LHC production of forward-center and forward-forward di-jets in the \( k_t \)-factorization un-integrated parton distribution frameworks

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Abstract

The present work is devoted to study the high-energy QCD events, such as the di-jet productions from proton-proton inelastic collisions at the LHC in the forward-center and the forward-forward configurations, using the unintegrated parton distribution functions (UPDF) in the $k_t$-factorization framework. The UPDF of Kimber et. al. (KMR) and Martin et.al. (MRW) are generated in the leading order (LO) and next-to-leading order (NLO), using the Harland – Lang et al. (MMHT2014) PDF libraries. While working in the forward-center and the forward-forward rapidity sectors, one can probe the parton densities at very low longitudinal momentum fractions ($x$). Therefore, such a computation can provide a valuable test-field for these UPDF. We find very good agreement with the corresponding di-jet production data available from LHC experiments. On the other hand, as we have also stated in our previous works, (i.e. the protons longitudinal and transverse structure function as well as hadron-hadron LHC W/Z production), the present calculations based on the KMR prescriptions show a better agreement with the corresponding experimental data. This conclusion is achieved, due to the particular visualization of the angular ordering constraint (AOC), despite the fact that the LO-MRW and the NLO-MRW formalisms both employ better theoretical descriptions of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation, and hence are expected to produce better results. The form of the AOC in the KMR prescription automatically includes the re-summation of the higher-order $\ln(1/x)$ type contributions, i.e. the Balitski-Fadin-Kuraev-Lipatov (BFKL) logarithms, in the LO-DGLAP evolution equation.

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Analyzing the raw data, which comes pouring out of the LHC, presents a challenge of considerable proportions, given that the dynamics of the true players in the hadronic inelastic collisions, i.e. partons, are shadowed by the laws of strong interactions. However, to understand the nature of our universe, it is paramount to enlighten the behavior of these fundamental substances. Amazingly, an answer came a few decades ago, in the form of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations,\(^1\)\(^-\)\(^4\),

\[
\frac{d}{d \log(Q^2)} g(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} \left[ P_{gg}^{(LO)}(\frac{x}{z}) g(z, Q^2) + P_{qg}^{(LO)}(\frac{x}{z}) \sum_q q(z, Q^2) \right],
\]

\[
\frac{d}{d \log(Q^2)} q(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} \left[ P_{qg}^{(LO)}(\frac{x}{z}) g(z, Q^2) + P_{qq}^{(LO)}(\frac{x}{z}) q(z, Q^2) \right].
\]

(1)

g(x, Q^2) and q(x, Q^2) as the solutions of the DGLAP evolution equations, are single-scale parton density functions (PDF), corresponding respectively to gluons and quarks. They depend on the fraction of the longitudinal momentum of parent hadron (\(x\)) and an ultra-violet cutoff (\(Q^2\)), which denotes the virtuality of the particle that is being exchanged throughout the inelastic scattering (IS). \(P_{ab}^{(LO)}\) are the LO splitting functions (see the section II). \(\alpha_s\) represents the LO running coupling constant of the strong interaction, conventionally approximated as:

\[
\alpha_s(Q^2) \simeq \frac{12\pi}{(33 - 2n_f) \log(Q^2/\Lambda_{QCD}^2)},
\]

where \(n_f\) is the number of involving flavors in the given strong interaction and \(\Lambda_{QCD}\) is the QCD fundamental low energy scale. The value of the \(\Lambda_{QCD}\) can be effectively extracted from experiment, around 300 MeV. The terms on the right-hand side of the equation (1), correspond to the real emission and the virtual contributions, respectively.

The main postulation in the DGLAP evolution equation, i.e. the strong ordering hypothesis, is to neglect the transverse momenta of the partons along the evolution ladder, and to sum over the \(\alpha_s \ln(Q^2)\) contributions. One finds out that neglecting the contributions that come from this transverse dependency may harm the precision of the calculations, particularly in the high-energy processes and in the small-\(x\) region\(^1\)\(^-\)\(^5\). Hence, the need for introducing some transverse momentum dependent (TMD) evolution equation becomes apparent. This gave rise to the Ciafaloni-Catani-Fiorani-Marchesini (CCFM) and the Balitski-Fadin-Kuraev-Lipatov (BFKL) evolution equations\(^16\)\(^-\)\(^25\).
One of the main features of the $CCFM$ evolution equation is that it employs a physical constraint, to ensure that the gluons emissions are accompanied by constant increase in the angle of the emission. This feature which is known as the angular ordering constraint ($AOC$), is related to the color coherent radiations of the gluons. The solutions of the $CCFM$ equation, $f(x, k_t^2, \mu^2)$ is a double-scaled $TMD \, PDF$, which in addition to the $x$ and $Q$, depends on the transverse momentum of the incoming partons, $k_t$. The idea behind the $CCFM$ evolution equation (to make the use of the $AOC$ in the evolution ladder) is valid only in the case of gluon-dominant processes, i.e. in the small-$x$ sector. If the proper physical boundaries are inserted, the $CCFM$ equation will reduce to the conventional $DGLAP$ and $BFKL$ evolutions [26].

Mathematically speaking, solving the $CCFM$ equation is rather difficult, usually possible with the help of Monte Carlo event generators, references [27, 28]. On the other hand, the main feature of the $CCFM$ equation, i.e. the $AOC$, can be used only for the gluon evolution and therefore, producing convincing quark contributions in this framework is only a recent development, see the references [29, 31]. Given these complexities, Martin et al, employed the idea of last-step evolution along the $k_t$-factorization framework, [5, 10], and developed the Kimber-Martin-Ryskin (KMR) and the Martin-Ryskin-Watt (MRW) approaches [11, 12]. Both of these formalisms are constructed around the solutions of the $LO \, DGLAP$ evolution equations and modified with different visualizations of the angular ordering constraint. Although the $unintegrated$ parton distribution functions ($UPDF$) of $MRW$ in the leading order ($LO$) and next-to-leading order ($NLO$) have been defined to improve the compatibility of the $KMR$ approach with the theory of the $LO \, DGLAP$ and extend it to a higher order $QCD$, the recent work suggests that the $KMR$ framework is more successful (or at least as successful) in describing experimental data, see for example the references [32, 40]. Nevertheless, to utter a rigid statement on this matter, further investigation is required.

One extraordinary test-ground for the $UPDF$ of the $k_t$-factorization is the probe of the forward-center and forward-forward rapidity sectors in the hadronic collisions, given that it involves the dynamics of the small-$x$ region, e.g. $x \sim 10^{-4} - 10^{-5}$, where the gluon density dominates. Since the decisive difference between the $UPDF$ of $KMR$ and $MRW$ is in the different manifestations of the $AOC$, one could argue that working in such phenomenological setups could potentially exploit this diversity and unveil the true capacities of the presumed
frameworks. For this propose, we have calculated the process of production of di-jets in the inelastic proton-proton collisions from the forward-center and the forward-forward rapidity regions, utilizing the UPDF of KMR and MRW in the LO and the NLO. Comparing these results with each other, and the results of the similar calculations in other frameworks, namely the linear and non-linear KS formalisms, and with the experimental data from the CMS collaboration, would provide an excellent opportunity to study the strength and the weaknesses of the UPDF in the \( k_t \)-factorization framework.

The outlook of this paper is as follows: In the section II we present a brief introduction to the framework of \( k_t \)-factorization and develop the required prescriptions for the KMR and the MRW UPDF, stressing their key differences regarding the involvement of the AOC in their definitions. The UPDF will be prepared in their proper \( k_t \)-factorization schemes using the PDF of Harland – Lang et al. \((\text{MMHT}2014)\) in the LO and the NLO. The section III contains a comprehensive description over the utilities and the means for the calculation of the \( k_t \)-dependent cross-section of the di-jets production in the p-p IS processes. The necessary numerical analysis will be presented in the section IV, after which a thorough conclusion will follow in section V.

II. \textit{The \textit{UPDF Calculations in the \( k_t \)-Factorization Framework}}

During a high energy hadronic collision, the involving partons, i.e. the partons that appear at the top of their respective evolution ladders, carry some inherently induced transverse momentum, as the remnant of the successive (an potentially infinite) number of evolution steps. When working within the framework of collinear factorization, such transverse momentum dependency is conventionally neglected, due to the assumption of the strong ordering that is embedded in the \textit{LO DGLAP} evolution equation,

\[
    k_{t,i-2}^2 \ll k_{t,i-1}^2 \ll k_{t,i}^2 \ll \cdots \ll k_{t,n}^2 \ll \mu^2.
\]

Avoiding such assumption, one can include the contributions coming from the transverse momentum distributions of the partons, using either the solutions of the \textit{CCFM} evolution equation or unify the \textit{BFKL} and the \textit{DGLAP} single-scaled evolution equations to form a properly tuned \( k_t \)-dependent framework. Utilizing these methods does not always come easy, since these frameworks are mathematically complex and in the case of \textit{CCFM},
not enough to include all of the contributing sub-processes. Alternatively, the single-scaled PDF of the DGLAP evolution equation can be convoluted with the required $k_t$-dependency during the last step of the evolution \cite{14}, postulating that:

$$k_{t,i-2}^2 \ll k_{t,i}^2 \ll \cdots \ll k_{t,n}^2 \sim \mu^2.$$  

Consequently, one may use the defining identity of the $k_t$-factorization,

$$a(x, \mu^2) = \int \mu^2 \frac{dk_t^2}{k_t^2} f_a(x, k_t^2, \mu^2),$$  \hspace{1cm} (2)

to define the UPDF, $f_a(x, k_t^2, \mu^2)$, with $a(x, \mu^2)$ being the solutions of the DGLAP equation times $x$ (i.e. $xq(x, Q^2)$ and $xg(x, Q^2)$). we should make this comment here that in the more precise definition, one should use the generalized UPDF \cite{5–10}, i.e. the double-UPDF (DUPDF), such that they take into account both quarks and gluons. Then we should write (compare with equation (2)):

$$a(x, \mu^2) = \int \mu^2 \frac{dk_t^2}{k_t^2} \int_x^1 f_a(x, z, k_t^2, \mu^2).$$

However, in this work we continue our calculations by using the UPDF. Afterwards, one can easily derive the direct expressions for the UPDF of the $k_t$-factorization, $f_a(x, k_t^2, \mu^2)$. Furthermore, in order to avoid the soft-gluon singularities, it is necessary to impose some physical constraint into this definition in the form of the AOC. Naturally, imposing different visualizations of the AOC will from different formalisms for the UPDF.

The first choice is the so called the KMR prescription. Introducing the virtual (loop) contributions via the Sudakov form factor,

$$T_a(k_t^2, \mu^2) = \exp \left( - \int \frac{\alpha_s(k^2)}{2\pi} \frac{dk^2}{k^2} \sum_{b=q,g} \int_{0}^{1-\Delta} dz P_{ab}^{(LO)}(z') \right),$$  \hspace{1cm} (3)

and utilizing the LO splitting functions, $P_{ab}^{(LO)}(z = x/x')$,

$$P_{gg}^{(LO)}(z) = 6 \left( z(1-z) + \frac{1-z}{z} + \frac{z}{1-z} \right),$$

$$P_{qq}^{(LO)}(z) = \frac{4}{3} \left( \frac{1+z^2}{1-z} \right),$$

$$P_{qg}^{(LO)}(z) = \frac{1}{2} \left( z^2 + (1-z)^2 \right),$$

$$P_{gq}^{(LO)}(z) = 4 \frac{1+(1-z)^2}{z} \hspace{1cm} (4)$$
as the probability of the emission of a parton \( a \) (with the longitudinal momentum fraction \( x \)) from a parent parton \( b \) (with the longitudinal momentum fraction \( x' \)), Kimber et al have defined the UPDF of KMR as follows:

\[
    f_a(x, k_t^2, \mu^2) = T_a(k_t^2, \mu^2) \sum_{b=q,g} \left[ \frac{\alpha_s(k_t^2)}{2\pi} \int_x^{1-\Delta} dz P_{ab}^{(LO)}(z) b \left( \frac{x}{z}, k_t^2 \right) \right].
\]  

(5)

The LO splitting functions parameterize the probability of evolving from a scale \( k_t \) to a higher scale \( \mu \) without any parton emissions. Naturally, the NLO extensions of these functions would take more complicated forms, see the following equation (10) in relation to the MRW prescriptions. The infra-red cut-off \( \Delta = k_t / (\mu + k_t) \) represents a visualization of the AOC, which automatically excludes the \( x = x' \) point from the range of \( z \)-integration blocking the soft gluon singularities that arise form the \( 1/(1-z) \) terms in the splitting functions.

One immediately notes that throughout the above definition, the \( k_t \)-dependency gets introduced into the UPDF, only at the last step of the evolution. In order to produce these UPDF, the single scaled \( b(x, k_t^2) \) functions can be obtained from the MMHT2014 library, [48], where the calculation of the single-scaled functions have been carried out using the IS data on the \( F_2 \) structure function of the proton. Additionally, using the constraint,

\[
    T_a(k_t^2 \geq \mu^2, \mu^2) = 1,
\]

provides the KMR formalism with a smooth behavior over the small-\( x \) region, where the \( \alpha_s \ln(1/x) \) effects dominate and the BFKL evolution equation becomes important. The reader should notice that in the \( k_t > \mu \) domain, the unintegrated quark densities of the KMR approach are non-vanishing, these parton density functions are considered to be in the LO level.

The second option is the MRW procedure. The UPDF of KMR, despite being proven to have physical value, suffers a miss-alignment with the theory of the color coherent radiations, since the AOC is a by-product of the successive gluonic emissions, therefore, its manifestation (the infra-red cut-off \( \Delta \), should only act on \( P_{qq}(z) \) and \( P_{qg}(z) \) splitting functions, i.e. the terms including the on-shell gluon emissions. Correcting this problem, Martin et al defined the MRW unintegrated densities in the LO through the following definitions [12]

\[
    f_{q}^{LO}(x, k_t^2, \mu^2) = T_q(k_t^2, \mu^2) \frac{\alpha_s(k_t^2)}{2\pi} \int_x^{1-\Delta} dz \left[ P_{qq}^{(LO)}(z) \frac{x}{z} q \left( \frac{x}{z}, k_t^2 \right) \Theta \left( \frac{\mu}{\mu + k_t} - z \right) + P_{qg}^{(LO)}(z) \frac{x}{z} g \left( \frac{x}{z}, k_t^2 \right) \right],
\]

(6)
and

\[ f_{g}^{LO}(x, k_t^2, \mu^2) = T_g(k_t^2, \mu^2) \frac{\alpha_s(k_t^2)}{2\pi} \int_x^1 \frac{dz}{z} \left[ P_{gg}^{(LO)}(z) \sum_q \frac{x}{z} q \left( \frac{x}{z}, k_t^2 \right) + P_{gg}^{(LO)}(z) \frac{x}{z} g \left( \frac{x}{z}, k_t^2 \right) \right], \tag{7} \]

with the modified loop contributions

\[ T_g(k_t^2, \mu^2) = \exp \left( -\int_{k_t^2}^{\mu^2} \frac{\alpha_s(k_t^2)}{2\pi} \frac{dk_t^2}{k_t^2} \int_0^{z_{max}} dz' P_{qq}^{(LO)}(z') \right), \tag{8} \]

and

\[ T_g(k_t^2, \mu^2) = \exp \left( -\int_{k_t^2}^{\mu^2} \frac{\alpha_s(k_t^2)}{2\pi} \frac{dk_t^2}{k_t^2} \left[ \int_{z_{min}}^{z_{max}} dz' z' P_{gg}^{(LO)}(z') + n_f \int_0^1 dz' P_{gg}^{(LO)}(z') \right] \right), \tag{9} \]

where \( z_{max} = 1 - z_{min} = \mu/(\mu + k_t) \) \[52\]. To a good approximation, include the main kinematics of partonic evolution are included in both of the UPDF of KMR and MRW. Interestingly, the particular choice of the AOC in the KMR formalism, despite being of the LO, includes some higher order contributions, i.e. from the \( \ln(1/x) \)-dominant sector. On the other hand, in the MRW case, the extension to the higher order must be inserted by the means of extra constraints.

To include the NLO corrections into the LO MRW framework, one needs to define the NLO splitting functions as,

\[ \tilde{P}_{ab}^{(LO+NLO)}(z) = \tilde{P}_{ab}^{(LO)}(z) + \frac{\alpha_s}{2\pi} \tilde{P}_{ab}^{(NLO)}(z), \tag{10} \]

with

\[ \tilde{P}_{ab}^{(i)}(z) = P_{ab}^{(i)}(z) - \Theta(z - (1 - \Delta')) \delta_{ab} F_{ab}^{(i)} P_{ab}(z), \tag{11} \]

with \( i = 0 \) corresponding to the LO and \( i = 1 \) to the NLO levels (It has been argued that, applying the approximation \( P^{(LO+NLO)}(z) \sim P^{(LO)}(z) \) will simplify the NLO prescription and have a negligible effect on the outcome \[12\], therefore we do not need to express the exact forms of the NLO splitting functions) . Consequently, the introduction of the AOC into the NLO MRW formalism is through the extended splitting functions and the \( \Theta(z - (1 - \Delta')) \) constraint, with \( \Delta' \) being defined as:

\[ \Delta' = \frac{k\sqrt{1 - z}}{k\sqrt{1 - z} + \mu}. \]
Additionally, one have to cut off the tail of the probability into the \( k_t > \mu \) region by inserting a secondary AOC related term into the body of the real emission sector,

\[
f^{NLO}_a(x, k_t^2, \mu^2) = \int_x^1 dz T_a \left( k^2 = \frac{k_t^2}{(1-z)}, \mu^2 \right) \frac{\alpha_S(k^2)}{2\pi} \sum_{b=q,g} \tilde{P}^{(LO+NLO)}_{ab}(z) \times b^{NLO} \left( \frac{x}{z}, k^2 \right) \Theta \left( 1 - z - \frac{k_t^2}{\mu^2} \right). \tag{12}\]

The Sudakov form factors in this framework are formulated as:

\[
T_q(k^2, \mu^2) = \exp \left( - \int_{k^2}^{\mu^2} \frac{\alpha_S(q^2)}{2\pi} dq^2 \int_0^1 dz' z' \left[ \tilde{P}^{(0+1)}_{qq}(z') + \tilde{P}^{(0+1)}_{gq}(z') \right] \right), \tag{13}\]

\[
T_g(k^2, \mu^2) = \exp \left( - \int_{k^2}^{\mu^2} \frac{\alpha_S(q^2)}{2\pi} dq^2 \int_0^1 dz' z' \left[ \tilde{P}^{(0+1)}_{gg}(z') + 2n_f \tilde{P}^{(0+1)}_{qg}(z') \right] \right). \tag{14}\]

The reader can find a comprehensive description of the NLO splitting functions in the references \[12, 53\].

In the figure 1, the UPDF of the \( k_t \)-factorization are plotted against the fractional longitudinal momentum of the parent hadron (\( x \)) and the transverse momentum of the parton, appearing on the top of the evolution ladder (\( k_t \)). The obvious difference in the behavior of the UPDF in different frameworks is a direct consequence of employing different manifestations of the AOC in their respective definitions.

III. THE DI-JET PRODUCTION IN THE P-P COLLISIONS AT THE LHC

Generally speaking, the main contributions into the hadronic cross-section of the di-jet productions at the LHC, i.e.,

\[ P_1 + P_2 \rightarrow J_1 + J_2 + X, \]

are the LO partonic sub-processes:

\[
g(k_1) + g^*(k_2) \rightarrow g(p_1) + g(p_2),
\quad g(k_1) + g^*(k_2) \rightarrow q(p_1) + \bar{q}(p_2),
\quad q(k_1) + g^*(k_2) \rightarrow q(p_1) + g(p_2). \tag{15}\]

Since we are considering the forward sector for the partons that are produced in the \( k_t \)-factorization, the stared partons in the equation (15), one can safely neglect the \( qq \) and
qq sub-processes. In the collinear factorization framework, the cross-section of a hadronic ISR can be written as a sum over all of the involving partonic cross-sections, times the probability of appearing the particular partonic configuration at top of the evolution ladder of the individual hadrons, i.e.,

\[
\sigma_{\text{Hadron-Hadron}} = \sum_{a_1, a_2 = q, g} \int_0^1 dx_1 \int_0^1 dx_2 \frac{a_1(x_1, \mu^2_1)}{x_1} \frac{a_2(x_2, \mu^2_2)}{x_2} \times \hat{\sigma}_{a_1 - a_2}(x_1, k^2_{1,t} = 0, \mu^2_1; x_2, k^2_{2,t} = 0, \mu^2_2),
\]

(16)

where \(\hat{\sigma}_{a_1 - a_2}\) denotes the cross-section of the incoming partons \(a_1\) and \(a_2\), respectively with the longitudinal momentum fractions \(x_1\) and \(x_2\), the hard scales \(\mu_1\) and \(\mu_2\) and neglected transverse momenta. \(\hat{\sigma}_{a_1 - a_2}\) may be defined as follows:

\[
d\hat{\sigma}_{a_1 a_2} = d\phi_{a_1 a_2} \frac{|M_{a_1 a_2}|^2}{F_{a_1 a_2}},
\]

(17)

with the multi-particle phase space \(d\phi_{a_1 a_2}\),

\[
d\phi_{a_1 a_2} \equiv \prod_i \frac{d^3 p_i}{2E_i} \delta^{(4)} \left(\sum p_{\text{in}} - \sum p_{\text{out}}\right),
\]

(18)

and the flux factor \(F_{a_1 a_2}\),

\[F_{a_1 a_2} \equiv x_1 x_2 s.
\]

(19)

\(s\) is the center of mass energy squared,

\[s = (P_1 + P_2)^2 = 2P_1 P_2,
\]

with \(P_1\) and \(P_2\) being the 4-momenta of the incoming hadrons, where we have neglected the mass of the proton, while working in the infinite momentum frame. \(M_{a_1 a_2}\) in the equation (17) are the matrix elements of the partonic sub-processes, the equations (15). To calculate these quantities, one must first understand the exact kinematics that rule over the corresponding partonic sub-processes.

To include the contributions coming from the transverse momentum dependency of the probability functions, one can use the definition of the UPDF in the framework of \(k_t\)-factorization, the equation (2) and rewrite the equation (16) as follows:

\[
\sigma_{\text{Hadron-Hadron}} = \sum_{a_1, a_2 = q, g} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^{\infty} \frac{dk^2_{1,t}}{k^2_{1,t}} \int_0^{\infty} \frac{dk^2_{2,t}}{k^2_{2,t}} f_{a_1}(x_1, k^2_{1,t}, \mu^2_1) f_{a_2}(x_2, k^2_{2,t}, \mu^2_2) \times \hat{\sigma}_{a_1 a_2}(x_1, k^2_{1,t}, \mu^2_1; x_2, k^2_{2,t}, \mu^2_2).
\]

(20)
Now, it is convenient to characterize $d\phi_{a_1a_2}$ in terms of the transverse momenta of the product particles, $p_{i,t}$, their rapidities, $y_i$, and the azimuthal angles of the emissions, $\varphi_i$,

$$\frac{d^3p_i}{2E_i} = \frac{\pi}{2} dp_{i,t}^2 dy_i d\varphi_i. \quad (21)$$

Working in the proton-proton center of mass frame, one may use below kinematics,

$$P_1 = \frac{\sqrt{s}}{2} (1, 0, 0, 1), \quad P_2 = \frac{\sqrt{s}}{2} (1, 0, 0, -1),$$

$$k_i = x_i P_i + k_{i,\perp}, \quad k_{i,\perp}^2 = -k_{i,t}^2, \quad i = 1, 2, \quad (22)$$

where the $k_i$ are the 4-momenta of the partons that enter the semi-hard process. Then, for each partonic sub-process, the conservation of the transverse momentum reads as,

$$k_{1,\perp} + k_{2,\perp} = p_{1,\perp} + p_{2,\perp}. \quad (23)$$

Afterwards, one can simply define,

$$x_1 = \frac{1}{\sqrt{s}} \left( p_{1,t} e^{+y_1} + p_{2,t} e^{+y_2} \right),$$

$$x_2 = \frac{1}{\sqrt{s}} \left( p_{1,t} e^{-y_1} + p_{2,t} e^{-y_2} \right). \quad (24)$$

The figure 2 illustrates the schematics for a proton-proton deep inelastic collision in the forward-center (or the forward-forward) rapidity sector in a particular partonic sub-process, i.e. $g^* + g \rightarrow q + \bar{q}$. Working within the boundaries of the forward-center or the forward-forward rapidity sector, without damaging the main assumptions, one can assume that $x_1 \sim 1$ and $x_2 \ll 1$. In the direct consequent of such approximation, we can safely neglect the transverse momentum dependency of the first parton entering the hard process (shift it to the collinear domain), and rewrite the equation (20) as,

$$\sigma_{\text{Hadron-Hadron}} = \sum_{a_1,a_2=q,g} \int_0^1 \frac{dx_1}{x_1} \int_0^1 \frac{dx_2}{x_2} \int_0^\infty \frac{dk_t^2}{k_t^2} a_1(x_1, \mu_1^2) f_{a_2}(x_2, k_t^2, \mu_2^2)$$

$$\times \hat{\sigma}_{a_1a_2}(x_1, \mu_1^2; x_2, k_t^2, \mu_2^2), \quad (25)$$

with the $k_t$ being defined as,

$$k_t = \left[ p_{1,t}^2 + p_{2,t}^2 + 2p_{1,t}p_{2,t} \cos(\Delta \varphi) \right]^{1/2}, \quad (26)$$

and $\Delta \varphi = \varphi_1 - \varphi_2$. 

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After determining the kinematics of the involving processes, it is possible to calculate their matrix elements, i.e. \( M_{a_1a_2} \). To this end, one have to sum over the \( dk_t^2/k_t^2 \) terms only from the ladder-type diagrams, and somehow systematically dispose the interference (the non-ladder) diagrams, e.g. by using a physical gauge for the gluons,

\[
d_{\mu\nu}(k) = -g_{\mu\nu} + n_\mu n_\nu + n_\mu k_\nu/k.n.
\]

(27)

Note that \( n = x_1 P_1 + x_2 P_2 \) is the gauge-fixing vector. One might expect that neglecting the contributions coming from the non-ladder diagrams, i.e. the diagrams where the production of the jets is a by-product of the hadronic collision (see the reference [40, 54]), would have a numerical effect on the results. Hence, using the equation (27) as our choice for the axial gauge for the gluons, we can safely subtract the "unfactorizable" contributions coming from the non-ladder type diagrams. Thus, using the regular Feynman rules, inserting the "non-sense" polarization for the incoming gluons,

\[
\sum \epsilon_\mu(k_i)\epsilon^\nu(k_i) = \frac{k_{\mu i,t} k_{\nu i,t}}{k_{i,t}^2},
\]

(28)

and imposing the "eikonal" approximation to justify the use of an on-shell prescription for the off-shell particles (via neglecting the exchanged momenta in the quark-gluon vertices and preserving the spin of the gluons, see the references [40, 54, 55]),

\[
- i \bar{u}(p_i) \gamma^\mu u(p_i) \to \frac{-2i}{k_{i,t}^2} P_\mu^i,
\]

(29)

one can manage to extract the matrix element, corresponding to the processes of the equation [15], see the appendix A.

Now, using the above equations, one can derive the master equation for the total cross-section of the production of di-jets in the framework of \( k_t \)-factorization,

\[
\sigma_{p-p}(P_1 + P_2 \to J_1 + J_2) = \sum_{a,c,d=q,g} \frac{1}{1 + \delta_{cd}} \int \frac{dy_1dy_2}{8\pi^2(x_1 x_2 s)^2} \frac{dp_{1,t}dp_{2,t}}{k_t^2} \frac{dA}{\Delta\phi} a(x_1, \mu^2)
\]

\[
\times f_g(x_2, k_t^2, \mu^2) \left| M_{a+g\to c+d}(x_1, \mu^2; x_2, k_t^2) \right|^2.
\]

(30)

The term \( 1/(1 + \delta_{cd}) \) restrains the over-counting indices. Note that, the existence of the term \( k_t^{-2} \) in the equation (30) is the remnant of the re-summation factor, \( dk_t^2/k_t^2 \), from the equation (2) and since we are interested to look for the transverse momentum dependent jets with \( p_{i,t} > 20 GeV \), the presence of such denominator would not cause any complication.
in the master equation. Additionally, we have to decide how to validate our UPDF in the non-perturbative region, i.e. where \( k_t < \mu_0 \) with \( \mu_0 = 1 \) GeV. A natural option would be to fulfill the requirement that:

\[
\lim_{k_t^2 \to 0} f_g(x, k_t^2, \mu^2) \sim k_t^2,
\]

and therefore, one can safely choose the following approximation for the non-perturbative region:

\[
f_g(x, k_t^2 < \mu_0^2, \mu^2) = \frac{k_t^2}{\mu_0^2} g(x, \mu_0^2) T_g(\mu_0^2, \mu^2).
\] (31)

In the next section, we will introduce some of the numerical methods that have been used for the calculation of the cross-section of the production of di-jets, using the UPDF of KMR and MRW.

IV. THE NUMERICAL ANALYSIS

We perform the 5-fold integration of the master equation (30), using the VEGAS algorithm in Monte-Carlo integration. To do this, we have selected the hard-scale of the UPDF as the share of each of the parent hadrons from the total energy of the center-of-mass frame:

\[
\mu = \frac{1}{2} E_{CM}.
\] (32)

Variating this normalization value around a factor of 2, will provide each framework with a decent uncertainty bound. One would also set the upper boundaries on the transverse momentum integrations to \( p_{t,\text{max}} = 4\mu \), noting that increasing this upper value does not have any effect on the outcome.

The forward rapidity sectors is conventionally defined as,

\[
3.2 < |\eta_f| < 4.8,
\] (33)

where \( \eta \) denotes the pseudorapidity of a produced particle,

\[
\eta = -ln \left[ tan \left( \frac{\theta}{2} \right) \right],
\]

with \( \theta \) being the angle between the propagation axis and the momentum of the particle. Alternatively, to work in the central rapidity sector, one have to choose,

\[
|\eta_c| < 2.8.
\] (34)
Therefore, while working in the infinite momentum frame i.e. where $\eta \simeq y$, to perform our calculations in the forward-center region, we set:

$$y_1 = \eta_c, \quad y_2 = \eta_f.$$ 

Trivially, the choice

$$y_1 = \eta_f, \quad y_2 = \eta_f,$$

marks the forward-forward region. Such framework should be ideal to describe the inclusive CMS data regarding the forward-center di-jet measurements for $p_{i,t} > 35$ GeV. After confirming that, one can go further, producing predictions in the framework of forward-forward di-jet production for the LHC.

Moreover, as a consequence of employing the inclusive scenario (i.e. $p_{i,t} > 35$ GeV and limiting the rapidity integrations to the forward or central regions), one must assure that the produced jets must lie within this specific region. Thus, in order to cut-off the collinear and the soft singularities, it is conventional to use the anti-$k_t$ algorithm [56], with radius $R = 1/2$, bounding the jets to this particular initial setup, through inserting a constraint on the $y - \varphi$ plane:

$$R > \left[ (\Delta \varphi)^2 + (y_2 - y_1)^2 \right]^{1/2}. \quad (35)$$

Introducing the anti-$k_t$ jet constraint ensures the production of 2 separated jets and rejects any single-jet scenarios.

V. RESULTS, DISCUSSIONS AND CONCLUSIONS

Having in mind the theory and the notions of the previous sections, we are able to calculate the production rates belonging to the di-jets in the forward-center and the forward-forward rapidity sectors, from the perspective of the $k_t$-factorization framework, utilizing the UPDF of KMR and MRW. The PDF of Harland–Lang et al. [48], MMHT2014, in the LO and NLO levels, are used as the input functions for the unintegrated gluon densities, i.e., the equations [5], [7] and [12]. Additionally, they are fit to be used as the solutions of the DGLAP, the PDF of the collinear factorization, directly in the master equation [30]. We tend to perform the above calculations in any of our presumed frameworks, the KMR, the LO MRW and the NLO MRW, then compare the results to each other, to the similar
calculations in other frameworks and to the existing experimental data, in the case of the forward-center.

So, the figures 3, 4 and 5 present the reader with the differential cross-section for the production of well-separated forward-central di-jets \(d^2\sigma/dp_t d\eta\), plotted against the transverse momentum of the corresponding jets \(p_t\) in the KMR, the LO MRW and the NLO MRW schemes respectively. The uncertainty bounds are calculated, varying the hard scale of the UPDF with a factor of 2, since this is the only arbitrary physical parameter in the framework of \(k_t\)-factorization. The blue-hatched pattern, the green-checkered and the red-vertically stripped patterns illustrate the individual contributions of the partonic sub-processes from the equation (15), corresponding to the \(g^* + g \rightarrow g + g\), \(g^* + g \rightarrow q + \bar{q}\) and \(g^* + q \rightarrow g + q\) processes respectively. The black-horizontally stripped pattern represents the sum of the sub-contributions. The calculations have been compared against the experimental data of the CMS collaboration, the reference [46]. One immediately notices that the share of the \(g^* + g \rightarrow g + g\) sub-process dominates, relative to the negligible shares of the remaining two sub-processes. Although all of these frameworks are relatively successful in describing the experimental data, see the figure 6, it is interesting to find that the UPDF of KMR do as well as (if not better than) the UPDF of MRW in predicting the experimental results. The closeness of the behavior of different frameworks is a consequence of our choice for the hard scale of the UPDF, the equation (32). In order to enlighten this point, the figure 7 illustrates the result of making different choices in such calculations, using the UPDF of the KMR. To demonstrate the effect of changing the hard scale of the UPDF in the outcome, the histograms are calculated utilizing the following hard scale prescriptions

\begin{align*}
a) \quad & \mu = \frac{1}{2} (p_{1,t} + p_{2,t}) , \\
b) \quad & \mu = \frac{1}{2} \left( p_{1,t}^2 + p_{2,t}^2 \right)^{1/2} , \\
c) \quad & \mu = \text{Max}(p_{1,t}, p_{2,t}) , \\
d) \quad & \mu = \frac{1}{4} E_{CM} , \\
e) \quad & \mu = \frac{1}{2} E_{CM} , \\
f) \quad & \mu = E_{CM} , \\
\end{align*}

where \(\text{Max}(p_{1,t}, p_{2,t})\) returns the higher value between the transverse momenta of the produced jets. To save computation time, we only considered the contributions coming form
the dominant \( g^* + g \to g + g \) sub-processes. The choice \( a \), which have been used in the similar calculations (e.g., the references \([41,45]\) in the high energy factorization, from the point of view of the \( UPDF \) of the color gloss condensation, \((CGC)\)) proves to be in contrast with the particular manifestation of the \( AOC \), specially in the case of \( NLO MRW UPDF \). This is in addition to the considerable off-shoot of the results in the smaller values of the transverse momenta belonging to the produced jets. In the figure [6] the yellow-checkered and the purple-vertically stripped patters represent the calculations in the linear and the non-linear \( KS \) frameworks, respectively. The above separation between the predictions of the \( KS \) framework and the experimental data is apparent. To avoid such complications, we have chosen the condition \( e \), in the equation (36), as the primary prescription for the hard scale of our \( UPDF \) throughout this work, see the section [IV].

Having a closer look into the figure [6], one notices that such off-shooting results also appear in our settings for the production of di-jets. This is perhaps because of the over-simplified dynamics that have been used to derive these measurements. An increase in the precision may be realized via including higher order diagrams and introducing the final state parton showers in this frameworks \([57]\). Beside this point, note that our results show an acceptable agreement with the experimental data of the \( CMS \) collaboration, reference \([46]\). Another interesting observation is that in the large \( k_t \), where the higher order corrections become important, the calculations in the \( KMR \) approach start to separate from the \( LO MRW \) and behave similar to the \( NLO MRW \). The reason is that the inclusion of the non-diagonal splitting functions into the domain of the \( AOC \) introduces some corrections from the \( NLO \) region (in the form of \( \ln(1/x) \) re-summations) into the \( KMR \) formalism.

A recent report from the \( CMS \) collaboration, the reference \([47]\), concerns the angular distribution of the produced jets in the forward-center rapidity sector from a deep inelastic event at the \( LHC \). Making use of this new information, we have calculated the differential cross-section of the forward-central di-jet production \( (d\sigma/d\Delta\varphi) \), plotted in the figure [8] against the angular difference of the produced partons (or equivalently the angular difference of the produced jets, \( \Delta\varphi \)). The panels (a), (b) and (c) in this figure illustrate the details of the calculations in each framework, consisting of the individual contributions of the sub-processes and the corresponding uncertainty bounds. The panel (d) presents the reader with the comparison of the total amounts in the presumed formalisms to each other and to the data from the reference \([47]\). Again, the results in the \( KMR \) approach seems to be equally
good (or better than) those from the MRW in the LO or the NLO.

After proving the success of our formalism in describing the experimental data for the production of di-jets in the forward-center rapidity region, we can move forward with the prediction of a similar event, in the forward-forward sector, i.e. by choosing the rapidity of the produced jets \((y_1 \text{ and } y_2)\) to be both in the boundaries that where specified within the equation \([33]\). Therefore, in the figure \([9]\) the reader is presented with our predictions regarding the dependency of the differential cross-section of the forward-forward di-jet production \((d\sigma_f/dp_t^f)\) to the transverse momenta of the produced jets \((p_t)\), in the framework of \(k_t\)-factorization. The panels (a), (b) and (c) of the figure illustrate these predictions in the KMR, the LO MRW and the NLO MRW formalisms, respectively. The contributions of the individual partonic sub-processes are included. These contributions have the same general behavior as in the forward-central case, in spite of the fact that the measured contribution for the \(g^* + g \rightarrow g + g\) and the \(g^* + q \rightarrow g + q\) sub-processes are closer, compared to their counterparts from the forward-center region,

\[
\hat{\sigma}_{F-C}(g^* + g \rightarrow g + g) \gg \hat{\sigma}_{F-C}(g^* + q \rightarrow g + q) \gg \hat{\sigma}_{F-C}(g^* + g \rightarrow q + \bar{q}),
\]

\[
\hat{\sigma}_{F-F}(g^* + g \rightarrow g + g) \gg \hat{\sigma}_{F-F}(g^* + q \rightarrow g + q) \gg \hat{\sigma}_{F-F}(g^* + g \rightarrow q + \bar{q}).
\]

(37)

In addition, one can clearly perceive the effect of the \(\Theta(1 - z - (k_t^2/\mu^2))\) constraint in the NLO MRW results, causing a steep descend in the corresponding histograms, in contrast with the behaviors of the results of the KMR and the LO MRW formalisms. Again, the similarity of the predictions of the KMR and the LO MRW schemes are a consequence of our choice of the hard scale, \(\mu\). Such similarity was also observed else where, e.g. the references \([38–40]\), specially in the smaller \(x\) domains.

The panel (d) of the figure \([9]\) represents a comparison between the results of the \(k_t\)-factorization with the results from other frameworks, namely the Balitsky-Kovchegov TMD PDF convoluted with the running coupling corrections \((rcBK, \text{see the references } [58, 59])\) and the Kutak-Sapeta TMD PDF \((KS)\), the reference \([45]\). Both of these frameworks are specially designed to describe the behavior of the small-x region, incorporating the non-linear evolution of the unintegrated parton densities with the KS framework and the high energy factorization \((HEF)\) formalism, in accordance with the BFKL iterative evolution equation. In the absence of any experimental data, we refrain ourselves from any assessments regarding these results. Nevertheless, the predictions of the KMR scheme (because of its previous
success) may provide a base line for a sound comparison. Also, the singular behavior of the 
NLO MRW results may appear undesirable.

Similar predictions are presented in the figures[10] and[11], describing the dependency of 
the differential cross-section of the forward-forward di-jet production, to the angle of the 
produced jets \((d\sigma_f/d\Delta\varphi)\) to \(\Delta\varphi\) in the figure[10] and to their rapidity \((d\sigma_f/d\eta_f)\) to \(\eta_f\) in the 
figure[11]. The notions of these diagrams are as in the figure[9]. The panel (d) of each figure 
includes the comparison of the \(k_t\)-factorization results to the existing results in the \(rcBK\) 
and the \(KS\) frameworks. The irregular behavior of the \(NLO\) MRW scheme in both cases, 
manifests itself in the form of lower values of the predicted differential cross-section. Again, 
the reliability of these predictions lies within the excellent credit of the \(KMR\ UPDF\) in 
describing the high energy \(QCD\) events.

In summary, throughout this work, we have tested the \(UPDF\) of the \(k_t\)-factorization, 
namely the \(KMR\) and \(MRW\) formalisms in the \(LO\) and the \(NLO\), calculating the produ-
tion rate of the di-jet pairs at the deep inelastic \(QCD\) collisions in the forward-center 
rapidity sector, compared the results to the existing experimental data of the \(CMS\) collabor-
ations and to the results of other frameworks. Through our analysis we have suggested that 
despite the theoretical advantages of the \(MRW\) formalism, the \(KMR\) approach performs as 
good as (if not better) behavior toward describing the experimental data. This is in general 
agreement with our previous findings, the references [32–40]. Additionally, one can clearly 
see that the \(KMR\) or \(MRW\) prescription work better than the \(KS\) in describing the exper-
iment. Based on these observations one concludes that the hard-scale dependence should be 
necessarily included in \(TMD\) analysis. Furthermore, we have predicted the results of the 
similar events in the forward-forward rapidity region, relying on the previous success of the 
\(UPDF\) of the \(k_t\)-factorization.

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Appendix A: The matrix elements of the partonic sub-processes

Assuming that $\mu_1 = \mu_2 \equiv \mu$, the matrix element squares, $|\mathcal{M}_{a_1+a_2\to b_1+b_2}|^2$, corresponding to the equations (15) can be defined for a QCD IS event as follows (also see the reference [45])

$$|\mathcal{M}_{g+g\to g}(x_1, \mu; x_2, k_t^2)|^2 = C_1 A_1,$$
$$|\mathcal{M}_{g+g\to q+\bar{q}}(x_1, \mu; x_2, k_t^2)|^2 = C_2 A_2 + C_2' A_2',$$
$$|\mathcal{M}_{q+g\to q}(x_1, \mu; x_2, k_t^2)|^2 = C_3 A_3 + C_3' A_3', \quad (A1)$$

with

$$C_1 = \frac{9}{8} \left( \frac{\alpha_S(\mu^2)}{4\pi} \right)^2,$$
$$C_2 = \frac{1}{6} \left( \frac{\alpha_S(\mu^2)}{4\pi} \right)^2, \quad C_2' = \frac{1}{8} C_2,$$
$$C_3 = \frac{4}{9} \left( \frac{\alpha_S(\mu^2)}{4\pi} \right)^2, \quad C_3' = \frac{1}{8} C_3, \quad (A2)$$

and

$$A_1 = \frac{2(e^{\Delta y} R_t + 1)^2(R_t e^{-\Delta y}(R_t e^{-\Delta y} + 1) + 1)^2(cos(\Delta \varphi) + 2cosh(\Delta y))}{R_t^2(R_t e^{-\Delta y} + 1)^2(cos(\Delta \varphi) + cos(\Delta y))},$$
$$A_2 = \frac{(R_t + e^{-\Delta y})^2(R_t^2 e^{-\Delta y} + e^{\Delta y})}{R_t(R_t e^{-\Delta y} + 1)^2},$$
$$A_2' = \frac{(R_t + e^{-\Delta y})^2(R_t^2 e^{-\Delta y} + e^{\Delta y})}{R_t(R_t e^{-\Delta y} + 1)^2(cos(\Delta y) - cos(\Delta \varphi))} cos(\Delta \varphi),$$
$$A_3 = \frac{(R_t + e^{-\Delta y})^2((R_t + e^{\Delta y})^2 + R_t^2)}{2R_t(R_t e^{-\Delta y} + 1)(cos(\Delta y) - cos(\Delta \varphi))},$$
$$A_3' = 2e^{-\Delta y}(e^{-\Delta y} - cos(\Delta \varphi)) \frac{(R_t + e^{-\Delta y})^2((R_t + e^{\Delta y})^2 + R_t^2)}{2R_t(R_t e^{-\Delta y} + 1)(cos(\Delta y) - cos(\Delta \varphi))}, \quad (A3)$$

where

$$\Delta y = y_2 - y_1, \quad R_t = \frac{p_{1,t}}{p_{2,t}}.$$
Using the above information, one can calculate the cross-sections of the equation [15].

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FIG. 1: The gluonic $UPDF$ of the $k_t$-factorization versus the fractional longitudinal momentum of the parent hadron ($x$) and the transverse momentum of the parton, appearing on the top of the evolution ladder ($k_t$) at $\mu = 100$ GeV. The difference in the behavior of the $UPDF$ in different frameworks is a direct consequence of employing different manifestations of the $AOC$ in their respective definitions. To plot these diagrams we have used the $PDF$ libraries of $MMHT2014$ in the $LO$ and the $NLO$ as the input for the equations $[5], [7]$ and $[12]$. 
FIG. 2: The deep inelastic scattering of two protons in the forward-center configuration. The diagram shows the $g^* + g \rightarrow q + \bar{q}$ sub-process, assuming that one of the quarks is being produced in the forward sector (bounded by $3.2 < |\eta_f| < 4.7$) and the other in the center sector (bounded by $|\eta_c| < 2.8$). The parton density related to the first proton is being described with the *integrated* PDF while the second parton is prepared using the UPDF in one of our presumed frameworks.
FIG. 3: The differential cross-section for the production of di-jets in the forward-center rapidity sector, calculated in the KMR framework for $E_{CM} = 7 \, TeV$. The contributions from each of the involving sub-processes form the equation (15) have been plotted separately. The black-oblique patterned histograms illustrate the sum of the partonic contributions. To determine the uncertainty of the calculations, we have manipulated the hard scale of the UPDF, $\mu = E_{CM}/2$, by a factor of 2. The data point are from the measurements of the CMS collaboration, the reference [46].

FIG. 4: The differential cross-section for the production of di-jets in the forward-center rapidity sector, calculated in the LO MRW framework. The notion of the diagrams are as in the figure 3.
FIG. 5: The differential cross-section for the production of di-jets in the forward-center rapidity sector, calculated in the \textit{NLO MRW} framework. The notion of the diagrams are as in the figure.

FIG. 6: The comparison between the differential cross-sections of the production of di-jets from the forward-center rapidity sector, in the different frameworks of the \( k_t \)-factorization. The results have been prepared as the numerical solutions the equation (30), using the \textit{UPDF} of KMR and MRW in the \textit{LO} and \textit{NLO} with \( E_{CM} = 7 \text{ TeV} \). The data points are from the \textit{CMS} report \[46\]. The yellow-checkered and the purple-vertically stripped patterns represent the calculations in the linear and non-linear \textit{KS} frameworks, respectively, see the reference \[41\].
FIG. 7: The differential cross-section for the production of di-jets in the forward-center rapidity sector, for different choices of the hard scale and from the dominant $g^* + g \rightarrow g + g$ sub-process. The calculations have been carried on in the KMR framework for $E_{CM} = 7\ TeV$. The histograms $a$ through $f$ have been calculated using the conditions from the equation (36). We have chosen the condition $e$ (the black-continues histograms), i.e. the equation (32), as the primary prescription throughout this work.
FIG. 8: The differential cross-section for the production of di-jets versus the angle of the outgoing jets, $\Delta \varphi$. The calculations are in the forward-center rapidity sector for $E_{CM} = 7$ TeV. The panels (a), (b) and (c) illustrate the calculations, utilizing the $UPDF$ of KMR, LO MRW and NLO MRW, respectively. The contributions from each of the involving sub-processes are shown separately. The panel (d) presents the comparison of these measurements against each other as well as the experimental data of the CMS collaboration, the reference [47]. The uncertainty of the calculations are provided through manipulating the hard scale of the $UPDF$ by a factor of 2.
FIG. 9: The calculated predictions for the production of forward-forward di-jets in the framework of $k_t$-factorization with the central-mass energy of 7 $TeV$. The differential cross-section for the production of di-jets are plotted against the transverse momenta of the produced jets, in the KMR, LO MRW and NLO MRW schemes (i.e. the panels (a), (b) and (c), respectively), demonstrating the contributions of the individual sub-processes. The uncertainty bound is determined by manipulating the hard scale of the UPDF, $\mu = E_{CM}/2$, by a factor of 2. The panel (d) represents a comparison between the results of the $k_t$-factorization with the results from other frameworks, namely the Balitsky-Kovchegov TMD PDF convoluted with running coupling corrections ($rcBK$, see the references [58, 59]) and the Kutak-Sapeta TMD PDF (KS), reference [45].
FIG. 10: The calculated predictions regarding the dependency of the differential cross-section for the production of forward-forward di-jets to $\Delta \varphi$ using the UPDF of $k_t$-factorization for $E_{CM} = 7 \, TeV$. The notion on the diagrams are as in the figure 9. In the panel (d), we have compared our results with the predictions made using the KS TMD PDF from the reference [45].
FIG. 11: The calculated predictions regarding the dependency of the differential cross-section for the production of forward-forward di-jets to rapidity of the produced jets, using the UPDF of $k_t$-factorization for $E_{CM} = 7$ TeV. The notion on the diagrams are as in the figure. In the panel (d), we have compared our results with the predictions made using the rcBK and $KS TMD PDF$ from the reference.