The study of stationary flow in the core of thin vortex ring

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Abstract. The theoretical investigation of stationary vortex ring in an ideal incompressible fluid is carried out. To obtain the stationary solution for vortex ring with uniform vorticity distribution, Fraenkel’s procedure is used. Further, based on finding solution Fraenkel’s procedure is modified to receive stationary vortex ring with isochronous vorticity distribution. The solutions of the uniform and isochronous problems are obtained in the form of an asymptotic expansion with respect to the small parameter characterised by the ratio of the core and ring diameters.

1. Introduction

The study of a vortex ring dynamics is of great interest due to the fact that it allows to investigate the complicated mechanisms of instabilities, which can play a key role in the problem of turbulence excitation.

Vortex ring can serve as a simple dynamic model of the large-scale structures observed in shear flows. Moreover, it is probably the easiest vortex element that can be created experimentally and studied theoretically. At the same time a vortex ring is 3-dimensional flow in which the effects connected with curvature of vortex lines become apparent. This fact permits one to investigate directly many problems of vortex dynamic and acoustic including processes of formation of perturbations, transfer to turbulence near the core, the connection of sound field with incompressible vortex dynamic.

As it is known a vortex ring is a vortex torus, moving with ellipsoidal region filled with fluid, called an atmosphere of the ring. In a case of a laminar vortex ring there is potential flow in the region of an atmosphere and in a case of a turbulent vortex ring the region of an atmosphere is filled with turbulent perturbations. The careful description of a model of vortex rings was made in [1], where, first of all, a self-induced movement of a vortex thread, rolling in a ring, was examined. The representation of a vortex as a thin circular thread in an ideal fluid is in accordance with a known hydrodynamic phenomena and it allows one to connect the middle parameters of the vortex ring – its shape and the vorticity with the propagation velocity. In a case of a laminar ring in an ideal fluid these parameters prove to be constant during a movement of the vortex that means the vortex will live unlimitedly long, not changing its form and the velocity.
The existence and the uniqueness of solving for a stationary thin vortex ring were proved by Fraenkel in [2]. The proof relies on mapping of 2D vortex flow with circular current lines to 3D flow with some another distribution of a vorticity inside the core. This mapping is sought as a series in a parameter of thickness determined as a ratio of the core cross-section dimension $a$ to the ring radius $R$ (fig.1) $\mu = \frac{a}{R} \ll 1$, which is supposed to be small. The examples of solving this problem for different distributions of a vorticity in the core of a vortex ring were considered in [3]. In particular, in a case of the uniform vortex ring (accordingly to [4] in uniform distribution the magnitude of the vorticity is proportional to the distance to the ring symmetry axis) the velocity field and the boundary shape were found in two approximations in the parameter of thickness [3]. However, a uniform vortex ring is not the isochronous one: more detail solution shows that the period of rotation of fluid particles inside the core of the vortex appears not to be the same. The main aim of this paper is to find the algorithm of solving the task for the isochronous vortex ring. Importance of this investigation is that all non-trivial eigen-oscillations of the isochronous vortex ring relates to a discrete spectrum while rings with other distributions of a vorticity contain continuous part. This fact means that the isochronous vortex ring is the simplest 3D vortex to study the problem of vortex ring stability.

Because of the fact that concrete 2D flow with circular streamlines which turns into the isochronous vortex ring is unknown in advance, Fraenkel’s procedure needs to be essentially modified on every step connected with calculation a 2D profile of the vorticity. The flow with the isochronous distribution of the vorticity differs from the flow with the uniform one only in the second approximation in the parameter of thickness and it gives an opportunity to choose the uniform distribution as a start point to build the solution for the isochronous ring. In the first section of this paper the uniform solution is built in which the terms are found up to the third approximation in the small parameter. In the second part the flow with the isochronous distribution of the vorticity is found in the same approximation. Higher approximations in the small parameter than was obtained in [4] will be required to investigate eigen-oscillations near the accumulation point.

2. The statement of the problem
The coordinate system moving with the vortex ring is used. Consider the cylindrical coordinates $r, \theta, z$ which coincide with the axis of the vortex ring and the polar coordinates $\rho, \phi$ in the cross-section of the vortex with the centre in the stagnation point (fig.1). Taking into account the axis symmetry of the flow, components of the velocity field are expressed through the $\psi$ called the stream function

$$v(r,z) = \left(-\frac{1}{r}\frac{\partial \psi}{\partial z}, 0, \frac{1}{r}\frac{\partial \psi}{\partial r}\right)$$

Figure 1. Vortex ring.
There is a connection between the stream function and the vorticity in accordance with [1]

$$\psi(r, z) = -\frac{1}{2} W r^2 + \frac{1}{4\pi} \int_0^{2\pi} \int_{0}^{\infty} \omega r \frac{d\sigma}{r^2} \left( \frac{\cos(\theta)}{(z - z')^2 + r^2 + 2rr' \cos(\theta)} \right)^{1/2} d\theta$$  \hspace{1cm} (1.2)

Where $\omega$ is a contravariant component of the vorticity. The velocity $\vec{W}$ of the ring and the region $\sigma$ of the vortex core are defined during solving this problem.

For the stationary field of the vorticity $\Omega = (0, \omega, 0)$ the Helmholtz equation is $(\nabla \times \nabla) \Omega = 0$. Due to the axis symmetry this equation can be rewritten as $\vec{v} \cdot \nabla r \left( \frac{\omega}{r} \right) = 0$, from which the equation

$$\frac{\omega}{r} = F(\psi)$$

follows, where $\psi$ is determined in (1.2). In particular, for the uniform distribution of the vorticity one can obtain $F(\psi) = \text{const}$

In this investigation only thin vortex rings are considered. Determine the parameter of thickness $\varepsilon$ which is supposed to be small, $\varepsilon << 1$. Analogously to [2] determine the coordinates $s, \varphi$, where $\varepsilon RS = \rho$. These coordinates are connected with the cylindrical coordinates with the axis $z$ coinciding with the axis of the ring by the following relations

$$r(\varepsilon, s, \varphi) = R(1 - \varepsilon S \cos \varphi), \quad z(\varepsilon, S, \varphi) = \varepsilon SR \sin \varphi$$  \hspace{1cm} (1.3)

Determine streamlines using parameter $s$

$$S = s + q(\varepsilon, s, \varphi)$$  \hspace{1cm} (1.4)

So that on the streamlines $s = \text{const}$, $0 \leq s \leq 1$, where $s = 1$ corresponds to the core boundary of the vortex ring. Note the streamlines in the first approximation in the case of the thin vortex ring are circular and $q(\varepsilon, s, \varphi) = O(\varepsilon)$

Following [2], introduce the contravariant component of the vorticity in the form $\frac{\omega}{r} = \frac{\Omega(\varepsilon)}{R}$, $s \leq 1$

and represent the stream function as $\psi = \varepsilon^2 R^2 \left( \Psi(s) + B(\varepsilon) \right)$, where

$$S = s + q(\varepsilon, s, \varphi), \quad |s - \hat{s}| = \left( s^2 + \hat{s}^2 - 2s\hat{s} \cos(\varphi - \hat{\varphi}) \right)^{1/2}$$  \hspace{1cm} (1.5)

Where $\vec{W} = \varepsilon^2 RW(\varepsilon)$. Fraenkel proved in [3] that for each two-dimensional distribution of vorticity $\Omega(s)$, the only stationary solutions turn out to exist. Transfer to the coordinates $s, \varphi$ in (1.2)

$$\Psi(s) + B(\varepsilon) = -\frac{1}{2} \frac{\Omega(s)}{R} \left(1 - \varepsilon (s + q) \cos \varphi \right)^2 + \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} G(\varepsilon, s + q, \varphi, \hat{s} + \hat{\varphi}, \hat{\varphi}) \left( 1 + \frac{\hat{g}}{\varepsilon S} \right) \hat{\varphi} d\hat{s} d\hat{\varphi}$$

$$G(\varepsilon, S, \varphi, \hat{S}, \hat{\varphi}) = \frac{1}{R^2} \left( \frac{2}{k} - k \right) \left( \frac{2}{k} - k \right) (E(k) - \frac{2}{k} K(k))$$

$$k = \left( \frac{4r(\varepsilon, S, \varphi) r(\varepsilon, \hat{S}, \hat{\varphi})}{\left( r(\varepsilon, \hat{S}, \hat{\varphi}) + r(\varepsilon, S, \varphi) \right)^2 + \left( z(\varepsilon, \hat{S}, \hat{\varphi}) - z(\varepsilon, S, \varphi) \right)^2} \right)^{1/2}$$
Where \( E(k) \) and \( K(k) \) – are complete elliptic integrals of the first and second kinds. Introduce all initial magnitudes in series expansion in the parameter of thickness \( \varepsilon \). We relate terms \( \varepsilon^n \) and \( \varepsilon \ln \varepsilon \) to the same \( O(\varepsilon^n) \).

\[
B(\varepsilon) = \sum_{m=0}^{\infty} B_m \varepsilon^m
\]

\[
\frac{1}{2} W(\varepsilon) \left( 1 - \varepsilon (s + q) \cos \varphi \right)^2 = \sum_{m=0}^{\infty} W_m \varepsilon^m
\]

\[
q(\varepsilon, s) = \sum_{m=1}^{\infty} q_m \varepsilon^m
\]

3. The algorithm of finding the stationary flow
The condition of constancy of the parameter \( s \) on the streamlines and independence of the left part (1.6) from the coordinate \( \varphi \) allows us to get secular equations to find the initial approximations for the shape of the streamlines of the stationary flow \( q_n(s, \varphi) \), which will be signed for the brevity as \( q_n \) and the constants \( B_n(\varepsilon) \), \( W_n(\varepsilon) \). Getting the secular equations consists in linearization of (1.6) in the small parameter \( \varepsilon \) and this way was described by Fraenkel in [3]. The problem of finding a concrete solution with certain 2 – dimensional distribution of a vorticity \( \Omega(s) \) consists in obtaining the magnitudes \( q, B, W \).

4. Calculating the period of rotation of fluid particles inside the core
It is convenient to use a new parameter \( \mu \), which is defined as a ratio of vortex core equivalent radius

\[
a = \frac{\sqrt{s}}{\pi} \quad (S \text{ is an area of vortex ring core})\text{to the vortex ring radius } R, \text{ so that }\]

\[
\varepsilon(\mu) = \mu - \frac{61}{128} \mu^3 + O(\mu^5).\]

Following [4] find the period of rotation of fluid particles on different streamlines. Further, we define the coordinates \( \sigma(\rho, \varphi), \psi(\rho, \varphi) \) in the core cross-section of the ring, which slightly differ from the coordinates \( \rho, \varphi \), respectively. We require that for the coordinates \( \sigma(\rho, \varphi) \) and \( \psi(\rho, \varphi) \) the following equations to be fulfilled

\[
V_0^\sigma = 0, \quad V_0^\psi = V_0^\psi(\sigma), \quad |g|^{\frac{1}{2}} = \sigma
\]

Where \( V_0^\sigma \), \( V_0^\psi \) are the contravariant components of the velocity field, \( g_{ij} \) is the metric tensor of the coordinate system \( \sigma, \psi, p \), where \( p = R\theta \) (Figure 1).

Making use of the equations

\[
V_0^\sigma = \frac{\partial \sigma}{\partial \rho} V_0^\rho + \frac{\partial \sigma}{\partial \varphi} V_0^\varphi, \quad V_0^\psi = \frac{\partial \psi}{\partial \rho} V_0^\rho + \frac{\partial \psi}{\partial \varphi} V_0^\varphi,
\]

\[
|g(\rho, \varphi, p)|^{\frac{1}{2}} = \frac{\partial(\sigma, \psi)}{\partial(\rho, \varphi)} |g(\sigma, \psi, p)|^{\frac{1}{2}}, \quad \text{where } p = R\theta \quad \text{(Figure 1)}
\]

we can get explicit expressions for \( \sigma, \psi \).

It is not difficult to obtain that on the different streamlines \( \sigma, p = \text{const} \) the period of rotation of fluid particles proves to be
The coordinates $\sigma, \psi$ for the uniform vortex ring ($\Omega(s) = \Omega_0 = \text{const}$) can be get as series in $\mu$

$$\sigma = \rho - \frac{5}{8} \mu \rho^2 \cos \varphi + \mu^2 \left( \frac{45\rho^3}{256a^3} + \left( \frac{3}{8} \rho \ln \frac{8}{\mu} - \frac{15}{64} \rho^3 \right) \cos 2\varphi \right) + \mu^3 \sigma_3 + O(\mu^4)$$

$$\psi = \varphi + \frac{7}{8} \mu \rho \sin \varphi + \mu^2 \left( \frac{15}{64} - \frac{3}{8} \ln \frac{8}{\mu} + \frac{11\rho^2}{128a^2} \right) \sin 2\varphi + \mu^3 \psi_3 + O(\mu^4)$$

Where

$$\sigma_3 = -\frac{\cos \varphi \left( a^2 \rho^2 \left( 96 \ln \frac{8}{\mu} - 60 \right) - 1219 \rho^4 \right) + \cos 3\varphi \left( a^2 \rho^2 \left( 192 \ln \frac{8}{\mu} - 264 \right) - 13 \rho^4 \right)}{4096a^3}$$

$$\psi_3 = -\frac{3 \left[ \sin \varphi \left( a^2 \rho \left( 368 \ln \frac{8}{\mu} - 230 \right) - 356 \rho^3 \right) + \sin 3\varphi \left( a^2 \rho \left( 56 \ln \frac{8}{\mu} - 47 \right) - 4 \rho^3 \right) \right]}{1024a^3}$$

In the coordinates $\sigma, \psi$ for the contravariant components of the velocity field and the metric tensor we obtain

$$V_0^\sigma = 0, \quad V_0^\psi = \frac{\Omega_0}{2} - \frac{64\sigma^2}{64a^2} \mu^2 + O(\mu^4), \quad |g|^{\frac{1}{2}} = \sigma$$

Using (4.1) and (4.2) we get

$$T = \frac{2\pi}{V_0^\psi(\sigma)}$$

As it is seen for the uniform vortex ring the period of fluid particles orbits is not a constant over the whole vortex core cross-section and it is in full accordance with [5]. Thus, the flow with the uniform vorticity is not isochronous.

5. The isochronous vortex ring

Determine the stationary flow for the isochronous vortex ring. In the case of the uniform vortex ring the distribution of the vorticity $\Omega(s)$ is assigned beforehand in accordance with (1.5) but to find the stationary flow of the isochronous vortex ring we need to determine $\Omega(s)$ for this case from the condition of constancy of the contravariant component of the velocity field $V_0^\psi$ on different streamlines $\sigma$, $p = \text{const}$.

Since the period of fluid particle rotation in the uniform vortex ring differs from the constant in the second approximation, an amendment to the vorticity for the isochronous vortex ring has the order $\mu^2$. Obviously, the coordinates $\sigma, \psi$ also will be changed. Determine new coordinates $\sigma', \psi'$, which corresponds to the conditions $V^\sigma = 0$, $V^\psi = \text{const}$ and $|g|^{\frac{1}{2}} = \sigma'$. In the first approximation these coordinates for the isochronous vortex ring match with $\sigma, \psi$ for the uniform vortex ring founded earlier. So we seek the vorticity as $\Omega^\prime = \Omega_0 + \mu^2 \omega_2 (\sigma')$.
Using \( \sigma^* = \rho - \mu \frac{5 \rho^2 \cos \varphi}{8a} + O\left(\mu^3\right) \) we obtain
\[ \Omega_\mu = \Omega_0 + \mu^2 \omega_2 (\rho) + O\left(\mu^3\right) \] (5.1)

It appears to be convenient to find the contravariant components of the velocity field in the following form
\[ V^\alpha = \mu \Omega_0 \frac{5 \rho^2}{16a} \sin \phi + \mu^2 g_2 \sin 2\phi + O\left(\mu^3\right) \]
\[ V^b = \frac{\Omega_0}{2} + \mu \Omega_0 \frac{7 \rho}{16a} \cos \phi + \mu^2 h_2 \cos 2\phi + \mu^2 h_2 + O\left(\mu^3\right) \] (5.2)

Taking into account (5.1), (5.2) and that
\[ \omega_2 (\rho) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho^2 \hat{h}_2\right) \] (5.3)

Analogously to (5.2) we represent the coordinates \( \sigma^*, \psi^* \) as
\[ \sigma^* (\rho, \varphi) = \rho - \mu \frac{5 \rho^2}{8a} \cos \varphi + \mu^2 a_2 \cos 2\varphi + \mu^2 \hat{a}_2 + O\left(\mu^3\right) \]
\[ \psi^* (\rho, \varphi) = \varphi + \mu \frac{7 \rho}{8a} \sin \varphi + \mu^2 b_2 \sin 2\varphi + O\left(\mu^3\right) \] (5.4)

Using the rules of the transformation for the contravariant components of the velocity field, the metric tensor and ((5.1), (5.2), (5.3), (5.4)), we can obtain \( a_2, \hat{a}_2, b_2 \) and \( \hat{h}_2 \). As a result
\[ \omega_2 (\rho) = \frac{21 \rho^2 \Omega_0}{16a^2} \]. Thus, we get
\[ \Omega (s) = \Omega_0 \left(1 + \varepsilon^2 \frac{21}{16} s^2\right) + O\left(\varepsilon^3\right) \] (5.5)

As it is seen, the vorticity inside the core of the vortex ring can be represented as a series in \( \varepsilon : \Omega_0 (s) + \varepsilon \Omega_1 (s) + \ldots \). Then in the left part of (2.4) adding terms \( \Psi_\mu (s) \) appear and they are determined by the expression (1.5) from adding terms of the vorticity \( \Omega_0 (s) + \varepsilon \Omega_1 (s) + \ldots \).

Knowing the distribution of the vorticity in the core we can obtain the boundary shape of the core and the velocity field in the core (the algorithm of finding these magnitudes was described above). Note that the expression for the vorticity (5.5) determines the flow in the isochronous vortex ring up to the third approximation \( T = \frac{4\pi}{\Omega_0} + O\left(\mu^4\right) \).

Introduce also the mapping of the coordinates which corresponds to the isochronous distribution of the vorticity in the vortex ring
\[ \sigma^*_3 = -\frac{\rho^3}{4096a^3} \left[ 3 \left(369 \rho^2 + 20a^2 - 32a^2 \ln \frac{8}{\mu} \right) \cos \varphi + \left(13 \rho^2 + 264a^2 - 192a^2 \ln \frac{8}{\mu}\right) \cos 3\varphi \right] 
\[ \sigma^*_3 = -\frac{\rho^3}{4096a^3} \left[ 3 \left(369 \rho^2 + 20a^2 - 32a^2 \ln \frac{8}{\mu} \right) \cos \varphi + \left(13 \rho^2 + 264a^2 - 192a^2 \ln \frac{8}{\mu}\right) \cos 3\varphi \right] 
\]
\[
\psi^* = \phi + \mu \frac{7 \rho \sin \varphi}{8a} + \mu^2 \left( \frac{11 \rho^2 + 30a^2 - 48a^2 \ln \frac{8}{\mu}}{128a^2} \right) \sin 2\varphi + \mu^3 \psi_3^* + O(\mu^4)
\]

\[
\psi_3^* = \frac{\rho}{1024 a^3} \left( 2 \left( 464 \rho^2 + 345a^2 - 552a^2 \ln \frac{8}{\mu} \right) \sin \varphi + 3 \left( 4 \rho^2 + 47a^2 - 56a^2 \ln \frac{8}{\mu} \right) \sin 3\varphi \right)
\]

6. Conclusion
The vortex ring with uniform and isochronous distributions of the vorticity was examined in this paper. Based on Fraenkel’s procedure for both cases the shape of the streamlines and the velocity field inside the core of the vortex ring are calculated up to \( \mu^3 \).

Fraenkel’s procedure [2,3] which allows to find the stationary flow in a thin vortex ring was essentially modified to determine the flow in the isochronous vortex ring. The main aim of finding the following terms is to obtain more precise expressions for oscillations of the vortex ring to reveal eigen-frequencies nearer the stagnation point obtained in [4]. There is a reasonable assumption that in this region of frequencies a new instability exists [5]. Earlier it was impossible to confirm this assumption because of shortage of finding approximations.

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