Numerical estimation of the curvature of a light wavefront in a weak gravitational field

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Abstract

The geometry of a light wavefront evolving in the 3-space associated with a post-Newtonian relativistic spacetime from a flat wavefront is studied numerically by means of the ray tracing method. For a discretization of the bidimensional wavefront, the surface fitting technique is used to determine the curvature of this surface at each vertex of the mesh. The relationship between the curvature of a wavefront and the change of the arrival time at different points on the Earth is also numerically discussed.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The description of the propagation of light in a gravitational field is even today a central problem in the general theory of relativity. The deflection of light rays and time delays of electromagnetic signals due to the presence of a gravitational field are phenomena detectable with current experimental techniques which allow designing new tests for general relativity. In this line, Samuel [1] has recently proposed a method for the direct measurement of the curvature of a light wavefront initially flat, curved when light crosses regions where the gravitational field is non-vanishing. He found a relationship between the differences of arrival time recorded at four points on the Earth measured by employing techniques of very long base interferometry and the volume of a parallelepiped determined by four points in the curved wavefront surface. This surface is described by means of a polynomial approximation of the eikonal in a Schwarzschild gravitational field. For more complex gravitational models, such as those considered by Klioner and Peip [2], de Felice et al [3] or Kopeikin and Schäfer [4] in studies of light propagation in the solar system, the use of numerical methods for the determination of the geometry of the wavefront surface would also be required. An
analytical approach to the relativistic modeling of light propagation has also been developed recently by Le Poncin-Lafitte et al [5] and Teyssandier and Le Poncin-Lafitte [6], where they present methods based on Synge’s world function and the perturbative series of powers of the Newtonian gravitational constant to determine the post-Minkowskian expansions of the time transfer functions.

Nowadays there are numerous techniques in computational differential geometry which allow one to analyze geometric properties of surfaces embedded in the ordinary Euclidean space. Techniques of this type are widely applied in different areas such as computational geometry, computer vision or seismology. In one of these methods, developed in works by Garimella and Swartz [7] and Cazals and Pouget [8], the estimation of differential quantities is established using a fitting of the local representation of the surface by means of a height function given by a Taylor polynomial. A survey of methods for the extraction of quadric surfaces from triangular meshes is found in Petitjean [9].

In this work, we consider a discretization of the wavefront surface replacing this surface by a polyhedral whose faces are equilateral triangles. At initial time, the surface is assumed to be flat and far enough from a gravitational source (say, the Sun) and moving towards this source. We study the deformation of the instantaneous polyhedral representing the wavefront when crossing a region in the relativistic 3-space near the Sun due to the bending of light rays by the gravitational field. In this study, we apply the ray tracing method with initial values on the vertices of the triangular mesh to obtain the corresponding discrete surface at each instant of time. Then we apply the techniques given in [7] and [8] to describe the wavefront as a surface embedded in the Riemannian 3-space of the post-Newtonian formalism of general relativity. For each vertex in the instantaneous mesh, we obtain a quadric which represents locally the surface by applying the least-squares method to the immediate neighboring vertex around the considered point which is represented in normal coordinates adapted to the light rays.

The structure of this paper is as follows. In section 2, we briefly introduce the basic model for the wavefront propagation in the post-Newtonian formalism. In section 3, we establish a discretized model of the wavefront surface by means of a regular triangulation, and we describe the method employed in this work for the study of the curvature of this surface. In section 4, a numerical estimation of the curvature of the surface is derived using the ray tracing method. For the numerical integration of the light ray equation, we use the Taylor algorithm implemented by Jorba and Zou [10] which is based on the Taylor series method for the integration of ordinary differential equations and which allows the use of high-order numerical integrators and arbitrary arithmetic accuracy as is required to describe the influence of weak gravitational perturbations on the bending of light rays. Finally, in section 5, we apply the method discussed above to study the effect of the wavefront curvature on the variation of the arrival time of the light at points on the Earth’s surface, following the model proposed in Samuel’s test [1]. The paper concludes by giving another approach to the estimation of the curvature of the wavefront derived from an approximation of the Wald curvature [12] associated with a quadruple of points in the wavefront.

2. Light propagation in a gravitational field

Let us consider a spacetime $(\mathcal{M}, g)$ corresponding to a weak gravitational field and choose a coordinate system $(z, ct)$ such that the coordinate representation of the metric tensor is

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad \text{with} \quad \eta_{\alpha\beta} = \text{diag} (1, 1, 1, -1).$$

(1)
where the coordinate components of the metric deviation $h_{ab}$ are given by
\[ h_{ab} = 2c^{-2}k \| z \|^{-1} \delta_{ab}, \quad h_{a4} = -4c^{-3}k \| z \|^{-1} Z_{a}, \quad h_{44} = 2c^{-2}k \| z \|^{-1}, \] (Greek indices run from 1 to 4 and Latin indices from 1 to 3) where $k := GM$ represents the gravitational constant of a monopolar distribution of matter (say the Sun) located at $Z^a(t)$ and $c$ represents the light speed.

In the post-Newtonian framework, one may consider a simultaneity space $\Sigma_t$ at each coordinate time $t$. From the fundamental equation of the geometrical optics for the phase \( \psi(z,t) \) of an electromagnetic wave \cite{11}:
\[ g^{ab} \frac{\partial \psi}{\partial z^a} \frac{\partial \psi}{\partial z^b} = 0, \] (3)
and Cauchy data \( \psi(z,0) = \text{const} \), given on a spacelike surface \( \mathcal{D}_0 := \{ (z,0) | \phi(z) = 0 \} \subset \Sigma_0 \) one obtains the characteristic hypersurface (light cone) \( \Omega := \{ (z,t) | \psi(z,t) = \text{const} \} \subset \mathcal{M} \) of the light propagation. The intersection of the characteristic hypersurface and the corresponding simultaneity space is the spacelike wavefront at time $t$ which will be denoted by $\mathcal{F}_t := \Omega \cap \Sigma_t$.

An alternative formulation of the problem of light propagation in the spacetime \( (\mathcal{M},g) \) is based on the determination of the bicharacteristics generated by the isotropic vectors $k := \nabla \psi$. From both the equation for the null geodesics $z(t) = (z(t), t)$ expressed in terms of the coordinate time and the isotropy condition $g(\dot{z}, \dot{z}) = 0$, one obtains in the post-Newtonian approach to general relativity that, neglecting terms of order $O(c^{-2})$, the null geodesics of \( (\mathcal{M},g) \) must satisfy the equations
\[ \ddot{z}^a = \varphi_a(z, \dot{z}, t) \quad \text{and} \quad 0 = g_{ab} \dot{z}^a \dot{z}^b. \] (4a)
(4b)

In (4a) the components of the acceleration $\varphi_a(z, \dot{z}, t)$ are given by (see \cite{13})
\[ \varphi_a(z, \dot{z}, t) = \frac{1}{c^2} h_{44,a} - \left[ \frac{1}{2} h_{44,\dot{z}^j} + h_{3k,l} + c(h_{4k,l} - h_{4l,k}) \right] \dot{z}^j \dot{z}^l - \left( h_{44,k} \delta^j_{l} + h_{ak,l} - \frac{1}{4} h_{kl,\dot{z}} \right) \dot{z}^j \dot{z}^l - \left( c^{-1} h_{4k,l} - \frac{1}{4} c^{-2} h_{jkl} \right) \dot{z}^j \dot{z}^l, \] (5)
where the first and third terms on the right-hand side of (5) are of order $O(1)$, while the remaining terms are $O(c^{-1})$.

For initial values \( z_0, \dot{z}_0 \), with $z_0 \in \mathcal{D}_0$ and \( (z_0/c, l) \), satisfying the condition (4b), the integration of the initial value problem corresponding to (4a) on an interval [0, $T$] allows us to determine the spacelike wavefront $\mathcal{F}_T$. This surface is embedded in the Riemannian 3-space \( (\Sigma_T, \hat{g}) \) where the components of the metric tensor are given by
\[ \hat{g}_{ab} := g_{ab} - \frac{g_{4a}g_{4b}}{g_{44}}, \] (6)
and at time $T$ the tangent vectors $\dot{z}(T)$ to the integral curves $z(t)$ are $\hat{g}$-orthogonal to $\mathcal{F}_T$.

3. Numerical description of a spacelike bidimensional wavefront

Hereafter, we consider the simplest gravitational model generated by a static point mass. Let $\mathcal{E}$ denote the quotient space of $\mathcal{M}$ by the global timelike vector field $\partial_t$ associated with the global coordinate system used in the post-Newtonian formalism. We will consider a region of the wavefront in $\mathcal{E}$ described by a coordinate chart \( \{ z \} \). Further, we assume that the Riemannian manifold \( (\mathcal{E}, \hat{g}) \) is almost flat and that the metric corresponding to $\hat{g}$ is quasi-Cartesian in the chosen coordinates.
3.1. Discretization of the initial wavefront

Given an asymptotically Cartesian coordinate system, we consider a set \( \mathcal{S}_0 \) formed by points with coordinates \((z_1, z_2, -\xi)\) where \( \xi > 0 \) is a number large enough so that \( \mathcal{S}_0 \) may be considered as a flat surface. The direction determined by the point \( O^* : (0, 0, -\xi) \) in \( \mathcal{S}_0 \) and the center of the Sun \( O : (0, 0, 0) \) is perpendicular to \( \mathcal{S}_0 \). We will study the geometry of a region \( \mathcal{C} \subset \mathcal{S}_0 \) determined by points \( P \) whose Euclidean distances \( d(P, O^*) \) to the point \( O^* \) satisfy \( R_0 \leq d(P, O^*) \leq 2R_0 \), where \( R_0 \) is the radius of the Sun.

For the discretization of the problem we consider, in the first place, the set of points \((a_1, a_2, a_3) \in \mathcal{A}^3\), where \( \mathcal{A} = [0, N_1, N_1 + 1, \ldots, N_2] \) and \( N_1 < N_2 \) are two natural numbers (see figure 1(a)). In \( \mathcal{A}^3 \) the point \((0, 0, 0)\) is excluded. Next, we construct in the complex plane a regular triangular mesh whose vertices are located between two hexagons as shown in figure 1(b), and the edges have length \( \ell \). The inner hexagon has sides of length \( \ell N_1 \) and the outer hexagon sides of length \( \ell N_2 \). In this triangulation each vertex is represented by a complex number of the set (see [14])

\[
\mathcal{F} := \{ z = a_1 + a_2 \omega + a_3 \omega^2 \mid a_1, a_2, a_3 \in \mathcal{A}, \ \omega := \exp(2\pi i/3) \},
\]

where \( i := \sqrt{-1} \). The vertices \((a_1, a_2, a_3)\) with some of their components equal to \( N_1 \) (resp. \( N_2 \)) are located on the inner (resp. outer) boundary of the mesh \( \mathcal{F} \). We establish an enumeration of the vertices as shown in figure 1(b), in such a way that the inner vertices \( z_j \) have subscripts \( j = 1, \ldots, J \). Finally, we apply the change of scale:

\[
\mathcal{F} \to \mathcal{F}^*, \quad z \mapsto \frac{z r}{N_1},
\]

so that the inner boundary is a hexagon of radius \( r \). In consequence, the length of each edge in this triangulation is equal to \( r/N_1 \).

The complex plane and the plane \( \mathcal{S}_0 \) may be identified by means of the mapping \( \iota : z \mapsto (\Re(z), \Im(z), -\xi) \). Thus, one obtains a discretization of the initial region of \( \mathcal{S}_0 \) which we are considering here, triangulated with vertices given by \( \iota(z_j) \) whose corresponding mesh on \( \mathcal{S}_0 \) will also be denoted by \( \mathcal{F}^* \).
3.2. Normal coordinates around a point

Now we assume that at each vertex in $\mathcal{V}^*$ a photon with velocity $\mathbf{z}_0 := (0, 0, c)$ is located. The null geodesics equation (4a) may be written as a first-order differential system $\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}, t)$ in phase space $\mathbf{u} := (\mathbf{z}, \dot{\mathbf{z}})$ which determines a flow in $\mathcal{E}$:

$$\mathbf{z}(t) = \Phi_t(\mathbf{z}_0, \dot{\mathbf{z}}_0),$$

in terms of the initial values $\mathbf{z}_0 := z(0) \in \mathcal{V}^*$, $\dot{\mathbf{z}}_0 := \dot{z}(0)$. Then, for each time $t$ there is a surface $s_t$ image of $s_0$ under the flow (9). At a point $z \in s_t$ the normal vector $\tau(z)$ coincides with the tangent vector to the curve $\mathbf{z}(t)$ at that point. Furthermore, before reaching the focal points of the beam of light, the triangulation $\mathcal{V}^*$ induces a triangulation $\mathcal{V}$ on the wavefront $s_t$ whose vertices we enumerate using the same labels used for the corresponding vertices in $s_0$.

In order to simplify the description of the geometry of the surface $s_t$ on a neighborhood of a point $P \in s_t$ we use a $\tilde{\gamma}$-orthonormal reference frame $\{e_i\}_{i=1}^3$ centered on that point, where one of its vectors, say $e_3$, is parallel to the vector $\mathbf{n}(P) := \tau(P)/\sqrt{\gamma(\tau, \tau)}$ tangent to the ray passing through that point. Let $\{y^j\}_{j=1}^3$ be a normal coordinate system with pole at point $P$ and associated normal reference frame $\{e_i\}_{i=1}^3$.

By using the classic formulae of Riemannian geometry (see [15] section 18, and [16]), the coordinate transformation $y^j \mapsto z^a$, from normal to post-Newtonian coordinates, is determined by

$$y^j = (\Lambda^{-1})^j_a (z^a - z_0^a + \frac{1}{2} (\Gamma^a_{bc})_0 (z^b - z_0^b) (z^c - z_0^c)),$$

where $\Lambda$ is a non-singular constant matrix and $(\Gamma^a_{bc})_0 := \tilde{\Gamma}^a_{bc}(P)$ are the Christoffel symbols at point $P$. By neglecting terms of order higher than $\Gamma^a_{bc}$, the inverse transformation may be approximated by

$$z^a = a_0^a + \Lambda^a_j (y^j - \frac{1}{2} (\Gamma^j_{ab})_0 y^b),$$

whose corresponding Jacobian determinant is

$$\frac{\partial z^a}{\partial y^j} = \Lambda^a_j (\delta^a_j - (\Gamma^a_{jb})_0 y^b).$$

In normal coordinates, a metric tensor $\gamma$ on the space $\mathcal{E}$ is determined from $\tilde{\gamma}$ by

$$\gamma_{ij} = \frac{\partial z^a}{\partial y^i} \frac{\partial z^b}{\partial y^j} \tilde{\gamma}_{ab},$$

and, as is well known, at point $P$ the tensor $\gamma_{ij}$ is reduced to $\delta_{ij}$ and the associated Christoffel symbols at this point are $\Gamma^a_{jk} = 0$.

3.3. Local approximation of the wavefront

From the triangulation $\mathcal{V}^*$ of the initial wavefront, the ray tracing method furnishes a discrete surface determined by the mesh $\mathcal{V}$. To compute differential magnitudes of the wavefront surface corresponding to this mesh, one needs to define a discrete neighborhood of each vertex in $\mathcal{V}$. For this, we consider first at each inner vertex $z^*_j \in \mathcal{V}^*$, $j = 1, \ldots, J$ a neighborhood (named hereafter $1$-ring in the terminology of computational geometry, e.g. [18]) formed by the six vertices $z^*_{jk}$ closest to $z^*_j$:

$$[z^*_{jk}, z^*_{jk}];_{j=1,\ldots,J,k=0,\ldots,5}, \text{ where } z^*_{jk} := z_j + \exp(k\pi i/3).$$

Then, for each 1-ring in (14) one may determine on the mesh $\mathcal{V}$ a corresponding 1-ring formed by the image of the points $z^*_j, z^*_{jk} \in s_0$ under the flow (9)

$$[z_j; z_{jk}] := [\Phi_t(z^*_j); \Phi_t(z^*_{jk})].$$
In a neighborhood of a point \( z_j \) the wavefront can be approximated by a height function on the orthogonal plane to \( e_3 \) determined by means of a least-squares fitting of the data (15) expressed in normal coordinates \( \{ y_j^i \}_{i=1}^{3} \). As a model for this surface, we chose a quadric passing through the coordinate origin whose gradient at this point is parallel to \( e_3 \),

\[
y^3 = f(y^1, y^2) := \frac{1}{2}a_1(y^1)^2 + a_2y^1y^2 + \frac{1}{2}a_3(y^2)^2,
\]

where \( a_1, a_2 \) and \( a_3 \) are the indeterminate coefficients to be obtained by the least-squares method.

The quadric (16) provides a surface \( \tilde{S}_j \) that approximates the surface \( S_t \) in a neighborhood of point \( z_j \) and is defined in the parametric form

\[
y_i = y_i(x^A), \quad A = 1, 2,
\]

(17)

In coordinates \( (x^1, x^2) \), the metric \( \gamma \) on \( S_t \) induces a metric on \( \tilde{S}_j \) whose associated metric tensor \( g \) has the form

\[
g_{AB} := \gamma_{ij} \frac{\partial y^i}{\partial x^A} \frac{\partial y^j}{\partial x^B}.
\]

Moreover, on the tangent plane to \( S_t \) at \( z_j \) one defines a tensor \( B \) associated with the normal \( n \) at that point as

\[
B : T_{z_j} S_t \times T_{z_j} S_t \rightarrow (T_{z_j} S_t)^\perp, \quad B(\partial A, \partial B) = \frac{\partial^2 y^3}{\partial x^A \partial x^B} n.
\]

Therefore, if \( \{ v_1, v_2 \} \) represents an orthonormal basis for the vector space \( T_{z_j} S_t \) consisting of eigenvectors of \( B \) with associated eigenvalues \( \lambda_1 \) and \( \lambda_2 \), and \( II \) denotes the second fundamental form, then the difference of sectional curvatures \( K \) and \( \bar{K} \) associated with the plane generated by \( \{ v_1, v_2 \} \), in \( S_t \) and \( S \) respectively, is given by a generalized Gauss formula (see [19], p 131)

\[
K_{rel} := K(v_1, v_2) - \bar{K}(v_1, v_2) = \lambda_1 \lambda_2,
\]

(20)

which is named (see [15]) relative sectional curvature \( K_{rel} \), whereas the mean curvature is determined by half the trace of \( II \):

\[
H = \frac{1}{2}(\lambda_1 + \lambda_2).
\]

(21)

4. Numerical estimation of the curvature of a wavefront

4.1. Integration of the equations of light rays

Here, we deal with the problem of the numerical integration of the initial value problem corresponding to (4a)–(4b). Equation (4a) for the light propagation in the gravitational field generated by a static material point may be rewritten in terms of new variables defined as

\[
u := (u_1, \ldots, u_6), \quad \text{with } u_i := z_i \text{ and } u_{i+3} := \dot{z}_i, \quad (i = 1, 2, 3),
\]

(22)

obtaining the following first-order differential system:

\[
\dot{u}_1 = u_4,
\]
\[
\dot{u}_2 = u_5,
\]
\[
\dot{u}_3 = u_6,
\]
\[
\dot{u}_4 = -\frac{c^2}{c^2 u_1^2 - 3u_1 u_2^2 - 4u_2 u_4 u_5 - 4u_3 u_4 u_6 + u_1 u_2^2 + u_1 u_6^2} \frac{1}{(u_1^2 + u_2^2 + u_3^2)^{3/2}},
\]

(23)
Figure 2. Semi-logarithmic representation of the difference $F(\tilde{u}_n) - F(u_0)$ along the numerical solution of the light ray equation obtained by means of the Taylor algorithm. The maximum value of $\Delta F$ is reached at the time when the photon is nearest to the Sun.

The solution $\mathbf{u}(t)$ must satisfy the constraint (4b) which, in the notation (22), may be expressed as

$$F(\mathbf{u}) := u_4^2 + u_5^2 + u_6^2 - c^2 + \frac{2\kappa}{c^2 \sqrt{u_4^2 + u_5^2 + u_6^2}}(u_4^2 + u_5^2 + u_6^2 + c^2) = 0. \quad (24)$$

To obtain a numerical solution of the initial value problem given by (23) and the initial data $\mathbf{u}(0) = (z(0), \dot{z}(0))$, we use the Taylor integrator developed by Jorba and Zhou [10], based on the classic Taylor series method for ordinary differential equations. In this method a Taylor expansion of the vector field $\dot{\mathbf{u}}$ defined in (23) is made using techniques of automatic differentiation to obtain the corresponding Taylor coefficients. The Taylor integrator allows the control of both the order and the step size employed in the method. Furthermore, the Taylor integrator is implemented so that one may use extended precision arithmetic for the highly accurate computation required in this problem.

From now on, we use normalized units taking the radius and the mass of the Sun as units of length and mass, respectively. Then, the initial values corresponding to a photon initially located at 100 astronomical units from the Sun are given by

$$u_0 := u(0) = (1.0, 0.0, -21494.6550, 0.0, 0.0, 0.4307). \quad (25)$$

For the integration over a period of time of 50400 s, using a precision of 120 binary digits and a tolerance Tol = $10^{-20}$, the Taylor algorithm gives the solution in terms of Taylor polynomials of degree 24. As a test of the accuracy of the method, in figure 2 the behavior of the function $F(\mathbf{u})$ appearing in the isotropy constraint (4b) is shown. In this figure we see that at the arrival point, the isotropy constraint on the tangent vector to the light ray is not satisfied with the same degree of accuracy required at the initial position, reaching this deviation its maximum value at the instant when the photon is nearest to the Sun.

In order to obtain from the numerical solution $\tilde{u}_n$ given by Taylor another value $u_n$ satisfying the condition (24) and such that the function $\|u_n - \tilde{u}_n\|$ reaches a minimum, we will apply at each step in the Taylor integrator the method of standard projection [20] to project $\tilde{u}_n$ on the manifold $F(\mathbf{u}) = 0$. This leads to a constrained extremum problem with a
Lagrangian function $L(u, \lambda) := \frac{1}{2} \| u - \tilde{u} \|^2 - F(u_n)^T \lambda$, where $\lambda$ is a Lagrange multiplier. The necessary condition of extremum and the constraint condition lead to

$$u_n = \tilde{u}_n + \nabla F(u_n)^T \lambda$$

$$0 = F(u_n)$$  \hspace{1cm} (26a, 26b)

and replacing (26a) with (26b) the following nonlinear equation for $\lambda$ is obtained:

$$F(\tilde{u}_n + \nabla_u F(u_n(\tilde{u}_n, \lambda))^T \lambda) = 0.$$  \hspace{1cm} (27)

This equation may be solved by applying the simplified Newton method. The projection stage in the algorithm spent a 3% of the 0.06 seconds-CPU employed by an Intel Core 2 Quad processor to determine the trajectory of a photon in the time interval we are considering.

4.2. Deformation of a wavefront by a static gravitational field

Now we apply the method described in section 3 to that region of a wavefront propagating along a tubular neighborhood around the $Oz$-axis and whose radius is $2R_\odot$. Initially the wave surface $\mathcal{S}_0$ is flat, perpendicular to the $Oz$-axis and with position and velocity given in (25). The wavefront $\mathcal{S}_T$ is then determined after a trip of 101 astronomical units.

4.2.1. Curvature at a point

First, we obtain an estimation of the curvature at a point in $\mathcal{S}_T$ by using a sequence of 1-rings with decreasing radii until a stable value of the curvature wavefront at that point is reached. Let us consider the 1-ring of radius $r$ centered at point $V_0^{(0)} := (1.0, 0.0, -21494.6550)$, determined by the three first components in (25) whose vertices are given by

$$V_0^{(k+1)} := (1 + r \cos(k\pi/3), \sin(k\pi/3), -21494.6550), \quad (k = 0, \ldots, 5).$$  \hspace{1cm} (28)

Now, we apply the Teyloro algorithm for the values of tolerance and arithmetical precision pointed out in section 4.1 to determine the images $V^{[k]}$ of the vertices $V_0^{(k)}$, $k = 0, \ldots, 6$, by solving (23) and taking for all vertices the same initial velocity $(0.0, 0.0, 0.4307)$ (in the normalized units we are considering). After a time $T = 54400$ s, the 1-ring $[V^{[0]}, V^{[1]}, \ldots, V^{[6]}]$ provides a discretized neighborhood of the point $V^{[0]} \in \mathcal{S}_T$ (see figure 3).
Applying the scheme developed in section 3.3 we obtain that the mean and the relative total curvatures at point \( V^{(0)} \) take, for both the different values of the radius \( r \) and the tolerances in the Taylor algorithm, the values shown in table 1, where one observes that for values \( \text{Tol} = 10^{-20} \) and \( r = R_\odot/50 \) the three first significant decimal numbers are correct.

| Radius \( R_\odot \) | \( K_{rel} \) \( 10^{-10} \) | \( H \) \( 10^{-10} \) | \( K_{rel} \) \( 10^{-20} \) | \( H \) \( 10^{-20} \) |
|----------------------|---------------------|---------------------|---------------------|---------------------|
| \( R_\odot/10 \)     | \(-0.72751 \times 10^{-10} \) | \( 0.15524 \times 10^{-7} \) | \(-0.72752 \times 10^{-10} \) | \( 0.14927 \times 10^{-7} \) |
| \( R_\odot/20 \)     | \(-0.72183 \times 10^{-10} \) | \( 0.16073 \times 10^{-7} \) | \(-0.72210 \times 10^{-10} \) | \( 0.15350 \times 10^{-7} \) |
| \( R_\odot/30 \)     | \(-0.72083 \times 10^{-10} \) | \( 0.16143 \times 10^{-7} \) | \(-0.72110 \times 10^{-10} \) | \( 0.15394 \times 10^{-7} \) |
| \( R_\odot/40 \)     | \(-0.72075 \times 10^{-10} \) | \( 0.15362 \times 10^{-7} \) | \(-0.72075 \times 10^{-10} \) | \( 0.15407 \times 10^{-7} \) |
| \( R_\odot/50 \)     | \(-0.72058 \times 10^{-10} \) | \( 0.15372 \times 10^{-7} \) | \(-0.72059 \times 10^{-10} \) | \( 0.15413 \times 10^{-7} \) |
| \( R_\odot/60 \)     | \(-0.72048 \times 10^{-10} \) | \( 0.15354 \times 10^{-7} \) | \(-0.72050 \times 10^{-10} \) | \( 0.15415 \times 10^{-7} \) |
| \( R_\odot/70 \)     | \(-0.72042 \times 10^{-10} \) | \( 0.15320 \times 10^{-7} \) | \(-0.72044 \times 10^{-10} \) | \( 0.15416 \times 10^{-7} \) |
| \( R_\odot/80 \)     | \(-0.72039 \times 10^{-10} \) | \( 0.15313 \times 10^{-7} \) | \(-0.72041 \times 10^{-10} \) | \( 0.15416 \times 10^{-7} \) |
| \( R_\odot/90 \)     | \(-0.72037 \times 10^{-10} \) | \( 0.15298 \times 10^{-7} \) | \(-0.72039 \times 10^{-10} \) | \( 0.15416 \times 10^{-7} \) |
| \( R_\odot/100 \)    | \(-0.72036 \times 10^{-10} \) | \( 0.15274 \times 10^{-7} \) | \(-0.72037 \times 10^{-10} \) | \( 0.15416 \times 10^{-7} \) |

4.2.2. Wavefront surface To obtain an estimation of the curvature of a region of the wavefront propagating along the \( Oz^3 \)-axis, we apply the ray tracing method with initial values on the wavefront surface described at the beginning of section 4.2. We use a regular discretization of the wavefront surface as that described in section 3.1; furthermore, taking into account the results shown in table 1 we choose the length of the edges in the corresponding mesh equal to \( R_\odot/50 \).

In the numerical model that we are studying, one assumes that the Sun is a point and we consider a hexagonal annular region on the initial wavefront similar to that shown in figure 1(b) where the inner and outer hexagons have radii of lengths \( R_\odot/25 \) and \( 2R_\odot \) respectively. Therefore a number of \( N := 28044 \) vertices are required. The Taylor algorithm with projection in a time interval \([0, T]\) (the time required to run a path of length equal to 101 AU) is applied to each photon located at an initial vertex; the CPU-time employed to carry out this computation is of 1544 s.

For the numerical solution \( u_n, n = 1, \ldots, N \), of (23), both the mean and relative-total curvatures at each inner vertex of the mesh on the surface \( \mathcal{S}_T \) are computed by applying the method described in section 3, schematized in pseudocode in table 2.

In figure 4, the surface \( \mathcal{S}_T \) at the time when the wavefront arrives at the Earth is shown using a grayscale to represent the mean curvature (note we have used a different scale on the \( Oz^3 \)-axis). In this figure, one observes that the absolute value of the mean curvature function defined on \( \mathcal{S}_T \) increases as the distance between the photon and the \( Oz^3 \)-axis decreases.

4.3. Curvature of a wavefront in the PPN formalism

To derive the deflection angle for light rays, instead of assuming the validity of general relativity, one may consider a more general expression for the metric generated by a spherical central body that is valid for different gravitational metric theories. In the parameterized post-Newtonian formalism, the expression of a spherically symmetric metric, written at
Table 2. Pseudocode for a method to compute the curvatures of a wavefront.

Data: $u_n^* := (z_n^*, \dot{z}_n^*)$, $n = 1, \ldots, N$

for $n = 1, \ldots, N$ do

$u_n := Taylor(t, u_n^*)$ // rays tracing
$y_n := NormalCoordinates(z_n)$ // see equation (10)

for $i = 0, \ldots, 6$ do

$y_{ni} := Ring(y_n)$ // see equation (15)

end

$(a_1, a_2, a_3) = LeastSquares(y_n)$ // see equation (17)
$\gamma_{AB}(x_n) := Metric(a_1, a_2, a_3, x_n)$ // see equation (18)
$B = SecondFundamentalForm(x_n)$ // see equation (19)
$(\lambda_1, \lambda_2) = Diagonalize(B)$
$(K_{\text{rel}}, H) = Curvature(\lambda_1, \lambda_2)$ // see equations (20) and (21)

end

the linearized order we are considering here, contains one parameter $\gamma$ which is usually interpreted as a modification of the curvature of the space. In the parameterized post-Newtonian formalism, the total relativistic deflection angle of light rays passing near the limb of the Sun is given approximately as (see [21])

$$\Delta \phi \simeq 2(1 + \gamma)1^\circ.75.$$ (29)

Using very long baseline interferometry techniques one may obtain high-precision general relativistic measurements for the deflection of radio signals from distant radio sources with an accuracy at the 0.02% level. In [22] an estimation of $0.9998 \pm 0.0004$ is given for the post-Newtonian parameter $\gamma$.

Here, we consider a gravitational field depending on the Eddington parameter $\gamma$ and adapt the numerical method described above to determine the dependence of the wavefront curvature
We take values of $\gamma$ in the interval $[0.9992, 1.0012]$ in which the experimental values given in [22] are included and consider 11 nodes in the numerical code developed above obtaining the results shown in figure 5 for the curvature of the light wavefront at a point near the Earth for a light ray grazing the Sun. In this figure, one observes a linear dependence of the total and mean curvature with respect to the parameter $\gamma$. For these values of $\gamma$ the variations of $K_{\text{rel}}$ and $H$ are in the second and third significant decimal digit, respectively. In consequence, the numerical treatment we applied gives account of the variations of the geometric model when a change of the PPN Eddington parameter is made.

5. Variation of the time of arrival and curvature of the wavefront

In this section, we consider the problem treated by Samuel [1] where the curvature of a wavefront in a gravitational field is detected by measuring the arrival times $T_{a,a}$ at four receiving stations located at points $E_a$ on an Earth hemisphere. The arrival time differences between these stations depend on the curvature of the wavefront and are related to the volume of a parallelepiped formed by the vectors $\vec{E}_0\vec{E}_j - c(T_j - T_0)n_j$, $j = 1, 2, 3$, where $n := (0, 0, 1)$. Samuel proposed the measurement of this non-zero volume as a new test of the general relativity.

5.1. Volume of a tetrahedron with vertices on the wavefront surface

Here we compute the volume and the arrival time differences associated with a simplex determined by four points on the wavefront numerically determined above. For this, we
consider a light ray reaching the reference station $E_0$ (see figure 6) at the instant of time $T_0$ and whose tangent vector at this point is $n_0$. Let $P_0^*$ be the position of this photon at a previous instant $T_0^*$ such that $c(T_0 - T_0^*) = 101$ AU. This position may be obtained by a backward integration of the equations of motion (23) with initial values $(P_0, -cn_0)$. On the plane determined by $(P_0^*, n_0^*)$, where $n_0^*$ is the tangent vector to the ray at point $P_0^*$, we consider the 1-ring $[P_0^*, P_j^*]$, $j = 1, \ldots, 6$, centered at $P_0^*$. Then, from (23) and the initial values $(P_0^*, cn_0^*)$ we determine the 1-ring $[Q_0^*, Q_i]$ image of $[P_0^*, P_j^*]$ by the flow (9) integrating again over the interval $[0, T_0 - T_0^*]$.  

In the numerical model we are considering, the coordinates of $P_0$ are $(2R_\odot, 0, 1$ AU) and $n_0 = (0, 0, 1)$. The original position $P_0^*$ and direction $n_0^*$ of the photon are then (expressed here using only three decimal digits)

$$P_0^* := (1.909, 0.0, -21494.224), \quad n_0^* = (0.183 \times 10^{-5}, 0.0, 0.431). \quad (31)$$

We choose as vertices $P_j^*$ the points

$$P_j^* = R(P_0^*, e_3, \vartheta)(r \cos(\pi j/3), r \sin(\pi j/3), -100 \text{ AU}) \quad j = 1, \ldots, 6, \quad (32)$$

where $R(P_0^*, e_3, \vartheta)$ represents a rotation around axis $(P_0^*, e_3)$ through an angle $\vartheta := e_3 \cdot n_0^*/\|n_0^*\|$.  

Following the same method employed in section 4.2.2 we perform a least-squares fitting of the data $(Q_{a})_{a=0}^{6}$ to get an approximation of the wavefront surface in a neighborhood of $Q_0$ given by a quadratic surface $Q$ which, in the normal coordinates corresponding to ray $(P_0, n_0)$, takes the form

$$Q(z) := z_3 = -0.1060 \times 10^{-5}z_1^2 + 0.6 \times 10^{-23}z_1z_2 + 0.1065 \times 10^{-5}z_2^2. \quad (33)$$

Now, on the Earth’s surface we choose four points $E_a$ with geocentric coordinates $(0, 0, -R_\oplus)$, $(R_\oplus, 0, 0)$, $(\frac{\sqrt{2}}{2}R_\oplus, \frac{\sqrt{2}}{2}R_\oplus, 0)$ and $(\frac{\sqrt{7}}{2}R_\oplus, -\frac{1}{2}R_\odot, 0)$ respectively, which will be transformed into normal coordinates. Assuming that the geometry of the 3-space in the vicinity of the Earth is Euclidean, one may determine the points $Q_a \in \mathcal{Q}$ whose distances to the corresponding stations $E_a$ are minima.  

Once the points $Q_a \in \mathcal{Q}$ corresponding to the images of the stations $E_a$ on the Earth are determined, we may compute the differences $\tau_a = \text{dist}(Q_a, Q_a^*)/c$ between the arrival times measured at each station under the assumptions that the wavefront surface is either a curved
surface $\mathcal{D}$ or a plane $\mathcal{P}$ determined by the pair $(P_0, n_0)$. These differences are given, for the data provided above, by
\[ \tau_0 = 0.978 724 814 93, \quad \tau_1 = 0.999 999 990 32, \quad \tau_2 = \tau_3 = 0.999 999 990 21, \quad (34) \]
whereas the distances \( l_{ij} := \text{dist}(Q_i, Q_j), i, j = 0, 1, 2, 3 \), between the projected points \( Q_a \) on \( \mathcal{Q} \) are (in the normalized units we are using)
\[
\begin{align*}
l_{01} &= 0.916 4245 \times 10^{-2}, & l_{02} &= 0.916 4249 \times 10^{-2}, & l_{03} &= 0.916 4249 \times 10^{-2}, \\
l_{12} &= 0.474 3770 \times 10^{-2}, & l_{13} &= 0.474 3770 \times 10^{-2}, & l_{23} &= 0.916 4262 \times 10^{-2},
\end{align*}
\]
where \( \text{dist}(A, B) \) denotes the Euclidian distance between two arbitrary points \( A \) and \( B \), and \( Q_a \) represents the points of minimum distance from the station \( E_a \) to the plane \( \mathcal{P} \). The volume of the tetrahedron determined by the points \( Q_a \) is proportional to the Cayley–Menger determinant [12] defined in terms of the lengths \( l_{ij} \) of the edges, and it is given by
\[
\text{Vol} = \frac{\sqrt{2}}{24} \left| \begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & l_{01}^2 & l_{02}^2 & l_{03}^2 \\
1 & l_{10}^2 & l_{12}^2 & l_{13}^2 \\
1 & l_{20}^2 & l_{21}^2 & l_{23}^2 \\
1 & l_{30}^2 & l_{31}^2 & l_{32}^2
\end{array} \right|, \quad (35)
\]
which for the lengths (35) takes the value \( \text{Vol} = 0.7 \times 10^{-7} \). Therefore the metric quadruple \((Q_a, l_{ab})\) determines a non-degenerate simplex in \( \mathbb{R}^3 \) (see for instance Saucan [17]). This is equivalent to saying that the metric quadruple \((Q_a, l_{ab})\) is not congruent to any quadruple of points in the Euclidean plane and consequently the curvature of the wavefront surface at the point \( Q_a \) is non-vanishing.

5.2. Estimation of the wavefront curvature from arrival time measurements

Now we assume that the points \( \{Q_a\}_{a=0}^3 \) on the wavefront are directly determined through measurements of arrival times. Here, we consider the problem of determining an approximation of the wavefront curvature in a region far enough from the Sun (say the Earth), without resorting to the ray tracing method.

An estimation of the Gaussian curvature of the wavefront surface can be obtained using the notion of the Wald curvature of a metric space established in the distance geometry (see [12]) that in the case of two-dimensional manifolds agrees with the Gaussian curvature. The Wald curvature is determined as the limit of the embedding curvatures of metric quadruples \((Q_a, l_{ab})\) that in the case of two-dimensional manifolds agrees with the Gaussian curvature. The notion of the Wald curvature of a metric space established in the distance geometry (see [12], p 274). This is equivalent to saying that the metric quadruple \((Q_a, l_{ab})\) is not congruent to any quadruple of points in the Euclidean plane and consequently the curvature of the wavefront surface at the point \( Q_a \) is non-vanishing.

In the hyperbolic plane \( \mathbb{H}^2_r \) of curvature \(-1/r^2\), represented by the Blumenthal model ([12], p 19) we consider the metric quadruple \((Q_a, l_{ab})\) defined in section 5.1. The curvature of a hyperbolic plane on which there exists a quadruple congruent with \( Q_j \) must fulfill both of the following conditions:
\[
A(r) := \begin{pmatrix}
1 & \cosh(l_{01}/r) & \cosh(l_{02}/r) & \cosh(l_{03}/r) \\
\cosh(l_{01}/r) & 1 & \cosh(l_{12}/r) & \cosh(l_{13}/r) \\
\cosh(l_{02}/r) & \cosh(l_{12}/r) & 1 & \cosh(l_{23}/r) \\
\cosh(l_{03}/r) & \cosh(l_{13}/r) & \cosh(l_{23}/r) & 1
\end{pmatrix} = 0, \quad (36)
\]
and each non-zero principal minor of \( A(r) \) of order \( m+1 \) has the sign \((-1)^m \) (see [12], p 274). For small values of the curvature \( 1/r \) the determinant \( \det(A(r)) \) can be approximated by a Taylor polynomial. Using the symbolic processor Maple and employing numerical precision
of Digits = 50 to perform the Taylor expansion, we obtain that (36) may be approximated by
the following algebraic equation for $\rho := 1/r$

$$0.235 \times 10^{-22} \rho^{10} + 0.481 \times 10^{-18} \rho^8 - 0.756 \times 10^{-29} \rho^6 = 0$$ (37)

having only one positive real root. This approximated solution is taken as an initial guessed
solution to solve the transcendental equation (36) by means of the command solve in Maple.
On the other hand, the conditions imposed on the sign of the principal minors are also satisfied.
Therefore, the Wald curvature $K_W$ associated with the chosen quadruple $\{Q_i\}$ is

$$K_W := -\frac{1}{r^2} = -0.16 \times 10^{-10}.$$ (38)

This result gives an approximation of the total curvature of the wavefront surface under the
assumption that locally this surface may be identified with a hyperbolic plane in which the
quadruple considered is isometrically embedded.

5.3. Scheme of the method

In this section, we present an outline of the construction of the method we used above to obtain
the Wald curvature corresponding to four receiving stations located at points $E_a$ and the arrival
time differences $\tau_a$.

- Define on the wavefront the points $P_a$ corresponding to the stations $E_a$: $OE_a + c_\tau_a n$, (39)

- Determine the relative distances $l_{ij}$ between the points $Q_i$:

$$l_{ij}^2 = L_{ij}^2 + 2c(\tau_j - \tau_i)(OE_j - OE_i) \cdot n + c^2(\tau_j - \tau_i)^2,$$ (40)

where $L_{ij}$ are the Euclidean distances between the stations $E_i$ and $E_j$.

- For the point $Q_a$ so obtained, establish the nonlinear equation (36) in terms of the curvature
of the surface.

- Solve the nonlinear equation (36) for the unknown $r$ to obtain an estimation of the curvature
of the wavefront in a neighborhood of the station $E_0$ by means of the Wald curvature $K_W$
of this surface modeled as a hyperbolic plane.

6. Conclusions

The ray tracing numerical method provides a useful tool for the description of spacelike
bidimensional wavefronts within the framework of the general relativity. We have studied a
method based on techniques of computational geometry that allows us to estimate the curvature
properties of the surface by making a least-squares fitting of the wavefront surface by a quadric
surface in the neighborhood of each point of this surface. The computation of the light rays
is carried out using an algorithm based on the Taylor method for the solution of differential
equations and employing high arithmetic precision. Further, we have applied a projection
at each step of the numerical integration process that allows us to guarantee the fulfillment
(at machine precision) of the isotropy condition for the tangent vector to the light ray. We
have also studied numerically the dependence of the curvature properties of the wavefront
surface on the value of the Eddington parameter $\gamma$. On the other hand, we have employed a
geometric computational approach to the study of the model proposed by Samuel as a new
general relativity test, by determining a numerical approximation of the volume corresponding
to a tetrahedron formed by four points on the wavefront that reaches four receiving stations.
on the Earth’s surface. Finally, we have obtained an estimation of the Wald curvature for the wavefront in the vicinity of the Earth by using the differences of arrival time recorded at four receiving stations on the Earth.

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