Choiceless cardinals and the continuum problem

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Abstract

Under large cardinal hypotheses beyond the Kunen inconsistency — hypotheses so strong as to contradict the Axiom of Choice — we solve several variants of the generalized continuum problem and identify structural features of the levels \( V_\alpha \) of the cumulative hierarchy of sets that are eventually periodic, alternating according to the parity of the ordinal \( \alpha \). For example, if there is an elementary embedding from the universe of sets to itself, then for sufficiently large ordinals \( \alpha \), the supremum of the lengths of all wellfounded relations on \( V_\alpha \) is a strong limit cardinal if and only if \( \alpha \) is odd.

1 Introduction

We write \( X \leq^* Y \) to abbreviate the statement that there is a surjective partial function from \( Y \) to \( X \). The Lindenbaum number of a set \( X \) is the cardinal \( \mathcal{N}^*(X) = \sup\{\eta + 1 : \eta \leq^* X\} \). (The term is due to Karagila.) For each ordinal \( \alpha \), we define the \( \alpha \)-th Lindenbaum number:

\[
\theta_\alpha = \sup_{x \in V_\alpha} \mathcal{N}^*(x)
\]

If \( \alpha \) is a successor ordinal, then \( \theta_\alpha = \mathcal{N}^*(V_{\alpha-1}) \) is the supremum of the lengths of all wellfounded relations on \( V_{\alpha-1} \), and if \( \gamma \) is a limit ordinal, then \( \theta_\gamma = \sup_{\xi < \gamma} \theta_\xi \). Thus \( \theta_\omega = \omega \) and \( \theta_{\omega+1} = \omega_1 \).

The Lindenbaum numbers provide a rough measure of the size of the levels of the cumulative hierarchy. Assuming the Axiom of Choice, \( \mathcal{N}^*(X) = |X|^+ \) for all sets \( X \), and so the generalized continuum hypothesis is equivalent to the assertion that \( \theta_{\omega+\xi} = \aleph_\xi \) for all ordinals \( \xi \). In this paper, however, we will avoid the Axiom of Choice in order to study large cardinal hypotheses so strong as to contradict it. Our main theorem shows that these choiceless large cardinal hypotheses imply a dramatic failure of the generalized continuum hypothesis at alternating levels of the cumulative hierarchy:

**Theorem.** Assume there is an elementary embedding from the universe of sets to itself. Then for all sufficiently large limit ordinals \( \gamma \) and all natural numbers \( n \), the following hold:

1. \( \theta_{\gamma+2n} \) is a strong limit cardinal: that is, for no \( \eta < \theta_{\gamma+2n} \) is \( \theta_{\gamma+2n} \leq^* P(\eta) \).
2. \( \theta_{\gamma+2n+1} \) is not a strong limit cardinal: in fact, \( \theta_{\gamma+2n+1} \leq^* P(\theta_{\gamma+2n}) \).

This is the content of Theorems 8.3 and 9.3. The main plan of attack is sketched in Section 3 following the statements of Theorem 3.3 and Theorem 3.4.
An ordinal $\epsilon$ is even if $\epsilon = \gamma + 2n$ for some limit ordinal $\gamma$ and natural number $n$; all other ordinals are odd. The theorem above highlights a key difference between the structure of the even and odd levels of the cumulative hierarchy under choiceless large cardinal hypotheses. It also indicates an analogy with the Axiom of Determinacy (AD), which has similar ramifications for much smaller Lindenbaum numbers: under AD, the cardinal $\theta_{\omega+2}$, which is usually denoted by $\Theta$, is a strong limit cardinal whereas $\theta_{\omega+3} = \Theta^+$. The former is a theorem of Moschovakis [1], while the latter is Proposition 2.2, a simple observation of the author’s.

The first instances of the periodicity phenomena around choiceless cardinals were discovered by Schlutzenberg and the author independently [2].

Theorem 1.1 (Goldberg, Schlutzenberg). Suppose $\alpha$ is an ordinal. If $j : V_\alpha \rightarrow V_\alpha$ is an elementary embedding, $j$ is definable over $V_\alpha$ from parameters if and only if $\alpha$ is odd. 

A notable distinction between this theorem and the main theorem of this paper is that the periodic properties identified here make no reference to metamathematical notions like elementary embeddings and definability nor to any large cardinal theoretic concepts whatsoever.

The calculation of the Lindenbaum numbers requires the development of a good deal of machinery for choiceless cardinals, which yields some other theorems. For example, recall that the Hartogs number of a set $X$, denoted by $\aleph(X)$, is the least ordinal that admits no injection into $X$.

Theorem 5.1. If $\epsilon$ is even and there is an elementary embedding from $V_{\omega+3}$ to itself, then $\aleph(V_{\omega+2}) = \theta_{\epsilon+2}$.

In other words, there is no $\theta_{\epsilon+2}$-sequence of distinct subsets of $V_{\epsilon+1}$.

Theorem 7.1. If there is an elementary embedding from the universe of sets to itself, then for some cardinal $\kappa$, for all even $\epsilon \geq \kappa$, if $\eta < \theta_{\epsilon+2}$, then the set of $\kappa$-complete ultrafilters on $\eta$ has cardinality less than $\theta_{\epsilon+2}$.

These theorems are analogous to consequences of the determinacy of real games (AD$_R$), which implies that $\aleph(V_{\omega+2}) = \Theta$ and that for all $\eta < \Theta$, the set of ultrafilters on $\eta$ has cardinality less than $\Theta$. The latter is a theorem of Kechris [3], while the former is part of the folklore, perhaps first observed by Solovay. Both these theorems require the hypothesis AD$_R$ since their conclusions are false in $L(\mathbb{R})$. Similarly, our theorems quoted above are false in $L(V_{\epsilon+1})$.

2 Background: limit Lindenbaum numbers and AD

The following lemma shows that for limit levels of the cumulative hierarchy, our main theorem is a simple consequence of ZF.

Lemma 2.1. Suppose $\gamma$ is a limit ordinal. Then $\theta_\gamma$ is a strong limit cardinal and $\theta_{\gamma+1} = \theta^+_{\gamma}$.

Proof. To see that $\theta_\gamma$ is a strong limit cardinal, fix $\eta < \theta_\gamma$. For some $\alpha < \gamma$, $\eta \leq^* V_\alpha$, and so $P(\eta) \leq V_{\alpha+1}$. Therefore if $\nu \leq^* P(\eta)$, then $\nu \leq^* V_{\alpha+1}$, and hence $\nu < \theta_\gamma$.

To see that $\theta_{\gamma+1} = \theta^+_{\gamma}$, suppose $f : V_\gamma \rightarrow \nu$ is a surjection, and we will show that $|\nu| \leq \theta_\gamma$. Note that

$$\nu = \bigcup_{\alpha < \gamma} f[V_\alpha]$$
and it is easy to see that $|\nu| \leq \gamma \cdot \sup_{\alpha < \gamma} \text{ot}(f[V\alpha])$. Since $f[V\alpha]$ is a wellorderable set that is the surjective image of $V\alpha$, $\text{ot}(f[V\alpha]) < \theta_\gamma$. Hence $|\nu| \leq \gamma \cdot \theta_\gamma = \theta_\gamma$. □

If $\epsilon$ is an even successor ordinal, we do not have nearly enough slack to generalize the proof above that $\theta_\gamma$ is a strong limit cardinal to $\theta_\epsilon$. Moreover, to prove $\theta_{\epsilon+1} = \theta_\epsilon^+$ would seem to require a hierarchy on $V_{\epsilon}$ similar to the rank hierarchy on $V_\gamma$. Under the Axiom of Determinacy, such a hierarchy naturally presents itself:

**Proposition 2.2 (AD).** Let $\Theta = \theta_{\omega+2}$. Then $\theta_{\omega+3} = \Theta^+$.

**Proof.** For simplicity, assume the Axiom of Dependent Choice. Then by the Martin-Monk theorem [4], the subsets of $V_{\omega+1}$ are arranged in a wellfounded hierarchy according to their position in the Wadge order of continuous reducibility. For $\alpha < \Theta$, let $\Gamma_\alpha$ be the set of subsets of $V_{\omega+1}$ of Wadge rank $\alpha$. By Wadge’s lemma, $\Gamma_\alpha \leq^* V_{\omega+1}$: indeed, for any $A \subseteq V_{\omega+1}$ outside $\Gamma_\alpha$, $\Gamma_\alpha$ is contained in the set of continuous preimages of $A$.

Suppose $\nu \leq^* V_{\omega+2}$ is an ordinal, and we will show $\nu < \Theta^+$. Fixing a surjection $f : V_{\omega+2} \rightarrow \nu$, $\nu = \bigcup_{\alpha \in \Theta} f[\Gamma_\alpha]$. Since $f[\Gamma_\alpha] \leq^* V_{\omega+1}$, $\text{ot}(f[\Gamma_\alpha]) < \Theta$. It follows that $|\nu| \leq \Theta$, so $\nu < \Theta^+$.

To prove the proposition without Dependent Choice, for each $A \subseteq V_{\omega+1}$, let $\delta(A)$ be the supremum of the lengths all wellfounded relations on $V_{\omega+1}$ that are continuously reducible to $A$, and let $\Lambda_\alpha$ be the set of all $A \subseteq V_{\omega+1}$ with $\delta(A) \leq \alpha$. As above, Wadge’s lemma implies that for $\alpha < \Theta$, $\Lambda_\alpha \leq^* V_{\omega+1}$. The argument of the previous paragraph can be pushed through by replacing $\Gamma_\alpha$ with $\Lambda_\alpha$. □

### 3 Periodicity for embeddings

The proof in [2] of Theorem [1] rests on the natural attempt to define, for each ordinal $\alpha$, the extension “by continuity” of a $\Sigma_1$-elementary embedding $j : V_\alpha \rightarrow V_\alpha$. To prove the proposition without Dependent Choice, for each $A \subseteq V_{\omega+1}$, let $\delta(A)$ be the supremum of the lengths all wellfounded relations on $V_{\omega+1}$ that are continuously reducible to $A$, and let $\Lambda_\alpha$ be the set of all $A \subseteq V_{\omega+1}$ with $\delta(A) \leq \alpha$. As above, Wadge’s lemma implies that for $\alpha < \Theta$, $\Lambda_\alpha \leq^* V_{\omega+1}$. The argument of the previous paragraph can be pushed through by replacing $\Gamma_\alpha$ with $\Lambda_\alpha$.

First, let $H_\alpha$ be the union of all transitive sets $M$ such that $M \leq^* V_\alpha$ for some $\gamma < \alpha$. The embedding $j$ can be extended to act on $H_\alpha$ as follows. Fix $x \in H_\alpha$, and let $R \in V_\alpha$ code a wellfounded relation whose transitive collapse is the transitive closure of $\{x\}$. Then define $j(x)$ to be the unique set $y$ such that the transitive collapse of $j(R)$ is the transitive closure of $\{y\}$. This extends $j$ to a well-defined $\Sigma_0$-elementary embedding $j : H_\alpha \rightarrow H_\alpha$. If $j$ is $\Sigma_n$-elementary on $V_\alpha$, then its extension to $H_\alpha$ is $\Sigma_n$-elementary. Throughout this paper, when faced with an elementary embedding of $V_\alpha$, we will often make use of its extension to $H_\alpha$ without comment.

Second, we attempt to extend $j$ to $V_{\alpha+1}$. If $j : V_\alpha \rightarrow V_\alpha$ is a $\Sigma_1$-elementary embedding, the **canonical extension** of $j$ is the map $j^+ : V_{\alpha+1} \rightarrow V_{\alpha+1}$ defined by

$$j^+(A) = \bigcup_{N \in H_\alpha} j(A \cap N)$$

It is not hard to show that the graph of $j^+$ is definable over $V_{\alpha+1}$ from (a code for) $j$. Theorem [1] is proved by an induction that establishes:

**Theorem 3.1.** Suppose $\epsilon$ is an even ordinal.

1. If $i : V_\epsilon \rightarrow V_\epsilon$ is elementary, then $i$ is cofinal: $H_\epsilon = \bigcup_{N \in H_\epsilon} i(N)$.
2. If $j : V_{\epsilon+1} \rightarrow V_{\epsilon+1}$ is elementary, then $j = (j \upharpoonright V_\epsilon)^+$.
Lemma 3.2. Suppose $j : M \to N$ is an elementary embedding between two transitive structures. Suppose $X, Y \in M$ and $M$ satisfies $X \leq^* Y$. If $j[Y] \in N$, then $j[X]$ is definable over $N$ from $j[Y]$ and parameters in $j[M]$. \hfill $\Box$

In particular, for all $\eta \leq^* Y$, $j[\eta]$ is definable over $N$ from $j[Y]$ and parameters in $j[M]$. Our strong undefinability theorem is as follows:

Theorem 3.3. If $\epsilon \leq \alpha$ are ordinals, $\epsilon$ is even, and $j : V_{\alpha} \to V_{\alpha}$ is elementary, then for all $\gamma < \epsilon$, $j[\theta_\epsilon]$ is not definable over $\mathcal{H}_\alpha$ from $j[V_\gamma]$ and parameters in $j[\mathcal{H}_\alpha]$.

The rough idea behind our analysis of the even Lindenbaum numbers (Theorem 3.3) is to show that given an even ordinal $\epsilon$ and an elementary embedding $j : V_{\epsilon+2} \to V_{\epsilon+2}$, for $\eta < \theta_{\epsilon+2}$, $j[P(\eta)]$ is definable over $\mathcal{H}_{\epsilon+2}$ from $j[V_{\epsilon+1}]$ and parameters in $j[\mathcal{H}_{\epsilon+2}]$, and therefore $\theta_{\epsilon+2} \not\leq^* P(\eta)$: otherwise by Lemma 3.2, $j[\theta_{\epsilon+2}]$ would be definable over $\mathcal{H}_{\epsilon+2}$ from $j[V_{\epsilon+1}]$ and parameters in $j[\mathcal{H}_{\epsilon+2}]$, contrary to Theorem 3.3.

Theorem 3.4 follows from a general ultrafilter-theoretic fact:

Theorem 3.4. Suppose $U$ is an ultrafilter on a set $X$ such that $X \times X \leq^* X$ and $\kappa = \text{crit}(U)$. Then for any ordinal $\eta$, if $j_U[\eta] \in M_U$, then $\eta \leq^* X$.

The rough idea behind our analysis of the odd Lindenbaum numbers is to show that given an even ordinal $\epsilon$ and an elementary embedding $j : V_{\epsilon+2} \to V_{\epsilon+2}$, for $\eta < \theta_{\epsilon+2}$, $j[P(\eta)]$ is definable over $\mathcal{H}_{\epsilon+2}$ from $j[V_{\epsilon+1}]$ and parameters in $j[\mathcal{H}_{\epsilon+2}]$, and therefore $\theta_{\epsilon+2} \not\leq^* P(\eta)$: otherwise by Lemma 3.2, $j[\theta_{\epsilon+2}]$ would be definable over $\mathcal{H}_{\epsilon+2}$ from $j[V_{\epsilon+1}]$ and parameters in $j[\mathcal{H}_{\epsilon+2}]$, contrary to Theorem 3.3.

Theorem 3.4 follows from a general ultrafilter-theoretic fact:

Proof of Theorem 3.3. Assume towards a contradiction that for some $\gamma < \epsilon$ and $p \in \mathcal{H}_{\alpha}$, $j[\theta_\epsilon]$ is definable over $\mathcal{H}_{\alpha}$ from $j[V_\gamma]$ and $j(p)$. By Theorem 1.1, if $\epsilon = \gamma + 1$, then $j[V_\gamma]$ is definable over $\mathcal{H}_{\alpha}$ from $j[V_{\gamma-1}]$, and so we may assume $\gamma + 1 < \epsilon$ by replacing $\gamma$ by $\gamma - 1$ if necessary.

Let $U$ be the ultrafilter on $V_{\gamma+1}$ derived from $j[V_\gamma]$, and let $k : \text{Ult}(\mathcal{H}_{\alpha}, U) \to \mathcal{H}_{\alpha}$ be the factor embedding given by $k([f]_U) = j(f)(j[V_\gamma])$. Fix a formula $\varphi$ such that

$$j[\theta_\epsilon] = \{ \xi < \theta_{\alpha} : \mathcal{H}_{\alpha} \models \varphi(\xi, j[V_\gamma], j(p)) \}$$

For each $x \in V_{\gamma+1}$, let $f(x) = \{ \xi < \theta_{\alpha} : \mathcal{H}_{\alpha} \models \varphi(\xi, x, p) \}$, so that

$$[f]_U = k^{-1}[j[\theta_\epsilon]] = j_U[\theta_\epsilon]$$

But then by Theorem 3.4, $\theta_\epsilon \leq^* V_{\gamma+1}$, contrary to the fact that $\gamma + 1 < \epsilon$. \hfill $\Box$

Though Theorem 3.4 itself is purely combinatorial, its proof uses the techniques of ordinal definability and forcing. Suppose $Y$ is a set and $X$ is ordinal definable from $Y$. Let $\aleph_\gamma^Y(X)$ denote the least ordinal $\delta$ such that there is no surjection from $X$ to $\delta$ that is definable from $Y$ and ordinal parameters.

Lemma 3.5 (Vopenka). Suppose $Y$ is a set and $X$ is ordinal definable from $Y$. Then for any $x \in X$, $\text{HOD}_{Y,x}$ is a $\aleph_\gamma^Y(X)$-cc generic extension of $\text{HOD}_Y$. \hfill $\Box$
We will only use Lemma 3.5 to conclude that $\mathcal{N}_x^+(X)$ is regular in $\text{HOD}_{Y,x}$ for all $x \in X$ and every stationary subset of $\mathcal{N}_x^+(X)$ in $\text{HOD}_Y$ remains stationary in $\text{HOD}_{Y,x}$. Both these properties are easy to verify combinatorially assuming $\mathcal{N}_x^+(X)$ is regular in $\text{HOD}_Y$. The $\text{HOD}_Y$-regularity of $\mathcal{N}_x^+(X)$ seems to require some assumption on $Y$; we will assume that there is an $\text{OD}_Y$ surjection from $X$ onto $X \times X$.

**Proof of Theorem 3.4.** Fix a function $f$ on $X$ such that $[f]_U = j_U[\eta]$ and a surjection $p : X \to X \times X$, and let $Y$ be a set from which $f$ and $p$ are ordinal definable. Let $\delta = \mathcal{N}_x^+(X)$. Then $\delta$ is regular in $M = \text{HOD}_Y$. Assume towards a contradiction that $\eta \geq \delta$.

For each $x \in X$, let $M_x = \text{HOD}_{Y,x}$, so that $M_x$ is a $\delta$-cc generic extension of $M$ for all $x \in X$. Let $N = \prod_{x \in X} M_x/U$ be the ultraproduct formed using only those functions on $X$ that are ordinal definable from $Y$. For each such function $g$, let $[g]$ denote the element of $N$ it represents. Then this ultraproduct satisfies Loš’s theorem in the sense that $N \models \varphi([g])$ if and only if $M_x \models \varphi(g(x))$ for $U$-almost all $x \in X$.

Since $M \subseteq M_x$ for all $x \in X$, we can define a function $i$ on $M$ by $i(B) = [c_B]$ where $c_B : X \to \{B\}$ is the constant function. Then $i$ is an elementary embedding from $M$ to

$$H = \{a \in N : \exists B \in M a \in i(B)\}$$

Identifying the wellfounded part of $N$ with its transitive collapse, $\delta + 1 \subseteq N$ since $i[\delta] = [f] \cap i(\delta)$ belongs to $N$. Clearly $\delta$ is regular in $M_x$ for all $x \in X$, and so $i(\delta)$ is regular in $N$. Moreover $i(\delta) = \sup i[\delta]$ since there is no $\text{OD}_Y$ function from $X$ to an unbounded subset of $\delta$. Since $i[\delta]$ belongs to $N$ and is an unbounded subset of the $N$-regular cardinal $i(\delta)$, we must have $\text{ot}(i[\delta]) = i(\delta)$. In other words, $i(\delta) = \delta$.

Working in $M$, fix a partition $S = (S_\alpha : \alpha < \delta)$ of $(S_\delta^\delta)^M$ into stationary sets. Let $T = i(S)$, so $T = (T_\alpha : \alpha < \delta)$ is a partition of $(S_\delta^\delta)^H$ into $H$-stationary sets. For each $x \in X$, any set in $P(\{\delta\}) \cap M$ that is stationary in $M$ remains stationary in $M_x$ by Lemma 3.5 and so by Loš’s theorem, any set in $P(\{\delta\}) \cap H$ that is stationary in $H$ remains stationary in $N$. Since $i$ is continuous at ordinals of cofinality $\omega$, the set $i[\delta]$ is an $\omega$-closed unbounded set in $N$. It follows that $T_\kappa \cap i[\delta] \neq \emptyset$ where $\kappa = \text{crit}(j_U) = \text{crit}(i)$. But if $i(\xi) \in T_\kappa$, then $\xi \in S_\alpha$ for some $\alpha < \delta$, and hence $\xi \in T_{i(\alpha)}$, which contradicts that $T_\kappa$ and $T_{i(\alpha)}$ are disjoint. □

## 4 The coding lemma

In this section, we establish an analog of the Moschovakis coding lemma [11] under choiceless large cardinal hypotheses. The proof is a slight twist on an argument of Woodin [5] establishing a similar property of $L(V_{\lambda+1})$ under $I_\vartheta$ in the context of ZFC.

For ordinals $\lambda \leq \epsilon$, let $I(\lambda, \epsilon)$ abbreviate the statement that for all $\alpha < \lambda$ and $A \subseteq V_{\epsilon+1}$, for some $A' \subseteq V_{\epsilon+1}$, there are elementary $j_0, j_1 : (V_{\epsilon+1}, A') \to (V_{\epsilon+1}, A)$ whose distinct critical points lie between $\alpha$ and $\lambda$. Admittedly, this is a somewhat contrived large cardinal hypothesis, but it is useful because if $\epsilon$ is even, $I(\lambda, \epsilon)$ is downwards absolute to inner models containing $V_{\epsilon+1}$ and follows from the existence of an elementary embedding $j : V_{\epsilon+2} \to V_{\epsilon+2}$ with critical point $\kappa$ such that $\lambda = \sup \{\kappa, j(\kappa), j(j(\kappa)), \ldots\}$.

**Lemma 4.1.** If $\lambda = \sup \{\kappa, j(\kappa), j(j(\kappa)), \ldots\}$ for some elementary $j : V_{\epsilon+2} \to V_{\epsilon+2}$ with critical point $\kappa$, then $I(\lambda, \epsilon)$ holds.

**Proof.** Let $\alpha < \lambda$ be the least ordinal such that $I(\lambda, \epsilon)$ fails for $\alpha$ in the sense that there is some set $A \subseteq V_{\epsilon+1}$ such that there is no $A' \subseteq V_{\epsilon+1}$ admitting elementary embeddings
$j_0, j_1 : (V_{\varepsilon +1}, A') \to (V_{\varepsilon +1}, A)$ whose distinct critical points lie between $\alpha$ and $\lambda$. Note that $\alpha$ is definable over $V_{\varepsilon +2}$, and so $j(\alpha) = \alpha$. Since $\lambda$ is the least fixed point of $j$ above its critical point $\kappa$, $\kappa > \alpha$.

Fix a set $A \subseteq V_{\varepsilon +1}$ such that $(\alpha, A)$ witnesses the failure of $I(\lambda, \epsilon)$. Let $A_1 = j(A)$ and $A_2 = j(j(A))$. By the elementarity of $j \circ j$, $(\alpha, A_2)$ also witnesses the failure of $I(\lambda, \epsilon)$. Since $j_0 = j \upharpoonright V_{\varepsilon +1}$ is an elementary embedding from $(V_{\varepsilon +1}, A)$ to $(V_{\varepsilon +1}, A_1)$, the elementarity of $j$ implies that $j_1 = j(j_0)$ is an elementary embedding from $(V_{\varepsilon +1}, A_1)$ to $(V_{\varepsilon +1}, A_2)$. But note that $j_0$ is also an elementary embedding from $(V_{\varepsilon +1}, A_1)$ to $(V_{\varepsilon +1}, A_2)$ since $j(A_1) = A_2$.

We have $\alpha < \text{crit}(j_0) = \kappa < j(\kappa) = \text{crit}(j_1) < \lambda$. Therefore $j_0$ and $j_1$ witness $I(\lambda, \epsilon)$ for $(\alpha, A_2)$, which contradicts that $(\alpha, A_2)$ witnesses the failure of $I(\lambda, \epsilon)$.

Throughout this section, we fix an even ordinal $\epsilon$. We will establish a (slightly technical) general result with the following consequence:

**Theorem 4.2.** Fix a class $A$ and let $M = L(V_{\varepsilon +1})[A]$ or $M = \text{HOD}_{V_{\varepsilon +1}, A}$. If $M$ satisfies $I(\lambda, \epsilon)$ for some $\lambda \leq \epsilon$, then for all $\eta < \theta_{\varepsilon +2}$, $M$ satisfies $P(\eta) \cap M \leq^* V_{\varepsilon +1}$.

**Definition 4.3.** A set $\Gamma \subseteq V_{\varepsilon +2}$ is an elementary pointclass if $\Gamma \leq^* V_{\varepsilon +1}$ and for all $\Sigma_1$-elementary embeddings $j : V_{\varepsilon} \to \varepsilon$ and all $A \in \Gamma$, the image and preimage of $A$ under the canonical extension of $j$ to $V_{\varepsilon +1}$ belongs to $\Gamma$.

- Let $\text{EP}$ denote the set of all elementary pointclasses.
- If $\lambda$ is an ordinal, then $\lambda$-pointclass choice holds if for any any total relation $R$ on $\text{EP} \times V_{\varepsilon +2}$, there is a sequence $(\Gamma_\alpha)_{\alpha < \lambda} \in \text{EP}^\lambda$ such that $\bigcup_{\alpha < \lambda} \Gamma_\alpha \leq^* V_{\varepsilon +1}$ and for all $\beta < \lambda$, there is some $A \in \Gamma_\beta$ such that $R((\bigcup_{\alpha < \beta} \Gamma_\alpha), A)$.
- The weak coding lemma states that for any surjective $\varphi : V_{\varepsilon +1} \to \eta$, there is an elementary pointclass $\Gamma$ such that every binary relation $R$ on $V_{\varepsilon +1}$ with $\sup \varphi[\text{dom}(R)] = \eta$ has a subrelation $S \in \Gamma$ such that $\sup \varphi[\text{dom}(S)] = \eta$.

Any model $M$ as in Theorem 4.2 satisfies $\lambda$-pointclass choice since $M$ contains a well-ordered set of elementary pointclasses whose union is $V_{\lambda +2}$.

**Lemma 4.4.** Assume $I(\lambda, \epsilon)$ and $\lambda$-pointclass choice. Then the weak coding lemma holds.

**Proof.** Suppose the weak coding lemma fails. Fix a surjection $\varphi : V_{\varepsilon +1} \to \eta$ witnessing this.

By $\lambda$-pointclass choice, there is a sequence $(\Gamma_\alpha)_{\alpha < \lambda} \in \mathcal{H}_{\varepsilon +2}$ such that for each $\beta < \lambda$, there is a binary relation $R$ on $V_{\varepsilon +1}$ in $\Gamma_\beta$ with $\sup \varphi[\text{dom}(R)] = \eta$ that has no subrelation $S \in \bigcup_{\alpha < \beta} \Gamma_\alpha$ such that $\sup \varphi[\text{dom}(S)] = \eta$.

Let $M \in \mathcal{H}_{\varepsilon +2}$ be a transitive set such that $V_{\varepsilon +1}, \eta \in M$ and $(\Gamma_\alpha)_{\alpha < \lambda}$ and $\varphi$ belong to $M$. It is left as an exercise for the reader to check that $I(\lambda, \epsilon)$ implies the existence of a transitive set $M'$ with $V_{\varepsilon +1}, \eta \in M'$ admitting an elementary embeddings $j_0, j_1 : M' \to M$ such that $\text{crit}(j_0) < \text{crit}(j_1) < \lambda$, $j_0(\eta) = j_1(\eta) = \eta$, and for some $(\Gamma'_\alpha)_{\alpha < \lambda}$ and $\varphi'$ in $M'$, $j_i((\Gamma'_\alpha)_{\alpha < \lambda}) = (\Gamma_\alpha)_{\alpha < \lambda}$ and $j_i(\varphi') = \varphi$ for $i = 0, 1$.

Let $\kappa = \text{crit}(j_0)$. By elementarity, there is a relation $R$ in $\Gamma_\kappa$ with $\sup \varphi'[\text{dom}(R)] = \eta$ that has no subrelation $S \in \bigcup_{\alpha < \kappa} \Gamma_\alpha$ such that $\sup \varphi'[\text{dom}(S)] = \eta$. Let $R_0 = j_0(R)$, and note that $R_0 \in \Gamma_{j_0(\kappa)}$ has no subrelation in $\Gamma_\kappa$ such that $\sup \varphi[\text{dom}(S)] = \eta$.

Since $j_1(\kappa) = \kappa$ and $R \in \Gamma_\kappa$, $j_1(R) \in \Gamma_\kappa$. Since $\Gamma_\kappa$ is an elementary pointclass, $R = j_1^{-1}[j_1(R)] \in \Gamma_\kappa$, and hence $j_0[R] \in \Gamma_\kappa$. But $S = j_0[R]$ is a subrelation of $j_0(R)$ such that $\sup \varphi'[\text{dom}(S)] = \sup j \circ \varphi'[\text{dom}(R)] = \eta$. 


and this is a contradiction. □

**Definition 4.5.**

- The **local collection principle** states that every total binary relation on $V_{\epsilon+1}$ has a total subrelation $S$ such that $\text{ran}(S) \leq^* V_{\epsilon+1}$.

- The **coding lemma** states that for any $\eta < \theta_{\epsilon+2}$ and any surjective $\varphi : V_{\epsilon+1} \to \eta$, there is an elementary pointclass $\Gamma$ such that every binary relation $R$ on $V_{\epsilon+1}$ with $\varphi[\text{dom}(R)] = \eta$ has a subrelation $S \in \Gamma$ such that $\varphi[\text{dom}(S)] = \eta$.

Any model $M$ as in Theorem 4.2 satisfies the local collection principle.

**Theorem 4.6.** Assume the local collection principle and the weak coding lemma. Then the coding lemma holds.

**Proof.** The proof is by induction on $\eta$. Let $\varphi : V_{\epsilon+1} \to \eta$ be a surjection. By the local collection principle and our induction hypothesis, there is an elementary pointclass $\Gamma = \{A_e\}_{e \in V_{\epsilon+1}}$ that witnesses the coding lemma for all $\gamma < \eta$.

Let $\Lambda$ be an elementary pointclass containing $U = \{(e, y) \in V_{\epsilon+1} \times V_{\epsilon+1} : y \in A_e\}$ that witnesses the weak coding lemma for $\eta$ and is closed under compositions of binary relations. We will show that $\Lambda$ witnesses the coding lemma for $\eta$.

Let $R$ be a binary relation on $V_{\epsilon+1}$ such that $\varphi[\text{dom}(R)] = \eta$. Let $\hat{R}(x, e)$ hold if $A_e$ is a subrelation of $R$ such that $\varphi[\text{dom}(A_e)] = \varphi(x)$. By our induction hypothesis, $\varphi[\text{dom}(R)] = \eta$, and so by the weak coding lemma, $\hat{R}$ has a subrelation $\hat{S} \in \Lambda$ such that $\varphi[\text{dom}(\hat{S})] = \eta$.

Now $S = U \circ \hat{R}$ is a subrelation of $R$ in $\Lambda$ such that $\varphi[\text{dom}(S)] = \eta$. □

## 5 The Hartogs number of $V_{\epsilon+2}$

The **Hartogs number** of a set $X$, denoted by $\aleph(X)$, is the least ordinal $\eta$ such that there is no $\eta$-sequence of distinct elements of $X$. The main theorem of this section computes the Hartogs number of the even levels of the cumulative hierarchy:

**Theorem 5.1.** Suppose $\epsilon$ is an even ordinal and there is an elementary embedding from $V_{\epsilon+3}$ to itself. Then $\aleph(V_{\epsilon+2}) = \theta_{\epsilon+2}$.

It is easily provable in ZF that $\aleph(V_{\epsilon+2}) \geq \aleph(V_{\epsilon+1}) = \theta_{\epsilon+2}$, so the main content of Theorem 5.1 is that there is no $\theta_{\epsilon+2}$-sequence of distinct subsets of $V_{\epsilon+1}$. This does not follow from the existence of an elementary embedding from $V_{\epsilon+2}$ to itself.

We begin by proving the following weak version of Theorem 5.1.

**Proposition 5.2.** Suppose $\epsilon$ is an even ordinal and there is an elementary $j : V_{\epsilon+3} \to V_{\epsilon+3}$. Then there is no sequence $\varphi = \langle \varphi_\eta : \eta < \theta_{\epsilon+2} \rangle$ such that for all $\eta < \theta_{\epsilon+2}$, $\varphi_\eta$ is a surjection from $V_{\epsilon+1}$ onto $\eta$.

**Proof.** Assume towards a contradiction that there is such a sequence. This implies $\theta_{\epsilon+2}$ is regular by the standard ZFC argument that $\aleph(X)$ is regular for any set $X$.

Let $\psi = j(\varphi)$. For any $\eta \in j[\theta_{\epsilon+2}]$, $j[\theta_{\epsilon+2}] \cap \eta = \psi_\eta \circ j^+[V_{\epsilon+1}]$.
It follows that \( j[\theta_{e+2}] \) is the unique \( \omega \)-closed unbounded subset \( C \subseteq \theta_{e+2} \) such that for all \( \eta \in C, C \cap \eta = \psi_\eta \circ j^+[V_{\xi+1}] \); here we use that any other such \( \omega \)-closed unbounded set has unbounded intersection with \( j[\theta_{e+2}] \), which is a consequence of the regularity of \( \theta_{e+2} \).

It follows that \( j[\theta_{e+2}] \) is definable in \( \mathcal{H}_{e+3} \) from \( j[V_\xi] \) and \( \bar{\psi} \in \text{ran}(j) \), which contradicts Theorem 5.3.

**Corollary 5.3.** Suppose \( \epsilon \) is an even ordinal and there is an elementary \( j : V_{\epsilon+3} \to V_{\epsilon+3} \).

Then for any class \( A \), letting \( M = \text{HOD}_{V_{\epsilon+1}, A} \), \( \theta^M_{\epsilon+3} < \theta_{\epsilon+2} \).

**Proof.** By incorporating \( j \) into \( A \), we may assume without loss of generality that \( j \upharpoonright M \in M \). We then have \( j(V_{\epsilon+2} \cap M) = V_{\epsilon+2} \cap M \).

Using Theorem 1.2, fix a sequence \( \langle \varphi_\eta : \eta < \theta^M_{\epsilon+2} \rangle \in M \) such that \( \varphi_\eta : V_{\epsilon+1} \to P(\eta) \cap M \) is a surjection. Note that \( j \upharpoonright P(\eta) \cap M \) is uniformly definable from \( j(\varphi_\eta) \) and \( j[V_\xi] \) in \( \mathcal{H}_{e+2} \) for \( \eta < \theta^M_{\epsilon+2} \), and as a consequence \( j \upharpoonright P_{bd}(\theta^M_{\epsilon+2}) \cap M \) is definable from \( j(\langle \varphi_\eta : \eta < \theta^M_{\epsilon+2} \rangle) \) and \( j \upharpoonright \theta^M_{\epsilon+2} \) in \( \mathcal{H}_{e+2} \). Since \( \theta^M_{\epsilon+2} < \theta_{\epsilon+2} \), it follows that \( j \upharpoonright P_{bd}(\theta^M_{\epsilon+2}) \cap M \) is definable in \( \mathcal{H}_{e+2} \) from \( j[V_\xi] \) and parameters in the range of \( j \).

Let \( E \) be the \( \mathcal{E} \)-extender of length \( \theta^M_{\epsilon+2} \) derived from \( j \), so \( E \) and \( j \upharpoonright P_{bd}(\theta^M_{\epsilon+2}) \cap M \) are essentially the same object. Let \( i : \theta^M_{\epsilon+3} \to \theta^M_{\epsilon+3} \) be the ultrapower associated to \( E \) (using only functions in \( M \)). We claim \( i = j \upharpoonright \theta^M_{\epsilon+3} \). Let \( k : \text{Ult}(\theta^M_{\epsilon+3}, E) \to \text{Ord} \) be the factor embedding defined by \( k([f, a]_E) = j(f)(a) \) for \( a \in [\theta^M_{\epsilon+2}]^{<\omega} \) and \( f \in M \) a function from an ordinal less than \( \theta^M_{\epsilon+2} \) into \( \theta^M_{\epsilon+3} \). We will show that \( k \) is the identity, or equivalently that \( k \) is surjective.

Fix \( \xi < \theta^M_{\epsilon+3} \), and let us show \( \xi \in \text{ran}(k) \). Let \( R \in M \) be a prewellorder of \( V_{\epsilon+2} \cap M \) of length greater than \( \xi \). Then \( j(R) \in M \) is a prewellorder of \( V_{\epsilon+2} \cap M \) of length greater than \( \xi \). Fix \( A \in V_{\epsilon+2} \cap M \) such that \( \text{rank}_{j(R)}(A) = \xi \). For each extensional \( \sigma \subseteq V_\xi \), let \( j_\sigma : V_\xi \to V_\xi \) denote the inverse of the transitive collapse of \( \sigma \) and let \( A_\sigma \subseteq V_{\epsilon+1} \) denote the preimage of \( A \) under the canonical extension of \( j_\sigma \). Define a partial function \( g : V_{\xi+1} \to \theta^M_{\epsilon+3} \) by \( g(\sigma) = \text{rank}_R(A_\sigma) \). Then \( g \in M \) and \( j(g)(j[V_\xi]) = \xi \). The ordertype \( \nu \) of \( \text{ran}(g) \) is less than \( \theta^M_{\epsilon+2} \). Let \( h : \nu \to \text{ran}(g) \) be the increasing enumeration. Then \( h \in M \) is a function from an ordinal less than \( \theta^M_{\epsilon+2} \) into \( \theta^M_{\epsilon+3} \), and since \( \xi \in \text{ran}(g) \), \( \text{ran}(j(h)) \) is some \( \alpha < j(\nu) \) such that \( j(h)(\alpha) = \xi \). This shows \( \xi \in \text{ran}(k) \), since \( \xi = k([h, a]_E) \).

It follows that \( i = j \upharpoonright \theta^M_{\epsilon+3} \), and so \( j[\theta^M_{\epsilon+3}] \) is definable in \( \mathcal{H}_{\epsilon+2} \) from \( j[V_\xi] \) and parameters in the range of \( j \). By Theorem 5.3, \( \theta^M_{\epsilon+3} < \theta_{\epsilon+2} \).

**Proof of Theorem 5.7.** Assume towards a contradiction that \( A = \langle A_\alpha : \alpha < \theta_{\epsilon+2} \rangle \) is a sequence of distinct subsets of \( V_{\epsilon+1} \). Let \( M = \text{HOD}_{V_{\epsilon+1}, A} \). Since \( A \in M \), \( \theta^M_{\epsilon+3} \geq \theta_{\epsilon+2} \), contradicting Corollary 5.3.

6 Rank Berkeley and rank reflecting cardinals

A cardinal \( \lambda \) is **rank Berkeley** if for all ordinals \( \alpha < \lambda \leq \beta \), there is an elementary embedding from \( V_\beta \) to itself with critical point between \( \alpha \) and \( \lambda \). The term is due to Schlutzenberg who noticed that if there is an elementary embedding from the universe of sets to itself, then there is a rank Berkeley cardinal. (This was realized independently and earlier by Woodin.)

Indeed, if \( j : V \to V \) is an elementary embedding with critical point \( \kappa \), then we claim \( \lambda = \sup\{\xi, j(\kappa), j(j(\kappa)), \ldots\} \) is rank Berkeley. Assume not, towards a contradiction, and note that \( j \) fixes the lexicographically least pair \( (\alpha, \beta) \) of ordinals \( \alpha < \lambda \leq \beta \) such that \( V_\beta \) admits no elementary embedding into itself with critical point between \( \alpha \) and \( \lambda \). Since \( \lambda \) is
the least fixed point of \( j \) above its critical point \( \kappa \), we have \( \alpha < \kappa \). Therefore \( j \restriction V_\beta \) witnesses
that there is an elementary embedding from \( V_\beta \) to itself with critical point between \( \alpha \) and
\( \lambda \), contrary to our choice of \((\alpha, \beta)\).

We prefer to work with rank Berkeley cardinals over elementary embeddings from the
universe of sets to itself, partly because the former notion is first-order and seems to capture
all of the set-theoretic content of the latter. Another reason for our preference is that the
least rank Berkeley cardinal is an important threshold in the choiceless theory of large
cardinals, as we now explain.

A cardinal \( \kappa \) is almost supercompact if for all ordinals \( \xi < \kappa < \beta \), for some
ordinal \( \bar{\beta} \) between \( \xi \) and \( \kappa \), there is an elementary embedding from \( V_{\bar{\beta}} \)
into \( V_\beta \) that fixes \( \xi \). In the context of ZFC, every almost supercompact cardinal is either a
supercompact cardinal or a limit of supercompact cardinals.

An apparently much weaker notion is that of a rank reflecting cardinal, a cardinal \( \kappa \) such
that for all ordinals \( \xi < \kappa < \beta \) and all formulas \( \varphi(x) \) in the language of set theory, there is
an ordinal \( \bar{\beta} \) between \( \xi \) and \( \kappa \) such that \( V_{\bar{\beta}} \models \varphi(\xi) \) if and only if \( V_\beta \models \varphi(\xi) \). Every almost
supercompact cardinal is rank reflecting, and every supercompact cardinal is a limit of rank
reflecting cardinals. In fact, the existence of a proper class of rank reflecting cardinals is
provable in ZF as an easy consequence of the Lévy-Montague reflection theorem.

We will take advantage of the following classification of almost supercompact cardinals,
which shows that rank reflection need not be so much weaker than almost supercompactness
after all:

**Theorem 6.1.** Let \( \lambda \) denote the least rank Berkeley cardinal.

(1) A cardinal \( \kappa \leq \lambda \) is almost supercompact if and only if it is either supercompact or a
limit of supercompact cardinals.

(2) A cardinal \( \kappa \geq \lambda \) is almost supercompact if and only if it is rank reflecting.

Justified by Theorem 6.1, the results of the following sections are stated in terms of
rank reflecting cardinals even though we will employ theorems from [6] concerning almost
supercompact cardinals.

**Proof of Theorem 6.1.** Since (1) will not be needed in our applications, we will only sketch
its proof.

Clearly any cardinal that is either supercompact or a limit of supercompact cardinals
is almost supercompact, so we focus on the converse. Suppose \( \xi \) is an ordinal and \( \eta = \eta_\xi \)
is the least ordinal greater than \( \xi \) such that for all \( \beta \geq \eta \), there is some \( \bar{\beta} \) between \( \xi \) and
\( \eta \) admitting an elementary embedding \( \pi : V_{\bar{\beta}} \rightarrow V_\beta \) such that \( \pi(\xi) = \xi \). Assuming there
is no rank Berkeley cardinal less than \( \eta \), we will show that \( \eta \) is supercompact. Since any
almost supercompact cardinal \( \kappa \) is the supremum of the set \( \{\eta_\xi : \xi < \kappa\} \), it will follow that
any almost supercompact cardinal less than or equal to the least rank Berkeley cardinal is
either a supercompact cardinal or a limit of supercompact cardinals.

For all \( \delta < \eta \), for all sufficiently large ordinals \( \alpha \), there is no \( \bar{\alpha} \) between \( \xi \) and \( \delta \) admitting
an elementary embedding \( \pi : V_{\bar{\alpha}} \rightarrow V_\alpha \) such that \( \pi(\xi) = \xi \) and \( \delta \in \text{ran}(\pi) \): otherwise, one
can show that for all ordinals \( \beta \), there some \( \bar{\beta} \) between \( \xi \) and \( \delta \) admitting an elementary
embedding \( \pi : V_{\bar{\beta}} \rightarrow V_\beta \) such that \( \pi(\xi) = \xi \), contrary to the minimality of \( \eta \).

Fix an ordinal \( \alpha \) large enough that no cardinal less than \( \eta \) is rank Berkeley in \( V_\alpha \) and for
all \( \delta < \eta \), there is no \( \bar{\alpha} \) between \( \xi \) and \( \delta \) admitting an elementary embedding \( \pi : V_{\bar{\alpha}} \rightarrow V_\alpha \).
fixing \( \xi \) with \( \delta \in \text{ran}(\pi) \). Suppose \( \tilde{\eta} < \tilde{\alpha} < \eta \) and \( \pi : V_{\tilde{\alpha}+1} \to V_{\alpha+1} \) is an elementary embedding with \( \pi(\tilde{\eta}) = \eta \). We will show that \( \text{crit}(\pi) = \tilde{\eta} \).

We first claim that \( \pi[\tilde{\eta}] \subseteq \tilde{\eta} \). Otherwise, let \( \delta < \tilde{\eta} \) be such that \( \pi(\tilde{\delta}) > \tilde{\eta} \). Let \( \delta = \pi(\tilde{\delta}) \). Since \( \delta < \eta \), our choice of \( \alpha \) implies that \( \tilde{\alpha} > \delta \). By elementarity, there is some \( \tilde{\alpha} < \tilde{\eta} \) admitting an elementary embedding \( \bar{\pi} : V_{\tilde{\alpha}} \to V_{\alpha} \) with \( \delta \in \text{ran}(\pi) \). Now \( \tilde{\alpha} < \tilde{\eta} < \delta \) and \( \pi \circ \bar{\pi} : V_{\tilde{\alpha}} \to V_{\alpha} \) is an elementary embedding with \( \delta \in \text{ran}(\pi \circ \bar{\pi}) \), contrary to our choice of \( \alpha \).

Assume towards a contradiction that \( \text{crit}(\pi) < \tilde{\eta} \). We will show that in \( V_\alpha \), there is a rank Berkeley cardinal less than \( \tilde{\eta} \), contrary to our choice of \( \alpha \). Since \( \pi[\tilde{\eta}] \subseteq \tilde{\eta} \) while \( \pi(\tilde{\eta}) = \eta \), \( \tilde{\eta} \) must have uncountable cofinality. Let \( \lambda = \sup\{\eta, \pi(\eta), \pi(\pi(\eta)), \ldots\} \), so \( \lambda \) has countable cofinality, and hence \( \lambda < \tilde{\eta} \). It is easy to see that \( \lambda \) is rank Berkeley in \( V_{\tilde{\eta}} \); consider the least \( \alpha < \beta \) with \( \alpha < \lambda \leq \beta < \tilde{\eta} \) such that there is no elementary embedding from \( V_\beta \) to itself with critical point between \( \alpha \) and \( \lambda \), and note that \( \pi(\alpha) = \alpha \) and \( \pi(\beta) = \beta \), and so \( \pi \downarrow V_\beta \) contradicts the definition of \( \alpha \) and \( \beta \). By elementarity, \( \lambda \) is rank Berkeley in \( V_\eta \), and hence \( \lambda \) is rank Berkeley in \( V_\tilde{\eta} \). By elementarity, \( \lambda \) is rank Berkeley in \( V_\alpha \), and this is a contradiction.

Turning to [2], the fact that almost supercompact cardinals are rank reflecting is immediate from the definition. Towards the converse, let us make some definitions and observations.

Fix an ordinal \( \xi \). We define an increasing continuous sequence of ordinals \( (\nu_i)_{i<\gamma_\xi} \) by letting \( \nu_0 = \xi \) and \( \nu_{i+1} \) the least ordinal \( \nu > \nu_i \) such that for all \( \nu \) between \( \xi \) and \( \nu_i \), there is no elementary embedding from \( V_\nu \) into \( V_\nu \) that fixes \( \xi \). The process terminates if \( \nu_{i+1} \) does not exist, in which case \( \gamma_\xi = i+1 \).

We claim that \( \gamma_\xi \) exists and is less than the least rank Berkeley cardinal \( \lambda \). To see this, assume not, and let \( j : V_{\nu_\lambda} \to V_{\nu_\alpha} \) be an elementary embedding fixing \( \xi \) whose critical point \( \eta \) is less than \( \lambda \). Then \( j \downarrow V_{\nu_{\eta+1}} \) witnesses that there is an elementary embedding from \( V_{\nu_{\eta+1}} \) to \( V_{\nu_{\gamma_{\nu_{\eta+1}}}+1} \) that fixes \( \xi \). By the elementarity of \( j \), for some \( \nu \) between \( \xi \) and \( \nu_{\gamma_{\nu_{\eta+1}}} \), there is an elementary embedding from \( V_\nu \) into \( V_{\nu_{\eta+1}} \) that fixes \( \xi \). This contradicts the definition of \( \nu_{\gamma_{\nu_{\eta+1}}} \).

Now suppose \( \kappa \geq \lambda \) is a rank reflecting cardinal above \( \xi \). Since \( \kappa \) is rank reflecting, there is some ordinal \( \beta < \kappa \) such that \( V_\beta \) correctly computes \( \gamma_\xi \). It follows that \( V_\beta \) correctly computes \( (\nu_i)_{i<\gamma_\xi} \), and hence \( \delta = \nu_{\gamma_\xi}-1 \) is less than \( \kappa \). But \( \delta \) has the property that for all \( \alpha \geq \delta \), there is some \( \tilde{\alpha} < \delta \) and an elementary embedding \( \pi : V_{\tilde{\alpha}} \to V_{\alpha} \) such that fixing \( \xi \). Since \( \xi < \kappa \) was arbitrary, \( \kappa \) is almost supercompact. \( \square \)

### 7 The number of ultrafilters on an ordinal

If \( X \) is a set and \( \kappa \) is a cardinal, then \( \beta_\kappa(X) \) denotes the set of \( \kappa \)-complete ultrafilters on \( X \). The main theorem of this section bounds the size of this set:

**Theorem 7.1.** Suppose \( \lambda \) is rank Berkeley, \( \kappa \geq \lambda \) is rank reflecting, and \( \epsilon \geq \kappa \) is an even ordinal. Then for all \( \eta < \theta_\epsilon \), \( |\beta_\kappa(\eta)| < \theta_\epsilon \).

This should be contrasted with the following fact [3] Theorem 4.14], which combined with Theorem 3.2 implies that \( \beta_\kappa(\eta) \) is quite large:

**Theorem 7.2.** Suppose \( \lambda \) is rank Berkeley and \( \kappa \geq \lambda \) is rank reflecting. Then every \( \kappa \)-complete filter on an ordinal extends to a \( \kappa \)-complete ultrafilter. \( \square \)

In the context of ZFC, if every \( \kappa \)-complete filter on an ordinal extends to a \( \kappa \)-complete ultrafilter, then \( \kappa \) is strongly compact, and in particular for all cardinals \( \eta \) of cofinality at least \( \kappa \), \( |\beta_\kappa(\eta)| = 2^{\kappa^+} \). Theorem 7.1 tells a very different story.
The tendency of sufficiently complete ultrafilters on ordinals to behave like ordinals themselves is the key to the results of this section. The following theorem [6, Lemma 3.6] is one example of this behavior:

**Theorem 7.3.** Suppose $\lambda$ is rank Berkeley and $\kappa \geq \lambda$ is rank reflecting. Then for all $\eta \geq \kappa$, $\beta_\kappa(\eta)$ is wellorderable.

In Theorem 8.3 we will use Theorem 7.2 and Theorem 7.3 in concert to get a handle on $\kappa$-complete filters on ordinals, but for our purposes one could do without Theorem 7.2 by uniformly replacing $\beta_\kappa(\eta)$ with a fixed wellorderable set $S$ of ultrafilters such that every $\kappa$-complete filter extends to an ultrafilter in $S$. The existence of such a set $S$ is much easier to prove than Theorem 7.2.

**Lemma 7.4.** Suppose $\lambda$ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and $\epsilon \geq \kappa$ is an even ordinal. Then for all $\eta < \theta_{\epsilon+3}$ and all surjections $\varphi : V_{\epsilon+1} \to \eta$, there is a wellordered sequence $\mathcal{F} \in \text{HOD}_{V_{\epsilon+1}, \varphi}$ of subsets of $P(\eta)$ with the following properties:

1. Every $U \in \beta_\kappa(\eta)$ extends some $B \in \mathcal{F}$.
2. For each $B \in \mathcal{F}$, $|\{U \in \beta_\kappa(\eta) : B \subseteq U\}| < \lambda$.

**Proof.** For each $\alpha \in \text{Ord}$, let $E_\alpha$ denote the set of elementary embeddings from $V_\alpha$ to itself. For $\xi < \theta_{\epsilon+3}$, let

$$B_\xi = \{\text{ran}(j) \cap \eta : j \in E_{\epsilon+3}, \varphi \in j[H_{\epsilon+2}], j(\xi) = \xi\}$$

We let $\mathcal{F} = \langle B_\xi \rangle_{\xi < \theta_{\epsilon+3}}$. Note that $\mathcal{F}$ is OD$_\varphi$. Also, each $B_\xi$ is included in HOD$_{V_{\epsilon+1}, \varphi}$: for $j \in E_{\epsilon+3}$ with $\varphi \in j[H_{\epsilon+2}]$, ran$(j) \cap \eta = \text{ran}(\varphi \circ j^+)$, and $j^+$ is definable over $V_{\epsilon+1}$ by Theorem 7.1. It follows that $\mathcal{F} \in \text{HOD}_{V_{\epsilon+1}, \varphi}$.

By the proof of [6, Lemma 4.6], if $U \in \beta_\kappa(\eta)$ has Ketonen rank $\lambda$, then $B_\xi \subseteq U$. This shows (1).

For any $j : V_{\epsilon+4} \to V_{\epsilon+4}$, $B_\xi$ belongs to the normal fine ultrafilter on $P(\eta)$ derived from $j$ using $j[\eta]$. By [6, Theorem 3.12], it follows that for each $\xi < \theta_{\epsilon+3}$, there is a partition $\mathcal{P}$ of $\eta$ with $|\mathcal{P}| < \lambda$ such that for each $A \in \mathcal{P}$, $B_\xi \cup \{A\}$ extends uniquely to an element of $\beta_\kappa(\eta)$. As a consequence, $|\{U \in \beta_\kappa(\eta) : B \subseteq U\}| < \lambda$, establishing (2).

**Proof of Theorem 7.4.** Fix a family $\mathcal{F}$ as in Lemma 7.4. In $M = \text{HOD}_{V_{\epsilon+1}, \varphi}$, $\mathcal{F}$ is a wellordered family of subsets of $P(\eta)$ and $P(\eta) \leq^* V_{\epsilon+1}$ by Theorem 7.2. Therefore $|\mathcal{F}| < \theta_{\epsilon+3}^M$. By Corollary 8.3, $\theta_{\epsilon+3}^M < \theta_{\epsilon+2}$. Fix an enumeration $\langle B_\alpha \rangle_{\alpha < |\mathcal{F}|}$ of $\mathcal{F}$. For $\alpha < |\mathcal{F}|$, let $\mathcal{U}_\alpha = \{U \in \beta_\kappa(\eta) : B_\alpha \subseteq U\}$, so that $|\mathcal{U}_\alpha| < \lambda$ for all $\alpha < |\mathcal{F}|$. Then $\beta_\kappa(\eta) = \bigcup_{\alpha < |\mathcal{F}|} \mathcal{U}_\alpha$, and so $|\beta_\kappa(\eta)| \leq |\mathcal{F}| \cdot \lambda < \theta_{\epsilon+2}$.

**8 The even levels**

If $U$ is an ultrafilter on a set $X$ and $h$ and $g$ are functions on $X$, write $h \subseteq_U g$ if there is a function $e$ such that $h(x) = e \circ g(x)$ for $U$-almost all $x \in X$.

Suppose $\lambda$ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and $\eta \geq \kappa$ is an ordinal. Our first lemma allows us to code any $\kappa$-wellfounded ultrafilter on $\eta$ as a pair $(D, F)$ where $D$ is an ultrafilter on an ordinal less than $\kappa$ and $F$ is a $\kappa$-complete filter.
Lemma 8.1. Suppose $\lambda$ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and $U$ is a $\kappa$-wellfounded ultrafilter on an ordinal $\eta$.

(1) There is a $\subseteq_U$-maximal function $g$ among all functions from $\eta$ to $\nu$ with $\nu < \kappa$.

(2) Letting $D = h_\kappa(U)$ and $k : \text{Ult}(P(\eta), D) \rightarrow \text{Ult}(P(\eta), U)$ be the factor embedding, the $\text{Ult}(P(\eta), D)$-ultrafilter $B$ derived from $k$ using $[\text{id}]_U$ generates a $\kappa$-complete filter.

We will use the wellordered collection lemma [6 Corollary 2.22].

Theorem 8.2. Suppose $\lambda$ is rank Berkeley and $\kappa \geq \lambda$ is rank reflecting. Then for any family $\mathcal{F}$ of nonempty sets with $|\mathcal{F}| < \kappa$, there is a sequence $\langle a_\alpha : \alpha \in V_\kappa \rangle$ such that for all $A \in \mathcal{F}$, there is some $x \in V_\kappa$ with $a_\alpha \in A$.

Proof of Lemma 8.1. We may assume $\kappa$ has cofinality $\omega$: otherwise $\kappa$ is a limit of rank reflecting cardinals of cofinality $\omega$, and one obtains the result by applying the lemma to these smaller cardinals combined with a simple regressive function argument. We omit the details since in our applications, we will only need the special case of the lemma in which $\kappa$ is the least rank reflecting cardinal greater than or equal to $\lambda$.

We begin with (1). Using the wellordered collection lemma (Theorem 8.2), fix functions $\langle f_x : x \in V_\kappa \rangle$ from $\eta$ to $\kappa$ such that for each $\alpha < \kappa$, there is some $x \in V_\kappa$ such that $\alpha = |f_x|_U$. Then define $g : \eta \rightarrow \kappa_{|x_\alpha}$ by $g(\xi) = \langle f_x(\xi) : x \in V_\kappa \rangle$.

For any $\nu < \kappa$ and $h : \eta \rightarrow \nu$, there is some $x \in V_\kappa$ such that for $U$-almost all $\xi < \eta$, $h(\xi) = f_x(\xi) = ev_x \circ g(\xi)$ where $ev_x : g[\eta] \rightarrow \kappa$ is given by $ev_x(s) = s(x)$. To finish, it suffices to show that there is some $A \in \mathcal{U}$ such that $|g[A]| < \kappa$.

We claim $\mathcal{N}(V_{\kappa+1}) = \kappa^+$: [6 Theorem 3.13] implies that $\kappa^+$ is measurable, but the proof that successor cardinals cannot be measurable under the Axiom of Choice shows that if a set $X$ carries a nonprincipal ultrafilter that is closed under $Y$-indexed intersections, then there is no injection from $X$ to $P(Y)$: since $\kappa$ is rank reflecting, $\kappa = \theta_\kappa$, and so any $\kappa^+$-complete ultrafilter on $\kappa^+$ is closed under $V_\kappa$-indexed intersections by [6 Lemma 3.5].

Since $\mathcal{N}(V_{\kappa+1}) = \kappa^+$, $|g[\eta]| \leq \kappa$. Since $\text{cf}(\kappa) = \omega$ and $U$ is countably complete, $|g[A]| < \kappa$ for some $A \in \mathcal{U}$.

We now turn to (2). Fix $g$ as in (1). We will use the fact that if $f : \nu \rightarrow P(\eta)$, then $[f]_B \in \mathcal{B}$ if and only if for $U$-almost every $\xi < \eta$, $\xi \in f \circ g(\xi)$.

Suppose $\gamma < \kappa$ and $\langle A_\beta : \beta < \gamma \rangle$ belong to the filter generated by $\mathcal{B}$. We will show that there is a set $A \in \mathcal{B}$ such that $A \subseteq \bigcap_{\xi < \gamma} A_\xi$. Applying the wellordered collection lemma (Theorem 8.2), let $\langle f_x : x \in V_\kappa \rangle$ be functions representing sets in $\mathcal{B}$ such that for all $\beta < \gamma$, there is some $x \in V_\kappa$ such that $[f_x]_D \subseteq A_\beta$.

Let $h : \eta \rightarrow V_{\kappa+1}$ be the function $h(\xi) = \{x \in V_\kappa : \xi \in f_x \circ g(\xi)\}$

For all $x \in V_\kappa$, the fact that $[f_x]_D \in \mathcal{B}$ implies that for $U$-almost all $\xi < \eta$, $x \in h(\xi)$. Since $\mathcal{N}(V_{\kappa+1}) = \kappa^+$, $|h[\eta]| \leq \kappa$, and so since $g$ is $\subseteq_U$-maximal, there is some $e : \nu \rightarrow V_{\kappa+1}$ such that $e \circ g(\xi) = h(\xi)$ for $U$-almost all $\xi$. For all $x \in V_\kappa$, since $x \in h(\xi)$ for $U$-almost all $\xi < \eta$, $x \in e(\alpha)$ for $D$-almost all $\alpha < \kappa$.

Define $f : \kappa \rightarrow P(\eta)$ by setting $f(\alpha) = \bigcap_{x \in e(\alpha)} f_x(\alpha)$.
and let $A = [f]_D$. For all $x ∈ V_κ$, for $D$-almost all $α < κ$, $f(α) ⊆ f_x(α)$, and hence $A ⊆ [f]_D$. This means $A ⊆ \bigcap_{β < γ} A_β$.

Finally, we claim $A ∈ B$. To see this, note that by the definition of $h$, for all $ξ < η$, $ξ ∈ \bigcap_{x ∈ h(ξ)} f_x o g(ξ)$. By our choice of $e$, for $U$-almost all $ξ < η$,

$$f o g(ξ) = \bigcap_{x ∈ o g(ξ)} f_x o g(ξ) = \bigcap_{x ∈ h(ξ)} f_x o g(ξ)$$

Hence for $U$-almost all $ξ < η$, $ξ ∈ f o g(ξ)$, which means that $A = [f]_D ∈ B$.

**Theorem 8.3.** Suppose $λ$ is rank Berkeley, $κ ≥ λ$ is rank reflecting, and $ε ≥ κ$ is an even ordinal. Then $θ_ε$ is a strong limit cardinal.

**Proof.** Fix an ordinal $η < θ_ε$. Let $j : V_{ε+1} → V_{ε+1}$ be an elementary embedding such that $j(η) = η$. Let $ν = |β_κ(η)|$ and fix a wellorder $≤$ of $β_κ(η)$ in $j[\mathcal{H}_{ε+1}]$. (Note that $P(ν)$ and $β_κ(η)$ belong to $\mathcal{H}_{ε+1}$.)

For each $ξ < η$, let $Eξ$ be the ultrafilter on $η$ derived from $j$ using $ξ$. Let $Dξ = g_*(Eξ)$ where $g : η → κ$ is minimal modulo $Eξ$ among all $Eξ$-maximal functions, and let $kξ : \text{Ult}(P(ν), Dξ) → \text{Ult}(P(ν), Eξ)$ be the factor embedding. Let $δξ = j(g)(ξ)$, so that $Dξ = Eδξ$. Note that $Dξ$ and $kξ$ are independent of the choice of $g$. Let $Bξ$, $Eξ$, and $μξ$ be the $\text{Ult}(P(ν), Dξ)$-ultrafilter derived from $kξ$. By Lemma 8.1 (2), $Bξ$ generates a $κ$-complete filter. Applying Theorem 7.2 let $Uξ$ be the $≤$-least $κ$-complete ultrafilter extending $Bξ$, and let $vξ$ be the $≤$-rank of $Uξ$.

Now $Eξ$ is uniformly definable in $\mathcal{H}_{ε+1}$ from $≤$; $j \upharpoonright P_{bd}(κ)$, $δξ$, and $μξ$. Specifically, $Dξ = j_0[U]$ where $D$ is the ultrafilter on $κ$ derived from $j \upharpoonright P_{bd}(κ)$ using $δξ$ and $U$ is the $vξ$-th element of $β_κ(η)$ in the wellorder $≤$. It follows that $(Eξ : ξ < η)$ is definable in $\mathcal{H}_{ε+1}$ from $≤$, $j \upharpoonright P_{bd}(κ)$, and the sequence of pairs of ordinals $s = ((δξ, νξ) : ξ < η)$.

We claim that $s$ is definable in $\mathcal{H}_{ε+1}$ from $j[V_κ]$ and parameters in $j[\mathcal{H}_{ε+1}]$. Since $s$ is a function from $η$ to $κ \times \text{ot}(≤)$ and $\text{ot}(≤) < θ_ε$ by Theorem 7.1, $s$ can be coded by a set of ordinals $A$ whose supremum $α$ is strictly less than $θ_ε$. Fixing $γ < ε$ and a surjection $ψ : V_γ → α$, $j[α]$ is definable as usual in $\mathcal{H}_{ε+1}$ from $j[V_κ]$ and $j(ψ)$. Also $A = j^{-1}[j(A)]$ is definable from $j[α]$ and $j(A)$. This means that $A$ is definable in $\mathcal{H}_{ε+1}$ from $j[V_κ]$ and parameters in $j[\mathcal{H}_{ε+1}]$, and hence so is $s$.

Putting everything together, $(Eξ : ξ < η)$ is definable in $\mathcal{H}_{ε+1}$ from $j[V_κ]$ and parameters in $j[\mathcal{H}_{ε+1}]$, and hence so is $j[P(ν)]$, noting that for any $S ⊆ η$,

$$j(S) = \{ξ < η : S ∈ Eξ\}$$

Since $j[P(ν)]$ is definable in $\mathcal{H}_{ε+1}$ from $j[V_κ]$ and parameters in $j[\mathcal{H}_{ε+1}]$, there can be no surjection $f : P(ν) → θ_ε$, lest

$$j[θ_ε] = j(f)[j[P(ν)]]$$

be definable in $\mathcal{H}_{ε+1}$ from $j[V_κ]$ and parameters in $\mathcal{H}_{ε+1}$, contrary to Theorem 3.3.

**9 The odd levels**

The following fact is implicit in the proof of Corollary 5.3.

**Lemma 9.1.** Suppose $ε$ is an even ordinal and $j : V_{ε+1} → V_{ε+1}$ is an elementary embedding. Let $E$ be the extender of length $θ_ε$ derived from $j$. Then $j_E(A) = j(A)$ for any bounded subset $A$ of $θ_{ε+1}$.
Corollary 9.2. If $\epsilon$ is an even ordinal and $j : V_{\epsilon+1} \rightarrow V_{\epsilon+1}$ is an elementary embedding with critical point $\kappa$, then the set of regular cardinals in the interval $(\theta_\epsilon, \theta_{\epsilon+1})$ has cardinality less than $\kappa$.

Proof. The set $R$ of regular cardinals in the interval $(\theta_\epsilon, \theta_{\epsilon+1})$ does not have cardinality exactly $\kappa$ since $|R|$ is definable over $\mathcal{H}_{\epsilon+1}$ while $\kappa$, being the critical point of the elementary embedding $j : \mathcal{H}_{\epsilon+1} \rightarrow \mathcal{H}_{\epsilon+1}$, is not.

Assume towards a contradiction $|R| > \kappa$ and that $\delta$ is the $\kappa$-th regular cardinal in the interval $(\theta_\epsilon, \theta_{\epsilon+1})$. Since $j : \mathcal{H}_{\epsilon+1} \rightarrow \mathcal{H}_{\epsilon+1}$ is elementary, $j(\delta)$ is the $j(\kappa)$-th regular cardinal in the same interval.

Letting $E$ be the extender of length $\theta_\epsilon$ derived from $j$, $j_E(\delta) = j(\delta)$, and $j_E$ is continuous at $\delta$ since each measure of $j_E$ lies on a cardinal smaller than $\delta$. It follows that $\text{cf}(j(\delta)) = \text{cf}(\sup j_E[\delta]) = \delta$, which contradicts that $j(\delta)$ is a regular cardinal larger than $\delta$. $\square$

Theorem 9.3. If $\lambda$ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and $\epsilon \geq \kappa$ is an even ordinal, then there is a surjection from $P(\theta_\epsilon)$ onto $\theta_{\epsilon+1}$.

Proof. If $j : V_{\epsilon+1} \rightarrow V_{\epsilon+1}$ is an elementary embedding, then the extender $E$ of length $\theta_\epsilon$ derived from $j$ is definable over $\mathcal{H}_{\epsilon+1}$ from $j[\theta_\epsilon \cup j \upharpoonright P_{\text{bd}}(\kappa)]$ and parameters in $j[\mathcal{H}_{\epsilon+1}]$: this is as in Theorem 8.3 using the fact that $|\bigcup_{\eta < \theta_\epsilon} \beta_\eta(\eta)| = \theta_\epsilon$ by Theorem 7.1. Since $j[\theta_{\epsilon+1}]$ can be recovered from $E$, we have that $j[\theta_{\epsilon+1}] \in \text{Ult}(V,U)$ where $U$ is the ultrafilter on $P(P_{\text{bd}}(\kappa) \cup \theta_\epsilon)$ derived from $j$ using $j[\theta_\epsilon \cup j \upharpoonright P_{\text{bd}}(\kappa)]$. Therefore by Theorem 8.1 there is a surjection from $P(P_{\text{bd}}(\kappa) \cup \theta_\epsilon)$ to $\theta_{\epsilon+1}$.

We may assume that $\epsilon \geq \kappa + 2$ since by Lemma 2.1 $\theta_{\kappa+1} = \kappa^+ \leq* P(\kappa)$. Since $P(P_{\text{bd}}(\kappa) \cup \theta_\epsilon) \leq V_{\kappa+1} \times P(\theta_\epsilon)$, there is a surjection $f : V_{\kappa+1} \times P(\theta_\epsilon) \rightarrow \theta_{\epsilon+1}$. For each $A \subseteq \theta_\epsilon$, let $S_A = \{f(x,A) : x \in V_{\kappa+1}\}$, let $\nu_A = \text{ot}(S_A)$, and let $g_A : \nu_A \rightarrow \theta_{\epsilon+1}$ be the increasing enumeration of $S_A$. Since $S_A \leq^* V_{\kappa+1}$, $\nu_A < \theta_\kappa$. Moreover, $\theta_{\epsilon+1} = \bigcup_{A \subseteq \theta_\epsilon} S_A$, and so the function $g : P(\theta_\epsilon) \times \theta_{\kappa+2} \rightarrow \theta_{\epsilon+1}$ defined by $g(A,\alpha) = g_A(\alpha)$ is a surjection. Obviously $|P(\theta_\epsilon) \times \theta_{\kappa+2}| = |P(\theta_\epsilon)|$, and so the proof is complete. $\square$

10 Questions

Assume there is an elementary embedding from the universe of sets to itself.

Question 10.1. For sufficiently large even ordinals $\epsilon$, is $\theta_{\epsilon+1} = \theta_{\epsilon+1}^+$?

This is probably the most glaring question left open by our theorems here. But many other combinatorial questions remain:

Question 10.2. For sufficiently large even ordinals $\epsilon$, is it true that for all $\eta < \theta_{\epsilon+2}$, there is a surjection from $V_{\epsilon+1}$ onto $P(\eta)$? Does the coding lemma hold?

Define a sequence of ordinals $\langle \nu_\alpha : \alpha \in \text{Ord} \rangle$ by setting $\nu_{\alpha+1} = \kappa^*(P(\nu_\alpha))$ and $\nu_\gamma = \sup_{\alpha < \gamma} \nu_\alpha$ for $\gamma$ a limit ordinal. The arguments of this paper show that if there is an elementary embedding from the universe of sets to itself, then $\nu_\alpha$ is a strong limit cardinal for all sufficiently large ordinals $\alpha$.

Question 10.3. Is $\theta_{\gamma+2n} = \nu_{\gamma+n}$ for all sufficiently large limit ordinals $\gamma$ and all $n < \omega$?

Certain arguments in this paper require simulating the Axiom of Choice using rank reflecting cardinals, and for this reason the following question remains open:
**Question 10.4.** If $\lambda$ is rank Berkeley — or just assuming there is an elementary embedding from $V_{\lambda+2}$ to itself — is $\theta_{\lambda+2}$ a strong limit cardinal?

By analogy with the choiceless axioms, it is natural to speculate that perhaps the right higher-order generalization of the Axiom of Determinacy is some kind of regularity property for subsets of $V_{\omega+2n+1}$ for $n < \omega$.

**Question 10.5.** Is there an extension of the Axiom of Determinacy that implies $\theta_{\omega+4}$ is a strong limit cardinal and $\theta_{\omega+5}$ is its successor? If $\theta_{\omega+2}$ is a strong partition cardinal, does this hold?

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