The kinetic one-dimensional equation with frequency of collisions, affine depending on the module molecular velocity

A. L. Bugrimov\textsuperscript{1}, A. V. Latyshev\textsuperscript{2} and A. A. Yushkanov\textsuperscript{3}

Faculty of Physics and Mathematics,
Moscow State Regional University, 105005,
Moscow, Radio str., 10A

Abstract

The one-dimensional kinetic equation with integral of collisions type BGK (Bhatnagar, Gross and Krook) and frequency of collisions affine depending on the module of molecular velocity is constructed.

Laws of preservation of number of particles, momentum and energy at construction equation are used.

Separation of variables leads to the characteristic equation. The system of the dispersion equations is entered. Its determinant is called as dispersion function. It is investigated continuous and discrete spectra of the characteristic equation.

The set of zero of the dispersion equation makes the discrete spectrum of the characteristic equation. The eigen solutions of the kinetic equation corresponding to discrete spectrum are found.

The solution of the characteristic equation in space of the generalized functions leads to eigen functions corresponding to the continuous spectrum.

Results of the spent analysis in the form of the theorem about structure of the general solution of the entered kinetic equation are formulated.

Key words: one-dimensional kinetic equation, affine dependence of collision frequency, laws preservation, separation of variables, characteristic equation, dispersion equation, discrete and continuous spectra, eigen functions of characteristic equation.

PACS numbers: 05.60.-k Transport processes, 51.10.+y Kinetic and transport theory of gases,

\textsuperscript{1}fakul fm@mgou.ru
\textsuperscript{2}avlatyshev@mail.ru
\textsuperscript{3}yushkanov@inbox.ru
Introduction. Statement of problem and basic equations

Analytical solutions of whole some boundary problems (temperature and density jumps, various slidings, et cetera) the kinetic theory of gas with use of BGK equation with constant frequency of collisions are received to the present time [1]-[5].

Approximation of a constancy of frequency of collisions far not always is possible to consider as adequate to a problem. In this connection attempts become to consider more the general, than BGK, models. In particular, the problem about isothermal sliding is considered for enough a wide class of BGK-models [1].

The case with the frequency of collisions proportional to the module of molecular velocity (i.e. with constant length of the free path) is considered also. In this approach problems about jumps of temperature and concentration [6], and as more the general, than BGK, models (see, for example, [7]-[9]) are considered.

At the same time there is an unresolved problem about temperature jump and concentration with use of the BGK-equation with any dependence of frequency on velocity, in spite on obvious importance of the solution of a problem in similar statement.

In the present work attempt to promote in this direction becomes. Here the case of the affine dependence of frequency of collisions on module velocity of molecules in models of one-dimensional gas is considered. Model of one-dimensional gas widely it was used in a number of works [10]-[12] for gave the good the consent with experiment.

Let us begin with the general statement. Let gas occupies half-space \( x > 0 \). The surface temperature \( T_s \) and concentration of sated steam of a surface \( n_s \) are set. Far from a surface gas moves with some velocity \( u \), being velocity of evaporation (or condensation), also has the temperature gradient

\[
g_T = \left( \frac{d \ln T}{dx} \right)_{x=+\infty}.
\]
It is necessary to define jumps of temperature and concentration depending on velocity and temperature gradient.

In a problem about weak evaporation it is required to define temperature and concentration jumps depending on velocity, including a temperature gradient equal to zero, and velocity of evaporation (condensation) is enough small. The last means, that

\[ u \ll v_T. \]

Here \( v_T \) is the heat velocity of molecules, having order of sound velocity order,

\[ v_T = \frac{1}{\sqrt{\beta_s}}, \quad \beta_s = \frac{m}{2k_B T_s}, \]

\( m \) is the mass of molecule, \( k_B \) is the Boltzmann constant.

In a problem about temperature jump it is required to define temperature and concentration jumps depending on a temperature gradient, thus evaporation (condensation) velocity it is considered equal to zero, and the temperature gradient is considered as small. It means, that

\[ lgT \ll 1, \quad l = \tau v_T, \quad \tau = \frac{1}{\nu_0}, \]

where \( l \) is the mean free path of gas molecules, \( \tau \) is the mean relaxation time, i.e. time between two consecutive collisions of molecules.

Let us unite both problems (about weak evaporation (condensation) and temperature jump) in one. We will assume that the gradient of temperature is small (i.e. relative difference of temperature on length of mean free path is small) and the velocity of gas in comparison with sound velocity is small. In this case the problem supposes linearization and distribution function it is possible to search in the form

\[ f(x, v) = f_0(v)(1 + h(x, v)), \]

where

\[ f_0(v) = n_s \left( \frac{m}{2\pi k_B T_s} \right)^{1/2} \exp \left[ - \frac{mv^2}{2k_B T_s} \right]. \]
is the absolute Maxwellian.

We take the linear kinetic equation which has been written down rather functions $h(x, v)$, with integral of collisions of relaxation type, in integral of collisions BGK named also (Bhatnagar, Gross and Krook), and having the following form

$$v \frac{\partial h}{\partial x} = \nu(v) \left[ l_0[v] + 2 \frac{v}{v_T} l_1[v] + \left( \frac{v^2}{v_T^2} - \beta \right) l_2[v] - h(x, v) \right]. \quad (1.1)$$

Here $l_\alpha[h]$ ($\alpha = 0, 1, 2$) is the any constants, subject to definition from laws of preservation of number of particles (numerical density), an momentum and energy, $\nu(v)$ is the collision frequency affine depending on module molecular velocity,

$$\nu(v) = \nu_0 \left( 1 + \sqrt{\pi a} \sqrt{\frac{m}{2k_B T_s}} |v| \right),$$

$a$ is the positive parameter, $0 \leq a < +\infty$.

The right part of the equation (1.1) is the linear integral of collisions, spread out on collision invariants

$$\psi_0(v) = 1,$$

$$\psi_1(v) = 2 \sqrt{\frac{m}{2k_B T_s}} v,$$

$$\psi_2(v) = \frac{mv^2}{2k_B T_s} - \beta.$$ 

The constant $\beta$ is finding from an orthogonality condition of invariants $\psi_0(v)$ and $\psi_2(v)$. Orthogonality here it is understood as equality to zero of scalar product with weight $\rho(v) = \nu(v) \exp\left( \frac{-mv^2}{2k_B T_s} \right)$

$$(f, g) = \int_{-\infty}^{\infty} \nu(v) \exp\left( \frac{-mv^2}{2k_B T_s} \right) f(v) g(v) dv.$$ 

Let us pass in the equation (1.1) to dimensionless velocity

$$C = \sqrt{\beta v} = \frac{v}{v_T}$$
and dimensionless coordinate

\[ x' = \nu_0 \sqrt{\frac{m}{2k_B T_s}} \frac{x}{l} \]

The variable \( x' \) let us designate again through \( x \).

In the dimensionless variables we will rewrite the equation (1.1) in the form

\[ C \partial h \partial x = (1 + \sqrt{\pi a|C|}) \left[ l_0[h] + 2Cl_1[h] + (C^2 - \beta)l_2[h] - h(x, C) \right]. \quad (1.2) \]

The constant \( \beta \) is defined, how it was already specified, from the condition

\[ (\psi_0, \psi_2) = \nu_0 \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a|C|})(C^2 - \beta) dC = 0. \]

From here we receive, that

\[ \beta = \beta(a) = \frac{2a + 1}{2(a + 1)}. \]

2. Laws of preservation and transformation of the kinetic equation

The modelling integral of collisions should satisfy to laws preservations of number of particles (numerical density), momentum and energy

\[ (\psi_0, M[h]) \equiv \nu_0 \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a|C|})M[h]\psi_0(C) dC = 0, \quad (2.1) \]

where \( \alpha = 0, 1, 2 \), \( \rho(C)M[h] \) is the model collision integral,

\[ M[h] = l_0[h] + 2Cl_1[h] + (C^2 - \beta)l_2[h] - h(x, C). \]

From the first equation from (2.1), i.e. preservation law of number of particles \( (\psi_0, M[h]) = 0 \) we receive that

\[ l_0[h] = \frac{(1, h)}{(1, 1)}. \]
Here

\[(1, 1) = \nu_0 \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a |C|}) dC = \nu_0 \sqrt{\pi} (a + 1),\]

\[(1, h) = \nu_0 \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a |C|}) h(x, C) dC.\]

It means, that

\[l_0[h] = \frac{1}{\sqrt{\pi} (a + 1)} \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a |C|}) h(x, C) dC.\]

From second equation from (2.1), i.e. preservation law of momentum \((\psi_1, M[h]) = 0\) we receive that

\[2l_2[h] = \frac{(C, h)}{(C, C)},\]

where

\[(C, C) = \nu_0 \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a |C|}) C^2 dC = \nu_0 \frac{\sqrt{\pi}}{2} (2a + 1),\]

\[(C, h) = \nu_0 \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a |C|}) C h(x, C) dC.\]

Therefore,

\[2l_1[h] = \frac{2}{\sqrt{\pi} (2a + 1)} \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a |C|}) C h(x, C) dC.\]

From third equation from (2.1), i.e. preservation law of energy \((\psi_2, M[h]) = 0\) we receive that

\[(\psi_2, M[h]) = \nu_0 \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a |C|}) (C^2 - \beta)^2 dC.\]
\[-\nu_0 \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a|C|})(C^2 - \beta) h(x, C) dC = 0,\]

whence

\[l_2[h] = \frac{(C^2 - \beta, h)}{(C^2 - \beta, C^2 - \beta)}.\]

Here

\[(C^2 - \beta, C^2 - \beta) = \nu_0 \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a|C|})(C^2 - \beta)^2 dC =\]

\[= \nu_0 \sqrt{\pi} \frac{4a^2 + 7a + 2}{4(a + 1)},\]

\[(C^2 - \beta, h) = \nu_0 \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a|C|})(C^2 - \beta) h(x, C) dC.\]

From last equalities it is had

\[l_2[h] = \frac{4(a + 1)}{\sqrt{\pi} (4a^2 + 7a + 2)} \int_{-\infty}^{\infty} e^{-C^2} (1 + \sqrt{\pi a|C|})(C^2 - \beta) h(x, C) dC.\]

Let us return to the equation (1.2) and by means of received above equalities let us transform this equation to the form

\[C' \frac{\partial h}{\partial x} + (1 + \sqrt{\pi a|C'|}) h(x, C') =\]

\[= (1 + \sqrt{\pi a|C'|}) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-C'^2} (1 + \sqrt{\pi a|C'|}) q(C, C', a) h(x, C') dC'. \quad (2.2)\]

Here \(q(C, C', a)\) is the kernel of equation,

\[q(C, C', a) = r_0(a) + r_1(a) CC' + r_2(a)(C^2 - \beta(a))(C'^2 - \beta(a)),\]

\[r_0(a) = \frac{1}{a + 1}, \quad r_1(a) = \frac{2}{2a + 1}, \quad r_2(a) = \frac{4(a + 1)}{4a^2 + 7a + 2}.\]

**Corollary.** Let us notice, that at \(a \to 0\) the equation (2.2) passes in the equation
\[ C \frac{\partial h}{\partial x} + h(x, C) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-C'^2 q(C, C', 0)h(x, C)} dC \]

with kernel

\[ q(C, C', 0) = 1 + 2CC' + 2\left(C^2 - \frac{1}{2}\right)\left(C'^2 - \frac{1}{2}\right). \]

This equation is one-dimensional BGK-equation with constant frequency of collisions.

Let us consider the second limiting case of the equation (2.2). We will return to expression of frequency of collisions also we will copy it in the form

\[ \nu(C) = \nu_0(1 + \sqrt{\pi a|C|}) = \nu_0 + \nu_1|C|, \]

where

\[ \nu_1 = \sqrt{\pi} \nu_0 a. \]

Let us tend \( \nu_0 \) to zero. In this limit the quantity \( a \) tends to \(+\infty\), because

\[ a = \frac{\nu_1}{\sqrt{\pi} \nu_0}. \]

It is easy to see, that in this limit

\[ \lim_{a \to +\infty} (1 + \sqrt{\pi a|C'|})q(C, C', a) = \sqrt{\pi}|C'|q_1(C, C'), \]

where

\[ q_1(C, C') = 1 + CC' + (C^2 - 1)(C'^2 - 1). \]

The equation (2.2) will thus be copied in the form

\[ C \frac{\partial h}{|C| \partial x_1} + h(x_1, C) = \int_{-\infty}^{\infty} e^{-C'^2 |C'|q_1(C, C')} dC'. \]

In this equation

\[ x_1 = \nu_1 \sqrt{\beta_s} x = \frac{x}{l_1}, \quad l_1 = v_T \tau_1, \quad \tau_1 = \frac{1}{\nu_1}. \]
This equation is the one-dimensional kinetic equation with the frequency of collisions proportional to the module of the molecular velocity.

In the equation (2.2) we will carry out variable replacement $\sqrt{\pi}a \rightarrow a$ and transform the received equation in the form

\[
\frac{C}{1 + a|C|} \frac{\partial h}{\partial x} + h(x, C) = \int_{-\infty}^{\infty} e^{-C'^2}(1 + a|C'|)q(C, C', a)h(x, C')dC'.
\]  

(2.3)

In this equation $q(C, C', a)$ is the kernel of equation,

\[q(C, C', a) = r_0(a) + r_1(a)CC' + r_2(a)(C^2 - \beta(a))(C'^2 - \beta(a)),\]

where

\[r_0(a) = \frac{1}{a + \sqrt{\pi}}, \quad r_1(a) = \frac{2}{2a + \sqrt{\pi}}, \quad r_2(a) = \frac{4(a + \sqrt{\pi})}{4a^2 + 7\sqrt{\pi}a + 2\pi},\]

\[\beta(a) = \frac{12a + \sqrt{\pi}}{2a + \sqrt{\pi}}.\]

Let us make in the equation (2.3) replacement of variable $C = C(\mu), C' = C(\mu')$, where

\[C(\mu) = \frac{\mu}{1 - a|\mu|}, \quad |\mu| < \alpha, \quad \alpha = \frac{1}{a}.\]

We denote the function $h(x, C)$ again through $h(x, \mu)$. The equation (2.3) passes into following equation, standard for transport equation

\[\mu \frac{\partial h}{\partial x} + h(x, \mu) = \int_{-\alpha}^{\alpha} \rho(\mu')q(\mu, \mu')h(x, \mu')d\mu',\]  

(2.4)

where

\[\rho(\mu') = \exp \left[ - \left( \frac{\mu'}{1 - a|\mu'|} \right)^2 \right] \frac{1}{(1 - a|\mu'|)^3},\]

\[q(\mu, \mu') = r_0(a) + r_1(a)\frac{\mu}{1 - a|\mu|} \frac{\mu'}{1 - a|\mu'|} + \]

\[\quad + \frac{r_2(a)(C^2 - \beta(a))(C'^2 - \beta(a))}{4a^2 + 7\sqrt{\pi}a + 2\pi} \frac{1}{1 - a|\mu'|} \frac{\mu'}{1 - a|\mu'|},\]
\[ + r_2(a) \left[ \left( \frac{\mu}{1 - a|\mu|} \right)^2 - \beta(a) \right] \left[ \left( \frac{\mu'}{1 - a|\mu'|} \right)^2 - \beta(a) \right]. \]

Let us notice, that on the ends of interval of integration

\[ \rho(\pm \alpha) = 0, \]

and, besides,

\[ \lim_{\mu \to \pm \alpha} \rho(\mu) C^n(\mu) = 0 \]

for any natural number \( n \).

### 3. Eigen functions and eigen values

Separation of variables in the equation (2.4), taken in the form

\[ h_\eta(x, \mu) = \exp \left( - \frac{x}{\eta} \right) \Phi(\eta, \mu), \quad \eta \in \mathbb{C}, \quad (3.1) \]

transforms equation (3.1) to characteristic equation

\[ (\eta - \mu)\Phi(\eta, \mu) = \eta Q(\eta, \mu), \quad \eta, \mu \in (-\alpha, +\alpha), \quad (3.2) \]

where

\[ Q(\eta, \mu) = r_0(a)n_0(\eta) + r_1(a)C'(\mu)n_1(\eta) + \]

\[ + r_2(a) \left( C^2(\eta) - \beta(a) \right) \left( C^2(\mu) - \beta(a) \right). \]

Here

\[ n_\alpha(\eta) = \int_{-\alpha}^{\alpha} \Phi(\eta, \mu) C^\alpha(\mu) \rho(\mu) d\mu, \quad \alpha = 0, 1, 2, \quad (3.3) \]

is the zero, first and second moments of eigen function with weight \( \rho(\mu) \).

Eigen functions of the continuous spectrum filling by the continuous fashion an interval \((-\alpha, \alpha)\), we find [13] in space of the generalized functions

\[ \Phi(\eta, \mu) = \eta Q(\eta, \mu) P \frac{1}{\eta - \mu} + g(\eta) \delta(\eta - \mu), \quad \eta \in (-\alpha, \alpha). \quad (3.4) \]
Here \( g(\eta) \) is the unknown function, defined from equations (3.3), \( P x^{-1} \) is the distribution, meaning principal value of integral by integration \( x^{-1} \), \( \delta(x) \) is the Dirac delta-function.

Let us substitute eigen functions (3.4) in normalization equalities (3.3). We will receive the following system of the dispersion equations

\[
n_\alpha(\eta) + \eta \int_{-\alpha}^{\alpha} Q(\eta, \mu) C^{\alpha}(\mu) \rho(\mu) \frac{d\mu}{\mu - \eta} = g(\eta) \rho(\eta) C^{\alpha}(\eta), \quad (3.5)
\]

\( \alpha = 0, 1, 2. \)

We denote

\[
t_n(\eta) = \eta \int_{-\alpha}^{\alpha} C^n(\mu) \frac{\rho(\mu) d\mu}{\mu - \eta}, \quad n = 0, 1, 2, 3, 4.
\]

Now system of the dispersion equations (3.5) it is possible transform to the form

\[
n_\alpha(\eta) + r_0(a) n_0(\eta) t_\alpha(a) + r_1(a) n_1(\eta) t_1(\eta) +
+ (n_2(\eta) - \beta(a) n_0(\eta))(t_{\alpha+2}(\eta) - \beta(a) t_\alpha(\eta)) = g(\eta) \rho(\eta) C^\alpha(\eta), \quad (3.6)
\]

where \( \alpha = 0, 1, 2. \)

Let us write down the equations (3.6) in the vector form

\[
\Lambda(\eta) n(\eta) = g(\eta) \rho(\eta) \begin{bmatrix} 1 \\ C(\eta) \\ C^2(\eta) \end{bmatrix}. \quad (3.7)
\]

Here \( \Lambda(\eta) \) is the dispersion matrix-function with elements

\[
\lambda_{ij}(\eta) \quad (i, j = 1, 2, 3),
\]

\( n(\eta) \) is the normalization vector with elements \( n_\alpha(\eta) \quad (\alpha = 0, 1, 2). \)

Elements of the dispersion matrix in the explicit form will more low be necessary

\[
\lambda_{11}(z) = 1 + \left[ r_2(a) + \beta^2(a) r_2(a) \right] t_0(z) - \beta(a) r_2(a) t_2(z),
\]
\begin{align*}
\lambda_{12}(z) &= r_1(a)t_1(z), \\
\lambda_{13}(z) &= r_2(a)\left[-\beta(a)t_0(z) + t_2(z)\right], \\
\lambda_{21}(z) &= \left[r_0(a) + \beta^2(a)r_2(a)\right]t_1(z) - \beta(a)r_2(a)t_3(z), \\
\lambda_{22}(z) &= 1 + r_1(a)t_3(z), \\
\lambda_{23}(z) &= r_2(a)\left[-\beta(a)t_1(z) + t_3(z)\right], \\
\lambda_{31}(z) &= \left[r_0(a) + \beta^2(a)r_2(a)\right]t_2(z) - \beta(a)r_2(a)t_4(z), \\
\lambda_{32}(z) &= r_1(a)t_3(z), \\
\lambda_{33}(z) &= 1 + r_2(a)\left[-\beta(a)t_2(z) + t_4(z)\right].
\end{align*}

We introduce the dispersion function \(\lambda(z)\), \(\lambda(z) = \det \Lambda(z)\). In the explicit form we have

\[ \lambda(z) = \lambda_{11}(z)\lambda_{22}(z)\lambda_{33}(z) + r_1(a)t_3(z)\lambda_{13}(z)\lambda_{21}(z) + \]

\[ + r_1(a)t_1(z)\lambda_{31}(z)\lambda_{23}(z) - \lambda_{13}(z)\lambda_{22}(z)\lambda_{31}(z) - \]

\[ - r_1(a)t_3(z)\lambda_{11}(z)\lambda_{23}(z) - r_1(a)t_1(z)\lambda_{21}(z)\lambda_{33}(z). \]

From vector equation (3.7) we find

\[ n_{\alpha}(\eta) = g(\eta)\rho(\eta)\frac{\Lambda_{\alpha}(\eta)}{\lambda(\eta)}, \quad \alpha = 0, 1, 2, \quad (3.8) \]

where \(\Lambda_{\alpha}(\eta)\) is the determinant received from determinant of system (3.6) by replacement in it \(\alpha\)-th column by the column from free members of this system. We will write out these determinants in the explicit form

\[ \Lambda_0(z) = \Lambda_{11}(z) - C(z)\Lambda_{21}(z) + C^2(z)\Lambda_{31}(z) = \lambda_{22}(z)\lambda_{33}(z) - \]

\[ - r_1(a)t_3(z)\lambda_{23}(z) - C(z)r_1(a)\left[t_1(z)\lambda_{33}(z) - t_2(z)\lambda_{13}(z)\right] + \]

\[ + \text{additional terms.} \]
\[ + C^2(z) \left[ r_1(a)t_1(z)\lambda_{23}(z) - \lambda_{22}(z)\lambda_{13}(z) \right], \]

\[ \Lambda_1(z) = \Lambda_{12}(z) + C(z)\Lambda_{22}(z) - C^2(z)\Lambda_{32}(z) = \lambda_{21}(z)\lambda_{33}(z) + \]

\[ + \lambda_{31}(z)\lambda_{33}(z) + C(z) \left[ \lambda_{11}(z)\lambda_{33}(z) - \lambda_{31}(z)\lambda_{13}(z) \right] - \]

\[ - C^2(z) \left[ \lambda_{11}(z)\lambda_{23}(z) - \lambda_{21}(z)\lambda_{13}(z) \right], \]

\[ \Lambda_2(z) = \Lambda_{31}(z) - C(z)\Lambda_{32}(z) + C^2(z)\Lambda_{33}(z) = \]

\[ = r_1(a)t_3(z)\lambda_{21}(z) - \lambda_{31}(z)\lambda_{22}(z) - C(z)r_1(a) \left[ t_3(z)\lambda_{11}(z) - t_1(z)\lambda_{33}(z) \right] + \]

\[ + C^2(z) \left[ \lambda_{11}(z)\lambda_{22}(z) - r_1(a)t_1(z)\lambda_{21}(z) \right]. \]

Here \( \Lambda_{ij}(z) \) is the minor of element \( \lambda_{ij}(z) \).

By means of equalities (3.8) we will transform equality for \( Q(\eta,\mu) \) to the form

\[ Q(\eta,\mu) = \tilde{Q}(\eta,\mu) \frac{g(\eta)}{\lambda(\eta)}\rho(\eta), \quad (3.9) \]

where

\[ \tilde{Q}(\eta,\mu) = r_0(a)\Lambda_0(\eta) + r_1(a)C(\mu)\Lambda_1(\eta) + \]

\[ + r_2(a) \left[ C^2(\mu) - \beta(a) \right] \left[ \Lambda_2(\eta) - \beta(a)\Lambda_0(\eta) \right]. \]

By means of equality (3.9) we will transform expression (3.4) for eigen functions

\[ \Phi(\eta,\mu) = \tilde{\Phi}(\eta,\mu)g(\eta), \quad (3.10) \]

where

\[ \tilde{\Phi}(\eta,\mu) = \eta \frac{\tilde{Q}(\eta,\mu)}{\lambda(\eta)}\rho(\eta)P \frac{1}{\eta - \mu} + \delta(\eta - \mu). \quad (3.11) \]

From equality (3.10) it is visible, that eigen functions are defined accurate within to coefficient – any function \( g(\eta) \), identically not equal to zero. Owing to uniformity of the initial kinetic equation it is possible to consider this function identically equal to unit \( (g(\eta) \equiv 1) \) and further
in quality eigen function corresponding to continuous spectrum, it is possible to consider the functions defined by equality (3.11). Apparently from the solution of the characteristic equation, continuous spectrum of the characteristic equation is the set

\[ \sigma_c = \{ \eta : -\alpha < \eta < +\alpha \}. \]

By definition the discrete spectrum of the characteristic equation consists of set of zero of dispersion function.

Expanding dispersion function in Laurent series in a vicinity infinitely remote point, we are convinced, that it in this point has zero of the fourth order. Applying an argument principle \[14\] from the theory of functions complex variable, it is possible to show, that other zero, except \( z_i = \infty \), dispersion function not has. Thus, the discrete spectrum of the characteristic equations consists of one point \( z_i = \infty \), multiplication factor which it is equal four,

\[ \sigma_d = \{ z_i = \infty \}. \]

To point \( z_i = \infty \), as to the 4-fold point of discrete spectrum, corresponds following four discrete (partial) solutions of the kinetic decision (2.4)

\[ h_0(x, \mu) = 1, \]
\[ h_1(x, \mu) = C(\mu), \]
\[ h_3(x, \mu) = C^2(\mu) - \frac{1}{2}, \]
\[ h_3(x, \mu) = (x - \mu) \left( C^2(\mu) - \frac{3}{2} \right). \]

Let us result formulas Sokhotsky for the difference and the sum of the boundary values of dispersion function from above and from below on the \((-\alpha, +\alpha)\)

\[ \lambda^+(\mu) - \lambda^-(\mu) = 2\pi i \rho(\mu) \widetilde{Q}(\mu, \mu), \quad \mu \in (-\alpha, +\alpha), \]

and

\[ \frac{\lambda^+(\mu) + \lambda^-(\mu)}{2} = \lambda(\mu), \quad \mu \in (-\alpha, +\alpha). \]
4. The structure of general solution of kinetic equation

Here we will sum up the done analysis.

We actually prove the theorem of general solution structure of the equations (2.4).

**Theorem.** The general solution of the equation (2.4) is the sum of the linear combinations of discrete (partial) solutions of this equation with any coefficients and integral on the continuous spectrum from eigen functions correspond to the continuous spectrum, with unknown coefficients

\[ h(x, \mu) = A_0 h_0(x, \mu) + A_1 h_2(x, \mu) + A_2 h_2(x, \mu) + A_3 h_3(x, \mu) + \alpha \int_{-\alpha}^{\alpha} \exp \left( -\frac{x}{\eta} \right) \tilde{\Phi}(\eta, \mu) A(\eta) d\eta. \]  

(4.1)

In equality (4.1) \( A_\alpha \quad (\alpha = 0, 1, 2, 3) \) is the coefficients correspond to the discrete spectrum, and the unknown function \( A(\eta) \) is the coefficient corresponds to continuous spectrum.

Coefficients of discrete and continuous spectra are subject to finding from boundary conditions. In following works authors assume to solve a number of substantial boundary problems of the kinetic theories.

Let us consider two partial limiting cases the kinetic equations.

Let us begin with the case \( a = 0 \). This case corresponds constant collision frequency of molecules. In this case essentially becomes simpler expression for eigen functions of the continuous spectrum

\[ \tilde{\Phi}(\eta, \mu) = \frac{1}{\sqrt{\pi}} \eta \left( \frac{3}{2} - \mu^2 \right) P \frac{1}{\eta - \mu} + e^{\eta^2} \lambda(\eta) \delta(\eta - \mu), \]

where \( \lambda(z) \) is the dispersion function, expression for which also essentially becomes simpler and has the following form

\[ \lambda(z) = -\frac{1}{2} - (z^2 - \frac{3}{2}) \lambda C(z), \]
$\lambda_C(z)$ is the dispersion function of plasma, entered by Van Kampen in 1955,

$$\lambda_C(z) = 1 + \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2}d\mu}{\mu - z} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2}d\mu}{\mu - z},$$

or

$$\lambda_C(z) = 1 - 2z e^{-z^2} \int_{0}^{\infty} e^{u^2} du + (-)\sqrt{\pi}iz^{-z^2}, \quad \text{Im } z > 0 \ (\text{Im } z < 0).$$

In numerical calculations instead of last it is convenient to use the formula

$$\lambda_C(z) = 1 - 2z^2 \int_{0}^{1} e^{-z^2(1-t^2)} dt + (-)\sqrt{\pi}iz^{-z^2}, \quad \text{Im } z > 0 \ (\text{Im } z < 0).$$

Fig. The real (curve 1) and imaginare (curve 2) parts of boundary values $\lambda^+(x)$ of dispersion function on real axis, $a = 0$. 
We investigate also other limiting case, when frequency of collisions is proportional to the module of molecular velocity, i.e. \( \nu(C) = \nu_1 |C| \).

In this case the kinetic equation passes in the equation

\[
\text{sign}C \frac{\partial h}{\partial x_1} + h(x_1, C) = \int_{-\infty}^{\infty} e^{-C'^2}|C'|q(C, C')h(x_1, C)dC',
\]  

where

\[
q(C, C') = 1 + CC' + (C^2 - 1)(C'^2 - 1).
\]

Further the variable \( x_1 \) we will again designate through \( x \).

Let us search for the solution of the equation (4.2) in the form

\[
h(x, C) = a_1(x) + \tilde{a}_1(x)\text{sign}C + a_2(x)C + \tilde{a}_1(x)Cs\text{ign}C +

+a_3(x)(C^2 - 1) + \tilde{a}_3(x)(C^2 - 1)\text{sign}C.
\]  

(4.3)

After substitution (4.3) in the equation (4.2), we receive six the linear differential equations of the first order

\[
a_1'(x) + \tilde{a}_1(x) = 0,
\]

\[
\tilde{a}_1'(x) = \frac{\sqrt{\pi}}{2}\tilde{a}_2(x),
\]

\[
a_2'(x) + a_2(x) = 0,
\]

\[
\tilde{a}_2'(x) = \frac{\sqrt{\pi}}{2}\tilde{a}_1(x) + \sqrt{\pi}a_3(x),
\]

\[
a_3'(x) + a_3(x) = 0,
\]

\[
\tilde{a}_3'(x) = \frac{\pi}{2}\tilde{a}_2(x).
\]

We solve this system and we find unknown functions \( a_j(x) \) and \( \tilde{a}_j(x) \) (\( j = 1, 2, 3 \))

\[
a_1(x) = -\frac{1}{\sqrt{3}\alpha_0}A_0e^{-\alpha_0x} - \tilde{A}_1x + A_1,
\]

\[
\tilde{a}_1(x) = -\frac{1}{\sqrt{3}\alpha_0}A_0e^{-\alpha_0x} + A_1,
\]

\[
a_2(x) = \frac{1}{\alpha_0}A_0e^{-\alpha_0x} + A_2,
\]

\[
\tilde{a}_2(x) = \frac{\alpha_0}{\alpha_0}A_0e^{-\alpha_0x}.
\]
\( \tilde{a}_2(x) = A_0 e^{-\alpha_0 x}, \)

\( a_3(x) = -\frac{1}{\sqrt{3} \alpha_0} A_0 e^{-\alpha_0 x} - \tilde{A}_3 x + A_3, \)

\( \tilde{a}_3(x) = -\frac{1}{\sqrt{3} \alpha_0} A_0 e^{-\alpha_0 x} + \tilde{A}_3. \)

Here \( A_0, A_1, A_2, A_3, \tilde{A}_1, \tilde{A}_3 \) are arbitrary constants, \( \alpha_0 = \frac{\sqrt{3\pi}}{2} \).

Thus, the required general solution of the equation (4.2) is constructed and has the following form

\[
h(x, C) = A_0 e^{-\alpha_0 x} \left[ -\frac{1}{\sqrt{3} \alpha_0} - \frac{1}{\sqrt{3}} \text{sign} C + \frac{1}{\alpha_0} C + C \text{sign} C - \frac{1}{\sqrt{3} \alpha_0} (C^2 - 1) - \frac{1}{\sqrt{3}} (C^2 - 1) \text{sign} C \right] - \tilde{A}_1 x + A_1 + \tilde{A}_1 \text{sign} C + A_2 C + (-\tilde{A}_3 x + A_3)(C^2 - 1) + \tilde{A}_3 (C^2 - 1) \text{sign} C.
\]

**4. Conclusion**

In the present work the one-dimensional kinetic equation with integral of collisions relaxation type BGK (Bhatnagar, Gross and Krook) is constructed. Frequency of collisions of molecules as affine depending on the module molecular velocity is considered.

At construction the equations are used laws of preservation of number of particles (the numerical density), momentum and energy. The constructed equation will be transformed to standard kind of the equation of type of the equation carrying over with polynomial kernel.

Separation of variables leads to the characteristic equation. By means of normalizing equalities the system of dispersion equations is entered. Its determinant is called as dispersion function. It is investigated continuous and discrete spectra of the characteristic equation.

The set of zero of the dispersion equation makes the discrete spectrum of the characteristic equation. The eigen solutions of the initial kinetic
equation corresponds to discrete spectrum are found. These solutions so-called discrete (or partial) solutions.

The solution of the characteristic equation in space of the generalized functions leads to eigen functions correspond to the continuous spectrum.

Results of the spent analysis are formulated in the form of the theorem about structure of the general solution of the entered kinetic equation.

REFERENCES

[1] Cercignani C. Theory and application of the Boltzmann equation. 1975. Scottish Academic Press. Edinburg and London.

[2] Latyhev A.V. Analytical methods of model kinetic equations and their applications//Dissertation of Doctor Science. Moscow: Keldysh Institute of Applied Mathematics of Russian Academy of Science. 1993.

[3] Latyshev A.V., Yushkanov A.A. Analytical solution of the problem about strong evaporation (condensation)// Izvestiya Russian Academy of Science. Ser. Mechanika, Fluid and Gas (Russian "Fluids Dynamics"). 1993. №6. 143-155 pp.

[4] Latyshev A.V., Yushkanov A.A. The theory and accurate solutions of problems of the slip of a binary gas along a plane surface// Comput. Maths. Math. Phys. 1991. V. 31 (8), p.p. 58–65.

[5] Latyshev A.V., Moisseev A.V. The solution of the boundary – value problems for the equation of radiation transfer// Comput. Maths. Math. Phys. 1994. V. 34 (2), p.p. 193–203.
[6] Cercignani C. The method of elementary solutions for kinetic models with velocity dependent collision frequency//Ann. Phys. 1966. V.40. 469-481 P.

[7] Latyshev A.V., Yushkanov A.A. Kinetic equations type Williams and their exact solutions. Monograph. M.: MGOU (Moscow State Regional University), 2004, 271 p.

[8] Latyshev A.V., Yushkanov A.A. Analytical methods in kinetic theory. Monograph. M.: MGOU, 2008, 280 p.

[9] Latyshev A.V., Yushkanov A.A. Boundary value problems for quantum gases. Monograph M.: MGOU, 2012, 266 p.

[10] Latyshev A.V., Yushkanov A.A. Analytical solution of one-dimensional problem about moderate strong evaporation (and condensation) in half-space// Appl. mech. and tech. physics. 1993. №1. 102-109 p. [russian]

[11] Siewert C.E., Thomas J.R., jr. Strong evaporation into a half-space//J. Appl. Math. Phys. 1981. V.32. №4. 421-433 P.

[12] Cercignani C., Frezzoti A. Linearized analysis of a one-speed B.G.K. model in the case of strong condensation// Bulgarian Academy of sci. theor. appl. mech. Sofia. 1988. V.XIX. №3. 19-23 P.

[13] Vladimirov V.S., Zharinov V.V. Equations of mathematical physics. M.: Fizmatlit. 2000. 399 c.

[14] Gakhov F.D. Boundary value problems. M.: Nauka. 640 p.[russian]