Detection of multipartite entanglement in the vicinity of symmetric Dicke states

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We present methods for detecting entanglement around symmetric Dicke states. In particular, we consider \(N\)-qubit symmetric Dicke states with \(N/2\) excitations. In the first part of the paper we show that for large \(N\) these states have the smallest overlap possible with states without genuine multi-partite entanglement. Thus these states are particularly well suited for the experimental examination of multi-partite entanglement. We present fidelity-based entanglement witness operators for detecting multipartite entanglement around these states. In the second part of the paper we consider entanglement criteria, somewhat similar to the spin squeezing criterion, based on the moments or variances of the collective spin operators. Surprisingly, these criteria are based on an upper bound for variances for separable states. We present both criteria detecting entanglement in general and criteria detecting only genuine multi-partite entanglement. The collective operator measured for our criteria is an important physical quantity: Its expectation value essentially gives the intensity of the radiation when a coherent atomic cloud emits light.

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I. INTRODUCTION

The nonclassical effects of quantum mechanics have already been studied theoretically for more than 50 years [1]. Which quantum states can lead to phenomena that are strikingly nonclassical? Which quantum states are useful for quantum information processing applications? The answers to these questions lead to the definition of separability, entanglement [2], and multi-partite entanglement [3, 4].

In the last decade, with the rapid development of quantum control [5] it has become possible to examine the nonclassicality of quantum mechanics experimentally by creating multi-qubit quantum states of photons [6, 7, 8, 9, 10, 11, 12, 13], trapped ions [3], and cold atoms on an optical lattice [14]. The first multi-qubit experiments concentrated on Greenberger-Horne-Zeilinger [15] (GHZ) states. As maximally entangled multi-qubit states, they are intensively studied and have been realized in numerous experiments [3, 6, 7, 8]. Other quantum states targeted in experiments due to their interesting properties are, for example, cluster states [11, 12, 13, 16, 17] and many-body singlet states [18].

In this paper we will discuss some of the advantages of using Dicke states [19] to study the nonclassical phenomena of quantum mechanics. In his seminal paper Ref. [19], Dicke considered the spontaneous emission of light by a cloud of two-state atoms which are coupled to the electromagnetic field as electric dipoles. He found that when the cloud acts as a coherent quantum system, the maximal light intensity is roughly proportional to the square of the number of atoms. This Dicke called superradiance. The highly correlated Dicke states, defined to describe the system above, are the simultaneous eigenstates of the collective angular momentum, \(J\) and its \(z\)-component, \(J_z\). In a typical many-qubit experiment, in which the qubits cannot be individually accessed, both the initial state and the dynamics are symmetric under the permutation of qubits. Thus in this paper we will consider only symmetric Dicke states. These are also the states with maximal \(J\). An \(N\)-qubit symmetric Dicke state with \(m\) excitations is defined as [20]

\[
|m, N\rangle := \left( \frac{N}{m} \right)^{\frac{m}{2}} \sum_k P_k(\{1, 1, \ldots, 1_m, 0_{m+1}, \ldots, 0_N\}),
\]

where \(P_k\) is the set of all distinct permutations of the spins. \(|1, N\rangle\) is the well known \(N\)-qubit W state.

Several proposals have been presented in the literature for the experimental creation of Dicke states. In Ref. [21] a scheme is considered for creating Dicke states in trapped ions using an adiabatic process. A method for the realization of arbitrary superposition of symmetric Dicke states by detecting the photons leaving a cavity is described in Ref. [22]. A novel scheme has been proposed for obtaining Dicke states based on creating closed subspaces for the quantum dynamics of an ion chain [23]. Other proposals are described for example in Refs. [24, 25, 26, 27].

On the experimental side, we have to mention that a three-qubit W state has been created in a photonic system [10, 28, 29]. Also, an eight-qubit W state has been prepared with trapped ions [30]. Very recently, a four-qubit Dicke state with two excitations has been created in a photonic system [31]. It turned out that this is one of the quantum states which can be obtained in a photonic experiment with a very good fidelity. Future experiments will most certainly lead to creation of Dicke states with multiple excitations in other physical systems. At this point it is important to ask the question: Are such states...
more useful than others from the point of view of quantum information processing? In Ref. [31] it has already been discussed that the Dicke state prepared in the experiment is useful for telecloning.

In this paper we demonstrate that Dicke states with multiple excitations are also good candidates for the experimental examination of genuine multipartite entanglement [4]. In particular, we discuss how to detect entanglement close to $|N/2, N\rangle$, i.e., an $N$-qubit symmetric Dicke state with $N/2$ excitations. We find that, similarly to GHZ [15] and cluster states [16], for large $N$ such states have the smallest overlap possible with states without multipartite entanglement.

In the second part of the paper entanglement detection schemes requiring only collective measurements are discussed [32, 33, 34, 35, 36, 37, 38]. Entanglement detection with collective measurements is important since in many experiments the qubits cannot be accessed individually. Even if the qubits can be individually accessed, our measurement schemes are still useful since they need a small experimental effort [31]. The schemes presented are based on an upper bound on the variances of collective observables for separable states. Any state violating this bound is detected as entangled. We present schemes for entanglement detection in general and also schemes for detecting only genuine multi-partite entanglement.

$|N/2, N\rangle$ is exactly the quantum state for which Dicke found that the superradiance is the strongest [19] for even $N$. We will show that if our schemes are applied to a system described in Dicke’s original paper [19] then the measurement of the collective observables of our scheme is essentially equivalent to the measurement of light intensity emitted by the atoms.

Our paper is organized as follows. In Sec. II we show that for a fidelity-based detection of multipartite entanglement the required fidelity is low for this state. In Sec. III, we discuss entanglement detection with collective observables close to the state $|N/2, N\rangle$. In the Appendix we present some calculations for Sec. II.

II. FIDELITY-BASED ENTANGLEMENT CRITERIA

Before starting our main discussion, let us first review the basic terminology of the field. An $N$-qubit state is called fully separable if its density matrix can be written as the mixture of product states

$$\rho = \sum_l p_l \rho_l^{(1)} \otimes \rho_l^{(2)} \otimes \ldots \otimes \rho_l^{(N)},$$  \hspace{1cm} (2)

where $\sum_l p_l = 1$ and $p_l > 0$. Otherwise the state is called entangled. Quantum optics experiments aim to create entangled states, since these are the quantum states which lead to phenomena very different from classical physics [5].

In a multi-qubit experiment it is important to detect genuine multi-qubit entanglement [10]: We have to show that all the qubits were entangled with each other, not only some of them. An example of the latter case is a state of the form

$$|\Phi\rangle = |\Phi_{1,m}\rangle \otimes |\Phi_{m+1,N}\rangle$$  \hspace{1cm} (3)

Here $|\Phi_{1,m}\rangle$ denotes the state of the first $m$ qubits while $|\Phi_{m+1,N}\rangle$ describes the state of the remaining qubits. Note that the state given by Eq. (3) might be entangled, but it is separable with respect to the partition $(1, 2, \ldots, m)(m + 1, m + 2, \ldots, N)$. Such states are called biseparable [4] and can be created from product states such that two groups of qubits do not interact. These concepts can be extended to mixed states. A mixed state is biseparable if it can be created by mixing biseparable pure states of the form Eq. (3). Note that we get mixed biseparable states even when mixing pure biseparable states which are separable with respect to different partitions (e.g., when mixing $(|00\rangle + |11\rangle)/\sqrt{2}$ and $|0\rangle(|00\rangle + |11\rangle)/\sqrt{2}$). An $N$-qubit state is said to have genuine $N$-partite entanglement if it is not biseparable.

Now we will present conditions for the detection of genuine multipartite entanglement. These will be criteria based on entanglement witness operators [39, 40, 41, 42, 43, 44, 45, 46]. In other words, these are criteria which are linear in operator expectation values [47]. Based on Ref. [10] we know that for biseparable states $\rho$

$$Tr(\rho|\Psi\rangle\langle\Psi|) \leq C_{\Psi},$$  \hspace{1cm} (4)

Here $|\Psi\rangle$ is a multi-qubit entangled state and $C_{\Psi}$ is the square of the maximal overlap of $|\Psi\rangle$ with biseparable states [10]

$$C_{\Psi} := \max_{\phi \in \mathcal{B}} |\langle \Psi|\phi\rangle|^2,$$  \hspace{1cm} (5)

where $\mathcal{B}$ denotes the set of biseparable pure states. Any state $\rho$ violating Eq. (4) is necessarily genuine multipartite entangled. The bound in Eq. (4) is sharp, that is, it is the lowest possible bound. Computing Eq. (5) seems to be a complicated optimization problem. Fortunately, it turns out that $C_{\Psi}$ equals the square of the maximum of the Schmidt coefficients of $|\Psi\rangle$ with respect to all bipartitions [10]. Thus $C_{\Psi}$ can be determined easily, without the need for multi-variable optimization.

The use of criteria of the type Eq. (4) are the following. Let us say that in an experiment one aims to prepare the state $|\Psi\rangle$. This preparation is not perfect, however, one might still expect that the state prepared in the experiment is close to $|\Psi\rangle$. Thus a fidelity-based entanglement criterion of the type Eq. (4) can be used to detect its entanglement. The smaller the required minimal fidelity $C_{\Psi}$, the better the criterion from practical point of view.

Now we will present criteria of the form Eq. (4) for detecting entanglement around symmetric Dicke states. Theorem 1. For biseparable quantum states $\rho$

$$Tr(\rho|N/2, N\rangle\langle N/2, N|) \leq \frac{N}{2N-1} =: C_{N/2,N}.$$  \hspace{1cm} (6)
This condition detects entanglement close to an \( N \)-qubit symmetric Dicke state with \( N/2 \) excitations. Here \( N \) is assumed to be even.

**Proof.** The Schmidt decomposition of \(|m, N\rangle\) according to the partition \((1, 2, \ldots, N_1)(N_1+1, N_1+2, \ldots, N)\) is [20]

\[
|m, N\rangle = \sum_k \lambda_k |k, N_1\rangle \otimes |m-k, N-N_1\rangle,
\]

where the Schmidt coefficients are

\[
\lambda_k = \left( \frac{N}{m} \right)^{-\frac{k}{2}} \left( \frac{N_1}{k} \right)^{\frac{k}{2}} \left( \frac{N-N_1}{m-k} \right)^{\frac{k}{2}}.
\]

We do not have to consider other partitions due to the permutational symmetry of our Dicke states. For \(|N/2, N\rangle\) we have \( m = N/2 \). Now we use that

\[
\left( \frac{N_1}{k} \right) \left( \frac{N-N_1}{N/2-k} \right) \leq \left( \frac{2}{1} \right) \left( \frac{N/2-1}{N-1} \right).
\]

The proof of Eq. (9) can be found in the Appendix. Thus we find that the maximal Schmidt coefficient can be obtained for \( N_1 = 2 \) and \( k = 1 \). For \( N_1 = 2 \) we obtain \( \lambda_1^2 = N(N-1)/2 \).

Thus we find that \( C_{N/2, N} \approx 1/2 \) for large \( N \). This makes the detection of multipartite entanglement around the state \(|N/2, N\rangle\) relatively easy. This property is quite remarkable: Up to now only GHZ [15], cluster [16] and graph states [48] known to have \( C = 1/2 \) [49, 50].

Considered to the previous paragraph, it is important to check how much our entanglement criterion is robust against noise. In order to see that let us consider a \(|N/2, N\rangle\) state mixed with white noise:

\[
\varrho(p) = p_{\text{noise}} \frac{1}{2N} + (1-p_{\text{noise}}) |N/2, N\rangle \langle N/2, N|,
\]

where \( p_{\text{noise}} \) is the ratio of noise. Our criterion is very robust: It detects a state of the form Eq. (10) as true multipartite entangled if

\[
p_{\text{noise}} < \frac{1}{2} \left[ \frac{N-2}{(N-1)(1-2^{-N})} \right].
\]

For large \( N \) we have \( p_{\text{noise}} \leq 1/2 \).

Note that the situation is very different for a W state. A condition which can be obtained for detecting genuine multi-partite entanglement around a W state is [10, 30]

\[
Tr(\varrho|1, N\rangle \langle 1, N|) \leq \frac{N-1}{N} =: C_{1,N}.
\]

Any state violating this condition is multi-partite entangled. However, note that with an increasing \( N \), \( C_{1,N} \) approaches rapidly 1. This makes multipartite entanglement detection based on Eq. (12) challenging.

### III. ENTANGLEMENT DETECTION WITH COLLECTIVE MEASUREMENTS

In Sec. II, for Theorem 1 we needed the measurement of the expectation value of \(|N/2, N\rangle \langle N/2, N|\). In order to measure this operator, it must be decomposed into the sum of multi-qubit correlation terms of the form \( A^{(1)} \otimes A^{(2)} \otimes A^{(3)} \otimes \ldots \) [10, 49, 50, 51, 52, 53, 54], where \( A^{(k)} \) acts on qubit \( k \). For measuring the expectation value of such correlation terms, we must be able to access the qubits individually.

However, in certain physical systems (e.g., optical lattices of bosonic two-state atoms [14]) only the measurement of collective quantities is possible. In this section we present entanglement criteria for detecting entanglement with collective measurements [32, 33, 34, 35, 36, 37, 38]. Our entanglement conditions will be built using the collective spin operators

\[
J_{x/y/z} := \frac{1}{2} \sum_{k=1}^{N} \sigma^{(k)}_{x/y/z},
\]

where \( \sigma^{(k)}_{x/y/z} \) denote Pauli spin matrices acting on qubit \( k \).

**Lemma 1.** For separable states the maximum of the expression

\[
a_x J_x^2 + a_y J_y^2 + a_z J_z^2 + b_x J_x + b_y J_y + b_z J_z \quad (14)
\]

with \( a_{x/y/z} \geq 0 \) and real \( b_{x/y/z} \) is the same as its maximum for translationally invariant product states (i.e., for product states of the form \( |\Psi\rangle = |\psi\rangle \otimes |\nu\rangle \)). In particular, if \( b_x = b_y = b_z = 0 \) then this expression is bounded from above by

\[
B := (a_x + a_y + a_z) \frac{N}{4} + \max(a_x, a_y, a_z) \frac{N}{2} \left( \frac{N}{2} - \frac{1}{2} \right).
\]

**Proof.** Due to the convexity of the set of separable states, it is enough to look for the maximum for pure product states. For technical reasons, let us consider a mixed product state of the form \( \rho = \otimes_{k=1}^{N} \rho^{(k)} \) and use the notation \( s^{(k)}_{x/y/z} := Tr(\rho^{(k)} \sigma^{(k)}/2) \). Hence we have to maximize

\[
f := (a_x + a_y + a_z) N + \sum_{l=x,y,z} a_l \left[ \sum_k s^{(k)}_l \right]^2 - \sum_k \left( s^{(k)}_l \right)^2 + b_l \sum_k s^{(k)}_l.
\]

Let us consider the constraints

\[
\sum_k s^{(k)}_l = K_l.
\]
for \( l = x, y, z \) where \( K_l \) are some constants. Note that \( f \) can be written as \( f = (a_x + a_y + a_z) N + a_x f_x + a_y f_y + a_z f_z \).

Now let us first take \( f_x \), that is, the part which depends only on the \( s_x^{(k)} \) coordinates. It can be written as

\[
f_x = \left( \sum_k s_x^{(k)} \right)^2 - \sum_k \left( s_x^{(k)} \right)^2 + \alpha \sum_k s_x^{(k)},
\]

where \( \alpha = 2a_x/a_x \). We build the constraint Eq. (17) into our calculation by the substitution

\[
s_x^{(N)} = K_x - \sum_{k=1}^{N-1} s_x^{(k)}.
\]

Then for any \( m < N \) we obtain the derivatives as

\[
\frac{\partial f_x}{\partial s_x^{(m)}} = -2s_x^{(m)} + 2(K_x - \sum_{k=1}^{N-1} s_x^{(k)}).
\]

In an extreme point this should be zero. Hence it follows that for all \( m < N \)

\[
s_x^{(m)} = s_x^{(N)},
\]

thus \( f_x \) takes its extremum when all \( s_x^{(m)} \) are equal. Let us now see whether this extreme point is a maximum. For any \( m, n < N \)

\[
\frac{\partial^2 f_x}{\partial s_x^{(m)} \partial s_x^{(n)}} = -2 - 2\delta_{mn},
\]

where \( \delta_{mn} \) is the Kronecker symbol. It is easy to see, that the matrix containing the second order derivatives is negative definite, thus our extremum is a maximum. It is also a global maximum, since based on Eq. (18) and the constraint Eq. (17) it is obvious that if any \( |s_x^{(m)}| \to \infty \) then \( f_x \to -\infty \). Similar calculations can be carried out for the part of \( f \) depending on the \( y \) and \( z \) coordinates.

We have just proved that for given \( K_x/y/z \), \( f \) given in Eq. (16) takes its maximum for translationally invariant product states for which \( s_x^{(k)} = K_x/y/z/N \). This maximum we will denote by \( f_{\text{max}}(K_x, K_y, K_z) \).

Let us now look for the \( K_x \), \( K_y \) and \( K_z \) for which \( f_{\text{max}} \) maximal. The condition for getting a physical state is \( \sum_i (K_i/N)^2 \leq 1/4 \) where the equality holds for pure product states. We find that \( f_{\text{max}} \) is a convex function thus it takes its maximum at the boundary of the domain allowed for \( K_x/y/z \), i.e., it takes its maximum for pure translationally invariant product states. Hence the upper bound Eq. (15) for \( f \) follows.

In general it is very hard to find the maximum for an operator expectation value for separable states [55, 56, 57, 58]. We have just proved that for operators of the form Eq. (14) which are constructed from first and second moments of the angular momentum coordinates this problem is easy: It can be reduced to a maximization over states of the form \( |\psi \rangle \otimes N \), i.e., to a maximization over three real variables \( s_x/y/z \). Note that it is not at all clear from the beginning that this simplification is possible. For example, when looking for the minimum of \( J_x^2 + J_y^2 + J_z^2 \) for pure product states, it turns out that the expression is minimized not by translationally invariant product states. To be more specific, for \( N = 2 \) when we minimize this expression for product states, the minimum is obtained for the state \( |1 \rangle - |1 \rangle \).

**Theorem 2.** As a special case of the previous criterion, we have that for separable states [59]

\[
\langle J_x^2 \rangle + \langle J_y^2 \rangle \leq \frac{N}{2} \left( \frac{N}{2} + \frac{1}{2} \right).
\]

For even \( N \), the left hand side is the maximal \( N/2 (N/2 + 1) \) only for an \( N \)-qubit symmetric Dicke state with \( N/2 \) excitations. Based on Lemma 1, the proof of this theorem is obvious. It can also be seen that the bound in Eq. (23) is sharp since a separable state of the form

\[
|\Psi_{xy} \rangle := 2^{-N/2} (|0 \rangle + |1 \rangle e^{i\phi}) \otimes N
\]

for any real \( \phi \) saturates the bound.

Based on Eq. (23), it is easy to see that for separable states we also have

\[
(\Delta J_x)^2 + (\Delta J_y)^2 \leq \frac{N}{2} \left( \frac{N}{2} + \frac{1}{2} \right).
\]

Thus \( J_{x/y}^2 \) could be replaced by the corresponding variances. Any state violating Eq. (25) is entangled. Note the curious nature of our criterion: A state is detected as entangled, if the uncertainties of the collective spin operators are larger than a bound.

How can we intuitively understand the criterion Eq. (23)? Using the notation \( \vec{J} = (J_x, J_y, J_z) \), one can rewrite it as [60]

\[
\langle \vec{J}^2 \rangle - \frac{N}{2} \left( \frac{N}{2} + \frac{1}{2} \right) \leq \langle \vec{J}^2 \rangle.
\]

For a given \( \langle \vec{J}^2 \rangle \), in order to violate Eq. (26), \( \langle \vec{J}^2 \rangle \) must be sufficiently low. For symmetric states (i.e., for states which could be used for describing two-state bosons) we have \( \langle \vec{J}^2 \rangle = \frac{N}{2} \left( \frac{N}{2} + 1 \right) \) and Eq. (26) turns into the condition

\[
\frac{N}{4} \leq \langle \vec{J}^2 \rangle.
\]

A condition similar to Eq. (27) has already been presented for the detection of two-qubit entanglement for symmetric states in Refs. [37, 38].

Criterion Eq. (23) detects the state of the form Eq. (10) as entangled if \( p_{\text{noise}} < 1/N \). Note that the limit on \( p_{\text{noise}} \) decreases rapidly with \( N \). Let us now consider a different type of noise:

\[
\varrho(p) = p_{\text{noise}} |\Psi_{xy} \rangle \langle \Psi_{xy} | + (1 - p_{\text{noise}}) |N/2, N \rangle \langle N/2, N |.
\]


where $\Psi_{xy}$ is defined in Eq. (24). Then criterion Eq. (23) detects the state as entangled for any $p_{\text{noise}} < 1$. Thus the usefulness of our criteria depends strongly on the type of the noise appearing in an experiment.

Criteria can also be obtained which detect entanglement around other multi-qubit Dicke states. For example, the expression [21]

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - 2m\langle J_z \rangle$$

(29)
takes its maximum at a Dicke state $|m + N/2, N\rangle$. The maximum for separable states can be obtained from Lemma 1.

Up to now we discussed how to detect entanglement with the measurement of collective observables. Now we show that a criterion similar to the one in Theorem 2 can be used to detect genuine multipartite entanglement. Such a criterion has already been presented for three qubits in Ref. [61]. For biseparable three-qubit states

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \leq 2 + \sqrt{5}/2 \approx 3.12.$$  

(30)

Both the state $|W\rangle = |1, 3\rangle$ and the state $|\overline{W}\rangle = |2, 3\rangle$ give the maximal 3.75 for the left-hand side of Eq. (30).

Now let us look for criteria for larger systems. In order to proceed, we will need the following:

**Lemma 2.** For a two-qubit quantum state

$$\langle M_1 \rangle^2 + \langle M_2 \rangle^2 + \langle M_3 \rangle^2 \leq \frac{16}{3},$$

where

$$M_1 := \sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)},$$  

$$M_2 := \sigma_x^{(1)} + \sigma_x^{(2)},$$  

$$M_3 := \sigma_y^{(1)} + \sigma_y^{(2)}.$$  

*(32)*

**Proof.** The proof is rather technical. Let us consider the vector $v := (\langle M_1 \rangle, \langle M_2 \rangle, \langle M_3 \rangle)$. We want to find an upper bound on $|v|$. We can easily write

$$|v|^2 = \langle M_1 \rangle^2 + \langle M_2 \rangle^2 + \langle M_3 \rangle^2.$$  

(33)

We have to look for the maximum of this expression for quantum states. The problem is that it is nonlinear in operator expectation values. Because of that we will employ the following equality

$$|v| = \max_{|n| = 1} vn,$$  

(34)

where $n$ is a real unit vector. The meaning of Eq. (34) is clear: The length of a vector equals to the maximum of its scalar product with a unit vector. Now the right hand side of Eq. (34) can be rewritten as

$$|v| = \max_{|n| = 1} \langle n_1 M_1 + n_2 M_2 + n_3 M_3 \rangle.$$  

(35)

The advantage of this expression is that it is linear in operator expectation values. The disadvantage is that we have to maximize over $n$. Now we will find an upper bound on the right hand side of Eq. (35). We will use the fact that for an operator $A$ the expectation value is bounded as $\langle A \rangle \leq \Lambda_{\max}(A)$. Here $\Lambda_{\max}(A)$ denotes the largest eigenvalue of operator $A$. Thus

$$|v| \leq \max_{|n| = 1} \Lambda_{\max}(n_1 M_1 + n_2 M_2 + n_3 M_3).$$

(36)

The eigenvalues of $(n_1 M_1 + n_2 M_2 + n_3 M_3)$ can easily be obtained analytically as the function of $n_k$. They are

$$\lambda_1 = 0,$$  

$$\lambda_2 = -2n_1,$$  

$$\lambda_{3/4} = n_1 \pm \sqrt{n_1^2 + 4n_2^2 + 4n_3^2}.$$  

(37)

Assuming $|n| = 1$, the eigenvalues given in Eq. (37) are bounded from above by $\sqrt{12}/3$. Hence, on based on Eq. (36) we obtain $|v|^2 \leq 16/3$ and Eq. (31) follows. □

Using Lemma 2, we can state the following:

**Theorem 3.** For a four-qubit biseparable state

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \leq \frac{7}{2} + \sqrt{3} \approx 5.23$$  

(38)

For the left hand side of Eq. (38) the maximum is 6 and it is obtained uniquely for the $|2, 4\rangle$ state.

**Proof.** First we present the proof for biseparable pure states with a $(12)(34)$ partition. For these $\langle J_x^2 \rangle + \langle J_y^2 \rangle = 2 + v_1 v_2/2$ where

$$v_1 := (x_1 x_2 + y_1 y_2, x_1 + x_2, y_1 + y_2, 1),$$  

$$v_2 := (1, x_3 + x_4, y_3 + y_4, x_3 x_4 + y_3 y_4).$$

(39)

Here we used the notation $x_1 x_2 = \langle \sigma_x^{(1)} \sigma_x^{(2)} \rangle$. Hence a bound can be obtained using the Cauchy-Schwarz inequality as $\langle J_x^2 \rangle + \langle J_y^2 \rangle \leq 2 + |v_1||v_2|/2 \leq 31/6 \approx 5.17$, where we used that due to Lemma 2 we have $|v|^2 \leq 16/3$.

Note that the upper bound we have just obtained for $\langle J_x^2 \rangle + \langle J_y^2 \rangle$ is smaller than the bound in Eq. (38) thus biseparable pure states with a $(12)(34)$ partition fulfill Eq. (38).

Now let us take biseparable states with the partition (1)(234). We will follow similar steps as in the proof of Lemma 2. Let us define the matrices

$$Q_a := \sigma_a^{(2)} + \sigma_a^{(3)} + \sigma_a^{(4)}, \quad a = x, y,$$

$$R := \sum_{l=x, y} \sigma_l^{(2)}\sigma_l^{(3)} + \sigma_l^{(4)}\sigma_l^{(4)}.$$  

(40)

Using these matrices we can write

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle = 2 + \frac{1}{2}(x_1 \langle Q_x \rangle + y_1 \langle Q_y \rangle + \langle R \rangle)$$

$$\leq 2 + \frac{1}{2} \max_{x_1, y_1 \leq 1} \Lambda_{\max}(x_1 Q_x + y_1 Q_y + R).$$

(41)
Now again for finding an upper bound we need the eigenvalues of \((x_1 Q_z + y_1 Q_y + R)\). These are
\[
\begin{align*}
\lambda_{1,2} &= -2 + X, \\
\lambda_{3,4} &= -2 - X, \\
\lambda_{5,6} &= 2 + X + 2\sqrt{1 + X + X^2}, \\
\lambda_{7,8} &= 2 - X + 2\sqrt{1 - X + X^2},
\end{align*}
\]
where \(X = \sqrt{x_1^2 + y_1^2}\). Assuming \(|X| \leq 1\), the upper bound of the eigenvalues in Eq. (42) is \(3 + 2\sqrt{3}\). Thus based on Eq. (41) we obtain Eq. (38) for biseparable states with a \((1)(234)\) partition.

Since the measured operators are symmetric under the permutation of qubits, this also proves that Eq. (38) holds for any biseparable pure state. Due to the convexity of biseparable states, it also holds for mixed biseparable states.

Criterion Eq. (30) and Theorem 3 have already been used in the experiment with photons described in Ref. [31] for detecting multipartite entanglement in three-qubit and four-qubit systems. Let us now briefly outline how to detect multiparticle entanglement for more than four qubits. For many qubits detecting multiparticle entanglement becomes difficult with collective observables, since (i) the robustness to noise is decreasing as the number of qubits is increasing and (ii) it is very hard to obtain the bound for biseparable states for an operator expectation value. The first problem can be handled building entanglement criteria which use higher order moments of the angular momentum coordinates \(J_x/y\). This makes the robustness to noise somewhat better. The second problem can be overcome, for example, by using the method applied in Refs. [49, 50]. This makes it possible to find upper bounds for operator expectation values for biseparable states for large number of qubits.

Finally, let us discuss how our entanglement conditions Eqs. (23,30,38) are connected to superradiance. The left hand side of Eq. (23) is the same expression which appears in Eq. (28) of Dicke’s original paper [19] giving the intensity of the superradiant light during spontaneous emission in a cloud of atoms. To be more precise, the light intensity is \(I := I_0 \langle J_x^2 + J_y^2 + J_z \rangle\) where \(I_0\) is the radiative rate of one atom in its excited state. Criterion Eq. (23) shows that if \(I/I_0 - \langle J_z \rangle\) is larger than a bound then the system is entangled. We can also see that there are separable states [e.g., the state presented in Eq. (24)] for which the light intensity scales roughly with the square of the number of qubits.

IV. CONCLUSION

We presented several methods for detecting entanglement in the vicinity of symmetric Dicke states with multiple excitations. In particular, we focused on \(N\)-qubit symmetric Dicke states with \(N/2\) excitations. We showed that they are well suited for experiments aiming to create and detect multi-partite entanglement. We presented fidelity-based criteria for detecting genuine multi-qubit entanglement in the vicinity of these states. We also considered entanglement criteria based on the measurement of collective observables. The relation of our entanglement conditions to superradiance was also discussed.

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APPENDIX: PROOF OF EQ. (9)

Here we present the proof of Eq. (9). First let us fix \(N_1\) and look for the maximum of the left hand side of Eq. (9) as the function of \(k\). (Without loss of generality, we consider \(N_1 \leq N/2\).) We define
\[
g_k := \left( \frac{N_1}{k} \right) \left( \frac{N - N_1}{N/2 - k} \right).
\]
Let us look for the \(k\) for which it is maximal. For that we compute the ratio of two consecutive \(g_k\)
\[
g_{k-1} = \frac{k(N/2 - N_1 + k)}{(N_1 - k + 1)(N/2 - k + 1)}. \tag{44}
\]
The right hand side of Eq. (44) equals 1 for \(k_m = (N_1 + 1)/2\). Thus for \(k < k_m\) we know that \(g_k/g_{k-1} \geq 1\) while for \(k > k_m\) we have \(g_k/g_{k-1} \leq 1\). Simple calculation shows that the integer value for which \(g_k\) is maximal is \(k = N_1/2\) for even \(N_1\) and \(k = (N_1 \pm 1)/2\) for odd \(N_1\).

Now we know that the maximum of the left hand side of Eq. (9) for a given \(N_1\) is
\[
h_{N_1} := \left( \frac{N_1}{N/2} \right) \left( \frac{N - N_1}{N/2 - N_1/2} \right), \tag{45}
\]
where \([x]\) denotes the integer part of \(x\). We find that for even \(N_1\)
\[
\frac{h_{N_1}}{h_{N_1+1}} \geq 1. \tag{46}
\]
Hence \(h_{N_1}\) must be maximized for some even \(N_1\). Further calculation shows that for even \(N_1\)
\[
\frac{h_{N_1}}{h_{N_1-2}} = \frac{N_1 - 1}{N_1 - N_1 + 2} \leq 1. \tag{47}
\]
Hence we know that \(h_{N_1}\) is maximized by \(N_1 = 2\). Thus we proved that the left hand side of Eq. (9) is maximized for \(N_1 = 2\) and \(k = 1\).
[1] A. Einstein, B. Podolsky and N. Rosen, "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?", Phys. Rev. 47, 777-780 (1935).

[2] R.F. Werner, "Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model", Phys. Rev. A 40, 4277-4281 (1989).

[3] R.F. Werner, "Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model", Phys. Rev. A 40, 4277-4281 (1989).

[4] D.M. Greenberger, M.A. Horne, A. Shimony, and A. Zeilinger, "Entanglement and Undistillability of a Mixed Two-Qubit State", Phys. Rev. Lett. 87, 040401 (2001).

[5] D. Bouwmeester, A. Ekert, and A. Zeilinger, The Physics of Quantum Information (Springer, Berlin, 2000).

[6] J.W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, "Experimental test of quantum nonlocality in three-photon Greenberger-Horne-Zeilinger entanglement", Nature 403, 515-519 (2000).

[7] D. Bouwmeester, A. Ekert, and A. Zeilinger, The Physics of Quantum Information (Springer, Berlin, 2000), page 209.

[8] Z. Zhao, T. Yang, Y.-A. Chen, A.-N. Zhang, M. Żukowski, and J.-W. Pan, "Experimental Violation of Local Realism by Four-Photon Greenberger-Horne-Zeilinger Entanglement", Phys. Rev. Lett. 91, 180401 (2003).

[9] Z. Zhao, Y.-A. Chen, A.-N. Zhang, T. Yang, H.J. Briegel, and J.-W. Pan, "Experimental demonstration of five-photon entanglement and open-destination teleportation", Nature 430, 54-58 (2004).

[10] M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, "Experimental Detection of Multiparticle Entanglement using Witness Operators", Phys. Rev. Lett. 92, 080502 (2004).

[11] P. Walther, K.J. Resch, T. Rudolph, E. Schenck, H. Weinfurter, V. Vedral, M. Aspelmeyer, and A. Zeilinger, "Experimental one-way quantum computing", Nature 434, 169-176 (2005).

[12] P. Walther, M. Aspelmeyer, K.J. Resch, and A. Zeilinger, "Experimental Violation of a Cluster State Bell Inequality", Phys. Rev. Lett. 95, 020403 (2005).

[13] N. Kiesel, C. Schmid, U. Weber, O. Gühne, G. Tóth, R. Ursin, and H. Weinfurter, "Experimental Analysis of a Four-Photon Cluster State", Phys. Rev. Lett. 95, 210502 (2005).

[14] O. Mandel, M. Greiner, A. Widera, T. Rom, T.W. Hänsch, and I. Bloch, "Controlled collisions for multiparticle entanglement of optically trapped atoms", Nature 425, 937-940 (2003).

[15] D.M. Greenberger, M.A. Horne, A. Shimony, and A. Zeilinger, "Bell's theorem without inequalities", Am. J. Phys. 58, 1131-1143 (1990).

[16] H.J. Briegel and R. Raussendorf, "Persistent Entanglement in Arrays of Interacting Particles", Phys. Rev. Lett. 86, 910-913 (2001).

[17] R. Raussendorf and H.J. Briegel, "A One-Way Quantum Computer", Phys. Rev. Lett. 86, 5188-5191 (2001).

[18] M. Eibl, S. Gaertner, M. Bourennane, C. Kurtsiefer, M. Żukowski, and H. Weinfurter, "Experimental Observation of Four-Photon Entanglement from Parametric Down-Conversion", Phys. Rev. Lett. 90, 200403 (2003).

[19] R.H. Dicke, "Coherence in Spontaneous Radiation Processes", Phys. Rev. 93, 99-110 (1954).

[20] J.K. Stockton, J.M. Geremia, A.C. Doherty, and H. Mabuchi, "Characterizing the entanglement of symmetric many-particle spin-1/2 systems", Phys. Rev. A 67, 022112 (2003).

[21] R.G. Unanyan and M. Fleischhauer, "Decoherence-Free Generation of Many-Particle Entanglement by Adiabatic Ground-State Transitions", Phys. Rev. Lett. 90, 133601 (2003).

[22] L.-M. Duan and H.J. Kimble, "Efficient Engineering of Multiatom Entanglement through Single-Photon Detections", Phys. Rev. Lett. 90, 253601 (2003).

[23] A. Retzker, E. Solano, and B. Reznik, "Tavis-Cummings model and collective multi-qubit entanglement in trapped ions", http://arxiv.org/quant-ph/0605048.

[24] J.K. Stockton, R. van Handel, and H. Mabuchi, "Deterministic Dicke-state preparation with continuous measurement and control", Phys. Rev. A 70, 022106 (2004).

[25] X. Zou, K. Pahlke, and W. Mathis, "Generation of arbitrary superpositions of the Dicke states of excitons in optically driven quantum dots", Phys. Rev. A 68, 034306 (2003).

[26] Y.-F. Xiao, Z.-F. Han, J. Gao, and G.-C. Guo, "Generation of Multi-Atom Dicke States with Quasi-unit Probability through the Detection of Cavity Decay", http://arxiv.org/quant-ph/0412202.

[27] V.N. Gorbachev and A.I. Trubilko, "On multiparticle W states, their implementations and application in the quantum informational problems", Laser Phys. Lett. 3, 59-70 (2006).

[28] M. Eibl, N. Kiesel, M. Bourennane, C. Kurtsiefer, and H. Weinfurter, "Experimental Realization of a Three-Qubit Entangled W State", Phys. Rev. Lett. 92, 077901 (2004).

[29] H. Mikami, Y. Li, K. Fukuoka, and T. Kobayashi, "New High-Efficiency Source of a Three-Photon W State and its Full Characterization Using Quantum State Tomography", Phys. Rev. Lett. 95, 150404 (2005).

[30] H. Häffner, W. Hänsel, C. Roos, J. Benhelm, D. Chek-al-kar, M. Chwalla, T. Körber, U. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Gühne, W. Dür, and R. Blatt, "Scalable multiparticle entanglement of trapped ions", Nature 438, 643-646 (2005).

[31] N. Kiesel, C. Schmid, G. Tóth, E. Solano, and H. Weinfurter, "Experimental Observation of Four-Photon Entangled Dicke State with High Fidelity", http://arxiv.org/quant-ph/0606234.

[32] A. Sørensen and K. Mølmer, "Spin-Spin Interaction and Spin Squeezing in an Optical Lattice", Phys. Rev. Lett. 86, 256-259 (2000).

[33] A. Sørensen, L.-M. Duan, J. I. Cirac, and P. Zoller, "Many-particle entanglement with Bose-Einstein condensates", Nature 403, 2274-2277 (1999).

[34] A. Sørensen and K. Mølmer, "Entanglement and Extreme Spin Squeezing", Phys. Rev. Lett. 86, 4431-4434 (2001).

[35] X.G. Wang, A. Sørensen, and K. Molmer, "Spin squeezing in the Ising model", Phys. Rev. A 64, 053815 (2001).

[36] G. Tóth, "Entanglement detection in optical lattices of
bosonic atoms with collective measurements”, Phys. Rev. A 69, 052327 (2004).

[37] J. Korbicz, J.I. Cirac, and M. Lewenstein, ”Spin Squeezing Inequalities and Entanglement of N Qubit States”, Phys. Rev. Lett. 95, 120502 (2005).

[38] J. Korbicz O. Gühne, M. Lewenstein, H. Häffner, C. F. Roos, and R. Blatt, ”Generalized spin squeezing inequalities in N qubit systems: theory and experiment”, http://arxiv.org/quant-ph/0601038.

[39] J. Korbicz O. Gühne, M. Lewenstein, H. Häffner, C. F. Roos, and R. Blatt, ”Generalized spin squeezing inequalities in N qubit systems: theory and experiment”, http://arxiv.org/quant-ph/0601038.

[40] M. Horodecki, P. Horodecki, and R. Horodecki, ”Separability of mixed states: necessary and sufficient conditions”, Phys. Lett. A 223, 1-8 (1996).

[41] B.M. Terhal, ”Bell inequalities and the separability criterion”, Phys. Lett. A 271, 319-326 (2000).

[42] M. Lewenstein, B. Kraus, J.I. Cirac, and P. Horodecki, ”Optimization of entanglement witnesses”, Phys. Rev. A 62, 052310 (2000).

[43] K. Chen and L.-A. Wu, ”Test for entanglement using physically observable witness operators and positive maps”, Phys. Rev. A 69, 022312 (2004).

[44] S. Yu and N. Liu, ”Entanglement Detection by Local Orthogonal Observables”, Phys. Rev. Lett. 95, 150504 (2005).

[45] L.-A. Wu, S. Bandyopadhyay, M.S. Sarandy, and D.A. Lidar, ”Entanglement observables and witnesses for interacting quantum spin systems”, Phys. Rev. A 72, 032300 (2005).

[46] F.G.S.L. Brandão, ”Quantifying entanglement with witness operators”, Phys. Rev. A 72, 022310 (2005).

[47] For entanglement criteria which are nonlinear in operator expectation values see for example O. Gühne, Phys. Rev. Lett. 92, ”Characterizing Entanglement via Uncertainty Relations”, 117903 (2004) and references therein.

[48] M. Hein, J. Eisert, and H.J. Briegel, ”Multipartite entanglement in graph states”, Phys. Rev. A 69, 062311 (2004).

[49] G. Tóth and O. Gühne, ”Detecting Genuine Multipartite Entanglement with Two Local Measurements”, Phys. Rev. Lett. 94, 060501 (2005).

[50] G. Tóth and O. Gühne, ”Entanglement detection in the stabilizer formalism”, Phys. Rev. A 72, 022340 (2005).

[51] O. Gühne, P. Hyllus, D. Bruss, A. Ekert, M. Lewenstein, C. Macchiavello, A. Sanpera, ”Detection of entanglement with few local measurements”, Phys. Rev. A 66, 062305 (2002).

[52] B.M. Terhal, ”Detecting Quantum Entanglement”, Theoret. Comput. Sci. 287, 313-335 (2002).

[53] O. Gühne and P. Hyllus, ”Investigating three qubit entanglement with local measurements”, Int. J. Theor. Phys. 42, 1001-1013 (2003).

[54] Arthur O. Pittenger and Morton H. Rubin, ”Geometry of entanglement witnesses and local detection of entanglement”, Phys. Rev. A 67, 012327 (2003).

[55] F. G. S. L. Brandão and R. O. Vianna, ”Separable Multipartite Mixed States: Operational Asymptotically Necessary and Sufficient Conditions”, Phys. Rev. Lett. 93, 220503 (2004).

[56] F. G. S. L. Brandão and R. O. Vianna, ”Robust semidefinite programming approach to the separability problem”, Phys. Rev. A 70, 062309 (2004).

[57] J. Eisert, P. Hyllus, O. Gühne, and M. Curty, ”Complete hierarchies of efficient approximations to problems in entanglement theory”, Phys. Rev. A 70, 062317 (2004).

[58] A.C. Doherty, P.A. Parrilo, and F.M. Spedalieri, ”Detecting multipartite entanglement”, Phys. Rev. A 71, 032333 (2005).

[59] Note that this condition can also be written in a more general way with the eigenvalues of the correlation matrix \( \mathbf{C} \) defined as \( c_{ab} := \langle J_a J_b + J_b J_a \rangle / 2 \). Assuming \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \), Eq. (23) can be reformulated as \( \lambda_1 + \lambda_2 \leq N(N + 1)/4 \).

[60] E. Solano, Max Planck Institute for Quantum Optics, Munich, Germany (private communication, 2005).

[61] Appendix C of Ref. [50].