Nonholonomic Ricci Flows: 
II. Evolution Equations and Dynamics

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Abstract

This is the second paper in a series of works devoted to nonholonomic Ricci flows. By imposing non–integrable (nonholonomic) constraints on the Ricci flows of Riemannian metrics we can model mutual transforms of generalized Finsler–Lagrange and Riemann geometries. We verify some assertions made in the first partner paper and develop a formal scheme in which the geometric constructions with Ricci flow evolution are elaborated for canonical nonlinear and linear connection structures. This scheme is applied to a study of Hamilton’s Ricci flows on nonholonomic manifolds and related Einstein spaces and Ricci solitons. The nonholonomic evolution equations are derived from Perelman’s functionals which are redefined in such a form that can be adapted to the nonlinear connection structure. Next, the statistical analogy for nonholonomic Ricci flows is formulated and the corresponding thermodynamical expressions are found for compact configurations. Finally, we analyze two physical applications: the nonholonomic Ricci flows associated to evolution models for solitonic pp–wave solutions of Einstein equations, and compute the Perelman’s entropy for regular Lagrange and analogous gravitational systems.

Keywords: Ricci flows, nonholonomic Riemann manifold, nonlinear connections, generalized Lagrange and Finsler geometry, Perelman’s functionals.

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1 Introduction

Current important and fascinating problems in modern geometry and physics involve the finding of canonical (optimal) metric and connection structures on manifolds, search for possible topological configurations and to find the relevant physical applications. In the past three decades, the Ricci flow theory has addressed such issues for Riemannian manifolds [1, 2, 3, 4, 5]; the reader can find existing reviews on Hamilton–Perelman theory [6, 7, 8, 9]. How to formulate and generalize these constructions for non–Riemannian manifolds and physical theories is a challenging topic in mathematics and physics. The typical examples arise in string/brane gravity containing non-trivial torsion fields, in modern mechanics, and field theory whose geometry is based in terms of symplectic and/or generalized Finsler (Lagrange or Hamilton) structures.

Our main concern is to prove that the Ricci flow theory can be generalized to various geometries and applied to solutions of fundamental problems in classical and quantum physics [10, 11, 12, 13] and geometric mechanics [14]. In generalized (non–Riemannian) geometries and a number of physical theories, the nonholonomic constraints are important for all questions related to the motion/ field equations, symmetries, invariants and conservation laws, in constructing exact solutions and choosing a procedure of quantization. Such geometric approaches are related to non–Riemannian geometric structures which require different generalizations to treat matter fields and to study spacetime geometries and construct the Ricci flows of the geometric and physical quantities on such spaces.

The first goal of this work is to investigate the geometry of the evolution equations under non–integrable (equivalently, nonholonomic/ anholonomic) constraints resulting in nonholonomic Riemann–Cartan and generalized Finsler–Lagrange configurations. The first partner paper [10] was devoted to a study of nonholonomic Ricci flows using the geometric constructions corresponding to the Levi Civita connection. Here we develop an approach that is adapted to the nonlinear connection structure. In this case, it is possible to elaborate an alternative geometric formalism by working with the canonical distinguished connection and which is also a metric compatible linear connection but contains a nonholonomically induced torsion (by ’off–diagonal’ metric coefficients). The second purpose is to study certain applications of the nonholonomic Ricci flow theory in modern gravity, Lagrange mechanics and analogous gravitational systems.

This paper is organized as follows: Section 2 is devoted to a study of Ricci flows on nonholonomic manifolds provided with nonlinear connection
structure (briefly, we refer to the terms N–connection and N–anholonomic manifold; the reader is urged to consult in advance the geometric formalism in \([10, 15]\)) and the introduction to monograph \([16]\) in order to study the evolution equations for the Levi Civita connection and the canonical distinguished connection). We analyze some examples of nonholonomic Ricci flows for the case of nonholonomic Einstein spaces and N–anholonomic Ricci solitons. We prove the existence and uniqueness of the N–anholonomic evolution.

In section 3, we define the Perelman’s functionals on N–anholonomic manifolds and construct the N–adapted variational calculus which provides a geometrical proof of the evolution equations for generalized Finsler–Lagrangian and nonholonomic metrics. We investigate the properties of the associated energy for nonholonomic configurations and formulate certain rules which allow us to extend the proofs for the Levi Civita connections to the case of canonical distinguished connections.

The statistical analogy for the nonholonomic Ricci flows is proposed in section 4 where we study certain important properties of the N–anholonomic entropy and define the related thermodynamical expressions.

Section 5 is devoted to the applications of the nonholonomic Ricci flow theory: we construct explicit solutions describing the Ricci flow evolutions of the Einstein spaces associated to the solitonic pp–waves. Then we compute the Perelman’s entropy for the Lagrangian mechanical systems and the related models of analogous gravitational theories. The concluding remarks are reserved for section 6.

2 Hamilton’s Ricci Flows on N–anholonomic Manifolds

In this section, we present some basic materials on Ricci flows on nonholonomic manifolds generalizing the results from \([11, 2]\) and outlined in sections 1.1–1.4 of \([6]\). On the geometry of N–anholonomic manifolds (i.e. manifolds enabled with nonholonomic distributions defining nonlinear connection, N–connection, structures), we follow the conventions from the first partner work \([10]\) and \([16, 17]\). We shall use boldface symbols for spaces/geometric objects enabled/adapted to N–connection structure.
2.1 On nonholonomic evolution equations

A nonholonomic manifold is defined as a pair \( V = (M, D) \), where \( M \) is a manifold and \( D \) is a non-integrable distribution on \( M \). For certain important geometric and physical cases, one considers N–anholonomic manifolds when the nonholonomic structure of \( V \) is established by a nonlinear connection (N–connection), equivalently, a Whitney decomposition of the tangent space into conventional horizontal (h) subspace, \((hV)\), and vertical (v) subspace, \((vV)\).

\[
T V = hV \oplus vV.
\]

Locally, a N–connection \( N \) is defined by its coefficients \( N^a_i(u) \),

\[
N = N^a_i(u) dx^i \otimes \frac{\partial}{\partial y^a},
\]

The Ricci flow equations were introduced by R. Hamilton as evolution equations

\[
\frac{\partial g_{\alpha\beta}(\chi)}{\partial \chi} = -2 \bar{R}_{\alpha\beta}(\chi)
\]

for a set of Riemannian metrics \( g_{\alpha\beta}(\chi) \) and corresponding Ricci tensors \( \bar{R}_{\alpha\beta}(\chi) \) parametrized by a real \( \chi \).

The normalized (holonomic) Ricci flows, with respect to the coordinate base \( \partial_\alpha = \partial/\partial u^\alpha \), are described by the equations

\[
\frac{\partial}{\partial \chi} g_{\alpha\beta} = -2 \, R_{\alpha\beta} + 2r \frac{g_{\alpha\beta}}{5},
\]

where the normalizing factor \( r = \int RdV/dV \) is introduced in order to preserve the volume \( V \). For N–anholonomic Ricci flows, the coefficients \( g_{\alpha\beta} = \)

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1In this series of works, we assume that the geometric/physical spaces are smooth and orientable manifolds.

2Usually, we consider a \((n+m)\)-dimensional manifold \( V \), with \( n \geq 2 \) and \( m \geq 1 \) (equivalently called to be a physical and/or geometric space). In a particular case, \( V = TM \), with \( n = m \) (i.e. a tangent bundle), or \( V = E = (E, M) \), \( \dim M = n \), is a vector bundle on \( M \), with total space \( E \). We suppose that a manifold \( V \) may be provided with a local fibred structure into conventional "horizontal" and "vertical" directions. The local coordinates on \( V \) are denoted in the form \( u = (x, y) \), or \( u^\alpha = (x', y') \), where the "horizontal" indices run the values \( i, j, k, \ldots = 1, 2, \ldots, n \) and the "vertical" indices run the values \( a, b, c, \ldots = n + 1, n + 2, \ldots, n + m \).

3For our further purposes, on generalized Riemann–Finsler spaces, it is convenient to use a different system of denotations than those considered by R. Hamilton or Grisha Perelman on holonomic Riemannian spaces.
\( g_{\alpha\beta} \) of any metric
\[
g = g_{\alpha\beta}(u) \, du^\alpha \otimes du^\beta
\]
can be parametrized in the form
\[
g_{\alpha\beta} = \begin{bmatrix}
g_{ij} + N_i^a N_j^b h_{ab} & N_i^c g_{ae}
g_{\alpha\beta} & g_{\alpha\beta}
\end{bmatrix}.
\]

With respect to the N–adapted frames \( e_\nu = (e_i, e_a) \) and coframes \( e^\alpha = (e^i, e^a) \), i.e. vielbeins adapted to the N–connection structure, for
\[
e_{\nu} = \left( e_i = \frac{\partial}{\partial x^i} - N_i^a(u) \frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a} \right),
\]
\[
e^\alpha = \left( e^i = dx^i, e^a = dy^a + N_i^a(u) dx^i \right),
\]
there are defined the frame transforms
\[
e_\alpha(\chi) = e_\alpha^\nu(\chi) \partial_\nu \quad \text{and} \quad e^\alpha(\chi) = e^\alpha_\alpha(\chi) du^\alpha,
\]
respectively parametrized in the form
\[
e_\alpha^\nu(\chi) = \begin{bmatrix}
\delta_i^i & \delta_i^a \\
0 & \delta_a^a
\end{bmatrix},
\]
\[
e^\alpha_\alpha(\chi) = \begin{bmatrix}
\delta_i^i & 0 \\
-\delta_a^b & \delta_a^a
\end{bmatrix},
\]
where \( \delta_i^i \) is the Kronecker symbol.

The Ricci flow equations (4) can be written in an equivalent form
\[
\frac{\partial}{\partial \chi} g_{ij} = 2\left[ N_i^a N_j^b \left( R_{ab} - \lambda g_{ab} \right) - R_{ij} + \lambda g_{ij} \right] - \frac{\partial}{\partial \chi} (N_i^a N_j^d), \tag{10}
\]
\[
\frac{\partial}{\partial \chi} g_{ab} = -2 \left( R_{ab} - \lambda g_{ab} \right), \tag{11}
\]
\[
\frac{\partial}{\partial \chi} (N_j^e g_{ae}) = -2 \left( R_{ia} - \lambda N_j^e g_{ae} \right), \tag{12}
\]
where \( \lambda = r/5 \) and the coefficients are defined with respect to local coordinate basis. Heuristic arguments for such equations both on holonomic and nonholonomic manifolds are discussed in Refs. \[10, 11, 12, 13\].

In N–adapted form, the tensor coefficients are defined with respect to tensor products of vielbeins (7) and (8). They are called respectively distinguished tensors/ vectors /forms, in brief, d–tensors, d–vectors, d–forms.
A distinguished connection (d–connection) $D$ on a $N$–anholonomic manifold $V$ is a linear connection conserving under parallelism the Whitney sum (1). In local form, a d–connection $D$ is given by its coefficients $\Gamma^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{be})$, where $^hD = (L^i_{jk}, L^a_{bk})$ and $^vD = (C^i_{jc}, C^a_{be})$ are respectively the covariant $h$– and $v$–derivatives. Such a d–connection $D$ is compatible to a metric $g$ if $Dg = 0$. The nontrivial $N$–adapted coefficients of the torsion of $D$, with respect to (7) and (8),

$$T = \{ T^\alpha_{\beta\gamma} = -T^\alpha_{\gamma\beta} = \left( T^i_{jk}, T^i_{ja}, T^a_{jk}, T^b_{ja}, T^b_{ca} \right) \}
$$

are given by formulas (A.9) in Ref. [10].

A distinguished metric (in brief, d–metric) on a $N$–anholonomic manifold $V$ is a second rank symmetric tensor $g$ which in $N$–adapted form is written

$$g = g_{ij}(x,y) \ e^i \otimes e^j + g_{ab}(x,y) \ e^a \otimes e^b. \quad (13)
$$

In brief, we write $g = h g \oplus N v g = [^h g, ^v g]$. Any metric $g$ on $V$ can be written in two equivalent forms as (5), with coefficients (6) with respect to a coordinate dual basis, or as (13) with $N$–adapted coefficients $g_{\alpha\beta} = [g_{ij}, g_{ab}]$ with respect to (5).

There are two classes of preferred linear connections defined by the coefficients $\{g_{\alpha\beta}\}$ of a metric structure $g$ (equivalently, by the coefficients of corresponding d–metric $(g_{ij}, h_{ab})$ and $N$–connection $N^i_a$ : we shall emphasize the functional dependence on such coefficients in some formulas):

- The unique metric compatible and torsionless Levi Civita connection $\nabla = \{ \Gamma^\gamma_{\alpha\beta}(g_{ij}, h_{ab}, N^a_i) \}$, for which $T^\alpha_{\beta\gamma} = 0$ and $\nabla g = 0$. This is not a d–connection because it does not preserve under parallelism the $N$–connection splitting (11).

- The unique metric canonical d–connection $\hat{D} = \{ \hat{\Gamma}^\gamma_{\alpha\beta}(g_{ij}, h_{ab}, N^a_i) \}$ is defined by the conditions $\hat{D} g = 0$ and $\hat{T}_i^i j k = 0$ and $\hat{T}^b_{ca} = 0$, but in general $\hat{T}^\gamma_{\alpha\beta} \neq 0$. The $N$–adapted coefficients $\hat{\Gamma}^\gamma_{\alpha\beta}$ and the deformation tensor $\hat{Z}^\gamma_{\alpha\beta}$, when

$$\hat{\Gamma}^\gamma_{\alpha\beta}(g_{ij}, h_{ab}, N^a_i) = \hat{\Gamma}^\gamma_{\alpha\beta}(g_{ij}, h_{ab}, N^a_i) + \hat{Z}^\gamma_{\alpha\beta}(g_{ij}, h_{ab}, N^a_i)
$$

are given by formulas (A.15)–(A.18) in [10].
In order to consider N–adapted Ricci flows, we have to change \( \nabla \rightarrow \hat{D} \) and, respectively, \( R_{\alpha \beta} \rightarrow \hat{R}_{\alpha \beta} \) in (10)–(12). The N–adapted evolution equations for Ricci flows of symmetric metrics, with respect to local coordinate frames, are written

\[
\frac{\partial}{\partial \chi} g_{ij} = 2 \left[ N^a_i N^b_j \left( \hat{R}_{ab} - \lambda g_{ab} \right) - \hat{R}_{ij} + \lambda g_{ij} \right] - g_{cd} \frac{\partial}{\partial \chi} (N^c_i N^d_j), \tag{14}
\]

\[
\frac{\partial}{\partial \chi} g_{ab} = -2 \left( \hat{R}_{ab} - \lambda g_{ab} \right), \tag{15}
\]

\[
\hat{R}_{ia} = 0 \quad \text{and} \quad \hat{R}_{ai} = 0, \tag{16}
\]

where the Ricci coefficients \( \hat{R}_{ij} \) and \( \hat{R}_{ab} \) are computed with respect to coordinate coframes.

We emphasize that, in general, under nonholonomic Ricci flows symmetric metrics may evolve in nonsymmetric ones. The Hamilton–Perelman theory of Ricci flows was constructed following the supposition that (pseudo) Riemannian metrics evolve only into other (pseudo) Riemannian metrics. In our approach, we consider Ricci flow evolutions of metrics subjected to certain classes of nonholonomic constraints, which may result in locally anisotropic geometric structures (like generalized Finsler–Lagrange metrics and connections) and even geometries with nonsymmetric metrics. For simplicity, in this work, we shall analyse nonholonomic evolutions when Ricci flows result only in symmetric metrics. This holds true, for instance, if the equations (16) are satisfied.

\textbf{Definition 2.1} Nonholonomic deformations of geometric objects (and related systems of equations) on a N–anholonomic manifold \( V \) are defined for the same metric structure \( g \) by a set of transforms of arbitrary frames into N–adapted ones and of the Levi Civita connection \( \nabla \) into the canonical d–connection \( \hat{D} \), locally parametrized in the form

\[
\partial_\alpha = (\partial_i, \partial_a) \rightarrow e_\alpha = (e_i, e_a); \quad g_{\alpha \beta} \rightarrow [g_{ij}, g_{ab}, N^a_i]; \quad \Gamma^\gamma_{\alpha \beta} \rightarrow \hat{\Gamma}^\gamma_{\alpha \beta}.
\]

It should be noted that the heuristic arguments presented in this section do not provide a rigorous proof of evolution equations with \( \hat{D} \) and \( \hat{R}_{\alpha \beta} \) all defined with respect to N–adapted frames (7) and (8). For instance, in Ref. [11], for five dimensional diagonal d–metric ansatz (13) with \( g_{ij} = \)

\[\text{In Refs. [10] [14], we discuss this problem related to the fact that the tensor } \hat{R}_{\alpha \beta} \text{ is not symmetric which results, in general, in Ricci flows of nonsymmetric metrics.}\]
$diag[\pm 1, g_2, g_3]$ and $g_{ab} = diag[g_4, g_5]$, we constructed exact solutions of the system

$$\frac{\partial}{\partial \chi} g_{ii} = -2 \left( \tilde{R}_{ii} - \lambda g_{ii} \right) - g_{cc} \frac{\partial}{\partial \chi} (N_i^c)^2,$$  \hspace{1cm} (17)

$$\frac{\partial}{\partial \chi} g_{aa} = -2 \left( \tilde{R}_{aa} - \lambda g_{aa} \right),$$  \hspace{1cm} (18)

$$\tilde{R}_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta,$$  \hspace{1cm} (19)

with the coefficients defined with respect to N–adapted frames (7) and (8). By nonholonomic deformations (equivalently, transforms, see Definition 2.1), the system (14)–(16) can be transformed into (17)–(19). A rigorous proof for nonholonomic evolution equations is possible following a N–adapted variational calculus for the Perelman’s functionals presented (below) for Theorems 3.1 and 4.1.

Having prescribed a nonholonomic $n + m$ splitting with coefficients $N_i^a$ on a (semi) Riemannian manifold $V$ provided with metric structure $g_{\alpha\beta}$, we can work with N–adapted frames (7) and (8) and the equivalent d–metric structure $(g_{ij}, g_{ab})$ (13). On $V$, one can be introduced two (equivalent) canonical metric compatible (both defined by the same metric structure, equivalently, by the same d–metric and N–connection) linear connections: the Levi Civita connection $\nabla$ and the canonical d–connection $\hat{D}$. In order to perform geometric constructions in N–adapted form, we have to work with the connection $\hat{D}$ which contains nontrivial torsion coefficients $\hat{T}^i_{ja}, \hat{T}^a_{jk}, \hat{T}^b_{ja}$ induced by the “off diagonal” metric / N–connection coefficients $N_i^a$ and their derivatives, see formulas (A.9) in Ref. [10]. In an alternative way, we can work equivalently with $\nabla$ by redefining the geometric objects, see Proposition 4.3 in Ref. [10].

We conclude that the geometry of a N–anholonomic manifold $V$ can be described equivalently by data $\{g_{ij}, g_{ab}, N_i^a, \nabla\}$, or $\{g_{ij}, g_{ab}, N_i^a, \hat{D}\}$. Of course, two different linear connections, even defined by the same metric structure, are characterized by different Ricci and Riemann curvatures tensors and curvature scalars. In this work, we shall prefer N–adapted constructions with $\hat{D}$ but also apply $\nabla$ if the proofs for $\hat{D}$ will be cumbersome. The idea is that if a geometric Ricci flow construction is well defined for one of the connections, $\nabla$ or $\hat{D}$, it can be equivalently redefined for the second one by considering the distortion tensor $Z^\gamma_{\alpha\beta}$.
2.2 Examples of N–anholonomic Ricci flows

We consider some classes of solutions [15, 16, 18] with nonholonomic variables in order to understand some properties of N–anholonomic Ricci flows [10, 11, 12, 13]. Nonholonomic Ricci solitons will be defined.

2.2.1 Nonholonomic Einstein spaces

Such spaces are defined by d–metrics constructed as solutions of the Einstein equations for the connection \( \hat{D} \) with nonhomogeneous horizontal and vertical cosmological 'constants', \( h_\lambda(x^k, y^a) \) and \( v_\lambda(x^k) \),

\[
\begin{align*}
\hat{R}^i_j & = v_\lambda(x^k) \delta^i_j, \\
\hat{R}^a_b & = h_\lambda(x^k, y^c) \delta^a_b, \\
\hat{R}_{aj} & = 0, \quad \hat{R}_{ja} = 0.
\end{align*}
\]

These equations can be integrated for certain general metric ansatz (13) and, equivalently, (6). For splitting 3+2 with coordinates \( u^\alpha = (x^1, x^2, x^3, y^4, y^5) \), \( \partial_i = \partial/\partial x^i, \partial_v = \partial/\partial v \), a class of exact solutions of the system (20) are parametrized (see details in Refs. [19, 16, 17]) in the form

\[
g = \epsilon_1(dx^1)^2 + \epsilon_2 g_2(x^2, x^3)(dx^2)^2 + \epsilon_3 g_3(x^2, x^3)(dx^2)^2 + \epsilon_4 h_0(x^2, x^3)[\partial_v f(x^i, v)]^2 |\varsigma_4| (e^4)^2 + \epsilon_5 [f(x^i, v) - f_0(x^i)]^2 (e^5)^2, \\
e^4 = dv + w_k(x^k, v)dx^k, \\
e^5 = dy^5 + n_k(x^k, v)dx^k,
\]

(21)

where the N–connection coefficients \( N^4_k = w_k \) and \( N^5_k = n_k \) are computed

\[
w_i = -\partial_i \varsigma_4(x^k, v)/\partial_v \varsigma_4(x^k, v), \\
n_k = n_k[1](x^i) + n_k[2](x^i) \int \frac{[\partial_v f(x^i, v)]^2 \varsigma_4(x^k, v)}{[f(x^i, v) - f_0(x^i)]^3} dv,
\]

for

\[
\varsigma_4(x^k, v) = \varsigma_{4[0]}(x^k) - \frac{\epsilon_4}{8} h_0^2(x^k) \int h_\lambda(x^k, v) [f(x^i, v) - f_0(x^i)] dv.
\]

In the ansatz (21), the values \( \epsilon_\alpha = \pm 1 \) state the signature of solution, the functions \( g_2 \) and \( g_3 \) are taken to solve two dimensional equations \( \hat{R}^2_2 = \hat{R}^3_3 = v_\lambda(x^2, x^3) \) and the generation function \( f(x^i, v) \) satisfies the condition \( \partial_v f \neq 0 \). The set of integration functions \( h_0^2(x^i), f_0(x^i), n_k[1](x^i) \) and \( n_k[2](x^i) \) depend on h–variables and can be defined in explicit form if certain boundary/initial conditions are imposed. Four dimensional solutions
can be generated by eliminating the dependence on variable $x^1$. There are certain classes of constraints defining foliated structures when with respect to a preferred system of reference, $\Gamma^\gamma_{\alpha\beta} = \tilde{\Gamma}^\gamma_{\alpha\beta}$ defining a subclass of Ricci flows with integrable ‘anisotropic’ structure, see details in Refs. [19, 15, 16, 18].

Let us consider an initial d–metric ($\mathbf{13}$), $^0g_{\alpha\beta} = [^0g_{ij} = g_{ij}(u, 0), ^0g_{ab} = g_{ab}(u, 0)]$, with constant scalar curvatures $^hR$ and $^vR$ for $^h\tilde{R} = g^{ij}\tilde{R}_{ij}$ and $^v\tilde{R} = g^{ab}\tilde{R}_{ab}$, see formula (A13) in Ref. [10], written for the $d$–connection $\tilde{D}$. We suppose that this holds for some $^v\lambda_0 = ^v\lambda(x^k) - \lambda = \text{const} > 0$ and $^h\lambda_0 = ^h\lambda(x^k, y^c) - \lambda = \text{const} > 0$ in [20] introduced in formulas for coefficients of [21]. For a set of $d$–metrics of this type, $g(\chi)$, the equations (17) and (18) transform into

$$\frac{\partial g^i}{\partial \chi_i} = -2^h\lambda_0g^i_\chi - \left[ g_4\frac{\partial (w^i_\chi)}{\partial \chi} + g_5\frac{\partial (n^i_\chi)}{\partial \chi} \right] , \text{ for } i = 2, 3; \quad (23)$$

$$\frac{\partial g_a}{\partial \chi} = -2^v\lambda_0g_a, \text{ for } a = 4, 5, \quad (24)$$

where, for simplicity, we put all $\epsilon_\alpha = 1$. Parametrizing

$$g^{ij}(u, \chi) = ^h\varrho^2(\chi)^0g^{ij}_\chi \text{ and } g_{ab}(u, \chi) = ^v\varrho^2(\chi)^0g_{ab}$$

and considering a fixed nonholonomic structure for all $\chi$, when $w^i_\chi(u, \chi) = w^i_\chi(u, 0)$ and $n^i_\chi(u, \chi) = n^i_\chi(u, 0)$, the solutions of (23) and (24) are respectively defined by two evolution factors

$$^h\varrho^2(\chi) = 1 - 2^h\lambda_0\chi \text{ and } ^v\varrho^2(\chi) = 1 - 2^v\lambda_0\chi.$$  

There are two shrinking points, one for the $h$–metric, $\chi \to 1/2^h\lambda_0$ when the scalar $h$–curvature $^h\tilde{R}$ becomes infinite like $1/(1/2^h\lambda_0 - \chi)$, and another one for the $v$–metric, $\chi \to 1/2^v\lambda_0$ when the scalar $h$–curvature $^v\tilde{R}$ becomes infinite like $1/(1/2^v\lambda_0 - \chi)$. Contrary, if the initial $d$–metric is with negative scalar curvatures, the components will expand homothetically for all times and the curvature will fall back to zero like $-1/\chi$. For integrable structures, for instance, for $w^i_\chi = n^i_\chi = 0$, and $^h\lambda_0 = ^v\lambda_0$, we get typical solutions for (holonomic) Ricci flows of Einstein spaces. There are more complex scenarios for nonholonomic Ricci flows. One can be considered situations when, for instance, $^h\lambda_0 > 0$ but $^v\lambda_0 < 0$, or, inversely, $^h\lambda_0 < 0$ but $^v\lambda_0 > 0$. Various classes of nonholonomic Ricci flow solutions with variable on $\chi$ components of $N$–connection (for instance, on three and four dimensional pp–wave and/or solitonic, or Taub NUT backgounds) were constructed and analyzed in Refs. [11,12,13].
2.2.2 N–anholonomic Ricci solitons

Let us consider how the concept of Ricci soliton \[20\] can be extended for the connection \(\tilde{\nabla} = (h\tilde{\nabla}, v\tilde{\nabla})\). We call a steady \(h\)–soliton (\(v\)–soliton) a solution to a nonholonomic horizontal (vertical) evolution moving under a one–parameter \(\chi\) subgroup of the symmetry group of the equation. A solution of the equation \([17]\) (or \([18]\)) parametrized by the group of N–adapted diffeomorphisms \(h^\varphi_\chi\) (or \(v^\varphi_\chi\)) is called a steady Ricci h–soliton (v–soliton). We can introduce the concept of Ricci distinguished soliton (d–soliton) as a N–adapted pair of a h–soliton and v–soliton. In the simplest case, the N–connection coefficients do not evolve on \(\chi\), \(N = 0\), but only \(g_{ij}(u,\chi)\) and \(g_{ab}(u,\chi)\) satisfy some simplified evolution equations (on necessity, in our further constructions we shall analyze solutions with \(N^a(u,\chi)\)).

Steady gradient Ricci d–soliton:

**Definition 2.2** For a d–vector \(X = (hX, vX)\) generating the d–group, the Ricci d–soliton on \(V\) is given by

\[
g_{ij}(u,\chi) = h^\varphi_\chi g_{ij}(u,0) \quad \text{and} \quad g_{ab}(u,\chi) = v^\varphi_\chi g_{ab}(u,0).
\]

(25)

This Definition implies that the right sides of \([17]\) and \([18]\) are equal respectively to the N–adapted Lie derivative \(L_X g = L_{hX} hg + L_{vX} vg\) of the evolving d–metric \(g_{\phi_N}(\chi) = hg(\chi) \oplus v_N vg(\chi)\). We give a particular important example when the initial d–metric \(g_{\phi_N}(0)\) is a solution of the steady Ricci d–solitonic equations

\[
2\tilde{R}_{ij} + g_{ik}\tilde{D}_j X^k + g_{jk}\tilde{D}_i X^k = 0,
\]
\[
2\tilde{R}_{ab} + g_{ac}\tilde{D}_b X^c + g_{bc}\tilde{D}_a X^c = 0.
\]

For \(X = \tilde{D}\varphi\), i.e. a d–gradient of a function \(\varphi\), we get a steady gradient Ricci d–soliton defined by the equations

\[
\tilde{R}_{ij} + \tilde{D}_i \tilde{D}_j \varphi = 0 \quad \text{and} \quad \tilde{R}_{ab} + \tilde{D}_a \tilde{D}_b \varphi = 0.
\]

(26)

It is obvious that a d–metric satisfying \([26]\) defines a steady gradient Ricci d–soliton \([25]\).
**Homothetically shrinking/expanding Ricci d–solitons:** There are d–metrics which move as N–adapted diffeomorphisms and shrink (or expands) by a $\chi$–dependent factor and solve the equations

\[
2\hat{R}_{ij} + g_{ik}\hat{D}_j X^k + g_{jk}\hat{D}_i X^k - 2 h\lambda(x^k, y^c)g_{ij} = 0,
\]
\[
2\hat{R}_{ab} + g_{ac}\hat{D}_b X^c + g_{bc}\hat{D}_a X^c - 2 v\lambda(x^k)g_{ab} = 0.
\]

We obtain the equations for homothetic gradient Ricci d–solitons,

\[
\hat{R}_{ij} + \hat{D}_i \hat{D}_j \varphi - 2 h\lambda_0 g_{ij} = 0, \tag{27}
\]
\[
\hat{R}_{ab} + \hat{D}_a \hat{D}_b \varphi - 2 v\lambda_0 g_{ab} = 0,
\]

for $X = \hat{D}\varphi$ and two homothetic constants $h\lambda_0$ and $v\lambda_0$. Such d–solitons are characterized by their h- and v–components. For instance, the h–component is shrinking/expanding/steady for $h\lambda_0 >, <, = 0$. One obtains Einstein d–metrics for $X = 0$.

**Proposition 2.1** On a compact N–anholonomic manifold $V$, any gradient Ricci d–soliton with h- and v–components being steady or expanding solutions is necessarily a locally anisotropic Einstein metric.

**Proof.** One should follow for the h- and v–components and the connection $\hat{D} = (h\hat{D}, v\hat{D})$ the arguments presented, for holonomic configurations, in [2] and in the Proof of Proposition 1.1.1 in [6].□

A number of noholonomic solitonic like solutions of the Einstein and Ricci flow equations were constructed in Refs. [15, 16, 18, 19, 21, 11, 12, 13]. Those solutions are with nonolonomic solitonic backgrounds and in the bulk define Einstein spaces for $\hat{D}$, with possible restrictions to $\nabla$. They are different from the above considered Ricci d–solitons which also define extensions of the Einstein d–metrics.

**2.3 Existence and uniqueness of N–anholonomic evolution**

For holonomic Ricci flow equations, the short–time existence and uniqueness of solutions were proved for compact manifolds in Refs. [11 [22] and extended to noncompact ones in Ref. [23]. Similar proofs hold true for N–anholonomic manifolds with the N–connection coefficients completely defined by ”off–diagonal” terms in the metrics of type (6). In general, there are two possibilities to obtain such results for nonholonomic configurations. In the first case, we can follow the idea to ”extract” nonholonomic flows form well defined Riemannian ones in any moment of ”time” $\chi$. In the second case,
we should change $\nabla \to \widehat{\nabla} = (h\widehat{\nabla}, v\widehat{\nabla})$ and perform an additional analysis if the nonholonomically induced torsion does not change substantially the method elaborated for the Levi Civita connection.

Usually, the geometric constructions defined by $\widehat{\nabla}$, with respect to any local frame, contain certain additional finite terms; such terms are contained in similar finite combinations of formulas for the Levi Civita connection. For simplicity, in our further proofs, we shall omit details if one of the mentioned possibilities constructions is possible (sure, in the N–anholonomic cases, the techniques is more cumbersome because we have to work with h– and v–subspaces and additional nonholonomic effects). Usually, we shall sketch the idea of the proof and key points distinguishing the nonholonomic objects.

**Lemma 2.1** Both the evolution equations (10)–(12) and theirs nonholonomic transforms (14)–(16) can be modified in a strictly parabolic system.

**Proof.** The proof for the first system is just that for the Levi Civita connection for a metric parametrized in the form (6), see details in Ref. [6] (proof of Lemma 1.2.1). We can nonholonomically deform such formulas taking into account that on N–anholonomic manifolds the coordinate transforms must be adapted to the splitting (1). For such deformations, we use a d–vector $X_\alpha = g_{\alpha\beta}g^{\gamma\tau}(\hat{\Gamma}_\beta^{\gamma\tau} - \hat{0}_\gamma^\beta_{\tau\gamma})$, where $\hat{0}_\gamma^\beta_{\tau\gamma}$ is the canonical d–connection of the initial d–metric $0g_{\alpha\beta}$. Putting (for simplicity) $\lambda = 0$, with respect to a local coordinate frame for the equations of nontrivial d–metric coefficients, we modify (14)–(16) to

$$
\frac{\partial}{\partial \chi}g_{ij} = 2\left(N_i^a N_j^b \hat{R}_{ab} - \hat{R}_{ij}\right) - g_{cd}(N_i^c N_j^d) + \hat{D}_i X_j + \hat{D}_j X_i,
$$

$$
\frac{\partial}{\partial \chi}g_{ab} = -2\hat{R}_{ab} + \hat{D}_a X_b + \hat{D}_b X_a,
$$

$$
\hat{R}_{ia} = -\hat{D}_i X_a - \hat{D}_a X_i, \quad \hat{R}_{ai} = \hat{D}_i X_a + \hat{D}_a X_i,
$$

$$
g_{\alpha\beta}(u, 0) = 0g_{\alpha\beta}(u). \tag{28}
$$

We can chose such systems of N–adapted coordinates when $\hat{D}_i X_a + \hat{D}_a X_i = 0$. Finally, we got a strictly parabolic system for the coefficients of d–connection when the equations $\hat{R}_{ia} = \hat{R}_{ai} = 0$ can be considered as some constraints defined by ”off–diagonal” h–v–components of the system of vacuum Einstein equations for $\hat{\nabla}$.

The system (28) is strictly parabolic in the evolution part and has a solution for a short time [24].
Using the connection $\hat{\mathbf{D}}$ instead of $\nabla$, and re-writing the initial value problem on $h$– and $v$–components (for the Levi Civita considerations, see considerations related to formulas (1.2.6) in [6]), we prove

**Theorem 2.1** For a compact region $U$ on a $N$–anholonomic manifold $V$ with given $g_{\alpha\beta}(u,0) = 0$ and $\hat{\mathbf{D}}$, there exists a constant $\chi_T > 0$ such that the initial value problem,

$$
\frac{\partial}{\partial \chi} g_{ij} = 2\left(N_i^a N_j^b \hat{R}_{ab} - \hat{R}_{ij}\right) - g_{cd} \frac{\partial}{\partial \chi}(N_i^c N_j^d) \quad \text{and} \quad \frac{\partial}{\partial \chi} g_{ab} = -2 \hat{R}_{ab},
$$

with constraints $\hat{R}_{ia} = 0$ and $\hat{R}_{ai} = 0$, has a unique smooth solution on $U \times [0, \chi_T)$.

**Proof.** It follows from the constructions for the Levi Civita connection $\nabla$, when $g_{\alpha\beta}(u,\chi)$ is defined for any $\chi \in [0, \chi_T)$. This allows us to define $N_i^a(\chi, u)$ and $[g_{ij}(\chi, u), g_{ab}(\chi, u)]$.

It should be noted that there are similar existence and uniqueness results for noncompact manifolds, for holonomic ones see Ref. [23]. They can be redefined for $N$–anholonomic ones but the proofs are more complicated and involve a huge amount of techniques from the theory of partial differential equations and geometry of nonholonomic spaces. We do not provide such constructions in this work.

One holds similar constructions to those summarized in sections 1.3 and 1.4 in Ref. [6] for curvature coefficients, orthonormalized frames and derivative estimates. In both cases, for the connections $\nabla$ and $\hat{\mathbf{D}}$, the same set of terms and products of components of $g_{\alpha\beta}$ and their derivatives are contained in the formulas under examination but for different connections there are different groups of such terms. Because there is a proof that such terms are bounded under evolution for $\nabla$–constructions, it is possible always to show that re-grouping them we get also a bounded value for $\hat{\mathbf{D}}$–constructions. This property follows from the fact that for both linear connections the geometric objects are defined by the coefficients of the metric (6). For simplicity, in this work we omit a formal dubbing in a "$\hat{\mathbf{D}}$–fashion" of formulas given in [6, 7, 8, 9] if the constructions are similar to those for $\nabla$.

We conclude this section with the remark that the Ricci flow equations, both on holonomic and $N$–anholonomic manifolds are heat type equations. The uniqueness of such equations on a complete noncompact manifold is not always held if there are not further restrictions on the growth of the solutions. For $N$–anholonomic configurations, this imposes the conditions that
the curvature and N–connection coefficients must be bounded under evolution. The equations for evolution of curvature on N–anholonomic manifolds is analyzed in section 4.3 of Ref. [10].

3 The Perelman’s Functionals on N–anholonomic Manifolds

Following G. Perelman’s ideas [3], the Ricci flow equations can be derived as gradient flows for some functionals defined by the Levi Cività connection $\nabla$. The functionals are written in the form (we use our system of denotations)

$$F(g, \nabla, f) = \int_V \left( R + |\nabla f|^2 \right) e^{-f} \, dV,$$

(29)

$$W(g, \nabla, f, \tau) = \int_V \left[ \tau ( R + |\nabla f|^2 + f - (n + m) \right] \, dV,$$

where $dV$ is the volume form of $g$, integration is taken over compact $V$ and $R$ is the scalar curvature computed for $\nabla$. For a parameter $\tau > 0$, we have $\int_V \mu dV = 1$ when $\mu = (4\pi\tau)^{-(n+m)/2} e^{-f}$. Following this approach, the Ricci flow is considered as a dynamical system on the space of Riemannian metrics and the functionals $F$ and $W$ are of Lyapunov type. Ricci flat configurations are defined as "fixed" on $\tau$ points of the corresponding dynamical systems.

In Ref. [14], we proved that the functionals (29) can be also re–defined in equivalent form for the canonical $d$–connection, in the case of Lagrange–Finsler spaces. In this section, we show that the constructions can be generalized for arbitrary N–anholonomic manifolds, when the gradient flow is constrained to be adapted to the corresponding N–connection structure.

Claim 3.1 For a set of N–anholonomic manifolds of dimension $n + m$, the Perelman’s functionals for the canonical $d$–connection $\hat{D}$ are defined

$$\hat{F}(g, \hat{D}, \hat{f}) = \int_V \left( \hat{R} + v \hat{R} + \left| \hat{D} \hat{f} \right|^2 \right) e^{-\hat{f}} \, dV,$$

(30)

$$\hat{W}(g, \hat{D}, \hat{f}, \hat{\tau}) = \int_V \left[ \hat{\tau} \left( \hat{R} + v \hat{R} + \left| \hat{D} \hat{f} \right| + \left| v D \hat{f} \right| \right)^2 + \hat{f} - (n + m) \mu \right] dV,$$

(31)
where $dV$ is the volume form of $^tg$; $R$ and $S$ are respectively the $h$- and $v$-components of the curvature scalar of $\hat{D}$ when $^*\hat{\mathbf{R}} = g^{ab} R_{ab} = h\hat{R} + v\hat{R}$, for $\hat{D}_\alpha = (D_i, D_a)$, or $\hat{D}_\alpha = (hD, vD)$ when $|\hat{D}\hat{f}|^2 = |hD\hat{f}|^2 + |vD\hat{f}|^2$, and $\hat{f}$ satisfies $\int_V \hat{\mu} dV = 1$ for $\hat{\mu} = (4\pi\tau)^{-(n+m)/2} e^{-f}$ and $\tau > 0$.

**Proof.** We introduce a new function $\hat{f}$, instead of $f$, in formulas (29) (in general, one can be considered non–explicit relations) when

\[
\left( R + |\nabla f|^2 \right) e^{-f} = \left( h\hat{R} + v\hat{R} + hD\hat{f} + vD\hat{f} \right) e^{-\hat{f}} + \Phi
\]

and re–scale the parameter $\tau \to \hat{\tau}$ to have

\[
\tau \left( R + |\nabla f|^2 \right) + f - n - m = \mu
\]

\[
\hat{\tau} \left( h\hat{R} + v\hat{R} + hD\hat{f} + vD\hat{f} \right) + \hat{f} - n - m = \hat{\mu} + \Phi_1
\]

for some $\Phi$ and $\Phi_1$ for which $\int V \Phi dV = 0$ and $\int V \Phi_1 dV = 0$. We emphasize, that we consider one parameter $\hat{\tau}$ both for the $h$– and $v$–subspaces. In general, we can take a couple of two independent parameters when $\hat{\tau} = (h\tau, v\tau)$. □

### 3.1 N–adapted variations

Elaborating an $N$–adapted variational calculus, we shall consider both variations in the so–called $h$– and $v$–subspaces stated by decompositions (1). For simplicity, we consider the $h$–variation $h\delta g = v_{ij}$, the $v$–variation $v\delta g = v_{ab}$, for a fixed $N$–connection structure in (13), and $h\delta \hat{f} = h\hat{f}$, $v\delta \hat{f} = v\hat{f}$.

A number of important results in Riemannian geometry can be proved by using normal coordinates in a point $u_0$ and its vicinity. Such constructions can be performed on a $N$–anholonomic manifold $V$.

**Proposition 3.1** For any point $u_0 \in V$, there is a system of $N$–adapted coordinates for which $\hat{\Gamma}^\gamma_{\alpha\beta}(u_0) = 0$.

**Proof.** In the system of normal coordinates in $u_0$, for the Levi Civita connection, when $\Gamma^\gamma_{\alpha\beta}(u_0) = 0$, we chose $e_\alpha g_{\beta\gamma} |_{u_0} = 0$. Following formulas similar computations for a d–metric (13), equivalently (5), we get $\hat{\Gamma}^\gamma_{\alpha\beta}(u_0) = 0$. □
We generalize for arbitrary N–anholonomic manifolds the Lemma 3.1 from [14] (considered there for Lagrange–Finsler spaces):

**Lemma 3.1** The first N–adapted variations of (30) are given by

$$\delta \hat{F}(v_{ij}, v_{ab}, h f, v f) = \left(32\right)$$

$$\int_{V} \left\{ \left[ -v^{ij}(\hat{R}_{ij} + \hat{D}_i \hat{D}_j \hat{f}) + \left(\frac{h v}{2} - h f\right) \left(2 h \Delta \hat{f} - |h D \hat{f}|^2\right) + \frac{h \hat{R}}{2} \right] + 
\left[ -v^{ab}(\hat{R}_{ab} + \hat{D}_a \hat{D}_b \hat{f}) + \left(\frac{v v}{2} - v f\right) \left(2 v \Delta \hat{f} - |v D \hat{f}|^2\right) + v \hat{R} \right] \right\} e^{-\hat{f}} dV$$

where $h \Delta = \hat{D}_i \hat{D}_i$ and $v \Delta = \hat{D}_a \hat{D}^a$, for $\hat{\Delta} = h \Delta + v \Delta$, and $h v = g^{ij} v_{ij}, v v = g^{ab} v_{ab}$.

**Proof.** It is an N–adapted calculus similar to that for Perelman’s Lemma in [3] (details of the proof are given, for instance, in [6], Lemma 1.5.2). In N–adapted normal coordinates a point $u_0 \in V$, we have

$$\delta \hat{R}^\alpha_{\beta\gamma\tau} = e_\beta \left( \delta \hat{\Gamma}^\alpha_{\gamma\tau} \right) - e_\gamma \left( \delta \hat{\Gamma}^\alpha_{\beta\tau} \right),$$

where

$$\delta \hat{\Gamma}^\alpha_{\gamma\tau} = \frac{1}{2} g^{\alpha\varphi} \left( \hat{D}_\gamma v_{\tau\varphi} + \hat{D}_\tau v_{\gamma\varphi} - \hat{D}_\varphi v_{\gamma\tau} \right).$$

Contracting indices, we can compute $\delta \hat{R} = \delta \hat{R}^\beta_{\beta\gamma\alpha}$ and

$$\delta \hat{R} = \delta(g^{\beta\gamma} \hat{R}_{\beta\gamma}) = \delta(g^{ij} \hat{R}_{ij} + g^{ab} \hat{R}_{ab})$$

$$\delta = \delta(g^{ij} \hat{R}_{ij}) + \delta(g^{ab} \hat{R}_{ab}) = \delta \left( h \hat{R} \right) + \delta \left( v \hat{R} \right),$$

where

$$\delta \left( h \hat{R} \right) = - h \Delta \left( h v \right) + \hat{D}_i \hat{D}_j v^{ij} - v^{ij} \hat{R}_{ij}$$

and

$$\delta \left( v \hat{R} \right) = - v \Delta \left( v v \right) + \hat{D}_a \hat{D}_b v^{ab} - v^{ab} \hat{R}_{ab}.$$
tensor for \( \hat{D} \)). In this work, we try to keep our constructions on Riemannian spaces, even they are provided with \( N \)–anholoronomic distributions, and avoid to consider generalizations of the so–called Lagrange–Eisenhart, or Finsler–Eisenhart, geometry analyzed, for instance in Chapter 8 of monograph [25] (for nonholonomic Ricci flows, we discuss the problem in [10]). The first \( N \)–adapted variation of the functional (30) is

\[
\delta \hat{F} = \delta \int_V e^{-\hat{f}} \left( h \hat{R} + v \hat{R} + \left| \hat{D} \hat{f} \right|^2 \right) dV =
\]

\[
\int_V \delta \left[ \delta ( h \hat{R})(v_{ij}) + \delta ( v \hat{R})(v_{ab}) + \delta (g^{ij} \hat{D}_i \hat{f} \hat{D}_j \hat{f}) + \delta (g^{ab} \hat{D}_a \hat{f} \hat{D}_b \hat{f}) \right] + [(h \hat{R} + g^{ij} \hat{D}_i \hat{f} \hat{D}_j \hat{f})(-h \delta \hat{f} + \frac{1}{2}g^{ij} v_{ij}) + (v \hat{R} + g^{ab} \hat{D}_a \hat{f} \hat{D}_b \hat{f})]dV
\]

\[
= \int_V e^{-\hat{f}} \left[ -h \Delta (h v) + \hat{D}_i \hat{D}_j v^{ij} - v^{ij} \hat{R}_{ij} - v^{ij} \hat{D}_i \hat{f} \hat{D}_j \hat{f} + 2g^{ij} \hat{D}_i \hat{f} \hat{D}_j \hat{f} \right] dV.
\]

For the \( h \)– and \( v \)–components, one holds respectively the formulas

\[
\int_V \left( \hat{D}_i \hat{D}_j v^{ij} - v^{ij} \hat{D}_i \hat{f} \hat{D}_j \hat{f} \right) e^{-\hat{f}} dV =
\]

\[
\int_V \left( \hat{D}_i \hat{f} \hat{D}_j v^{ij} - v^{ij} \hat{D}_i \hat{f} \hat{D}_j \hat{f} \right) e^{-\hat{f}} dV = \int_V v^{ij} \hat{D}_i \hat{f} \hat{D}_j \hat{f} e^{-\hat{f}} dV,
\]

\[
\int_V g^{ij} \hat{D}_i \hat{f} \hat{D}_j \hat{f} h \hat{f} e^{-\hat{f}} dV = \int_V h \hat{f} \left[ g^{ij} \hat{D}_i \hat{f} \hat{D}_j \hat{f} - h \Delta \hat{f} \right] e^{-\hat{f}} dV,
\]

\[
\int_V h \Delta (h v) e^{-\hat{f}} dV = \int_V g^{ij} \hat{D}_i \hat{f} \hat{D}_j \left( h v \right) e^{-\hat{f}} dV
\]

\[
= \int_V h v \left[ g^{ij} \hat{D}_i \hat{f} \hat{D}_j \hat{f} - h \Delta \hat{f} \right] e^{-\hat{f}} dV
\]

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Putting these formulas into (33) and re-grouping the terms, we get (32) which complete the proof. □

**Definition 3.1** A d–metric $g$ evolving by the (nonholonomic) Ricci flow is called a (nonholonomic) breather if for some $\chi_1 < \chi_2$ and $\alpha > 0$ the metrics $\alpha g(\chi_1)$ and $\alpha g(\chi_2)$ differ only by a diffeomorphism (in the N–anholonomic case, preserving the Whitney sum (1)). The cases $\alpha (=, <, > 1$ define correspondingly the (steady, shrinking) expanding breathers.

The breather properties depend on the type of connections which are used for definition of Ricci flows. For N–anholonomic manifolds, one can be the situation when, for instance, the h–component of metric is steady but the v–component is shrinking.

### 3.2 Evolution equations for N–anholonomic d–metrics

Following a N–adapted variational calculus for $\hat{F}(g, \hat{f})$, see Lemma 3.1, with Laplacian $\hat{\Delta}$ and h– and v–components of the Ricci tensor, $\hat{R}_{ij}$ and $\hat{R}_{ab}$, defined by $\hat{D}$ and considering parameter $\tau(\chi), \partial\tau/\partial\chi = -1$ (for simplicity, we shall not consider the normalized term and put $\lambda = 0$), one holds

**Theorem 3.1** The Ricci flows of d–metrics are characterized by evolution
equations

\[ \frac{\partial g_{ij}}{\partial \chi} = -2\widehat{R}_{ij}, \quad \frac{\partial g_{ab}}{\partial \chi} = -2\widehat{R}_{ab}, \]

\[ \frac{\partial \widehat{f}}{\partial \chi} = -\Delta \widehat{f} + \left| \widehat{D} \widehat{f} \right|^2 - h\widehat{R} - v\widehat{R} \]

and the property that

\[ \frac{\partial}{\partial \chi} \widehat{F}(g(\chi), \widehat{f}(\chi)) = 2 \int_{V} \left[ \widehat{R}_{ij} + \widehat{D}_i \widehat{D}_j \widehat{f}^2 + \left| \widehat{R}_{ab} + \widehat{D}_a \widehat{D}_b \widehat{f} \right|^2 \right] e^{-\widehat{f}} dV \]

and \( \int e^{-\hat{f}} dV \) is constant. The functional \( \widehat{F}(g(\chi), \widehat{f}(\chi)) \) is nondecreasing in time and the monotonicity is strict unless we are on a steady \( d \)-gradient solution.

**Proof.** For Riemannian spaces, a proof was proposed by G. Perelman \[3\] (details of the proof are given for the connection \( \nabla \) in the Proposition 1.5.3 of \[6\]). For Lagrange–Finsler spaces, we changed the status of such statements to a Theorem (see Ref. \[14\], where similar results are proved with respect to coordinate frames using values \( g_{ij}, g_{ab} \) and \( \widehat{R}_{ij}, \widehat{R}_{ab} \)) because for nonholonomic configurations there are not two alternative ways (following heuristic or functional approaches) to define Ricci flow equations in N–adapted form.

Using the formula (32), we have

\[
\frac{\partial}{\partial \chi} \widehat{F} = \int_{V} \left[ 2\widehat{R}_{ij}(\widehat{R}_{ij} + \widehat{D}_i \widehat{D}_j \widehat{f}^2 + \left| \widehat{R}_{ab} + \widehat{D}_a \widehat{D}_b \widehat{f} \right|^2 \right] e^{-\widehat{f}} dV
\]

\[ + \left( h\widehat{R} + v\widehat{R} - \frac{\partial \widehat{f}}{\partial \chi} \right)(-h\Delta \widehat{f} - v\Delta \widehat{f}) \]

\[ + g^{ij} \widehat{D}_i \widehat{f} \widehat{D}_j \widehat{f} + g^{ab} \widehat{D}_a \widehat{f} \widehat{D}_b \widehat{f} - h\widehat{R} - v\widehat{R} \int_{V} e^{-\widehat{f}} dV \]

\[ = \int_{V} \left[ 2g^{\alpha\beta} g^{\beta\gamma} \overline{\widehat{R}}_{\alpha\beta}(\overline{\widehat{R}}_{\alpha\beta} + \widehat{D}_\alpha \widehat{D}_\beta \widehat{f}^2 \right] \right) e^{-\widehat{f}} dV \]

\[ + \left( \overline{\widehat{R}} - \frac{\partial \hat{f}}{\partial \chi} \right)(-\Delta \widehat{f} + \left| \widehat{D} \widehat{f} \right|^2 - \overline{\widehat{R}}) \int_{V} e^{-\widehat{f}} dV. \]
At the next step, we consider formulas

\[
\int \left( \hat{\Delta} \hat{f} - |\hat{D} \hat{f}|^2 \right) (2 \hat{\Delta} \hat{f} - |\hat{D} \hat{f}|^2) e^{-\hat{f}} dV
\]

\[
= \int [g^{\alpha\beta}(\hat{D}_\alpha \hat{f})\hat{D}_\beta(-2\hat{\Delta} \hat{f} + |\hat{D} \hat{f}|^2)] e^{-\hat{f}} dV
\]

\[
= 2 \int (\hat{D}^\alpha \hat{f})[-\hat{D}^\beta(\hat{D}_\alpha \hat{D}_\beta \hat{f}) + \hat{R}_{\alpha\gamma}(\hat{D}^\gamma \hat{f}) + (\hat{D}^\beta \hat{f})(\hat{D}_\alpha \hat{D}_\beta \hat{f})] e^{-\hat{f}} dV
\]

\[
= 2 \int \{ [\hat{D}^\alpha \hat{D}^\beta \hat{f} - (\hat{D}^\alpha \hat{f})(\hat{D}^\beta \hat{f})](\hat{D}_\alpha \hat{D}_\beta \hat{f}) + \hat{R}_{\alpha\gamma}(\hat{D}^\alpha \hat{f})(\hat{D}^\gamma \hat{f})
\]

\[
+ (\hat{D}^\beta \hat{f})(\hat{D}_\alpha \hat{D}_\beta \hat{f})(\hat{D}^\alpha \hat{f}) \} e^{-\hat{f}} dV
\]

\[
= 2 \int [g^{\alpha\beta}\hat{D}_\alpha \hat{D}_\beta \hat{f}]^2 + g^{\alpha\alpha'}g^{\beta\beta'}\hat{R}_{\alpha\beta}(\hat{D}_\alpha \hat{f})(\hat{D}_\beta \hat{f}) e^{-\hat{f}} dV
\]

and (using the contracted second Bianchi identity)

\[
\int \left( \hat{\Delta} \hat{f} - |\hat{D} \hat{f}|^2 \right) \hat{R} e^{-\hat{f}} dV
\]

\[
= - \int g^{\alpha\beta}(\hat{D}_\alpha \hat{f}) \hat{D}_\beta(\hat{R} e^{-\hat{f}}) dV =
\]

\[
2 \int g^{\alpha\alpha'}g^{\beta\beta'}\hat{R}_{\alpha\beta}(\hat{D}_\alpha \hat{D}_\beta \hat{f}) - (\hat{D}_\alpha \hat{f})(\hat{D}_\beta \hat{f})] e^{-\hat{f}} dV.
\]

Putting the formulas into (34), we get

\[
\frac{\partial}{\partial \chi} \hat{F}(g(\chi), \tilde{f}(\chi))
\]

\[
= 2 \int g^{\alpha\alpha'}g^{\beta\beta'}(\hat{R}_{\alpha\beta} + \hat{D}_\alpha \hat{D}_\beta \hat{f})
\]

\[
+ (\hat{D}_\alpha \hat{D}_\beta \hat{f})(\hat{D}_\alpha \hat{D}_\beta \hat{f}) e^{-\hat{f}} dV
\]

\[
= 2 \int g^{\alpha\alpha'}g^{\beta\beta'}(\hat{R}_{\alpha\beta} + \hat{D}_\alpha \hat{D}_\beta \hat{f}) (\hat{R}_{\alpha\beta} + \hat{D}_\alpha \hat{D}_\beta \hat{f}) e^{-\hat{f}} dV
\]

\[
= 2 \int \left[ \hat{R}_{ij} + \hat{D}_i \hat{D}_j \hat{f} \right]^2 + \left| \hat{R}_{ab} + \hat{D}_a \hat{D}_b \hat{f} \right|^2 e^{-\hat{f}} dV.
\]
The final step is to prove that \( \int e^{-\hat{f}} dV = \text{const.} \). In our case, we take the volume element
\[
dV = \sqrt{|\det[g_{\alpha\beta}]|} e^1 e^2 \ldots e^{n+m} = \sqrt{|\det[g_{\alpha\beta}]|} dx^1 \ldots dx^n dy^{n+1} \ldots dy^{n+m}
\]
and compute the N–adapted values
\[
\frac{\partial}{\partial \chi} dV = \frac{\partial}{\partial \chi} \left( \sqrt{|\det[g_{\alpha\beta}]|} e^1 e^2 \ldots e^{n+m} \right)
\]
\[
= \frac{1}{2} \left( \frac{\partial}{\partial \chi} \log |\det[g_{\alpha\beta}]| \right) dV = \frac{1}{2} \left( \frac{\partial}{\partial \chi} \log |\det[g_{ij}] + \det[g_{ab}]| \right) dV
\]
\[
= \frac{1}{2} \left( g^{\alpha\beta} \frac{\partial}{\partial \chi} g_{\alpha\beta} \right) dV = \frac{1}{2} \left( g^{ij} \frac{\partial}{\partial \chi} g_{ij} + g^{ab} \frac{\partial}{\partial \chi} g_{ab} \right) dV
\]
\[
= - s\hat{R} dV = (- h\hat{R} - v\hat{R}) dV.
\]

Hence, we can compute
\[
\frac{\partial}{\partial \chi} (e^{-\hat{f}} dV) = e^{-\hat{f}} \left( - \frac{\partial}{\partial \chi} \hat{f} - s\hat{R} \right) dV
\]
\[
= \left( \hat{\Delta} \hat{f} - |\hat{D}\hat{f}|^2 \right) e^{-\hat{f}} dV = - \hat{\Delta}(e^{-\hat{f}}) dV.
\]

It follows
\[
\frac{\partial}{\partial \chi} \int_V e^{-\hat{f}} dV = - \int_V \hat{\Delta} \left( e^{-\hat{f}} \right) dV = 0.
\]

The proof of theorem is finished. \( \square \)

### 3.3 Properties of associated d–energy

On N–anholonomic manifolds, we define the associated d–energy
\[
\hat{\lambda}(g, \hat{D}) \doteq \inf \{ \hat{\mathcal{F}}(g(\chi), \hat{f}(\chi)) | \hat{f} \in C^\infty(V), \int_V e^{-\hat{f}} dV = 1 \}. \tag{35}
\]

This value contains information on nonholonomic structure on \( V \). It is also possible to introduce the associated energy defined by \( \mathcal{F}(g, \nabla, f) \) from (29) as it was originally considered in Ref. [3],
\[
\lambda(g, \nabla) \doteq \inf \{ \mathcal{F}(g(\chi), f(\chi)) | f \in C^\infty(V), \int_V e^{-f} dV = 1 \}.
\]
Both values $\lambda$ and $\tilde{\lambda}$ are defined by the same sets of metric structures $g(\chi)$ but, respectively, for different sets of linear connections, $\tilde{D}(\chi)$ and $\nabla(\chi)$. One holds also the property that $\lambda$ is invariant under diffeomorphisms but $\tilde{\lambda}$ possesses only $N$–adapted diffeomorphism invariance. In this section, we state the main properties of $\tilde{\lambda}$.

**Proposition 3.2** There are canonical $N$–adapted decompositions, to splitting (31), of the functional $\tilde{\mathcal{F}}$ and associated $d$–energy $\lambda$.

**Proof.** We express the first formula (30) in the form

$$\tilde{\mathcal{F}}(g, \tilde{D}, \tilde{f}) = h \tilde{\mathcal{F}}(g, hD, \tilde{f}) + v \tilde{\mathcal{F}}(g, vD, \tilde{f}),$$  \(36\)

where

$$h \tilde{\mathcal{F}}(g, hD, \tilde{f}) = \int V \left( h \tilde{R} + \left| hD\tilde{f} \right|^2 \right) e^{-\tilde{f}} dV,$$

$$v \tilde{\mathcal{F}}(g, vD, \tilde{f}) = \int V \left( v \tilde{R} + \left| vD\tilde{f} \right|^2 \right) e^{-\tilde{f}} dV.$$

Introducing (36) into (35), we get the formulas, respectively, for $h$–energy,

$$h \tilde{\lambda}(g, hD) \doteq \inf \{ h \tilde{\mathcal{F}}(g, \tilde{f}(\chi)) \mid \tilde{f}(\chi) \in C^\infty(V), \int V e^{-\tilde{f}} dV = 1 \},$$  \(37\)

and $v$–energy,

$$v \tilde{\lambda}(g, vD) \doteq \inf \{ v \tilde{\mathcal{F}}(g, \tilde{f}(\chi)) \mid \tilde{f}(\chi) \in C^\infty(V), \int V e^{-\tilde{f}} dV = 1 \},$$  \(38\)

where

$$\tilde{\lambda} = h \tilde{\lambda} + v \tilde{\lambda}$$

which complete the proof. $\Box$

It should be noted that the functional $\tilde{\mathcal{W}}$ (31) depends linearly on $\tilde{f}$ which does not allow a $N$–adapted decomposition for arbitrary functions similarly to (36). From this Proposition 3.2 one follows

**Corollary 3.1** The $d$–energy (respectively, $h$–energy or $v$–energy) has the property:
• \( \hat{\lambda} \) (respectively, \( h\hat{\lambda} \) or \( v\hat{\lambda} \)) is nondecreasing along the \( N \)-anholonomic Ricci flow and the monotonicity is strict unless we are on a steady distinguished (respectively, horizontal or vertical) gradient soliton;

• a steady distinguished (horizontal or vertical) breather is necessarily a steady distinguished (respectively, horizontal or vertical) gradient solution.

**Proof.** We present the formulas for distinguished values (we get respectively, the \( h \)– or \( v \)–components if the \( v \)– or \( h \)– components are stated to be zero). For \( \hat{u} = e^{-\hat{f}/2} \), we express

\[
\hat{F}(g, \hat{D}, \hat{f}) = \int_V \left( \hat{sR}\hat{u}^2 + 4 |\hat{D}\hat{u}|^2 \right) e^{-\hat{f}} dV,
\]

where

\[
h\hat{F}(g, hD, \hat{f}) = \int_V \left( h\hat{R}\hat{u}^2 + 4 |hD\hat{u}|^2 \right) e^{-\hat{f}} dV,
\]

\[
v\hat{F}(g, vD, \hat{f}) = \int_V \left( v\hat{R}\hat{u}^2 + 4 |vD\hat{u}|^2 \right) e^{-\hat{f}} dV,
\]

subjected to the constraint \( \int_V \hat{u}^2 dV = 1 \) following from \( \int e^{-\hat{f}} dV = 1 \). In this case, we can consider \( \hat{\lambda} \) (respectively, \( h\hat{\lambda} \) or \( v\hat{\lambda} \)) as the first eigenvalue of the operator \( -4\Delta + \hat{sR} \) (respectively, of \( -4h\Delta + h\hat{R} \), or \( -4v\Delta + v\hat{R} \)). We denote by \( \hat{u}_0 > 0 \) the first eigenfunction of this operator when

\[
\left( -4\Delta + \hat{sR} \right) \hat{u}_0 = \hat{\lambda}\hat{u}_0
\]

and \( \hat{f}_0 = -2 \log |\hat{u}_0| \) is a minimizer, \( \hat{\lambda}(g, \hat{D}) = \hat{F}(g, \hat{D}, \hat{f}_0) \), satisfying the equation

\[
-2\hat{\Delta}\hat{f}_0 + |\hat{D}\hat{f}_0|^2 - \hat{sR} = \hat{\lambda}.
\]

It should be noted that we can always solve the backward (in “time” \( \chi \)) heat equation

\[
\frac{\partial \hat{f}}{\partial \chi} = -\hat{\Delta}\hat{f} + |\hat{D}\hat{f}|^2 - h\hat{R} - v\hat{R},
\]

\[
\hat{f}|_{\chi=\chi_0} = \hat{f}_0,
\]

24
to obtain a solution $\hat{f}(\chi)$ for $\chi \leq \chi_0$ constrained to satisfy $\int_V e^{-\hat{f}} dV = 1$. This follows from the fact that the equation can be re-written as a linear equation for $\hat{u}^2$,

$$\frac{\partial \hat{u}^2}{\partial \chi} = -\hat{\Delta}(\hat{u}^2) + \hat{\mathcal{R}} \hat{u}^2$$

which gives the solution for $\hat{f}(\chi)$ when at $\chi = \chi_0$ the infimum $\hat{\lambda}$ is achieved by some $\hat{u}_0$ with $\int_V \hat{u}_0^2 dV = 1$. From Theorem 3.1 we have

$$\hat{\lambda}(g(\chi), \hat{D}(\chi)) \leq \hat{\mathcal{F}}(g(\chi), \hat{D}(\chi), \hat{f}(\chi)) \leq \hat{\lambda}(g(\chi_0), \hat{D}(\chi_0)).$$

Finally, we note that one could be different values $\hat{u}_0$ for the $h$–components and $v$–components if we consider the equations and formulas only on $h$– and, respectively, $v$–subspaces. $\square$

For holonomic configurations, the Corollary 3.1 was proven in Ref. [3] (see also the details of proof of Corollary 1.5.4 in Ref. [6]). In the case of $N$–anholonomic manifolds, the proof is more cumbersome and depends on properties of $h$– and $v$–components of the canonical $d$–connection.

### 3.4 On proofs of $N$–adapted Ricci flow formulas

In sections 3.1, 3.2 and 3.3 we provided detailed proofs of theorems and explained the difference between $N$–adapted geometric constructions with the canonical $d$–connection and the Levi Civita connection (both such linear connections being defined by the same metric structure). Summarizing our proofs and comparing with those outlined, for instance, in Ref. [6] for holonomic flows of (pseudo) Riemannian metrics, we get:

**Conclusion 3.1** One holds the rules:

- Any Ricci flow evolution formula for Riemannian metrics containing the Levi Civita connection $\nabla$ has its analogous in terms of the canonical $d$–connection $\hat{D} = (hD, vD)$ on $N$–anholonomic manifolds:

- A $N$–adapted tensor calculus with symmetric $d$–metrics can be generated from a covariant Levi Civita calculus by following contractions of operators with the (inverse) metric, for instance, in the form

$$g^{\alpha \beta} \hat{R}_{\alpha \gamma} = g^{\alpha \beta} \hat{R}_{\alpha \gamma} \to g^{\alpha \beta} \hat{\mathcal{R}}_{\alpha \gamma} = g^{ij} \hat{R}_{ij} + g^{ab} \hat{R}_{ab},$$
and formal changing of the coordinate (co) bases into $N$–adapted ones,

$$
\partial_\alpha \rightarrow e_\alpha = (e_i = \partial_i - N_i^b \partial_b, e_a = \partial_a),
$$

$$
du^\alpha \rightarrow e^\alpha = (e^i = dx^i, e^a = dy^a + N^a_k dx^k);$$

this allows to preserve a formal similarity between the formulas on Riemannian manifolds and their analogous on $N$–anholonomic manifolds, selecting symmetric values $\hat{R}^{ij}$ and $\hat{R}^{ab}$ even the connection $\hat{\nabla}$ is with nontrivial torsion and $\hat{\nabla}_{\beta\gamma} \neq \hat{\nabla}_{\gamma\beta}$.

- We get a formal dubbing on $h$– and $v$–subspaces of geometric objects on $N$–anholonomic manifolds but the $h$– and $v$–analogous are computed by different formulas and components of the $d$–metric and canonical $d$–connection. The $h$– and $v$–components are related by nonholonomic constraints and may result in different physical effects.

In our further considerations we shall omit detailed proofs if they can be obtained following the rules from Conclusion 3.1.

The $d$–energy $\tilde{\lambda} = h\tilde{\lambda} + v\tilde{\lambda}$ allowed us to define the properties of steady distinguished (respectively, into horizontal or vertical) gradient solutions. In order to consider expanding configurations, one introduces a scale invariant value

$$
\tilde{\lambda}(g, \hat{\nabla}) = \tilde{\lambda}(g, \hat{\nabla}) \ Vol(g_{\alpha\beta}),
$$

where $Vol(g_{\alpha\beta})$ is the volume of a compact $V$ defined with respect to $g_{\alpha\beta}$ which is the same both for the constructions with the Levi Civita connection and the canonical $d$–connection.

**Corollary 3.2** The scale invariant (si) $d$–energy $\tilde{\lambda} = h\tilde{\lambda} + v\tilde{\lambda}$ (respectively, hsi–energy, $h\tilde{\lambda}$, or vsi–energy, $v\tilde{\lambda}$) has the property:

- $\tilde{\lambda}$ (respectively, $h\tilde{\lambda}$ or $v\tilde{\lambda}$) is nondecreasing along the $N$–anholonomic Ricci flow whenever it is non–positive: the monotonicity is strict unless we are on a expanding distinguished (respectively, horizontal or vertical) gradient soliton;

- an expanding breather is necessarily an expanding $d$–gradient (respectively, $h$–gradient or $v$–gradient) soliton.

**Exercise 3.1** For holonomic configurations, the Corollary 3.2 transforms into a similar one for the Leivi Civita connection, see [3]. We suggest the
reader to perform the proof for \( N \)-anholonomic manifolds following the details given in the proof of Corollary 1.5.5 in Ref. [6] but applying the rules from Conclusion [3, 4].

The second points of Corollaries 3.1 and 3.2 state that all compact steady or expanding Ricci \( d \)-solitons are \( d \)-gradient ones and such properties should be analyzed separately on \( h \)- and \( v \)-subspaces (see section 2.2.2 on \( d \)-solitons derived for noholonomic Ricci flows). This results in a \( N \)-anholonomic version of Perelman’s conclusion about Einstein metrics and Ricci flows:

**Proposition 3.3** On a compact \( N \)-anholonomic manifold, a steady or expanding breather is necessary an Einstein \( d \)-metric satisfying the Einstein equations for the connection \( \hat{\nabla} = (hD, vD) \) with, in general, anisotropically polarized cosmological constant.

Various types of exact solutions of the nonholonomic Einstein and Ricci flow equations with anisotropically polarized cosmological constants were constructed and analyzed in Refs. [15, 16, 12, 13, 19].

Finally, we note that in order to handle with shrinking solutions, it is convenient to use the second Perelman’s functional \( \hat{\mathcal{W}} \), see second formula in (29) and its \( N \)-anholonomic version \( \hat{\mathcal{W}} \) (31). There are more fundamental consequences from such functionals which we shall analyze in the next section.

## 4 Statistical Analogy for Nonholonomic Ricci flows

Grisha Perelman showed that the functional \( \mathcal{W} \) is in a sense analogous to minus entropy [3]. We argue that this property holds true for nonholonomic Ricci flows which provides statistical models for nonholonomic geometries, in particular, for regular Lagrange (Finsler) systems. The aim of this section is to develop the constructions from sections 4 and in [14] to general \( N \)-anholonomic spaces provided with canonical \( d \)-connection structure.

### 4.1 Properties of \( N \)-anholonomic entropy

For any positive numbers \( h_a \) and \( v_a \), \( \tilde{a} = h_a + v_a \), and \( N \)-adapted diffeomorphisms on \( V \), denoted \( \tilde{\varphi} = (h \varphi, v \varphi) \), we have

\[
\hat{\mathcal{W}}(\, h_a h \varphi^* g_{ij}, v_a v \varphi^* g_{ab}, \tilde{\varphi}^* \hat{\nabla}, \tilde{\varphi}^* \hat{f}, \tilde{\alpha}, \tilde{\tau} \) = \hat{\mathcal{W}}(g, \hat{\nabla}, \hat{f}, \hat{\tau})
\]
which mean that the functional $\widehat{W}$ is invariant under $N$–adapted parabolic scaling, i.e. under respective scaling of $\widehat{\tau}$ and $g_{\alpha\beta} = (g_{ij}, g_{ab})$. For simplicity, we can restrict our considerations to evolutions defined by $d$–metric coefficients $g_{\alpha\beta}(\widehat{\tau})$ with not depending on $\widehat{\tau}$ values.

In a similar form to Lemma 3.1, we get the following first $N$–adapted variation formula for $\widehat{W}$:

**Lemma 4.1** The first $N$–adapted variations of (31) are given by
\[
\delta \widehat{W}(v_{ij}, v_{ab}, h f, v f, \widehat{\tau}) = \int_V \left\{ \widehat{\tau} \left[ -v^{ij}(\widehat{R}_{ij} + \widehat{D}_i \widehat{D}_j \widehat{f} - \frac{g_{ij}}{2\widehat{\tau}}) - v^{ab}(\widehat{R}_{ab} + \widehat{D}_a \widehat{D}_b \widehat{f} - \frac{g_{ab}}{2\widehat{\tau}}) \right] + \frac{1}{2} \widehat{\eta} \left[ h \widehat{R} + 2 h \Delta \widehat{f} - h D_i \widehat{D}^i \widehat{f} - h f - n - 1 \right] \right\} (4\pi \widehat{\tau})^{-\frac{n+m}{2}} e^{-\widehat{f}} dV,
\]

where $\widehat{\eta} = \delta \widehat{\tau}$.

**Proof.** It is similar to the proof of Lemma 1.5.7 presented in [6] but $N$–adapted following the rules stated in Conclusion 3.1. □

For the functional $\widehat{W}$, one holds a result which is analogous to Theorem 3.1

**Theorem 4.1** If a $d$–metric $g(\chi)$ and functions $\widehat{f}(\chi)$ and $\widehat{\tau}(\chi)$ evolve according the system of equations
\[
\frac{\partial g_{ij}}{\partial \chi} = -2\widehat{R}_{ij}, \quad \frac{\partial g_{ab}}{\partial \chi} = -2\widehat{R}_{ab},
\]
\[
\frac{\partial \widehat{f}}{\partial \chi} = -\widehat{\Delta} \widehat{f} + \left| \widehat{D} \widehat{f} \right|^2 - h \widehat{R} - v \widehat{R} + \frac{n + m}{\widehat{\tau}},
\]
\[
\frac{\partial \widehat{\tau}}{\partial \chi} = -1
\]

and the property that
\[
\frac{\partial}{\partial \chi} \widehat{W}(g(\chi), \widehat{f}(\chi), \widehat{\tau}(\chi)) = 2 \int_V \widehat{\tau} \left[ \widehat{R}_{ij} + D_i D_j \widehat{f} - \frac{1}{2\widehat{\tau}} g_{ij} \right]^2 + \left| \widehat{R}_{ab} + D_a D_b \widehat{f} - \frac{1}{2\widehat{\tau}} g_{ab} \right|^2 (4\pi \widehat{\tau})^{-\frac{n+m}{2}} e^{-\widehat{f}} dV.
\]
and \( \int \frac{(4\pi)^{-(n+m)/2}}{V} e^{-\tilde{f}} dV \) is constant. The functional \( \hat{W} \) is \( h- \) (\( v- \)) nondecreasing in time and the monotonicity is strict unless we are on a shrinking \( h- \) (\( v- \)) gradient soliton. This functional is \( N- \) adapted nondecreasing if it is both \( h- \) and \( v- \) nondecreasing.

**Proof.** The statements and proof consist a \( N- \) adapted modification of Proposition 1.5.8 in [6] containing the details of the original result from [3]. For such computations, one has to apply the rules stated in Conclusion 3.1. □

In should be noted that a similar Theorem was formulated for Ricci flows of Lagrange–Finsler spaces [14] (see there Theorem 4.2), where the evolution equations were written with respect to coordinate frames. In this work, for Theorem 4.1 the evolution equations are written with respect to \( N- \) adapted frames. If the \( N- \) connection structure is fixed in "time" \( \chi \), or \( \hat{\tau} \), we do not have to consider evolution equations for the \( N- \) anholonomic frame structure. For more general cases, the evolutions of preferred \( N- \) adapted frames (a proof for coordinate frames is given in Ref. [10]; in \( N- \) adapted form, we have to follow the rules from Conclusion 3.1):

**Corollary 4.1** The evolution, for all time \( \tau \in [0, \tau_0) \), of preferred frames on a \( N- \) anholonomic manifold

\[
\mathbf{e}_\alpha(\tau) = \mathbf{e}_\alpha^\beta(\tau, u) \partial_{\hat{\alpha}}
\]

is defined by the coefficients

\[
\begin{align*}
\mathbf{e}_\alpha^\beta(\tau, u) &= \begin{bmatrix}
e_i^j(\tau, u) & N_i^j(\tau, u) e_\alpha^b(\tau, u) \\
0 & e_\alpha^a(\tau, u)
\end{bmatrix}, \\
\mathbf{e}_\alpha^\alpha(\tau, u) &= \begin{bmatrix}
e_i^i = \delta_i^i & e_i^b = -N_i^b(\tau, u) & e_\alpha^a = \delta_\alpha^a \\
e_i^a = 0 & e_\alpha^a = \delta_\alpha^a
\end{bmatrix}
\end{align*}
\]

with

\[
g_{ij}(\tau) = e_i^j(\tau, u) e_j^i(\tau, u) \eta_{ij} \text{ and } g_{ab}(\tau) = e_a^\alpha(\tau, u) e_b^\beta(\tau, u) \eta_{ab},
\]

where \( \eta_{ij} = \text{diag}[\pm 1, \ldots, \pm 1] \) and \( \eta_{ab} = \text{diag}[\pm 1, \ldots, \pm 1] \) establish the signature of \( g_{\alpha\beta}(u) \), is given by equations

\[
\frac{\partial}{\partial \tau} e_\alpha^\alpha = g^{\alpha\beta} \hat{R}_{\beta\gamma} e_\gamma^\alpha \tag{39}
\]

if we prescribe that the geometric constructions are derived by the canonical \( \tilde{d}- \) connection.
It should be noted that $g^{\alpha\beta} \tilde{R}_{\beta\gamma} = g^{ij} \tilde{R}_{ij} + g^{ab} \tilde{R}_{ab}$ in (39) selects for evolution only the symmetric components of the Ricci $d$–tensor for the canonical $d$–connection. This property was not stated in a similar Corollary 4.1 in Ref. [10].

4.2 Thermodynamic values for N–anholonomic Ricci flows

We follow the section 5 in [3] and prove that certain statistical analogy can be proposed for N–anholonomic manifolds (we generalize the results for Ricci flows of Lagrange–Finsler spaces [14]).

The partition function $Z = \int \exp(-\beta E) d\omega(E)$ for the canonical ensemble at temperature $\beta^{-1}$ is defined by the measure taken to be the density of states $\omega(E)$. The thermodynamical values are computed in the form: the average energy, $<E> = -\partial \log Z/\partial \beta$, the entropy $S = \beta <E> + \log Z$ and the fluctuation $\sigma = <(E-<E>)^2> = \partial^2 \log Z/\partial \beta^2$.

Let us consider a set of $d$–metrics $g(\tilde{\tau})$, $N$–connections $N^a_i(\tilde{\tau})$ and related canonical $d$–connections and $\tilde{D}(\tilde{\tau})$ subjected to the conditions of Theorem 4.1. One holds

**Theorem 4.2** Any family of N–anholonomic geometries satisfying the evolution equations for the canonical $d$–connection is characterized by thermodynamic values

\[
<\tilde{E}> = -\frac{\tilde{\tau}^2}{\mu} \int_V \left( h\tilde{R} + v\tilde{R} + |h\tilde{D}\tilde{f}|^2 + |v\tilde{D}\tilde{f}|^2 - \frac{n+m}{2\tilde{\tau}} \right) \mu dV,
\]

\[
\tilde{S} = -\int_V \left[ \tilde{\tau} \left( h\tilde{R} + v\tilde{R} + |h\tilde{D}\tilde{f}|^2 + |v\tilde{D}\tilde{f}|^2 \right) + \tilde{f} - n - m \right] \mu dV,
\]

\[
\tilde{\sigma} = 2 \frac{\tilde{\tau}^4}{\mu} \int_V \left[ \tilde{R}_{ij} + \tilde{D}_i \tilde{D}_j \tilde{f} - \frac{1}{2\tilde{\tau}} |g_{ij}|^2 + |\tilde{R}_{ab} + \tilde{D}_a \tilde{D}_b \tilde{f} - \frac{1}{2\tilde{\tau}} |g_{ab}|^2 \right] \mu dV.
\]

**Proof.** It follows from a straightforward computation for

\[
\tilde{Z} = \exp \left\{ \int_V \left[ -\tilde{f} + \frac{n+m}{2} \right] \tilde{\mu} dV \right\}.
\]

□

We note that similar values $<\tilde{E}>$, $\tilde{S}$ and $\tilde{\sigma}$ can computed for the Levi Civita connections $\nabla(\tilde{\tau})$ also defined for the metrics $g(\tilde{\tau})$, see functionals (29).
Corollary 4.2 A $N$–anholonomic geometry defined by the canonical d–connection $\hat{D}$ is thermodynamically more (less, equivalent) convenient than a similar one defined by the Levi Civita connection $\nabla$ if $\hat{S} < S$ ($\hat{S} > S, \hat{S} = S$).

Following this Corollary, we conclude that such models are positively equivalent for integrable $N$–anholonomic structures with vanishing distortion tensor. There are necessary explicit computations of the thermodynamical values for different classes of exact solutions of nonholonomic Ricci flow equations [11, 12, 13] or of the Einstein equations with nonholonomic/ noncommutative/ algebroid variables [15, 16, 18, 19] in order to conclude which configurations are physically more convenient for $N$–anholonomic or (pseudo) Riemannian configurations. A number of exact solutions constructed in the cited works can be restricted to foliated configurations when the Ricci tensor of the canonical d–connection is equal to the Ricci tensor for the Levi Civita connection even the mentioned linear connections are different. From viewpoint of observable classical effects such spaces are equivalent, but thermodynamically the foliated structure can be with lower/higher energy and entropy because of terms $|hD\tilde{f}|^2$, $|hD\hat{f}|^2$ and $\hat{D}_i\hat{D}_j\tilde{f}$, $\hat{D}_a\hat{D}_b\tilde{f}$ which provide different contributions if to compare to similar terms defined by the Levi Civita connection.

5 Applications of Ricci Flow Theory in Einstein Gravity and Geometric Mechanics

In this section, there are provided two examples: 1) we construct a class of exact solutions defining constrained Ricci flows of solitonic pp–waves in general relativity and 2) show how a statistical model and an effective thermodynamics can be provided for Ricci flows in geometric mechanics and analogous gravity.

5.1 Nonholonomic Ricci flow evolution of solitonic pp–waves and Einstein gravity

Let us consider a four dimensional (pseudo) Riemannian metric imbedded trivial into a five dimensional (5d) spacetime of signature ($\epsilon_1 = \pm, -, -, -,$, +)

$$\delta s_{[3]}^2 = \epsilon_1 \, d\zeta^2 - dx^2 - dy^2 - 2\kappa(x, y, p) \, dp^2 + dv^2/8\kappa(x, y, p), \quad (40)$$
where the local coordinates are labelled $x^1 = \kappa$, $x^2 = x$, $x^3 = y$, $x^4 = p$, $x^5 = v$, with $\kappa$ being the extra dimension coordinate, and the nontrivial metric coefficients parametrized

$$
\tilde{g}_1 = \epsilon_1 = \pm 1, \quad \tilde{g}_2 = -1, \quad \tilde{g}_3 = -1,
$$
$$
\tilde{h}_4 = -2\kappa(x, y, p), \quad \tilde{h}_5 = 1/8 \kappa(x, y, p).
$$

The metric (40) defines a trivial 5d extension of the vacuum solution of the Einstein equation defining pp–waves [36] for any $\kappa(x, y, p)$ solving

$$
\kappa_{xx} + \kappa_{yy} = 0,
$$

with $p = z + t$ and $v = z - t$, where $(x, y, z)$ are usual Cartesian coordinates and $t$ is the time like coordinates. The simplest explicit examples of such solutions are

$$
\kappa = (x^2 - y^2) \sin p,
$$

defining a plane monochromatic wave, or

$$
\kappa = \frac{xy}{(x^2 + y^2)^2 \exp[p_0^2 - p^2]}, \quad \text{for } |p| < p_0;
$$
$$
\kappa = 0, \quad \text{for } |p| \geq p_0,
$$

for a wave packet travelling with unit velocity in the negative $z$ direction.

A special interest for pp–waves in general relativity is related to the fact that any solution in this theory can be approximated by a pp–wave in vicinity of horizons. Such solutions can be generalized by introducing nonlinear interactions with solitonic waves [37, 38, 39, 40, 41] and nonzero sources with nonhomogeneous cosmological constant induced by an ansatz for the antisymmetric tensor fields of third rank. A very important property of such nonlinear wave solutions is that they possess nontrivial limits defining new classes of generic off–diagonal vacuum Einstein spacetimes and can be generalized for Ricci flows induced by evolutions of N–connections.

We construct a new class of generic off–diagonal solutions by considering an ansatz of type (21) when some coefficients depend on Ricci flow parameter $\chi$,

$$
\delta s^2_{[5]} = \epsilon_1 \, d\kappa^2 - e^{\psi(x, y)} \left( dx^2 + dy^2 \right) - 2\kappa(x, y, p) \eta_4(x, y, p, \chi) \delta p^2 + \frac{\eta_5(x, y, p, \chi)}{8\kappa(x, y, p)} \delta v^2,
$$
$$
\delta p = dp + w_2(x, y, p)dx + w_3(x, y, p)dy,
$$
$$
\delta v = dv + n_2(x, y, p)dx + n_3(x, y, p)dy.
$$
For trivial polarizations $\eta_\alpha = 1$ and $w_{2,3} = 0$, $n_{2,3} = 0$, the metric (42) is just the pp-wave solution (40).

Considering an ansatz (42) with $g_2 = -e^{\psi(x^2,x^3)}$ and $g_3 = -e^{\psi(x^2,x^3)}$, we can restrict the solutions of the system (23), (24) and (27) to define Ricci flows solutions with the Levi Civita connection (see formulas (49) in Ref. [42]) if

$$\psi^{\bullet\bullet} + \psi'' = -\lambda$$

$$h_5^* \phi / h_4 h_5 = \lambda,$$

$$w'_2 - w'_3 + w^*_3 w'_2 - w_2 w^*_3 = 0,$$

$$n'_2(\chi) - n^*_3(\chi) = 0,$$

for

$$w_i = \partial_i \phi / \phi^*, \text{ where } \phi = -\ln \left| \sqrt{|h_4 h_5|/|h_5^*|} \right|,$$  \hspace{1cm} (44)

for $\hat{i} = 2,3$, where, for simplicity, we denote $\psi' = \partial \psi / \partial x$, $\psi^* = \partial \psi / \partial y$ and $\eta^* = \partial \eta / \partial p$ etc.

Let us show how the anholonomic frame method can be used for constructing 4d metrics induced by nonlinear pp-waves and solitonic interactions for vanishing sources and the Levi Civita connection. For an ansatz of type (42), we write

$$\eta_5 = 5\kappa b^2$$

and

$$\eta_4 = h_0^2 (b^*)^2 / 2\kappa.$$

A class of solitonic solutions can be generated if $b$ is subjected to the condition that $\eta_5 = \eta(x,y,p)$ are solutions of 3d solitonic equations,

$$\eta^{\bullet\bullet} + \epsilon (\eta' + 6\eta \eta^* + \eta^{***})^* = 0, \quad \epsilon = \pm 1,$$  \hspace{1cm} (45)

or other nonlinear wave configuration, and $\eta_2 = \eta_2 = e^{\psi(x,y,\chi)}$ is a solution in the first equation in (45). We chose a parametrization when

$$b(x,y,p) = \tilde{b}(x,y)q(p)k(p),$$

for any $\tilde{b}(x,y)$ and any pp-wave $\kappa(x,y,p) = \tilde{\kappa}(x,y,k(p)$ (we can take $\tilde{\kappa} = \tilde{\kappa}$), where $q(p) = 4 \tan^{-1}(e^{\pm p})$ is the solution of "one dimensional" solitonic equation

$$q^{**} = \sin q.$$  \hspace{1cm} (46)

In this case,

$$w_2 = \left[ (\ln |qk|)^* \right]^{-1} \partial_x \ln |\tilde{b}|$$

and

$$w_3 = \left[ (\ln |qk|)^* \right]^{-1} \partial_y \ln |\tilde{b}|.$$  \hspace{1cm} (47)
The final step in constructing such Einstein solutions is to choose any two functions $n_{2,3}(x, y)$ satisfying the conditions $n_{2}^{\ast} = n_{3}^{\ast} = 0$ and $n_{2}^{\prime} - n_{3}^{\prime} = 0$ which are necessary for Riemann foliated structures with the Levi Civita connection, see discussion of formulas (42) and (43) in Ref. [42] and conditions (43). This means that in the integrals of type (22) we have to fix the integration functions $n_{2,3}^{[1]} = 0$ and take $n_{2,3}^{[0]}(x, y)$ satisfying $(n_{2}^{[0]})^{\prime} - (n_{3}^{[0]})^{\ast} = 0$.

Now, we generalize the 4d part of ansatz (42) in a form describing normalized Ricci flows of the mentioned type vacuum solutions extended for a prescribed constant $\lambda$ necessary for normalization. Following the geometric methods from [42] (we omit computations and present the final result), we construct a class of 4d metrics

$$
\delta s_{[4d]}^{2} = -e^{|x|y|} (dx^2 + dy^2) - h_{0}^{2} b_{2}(\chi)[(|qk|)k^{2}] \delta p^{2} + \bar{b}_{2}^{2}(\chi)(qk)k^{2} \delta v^{2},
$$

$$
\delta p = dp + [(\ln |qk|)]^{\ast} \partial_{x} \ln \bar{b} \ dx + [(\ln |qk|)]^{\ast} \partial_{y} \ln \bar{b} \ dy,
$$

$$
\delta v = dv + n_{2}^{[0]}(\chi) dx + n_{3}^{[0]}(\chi) dy,
$$

where we introduced a parametric dependence on $\chi$ for

$$
b(x, y, p, \chi) = \bar{b}(x, y)q(p)k(p)b_{r}(\chi), \ n_{2,3}^{[0]}(\chi) = s n_{2,3}(x, y), \ n_{2,3}(\chi),
$$

for any functions such that $(s n_{3})^{\prime} = (s n_{2})^{\ast}$ and

$$
2\lambda = -\bar{b}(qk)^{2}(s n_{2,3}) \frac{d(s n_{2,3})}{d\chi},
$$

in order to solve the equations (23), (24) and (27), for $\lambda = \lambda_{0}$, defining steady homothetic gradient Ricci d–solutions of Einstein d–metrics for $X = 0$.

We emphasize that we constructed various classes of solitonic pp–wave configurations and their Ricci flow evolutions subjected to different type of nonholonomic constraints in Refs. [11, 12, 43], which are different from the flows of metrics of type (48). For simplicity, in this section, we have analyzed only a minimal extension of vacuum Einstein solutions in order to describe nonholonomic flows of the v–components of metrics adapted to the flows of N–connection coefficients $n_{2,3}^{[0]}(\chi)$. Such nonholonomic constraints on metric coefficients define Ricci flows of families of Einstein solutions defined by nonlinear interactions of a 3D soliton and a pp–wave.

5.2 Thermodynamic entropy in geometric mechanics and analogous gravity

In this section, we apply the statistical analogy for nonholonomic Ricci flows formulated in section [4.2] for computing thermodynamical values de-
fined by a regular Lagrange (Finsler) generating function in mechanics and geometric modelling of analogous gravity.

Any regular Lagrange mechanics $L(x, y) = L(x^i, y^a)$ can be geometrized on a nonholonomic manifold $V$, dim $V = 2n$, enabled with a d–metric structure

$$L^g_{ij}(x, y) = \frac{1}{2} \partial^2 L \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j},$$

(50)

with $L^e_i$ computed following formulas (8) for the canonical N–connection structure

$$L^N_a = \frac{\partial G^a}{\partial y^i}, \quad G^i = \frac{1}{4} L^g_{ij} \left( \frac{\partial^2 L}{\partial y^j \partial x^k} y^k - \frac{\partial L}{\partial x^i} \right),$$

see details in Refs. [17, 16, 10, 14]. Here we note that originally the Lagrange geometry was elaborated on the tangent bundle $TM$ of a manifold $M$, i.e. $V = TM$, following the methods of Finsler geometry [25, 29] (Finsler configurations can be obtained in a particular case when $L(x, y) = F^2(x, y)$ for a homogeneous fundamental function $F(x, \lambda y) = |\lambda| F(x, y)$ for any non–vanishing $\lambda \in \mathbb{R}$; for simplicity, we consider here only Lagrange configurations. The Hessian (50) defines the so–called Lagrange quadratic form and the corresponding Sasaki type lift to a d–metric (49) which is a particular case of metric (48). For $L^g$, we can compute the corresponding canonical d–connection $L^D$ and respective curvature, $L^R_{\alpha\beta\gamma\tau}$, and Ricci, $L^R_{\alpha\beta}$, d–tensors. In brief, we can say that a regular Lagrange geometry can be always modelled as a nonholonomic Riemann–Cartan space with canonically induced torsion $L^T$ completely defined by the d–metric, $L^g$, and N–connection, $L^N$, structures. We can define also an equivalent Riemannian geometric model defined by corresponding $[L^\nabla, L^g_{\alpha\beta}]$, where the Levi Civita connection $L^\nabla$ is computed for a generic off–diagonal metric (5) with coefficients (6) computed following re–definition of (49) with respect to a coordinate basis.

One holds true the inverse statements that any (pseudo) Riemannian space of even dimension can be equivalently modelled as a nonholonomic Riemann–Cartan manifold with effective torsion induced by the ”off–diagonal” metric components and corresponding analogous ”mechanical” model of Lagrange geometry with effective Lagrange variables, see discussions from Refs. [26, 27, 28]. We note here that the Einstein gravity can be rewritten equivalently in Lagrange and/or almost Kähler variables which is im-
portant for elaborating deformation quantization models of quantum gravity and analogous theories of gravity defined by data $[L\nabla, Lg_{\alpha\beta}]$ and/or $[L\hat{D}, Lg, LN]$.

Let us suppose that a set of regular mechanical systems with Lagrangians $L(\hat{\tau}, x, y)$ is described by respective d–metrics $Lg(\hat{\tau})$ and N–connections $LN^a(\hat{\tau})$ and related canonical linear connections $L\nabla(\hat{\tau})$ and $L\hat{D}(\hat{\tau})$ subjected to the conditions of Theorem 4.2. We conclude that any Ricci flow for a family of regular Lagrange systems (mechanical ones, or effective, for analogous gravitational interactions) the canonical d–connection is characterized by thermodynamic values

$$<L\hat{E}> = -\hat{\tau}^2 \int_V \left( L\hat{R} + L\hat{S} + \left| h L\hat{D}\hat{f} \right|^2 + \left| v L\hat{D}\hat{f} \right|^2 - \frac{n}{\hat{\tau}} \right) \hat{\mu} \, dV,$$

$$L\hat{S} = -\int_V \left[ \hat{\tau} \left( L\hat{R} + L\hat{S} + \left| h L\hat{D}\hat{f} \right|^2 + \left| v L\hat{D}\hat{f} \right|^2 \right) + \hat{f} - 2n \right] \hat{\mu} \, dV,$$

$$L\hat{\sigma} = 2 \hat{\tau}^4 \int_V \left[ \left| L\hat{R}_{ij} + L D_i L D_j \hat{f} - \frac{1}{2\hat{\tau}} L g_{ij} \right|^2 + | L\hat{R}_{ab} + L D_a L D_b \hat{f} - \frac{1}{2\hat{\tau}} L g_{ab}|^2 \right] \hat{\mu} \, dV.$$

The simplest examples of such mechanical (effective gravitational) families of Lagrangians can be obtained if the constants of the theory (masses, charges, electromagnetic and/or gravitational constants etc) are supposed to run on a real parameter $\hat{\tau}$ (Dirac’s hypothesis). Additionally to field (motion) equations and corresponding symmetries and conservation laws, such models are characterized by effective thermodynamical values of type $<L\hat{E}>, L\hat{S}, L\hat{\sigma},...$ stating not only optimal spacetime topological configurations for the 3d space (which follows from the Poincare hypothesis) but certain effective ”energies”, ”entropies”,... derived from the Perelman’s functionals.

6 Conclusions

In this paper we have developed the formal theory of Ricci flows for N–anholonomic manifolds, i.e. nonholonomic manifolds provided with a nonlinear connection (N–connection) structure. Such manifolds can be effectively considered in any model of gravity with metric and linear connection fields if we impose nonholonomic constraints on the frame structure.
The concept of nonholonomic manifold provides a unified geometric arena for Riemann–Cartan and Finsler–Lagrange geometries. Such developments lead to general expressions for the evolution of geometrical objects under Ricci flows with constraints and when Riemannian configurations transform into generalized Finsler like ones and vice versa.

It is worth remarking that the constructions with the canonical distinguished (d) connection, in abstract form, are very similar to those for the Levi Civita connection. The geometric formalism does not contain those difficulties which are characteristic of nonmetric connections and arbitrary torsion. The bulk of Hamilton’s results seem to have generalizations for N–anholonomic manifolds. This is possible because in our approach a subset of the ”off–diagonal” metric coefficients can be transformed into the coefficients of a N–connection structure. Even such nonholonomic transforms induce nontrivial torsion coefficients for the canonical d–connection, the condition selecting symmetric metrics (semi–Riemannian, Lagrange or Finsler ones....) allows us to preserve a formal similarity to the ‘standard’ Riemannian case.

The Grisha Perelman’s functional approach is discussed for nonholonomic Ricci flow models. A clear distinction is made between the constructions with the Levi Civita and canonical d–connection. We can work equivalently with connections of both type but the second one allows to perform a rigorous calculation and find proofs which are adapted to the N–connection structure. The reason being that we can apply a number of geometric methods formally elaborated in Finsler geometry and geometric mechanics which are very efficient in investigating nonholonomic configurations in modern gravity and field theory.

This framework is applied to the development of a statistical analogy of nonholonomic Ricci flows. We have already tested it for Lagrange–Finsler systems [14] but the constructions seem to work for arbitrary nonholonomic splitting of dimension \( n + m \), when \( n \geq 2 \) and \( m \geq 1 \). Here, we would like to mention that there are alternative approaches to geometric and non–equilibrium thermodynamics, locally anisotropic kinetics and kinetic processes in terms of Riemannian and Finsler like objects on phase and thermodynamic spaces, see reviews of results and bibliography in Refs. [30, 31, 32, 33, 34, 35] and Chapter 6 from [16]. Those models are not related to Ricci flows of geometric objects and do not seem related to the statistical thermodynamics of metrics and connections which can be derived from the holonomic or anholonomic Perelman’s functionals.

We would like to discuss possible connections of the Perelman’s functional approach to the black hole geometry and thermodynamics. We proved
that the Ricci flow statistical analogy holds true for various types of Einstein spaces, Lagrange–Finsler geometries and nonholonomic configurations. In general relativity, it was recently constructed a new class of spherically symmetric solutions of the Einstein equations associated to a delta function point mass source at $r = 0$ and which are different (not diffeomorphic) from the well known Hilbert-Schwarzschild solutions to the static spherically symmetric vacuum solutions of Einstein’s equations. The last variant has a well known black hole thermodynamical interpretation but other classes of solutions can not be considered in the framework of Hawking’s theory.

Finally, we emphasize that the Perelman’s functionals can be used in order to derive thermodynamical expressions for various classes of solutions but the problem of the physical interpretation of such expressions is still an open question. Certain applications of the nonholonomic Ricci flow theory in gravity theories and geometric mechanics were considered in our recent works [11, 12, 13, 14, 21, 42, 43].

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