Crossed morphisms, integration of post-Lie algebras and the post-Lie Magnus expansion

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ABSTRACT

This letter is divided in two parts. In the first one it will be shown that the datum of a post-Lie product is equivalent to the one of an invertible crossed morphism between two Lie algebras. Moreover it will be argued that the integration of such a crossed morphism yields the post-Lie Magnus expansion associated to the original post-Lie algebra. The second part is devoted to present two combinatorial methods to compute the coefficients of this remarkable formal series. Both methods are based on special tubings on planar trees.

1. Introduction

Pre and post-Lie algebras form two classes of non-associative algebras which play an important role both in pure and in applied mathematics. Pre-Lie algebras, also known in the literature under the name of left-symmetric and Vinberg algebras, were introduced in the mid-sixty by Gerstenhaber, see [27] and, independently, by Vinberg, see [46]. Since then they appeared unexpectedly in almost every area of modern mathematics, from differential geometry [4, 34, 42], to combinatorics [5, 14, 15], from mathematical physics [24], to numerical analysis [10, 28], see [11, 25, 33] for comprehensive reviews. Post-Lie algebras have been introduced more recently by Vallette in [45] and, independently, by Lundervold and Munthe-Kaas in [32]. Since then they have been thoroughly investigated, see for example [3, 12, 17, 20, 22, 35, 36] and also [23, 26]. Recall that a post-Lie Lie algebra is a pair \((\mathfrak{h}, \triangleright)\) of a Lie algebra \(\mathfrak{h}\), whose Lie bracket will be denoted by \([\cdot, \cdot]\), and a bilinear map \(\triangleright: \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}\) called post-Lie product, satisfying the following two properties:

\[(\text{PL1})\quad x \triangleright [y, z] = [x \triangleright y, z] + [y, x \triangleright z],\quad \text{and} \]

\[(\text{PL2})\quad [x, y] \triangleright z = a_c(x, y, z) - a_c(y, x, z), \quad \text{for all } x, y \text{ and } z \text{ in } \mathfrak{h}.\]

In the right-hand side of (PL2)

\[a_c(x, y, z) = x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z, \quad \forall x, y, z \in \mathfrak{h}\]
denotes the associator defined by the bilinear product $\triangleright$. A post-Lie algebra whose Lie bracket is trivial is a pre-Lie algebra. On the other hand, a post-Lie algebra $(\mathfrak{b}, \triangleright)$ gives rise to a Lie algebra $\overline{\mathfrak{b}}$ with same underlying vector space as $\mathfrak{b}$ and whose Lie bracket $[\cdot, \cdot] : \mathfrak{b} \otimes \mathfrak{b} \to \mathfrak{b}$ is defined by

$$[[x, y]] = x \triangleright y - y \triangleright x + [x, y], \quad \forall x, y \in \mathfrak{b}. \quad (1.2)$$

Moreover, there is a morphism of Lie algebras $\nu : \overline{\mathfrak{b}} \to \text{Der}(\mathfrak{b})$, defined by

$$\nu_x(y) = x \triangleright y, \quad \forall x, y \in \overline{\mathfrak{b}}, \quad (1.3)$$

where $\text{Der}(\mathfrak{b})$ denotes the Lie algebra of the derivations of $\mathfrak{b}$.

The enveloping algebra $\mathcal{U}(\mathfrak{b})$ of a post-Lie algebra was analyzed in depth in [20], where it was shown, along the same line of [38, 39], that on $\mathcal{U}(\mathfrak{b})$ can be defined a new associative product $\ast : \mathcal{U}(\mathfrak{b}) \otimes \mathcal{U}(\mathfrak{b}) \to \mathcal{U}(\mathfrak{b})$, called the Grossman-Larson product, which is compatible with both the original antipode and the original coalgebra structure. These results entail the existence of new Hopf algebra $\mathcal{U}_*(\mathfrak{b})$, defined on the underlying vector space of $\mathcal{U}(\mathfrak{b})$, which turns out to be isomorphic to the Hopf algebra $\mathcal{U}(\overline{\mathfrak{b}})$. After a suitable completion of the Hopf algebras involved, the latter isomorphism defines an isomorphism between the (completed) Lie algebras $\overline{\mathfrak{b}}$ and $\mathfrak{b}$, whose inverse $\chi : \mathfrak{b} \to \overline{\mathfrak{b}}$, called the \textit{post-Lie Magnus expansion}, pLMe hereafter, is one of the main concerns of the present note.

The pLMe has two predecessors, the pre-Lie and the classical Magnus expansions, see [8]. The \textit{pre-Lie Magnus expansion} $\chi$ appeared at the beginning of the eighties in the work of Agrachev and Gramkelidze, see [1]. However it has been dubbed as such only in [21], where the classical Magnus expansion was extensively explored from the viewpoint of pre-Lie and dendriform algebras. Finally in [15] was presented a formula expressing $\chi$ in terms of the so called Grossman-Larson product which reads as

$$\chi(x) = \log_{\ast}(\exp(x)), \quad \text{see also [5]}. \quad \text{(1.1)}$$

On the other hand, the pLMe was introduced in [20] in connection with a particular class of iso-spectral flow equations. There it is was shown that for every $x \in \mathfrak{b}$, $\chi_x(t) := \chi(tx) \in \mathfrak{b}[[t]]$ satisfies the following non-linear ODE

$$\dot{\chi}_x(t) = (d \exp_{\ast})_{-\chi_x(t)} \left( \exp_{\ast}(-\chi_x(t)) \triangleright x \right),$$

and that, the \textit{non-linear post-Lie differential equation}

$$\dot{x}(t) = -x(t) \triangleright x(t),$$

for $x = x(t) \in \mathfrak{b}[[t]]$, with initial condition $x(0) = x_0 \in \mathfrak{b}$, has as a solution

$$x(t) = \exp_{\ast}(-\chi_{x_0}(t)) \triangleright x_0.$$ 

In [16] it was underlined the relevance of the pLMe to the theory of Lie group integrators and in [35] it was proven that on a post-Lie algebra, in analogy to what happens on every pre-Lie algebra, see [1] and also [4, 18], the pLMe provides an isomorphism between the group of formal flows and the BCH-group defined on $\overline{\mathfrak{b}}$.

The aim of this letter is twofold. In the first place, starting from the integration result presented in [35], we give a more Lie-theoretic interpretation of the pLMe, in terms of the so called crossed morphisms of Lie groups and Lie algebras, see for example [7, 29, 30, 40].

More precisely, first we show that a post-Lie algebra structure on $\mathfrak{b}$ is equivalent to the datum $(\text{id}, \nu)$, where $\text{id} : \overline{\mathfrak{b}} \to \mathfrak{b}$, the identity map, is a crossed morphism relative to $\nu \in \text{Hom}_{\text{Lie}}(\overline{\mathfrak{b}}, \text{Der}(\mathfrak{b}))$. Then we argue that the pLMe is the inverse of a crossed morphism between the corresponding (local) Lie groups $\overline{\mathcal{G}}$ and $\mathcal{G}$ obtained \textit{integrating} id.
We would like to stress that while the local existence of the pLMe, at the level of the Lie groups \( H \) and \( \overline{H} \), is guaranteed by general Lie theory, its global existence is obstructed, see [37], and its explicit expression seems, from this viewpoint, really difficult to obtain. On the other hand, working formally at the level of the completed enveloping algebras of \( \mathfrak{h} \) and \( \mathfrak{h} \), one first can prove the existence of the pLMe and then, using a formal integration process, one can show that the inverse of the pLMe is a crossed morphism between the corresponding local Lie groups.

This line of thoughts opens the door to a categorical interpretation of the various approaches to post-Lie and pre-Lie algebras which one finds in the literature, see for example [3, 12, 32] and references therein. On the other hand, it makes clear the universal nature of the pLMe, asking for a (more systematic) method to compute the coefficients of this expansion. This goal is achieved in the second part of the paper, where such a method, based in the so called tubings, see [13], is presented.

**Relations with other works.** Post-Lie algebras appeared recently as central objects in the study of the so called \( O \)-operators, first introduced in [31], which are particular extensions of the classical \( r \)-matrices, playing an important role in the theory of the generalized Lax pair representations introduced in [9]. The notion of \( O \)-operator was further extended in [3], where the concepts of \( O \)-operator of weight \( \lambda \) and, respectively, of extended \( O \)-operator, were introduced. It is in this framework that the relation between (generalized) Lax representations and post-Lie algebras crystallized. In particular in [3] it was shown that the post-Lie algebra structures on a Lie algebra \( \mathfrak{g} \) are in one-to-one correspondence with the pairs \((\mathfrak{v}, \mathfrak{h}), O\) where \( \mathfrak{h} \) is a Lie algebra, \( v \in \text{Hom}_{Lie}(\mathfrak{g}, \text{Der}(\mathfrak{h})) \) and \( O : \mathfrak{h} \rightarrow \mathfrak{g} \) is an invertible \( O \)-operator of weight 1, see [3, Corollary 5.5]. This result should be compared with Proposition 7 of the present work, see also 1 in Remark 11.

Another instance where the notion of post-Lie algebra arises naturally is the theory of the so called simply-transitive \textit{NIL-affine actions} of nilpotent Lie groups, see [12]. In this reference it was shown that given \((G, \mathfrak{n})\), a pair of connected and simply-connected nilpotent Lie groups, there exists a simply transitive NIL-action of \( G \) on \( N \) if and only if there exists a Lie algebra \( \mathfrak{g}' \sim \mathfrak{g} \) such that the pair \((\mathfrak{g}', \mathfrak{n})\) carries a structure of a post-Lie algebra, see [12, Theorem 2.5]. The proof of this result is based on the observation that a pair of Lie algebras \((\mathfrak{g}, \mathfrak{n})\) carries a structure of a post-Lie algebra if and only if there is a faithful morphism of Lie algebras \( \varphi : \mathfrak{g} \rightarrow \mathfrak{n} \triangleleft \text{Der}(\mathfrak{n}) \) of the form \( \varphi(x) = (x, L(x)) \), for all \( x \in \mathfrak{g} \), see [12, Proposition 2.11] for the precise statement. This result should be compared with 2 in Remark 11 of the present work.

**Plan of the present work.** In Sect. 2 is recalled the notion of crossed morphism of Lie groups and of Lie algebras. Moreover, the relation between \textit{invertible} crossed morphisms of Lie algebras and post-Lie algebras is explained. This section closes with a few remarks about \textit{local} Lie groups.

In Sect. 3 it is shown that the datum of a crossed morphism between two Lie algebras yields a morphism between the associated universal enveloping algebras, which, when the crossed morphism is invertible, provides an isomorphism giving rise to the Grossman-Larson product. After introducing a suitable \textit{integration functor}, the last part of this section is devoted to the analysis of the pLMe from the categorical view-point mentioned above.

In Sect. 4 two combinatorial interpretations of the coefficients of the pLMe are given. Both interpretations are based on the notion of nested tubing. More precisely, the first method is based on the so called \textit{vertical} nested tubings and it allows to compute recursively the coefficients associated to any forest. The second method is based on the concept of \textit{horizontal} nested tubing and it yields a closed formula for these coefficients. This section is divided into six parts. The first four parts 4.1–4.4 are essentially a reminder; they serve to set up conventions and to introduce the adequate combinatorics in order to handle the pLMe. More in details, the first part sets up conventions and notations used to describe planar trees and forests and it introduces specific grafting operations for these combinatorial structures. The second part is a brief reminder on the combinatorial operad \( PSB \), a model of the operad \( PostLie \), which serves as a combinatorial base to handle operations on the free post-Lie algebra (on one generator) and on its universal
enveloping algebra. These last two algebras are the subject of the third and of the fourth parts. The fifth part is dedicated to the notions of vertical and horizontal nested tubings which are the last essential ingredient to compute the pLMe. Finally, the last part is devoted to the computation of this expansion, first in terms of vertical, then in terms of horizontal nested tubings.

1.1. Conventions

Throughout the paper \( \mathbb{K} \) will denote a field of characteristic zero. The tensor product will be taken over \( \mathbb{K} \). In particular the tensor product of two \( \mathbb{K} \)-vector spaces \( V \) and \( W \) will be denoted by \( V \otimes W \). The space of \( \mathbb{K} \)-linear morphisms, say from a vector space \( V \) to \( W \), will be denoted by \( \text{Hom}_{\mathbb{K}}(V, W) \). Likewise, \( \text{Hom}_{\text{Lie}}(\mathfrak{g}, \mathfrak{h}) \) will denote the space of morphisms of Lie algebras, \( \text{Hom}_{\text{GpLie}}(G, H) \) the space of morphisms of Lie groups, and so on. All Lie groups considered will be connected and simply-connected. The category of the post-Lie algebras and their morphisms will be denoted by \( \text{PostLie} \) while the category of the pre-Lie algebras and their morphisms will be denoted by \( \text{PreLie} \). For \( n \geq 1 \), the symmetric group on \( n \) letters is denoted by \( S_n \).

2. Crossed morphisms

In this section we will recall the concepts of post-Lie algebra and of crossed morphism, in the case of Lie algebras and of Lie groups, and we will comment on how these relate to each other.

2.1. Crossed morphisms of Lie algebras

**Definition 1.** Let \( \mathfrak{g} \) and \( \mathfrak{h} \) be two Lie algebras, and let \( \nu : \mathfrak{g} \to \text{Der}(\mathfrak{h}) \) be a morphism of Lie algebras. A **crossed morphism relative to** \( \nu \) is a map \( \phi \in \text{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{h}) \) that satisfies

\[
\phi([x, y]_\mathfrak{g}) = \nu_x(\phi(y)) - \nu_y(\phi(x)) + [\phi(x), \phi(y)]_\mathfrak{h}, \quad \forall x, y \in \mathfrak{g}. \tag{2.1}
\]

The set of crossed morphisms from \( \mathfrak{g} \) to \( \mathfrak{h} \) relative to \( \nu \) is denoted with \( \text{Cross}^\nu(\mathfrak{g}, \mathfrak{h}) \). The subset of the invertible crossed morphisms is denoted with \( \text{Cross}^\nu_{\text{inv}}(\mathfrak{g}, \mathfrak{h}) \).

**Example 2.** If \( \mathfrak{h} \) is abelian, i.e. if \( [\cdot, \cdot]_\mathfrak{h} = 0 \), then \( \text{Der}(\mathfrak{g}) = \text{End}_{\mathbb{K}}(\mathfrak{g}) \). In this case \( \phi \) is a crossed morphism from \( \mathfrak{g} \) to \( \mathfrak{h} \), relative to \( \nu \in \text{End}_{\mathbb{K}}(\mathfrak{g}) \), if and only if

\[
\phi([x, y]_\mathfrak{g}) = \nu_x(\phi(y)) - \nu_y(\phi(x)), \quad \forall x, y \in \mathfrak{g}.
\]

**Example 3.** If \( f \in \text{Hom}_{\text{Lie}}(\mathfrak{g}, \mathfrak{h}) \) then \( \nu_f : \mathfrak{g} \to \text{Der}_{\text{Lie}}(\mathfrak{h}) \) defined by

\[
\nu_f(x)(a) = [f(x), a]_\mathfrak{h}, \quad \forall a \in \mathfrak{h},
\]

is a morphism of Lie algebras and \( \phi \in \text{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{h}) \) belongs to \( \text{Cross}^\nu(\mathfrak{g}, \mathfrak{h}) \) if and only if \( f + \phi \in \text{Hom}_{\text{Lie}}(\mathfrak{g}, \mathfrak{h}) \).

**Example 4.** If \( \mathfrak{g} = \mathfrak{h} \) and \( \phi \) is the identity, then \( \mathfrak{h} \) has another Lie algebra structure, given by

\[
[x, y]_\mathfrak{h} := \nu_y(x) - \nu_x(y) + [x, y]_\mathfrak{h}, \quad \forall x, y \in \mathfrak{h}. \tag{2.2}
\]

The resulting Lie algebra is denoted by \( \overline{\mathfrak{h}} = (\mathfrak{h}, [\cdot, \cdot], - , - ) \).

**Definition 5.** The category \( \text{CM} \) is as follows. The objects are the tuples \((\mathfrak{g}, \mathfrak{h}, \nu, \phi)\) of two Lie algebras \( \mathfrak{g} \) and \( \mathfrak{h} \) and \((\nu, \phi) \in \text{Hom}_{\text{Lie}}(\mathfrak{g}, \text{Der}(\mathfrak{h})) \times \text{Cross}^\nu(\mathfrak{g}, \mathfrak{h}) \). The morphisms between \((\mathfrak{g}, \mathfrak{h}, \nu, \phi)\) and \((\mathfrak{g}', \mathfrak{h}', \nu', \phi')\) are pairs \((f, g) \in \text{Hom}_{\text{Lie}}(\mathfrak{g}, \mathfrak{g}') \times \text{Hom}_{\text{Lie}}(\mathfrak{h}, \mathfrak{h}')\) such that
(M1) $g \circ \phi = \phi' \circ f$ and
(M2) $g(v_x(a)) = v'_{f(x)}(g(a))$, for all $x \in \mathfrak{g}$ and $a \in \mathfrak{h}$.

The subcategory $\mathbf{CM}_{\text{inv}} \subset \mathbf{CM}$ is the one of those tuples $(\mathfrak{g}, \mathfrak{h}, v, \phi)$ such that $\phi$ is an invertible crossed morphism, i.e. $\phi \in \text{Cross}^1_{\text{inv}}(\mathfrak{g}, \mathfrak{h})$. The subcategory $\iota : \mathbf{CM}_{\text{id}} \subset \mathbf{CM}_{\text{inv}}$ is the one of the tuples of the form $(\overline{\mathfrak{h}}, \mathfrak{h}, v, \text{id})$, where $\overline{\mathfrak{h}}$ is as in Example 4.

Let $R : \mathbf{CM}_{\text{inv}} \rightarrow \mathbf{CM}_{\text{id}}$ be the functor given by
\begin{equation}
R(\mathfrak{g}, \mathfrak{h}, v, \phi) = (\overline{\mathfrak{h}}, \mathfrak{h}, v \circ \phi^{-1}, \text{id}) \quad \text{and} \quad R(f, g) = (g, g).
\end{equation}
Note that in the first tuple, the Lie algebra $\overline{\mathfrak{h}}$ is determined by $(\mathfrak{h}, v \circ \phi^{-1})$ so that its Lie bracket is given by $[x, y] = v_{\phi^{-1}(x)}(y) - v_{\phi^{-1}(y)}(x) + [x, y]_{\mathfrak{h}}$ for all $x, y \in \mathfrak{h}$, according to (2.2).

**Proposition 6.** The two categories $\mathbf{CM}_{\text{id}}$ and $\mathbf{CM}_{\text{inv}}$ are adjoint equivalent.

**Proof.** The inclusion functor is full and essentially surjective which proves the equivalence. It remains to show that the functor $R$ is a right adjoint to $\iota$. To do this it is enough to check that the unit $\eta : \text{id} \rightarrow R \circ \iota$ and counit $\epsilon : \iota \circ R \rightarrow \text{id}$ transformations satisfy the triangle relations:

\[ R \circ \eta = \iota, \quad \eta R \circ \iota = \text{id}, \quad \iota R \circ \iota = \text{id}, \quad \text{and} \quad \text{id} \circ \iota = \epsilon. \]

To a tuple $(\overline{\mathfrak{h}}, \mathfrak{h}, v, \text{id}) \in \mathbf{CM}_{\text{id}}$ one may associate the Post-Lie algebra $(\mathfrak{h}, \triangleright)$ where
\begin{equation}
x \triangleright y := v_{x}(y) \quad \text{for all} \quad x, y \in \mathfrak{h}.
\end{equation}
Indeed, (PL1) is clear since $v_{x}$ is a derivation of $\mathfrak{h}$, and (PL2) results from the fact that $v : \overline{\mathfrak{h}} \rightarrow \text{Der}(\mathfrak{h})$ is a Lie morphism: for all $x, y$ and $z$ in $\mathfrak{h}$, one has
\begin{equation}
[x, y] \triangleright z = v_{[x, y]}(z) = v_{v_{x}(y) + v_{y}(x)}(z) = v_{x}(v_{y}(z)) - v_{y}(v_{x}(z)) - v_{v_{x}(y)}(z) + v_{v_{y}(x)}(z).
\end{equation}
The following is straightforward.

**Proposition 7.** The two categories $\mathbf{CM}_{\text{id}}$ and $\text{PostLie}$ are isomorphic.

**Remark 8.** The tuples $(\mathfrak{g}, \mathfrak{h}, v, \phi)$ of $\mathbf{CM}_{\text{inv}}$, where $\mathfrak{h}$ is an abelian Lie algebra form a full subcategory of $\mathbf{CM}_{\text{inv}}$, denoted, hereafter, by $\mathbf{CM}_{\text{pl}}$. Moreover, the full subcategory $\mathbf{CM}_{\text{pl}, \text{id}} \subset \mathbf{CM}_{\text{pl}}$ whose objects are the tuples $(\overline{\mathfrak{h}}, \mathfrak{h}, v, \text{id})$ and which is adjoint equivalent to $\mathbf{CM}_{\text{pl}}$, turns out to be isomorphic to $\text{PreLie}$, recovering the result of [2], see also [6] and references therein. To be more explicit, it is worth noting that if $(\mathfrak{g}, \mathfrak{h}, v, \phi)$ is an object in $\mathbf{CM}_{\text{pl}}$, then
\[ \phi([x, y]_{\mathfrak{g}}) = v_{x}(\phi(y)) - v_{y}(\phi(x)), \quad \forall x, y \in \mathfrak{g}, \]
i.e. $\phi$ is a bijective 1-cocycle (of the Chevalley-Eilenberg cohomology) of $\mathfrak{g}$ with values in $\mathfrak{h}$.

**Example 9 ([7, 32]).** For a given Lie group $K$ whose Lie algebra is $\mathfrak{t}$, let $\mathfrak{g} = X(K)$ be the Lie algebra of the vector fields on $K$ and $\mathfrak{h} = C^\infty(K, \mathfrak{t})$ with the Lie bracket defined by $[[f, g]](k) = [f(k), g(k)]$, for all $f, g \in \mathfrak{h}$ and $k \in K$. Then
\begin{equation}
v_{X_f}(k) := k f \circ k, \quad \forall X \in \mathfrak{g}, f \in \mathfrak{h}
\end{equation}
is a morphism of Lie algebras from $\mathfrak{g}$ to $\text{Der}(\mathfrak{h})$, see Remark 10 below.

**Remark 10.** Note that $\mathfrak{h}$ can be identified with $C^\infty(K) \otimes \mathfrak{t}$. More precisely, if $\{e_i\}_{i=1, \ldots, \dim K}$ is a basis of $\mathfrak{t}$, every $f \in \mathfrak{h}$ can be written as $f = \sum_{i=1}^{\dim K} f_i \otimes e_i$, for some $f_i \in C^\infty(K)$. Under this identification, for all $X \in \mathfrak{g}$, one has $X(f) = \sum_i (X f_i) \otimes e_i$, i.e.
(Xf)(k) = X_k f = \sum_i X_k(f_i) e_i, \ \forall k \in K,

see for example [43, Sect. 6 in Chapter 5].

Furthermore, note that \( \theta \in \Omega^1(K, t) \), defined using the left-translations \( \{ L_k \}_{k \in K} \) by \( \theta_k(v) = (L_{k^{-1}})_{s, k}(v) \) for all \( k \in K \) and \( v \in T_k K \), induces an application \( \phi : g \to h \), defined by

\[
\phi(X) = i_X \theta, \ \forall X \in g,
\]

where \( i_X \theta \) is the element in \( h \) such that

\[
i_X \theta(k) = \theta_k(X(k)) = (L_{k^{-1}})_{s, k}(X(k)), \ \forall k \in K.
\]

Computing \( i_{\{ X, Y \}} \theta = \mathcal{L}_X(i_Y \theta) - i_Y(\mathcal{L}_X \theta) \), where \( \mathcal{L}_X \) denotes the operation of Lie derivative in the direction \( X \), and recalling that \( \theta \) satisfies the Maurer-Cartan equation, i.e. \( d\theta + \frac{1}{2}[\theta, \theta] = 0 \), one obtains

\[
i_{\{ X, Y \}} \theta = i_X(d_Y \theta) - i_Y(d_X \theta) + \| i_X \theta, i_Y \theta \|, \ \forall X, Y \in g,
\]

i.e. \( \phi \in \text{Cross}_{\text{inv}}^\omega(g, h) \). The application \( \phi \) is an invertible \( C^\infty(K) \)-linear map, such that \( \phi(X_x) = x \) for all \( x \in t \), where \( X_x \) is the left-invariant vector field defined by the element \( x \in t \). Applying the functor \( R \) defined in (2.3), one concludes that

\[
f \triangleright g := \phi^{-1}(f) g, \ \forall f, g \in h
\]

makes \( (h, \| - , - \|, \triangleright) \) into a post-Lie algebra, see Remark 10 for the definition of the right-hand side of (2.7). Moreover, \( \mathfrak{h} \) is the Lie algebra whose underlying vector space is \( C^\infty(K, t) \) and whose Lie bracket is

\[
\| f, g \| = \phi^{-1}(f) g - \phi^{-1}(g) f + \| f, g \|, \ \forall f, g \in C^\infty(K, t).
\]

Pulling back (2.7) to \( g \), one obtains a \( K \)-bilinear application \( \triangleright : g \otimes g \to g \), defined by

\[
X \triangleright Y = \phi^{-1}(\phi(X) \triangleright \phi(Y)), \ \forall X, Y \in g,
\]

which is \( C^\infty(K) \)-linear with respect to the first entry and such that

\[
X \triangleright (\xi Y) = X(\xi) Y + \xi X \triangleright X, \ \forall \xi \in C^\infty(K), \ X, Y \in \mathfrak{X}(K).
\]

This last identity, together with (2.7), implies that \( X \triangleright Y = 0 \) for all \( X \in \mathfrak{X}(K) \) and all \( Y \) left-invariant. In other words, \( \triangleright \) defines a flat linear connection on \( TK \), whose flat sections are the left-invariant vector fields, and whose torsion is easily shown to be parallel since \( T(X_x, X_y) = -X_{[x, y], h} \), for all \( x, y \in t \).

**Remark 11.** A couple of remarks are now in order.

1. Keeping the same notations introduced above, \( r \in \text{Hom}_K(h, g) \) is called an \( O \)-operator of weight \( \lambda \in \mathbb{R} \), if

\[
[r(x), r(y)]_g = r(v_{r(x)} y - v_{r(y)} x + \lambda [x, y]_h), \ \forall x, y \in h.
\]

The tuples \( (g, h, v, r) \) where \( r \) satisfies (2.9) form a category \( \text{CM}_{O, \lambda} \). The morphisms between \( (g, h, v, r) \) and \( (g', h', v', r') \) are pairs \( (f, g) \in \text{Hom}_\text{Lie}(g, g') \times \text{Hom}_\text{Lie}(h, h') \), satisfying (M2) and the analogue of (M1), i.e. \( f \circ r = r' \circ g \). The full subcategory of \( \text{CM}_{O, \lambda=1} \) of those tuples \( (g, h, v, r) \) whose \( r \) is invertible is isomorphic to \( \text{CM}_{\text{inv}} \); therefore, by Propositions 6 and 7, it is adjoint equivalent to \textbf{PostLie}. In this way we recover the description of post-Lie algebras given in [3].
(2) Let \( \text{CM}_b \) be the category whose objects are the tuples \((g, b, \varphi)\), where \( \varphi \in \text{Hom}_{\text{Lie}}(g, b \rtimes \text{Der}(b)) \) and the Lie bracket in \( b \rtimes \text{Der}(b) \) is defined by the formula
\[
\{ (h_1, d_1), (h_2, d_2) \} = ([h_1, h_2]_b + d_1(h_2) - d_2(h_1), [d_1, d_2])
\]
Note that composing \( g : g \to b \rtimes \text{Der}(b) \) with the canonical projections \( \pi_2 : b \rtimes \text{Der}(b) \to \text{Der}(b) \) and \( \pi_1 : b \rtimes \text{Der}(b) \to b \), one gets \( \psi \in \text{Hom}_{\text{Lie}}(g, \text{Der}(b)) \) and, respectively, \( \phi \in \text{Cross}^\psi(g, b) \). A morphism between two objects \((g, b, \varphi)\) and \((g', b', \varphi')\) in \( \text{CM}_b \) is a bijective linear map, is easily shown to be isomorphic to \( \text{CM}_{\text{inv}} \). Analogously the full subcategory \( \text{CM}_{b, \text{inv}} \subset \text{CM}_b \) whose objects are \((g, b, \varphi)\), where \( \phi \) is a bijective linear map, is easily shown to be isomorphic to \( \text{CM}_{\text{inv}} \). Analogously the full subcategory \( \text{CM}_{b, \text{id}} \subset \text{CM}_{b, \text{inv}} \) whose objects are \((g, b, \varphi)\), where \( g \) and \( b \) are defined on the same underlying vector space and \( \phi = \text{id} \), turns out to be isomorphic to \( \text{CM}_{\text{id}} \). In this way one recovers the description of \( \text{PostLie} \) given in [12].

2.2. Crossed morphism of Lie group type objects

In analogy to the Lie algebra case discussed in Subsection 2.1, one can define the notion of crossed morphism between two Lie groups.

First, recall that for \( H \) a Lie group, \( \text{Aut}(H) \) denotes the group of automorphisms of \( H \), whose elements are the \( \phi : H \to H \) such that
\[
\begin{align*}
(i) & \quad \phi \text{ is an isomorphism of abstract groups,} \\
(ii) & \quad \phi \text{ is a diffeomorphism.}
\end{align*}
\]

Definition 12. Let \( G \) and \( H \) be two Lie groups and let \( \Upsilon : G \to \text{Aut}(H) \) be a morphism of Lie groups. A crossed morphism relative to \( \Upsilon \) is a smooth map \( \Phi : G \to H \) that satisfies
\[
\Phi(gh) = \Phi(g) \Upsilon_g(\Phi(h)), \quad \forall g, h \in G. \quad (2.10)
\]
We let \( \text{Cross}^\Upsilon(G, H) \) denote the set of the crossed morphisms relative to \( \Upsilon \).

By changing the category of Lie algebras by the one of Lie groups in the definition of \( \text{CM} \), one obtains the following

Definition 13. The category \( \text{CM}_{\text{id}} \) is as follows. The objects are the tuples \((G, H, \Upsilon, \Phi)\) of two Lie groups \( G \) and \( H \) and \((\Upsilon, \Phi) \in \text{Hom}_{\text{LieGp}}(G, \text{Aut}(H)) \times \text{Cross}^\Upsilon(G, H) \). The morphisms between \((G, H, \Upsilon, \Phi)\) and \((G', H', \Upsilon', \Phi')\) are pairs \((f, g) \in \text{Hom}_{\text{LieGp}}(G, G') \times \text{Hom}_{\text{LieGp}}(H, H') \) such that
\[
g \circ \Phi = \Phi' \circ f \quad \text{and} \quad g(\Upsilon_x(a)) = \Upsilon'_{f(x)}(g(a)) \quad \text{for all} \ x \in G \text{ and} \ a \in H.
\]

In analogy to the Lie algebra case, one can define \( \text{CMGp}_{\text{id}} \) and \( \text{CMGp}_{\text{inv}} \), two subcategories of \( \text{CM}_{\text{id}} \) such that \( \text{CMGp}_{\text{id}} \subset \text{CMGp}_{\text{inv}} \), and one can prove that they are adjoint equivalent. Note that the projection functor
\[
P : \text{CMGp}_{\text{inv}} \to \text{CMGp}_{\text{id}}
\]
sends any tuple \((G, H, \Upsilon, \Phi)\) to \((\overline{H}, H, \Upsilon \circ \Phi^{-1}, \text{id})\), where the product of \( \overline{H} = (H, *) \) is given by
\[
h_1 * h_2 = h_1 \Upsilon^{-1}_{\Phi^{-1}(h_1)}(h_2) \quad \text{for all} \ h_1, h_2 \in H. \quad (2.11)
\]

The classical Lie functor gives rise to a functor
\[
T_e : \text{CMGp} \to \text{CM}
\]
that sends \((G, H, \Upsilon, \Phi)\) to \((g, b, \Upsilon_{\Phi}, \Phi_{\Upsilon}, \Phi_{\Upsilon})\). It restricts to the subcategories of invertible crossed morphisms.
It is worth observing that every Lie group is associated to a Lie algebra whose underlying vector space is the tangent space at \( 0 \) of the group. Proposition 14.

**Proposition 14.** \( R \circ T_e = T_e \circ P. \)

The previous constructions and remarks can be adapted almost verbatim to the case of local Lie groups. Instead of recalling the formal definition of this structure, we simply remind that a local Lie group is a smooth manifold \( M \) with a distinguished point \( e \) and two operations \( \mu \) and \( i \) only partially defined, i.e. defined on a suitable neighborhood of \( e \) and satisfying the following compatibility conditions

1. \( \mu(e,x) = x = \mu(x,e) \)
2. \( \mu(x,i(x)) = e = \mu(i(x),x) \)
3. \( \mu(\mu(x,y),z) = \mu(x,\mu(y,z)) \), for all \( x,y,z \in M \) sufficiently close to \( e \in M \).

To every local Lie group can be associated a Lie algebra whose underlying vector space is the tangent space at \( e \) and whose Lie bracket is defined restricting the canonical Lie bracket of \( \mathfrak{g} \) to \( \mathfrak{h} \), see Remark 20 below. Furthermore, there is an isomorphism of bialgebras \( \Theta \) which turns out to be responsible for the existence of the post-Lie Magnus expansion, which, for this reason, will be considered as a formal series. The categories introduced in the first part of this section have local analogues. More precisely, exchanging Lie groups for local Lie groups, one can define \( \text{CMGp}_{\text{loc}}, \text{CMGp}_{\text{inv}}^{\text{loc}} \) and, respectively, \( \text{CMG}_{\text{id}}^{\text{loc}} \). All the observations made about the categories \( \text{CMGp}, \text{CMGp}_{\text{inv}} \) and, respectively, \( \text{CMG}_{\text{id}} \) can be extended, verbatim, to their local versions.

### 3. Universal enveloping algebras and the post-Lie Magnus expansion

Recall that to a post-Lie algebra \( (\mathfrak{h}, \triangleright) \) one may associate two universal enveloping algebras \( U(\mathfrak{h}) \) and \( U' \). The latter inherits a non-canonical structure of bialgebra if endowed with the Grossman-Larson product *, which is defined by a suitable extension of the post-Lie product to \( U(\mathfrak{h}) \), see Remark 20 below. Furthermore, there is an isomorphism of bialgebras \( \Theta : U(\mathfrak{h}) \rightarrow (U(\mathfrak{h}), *) \) which turns out to be responsible for the existence of the \( p \text{LM}_{\text{p}} \) function \( \chi : H \rightarrow \mathfrak{h} \). It is worth noting that the two local Lie groups \( H \) and \( \mathfrak{h} \) share the same underlying vector space, which is also the vector space underlying the Lie algebras \( \mathfrak{h} \) and \( \mathfrak{h} \). For this reason \( \chi \) can be thought as a map from \( H \) to \( \mathfrak{h} \) (or from \( \mathfrak{h} \) to \( \mathfrak{h} \), see the Introduction), see also [35, Remark 33].

This section is aimed to present a categorification of the previous picture. More precisely, first is defined a functor \( U : \text{CM} \rightarrow \text{Pbialg} \), which provides the above data \( (U(\mathfrak{h}), U(\mathfrak{h}), \Theta) \) when restricted to \( \text{CM}_{\text{id}} \). Then is defined an integration functor which gives rise to the \( p \text{LM}_{\text{e}} \). The following diagram gives an overview of the functors considered in the previous and present sections; the bottom line corresponds to the above discussion.
Notation 15. The universal enveloping algebra of a post-Lie algebra \((\mathfrak{h}, \triangleright)\) is the universal enveloping algebra of the underlying Lie algebra \(\mathfrak{h}\). It is a bialgebra when endowed with the shuffle coproduct \(\Delta_{\mathfrak{h}} : \mathcal{U}(\mathfrak{h}) \rightarrow \mathcal{U}(\mathfrak{h})^{\otimes 2}\). It may be useful to consider Sweedler’s notation (without sum):

\[
\Delta_{\mathfrak{h}}(X) = X_1 \otimes X_2
\]

for all \(X \in \mathcal{U}(\mathfrak{h})\).

Definition 16. The category \(\textbf{Pbialg}\) is as follows. The objects are tuples \((A, B, \theta)\) where \(A\) and \(B\) are bialgebras, \(B\) is an \(A\)-module and \(\theta : A \rightarrow B\) is a morphism of \(A\)-modules and coalgebras. The morphisms from \((A, B, \theta)\) to \((A', B', \theta')\) are pairs \((f, g)\) such that:

- \(f : A \rightarrow A'\) is a morphism of bialgebras;
- \(g : B \rightarrow B'\) is a morphism of coalgebras satisfying \(g(a \cdot b) = f(a) \cdot g(b)\) for all \(a \in A\) and \(b \in B\); and
- \(g \circ \theta = \theta' \circ f\).

Let \(\textbf{Pbialg}_{\text{inv}}\) be the subcategory of \(\textbf{Pbialg}\) of those tuples \((A, B, \theta)\) such that \(\theta\) is an isomorphism.

Remark 17. If \((A, B, \theta)\) belongs to \(\textbf{Pbialg}_{\text{inv}}\) then \(B\) has another bialgebra structure, given by \(b * b' := \theta^{-1}(b) \cdot b\) for all \(b, b' \in B\). Moreover, \(\theta : A \rightarrow (B, *)\) is an isomorphism of bialgebras.

Let \(\mathcal{U} : \text{CM} \rightarrow \text{Pbialg}\) be the functor that associates to each tuple \((g, \mathfrak{h}, v, \varphi)\) the tuple \((\mathcal{U}(g), (\mathcal{U}(\mathfrak{h}), M_{(v, \varphi)}, \Theta_{(v, \varphi)})\).

The constructions of the action \(M_{(v, \varphi)} : \mathcal{U}(g) \rightarrow \text{End}_K(\mathcal{U}(\mathfrak{h}))\) and of the morphism \(\Theta_{(v, \varphi)}\) recalled here below are straightforward generalizations of analogue ones presented in [35, Section 5], to which we refer the reader for further details. Since \(v\) takes values in \(\text{Der}(\mathfrak{h})\), it can be extended to an application with values in the derivations of (the associative algebra) \(\mathcal{U}(\mathfrak{h})\). By keeping the same notation for this extension, this means that \(v_x(XY) = Xv_x(Y) + v_x(X)Y\) for each \(x \in g\) and \(X, Y \in \mathcal{U}(\mathfrak{h})\). Let \(\sigma^\phi : g \rightarrow \text{End}_K(\mathcal{U}(\mathfrak{h}))\) be the linear application defined by

\[
\sigma^\phi(x)(X) = \phi(x) \cdot X \text{ for all } x \in g \text{ and } X \in \mathcal{U}(\mathfrak{h}),
\]

and let \(M_{(v, \varphi)} : g \rightarrow \text{End}_K(\mathcal{U}(\mathfrak{h}))\) be the linear map defined by

\[
M_{(v, \varphi)}(x) = v_x + \sigma^\phi_x, \text{ for all } x \in g.
\]  

The following result shows that \(M_{(v, \varphi)}\) extends to a morphism of associative algebras \(M_{(v, \varphi)} : \mathcal{U}(g) \rightarrow \text{End}_K(\mathcal{U}(\mathfrak{h}))\), providing the desired action map.
Lemma 18. For all \(x, y \in \mathfrak{g}\), one has
\[
M_{(v, \phi)}([x, y]) = [M_{(v, \phi)}(x), M_{(v, \phi)}(y)].
\] (3.2)

In other words, \(\mathcal{U}(\mathfrak{h})\) carries a structure of a \((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})\)-module defined by (3.1).

Proof. For every \(x, y \in \mathfrak{g}\) and \(a \in \mathfrak{h}\), it suffices to compare
\[
M_{(v, \phi)}([x, y])_{\mathfrak{g}}(a) \quad \text{with} \quad [M_{(v, \phi)}(x), M_{(v, \phi)}(y)](a),
\]
recalling that \(\phi\) satisfies (2.1) and \(v \in \text{Hom}_{\text{Lin}}(\mathfrak{g}, \text{Der}(\mathfrak{h}))\).

The map \(\Theta_{(v, \phi)} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})\) is defined first on every monomial \(X \in \mathcal{U}(\mathfrak{g})\) by
\[
\Theta_{(v, \phi)}(X) = M_{(v, \phi)}(X)(1)
\]
and then it is extended to all \(\mathcal{U}(\mathfrak{g})\) by linearity. It is both a morphism of coalgebras and of left \(\mathcal{U}(\mathfrak{g})\)-modules; see [19] and also [35, Proposition 28]. In order to simplify the notation used in the next result, where it is shown that \(\mathcal{U}\) is functor, we will write \(M_A\) for \(M_{(v, \phi)}(A)\) and \(M'_B\) for \(M_{(v', \phi')}(B)\) for any \(A \in \mathcal{U}(\mathfrak{g})\) and \(B \in \mathcal{U}(\mathfrak{h})\).

Lemma 19. Let \((f, g) : (\mathfrak{g}, \mathfrak{h}, v, \phi, \{a, b\}) \to (\mathfrak{g}', \mathfrak{h}', v', \phi').\) For all \(A \in \mathcal{U}(\mathfrak{g})\) and \(B \in \mathcal{U}(\mathfrak{h})\), one has
\[
\mathcal{U}(g)(M_A(B)) = M'_{\mathcal{U}(f)(A)}(\mathcal{U}(g)(B)).
\]
In particular, one has \(\mathcal{U}(g) \circ \Theta_{(v, \phi)} = \Theta_{(v', \phi')} \circ \mathcal{U}(f)\).

Proof. By linearity, it is enough to show the result for \(A = a_1 \cdots a_m \in \mathcal{U}(\mathfrak{g})\) and \(B = b_1 \cdots b_n \in \mathcal{U}(\mathfrak{h})\) being two monomials. The proof is by induction on \(m\). For \(m = 1\), one has
\[
\mathcal{U}(g)(M_A(B)) = \mathcal{U}(g)(v_A(B) + \phi(A)B) = \mathcal{U}(g)\left(\sum_{1 \leq i \leq n} b_1 \cdots b_{i-1}v_A(b_i)b_{i+1} \cdots b_n + \phi(A)B\right) = \sum_{1 \leq i \leq n} g(b_1) \cdots g(b_{i-1})v'_{f(a)}(g(b_i))g(b_{i+1}) \cdots g(b_n) + \phi'(f(a))\mathcal{U}(g)(B) = v'_{f(a)}(\mathcal{U}(g)(B)) + \phi'(f(a))\mathcal{U}(g)(B) = M'_{\mathcal{U}(f)(a)}(\mathcal{U}(g)(B)).
\]
Let \(m \geq 2\). Remark that \(v_{a_1}(M_{a_2}(\cdots (M_{a_m}(B)) \cdots)\) can be written as a sum of terms of the form \(C_1v_{a_1}(C_2)C_3\) where each \(C_i \in \mathcal{U}(\mathfrak{g})\) are monomial of the following form. By writing \(C_i\) as \(c_{k_1} \cdots c_{k_r}\), the term \(c_r\) is of the form \(\phi(a_i)\), or \(v_{a_{j_1}}(v_{a_{j_2}}(\cdots v_{a_{j_r}}(\phi(a_{j_{r+1}})) \cdots))\) or \(v_{a_{j_1}}(v_{a_{j_2}}(\cdots v_{a_{j_{r'}}}(B) \cdots))\) for some indices \(\{j_1, \ldots, j_{r+1}\} \subset \{1, \ldots, m\}\). Consequently, one has
\[
\mathcal{U}(g)(v_{a_1}(M_{a_2}(\cdots (M_{a_m}(B)) \cdots)) = v'_{f(a_1)}(M'_{f(a_2)}(\cdots (M'_{f(a_m)}(\mathcal{U}(g)(B)) \cdots))
\]
Therefore, one has
\[
\mathcal{U}(g)(M_{a_1 \cdots a_m}(B)) = \mathcal{U}(g)(v_{a_1}(M_{a_2 \cdots a_m}(B)) + \phi(a_1)M_{a_2 \cdots a_m}(B)) = M'_{\mathcal{U}(f)(a_1 \cdots a_m)}(\mathcal{U}(g)(B))
\]
Note that if \(\phi\) is invertible, then \(\Theta_{(v, \phi)}\) is invertible as well; see [35, Theorem 29]. Therefore, \(\mathcal{U}\) restricts to a functor
\[
\mathcal{U} : \text{CM}_{\text{inv}} \to \text{Pbialg}_{\text{inv}}.
\]
Since \((t, R)\) is an adjoint equivalence, the counit provides a natural isomorphism \(\Psi = U\epsilon : RU \rightarrow U\). In particular one has

\[
\Theta_{(\circ \phi^{-1}, id)} = \Theta_{(\phi, \phi)} \circ U(\phi^{-1}).
\]  

(3.4)

**Remark 20.** Remark 17 entails that \(\Theta_{(\circ \phi^{-1}, id)}\) is a morphism of bialgebras, so that one recovers the initial viewpoint of [19], see also [38, 39]. In particular, the resulting * product on \(U(\mathfrak{g})\) is the Grossman-Larson product, whose construction is recalled below. The post-Lie product on \(\mathfrak{h}\), defined via (2.4), can be extended to a map \(\star : U(\mathfrak{h})^\otimes 2 \rightarrow U(\mathfrak{h})\) with the following properties. For all \(X, Y\) and \(Z\) in \(U(\mathfrak{h})\) and \(x \in \mathfrak{h}\), one has:

\[
\begin{align*}
(\text{D1}) & \quad 1 \triangleright X = X \quad \text{and} \quad X \triangleright 1 = 0; \\
(\text{D2}) & \quad X \triangleright (Y \cdot Z) = (X_1 \triangleright Y) \cdot (X_2 \triangleright Z); \quad \text{and}, \\
(\text{D3}) & \quad (x \cdot X) \triangleright y = x \cdot (X \triangleright y) = (x \triangleright X) \triangleright y.
\end{align*}
\]

This extended product gives rise to a \(D\)-bialgebra \((U(\mathfrak{h}), \Delta_{\mathfrak{h}}, \triangleright)\); see [35, Definition 19]. The Grossman-Larson product is given by

\[
\star : U(\mathfrak{h})^\otimes 2 \rightarrow U(\mathfrak{h})
\]

\[
X \otimes Y \mapsto X_1(X_2 \triangleright Y).
\]

(3.5)

### 3.1. Integration of post-Lie algebras

Let \(CM_{\text{fin}}^\text{inv}\) denote the subcategory of \(CM_{\text{inv}}\) of those tuples whose Lie algebras are finite dimensional and let \(Pbialg_{\text{inv}}\) be the category obtained from \(Pbialg_{\text{inv}}\) by requiring that the corresponding bialgebras are complete. After completion, the functor \(U : CM_{\text{fin}}^\text{inv} \rightarrow Pbialg_{\text{inv}}\) induces a functor \(\tilde{U} : CM_{\text{fin}}^\text{inv} \rightarrow Pbialg_{\text{fin}}\) whose image defines a category \(\text{Im}(\tilde{U})\). More precisely, the objects of \(\text{Im}(\tilde{U})\) are tuples of the form \((\tilde{U}(\mathfrak{g}), (\tilde{U}(\mathfrak{h}), M_{(v, \phi)}), \Theta_{(v, \phi)}) = \tilde{U}(\mathfrak{g}, \mathfrak{h}, v, \phi)\), and its morphisms are of the form \(\tilde{U}(f, g)\), where \((f, g)\) are morphisms in \(CM_{\text{fin}}^\text{inv}\).

**Remark 21.** Hereafter we will work with local groups and with completed universal enveloping algebras. In particular if \(G\) is the local group \((\mathfrak{g}, \text{BCH}\_\mathfrak{g})\), we will denote by \(\exp_{\mathfrak{g}}\) the map between \(G\) and \(\tilde{U}(\mathfrak{g})\) which associates to every \(x \in G\) the group-like element \(\exp_{\mathfrak{g}}(x) = \sum_{j \geq 0} \frac{x^j}{j!}\). This exponential map is an isomorphism (of groups) from \(G\) to the group of group-like elements in \(\tilde{U}(\mathfrak{g})\), whose inverse will be denoted by \(\log_{\mathfrak{g}}\), see [26, 41, Appendix A] and [35, Section 5].

Finally, we can introduce the integration functor

\[
\text{Int} : \text{Im}(\tilde{U}) \rightarrow CM_{\text{loc}}^\text{Gp},
\]

(3.6)

which associates to any tuple \((\tilde{U}(\mathfrak{g}), (\tilde{U}(\mathfrak{h}), M_{(v, \phi)}), \Theta_{(v, \phi)})\) in \(\text{Im}(\tilde{U})\) the tuple \((G, H, Y_{(v, \phi)}, \Phi_{(v, \phi)})\) in \(CM_{\text{loc}}^\text{Gp}\). More precisely, the map \(Y_{(v, \phi)} : G \rightarrow \text{Aut}(H)\) is given by

\[
Y_{(v, \phi)}(y) = \text{Exp}(v),
\]

where \(v_\mathfrak{g}(y) := M(x)(y) - \Theta(x) \cdot y\) for all \(x \in \mathfrak{g}\) and \(y \in \mathfrak{h}\).

**Lemma 22.** \(Y_{(v, \phi)}\) is a morphism of local groups.

**Proof.** Note that if \(d \in \text{Der}(\mathfrak{g})\), then \(\text{Exp}(d) \in \text{Aut}(\mathfrak{g}) \subseteq \text{Aut}(G)\). Furthermore, if \(\mathfrak{g}\) and \(\mathfrak{h}\) are two (finite dimensional) Lie algebras and \(v \in \text{Hom}_{\text{Lie}}(\mathfrak{g}, \text{Der}(\mathfrak{h}))\), then one has
\[
\text{Exp} \left( v_x \right) \text{Exp} \left( v_y \right) = \text{Exp} \left( \text{BCH}_{\text{End}(h)}(v_x, v_y) \right) = \text{Exp} \left( v_{\text{BCH}(x, y)} \right), \quad \forall x, y \in \mathfrak{g},
\]
which gives the result. \hfill \blacksquare

The map \( \Phi_{(v, \phi)} : \mathcal{G} \rightarrow \mathcal{H} \) is defined by
\[
\Phi_{(v, \phi)} = \log_h \circ \Theta_{(v, \phi)} \circ \exp_h.
\]

Lemma 23. \( \Phi_{(v, \phi)} \) is a crossed morphism of local groups.

Proof. It is a direct consequence of Theorem 25 stated hereafter. Indeed formula (3.8) can be written as
\[
\Phi_{(v, \phi)}(\text{BCH}_h(x, y)) = \text{BCH}_h(\Phi_{(v, \phi)}(x), \text{Exp}(v_x \Phi_{(v, \phi)}(y))).
\]

After these preliminary remarks, a direct verification shows that (3.6) is indeed a functor.

3.2. The post-Lie Magnus expansion

Observe that since \( \phi \) is invertible, so is \( \Phi_{(v, \phi)} \). Its inverse \( \chi_{(v, \phi)} : \mathcal{H} \rightarrow \mathcal{G} \) is therefore given by
\[
\chi_{(v, \phi)} = \log_h \circ (\Theta_{(v, \phi)})^{-1} \circ \exp_h. \tag{3.7}
\]

Definition 24. The map \( \chi_{(v, \phi)} \) is called the post-Lie Magnus expansion associated to \((g, h, v, \phi) \in \text{CM}_{\text{inv}}\).

In analogy to [35, Proposition 39], one can prove the following result.

Theorem 25. For all \( a, b \in \mathfrak{h} \), one has
\[
\text{BCH}_h(\chi_{(v, \phi)}(a), \chi_{(v, \phi)}(b)) = \chi_{(v, \phi)}(\text{BCH}_h(a, \text{Exp}(v_{\chi_{(v, \phi)}(a)}b))). \tag{3.8}
\]

The proof of this result is based on the following two preliminary lemmas. First recall that the bialgebra \( \mathcal{U}(\mathfrak{h}) \) can be endowed with another product \( \ast \), see Remark 17. Also recall that, by (2.4), the map \( \triangleright : a \otimes b \mapsto v_{\phi^{-1}(a)}(b) \) defines a post-Lie product on \( \mathfrak{h} \).

Let \( z : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{h} \) be defined by
\[
a \triangleleft z b = \log_h(\text{exp}_h(a) \ast \text{exp}_h(b)) \quad \text{for all } a, b \in \mathfrak{h}.
\]

Lemma 26. For all \( a, b \in \mathfrak{h} \) one has
\[
a \triangleleft z b = \text{BCH}_h(a, \text{exp}_h(a) \triangleright b).
\]

Proof. This result was proven in [26] and that proof extends without modification to this context. \hfill \blacksquare

Lemma 27. For all \( a, b \in \mathfrak{h} \) one has
\[
\text{exp}_z(a) \triangleright b = \text{Exp}(v_{\chi_{(v, \phi)}(a)})b, \tag{3.9}
\]
where the right hand side of the previous formula reads as
\[
b + v_{\chi_{(v, \phi)}(a)}(b) + \frac{1}{2} v_{\chi_{(v, \phi)}(a)}(v_{\chi_{(v, \phi)}(a)}(b)) + \frac{1}{3!} v_{\chi_{(v, \phi)}(a)}(v_{\chi_{(v, \phi)}(a)}(v_{\chi_{(v, \phi)}(a)}(b))) + \cdots
\]
Proof. Recall that the extension of the post-Lie product to $U(h)$ endowed the latter with a structure of a $D$-bialgebra, see Remark 20 and reference therein. In this case one has

(i) $a \ast A = a \cdot A + a \triangleright A$, for all $a \in h$, see Formula (4.20) pag. 570 in [35];
(ii) $(a \cdot A) \triangleright a' = a \triangleright (A \triangleright a') - (a \triangleright A) \triangleright a'$, see Formula (D.5) pag. 566 in [35],

for all $a, a' \in h$ and $A \in U(h)$. Plugging (ii) into (i) one obtains

$$(a \ast A) \triangleright a' = a \triangleright (a \triangleright A).$$

The proof of the statement follows now from a simple induction on the length of the monomials in the right-hand side of (3.9), applying (i) and (ii) above recalled.

After these observations the proof of Theorem 25 follows along the lines of the proof of the analogue result presented in [35] and for this reason it will not be proposed here again.

Before closing this section it is worth observing that the functoriality of the pLMe's construction entails the existence of relations between pLMe defined by morphic objects in $\mathcal{CM}^{\text{fin}}_{\text{inv}}$. More precisely let $X = (g, h, v, \phi)$ be an object of $\mathcal{CM}^{\text{fin}}_{\text{inv}}$. Then, the application of Int to (3.4) yields

$$\Phi_{(v \circ \phi^{-1}, \text{id})} = \Phi_{(v, \phi)} \circ \text{Int}(\Upsilon(\phi^{-1})).$$

In other words, one has

$$\Upsilon_{(v, \phi)} = \phi^{-1} \circ \Upsilon_{(v \circ \phi^{-1}, \text{id})}.$$

Let $(f, g) : (g, h, v, \phi) \to (g', h', v', \phi')$ be a morphism in $\mathcal{CM}^{\text{fin}}_{\text{inv}}$. By applying $\text{Int} \circ \Upsilon \circ R$, one obtains

$$\Phi_{(v \circ \phi^{-1}, \text{id})} \circ \text{Int}(\Upsilon(g)) = \text{Int}(\Upsilon(g)) \circ \Phi_{(v \circ \phi^{-1}, \text{id})}.$$

In particular, if $(f, g)$ is an isomorphism one has

$$\Upsilon_{(v \circ \phi^{-1}, \text{id})} = g^{-1} \circ \Upsilon_{(v \circ \phi^{-1}, \text{id})} \circ g.$$

4. Computing the post-Lie Magnus expansion

In this section we aim to present two methods to compute the coefficients of the pLMe. Both rely on the notion of nested tubing which, in turn, is based on the combinatorial description of the free post-Lie algebra in terms of planar rooted trees. More precisely, the first method makes use of the vertical nested tubings and it allows to compute, recursively, the coefficients associated to any forest. On the other hand, the second method utilizes the horizontal nested tubings and it allows to express the coefficients of the pLMe in a closed form. This section is divided in two parts. The first encloses the necessary background to introduce the nested tubings, which will be the subject of the second part, where it will be explained how they can be used to compute the coefficients of the pLMe.

4.1. Planar trees and forests

Definition 28. A planar rooted tree is an isomorphism class of contractible graphs, embedded in the plane and endowed with a distinguished vertex, called the root, to which is attached an adjacent half-edge, called the root-edge of the planar tree.
Pictorially, our trees are drawn with the root at the bottom and the order on the set of the incoming edges of a vertex is given by the clockwise direction, i.e. from the left to the right.

For a planar rooted tree $T$, we let $V(T)$ be the set of all its vertices. On it, we consider two orders:

- The level partial order $<$ is defined by orienting the edges of $T$ toward the root, except the root-edge. Given two vertices $u$ and $v$ of $V(T)$, we write $v < u$ if there is a string of oriented edges from $v$ to $u$. In particular, the root is maximal for this partial order.
- The canonical linear order $<$ is defined as follows. The vertices are listed in increasing order, accordingly with their first appearance in the path starting from the root vertex and running along the tree in the clockwise direction. In particular, the root vertex is the minimal element of $<$. 

**Example 29.** The set of the vertices of the following planar rooted tree $T$,

![Tree](image)

is $V(T) = \{a, b, c, d, e, f\}$. The level partial order is given by $f < c, e < c, c < a, d < a$ and $b < a$. The canonical linear order is given by $a < c < f < e < d < b$.

From now on, when there is no ambiguity, planar rooted trees are simply called trees.

**Example 30.** For every two trees $R$ and $S$, let $C(\bullet; R, S)$ be the corolla with an unlabeled vertex $v$ of arity 2 as root; the roots of $R$ and $S$ are input edges of $v$ in this order.

**Definition 31.** Let $T$ be a tree and let $v$ be one of its vertices. Consider a small disk centered in $v$. The outgoing and incoming edges of $v$ cut the disk into connected components. If $v$ has at least one incoming edge, the left side of $v$ is the connected component delimited by the outgoing edge and the first incoming edge of $v$. Otherwise, its left side is the unique connected component of the cut disk.

**Example 32.** A vertex $v$ and its left side (the darkest gray region):

![Tree](image)

**Definition 33.** A forest is a (non commutative) word of trees. For $n \geq 1$, the forest of $n$ times the tree with one vertex is denoted by $\bullet^{\times n}$ and is called horizontal.

Trees and forests can be grafted at vertices as follows.

**Notation 34.**

1. For any two trees $R$ and $T$ we let $R \triangleright_v T$ be the tree obtained by grafting the root-edge of $R$ at the vertex $v$ on its left side.

2. In order to define the grafting of two forests it is convenient to introduce some notations. Let $E$ be a forest composed by $n$ trees. Every partition of $n$, $n = n_0 + n_1 + \ldots + n_k$, such that $n_i \geq 1$ for $1 \leq i \leq k$ and $n_0 \geq 0$, defines a partition of $E$ into $k + 1$ subforests $E_0, \ldots, E_k$ defined as follows. For every $l \in \{0, 1, \ldots, k\}$, the subforest $E_l$ is obtained from $E$ by deleting the first $n_0 + \cdots + n_{l-1}$ trees and the last $n_{l+1} + \cdots + n_k$ trees. Let $F$ be a forest, $v_1, \ldots, v_k$
be \( k \) vertices of \( F \) and \( n = n_0 + n_1 + \ldots + n_k \) be a partition as above. The forest \( E^{n_0,n_1,\ldots,n_k}F \) is obtained from \( E \) and \( F \) as follows:
- concatenating the subforest \( E_0 \) to the left of \( F \);
- for \( 1 \leq r \leq k \) grafting all the roots of \( E_r \) to the left-side of \( v_r \).

In particular,
- for \( k = 0 \), the operation \( \times \) is the concatenation operation that we simply denote by \( \times \);
- for \( k = 1 \) and \( n_0 = 0 \), the operation \( \triangleright \) is the grafting of all the roots to a single vertex \( v \) that we simply denote by \( \triangleright v \);
- for \( n_0 = 0 \), we write \( \triangleright n_1,\ldots,n_k \) as \( \triangleright v_1,\ldots,v_k \), as this operation involves only grafting and no concatenation.

For instance, one has (colors are for illustration purposes only)

\[
\begin{align*}
\triangleright v & = \quad \text{and} \quad \triangleright v' = \quad \text{and} \quad \times_{v_1,v_2}^{1,2,1} (v_1, v_2) = \\
\end{align*}
\]

In the first case \( k = 1 \), \( n_0 = 0 \) and \( n_1 = 1 \); in the second case one has \( k = 1 \), \( n_0 = 0 \) and \( n_1 = 2 \); in the last case one has \( k = 2 \), \( n_0 = 1 \), \( n_1 = 2 \) and \( n_3 = 1 \).

We will also be led to consider trees with labelings, or more in general, with partial labelings.

**Definition 35.** Let \( T \) be a tree and let \( U \) be a subset of \( V(T) \). A \( U \)-label of \( T \) is a bijection \( l : U \to \{1, \ldots, n\} \). A tree \( T \) equipped with a \( U \)-label is called partially labeled.

**Example 36.** Examples of partially labeled trees:

**4.2. Definition of \( \mathcal{PSB} \)**

To make this paper as self-contained as possible, in this subsection is collected a few background information about the operad \( \mathcal{PSB} \); for more details we refer the reader to [35, Section 3].

For \( n \geq 1 \), let \( \mathcal{L}(n) \) be the \( \mathbb{K} \)-vector space generated by the fully labeled trees with \( n \) vertices. For each \( n \geq 2 \) let \( \mathcal{W}(n) \) be the \( \mathbb{K} \)-vector space generated by trees \( T \) with partial labeling \( l : U \to \{1, \ldots, n\} \) that satisfy:

\( \begin{align*}
(\text{a}) \quad & \text{the root of } T \text{ is unlabeled;} \\
(\text{b}) \quad & \text{if a vertex of } T \text{ is unlabeled, then so is its } \prec \text{-successor;} \\
(\text{c}) \quad & \text{each unlabeled vertex of } T \text{ has exactly two incoming edges.}
\end{align*} \)

Let

\[
\mathcal{LW}(1) := \mathcal{L}(1) \quad \text{and} \quad \mathcal{LW}(n) := \mathcal{L}(n) \oplus \mathcal{W}(n) \quad \text{for } n \geq 2.
\]

In [35], a structure of operad was provided on the collection \( \{\mathcal{LW}(n)\}_n \). One may therefore consider the following ideal \( \mathcal{I} \subset \mathcal{LW} \) generated by

\[
\left\{ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3
\end{array}
\begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} \right\}.
\]
For each \( n \geq 1 \), we let \( \mathcal{PSB}(n) := \mathcal{LW}(n)/\mathcal{I}(n) \).

**Theorem 37.** [35] The collection \( \{ \mathcal{PSB}(n) \}_{n} \) is endowed with a structure of symmetric operad which makes it isomorphic to the operad \( \mathcal{PostLie} \).

By way of example, hereafter we spell out the operadic composition of \( \mathcal{PSB} \) in the case of two fully labeled trees \( T \in \mathcal{PSB}(m) \) and \( R \in \mathcal{PSB}(n) \). Let \( v \) be the vertex of \( T \) labeled by \( i \) and let \( k \) be the number of its incoming edges. For a given map \( \phi : \{1, ..., k\} \rightarrow V(R) \), let \( T \circ_i^\phi R \) be the tree obtained by replacing the vertex labeled by \( i \) with the tree \( R \), and then grafting the incoming edges of \( i \) to the labeled vertices of \( R \) following the map \( \phi \). The grafting is performed respecting the natural order of each fiber of \( \phi \). This means that if \( \phi^{-1}(v) = \{i_1 < i_2 < \cdots < i_k\} \subset \{1 < \cdots < k\} \), then, in the resulting tree, the incoming edge, obtained from the grafting of the \( i \)th incoming edge, is the \( r \)th incoming edge of \( v \). Finally, the labeling of \( T \circ_i^\phi R \) is given by classical re-indexation and the operadic composition of \( T \) and \( R \) at \( i \) is:

\[
T \circ_i R = \sum_{\phi} T \circ_i^\phi R,
\]

where \( \phi \) runs through the set of maps from \( \{1, ..., k\} \) to \( V(R) \). For instance, one has

\[
\begin{align*}
2 & \quad \cdot \quad 3 & \quad \circ_i & \quad 2 & \quad 1 \\
1 & \quad 1 & & & \\
\end{align*} = \begin{align*}
3 & \quad \cdot \quad 4 & \quad 2 & \quad + & \quad 3 & \quad \cdot \quad 2 & \quad + & \quad 4 & \quad 2 & \quad + & \quad 3 & \quad 2 & \quad 1 \\
1 & \quad 1 & & & \\
\end{align*}
\]

### 4.3. The free post-Lie algebra

Given an operad \( O \) and a vector space \( V \), we denote by \( O(V) \) the free \( O \)–algebra generated by \( V \). It is explicitly given by \( O(V) = \bigoplus_{n \geq 0} O(n) \otimes_{\mathbb{S}_n} V^\otimes n \). By Theorem 37, we know that \( \mathcal{PSB}(\mathbb{K}) \) is the free post-Lie algebra on \( \mathbb{K} \), which is the vector space generated by the trees of \( \mathcal{PSB} \) with a unique label. In other words, if we let \( \mathbb{K} = \mathbb{K} < \bullet > \) for a generator \( \bullet \), then \( \mathcal{PSB}(\mathbb{K}) \) is generated by the set

\[
G = \left\{ \begin{array}{c}
\begin{array}{c}
\circ \quad 1 \\
\bullet \\
\end{array},
\begin{array}{c}
\circ \quad 1 \\
\bullet \\
\end{array},
\begin{array}{c}
\circ \quad 1 \\
\bullet \\
\end{array}
\end{array} \right\}.
\]

Here, by abuse of notation, we represented classes of trees as trees. Let us distinguish the subset \( G_\bullet \) of those classes of trees that have at least one round-shape vertex (i.e. the generating set of the Lie elements); let \( G_\circ \) be its complementary.

The operadic structure of \( \mathcal{PSB} \) provides both the Lie and the post-Lie product of any two elements. Explicitly, the Lie product of two generators \( R \) and \( S \) is the class of the tree \( C(\bullet; R,S) \) (see Example 30); their post-Lie product \( R \triangleright S \) is as follows.

**Definition 38.** For a tree \( T \in G \), let \( V_\bullet(T) \) and \( V_\circ(T) \) be the sets of round-shape and square-shape vertices of \( T \), respectively.

If \( R \) is a tree in \( G \setminus G_\circ \) and \( S \in G \), then the post-Lie product of \( R \) and \( S \) is given by

\[
R \triangleright S = \sum_{\nu \in V_\circ(S)} R \triangleright_\nu S.
\]

If \( R \) is in \( G_\circ \), in order to express the post-Lie product \( R \triangleright S \) we introduce a few auxiliary operations. Let us recall from [35, Section 3.3.1] the vertex-wise action of symmetric groups. For a tree \( T \) in \( \mathcal{W}(n) \), the symmetric group \( \mathbb{S}_2 \) acts on a round-shape vertex of \( T \) by switching its two
inputs. More precisely, for a round-shape vertex \( v \) of \( T \), the maximal subtree of \( T \) with root \( v \) can be written as \( C(v; T_1, T_2) \). The vertex-wise permutation of \( v \) by \( \sigma \in S_2 \) is given by replacing in \( T \) the subtree \( C(v; T_1, T_2) \) with \( C(v; T_{\sigma(1)}, T_{\sigma(2)}) \). More generally, if \( T \) has \( q = |V_\bullet(T)| \) round-shape vertices, each tuple of transpositions \( \sigma = (\sigma_1, \ldots, \sigma_q) \in S_2^q \) defines a tree \( T_\sigma \) as follows. Consider the ordered subset \( V_\bullet(T) \) of \( V(T) \) endowed with the canonical linear order. One first permutes the inputs of the \( q \)th round-shape vertex of \( T \) according to \( \sigma_q \); this gives a tree \( T' \). Then, one permutes the \((q-1)\)th vertex of \( V_\bullet(T') \) according to \( \sigma_{q-1} \); this gives a tree \( T'' \), and so on. The resulting tree is denoted by \( T_{\sigma} \). For instance, if \( \sigma = (\tau, \tau) \) and \( \sigma' = (\tau, id) \), where \( \tau \) denotes the non-trivial transposition of \( S_2 \), then for

\[
T = \begin{array}{c}
\tau_2 \\
v_1
\end{array}
\quad \text{one has} \quad T_{\sigma} = \begin{array}{c}
\tau_3 \\
v_2
\end{array}
\quad \text{and} \quad T_{\sigma'} = \begin{array}{c}
\tau_2 \\
v_1
\end{array}
\begin{array}{c}
\tau_3 \\
v_2
\end{array}.
\]

If, for each \( q \geq 1 \), one lets \( \mathcal{W}(n)_q \) be the sub vector space of \( \mathcal{W}(n) \) generated by the trees with \( q \) round-shape vertices, then for each element of \( S_2^q \) one obtains a linear map \( \mathcal{W}(n)_q \to \mathcal{W}(n)_q \) by extending linearly the above assignation \( T \mapsto T_{\sigma} \).

Recall also that any tree \( T \) in \( \mathcal{W}(n) \) can be contracted into a tree \( Con(T) \) with only one round-shape vertex; the latter is obtained by contracting all the edges between round-shape vertices. Note that this vertex may have more than two inputs, so the tree \( Con(T) \) may not belong to \( \mathcal{W}(n) \). One extends such an assignment by linearity to a map from \( \mathcal{W}(n) \) to the vector space generated by trees with a \( U \)-labeling \( l : U \to \{1, \ldots, n\} \). For instance,

\[
Con\left(\begin{array}{c}
\bullet \\
\bullet
\end{array}\right) = \begin{array}{c}
\bullet \\
\bullet
\end{array}.
\]

Given two trees \( T \) and \( T' \) and a vertex \( v \) of \( T' \), we let \( T \triangleright_v Con T' \) be the tree obtained from \( T \triangleright_v T' \) by contracting the edge between the root of \( T \) and \( v \); the resulting vertex inherits the shape of \( v \).

For \( R \) in \( G_\bullet \) and \( S \in G \), one has

\[
R \triangleright S = \sum_{v \in V_\bullet(S)} \sum_{\sigma \in S_2^{|V_\bullet(R')|}} \epsilon(\sigma) Con(R'_\sigma) \triangleright_v Con S,
\]

where \( R' \) is any representative of \( R \), and the sign \( \epsilon(\sigma) \) is the product of the signatures \( sgn(\sigma_1) \cdots sgn(\sigma_q) \) for \( \sigma = (\sigma_1, \ldots, \sigma_q) \). We refer to [35, Section 3.1] for more details.

Let us interpret \( Con(R'_\sigma) \triangleright_v Con S \) in terms of grafting of forests: If \( R \) has \( k \) round-shape vertices, then \( Con(R') \) is a corolla of \( k \) trees \( T_1, \ldots, T_k \) of \( G_\bullet \). Therefore, for each \( \sigma \) as above, one has

\[
Con(R'_\sigma) \triangleright_v Con S = (T_{\sigma(1)} \cdots T_{\sigma(q)}) \triangleright_v S.
\]

4.4. The universal enveloping algebra of the free post-Lie algebra

**Definition 39.** Let \( n, k \geq 1 \) and let \( q_1 + \cdots + q_k = n \) be a partition of \( n \) into positive integers \( q_i \geq 0 \). A \((q_1, \ldots, q_k)\)-shuffle is a partition of \( \{1 < \cdots < n\} \) by \( k \) ordered sets of cardinality \( q_i \), for each \( 1 \leq i \leq k \).

Note that the number of \((q_1, \ldots, q_k)\)-shuffles is \( sh_{q_1, \ldots, q_k} := \frac{(q_1 + \cdots + q_k)!}{q_1! \cdots q_k!} \).
Recall that if \((g, \triangleright)\) is a post-Lie algebra, its universal enveloping algebra \(\mathcal{U}(g)\), with its standard associative product, becomes a bialgebra if endowed with the shuffle coproduct \(\Delta_{sh} : \mathcal{U}(g) \to \mathcal{U}(g)^{\otimes 2}\). The Lie algebra of the primitive elements of this bialgebra is \(g \subset \mathcal{U}(g)\), i.e. \(\Delta_{sh}(l) = l \otimes 1 + 1 \otimes l\) if and only if \(l\) is a Lie element, i.e. \(l \in g\). Therefore, for every Lie element \(l\) and for every integers \(i \geq 1\) and \(k \geq 2\), one has

\[
\Delta_{sh}^{(k)}(l) = \sum_{i_1 + \cdots + i_k = i} \text{sh}_{i_1, \ldots, i_k} l_1 \otimes \cdots \otimes l_k,
\]

where \(l^i = l \cdots l\), for all \(l \in g\). Furthermore, recall that the Grossman-Larson product on \((\mathcal{U}(g), \Delta_{sh})\) is given by \(* : X \otimes Y \mapsto X_{(1)}(X_{(2)} \triangleright Y)\) for all \(X, Y \in \mathcal{U}(g)\), where \(\triangleright : \mathcal{U}(g)^{\otimes 2} \to \mathcal{U}(g)\) is the extension of the post-Lie product and it satisfies the properties 1, 2 and 3, see Remark 20. The left-side extension of the post-Lie product to \(\mathcal{U}(g)\) was described in [35] in terms of post-symmetric braces, which are operations encoded by the corollas in \(\mathcal{P}SB\). This means that, for \(X = x_1 \cdots x_n \in \mathcal{U}(g)\) and \(y \in g\), one has

\[
X \triangleright y = \begin{array}{c}
1 \\
2 \\
\vdots \\
n \\
n + 1 \\
x_1 \otimes \cdots \otimes x_n \otimes y.
\end{array}
\]

The rest of this section is dedicated to the universal enveloping algebra of the free post-Lie algebra \(\mathcal{P}SB(\mathbb{K})\). Remark that, since the product of the free associative algebra on \(\mathcal{P}SB(\mathbb{K})\) is given by concatenation of trees, its underlying vector space is generated by the forests of \(G\). The universal enveloping algebra \((\mathcal{U}(\mathcal{P}SB(\mathbb{K})), \ast)\) is the vector space generated by the forests on \(G\), modded out by the ideal generated by \(RS - SR - C(\bullet; R, S)\) for every \(R\) and \(S\) in \(G\). For later use, let us describe the product \(E \ast F\) for a few particular forests \(E\) and \(F\).

**Lemma 40.** For every Lie element \(l\) and every forest \(F\), one has \(l \ast F = lF + l \triangleright F\).

**Proof.** Recall that Lie elements are primitive elements for the shuffle coproduct. We conclude by observing that \(l \ast F := l_{(1)}(l_{(2)} \triangleright F)\).

**Lemma 41.** For every \(n \geq 1\) and for every tree \(T\), one has

\[
\square \ast^n \triangleright T = \sum_{1 \leq k \leq |T|} \sum_{n_1 + \cdots + n_k = n, n_i > 0} \text{sh}_{n_1, \ldots, n_k} \triangleright^n \triangleright_{v_1, \ldots, v_k} T.
\]

**Proof.** Recall from (4.6) that \(\square \ast^n \triangleright T\) is given by

\[
\left( \cdots \left( \begin{array}{c} 1 \\
2 \\
\vdots \\
n \\
n + 1 \end{array} \right) \ast_{n+1} T \right) \ast_{n+2} \cdots \ast_n \square.
\]

From the definition of the operadic structure of \(\mathcal{P}SB\), see (4.1), we know that this is the sum of \(\square \ast^n \triangleright_{v_1, \ldots, v_k} T\) over all distinct vertices \(v_1, \ldots, v_k\) and all the maps \(\phi : \{1, \ldots, n\} \to \{1, \ldots, k\}\) such that \(|\phi^{-1}(i)| = n_i\) for \(1 \leq i \leq k\).

**Lemma 42.** Let \(T_1 \cdots T_k\) be a forest of \(k\) trees and let \(n \geq 1\). One has

\[
\square \ast^n \ast \cdots \ast \sqrt{n} \triangleright T_1 \cdots \sqrt{n} \triangleright T_k.
\]
Proof. By (4.5), one has
$$\sum_{j_0 + j_1 = n, j_0 \geq 0} sh_{j_0,j_1} \cdot x^{j_1} (x^{j_1} \triangleright F),$$
and in turn, by 2 and (4.5), one has
$$\sum_{i_1 + \ldots + i_k = i} sh_{i_1,\ldots,i_k} (x^{i_1} \triangleright T_1) \cdot \ldots \cdot (x^{i_k} \triangleright T_k).$$

4.5. Nested tubings

In this subsection are presented two notions of nested tubings of forests, the vertical and the horizontal ones. As far as the terminology is concerned we mention that the term tubing is borrowed from [13] though the present definition differs from the original one. Hereafter $<$ and $<$ denote the level partial order and, respectively, the canonical linear order introduced in Sect. 4.1. For a vertex $v$ of a tree $T$, let $b_v \subseteq V(T)$ be the subset of the $<$-predecessors of $v$; it inherits the order $<$. The set of roots $\text{Root}(F)$ of a forest $F$ has a horizontal order $<_{h}$ that is increasing as one goes from left to right: for a forest $ST$, one has $v <_{h} w$ for $v$ the root of $S$ and $w$ the root of $T$.

Recall that $\mathcal{G}_n$ is the subset of $\mathcal{G}$ of those trees that have only square-shape vertices $\blacksquare$. Let $\mathcal{F}$ be the set of the forests of $\mathcal{G}_n$, whose elements are forests with exactly $n$ vertices. Let $\mathcal{F}_n$ be the set $\mathcal{F}_n \setminus \{\blacksquare^n\}$, where $\blacksquare^n$ denotes the horizontal forest of $n$ trees.

**Definition 43.** A higher set of a poset $(\mathcal{P}, <)$ is a subset of $\mathcal{P}$ that contains the $<$-successors of each of its elements.

**Definition 44.** A tube of a tree $T$ is a connected higher set $t$ of $(V(T), <)$, such that, for each $v \in t$, $t \cap b_v$ is a higher set of $(b_v, <)$.

**Definition 45.** A tube of a forest $F \in \mathcal{F}$ is a subset of $V(F)$ such that its intersection with $(\text{Root}(F), <_{h})$ is a higher set and such that it intersects each tree of $F$ into a (possibly empty) tube.

**Remark 46.** A tube of a forest $F$ can be identified with a subforest of $F$; we will often use this identification implicitly.

**Definition 47.** Two tubes are nested if one of them is a subset of the other.

**Definition 48.** A nested tubing of $F \in \mathcal{F}$ is a collection of pairwise nested, non-empty tubes of $F$ such that:

(i) it contains at least two tubes;
(ii) it contains the maximal tube (the tube that is the whole set of the vertices of the forest).

For a nested tubing $t = \{t_i\}_{i \in I}$ of $F$, the boundary of the tube $t_i$ is $\partial t_i = t_i \setminus \{t_j \nsubseteq t_i\}$.

4.5.1. Vertical nested tubings

**Definition 49.** A vertical nested tubing is a nested tubing such that:
the boundary of each tube is not a horizontal forest made of more than one tree; and,
(2) if the boundary of a tube is a forest, then either all the roots of this forest are connected to
a single vertex of a subtube, or none of the roots are connected to any subtube.

Definition 50. For $F$ in $\mathcal{F}or'$, let $\text{Tub}(F)$ be the set of its vertical nested tubings.

Example 51. (a) is a tube, (b) is a higher set that is not a tube (condition on $b_v$ unsatisfied), (c),
(d) and (e) are not vertical nested tubings (condition 1 is not satisfied; in addition for (d), condition
2 is not satisfied either). The last three examples (f), (g) and (h) are vertical nested tubings.

Example 52. Here below we collect all the vertical nested tubings of

4.5.2. Horizontal nested tubings

Definition 53. A horizontal nested tubing of $F \in \mathcal{F}or$ is a nested tubing of $F$ such that the
boundary of each tube is a horizontal forest.

We let $h\text{Tub}(F)$ denote the set of horizontal nested tubings of $F$. For every $p_1 + \cdots + p_k = N$
such that $p_i > 0$, and $F \in \mathcal{F}or'_N$, we let $h\text{Tub}(F)_{p_1,\ldots,p_k}$ be the subset of $h\text{Tub}(F)$ of those horizon-
tal nested tubings $t = t_1 \supset t_2 \supset \cdots \supset t_k$ such that $|\partial t_i| = p_i$ for each $1 \leq i \leq k$. One has

$$h\text{Tub}(F) = \bigsqcup_{p_1 + \cdots + p_k = N, p_i > 0} h\text{Tub}_{p_1,\ldots,p_k}(F).$$

(4.7)

Example 54. In Example 51, (a) is a tube whose boundary is not a horizontal forest, (c) to (g)
are horizontal nested tubings, and (h) is not horizontal.

Example 55. Here below we collect all the horizontal nested tubings of

4.6. Post-Lie Magnus expansion in terms of nested tubings

For a post-Lie algebra $\mathfrak{g}$, the pLMe of $x \in \mathfrak{g}$ is the element $\chi(x)$ that satisfies the equation

$$\exp \cdot (x) = \exp \ast (\chi(x)).$$

In particular, in $\dot{U}_\ast(\mathcal{P}SB(\mathbb{K}))$, it is a sum over all forests in $\mathcal{F}or'$:

$$\chi(\blacksquare) = \sum_{F \in \mathcal{F}or} c_F F.$$  

(4.8)
Hereafter we present two methods to compute the coefficients $c_{F}$ of (4.8), the first based on vertical and the second on horizontal nested tubings.

4.6.1. Post-Lie Magnus expansion via vertical nested tubings

As showed in [23, Equation (81)] the pLMe can be expressed as a sum $\chi = \sum_{n \geq 1} \chi_n$, where

$$\chi_n(x) = \frac{x^n}{n!} - \sum_{p_1 + \cdots + p_k = n, \forall i : p_i > 0} \frac{1}{k!} \chi_{p_1}(x) \cdots \chi_{p_k}(x) \quad \text{for all } x \in \mathfrak{g}.$$  

We will describe $\chi_n : \mathcal{PSB}(K) \to \mathcal{PSB}(K) \subset \mathcal{U}_n(\mathcal{PSB}(K))$ for the free post-Lie algebra on $K = K < \mathfrak{g} >$.

Note that $\chi_n(K)$ is a homogeneous Lie polynomial of degree $n$. In particular, in $\mathcal{U}_n(\mathcal{PSB}(K))$ it is a sum over all forests in $\mathcal{F}$ for with $n$ vertices:

$$\chi_n(K) = \sum_{F \in \mathcal{F}} c_F.$$  

**Definition 56.** Given two integers $n, k \geq 2$, a partition $p_1 + \cdots + p_k = n$ by strictly positive integers and $F$ in $\mathcal{F}$ for $n$, let $\mathcal{D}(F)_{p_1, \cdots, p_k}$ be the set of all the possible decompositions of $F$ of the form

$$F = F_1 \times (\cdots (K_{k-3}(F_{k-2}K_{k-2}(F_{k-1}K_{k-1}F_k))) \cdots),$$  

where $F_i$ runs through the forests with $p_i$ vertices that are not horizontal and where $K_i$ is either the concatenation $\times$ or the one vertex grafting operation $\triangleright_v$, for some vertex $v$.

**Lemma 57.** There is a bijection between $\mathcal{D}(F)_{p_1, \cdots, p_k}$ and the set of all the vertical nested tubings $t = t_1 \supset t_2 \supset \cdots \supset t_k$ of $F$ such that $|\partial t_i| = p_i$ for each $1 \leq i \leq k$.

**Proof.** Since concatenation and grafting remove neither vertices nor edges, the decomposition (4.9) provides an embedding of $F_1, \ldots, F_k$ into $F$, which we claim, can be represented by a vertical nested tubing. Explicitly, the tube $t_k$ is $F_k$, which is the most right sided subforest of $F$; the tube $t_{k-1}$ is the subforest $F_{k-1}K_{k-1}F_k$ of $F$ and it contains $F_k$, etc. For example, one has

This assignment is well-defined:

- grafting is on the left side of a vertex; this is condition on $b_v$ in Definition 44;
- one vertex grafting of forests corresponds to the condition (2) of Definition 49;
- concatenations with right most parentheses correspond to the higher set condition for the order $<_h$.

Let us show that such assignment is surjective. Firstly, the above discussion shows that the tubings are such that they do not encode any other type of operations (other than concatenations and one vertex graftings with right most parenthesis). Secondly, note that condition (2) of Definition 49 ensures that the whole forest is decomposed. Moreover, since the tubes $t_i$ are such that $|\partial t_i| = p_i$, their boundary $\partial t_i$ corresponds to subforests of $F$ that are in $\mathcal{F}$ for $p_i$. Finally, condition (1) of Definition 49 corresponds to the absence of the forest $^v\mathcal{F}$.
For each vertical nested tubings $F$ in $\mathcal{F}or'_N$, one has $c_F = \sum_{t \in \text{Tub}(F)} c_t$, where $c_t = \frac{1}{|t|} \prod_{\partial t} c_{\partial t}$ and $c_\bullet = 1$.

**Proof.** As $\chi_j(\bullet) = \bullet$, the first coefficient $c_\bullet$, which is the coefficient of the unique tubing of $\bullet$, is 1. Let $n \geq 2$ and let $F$ be a forest in $\mathcal{F}or'_N$ that is not $\bullet^n$. To compute $c_F$, let us remark that $F$ appears in $\frac{1}{k!} \chi_{p_1}(x) * \cdots * \chi_{p_k}(x)$ for some $k$-partitions $p_1 + \cdots + p_k = n$. For each of these partitions, $F$ is obtained by shuffle concatenations and/or graftings of $k$ forests, say $F_1, \ldots, F_k$, that belong to $\chi_{p_1}(x), \ldots, \chi_{p_k}(x)$ respectively (shuffles arise from 2). Note that there are several operations of these types that we can exclude. Indeed, since for all $p$ the element $\chi_p(\bullet)$ is a Lie polynomial, for each $p$, every $F_i \in \chi_{p_i}$ is either a tree or it belongs to a commutator. Therefore, thanks to (4.2), (4.3), (4.4) and to Lemma 40, to describe the product $F_i * F_{i+1}$ it is enough to consider the concatenation $F_i; F_{i+1}$ and the grafting $F_i \bowtie F_{i+1}$ for each vertex $v$ of $F_{i+1}$. Moreover, since * is associative, we can restrict ourselves to applying the concatenation and the one vertex grafting operations with the right most parentheses. In other words, it is enough to consider all the expressions of the form (4.9) which, thanks to Lemma 57, correspond to vertical nested tubings. For each vertical nested tubings $t = t_1 \supset t_2 \supset \cdots \supset t_k$ of $F$ such that $|\partial t_i| = p_i$ for each $1 \leq i \leq k$, we let $c_t = \frac{1}{k!} c_{t_1} c_{t_2} \cdots c_{t_k}$. By summing over all the possible tubings, one obtains $c_F = \sum_{t \in \text{Tub}(F)} c_t$; note that condition (1) ensures that tubings encode non-trivial decompositions. In addition, note that each $c_{t_i}$ itself is given by $c_{\partial t_i}$, which gives the result. Note that Lemma 57 stands for forests $F_i$ in $\mathcal{F}or'_N$, some of which may not be in $\chi_{p_i}(\bullet)$, that is, there are forests $F_i$ such that $c_{F_i} = 0$. This is not an issue since if $c_{F_i} = 0$ for some $i$, then $c_t = 0$. □

### 4.6.2. Post-Lie Magnus expansion via horizontal nested tubings

This section is devoted to the computation of the pMe using horizontal nested tubings. While the previous method is recursive, the present method allows to compute the coefficient $c_F$ of any forest $F \in \mathcal{F}or'_N$ for $N \geq 2$ in a closed form. To this end, we will use the following formula for $\chi$:

$$\chi(x) = \log * (\exp (x)) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \left( \sum_{j \geq 1} \frac{x^j}{j!} \right)^k.$$

(4.10)

In particular we will be led to investigate elements of the form

$$\bullet^{x_{p_1}} * (\cdots (\bullet^{x_{p_{k-1}}} * \bullet^{x_{p_k}}) \cdots),$$

(4.11)

for partitions $p_1 + \cdots + p_k = N$ with $p_i > 0$.

**Definition 59.** Given two integers $N, k \geq 2$, a partition $p_1 + \cdots + p_k = N$ by strictly positive integers and $F$ in $\mathcal{F}or'_N$, let $h\Xi(F)_{p_1, \ldots, p_k}$ be the set of all the possible decompositions of $F$ of the form

$$F = \bullet^{x_{p_1}} \bowtie_1 \cdots \bowtie_{k-1} (\bowtie_{k-1} (\bowtie_{k-2} (\bowtie_{k-2} (\bowtie_{k-1} (\bowtie_{k-1} \cdots \bowtie_{k-1} \bullet^{x_{p_k}})) \cdots)) \cdots),$$

(4.12)

in which each $\bowtie_i$ is an operation of the form $\bowtie_{n_{i_1}, n_{i_2}, \ldots, n_{i_k}}$ as introduced in Notation 34, item (2).

**Lemma 60.** For each $N \geq 2$ and each $p_1 + \cdots + p_k = N, p_i > 0$, there is a bijection between $h\Xi(F)_{p_1, \ldots, p_k}$ and $h\text{Tub}_{p_1, \ldots, p_k}(F)$.

**Proof.** Since the operations $\bowtie_{n_{i_1}, n_{i_2}, \ldots, n_{i_k}}$ remove neither vertices nor edges, the decomposition (4.12) provides an embedding of $\bullet^{x_{p_1}}, \ldots, \bullet^{x_{p_k}}$ into $F$, which, we claim, can be represented by a horizontal nested tubing. Explicitly, the tube $t_k$ is $\bullet^{x_{p_k}}$, which is the most right sided horizontal subforest $\bullet^{x_{p_k}}$ of $F$; the tube $t_{k-1}$ is the subforest $\bullet^{x_{p_{k-1}}} \bowtie_{k-1} \bullet^{x_{p_k}}$ of $F$, etc. For example, one has
since grafting is on the left side of a vertex, the tubes satisfy condition on \( b_\gamma \) in Definition 44;
- since operations are performed with right most parentheses, the tubes satisfy the higher set condition for the order \( <_h \).

To close the proof, we use the argument below which yields the existence of an inverse of this map. Each pair of tubes \((t_{i-1}, t_i)\) with \( t_{i-1} \supset t_i \) determines an operation of the form \( \kappa_{n_0, n_1, \ldots, n_k}^{v_{i_1}, \ldots, v_{i_k}} \); the integer \( n_0 \) is the number of roots of \( \partial t_{i-1} \) that are not attached to any vertex of \( t_i \) (they are the most left sided \( n_1 \) roots of \( \partial t_{i-1} \) because of the higher set condition for \( <_h \)); the integer \( n_1 \) is the number of next roots (from left to right) of \( \partial t_{i-1} \) that are attached to a same vertex, which is \( v_i \), etc.

\underline{Notation 61.} Let \( F \) be a forest and let \( t \) be a horizontal nested tubing of \( F \). For each non minimal tube \( s \) of \( t \), we let \( s' \subset s \) be its predecessor in \( t \); so, \( \partial s = s \setminus s' \). Recall that \( s' \) is a forest, say of trees \( T_{1}, \ldots, T_{\text{len}(s')} \). In particular, \( \text{len}(s') \) denotes the number of trees in the tube \( s' \).

\begin{itemize}
  \item We let \( j(s)_0 \geq 0 \) be the number of roots of \( F \) in \( \partial s \), that is the number of vertices that are not attached to any vertex of \( s' \).
  \item For \( 1 \leq a \leq \text{len}(s') \), we let \( j(s)_a \) be the number of vertices of \( \partial s \) that are attached to vertices of \( T_a \).
  \item For each tree \( T_a \) of \( s' \) and each vertex \( v \) of \( T_a \), we let \( f^i(v) \) be the cardinal of its fiber in \( \partial s \), that is the cardinal of the set \( b_\gamma \cap \partial s \) of vertices in \( \partial s \) that are attached to \( v \).
  \item We let \( k_a \geq 0 \) be the number of vertices \( v \) of \( T_a \) such that \( f^i(v) \neq 0 \) and we let \( v_1^a, \ldots, v_{k_a}^a \) be the collection of such vertices.
\end{itemize}

\underline{Example 62.} Consider the horizontal nested tubing of (4.13); let \( s \) be its maximal tube. The boundary \( \partial s \) is a forest of five trees; the first two trees (i.e. left most sided) are not attached to \( s' \), and the next three trees are attached to the same tree of \( s' \). Therefore one has \( j(s)_0 = 2, j(s)_1 = 3 \) and \( j(s)_2 = 0 \). For the first tree \( T_1 \) of \( s' \) (the corolla with 3 vertices, the root \( v_n \), the most left-sided vertex \( v_1 \) and the other one \( v_2 \)), one has \( f^1(v_1) = 2, f^1(v_2) = 1 \) and \( f^1(v_2) = 0 \).

For \( F \in \mathcal{F}_{\mathbf{ord}_N} \) and \( t \in \mathcal{H}_{\text{Tub}}(F)_{p_1, \ldots, p_s} \), we let \( A(t) \) be the number of times the expression that corresponds to \( t \) via Lemma 60 appears in \( \boxtimes_{p_1} \ast \cdots \ast (\boxtimes_{p_{k-1}} \ast \boxtimes_{p_k}) \cdots \).

\underline{Lemma 63.}

\[ A(t) = \prod_{s \in t, s \text{ not minimal}} \text{sh}_{j(s)_0, \ldots, j(s)_{\text{len}(s')}} \prod_{1 \leq a \leq \text{len}(s')} \text{sh}_{f^a(v_1), \ldots, f^a(v_{k_a})}. \]

\underline{Proof.} The proof is by induction. We let \( t = t_k \supset t_{k-1} \supset \cdots \supset t_1 \). Consider \( t_2 \supset t_1 \) as a horizontal nested tubing for the forest \( t_2 \). Recall that \( \text{len}(t_i) \) is the number of trees in the tube \( t_i \). One has
Lemma 63, gives
\[\text{If we let } D_p, \text{ then one has } \sum_{t \in \text{hTub}(F)} \frac{(-1)^{|t|}}{|t|!} \prod_{s \in t} \text{sh}_{j(t_1), \ldots, j(t_{|t|})} \prod_{1 \leq a \leq |t|} \text{sh}_{f(t_{a})}, \ldots, f(t_{a}), \text{ times } A(t), \text{ where } t' := t_{k-1} \supset t_{k-2} \supset \cdots \supset t_1. \text{ Hence the result.} \]

**Theorem 64.** With Notation 61, for each \( F \) in \( \mathcal{F}or'_N \) with \( N \geq 2 \), the coefficient \( c_F \) is
\[
\sum_{t \in \text{hTub}(F)} \frac{(-1)^{|t|}}{|t|!} \prod_{s \in t} \text{sh}_{j(s_1), \ldots, j(s_{|t|})} \prod_{1 \leq a \leq \text{len}(s')} \text{sh}_{f(t_{a})}, \ldots, f(t_{a}), \text{ where } s' \text{ is minimal.}
\]

**Proof.** Let \( F \) be in \( \mathcal{F}or'_{N} \) for \( N \geq 2 \). By using Eq. (4.10) one can write \( F(\mathcal{G}) \) as
\[
\sum_{N \geq 1} \sum_{p_1, \ldots, p_N = N, \ p_i > 0} \frac{(-1)^{k-1}}{k} \frac{1}{p_1! p_2! \cdots p_k!} \times p_1 \ast \cdots \ast (\times p_{k-1} \ast \times p_k) \ast \cdots.
\]

If we let \( D_{p_1, \ldots, p_k} \) denote the number of times the forest \( F \) appears in \( \times p_1 \ast \cdots \ast (\times p_{k-1} \ast \times p_k) \ast \cdots \), then one has \( c_F = \sum_{p_1, \ldots, p_k = N} \frac{(-1)^{k-1}}{k} \frac{1}{p_1! p_2! \cdots p_k!} D_{p_1, \ldots, p_k}, \)

Let us compute \( D_{p_1, \ldots, p_k} \). Consider a decomposition of \( F \) of the form (4.12); by Lemma 60 this amounts to considering \( t \in \text{hTub}(F)_{p_1, \ldots, p_k}(F) \). Such a decomposition appears exactly \( A(t) \) times in \( \times p_1 \ast \cdots \ast (\times p_{k-1} \ast \times p_k) \ast \cdots \). Therefore one has \( D_{p_1, \ldots, p_k} = \sum_{t \in \text{hTub}(F)_{p_1, \ldots, p_k}} A(t), \) which by Lemma 63, gives
\[
D_{p_1, \ldots, p_k} = \sum_{t \in \text{hTub}(F)_{p_1, \ldots, p_k}} \prod_{s \in t} \text{sh}_{j(s_1), \ldots, j(s_{|t|})} \prod_{1 \leq a \leq \text{len}(s')} \text{sh}_{f(t_{a})}, \ldots, f(t_{a})
\]

Finally, using the decomposition (4.7) of \( \text{hTub}(F) \) one obtains the result. \( \square \)

**Example 65.** Here is presented the computation of \( c_F \) for \( F = \)

Let us list all the possible horizontal nested tubings, which are of the form \( (p_k, p_{k-1}, \ldots, p_1) \) for \( 1 \leq k \leq 5 \). There are only four possibilities, which corresponds to \( (1, 1, 1, 1, 1) \), \( (1, 2, 1, 1) \), \( (2, 1, 1, 1) \) and \( (3, 1, 1) \):
In all those cases there are no shuffles involved because all tubes are trees and there is only one vertex which has a non trivial fiber. One obtains

\[ c_F = \frac{1}{5} (1 \times 1 \times 1 \times 1 \times 1) - \frac{1}{4} \left( 1 \times 1 \times \frac{1}{2!} \times 1 + 1 \times 1 \times 1 \times \frac{1}{2!} \right) + \frac{1}{3} \left( 1 \times 1 \times \frac{1}{3!} \right) = \frac{1}{180}. \]

**Example 66.** Here is the computation of \( c_F \) for \( F = \). The horizontal nested tubings are listed in Example 55 and correspond to \((2, 2), (2, 1, 1), (1, 2, 1), (1, 1, 2)\) (two possible horizontal tubings), and \((1, 1, 1, 1)\) (three possibilities). In the first tubing, for \( s \) the maximal tube, its predecessor \( s' \) has two trees \( T_1 \) and \( T_2 \); one has \( j(s)_0 = 0 \) (there is no unattached vertices in \( s \)), \( j(s)_1 = 1 \) and \( j(s)_2 = 1 \); and, \( f'(v_1') = 1 \) and \( f'(v_2') = 1 \), where \( v_1' \) is the unique vertex of \( T_i \). Therefore

\[ sh_{j(s)_0}, \ldots, j(s)_0 \sum_{i=1}^{\ell(s)} sh_{j(s)_1}, \ldots, j(s)_1 \times sh_{1} \times sh_{1} = 2. \]

Doing this for each tube and tubing, one obtains

\[ c_F = -\frac{1}{2} \left( \frac{1}{2!} \times \frac{1}{2!} \times 2 \right) + \frac{1}{3} \left( \frac{1}{2!} \times 2 + \frac{1}{2!} \times 2 + \frac{1}{2!} \right) - \frac{1}{4} (1 + 1 + 1) = 0. \]

Of course, since \( \chi(\square) \) is a Lie element, we already knew that \( c_F = 0. \)

To end this part we give the first four terms of \( \chi(\square): \)

\[ \chi_1(\square) = -\frac{1}{2}, \quad \chi_2(\square) = -\frac{1}{2}, \quad \chi_3(\square) = \frac{1}{3} \quad \text{and} \quad \chi_4(\square) = -\frac{1}{4} - \frac{1}{12} + \frac{1}{24} + \frac{1}{12}. \]

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