Theoretical and Practical Bounds on the Initial Value of Skew-Compensated Clock for Clock Skew Compensation Algorithm Immune to Floating-Point Precision Loss

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Abstract—A clock skew compensation algorithm was recently proposed based on the extension of Bresenham’s line drawing algorithm (Kim and Kang, IEEE Commun. Lett., vol. 26, no. 4, pp. 902–906, Apr. 2022), which takes into account the discrete nature of clocks in digital communication systems and mitigates the effect of limited floating-point precision on clock skew compensation. It lacks, however, a theoretical analysis of the range of the initial value of skew-compensated clock, which is also an initial condition for the proposed algorithm. In this letter, we provide practical as well as theoretical bounds on the initial value of skew-compensated clock based on a systematic analysis of the errors of floating-point operations, which replace the approximate bounds in Theorem 1 of the prior work.

Index Terms—Clock skew compensation, floating-point arithmetic, error bounds, wireless sensor networks.

I. INTRODUCTION

A clock skew compensation algorithm immune to floating-point precision loss was proposed in [1], which takes into account the discrete nature of clocks in digital communication systems and thereby mitigates the effect of limited floating-point precision on clock skew compensation based on the extension of Bresenham’s line drawing algorithm [2]. The extended bounds on the initial value of skew-compensated clock in Theorem 1 of [1], which are closely related with the initial condition for the proposed algorithm, include a term for floating-point operation error due to precision loss (i.e., ε in (11) of [1]). The value of ε is not explicitly mentioned in the theorem but approximately set for numerical examples based on the property of the single-precision floating-point format as defined in the IEEE standard for floating-point arithmetic (IEEE 754-2008) [3], which may result in unexpected behaviors during the refinement of the skew-compensated clock based on the proposed algorithm due to the lack of guarantee of the approximate bounds on the initial condition.

In this letter, we revisit Theorem 1 of [1] and provide practical as well as theoretical bounds on the initial value of skew-compensated clock based on a systematic analysis of the errors of floating-point operations.

II. PRELIMINARY: RELATIVE ERRORS OF FLOATING-POINT OPERATIONS

We briefly review the results of [4] on relative errors of floating-point operations, which are a basis for our work on both theoretical and practical error bounds.

As the exponent range is no limiting factor in clock skew compensation, we can define an associated set $\mathbb{F}$ of floating-point numbers with a base β and a precision $p$ as follows:

$$\mathbb{F} = \{0\} \cup \{ M \beta^e | M, e \in \mathbb{Z}, \beta^{p-1} \leq |M| < \beta^p \} .$$ (1)

We also denote a round-to-nearest function by $\mathsf{fl} : \mathbb{R} \rightarrow \mathbb{F}$, which satisfies

$$| t - \mathsf{fl}(t) | = \min_{f \in \mathbb{F}} | t - f | , \quad t \in \mathbb{R} .$$ (2)

Based on $\mathbb{F}$ and $\mathsf{fl}$, we can define two relative errors for $t \in \mathbb{R}$ and $t \neq 0$:

$$E_1(t) \triangleq \frac{| t - \mathsf{fl}(t) |}{| t |} ,$$ (3)

$$E_2(t) \triangleq \frac{| t - \mathsf{fl}(t) |}{| \mathsf{fl}(t) |} .$$ (4)

where $E_1$ and $E_2$ are the errors relative to $t$ and $\mathsf{fl}(t)$, respectively; as discussed in [4], the relative errors may be defined to be zero when $t=0$.

The optimal relative error bounds on various floating-point operations are systematically analyzed in [4], and we summarize part of the results relevant to our work in Table I where $x, y \in \mathbb{F}$ and $u = \frac{1}{2} \beta^{1-p}$ is the unit roundoff associated with $\mathsf{fl}$ and $\mathbb{F}$. Note that the first row in Table I is for rounding a real number $t \in \mathbb{R}$ to a floating-point number $x \in \mathbb{F}$, whose relative errors become zero when $t \in \mathbb{F}(\subset \mathbb{R})$.

| $t$ | bound on $E_1(t)$ | bound on $E_2(t)$ |
|-----|------------------|------------------|
| real number | $\frac{| x - \mathsf{fl}(x) |}{| x |}$ | $\frac{| x - \mathsf{fl}(x) |}{| \mathsf{fl}(x) |}$ |
| $x \cdot y$ | $\frac{| u - 2u^2 |}{| u |}$ | if $\beta = 2$, $\frac{| u - 2u^2 |}{| u |}$ if $\beta > 2$ |
| $x/y$ | $\frac{| u - 2u^2 |}{| u |}$ if $\beta = 2$, $\frac{| u - 2u^2 |}{| u |}$ if $\beta > 2$ |

As in [1], we confine our discussions to a network with one head node and one sensor node, where we focus only on the clock skew compensation at the sensor node. In this case, we describe the hardware clock $T$ of the sensor node with respect

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1For instance, in the time synchronization schemes based on the reverse two-way message exchange, the clock offset is independently compensated for at the head node, while the sensor node only synchronizes the frequency of its logical clock to that of the reference clock [5].
to the reference clock \( t \) of the head node using the first-order affine clock model of [6] but without clock offset, i.e.,

\[
T(t) = (1 + \epsilon)t,
\]

where \( \epsilon \in \mathbb{R} \) denotes the clock skew. To compensate for the clock skew and estimate the reference clock \( t \) based on the hardware clock \( T(t) \), we need to calculate \( \frac{1}{1 + \epsilon} \) based on an estimate of the clock skew \( \epsilon \).

Because the division of floating-point numbers incurs substantial precision loss at resource-constrained wireless sensor network (WSN) platforms with 32-bit single precision [7], we propose a clock skew compensation algorithm using only integer addition/subtraction and comparison based on the extension of Bresenham’s algorithm in [1]. As discussed in [1], we assume that two positive integers \( D \) and \( A \) are given so that \( \frac{D}{A} \) be an estimate of \( \frac{1}{1 + \epsilon} \). Theorem 1 of [1] states that, given a hardware clock \( i \), its skew-compensated clock \( j \)—both \( i \) and \( j \) are non-negative integers—satisfies the following condition: For \( \frac{iD}{A} < 1 \),

\[
\frac{i}{A}D - 1 - \epsilon < j < \frac{i}{A}D + 1 + \epsilon,
\]

where \( \epsilon \geq 0 \) indicates the error due to the precision loss in computing \( \frac{iD}{A} \); the value of \( \epsilon \) is approximately set to \( 10^{-7}i \) for numerical examples, which, however, is not based on a systematic and quantitative analysis of the errors of floating-point operations in \( \frac{iD}{A} \).

Lemma 1 in this regard, provides the optimal bounds on the initial value of skew-compensated clock based on the floating-point operations of non-negative real numbers.

Lemma 1. For \( t = x \frac{y}{z} \), where \( x, y, z \in \mathbb{R} \) with \( x, y \geq 0 \) and \( z > 0 \), \( \text{fl}(t) \) satisfies

\[
\frac{1 - u + 2u^2}{(1 + u)^2(1 + 2u)}t \leq \text{fl}(t) \leq \frac{(1 + 2u)^3(1 + u - 2u^2)}{(1 + u)^2}t. \tag{7}
\]

Proof: Because we consider rounding errors as well as multiplication and division errors, \( \text{fl}(t) \) is given by

\[
\text{fl}(t) = \text{fl}(\text{fl}(x) \text{fl}\left(\frac{y}{z}\right)) = \frac{y(1 + \delta_2)}{z(1 + \delta_3)}(1 + \delta_4), \tag{8}
\]

where \( \delta_1, \delta_2, \delta_3 \) are \( E_1 \) relative errors in rounding a real number to a floating-point number, \( \delta_4 \) is \( E_1 \) relative error in the division of two floating-point numbers, and \( \delta_5 \) is \( E_1 \) relative error in the multiplication of two floating-point numbers. Table 1 shows that these relative errors for \( \beta = 2 \) are bounded as follows:

\[
|\delta_1|, |\delta_2|, |\delta_3|, |\delta_5| \leq \frac{u}{1 + u}, \quad |\delta_4| \leq u - 2u^2. \tag{9}
\]

From (9), we have

\[
\frac{1}{1 + u} \leq 1 + \delta_1 \leq \frac{1 + 2u}{1 + u}, \quad \frac{1}{1 + u(1 + 2u)} \leq (1 + \delta_1)(1 + \delta_2) \leq \frac{(1 + 2u)^2}{1 + u}, \quad \frac{1 - u + 2u^2}{(1 + u)(1 + 2u)} \leq (1 + \delta_1)(1 + \delta_2)(1 + \delta_4) \leq \frac{(1 + 2u)^2(1 + u - 2u^2)}{1 + u}, \tag{10}
\]

From (8) and (10), we obtain (7).

Now we can extend Theorem 1 of [1] based on Lemma 1.

Theorem 1. Given a hardware clock \( i \), we can obtain its skew-compensated clock \( j \) as follows:

Case 1. \( \frac{iD}{A} < 1 \): The skew-compensated clock \( j \) satisfies

\[
\left\lfloor \frac{iD}{A} \right\rfloor \leq j \leq \left\lceil \frac{iD}{A} \right\rceil. \tag{11}
\]

Because we cannot know the exact value of \( \frac{iD}{A} \) due to limited floating-point precision, however, we extend [1] to include the effect of the precision loss: For floating-point numbers with a base \( \beta = 2 \) and a precision \( p \),

\[
\left\lfloor \frac{1 - u + 2u^2}{(1 + u)^2(1 + 2u)}t \right\rfloor \leq j \leq \left\lceil \frac{(1 + 2u)^3(1 + u - 2u^2)}{(1 + u)^2}t \right\rceil. \tag{12}
\]

where \( t = i \frac{D}{A} \) and \( u = 2^{-p} \).

Let \( k, \ldots, k + l \) be the candidate values of \( j \) satisfying (12). We determine \( j \) by starting from the point \( (i - 1, k) \) and applying the extended Bresenham’s algorithm with \( \nabla_j(i, k) \) defined in [1] and on; \( j \) is determined by the \( y \)-coordinate of the valid point whose \( x \) coordinate is \( i \).

Case 2. \( \frac{iD}{A} > 1 \): In this case, we can decompose the skew-compensated clock \( j \) into two components as follows:

\[
\frac{iD}{A} = i + \frac{D - A}{A}. \quad \tag{13}
\]

Now that \( \frac{D - A}{A} < 1 \), we can apply the same procedure of Case 1 to the second component in (13) by setting \( \Delta a \) and \( \Delta b \) to \( A \) and \( D - A \), respectively. In this case, \( t \) in (17) is equal to \( \frac{D - A}{A} \).

Let \( j \) be the result from the procedure. The skew-compensated clock \( j \) is given by \( i + j \) as per (13).

Proof: (12) is the result of the application of Lemma 1 to (11). The rest of Theorem 1 is identical to Theorem 1 of [1].
A. Loosening Bounds for Practical Implementation

Though Theorem 1 provides theoretically-guaranteed lower and upper bounds on the initial value of skew-compensated clock $j$, obtaining the exact values of bounds—i.e., the lhs and the rhs of (12)—could be a challenge, especially at resource-constrained sensor nodes with limited floating-point precision.

For its practical implementation based on limited floating-point precision, therefore, we can loosen the lhs and the rhs of (12) as follows:

$$\frac{1-u+2u^2}{(1+u)^2(1+2u)} t \leq j \leq \frac{(1+2u)^3(1-u-2u^2)}{(1+u)^2} t,$$

$$\frac{1-u}{(1+u)^2(1+2u)} t \leq j \leq \frac{(1+2u)^3(1+u)}{(1+u)^2} t,$$

$$\frac{1-u}{(1+2u)^2(1+2u)} t \leq j \leq \frac{(1+2u)^3}{1+u} t,$$

(14)

Note that the lhs and the rhs of (14) consist only of the elements of $F$, i.e.,

$$1 - u = (2^p - 1)2^{-p} \in F,$$

$$1 + 2u = (2^{p-1} + 1)2^{-p} \in F,$$

which eliminates the rounding errors for those terms not belonging to $F$ in (12).

IV. Numerical Examples

We first investigate the effect of floating-point precision on the calculation of various bounds on the initial value of skew-compensated clock through numerical experiments. Table II summarizes the results of comparison of the theoretical bounds of (12) calculated based on the floating-point formats of single precision and double precision of IEEE 754-2008 together with the practical bounds of (14) and the approximate bounds of (11) of [1] based on single precision; the calculated bounds are compared to those based on binary512 floating-point format providing 489 precision in bits, which serve as a reference for the comparison. As in [1], we set the value of $\varepsilon$ and $D$ to $10^{-7} t$ and 1,000,000, respectively, and generate one million samples of $A$ corresponding to clock skew uniformly distributed in the range of $[-100 \text{ppm}, 100 \text{ppm}]$.

Table II shows that the double precision is enough for the calculation of the theoretical bounds of (12), while the results for both theoretical bounds of (12) and the approximate bounds of (11) of [1] based on single precision violate the reference bounds of (14) based on binary512 due to the floating-point precision loss as indicated by the negative values of $\Delta LB$ and $\Delta UB$; this implies that the results of the clock skew compensation based on Theorem 1 of [1] with the approximate bounds cannot be theoretically guaranteed to be correct. Though being loose as indicated by the positive values of $\Delta LB$ and $\Delta UB$, the practical bounds of (14) based on single precision, on the other hand, do not violate the reference bounds in spite of the limited floating-point precision; it turns out that the loosening of the bounds discussed in Section III-A counteracts the effect of limited floating-point precision on the calculation of (14).

We also compare the results of clock skew compensation algorithms again under the same condition as [1], which are summarized in Table III For the randomly-generated one million samples of $A$, the clock skew compensation algorithm of [1] with both approximate and practical bounds provides exactly the same bounded compensation errors compared to the unbounded errors for the single-precision algorithm, while the numbers of iterations with the practical bounds are larger than those with the approximate bounds for $i \geq 1e7$. This indicates that the approximate bounds of (11) of [1] could provide valid initial conditions for the cases considered in these examples with less numbers of iterations than those for the practical bounds of (14) but at the expense of the lack of theoretical guarantee.

V. Concluding Remarks

In this letter, we have revisited Theorem 1 of [1] and derived theoretical and practical bounds on the initial value of skew-compensated clock based on a systematic analysis of the errors of floating-point operations. The theoretical bounds provide a reference framework for the analysis and comparison of bounds on the initial value of skew-compensated clock, while the practical bounds could replace the approximate bounds suggested in [1] as an implementation option for limited floating-point precision. The numerical examples demonstrate that, unlike the approximate bounds of [1], the proposed practical bounds based on single-precision floating-point format do not violate the theoretical bounds and thereby can guarantee the correctness of the clock skew compensation even on resource-constrained computing platforms like wireless sensor nodes.

REFERENCES

[1] K. S. Kim and S. Kang, “Clock skew compensation algorithm immune to floating-point precision loss,” IEEE Commun. Lett., vol. 26, pp. 1–1, Jan. 2022, Early Access.

[2] J. E. Bresenham, “Algorithm for computer control of a digital plotter,” IBM Systems Journal, vol. 4, no. 1, pp. 25–30, 1965.

[3] IEEE Computer Society, IEEE Std 754™-2008, IEEE standard for floating-point arithmetic, Std., Aug. 2008.

[4] C.-P. Jeannerod and S. M. Rump, “On relative errors of floating-point operations: Optimal bounds and applications,” Mathematics of computation, vol. 87, no. 310, pp. 803–819, 2018. [Online]. Available: http://hdl.handle.net/11420/2654

[5] K. S. Kim, S. Lee, and E. G. Lim, “Energy-efficient time synchronization based on asynchronous source clock frequency recovery and reverse two-way message exchanges in wireless sensor networks,” IEEE Trans. Commun., vol. 65, no. 1, pp. 347–359, Jan. 2017.

[6] R. T. Rajan and A.-J. van der Veen, “Joint ranging and clock synchronization for a wireless network,” in Proc. CAMSAP 2011, Dec. 2011, pp. 297–300.

[7] X. Huan and K. S. Kim, “On the practical implementation of propagation delay and clock skew compensated high-precision time synchronization schemes with resource-constrained sensor nodes in multi-hop wireless sensor networks,” Computer Networks, vol. 166, pp. 1–8, Jan. 2020.
### TABLE II

**Comparison of Bounds on the Initial Value of Skew-Compensated Clock.**

| Bounds                          | $i$  | $\Delta LB$ | $\Delta UB$ |
|--------------------------------|------|--------------|--------------|
|                                | Min. | Max. | Avg. | Min. | Max. | Avg. |
| Theoretical bounds of (12)     | 1e6  | 0    | 0    | 0    | 0    | 0    |
| based on double precision      | 1e7  | 0    | 0    | 0    | 0    | 0    |
|                                | 1e8  | 0    | 0    | 0    | 0    | 0    |
|                                | 1e9  | 0    | 0    | 0    | 0    | 0    |
| Theoretical bounds of (12)     | 1e6  | 0    | 0    | 0    | 0    | 0    |
| based on single precision      | 1e7  | 0    | 0    | 0    | 0    | 0    |
|                                | 1e8  | -18  | 0    | -7.9666 | 0   | 6   | 1.7206 |
|                                | 1e9  | -125 | 0    | -4.7102e1 | 0  | 104 | 3.6533e1 |
| Practical bounds of (14)       | 1e6  | 0    | 0    | 0    | 0    | 0    |
| based on single precision      | 1e7  | 0    | 0    | 0    | 0    | 0    |
|                                | 1e8  | 12   | 0    | 5.98980 | 0   | 6   | 2.9912 |
|                                | 1e9  | 120  | 0    | 5.9890e1 | 0  | 60  | 2.9880e1 |
| Approximate bounds of (11) of [I] based on single precision† | 1e6  | 0    | 1    | 4.9696e-1 | 0  | 0    | 0    |
|                                | 1e7  | -2   | 1    | -5.0346e-1 | -2 | 1   | -2.8513e-1 |
|                                | 1e8  | -20  | 10   | -5.0063 | -20 | 10  | -4.8369 |
|                                | 1e9  | -199 | 100  | -4.9696e-1 | -199 | 100 | -4.9645e1 |

$\Delta LB = LB_{binary512} - LB$, where $LB_{binary512}$ is the lower bound of (12) based on binary512.

$\Delta UB = UB - UB_{binary512}$, where $UB_{binary512}$ is the upper bound of (12) based on binary512.

† With $\varepsilon = 10^{-7}$.

### TABLE III

**Comparison of Clock Skew Compensation Algorithms.**

| Algorithm                                | $i$  | Compensation error | # of iterations |
|------------------------------------------|------|--------------------|-----------------|
|                                          | Min. | Max. | Avg. | Min. | Max. | Avg. |              |
| Single precision†                        | 1e6  | 0    | 0    | 0    | -   | -   | -   |
|                                          | 1e7  | 0    | 0    | 0    | -   | -   | -   |
|                                          | 1e8  | -4   | 1    | -1.9842 | -   | -   | -   |
|                                          | 1e9  | -199 | 44   | 1.2502e1 | -   | -   | -   |
| Clock skew compensation with practical bounds of (14) | 1e6  | -1   | 0    | -4.9663e-1 | 1  | 2   | 1.5034 |
|                                          | 1e7  | -1   | 0    | -4.9668e-1 | 1  | 8   | 4.5232 |
|                                          | 1e8  | -1   | 0    | -5.0220e-1 | 1  | 72  | 3.6710e1 |
|                                          | 1e9  | -1   | 0    | -4.9734e-1 | 1  | 832 | 4.1872e2 |
| Clock skew compensation with approximate bounds of (11) of [I]† | 1e6  | -1   | 0    | -4.9663e-1 | 2  | 2   | 2    |
|                                          | 1e7  | -1   | 0    | -4.9668e-1 | 3  | 4   | 3.0050 |
|                                          | 1e8  | -1   | 0    | -5.0220e-1 | 21 | 22  | 2.1005e1 |
|                                          | 1e9  | -1   | 0    | -4.9734e-1 | 201| 202 | 2.0101e2 |

† With respect to $\lfloor i \frac{DA}{i} \rfloor$ based on double precision.

† $\lfloor i \frac{DA}{i} \rfloor$ based on single precision.