Constrained 4D Gauss-Bonnet gravity and its cosmological implications

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Recently a new 4D Einstein-Gauss-Bonnet theory has been introduced [Phys. Rev. Lett. 124 (2020) 081301] with a serious debate that it does not possess a covariant equation of motion in 4D. In this note, we will present a new proposal to make this happen, by introducing a Lagrange multiplier to the action which eliminates the higher dimensional term from the equation of motion. The theory has then a covariant 4D equation of motion which is useful to study the less symmetric metrics. On top of FRW universe, the constraint theory is equivalent to the original 4D Einstein-Gauss-Bonnet gravity. We will also consider the anisotropy of the model and compare the theory with observational data. We will see that the theory becomes non-conservative and the matter density abundance falls more rapidly at larger redshifts compared to the conservative matter sources.

INTRODUCTION

The Einstein’s theory of general relativity has witnessed serious observational and theoretical challenges since its birth in 1915 as a way to describe the gravitational interaction. At the beginning, the theory has been challenged from reconciling the common sense with the predictions of the theory and afterward from solar system/local tests to cosmological/large scale experiments. However, one of the most controversial issues about the theory is to correctly describing the late time accelerated expansion of the universe. The first proposal was given by Einstein himself, considering the cosmological constant to the original Einstein-Hilbert action [1]. This idea together with the inclusion of cold dark matter to elucidate local observations, is now well-known as the ΛCDM model

\[ S = \int d^4x \sqrt{-g}(R - 2\Lambda) + S_m, \]  

where \( \Lambda \) is the cosmological constant, and \( S_m \) is the action for all baryonic matter fields plus dark matter.

Nowadays, we believe that the cosmological constant, suffers from theoretical/phenomenological problems [1], tempting cosmologists to consider other options, generally known as dark energy models. The simplest possibility of adding a dark energy to the Einstein’s theory is to promote the cosmological constant to a dynamical field. This field could be a scalar, vector or in general some higher spin field [2]. However, one can consider some modifications of the Einstein-Hilbert action itself which could involve higher dimensions [3], granting mass to the mass-less graviton [4], considering richer geometries [5], or changing the gravitational interactions by modifying the Ricci scalar in the Einstein-Hilbert action [6]. All of these ideas are generally known as modified theories of gravity.

Changing Ricci scalar in the action could be done in numerous ways. The simplest way is to substitute the Ricci scalar with an arbitrary function of \( R \), resulting in a so-called \( f(R) \) theory of gravity [6]. The theory can be easily proven to be equivalent to a specific class of Brans-Dicke theory and hence it is free of Ostrogradski ghost. As a result it can be safely considered as a geometrical candidate for dark energy. Another possibility to generalize the Ricci scalar in the Einstein-Hilbert action is to add some higher order terms constructed from Ricci and Riemann tensors. In general these higher order terms possess extra unhealthy degrees of freedom which will eradicate the validity of the theory. It is well-known that the only terms which cancels the ghost degrees of freedom would be obtained from the Lovelock prescription [7]. However, Lovelock terms are all vanished in four dimensional space-times and as a result only the Ricci scalar remains as a healthy candidate to describe gravitational interactions. The first non-trivial Lovelock term is the Gauss-Bonnet invariant which is non-zero for \( D > 4 \), where \( D \) is the space-time dimension. An interesting idea to put the effects of this term in the theory is to assume that the universe is fundamentally five dimensional. Then we will project the extra dimension to our ordinary four dimensional space-time and the effect of Gauss-Bonnet term will appear in our 4D universe [8].

Recently, an idea appears in the literature, presenting a new way to make some non-vanishing effect of the Gauss-Bonnet Lagrangian in 4D [9]. The idea comes from the fact that the vanishing of the Gauss-Bonnet invariant in 4D is due to the presence of an overall factor \( D - 4 \). The author then claimed that if one considers a \( D \)-dimensional abstract space-time and re-scales the Gauss-Bonnet parameter \( \alpha \) to \( \alpha/(D - 4) \), one then obtain some non-vanishing contributions of the Gauss-Bonnet invariant in 4D by taking the limit \( D \to 4 \). The consequences of the theory is widely investigated in the literature, exploring its effects in black-holes [10], wormholes [11], compact stars [12] and also in cosmology [13]. Also, many works has been done to generalize the idea to higher/lower dimensions, relations to other theories.
and also its quantum aspects of the theory [14]. For example, in [15], the authors showed that in order to have a well-defined linearized theory the metric should be locally conformally flat. Also, many works have been done to constraint the new re-scaled Gauss-Bonnet parameter from observational data [16]. In [17] the authors obtained the so far strongest constraint on the theory parameter, namely, \( (\alpha = 2.69 \pm 11.67) \times 10^{48} \text{eV}^{-2} \).

However, the most important issue related to the 4D Einstein-Gauss-Bonnet theory is that despite the fact that the Lagrangian has an overall factor \( D - 4 \), this is not the case for the equation of motion. As have been shown in the literature [18] and we will also review in the next section, the equation of motion can be decomposed to two separate parts, one has the \( D - 4 \) factor and the other which is proportional to the Weyl tensor, does not have the factor. As a result, for a generic non-symmetric space-time, the second part will make the equation of motion divergent. Also, it has been pointed out that for non-symmetric space-times, there is no canonical way to define a higher dimensional metric, and the theory seems ambiguous for generic space-times and the limiting process is not well-defined. As a result, the theory does not have a covariant 4D equation of motion.

There are some attempts in the literature to resolve these problems, for example to couple the theory to a scalar field [18]. In this paper, we put forward a new idea to deal with the above issues. As we have seen above, the main difficulty for having a covariant equation of motion is that the Gauss-Bonnet term has a term without the factor \( D - 4 \). As a result in the limiting process this term causes problems for generic space-times. We will covariantly set this part to zero from the action by imposing a Lagrange multiplier. This will allow us to have a covariant 4D equation of motion and the limiting process would be unambiguous.

In this paper, we will show that the isotropic cosmology of the new model would be identical to the original theory. We will then consider the anisotropic universe in this model. It should be noted that the same process could not be done in the original theory since, as we discussed above, there is an ambiguity in defining the higher dimensional metric. In our new framework, however, there is no need for higher dimensional metric since we have a covariant 4D equation of motion.

**THE MODEL**

Let us start with the 4D Gauss-Bonnet gravity theory introduced in [9]. The action in \( D \) dimensions can be written as

\[
S = \int d^D x \sqrt{-g} \left[ \kappa^2 (R - 2\Lambda) + \mathcal{L}_m + \beta \mathcal{G} \right],
\]

where \( \Lambda \) is the cosmological constant, \( \mathcal{L}_m \) is the matter Lagrangian and \( \mathcal{G} \) is the Gauss-Bonnet term defined as

\[
\mathcal{G} = R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} - 4R^{\mu\nu} R_{\mu\nu} + R^2.
\]

The 4D Gauss-Bonnet recipe [9] is to rescale the Gauss-Bonnet coupling constant \( \beta \) to \( \alpha/(D - 4) \), so that the Gauss-Bonnet term acquire a non-vanishing part in the limit \( D \to 4 \).

The equation of motion of the metric field in \( D \) dimensions can be obtained from the action (2) as

\[
\kappa^2 (G_{\mu\nu} + \Lambda g_{\mu\nu}) + \frac{\alpha}{D - 4} g_{\mu\nu} = \frac{1}{2} T_{\mu\nu},
\]

where we have defined the Gauss-Bonnet tensor

\[
G_{\mu\nu} = 2 R R_{\mu\nu} - 4 R_{\mu\alpha\beta\gamma} R^\alpha_{\mu\beta\gamma} - 4 R_{\mu\nu\alpha\beta} R^{\alpha\beta} - 2 R_{\mu\alpha\beta\gamma} R^\alpha_{\nu\beta\gamma} - \frac{1}{2} g_{\mu\nu} \mathcal{G}.
\]

Here, \( T_{\mu\nu} \) is the energy-momentum tensor defined as

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}.
\]

In order to have a covariant 4D equation of motion, it is needed that the tensor \( G_{\mu\nu} \) has an overall factor \( D - 4 \). In fact it can be easily verified that the trace of the Gauss-Bonnet tensor has the above property and as a result the equation of motion (4) in \( D \) dimensions is

\[
\kappa^2 (D - 2) R - 2D\kappa^2 \Lambda = -T - \alpha \mathcal{G}.
\]

However, in general, this is not the case for the tensor \( G_{\mu\nu} \). Specifically, one can decompose the Gauss-Bonnet part \( G_{\mu\nu} \) [18] as

\[
G_{\mu\nu} = C_{\mu\nu} + (D - 4) S_{\mu\nu},
\]

where

\[
C_{\mu\nu} = 2 W_{\mu\alpha\beta\gamma} W^\alpha_{\nu\beta\gamma} - \frac{1}{2} W_{\alpha\beta\gamma\delta} W^{\alpha\beta\gamma\delta} g_{\mu\nu},
\]

is the Lanczos-Bach tensor where \( W_{\alpha\beta\gamma\delta} \) is the Weyl tensor, and

\[
S_{\mu\nu} = \frac{2}{(D - 2)^2} \left[ D R R_{\mu\nu} - 2(D - 2) R_{\mu\nu\alpha\beta} R^{\alpha\beta} + (D - 1) g_{\mu\nu} \left( R_{\alpha\beta} R^{\alpha\beta} + \frac{D^2 - D + 2}{4(D - 1)^2} R^2 \right) - 2(D - 1) R_{\mu\alpha\nu\beta} R^{\alpha\beta} \right].
\]

It is then obvious that in the limit \( D \to 4 \) of the equation of motion (4), the term \( S_{\mu\nu} \) remains finite and gives us a non-trivial contribution of the Gauss-Bonnet Lagrangian in \( 4D \). However, the term \( C_{\mu\nu} \) in the field equation (4) which is proportional to the Weyl tensor does not have a
As a result, we cannot obtain a generally covariant 4D energy-momentum tensor. More precisely, in the case introduced above, but results in the non-conservation of part of the Gauss-Bonnet term. This solves the problems there is no degeneracy in taking the limit \( D \to 4 \) rendering its vanishing in 4D.

In the new approach presented above, we start with a higher dimensional metric, where the tensor \( C_{\mu\nu} \) is non-vanishing in general, and then tend \( D \to 4 \). This causes a two-fold problem. Firstly, we are only interested in the 4D space-times with 4D metric field. But there is no canonical way to define a higher dimensional metric and as a result, taking the limit \( D \to 4 \) is not unique. This problem gets worth when the 4D metric is asymmetric, like Bianchi space-times. Here, there is no preferred way to define higher dimensional metric.

Secondly, the limit \( D \to 4 \) is not continuous. This can be seen by considering the fact that the tensor \( C_{\mu\nu} \) is identically zero in 4D but it is not in higher dimensions. So we got \( \frac{g}{D} \) when we naively take the limit \( D \to 4 \) and as a result we cannot obtain a generally covariant 4D equation of motion for Gauss-Bonnet theory.

It should be noted that the above problems seem unimportant for symmetric spacetimes, i.e. when one considers FRW cosmology, blackholes, compact stars, even scalar perturbations around FRW space times, since there is a canonical way to define a higher dimensional metric and for that metric the tensor \( C_{\mu\nu} \) vanishes. So, there is no degeneracy in taking the limit \( D \to 4 \) of the field equation (4). However, this does not imply that the theory has a well-defined covariant equation of motion in 4D.

It should be mentioned that there is another possibility that we set \( C_{\mu\nu} = 0 \) by hand and only consider the \( s_{\mu\nu} \) part of the Gauss-Bonnet term. This solves the problems introduced above, but results in the non-conservation of the energy-momentum tensor. More precisely, in the case of vanishing \( C_{\mu\nu} \), the covariant divergence of the field equation (4) gives

\[
\nabla_\alpha T^\alpha_\nu = \alpha \left[ \nabla_\nu \left( R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{6} R^2 \right) + R_{\nu\alpha} \nabla^\alpha R - 2 R^{\alpha\beta} \nabla_\beta R_{\nu\alpha} + 4 R_{\nu\alpha\beta\gamma} \nabla^\gamma R^{\alpha\beta} \right].
\]

which is non-zero. In this paper, we put forward the above argument by imposing the vanishing of the tensor \( C_{\mu\nu} \) to the action (2) through a Lagrange multiplier. The new constrained action becomes

\[
S = \int d^D x \sqrt{-g} \left( \kappa^2 (R - 2\Lambda) + \mathcal{L}_m \right) + \frac{\alpha}{D - 4} G + \frac{1}{2} \lambda^{\mu\nu} C_{\mu\nu},
\]

where the symmetric tensor \( \lambda^{\mu\nu} \) is a Lagrange multiplier. The procedure here is the same as [9]; after obtaining the field equations, we perform the limit \( D \to 4 \) to obtain a 4D covariant equation of motion.

The variation should be performed with respect to the metric and the Lagrange multiplier \( \lambda_{\mu\nu} \). As a result the theory would have two equations of motions for \( g_{\mu\nu} \) and \( \lambda_{\mu\nu} \) which can be used to determine all the variables. In our case, the variation of the action (12) with respect to \( \lambda_{\mu\nu} \) gives \( C_{\mu\nu} = 0 \). We then consider the limit \( D \to 4 \) of this equation. However, \( C_{\mu\nu} \) identically vanishes in 4D and there is no equation of motion for the Lagrange multiplier in 4D. As a result in this model, the Lagrange multiplier would become an arbitrary tensor which could be determined by physical considerations.

The equation of motion of the metric tensor after taking the limit becomes

\[
\kappa^2 (G_{\mu\nu} + \Lambda g_{\mu\nu}) + \alpha s_{\mu\nu} + L_{\mu\nu} = \frac{1}{2} T_{\mu\nu},
\]

where we have defined

\[
s_{\mu\nu} = \lim_{D \to 4} S_{\mu\nu} = \frac{1}{2} \left[ 4 R R_{\mu\nu} - 4 R_{\mu\alpha\nu\beta} R^{\alpha\beta} + 3 g_{\mu\nu} \left( R_{\alpha\beta} R^{\alpha\beta} + \frac{7}{18} R^2 \right) - 6 R_{\mu\alpha} R_{\nu}^\alpha \right],
\]

and
\[ L_{\mu \nu} = \frac{1}{4} W^2 \lambda_{\mu \nu} + \lambda^{\alpha \beta} W_{\mu \alpha} \gamma^\beta W_{\nu \gamma \delta} + 3 \lambda^{\gamma \delta} R_{\gamma \delta}^{\alpha \beta} W_{\mu \nu} - R^\alpha_{\mu \nu} W_{\beta \gamma \delta} - \frac{1}{2} \lambda \eta_{\mu \nu} R_{\beta \gamma \delta}^{\alpha \beta} W_{\lambda \nu} W_{\alpha \beta \gamma} + W_{\beta \gamma \delta} R^\alpha_{\beta \gamma \delta} \]

is the contribution from the new Lagrange multiplier term in the action. In the above expression we have defined

\[ W^2 = W_{\alpha \beta \gamma \delta} W^{\alpha \beta \gamma \delta}, \quad \lambda = \lambda_{\mu \nu}. \] (16)

Some points should be clarified in this step. First, as we have discussed above, we do not have an equation to determine the Lagrange multiplier \( \lambda_{\mu \nu} \). Secondly, one can easily verify that the tensor \( L_{\mu \nu} \) vanishes in 4D. This is because this tensor is obtained from variation of the Lanczos-Bach tensor in \( D \) dimensions and then taking the \( D \rightarrow 4 \). Since this limiting process is continues for this term, the tensor \( L_{\mu \nu} \) could be seen as a variation of the Lanczos-Bach tensor in 4D. In 4D the Lanczos-Bach tensor vanishes identically, the \( L_{\mu \nu} \) does not contribute to the above equations of motion. Thirdly, the energy-momentum tensor is not conserved in this model as this limiting process is continues for this term, the tensor \( L_{\mu \nu} \) could be seen as a variation of the Lanczos-Bach tensor in 4D. In 4D the Lanczos-Bach tensor vanishes identically, the \( L_{\mu \nu} \) does not contribute to the above equations of motion. Lastly, the energy-momentum tensor is not conserved by taking the covariant divergence of the equation of motion (13).

The resulting equation of motion in for 4D Gauss-Bonnet theory in four dimensions can be written as

\[ \kappa^2 (G_{\mu \nu} + \Lambda g_{\mu \nu}) + \frac{\alpha}{2} \left[ 4 R_{\mu \nu} - 4 R_{\mu \alpha \beta} R^{\alpha \beta} \right] + 3 g_{\mu \nu} \left( R_{\alpha \beta} R^{\alpha \beta} + \frac{7}{18} R^2 \right) - 6 R_{\mu \alpha} R_{\nu}^{\alpha} \right] = \frac{1}{2} T_{\mu \nu}, \] (17)

Actually, if one assumed that the Lanczos-Bach tensor where vanishing from the first place, the same result would appear. In our procedure, this by hand cancellation is replaced by a dynamical cancellation through the Lagrangian.

**ISOTROPIC COSMOLOGY**

As an example of a symmetric space-time, let us try the new constrained 4D Gauss-Bonnet theory on the FRW space-time. Assume that the line element is

\[ ds^2 = a(t)^2 \eta_{\mu \nu} dx^\mu dx^\nu, \] (18)

where \( a(t) \) is the scale factor and \( \eta_{\mu \nu} \) is the Minkowski metric. We consider a perfect fluid which is characterized by energy density \( \rho \) and thermodynamic pressure \( p \), with the energy-momentum tensor given by

\[ T^{\mu \nu} = (\rho + p) u^\mu u^\nu + pg^{\mu \nu}. \] (19)

The Friedmann and Raychaudhuri equations can be written as

\[ 3 \kappa^2 H^2 = \frac{1}{2} a^2 \rho + \kappa^2 \Lambda a^2 - \frac{3 a^2}{a^2} H^4, \] (20)

and

\[ 2 \kappa^2 \dot{H} = \frac{2}{3} \kappa^2 \Lambda a^2 - \frac{1}{6} a^2 (\rho + 3p) + \frac{2 a H^2}{a^2} (H^2 - 2 \dot{H}). \] (21)

The conservation of the energy-momentum tensor can be written as

\[ \dot{\rho} + 3H (\rho + p) = 0. \] (22)

**ANISOTROPIC COSMOLOGY**

In this section, we will consider the cosmological implications of the anisotropic universe in the constraint 4D Gauss-Bonnet theory. The metric is taken as

\[ ds^2 = -dt^2 + a_1^2 dx^2 + a_2^2 (dy^2 + dz^2), \] (23)

where \( a_1, a_2 \) are directional time dependent scale factors. We assume that the universe is filled with an anisotropic universe in the constraint 4D Gauss-Bonnet theory. The metric is taken as

\[ ds^2 = -dt^2 + a_1^2 dx^2 + a_2^2 (dy^2 + dz^2), \] (23)

where \( a_1, a_2 \) are directional time dependent scale factors. We assume that the universe is filled with an anisotropic matter content with energy-momentum tensor of the form

\[ T^{\mu \nu} = \text{diag}(-\rho, p, p, p), \] (24)

where \( \rho \) and \( p \) are energy density and thermodynamic pressures of the fluid respectively.

Let us define the mean Hubble and the anisotropy parameter as

\[ H_i = \frac{\dot{a}_i}{a_i}, \quad \Delta H_i = H - H_i \quad i = 1, 2, \]

\[ 3H = H_1 + 2H_2, \]

\[ 3A = \left( \frac{\Delta H_1}{H} \right)^2 + 2 \left( \frac{\Delta H_2}{H} \right)^2. \] (25)
Also we define the deceleration parameter as
\[ q = -1 + \frac{d}{dt} \left( \frac{1}{H} \right). \tag{26} \]

It should be noted that in the case of isotropic universe with \( a_1 = a_2 \), the anisotropy parameter vanishes and the mean Hubble and deceleration parameters become the standard Hubble and deceleration parameters of isotropic universe. Since the universe becomes isotropic at late times, we will compare the mean Hubble and deceleration parameters of the anisotropic 4D Gauss-Bonnet universe with late time observational data. The Friedman and Raychaudhuri equations can be written as

\[
3\kappa^2(1 - B^2)H^2 = \kappa^2\Lambda + \frac{1}{2}\rho - 3\alpha \left[ (1 - 15B^2 + 36B^3 + 3B^4)H^4 + 2B(1 + 2B)\dot{B}H^3 \right. \\
+ 2B\ddot{B}H + B^2\dot{H}^2 + (\dot{B}^2 + B^2(1 + 2B)\dot{H})H^2 \left. \right], 
\tag{27} \]

\[
\kappa^2 \left[ 2(1 - B)\dot{H} + 3(1 - B)^2H^2 - 2H\dot{B} \right] = -\frac{1}{2}p + \kappa^2\Lambda - \alpha \left[ 3(5B^4 + 7B^2 - 4B + 1)H^4 - 2(2 + B)(1 - 4B)\dot{B}H^3 \right. \\
+ 6B\ddot{B}H + 3B^2\dot{H}^2 + (3\dot{B}^2 + 2(4B^3 + 9B^2 - 6B + 2)\dot{H})H^2 \left. \right], 
\tag{28} \]

\[
\kappa^2 \left[ 3(1 + B + B^2)H^2 + H\dot{B} + (2 + B)\dot{H} \right] = -\frac{1}{2}p + \kappa^2\Lambda + \alpha \left[ 3(7B^4 + 6B^3 + 8B^2 - 2B - 1)H^4 + 12B\ddot{B}H\dot{H} + 6B^2\dot{H}^2 \right. \\
+ 2(5B^2 + 14B - 1)\dot{B}H^3 + (6\dot{B}^2 + 2(5B^3 + 18B^2 - 3B - 2)\dot{H})H^2 \left. \right]. 
\tag{29} \]

Also, the conservation equation of the matter content can be written as
\[
\dot{\rho} + 3H(\rho + p) = -12\alpha \left( B(1 + 2B)H^2 + \frac{d}{dt}(BH) \right) \times \left( 3B(1 - B)(1 + 2B)H^3 + (4 + B)\ddot{B}H^2 + 2\dddot{B}H + 2\dot{B}\dot{H} + (\dot{B} + B(5 + B)\dot{H})H \right), \tag{30} \]

where we have defined \( B \equiv \sqrt{A/2} \). It should be explicitly checked that the tensor \( L_{\mu\nu} \) in equation (15) vanishes in the case of Bianchi universe. Also, the above conservation equation shows that contrary to the isotropic FRW universe, the matter sector is no longer conserved in this case.

In order to solve the above set of dynamical equations, we assume that the universe is filled with radiation, with energy density \( \rho_r \) and pressure \( p_r = \rho_r/3 \) and also pressure-less dust with energy density \( \rho_m \). We then define the following set of dimensionless parameters
\[
\rho = \rho_r + \rho_m, \quad H = H_0h, \quad \bar{\alpha} = \alpha\kappa^2H_0^2, \\
\Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \quad \bar{\rho}_i = \frac{\rho_i}{6\kappa^2H_0^2}, \quad i = r, m \tag{31} \]

where \( H_0 \) is the current value of the Hubble parameter.

In order to compare the model with observations it is customary to transform the field equations to the redshift coordinates, defined as
\[
1 + z = \frac{1}{a}. \tag{32} \]

We will assume that all the non-conservative sources is handled by dust. As a result the radiation component
of the cosmic fluid would be conserved. One can then obtain

$$\bar{\rho}_r = (1 + z)^4 \Omega_{r0},$$

(33)

where $\Omega_{r0}$ is the current radiation density abundance.

Considering the Friedman and Raychaudhuri equations at $z = 0$ and denoting that by definition $h(z = 0) = 1$, one can obtain the cosmological constant as

$$\Omega_{\Lambda} = 1 + \beta - \Omega_{r0} - \Omega_{m0},$$

(34)

where $\Omega_{m0}$ is the current value of the dust density abundance. Also, one can obtain the following constraint on the derivatives of the function $B$ and $h$ as

$$B'(0) = 0, \quad h'(0) = \frac{3\Omega_{m0} + 4\Omega_{r0}}{2 + 4\beta}.$$

(35)

In figure (1), we have plotted the mean Hubble parameter and also the deceleration parameter as a function of the redshift for different values of $\beta = 0.001$ (dashed), 0.05 (dotted), 0.1 (dot-dashed) and 0.5 (long dashed). The red solid line represents the evolution in the $\Lambda$CDM model. The error bars are associated with the observational data on the Hubble parameter [19]. One can see from the figures that the Hubble parameter increases as the parameter $\beta$ decreases. As a result the model predicts smaller universe for smaller values of $\beta$. This can also be seen from the deceleration parameter where smaller values of $\beta$ imply the universe has more deceleration at larger redshift. In figure (2), we have plotted the anisotropy parameter $B$ as a function of redshift. One can see from the figure that the universe becomes isotropic at late times. Also, in figure (3), we have depicted the dust matter density abundance as a function of $z$. The red solid curve corresponds to the conservative case. It should be noted that in the 4D-Gauss-Bonnet theory, the matter sector is not conserved. This can be seen from the figure, where the curves fall more rapidly at larger redshifts compare to the conservative case. In summary, we note that the present theory can in principle explain late time observational data. However, more analysis would be needed to determine the viability of the theory as an alternative theory of general relativity.

CONCLUSIONS

In this paper, we have considered a new way to resolve the problem of having an equation of motion in 4D Einstein-Gauss-Bonnet gravity by introducing a Lagrange multiplier term to the action. This new term will
remove the unwanted part of the Gauss-Bonnet equation of motion which is not proportional to the factor $D - 4$ and as a result enabled us to write a covariant equation of motion for the theory. This new term however, make the 4D matter field to be non-conservative. In four space-time dimensions, the equation of motion of the Lagrange multiplier satisfied identically and as a result we have an arbitrary tensor field which is not determined in the theory. This signaling a new symmetry which could be unraveled in the future. For symmetric space-times, the new constraint theory is equivalent to the original theory. So, all the works done in the original context could be trivially applied to the new constrained version. The richness of this new constrained 4D Gauss-Bonnet theory is that the field equation is now four dimensional and as a result we could consider less symmetric space-times like Bianchi types or rotating black hole solutions in this context. Importantly, the analysis of perturbations, which needs broken homogeneity and isotropy, should be done in our framework. The same analysis could not be done in the original version of the theory because of the ambiguity of defining higher dimensional metrics for these space-times. We have then considered an anisotropic universe describing by the Bianchi type I universe as an example. We have solved the equations numerically and see that the mean Hubble parameter tends to the current Hubble parameter at late times. More precisely, at redshifts smaller than unity, there is no significant deviations between the mean Hubble parameter and the observational data. The anisotropy parameter is also an increasing function of the redshift, predicting that the universe becomes isotropic at late times. Overall, we have shown that the constraint 4D Gauss-Bonnet theory can explain observational data on the Hubble parameter.

It should be noted that in the Bianchi space-time, the matter energy-momentum tensor is no longer conserved. In figure (3), we have plotted the evolution of the matter density abundance as a function of redshift $z$ for $\beta = 0.001$ (dashed), $0.05$ (dotted), $0.1$ (dot-dashed) and $0.5$ (long dashed). The red solid line represents the evolution of the conservative matter source.

![Graph](image-url)

**FIG. 3.** The evolution of the dust matter density abundance as a function of redshift $z$ for $\beta = 0.001$ (dashed), $0.05$ (dotted), $0.1$ (dot-dashed) and $0.5$ (long dashed). The red solid line represents the evolution of the conservative matter source.

At last, we have to say that more analysis would be needed to fully understand the nature of theory together with is observational constraints.

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