Analytical Solution for the SU(2) Hedgehog Skyrmion and Static Properties of Nucleons

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Abstract
An analytical solution for symmetric Skyrmion was proposed for the SU(2) Skyrme model, which take the form of the hybrid form of a kink-like solution and that given by the instanton method. The static properties of nucleons was then computed within the framework of collective quantization of the Skyrme model, with a good agreement with that given by the exact numeric solution. The comparisons with the previous results as well as the experimental values are also given.

Keywords: Skyrme Model, Soliton, Nucleons

PACS: 12.38.-t, 11.15.Tk, 12.38.Aw

1. Introduction
The Skyrme model [1] is an effective field theory of mesons and baryons in which baryons arise as topological soliton solutions, known as Skyrmions. The model is based on the pre-QCD nonlinear σ model of the pion meson and was usually regarded to be consistent with the low-energy limit of large-N QCD [2]. For this reason, among others, it has been extensively revisited in recent years [3, 4, 5, 6, 8, 9] (see, [10, 11], for a review). Owing to the high nonlinearity, the solution to the Skyrme model was mainly studied through the numerical approach. It is worthwhile, however, to seek the analytic solutions [12, 13, 14] of Skyrmions due to its various applications in baryon phenomenology. One of noticeable analytic method for studying the Skyrmion solutions is the instanton approach proposed by Atiyah and Manton [12] which approximates critical points of the Skyrme energy functional.

In this Letter, we address the static solution of hedgehog Skyrmion in the SU(2) Skyrme model without pion mass term and propose an analytical solution for the hedgehog Skyrmion by writing it as the hybrid form of a kink-like solution and the analytic solution obtained by the instanton method [18]. Two lowest order of Padé approximations was used and the corresponding solutions for Skyrmion profile are given explicitly by using the downhill simplex method. The Skyrmion mass and static properties of nucleon as well as delta was computed and compared to the previous results.

2. Analytic solution to Skyrme model
The SU(2) Skyrme action [1] without pion mass term is given by

\[ S^K = \int d^3x \left[ -\frac{f_\sigma^2}{4} Tr(L_\mu)^2 + \frac{1}{32e^2} Tr([L_\mu, L_\nu]^2) \right] \]  

in which \( L_\mu = U^\dagger \partial_\mu U \), \( U(x, t) \in SU(2) \) is the nonlinear realization of the chiral field describing the \( \sigma \) field and \( \pi \) mesons with the unitary constrain \( U^\dagger U = 1 \), \( 2f_\pi \) the pion decay constant, and \( e \) a dimensionless constant characterizing nonlinear coupling. The Cauchy-Schwartz inequality for (1) implies [15]

\[ E^K \geq 6\pi^2 (f_\pi/e) |B|, \quad \text{where} \quad B = (1/24\pi^2) \int d^4x e^{i/K} Tr(L_\mu L_\nu) \]

is the topological charge, known as baryon number. Using the hedgehog ansatz, \( U(x) = \cos(F) - i(\hat{x} \cdot \hat{\sigma}) \sin(F) \) (\( \hat{\sigma} \) are the three Pauli matrices) with \( F \equiv F(r) \) depending merely on the radial coordinate \( r \), the static energy for \( 1 \) becomes

\[ E^K = 2\pi \int dx \left[ x^2 F'_x + 2 \sin^2(F) (1 + F'_x^2) + \frac{\sin^2(F)}{x^2} \right] \]

with \( x = e f_\pi r \) a dimensionless variable and \( F_x \equiv dF(x)/dx \). The equation of motion of \( 2 \) is

\[ \left( 1 + 2 \frac{\sin^2(F)}{x^2} \right) F_{xx} + 2 \frac{\sin(2F)}{x^2} \left( F_x^2 - 1 - \frac{\sin^2(F)}{x^2} \right) = 0, \]  

where the boundary condition \( F(0) = \pi, F(\infty) = 0 \) will be imposed so that it corresponds to the physical vacuum for \( U = \pm 1 \). The equation (3) is usually solved numerically due to its high nonlinearity [3, 4, 6, 8].

A kink-like analytic solution was given by [8]

\[ F_1(x) = 4 \arctan[\exp(-x)], \]

with \( E^K = 1.24035(6n^2 f_\pi/e) \), while an alternative Skyrmion profile, proposed based on the instanton method, takes the form [8]

\[ F_2(x) = \pi \left[ 1 - (1 + \frac{A}{x^2})^{-1/2} \right], \]

with \( A = 4 \arctan[\exp(-x)] \).
with corresponding energy $1.2432(6\pi^2 f_2/e)$ for the numeric factor $\lambda = 2.109$. The singularity at $r = 0$ is gauge dependent and can be gauged away without affecting the value for the Skyrme field.

To find the more accurate analytic solution, we first improve the solution (2) into $4 \arctan[\exp(-c x)]$ with $a$ a numeric factor and then take $\lambda$ to be a $x$-dependent function: $\lambda \to \lambda(x)$. Hence, we propose a Skyrmon profile function in the hybrid form mixing (4) and (5)

$$F_w(x) = 4w \arctan[\exp(-c x)] + \pi(1 - w) \left[1 - (1 + \frac{\lambda(x)}{x^2})^{-1/2}\right]$$

with $w \in [0, 1]$ being a positive weight factor. In principle, one can find the governing equation for the unknown function $w$ since it has as equal potential as series in approximating a continuous function. Note that we have already written $\lambda(x)$ as function of $x^2$ instead of $x$ since so is $F_w(x)$ in (5). The simplest nontrivial case of the above Padé approximation to parameterize $\lambda(x)$

$$\lambda(x) = \lambda_0 \frac{1 + ax^2}{1 + bx^2}$$

since it has as equal potential as series in approximating a continuous function. The result for the optimized parameters is given by

$$a = 0.30218, b = 1.331975, c = 2.094056, w = 0.286566, \lambda_0 = 7.323787$$

with $E^{2k} = 1.23152(6\pi^2 f_2/e)$. The solution (6), with $\lambda(x)$ given by (7) and the parameters (9), is referred as solution Hyb(2/2) for short in this paper and is plotted in Fig. 1 compared to the solutions (4) and (5), and the numerical solution (Num.) to the equation (3). We also include the analytic solutions given by (9) and (10).

The minimization of the energy (2) with the trial function (6) with respect to the variational parameters ($a, b, c, w, \lambda_0$) was carried out numerically for the [2/2] Padé approximant (11) using the downhill simplex method (the Nelder-Mead algorithm). The result for the optimized parameters is given by

$$a = 0.2598, b = 0.5446, c = 1.9932, \lambda_0 = 0.0538, w = 0.1226, \lambda_0 = 0.39439, \lambda_0 = 0.39439$$

The solution (6) with $\lambda(x)$ specified by (11) and (12) will be referred as the Hyb(4/4) in this paper and is also plotted in Fig. 1. The Fig.2 shows the behaviors of $F(x)$ at large $x$ for Hyb(4/2) as well as Hyb(4/4), and the numeric solution. The asymptotic expansion of the solution Hyb(4/4) shows that for small $x \ll 1$ the profile becomes

$$F_w(x) = \pi - 2.028368x + 0.518855x^3 - 0.641539x^5 + \cdots.$$ (10)

One can see that (11) agrees well with (10) up to $x^3$. For large $x \to \infty$ the series solution for $F(x)$ can be obtained by solving (3) with $x$ replaced by $1/y$ and using the series expansion for small $y$. After re-changing $y$ to $x$, one finds

$$F(x) = \frac{2.1596}{x^2} \left\{1 - \frac{0.222}{x^4} - \frac{116.0}{x^6} + \frac{0.113}{x^8} + \frac{2.71}{x^{10}} + \cdots\right\}.$$ (12)

On the other hand, the solution Hyb(2/2), at large $x$, has the asymptotic form

$$F_w(x) = \frac{2.0348}{x^2} \left\{1 + \frac{0.91576}{x^2} - \frac{5.8524}{x^4} + \frac{9.6819}{x^6} + \frac{3.1677}{x^8} - \frac{51.943}{x^{10}} + \cdots\right\},$$ (13)

which agrees globally with (12) except for a small bit differences. The detailed differences between (12) and (13) at large $x$ can be due to the fact that the variationally-obtained solution (6) approximates the Skyrmion profile globally may produce large errors in local region, for instance, $F_w(50) = 8.14 \times 10^{-4}$ while $F(50) = 8.638 \times 10^{-4}$.

The disagreement can be improved by employing the Padé approximant of higher order than (6), for example, the [4/4] approximant

$$\lambda(x) = \lambda_0 \frac{1 + ax^2 + bx^4}{1 + bx^2 + bx^4}.$$ (14)

The minimization of (2) using (14), as done for the [2/2] approximant, yields the numerically optimal parameters,

$$a = 0.2598, b = 0.5446, c = 1.9932, \lambda_0 = 0.0538, \lambda_0 = 0.39439, \lambda_0 = 0.39439$$

The solution (6) with $\lambda(x)$ specified by (11) and (12) will be referred as the Hyb(4/4) in this paper and is also plotted in Fig. 1. The Fig.2 shows the behaviors of $F(x)$ at large $x$ for Hyb(2/2) as well as Hyb(4/4), and the numeric solution. The asymptotic expansion of the solution Hyb(4/4) shows that for small $x \ll 1$ the profile becomes

$$F_w(x) = \pi - 2.0243x + 0.4654x^3 - 0.4270x^5 + \cdots.$$ while for $x \to \infty$

$$F_w(x) = \frac{2.220}{x^2} \left\{1 - \frac{0.9150}{x^2} - \frac{9.5908}{x^4} + \frac{65.358}{x^6} + \frac{264.78}{x^8} + \frac{821.88}{x^{10}} + \cdots\right\},$$

Here, a better value $F_w(50) = 8.877 \times 10^{-4}$ is obtained for the latter asymptotic profile in contrast with the solution Hyb(2/2). The computed Skyrmon energies (4), measured in the unit of $2f_2/e$, are listed in Table I, including the corresponding results obtained by the numeric solution and obtained in the relevant references as indicated.
In solving (3) numerically, we employ the nonlinear shoot algorithm for the boundary values at $x = 0.001$ and $x = 1000$ based on the asymptotic formulas (10) and (12) of the chiral angle $F(x)$.

3. The static properties of nucleons at low energy

The static properties of nucleons can be extracted by semi-classically quantizing the spinning modes of Skyrme Lagrangian using the collective variables [3]. Here, we will use the solution Hyb(2/2) and Hyb(4/4) to compute the static properties of nucleons and nucleon-isobar($\Lambda$) within the framework of the bosonic quantization of Skyrme model.

Following Adkin et al. [4], one can choose $SU(2)$-variable $A(t)$ as the collective variables, and substitute $U = A(t)U_{0}(x)A(t)^{\dagger}$ into (11). In the adiabatic limit, one has

$$S^{\Sigma K} = S^{\Sigma K}_{0} + i_0 \int dt Tr \left[ \frac{\partial A}{\partial t} \frac{\partial A^\dagger}{\partial t} \right],$$

(16)

with $S^{\Sigma K}_{0}$ the action for the static hedgehog configuration, $I_0 = \pi/(3e^2 f_\pi)$, and

$$\Lambda = 8 \int_{0}^{\infty} x^2 dx \sin^2 F[1 + F_x^2 + \sin^2 F/x^2].$$

(17)

which is independent of $f_\pi$ and $e$. The Hamiltonian associated to (15), when quantized via the procedure described in terms of collective coordinates, yields an eigenvalue $(H = M + J(J + 1))/(2I_0\Lambda)$, with $M = E^{\Sigma K}$ being the soliton energy of the Skyrmion. This yields the masses of the nucleon and $\Lambda$-isobar

$$M_N = M + \frac{3}{8I_0\Lambda}, \quad M_\Lambda = M + \frac{15}{8I_0\Lambda}.$$  

(18)

By adjusting $f_\pi$ and $e$ to fit the nucleon and delta masses in (18), one can fix the model parameters $f_\pi$ and $e$ using the calculated $M$ and $\Lambda$ through (2) and (17).

The isoscalar root mean square(r.m.s) radius and isoscalar magnetic r.m.s radius are given by

$$e_{f_\pi}(r^2)^{1/2}_{J=0} = \left(-\frac{2}{\pi} x^2 \sin^2 F x \right)^{1/2}$$

$$e_{f_\pi}(r^2)^{1/2}_{J=0} = \left(\int_{0}^{\infty} x^2 \sin^2 F x dx \right)^{1/2}$$

(19)

respectively. Combining with the masses of nucleon and the delta, one can evaluate the magnetic moments for proton and neutron via the following formula

$$\mu_{p,n} = \mu_{p,n}^{J=0} + \mu_{p,n}^{J=1} = \frac{(r^2)^{J=0}}{9} M_N(M_\Lambda - M_N) \pm \frac{M_N}{2(M_\Lambda - M_N)},$$

(20)

where plus and minus correspond to proton and neutron, respectively. The calculated results for these quantities using two solution schemes (Hyb(2/2) and Hyb(4/4)) are shown explicitly in Table II, compared to the experimental values as well as that computed by the numerical solution for $F(x)$. The corresponding results from other predictions are also shown in this table. Here in Table II, we use the experimental values $M_N = 938.9MeV$, $M_\Lambda = 1232MeV$ for fixing $e$ and $f_\pi$ through (15), in contrast with the input $M_N = 938MeV$, $M_\Lambda = 1232MeV$ used by Ref. [3] and Ref. [4].

To check the solution further, we also list, in the Table II, the axial coupling constant and the $nn$-coupling, which are given by

$$g_A = -\frac{\pi}{3e^2} G, \quad g_{nn} = \frac{M_N}{f_\pi}$$

(21)

respectively. Here, the numeric factor $G$ is

$$g = 4 \int_{0}^{\infty} dx \sin^2 F \left[ F_x + \frac{\sin 2F}{x} \right] \left[ F_x + \frac{\sin 2F}{x} \right]^2$$

(22)

4. Concluding remarks

We show that the hybrid form of a kink-like solution and that given by the instanton method are suited to approximate the exact solution for the hedgehog Skyrmion, when combining with Padé approximation. The resulted analytic solution (6) has two remarkable features: (1) it is simple in the sense that it is globally given in whole region; (2) it well approaches the asymptotic behavior of the exact solution. We note that the further generalization of (6), made by approximating $c$ in (6) via Padé approximation, does not exhibit remarkable improvement, particularly in the asymptotic behavior of the chiral angle $F(x)$ at infinity. We expect that our solution can be useful in the dynamics study of the Skyrmion evolution and interactions.

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Table I

| Work | Ref.[3] | Ref.[13] | Ref.[6] | Ref.[14] | Ref.[12] | Hyb(2/2) | Hyb(4/4) | Num. |
|------|---------|---------|-------|----------|---------|----------|----------|-----|
| &times;3 &times;10^{-3} | 36.5 | 36.47 | 36.484 | 36.47 | 36.4638 | 36.4638 | 36.4616 |
| &times;3 &times;10^{-4} | 1.233 | 1.2317 | 1.2322 | 1.23186 | 1.2317 | 1.23152 | 1.23146 | 1.23145 |

Table II

| Quantities | Ref.[3] | Ref.[14] | Hyb.(2/2) | Hyb.(4/4) | Num. | Expt. |
|------------|---------|---------|----------|----------|-----|------|
| M/(2f_p/e) | 36.5 | 36.5 | 36.4638 | 36.4638 | 36.4616 | – |
| 2f_p(MeV) | 129 | 130 | 128.730 | 129.453 | 129.260 | 186 |
| e | 5.45 | 5.48 | 5.4229 | 5.4527 | 5.4446 | – |
| Λ | 50.9 | 52.2 | 50.1467 | 51.2830 | 50.9782 | – |
| &times;3 &times;10^{-12}(fm) | 0.59 | 0.586 | 0.5985 | 0.5920 | 0.5938 | 0.72 |
| &times;3 &times;10^{-12}(fm) | 0.92 | 0.920 | 0.9258 | 0.9208 | 0.9222 | 0.81 |
| μ_p | 1.87 | – | 1.8825 | 1.8764 | 1.8781 | 2.79 |
| μ_n | –1.31 | – | –1.3209 | –1.3269 | –1.3253 | –1.91 |
| | | | 1.43 | – | 1.4252 | 1.4141 | 1.4171 | 1.46 |
| g_A | 0.61 | – | 0.6332 | 0.5992 | 0.6081 | 1.23 |
| g_{σNN} | 8.9 | – | 9.2364 | 8.6921 | 8.8343 | 13.5 |

Fig.1.

Fig.2.
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1. Introduction

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In this Letter, we address the static solution of hedgehog Skyrmion in the SU(2) Skyrme model without pion mass term and propose an analytical solution for the hedgehog Skyrmion by writing it as the hybrid form of a kink-like solution and the analytic solution obtained by the instanton method\cite{8}. Two lowest order of Padé approximations were used and the corresponding solutions for Skyrmion profile are given explicitly by using the downhill simplex method. The Skyrmion mass and static properties of nucleons as well as delta was computed and compared to the previous results.

2. Analytic solution to Skyrme model

The SU(2) Skyrme action\cite{1} without pion mass term is given by

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in which $L_{\mu} = U^\dagger \partial_{\mu} U$, $U(x, t) \in SU(2)$ is the nonlinear realization of the chiral field describing the $\sigma$ field and $\pi$ mesons with the unitary constrain $U^\dagger U = 1$, $2f_\pi$ the pion decay constant, and $e$ a dimensionless constant characterizing nonlinear coupling. The Cauchy-Schwartz inequality for \cite{1} implies

$$ E^{SK} \geq 6\pi^2(f_\pi/e)B, $$

where $B \equiv (1/24\pi^2) \int d^4x e^{ik/2} Tr(L_{\mu}L_{\nu})$ is the topological charge, known as baryon number. Using the hedgehog ansatz, $U(x) = \cos(F) - i(\hat{\sigma} \cdot \partial) \sin(F) / F$ (the $\hat{\sigma}$ are the three Pauli matrices) with $F \equiv F(r)$ depending merely on the radial coordinate $r$, the static energy for \cite{1} becomes

$$ E^{SK} = 2\pi\int_e (x^2 F_x^2 + 2 \sin^2(F)(1 + F_x^2) + \sin(F)^2) dx $$

with $x = e f_\pi r$ a dimensionless variable and $F_x \equiv dF(x)/dx$. The equation of motion of \cite{2} is

$$ \left( 1 + 2 \frac{\sin^2(F)}{x^2} \right) F_{xx} + 2 \frac{\sin(2F)}{x^2} \left( F_x^2 - 1 - \frac{\sin^2 F}{x^2} \right) = 0, $$

where the boundary condition $F(0) = \pi, F(\infty) = 0$ will be imposed so that it corresponds to the physical vacuum for $U = \pm 1$. The equation \cite{2} is usually solved numerically due to its high nonlinearity\cite{3, 4, 8, 9}.

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$$F_w(x) = 4w\arctan[\exp(-c x)] + \pi(1 - w) \left[1 - (1 + \frac{\lambda(x)}{x^2})^{-1/2}\right]$$

with $w \in [0, 1]$ being a positive weight factor. In principle, one can find the governing equation for the unknown $\lambda(x)$ by substituting (6) into (3) and obtain a series solution of $\lambda(x)$ by solving the governing equation. Here, however, we choose the Padé approximation to parameterize $\lambda(x)$

$$\lambda(x) = \lambda_0 + \frac{1 + ax^2 + \cdots}{1 + bx^2 + \cdots},$$

since it has as equal potential as series in approximating a continuous function. Note that we have already written $\lambda(x)$ as function of $x^2$ instead of $x$ since so is $F_w(x)$ in (5). The simplest nontrivial case of the above Padé approximation is the [2/2] approximant

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The minimization of the energy (2) with the trial function (6) with respect to the variational parameters $(a, b, c, w, \lambda_0)$ was carried out numerically for the [2/2] Padé approximant (8) using the downhill simplex method (the Nelder-Mead algorithm). The result for the optimized parameters is given by

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To check how well the asymptotic behavior of (12) is we apply the asymptotic expansion analysis on the profile $F(x)$. For small $x \ll 1$ the solution $F(x)$ to the equation (3) is given by

$$F(x) = \pi + Ax + Bx^3 + Cx^5 + \cdots$$

$$= \pi - 2.007528x + 0.358987x^3 - 0.146499x^5 + \cdots,$$

(see also [12], where the variable $x$ used is twice of $x$ in this paper) while the analytic solution (6), when (8) and (9) is used, behaves like

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One can see that (11) agrees well with (10) up to $x^5$. For large $x \rightarrow \infty$ the series solution for $F(x)$ can be obtained by solving (3) with $x$ replaced by $1/y$ and using the series expansion for small $y$. After re-changing $y$ to $x$, one finds

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The disagreement can be improved by employing the Padé approximant of higher order than (6), for example, the [4/4] approximant

$$\lambda(x) = \lambda_0 + \frac{1 + ax^2 + a_4x^4}{1 + bx^2 + b_4x^4}.$$ (14)

The minimization of (2) using (14), as done for the [2/2] approximant, yields the numerically optimal parameters,

$$a = 0.2598, b = 0.5446, c = 1.9932, a_4 = 0.0538, b_4 = 0.1226, w = 0.1839, \lambda_0 = 3.9439.$$

The solution (6) with $\lambda(x)$ specified by (14) and (15) will be referred as the Hybrid(4/4) in this paper and is also plotted in Fig.1. The Fig.2 shows the profiles of $F(x)$ at large $x$ for Hyb(2/2) as well as Hybrid(4/4), and the numeric solution. The asymptotic expansion of the solution Hybrid(4/4) shows that for small $x \ll 1$ the profile becomes

$$F_w(x) = \pi - 2.0424x + 0.4654x^3 - 0.4270x^5 + \cdots$$

while for $x \rightarrow \infty$

$$F_w(x) = \frac{2.220}{x^2} \left\{1 - \frac{0.9150}{x^2} - \frac{9.5908}{x^4} - \frac{65.358}{x^6} + \frac{264.78}{x^8} - \frac{821.88}{x^{10}} + \cdots\right\},$$

Here, a better value $F_w(50) = 8.877 \times 10^{-4}$ is obtained for the latter asymptotic profile in contrast with the solution Hybrid(2/2). The computed Skyrmon energies (6), measured in the unit of $2f_e/e$, are listed in Table 1, including the corresponding results obtained by the numeric solution and obtained in the relevant references as indicated.
In solving (3) numerically, we employ the nonlinear shoot algorithm for the boundary values at \( x = 0.001 \) and \( x = 1000 \) based on the asymptotic formulas (10) and (12) of the chiral angle \( F(x) \).

3. The static properties of nucleons at low energy

The static properties of nucleons can be extracted by semi-classically quantizing the spinning modes of Skyrme Lagrangian using the collective variables \( S \). Here, we will use the solution Hyb(2/2) and Hyb(4/4) to compute the static properties of nucleons and nucleon-isobar(\( \Delta \)) within the framework of the bosonic quantization of Skyrme model.

Following Adkin et al. [3], one can choose \( SU(2) \)-variable \( A(t) \) as the collective variables, and substitute \( U = A(t)U_{0}(x)A(t)^{\dagger} \). In the adiabatic limit, one has

\[
S^{SK} = S_{0}^{SK} + i\hbar\Lambda\int dtTr\left[\frac{\partial A}{\partial t} \frac{\partial A^{\dagger}}{\partial t}\right],
\]

with \( S_{0}^{SK} \) the action for the static hedgehog configuration, \( \hbar = (3e^{3}f_{A}) \), and

\[
\Lambda = 8\int_{0}^{\pi}x^{2}\sin^{2} F[1 + F_{x}^{2} + \sin^{2} F/x^{2}].
\]

which is independent of \( f_{A} \) and \( e \). The Hamiltonian associated to (16), when quantized via the quantization procedure in terms of collective coordinates, yields an eigenvalue (\( H = M + J(J + 1)/(2\hbar\Lambda) \), with \( M = E^{SK} \) being the soliton energy of the Skyrmion. This yields the masses of the nucleon and \( \Delta \)-isobar

\[
M_{N} = M + \frac{3}{8\hbar\Lambda}, \quad M_{\Delta} = M + \frac{15}{8\hbar\Lambda}.
\]

By adjusting \( f_{A} \) and \( e \) to fit the nucleon and delta masses in (18), one can fix the model parameters \( f_{A} \) and \( e \) using the calculated \( M \) and \( \Lambda \) through (2) and (17).

The isoscalar root mean square(r.m.s) radius and isoscalar magnetic r.m.s radius are given by

\[
e_{f_{A}}(r^{2})_{I=0}^{1/2} = \sqrt{\frac{2}{\pi}}x^{2}\sin^{2} F_{x}
\]

\[
e_{f_{A}}(r^{2})_{M=0}^{1/2} = \frac{1}{2}\left[\frac{1}{x^{2}}\sin^{2} F_{x}dx\right]^{1/2}
\]

respectively. Combining with the masses of nucleon and the delta, one can evaluate the magnetic moments for proton and neutron via the following formula

\[
\mu_{p,r} = \mu_{p,r}^{I=0} + \mu_{p,r}^{I=1} = \frac{(r^{2})_{I=0}}{9}M_{N}(M_{\Delta} - M_{N}) \pm \frac{M_{N}}{2}M_{\Delta} - M_{N})\]

where plus and minus correspond to proton and neutron, respectively. The calculated results for these quantities using two solution schemes (Hyb(2/2) and Hyb(4/4)) are shown explicitly in Table II, compared to the experimental values as well as that computed by the numeric solution for \( F(x) \). The corresponding results from other predictions are also shown in this table. Here in Table II, we use the experimental values \( M_{N} = 938.9MeV, \quad M_{\Delta} = 1232MeV \) for fixing \( e \) and \( f_{A} \) through (18), in contrast with the input \( M_{N} = 938MeV, \quad M_{\Delta} = 1232MeV \) used by Ref. [3] and Ref. [14].

To check the solution further, we also list, in the Table II, the axial coupling constant and the \( nN \)-coupling, which are given by

\[
g_{A} = -\frac{\pi}{3e^{2}}G, \quad g_{nN} = \frac{M_{N}}{\pi f_{A}}
\]

respectively. Here, the numeric factor \( G \) is

\[
G = 4\int_{0}^{\infty}dxx^{2}\left[F_{x} + \frac{\sin 2F}{x} + \frac{\sin 2F}{x}(F_{x})^{2}
\right.

\]

\[
+ \frac{2\sin^{2} F}{x^{2}}F_{x} + \frac{\sin^{2} F}{x^{2}}\sin 2F\right].
\]

4. Concluding remarks

We show that the hybrid form of a kink-like solution and that given by the instanton method are suited to approximate the exact solution for the hedgehog Skyrmion, when combining with Padé approximation. The resulted analytic solution \( \xi \) has two remarkable features: (1) it is simple in the sense that it is globally given in whole region; (2) it well approaches the asymptotic behavior of the exact solution. We note that the further generalization of \( \xi \), made by approximating \( c \) in \( \xi \) via Padé approximation, does not exhibit remarkable improvement, particularly in the asymptotic behavior of the chiral angle \( F(x) \) at infinity. We expect that our solution can be useful in the dynamics study of the Skyrmion evolution and interactions.

Acknowledgements

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Table I

| Work | Ref. [3] | Ref. [13] | Ref. [6] | Ref. [14] | Hyb(2/2) | Hyb(4/4) | Num. |
|------|----------|----------|----------|----------|----------|----------|------|
| $\frac{E}{(2f_\pi/e)}$ | 36.5 | 36.47 | 36.484 | 36.47 | 36.4638 | 36.4638 | 36.4616 |
| $\frac{M}{(6f_\pi/e)}$ | 1.233 | 1.2317 | 1.2322 | 1.23186 | 1.2317 | 1.23152 | 1.23146 |

Table II

| Quantities | Ref. [3] | Ref. [14] | Hyb(2/2) | Hyb(4/4) | Num. | Expt. |
|------------|----------|----------|----------|----------|------|-------|
| $2f_\pi$ (MeV) | 129 | 130 | 128.730 | 129.453 | 129.260 | 186 |
| $e$ | 5.45 | 5.48 | 5.4229 | 5.4527 | 5.4446 | –  |
| $\Lambda$ | 50.9 | 52.2 | 50.1467 | 51.2830 | 50.9782 | –  |
| $(r^2)_{\text{MeV}}$ (fm) | 0.59 | 0.586 | 0.5985 | 0.5920 | 0.5938 | 0.72 |
| $(r^2)_{\text{MeV}}$ (fm) | 0.92 | 0.920 | 0.9258 | 0.9208 | 0.9222 | 0.81 |
| $\mu_p$ | 1.87 | – | 1.8825 | 1.8764 | 1.8781 | 2.79 |
| $\mu_n$ | –1.31 | – | –1.3209 | –1.3269 | –1.3253 | –1.91 |
| $|\mu_p/\mu_n|$ | 1.43 | – | 1.4252 | 1.4141 | 1.4171 | 1.46 |
| $g_A$ | 0.61 | – | 0.6332 | 0.5992 | 0.6081 | 1.23 |
| $g_{\pi NN}$ | 8.9 | – | 9.2364 | 8.6921 | 8.8343 | 13.5 |

![Fig.1](image1.png)

![Fig.2](image2.png)
Analytical Solution for the SU(2) Hedgehog Skyrme Model and Static Properties of Nucleons

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Abstract
An analytical solution for symmetric Skyrmion was proposed for the SU(2) Skyrme model, which take the form of the hybrid form of a kink-like solution and that given by the instanton method. The static properties of nucleons was then computed within the framework of collective quantization of the Skyrmie model, with a good agreement with that given by the exact numeric solution. The comparisons with the previous results as well as the experimental values are also given.

Keywords: Skyrme Model, Soliton, Nucleons
PACS: 12.38.-t, 11.15.Tk, 12.38.Aw

1. Introduction
The Skyrme model [1] is an effective field theory of mesons and baryons in which baryons arise as topological soliton solutions, known as Skyrmions. The model is based on the pre-QCD nonlinear σ model of the pion meson and was usually regarded to be consistent with the low-energy limit of large-N QCD [2]. For this reason, among others, it has been extensively revisited in recent years [3, 4, 5, 6, 7, 8] (see, [9, 10, 11], for a review). Owing to the high nonlinearity, the solution to the Skyrme model was mainly studied through the numerical approach. It is worthwhile, however, to seek the analytic solutions [12, 13, 14] of Skyrmions due to its various applications in baryon phenomenology. One of noticeable analytic method for studying the Skyrmion solutions is the instanton approach proposed by Atiyah and Manton [8] which approximates critical points of the Skyrmie energy functional.

In this Letter, we address the static solution of hedgehog Skyrmion in the SU(2) Skyrme model without pion mass term and propose an analytical solution for the hedgehog Skyrmion by writing it as the hybrid form of a kink-like solution and the analytic solution obtained by the instanton method [8]. Two lowest order of Padé approximations was used and the corresponding solutions for Skyrmion profile are given explicitly by using the downhill simplex method. The Skyrmion mass and static properties of nucleon as well as delta was computed and compared to the previous results.

2. Analytic solution to Skyrme model
The SU(2) Skyrme action [1] without pion mass term is given by

\[ S^{SK} = \int \frac{d^4x}{e} [\frac{F_x^2}{4} Tr(L_{\mu}^2) + \frac{1}{32\pi^2} Tr([L_{\mu}, L_{\nu}])^2] \]

in which \( L_{\mu} = U^* \partial_{\mu} U \). \( U(x, t) \in SU(2) \) is the nonlinear realization of the chiral field describing the σ field and π mesons with the unitary constrain \( U^* U = 1 \), \( 2f_\pi \), the pion decay constant, and \( e \) is a dimensionless constant characterizing nonlinear coupling. The Cauchy-Schwartz inequality for (1), implies [15] \( E^{SK} \geq 6\pi^2(f_\pi/e)|B| \), where \( B \equiv (1/24\pi^5) \int d^4x e^{iF} Tr(L_{i}\dot{L}_{j}\dot{L}_{k}) \) is the topological charge, known as baryon number. Using the hedgehog ansatz, \( U(x) = \cos(F) - i(\dot{x} \cdot \vec{\sigma}) \sin(F) (\vec{\sigma} \text{are the three Pauli matrices}) \) with \( F \equiv F(r) \) depending merely on the radial coordinate \( r \), the static energy for (1) becomes

\[ E^{SK} = 2\pi(f_\pi/e) \int dx \left[ x^2 F_x^2 + 2\sin^2(F)(1 + F_x^2) + \frac{\sin^4(F)}{x^2} \right] \]

with \( x = ef_\pi r \) a dimensionless variable and \( F_x \equiv dF(x)/dx \). The equation of motion of (2) is

\[ \left( 1 + 2\frac{\sin^2 F}{x^2} \right) F_{xx} + \frac{2}{x} F_x + \frac{\sin(2F)}{x^2} \left( F_x^2 - 1 - \frac{\sin^2 F}{x^2} \right) = 0 \]

where the boundary condition \( F(0) = \pi, F(\infty) = 0 \) will be imposed so that it corresponds to the physical vacuum for (1): \( U = \pm 1 \). The equation (3) is usually solved numerically due to its highly-nonlinearity, as done in most calculations [3, 4, 5, 6, 7, 8].

A kink-like analytic solution was given by [8]

\[ F_1(x) = 4 \arctan[\exp(-x)], \]

with \( E^{SK} = 1.24035(6\pi^2f_\pi/e) \), while an alternative Skyrmion profile, proposed based on the instanton method, takes the form [8]

\[ F_2(x) = \pi \left[ 1 - (1 + \frac{A}{x^2})^{-1/2} \right], \]

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with corresponding energy $1.2432(6n^2f_e/e)$ for the numeric factor $\lambda = 2.109$. The singularity at $r = 0$ is gauge dependent and can be gauged away without affecting the value for the Skyrmion field. To find the more accurate analytic solution, we first improve the solution (2) into 4 arctan[exp(−cx)] with c a numeric factor and then take $\lambda$ to be a $x$-dependent function: $\lambda \rightarrow \lambda(x)$. Hence, we propose a Skyrmion profile function in the hybrid form mixing (4) and (5)

$$F_w(x) = 4w \arctan[\exp(-cx)] + \pi(1 - w) \left[1 - (1 + \frac{\lambda(x)}{x^2})^{1/2}\right]$$

(6)

with $w \in [0,1]$ being a positive weight factor. In principle, one can find the governing equation for the unknown $\lambda(x)$ by substituting (6) into (3) and obtain a series solution of $\lambda(x)$ by solving the governing equation. Here, however, we choose the Padé approximation to parameterize $\lambda(x)$

$$\lambda(x) = \lambda_0 + \frac{a x^2}{1 + b x^2} + \cdots,$$

(7)

since it has as equal potential as series in approximating a continuous function. Note that we have already written $\lambda(x)$ as function of $x^2$ instead of $x$ since so is $F_w(x)$ in (5). The simplest nontrivial case of the above Padé approximation is the [2/2] approximant

$$\lambda(x) = \lambda_0 + \frac{a x^2}{1 + b x^2}.$$

(8)

The minimization of the energy (3) with the trial function (5) with respect to the variational parameters ($a$, $b$, $c$, $w$, $\lambda_0$) was carried out for the [2/2] Padé approximant (8) using the downhill simplex method (the Nelder-Mead algorithm). The result for the numerically optimized parameters is given by

$$a = 0.330218, b = 1.331975, c = 2.094056,$$

$$w = 0.286566, \lambda_0 = 7.323877.$$

(9)

with $E_s^K = 1.23152(6n^2f_e/e)$. The solution (6), with $\lambda(x)$ given by (7) and the parameters (9), is referred as solution Hyb(2/2) for short in this paper and is plotted in Fig.1, compared to the solutions (4) and (5), and the numerical solution (Num.) to the equation (3). We also include the analytic solutions given by (14) and the solution in the form of the purely Padé approximant (13) for comparison. A quite well agreement of our solution with the numerical solution can be seen from this plot. We note that the inequality $E_s^K /[6n^2f_e/e] \geq |B| = 1$ is fulfilled for all of cited results of the energy(see Table I).

To check how well the asymptotic behavior of (6) is we apply the asymptotic expansion analysis on the profile $F(x)$. For small $x \ll 1$ the solution $F(x)$ to the equation (3) can be given by solving the equation via series expansion, which is

$$F(x) = \pi + Ax + Bx^3 + Cx^5 + \cdots$$

$$= \pi - 2.007528 x + 0.358987 x^3 - 0.146499 x^5 + \cdots,$$

(10)

(see also [12], where the variable x used is twice of x in this paper) while the analytic solution (6), when (5) is used, behaves like

$$F_w(x) = \pi - 2.028368 x + 0.518855 x^3 - 0.641539 x^5 + \cdots.$$  

(11)

One can see that (11) agrees well with (10) up to $x^8$. For large $x \rightarrow \infty$ the series solution for $F(x)$ can be obtained by solving (3) with $x$ replaced by $1/y$ and using the series expansion for small $y$. After re-changing $y$ to $x$, one finds

$$F(x) = \frac{2.1596}{x^2} \left[1 - \frac{0.222}{x^4} - \frac{116.0}{x^6} + \frac{0.113}{x^8} + \frac{2.71}{x^{10}} + \cdots\right].$$

(12)

On the other hand, the solution Hyb(2/2), at large $x$, has the asymptotic form

$$F_w(x) = \frac{2.0348}{x^2} \left[1 - \frac{0.91576}{x^2} - \frac{5.8524}{x^4} + \frac{9.6819}{x^6} + \frac{3.1677}{x^8} - \frac{51.943}{x^{10}} + \cdots\right],$$

(13)

which agrees globally with (12) except for a small bit differences. The detailed differences between (12) and (13) at large $x$ can be due to the fact that the variationally-obtained solution (6) approximates the Skyrmion profile globally and may produce small errors in local region, for instance, $F_w(50) = 8.142 \times 10^{-4}$ while $F(50) = 8.638 \times 10^{-4}$.

The disagreement can be improved by employing the Padé approximant of higher order than (6), for example, the [4/4] approximant

$$\lambda(x) = \lambda_0 + \frac{a x^2 + a x^4}{1 + b x^2 + b x^4}.$$

(14)

The minimization of (2) using (14), as done for the [2/2] approximant, yields the numerically optimal parameters,

$$a = 0.2598, b = 0.5446, c = 1.9932, a_4 = 0.0538, b_4 = 0.1226, w = 0.1839, \lambda_0 = 3.9439,$$

(15)

The solution (6) with $\lambda(x)$ specified by (14) and (15) will be referred as the Hyb(4/4) in this paper and is also plotted in Fig.1. The Fig.2 shows the profiles of $F(x)$ at large $x$ for Hyb(2/2) as well as Hyb(4/4), and the numeric solution. The asymptotic expansion of the solution Hyb(4/4) shows that for small $x \ll 1$ the profile becomes while for $x \rightarrow \infty$
In solving (3) numerically, we employ the nonlinear shoot algorithm for the boundary values at \( x = 0.001 \) and \( x = 1000 \) based on the asymptotic formulas (10) and (12) of the chiral angle \( F(x) \).

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The static properties of nucleons can be extracted by semi-classically quantizing the spinning modes of Skyrme Lagrangian using the collective variables [3]. Here, we will use the solution Hyb(2/2) and Hyb(4/4) to compute the static properties of nucleons and nucleon-isobar(\( \Delta \)) within the framework of the bosonic quantization of Skyrme model.

Following Adkin et al. [3], one can choose \( SU(2) \)-variable \( A(t) \) as the collective variables, and substitute \( U = A(t)U_0(x)A(t) \) into (1). In the adiabatic limit, one has

\[
S^{SK} = S_0^{SK} + i_0 \Lambda \int dt Tr \left[ \frac{\partial A}{\partial t} \frac{\partial A^\dagger}{\partial t} \right],
\]

with \( S_0^{SK} \) the action for the static hedgehog configuration, \( i_0 = \pi/(3e^2 f_\pi) \), and

\[
\Lambda = 8 \int_0^\infty x^2 dx \sin^2 F[1 + F_x^2 + \sin^2 F/x^2].
\]

which is independent of \( f_\pi \) and \( e \). The Hamiltonian associated to (16), when quantized via the quantization procedure in terms of collective coordinates, yields an eigenvalue \( \langle H \rangle = M + J(J + 1)/(2i_0 \Lambda) \), with \( M = E^{SK} \) being the soliton energy of the Skyrmion. This yields the masses of the nucleon and \( \Delta \)-isobar

\[
M_N = M + \frac{3}{8i_0 \Lambda}, \quad M_\Delta = M + \frac{15}{8i_0 \Lambda}.
\]

By adjusting \( f_\pi \) and \( e \) to fit the nucleon and delta masses in (18), one can fix the model parameters \( f_\pi \) and \( e \) using the calculated \( M \) and \( \Lambda \) through (2) and (17).

The isoscaler root mean square(r.m.s) radius and isoscaler magnetic r.m.s radius are given by

\[
e_{fs}(r^2)_{I=0}^{1/2} = \frac{3}{\pi} \int_0^\infty x^2 F_x^2 dx \]

\[
e_{fs}(r^2)_{M=0}^{1/2} = \left( \frac{1}{4} \right) \int_0^\infty x^2 \sin^2 F_x dx \]

respectively. Combining with the masses of nucleon and the delta, one can evaluate the magnetic moments for proton and neutron via the following formula

\[
\mu_{p,n} = \mu_{p,n}^{I=0} + \mu_{p,n}^{J=1} = \frac{9}{M_N(M_\Delta - M_N)} \frac{M_N}{(2M_\Delta - M_N)} \).
\]

where plus and minus correspond to proton and neutron, respectively. The calculated results for these quantities using two solution schemes (Hyb(2/2) and Hyb(4/4)) are shown explicitly in Table II, compared to the experimental values as well as that computed by the numeric solution for \( F(x) \). The corresponding results from other predictions are also shown in this table. Here in Table II, we use the experimental values \( M_N = 938.9 MeV \), \( M_\Delta = 1232 MeV \) for fixing \( e \) and \( f_\pi \) through (15), in contrast with the input \( M_N = 938 MeV \), \( M_\Delta = 1232 MeV \) used by Ref. [3] and Ref. [14].

To check the solution further, we also list in the table, the axial coupling constant and the \( nn \)-coupling, which are given by

\[
g_A = -\frac{\pi}{3e^2}, \quad g_{NN} = \frac{M_N}{f_\pi} \frac{g_A}{A}
\]

respectively. Here, the numeric factor \( G \) is

\[
G = 4 \int_0^\infty dx \left\{ \frac{F_x^2}{x^2} + \frac{2 \sin^2 F_x}{x^2} \right\}.
\]

4. Concluding remarks

We show that the hybrid form of a kink-like solution and that given by the instanton method are suited to approximate the exact solution for the hedgehog Skyrmion, when combining with Padé approximation. The resulted analytic solution (6) has two remarkable features: (1) it is simple in the sense that it is globally given in whole region; (2) it well approaches the asymptotic behavior of the exact solution. We note that the further generalization of (6), made by approximating \( c \) in (6) via Padé approximation, does not exhibit remarkable improvement, particularly in the asymptotic behavior of the chiral angle \( F(x) \) at infinity. We expect that our solution can be useful in the dynamics study of the Skyrmion evolution and interactions.

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Table I

| Work     | [3] | [13] | [6] | [14] | [12] | Hyb(2/2) | Hyb(4/4) | Num.    |
|----------|-----|------|-----|------|------|----------|----------|---------|
| $\frac{E}{2f_{\pi}/e}$ | 36.5 | 36.47 | 36.484 | 36.47 | 36.47 | 36.4638 | 36.4638 | 36.4616 |
| $\delta f_{\pi}/\psi$ | 1.233 | 1.2317 | 1.2322 | 1.23186 | 1.2317 | 1.23152 | 1.23146 | 1.23145 |

Table II

| Quantities | Ref [3] | Ref [14] | Hyb.(2/2) | Hyb.(4/4) | Num. | Expt. |
|------------|---------|----------|-----------|-----------|------|-------|
| $M/(2f_{\pi}/e)$ | 36.5 | 36.5 | 36.4638 | 36.4638 | 36.4616 | – |
| $2f_{\pi}(MeV)$ | 129 | 130 | 128.730 | 129.453 | 129.260 | 186 |
| $\epsilon$ | 5.45 | 5.48 | 5.4229 | 5.4527 | 5.4446 | – |
| $\Lambda$ | 50.9 | 52.2 | 50.1467 | 51.2830 | 50.9782 | – |
| $g^{T_{1/2}}_{\pi NN}(fm)$ | 0.59 | 0.586 | 0.5985 | 0.5920 | 0.5938 | 0.72 |
| $(r^2)_{\pi NN}^{T_{1/2}}(fm)$ | 0.92 | 0.920 | 0.9258 | 0.9208 | 0.9222 | 0.81 |
| $\mu_p$ | 1.87 | – | 1.8825 | 1.8764 | 1.8781 | 2.79 |
| $\mu_n$ | –1.31 | – | –1.3209 | –1.3269 | –1.3253 | –1.91 |
| $|\mu_p/\mu_n|$ | 1.43 | – | 1.4252 | 1.4141 | 1.4171 | 1.46 |
| $g_A$ | 0.61 | – | 0.6332 | 0.5992 | 0.6081 | 1.23 |
| $g_{\pi NN}$ | 8.9 | – | 9.2364 | 8.6921 | 8.8343 | 13.5 |

![Fig.1](image1.png)

![Fig.2](image2.png)