Solutions of Painlevé II on real intervals: novel approximating sequences

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Abstract

Novel sequences of approximants to solutions of Painlevé II on finite intervals of the real line, with Neumann boundary conditions, are constructed. Numerical experiments strongly suggest convergence of these sequences in a surprisingly wide range of cases, even ones where ordinary perturbation series fail to converge. These sequences are here labeled extraordinary because of their unusual properties. Each element of such a sequence is defined on its own interval. As the sequence (apparently) converges to a solution of the corresponding boundary value problem for Painlevé II, these intervals themselves (apparently) converge to the defining interval for that problem, and an associated sequence of constants (apparently) converges to the constant term in the Painlevé II equation itself. Each extraordinary sequence is constructed in a nonlinear fashion from a perturbation series approximation to the solution of a supplementary boundary value problem, involving a generalization of Painlevé II that arises in studies of electrodiffusion.

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1 Introduction

Painlevé’s second nonlinear ordinary differential equation PII [Painlevé 02] continues to be studied widely, not only because of its intrinsic mathematical significance (see [Ablowitz and Segur 77; Joshi and Kruskal 94; Bassom et al. 98; Sakai 01; Clarkson 06; Kajiwara et al. 17] and references therein), but also because of its association with nonlinear systems of interest in applications (see [Bass 64; Rogers et al. 99; Zaltzman and Rubinstein 07; Bass et al. 10; Bracken and Bass 18] and references therein). Two families of solutions of PII are known in special cases [Bass et al. 10], expressed in terms of ratios of polynomials in the one [Yablonskii 59; Vorob’ev 65], and ratios of Airy functions in the other [Lukashevich 71; Clarkson 16], but in general solutions have not been expressed in terms of more familiar functions and are referred to as Painlevé transcendents [Ince 56; Clarkson 06].

Consider the boundary value problem (BVP) for PII in standard form on the real interval \([a, b]\), with Neumann boundary conditions (BCs), defined by

\[
y''(z) = 2y(z)^3 + zy(z) + C, \quad a < z < b, \quad y'(a) = 0 = y'(b). \tag{1}
\]

In particular, consider the case where

\[
a = -15.650 \ldots, \quad b = -14.911 \ldots, \quad C = -1.468 \ldots \tag{2}
\]

The reasons for this choice of values will be made clear below, where it will also be explained why they are only numerically determined, and how we know that in this case there exists a solution \(y(z)\) that is free from singularities, monotonically decreasing and everywhere positive on the interval of interest.

In what follows it is shown that for this particular BVP and uncountably many others of the form (1), there exists a novel type of approximating sequence, to be denoted by \(y_E^{(n)}(z)\), \(n = 1, 2, \ldots\), which appears to converge to a solution \(y(z)\). For the example defined by (1) and (2), Fig. 1 on the left and right shows plots of \(y_E^{(n)}(z)\) for \(n = 1, 2, 3, 4\) and for \(n = 8, 9, 10, 11\), respectively, together with \(y(z)\).

This type of sequence is highly unusual, and henceforth it will be labeled extraordinary. In the first place, in such a sequence each approximant \(y_E^{(n)}(z)\) is defined on its own interval \([a_n, b_n]\), and is accompanied by a constant \(C_n\). In the (apparently) convergent cases, the \(a_n\), \(b_n\) and \(C_n\) values converge to values \(a\), \(b\) and \(C\) while the approximants converge to a solution of (1) with these limiting parameter values. This behaviour is partially evident in Fig. 1. To make it clearer, Fig. 2 on the left shows the progression of the values...
$a_n$ and $b_n$ towards $a$ and $b$, while Fig. 2 on the right shows the progression of $C_n$ values towards $C$, with $a$, $b$, and $C$ as in [2].

In the second place, the values $a_n$, $b_n$ and $C_n$ at each stage, as well as their limiting values $a$, $b$ and $C$, can only be estimated numerically, and are not known exactly $a$ priori.

In the third place, and perhaps most remarkable of all, is the complicated way in which such an extraordinary sequence is constructed. Perturbation theory is first applied to a different, supplementary BVP — one that involves a generalization of PII arising in an application to electrodiffusion — to obtain an (ordinary) sequence of approximants to the solution of that supplementary BVP. A nonlinear procedure is then applied to convert this ordinary sequence into an extraordinary sequence of approximants to a solution of a BVP [1] involving PII. It does not seem to be possible to construct such extraordinary sequences directly from BVPs of the form [1].

2 Constructing extraordinary sequences

The supplementary BVP to be used, derives from a system of coupled first-order nonlinear ODEs that govern a model of two-ion electrodiffusion. The model and the ensuing BVP have been known for over 50 years [Bass 64], but have been the subject of renewed interest and much associated research in recent years [Rogers et al. 99; Bass et al. 10; Amster et al. 11; Bracken...
at al. 12; Bass and Bracken 14; Bracken and Bass 16; Bracken and Bass 18].

The BVP consists of the ODE

\[
2\nu E''(x) = \nu E(x)^3 + \left\{4\sigma + \nu [E(0)^2 - E(1)^2]\right\} x E(x) \\
+ \left\{2 - 2\sigma - \nu E(0)^2\right\} E(x) + \nu \tau \left\{E(0)^2 - E(1)^2\right\} - 4\mu
\]

for \(0 < x < 1\), with Neumann BCs

\[
E'(0) = 0 = E'(1).
\]

The parameters \(\nu, \sigma, \tau\) and \(\mu\) appearing here take any chosen constant values, subject to

\[
0 < \nu, \quad 0 < \sigma < 1, \quad -1 < \tau < 1, \quad -\infty < \mu < \infty.
\]

Their interpretation in the context of electrodiffusion and that of the variables \(x\) and \(E(x)\) appearing in (3), is not important to the present work. Interested readers are referred to earlier works, in particular to Eqn. (14) of [Bracken and Bass 18], where \(\sigma, \tau\) and \(\mu\) are denoted \(1 - 2c_0 (= 2c_1 - 1)\), \(1 - 2\tau_+ (= 2\tau_- - 1)\) and \(j_1\), respectively, because of that interpretation.

Note that as well as involving a cubic nonlinearity and an \(x\)-dependent term similar to those found in PII in standard form [1], the ODE (3) has the...
unusual complicating feature that the unknown values \(E(0)\) and \(E(1)\) appear nonlinearly in it, and must be found together with \(E(x)\) for \(0 < x < 1\), as part of any solution. Despite this complication, it is known [Thompson 94; Park and Jerome 97; Amster et al. 11; Bracken et al. 12; Bracken and Bass 16; Bracken and Bass 18] that there exists a unique solution of the BVP involving this generalized version \(\text{(3)}\) of PII, and that this solution is singularity-free and either monotonically decreasing and positive (Type A), or monotonically increasing and negative (Type B), everywhere on the interval \([0, 1]\). From this it follows in particular that in every solution

\[ E(0)^2 > E(1)^2. \] (6)

A series expansion of \(E(x)\) satisfying \(\text{(3)}\) and \(\text{(4)}\), including the end point values \(E(0)\) and \(E(1)\), has been obtained [Bracken and Bass 18] by perturbing away from the solution \(E(x) = 0\) when \(\mu = 0\) in \(\text{(3)}\). Numerical experiments in that work strongly suggest convergence of the series for a wide range values of the parameters \(\text{(5)}\), much wider than expected on the basis of earlier studies [Bass 64] and approximation schemes [MacGillivray 68], and in particular for cases with \(\sigma = 1/3, \tau = -0.2\), and

\[ 0 < \nu \leq 10, \quad -2 < \mu < 2. \] (7)

(For values of \(|\mu|\) greater than about 2, the series typically appears to diverge, whatever the values of the other parameters.)

To summarize the method, the series expansion is obtained by introducing a book-keeping parameter \(\epsilon\) that can later be set equal to 1, replacing \(\mu\) by \(\epsilon\mu\) in \(\text{(3)}\), and seeking the solution in the form

\[ E(x) = 0 + \epsilon E_1(x) + \epsilon^2 E_2(x) + \ldots , \] (8)

while emphasizing that such an expansion is also applied to the end-point values \(E(0)\) and \(E(1)\) appearing in \(\text{(3)}\). Equating terms of the same degree in \(\epsilon\), it is found that \(E_n(x)\) satisfies

\[ \nu E_n''(x) = (1 - \sigma + 2\sigma x) E_n(x) + R_n(x), \quad E_n'(0) = 0 = E_n'(1), \] (9)

where \(R_n(x)\) depends only on the \(E_k(x), k = 1, 2, \ldots, n-1,\) including their end-point values. (For further details, see Sec. 4 in [Bracken and Bass 18].)

Linearly-independent solutions of the homogeneous ODE \((R_n = 0)\) in \(\text{(9)}\) are provided by

\[ A(x) = Ai(s), \quad B(x) = Bi(s), \quad s = (1 - \sigma + 2\sigma x)/(4\nu\sigma^2)^{1/3}, \] (10)
where $A_i$ and $B_i$ are Airy functions of the first and second kind [Abramowitz and Stegun 64]. Because the Wronskian of $A_i$ and $B_i$ is given by $1/\pi$, that of $A$ and $B$ is given by

$$W = \left[2\sigma/(\pi^3 \nu)\right]^{1/3},$$

and the method of variation of parameters gives the general solution of the ODE in (9) as

$$E_n(x) = -\frac{1}{\nu W} \left\{ A(x) \int_0^x R_n(y) B(y) \, dy - B(x) \int_0^x R_n(y) A(y) \, dy \right\} + d_{n,A} A(x) + d_{n,B} B(x)$$

(12)

with $d_{n,A}$, $d_{n,B}$ arbitrary constants. Imposing the BCs in (9) then gives

$$d_{n,A} = -\frac{B'(0)}{A'(1) B'(0) - A'(0) B'(1)} \nu W \left\{ A'(1) \int_0^1 R_n(y) B(y) \, dy - B'(1) \int_0^1 R_n(y) A(y) \, dy \right\},$$

$$d_{n,B} = -A'(0) d_{n,A} / B'(0).$$

(13)

It may be noted in passing that the first non-zero term $E_1(x)$ in (8) was found many years ago [Bass 64] in the form (12) as the solution of the linearised version of (3).

The next step is to set

$$E(x) = \sum_{k=1}^n E_k(x), \quad n = 1, 2, \ldots$$

(14)

so defining a sequence of approximants to $E(x)$. In particular, this defines $n$th approximations $E^{(n)}(0)$ to $E(0)$ and $E^{(n)}(1)$ to $E(1)$. The accuracy of the $n$th approximation is tested by introducing the error measure

$$\Delta_n = \max \left\{ |E^{(n)}(x) - E(x)| + |E^{(n)'}(x) - E'(x)| \right\}$$

(15)

over all $x \in [0, 1]$.

Critically important in what follows is that $E(x)$ satisfying (3) can be converted by a nonlinear transformation involving $E(0)$ and $E(1)$ into a solution $y(z)$ of a corresponding BVP [1] for PII [Bass 64]. The conversion
is defined by formulas (3.9) – (3.12) in [Bass et al. 10] and formulas (13) in [Bracken and Bass 18], which set

\[ y(z) = \frac{1}{2\beta} E \left( \frac{z - \gamma}{\beta} \right), \quad \gamma = a \leq z \leq b = \gamma + \beta, \]

\[ C = \frac{\nu \tau [E(0)^2 - E(1)^2] - 4\mu}{4\nu \beta^3}, \quad (16) \]

where

\[ \beta = \left( \frac{2\sigma}{\nu} + \frac{1}{2} [E(0)^2 - E(1)^2] \right)^{1/3}, \quad \gamma = \frac{1}{\nu \beta^2} \left[ 1 - \sigma - \frac{1}{2} \nu E(0)^2 \right]. \quad (17) \]

Note that \( \beta > 0 \) as a consequence of (5) and (6) (implying \( b > a \)), and also from (16) that the monotonicity and definite sign of the solution \( E(x) \) translates into similar properties for \( y(z) \).

Corresponding to (14), a sequence of approximants \( y^{(n)}_E(z) \) to \( y(z) \) is obtained in the form

\[ y^{(n)}_E(z) = \frac{1}{2\beta_n} E^{(n)} \left( \frac{z - \gamma_n}{\beta_n} \right), \quad \gamma_n = a_n \leq z \leq b_n = \gamma_n + \beta_n, \quad (18) \]

with

\[ \beta_n = \left( \frac{2\sigma}{\nu} + \frac{1}{2} [E^{(n)}(0)^2 - E^{(n)}(1)^2] \right)^{1/3}, \]

\[ \gamma_n = \frac{1}{\nu \beta_n^2} \left[ 1 - \sigma - \frac{1}{2} \nu E^{(n)}(0)^2 \right]. \quad (19) \]

As a consequence of the BCs (4), these approximants satisfy

\[ y^{(n)}_E'(a_n) = 0 = y^{(n)}_E'(b_n). \quad (20) \]

Thus the function \( y(z) \) is (potentially) approached by a sequence of approximants \( y^{(n)}_E(z) \) defined on intervals \([a_n, b_n]\) that can differ from one value of \( n \) to the next and that approach an interval \([a, b]\) determined only as \( n \to \infty \). Furthermore, corresponding to the definition of \( C \) in (16), we have

\[ C_n = \frac{\nu \tau [E^{(n)}(0)^2 - E^{(n)}(1)^2] - 4\mu}{4\nu \beta_n^3}, \quad (21) \]

so that the value of \( C \) in the ODE (1) satisfied by \( y(z) \), and hence the ODE itself, is only determined in the limit.
Whenever the sequence of approximants $E^{(n)}(x)$ converges to $E(x)$ satisfying (3) and (4), as it appears to do in a surprisingly wide variety of cases [Bracken and Bass 18], it follows that the sequence of approximants $y_{E}^{(n)}(z)$ converges to $y(z)$ satisfying (1) for some corresponding set of values for $a$, $b$ and $C$.

3 Illustrative numerical examples

[Remark: Numerical approximations have been used for all functions involved in the figures appearing here and above, and in the evaluation of the error measure (15). These approximations were obtained using commercial packages [MATLAB 16] to solve the BVP (3), (4), and to evaluate the integrals and Airy functions in the formulas (12), (13) for $E_{n}(x)$. As in [Bracken and Bass 18], the conservative view is adopted that numerical calculations are accurate to 1 part in $10^7$ in the determination of $E(x)$, and also of $E^{(n)}(x)$ and $\Delta_n$, up to $n = 500$.]

Two examples are now considered, where BVPs of the form (3), (4) with different parameter values lead to corresponding BVPs of the form (1). The sequence of approximants (14) has been considered previously for both cases (see the fifth and first entries in Table 1, and Figs. 5 and 3 in [Bracken and Bass 18], where arguments for convergence have also been presented).

For each example, $E(x)$ is first determined, including $E(0)$ and $E(1)$, and then (16) and (17) are used to get the values $a$, $b$, and $C$ that complete the definition of the corresponding BVP (1). Because they are determined numerically, these values and so the BVP itself, are only known approximately. Also from $E(x)$, (16) and (17), a solution $y(z)$ of (1) is constructed.

Next successive $E^{(n)}(x)$ are determined, including $E^{(n)}(0)$ and $E^{(n)}(1)$, from which the values $a_n$, $b_n$, and $C_n$ are determined using (18) and (19). Also from $E^{(n)}(x)$, successive approximants $y_{E}^{(n)}(z)$ to $y(z)$ are determined using (18), with each approximant defined on its corresponding interval $[a_n, b_n]$.

The first example concerns the BVP (3), (4) with parameter values

$$\sigma = 1/3, \quad \tau = -0.2, \quad \nu = 3.5, \quad \mu = 2.0.$$  \hspace{1cm} (22)

This BVP is known [Bracken and Bass 18] to have a unique solution $E(x)$ of Type A, as shown in Fig. 3, where plots of $E(x)$ and $E^{(n)}(x)$ are shown, for $n = 1, 2, 3$ on the left, and for $n = 7, 8, 9$ on the right, suggesting convergence. (Note the different scales on the vertical axes for the two sets of plots.) This suggestion is reinforced by Fig. 4 on the left and in the center, showing the decrease of $\log_{10}(\Delta_n)$ to less than $-7$ by $n = 43$ and beyond,
out to $n = 500$, and on the right, the corresponding agreement of $E^{(44)}(x)$ and $E(x)$ to better than 1 part in $10^7$.

Figure 3: The case $\sigma = 1/3$, $\tau = -0.2$, $\nu = 3.5$, $\mu = 2.0$. Plots of $E(x)$ and $E^{(n)}(x)$ for selected values of $n$, suggesting convergence.

For this example, it is found that $E(0) = 4.180\ldots$ and $E(1) = 4.129\ldots$ (see Fig. 4), leading to $a$, $b$ and $C$ as in (2). That these values are only known approximately is now seen to be a result of the numerical manipulations. The solution $y(z)$ of (1), (2) constructed from $E(x)$ using (16) and (17) is shown in Fig. 1, together with $y^{(n)}_E(z)$ for $n = 1, 2, 3, 4$ and for $n = 8, 9, 10, 11$, as constructed from $E^{(n)}(x)$. Convergence of the $y^{(n)}_E(z)$ to $y(z)$ will follow from convergence of the $E^{(n)}(x)$ to $E(x)$, assuming that convergence holds in the latter case as strongly suggested by the numerical computations.

The parameter values $a_n$, $b_n$ and $C_n$, constructed from $E^{(n)}(0)$ and $E^{(n)}(1)$ using (18), (19) and (21), are shown in Fig.2.

Note the very different character of the approach of the extraordinary sequence of approximants $y^{(n)}_E(z)$ to $y(z)$, as shown in Fig. 1 when compared with the approach of the (ordinary) sequence of approximants $E^{(n)}(x)$ to $E(x)$, as shown in Fig. 3.

The second example concerns the BVP (3), (4) with parameter values

$\sigma = 1/3, \quad \tau = -0.2, \quad \nu = 0.1, \quad \mu = -0.5,$

(23)

for which there is known to exist a unique solution of Type B. The corre-
Figure 4: The case $\sigma = 1/3$, $\tau = -0.2$, $\nu = 3.5$, $\mu = 2.0$. On the left and in the centre, plots of $\log_{10} \Delta_n$ versus $n$, out to $n = 500$. On the right, plot of $E(x)$ (solid line) and $E^{(44)}(x)$ (circles) showing strong agreement.

The corresponding BVP (1) in this case is found to have
\[ a = 1.645 \ldots , \ b = 3.554 \ldots , \ c = 0.714 \ldots . \] (24)

Fig. 5 shows a plot of $y(z)$, together with $y_E^{(n)}(z)$ for $n = 1, 2$. In this case, apparent convergence is much more rapid than in the previous example. At the level of resolution in the figure the curve for $n = 2$ is difficult to distinguish, and curves for larger values of $n$ cannot be distinguished at all, from the curve for $y(z)$.

The different intervals of support for $y_E^{(1)}(z)$, $y_E^{(2)}(z)$ and $y(z)$ are barely discernible in Fig. 5, but (apparent) convergence of $a_n$, $b_n$, and $C_n$ values to $a$, $b$ and $C$ as in (24) is shown in Fig. 6. The (apparent) rapid convergence of the extraordinary sequence of approximants to $y(z)$ in this case follows from the (apparent) rapid convergence of the corresponding sequence of approximants $E^{(n)}(x)$ to $E(x)$, as seen in Fig. 7, which shows $\log_{10}(\Delta_n)$ decreasing to less than $-7$ by $n = 7$ and beyond, out to $n = 500$, and accordingly the strong agreement of $E(x)$ and $E^{(8)}(x)$ to better than 1 part in $10^7$.

A perplexing feature of the construction of extraordinary sequences of approximants is the inability to determine a priori the appropriate parameter values in the BVP (3), (4) that will lead to to a given choice of $a$, $b$ and $C$ in (1). It is the converse procedure that is more direct: Choose parameters in (3), (4), solve that BVP numerically, and use the solution to determine values
for the parameters in a BVP (1). Then determine the $E^{(n)}(x)$ and hence approximate values for $C_n$ and for the intervals $[a_n, b_n]$ on which successive approximants $y^{(n)}_E(z)$ to the solution of (1) are defined.

Given that, and given that it is not known a priori for which values of $a, b$ and $C^{(1)}$ posseses a singularity-free solution, it becomes important to determine those values of these constants that correspond to values of the parameters appearing in (3), in the ranges (5), because each such choice of those parameters is known to determine a singularity-free solution of (3), (4), from which a corresponding singularity-free solution of (1) follows using (16) and (17). Unfortunately, here it must be recognized that $a, b$ and $C$ must properly be regarded as functions of all four parameters appearing in (3). Ideally these three functions should be evaluated over the full four-dimensional space of points $(\sigma, \tau, \nu, \mu)$ defined by (5), perhaps restricted as in (7), to determine for which BVPs (1), (apparently) convergent extraordinary approximating sequences can be found. Unfortunately, it is difficult to explore such a four-dimensional space by numerical means: there is “a tyranny of dimensionless parameters” [Montroll and Shuler 79].

The variation of $a, b$ and $C$ can at least be found when three of the parameters $(\sigma, \tau, \nu, \mu)$ are held fixed, while one, say $\mu$, is allowed to vary. Fig. 8 shows plots of $a, b$ and $C$ values determined by varying $\mu$ values for $\sigma = 1/3, \tau = -0.2$ and $\nu = 3.5$, including those for the first example considered.
above, corresponding to \( \mu = 2.0 \), shown as circled points. Similarly, Fig. 9 shows corresponding plots when \( \sigma = 1/3, \tau = -0.2 \) and \( \nu = 0.1 \), including those for the second example considered above, corresponding to \( \mu = -0.5 \), again shown as circled points.

It can also be noted again that when \( \mu = 0 \), the solution of (3) and (4) is \( E(x) = 0 \). Then (16) gives \( C = 0 \) and \( y(z) = 0 \), which is the solution of (1) whatever the values of \( a \) and \( b \) when \( C = 0 \). As \( \mu \) approaches 0, and so also \( E(x) \), it follows from (16) that

\[
a \to (1 - \sigma)/(4\nu\sigma^2)^{1/3}, \quad b \to a + (2\sigma/\nu)^{1/3}.
\]

With \( \sigma = 1/3 \) this gives \( a = 0.575 \ldots \), \( b = 1.150 \ldots \) when \( \nu = 3.5 \), and \( a = 1.882 \ldots \), \( b = 3.764 \ldots \) when \( \nu = 0.1 \), providing checks on Figs. 8 and 9 at the points where \( \mu = 0 \).

4 Concluding remarks

Complicated approximating sequences to solutions of BVPs may seem of little value when fast and accurate numerical solutions are readily attainable. The real value of the extraordinary sequences described above lies not in their utility as approximating schemes, but in their very existence as explicit expansions of Painlevé transcendents, whose properties are still very much being explored, in terms of familiar Airy functions. Connections of PII with Airy functions have been obtained before [Lukashevich 71; Ablowitz and Segur 77; Clarkson 16], in particular in sequences of special solutions, as
Figure 7: The case $\sigma = 1/3$, $\tau = -0.2$, $\nu = 0.1$, $\mu = -0.5$. On the left and in the centre, plots of $\log_{10} \Delta_n$ versus $n$, out to $n = 500$. On the right, plot of $E(x)$ (solid line) and $E^{(8)}(x)$ (circles) showing strong agreement.

mentioned earlier, but the expansions obtained here are very different in character from earlier results.

Perhaps the most remarkable aspect of the construction of extraordinary sequences is the apparent need to involve the unusual supplementary ODE (3) and its solutions. It seems to be impossible to construct these sequences by working entirely in the framework of the BVP (1), so avoiding this indirect approach.

The reader may well wonder if a more obvious approach would provide a much simpler and more direct way to obtain sequences of approximants to solutions of (1), by perturbing away from the trivial solution which applies when $C = 0$. This approach was considered, first replacing $C$ by $\epsilon C$ in (1) with the introduction of a book-keeping parameter $\epsilon$, and then expanding $y(z)$ as

$$y(z) = 0 + \epsilon y_1(z) + \epsilon^2 y_2(z) + \ldots.$$  \hfill (26)

Substituting (26) in the ODE and equating powers of $\epsilon$ leads to a linear BVP for each successive $y_n(z)$, one that is explicitly solvable, again in terms of Airy functions. A sequence of approximants can then be defined by

$$y^{(n)}(z) = \sum_{k=1}^{n} y_n(z), \quad n = 1, 2, \ldots$$  \hfill (27)
It was found that this sequence apparently converges rapidly to a solution in the second case considered above, defined by (1) and (24). However, in the first case, defined by (1) and (2), the sequence apparently diverges.

It is not hard to pinpoint a key difference between the two cases. From Fig. 5 it can be seen that in the second case \(0.23 < |y(z)| < 0.33\) for all \(a < z < b\). It follows that in this case

\[
|2y(z)^3| \ll |zy(z)|, \quad |2y(z)^3| \ll |C|, \quad a < z < b, \quad (28)
\]

so the nonlinear term in the ODE is everywhere small compared with the other terms on the RHS, and it is no surprise that the perturbation sequence (apparently) converges. In contrast, Fig. 1 shows that in the first case \(|y(z)| > 2.79\) for all \(a < z < b\), so the nonlinear term is not small compared with the other terms on the RHS and the sequence (apparently) diverges.

It is quite unclear why the much more complicated approach described in Sec. 2 produces an apparently convergent sequence, especially in cases like that in the first example above where ordinary perturbation fails. Indeed, it remains a mystery why the sequence of functions \(E^{(n)}(x)\) in [14] apparently converges to the solution \(E(x)\) of the BVP \([3], [4]\) in such a wide variety of cases [Bracken and Bass 18], leading to apparently convergent extraordinary sequences of approximants to solutions of BVPs \([1]\) in a correspondingly wide range of cases.
Figure 9: Ranges of $a$, $b$ and $C$ values occurring for $\sigma = 1/3$, $\tau = -0.2$, $\nu = 0.1$ and variable $\mu$. Values at $\mu = -0.5$ are circled.

Evidently the results presented above, being largely based on numerical experiments, pose important unanswered questions warranting further analysis. But such analysis is beyond the scope of the present work.

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