Abstract

In this work we investigate the role of multivalued fields in the formulation of monopole operators and its connection with topological defects. In quantum field theory it is known that certain states describe collective modes of the fundamental fields and are created by operators that are often non-local, being defined over lines or over higher dimensional surfaces, and for this reason may be sensitive to global, topological, properties of the system and depends on nonperturbative data. Such operators are generally known as monopole operators. Also, sometimes, they act as disorder operators because its nonzero expectation values define a disordered vacuum associated with a condensate of the collective modes, also known as defects. In this work we investigate the definition of these operators and its relation with the multivalued properties of the fundamental fields. We study some examples of scalar field theories, generalize the discussion to $p$-forms and applications to superconducting systems. By splitting the fields in their regular and singular parts we identify a ambiguity that can be explored, much like gauge symmetry, in order to define observables.

1 Introduction

In this work we undertake an investigation of multivalued fields and its relation with so-called monopole operators. We aim to develop a formalism for dealing with multivalued fields and to study its application to describing defects in field theory.

Quantum field theory can be defined by the algebra of local operators [1]. Modulo superselection sectors, the theory can also be defined by the correlation functions of local operators, which are usually the quantum versions of the fundamental fields, and composite operators constructed from them, appearing in a classical Lagrangian formulation of the theory. Locality of the Lagrangian formulation demands that such operators are local, defined at a point. But there are
important information also in the correlation function of non-local operators, that are defined on extended regions such as lines or surfaces \[2,3,4,5\]. Following current nomenclature, we will generically call them **monopole operators** \[6,7\]. The name comes from the fact that in compact electrodynamics the operator that creates a monopole state is non-local in the sense of demanding a (Dirac) string for its definition with respect to the original gauge field, but it has received other names such as surface operators, disorder operators, vortex operators, etc... In fact, they are akin to disorder operators in lattice Ising systems, where the original variables are the spin operators localized on sites and the disorder operator is defined by a set of links cut by a line. Each cut link changes the coupling between the spins of the sites it connects and it is said that this configuration represents a defect in the system. Since it involves a collection of spins in different sites, in terms of the original spin variables the corresponding operator is non-local \[8,9\]. Alternatively, one may devise a local formulation in terms of dual variables in a dual lattice \[10\]. This is the usual order-disorder duality. In a quantum field theory language the local operators creates local excitations, particles, and the non-local extended operators creates collective excitations of the original particles or localized states in a dual formulation. Monopole operators are tricky to study because the very definition of a fundamental excitation is perturbative, but duality exchanges weak and strong couplings and if the original system is in the perturbative regime, the monopole operators will be local in a formulation that is in a strongly coupled regime. Therefore these operators are usually study in their non-local formulation.

The theory of monopole operators has been developed for many years. Its origins may be traced to the work of Kadanoff and Ceva \[8\] in the context of Lattice theories and was subsequently introduced in field theory by Mandelstam and ’t Hooft \[3\]. The concept was further developed by many authors, notably more related to our approach, Ezawa \[11\], Polyakov \[12\], Polchinski \[13\] and Marino and Swieca \[9\]. Recently the subject has received great attention in connection with the Langlands program \[14\]. The subject has been developed more recently with a systematic study of the many uses of monopole operators in gauge theories \[15,6,7,16,17,18\].

The non-local nature of monopole operators demands an extension of the usual definition of fields. In a Lagrangian setting it is generally assumed that fields are regular functions, without singularities and infinitely differentiable. But it is well known since Dirac’s formulation of monopoles in electromagnetism that it may be useful to allow for multivalued fields. More recently this has been understood as part of mathematical structure that sees the fields as defined over non-trivial spaces \[19,20,21,22,23\]. But there are circumstances where to allow for the functions to be multivalued may lead to a more direct description of monopole operators and the study of these instances is the purpose of this work (this is similar to the dichotomy between Dirac’s string formulation \[24\] and Wu-Yang’s formulation \[25\] of magnetic monopoles). We will describe multivalued fields with the help of surfaces whose placement in space will characterize the multivalued character of the fields. The formulation here presented is restricted to examples where the field may be expressed as a direct sum of a regular, single valued, part and a singular, multivalued, part. The different field configurations are thus specified by the continuous field representing the regular part along with the singular part that is characterized by distribution-valued fields that may be identified with surfaces generically called branes. Thus, in the path integral formulation of the system, one regard the ensemble as mixture of continuous variables to be integrated over and
branes configurations to be summed over. In this work we will not try to specify a well defined prescription to evaluate the sum over brane configurations, but the subject has been developed by many authors and one interesting recent take on this can be found in [26].

The importance of multivalued fields and its connection with large gauge symmetry with the explicit use of singular surfaces, along the lines presented here, has been pointed out by Kleinert [27, 28]. The concept of singular surfaces and its importance for the structure of the non-perturbative vacuum in gauge theories taking into account their collective behavior has been discussed in [29, 30] and the subject was subsequently further developed with many applications leading to the definition of brane symmetry [31, 32, 33, 34, 35].

The main result of this paper is to provide a framework for the study of monopole operators with the use of multivalued fields and to explore this framework in the analysis of the electromagnetic response of superconducting states of matter. This work is organized as follows: in section 2 we provide a general definition of what we mean by multivalued fields, in section 3 we start the discussion of monopole operators in the path integral formulation, this will be carried out mostly for scalar theories in $D = 2$ dimensions for simplicity, but some comments about general $p$-form theories in $D$-dimensions are provided at the end of the section. Section 4 deals with the concept of brane symmetry and its relation with the other symmetries of the system. We will first study the case of the scalar field and its global shift symmetry and then proceed with the study of gauge symmetries. The study of gauge symmetry will naturally lead to the study of brane symmetry as a manifestation of the so-called singular or large gauge transformation. We end this section with a discussion of the important interplay of all these symmetries in understanding the electromagnetic response of superconductors. In section 5 we present our concluding remarks.

2 Multivalued fields

2.1 Scalar fields

In this section we study several properties of scalar fields systems when multivalued (singular) contributions are allowed in its definition. We start by considering a scalar field $\phi(x)$ and a closed curve $C$ in such a way that

$$\oint_C dx^\mu \partial_\mu \phi(x) = \alpha,$$  

(1)

where $\alpha$ is a constant. If $\alpha$ is nonzero we see that the field $\phi$ cannot be regular everywhere. One can understand this better using Stokes theorem in order to write

$$\oint_C dx^\mu \partial_\mu \phi(x) = \int_S d\sigma_1 d\sigma_2 \varepsilon^{a_1 a_2} \frac{\partial y^{\mu_1}}{\partial \sigma_{a_1}} \frac{\partial y^{\mu_2}}{\partial \sigma_{a_2}} \partial_{\mu_1} \partial_{\mu_2} \phi = \alpha,$$  

(2)

where $S$ is the surface bounded by the curve $C$. Thus, if $\alpha \neq 0$, it must be that $\partial_{[\mu_1} \partial_{\mu_2]} \phi(x) \neq 0$ at least at some point in $S$. This implies that there is a singular contribution to $\phi(x)$ which can
We see that we are evaluating the field, and different paths will lead to different results. This defines $K$ as a $(D-1)$-brane. We define a $p$-brane as a singular $p$-form defining a $p$-surface embedded on a $D$-dimensional manifold.

One can note that the integral defining the singular part of the field in (3) is path dependent, that is, we must choose a particular path to integrate over ending on the point in space where we are evaluating the field, and different paths will lead to different results. This defines $\phi$ as a multivalued field. The $(D-1)$-brane $K$ thus define a co-dimension 1-surface in the $D$-dimensional space such that the intersection number with the line defining the path of integration will provide a nonzero contribution proportional to $\beta$. Explicitly we have

$$\oint_C dx^\mu \partial_\mu \phi = \beta \oint_C dx^\mu \tilde{K}_\mu$$

$$= \beta \oint_C d\tau \int_\Sigma d\tau_1 \ldots d\tau_{D-1} \varepsilon_{\mu_1 \ldots \mu_{D-1}} \frac{\partial x^\mu}{\partial \tau_1} \ldots \frac{\partial x^D}{\partial \tau_{D-1}} \delta^D(x - y(\tau_1, \ldots, \tau_{D-1}))$$

$$= \beta n. \quad (5)$$

Where $n \in \mathbb{Z}$ is the number of times that the line $C$ crosses the surface $\Sigma$. Choosing $\beta = \frac{2}{n}$, we obtain the desired property (1). Here we note that the complete information about singularities and the multivalueness of the field is encoded in the line integral of $\tilde{K}$. This is due to the fact that the singular part contributes additively to the expression.

### 2.2 1-forms

Another well known example is the vector field. The analog of (1) is

$$\oint_S d\sigma_1 d\sigma_2 \varepsilon^{a_1 a_2} \frac{\partial y^{\mu_1}}{\partial \sigma_{a_1}} \frac{\partial y^{\mu_2}}{\partial \sigma_{a_2}} F_{\mu_1 \mu_2} = \int_V d\tau_1 d\tau_2 d\tau_3 \varepsilon^{a_1 a_2 a_3} \frac{\partial y^{\mu_1}}{\partial \sigma_{a_1}} \frac{\partial y^{\mu_2}}{\partial \sigma_{a_2}} \frac{\partial y^{\mu_3}}{\partial \sigma_{a_3}} \partial_\mu_1 \partial_\mu_2 \partial_\mu_3 \sigma_{a_1} \sigma_{a_2} \sigma_{a_3} = \alpha, \quad (6)$$

where $F_{\mu \nu} = \partial_\mu A_\nu$ and $\alpha$ is a constant, identified in this case with the magnetic charge. We recognize the left hand side as the magnetic flux over the closed surface $S$ that encloses the volume.
This is just the expression for the existence of a magnetic charge. For this flux to be nonzero we have to allow $A$ to display a singular part. We define the multivalued form

$$A_\mu(x) = A^\text{reg}_\mu(x) + \frac{\beta}{2} \int^x dy' \tilde{M}_{\nu\mu}(y)$$

and therefore

$$F_{\mu\nu}(x) = F^\text{reg}_{\mu\nu}(x) + \beta \tilde{M}_{\mu\nu}(x),$$

where we thus recognize $\tilde{M}$ as the Dirac string.

This discussion can of course be conducted at a more modern level recognizing this structure as a fiber bundle with connection and the nonzero flux of the 2-form $F$ as the second Chern number of the bundle. The relevant mathematical structure to describe what we are about to encounter will naturally lead to concepts such as the Cheeger-Simons group of differential characters and Deligne-Bailinson cohomology classes (see for instance [19, 20, 21, 22, 23] and related works), where the singular parts are naturally incorporated by endowing the mathematical spaces with nontrivial topological properties. But it pays off to maintain the discussion in terms of singular parts because it is more convenient for the present purpose as it makes explicit the contribution of points, strings and p-branes that will lead to the construction of the monopole operators. Because of the connection with the original Dirac string, we will call such singular terms as Dirac branes.

The generalization to $p$-forms is straightforward and we will discuss this general case also in the sections ahead.

### 3 Quantum properties

#### 3.1 Multivalued fields and path integrals

So far the discussion has taken place in the classical setting. A very useful application of Dirac branes is on the definition of hyper-surface operators in quantum field theory, also known as monopole operators. These are nonlocal operators that demand an extended region to be defined (as opposed to local operators that are defined at a point). The extended region may be provided by the Dirac branes. The most famous example of such operators is the Wilson operator [36]. It has a closely related operator that goes by the name of ‘t Hooft operator [3]. They have enjoyed renewed interest in recent studies of the Langlands program with its connection with electromagnetic duality and also applications as probes to topological states of matter [4, 5, 6, 7, 15, 16, 17, 18].

Let’s explore the meaning of monopole operators studying a simple example. We consider a scalar field theory in Euclidean space and the operator supported on points

$$\hat{\mathcal{W}}_J = e^{ia} \int d^Dx \phi(x)J(x)$$

(9)
Where $J(x)$ is a 0-brane configuration (for instance, $J(x) = \sum_i \delta^D(x - x_i)$). We introduce the operator $\hat{\Pi}(x)$ that satisfies

$$\left[ \hat{\phi}(x), \hat{\Pi}(y) \right] = i\delta^D(x - y)$$  \hspace{1cm} (10)

a possible realization of this operator is the functional derivative operator $\hat{\Pi}(x) \rightarrow -i\frac{\delta}{\delta \phi(x)}$ in the field representation, where the operator $\hat{\phi}(x)$ acts by multiplication by $\phi(x)$. It is easy to deduce the commutator eq.\( \text{(10)} \) in this representation by studying its action on a arbitrary functional of $\phi(x)$. It follows

$$\left[ \hat{W}_J, \hat{\Pi}(y) \right] = \left[ e^{i\alpha \int d^D x \phi(x)J(x)}, \hat{\Pi}(y) \right] = i\frac{\delta}{\delta \phi(y)} e^{i\alpha \int d^D x \phi(x)J(x)} = -\alpha J(y) \hat{W}_J$$ \hspace{1cm} (11)

From this we can conclude that $\hat{W}_J$ acts as a shift operator for $\hat{\Pi}(x)$

$$\hat{W}_J \hat{\Pi}(x) \hat{W}_J^{-1} = \hat{\Pi}(x) - \alpha J(x)$$ \hspace{1cm} (12)

Similarly, since $\hat{\phi}(x)$ is the generator of shifts in $\hat{\Pi}(x)$, the canonical relation given by eq.\( \text{(10)} \) also implies that $\hat{\Pi}(x)$ is a generator of shifts in $\hat{\phi}(x)$ and we are lead to define the operator $\hat{T}_Q$ such that

$$\hat{T}_Q \hat{\phi}(x) \hat{T}_Q^{-1} = \hat{\phi}(x) - \beta Q(x)$$ \hspace{1cm} (13)

where we take $Q(x)$ to be a 0-brane. $\hat{T}_Q$ is known as the ’t Hooft operator. It follows that

$$\hat{T}_Q = e^{-i\beta \int d^D x Q(x)\hat{\Pi}(x)} \rightarrow e^{-\beta \int d^D x Q(x)\frac{\delta}{\delta \phi(x)}}$$ \hspace{1cm} (14)

It is clear that acting of any functional $F(\hat{\phi}(x))$

$$\hat{T}_Q F(\hat{\phi}(x)) \hat{T}_Q^{-1} = F(\hat{\phi}(x) - \beta Q)$$ \hspace{1cm} (15)

All these properties of $\hat{T}_Q$ must be understood by its action on arbitrary functionals of $\hat{\phi}(x)$ in the sense that both sides of eq.$\text{(13)}$, eq.$\text{(14)}$ and eq.$\text{(15)}$ are operators acting on functionals of $\hat{\phi}(x)$.

A relation between $\hat{T}_Q$ and $\hat{W}_J$ can thus be obtained

$$\hat{T}_Q \hat{W}_J(\hat{\phi}) \hat{T}_Q^{-1} = \hat{W}_J(\hat{\phi} - \beta Q) = e^{i\alpha \beta \int d^D x Q(x)J(x)} \hat{W}_J(\hat{\phi})$$ \hspace{1cm} (16)

or

$$\hat{T}_Q \hat{W}_J = e^{i\alpha \beta \int d^D x Q(x)J(x)} \hat{W}_J \hat{T}_Q$$ \hspace{1cm} (17)

This is the well known algebra of ’t Hooft and Wilson operators, whose general structure can be generalized even to non-abelian gauge theories.
The computation of correlation functions with insertions of \( \hat{W}_J \) has the straightforward definition

\[
\langle \hat{W}_J F(\hat{\phi}) \rangle = \frac{1}{Z} \int \mathcal{D}\phi \ W_J F(\phi) \ e^{-S} = \frac{1}{Z} \int \mathcal{D}\phi \ F(\phi) \ e^{-S+i\alpha \int d^D x \ \phi(x) J(x)} = \langle F(\hat{\phi}) \rangle_J
\]  

(18)

Where \( Z = \int \mathcal{D}\phi \ e^{-S} \). That is, the rule to compute correlations with \( \hat{W}_J \) insertions is to perform the path integral with the action modified by the addition of the term \( i\alpha \int d^D x \ \phi(x) J(x) \).

In order to define the correlation functions with \( \hat{T}_Q \) insertions we note that

\[
\langle F(\hat{\phi} - \beta Q) \rangle = \frac{1}{Z} \int \mathcal{D}\phi \ F(\phi - \beta Q) \ e^{-S} = \frac{1}{Z} \int \mathcal{D}\phi' \ F(\phi') \ e^{-S(\phi' + \beta Q)} = \langle F(\hat{\phi}) \rangle^Q
\]  

(19)

where \( \phi' = \phi - \beta Q \). This implies the definition

\[
\langle \hat{T}_Q \rangle = \frac{1}{Z} \int \mathcal{D}\phi \ e^{-S(\phi + \beta Q)}
\]  

(20)

and we are led to conclude that the computation of correlation functions of \( \hat{T}_Q \) amounts to a modification of the action by shifting the field by a singular contribution, this could also be read from (14). \( \hat{T}_Q \) and \( \hat{W}_J \) are an example of a pair of order and disorder operators. We will study this example further in the next subsection.

**Dual representation of the Wilson operator for the scalar field in \( D = 2 \)**

In order to develop a better grasp for the definition of monopole operators we will explicitly obtain the dual representation of the Wilson operator in the 2\( D \) Euclidean case. We note that computation of the path integral defining the expectation value of \( \hat{W}_J \) can be performed with other variables (up to normalization constants)

\[
\langle \hat{W}_J \rangle \sim \int \mathcal{D}\phi \ e^{-\int d^2x (\frac{1}{2} \partial_\mu \phi \partial_\mu \phi - i\alpha \phi J)} \sim \int \mathcal{D}\phi \ \mathcal{D}\xi \ e^{-\int d^2x (i\xi_\mu \partial_\mu \phi + \frac{i}{2} \xi^2 - i\alpha \phi J)}
\]

\[
\sim \int \mathcal{D}\xi \ \delta(\partial_\mu \xi_\mu - \alpha J) e^{-\int d^2x (\frac{i}{2} \xi^2)}
\]  

(21)

Consider the configuration

\[
J(x) = \delta^2(x - x_0) - \delta^2(x - x'_0)
\]  

(22)

We can take the points \( x_0 \) and \( x'_0 \) as the boundary of a line and write

\[
J(x) = \partial_\mu K^\mu
\]  

(23)

with

\[
K^\mu(x) = \int_{\sigma_{x_0}}^{\sigma_{x'_0}} d\sigma \ \frac{dy^\mu}{d\sigma} \delta^2(x - y(\sigma))
\]  

(24)

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where \( \Upsilon_{x_0}^{x_0'} \) is the line connecting \( x_0 \) and \( x_0' \). We can then solve the constraint of the delta-function in the path integral \( \langle 21 \rangle \) as

\[
\xi^\mu = \varepsilon^{\mu \nu} \partial_\nu \psi^{\text{reg}} + \alpha K^\mu
\]  

(25)

where \( \psi^{\text{reg}} \) is a single-valued field. From this we can conclude that \( \xi^\mu \) is the derivative of a multivalued function \( \xi^\mu = \varepsilon^{\mu \nu} \partial_\nu \psi \)

\[
\psi(x) = \psi^{\text{reg}}(x) - \alpha \int^x dy \varepsilon_{\mu \nu} K^\nu(y) = \psi^{\text{reg}}(x) - \alpha \int d^2 z L^\mu(z; x, -\infty) \varepsilon_{\mu \nu} K^\nu(z; x_0, x_0')
\]  

(26)

where \( L^\mu(z; x, -\infty) \) is the line connecting \(-\infty\) to the point \( x \).

\[
L^\mu(z; x, -\infty) = \int_{\Gamma_{-\infty}^{x}} d\tau \frac{dy^\mu}{d\tau} \delta^2(z - y(\tau))
\]  

(27)

The multivaluedness of \( \psi(x) \) is encoded in the fact that its definition depends on the path \( \Gamma_{-\infty}^{x} \) from \(-\infty\) to \( x \). The integral in \( \langle 26 \rangle \) counts the number of times the line \( \Gamma_{-\infty}^{x} \) crosses the line \( \Upsilon_{x_0}^{x_0'} \), obviously this number depends on the choice of \( L \), which is the same as to choose a branch cut \( (x \) is the branch point). Note that

\[
\partial^{(z)} L^\mu(z; x, -\infty) = \delta^\mu_\nu \delta^2(z - x)
\]  

(28)

also

\[
\partial^{(z)} L^\mu(z; x, -\infty) = -\delta^2(z - x)
\]  

(29)

This basically defines the branch point associated with \( L \) and is therefore the invariant part. From it we can infer the ambiguity in the path,

\[
L^\mu(z; x, -\infty) \rightarrow L^\mu(z; x, -\infty) + \partial^{(z)} M^{\rho \mu}(z)
\]  

(30)

where

\[
M^{\rho \mu}(z) = \varepsilon^{\rho \mu} \int_{\Omega} d^2 y \delta^2(z - y) = \varepsilon^{\rho \mu} \Theta(\Omega, z)
\]  

(31)

with \( \Omega \) the area swept out by the line \( \Gamma_{-\infty}^{x} \) as it changes (keeping \( x \) fixed). The Heaviside function is defined as

\[
\Theta(\Omega, x) = \begin{cases} 
1 & \text{if } x \in \Omega \\
0 & \text{if } x \notin \Omega
\end{cases}
\]  

(32)

Similarly and more important, eq.(23) implies an ambiguity

\[
K^\mu(z; x_0, x_0') \rightarrow K^\mu(z; x_0, x_0') + \partial^{(z)} N^{\rho \mu}(z)
\]  

(33)
where

\[ N_{\mu}^{\rho\nu}(z) = \varepsilon_{\mu}^{\rho\nu} \int_{\Sigma} d^2y \delta^2(z - y) = \varepsilon_{\mu}^{\rho\nu} \Theta(\Sigma, z) \] (34)

with \( \Sigma \) the area swept out by the line \( Y_{x_0}^{x'_0} \) as it changes (keeping \( x_0 \) and \( x'_0 \) fixed).

It is interesting to define the intersection number between \( \Gamma_{x}^{-\infty} \) and \( Y_{x_0}^{x'_0} \)

\[ n(\Gamma_{x}^{-\infty}, Y_{x_0}^{x'_0}) = \int d^2z L^\mu(z; x, -\infty) \varepsilon_{\mu\nu} K^\nu(z; x_0, x'_0) \] (35)

which is a integer \( n(\Gamma_{x}^{-\infty}, Y_{x_0}^{x'_0}) \in \mathbb{Z} \). Under transformations (30) and (33) we have

\[ \delta \Gamma n(\Gamma_{x}^{-\infty}, Y_{x_0}^{x'_0}) = \int d^2z \delta \Gamma L^\mu(z; x, -\infty) \varepsilon_{\mu\nu} K^\nu(z; x_0, x'_0) \]

\[ = \int d^2z \Theta(\Omega, z) \partial_{\rho}^{(z)} K^\rho(z; x_0, x'_0) = \Theta(\Omega, x_0) - \Theta(\Omega, x'_0) \] (36)

and

\[ \delta \Sigma n(\Gamma_{x}^{-\infty}, Y_{x_0}^{x'_0}) = \int d^2z L^\mu(z; x, -\infty) \varepsilon_{\mu\nu} \partial_{\rho}^{(z)} N_{\rho}^{\mu\nu}(z) \]

\[ = - \int d^2z \Theta(\Sigma, z) \partial_{\rho}^{(z)} L^\rho(z; x, -\infty) = \Theta(\Sigma, x) \] (37)

Eq.(26) can then be written as

\[ \psi(x) = \psi^{\text{reg}}(x) - \alpha n(\Gamma_{x}^{-\infty}, Y_{x_0}^{x'_0}) \] (38)

Note that under the transformation (37) we have

\[ \delta \Sigma \psi(x) = \delta \Sigma \psi^{\text{reg}}(x) - \alpha \Theta(\Sigma, x) \] (39)

The field \( \psi(x) \) will be invariant under this transformation if its regular part changes as

\[ \delta \Sigma \psi^{\text{reg}}(x) = \alpha \Theta(\Sigma, x) \] (40)

Note that \( \partial_{\mu} \partial_{\nu} \Theta(\Sigma, x) = 0 \) so that regularity of \( \psi^{\text{reg}}(x) \) is maintained by this transformation.

Therefore the path integral over \( \xi_{\mu} \) translates into a path integral over \( \psi^{\text{reg}} \)

\[ \int D\phi \, e^{-\int d^2x \left( \frac{i}{2} \partial_{\mu} \phi \partial_{\nu} \phi - i A \phi J \right)} \sim \int D\psi^{\text{reg}} \, e^{-\int d^2x \frac{1}{2} \left( \partial_{\mu} \psi^{\text{reg}} - \alpha \varepsilon_{\mu\nu} K^\nu \right)^2} \] (41)

Therefore the insertion of the Wilson operator manifests itself in the dual formulation as a path integral over multivalued (compact) fields \( \psi \). The multivalueness of this field is probed by the integral of a path circling locations of the insertions \( (x_0 \text{ and } x'_0) \). We note that the line \( L \) defining
the multivalueness of $\psi$ has completely disappeared from the path integral. Note also that the left hand side is explicitly invariant under \( \partial \mu K^\mu \), the same is true for the right hand side because of \( \partial \mu Q \).

Note that from the point of view of the right-hand side of eq.\((41)\) the procedure amounts to a path integral over a field that underwent a shift $\psi^{reg}(x) \to \psi^{reg}(x) - \alpha \int^x dy^\mu \varepsilon_\mu K^\nu(y)$. This shift, as we know, can be ascribed to the action of an operator $\hat{T}_Q$, with $Q(x) = \int^x dy^\mu \varepsilon_\mu K^\nu(y) = \int d^2z L^\mu(z;x, -\infty) \varepsilon_\mu K^\nu(z;x_0, x'_0)$. This is the prescription for the computation of $\langle \hat{T}_Q \rangle$. We conclude that to compute $\langle W_f \rangle$ in the original theory is the same as to compute $\langle \hat{T}_Q \rangle$ in the dual theory, where $J = \partial\mu K^\mu = -\varepsilon^{\mu\nu} \partial\mu \partial\nu Q$.

$p$-forms

We can follow the same lines when discussing general $p$-forms em $D$-dimensions. We define a multivalued $p$-form as

$$A_{\mu_1 \cdots \mu_p}(x) = A^{reg}_{\mu_1 \cdots \mu_p}(x) + \frac{\alpha}{(p + 1)!} \int^x dy^\mu \tilde{K}_{\mu_1 \cdots \mu_p}(y; \Sigma)$$

where

$$\tilde{K}_{\mu_1 \mu_2 \cdots \mu_{p+1}}(x; \Sigma) = \varepsilon_{\mu_1 \cdots \mu_{p+1} \nu_{p+2} \cdots \nu_D} \int_\Sigma d\tau_1 \cdots d\tau_{D-p-1} \frac{\partial y^{\nu_{p+2}}}{\partial \tau_1} \cdots \frac{\partial y^{\nu_D}}{\partial \tau_{D-p-1}} \varepsilon^{a_1 \cdots a_D} \delta^D(x - y(\tau_1, \cdots, \tau_{D-p-1}))$$

$$= \frac{1}{(D - p - 1)!} \varepsilon_{\mu_1 \cdots \mu_{p+1} \nu_{p+2} \cdots \nu_D} K^{\nu_{p+2} \cdots \nu_D}(x; \Sigma)$$

(43)

There is a brane symmetry associated with $K$. The observable is the generalized magnetic ($D - p - 2$)-current

$$J^{\mu_1 \mu_2 \cdots \mu_{D-p-2}}(x; \partial\Sigma) = \partial\mu K^{\mu_1 \cdots \mu_{D-p-2}}(x; \Sigma)$$

(44)

such that

$$\varepsilon^{\mu_1 \mu_2 \cdots \mu_{D-p-2} \mu D-p-1 \cdots \mu_D} \partial_{\mu_{D-p-1}} \partial_{\mu_D} A_{\mu D-p+1 \cdots \mu_D} = \alpha J^{\mu_1 \mu_2 \cdots \mu_{D-p-2}}(x; \partial\Sigma)$$

(45)

Therefore, the brane symmetry is

$$K^{\mu_1 \cdots \mu_{D-p-1}}(x; \Sigma) \to K^{\mu_1 \cdots \mu_{D-p-1}}(x, \Sigma) + \partial\mu M^{\mu_1 \cdots \mu_{D-p-1}}(x; \Omega)$$

(46)

in such a way that $\partial\Omega = \Sigma' - \Sigma$.

Following our considerations from before, if the multivalued $p$-form is to be made invariant under this transformation (this demand is only called for if the $p$-form is an observable, we will
see that when the $p$-form is a gauge field only the gauge invariant forms are to be made invariant under brane symmetry as well), we must have

$$\delta_{\Sigma} A_{\mu_1 \cdots \mu_p}^{\text{reg}}(x) = -\alpha \int^x dy^\mu \varepsilon_{\mu \mu_1 \cdots \mu_p \nu_{p+1} \cdots \nu_{D-1}} \frac{1}{(D-p-1)!} \partial_\rho M^{\rho \nu_{p+1} \cdots \nu_{D-1}}(y; \Omega)$$

$$= -\alpha \varepsilon_{\mu \mu_1 \cdots \mu_p \nu_{p+1} \cdots \nu_{D-1}} \frac{1}{(D-p-1)!} \int d^D z L^\mu(z; \Gamma^x) \partial_\rho M^{\rho \nu_{p+1} \cdots \nu_{D-1}}(z; \Omega)$$

$$= -\alpha P_{\mu_1 \cdots \mu_p}(x)$$

(47)

This cannot be written as a simple Heaviside function as before. But we have the properties

$$\partial_{\mu_1} P_{\mu_1 \cdots \mu_p}(x) = 0$$

$$\varepsilon_{\mu_1 \mu_2 \cdots \mu_p \nu_{p+1} \cdots \nu_{D-1}} \partial_{\nu_{p+1}} \partial_{\nu_{p+2}} P_{\mu_1 \cdots \mu_p}(x) = 0$$

(48)

where the second property implies that regularity is maintained by the brane transformation.

4 Symmetries

4.1 Turning global symmetry into brane symmetry

We may devise a general rule for the construction of disorder ’t Hooft operators using symmetry as guide. This idea was explored in [9] where the procedure of finding the disorder operator was based on a introduction of a counter-term Lagrangian and restricting the original symmetry of the system to a region of the space. This procedure was based on the original lattice studies of [8]. We note that the free massless scalar action 2-dimensions has a global symmetry under

$$\phi(x) \rightarrow \phi(x) + \alpha$$

(49)

where $\alpha$ is a constant. Consider defining this symmetry effective only over a region $\Sigma$, a 2-dimensional surface. We can thus write, instead of (49)

$$\phi(x) \rightarrow \phi(x) + \alpha \Theta(x, \Sigma)$$

(50)

Imposing this symmetry in the action, we already know how to proceed. We have to modify the field $\phi$ to allow for a multivalued contribution (as in eq(38) for $\psi$), such that (50) is the transformation of the regular part of the field and $\Sigma$ is the surface swept as the line $\Upsilon$ defining $K$ (as in eq.(24)) is changed. We thus recover what has been discussed previously.

The generalization of this idea to tackle other global symmetries is straightforward if the symmetry can be cast as a shift in some field. Consider for instance a complex scalar field with $U(1)$ symmetry. In this case we have exactly the same situation as above for the phase of the complex field, in fact if $\Phi$ is a complex scalar field the $U(1)$ symmetry is

$$\Phi(x) \rightarrow e^{i\alpha} \Phi(x)$$

(51)
in terms of the phase $\theta(x)$ of $\Phi(x) = |\Phi|(x)e^{i\theta(x)}$ we obtain

$$\theta(x) \to \theta(x) + \alpha$$

(52)

As before, we turn this global symmetry into a brane symmetry

$$\theta(x) \to \theta(x) + \alpha \Theta(x, \Sigma)$$

(53)

This amounts to consider a multivalued $\theta$

$$\theta(x) = \theta^\text{reg}(x) + \alpha \int^x dy^\mu \tilde{K}_\mu(y, \Upsilon)$$

(54)

such that (53) becomes the transformation of $\theta^\text{reg}$ that is compensated by the transformation of $K$ (the deformation of the surface $\Upsilon$)

$$\theta^\text{reg}(x) \to \theta^\text{reg}(x) + \alpha \Theta(x, \Sigma)$$

(55)

$$\int^x dy^\mu \tilde{K}_\mu(y, \Upsilon) \to \int^x dy^\mu \tilde{K}_\mu(y, \Upsilon') - \Theta(\Sigma, x)$$

(56)

where $\partial \Sigma = \Upsilon' - \Upsilon$. If the original action of the scalar field has the form

$$S = \int d^2x |\partial_\mu \Phi|^2 + V(\Phi)$$

(57)

since now $\Phi(x) = \Phi^\text{reg}(x)e^{i\alpha \int^x dy^\mu \tilde{K}_\mu(y, \Upsilon)}$, we have

$$\partial_\mu \Phi = \left(\partial_\mu \Phi^\text{reg}(x) + i\alpha \tilde{K}_\mu(x, \Upsilon)\Phi^\text{reg}(x)\right)e^{i\alpha \int^x dy^\mu \tilde{K}_\mu(y, \Upsilon)}$$

(58)

It follows that the new action, representing the insertion of the disorder operator (or, which is the same, the imposition of brane symmetry) is

$$S \to \int d^2x |\partial_\mu \Phi^\text{reg}(x) + i\alpha \tilde{K}_\mu(x, \Upsilon)\Phi^\text{reg}(x)|^2 + V(\Phi) = \int d^2x |\hat{D}_\mu \Phi^\text{reg}(x)|^2 + V(\Phi)$$

(59)

where

$$\hat{D}_\mu \Phi^\text{reg}(x) = \partial_\mu \Phi^\text{reg}(x) + i\alpha \tilde{K}_\mu(x, \Upsilon)\Phi^\text{reg}(x)$$

(60)

is the brane covariant derivative.

We learn that in order to compute, for instance, the 2-point function of the disorder operator, with the insertion points in $x_0$ and $x'_0$ in the complex scalar theory in $2D$, we just compute the path integral with the above action where $\Upsilon$ is the line connecting $x_0$ and $x'_0$. Due to the brane symmetry, this 2-point function does not depend on which $\Upsilon$ is actually chosen to perform the computation

$$\langle \mu^*(x_0) \mu(x'_0) \rangle = \frac{1}{Z} \int D\Phi^\text{reg} D\Phi^\text{reg} e^{-\int d^2x |\hat{D}_\mu \Phi^\text{reg}(x)|^2 + V(\Phi)}$$

(61)

In order to compute a general $n$-point function we just choose appropriate $\Upsilon$ ending on the point of insertions (if it is a 1-point function for instance we take $\Upsilon$ as the line from infinity to the point of insertion). The complex nature of the operator amounts to endow $\Upsilon$ with an orientation.
4.2 Gauge invariance and brane symmetry

These considerations are suitable to be applied to the case when there is a gauge redundancy in the system. Consider an abelian 1-form gauge theory in $D$-dimensions. It is described by a gauge field $A_\mu$ and the theory is such that it is invariant under

$$A_\mu \rightarrow A_\mu + \partial_\mu \phi$$  \hspace{1cm} (62)

The field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the natural invariant constructed from $A_\mu$. But note that in order for $F_{\mu\nu}$ to be an invariant under (62), $\phi$ must be single-valued. If $\phi$ is multivalued we would have

$$A_\mu \rightarrow A_\mu + \partial_\mu \phi^{reg} + \alpha \tilde{K}_\mu$$  \hspace{1cm} (63)

and the field strength would change by

$$F_{\mu\nu} \rightarrow F_{\mu\nu} + \alpha \left( \partial_\mu \tilde{K}_\nu - \partial_\nu \tilde{K}_\mu \right)$$  \hspace{1cm} (64)

In (63) we have what one may call a singular gauge transformation, which is a combined regular gauge and brane transformations. In order to make sense of this proposal, we have to allow $A_\mu$ to be multivalued.

$$A_\mu(x) = A^{reg}_\mu(x) + \frac{\alpha}{2} \int_x^y dy' \tilde{M}_{\mu\nu}(y)$$  \hspace{1cm} (65)

This leads to the definition of the gauge and brane invariant

$$F_{\mu\nu} \equiv F^{reg}_{\mu\nu} + \alpha \tilde{M}_{\mu\nu}$$  \hspace{1cm} (66)

Now, eq. (64) becomes the transformation for $F^{reg}_{\mu\nu}$ and the full set of brane symmetries is

$$\tilde{M}_{\mu\nu} \rightarrow \tilde{M}_{\mu\nu} - \left( \partial_\mu \tilde{K}_\nu - \partial_\nu \tilde{K}_\mu \right)$$  \hspace{1cm} (67)

$$F^{reg}_{\mu\nu} \rightarrow F^{reg}_{\mu\nu} + \alpha \left( \partial_\mu \tilde{K}_\nu - \partial_\nu \tilde{K}_\mu \right)$$  \hspace{1cm} (68)

Also, eq. (63) becomes the transformation for $A^{reg}_\mu$

$$A^{reg}_\mu \rightarrow A^{reg}_\mu + \partial_\mu \phi^{reg} + \alpha \tilde{K}_\mu$$  \hspace{1cm} (69)

while $A_\mu$ in eq. (65) transforms as

$$A_\mu \rightarrow A_\mu + \partial_\mu \phi^{reg} + \alpha \tilde{K}_\mu - \alpha \tilde{P}_\mu$$  \hspace{1cm} (70)

where

$$\tilde{P}_\mu = \frac{1}{2} \int_x^y dy' \partial_{[\mu} \tilde{K}_{\nu]}(y)$$  \hspace{1cm} (71)
Note that this in fact maintains \( F_{\mu \nu} \) on eq. (66), invariant, as it must by construction.

It is important to note that this set of transformations is different from the ones discussed in eqs. (47) and (48). There we had a pure brane symmetry that left \( A_\mu \) invariant, because it is an observable in that case. The main difference is that the pure brane symmetry did not shift the longitudinal part, only the transverse part. In fact for the present case we would have simply

\[
A_\mu^{\text{reg brane}} \rightarrow A_\mu^{\text{reg}} - \alpha P_\mu
\]  

(72)

Note that \( \partial_\mu P_\mu = 0 \), and therefore this is a transformation that shifts only the transverse part of \( A_\mu^{\text{reg}} \), as opposed to a regular gauge transformation that shifts only the longitudinal part. But when we consider a singular gauge transformation, we include also a transverse shift because of the presence of the extra term \( \tilde{K}_\mu \) in the expression of \( \partial_\mu \phi \). The appearance of \( P_\mu \) has the role of restoring the pure longitudinal shift that a gauge symmetry is supposed to have, in fact

\[
\partial_\mu \tilde{K}_\mu - \partial_\nu \tilde{P}_\mu = 0
\]  

(73)

so that the transverse part suffers no shift under the singular gauge transformation, which can also be seen as a combination of a regular gauge transformation (shift by \( \partial_\mu \phi^{\text{reg}} \)) and a brane transformation that only shifts the longitudinal part (shift by \( \alpha \tilde{K}_\mu - \alpha \tilde{P}_\mu \)).

Summarizing, we have just constructed the elements to describe the theory of magnetic monopoles in an abelian gauge theory. We learned that allowing for multivalued gauge transformations (multivalued \( \phi \)) and imposing that the theory remains invariant we are led to introduce multivalued gauge fields. This leads to the definition of the (brane and gauge) invariant \( F_{\mu \nu} \) (66). We thus obtain the identity

\[
\partial_\mu \tilde{F}^{\mu \mu_2 \cdots \mu_{D-2}} = \alpha \partial_\mu \tilde{M}^{\mu \mu_2 \cdots \mu_{D-2}} = \alpha J_{m}^{\mu \mu_2 \cdots \mu_{D-2}}
\]  

(74)

Where \( J_{m}^{\mu \mu_2 \cdots \mu_{D-2}} = \partial_\mu \tilde{M}^{\mu \mu_2 \cdots \mu_{D-2}} \) is the magnetic \((D - 3)\)-current with \( M^{\mu \mu_2 \cdots \mu_{D-2}} \) the Dirac \((D - 2)\)-brane. In \( D = 4 \) we have the familiar equation

\[
\partial_\mu \tilde{F}^{\mu \nu} = \alpha \partial_\mu \tilde{M}^{\mu \nu} = \alpha J_{m}^{\nu}
\]  

(75)

Where \( J_{m}^{\nu} = \partial_\mu M^{\mu \nu} \) is the magnetic current with \( M^{\mu \nu} \) the worldvolume of the Dirac string.

**Turning global symmetry into a singular gauge symmetry**

Let’s return to our example of the complex scalar field with \( U(1) \) global symmetry acting as

\[
\Phi(x) \rightarrow e^{i\alpha} \Phi(x)
\]  

(76)

or, in terms of the phase \( \theta(x) \) of \( \Phi(x) = |\Phi|(x)e^{i\theta(x)} \)

\[
\theta(x) \rightarrow \theta(x) + \alpha
\]  

(77)
If we now turn this into a local symmetry $\alpha \rightarrow \alpha \phi(x)$ we need to introduce a gauge field and redefine the derivative to a covariant one

$$D_\mu \Phi(x) = \partial_\mu \Phi(x) - i\alpha A_\mu(x)\Phi(x)$$

(78)

such that the gauge symmetry manifests itself as

$$\theta(x) \rightarrow \theta(x) + \alpha \phi(x)$$
$$A_\mu \rightarrow A_\mu + \partial_\mu \phi$$

(79)

so that the covariant derivative transforms as a normal derivative would if the symmetry was global

$$D_\mu \Phi(x) \rightarrow e^{i\alpha \phi(x)} D_\mu \Phi(x)$$

(80)

Now, what if $\phi(x)$ is a multivalued field? This would correspond to a so-called large gauge transformation. When we don’t consider the multivalued nature of $\phi$ we are in fact regarding it as an infinitesimal parameter and in this case $\phi$ takes value in $\mathbb{R}$, so that this is a $\mathbb{R}$ gauge symmetry. But $\phi$ has the nature of an angular variable since it is a shift in a field that takes values in the circle $S^1$. So in order to have a true $U(1)$ gauge transformation we must allow $\phi$ to take values in $S^1$, thus defining a $U(1)$ gauge symmetry. This case is also called compact gauge symmetry (because $\phi$ takes values in a compact space) and the former case is called non-compact gauge symmetry. We therefore write

$$\phi(x) = \phi^{reg}(x) + \beta \int^x dy^\mu \tilde{K}_\mu(y)$$

(81)

with $\beta$ a new parameter. The previous analysis of the singular gauge symmetry follows. We already learned that to make sense of a singular gauge transformation we have to introduce a multivalued gauge field

$$A_\mu(x) = A_\mu^{reg}(x) + \frac{\beta}{2} \int^x dy^\nu \tilde{M}_{\nu\mu}(y)$$

(82)

The singular gauge transformation then manifests itself as

$$\theta(x) \rightarrow \theta(x) + \alpha \phi^{reg}(x) + \alpha \beta \int^x dy^\mu \tilde{K}_\mu(y)$$
$$\tilde{M}_{\mu\nu} \rightarrow \tilde{M}_{\mu\nu} - \left(\partial_\mu \tilde{K}_\nu - \partial_\nu \tilde{K}_\mu\right)$$
$$A_\mu^{reg} \rightarrow A_\mu^{reg} + \partial_\mu \phi^{reg} + \beta \tilde{K}_\mu$$

(83)

In order to have the expected property, the covariant derivative must be defined with respect to the regular part of the gauge field

$$D_\mu \Phi(x) = \partial_\mu \Phi(x) - i\alpha A_\mu^{reg}(x)\Phi(x)$$

(84)
This follows from observing that the relevant invariant structure is

$$
\partial_\mu \theta(x) - \alpha A^\text{reg}_\mu(x)
$$

(85)

It is thus immediate to see that under eq. (83)

$$
D_\mu \Phi(x) \to e^{i\alpha \phi(x)} D_\mu \Phi(x)
$$

(86)

It is useful to write the covariant derivative separating the regular parts from the singular ones

$$
D_\mu \Phi(x) = e^{i\gamma \int_x dy^\mu \tilde{R}_\mu(y)} \left( \partial_\mu \Phi^\text{reg}(x) + i\gamma \tilde{R}_\mu(x) \Phi^\text{reg}(x) - i\alpha A^\text{reg}_\mu(x) \Phi^\text{reg}(x) \right)
$$

(87)

where we defined \( \Phi(x) = \Phi^\text{reg}(x)e^{i\gamma \int_x dy^\mu \tilde{R}_\mu(y)} \), which amounts to make explicit the singular part of the phase field

$$
\theta(x) = \theta^\text{reg}(x) + \gamma \int_x dy^\mu \tilde{R}_\mu(y)
$$

(88)

This is such that under the singular gauge transformation we have

$$
\theta^\text{reg}(x) \to \theta^\text{reg}(x) + \alpha \phi^\text{reg}(x)
$$

$$
\tilde{R}_\mu \to \tilde{R}_\mu + \frac{\alpha \beta}{\gamma} \tilde{K}_\mu
$$

(89)

Note that we defined a seemingly independent parameter \( \gamma \) for the singular part of \( \theta \), but in fact it does not make sense to have \( \gamma = 0 \) and \( \beta \neq 0 \), because even if we start with a regular \( \theta \) it will become singular under the multivalued gauge transformation. Therefore \( \gamma \) is not independent from \( \beta \) and we can simply identify them \( \gamma = \beta \).

An important property to note is that the combination \( B_\mu \equiv A^\text{reg}_\mu(x) - \alpha \tilde{R}_\mu(x) \) transforms as a regular gauge field under the singular gauge transformation

$$
B_\mu \to B_\mu + \partial_\mu \phi^\text{reg}
$$

(90)

and the singular gauge covariant derivative assumes the form

$$
D_\mu \Phi(x) = e^{i\gamma \int_x dy^\mu \tilde{R}_\mu(y)} \left( \partial_\mu \Phi^\text{reg}(x) + i\alpha B_\mu(x) \Phi^\text{reg}(x) \right)
$$

(91)

Such that all the singular character of this derivative is isolated in an overall phase that does not contribute in the action. This is an important property because when computing quantum corrections generated by fluctuations of \( \Phi \), the result will be as if there where no singular parts and the gauge field is \( B_\mu \). We can also write the covariant derivative as

$$
D_\mu \Phi(x) = e^{i\gamma \int_x dy^\mu \tilde{R}_\mu(y)} \left( e^{i\theta^\text{reg}} \partial_\mu |\Phi^\text{reg}(x)| + (i\partial_\mu \theta^\text{reg} - i\alpha B_\mu(x)) \Phi^\text{reg}(x) \right)
$$

(92)

We can identify three layers of structure for the \( U(1) \) scalar field theory:
1. Configurations with small $\theta(x)$ and also small gauge transformations $\phi(x)$. This corresponds to the textbook case where only small fluctuations of $\theta(x)$ are relevant and the gauge group is effectively $\mathbb{R}$, also known as non-compact gauge theory. The analysis can proceed with trivial perturbation theory with all fields being regular. The key quantities to analyze are the gauge invariant terms. Consider the covariant derivative, for instance. The relevant quantity is

$$\partial_\mu \theta^{\text{reg}} - \alpha A^{\text{reg}}_\mu(x) + \beta \tilde{R}_\mu(x)$$

with $\theta(x)$ small, we have $\tilde{R}_\mu(x) = 0$ and this is just a term gauge equivalent to $A^{\text{reg}}_\mu(x)$. Also, since the gauge transformation is small, the gauge field is single-valued and $A_\mu = A^{\text{reg}}_\mu$ so that $F_{\mu\nu}(A) = F_{\mu\nu}(A^{\text{reg}})$ and (shifting $A^{\text{reg}}_\mu \rightarrow A^{\text{reg}}_\mu + \frac{1}{\alpha} \partial_\mu \theta^{\text{reg}}$)

$$F_{\mu\nu}(A^{\text{reg}}) + \frac{1}{\alpha} \partial_\mu \theta^{\text{reg}} = F_{\mu\nu}(A^{\text{reg}})$$

and no nontrivial fluxes are present.

2. Configurations with large $\theta(x)$ and small gauge transformations $\phi(x)$. In this case the field $\theta$ takes values in the circle $S^1$ (its image is defined in $[0, 2\pi]$), but we only consider infinitesimal gauge transformations. This corresponds to the case where there are vortices in the system associated with closed flux lines. Since $\theta(x)$ is multivalued we write

$$\theta(x) = \theta^{\text{reg}}(x) + \beta \int^x dy^\mu \tilde{R}_\mu(y)$$

and there is a brane symmetry under which $\theta$ is invariant

$$\theta^{\text{reg}}(x) \rightarrow \theta^{\text{reg}}(x) + \beta \Theta(x)$$

$$\tilde{R}_\mu \rightarrow \tilde{R}_\mu - \partial_\mu \Theta(x)$$

This transformation has nothing to do with the gauge symmetry that remains the trivial one $\phi = \phi^{\text{reg}}$ and there is no need to consider a multivalued $A_\mu$, so that $A_\mu = A^{\text{reg}}_\mu$ (note that the gauge group is still effectively $\mathbb{R}$). But now with nonzero $\tilde{R}_\mu$, we have nontrivial fluxes. In fact, with $B_\mu(x) = A^{\text{reg}}_\mu(x) - \frac{2}{\alpha} \tilde{R}_\mu(x)$

$$F_{\mu\nu}(A) = F_{\mu\nu}(B + \frac{\beta}{\alpha} \tilde{R})$$

And the fluxes are closed because we still have

$$\partial_\mu * F^{\mu\nu}(A) = 0$$

3. Configurations with large $\theta(x)$ and large gauge transformations $\phi(x)$. In this case, both $\theta(x)$ and $\phi(x)$ are compact variables and now both functions are multivalued. $\phi(x)$ defines a
compact gauge group $U(1)$. The multivalued gauge transformation demands that $A_\mu$ is also multivalued and we have the following set of brane-gauge fields

$$\theta(x) = \theta^{reg}(x) + \beta \int^x dy^\mu \tilde{R}_\mu(y)$$

$$A_\mu(x) = A^{reg}_\mu(x) + \frac{\beta}{2} \int^x dy^\nu \tilde{M}_{\nu\mu}(y)$$

(99)

The multivalued gauge transformation

$$\phi(x) = \phi^{reg}(x) + \beta \int^x dy^\mu \tilde{K}_\mu(y)$$

(100)

acts on these fields as

$$\theta^{reg}(x) \to \theta^{reg}(x) + \alpha \phi^{reg}(x)$$

$$\tilde{R}_\mu \to \tilde{R}_\mu + \alpha \tilde{K}_\mu$$

$$A^{reg}_\mu \to A^{reg}_\mu + \partial_\mu \phi^{reg} + \beta \tilde{K}_\mu$$

$$\tilde{M}_{\mu\nu} \to \tilde{M}_{\mu\nu} - \left( \partial_\mu \tilde{K}_\nu - \partial_\nu \tilde{K}_\mu \right)$$

(101)

The covariant derivative remains the same. The term written in (93) is invariant under the multivalued gauge transformation. But now we have $A_\mu \neq A^{reg}_\mu$ and the field strength has nontrivial open fluxes

$$F_{\mu\nu}(A) = F_{\mu\nu}(A^{reg}) + \beta \tilde{M}_{\mu\nu}$$

(102)

Writing this in terms of the brane invariant gauge field

$$B_\mu(x) = A^{reg}_\mu(x) - \frac{\beta}{\alpha} \tilde{R}_\mu(x)$$

(103)

we have

$$F_{\mu\nu}(A) = F_{\mu\nu}(B + \frac{\beta}{\alpha} \tilde{R}) + \beta \tilde{M}_{\mu\nu} = F_{\mu\nu}(B) + \beta \left( \tilde{M}_{\mu\nu} + \frac{1}{\alpha} \partial_\nu \tilde{R}_\mu \right)$$

(104)

The last term represents brane invariant flux open lines. That they are open can be seen by noting

$$\partial^\mu F^{\mu\nu}(A) = \beta \partial_\mu M^{\mu\nu} = \beta J^\nu_m$$

(105)

### 4.3 Defects structure in superconductors

An interesting application of our discussion is the physics of the electromagnetic response in superconductor. The effective theory in the London limit is

$$S_{SC} = \int d^4x \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{q^2 M^2}{2} \left( A_\mu + \frac{1}{q} \partial_\mu \theta \right)^2 \right).$$

(106)
This theory is obtained as a limit of the Ginzburg-Landau model such that the field $\theta$ is the phase of the complex scalar field effectively describing the condensate of Cooper pairs. The discussion in the previous section follows almost immediately then. Including vortex contribution to this action amounts to allow $\theta$ to be multivalued leading to closed fluxes contributions as we saw in the item 2 above. We can also open these flux lines by introducing monopoles contribution making the gauge transformation multivalued as in (100), this corresponds to item 3 above. The action is changed to have the form

$$S_{SC}^{Mon} = \int d^4x \left( \frac{1}{4} \left( F_{\mu\nu}(B) + \beta \left( \tilde{M}_{\mu\nu} + \frac{1}{\alpha} \partial_{[\mu} \tilde{R}_{\nu]} \right) \right)^2 + \frac{q^2 M^2}{2} \left( B_{\mu} + \frac{1}{q} \partial_{\mu} \theta^{reg} \right)^2 \right), \quad (107)$$

The last term is a mass term an we see that $\theta^{reg}$ is a Goldstone mode setting the longitudinal part of the gauge field exactly to zero inside the superconductor, while the transverse part decays exponentially within a distance $\sim 1/M$. In fact, variation of the action (107) with respect to changes in $\theta^{reg}$ leads to

$$qM^2 \partial^\mu \left( B_{\mu} + \frac{1}{q} \partial_{\mu} \theta^{reg} \right) = 0 \quad (108)$$

which is the expression for the conservation of the supercurrent

$$j^\mu_s = qM^2 \left( B_{\mu} + \frac{1}{q} \partial_{\mu} \theta^{reg} \right) \quad (109)$$

The action (107) describes magnetic monopoles inside a superconductor and it is suitable for the computation of the correlation function of monopole operators

$$\langle \mu(x_0)\mu(x'_0) \rangle = \frac{1}{Z} \sum_{\tilde{R}} \int D\tilde{B} \ e^{-S_{SC}^{Mon}} \quad (110)$$

Where $\mu(x_0)$ denotes the insertion of a monopole at the position $x_0$. $\tilde{M}$ is the Dirac string connecting the monopoles (or extending to infinity in the case of one monopole) and sum over $\tilde{R}$ span the closed flux lines configurations and will effectively amount to a sum over all possible lines connecting the monopoles. This correlation function is well known and it is just the ’t Hooft loop for confined monopoles; it will have a area law asymptotically. A nice way to see this is to perform a dual transformation obtaining the the massive Kalb-Ramond field theory minimally coupled to the line $\tilde{M}_{\mu\nu} + \frac{1}{\alpha} \partial_{[\mu} \tilde{R}_{\nu]}$, thus showing that the line carries energy and its preferred configuration will be the line minimizing this energy, corresponding to a confining string between monopoles.

It’s tempting to consider $\mu(x_0)$ as the disorder operator for a superconductor, but this is misleading because as argued in [37] there is no proper local order parameter for superconductors and the system is better described as a topological state of matter. The main argument of [37] is that the would be order parameter, the complex field in the Ginzburg-Landau model, is gauge dependent. In the present picture the would be disorder field $\mu(x_0)$ is also nonlocal, since it comes with a choice of the line $\tilde{M}$. In the non-superconducting phase, the Dirac string $\tilde{M}$ carries no energy and is a trivial redundancy (what we called brane symmetry above), with the closed
vortices $\vec{R}$ decoupling (as can be easily seen with $M = 0$ in (107) and rewriting the action in terms of $A^{reg}$). In the superconducting phase the Dirac string $\hat{M}$ combines with the closed vortices $\vec{R}$ to form line configurations that carries energy, but still maintaining brane symmetry, we may call this situation *brane symmetry breaking*, due to its similarity with the gauge symmetry breaking. But, as in the Higgs mechanism, of course there is no true symmetry breaking and the brane symmetry is just hidden. Gauge symmetry is more properly understood as a redundancy in the variables describing the system and so is the brane symmetry, both are never truly broken. As a result there is no true local disorder parameter.

**Topological superconductor**

Another interesting model is provided by the effective theory describing the electromagnetic response of topological superconductor. It was proposed in [38] and further studied in [39] and [40]. Here we follow the notation of [40] where this action was obtained from symmetry considerations and a careful analysis of the relevant degrees of freedom

$$S_{TSC} = \int d^4x \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{q^2 M^2}{2} \left( A_\mu + \frac{1}{q} \partial_\mu \theta \right)^2 + \frac{m^2}{2} \left( \partial_\mu \theta \right)^2 \right.$$

$$- \left. \frac{q^2 M^4 m^2}{\Lambda^6} \partial_\mu \bar{\theta} \varepsilon^{\mu\nu\rho\sigma} \left( A_\nu + \frac{1}{q} \partial_\nu \theta \right) \partial_\rho \left( A_\sigma + \frac{1}{q} \partial_\sigma \theta \right) + \bar{\rho} \cos (\bar{\theta}) \right) , \quad (111)$$

In (111) we see that the first two terms characterize the usual electromagnetic response of a superconductor in the London limit, with the real field $\theta$ as the phase of a complex scalar field in the Ginzburg-Landau theory. The other terms characterize the system as a topological superconductor associated with two geometrically disconnected Fermi surfaces (two Fermi surfaces for short). The field $\bar{\theta}$ describes the phase difference between these surfaces and is associated to a charge exchange induced by instantons (see [40] for details). The massive parameter $M$ quantifies the inverse of the penetration length characterizing the Meissner effect. The Axion-like $\bar{\theta}$ excitations are massive with a mass given by $\sqrt{\frac{q^2 M^4 m^2}{\Lambda^6}}$. The scale $\Lambda$ in the axionic interaction turns out to be not independent, for topological reasons, and given by a integer multiple of $(8\pi M^4 m^2)^{1/6}$. The last term is a Josephson term.

The same analysis including multivalued fields can be carried out for this model. The inclusion of closed vortices $\vec{R}$, turning $\theta$ into a multivalued field, is straightforward and the introduction of monopoles, opening the flux lines, is also simple, replacing the field $A$ by $B$ everywhere. The result is

$$S_{TSC}^{Mon} = \int d^4x \left( \frac{1}{4} \left( F_{\mu\nu}(B) + \beta \left( M_{\mu\nu} + \frac{1}{\alpha} \partial_\mu \vec{R}_\nu \right) \right)^2 + \frac{q^2 M^2}{2} \left( B_\mu + \frac{1}{q} \partial_\mu \theta^{reg} \right)^2 \right.$$

$$+ \frac{m^2}{2} \left( \partial_\mu \bar{\theta} \right)^2 - \left. \frac{q^2 M^4 m^2}{\Lambda^6} \partial_\mu \bar{\theta} \varepsilon^{\mu\nu\rho\sigma} \left( B_\nu + \frac{1}{q} \partial_\nu \theta^{reg} \right) \partial_\rho \left( B_\sigma + \frac{1}{q} \partial_\sigma \theta^{reg} \right) + \bar{\rho} \cos (\bar{\theta}) \right) , \quad (112)$$

But now there is also the possibility of vortices contributions coming from $\bar{\theta}$. These are called *chiral vortices*. There is no corresponding monopoles for these vortices lines since they don’t
carry any flux. But they nevertheless have an important contribution to the physics of such superconductors. Computing the variation of the action (112) with respect to changes in $\theta^{\text{reg}}$ leads to

$$ qM^2 \partial^\mu \left( B_\mu + \frac{1}{2} \partial_\mu \theta^{\text{reg}} \right) + i \frac{qM^4 m^2}{\Lambda^6} \partial_\mu \partial_\nu \bar{\theta} \varepsilon^{\mu\nu\rho\sigma} \partial_\rho B_\sigma = 0 \quad (113) $$

We note that if $\bar{\theta}$ is regular, this becomes simply the expression for the conservation of the supercurrent $j_\mu^s = qM^2 \left( B_\mu + \frac{1}{2} \partial_\mu \theta^{\text{reg}} \right)$ of a normal superconductor (108). Now, consider that $\bar{\theta}$ is multivalued. In that case we have

$$ \bar{\theta}(x) = \theta^{\text{reg}}(x) + \lambda \int^x dy^\mu \tilde{N}_\mu(y), \quad (114) $$

and (113) becomes

$$ \partial_\nu j_\nu^s = -i \frac{qM^4 m^2}{\Lambda^6} \lambda \partial_\nu \tilde{N}_\mu \varepsilon^{\mu
\nu\rho\sigma} \partial_\rho B_\sigma = i \frac{qM^4 m^2}{2\Lambda^6} \lambda \tilde{J}_\nu^\rho F_{\rho\sigma}(B), \quad (115) $$

where

$$ \tilde{N}_\mu(x) = \frac{1}{3!} \varepsilon_{\mu\nu\rho\sigma} N^{\nu\rho\sigma}(x) $$

$$ = \varepsilon_{\mu\nu\rho\sigma} \int_\Sigma d\tau_1 d\tau_2 d\tau_3 \frac{\partial y^\mu}{\partial \tau_1} \frac{\partial y^\nu}{\partial \tau_2} \frac{\partial y^\rho}{\partial \tau_3} \delta^4(x - y(\tau_1, \tau_2, \tau_3)) \quad (116) $$

and the chiral vortex current is defined by

$$ \tilde{J}_\mu^\nu = \partial_\mu N^{\nu\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} \partial_\rho \tilde{N}_\sigma \quad (117) $$

We thus see that the longitudinal part is not exactly zero inside a topological superconductor, but lives only at vortices loci.

## 5 Conclusion

In this work we explored the application of multivalued fields in the formulation of monopole operators. These operators have a non-local structure that encapsulates collective properties of the original degrees of freedom. The main element in our construction was the introduction of defects through the representation of the fundamental fields as the direct sum of a regular part and a singular part that characterizes its multivaluedness. To the singular part one can associate a geometrical picture in terms of surfaces, or branes, whose arbitrary placement in space is the embodiment of the multivalued nature of the field. It is thus possible to identify different types of ambiguities in the placement of the brane related to the nature of the field: If the field is an observable, that is, if it creates a physical state, there must be no ambiguity and the field is invariant with respect to changes in the brane position, it is said to be brane invariant. The case
of gauge fields is more interesting because the fundamental fields have unphysical components related to the gauge ambiguity. This leads to an interplay between gauge transformation and brane transformation which is the manifestation of the well known concepts of small and large gauge transformations. These concepts becomes very clearly stated in the language of multivalued fields and branes.

The elements presented in this work thus lead to a natural description of defects. We discuss the well known result that the insertion of defects in the system amounts to a deformation of the original action with the introduction of terms representing singularities in the domain of the fundamental fields. This result, cast in the language of multivalued fields and branes, naturally furnishes the correct formulation. In that way we were able to reproduce the results of [9] for the computation of correlation functions of disorder operators. Also the concepts of Wilson and t’ Hooft operators in abelian theories can be naturally cast in this language.

The interplay of gauge symmetry and brane symmetry is important for the study of superconductors and its ensuing vortices. We showed that open and closed vortices can be introduced and its corresponding correlation functions computed. In fact, for the case of topological superconductors, we saw that in the description of the electromagnetic response the so called chiral vortices are an important aspect of the phenomenology. These vortices can be naturally incorporated taking into account the multivaluedness of the axion-like field in the effective theory introduced in [38].

All the results presented here are valid for the case where the multivalued character of the field can be expressed as a direct sum of a regular part and a singular part. For the case of gauge theories it is thus suitable for application to abelian systems only. The definition of the covariant derivative also shows that the introduction of matter is straightforward for bosons and fermions. But it would be interesting to explore further the introduction of fermions due to the matrix nature of the ”polar decomposition” of a fermionic field. The generalization of our prescription to non-abelian system is less straightforward because of the mixing between the singular and regular parts and the interpretation of such term becomes less clear.

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