On the Singularities in the Susceptibility Expansion for the Two-Dimensional Ising Model

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Abstract

For temperatures below the critical temperature, the magnetic susceptibility for the two-dimensional isotropic Ising model can be expressed in terms of an infinite series of multiple integrals. With respect to a parameter related to temperature and the interaction constant, the integrals may be extended to functions analytic outside the unit circle. In a groundbreaking paper, B. G. Nickel [10] identified a class of singularities of these integrals on the unit circle. In this note we show that there are no other singularities on the unit circle.

I. Introduction

For the two-dimensional zero-field Ising model on a square lattice, the magnetic susceptibility as a function of temperature is usually studied through its relation with the zero-field spin-spin correlation function:

\[ \beta^{-1}\chi = \sum_{M,N \in \mathbb{Z}} \{ \langle \sigma_{0,0} \sigma_{M,N} \rangle - \mathcal{M}^2 \} \]

(1)

where \( \beta = (k_B T)^{-1} \), \( T \) is temperature, \( k_B \) is Boltzmann’s constant and \( \mathcal{M} \) is the spontaneous magnetization (see, e.g., [8]). Fisher [4] in 1959 initiated the analysis of the analytic structure of \( \chi \) near the critical temperature \( T_c \) by relating it to the long-distance asymptotics of the correlation function at \( T_c \) (a result known to Kaufman and Onsager). Subsequently Wu et al. [15] derived the exact form factor expansion of \( \chi \) which has the structure of an infinite series whose \( n \)th order term is an \( n \)-dimensional integral. In later work [13, 16, 12] the structure of the integrands of these \( n \)-dimensional integrals was simplified.
The analysis of $\chi$ as a function of the complex variable $T$ was initiated by Guttmann and Enting \[5\] where, by the use of high-temperature series expansions, they were led to conjecture that $\chi$, as a function of $T$, possesses a natural boundary. In two groundbreaking papers, Nickel \[10, 11\] analyzed the $n$-dimensional integrals appearing in the form factor expansion of $\chi$ and identified a class of complex singularities, now called Nickel singularities, that lie on a curve and which become ever more dense with increasing $n$. This work of Nickel provides very strong support for the existence of a natural boundary for $\chi$. For further developments see Chan et al. \[3\] and the review article \[9\].

We recall that if $T_c$ denotes the critical temperature, then for the isotropic Ising model, where horizontal and vertical interaction constants have the same value $J$, the spontaneous magnetization is given for $T < T_c$ by \[17, 8\]

$$M = (1 - k^2)^{1/8},$$

where $k := (\sinh 2\beta J)^{-2}$; and $M$ is zero for $T > T_c$. Thus $k = 1$ defines the critical temperature $T_c$ and $0 < k < 1$ is the region $0 < T < T_c$. Boukraa et al. \[1\] (building on work of Lyberg and McCoy \[7\]) introduced a simplified model for $\chi$, called the diagonal susceptibility $\chi_d$ which has the following analogous representation to (1):

$$\beta^{-1}\chi_d = \sum_{N \in \mathbb{Z}} \{\langle \sigma_{0,0}\sigma_{N,N} \rangle - M^2 \}.$$ 

By an analysis similar to that of Nickel, they are led to conjecture a natural boundary for $\chi_d$; which in terms of the complex variable $k$, is the unit circle $|k| = 1$. This conjecture thus says that the low temperature phase, $T < T_c$, is separated from the high-temperature phase $T > T_c$ by the natural boundary $|k| = 1$. This conjecture for $\chi_d$ is precisely the same as the conjectured natural boundary for $\chi$. In the low-temperature phase, the present authors proved that $|k| = 1$ is a natural boundary for $\chi_d$ \[14\] thus adding additional support for the conjecture for $\chi$.

We now state the results of this paper. We set

$$s = 1/\sqrt{k} = \sinh 2\beta J,$$

so that the low-temperature phase corresponds to $s > 1$. If we define

$$D(x, y; s) = s + s^{-1} - (x + x^{-1})/2 - (y + y^{-1})/2,$$ 

then we have the expansion

$$\beta^{-1}\chi = 1 - M^2 + 2M^2 \sum_{n=1}^{\infty} \chi^{(2n)},$$

As Nickel noted, for a rigorous proof one must show that there are no cancellations of the singularities in the infinite sum.
where
\[ \chi^{(n)} = \frac{1}{n!} \frac{1}{(2\pi i)^n} \int_{C_r} \cdots \int_{C_r} \frac{\prod_j x_j^{-1} + \prod_j y_j^{-1}}{(1 - \prod_j x_j) (1 - \prod_j y_j)} \prod_{j<k} \frac{x_j - x_k y_j - y_k}{x_j x_k - 1 y_j y_k - 1} \prod_j D(x_j, y_j; s). \]

Here \( C_r \) denotes the circle with center zero and radius \( r < 1 \) and \( r \) sufficiently close to 1 (depending on \( s \)). All indices in the integrand run from 1 to \( n \). A derivation of this representation will be given in Appendix A.

We extend \( \chi^{(n)} \) to a function of the complex variable \( s \) with \( |s| > 1 \). A Nickel singularity is a point \( s^0 \) on the unit circle \( T \) such that the real part of \( s^0 \) is the average of the real parts of two \( n \)th roots of unity.

We shall show that for \( n \) even these are the only singularities of \( \chi^{(n)} \). More precisely, \( \chi^{(n)} \) extends from the exterior of \( T \) to a \( C^\infty \) function on \( T \) except for the Nickel singularities.\(^2\)

Here we use the term “singularity” to denote a point in no neighborhood of which a function is \( C^\infty \). In the physics literature it usually means a point beyond which a function cannot be continued analytically. It appears that \( \chi^{(n)} \) satisfies a linear differential equation with only regular singular points (although the authors admit not having seen a derivation of this that they understand).\(^3\) At a regular singular point the function has a series expansion whose leading term is a fractional or negative power, or a power times a power of the logarithm. Such a function cannot extend from outside \( T \) to be \( C^\infty \) in a neighborhood of that point. Therefore we get the stronger result that for \( n \) even \( \chi^{(n)} \) extends analytically across the unit circle except at the Nickel singularities.

II. Outline of the proof

With the notations
\[ F(x) = \frac{1}{1 - \prod_j x_j}, \quad F(y) = \frac{1}{1 - \prod_j y_j}, \quad F_{jk}(x) = \frac{1}{1 - x_j x_k}, \quad F_{jk}(y) = \frac{1}{1 - y_j y_k}, \]
\[ G_j(x, y; s) = \frac{1}{D(x_j, y_j; s)}, \quad \Delta(x, y) = \left( \prod_j x_j^{-1} + \prod_j y_j^{-1} \right) \prod_{j<k} (x_j - x_k) (y_j - y_k), \]
all thought of as functions on \( \mathbb{R}^n \times \mathbb{R}^n \), the integral becomes
\[ \int_{C_r^n} \int_{C_r^n} F(x) F(y) \prod_{j<k} F_{jk}(x) \prod_{j<k} F_{jk}(y) \prod_j G_j(x, y; s) \Delta(x, y) \, dx \, dy. \]

\(^2\)For \( n \) odd our argument leaves open the possibility of other singularities. See footnote \(^5\).
\(^3\)The equations for \( n \leq 6 \) have been found \([2]\), and all their singularities are regular.
This equals $r^{2n}$ times

$$\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} F(rx) F(ry) \prod_{j<k} F_{jk}(rx) \prod_{j<k} F_{jk}(ry) \prod_j G_j(rx, ry; s) \Delta(rx, ry) \, dx \, dy. \quad (4)$$

A partition of unity allows us to localize. Near any given point $(x^0, y^0) \in \mathbb{T}^n \times \mathbb{T}^n$ some of the $F$-factors may become singular as $r \to 1$, and after letting $r \to 1$ some of the $G$-factors may become singular as $s \to s^0 \in \mathbb{T}$. We represent each of these potentially singular factors as an exponential integral over $\mathbb{R}^+$. The gradient of the exponent in the resulting integrand is approximately a linear combination with positive coefficients of certain vectors, one from each factor. Unless $s^0$ is a Nickel singularity, the convex hull of these vectors does not contain 0, a fact that allows us to find a lower bound for the length of the gradient. (This is the crucial point in the proof\footnote{Each of the limiting factors $F(x_0, y_0)$, $F_{jk}(x_0)$, $F_{jk}(y_0)$, $G_j(x, y; s^0)$ may be interpreted as a distribution on $\mathbb{T}^n \times \mathbb{T}^n$. That 0 is not in the convex hull of the vectors is precisely the condition that allows one to define the product of these distributions as a distribution \[6\]. This is what led us to the present proof.}.) Then several applications of the divergence theorem give the bound $O(1)$ for the integral, uniformly in $s$ and $r$. The same is true after differentiating with respect to $s$ any number of times. This will imply that $\chi^{(s)}$ extends to a $C^\infty$ function on $\mathbb{T}$ excluding these points.

### III. The proof

For a given point $(x^0, y^0) = ((x^0_j), (y^0_j)) \in \mathbb{T}^n \times \mathbb{T}^n$ some of the factors in (4) become singular as $r \to 1$ and $s \to s^0$, as described above. For example $F(rx)$ becomes singular when $\prod_j x^0_j = 1$ and $G_j(rx, ry; s)$ becomes singular when

$$\Re x^0_j + \Re y^0_j = 2 \Re s^0.$$

There is a neighborhood of $(x^0, y^0)$ in which no other factors become singular, so that outside this neighborhood the rest of the integrand is a smooth function of $x$ and $y$ and bounded for $s$ in a neighborhood of $s^0$, together with each of its derivatives with respect to $s$. Let $\psi(x, y)$ be a $C^\infty$ function with support in this neighborhood. (Eventually the support will be taken even smaller.) We shall show that the integral (4), with the function $\psi(x, y)$ inserted in the integrand, is uniformly bounded for $s$ in a neighborhood of $s^0$, together with each derivatives with respect to $s$, when $r$ is taken close enough (depending on $s$) to 1.

In our neighborhood we make the variable changes

$$x_j = x^0_j e^{i\theta_j}, \quad y_j = y^0_j e^{i\varphi_j}.$$
Below we give the behavior of the reciprocals of the $F$-factors, in terms of the $\theta_j$, $\varphi_j$, if the factors become singular at $(x^0, y^0; s^0)$.

\[
1/F(rx) = -i \sum_j \theta_j + O\left((1 - r) + \sum \theta_j^2\right),
\]

\[
1/F(ry) = -i \sum_j \varphi_j + O\left((1 - r) + \sum \varphi_j^2\right),
\]

\[
1/F_{jk}(rx) = -i (\theta_j + \theta_k) + O\left((1 - r) + \theta_j^2 + \theta_k^2\right),
\]

\[
1/F_{jk}(ry) = -i (\varphi_j + \varphi_k) + O\left((1 - r) + \varphi_j^2 + \varphi_k^2\right).
\]

We note that the real parts of these reciprocals are at least $1 - r$, and so are all positive.

For any singular $G$-factors we have

\[
i/G_j(rx, ry; s) = -i (\alpha_j \theta_j + \beta_j \varphi_j) - i [s + s^{-1} - (s^0 + s^{0-1})] + O\left((1 - r) + \theta_j^2 + \varphi_j^2\right),
\]

where

\[
\alpha_j = \text{Im} x_j^0, \quad \beta_j = \text{Im} y_j^0.
\]

The reason we put the factor $i$ on the left is that now the real part of the right side, which is equal to the imaginary part of the expression in brackets, is positive when $\text{Im} s > 0$ and $r$ is sufficiently close to 1 (depending on $s$). This we assume. (Otherwise we replace the factor $i$ by $-i$ and change signs in the definitions of $\alpha_j$ and $\beta_j$.)

All estimates are consistent with differentiation. For example, the result of differentiating $1/F(rx)$ with respect to $\theta_k$ is $-i + O((1 - r) + \sum |\theta_j|)$.

In what follows we exclude $s^0 = \pm 1, \pm i$, which are Nickel singularities for even $n$. Thus we assume $(\alpha_j, \beta_j) \neq (0, 0)$.

Because all real parts of the reciprocals are positive they may be represented as integrals over $\mathbb{R}^+$. Thus, we have for any singular factor,

\[
F(rx) = \int_{\mathbb{R}^+} e^{i\xi (\sum_j \theta_j + \text{correction})} d\xi,
\]

\[
F(ry) = \int_{\mathbb{R}^+} e^{i\eta (\sum_j \varphi_j + \text{correction})} d\eta,
\]

\[
F_{jk}(rx) = \int_{\mathbb{R}^+} e^{i\xi_{jk} (\theta_j + \theta_k + \text{correction})} d\xi_{jk},
\]

\[
F_{jk}(ry) = \int_{\mathbb{R}^+} e^{i\eta_{jk} (\varphi_j + \varphi_k + \text{correction})} d\eta_{jk}.
\]
\[ F_{jk}(ry) = \int_{\mathbb{R}^+} e^{i\eta_{jk} (\varphi_j + \varphi_k + \text{correction})} d\eta_{jk}, \]
\[ G_j(x; y; s) = i \int_{\mathbb{R}^+} e^{i\zeta_j (\alpha_j \theta_j + \beta_j \varphi_j + s + s^{-1} - s^0 - s^{0-1} + \text{correction})} d\zeta_j. \]

In all of these, “correction” denotes \( i \) times the \( O \) terms above.

Thus, the integral (4) is replaced by one in which the cut-off function \( \psi(x, y) \) is inserted into the integrand and each eventually singular factor is replaced by an integral over \( \mathbb{R}^+ \). Denote the number of these factors (and so the number of \((\xi, \eta, \zeta)\)-integrations) by \( N \). We change the order of integration and integrate first with respect to the \( \theta_j, \varphi_j \). We want to apply the divergence theorem so that we eventually get a bound \( O(R^{-N-1}) \), where \( R \) is the radial variable in the \( N \)-dimensional \((\xi, \eta, \zeta)\)-space.

To do this we have to find a lower bound for the length of the gradient of the sum of the exponents coming from the \((\xi, \eta, \zeta)\)-integrations.

We define the following vectors in \( \mathbb{R}^n \times \mathbb{R}^n \):
\[
X = (1 \ 1 \ \cdots \ 1 \ 0 \ 0 \ \cdots \ 0) \\
Y = (0 \ 0 \ \cdots \ 0 \ 1 \ 1 \ \cdots \ 1) \\
X_{jk} = (0 \ \cdots \ 1 \ \cdots \ 1 \ \cdots \ 0 \ 0 \ \cdots) \\
Y_{jk} = (\cdots \ 0 \ 0 \ \cdots \ 1 \ \cdots) \\
Z_j = (0 \ \cdots \ \alpha_j \ 0 \ \cdots \ \beta_j \ \cdots \ 0). 
\]

Let us explain. The first \( n \) components are the \( \theta_j \) components, the last \( n \) the \( \varphi_j \) components. For \( X \) the ones are the first \( n \) components and the zeros are the rest, and for \( Y \) these are reversed. For \( X_{jk} \) the ones are components \( j \) and \( k \) and the others are zero, and for \( Y_{jk} \) the ones are components \( n + j \) and \( n + k \) and the others are zero. For \( Z_j \) component \( j \) is \( \alpha_j \) and component \( n + j \) is \( \beta_j \), and the others are zero.

Aside from the factor \( i \) and the correction term from each summand, the gradient of the sum of the exponents is the subsum of
\[
\xi X + \eta Y + \sum_{j<k} \xi_{jk} X_{jk} + \sum_{j<k} \eta_{jk} Y_{jk} + \sum \zeta_j Z_j \tag{5} 
\]
containing the \( N \) \((\xi, \eta, \zeta)\)-variables that actually appear.

**Lemma 1.** Suppose that \( n \) is even and that \( s^0 \) is not a Nickel singularity. Then 0 is not in the convex hull of those of the vectors \( X, Y, X_{jk}, Y_{jk}, Z_j \) that appear in the subsum of (5).

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Proof. We show that if a linear combination of these vectors with nonnegative coefficients is zero, but not all the coefficients are zero, then $s^0$ is a Nickel singularity. We say that a vector “appears” in the linear combination if its coefficient is nonzero. Some $Z_j$ must appear since all the others have nonnegative components and at least one positive component. (Recall that $Z_j$ appears when $\Re e x^0_j + \Re e y^0_j = 2 \Re e s^0$.)

If $X_{jk}$ appears then so must $Z_j$ and $Z_k$ and $\alpha_j, \alpha_k < 0$, to cancel the nonzero components of $X_{jk}$. But $X_{jk}$ appears only when $x^0_j x^0_k = 1$, so $\alpha_j + \alpha_k = 0$, which is a contradiction. Thus no $X_{jk}$ appears. Similarly no $Y_{jk}$ appears.

Since some $Z_j$ appears either $X$ or $Y$ must. Suppose that $X$ appears. (In particular $\prod x^0_j = 1$.) Then all $\alpha_j < 0$, and if the coefficient of $X$ is $c_X$ the coefficient of $Z_j$ must be $-c_X/\alpha_j$.

There are two subcases:

(i) $Y$ appears: (In particular $\prod y^0_j = 1$.) In analogy with the above, if the coefficient of $Y$ is $c_Y$ then the coefficient of $Z_j$ is $-c_Y/\beta_j$. Thus $\alpha_j/\beta_j = c_X/c_Y$ for all $j$. We claim that this implies that all $x_j$ are equal and all $y_j$ are equal. Consider pairs $(x, y)$ with both in the lower half-plane, and $\Re e x + \Re e y = 2 \Re e s^0$. Set $x = e^{i\theta}$, $y = e^{i\varphi}$.

It is an exercise in calculus to show that as $\theta$ increases while $\cos \theta + \cos \varphi$ remains constant the ratio $\Im m x/\Im m y = \sin \theta / \sin \varphi$ strictly decreases if $\Re e s^0 > 0$ and strictly increases if $\Re e s^0 < 0$. Therefore this ratio determines $\theta$, and so $x$. Similarly the ratio determines $y$. So all $x_j$ are equal and all $y_j$ are equal, as claimed. They must both be $n$th roots of unity, so $s^0$ is a Nickel singularity.

(ii) $Y$ does not appear: Since all $Z_j$ appear, we must have all $\beta_j = 0$ in this case. So all $y^0_j = \pm 1$. If some $y^0_j = 1$ then $\Re e s^0 > 0$, because if $\Re e s^0$ were negative it could not be the average of 1 and some $\Re e x^0_j$. Then all $y^0_j = 1$, for the same reason. Hence each $\Re e x_j = 2 \Re e s - 1$, and since all $\alpha_j < 0$ this implies that all $x_j$ are equal, and equal to some $n$th root of unity. Thus $s^0$ is a Nickel singularity. If some $y^0_j = -1$, and therefore all $y_j = -1$, this again implies that all $x_j$ equal some $n$th root of unity. Since $n$ is even $s^0$ is again a Nickel singularity. □

If 0 is not in the convex hull of vectors then there is a lower bound for linear combinations of them with nonnegative coefficients, even when the vectors are perturbed.

Lemma 2. Assume 0 is not in the convex hull of the vectors $V_1, \ldots, V_N$. Then for sufficiently small $\varepsilon > 0$ there is a $\delta > 0$ such that, for vectors $U_j$ with $|U_j - V_j| < \varepsilon$ and coefficients $c_j \geq 0$, we have

\[
\left| \sum_j c_j U_j \right| \geq \delta \sum_j c_j. \tag{6}
\]

Proof. Suppose the result is not true. Then there is a sequence $\varepsilon_k \to 0$, vectors $U_{j,k}$

Since $-1$ is not an $n$th root of unity when $n$ is odd, these $s^0$ are not Nickel singularities.
with \(|U_{j,k} - V_j| \leq \varepsilon_k\), and coefficients \(c_{j,k} \geq 0\) such that for each \(k\),

\[
\left| \sum_j c_{j,k} U_j \right| < \frac{1}{k} \sum_j c_{j,k}.
\]

By homogeneity we may assume that each \(\sum_j c_{j,k} = 1\). Then, by taking subsequences, we may assume that each \(c_{j,k}\) converges as \(k \to \infty\) to some \(c_j\). Then \(\sum_j c_j = 1\), and each \(U_{j,k} \to V_j\), so \(\sum_j c_j V_j = 0\). This is a contradiction. 

**Lemma 3.** Assume \(n\) is even and \(s^0\) is not a Nickel singularity. There is a neighborhood of \((x^0, y^0)\) such that if \(\psi(x, y)\) is a \(C^\infty\) function with support in that neighborhood then the integral (4), with \(\psi\) inserted in the integrand and \(r\) sufficiently close to 1 (depending on \(s\)), is bounded in a neighborhood of \(s = s^0\); and the same is true for each derivative with respect to \(s\).

**Proof.** With the same \(\psi\) as above, we combine Lemmas 1 and 2 to deduce that if \(r\) is close enough to 1, and the support of \(\psi\) is small enough, then the length of the gradient of the exponent in the integral is at least a constant times the sum of the coefficients in the subsum of (5) that arises. Therefore \(N + 1\) applications of the divergence theorem shows that the integral over the \(\theta_j, \varphi_j\) has absolute value at most a constant times \(1/R^{N+1}\), where \(R\) is the radial variable in the \(N\)-dimensional \((\xi, \eta, \zeta)\)-space.

Therefore the integral (4) with \(\psi(x, y)\) inserted in the integrand, which results after integration over the \((\xi, \eta, \zeta)\), is \(O(1)\) uniformly for \(s\) in a neighborhood of \(s^0\). (The integral over \(R < 1\) is clearly bounded.) Differentiating with respect to \(s\) any number of times just brings down powers of the \(\zeta_j\), and so only requires more applications of the divergence theorem.

**THEOREM.** When \(n\) is even \(\chi^{(n)}\) extends to a \(C^\infty\) function on \(\mathbb{T}\) except at the Nickel singularities.

**Proof.** Assume \(s^0\) is not a Nickel singularity. Each \((x^0, y^0)\) has a neighborhood given by Lemma 3. Finitely many of these neighborhoods, cover \(\mathbb{T}^n \times \mathbb{T}^n\). We can find a \(C^\infty\) partition of unity \(\{\psi_i(x, y)\}\) such that the support of each \(\psi_i\) is contained in one of these neighborhoods. Each integral (4) with \(\psi_i(x, y)\) inserted in the integrand and \(r\) sufficiently close to 1, together with each derivative with respect to \(s\), is bounded in a neighborhood of \(s = s^0\). Therefore the same is true of (4) itself, and therefore for \(r^n\) times (4), which is independent of \(r\), and therefore for \(\chi^{(n)}\). This implies that \(\chi^{(n)}\) extends to a \(C^\infty\) function on \(\mathbb{T}\) in a neighborhood of \(s^0\). 

\(^6\)We explain this in Appendix B.

\(^7\)We explain this in Appendix C.
Appendix A

For $T < T_c$ and $N \geq 0$ we have the following Fredholm determinant representation of the spin-spin correlation function (see [13, p. 375] or [12, p. 142]):

$$\langle \sigma_{00} \sigma_{MN} \rangle = \mathcal{M}^2 \det(I + g_{MN}).$$

The operator has kernel

$$g_{MN}(\theta_1, \theta_2) = e^{iM\theta_1 - N\gamma(e^{i\theta_1})} h(\theta_1, \theta_2),$$

where

$$h(\theta_1, \theta_2) = \frac{\sinh \frac{1}{2}(\gamma(e^{i\theta_1}) - \gamma(e^{i\theta_2}))}{\sin \frac{1}{2}(\theta_1 + \theta_2)},$$

and $\gamma(z)$ is defined by

$$\cosh \gamma(z) = s + s^{-1} - (z + z^{-1})/2,$$

with the condition that $\gamma(z)$ is real and positive for $|z| = 1$. The operator acts on $L^2(-\pi, \pi)$ with weight function

$$\frac{1}{2\pi \sinh \gamma(e^{i\theta})}.$$

Using the identity (see [13, (5.5)] or [12, (2.69)])

$$\det(h(\theta_j, \theta_k)) = \prod_{j<k} [h(\theta_j, \theta_k)]^2,$$

and the Fredholm expansion we obtain that $\langle \sigma_{00} \sigma_{MN} \rangle$ equals

$$\mathcal{M}^2 \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{1}{(2\pi)^{2n}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{j<k} [h(\theta_j, \theta_k)]^2 \prod_{j} e^{iM\theta_j - N\gamma(e^{i\theta_j})} \frac{d\theta_j}{\sinh \gamma(e^{i\theta_j})}. \quad (7)$$

Here all indices run from 1 to $2n$. We used the fact that since the matrix $(h(\theta_j, \theta_k))$ is antisymmetric its odd-order determinants vanish.

We have the identity, observed in [11],

$$\frac{\sinh \left(\frac{1}{2}(\gamma(e^{i\theta_1}) - \gamma(e^{i\theta_2}))\right)}{\sin \left(\frac{1}{2}(\theta_1 + \theta_2)\right)} = \frac{\sin \left(\frac{1}{2}(\theta_1 - \theta_2)\right)}{\sinh \left(\frac{1}{2}(\gamma(e^{i\theta_1}) + \gamma(e^{i\theta_2}))\right)}$$

Therefore, with $x_j = e^{i\theta_j}$,

$$[h(\theta_1, \theta_2)]^2 = \frac{e^{-\gamma(x_1)} - e^{-\gamma(x_2)}}{1 - e^{-\gamma(x_1) - \gamma(x_2)}} \frac{x_1 - x_2}{1 - x_1 x_2}.$$
With $D(x, y; s)$ defined by (2) a short calculation shows that

$$y \, D(x, y; s) = -\frac{1}{2} (y - e^{-\gamma(x)}) (y - e^{\gamma(x)}).$$

Thus inside the unit circle $1/(y \, D(x, y; s))$ has a pole at $y = e^{-\gamma(x)}$ with residue $1/\sinh \gamma(x)$. It follows that

$$\frac{1}{(2\pi)^{2n}} \int_{C_r} \cdots \int_{C_r} \prod_{j<k} \frac{y_j - y_k}{1 - y_j y_k} \prod_j \frac{y_j^{N-1} \, dy_j}{D(x_j, y_j; s)} = \prod_j \frac{e^{-N\gamma(x_j)}}{\sinh \gamma(x_j)} \prod_{j<k} \frac{e^{-\gamma(x_j)} - e^{-\gamma(x_k)}}{1 - e^{-\gamma(x_j)} - e^{-\gamma(x_k)}}.$$

We deduce that the integral in (7) equals

$$\frac{1}{(2\pi)^{2n}} \int_{C_r} \cdots \int_{C_r} \prod_{j<k} \frac{y_j - y_k}{1 - y_j y_k} \prod_j \frac{x_j^M \, y_j^N}{D(x_j, y_j; s)} \prod_j \frac{dx_j \, dy_j}{x_j \, y_j}. \quad (8)$$

It remains to compute

$$\sum_{M,N \in \mathbb{Z}} \left\{ \langle \sigma_{0,0} \sigma_{M,N} \rangle - \mathcal{M}^2 \right\}.$$

Since $\langle \sigma_{0,0}^2 \rangle = 1$ the $(0,0)$ term equals $1 - \mathcal{M}^2$. For the remaining terms, subtracting $\mathcal{M}^2$ in the summand is the same as taking the sum in (7) only over $n > 0$.

To compute the sum over $(M, N) \neq (0,0)$ we use the fact that $\langle \sigma_{0,0} \sigma_{M,N} \rangle$ is even in $M$ and in $N$, so

$$\sum_{(M,N)\neq(0,0)} = 4 \sum_{M,N \geq 0} -2 \sum_{M=0,N \geq 0} -2 \sum_{N=0,M \geq 0}$$

and find that after summing, the factor $\prod_j x_j^M \, y_j^N$ in the integrand in (8) gets replaced by

$$2 \frac{\prod_j x_j + \prod_j y_j}{(1 - \prod_j x_j) (1 - \prod_j y_j)}.$$

This gives (3).

**Appendix B**

Suppose $f$ and $g$ are $C^\infty$ functions on $\mathbb{R}^d$, with $f$ having compact support, and we have an integral

$$\int f(\theta) \, e^{g(\theta)} \, d\theta.$$
We write it as
\[
\int f(\theta) \frac{\nabla g(\theta)}{|\nabla g(\theta)|^2} \cdot \nabla e^{g(\theta)} \, d\theta.
\]

If define the operator \( L \) by
\[
(Lf)(\theta) = -\nabla \cdot f(\theta) \frac{\nabla g(\theta)}{|\nabla g(\theta)|^2},
\]
then \( q \) applications of the divergence theorem show that the integral equals
\[
\int (L^q f)(\theta) e^{g(\theta)} \, d\theta.
\]

Now we have
(a) \( L^q f \) is a linear combination of (partial) derivatives of \( f \) with coefficients that are homogeneous polynomials of degree \( q \) in derivatives of the components of \( \nabla g/|\nabla g| \);
(b) each \( p \)th derivative of each component of \( \nabla g/|\nabla g| \) equals \( 1/|\nabla g|^{2p+2} \) times a homogeneous polynomial of degree \( 2p + 1 \) in derivatives of \( g \).

Assume that we also have
(c) \( |\nabla g(\theta)| \geq \mu \) and each derivative of \( g(\theta) \) is \( O(\mu) \);
(d) each derivative of \( f(\theta) \) is \( O(1) \).

Then assuming that \( \Re e g \) is uniformly bounded above, we can conclude that
\[
\int_{\mathbb{R}^d} f(\theta) e^{g(\theta)} \, d\theta = O(\mu^{-q}) \quad \text{for all } q.
\]

In the application in Lemma 3 we have \( d = 2n \), \( g \) is the sum of the exponents in the integrals, \( f \) is the product of other integrands, and \( \mu \) can be taken to be a small constant times the sum of the coefficients in the subsum of (5).

**Appendix C**

Suppose \( \mathcal{U} \) is an open set in \( \mathbb{T} \), that \( f \) is analytic in the region
\[
\Omega = \{ Rs : s \in \mathcal{U}, \ 1 < R < 1 + \delta \},
\]
and that \( f \) and each of its derivatives is bounded in \( \Omega \). We show that \( f \) extends to a \( C^\infty \) function on \( \Omega \cup \mathcal{U} \).

Pick any \( s_0 \in \Omega \). We have for each \( k \geq 0 \) and \( s' \in \Omega \),
\[
f^{(k)}(s') = f^{(k)}(s_0) + \int_{s_0}^{s'} f^{(k+1)}(t) \, dt.
\]
Since $f^{(k+1)}$ is bounded, this shows that that $f^{(k)}$ extends continuously to $\Omega \cup U$. Denote by $f_k(s)$ this extension. In particular $f_0$ is the continuous extension of $f$. We show that it belongs to $C^\infty$.

We show by induction that $f_0 \in C^k$. We know this for $k = 0$. Assuming this for $k$, we see that for $s \in U$,

$$\frac{d^k}{ds^k} f_0(s) = \lim_{s' \to s} \frac{d^k}{ds^k} f(s') = f^{(k)}(s_0) + \int_{s_0}^s f_{k+1}(t) \, dt.$$

It follows that $f_0$ is $k + 1$ times differentiable and

$$\frac{d^{k+1}}{ds^{k+1}} f_0(s) = f_{k+1}(s) = \lim_{s' \to s} \frac{d^{k+1}}{ds^{k+1}} f(s').$$

This gives the assertion.

**Acknowledgments**

That authors thank Tony Guttmann, Masaki Kashiwara, Jean-Marie Maillard, Bernie Nickel, Jacques Perk, and, especially, Barry McCoy for helpful communications.

This work was supported by National Science Foundation grants DMS–1207995 (first author) and DMS–0854934 (second author).

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