Application of the discrete quantum–classical system to the information transfer.

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Abstract

Discussion about properties of a transmitter is performed. It gives the possibility to qualify the quality of a transmission. The main idea of this paper is to investigate the influence of detectors onto the information transfer. The detector model is described within the open system theory using the event–enhanced quantum theory. The special attention is paid to the role of the classical part of the coupled quantum–classical system. The theory of quantum–classical detectors is developed and the optimization of the detectors is performed. The measurement device with the 100% efficiency is described. Additionally the detector which efficiency does not depend on the number of distinguishable states is obtained. A new approach to the coding problem is proposed. The coding is based on the measurement events instead of the mathematical features of the quantum theory. Within this approach the optimum coding is proposed. Finally the whole system is examined and it is shown that the amount of quantum states required for a information transfer is about tens.

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1 Introduction

The issue of sending information through a quantum system became very popular lately [1] — [14].

We may set great hopes on this method of the transmission [3] [8] [10] e.g. the transmission is said to be prove again eavesdropping [8], there is a well-known "no-cloning theorem" saying that unknown quantum state cannot be cloned [31] [35], as well as special method wich secure communication between two users [30]; the speed of a transmission could be bigger than in classical systems [3]; to make channels capacity bigger [10]; to lower amount of energy required for generating a signal [10].

This subject has been studied along the following two lines:

a. Investigation of the properties of information channels (i.e. within information theory) [3] [10] [12] [13].

b. Creation and investigation of the quantum systems which may work as a transmitter (theoretically build systems [4] [5] [9] [14] as well as experimental systems [15] — [17].

It should be noticed that most of works are focused on creation and investigation of pure quantum systems, leaving untouched the question of the role of the measurement of the quantum systems. The presence and influence of the classical system onto the quantum system is skipped, even in the articles devoted to the role of the measurement. This quastion is very important, especially when we consider an efficiency of such a detector. (A very important parameter when we think about receiving messages.) Such a approach omit the process of a transmition of the logical value into the quantum state and the inverse process – when the quantum state is changed into the classical state (the logical symbol).

In this work I pay a special attention to the role and property of the detectors being used as a part of a transmitter, which generate and detect signal transmitted through quantum channels.\footnote{As a detector I understand here every device which can measure or change the quantum state.}

\footnote{The role of a detector has an important influence into two important cases: the quantum information transfer and the quantum data storage. The result obtained in this paper can be transferred to the case of storage of information and could give an estimation of the amount of quantum states required by the task.}
The main point of this paper is a qualification of a way of coding, which fully avail properties of a system as a discrete coding and designation of the form of operators which allow to maximize the efficiency of the detector (e.g. the operators allow to achieve the probability of registration of a quantum state equal one).

I think that the main novum of the article is an examining of the classical properties of the quantum – classical system, which may be used as a transmitter, basing on the theory of continuous measurement. Also I’ve proved that the asymptotic efficiency of a detector does not depend on the number of recognizable states. This may give a great possibility to increase the speed of information transfer by the usage of n–state encoding. In the example of transmitter we see that within the chosen method of encoding the illegibility of transmission is not very sensitive to the efficiency of the detector. It is enough to send about 60 quantum states to achieve the confidence level bigger than 55%. It is done with the detectors with the efficiency 90% as well as 45%.

This paper is organized as follows. The section 2 mathematically formulates the basic features of the event–enhanced theory which are necessary for the later investigation. The section 3 is the main section of this article. The part 3.1 analyzes the properties of a system from a practical point of view. The next two parts 3.2 3.3 define and describe the transmitter and properties of codes which match the features of quantum state transmission. The subsections 3.4, 3.5, 3.6 formulate and discuss two possible ways of encoding classical state. The special care is taken of the decoding problem. In both cases of encoding the optimum receiver is designed. Theirs properties and abilities to receive the classical message are investigated in the subsection 3.5.3, 3.6.2 and 3.6.3 (3.5.3 – the first encoding method; 3.6.2, 3.6.3 – the second way of encoding). Finally one of this method is eliminated, because it is too difficult to avoid errors during applying this method. The explanation of this elimination is given in the section 3.5.4. Eventually I designate within the discrete coding the method how to assign the logical value to measurement events. The section 3.6 describes the two state detector, which can be used as a receiver. The designed measuring device has an ability to detect 100% of sended quantum states. The section 3 show also that the efficiency of the one state and two states detector could be the same and the appendix 6 proof that the efficiency of the measuring device does not depend on the number of distinguished states. The section 4 illustrates the developed the-
ory. There is an example of a system which can be used as a transmitter. The system consists of two detectors. The first detector (described in 4.1) performs the non-demoliation measurement and together with a source form the sender. The second detector (part 4.2) illustrates numerically the activity of the system and compares the work of two kinds of detectors (with the efficiency 90% and 45%). Within the model the simplest classical error correction method is investigated and the number of states required for an intelligible transmission is considered. The end of this paper (section 4.4) suggest several possible ways of alteration of the system allowing to improve some of their features.

2 Mathematical background

As it was mentioned in the introduction the most suitable way of describing the influence of a detector onto evolution of a transmitter is the theory of coupled quantum-classical systems which was proposed and developed by prof. Ph. Blanchard and prof. A. Jadczyk [18] [20] [23] [24] [26].

I use this theory on account of:

- the straightforward description of a classical system (usually a measurement device), which permits an internal evolution of the classical system;

- it enable communication of the systems in both directions i.e.:
  - flow of information from quantum to classical system,
  - control of quantum states and processes by classical parameters (as it is done in real experiments).

This presentation is based on the model proposed by prof. Ph. Blanchard and prof. A. Jadczyk in [20]. I restrict this presentation to the model where the classical system is a discrete one.

The main idea is to describe the evolution of the system as given by completely positive semigroups. The system is constructed by coupling the classical and quantum one. The most natural way of doing this is to use a

\[^3\] The interested readers may find a more precise description of the theory in [18] [19] [23].
tensor product of classical and quantum spaces. The evolution is governed by completely positive semigroups rather than a unitary evolution (by coupling classical and quantum system I obtain the system with a dissipation). I assume that the pure states of the quantum system are given by rays in a complex, finite or infinite dimensional Hilbert space $H_q$. The observable algebra of the quantum system is the algebra $A_q = L(H_q)$ of all bounded operators on $H_q$. The statistical states of the quantum system are given by positive, weakly continuous functionals $w$ on $A_q$ with $w(I) = 1$. Let $S_q$ be a convex set of these states. The elements of $S_q$ are positive operators on $H_q$ of trace 1.

Let $X_{cl}$ denote the set of pure states of a classical system. I restrict the model to the case where $X_{cl}$ is a finite set (with $n + 1$ elements). $S_{cl}$ is a set of statistical states of the classical system, which is the space of probability measures on $X_{cl}$. In this case the state $P \in S_{cl}$ are $n + 1$ tuples $P = (p_0, \ldots, p_n)$, where $p_\alpha \geq 0, \sum_\alpha p_\alpha = 1$. The observable algebra of the classical system $A_{cl}$ is the abelian algebra of complex function on $X_{cl}$, i. e. $A_{cl} \cong C^{n+1}$.

The total system has as its algebra:

$$A_{tot} = A_q \otimes A_{cl} = L(H_q) \otimes C^{n+1} \quad (2.1)$$

It is convenient to realize $A_{tot}$ as an algebra of operators on some auxiliary Hilbert space:

$$H_{tot} = H_q \otimes C^{n+1} \quad (2.2)$$

The algebra $A_{tot}$ is then isomorphic to the algebra of block-diagonal matrices $A = diag(a_0, \ldots, a_n)$, whose entries $a_\alpha$ are bounded linear operators on $H_q$.

There are four special operations on this algebra.

- The embedding of the quantum and classical algebras into $A_{tot}$. They are respectively:

  $$i_q : a \in L(H_q) \to a \otimes I = diag(a, \ldots, a) \quad (2.3)$$

  $$i_c : \lambda = (\lambda_0, \ldots, \lambda_n) \to diag(\lambda_0 I, \ldots, \lambda_n I) \quad (2.4)$$

  $\lambda_\alpha \in C$

  so the states of $A_{tot}$ are represented by block-diagonal matrices

  $$\rho = diag(\rho_0, \ldots, \rho_n) \quad (2.5)$$

\[4\] There is a possibility to build the model where the $X_{cl}$ is a infinite set [23].
\( \rho_\alpha \in L(H_q) \) and \( \sum_\alpha \text{Tr}(\rho_\alpha) = 1 \)

For the expectation value of an observable \( A \in A_{\text{tot}} \) in a state \( \Omega \in S_{\text{tot}} \)
I have \( \Omega(A) = \sum_\alpha \text{Tr}(w_\alpha a_\alpha) \). I shall identify states \( \Omega \) with operators \( W \) representing them.

- The projectors, which projects states of \( A_{\text{tot}} \) onto the states of \( A_q \) and \( A_{\text{cl}} \). They are defined as follows:

\[
\begin{align*}
\pi_q(\rho) &= \sum_\alpha \rho_\alpha \quad (2.6) \\
\pi_c(\rho) &= (\text{Tr}(\rho_0), \ldots, \text{Tr}(\rho_n)) \quad (2.7)
\end{align*}
\]

Thus

\[ \text{Tr}(\rho \ i_q(a)) = \text{Tr}(\pi_q(\rho)a) \quad (2.9) \]

and

\[ \text{Tr}(\rho \ i_c(\lambda)) = \sum_\alpha \pi_c(\rho)_\alpha \lambda_\alpha \quad (2.10) \]

If we have states \( P = (p_0, \ldots, p_n) \in S_{\text{cl}} \) and \( w \in S_q \) the state of the joint system may be build in the way:

\[ w \otimes P = \text{diag}(p_0w, \ldots, p_n w) \quad (2.11) \]

This is very useful because it allows us to build an initial state. This state represents the situation when there are no correlations between the states.

As I said the time evolution was given by completely positive semigroups. It is a CP (completely positive) semigroup \( \alpha_t, t \geq 0 \) of CP maps \( \alpha_t \) of the algebra of observables. The time evolution of states is given by the one parameter semigroup of dual maps \( \alpha_t : S_{\text{tot}} \to S_{\text{tot}} \) with:

\[ \alpha_t(\rho)(A) = \rho(\alpha_t(A)) \quad (2.12) \]

It follows directly from the definition that \( \alpha_t \) maps states into states, preserving their positivity and normalisation. Owing to the theorems by Stinespring

\[ ^5 \text{A brief discussions why this mathematical structure is used here can be found in [19], [21], [27], [28]} \]
and Lindblad [21] [22] any norm continuous semigroup of CP maps $\alpha^t$ must be of the form

$$\alpha^t = \exp(tL) \quad (2.13)$$

with

$$L(A) = i[H, A] + \sum_{i=1}^{N} V_i AV_i^* - \frac{1}{2}\{\sum_i V_i V_i^*, A\} \quad (2.14)$$

where

$$\sum_i V_i V_i^* \in A_{tot} \quad (2.15)$$

and

$$V_i AV_i^* \in A_{tot} \text{ whenever } A \in A_{tot} \quad (2.16)$$

$V_i$ will be denominated as a coupling operators, because they define the coupling between classical and quantum system. $H$ is an arbitrary Hermitian operator in $A_{tot}$: $H = H^* \in A_{tot}$. Let me denote $\rho(t) = \alpha_t(\rho)$, so the evolution [2.14] of observables of the system leads to the Liouville evolution equation for states:

$$\dot{\rho}(t) = -i[H, \rho(t)] + \sum_i V_i^* \rho(t)V_i - \frac{1}{2}\{\sum_i V_i V_i^*, \rho(t)\} \quad (2.17)$$

### 3 Optimization of the coupling operators.

#### 3.1 Features of the transmission.

##### 3.1.1 Speed of the transmission.

I define this quantity as follows:

$$\eta = \frac{\text{size of the signal}}{\text{time of the transmission}} \quad (3.1)$$

*Size of the signal* is measured as a number of bits in binary coding. This measure is being used by engineers when they describe classical transmissions (for instance in computer nets).

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6 It is important to observe that the operators $V_i$ do not need to belong to the $A_{tot}$. 7
3.1.2 Intelligibility of the transmission.

This is one of the most important features of the quantum transmission. The basic features, which influence on the intelligibility are:

a. resistance to errors,

b. ability to generate the require signal (for a sender),

c. ability to receive sended signal (for receiver).

This quality may be defined as follows:

\[ \zeta = \frac{\text{input signal} - \text{output signal}}{\text{size of the signal}} \]  \hspace{1cm} (3.2)

Difference between input and output signal is measured as a number of bits which are different in input and output information.

Within this part of discussion I want to point out some important questions, which should be answered when the particular system is examined.

a. What is the probability of occurrence of errors during the transmission?

b. Does this probability depend on the kind of transmitted signal?

c. What is the probability of the proper generation of the quantum state?

d. What is the probability of the proper decoding of the generated quantum state?

3.1.3 Confidence level.

I want to point out here the question about resistance to eavesdropping, what is very important from a practical point of view. In which way the output will change in the presence of the eavesdropping (the increase of noise, what kind of noise and so on).

3.1.4 Resistance to noise and disturbance.

This can be defined as a ratio of the properly received signals to the amount of all signals. Of course the best value is one, because this is the case of the ideal receiver.
3.2 Structure of the transmitter.

The simplest version of the applicable system is:

\[ S \leftrightarrow R \]

sender  channel  receiver

transmitter

**Sender** – the quantum–classical system able to generate quantum state, which may be discerned by receiver. This system should generate \( n \) distinguishable quantum states. (It makes possible to use \( n \) – state encoding.)

**Receiver** – the quantum–classical system, which can distinguish the quantum states generated by sender end matching the logical value.

**Channel** – the quantum system transmitting the quantum signal between sender and receiver.

3.3 Usage of codes.

In the following I consider the case of discrete coding. This is the most natural way of transmitting information thorough the quantum channels. The reason of such a situation is a measurement of a quantum system. As a result of such a measurement we do not obtain unequivocal result, but one value is chosen from the all possible results (with some probability). Consequences of this kind of measurement are very important for choosing the proper way of encoding. Therefore it is extremally difficult to construct the system transmitting information in the other way of encoding – the continuous encoding (sometimes it is called as a analogue encoding). This method would require a very sophisticated error correction system. What more the probability of obtaining the value of the "quantum variable" with infinite precision is zero.

\[ \text{Within this article I leave the influence of the channels onto the action of the transmitter, because I want to concentrate on the property of the sender and receiver. In other words I assume that the channel is ideal.} \]
Of course the solution is very simple – divide the range of the variable into intervals and assign them the logical value, but it means to come back to discrete encoding. In addition, considering the practical requirements of the transmitter it is obvious that the discrete code will satisfy them.

3.4 Transmission of the binary signal.

Accomplishment of the binary code demand to distinguish two different states of the quantum system, which can be understand as a logical value (respectively 0 or 1).

This condition fulfil two kinds of systems:

a. We make use of the following measurement events:
   - no registration of the quantum state – 0,
   - registration of the quantum state – 1.

b. The detector discern quantum states and assign the logical value to the quantum state.

3.5 Investigation of the properties of the system based on the encoding: registration and no registration.

3.5.1 Requirements of the system.

The main problem arise out of the events when the state is properly generated but is not registered. It is very difficult to find out that this kind of the error has been occured. Moreover the probability of this error became greater when the information is longer.

So I have got two following condition:

- the probability of registration of the quantum state should be \( P \cong 1 \),
- the work of the sender should be fully controlled.
3.5.2 Building up the receiver.

Firstly I check the conditions for building up the receiver. The main point of the construction of the receiver is to realize what kind of events can be understood by observer. I have chosen two events: registration and no registration of the quantum state.

It is possible to use two kinds of the detectors: the classical state shows the fact of registration of the quantum state or the classical state shows the number of the registered quantum states, but the last case can be identified with a system of $n$ detectors changing their states one by one as the registration appears, so the first system will be investigated.

3.5.3 Description of the receiver.

The classical state is a probability space. Let $\alpha$ denotes the event and $p_\alpha$ the probability of the event $\alpha$, then from the probability theory:

$$\sum_\alpha p_\alpha = 1 \quad (3.1)$$

In the case of the detector there are two elementary events: registration of the state and the complementary event (no registration of the quantum state).

**Remark**

It should be noticed that the complementary event plays the special role. It is natural to make as a initial state

$$p_0(t = 0) = 1 \quad (3.2)$$

It means that the initial state of the detector is fully known. It is obvious that in the case of more complicated systems the complementary event is only one, so should play a special role.

I will use the notation:

- $p_0$ – the probability of the complementary events,
- $p_1, \ldots, p_n$ – the probability that $1, \ldots, n$ event has occurred.

It follows from the above consideration that the system which distinguishes $n$ different states should be described within $n + 1$ - dimensional probability space.

The quantum space is described as a $k$ - dimensional ($k \geq 1$) Hilbert space.

Summing up the assumption I have:
• 2 – dim classical space,
• $k$ – dim quantum space,

$$\lim_{t \to \infty} P = 1 \quad (3. 3)$$

($P$ – is the probability of registration of generated quantum state.)

The total space.

Following the section [2]:

$A_q$ – algebra of the quantum system,

$A_c$ – algebra of the classical system.

So the algebra of the total system is:

$$A_{tot} = A_q \otimes A_c = L(H_q) \otimes C^2 \quad (3. 4)$$

The states of the system are block – diagonal matrixes:

$$\rho \in S_{tot} \rho = diag(\rho_0, \ldots, \rho_n) \quad (3. 5)$$

where $\rho_0, \ldots, \rho_n \in L(H_q)$

Following the assumption [3. 2]:

$$\sum_{i=0}^{n} Tr \rho_i = 1 \quad (3. 6)$$

$$\forall_i Tr \rho_i \geq 0 \quad (3. 7)$$

The evolution of the states is described by the equation [2. 17]. Because of the assumption about the classical space operators $V_i$ are:

$$V_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \quad (3. 8)$$

The operators $V_i$ should satisfy the conditions [2. 15], [2. 16] and using the form of $V_i$ [3. 8] I obtain

$$V_i V_i^* = \begin{pmatrix} a_i a_i^* + b_i b_i^* & a_i c_i^* + b_i d_i^* \\ c_i a_i^* + d_i b_i^* & c_i c_i^* + d_i d_i^* \end{pmatrix} \quad (3. 9)$$
\[ V_i^* \rho(t) V_i = \begin{pmatrix} a_i^* \rho_0(t) a_i + c_i^* \rho_1(t) c_i & a_i^* \rho_0(t) b_i + c_i^* \rho_1(t) d_i \\ b_i^* \rho_0(t) a_i + d_i^* \rho_1(t) c_i & b_i^* \rho_0(t) b_i + d_i^* \rho_1(t) d_i \end{pmatrix} \]  \tag{3. 10}

and finally:

\[
\begin{cases}
\sum_i a_i c_i^* + b_i d_i^* = 0 \\
\sum_i c_i a_i^* + d_i b_i^* = 0 \\
\sum_i a_i^* \rho_0(t) b_i + c_i^* \rho_1(t) d_i = 0 \\
\sum_i b_i^* \rho_0(t) a_i + d_i^* \rho_1(t) c_i = 0
\end{cases}
\]  \tag{3. 11}

Reducing the conjugate equation:

\[
\begin{cases}
\sum_i a_i c_i^* + b_i d_i^* = 0 \\
\sum_i a_i^* \rho_0(t) a_i + c_i^* \rho_1(t) d_i = 0
\end{cases}
\]  \tag{3. 12}

If we consider the system which is investigated, we see that the result of the action of this system depends on the initial state and with the assumption that the operators \( V_i \) do not depend on the time it is obvious that \( \rho_0 \) and \( \rho_1 \) should be eigenvectors of the operators \( a_i, b_i, c_i, d_i \).

I assume for the simplicity \( a_i, b_i, c_i, d_i \) are mutualy orthogonal (in relation to the parameter \( i \)).

The equation \[3. 12\] takes the form:

\[
\forall i \begin{cases}
 a_i c_i^* + b_i d_i^* = 0 \\
 a_i^* \rho_0(t) a_i + c_i^* \rho_1(t) d_i = 0
\end{cases}
\]  \tag{3. 13}

With the assumption that \[3. 13\] should be fulfilled \( \forall \rho_0(t), \rho_1(t) \) I obtain the condition:

\[ ((a = 0 \lor b = 0) \land (c = 0 \lor d = 0)) \land ((a = 0 \lor c = 0) \land (b = 0 \lor d = 0)) \]  \tag{3. 14}

The condition \[3. 14\] gives the following possibilities:

\[ a = 0, d = 0 \Rightarrow V = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \]  \tag{3. 15}

\[ a = 0, b = 0, c = 0 \Rightarrow V = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \]  \tag{3. 16}

\[ a = 0, c = 0, d = 0 \Rightarrow V = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \]  \tag{3. 17}

\[ b = 0, c = 0 \Rightarrow V = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \]  \tag{3. 18}
Using the properties of the trace:

\[ \rho_0(t) = -i[H, \rho_0(t)] + \sum_i (c_i^* \rho_1(t) c_i) - \frac{i}{2} \sum_i (\rho_0(t) b_i b_i^* + b_i^* b_i \rho_0(t)) \]
\[ \rho_1(t) = -i[H, \rho_1(t)] + \sum_i (b_i^* \rho_0(t) b_i) - \frac{i}{2} \sum_i (\rho_1(t) c_i c_i^* + c_i c_i^* \rho_1(t)) \]

(3. 21)

So the evolution of the classical part of the system is given by (following [2, 7]):

\[ \begin{align*}
\text{Tr} \dot{\rho}_0(t) &= \text{Tr} \{-i[H, \rho_0(t)] + \sum_i (c_i^* \rho_1(t) c_i) - \frac{i}{2} \sum_i (\rho_0(t) b_i b_i^* + b_i^* b_i \rho_0(t)) \} \\
\text{Tr} \dot{\rho}_1(t) &= \text{Tr} \{-i[H, \rho_1(t)] + \sum_i (b_i^* \rho_0(t) b_i) - \frac{i}{2} \sum_i (\rho_1(t) c_i c_i^* + c_i c_i^* \rho_1(t)) \}
\end{align*} \]

(3. 22)

Using the properties of the trace:

\[ \begin{align*}
\text{Tr} \dot{\rho}_0(t) &= -\sum_i \text{Tr} (b_i b_i^* \rho_0(t) - c_i c_i^* \rho_1(t)) \\
\text{Tr} \dot{\rho}_1(t) &= \sum_i \text{Tr} (b_i^* \rho_0(t) b_i - c_i c_i^* \rho_1(t))
\end{align*} \]

(3. 23)

We see here a very important feature of the system:

\[ \text{Tr} \dot{\rho}_0(t) = -\text{Tr} \dot{\rho}_1(t) \]

(3. 24)

Now I consider the range of "i". This problem can be solved in two ways: to use algebraic methods and check if it is possible to find such b and c that: \[ \sum_{i=1}^{N} b_i b_i^* = b b^* \] and \[ \sum_{i=1}^{N} c_i c_i^* = c c^* \]. It is a quite simple task. If we use the assumption about orthogonality of the operators \( b_i, c_i \) it is easy to satisfy the above condition by substitution: \( c = \sum_{i=1}^{N} c_i, b = \sum_{i=1}^{N} b_i \).

The other way of arguing is to see that it is enough to take only one operator \( V \) to fulfil the condition [3, 3]. So the equation [3, 21] take the form:

\[ \begin{align*}
\dot{\rho}_0(t) &= -i[H, \rho_0(t)] + (c^* \rho_1(t) c) - \frac{i}{2} (\rho_0(t) b b^* + b^* b \rho_0(t)) \\
\dot{\rho}_1(t) &= -i[H, \rho_1(t)] + (b^* \rho_0(t) b) - \frac{i}{2} (\rho_1(t) c c^* + c c^* \rho_1(t))
\end{align*} \]

(3. 25)

This six instance we may divide into two groups: with the zeros on the diagonal and zeros on the antidiagonal.

The case of the zeros on the antidiagonal is not interesting because all equation separates into the independent equations.

In the case of the zeros on the diagonal I will investigate the case [3, 15] because [3, 17] and [3, 20] can be understand as a special case of [3, 15].

The equation [2, 17] may be rewritten:

\[ \begin{align*}
\dot{\rho}_0(t) &= -i[H, \rho_0(t)] + \sum_i (c_i^* \rho_1(t) c_i) - \frac{i}{2} \sum_i (\rho_0(t) b_i b_i^* + b_i^* b_i \rho_0(t)) \\
\dot{\rho}_1(t) &= -i[H, \rho_1(t)] + \sum_i (b_i^* \rho_0(t) b_i) - \frac{i}{2} \sum_i (\rho_1(t) c_i c_i^* + c_i c_i^* \rho_1(t))
\end{align*} \]

(3. 21)
and the equation \(3.23\)

\[
\begin{align*}
T \dot{\rho}_0(t) &= -Tr(cc^*\rho_0(t) - bb^*\rho_1(t)) \\
T \dot{\rho}_1(t) &= -Tr(cc^*\rho_0(t) - bb^*\rho_1(t))
\end{align*}
\] (3.26)

I shall use the notation:

\[
\begin{align*}
p_0(t) &= Tr\rho_0(t) \\
p_1(t) &= Tr\rho_1(t)
\end{align*}
\] (3.27)

so

\[
\begin{align*}
\dot{p}_0(t) &= -Tr(cc^*\rho_0(t) - bb^*\rho_1(t)) \\
\dot{p}_1(t) &= -Tr(cc^*\rho_0(t) - bb^*\rho_1(t))
\end{align*}
\] (3.28)

**Remark**

The evolution of the system (nontrivial) remains until the balance is obtained:

\[Tr cc^*\rho_1(t) = Tr bb^*\rho_0(t)\] (3.29)

I assume that the operators \(b, c\) are the base vector of the quantum Hilbert space. There are two interesting possibility of chosing \(b, c\):

a. they differ in coupling constant,

b. they are mutually orthogonal.

Solution of the case [b] require knowledge of the Hamiltonian of the system, what more except the case of the nonlinear Hamiltonian the balance of the system can not be achieved in a finite time. So it is easier to investigate the system where the operators \(b, c\) are of the form [a].

I shall use the notation:

\[
\begin{align*}
b &= k_1e \\
c &= k_2e
\end{align*}
\] (3.30)

where \(e\) satisfy:

\[
\begin{align*}
Tr e &= 1 \\
e^* &= e = ee
\end{align*}
\] (3.31)

---

8 It does not affect to the possibility of chosing the operators \(b, c\), because we may change the base vector in order to obtain the proper shape of \(b, c\).

The operators \(b, c\) may be chosen to fulfill some requirements e.g. to obtain the maximum of information of quantum states \(33\).
The operator $e$ may act in two ways:

\begin{align*}
e \rho_0(t) &= \rho_0(t) \\
e \rho_1(t) &= \rho_1(t)
\end{align*}

(3. 32)

or

\begin{align*}
e \rho_0(t) &\ne \rho_0(t) \\
e \rho_1(t) &\ne \rho_1(t)
\end{align*}

(3. 33)

Firstly I investigate the case [3. 32].

Using the notation [3. 27] the equations [3. 25] take the form:

\begin{alignat*}{2}
\dot{p}_0(t) &= \quad (k_1^2 p_0(t) - k_2^2 p_1(t)) \\
\dot{p}_1(t) &= k_1^2 p_0(t) - k_2^2 p_1(t)
\end{alignat*}

(3. 34)

Now I obtain the asymptotic form of $p_0, p_1$. The condition for the asymptotic solution $(p_0(\infty), p_1(\infty))$ is:

\begin{align*}
\dot{p}_0 = \dot{p}_1 = 0
\end{align*}

(3. 35)

I use [3. 34], [3. 35] and the assumption [3. 1] to obtain the solutions:

\begin{align*}
p_0(\infty) &= \frac{k_2^2}{k_1^2 + k_2^2} \\
p_1(\infty) &= \frac{k_1^2}{k_1^2 + k_2^2}
\end{align*}

(3. 36)

The maximal value of the efficiency of the detector is obtained when $k_2 = 0$ and $k_1 \neq 0$. In the special case $k_1 = k_2 \neq 0$ $p_1 = \frac{1}{2}$.

The case [3. 33]

I use the decomposition:

\begin{align*}
\rho_0 &= \rho_{0\parallel} + \rho_{0\perp} \\
\rho_1 &= \rho_{1\parallel} + \rho_{1\perp}
\end{align*}

(3. 37)

where

\begin{align*}
e \rho_{0\parallel} &= \rho_{0\parallel} \\
e \rho_{0\perp} &= 0 \\
e \rho_{1\parallel} &= \rho_{1\parallel} \\
e \rho_{1\perp} &= 0
\end{align*}

(3. 38)

Applying [3. 38] to the equations [3. 25], [3. 26]:

\begin{align*}
\begin{cases}
\dot{\rho}_{0\parallel}(t) + \dot{\rho}_{0\perp}(t) = -i[H, \rho_{0\parallel}(t) + \rho_{0\perp}(t)] + k_2^2 e \rho_{0\parallel} + e - \frac{1}{2} (\rho_{0\parallel} e + e \rho_{0\parallel}) \\
\dot{\rho}_{1\parallel}(t) + \dot{\rho}_{1\perp}(t) = -i[H, \rho_{1\parallel}(t) + \rho_{1\perp}(t)] + k_1^2 e \rho_{0\parallel} + e - \frac{1}{2} k_2^2 (\rho_{1\parallel} e + e \rho_{1\parallel})
\end{cases}
\end{align*}

(3. 39)
I assume that the commutator of the Hamiltonian and \(\rho\) does not change the decomposition of \(\rho\).

The equation \([3.39]\) may be rewritten:

\[
\begin{align*}
\dot{\rho}_{0\perp}(t) &= -i[H, \rho_{0\perp}(t)] \\
\dot{\rho}_{1\perp}(t) &= -i[H, \rho_{1\perp}(t)] \\
\dot{\rho}_{0\parallel}(t) &= -i[H, \rho_{0\parallel}(t)] + k_2^2 \rho_{1\parallel} - k_1^2 \rho_{0\parallel} \\
\dot{\rho}_{1\parallel}(t) &= -i[H, \rho_{1\parallel}(t)] + k_1^2 \rho_{0\parallel} - k_2^2 \rho_{1\parallel}
\end{align*}
\]

so

\[
\begin{align*}
\text{Tr}\dot{\rho}_{0\perp}(t) &= 0 \\
\text{Tr}\dot{\rho}_{1\perp}(t) &= 0
\end{align*}
\]

Let me take the notation:

\[
\begin{align*}
b_0(t) &= \dot{\rho}_{0\perp}(t) \\
b_1(t) &= \dot{\rho}_{1\perp}(t) \\
a_0(t) &= \dot{\rho}_{0\parallel}(t) \\
a_1(t) &= \dot{\rho}_{1\parallel}(t)
\end{align*}
\]

The condition \([3.1]\) may be rewritten:

\[
b_0(t) + b_1(t) + a_0(t) + a_1(t) = 1
\]

I obtain the condition from \([3.34]\):

\[
k_1^2 a_0(\infty) = k_2^2 a_1(\infty)
\]

applying \([3.43]\) to \([3.44]\):

\[
a_1(\infty) = \frac{k_1^2}{k_1^2 + k_2^2} (1 - b_0(\infty) - b_1(\infty))
\]

By rewritting the initial condition \([3.2]\) in the notation \([3.42]\) I get:

\[
\begin{align*}
a_0(0) + b_0(0) &= 1 \\
a_1(0) + b_1(0) &= 0
\end{align*}
\]

\(^9\)It is possibile to allow the Hamiltonian to change the decomoposition of \(\rho\) but this change should be negligible in comparison to the influence of the classical part of the system.
Applying properties of density matrix:

\[ a_1(t) \geq 0, \quad b_1 \geq 0 \Rightarrow a_1(0) = b_1(0) = 0. \]

I receive with condition [3.41]:
\[ b_1(t) = \text{const}, \quad b_0(t) = \text{const}, \]

so [3.45] take the form:
\[ a_1(\infty) = \frac{k_1^2}{k_1^2 + k_2^2}(1 - b_0) \quad (3.47) \]

We see that the efficiency of the detector depends on the initial state of the measured quantum system. This is, somehow, obvious, because if the device is not sensitive to some parameter it can not register the system without this property.

### 3.5.4 Time of detection.

In the previous paragraph I have investigated the efficiency of the detector and the resistance to noise. This section is devoted to the time of detection.

In the case of the decomposition [3.32] probabilities are governed by the equations [3.34]. The solution of this equation has the form:

\[
\begin{align*}
p_0(t) &= \frac{k_1^2}{k_1^2 + k_2^2} e^{-(k_1^2 + k_2^2)t} + \frac{k_2^2}{k_1^2 + k_2^2} \\
p_1(t) &= \frac{k_2^2}{k_1^2 + k_2^2} (1 - e^{-(k_1^2 + k_2^2)t})
\end{align*}
\]

(3.48)

If the \( \rho \) is of the form [3.33] the solution is:

\[
\begin{align*}
p_0(t) &= \frac{ak_1^2}{k_1^2 + k_2^2} e^{-(k_1^2 + k_2^2)t} + \frac{bk_1^2}{k_1^2 + k_2^2} + b \\
p_1(t) &= \frac{ak_2^2}{k_1^2 + k_2^2} (1 - e^{-(k_1^2 + k_2^2)t})
\end{align*}
\]

(3.49)

where \( a = a_0(0), \quad b = b_0(0). \)

We see from the solutions [3.48] [3.49] that the efficiency of the detector depends on the time of coupled evolution of the system. We see also that by changing the value of \( k_1, k_2 \) – the coupling constants is possible to change the time required to obtain the particular efficiency of the detector.

As I have already mentioned in the section [3.1.2] the most important feature of the system is the intelligibility. In this method of coding the resistance to errors is not good enough. First of all the ideal efficiency (100%)
is achieved after the infinity time of the measurement and it is possible only when the quantum state is the eigenvalue of the operator \(e\). This means that the contribution of a noise and disturbance during the generation and transmission of the signal is not allowed. Summing up, because of the difficulties: the finite time of the measurement, errors arising during generation and transmission of the signal the efficiency is always lower than 100%. The result of this is that not every signal generated by sender is detected by receiver this means that some of the signal sended as a logical value ”1” may be understand as a ”0”. This kind of errors is very difficult to eliminate and require sophisticated error correction methods, but usage of such methods strongly lower speed of the transmission. What more the probability of occurring this error become bigger when the message is longer. The problem described above is not the only difficulty of the system. The very important aspect of this method is a sequential of the signal i.e. if the ”no registration” signal is observed does it means that one, two or more zeros have been received? We see that this system has serious difficulties and solving them may require hard work to do. In the next section I present the other way of coding, which is not so sensitive for properties of the detector and give us a chance to build up properly working system with high level of confidence.

3.6 Application of the multistate detector to the information transfer.

The multistate detector, in other words the system distinguishing \(n\) quantum states, is a more applicable system than the detector described in the section 3.5.3, because it is comparatively easy to improve its efficiency up to requested level. It can be done just by repeating the generation of the signal. What more the number of repetition required to achieve high confidence level does not have to be very big. It gives hope for high speed of transmission. Of course it will be a small probability of misunderstanding of the signal because of a noise but this aspect may be improved by introducing a proper filter.

3.6.1 Two state detector.

For the sake of simplicity I will investigate the two–state detector, which is sufficient for receiving binary code.
The system consist of:

- the classical space which is the 3 - dim probability space $p(t) = (p_0(t), p_1(t), p_2(t))$ with the initial condition
  
  $$p_0(0) = 1, p_1(0) = 0, p_2(0) = 0; \quad (3.1)$$

- the quantum space, which is a Hilbert space.

Following the section [2] the states of the coupled system are of the form:

$$\rho(t) = \text{diag}(\rho_0(t), \rho_1(t), \rho_2(t)) \quad (3.2)$$

where the initial condition is:

$$\rho(0) = \text{diag}(\rho_0(0), \rho_1(0), \rho_2(0)) \quad (3.3)$$

$\rho_q \in H$ – the initial state of the quantum system. Operators in the equation [2.17] are of the form:

$$V_i = \begin{pmatrix}
  a_{11} & a_{12}^i & a_{13}^i \\
  a_{21} & a_{22}^i & a_{23}^i \\
  a_{31} & a_{32}^i & a_{33}^i
\end{pmatrix} \quad (3.4)$$

The entries of $V_i$ belongs to the $L(H)$ (as it was explained in section [2]).

From the condition [2.15] I obtain the equations:

$$\begin{cases}
  \sum_i a_{11}^i a_{21}^i + a_{12}^i a_{22}^i + a_{13}^i a_{23}^i = 0 \\
  \sum_i a_{11}^i a_{31}^i + a_{12}^i a_{32}^i + a_{13}^i a_{33}^i = 0 \\
  \sum_i a_{21}^i a_{31}^i + a_{22}^i a_{32}^i + a_{23}^i a_{33}^i = 0 \\
  \sum_i a_{11} a_{31}^i + a_{12} a_{32}^i + a_{13} a_{33}^i = 0 \\
  \sum_i a_{21} a_{31}^i + a_{22} a_{32}^i + a_{23} a_{33}^i = 0 \\
  \sum_i a_{31} a_{11}^i + a_{32} a_{12}^i + a_{33} a_{13}^i = 0 \\
  \sum_i a_{31} a_{21}^i + a_{32} a_{22}^i + a_{33} a_{23}^i = 0
\end{cases} \quad (3.5)$$

I obtain by reducing the conjugate equations:

$$\begin{cases}
  \sum_i a_{11}^i a_{21}^i + a_{12} a_{22}^i + a_{13} a_{23}^i = 0 \\
  \sum_i a_{11}^i a_{31}^i + a_{12} a_{32}^i + a_{13} a_{33}^i = 0 \\
  \sum_i a_{21} a_{31}^i + a_{22} a_{32}^i + a_{23} a_{33}^i = 0 \quad (3.6)
\end{cases}$$
Using the condition (3.16) and after the reduction of the conjugate equations I have:

\[
\begin{align*}
\sum_i a_{11}^i \rho_0(t)a_{12}^i + a_{21}^i \rho_1(t)a_{22}^i + a_{31}^i \rho_2(t)a_{32}^i &= 0 \\
\sum_i a_{11}^i \rho_0(t)a_{13}^i + a_{21}^i \rho_1(t)a_{23}^i + a_{31}^i \rho_2(t)a_{33}^i &= 0 \\
\sum_i a_{12}^i \rho_0(t)a_{13}^i + a_{22}^i \rho_1(t)a_{23}^i + a_{32}^i \rho_2(t)a_{33}^i &= 0
\end{align*}
\]

(3.7)

The operators \(V_i\) should satisfy the equations (3.6) and (3.7). Following the same argumentation as in the section 3.5.3, equations (3.6), (3.7) may be rewritten:

\[
\forall_i \begin{cases}
 a_{11}^i a_{21}^i + a_{12}^i a_{22}^i + a_{13}^i a_{23}^i = 0 \\
 a_{11}^i a_{31}^i + a_{12}^i a_{32}^i + a_{13}^i a_{33}^i = 0 \\
 a_{21}^i a_{31}^i + a_{22}^i a_{32}^i + a_{23}^i a_{33}^i = 0 \\
 a_{11}^i \rho_0(t)a_{12}^i + a_{21}^i \rho_1(t)a_{22}^i + a_{31}^i \rho_2(t)a_{32}^i = 0 \\
 a_{11}^i \rho_0(t)a_{13}^i + a_{21}^i \rho_1(t)a_{23}^i + a_{31}^i \rho_2(t)a_{33}^i = 0 \\
 a_{12}^i \rho_0(t)a_{13}^i + a_{22}^i \rho_1(t)a_{23}^i + a_{32}^i \rho_2(t)a_{33}^i = 0
\end{cases}
\]

(3.8)

The equations (3.8) should be satisfied for every \(\rho_0(t), \rho_1(t), \rho_2(t)\). It could happen when:

a. all nondiagonal entries are zeros, but this case is not interesting for applications;

b. the following condition are satisfied:

\[
(a_{31} = 0 \lor a_{23} = 0) \land (a_{21} = 0 \lor a_{23} = 0) \land (a_{12} = 0 \lor a_{13} = 0) \land \\
(a_{12} = 0 \lor a_{32} = 0) \land (a_{23} = 0 \lor a_{13} = 0) \land (a_{13} = 0 \lor a_{23} = 0)
\]

(3.9)

The condition (3.9) gives the eleven possible shapes of the operator \(V\):

\[
\begin{align*}
a_{31} = a_{23} = a_{12} = 0 &\Rightarrow \begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix} \\
a_{32} = a_{21} = a_{13} = 0 &\Rightarrow \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix} \\
a_{32} = a_{21} = a_{12} = 0 &\Rightarrow \begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix}
\end{align*}
\]

(3.10), (3.11), (3.12)
\(a_{32} = a_{23} = a_{13} = 0 \Rightarrow \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \quad (3.13)\)

\(a_{31} = a_{21} = a_{12} = a_{23} = 0 \Rightarrow \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & b & 0 \end{pmatrix} \quad (3.14)\)

\(a_{31} = a_{21} = a_{12} = a_{13} = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & b & 0 \end{pmatrix} \quad (3.15)\)

\(a_{31} = a_{21} = a_{13} = a_{32} = 0 \Rightarrow \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \quad (3.16)\)

\(a_{31} = a_{23} = a_{13} = a_{12} = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \quad (3.17)\)

\(a_{31} = a_{23} = a_{13} = a_{32} = 0 \Rightarrow \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.18)\)

\(a_{32} = a_{21} = a_{12} = a_{23} = 0 \Rightarrow \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \quad (3.19)\)

\(a_{32} = a_{21} = a_{12} = a_{23} = 0 \Rightarrow \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \quad (3.20)\)
3.6.2 Optimization of the operators $V_i$.

Making the allowance for the form of $V_i$ [3. 4] the equation [2. 17] takes the form:

$$
\begin{align*}
\dot{\rho}_0(t) &= -i[H, \rho_0(t)] + \sum_i (a^i_{11} \rho_0(t) a^i_{11} + a^i_{21} \rho_1(t) a^i_{21} + a^i_{31} \rho_2(t) a^i_{31}) - \\
&\frac{1}{2} \{ \sum_i (a^i_{11} + a^i_{12} + a^i_{13}), \rho_0(t) \} \\
\dot{\rho}_1(t) &= -i[H, \rho_1(t)] + \sum_i (a^i_{12} \rho_0(t) a^i_{12} + a^i_{22} \rho_1(t) a^i_{22} + a^i_{32} \rho_2(t) a^i_{32}) - \\
&\frac{1}{2} \{ \sum_i (a^i_{21} + a^i_{22} + a^i_{23}), \rho_1(t) \} \\
\dot{\rho}_2(t) &= -i[H, \rho_2(t)] + \sum_i (a^i_{13} \rho_0(t) a^i_{13} + a^i_{23} \rho_1(t) a^i_{23} + a^i_{33} \rho_2(t) a^i_{33}) - \\
&\frac{1}{2} \{ \sum_i (a^i_{31} + a^i_{32} + a^i_{33}), \rho_2(t) \}
\end{align*}
$$

(3. 21)

and for the classical subsystem:

$$
\begin{align*}
\dot{\rho}_0(t) &= \sum_i Tr(-a^i_{13} a^i_{13} \rho_0(t) + a^i_{21} a^i_{21} \rho_1(t) + a^i_{31} a^i_{31} \rho_2(t)) \\
\dot{\rho}_1(t) &= \sum_i Tr(a^i_{12} a^i_{12} \rho_0(t) - (a^i_{21} + a^i_{23} + a^i_{32}) \rho_1(t) + a^i_{32} a^i_{32} \rho_2(t)) \\
\dot{\rho}_2(t) &= \sum_i Tr(a^i_{13} a^i_{13} \rho_0(t) + a^i_{23} a^i_{23} \rho_1(t) - (a^i_{31} + a^i_{32} + a^i_{33}) \rho_2(t))
\end{align*}
$$

(3. 22)

Taking under consideration the forms of the operators $V$ [3. 10] [3. 20], the equations [3. 22] take the form:

$$
\begin{align*}
\dot{p}_0(t) &= \sum_i Tr(-a^i_{13} a^i_{13} \rho_0(t) + a^i_{21} a^i_{21} \rho_1(t)) \\
\dot{p}_1(t) &= \sum_i Tr(-a^i_{21} a^i_{21} \rho_1(t) + a^i_{32} a^i_{32} \rho_2(t)) \\
\dot{p}_2(t) &= \sum_i Tr(a^i_{13} a^i_{13} \rho_0(t) - a^i_{32} a^i_{32} \rho_2(t))
\end{align*}
$$

(3. 23)

$$
\begin{align*}
\dot{p}_0(t) &= \sum_i Tr(-a^i_{12} a^i_{12} \rho_0(t) + a^i_{31} a^i_{31} \rho_2(t)) \\
\dot{p}_1(t) &= \sum_i Tr(a^i_{12} a^i_{12} \rho_0(t) - a^i_{23} a^i_{23} \rho_1(t)) \\
\dot{p}_2(t) &= \sum_i Tr(a^i_{13} a^i_{13} \rho_0(t) - a^i_{31} a^i_{31} \rho_2(t))
\end{align*}
$$

(3. 24)

In the equations [3. 23], [3. 24] operator $V$ gives the cascade–like connection of the evolution of the probabilities.

$$
\begin{align*}
\dot{p}_0(t) &= \sum_i Tr(-a^i_{13} a^i_{13} \rho_0(t) + a^i_{21} a^i_{21} \rho_1(t) + a^i_{31} a^i_{31} \rho_2(t)) \\
\dot{p}_1(t) &= \sum_i Tr(-a^i_{21} a^i_{21} \rho_1(t)) \\
\dot{p}_2(t) &= \sum_i Tr(a^i_{13} a^i_{13} \rho_0(t) - a^i_{31} a^i_{31} \rho_2(t))
\end{align*}
$$

(3. 25)

$$
\begin{align*}
\dot{p}_0(t) &= \sum_i Tr(-a^i_{12} a^i_{12} \rho_0(t) + a^i_{21} a^i_{21} \rho_1(t)) \\
\dot{p}_1(t) &= \sum_i Tr(a^i_{12} a^i_{12} \rho_0(t) - a^i_{21} a^i_{21} \rho_1(t)) \\
\dot{p}_2(t) &= \sum_i Tr(-a^i_{31} a^i_{31} \rho_2(t))
\end{align*}
$$

(3. 26)
In the above equations (3.25 – 3.27) there is a probability which evolves independently.

\[ \begin{align*}
\dot{p}_0(t) &= \sum_i \text{Tr}(-a_{13}^i a_{13}^{*i} p_0(t)) \\
\dot{p}_1(t) &= \sum_i \text{Tr}(a_{32}^i a_{32}^{*i} p_2(t)) \\
\dot{p}_2(t) &= \sum_i \text{Tr}(a_{13}^i a_{13}^{*i} p_0(t) - a_{32}^i a_{32}^{*i} p_2(t))
\end{align*} \] (3.27)

In the case of the initial condition (3.38) the equations (3.26) gives the constant value of probabilities.

\[ \begin{align*}
\dot{p}_0(t) &= 0 \\
\dot{p}_1(t) &= \sum_i \text{Tr}(-a_{23}^i a_{23}^{*i} p_1(t) + a_{32}^i a_{32}^{*i} p_2(t)) \\
\dot{p}_2(t) &= \sum_i \text{Tr}(a_{23}^i a_{23}^{*i} p_1(t) - a_{32}^i a_{32}^{*i} p_2(t))
\end{align*} \] (3.28)

The equations (3.29 – 3.31) have the property that the only two entries of the state \( \rho \) are included in the equations.

\[ \begin{align*}
\dot{p}_0(t) &= \sum_i \text{Tr}(a_{21}^i a_{21}^{*i} p_1(t)) \\
\dot{p}_1(t) &= \sum_i \text{Tr}(-a_{21}^i a_{21}^{*i} p_1(t) + a_{32}^i a_{32}^{*i} p_2(t)) \\
\dot{p}_2(t) &= \sum_i \text{Tr}(a_{32}^i a_{32}^{*i} p_2(t))
\end{align*} \] (3.29)

\[ \begin{align*}
\dot{p}_0(t) &= \sum_i \text{Tr}(a_{31}^i a_{31}^{*i} p_2(t)) \\
\dot{p}_1(t) &= \sum_i \text{Tr}(a_{32}^i a_{32}^{*i} p_1(t)) \\
\dot{p}_2(t) &= \sum_i \text{Tr}(a_{23}^i a_{23}^{*i} p_1(t) - a_{31}^i a_{31}^{*i} p_2(t))
\end{align*} \] (3.30)

\[ \begin{align*}
\dot{p}_0(t) &= \sum_i \text{Tr}(a_{31}^i a_{31}^{*i} p_2(t)) \\
\dot{p}_1(t) &= \sum_i \text{Tr}(a_{32}^i a_{32}^{*i} p_1(t)) \\
\dot{p}_2(t) &= \sum_i \text{Tr}(a_{23}^i a_{23}^{*i} p_1(t) - a_{31}^i a_{31}^{*i} p_2(t))
\end{align*} \] (3.31)

\[ \begin{align*}
\dot{p}_0(t) &= \sum_i \text{Tr}(-a_{12}^i a_{12}^{*i} p_0(t) + a_{21}^i a_{21}^{*i} p_1(t)) \\
\dot{p}_1(t) &= \sum_i \text{Tr}(a_{12}^i a_{12}^{*i} p_0(t) - a_{21}^i a_{21}^{*i} p_1(t)) \\
\dot{p}_2(t) &= 0
\end{align*} \] (3.32)

\[ \begin{align*}
\dot{p}_0(t) &= \sum_i \text{Tr}(a_{13}^i a_{13}^{*i} p_0(t) + a_{31}^i a_{31}^{*i} p_2(t)) \\
\dot{p}_1(t) &= 0 \\
\dot{p}_2(t) &= \sum_i \text{Tr}(a_{13}^i a_{13}^{*i} p_0(t) - a_{31}^i a_{31}^{*i} p_2(t))
\end{align*} \] (3.33)
The last two equations \[3.32\], \[3.33\] are the most interesting, because of their simplicity. Theirs action concentrate on the evolution of one of the probabilities.

### 3.6.3 Implementation of the operators \(V_1, V_2\)

I have not answered to the question about the range of the parameter \(i\). I am not going to give a full algebraic analysis, but I show that when \(i \in \{1, 2\}\) and \(V_1, V_2\) are of the form \[3.18, 3.19\] it is possible to cover the whole range of the efficiency of the detector.

Designating the shape of operators \(a_{12}, a_{21}, a_{13}, a_{31}\) I take under the consideration the following aspects:

a. The pairs \((a_{12}, a_{21})\) and \((a_{13}, a_{31})\) should significantly differ, otherwise \(p_1\) and \(p_2\) give the similar result and distinguishing quantum states could become impossible. This can make the transmission unreadable.

The best situation is when they are mutually orthogonal.

b. Make the efficiency of the detector close to one as much as possible.

We should perceive, that the operators \[3.18, 3.19\] give the same evolution as a detector considered in the section \[3.5.3\], so I apply the result of this section and investigate their joint-action.

I use the notation:

\[
\begin{align*}
a_{12} &= k_1 e_2 \\
a_{21} &= k_2 e_2 \\
a_{13} &= n_1 e_3 \\
a_{31} &= n_2 e_3
\end{align*}
\]

(3.34)

where:

\[
\begin{align*}
k_1, k_2, n_1, n_2 &\in \mathbb{R} \\
(e_2, e_3) &= 0 \\
e_2^* &= e_2 = e_2^2 \\
e_3^* &= e_3 = e_3^3
\end{align*}
\]

So the equation \[3.22\] may be rewritten:

\[
\begin{align*}
\dot{p}_0(t) &= -Tr((k_1^2 e_2^2 + n_1^2 e_3^2)\rho_0(t) - k_2^2 e_2\rho_1(t) - n_2^2 e_3\rho_2(t)) \\
\dot{p}_1(t) &= Tr(k_1^2 e_2\rho_0(t) - k_2^2 e_2\rho_1(t)) \\
\dot{p}_2(t) &= Tr(n_1^2 e_3\rho_0(t) - n_2^2 e_3\rho_2(t))
\end{align*}
\]

(3.35)

Using the notation \[3.3\] I consider the following initial states:
a. \( e_2 \rho_q = \rho_q \)
\( e_3 \rho_q = 0 \)

b. \( e_2 \rho_q = 0 \)
\( e_3 \rho_q = \rho_q \)

c. \( \rho_q = a e_2 + b e_3 \) this case could be modified in the way: \( \rho_q = \sum_i a_i e_i \)
then
\( e_2 \rho_q = a_2 e_2 \)
\( e_3 \rho_q = a_3 e_3 \).

The case a),b) can be considered jointly, because by changing the notation we can transform one case into another obtaining the same equations. What more all this cases are included in the c). (We can obtain them by proper choice of coefficients \( a_2, a_3 \).)

From the equation \([3. 21]\) I obtain:

\[
\begin{align*}
\dot{\rho}_0(t) &= -i[H, \rho_0(t)] + k_2^2 e_2 \rho_1(t) e_2 + n_2^2 e_3 \rho_2(t) e_3 - \frac{1}{2} \{k_2^2 e_2 + n_1^2 e_3, \rho_0(t)\} \\
\dot{\rho}_1(t) &= -i[H, \rho_1(t)] + k_1^2 e_2 \rho_0(t) e_2 - \frac{1}{2} \{k_2^2 e_2, \rho_1(t)\} \\
\dot{\rho}_2(t) &= -i[H, \rho_2(t)] + n_1^2 e_3 \rho_0(t) e_3 - \frac{1}{2} \{n_2^2 e_3, \rho_2(t)\}
\end{align*}
\]

Using the decomposition c) and assuming, that only \( a_2, a_3 \) depend on time and

\[
\forall_{i \neq 2} ([H, e_2], e_i) = 0 \quad \text{and} \quad \forall_{i \neq 3} ([H, e_3], e_i) = 0
\]

I get

\[
\begin{align*}
\dot{a}_0(t)e_2 &= -i[H, a_0(t)e_2] + (k_2^2 a_1(t) - k_1^2 a_0(t))e_2 \\
\dot{a}_1(t)e_2 &= -i[H, a_1(t)e_2] + (k_1^2 a_0(t) - k_1^2 a_1(t))e_2 \\
\dot{a}_2(t)e_2 &= -i[H, a_2(t)e_2] \\
\dot{b}_0(t)e_3 &= -i[H, b_0(t)e_3] + (n_2^2 b_2(t) - n_1^2 b_0(t))e_3 \\
\dot{a}_1(t)e_3 &= -i[H, a_1(t)e_3] + (n_1^2 a_0(t) - n_2^2 a_1(t))e_3 \\
\dot{b}_1(t)e_3 &= -i[H, b_1(t)e_3] \\
\dot{b}_2(t)e_3 &= -i[H, b_2(t)e_3] - (n_1^2 b_0(t) - n_2^2 b_2(t))e_3
\end{align*}
\]
After the trace operation:

\[
\begin{align*}
\dot{a}_0(t) &= k_2^2 a_1(t) - k_1^2 a_0(t) \\
\dot{b}_0(t) &= n_2^2 b_2(t) - n_1^2 b_0(t) \\
\dot{a}_1(t) &= k_1^2 a_0(t) - k_2^2 a_1(t) \\
\dot{b}_1(t) &= 0 \\
\dot{a}_2(t) &= 0 \\
\dot{b}_2(t) &= n_1^2 b_0(t) - n_2^2 b_2(t)
\end{align*}
\]  

(3. 39)

from [3. 39] we see:

\[
\begin{align*}
a_2 &= \text{const} \\
b_1 &= \text{const}
\end{align*}
\]  

(3. 40)

The equation [3. 39] may be divided into two sets:

\[
\begin{align*}
\dot{a}_0(t) &= k_2^2 a_1(t) - k_1^2 a_0(t) \\
\dot{a}_1(t) &= k_1^2 a_0(t) - k_2^2 a_1(t)
\end{align*}
\]  

(3. 41)

and

\[
\begin{align*}
\dot{b}_0(t) &= n_2^2 b_2(t) - n_1^2 b_0(t) \\
\dot{b}_2(t) &= n_1^2 b_0(t) - n_2^2 b_2(t)
\end{align*}
\]  

(3. 42)

The solution of the equations [3. 41] and [3. 42] is:

\[
\begin{align*}
a_0(t) &= -Ae^{(k_2^2+k_1^2)t} + \frac{k_2^2}{k_1^2} C \\
a_1(t) &= Ae^{-(k_2^2+k_1^2)t} + C \\
b_0(t) &= Be^{-(n_2^2+n_1^2)t} + \frac{n_2^2}{n_1^2} D \\
b_2(t) &= -Be^{-(n_2^2+n_1^2)t} + D
\end{align*}
\]  

(3. 43)

The conditions for the coefficients arise from [3. 43], [3. 2] and they are:

\[
\begin{align*}
\frac{n_2^2}{n_1^2} D + \frac{k_2^2}{k_1^2} C &= A + B + 1 \\
A + C &= 0 \\
B + D &= 0
\end{align*}
\]  

(3. 44)

The exact solution can be found when the initial state is known. I consider some important cases:
a. \( D = 0 \Rightarrow p_1(t \to \infty) = \frac{k_1^2}{k_1^2 + k_2^2} \) The maximal value is obtained for
\( k_1 \neq 0, \ k_2 = 0 \) (like in the section \[3.5.3\]).

b. \( C = 0 \Rightarrow p_1(t \to \infty) = 0 \ \ \ \ p_2(t \to \infty) = \frac{k_2^2}{k_1^2 + k_2^2} \)
As above the maximum of the efficiency of the detector is obtained for:
\( n_1 \neq 0, \ n_2 = 0 \)
It is interesting to see what has happened when none of the components
of \( \rho \) is distinguished:
\[
C = D \Rightarrow \quad p_1(t \to \infty) = p_2(t \to \infty) = \frac{k_1^2n_1^2}{n_1^2(k_1^2+k_2^2)+k_1^2(n_1^2+n_2^2)}
\]
So the maximum is obtained when:
\( k_1 \neq 0, \ n_1 \neq 0, \ k_2 = n_2 = 0 \) and \( p_1(t \to \infty) = p_2(t \to \infty) = \frac{1}{2} \)
The maximum of the efficiency of the detector for any signal is obtained when:
\( p_1(t \to \infty) + p_2(t \to \infty) = 1 \)
so \( k_2 = n_2 = 0 \).

3.6.4 Conclusion.

The maximal value of the efficiency depends on the initial state of the quantum system (as in the section \[3.5.3\]), but it is also possible to control this value by the choice of the coefficients: \( n_1, \ n_2, \ k_1, \ k_2 \). The system may be simplified by the assumption: \( n_1 = k_1, \ n_2 = k_2 \). The efficiency of the system may still achieve the value one. This choice has one more advantage, it guarantee the same efficiency for both kind of signal.

4 Example of the transmitter.

I investigate the simplest transmitter, which uses the quantum state as a carrier.
I use the following system:
S – source of the quantum states;

D I – the first detector (working also as a filter of a signal);

D II – the second detector (it plays the role of the receiver).

In the notation of the section 3.2 the source and detector D I together form the sender.

I assume that the source generate the quantum state which has the required properties (some level of the noise is allowed).

The detectors D I, D II are the open systems so theirs evolutions is described by equation [2.17].

I investigate the following initial quantum state:

a. $\rho(0) = \rho e$

b. $\rho(0) = \sum_i \rho_i e_i$

c. $\rho(0) = \sum_i \rho_i e_i + \sum_{i \neq j} \rho_{ij} e_{ij}$

$e_i$ – a normalized projector $e_i \in L(H)$ (it describes the diagonal elements);

$e_{ij}$ – a projector which describes the non diagonal elements.

4.1 Properties of the detector D I.

This detector apart from a registration of the states should pass the states in unchanged states (unchanged as much as possible). I assume the following form of the coupling operator $V$:

$$ V = \sqrt{k} \begin{pmatrix} 0 & e_1 \\ e_1 & 0 \end{pmatrix} $$

(4.1)
\( \sqrt{k} \) – coupling constant,
\( e_1 \) – normalized projector.

For the sake of simplicity I assume \([H, \rho(t)] = 0\).
This assumption should be understood that the evolution generated by Hamiltonian operator must be negligible in comparison to the evolution governed by the coupling. This is quite easy to achieve, the coupling constant may be increased due to shorter the influence of the Hamiltonian evolution.

The probability space is a 2 – dim one, so the equation \([2.17]\) has the solution (following \([20]\)):

\[
\begin{align*}
\rho_0(t) &= p_0\left(\frac{1}{2}e\rho q e + f\rho q f\right) + \frac{1}{2}p_1e\rho q e + e^{-\frac{1}{2}kt}p_0\left(e\rho q f + f\rho q e\right) + e^{-kt}p_0\left(e\rho q e + e\rho q e\right) + \frac{1}{2}e^{-2kt}(p_0e\rho q e + p_1e\rho q e) \\
\rho_1(t) &= \frac{1}{2}p_0e\rho q e + p_1(\rho q + \frac{3}{2}e\rho q e - \{e, \rho q\}) + e^{-\frac{1}{2}kt}p_1(\{e, \rho q\} - 2e\rho q e) + \frac{1}{2}e^{-2kt}(p_1e\rho q e - p_0e\rho q e)
\end{align*}
\]  
(4.2)

where:
\( f = 1 - e \)
\( \forall \rho_i(0) = p_i\rho q \)

Applying the projector \([2.6]\) to the solution \([4.2]\) I obtain:

\[
\rho_q(t) = \rho_q + 2e\rho q e + \{e, \rho q\} + e^{-\frac{1}{2}kt}\{e, \rho q\} - 2e\rho q e
\]  
(4.3)

In the cases of the initial condition a) – c) the solution \([1.3]\) gives

a. \( \rho_q(t) = \rho_q \)

b. \( \rho_q(t) = \rho_q \)

c. \( \rho_q(t) = \sum_i \rho_i e_i + \sum_{ij}[1 + (\delta_{i1} + \delta_{j1})(e^{-\frac{1}{2}kt} - 1)]\rho_{ij}e_{ij} \)

Applying the projector \([2.7]\) onto the classical subsystem I obtain:

\[
\begin{align*}
p_0(t) &= \frac{1}{2}(p_1 - p_0)q_1 + p_0 + \frac{1}{2}(p_0 - p_1)q_1e^{-2kt} \\
p_1(t) &= \frac{1}{2}(p_0 - p_1)q_1 + p_1 + \frac{1}{2}(p_1 - p_0)q_1e^{-2kt}
\end{align*}
\]  
(4.4)

where
\( q_1 = Tr e_1\rho_q \)
For the initial state [3.2], with the assumption [3.1] I obtain for the cases a) – c):

\[
\begin{align*}
\text{a)} & \quad \begin{cases} 
    p_0(t) = \frac{1}{2}(1 - e^{-2kt}) \\
    p_1(t) = \frac{1}{2}(1 + e^{-2kt})
\end{cases} \\
\text{b)} & \quad \begin{cases} 
    p_0(t) = 1 + \frac{1}{2}(e^{-2kt} - 1)\rho_1 \\
    p_1(t) = \frac{1}{2}(1 - e^{-2kt})\rho_1
\end{cases} \\
\text{c)} & \quad \text{The result are the same as in b). The quantum states b), c) are identical from the point of view of the detector.}
\end{align*}
\]

4.2 Properties of the detector D II

The system working as a detector D II should satisfy the following condition: to be able to distinguish at least two different quantum states, the efficiency of the detector should be as big as possible. Taking under consideration the above condition I shall use the system described in the section [3.6.1] which satisfy:

- \( k_1 = n_1 = n \) should be big enough to satisfy the conditions of the section [4.1]
- \( k_2 = n_2 = 0 \)

4.3 Implementation of the system.

In the following I consider the properties of the transmission thorough such a system.

I make the assumption:

- The source generates the signal \( \rho \) such that \( Tr(e_1\rho) \geq 0, 8 \) (or \( Tr(e_2\rho) \geq 0, 8 \)) (the choice depends on the value which is being transmitted).

- The parameters of the detectors should be chosen in such a way that the real efficiency of the detectors achieves 90% of the maximum one.

- The classical subsystem satisfy the initial condition [3.2].

\[10\] This assumption say that the source generate a good quality signal but the noise is allowed also.
Under the above condition I receive the probabilities of the registration of the signal:

\[
DI \quad p_1(t) = \frac{1}{2}(1 - e^{-2kt})\rho_1 \tag{4.5}
\]
\[
DII \quad p_1(t) = (1 - e^{-n^2t})\rho_1, \quad p_2(t) = (1 - e^{-n^2t})\rho_2 \tag{4.6}
\]

I would like to stress that this equation describe the evolution of the probabilities of the registration of the state. What more the probability of an event may be specified after an infinity serious of experiments. However the probability of the registration of a state may be specified with some level of confidence. I define the confidence level as a 90%, this means that with the probability of 90% the measured value is the real value.

The aim of the work of the detectors:

D I This detector check if the signal has been generated. So I have to designate how many states should be generated by the source in order to be sure that the detector register at least one state.

D II It recognize the kind of the signal. In order to do this job, the detector should designate if the value \(\rho_1\) belongs to the interval \((\rho_1 - a, \rho_1 + a)\) within the confidence level.

4.3.1 Detector D I.

For producing an effect by the system D I is required a very small number of states. I assume that the act of generation of a state is an independent event, so the probability of registration at least one of the generated states could be found as a probability of complementary event to the no registration of the states. I have:

\[
P(\text{no registration}) + P(\text{registration}) = 1 \tag{4.7}
\]

and from \([4.6]\)

\[
P(\text{no registration}) = \left(\frac{1}{2}(1 - e^{-2kt_0})\rho_1\right)^n \tag{4.8}
\]

I understand as a signal a quantum state with some (specified) properties.
where:
\( n \) – number of generated states,
\( t_0 \) – time of evolution of coupled systems.

Applying equations [4. 7] [4. 8] we can find out, that it is enough, with the assumed confidence level, to generate 4 state to be sure that the detector D I confirm the fact of sending the signal.

### 4.3.2 Detector D II.

The situation of the detector D II is more complicated than with D I. It is not enough to confirm the fact of receiving the state, but the value of \( \rho_1 \) should be estimated, otherwise we do not know what kind of signal is being received.

I evaluate the probability of registration the state, which has the coefficient of the vector \( e_1 \) belonging to the interval between \( w_1 = \rho_1 - a \), \( w_2 = \rho_1 + a \).

\( a \) – the measuring accuracy,
\( \rho_1 \) – the real value,
\( w_1, w_2 \) – respectively the infimum and supremum of the interval.

The following value is obtained from the experiment:

\[
\hat{p}(t) = \frac{i}{m} 
\]

\( i \) – number of state registered by the detector D II as a particular logical value (let it be "1"),
\( m \) – number of the states generated by the sender in order to send the message.

I introduce two auxiliary quantity:

\( p^{-}(t_0) \) – the probability of registration of the state during the measurement in time \( t_0 \), where the decomposition of \( \rho \) is \( \rho = w_1 e_1 + \sum_{i=2} \rho_i e_i \),
\( p^{+}(t_0) \) – the probability of registration the state \( \rho \), where \( \rho = w_2 e_1 + \sum_{i=2} \rho_i e_i \).

The expectation value is respectively:

\[
\begin{align*}
i^- &= m \ p^{-}(t_0) = m \ p_1(t_0) - (1 - e^{-nt_0}) a m \\
i^+ &= m \ p^{+}(t_0) = m \ p_1(t_0) + (1 - e^{-nt_0}) a m
\end{align*}
\]

\(^{12}\)It is very important to know the number of states generated by the sender in time.
I solve the following two aspects:

a. What is the smallest number of states required for obtaining such \( \rho_1 \) that estimated \( \rho_1 \in (w_1, w_2) \)?

b. What is the smallest number of states, that with the certain confidence level is possible to say that \( \rho_1 \in (w_1, w_2) \)?

**The point a)**

It is possible to obtain the result which belong to the specified interval, when the expectation value \( i^-, i^+ \) satisfy the condition:

\[
i^- \leq [i^+]
\]

\([i^+] \) – the integer part of \( i^+ \)

so the intersection \([i^-, i^+] \) \( \cap \mathcal{N} \neq \emptyset \)

\( \mathcal{N} \) – the set of integer numbers.

The above condition I rewrite in the more useful form:

\[
i^+ - i^- \geq 1
\]

so for the detector D II I obtain:

\[
m \geq \frac{1}{2(1 - e^{-n^2\omega})a} \quad (4.11)
\]

The value obtained in (4.11) specify the lower limit of the number of generated states required to change the classical state of the detector. It can be seen that after a few generated states it is possible to receive the message properly.

**The point b).**

I use the Bernoulli distribution for evaluating the probability of registration of "\( i \)" states among "\( m \)" received is:

\[
P(i, m) = \binom{m}{i} (p_1(t))^i (1 - p_1(t))^{m-i} \quad (4.12)
\]

For obtaining the probability of evaluation \( \hat{p}_1 \in (p^-_1, p^+_1) \) The distribution should be sum up over the advantageous events.

I introduce the function:
\[ [x]_– \text{ the integer number which } [x]_– \leq x \]
\[ [x]_+ \text{ the integer number which } [x]_+ \geq x \]

so the probability is:

\[ P(m) = \sum_{i=[i^-]+}^{[i^+]_-} P(i, m) \]  \hspace{1cm} (4. 13)

**Remark**

It is noticeable, that with the increasing number of the generated states the probability of obtaining the proper value does not have to increase. It may happen that despite increasing the number of sented states the probability decrease. This fact should be taken under consideration during the real experiment.

**4.3.3 Examples:**

In the following examples I assume:
\( \rho_1 = 0.8, \)
\( t_0 \text{ is such that } 1 - e^{-n^2t_0} = 0.9 \)

the measuring accuracy of \( \rho_1: a = 0.05 \)

so I obtain:

from the equation [4. 11] the smallest number of states, which should be generated:

\[ m \geq 12 \]

In this case the expectation value \( i^-, i^+ \) are:

\[ i^- = 8, 1 \]
\[ i^+ = 9, 18 \]

We see that the only advantageous event is to register 9 states, then
\( p_1 = 0.75 \) and \( p_1 = 0.72 \)
The probability of obtaining this result is:

\[ P(12) = P(9, 12) = \left( \frac{12}{9} \right) (0, 72)^9(0, 28)^3 = 0.25 \]  \hspace{1cm} (4. 14)

Example to the remark.

For \( m = 12 \) the adventegous event is to measure 9 states and the probability of such event is:
For $m = 15$ the advantageous event is to measure 11 states, then:
$P(15) = 0.22$
We see that the probability of obtaining the proper result decreases despite increasing the number of the generated states.

Achieving the confidence level higher than 60% is possible for $m = 62$. The advantageous events are:

$$i = \{42, 43, 44, 45, 46, 47\} \text{ and } P(62) = 0.603$$

It is interesting that the efficiency of the detector is not extremely important because for the detector working with the efficiency 0.45 (it is half of the efficiency assumed above), for $m = 66$ the system achieves similar confidence level. In this case:

$$i = \{21, 22, 23, 24, 25, 26\} \text{ and } P(66) = 0.56$$

So very close to the above example.

### 4.4 Conclusion

This calculation could be simplified by replacing binomial distribution by normal distribution, but this can be done only for big number of events, whereas the aim of the work was to estimate the smallest number of states required for achieving the assumed quality of transmission.

The investigation presented in whole work does not exhaust the range of improvement of such a transmission e.g. the control signal can be introduced in order to increase the confidence level or resistance to eavesdropping; use the sophisticated way of coding including several possibilities of quantum error correction codes [1, 29, 30, 31] and so on. The number of possible improvements is great.

The last question which I’d like to consider is the resistance to noise. In the previous example I’ve used the signal which had as its part a 20% of noise and despite of it the transmission was possible. I admit that usage of more than 2-state coding may considerably increase the speed of the transmission. The boundaries of n-state coding may be the properties of the quantum system (the Hamiltonian of this system) and of the channel. I’ve omitted the role of the channels, but in the farther investigation its role should be considered also.
5 Aknowledgments

I'd like to thank prof. A. Jadczyk for his helpful discussions and hints during the work on this subject.

6 Appendix

The maximum efficiency of the n–state detector.

I assume the following coupling operator \( V_i \):

\[
V_i = \sqrt{k} \left( \begin{array}{cccc}
0 & \cdots & e_i & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0
\end{array} \right)
\] (6.1)

Where \( e_i \) are projectors operators.

The operator [6.1] satisfy the conditions [2.15] and [2.16]. The equation [2.17] in this case take the form:

\[
\begin{aligned}
\dot{\rho}_0(t) &= -i[H, \rho_0(t)] - \frac{1}{2}k(\sum e_i e_i \rho_0(t) + \rho_0(t) \sum e_i e_i) \\
\dot{\rho}_i(t) &= -i[H, \rho_i(t)] + k e_i \rho_0(t) e_i
\end{aligned}
\] (6.2)

Using the trace operations:

\[
\begin{aligned}
\dot{\rho}_0(t) &= -k Tr \sum e_i \rho_0(t) \\
\dot{\rho}_i(t) &= k Tr e_i \rho_0(t)
\end{aligned}
\] (6.3)

I assume that the commutator of the Hamiltonian and \( \rho \) does not change the decomposition of \( \rho \). So the equations [6.1] may be rewritten in the form similar to [3.40].

Let

\[
\rho_0(0) = a_j e_j
\] (6.4)

So I obtain

\[
\begin{aligned}
\dot{\rho}_0(t) &= -k Tr \sum a_j(t) e_j \\
\dot{\rho}_j(t) &= k Tr a_j(t) e_j \\
\dot{\rho}_{i \neq j} &= 0
\end{aligned}
\] (6.5)
The solution of the equations (6.3) are:

\[
\begin{align*}
  p_0 &= Ae^{-kt} \\
  p_j &= C - Be^{-kt} \\
  p_{i\neq j} &= \text{const}
\end{align*}
\] (6.6)

Applying the initial condition (3.2):
\[A = 1\]
Assuming that \(p_j(\infty) = 1\) (the ideal efficiency):
\[C = 1\]
and from the probability theory:
\[B = A = 1\]
So the solution take the form:
\[
\begin{align*}
  p_0 &= e^{-kt} \\
  p_j &= 1 - e^{-kt} \\
  p_{i\neq j} &= 0
\end{align*}
\] (6.7)

The asymptotic value of the efficiency of the detector is one, of course when the initial state has the required form.

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