A Study on Integer Additive Set-Graceful Graphs

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Abstract

A set-labeling of a graph $G$ is an injective function $f : V(G) \rightarrow \mathcal{P}(X)$, where $X$ is a finite set and a set-indexer of $G$ is a set-labeling such that the induced function $f^{\oplus} : E(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$ defined by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective. An integer additive set-labeling is an injective function $f : V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$, $\mathbb{N}_0$ is the set of all non-negative integers and an integer additive set-indexer is an integer additive set-labeling such that the induced function $f^+ : E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective. In this paper, we extend the concepts of set-graceful labeling to integer additive set-labelings of graphs and provide some results on them.

Key words: Set-indexers, integer additive set-indexers, set-graceful graphs, integer additive set-graceful labeling, integer additive set-graceful graphs.

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1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [18] and for more about graph labeling, we refer to [13]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

All sets mentioned in this paper are finite sets of non-negative integers. We denote the cardinality of a set $A$ by $|A|$. We denote, by $X$, the finite

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ground set of non-negative integers that is used for set-labeling the elements of $G$ and cardinality of $X$ by $n$.

The research in graph labeling commenced with the introduction of $\beta$-valuations of graphs in [20]. Analogous to the number valuations of graphs, the concepts of set-labelings and set-indexers of graphs are introduced in [1] as follows.

Let $G$ be a $(p,q)$-graph. Let $X$, $Y$ and $Z$ be non-empty sets and $P(X)$, $P(Y)$ and $P(Z)$ be their power sets. Then, the functions $f : V(G) \rightarrow P(X)$, $f : E(G) \rightarrow P(Y)$ and $f : V(G) \cup E(G) \rightarrow P(Z)$ are called the set-assignments of vertices, edges and elements of $G$ respectively. By a set-assignment of a graph, we mean any one of them. A set-assignment is called a set-labeling or a set-valuation if it is injective.

A graph with a set-labeling $f$ is denoted by $(G, f)$ and is referred to as a set-labeled graph. For a $(p,q)$- graph $G = (V, E)$ and a non-empty set $X$ of cardinality $n$, a set-indexer of $G$ is defined as an injective set-valued function $f : V(G) \rightarrow P(X)$ such that the function $f^\oplus : E(G) \rightarrow P(X) - \{\emptyset\}$ defined by $f^\oplus(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where $P(X)$ is the set of all subsets of $X$ and $\oplus$ is the symmetric difference of sets.

**Theorem 1.1.** [1] Every graph has a set-indexer.

Analogous to graceful labeling of graphs, the concept of set-graceful labeling of a graph is defined in [1] as follows.

Let $G$ be a graph and let $X$ be a non-empty set. A set-indexer $f : V(G) \rightarrow P(X)$ is called a set-graceful labeling of $G$ if $f^\oplus(E(G)) = P(X) - \{\emptyset\}$. A graph $G$ which admits a set-graceful labeling is called a set-graceful graph.

Let $N_0$ be the set of all non-negative integers. An integer additive set-labeling (IASL, in short) is an injective function $f : V(G) \rightarrow P(N_0)$. A graph $G$ which admits an IASL is called an IASL graph. An integer additive set-indexer $f$ is an integer additive set-indexer (IASI, in short) if the induced function $f^+ : E(G) \rightarrow P(N_0)$ defined by $f^+(uv) = f(u) + f(v)$ is injective. A graph $G$ which admits an IASI is called an IASI graph.

The cardinality of the set-label of an element (vertex or edge) of a graph $G$ is called the set-indexing number of that element. An IASL (or an IASI) is said to be a $k$-uniform IASL (or $k$-uniform IASI) if $|f^+(e)| = k$ $\forall$ $e \in E(G)$. The vertex set $V(G)$ is called $l$-uniformly set-indexed, if all the vertices of $G$ have the set-indexing number $l$.

In this paper, we extend the concepts of set-graceful labelings to integer additive set-labels of a given graph $G$ and establish some results on them.
2 Integer Additive Set-Graceful Graphs

Note that under an integer additive set-labeling, no element of a given graph can have ∅ as its set-labeling. Hence, we need to consider only non-empty subsets of $X$ for set-labeling the elements of $G$.

**Remark 2.1.** Let $X$ be a finite set of non-negative integers and let $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ be an integer additive set-labeling on a graph $G$. Note that, here the induced function $f^+ : E(G) \to \mathcal{P}(X) - \{\emptyset\}$ defined by $f^+(uv) = f(u) + f(v)$, is the sum set of the sets $f(u)$ and $f(v)$. Hence, \{0\} can not be the set-label of any edge of $G$.

In view of Remark 2.1 we introduce the following notion.

**Definition 2.2.** Let $G$ be a graph and let $X$ be a non-empty set. An integer additive set-indexer $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ is called a integer additive set-graceful labeling (IASGL, in short) of $G$ if $f^+(E(G)) = \mathcal{P}(X) - \{\emptyset, \{0\}\}$. A graph $G$ which admits an integer additive set-graceful labeling is called an integer additive set-graceful graph (in short, iasg-graph).

An iasg-graph is illustrated in Figure 1.

![Figure 1](image-url)
Proposition 2.3. If \( f : V(G) \to \mathcal{P}(X) - \{\emptyset\} \) is an integer additive set-graceful labeling on a given graph \( G \), then \( \{0\} \) must be a set-label of one vertex of \( G \).

Proof. If possible, let \( \{0\} \) is not the set-label of a vertex in \( G \). Since \( X \) is the set of non-negative integers, it contains at least one element, say \( x \) which is not the sum of any two elements in \( X \). Hence, \( \{x\} \) can not be the set-label of any edge of \( G \). This is a contradiction to the hypothesis that \( f \) is an integer additive set-graceful labeling.

Remark 2.4. If \( f : V(G) \to \mathcal{P}(X) - \{\emptyset\} \) is an integer additive set-graceful labeling on a given graph \( G \), then the ground set \( X \) must contain the element 0.

Observation 2.5. Let \( f : V(G) \to \mathcal{P}(X) - \{\emptyset\} \) be an integer additive set-graceful labeling on a given graph \( G \). Then, the vertices of \( G \), whose set-labels, containing the element 0, are not the sumsets of subsets of \( x \) must be adjacent to the vertex \( v \) that has the set-label \( \{0\} \).

Proof. let \( A_i \subset X \), containing the element 0, is the set-label of a vertex say \( v_i \) of \( G \). If \( A_i \) is not a sumset of two subsets of \( X \), then \( A_i \) is a set-label of a vertex of \( G \) if it is adjacent to the vertex \( v \) with set-label \( \{0\} \).

Proposition 2.6. Let \( x_i \) be a non-zero element of \( X \), which is not the sum of two elements in \( X \). Then, the vertex with the set-label \( \{x_i\} \) must be adjacent to the the vertex having the set-label \( \{0\} \).

Proof. Let \( x_i \in X \) is not the sum of any two elements in \( X \). Since, \( f \) is an integer additive set-graceful labeling, \( \{x_i\} \) must be the set-label of one edge, say \( e \) of \( G \). This possible only when one end vertex of \( e \) has the set-label \( \{0\} \) and the other end vertex has the set-label \( \{x_i\} \).

Corollary 2.7. Let \( f : V(G) \to \mathcal{P}(X) - \{\emptyset\} \) be an integer additive set-graceful labeling on a given graph \( G \) and let \( x_1 \) and \( x_2 \) be the minimal and second minimal non-zero element of \( X \). Then, the vertices of \( G \) that have the set-labels \( \{x_1\} \) and \( \{x_2\} \), must be adjacent to the vertex \( v \) that has the set-label \( \{0\} \).

Proof. Since \( x_1 \) and \( x_2 \) are the two minimal elements of \( X \), they are not the sum of any two elements in \( X \). Then by Proposition 2.6, the vertices of \( G \) that have the set-labels \( \{x_1\} \) and \( \{x_2\} \), must be adjacent to the vertex \( v \) that has the set-label \( \{0\} \).
Proposition 2.8. Let $G$ be an iasg-graph. Then, there are at least $1 + 2^{n-1}$ vertices of $G$ adjacent to the vertex having the set-label $\{0\}$, where $n$ is the cardinality of the ground set $X$. That is, at least $2^{|X|-1} + 1$ edges incident on the vertex that is labeled by $\{0\}$.

Proof. Let $G$ be an integer additive set-graceful graph. Then, by Proposition 2.5, the vertices of $G$ whose set-labels contain the element 0 must be adjacent to the vertex, say $v$, having set-label $\{0\}$.

Let $X_i \subset X$ which contains the element 0 and $X_i \neq \{0\}$. Then, $X_i$ is a set-label of an edge $e$ of $G$ if and only if one end vertex of $e$ is $v$ and the other end vertex has the set-label $X_i$. Note that the number of subsets of $X$ that contain 0 is $2^n - 1$. Hence, the number of vertices whose set-labels contain 0 and are adjacent to the vertex with set-label $\{0\}$ is $2^n - 1$. Also, by Proposition 2.7, the vertex with set-label $\{x_1\}$ and the vertex with set-label $\{x_2\}$, where $x_1$ and $x_2$ are the minimal and the second minimal non-zero elements of $X$, are also adjacent to $v$. Therefore, the minimum number of edges that are adjacent to $v$ is $2 + (2^n - 1) = 1 + 2^{n-1}$.

Proposition 2.9. Let $f : V(G) \rightarrow P(X) - \{\emptyset\}$ be an integer additive set-graceful labeling on a given graph $G$ and let $x_n$ be the maximal element of $X$. Then, $x_n$ is an element of the set-label of a vertex $v$ of $G$ if $v$ is a pendant vertex that is adjacent to the vertex labeled by $\{0\}$.

Proof. Let $v$ be a vertex of $G$ that has a set-label containing $x_n$. If $v$ is adjacent to a vertex, say $u$, with a set-label containing a non-zero element, say $x_1$, then $f^+(uv)$ contains the element $x_n + x_1$ which is not an element of $X$, which is a contradiction to the hypothesis that $G$ is an iasg-graph.

Proposition 2.10. Let $A_i$ and $A_j$ are two distinct subsets of the ground set $X$ and let $x_i$ and $x_j$ be the maximal elements of $A_i$ and $A_j$ respectively. Then, $A_i$ and $A_j$ are the set-labels of two adjacent vertices of an iasg-graph $G$ is that $x_i + x_j \leq x_n$, the maximal element of $X$.

Proof. Let $v$ be a vertex of $G$ that has a set-label $A_i$ whose maximal element $x_i$. If $v$ is adjacent to a vertex, say $u$, with a set-label $A_j$ whose maximal element is $x_j$, then $f^+(uv)$ contains the element $x_i + x_j$. Therefore, $x_i + x_j \in X$. Hence, $x_i + x_j \leq x_n$.

Corollary 2.11. If $G$ is a graph without pendant vertices, then no vertex of $G$ can have a set-label consisting of the maximal element of the ground set $X$. 

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Remark 2.12. Invoking the above results, we note that the vertex with the set-label \{0\} has the maximum degree in \(G\).

The following results establish the relation between the size of an iasg-graph and the cardinality of its ground set.

**Theorem 2.13.** A graph \(G\) admits an integer additive set-graceful labeling, then it has even number of edges.

*Proof.* Let \(f\) be an integer additive set-graceful labeling defined on \(G\). Then, \(f^+(E(G)) = \mathcal{P}(X) - \{\emptyset, \{0\}\}\). Therefore, \(|E(G)| = |\mathcal{P}(X)| - 2 = 2^{|X|} - 2 = 2(2^{|X| - 1} - 1).\)

**Theorem 2.14.** Let \(G\) be an iasg-graph, with an integer additive set-graceful labeling \(f\). Then, the cardinality of the ground set \(X\) is \(\log_2(|E(G)| + 2)\).

*Proof.* Let \(G\) be an iasg-graph. Due to theorem \[2.13\]

\[
|E(G)| = 2^{|X|} - 2 \\
|E(G)| + 2 = 2^{|X|} \\
\therefore \log_2(|E(G)| + 2) = |X|
\]

This completes the proof.

The conditions for certain graphs and graph classes to admit a integer additive set-graceful labeling are established in following discussions.

**Theorem 2.15.** A star graph \(K_{1,m}\) admits an integer additive set-graceful labeling if and only if \(m = 2^n - 2\) for any integer \(n > 1\).

*Proof.* Let \(v\) be the vertex of degree greater than 1. Let \(m = 2^n - 2\) and \(\{v_1, v_2, \ldots, v_m\}\), be the vertices in \(K_{1,m}\) which are adjacent to \(v\). Let \(X\) be a set of non-negative integers containing 0.

First, assume that \(K_{1,m}\) admits an integer additive set-graceful labeling, say \(f\). Then, by Theorem \[2.13\] \(|E(G)| = m = 2^{|X|} - 2\). Therefore, \(m = 2^n - 2\), where \(n = |X| > 1\).

Conversely, assume that \(m = 2^n - 2\) for some integer \(n > 1\). Label the vertex \(v\) by the set \(\{0\}\) and label the remaining \(m\) vertices of \(K_{1,m}\) by the remaining \(m\) distinct non-empty subsets of \(X\). Clearly, this labeling is an integer additive set-graceful labeling for \(K_{1,m}\).
Figure 2 illustrates the admissibility of the star graph $K_{1,6}$.

The following theorem checks whether a tree can be an iasg-graph.

**Proposition 2.16.** If a tree on $m$ vertices admits an integer additive set-graceful labeling, then $1 + m = 2^n$, for some positive integer $n > 1$.

*Proof.* Let $G$ be a tree on $m$ vertices. Then, $|E(G)| = m - 1$. Assume that $G$ admits an integer additive set-graceful labeling, say $f$. Then, by Theorem 2.13 for a ground set $X$ of cardinality $n$,

\begin{align*}
    m - 1 &= 2^{|X|} - 2 \\
    m + 1 &= 2^{|X|} \\
    m + 1 &= 2^n.
\end{align*}

\[\square\]

**Corollary 2.17.** Let $G$ be a tree on $m$ vertices. For a ground set $X$, let $f : V(G) \to \mathcal{P}(X)$ be an integer additive set-graceful labeling on $G$. Then, $|X| = \log_2(n + 1)$.

*Proof.* By Theorem 2.16, $m + 1 = 2^{|X|} \Rightarrow |X| = \log_2(n + 1)$.

\[\square\]

**Theorem 2.18.** A tree $G$ is an iasg-graph if and only if it is a star $K_{1,2^n-2}$, for some positive integer $n$. 

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Proof. If $G = K_{1,2^n-2}$, then by Theorem $2.15$, $G$ admits an integer additive set-graceful labeling. Conversely, assume that the tree $G$ on $m$ vertices admits an integer additive set-graceful labeling, say $f$ with respect to a ground set $X$ of cardinality $n$. Therefore, all the $2^n - 1$ non-empty subsets of $X$ must be required for labeling the vertices of $X$. Also, note that $0 \in X$ and $\{0\}$ can not be a set-label of any edge of $G$. Hence, all the remaining $2^n - 2$ non-empty subsets of $X$ are required for the labeling the edges of $G$.

Clearly, the vertices, whose set-labels containing 0, must be adjacent to the vertex $v$ which has the set-label $\{0\}$. The number of vertices of $G$ that have set-labels containing 0 is $2^n - 1$.

Also, by Proposition $2.9$, the vertices, whose set-labels containing the maximal element $x_n$ of $X$ must also be adjacent to the vertex labeled by $\{0\}$. Also, note the number of vertices of $G$ that have set-labels containing $x_n$ is $2^n - 1$.

The number of set-labels that have both 0 and $x_n$ is $2^n - 2$. Hence, the number of vertices that must be adjacent to $v$ is $2^{n-1} + 2^{n-1} - 2^{n-2} = 2^{n-2}$. Let $X_i$ be a subset of $X$ which contains either 0 or $x_n$ and let $v_i$ be the vertex of $G$ that has the set-label $X_i$. Then, the set-label of the edge $vv_i$ is also $X_i$. Let $X_j$ be a subset of $X$ that contains neither 0 nor $x_n$. Then, if we label a vertex $v_j$, not adjacent to $v$, by $X_j$, subject to the condition provided in $2.10$, no edge of $G$ can have the set-label $X_j$. Hence, all vertices of $G$ with non-empty subsets of $X$ as set-labels must be adjacent to the vertex $v$. Hence $G$ is a star graph $\deg(v) = 2^n - 2$.

We now check the admissibility of integer additive set-graceful labeling by path graphs and cycle graphs.

**Corollary 2.19.** For a positive integer $m > 2$, the path $P_m$ does not admit an integer additive set-graceful labeling.

**Proof.** Every path is a tree and no path other than $P_2$ is a star graph. Hence, by Theorem $2.18$, $P_m$, $m > 2$ is not an iasg-graph.

**Proposition 2.20.** For any positive integer $m > 3$, the cycle $C_m$ does not admit an integer additive set-graceful labeling.

**Proof.** Let $X$ be a ground set with $n$ elements. Since $C_m$ has $m$ edges, by Theorem $2.13$,

$$m = 2^n - 2 \quad (2.0.1)$$

Since $C_m$ has no pendant vertices, by Proposition $2.9$, the maximal element, say $x_n$, will not be an element of any set-label of the vertices of $C_m$. Therefore,
only \(2^{n-1} - 1\) non-empty subsets of \(X\) are available for labeling the vertices of \(C_m\). Hence,

\[ m \leq 2^{n-1} - 1 \quad (2.0.2) \]

Clearly, Equation \(2.0.1\) and Equation \(2.0.2\) do not hold simultaneously. Hence, \(C_m\) does not admit an integer additive set-graceful labeling.

An interesting question we need to address here is whether complete graphs admit integer additive set-graceful labeling. We investigate the conditions for a complete graph to admit an integer additive set-graceful labeling and based on these conditions check whether the complete graphs are \(iasg\)-graphs.

**Theorem 2.21.** A complete graph \(K_m\), does not admit an integer additive set-indexer.

**Proof.** Since \(K_2\) has only one edge and \(K_3\) has three edges, by Theorem \(2.13\) \(K_1\) and \(K_2\) do not have an integer additive set-graceful labeling. So we need to consider the complete graphs on more than three vertices.

Assume that a complete graph \(K_m, m > 3\) admits an integer additive set-graceful labeling. Then, by Theorem \(2.13\)

\[ |E(G)| = \frac{m(m-1)}{2} \]

\[ 2^{|X|} - 2 = \frac{m(m-1)}{2} \]

\[ 2^{|X|-1} - 1 = \frac{m(m-1)}{4}. \]

Since \(|X| > 1\), \(2^{|X|-1} - 1\) is a positive integer. Hence, \(m(m-1)\) is a multiple of 4. This is possible only when either \(m\) or \((m-1)\) is a multiple of 4.

Since \(|X| > 1\), \(2^{|X|-1} - 1\) is a positive odd integer. Hence, for an odd integer \(k\), either \(m = 4k\) or \(m - 1 = 4k\). Therefore, \(2^{|X|-1} - 1 = \frac{4k(4k-1)}{4} = k(4k-1)\) or \(2^{|X|-1} - 1 = \frac{4k(4k-1)}{4} = k(4k+1)\). That is, \(2^{|X|-1} = 1 + k(4k \pm 1)\).

That is, a complete graph \(K_m\) admits an integer additive set-graceful labeling, if, there exist an integral solution for the equation

\[ 4k^2 \pm k + 1 = 2^n \quad (2.0.3) \]

where \(k\) is an odd non-negative integer and \(n > 3\) be a positive integer.

Since we can not find an odd integer \(k\) satisfying Equation \(2.0.3\), \(K_m\) does not admit an integer additive set-graceful labeling. \(\square\)
Since all graphs are not iasg-graphs in general, it is necessary to find the condition for a graph to admit an integer additive set-graceful labeling. The following results answers the questions regarding the admissibility of an integer additive set-graceful labeling by a graph.

**Theorem 2.22.** An iasg-graph \( G \) has at least \( 2^{|X|} - 1 - \mu \) pendant vertices which are adjacent to the vertex having the set-label \( \{0\} \) and degree \( 1 + 2^{n_1 - 1} \), \( \mu \) is the number of subsets of \( X \) containing the maximal element of \( X \), which are sumsets of two subsets of \( X \).

**Proof.** By Proposition 2.9 the vertices of \( G \), whose set-labels contain the maximal element \( x_n \) of \( X \), are pendant vertices and must be adjacent to the vertex having set-label \( \{0\} \). The number of subsets containing the elements \( x_n \) is \( 2^{n_1 - 1} \). But all the subsets of \( X \) containing \( x_n \) need not be the set-labels of the vertices of \( G \).

Let \( X_r \) be a subset of \( X \) that contains the element \( x_n \). \( X_r \) need not be the set-label of any vertex of \( G \) if there exist two set-labels \( X_i \) and \( X_j \) of two vertices \( v_i \) and \( v_j \) respectively, such that the sumset \( X_i + X_j = X_r \). Let \( \mu \) be the number of subsets of \( X \) containing \( x_n \), which are sumsets of two subsets of \( X \). Hence, the minimum number of pendant vertices in \( G \) is \( 2^{n_1 - 1} - \mu \).

The converse of Theorem 2.22 is also true. Hence, we have,

**Theorem 2.23.** A graph \( G \) admits an integer additive set-graceful labeling if and only if it has at least \( 2^{|X|} - 1 - \mu \) pendant vertices which are adjacent to the vertex having the set-label \( \{0\} \) and degree \( d(V) = 1 + 2^{n_1 - 1} \), where \( \mu \) is the number of subsets of \( X \) containing the maximal element of \( X \), which are sumsets of two subsets of \( X \).

**Proof.** Necessary part of the theorem follows from Theorem 2.22.

Conversely, assume that a graph \( G \) has \( 2^{|X|} - 1 - \mu \) pendant vertices which are adjacent to the vertex \( v \), where \( \mu \) is the number of subsets of \( X \) containing the maximal element of \( X \), which are sumsets of two subsets of \( X \). Label the vertex \( v \) by \( \{0\} \).

Now, \( X \) has \( 2^{|X|} - 1 \) subsets containing the maximal element \( x_n \) of \( X \) among which \( \mu \) are sumsets of some subsets of \( X \). Hence, assign subsets of \( X \) containing \( x_n \), that is not a sumset of two subsets of \( X \), to the pendant vertices of \( G \) which are adjacent to \( v \). Now, label other vertices of \( G \) satisfying the conditions mentioned in Propositions 2.10, 2.9, and 2.5. This labeling is an integer additive set-graceful labeling for \( G \).
3 Conclusion

In this paper, we have discussed an extension of set-graceful labeling of graphs to sum-set labelings and have done a characterisations based on this labeling. Certain problems in this area are still open.

We note that the admissibility of integer additive set-indexers by the graphs by the graphs depends upon the nature of elements in $X$. A graph may admit an IASGL for some ground sets and may not admit an IASGL for some other ground sets. Hence, choosing a ground set is very important to discuss about $iasg$-graphs.

Some of the areas which seem to be promising for further studies are listed below.

Problem 3.1. Characterise different graph classes which admit integer additive set-graceful labelings.

Problem 3.2. Verify the existence of integer additive set-graceful labelings for different graph operations and graph products.

Problem 3.3. Analogous to set-sequential labelings, define integer additive set-sequential labelings of graphs and its properties.

Problem 3.4. Characterise different graph classes which admit integer additive set-sequential labelings.

Problem 3.5. Verify the existence of integer additive set-sequential labelings for different graph operations and graph products.

The integer additive set-indexers under which the vertices of a given graph are labeled by different standard sequences of non negative integers, are also worth studying. All these facts highlight a wide scope for further studies in this area.

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