Mobile Robot Path Following Control in 2D Using a 3D Guiding Vector Field: Singularity Elimination and Global Convergence

Weijia Yao, Héctor García de Marina and Ming Cao

Abstract—In vector-field-based path following (VF-PF), a vector field is designed and utilized to guide a robot to follow a desired path. VF-PF algorithms have been shown to achieve high path-following precision under small control effort compared with many other path following algorithms. However, due to the existence of singular points where the vector field vanishes, the global convergence of VF-PF algorithms to the desired path cannot be guaranteed. Moreover, as it holds for most of the existing path following algorithms, VF-PF algorithms may fail to enable following a self-intersected desired path. In this paper, we propose a method to assist generic VF-PF algorithms in their global convergence. In particular, our method creates vector fields free of singular points in a higher-dimensional space containing the lower-dimensional desired path. Subsequently, the projection of our vector fields on the lower-dimensional space of the desired path can be exploited to guarantee the global convergence of the robot to the desired path, including those self-intersected. Finally, we show that our algorithm combines and fairly extends features from existing path following algorithms and a nonlinear trajectory tracking algorithm. The theoretical results have been validated by experiments and simulations.

I. INTRODUCTION

The path following control problem deals with finding suitable guidance for a mobile robot to converge to and move along a desired geometric path. Path following is a basic function for many mobile robots [1], and recently related new applications have emerged, e.g., to probe atmospheric phenomena by drones [2]. Among many path following algorithms, vector-field-based path following (VF-PF) algorithms are promising as they can achieve small path-following errors and use less control effort [3]. In VF-PF algorithms, a vector field is carefully designed such that its integral curves converge to the desired path [4], [5]. Such a vector field is also known as a guiding vector field (GVF) [4], since a desired velocity at each point of the field guides the robot.

One can find in the literature a variety of robot-specific VF-PF algorithms [3], [6], [7], [8], [9], [5], [10]. However, the encountering of singular points where a vector field becomes zero may compromise the global convergence to the desired path. Furthermore, the presence of such singular points increases the difficulty in the analysis of the algorithm. For example, if a 2D desired path is a simple closed curve, then this limitation is inherent: there is always at least one singular point within the region enclosed by the desired path due to the Poicaré-Bendixson theorem [11]. Therefore, if the desired path is a circle, a robot cannot start and then escape from the centre since it is a singular point of the 2D guiding vector field [4]. In addition, normalization of the vector field at this point is not well-defined. To the best of our knowledge, few studies deal with singular points of guiding vector fields. Some studies, e.g., [8], [10], assume that the singular points are repulsive to simplify the analysis. This assumption is dropped in [4] for the 2D case, and the study concludes that the extensibility [12] of the integral curves might be finite if there are singular points. In general, path following algorithms (e.g. [13]) only guarantee convergence locally within the vicinity of the desired path.

Following self-intersected desired paths is not feasible for generic VF-PF algorithms. This is because the crossing points of the desired path are also singular points of the vector field; i.e., no valid guiding directions are available at the intersections. Thus, a robot gets stuck at the crossing points on the desired path (see Fig. 1). The algorithms in [4], [5], [8] are ineffective simply due to the violation of the assumption: no singular points are allowed on the desired path. In fact, many other path following algorithms are only applied to simple desired paths such as a straight line or a circle [3], [14], [6], and have not addressed the problem of following a self-intersected desired paths. For example, the line-of-sight (LOS) method [13] is not applicable in this case as there is not a unique projection point in the vicinity of a crossing point of the desired path.

The contributions of our paper are summarized as follows: we propose a new idea to eliminate singular points in guiding vector fields. Consequently, the proposed VF-PF algorithm deals with self-intersected desired paths successfully. Therefore, we substantially extend and improve the performance of the VF-PF algorithms in [4], [5]. In particular, we construct a 3D vector field that is free of singular points to assist the robot in following a 2D self-intersected desired path. Thanks to the singularity-free vector field, global convergence to the desired path, self-intersected or not, is rigorously guaranteed. Moreover, we show that our proposed VF-PF algorithm is a significant extension of both existing VF-PF algorithms and trajectory tracking algorithms. Due to the page limit, we only consider planar self-intersected desired paths. However, the proposed idea can be easily generalized to desired paths in a higher-dimensional space.

This paper is organized as follows. Section II introduces the 2D vector field and the motivation of our work. Then in Section III, the 3D vector field and the problem definition are presented. The main results are elaborated in Section IV. In Section V, an experiment is carried out, and the comparison
with a trajectory tracking algorithm is presented in Section VI. Finally, Section VII concludes the paper.

II. Motivation

In the VF-PF problem, the desired path to follow is usually described by the zero-level set of a sufficiently smooth function. In particular, in the 2D Euclidean space $\mathbb{R}^2$, the desired path is described by the following set [4]:

$$\Phi = \{(x, y) \in \mathbb{R}^2 : \bar{\phi}(x, y) = 0\}, \quad (1)$$

where $\bar{\phi} : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable to guarantee the existence and uniqueness of the integral curves of the guiding vector field [11, Theorem 3.1] introduced in the sequel. Note that this description of the desired path without any parametrization is common in the field of VF-PF control [5], [8], [10], [15], [16], [17], [18]. We can exploit the description (1) by using the absolute value of the level set $|\bar{\phi}(t)|$ instead of the Euclidean distance $\text{dist}(\xi, \Phi) := \inf\{||\xi - p|| : p \in \Phi\}$ between a robot’s position $\xi \in \mathbb{R}^2$ and the desired path $\Phi$ to measure how far the robot is away from the desired path. For example, when $|\bar{\phi}(\xi)| = 0$, the robot is precisely on the desired path. We can see the 2D VF-PF problem as the design of a continuously differentiable vector field $\nabla \phi : \mathbb{R}^2 \to \mathbb{R}^2$ to show up on the right-hand side of the differential equation $\dot{x}(t) = \nabla \phi(x(t))$. The design requires to satisfy two conditions: (1) There exists a neighborhood $\mathcal{D} \subset \mathbb{R}^2$ of the desired path $\Phi$ in (1) such that for all initial conditions $\xi(0) \in \mathcal{D}$, the distance $\text{dist}(\xi(t), \Phi)$ between the solution $\xi(t)$ and the desired path $\Phi$ approaches zero as time $t \to \infty$; that is, $\lim_{t \to \infty} \text{dist}(\xi(t), \Phi) = 0$. (2) If a solution starts from the desired path, then the solution stays on the path for $t \geq 0$ (i.e., $\xi(0) \in \Phi \implies \xi(t) \in \Phi$ for all $t \geq 0$). Furthermore, the vector field on the desired path is nonzero.

The solutions to the differential equation $\dot{x}(t) = \nabla \phi(x(t))$ are the integral curves of the vector field $\nabla \phi$. Therefore, one aims to find a suitable vector field $\nabla \phi$ such that the integral curves approach the desired path eventually and stay on the desired path once it starts from it. For simplicity, we say that the integral curves of the vector field that satisfy the above two conditions assist a robot in converging to and travelling along the desired path. One example of such a vector field is given in [4]:

$$\nabla \phi(x, y) = E \nabla \bar{\phi}(x, y) - k \psi(\bar{\phi}(x, y)) \nabla \bar{\phi}(x, y), \quad (2)$$

where $E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is the $90^\circ$ rotation matrix, $k$ is a positive constant, $\psi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $\psi(0) = 0$, and $\nabla (\cdot)$ denotes the gradient of a scalar function $(\cdot)$. Note that one of the assumptions is that there are no singular points of the vector field on the desired path. However, the vector field corresponding to a self-intersected desired path must have at least one singular points on the desired path as stated in the following proposition.

**Proposition 1.** Denote a crossing point of the desired path $\Phi$ in (1) by $c \in \Phi$. Then $c$ is a singular point of the vector field in (2); that is, $\nabla \phi(c) = 0$.

**Proof.** Since $c \in \Phi$, we have $\bar{\phi}(c) = 0$, and thus $\chi(c) = E \nabla \bar{\phi}(c)$ in view of (2). Next we show that the gradient at the crossing point $\nabla \bar{\phi}(c)$ is zero; hence $\chi(c) = 0$. Suppose, on the contrary, the gradient is not zero. Since $\nabla \phi$ is continuously differentiable (as $\bar{\phi} \in C^2$), one can use the implicit function theorem [19] to conclude that there is a unique curve in a neighborhood of $c$ satisfying $\bar{\phi}(x, y) = 0$. But this contradicts the fact that $\Phi$ is a self-intersected path. Therefore, the gradient at the crossing point is indeed zero, resulting in $\chi(c) = 0$. \qed

In Fig. 1, one can check that the vector field at the crossing point is zero. Intuitively, this is due to the loss of directionality information at the crossing point; there is no obvious preference regarding which direction the trajectory should go towards at the crossing point. Thus, it is impossible to create a 2D vector field of which the integral curves follow the self-intersected desired path. In the following section, we propose a solution to this inherent limitation by designing a higher-dimensional (3D) guiding vector field.

III. Problem Formulation

We propose to add an additional dimension to the original 2D guiding vector field such that we can “record” the directionality information. Thus, even at the crossing points, the additional dimension of the vector field can indicate the preferred direction of motion.

Starting from the description of the 2D desired path $\Phi$ in (1), the (virtual) 3D desired path $\Phi$ in $\mathbb{R}^3$ is naturally characterized by adding an additional constraint in $\mathbb{R}^3$ as follows:

$$\Phi = \{\xi \in \mathbb{R}^3 : \phi_1(\xi) = 0, \phi_2(\xi) = 0\}, \quad (3)$$

where $\phi_1, \phi_2 \in C^2$, $\xi = (x, y, w) \in \mathbb{R}^3$ and $w$ is the additional dimension of the vector field. This implies that the desired path $\Phi$ is the intersection of two surfaces described by $\phi_i = 0, i = 1, 2$. In our previous work [5], the corresponding 3D guiding vector field $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ that solves the path following problem is

$$\chi(\xi) = \nabla \phi_1(\xi) \times \nabla \phi_2(\xi) - \sum_{i=1}^{2} k_i \phi_i(\xi) \nabla \phi_i(\xi). \quad (4)$$

The singular set of the vector field which is a set of all singular points will be discussed in the sequel, and is formally given below:

$$C = \{c \in \mathbb{R}^3 : \chi(c) = 0\}. \quad (5)$$

![Fig. 1](image-url)
Note that the guiding vector field in (4) is nonlinear, and thus the differential equation \( \xi(t) = \chi(\xi(t)) \) is generally difficult to analyze. However, as shown in [5], under some mild assumptions, the integral curves of the 3D vector field only have two outcomes; this is stated in the following lemma.

**Lemma 1** ([5]). The integral curves of the guiding vector field \( \chi \) in (4) either converge to the desired path \( P \) in (3), or converge to the singular set \( C \) in (5).

Since the final objective is to follow the 2D desired path \( P' \), one needs to project the (virtual) higher-dimensional desired path \( P' \) to reduce the degrees of freedom of its ambient space. More specifically, a linear projection operator \( P_a : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is defined; this operator is a linear transformation that can project a vector to the hyperplane orthogonal to a given nonzero vector \( a \in \mathbb{R}^3 \). In particular,

\[
P_a = I - \hat{a} \hat{a}^T,
\]

where \( I \) is the identity matrix of suitable dimensions and \( \hat{a} := a/\|a\| \). Now we can define the projected desired path and the projected singular set as follows:

\[
P \prime = \{ q \in \mathbb{R}^3 : q = P_a \xi, \xi \in P \} \quad \text{(7)}
\]

\[
C' = \{ q \in \mathbb{R}^3 : q = P_a \xi, \xi \in C \}. \quad \text{(8)}
\]

Therefore, to get the integral curves of a guiding vector field follow a self-intersected 2D desired path, it is desirable that there are no singular points in the (higher-dimensional) guiding vector field (due to Proposition 1), which also implies the appealing feature of global convergence to the desired path (since \( C = \emptyset \) in Lemma 1). To sum up, the problem is formally formulated as follows:

**Problem 1.** Given a (possibly self-intersected) desired path \( P \subset \mathbb{R}^2 \), one aims to find a higher-dimensional desired path \( P \subset \mathbb{R}^3 \), which satisfies the following conditions:

1) There exist functions \( \phi_i(\cdot) \in C^2 \) such that \( P \) is described by (3).

2) There exists a projection operator \( P_a \) in (6) such that the projected desired path \( P' \) in (7) satisfies \( P' = \{ (x, y, 0) \in \mathbb{R}^3 : (x, y) \in P \} \).

3) The singular set \( C \) of the corresponding higher-dimensional vector field \( \chi : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) in (4) is empty.

In the next section, we propose a new idea to seek such a higher-dimensional (virtual) desired path \( P \subset \mathbb{R}^3 \). Once such a 3D desired path is obtained, the 3D vector field will be automatically generated by (4).

**IV. SELF-INTERSECTED DESIRED PATH FOLLOWING**

This section is split into three subsections regarding the solution to Problem 1. Firstly, we show that if a vector field \( \chi \) converges to the 3D (virtual) desired path \( P \), then its projected dynamics converges to the 2D desired path \( P' \).

Secondly, we provide the detailed construction of the 3D (virtual) desired path satisfying the conditions in Problem 1. Thirdly, based on the 3D vector field, a control algorithm is designed for a unicycle robot to follow a self-intersected desired path.

### A. Projected Dynamics

A general result about the integral curves of the projected vector field \( P_a \chi \) is presented as below.

**Lemma 2** (Projected dynamics). Let \( \chi : \mathcal{D} \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be a vector field that is locally Lipschitz continuous. Suppose \( \xi(t) \) is the unique solution to the initial value problem \( \xi(t) = \chi(\xi(t)), \xi(0) = \xi_0 \in \mathcal{D} \). Then \( \xi(t), \eta(t) \), where \( \eta(t) = P_a \xi(t) \) and \( P_a \) is the projection operator in (6) associated with a given nonzero vector \( a \in \mathbb{R}^3 \), is the unique solution to the following initial value problem:

\[
\begin{cases}
\dot{\xi}(t) = \chi(\xi(t)) \\
\dot{\eta}(t) = P_a \chi(\xi(t))
\end{cases}
\]

Moreover, if the solution \( \xi(t) \) converges to some set \( \mathcal{A} \neq \emptyset \subset \mathbb{R}^3 \), then the projected solution \( \eta(t) \) converges to the projected set \( \mathcal{A}' = \{ q \in \mathbb{R}^3 : q = P_a \xi, \xi \in \mathcal{A} \} \).

### Proof.

Since \( \chi \) is locally Lipschitz and \( \| P_a \| = 1 \), where \( \| \cdot \| \) is the induced matrix 2-norm, it follows that \( P_a \chi \) is also locally Lipschitz continuous. Therefore, one observes that \( (\xi(t), \eta(t)) \), where \( \eta(t) = P_a \xi(t) \), is the unique solution to (9) ([12]). Fix \( t \), then dist\((\eta(t), \mathcal{A}') = \inf\{\| P_a(\xi(t) - q) \| : q \in \mathcal{A} \} \leq \inf\{\| P_a \| \| \xi(t) - q \| : q \in \mathcal{A} \} = \text{dist}(\xi(t), \mathcal{A}). \)

Since \( \xi(t) \) converges to \( \mathcal{A} \), dist\((\xi(t), \mathcal{A}) \rightarrow 0 \) as \( t \rightarrow \infty \). Namely, for any \( \epsilon > 0 \), there exists a \( T > 0 \), such that for all \( t \geq T \), dist\((\xi(t), \mathcal{A}) < \epsilon \); hence dist\((\eta(t), \mathcal{A}') \leq \text{dist}(\xi(t), \mathcal{A}) < \epsilon \). Therefore, dist\((\eta(t), \mathcal{A}') \rightarrow 0 \) as \( t \rightarrow \infty \).

Thus the projected solution \( \eta(t) \) converges to the projected set \( \mathcal{A}' \).

One of the simplest projection operators \( P_a \) is associated with the vector \( a = (0, 0, 1)^T \). This can be used to project a three-dimensional vector to a 2D plane by “zeroing” the third coordinates. For the problem of path following, we can therefore design a suitable 3D vector field such that the integral curves of the first two components of the 3D vector field follow the projected 2D desired path. The advantage of this method is that singular points can be eliminated and the global convergence to the desired path is guaranteed as discussed in the following subsection.

### B. Construction of the 3D Virtual Desired Path

Suppose a 2D desired path \( \mathcal{P} \) is parameterized by

\[
x = f_1(w), \quad y = f_2(w),
\]

where \( w \in \mathbb{R} \) is the parameter of the path and \( f_i \in C^2, i = 1, 2 \). Then, we can simply let

\[
\phi_1(x, y, w) = x - f_1(w), \quad \phi_2(x, y, w) = y - f_2(w),
\]

so that the 3D virtual desired path is described by (3) and (11). Intuitively, the 3D desired path \( P \) is obtained by
stretches the 2D desired path $\mathcal{P}$ along the virtual $w$-axis (see Fig. 2). One can check that $\nabla \phi_1 = (1, 0, -f_1'(w))^\top$ and $\nabla \phi_2 = (0, 1, -f_2'(w))^\top$, where $f_i'(w) := \frac{df_i(w)}{dw}, i = 1, 2$. Thus

$$\nabla \phi_1 \times \nabla \phi_2 = (f_1'(w), f_2'(w), 1)^\top.$$ 

It is interesting to note that the third coordinate of this vector is a constant 1 regardless of the specific form of the desired path. This means that $\|\nabla \phi_1 \times \nabla \phi_2\| \neq 0$ in $\mathbb{R}^3$ globally. A closer examination of the guiding vector field (4) reveals that the first term $\nabla \phi_1 \times \nabla \phi_2$ is orthogonal to the second term $\sum_{i=1}^2 k_i \phi_i^2 \nabla \phi_i$ due to the property of the cross product. Therefore, since $\|\nabla \phi_1 \times \nabla \phi_2\| = 0$ in $\mathbb{R}^3$ globally, an appealing property is that the vector field $\chi(\xi)$ is not globally in $\mathbb{R}^3$. This means that the guiding vector field $\chi$ has no singular points. Furthermore, one also observes that the 2D desired path $\mathcal{P}$ is the projection of the 3D (virtual) desired path $\mathcal{P}$ on the plane $w = 0$. Therefore, the projection operator $P_w$ can be chosen to associate with the vector $\alpha = (0, 0, 1)^\top$. Thus, the conditions in Problem 1 are all satisfied. This result is formally stated in the following theorem.

**Theorem 1.** Consider a 2D desired path $\mathcal{P} \subset \mathbb{R}^2$ parametrized by (10). Let $\phi_1$ and $\phi_2$ be chosen as in (11). Then, there are no singular points in the corresponding three-dimensional vector field $\chi : \mathbb{R}^3 \to \mathbb{R}^3$. Let $\alpha = (0, 0, 1)^\top$ for the projection operator $P_w$. Suppose the projected solution to (9) is $\eta(t) := (x(t), y(t), w(t))^\top$. Then the 2D trajectory $\eta_2(t) := (x(t), y(t))^\top$ will globally asymptotically converge to the 2D desired path $\mathcal{P}$ as $t \to \infty$.

**Proof.** From (4) and (11), the 3D vector field $\chi$ is

$$\chi(x, y, w) = \begin{bmatrix} f_1'(w) - k_1 \phi_1 \\ f_2'(w) - k_2 \phi_2 \\ 1 + k_1 \phi_1 f_1'(w) + k_2 \phi_2 f_2'(w) \end{bmatrix}.$$ 

As discussed before, we have that the singular set $\mathcal{C} = \emptyset$. According to Lemma 1 and Lemma 2, together with $\mathcal{C} = \emptyset$, then the projected solution $\eta(t)$ will globally asymptotically converge to the projected desired path $\mathcal{P}' := \{q \in \mathbb{R}^3 : q = P_w \xi, \xi \in \mathcal{P}\}$, where $\mathcal{P}$ is defined by (3) and (11). Since $a^\top \eta = a^\top P_w \xi = 0$, the third coordinate of the projected solution $w(t) \equiv 0$, meaning that the trajectory $\eta_2(t)$ lies on the $XY$-plane. Therefore, the 2D trajectory $\eta_2(t) := (x(t), y(t))^\top$ will globally asymptotically converge to the 2D desired path $\mathcal{P}$. \hfill $\Box$

**Remark 1.** The additional coordinate $w$ is reminiscent of the time variable $t$ in trajectory tracking algorithms, but there are major differences. In trajectory tracking, a robot tracks a desired trajectory point which moves independently of the robot states along the desired trajectory. namely, the dynamics of time is open loop $\dot{t} = 1$. In our proposed approach, the point $(f_1(w), f_2(w))$ can be roughly regarded as a counterpart of the trajectory point, but note that the dynamics of $w$, i.e., $\dot{w}(t) = \gamma_3(\xi(t))$, where $\gamma_3$ is the third entry of the vector field, is dependent on the robot state, and acts in a closed-loop manner. We also note that the robot is not tracking the point $(f_1(w), f_2(w))$. The details and a comparison study are presented in Section VI.

**Remark 2.** It is important to highlight three advantages of our approach.

1) Our approach has successfully transformed a possibly self-intersected 2D desired path $\mathcal{P}$ to a non-self-intersected 3D virtual desired path $\mathcal{P}$ by stretching along the virtual coordinate $w$.

2) All the singular points that might exist in the original 2D guiding vector field $\mathcal{X}$ vanish in the 3D guiding vector field $\chi$. Due to the singularity-free vector field $\chi$, the global convergence to the 3D virtual desired path is guaranteed, so it is for the 2D desired path by the projected dynamics. We remark that most of the path-following algorithms in the literature can only guarantee local convergence.

3) Our approach enables one to easily find out the surface functions $\phi_i$ of which the intersection of the zero-level sets is the 3D desired path $\mathcal{P}$ (see (3)), once a parametric form of the desired path is given.

### C. Control Algorithm Design for a Unicycle Robot

We now exploit the property of global convergence to the 2D desired path by the integral curves of the guiding vector field. In particular, we will design a control algorithm to command a mobile robot such that it eventually aligns with our proposed guiding vector field. Note that this idea is applicable to any robots whose motion is essentially determined by its orientation, such as the unicycle model (including the Dubin’s car model), the car-like model and the underwater glider model [1]. For simplicity, we consider the unicycle robot model as follows:

$$\dot{x} = v_u \cos \theta \quad \dot{y} = v_u \sin \theta \quad \dot{\theta} = \omega_u,$$ 

where $(x, y)$ is the position, $\theta$ is the orientation, $v_u$ is the speed control input and $\omega_u$ is the angular velocity control input. Since the 3D vector field will be used, the generalized 3D velocity vector of the robot needs to be defined as $\xi = (\dot{x}, \dot{y}, \dot{w})^\top$, where $(\dot{x}, \dot{y})$ is the actual velocity of the robot as defined in (12) and $\dot{w}$ is the virtual velocity in the additional coordinate that is to be determined later. We aim to design
A control input $v_u$, $\omega_u$ and the virtual velocity $\dot{w}$ such that the orientation and the length of the generalized velocity $\xi = (\dot{x}, \dot{y}, \dot{w})^T$ will asymptotically be the same as the scaled vector field $s \hat{x}$ (i.e., $\xi \rightarrow s \hat{x}$ as $t \rightarrow \infty$), where $s$ is a given positive constant. Thus the actual robot trajectory $(x(t), y(t))$ is proved to converge to a given 2D desired path $\mathcal{P}$ as the following proposition states.

**Proposition 2.** Suppose a 2D parameterized desired path $\mathcal{P} \subset \mathbb{R}^2$ and the corresponding 3D vector field $\chi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as constructed in Section IV are given, and the robot model in (12) is considered. Assume that the vector field satisfies $\chi_1(\xi)^2 + \chi_2(\xi)^2 \neq 0$ for $\xi \in \mathbb{R}^3$, where $\chi_i$ denotes the $i$-th entry of the vector field $\chi$. Let

$$\dot{w} = s \hat{x}_3, \quad (13a)$$

$$v_u = s \sqrt{\hat{x}_1^2 + \hat{x}_2^2}, \quad (13b)$$

$$\omega_u = \hat{\theta}_d - k_0 \hat{h}^T E \hat{g}^{'}, \quad (13c)$$

$$\hat{\theta}_d = \left( \frac{-1}{\|g\|} g^T E J(g) \hat{\xi} \right), \quad (13d)$$

where $\hat{\cdot}$ is the normalization operator, $s$ and $k_0$ are positive constants, $h' = v_u (\cos \theta, \sin \theta) \hat{\xi}^T$, $g' = s(\hat{x}_1, \hat{x}_2)^T$. $E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $J(g')$ is the Jacobian matrix of $g'$ with respect to the generalized position $\xi \in \mathbb{R}^3$ and $\hat{\xi} = (\dot{x}, \dot{y}, \dot{w})^T$ is the generalized velocity. Denote the angle directed from $g'$ to $\hat{h}'$ by $\beta \in (-\pi, \pi)$. If the initial angle $\beta(0) \in (-\pi, \pi)$, then the generalized velocity $\hat{\xi}$ will converge asymptotically to the scaled vector field $s \hat{x}$ as $t \rightarrow 0$. Furthermore, the actual robot trajectory $(x(t), y(t))$ will converge to the desired path $\mathcal{P} \subset \mathbb{R}^2$ asymptotically as $t \rightarrow \infty$.

**Proof.** We define the error (difference) between the generalized velocity $\hat{\xi}$ and the scaled vector field $s \hat{x}$ as below:

$$e(t) = \hat{\xi} - s \hat{x} = \begin{bmatrix} v_u \cos \theta - s \hat{x}_1 \\ v_u \sin \theta - s \hat{x}_2 \\ 0 \end{bmatrix} = \begin{bmatrix} h' - g' \\ 0 \end{bmatrix},$$

where the 0 entry is due to (13a). Thus we only need to focus on the first two entries of $e(t)$. Note that $\|h'\| = s \sqrt{\hat{x}_1^2 + \hat{x}_2^2} = \|g'\|$, thus it is possible to proceed to show that $e(t) \rightarrow 0$ asymptotically. Let $e' = \hat{h}' - \hat{g}'$, where $\hat{h}' = (\cos \theta, \sin \theta)^T$ and $\hat{g}' = (\hat{x}_1, \hat{x}_2)^T / \sqrt{\hat{x}_1^2 + \hat{x}_2^2}$. Choose the Lyapunov function candidate $V = 1/2 e'^T e'$ and its time derivative is

$$\dot{V} = e'^T e' = (\hat{\theta} E \hat{h}' - \hat{\theta}_d \hat{E}_g')^T (\hat{h}' - \hat{g}')$$

$$= (\hat{\theta} - \hat{\theta}_d)(\hat{h}'^T E \hat{g}')$$

$$= -k_0 (\hat{h}'^T E \hat{g}')^2 \leq 0. \quad (14)$$

The second equation makes use of the identities: $\hat{h}' = \hat{\theta} E \hat{h}'$ and $\hat{g}' = \hat{\theta}_d \hat{E}_g'$, where $\hat{\theta}_d$ is defined in (13d). The third equation is derived by exploiting the facts that $E^T = -E$ and $a^T Ea = 0$ for any vector $a \in \mathbb{R}^2$. Note that $\dot{V} = 0$ if and only if the angle difference between $\hat{h}'$ and $\hat{g}'$ is $\beta = 0$ or $\beta = \pi$. Since it is assumed that the initial angle difference $\beta(t = 0) \neq \pi$, it follows that $\dot{V}(t = 0) < 0$, and hence there exists a sufficiently small $\epsilon > 0$ such that $V(t = \epsilon) < V(t = 0)$. It can be shown by contradiction that $|\beta(t)|$ is monotonically decreasing with respect to time $t$. Suppose there exist $0 < t_1 < t_2$ such that $|\beta(t_1)| < |\beta(t_2)|$. It can be calculated that $V(t) = 1 - \cos \beta(t)$, and thus $V(t_1) < V(t_2)$, contradicting the decreasing property of $V$. Thus $|\beta(t)|$ is indeed monotonically decreasing. By (14), one observes that $|\beta(t)|$ and $V(t)$ tends to 0, implying that the generalized velocity $\hat{\xi}$ will converge asymptotically to the scaled vector field $s \hat{x}$. Therefore, the generalized trajectory $(x(t), y(t))$ will converge to the higher-dimensional desired path $\mathcal{P}$ constructed in Section IV. Then by Theorem 1, the actual robot trajectory (i.e., the projected solution) $(x(t), y(t))$ will converge to the desired path $\mathcal{P} \subset \mathbb{R}^2$ asymptotically as $t \rightarrow \infty$. □

**Remark 3.** A major feature of this algorithm is the introduction of a virtual coordinate and its dynamics. These do not correspond to any physical quantities in practice, but it is important to form a higher-dimensional dynamics as shown in Lemma 2. By setting the third entry of the generalized velocity to be the same as the corresponding entry of the scaled vector field, it has been ensured that the third entries are always “aligned” and thus the algorithm mainly deals with the alignment of the other two entries with the scaled vector field.

**V. Experiment**

We use an e-puck robot [20] to follow a self-intersected desired path: the projection of a trefoil knot that is parameterized by $x = \cos(0.02w)(80 \cos(0.03w) + 160) + 600$, $y = \sin(0.02w)(80 \cos(0.03w) + 160) + 350$, where $w$ is the parameter of the desired path. For convenience, we use pixels as the distance unit. The robot has on top a data matrix, which is recognized by an overhead camera to obtain the robot’s position and orientation. The camera is connected to a computer, on which the control algorithm is implemented. Then, the linear velocity and angular velocity input $v_u$ and $\omega_u$ are transmitted by the computer via a Bluetooth module.

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**Algorithm 1: Proposed VF-PF control algorithm**

**Input:** parametrization of the desired path in (10);
1. initialized the virtual coordinate $u(0) = u_0$;
2. while stop signal not received do
3. obtain robot states $(x, y, \theta)$ and velocity $(\dot{x}, \dot{y})$;
4. calculate 3D vector field $\chi$ by (4) and (11);
5. calculate $\dot{w}$ by (13a) and let $\hat{\xi} = (\dot{x}, \dot{y}, \dot{w})^T$;
6. calculate $v_u$ and $\omega_u$ by (13b) and (13c);
7. apply the control inputs $v_u$ and $\omega_u$ to the robot;
8. update $w \leftarrow w + \omega \Delta t$, where $\Delta t$ is the iteration lapse.

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#### V. EXPERIMENT

We use an e-puck robot [20] to follow a self-intersected desired path: the projection of a trefoil knot that is parameterized by $x = \cos(0.02w)\cdot(80 \cos(0.03w) + 160) + 600$, $y = \sin(0.02w)\cdot(80 \cos(0.03w) + 160) + 350$, where $w$ is the parameter of the desired path. For convenience, we use pixels as the distance unit. The robot has on top a data matrix, which is recognized by an overhead camera to obtain the robot’s position and orientation. The camera is connected to a computer, on which the control algorithm is implemented. Then, the linear velocity and angular velocity input $v_u$ and $\omega_u$ are transmitted by the computer via a Bluetooth module.
at a fixed frequency of 20 Hz to the robot. The initial configuration of the robot is: \((x(0), y(0), \theta(0)) = (923, 545, \pi)\) and the initial value of the additional coordinate is \(w(0) = 0\). The rest of the parameters are chosen as \(s = 10, k_1 = 0.5, k_2 = 0.2, k_c = 50\). The implementation procedure is listed in Algorithm 1, and the experiment results are shown in Fig. 3. At time \(t = 87s\), the robot was manually moved away from the desired path. However, after that, it headed towards the desired path again to decrease the path-following error. As shown in Fig. 3(c), the path-following error characterized by \(||(\phi_1, \phi_2)||\) decreased significantly and fluctuated around zero eventually due to practical reasons, such as sensor noise.

VI. DISCUSSION. PATH OR TRAJECTORY TRACKING?

In this section, we discuss whether our presented algorithm falls into the category of path tracking (i.e., path following) or trajectory tracking. In fact, while our generated guiding vector field is the standard output for the path tracking approach, we will argue that our algorithm can also be seen as a fair substantial extension of a trajectory tracking approach. Therefore, our guiding approach combines and extends elements from both approaches.

Our proposed algorithm considers the desired path as a geometric object as commonly assumed in path tracking. In contrast, a trajectory tracking algorithm is designed around a desired trajectory that specifies desired positions, velocities, etc. at specific time instants. In the latter approach, the “drawn path” consists of a sequence of desired trajectory points; namely, the time evolution of the desired position. In general, for VF-PF algorithms, a parametrization of the desired path is not required to generate a desired trajectory point. However, we exploit the parametrization in Theorem 1 to facilitate the expression of the functions \(\phi_i\), of which the zero-level sets are two surfaces, and the intersection of these two surfaces are exactly the (3D) desired path. Note that the subsequent derivation of the higher-dimensional guiding vector field depends merely on \(\phi_i\), independent of the specific parametrization of the desired path.

The additional coordinate \(w\) in our proposed VF-PF algorithm seems similar to but essentially different from the time \(t\) in trajectory tracking. In trajectory tracking, we remind that \(t = 1\); i.e., a desired trajectory \(r(t)\) is prescribed and it evolves as time elapses in an open-loop manner independent of the states of the robot. However, in our approach, the dynamics of the additional coordinate \(\dot{w}(t) = s\xi_3(\xi(t))\) in (13a) are in the closed loop with the states of the robot (i.e., \(\xi(t)\)). Consequently, we improve the performance of our algorithm, e.g., under noisy measurements, with respect to the standard trajectory tracking approach. This claim is further justified by numerical experiments and theoretical studies as follows.

Using the unicycle robot model in (12), we compare the proposed VF-PF algorithm in Section IV with the nonlinear trajectory tracking algorithm introduced in the classical monograph [1, p506]. For path following, the desired path is the projection of a Lissajous knot [21] parameterized as follows: \(x = 250 \cos(0.06w + 0.1) + 600, \ y = 250 \cos(0.08w + 0.7) + 350\), where \(w\in \mathbb{R}\) is the parameter. Then we use (11) to create a 3D vector field and (13) to calculate the control inputs \(u_v\) and \(\omega_u\). The control gains are \(k_1, k_2\) for the converging term of the vector field (i.e., \(\sum i=1 \xi_i \phi_i \nabla \phi_i\)) and \(k_9\) for the angular control input \(\omega_u\). For trajectory tracking, the prescribed trajectory is simply obtained by replacing the path parameter \(w\), of which the closed-loop dynamics are defined in (13a), by time \(t\), of which the dynamics are trivially the open-loop \(t = 1\). For ease of explanation, the desired trajectory is denoted by \((x_{\text{trj}}(t), y_{\text{trj}}(t))\), and the feasible desired heading \(\theta_{\text{trj}}(t)\) is computed from \((x_{\text{trj}}(t), y_{\text{trj}}(t))\) [1, p503]. The control inputs for the trajectory tracking algorithm are as follows:

\[
e' := \begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x - x_{\text{trj}} \\ y - y_{\text{trj}} \end{bmatrix} \]  
\[u_1 = -k_{trj} e_1', \ u_2 = -k_{trj} v_d e_2' \sin e_3'/e_3 - k_{trj} e_3' \]  
\[v_u = v_d \cos e_3' - u_1, \ \omega_u = \omega_d - u_2, \ 
\]

where \(v_d(t) = \sqrt{\dot{x}_d(t)^2 + \dot{y}_d(t)^2}, \ \omega_d(t) = (\dot{y}_d(t)\dot{x}_d(t) - \dot{x}_d(t)\dot{y}_d(t))/\dot{x}_d(t)^2 + \dot{y}_d(t)^2\), and \(k_{trj}^1, k_{trj}^2, k_{trj}^3\) are positive control gains. In both numerical experiments, the initial positions \((x(0), y(0))\) and orientations \(\theta(0)\) of the robot are the same. For path following, the initial value of the additional coordinate is \(w(0) = 0\), while correspondingly for trajectory tracking, the initial time instant is \(t = 0\). In addition, we choose the same control gains for these two simulations; namely, \(k_1 = k_2 = 0.05, k_9 = 1\) for the path following algorithm and \(k_{trj}^1 = k_{trj}^2 = 0.05, k_{trj}^3 = 1\) for the trajectory tracking algorithm\(^2\).

\(^2\)Supplementary material: http://tiny.cc/cdc20_yao.
If the measurements of the robot positions are accurate (i.e., no white noise is added), then both algorithms enable the robot to follow (track) the desired path (trajectory) successfully (see the green dash lines in Fig. 4). However, by observing the robot positions labeled by (1), (2) and (3) in Fig. 4(a) and Fig. 4(b), we note that the robot behaviors in the beginning are quite different, despite that the two simulations apply the same initial conditions. In the trajectory tracking case, the robot revolves before aligning with the desire path, while in the VF-PF case, the robot moves more “naturally”: it “aligns” with the desired path immediately and decreases its distance to the desired path gradually. The “unnatural” movement for the trajectory tracking algorithm is due to the “open-loop” dynamics of the desired trajectory point. In the beginning, the desired trajectory point \((x_d(0), y_d(0))\) is far away from the robot’s initial position (see the red point labeled by \(\circ\) in Fig. 4(b)), and thus the robot needs to approach it to decrease the tracking error. Since the movement of the desired trajectory point is independent of the robot position, the robot needs to steer its heading continually as the trajectory point moves along the desired path; the desired trajectory points at time \(t = 2\) s and \(t = 30\) s are illustrated by two red points labeled by \(\bigcirc\) and \(\bigtriangledown\) respectively in Fig. 4(b). In view of (11), it is suggestive to call \((f_1(w(t)), f_2(w(t)))\) the guiding point as a counterpart of the desired trajectory point. By contrast, although the guiding point \((f_1(w(t)), f_2(w(t)))\) is also on the far right side at \(t = 0\) (the same as the desired trajectory point), it “rushes” to the vicinity of the robot’s initial position shortly after 2 seconds (the red point labeled by \(\triangle\) in Fig. 4(a)), activated by the closed-loop dynamics \(\dot{w}(t) = s\hat{\theta}_3(\xi(t))\) and the converging property of the integral curves of the higher-dimensional (3D) vector field. This enables the robot to approach the “nearest point” of the desired path, and thus avoids the unnatural revolving behavior in the beginning.

To experimentally compare the difference of two algorithms under environmental perturbation, we intentionally add a significant amount of band-limited white noise (power: 10, sampling time: 0.1 s) to the positions perceived by the robot. As seen from the magenta lines in Fig. 4, the robot trajectory for the path following algorithm is visually more smooth that that for the trajectory tracking algorithm. This is partly attributed to the first term in (4), which guarantees that there is always a nonzero tangential term to the level curves of the desired path, not being affected by the second term due to orthogonality (see Appendix for some theoretical analysis). This property makes our algorithm appealing for fixed-wing aircraft [10], [22], since we can guarantee that, in the presence of noise, the control will not demand the robot to vibrate intensively in its heading, or even move backwards, by tuning the gains of the vector field such that the first term dominates. On the other hand, there is not such a tangential term in the trajectory tracking algorithm, and thus the robot heading vibrates as the perceived robot position varies. In some extreme cases, the robot might move backwards if the noisy measured robot position overtook the desired trajectory points. One might decrease the gains of the trajectory tracking algorithm to reduce the vibration, but it also comprises the tracking performance and leads to higher tracking errors.

VII. CONCLUSION AND FUTURE WORK

We have proposed a 3D singularity-free guiding vector field to follow a self-intersected desired path in 2D. Moreover, we have rigorously proven, and experimentally validated with a unicycle robot, the global convergence of the integral curves of our guiding vector field to the desired path. Several advantages of the proposed vector field are summarized in Remark 2. We have also compared our proposed VF-PF algorithm with a nonlinear trajectory tracking algorithm and the existing VF-PF algorithms, and concluded that our approach, albeit sharing some common features, is a significant extension. Note that our proposed algorithm is not restricted to 2D desired paths; in fact, 3D or even higher-dimensional desired paths (possibly in the robot configuration space) is still applicable using the same idea. We will present such extension in future work.

APPENDIX

We can decompose the vector field (4) as below:

\[
\chi(\xi) = \nabla \phi_1(\xi) \times \nabla \phi_2(\xi) + \sum_{i=1}^{2} -k_i \phi_i(\xi) \nabla \phi_i(\xi),
\]

where the first term \(\tau(\xi)\) is tangent to the level curves of \(\phi_i\), and thus provide a propagation direction along the desired path, and the latter term \(\iota(\xi)\) is a converging term to push the trajectory towards the desired path. In the presence of considerable noise, the converging term \(\iota(\xi)\) affects the “smoothness” of the actual trajectory greatly as the perceived robot position fluctuates. However, the propagation term \(\tau(\xi)\) alleviates this negative effect, by always providing a forward direction. In addition, in the vicinity of the desired path,
\[ \phi_1 \approx 0, \text{ and hence the converging term } \epsilon(\xi) \approx 0, \text{ and the propagation term } \tau(\xi) \text{ dominates the robot motion. In applications where backward motion is not desirable (e.g., UAV), one could manually add a gain to the propagation term } \tau(\xi) \text{ such that it always dominates the converging term (or decrease the gain for the converging term). Therefore, roughly speaking, the robot trajectory is less sensitive to the noise perturbation using our approach.}

In fact, it can be proved that our approach has a robustness property due to the closed-loop property of the dynamics of } \psi; \text{ precisely, the path-following error dynamics is locally input-to-state stable. Suppose that the disturbance is denoted by } d, \text{ and we define a set where } \mathcal{E}_\alpha := \{ \xi \in \mathbb{R}^3 : ||(\phi_1(\xi), \phi_2(\xi))|| < \alpha \} \text{ for a positive constant } \alpha. \text{ Due to the noise perturbation on the perceived robot position, it is further introduced into the vector field. Hence the perturbed system is:}

\[ \ddot{\xi}(t) = \chi(\xi(t)) + d(t), \quad (16) \]

where } d : \mathbb{R}_+^n \to \mathbb{R}^n, \text{ is assumed to be a piecewise continuous and bounded function of time } t, \text{ for } t \geq 0. \text{ It can be shown that the path-following error dynamics } \dot{e}(t) = N^T(\xi(t))(\chi(\xi(t)) + d(t)) \text{ is locally input-to-state stable. This leads to the following proposition.}

**Proposition 3.** Consider the perturbed system (16). If } |f_i(w)| \text{ is upper bounded in } \mathcal{E}_\alpha \text{ for } i = 1, 2, \text{ then the error dynamics is locally input-to-state stable. Namely, there exists positive constants } \delta \text{ and } r, \text{ such that if the initial condition } \xi(0) \in \mathcal{E}_\delta \text{ and the disturbance is uniformly bounded by } r \text{ (i.e., } d(t) \leq r \text{ for } t \geq 0), \text{ then the path-following error } ||e(t)|| \text{ is uniformly ultimately bounded. In addition, if the disturbance is vanishing: that is, } d(t) \xrightarrow{t \to \infty} 0, \text{ then the path-following error } ||e(t)|| \text{ is also vanishing.}

**Proof.** The proof is similar to [23, Theorem 3].

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