Construction of Szász-Mirakjan-type operators which preserve $a^x$, $a > 1$

Rishikesh Yadav$^{1,*}$, Vishnu Narayan Mishra$^{2,+}$

1 Applied Mathematics and Humanities Department, Sardar Vallabhbhai National Institute of Technology, Surat, Surat-395 007 (Gujarat), India.
2 Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak 484 887, Anuppur, Madhya Pradesh, India.

*rishikesh2506@gmail.com, †vishnunarayanmishra@gmail.com

Abstract. In this paper, we introduce a new type of Szász-Mirakjan operators, which preserve $a^x$, $a > 1$ fixed and $x \geq 0$. We study uniform convergence of the operators by using some auxiliary result and also error estimation is given. The convergence of said operators are shown and analyzed by graphics, also in same direction we find better rate of convergence than Szász-Mirakjan operators by analyzing the graphics. Voronovskaya-type theorem is studied and a comparison is shown under sense of convexity with Szász-Mirakjan operators. In last section, modified sequence is constructed in the space of integral function.

Keywords: Szász-Mirakjan operators, King’s operators, modulus of continuity, Voronovskaya-type theorem, convexity of function.

1. Introduction

In 1941, Mirakjan [6] defined the operators $S_n : C_2[0, \infty) \rightarrow C[0, \infty)$ for any $x \in [0, \infty)$ and for any $n \in \mathbb{N}$ given by

(1.1) $S_n(f ; x) = \sum_{k=0}^{\infty} s_{n,k}(x)f \left( \frac{k}{n} \right),$

where $s_{n,k}(x) = e^{-nx} \left( \frac{nx}{k!} \right)^k$, $x \in [0, \infty)$,

and

$C_2[0, \infty) = \{ f \in C[0, \infty) : \lim_{x \rightarrow 0} \frac{f(x)}{1+x^2} \text{ exists and is finite} \},$

which is a Banach space endowed with

\[ \| f \| = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1+x^2}. \]

The operators $S_n(f ; x)$ are called Szász-Mirakjan operators, where $s_{n,k}(x)$ are Szász’s basis functions. They were extensively studied in 1950 by Szász [13].

The operators $S_n$ have many more properties similar to classical Bernstein operators. Both are positive and linear operators. Most of the approximating operators $L_n$ (say) preserve $e_i(x) = x^i \ (i = 0, 1)$ i.e. $L_n(e_0) = e_0(x)$ and $L_n(e_1) = e_1(x)$, $n \in \mathbb{N}$. These conditions hold for the Bernstein polynomials, the Szász-Mirakjan operators, the Baskakov operators (see [14], [2], [15], [16]) but $L_n(e_2) \neq e_2$, for any of these operators.

In 2003 King [9] approached a sequence $\{ V_n \}$ of linear positive operators which modify the Bernstein operator and approximate each continuous function on $[0,1]$. These operators preserve the test functions $e_0$ and $e_2$ and have better rate of convergence than classical Bernstein operators in $0 \leq x \leq \frac{1}{2}$. King’s approach was further investigated by several authors. In [10] the authors investigated some approximation result on the Meyer-König and Zeller type operators which preserve $x^2$. 
In 2006, Morales et al. [3] considered an operators $B_{n,a}$ which fix $e_0$ and $e_0 + ae_2$, where $a \in [0, \infty)$ and after that some extension appeared in paper [7]. In this paper the authors assume $\tau_n = (B_n\tau)^{-1} \circ \tau$, where $\tau$ is any strictly continuous function defined on $[0,1]$ such that $\tau(0) = 0$ and $\tau(1) = 1$, i.e., they studied a sequence $V^\tau_n : C[0,1] \to C[0,1]$, defined by

$$V^\tau_n = B_n f \circ \tau_n = B_n f \circ (B_n \tau)^{-1} \circ \tau.$$  

Here the operators $V^\tau_n$ preserve $e_0$ and $\tau$.

Moreover a general extension is seen in paper [8] by considering operators $(B_{n,0,j}f)$ which fix $e_0$ as well as $e_j$ where $1 < j \leq n$ and are defined as

$$B_{n,0,j}f(x) = \sum_{k=0}^{n} \binom{n}{k} x^n (1-x)^{n-k} f\left(\frac{k(k-1)\cdots(k-j+1)}{n(n-1)\cdots(n-j+1)}\right), \quad f \in C[0,1].$$

But in whole of the process these operators (all above) are defined on a finite interval.

On the other hand, Duman and Özarslan [12] constructed Szász-Mirakjan-type operators which reproduce $e_0$, $e_2$ and having better error estimation than the classical Szász-Mirakjan operators.

In 2016, a modification of Szász-Mirakjan-type operators which reproduce $e_0$ and $e^{2ax}$, $a > 0$, were constructed by Acar et al. [17] and they studied important properties related to these operators. They proved uniform convergence, order of approximation with the help of a certain weighted modulus of continuity, and also discussed some shape preserving properties of the defined operators.

2. Construction of the operators

Let $u_n(x)$ be a continuous sequence of function defined on $[0, \infty)$, with $0 \leq u_n(x) < \infty$, then by (1.1), we have

$$L^*_n(f; u_n(x)) = e^{-u_n(x)} \sum_{k=0}^{\infty} \frac{(nu_n(x))^k}{k!} f\left(\frac{k}{n}\right),$$

for every $f \in C[0,\infty)$ and each $x \in [0,\infty)$, where $n \in \mathbb{N}$.

The set $\{e_0, e_1, e_2\}$ is a $K_+$-subset of $C_{\rho}[0,\infty)$ for $\rho \geq 2$. This space $C_{\rho}[0,\infty)$ is isomorphic to $C[0,1]$, where $K_+$ is the Korovkin set (see [1] for details).

Now, if $u_n(x)$ is replaced by $s_n(x)$ defined as

$$s_n(x) = \frac{x \log a}{(-1 + a \varpi)n}, \quad \forall \ x \geq 0, \ a > 1,$$

then we have the following linear positive operators:

$$S^*_n,a(f; x) = \sum_{k=0}^{\infty} a \left(\frac{-x}{-1 + \varpi}\right)^k \frac{(x \log a)^k}{(-1 + a \varpi)^k k!} f\left(\frac{k}{n}\right)$$

where $f \in C_{\rho}[0,\infty), \ \rho > 0$ and $x \geq 0$.

**Remark 2.1.** We have

$$s_n(x) = \frac{x \log a}{(-1 + a \varpi)n}, \quad \forall \ x \geq 0, \ a > 1.$$

Now taking limit as $n \to \infty$, then

$$\lim_{n \to \infty} s_n(x) = \lim_{n \to \infty} \frac{x \log a}{(-1 + a \varpi)n} = x,$$

i.e. the given sequence of functions (2.2) converges to $x$, then equation (2.1) reduces to the well known Szász-Mirakjan operators (1.1).
3. Auxiliary results

In this section, we shall give some properties of the operators (2.1) which we shall use in the proof of the main theorem.

**Lemma 3.1.** Let \( e_i(x) = x^i, \ i = 0, 1, 2, 3, 4 \). Then for all \( x \geq 0 \), we have

1. \( S_{n,a}(e_0; x) = 1 \),
2. \( S_{n,a}(e_1; x) = \frac{x \log a}{(-1 + a \frac{1}{n})} \),
3. \( S_{n,a}(e_2; x) = \frac{x \log a \left(-1 + a \frac{1}{n} + x \log a\right)}{(-1 + a \frac{1}{n})^2} \),
4. \( S_{n,a}(e_3; x) = \frac{x \log a \left(-1 + a \frac{1}{n} + 3 \left(-1 + a \frac{1}{n}\right) x \log a + x^2 (\log a)^2\right)}{(-1 + a \frac{1}{n})^3} \),
5. \( S_{n,a}(e_4; x) = \frac{x \log a \left(-1 + a \frac{1}{n} + 7 \left(-1 + a \frac{1}{n}\right)^2 x \log a + 6 \left(-1 + a \frac{1}{n}\right) (x \log a)^2 + (x \log a)^3\right)}{(-1 + a \frac{1}{n})^4} \).

**Proof.** We have \( f \in C[0, \infty) \) and for each \( x \geq 0 \), then

1. \( S_{n,a}(e_0; x) = \sum_{k=0}^{\infty} a \left(-\frac{x}{1 + a \frac{1}{n}}\right) \frac{(x \log a)^k}{(-1 + a \frac{1}{n})^k k!} = a \left(-\frac{x}{1 + a \frac{1}{n}}\right) a \left(-\frac{x}{1 + a \frac{1}{n}}\right) = 1 \).
2. \( S_{n,a}(e_1; x) = \sum_{k=0}^{\infty} a \left(-\frac{x}{1 + a \frac{1}{n}}\right) \frac{(x \log a)^k}{(-1 + a \frac{1}{n})^k k!} \frac{k}{n} = a \left(-\frac{x}{1 + a \frac{1}{n}}\right) \sum_{k=0}^{\infty} \frac{(x \log a)^k}{(-1 + a \frac{1}{n})^k (k-1)!} \frac{k}{n} = a \left(-\frac{x}{1 + a \frac{1}{n}}\right) a \left(-\frac{x}{1 + a \frac{1}{n}}\right) \frac{(x \log a)}{(-1 + a \frac{1}{n})} \) \( = \frac{(x \log a)}{(-1 + a \frac{1}{n})} \).

Similarly, the other equalities can be proved. \( \square \)

**Lemma 3.2.** Let \( \lambda \geq 0 \). Then we have

\[ S_{n,a}(e^{\lambda x}; x) = \sum_{k=0}^{\infty} a \left(-\frac{x}{1 + a \frac{1}{n}}\right) \frac{(x \log a)^k}{(-1 + a \frac{1}{n})^k k!} e^{\lambda x} \] 
\[ = a \left(-\frac{x}{1 + a \frac{1}{n}}\right) e^{\lambda x} \]
4. Convergence theorems

**Theorem 4.1.** Let \( a > 1 \) then for all \( x \in [0, \infty) \) we have

1. \( S_{n,a}^*(f; x) \) is linear and positive on \( C_B[0, \infty) \),
2. \( S_{n,a}^*(a^i; x) = a^x \),
3. \( \lim_{n \to \infty} S_{n,a}^*(f; x) \to f \) uniformly on \([0, b]\), \( b > 0 \), provided \( f \in C_{\rho}[0, \infty) \), \( \rho \ge 2 \),

where \( C_B[0, \infty) \) is the space of all continuous and bounded functions defined on \([0, \infty)\).

**Theorem 4.2.** Let \( K \) be subspace of the Banach lattice \( C_2[0, \infty) \) defined as

\[ K = \{ f \in C[0, \infty) : \lim_{n \to \infty} f(x) \text{ is finite} \}. \]

Then \( \lim_{n \to \infty} S_{n,a}^*(f) \to f \) uniformly on \([0, \infty)\) provided \( f \in K \).

**Proof of Theorem 4.1.**

1. By above operators (2.3) we can see that \( S_{n,a}^*(f; x) \) is linear and positive.

2. Since we have the operators

\[
S_{n,a}^*(f; x) = \sum_{k=0}^{\infty} a \left( -\frac{x}{1+a} \right)^k \frac{(x \log a)^k}{(-1 + a^x)^k k!} f \left( \frac{k}{a} \right)
\]

then

\[
S_{n,a}^*(a^i; x) = \sum_{k=0}^{\infty} a \left( -\frac{x}{1+a} \right)^k \frac{(x \log a)^k}{(-1 + a^x)^k k!} a^x
\]

\[
= a \left( -\frac{x}{1+a} \right)^k \sum_{k=0}^{\infty} \frac{(xa^x \log a)^k}{(-1 + a^x)^k k!}
\]

\[
= a \left( -\frac{x}{1+a} \right)^k \exp \left( \frac{x a^x \log a}{-1 + a^x} \right).
\]

3. Before proving the third result or Theorem (4.2), we wish to generalize some properties, since we have to prove uniform convergence of the operators over any compact set \([0, b]\), \( b > 0 \) and extensively on \([0, \infty)\), then for \( b > 0 \) (fixed), consider the homomorphism \( F_b : C[0, \infty) \to C[0, b] \) (lattice homomorphism) defined as \( F_b(f) = f|_{[0,b]} \) for every \( f \in C[0, \infty) \). Moreover, in this case we can see that

\[
\lim_{n \to \infty} F_b(S_{n,a}^*(e_i)) = F_b(e_i), \quad \forall \ i = 0, 1, 2 \quad \text{and} \quad \lim_{n \to \infty} F_b(S_{n,a}^*(a^i)) = F_b(a^i), \quad a > 1.
\]

Now, the well known Korovkin type property with respect to monotone operators (Theorem 4.1.4 (vi) of [1], p. 199) is used. Let \( H \) be a cofinal subspace of \( C(Y) \), where \( Y \) is a compact set. If \( S : C(Y) \to C_2[0, \infty) \) is a lattice homomorphism and if \( \{L_n\} \) is a sequence of linear positive operators from \( C(Y) \) into \( C_2[0, \infty) \) such that \( \lim_{n \to \infty} L_n(h) = S(h), \quad \forall h \in H \), then \( \lim_{n \to \infty} L_n(f) = f \) provided that \( f \) belongs to the Korovkin closure of \( H \).

Hence, by (4.1) and above properties, we have (3) of Theorem (4.1) but extensively to get uniform convergence on \([0, \infty)\) of the sequence \( \{S_{n,a}^*(f)\} \), we have to add some other properties.

\( K \) is a subspace of the Banach lattice \( C_2[0, \infty) \) endowed with the sup-norm.

For a given \( \alpha > 0 \), let us suppose that the function \( f_{\alpha}(x) = a^{-\alpha x}, \quad (a > 1 \text{ and } x \ge 0) \), then we have

\[
S_{n,a}^*(f_{\alpha}; x) = \sum_{k=0}^{\infty} a \left( -\frac{x}{1+a} \right)^k \frac{(x \log a)^k}{(-1 + a^x)^k k!} a^{-\alpha x}
\]

\[
= a \left( -\frac{x}{1+a} \right)^k \sum_{k=0}^{\infty} \frac{(xa^x \log a)^k}{(-1 + a^x)^k k!}
\]

\[
= \exp \left( \frac{x a^x \log a}{-1 + a^x} \right).
\]

\[
= a \left( -\frac{x}{1+a} \right)^k \frac{(x \log a)^k}{(-1 + a^x)^k k!}.
\]
Observe that

\[ S_{n,a}(f_\alpha;x) = a \frac{(-1 - x \alpha)}{-1 + x \alpha} . \]

On the other hand, if we consider \( f_\alpha = \exp(-\alpha x) \), \( x \geq 0 \), then with the help of Lemma (3.2), we have

\[ S_{n,a}(f_\alpha;x) = a \frac{(-1 - x \alpha)}{-1 + x \alpha} . \]

Observe that

\[ \lim_{n \to \infty} S_{n,a}(f_\alpha) = f_\alpha \text{ uniformly on } [0, \infty). \]

Hence using this limit and applying the Proposition 4.2.5-(7) of ([1], p.215), we can obtain Theorem (4.2).

5. Error estimation

In this section, we will compute the rate of convergence of the given operators (2.3) and other equalities.

Let us define a function \( \phi_x(t) \), \( x \geq 0 \) by \( \phi_x(t) = (t - x) \) then by Lemma 3.1, we can get the following results:

**Lemma 5.1.** For each \( x \geq 0 \), we have:

1. \( S_{n,a}^*(\phi_x(t);x) = -\frac{1}{(-1 + \alpha^n)} x \left( -n - a^n \right) \),

2. \( S_{n,a}^*(\phi_x^2(t);x) = \frac{1}{(-1 + \alpha^n)^2} x \left( -1 - a^n \right)^2 n^2x - \left( 1 - 2n \right) \log a + x (\log a)^2 \),

3. \( S_{n,a}^*(\phi_x^3(t);x) = \frac{x}{(-1 + \alpha^n)^3n^3} \left( -1 + a^n \right)^2 - (-1 - a^n)^2 (1 - 3nx + 3n^2x^2) \log a \)

   \[ + 3(-1 + \alpha^n)(-1 + nx)(\log a)^2 - x^2 (\log a)^3, \]

4. \( S_{n,a}^*(\phi_x^4(t);x) = \frac{x}{(-1 + \alpha^n)^4n^4} \left( -1 + a^n \right)^4n^4 \)

   \[ - (-1 + a^n)^3 (1 + 4nx - 6n^2x^2 + 4n^3x^3) \log a \]

   \[ + (-1 + a^n)^2x(7 - 12nx + 6n^2x^2)(\log a)^2 \]

   \[ - 2(-1 + a^n)x(-3 + 2nx(\log a)^3 + x^3 (\log a)^4). \]

**Proof.** Since for each \( x \geq 0 \), we have

1. \( S_{n,a}^*(\phi_x(t);x) = \frac{\sum_{k=0}^{\infty} a \left( \frac{-x}{-1 + \alpha^n} \right) (x \log a)^k}{(-1 + \alpha^n)^k} \left( \frac{k}{n} - x \right) \)

   \[ = \frac{a \left( \frac{-x}{-1 + \alpha^n} \right)}{n} \sum_{k=0}^{\infty} \frac{(x \log a)^k}{(-1 + \alpha^n)^k(k - 1)!} - xa \left( \frac{-x}{-1 + \alpha^n} \right) \sum_{k=0}^{\infty} \frac{(x \log a)^k}{(-1 + \alpha^n)^kk!} \]

   \[ = \frac{a \left( \frac{-x}{-1 + \alpha^n} \right)}{n} \left( \frac{x \log a}{-1 + \alpha^n} \right) - xa \left( \frac{-x}{-1 + \alpha^n} \right) a \left( \frac{-x}{-1 + \alpha^n} \right) \]

   \[ = \frac{(x \log a)}{(-1 + \alpha^n)n} - x. \]
2. \( S_{n,a}^* \left( \phi_2^2(t); x \right) = \sum_{k=0}^{\infty} a \left( -\frac{x}{1 + a^\frac{1}{k}} \right) \frac{(x \log a)^k}{(1 + a^\frac{1}{k})^k k!} \left( \frac{k}{n} - x \right)^2 \)

\[
= a \left( -\frac{x}{1 + a^\frac{1}{n}} \right) \frac{n^2}{\sum_{k=0}^{\infty} (x \log a)^k k^2 (1 + a^\frac{1}{k})^k k!} - 2ax \left( -\frac{x}{1 + a^\frac{1}{n}} \right) \sum_{k=0}^{\infty} (x \log a)^k k^2 (1 + a^\frac{1}{k})^k k! 
+x^2 a \left( -\frac{x}{1 + a^\frac{1}{n}} \right) \sum_{k=0}^{\infty} (x \log a)^k (1 + a^\frac{1}{k})^k k! 
\]

\[
= a \left( -\frac{x}{1 + a^\frac{1}{n}} \right) \frac{n^2}{\sum_{k=0}^{\infty} (x \log a)^k k^2 (1 + a^\frac{1}{k})^k k!} \left( \frac{x \log a}{1 + a^\frac{1}{n}} \right) a \left( -\frac{x}{1 + a^\frac{1}{n}} \right) + x^2 
\]

\[
= \frac{x}{(1 + a^\frac{1}{n})^2} \sum_{k=0}^{\infty} (x \log a)^k (1 + a^\frac{1}{k})^k k! 
\]

Similarly, the other identities can be proved. \( \square \)

Let \( f \in C_B[0, \infty) \) and \( x \geq 0 \). Then the modulus of continuity of \( f \) is defined to be
\[
\omega(f, \delta) = \sup_{|t-x| \leq \delta} |f(t) - f(x)|, \quad t \in [0, \infty).
\]

Based on the modulus of continuity, we have the theorem:

**Theorem 5.1.** For every \( f \in C_B[0, \infty) \), the space of all continuous and bounded functions defined on \([0, \infty), x \geq 0 \) and \( n \in \mathbb{N} \), we have
\[
|S_{n,a}^*(f; x) - f(x)| \leq 2\omega(f, \delta_{n,x}),
\]
where \( \delta_{n,x} = \sqrt{l} \) and the value of \( l \) is given by (2) of Lemma 5.1.

**Proof.** Let \( f \in C_B[0, \infty) \) and \( x \geq 0 \), we have
\[
|f(t) - f(x)| \leq \omega(f, \delta_{n,x}) \left( \frac{|t-x|}{\delta_{n,x}} + 1 \right),
\]

now applying the operators \( S_{n,a}^* \) to both sides, we have
\[
S_{n,a}^*|f(t) - f(x)| \leq \omega(f, \delta_{n,x}) S_{n,a}^* \left( \frac{|t-x|}{\delta_{n,x}} + 1 \right)
\]

\[
\leq \omega(f, \delta_{n,x}) \left( S_{n,a}^* \frac{|t-x|}{\delta_{n,x}} + S_{n,a}^* \right)
\]

\[
= \omega(f, \delta_{n,x}) \left( \delta_{n,x}^{-1} S_{n,a}^* |t-x| + 1 \right)
\]

\[
\leq \omega(f, \delta_{n,x}) \left( \delta_{n,x}^{-1} \sqrt{S_{n,a}^* (t-x)^2} + 1 \right), \quad (\text{Using the Cauchy-Schwarz Inequality})
\]

\[
= \omega(f, \delta_{n,x}) \left( \delta_{n,x}^{-1} \sqrt{l} + 1 \right),
\]

where \( l \) is given by (2) of lemma (5.1).

\[
= \omega(f, \delta_{n,x}) \left( \delta_{n,x}^{-1} \delta_{n,x} + 1 \right), \quad \text{where} \ \delta_{n,x} = \sqrt{l}
\]
\[ \leq 2\omega(f, \delta_{n,x}). \]

6. Example, Graphical approach and analysis

In this section, we shall discuss the convergence properties by graphs of the defined operators (2.3) to given functions, and analyze the rate of convergence of the defined operators. At last, we shall show that the rate of convergence of the defined operators is better than Százs-Mirakjian operators \( S_n(f; x) \).

**Example 6.1.** Let \( f(x) = x(x - \frac{1}{4})(x - \frac{1}{4}) \), choosing \( a = 1.5 \), \( n = 1 \) to 10 and \( n = 100, 500 \) then the convergence to the function \( f(x) \) by the operators \( S^*_{n,a}(f; x) \) are illustrated in the figures. In that order, we shall see, the convergence of the operators \( S^*_{n,a}(f; x) \) to the function \( f(x) \) for particular value of \( n = 10 \) for different values of \( a > 1 \).

![Figure 1. Convergence of the operators \( S^*_{n,a}(f; x) \) to \( f(x) \)](image)

**Example 6.2.** For \( n = 5, 30, 300 \), the comparison of convergence of the defined operators \( S^*_{n,a}(f; x) \) and Százs-Mirakjian operators \( S_n(f; x) \) to \( f(x) = x^2 \exp 4x \) for fix value of \( a \) are shown in figures. But for same function \( f(x) \) the comparison is also take place by choosing different values of \( a > 1 \) as \( a = 15, 150, 1500 \).

![Figure 2. Convergence of the operators \( S^*_{n,a}(f; x) \) to \( f(x) \)](image)

![Figure 3. Comparison of the operators \( \hat{S}^*_{n,a}(f; x) \) and \( S_n(f; x) \)](image)
example. 6.3. Let \( f(x) = x^2 \cos 4x \) and \( a = 1500 \), for \( n = 10, 1000 \), the comparison of convergence of \( S_{n,a}^*(f; x) \) and Szász-Mirakjan operators \( S_n(f; x) \) are illustrated in figures.

**Conclusion** The convergence of the operators \( S_{n,a}^*(f; x) \) to the given function \( f(x) \) is shown by above figures by taking different values of \( n \) and fix \( a \). The rate of convergence is also effected by taking different values of \( a \), it means rate of convergence is also depend on \( a > 1 \), it can be observed by Figure 1, additionally, in figure 2, for fixing \( a \) as 1000, the convergence of the defined operators is took place. In Example 6.2, we put a comparison between \( S_{n,a}^*(f; x) \) and Szász-Mirakjan operators \( S_n(f; x) \) in sense of rate of convergence by both cases as one of them \( a > 1 \) is fix and in other case (in Figure 4) by taking different values of \( a > 1 \) (even though fixed), we compute the rate of convergence and for large value of \( n \) these operators converge to the function \( f(x) \). For \( n = 10, 1000 \), the rate of convergence of the defined operators is good as Szász-Mirakjan operators \( S_n(f; x) \), it can be seen in above figures (3, 4, 5).

7. A Voronovskaya-type theorem

To examine the asymptotic behavior of the operators (2.3), we will prove a Voronovskaya-type theorem for the operators \( S_{n,a}^* \). Before going to the main Theorem 7.1, we have the lemma:

**Lemma 7.1.**  \( \lim_{n \to \infty} n^2 S_{n,a}^*(\phi_2^4(t); x) = 3x^2 \), uniformly for each \( x \in [0, b] \), \( b > 0 \).

**Proof.** By Lemma 5.1, we have

\[
S_{n,a}^*(\phi_2^4(t); x) = \left( \frac{x}{(-1 + a \frac{x}{n})^4 n^3} \right) ((-1 + a \frac{x}{n})^4 n^4 x^3 \neg (-1 + a \frac{x}{n})^3 (-1 + 4nx - 6n^2 x^2 + 4n^3 x^3) \log a + (-1 + a \frac{x}{n})^2 x (7 - 12nx + 6n^2 x^2) (\log a)^2 - 2(-1 + a \frac{x}{n})^3 x^2 (-3 + 2nx) (\log a)^3 + x^3 (\log a)^4)
\]

Using Mathematica, we get

\[
\lim_{n \to \infty} n^2 S_{n,a}^*(\phi_2^4(t); x) = \lim_{n \to \infty} n^2 \left( \frac{x}{(-1 + a \frac{x}{n})^4 n^3} \right) ((-1 + a \frac{x}{n})^4 n^4 x^3
\]
where
\[ (7.3) \]

By inequality \((7.4)\)\footnote{\( \left( \frac{\phi(x)(7.2)}{f(t)} \right) \)}, we get
\[ (7.4) \]

Applying the Cauchy-Schwarz theorem to \( S_{n,a}(\phi(t,x)) \), we get
\[ (7.4) \]

Let \( \xi(t,x) = \xi(t;x) \) and \( \zeta(x,x) = 0 \) as \( \xi(x,x) = 0 \) and \( \zeta(x,x) \in C_\rho[0,\infty) \). But we know that (already proved)
\[ \lim_{n \to \infty} n S_{n,a}(f;x) = f(x) \text{ uniformly for each } x \in [0,b] \text{ and } b > 0. \]

So,
\[ \lim_{n \to \infty} n S_{n,a}(\xi(t;x)) = \lim_{n \to \infty} S_{n,a}(\zeta(t;x)) = \zeta(x,x) = 0. \]

By inequality (7.4), we get
\[ (7.5) \]

Now by (7.3), we have
\[ \lim_{n \to \infty} n \left( S_{n,a}(f;x) - f(x) \right) = \left( \frac{\phi(t,x)}{f(t)} \right) \]
Where,

\[
I_1 = \lim_{n \to \infty} n \frac{(-x(-n + a^n n - \log a))}{(-1 + a^n)n} = \lim_{n \to \infty} \frac{(-x(-1 + a^n - \log a))}{(-1 + a^n) \frac{1}{n}} = \lim_{p \to 0} \frac{(-x(-1 + a^p - p \log a))}{(-1 + a^p)p}, \text{ on replacing } \frac{1}{n} \text{ by } p.
\]

Using L. hospital rule (as \(\frac{0}{0}\) form) two times, we have

\[
I_1 = \lim_{p \to 0} \frac{-x a^p(\log a)^2}{a^p(\log a)^2 p + 2a^p \log a} = (-\frac{1}{2} x \log a),
\]

and

\[
I_2 = \lim_{n \to \infty} \frac{n}{2} \left( x \frac{(-1 + a^n)^2 n^2 x - (-1 + a^n)(-1 + 2nx) \log a + x(\log a)^2}{(-1 + a^n)^2 n^2} \right) = \lim_{l \to 0} \frac{1}{2} \left( x \frac{(-1 + a^l)^2 x - (-1 + a^l)(-l^2 + 2lx) \log a + x l^2(\log a)^2}{(-1 + a^l)^2 l} \right), \text{ on replacing } \frac{1}{n} \text{ by } l.
\]

Using L. hospital rule (as \(\frac{0}{0}\) form) three times, we have

\[
I_2 = \lim_{l \to 0} \frac{x (6a^2 x (\log a)^3 + (-1 + a^l) x a^l (\log a)^3 - a^l (\log a)^4 (-l^2 + 2lx) - 3a^l (\log a)^3 (-2l + 2x) + 6a^l (\log a)^2)}{2 (6a^2 (\log a)^3 l + 6a^2 (\log a)^2 + 6(-1 + a^l) a^l (\log a)^2 + 2(-1 + a^l) l a^l (\log a)^3)} = \frac{x}{2}.
\]

Using \(I_1\) and \(I_2\) in above equation, we have

\[
(7.6) \quad \lim_{n \to \infty} n (S_{n,a}^*(f; x) - f(x)) = -\frac{x}{2} (f'(x) \log a - f''(x)) \quad \square
\]

But for an unbounded interval, we generalize the above theorem by a corollary, given below:

**Corollary 7.1.** For each \(f \in C[0, \infty)\) such that \(f, f', f'' \in K\), we have

\[
\lim_{n \to \infty} n (S_{n,a}^*(f; x) - f(x)) = -\frac{x}{2} (f'(x) \log a - f''(x)),
\]

holds uniformly for all \(x \in [0, \infty)\).

**8. Comparison of new operators \(S_{n,a}^* f\) with the classical Szász-Mirakjan operators with respect to convexity**

A comparison of the new operators given by (2.3) with the Szász-Mirakjan operators will take place. And we will show that the present operators (2.3) have better rate of convergence under certain conditions such as generalized convexity.
DEFINITION 8.1. A function \( f(x) \) defined on \((a, b)\) is said to be convex with respect to \((e_0, \sigma_a(x))\) i.e. 
\((1, \sigma_a(x))\)-convex provided 
\[
\begin{vmatrix}
1 & 1 & 1 \\
\sigma_a(x_1) & \sigma_a(x_2) & \sigma_a(x_3) \\
f(x_1) & f(x_2) & f(x_3)
\end{vmatrix} \geq 0, \quad a < x_1 < x_2 < x_3 < b,
\]

Note that if such an \( f(x) \in C[a, b] \), then above inequality will hold, by continuity, for all \( a \leq x_1 \leq x_2 \leq x_3 \leq b \).
The set of all functions which satisfy (8.1) is denoted by \( \ell(1, \sigma_a(x)) \).

In general for strict convexity i.e. a function \( f \) is strictly \((1, \sigma_a(x))\) convex if above inequality is strictly greater than 0.

REMARK 8.1. A function \( f \in C^2[0, \infty) \) is convex with respect to the function \( \sigma_a(x) = a^x \), \( a > 1 \), iff
\[
f''(x) \geq (\log(a)) f'(x), \quad x \geq 0.
\]

PROOF. Using Remark (3.1) of [11], for which a function \( f \in C^2[0, 1] \) is convex with respect to any continuous and strictly increasing function \( \tau_1(x) \) iff
\[
f''(x) \geq \frac{\tau''_1(x)}{\tau'_1(x)} f'(x), \quad x \in [0, 1],
\]
but extensively on infinite interval, Acar et al. [17] shown a relation of the given function \( f \in C^2[0, \infty) \) with respect to exponential function.

Now replacing \( \tau_1(x) \) by \( a^x \), we have the above result.

By Corollary 7.1 and Remark 8.1, we have

COROLLARY 8.1. If given \( f \in C^2[0, \infty) \) is \((1, \sigma_a(x))\)-convex or \( \sigma_a(x)\)-convex with respect to \( \sigma_a(x) = a^x \), \( a > 1 \), for all \( x \in [0, \infty) \) then \( \exists N \in \mathbb{N} \), so that we have
\[
f(x) \leq S^*_n, a(f; x), \quad n \geq N.
\]
Here \( N \) is dependent on \( x \).

For main theorems, we have following theorem of Cheney and Sharma [5] (also see [4]).

THEOREM 8.2. If \( f \in C[0, \infty) \) is convex then we have
\[
f(x) \leq S_{n+1}(f; x) \leq S_n(f; x), \quad x \geq 0, \quad n \geq 1,
\]
as well as if \( f \) is increasing (decreasing), then \( S_n(f; x) \) is increasing (decreasing).

THEOREM 8.3. Let \( f \in C[0, \infty) \) is decreasing and convex for any \( x \in [0, \infty) \), then there exists \( N \in \mathbb{N} \) such that
\[
S^*_n, a(f; x) \geq S^*_{n+1, a}(f; x) \geq f(x), \quad n \geq N.
\]

PROOF. Before passing to the direct proof of the above theorem, we shall express the given operators (2.3) in form of Szász-Mirakjan operators. We can write \( S^*_n, a(f; x) = S_n(f; l_n) \), where
\[
l_n = (S_n(a^x; x))^{-1} \circ a^x.
\]
Since here \( a^x \), \( 0 \leq x < \infty \), \( a > 1 \) is a convex function and using Theorem 8.2, then we have
\[
S_n a^t \geq S_{n+1} a^t, \quad (S_{n+1} a^t)^{-1} \geq (S_n a^t)^{-1}, \quad l_{n+1} = (S_{n+1} a^t)^{-1} \circ a^x \geq (S_n a^t)^{-1} \circ a^x = l_n.
\]
i.e.,
\[
l_n(x) \leq l_{n+1}(x)
\]
Since \( f \in C[0, \infty) \) is convex and decreasing then we have
\[
f(x) \leq S_n(f; x) \leq S_{n+1}(f; x)
\]
\[
(8.3)
\]

Using (8.2), we have
\[ S^*_n,a(f; x) - S^*_{n+1,a}(f; x) = (S_n f) \circ (l_n(x)) - (S_{n+1} f) \circ (l_{n+1}(x)) \]
\[ = (S_n f) \circ (l_n(x)) - (S_{n+1} f) \circ (l_n(x)) + (S_{n+1} f) \circ (l_{n+1}(x)) \]
\[ - (S_{n+1} f) \circ (l_n(x)) \]
\[ \geq 0. \]

And hence
\[ S^*_n,a(f; x) \geq S^*_{n+1,a}(f; x), \quad \forall \ x \in [0, \infty), \ a > 1. \] (8.4)

Since last inequality of the theorem is a consequence of inequality (8.4) and Theorem 4.2.

**Theorem 8.4.** Let \( f \in C[0, \infty) \) be an increasing and \((1, a^x)\) convex with respect to \( a^x \), \( a > 1 \). Then
\[ f(x) \leq S^*_n,a(f; x), \quad x \in [0, \infty), \ a > 1, \ n \geq 1, \]
but also \( S_n(a^t) \geq a^x \) as \( a^x \) is convex and \((S_n(a^t))^{-1}\) is increasing, so we have
\[ (S_n(a^t))^{-1} \circ S_n(a^t) \geq (S_n(a^t))^{-1} \circ a^x, \]
\[ \Rightarrow x \geq ((S_n(a^t))^{-1} \circ a^x)(x), \]
(8.5)

Then from (8.5), it directly follows
\[ S^*_n,a(f; x) \leq S_n(f; x). \]

**9. An Extension**

In a further investigation we obtain approximations by linear positive operators using new Kantorovich-type operators in the space of integrable function. The Kantorovich type operators of the operators (2.3) are given as bellow:

\[ \tilde{S}^*_n,a(f; x) = n \sum_{k=0}^{\infty} a \left( \frac{-x^{1+ \frac{1}{a}}}{1+ \frac{1}{a}} \right) \frac{(x \log a)^k}{(-1 + a^x)^k k!} \int_{\frac{1}{a}}^{x} f(t) dt. \] (9.1)

Here we get the following identities as given bellow.

1. \( \tilde{S}^*_n,a(e_0; x) = 1, \)
2. \( \tilde{S}^*_n,a(e_1; x) = \frac{1}{2n} + \frac{x \log a}{(-1 + a^x)n}, \)
3. \( \tilde{S}^*_n,a(e_2; x) = \frac{1}{3n^2} + \frac{2x \log a}{(-1 + a^x)n^2} + \frac{x^2 \log a^2}{(-1 + a^x)^2 n^2}. \)

With regard of the above integral operators (9.1), we raise the problem to investigate their convergence in \( L_p \)-spaces.

Here, by the above identities, we get
\[ \lim_{n \to \infty} \tilde{S}^*_n,a(e_i) = e_i, \quad e_i = x^i, \ i = 0, 1, 2. \]
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