New Construction of $\Delta$-Operator in Field-Antifield Formalism

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It is proven that the nilpotent $\Delta$-operator in the field-antifield formalism can be constructed in terms of an antisymplectic structure only.

1 Introduction

The field-antifield formalism, developed by Batalin and Vilkovisky [1], provides a unique closed approach to covariant quantization of general gauge theories, based on a special kind of global supersymmetry, the so-called BRST symmetry [2]. Generally speaking, the BRST symmetry is expressed in the nilpotency property of odd second-order differential $\Delta$-operator constructed explicitly [1] in the Darboux coordinates. Investigation of geometrical meaning of the field-antifield formalism [3] requires to consider all basic objects defined invariantly on a supermanifold equipped with an antisymplectic structure (for definition, see below). First of all it concerns the $\Delta$-operator. Up to now two definitions of $\Delta$-operator are known [4, 5]. Khudaverdian’s construction of $\Delta$-operator includes an antisymplectic structure and a measure density. The requirement of nilpotency for $\Delta$-operator in this approach leads to a compatibility condition between these two structures. Batalin-Bering’s definition of $\Delta$-operator involves additionally an odd scalar curvature. In this case the compatibility condition between an antisymplectic structure and a measure density can be omitted, but the definition operates with three independent structures.

Our aim of this work is to study the re-definition of $\Delta$-operator acting on an arbitrary antisymplectic supermanifold. The re-definition is based on the remarkable fact proved in [7] that there exists the only symmetric connection which is compatible with a given antisymplectic structure. Then $\Delta$-operator can be constructed in the form likes to the usual Laplacian in the Riemannian geometry.

The paper is organised as follows. In Sect. 2, we remind the definition and basic properties of the antibracket using an antisymplectic structure which is used for construction of the antibracket being the basic object in the field-antifield quantization of general gauge theories in Lagrangian formalism [1].

2 Antisymplectic structure

In this section, we remind the notation of an antisymplectic structure which is used for construction of the antibracket being the basic object in the field-antifield quantization of general gauge theories in Lagrangian formalism [1].
Let us consider a supermanifold $M$ with an even dimension $\text{dim} M = 2N$. In the vicinity of each point $P \in M$ local coordinates $x^i, \epsilon(x^i) = \epsilon_i$ can be introduced. Let $\Omega^{ij}$ ($\epsilon(\Omega^{ij}) = \epsilon_i + \epsilon_j + 1$) be an odd non-degenerate second-rank tensor field of type $(2, 0)$ on $M$ obeying the generalized symmetry property

$$\Omega^{ij} = (-1)^{\epsilon_i \epsilon_j} \Omega^{ji}$$

and satisfying the relations

$$\Omega^{in} \frac{\partial \Omega^{jk}}{\partial x^n} (-1)^{\epsilon_i (\epsilon_{k+1})} + \text{cycle}(i, j, k) \equiv 0.$$  \hspace{1cm} (2)

The inverse tensor field $\Omega_{ij}$ has the generalized symmetry property

$$\Omega_{ij} = (-1)^{\epsilon_i \epsilon_j} \Omega_{ji}.$$ \hspace{1cm} (3)

and the relations (2) can be rewritten in terms of $\Omega_{ij}$ as (see, [7])

$$\Omega_{ij, k} (-1)^{\epsilon_k (\epsilon_{i+1})} + \text{cycle}(i, j, k) \equiv 0.$$ \hspace{1cm} (4)

Any odd non-degenerate tensor field $\Omega^{ij}$ obeying the properties (1) and (2) (or, equivalently, $\Omega_{ij}$ with the properties (3) and (4)) is referred as an antisymplectic structure on a supermanifold $M$.

Having an antisymplectic structure $\Omega^{ij}$, one can define the following odd bilinear operation for any two scalar functions $A$ and $B$ on $M$

$$(A, B) = \frac{\partial A}{\partial x^i} (-1)^{\epsilon_i} \Omega^{ij} \frac{\partial B}{\partial x^j}, \hspace{1cm} \epsilon((A, B)) = \epsilon(A) + \epsilon(B) + 1.$$ \hspace{1cm} (5)

This definition leads to the invariance of the operation (5) under local coordinate transformations $x \to \bar{x}, (\bar{A}, \bar{B}) = (A, B)$. From Eqs. (1) and (5) it follows

$$(A, B) = -(-1)^{(\epsilon(A)+1)(\epsilon(B)+1)} (B, A),$$ \hspace{1cm} (6)

i.e., the generalized symmetry property for the operation introduced. In its turn, from Eq. (1) one can obtain the following identity

$$(A, (B, C))(-1)^{(\epsilon(A)+1)(\epsilon(C)+1)} + \text{cycle}(A, B, C) \equiv 0,$$ \hspace{1cm} (7)

i.e., the Jacobi identity for the operation. It means that $(A, B)$ (5) is the antibracket. The pair $(M, \Omega)$ defines an antisymplectic supermanifold.

3 Connection

We now equip an antisymplectic supermanifold with a symmetric connection (a covariant derivative). Let an antisymplectic structure $\Omega^{ij}$ be covariant constant

$$\Omega^{ij} \nabla_k = 0.$$ \hspace{1cm} (8)

Then the inverse tensor field $\Omega_{ij}$ will be covariant constant too

$$\Omega_{ij} \nabla_k = 0, \hspace{1cm} \Omega_{ij, k} - \Omega_{il} \Delta^l_{jk} - \Omega_{jl} \Delta^l_{ik} (-1)^{\epsilon_i \epsilon_j} = 0,$$ \hspace{1cm} (9)

where $\Delta^i_{jk}$ ($\epsilon(\Delta^i_{jk}) = \epsilon_i + \epsilon_j + \epsilon_k$) is a symmetric connection and the symmetry property of $\Omega_{ij}$ was used. In [7] it was proven that there exists the unique symmetric connection $\Delta^i_{jk}$ compatible with a given antisymplectic structure. The result is

$$\Delta^i_{jk} = \frac{1}{2} \Omega^{ij} \left( \Omega_{ij, k}(-1)^{\epsilon_i \epsilon_j} + \Omega_{jk, i}(-1)^{\epsilon_i \epsilon_j} - \Omega_{ki, j}(-1)^{\epsilon_i \epsilon_j} \right) (-1)^{\epsilon_j \epsilon_i + \epsilon_i}.$$ \hspace{1cm} (10)

We see that in the case of an antisymplectic supermanifold $(M, \Omega)$ there is the only symmetric connection compatible with a given antisymplectic structure. It looks like Riemannian geometry.
Let us consider the antisymplectic curvature tensor

\[ R_{ijkl} = \Omega_{i n} R_{n jkl}^n, \quad \epsilon(R_{ijkl}) = \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l + 1, \quad (11) \]

where \( R_{n jkl}^n \) is the curvature tensor of a symmetric connection \( \Delta^i_{jk} \). This leads to the following representation,

\[ R_{nljk} = -\Delta_{nlj,k} + \Delta_{nlk,j} \left( -1 \right)^{\epsilon_i \epsilon_j \epsilon_k \epsilon_l} \epsilon_n \epsilon_i + \epsilon_k \epsilon_l + 1, \quad (12) \]

where

\[ \Delta_{ijk} = \Omega_{in} \Delta_{njkl}^n, \quad \epsilon(\Delta_{ijk}) = \epsilon_i + \epsilon_j + \epsilon_k + 1. \quad (13) \]

Using this, the relation (9) reads

\[ \Omega_{ij,k} = \Delta_{ijk} + \Delta_{jik} \left( -1 \right)^{\epsilon_i \epsilon_j \epsilon_k}, \quad (14) \]

This relation plays very important role in deriving the symmetry properties of the antisymplectic curvature tensor [7].

The antisymplectic curvature tensor obeys the following generalized symmetry properties

\[ R_{ijkl} = -(-1)^{\epsilon_i \epsilon_j} R_{ijlk}, \quad R_{ijkl} = -(-1)^{\epsilon_i \epsilon_j} R_{jikl}, \quad R_{ijkl} = R_{klij} \left( -1 \right)^{\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l}. \quad (15) \]

and the Jacobi identity

\[ (-1)^{\epsilon_i \epsilon_j} R_{ijkl} + (-1)^{\epsilon_i \epsilon_k} R_{iklj} + (-1)^{\epsilon_k \epsilon_j} R_{iklj} = 0. \quad (16) \]

Ricci tensor can be defined by contracting two indices of curvature tensor

\[ R_{ij} = R^k_{ikj} \left( -1 \right)^{\epsilon_i + \epsilon_j} = \Omega_{kn} R_{nikj} \left( -1 \right)^{\epsilon_i + \epsilon_k + \epsilon_l}, \quad \epsilon(R_{ij}) = \epsilon_i + \epsilon_j. \quad (17) \]

Ricci tensor is a generalized symmetric tensor

\[ R_{ij} = R_{ji} \left( -1 \right)^{\epsilon_i \epsilon_j}. \quad (18) \]

The further contraction between a given antisymplectic structure and Ricci tensor gives scalar curvature

\[ R = \Omega^{ij} R_{ij} \left( -1 \right)^{\epsilon_i + \epsilon_j}, \quad \epsilon(R) = 1. \quad (19) \]

which, in general, is not equal to zero. Notice that the scalar curvature tensor squared is identically equal to zero, \( R^2 = 0 \). The scalar curvature tensor was quite recently used in [5] for generalization of Khudaverdian’s construction of \( \Delta \)-operator [4].

4 \quad \Delta\text{-operator}

Having an antisymplectic structure \( \Omega^{ij} \) and the unique symmetric connection (the covariant derivative) compatible with this structure there is the intrinsic definition of an odd second-order differential operator \( \Delta \) (the odd Laplacian)

\[ \Delta = \frac{1}{2} \Omega^{ij} \nabla_j \nabla_i \left( -1 \right)^{\epsilon_i + \epsilon_j} = \frac{1}{2} \nabla_j \nabla_i \Omega^{ij} \left( -1 \right)^{\epsilon_i + \epsilon_j}, \quad \epsilon(\Delta) = 1. \quad (20) \]

acting (from the right) on any tensor field defined on a supermanifold \( M \) as a scalar operator. Note that the definition (20) is completely given in terms of an antisymplectic structure and its partial derivatives only. The operator \( \Delta \) is obviously nilpotent

\[ \Delta^2 = 0. \quad (21) \]
Action of $\triangle$ on product of two scalar function reproduces the antibracket
\[(A \cdot B)\triangle = A \cdot (B\triangle) + (A\triangle) \cdot B(-1)^{\epsilon(B)} + (A,B)(-1)^{\epsilon(B)}.\]  
(22)

Above mentioned properties of $\triangle$ allows us to speak of new definition of the $\triangle$-operator in the field-antifield formalism. The quantum master equation
\[\exp \left\{ \frac{i}{\hbar} W \right\} \triangle = 0 \]  
(23)
is not modified
\[\frac{1}{2}(W,W) = i\hbar W\triangle \]  
(24)
in contrast with [5], where generalization of $\triangle$-operator due to the odd scalar curvature led to modification of the quantum master equation written in terms of the antibracket.

5 Conclusion

We have introduced the definition of $\Delta$-operator in the field-antifield formalism in terms of an antisymplectic structure and its partial derivatives only. It became possible because of the remarkable fact: there is the unique symmetric connection compatible with a given antisymplectic structure. It was shown that the $\Delta$-operator is nilpotent, reproduces the antibracket in standard way, gives the usual form of the quantum master equation.

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