CONTINUITY OF WEAK SOLUTIONS TO ROUGH INFINITELY DEGENERATE EQUATIONS

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Abstract. We obtain a generalization of the DeGiorgi Lemma to the infinitely degenerate regime and apply it to obtain continuity of weak solutions to certain infinitely degenerate equations. This reproduces the continuity result obtained in [KoRiSaSh1] via Moser iteration, but only for homogeneous equations. However, the proofs are much less technical and more transparent.

Contents

1. Introduction 1
1.1. Preliminaries and definitions 2
1.2. Control balls 3
1.3. Higher dimensions 5
2. Proportional vanishing $L^1$-Sobolev inequality 5
2.1. Orlicz-Sobolev inequality 7
2.2. The DeGiorgi Lemma 8
3. Continuity of locally bounded weak solutions 8
3.1. Local boundedness 8
3.2. Caccioppoli inequality 9
3.3. Proof of Theorem 2 9
References 12

1. Introduction

In [KoRiSaSh2], building on work from [KoRiSaSh1], local boundedness was established for weak subsolutions to certain infinitely degenerate elliptic divergence form equations, motivated by the pioneering work of Fedii [Fe], Kusuoka and Strook [KuStr], Morimoto [Mor] and Christ [Chr]. The main theorem on local boundedness in [KoRiSaSh2] included this.

Theorem 1 [KoRiSaSh2]. Suppose that $D \subset \mathbb{R}^n$ is a domain in $\mathbb{R}^n$ with $n \geq 3$ and that

$$Lu \equiv \text{div} A(x,u) \nabla u, \quad x = (x_1, ..., x_n) \in D,$$

where $A(x,z) \sim \begin{bmatrix} I_{n-1} & 0 \\ 0 & f(x_1)^2 \end{bmatrix}$, $I_{n-1}$ is the $(n-1) \times (n-1)$ identity matrix, $A$ has bounded measurable components, and the geometry $F = -\ln f$ satisfies the structure conditions in Definition 6 below.

1. If $F \leq D_\sigma$ for some $0 < \sigma < 1$, then every weak solution to $Lu = \phi$ with $A$-admissible $\phi$ is locally bounded in $D$.

2. Conversely, if $n \geq 3$ and $\sigma > 1$, then there exists an unbounded weak solution $u$ in a neighbourhood of the origin in $\mathbb{R}^n$ to the equation $Lu = 0$ with geometry $F = D_\sigma$.

Where geometry $D_\sigma$ is defined as $D_\sigma(x) \equiv \left(\frac{1}{|x|}\right)^\sigma$, $x > 0$.

The purpose of this paper is to improve the local boundedness conclusion in part (1) of Theorem 1 to include continuity. For the geometric continuity theorem we need to consider a less degenerate family of geometries. For $k \geq 0$ and $0 < \sigma < \infty$, define $F_{k,\sigma}(r) = (\ln \frac{1}{r}) \left(\ln^{(k)} \frac{1}{r}\right)^\sigma$ and $f_{k,\sigma}(r) = e^{-F_{k,\sigma}(r)}$.

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\( r^{(\ln k)^{\frac{1}{\sigma}}} \). Note that \( F_{\sigma}(r) = (\ln \frac{1}{r})^{\frac{1}{\sigma}} \) and \( F_{\sigma}(r) = \frac{1}{r^\sigma} \) for \( 0 < \sigma < \infty \) are essentially the same families of geometries.

**Theorem 2.** Suppose that \( \Omega \subset \mathbb{R}^n \) is a domain in \( \mathbb{R}^n \) with \( n \geq 2 \) and that

\[
Lu = \text{div} A(x, u) \nabla u, \quad x = (x_1, \ldots, x_n) \in \Omega,
\]

where \( A(x, z) \sim \begin{bmatrix} I_{n-1} & 0 \\ 0 & f(x_1) \end{bmatrix} \), \( I_{n-1} \) is the \((n-1) \times (n-1)\) identity matrix, \( A \) has bounded measurable components, and the geometry \( F = -\ln f \) satisfies the structure conditions in Definition [6].

1. If \( F \leq F_{3, \sigma} \) for some \( 0 < \sigma < 1 \), then every weak solution to \( Lu = 0 \) is continuous in \( \Omega \).
2. On the other hand, if \( n \geq 3 \) and \( \sigma \geq 1 \), then there exists a locally unbounded weak solution \( u \) in a neighbourhood of the origin in \( \mathbb{R}^n \) to the equation \( Lu = 0 \) with geometry \( F = F_{0, \sigma} \).

### 1.1. Preliminaries and definitions.** We recall some of the terminology and definitions from [KoRiSaSh1] and [KoRiSaSh2] that we use here. Let \( A(x) \) be a nonnegative semidefinite \( n \times n \) matrix valued function in a bounded domain \( \Omega \subset \mathbb{R}^n \). We consider the second order special quasilinear equation (‘special’ because only \( u \), and not \( \nabla u \), appears nonlinearly),

\[
Lu \equiv \nabla^T A(x, u(x)) \nabla u = \phi, \quad x \in \Omega,
\]

and we assume the following quadratic form condition on the quasilinear matrix \( A(x, u(x)) \),

\[
k \xi^T A(x) \xi \leq \xi^T A(x, z) \xi \leq K \xi^T A(x) \xi,
\]

for a.e. \( x \in \Omega \) and all \( z \in \mathbb{R}, \xi \in \mathbb{R}^n \). Here \( k, K \) are positive constants and we assume that \( A(x) = B(x)^T B(x) \) where \( B(x) \) is a Lipschitz continuous \( n \times n \) real-valued matrix defined for \( x \in \Omega \). We also consider the linear equation

\[
Lu \equiv \nabla^T A(x) \nabla u = \phi, \quad x \in \Omega,
\]

and define the \( A \)-gradient by

\[
\nabla_A B(x) \nabla.
\]

**Definition 3.** The degenerate Sobolev space \( W^{1,2}_A(\Omega) \) is normed by

\[
\|v\|_{W^{1,2}_A(\Omega)} = \sqrt{\int_{\Omega} \left( |v|^2 + \nabla v^T A \nabla v \right)} = \sqrt{\int_{\Omega} \left( |v|^2 + |\nabla_A v|^2 \right)}.
\]

**Definition 4.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Assume that \( \phi \in L^2_{\text{loc}}(\Omega) \). We say that \( u \in W^{1,2}_A(\Omega) \) is a weak solution to \( Lu = \phi \) provided

\[- \int_{\Omega} \nabla w(\nabla^T A(x, u(x)) \nabla u = \int_{\Omega} \phi w \]

for all \( w \in (W^{1,2}_A(\Omega), \text{ where } (W^{1,2}_A(\Omega) \) denotes the closure in \( W^{1,2}_A(\Omega) \) of the subspace of Lipschitz continuous functions with compact support in \( \Omega \).

Note that our quadratic form condition \( [1.1] \) implies that the integral on the left above is absolutely convergent, and our assumption that \( \phi \in L^2_{\text{loc}}(\Omega) \) implies that the integral on the right above is absolutely convergent. Weak sub and super solutions are defined by replacing \( = \) with \( \geq \) and \( \leq \) respectively in the display above.

Given a geometry \( F = -\ln f \), we define the balls \( B \) to be the control balls associated with the \( n \times n \) matrix \( M_F(x) = \begin{bmatrix} I_{n-1} & 0 \\ 0 & f(x_1) \end{bmatrix} \). Assuming the structure conditions in Definition [6] below, we recall from [KoRiSaSh1] that the Lebesgue measure of the two dimensional ball \( B_{2D}(x, r) \) centered at \( x \in \mathbb{R}^2 \) with radius \( r > 0 \) satisfies

\[
|B_{2D}(x, r)| \approx \begin{cases} r^2 f(x_1) & \text{if } r \leq \frac{1}{|F'(x_1)|} \\ \frac{1}{|F'(x_1)|^2} & \text{if } r \geq \frac{1}{|F'(x_1)|} \end{cases}
\]
Definition 5. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and let $A(x)$ be a nonnegative semidefinite $n \times n$ matrix valued function as above. Fix $x \in \Omega$ and $\rho > 0$. We say $\phi$ is $A$-admissible at $(x, \rho)$ if

$$\|\phi\|_{X(B(x,\rho))} \equiv \sup_{v \in (W^{1,1}_A)_0(B(x,\rho))} \int_{B(x,\rho)} |v\phi| \, dy \leq \infty.$$ 

Definition 6 (structure conditions). We refer to the following five conditions on $F : (0, \infty) \to \mathbb{R}$ as structure conditions:

1. $\lim_{x \to 0^+} F(x) = +\infty$;
2. $F'(x) < 0$ and $F''(x) > 0$ for all $x \in (0, R)$;
3. $\frac{1}{2} |F'(r)| \leq |F'(x)| \leq C |F'(r)|$ for $\frac{1}{2} r < x < 2r < R$;
4. $\frac{1}{2} |F'(x)|$ is increasing in the interval $(0, R)$ and satisfies $\frac{1}{2} |F'(x)| \leq \frac{1}{\varepsilon}$ for $x \in (0, R)$;
5. $\frac{F'(x)}{F''(x)} \approx \frac{1}{2}$ for $x \in (0, R)$.

Remark 7. We make no smoothness assumption on $f$ other than the existence of the second derivative $f''$ on the open interval $(0, R)$. Note also that at one extreme, $f$ can be of finite type, namely $f(x) = x^\alpha$ for any $\alpha > 0$, and at the other extreme, $f$ can be of strongly degenerate type, namely $f(x) = e^{-\frac{x}{1-x}}$ for any $\alpha > 0$. Assumption (1) rules out the elliptic case $f(0) > 0$.

Notation 8. We refer to a function $F$ satisfying the structure conditions in Definition 6 as a ‘geometry’ since $F = -\ln f$ then specifies the nonnegative semidefinite matrix $M_F = \left[ \begin{array}{cc} 1_{n-1} & 0 \\ 0 & f(x_1)^2 \end{array} \right]$ and hence the geometry of the associated control balls. The class of degenerate elliptic linear operators

$$Lu = \text{div} A(x) \nabla u, \quad A(x) \sim M_F(x),$$

is also specified along with the associated class of quasilinear operators

$$L_u = \text{div} A\nabla u, \quad A(x, z) \sim M_F(x).$$

1.2. Control balls. We now recall further notation from [KoRiSaSh1] and [KoRiSaSh2], beginning with the case of $n = 2$ dimensions. Let $d(x, y)$ be the control metric on an open subset $\Omega$ of the plane $\mathbb{R}^2$ that is associated with the matrix $A$, and refer to the associated balls as control balls, subunit balls, or $A$-balls. Now we recall the definition of “height” of an arbitrary $A$-ball. Let $X = (x_1, 0)$ be a point on the positive $x$-axis and let $r$ be a positive real number. Let the upper half of the boundary of the ball $B(X, r)$ be given as the graph of the function $\varphi(x)$, $x_1 - r < x < x_1 + r$. Denote by $\beta_{X,P}$ the geodesic that meets the boundary of the ball $B(X, r)$ at the point $P = (x_1 + r^*, h)$ where $\beta_{X,P}$ has a vertical tangent at $P$, $r^* = r^*(x_1, r)$ and $h = h(x_1, r) = \varphi(x_1 + r^*)$. Here both $r^*$ and $h$ are functions of the two independent variables $x_1$ and $r$, but we will often write $r^* = r^*(x_1, r)$ and $h = h(x_1, r)$ for convenience. We refer to $h = h(x_1, r)$ as the height of the ball $B(x_1, 0, r)$. In [KoRiSaSh1] the authors proved the following estimates on the height.

Proposition 9. Let $\beta_{X,P}$, $r^*$ and $h$ be defined as above. Define $\lambda(x)$ implicitly by

$$r = \int_{x_1}^x \frac{\lambda(x)}{\sqrt{\lambda(x)^2 - f(u)^2}} \, du.$$ 

Then

1. For $x_1 - r < x < x_1 + r$ we have $\varphi(x) \leq \varphi(x_1 + r^*) = h$.
2. If $r \geq \frac{1}{|F'(x_1)|}$, then

$$h \approx \frac{f(x_1 + r)}{|F'(x_1 + r)|} \quad \text{and} \quad r - r^* \approx \frac{1}{|F'(x_1 + r)|}.$$ 
3. If $r \leq \frac{1}{|F'(x_1)|}$, then

$$h \approx rf(x_1) \quad \text{and} \quad r - r^* \approx r.$$ 

Now consider a sequence of metric balls $\{B(x, r_k)\}_{k=1}^\infty$ centered at $x \in \Omega$ with radii $r_k \searrow 0$ such that $r_0 = r$ and

$$|B(x, r_k) \setminus B(x, r_{k+1})| \approx |B(x, r_{k+1})|, \quad k \geq 1,$$
so that \( B(x, r_k) \) is divided into two parts having comparable area. We may in fact assume that

\[
(1.4) \quad r_{k+1} = \begin{cases} 
    r^* (x_1, r_k) & \text{if } r_k \geq \frac{1}{F(x_1)} \\
    \frac{1}{2} r_k & \text{if } r_k < \frac{1}{F(x_1)}
\end{cases}
\]

where \( r^* \) is defined in Proposition [2]. Now for \( x_1, t > 0 \) define

\[
h^* (x_1, t) = \int_{x_1}^{x_1 + t} \frac{f^2 (u)}{\sqrt{f^2 (x_1 + t) - f^2 (u)}} du,
\]

so that \( h^* (x_1, t) \) describes the ‘height’ above \( x_2 \) at which the geodesic through \( x = (x_1, x_2) \) curls back toward the \( y \)-axis at the point \((x_1 + t, x_2 + h^* (x_1, t))\). Then in the case \( r_k \geq \frac{1}{F(x_1)} \), we have \( h^* (x_1, r_k + 1) = h(x_1, r_k), k \geq 0 \), where \( h(x_1, r_k) \) is the height of \( B(x, r_k) \). In the opposite case \( r_k < \frac{1}{F(x_1)} \), we have \( r_{k+1} = \frac{1}{2} r_k \) instead, and we will estimate differently.

For \( k \geq 0 \) define

\[
E (x, r_k) = \begin{cases} 
    \{ y : x_1 + r_{k+1} \leq y_1 < x_1 + r_k, \, |y_2| < h^* (x_1, y_1 - x_1) \} & \text{if } r_k \geq \frac{1}{F(x_1)} \\
    \{ y : x_1 + r_k + 1 \leq y_1 < x_1 + r_k, \, |y_2| < h^* (x_1, r_k) = h(x_1, r_k) \} & \text{if } r_k < \frac{1}{F(x_1)}
\end{cases},
\]

where we have written \( r^*_k = r^* (x_1, r_k) \) for convenience. In [KoRiSaSh1] it was shown that

\[
(1.5) \quad |E(x, r_k)| \approx |E(x, r_k) \cap B(x, r_k)| \approx |B(x, r_k)| \quad \text{for all } k \geq 1,
\]

and hence that

\[
|E(x, r_k) \cap B(x, r_k)| \geq \frac{1}{2} c r_k f(x_1) r_k \approx |B(x, r_k)| \geq |E(x, r_k) \cap B(x, r_k)|.
\]

Now define \( \Gamma (x, r) \) to be the set

\[
\Gamma (x, r) = \bigcup_{k=1}^{\infty} E(x, r_k).
\]

The following lemma was proved in [KoRiSaSh1].

**Lemma 10.** With notation as above, in particular with \( r_0 = r \) and \( r_1 \) given by (1.4), and assuming \( \int_{E(x, r_1)} w = 0 \), we have the subrepresentation formula

\[
(1.6) \quad w(x) \leq C \int_{\Gamma(x, r)} |\nabla_A w(y)| \frac{\tilde{d}(x, y)}{|B(x, d(x, y))|} dy,
\]

where \( \nabla_A \) is as in (1.2) and

\[
\tilde{d}(x, y) \equiv \min \left\{ d(x, y), \frac{1}{F'(x_1 + d(x, y))} \right\}.
\]

Note that when \( f(r) = r^N \) is finite type, then \( \tilde{d}(x, y) \approx d(x, y) \). Now define

\[
(1.7) \quad K_r (x, y) \equiv \frac{\tilde{d}(x, y)}{|B(x, d(x, y))|} 1_{\Gamma(x, r)} (y),
\]

and for

\[
y \in \Gamma(x, r) = \{ y \in B(x, r) : x_1 \leq y_1 \leq x_1 + r, \, |y_2 - x_2| < h_{x,y} \},
\]

let \( h_{x,y} = h^* (x_1, y_1 - x_1) \). Denote the dual cone \( \Gamma^*(y, r) \) by

\[
\Gamma^*(y, r) \equiv \{ x \in B(y, r) : y \in \Gamma(x, r) \}.
\]

Then we have

\[
(1.8) \quad \Gamma^*(y, r) = \begin{cases} 
    \{ x \in B(y, r) : x_1 \leq y_1 \leq x_1 + r, \, |y_2 - x_2| < h_{x,y} \} & \text{if } y \in \Gamma(x, r) \\
    \{ x \in B(y, r) : y_1 - r \leq x_1 \leq y_1, \, |x_2 - y_2| < h_{x,y} \} & \text{if } y \not\in \Gamma(x, r)
\end{cases},
\]

and consequently we get the ‘straight across’ estimate,

\[
(1.9) \quad \int K_r (x, y) \, dx \approx \int_{y_1 - r}^{y_1} \left\{ \int_{y_2 - h_{x,y}}^{y_2 + h_{x,y}} \frac{1}{h_{x,y}} \, dx_2 \right\} \, dx_1 \approx \int_{x_1}^{x_1 + r} dy_1 = r.
\]
1.3. **Higher dimensions.** Recall that in the two dimensional case, we have

$$|B_{2D} (x, d (x, y))| \approx h_{x,y} \hat{d} (x, y) \approx h_{x,y} \min \left\{ d (x, y), \frac{1}{|F' (x_1 + d (x, y))|} \right\}. $$

In the three dimensional case, the quantities $h_{x,y}$ and $\hat{d} (x, y)$ remain formally the same (see Chapter 10 of [KoRiSaSh1]) and we can write a typical geodesic in the form

$$\begin{align*}
x_2 &= C_2 + k \int_0^{x_1} \frac{\lambda}{\sqrt{\lambda^2 - |f (u)|^2}} \, du \\
x_3 &= C_3 + \int_0^{x_1} \frac{|f (u)|^2}{\sqrt{\lambda^2 - |f (u)|^2}} \, du,
\end{align*}$$

so that a metric ball centered at $y = (y_1, y_2, y_3)$ with radius $r > 0$ is given by

$$B (y, r) \equiv \left\{ (x_1, x_2, x_3) : (x_1, x_3) \in B_{2D} \left( (y_1, y_3), \sqrt{r^2 - |x_2 - y_2|^2} \right) \right\},$$

where $B_{2D} (a, s)$ denotes the 2-dimensional control ball centered at $a$ in the plane parallel to the $x_1, x_3$-plane with radius $s$ that was associated with $f$ above (see Corollaries 107 and 108 in [KoRiSaSh1] and the subsequent paragraph).

In dimension $n \geq 4$, the same arguments show that a typical geodesic has the form

$$\begin{align*}
x_2 &= C_2 + k \int_0^{x_1} \frac{\lambda}{\sqrt{\lambda - |f (u)|^2}} \, du \\
x_3 &= C_3 + \int_0^{x_1} \frac{|f (u)|^2}{\lambda - |f (u)|^2} \, du,
\end{align*}$$

where $x_2, C_2, k \in \mathbb{R}^{n-2}$ are now $(n-2)$-dimensional vectors, so that a metric ball centered at

$$y = (y_1, y_2, y_3) \in \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R} = \mathbb{R}^n,$

with radius $r > 0$ is given by

$$B (y, r) \equiv \left\{ (x_1, x_2, x_3) : (x_1, x_3) \in B_{2D} \left( (y_1, y_3), \sqrt{r^2 - |x_2 - y_2|^2} \right) \right\},$$

where $B_{2D} (a, s)$ denotes the 2-dimensional control ball centered at $a$ in the plane parallel to the $x_1, x_3$-plane with radius $s$ that was associated with $f$ above. The following lemma was proved in [KoRiSaSh2], correcting Lemma 109 in Chapter 10 of [KoRiSaSh1].

**Lemma 11.** The Lebesgue measure of the three dimensional ball $B_{3D} (x, r)$ satisfies

$$|B_{3D} (x, r)| \approx \begin{cases} r^3 f (x_1) & \text{if } r \leq \frac{1}{|F' (x_1)|} \\
\frac{f (x_1 + r)}{|F' (x_1 + r)|} \sqrt{r |F' (x_1 + r)|} & \text{if } r \geq \frac{1}{|F' (x_1)|}, \end{cases}$$

and that of the $n$-dimensional ball $B_{nD} (x, r)$ satisfies

$$|B_{nD} (x, r)| \approx \begin{cases} r^n f (x_1) & \text{if } r \leq \frac{2}{|F' (x_1)|} \\
\frac{f (x_1 + r)}{|F' (x_1 + r)|} (r |F' (x_1 + r)|)^{\frac{n-1}{2}} & \text{if } r \geq \frac{2}{|F' (x_1)|}. \end{cases}$$

2. **Proportional vanishing $L^1$-Sobolev inequality.**

Our geometric continuity theorem requires the proportional vanishing $L^1$-Sobolev inequality, which we will now establish. For simplicity we consider first the 2-dimensional case.

Define

$$K_r (x, y) \equiv \frac{\hat{d} (x, y)}{|B (x, d (x, y))|} 1_{\Gamma (x, r)} (y),$$

and

$$\Gamma (x, r) = \{ y \in B (x, r) : x_1 \leq y_1 \leq x_1 + r, |y_2 - x_2| < h^* (x_1, y_1 - x_1) \},$$

and for $y \in \Gamma (x, r)$ let $h_{x,y} = h^* (x_1, y_1 - x_1)$. Using the estimate $|B (x, d (x, y))| \approx h_{x,y} \hat{d} (x, y)$ from Section 1.3 we have

$$K_r (x, y) \approx \frac{1}{h_{x,y}} 1_{\{ (x,y) : x_1 \leq y_1 \leq x_1 + r, |y_2 - x_2| < h_{x,y} \}} (x, y).$$
Now denote the dual cone $\Gamma^*(y, r)$ by
\[
\Gamma^*(y, r) \equiv \{ x \in B(y, r) : y \in \Gamma(x, r) \}.
\]

Then we have
\[
\begin{align*}
\Gamma^*(y, r) &= \{ x \in B(y, r) : x_1 \leq y_1 \leq x_1 + r, \ |y_2 - x_2| < h_{x,y} \} \\
&= \{ x \in B(y, r) : y_1 - r \leq x_1 \leq y_1, \ |x_2 - y_2| < h_{x,y} \},
\end{align*}
\]
and consequently we get the ‘straight across’ estimate in $n = 2$ dimensions,
\[
\int K_r(x, y) \ dx \simeq \int_{y_1 - r}^{y_1 + r} \left\{ \int_{y_2 - h_{x,y}}^{y_2 + h_{x,y}} \frac{1}{h_{x,y}} \ dx_2 \right\} \ dx_1 \simeq \int_{x_1}^{x_1 + r} \ dy_1 = r .
\]

Turning now to the case of $n \geq 3$ dimensions, we have using Lemma [11] that
\[
K_{B(0, r_0)}(x, y) \approx \begin{cases} 
\frac{1}{r^{n-1}} f(x_1) 1_{\Gamma(x, r_0)}(y), & 0 < r = y_1 - x_1 < \frac{2}{|F'(x_1)|}, \\
\frac{|F'(x_1 + r)|^{n-1}}{f(x_1 + r) \lambda(x_1, r)^{n-2}} 1_{\Gamma(x, r_0)}(y), & R \geq r = y_1 - x_1 > \frac{2}{|F'(x_1)|},
\end{cases}
\]
where $\lambda(x_1, r) \equiv \sqrt{r \ |F'(x_1 + r)|}$. We denote the size of the kernel $K_{B(0, r_0)}(x, y)$ as
\[
\frac{1}{s_{y_1 - x_1}} = \begin{cases} 
\frac{1}{r^{n-1}} f(x_1), & 0 < r = y_1 - x_1 < \frac{2}{|F'(x_1)|}, \\
\frac{|F'(x_1 + r)|^{n}}{f(x_1 + r) \lambda(x_1, r)^{n-2}} , & 0 < r = y_1 - x_1 \geq \frac{2}{|F'(x_1)|},
\end{cases}
\]
and where the quantity $s_r$ can be, roughly speaking, thought of a cross sectional volume analogous to the height $h_r$ in the two dimensional case. We have
\[
\begin{align*}
\int_{B_+(0, r_0)} K_{B(0, r_0)}(x, y) \ dy &= \sum_{k=0}^{\infty} \int_{x_1 + r_k}^{x_1 + r_{k+1}} \left[ \int_{y_2 \leq \sqrt{r_k^2 - r_{k+1}^2}} \left\{ \int_{x_3 - h^*(x_1, r_k)}^{x_3 + h^*(x_1, r_k)} \frac{1}{s_{y_1 - x_1}} |B(0, r_0)| \ dy_3 \right\} \ dy_2 \right] \ dy_1 \\
&= \sum_{k=0}^{\infty} \int_{x_1 + r_k}^{x_1 + r_{k+1}} \left[ \left( \sqrt{r_k^2 - r_{k+1}^2} \right)^{n-2} \frac{2 h^*(x_1, r_k)}{s_{y_1 - x_1}} \right] \ dy_1 \\
&\approx \sum_{k=0}^{\infty} \frac{1}{s_{y_1 - x_1}} \ dy_1 = \int_{x_1}^{x_1 + r_0} \ dy_1 = r_0,
\end{align*}
\]
where the approximation in the fourth line above comes from the estimates
\[
\left( \sqrt{r_k^2 - r_{k+1}^2} \right)^{n-2} 2 h^*(x_1, r_k) (r_k - r_{k+1}) \approx |E(x, r_k)| \approx |B(x, r_k)| \approx s_{r_k} (r_k - r_{k+1}),
\]
for $x_1 + r_k \leq y_1 < x_1 + r_{k+1}$.
\[
\text{This gives the } n\text{-dimensional ‘straight across’ estimate,}
\]
\[
\int_{B(0, r_0)} K_{B(0, r_0)}(x, y) \ dy \approx r_0 .
\]

We can now prove the proportional vanishing $L^1$-Sobolev inequality by appealing to the the $(1, 1)$ Poincaré inequality in [KoRiSaSh2]. We recall it here for convenience

**Proposition 12.** Let the balls $B(0, r)$ and the degenerate gradient $\nabla_A$ be as above. There exists a constant $C$ such that the Poincaré Inequality
\[
\int_{B(0, r)} |w(x) - \bar{w}| \ dx \leq C r \int_{B(0, 2r)} |\nabla_A w| \ dx
\]
Next, it follows from the proof of Proposition 12 that
\[ (2.9) \]
where the particular family of Young functions \( \Phi \) we are interested in, is defined as follows
\[ (2.5) \]
\[ (2.6) \]
holds for any Lipschitz function \( w \) that vanishes on a subset \( E \) of the ball \( B(0,r) \) with \( |E| \geq \frac{1}{2} |B(0,r)| \), and all sufficiently small \( r > 0 \).

Proof. We have
\[
\int_{B(0,r)} |w| \, dx = \int_{B(0,r)} \left| w(x) - \frac{1}{|E \cap B|} \int_{E \cap B} w(y) \, dy \right| \, dx \\
\leq \frac{1}{|E \cap B|} \int \int_{B \times E \cap B} |w(x) - w(y)| \, dx \, dy.
\]

Next, it follows from the proof of Propositions 12 that
\[ (2.7) \]
Estimate (2.5) follows from (2.6) and (2.7).

2.1. Orlicz-Sobolev inequality. In this section we state the Orlicz-Sobolev inequality proved in [KoRiSaSh1] and [KoRiSaSh2]
\[ (2.8) \]
\[ (2.9) \]
where the particular family of Young functions \( \Phi \) we are interested in, is defined as follows
\[ (2.9) \]
This is Proposition 70 in [KoRiSaSh2]

Proposition 14. Let \( n \geq 2 \). Assume that for some \( C > 0 \) the function
\[ (2.10) \]
\[ (2.11) \]
satisfies \( \lim_{r \to 0} \varphi(r) = 0 \). Assume in addition that geometry \( F \) satisfies
\[ (2.11) \]
Then:
(1) the \( (\Phi, \varphi) \)-Sobolev inequality \[ (2.3) \] holds with geometry \( F \), with \( \varphi \) as in \[ (2.10) \], and with \( \Phi \) as in \[ (2.7) \], \( N > 1 \),
(2) and if \( \varphi_{\max}(r) \equiv \sup_{0 < s < r} \varphi(s) < \infty \) is a finite constant function, then the \( (\Phi, \varphi_{\max}) \)-Sobolev inequality \[ (2.3) \] holds with geometry \( F \), with \( \varphi \) as in \[ (2.10) \], and with \( \Phi \) as in \[ (2.7) \], \( N > 1 \),
(3) in particular, if for some \( \varepsilon > 0 \) we have
\[ (2.12) \]
then the \( (\Phi, \varphi_{\max}) \)-Sobolev inequality \[ (2.3) \] holds with geometry \( F \) and \( \varphi_{\max}(r) \equiv C \).
2.2. The DeGiorgi Lemma. Here is an infinitely degenerate variation on the DeGiorgi Lemma in Lemma 1.4 of Caffarelli and Vasseur [CaVa], but yielding an estimate different from that of Caffarelli and Vasseur - one that does not involve an isoperimetric inequality. For convenience we recall the proportional vanishing $L^1$-Sobolev inequality (2.5) from the previous section:

\[ \int_B |w| \leq C r(B) \int_{2B} |\nabla_A w|, \]

for all Lipschitz $w$ supported in $2B$ that vanish on a subset $E$ of a ball $B$ with $|E| \geq \frac{1}{2} |B|$.

**Lemma 15.** Suppose that the proportional vanishing $L^1$-Sobolev inequality (2.13) holds. Fix $x$ and $r$ and suppose that $w$ satisfies $\int_{B(x,2r)} |\nabla_A w(y)|^2 \, dy \leq C_0$. Set

\[ A = \{ y \in B(x,r) : w(y) \leq 0 \}, \]
\[ C = \{ z \in B(x,r) : w(z) \geq 1 \}, \]
\[ D = \{ y \in B(x,2r) : 0 < w(y) < 1 \}. \]

Then if $|A| \geq \frac{1}{2} |B(x,r)|$, we have

\[ C_0 |D| \geq C_1 \left( \frac{|A||C|}{r|B(x,r)|} \right)^2. \]

**Proof.** Let $\overline{w}(y) \equiv \max \{0, \min \{1, w(y)\}\}$, and note that $\overline{w}(z) = 1$ for $z \in C$. Then applying (2.13) with $w = \overline{w}$, $B = B(x,r_0)$ and $E = A$, we have that

\[ |C| |A| = \int_C \overline{w}(z) \, dz \leq \int_{B(x,r)} \overline{w}(z) \, dz \leq C r |B(x,r)| \int_{B(x,2r)} |\nabla_A \overline{w}(y)| \, dy = C r |B(x,r)| \int_D |\nabla_A w(y)| \, dy \lesssim r_0 |B(x,r)| \sqrt{|D|} \|\nabla_A w\|_{L^2(B(x,2r))} \lesssim C_0 r_0 |B(x,r)| \sqrt{|D|}. \]

Thus we obtain

\[ |C| |A| \lesssim r |B(x,r)| \sqrt{C_0 |D|}, \]

or

\[ C_0 |D| \geq C_1 \left( \frac{|A||C|}{r|B(x,r)|} \right)^2. \]

3. Continuity of locally bounded weak solutions

3.1. Local boundedness. We first recall Corollary 23 from [KoRiSaSh2] to the local boundedness result that will be used in the proof of continuity theorem.

**Corollary 16.** Suppose all the assumptions of Theorem 3 are satisfied. Then

\[ \|u_+\|_{L^\infty(\frac{1}{3}B)} \leq A_{N,\epsilon}(3r) \left( \frac{1}{|3B|} \int_B u_+^2 \right)^{\frac{1}{2}} + \|\phi\|_X, \]

\[ \text{where } A_{N,\epsilon}(r) = C_1 \exp \left( C_2 \left( \frac{\varphi(r)}{r} \right)^{\frac{1}{N-\epsilon}} \right). \]
3.2. Caccioppoli inequality. Recall from [KoKiSaSh2, see (3.2)] that if $u$ is a weak subsolution to $Lu = 0$, then we have the standard Caccioppoli inequality

$$
\int_B |\nabla A(\psi u^+)|^2 \leq C \left( \|\psi\|_{L^\infty} + \|\nabla A\psi\|_{L^\infty} \right)^2 \int_{B \cup \text{supp}\psi} u^+_2.
$$

We repeat the proof here, which simplifies in the homogeneous setting. From the assumption that $\nabla A \nabla u = 0$ in the weak sense we obtain

$$
0 = \int (\psi^2 u^+) \nabla A \nabla u = -\int (\psi^2 u^+) A \nabla u + \int 2\psi u_+ (\nabla \psi) A \nabla u + \int \psi^2 (\nabla u_+) A \nabla u_+,
$$

which gives

$$
\int \psi^2 |\nabla A u_+|^2 = -\int 2\psi u_+ (\nabla \psi) A \nabla u \leq 2 \left( \int |\psi \nabla A u_+|^2 \right)^{\frac{1}{2}} \left( \int |\nabla A \psi|^2 u_+^2 \right)^{\frac{1}{2}},
$$

hence

$$
\int |\psi \nabla A u_+|^2 \leq 4 \int |\nabla A \psi|^2 u_+^2.
$$

We now use

$$
\int_B |\nabla A(\psi u^+)|^2 = \int_B |u_+ \nabla A \psi + \psi \nabla A u_+|^2 \leq 2 \left\{ \int_B |u_+ \nabla A \psi|^2 + \int_B |\psi \nabla A u_+|^2 \right\}
$$

to complete the proof of Caccioppoli.

3.3. Proof of Theorem 2. We can now prove Theorem 2 for weak solutions to a homogeneous degenerate equation. In fact it is easily seen, using Corollary 16 that it suffices to prove the following local statement with $\frac{1}{2\sqrt{\delta(r)}} = A_{N,\epsilon}(3r)$, where $A_{N,\epsilon}(r)$ is the constant in the local boundedness inequality (3.1) defined in (3.2).

**Proposition 17.** Let $B_r = B(x, r)$. Suppose that (3.14) holds, and also that for some $\delta(r) > 0$, we have the following local boundedness inequality,

$$
\|u_+\|_{L^\infty(B_{2r}^+)} \leq \frac{1}{2\sqrt{\delta(r)}} \left( \frac{1}{|B_{3r}|} \int_{B_r} u_+^2 \right)^{\frac{1}{2}}, \quad \text{whenever } Lu = 0 \text{ in } B_r,
$$

for all $0 < r < r_0$. Moreover we assume the summability condition

$$
\sum_{j=1}^\infty \lambda_j = \infty,
$$

where $r_j = \frac{r_0}{2^j}$ and $\lambda_j \equiv \frac{1}{2^{j+\frac{1}{2}+\alpha(r_j)}}$ for $j \geq 1$, and where $C_3$ is a positive constant. Then if $u$ is a weak solution to $Lu = 0$ in $B_{r_0}$, we conclude that $u$ is continuous at $x$.

We now reduce the proof of Proposition 17 to proving Proposition 19 below, which shows that if a solution $v$ is bounded by 1 in a ball $B_{3r}$, and is nonpositive on a ‘sufficiently large’ subset of the smaller ball $B_r$, then $v$ is in fact bounded by a constant less than 1 in the smaller ball $B_r$. This reduction is achieved in two steps by way of the following lemma.

**Lemma 18.** Suppose that the local boundedness property in Proposition 17 holds, equivalently (3.4). Let $u$ be a weak solution of $Lu = 0$ in $B_{r_0}$. Then with $\lambda(r) \equiv \frac{1}{2^{j+\frac{1}{2}+\alpha(r_j)}}$ we have

$$
\text{osc } u \leq \left( 1 - \frac{\lambda(r)}{2} \right) \text{osc } u, \quad 0 < r < r_0.
$$
Indeed, iterating Lemma 18 gives
\[
\operatorname{osc}_{B_r} u \leq \left\{ \prod_{j=1}^{\ell} \left( 1 - \frac{\lambda \left( \frac{r}{2^j} \right)}{2} \right) \right\} \operatorname{osc}_{B_r} u, \quad \ell \geq 1,
\]
which implies \( \lim_{\ell \to \infty} \operatorname{osc}_{B_{r^{1/\ell}}} u = 0 \) since the infinite product above vanishes if the summability condition (3.6) holds. Hence \( u \) is continuous at \( x \), which proves Proposition 17.

Next we note that Lemma 18 is in turn an easy consequence of this proposition.

**Proposition 19.** Suppose that the local boundedness property (3.4) holds. With \( r \) sufficiently small, let \( \lambda (r) \equiv \frac{1}{2^{1+\frac{\beta_r}{2(\gamma+r)}}} \in (0, 1) \).

Let \( v \leq 1 \) and \( \mathcal{L} v = 0 \) in \( B_{3r} \). Assume that \( |B_r \cap \{ v \leq 0 \}| \geq \frac{1}{2} |B_r| \). Then
\[
\sup_{B_r} v \leq 1 - \lambda (r).
\]

Indeed, to prove Lemma 18 from Proposition 19 let
\[
v (x) \equiv \frac{2}{\operatorname{osc}_{B_r} u} \left\{ u (x) - \frac{\sup_{B_r} u + \inf_{B_r} u}{2} \right\},
\]
so that \( -1 \leq v \leq 1 \) on \( B_r \). If \( |\{ v \leq 0 \} \cap B_r| \geq \frac{1}{2} |B_r| \), then Proposition 19 gives \( \operatorname{osc}_{B_r} v \leq 2 - \lambda (r) \), which implies
\[
\operatorname{osc}_{B_r} u \leq \left( 1 - \frac{\lambda (r)}{2} \right) \operatorname{osc}_{B_r} u.
\]

If instead we have \( |\{ v \geq 0 \} \cap B_r| \geq \frac{1}{2} |B_r| \), we obtain the same inequality by applying the above argument to \( -v \). This completes the proof of Lemma 18.

Thus matters have been reduced to proving Proposition 19.

**Proof of Proposition 19.** Define
\[
w_k = 2^k \left( v - (1 - 2^{-k}) \right)
\]
and let \( \varphi \in C_0^\infty (B_{3r}) \) be such that \( \varphi = 1 \) on \( B_{2r} \) and \( \| \nabla A \varphi \|_{L^\infty (B_{3r})} \leq \frac{2}{r} \). Using \( \mathcal{L} w_k = 0 \) in \( B_{3r_0} \) and the standard Caccioppoli inequality (3.3) with \( \phi = 0 \) we have
\[
\int_{B_{2r}} |\nabla A (w_k)_+|^2 \leq \int_{B_{3r}} |\nabla A (\varphi (w_k)_+)|^2 \leq C \| \nabla A \varphi \|_{L^\infty}^2 \int_{B_{3r} \cap \supp \varphi} (w_k)^2 \leq 4C |B_{3r}| r^{-2},
\]
where the last inequality follows from the fact that \( w_k \leq 1 \), which in turn follows from the assumption that \( v \leq 1 \). Define
\[
A_k = \{ 2w_k \leq 0 \} \cap B_r, \quad C_k = \{ 2w_k \geq 1 \} \cap B_r, \quad D_k = \{ 0 \leq 2w_k < 1 \} \cap B_{2r}.
\]
Also note that \( v \leq 0 \) implies \( w_k \leq 0 \), and therefore we have
\[
|A_k| = |\{ 2w_k \leq 0 \} \cap B_r| = |\{ w_k \leq 0 \} \cap B_r| \geq \frac{1}{2} |B_r|,
\]
so that the hypothesis \( |A_k| \geq \frac{1}{2} |B_r| \) in the proportional vanishing \( L^1 \)-Sobolev inequality (2.13) holds.

We will apply Lemma 15 with \( w = 2w_k \) recursively for \( k = 0, 1, 2, \ldots \) below, but only as long as
\[
\int_{B_r} (w_{k+1})^2 \leq \delta (r),
\]
where \( \delta (r) \) is the positive constant in Proposition 17. We use both (3.6) and (3.7), and the fact that
\[
w_{k+1} = 2w_k - 1
\]
implies $C_k = \{w_{k+1} \geq 0\} \cap B_r$, and hence

$$|C_k| \geq \int_{C_k} (w_{k+1})^+ dx = \int_{B_r} (w_{k+1})^+ dx \geq \delta (r) |B_{3r}|,$$

to obtain from Lemma 15 that

$$\left| \left\{ 0 < w_k < \frac{1}{2} \right\} \cap B_{2r} \right| = \left| \{0 < 2w_k < 1\} \cap B_{2r} \right| = |D_k| \geq \frac{C_1}{C |B_{3r}| r^{-2}} \left( \frac{1}{2} \frac{|B_r| \delta (r) |B_{3r}|}{r |B_r|} \right)^2 = \frac{C_1}{4C} \delta (r)^2 |B_{3r}| = \alpha (r),$$

where

$$\alpha (r) \equiv \frac{C_1}{4C} \delta (r)^2 |B_{3r}| > 0$$

depends on $r$, but not on $k$ or $v$. This gives

$$|B_{2r}| \geq |\{w_k \leq 0\} \cap B_{2r}| = |\{2w_{k-1} \leq 1\} \cap B_{2r}|$$

$$= |\{w_{k-1} \leq 0\} \cap B_{2r}| + \left| \left\{ 0 < w_{k-1} \leq \frac{1}{2} \right\} \cap B_{2r} \right|$$

$$\geq |\{w_{k-1} \leq 0\} \cap B_{2r}| + \alpha$$

$$\vdots$$

$$\geq |\{w_0 \leq 0\} \cap B_{2r}| + k \alpha (r) \geq k \alpha (r),$$

The above inequality, namely $|B_{2r}| \geq k \alpha (r)$, must fail for a finite $k$ independent of $v$, in fact it fails for the unique integer $k_0 \geq 0$ satisfying $\frac{|B_{2r}|}{\alpha (r)} < k_0 \leq \frac{|B_{2r}|}{\alpha (r)} + 1$, and for this $k_0$ we must then have

$$\int_{B_r} (w_{k_0+1})^+ dx < \delta (r) |B_{3r}|.$$  

By the local boundedness inequality (3.4), we conclude that in the ball $B_{\frac{r}{2}}$, we have

$$w_{k_0+1} \leq \left\| (w_{k_0+1})^+ \right\|_{L^\infty} (B_{\frac{r}{2}}) \leq \frac{1}{2 \sqrt{\delta (r)}} \left( \frac{1}{|B_{3r}|} \int_{B_r} (w_{k_0+1})^+ dx \right)^{\frac{1}{2}} \leq \frac{1}{2 \sqrt{\delta (r)}} \sqrt{\delta (r)} = \frac{1}{2}.$$  

Rescaling back to $v$ now gives

$$2^{k_0+1} \left[ v - \left( 1 - \frac{1}{2^{k_0+1}} \right) \right]^+ \leq \frac{1}{2} \text{ in } B_{\frac{r}{2}}$$

$$\implies v \leq 1 - \frac{1}{2^{k_0+1}} + \frac{1}{2} \frac{1}{2^{k_0+1}} = 1 - \frac{1}{2^{k_0+2}} \text{ in } B_{\frac{r}{2}}.$$  

Finally, we note that (3.8) gives $\alpha (r) = \frac{C_1}{4C} \delta (r)^2 |B_{3r}|$, and hence

$$k_0 \leq \frac{|B_{2r}|}{\alpha (r)} + 1 = \frac{|B_{2r}|}{\frac{C_1}{4C} \delta (r)^2 |B_{3r}|} + 1 \leq \frac{C_3}{\delta (r)^2} + 1,$$

and thus that

$$\frac{1}{2^{k_0+2}} \geq \frac{1}{2^{3 + \frac{C_3}{\delta (r)^2}}} = \lambda (r).$$

We can now obtain our main result, Theorem 2, directly from Part (1) of Theorem 1 and Proposition 17. To check the summability condition 3.3 we first note that for the geometry $F_{\sigma,k}$, i.e. $F (r) = F_{\sigma,k} = (\ln \frac{1}{r}) \left( \ln (k) \frac{1}{r} \right)^\sigma$, we have

$$|F' (r)| \approx \frac{1}{r} \left( \ln (k) \frac{1}{r} \right)^\sigma, \quad F'' (r) \leq (1 + \varepsilon)\frac{|F' (r)|}{r},$$


where \( \varepsilon \) can be made arbitrarily small by choosing \( r_0 \) sufficiently small. Thus the conditions of Proposition (14) are satisfied and we have the following estimate for the superradius:

\[
\varphi(r) \leq Cr \left( \ln^{(k)} \frac{1}{r} \right)^{\sigma N}.
\]

Now we use the formula (3.2),

\[
A_{N,\varepsilon}(r) \equiv C_1 \exp \left\{ C_2 \left( \frac{\varphi(r)}{r} \right)^{\frac{1}{1-\varepsilon}} \right\},
\]

together with \( A_{N,\varepsilon}(3r) = \frac{1}{2\sqrt{\delta(r)}} \), to obtain that for the geometry \( F = F_{k,\sigma} \) we have

\[
\delta(r) = \frac{1}{4A_{N,\varepsilon}(3r)^2} = \frac{1}{4C_2^2} e^{-2C_2 \left( \frac{\varphi(r)}{r} \right)^{\frac{1}{1-\varepsilon}}} \geq e^{-C'(\ln^{(k)}(\frac{1}{r}))^{\frac{1}{1-\varepsilon}}}.
\]

Note that if \( \sigma < 1 \) we can find \( N > 1 \) satisfying

\[
N > \frac{1 + \varepsilon}{1 - \sigma},
\]

which implies

\[
\gamma = \frac{\sigma N}{N - 1 - \varepsilon} < 1.
\]

Now choose \( r_j = 4^{-j}r_0 \) and let \( k = 3 \), we thus have

\[
\frac{1}{\delta(r_j)} \leq e^{C'' \left[ \ln^{(2)} \left( \frac{1}{r_j} \right) \right]}.
\]

Now we use \( \lambda_j \equiv \frac{1}{2 \sqrt{\delta(r_j)^3}} \) from (3.6), together with the inequality

\[
\ln^{(k)} a \leq \varepsilon \ln^{(k)} a, \quad \text{for } a \text{ sufficiently large depending on } \gamma < 1 \text{ and } \varepsilon > 0,
\]

to obtain

\[
\sum_{j=1}^{\infty} \lambda_j = \sum_{j=1}^{\infty} \frac{1}{2^{3+\varepsilon} r_j^{\varepsilon}} \approx \sum_{j=1}^{\infty} \frac{1}{2^{3+C_3\varepsilon} \varepsilon^{\gamma} \ln^{(2)} \ln^{(3)} \left( \frac{1}{r_j} \right)} \geq \sum_{j=1}^{\infty} 2^{-C'' \varepsilon \ln^{(2)} \ln^{(3)} \left( \frac{1}{r_j} \right)} \geq \sum_{j=1}^{\infty} e^{-C''' \varepsilon \ln^{(2)} \ln^{(3)} \left( \frac{1}{r_j} \right)} = \sum_{j=1}^{\infty} \frac{1}{j^{C'''' \varepsilon}} = \infty,
\]

if \( \varepsilon > 0 \) is chosen sufficiently small. This establishes the summability condition and finishes the proof of Theorem 2.

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