EQUIVARIANT CERF THEORY AND PERTURBATIVE SU(n) CASSON INVARIANTS

SHAOYUN BAI AND BOYU ZHANG

Abstract. We develop an equivariant Cerf theory for Morse functions on finite-dimensional manifolds with group actions, and adapt the technique to the infinite-dimensional setting to study the moduli space of perturbed flat SU(n)–connections. As a consequence, we prove the existence of perturbative SU(n) Casson invariants on integer homology spheres for all $n \geq 3$, and write down an explicit formula when $n = 4$. This generalizes the previous works of Boden-Herald [BH98] and Herald [Her06].

1. Introduction

The Casson invariant is an invariant for oriented integer homology 3-spheres introduced by Casson in 1985 (see [AM90] or [Mar88]). Taubes [Tau90] proved that the Casson invariant is equal to half of the number of points (counted with signs) of the moduli space of irreducible critical points of the perturbed Chern-Simons functionals with SU(2)–gauge. Boden and Herald [BH98] studied the case when the gauge group is SU(3) and constructed a perturbative SU(3) Casson invariant for integer homology spheres. Variations of the SU(3) Casson invariant were later given by Boden-Herald-Kirk [BHK01] and Cappell-Lee-Miller [CLM02]. However, the construction of perturbative SU(n) Casson invariants for $n \geq 4$ has remained open since then.

A different approach of generalizing the Casson invariant was introduced by Boyer-Nicas [BN90] and Walker [Wal92], where one studies the intersection of representation varieties of handlebodies. Cappell-Lee-Miller [CLM90] announced a program of extending the Casson invariant to all oriented closed 3-manifolds and all semi-simple Lie groups using Bierstone transversality. The program was carried out in detail for SO(3), U(2), Spin(4), and SO(4) by Curtis [Cur94].

The main difficulty of constructing SU(n) Casson invariants using the Chern-Simons functional is that one has to perturb so that the moduli space of irreducible critical points is cut out transversely, but the signed count of irreducible critical points depend on the perturbation. Therefore, one has to study the moduli space of both reducible and irreducible critical points, and understand the difference of the critical sets between different choices of perturbations, in order to write down a counting of critical points that is independent of the perturbation.

In the SU(3) case, Boden and Herald [BH98] showed that the moduli space of critical points over a generic 1-parameter family of perturbations can only admit a particular type of bifurcation, and that one can write down a weighted counting of critical points using spectral flow so that the counting does not change under the bifurcation, therefore the SU(3) Casson invariant is constructed. The constructions in [BHK01] and [CLM02] were based on the same line of argument but used...
different weight functions to make the resulting invariants behave better. Later, Herald [Her06] studied the bifurcations of moduli spaces for general gauge groups and characterized the possible bifurcations for SU(4), but the relation between the bifurcations and the spectral flows was not given, and the SU(4) Casson invariant was not discussed. The argument of [Her06] relies on a technical property of SU(n)-connections called “sphere transitivity”, which is not satisfied by SU(n)-connections in general.

In this paper, we give a complete description of the possible changes of the moduli space of critical points with different perturbations when the gauge group is SU(n), for all \( n \geq 3 \). We also compute the corresponding changes of the spectral flows. As a consequence, we prove that perturbative SU(n) Casson invariants exist for all \( n \geq 3 \) on integer homology spheres. We also write down an explicit formula of Casson invariant when \( n = 4 \). Most of the arguments work for general three-manifolds and for arbitrary simple compact gauge groups, and we will state the results in the more general setting whenever possible. In fact, the only place that requires the manifold to be an integer homology sphere and that the gauge group to be SU(n) is in the construction of the equivariant index in Section 5.3.

In order to define the SU(n) Casson invariant, one needs to study the reducible connections with different possible stabilizers simultaneously. Figure 1 illustrates some possible bifurcations of the moduli space over a 1-parameter family of perturbations when the gauge group is SU(5). Here, the notation \( \mathbb{C}^5 \) means that the

![Figure 1. A possible bifurcation diagram for SU(5)](image-url)
corresponding connection is irreducible; \( \mathbb{C}^4 \oplus \mathbb{C} \) means it is given by the direct sum of an irreducible connection on the trivial \( \mathbb{C}^4 \)-bundle and a connection on the trivial \( \mathbb{C} \)-bundle; \( \mathbb{C}^3 \oplus \mathbb{C}^2 \) means it is given by the direct sum of an irreducible connection on the trivial \( \mathbb{C}^3 \)-bundle and an irreducible connection on the trivial \( \mathbb{C}^2 \)-bundle; and \( (\mathbb{C}^2)^{\oplus 2} \oplus \mathbb{C} \) means the connection is given by \( B_1 \oplus B_1 \oplus B_2 \), where \( B_1 \) is an irreducible connection on the trivial \( \mathbb{C}^2 \)-bundle, and \( B_2 \) is a connection on the trivial \( \mathbb{C} \)-bundle.

In the definition of the SU(3)-Casson invariants [BH98, BHK01, CLM02], the weights on the reducible connections are given by the spectral flow, which assigns an integer to each critical point. However, notice that in Figure 1, the moduli space of reducible connections of the form \( (\mathbb{C}^2)^{\oplus 2} \oplus \mathbb{C} \) has two possible bifurcations: it can either bifurcate a reducible connection of the form \( \mathbb{C}^4 \oplus \mathbb{C} \), or a reducible connection of the form \( \mathbb{C}^3 \oplus \mathbb{C}^2 \). Therefore, in order to keep track of this, we need to use a refinement of the spectral flow that can take values in higher-dimensional spaces. In Section 5.2, we will define an equivariant spectral flow, which assigns to each critical point an element in the representation ring of the stabilizer. Similar to the classical spectral flow, the equivariant spectral flow is not gauge invariant. In Section 5.3, we add the equivariant spectral flow by another term given by the Chern-Simons functional to cancel the gauge ambiguity. As a result, we associate an equivariant index to each critical orbit. The equivariant index takes value in \( \tilde{\mathbb{R}}_{SU(n)} \), which is a space defined by Definition 3.39 using the representation rings of the subgroups of SU(\( n \)). The equivariant index of the orbit of \( B \) will be denoted by \( \text{ind} B \).

We say that a critical point of the perturbed Chern-Simons functional is non-degenerate, if the Hessian of the perturbed Chern-Simons functional at the point has the minimum possible kernel. The precise definition will be given by Definition 4.13. The next result will be proved as an immediate consequence of Theorem 5.13:

**Theorem 1.1.** For every \( n \geq 3 \), there exists a function

\[
w : \tilde{\mathbb{R}}_{SU(n)} \to \mathbb{C}
\]

with the following property. Suppose \( Y \) is an integer homology sphere, let

\[P = SU(n) \times Y\]

be the trivial SU(\( n \))-bundle over \( Y \), let \( \theta \) be the trivial connection of \( P \). Then for a generic holonomy perturbation \( \pi \), the critical set of the perturbed Chern-Simons functional consists of finitely many non-degenerate orbits. Let \( M_\pi \) be the moduli space of critical points of the Chern-Simons functional perturbed by \( \pi \), and decompose \( M_\pi \) as

\[M_\pi = M^*_\pi \sqcup M^r_\pi,\]

where \( M^*_\pi \) consists of irreducible critical orbits, and \( M^r_\pi \) consists of reducible critical orbits. Then for \( \pi \) sufficiently small, the sum

\[
\lambda_w := \sum_{[B] \in M^*_\pi} (-1)^{S_f(B, \pi)} + \sum_{[B] \in M^r_\pi} e^{\pi i \cdot CS(\hat{B})/(\pi^2)} \cdot w(\text{ind} B)
\]

(1.1)

is independent of \( \pi \), where \( S_f(B, \pi) \in \mathbb{Z} \) is the (classical) spectral flow from \( K_{B, \pi} \) to \( K_{\theta, \pi} \) (defined by Definition 4.12) via the linear homotopy, and \( \hat{B} \) is a flat connection close to \( B \).

The term \( \pi^2 \) in (1.1) comes from the normalization convention of the definition of the Chern-Simons functional in Equation (4.4).
When $n = 3$, the SU(3)–Casson invariant of Boden-Herald [BH98] can be arranged into the form of (1.1).

The function $w$ in Theorem 1.1 is constructed by induction and hence it is possible to write down the formula for any given value of $n$. We will write down an explicit formula when $n = 4$.

The proof of the main result is organized as follows: In Section 2 and Section 3, we prove an analogous result on finite-dimensional manifolds. In Section 4, we develop the necessary transversality properties for the holonomy perturbations. In Section 5, we apply the Kuranishi reduction argument to prove the main results by reducing to the finite-dimensional case. In Section 6, we characterize all the possible bifurcations of moduli space in SU($n$)–gauge and write down an explicit formula for the SU(4) Casson invariant.

The finite-dimensional results established by Section 2 and Section 3 can be thought of as an equivariant version of Cerf theory [Cer70] and may be of independent interest. We briefly summarize the main result here. Suppose $G$ is a compact Lie group acting on a smooth closed oriented manifold $M$. From [Was69], a smooth $G$–invariant function $f : M \to \mathbb{R}$ is called $G$–Morse if for all $p \in M$ being a critical point of $f$, the kernel of the Hessian of $f$ at $p$ is equal to the tangent space of the $G$–orbit passing through $p$. Suppose $p$ is a critical point of a $G$–Morse function, we will define the equivariant index of $p$ in Definition 2.27, which is given by the subspace of $T_pM$ spanned by the negative eigenvectors of Hess$_p f$, as a representation of the stabilizer of $p$. We use ind$p \in \mathcal{R}_G$ to denote the equivariant index of $p$, where $\mathcal{R}_G$ is the space given by Definition 2.9 that consists of isomorphism classes of representations. It will be shown in Section 2 that the equivariant index is constant on the orbit of $p$. Let Conj($G$) be the set of conjugation classes of closed subgroups of $G$, and let $\mathbb{Z}\text{Conj}(G)$ be the free abelian group generated by Conj($G$). The set Conj($G$) embeds canonically in $\mathcal{R}_G$, and can be regarded as a subset of $\mathcal{R}_G$. Theorem 3.10 will give a complete description of the possible difference of critical sets of different $G$–Morse functions on $M$. We will then prove the following result using Theorem 3.10.

**Theorem 1.2.** Given $G$, there exists a unique map $\eta : \mathcal{R}_G \to \mathbb{Z}\text{Conj}(G)$ with $\eta(\sigma) = \sigma$ for all $\sigma \in \text{Conj}_G$, such that the following holds. For every closed $G$–manifold $M$, let $f$ be a $G$–Morse function on $M$, let $\text{Crit}(f)$ be the set of critical orbits of $f$, then the sum

$$\sum_{[p] \in \text{Crit}(f)} \eta(\text{ind} p)$$

is independent of the function $f$.

We finish the introduction with several remarks.

For the proofs of both Theorem 1.1 and Theorem 1.2, it is crucial to establish certain transversality results under group actions (Lemmas 3.21, 3.23, 4.17, and 4.19). Our argument is adapted from an equivariant transversality argument of Wendl [Wen16, Theorem D], which can be further dated back to Taubes [Tau96]. This argument seems to have simplified an earlier argument of Herald [Her06]: while the argument in [Her06] requires a high-order abundance of the holonomy perturbations (cf. [Her06, Definition 20]), the argument adapted from [Wen16] only depends on the second-order abundance.
The sum over the reducible connections in (1.1) can be thought of as a “correction term” for the counting of irreducible connections. Although the construction of correction terms has its root in many papers on gauge theory, for example in [BH98] and [MRS11], this paper shows that it is possible to construct correction terms for reducible connections on all strata simultaneously. Similar correction terms are also being sought for in other fields, for example, in the construction of enumerative invariants of calibrated 3-manifolds in G2 manifolds [DW17], and in the construction of integer-valued refinements of Gromov-Witten invariants. We hope this paper can provide some insight into those questions as well.

Nakajima [Nak16] conjectured that SU(n) Casson invariants are related to the counting of solutions to the generalized Seiberg-Witten equations. In particular, the bifurcation phenomenon of the moduli space of perturbed flat SU(n)–connections is conjecturally related to the non-compactness of the moduli space of generalized Seiberg-Witten equations. By this conjecture, the correction term in (1.1) could potentially be related to the (conjectured) correction terms in the generalized Seiberg-Witten theory discussed in [Hay12].

It is also a natural question to ask about the properties of the SU(n) Casson invariants. In particular, it would be interesting to see if the invariants admit any surgery formulas. One can also ask about the asymptotic behavior of the SU(n) Casson invariant when \( n \to \infty \). These questions will not be discussed in the current paper.

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## 2. \( G \)–Morse functions

Suppose \( M \) is a closed smooth manifold, let \( f_0, f_1 \) be two Morse functions on \( M \). Cerf’s theorem [Cer70] states that a generic 1-parameter family from \( f_0 \) to \( f_1 \) has finitely many degeneracies, and each degeneracy corresponds to a birth-death transition on the critical set. More precisely, suppose
\[
F : [0, 1] \times M \to \mathbb{R}
\]
is a generic smooth map such that \( F(0, x) = f_0(x) \) and \( F(1, x) = f_1(x) \), then \( F(t, \cdot) \) is Morse for all but finitely many values of \( t \); for every \( t_0 \) such that \( F(t_0, \cdot) \) is not Morse, there is exactly one degeneracy point where \( F \) is locally conjugate to
\[
c + x_1^3 + \epsilon_1(t - t_0)x_1 + \epsilon_2x_2^2 + \cdots + \epsilon_nx_n^2,
\]
with \( \epsilon_i \in \{-1, 1\} \) and \( c \in \mathbb{R} \) being constants.

Suppose \( p \) is a critical point of a Morse function \( f \), recall that the index of \( f \) at \( p \) is defined to be the number of negative eigenvalues of the Hessian of \( f \) at \( p \) counted with multiplicities. By (2.1), each time \( F \) goes through a degeneracy point, it creates or cancels a pair of critical points with consecutive indices. Let \( n_k(f) \) be the number of critical points of a Morse function \( f \) with index \( k \). Then by Cerf’s
theorem, for any two Morse functions $f_0$ and $f_1$ on $M$, we have
\[
\sum_{k=0}^{\dim M} (-1)^k n_k(f_0) = \sum_{k=0}^{\dim M} (-1)^k n_k(f_1).
\]
As a consequence, the value of
\[
\sum_{k=0}^{\dim M} (-1)^k n_k(f)
\]
does not depend on the Morse function $f$ and hence is an invariant of $M$. It is well-known that (2.2) is equal to the Euler number of $M$.

The purpose of Section 2 and Section 3 is to generalize the results above and establish a Cerf theory for manifolds with group actions. Section 2 will introduce the necessary terminologies, and Section 3 will state and prove the main results.

### 2.1. The equivariant topology of $M$

For the rest of Section 2 and Section 3, $G$ will denote a compact Lie group, and $M$ will denote a compact smooth manifold possibly with boundary. We also fix a smooth $G$–action and a smooth $G$–invariant Riemannian metric on $M$. By the invariance of domain, if $\partial M \neq \emptyset$, then $\partial M$ is preserved by the $G$–action.

This subsection establishes the basic topological properties of the $G$–action on $M$. Most of the results are standard, and the reader may refer to, for example, [Was69], for a more complete discussion.

**Lemma 2.1.** $\partial M$ has a tubular neighborhood that is $G$–equivariantly diffeomorphic to $(-1,0] \times \partial M$, where $G$ acts on $(-1,0]$ trivially.

**Proof.** Let $\nu(\partial M)$ be the orthogonal complement of $T(\partial M)$ in $TM|_{\partial M}$. Since the metric on $M$ is $G$–invariant, the exponential map on $\nu(\partial M)$ gives a $G$–equivariant embedding from $(-\epsilon,0] \times \partial M$ to $M$ for $\epsilon$ sufficiently small. \qed

**Definition 2.2.** For $p \in M$, define
\[
\text{Stab}(p) := \{g \in G | g(p) = p\},
\]
\[
\text{Orb}(p) := \{g(p) \in M | g \in G\}.
\]
Then $\text{Stab}(p)$ is a closed subgroup of $p$, and $\text{Orb}(p)$ is diffeomorphic to $G/\text{Stab}(p)$. If $p \in M - \partial M$, then $\text{Orb}(p)$ is a closed submanifold of $M$; if $p \in \partial M$, then $\text{Orb}(p)$ is a closed submanifold of $\partial M$.

**Definition 2.3.** Suppose $p \in M - \partial M$, let $S_p \subset T_pM$ be the orthogonal complement of $T_p\text{Orb}(p)$ in $T_pM$.

Since the metric on $M$ is $G$–invariant, $S_p$ is invariant under the action of $\text{Stab}(p)$ and hence can be regarded as an orthogonal representation of $\text{Stab}(p)$. Viewing $G$ as a principal $\text{Stab}(p)$–bundle over $\text{Orb}(p)$, this representation defines $G \times_{\text{Stab}(p)} S_p$ as an associated vector bundle over $\text{Orb}(p)$. The next lemma is a well-known property of compact Lie group actions, and the reader may refer to, for example, [DK12, Theorem 2.4.1], for a proof.

**Lemma 2.4.** Suppose $p \in M - \partial M$, then $\text{Orb}(p)$ has a $G$–invariant open tubular neighborhood that is $G$–equivariantly diffeomorphic to a $G$–invariant open neighborhood of the zero section of $G \times_{\text{Stab}(p)} S_p$, where $\text{Orb}(p)$ embeds as the zero section. \qed
We introduce the following definitions:

**Definition 2.5.** Suppose $p \in M - \partial M$. Let $D$ be an embedded closed disk in $M$ with dimension equal to $S_p$. Then $D$ is called a slice of $p$ if the following hold:

1. $D$ intersects $\text{Orb}(p)$ transversely at $p$, and $D$ is invariant under the $\text{Stab}(p)$–action;
2. $D$ is $\text{Stab}(p)$–equivariantly diffeomorphic to a closed ball in $S_p$ where $p \in D$ is mapped to 0;
3. The map $\varphi_D : G \times_{\text{Stab}(p)} D \to M$
$$[g, x] \mapsto g(x)$$
is a smooth embedding.

By Lemma 2.4, every interior point of $M$ has a slice.

**Definition 2.6.** Suppose $D$ is a slice of $p$, and suppose $\varphi_D$ is the diffeomorphism given by Definition 2.5. Define $U_p(D)$ to be the image of $\varphi_D$ in $M$.

**Remark 2.7.** By definition, $U_p(D)$ is a closed $G$–invariant neighborhood of $\text{Orb}(p)$. The set of $G$–invariant functions on $U_p(D)$ is in one-to-one correspondence with the set of $\text{Stab}(p)$–invariant functions on $D$ via restrictions to $D$.

**Lemma 2.8.** Suppose $H_1, H_2$ are closed subgroups of $G$, and suppose there exists $u \in G$ such that $H_1 = uH_2u^{-1}$.

Let
$$\rho_1 : H_1 \to \text{Hom}(V, V)$$
be a representation of $H_1$, and let
$$\rho_2 : H_2 \to \text{Hom}(V, V)$$
be the representation of $H_2$ defined by
$$\rho_2(h) = \rho_1(uhu^{-1}).$$
Then $G \times_{\rho_1} V$ is $G$–equivariantly diffeomorphic to $G \times_{\rho_2} V$.

**Proof.** Consider the map
$$\varphi : G \times V \to G \times_{\rho_2} V$$
$$(g, v) \mapsto [gu, v].$$
For $h \in H_1$, we have
$$\varphi(gh, v) = [guh, v] = [gu \cdot u^{-1}hu, v]$$
$$= [gu, \rho_2(u^{-1}hu)v] = [gu, \rho_1(h)v] = \varphi(g, \rho_1(h)v).$$
Therefore $\varphi$ induces a map $\tilde{\varphi}$ from $G \times_{\rho_1} V$ to $G \times_{\rho_2} V$. It is straightforward to verify that $\tilde{\varphi}$ is a $G$–equivariant diffeomorphism.

Lemma 2.4 and Lemma 2.8 are the motivations of the following definition:
**Definition 2.9.** Let \( R_G \) be the set of isomorphism classes of \((H, V, \rho)\), where \( H \) is a closed subgroup of \( G \), and \( \rho : H \to \text{Hom}(V, V) \) is a finite-dimensional representation of \( H \). We say that \((H, V, \rho)\) is isomorphic to \((H', V', \rho')\), if there exists \( g \in G \) and an isomorphism \( \phi : V \to V' \), such that
\[ H' = ghg^{-1}, \]
and
\[ \rho'(ghg^{-1}) = \phi \circ \rho(h) \circ \phi^{-1}. \]

We also introduce the following notations for later reference:

**Definition 2.10.** Let \( \text{Conj}(G) \) be the set of conjugation classes of closed subgroups of \( G \). Suppose \( H \) is a closed subgroup of \( G \), we will use \([H] \in \text{Conj}(G)\) to denote the conjugation class of \( H \).

**Definition 2.11.** Suppose \([H] \in \text{Conj}(G)\). Define \( R_G([H]) \) to be the subset of \( R_G \) consisting of elements represented by the representations of \( H \).

Let \( \sigma_1, \sigma_2 \in R_G([H]) \), and suppose \( \sigma_i \) is represented by \((H, V_i, \rho_i)\) for \( i = 1, 2 \). Then the direct sum of \( \sigma_1 \) and \( \sigma_2 \) is in general not well-defined, because there may exist an element \( g \in G \) with \( H = ghg^{-1} \), such that \( h \mapsto \rho_1(h) \oplus \rho_2(h) \) and \( h \mapsto \rho_1(h) \oplus \rho_2(ghg^{-1}) \) are non-isomorphic representations of \( H \) on \( V_1 \oplus V_2 \). On the other hand, suppose \( \tau \in R_G([G]) \) is given by \((G, V_3, \rho_3)\), then it is straightforward to verify that the element
\[ [H, V_1 \oplus V_3, \rho_1 \oplus (\rho_3|_H)] \in R_G([H]) \]
does not depend on the choice of the representatives. Therefore the following definition is well-defined.

**Definition 2.12.** Let \( H \) be a closed subgroup of \( G \), define
\[ \oplus : R_G([H]) \times R_G([G]) \to R_G([H]) \]
as follows. Suppose \( \sigma \in R_G([H]) \) is represented by \((H, V, \rho, \sigma)\), and \( \tau \in R_G([G]) \) is represented by \((G, V', \rho')\), then \( \sigma \oplus \tau \in R_G([H]) \) is defined to be the element represented by
\[ (H, V \oplus V', \rho \oplus (\rho'|_{V'})). \]

Let \( \mathbb{Z}R_G \) denote the free abelian group generated by \( R_G \). Then the direct sum operator extends linearly to
\[ \oplus : \mathbb{Z}R_G \times R_G([G]) \to \mathbb{Z}R_G. \quad (2.3) \]

**Definition 2.13.** Suppose \( H \) is a closed subgroup of \( G \). Define
\[ i^H_G : R_H \to R_G \]
to be the tautological map by viewing subgroups of \( H \) as subgroups of \( G \). Then \( i^H_G \) induces an homomorphism from \( \mathbb{Z}R_H \) to \( \mathbb{Z}R_G \), which we also denote by \( i^H_G \).

**Definition 2.14.** Suppose \( H \) is a compact Lie group, and let \( \rho : H \to \text{Hom}(V, V) \) be a finite-dimensional real representation of \( H \).

1. We say that \( \rho \) is trivial, if \( \rho(h) = \text{id} \) for all \( h \in H \).
2. We say that \( \rho \) has no trivial component, if the isotypic decomposition of \((V, \rho)\) has no trivial component, or equivalently, if \( \rho \) does not have non-zero fixed point.
We now return to the discussion of the topology of $M$.

**Definition 2.15.** For each $\sigma \in \mathcal{R}_G$, define $M_\sigma$ to be the set of $p \in M - \partial M$ such that the $\text{Stab}(p)$–representation $S_p$ represents the isomorphism class $\sigma$.

The following lemma is another standard property of compact Lie group actions, and the proof is essentially the same as [DK12, Theorem 2.7.4].

**Lemma 2.16.** The decomposition

$$M - \partial M = \bigcup_{\sigma \in \mathcal{R}_G} M_\sigma$$

has the following properties:

1. For every $\sigma$, the set $M_\sigma$ is a (not necessarily closed) $G$–invariant submanifold of $M - \partial M$.
2. Suppose $p \in M_\sigma$, then the action of $\text{Stab}(p)$ on $T_p M_\sigma / T_p \text{Orb}(p)$ is trivial, and the action of $\text{Stab}(p)$ on $T_p M / T_p M_\sigma$ has no trivial component.
3. Suppose $p \in M_\sigma$, let $[p]$ be the image of $p$ in $M_\sigma / G$. Let $D$ be a slice of $p$, let $D^0 \subset D$ be the fixed-point subset of the $\text{Stab}(p)$–action. Then $M_\sigma / G$ is a manifold, and $D^0$ maps diffeomorphically to a closed neighborhood of $[p]$ in $M_\sigma / G$.
4. There are only finitely many $\sigma$ such that $M_\sigma \neq \emptyset$.

**Proof.** Let $p \in M - \partial M$ and $g \in G$, then $\text{Stab}(gp) = g \text{Stab}(p) g^{-1}$, and $S_{gp} = (Tg)(S_p)$, where $Tg$ is the tangent map of the action by $g$. Therefore $S_p$ and $S_{gp}$ represent the same element in $\mathcal{R}_G$, and hence $M_\sigma$ is $G$–invariant.

Let $D$ be a slice of $p$, and let $U_p(D)$ be the neighborhood given by Definition 2.6. Let $D^0$ be the fixed-point subset of $D$ with respect to the $\text{Stab}(p)$–action. Then $M_\sigma \cap U_p(D)$ is given by $G \times_{\text{Stab}(p)} D^0$. Therefore $M_\sigma$ is a submanifold of $M$, and Parts (1), (2), (3) of the lemma are proved.

To prove Part (4) of the lemma, we apply induction on $\dim M$. The statement is obvious if $\dim M = 0$. Now suppose the statement is true for $\dim M < k$, we show that it also holds for $\dim M = k$. For $p \in M$, let $U_p$ be as above, let $M'$ be the unit sphere of $S_p$. By the induction hypothesis on $\text{(Stab}(p), M')$, we conclude that there are only finitely many $\sigma \in \mathcal{R}_G$ such that

$$M_\sigma \cap (U_p - \text{Orb}(p)) \neq \emptyset.$$  

Therefore there are only finitely many $\sigma \in \mathcal{R}_G$ such that $M_\sigma \cap U_p \neq \emptyset$. The statement then follows from the compactness of $M$ and Lemma 2.1. \qed

**Definition 2.17.** Let $\nu(M_\sigma)$ be the orthogonal complement of $TM_\sigma$ in $TM |_{M_\sigma}$.

By definition, $\nu(M_\sigma)$ is a $G$–equivariant vector bundle over $M_\sigma$. By Part (2) of Lemma 2.16, for each $p \in M_\sigma$, the action of $\text{Stab}(p)$ on $\nu(M_\sigma) |_p$ has no trivial component.

2.2. $G$–Morse functions. Suppose $f$ is a $G$–invariant $C^2$ function on $M$, then the critical set of $f$ is invariant under the $G$–action. An orbit $\text{Orb}(p) \subset M$ is called a critical orbit if it consists of critical points. Let $p$ be a critical point of $f$. We use $\text{Hess}_p f : T_p M \to T_p M$ to denote the Hessian of $f$ at $p$. Then $\text{Hess}_p f$ is a $\text{Stab}(p)$–equivariant self-adjoint map, and

$$\text{Hess}_p f (T_p \text{Orb}(p)) = \{0\},$$
Hess$_p f (S_p) \subset S_p$.

We can now introduce the definition of $G$–Morse functions. Our definition of $G$–Morse functions is equivalent to the definition of Morse functions in [Was69].

In the following definition, $N$ denotes a smooth manifold possibly with boundary, and it is endowed with a smooth $G$–action and a smooth $G$–invariant Riemannian metric. The manifold $N$ is allowed to be non-compact.

**Definition 2.18.** Let $f$ be a $G$–invariant $C^2$ function on $N$. We say that $f$ is $G$–Morse, if

1. $\nabla f \neq 0$ everywhere on $\partial N$,
2. $\ker \text{Hess}_p f = T_p \text{Orb}(p)$ for all critical points $p$ of $f$.

**Remark 2.19.** Suppose $f$ is a $G$–Morse function on $N$, then the critical orbits are discrete. If $N$ is compact, then $f$ has only finitely many critical orbits.

**Remark 2.20.** Let $C^\infty_G(M)$ be the space of $G$–invariant $C^\infty$ functions on $M$, and recall that $M$ is compact. By [Was69], Lemma 4.8, if $\partial M = \emptyset$, then smooth $G$–Morse functions are dense in $C^\infty_G(M)$. If $\partial M \neq \emptyset$, one can construct a $G$–Morse function on $M$ by taking a $G$–Morse function $f$ on the double of $M$ such that $\nabla f \neq 0$ on $\partial M$. Therefore $G$–Morse functions always exist on $M$.

**Remark 2.21.** Let $C^2_G(M)$ be the Banach space of $G$–invariant $C^2$ functions on $M$, then the set of $G$–Morse functions is open in $C^2_G(M)$.

**Remark 2.22.** Suppose $p \in M - \partial M$, let $D$ be a slice of $p$, let $U_p(D)$ be the neighborhood of $p$ defined by 2.6. Then the set of $G$–Morse functions on $U_p(D)$ is in one-to-one correspondence with the set of $\text{Stab} (p)$–Morse functions on $D$ via restrictions to $D$.

For $\sigma \in \mathcal{R}_G$, recall that $\nu(M_\sigma)$ denotes the orthogonal complement of $TM_\sigma$ in $TM|_{M_\sigma}$. Suppose $f$ is a $G$–invariant function on $M$, and suppose $p \in M_\sigma$. Since $\text{Hess}_p f$ restricts to a $\text{Stab} (p)$–equivariant map on $S_p$, by Schur’s lemma and Part (2) of Lemma 2.16, we have

$$(\text{Hess}_p f) \nu(M_\sigma)|_p \subset \nu(M_\sigma)|_p.$$

**Definition 2.23.** Suppose $M_\sigma \neq \emptyset$, let $f$ be a $G$–invariant $C^2$ function on $M$. Define

$$\text{Hess}_\sigma f : \nu(M_\sigma) \to \nu(M_\sigma)$$

to be the bundle map given by the Hessian of $f$.

By definition, $\text{Hess}_\sigma f$ is self-adjoint and $G$–equivariant.

**Lemma 2.24.** A $G$–invariant function $f$ is $G$–Morse if and only if the following holds:

1. $\nabla f \neq 0$ on $\partial M$,
2. for every $\sigma \in \mathcal{R}_G$ such that $M_\sigma \neq \emptyset$, the function $f|_{M_\sigma}$ reduces to a (classical) Morse function on $M_\sigma/G$, and $\text{Hess}_\sigma f$ is non-degenerate at all critical points of $f|_{M_\sigma}$.

**Proof.** Let $f$ be a $G$–invariant function, and take $p \in M_\sigma$. By Lemma 2.4, $T_p M$ decomposes as a $\text{Stab}(p)$–representation into

$$T_p M = T_p \text{Orb}(p) \oplus T_p M_\sigma / T_p \text{Orb}(p) \oplus T_p M / T_p M_\sigma,$$
Suppose lemmas are essentially contained in [Was69].

When we establish the equivariant transversality properties. The proofs of these lemmas are essentially contained in [Was69].

The next two lemmas will be used in the proofs of Lemma 3.21 and Lemma 3.23 when we establish the equivariant transversality properties. The proofs of these lemmas are essentially contained in [Was69].

**Lemma 2.25.** Suppose \( M_\sigma \neq \emptyset \), and suppose \( F \subset M \) is a \( G \)-invariant compact subset disjoint from \( M_\sigma \). Let \( f_\sigma \) be a \( G \)-invariant smooth function on \( M_\sigma \), and let \( H_\sigma : \nu(M_\sigma) \to \nu(M_\sigma) \) be a \( G \)-equivariant, smooth, self-adjoint bundle map. Suppose \( f_\sigma \) and \( H_\sigma \) are compactly supported. Then there exists a \( G \)-invariant smooth function \( f \) on \( M \), such that

1. \( f|_{M_\sigma} = f_\sigma \), \( \text{Hess}_\sigma f = H_\sigma \),
2. \( f = 0 \) on a neighborhood of \( F \).

**Proof.** Let \( A, B \) be \( G \)-invariant open subsets of \( M_\sigma \), such that

\[
\text{supp } f_\sigma \cup \text{supp } H_\sigma \subset A \subset \overline{A} \subset B \subset \overline{B} \subset M_\sigma.
\]

Let \( \exp : \nu(M_\sigma)|_B \to M \) be the exponential map, then \( \exp \) is a diffeomorphism near the zero section. Let \( U \subset \nu(M_\sigma)|_B \) be a \( G \)-invariant open neighborhood of the zero section such that \( \exp \) is a diffeomorphism on \( U \). We may choose \( U \) sufficiently small such that \( \exp \) is disjoint from \( F \). Let \( \chi \) be a \( G \)-invariant smooth cut-off function that are supported in \( U \) and is equal to 1 on \( A \).

Let \( \pi : \nu(M_\sigma) \to M_\sigma \) be the projection map, let

\[
\tilde{f}(x) := f_\sigma \left( \pi \left( (\exp|_U)^{-1}(x) \right) \right)
\]

be the pull-back of \( f_\sigma \) to \( U \). Endow \( U \) with the pull-back metric, define a function \( \tilde{f} \) on \( U \) by

\[
\tilde{f}(x) := \tilde{f}(x) - \langle (\text{Hess}_\sigma \tilde{f}) x, x \rangle + (H_\sigma x, x).
\]

Extending the function \( \langle \chi \cdot \tilde{f} \rangle \circ (\nu|_U)^{-1} \) to \( M \) by zero yields the desired \( f \).

**Lemma 2.26.** Let \( \sigma_1, \sigma_2 \in \mathcal{R}_G \). For \( i = 1, 2 \), let \( A_i \subset M_{\sigma_i} \) be \( G \)-invariant open sets such that \( A_1 \subset M_{\sigma_i} \). If \( \sigma_1 = \sigma_2 \), we further assume that \( A_1 \cap A_2 = \emptyset \). Let \( f_i \) be a \( G \)-invariant smooth function on \( M_{\sigma_i} \), and let

\[
H_i : \nu(M_{\sigma_i}) \to \nu(M_{\sigma_i})
\]

be a \( G \)-equivariant smooth self-adjoint bundle map. Suppose \( \text{supp } f_1 \cup \text{supp } H_1 \subset A_1 \). Then there exists a \( G \)-invariant smooth function \( f \) on \( M \), such that \( f|_{A_i} = f_i \), and \( \text{Hess}_\sigma f|_{A_i} = H_i \) for \( i = 1, 2 \).

**Proof.** If \( \sigma_1 = \sigma_2 \), then the statement follows from Lemma 2.25. If \( \sigma_1 \neq \sigma_2 \), then by Lemma 2.25, there exist smooth \( G \)-invariant functions \( f^{(1)} \) and \( f^{(2)} \) on \( M \), such that

1. \( f^{(1)}|_{A_1} = f_1 \), \( \text{Hess}_\sigma f^{(1)}|_{A_1} = H_1 \),
2. \( f^{(2)}|_{A_2} = f_2 \), \( \text{Hess}_\sigma f^{(2)}|_{A_2} = H_2 \),
3. \( f^{(1)} = 0 \) on a neighborhood of \( A_2 \),
(4) \( f^{(2)} = 0 \) on a neighborhood of \( \overline{A_1} \).

Therefore, the function \( f = f^{(1)} + f^{(2)} \) satisfies the desired conditions. \( \square \)

2.3. **Equivariant indices of \( G \)-Morse functions.** We introduce the notion of equivariant index, which will play a crucial role in the equivariant Cerf theory in Section 3.

**Definition 2.27.** Suppose \( f \) is a \( G \)-Morse function on \( M \), suppose \( O \) is a critical orbit of \( f \). Let \( p \in O \), let \( \text{Hess}_p f \) be the subspace of \( T_p M \) spanned by the negative eigenvectors of \( \text{Hess}_p f \), then \( \text{Hess}_p f \) is \( \text{Stab}(p) \)-invariant. Define the equivariant index of \( O = \text{Orb}(p) \) to be the element in \( \mathbb{R}^G \) represented by \( \text{Hess}_p f \) as a \( \text{Stab}(p) \)-representation.

It is straightforward to verify that the equivariant index does not depend on the choice of the point \( p \in O \).

**Definition 2.28.** Recall that \( \mathbb{Z}^G \) denotes the free abelian group generated by \( R^G \). Suppose \( f \) is a \( G \)-Morse function on \( M \). Define the total index of \( f \) to be the element

\[
\sum_{\sigma \in R^G} n_\sigma(f) \cdot \sigma \in \mathbb{Z}^G,
\]

where \( n_\sigma(f) \) denotes the number of critical orbits of \( f \) with equivariant index \( \sigma \).

We will use ind \( f \) to denote the total index of a \( G \)-Morse function \( f \).

**Lemma 2.29.** Suppose \( f_t : M \to \mathbb{R} \) is a smooth 1-parameter family of \( G \)-Morse functions on \( M \), then ind \( f_t \) is constant with respect to \( t \).

**Proof.** If \( G \) is trivial, then the result is a standard property of (classical) Morse functions. The general case follows from the classical case and Lemma 2.24. \( \square \)

3. **Equivariant Cerf theory**

This section establishes the equivariant Cerf theory. All representations will be finite-dimensional in this section unless otherwise specified.

3.1. **Local bifurcation models.** If \( G \) is the trivial group, then by Cerf’s theorem, the critical points of two different \( G \)-Morse functions are related to each other by a sequence of isotopies and birth-death transitions.

When \( G \) is non-trivial, it is possible to have more complicated bifurcations. This subsection constructs several examples of bifurcations that will serve as local models later in Section 3.2.

**Example 3.1.** Let \( G = \mathbb{Z}/2 \), consider the action of \( G \) on \( M = [-1,1] \) such that the generator of \( G \) acts by \( x \mapsto -x \). Define a 1-parameter family of functions on \( M \) by

\[
f_t(x) = tx^2 - x^4, \quad t \in [-1,1].
\]

The function \( f_t \) is \( G \)-Morse when \( t \neq 0 \). When \( t < 0 \), the function \( f_t \) has one critical orbit with stabilizer \( \mathbb{Z}/2 \); when \( t > 0 \), the function \( f_t \) has two critical orbits, one with stabilizer \( \mathbb{Z}/2 \) and the other with stabilizer \{1\}.

The next example generalizes Example 3.1.
Example 3.2. Let $H$ be an arbitrary compact Lie group, let $V$ be a finite-dimensional Euclidean space, and let $\rho_V : H \to \text{Hom}(V, V)$ be an orthogonal representation of $H$. We construct a 1-parameter family of $H$–invariant functions as follows. Let $g$ be a smooth $H$–Morse function on the unit sphere of $V$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth non-negative function supported in $[1/2, 2]$ such that $\chi(1) = 1$ and $\chi'(1) = 0$. Let $B_V(r)$ be the closed ball in $V$ centered at 0 with radius $r$. Define

$$F_V : [-1, 1] \times B_V(2) \to \mathbb{R}$$

by

$$F_V(t, x) = \begin{cases} t\|x\|^2 - \|x\|^4 & \text{if } t \leq 0, \\ t\|x\|^2 - \|x\|^4 + e^{-1/t} \cdot \chi\left(\|x\| / \sqrt{\frac{1}{2}}\right) \cdot g(x/\|x\|) & \text{if } t > 0. \end{cases} \quad (3.1)$$

The function $F_V(t, \cdot)$ is $H$–Morse for $t \neq 0$. When $t < 0$, the function $F_V(t, \cdot)$ has one critical orbit at 0; when $t > 0$, the function $F_V(t, \cdot)$ has one critical orbit at 0 and a finite set of critical orbits on $\partial B_V\left(\sqrt{\frac{1}{2}}\right)$ corresponding to the critical orbits of $g$.

Example 3.2 can be further generalized as follows.

Example 3.3. Let $H, V, g, F_V$ be as in Example 3.2. Let $V'$ be another orthogonal representation of $H$. Let $B_{V \oplus V'}(r)$ be the closed ball in $V \oplus V'$ centered at 0 with radius $r$, and define $B_V(r)$ and $B_{V'}(r)$ similarly. We define a 1-parameter family of $H$–invariant functions on $B_{V \oplus V'}(2)$ as follows. Let $h : V' \to \mathbb{R}$ be a $H$–Morse function on $B_{V'}(2)$ such that 0 is the unique critical point of $h$. Define

$$F_{V \oplus V'} : [-1, 1] \times B_{V \oplus V'}(2) \to \mathbb{R}$$

$$(t, (x, y)) \mapsto F_V(t, x) + h(y), \quad (3.2)$$

where $x \in V$, $y \in V'$. The function $F_{V \oplus V'}(t, \cdot)$ is $H$–Morse for $t \neq 0$, and $(x, y)$ is a critical point of $F_{V \oplus V'}(t, \cdot)$ if and only if $y = 0$ and $x$ is a critical point of $F_V(t, \cdot)$.

Example 3.3 can be extended to general $G$–manifolds using local slices.

Example 3.4. Suppose $M$ is a $G$–manifold and $p \in M$. Let $D$ be a slice of $p$, recall that $U_p(D)$ is the closed $G$–invariant tubular neighborhood of $\text{Orb}(p)$ defined by Definition 2.6. Then $G$–invariant functions on $U_p(D)$ are in one-to-one correspondence with $\text{Stab}(p)$–invariant functions on $D$ by restricting to $D$, and $G$–Morse functions on $U_p(D)$ correspond to $\text{Stab}(p)$–Morse functions on $D$. Let $H = \text{Stab}(p)$, then Example 3.3 gives rise to 1-parameter families of $G$–invariant functions on $U_p(D)$.

Definition 3.5. Suppose $f_t$ is a smooth 1-parameter family of $G$–invariant functions on $M$. We say that $f_t$ has an irreducible bifurcation at $t = t_0$, if there exists $p \in M - \partial M$ such that the following holds:

1. There exists $\epsilon > 0$, such that $f_t$ is $G$–Morse for

$$t \in [t_0 - \epsilon, t_0) \cup (t_0, t_0 + \epsilon].$$

2. The function $f_{t_0}$ has exactly one degenerate critical orbit at $\text{Orb}(p)$, and $\nabla f_{t_0} \neq 0$ on $\partial M$.

3. $S_p$ decomposes as $V \oplus V'$ as an orthogonal $\text{Stab}(p)$–representation, where $V$ is non-trivial and irreducible.
(4) There exists a slice $D$ at $p$, a $\text{Stab}(p)$–Morse function $g$ on the unit sphere of $V$, and a smooth $\text{Stab}(p)$–Morse function $h$ on $B_{V'}(2)$ with $0$ being the only critical point of $h$, such that $f_{t-t_0}$ or $f_{t_0-t}$ is locally given by the family $F_{V \oplus V'}$ defined by (3.2) on $D$.

Recall that Definition 2.28 defines the total index of a $G$–Morse function as an element in $Z\mathcal{R}_G$. We compute the change of total indices under an irreducible bifurcation.

Suppose $f_t$ is a 1-parameter family of $G$–invariant functions on $M$, such that

(1) $f_t$ is $G$–Morse for $t \neq 0$,
(2) $f_t$ has an irreducible bifurcation at $t = 0$ on the orbit $\text{Orb}(p)$.

Without loss of generality, assume there is a slice $D$ of $p$ such that $f_t$ (instead of $f_{-t}$) is locally given by $F_{V \oplus V'}$ on $D$.

Let $H = \text{Stab}(p)$, let $S_p \cong V \oplus V'$ be the decomposition of $S_p$ in Part (2) of Definition 3.5, and let $h$ and $g$ be as in Part (4) of Definition 3.5. Let

$$\rho_V : H \rightarrow \text{Hom}(V, V)$$

be the representation of $H$ on $V$.

Let $\text{Hess}^- h \subset V'$ be the subspace spanned by the negative eigenvectors of the Hessian of $h$ at the origin. Let

$$\rho_h : H \rightarrow \text{Hom}(\text{Hess}^- h, \text{Hess}^- h)$$

be the representation of $H$ on $\text{Hess}^- h$.

For $t = -\epsilon$ with $\epsilon > 0$ and sufficiently small, the total index of $f_t$ on $U_p(D)$ is given by

$$[H, \text{Hess}^- h \oplus V, \rho_h \oplus \rho_V] \in Z\mathcal{R}_G. \quad (3.3)$$

To express the total index of $f_t$ for $t > 0$, suppose the total index of $g$ as an $H$–Morse function is given by

$$\text{ind} g \in Z\mathcal{R}_H.$$

Notice that if the action of $H$ is transitive on the unit sphere of $V$, then

$$\text{ind} g = [H, 0],$$

where $[H, 0] \in \mathcal{R}_H$ is the element given by the zero representation of $H$. For $t = \epsilon$, with $\epsilon > 0$ and sufficiently small, the total index of $f_t$ on $U_p(D)$ is given by

$$[H, \text{Hess}^- h, \rho_h] + i_H^H(\text{ind} g \oplus \mathbb{R} \oplus [H, \text{Hess}^- h, \rho_h]) \in Z\mathcal{R}_G, \quad (3.4)$$

where $\mathbb{R} \in \mathcal{R}_H([H])$ is given by the trivial representation of $H$ on $\mathbb{R}$, the direct sum operator is taken in $\mathcal{R}_H$ and is defined by (2.3), and the map $i_H^H$ is defined by Definition 2.13.

Notice that the total indices given by (3.3) and (3.4) only depend on $H$, $V$, $\rho_V$, $\text{Hess}^- h$, $\rho_h$, and the $H$–Morse function $g$. Therefore we make the following definition.

**Definition 3.6.** Suppose $H$ is a closed subgroup of $G$, and $V$ is a non-trivial irreducible orthogonal representation of $H$. Let $g$ be a $H$–Morse function of $g$ on the unit sphere of $V$, and let $V'$ be a real representation of $H$. Suppose the total index of $g$ is given by $\text{ind} g \in \mathcal{R}_H$, and let $\rho_V$, $\rho_{V'}$ be the actions of $H$ on $V$, $V'$ respectively. Define

$$\xi_H(V, V', g) := [H, V \oplus V', \rho_V \oplus \rho_{V'}] \in Z\mathcal{R}_H,$$
and
\[ \xi_H^+(V, V', g) := [H, V', \rho_{V'}] + \text{ind } g \otimes \mathbb{R} \otimes [H, V', \rho_{V'}] \in \mathbb{Z} \mathcal{R}_H, \]
where \( \mathbb{R} \) denotes the trivial representation of \( H \) on \( \mathbb{R} \). Define
\[ \xi_H(V, V', g) := \xi_H^-(V, V', g) - \xi_H^+(V, V', g). \]

By (3.3) and (3.4), an irreducible bifurcation changes the total index by
\[ \pm i_H^H(\xi_H(V, V', g)), \]
where \( H, V, g \) are as in Definition 3.5, and \( V' \) is given by \((\text{Hess } h, \rho_h)\).

By the definitions, we have
\[ \xi_H^\pm(V, V', g) = \xi_H^\pm(V, 0, g) \oplus [V'], \quad \text{(3.5)} \]
\[ \xi_H(V, V', g) = \xi_H(V, 0, g) \oplus [V'], \quad \text{(3.6)} \]
where 0 denotes the zero representation of \( H \).

We now define the birth-death bifurcations with the presence of \( G \) action.

**Definition 3.7.** Suppose \( f_t \) is a smooth 1-parameter family of \( G \)–invariant functions on \( M \). We say that \( f_t \) has a birth-death bifurcation at \( t = t_0 \), if there exists \( p \in M \), such that following holds:

1. There exists \( \epsilon > 0 \), such that \( f_t \) is \( G \)–Morse for \( t \in [t_0 - \epsilon, t_0] \cup (t_0, t_0 + \epsilon) \).
2. The function \( f_{t_0} \) has exactly one degenerate critical orbit at \( \text{Orb}(p) \), and \( \nabla f_{t_0} \neq 0 \) on \( \partial M \).
3. Suppose \( p \in M_\sigma \), then Hess\( f_{t_0} \) (from Definition 2.23) is non-degenerate at \( p \).
4. The 1-parameter family of functions on \( M_\sigma/G \) induced by \( f_t|_{M_\sigma} \) has a birth-death singularity (in the classical sense) at \((t_0, [p])\), where \([p]\) is the image of \( p \) in \( M_\sigma/G \).

Since the classical birth-death singularity creates or cancels a pair of critical points with consecutive indices on \( M_\sigma/G \), the birth-death bifurcation in Definition 3.7 creates or cancels a pair of critical orbits whose indices are given by \( \sigma \) and \( \sigma \oplus \mathbb{R} \) for some \( \sigma \in \mathcal{R}_G \), where \( \mathbb{R} \in \mathcal{R}_G([G]) \) is the element represented by the trivial representation of \( G \) on \( \mathbb{R} \). Therefore a birth-death bifurcation changes the total index by \( \pm (\sigma + \sigma \oplus \mathbb{R}) \).

**Definition 3.8.** Let \( \text{Bif}_G \subset \mathbb{Z} \mathcal{R}_G \) be the subgroup generated by
\[ i_G^H(\xi_H(V, V', g)) \text{ and } \sigma + \sigma \oplus \mathbb{R} \]
for all possible choices of \( H, V, V', g \), and for all \( \sigma \in \mathcal{R}_G \).

The following lemma follows immediately from the definitions and (3.6).

**Lemma 3.9.** Suppose \( \sigma \in \mathcal{R}_G([G]) \), then \( \text{Bif}_G \oplus \sigma \subset \text{Bif}_G \). \( \square \)
3.2. Statement of the equivariant Cerf theorem. The main result of this section is the following theorem, which states that irreducible bifurcations and birth-death bifurcations generate all the possible changes on the total index.

**Theorem 3.10.** Suppose $f_0$ and $f_1$ are $G$–Morse functions on $M$, and suppose there exists a smooth 1-parameter family $f_t$, $t \in [0, 1]$ connecting $f_0$ and $f_1$ such that
\[ \nabla f_t \neq 0 \text{ on } \partial M \]
for all $t$. Then $\text{ind } f_0 - \text{ind } f_1 \in \text{Bif}_G$.

**Remark 3.11.** Unlike the classical Cerf theorem, we do not claim that a generic 1-parameter family only contains irreducible bifurcations and birth-death bifurcations. In fact, this claim is not true: let $G = \{ u \in \mathbb{C} | u^3 = 1 \}$, and consider the action of $G$ on $\mathbb{C}$ by multiplications. Consider the 1-parameter family of $G$–invariant functions on $\mathbb{C}$ given by
\[ f_t(z) := t|z|^2 + \text{Re}(z^3), \]
then $f_t(z)$ is $G$–Morse for $t \neq 0$, and when $t = 0$ it has exactly one degenerate critical point at $z = 0$. However, the change of $\text{ind } f_t$ across $t = 0$ cannot be given by an irreducible bifurcation or a birth-death bifurcation. One can also prove that a sufficiently small perturbation of $f_t$ still has only one degenerate critical point, therefore it is neither an irreducible bifurcation nor a birth-death bifurcation. Nevertheless, the change of $\text{ind } f_t$ across $t = 0$ is given by the composition of an irreducible bifurcation and a birth-death bifurcation, therefore we still have $\text{ind } f_{t_1} - \text{ind } f_{t_2} \in \text{Bif}_G$ for $t_1, t_2 \neq 0$.

The proof of Theorem 3.10 will be divided into two parts. The first part of the proof applies an equivariant transversality argument to obtain the properties of a generic 1-parameter family of $G$–invariant functions. The second part of the proof further modifies the generic family so that it can be compared with the irreducible bifurcations and the birth-death bifurcations. We will first lay down several lemmas in linear algebra in Section 3.3 as preparation, then establish the transversality results in Section 3.4, and finish the proof of Theorem 3.10 in Section 3.5.

3.3. Preliminaries on linear algebra. Notice that if $H$ is a compact Lie group and $V$ is an irreducible representation of $H$, then $\text{Hom}_H(V, V)$ is an associative division algebra over $\mathbb{R}$, therefore $\text{Hom}_H(V, V) \cong \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

**Definition 3.12.** Suppose $H$ is a compact Lie group and $V$ is an irreducible representation of $H$. We say that $V$ is of type $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, if
\[ \text{Hom}_H(V, V) \cong \mathbb{R}, \mathbb{C}$ or $\mathbb{H}. \]
respectively.

**Definition 3.13.** Suppose $H$ is a compact Lie group and $V$ is an orthogonal representation of $H$, define $\text{Sym}_H(V)$ to be the subspace of $\text{Hom}_H(V, V)$ consisting of symmetric maps.

**Definition 3.14.** Suppose $\mathbb{K} \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \}$. Let $r$ be a non-negative integer. Define $d_\mathbb{K}(r)$ to be the dimension of self-dual $\mathbb{K}$–linear maps on $\mathbb{K}^r$. In other words, we have
\[ d_\mathbb{K}(r) = \begin{cases} \frac{1}{2}r(r + 1) & \text{if } \mathbb{K} = \mathbb{R}, \\ r^2 & \text{if } \mathbb{K} = \mathbb{C}, \\ 2r^2 - r & \text{if } \mathbb{K} = \mathbb{H}. \end{cases} \]
The following statement is a straightforward consequence of Schur’s lemma.

**Lemma 3.15.** Suppose $H$ is a compact Lie group, $V$ is an orthogonal representation of $H$, and suppose the isotypic decomposition of $V$ is given by

$$V \cong V_{i_1}^{a_{i_1}} \oplus \cdots \oplus V_{i_m}^{a_{i_m}},$$

where $V_i$ is of type $K_i$ for $K_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Then

$$\dim \mathbb{R} \operatorname{Sym}_H(V) = \sum_{i=1}^{m} d_{K_i}(a_i).$$

□

**Definition 3.16.** Suppose $\sigma \in R_G$ is represented by an orthogonal representation $(H, V)$. Define

$$d(\sigma) := \dim \mathbb{R} \operatorname{Sym}_H(V).$$

By Lemma 3.15, $d(\sigma)$ does not depend on the choice of $(H, V)$ or the Euclidean structure of $V$.

**Definition 3.17.** Let $H$ be a compact Lie group. Suppose $V$ is an orthogonal $H$–representation. For $\sigma \in R_G([H])$, let $\operatorname{Sym}_{H,\sigma}(V) \subset \operatorname{Sym}_H(V)$ be the subspace consisting of $s \in \operatorname{Sym}_H(V)$ such that $ker s$ represents $\sigma$.

**Lemma 3.18.** $\operatorname{Sym}_{H,\sigma}(V)$ is a submanifold of $\operatorname{Sym}_H(V)$ with codimension $d(\sigma)$. Moreover, suppose $s \in \operatorname{Sym}_{H,\sigma}(V)$, and suppose $L \subset \operatorname{Sym}_H(V)$ is a linear subspace, let $\Pi : V \to ker s$ be the orthogonal projection onto $ker s$, then $s + L$ is transverse to $\operatorname{Sym}_{H,\sigma}(V)$ if and only if the map

$$\varphi : L \to \operatorname{Sym}_H(ker s)$$

defined by

$$\varphi(l)(x) := \Pi(l(x))$$

is a surjection.

**Proof.** Suppose $s \in \operatorname{Sym}_{H,\sigma}(V)$. Let $V_1 = ker s$, and let $V_2$ be the orthogonal complement of $V_1$, then $s$ restricts to an invertible self-adjoint map on $V_2$. Suppose $s' \in \operatorname{Sym}_H(V)$ is close to $s$, then under the decomposition $V = V_1 \oplus V_2$, the map $s'$ is decomposed as

$$s' = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

where $S_{ij} : V_i \to V_j$ is an $H$–equivariant map. For $s'$ sufficiently close to $s$, the map $S_{22}$ is invertible, and we have

$$\begin{pmatrix} \text{id} & -S_{12} \circ S_{22}^{-1} \\ 0 & \text{id} \end{pmatrix} \cdot \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} \text{id} & 0 \\ -S_{22}^{-1} \circ S_{21} & \text{id} \end{pmatrix} \cdot \begin{pmatrix} S_{11} - S_{12} \circ S_{22}^{-1} \circ S_{21} & 0 \\ 0 & S_{22} \end{pmatrix}. \tag{3.7}$$

Hence $s' \in \operatorname{Sym}_{H,\sigma}(V)$ if and only if

$$S_{11} - S_{12} \circ S_{22}^{-1} \circ S_{21} = 0.$$

As a result, $\operatorname{Sym}_{H,\sigma}$ is a manifold near $s$, and its tangent space at $s$ is given by $S_{11} = 0$. Therefore the lemma is proved. □
3.4. Transversality. Let $f_0$ and $f_1$ be as in Theorem 3.10, this subsection studies the property of a generic path from $f_0$ to $f_1$.

Let $C^\infty_G(M)$ be the space of $G$-invariant $C^\infty$ functions $f$ on $M$, then $C^\infty_G(M)$ is a closed subspace of $C^\infty(M)$. Endow $C^\infty_G(M)$ with the standard $C^\infty$-topology.

Let $f_t$ be as in Proposition 3.28. Since $\nabla f_t \neq 0$ on $\partial M$, there exists $0 = t_1 < t_2 < \cdots < t_m = 1$ such that

$$uf_{t_i} + (1 - u)f_{t_{i+1}}$$

have non-vanishing gradients on $\partial M$ for all $i = 1, \cdots, m - 1$ and $u \in [0, 1]$. Let $\{g_0, g_1, \cdots\}$ be a countable dense subset of $C^\infty_G(M)$ that contains $f_{t_i}$ for all $i = 1, \cdots, m$. For $m \in \mathbb{Z}^+$, let

$$N_m := \sup\{\|g_0\|, \cdots, \|g_m\|\}.$$

Let $F$ be the Banach space defined by

$$F := \{ (a_0, a_1, \cdots) | \sum_{m \geq 0} N_m |a_m| < +\infty \}.$$

Then the map

$$\iota : F \to C^\infty_G(M)$$

$$(a_0, a_1, \cdots) \mapsto \sum_{m \geq 0} a_m g_m$$

is a continuous linear map with dense image.

Let

$$F^0 := \{ p \in F | \nabla \iota(p) \neq 0 \text{ on } \partial M \},$$

then $F^0$ is an open subset of $F$. Take $p_0, p_1 \in F$ such that $\iota(p_0) = f_0, \iota(p_1) = f_1$, then $p_0$ and $p_1$ are in the same connected component of $F^0$ by the construction of $F$.

**Definition 3.19.** Suppose $M$ is a Banach manifold and $d$ is a non-negative integer. A subset $S \subset M$ is called a $C^\infty$-subvariety with codimension at least $d$, if $S$ can be covered by the image of countably many smooth Fredholm maps with index $-d$.

**Definition 3.20.** Suppose $\sigma \in \mathcal{R}_G$. Let

$$F_\sigma := \{ u \in F^0 | \exists p \in M \text{ such that } \nabla_{p(t)}(u) = 0, \text{ and } \ker \text{Hess}_p \iota(u)/T_p \text{Orb } p \text{ represents } \sigma \}.$$

**Lemma 3.21.** $F_\sigma$ is a $C^\infty$-subvariety of $F^0$ with codimension at least $d(\sigma)$.

**Proof.** Let

$$\tilde{F}_\sigma := \{ (u, p) \in F^0 \times M | \nabla \iota(u)(p) = 0, \ker \text{Hess}_p \iota(u)/T_p \text{Orb } p \text{ represents } \sigma \}.$$

Suppose $(u, p) \in \tilde{F}_\sigma$, let $D$ be a slice of $p$, we claim that there exists an open neighborhood $U$ of $(u, p)$ in $F^0 \times D$, such that

1. $F_\sigma \cap U$ is a Banach manifold,
2. The projection of $\tilde{F}_\sigma \cap U$ to $F^0$ is Fredholm and has index $-d(\sigma)$. 

The result then follows from the above claim and the separability of $F^0 \times M$.

To prove the claim, notice that changing the $G$–invariant metric of $M$ does not change the set $\tilde{F}_\sigma$, therefore we may assume without loss of generality that $D$ is $\text{Stab}(p)$–equivariantly diffeomorphic to a closed disk of $S_p$, via an isometry, therefore $TD$ is canonically trivialized by $S_p$ via parallel translations. Let $D^0$ be the fixed-point subset of $D$ with respect to the $\text{Stab}(p)$–action, let $S^0_p$ be the fixed-point subset of $S_p$.

Then

$$\tilde{F}_\sigma \cap (F^0 \times D) = \tilde{F}_\sigma \cap (F^0 \times D^0),$$

and $\tilde{F}_\sigma \cap (F^0 \times D^0)$ is given by the pre-image of $\{0\} \times \text{Sym}_{\text{Stab}(p),\sigma}(S_p)$ of the map

$$\varphi : F^0 \times D^0 \to (S^0_p)^* \times \text{Sym}_{\text{Stab}(p)}(S_p)$$

$$(v, q) \mapsto (\nabla_q \iota(u)|_{D^0}, \text{Hess} \iota(u)|_{S^0})_p.$$ 

By Lemma 2.25 and the density of $\text{Im} \iota$ in $C^r_G(M)$, the tangent map of $\varphi$ is surjective at $(u, p)$. Therefore by Lemma 3.18,

$$\varphi^{-1}(\{0\} \times \text{Sym}_{\text{Stab}(p),\sigma}(S_p))$$

is a Banach manifold near $(u, p)$.

The embedding of $\varphi^{-1}(\{0\} \times \text{Sym}_{\text{Stab}(p),\sigma}(S_p))$ in $F^0 \times D^0$ is Fredholm with index $-\dim D^0 - d(\sigma)$, and the projection of $F^0 \times D^0$ to $F^0$ is Fredholm with index $\dim D^0$. Therefore the projection of $\varphi^{-1}(\{0\} \times \text{Sym}_{\text{Stab}(p),\sigma}(S_p))$ to $F^0$ is Fredholm and has index $-d(\sigma)$. \hfill \Box

Notice that $d(\sigma) = 0$ if and only if $\sigma$ is given by a zero representation. Therefore by Lemma 3.21, the function $\iota(u)$ is $G$–Morse for a generic $u \in F$. Hence we have recovered the existence theorem of $G$–Morse functions by Wasserman [Was69].

The following discussion shows that for a generic 1-parameter family $f_t$, there is at most one degeneracy orbit at any given $t$.

**Definition 3.22.** Suppose $\sigma_1, \sigma_2 \in R_G$. Let $F_{\sigma_1, \sigma_2}$ to be the set of $u \in F^0$, such that there exist $p, q \in M$ with the following properties:

1. $\text{Orb}(p) \neq \text{Orb}(q)$,
2. $\nabla_p \iota(u) = 0, \nabla_q \iota(u) = 0$,
3. $\ker \text{Hess}_p \iota(u)/T_p \text{Orb } p$ represents $\sigma_1$,
4. $\ker \text{Hess}_q \iota(u)/T_q \text{Orb } q$ represents $\sigma_2$.

**Lemma 3.23.** $F_{\sigma_1, \sigma_2}$ is a $C^\infty$–subvariety of $F^0$ with codimension at least $d(\sigma_1) + d(\sigma_2)$.

**Proof.** The proof is essentially the same as Lemma 3.21. Let

$$\tilde{F}_{\sigma_1, \sigma_2} := \{(u, p, q) \in F^0 \times M \times M \mid \text{Orb}(p) \neq \text{Orb}(q),$$

$$\nabla_p \iota(u) = 0, \ker \text{Hess}_p \iota(u)/T_p \text{Orb } p \text{ represents } \sigma_1,$$

$$\nabla_q \iota(u) = 0, \ker \text{Hess}_q \iota(u)/T_q \text{Orb } q \text{ represents } \sigma_2\}.$$ 

Suppose $(u, p, q) \in \tilde{F}_\sigma$, let $D_p, D_q$ be a slice of $p$ and $q$ respectively such that the images of $D_p$ and $D_q$ are disjoint in the quotient set $M/G$. we claim that there exists an open neighborhood $U$ of $(u, p, q)$ in $F^0 \times D_p \times D_q$, such that

1. $\tilde{F}_{\sigma_1, \sigma_2} \cap U$ is a Banach manifold,
2. The projection of $\tilde{F}_{\sigma_1, \sigma_2} \cap U$ to $F^0$ is Fredholm and has index $-d(\sigma_1) - d(\sigma_2)$.
The result then follows from the above claim and the separability of $\mathcal{F}^0 \times M \times M$.

To prove the claim, notice that changing the $G$–invariant metric of $M$ does not change the set $\mathcal{F}_{\pi_1,\pi_2}$, therefore we may assume without loss of generality that $D_p$ is $\text{Stab}(p)$–equivariantly diffeomorphic to a closed disk of $S_p$ via an isometry, and also $D_q$ is $\text{Stab}(q)$–equivariantly diffeomorphic to a closed disk of $S_q$ via an isometry. Therefore $TD_p$ and $TD_q$ are canonically trivialized by parallel translations. Let $D^0_p$ be the fixed-point subset of $D_p$ with respect to the $\text{Stab}(p)$–action, let $S^0_p$ be the fixed-point subset of $S_p$, and define $D^0_q$, $S^0_q$ similarly. Then

$$\mathcal{F}_{\pi_1,\pi_2} \cap (\mathcal{F}^0 \times D_p \times D_q) = \mathcal{F}_{\pi_1,\pi_2} \cap (\mathcal{F}^0 \times D^0_p \times D^0_q),$$

and the intersection is given by the pre-image of

$$\{0\} \times \text{Sym}_{\text{Stab}(p),\pi_1}(S_p) \times \text{Sym}_{\text{Stab}(q),\pi_1}(S_q)$$

of the map

$$\varphi : \mathcal{F}^0 \times D^0_p \times D^0_q \to (S^0_p)^* \times \text{Sym}_{\text{Stab}(p)}(S_p) \times (S^0_q)^* \times \text{Sym}_{\text{Stab}(q)}(S_q)$$

$$(u, s, t) \mapsto (\nabla_s t(u)|_{D^0_p}, \text{Hess } t(u)|_{S_p}, \nabla_s t(u)|_{D^0_q}, \text{Hess } t(u)|_{S_q}).$$

By Lemma 2.26 and the density of $\text{Im } \iota$ in $C^\infty_G(M)$, the tangent map of $\varphi$ is surjective at $(u, p, q)$, therefore by Lemma 3.18,

$$\varphi^{-1}(\{0\} \times \text{Sym}_{\text{Stab}(p),\pi_1}(S_p) \times \text{Sym}_{\text{Stab}(q),\pi_1}(S_q))$$

is a Banach manifold near $(u, p, q)$.

The embedding of (3.8) to $\mathcal{F}^0 \times D^0_p \times D^0_q$ is Fredholm with index

$$- \dim D^0_p - \dim D^0_q - d(\sigma_1) - d(\sigma_2),$$

and the projection from $\mathcal{F}^0 \times D^0_p \times D^0_q$ to $\mathcal{F}^0$ is Fredholm with index $\dim D^0_p + \dim D^0_q$. Therefore the projection of (3.8) to $\mathcal{F}^0$ is Fredholm and has index $-d(\sigma_1) - d(\sigma_2)$. $\square$

Lemma 3.21 and Lemma 3.23 have the following immediate corollary.

**Corollary 3.24.** Suppose $p_t$, $t \in [0, 1]$ is a generic path from $p_0$ to $p_1$ in $\mathcal{F}^0$ that intersects all $\mathcal{F}_t$ and $\mathcal{F}_{\pi_1,\pi_2}$ transversely, and let $f_t = \iota(p_t)$. Then there are at most countably many $t$ such that $f_t$ is not $G$–Morse; for every such $t$, there is exactly one critical orbit $\text{Orb}(p)$ of $f_t$, and $\ker \text{Hess } f_t/T_p \text{Orb}(p)$ is an irreducible representation of $\text{Stab}(p)$. $\square$

### 3.5. Proof Theorem 3.10.

This subsection finishes the proof of Theorem 3.10 using the previous results and an induction argument.

Define $\mathcal{M} := \mathbb{Z} \geq 0 \times \mathbb{Z} \geq 0$, and let $\succeq$ be the lexicographical order on $\mathcal{M}$. Namely, for $(a, b), (a', b') \in \mathcal{M}$, we have $(a, b) \succeq (a', b')$ if and only if $a \geq a'$, or $a = a'$ and $b \geq b'$. We write $(a, b) \prec (a', b')$ if $(a, b) \succeq (a', b')$ and $(a, b) \neq (a', b')$. Define $\prec$ and $\preceq$ to be the reverses of $\succeq$ and $\succeq$ respectively.

**Definition 3.25.** Suppose $H$ is a compact Lie group, define

$$m(H) := (\dim H, \# \pi_0(H)) \in \mathcal{M},$$

where $\# \pi_0(H)$ equals the number of elements in $\pi_0(H)$, or equivalently, $\# \pi_0(H)$ equals the number of connected components of $H$.

We define the following filtration on $\text{Bif}_G$. 
Definition 3.26. Suppose \( k \in \mathcal{M} \). Define \( \text{Bif}^{(k)}_G \subset \mathbb{Z} \mathcal{R}_G \) to be the subgroup generated by
\[
i^H_G(\xi_H(V,V',g)) \quad \text{and} \quad \sigma + \sigma \oplus \mathbb{R},
\]
for all possible choices of \( H \) such that \( m(H) \preceq k \), and for all possible choices of \( V, V', g, \) and \( \sigma \in \mathcal{R}_G([H]) \).

Then Lemma 3.9 also holds for the filtrations:

Lemma 3.27. Let \( k \in \mathcal{M} \). Suppose \( \sigma \in \mathcal{R}_G([G]) \), then \( \text{Bif}^{(k)}_G \oplus \sigma \subset \text{Bif}^{(k)}_G \). \( \square \)

Notice that \( (\mathcal{M}, \succeq) \) is a totally ordered set, and every non-empty subset of \( \mathcal{M} \) has a minimum element. Therefore one can apply induction on \( (\mathcal{M}, \succeq) \). We prove the following stronger version of Theorem 3.10 using induction on \( k \in \mathcal{M} \):

Proposition 3.28. Suppose \( f_0 \) and \( f_1 \) are \( G \)-Morse functions on \( M \), and suppose there exists a smooth 1-parameter family \( f_t, t \in [0,1] \) connecting \( f_0 \) and \( f_1 \) such that
\[
\nabla f_t \neq 0 \text{ on } \partial M
\]
for all \( t \). Let
\[
k = \max_{p \in M} m(\text{Stab}(p)),
\]
Then \( \text{ind } f_0 - \text{ind } f_1 \in \text{Bif}^{(k)}_G \).

Notice that \( m(H) \succeq (0, 1) \) for all compact Lie groups \( H \). If
\[
\max_{p \in M} m(\text{Stab}(p)) = (0, 1),
\]
then \( \text{Stab}(p) \) is trivial for all \( p \in M \), thus \( M/G \) is a manifold, and \( G \)-equivariant functions on \( M \) are equivalent to smooth functions on \( M/G \). Therefore Proposition 3.28 follows from the classical Cerf’s theorem.

Now suppose \( k \succ (0, 1) \), and suppose Proposition 3.28 is true for all \( (M, G) \) such that
\[
\max_{p \in M} m(\text{Stab}(p)) \prec k.
\]
We prove Proposition 3.28 when
\[
\max_{p \in M} m(\text{Stab}(p)) = k.
\]

We start with several technical lemmas.

Lemma 3.29. Let \( H \) be a compact Lie group, and suppose \( V \) is an orthogonal representation of \( H \) without trivial components. Let \( B(r) \) be the closed ball in \( V \) centered at \( 0 \) with radius \( r \). Suppose \( f_i \) with \( t \in [0,1] \) is a smooth 1-parameter family of \( H \)-invariant functions on \( B(1) \), such that \( \text{Hess } f_i \) is positive definite at \( 0 \) for \( i = 0, 1 \). Then there exists a family of \( H \)-invariant functions \( \tilde{f}_t, t \in [0,1] \) on \( B(1) \), such that
\( \tilde{f}_t = f_t \) for \( t = 0, 1 \),
(2) there exists a neighborhood \( N(\partial B(1)) \) of \( \partial B(1) \), such that \( \tilde{f}_t = f_t \) on \( N(\partial B(1)) \) for all \( t \),
(3) \( \text{Hess } \tilde{f}_t \) is positive definite at \( 0 \) for all \( t \in [0,1] \).
Proof. Let $\chi : B(1) \to \mathbb{R}$ be a $G$–invariant cut-off function that equals 1 near 0 and equals 0 near $\partial B(1)$. Then the family

$$ \tilde{f}_t := (1 - \chi) \cdot f_t + \chi \cdot ((1 - t)f_0 + tf_1) $$

satisfies the desired conditions. \qed

**Lemma 3.30.** Let $H$ be a closed subgroup of $G$, let $V$ be a finite-dimensional orthogonal $H$–representation, and let $B(1)$ be the closed unit ball of $V$. Let $M = G \times_H B(1)$, let $f_0$ and $f_1$ be $G$–Morse functions on $M$. Let $D = H \times B(1) \subset M$.

Then $f_0|_D$ and $f_1|_D$ are $H$–Morse functions. Suppose $\text{ind} f_0|_D - \text{ind} f_1|_D \in \text{Bif}_H^{(k)}$, then $\text{ind} f_0 - \text{ind} f_1 \in \text{Bif}_G^{(k)}$.

**Proof.** We have

$$ i^H_G(\text{Bif}_H^{(k)}) \subset \text{Bif}_G^{(k)} $$

and

$$ i^H_G(\text{ind} f_0|_D - \text{ind} f_1|_D) = \text{ind} f_0 - \text{ind} f_1. $$

Hence the lemma is proved. \qed

The following lemmas verify several special cases.

**Lemma 3.31.** Suppose $m(G) \leq k$, and let $V$ be an orthogonal representation of $G$ without trivial component. Let $B(r)$ be the closed ball in $V$ centered at 0 with radius $r$. Suppose $f_t$, $t \in [0, 1]$ is a smooth 1-parameter family of $H$–invariant functions on $B(1)$, such that

1. $f_0$ and $f_1$ are $G$–Morse,
2. $\nabla f_t \neq 0$ on $\partial B(1)$ for all $t$.

Then $\text{ind} f_0 - \text{ind} f_1 \in \text{Bif}_G^{(k)}$.

**Proof.** By the construction of Example 3.3 and a standard patching argument, there exist $\hat{f}_i$, $i = 0, 1$, such that

1. $\hat{f}_i = f_i$ on a neighborhood of $\partial B(1)$,
2. $\hat{f}_i$ is $G$–Morse,
3. $\hat{f}_i$ is obtained from $f_i$ by a finite sequence of irreducible bifurcations at 0,
4. $\text{Hess} \hat{f}_i$ is positive definite at 0.

By definition, we have

$$ \text{ind} f_i - \text{ind} \hat{f}_i \in \text{Bif}_G^{(k)}, \quad i = 0, 1. $$

By Lemma 3.29, $\hat{f}_0$ and $\hat{f}_1$ can be connected by a 1-parameter family of $G$–invariant functions $f_t$, such that $\nabla \hat{f}_t \neq 0$ on $\partial B(1)$, and $\text{Hess} \hat{f}_t$ is positive definite at 0 for all $t$. Therefore, there exists $\epsilon$ sufficiently small, such that for all $t$, we have $\nabla \hat{f}_t \neq 0$ on $\partial B(\epsilon)$, and $\hat{f}_t|_{B(\epsilon)}$ is $G$–Morse with a unique critical point at 0.

Since $V$ has no trivial component, we have

$$ k' := \max_{p \in B(1) - B(\epsilon)} m(\text{Stab}(p)) < k. $$

Applying the induction hypothesis on the closure of $B(1) - B(\epsilon)$ yields

$$ \text{ind} \hat{f}_0 - \text{ind} \hat{f}_1 \in \text{Bif}_G^{(k')} \subset \text{Bif}_G^{(k)}, $$

therefore the lemma is proved. \qed
Lemma 3.32. Suppose $f_0$ and $f_1$ are connected by a smooth family $f_t$, $t \in [0, 1]$ of $G$-invariant functions $M$, with the following properties:

1. there exists $t_0 \in (0, 1)$, such that $f_t$ is Morse when $t \neq t_0$,
2. the function $f_{t_0}$ has exactly one degenerate critical orbit at $\text{Orb}(p)$, and $\nabla f_{t_0} \neq 0$ on $\partial M$,
3. $\ker \text{Hess}_{f_{t_0}}/T_p \text{Orb}(p)$ contains no trivial component as a representation of $\text{Stab}(p)$,
4. $m(\text{Stab}(p)) \leq k$.

Then $\text{ind } f_0 - \text{ind } f_1 \in \text{Bif}^{(k)}_G$.

Proof. Since the critical orbits of $f_{t_0}$ are discrete on $M - \text{Orb}(p)$, there exists a slice $D$ of $p$, such that $\nabla f_{t_0} \neq 0$ on $\partial D$. By shrinking the interval of $t$ if necessary and invoking Lemma 2.29, we may assume without loss of generality that $\nabla f_t \neq 0$ on $\partial D$ for all $t$.

Recall that $U_p(D)$ is the neighborhood of $\text{Orb}(p)$ defined by Definition 2.6. By Lemma 2.29 again, we only need to compare the total indices of $f_i$ on $U_p(D)$ for $i = 0, 1$. By Lemma 3.30, we may further assume without loss of generality that $M = D$ and $\text{Stab}(p) = G$.

Identify $D$ with the closed unit ball of $S_p$, and decompose $S_p$ as $S_p = S'_p \oplus S''_p$, where $S'_p = \ker \text{Hess}_{f_{t_0}}$, and $S''_p$ is the orthogonal complement of $S'_p$ in $S_p$. By the assumptions, the Hessian of $f_{t_0}$ is non-degenerate on $S''_p$.

Let $B_{V'}(r)$ be the closed ball in $V'$ centered at 0 with radius $r$. Let $\pi', \pi''$ be the orthogonal projections of $S_p$ onto $S'_p, S''_p$ respectively. Define

$$M_{t, \epsilon} := \{x \in D| \pi'' \nabla f_t(x) = 0, \|\pi' x\| \leq \epsilon\}.$$

By the implicit function theorem, for $t$ sufficiently close to $t_0$ and $\epsilon$ sufficiently small, and by shrinking $D$ if necessary, $M_{t, \epsilon}$ is $G$-equivariantly diffeomorphic to $B_{V'}(\epsilon)$ via $\pi'$.

Let $\sigma \in \mathcal{R}_G([G])$ be represented by the subspace of $S''_p$ generated by the eigenvectors of $\text{Hess}_{f_{t_0}}|_{S''_p}$ as a $G$-representation. Let $\hat{f}_t$ be the restrictions of $f_t$ on $M_{t, \epsilon}$. Under the assumption that $M = D$ and $(\epsilon, t)$ being sufficiently close to $(0, t_0)$ with $t \neq t_0$, we claim that $\hat{f}_t$ is $G$-Morse on $M_{t, \epsilon}$, and

$$\text{ind } f_t = \text{ind } \hat{f}_t \oplus \sigma.$$

The desired result then follows from this claim by Lemma 3.27 and Lemma 3.31.

To prove the claim, suppose $(\epsilon, t)$ is sufficiently close to $(0, t_0)$ such that $M_{t, \epsilon}$ is a $G$-manifold, and suppose $q \in M_{t, \epsilon}$ is a critical point of $\hat{f}_t$ on $M_{t, \epsilon}$, then $\text{Orb}(q) \subset M_{t, \epsilon}$. Recall that $S_q \subset T_qD$ is the orthogonal complement of $T_q\text{Orb}(q)$ in $T_qD$. Let $S'_q$ be the orthogonal complement of $T_q\text{Orb}(q)$ in $T_qM_{t, \epsilon}$, and let $S''_q$ be the orthogonal complement of $T_qM_{t, \epsilon}$ in $T_qD$. Then we have

$$S_q = S'_q \oplus S''_q.$$

Suppose $\text{Hess}_q f_t : S_q \to S_q$ is given by the matrix $\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$ under the decomposition above, then

$$N_{11} : S'_q \to S'_q.$$
equals $\text{Hess}_q \hat{f}_t|_{S_q}$. Recall that $M = D$, which is identified with a closed ball in a linear space. For each $v \in S_q'$, we have

$$N_{11}(v) + N_{21}(v) = (\text{Hess}_q f_1)(v) = \frac{\partial}{\partial v}(\nabla f_1) \in \ker \pi'' = S_q'$$

by the definition of $M_{t,s}$. Therefore, there exist constants $z_1$ and $\epsilon_1$, such that for all $(\epsilon, t) \in (0, \epsilon_1) \times (t_0 - \epsilon_1, t_0 + \epsilon_1)$, we have

$$z_1 \cdot \|N_{11}(v)\| \geq \|N_{21}(v)\|.$$  \hspace{1cm} (3.9)

In particular, we have $\ker N_{11} \subset \ker N_{21}$, thus

$$\ker N_{11} \subset \ker \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}.$$  

By the assumptions, $f_t$ is $G$–Morse for all $t \neq t_0$, which implies $\ker \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} = 0$ for all $t \neq t_0$. Therefore $N_{11}$ is invertible, and hence $\hat{f}_t$ is $G$–Morse, when $\epsilon < \epsilon_1$ and $t \in (t_0 - \epsilon_1, t_0) \cup (t_0, t_0 + \epsilon_1)$.

For $(\epsilon, t)$ sufficiently close to $(0, t_0)$, the map

$$N_{22} : \pi'' \rightarrow \pi''$$

is an approximation of $\text{Hess}_q f_{t_0}|_{S_{q'}'}$, which is invertible. By the continuity of $\text{Hess}_q f_t$, we have

$$\lim_{(\epsilon, t) \rightarrow (0, t_0)} (\|N_{11}\| + \|N_{21}\| + \|N_{12}\|) = 0.$$  

Therefore, there exist constants $z_2, \epsilon_2$, such that for all

$$(\epsilon, t) \in (0, \epsilon_2) \times (t_0 - \epsilon_2, t_0 + \epsilon_2),$$

we have

$$\|N_{22}^{-1}\| \leq z_2, \quad \|N_{12}\| \leq \frac{1}{2z_1 z_2}.$$  

Let $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$, and suppose $\epsilon < \epsilon_0$, $t \in (t_0 - \epsilon_0, t) \cup (t, t_0 + \epsilon_0)$. Let $s \in [0, 1]$. Notice that

$$\begin{pmatrix} \text{id} & -sN_{12} \circ N_{22}^{-1} \\ 0 & \text{id} \end{pmatrix} \cdot \begin{pmatrix} N_{11} & sN_{12} \\ sN_{21} & N_{22} \end{pmatrix} \cdot \begin{pmatrix} \text{id} & 0 \\ -sN_{22}^{-1} \circ N_{21} & \text{id} \end{pmatrix}$$

$$= \begin{pmatrix} N_{11} - s^2N_{12} \circ N_{22}^{-1} \circ N_{21} & sN_{12} \\ 0 & N_{22} \end{pmatrix}.$$  \hspace{1cm} (3.10)

For every $v \in S_q'$, we have

$$\| (N_{11} - s^2N_{12} \circ N_{22}^{-1} \circ N_{21})(v) \| \geq \|N_{11}(v)\| - s^2\|N_{12}\| \cdot \|N_{22}^{-1}\| \cdot \|N_{21}(v)\|$$

$$\geq \|N_{11}(v)\| - s^2\|N_{12}\| \cdot \|N_{22}^{-1}\| \cdot (z_1\|N_{11}(v)\|)$$

$$\geq \|N_{11}(v)\| - s^2 \cdot \frac{1}{2z_1 z_2} \cdot z_2 \cdot (z_1\|N_{11}(v)\|)$$

$$\geq \frac{1}{2} \|N_{11}(v)\|.$$  

Since $N_{11}$ is injective, the estimates above imply that $N_{11} - s^2N_{12} \circ N_{22}^{-1} \circ N_{21}$ is injective, therefore it is invertible for all $s \in [0, 1]$. By (3.10), the map

$$\begin{pmatrix} N_{11} & sN_{12} \\ sN_{21} & N_{22} \end{pmatrix}$$  \hspace{1cm} (3.11)
is invertible for all \( s \in [0, 1] \). Moreover, the family (3.11) is \( \text{Stab}(q) \)-equivariant for all \( s \in [0, 1] \). Therefore the equivariant index of \( f_t \) at \( q \) is represented by the subspace of \( S_q \) generated by the negative eigenvectors of \( N_{11} \) and \( N_{22} \), and hence the claim is proved.

**Lemma 3.33.** Suppose \( f_0 \) and \( f_1 \) are connected by a smooth family \( f_t, t \in [0, 1] \) of \( G \)-invariant functions \( M_t \), with the following properties:

1. \( \nabla f_t \neq 0 \) on \( \partial M \) for all \( t \),
2. \( \text{Hess}_t f_t \) (from Definition 2.23) is non-degenerate for all \( t \) and all \( \sigma \).

Suppose also that

\[
\max_{p \in M} m(\text{Stab}(p)) = k.
\]

Then \( \text{ind} f_0 - \text{ind} f_1 \in \text{Bif}_G^{(k)} \).

**Proof.** Suppose \( \sigma \in \mathcal{R}_G \). By the assumptions, \( \text{Hess}_t f \) is always non-degenerate, and \( f_t \) induces a family of smooth functions on \( M_\sigma / G \). Therefore, the result follows from the classical Cerf theory on \( M_\sigma / G \). In fact, \( \text{ind} f_0 - \text{ind} f_1 \) is contained in the subgroup of \( \mathbb{Z}\mathcal{R}_G \) generated by

\[
\sigma + \sigma \oplus \mathbb{R}
\]

for \( \sigma \in \bigcup_{p \in M} \mathcal{R}_G([\text{Stab}(p)]) \). \( \square \)

We can now finish the proof of Proposition 3.28.

**Proof of Proposition 3.28.** Recall that we defined \( \mathbf{p}_0, \mathbf{p}_1 \in \mathcal{F}^0 \) such that \( \iota(\mathbf{p}_0) = f_0 \), \( \iota(\mathbf{p}_1) = f_1 \), and \( \mathbf{p}_0 \) and \( \mathbf{p}_1 \) are on the same connected component of \( \mathcal{F}^0 \).

Let \( \mathbf{p}_t, t \in [0, 1] \) be a generic path from \( \mathbf{p}_0 \) to \( \mathbf{p}_1 \) that is transverse to all \( \mathcal{F}_\sigma \) and \( \mathcal{F}_{\sigma_1, \sigma_2} \). Let \( f_t = \iota(\mathbf{p}_t) \). By Corollary 3.24, for every \( t_0 \) such that \( f_{t_0} \) is not Morse, the function \( f_{t_0} \) has exactly one degenerate critical orbit \( \text{Orb}(p) \) where \( \text{Hess}_p f_{t_0} / T_p \text{Orb}(p) \) is an irreducible \( \text{Stab}(p) \)-representation.

Let \( \mathcal{S} \) be the set of \( t \in [0, 1] \) such that \( f_t \) is not \( G \)-Morse, then \( \mathcal{S} \) is a closed subset of \( [0, 1] \). For each \( \sigma \in \mathcal{R}_G \), let \( \mathcal{S}_\sigma \) be the subset of \( t \in \mathcal{S} \) such that \( f_t \) is degenerate at a critical orbit \( \text{Orb}(p) \subset M_\sigma \) where \( \ker \text{Hess}_p f_{t_0} / T_p \text{Orb}(p) \) is in the isomorphism class \( \sigma \).

Suppose \( \sigma \) is given by a non-trivial irreducible representation. For each \( t_0 \in \mathcal{S}_\sigma \), let \( \text{Orb}(p) \) be the critical orbit of \( f_{t_0} \), and let \( D \) be a slice of \( p \), let \( D^0 \subset D \) be the fixed-point set of the \( \text{Stab}(p) \) action. Then \( f_{t_0} \) induces a Morse function on \( M_\sigma / G \), therefore for \( \epsilon > 0 \) sufficiently small, there is a unique smooth map

\[
p(t) : (t_0 - \epsilon, t_0 + \epsilon) \to D_0,
\]

such that \( p(t_0) = p \), and \( p(t) \) is a critical point of \( f_t \). Since the path \( \mathbf{p}_t \) intersects \( \mathcal{F}_\sigma \) transversely (or more precisely, it intersects the projection of \( \mathcal{F}_\sigma \) in the proof of Lemma 3.21 transversely), we have

\[
\frac{d}{dt} |_{t=0} \text{Hess}_{p(t)} f_t \neq 0 \text{ on } \ker \text{Hess}_p f \cap S_p.
\]

Therefore \( t_0 \) is an isolated point of \( \mathcal{S} \). As a result, \( \mathcal{S}_\sigma \) is a finite set and is isolated in \( \mathcal{S} \).

Let \( \mathcal{S}' \subset \mathcal{S} \) be the union of \( \mathcal{S}_\sigma \) for all \( \sigma \in \mathcal{R}_G \) that are given by non-trivial representations. Then by the previous argument, we can divide the interval \([0, 1]\) into finitely many sub-intervals, such that each interval is either disjoint from \( \mathcal{S}' \), or contains exactly one point of \( \mathcal{S} \) which is also in \( \mathcal{S}' \). In the former case, the
sub-interval defines a family of $G$–invariant functions that satisfies the conditions of Lemma 3.33; in the latter case, the sub-interval defines a family that satisfies the conditions of Lemma 3.32. Therefore, Proposition 3.28 is proved.

3.6. Invariant counting of critical orbits. This section studies the weighted counting of critical orbits of $G$–Morse functions and prove Theorem 1.2. By Theorem 3.10, suppose

$$w : \mathbb{Z}R_G \to \mathbb{Z}$$

is a homomorphism such that $w = 0$ on $\text{Bif}_G$, then the value of $w(\text{ind } f)$ does not depend on the choice of the $G$–Morse function $f$. We will classify all such functions $w$ by investigating the group structure of $\mathbb{Z}R_G / \text{Bif}_G$. We start with the following definition.

**Definition 3.34.** Let $R_G^{(0)} \subset R_G$ be the subset represented by zero representations.

By definition, $R_G^{(0)}$ is in one-to-one correspondence with $\text{Conj}(G)$.

**Lemma 3.35.** Let $H$ be a compact Lie group, let $k = m(H) \in \mathbb{N}$, and assume $k \succ (0,1)$. Suppose $V$ is a non-trivial, irreducible, orthogonal representation of $H$, and let $V'$ be another representation of $H$. Let $g_1, g_2$ be two $H$–Morse functions on the unit sphere of $V$. Then there exists $k' \prec k$, such that

$$\xi_H(V, V', g_1) - \xi_H(V, V', g_2) \in \text{Bif}_H^{(k')}.$$

**Proof.** By (3.6) and Lemma 3.27, we only need to prove the lemma for $V' = 0$. Let $S(V)$ be the unit sphere of $V$, and let

$$k' = \max_{p \in S(V)} m(\text{Stab}(p)).$$

Since $V$ is a non-trivial irreducible representation, we have

$$k' \prec m(H).$$

By Proposition 3.28, we have

$$\text{ind } g_1 - \text{ind } g_2 \in \text{Bif}_H^{(k')}.$$

Therefore the result follows from the definition of $\xi_H(V, V', g_i)$.

**Lemma 3.36.** Let $H$ be a compact Lie group, let $k = m(H) \in \mathbb{N}$, and assume $k \succ (0,1)$. Suppose $V_1, V_2$ are two non-trivial, irreducible, orthogonal representations of $H$, and let $V'$ be another orthogonal representation of $H$. Let $g_1, g_2$ be $H$–Morse functions on the unit spheres of $V_1, V_2$ respectively. Then there exists $k' \prec k$, such that

$$\xi_H(V_1, V_2 \oplus V', g_1) + \xi_H(V_2, V', g_2) - \xi_H(V_2, V_1 \oplus V', g_2) - \xi_H(V_1, V', g_1) \in \text{Bif}_H^{(k')}.$$

**Proof.** By (3.6) and Lemma 3.27, we only need to prove the lemma for $V' = 0$. Let $V = V_1 \oplus V_2$. Let $B_V(r)$ be the closed ball in $V$ centered at $0$ with radius $r$. Let $f = -\|x\|^2$ be defined on $B_V(2)$. By the constructions of Example 3.2 and Example 3.3, one can obtain an $H$–Morse function $f_1$ from $f$ by two irreducible bifurcations at $0$ such that

1. $\text{ind } f_1 = -\xi_H(V_1, V_2, g_1) - \xi_H(V_2, 0, g_2)$,
2. Hess $f_1$ is positive definite at $0$.

Similarly, by switching the roles of $V_1$ and $V_2$, one obtains a function $f_2$ such that

1. $\text{ind } f_2 = -\xi_H(V_2, V_1, g_2) - \xi_H(V_1, 0, g_1)$,
By definition, \( f_1 \) and \( f_2 \) can be connected by a smooth \( G \)-invariant functions \( f_t, t \in [1, 2] \), such that \( \nabla f_t \neq 0 \) on \( \partial B(2) \).

Let \( k' = \max_{p \in V - \{0\}} m(\text{Stab}(p)) \).

Since \( V \) does not contain trivial component, we have \( k' \prec m(H) \).

By Lemma 3.29 and Proposition 3.28, we have

\[ \xi_H(V_1, V_2, g_1) + \xi_H(V_2, 0, g_2) - \xi_H(V_2, V_1, g_2) - \xi_H(V_1, 0, g_1) = \text{ind} f_2 - \text{ind} f_1 \in \text{Bif}^{(k')} \]

Therefore the lemma is proved.

\[ \square \]

**Theorem 3.37.** The composition of the homomorphisms

\[ \Phi : \mathbb{Z}R_{G}^{(0)} \to \mathbb{Z}R_{G} \to \mathbb{Z}R_{G} / \text{Bif}_{G} \]

is an isomorphism.

**Proof.** We first prove that \( \Phi \) is a surjection. Let \( \pi : \mathbb{Z}R_{G} \to \mathbb{Z}R_{G} / \text{Bif}_{G} \) be the projection map. If \( H \) is a closed subgroup of \( G \) and \( \rho : H \to \text{Hom}(V, V) \) is a representation of \( H \), let \([H, V, \rho] \in R_{G}\) be the element represented by \((H, V, \rho)\).

We deduce a contradiction assuming \( \Phi \) is not surjective. Let \((H, V, \rho)\) a representation such that

\[ \pi([H, V, \rho]) \notin \text{Im} \Phi, \quad (3.12) \]

and choose one with the minimum value of \( m(H) \). If there are multiple such representations with the same value of \( m(H) \), we choose one with the minimum value of \( \dim V \).

If \( \dim V = 0 \), then \([H, V, \rho] \in R_{G}^{(0)}\) and hence (3.12) contradicts the definition of \( \Phi \). If \( \dim V > 0 \), decompose \( V \) as \( V = V_1 \oplus V_2 \), where \( V_1 \) is an irreducible orthogonal representation of \( H \). If \( V_1 \) is trivial, then we have

\[ \pi([H, V, \rho]) = -\pi([H, V_2, \rho|_{V_2}]), \]

therefore \( \pi([H, V_2, \rho|_{V_2}]) \notin \text{Im} \Phi \), contradicting the definition of \( (H, V, \rho) \). If \( V_1 \) is non-trivial, let \( g \) be an \( H \)-Morse function on the unit sphere of \( V_1 \). We have

\[ [H, V, \rho] = i^H_G (\xi_H(V_1, V_2, g)), \]

therefore by the definition of \( \text{Bif}_{G} \), we have

\[ \pi([H, V, \rho]) = \pi \circ i^H_G (\xi_H(V_1, V_2, g)). \]

Notice that \( i^H_G (\xi_H(V_1, V_2, g)) \) is given by a linear combination of \([H, V_2, \rho|_{V_2}]\) and elements of \( R_H \) given by the representations of groups \( K \) with \( m(K) \prec m(H) \), which yields a contradiction to the definition of \( (H, V, \rho) \). In conclusion, the map \( \Phi \) is surjective.

To prove the injectivity of \( \Phi \), we construct a projection

\[ p : \mathbb{Z}R_{G} \to \mathbb{Z}R_{G}^{(0)}, \]

such that \( \ker p \supset \text{Bif}_{G} \), and \( p \) restricts to the identity map on \( \mathbb{Z}R_{G}^{(0)} \). Suppose \([H, V, \rho] \in R_{G}\), we define \( p([H, V, \rho]) \) by induction on \( m(H) \) and \( \dim V \) as follows.

If \( \dim V = 0 \), then \([H, V, \rho] \in R_{G}^{(0)}\), and we define \( p([H, V, \rho]) = [H, V, \rho] \).
If \( \dim V > 0 \), suppose \( p([H', V', \rho']) \) is already defined when \( m(H') \prec m(H) \), and when \( m(H') = m(H) \), \( \dim V' < \dim V \), such that \( p = 0 \) on \( \text{Bif}^{(k')}_{G} \) for all \( k' \prec m(H) \). Decompose \( V \) as \( V = V_1 \oplus V_2 \), where \( V_1 \) is an irreducible orthogonal representation of \( H \). If \( V_1 \) is trivial, then we define
\[
p([H, V, \rho]) := -p([H, V_2, \rho]).
\]
If \( V_1 \) is non-trivial, let \( g \) be an \( H \)–Morse function on the unit sphere of \( V_1 \). Define
\[
p([H, V, \rho]) := p \circ \iota^H_{G} (\xi^+_H(V_1, V_2, g)).
\]
By Lemma 3.35, Lemma 3.36, and the induction hypothesis, the definition of \( p \) does not depend on the choice of the decomposition of \( V \) or the choice of the function \( g \), and \( p \) restricts to zero on \( \text{Bif}^{(m(H))}_{G} \).

**Proof of Theorem 1.2.** By Theorem 3.10 and the constructions of Section 3.1, the sum
\[
\sum_{[\rho] \in \text{Crit}(f)} \eta(\text{ind } p)
\]
is independent of the function \( f \) for all closed \( G \)–manifolds \( M \), if and only if \( \eta \) is identically zero on \( \text{Bif}^{G} \). Therefore the desired result is an immediate consequence of Theorem 3.37. \( \square \)

**Corollary 3.38.** Every map
\[
w : R^G_G(0) \to \mathbb{Z}
\]
can be uniquely extended to a homomorphism
\[
w : \mathbb{Z} R^G_G \to \mathbb{Z},
\]
such that \( w = 0 \) on \( \text{Bif}^{G} \). \( \square \)

### 3.7. Extending \( R^G_G \) over \( \mathbb{R} \)

In the gauge-theoretic setting, the space spanned by the negative eigenvalues of the Hessian is always infinite-dimensional, and one has to use the spectral flow to define an analogue of the equivariant index. However, the spectral flow is in general not gauge invariant, and one way to cancel the gauge ambiguity is to add the spectral flow by a term given by the Chern-Simons functional. Since the Chern-Simons functional takes values in \( \mathbb{R} \), we need to extend the set \( R^G_G \) to a space over \( \mathbb{R} \).

Notice that the definition of \( R^G_G \) can be reformulated as follows. Suppose \( H \) is a compact Lie group, let \( R(H) \) be the representation ring of \( H \). In the following, we will only use the abelian group structure of \( R(H) \). Let \( R^+(H) \subset R(H) \) be the subset of elements given by \([H, V, \rho] - [H, 0]\), where \([H, 0]\) is the isomorphism class of the zero representation of \( H \).

Consider the disjoint union \( \sqcup_H (R(H)) \), where \( H \) runs over all closed subgroups of \( G \). The group \( G \) acts on \( \sqcup_H (R(H)) \) by conjugation as follows. Suppose
\[
\rho : H \to \text{Hom}(V, V)
\]
is a representation of \( H \). Let \( g \in G \). Define the action of \( g \) on \([H, V, \rho]\) to be the isomorphism class of \((g H g^{-1}, V, g(\rho))\), where \( g(\rho) \) is given by
\[
g(\rho) : g H g^{-1} \to \text{Hom}(V, V)
\]
h \mapsto \rho(g^{-1} h g).
Then the conjugation action of \( g \) preserves \( \sqcup_H (\mathcal{R}^+(H)) \), and it extends linearly as maps from \( \mathcal{R}(H) \) to \( \mathcal{R}(gHg^{-1}) \). The set \( \mathcal{R}_G \) equals the quotient set of \( \sqcup_H (\mathcal{R}^+(H)) \) by the conjugation action of \( G \).

**Definition 3.39.** Define \( \mathcal{R}_G^{(0)} \) to be the quotient set of \( \sqcup_H \mathcal{R}(H) \otimes \mathbb{R} \), where \( H \) runs over all closed subgroups of \( G \), by the conjugation action of \( G \).

Let \( \mathcal{R}_G^{(0)} \) be the subset of \( \mathcal{R}_G \) consisting of the elements represented by

\[
a_1 \cdot [H, V_1, \rho_1] + \cdots + a_k \cdot [H, V_k, \rho_k] \in \mathcal{R}(H) \otimes \mathbb{R},
\]

where \( H \) is a closed subgroup of \( G \), and \( (V_i, \rho_i) \) are non-isomorphic irreducible representations of \( H \), and \( a_i \in [0, 1) \).

Suppose \( H \) is a closed subgroup of \( G \), let \( \mathcal{R}_G([H]) \) be the image of \( \sqcup_H \mathcal{R}(H) \otimes \mathbb{R} \) in \( \mathcal{R}_G \).

The conjugation action of \( G \) is trivial on \( \mathcal{R}(G) \otimes \mathbb{R} \). Therefore similar to Definition 2.12, we have a well-defined direct sum operation

\[
\oplus : \mathcal{R}_G \times (\mathcal{R}(G) \otimes \mathbb{R}) \rightarrow \mathcal{R}_G
\]
given as follows. Suppose \( \sigma \in \mathcal{R}_G \) is given by

\[
a_1 \cdot [H, V_1, \rho_1] + \cdots + a_k \cdot [H, V_k, \rho_k] \in \mathcal{R}(H) \otimes \mathbb{R},
\]

and suppose \( \tau \in \mathcal{R}(G) \otimes \mathbb{R} \) be given by

\[
b_1 \cdot [G, W_1, \eta_1] + \cdots + b_s \cdot [G, W_s, \eta_s] \in \mathcal{R}(G) \otimes \mathbb{R},
\]

then \( \sigma \oplus \tau \) is defined to be the element represented by

\[
\sum_{i=1}^k \sum_{j=1}^s a_i b_j \cdot [H \oplus G, V_i \oplus W_j, \rho_i \oplus \eta_j |_{H}] \in \mathcal{R}(H) \otimes \mathbb{R}.
\]

**Definition 3.40.** Suppose \( H \) is a closed subgroup of \( G \), and \( (V, \rho_V) \) is a non-trivial irreducible orthogonal representation of \( H \). Let \( g \) be an \( H \)-Morse function on the unit sphere of \( V \), and let \( \widetilde{V} \in \mathcal{R}(H) \otimes \mathbb{R} \). Define

\[
\xi_H(V, \widetilde{V}, g) := \xi_H(V, 0, g) \oplus \widetilde{V}.
\]

**Definition 3.41.** Let \( \mathbb{Z} \text{Bif}_G \) be the subgroup of \( \mathbb{Z} \mathcal{R}_G \) generated by all the elements given by \( \mathbb{Z} \mathcal{R}_G^{(0)} \) and \( \sigma + \sigma \otimes \mathbb{R} \), where \( \widetilde{V} \in \mathcal{R}(H) \otimes \mathbb{R} \), and \( \sigma \in \mathcal{R}_G \).

The proof of Theorem 3.37 can be easily modified to prove the following result.

**Proposition 3.42.** The composition of the homomorphisms

\[
\Phi : \mathbb{Z} \mathcal{R}_G^{(0)} \hookrightarrow \mathbb{Z} \mathcal{R}_G \twoheadrightarrow \mathbb{Z} \mathcal{R}_G/\text{Bif}_G
\]

is an isomorphism.

4. **Holonomy perturbations and transversality**

The rest of this paper generalizes the results in Section 3 to the gauge-theoretic setting. This section establishes the transversality properties that are analogous to Lemma 3.21 and Lemma 3.23.
4.1. **Preliminaries.** Let $Y$ be a smooth, oriented, closed 3–manifold. Let $G$ be a compact, simply-connected simple Lie group, and let $\mathfrak{g}$ be the Lie algebra of $G$. By Cartan’s theorem [Car52, Bot56], we have

$$
\pi_1(G) = \pi_2(G) = 0, \quad \pi_3(G) \cong \mathbb{Z},
$$

therefore every principal $G$–bundle over $Y$ is trivial. Let $P = Y \times G$ be the trivial bundle. We will abuse notation and also use $\mathfrak{g}$ to denote the trivial $\mathfrak{g}$–bundle over $Y$ when there is no source of confusion.

Fix an integer $k \geq 2$. Let $C$ be the space of $L^2_k$–connections over $P$, then $C$ is an affine space over $L^2_k(T^*Y \otimes \mathfrak{g})$. Let $\mathcal{G}$ be the $L^2_{k+1}$–gauge group of $P$, then $\mathcal{G}$ is identified with the set of $L^2_{k+1}$–maps from $Y$ to $G$.

Let $\theta$ be the trivial connection associated to the product structure (4.2), then $C = \theta + L^2_k(T^*Y \otimes \mathfrak{g})$, and the action of $g \in G$ on $\theta + b \in C$ is given by

$$
g(\theta + b) = \theta + \text{Ad}_g(b) - g^{-1}dg.
$$

By the Sobolev multiplication theorem, $\mathcal{G}$ is a Banach Lie group that acts smoothly on $C$.

By (4.1), we have

$$
\pi_0(\mathcal{G}) \cong \mathbb{Z},
$$

where the group structure on $\pi_0(\mathcal{G})$ is induced from the group structure of $G$. Since $Y$ is oriented, one can fix a canonical choice of the isomorphism (4.3). For $g \in \mathcal{G}$, we will use $\text{deg} g$ to denote the image of $g$ in $\mathbb{Z}$ under this isomorphism.

Fix a Riemannian metric on $Y$, and define the inner product on $\mathfrak{g}$ by the Killing form. The **Chern-Simons functional** on $C$ is given by

$$
\text{CS}(\theta + b) := \frac{1}{2} \langle *d\theta b, b \rangle_{L^2_k} + \frac{1}{3} \langle *[b \wedge b], b \rangle_{L^2_k}.
$$

(4.4)

CS($\theta + b$) is independent of the choice of the Riemannian metric. Let $\text{grad} \text{CS}$ be the formal gradient of $\text{CS}$, then we have

$$
\text{CS}(\theta) = 0,
$$

then we have

$$
(\text{grad} \text{CS})(B) = *F_B \text{ for all } B \in C.
$$

The Chern-Simons functional is only invariant under the identity component of $\mathcal{G}$. In general, there exists a constant $c$ depending on $G$ such that

$$
\text{CS}(g(B)) - \text{CS}(B) = c \text{ deg}(g)
$$

for all $B \in C$ and $g \in \mathcal{G}$. Therefore, the Chern-Simons functional defines a $\mathcal{G}$–invariant map from $C$ to $\mathbb{R}/c\mathbb{Z}$.

We make the following definition analogous to Definition 2.2.

**Definition 4.1.** Suppose $B \in C$. Define

$$
\text{Stab}(B) := \{ g \in \mathcal{G} | g(B) = B \}
$$

and

$$
\text{Orb}(B) := \{ g(B) \in C | g \in \mathcal{G} \}.
$$
Notice that although $G$ is an infinite dimensional group, the stabilizer $\text{Stab}(B)$ is always finite dimensional for any $B \in \mathcal{C}$. In fact, let $y_0 \in Y$ be a fixed point, and let

$$G_{y_0} := \{ g \in G | g = \text{id} \text{ on } y_0 \},$$

then the action of $G_{y_0}$ on $\mathcal{C}$ is free. Since $G_{y_0}$ is a normal subgroup of $G$ and $G/G_{y_0} \cong G$, the stabilizer group $\text{Stab}(B)$ maps isomorphically to a closed subgroup of $G$ by restricting to $y_0$.

The following is a standard result in gauge theory and is analogous to Lemma 2.4. The reader may refer to, for example, [DK90, Section 4.2.1], for more details.

**Proposition 4.2** (Slice theorem). For $B \in \mathcal{C}$, let

$$S_{B,\epsilon} := \{ B + b | d_B^* b = 0, \|b\|^2_{L^2_{x,B}} < \epsilon \},$$

where

$$\|b\|^2_{L^2_{x,B}} := \sum_{i=0}^{k} \|\nabla_i b\|^2_{L^2}.$$

Then for each $B$, there exists $\epsilon > 0$ depending on $B$, such that

$$\mathcal{G} \times_{\text{Stab}(B)} S_{B,\epsilon} \to \mathcal{C}$$

$$[g, B + b] \mapsto g(B + b)$$

is a diffeomorphism onto an open neighborhood of $\text{Orb}(B)$.

### 4.2. Holonomy perturbations

Holonomy perturbations are widely used in gauge theory as a family of perturbations of the Chern-Simons functional. They were first used by Donaldson [Don87] in the study of 4–dimensional Yang-Mills theory and Floer [Flo88] in the construction of instanton Floer homology for 3–manifolds. For our purpose, we briefly review the construction following the notation of [KM11].

**Definition 4.3.** We regard the circle $S^1$ as $\mathbb{R}/\mathbb{Z}$, and let $D^2$ be the open unit disk in the plane. A cylinder datum is a tuple $(q_1, \cdots, q_m, \mu, h)$ with $m \in \mathbb{Z}^+$ that satisfies the following conditions.

1. $q_i : S^1 \times D^2 \to Y$ is a smooth immersion for $i = 1, \cdots, m$;
2. there exists $\epsilon > 0$ such that $q_1, \ldots, q_m$ coincide on $(-\epsilon, \epsilon) \times D^2$;
3. $\mu$ is a non-negative, smooth, compactly supported 2-form on $D^2$, such that

$$\int_{D^2} \mu = 1;$$

4. $h : G^m \to \mathbb{R}$ is a smooth function that is invariant under the diagonal action of $G$ by conjugations.

Suppose $B \in \mathcal{C}$, let $q = (q_1, \cdots, q_m, \mu, h)$ be a cylinder datum, then for $z \in D^2$ and $i = 1, \cdots, m$, the holonomy of $B$ at $q_i(0, z)$ along $q_i(S^1 \times \{z\})$ defines a map on the fiber $P_{q_i(0, z)}$. Under the trivialization of $P$ given by (4.2), this map is given by a left multiplication of an element in $G$, and we use $\text{Hol}_{q_i,z}(B) \in G$ to denote this element. Therefore we have a map

$$\text{Hol}_q(B) : D^2 \to G^m$$

$$z \mapsto (\text{Hol}_{q_1,z}(B), \cdots, \text{Hol}_{q_m,z}(B)).$$
The *cylinder function* associated to \( q \) is defined to be
\[
f_q : C \to \mathbb{R}
\]
\[
B \mapsto \int_{D^2} h(\text{Hol}_q(B)) \mu.
\]
By definition, \( f_q \) is a \( G \)-invariant function on \( C \).

Let \( T \) be the tangent bundle of \( C \). For \( B \in C \), recall that the *formal gradient* of \( f_q \) at \( B \) is defined to be the unique vector
\[
b \in T|_B = L^2_k(T^*Y \otimes g)
\]
such that for every \( b' \in L^2_k(T^*Y \otimes g) \), we have
\[
\frac{d}{dt} f_q(B + tb') = \langle b, b' \rangle_{L^2}.
\]
By [KM11, Proposition 3.5 (i)], the formal gradient of \( f_q \) exists and is a smooth section of \( T \). We use \( \text{grad} f_q \) to denote the formal gradient of \( f_q \).

Let \( \{q_i\}_{i \in \mathbb{N}} \) be a fixed sequence of cylinder data, such that for every cylinder datum
\[
q = (q_1, \ldots, q_m, \mu, h),
\]
there is a subsequence of \( \{q_i\}_{i \in \mathbb{N}} \) so that it consists of elements with the same value of \( m \) and converges to \( q \) in \( C^\infty \). By the discussion before [KM11, Definition 3.6], there exists a sequence \( \{C_i\}_{i \in \mathbb{N}} \) of positive real numbers, such that for every sequence \( \{a_i\}_{i \in \mathbb{N}} \subset \mathbb{R} \) satisfying
\[
\sum |a_i| C_i < +\infty,
\]
the series \( \sum a_i f_{q_i} \) converges to a smooth function on \( C \), and the series
\[
\sum a_i \text{grad} f_{q_i}
\]
converges to a smooth section of \( T \) that equals the formal gradient of \( \sum a_i f_{q_i} \).

**Definition 4.4.** Let \( P \) be the Banach space of sequences of real numbers \( \{a_i\}_{i \in \mathbb{N}} \) such that \( \sum |a_i| C_i < +\infty \). For \( \pi = \{a_i\}_{i \in \mathbb{N}} \in P \), define its norm in \( P \) by
\[
\|\pi\|_P := \sum |a_i| C_i.
\]
Define
\[
f_\pi := \sum a_i f_{q_i},
\]
\[
V_\pi := \sum a_i \text{grad} f_{q_i}.
\]

For \( \pi \in P \), let \( DV_\pi \) be the derivative of \( V_\pi \). Then for \( B \in C \), the derivative \( DV_\pi(B) \) defines a linear endomorphism on
\[
T|_B = L^2_k(T^*Y \otimes g).
\]
Let
\[
\text{Sym}(T|_B)
\]
be the Banach space of linear endomorphism on \( T|_B \) that are bounded with respect to the \( L^2_k \)-norm and symmetric with respect to the \( L^2 \)-norm, and define the norm on \( \text{Sym}(T|_B) \) as the \( L^2 \)-operator norm. Then \( \text{Sym}(T|_B) \) for \( B \in C \) form a (trivial) Banach vector bundle \( \text{Sym}(T) \) over \( C \). By [KM11, Proposition 3.7 (ii)], the map \( B \mapsto DV_\pi(B) \) is a smooth section of \( \text{Sym}(T) \).
Fix a base point $y_0$ on $Y$, and let $\gamma$ be a smooth arc from $y_0$ to $y \in Y$. For $B \in \mathcal{C}$, the holonomy of $B$ along $\gamma$ is a map from $P|_{y_0}$ to $P|_y$. Since $P$ is the trivial bundle, the map is given by the left multiplication of an element in $G$. We use

$$\text{Hol}_\gamma(B) \in G$$

to denote the corresponding element. Suppose $b \in L^2_k(T^*Y \otimes \mathfrak{g})$, then

$$\frac{d}{dt} \bigg|_{t=0} \text{Hol}_\gamma(B + tb)$$

defines a tangent vector of $G$ at $\text{Hol}_\gamma(B) \in G$.

The rest of this subsection proves that the holonomy perturbations are sufficiently flexible so that it can realize any $G$-invariant jet on any finite dimensional subspace of $\mathcal{C}$. The precise statement is given by Proposition 4.9. This property will be used in the transversality argument in Section 4.4.

**Lemma 4.5.** Let $B_1, B_2 \in \mathcal{C}$, and suppose there exists $u \in G$, such that

$$\text{Hol}_\gamma(B_1) = u \text{Hol}_\gamma(B_2)u^{-1}$$

(4.5)

for all immersed loops $\gamma$ that start and end at $y_0$. Then there exists $g \in \mathcal{G}$, such that $g(B_1) = B_2$.

**Proof.** Take $g_0 \in \mathcal{G}$ such that $g_0|_{y_0} = u$, then

$$\text{Hol}_\gamma(g_0(B_1)) = u \text{Hol}_\gamma(B_1)u^{-1}$$

for all loops $\gamma$ based at $y_0$. Therefore, by replacing $B_1$ with $g_0(B_1)$, we may assume without loss of generality that $u = \text{id} \in G$.

Define a gauge transformation $g$ of $P$ as follows. For $y \in Y$, take an arc $\gamma$ from $y_0$ to $y$, and define the value of $g$ at $y$ to be

$$g|_y = \text{Hol}_\gamma(B_2) \cdot \text{Hol}_\gamma(B_1)^{-1}.$$ 

Since (4.5) holds with $u = \text{id}$, the definition of $g$ is independent of the choice of $\gamma$, and we have $g(B_1) = B_2$. Since $B_1$ and $B_2$ are both $L^2_k$-connections, the standard regularity argument implies that $g$ is a $L^2_{k+1}$-gauge transformation, and hence $g \in \mathcal{G}$.

The following two lemmas prove an infinitesimal version of Lemma 4.5.

**Lemma 4.6.** Let $B \in \mathcal{C}$. Suppose $b \in L^2_k(T^*Y \otimes \mathfrak{g})$ satisfies

$$\frac{d}{dt} \bigg|_{t=0} \text{Hol}_\gamma(B + tb) = 0$$

(4.6)

for all immersed loops $\gamma$ that start and end at $y_0$, then there exists $s \in L^2_k(\mathfrak{g})$, such that $s(y_0) = 0$ and $d_{sB}s = b$.

**Proof.** We define the section $s$ as follows. Let $y \in Y$, take a smooth arc $\gamma$ from $y_0$ to $y$, and let $s(y) \in \mathfrak{g}$ be the unique element such that

$$\frac{d}{dt} \bigg|_{t=0} \text{Hol}_\gamma(B + tb) = \frac{d}{dt} \bigg|_{t=0} \left( \exp \left( -t s(y) \right) \cdot \text{Hol}_\gamma(B) \right).$$

(4.7)

By (4.6), the value of $s(y)$ does not depend on the choice of the arc $\gamma$. By the Sobolev embedding theorems, $B - \theta$ is continuous, therefore (4.7) implies that $s$ is $C^1$ and $d_{sB}s = b$. It then follows from the standard bootstrapping argument that $s \in L^2_k(\mathfrak{g})$. 

\[\square\]
Lemma 4.7. Let $B \in \mathcal{C}$, $\xi \in \mathfrak{g}$. Suppose $b \in L^2_k(T^*Y \otimes \mathfrak{g})$ satisfies
\[
\left. \frac{d}{dt} \right|_{t=0} \text{Hol}_{\gamma}(B + tb) = \left. \frac{d}{dt} \right|_{t=0} (\exp(t\xi) \cdot \text{Hol}_{\gamma}(B) \cdot \exp(-t\xi)) 
\] (4.8)
for all immersed loops $\gamma$ that start and end at $y_0$, then there exists $s \in L^2_{k+1}(\mathfrak{g})$, such that $s(y_0) = \xi$, and $d_B s = b$.

Proof. Let $g_t \in \mathcal{G}$, $t \in (-1, 1)$, be a smooth family of gauge transformations such that $g_0 = \text{id}$ and $\left. \frac{d}{dt} \right|_{t=0} g_t(y_0) = \xi$.

Let $s_0 = \left. \frac{d}{dt} \right|_{t=0} g_t$, then $s_0$ is a smooth section of $\mathfrak{g}$. Let $b' = b - d_B s_0$,

then
\[
\left. \frac{d}{dt} \right|_{t=0} \text{Hol}_{\gamma}(B + tb') = 0,
\]
and the desired result follows from Lemma 4.6. $\square$

Lemma 4.5 and Lemma 4.7 have the following immediate consequence.

Lemma 4.8. Let $m \in \mathbb{Z}^+$, let $G$ act on $G^m$ diagonally by conjugation on each component. Suppose $B_1, B_2 \in \mathcal{C}$ are not gauge equivalent. For $i = 1, 2$, let $L_i \subset T|_{B_i} = L^2_k(T^*Y \otimes \mathfrak{g})$ be a finite dimensional linear space such that

$L_i \cap \text{Im} \ d_B_i = \{0\}$.

Then there exists
\[
\vec{\gamma} = (\gamma_1, \cdots, \gamma_m),
\]
where each $\gamma_j$ is an immersed loop in $Y$ based at $y_0$, such that the map
\[
\text{Hol}_{\vec{\gamma}} : \mathcal{C} \to G^m
\]
\[
B \mapsto (\text{Hol}_{\gamma_1}(B), \cdots, \text{Hol}_{\gamma_m}(B))
\]
satisfies the following conditions:

(1) $\text{Hol}_{\vec{\gamma}}(B_1) \neq \text{Hol}_{\vec{\gamma}}(B_2)$ in the quotient set $G^m/G$,

(2) for $i = 1, 2$, let $p_i = \text{Hol}_{\gamma_i}(B_i)$, then the tangent map of $\text{Hol}_{\vec{\gamma}}$ maps $L_i$ injectively into $T_{p_i}G^m/T_{p_i}\text{Orb}(p_i)$.

Proof. Since $B_1$ and $B_2$ are not gauge equivalent, by Lemma 4.5, there exists
\[
\vec{\gamma}^{(0)} = (\gamma_1^{(0)}, \cdots, \gamma_m^{(0)}),
\]
where each $\gamma_j$ is an immersed loop in $Y$ based at $y_0$, such that $\text{Hol}_{\gamma^{(0)}}(B_1)$ and $\text{Hol}_{\gamma^{(0)}}(B_2)$ are not conjugate in $G^{m_0}$.

By Lemma 4.7, for $i = 1, 2$, there exists
\[
\vec{\gamma}^{(i)} = (\gamma_1^{(i)}, \cdots, \gamma_m^{(i)}),
\]
where each \( \gamma_j \) is an immersed loop in \( Y \) based at \( y_0 \), such that if we write

\[
p'_i := \text{Hol}_{p'_i}(B_i) \in G^{m_i},
\]
then the tangent map of \( \text{Hol}_{p'_i} \) maps \( L_i \) injectively into \( T_{p'_i}G^{m_i}/T_{p'_i}\text{Orb}(p_i) \).

As a consequence, the tuple

\[
\gamma = (\gamma_1^{(0)}, \cdots, \gamma_{m_1}^{(0)}), \gamma_1^{(1)}, \cdots, \gamma_{m_1}^{(1)}, \gamma_1^{(2)}, \cdots, \gamma_{m_2}^{(2)}
\]
satisfies the desired conditions. \( \square \)

Lemma 4.8 implies the following property of holonomy perturbations, which will be used in the transversality argument in Section 4.4.

**Proposition 4.9.** Suppose \( B_1, B_2 \in C \) are not gauge equivalent. For \( i = 1, 2 \), let

\[
L_i \subset T|B_i = L^2_i(T^*Y \otimes g)
\]
be finite dimensional linear spaces that are invariant under the action of \( \text{Stab}(B) \), such that

\[
L_i \cap \text{Im } d_{B_i} = \{0\}.
\]
We further assume that \( B_1, B_2 \) are smooth and that \( L_1, L_2 \) are spanned by smooth sections of \( T^*Y \otimes g \).

For \( r \in \mathbb{Z}^+ \), let \( J_i(r) \) be the linear space of \( r \)-jets of \( \text{Stab}(B_i) \)-invariant functions on the affine space \( B_i + L_i \) at \( B_i \), then every \( \pi \in P \) defines an element in \( J_i(r) \) by restricting \( f_\pi \) to \( B_i + L_i \). Let

\[
\Phi : P \to J_1(r_1) \oplus J_2(r_2)
\]
be given by the restriction maps. Then \( \Phi \) is surjective for all \( r_1, r_2 \).

**Proof.** Since \( J_1(r_1) \oplus J_2(r_2) \) is a finite dimensional linear space and (4.9) is a linear map, we only need to show that the image is dense.

Suppose \( (j_1, j_2) \in J_1(r_1) \oplus J_2(r_2) \). As in Lemma 4.8, for \( m \in \mathbb{Z}^+ \), let \( G \) act on \( G^m \) diagonally by taking conjugation at each component. By Lemma 4.8, there exists

\[
\overline{\gamma} = (\gamma_1, \cdots, \gamma_m)
\]
where each \( \gamma_j \) is an immersed loop in \( Y \) based at \( y_0 \), and a smooth \( G \)-invariant function \( h \) on \( G^m \), such that the map

\[
\text{Hol}_\overline{\gamma} : C \to G^m
\]

\[
B \mapsto (\text{Hol}_{\gamma_1}(B), \cdots, \text{Hol}_{\gamma_m}(B))
\]
satisfies the statement of Lemma 4.8. Since \( B_1, B_2 \) are smooth and that \( L_1, L_2 \) are spanned by smooth sections of \( T^*Y \otimes g \), the map \( \text{Hol}_\overline{\gamma} \) is smooth on the affine spaces \( B_i + L_i \) \( (i = 1, 2) \). Therefore, by Lemma 2.26, there exists a smooth \( G \)-invariant function

\[
h : G^m \to \mathbb{R},
\]
such that the map

\[
f_{\overline{\gamma}, h} : C \to \mathbb{R}
\]

\[
B \mapsto h(\text{Hol}_{\gamma_1}(B), \cdots, \text{Hol}_{\gamma_m}(B))
\]
restricts to \( (j_1, j_2) \) in \( J_1(r_1) \oplus J_2(r_2) \). For \( i = 1, \cdots, m \), let the map \( q_i \) be given by

\[
q_i : S^1 \times D^2 \to Y
\]

\[
(s, z) \mapsto \gamma_i(s),
\]
let $\mu$ be an arbitrary smooth, non-negative, compactly supported 2-form on $D^2$ that integrates to 1, and let $h$ be given by (4.10). Recall that $\{q_i\}_{i \in \mathbb{N}}$ is the sequence of cylinder data used in Definition 4.4. Then by definition, there exists a subsequence of $\{q_i\}_{i \in \mathbb{N}}$ that converges to $(q_1, \cdots, q_m, \mu, h)$ in $C^\infty$. Let $\{q_n\}_{i \in \mathbb{N}}$ be such a subsequence, then we have
\[
\lim_{i \to \infty} \Phi(f_{q_{n_i}}) = (j_1, j_2),
\]
and hence the result is proved. \hfill \square

4.3. Hessians of perturbed flat connections. This subsection defines the Hessians of perturbed Chern-Simons functionals at critical points.

Let $\pi \in \mathcal{P}$, let $\text{grad}(CS + f_\pi)(B)$ be the formal gradient of $CS + f_\pi$ at $B$. Then
\[
\text{grad}(CS + f_\pi)(B) = *F_B + V_\pi(B),
\]
and the derivative of $\text{grad}(CS + f_\pi)$ at $B$ is given by the operator $*d_B +DV_\pi(B)$.

**Definition 4.10.** A connection $B$ is called $\pi$-flat, if
\[
*F_B + V_\pi(B) = 0.
\]

**Lemma 4.11.** If $B$ is $\pi$-flat, then the operator $*d_B +DV_\pi(B)$ is identically zero on $\text{Im}d_B \cap L^2_k(T^*Y \otimes \mathfrak{g})$.

**Proof.** Notice that $\text{Im}d_B \cap L^2_k(T^*Y \otimes \mathfrak{g})$ is the tangent space of $\text{Orb}(B)$. Suppose $s \in \text{Im}d_B \cap L^2_k(T^*Y \otimes \mathfrak{g})$, let $B(t)$ be a smooth curve in $\text{Orb}(B)$ such that
\[
\frac{d}{dt}B(t) = s \quad \text{at } t = 0.
\]
Then by the gauge invariance of Equation (4.11), we have
\[
*F_{B(t)} + V_\pi(B(t)) = 0
\]
for all $t$, therefore
\[
d Bs + DV_\pi(B)(s) = \left. \frac{d}{dt} \right|_{t=0} \left( *F_{B(t)} + V_\pi(B(t)) \right) = 0,
\]
and hence the lemma is proved. \hfill \square

Suppose $B \in \mathcal{C}$ and $\pi \in \mathcal{P}$. Define the operator
\[
K_{B,\pi} : L^2_k(\mathfrak{g}) \oplus L^2_k(T^*Y \otimes \mathfrak{g}) \to L^2_k(\mathfrak{g}) \oplus L^2_{k-1}(T^*Y \otimes \mathfrak{g})
\]
by
\[
K_{B,\pi}(\xi, b) := (d_B^*b, dB\xi + *d_Bb + DV_\pi(B)(b)).
\]
Then $K_{B,\pi}$ is self-adjoint and elliptic, therefore it is Fredholm with index zero, and its spectrum is discrete and is contained in $\mathbb{R}$.

The domain of $K_{B,\pi}$ can be orthogonally decomposed as
\[
L^2_k(\mathfrak{g}) \oplus \left( \text{Im}d_B \cap L^2_k(T^*Y \otimes \mathfrak{g}) \right) \oplus (\ker d_B^* \cap L^2_k(T^*Y \otimes \mathfrak{g})),
\]
and range of $K_{B,\pi}$ can be orthogonally decomposed as
\[
L^2_{k-1}(\mathfrak{g}) \oplus \left( \text{Im}d_B \cap L^2_{k-1}(T^*Y \otimes \mathfrak{g}) \right) \oplus (\ker d_B^* \cap L^2_{k-1}(T^*Y \otimes \mathfrak{g})).
\]
If $B$ is $\pi$-flat, then by Lemma 4.11 and the fact that $K_{B,\pi}$ is self-adjoint, the operator $K_{B,\pi}$ is given by
\[
\begin{pmatrix}
0 & d_B^* & 0 \\
-d_B & 0 & 0 \\
0 & 0 & *d_B + DV_\pi(B)
\end{pmatrix}
\]
(4.15)
under the decompositions (4.13) and (4.14).

**Definition 4.12.** Suppose $B$ is $\pi$–flat, define $\text{Hess}_{B,\pi}$ to be the operator

$$\text{Hess}_{B,\pi} : \ker d_B^* \cap L^2_k(T^*Y \otimes g) \to \ker d_B^* \cap L^2_{k-1}(T^*Y \otimes g)$$

given by

$$\text{Hess}_{B,\pi} := *d_B + DV_{\pi}(B).$$

By definition, $\text{Hess}_{B,\pi}$ is a self-adjoint Fredholm operator with index zero.

**Definition 4.13.** We say that a $\pi$–flat connection $B$ non-degenerate, if $\text{Hess}_{B,\pi}$ is an isomorphism. We say that $\pi \in \mathcal{P}$ is non-degenerate, if all critical points of $CS + f_\pi$ are non-degenerate as $\pi$–flat connections.

### 4.4. Equivariant transversality

This subsection establishes the transversality properties of holonomy perturbations that are analogous to Lemma 3.21 and Lemma 3.23.

Recall that for each $y \in Y$, the restriction to $y$ gives a map $r_y : \text{Stab}(B) \to G$ that sends $\text{Stab}(B)$ to a closed subgroup of $G$. Suppose $y_1, y_2 \in Y$, let $\gamma$ be an arc from $y_1$ to $y_2$. Then

$$r_{y_2}(g) = \text{Hol}_\gamma(B) \cdot r_{y_1}(g) \cdot \text{Hol}_\gamma(B)^{-1}$$
for all $g \in \text{Stab}(B)$.

As a consequence, every finite dimensional representation of $\text{Stab}(B)$ defines an element of $R_G$.

Suppose $\pi \in \mathcal{P}$ and let $\text{Orb}(B)$ be a critical orbit of $CS + f_\pi$, then $\ker \text{Hess}_{B,\pi}$ defines an element in $R_G$ as a $\text{Stab}(B)$–representation. It is straightforward to verify that this element is invariant under gauge transformations of $B$.

**Definition 4.14.** A linear map $P$ defined on a linear subspace of $L^2(T^*Y \otimes g)$ is called symmetric, if $\langle Px, y \rangle = \langle x, Py \rangle$ for all $x, y$ in the domain of $P$.

Let $\mathcal{H}(B)$ be the Banach space of $\text{Stab}(B)$–invariant, symmetric, bounded, linear operators from the Banach space

$$\ker d_B^* \cap L^2_k(T^*Y \otimes g)$$
to the Banach space

$$\ker d_B^* \cap L^2_{k-1}(T^*Y \otimes g).$$

Define the norm on $\mathcal{H}(B)$ as the operator norm, then $\mathcal{H}(B)$ becomes a Banach space. For $\sigma \in R_G$, let $\mathcal{H}_\sigma(B) \subset \mathcal{H}(B)$ consist of the elements that are Fredholm with index 0 such that their kernels represent $\sigma$.

Recall that the dimension $d(\sigma)$ is defined by Definition 3.16. Suppose $s \in \mathcal{H}_\sigma(B)$, recall that we use $\text{Sym}_{\text{Stab}(B)} \ker s$ to denote the space of $\text{Stab}(B)$–equivariant symmetric maps on $\ker s$.

**Lemma 4.15.** $\mathcal{H}_\sigma(B)$ is a submanifold of $\mathcal{H}(B)$ with codimension $d(\sigma)$. Moreover, suppose $s \in \mathcal{H}_\sigma(B)$, let $\Pi$ be the $L^2$–orthogonal projection from

$$\ker d_B^* \cap L^2_k(T^*Y \otimes g)$$
to $\ker s$. Suppose $L \subset \mathcal{H}(B)$ is a linear subspace. Then $s + L$ is transverse to $\mathcal{H}_\sigma(B)$ if and only if the linear map

$$L \to \text{Sym}_{\text{Stab}(B)} \ker s$$
$l \mapsto \Pi \circ (l|_{\ker s})$

is surjective.

Proof. The proof is similar to the argument of Lemma 3.18. Suppose $s \in \mathcal{H}_{\sigma}(B)$. For $i = k, k - 1$, let $V^{(i)}$ be the $L^2$-orthogonal complement of $\ker s$ in $\ker d_B^* \cap L^2(T^*Y \otimes g)$. Then the domain and range of $s$ decompose as $\ker s \oplus V^{(k)}$ and $\ker s \oplus V^{(k-1)}$ respectively. By the assumptions, the map $s$ restricts to an isomorphism from $V^{(k)}$ to $V^{(k-1)}$.

There exists an open neighborhood $U$ of $s$ in $\mathcal{H}(B)$ such that all $s' \in U$ are Fredholm and have index zero. Suppose $s'$ decomposes as a map from $\ker s \oplus V^{(k)}$ to $\ker s \oplus V^{(k-1)}$ as

$$s' = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

then after shrinking $U$ if necessary, the map

$$S_{22} : V^{(k)} \to V^{(k-1)}$$

is always invertible. By the same computation as (3.7), we have $s' \in \mathcal{H}_{\sigma}(B)$ if and only if

$$S_{11} - S_{12} \circ S^{-1}_{22} \circ S_{21} = 0.$$

Therefore $\mathcal{H}_{\sigma}(B)$ is a submanifold of $\mathcal{H}(B)$ near $s$, and its tangent space at $s$ is given by $S_{11} = 0$. Hence the lemma is proved. $\square$

**Definition 4.16.** Suppose $\sigma \in \mathcal{R}_G$. Let

$$\mathcal{P}_{\sigma} := \{ \pi \in \mathcal{P} | \exists B \in \mathcal{C} \text{ such that } *F_B + V_\pi = 0, \text{ and } \ker \text{Hess}_{B,\pi} \text{ represents } \sigma \}.$$

**Lemma 4.17.** $\mathcal{P}_{\sigma}$ is a $C^\infty$-subvariety of $\mathcal{P}$ with codimension at least $d(\sigma)$.

Recall that the concept of $C^\infty$-subvariety was introduced by Definition 3.19.

Proof. Let

$$\widetilde{\mathcal{P}}_{\sigma} := \{ (\pi, B) \in \mathcal{P} \times \mathcal{C} | *F_B + V_\pi = 0, \text{ and } \ker \text{Hess}_{B,\pi} \text{ represents } \sigma \}.$$

Suppose $(\pi, B) \in \widetilde{\mathcal{P}}_{\sigma}$. By elliptic regularity, $B$ is smooth and $\ker \text{Hess}_{B,\pi}$ is spanned by smooth sections of $T^*Y \otimes g$ after a gauge transformation.

Let $S_{B,\epsilon}$ be a slice of $B$ given by Proposition 4.2. We claim that there exists an open neighborhood $U$ of $(\pi, B)$ in $\mathcal{P} \times S_{B,\epsilon}$, such that

1. $\widetilde{\mathcal{P}}_{\sigma} \cap U$ is a Banach manifold,
2. The projection of $\widetilde{\mathcal{P}}_{\sigma} \cap U$ to $\mathcal{P}$ is Fredholm and has index $-d(\sigma)$.

The result then follows from the above claim and the separability of $\mathcal{P} \times \mathcal{C}$.

To prove the claim, let $S^0_{B,\epsilon}$ be fixed-point subspace of $S_{B,\epsilon}$ under the action of $\text{Stab}(B)$, and let $S^0_{B,\epsilon}$ be the closure of $S^0_{B,\epsilon}$ in $L^2(T^*Y \otimes g)$. Then

$$\widetilde{\mathcal{P}}_{\sigma} \cap \mathcal{P} \times S_{B,\epsilon} = \widetilde{\mathcal{P}}_{\sigma} \cap \mathcal{P} \times S^0_{B,\epsilon}. \quad (4.16)$$

Suppose $B' \in S^0_{B,\epsilon}$, then we have $\text{Stab}(B') = \text{Stab}(B)$. Suppose $i = k$ or $k - 1$, then the spaces

$$\ker d_B^* \cap L^2(T^*Y \otimes g)$$
form a (trivial) \( \text{Stab}(B) \)-equivariant Banach vector bundle over \( S^0_{B,\epsilon} \). We fix a \( \text{Stab}(B) \)-equivariant trivialization of this vector bundle near \( B \). One way of constructing the trivialization is to take the \( L^2 \)-orthogonal projection onto \( \ker d^*_B \cap L^2_1(T^* Y \otimes g) \).

Under the trivialization above, the set (4.16) is given by the pre-image of \( \{0\} \times \mathcal{H}_\sigma(B) \) of the map

\[
\varphi : \mathcal{P} \times S^0_{B,\epsilon} \to \widehat{S}_{B,\epsilon} \times \mathcal{H}(B) \\
(\pi', B') \mapsto (\ast F_{B'} + V_{\pi'}, \text{Hess}_{B',\pi'}). 
\]

Therefore it follows from Lemma 4.15 and Proposition 4.9 that

\[
\varphi^{-1}(\{0\} \times \mathcal{H}_\sigma(B))
\]

is a Banach manifold near \((\pi, B)\).

We now compute the tangent map of the projection of \( \varphi^{-1}(\{0\} \times \mathcal{H}_\sigma(B)) \) to \( \mathcal{P} \). Let \( \Pi \) be the \( L^2 \)-orthogonal projection to \( \ker \text{Hess}_{B,\pi} \), and let \( \Pi^\perp = \text{id} - \Pi \). For each \( \eta \in \mathcal{P} \), let \( j_1(\eta) = \Pi(V_\eta) \), let

\[
j_2(\eta) = \Pi \circ (DV_\eta|_{\ker \text{Hess}_{B,\pi}}),
\]

then \( j_2(\eta) \) is a symmetric endomorphism on \( \ker \text{Hess}_{B,\pi} \). Let \( T^0 \), \( \tilde{T}^0 \) be the tangent spaces of \( S^0_{B,\epsilon}, S^0_{B,\epsilon} \) at \( B \) respectively. By Lemma 4.15, the tangent space of \( \varphi^{-1}(\{0\} \times \mathcal{H}_\sigma(B)) \) at \((\pi, B)\) is given by

\[
\tilde{T} = \{(\eta, b) \in \mathcal{P} \times T^0_\mathcal{P} | \text{Hess}_{B,\pi}(b) + V_\eta = 0, \Pi \circ \ast \text{ad}(b) + j_2(\eta) = 0 \}.
\]

Decompose \( b \) as \( b_1 + b_2 \) where \( b_1 = \Pi(b) \) and \( b_2 = \Pi^\perp(b) \), and let \( \text{Hess}_{B,\pi}^{-1} \) be the inverse of \( \text{Hess}_{B,\pi} \) on \( (\ker \text{Hess}_{B,\pi})^\perp \). Then \((\eta, b) \in \tilde{T} \) if and only if

1. \( j_2(\eta) = -\Pi \circ \ast \text{ad}(b) \),
2. \( j_1(\eta) = 0 \),
3. \( b_2 = -\text{Hess}_{B,\pi}^{-1}\Pi^\perp(V_\eta) \).

Therefore by Proposition 4.9, the linear map

\[
\tilde{T} \to \mathcal{P} \times \ker \text{Hess}_{B,\pi} \\
(\eta, b) \mapsto (\eta, b_1)
\]

is injective with closed image, and the codimension of its image equals \( \dim \ker \text{Hess}_{B,\pi} + d(\sigma) \).

Since the projection from \( \mathcal{P} \times \ker \text{Hess}_{B,\pi} \) to \( \mathcal{P} \) is Fredholm and with index equal to \( \dim \ker \text{Hess}_{B,\pi} \), we conclude that the projection of \( \tilde{T} \) to \( \mathcal{P} \) is Fredholm with index \(-d(\sigma)\), and hence the result is proved. \( \square \)

**Definition 4.18.** Suppose \( \sigma_1, \sigma_2 \in \mathcal{R}_G \). Let \( \mathcal{P}_{\sigma_1,\sigma_2} \) to be the set of \( \pi \in \mathcal{P} \), such that there exist \( B_1, B_2 \in \mathcal{C} \) with the following properties:

1. \( \text{Orb}(B_1) \neq \text{Orb}(B_2) \),
2. \( \ast F_{B_1} + V_{\pi_1} = 0 \), \( \ast F_{B_2} + V_{\pi_2} = 0 \),
3. \( \ker \text{Hess}_{B_1,\pi} \) represents \( \sigma_1 \),
4. \( \ker \text{Hess}_{B_2,\pi} \) represents \( \sigma_2 \).

**Lemma 4.19.** Suppose \( \sigma_1, \sigma_2 \in \mathcal{R}_G \). Then \( \mathcal{P}_{\sigma_1,\sigma_2} \) is a \( C^\infty \)-subvariety of \( \mathcal{P} \) with codimension at least \( d(\sigma_1) + d(\sigma_2) \).
Proof. The proof is essentially the same as Lemma 4.17. Let
\[ \mathcal{P}_{\sigma_1, \sigma_2} := \{ (\pi, B_1, B_2) \in \mathcal{P} \times \mathcal{C} \times \mathcal{C} \mid \text{Orb}(B_1) \neq \text{Orb}(B_2), \]
\[ \ast F_{B_1} + V_{\pi_1} = 0, \text{ ker Hess}_{B_1, \pi} \text{ represents } \sigma_1, \]
\[ \ast F_{B_2} + V_{\pi_2} = 0, \text{ ker Hess}_{B_2, \pi} \text{ represents } \sigma_2 \}. \]

Suppose \((\pi, B_1, B_2) \in \mathcal{P}_{\sigma_1, \sigma_2}\). By elliptic regularity, \(B_i (i = 1, 2)\) are smooth and \(\text{ker Hess}_{B_i, \pi}\) are spanned by smooth sections of \(T^* Y \otimes \mathfrak{g}\) after gauge transformations.

For \(i = 1, 2\), let \(S_{B_i, \epsilon_i}\) be a slice of \(B_i\) given by Proposition 4.2. We claim that there exists an open neighborhood \(U \) of \((\pi, B_1, B_2)\) in \(\mathcal{P} \times S_{B_1, \epsilon_1} \times S_{B_2, \epsilon_2}\), such that

1. \( \mathcal{P}_{\sigma_1, \sigma_2} \cap U \) is a Banach manifold,
2. The projection of \( \mathcal{P}_{\sigma_1, \sigma_2} \cap U \) to \( \mathcal{P} \) is Fredholm and has index \(-d(\sigma_1) - d(\sigma_2)\).

The result then follows from the above claim and the separability of \( \mathcal{P} \times \mathcal{C} \times \mathcal{C} \).

To prove the claim, let \( S_{B_1, \epsilon_1}^0, S_{B_2, \epsilon_2}^0 \) be as in the proof of Lemma 4.17. Then
\[ \mathcal{P}_{\sigma_1, \sigma_2} \cap \mathcal{P} \times S_{B_1, \epsilon_1} \times S_{B_2, \epsilon_2} = \mathcal{P}_{\sigma_1, \sigma_2} \cap \mathcal{P} \times S_{B_1, \epsilon_1}^0 \times S_{B_2, \epsilon_2}^0, \]
and it is given by the pre-image of \(\{0\} \times \mathcal{H}_{\sigma_1}(B_1) \times \{0\} \times \mathcal{H}_{\sigma_2}(B_2)\) of the map
\[ \varphi : \mathcal{P} \times S_{B_1, \epsilon_1}^0 \times S_{B_2, \epsilon_2}^0 \to \mathcal{P}_{\sigma_1, \sigma_2} \cap \mathcal{P} \times S_{B_1, \epsilon_1} \times S_{B_2, \epsilon_2}, \]
\[ \varphi(\pi', B_1', B_2') = (\pi, B_1, B_2, \ast F_{B_1} + V_{\pi_1}, \ast F_{B_2} + V_{\pi_2}, \text{ Hess}_{B_1, \pi_1}, \text{ Hess}_{B_2, \pi_2}). \]

By Lemma 4.15 and Proposition 4.9,
\[ \varphi^{-1}(\{0\} \times \mathcal{H}_{\sigma_1}(B_1) \times \{0\} \times \mathcal{H}_{\sigma_2}(B_2)) \]
(4.17)
is a Banach manifold near \((\pi, B_1, B_2)\). By the same argument as the proof of Lemma 4.17, the projection of (4.17) to \( \mathcal{P} \) is Fredholm with index \(-d(\sigma_1) - d(\sigma_2)\) near \((\pi, B_1, B_2)\).

Lemma 4.17 and Lemma 4.19 have the following immediate corollaries. Recall that by Definition 4.13, \(\pi \in \mathcal{P}\) is called non-degenerate if all the critical points of \( CS + f_\pi \) are non-degenerate.

**Corollary 4.20.** The set \( \mathcal{P}^{\text{reg}} \subset \mathcal{P} \) of non-degenerate holonomy perturbations is of Baire second category.

**Corollary 4.21.** For any pair \(\pi_0, \pi_1 \in \mathcal{P}^{\text{reg}}\), one can find a generic smooth path \(\pi_t : [0, 1] \to \mathcal{P}\) from \(\pi_0\) to \(\pi_1\), such that there are only countably many \(t\) where \(\pi_t\) is degenerate. Moreover, for every such \(t\) there is exact one degenerate critical orbit \(\text{Orb}(B)\) of \(CS + f_\pi\), and the kernel of \(\text{Hess}_{B, \pi_t}\) is an irreducible representation of \(\text{Stab}(B)\).

5. SU(n) Casson invariants for integer homology spheres

This section proves Theorem 1.1. From now on, we assume that \(G = \text{SU}(n)\) and \(Y\) is an integer homology sphere.
5.1. **Classification of stabilizers.** When $G = SU(n)$, there is a combinatorial classification of all possible stabilizers on $C$.

Recall that $P$ is the trivial $SU(n)$–bundle given by (4.2). Let $B$ be a connection on $P$. Let $y_0 \in Y$, then $\text{Stab}(B)$ embeds as a closed subgroup of $SU(n)$ by restricting to $y_0$. Let $\text{Hol}_{y_0}(B) \subset G$ be the holonomy group of $B$ at $y_0$, then $\text{Hol}_{y_0}(B)$ is a closed subgroup of $SU(n)$, and the restriction of $\text{Stab}(B)$ to $y_0$ equals the commutator subgroup of $\text{Hol}_{y_0}(B)$.

View $C^n$ is a representation of $\text{Hol}_{y_0}(B) \subset SU(n)$, then an element $g \in SU(n)$ is in the commutator group of $\text{Hol}_{y_0}(B)$ if and only if it gives a $\text{Hol}_{y_0}(B)$–module homomorphism on $C^n$. Suppose the isotypic decomposition of $C^n$ as a representation of $\text{Hol}_{y_0}(B)$ is given by

$$C^n \cong V(n_1)^{\oplus m_1} \oplus \cdots \oplus V(n_r)^{\oplus m_r}, \quad (5.1)$$

where $V(n_i), 1 \leq i \leq r$ are $C$–vector spaces of dimension $n_i$. Then by Schur’s lemma, every unitary $\text{Hol}_{y_0}(B)$–homomorphism $g$ can be decomposed as

$$g = \text{Diag}(g_1, \ldots, g_r),$$

where

$$g_i \in U(m_i) \otimes \text{id}_{V(n_i)}.$$  

Hence the commutator subgroup of $\text{Hol}_{y_0}(B)$ is given by the kernel of

$$U(m_1) \times \cdots \times U(m_r) \to U(1)$$

$$(u_1, \ldots, u_r) \mapsto \det(u_1)^{n_1} \cdots \det(u_r)^{n_r},$$

which will be denoted by

$$\text{S}(U(m_1)^{n_1} \times \cdots \times U(m_r)^{n_r}). \quad (5.2)$$

Let $E$ be the unitary $C^n$–bundle associated to $P$. By taking parallel translations with respect to $B$, the decomposition (5.1) at $y_0$ gives a decomposition of $E$. Since $Y$ is an integer homology sphere, every unitary complex vector bundle over $Y$ is trivial, therefore we may write $E$ as

$$E = E(n_1)^{\oplus m_1} \oplus \cdots \oplus E(n_r)^{\oplus m_r},$$

where every $E(n_i)$ is a trivial $C$–vector bundle of rank $n_i$, and the connection $B$ decomposes as the direct sum of irreducible unitary connections on $E(n_i)$.

We summarize the previous discussions as follows.

**Definition 5.1.** Let $\Sigma_n$ be the set of tuples of positive integers

$$((n_1, m_1), \ldots, (n_r, m_r)),$$

such that

1. $n = \sum_{i=1}^r m_i n_i$,
2. $n_1 \leq n_2 \leq \cdots \leq n_r$,
3. the $m_i$’s are in non-decreasing order if the corresponding $n_i$’s are the same.

**Definition 5.2.** Suppose $\sigma = ((n_1, m_1), \ldots, (n_r, m_r)) \in \Sigma_n$. Define $C_\sigma$ to be the subset of $C$ consisting of $B \in C$, such that $P \times_{SU(n)} C^n$ decomposes as

$$E = E(n_1)^{\oplus m_1} \oplus \cdots \oplus E(n_r)^{\oplus m_r},$$

where every $E(n_i)$ is a trivial $C$–vector bundle of rank $n_i$, and the connection $B$ decomposes as the direct sum of irreducible unitary connections on $E(n_i)$.  


By the previous discussions, $\mathcal{C}$ is stratified by the union of $\mathcal{C}_\sigma$ for $\sigma \in \Sigma_n$.

**Definition 5.3.** Suppose $\sigma_1, \sigma_2 \in \Sigma_n$. We write $\sigma_1 \prec \sigma_2$ if and only if $\mathcal{C}_{\sigma_1}$ is in the closure of $\mathcal{C}_{\sigma_2}$. Then $\prec$ defines a partial order on $\Sigma_n$.

**Definition 5.4.** Let $\sigma = ((m_1, m_1), \ldots, (n_r, m_r)) \in \Sigma_n$. Define
\[
H_\sigma := S(U(m_1)^{m_1} \times \cdots \times U(m_r)^{m_r}).
\]

Let $H$ be the subgroup of $SU(n)$ associated to $\sigma$.

**Definition 5.5.** Suppose $H$ is a closed subgroup of $SU(n)$. Recall that $\mathcal{G}$ is identified with the set of $L^2_{k+1}$-maps from $Y$ to $SU(n)$. Let $\mathcal{G}_H$ be the subgroup of $\mathcal{G}$ consisting of constant maps to $H$. Define $\mathcal{C}_H \subset \mathcal{C}$ to be the fixed point set of $\mathcal{G}_H$.

By definition, $\mathcal{C}_H$ is an affine space, and $\theta \in \mathcal{C}_H$ for all $H$. Suppose $B \in \mathcal{C}_\sigma$, then there exists a gauge transformation $g \in \mathcal{G}$ such that $g(B) \in \mathcal{C}_H$, and the linear homotopy from $g(B)$ to $\theta$ remains in $\mathcal{C}_H$.

Suppose $\sigma_1 \prec \sigma_2$, then there exists $g \in \mathcal{G}$ such that $H_{\sigma_2} \subset g \cdot H_{\sigma_1} \cdot g^{-1}$. We abuse the notation and also use $g \in \mathcal{G}$ to denote the constant map from $Y$ to $g \in G$, then
\[
g(\mathcal{C}_{H_{\sigma_1}}) \subset \mathcal{C}_{H_{\sigma_2}}.
\]

5.2. **Equivariant spectral flow.** Let $\mathcal{V}$ be a Hilbert space, and let $\mathcal{D} \subset \mathcal{V}$ be a dense subspace. Suppose $f_t : \mathcal{D} \to \mathcal{V}, t \in [0, 1]$, is a smooth family of self-adjoint operators on $\mathcal{V}$, such that the spectra of $f_t$ is discrete on $\mathbb{R}$ for all $t$, and that 0 is not in the spectra of $f_0$ and $f_1$. Let $\lambda_0 > 0$ be the minimum of absolute values of the eigenvalues of $f_0$ and $f_1$. For a generic $c \in (-\lambda_0, \lambda_0)$, the eigenvalues of the family $f_t + c \cdot \text{id}$ cross zero transversely. The spectral flow of $f_t$, $t \in [0, 1]$, is defined to be the number of times where a negative eigenvalue of $f_t + c \cdot \text{id}$ crosses zero and becomes a positive eigenvalue, minus the number of times where a positive eigenvalue of $f_t + c \cdot \text{id}$ crosses zero and becomes a negative eigenvalue, as $t$ goes from 0 to 1.

Suppose $H$ is a compact Lie group that acts on $\mathcal{V}$, and suppose the family $f_t$ is $H$–equivariant, then we can refine the definition of spectral flow and obtain an element in the representation ring of $H$ as follows. Recall that the representation ring of $H$ is denoted by $\mathcal{R}(H)$. Let $\mathcal{R}^{\text{irr}}(H)$ be the set of isomorphism classes of irreducible representations of $H$. Then for each $W \in \mathcal{R}^{\text{irr}}(H)$, $f_t$ defines a family of self-adjoint operators on $\text{Hom}_H(W, \mathcal{V})$. Let $n_W \in \mathbb{Z}$ be the spectral flow of the induced operators on $\text{Hom}_H(W, \mathcal{V})$ by $f_t$, then we define the equivariant spectral flow of the family $f_t$ to be
\[
\sum_{W \in \mathcal{R}^{\text{irr}}(H)} n_W \cdot [W] \in \mathcal{R}(H).
\]

Alternatively, the equivariant spectral flow can be described as follows. As before, let $\lambda_0 > 0$ be the minimum of absolute values of the eigenvalues of $f_0$ and $f_1$, and take $c \in (-\lambda_0, \lambda_0)$ such that the eigenvalues of the family $f_t + c \cdot \text{id}$ cross zero transversely. Suppose $f_t + c \cdot \text{id}$ has eigenvalue zero for $t = t_1, \ldots, t_r$. We may further perturb $c$ such that at each $t_i$, the eigenvalues either cross zero from the negative side to the positive side, or from positive the positive side to the negative side, but not in both directions. Let $\eta_i = 1$ if the eigenvalues cross zero from the negative side to the positive side at $t_i$, and let $\eta_i = -1$ if the eigenvalues cross from
the positive side to the negative side at \( t_i \). At each \( t_i \), the kernel of \( f_{t_i} + c \cdot \text{id} \) is finite-dimensional and \( H \)-invariant, and hence it defines an element \([W_i] \in \mathcal{R}(H)\). Then the equivariant spectral flow of \( f_i \) is given by

\[
\sum_{i=1}^{r} \eta_i \cdot [W_i].
\]

If \( f_0 \) or \( f_1 \) have non-trivial kernel, we define the *equivariant spectral flow* to be the spectral flow from \( f_0 + \epsilon \cdot \text{id} \) to \( f_1 + \epsilon \cdot \text{id} \), for \( \epsilon \) positive and sufficiently small.

**Remark 5.6.** When \( f_0 \) or \( f_1 \) have non-trivial kernel, our convention of the spectral flow is different from [BH98, Definition 4.1]. The current convention is slightly more convenient for the later discussions.

**Definition 5.7.** Suppose \( H \) is a closed subgroup of \( \text{SU}(n) \), let \( B \in \mathcal{C}^H \), and let \( \pi \in \mathcal{P} \). Recall that the operator \( K_{B,\pi} \) is defined by Equation (4.12). Define

\[
S_{f_H}(B, \pi) \in \mathcal{R}(H)
\]

to be the \( H \)-equivariant spectral flow from \( K_{B,\pi} \) to \( K_{\theta,0} \), by the linear homotopy from \((B, \pi)\) to \((\theta,0)\).

Suppose \( H_1 \subset H_2 \), let

\[
r_{H_2}^{H_1} : \mathcal{R}(H_2) \to \mathcal{R}(H_1)
\]

be the homomorphism given by restrictions of representations of \( H_2 \) to representations of \( H_1 \). Suppose \( B \in \mathcal{C}^{H_2} \), then we have

\[
S_{f_{H_1}}(B, \pi) = r_{H_2}^{H_1}(S_{f_{H_2}}(B, \pi)).
\]

If \( \sigma_1, \sigma_2 \in \Sigma_n \) satisfies \( \sigma_1 \prec \sigma_2 \) (see Definition 5.3), let \( g \) be an element of \( G \) such that \( H_{\sigma_2} \subset g \cdot H_{\sigma_1} \cdot g^{-1} \). Then there is a homomorphism

\[
r_g : \mathcal{R}(H_{\sigma_1}) \to \mathcal{R}(H_{\sigma_2})
\]

that takes the isomorphism class of a representation

\[\rho : H_{\sigma_1} \to \text{Hom}(V,V)\]

to the isomorphism class of

\[H_{\sigma_2} \to \text{Hom}(V,V)\]

\[h \mapsto \rho(g^{-1}hg)\].

Suppose \( B \in \mathcal{C}^{H_{\sigma_1}} \). We abuse the notation and let \( g \in \mathcal{G} \) denote the constant map from \( Y \) to \( g \), then \( g(B) \in \mathcal{C}^{H_{\sigma_2}} \), and we have

\[
S_{f_{H_{\sigma_2}}}(g(B)) = r_g(S_{f_{H_{\sigma_1}}}(B)).
\]

Notice that \( S_{f_H} \) is in general not gauge invariant: if \( B \in \mathcal{C}^H \), and \( g \in \mathcal{G} \) is a gauge transformation such that \( g(B) \in \mathcal{C}^H \), then \( S_{f_H}(B) \) may not be equal to \( S_{f_H}(g(B)) \). This issue will be discussed in Section 5.3.
5.3. **Index of non-degenerate perturbed flat connections.** This subsection constructs a correction term that cancels the gauge ambiguity of the equivariant spectral flow.

Let \( A \subset C \) be the space of flat connections on \( P \). Then there exists a \( G \)-invariant open neighborhood \( U \) of \( A \), such that for each \( \sigma \in \Sigma_n \), the inclusion
\[
A \cap C^{H_{\sigma}} \hookrightarrow U \cap C^{H_{\sigma}}
\]
induces a one-to-one correspondence on the set of connected components. By the Uhlenbeck compactness theorem, the quotient set \( A/\hat{\mathcal{G}} \) is compact, therefore there exists \( r_0 > 0 \) depending on \( Y \), such that if \( \|\pi\|_{\mathcal{P}} < r_0 \), then all critical points of \( CS + f_\pi \) lie in \( U \).

Take \( \sigma = ((n_1, m_1), \ldots, (n_r, m_r)) \in \Sigma_n \), and suppose \( B \in U \cap C^{H_{\sigma}} \). Then \( E = P \times_{SU(n)} \mathbb{C}^n \) is decomposed as
\[
E = E(n_1)^{\oplus m_1} \oplus \cdots \oplus E(n_r)^{\oplus m_r},
\]
where \( E(n_i) \) have rank \( n_i \), and \( E(n_i) \) are constant subbundles of \( E \) with respect to the trivialization \( (4.2) \). The connection \( B \) is given by the direct sum of irreducible connections on each \( E(n_i) \). Since \( B \in U \cap C^{H_{\sigma}} \), there exists \( \hat{B} \in A \cap C^{H_{\sigma}} \), such that \( \hat{B} \) and \( B \) are in the same connected component of \( U \cap C^{H_{\sigma}} \). The connection \( \hat{B} \) is also given by the direct sum of flat connections on each \( E(n_i) \). Let \( \hat{B} \) be the restriction of \( \hat{B} \) to \( E(n_i) \). Since the Chern-Simons functional is constant on the connected components of \( A \), the value of \( CS(\hat{B}_i) \) is independent of the choice of \( \hat{B} \).

Suppose \( g : E \to E \) is a gauge transformation that decomposes as the direct sum of \( g_1, \ldots, g_r \), where \( g_i : E(n_i) \to E(n_i) \) is a unitary bundle map. Then each \( g_i \) is given by a map from \( Y \) to \( U(n_i) \). Since \( Y \) is an integer homology sphere, \( g_i \) is homotopic to a constant map if \( n_i = 1 \), and when \( n_i \geq 2 \), then the homotopy class of \( g_i \) is classified by the induced map \( H_3(Y; \mathbb{Z}) \to H_3(U(n_i)) \cong \mathbb{Z} \).

Recall that \( Y \) is oriented and fix an isomorphism from \( H_3(U(n_i)) \to \mathbb{Z} \), the map \( (5.5) \) identifies the homotopy classes of \( g_i \) with \( \mathbb{Z} \). We use \( \deg g_i \in \mathbb{Z} \) to denote the image of the homotopy class of \( g_i \) in \( \mathbb{Z} \).

By the excision property of index, there exist \( \tau_1, \ldots, \tau_r \in \mathcal{R}(H_\sigma) \) that only depend on \( n \) and \( \sigma \), such that the \( H_\sigma \)-equivariant spectral flow from \( K_{B, \pi} \) to \( K_{g(B), \pi} \) is given by
\[
\sum_{n_i \geq 2} \sum_{n_i \geq 2} \deg(g_i) \cdot \tau_i.
\]

On the other hand, let \( B_i \) be the restriction of \( B \) to \( E(n_i) \), then for each \( n_i \geq 2 \), we have
\[
CS(g_i(B_i)) - CS(B_i) = 4\pi^2 n_i \cdot \deg(g_i).
\]

**Definition 5.8.** Suppose \( B \in U \cap C^{H_{\sigma}} \). Define
\[
CS_\sigma(B) := \sum_{n_i \geq 2} \frac{CS(B_i)}{4\pi^2 n_i} \cdot \tau_i \in \mathcal{R}(H_\sigma) \otimes \mathbb{R}.
\]

Now suppose \( \pi \in \mathcal{P} \) satisfies \( \|\pi\|_{\mathcal{P}} < r_0 \) so that all the critical points of \( CS + f_\pi \) are contained in \( U \), and suppose \( B \) is \( \pi \)-flat. We define a gauge-invariant equivariant index of \( B \) that takes value in \( \tilde{R}_{SU(n)} \) (see Definition 3.39).
**Definition 5.9.** Let \( \pi, B \) be as above. Take \( \sigma \in \Sigma_n \) such that \( B \in C_{\sigma} \), and take \( g \in G \) such that \( g(B) \in CH_{\sigma} \). Define \( \text{ind}(B, \pi) \in \hat{R}_{SU(n)}([H_\sigma]) \) to be the element represented by

\[
S_f H_{\sigma}(g(B), \pi) - [\ker d_{g(B)}] - CS_\sigma(g(B)) \in \mathcal{R}(H_\sigma) \otimes \mathbb{R},
\]

(5.6)

where \([\ker d_{g(B)}] \in \mathcal{R}([H_\sigma])\) is given by \( \ker d_{g(B)} \subset L^2_k(\mathfrak{g}) \) as an \( H_\sigma \)-representation.

**Remark 5.10.** The extra term \([\ker d_{g(B)}]\) is necessary for the proof of Lemma 5.15.

The definition of \( \text{ind}(B) \) is independent of the choice of \( g \) and therefore is gauge-invariant.

Notice that since \( Y \) is an integer homology sphere, the Chern-Simons functional of any flat connection on a line bundle over \( Y \) equals zero. Therefore we have

\[
\frac{\text{CS}(\hat{B})}{4\pi^2 n} = \sum_{n_i \geq 2} \frac{\text{CS}(\hat{B}_i)}{4\pi^2 n_i} m_i,
\]

(5.7)

where the normalizing constants on the denominators come from the convention in the definition of the Chern-Simons functional (4.4).

If \( \sigma_1, \sigma_2 \in \Sigma_n \) satisfies \( \sigma_1 \prec \sigma_2 \) (see Definition 5.3), let \( g \) be an element of \( G \) such that \( H_{\sigma_2} \subset g \cdot H_{\sigma_1} \cdot g^{-1} \). The homomorphism \( r_g \) given by (5.3) extends linearly to a homomorphism

\[
r_g : \mathcal{R}(H_{\sigma_1}) \otimes \mathbb{R} \to \mathcal{R}(H_{\sigma_2}) \otimes \mathbb{R}.
\]

Suppose \( B \in CH_{\sigma_1} \). We abuse the notation and let \( g \in G \) be the constant map from \( Y \) to \( g \), then \( g(B) \in CH_{\sigma_2} \), and (5.7) implies that

\[
\text{CS}_{\sigma_2}(g(B)) = r_g(\text{CS}_{\sigma_1}(B)).
\]

(5.8)

Recall that by Uhlenbeck’s compactness theorem, the moduli space of \( \pi \)-flat connections is compact for all \( \pi \in \mathcal{P} \). If \( \pi \) is non-degenerate, then the critical set is finite.

**Definition 5.11.** Suppose \( \pi \in \mathcal{P} \) satisfies \( \|\pi\|_{\mathcal{P}} < r_0 \) so that all the critical points of \( \text{CS} + f_\pi \) are contained in \( \mathcal{U} \), and suppose that \( \pi \) is non-degenerate. Define the total index of \( \pi \) by

\[
\text{ind}(\pi) := \sum_{\text{Orb}(B) \text{ is } \pi-\text{flat}} \text{ind}(B, \pi) \in \mathbb{Z}\hat{R}_{SU(n)}
\]

We also introduce the following refinement of Definition 5.11.

**Definition 5.12.** Suppose \( \pi \in \mathcal{P} \) satisfies \( \|\pi\|_{\mathcal{P}} < r_0 \) so that all the critical points of \( \text{CS} + f_\pi \) are contained in \( \mathcal{U} \), and suppose that \( \pi \) is non-degenerate. Let \( \eta \) be a connected component of \( \mathcal{U} \), define

\[
\text{ind}_\eta(\pi) := \sum_{\text{Orb}(B) \text{ is } \pi-\text{flat} \atop \text{Orb}(B) \cap \eta \neq \emptyset} \text{ind}(B, \pi) \in \mathbb{Z}\hat{R}_{SU(n)}.
\]

(5.4) **Comparison of the total index.** Suppose \( \pi_0, \pi_1 \in \mathcal{P} \) are non-degenerate and sufficiently small such that \( \text{ind}(\pi_0), \text{ind}(\pi_1) \) are defined. Let \( \mathcal{A}, \mathcal{U} \) be as in Section 5.3. Suppose \( \eta \) is a connected component of \( \mathcal{U} \).

The main result of this subsection is the following theorem.

**Theorem 5.13.** \( \text{ind}_\eta(\pi_0) - \text{ind}_\eta(\pi_1) \in \overline{\text{Bif}}_{SU(n)} \).
And we have the following immediate corollary.

**Corollary 5.14.** \( \text{ind}(\pi_0) - \text{ind}(\pi_1) \in \widetilde{\text{Bif}}_{SU(n)} \).

**Proof of Theorem 5.13.** We use the Kuranishi reduction argument to reduce to the finite-dimensional case so that we can invoke Theorem 3.10.

Take a generic smooth path \( \pi_t \) \( (t \in [0, 1]) \) from \( \pi_0 \) to \( \pi_1 \) in the sense of Corollary 4.21. To simplify the notation, we will denote \( \text{CS} + f_{\pi_t} \) by \( \text{CS}_t \).

Suppose \( \text{CS}_{t_0} \) is degenerate at \( B_0 \) where \( B_0 \in \eta \). Let \( V_0 \subset \ker d_{B_0}^* \) be the kernel of \( \text{Hess}_{B_0, \pi_0} \). For \( m = k, k - 1 \), let \( V_0^{\perp(m)} \) be the orthogonal complement of \( V_0 \) in \( \ker d_{B_0}^* \cap L_{B_0}^2(T^*Y \otimes g) \). Then \( \text{Hess}_{B_0, \pi_t} \) restricts to an isomorphism from \( V_0^{\perp(k)} \) to \( V_0^{\perp(k-1)} \).

Let

\[
\Pi_0 : L_0^2(T^*Y \otimes g) \to V_0
\]

be the \( L^2 \)-orthogonal projection onto \( V_0 \), and let

\[
\Pi_0^{\perp(m)} : L_0^2(T^*Y \otimes g) \to V_0^{\perp(m)}
\]

be the \( L^2 \)-orthogonal projection onto \( V_0^{\perp(m)} \) for \( m = k, k - 1 \).

For \( \epsilon > 0 \) sufficiently small, let \( S_{B_0, \epsilon} \) be the slice of \( B_0 \) given by Proposition 4.2, and define

\[
M_{t, \epsilon} := \{ B \in S_{B_0, \epsilon} | \Pi_0^{(k-1)}(*F_B + V_{\pi_t}) = 0 \}. \tag{5.9}
\]

Then by the implicit function theorem, there exists \( \epsilon_0 > 0 \), such that for all \( \epsilon \in (0, \epsilon_0) \) and all \( t \in (t_0 - \epsilon_0, t_0 + \epsilon_0) \), the set \( M_{t, \epsilon} \) is an embedded manifold with dimension \( \dim V_0 \), and the \( L^2 \)-orthogonal projection of \( M_{t, \epsilon} \) to \( V_0 \) is a smooth embedding with open image.

Let \( H_0 \) be the stabilizer of \( B_0 \), then \( H_0 \) acts on \( M_{t, \epsilon} \). Let \( U_\epsilon(B_0) \) be the image of \( S_{B_0, \epsilon} \) under the action of \( \mathcal{G} \). By Proposition 4.2, \( U_\epsilon(B_0) \) is a \( \mathcal{G} \)-invariant open neighborhood of \( B_0 \). For all \( \epsilon \in (0, \epsilon_0) \) and all \( t \in (t_0 - \epsilon_0, t_0 + \epsilon_0) \), the critical orbit of \( \text{CS}_0 \) on \( U_\epsilon(B_0) \) is in one-to-one correspondence with the critical orbits of the restriction of \( \text{CS}_t \) to \( M_{t, \epsilon} \). Since \( \text{CS}_{t_0} \) has exactly one degenerate critical orbit \( B_0 \), it has at most countably many critical orbits, and thus we may choose \( \epsilon \) such that \( \partial U_\epsilon(B_0) \) contains no critical orbit of \( \text{CS}_{t_0} \). By Uhlenbeck’s compactness theorem, there are only finitely many critical points of \( \text{CS}_{t_0} \) on \( \mathcal{C} - U_\epsilon(B_0) \). Also recall that \( \pi_t \) is non-degenerate except for countably many values of \( t \). Therefore there exist \( t_+ \in (t_0, t_0 + \epsilon_0) \) and \( t_- \in (t_0 - \epsilon_0, t_0) \), such that

1. for all \( t \in (t_-, t_+) \), the boundary \( \partial U_\epsilon(B_0) \) contains no critical orbit of \( \text{CS}_t \);
2. for all \( t \in (t_-, t_+) \), all the critical points of \( \text{CS}_t \) on \( \mathcal{C} - U_\epsilon(B_0) \) are non-degenerate.

As a consequence, take \( t_- \in (t_-, t_0) \) and \( t_+ \in (t_0, t_+) \) such that \( \pi_{t_-'} \) and \( \pi_{t_+'} \) are non-degenerate, then the difference

\[
\text{ind} \pi_{t_-'} - \text{ind} \pi_{t_+'}
\]

is given by the difference of the total indices of \( \text{CS}_{t_-'} \) and \( \text{CS}_{t_+'} \) on \( U_\epsilon(B_0) \). We claim that:

**Lemma 5.15.** Let \( H_0, \text{CS}_t \) and \( M_{t, \epsilon} \) be as above, and suppose \( B \in M_{t, \epsilon} \) is a critical point of the restriction of \( \text{CS}_t \) to \( M_{t, \epsilon} \). Suppose \( (\epsilon, t) \) is sufficiently close to \( (0, t_0) \). Then \( B \) is non-degenerate as a critical point of \( M_{t, \epsilon} \) when regarding \( M_{t, \epsilon} \) as
a finite dimensional $H_0$–manifold, if and only if $B$ is non-degenerate as a $\pi_t$–flat connection. If $B$ is non-degenerate, let $\text{ind} \epsilon, B \in \mathcal{R}_{H_0}$ be the index of the critical orbit of $B$ as a point on $M_{t, \epsilon}$, then $\text{ind}(B, \pi_t) \in \widetilde{\mathcal{R}}_{\text{SU}(n)}$ is given by

$$\text{ind}(B, \pi_t) = i_{\text{SU}(n)}^{H_0} \left( \text{ind} B_0 + \text{ind} \epsilon, B \right).$$

We will postpone the proof of Lemma 5.15 to the next subsection. Lemma 5.15 and Theorem 3.10 imply that

$$\text{ind} \pi_t' + \text{ind} \pi_t' - \text{ind} \pi_t \in \widetilde{\mathcal{B}}_{\text{SU}(n)}.$$

Now consider

$$\mathcal{S} := \{ t \in (0, 1) | \pi_t \text{ is degenerate} \}.$$  

The value of $\text{ind} \pi_t$ is constant on any open interval in $[0, 1] - \mathcal{S}$. For each $t_0 \in \mathcal{S}$, let $I_{t_0} := (t_-, t_+)$ be the open interval given as above, then by the previous argument, the image of $\text{ind} \pi_t$ in

$$\widetilde{\mathcal{R}}_{\text{SU}(n)}/\widetilde{\mathcal{B}}_{\text{SU}(n)}$$

is constant on $I_{t_0} - \mathcal{S}$. Since $\mathcal{S}$ is countable, $I_{t_0} - \mathcal{S}$ is a dense subset of $I_{t_0}$. Since $\mathcal{S}$ is compact, there exists a finite subset of $\mathcal{S}$ such that the corresponding open intervals $I_{t_0}$ cover $\mathcal{S}$, therefore the theorem is proved. \hfill $\Box$

We can now prove Theorem 1.1 as a straightforward consequence of Theorem 5.13 and Proposition 3.42.

**Proof of Theorem 1.1.** Let $\{1\}$ be the trivial group, then $\mathcal{R}(\{1\}) \otimes \mathbb{R} \cong \mathbb{R}$, therefore $\mathbb{R}$ canonically embeds in $\widetilde{\mathcal{R}}_{\text{SU}(n)}$ as the image of $\mathcal{R}(\{1\}) \otimes \mathbb{R}$. Under this identification, we have $\mathbb{R} \cap \widetilde{\mathcal{R}}_{\text{SU}(n)}^{(0)} = [0, 1)$. Let $w$ be an arbitrary function from $\widetilde{\mathcal{R}}_{\text{SU}(n)}^{(0)}$ to $\mathbb{C}$ such that $w(s) = e^{\pi i s}$ on $[0, 1)$. By Proposition 3.42, the function $w$ can be uniquely extended to a homomorphism

$$w : \mathbb{Z} \widetilde{\mathcal{R}}_{\text{SU}(n)} \rightarrow \mathbb{C}.$$  

such that $w = 0$ on $\widetilde{\mathcal{B}}_{\text{SU}(n)}$.

We claim that $w$ satisfies the desired condition. Since $w = 0$ on $\widetilde{\mathcal{B}}_{\text{SU}(n)}$, we have $w(s + 1) = -w(s)$ on the image of $\mathbb{R}$ in $\widetilde{\mathcal{R}}_{\text{SU}(n)}$, therefore $w(s) = e^{\pi i s}$ on the image of $\mathbb{R}$.

Let $\mathcal{A}, \mathcal{U}$ be as in Section 5.3, and suppose $\eta$ is a connected component of $\mathcal{U}$. By Theorem 5.13, the sum

$$\lambda_{\eta, w} := \sum_{[B] \in \mathcal{M}^+} w(\text{ind} B) + \sum_{[B] \in \mathcal{M}^+} w(\text{ind} B)$$

is independent of the perturbation $\pi$. Notice that for $[B] \in \mathcal{M}^+$, the equivariant index of $B$ is an element of $\mathcal{R}(\{1\}) \otimes \mathbb{R} \cong \mathbb{R}$ given by

$$Sf(B, \pi) - \frac{\text{CS}(B_\eta)}{\pi^2},$$

where $B_\eta$ is a flat connection in the connected component $\eta$, and $Sf(B, \pi) \in \mathbb{Z}$ is the (classical) spectral flow from $K_{B, \pi}$ to $K_{\theta, 0}$ via the linear homotopy. By the
definition of \( \mathcal{U} \), the value of \( \text{CS}(B_0) \) is independent of the choice of \( B_0 \). Therefore

\[
e^{\pi i \text{CS}(B_0) / (4\pi^2 n)} : \lambda_{\eta,w} = \sum_{B \in \mathcal{M}^*} (-1)^{S_f(B, \pi)} + \sum_{B \in \mathcal{M}^*} e^{\pi i \text{CS}(B_0) / (4\pi^2)} \cdot \mu(\text{ind } B)
\]

is independent of the choice of \( \pi \), and hence the theorem is proved.  \( \square \)

5.5. **Proof of Lemma 5.15.** This subsection is devoted to the proof of Lemma 5.15. As in the proof of Theorem 5.13, let \( V_0 \) be the kernel of \( \text{Hess}_{B_0_0} \). For \( m = k, k - 1 \), let \( V_0^{(m)} \) be the \( L^2 \)-orthogonal complement of \( V_0 \) in \( \ker d_{B_0_0} \cap L^2_m(T^*Y \otimes g) \). Let \( H_0 = \text{Stab}(B_0) \), \( H = \text{Stab}(B) \).

Notice that \( \ker d_{B_0} \) is the tangent space of \( \text{Stab}(B_0) \), and \( \ker d_B \) is the tangent space of \( \text{Stab}(B) \). Since \( B \in S_{B_0, \epsilon} \), by Proposition 4.2, we have \( \text{Stab}(B) \subset \text{Stab}(B_0) \), therefore \( \ker d_B \subset \ker d_{B_0} \).

Let \( V_1(B) \) be the \( L^2 \)-orthogonal complement of \( \ker d_B \) in \( \ker d_{B_0} \). Then \( V_1(B) \) is a finite dimensional subspace of \( \ker d_{B_0} \), and we have

\[
T_{id} H_0 = T_{id} H \oplus V_1(B).
\]

Since \( B_0 \) is in \( L^2_k \), it follows from the standard bootstrapping argument that

\[
\ker d_{B_0} \subset L^2_{k+1}(g).
\]

Let

\[
V_2(B) := d_B(\ker d_{B_0}) \subset L^2_k(T^*Y \otimes g),
\]

then \( V_2(B) \) is a finite dimensional subspace of \( \text{Im } d_B \). Notice that \( V_2(B) \) is the tangent space of the \( H_0 \)-orbit of \( B \). Therefore \( V_2(B) \subset \ker d_{B_0} \), and the map

\[
d_B : V_1(B) \to V_2(B)
\]

is an isomorphism.

For \( m = k, k - 1 \), let \( V_2(B)^{(m)} \) be the \( L^2 \)-orthogonal complement of \( V_2(B) \) in \( \text{Im } d_B \cap L^2_m(T^*Y \otimes g) \).

Let \( \Pi_B \) be the \( L^2 \)-orthogonal projection onto \( \ker d_B^* \), and define

\[
V_3(B) := \Pi_B(V_2(B)^{\perp}),
\]

where \( V_2(B)^{\perp} \) is the \( L^2 \)-orthogonal complement of \( V_2(B) \) in \( T_B M_{\epsilon,t} \). Then \( V_3(B) \) is finite dimensional, and we have

\[
\text{dim } V_0 = \text{dim } T_B M_{\epsilon,t} = \text{dim } V_2(B) + \text{dim } V_3(B).
\]

For \( m = k, k - 1 \), let \( V_2(B)^{(m)} \) be the orthogonal complement of \( V_3(B) \) in \( \ker d_B^* \cap L^2_m(T^*Y \otimes g) \).

To simplify the notation, we make the following definitions.

**Definition 5.16.** We say that a function \( f(B) \) of \( B \) converges to \( c \) as \( (\epsilon, t) \to (0, t_0) \), if for every \( \delta > 0 \), there exists \( \epsilon_1 > 0 \) depending on \( B_0 \), such that whenever \( \epsilon < \epsilon_1 \) and \( t \in (t_0 - \epsilon_1, t_0 + \epsilon_1) \), we have \( |f(B) - c| < \delta \).

**Definition 5.17.** Suppose \( W, W' \) are Banach spaces, and \( \iota_V : V \hookrightarrow W, \iota_{V'} : V' \hookrightarrow W' \) are embeddings of fixed closed subspaces. Suppose \( V(B) \subset W, V'(B) \subset W' \) are closed subspaces depending on \( B \).
(1) We say that $V(B)$ converges to $V$ as $(\epsilon, t) \to (0, t_0)$, if for every $\delta > 0$, there exists $\epsilon_1 > 0$ depending on $B_0$, such that whenever $\epsilon < \epsilon_1$ and $t \in (t_0 - \epsilon_1, t_0 + \epsilon_1)$, there exists a bounded linear operator $\varphi : V \to W$, such that $\varphi(V) = V(B)$, and
\[
\|\varphi - \iota_V\| < \delta,
\]
where $\| \cdot \|$ denotes the operator norm.

(2) Suppose $H : V \to V'$ is a bounded linear operator, and $H(B) : V(B) \to V'(B)$ is a bounded linear operator that depends on $B$. We say that $H(B)$ converges to $H$ as $(\epsilon, t) \to (0, t_0)$, if for every $\delta > 0$, there exists $\epsilon_1 > 0$ depending on $B_0$, such that whenever $\epsilon < \epsilon_1$ and $t \in (t_0 - \epsilon_1, t_0 + \epsilon_1)$, there exist bounded linear operators $\varphi : V \to W$, $\varphi' : V' \to W'$, such that
\[
\varphi(V) = V(B), \quad \varphi'(V') = V'(B), \quad \|\varphi - \iota_V\| < \delta, \quad \|\varphi' - \iota_{V'}\| < \delta, \quad \|H - \varphi^{-1} \circ H(B) \circ \varphi\| < \delta.
\]

Lemma 5.18. Suppose $m = k$ or $k - 1$. Then $V_2'(B)^{(m)}$ converges to $\text{Im} d_{B_0} \cap L^2_m(T^*Y \otimes g)$, and $V_3'(B)^{(m)}$ converges to $V_0^{(m)}$, as $(\epsilon, t) \to (0, t_0)$.

**Proof.** Let $(\ker d_{B_0})^\perp \subset L^2(g)$ be the $L^2$–orthogonal complement of $\ker d_{B_0}$. Then
\[
d_{B_0} : (\ker d_{B_0})^\perp \cap L^2_{m+1}(g) \to \text{Im} d_{B_0} \cap L^2_m(T^*Y \otimes g)
\]
is an isomorphism. Let $d_{B_0}^{-1} : \text{Im} d_{B_0} \cap L^2_m(T^*Y \otimes g) \to (\ker d_{B_0})^\perp \cap L^2_{m+1}(g)$ be its inverse map. Let $\Pi_2$ be the $L^2$–orthogonal projection to $V_2(B)$, and let $\Pi_2^\perp = \text{id} - \Pi_2$. Then
\[
V_2'(B)^{(m)} = \Pi_2^\perp \circ d_B\left((\ker d_{B_0})^\perp \cap L^2_{m+1}(g)\right) = \Pi_2^\perp \circ d_B \circ d_{B_0}^{-1} \left(\text{Im} d_{B_0} \cap L^2_m(T^*Y \otimes g)\right).
\]

Let $\iota : \text{Im} d_{B_0} \cap L^2_m(T^*Y \otimes g) \hookrightarrow L^2_m(T^*Y \otimes g)$ be the inclusion map. Since $V_2(B) \subset \ker d^*_B$, we have $\Pi_2 = 0$ on $\text{Im} d_{B_0} \cap L^2_m(T^*Y \otimes g)$. Therefore there exists a constant $z_1$, such that
\[
\Pi_2^\perp \circ d_B \circ d_{B_0}^{-1} - \iota \|
\leq \Pi_2^\perp \circ d_B \circ d_{B_0}^{-1} - \Pi_2^\perp \circ d_{B_0} \circ d_{B_0}^{-1} + \Pi_2^\perp \circ d_{B_0} - \Pi_2^\perp \circ d_{B_0} \circ d_{B_0}^{-1} + \Pi_2^\perp - \iota
\]
\[
= \Pi_2^\perp \circ d_B \circ d_{B_0}^{-1} - \Pi_2^\perp \circ d_{B_0} \circ d_{B_0}^{-1} + \Pi_2^\perp - \iota
\]
\[
\leq z_1 \cdot \|B - B_0\|_{L^2_{m+1}} \|d_{B_0}^{-1}\| \leq \epsilon z_1 \|d_{B_0}^{-1}\|.
\]

Hence $V_2'(B)^{(m)}$ converges to $\text{Im} d_{B_0} \cap L^2_m(T^*Y \otimes g)$ as $(\epsilon, t) \to (0, t_0)$.

Recall that $V_2(B)^\perp$ denotes the $L^2$–orthogonal complement of $V_2(B)$ in $T_B M_{\epsilon,t}$, and $\Pi_B$ denotes the $L^2$–orthogonal projection onto $\ker d^*_B$. By definition,
\[
V_3(B) = \Pi_B(V_2(B)^\perp).
\]

Let $\Pi_B^\perp = \text{id} - \Pi_B$, then $\Pi_B^\perp$ is the orthogonal projection onto $\text{Im} d_B \cap L^2_k(T^*Y \otimes g)$, which is spanned by $V_2(B)$ and the space
\[
d_B\left((\ker d_{B_0})^\perp \cap L^2_{k+1}(T^*Y \otimes g)\right).
\]
Moreover, as \((\epsilon, t) \to (0, t_0)\), the space (5.10) converges to
\[
d_{B_0}(\text{ker } d_{B_0}^\perp \cap L^2_{k+1}(T^*Y \otimes g)) = V_0^{(k)}.
\]
Since \(V_2(B)^\perp \subset T_B M_{t, \epsilon}\) which converges to \(V_0\), and \(V_0\) is orthogonal to \(V_0^{(k)}\), we conclude that
\[
\lim_{(\epsilon, t) \to (0, t_0)} \|\Pi_B^{\perp} |_{V_2(B)^\perp}\| = 0.
\]
and
\[
\lim_{(\epsilon, t) \to (0, t_0)} \|\Pi_B |_{V_0^{(k)}}\| = 0.
\]

By (5.11), \(V_3(B)\) gets arbitrarily close to \(V_2(B)^\perp\) as \((\epsilon, t) \to (0, t_0)\). Therefore \(V_3(B)\) is transverse to \(V_0^{(m)}\). Let \(\Pi_3\) be the \(L^2\)-orthogonal projection to \(V_3(B)\), and let \(\Pi_3^\perp = \text{id} - \Pi_3\), we then have
\[
V_3'(B)^{(m)} = \Pi_3^{\perp} \circ \Pi_B^{\perp} (V_0^{(m)}).
\]

By (5.12), the space \(\Pi_B^{\perp} (V_0^{(m)})\) converges to \(V_0^{(m)}\) as \((\epsilon, t) \to (0, t_0)\). Since \(V_2(B)^\perp\) is tangent to \(T_B M_{t, \epsilon}\), which is orthogonal to \(V_0^{(m)}\), we have that \(\Pi_3^{\perp} (V_0^{(m)})\) converges to \(V_0^{(m)}\) as \((\epsilon, t) \to (0, t_0)\). Therefore the desired result follows from (5.13).

Now we return to the proof Lemma 5.15. For \(m = k, k - 1\), let \((\text{ker } d_{B_0})_{(m)}^{\perp}\) be the orthogonal complement of \(\text{ker } d_{B_0}\) in \(L^2_{m}(g)\). Then the domain of the operator \(K_{B_0, \pi_{t_0}}\) is orthogonally decomposed as
\[
\ker d_{B_0} \oplus (\ker d_{B_0})_{(k+1)}^{\perp} \oplus (\text{Im } d_{B_0} \cap L^2_k (T^*Y \otimes g)) \oplus V_0 \oplus V_0^{(k)},
\]
and the range of \(K_{B_0, \pi_{t_0}}\) is orthogonally decomposed as
\[
\ker d_{B_0} \oplus (\ker d_{B_0})_{(k)}^{\perp} \oplus (\text{Im } d_{B_0} \cap L^2_{k-1} (T^*Y \otimes g)) \oplus V_0 \oplus V_0^{(k-1)}.
\]
Under this decomposition, the operator \(K_{B_0, \pi_{t_0}}\) is given by the matrix
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{B_0}^L & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]
Moreover, the restricted maps
\[
d_{B_0} : (\ker d_{B_0})_{(k)}^{\perp} \to \text{Im } d_{B_0} \cap L^2_k (T^*Y \otimes g)
\]
and
\[
\text{Hess}_{B_0, \pi_{t_0}} : V_0^{(k)} \to V_0^{(k-1)}
\]
are isomorphisms.

Now we study the operator \(K_{B, \pi_t}\). Decompose the domain of \(K_{B, \pi_t}\) as
\[
\ker d_B \oplus V_1(B) \oplus (\ker d_{B_0})_{(k+1)}^{\perp} \oplus V_2(B) \oplus V_2'(B) \oplus V_3(B) \oplus V_3'(B),
\]
and decompose the range of \(K_{B, \pi_t}\) as
\[
\ker d_B \oplus V_1(B) \oplus (\ker d_{B_0})_{(k)}^{\perp} \oplus V_2(B) \oplus V_2'(B) \oplus V_3(B) \oplus V_3'(B).
Then by Equation (4.15), the operator is given by a matrix of the form
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{11}^* & M_{21}^* & 0 & 0 \\
0 & 0 & 0 & M_{12}^* & M_{22}^* & 0 & 0 \\
0 & M_{11}^* & M_{12}^* & 0 & 0 & 0 & 0 \\
0 & M_{21}^* & M_{22}^* & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & N_{11} & N_{12} & 0 \\
0 & 0 & 0 & 0 & N_{21} & N_{22} & 0 
\end{pmatrix}.
\]
(5.17)

By the previous arguments, the operator \( K_{B, \pi} \) maps \( V_1(B) \) isomorphically to \( V_2(B) \). Therefore \( M_{11} \) is invertible, and \( M_{21} = 0 \). By Lemma 5.18, \( M_{22} \) converges to the isomorphism (5.15) as \((\epsilon, t) \to (0, t_0)\), therefore \( M_{22} \) is an isomorphism when \(|\epsilon| \) and \(|t - t_0|\) are sufficiently small.

We now study the matrix
\[
\begin{pmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22} 
\end{pmatrix}.
\]

**Lemma 5.19.** There exists a constant \( \epsilon_1 > 0 \) depending only on \( B_0 \), such that the following statements hold when \(|\epsilon|, |t - t_0| < \epsilon_1\):

1. The operator
\[
\begin{pmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22} 
\end{pmatrix}
\]
is invertible if and only if \( N_{11} \) is invertible.

2. Suppose \( N_{11} \) is invertible, then
\[
\begin{pmatrix}
N_{11} & s N_{12} \\
N_{21} & s N_{22} 
\end{pmatrix}
\]
is invertible for all \( s \in [0, 1] \).

**Proof.** By Lemma 4.11 and the definitions, the image of \( K_{B, \pi} \) on \( V_3(B) \) is given by the directional derivatives of \( \text{grad}_{\epsilon, t} \) on \( M_{t, \epsilon} \). By the definition of \( M_{t, \epsilon} \) from (5.9), we have
\[
\Pi_0^{(k-1)}(K_{B, \pi}(V_3(B))) = 0.
\]
By Lemma 5.18, the space \( V_3^*(B)^{(k-1)} \) converges to \( V_3^0 \) as \((\epsilon, t) \to (0, t_0)\). Therefore, there exists a constant \( z_3 \) independent of \( B \), such that
\[
z_1 \| N_{11}(v) \| \geq \| N_{21}(v) \|
\]
for all \( v \in V_3(B) \) when \((\epsilon, t)\) is sufficiently close to \((0, t_0)\).

Recall that \( N_{11} \) is a linear endomorphism on the finite dimensional space \( V_3(B) \). If \( N_{11} \) is non-invertible, then ker \( N_{11} \neq 0 \). Let \( v \in V_3(B) \) be a non-zero vector in ker \( N_{11} \), then by (5.18), we have \( N_{21}(v) = 0 \), thus
\[
\begin{pmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22} 
\end{pmatrix}
\]
is non-invertible.

We now assume \( N_{11} \) is invertible and prove Part (2). By taking \( s = 1 \), Part (2) implies Part (1) of the lemma in the case that \( N_{11} \) is invertible.

Since the operator \( K_{B, \pi} \) is self-adjoint and depends continuously on \((B, \pi)\), we have
\[
\lim_{(\epsilon, t) \to (0, t_0)} \| N_{12} \| = \lim_{(\epsilon, t) \to (0, t_0)} \| N_{21} \| = 0.
\]
By Lemma 5.18, the operator $N_{22}$ converges to the isomorphism (5.16) as $(\epsilon, t) \to (0, t_0)$. Therefore, there exist constants $z_4$, $\epsilon_2$ only depending on $B_0$, such that when $\epsilon < \epsilon_2$, $t \in (t_0 - \epsilon_2, t_0 + \epsilon_2)$, we have
\[
\|N_{22}^{-1}\| \leq z_2, \quad \text{and} \quad \|N_{12}\|, \|N_{21}\| \leq \frac{1}{2z_1 z_2}.
\]
Notice that
\[
\begin{pmatrix}
\text{id} & -sN_{12} \circ N_{22}^{-1} \\
0 & \text{id}
\end{pmatrix}
\begin{pmatrix}
N_{11} & sN_{12} \\
N_{21} & N_{22}
\end{pmatrix}
\begin{pmatrix}
\text{id} & 0 \\
-sN_{22}^{-1} \circ N_{21} & \text{id}
\end{pmatrix}
= \begin{pmatrix}
N_{11} - s^2 N_{12} \circ N_{22}^{-1} \circ N_{21} & 0 \\
0 & N_{22}
\end{pmatrix}.
\]
(5.19)
For every $v \in V_3(B)$ and $s \in [0, 1]$, we have
\[
\| (N_{11} - s^2 N_{12} \circ N_{22}^{-1} \circ N_{21})(v) \| \geq \|N_{11}(v)\| - s^2 \|N_{12}\| \cdot \|N_{22}^{-1}\| \cdot \|N_{21}(v)\|
\geq \|N_{11}(v)\| - s^2 \|N_{12}\| \cdot \|N_{22}^{-1}\| \cdot (z_1 \|N_{11}(v)\|)
\geq \|N_{11}(v)\| - s^2 \cdot \frac{1}{2z_1 z_2} \cdot \frac{z_2}{z_1} \|N_{11}(v)\|
\geq \frac{1}{2} \|N_{11}(v)\|.
\]
Since $N_{11}$ is injective, the estimates above imply that $N_{11} - s^2 N_{12} \circ N_{22}^{-1} \circ N_{21}$ is injective, therefore it is invertible for all $tsin[0, 1]$. By (5.19), the operator
\[
\begin{pmatrix}
N_{11} & sN_{12} \\
N_{21} & N_{22}
\end{pmatrix}
\]
is invertible for all $s \in [0, 1]$.

We can now finish the proof of Lemma 5.15. By definition, $B$ is non-degenerate as a $\pi_t$–flat connection if and only if $\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$ is invertible. By Part (1) of Lemma 5.19, this is equivalent to $N_{11}$ being invertible. On the other hand, $N_{11}$ is conjugate to the Hessian of $CS_t$ as a function on $M_{t, \epsilon}$ restricted to the normal direction of the $H_0$–orbit of $B$. Therefore $B$ is non-degenerate as a $\pi_t$–flat connection if and only if it is non-degenerate as a critical point of $CS_t$ on the $H_0$–manifold $M_{t, \epsilon}$.

To compare the indices of $B_0$ and $B$ as perturbed flat connections when $B$ is non-degenerate, we need to compute the $H$–equivariant spectral flow from the operator (5.17) to the operator (5.14).

Recall that $M_{11}$ is invertible and $M_{12} = 0$. Therefore by Lemma 5.19, the linear deformation from (5.14) to
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M_{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & N_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_{22}
\end{pmatrix}
\]
has zero spectral flow.
We then deform (5.20) to the following operator via a linear deformation:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M_{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & N_{22}
\end{pmatrix}.
\]

(5.21)

Let \( V^{-3}(B) \subset V_3(B) \) be the subspace generated by the negative eigenvectors of \( N_{11} \), then the \( H \)-equivariant spectral flow from (5.20) to (5.21) is given by

\[
[V_2(B)] + [V^{-3}_3(B)] \in \mathcal{R}(H).
\]

Finally, notice that all the maps constructed in Lemma 5.18 are \( H \)-equivariant, therefore when \((\epsilon, t)\) is sufficiently close to \((0, t_0)\), the linear homotopy from (5.21) to (5.14) has zero spectral flow. In conclusion, the \( H \)-equivariant spectral flow from \( K_{B, \pi t} \) to \( K_{B_0, \pi t_0} \) is represented by the \( H \)-representation \( V_2(B) \oplus V^{-3}_3(B) \). Since \( V^{-3}_3(B) \) also represents the equivariant index of \( B \) as a critical point on \( M_{\epsilon, t} \), and

\[
[V_2(B)] = [\ker d_{B_0}] - [\ker d_B]
\]

in the representation ring of \( H \), the desired result follows from the definition of \( \text{ind}(B, \pi) \) and Equations (5.4) and (5.8).

6. Computations and examples

6.1. Characterization of irreducible bifurcations on \( C \). Notice that although the definition of \( R_G \) is given by the representations of all closed subgroups of \( G \), for a fixed \( G \)-manifold \( M \), there are only finitely many possible conjugation classes of \( \text{Stab}(p) \) for \( p \in M \), and there are only finitely many representations (up to conjugations by \( G \)) that can represent the equivariant index of critical points.

Similarly, for the perturbed Chern-Simons functionals on \( C \), only finitely many stabilizer groups (up to conjugations) and irreducible representations arise in the description of bifurcations. We already classified all the possible stabilizer groups in Section 5.1, this subsection classify all the possible irreducible representations, and characterize the corresponding bifurcations.

Recall that the set \( \Sigma_n \) is defined by Definition 5.1. Let

\[
\sigma = ((n_1, m_1), \ldots, (n_r, m_r)) \in \Sigma_n,
\]

and let \( B \in \mathcal{C}_\sigma \). After a gauge transformation, we may assume that \( B \in \mathcal{C}^{H_\sigma} \). Then \( E = P \times_{\text{SU}(n)} \mathbb{C}^n \) decomposes as

\[
E = E(n_1)^{\oplus m_1} \oplus \cdots \oplus E(n_r)^{\oplus m_r},
\]

(6.1)

where \( E(n_i) \) are constant subbundles of \( E \) with rank \( n_i \), and \( B \) is given by the direct sum of irreducible connections on each \( E(n_i) \).

Recall that \( T \) denotes the tangent bundle of \( C \), and we have

\[
T|_B = L^2_k(T^*Y \otimes \mathfrak{g}).
\]

The action of \( \text{Stab}(B) \cong H_\sigma \) on \( T|_B \) is given pointwise on \( \mathfrak{g} \) by the adjoint action of \( H_\sigma \). Therefore, we only need to find all the irreducible components of \( \mathfrak{g} \) as \( H_\sigma \)-representations.
Decompose $\mathfrak{g} = \mathfrak{su}(n)$ as

$$\mathfrak{g} = \mathfrak{g}_\sigma \oplus \mathfrak{g}_\sigma^\perp,$$

where $\mathfrak{g}_\sigma$ is the Lie algebra of the subgroup of SU($n$) that preserves the decomposition (6.1), and $\mathfrak{g}_\sigma^\perp$ is the orthogonal complement of $\mathfrak{g}_\sigma$. Then the action of $H_\sigma$ on $\mathfrak{g}_\sigma$ is trivial, and one only needs to compute the irreducible components of $\mathfrak{g}_\sigma^\perp$ as a representation of $H_\sigma$. The space $\mathfrak{g}_\sigma^\perp \subset \mathfrak{su}(n)$ consists of the matrices

$$\begin{pmatrix}
W_{11} & W_{12} & \ldots & W_{1r} \\
W_{21} & W_{22} & \ldots & W_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
W_{r1} & W_{r2} & \ldots & W_{rr}
\end{pmatrix},$$

(6.2)

such that for $1 \leq p, q \leq r$:

1. if $p \neq q$, $W_{pq}$ is a $C$-valued $m_p n_p \times m_q n_q$ matrix, and $W_{pq} = -W_{qp}$;
2. if $p = q$, $W_{pp}$ is given by

$$\begin{pmatrix}
W_{11}^{(p)} & W_{12}^{(p)} & \ldots & W_{1m_p}^{(p)} \\
W_{21}^{(p)} & W_{22}^{(p)} & \ldots & W_{2m_p}^{(p)} \\
\vdots & \vdots & \ddots & \vdots \\
W_{m_p 1}^{(p)} & W_{m_p 2}^{(p)} & \ldots & W_{m_p m_p}^{(p)}
\end{pmatrix},$$

where

$$W_{11}^{(p)}, \ldots, W_{m_p m_p}^{(p)} \in \mathfrak{su}(n_p),$$

$$W_{11}^{(p)} + \cdots + W_{m_p m_p}^{(p)} = 0,$$

and

$$W_{i,j}^{(p)} = -(W_{j,i}^{(p)})^*$$

for all $i, j$.

For $1 \leq p \leq q \leq r$, let $\mathfrak{g}_\sigma^\perp(p, q)$ be the subspace of $\mathfrak{g}_\sigma^\perp$ consisting of matrices in the form (6.2) such that $W_{ij} = 0$ unless $(i, j) = (p, q)$ or $(q, p)$. Then $\mathfrak{g}_\sigma^\perp(p, q)$ is invariant under the action of $H_\sigma$.

For $p = 1, \ldots, r$, let

$$\varphi_p : H_\sigma \to U(m_p)$$

be given by the restriction of $H_\sigma$ to $E(n_p)^{m_p}$. Then

$$\text{Im} \varphi_p = \begin{cases} 
U(m_p) & \text{if } r \geq 2, \\
SU(m_p) & \text{if } r = 1.
\end{cases}$$

(6.3)

Moreover, for $1 \leq p < q \leq r$, the image of

$$\varphi_p \times \varphi_q : H_\sigma \to U(m_p) \times U(m_q)$$

is given by

$$\text{Im}(\varphi_p \times \varphi_q) = \begin{cases} 
U(m_p) \times U(m_q) & \text{if } r \geq 3, \\
S \left( U(m_p)^{n_r} \times U(m_q)^{n_q} \right) & \text{if } r = 2.
\end{cases}$$

(6.4)

where the group $S \left( U(m_p)^{n_r} \times U(m_q)^{n_q} \right)$ is defined by (5.2).

For $1 \leq p \leq r$, let $V_p$ be the representation of $H_\sigma$ on $\mathfrak{su}(m_p)$ given by the composition of $\varphi_p$ and the adjoint action of $U(m_p)$. For $1 \leq p < q \leq r$, let $V_{p,q}$
be the representation of $H_\sigma$ on $\text{Mat}_{m_p \times m_q}(\mathbb{C})$, where the action of $h \in H_\sigma$ on $x \in \text{Mat}_{m_p \times m_q}(\mathbb{C})$ is given by $\varphi_p(h) \cdot x \cdot \varphi_q(h)^{-1}$.

Then the following lemma gives a complete description of the isotypic decomposition of $g_+^\perp$ as an $H_\sigma$-representation.

**Lemma 6.1.** Suppose $1 \leq p < q \leq r$.

1. $V_p$ and $V_{p,q}$ are irreducible representations of $H_\sigma$.
2. The representation of $H_\sigma$ on $g_+^\perp(p, q)$ is given by the direct sum of $n_p \cdot n_q$ copies of $V_{p,q}$.
3. The representation of $H_\sigma$ on $g_+^\perp(p, p)$ is given by the direct sum of $n_p^2$ copies of $V_p$.

**Proof:** The irreducibility of $V_p$ and $V_{p,q}$ follows from (6.3) and (6.4). The rest of the lemma is a straightforward consequence of the definition of $g_+^\perp(p, q)$. □

**Definition 6.2.** Suppose

$$\sigma = ((m_1, m_1), \ldots, (m_r, m_r)) \in \Sigma_n,$$

and let $V_p$ and $V_{p,q}$ be given as above. We say that $\sigma' \in \Sigma_n$ bifurcates from $\sigma$, if at least one of the following holds:

1. There exist $p$ and $0 \neq x \in V_p$, such that $\text{Stab}(x) \subset H_\sigma$ is conjugate to $H_{\sigma'}$ in $\text{SU}(n)$.
2. There exist $p < q$ and $0 \neq x \in V_{p,q}$, such that $\text{Stab}(x) \subset H_\sigma$ is conjugate to $H_{\sigma'}$ in $\text{SU}(n)$.

Notice that for every $x \in V_p$, there exists $h \in H_\sigma$ such that $h(x)$ is given by a diagonal matrix. Similarly, for every $x \in V_{p,q}$, there exists $h \in H_\sigma$, such that $h(x) \in \text{Mat}_{m_p \times m_q}(\mathbb{C})$ has the form

$$\begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\
0 & 0 & \cdots & \lambda_{m_p} & 0 & \cdots & 0
\end{pmatrix}.$$

Therefore the following lemma follows from a straightforward computation in linear algebra.

**Lemma 6.3.** Suppose $\sigma = ((m_1, m_1), \ldots, (m_r, m_r)) \in \Sigma_n$. Then $\sigma'$ bifurcates from $\sigma$ if and only if $\sigma'$ is given by one of the following, after a permutation of the entries of $\sigma'$:

1. replacing a pair $(n_p, m_p)$ in $\sigma$ by a sequence $(n_p, m'_1), \ldots, (n_p, m'_j)$, such that $m_p = m'_1 + \cdots m'_j$,
2. replacing two pairs $(n_p, m_p), (n_q, m_q)$ in $\sigma$, where $m_p \leq m_q$, by a sequence

$$(n_q, m_q - m_p), (n_p + n_q, m'_1), \ldots, (n_p + n_q, m'_j),$$

or

$$(n_q, m_q - m_p), (n_p + n_q, m'_1), \ldots, (n_p + n_q, m'_{j-1}), (n_p, m'_j),$$

such that $m_p = m'_1 + \cdots m'_j$. □
6.2. A closed formula of SU(4) Casson invariant. We write down an explicit closed formula of SU(4) Casson invariant using the previous computations.

The set $\Sigma_4$ has 5 elements. For each $\sigma \in \Sigma_4$, we list the irreducible components that may appear in the isotypic decomposition of the equivariant spectral flows on $C^{H_\sigma}$, and introduce a notation for each component of the isotypic decomposition.

1. If $\sigma = ((4,1),),$ the equivariant spectral flow is given by trivial representations of $\mathbb{Z}/4$. Denote the equivariant spectral flow and the equivariant index by $Sf_{(4,1)}$ and $\text{ind}_{(4,1)}$ respectively.

2. If $\sigma = ((1,1),)$, then $H_\sigma \cong U(1)$, and it is given by

$$\text{Diag}(e^{i\alpha}, e^{i\beta}, e^{i\beta})$$

where $\alpha + 3\beta \equiv 0 \mod 2\pi$. Taking $\beta$ to be the coordinate on $U(1)$, the Lie algebra $\mathfrak{su}(4)$ is decomposed as $\mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(3)) \oplus \mathbb{C}^4$ such that $U(1)$ acts trivially on the first component and acts by $e^{i\beta}$ on each $\mathbb{C}$ component. Write the latter irreducible representation as $(U(1),((1,1),),\mathbb{C},\rho_4)$.

Decompose the spectral flow $Sf_{H_\sigma}$ as

$$Sf_{((1,1),,)} \oplus Sf_{((1,1),,)^{-}}$$

where $Sf_{((1,1),,)}$ is given by the trivial components, and $Sf_{((1,1),,)^{-}}$ is given by the components that are isomorphic to $(U(1),((1,1),,),\mathbb{C},\rho_4)$. Decompose the equivariant index (5.6) similarly as $\text{ind}_{((1,1),,)} \oplus \text{ind}_{((1,1),,)^{-}}$.

3. If $\sigma = ((2,1),)$, then $H_\sigma \cong U(1)$, and it is given by

$$\text{Diag}(e^{i\alpha}, e^{i\alpha}, e^{i\beta}, e^{i\beta})$$

with $\alpha + \beta \equiv 0 \mod 2\pi$. Taking $\beta$ to be the coordinate on $U(1)$, the Lie algebra $\mathfrak{su}(4)$ is decomposed as $\mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(2)) \oplus \mathbb{C}^4$ such that $U(1)$ acts trivially on the first component and acts by $e^{i\beta}$ on each $\mathbb{C}$ component. Write the latter irreducible representation as $(U(1),((2,1),),\mathbb{C},\rho_2)$.

Decompose the spectral flow $Sf_{H_\sigma}$ as

$$Sf_{((2,1),,)} \oplus Sf_{((2,1),,)^{-}}$$

where $Sf_{((2,1),,)}$ is given by the trivial components, and $Sf_{((2,1),,)^{-}}$ is given by the components that are isomorphic to $(U(1),((2,1),),\mathbb{C},\rho_2)$. Decompose the equivariant index (5.6) similarly as $\text{ind}_{((2,1),,)} \oplus \text{ind}_{((2,1),,)^{-}}$.

4. If $\sigma = (2,2)$, then $H_\sigma \cong SU(2)$, and it consists of the matrices of determinant 1 with the form

$$\begin{pmatrix}
u_{11} \text{id} & \nu_{12} \text{id} \\
\nu_{21} \text{id} & \nu_{22} \text{id}
\end{pmatrix},$$

where id is the $2 \times 2$ identity matrix and

$$\begin{pmatrix}
u_{11} & \nu_{12} \\
\nu_{21} & \nu_{22}
\end{pmatrix}$$

is a unitary $2 \times 2$ matrix. The Lie algebra $\mathfrak{su}(4)$ is decomposed as $\mathfrak{l}_{(2,2)} \oplus \mathfrak{su}(2)^{\oplus 4}$ such that $H_\sigma$ acts on the first component trivially and acts on each $\mathfrak{u}(2)$ component by restricting the adjoint action of $U(2)$. Write the latter irreducible representation as $(SU(2),\mathfrak{su}(2),\text{Ad})$. 


Decompose the equivariant spectral flow $S_{f_{H_\sigma}}$ as $S_{f_{(2,2)}} \oplus S_{f_{(2,2)^+}}$, where $S_{f_{(2,2)}}$ is given by the trivial components, and $S_{f_{(2,2)^+}}$ is given by the components that are isomorphic to $(\text{SU}(2), \mathfrak{su}(2), \text{Ad})$. Decompose the equivariant index (5.6) similarly as $\text{ind}_{(2,2)} \oplus \text{ind}_{(2,2)^+}$.

(5) If $\sigma = ((1, 2), (2, 1))$, then $H_{\sigma}$ consists of matrices of the form

$$\begin{pmatrix}
 u_{11} & u_{12} & 0 & 0 \\
 u_{21} & u_{22} & 0 & 0 \\
 0 & 0 & 0 & e^{i \alpha} \\
 0 & 0 & 0 & e^{i \alpha}
\end{pmatrix} \in \text{SU}(4).$$

The Lie algebra $\mathfrak{su}(4)$ is decomposed as $\mathfrak{I}_{((1,2),(2,1))} \oplus \mathfrak{su}(2) \oplus \text{Mat}_{2 \times 1}(\mathbb{C})^{\oplus 2}$. Write the last irreducible representation as $(\text{SU}(2) \times \text{U}(1)), \text{Mat}(\mathbb{C})_{2 \times 1}, \text{mult})$.

Decompose the equivariant spectral flow $S_{f_{H_\sigma}}$ as $S_{f_{((1,2),(2,1))}} \oplus S_{f_{((1,2),(2,1))^+}} \oplus S_{f_{((2,1),(2,1))^+}}$, where $S_{f_{((1,2),(2,1))}}$ is given by the trivial components, $S_{f_{((1,2),(2,1))^+}}$ is given by the components that are isomorphic to the action of $H_{\sigma}$ on $\mathfrak{su}(2)$, and $S_{f_{((2,1),(2,1))^+}}$ is given by the components that are isomorphic to $(\text{SU}(2) \times \text{U}(1)), \text{Mat}(\mathbb{C})_{2 \times 1}, \text{mult})$.

Decompose the equivariant index $(5.6)$ as $\text{ind}_{((1,2),(2,1))} \oplus \text{ind}_{((1,2),(2,1))^+} \oplus \text{ind}_{((2,1),(2,1))^+}$.

The possible irreducible bifurcations on $C$ are given by

$$((1, 1), (3, 1)) \rightarrow ((4, 1)), \quad ((2, 1), (2, 1)) \rightarrow ((4, 1)),$$

$$((2, 2)) \rightarrow ((2, 1), (2, 1)), \quad ((1, 2), (1, 2)) \rightarrow ((1, 1), (3, 1)).$$

Note that the isotypical piece $\mathfrak{su}(2)$ corresponding to type $((1, 2), (2, 1))$ would induce the bifurcation $((1, 2), (2, 1)) \rightarrow ((1, 1), (1, 1), (2, 1))$, but the latter does not exist for perturbed-flat SU(4)-connections over an integer homology sphere for small perturbations. In the case of $\text{SU}(4)$, the stabilizers act transitively on the unit spheres of the relevant irreducible representations. Therefore, the only equivariant Morse functions on the unit spheres are the constant functions.

Let $\rho_0$ denote the trivial representations, and let 0 denote the zero representations. Then the following are the corresponding of values of $\xi_H(V, 0, g)$ for the bifurcations above:

$$[U(1)_{((1,1),(3,1))}, \mathbb{C}, \rho_4] - [U(1)_{((1,1),(3,1))}, 0] - [\mathbb{Z}/4, \mathbb{R}, \rho_0],$$

$$[U(1)_{((2,1),(2,1))}, \mathbb{C}, \rho_2] - [U(1)_{((2,1),(2,1))}, 0] - [\mathbb{Z}/4, \mathbb{R}, \rho_0],$$

$$[\text{SU}(2), \mathfrak{su}(2), \text{Ad}] - [\text{SU}(2), 0] - [U(1)_{((2,1),(2,1))}, \mathbb{R}, \rho_0],$$

$$[S(\text{U}(2) \times \text{U}(1)), \text{Mat}(\mathbb{C})_{2 \times 1}, \text{mult}] - [S(\text{U}(2) \times \text{U}(1)), 0] - [\text{U}(1)_{((1,1),(3,1))}, \mathbb{R}, \rho_0].$$

Suppose $H$ is a compact Lie group, let

$$\text{dim} : \mathcal{R}(H) \rightarrow \mathbb{Z}$$

be the map given by taking formal dimensions. Then the map extends linearly to

$$\text{dim} : \mathcal{R}(H) \otimes \mathbb{R} \rightarrow \mathbb{R},$$

and hence it defines a map

$$\bar{\text{dim}} : \bar{\mathcal{R}}_G \rightarrow \mathbb{R}$$
for any compact Lie group $G$.

Choose $\pi \in \mathcal{P}$ to be a generic small perturbation and let $\mathcal{M}_\pi$ be the moduli space of $\pi$-flat connections. Then the following formula defines an SU(4)–Casson invariant:

$$\sum_{[B] \in \mathcal{M}_\pi(4,1)} (-1)^{\dim Sf(4,1)(B)}$$

$$+ \sum_{[B] \in \mathcal{M}_\pi((1,1),(3,1))} (-1)^{\dim Sf((1,1),(3,1))(B)} \cdot \frac{1}{2} \dim \ker d_{\pi}(B)$$

$$+ \sum_{[B] \in \mathcal{M}_\pi((2,1),(2,1))} (-1)^{\dim Sf((2,1),(2,1))(B)} \cdot \frac{1}{2} \dim \ker d_{\pi}(B)$$

$$+ \frac{1}{2} \sum_{[B] \in \mathcal{M}_\pi((1,2),(2,1))} (-1)^{\dim Sf((1,2),(2,1))(B)} \cdot \frac{1}{3} \dim \ker d_{\pi}(B)$$

$$\cdot \left( \frac{1}{3} \dim \ker d_{\pi}(B) + 1 \right)$$

$$+ \frac{1}{2} \sum_{[B] \in \mathcal{M}_\pi((1,2),(2,1))} (-1)^{\dim Sf((1,2),(2,1))(B)} \cdot \frac{1}{4} \dim \ker d_{\pi}(B)$$

$$\cdot \left( \frac{1}{4} \dim \ker d_{\pi}(B) + 1 \right).$$

**Remark 6.4.** The extra $(-1)$'s on the exponents of $(-1)$ in the above formula comes from the trivial components of the term $-\ker d_{\pi}(B)$ in Definition 5.9

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, NEW JERSEY 08544, USA
E-mail address: shaoyunb@math.princeton.edu

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, NEW JERSEY 08544, USA
E-mail address: bz@math.princeton.edu