Application of Neural Network Machine Learning to Solution of Black-Scholes Equations

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Abstract

This paper presents a novel way to predict options price for one day in advance, utilizing the method of Quasi-Reversibility for solving the Black-Scholes equation. The Black-Scholes equation is solved forwards in time, which is an ill-posed problem. Thus, Tikhonov regularization via the Quasi-Reversibility Method is applied. This procedure allows to forecast stock option prices for one trading day ahead of the current one. To enhance these results, the Neural Network Machine Learning is applied on the second stage. Real market data are used. Results of Quasi-Reversibility Method and Machine Learning method are compared in terms of accuracy, precision and recall.

Keywords:
The Black-Scholes equation, Ill-posed problem, regularization method, parabolic equation with the reversed time, Machine Learning, neural network.

1 Introduction

This paper discusses a new empirical mathematical model for generating more accurate option trading strategy using initial and boundary conditions for the underlying stock. The idea was initially proposed in [5]. The basis for this idea is the Black-Scholes equation. In mathematical finance, the Black-Scholes equation is a parabolic partial differential equation that determines the dynamics of the price of European options [7].

The time at a given time $t$ will occur is $\tau$,

$$\tau = T - t.$$  \hspace{1cm} (1.1)

$f(s)$ be the payoff function of that option at the maturity time $t = T$ and $s$ is the stock price. Let’s assume that the risk-free interest rate equals zero. The function $u(s, \tau)$ is the price of that option and the variable $\tau$ is the one defined in (1.1). Let’s assume that
this function $u(s, \tau)$ satisfies the Black-Scholes equation with the volatility coefficient $\sigma$ [11] Chapter 7, Theorem 7.7:

$$\frac{\partial u(s, \tau)}{\partial \tau} = \frac{\sigma^2 s^2}{2} \frac{\partial^2 u(s, \tau)}{\partial s^2},$$

$$u(s, 0) = f(s),$$

(1.2)

The payoff function is $f(s) = \max(s - K, 0)$, where $K$ is the strike price [11] and $s > 0$.

The option price function is defined by the Black-Scholes formula:

$$u(s, \tau) = s \Phi(\theta_+(s, \tau)) - e^{-r\tau} K \Phi(\theta_-(s, \tau)),$$

(1.3)

Based on the Itô formula, we have:

$$du = \left( - \frac{\partial u(s, T - t)}{\partial \tau} + \frac{\sigma^2 s^2}{2} \frac{\partial^2 u(s, T - t)}{\partial s^2} \right) dt + \sigma s \frac{\partial u(s, T - t)}{\partial s} dW.$$  

(1.4)

If equation (1.2) is solved forwards in time to forecast prices of stock options is an ill-posed inverse problem. For this reason, we used the Method of Quasi- Reversibility (QRM) that is a version of the Tikhonov regularization method. Uniqueness, stability and convergence theorems for this method were formulated in [5] and [3], also, see [10] for proofs.

We have four major questions that we raise in this paper:

1. What is the forecast interval of the options prices?
2. What are the boundary and initial conditions on the interval for the Black-Scholes equation?
3. What are the values of the volatility coefficient in the future?
4. How to solve the Black-Scholes equation forwards in time $t$?

The first three questions are addressed in our new mathematical model. We use the regularization method of [4] to address the fourth question. Theorems about stability and convergence of this method are formulated. These theorems were proven in [4] for a general parabolic equation of the second order where the main key of this method is based on the method of Carleman estimates.

Let the function $f \in L^2(0, \pi)$ and let $T = \text{const.} > 0$. To demonstrate that our problem is ill-posed, we consider the example of based on the problem for the heat equation with the reversed time

$$u_t + u_{xx} = 0, (x, t) \in (0, \pi) \times (0, T),$$

(1.5)

with Dirichlet boundary conditions

$$u(0, t) = 0, u(\pi, t) = 0,$$

(1.6)

and the initial condition

$$u(x, 0) = f(x).$$

(1.7)
The unique solution of this problem is:

\[ u(x, t) = \sum_{n=1}^{\infty} f_n \sin (nx)e^{n^2t} . \]

Consider

\[ u_N(x, t) = \sum_{n=1}^{N} f_n \sin (nx)e^{n^2t} . \]

Then

\[ ||u_N(x, T)||^2_{L^2(0,\pi)} = \sum_{n=1}^{N} f_n^2e^{2n^2T} \approx f_N^2e^{2N^2T} \to \infty \]

as \( N, T \to \infty \). Hence, the problem (1.5)-(1.7) is severely unstable.

We conclude therefore that to obtain a more or less accurate solution of the Black-Scholes equation forwards in time, we need to solve it on a short time interval \((0, T)\). To get better accuracy, the regularization method works only for a short time interval.

Section 3 presents our mathematical model, the method of Quasi Reversibility as well as the trading strategy. The Quasi Reversibility Method is based on the minimization of a Tikhonov-like functional \( J_\beta(u) \). We do this using conjugate gradient method. The minimization process was performed by Hyak Next Generation Supercomputer of the research computing club of University of Washington. The code was parallelized in order to maximize the performance on supercomputer clusters.

The historical data for stock options was collected from the Bloomberg terminal \([2]\) of University of Washington. From this data, we obtained about 177,000 minimizers.

Due to ill-posedness of the problem the solution is very sensitive to the noise in the initial data (stock and option prices for the three days preceding the day of forecast). Given results of the Quasi Reversibility Method, we apply on the second stage Machine Learning to reduce the probability of non-profitable trades caused by wrong option price prognosis because of the noise in input data.

Section 4 is dedicated to application of binary classification and regression Neural Network Machine Learning.

Sections 5 and 6 present our results and the summary.

Python with the SciPy and Torch modules were used for implementation of the method of Quasi- Reversibility and Neural Network Machine Learning (binary classification and regression).

2 The new mathematical model and the method of Quasi-Reversibility

Let’s denote \( s \) as the stock price, \( t \) as the time and \( \sigma(t) \) as the volatility of the option. The historical implied volatility listed on the market data of \([2]\) is used in our particular case. We assume that \( \sigma = \sigma(t) \) to avoid other historical data for the volatility. Let’s call \( u_b(t) \) and \( u_a(t) \) the bid and ask prices of the options at the moment of time \( t \) and \( s_b(t) \) and \( s_a(t) \) the bid and ask prices of the stock at the moment of time \( t \). It is also known that
Let's introduce

\[ f_s(t) = \frac{s_a(t)}{s_b(t)} - 1 \] (2.3)

and

\[ f_u(t) = \frac{u_a(t)}{u_b(t)} - 1 \] (2.4)

Based on real market data we have observed that usually

\[ 0 \leq f_s(t) \leq 0.003 \] (2.5)

and

\[ 0 \leq f_u(t) \leq 0.27 \] (2.6)

The idea is to approximate the Black-Scholes equation solutions

\[ Lu = u_t + \frac{\sigma^2(t)}{2} s^2 u_{ss} = 0, (s, t) \in (s_b(0), s_a(0)) \times (0, 2\tau) = X_{2\tau}, \] (2.7)

with Dirichlet boundary conditions

\[ u(s_b, t) = u_b(t), u(s_a, t) = u_a(t), t \in [0, 2\tau], \] (2.8)

and the initial condition

\[ u(s, 0) = f(s), s \in [s_b(0), s_a(0)]. \] (2.9)

Where \( L \) is the partial differential operator of the Black-Scholes equation. Based on Bloomerg terminal we used with End of Day Underlying Price Last, End of Day Underlying Price Bid, End of Day Underlying Price Ask, \( t \) is time, \( \sigma(t) \) is the volatility of the stock option. It was used Implied Volatility Using Last Trade Price (IVOL).

\( u(s, t) \) is the price of the stock option. End of Day Option Price Last, End of Day Option Price Bid and End of Day Option Price Ask are the notation that we applied in our algorithm.

**Problem 1.** Find the function \( u \in H^2(X_{2\tau}) \) satisfying conditions (2.7)-(2.9).

This problem considers as ill-posed since we solve equation (2.7) forwards in time.

**Remarks 3.1:** We increase here the required smoothness of the solution from \( H^{2,1}(X_{2\tau}) \) to \( H^2(X_{2\tau}) \).

Our algorithm based on solving the inverse problem for the Black-Scholes with reversed time equation has five steps:

**Step 1 (Dimensionless variables).**
We require to make our equation dimensionless. \( s_b < s_a \). Let’s denote \( s_b = s_b(0) \), \( s_a = s_a(0) \). Dimensionless variables were applied \( x, t' \) such that

\[
x = \frac{s - s_b}{s_a - s_b}
\]

\[
t' = \frac{t}{255}
\]

and now we can say that \( s \) is \( x \) and \( t \) is \( t' \).

According to these substitutions, the equation becomes

\[
Ru = u_t + \sigma^2(t) A(x) u_{xx},
\]

where

\[
A(x) = \frac{255 [x(s_a - s_b) + s_b]^2}{2 (s_a - s_b)^2}
\]

\[
X_{2\tau} = \{(x, t) \in (0, 1) \times (0, 2\tau)\}.
\]

\[
u(x, 0) = g(x), x \in (0, 1)
\]

\[
u(0, t) = u_b(t), u(1, t) = u_a(t).
\]

And the operator \( L \) in (2.7) is the operator \( R \)

Step 2 (Interpolation and extrapolation).

Our goal is to forecast option price from 'today' to 'tomorrow' and 'the day after tomorrow'. We do have 255 trading days annually. For this reason, let’s introduce \( \tau > 0 \) as our unit of time for which we want to make our prediction the option price. Because we predict option prices having the information of these prices, as well as of other parameters for 'today', 'yesterday' and 'the day before yesterday', we consider \( \tau \) is one trading day. 'One day' \( \tau = 1/255 \). 'Today' \( t = 0 \). 'Tomorrow' \( t = \tau \). 'The day after tomorrow' \( t = 2\tau \). The variable \( s \) is for interval, i.e \( s \in [s_b(0), s_a(0)] \). We applied the idea associated with interpolation discrete values of functions \( u_b(t) \), \( u_a(t) \), and \( \sigma(t) \) between these three points (the day before yesterday, yesterday and today) and then extrapolation functions \( u_b(t) \), \( u_a(t) \) between three points (today, tomorrow and the day after tomorrow). Where \( t = -2\tau \) is "the day before yesterday", \( t = -\tau \) is "yesterday" and \( t = 0 \) is "today". We used quadratic polynomials for both approximation and extrapolation of values of functions. Thus, these three functions \( u_b(t) \), \( u_a(t) \), and \( \sigma(t) \) was obtained for a small future time interval, i.e \( (0, 2\tau) \). \( X_{2\tau} \). Where \( u_b(t) \), \( u_a(t) \) were applied for boundary conditions and \( \sigma(t) \) is coefficient function for our problem. The initial condition was set as \( u(x, 0) = g(x) = x(u_a(0) - u_b(0))x + u_b(0) \). This function is the result of approximation by linear function due to the fact that the interval between bid and ask prices is relatively small. The domain was \( X_{2\tau} = \{(x, t) : x \in (0, 1), t \in (0, 2\tau)\} \).

Step 3 (Statement of the Problem).

Problem 2. Assume that functions

\[
\begin{align*}
\ u_b(t), u_a(t) \in H^2[0, 2\tau], \sigma(t) \in C^1[0, 2\tau].
\end{align*}
\]
Find the solution \( u \in H^2(X_{2\tau}) \) of the following initial boundary value problem:

\[
Ru = 0 \text{ in } X_{2\tau},
\]

\[
u(0, t) = u_b(t), \quad u(1, t) = u_a(t), \quad t \in (0, 2\tau),
\]

\[
\nu(x, 0) = g(x), \quad x \in (0, 1),
\]

where the partial differential operator \( R \) is defined in (2.12), the function \( A(x) \) is defined in (2.13), the initial condition \( g(x) \) is defined in (2.15), and the domain \( X_{2\tau} \) is defined in (2.14).

**Theorem 2.1.** The following problem (2.12)-(2.15) has one solution \( u \in H^{2,1}(X_{2\tau}) \).

The proof of this theorem is [3].

**Step 4 (Numerical method of solving the problem. Regularization).**

Due to the ill-posedness of the problem, we can not say about existence of the solution. Thus, it was applied the regularization method:

Let’s consider function \( F(x, t) = x(u_a(t) - u_b(t)) + u_b(t), \quad (x, t) \in X_{2\tau} \). This function \( F \in H^2(X_{2\tau}) \). It follows from (2.15) and (2.16) that

\[
F(x, 0) = g(x),
\]

\[
F(0, t) = u_b(t), \quad F(1, t) = u_a(t).
\]

We used an unbounded differential operator \( R : H^{2,1}(X_{2\tau}) \to L^2(X_{2\tau}) \), where \( H^{2,1}(X_{2\tau}) \) is a dense linear set in the space \( L^2(X_{2\tau}) \). Where

\[
Ru = u_t + \sigma^2(t) A(x) u_{xx}
\]

Let’s introduce Tikhonov-like functional as:

\[
J_\beta(u) = \int_{X_{2\tau}} (Ru)^2 \, ds dt + \beta \|u\|_{H^2(X_{2\tau})}^2,
\]

where \( \beta \in (0, 1) \) is the parameter of regularization. To solve the problem, we minimized the functional \( J_\beta(u) \) on the set

\[
V = \{ u \in H^2(X_{2\tau}) : u(0, t) = u_b(t), \quad u(1, t) = u_a(t), \quad u(x, 0) = g(x) \}.
\]

**Step 5 (Minimization Problem).**

**Minimization Problem 1.** \( J_\beta : H^2(X_{2\tau}) \to \mathbb{R} \) is the regularization Tikhonov functional.

We have used the converting of our partial derivatives from (2.24) into finite differences. A finite difference grid was applied to cover the domain \( X_{2\tau} \). The minimization process was to differentiate our functional \( J_\beta(u) \) with respect to the values of the function \( u(x, t) \) at each grid points via conjugate gradient method. The point \( u = 0 \) was used for the
starting point. Based on computational study with simulated data we have realized that the optimal value of the regularization parameter would be $\beta = 0.01$.

Minimization Problem 1 is a QRM for Problem 2. This is an version of the QRM for problem \([2.18]-[2.20]\). In section 4 we discuss the theory of this specific version of the QRM. In particular, Theorem 4.2 of section 4 presents uniqueness of the solution $u \in H^2(X_{2r})$ of Problem 2 and implies an estimate of the stability of this solution with respect to the noise in the data. Theorem 4.3 of section 4 shows existence and uniqueness of the minimizer $u_{\beta} \in H^{2,1}(X_{2r})$ of the functional $J_\beta(u)$ on the set $V$ defined in \([2.25]\). We call such a minimizer “regularized solution” [9]. Theorem 4.4 estimates convergence rate of regularized solutions to the exact solution of Problem 2 with the noiseless data. Such estimates depend on the noise level in the data. All proof of these theorems are presented in [10].

3 Analysis

This section is devoted to convergence analysis for Problem 2 of subsection 3.2. This problem is the initial boundary value problem for parabolic equation \([2.18]\) with the reversed time. The QRM and convergence analysis for this problem for a more general parabolic operator in $\mathbb{R}^n$ with arbitrary variable coefficients was proposed in [4]. Then theorems were presented in [5]. However a stability estimate was not a part of [5], such an estimate was proven in [4]. The same is true for the convergence theorems of QRM in [4, 5]. The smallness assumption was lifted in [5] via a new Carleman estimate. Results of [5] for a 1-D case were significantly modified in this section. Our computations below on a small time interval $(0, 2\tau) = (0, 0.00784)$ (see [4, 5], [8, Theorem 1 of section 2 in Chapter 4] might result in the requirement of even a smaller length of that interval.

3.1 Problem statement

Let’s consider a number $T > 0$ and introduce $Q_T$ as:

$$Q_T = \{(x, t) \in (0, 1) \times (0, T)\}.$$ Consider two numbers $b_0, b_1 > 0$ and $b_0 < b_1$. Let the function $b(x, t) \in C^1(\overline{Q_T})$ satisfies:

$$\|b\|_{C^1(\overline{Q}_T)} \leq b_1, \; b(x, t) \geq b_0 \text{ in } Q_T. \quad (3.1)$$

We also have functions $\psi_0(t), \psi_1(t) \in H^2(0, T)$. In the above case of subsection 3.2,

$$T = 2\tau, \; b(x, t) = \sigma^2(t) A(x), \; \psi_0(t) = u_b(t), \; \psi_1(t) = u_a(t).$$

We now formulate Problem 3, which is a slight generalization of Problem 2.

**Problem 3.** Find a solution $v \in H^2(Q_T)$ of the following (IBVP):

$$Nv = v_t + b(x, t) v_{xx} = 0 \text{ in } Q_T, \quad (3.2)$$

$$v(0, t) = \psi_0(t), \; v(1, t) = \psi_1(t), \; t \in (0, T), \quad (3.3)$$

$$v(x, 0) = z(x) = \psi_0(0) (1 - x) + \psi_1(0) x, \; x \in (0, 1). \quad (3.4)$$
Remark 4.1. Because Problem 2 is less general than Problem 3, then this analysis of convergence for Problem 3 also works for Problem 2.

We use the linear function for \( v(x,0) \) in (3.4) is to simplify the initial condition in (2.20). Now problem 3 is an IBVP for the parabolic equation (3.2) with the reversed time. For this reason, the problem can be considered as ill-posed. Assume that the boundary with a noise of the level \( \nu > 0 \) in (3.3) are in place. Here \( \nu \) is a sufficiently small number, i.e.

\[
\| \psi_0 - \psi^*_0 \|_{H^1(0,T)} < \nu, \| \psi_1 - \psi^*_1 \|_{H^1(0,T)} < \nu,
\]

(3.5)

where functions \( \psi^*_0, \psi^*_1 \in H^2(0,T) \) are “ideal” noiseless data. We assume that there exists an exact solution \( v^* \in H^2(Q_T) \) of problem (3.2)-(3.4) with these noiseless data (based on the theory of Ill-Posed problems). Below we present estimates how this noise affects the accuracy of the solution of Problem 3 and also discuss the convergence rate of numerical solutions obtained by QRM to the exact one as \( \nu \to 0 \).

Let’s introduce the version of functional (2.24):

\[
I_\beta(v) = \int_{Q_T} (Nv)^2 \, dxdt + \beta \| v \|^2_{H^2(Q_T)}.
\]

(3.6)

We also have the set \( W \subset H^2(Q_T) \),

\[
W = \{ v \in H^2(Q_T) : v(0,t) = \psi_0(t), v(1,t) = \psi_1(t), v(x,0) = z(x) \}.
\]

(3.7)

The solution of Problem 3 is approximate solution by solving the following problem:

**Minimization Problem 2.** Minimize the functional \( I_\beta(v) \) on the set \( W \) given in (3.7).

Minimization Problem 2 is QRM for Problem 3.

### 3.2 Theorems

This subsection presents four theorems for Problem 3. All proofs might be found in [10]. First, let’s introduce the Carleman Weight Function \( \phi_\alpha(t) \) with \( \alpha > 2 \) for the operator \( \partial_t + b(x,t) \partial_x^2 \) as:

\[
\phi_\alpha(t) = e^{(T+1-t)\alpha}, \quad t \in (0,T).
\]

(3.8)

As a result, the function \( \phi_\alpha(t) \) is decreasing on \( [0,T] \), \( \phi'_\alpha(t) < 0 \),

\[
\max_{[0,T]} \phi_\alpha(t) = \psi_\alpha(0) = e^{(T+1)\alpha}, \quad \min_{[0,T]} \phi_\alpha(t) = \phi_\alpha(T) = e.
\]

(3.9)

Denote

\[
H^2_0(Q_T) = \{ u \in H^2(Q_T) : u(0,t) = u(1,t) = 0 \}.
\]

(3.10)

\[
H^2_{0,0}(Q_T) = \{ u \in H^2_0(Q_T) : u(x,0) = 0 \}.
\]

(3.11)

**Theorem 4.1** (Carleman estimate). *Let the coefficient \( b(x,t) \) of the operator \( N \) satisfies conditions (3.1). Then there exist a sufficiently large number \( \alpha_0 = \alpha_0(T,b_0,b_1) > 2 \) and a constant \( C = C(T,b_0,b_1) > 0 \), both depending only on listed parameters, such that the following Carleman estimate holds for the operator \( N :*

\[
\int_{Q_T} (Nu)^2 \phi_\alpha^2 \, dxdt \geq C \sqrt{\alpha} \int_{Q_T} u_x^2 \psi_\alpha^2 \, dxdt + C \alpha^2 \int_{Q_T} u^2 \phi_\alpha^2 \, dxdt
\]
\[-C \sqrt{\alpha} \|u\|_{H^2(\Omega_T)}^2 - C \lambda (T + 1)^\alpha \epsilon^2 (T + 1)^\alpha \|u(x,0)\|_{L^2(0,1)}^2, \quad \forall \alpha \geq \alpha_0, \forall u \in H^1_0(\Omega_T). \quad (3.12)\]

Carleman estimate \([3.12]\) is the MAIN TOOL to proofs of Theorems 4.2, 4.4.

**Theorem 4.2** (Hölder stability estimate for Problem 3 and uniqueness). Let the coefficient \(b(x,t)\) of the operator \(N\) satisfies conditions \([3.1]\). Let’s assume that the functions \(v \in H^2(\Omega_T)\) and \(v^* \in H^2(\Omega_T)\) are solutions of Problem 3 with the vectors of data \((\psi_0(t), \psi_1(t))\) and \((\psi_0^*(t), \psi_1^*(t))\) respectively, where \(\psi_0, \psi_1, \psi_0^*, \psi_1^* \in H^2(0,T)\). Assume also that error estimates \((3.5)\) of the boundary data is in place. Choose an arbitrary number \(\epsilon \in (0,T)\). Denote

\[
\lambda = \lambda(T, \epsilon) = \frac{\ln (T + 1 - \epsilon)}{\ln (T + 1)} \in (0,1). \quad (3.13)
\]

Then there exists a sufficiently small number \(\nu_0 = \nu_0(T, b_0, b_1) \in (0,1)\) and a constant \(C_1 = C_1(T, b_0, b_1, \epsilon) > 0\), both depending only on listed parameters, such that the following stability estimate holds for all \(\nu \in (0, \nu_0)\):

\[
\|v_{x} - v_{x}^*\|_{L^2(\Omega_{T-\epsilon})} + \|v - v^*\|_{L^2(\Omega_{T-\epsilon})} \leq C_1 \left(1 + \|v - v^*\|_{H^2(\Omega_T)}\right) \exp \left[-(\ln \nu^{-1/2})^\lambda\right]. \quad (3.14)
\]

Below \(C = C(T, b_0, b_1) > 0\) and \(C_1 = C_1(T, a_0, b_1) > 0\) denote different constants depending only on listed parameters.

**Corollary 4.1** (uniqueness). Let the coefficient \(b(x,t)\) of the operator \(N\) satisfies conditions \([3.1]\). Then Problem 3 has at most one solution (uniqueness).

**Proof.** If \(\nu = 0\), then \((3.14)\) implies that \(v(x,t) = v^*(x,t)\) in \(\Omega_{T-\epsilon}\). Since \(\epsilon \in (0,T)\) is an arbitrary number, then \(v(x,t) \equiv v^*(x,t)\) in \(\Omega_T\). \(\square\)

**Theorem 4.3** (existence and uniqueness of the minimizer). Let functions \(\psi_0(t), \psi_1(t) \in H^2(0,T)\). Let \(W\) be the set defined in \((3.7)\). Then there exists unique minimizer \(v_{\min} \in W\) of functional \((3.6)\) and

\[
\|v_{\min}\|_{H^2(\Omega_T)} \leq \frac{C}{\sqrt{\beta}} \left(\|\psi_0\|_{H^2(0,T)} + \|\psi_1\|_{H^2(0,T)}\right). \quad (3.15)
\]

In the theory of Ill-Posed Problems, this minimizer \(v_{\min}\) is called “regularized solution” of Problem 3 \([9]\). According to the theory of Ill-Posed problems, it is important to establish convergence rate of regularized solutions to the exact one \(v^*\). In doing so, one should always choose a dependence of the regularization parameter \(\beta\) on the noise level \(\nu\), i.e. \(\beta = \beta(\nu) \in (0,1)\) \([2]\).

**Theorem 4.4** (convergence rate of regularized solutions). Let \(v^* \in H^2(\Omega_T)\) be the solution of Problem 3 with the noiseless data \((\psi_0^*(t), \psi_1^*(t))\). Let functions \(\psi_0, \psi_1, \psi_0^*, \psi_1^* \in H^2(0,T)\). Let \(v_{\min} \in W\) be the unique minimizer of functional \((3.6)\) on the set \(W\). Assume that error estimates \((3.5)\) hold. Choose an arbitrary number \(\epsilon \in (0,T)\). Let \(\lambda = \lambda(T, \epsilon) \in (0,1)\) be the number defined in \((3.13)\) and let

\[
\beta = \beta(\nu) = \nu^2, \quad (3.16)
\]
Then there exists a sufficiently small number $\nu_0 = \nu_0(T, b_0, b_1) \in (0, 1)$ depending only on listed parameters such that the following convergence rate of regularized solutions $v_{\min}$ holds for all $\nu \in (0, \nu_0)$:

$$\| \partial_x v_{\min} - \partial_x v^* \|_{L_2(Q_{T-\epsilon})} + \| v_{\min} - v^* \|_{L_2(Q_{T-\epsilon})} \leq C_1 \left( 1 + \|v^*\|_{H^2(Q_T)} + \|\psi_0^*\|_{H^2(0,T)} + \|\psi_1^*\|_{H^2(0,T)} \right) \exp \left[ - \left( \ln \nu^{-1/2} \right) \lambda \right].$$

### 3.3 Trading Strategy:

We use minimizers obtained from the method of Quasi-Reversibility to build a strategy for trading options. Let’s define

$$REAL(0) = \frac{u_a(0) + u_b(0)}{2} \quad (3.18)$$

$$REAL(\tau) = \frac{u_a(\tau) + u_b(\tau)}{2} \quad (3.19)$$

$$EST(\tau) = u_\beta(1/2, k\tau) \quad (3.20)$$

or if it was not applied dimensionless

$$EST(\tau) = u_\beta \left( \frac{s_a + s_b}{2}, k\tau \right) \quad (3.21)$$

where $k = 1$

Here $EST(\tau)$ means minimizer.

Let’s buy an option if the following holds

$$EST(\tau) \geq REAL(0) \quad (3.22)$$

The predicted outcome of option trade is Positive if

$$EST(\tau) \geq REAL(0) \quad (3.23)$$

**Definition 1.**

It is True Positive if

$$EST(\tau) \geq REAL(0) \quad (3.24)$$

and

$$REAL(\tau) \geq REAL(0) \quad (3.25)$$

**Definition 2.**

It is True Negative if

$$EST(\tau) < REAL(0) \quad (3.26)$$

and

$$REAL(\tau) < REAL(0) \quad (3.27)$$
**Definition 3.**
It is False Positive if
\[ EST(\tau) \geq REAL(0) \] (3.28)
and
\[ REAL(\tau) < REAL(0) \] (3.29)

**Definition 4.**
It is False Negative if
\[ EST(\tau) < REAL(0) \] (3.30)
and
\[ REAL(\tau) \geq REAL(0) \] (3.31)

The accuracy of trading strategy is defined as
\[
Accuracy = \frac{TP + TN}{\sum \text{options}}
\] (3.32)

where \( TP \) is a summation of True Positive and \( TN \) is a summation of True Negative and \( \sum \text{options} \) is a summation of options in data set.

The precision of trading strategy is defined as
\[
Precision = \frac{TP}{TP + FP}
\] (3.33)

where \( FP \) is a summation of False Positive.

The recall of trading strategy is defined as
\[
Recall = \frac{TP}{TP + FN}
\] (3.34)

where \( FN \) is a summation of False Negative.

The average relative error of trading strategy is defined as
\[
Error = \frac{1}{N} \sum \left| \frac{EST(\tau) - REAL(\tau)}{REAL(\tau)} \right|
\] (3.35)
4 Application of Neural Network Machine Learning

The Black-Scholes equation gives fair value of options in perfect market. However, real options prices contain some level of noise. We try to filter mispredictions (i.e. where minimizers result in False Positive or False Negative) caused by input noise using Machine Learning to improve accuracy, precision and recall of the trading strategy. We built a neural network with 13 element input vector and 3 fully connected hidden layers. (See Fig 1). Input vector consists of minimizers (for \( t = \tau, 2\tau \)) obtained from the method of Quasi-Reversibility, stock ask and bid price (for \( t = 0 \)), option ask and bid price and volatility (for \( t = -2\tau, -\tau, 0 \)).

All vectors and labels are split into three parts: training, validation and test sets. The training set is used for weight learning. Validation set is used for tuning of the neural network hyper-parameters. Test set is for generating the outcomes of trading strategy.

We collected historical option and stock prices along with implied volatility on companies consisting of Russel 2000 index [6].

We compared the profitability of the trading strategy based on the original minimizer set with the profitability of the output of Machine Learning.

4.1 Machine Learning Input Vector Normalization

\[
\mu = \frac{u_a(0) + u_a(-\tau) + u_a(-2\tau) + u_b(0) + u_b(-\tau) + u_b(-2\tau)}{6} \tag{4.1}
\]

\[
op_n = \frac{u(t) - \mu}{\sigma} \tag{4.2}
\]

\[
s_n = \frac{(s - st) - \mu}{\sigma} \tag{4.3}
\]
where \( op_n \) is a normalized option price, \( s_n \) is a normalized stock price normalization, \( s \) is the stock, \( st \) is the strike and \( \sigma \) is the standard deviation.

### 4.2 Binary classification

Supervised Machine Learning has been applied to the neural network for the Cross Entropy Loss function with regularization:

\[
L(\theta) = \frac{1}{m} \sum_{i=1}^{m} [-y^{(i)} \log(h_\theta(x^{(i)}) - (1 - y^{(i)}) \log(1 - h_\theta(x^{(i)}))] + \frac{\lambda}{2m} \sum_{j=1}^{n} \theta_j^2
\]

Where \( \theta \) are weights which are optimized by minimizing the loss function using the method of gradient descent. \( \lambda \) is a parameter of regularization. \( x^{(i)} \) is our normalized 13-dimensional vectors. \( h_\theta \) is output of the neural network. \( m \) is the number of vectors in the training set. \( y^{(i)} \) is our labels (the ground truth). The trading strategy is defined by

\[
H_c = \begin{cases} 
1, & \text{if } h_\theta > c \\
0, & \text{otherwise.}
\end{cases}
\]

where \( c \) is the threshold obtained by maximizing accuracy on validation set. The labels are set to 1 for profitable trades and 0 otherwise.

### 4.3 Regression model

Similarly, instead of using binary classification, we can use the same ML architecture to predict the option price for tomorrow (\( \tau \)). We have the same input features as the classification neural network. Regression learning uses mean squared error as the loss function:

\[
L(\tilde{h}_\theta, \tilde{y}) = \frac{1}{m} \sum_{n}(\tilde{h}_\theta - \tilde{y}_n)^2
\]

Where \( m \) is the size of the data set, \( \tilde{h}_\theta \) is the predicted value and \( \tilde{y} \) is the real value (\( Real(\tau) \)).

### 5 Results

The following graph shows the accuracy of the results on validation set. We use it to determine the optimal value of hyper-parameter \( c \) (the threshold value of binary classification).
Observation 1.

The accuracy was improved by both Machine Learning methods compared to the method of Quasi-Reversibility. Based on this graph we set $c = 0.5$.

The next graph presents Recall and Precision diagram built on validation data set.

Figure 2: Accuracy. Threshold dependency
Figure 3: Precision vs Recall. Threshold dependency. Here X indicates the position of Recall and Precision produced by Regression Neural Network and ◇ indicates the position of Recall and Precision produced by QRM.

**Observation 2.**

Binary Classification and Regression produced similar results that improved both precision and recall compared to the method of Quasi-Reversibility.
Further we divided our test data into bins where (horizontal axis, see Figure 4) each bin is determined by \( \frac{s - st}{\Delta} \) with step size 0.1. for each bin we calculated precision (see Figure 4).

![Graph](image)

**Figure 4:** Binary Classification, Regression NN and the method of Quasi-Reversibility.

**Observation 3.**

To our surprise, when stock price was close to the strike price Machine Learning and the method of Quasi-Reversibility give similar precision (bin 0). With stock price diverging from the strike price Machine Learning produced better precision.
The following tables summarize the accuracy, precision and recall for all methods on test data.

Table 2. Final results on Test Data.

| Method             | Accuracy | Precision | Recall | Error |
|--------------------|----------|-----------|--------|-------|
| QRM                | 49.77%   | 55.77%    | 52.43% | 12%   |
| Binary Classification | 56.36%   | 59.56%    | 70.22% | NA    |
| Regression NN      | 55.42%   | 60.32%    | 61.29% | NA    |

Table 3. Percentages of options with profits/losses for three different methods.

| Method             | Profitable options | Options with loss |
|--------------------|--------------------|-------------------|
| QRM                | 55.77%             | 44.23%            |
| Binary Classification | 59.56%             | 40.44%            |
| Regression NN      | 60.32%             | 39.68%            |

6 Summary

To predict prices of stock options, we used two empirical mathematical models for Black-Scholes equation. The results achieved by solving the equation forwards in time (as an ill-posed problem) and applying Supervised Machine Learning (Binary Classification and Regression Neural Network, and using these methods with the real market data, show that this methodology produce promising results, potential applications within real-world trading and investment strategies.

The comparison of our methods resulted in the following two conclusions:

1. The predictions of the method of Quasi-Reversibility ended up being profitable for 55.77% of the options. Compare this to a 59.56% profitability rate for the Binary Classification method, and a 60.32% profitability rate for Regression Neural Network, used on the same data set and with the same trading strategy.

2. As shown by figures in section 5, option price forecasting using Machine Learning gives us significant accuracy and profit improvements over the method of Quasi-Reversibility. However, when stock price is close to the strike price both models give similar results.

The authors hypothesize that options traders can generate significant profits using trading strategies reliant on predictions generated with these methods.
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