Summation formulas involving generalized harmonic numbers

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**Abstract**

By means of the derivative operator and three hypergeometric series identities, several interesting summation formulas involving generalized harmonic numbers are established.

**1. Introduction**

For a complex number $x$, define the shifted-factorial to be

$$(x)_0 = 0 \quad \text{and} \quad (x)_n = x(x + 1) \cdots (x + n - 1) \quad \text{when} \quad n \in \mathbb{N}.$$ 

Following Bailey [4], define the hypergeometric series by

$$
_{1+r}F_s \left[ \begin{array}{c} a_0, a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array} \vline \begin{array}{c} z \\ \end{array} \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{(1)_k (b_1)_k \cdots (b_s)_k} z^k,
$$

where $\{a_i\}_{i \geq 0}$ and $\{b_j\}_{j \geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then Dougall’s $_5F_4(1)$-series formula (cf. [4, p. 27]) can be stated as

$$
_{5}F_{4} \left[ \begin{array}{c} a, 1 + \frac{a}{2}, b, c, d \\ \frac{1}{2}, 1 + a - b, 1 + a - c, 1 + a - d \end{array} \vline \begin{array}{c} 1 \\ \end{array} \right] = \frac{\Gamma(1 + a - b) \Gamma(1 + a - c) \Gamma(1 + a - d) \Gamma(1 + a - b - c - d)}{\Gamma(1 + a) \Gamma(1 + a - b - c) \Gamma(1 + a - b - d) \Gamma(1 + a - c - d)},
$$

(1)
where the parameters satisfy $\text{Re}(1 + a - b - c - d) > 0$ and $\Gamma(x)$ is the well-known gamma function
\[
\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \, dt \quad \text{with} \quad \text{Re}(x) > 0.
\]

When $d = a/2$, it reduces to Dixon’s $3F_2(1)$-series formula (cf. [4, p. 13]):

\[
3F_2 \left[ \begin{array}{c} a, b, c \\ 1 + a - b, 1 + a - c \\ \end{array} \right] = \frac{\Gamma(1 + \frac{a}{2}) \Gamma(1 + a - b) \Gamma(1 + a - c) \Gamma(\frac{a}{2} - b - c)}{\Gamma(1 + a) \Gamma(1 + a - b) \Gamma(1 + a - c) \Gamma(1 + a - b - c)}
\]

(2)

provided that $\text{Re}(1 + \frac{a}{2} - b - c) > 0$. A Dixon-like formula that will appear in Section 3 is

\[
3F_2 \left[ \begin{array}{c} a, b, c \\ 1 + a - b, a - c \\ \end{array} \right] = \frac{1}{2^{1+c}} \frac{\Gamma(1 + a - b) \Gamma(\frac{1+a}{2} - b - c) \Gamma(\frac{a-c}{2} \Gamma(\frac{1+a-c}{2})}{\Gamma(1 + a - b - c) \Gamma(\frac{1+a-2}{2} - b) \Gamma(\frac{a-c}{2})}
\]

+ \frac{1}{2^{1+c}} \frac{\Gamma(1 + a - b) \Gamma(\frac{2+a}{2} - b - c) \Gamma(\frac{a-c}{2}) \Gamma(\frac{1+a-c}{2})}{\Gamma(1 + a - b - c) \Gamma(\frac{1+a-2}{2} - b) \Gamma(\frac{a-c}{2})}
\]

(3)

provided that $\text{Re}(\frac{1+a}{2} - b - c) > 0$.

For a complex number $x$ and a positive integer $\ell$, define generalized harmonic numbers of $\ell$-order to be

\[
H_0^{(\ell)}(x) = 0 \quad \text{and} \quad H_n^{(\ell)}(x) = \sum_{k=1}^{n} \frac{1}{(x + k)^{\ell}} \quad \text{with} \quad n \in \mathbb{N}.
\]

When $x = 0$, they become harmonic numbers of $\ell$-order

\[
H_0^{(\ell)} = 0 \quad \text{and} \quad H_n^{(\ell)} = \sum_{k=1}^{n} \frac{1}{k^{\ell}} \quad \text{with} \quad n \in \mathbb{N}.
\]

Setting $\ell = 1$ in $H_0^{(\ell)}(x)$ and $H_n^{(\ell)}(x)$, we obtain generalized harmonic numbers

\[
H_0(x) = 0 \quad \text{and} \quad H_n(x) = \sum_{k=1}^{n} \frac{1}{x + k} \quad \text{with} \quad n \in \mathbb{N}.
\]

When $x = 0$, they reduce to classical harmonic numbers

\[
H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{with} \quad n \in \mathbb{N}.
\]

For a differentiable function $f(x)$, define the derivative operator $D_x$ by

\[
D_x f(x) = \frac{d}{dx} f(x).
\]
Then it is not difficult to show that
\[
D_x \left( \begin{array}{c} x + s \\ t \end{array} \right) = \left( \begin{array}{c} x + s \\ t \end{array} \right) \left[ H_s(x) - H_{s-t}(x) \right],
\]
\[
D_x H_n^{(\ell)}(x) = -\ell H_n^{(\ell+1)}(x),
\]
where \( s, t \in \mathbb{N}_0 \) with \( t \leq s \).

As pointed out by Richard Askey (cf. [3]), expressing harmonic numbers in accordance with differentiation of binomial coefficients can be traced back to Issac Newton. In 2003, Paule and Schneider [22] computed the family of series:
\[
W_n(\alpha) = \sum_{k=0}^{n} \binom{n}{k} \alpha \left[ 1 + \alpha(n-2k)H_k \right]
\]
with \( \alpha = 1, 2, 3, 4, 5 \) according to this approach and Zeilberger’s algorithm for definite hypergeometric sums. Subsequently, Chu and Donno [15] verified Paule and Schneider’s results, gave Paule-Schneider type identities with \( \alpha = -2, -1, 6 \) and derived a lot of different conclusions by applying the derivative operator to Gauss’ \( _2F_1(1) \)-series formula, Saalschütz’s \( _3F_2(1) \)-series formula, (1) and Whipple’s transformation between a \( _7F_6(1) \)-series and a \( _4F_3(1) \)-series. General Paule-Schneider type identities with \( \alpha \) being an integer were explored by Krattenthaler et al. [19] and Wei et al. [31] in terms of this way and the hypergeometric form of Andrews’ \( q \)-series transformation. Recently, Wang and Wei [30] got two families of summation formulas involving generalized harmonic numbers by using the derivative operator and Bailey’s \( _2F_1(1/2) \)-series formula. Many results from the higher derivatives of Gauss’ \( _2F_1(1) \)-series formula, (2) and (1) can be seen in the papers [14,16,29,32]. For several conclusions from differentiation of binomial coefficients which are not related to known hypergeometric series, the reader may refer to [24,26,33].

By combining the comparing coefficient method with Gauss’ \( _2F_1(1) \)-series formula, Kummer’s \( _2F_1(-1) \)-series formula, (2), Dixon’s \( _4F_3(-1) \)-series formula and (1), a lot of nice harmonic number identities were established in the papers [5,6,9–11,21,35]. Through an extension of Zeilberger’s algorithm, Chyzak [17] confirmed the identity:
\[
\sum_{k=1}^{n} k^2H_{n+k} = \frac{n(n+1)(2n+1)}{6} \left( 2H_{2n} - H_n \right) - \frac{n(n+1)(10n-1)}{36}
\]
which is the bonus problem 69 proposed by Graham et al. [18, Chapter 6]. Schneider [23] also verified it by means of an algorithm built on Karr’s difference field theory. Several years ago, Chen et al. [7] proved this formula and some other concise results in accordance with the Abel-Zeilberger algorithm.

Except for the three approaches just displayed, there exist other ways that are valid to harmonic number identities. According to the WZ method, Ahlgren et al. [1] confirmed the beautiful identity:
\[
\sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k} \left[ 1 + 2kH_{n+k} + 2kH_{n-k} - 4kH_k \right] = 0
\]
which implies Beukers’ conjecture on Apéry numbers (cf. [2]). It was also proved by Chu [12] in terms of the partial fraction decomposition approach. Different conclusions from the same way can be found in the papers [13,34]. Cheon et al. [8] and Wang [28] studied harmonic number identities by utilizing Riordan arrays. Kronenburg [20] gained two kinds of formulas by employing the difference operator. Sofo [25] deduced several quadratic alternating harmonic number sums through the integral method. It should be mentioned that Sun [27] showed recently some congruence relations concerning harmonic numbers to us.

Although (1) and (2) have played an important role in the development process of harmonic number identities, new results can be offered when we add clever tricks. With the aid of the reversal techniques and bisection method, we shall establish several interesting summation formulas involving generalized harmonic numbers by applying the derivative operator to (1)–(3). They can produce numerous harmonic number identities when the parameters are specified. For making the reader have a taste, we select, above all, the following two ones:

\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{H_k^{(2)}}{k} = \frac{1 + 2n}{2 + 2n} H_n^{(2)},
\]

\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} H_k^2 = \frac{1 + 2n}{2 + 2n}\left\{H_n^2 - \frac{H_{1+2n}}{1+n} - H_{1+2n}^{(2)} + H_{1+n}^{(2)}\right\},
\]

where the first equation comes from the case \( p = 0 \) of (7) and the second equation is exactly Proposition 3.

2. Dixon’s identity, reversal techniques and summation formulas involving generalized harmonic numbers

**Theorem 1:** Let \( x \) and \( y \) be complex numbers. Then

\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{(x+k)(y+k)}{(x+2n)_k(y+2n)_k} H_k(x) = \frac{1}{2} \left\{\binom{x+n}{n}\binom{y+n}{n}\binom{1+x+y+2n}{2n} H_n(x) - H_n(1+x+y) + H_{2n}(1+x+y)\right\}.
\]

**Proof:** Perform the replacements \( a \to -2n, b \to 1+x, c \to 1+y \) in (2) to achieve

\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{(x+k)(y+k)}{(x+2n)_k(y+2n)_k} = \frac{(x+n)(y+n)(1+x+y+2n)}{(x+2n)(y+2n)(1+x+y+n)}.
\]

Applying the derivative operator \( D_x \) to both sides of (4), we attain

\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{(x+k)(y+k)}{(x+2n)_k(y+2n)_k} \{H_k(x) + H_{2n-k}(x)\}.
\]
Theorem 2: Let $x$ be a complex number. Then
\[
\frac{(x+y_n)(y+n)}{(x+2n)(y+2n)(1+x+y+n)} \{ H_n(x) - H_n(1+x+y) + H_{2n}(1+x+y) \}.
\]

By means of the reversal techniques, it is easy to see that
\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{x+k}{k} H_k(x) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} H_{2n-k}(x) \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{y+k}{k} H_k(x).
\]

Therefore, we derive Theorem 1 to complete the proof.

Taking $x = p, y = q$ with $p, q \in \mathbb{N}_0$ in Theorem 1 and using (4), we have the summation formula on harmonic numbers:
\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{p+k}{k} \binom{q+k}{k} H_{p+k} = \frac{1}{2} \binom{p+n}{2n} \binom{q+n}{2n} \binom{1+p+q+2n}{2n} \{ H_p + H_{p+n} - H_{1+p+q+n} + H_{1+p+q+2n} \}.
\]

Theorem 2: Let $x$ be a complex number. Then
\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{x+k}{k}^2 H_k^{(2)}(x) = \frac{1}{2} \binom{x+n}{2n}^2 \binom{1+2x+2n}{2n} \binom{2n}{H_n^{(2)}(x)}.
\]

Proof: Applying the derivative operator $D_y$ to Theorem 1 and then fixing $y = x$, we obtain
\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{x+k}{k}^2 \{ H_k^2(x) + H_k(x)H_{2n-k}(x) \} = \frac{1}{2} \binom{x+n}{2n}^2 \binom{1+2x+2n}{2n} \binom{2n}{H_n^{(2)}(1+2x) - H_{2n}(1+2x) + [H_n(1+2x) - H_{2n}(1+2x) - H_n(x)]^2}.
\]

Applying the derivative operator $D_x$ to Theorem 1 and then choosing $y = x$, we get
\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{x+k}{k}^2 \{ H_k^2(x) + H_k(x)H_{2n-k}(x) - H_{k}^{(2)}(x) \} = \frac{1}{2} \binom{x+n}{2n}^2 \binom{1+2x+2n}{2n} \binom{2n}{H_n^{(2)}(1+2x) - H_{2n}(1+2x) - H_n^{(2)}(x) + [H_n(1+2x) - H_{2n}(1+2x) - H_n(x)]^2}.
\]

The difference of (6) and the last equation gives Theorem 2.
Setting $x = p$ with $p \in \mathbb{N}_0$ in Theorem 2 and utilizing (4), we gain the summation formula on harmonic numbers:

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{p+k}{k}^2 H^{(2)}_{p+k} = \frac{1}{2} \left( \binom{p+n}{2n} \binom{1+2p+2n}{2n} \right) \{ H^{(2)}_{p+n} + H^{(2)}_p \}. \tag{7}$$

**Proposition 3 (Harmonic number identity):**

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} H_k^2 = \frac{1+2n}{2+2n} \left\{ H_{1+2n}^2 - \frac{H_{1+2n}}{1+n} - H_{1+2n}^{(2)} + H_{1+n}^{(2)} \right\}.$$  

**Proof:** The case $x = 0$ of (6) reads as

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \{ H_k^2 + H_k H_{2n-k} \} = \frac{1+2n}{2+2n} \left\{ \left( H_{1+2n} + \frac{1}{1+n} \right)^2 - H_{1+2n}^{(2)} + H_{1+n}^{(2)} \right\}. $$

The case $p = n$ of Wei et al. [32, Corollary 21] is

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} H_k H_{2n-k} = \frac{1+2n}{2(1+n)^2} \left\{ \frac{1}{1+n} - H_{1+2n} \right\}.$$  

The difference of the last two equations offers Proposition 3. □

### 3. Dixon-like identity, bisection method and summation formulas involving generalized harmonic numbers

**Lemma 4 (Dixon-like identity):**

$$\text{3F}_2 \left[ \begin{array}{c} a,b,c \\ 1+a-b,a-c \\ 1 \end{array} \right] = \frac{\Gamma(1+a-b)\Gamma \left( \frac{1+a}{2} - b - c \right)\Gamma(\frac{a-c}{2})\Gamma(\frac{1+a-c}{2})}{2^{1+c} \Gamma(1+a-b-c)\Gamma \left( \frac{a}{2} \right)\Gamma \left( \frac{1+a}{2} - b \right)\Gamma \left( \frac{1+a-c}{2} \right)} + \frac{\Gamma(1+a-b)\Gamma \left( \frac{2+a}{2} - b - c \right)\Gamma(\frac{a-c}{2})\Gamma(\frac{1+a-c}{2})}{2^{1+c} \Gamma(1+a-b-c)\Gamma \left( \frac{1+a}{2} \right)\Gamma \left( \frac{2+a}{2} - b \right)\Gamma \left( \frac{a}{2} - c \right)}$$

provided that $\text{Re}(\frac{1+a}{2} - b - c) > 0$.

**Proof:** Recall Whipple’s $\text{3F}_2$-series identity (cf. [4, p. 16]):

$$\text{3F}_2 \left[ \begin{array}{c} a,1-a,b \\ c,1+2b-c \\ 1 \end{array} \right] = \frac{\pi 2^{1-2b} \Gamma(c)\Gamma(1+2b-c)}{\Gamma(\frac{a+c}{2})\Gamma(\frac{1+a-c}{2} + b)\Gamma(\frac{1-a-c}{2} + b)},$$

where $\text{Re}(b) > 0$. Employ the substitution $a \to 1+a$ in the last equation to achieve

$$\text{3F}_2 \left[ \begin{array}{c} 1+a,-a,b \\ c,1+2b-c \\ 1 \end{array} \right] = \frac{\pi 2^{1-2b} \Gamma(c)\Gamma(1+2b-c)}{\Gamma(\frac{1+a+c}{2})\Gamma(\frac{2+a-c}{2} + b)\Gamma(\frac{-a+c}{2})\Gamma(\frac{1-a-c}{2} + b)}.$$
The linear combination of the last two equations produces
\[
\begin{align*}
3F_2 \left[ \begin{array}{c}
 a, -a, b \\
 c, 1 + 2b - c \\
 \end{array} \right] 1 & = \frac{1}{2} \frac{\pi 2^{1-2b} \Gamma(c) \Gamma(1 + 2b - c)}{\Gamma\left(\frac{a + c}{2}\right) \Gamma\left(\frac{1 + a - c + b}{2}\right) \Gamma\left(\frac{1 - a - c + b}{2}\right)} \\
& \quad + \frac{1}{2} \frac{\pi 2^{1-2b} \Gamma(c) \Gamma(1 + 2b - c)}{\Gamma\left(\frac{1 + a + c}{2}\right) \Gamma\left(\frac{2 + a - c + b}{2}\right) \Gamma\left(\frac{-a + c}{2} + b\right)} \\
& \quad \times \frac{\pi 2^{1-2b} \Gamma(c) \Gamma(1 + 2b - c)}{\Gamma\left(\frac{1 + a - c}{2}\right) \Gamma\left(\frac{2 + a - c + b}{2}\right) \Gamma\left(\frac{-a + c}{2}\right) + b}.
\end{align*}
\]
(8)

In accordance with Kummer’s transformation formula (cf. [4, p. 98]):
\[
3F_2 \left[ \begin{array}{c}
 a, b, c \\
 d, e \\
 \end{array} \right] 1 = \frac{\Gamma(e) \Gamma(1 + a - b - c)}{\Gamma(d) \Gamma(1 + a - b - c)} 3F_2 \left[ \begin{array}{c}
 a, d - b, d - c \\
 e, e - b - c \\
 \end{array} \right] 1,
\]
we attain
\[
3F_2 \left[ \begin{array}{c}
 c, a, b \\
 a - c, 1 + a - b \\
 \end{array} \right] 1 = \frac{\Gamma(1 + a - b) \Gamma(1 + a - 2b - 2c)}{\Gamma(1 + a - b - c) \Gamma(1 + a - 2b - c)} \times 3F_2 \left[ \begin{array}{c}
 c, -c, a - b - c \\
 a - c, 1 + a - 2b - c \\
 \end{array} \right] 1.
\]

Evaluating the series on the right hand side by (8), we obtain Lemma 4 to finish the proof.

\[\square\]

**Theorem 5:** Let \( x \) be a complex number. Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2x+k)}{(2x+n+k)} H_k(x) = \frac{4}{n} \left( \frac{x+n}{n} \right) \left( \frac{H_n(x) + H_n - 2H_{2n}}{(2x+2n)(x+n)} \right).
\]

**Proof:** The case \( c = -n \) of Lemma 4 can be written as
\[
3F_2 \left[ \begin{array}{c}
 a, b, -n \\
 1 + a - b, a + n \\
 \end{array} \right] 1 = 2^{n+1} \frac{(\frac{1+n}{2})_n(\frac{2+a}{2} - b)_n}{(a+n)_n(1 + a - b)_n} + 2^{n-1} \frac{(\frac{n}{2})_n(\frac{1+a}{2} - b)_n}{(a+n)_n(1 + a - b)_n}.
\]

Replace respectively \( a \) and \( b \) by \( 1 + x \) and \( 1 + y \) in the last equation to get
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(x+k)}{(x+n+k)} \frac{(y+k)}{(y+n+k)} = 2^{n+1} \frac{\binom{\frac{3}{2}+n}{n}(\frac{2}{2} - y+n)}{(x+2n)(x-y+n)} + 2^{n-1} \frac{\binom{\frac{3}{2}+n}{n}(\frac{2}{2} - y+n)}{(x+2n)(x-y+n)}.
\]
(9)
Applying the derivative operator $D_y$ to both sides of (9), we gain

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(x+k)}{x+n+k} \frac{(y+k)}{x-y+k} \left\{ H_k(y) + H_k(x-y) \right\}
\]

\[
= 2^{2n-1} \binom{n}{\frac{x+y}{n}} \frac{n}{x+2n} \frac{y-n}{x-2n} \left\{ H_n(x-y) - H_n(\frac{x}{2} - y) \right\}
\]

\[
+ 2^{2n-1} \binom{n}{\frac{x+y}{n}} \frac{n}{x+2n} \frac{y-n}{x-2n} \left\{ H_n(x-y) - H_n(\frac{x}{2} - y) \right\}
\]

According to the relation

\[
H_n(\frac{x-1}{2} - y) = 2H_n(x - 2y) - H_n(\frac{x}{2} - y),
\]

(10) can be reformulated as

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(x+k)}{x+n+k} \frac{(y+k)}{x-y+k} \left\{ H_k(y) + H_k(x-y) \right\}
\]

\[
= 2^{2n-1} \binom{n}{\frac{x+y}{n}} \frac{n}{x+2n} \frac{y-n}{x-2n} \left\{ H_n(x-y) - H_n(\frac{x}{2} - y) - 2H_2n(x - 2y) \right\}
\]

\[
+ 2^{2n-1} \binom{n}{\frac{x+y}{n}} \frac{n}{x+2n} \frac{y-n}{x-2n} \left\{ H_n(x-y) - H_n(\frac{x}{2} - y) + \frac{2}{x - 2y + 2n} \right\}
\]

\[
- 2^{2n-1} \binom{n}{\frac{x+y}{n}} \frac{n}{x+2n} \frac{y-n}{x-2n} \frac{2}{x - 2y + 2n}.
\]

Substitute respectively $x$ and $y$ by $2x$ and $x$ in the last equation to give Theorem 5.

Taking $x = p$ with $p \in \mathbb{N}_0$ in Theorem 5 and exploiting (9), we achieve the summation formula on harmonic numbers:

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2p+k)}{2p+n+k} \left\{ H_{p+k} \right\}
\]

\[
= 4^{n-1} \binom{n}{p} \left\{ H_{p+n} + H_p + H_n - 2H_{2n} \right\} - \frac{4^{n-1}}{n} \binom{p+n-1}{2p+2n} \binom{p+n}{p+n}.
\]

(11)
**Theorem 6:** Let \( x \) be a complex number. Then

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\left(\frac{5}{2} + k\right) \left(\frac{x - \frac{1}{2} + k}{\frac{5}{2} - k}\right) H_{2k}(x)}{\binom{5}{2} + k \left(\frac{x - \frac{1}{2} + n + k}{\frac{5}{2} - k}\right)} = 4^{n-1} \left(\frac{\frac{5}{2} - \frac{1}{2} + n}{\frac{5}{2} - n}\right) \{H_n(x) + H_n\left(\frac{x - \frac{1}{2}}{2}\right)\}
\]

**Proof:** Perform the replacements \( x \to x - \frac{1}{2}, y \to \frac{x}{2} \) in (10) to attain

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\left(\frac{5}{2} + k\right) \left(\frac{x - \frac{1}{2} + k}{\frac{5}{2} - k}\right)}{\binom{5}{2} + k \left(\frac{x - \frac{1}{2} + n + k}{\frac{5}{2} - k}\right)} \{H_k\left(\frac{5}{2}\right) + H_k\left(\frac{x - \frac{1}{2}}{2}\right)\}
\]

\[
= 2^{2n-1} \left(\frac{\frac{5}{2} - \frac{1}{2} + n}{\frac{5}{2} - n}\right) \{H_n(x) + H_n\left(\frac{x - \frac{1}{2}}{2}\right)\}
\]

In terms of the relation

\[
H_k\left(\frac{5}{2}\right) + H_k\left(\frac{x - \frac{1}{2}}{2}\right) = 2H_{2k}(x), \quad (12)
\]

the last equation can be expressed as Theorem 6 to complete the proof. **□**

Fixing \( x = p \) with \( p \in \mathbb{N}_0 \) in Theorem 6 and using (9), we obtain the summation formula on harmonic numbers and generalized harmonic numbers:

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\left(\frac{p}{2} + k\right) \left(\frac{p - \frac{1}{2} + k}{\frac{p}{2} - k}\right) H_{p+2k}}{\binom{p}{2} + k \left(\frac{p - \frac{1}{2} + n + k}{\frac{p}{2} - k}\right)}
\]

\[
= 4^{n-1} \left(\frac{\frac{p}{2} - \frac{1}{2} + n}{\frac{p}{2} - n}\right) \left(\frac{\frac{3}{2} + n}{\frac{p}{2} - \frac{1}{2} + 2n}\right) \{2H_p + H_n\left(\frac{p - 1}{2}\right)\}
\]

\[
+ 4^{n-1} \left(\frac{\frac{p}{2} - \frac{3}{2} + n}{\frac{p}{2} - n}\right) \left(\frac{\frac{5}{2} + n}{\frac{p}{2} - \frac{1}{2} + 2n}\right) \{2H_p + H_n\left(\frac{p - 1}{2}\right)\}. \quad (13)
\]
4. Dougall’s identity, bisection method and summation formulas involving generalized harmonic numbers

**Theorem 7:** Let $x$ and $y$ be complex numbers. Then

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2x+k)(2y+k)}{(2x+n+k)(2x-2y+k)} \frac{1 + 2x + 2k}{1 + 2x + n + k} H_k(x) = \frac{1}{2} \frac{(2x+n)}{(x+y+n)} \left\{ H_n(x) - H_n(x - 2y - 1) \right\}.
$$

**Proof:** Employ the substitutions $a \rightarrow 1 + x$, $b \rightarrow 1 + y$, $c \rightarrow 1 + z$ in (1) to get

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(x+k)(y+k)(z+k)}{(x+n+k)(x-y+k)(x-z+k)} \frac{1 + x + 2k}{1 + x + n + k} \left\{ H_k(z) + H_k(x - z) \right\} = \frac{(x+n)}{(x-y+n)} \left\{ H_n(x - z) - H_n(x - y - z - 1) \right\}.
$$

Applying the derivative operator $D_z$ to both sides of (14), we gain

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(x+k)(y+k)(z+k)}{(x+n+k)(x-y+k)(x-z+k)} \frac{1 + x + 2k}{1 + x + n + k} \left\{ H_k(z) + H_k(x - z) \right\} = \frac{(x+n)}{(x-y+n)} \left\{ H_n(x - z) - H_n(x - y - z - 1) \right\}.
$$

Replacing respectively $x$, $y$ and $z$ by $2x$, $2y$ and $x$ in the last equation to offer Theorem 7. □

Choosing $x = p$, $y = \frac{q}{2}$ with $p, q \in \mathbb{N}_0$ in Theorem 7 and utilizing (14), we achieve the summation formula on harmonic numbers:

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2p+k)(q+k)}{(2p+n+k)(2p-q+k)} \frac{1 + 2p + 2k}{1 + 2p + n + k} H_{p+k} = \frac{1}{2} \frac{(2p+n)}{(p+q+n-1)} \left\{ H_p + H_{p+n} + H_{p-q-1} - H_{p-q+n-1} \right\}.
$$

**Theorem 8:** Let $x$ and $y$ be complex numbers. Then

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(x+k)(x-\frac{1}{2}+k)(y+k)}{(x-\frac{1}{2}+n+k)(x-y+\frac{1}{2}+k)} \frac{1 + 2x + 4k}{1 + 2x + 2n + 2k} H_{2k}(x) = \frac{1}{2} \frac{(x-\frac{1}{2}+n)}{(x-y-\frac{1}{2}+n)} \left\{ H_n(x-\frac{1}{2}) - H_n(x-\frac{3}{2} - y) \right\}.
$$
Applying the derivative operator \( D \) to both sides of the last equation, we get

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2x)^k}{(2x)^k} \frac{(2y)^k}{(2y)^k} \frac{1 + 2x + 2k}{1 + 2x + 2n + 2k} H_k^{(2)}(x) \left\{ H_k(x) + H_k(2x - 2y) \right\} = \frac{1}{2(x - 2y + n)} \left\{ (x - 2y) A_n(x, y) - \frac{n B_n(x, y)}{x - 2y + n} - \frac{2n}{(x - 2y + n)^2} \right\},
\]

where the symbols on the right hand side stand for

\[
A_n(x, y) = \left[ H_n(x) - H_n(x - 2y) \right] \left[ H_n(2x - 2y) - H_n(x - 2y) \right] - H_n^{(2)}(x - 2y),
B_n(x, y) = H_n(x) + H_n(2x - 2y) - 2H_n(x - 2y).
\]
Perform the replacement \( y \rightarrow \frac{x}{2} \) in the last equation to produce Theorem 9.

Taking \( x = p \) with \( p \in \mathbb{N}_0 \) in Theorem 9 and using (14), (16) and the relation

\[
H_k^2(p) = (H_{p+k} - H_p)^2 = H_{p+k}^2 - 2H_pH_{p+k} + H_p^2,
\]

we gain the summation formula on harmonic numbers:

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{2p+k}{k} \right) \frac{1+2p+2k}{1+2p+n+k} H_{p+k}^2
\]

\[
= \frac{1}{2n} \left( \sum_{n} \binom{n}{p} \sum_{m} \binom{m}{n} \right) \left\{ H_{n-1}^2 - H_p - H_{p+n} \right\}.
\]

**Theorem 10:** Let \( x \) and \( y \) be complex numbers. Then

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{x+1}{k} \right) \frac{1+x+2k}{1+x+n+k} \left( H_k(y) + H_k(x-y) \right) \left( H_k(z) + H_k(x-z) \right)
\]

\[
= \frac{1}{4} \left( \frac{x+y-z+1}{n} \right) \left\{ H_n(x, y) - H_n^{(2)}(x - y - z - 1) \right\},
\]

where \( C_n(x, y) = [H_n(x - y) - H_n(x - y - z - 1)][H_n(x - z) - H_n(x - y - z - 1)].

Employ the substitutions \( x \rightarrow x - \frac{1}{2}, y \rightarrow \frac{x}{2}, z \rightarrow \frac{x}{2} \) in the last equation to attain

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{x+1}{k} \right) \frac{1+2x+4k}{1+2x+2n+2k} \left\{ H_k \left( \frac{x}{2} \right) + H_k \left( \frac{x-1}{2} \right) \right\}^2
\]

\[
= \frac{1}{2n} \left( \sum_{n} \binom{n}{p} \sum_{m} \binom{m}{n} \right) \left\{ H_n \left( \frac{x-1}{2} \right) - H_n \left( -\frac{3}{2} \right) \right\}^2 - H_n^{(2)} \left( -\frac{3}{2} \right).
\]

In accordance with (12), the last equation can be expressed as Theorem 10 to complete the proof.

Fixing \( x = p \) with \( p \in \mathbb{N}_0 \) in Theorem 10 and utilizing (14), (17) and the relation

\[
H_{2k}^2(p) = (H_{p+2k} - H_p)^2 = H_{p+2k}^2 - 2H_pH_{p+2k} + H_p^2,
\]
we deduce the summation formula on harmonic numbers and generalized harmonic numbers:

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\left(\frac{p-1}{2}+k\right)^2 \left(p-\frac{1}{2}+k\right) \left(p-\frac{1}{2}+n+k\right)^2}{1 + 2p + 2n + 2k} H_{p+2k}^2
\]

\[
= \frac{1}{4} \frac{(p-\frac{1}{2}+n)^2}{\left(\frac{p-1}{2}+n\right)^2} \left\{ \left[ H_n \left(\frac{p-1}{2}\right) - H_n \left( -\frac{3}{2}\right) \right]^2 - H_n^{(2)} \left( -\frac{3}{2}\right) \right\}
\]

\[+ 4H_p \left[ H_p + H_n \left(\frac{p-1}{2}\right) - H_n \left( -\frac{3}{2}\right) \right]. \quad (19)\]

With the change of the parameters \(p\) and \(q\), (5), (7), (11), (13) and (16)–(19) can create a lot of concrete harmonic number identities. The corresponding results will not be laid out here.

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