Topological Quantum Computation by Manipulating Quantum Tunneling Effect of the Toric Codes

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Quantum computers are predicted to utilize quantum states to perform memory and to process tasks far faster than those of conventional classical computers. Various designs have been proposed to establish a quantum computer, including manipulating electrons in a quantum dot or phonon by using ion traps, cavity QED, or nuclear spin by NMR techniques. Recently, people find that it may be possible to incorporate intrinsic fault tolerance into a quantum computer - topological quantum computation (TQC) which has the debilitating effects of decoherence and free from errors. The key point is to store and manipulate quantum information in a “non-local” way, as means that the “non-local” properties of a quantum system remain unchanged when one does local operations on it. An interesting idea to realize fault-tolerant quantum computation is anyon-braiding proposed by Kitaev [1, 2]. He pointed out that the degenerate ground states of a topological order make up a protected code subspace (the topological qubit) free from errors [3-4].

It is known that there are two types of topological orders in two dimensions spin models - non-Abelian topologically ordered state and Z_2 topologically ordered state. Z_2 topological order is the simplest topologically ordered state with three types of quasiparticles: Z_2 charge, Z_2 vortex, and fermions [5]. Z_2 charge and Z_2 vortex are all bosons with mutual π statistics between them. The fermions can be regarded as bound states of a Z_2 charge and a Z_2 vortex. Recently, several two dimensional exactly solved spin models with Z_2 topological orders were found, such as the toric-code model [1], the Wen plaquette model [3, 4] and the Kitaev model on a honeycomb lattice [2]. One can operate on the protected code subspace of Z_2 topological order by creating an excitation pair, moving one of the excitations around the torus, and annihilating it with the other. However, the non-local operations do not form a complete basis.

On the other hand, in non-Abelian topological orders the elementary excitation becomes non-Abelian anyon with nontrivial statistics. Now people focus on realizing TQC by braiding non-Abelian anyons. The degenerate states undergo a nontrivial unitary transformation when a non-Abelian anyon moves around the other. One can initialize, manipulate and measure the degenerate ground states with several non-Abelian anyons [2, 5]. Along the road, the key point is to manipulate single quasi-particle which becomes a hot issue recently [6, 7, 8, 9, 10, 11, 12, 13].

Kitaev have noted the degenerate ground states (on a torus) as the toric code. In this paper We will design TQC by manipulating the degenerate ground states of Z_2 topological orders (the toric code), instead of that by braiding anyons in a non-Abelian topological order. The key point to manipulate the degenerate ground states is to tune quantum tunneling effect by controlling external field on spin models. Firstly the effective theory of the toric code in Z_2 topological orders is formalized. Secondly, by using the Wen-Paquette model as an example, we show how to control the toric code by tuning the tunneling of the degenerate ground states [8, 14, 15]. In this part, we will concentrate on the measurement of toric code. Finally we give a short discussion on the realization of the Wen-Paquette model in cold atoms.

The effective model of the toric code : For Z_2 topological orders on an even-by-even (e×e) lattice on a torus, the ground state degeneracy is 4[3, 14, 15]. The four degenerate ground states are denoted by | m, n⟩, m = 0, 1 and n = 0, 1. |m, n⟩ have different boundary conditions of fermion’s wave-functions ψ(x, y) as

\[ ψ(x, y) = (-1)^m ψ(x, y+L_y), \quad ψ(x, y) = (-1)^n ψ(x+L_x, y). \]

(1)

Physically, topological degeneracy arises from presence or the absence of π flux of fermions through the two holes of the torus. The degenerate ground states make up two qubits which can be mapped onto quantum states of two pseudo-spins 2 and 2, | 0, 0 ⟩ → | 1⟩ ⊗ | 2⟩, | 0, 0 ⟩ → | 1⟩ ⊗ | 2⟩, | 0, 1 ⟩ → | 1⟩ ⊗ | 1⟩, | 1, 0 ⟩ → | 1⟩ ⊗ | 1⟩.

It is known that the degenerate ground states with
topological orders have same energy in thermodynamic limit. In a finite system, the degeneracy of the ground states is (partially) removed due to tunneling processes, of which a virtual quasi-particle moves around the torus before annihilated with the other one. In $Z_2$ topological orders, there are nine tunneling processes denoted by $C_{e}^{x}, C_{o}^{y}, C_{e}^{+x}+y, C_{f}^{y}, C_{i}^{y}, C_{e}^{x}+y, C_{e}^{y}, C_{o}^{x}+y, C_{o}^{x}+y$, that correspond to virtual $Z_2$-vortex, fermion, $Z_2$ charge propagating along $\hat{e}_x, \hat{e}_y$ and $\hat{e}_x + \hat{e}_y$ (or $\hat{e}_y - \hat{e}_y$) directions around the torus, respectively. For example, the process $C_{o}^{y}$ becomes the unitary operation as

\[(|\uparrow\rangle_1, |\downarrow\rangle_1, |\downarrow\rangle_2 ) \rightarrow ( |\downarrow\rangle_1, |\uparrow\rangle_1, |\downarrow\rangle_2 , |\downarrow\rangle_2 ) . \tag{2}\]

Hence we can use the pseudo-spin operator $\tau^z_i \otimes 1$ to represent $C_{e}^{x}$. Similarly $C_{e}^{y}, C_{o}^{x}+y, C_{f}^{y}, C_{i}^{y}, C_{e}^{x}+y, C_{o}^{x}+y$, $C_{o}^{y}$ can be denoted by $1 \otimes \tau^z_i, \tau^y_i \otimes \tau^z_i, \tau^x_i \otimes \tau^z_i, \tau^y_i \otimes \tau^z_i, \tau^x_i \otimes 1, \tau^y_i \otimes 1$, respectively. Among them, one can choose four basic processes, $C_{e}^{x}, C_{e}^{y}, C_{i}^{y}, C_{f}^{y}$, and check the ground state degeneracy from the commutation relations between $C_{e}^{x}$ and $C_{i}^{y}$ that obeys the Heisenberg algebra, $C_{e}^{x} C_{i}^{y} = e^{i\pi} C_{i}^{y} C_{e}^{x}$.

Hence the effective Hamiltonian of the toric code is obtained in term of the pseudo-spin operators:

\[\mathcal{H}_{\text{eff}} = J_{xx} \tau^x_i \tau^x_j + J_{yy} \tau^y_i \tau^y_j + J_{zz} \tau^z_i \tau^z_j + J_{xz} \tau^x_i \tau^z_j + \hat{h}^x_i \tau^x_i + \hat{h}^z_i \tau^z_i + \hat{h}^x_j \tau^x_j + \hat{h}^z_j \tau^z_j. \tag{3}\]

Here $J_{xx}, J_{yy}, J_{zz}, J_{xz}$, $\hat{h}^x_i, \hat{h}^x_j, \hat{h}^y_i, \hat{h}^y_j$ are determined by the energy splitting of the degenerate ground states from the nine tunneling processes in a $Z_2$ topological order.

Manipulating the toric code by controlling tunneling splitting: To design a topological quantum computer, one needs to do arbitrary unitary operations on the toric code. To emphasize this point, we introduce a concept ‘Controllable Topological Order (CTO)’. In a CTO, quasi-particles’ dispersions and the energy splitting of the degenerate ground states can be manipulated.

In the following part, based on an example of controllable $Z_2$ topological order - the Wen-plaquette model, we demonstrate how to do TOC. The Hamiltonian of the Wen-plaquette model is

\[H_0 = -g \sum_i F_i. \tag{4}\]

Here $F_i = \sigma_i^x \sigma_{i+\hat{e}_x}^x + \sigma_i^x \sigma_{i+\hat{e}_y}^y + \sigma_i^y \sigma_{i+\hat{e}_y}^y$ and $g > 0$. $\sigma_i$ is Pauli matrices on sites, $i$. The ground state denoted by $F_i \equiv +1$ at each site is a $Z_2$ topological state with the ground state energy $E_g = -gN$ where $N$ is the total lattice number. On an $e \times e$ lattice, the four degenerate ground states make up two qubits. However, on even-by-odd ($e \times o$), odd-by-even ($o \times e$) and odd-by-odd ($o \times o$) lattices, the ground state degeneracy is two instead of four.

In this model $Z_2$ charge is defined as $F_i \in \text{even} = -1$ at even sub-plaquette and $Z_2$ vortex is $F_i \in \text{odd} = -1$ at odd sub-plaquette. The fermions can be regarded as bound states of a $Z_2$ charge and a $Z_2$ vortex on two neighbor plaquettes. These quasi-particles in such exactly solved model have flat band. In other words, the quasi-particles cannot move at all. Under the perturbation

\[H' = h^x \sum_i \sigma^x_i + h^y \sum_i \sigma^y_i, \tag{5}\]

the quasiparticles begin to hop. The term $h^x \sum_i \sigma^x_i$ drives the $Z_2$ vortex, $Z_2$ charge and fermion hopping along diagonal direction $\hat{e}_x + \hat{e}_y$. The term $h^y \sum_i \sigma^y_i$ drive fermion hopping along $\hat{e}_x$ and $\hat{e}_y$ directions without affecting $Z_2$ vortex and $Z_2$ charge. One can see the detailed description of the three kinds of quasi-particles in Ref.[15].

As a result, there exist five tunneling processes under the perturbation $H', C_{e}^{x}C_{o}^{y}, C_{f}^{y}, C_{i}^{y}, C_{o}^{x}+y$. Let us calculate the ground state energy splitting from a higher order (degenerate) perturbation approach. From the tunneling processes of $Z_2$ vortex and $Z_2$ charge $1, 2, 3, 4, C_{o}^{x}+y$, one can determine $J_{xx} = J_{yy} = J = g(h^x_y)^L$.

For a $L_x \times L_y$ lattice on a torus, $L$ is equal to $\frac{L_x L_y}{\xi}$ where $\xi$ is the maximum common divisor for $L_x$ and $L_y$. From the tunneling process of fermion, $C_{f}^{y}$ and $C_{f}^{x}+y$, one can obtain $\hat{h}^x_i \sim g(h^y_x)^L, \hat{h}^y_i \sim g(h^x_y)^L \xi$ and $J_{zz} \sim g(h^y_x)^2 L + g(h^x_y)^L \xi$. Other parameters are all zero, $J_{xx} = J_{zz} = \hat{h}^x_i = \hat{h}^y_i = 0$. Then the effective model is simplified into

\[\mathcal{H}_{\text{eff}} \simeq J (\tau^x_i \tau^x_j + \tau^y_i \tau^y_j + J_{zz} \tau^z_i \tau^z_j) + \hat{h}^x_i \tau^x_i + \hat{h}^y_i \tau^y_i + \hat{h}^z_i \tau^z_i. \tag{6}\]

On $e \times o$, $o \times e$ or $o \times o$ lattices, the effective model $\mathcal{H}_{\text{eff}}$ becomes more simple. For example, the two degenerate ground states on an $L \times L$ ($L$ is an odd number) lattice are $|\uparrow\rangle_1 \otimes |\downarrow\rangle_2$ and $|\downarrow\rangle_1 \otimes |\uparrow\rangle_2$ which are denoted as $|\uparrow\rangle$ and $|\downarrow\rangle$ in the following parts of this paper. And the pseudo-spin operator from different tunneling processes is denoted by
where γ = J_{zz}Δτ_γ, θ = J_{zz}Δτ_θ and φ = JΔτ_φ. For example, the Hadamard gate can be a special pseudo-spin rotation operator, \( U_θ,φ(γ = \frac{π}{2}, θ = \frac{π}{2}, φ = \frac{π}{2}) \). To design the Hadamard gate, we may apply the external field along z-direction at an interval \( Δτ_θ = \frac{π}{J_{zz}} \) firstly. Then, we swerve the external field along x-direction at an interval \( Δτ_φ = \frac{π}{J} \). Finally, the external field along z-direction is added at an interval \( Δτ_ν = \frac{π}{J_{zz}} \). Using similar method, one can reach certain quantum operations demanded by TQC and have the ability to carry out arbitrary gate onto the toric code at will, \( α|↑⟩ + βe^{iφ}|↓⟩ \) with \( α, β ≥ 0 \) \( (α^2 + β^2 = 1) \).

Thirdly let us discuss the measurement of an arbitrary state | \( \text{vac} \rangle \rangle = α|↑⟩ + βe^{iφ}|↓⟩ \rangle \). The interference from Aharonov–Bohm (AB) effect allows one to design an experimentally observable distinction between the processes with or without a \( π \)-flux inside the loop. To determine \( α, β \) and \( φ \), we need to observe both fermion interference and \( Z_2 \) vortex interference. Fig.2 is a scheme to show the AB interference on a torus.

Let’s explain how to determine \( α \) and \( β \) by AB effect from fermion-interference. To observe the AB interference, we add a small external field \( h^x \rightarrow 0 \) and \( h^z = 0 \). Now \( Z_2 \) vortex, \( Z_2 \) charge and fermion hop along diagonal direction \( ε_x + ε_y \). On a torus, there exist two symmetrical paths between two sites ( \( i \) and \( j \) ) on opposite positions on a torus, \( γ_1 \) and \( γ_2 \). Then the two trajectories will contribute to the transition amplitude \( T_{i,j} \) according to:

\[
T_{i,j} = |\Psi_{i,j}^γ|^2 + |\Psi_{i,j}^γ'|^2 + 2e^{iφ} |\Psi_{i,j}^γ|\Psi_{i,j}^γ'| \tag{9}
\]

where \( \Psi_{i,j}^γ \) and \( \Psi_{i,j}^γ' \) are the wave functions of fermions of the two trajectories. For the ground state \( |↑⟩ \rangle \), \( ε \) is unit, \( ε = 1 \). However, for the ground state \( |↓⟩ \rangle \), \( ε = -1 \). Then we can distinguish these two cases. For two symmetrical paths \( \Psi_{i,j}^γ = \Psi_{i,j}^γ' = t_f \), we get a probability \( α^2 \) for \( |↑⟩ \rangle \) with \( T_{i,j} = 4t_f^2 \) and a probability \( β^2 \) for \( |↓⟩ \rangle \) with \( T_{i,j} = 0 \).

On the other hand, one can determine the parameter \( φ \) by observing \( Z_2 \) vortex interference from \( i \) to site \( j \). The wave function of \( Z_2 \) vortex has a periodic boundary condition along \( x \) direction for the ground state \( |↑⟩' = \frac{1}{\sqrt{2}} |↑⟩ + |↓⟩ \rangle \) and an anti-periodic boundary condition for the ground state \( |↓⟩' = \frac{1}{\sqrt{2}} |↓⟩ + |↑⟩ \rangle \). Then an arbitrary state \( \text{vac} = α |↑⟩ + βe^{iφ} |↓⟩ \rangle \) re-written into:

\[
|\text{vac}⟩ = \sqrt{\frac{1}{2} + αβ cos φ cos φ' |↑⟩} + \sqrt{\frac{1}{2} - αβ cos φ cos φ' |↓⟩} \tag{10}
\]

where \( φ' = \arctan(\frac{sin φ}{\beta cos φ - α}) \) and \( φ'' = \arctan(\frac{sin φ}{\beta cos φ + α}) \). For two symmetrical paths \( \Psi_{i,j}^γ = \Psi_{i,j}^γ' = t_v \), we get a probability \( \frac{1}{2} + αβ cos φ \) for \( |↑⟩ \rangle \) with \( T_{i,j} = 4t_v^2 \) and a probability \( \frac{1}{2} - αβ cos φ \) for \( |↓⟩ \rangle \rangle \) with \( T_{i,j} = 0 \). As a result, we determine the parameters \( α, β \) and \( φ \) of an arbitrary state \( \text{vac} = α |↑⟩ + βe^{iφ} |↓⟩ \rangle \).

For \( Z_2 \) topological orders on an \( e \times e \) lattice, the protected subspace becomes two qubits. To do arbitrary

\[ H_{eff} = J(τ^x + τ^y) + J_{zz}τ^z. \tag{7} \]

Fig.1 shows the energy splitting (scaled by \( g \)) of the two degenerate ground states from the exact diagonalization numerical results. Then by adding the specific perturbations to the Wen-plaquette model, \( H' \), one can change different quasiparticles’ hopping and then manipulate the toric code by controlling tunneling splitting of degenerate ground states. Such dramatic property in the Wen-plaquette model gives an example of so-called controllable topological orders. Similar properties have been used to create and manipulate anyons in the Kitaev model[9, 10].

Topological quantum computation : In this section we focus on two degenerate ground states on an \( o \times o \) lattice \( (|↑⟩ \rangle \) and \( |↓⟩ \rangle \) \) and show the initialization, the unitary transformation and the measurement.

Firstly we initialize the system into the quantum state \( |↑⟩ \rangle \). This process will occur according to the Hamiltonian \( H' = h(t) \sum_i σ_z \) where \( h(t) = e^{-t/t_0} - 1 \). At first \( t \rightarrow -∞ \), the ground state is the spin-polarized state of \( σ_z \) eigenstates \( |ψ_0⟩ \rangle \). Since the effective Hamiltonian of the toric code is \( H_{eff} = h^z - τ^z \), the time-evolution operator \( U(t) = e^{-iH_{eff}t} \) in the topologically ordered phase becomes a projection operator of the pseudo-spin as \( U(t)|ψ⟩ \rangle \rightarrow |↑⟩ \rangle \kehrung. The system will eventually evolve adiabatically and continuously from \( |ψ_0⟩ \rangle \) into the final state \( U(t=0)|ψ_0⟩ \rangle \rightarrow |↑⟩ \rangle \). \( |↑⟩ \rangle \) becomes the initial state prepared for TQC.

Secondly we show unitary operations of the TQC[8, 9, 11]. A general pseudo-spin rotation operator is defined as:

\[ U_{θ,φ} = e^{-\frac{π}{2}τ^x} e^{-\frac{π}{2}e^{(τ^x+τ^y)}e^{-\frac{π}{2}θτ^x}} \tag{8} \]
unitary transformation on the protected subspace, one needs to apply the external field on a sub-lattice,

$$H' = h \sum_{i \text{ even}} \sigma_i^x + h \sum_{i \text{ even}} \sigma_i^y + h^z \sum_i \sigma_i^z. \quad (11)$$

This is because the perturbation $h \sum_{i \text{ even}} \sigma_i^x + h \sum_{i \text{ even}} \sigma_i^y$ drives only $Z_2$ vortex hopping without affecting $Z_h$. Furthermore, on a manifold with higher genus, the degenerate ground states become multi-qubit, which can be mapped onto multipseudo-spin model. By the same method one can do TQC on the multi-qubit.

**Realization of the Wen-plaquette model:** Finally we discuss the realization of the Wen-plaquette model in a optical lattice of cold atoms. Because the Wen-plaquette model can be regarded as an effective model of the Kitaev model on a two dimensional hexagonal lattice, one may realize the Kitaev model firstly. The Hamiltonian of the Kitaev model is

$$H = \sum_{j+l=\text{even}} (J_x \sigma_j^x \sigma_{j+1,l}^x + J_y \sigma_j^y \sigma_{j-1,l}^y + J_z \sigma_j^z \sigma_{j+1,l}^z). \quad (12)$$

where $j$ and $l$ denote the column and row indices of the lattice. In the limit $J_x \gg J_z \sim J_y$, the model is simplified into the Wen-plaquette model

$$H_0 = -\frac{J_x J_y}{16|J_z|^2} \sum_i \sigma_{\text{left}(i)}^x \sigma_{\text{right}(i)}^x \sigma_{\text{up}(i)}^y \sigma_{\text{down}(i)}^y. \quad (13)$$

Then one can use the Kitaev model in the limit $J_x \gg J_z \approx J_y$ on a torus to do the TQC. The realization of the Kitaev model on a two dimensional hexagonal lattice has been proposed in Ref. [10, 13, 16, 17]. The essential idea realizing the Kitaev model is to induce and control virtual spin-dependent tunneling between neighboring atoms in the lattice that results in a controllable Heisenberg exchange interaction.

**Summary and discussion:** By controlling the tunneling processes, one can do unitary transformation on a topological protected qubit (the toric code), as paves a new road towards TQC rather than by anyon-braiding. By using a designer Hamiltonian - the Wen-Plaquette model as an example, we show how to control the toric code to realize TQC. In particular, we give a proposal to the measurement.

In the end we give a short comment on the advantage and the disadvantage of TQC by tuning quantum tunneling effect. For TQC by anyon-braiding, one needs to manipulate single quasi-particle which demands new techniques. On the contrary, for the TQC by tuning tunneling, one needs only adjust the global field strength (or direction). However, it is a true challenge to realize the designed spin model on a manifold of higher genus in the optical lattice of cold atoms. Such unsolved issue will be worked out in the future.

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