GENERATION OF MUTUALLY UNBIASED BASES AS POWERS OF A UNITARY MATRIX IN 2-POWER DIMENSIONS

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Abstract. Let $q$ be a power of $2$. We show by representation theory that there exists a $q \times q$ unitary matrix of multiplicative order $q + 1$ whose powers generate $q + 1$ complex pairwise mutually unbiased bases in $\mathbb{C}^q$. When $q$ is a power of an odd prime, there is a $q \times q$ unitary matrix of multiplicative order $q + 1$ whose first $(q + 1)/2$ powers generate $(q + 1)/2$ complex pairwise mutually unbiased bases. We also show how the existence of these matrices implies the existence of special orthogonal decompositions of certain simple Lie algebras.

1. Introduction

Let $d$ be a positive integer and let $\mathbb{C}^d$ denote a vector space of dimension $d$ over the field $\mathbb{C}$ of complex numbers. Let $\langle u, v \rangle$ denote a positive definite hermitian form on $\mathbb{C}^d \times \mathbb{C}^d$, which is linear in the first variable and conjugate linear in the second.

Let $B_1 = \{ u_1, \ldots, u_d \}$ and $B_2 = \{ v_1, \ldots, v_d \}$ be orthonormal bases of $\mathbb{C}^d$. We say that the bases $B_1$ and $B_2$ are mutually unbiased if

$$|\langle u_i, v_j \rangle|^2 = \frac{1}{d}$$

for $1 \leq i, j \leq n$. If we identify the orthonormal bases $B_1$ and $B_2$ with $d \times d$ unitary matrices $U_1$ and $U_2$, respectively, then the condition that the bases are mutually unbiased is equivalent to saying that each entry of the product $U_1 U_2^\dagger = U_1 U_2^{-1}$, where $U_2^\dagger$ denotes the conjugate-transpose of $U_2$, has absolute value $1/\sqrt{d}$.

Following [7], let $N(d)$ denote the maximum number of orthonormal bases of $\mathbb{C}^d$ which are pairwise mutually unbiased. It has been proved that $N(d) \leq d + 1$, and furthermore the equality $N(d) = d + 1$ occurs whenever $d$ is a power of a prime. It is presently unknown if the equality $N(d) = d + 1$ ever happens when $d$ is not a power of a prime.

Explicit constructions of $d + 1$ mutually unbiased bases when $d$ is a power of a prime have been based on properties of finite fields. [1]. Somewhat less explicit constructions use the irreducible representations of extra-special $p$-groups to realize the bases as eigenspaces of finite abelian groups acting on $\mathbb{C}^d$. See, for example, [1] and the references of [7].

The main purpose of this paper is to show that when $d = 2^a$, we can construct $d + 1$ pairwise mutually unbiased bases of $\mathbb{C}^d$ as the powers of a unitary matrix of multiplicative order $d + 1$. This matrix arises as an automorphism of an extra-special 2-group which has a special action on the maximal abelian subgroups of the extra-special group. The equivalent construction when $d = p^a$, with $p$ an odd integer, is forthcoming.

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prime, yields a unitary matrix of multiplicative order \( d + 1 \) whose first \( (d + 1)/2 \) powers provide \( (d + 1)/2 \) pairwise mutually unbiased bases of \( \mathbb{C}^d \). We show that it is generally impossible to achieve \( d + 1 \) pairwise mutually unbiased bases of \( \mathbb{C}^d \) using the powers of a single matrix in the odd prime power case.

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2. Construction of a class of \( p \)-groups

We shall try to keep our exposition reasonably self-contained and work from basic principles of finite group theory. Let \( p \) be a prime and let \( q = p^a \), where \( a \) is a positive integer. Let \( F \) denote the finite field of order \( q^2 \). We define a multiplication on the set \( F \times F \) by putting

\[
(a, b)(c, d) = (a + c, a^q c + b + d)
\]

for all ordered pairs \((a, b)\) and \((c, d)\) in \( F \times F \). It is straightforward to see that \( F \times F \) is a finite group of order \( q^4 \), which we shall denote by \( G_q \). The identity element is \((0, 0)\) and the inverse of \((a, b)\) is \((-a, a^q + 1 - b)\). The center \( Z(G_q) \) of \( G_q \) consists of all elements \((0, b)\) and is elementary abelian of order \( q^2 \).

We set \( x = (a, b), y = (c, d) \) and let \([x, y]\) denote the commutator \( x^{-1} y^{-1} xy \). We find that

\[
[x, y] = (0, a^q c - c^q a).
\]

Note that any element \( e \) of \( F \) expressible in the form \( e = a^q c - c^q a \) satisfies \( e^q = -e \).

It follows that the commutator subgroup \( G_q' \), which is generated by the commutators \([x, y]\), consists of elements of the form \((0, e)\), where \( e^q = -e \), and hence \( |G_q'| = q \).

Assuming that \( a \neq 0 \), it is easy to check that the centralizer of \( x \) consists of all elements \((c, d)\), where \( d \) is an arbitrary element of \( F \) and \( a^q c = ac^q \). Thus the centralizer of \( x \) has order \( q^3 \).

We now consider the irreducible complex characters of \( G_q \).

Lemma 1. The group \( G_q \) has exactly \( q^3 \) irreducible complex characters of degree 1. All other irreducible complex characters of \( G_q \) have degree \( q \) and they vanish outside \( Z(G_q) \).

Proof. Since the commutator subgroup \( G_q' \) has order \( q \), \( G_q \) has exactly \( |G_q : G_q'| = q^3 \) different irreducible characters of degree 1. Now let \( x \) be any non-central element of \( G_q \). The second orthogonality relation implies that, as the centralizer of \( x \) has order \( q^3 \),

\[
q^3 = \sum |\chi(x)|^2,
\]

where the sum extends over all irreducible characters \( \chi \) of \( G_q \). The characters of degree 1 each contribute 1 to this sum, and since there are \( q^3 \) of them, we deduce that \( \psi(x) = 0 \) for any irreducible character \( \psi \) of degree greater than 1. Finally, Schur’s Lemma implies that \( |\psi(z)| = \psi(1) \) for all elements \( z \) of \( Z(G_q) \) and hence we obtain by the first orthogonality relation

\[
|G_q| = q^4 = \sum |\psi(z)|^2 = |Z(G_q)| \psi(1)^2 = q^2 \psi(1)^2,
\]
where the sum extends over all elements $z$ of $Z(G_q)$. We deduce that $\psi(1) = q$, as required.

Next, we construct an automorphism of $G_q$ which ultimately will be used to generate the mutually unbiased bases.

**Lemma 2.** Let $\alpha$ be an element of order $q+1$ in $\mathbb{F}$. Then the mapping $\sigma : G_q \to G_q$ given by

$$\sigma(a, b) = (\alpha a, b)$$

is an automorphism of order $q+1$ of $G_q$.

**Proof.** Given elements $x = (a, b), y = (c, d)$ in $G_q$, we have

$$\sigma(xy) = \sigma(a + c, a^q c + b + d) = (\alpha(a + c), a^q c + b + d)$$

and

$$\sigma(x)\sigma(y) = (\alpha a, b)(\alpha c, d) = (\alpha(a + c), \alpha^{q+1} a^q c + b + d).$$

We deduce that $\sigma(xy) = \sigma(x)\sigma(y)$, since $\alpha^{q+1} = 1$ by hypothesis. Thus, $\sigma$ is an automorphism of $G_q$ and it is clear that $\sigma$ has order $q+1$, since $\alpha$ has order $q+1$. □

Next, we construct a maximal abelian subgroup $A$ of $G_q$ by setting

$$A = \{(a, b) : a^q = a\},$$

where $b$ runs over $\mathbb{F}$. It is straightforward to verify that $A$ is indeed abelian and $|A| = q^3$. Applying the automorphism $\sigma$, we define subgroups $A_i$ so that

$$A_i = \sigma^{i-1}(A)$$

for $1 \leq i \leq q+1$ (thus $A_1 = A$).

**Lemma 3.** Suppose that $p = 2$. Then we have $A_i \cap A_j = Z(G_q)$ if $1 \leq i \neq j \leq q+1$, and $G_q$ is the union of $A_1, \ldots, A_{q+1}$.

**Proof.** Suppose that $(c, d) \in A_i \cap A_j$, where $1 \leq i, j \leq q + 1$. Then we have $c = \alpha^{i-1} a = \alpha^{j-1} a_1$, where $a^q = a$, $a_1^q = a_1$. Since we need only to investigate the case when $a$ and $a_1$ are both non-zero, we deduce that $(\alpha)^{q(i-j)} = \alpha^{i-j}$. However, as $\alpha^q = \alpha^{-1}$, we obtain that $\alpha^{2(i-j)} = 1$. Since $\alpha$ has odd order, this is only possible if $i = j$. □

The result above does not hold when $p$ is odd, since $\alpha^{(q+1)/2} = -1$, and consequently

$$A_{i+(q+1)/2} = A_i$$

for $1 \leq i \leq (q + 1)/2$. It is however easy to establish the following analogue of Lemma 3.

**Lemma 4.** Suppose that $p$ is odd. Then we have $A_i \cap A_j = Z(G_q)$ if $1 \leq i \neq j \leq (q + 1)/2$. 

3. The irreducible representations of the group $G_q$

Let $\chi$ be an irreducible character of $G_q$ of degree $q$ and let $X$ be an irreducible representation of $G_q$ with character $\chi$. We may assume that $X(G_q)$ consists of unitary matrices. Lemma 1 shows that $\chi(x) = 0$ if $x \not\in Z(G_q)$ and thus we see that no element outside $Z(G_q)$ is in the kernel of $X$. On the other hand, setting $Z = Z(G_q)$ for brevity, $X(Z)$ is certainly contained in the center of $X(G_q)$ and it is a non-trivial cyclic group consisting of scalar multiples of the identity by Schur’s Lemma. Since $Z$ is an elementary abelian $p$-group, we deduce that $|X(Z)| = p$ and $|\ker X| = q^2/p$. It follows that $X(G_q)$ has order $q^2p$. Of course, $X(G_q)$ is an extra-special $p$-group, but we do not need to know this.

Recall now the automorphism $\sigma$ of $G_q$. We define the conjugate representation $X^\sigma$ by

$$X^\sigma(x) = X(\sigma(x))$$

for all $x \in G_q$. Since $\sigma$ fixes the elements of $Z$ pointwise, and the character of $X$ vanishes outside $Z$, it is clear that $X^\sigma$ also has character $\chi$. It follows that $X$ and $X^\sigma$ are equivalent representations and thus there exists an invertible matrix $D$, say, satisfying

$$X^\sigma(x) = X(\sigma(x)) = D^{-1}X(x)D$$

for all $x \in G_q$. We include a proof of the following well known lemma for completeness.

**Lemma 5.** Assume the notation introduced above. Then multiplying the matrix $D$ by a non-zero complex scalar if necessary, we may arrange for $D$ to satisfy $D^{q+1} = I$, $\det D = 1$, and to be unitary. Furthermore, the resulting $D$ has all its entries in the field of definition of $X$.

**Proof.** We first note that we may assume that $D$ as defined above has its entries in the field of definition of $X$, since it defines an equivalence between two representations over that field. Now it is straightforward to verify that

$$D^{-1}X(x)D^i = X(\sigma^i(x))$$

for each integer $i$ and hence

$$D^{-(q+1)}X(x)D^{q+1} = X(x)$$

for all $x \in G_q$, since $\sigma$ has order $q + 1$. Therefore, $D^{q+1}$ commutes with all $X(x)$ and hence is a scalar multiple of the identity by Schur’s Lemma, say $D^{q+1} = \lambda I$.

Let $d$ be the determinant of $D$. Taking determinants in the equality $D^{q+1} = \lambda I$, we obtain

$$d^{q+1} = \lambda^q.$$  

We accordingly set $D_1 = d\lambda^{-1}D$ and calculate that

$$D_1^{q+1} = d^{q+1}\lambda^{-(q+1)}D^{q+1} = \lambda^{-1}D^{q+1} = I.$$  

Thus, if we replace $D$ by $D_1$, we obtain $D^{q+1} = I$. We note also that $\det D = 1$ in this case.

Finally, since $D$ normalizes $X(G_q)$, and the elements $X(x)$ are unitary, we see that

$$X(x)DD^\dagger X(x)^\dagger = X(x)DD^\dagger X(x)^{-1} = DD^\dagger$$

Thus $DD^\dagger$ commutes with all elements $X(x)$ and is therefore a scalar multiple of the identity, by Schur’s Lemma, say $DD^\dagger = \mu I$. But the diagonal entries of $DD^\dagger$
are sums of the squares of the moduli of complex numbers and are thus positive real numbers. This implies that μ is a positive real number. We also have $D^{q+1} = I$ and it follows that

$$(D^\dagger)^{q+1} = I = \mu^{q+1} D^{-(q+1)} = \mu^{q+1} I.$$  

We deduce that $\mu^{q+1} = 1$ and since $\mu$ is a positive scalar, $\mu = 1$. This means that D is unitary. □

We note that we may take the field of definition of $X$ to be $\mathbb{Q}(\sqrt{-1})$ when $p = 2$ and $\mathbb{Q}(\omega)$, where $\omega$ is a primitive $p$-th root of unity, when $p$ is odd, and consequently we may assume that $D$ has all entries in the corresponding field.

4. Construction of mutually unbiased bases using powers

Recall that the abelian subgroups $A_i$ of $G_q$ satisfy $|A_i : Z(G_q)| = q$ for all $i$. Let $x_{i,j}$, where $1 \leq j \leq q$, be a system of coset representatives of $Z(G_q)$ in $A_i$, where we choose $x_{i,1}$ to be the identity element. Let $X$ be a fixed irreducible complex representation of $G_q$ of degree $q$, consisting of unitary matrices. We set

$$C_i = \{ X(x_{i,j}) : 1 \leq j \leq q \}.$$  

The subsets $C_i$ consist of $q$ commuting unitary matrices and satisfy $C_i \cap C_j = I$ for $i \neq j$.

The trace inner product $f(U, V)$ of $q \times q$ complex matrices $U$ and $V$ is defined by

$$f(U, V) = \text{tr}(U^\dagger V),$$  

where $\text{tr}$ signifies the usual matrix trace. The trace inner product is a positive definite hermitian form on the space of complex matrices. Since for any element $x$ of $G_q$ outside $Z(G_q)$ we have $\text{tr} X(x) = 0$, it is straightforward to see that the non-identity elements of $C_i$ are orthogonal to the non-identity elements of $C_j$ with respect to the trace inner product.

We are now in a position to use a result of [1] to construct mutually unbiased bases.

**Theorem 1.** Let $q$ be a power of 2 and let $X$ be an irreducible complex representation of $G_q$ of degree $q$, consisting of unitary matrices. Let $D$ be a $q \times q$ matrix that satisfies $D^{q+1} = I$ and $D^{-1} X(x) D = X(\sigma(x))$ for all $x$ in $G_q$. Then the powers $D, D^2, \ldots, D^{q+1} = I$ define $q+1$ pairwise mutually unbiased bases. Furthermore, all entries of $D$ are in the field $\mathbb{Q}(\sqrt{-1})$.

**Proof.** The matrices in $C_1$ commute and there therefore exists an orthonormal basis of $\mathbb{C}^q$ consisting of simultaneous eigenvectors of the elements of $C_1$. This orthonormal basis determines a unitary matrix $B$, say. Now the matrix $D$ satisfies

$$D^{-(i-1)} C_1 D^{i-1} = C_i$$  

and thus the corresponding unitary matrix constructed of simultaneous orthonormal eigenvectors of $C_i$ is $D^{-(i-1)} B$. By Theorem 3.2 of [1], the unitary matrices $D^{-(i-1)} B$ for $1 \leq i \leq q+1$ define $q+1$ pairwise mutually unbiased bases. But we may multiply these matrices on the right by $B^{-1}$ and still retain the unbiased property. The matrices so obtained are simply the powers of $D$. □
Example. The matrix
\[
\frac{1 + i}{2} \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix},
\]
where \(i^2 = -1\), is unitary and has order 3. Its three different powers generate three mutually unbiased bases in \(\mathbb{C}^2\).

When \(q\) is a power of an odd prime, we can obtain half the maximum number of mutually unbiased bases from the powers of a unitary matrix, as a minor modification of the proof above shows.

**Theorem 2.** Let \(q\) be a power of an odd prime and let \(X\) be an irreducible complex representation of \(G_q\) of degree \(q\), consisting of unitary matrices. Let \(D\) be a \(q \times q\) matrix that satisfies \(D^{q+1} = I\) and \(D^{-1}X(x)D = X(\sigma(x))\) for all \(x\) in \(G_q\). Then the powers \(I, D^{-1}, \ldots, D^{-(q-1)/2}\) define \((q+1)/2\) mutually unbiased bases.

**Proof.** The proof is identical with the previous one, except that, since
\[
D^{-(q+1)/2}C_1D^{(q+1)/2} = C_1,
\]
we only obtain \((q+1)/2\) bases using the powers. \(\square\)

There is an essential difference between the generation of mutually unbiased bases in the 2-power dimension case and the odd prime power dimension case, as we shall soon show. We first prove a lemma on involutory unitary matrices.

**Lemma 6.** Let \(d\) be an odd positive integer and let \(U\) be a \(d \times d\) unitary matrix satisfying \(U^2 = I\). Suppose that each diagonal entry of \(U\) has absolute value \(1/\sqrt{d}\). Then \(d\) is a square.

**Proof.** Since we have
\[
UU^\dagger = U^2 = I,
\]
it follows that \(U = U^\dagger\), and thus \(U\) is hermitian. The diagonal entries of \(U\) are therefore real and consequently equal \(\pm 1/\sqrt{d}\). As \(U\) is an involution, its eigenvalues equal \(\pm 1\) and hence its trace is an odd integer, since \(d\) is odd by hypothesis. However the trace of \(U\) equals the sum of its diagonal entries and thus we must have
\[
\frac{r - s}{\sqrt{d}} = e,
\]
where \(e\) is an odd integer, \(r\) is the number of diagonal entries of \(U\) equal to \(1/\sqrt{d}\), and \(s\) is the number of diagonal entries of \(U\) equal to \(-1/\sqrt{d}\). This implies that \(\sqrt{d}\) is rational and hence \(d\) is a square. \(\square\)

**Corollary 1.** Let \(d\) be an odd positive integer which is not a square. Then there does not exist a \(d \times d\) unitary matrix \(D\) of even multiplicative order \(2r\), say, whose powers \(D, D^2, \ldots, D^{2r} = I\) generate \(2r\) pairwise mutually unbiased bases.

**Proof.** Suppose on the contrary that such a unitary matrix \(D\) exists. Then \(D^r\) satisfies the hypotheses of Lemma 6, which is impossible, since \(d\) is not a square. Thus no such \(D\) exists. \(\square\)

We now see that there can be no analogue of Theorem 1 when \(q\) is an odd power of an odd prime, and thus we probably cannot improve Theorem 2.
5. Reality questions for mutually unbiased bases

There is a concept of real mutually unbiased bases, which appears to be much more restrictive than its complex counterpart. Suppose that we have a positive definite symmetric bilinear form defined on $\mathbb{R}^d \times \mathbb{R}^d$. An orthonormal basis of $\mathbb{R}^d$ corresponds to a real $d \times d$ orthogonal matrix. Orthonormal bases $B_1$ and $B_2$ of $\mathbb{R}^d$ with corresponding $d \times d$ orthogonal matrices $O_1$ and $O_2$ are mutually unbiased when each entry of the product $O_1O_2^{-1}$ has absolute value $1/\sqrt{d}$. This means that $\sqrt{d}O_1O_2^{-1}$ is an Hadamard matrix. This condition alone is enough to show that pairs of real mutually unbiased bases can only exist for even values of $d$ (and, apart from $d = 2$, only for values of $d$ divisible by 4). See, for example, [2], for information on the existence problem for real mutually unbiased bases.

Let $N_{\mathbb{R}}(d)$ denote the maximum number of orthonormal bases of $\mathbb{R}^d$ which are pairwise mutually unbiased. It can be proved that $N_{\mathbb{R}}(d) \leq d/2 + 1$, and furthermore the equality occurs when $d$ is a power of 4. As we hinted above, in many cases the value of $N_{\mathbb{R}}(d)$ for specific $d$ is substantially less than the general inequality suggests. See [2] for much more information on this topic.

Constructions of families of real pairwise mutually unbiased bases meeting the upper bound $d/2 + 1$ when $d$ is a power of 4 generally use orthogonal geometry over the field of order 2 or real representations of extra-special 2-group. See, for example, Proposition 6 of [4] or Theorem 3.4 of [5]. We are interested in raising the question of whether there exists any analogue of Theorem 1 for real mutually unbiased bases. Setting $q = 2^{2m-1}$, we thus look for a real $2^{2m} \times 2^{2m}$ orthogonal matrix of order $q+1$ whose powers generate $q+1$ real mutually unbiased bases. At present, we have not succeeded in showing that such matrices exist, but we note the following $4 \times 4$ example.

Example. The matrix

$$
\frac{1}{2} \begin{pmatrix}
-1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1
\end{pmatrix}
$$

is orthogonal and has order 3. Its three different powers generate three mutually unbiased bases in $\mathbb{R}^2$. We obtain an Hadamard matrix if the matrix above is multiplied by 2.

6. Connections with orthogonal decompositions of simple Lie algebras

The paper [3] has shown a connection between the existence of $d+1$ mutually unbiased bases in $\mathbb{C}^d$ and orthogonal decompositions of the special linear Lie algebra $sl_d(\mathbb{C})$ into a direct sum of Cartan subalgebras. In this section we show how the automorphism $\sigma$ introduced above gives extra information on such a decomposition in the 2-power case. We also obtain similar information about an orthogonal decomposition of the symplectic Lie algebra of $d \times d$ matrices when $d$ is a 2-power.

We employ the notation of Section 4. Let $\mathcal{L}$ denote the special linear Lie algebra of all $q \times q$ complex matrices of trace zero. Recall that we have defined the subsets $\mathcal{C}_i$, which consist of $q-1$ commuting matrices of trace zero, together with the identity. These trace zero matrices are linearly independent, since they are pairwise orthogonal with respect to the trace inner product. Let $\mathcal{H}_i$ denote the linear span
of the non-identity elements in $C_i$. Clearly, $H_i$ is an abelian subalgebra of $L$ of dimension $q - 1$. Furthermore, as indicated in [3], Theorem 5.1, $H_i$ is self-normalizing in $L$. For, up to conjugation by a unitary matrix, which induces an automorphism of $L$, we may assume that $H_i$ is a $q - 1$-dimensional subspace of diagonal trace zero matrices, and hence equals the entire subspace of all such diagonal matrices, which is well known to be a Cartan subalgebra. We note also that $H_i$ is orthogonal to $H_j$ with respect to the Killing form of $L$, since this is a scalar multiple of the usual trace form, [3], Lemma 4.2.

We therefore obtain a decomposition
\[ L = H_1 \perp \cdots \perp H_{q+1} \]
of $L$ into an orthogonal direct sum of $q + 1$ Cartan subalgebras. Now conjugation by the matrix $D$ described in Theorem 1 induces an automorphism of order $q + 1$ of $L$. In the case that $q$ is a power of 2, this automorphism transitively permutes the $q + 1$ Cartan subalgebras, by the arguments of Theorem 1. (The automorphism of course preserves the Killing form.) Thus we may summarize our findings in this case.

**Theorem 3.** Let $q$ be a power of 2 and let $L$ denote the Lie algebra of all $q \times q$ complex matrices of trace 0. Then there is an orthogonal decomposition
\[ L = H_1 \perp \cdots \perp H_{q+1} \]
of $L$ into a direct sum of $q + 1$ Cartan subalgebras $H_i$, which are transitively permuted by an automorphism of $L$ of order $q + 1$.

We observe that a larger finite group of automorphisms permutes the summands of the decomposition above, while preserving the Killing form, since the group generated by $X(G_q)$ and $D$ also has this property.

This result bears a superficial resemblance to a deep result of Thompson, which shows that the complex Lie algebra $L$ of type $E_8$ admits an orthogonal decomposition into a direct sum of 31 Cartan subalgebras, which are transitively permuted by a group of automorphisms isomorphic to the Dempwolf group.

We do not know if the corresponding orthogonal decomposition of the special linear Lie algebra admits a similar transitive action of an automorphism of order $q + 1$ when $q$ is a power of an odd prime.

The decomposition described in Theorem 3 can be refined into a similar decomposition of the symplectic Lie algebra. We sketch some of the details here. The representation $X$ of $G_q$ described in Theorem 1 has a real-valued character but it cannot be defined over the real numbers. We say that it is of symplectic type, as the elements $X(x)$ preserve a non-degenerate symplectic form. This means that there is an invertible $q \times q$ skew-symmetric matrix $S$, say, such that
\[ X(x)SX(x)^T = S \]
for all $x \in G_q$. Furthermore, it is straightforward to see that, given our normalization of $D$, $D$ also satisfies $DSD^T = S$.

The symplectic Lie algebra $L$ consists of all $q \times q$ matrices $A$ which satisfy
\[ AS + SA^T = 0. \]
It has dimension $q(q + 1)/2$ and rank $q/2$. 
Suppose that \( x \) is an element of \( G_q \) such that \( X(x) \) has order 4. Then \( X(x)^2 = -I \) and hence
\[
X(x)^{-1} = -X(x).
\]
It follows that the equality \( X(x)SX(x)^T = S \) is equivalent to \( X(x)S + SX(x)^T = 0 \) in this case. Thus the elements of order 4 in \( X(G_q) \) are in the symplectic Lie algebra.

Consider now the abelian subgroup \( X(A_i) \) of \( X(G_q) \). It is straightforward to show that, if \( q = 2^m \), it is isomorphic to a direct product
\[
\mathbb{Z}_4 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2
\]
of cyclic groups and thus contains \( 2^m \) elements of order 4. Let \( D_i \) denote the linear span of the elements of order 4 in \( X(A_i) \). This is an abelian subalgebra of dimension \( q/2 \) of \( L \), which must be a Cartan subalgebra, as it consists of diagonalizable elements and has the appropriate dimension.

The \( q+1 \) subalgebras \( D_i \), where \( 1 \leq i \leq q+1 \), are all orthogonal with respect to the Killing form of \( L \), for the same reason as given in the proof of Theorem 3. Finally, \( D \) induces an automorphism of order \( q+1 \) of \( L \) by conjugation, since it preserves the symplectic form determined by \( S \) (indeed the group generated by \( X(G_q) \) and \( D \) has this property). This automorphism transitively permutes the Cartan subalgebras \( D_i \), since \( D \) maps the elements of order 4 in \( X(A_i) \) into the elements of order 4 in \( X(A_{i+1}) \).

Summarizing, we have proved the following.

**Theorem 4.** Let \( q \) be a power of 2 and let \( L \) denote the symplectic Lie algebra of \( q \times q \) complex matrices. Then there is an orthogonal decomposition
\[
L = H_1 \perp \cdots \perp H_{q+1}
\]
of \( L \) into a direct sum of \( q+1 \) Cartan subalgebras \( H_i \) which are transitively permuted by an automorphism of \( L \) of order \( q+1 \).

It is interesting to observe that at the time of this writing, orthogonal decompositions of the symplectic Lie algebra are only known in this 2-power rank case. The paper [6] contains related and more general constructions for orthogonal decompositions of Lie algebras using the geometry of finite vector spaces, and contains many references to earlier work.

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