Refined asymptotics around solitons for gKdV equations

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Abstract

We consider the generalized Korteweg-de Vries equation

\[ \partial_t u + \partial_x (\partial_x^2 u + f(u)) = 0, \quad (t, x) \in [0, T) \times \mathbb{R} \]  

(0.1)

with general \( C^2 \) nonlinearity \( f \). Under an explicit condition on \( f \) and \( c > 0 \), there exists a solution in the energy space \( H^1 \) of (0.1) of the type \( u(t, x) = Q_c(x - x_0 - ct) \), called soliton. Stability theory for \( Q_c \) is well-known.

In \( [11] \), \( [14] \), we have proved that for \( f(u) = u^p \), \( p = 2, 3, 4 \), the family of solitons is asymptotically stable in some local sense in \( H^1 \), i.e. if \( u(t) \) is close to \( Q_c \) (for all \( t \geq 0 \)), then \( u(t, \cdot + \rho(t)) \) locally converges in the energy space to some \( Q_{c_+} \) as \( t \to +\infty \), for some \( c_+ \sim c \). The main improvement in \( [14] \) is a direct proof, based on a localized Viriel identity on the solution \( u(t) \). As a consequence, we have obtained an integral estimate on \( u(t, \cdot + \rho(t)) - Q_{c_+} \) as \( t \to +\infty \).

In \( [9] \) and \( [15] \), using the indirect approach of \( [11] \), we could extend the asymptotic stability result under general assumptions on \( f \) and \( Q_c \). However, without Viriel argument directly on the solution \( u(t) \), no integral estimate is available in that case.

The objective of this paper is twofold.

The main objective is to prove that in the case \( f(u) = u^p \), \( p = 2, 3, 4 \), \( \rho(t) - c_+ t \) has limit as \( t \to +\infty \) under the additional assumption \( x_+ u \in L^2(\mathbb{R}) \), which is consistent with a counterexample in \( [14] \). This result persists for general nonlinearity if a Virial type estimate is assumed. The main motivation for this type of result is the determination of explicit shifts due to collision of two solitons in the nonintegrable case \( p = 4 \), see \( [16] \).

The second objective of this paper is to provide large time stability and asymptotic stability results for two soliton solutions for the case of general nonlinearity \( f(u) \), when the ratio of the speeds of the solitons is small. The motivation is to accompany the two papers \( [16] \), \( [17] \), devoted to collisions of two solitons in the nonintegrable case. The arguments are refinements of \( [22] \), \( [18] \) specialized to the case \( u(t) \sim Q_{c_1} + Q_{c_2} \), for \( 0 < c_2 \ll c_1 \).

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1 Introduction

We consider the generalized Korteweg-de Vries (gKdV) equations:

\[ \partial_t u + \partial_x(\partial_x^2 u + f(u)) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}, \tag{1.1} \]

for \( u(0) = u_0 \in H^1(\mathbb{R}) \), with a general \( C^2 \) nonlinearity \( f \). We assume that

\[ \text{for } p = 2, 3 \text{ or } 4, \quad f(u) = u^p + f_1(u) \quad \text{where } \lim_{u \to 0} \left| \frac{f_1(u)}{u^p} \right| = 0. \tag{1.2} \]

Denote \( F(s) = \int_0^s f(s')ds' \). Note that for (1.1), one can solve the Cauchy locally in time in \( H^1 \), using the arguments of Kenig, Ponce and Vega [5] (using the norms given for \( f(u) = u^2 \) in \( H^1 \)). Moreover, the following conservation laws holds for \( H^1 \) solutions:

\[
\int u^2(t) = \int u^2_0, \quad E(u(t)) = \frac{1}{2} \int (\partial_x u(t))^2 - \int F(u(t)) = \frac{1}{2} \int (\partial_x u_0)^2 - \int F(u_0). \]

Recall that if \( Q_c \) is a solution of

\[ Q''_c + f(Q_c) = cQ_c, \quad x \in \mathbb{R}, \quad Q_c \in H^1(\mathbb{R}), \tag{1.3} \]

then \( R_{c,t_0}(t, x) = Q_c(x - x_0 - ct) \) is solution of (1.1). We call soliton such nontrivial traveling wave solution of (1.1).

By well-known results on equation (1.3) (see [1], [15]), there exists \( c_*(f) > 0 \) such that

\[ c_*(f) = \sup\{c > 0 \text{ such that } \forall c' \in (0, c), \exists Q_{c'} \text{ positive solution of (1.3)}\}. \]

Note that for \( f(u) = u^p \), \( c_*(u^p) = +\infty \) and for all \( c > 0 \), \( Q_c(x) = c^{-1/p} Q(\sqrt{c}x) \), where \( Q(x) = Q_1(x) = \left( \frac{p+1}{2} \cosh^{-2} \left( \frac{p-1}{2} x \right) \right)^{\frac{1}{p-1}} \). Recall that if a solution \( Q_c > 0 \) of (1.3) exists then \( Q_c \) is the unique (up to translation) positive solution of (1.3) and can be chosen even on \( \mathbb{R} \) and decreasing on \( \mathbb{R}^+ \).

From Weinstein [22], the soliton \( Q_c \) is orbitally stable if

\[
\frac{d}{dc} \int Q_c^2(x)dx \bigg|_{c'=c} > 0. \tag{1.4} \]

Concerning asymptotic stability, we have proved in [15] the following general result

**Asymptotic stability** ([15], [14], [11]) Assume that \( f \) is \( C^2 \) and satisfies (1.2). Let \( 0 < c_0 < c_*(f) \). There exists \( \alpha_0 > 0 \) such that if \( u(t) \) is a global \( (t \geq 0) \) \( H^1 \) solution of (1.1) satisfying

\[ \forall t \geq 0, \quad \inf_{r \in \mathbb{R}} \| u(t) - Q_{c_0}(\cdot - r) \|_{H^1} < \alpha_0, \tag{1.5} \]

then the following hold.

1. **Asymptotic stability in the energy space.** There exist \( t \mapsto c(t) \in (0, c_*(f)) \), \( t \mapsto \rho(t) \in \mathbb{R} \) such that

\[ u(t) - Q_{c(t)}(\cdot - \rho(t)) \to 0 \quad \text{in } H^1(x > \frac{c_0 t}{10}) \text{ as } t \to +\infty. \tag{1.6} \]
2. Convergence of the scaling parameter. Assume further that there exists \( \sigma_0 > 0 \) such that \( c \mapsto \int Q_c^2 \) is not constant in any interval \( I \subset [c_0 - \sigma_0, c_0 + \sigma_0] \). Then, by possibly taking a smaller \( \sigma_0 > 0 \), there exists \( c^+ \in (0, c_*(f)) \) such that \( c(t) \to c^+ \) as \( t \to +\infty \). Moreover, \( \rho'(t) \to c^+ \) as \( t \to +\infty \).

Recall that the main improvement in [14] with respect to [11] is a direct proof, based on a localized Viriel etimate on the solution \( u(t) \), see Claim B.2 in the present paper for a similar result. As a consequence, we have obtained the following estimate: there exists \( K > 0 \) such that

\[
\int_0^{+\infty} \int (u(t, x) - Q_{c(t)}(x - \rho(t)))^2 e^{-\sqrt{\kappa t} \sqrt{|x - \rho(t)|}} dx dt \leq K \alpha_0^2, \tag{1.7}
\]

A Viriel identity was proved in the case \( f(u) = u^p \) for \( p = 2, 3, 4 \) see Proposition 6 in [11], and used under a localized form in [14] (see also the case \( p = 5 \) in [10]). In contrast, the proof of asymptotic stability in [9], [15] is for general \( f(u) \), but it is indirect, following the original approach of [11]. Thus, in this case, it is not known whether (1.7) holds. See further comments on this result in [15].

1.1 Refined asymptotics for power nonlinearities

Our main objective in this paper is to refine the convergence result for the power case, i.e. \( f(u) = u^p \) with \( p = 2, 3 \) and \( 4 \) concerning the behavior of \( \rho(t) \) as \( t \to +\infty \). In fact, the only requirement is that (1.7) holds, in particular, a virial type estimate around \( Q_c \) is sufficient. A typical result in this direction is the following.

**Theorem 1** Assume \( f(u) = u^p \), for \( p = 2, 3 \) or 4. Let \( c_0 > 0 \). There exists \( \alpha_0 > 0 \) such that if \( u(t) \) is an \( H^1 \) solution of (1.1), satisfying

\[
\inf_{r \in \mathbb{R}} \|u(0, \cdot + r) - Q_{c_0}\|_{H^1} < \alpha_0, \quad \int_{x > 0} x^2 u^2(0, x) dx < +\infty, \tag{1.8}
\]

then there exists \( c^+ > 0 \) and \( x^+ \in \mathbb{R} \) such that

\[
\lim_{t \to +\infty} c(t) = c^+, \quad \lim_{t \to +\infty} \rho(t) - c^+ t = x^+, \tag{1.9}
\]

where \( c(t) \) and \( \rho(t) \) are defined as in (1.6).

Recall that Pego and Weinstein [21] and Mizumachi [20] also obtained results of asymptotic stability in weighted spaces, where convergence of \( \rho(t) - t \) is proved. The results [21] and [20] depend on a spectral assumption which is proved only for \( p = 2, 3 \). Moreover, in [21], the initial data has to belong to an exponential weighted space. This condition has been relaxed in [20], where the assumption is \( \int_{x > 0} x^{11} u^2 < +\infty \).

We point out two main motivations for this kind of results:

• First, in [14], we gave the following counterexample:

For any \( \alpha > 0 \), there exists an \( H^1 \) solution \( u(t) \) of the KdV equation i.e. (1.1) with \( f(u) = u^2 \), such that \( \sup_{t \in \mathbb{R}} \|u(t) - Q(x - \rho(t))\|_{H^1} \leq \alpha \), and for some \( \kappa > 0 \), \( \rho(t) \),

\[
\lim_{t \to +\infty} \|u(t) - Q(x - \rho(t))\|_{H^1(x \geq t/2)} = 0, \quad \lim_{t \to +\infty} \frac{\rho(t) - t}{\sqrt{\log(t)}} = \kappa.
\]
The initial data used in this construction contains a series of small solitons, which converges in $H^1$ but not in $L^2(x^2dx)$. This proves that the convergence of $\rho(t) - c^+t$ as $t \to +\infty$ is not true in general and requires some additional decay on the solution. In this respect, the assumption $\int x^2u^2 < +\infty$ in Theorem [11, 14] seems rather weak and improve the results in [21] and [20]. When looking for an $L^2$ condition independent of $p$ for (1.1), we think that $\int x^2u^2$ is optimal. Indeed, it seems that the relevant quantity is the $L^1$ norm, see [10], where it proved that during the collision of two solitons, the shifts on the trajectories are related to $L^1$ norms.

Thus, the natural question left open by Theorem [11] is whether assuming $\int_{x>0}|u| < +\infty$ is sufficient to obtain convergence of $\rho(t) - c^+t$. This might require a much more refined analysis.

• A second motivation for estimating $\rho(t) - c^+t$ as $t \to \infty$ appears in the context of two soliton collisions. In a paper [16] concerning the collision of two solitons of different sizes for the gKdV equation (1.1) with $f(u) = u^4$, we were able to compute the shift on the trajectories of the solitons resulting from their collision. The explicit shift is obtained for a fixed time $T$, long after the collision. The proof of Theorem 1 allows us to quantify the variation of this shift due to long time i.e. for $t > T$.

We also point out that the proof of Theorem [11] relies on several refinements of the proof of the asymptotic stability in [11, 14] and is independent from the methods in [21, 20]. Let $\eta(t, x) = u(t, x) - Q_{c(t)}(x - \rho(t))$. Observe that the standard estimate

$$|\rho'(t) - c(t)| \leq K \left( \int \eta^2(t, x)e^{-\frac{x^2}{4\rho(t)}}dx \right)^{\frac{1}{2}},$$

and (1.7) do not give any conclusion.

We proceed as follows. First, we improve the monotonicity arguments on $u(t)$ used in [11], [18] and [14]. The improvement is to prove monotonicity results on $\eta(t)$, which are much more precise (see Claim [13, 3]). This argument allows us to prove that (1.7) implies $\int_0^{+\infty} |c(t) - c^+|dt < +\infty$. Second, the control of $\int_0^{+\infty}(\rho'(t) - c(t))dt$ is obtained through the equation of $\eta(t)$, by noting that at the first order $\rho'(t) - c(t)$ is the derivative of some bounded function of $t$. Note that we do not prove $\int_0^{+\infty} |\rho'(t) - c(t)|dt < +\infty$.

### 1.2 Large time behavior in the two soliton case

A second objective of this paper is to provide asymptotic analysis in large time related to two soliton solutions of (1.1).

From [8] (see also [18]), there exist solutions $u(t, x)$ of (1.1) which are asymptotic $N$-soliton solutions at $t \to -\infty$ in the following sense: let $N \geq 1$, $c_1 > \ldots > c_N > 0$, and $x_1, \ldots, x_N \in \mathbb{R}$, there exists a unique $H^1$ solution $U$ of (1.1) such that

$$\lim_{t \to -\infty} \left\| U(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j - c_jt) \right\|_{H^1(\mathbb{R})} = 0. \quad (1.10)$$

The behavior displayed by these solutions is stable in some sense. Considering for example the case of two solitons, there exist a large class of solutions such that as $t \sim -\infty$,

$$u(t, x) = Q_{c_1}(x-x_1-c_1t) + Q_{c_2}(x-x_2-c_2t) + \eta(t, x), \quad (1.11)$$
where \( c_1 > c_2 \) and \( \eta(t) \) is a dispersion term small in the energy space \( H^1 \) with respect to \( Q_{c_1}, Q_{c_2} \) (see [18]). From the Physics point of view, the two solitons \( Q_{c_1} \) and \( Q_{c_2} \) have collide at some time \( t_0 \). In the special case \( c_2 \ll c_1 \) (or equivalently, \( \|Q_{c_2}\|_{H^1} \ll \|Q_{c_1}\|_{H^1} \)) and \( \|\eta(t)\|_{H^1} \ll \|Q_{c_2}\|_{H^1} \), for \( t \) close to \(-\infty\), we have introduced in [16] explicit computations allowing to understand the collision at the main orders, using a new nonlinear “basis” to write and compute an approximate solution \( v(t,x) \) up to any order of size.

Recall that the problem of collision of two solitons is a classical question in nonlinear wave propagation. In the so-called integrable cases (i.e. \( f(u) = u^2 \) and \( f(u) = u^3 \)) it is well-known that there exist explicit multi-soliton solutions, describing the elastic collision of several solitons (see Hirota [3], Lax [6] and the review paper Miura [19]). Note that in experiments, or numerically for more accurate nonintegrable models (see Craig et al. [2], Li and Sattinger [7] and other references in [16]), this remarkable property is mainly preserved, i.e. the collision of two solitons is almost elastic, however, a (very small) residual part is observed after the collision. Equation (1.1) being not integrable (unless \( f(u) = u^2 \) and \( f(u) = u^3 \)), explicit \( N \)-soliton solutions are not available in this case. The results obtained in [16] and [17], using the present paper, are the first rigorous results concerning inelastic (but almost elastic) collision in a nonintegrable situation. We refer to the introduction and the references in [16] for an overview on these questions.

In [16] and [17], the approximate solution is adapted to treat a large but fixed time interval around the collision region, but not the large time asymptotics (see for example Proposition 3.1 in [16]). In Proposition 2 below, we give stability and asymptotic stability results required to control the asymptotics in large time of the solutions constructed in [16] and [17]. In particular, we need a sharp stability result in the case where one soliton is small with respect of the other. We claim the following.

**Proposition 2** Assume that \( f \) is \( C^2 \) and satisfies (1.2) for \( p = 2, 3 \) or \( 4 \). Let \( 0 < c_2 < c_1 < c_4 \) be such that (1.4) holds for \( c_1, c_2 \). Let \( 0 < c_1 < c_0(f) \) be such that (1.4) holds. There exist \( c_0(c_1) \) and \( K_0(c_1) > 0 \), continuous in \( c_1 \) such that for any \( 0 < c_2 < c_0(c_1) \) and for any \( \omega > 0 \), the following hold. Let \( T_{c_1,c_2} = c_1^2 \left( \frac{c_4}{c_1} \right)^{-\frac{3}{2} + \frac{1}{4\omega}} \). Let \( u(t) \) be an \( H^1 \) solution of (1.1) such that for some \( t_1 \in \mathbb{R} \) and \( \frac{3}{2} T_{c_1,c_2} \leq X_0 \leq \frac{3}{2} T_{c_1,c_2}, \)

\[
\|u(t_1) - Q_{c_1} - Q_{c_2}(\cdot + X_0)\|_{H^1} \leq c_2 \omega + \frac{1}{\omega + 1} + \frac{1}{4}. \tag{1.12}
\]

Then, there exist \( C^1 \) functions \( \rho_1(t), \rho_2(t) \) defined on \([t_1, +\infty[)\) such that

1. **Stability of the two solitons.**

\[
\sup_{t \geq t_1} \|u(t) - Q_{c_1}(\cdot - \rho_1(t)) - Q_{c_2}(\cdot - \rho_2(t))\|_{H^1} \leq K c_2 \omega + \frac{1}{\omega + 1} + \frac{1}{4}, \tag{1.13}
\]

\[
\forall t \geq t_1, \quad \frac{1}{2} c_1 \leq (\rho_1 - \rho_2)'(t) \leq \frac{3}{2} c_1, \quad |\rho_1(t_1)| \leq K c_2 \omega + \frac{1}{\omega + 1} + \frac{1}{4}, \quad |\rho_2(t_1) - X_0| \leq K c_2^2. \tag{1.14}
\]

2. **Asymptotic stability.** There exist \( c_1^+, c_2^+ > 0 \) such that

\[
\lim_{t \to +\infty} \|u(t) - Q_{c_1^+}(x - \rho_1(t)) - Q_{c_2^+}(x - \rho_2(t))\|_{H^1(x > \frac{3}{4} T_{c_1,c_2})} = 0. \tag{1.15}
\]

\[
\left| \frac{c_1^+}{c_1} - 1 \right| \leq K c_2 \omega + \frac{1}{\omega + 1} + \frac{1}{4}, \quad \left| \frac{c_2^+}{c_2} - 1 \right| \leq K c_2^2. \tag{1.16}
\]
Remark. The time $T_{c_1,c_2}$ corresponds to a time long after the collision of the two solitons (see [16]). In (1.12), since $X_0 > \frac{1}{2} T_{c_1,c_2}$, the two solitons are decoupled for any $t \geq t_1$.

Proposition 2 follows directly from known arguments in Weinstein [22], and in [18], [15]. The only new point is the fact that one soliton is small with respect to the other. However, these statements are essential in [16] and [17]. In those works, the point is to show that even in a nonintegrable situation, the two soliton structure is preserved by collision. The method in [16], [17] concerns the collision problem in $[-T_{c_1,c_2}, T_{c_1,c_2}]$. To obtain global in time results, it is essential to prove the global in time stability of the two soliton structure after the collision, i.e. for $t > T_{c_1,c_2}$, which is provided by Proposition 2. Proposition 2 is directly applied in [17]. A slightly more precise stability result is required in [16], see Proposition 4 in Section 2.

Now, we claim an extension of Theorem 1 to the case of two solitons, with a qualitative control on $\lim_{t \to +\infty} \rho(t) - c^+ t$.

**Theorem 3** Under the assumptions of Proposition 2, assume further that $f(u) = u^p$, for $p = 2, 3$ or $4$ and $\int_{x > 0} x^2 u^2(0,x) dx < +\infty$. Then, there exist $x_1^+$ and $x_2^+$ such that

$$\lim_{t \to +\infty} \rho_1(t) - c_1^+ t = x_1^+, \quad \lim_{t \to +\infty} \rho_2(t) - c_2^+ t = x_2^+. \quad (1.17)$$

In the case $p = 4$, if in addition, for some $\kappa > 0$,

$$\alpha < \kappa c_2^\frac{4}{7} \quad \text{and} \quad \int_{x > \frac{11}{12} \ln c_2} x^2 u^2(t_1, x) dx < \kappa c_2^\frac{5}{7} \quad (1.18)$$

then

$$|x_1^+ - \rho_1(0)| \leq K c_2^\frac{3}{7}, \quad |x_2^+ - \rho_2(0)| \leq K c_2^\frac{1}{12}. \quad (1.19)$$

The main motivation of Theorem 3 is the following: in [16], in the same context as before, we were able to compute the main order of the shift on the trajectories of the solitons due to the collision at time $t = T_{c_1,c_2}$. Theorem 3 proves that the shifts do change at the main order in large time (for example, at the main order, the shift of $Q_2$ is a nonzero constant independent of $c_2$, so that it is preserved by (1.19)). See proof of Theorem 1.2 in [16] for details.

The plan of the paper is as follows. In Section 2, we prove the stability part of Proposition 2. We focus on the case $f(u) = u^p$ for simplicity, the proof in the general case being exactly the same (see [15] and [18]). Moreover, by a scaling argument we consider only the case $c_1 = 1, c_2 = c$, where $c$ is small enough.

In Section 3, we prove Theorem 3. First, we use the methods of localized Viriel estimates as in [14] to obtain the equivalent of (1.7) for two solitons. Next, we prove (1.17) and (1.19). The proof of Theorem 1 follows directly from the arguments of Section 3, thus it will be omitted.

## 2 Stability for large time of 2-soliton like solutions

Recall that we restrict ourselves to the case $f(u) = u^p\ (p = 2, 3, 4)$ and $c_1 = 1, c_2 = c$ small enough. Let

$$T_c = c^{-\frac{4}{p-2} - \frac{4}{q}}, \quad q = \frac{1}{p-1} - \frac{1}{4}.$$
Denote for $v \in H^1(\mathbb{R})$, $\|v\|_{H^1} = \left( \int_{\mathbb{R}} \left( (v'(x))^2 + cv^2(x) \right) dx \right)^{\frac{1}{2}}$, which corresponds to the natural norm to study the stability of $Q_c$.

**Proposition 4 (Stability of two decoupled solitons)** There exists $K > 0$, $\alpha_0 > 0$, $c_0 > 0$ such that for any $0 < c < c_0$, $0 < \alpha < \alpha_0$, the following is true.

Let $u(t)$ be an $H^1$ solution of (1.1) such that for some $t_1 \in \mathbb{R}$ and $X_0 > 0$,

$$\|u(t_1) - Q - Q_c(\cdot + X_0)\|_{H^1} \leq \alpha c^{\frac{3}{4} + \frac{1}{2}}. \quad (2.1)$$

Then there exist $C^1$ functions $\rho_1(t)$, $\rho_2(t)$ defined on $[t_1, +\infty)$ such that

$$\sup_{t \geq t_1} \|u(t) - (Q(\cdot - \rho_1(t)) + Q_c(\cdot - \rho_2(t)))\|_{H^1} \leq K \alpha c^{\frac{3}{4} + \frac{1}{2}} + K \exp \left( -c^{-\frac{1}{2}} \right), \quad (2.2)$$

$$\forall t \geq t_1, \quad \frac{1}{2} \leq \rho_1(t) - \rho_2(t) \leq \frac{3}{2}, \quad (2.3)$$

$$|\rho_1(t_1)| \leq K \alpha c^{\frac{3}{4} + \frac{1}{2}}, \quad |\rho_2(t_1) - X_0| \leq K \alpha.$$

**Remark.** Note that the proof of Proposition 4 does not need any new arguments with respect to (1.8). We only need to check that the argument of (1.8) still applies to the situation where one soliton is small with respect to the other.

Since $\|Q_c\|_{L^2} = c^\alpha \|Q\|_{L^2}$ (see Claim A.2), the assumption (2.1) does not seem optimal by a factor $\sqrt{c}$. This is due to the fact that the appropriate norm for the stability of $Q_c$ is $\|\cdot\|_{H^1}$.

**Proof of Proposition 4** By time translation invariance, we may assume that $t_1 = 0$. Let $X_0 > T_c/2$ be such that

$$\|u(0) - Q - Q_c(\cdot + X_0)\|_{H^1} \leq \alpha c^{\frac{3}{4} + \frac{1}{2}}. \quad (2.4)$$

Let $D_0 > 2$ to be chosen later, $r = \frac{1}{400}$ and

$$t^* = \sup \left\{ t \geq 0 \mid \forall t' \in [0, t), \ \exists \tilde{\rho}_1, \tilde{\rho}_2 \in \mathbb{R} \mid \tilde{\rho}_1 - \tilde{\rho}_2 > \frac{1}{4} T_c \right. \quad \text{and} \quad \|u(t') - Q(-\tilde{\rho}_1) - Q_c(-\tilde{\rho}_2)\|_{H^1} \leq D_0 (\alpha c^{\frac{3}{4} + \frac{1}{2}} + \exp(-c^{-r})) \left\}. \right.$$  

Observe that $t^* > 0$ is well-defined since $D_0 > 2$, (2.4) and the continuity of $t \mapsto u(t)$ in $H^1$. The objective is to prove $t^* = +\infty$. For the sake of contradiction, we assume that $t^*$ is finite.

First, we decompose the solution on $[0, t^*)$ using modulation theory around the sum of two solitons (see proof of Claim A.1 in Appendix A.1).

**Claim 2.1 (Decomposition of the solution)** For $\alpha > 0$, $c > 0$ small enough, independent of $t^*$, there exist $C^1$ functions $\rho_1(t)$, $\rho_2(t)$, $c_1(t)$, $c_2(t)$, defined on $[0, t^*)$, such that the function $\eta(t)$ defined by

$$\eta(t, x) = u(t, x) - R_1(t, x) - R_2(t, x),$$
where for \( j = 1, 2 \), \( R_j(t, x) = Q_{c_j(t)}(x - \rho_j(t)) \), satisfies for all \( t \in [0, t^*] \),

\[
\int R_j(t) \eta(t) = \int (x - \rho_j(t)) R_j(t) \eta(t) = 0, \quad j = 1, 2,
\]

(2.5)

\[
||\eta(t)||_{H^1} + |c_1(t) - 1| + c^a \left| \frac{c_2(t)}{c} - 1 \right| \leq KD_0(\alpha c^\theta + c^{-\frac{1}{4}} \exp(-c^{-r})),
\]

(2.6)

|\rho'_2(t)| + |\rho'_1(t) - 1| \leq \frac{1}{10}, \quad |\rho_1(t) - \rho_2(t) \geq \frac{1}{2} T_c,
\]

(2.7)

\[
||\eta(0)||_{H^1} + |c_1(0) - 1| + c^a \left| \frac{c_2(0)}{c} - 1 \right| \leq K \alpha c^\theta + \frac{1}{2} T_c,
\]

(2.8)

\[
|\rho_1(0)| + c^{a+\frac{1}{2}}|\rho_2(0) - X_0| \leq K \alpha c^\theta + \frac{1}{2}.
\]

We define

\[
\psi(x) = \frac{2}{\pi} \arctan(\exp(-x/4)), \quad \text{so that} \quad \lim_{+\infty} \psi = 0, \quad \lim_{-\infty} \psi = 1,
\]

(2.9)

\[
\forall x \in \mathbb{R}, \quad \psi(-x) = 1 - \psi(x), \quad \psi'(x) = \frac{1}{4\pi \cosh(x/4)}, \quad |\psi''(x)| \leq \frac{1}{16} |\psi'(x)|.
\]

For \( m(t) = \frac{1}{2}(\rho_1(t) + \rho_2(t)) \), we set

\[
I(t) = \int u^2(t) \psi(x - m(t)) \, dx, \quad g(t) = \int (\eta^2(t, x) + (c + \psi(x - m(t))) \eta^2(t, x)) \, dx.
\]

(2.10)

Note that \( I(t) \) corresponds at the main order to the \( L^2 \) norm of the solution \( u(t) \) at the right of the slow soliton \( R_2(t) \), and the functional \( g(t) \) corresponds locally to the norm adapted to each soliton. In particular, we have \( g(t) \leq ||\eta(t)||_{H^1} \).

We expand \( u(t) = R_1(t) + R_2(t) + \eta(t) \) in the three quantities \( \int u^2(t), I(t) \) and \( E(u(t)) \).

**Lemma 2.1 (Expansion of energy type quantities)**  For all \( t \in [0, t^*] \),

\[
\left| \int u^2(t) - \left( c_1^q(t) + c_2^q(t) \right) \int Q^2 - \int \eta^2(t) \right| \leq Ke^{-\frac{q}{4}t} \exp(-2c^{-r}),
\]

(2.11)

\[
|I(t) - c_1^q(t) \int Q^2 - \int \eta^2 \psi(x - m(t))| \leq Ke^{-\frac{q}{2}t} \exp(-2c^{-r}),
\]

(2.12)

\[
\left| E(u(t)) - \left( E(R_1) + E(R_2) + \frac{1}{2} \int \eta^2(t) - p \left( R_1^{q-1}(t) + R_2^{q-1}(t) \right) \eta^2(t) \right) \right| \leq KD_0(\alpha + \exp(-\frac{1}{2} c^{-r})) g(t) + Ke^{-\frac{q}{4}t} \exp(-2c^{-r}),
\]

(2.13)

\[
\left| E(R_j(t)) - E(R_j(0)) + \frac{c_j(0)}{2} \left[ c_{1j}^q(t) - c_{1j}^q(0) \right] \int Q^2 \right| \leq Ke_j^{q+1}(0) \left( \frac{c_{1j}^q(t)}{c_{1j}^q(0)} - 1 \right)^2.
\]

(2.14)

**Lemma 2.1** is proved in Appendix A.2. In the rest of this section, we assume \( \alpha \) and \( c \) small enough so that

\[
||\eta(t)||_{H^1} + |c_1(t) - 1| + \left| \frac{c_2(t)}{c} - 1 \right| + D_0(\alpha + \exp(-\frac{1}{2} c^{-r})) \leq \frac{1}{100}.
\]

(2.15)

We next obtain a contradiction from the following lemma.
Lemma 2.2 There exists $D_0 > 0$ such that for $\alpha, c > 0$ small enough, independent of $t^*$,

$$\sup_{t \in [0, t^*]} \|u(t) - Q(., - \rho_1(t)) - Q_c(., - \rho_2(t))\|_{H^1} \leq \frac{1}{2} D_0 (\alpha e^{q \frac{1}{2}} + \exp(-c^{-\frac{1}{2}})).$$

(2.16)

If $t^* < +\infty$, then Lemma 2.2 and the continuity in $H^1$ of $u(t)$ contradict the definition of $t^*$. Therefore, we only have to prove Lemma 2.2.

Proof of Lemma 2.2. Step 1. Monotonicity result on $\mathcal{I}(t)$.

Claim 2.2 (Almost monotonicity property of $\mathcal{I}$) For $\alpha$ and $c$ small enough,

$$\forall t \in [0, t^*], \quad \mathcal{I}(t) - \mathcal{I}(0) \leq K \exp(-c^{-\frac{1}{2}}).$$

(2.17)

Proof of Claim 2.2. By standard calculations, we have

$$\mathcal{I}'(t) = -3 \int u^2 \chi'(x-m(t)) - m'(t) \int u^2 \chi'(x-m(t)) + \int u^2 \chi''(x-m(t))$$

$$+ \frac{2p}{p+1} \int u^{p+1} \chi'(x-m(t))$$

$$\leq -\frac{3}{20} \int u^2 \chi'(x-m(t)) + \frac{2p}{p+1} \int u^{p+1} \chi'(x-m(t))$$

(we have used $\chi' > 0, (2.9)$ and $m'(t) \geq 2/5$ by (2.7)).

In the nonlinear term $\int u^{p+1} \chi'(x-m(t))$, we expand $u(t) = R_1(t) + R_2(t) + \eta(t)$. We obtain

$$\int u^{p+1} \chi'(x-m(t)) \leq K' \int u^2 \left( R_1^{p-1} + R_2^{p-1} + \|\eta\|_{L^1} \right) \chi'(x-m(t))$$

$$\leq K' \int u^2 R_1^{p-1} \chi'(x-m(t)) + \frac{1}{10} \int u^2 \chi'(x-m(t))$$

for $\alpha$ and $c$ small enough, since $\|\eta\|_{L^1} \leq K \|\eta\|_{H^1} \leq K c$ and $0 < R_2^{p-1} \leq Kc$. Moreover, by calculations similar to the ones of Claim A.3 and $\|u\|_{L^2} \leq K$, we have

$$\int u^2 R_1^{p-1} \chi'(x-m(t)) \leq Ke^{-\frac{1}{10}} \exp(-c^{-\frac{1}{2}}).$$

Thus, for all $t' \in [0, t^*]$,

$$\mathcal{I}'(t') \leq Ke^{-\frac{1}{10}} \exp(-c^{-\frac{1}{2}}).$$

Let $t \in [0, t^*]$. By integration on $[0, t]$,

$$\mathcal{I}(t) - \mathcal{I}(0) \leq K \exp(-c^{-\frac{1}{2}}). \quad \square$$

Step 2. Estimates on the scaling parameters. Let

$$\Delta_j(t) = \frac{c_j^{2q}(t)}{c_j^{2q}(0)} - 1.$$

Claim 2.3 For all $t \in [0, t^*]$,

$$|\Delta_1(t)| + c^{2q+1} |\Delta_2(t)| \leq K (g(t) + g(0) + \exp(-2c^{-r})).$$

(2.18)
Proof of Claim 2.3. Since there are only two solitons, the proof follows only from the $L^2$ norm and the energy conservation, i.e. (2.11), (2.13) and (2.14). (When there are more than three solitons, the use of quantities such as $\mathcal{I}(t)$ is also needed, see [18].) Let $t \in [0, t^*].$

From (2.11) taken at time 0 and $t$, and $\int u^2(t) = \int u^2(0),$

\[
|c_1^{2q}(0)\Delta_1(t) + c_2^{2q}(0)\Delta_2(t)| \leq K \left( \int \eta^2(t) + \int \eta^2(0) + \exp(-2c^{-r}) \right) 
\leq \frac{K}{c} (g(t) + g(0) + \exp(-2c^{-r})).
\] (2.19)

From (2.13), $E(u(t)) = E(u(0))$ and

\[
\left| \int \eta^2(t) - p \left( R_{1}^{p-1}(t) + R_{2}^{p-1}(t) \right) \eta^2(t) \right| \leq Kg(t),
\]

we have

\[
|E(R_1(t)) - E(R_1(0)) + E(R_2(t)) - E(R_2(0))| \leq K(g(t) + g(0) + \exp(-2c^{-r})).
\]

Then, from (2.14), we obtain

\[
|c_1^{2q+1}(0)\Delta_1(t) + c_2^{2q+1}(0)\Delta_2(t)| \leq K(g(t) + g(0)) + \Delta_1^2(t) + c^{2q+1}\Delta_2^2(t) + K \exp(-2c^{-r}).
\] (2.20)

Multiplying (2.19) by $c_2(0)$ and combining with (2.20), from (2.15), we obtain

\[
c_1^{2q}(0)(c_1(0) - c_2(0))|\Delta_1(t)| \leq K(g(t) + g(0)) + \Delta_1^2(t) + c^{2q+1}\Delta_2^2(t) + K \exp(-2c^{-r}).
\]

By (2.15), we obtain

\[
|\Delta_1(t)| \leq K(g(t) + g(0)) + \Delta_1^2(t) + c^{2q+1}\Delta_2^2(t) + K \exp(-2c^{-r}).
\]

Using this estimate in (2.20), we obtain similarly

\[
c^{2q+1}|\Delta_2(t)| \leq K(g(t) + g(0)) + \Delta_1^2(t) + c^{2q+1}\Delta_2^2(t) + K \exp(-2c^{-r}).
\]

Therefore, for $\Delta_1(t), \Delta_2(t)$ small enough (by (2.6)), we obtain

\[
|\Delta_1(t)| + c^{2q+1}|\Delta_2(t)| \leq K(g(t) + g(0)) + K \exp(-2c^{-r}).
\]

\[
\Box
\]

Step 3. Main argument of the proof of stability.

For $t \in [0, t^*]$, as in [18] we set

\[
\mathcal{F}(u(t)) = E(u(t)) + \frac{c_2(0)}{2} \int u^2(t) + \frac{c_1(0) - c_2(0)}{2} \mathcal{I}(t).
\]

The functional $\mathcal{F}$ coincides in a neighborhood of $R_1$ (respectively, $R_2$) with the functional introduced by Weinstein in [22] to prove the stability of $R_1$ (resp., $R_2$).

We claim the following result on the quadratic part (in $\eta$) of $\mathcal{F}(u(t))$. 
Claim 2.4  Let
\[ H(t) = \frac{1}{2} \int \eta^2(t) + [c_2(0) + (c_1(0) - c_2(0))\psi(x-m(t))] \eta^2(t) - p\left(R^{p-1}_1(t) + R^{p-1}_2(t)\right) \eta^2(t). \]

There exists \( \lambda_0 > 0 \) independent of \( c \) such that, for all \( t \in [0, t^*] \),
\[ \lambda_0 g(t) \leq H(t) \leq \frac{1}{\lambda_0} g(t). \]  \( 2.21 \)

See Appendix A.3 for the proof of Claim 2.4.

On the one hand, using \( 2.11 - 2.14 \), we obtain the following estimate
\[ |F(u(t)) - F(u(0)) - (H(t) - H(0))| \leq KD_0(\alpha + \exp(-\frac{1}{2}c^{-r}))(g(t) + g(0)) + K\Delta_1^2(t) + c^{2q+1}\Delta_2^2(t) + K\exp(-2c^{-r}). \]  \( 2.22 \)

By \( H(0) \leq Kg(0) \), we obtain
\[ |F(u(t)) - F(u(0)) - H(t)| \leq Kg(0) + KD_0(\alpha + \exp(-\frac{1}{2}c^{-r}))g(t) + K\Delta_1^2(t) + c^{2q+1}\Delta_2^2(t) + K\exp(-2c^{-r}). \]  \( 2.23 \)

On the other hand, by conservation of \( E(u(t)) \) and \( \int u^2(t) \), and by the monotonicity of \( I(t) \) (see \( 2.17 \)), we have
\[ F(u(t)) - F(u(0)) \leq K\exp(-c^{1-r}), \]  \( 2.24 \)

and thus by Claims 2.4 and 2.3 we obtain
\[ g(t) \leq \frac{1}{\lambda_0} H(t) \leq \frac{1}{\lambda_0} (F(u(t)) - F(u(0))) + |F(u(t)) - F(u(0)) - H(t)| \]  \( 2.25 \)

\[ \leq Kg(0) + K[D_0(\alpha + \exp(-\frac{1}{2}c^{-r})) + \Delta_1(t) + \Delta_2(t)]g(t) + K\exp(-2c^{-r}). \]

For \( \alpha \) and \( c \) small enough, \( g(t) \), \( \Delta_1(t) \), \( \Delta_2(t) \) and \( D_0(\alpha + \exp(-\frac{1}{2}c^{-r})) \) are small, and from \( 2.3 \), we obtain the following.

Claim 2.5
\[ \|\eta(t)\|_{H^1}^2 \leq g(t) \leq Kg(0) + K\exp(-2c^{-r}) \leq K\alpha c^{2q+1} + K\exp(-2c^{-r}). \]  \( 2.26 \)

\[ |\Delta_1(t)| + c^{2q+1}|\Delta_2(t)| \leq K\exp(g(t) + g(0) + \exp(-2c^{-r})) \leq K\alpha c^{2q+1} + K\exp(-2c^{-r}). \]  \( 2.27 \)

\[ |c_1(t) - 1| + c^{q+\frac{1}{2}} \left| \frac{c_2(t)}{c} - 1 \right| \leq K\alpha c^{q+\frac{1}{2}} + K\exp(-2c^{-r}). \]  \( 2.28 \)

Note that \( 2.27 \) follows from Claim 2.3 and \( 2.26 \).

Now, we go back to \( u(t) \) to prove \( 2.22 \). From direct calculations (recall that \( \|Q_c\|_{H^1_c} \sim Kc^{q+\frac{1}{2}} \)) and \( 2.28 \), we have
\[ \|R_1(t) - Q(., \rho_1(t))\|_{H^1_c} \leq K|c_1(t) - 1| \leq K\alpha c^{q+\frac{1}{2}} + K\exp(-2c^{-r}), \]
\[ \| R_2(t) - Q_c(\cdot - \rho_2(t)) \|_{H^1_c} \leq Kc^{\alpha + \frac{1}{2}} \left( \frac{\rho_2(t)}{c} - 1 \right) \leq K\alpha c^{\alpha + \frac{1}{2}} + K \exp(-2c^{-r}). \]

Therefore, we obtain
\[
\| u(t) - Q(\cdot - \rho_1(t)) - Q_c(\cdot - \rho_2(t)) \|_{H^1_c} \\
\leq \| u(t) - R_1(t) - R_2(t) \|_{H^1_c} + \| R_1(t) - Q(\cdot - \rho_1(t)) \|_{H^1_c} + \| R_2(t) - Q_c(\cdot - \rho_2(t)) \|_{H^1_c} \\
\leq \| \eta(t) \|_{H^1_c} + K\alpha c^{\alpha + \frac{1}{2}} + K \exp(-c^{-r}) \leq K_1 \left( \alpha c^{\alpha + \frac{1}{2}} + \exp(-c^{-r}) \right),
\]

where \( K_1 \) is independent of \( \alpha, c \) and \( D_0 \).

Choose now
\[ D_0 = 4K_1, \]
and then choose \( \alpha > 0, c > 0 \) small enough, so that all the previous estimates hold. Then, for all \( t \in [0, t^*] \), we have
\[ \| u(t) - Q(\cdot - \rho_1(t)) - Q_c(\cdot - \rho_2(t)) \|_{H^1_c} \leq \frac{1}{4} D_0 \left( \alpha c^{\alpha + \frac{1}{2}} + \exp(-c^{-r}) \right). \]

### 3 Refined asymptotics for the 2-soliton structure

We claim the following.

**Proposition 5 (Asymptotic stability)** There exist \( K > 0, \alpha_0 > 0, \tau_0 > 0 \) such that for any \( 0 < c < \tau_0, 0 < \alpha < \alpha_0 \) the following if true.

Let \( u(t) \) be an \( H^1 \) solution of (1.1) such that for \( \frac{1}{2} T_c \leq X_0 \leq \frac{3}{2} T_c \),
\[ \| u(0) - Q - Q_c(\cdot + X_0) \|_{H^1} \leq \alpha c^{\alpha + \frac{1}{2}}, \tag{3.1} \]
so that Proposition \( \square \) applies with \( \rho_1(t), \rho_2(t) \). Then

1. **Convergence of \( u(t) \).** There exist \( c_1^+, c_2^+ > 0 \) such that
\[
\lim_{t \to +\infty} \| u(t) - Q_{c_1^+}(x - \rho_1(t)) - Q_{c_2^+}(x - \rho_2(t)) \|_{H^1(x > ct/10)} = 0. \tag{3.2}
\]

\[ |c_1^+ - 1| \leq K\alpha c^{\alpha + \frac{1}{2}} + K \exp(-c^{-\frac{1}{400}}), \quad \left| \frac{c_2^+}{c} - 1 \right| \leq K\alpha + K \exp(-c^{-\frac{1}{400}}), \tag{3.3} \]

2. **Assume further that** \( f(u) = u^p \), **for** \( p = 2, 3 \) **or** \( 4 \), **and** \( \int_{x > 0} x^2 u^2(0, x) dx < K_0 \). **Then,** **there exist** \( x_1^+ \) **and** \( x_2^+ \) **such that**
\[
\lim_{t \to +\infty} \rho_1(t) - c_1^+ t = x_1^+, \quad \lim_{t \to +\infty} \rho_2(t) - c_2^+ t = x_2^+. \tag{3.4}
\]

**In the case** \( p = 4 \), **if in addition,** **for some** \( \kappa > 0, \)
\[
\alpha < \kappa c^{\frac{1}{c^+}} \quad \text{and} \quad \int_{x > \frac{1}{\kappa c^+} \ln c} x^2 u^2(0, x) dx < \kappa c^{\frac{5}{4}} \tag{3.5}
\]

then
\[ |x_1^+ - \rho_1(0)| \leq Kc^{\frac{5}{4}}, \quad |x_2^+ - \rho_2(0)| \leq Kc^{\frac{1}{2}}. \tag{3.6} \]
Remark. To obtain the convergence of the translation parameters, one has to add an extra assumption on the initial data such as (3.5). Indeed, in the energy space, one can construct an explicit example where convergence does not hold (see [14]). Condition (3.5) is enough for our purposes and could be relaxed, and adapted for the cases $p = 2, 3$.

In what follows, we concentrate on the case $f(u) = u^p$ for $p = 2, 3$ or 4. The proof of the asymptotic stability (part 1 of Proposition 5) in the case of a general nonlinearity $f(u)$ follows from [13] and [15]. Note that estimate (3.3) is a direct consequence of (2.28).

In the proof of Proposition 5 we need another proof of the asymptotic stability for $u^p$, for $p = 2, 3$ or 4, which is derived from the direct arguments of [14]. The interest of this direct approach is to obtain an estimate on the convergence (see Lemma 3.1), which is fundamental in proving the convergence of the translation parameters. For a general nonlinearity, this kind of property is open.

Proof of Proposition 5.
1. The argument presented now is very similar to [14], proof of Theorem 1, Step 3. We keep the notation of the proof of Proposition 4 in particular, the decomposition of nonlinearity, this kind of property is open. Let $\eta \in C^1([-\infty, 0])$, $\eta < 0$ on $[2, +\infty)$, $e^{-x} \leq \Phi(x) \leq 3e^{-x}$, $\Phi' \leq 0$ on $\mathbb{R}^+$, for $A > 0$, $\eta \in C^1([-\infty, 0])$, $\Phi(y) = 0$ on $[0, 1]$; $\Phi(y)$ is an even smooth function such that $\Phi(x) = e^{-x}$ on $[2, +\infty)$, $e^{-x} \leq \Phi(x) \leq 3e^{-x}$, $\Phi' \leq 0$ on $\mathbb{R}^+$, $\Psi(x) = \int_{0}^{x} \Phi(y)dy$, $L_{0} = \int_{0}^{+\infty} \Phi$, $\frac{\Psi_{A}(x)}{A} = A\Psi \left( \frac{x}{A} \right)$ for $A > 0$, $\Psi_{j}(t, x) = \Psi_{A}(\sqrt[c_{j}(t)]{c_{j}(t)(x - \rho_{j}(t))})$, $\Theta_{1}(t, x) = \Psi_{A}(t, x) + L_{0}A$, $\Theta_{2}(t, x) = \Psi_{A}(t, x) - L_{0}A$.

Remark that $\Theta_{1} > 0, \Theta_{1}' > 0, \lim_{-\infty} \Theta_{1} = 0$ and $\Theta_{2} < 0, \Theta_{2}' < 0, \lim_{+\infty} \Theta_{2} = 0$. Let $K_{1}(t) = L_{0}A \int R_{2}^{2}(t) + \int \Theta_{1}(t)\eta^{2}(t)$, $K_{2}(t) = -L_{0}A \int R_{2}^{2}(t) + \int \Theta_{2}(t)\eta^{2}(t)$.

Claim 3.1 (Viriel estimate) There exist $A \geq 5$, $K > 0$, $\alpha_{0} > 0$ and $\tau_{0} > 0$ such that for $0 < \alpha < \alpha_{0}$, $0 < c < \tau_{0}$ and for all $t \in [0, +\infty)$,

$$\sqrt{c_{j}(t)}g_{j}(t) \leq K \left( \frac{d}{dt} c_{j}(t) + e^{-\frac{1}{8} \sqrt{\tau(t + \tau_{c})}} \right).$$

(3.7)
Thus Lemma 3.1 is proved. Note that this result is very similar to Lemma 2 in [14]. In [14], the identity was established in the case of one soliton. Here, to treat the two soliton situation, we have to use an additional term in $K_1(t)$, $K_2(t)$.

From now on, we fix $A \geq 5$ such that Claim 3.1 holds. Then $|\Theta_1(x)| + |\Theta_2(x)| \leq K = K(A)$. From (2.27) in Claim 2.5 and $\int R^2_j = c^2 q_j(t) \int Q^2$, we have, for $t \geq 0$,

$$|K_1(t) - K_1(0)| \leq K(c^2 q(t) - c^2 q(0)) + \|\eta(t)\|_2^2 + \|\eta(0)\|_2^2 \leq K(\alpha^2 c^{2q+1} + \exp(-2c^{-r})),$$

$$|K_2(t) - K_2(0)| \leq K(c^2 q(t) - c^2 q(0)) + \|\eta(t)\|_2^2 + \|\eta(0)\|_2^2 \leq K(\alpha^2 c^{2q} + \exp(-2c^{-r})).$$

Therefore, integrating (3.7) on $[0, +\infty)$ and using $T_\epsilon = c^{-\frac{1}{2} - \frac{1}{10}}$, we obtain ($r = \frac{1}{400}$),

$$\int_0^{+\infty} g_1(t) dt \leq K(\alpha^2 c^{2q+1} + \exp(-2c^{-r})), \quad (3.8)$$

$$\int_0^{+\infty} \sqrt{c} g_2(t) dt \leq K(\alpha^2 c^{2q} + \exp(-2c^{-r})). \quad (3.9)$$

Thus Lemma 3.1 is proved.

Now, we control the scaling parameters. Estimate (B.4) and Lemma 3.1 imply that $c_1(t)$ and $c_2(t)$ have limits as $t \to +\infty$, which we denote respectively by $c_1^+$ and $c_2^+$. By the stability result (2.28),

$$|c_1^+ - 1| \leq K(\alpha c^{q + \frac{1}{2}} + \exp(-c^{-r})), \quad \left|\frac{c_2^+}{c} - 1\right| \leq K(\alpha + \exp(-c^{-r})).$$

Now, we extend the convergence of $\eta$ to 0 in a large region in space, following the proof of Theorem 1 in [14]. We give a sketch the proof (see [14], proof of Theorem 1, Step 3, for more details).

From (3.5), (3.9) (Viriel argument), there exists a sequence $(t_n)$ with $t_n \in [n, n + 1)$ such that $g_1(t_n) + g_2(t_n) \to 0$ as $n \to +\infty$. Using $g_j(t) \leq K_j g_j(t)$ (by a direct computation using (B.1), (3.3), (3.4)), we obtain $\lim_{t \to +\infty} (g_j(t) + g_2(t)) = 0$.

The rest of the proof is based only on monotonicity arguments on $u(t)$ and $u_x(t)$ such as in Claim 2.2 applied on different regions. Set

$$\mathcal{I}_{\sigma,y_0}(t) = \int (u_x^2 + u^2)(t, x) \psi(\sqrt{\sigma}(x - \frac{a}{2} - y_0)) dx. \quad (3.10)$$

First, for $x_0 > 0$, let $y_0 = \rho_1(t_0) - \frac{a}{2} t_0 + x_0$ and $\sigma = 1$. Using $\mathcal{I}_{\sigma,y_0}(t_0) - \mathcal{I}_{\sigma,y_0}(0)$, we obtain

$$\limsup_{t \to +\infty} \int_{x > \rho_1(t_0) + x_0} (\eta_x^2 + \eta^2)(t, x) dx \leq K e^{-\frac{a y_0}{2}}.$$

Next, for $y_0 = \rho_2(t_0) - \frac{a}{2} t_0 + x_0$, $\sigma = c$, using $\mathcal{I}_{\sigma,y_0}(t_0) - \mathcal{I}_{\sigma,y_0}(\tilde{t}_0)$, with $\frac{a}{4} \tilde{t}_0 + y_0 = \rho_1(\tilde{t}_0) - x_0$ (note that $\tilde{t}_0 \geq \frac{a}{4} y_0$ for $t_0$ large), we deduce

$$\limsup_{t \to +\infty} \int_{x > \rho_2(t_0) + x_0} (\eta_x^2 + c \eta^2)(t, x) dx \leq K e^{-\frac{a y_0}{4}}.$$
Finally, by applying another monotonicity argument for \( y_0 = 0 \) and \( \sigma = \frac{2}{5}c \), we obtain

\[
\lim_{t \to +\infty} \int_{x > \frac{x_0}{2t}} (\eta_x^2 + \eta^2)(t, x)dx = 0. \tag{3.11}
\]

Estimate \((3.2)\) follows.

2. Now, we prove the second part of Proposition 5. Assume that

\[
\int_{x > 0} x^2 u^2(0, x)dx < K \quad \text{and} \quad \gamma_0(\alpha, c) = \int_{x > |\ln(\alpha e^{q+ \frac{1}{4}})|} x^2 u^2(0, x)dx < 1. \tag{3.12}
\]

Note that

\[
\rho_j(t) - \rho_j(0) - c_j^+ t = \int_0^t (c_j(s) - c_j^+) ds + \int_0^t (\rho_j'(s) - c_j(s)) ds. \tag{3.13}
\]

To prove that \( \rho_j(t) - c_j^+ t \) has a limit as \( t \to +\infty \), we will study separately the existence of limits as \( t \to +\infty \) of the two integrals above.

2a. Preliminary : Monotonicity results on \( \eta(t) \). We introduce monotonicity results on \( \eta(t) \) (and not on \( u(t) \) as before) that are refinement of Claim 2.2.

We define, for \( j = 1, 2 \),

\[
\mathcal{M}_j(t) = \int \eta^2 \psi_j, \quad \mathcal{E}_j(t) = \int \left[ \frac{1}{2} \eta_x^2 - \frac{1}{p+1} \left( (R_1 + R_2 + \eta)^{p+1} - (p+1)R_1^p \eta - (p+1)R_2^p \eta - (R_1 + R_2)^{p+1} \right) \right] \psi_j, \tag{3.14}
\]

where \( \psi_1(x) = \psi(\tilde{x}), \tilde{x} = x - \rho_1(t) + \frac{1}{2}(t - t_0) \) and \( \psi_2(x) = \psi(\sqrt{c} \tilde{x}_c), \tilde{x}_c = x - \rho_2(t) + \frac{2}{3}(t - t_0) \), for \( t_0 \geq 0 \) and \( \psi(x) \) is defined by \((2.9)\).

We claim the following monotonicity results (see the proof in Appendix B.2).

**Claim 3.2** For all \( t \geq t_0 \geq 0 \),

\[
\frac{d}{dt} \left( c_1^{2q}(t) \int Q^2 + \mathcal{M}_1(t) \right) \leq Ke^{\frac{-1}{16\sqrt{c}}(t-t_0)} g_1(t) + Ke^{\frac{-1}{12\sqrt{c}}(t+T_c)},
\]

\[
\frac{d}{dt} \left( -\frac{2q}{2q+1} c_1^{2q+1}(t) \int Q^2 + 2\mathcal{E}_1(t) + \frac{1}{100} \left( c_1^{2q}(t) \int Q^2 + \mathcal{M}_1(t) \right) \right) \leq Ke^{\frac{-1}{16\sqrt{c}}(t-t_0)} g_1(t) + Ke^{\frac{-1}{12\sqrt{c}}(t+T_c)},
\]

\[
\frac{d}{dt} \left( c_1^2(t) + c_2^2(t) \right) \int Q^2 + \mathcal{M}_2(t) \leq Ke^{\frac{-5\sqrt{c}}{100}(t-t_0)} \sqrt{c} g_2(t) + Ke^{\frac{-1}{12\sqrt{c}}(t+T_c)} ,
\]

\[
\frac{d}{dt} \left( -\frac{2q}{2q+1} \left( c_1^{2q+1}(t) + c_2^{2q+1}(t) \right) \int Q^2 + 2\mathcal{E}_2(t) + \frac{c}{100} \left( \left( c_1^{2q}(t) + c_2^{2q}(t) \right) \int Q^2 + \mathcal{M}_2(t) \right) \right) \leq Ke^{\frac{-5\sqrt{c}}{100}(t-t_0)} c^2 g_2(t) + Ke^{\frac{-1}{12\sqrt{c}}(t+T_c)} .
\]

**Remark.** The improvement of Claim 3.2 with respect to the monotonicity on \( u(t) \) in Claim 2.2 or Lemma 3 in [14] is that the upper bound can be integrated twice in time.

Claim 3.2 is one example of monotonicity result on localized energy type quantities on \( \eta(t) \). In Appendix B.2, we prove a slightly more general version of Claim 3.2 where \( \tilde{x} = x - \rho_1(t) + \frac{2}{3}(t - t_0) - x_0 \), where \( 0 < \sigma \leq \frac{1}{2} \) and \( x_0 \in \mathbb{R} \), and we claim other monotonicity results to be used in this paper.
Let
\[ \tilde{g}_1(t) = \int (\eta_1^2 + \eta_2^2)(t, x)\psi(x - \rho_1(t))dx, \quad \tilde{g}_2(t) = \int (\eta_1^2 + c\eta_2^2)(t, x)\psi(\sqrt{c}(x - \rho_2(t)))dx. \]

Note that there exists \( K > 0 \) such that \( \psi(x) \geq \frac{1}{K} e^{-\frac{|x|}{6}} \) on \( \mathbb{R} \). Thus, we have \( g_j(t) \leq K \tilde{g}_j(t) \).

We have the following consequence of Claim 3.2.

**Claim 3.3 (Control on the scaling parameters)** For all \( t \geq t_0 \),
\[
|c_1(t) - c_1(t_0)| \leq K(\tilde{g}_1(t) + \tilde{g}_1(t_0)) + K \int_{t_0}^t e^{-\frac{1}{2}(t' - t_0)}g_1(t')dt' + Ke^{-\frac{1}{2}\sqrt{c}(t_0 + T_c)},
\]
\[
|c_2^{2q+1}(t) - c_2^{2q+1}(t_0)| \leq 2|c_1(t) - c_1(t_0)| + K(\tilde{g}_2(t) + \tilde{g}_2(t_0)) + Kc^2 \int_{t_0}^t e^{-\frac{q^2}{32}(t' - t_0)}g_2(t')dt' + Ke^{-\frac{1}{2}\sqrt{c}(t_0 + T_c)}.
\]

**Proof of Claim 3.3** Integrating the conclusion of Claim 3.2 between \( t_0 \) and \( t \), we obtain
\[
c_1^{2q}(t) - c_1^{2q}(t_0) \leq \frac{1}{Q^2} (\mathcal{M}_1(t_0) - \mathcal{M}_1(t)) + K \int_{t_0}^t e^{-\frac{1}{2}(t' - t_0)}g_1(t')dt' + Ke^{-\frac{1}{2}\sqrt{c}(t_0 + T_c)},
\]
\[
\frac{2q}{2q + 1} \left( c_1^{2q+1}(t) - c_1^{2q+1}(t_0) \right) - \frac{1}{100} (c_1^{2q}(t) - c_1^{2q}(t_0)) \leq \frac{1}{Q^2} (2\mathcal{E}_1(t) - 2\mathcal{E}_1(t_0))
\]
\[
+ \frac{1}{100} \int_{t_0}^t e^{-\frac{1}{2}(t' - t_0)}g_1(t')dt' - Ke^{-\frac{1}{2}\sqrt{c}(t_0 + T_c)}.
\]

Since \( c_1 \sim 1 \) by (2.15), \( \mathcal{M}_1(t) \geq 0 \), \( \mathcal{E}_1(t) \geq -K\tilde{g}_1(t) \geq -K' \tilde{g}_1(t) \), and \( \mathcal{E}_1(t_0) \leq K \tilde{g}_1(t_0) \), \( \mathcal{M}_1(t_0) \leq K \tilde{g}_1(t_0) \), we obtain the first estimate of (3.15). The estimate on \( |c_2^{2q+1}(t) - c_2^{2q+1}(t_0)| \) is obtained in the same way using \( \mathcal{M}_2(t) \) and \( \mathcal{E}_2(t) \). \( \square \)

We claim the following lemma.

**Lemma 3.2** Assume that (3.12) holds. Then,
\[
\int_0^{+\infty} \tilde{g}_1(t)dt \leq K((\alpha c^{q+\frac{1}{2}})^{\frac{q}{2}} + \gamma_0(\alpha, c) + \alpha^{-\frac{1}{4}} \exp(-\frac{3}{2}c^{-r})),
\]
\[
\int_0^{+\infty} \tilde{g}_2(t)dt \leq K(\alpha \gamma c^{q+\frac{1}{2}} + \alpha^2 c^{2q+\frac{1}{2}} + \frac{1}{c} \gamma_0(\alpha, c) + \alpha^{-\frac{1}{4}} \exp(-\frac{3}{2}c^{-r}))
\]
\[
\lim_{t \to +\infty} t \int (\eta_1^2(t, x) + \eta_2^2(t, x))\psi(x - \delta t)dx = 0.
\]

For the proof see Appendix B.3. Note that the proof is based only on Lemma 3.1 and monotonicity arguments such as Claim 3.2. We follow the same steps as in the proof of Claim 3.2, using quantities \( \mathcal{M}_j(t) \), \( \mathcal{E}_j(t) \) instead of \( I_{\alpha, \gamma_0} \) on the same lines. The proof of the monotonicity is the same as the one of Claim 3.2.

2b. Estimate on \( \rho_j(t) - c_j^+ \).
Lemma 3.3 (Estimate on $c_j(t) - c_j^+$) Assume that (3.12) holds. Then,

$$
\int_0^{+\infty} |c_1(t) - c_1^+| dt \leq K((\alpha c^{q+\frac{1}{2}})\gamma_1 + \gamma_0(\alpha, c) + \alpha^{-\frac{1}{4}} \exp(-c^{-r})),$$

$$
\int_0^{+\infty} |c_2(t) - c_2^+| dt \leq K(\alpha^{\frac{7}{4}} c^{\frac{1}{4}q - \frac{1}{2}} + \alpha^{2} c^{\frac{1}{4}q - \frac{1}{2}} + c^{-2q-1} \gamma_0(\alpha, c) + \alpha^{-\frac{1}{4}} \exp(-c^{-r})).
$$

Proof of Lemma 3.3 Let $t \to +\infty$ in Claim 3.3 since $c_j(t) \to c_j^+$ and $\tilde{g}_j(t) \to 0$, we obtain

$$
|c_1^+ - c_1(t_0)| \leq K \tilde{g}_1(t_0) + \int_{t_0}^{+\infty} e^{-\frac{1}{\alpha}(t'-t_0)} g_1(t') dt' + Ke^{-\frac{1}{4\alpha} \sqrt{c(t_0+T_c)}},
$$

$c^{2q} |c_2^+ - c_2(t_0)| \leq K |c_1^+ - c_1(t_0)| + K \tilde{g}_2(t_0) + Ke^3 \int_{t_0}^{+\infty} e^{-\frac{1}{\alpha}(t'-t_0)} g_2(t') dt' + Ke^{-\frac{1}{4\alpha} \sqrt{c(t_0+T_c)}}. $

Thus, integrating on $[0, +\infty)$ and using Fubini Theorem, since $g_1(t) \leq K \tilde{g}_1(t)$, we obtain

$$
\int_0^{+\infty} |c_1(t) - c_1^+| dt \leq K \int_0^{+\infty} \tilde{g}_1(t) dt + K c^{2q} \exp(-2c^{-r}).
$$

Similarly,

$$
\int_0^{+\infty} |c_2(t) - c_2^+| dt \leq K \int_0^{+\infty} (\tilde{g}_1(t) + \tilde{g}_2(t)) dt + K c^{2q} \exp(-2c^{-r}).
$$

Thus Lemma 3.3 is a consequence of Lemma 3.2. □

Lemma 3.4 (Estimate on $\rho_j'(t) - c_j(t)$) Assume that (3.12) holds. For $j = 1, 2$, $\int_0^{+\infty} (\rho_j'(t) - c_j(t)) dt$ is defined and

$$
\int_0^{+\infty} (\rho_1'(t) - c_1(t)) dt \leq K((\alpha c^{q+\frac{1}{2}})\gamma_1 + \gamma_0(\alpha, c) + \exp(-c^{-r})),
$$

$$
\int_0^{+\infty} (\rho_2'(t) - c_2(t)) dt \leq K(\alpha^{\frac{7}{4}} c^{\frac{1}{4}q + \frac{1}{2}} + \alpha^{2} c^{\frac{1}{4}q - \frac{1}{2}} + \alpha^{3} c^{-\frac{1}{2} - \frac{3}{4}q} + \alpha^{4} c^{-\frac{1}{2} - \frac{3}{4}q} + \alpha^{5} c^{-\frac{1}{2} - \frac{3}{4}q}).
$$

Lemma 3.4 is proved in Appendix B.4. Note that it makes use of the following functional

$$
J_j(t) = c_j^{-2q}(t) \int \eta(t,x) \left( \int_{-\infty}^{x} \hat{R}_j(t,x') dx' \right) dx,
$$

where $\hat{R}_j(t,x) = \tilde{Q}_{c_j(t)}(x - \rho_j(t))$, which is an $L^1$-type quantity, already introduced in [13]. For $p = 3$, another argument can be used. From Claim B.4 (B.5),

$$
|\rho_j'(t) - c_j(t)| \leq Kg_j(t) + Ke^{-\frac{1}{8} \sqrt{c(t+T_c)}},
$$

where $g_j(t)$ is defined in (B.2). By (3.8)-(3.9), we obtain in this case:

$$
(p = 3) \quad \int_0^{+\infty} |\rho_1'(t) - c_1(t)| + c^{2} |\rho_2'(t) - c_2(t)| dt \leq K \alpha^{2} c^{2q+1} + K \exp(-2c^{-r}).
$$
Moreover, if \( u_0 \) and unique Lemma A.1 (Existence of modulation parameters)

\[ j \]

then, for \( \rho_j(t) - c_j^+ t \to x_j^+ \) as \( t \to +\infty \),

where

\[
|x_j^+ - \rho_1(0)| \leq K((\alpha c^{q + \frac{1}{2}})^\frac{q}{2} + \gamma_0^{\frac{1}{2}}(\alpha, c) + \alpha^{-\frac{1}{2}} \exp(-c^{-r})) ,
\]

\[
|x_2^+ - \rho_2(0)| \leq K(\alpha^\frac{q}{2} e^{-\frac{1}{2}q - \frac{1}{4}} + \alpha^\frac{q}{2} e^{-\frac{1}{2}q - \frac{1}{4}} + \alpha^2 c^{-\frac{1}{2}q - \frac{1}{2}} + c^{-2q-1} \gamma_0(\alpha, c) + (c^{-q + \frac{1}{4}} + \alpha^2 c^{-q - \frac{1}{2}})(\frac{1}{2} + \gamma_0(\alpha, c) + \alpha^{-\frac{1}{2}} \exp(-\frac{1}{4} c^{-r})).
\]

3. Control of the shifts under assumption (3.5) for \( p = 4 \). Now, we assume, for some \( \kappa > 1 \),

\[
\gamma_0 = \gamma_0(\kappa c^{\frac{1}{4}}, c) \leq \int_{x > \frac{1}{12} |\ln c|} x^2 u^2(0, x) dx < K c^{\frac{5}{8}}.
\]

We apply the previous estimate with \( \bar{\alpha} = \kappa c^{\frac{1}{4}} \). Then, from (3.17), using \( q = \frac{1}{12} \) in this case, we get

\[
|x_j^+ - \rho_1(0)| \leq K c^{\frac{5}{8} \left(\frac{1}{4} + \frac{1}{12}\right)} + K c^{\frac{5}{8}} + K c^{-\frac{1}{12}} \exp(-\frac{1}{2} c^{-r})) \leq K c^{\frac{5}{8}},
\]

\[
|x_2^+ - \rho_2(0)| \leq K c^{\frac{1}{12}},
\]

where the worst term in \( |x_2^+ - \rho_2(0)| \) is \( c^{-2q-1} \gamma_0(\alpha, c) \) \( \leq K c^{\frac{5}{12}} \).

A Appendix

A.1 Proof of Claim 2.1

We first state a preliminary result. Recall \( T_c = c^{-\frac{1}{2}(1+\frac{1}{2\alpha})} \). For \( \alpha, c > 0 \), we define

\[
\mathcal{U}(\alpha, c) = \left\{ u \in H^1(\mathbb{R}); \exists r_1, r_2 \in \mathbb{R} \mid |r_1 - r_2| > \frac{1}{2} T_c \right\}
\]

\[ \text{and} \quad \|u - Q(\cdot - r_1) - Q_c(\cdot - r_2)\|_{H^1} \leq \alpha c^\alpha \} \right. \}
\]

Lemma A.1 (Existence of modulation parameters) There exists \( \tau_0 > 0, \alpha_0 > 0, K > 0 \) and unique \( C^1 \) functions \( (c_1, c_2, \rho_1, \rho_2) : \mathcal{U}(\alpha_0, \tau_0) \to (0, +\infty)^2 \times \mathbb{R}^2 \), such that if \( u \in \mathcal{U}(\alpha_0, \tau_0) \), and

\[
\eta(x) = u(x) - Q_{c_1}(x - \rho_1) - Q_{c_2}(x - \rho_2),
\]

then, for \( j = 1, 2 \),

\[
\int Q_{c_j}(x - \rho_j) \eta(x) dx = \int Q'_{c_j}(x - \rho_j) \eta(x) dx = 0.
\]

Moreover, if \( u \in \mathcal{U}(\alpha, c) \), with \( 0 < \alpha < \alpha_0, 0 < c < \tau_0 \), then

\[
\|\eta\|_{H^1} + |c_1 - 1| \leq K \alpha c^\alpha, \quad \left| \frac{c_2}{c} - 1 \right| \leq K \alpha, \quad |\rho_1 - \rho_2| > \frac{1}{4} T_c.
\]
Moreover, by \( \nu\) and \( \alpha\), we give the proof of some of these estimates. First, note that

\[
V(\alpha) = V_{C, r_1, r_2}(\alpha) = \{ u \in H^1(\mathbb{R}), \|u - Q(. - r_1) - Q_c(. - r_2)\|_{H^1} \leq 2\alpha e^q \}. \tag{A.5}
\]

Let \( V(\alpha) = [0, \alpha] \times [-\alpha, \alpha]^2 \times V(\alpha) \subset \mathbb{R}^4 \times H^1(\mathbb{R}), \)

\[
M_0 = (0, 0, 0, 0, Q(. - r_1) + Q_c(. - r_2)), \quad \text{and any } M = (\lambda_1, \lambda_2, y_1, y_2, u) \in V(\alpha). \tag{A.6}
\]

Let also

\[
Q_1(x) = Q_{1+\lambda_1 c^q}(x - r_1 - e^q y_1), \quad Q_2(x) = Q_{c(1+\lambda_2)}(x - r_2 - e^q y_2),
\]

\[
w(M) = c^{-q}(u - Q_1 - Q_2),
\]

\[
\nu_1(M) = \int w(M)Q_1, \quad \nu_2(M) = c^{-q} \int w(M)Q_2,
\]

\[
\mu_1(M) = \int w(M)Q_1', \quad \mu_2(M) = c^{-(q+\frac{1}{2})} \int w(M)Q_2'.
\]

For any \( M \in V(\alpha) \), since \( \int Q_2^2 = c^{2q} \int Q^2 \) and \( \int (Q'_c)^2 = c^{2(q+\frac{1}{2})} \int (Q')^2 \), we have

\[
|\nu_1(M)| + |\nu_2(M)| + |\mu_1(M)| + |\mu_2(M)| \leq K\alpha. \tag{A.7}
\]

**Claim A.1** For any \( M \in V(\alpha) \), for any \( j, k = 1, 2, j \neq k \),

\[
\left| \frac{\partial \nu_j}{\partial y_j}(M) + \frac{5 - p}{4(p-1)} \int Q^2 \right| + \left| \frac{\partial \mu_j}{\partial y_j}(M) - \int (Q')^2 \right| + \left| \frac{\partial \nu_j}{\partial y_j}(M) \right| + \left| \frac{\partial \mu_j}{\partial y_j}(M) \right| \leq K\alpha,
\]

\[
\left| \frac{\partial \nu_j}{\partial y_k}(M) + \frac{\partial \mu_j}{\partial \lambda_k}(M) + \frac{\partial \nu_j}{\partial \lambda_k}(M) + \frac{\partial \mu_j}{\partial y_k}(M) \right| \leq K \exp(-c^{-r}).
\]

**Proof of Claim A.1** The claim follows from elementary calculations similar to the ones in [18], Appendix A. We give the proof of some of these estimates. First, note that

\[
\frac{\partial \nu_1}{\partial \lambda_1}(M) = -c^{-q} \int \frac{\partial Q_1}{\partial \lambda_1} Q_1 + \int w(M) \frac{\partial Q_1}{\partial \lambda_1}.
\]

Moreover, by \( Q_{c_0}(x) = c_0^{-p} Q(\sqrt{c_0}x) \), we have (recall \( \tilde{Q}_{c_0} = \frac{2}{p-1} Q_{c_0} + xQ_{c_0}' \))

\[
\frac{\partial Q_{c_0}}{\partial c_0} = \frac{\tilde{Q}_{c_0}}{2c_0} \quad \text{and thus} \quad \int \frac{\partial Q_{c_0}}{\partial c_0} Q_{c_0} = \frac{5 - p}{4(p-1)} c_0^{2q-1} \int Q_{c_0}^2.
\]

We have

\[
\frac{\partial Q_1}{\partial \lambda_1} = c^q \frac{1}{2(1 + \lambda_1 c^q)} \tilde{Q}_{1+\lambda_1 c^q}.
\]

Thus,

\[
\frac{\partial \nu_1}{\partial \lambda_1}(M) = -\frac{5 - p}{4(p-1)} (1 + \lambda_1 c^q)^{2q-1} \int Q^2 + \int w(M) \frac{\partial Q_1}{\partial \lambda_1},
\]

19
and by \( \|w\|_{L^2} \leq K\alpha \) and \( |\lambda_1| \leq \alpha \), we obtain

\[
\left| \frac{\partial \nu_1}{\partial \lambda_1}(M) + \frac{5-p}{4(p-1)} \int Q^2 \right| \leq K\alpha.
\]

Similarly,

\[
\frac{\partial \nu_2}{\partial \lambda_2}(M) = -c^{-2q} \int \frac{\partial Q_2}{\partial \lambda_2} Q_2 + c^{-q} \int w(M) \frac{\partial Q_2}{\partial \lambda_2}.
\]

We have

\[
\frac{\partial Q_2}{\partial \lambda_2} = \frac{1}{2(1 + \lambda_2)} \hat{Q}_{c(1+\lambda_2)} \quad \text{and so} \quad c^{-2q} \int \frac{\partial Q_2}{\partial \lambda_2} Q_2 = (1 + \lambda_2)^{2q-1} \frac{5-p}{4(p-1)} \int Q^2.
\]

From

\[
\left| c^{-q} \int w(M) \frac{\partial Q_2}{\partial \lambda_2} \right| \leq \|w\|_{L^2} c^{-q} \frac{\|\partial Q_2\|_{L^2}}{\partial \lambda_2} \leq K\|w\|_{L^2}^2,
\]

and \( |\lambda_2| \leq K\alpha \), we obtain

\[
\left| \frac{\partial \nu_2}{\partial \lambda_2}(M) + \frac{5-p}{4(p-1)} \int Q^2 \right| \leq K\alpha.
\]

Third, we have

\[
\frac{\partial \mu_2}{\partial y_2}(M) = -c^{-2q-\frac{1}{2}} \int \frac{\partial Q_2}{\partial y_2} Q_2 + c^{-q} \int w(M) \frac{\partial Q'_2}{\partial y_2}.
\]

But

\[
\frac{\partial Q_2}{\partial y_2} = -c^{\frac{1}{2}} Q'_2, \quad \frac{\partial Q'_2}{\partial y_2} = -c^{-\frac{1}{2}} Q''_2,
\]

and so

\[
-c^{-2q-\frac{1}{2}} \int \frac{\partial Q_2}{\partial y_2} Q_2 = c^{-2(q+\frac{1}{4})} \int (Q'_2)^2 = (1 + \lambda_2)^{2(q+\frac{1}{4})} \int (Q')^2.
\]

Moreover,

\[
c^{-q} \left| \int w \frac{\partial Q'_2}{\partial y_2} \right| \leq K\alpha.
\]

Fourth, we have

\[
\frac{\partial \nu_2}{\partial y_2}(M) = c^{-2q} \int \left( -\frac{\partial Q_2}{\partial y_2} \right) Q_2 + c^{-q} \int w \frac{\partial Q_2}{\partial y_2}.
\]

The first term in the right hand side is 0 by parity, the second term is controlled by \( K\alpha \).

Finally, we check a different term:

\[
\frac{\partial \nu_1}{\partial y_2}(M) = c^{-q-\frac{1}{2}} \int Q'_2 Q_1,
\]

since \( \frac{\partial Q_1}{\partial y_2} = 0 \). By \( \int Q'_2 Q_1 \leq K \exp(-2c^{-r}) \) (see proof of Claim \textit{A.3} for similar estimates), we obtain the desired estimate. \( \square \)
By Claim A.1 and A.7, and \((\nu_1, \nu_2, \mu_1, \mu_2)(M_0) = (0, 0, 0, 0)\), we apply the implicit function theorem to \((\nu_1, \nu_2, \mu_1, \mu_2)\): there exists \(c > 0\), \(a_0 > 0\) (chosen independent of \(c, r_1, r_2\)) such that, if \(0 < c < c_0\), \(0 < \alpha < a_0\), for all \(u \in V(\alpha)\), there exists \(\lambda_1(u), \lambda_2(u), y_1(u), y_2(u)\) unique such that if \(M(u) = \lambda_1(u), \lambda_2(u), y_1(u), y_2(u), u\), then

\[
\nu_1(M(u)) = \nu_2(M(u)) = \mu_1(M(u)) = \mu_2(M(u)) = 0. \tag{A.8}
\]

Moreover,

\[
|\lambda_1(u)| + |\lambda_2(u)| + |y_1(u)| + |y_2(u)| \leq K\alpha. \tag{A.9}
\]

Now, we set

\[
c_1(u) = 1 + \lambda_1(u)c^\alpha, c_2(u) = c(1 + \lambda_2(u)), y_1(u) = r_1 + y_1(u)c^\alpha, y_2(u) = r_2 + y_2(u)c^{-\frac{1}{2}}.
\]

For any \(u \in U(\alpha, c)\), there exist \(r_1, r_2\) satisfying \(|r_1 - r_2| \geq \frac{1}{2}T_c\) and \(|u - Q(\cdot - r_1) - Q(\cdot - r_2)||H^1| \leq 2\alpha c^\alpha\). Thus \(c_1(u), c_2(u), r_1(u), r_2(u)\) are defined as before for such \(u\). Uniqueness and regularity are consequences of the implicit function theorem. Note finally that (A.9) implies \(|c_1 - 1| \leq K\alpha^\alpha, |\frac{c}{c} - 1| \leq K\alpha\) and \(|x_1 - x_2| \geq \frac{1}{2}T_c\). \(\square\)

Proof of Claim 2.1. For \(t = 0\), using assumption (2.3), we apply Lemma A.1 to \(u(0)\) with \(\alpha c^\frac{1}{2}\) instead of \(\alpha\). We find \(\rho_1(0), \rho_2(0), c_1(0)\) and \(c_2(0)\) such that (2.5) and (2.8) hold.

For \(t > 0\), by the definition of \(t^*\), \(\|u(t)||H^1| \leq c^{-\frac{1}{2}}\|u(t)||H^1| \leq D_0(\alpha + c^{-\frac{1}{2}} \exp(-c^{-\gamma}))\). We apply Lemma A.1 to \(u(t)\) where \(\alpha\) is replaced by \(D_0(\alpha + c^{-\frac{1}{2}} \exp(-c^{-\gamma}))\) which is small for \(\alpha\) small depending on \(D_0\). We obtain directly (2.5)–(2.6). The estimates on \(\rho_1'(t)\) and \(\rho_2'(t)\) follow from the equation of \(\eta(t)\) written in Claim 2.1, see Claim B.1 below. Since \(\rho_1'(t) - \rho_2'(t) \geq \frac{1}{2}\) and \(\rho_1(0) - \rho_2(0) \geq \frac{1}{2}T_c\), we obtain (2.7). \(\square\)

A.2 Proof of Lemma 2.1

First, we recall well-known identities related to \(Q_c\) for \(f(u) = u^p\).

Claim A.2 (Identities for any \(p > 1\))

\[
\int Q^{p+1} = \frac{2(p + 1)}{p + 3} \int Q^2, \quad \int (Q')^2 = \frac{p - 1}{p + 3} \int Q^2.\n\]

\[
\int Q_c^2 = c^{2q} \int Q^2, \quad E(Q) = c^{2q + 1} E(Q) = -\frac{5 - p}{2(p + 3)} c^{2q + 1} \int Q^2.
\]

Proof of Lemma A.2. These are well-known calculations. We have \(Q^p = Q - Q''\) and \(\frac{2}{p+1} Q^{p+1} = Q^2 - (Q')^2\). Thus, by integration:

\[
\int Q^{p+1} = \int Q^2 + \int (Q')^2, \quad \frac{2}{p + 1} \int Q^{p+1} = \int Q^2 - \int (Q')^2.
\]

Therefore, \(\int Q^{p+1} = \frac{2(p+1)}{p+3} \int Q^2\) and \(\int (Q')^2 = \int Q^{p+1} - \int Q^2 = \frac{p - 1}{p + 3} \int Q^2\). Moreover, \(E(Q) = \frac{1}{2} \int (Q')^2 - \frac{1}{p+1} \int Q^{p+1} = \frac{p-5}{2(p+3)} \int Q^2\).
Since $Q_c(y) = e^{\frac{1}{2}Q(\sqrt{cy})}$ and $q = \frac{1}{r-1} - \frac{1}{4}$, we have
\[
\int Q_c^2(y)dy = e^{\frac{1}{2}Q} \int Q^2(\sqrt{cy})dy = c^{2q} \int Q^2.
\]
Similarly, \( \int (Q_c')^2 = c^{2q+1} \int (Q')^2 \) and \( \int Q_c^{p+1} = c^{2q+1} \int Q^{p+1} \), and so \( E(Q_c) = c^{2q+1}E(Q) \). □

Then, we claim the following estimates (recall $r = \frac{1}{46}$).

**Claim A.3** For all $t \in [0, t^*)$,
\[
\forall x \in \mathbb{R}, \quad R_1(t, x) \leq K\psi(x-m(t)); \quad (A.10)
\]
\[
0 \leq \int R_1(t, x)R_2(t, x)dx \leq Ke^{-\frac{\sqrt{c}}{2}t} \exp(-2e^{-r}); \quad (A.11)
\]
\[
0 \leq \int R_1(t, x)(1-\psi(x-m(t)))dx \leq Ke^{-\frac{\sqrt{c}}{2}t} \exp(-c^{-\frac{1}{2}}e^{-r}); \quad (A.12)
\]
\[
0 \leq \int R_2(t, x)\psi(x-m(t))dx \leq Ke^{-\frac{\sqrt{c}}{2}t} \exp(-2e^{-r}). \quad (A.13)
\]

**Proof of Claim A.3** First, note that since $c_1(t) > \frac{1}{2}$ and $c_2(t) > \frac{1}{2}$, we have
\[
0 \leq R_1(t, x) \leq Ke^{-\frac{1}{2}|x-p_1(t)|}, \quad 0 \leq R_2(t, x) \leq Ke^{-\frac{c}{2}|x-p_2(t)|}.
\]

**Proof of (A.10).** For $x > m(t)$, we have $\psi(x-m(t)) > \frac{1}{2}$ and so $R_1(t, x) \leq K\psi(x-m(t))$. For $x \leq m(t)$, by the definition of $\psi(x)$ and $m(t) < p_1(t)$, we have
\[
R_1(t, x) \leq Ke^{\frac{c}{2}|x-p_1(t)|} \leq Ke^{\frac{1}{2}(x-m(t)) + \frac{c}{2}(m(t)-p_1(t))} \leq K\psi(x-m(t)).
\]

**Proof of (A.11).** We have
\[
0 \leq R_1(t, x)R_2(t, x) \leq Ke^{-\frac{1}{2}|x-p_1(t)|}e^{-\frac{\sqrt{c}}{2}|x-p_2(t)|}
\]
\[
\leq Ke^{-\frac{c}{2}|x-p_1(t)|}e^{\frac{\sqrt{c}}{2}|x-p_1(t)|}e^{-\frac{\sqrt{c}}{2}|p_1(t)-p_2(t)|}
\]
\[
\leq Ke^{-\frac{c}{2}|x-p_1(t)|}e^{-\frac{\sqrt{c}}{2}t}e^{-\frac{c}{2}e^{-2r}} \leq Ke^{-\frac{c}{2}|x-p_1(t)|}e^{-\frac{\sqrt{c}}{2}t} \exp(-2e^{-r}).
\]
for $c$ small enough. Thus by integration in $x$, we obtain (A.11).

**Proof of (A.12).** For $x \geq m(t) + \frac{t}{8} + \frac{1}{32}T_c$, $1 - \psi(x-m(t)) \leq Ke^{\frac{c}{2}(x-m(t))} \leq Ke^{-\frac{1}{2}} \exp(-\frac{1}{64}T_c)$, and so $R_1(t)(1-\psi(x-m(t))) \leq Ke^{\frac{c}{2}|x-p_1(t)|}e^{-\frac{1}{2}T_c} \exp(-\frac{1}{64}T_c)$.

For $x \leq m(t) + \frac{t}{8} + \frac{1}{32}T_c$, $p_1(t) \leq \frac{t}{8} - \frac{1}{32}T_c$,
\[
R_1(t) \leq Ke^{-\frac{c}{2}|x-p_1(t)|} \leq Ke^{-\frac{c}{2}|x-p_1(t)|}e^{-\frac{1}{32}T_c} \exp(-\frac{1}{128}T_c),
\]
and $0 \leq 1 - \psi(x-m(t)) \leq 1$. Thus, by integration in $x$ and for $c$ small, we obtain (A.12).

**Proof of (A.13).** For $x \leq m(t) + \frac{t}{8} - \frac{1}{32}T_c$,
\[
\psi(x-m(t)) \leq Ke^{\frac{c}{2}(x-m(t))} \leq Ke^{-\frac{1}{2}} \exp(-\frac{1}{64}T_c),
\]
\[
R_2(t)\psi(x-m(t)) \leq Ke^{\frac{1}{2}T_c}e^{-\frac{\sqrt{c}}{2}|x-p_2(t)|} \exp(-\frac{1}{64}T_c).
\]
For $x \geq m(t) - \frac{t}{8} - \frac{1}{32}T_c \geq p_2(t) + \frac{t}{8} + \frac{1}{32}T_c$, we have
\[
R_2(t) \leq Ke^{\frac{1}{2}T_c}e^{-\frac{\sqrt{c}}{2}|x-p_2(t)|} \leq Ke^{\frac{1}{2}T_c}e^{-\sqrt{T_c}} \exp(-\frac{1}{64}T_c),
\]
and $0 \leq \psi(x-m(t)) \leq 1$. Thus, by integration in $x$ and for $c$ small, we obtain (A.13). □
Proof of Lemma 2.1. From \( \int R_j \eta = 0 \), we have
\[
\int u^2(t) = \int R_1^2(t) + \int R_2^2(t) + \int \eta^2(t) + 2 \int R_1(t)R_2(t).
\]
Thus (2.11) is now a consequence of (A.11) and \( \int R_j^2(t) = c_j^2(t) \int Q^2 \) (Claim A.2).

Proof of (2.13). First, we prove the following estimate:
\[
|E(u(t)) - \left\{ E(R_1) + E(R_2) + \frac{1}{2} \int \eta_x^2(t) - p \left( R_1^{p-1}(t) + R_2^{p-1}(t) \right) \eta(t) \right\}| \leq Ke^{-\frac{c}{t}} \exp(-2c^{-r}).
\]

Let \( R(t) = R_1(t) + R_2(t) \) (note that \( R^k \leq K(R_1^k + R_2^k) \)). By expanding \( u(t) \), we have
\[
E(u(t)) = \frac{1}{2} \int (R + \eta)_x^2 - \frac{1}{p+1} \int (R + \eta)^{p+1},
\]
and thus
\[
|E(u(t)) - \left\{ E(R) - \int (R'' + R^p) \eta + \frac{1}{2} \int \eta_x^2 - pR^{p-1}\eta^2 \right\}| \leq K \int (R'' - \frac{R}{2}) |\eta|^3 + \int |\eta|^{p+1}.
\]

By estimates similar to (A.11) related to the decay of \( Q \) and its derivatives,
\[
|E(R) - E(R_1) - E(R_2)| \leq Ke^{-\frac{c}{t}} \exp(-2c^{-r}).
\]

Since \( R_j'' + R_j^p = c_j(t)R_j \) and \( \int R_j \eta = 0 \), we have by (A.11):
\[
\left| \int (R'' + R^p) \eta \right| \leq \int |R'' - R_1'' + R_2''| |\eta| \leq Ke^{-\frac{c}{t}} \exp(-2c^{-r}).
\]

Similarly, we have
\[
\left| \int (R_1^{p-1} - R_1^{p-1} - R_2^{p-1}) \eta^2 \right| \leq Ke^{-\frac{c}{t}} \exp(-2c^{-r}).
\]

Thus, we obtain (A.14).

Now, we continue proving (2.13) by estimating \( \int \left( R_1^{p-2} + R_2^{p-2} \right) |\eta|^3 + \int |\eta|^{p+1} \). Let
\[
\beta = D_0(\alpha + c^{-q-\frac{q}{2}} \exp(-c^{-r})) \quad \text{so that by (2.6) } \|\eta(t)\|_{H^1} \leq K \beta e^q.
\]

(A.15)
Note that $\beta \leq D_0(\alpha + \exp(-\frac{1}{2}c^{-r}))$, for $c$ small enough. We have by the Gagliardo–Nirenberg inequality:

$$
\int |\eta|^{p+1} \leq K \left( \int \eta_x^2 \right)^{\frac{p-1}{4}} \left( \int \eta^2 \right)^{\frac{p+3}{4}} \leq K \beta^{p-1} \int \eta_x^2 + K \beta^{-\frac{(p-1)^2}{8-p}} \left( \int \eta^2 \right)^{\frac{p+3}{8-p}}.
$$

Since $\frac{p+3}{8-p} = 1 + \frac{1}{2q}, \frac{1}{q} - \frac{(p-1)^2}{8-p} = p - 1 (2q = \frac{5-p}{2(p-1)})$ and using (A.15), we obtain

$$
\beta^{-\frac{(p-1)^2}{8-p}} \left( \int \eta_x^2 \right)^{\frac{p+3}{8-p}} = \beta^{-\frac{(p-1)^2}{8-p}} \left( \int \eta^2 \right)^{\frac{1}{2q}} \int \eta^2 \leq \beta^{-\frac{1}{8-p}} c \int \eta^2 \leq K \beta^{p-1} c \int \eta^2
$$

Thus,

$$
\int |\eta|^{p+1} \leq K \beta^{p-1} \left[ \int \eta_x^2 + c \int \eta^2 \right].
$$

In addition, from (A.15),

$$
\int R_2^{p-2} |\eta|^3 \leq \beta \int R_2^{p-1} \eta^2 + \beta^{-(p-2)} \int |\eta|^{p+1} \leq K \beta \left[ \int \eta_x^2 + c \int \eta^2 \right],
$$

$$
\int R_1^{p-2} |\eta|^3 \leq K \|\eta\|_{L^\infty} \int R_1 \eta^2 \leq K \beta \int \eta^2 \psi(x - m(t)).
$$

Gathering these estimates and (A.14), we obtain (2.13) (note that $p \geq 2$).

Proof of (2.14). By Claim A.2 we have

$$
E(R_j(t)) - E(R_j(0)) = -\frac{5-p}{2(p+3)} (c^{2q+1}(t) - c^{2q+1}(0)) \int Q^2.
$$

But since $\frac{2q+1}{2q} = \frac{p+3}{8-p}$,

$$
\left| c_j^{2q+1}(t) - c_j^{2q+1}(0) - \frac{p+3}{5-p} c_j(0) [c^{2q}(t) - c^{2q}(0)] \right| \leq K c_j^{2q+1}(0) \left( \frac{c_j^{2q}(t)}{c_j^{2q}(0)} - 1 \right)^2.
$$

Thus (2.14) follows. \(\square\)

### A.3 Proof of Claim 2.4

The proof is based on the following well-known fact: There exists $\lambda_1 > 0$ such that if $v \in H^1(\mathbb{R})$ satisfies $\int Qv = \int xQv = 0$, then

$$
\int v_x^2 - pQ^{p-1}v^2 + v^2 \geq \lambda_1 \|v\|_{H^1}^2.
$$

(A.16)

It is similar to [18], Proof of Lemma 4. Set

$$
H_0(t) = \int \left( \eta_x^2 + [c_2(t) + c_1(t) \psi(x - m(t))] \eta^2 - p(R_1^{p-1} + R_2^{p-1}) \eta^2 \right).
$$

Note that $H_0(t)$ and $H(t)$ are easily compared. Indeed, we have

$$
|H(t) - H_0(t)| \leq \int |c_2(t) - c_2(0)| + |c_1(t) - c_1(0) - c_2(0)| \psi(x - m(t)) \eta^2.
$$
Let $\epsilon_0 > 0$. By (2.6), for $\alpha$ and $c$ small enough, we have
\[ |H(t) - H_0(t)| \leq \epsilon_0 \int [c + \psi(x-m(t))] \eta^2. \]
Thus, it is sufficient to prove (2.21) for $H_0(t)$ for some $\lambda_0 > 0$ independent of $c$ and $\alpha$.

First, we consider a function $\Phi \in C^2(\mathbb{R})$, $\Phi(x) = \Phi(-x)$, $\Phi' \leq 0$ on $\mathbb{R}^+$, with
\[ \Phi(x) = 1 \text{ on } [0,1]; \Phi(x) = e^{-x} \text{ on } [2, +\infty), e^{-x} \leq \Phi(x) \leq 3e^{-x} \text{ on } \mathbb{R}^+. \]
Let $\Phi_B(x) = \Phi(\frac{x}{B})$. We recall the following claim from [18], page 355 (and references therein):

There exists $B_0 > 0$ such that, for all $B > B_0$, if $v \in H^1(\mathbb{R})$ satisfies $\int Qv = \int xQv = 0$, then
\[ \int \Phi_B(x) \left(v_x^2 - pQ^{p-1}v^2 + v^2\right) dx \geq \frac{\lambda_1}{4} \int \Phi_B(x)(v_x^2 + v^2) dx. \quad (A.17) \]
This result is a localized version of (A.16), and is easily proved by direct calculations.

By a scaling argument, i.e. changing $x$ into $x\sqrt{c}$ and using the definition of $Q_c$, we have:
If $v \in H^1(\mathbb{R})$ satisfies $\int Q_cv = \int xQ_cv = 0$, then
\[ \int \Phi_{\sqrt{c}}(x) \left(v_x^2 - pQ^{p-1}v^2 + cv^2\right) dx \geq \frac{\lambda_1}{4} \int \Phi_{\sqrt{c}}(x)(v_x^2 + cv^2) dx. \quad (A.18) \]

Now, we consider $\eta$ as in the proof of Lemma [2.2], i.e. satisfying the orthogonality conditions $\int \eta R_j = \int \eta x R_j = 0$ for $j = 1,2$. Let $\Phi_1(t,x) = \Phi_B(\rho_1(t))$ and $\Phi_2(t,x) = \Phi_{\sqrt{c}}(x - \rho_2(t))$. We have
\[ H_0(t) = \int \Phi_2 \left(\eta_x^2 - pR_2^{p-1}\eta^2 + c_2(t)\eta^2\right) dx + \int \Phi_1 \left(\eta_x^2 - pR_1^{p-1}\eta^2 + c_1(t)\eta^2\right) dx \\
- p \int R_2^{p-1}(1 - \Phi_2)\eta^2 - p \int R_1^{p-1}(1 - \Phi_1)\eta^2 \\
+ \int (1 - \Phi_1 - \Phi_2)\eta_x^2 + [(1 - \Phi_2)c_2(t) + (1 - \Phi_1)c_1(t)\psi(x-m(t))] \eta^2. \]
Let $\lambda_2 = \min(\frac{\lambda_1}{2}, \frac{1}{2})$. Since $1 - \Phi_1 - \Phi_2 \geq 0$, $c_2(t) > \frac{\sqrt{c}}{2}$ and $c_1(t) \geq \frac{1}{2}$, we have by (A.17) and (A.18):
\[ \int \Phi_2 \left(\eta_x^2 - pR_2^{p-1}\eta^2 + c_2(t)\eta^2\right) dx + \int \Phi_1 \left(\eta_x^2 - pR_1^{p-1}\eta^2 + c_1(t)\eta^2\right) dx \\
+ \int (1 - \Phi_1 - \Phi_2)\eta_x^2 + [(1 - \Phi_2)c_2(t) + (1 - \Phi_1)c_1(t)\psi(x-m(t))] \eta^2 \\
\geq \lambda_2 \int \left[\eta_x^2 + [c + \psi(x-m(t))]\eta^2\right]. \]
Finally, since $|R_2^{p-1}(t,x)| \leq Kce^{-\frac{\sqrt{c}}{2}|x-\rho_2(t)|}$ and $\Phi_2(x) = 0$ for $|x-\rho_2(t)| \leq \frac{B}{\sqrt{c}}$, we have
\[ \int R_2^{p-1}(1 - \Phi_2)\eta^2 \leq Kce^{-\frac{\sqrt{c}}{2}B} \int \eta^2 \leq \epsilon_0 c \int \eta^2, \]
for $B$ large enough. Similarly, for $B$ large enough, since $e^{-\frac{\sqrt{c}}{2}|x-\rho_1(t)|} \leq K\psi(x-m(t))$, we have
\[ \int R_1^{p-1}(1 - \Phi_1)\eta^2 \leq Kce^{-\frac{\sqrt{c}}{2}B} \int \psi(x-m(t))\eta^2 \leq \epsilon_0 \int \psi(x-m(t))\eta^2. \]
Therefore, for $B$ large enough, we obtain

$$H_0(t) \geq \frac{\lambda_2}{2} \int (\eta_x^2 + [c + \psi(x-m(t))]\eta^2).$$

This completes the proof of Claim 2.4. □

**B Appendix**

**B.1 Proof of Claim 3.1 - Localized Viriel estimate**

By explicit calculations, $\eta(t)$ satisfies

$$\eta = (-\eta_{xx} - (R_1 + R_2 + \eta)^p + R_1^p + R_2^p) - \frac{c_j'}{2c_1} \tilde{R}_1 - \frac{c_j'}{2c_2} \tilde{R}_2 + (\rho'_1 - c_1) R_1 + (\rho'_2 - c_2) R_2x, \quad (B.1)$$

where $\tilde{R}_j(t, x) = \tilde{Q}_{c_j}(x - \rho_j(t))$ and $\tilde{Q}_c(x) = \frac{2}{p-1} Q_c + x Q'_c$.

**Step 1.** Control of the geometrical parameters.

**Claim B.1** Let

$$g_1(t) = \int (\eta_x^2 + \eta^2)(t, x)e^{-\frac{1}{2}|x - \rho_1(t)|} dx, \quad g_2(t) = \int (\eta_x^2 + c\eta^2)(t, x)e^{-\frac{1}{2}|x - \rho_2(t)|} dx. \quad (B.2)$$

Then, for all $t \geq 0$,

$$|\langle \rho'_j - c_j \rangle^2 \rangle | \leq K c_{j}^{q + \frac{1}{2}} \sqrt{g_j(t)} + Kg_j(t) + Ke^{-\frac{1}{8} \sqrt{c}(t + T_c)}, \quad (B.3)$$

$$|\langle c_j^{2q} \rangle | \leq K \sqrt{c_j} g_j(t) + Ke^{-\frac{1}{8} \sqrt{c}(t + T_c)}, \quad (B.4)$$

$$\left| \frac{1}{2} (\rho'_j - c_j) c_2^q \int Q^2 - (p - 3) \int \eta R_j^p \right| \leq Kg_j(t) + Ke^{-\frac{1}{8} \sqrt{c}(t + T_c)}. \quad (B.5)$$

**Proof of Claim [B.1]** Let $(j, k) = (1, 2)$ or $(j, k) = (2, 1)$. Since $\int Q \tilde{Q} = 2q \int Q^2$ and $\int QQ_x = 0$,

$$0 = \frac{d}{dt} \int \eta R_j = \int \eta R_j - \rho'_j \int \eta R_{jx} + \frac{c_j'}{2c_j} \int \eta \tilde{R}_j$$

$$= - \int (-\eta_{xx} + c_j \eta - pR_j^{p-1} \eta) R_{jx} - (\rho'_j - c_j) \int \eta R_{jx} + \frac{c_j'}{2c_j} \int \eta \tilde{R}_j$$

$$+ \int \left[(R_j + R_k + \eta)^p - R_j^p - R_k^p - pR_j^{p-1} \eta\right] R_{jx}$$

$$- q c_j \int R_j^2 - \frac{c_j'}{2c_k} \int \tilde{R}_k R_j + (\rho'_k - c_k) \int R_{kx} R_j. \quad (B.6)$$

First, we note that the first integral in $[B.6]$ is zero since $\mathcal{L}$ is self-adjoint and $\mathcal{L} Q' = 0$. Second, we have $\sqrt{c_j} |R_j(t, x)| + |R_{jx}(t, x)| \leq c_{p-1}^{\frac{1}{2}} e^{-\frac{1}{8} \sqrt{c}(x - \rho_j)}$, and $\int R_j^2 = c_j^{2q} \int Q^2, \int R_{jx}^2 =$
Thus, $\phi R \int Q_x^2$. Finally, since $\rho_1(t) - \rho_2(t) \geq \frac{1}{2}(t + T_c)$, by the proof of Claim A.3 all the terms containing a product $R_j R_k$ or their derivatives, are controlled by $e^{-\frac{1}{8}\sqrt{t}(t + T_c)}$. Thus,

$$
\left| \frac{1}{2}(c_j^{2q})'(t) \int Q^2 + (\rho_j - c_j) \int \eta R_j x - \frac{c_j}{2c_j} \int \eta \tilde{R}_j \right| - \int \left[ (R_j + \eta)^p - R_j^p - pR_{j-1}^p \eta \right] R_j x \leq Ke^{-\frac{1}{8}\sqrt{t}(t + T_c)}.
$$

(B.7)

Next, we note that

$$
| (R_j + \eta)^p - R_j^p - pR_{j-1}^p \eta | \leq K (| R_j |^p - 2 | \eta |^2 + | \eta |^p),
$$

(B.8)

and thus

$$
\left| (c_j^{2q})' \right| \leq K \left( c_j^{2q + \frac{1}{2}} | \rho_j - c_j | + | (c_j^{2q})' | \right) \left[ c_j^{-2q} \int \eta^2 e^{-\sqrt{t}j | x - \rho_j |} \right]^\frac{1}{2} + K \int (| R_j |^p - 2 | \eta |^2 + | \eta |^p) | R_j x | + Ke^{-\frac{1}{8}\sqrt{t}(t + T_c)}.
$$

We claim

$$
\int (| R_j |^p - 2 | \eta |^2 + | \eta |^p) | R_j x | \leq K \sqrt{c_j g_j(t)}.
$$

(B.9)

Indeed, first,

$$
\int | R_j |^p | R_j x | | R_j x | \eta^2 \leq K c_j^3 \int \eta^2 e^{-\frac{\sqrt{t}j}{2} (x - \rho_j)} \leq K c_j^\frac{1}{2} g_j(t).
$$

Second (for $p = 3$ and 4),

$$
\int | \eta |^p | R_j x | \leq \frac{1}{c_j^{p-1} + \frac{1}{2}} \| \eta^2 \phi_j \|_{L_c^2} \int \eta^2 e^{-\frac{\sqrt{t}j}{2} (x - \rho_j)}
$$

where $\phi_j(x) = e^{-\frac{\sqrt{t}j}{2} (x - \rho_j)}$. Since $\phi_j x = -\frac{\sqrt{t}j}{2} \phi_j$, we have

$$
\sup_{x \geq \rho_j(t)} | \eta^2(x) \phi_j(x) |^2 \leq \left[ \int_{x \geq \rho_j(t)} \left| 2 \eta x \phi_j + \eta^2 \frac{\phi_j}{2 \phi_j} \right| \right]^2 \leq K \int \eta^2 \phi_j \left( \int_{x \geq \rho_j(t)} \eta^2 \phi_j + c_j \eta^2 \phi_j \right).
$$

Thus, $\| \eta^2 \phi_j \|_{L_c^\infty} \leq K \left( \int \eta^2 \phi_j \right) g_j(t)$. We obtain

$$
\int | \eta |^p | R_j x | \leq \frac{1}{c_j^{p-1} + \frac{1}{2}} \left( \int \eta^2 e^{-\frac{\sqrt{t}j}{2} (x - \rho_j)} \right)^{\frac{p-2}{2}} g_j(t)
$$

$$
\leq K c_j^{\frac{1}{2} - p - 1} \left( \int \eta^2 e^{-\frac{\sqrt{t}j}{2} (x - \rho_j)} \right)^{\frac{p-2}{2}} g_j(t) \leq K c_j^{\frac{1}{2} - \frac{p}{2} + \frac{p}{2}} g_j(t) \leq K c_j^{\frac{1}{2}} g_j(t),
$$

using $\| \eta \|_{H^1} \leq K \alpha^q$. This proves (B.9).

Therefore, using again $\| \eta \|_{H^1} \leq K \alpha^q$, for $\alpha$ small,

$$
\left| (c_j^{2q})' \right| \leq K | \rho_j - c_j | c_j^{q + \frac{1}{2}} \left( \int \eta^2 e^{-\sqrt{t}j (x - \rho_j)} \right)^\frac{1}{2} + K \sqrt{c_j} g_j(t) + Ke^{-\frac{1}{8}\sqrt{t}(t + T_c)}.
$$

(B.10)
Using \( \int \eta(x - \rho_j) R_j = 0 \) and \( \int \eta R_j = 0 \), we have in a similar way

\[
0 = \frac{d}{dt} \int \eta(x - \rho_j) R_j = \int \eta_t(x - \rho_j) R_j - \rho'_j \int \eta(x - \rho_j) R_{jx} + \frac{c'_j}{2c_j} \int \eta(x - \rho_j) \dot{R}_j
\]

\[
= - \int (-\eta_{xx} + c_j \eta - p R_{j,1}^{-1}) ((x - \rho_j) R_j)_x - (\rho'_j - c_j) \int \eta(x - \rho_j) R_{jx} + \frac{c'_j}{2c_j} \int \eta(x - \rho_j) \dot{R}_j + \int \left[ (R_j + R_k + \eta R_{j,1}^p - R_{k,1}^p - p R_{j,1}^{-1}) ((x - \rho_j) R_j)_x \right.
\]

\[
+ (\rho'_j - c_j) \int R_{jx} (x - \rho_j) R_j - \frac{c'_k}{2c_k} \int \dot{R}_k (x - \rho_j) R_j + (\rho'_k - c_k) \int R_{kx} (x - \rho_j) R_j.
\]

From \( \mathcal{L}(\eta x Q)_x = -2Q - (p - 3) Q^p \) and (B.9), it follows that

\[
\left| \frac{1}{2} (\rho'_j - c_j) c_j^2 \int Q^2 - (p - 3) \int \eta R_j^p \right| \leq K g_j(t) + K \left( |\rho'_j - c_j| c_j^{2q} + c_j^{-\frac{1}{2}} |(c_j^{q})'| \right) c_j^{-q} \left( \int \eta^2 e^{-\sqrt{\phi_j} |x - \rho_j|} \right)^{\frac{1}{2}} + K e^{-\frac{1}{8} \sqrt{\phi_j} (t + T_c)},
\]

which implies by (B.10) and \( ||\eta||_{H^1} \leq K \alpha c^q \),

\[
|\rho'_j - c_j| c_j^{2q} \leq K c_j^{q + \frac{1}{2}} \sqrt{g_j(t)} + K g_j(t) + K e^{-\frac{1}{8} \sqrt{\phi_j} (t + T_c)}. \tag{B.12}
\]

Now, inserting (B.12) in (B.10), we have the following estimate

\[
|c_j^{2q})'| \leq K \sqrt{c_j} g_j(t) + K e^{-\frac{1}{9} \sqrt{\phi_j} (t + T_c)}. \tag{B.13}
\]

Finally, inserting (B.13) and (B.12) in (B.11), we obtain (B.5). \( \square \)

**Step 2.** Viriel identity in \( \eta \) and conclusion of the proof. For \( w \in H^1(\mathbb{R}) \), define

\[
H_\infty(w, w) = \frac{1}{2} \int (3w_x^2 + w^2 - p(Q_{p-1} - (p - 1)x Q_{x} Q_{p-2}) w^2),
\]

\[
H_\infty^*(w, w) = H_\infty(w, w) - \frac{2(p - 3)}{\int Q^2} \left( \int w_x Q_x \right) \left( \int w Q^p \right).
\]

Recall that the two quantities \( H_\infty(w, w) \) and \( H_\infty^*(w, w) \) were introduced in [11] for a Viriel type identity. Here, by analogy, we define,

\[
H_j(\eta, \eta) = \frac{1}{2} \int \left( 3\eta_x^2 \Phi_j + c_j \eta^2 \Phi_j - p(R_{j,1}^{p-1} \Phi_j - (p - 1) \Psi_j R_{jx} R_{j,1}^{p-2}) \eta^2 \right),
\]

\[
H_j^*(\eta, \eta) = H_j(\eta, \eta) - \frac{2(p - 3)}{\int R_j^2} \left( \int \eta \Psi_j R_{jx} \right) \left( \int \eta R_j^p \right),
\]

where \( \Phi_j(t, x) = \frac{\partial}{\partial x} \Psi_j(t, x) = \sqrt{c_j(t)} \Phi \left( \frac{\sqrt{c_j(t)}}{4} (x - \rho_j(t)) \right) \).

By straightforward calculations, we have

\[
\frac{d}{dt} K_j(t) = \frac{d}{dt} \int \Theta_j(t) \eta^2(t) + \Theta_j(0) (c_j^{2q})' \int Q^2,
\]

28
\[ \frac{1}{2} \frac{d}{dt} \int \Theta_j(t) \eta^2(t) = \int \Theta_j \eta \eta - \frac{c_j}{2} \int \Theta_j \eta^2 - \frac{1}{2} (\rho_j' - c_j) \int \Theta_{jx} \eta^2 + \frac{c_j}{4c_j} \int (x - \rho_j) \Theta_{jx} \eta^2. \]

Therefore, by using the equation of \( \eta \), we have, for \( j \neq k \),

\[ \frac{1}{2} \frac{d}{dt} K_j(t) = -H_j^s(\eta, \eta) + \frac{1}{2} \int \Phi_{jxx} \eta^2 - \frac{1}{2} (\rho_j' - c_j) \int \Phi_j \eta^2 + \frac{c_j}{4c_j} \int (x - \rho_j) \Phi_j \eta^2 \]

\[ + \int \left[ (R_j + R_k + \eta)^p - R_j^p - p R_j^{p-1} \eta \right] (\Theta_j \eta)_x \]

\[ + \frac{1}{2} \Theta_j(0)(c_j^2)^l_1 \int Q^2 - \frac{P}{2} \Theta_j(0) \int (R_j^{p-1})_x \eta^2 + \Theta_j(0)(\rho_j' - c_j) \int \eta R_jx \]

\[ - \frac{c_j}{2c_j} \int \tilde{R}_j \Theta_j \eta - \frac{c_j}{2c_j} \int \tilde{R}_k \Theta_j \eta + (\rho_k' - c_k) \int R_k \Theta_j \eta \]

\[ + \left[ (\rho_j' - c_j) - \frac{2(p - 3)}{R_j^2} \int R_j \eta \right] \int R_j \Psi_j \eta = E_1 + E_2 + E_3 + E_4 + E_5, \]

where we have used

\[ - \int (\eta_{xx} + p R_j^{p-1} \eta)_x \Theta_j \eta - \frac{c_j}{2} \int \Theta_j \eta^2 = -H_j(\eta, \eta) + \frac{1}{2} \int \Phi_{jxx} \eta^2 - \frac{P}{2} \Theta_j(0) \int (R_j^{p-1})_x \eta^2, \]

since \( \frac{\partial}{\partial x} \Psi_j = \frac{\partial}{\partial x} \Theta_j = \Phi_j \) and \( \Theta_j(t) = \Theta_j(0) + \Psi_j(t) \).

From this identity, we claim

**Claim B.2** There exist \( A \geq 5, \ k_0 > 0 \) such that, for \( \alpha \) small enough, and for all \( t \geq 0 \),

\[ \frac{d}{dt} K_j(t) \leq -k_0 \sqrt{c_j} \int (\eta_{xx}^2 + c_j \eta^2) e^{-\frac{\sqrt{c_j}}{4} |x - \rho_j|} + \frac{1}{k_0} e^{-\frac{1}{4k_0} |x - \rho_j|(t + T_j)}. \]  \hspace{1cm} (B.14)

By (2.15) and \( A \geq 5 \), we have \( \sqrt{A} / A < 1/4 \) and \( \sqrt{c_j} / A < \sqrt{c_j} / 4 \), and thus we obtain the conclusion of Claim B.1 from Claim B.2.

**Proof of Claim B.2** By Proposition 6 and localization arguments as proof of Proposition 6 (see also proof of Claim 2.4 in the present paper), there exists \( A_0 > 0, \ k_0 > 0 \), such that if \( A > A_0 \) then

\[ H_j^s(\eta, \eta) \geq k_0 \sqrt{c_j} \int (\eta_{xx}^2 + c_j \eta^2) e^{-\frac{\sqrt{c_j}}{4} |x - \rho_j|}. \]  \hspace{1cm} (B.15)

The claim means that all other terms in the previous identity can be absorbed by this term for some \( A \geq 5 \) for \( \alpha, c \) small enough up to and error term of size \( e^{-\frac{1}{4k_0} |x - \rho_j|(t + T_j)} \).

First, since \( \Phi_{jxx} = \frac{\partial^2}{\partial x^2} \Phi_j \), we have

\[ \left| \int \Phi_{jxx} \eta^2 \right| \leq \frac{K}{A^2 c_j^2} \int \eta^2 e^{-\frac{\sqrt{c_j}}{4} |x - \rho_j|} \leq \frac{\lambda_0}{100 c_j^3} \int \eta^2 e^{-\frac{\sqrt{c_j}}{4} |x - \rho_j|}, \]  \hspace{1cm} (B.16)

for \( A \) large enough. Now, \( A \) is fixed to such value.

Next, we have from \( \| \eta \|_H^1 \leq K \alpha C \) and (B.3)

\[ |\rho_j' - c_j| \leq K \alpha c_j^{1-q} \sqrt{g_j(t)} + K c_j^{-2q} \int (\eta_{xx}^2 + c_j \eta^2) e^{-\frac{\sqrt{c_j}}{4} |x - \rho_j|} + K e^{-\frac{1}{4k_0} |x - \rho_j|(t + T_j)}, \]
\[
\int \Phi_j \eta^2 \leq K \sqrt{c_j} \int \eta^2 e^{-\sqrt{\eta^2} |x-\rho_j|} \leq K \alpha c^q \sqrt{g_j} \quad \text{and} \quad \int \Phi_j \eta^2 \leq K \alpha^2 c_j^{2q+\frac{1}{2}}.
\]

Therefore,
\[
|\langle \rho'_j - c_j \rangle \int \Phi_j \eta^2 | \leq K \alpha \sqrt{c_j} \int (\eta^2 + c_j \eta^2) e^{-\sqrt{\eta^2} |x-\rho_j|} + K e^{-\frac{1}{8} \sqrt{\eta_j}(t+T_\epsilon)}.
\]

Thus, for \(\alpha\) small enough, \(|\langle \rho'_j - c_j \rangle \int \Phi_j \eta^2 | \leq \frac{\lambda_0}{100 \sqrt{c_j}} \int (\eta^2 + c_j \eta^2) e^{-\sqrt{\eta^2} |x-\rho_j|} + K e^{-\frac{1}{8} \sqrt{\eta_j}(t+T_\epsilon)}
\].

By \(|\eta|_{H^1} \leq K \alpha c^q, \|(x-\rho_j)\Phi_j | \leq K, \) \(B.4\), we have
\[
\left| \frac{c_j}{4c_j} \int (x-\rho_j) \Phi_j \eta^2 \right| \leq K \alpha^2 |(c_j^2)'| \leq \frac{\lambda_0}{100 \sqrt{c_j}} \int (\eta^2 + c_j \eta^2) e^{-\sqrt{\eta^2} |x-\rho_j|} + K e^{-\frac{1}{8} \sqrt{\eta_j}(t+T_\epsilon)},
\]

for \(\alpha\) small enough. Thus
\[
E_1 \leq -\frac{3\lambda_0}{4} \sqrt{c_j(t)} \int (\eta^2 + c_j \eta^2) e^{-\sqrt{\eta^2} |x-\rho_j|}.
\]

Now, we treat \(E_2\). First, all terms containing powers of \(R_k\) are controlled by \(K e^{-\frac{1}{8} \sqrt{\eta_j}(t+T_\epsilon)}\) since \(\Theta_j\) is exponentially small around \(\rho_k(t)\). Therefore, we need only estimate terms of the form
\[
r \in \{2, \ldots, p-1\}, \quad \int R_j^{p-r} \eta^r (\Theta_j \eta)_x \quad \text{and} \quad \int \eta^p (\Theta_j \eta)_x = \frac{p}{p+1} \int \eta^{p+1} \Phi_j, \quad (B.17)
\]

For the last term in \(B.17\), we have, arguing as in the proof of \(B.9\),
\[
\left| \int \eta^{p+1} \Phi_j \right| \leq K \sqrt{c_j} \int (\alpha^{p-1} c_j \eta^2 + \alpha^{p-5} c_j^{\frac{3}{4} - \frac{3}{4} \eta^2} e^{-\sqrt{\eta^2} |x-\rho_j|} \leq K \alpha^{p-1} \sqrt{c_j} \int (\eta^2 + c_j \eta^2) e^{-\sqrt{\eta^2} |x-\rho_j|} \]

for \(\alpha\) small enough. For the first term in \(B.17\), we integrate by parts, so that
\[
\int R_j^{p-r} \eta^r (\Theta_j \eta)_x = \frac{1}{r+1} \int (r R_j^{p-r} \Phi_j - (R_j^{p-r-1})_x \Theta_j) \eta^{r+1}.
\]

Finally, we treat all these terms similarly as in \(B.9\), so that, for \(\alpha\) small enough,
\[
|E_2| \leq \frac{\lambda_0}{100 \sqrt{c_j}} \int (\eta^2 + c_j \eta^2) e^{-\sqrt{\eta^2} |x-\rho_j|} + K e^{-\frac{1}{8} \sqrt{\eta_j}(t+T_\epsilon)}.
\]

For \(E_3\), note that by \(B.7\),
\[
|E_3| = \frac{1}{2} \Theta_j(0) \left( (c_j^2)' \int Q^2 - \frac{p}{2} \int (R_j^{p-1})_x \eta^2 + (\rho'_j - c_j) \int \eta R_{jx} \right)
\]

\[
\leq K \left| \frac{c_j}{c_j} \int \eta \tilde{R}_{jx} \right| + K \int |R_{jx}| (|R_j|^{p-3} \eta^3 + |\eta|^p).
\]

For the first term, since \(\|\tilde{R}_j\|_{L^2} \leq K c^q\) and \(|\eta|_{L^2} \leq K \alpha c^q\), we have \(|\int \tilde{R}_j \Theta_j \eta| \leq K \alpha c^{2q}\), and so by \(B.4\), and arguing for the other term as for \(E_2\), for \(\alpha\) small enough, we get
\[
|E_3| \leq \frac{\lambda_0}{100 \sqrt{c_j}} \int (\eta^2 + c_j \eta^2) e^{-\sqrt{\eta^2} |x-\rho_j|} + K e^{-\frac{1}{8} \sqrt{\eta_j}(t+T_\epsilon)}.
\]
For the first term in $E_4$, we proceed as for $E_3$. The last two terms in $E_4$ are controled by $K e^{-\frac{1}{8}\sqrt{\gamma}(t+T_c)}$ since they contain products of exponentially decaying functions centered around $\rho_j(t)$ and $\rho_k(t)$. Thus,

$$|E_4| \leq \frac{\lambda_0}{100} \sqrt{c_j} \int (\eta_x^2 + c_j \eta^2) e^{-\frac{\sqrt{\gamma}}{\lambda}|x-\rho_j|} + K e^{-\frac{1}{8}\sqrt{\gamma}(t+T_c)}.$$ 

Since $|\Psi_j| \leq K$, $\|R_{j,x}\|_{L^2} \leq K c_j^q + \frac{1}{2}$ and $\|\eta\|_{L^2} \leq K \alpha e^g$, we have $|\int R_{j,x} \Psi_j \eta| \leq K \alpha e^{2g + \frac{1}{4}}$, and then we obtain by (B.5)

$$|E_5| \leq \frac{\lambda_0}{100} \sqrt{c_j} \int (\eta_x^2 + c_j \eta^2) e^{-\frac{\sqrt{\gamma}}{\lambda}|x-\rho_j|} + K e^{-\frac{1}{8}\sqrt{\gamma}(t+T_c)},$$

for $\alpha$ small enough. Thus, the proof of Claim B.2 is complete. \hfill \square

### B.2 Proof of Claim 3.2 – Monotonicity results on $\eta(t)$

In this appendix, we prove monotonicity results for quantities defined in $\eta(t)$. Claim 3.2 is a direct consequence of Claim B.3 below for $x_0 = 0$. Recall that $\psi(x)$ is defined by (2.9).

For $0 \leq t_0 \leq t$, $x_0 \geq 0$, $j = 1, 2$, let

$$\mathcal{M}_j(t) = \int \eta_x^2 \psi_j,$$

$$\mathcal{E}_j(t) = \int \left[ \frac{1}{2} \eta_x^2 - \frac{1}{p+1} \left( (R_1 + R_2 + \eta)^{p+1} - (p+1) R_1^p \eta - (p+1) R_2^p \eta - (R_1 + R_2)^{p+1} \right) \right] \psi_j,$$

where $\psi_1(x) = \psi(\bar{x})$, $\bar{x} = x - \rho_1(t) + x_0 + \frac{1}{2}(t - t_0)$, and $\psi_2(x) = \psi(\sqrt{c} \bar{x}_c)$, $\bar{x}_c = x - \rho_2(t) + x_0 + \frac{\zeta}{2}(t - t_0)$.

For $0 \leq t \leq t_0$, $x_0 \geq 0$, $\bar{x} = x - \rho_1(t) - x_0 - \frac{1}{2}(t_0 - t)$, let

$$\mathcal{M}_0(t) = \int \eta_x^2 \psi(\bar{x}),$$

$$\mathcal{E}_0(t) = \int \left[ \frac{1}{2} \eta_x^2 - \frac{1}{p+1} \left( (R_1 + R_2 + \eta)^{p+1} - (p+1) R_1^p \eta - (p+1) R_2^p \eta - (R_1 + R_2)^{p+1} \right) \right] \psi(\bar{x}).$$

For $t_0 \geq \frac{2}{5} T_c$, let $\bar{t}_0 = \frac{\zeta}{8} t_0$, and for $\bar{t}_0 \leq t \leq t_0$, let $\bar{x}_c = x - \rho_2(t) - \sigma_0(t(t_0 - t))$, where $\sigma_0 = (\rho_1(t_0) - \rho_2(t_0))/(t_0 - t_0)$; note that $\frac{\zeta}{4} \leq \sigma_0 \leq \frac{\zeta}{2}$ by (2.8). Let

$$\mathcal{M}_{1,2}(t) = \int \eta_x^2 \psi(\sqrt{c} \bar{x}_c),$$

$$\mathcal{E}_{1,2}(t) = \int \left[ \frac{1}{2} \eta_x^2 - \frac{1}{p+1} \left( (R_1 + R_2 + \eta)^{p+1} - (p+1) R_1^p \eta - (p+1) R_2^p \eta - (R_1 + R_2)^{p+1} \right) \right] \psi(\sqrt{c} \bar{x}_c).$$
Claim B.3 Let $x_0 > 0$, $t_0 > 0$. (i) For all $t \geq t_0$,
\[
\frac{d}{dt} \left( c_1^{2q}(t) \int Q^2 + M_1(t) \right) \leq Ke^{-\frac{c}{2} (t-t_0+x_0)} g_1(t) + Ke^{-\frac{1}{100} \sqrt{c} (t+T_c)},
\]
\[
\frac{d}{dt} \left( -\frac{2q}{2q+1} c_1^{2q+1}(t) \int Q^2 + 2E_1(t) + \frac{1}{100} \left( c_1^{2q}(t) \int Q^2 + M_1(t) \right) \right) \leq Ke^{-\frac{c}{2} (t-t_0+x_0)} g_1(t) + Ke^{-\frac{1}{100} \sqrt{c} (t+T_c)}.
\]
\[
\frac{d}{dt} \left( c_1^{2q}(t) + c_2^{2q}(t) \right) \int Q^2 + M_2(t) \leq Ke^{-\frac{c}{2} (t-t_0)} e^{-\frac{c}{2} (t-t_0)} \sqrt{c} g_2(t) + Ke^{-\frac{1}{100} \sqrt{c} (t+T_c)},
\]
\[
\frac{d}{dt} \left( -\frac{2q}{2q+1} (c_1^{2q+1}(t) + c_2^{2q+1}(t)) \int Q^2 + 2E_2(t) + \frac{c}{100} \left( (c_1^{2q}(t) + c_2^{2q}(t)) \int Q^2 + M_2(t) \right) \right) \leq Ke^{-\frac{c}{2} (t-t_0)} e^{-\frac{c}{2} (t-t_0)} \sqrt{c} g_2(t) + Ke^{-\frac{1}{100} \sqrt{c} (t+T_c)}.
\]

(ii) For all $0 \leq t \leq t_0$,
\[
\frac{d}{dt} M_0(t) \leq Ke^{-\frac{c}{100} (t-t_0+x_0)} g_1(t) + Ke^{-\frac{1}{100} (t-t_0+x_0)} e^{-\frac{c}{100} (t+T_c)},
\]
\[
\frac{d}{dt} \left( E_0(t) + \frac{1}{100} M_0(t) \right) \leq Ke^{-\frac{c}{100} (t-t_0+x_0)} g_1(t) + Ke^{-\frac{1}{100} (t-t_0+x_0)} e^{-\frac{c}{100} (t+T_c)}.
\]

(iii) For all $0 \leq t_0 \leq t \leq t_0$,
\[
\frac{d}{dt} \left( c_1^{2q}(t) \int Q^2 + M_{1,2}(t) \right) \leq Ke^{-\frac{c}{100} (t-t_0)} \sqrt{c} g_2(t) + Ke^{-\frac{c}{100} (t-t_0)} g_1(t) + Ke^{-\frac{1}{100} \sqrt{c} (t+T_c)},
\]
\[
\frac{d}{dt} \left( -\frac{2q}{2q+1} c_1^{2q+1}(t) \int Q^2 + 2E_{1,2}(t) + \frac{c}{100} \left( c_1^{2q}(t) \int Q^2 + M_{1,2}(t) \right) \right) \leq Ke^{-\frac{c}{100} (t-t_0)} \sqrt{c} g_2(t) + Ke^{-\frac{c}{100} (t-t_0)} g_1(t) + Ke^{-\frac{1}{100} \sqrt{c} (t+T_c)}.
\]

Note that the monotonicity results on $E_j(t)$ requires the addition of some quantity related to $M_j(t)$ (here the constant $\frac{1}{100}$ is somewhat arbitrary, any small fixed positive constant would work). See also Lemma 1 in [8] for similar calculations in $u(t)$.

Proof of Claim B.3 We prove the part of Claim B.3 concerning $M_1(t), E_1(t), M_2(t), E_2(t)$. The rest is proved similarly.

Let $\tilde{x} = x - \rho_1(t) + \sigma(t-t_0) + x_0$, where $0 < \sigma \leq \frac{1}{2}$, $x_0 > 0$, $t \geq t_0 \geq 0$. First, we compute $\frac{1}{2} \frac{d}{dt} \int \eta^2 \psi(\tilde{x})$ using the equation of $\eta(t)$ (B.1).
\[
\frac{1}{2} \frac{d}{dt} \int \eta^2 \psi(\tilde{x}) = \int \eta \eta \psi(\tilde{x}) + (\sigma - \rho_1(t)) \int \eta^2 \psi'(\tilde{x})
\]
\[
= -\int \eta_{xx} \eta \psi(\tilde{x}) - \int (\rho_1 + R_1 + R_2 + \eta^p - R_1^p - R_2^p) \eta \psi(\tilde{x}) - \frac{c_1}{2c_1} \int \tilde{R}_1 \eta \psi(\tilde{x})
\]
\[
- \frac{c_1}{2c_2} \int \tilde{R}_2 \eta \psi(\tilde{x}) + (\rho_1 - c_1) \int R_1 \eta \psi(\tilde{x}) + (\rho_2 - c_2) \int R_2 \eta \psi(\tilde{x})
\]
\[
+ (\sigma - \rho_1(t)) \int \eta^2 \psi'(\tilde{x}).
\]
First, \(- \int \eta_{xx} \psi''(\bar{x}) = -\frac{3}{2} \int \eta^2_{xx} \psi''(\bar{x}) + \frac{1}{2} \int \eta^2 \psi''(\bar{x})\) and so by \(\psi'' \leq \frac{1}{4} \psi', \sigma \leq \frac{1}{8}\) and \(\rho_1(t) \geq \frac{3}{4}\), we obtain similarly as for \(u(t)\) in the proof of Claim 2.2.

\[- \int \eta_{xx} \psi(\bar{x}) + (\sigma - \rho_1(t) \int \eta^2 \psi'(\bar{x}) - \frac{3}{2} \int \eta^2 \psi'(\bar{x}) - \frac{1}{8} \int \eta^2 \psi'(\bar{x}).\]

Note that by the decay properties of \(R_2\) and \(\psi\), we have (see e.g. Claim A.3)

\(|\int \tilde{R}_2 \eta \psi(\bar{x}) + |\int R_{2x} \eta \psi(\bar{x}) + |\int ((R_1 + R_2 + \eta)^p - (R_1 + \eta)^p \eta \psi(\bar{x}) \leq Ke^{-\frac{1}{4} \sqrt[3]{(t + T_c)}}.\]

Thus,

\[\frac{1}{2} \frac{d}{dt} \int \eta^2 \psi(\bar{x}) \leq -\frac{3}{2} \int \eta^2 \psi'(\bar{x}) - \frac{1}{8} \int \eta^2 \psi'(\bar{x}) - \int ((R_1 + \eta)^p - R_1^p_x \eta \psi(\bar{x}) - \frac{c_1}{2c_1} \int \tilde{R}_1 \eta \psi(\bar{x}) + (\rho_1 - c_1) \int R_{1x} \eta \psi(\bar{x}) + Ke^{-\frac{1}{4} \sqrt[3]{(t + T_c)}}.\]

Note that

\[\int [(R_1 + \eta)^p - R_1^p - pR_1^{p-1}] R_{1x} = \int (R_1 + \eta)^p ((R_1 + \eta)_x - \eta_x) - \int (R_1^p)_x \eta,\]

and from (B.7),

\[\frac{1}{2} (c_1^{2q})' \int Q^2 \leq - (\rho_1 - c_1) \int \eta R_{1x} + \frac{c_1}{2c_1} \int \eta \tilde{R}_1 + \int ((R_1 + \eta)^p - R_1^p)_x \eta + Ke^{-\frac{1}{4} \sqrt[3]{(t + T_c)}}.\]

We obtain

\[\frac{1}{2} \frac{d}{dt} \left( c_1^{2q}(t) \int Q^2 + M_1(t) \right) \leq -\frac{3}{2} \int \eta^2 \psi'(\bar{x}) - \frac{1}{8} \int \eta^2 \psi'(\bar{x}) + Ke^{-\frac{1}{4} \sqrt[3]{(t + T_c)}} + \int ((R_1 + \eta)^p - R_1^p)_x \eta (1 - \psi(\bar{x})) + \frac{c_1}{2c_1} \int \tilde{R}_1 \eta (1 - \psi(\bar{x})) + (\rho_1 - c_1) \int R_{1x} \eta (1 - \psi(\bar{x})) + Ke^{-\frac{1}{4} (\sigma(t - t_0) + x_0)} g_1(t) + Ke^{-\frac{1}{4} \sqrt[3]{(t + T_c)}}.\]

Therefore, by \((1 - \psi(\bar{x})) e^{-\sqrt{c_1(t)(x - \rho_1(t)))} \leq Ke^{-\frac{1}{8} (\sigma(t - t_0) + x_0)},\)

\[\frac{1}{2} \frac{d}{dt} \left( c_1^{2q}(t) \int Q^2 + M_1(t) \right) \leq -\frac{3}{2} \int \eta^2 \psi'(\bar{x}) - \frac{1}{8} \int \eta^2 \psi'(\bar{x}) + Ke^{-\frac{1}{4} (\sigma(t - t_0) + x_0)} g_1(t) + Ke^{-\frac{1}{4} \sqrt[3]{(t + T_c)}}.\]
The proof is similar for $E_1(t)$, up to some additional algebraic cancellations. First,

$$
\frac{d}{dt}E_1 = \int \eta_{xx} \eta_x \psi(\tilde{x}) - \int \eta_t ((R_1 + R_2 + \eta)^p - R_1^p - R_2^p) \psi(\tilde{x})
- \int ((R_1 + R_2 + \eta)^p - pR_1^{p-1}\eta - (R_1 + R_2)^p)R_1\psi(\tilde{x})
- \int ((R_1 + R_2 + \eta)^p - pR_2^{p-1}\eta - (R_1 + R_2)^p)R_2\psi(\tilde{x})
+ (\sigma - \rho_1(t)) \left[ \int \left( \frac{1}{p+1} \eta_x^2 - \frac{1}{p+1} ((R_1 + R_2 + \eta)^{p+1} - (p+1)R_1^p \eta - R_1^{p+1}) \right) \psi(\tilde{x}) \right].
$$

All terms containing $R_2$, $R_2t$ are controlled by $Ke^{-\frac{1}{2} \sqrt{\tau(t+T_\varepsilon)}}$. Note that $R_{1t} = \frac{c_1'}{2c_1} \tilde{R}_1 - \rho_1 R_{1x}$.

Thus,

$$
\frac{d}{dt}E_1 \leq \int \eta_t(-\eta_{xx} - ((R_1 + R_2 + \eta)^p - R_1^p - R_2^p) \psi(\tilde{x}) - \int \eta_t \eta_x \psi(\tilde{x})
- \int ((R_1 + \eta)^p - pR_1^{p-1}\eta - R_1^p) (\frac{c_1'}{2c_1} \tilde{R}_1 - \rho_1 R_{1x}) \psi(\tilde{x})
+ (\sigma - \rho_1(t)) \left[ \int \left( \frac{1}{p+1} \eta_x^2 - \frac{1}{p+1} ((R_1 + \eta)^{p+1} - (p+1)R_1^p \eta - R_1^{p+1}) \right) \psi(\tilde{x}) \right] + Ke^{-\frac{1}{2} \sqrt{\tau(t+T_\varepsilon)}}.
$$

Thus, using the equation of $\eta$, we get

$$
\frac{d}{dt}E_1 \leq -\frac{1}{2} \int (-\eta_{xx} - ((R_1 + R_2 + \eta)^p - R_1^p - R_2^p)^2 \psi(\tilde{x})
- \frac{c_1'}{2c_1} \int \tilde{R}_1 (-\eta_{xx} - ((R_1 + \eta)^p - R_1^p)) \psi(\tilde{x})
+ (\rho_1(t) - c_1) \int R_{1x} (-\eta_{xx} - ((R_1 + \eta)^p - R_1^p)) \psi(\tilde{x})
+ \int (\eta_{xx} + (R_1 + \eta)^p - R_1^p) \eta_{xx} \psi(\tilde{x}) + \frac{c_1'}{2c_1} \int \tilde{R}_1 \eta_x \psi(\tilde{x})
- \int ((R_1 + \eta)^p - pR_1^{p-1}\eta - R_1^p) (\frac{c_1'}{2c_1} \tilde{R}_1 - \rho_1 R_{1x}) \psi(\tilde{x})
+ (\sigma - \rho_1(t)) \left[ \int \left( \frac{1}{p+1} \eta_x^2 - \frac{1}{p+1} ((R_1 + \eta)^{p+1} - (p+1)R_1^p \eta - R_1^{p+1}) \right) \psi(\tilde{x}) \right] + Ke^{-\frac{1}{2} \sqrt{\tau(t+T_\varepsilon)}}.
$$

First, terms containing $\psi(\tilde{x})$ and $R_1$ are all controlled by $Ke^{-\frac{1}{2} \sqrt{(\sigma(t-T_\varepsilon)) + T_\varepsilon}} g_1(t)$. Second, we note that

$$
\int \eta_{xxx} \eta_x \psi(\tilde{x}) + p \int \eta_x^2 \eta_{xx} \psi(\tilde{x}) + \int \eta_x^2 \eta_x \psi(\tilde{x}) \leq \frac{1}{2} \int \eta_x^2 \psi''(\tilde{x}) - \frac{1}{16} \int \eta_x^2 \psi''(\tilde{x}) \leq 0,
$$

by $\psi'' \leq \frac{1}{16} \psi'$, $\frac{1}{4} \leq \rho_1(t) - \sigma \leq 2$ and the fact that $p \int |\eta_x^2 \eta_{xx} \psi(\tilde{x})| \leq K ||\eta||_{H^1}^{p-1} \int \eta_x^2 \psi'(\tilde{x}) \leq \frac{1}{16} \int \eta_x^2 \psi'(\tilde{x})$ for $\alpha$ small enough.
Therefore,

\[
\frac{d}{dt} E_1 \leq \frac{2}{p + 1} \int |\eta|^{p+1} \psi'(\tilde{x}) - c_1' \frac{1}{2c_1} \int \tilde{R}_1(-\eta_{xx} + c_1 \eta - pR_1^{p-1} \eta) \psi(\tilde{x}) \\
+ (\rho'_1 - c_1) \int R_{1x}(-\eta_{xx} + c_1 \eta - pR_1^{p-1} \eta) \psi(\tilde{x}) + c_1 \int R_{1x}((R_1 + \eta)^p - pR_1^{p-1} \eta - R_1^p) \psi \\
+ c_1' \frac{1}{2c_1} \int \tilde{R}_1 \eta \psi - c_1(\rho'_1 - c_1) \int \eta R_{1x} \psi + Ke^{-\frac{1}{\tilde{c}}(\sigma(t-t_0)+x_0) g_1(t)} + Ke^{-\frac{1}{\tilde{c}}\sqrt{\tilde{c}}(t+T_c)}.
\]

Using the fact that \( \mathcal{L} \bar{Q} = -2Q \) and \( \mathcal{L} \bar{Q'} = 0 \), \((\bar{R}_1 \psi)_{xx} = \bar{R}_{1xx} \psi + 2\bar{R}_{1x} \psi' + \bar{R}_1 \psi''\), we obtain:

\[
\frac{d}{dt} E_1 \leq \frac{2}{p + 1} \int |\eta|^{p+1} \psi'(\tilde{x}) + c_1' \frac{1}{2c_1} \int \tilde{R}_1 \eta \psi - c_1(\rho'_1 - c_1) \int \eta R_{1x} \psi \\
+ c_1 \int R_{1x}((R_1 + \eta)^p - pR_1^{p-1} \eta - R_1^p) \psi + Ke^{-\frac{1}{\tilde{c}}(\sigma(t-t_0)+x_0) g_1(t)} + Ke^{-\frac{1}{\tilde{c}}\sqrt{\tilde{c}}(t+T_c)}.
\]

Using \( (\mathbf{1.7}) \) and \((1 - \psi(\tilde{x}))e^{-\sqrt{c_1(t-x-\rho(t))} \leq Ke^{-\frac{1}{\tilde{c}}(\sigma(t-t_0)+x_0)}, \) we obtain

\[
\frac{d}{dt} E_1 \leq \frac{1}{2} c_1(2^{2q})' \int Q^2 + \frac{2}{p + 1} \int |\eta|^{p+1} \psi'(\tilde{x}) + Ke^{-\frac{1}{\tilde{c}}(\sigma(t-t_0)+x_0) g_1(t)} + Ke^{-\frac{1}{\tilde{c}}\sqrt{\tilde{c}}(t+T_c)}.
\]

Since \( c_1(2^{2q})' = \frac{2q}{2q+1} c_1(2^{2q+1})' \), and \( \frac{2}{p + 1} \int |\eta|^{p+1} \psi'(\tilde{x}) \leq K|\eta|^{p-1} \int \eta^2 \psi'(\tilde{x}) \leq \frac{1}{100} \int \eta^2 \psi'(\tilde{x}) \) for \( \alpha \) small enough, by \( (\mathbf{1.18}) \), we obtain:

\[
\frac{d}{dt} \left( -\frac{2q}{2q+1} c_1(2^{2q+1})' \int Q^2 + 2E_1(t) + \frac{1}{100} \left( c_1'(t) \int Q^2 + \mathcal{M}_1(t) \right) \right) \\
\leq Ke^{-\frac{1}{\tilde{c}}(\sigma(t-t_0)+x_0)(\eta(\tilde{c}\tilde{x}_c))} g_1(t) + Ke^{-\frac{1}{\tilde{c}}\sqrt{\tilde{c}}(t+T_c)}.
\]

Thus the claim is proved for \( E_1(t) \).

For \( \mathcal{M}_2(t) \) and \( \mathcal{E}_2(t) \), the proof is the same. For example, we have, for \( t \geq t_0 \),

\[
\frac{1}{2} \frac{d}{dt} \mathcal{M}_2(t) \leq \int ((R_1 + \eta)^p - R_1^p) \eta(1 - \psi(\sqrt{c\tilde{x}_c})) + \int ((R_2 + \eta)^p - R_2^p) \eta(1 - \psi(\sqrt{c\tilde{x}_c})) \\
+ \frac{c'_1}{2c_1} \int \tilde{R}_1 \eta(1 - \psi(\sqrt{c\tilde{x}_c})) + \frac{c'_2}{2c_2} \int \tilde{R}_2 \eta(1 - \psi(\sqrt{c\tilde{x}_c})) + Ke^{-\frac{1}{\tilde{c}}\sqrt{\tilde{c}}(t+T_c)} \\
- (\rho'_1 - c_1) \int R_{1x} \eta(1 - \psi(\sqrt{c\tilde{x}_c})) - (\rho'_2 - c_2) \int R_{2x} \eta(1 - \psi(\sqrt{c\tilde{x}_c})) \\
\leq Ke^{-\frac{1}{\tilde{c}}(\sigma(t-t_0)+x_0)(\sqrt{c}g_2(t))} + Ke^{-\frac{1}{\tilde{c}}\sqrt{\tilde{c}}(t+T_c)},
\]

by Claim \( (\mathbf{3.3}) \) (as in the proof of Claim \( (\mathbf{3.2}) \), \( R_1(1 - \psi(\sqrt{c\tilde{x}_c})) \leq Ke^{-\frac{1}{\tilde{c}}\sqrt{\tilde{c}}(t+T_c)} \) and \( R_2(1 - \psi(\sqrt{c\tilde{x}_c})) \leq Ke^{-\frac{1}{\tilde{c}}\sqrt{\tilde{c}}(t+T_c)} \)).

### B.3 Proof of Lemma \( (\mathbf{3.2}) \)

We follow similar steps as in the proof of \( (\mathbf{2.5}) \), using monotonicity arguments on \( \eta(t) \) instead of \( u(t) \).
(i) Estimate in the region $x > \rho_1(t) + x_0$. We claim, for all $x_0 > 0$:

\[
\int_0^{+\infty} \int (\eta^2(t, x) + \eta^2(t, x)) \psi(x - \rho_1(t) - x_0) dx dt \\
\leq K(\alpha^2 c^{2q+1} + e^{-\frac{t}{4}x_0} + \exp(-2c^{-r})) + K \int_{x>x_0} x^2 u^2(0, x) dx.
\]  

(B.19)

Let us prove (B.19). By Claim B.3 using the estimate on $\frac{4}{\pi} \mathcal{M}_0(t)$ and integrating between 0 and $t_0$, we get the estimate:

\[
\int_0^{+\infty} \int (\eta^2(t_0, x) \psi(x - \rho_1(t_0) - x_0) dx dt \\
\leq \int_0^{+\infty} \int (\eta^2(0, x) \psi(x - \rho_1(0) - x_0) dx dt \\
+ \int_{t_0}^{+\infty} e^{-\frac{t_0}{4}x_0} g_1(t) dt + Ke^{-\frac{2}{4}c(t_0+T_x) - \frac{x_0}{4}x_0}.
\]  

(B.20)

Note that by Fubini Theorem, $|\rho_1(0)| \leq 1$, the expression of $\psi (2.9)$, and $\|\eta(0, x)\|^2 \leq K\alpha^2 c^{2q+1}$,

\[
\int_0^{+\infty} \int (\eta^2(0, x) \psi(x - \rho_1(0) - x_0 - \frac{x}{4}) dx dt = 2 \int \eta^2(0, x + \rho_1(0) + x_0) \left( \int_{-\infty}^{x} \psi \right) dx \\
\leq C\alpha^2 c^{2q+1} + K \int_{x>x_0} x\eta^2(0, x) dx.
\]

Next, by Lemma B.3 and $x_0 \geq 0$,

\[
\int_0^{+\infty} \int_0^{t_0} e^{-\frac{t(t-t_0)+x_0}{4}} g_1(t) dt dt_0 \leq \int_0^{+\infty} \int g_1(t) \left( \int_0^{+\infty} e^{-\frac{x}{4}t(-t_0)} dt \right) dt \\
\leq 16 \int_0^{+\infty} g_1(t) dt \leq K(\alpha^2 c^{2q+1} + \exp(-2c^{-r})).
\]

Finally, by $\int_0^{+\infty} e^{-\frac{x}{4}(t+\tau) - \frac{x_0}{4}x_0} dt \leq \exp(-2c^{-r})$ and integrating (B.20) for $t_0 \in [0, +\infty)$, we obtain

\[
\int_0^{+\infty} \int_0^{+\infty} \eta^2(t, x) \psi(x - \rho_1(t) - x_0) dx dt \\
\leq \int_{x>x_0} x\eta^2(0, x) dx + K(\alpha^2 c^{2q+1} + \exp(-2c^{-r})).
\]  

(B.21)

To control $\int_0^{+\infty} \int_0^{x} \eta^2(t, x) \psi(x - \rho_1(t) - x_0) dx dt$, we use the same argument on $\mathcal{E}_0(t)$. First, we claim as a consequence of a standard argument (Kato’s identity) applied to the equation of $\eta(t)$ that there exists $0 < \bar{t} \leq 1$, such that

\[
\int_{x>0} x (\eta^2_x + \eta^2) \bar{t}, dx \leq K\alpha^2 c^{2q+1} + K \int_{x>x_0} x^2 \eta^2(0, x) dx.
\]

(B.22)

Indeed, let $\lambda > 0$ to be fixed large enough and $I(t) = \int_0^{\eta^2(t, x) \left( \int_{-\infty}^{x-\lambda t} (\int_{-\infty}^{s} \psi) ds \right) dx$. Then, by the equation of $\eta$ (B.11), for some $K_0 > 0$,

\[
\frac{1}{2} I'(t) \leq -\frac{3}{2} \int_0^{\eta^2_x} \int_{-\infty}^{x-\lambda t-x_0} \psi - \lambda \int_0^{\eta^2} \int_{-\infty}^{x-\lambda t-x_0} \psi \\
+ \int \eta^2 \psi' (x - \lambda t - x_0) + K_0 \int \eta^2 \int_{-\infty}^{x-\lambda t-x_0} \psi \leq -\frac{1}{2} \int (\eta^2_x + \eta^2) \int_{-\infty}^{x-\lambda t-x_0} \psi,
\]

36
by $\psi' \leq \frac{1}{4} \int_{-\infty}^{x} \psi$ (from (2.15)) and choosing $\lambda > K_0 + 1$. Thus, $\int_0^1 \int (\eta_x^2 + \eta^2) \int_{-\infty}^{x-\lambda t-x_0} \psi dx dt \leq 2I(0)$, and there exists $t \in [0, 1]$ such that

$$\int \eta_x^2(t) \int_{-\infty}^{x-\rho_1(t)-x_0} \psi dx \leq K \int \eta_x^2(t) \int_{-\infty}^{x-\lambda t-x_0} \psi dx \leq KI(0) \leq Kq_2e^{2q+1} + K \int_{x>x_0} x^2\eta^2(0, x) dx. \quad (B.23)$$

By Claim [B.3], Lemma [B.1], [B.21] and $\|\eta(t)\|_{H^1} \leq Kq_2e^{2q+1}$ for $t \in [0, 1]$, we obtain

$$\int_0^{+\infty} \int \eta_x^2(t, x)\psi(x - \rho_1(t) - x_0) dx dt \leq \int_0^{+\infty} \int \eta_x^2(t, x)\psi(x - \rho_1(t) - x_0) dx dt + 2 \int_0^{+\infty} E_0(t) dt + K \int_0^{+\infty} \int \eta_x^2(t, x)\psi(x - \rho_1(t) - x_0) dx dt \leq K \int (\eta_x^2 + \eta^2)(t, x) + \rho_1(t) + x_0 \int_{-\infty}^{x} \psi dx + K(\alpha^2e^{2q+1} + \exp(-2c^r)). \quad (B.24)$$

From (B.23), we obtain, for all $x_0 > 0$,

$$\int_0^{+\infty} \int (\eta_x^2(t, x) + \eta^2(t, x))\psi(x - \rho_1(t) - x_0) dx dt \leq K(\alpha^2e^{2q+1} + \exp(-2c^r)) + K \int_{x>x_0} x^2\eta^2(0, x) dx. \quad (B.25)$$

Since $\eta(0, x) = u(0, x) - Q_{c_1(0)}(x - \rho_1(0)) - Q_{c_2(0)}(x - \rho_2(0))$, and $\rho_1(0) - \rho_2(0) \geq Tc/4, \quad (2.15)$, we have

$$\int_{x>x_0} x^2\eta^2(0, x) dx \leq K \int_{x>x_0} x^2u^2(0, x) dx + K \int_{x>x_0} x^2(Q^2_{c_1(t)}(x - \rho_1(0)) + Q^2_{c_2(t)}(x - \rho_2(0))) dx \leq K \int_{x>x_0} x^2u^2(0, x) dx + K((x_0^2 + 1)e^{-2\sqrt{c_1(0)x_0}} + \exp(-2c^r)) \leq K \int_{x>x_0} x^2u^2(0, x) dx + K(e^{-\frac{r}{4}x_0} + \exp(-2c^r)). \quad (B.26)$$

Thus, (B.19) is proved.

(ii) Estimate of $\tilde{g}_1(t)$. We claim

$$\int_0^{+\infty} \tilde{g}_1(t) dt \leq K((\alpha^q + \frac{3}{2})\gamma + \gamma_0(\alpha, c) + \alpha^{-\frac{1}{2}} \exp(-\frac{3}{2}c^r)), \quad (B.27)$$

where $\gamma_0(\alpha, c) = \int_{x>|\ln(\alpha^q + \frac{3}{2})|} x^2u^2(0, x) dx$. For $x_0 > 1$ to be fixed later, we observe that

$$0 < \psi(x) - \psi(x - x_0) = \int_{x-x_0}^{x} \psi'(s) ds \leq K \int_{x-x_0}^{x} e^{-\frac{|s|}{4}} ds \leq Ke^{-\frac{|x_0|}{4}} e^{x_0/4}. \quad (2.15)$$
Thus,

\[ \tilde{g}_1(t) \leq Ke^{\frac{c}{\alpha}}g_1(t) + \int (\eta^2 + \eta^2)(t, x)\psi(x - \rho_1(t) - x_0)dx. \]

Therefore, it follows from (B.19) and Lemma 3.1 that

\[ \int_0^{+\infty} \tilde{g}_1(t)dt \leq K \left( e^{\frac{c}{\alpha}^2} + e^{-\frac{c}{\alpha}x_0} \right) + K \int_{x > x_0} x^2u^2(0, x)dx + Ke^{\frac{c}{\alpha}x_0} \exp(-2c^{c_r}). \]

We obtain (B.27) by choosing \( x_0 = |\ln(\alpha t^{1/\beta})| \).

(iii) Estimate of \( \tilde{g}_2(t) \). We claim

\[ \int_0^{+\infty} \tilde{g}_2(t)dt \leq K(\alpha^{\frac{2}{5}}c^q^{q} + \alpha^2c^{2q+\frac{1}{2}} + \frac{1}{\alpha}\gamma_0(\alpha, c) + \frac{1}{\alpha}c \exp(-\frac{3}{2}c^{r}). \]

We estimate separately \( \int_0^{\frac{3}{2}T_c} \tilde{g}_2(t)dt \) and \( \int_{\frac{3}{2}T_c}^{+\infty} \tilde{g}_2(t)dt \). For the first integral, we use \( \tilde{g}_2(t) \leq \|\eta(t)\|^{2}_{H^2} \leq K\alpha^2c^{2q+1} + K \exp(-2c^{r}) \) by (2.26). Thus, by \( T_c \leq c^{-\frac{1}{2}}(1+q) \),

\[ \int_0^{\frac{3}{2}T_c} \tilde{g}_2(t)dt \leq \frac{K}{c}T_c\alpha^{2}c^{2q+1} \leq K\alpha^2c^{2q+\frac{1}{2}} + K \exp(-2c^{r}). \]

Now, we use Claim B.3. By integration in time on \([\frac{3}{8}T_c, t_0] \) of the estimates of \( \frac{d}{dt}\mathcal{M}_{1,2}(t) \) and \( \frac{d}{dx}\mathcal{E}_{1,2}(t) \) and using \( \mathcal{E}_{1,2}(t) + c_1(\frac{3}{8}T_c)\mathcal{M}_{1,2}(t) \geq \frac{1}{2}\tilde{g}_2(t) \) (see Claim 2.4), and then Claim 3.3 we obtain

\[ \tilde{g}_2(t_0) \leq K\tilde{g}_1(\frac{3}{8}T_c) + K|c_1(t_0) - c_1(\frac{3}{8}T_c)| + K \int_{\frac{3}{8}T_c}^{t_0} \left( e^{-\frac{c}{\alpha}^{\frac{3}{4}}(t_0-t)}c^2g_2(t) + e^{-\frac{c}{\alpha}^{\frac{3}{4}}(t-t_0)}g_1(t) \right) dt \\
+ Ke^{-\frac{1}{2}\sqrt{\alpha}^{\frac{3}{4}}(t_0+T_c)} \\
\leq K(\tilde{g}_1(\frac{3}{8}T_c) + \tilde{g}_1(t_0)) + K \int_{\frac{3}{8}T_c}^{t_0} \left( e^{-\frac{c}{\alpha}^{\frac{3}{4}}(t_0-t)}c^2g_2(t) + e^{-\frac{c}{\alpha}^{\frac{3}{4}}(t-t_0)}g_1(t) \right) dt + Ke^{-\frac{1}{2}\sqrt{\alpha}^{\frac{3}{4}}(t_0+T_c)}. \]

Integrating in \( t_0 \) on \([\frac{3}{8}T_c, +\infty) \), we get \( \bar{q} = \frac{3}{8}T_c \) by Fubini Theorem,

\[ \int_{\frac{3}{8}T_c}^{+\infty} \tilde{g}_2(t)dt \leq \text{\$\frac{8K}{c}\$} \int_{\frac{3}{8}T_c}^{+\infty} \tilde{g}_1(t)\bar{q} + \int_{\frac{3}{8}T_c}^{+\infty} \tilde{g}_1(t)dt + K \int_{\frac{3}{8}T_c}^{+\infty} g_2(t)c^{\frac{3}{2}} \int_{\frac{3}{8}T_c}^{+\infty} e^{-\frac{c}{\alpha}^{\frac{3}{4}}(t_0-t)}dt0dt dt \\
+ K \int_{\frac{3}{8}T_c}^{+\infty} g_1(t) \int_{\frac{3}{8}T_c}^{+\infty} e^{-\frac{c}{\alpha}^{\frac{3}{4}}(t_0-t)}dt0dt + K \exp(-2c^{r}) \\
\leq \frac{K}{c} \int_{0}^{+\infty} \tilde{g}_1(t)dt + K \int_{0}^{+\infty} g_2(t)dt + K \exp(-2c^{r}) \\
\leq \frac{K}{c} \left( (\alpha c^{\frac{1}{2}} + \gamma_0(\alpha, c)) + K\alpha^2c^{2q+\frac{1}{2}} + K\alpha^{-\frac{1}{2}} \exp(-\frac{3}{2}c^{r}). \right) 

by (B.27) and Lemma 3.1. Thus, (B.28) is proved.
(iv) **Pointwise estimates.** Now, we claim pointwise estimates, useful for the proof of Lemma 3.3.

\[
\lim_{t \to +\infty} t(\tilde{g}_1(t) + \tilde{g}_2(t)) = 0, \quad \lim_{t \to +\infty} t \int_0^t (\eta^2_2(t,x) + \eta^2(t,x))\psi(x - \frac{c}{10} t)dx = 0,
\]

\(\forall t \geq 0, \quad t \tilde{g}_1(t) \leq K \int_0^{+\infty} \tilde{g}_1(t)dt + K \exp(-2c^{-r}).\)  \(\text{(B.29)}\)

\(t \tilde{g}_2(t) \leq K \int_0^{+\infty} \tilde{g}_2(t)dt + \frac{K}{c} \int_0^{+\infty} \tilde{g}_1(t)dt + K \exp(-2c^{-r}).\)  \(\text{(B.30)}\)

We check the estimate for \(\tilde{g}_1(t)\), the other estimates follow from similar arguments.

Let \(t_1 \leq t_0 \leq t \leq 2t_1\). Integrating the conclusion of Claim 3.1 between \(t_0\) and \(t\), we get

\[
\mathcal{M}_1(t) - \mathcal{M}_1(t_0) \leq (c_1^2(t_0) - c_1^2(t)) \int Q^2 + K \int_{t_0}^t e^{-\frac{1}{36}(t'-t_0)}g_1(t')dt' + Ke^{-\frac{1}{32}\sqrt{c}(t_0 + T_c)},
\]

\[
2\mathcal{E}_1(t) - 2\mathcal{E}_1(t_0) + \frac{1}{100}(\mathcal{M}_1(t) - \mathcal{M}_1(t_0)) \leq K \int_{t_0}^t e^{-\frac{1}{36}(t'-t_0)}g_1(t')dt' + Ke^{-\frac{1}{32}\sqrt{c}(t_0 + T_c)}
+ \left[ \frac{-2q}{2q+1}(c_1^{2q+1}(t_0) - c_1^{2q+1}(t)) + \frac{1}{100}(c_1^{2q}(t_0) - c_1^{2q}(t)) \right] \int Q^2.
\]

Since \(c_1^{2q+1}(t_0) - c_1^{2q+1}(t) - \frac{2q+1}{2q} c_1(t_0)(c_1^{2q}(t_0) - c_1^{2q}(t)) \leq K|c_1(t) - c_1(t_0)|^2\), we obtain

\[
\mathcal{E}_1(t) + c_1(t_0)\mathcal{M}_1(t) \leq \mathcal{E}_1(t_0) + c_1(t_0)\mathcal{M}_1(t_0) + K|c_1(t) - c_1(t_0)|^2
+ K \int_{t_0}^t e^{-\frac{1}{36}(t'-t_0)}g_1(t')dt' + Ke^{-\frac{1}{32}\sqrt{c}(t_0 + T_c)}.
\]

Thus, by Claim 3.3,

\[
\mathcal{E}_1(t) + c_1(t_0)\mathcal{M}_1(t) \leq \mathcal{E}_1(t_0) + c_1(t_0)\mathcal{M}_1(t_0) + K \tilde{g}_1^2(t) + K \tilde{g}_1^2(t_0)
+ K \int_{t_0}^t e^{-\frac{1}{36}(t'-t_0)}g_1(t')dt' + Ke^{-\frac{1}{32}\sqrt{c}(t_0 + T_c)}.
\]

We clearly have \(\mathcal{E}_1(t_0) + c_1(t_0)\mathcal{M}_1(t_0) \leq K \tilde{g}_1(t_0)\) and by a variant of Claim 2.4 there exists \(\kappa_0 > 0\) such that \(\mathcal{E}_1(t) + c_1(t_0)\mathcal{M}_1(t) \geq \kappa_0 \int (\eta^2(t,x) + \eta^2(t,x))\psi(x)dx \geq \frac{1}{\kappa} \tilde{g}_1(t)\). Thus, we obtain

\[
\tilde{g}_1(t) \leq K \tilde{g}_1(t_0) + K \int_{t_0}^t e^{-\frac{1}{16}(t'-t_0)}g_1(t')dt' + Ke^{-\frac{1}{32}\sqrt{c}(t_0 + T_c)}.
\]

In particular, taking \(t = 2t_1\) and integrating in \(t_0 \in [t_1, 2t_1]\), we obtain

\[
t_1 \tilde{g}_1(2t_1) \leq \int_{t_1}^{2t_1} \tilde{g}_1(t)dt + K \int_{t_1}^{2t_1} \int_{t_0}^{2t_1} e^{-\frac{1}{16}(t'-t_0)}g_1(t')dt'dt_0 + K t_1 e^{-\frac{1}{32}\sqrt{c}(t_1 + T_c)}
\leq K \int_{t_1}^{2t_1} \tilde{g}_1(t)dt + K t_1 e^{-\frac{1}{32}\sqrt{c}(t_1 + T_c)}.
\]

Since \(\int_0^{+\infty} \tilde{g}_1(t)dt < +\infty\), we obtain \(\lim_{t \to +\infty} t \tilde{g}_1(t) = 0\) and the estimate on \(t \tilde{g}_1(t)\).
B.4 Proof of Lemma 3.4

We claim the following preliminary result on \( J_j(t) \) defined in (3.16).

Claim B.4 (i) Equation of \( J_j(t) \):

\[
\begin{align*}
\left| \frac{5-p}{2(p-1)} (\rho_1'(t) - c_1(t)) \int Q^2 + J_1'(t) \right| & \leq K \left( g_1(t) + e^{-\frac{3}{2} \sqrt{T_c(t)}} \right), \\
\left| \frac{5-p}{2(p-1)} (\rho_2'(t) - c_2(t)) \int Q^2 + J_2'(t) \right| & \leq Kc^{-\frac{3}{2p}} g_1(t) + Kc^{-\frac{1}{p}} \sqrt{T_c(t)} \tag{B.31}
\end{align*}
\]

\( + K \left( 1 + \alpha c^{-\frac{1}{4}} + c^{-q_1 + \frac{1}{4}} \gamma_0^2 (\alpha, c) + \alpha^{-\frac{1}{2}} \exp(-\frac{1}{2} c^{-r}) \right) e^{-2q} g_2(t) \).

(ii) Estimates on \( J_j(t) \):

\[
\begin{align*}
|J_1(0)| & \leq K((\alpha c^{q_1 + \frac{1}{4}})^{\frac{7}{2}} + \gamma_0^{\frac{1}{2}}(\alpha, c) + \exp(-c^{-r})), \\
|J_2(0)| & \leq K(\alpha^{q_1 + \frac{1}{4}} + c^{-q_1 + \frac{1}{4}} \gamma_0^2 (\alpha, c) + \exp(-\frac{1}{2} c^{-r})), \\
|J_1(t)| + |J_2(t)| & \to 0 \quad \text{as } t \to +\infty.
\end{align*}
\]

Assuming this claim, we finish the proof of Lemma 3.4. Integrating (B.31) in \( t \) on \([0, +\infty)\) since \( |J_1(t)| + |J_2(t)| \to 0 \) as \( t \to +\infty \), and using Lemma 3.1 we obtain

\[
\begin{align*}
\left| \int_0^{+\infty} (\rho_1'(t) - c_1(t)) dt \right| & \leq K|J_1(0)| + K \int_0^{+\infty} g_1(t) dt + K \exp(-2c^{-r}) \\
& \leq K((\alpha c^{q_1 + \frac{1}{4}})^{\frac{7}{2}} + \gamma_0^{\frac{1}{2}}(\alpha, c) + \exp(-c^{-r})), \\
\left| \int_0^{+\infty} (\rho_2'(t) - c_2(t)) dt \right| & \leq K|J_2(0)| + Kc^{-\frac{3}{2p}} \int_0^{+\infty} g_1(t) dt + K \exp(-2c^{-r}) \\
+ K \left( 1 + \alpha c^{-\frac{1}{4}} + c^{-q_1 + \frac{1}{4}} \gamma_0^2 (\alpha, c) + \alpha^{-\frac{1}{2}} \exp(-\frac{1}{2} c^{-r}) \right) e^{-2q} \int_0^{+\infty} g_2(t) dt \\
& \leq K(\alpha^{q_1 + \frac{1}{4}} + c^{-q_1 + \frac{1}{4}} \gamma_0^2 (\alpha, c) + \exp(-\frac{1}{2} c^{-r})).
\end{align*}
\]

Proof of Claim B.4 (i) Equation of \( J_j(t) \): We compute the time derivative of \( J_j(t) \):

\[
J_j(t) = c_j^{-2q} \int \eta(t, x) \left( \int_{-\infty}^{x} \tilde{R}_j(t, x') dx' \right) dx,
\tag{B.32}
\]

where \( \tilde{R}_j(t, x) = \tilde{Q}_{c_j(t)}(x - \rho_j(t)) \). The argument is the same as in the proof of Claim B.1

Let \((j, k) = (1, 2)\) or \((j, k) = (2, 1)\) and \( \tilde{Q} = \frac{2}{p-1} \tilde{Q} + x \tilde{Q}' \), \( \tilde{R}_j = c_j^{\frac{1}{p-1}}(t)\tilde{Q}(\sqrt{c_j(t)}(x - \rho_j(t))) \).
Then,

$$\frac{d}{dt}J_j = -2q\frac{c_j}{c_j}J_j + c_j^{-2q}\left(\int \eta \int_{-\infty}^{x} \bar{R}_j - \rho_j \int \eta \bar{R}_j + \frac{c_j}{2c_j} \int \eta \int_{-\infty}^{x} \tilde{\bar{R}}_j \right)$$

$$= -2q\frac{c_j}{c_j}J_j + c_j^{-2q}\left( -\int (\eta\bar{R}_j + c_j\eta - pR_j^{-1}\eta)\tilde{\bar{R}}_j - (\rho_j - c_j) \int \eta \bar{R}_j \right)$$

$$+ \int ((R_j + R_k + \eta)^p - R_j^p - R_k^p - pR_j^{-1}\eta)\tilde{\bar{R}}_j - \frac{c_j}{c_1} \int \tilde{R}_1 \int_{-\infty}^{x} \tilde{R}_j - \frac{c_j}{c_2} \int \tilde{R}_2 \int_{-\infty}^{x} \tilde{R}_j$$

$$- (\rho_j - c_1) \int R_1 \tilde{R}_j - (\rho_2 - c_2) \int R_2 \tilde{R}_j + \frac{c_j}{2c_j} \int \eta \int_{-\infty}^{x} \tilde{R}_j \right).$$

Note that $LQ = -2Q$, and thus $\int (\eta\bar{R}_j + c_j\eta - pR_j^{-1}\eta)\tilde{\bar{R}}_j = -2\int \eta\bar{R}_j = 0$, $\int R_j \tilde{R}_j = \frac{5-p}{2(p-1)}c_j^{2q} \int Q^2$, $\int \tilde{R}_j \int_{-\infty}^{x} \tilde{R}_j = -c_j^{-p-1}\left(\int Q\right)^2$, and finally $\left|\int \tilde{R}_1 \int_{-\infty}^{x} \tilde{R}_2 \right| \leq Kc^{-\frac{1}{2}+\frac{1}{p-1}}$. As in the proof of Claim [B.1] we obtain

$$\left|J_j(t) + \frac{5-p}{2(p-1)}(\rho_j (t) - c_j(t)) \int Q^2 \right| \leq K\left( c_j^{-q+\frac{5}{2}}h_j + |\rho_j - c_j|c_j^{-q-\frac{1}{2}} + c_j^{-2q}g_j + |c_j^{-1/p-1} + \frac{c_j}{c_j} e^{-\frac{1}{4}e^{\psi(t)T_c}} \right),$$

where $h_j(t) = \int |\eta(t, x)| \left( \int_{-\infty}^{x} e^{-\frac{\sqrt{c_j(t)}}{2}|x' - \rho_j(t)|} dx' \right) dx.$

Using Claim [B.3] we obtain

$$\left|J_j(t) + \frac{5-p}{2(p-1)}(\rho_j (t) - c_j(t)) \int Q^2 \right| \leq K\left( c_j^{-2q}g_j(t)(1 + e^{-\frac{1}{4}e^{\psi(t)T_c}}) + c_j^{-\frac{1}{p-1}} + e^{-\frac{1}{4}e^{\psi(t)T_c}} \right).$$

(ii) Estimates on $h_j$ and $J_j$. We begin with the estimates at $t = 0$. First, by Cauchy-Schwarz inequality, $|\rho_1(0)| \leq 1$, and for $x_0 > 1$, we have $(x_+ = \max(x, 0))$

$$|J_1(0)|^2 \leq Kh_1^2(0) \leq K\left( \int |\eta(0)|^2 \right)^2 \leq K\left( \int \eta^2(0, x)(1 + x_+^2)\psi(x) dx \right) \left( \int \frac{\psi(x)}{1 + x_+^2} dx \right)$$

$$\leq Kx_0^2|\eta(0)||L^2 + K\int_{x_0}^{x_+} x^2\eta^2(0, x) dx.$$

Choose $x_0 = |\ln(\alpha e\alpha^{-\frac{1}{2}})|$ so that by $|\eta(0)||L^2 \leq K\alpha e\alpha^{-\frac{1}{2}}$, we obtain $|J_1(0)|^2 \leq K((\alpha e\alpha^{-\frac{1}{2}})^2 + \gamma_0(\alpha) + \exp(-2c^{-1})).$

Next, by considering the three space regions $x < \rho_2(0), \rho_2(0) < x < \rho_1(0), \rho_1(0) < x,$ with $\rho_1(0) - \rho_2(0) \leq 2T_c,$ we have

$$e^{2q-\frac{1}{p-1}}|J_2(0)| \leq Kh_2(0) \leq K(e^{-\frac{1}{4}|\eta(0)||L^2 + T_c^2|\eta(0)||L^2 + h_1(0)),$$
By $e^{2q \frac{t^2}{T} - 1} = c^q \frac{t^2}{T} - 1$ and $\|\eta(0)\|_{L^2} \leq Ka^q \frac{t^2}{T} - \frac{1}{c}$, we obtain (using $T_c^2 \leq Kc^{-\frac{1}{4} (1 + q)}$),

$$\left| J_2(0) \right| \leq Kc^{-q + \frac{1}{4}}(\alpha c^q \frac{t^2}{T} - \frac{1}{c}) c^q \frac{t^2}{T} - \frac{1}{c} + \alpha^q \frac{t^2}{T} - \frac{1}{c} + \gamma_0^q (\alpha, c) + \exp(-c^{-r})$$

$$\leq Kc^{-q + \frac{1}{4}}(\alpha^q \frac{t^2}{T} - \frac{1}{c} + \alpha^q \frac{t^2}{T} - \frac{1}{c} + \gamma_0^q (\alpha, c) + \exp(-c^{-r}))$$

$$\leq (\alpha^q \frac{t^2}{T} - \frac{1}{c} + \gamma_0^q (\alpha, c) + \exp(-c^{-r})). \quad (B.34)$$

Now, we prove estimates on $h_j(t)$ to be inserted in $(B.33)$. By the properties of $\psi$,

$$h_1^2(t) \leq K \int \eta^2(t, x + r(t))(1 + x^2) \psi(x) dx \leq K \int \eta^2(t, x + r(t)) \left( \int_{-\infty}^{x} \int_{-\infty}^{x} \psi \right) dx.$$

By estimate $(B.20)$ and Lemma 3.1 for any $t_0$, $x_0 > 0$, we have

$$\int \eta^2(t_0, x + r(t_0)) \psi(x - x_0) dx$$

$$\leq \int \eta^2(0, x + r(0) + \frac{1}{2} t_0) \psi(x - x_0) dx + \int_{t_0}^{t_0} e^{\frac{\gamma}{\frac{1}{2} (t_0 - t + x_0)}} g_1(t) dt + K e^{\frac{\gamma}{\frac{1}{2} (t_0 + T_c) - \frac{1}{2} x_0}}$$

$$\leq \int \eta^2(0, x + r(0) + \frac{1}{2} t_0) \psi(x - x_0) dx + K e^{-\frac{\gamma}{\frac{1}{2}}} (e^{\frac{\gamma}{\frac{1}{2}\alpha^2 \frac{t^2}{T} - \frac{1}{c}}} + \sup_{t \in [\frac{1}{2} t_0, t_0]} g_1(t) + e^{-\frac{\gamma}{\frac{1}{2} x_0}}(t_0 + T_c)). \quad (B.35)$$

Note that $\int_{0}^{+\infty} \int_{y}^{+\infty} \psi(x - x_0) dx_0 dy = \int_{0}^{+\infty} \int_{-\infty}^{+y} \psi(x') ds' dy = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi(x') ds' ds$. Thus, by Fubini Theorem, integrating $(B.35)$ in $x_0$ on $[y, +\infty)$, for $y \geq 0$ and then integrating in $y \in [0, +\infty)$, we obtain

$$\int \int \eta^2(t_0, x + r(t_0)) \left( \int_{-\infty}^{x} \int_{-\infty}^{x} \psi \right) dx \leq \int \eta^2(0, x + r(0) + \frac{1}{2} t_0) \left( \int_{-\infty}^{x} \int_{-\infty}^{x} \psi \right) dx$$

$$+ K \left( e^{-\frac{\gamma}{\frac{1}{2}\alpha^2 \frac{t^2}{T} - 1}} + \sup_{t \in [\frac{1}{2} t_0, t_0]} g_1(t) + e^{-\frac{\gamma}{\frac{1}{2} t_0 + T_c}} \right).$$

Therefore, from the assumption on $u(0)$, it follows that $J_1(t) \to 0$ as $t \to +\infty$ and for all $t \geq 0$, $h_1(t) \leq K$. Estimate (ii) for $J_1^t$ then follows from $(B.33)$.

Now, we estimate $h_2(t)$. As before, since $r_1(t) - r_2(t) \leq K(t + T_c)$,

$$h_2(t) \leq Kc^{-\frac{1}{4}} \|\eta(t)\|_{L^2} + K \int_{r_2(t) < x < r_1(t)} |\eta(t, x)| dx + Kh_1(t)$$

$$\leq K \left( c^{-\frac{1}{4}} \|\eta(t)\|_{L^2} + c^{-\frac{1}{4}} (t + T_c) \frac{\sqrt{g_2(t)}}{c} + h_1(t) \right).$$

By Lemma 3.2, we obtain that $h_2(t) \to 0$ as $t \to +\infty$. Moreover, by the estimate on $h_1(t)$, $c^{-\frac{1}{4}} \|\eta(t)\|_{L^2} \leq K \alpha c^{-\frac{1}{4}}$, and $(B.30)$, we have

$$h_2(t) \leq K \left( 1 + \alpha c^{-\frac{1}{4}} + \frac{1}{\sqrt{c}} (t + T_c) \frac{\sqrt{g_2(t)}}{c} \right)$$

$$\leq K \left( 1 + \alpha c^{-\frac{1}{4}} + \frac{1}{\sqrt{c}} \left( \int_{0}^{+\infty} \tilde{g}_2(t) dt \right)^{\frac{1}{2}} + \frac{1}{c} \left( \int_{0}^{+\infty} \tilde{g}_1(t) dt \right)^{\frac{1}{2}} + \exp(-c^{-r}) \right).$$
Thus, from Lemma 3.2 and $-q + \frac{3}{4} \geq 0 \ (p = 2, 3, 4)$ we obtain
\[ c^{-q + \frac{3}{4}} h_2(t) \leq K \left( 1 + \alpha c^{-\frac{3}{4}} + c^{-q - \frac{1}{4}} \gamma_0 (\alpha, c) + \alpha^{-\frac{1}{8}} \exp\left( -\frac{1}{2} c^{-r} \right) \right). \]

This estimate, inserted in (B.33), gives (i) for $J'_2$.

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