( D_{s_2} \leq C\|\tilde{T}_s\|_{L^2} \leq C\|\tilde{T}\|_{L^2} + C\epsilon(s)\|\tilde{A}_M\|. \)

Since the number of summands in \((20)\) is at most \( C(\log M)^2 \), the lemma follows.

Lemma 14. We have:

\[ \|TT\|_{A_M} \leq C \left( \|T\|_{L^2} \|\tilde{T}\|_{A_M} + \|T\|_{A_M} \|\tilde{T}\|_{L^2} \right) + \epsilon_1(M) \|T\|_{A_M} \|\tilde{T}\|_{A_M}, \]

where \( \epsilon_1(M) \leq CM^{-\frac{\delta(\alpha-1+4)}{16a}}, \) and the constant \( C \) does not depend on \( M \).

Proof. We use the identity

\[ T \tilde{T} = \lambda \tilde{T} + \lambda T + \sum_s (K_s + H_s) \ast \tilde{T}_s + \sum_s (\tilde{K_s} + \tilde{H}_s) \ast \tilde{T}_s, \]

(\( T_s, \tilde{T}_s \) as in the previous Lemma). We apply Lemma \([13]\) and obtain the estimates in the case \( s \leq M \). The case \( s > M \) is immediate, since
then $\mathcal{H}_s$ vanish and by $\ell^2$ boundedness of $T_s, \tilde{T}_s$, the kernels $K_s * \tilde{T}_s, \hat{K}_s * T_s$ satisfy conditions $(i)_{C_s}...(iv)_{C_s}$ of definition 2 with appropriate norm control.

\[ \square \]

4. Proof of Lemma 6.

In this section we slightly abuse the notation and denote generic $s$ by $M$. We note that $H_M$, introduced in (14) is supported in $[-CM^\alpha, CM^\alpha]$. Denote $G_M = H_M * H_M$. The estimates (17) and (16) on $G_M$ have been proved in [18], the estimate (17) under additional restriction $M^{\delta_0} \leq |x|, |x + u|$ and the estimate (16) for any $x \neq 0$. In what follows we will prove (17) for the remaining case $M^{\alpha-1+\delta_L} \leq x, x + u \leq M^{\delta_0}$. Then the new function $\tilde{G}_M$ defined on the whole $\mathbb{Z}$ by $\tilde{G}_M(x) = G_M(x)$ for $|x| \geq M^{\alpha-1+\delta_L}$ and $\tilde{G}_M(x) = G_M([M^{\alpha-1+\delta_L}])$ for $|x| \leq M^{\alpha-1+\delta_L}$ satisfies (17). Since $G_M(x) = G_M(-x)$, for $|x| \geq M^{\alpha-1+\delta_L}$ we obviously have, for those $x$, $\tilde{G}_M(x) = G_M(x)$. We will denote $\tilde{G}_M$ again by $G_M$ and define $E_M(x)$ by equation (15) with additional condition $G_M(0) + E_M(0) = 0$. Then $E_M(x)$ obviously satisfies (16).

We will apply the method of trigonometric polynomials and we refer the reader to [9] for all background facts. We begin with some definitions used in the sequel.

**Definition.** Let $\delta > 0$ be small, and $\delta_0 = \frac{\delta}{100}$. We consider the partition of the interval $[0, 1)$ into intervals of the form

$$ I_r = \left[ \frac{r}{M^{\delta_0}}, \frac{r + 1}{M^{\delta_0}} \right) \subset [0, 1), \quad 0 \leq r < M^{\delta_0} $$

For a number $\Delta \in (0, 1)$ we will denote by $I(\Delta)$ the unique interval of the above form such that $\Delta \in I(\Delta)$. We will write $I_r = [a(I_r), b(I_r))$ and denote by $l(\Delta) = l(I(\Delta)) = b(I(\Delta)) - a(I(\Delta))$ the length of $I(\Delta)$.

Furthermore, we let $m(h, x, \Delta)$ be the unique, if it exists, non-negative solution of

$$ (m + h)^\alpha - m^\alpha = x + \Delta, $$

where $x, h \in \mathbb{N}$ and $0 \leq \Delta < 1$. Let

$$ H = \frac{x}{M_\alpha}, \quad x \in \mathbb{N}, \quad M^{\alpha-1+\delta_L} \leq x \leq M^{\delta_0}, $$

$$ \|w\| = \inf_{k \in \mathbb{Z}} |k - w|, \quad w \in \mathbb{R}. $$
We will consider the following condition for \((h, x, \Delta, k)\):

\[
\forall \, m, \quad m(h, x, \alpha(I(\Delta))) \leq m \leq m(h, x, b(I(\Delta))) \implies \\| \alpha \cdot k \cdot m^{\alpha-1} \| \geq M^{-\delta_0/2}.
\]

**Lemma 15.** If \(\frac{M}{2} \leq m \leq 2M\) and satisfies (22), and \(H, x, h, \Delta\) as above then

\[
C^{-1}H \leq h \leq CH
\]

for some constant \(C\) independent of \(M, x, h, \Delta\). Moreover we have the following estimates:

\[
m(h, x, b(I(\Delta))) - m(h, x, a(I(\Delta))) = c_\alpha \frac{l(I(\Delta))}{h} m(h, x, 0)^{2-\alpha} \left(1 + O(M^{-\delta_0})\right),
\]

\[
m(h, x, 0) = \left(\frac{x}{h^{\alpha}}\right)^\rho \left(1 + O(M^{-\delta_0})\right), \quad \text{where} \quad \rho = 1 - \frac{1}{\alpha-1}
\]

(27)

(28)

\[S = \sum_{H/\mathcal{C} \leq CH} \left( \frac{m(h, x, b_r)}{M} \right)^2 \frac{l(I_r)}{h} m(h, x, b_r)^{2-\alpha} = c_\alpha M^{2-\alpha} \left(1 + O(M^{-\delta_0})\right),\]

where the choice of \(b_r \in I_r\) is arbitrary, and \(\varphi \in C^\infty_c(\frac{1}{2}, 2)\).

**Proof.** The estimate (26) follows immediately from the Taylor’s formula. In order to prove (27) we use the mean value theorem and the definition of \(m(h, x, t)\):

\[
\frac{\partial m(h, x, t)}{\partial t} = \frac{\partial m(h, x, t)}{\partial x} = \frac{m(h, x, t)^{2-\alpha}}{\alpha(\alpha-1)h} \left(1 + O\left(\frac{h}{M}\right)\right) = O\left(\frac{M}{x}\right).
\]

(30)

\[
m(h, x, t)^{2-\alpha} - m(h, x, 0)^{2-\alpha} = \frac{1}{2} \frac{l(I_r)}{h} m(h, x, 0)^{2-\alpha} = c_\alpha M^{2-\alpha} \left(1 + O(M^{-\delta_0})\right).
\]

(31)

Hence:

\[
m(h, x, b(I(\Delta))) - m(h, x, a(I(\Delta))) = l(\Delta) \cdot \frac{\partial m(h, x, t)}{\partial x} = \frac{m(h, x, t)^{2-\alpha}}{\alpha(\alpha-1)h} \left(1 + O\left(\frac{h}{M}\right)\right) = l(\Delta) \frac{m(h, x, 0)^{2-\alpha}}{\alpha(\alpha-1)h} \left(1 + O\left(\frac{h}{M}\right)\right) \left(1 + O\left(\frac{1}{x}\right)\right).
\]

We now prove (28). Let \(x_1\) be such that

\[m(h, x_1, 0) = \left(\frac{x}{h^{\alpha}}\right)^\rho.\]
that is
\[ x_1 = \left( \frac{x}{h\alpha} \right)^{\alpha} + h - \left( \frac{x}{h\alpha} \right)^{\rho\alpha}. \]

Using the Taylor’s formula applied to (22) we obtain \(|x_1 - x| \leq xM^{-1/100}\).
We have:
\[
\left| \frac{m(h, x_1, 0) - m(h, x, 0)}{m(h, x_1, 0)} \right| \leq \frac{C_1}{M} \left| \frac{\partial m(h, x_1, b) |x_1 - x|}{\partial x_1} \right| \leq \frac{C_1 M}{M} |x - x_1| \leq M^{-1/100}.
\]

We now prove the last part, (29). Using the estimate (30) it is straightforward to check that
\[
S = \left( \sum_{H/C \leq h \leq CH} \varphi(\frac{m(h, x, 0)}{M})^2 \frac{m(h, x, 0)^{2-\alpha}}{h} l(I_r) \right) \left( 1 + O(M^{-(\alpha-\delta)}) \right)
\]
\[
= \left( \sum_{H/C \leq h \leq CH} \varphi(\frac{1}{M} \left( \frac{x}{\alpha h} \right)^{\rho} \left( \frac{x}{\alpha h} \right)^{\rho(2-\alpha)} \frac{1}{h} \right) \left( 1 + O(M^{-(\alpha-\delta)}) \right)
\]
\[
= \left( \int_0^\infty \varphi(\frac{1}{M} \left( \frac{x}{\alpha h} \right)^{\rho} \left( \frac{x}{\alpha h} \right)^{\rho(2-\alpha)} \frac{dh}{h} \right) \left( 1 + O(M^{-\delta}) \right).
\]

The last equality follows from (26), and the fact, that by (26)
\[
\varphi\left( \frac{1}{M} \left( \frac{x}{\alpha h} \right)^{\rho} \right) = 0 \quad \text{for } h \leq C^{-1} H \text{ or } h \geq CH,
\]
and the Taylor’s formula. Now, by the change of variables, the last integral equals to \(c_\alpha M^{2-\alpha}\) and (29) follows.

\[ \square \]

**Lemma 16.** Let \(M^{\alpha-1+\delta} \leq x \leq M^{\frac{10}{100}}\). We then have:
\[
M^2 H_M * H_M(x) = \sum_{H/C \leq h \leq CH, I_r \subset \{0,1\}} \varphi\left( \frac{m(h, x, a(I_r))}{M} \right)^2 \left( |J_{h,x,I_r}^-| + |J_{h,x,I_r}^+| \right) + Er(x),
\]
where \(J_{h,x,I_r}^-\) and \(J_{h,x,I_r}^+\) are sets satisfying the inclusions:
\[
J_{h,x,I_r}^- \supset \{ m \in [m(h, x, a(I_r)), m(h, x, b(I_r)) : \{m^\alpha \} \geq 1 - a(I_r) \},
\]
\[
J_{h,x,I_r}^- \subset \{ m \in [m(h, x, a(I_r)), m(h, x, b(I_r)) : \{m^\alpha \} \geq 1 - b(I_r) \},
\]
\[
J_{h,x,I_r}^+ \supset \{ m \in [m(h, x, a(I_r)), m(h, x, b(I_r)) : \{m^\alpha \} \leq 1 - b(I_r) \},
\]
\[
J_{h,x,I_r}^+ \subset \{ m \in [m(h, x, a(I_r)), m(h, x, b(I_r)) : \{m^\alpha \} \leq 1 - a(I_r) \}.
\]
Moreover, for the error function \( Er(x) \) we have \( |Er(x)| \leq CM^{1-\alpha}M^{2-\alpha} \) so it satisfies conditions \((16)\) and \((17)\) required for \( G \).

**Proof.** By the definition of \( H_M \), we have:

\[
M^2 H_M * H_M(x) = \sum_{m_1, m_2 \in \mathbb{Z}} \varphi\left(\frac{m_1}{M}\right) \varphi\left(\frac{m_2}{M}\right) \frac{M}{m_1} \delta_{\pm\lfloor m_1^\alpha \rfloor} * \delta_{\pm\lfloor m_2^\alpha \rfloor}(x)
\]

\[
= 2 \sum_{m_1, m_2 \in \mathbb{Z}} \tilde{\varphi}\left(\frac{m_1}{M}\right) \tilde{\varphi}\left(\frac{m_2}{M}\right) \delta_{\lfloor m_1^\alpha \rfloor - \lfloor m_2^\alpha \rfloor}(x) = (\dagger)
\]

where we have denoted \( \tilde{\varphi}(t) = \text{sgn}(t)|t|^{-1}\varphi(t) \), and used the fact that for \( m_1 > m_2 \) and \( 0 < x \leq M^{\frac{99}{100}} \) the equation \( \pm\lfloor m_1^\alpha \rfloor \pm \lfloor m_2^\alpha \rfloor = x \) can be solved only when \( \lfloor m_1^\alpha \rfloor - \lfloor m_2^\alpha \rfloor = x \).

We now fix \( h > 0 \) and consider solutions to the equation:

\[
x = \lfloor m_1^\alpha \rfloor - \lfloor m_2^\alpha \rfloor, \quad m_1 - m_2 = h, \quad \frac{M}{2} \leq m_1 \leq 2M.
\]

Each solution is a pair \( m_1, m_2 \), but it is determined uniquely by its larger component \( m_1 \). In the following we refer to \( m_1 \) as “the solution”.

The set \( J_{h,x,I_r}^+ \) consists of solutions with additional condition \( m_1^\alpha - m_2^\alpha = x + \Delta, \quad \Delta \in I_r \subset [0,1) \).

The complementary set, \( J_{h,x,I_r}^- \), consists of solutions with additional condition

\[
m_1^\alpha - m_2^\alpha = x - 1 + \Delta, \quad \Delta \in I_r \subset [0,1).
\]

It is immediate, that if \( \lfloor (m + h)^\alpha \rfloor - \lfloor m^\alpha \rfloor = x \) then

\[
(m + h)^\alpha - m^\alpha = x + \Delta,
\]

or

\[
(m + h)^\alpha - m^\alpha = x - 1 + \Delta,
\]

for some \( \Delta \in [0,1) \). Hence

\[
\left\{ \frac{1}{2}M \leq m \leq 2M : (\exists k) x = \lfloor m^\alpha \rfloor - \lfloor k^\alpha \rfloor \right\} = \bigcup_{H/C \leq h \leq CH} J_{h,x,I_r}^+ \cup J_{h,x,I_r}^-.
\]

Hence, we have

\[
(\dagger) = 2 \sum_{I_r \subset [0,1)} \sum_{H/C \leq h \leq CH} \sum_{m_1 \in J_{h,x,I_r}^+ \cup J_{h,x,I_r}^-} \tilde{\varphi}\left(\frac{m_1}{M}\right) \tilde{\varphi}\left(\frac{m_2}{M}\right) \delta_{\lfloor m_1^\alpha \rfloor - \lfloor m_2^\alpha \rfloor}(x) = (\ddagger)
\]
Since for $m_1 \in J^+_{h,x,\Delta} \cup J^-_{h,x,\Delta}$ we have by (27) $|m_1 - m(h,x,a(\Delta))| \leq CM^{2-\alpha}$, $|m_2 - m(h,x,a(\Delta))| \leq CM^{2-\alpha} + C|m_1 - m_2| \leq CM^{2-\alpha} + CH \leq CM^{2-\alpha}$, applying Taylor formula for $\varphi$ we get

$$
(\dagger) = 2 \sum_{I_r \subset [0,1)} \sum_{H/C \leq h \leq CH} \varphi \left( \frac{m(h,x,a(I_r))}{M} \right)^2 \sum_{m_1 \in J^+_{h,x,I_r} \cup J^-_{h,x,I_r}} 1 + Er(x)
$$

where the error term $Er(x)$ satisfies

$$(32) \quad |Er| \leq CM^{1-\alpha} \# \left\{ \frac{1}{2} M \leq m \leq 2M : (\exists k \in \mathbb{N}) x = [m^\alpha] - [k^\alpha] \right\} \leq CM^{1-\alpha} M^{2-\alpha}
$$

The last inequality, by [18] is true for every $x \in \mathbb{Z}$. The first statement of Lemma follows.

If for some $\Delta \in I(\Delta) \subset [0,1)$ we have

$$(m + h)^\alpha - m^\alpha = x + \Delta, \quad x \in \mathbb{N},
$$

and

$$\{m^\alpha\} \leq 1 - b(I(\Delta)),
$$

then

$$[(m + h)^\alpha] - [m^\alpha] = x.
$$

So

$$\{m^\alpha\} + \{(m + h)^\alpha - m^\alpha\} \leq 1 - b(I(\Delta)) + \Delta,
$$

and thus

$$\{(m + h)^\alpha\} = \{m^\alpha\} + \{(m + h)^\alpha - m^\alpha\} = \{m^\alpha\} + \Delta.
$$

So,

$$[(m + h)^\alpha] - [m^\alpha] = x + \Delta - \{(m + h)^\alpha\} - \{m^\alpha\} = x.
$$

Analogously:

$$\{m^\alpha\} \geq 1 - a(I(\Delta)) \Rightarrow \{m^\alpha\} + \{(m + h)^\alpha - m^\alpha\} > 1 \Rightarrow \{(m + h)^\alpha\} = \{m^\alpha\} + \Delta - 1,
$$

and then

$$[(m + h)^\alpha] - [m^\alpha] = x - 1.
$$

It follows, that

$$[(m + h)^\alpha] - [m^\alpha] = x \Rightarrow \{m^\alpha\} \leq 1 - a(I(\Delta)).
$$

The required inclusions now follow. \qed
Let us introduce the following 4 functions. Given an interval $I_r \subset [0,1)$ let

$$
\chi_1 = \chi_{[1-a(I_r),1-M^{-\delta_0}]}, \quad \chi_2 = \chi_{[1-b(I_r),1]}.
$$

Also, choose a function $\varphi$, smooth, even, positive, monotone on $\mathbb{R}^+$, with support contained in $[-M^{-\delta_0}, M^{-\delta_0}]$, and with integral 1. Extend these three functions as 1-periodic on $\mathbb{R}$ ($M^{-\delta_0} << 1$), and let

$$
\psi_{M,I_r}^- = \chi_1 * \varphi, \quad \psi_{M,I_r}^+ = \chi_2 * \varphi,
$$

where the convolutions are on the torus. Using Lemma 18 we have the following obvious estimates:

$$
\sum_{m(h,x,a(I_r)) \leq m \leq m(h,x,b(I_r))} \psi_{M,I_r}^-(m^\alpha) \leq \int_{J_{h,x,I_r}} dt,
$$

$$
\left| J_{h,x,I_r}^- \right| \leq \sum_{m(h,x,a(I_r)) \leq m \leq m(h,x,b(I_r))} \psi_{M,I_r}^+(m^\alpha).
$$

We now choose new

$$
\chi_1 = \chi_{[M^{-\delta_0},1-b(I_r)]}, \quad \chi_2 = \chi_{[0,1-a(I_r)]},
$$

and let

$$
\psi_{M,I_r}^+ = \chi_1 * \varphi, \quad \psi_{M,I_r}^- = \chi_2 * \varphi.
$$

In this case, we have

$$
\sum_{m(h,x,a(I_r)) \leq m \leq m(h,x,b(I_r))} \psi_{M,I_r}^+(m^\alpha) \leq \int_{J_{h,x,I_r}} dt,
$$

$$
\left| J_{h,x,I_r}^+ \right| \leq \sum_{m(h,x,a(I_r)) \leq m \leq m(h,x,b(I_r))} \psi_{M,I_r}^-(m^\alpha).
$$

It is straightforward to see, that if $\psi$ is any one of the above introduced functions we have the estimates:

$$
\sum_{k \in \mathbb{Z}} \left| \hat{\psi}(k) \right| \leq C \log M,
$$

$$
\sum_{|k| > M^{\delta_0}} \left| \hat{\psi}(k) \right| \leq C M^{-\delta_0}.
$$

**Lemma 17.** We have an estimate

$$
\left| \sum_{m(h,x,a(I_r)) \leq m \leq m(h,x,b(I_r))} \psi(m^\alpha)-(m(h,x,b(I_r))-m(h,x,a(I_r))) \int_0^1 \psi(t) dt \right| \leq \sum_{0 < |k| \leq M^{\delta_0}} \left| \hat{\psi}(k) \right| \left| S_k(h,x,I_r) \right| + \frac{C}{M^{\delta_0/4}} \left| m(h,x,b(I_r))-m(h,x,a(I_r)) \right|,
$$

$$
\sum_{m(h,x,a(I_r)) \leq m \leq m(h,x,b(I_r))} \psi(m^\alpha)-(m(h,x,b(I_r))-m(h,x,a(I_r))) \int_0^1 \psi(t) dt \right| \leq \sum_{0 < |k| \leq M^{\delta_0}} \left| \hat{\psi}(k) \right| \left| S_k(h,x,I_r) \right| + \frac{C}{M^{\delta_0/4}} \left| m(h,x,b(I_r))-m(h,x,a(I_r)) \right|,
$$

$$
\sum_{m(h,x,a(I_r)) \leq m \leq m(h,x,b(I_r))} \psi(m^\alpha)-(m(h,x,b(I_r))-m(h,x,a(I_r))) \int_0^1 \psi(t) dt \right| \leq \sum_{0 < |k| \leq M^{\delta_0}} \left| \hat{\psi}(k) \right| \left| S_k(h,x,I_r) \right| + \frac{C}{M^{\delta_0/4}} \left| m(h,x,b(I_r))-m(h,x,a(I_r)) \right|.
$$
where $\psi$ is any of the functions $\psi^{\pm}_{M,I_r}$, and

\[(35) \quad |S_k(h,x,I_r)| \leq \frac{1}{M^{4\delta_0/4}} |m(h,x,b(I_r)) - m(h,x,a(I_r))|\]

if $(h,x,\Delta,k)$ satisfies (25) and always

\[(36) \quad |S_k(h,x,I_r)| \leq C|m(h,x,b(I_r)) - m(h,x,a(I_r))|\]

**Proof.** Let us denote

\[(37) \quad \mathcal{J} = \{m(h,x,a(I_r)) \leq m \leq m(h,x,b(I_r))\}.

We have

\[
\left| \sum_{m \in \mathcal{J}} \psi(m^\alpha) - \sum_{m \in \mathcal{J}} \hat{\psi}(0) \right| \leq \sum_{0<|k| \leq M^{2\delta_0}} \left| \hat{\psi}(k) \right| \sum_{m \in \mathcal{J}} e^{2\pi i m^\alpha \cdot k} + |\mathcal{J}| \cdot \sum_{|k|>M^{2\delta_0}} \left| \hat{\psi}_{M,I_r}(k) \right|
\]

It follows from (34) that $II \leq |\mathcal{J}| M^{-\delta_0}$. We will estimate $I$. We have, as in the proof of Van der Corput’s difference lemma, [9]:

\[
\left| \sum_{m \in \mathcal{J}} e^{2\pi i m^\alpha k} \right| \leq \frac{1}{D} \sum_{m \in \mathcal{J}} \left| \sum_{s=0}^{D-1} e^{2\pi i ((m+s)^\alpha - m^\alpha) \cdot k} \right| + C \cdot D
\]

\[
\leq \frac{1}{D} \sum_{m \in \mathcal{J}} \left| \sum_{s=0}^{D-1} e^{2\pi iksam^{\alpha-1}} \right| + C |\mathcal{J}| \left( \frac{D^2 M^{2\delta_0}}{M^{2-\alpha}} + \frac{D}{|\mathcal{J}|} \right),
\]

with the second term of the last expression estimated by $|\mathcal{J}|(\frac{M^{4\delta_0}}{M^{2-\alpha}} + M^{5\delta_0-\frac{100}{3}}) \leq |\mathcal{J}| M^{-\delta_0}$ if we have $D = M^{\delta_0}$. We have used in the above the following obvious consequence of the Taylor’s formula

\[e^{2\pi i ((m+s)^\alpha - m^\alpha)} = e^{2\pi i osm^{\alpha-1}} + O\left(\frac{s^2}{m^{2-\alpha}}\right).\]

We continue the original estimate:

\[
\leq \frac{1}{D} \sum_{m \in \mathcal{J}} \min_{\|\alpha km^{\alpha-1}\|} \left\{ D, \frac{2}{\|\alpha km^{\alpha-1}\|} \right\} + C |\mathcal{J}| \frac{M^{2\delta_0}}{M^{\delta_0}}.
\]

Now, if $(h,x,\Delta,k)$ satisfies the (25) condition, then

\[
\frac{1}{D} \sum_{m \in \mathcal{J}} \min_{\|\alpha km^{\alpha-1}\|} \left\{ D, \frac{2}{\|\alpha km^{\alpha-1}\|} \right\} \leq M^{-\delta_0/2} |\mathcal{J}|.
\]
Lemma 18. Assume $|k| \leq M^{2\delta_0}$. We have the estimates

$$\sum_{1/CH \leq k \leq CH} |S_k(h, x, I_r)| \leq \frac{CH}{M^{\delta_0/4}} |m(h, x, b(I_r)) - m(h, x, a(I_r))| \leq C l(I_r) M^{2-\alpha-\delta_0/4}.$$

Proof. The last inequality is an obvious consequence of (27). Based on (35) and (36) it is enough to prove the estimate

$$\# \{h : (h, x, \Delta, k) \text{ does not satisfy (25)} \} \leq CH M^{-\delta_0/4}.$$

To do so, let us momentarily fix $h, x, \Delta, k$ which do not satisfy (25), and thus there exists $m \in J$ such that

$$\|\alpha k m^{\alpha-1}\| < M^{-\delta_0/2}.$$

Let $|k| \leq M^{2\delta_0}$. We will show the estimate

$$\alpha k m^{\alpha-1} = \frac{kx}{h} + O(M^{-\frac{\delta}{2}}),$$

Since $m \in J$, it satisfies the equation

$$(m + h)\alpha - m\alpha = x + \Delta, \quad a(I(\Delta)) \leq \Delta < b(I(\Delta)),$$

and by the mean-value theorem

$$\alpha h m^{\alpha-1} = x + \Delta + O\left(\frac{h^2 M^\alpha}{M^2}\right),$$

By (23) we have $M^\delta \leq H \leq M^{99/100}$ and consequently since $|k| \leq M^{2\delta_0}$ and $2\delta_0 < \delta/2$

$$\alpha k m^{\alpha-1} = \frac{kx}{h} + O(M^{-\delta/2}).$$

We have

$$\left\|\frac{kx}{h}\right\| \leq \|\alpha k m^{\alpha-1}\| + M^{-\delta/2} \leq 2M^{-\delta_0/2},$$

Now, let $w \in \mathbb{N}$ be the integer approximation of $\frac{kx}{h}$, thus

$$\frac{kx}{h} = w + e, \quad |e| \leq 2M^{-\delta_0/2}.$$
Now, since $|e_i h_i| < 2HM^{-\delta_0/2}$.

Now, for given number $z$ with $|z| < 2HM^{-\delta_0/2}$ we consider the set

$$A_z = \{h_i : kx = h_i w_i + z\}.$$  

If for each $z$ the number of elements of $A_z$ is $< \frac{1}{2}M^{6\delta_0/4}$, then the total number of $h_i$'s satisfying (38) would be $< \frac{1}{2}M^{6\delta_0/4} \cdot 2HM^{-\delta_0/2} = HM^{-\delta_0/4}$, which is a contradiction. Thus, there must be a $z$, for which

$$\#\{h_i : kx = h_i w_i + z\} \geq \frac{1}{2} M^{6\delta_0/4}.$$  

Now, since $|z| < \frac{C_{\epsilon, r}}{M^{0+\epsilon}}$, $k \neq 0$ we have $0 \neq |kx - z| \leq M^{6\delta_0/2+1}$ and by (39) $kx - z$ has at least $M^{6\delta_0/4}$ divisors, which is impossible by a well known estimate on the number of divisors.

**Corollary 19.** We have

$$\sum_{I_r} \sum_{h \sim H} \left| J_{h,x,I_r}^{+} - (m(h,x,b(I_r)) - m(h,x,a(I_r))) \right| \leq C M^{2-\alpha-\delta_0/4}.$$  

$$\sum_{I_r} \sum_{h \sim H} \varphi\left(\frac{m(h,x,a(I_r))}{M}\right)^2 \left( J_{h,x,I_r}^{+} \right) = S + O\left(M^{2-\alpha-\delta_0/4}\right)$$  

where $S$ is defined by (29).

**Proof.** The first formula is an immediate consequence of Lemmas (18) and (17). For the second formula we apply (27) and the first part.

5. A COUNTEREXAMPLE

In this section we prove the theorem (4). Fix $1 < \alpha < 1 + \frac{1}{1000}$, $0 < \delta \leq \frac{(\alpha-1)^2}{\alpha}$ and $\kappa = c\delta$, where $c$ will be specified later. Let $\{M_l\}_l$ be sequence of integers satisfying $10M_l \leq M_{l+1}^{-1-1.1\delta}$, $\varphi \in C^\infty_c(1,2)$ real valued. We put $\varphi_s = \varphi$ if for some $l$ we have (recall $s$ is dyadic) $s \in U_- = [M_l^{-1-1.1\delta}, M_{l+1}^{-1-\delta}]$ or $s \in U_+ = [M_l^{-1+1.1\delta}, M_{l+1}^{-1-1.1\delta}]$ and $\varphi_s = 0$ otherwise.

We will consider Hilbert transform $H_{M^\alpha} = \sum_{s: \alpha-1.1\delta < s \leq M} H_s$ (we use more convenient $H_s$ instead of $H_s$ ) corresponding to this sequence $\{\varphi_s\}$ and $\theta = \alpha - 1 - 1.1\delta$.  

Fix \( l \) and denote \( M = M_l \). By (3), \( \mathbb{H}_{M} \) contains two large blocks \( \mathbb{H}_+, \mathbb{H}_- \) corresponding to summation indices in \( U_+, U_- \) respectively. For \( P = M^{\alpha(1-\delta)} \) and an integer \( j \) satisfying, for \( C \) sufficiently large, 
\[
\frac{1}{C} M^{\alpha(2+0.9\delta-\alpha)} \leq j \leq CM^{\alpha(2+\delta-\alpha)} ,
\]
let \( I_j = [(j-1)P, (j+1)P] \). Consider \( A_j \), the set of \( n \in U_- \) such that for some \( x \in I_j \) the equation
\[
(m^n) \pm [n^n] = x
\]
has more than 1 solution (a pair \( m, n \) with \( m \in U_+ \) and \( n \in U_- \)), we allow the different choice of \( \pm \) signs for different solutions. Let \( m_1 \) and \( m_2 \) satisfy (10) possibly with different \( x_1, x_2 \in I_j \) and \( n_1, n_2 \in U_- \). We define \( h = m_1 - m_2 \) and estimate using \( m_1, m_2 \in U_+ \) and the Taylor’s formula
\[
|m_1^n - m_2^n| \leq P \Rightarrow hM^{(1-0.1\delta)(\alpha-1)} \leq CM^{\alpha(1-\delta)}
\]
Let \( H = \frac{C M^{\alpha(1-\delta)}}{M^{(1-0.1\delta)(\alpha-1)}} \), hence \( |h| \leq H \), that is \( m_1, m_2 \) are contained in the interval of length \( H \) containing some \( m_0 \) satisfying (10). If \( n_1 \in A_j \), then for some \( n_2 \neq n_1 \) we have two pairs \( m_1, n_1 \) and \( m_2, n_2 \) satisfying (10). In what follows we assume that the \( \pm \) signs corresponding to both pairs are minus. By (10) we obtain
\[
[n_1^n] - [n_2^n] = [m_1^n] - [m_2^n] = [m_1^n - m_2^n] + \Delta , \quad \Delta \in \{-1,0,1\}.
\]
We have:
\[
m_1^n - m_2^n = m_1^n - m_0^n + m_0^n - m_2^n = ah_1 m_0^{\alpha - 1} - ah_2 m_0^{\alpha - 1} + O(H^2 M^{\alpha - 2}), \quad H^2 M^{\alpha - 2} \leq 1.
\]
From this:
\[
[m_1^n - m_2^n] + \Delta = [\alpha (h_1 - h_2) m_0^{\alpha - 1}] + \Delta_1
\]
\[
\Delta_1 \in \{-2,-1,0,1,2\}, \quad -H \leq h_1, h_2 \leq H.
\]
There are at most \( 5(4H + 1) \) different numbers represented by right hand side of (42). By lemma 6 the number of solutions to
\[
[n_1^n] \pm [n_2^n] = k , \quad 0 < n_1, n_2 \leq M^{\alpha-1-\delta}
\]
is at most \( CM^{(\alpha-1-\delta)(2-\alpha)} \). Thus the number of pairs \( (n_1, n_2) \) with \( n_1, m_1 \) and \( n_2, m_2 \) satisfying (41), that is (40) for the same \( x \), does not exceed
\[
M^{(\alpha-1-\delta)(2-\alpha)} \cdot 21 H \leq C \cdot M^{\alpha-1-1.9\delta}.
\]
The case of other choices of ± signs follows exactly the same way. So we obtained $|A_j| \leq M^{\alpha - 1 - 1.95}$.

Let $x$ be of the form

$$x = [m^\alpha] \pm [n^\alpha], \quad n \notin A_j \cup A_{j-1} \cup A_{j+1}, [m^\alpha] \in I_j$$

Then one can easily verify, that the representation (44) is unique, and it remains unique if we drop the assumption $[m^\alpha] \in I_j$ (we remark that if $n \leq M^{\alpha - 1 - 1.95}$ than this statement is immediate and do not require an argument above ). In particular for $x, m, n$ related by (44)

$$\|H_+ * H_-(x)\| \geq \frac{1}{m \cdot n},$$

(45)

$$H_- * H_-(x) = 0,$$

Thus (we leave the proof for the reader)

$$\|H_+ * H_-\|_{L^p} \geq C\left(\frac{\delta \kappa}{100}\right)^{\frac{1}{p}} (\log M)^2, \quad p = 1 + \frac{1}{\log M}.$$  (46)

We will show the estimate

$$\|H_+ * H_+\|_{L^p} \leq C\kappa^{\frac{2}{p}} (\log M)^2$$  (47)

where $p$ is as in (46). We have $H_+ * H_+ = \sum_{s_1, s_2 \leq \text{dyadic}} H_{s_1} * H_{s_2}$. Since this expression contains at most $C\kappa^2 (\log M)^2$ summands, it suffices to prove that $\|H_{s_1} * H_{s_2}\|_{L^p} \leq C$. Assume $s_1 \geq s_2$. Since $H_{s_1} * H_{s_2}$ is supported in $[-Cs_1^\alpha, Cs_2^\alpha]$, by Cauchy-Schwartz, it suffices to have $\|H_{s_1} * H_{s_2}\|_{L^2} \leq Cs_1^\alpha$. We have $\|H_{s_1} * H_{s_2}\|_{L^2} = \langle H_{s_1} * H_{s_2} \rangle_{L^2} \leq C\left(\frac{1}{s_1 s_2} + \frac{s_2^2}{s_1 s_2} \right)$ where, since $H_{s_2} * H_{s_2}$ is supported in $[-Cs_2^\alpha, Cs_2^\alpha]$, the last estimate follows from the lemma 4. Fix sufficiently small $c > 0$ and $\kappa = c\delta$. From the (46), (47) and (13) we infer that the estimate

$$\|(H_+ + H_-) * (H_+ + H_-)\|_{L^p} \leq \frac{C}{p-1},$$

cannot hold uniformly with $M$ and $p > 1$. By the definition, $H_{M^\alpha}$ is antisymmetric with operator $\ell^2 \rightarrow \ell^2$ norm controlled independently of $M$, so it has purely imaginary spectrum contained in some fixed interval $D \subset i\mathbb{R}$. Let $\Gamma$ be a contour in $\mathbb{C}$ enclosing $D$. Then we have $\|((\lambda I + H_{M^\alpha})^{-1})_{\ell^2 \rightarrow \ell^2} \| \leq C$. Now, if we have $\|((\lambda I + H_{M^\alpha})^{-1})_{\ell^2 \rightarrow \ell^2} \| \leq C$, uniformly for $M$ and $\lambda \in \Gamma$, we should have $\|((\lambda I + H_{M^\alpha})^{-1})_{\ell^p \rightarrow \ell^p} \| \leq \frac{C}{p-1}$. 
The formula $H_{2M} = \frac{1}{2\pi i} \oint_{\Gamma} \lambda^2 (\lambda I + H_{M^\alpha})^{-1} d\lambda$ implies that the estimate
$$\|H_{2M}\|_{\ell^p \to \ell^p} \leq \frac{C}{p}$$
holds uniformly in $M$. A contradiction.

**Remark.** Now we return to a particular case of the result [14] announced in Remark (vi) of Section 2. We sketch the proof of the following fact: for $\lambda$ fixed and $|\lambda|$ sufficiently large, the operators $(\lambda + H_{M^\alpha})^{-1}$ are not of weak type $(1,1)$ uniformly in $M$. We will remove large $|\lambda|$ requirement in [14].

Recall, that we have
$$H_{M^\alpha} = H_+ + H_-$$
with both components comprised of summands with indices in $U_+$ and $U_-$ respectively.

**Lemma 20.** We have, for $l \geq 2$,
$$H^+_l = \sum_{s \geq M^\alpha(1-0.1s)} K^+_s, \quad H^-_l = \sum_{s \geq M^\alpha(1-1.1s)} K^-_s.$$ The kernels $K^+_s, K^-_s$ satisfy conditions (i)$_s \ldots$ (iv)$_s$ with the constant $|D_s| \leq C^l_0$, where $C_0$ is some universal constant.

**Proof.** Corollary of Lemma [12] \hfill \Box

**Lemma 21.** Let $k \geq 3$ and let us consider $H^{k-1}_+ H_-$ and $H^{k-1}_- H_+$. Then, for $p > 1$
$$\|H^{k-1}_+ H_-\|_{\ell^p \to \ell^p} \leq \frac{C^{k-1}_0}{(p-1)^2},$$
and similar estimate for $H^{k-1}_- H_+$.

**Proof.** It is immediate corollary of Lemma [20] \hfill \Box

**Corollary 22.** For $\lambda$ sufficiently large and fixed, we have, for all $M$ sufficiently large, $p = 1 + \frac{1}{\log M}$
$$\|(\lambda - H_{M^\alpha})^{-1}\|_{\ell^p \to \ell^p} \geq \frac{C}{(p-1)^2 |\lambda|^3}.$$ 

**Proof.**
$$\lambda - H_{M^\alpha} = \sum_{k=0}^{\infty} \frac{H^k_{M^\alpha}}{\lambda^{k+1}},$$
$$H^k_{M^\alpha} = (H_+ + H_-)^k = \sum_{j=0}^{k} \binom{k}{j} H^k_{H_+^j H_-^j}.$$
Using Lemmas 20 and 21 for $p > 1$ we have:

$$\left\| \sum_{k=3}^{\infty} \frac{\mathbb{H}_k^{M^\alpha}}{\lambda^{k+1}} \right\|_{\ell^p \to \ell^p} \leq \frac{C}{|\lambda|^4(p-1)^2},$$

moreover, by independent of $M$ near $\ell^1$ estimate $\| \mathbb{H}_M^{M^\alpha} \|_{\ell^p \to \ell^p} \leq C(p-1)^{-1}$,

$$\left\| \frac{\mathbb{H}_2^{M^\alpha}}{\lambda^3} + \frac{\mathbb{H}_1^{M^\alpha}}{\lambda^2} + \frac{\delta_0}{\lambda} \right\|_{\ell^p \to \ell^p} \geq \frac{1}{|\lambda|^3} \| \mathbb{H}_M^{M^\alpha} \|_{\ell^p \to \ell^p} - \frac{C}{|\lambda|^2(p-1) - |\lambda|}.$$

For the estimate $\| \mathbb{H}_M^{M^\alpha} \|_{\ell^p \to \ell^p} \leq C(p-1)^{-1}$ one does not need the weak type $(1,1)$ estimates on $\mathbb{H}_M^{M^\alpha}$. Classical interpolation argument based on the Fourier transform estimates of $\mathbb{H}_M^{M^\alpha}$ (7), produce constant $C$ independent on $M, \theta$. We leave the details for the interested reader.

Thus, for $\lambda$ and $M$ sufficiently large, $p = 1 + \frac{1}{\log M}$ we have

$$\| (\lambda - \mathbb{H}_M^{M^\alpha})^{-1} \|_{\ell^p \to \ell^p} \geq \frac{C}{(p-1)^2|\lambda|^3},$$

as in the proof of Theorem 4.

□

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