THE EDGE IDEALS OF THE JOIN OF SOME VERTEX
WEIGHTED ORIENTED GRAPHS

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ABSTRACT. In this paper, we describe primary decomposition of the edge ideal of the join of some graphs in terms of that information of the edge ideal of every weighted oriented graph. Meanwhile, we also study depth and regularity of symbolic powers and ordinary powers of such an edge ideal. We explicitly compute depth and regularity of ordinary powers of the edge ideal of the join of two graphs consisting of isolated vertices, and also provide upper bounds of regularity of symbolic powers of such an edge ideal. For the edge ideal of the join of two graphs with at least an oriented edge for per graph, we give the exact formulas for their depth and regularity, and also provide the upper bounds of regularity of ordinary powers of such an edge ideal. Some examples show that these upper bounds can be obtained, but may be strict.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a finite simple (no loops, no multiple edges) undirected graph. A weighted oriented graph $D$ whose underlying graph is $G$, is a triplet $(V(D), E(D), w)$ where $V(D) = V(G)$ is the vertex set, $E(D)$ is a directed edge set and $w$ is a function $w : V(D) \rightarrow \mathbb{N}^+$, where $\mathbb{N}^+ = \{1, 2, \ldots\}$. Specifically, $E(D)$ consists of ordered pairs $(x_i, x_j) \in V(D) \times V(D)$ where the pair $(x_i, x_j)$ represents a directed edge from $x_i$ to $x_j$. Some times for short we denote $V(D)$ and $E(D)$ by $V$ and $E$ respectively. For any $x_i \in V$, its weight is denoted by $w(x_i)$ or $w_i$. For any $x_i \in V(D)$, the sets $N^+_D(x_i) = \{x_j | (x_i, x_j) \in E(D)\}$ and $N^-_D(x_i) = \{x_j | (x_j, x_i) \in E(D)\}$ are called the out-neighbourhood and in-neighbourhood of $x_i$, respectively. Furthermore, the set $N_D(x_i) = N^+_D(x_i) \cup N^-_D(x_i)$ is called the neighbourhood of $x_i$. If $N_D(x_i) = N^+_D(x_i)$, then $x_i$ is called a source of $D$; If $N_D(x_i) = N^-_D(x_i)$, then $x_i$ is called a sink of $D$. Define $deg_D(x_i) = |N_D(x_i)|$ and $N_D[x_i] = N_D(x_i) \cup \{x_i\}$.

Let $D = (V(D), E(D), w)$ denote a weighted oriented graph with vertices $V(D) = \{x_1, \ldots, x_n\}$. By identifying the vertices with the variables in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$, we can associate to each weighted oriented graph $D$ a monomial ideal

$I(D) = \langle x_i^{w(x_j)} | (x_i, x_j) \in E(D) \rangle$.

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This ideal is called the edge ideal of $D$. The generators of $I(D)$ are independent of the weight assigned to a source vertex. Therefore, to simplify our formulas, throughout this paper, we shall assume that source vertices always have weight one.

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring in $n$ variables over a field $\mathbb{K}$ and $I$ be a nonzero homogeneous ideal of $S$. Then for $k \geq 1$, the $k$th symbolic power of $I$ is defined as $I^{(k)} = \bigcap_{P \in \Ass(I)} (I^k S_P) \cap S$, where $\Ass(I)$ is the set of associated prime ideals of $I$. Geometrically, the symbolic powers are important since they capture all the polynomials that vanish with a given multiplicity (see [8]). It is clear that $I^k \subseteq I^{(k)}$ for all $k \geq 1$ but the reverse containment may fail. We know that the regularity and depth are two central invariants associated to $I$. It is well known that $\reg(I^t)$ is asymptotically a linear function for $t \gg 0$, i.e., there exist constants $a, b$ and a positive integer $t_0$ such that for all $t \geq t_0$, $\reg(I^t) = at + b$ (see [7,14]). It was proved that if $I = I(G)$ is an edge ideal of a simple graph, then $a \leq 2$. Kumar et al. [15] studied the regularity of symbolic powers of edge ideals of the join of simple graphs, they show that:

**Theorem 1.1.** Let $r \geq 2$ be an integer and $G_1 * \cdots * G_r$ be the join of $r$ simple graphs $G_1, \ldots, G_r$ with pairwise disjoint vertex sets $V_1, \ldots, V_r$, respectively. Then

\[
\reg(S/I(G^{(t)})) = \max\{\reg(S/I(G_j^{(t)})) - i + t \mid 1 \leq i \leq t, 1 \leq j \leq r\}, \text{ for all } t \geq 1.
\]

In [19], Selvaraja defined following classes of graphs:

\[ A = \{G \mid \reg(S/(I(G)^{t+1} : u)) \leq \reg(S/I(G)), u \in G(I(G)^{t}), t \geq 1\}, \]

where $G(I(G)^{t})$ denotes the minimal monomial generating set of $I(G)^{t}$. He show that:

**Theorem 1.2.** Let $G_1, G_2 \in A$ be two graphs with disjoint vertex sets. Then for all $t \geq 1$, then

\[
\reg(S/I(G_1 * G_2)^{t}) \leq 2t + \reg(S/I(G_1 * G_2)) - 2.
\]

As far as we know, little is known about how to calculate regularities of symbolic powers and ordinary powers of edge ideals of some weighted oriented graphs (see [20,21,22,23]).

In [2], Brodmann showed that depth $(S/I^t)$ is a constant for $t \gg 0$, and this constant is bounded above by $n - \ell(I)$, where $\ell(I)$ is the analytic spread of $I$. It is shown in [12 Theorem 1.2] that depth $(S/I^t)$ is a nonincreasing function of $t$ when all powers of $I$ have a linear resolution and conditions are given in that paper under which all powers of $I$ will have linear quotients. In this regard, there has been an interest in determining the smallest value $t_0$ such that depth $(S/I^t)$ is a constant for all $t \geq t_0$. (see [16,12,17,20,21,22,23]).

In this article, we focus on algebraic properties corresponding to the irreducible decomposition (see Theorem 3.5), depth and regularity of symbolic powers and ordinary powers of edge ideals of the join of some weighted oriented graphs.

Let $r \geq 2$ be an integer and $D_1, \ldots, D_r$ be weighted oriented graphs over pairwise disjoint vertex sets $V_1, \ldots, V_r$, respectively. The join of $D_1, \ldots, D_r$, denoted by
$D_1 \cdots D_r$, is a weighted oriented graph over the vertex set $V_1 \sqcup \cdots \sqcup V_r$, whose edge set is

$$E_1 \sqcup \cdots \sqcup E_r \cup \{(x, y) : x \in V_i, y \in V_j \text{ with } i < j\}.$$ 

In particular, if $r = 2$, then the join $D_1 \ast D_2$ of $D_1$ and $D_2$ is a weighted oriented graph over the vertex set $V_1 \sqcup V_2$ whose edge set is $E_1 \cup E_2 \cup \{(x, y) : x \in V_1, y \in V_2\}$.

Our main results are as follows:

**Theorem 1.3.** Let $D := D_1 \ast D_2$ be the join of two weighted oriented graphs $D_1$ and $D_2$, where $D_i$ consists of isolated vertices with $V_i = \{x_{ij} | j \in [n_i]\}$ for $i = 1, 2$. Then, for all $t$, we have

1. $\text{depth}(S/I(D)\{t\}) = \text{depth}(S/I(D)\{t\}) = 1$;
2. $\text{reg}(S/I(D)\{t\}) = \text{reg}(S/I(D)\{t\}) = \sum_{x \in V(D)} w(x) - |V(D)| + 1 + (t-1)(w+1)$

where $w = \max\{w(x) | x \in V(D)\}$.

**Theorem 1.4.** Let $r \geq 2$ be an integer and let $D := D_1 \cdots D_r$ be the join of weighted oriented graphs $D_1, \ldots, D_r$, where $D_i$ consists of isolated vertices with $V_i = \{x_{ij} | j \in [n_i]\}$ for any $i \in [r]$. Then, for any $t \geq 1$, we have

1. $\text{depth}(S/I(D)\{t\}) = 1$;
2. $\text{reg}(S/I(D)\{t\}) \leq \sum_{x \in V} w(x) - |V| + 1 + (t-1)(w+1)$.

The equality holds if $w = \max\{w(x) | x \in V_2\}$, where $w = \max\{w(x) | x \in V\}$ and $V = V_1 \sqcup \cdots \sqcup V_r$.

**Theorem 1.5.** Let $r \geq 3$ be an integer, and let $D := D_1 \cdots D_r$ be the join of weighted oriented graphs $D_1, \ldots, D_r$, where the vertex set of $D_i$ is $V_i = \{x_{ij} | j \in [n_i]\}$ and $n_i = |V_i|$ for $i \in [r]$. If each $D_i$ contains at least an oriented edge. Then

1. $\text{depth}(S/I(D)) = 1$;
2. $\text{reg}(S/I(D)) = \max\{\text{reg}(S_i/I(D_i)) + \sum_{x \in T_i} w(x) - |T_i| : i \in [r]\}$, where $T_i = V(D) \setminus \bigcup_{j=1}^{i} V_j$ for any $i \in [r]$.

**Theorem 1.6.** Let $D := D_1 \ast D_2$ be the join of two weighted oriented graphs $D_1$ and $D_2$, where $D_1$ contains at least an oriented edge and $D_2$ is a weighted oriented complete graph. If $\text{reg}(S_1/I(D_1)\{t\}) \leq \text{reg}(S_1/I(D_1)) + (t-1)(w' + 1)$, where $S_1 = \mathbb{K}[x_{1i} : i \in V(D_1)]$ and $w' = \max\{w(x) | x \in V(D_1)\}$. Then, for all $t \geq 1$, we have

$$\text{reg}(S/I(D)\{t\}) \leq \text{reg}(S/I(D)) + (t-1)(w+1)$$

where $w = \max\{w(x) | x \in V(D_1) \cup V(D_2)\}$. The equality holds when $w' = w$ and $\text{reg}(S_1/(I(D_1)\{t\})) = \text{reg}(S_1/I(D_1)) + (t-1)(w' + 1)$.

Our results partially generalize the corresponding conclusion of symbolic powers and ordinary powers of edge ideals of the join of some simple graphs, since if $w(x) = 1$ for all $x \in V(D)$, then $I(D) = I(G)$.

Our paper is organized as follows. In the preliminary section, we collect the needed notations and basic facts from the literature. In section 2, we describe
primary decomposition of the edge ideal of the join of some graphs in terms of that information of the edge ideal of every weighted oriented graph. In section 3, we study depth and regularity of symbolic powers and ordinary powers of the edge ideal of the join of weighted oriented graphs, where each graph consists of isolated vertices. We provide some exact formulas for depth and regularity of ordinary powers and also provide upper bounds of regularity of symbolic powers of such an edge ideal.

In section 4, we study depth and regularity of symbolic powers and ordinary powers of the edge ideal of the join of weighted oriented graphs, where each graph contains at least an oriented edge, we give upper bounds of regularity of ordinary powers of such an edge ideal.

2. Preliminaries

In this section, we gather together the needed notations and basic facts, which will be used throughout this paper. Two important invariants we focus on are depth and regularity, we define them by means of local cohomology modules.

Let $S$ be a positively graded algebra and $m$ its maximal homogeneous ideal. Let $M$ be a finitely generated graded $S$-module. Let $H^i_m(M)$, for $i \geq 0$, denote the $i$-th local cohomology module of $M$ with respect to $m$. We define

$$\text{depth}(M) = \min \{ i \mid H^i_m(M) \neq 0 \}$$

$$\text{reg}(M) = \max \{ i + j \mid H^i_m(M)_j \neq 0 \}.$$

**Remark 2.1.** Let $a_i(M) := \max \{ j \mid H^i_m(M)_j \neq 0 \}$ with the convention that $a_i(M) := -\infty$ if $H^i_m(M) = 0$. Then

$$\text{depth}(M) = \min \{ i \mid a_i(M) \neq -\infty \},$$

$$\text{reg}(M) = \max \{ a_i(M) + i \mid i \geq 0 \}.$$

When $S$ is a polynomial ring over a field $K$ and $I$ is a nonzero proper homogeneous ideal in $S$, depth and regularity of $I$ are closely related to the minimal free resolution and graded Betti numbers of $I$ in the following way. Suppose that $I$ admits the following minimal free resolution

$$0 \to \bigoplus_j S(-j)^{\beta_{p,j}(I)} \to \bigoplus_j S(-j)^{\beta_{p-1,j}(I)} \to \cdots \to \bigoplus_j S(-j)^{\beta_{0,j}(I)} \to I \to 0,$$

where $S(-j)$ is an $S$-module obtained by shifting the degrees of $S$ by $j$. The $(i,j)$-th graded Betti number $\beta_{i,j}(I)$ is an invariant of $I$ that equals the number of minimal generators of degree $j$ in the $i$-th syzygy module of $I$.

Let $\text{pd}(I)$ denote the *projective dimension* of $I$. Then

$$\text{pd}(I) = \max \{ i \mid \beta_{i,j}(I) \neq 0 \text{ for some } j \},$$

$$\text{reg}(I) = \max \{ j - i \mid \beta_{i,j}(I) \neq 0 \},$$

$$\text{depth}(I) = n - \text{pd}(I).$$

By looking at the minimal free resolution and Auslander-Buchsbaum formula (see Theorem 1.3.3 of [3]), it is easy to see that
Lemma 2.2. ([10] Lemma 1.3) Let \( u \in S \) be a monomial of degree \( d \) and \( J = (u) \), and let \( I \subset S \) be a proper nonzero homogeneous ideal. Then

1. \( \text{reg} (I) = \text{reg} (S/I) + 1; \)
2. \( \text{reg} (S/J) = d - 1. \)

The following lemmas are often used for computing regularity and depth of a module.

Lemma 2.3. ([5] Theorem 2.5) \( S = \mathbb{K}[x_1, \ldots, x_n] \) be a polynomial ring and let \( I \subset S \) be a graded ideal with \( \dim (R/I) \leq 1 \). Then, for any finitely generated graded \( S \)-module \( M \), we have \( \text{reg} (IM) \leq \text{reg} (I) + \text{reg} (M). \)

Lemma 2.4. ([13] Lemmas 2.1 and 3.1) Let \( 0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0 \) be a short exact sequence of finitely generated graded \( S \)-modules. Then we have

1. \( \text{reg} (N) \leq \max \{\text{reg} (M), \text{reg} (P)\} \), the equality holds if \( \text{reg} (P) \neq \text{reg} (M) - 1. \)
2. \( \text{reg} (M) \leq \max \{\text{reg} (N), \text{reg} (P) + 1\} \), the equality holds if \( \text{reg} (N) \neq \text{reg} (P). \)
3. \( \text{depth} (N) \geq \min \{\text{depth} (M), \text{depth} (P)\} \), the equality holds if \( \text{depth} (P) \neq \text{depth} (M) - 1. \)
4. \( \text{depth} (M) \geq \min \{\text{depth} (N), \text{depth} (P) + 1\} \), the equality holds if \( \text{depth} (N) \neq \text{depth} (P). \)

Lemma 2.5. ([13] Lemma 2.2, Lemma 3.2) \( S_1 = \mathbb{K}[x_1, \ldots, x_m] \) and \( S_2 = \mathbb{K}[x_{m+1}, \ldots, x_n] \) be two polynomial rings over \( \mathbb{K} \), \( I \subset S_1 \) and \( J \subset S_2 \) be two nonzero homogeneous ideals. Let \( S = S_1 \otimes_{\mathbb{K}} S_2 \). Then we have

1. \( \text{reg} (S/(I + J)) = \text{reg} (S_1/I) + \text{reg} (S_2/J); \)
2. \( \text{depth} (S/(I + J)) = \text{depth} (S_1/I) + \text{depth} (S_2/J); \)
3. \( \text{reg} (S/II) = \text{reg} (S_1/I) + \text{reg} (S_2/J) + 1; \)
4. \( \text{depth} (S/II) = \text{depth} (S_1/I) + \text{depth} (S_2/J) + 1. \)

In particular, if \( u \) is a monomial of degree \( d \) such that \( \text{supp} (u) \cap \text{supp} (I) = \emptyset \), let \( J = (u) \), then \( \text{reg} (J) = d \) and \( \text{reg} (JI) = \text{reg} (I) + d. \)

Let \( I = Q_1 \cap \cdots \cap Q_m \) be a primary decomposition of the ideal \( I \). For \( P \in \text{Ass}(S/I) \), we denote \( Q_{\leq P} \) to be the intersection of all \( Q_i \) with \( \sqrt{Q_i} \subseteq P. \)

Lemma 2.6. ([6] Theorem 3.7) The \( k \)th symbolic power of a monomial ideal \( I \) is

\[
I^{(k)} = \bigcap_{P \in \text{Ass}(I)} Q_{\leq P}^k.
\]

For a positive integer \( n \), we set \( [n] = \{1, 2, \ldots, n\}. \)

Lemma 2.7. ([9] Theorem 5.6) Let \( \mathbb{K} \) be a field of \( char (\mathbb{K}) = 0, \) and let \( S_1 = \mathbb{K}[x_1, \ldots, x_m], S_2 = \mathbb{K}[x_{m+1}, \ldots, x_n] \) be two polynomial rings over \( \mathbb{K} \), and \( I \subset S_1, J \subset S_2 \) be two nonzero monomial ideals. Let \( S = S_1 \otimes_{\mathbb{K}} S_2. \) Then for any \( t \geq 1, \) we have

1. \( \text{reg} \left( \frac{S}{(x_{1}^{t+j}, \ldots, x_{m}^{t+j})} \right) = \max \left\{ \text{reg} \left( \frac{S}{(x_{1}^{t+j}, \ldots, x_{m}^{t+j})} \right), \text{reg} \left( \frac{S}{(x_{m+1}^{t+j}, \ldots, x_{n}^{t+j})} \right) \right\}. \)
(2) \( \text{depth}\left( \frac{S}{(I+J)^{[2]}_{i \in [l-1]}} \right) = \min \{ \text{depth}\left( \frac{S_{i}}{I^{[i-1]}} \right) + \text{depth}\left( \frac{S_{j}}{J^{[j-1]}} \right) + 1, \text{depth}\left( \frac{S_{i}}{I^{[i-1]}} \right) + \text{depth}\left( \frac{S_{j}}{J^{[j-1]}} \right) \} \).

Obviously, we also have \( \text{reg}(S/I^{(i)}) = \text{reg}(S_1/I^{(i)}) \) and \( \text{reg}(S/J^{(i)}) = \text{reg}(S_2/J^{(i)}) \) for any \( i \geq 1 \).

3. Primary decomposition of edge ideal of join of graphs

In this section, we provide primary decomposition of the edge ideal of the join of some graphs in terms of that information of the edge ideal of every weighted oriented graph. First, we give a definition of the join of weighted oriented graphs.

**Definition 3.1.** Let \( r \geq 2 \) be an integer and \( D_1, \ldots, D_r \) be weighted oriented graphs over pairwise disjoint vertex sets \( V_1, \ldots, V_r \), respectively. The join of \( D_1, \ldots, D_r \), denoted by \( D_1 \ast \cdots \ast D_r \), is a weighted oriented graph over the vertex set \( V_1 \sqcup \cdots \sqcup V_r \), whose edge set is

\[ E_1 \cup \cdots \cup E_r \cup \{(x, y) : x \in V_i, y \in V_j \text{ with } i < j \} \]

In particular, if \( r = 2 \), then the join \( D_1 \ast D_2 \) of \( D_1 \) and \( D_2 \) is a weighted oriented graph over the vertex set \( V_1 \sqcup V_2 \) whose edge set is \( E_1 \cup E_2 \cup \{(x, y) : x \in V_1, y \in V_2 \} \).

**Example 3.2.** The following are some typical examples of the join of two and three weighted oriented graphs, respectively.
Fig. 1 Some typical examples of some classes of weighted bipartite and 3-partite graphs

**Remark 3.3.** Let $D := D_1 \ast \cdots \ast D_r$ be the join of weighted oriented graphs $D_1, \ldots, D_r$, where the vertex set of $D_i$ is $V_i = \{x_{ij} \mid j \in [n_i]\}$ and $n_i = |V_i|$ for any $i \in [r]$. Let $S = \mathbb{K}[x_{ij} : i \in [r], j \in [n_i]]$. Then the edge ideal of $D$ is a monomial ideal of $S$

$$I(D) = \sum_{i=1}^{r} I(D_i) + (x_{ij}x_{k\ell}^{u_{ij\ell}} \mid j \in [n_i], \ell \in [n_k] \text{ where } 1 \leq i < k \leq r).$$

In particular, if each $D_i$ consists of isolated vertices, then

$$I(D) = (x_{ij}x_{k\ell}^{u_{ij\ell}} \mid j \in [n_i], \ell \in [n_k] \text{ where } 1 \leq i < k \leq r).$$

A monomial ideal is called *irreducible* if it cannot be written as a proper intersection of two other monomial ideals. It is called *reducible* if it is not irreducible. It is well known that a monomial ideal is irreducible if and only if it is generated by pure powers of variables, that is, it has the form $(x_{1i}^{a_1}, \ldots, x_{ki}^{a_k})$. The following lemma is a fundamental fact.

**Lemma 3.4.** ([11, Theorem 1.3.1]) Let $I \subset S$ be a monomial ideal. Then there exists a unique decomposition

$$I = Q_1 \cap \cdots \cap Q_r$$

such that none of the $Q_i$ can be omitted in this intersection and each $Q_i$ is an irreducible monomial ideal.

This decomposition is called *irredundant presentation* of $I$ and each $Q_i$ is called an irredundant component of $I$.

For a monomial ideal $I \subset S$, we denote by $\mathcal{G}(I)$ the unique minimal set of monomial generators of $I$. Let $u \in S$ be a monomial, we set $\text{supp}(u) = \{x_i : x_i | u\}$. If $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$, we set $\text{supp}(I) = \bigcup_{i=1}^{m} \text{supp}(u_i)$.

Now, we determine the irredundant presentation of the edge ideal $I(D_1 \ast \cdots \ast D_r)$ by using the irredundant presentation of the edge ideal $I(D_i)$ of each weighted oriented graph $D_i$.

**Theorem 3.5.** Let $r \geq 2$ be an integer, and let $D := D_1 \ast \cdots \ast D_r$ be the join of weighted oriented graphs $D_1, \ldots, D_r$, where the vertex set of $D_i$ is $V_i = \{x_{ij} \mid j \in [n_i]\}$ and $n_i = |V_i|$ for $i \in [r]$. Set $I_i = (x_{ij} \mid j \in [n_i]), I_{i}^{u_j} = (x_{ij}^{u_{ij}} \mid j \in [n_i]), \hat{I}_i = \sum_{j=1}^{i-1} I_j + \sum_{j=i+1}^{r} I_j^{u_j}$ and $\sum_{j=1}^{0} I_j = \sum_{j=r+1}^{r} I_j^{u_j} = (0)$ by convention. Let $I(D_i) = Q_{i1} \cap \cdots \cap Q_{it_i}$ be the irredundant presentation of $I(D_i)$. Then the irredundant presentation of $I(D)$ is

$$I(D) = \bigcap_{i=1}^{r} \bigcap_{j=1}^{t_i} (\hat{I}_i + Q_{ij}).$$
Proof. We first prove that $I(D) = \bigcap_{i=1}^{r} (\widehat{I}_i + I(D_i))$ by induction on $r$. If $r = 2$, then

$$(\widehat{I}_1 + I(D_1)) \cap (\widehat{I}_2 + I(D_2)) = \widehat{I}_1 \cap \widehat{I}_2 + \widehat{I}_1 \cap I(D_2) + \widehat{I}_2 \cap I(D_1) + I(D_1) \cap I(D_2)$$

$$= \widehat{I}_1 \widehat{I}_2 + I(D_1) + I(D_2)$$

$$= (x_{i_1}x_{2j}^{w_j} | i \in [n_1], j \in [n_2]) + I(D_1) + I(D_2) = I(D)$$

where the second equality holds because of $\text{supp} (\widehat{I}_1) \cap \text{supp} (\widehat{I}_2) = \emptyset$, $I(D_1) \subset \widehat{I}_2$ and $I(D_2) \subset \widehat{I}_1$.

Suppose that $r \geq 3$ and that the statement holds for $r - 1$, that is

$$I(D_1 \ast \cdots \ast D_{r-1}) = \bigcap_{i=1}^{r-1} (\widehat{I}_i + I(D_i)) = (I_{2}^{w_2} + \cdots + I_{r-1}^{w_{r-1}} + I(D_1)) \cap (I_1 + I_3^{w_3} + \cdots + I_{r-1}^{w_{r-1}} + I(D_2)) \cap \cdots \cap (I_1 + \cdots + I_{r-2} + I_{r-1} + I(D_{r-1}))$$

Thus

$$\bigcap_{i=1}^{r} (\widehat{I}_i + I(D_i)) = \bigcap_{i=1}^{r-1} (\widehat{I}_i + I(D_i)) \cap (\widehat{I}_r + I(D_r))$$

$$= (I_{2}^{w_2} + \cdots + I_{r}^{w_{r}} + I(D_1)) \cap (I_1 + I_3^{w_3} + \cdots + I_r^{w_r} + I(D_2))$$

$$\cap \cdots \cap (I_1 + \cdots + I_{r-2} + I_{r-1} + I(D_{r-1}))$$

$$= (I_r^{w_r} + I(D_1 \ast \cdots \ast D_{r-1})) \cap (I_1 + \cdots + I_{r-1} + I(D_r))$$

$$= \sum_{i=1}^{r-1} I_i \cap I_r^{w_r} + I_r^{w_r} \cap I(D_r) + I(D_1 \ast \cdots \ast D_{r-1}) \cap (I_1 + \cdots + I_{r-1})$$

$$+ I(D_1 \ast \cdots \ast D_{r-1}) \cap I(D_r)$$

$$= \sum_{i=1}^{r-1} I_i I_r^{w_r} + I(D_1 \ast \cdots \ast D_{r-1}) + I(D_r) = I(D)$$

where the penultimate equality holds because of $\text{supp} (I_i) \cap \text{supp} (I_r^{w_r}) = \emptyset$, $I(D_r) \subset I_r^{w_r}$ and $I(D_1 \ast \cdots \ast D_{r-1}) \subset (I_1 + \cdots + I_{r-1})$, the last equality holds because $I(D_1 \ast \cdots \ast D_{r-1}) = \sum_{i=1}^{r-1} I(D_i) + (x_{ij}x_{k\ell}^{w_{k\ell}} | j \in [n_i], \ell \in [n_k]$ where $1 \leq i < k \leq r - 1$).

Since each $I(D_i) = \bigcap_{j=1}^{t_i} Q_{ij}$, one has $\widehat{I}_i + I(D_i) = \widehat{I}_i + \bigcap_{j=1}^{t_i} Q_{ij} = \bigcap_{j=1}^{t_i} (\widehat{I}_i + Q_{ij})$. Note that $\text{supp} (\widehat{I}_i) \cap \text{supp} (Q_{ij}) = \emptyset$, it implies that the ideal $\widehat{I}_i + Q_{ij}$ is irreducible for any $i \in [r], j \in [n_i]$. It follows that

$$I(D) = \bigcap_{i=1}^{r} (\widehat{I}_i + I(D_i)) = \bigcap_{i=1}^{r} \bigcap_{j=1}^{t_i} (\widehat{I}_i + Q_{ij})$$

We finish the proof. □
Given an ideal $I \subset S$, we set
\[
\text{bight}(I) = \sup \{ \text{ht}(P) \mid P \text{ is a minimal prime ideal of } S \text{ over } I \}.
\]

As a direct consequence of the above theorem, one has the following corollary.

**Corollary 3.6.** Let $D := D_1 \ast \cdots \ast D_r$ be the join of weighted oriented graphs $D_1, \ldots, D_r$ as in Theorem 3.5. Then
\[
(a) \ \text{ht}(I(D)) = \min \{ \sum_{j=1, j \neq i}^r n_j + \text{ht}(I(D_i)) \mid i \in [r] \},
\]
\[
(b) \ \dim(S/I(D)) = \max \{ n_i - \text{ht}(I(D_i)) \mid i \in [r] \},
\]
\[
(c) \ \text{bight}(I(D)) = \max \{ \sum_{j=1, j \neq i}^r n_j + \text{ht}(I(D_i)) \mid i \in [r] \}.
\]

**Corollary 3.7.** Let $D := D_1 \ast \cdots \ast D_r$ be the join of weighted oriented graphs $D_1, \ldots, D_r$ as in Theorem 3.5. Then $I(D)$ is unmixed if and only if each $I(D_i)$ is unmixed and $n_j - \text{height}(I(D_j)) = n_k - \text{height}(I(D_k))$ for any different $j, k \in [r]$. \hfill \Box

**Proof.** By Theorem 3.5 and (a) and (c) in Corollary 3.6 we know that $I(D)$ is unmixed if and only if $\sum_{j=1, j \neq i}^r n_j + \text{ht}(Q_{ij}) = \sum_{j=1, j \neq i}^r n_j + \text{ht}(Q_{ik})$ for any different $j, k \in [t_i]$ and $i \in [r]$, and $\sum_{i=1, i \neq j}^r n_i + \text{ht}(I(D_j)) = \sum_{i=1, i \neq k}^r n_i + \text{ht}(I(D_k))$ for any $j, k \in [r]$. This implies that each $I(D_i)$ is unmixed and $n_j - \text{ht}(I(D_j)) = n_k - \text{ht}(I(D_k))$, as wished.

**Corollary 3.8.** Let $r \geq 2$ be a positive integer and let $D := D_1 \ast \cdots \ast D_r$ be the join of weighted oriented graphs $D_1, \ldots, D_r$, where $D_i$ consists of isolated vertices with the vertex set $V_i = \{ x_{ij} \mid j \in [n_i] \}$ and $n_i = |V_i|$ for any $i \in [r]$. Then $I(D)$ is unmixed if and only if $n_1 = n_2 = \cdots = n_r$. \hfill \Box

4. **Symbolic powers of the edge ideal of the join of graphs consisting of isolated vertices**

In this section, we study depth and regularity of symbolic powers and ordinary powers of the edge ideal of the join of weighted oriented graphs, where each graph consists of isolated vertices. For the edge ideal of the join of two weighted oriented graphs, we give the exact formulas for depth and regularity of their powers and symbolic powers. For the edge ideal of the join of $r(\geq 3)$ weighted oriented graphs, we provide the exact depth formulas and also give the upper bounds on regularity of symbolic powers.

**Lemma 4.1.** ([1] Lemma 4.4) Let $u_1, \ldots, u_m$ be a regular sequence of homogeneous polynomials in $S$ with $\deg(u_1) = \cdots = \deg(u_t) = d$. Let $I = (u_1, \ldots, u_m)$. Then we have $\text{reg}(I^t) = dt + (d - 1)(m - 1)$ for all $t \geq 1$.

First, we consider the case $r = 2$ and each $D_i$ consisting of isolated vertices.
Theorem 4.2. Let \( D := D_1 \ast D_2 \) be the join of two weighted oriented graphs \( D_1 \) and \( D_2 \), where \( D_i \) consists of isolated vertices with \( V_i = \{ x_{ij} \mid j \in [n_i] \} \), \( n_i = |V_i| \) for all \( i \in [2] \). Then, for all \( t \geq 1 \), we have

(1) \( \text{depth}(S/I(D)^t) = 1; \)
(2) \( \text{reg}(S/I(D)^t) = \sum_{x \in V} w(x) - |V| + 1 + (t-1)(w+1) \), where \( w = \max\{w(x) \mid x \in V\} \) and \( V = V_1 \sqcup V_2 \).

Proof. Note that \( I(D)^t = I^tJ^t \), where \( I = (x_{11}, \ldots, x_{1n_1}) \), \( J = (x_{21}^{w_2}, \ldots, x_{2n_2}^{w_2}) \) and \( \text{supp}(I) \cap \text{supp}(J) = \emptyset \). Then, by Lemma 2.5 (3) and (4), one has

\[
\text{depth}(S/I(D)^t) = \text{depth}(S/J^tI^t) = \text{depth}(S/I^t) + \text{depth}(S_2/J^t) + 1
\]

and

\[
\text{reg}(S/I(D)^t) = \text{reg}(S/J^tI^t) = \text{reg}(S_1/I^t) + \text{reg}(S_2/J^t) + 1
\]

Therefore, we obtain \( \text{depth}(S/I(D)^t) = 1 \), since both \( I \) and \( J \) are \( m_1 \)-primary and \( m_2 \)-primary in \( S_1 \) and \( S_2 \) respectively, where \( m_1 = (x_{11}, \ldots, x_{1n_1}) \) and \( m_2 = (x_{21}, \ldots, x_{2n_2}) \). By Lemmas 2.2 (1) and 4.3 and the following lemma, we obtain

\[
\text{reg}(S/I(D)^t) = t + \text{reg}(S_2/J^t) = \sum_{x \in V} w(x) - |V| + 1 + (t-1)(w+1),
\]

where \( w = \max\{w(x) \mid x \in V\} \).

Lemma 4.3. Let \( J = (x_{21}^{w_2}, \ldots, x_{2n_2}^{w_2}) \). Then, for any \( t \geq 1 \), we have

\[
\text{reg}(S/J^t) = \sum_{i=1}^{n_2} w_{2i} - n_2 + (t-1)w
\]

where \( w = \max\{w_{2i} \mid i \in [n_2]\} \).

Proof. We apply induction on \( n_2 \) and \( t \). Without loss of generality, we can assume that \( w = w_{2n_2} \). The case \( n_2 = 1 \) or \( t = 1 \) is obvious by Lemmas 2.2 (2) and 2.5 (1).

Now, we assume that \( n_2, t \geq 2 \). By some simple calculations, we obtain that \( J^t : x_{21}^{w_2} = J^{t-1} \) and \( (J^t, x_{21}^{w_2}) = (x_{22}^{w_2}, \ldots, x_{2n_2}^{w_2})^t + (x_{21}^{w_2}) \). By Lemma 2.5 (1) and inductive hypothesis, we obtain

\[
\text{reg}(S/(J^t, x_{21}^{w_2})) = \text{reg}(S/(x_{22}^{w_2}, \ldots, x_{2n_2}^{w_2})^t) + w_{21} - 1
\]

\[
= \sum_{i=2}^{n_2} w_{2i} - (n_2 - 1) + (t-1)w + w_{21} - 1
\]

\[
= \sum_{i=1}^{n_2} w_{2i} - n_2 + (t-1)w
\]

and

\[
\text{reg}(S/(J^t : x_{21}^{w_2})) = \text{reg}(S/J^{t-1}) = \sum_{i=1}^{n_2} w_{2i} - n_2 + (t-2)w.
\]

The desired result holds by Lemma 2.4 (1) and the following exact sequence

\[
0 \longrightarrow (S/J^t : x_{21}^{w_2})(-w_{21}) \longrightarrow S/J^t \longrightarrow S/(J^t, x_{21}^{w_2}) \longrightarrow 0. \]

Next, we consider the case \( r \geq 3 \) and each \( D_i \) consisting of isolated vertices.

**Lemma 4.4.** Let \( r \geq 3 \) be an integer, and let \( D := D_1 \ast \cdots \ast D_r \) be the join of weighted oriented graphs \( D_1, \ldots, D_r \), where \( D_i \) consists of isolated vertices with \( V_i = \{ x_{ij} | j \in [n_i] \} \) and \( n_i = |V_i| \) for any \( i \in [r] \). Assume that \( I_i = (x_{ij} | j \in [n_i]) \), 
\[
I_i^{w_i} = (x_{ij}^{w_{ij}} | j \in [n_i]) \quad \text{and} \quad \hat{I}_i = \sum_{j=1}^{r} I_j^{w_j} + \sum_{j=i+1}^{r} I_j^{w_j}.
\]

(1) \( \forall r \) follows from Remark 3.3.

By Theorem 3.5 in the case of all \( \exists \). Lemma 4.4.

Let \( L = \bigcap_{i=1}^{r} \hat{I}_i \), then \( L = I_1 + I(D_2 \ast \cdots \ast D_r) \).

**Proof.** (1) follows from Remark 3.3.

(2) We apply induction on \( r \). If \( r = 3 \), one has \( L = \hat{I}_2 \cap \hat{I}_3 = (I_1 + I_3^{w_3}) \cap (I_1 + I_2) = I_1 + I_2 \bigcap_{i=1}^{3} I_i \), as desired. If \( r \geq 4 \), then, by induction, we have
\[
\bigcap_{i=2}^{r-1} \hat{I}_i = (I_1 + I_3^{w_3} + \cdots + I_{r-1}^{w_{r-1}}) \cap (I_1 + I_2 + I_4^{w_4} + \cdots + I_{r-1}^{w_{r-1}}) \cap \cdots \cap (I_1 + I_2 + I_{r-2} + I_{r-1})
= I_1 + \sum_{i=2}^{r-2} \sum_{j=i+1}^{r-1} I_i I_j^{w_j}.
\]

By Theorem 3.5 in the case of all \( I(D_i) = 0 \), we obtain
\[
L = \bigcap_{i=2}^{r} \hat{I}_i = (I_1 + I_3^{w_3} + \cdots + I_r^{w_r}) \cap (I_1 + I_2 + I_4^{w_4} + \cdots + I_r^{w_r}) \cap \cdots \cap (I_1 + \cdots + I_{r-1})
= I_1 + (I_3^{w_3} + \cdots + I_r^{w_r}) \cap (I_2 + I_4^{w_4} + \cdots + I_r^{w_r}) \cap \cdots \cap (I_2 + \cdots + I_{r-1})
= I_1 + I(D_2 \ast \cdots \ast D_r).
\]

\( \square \)

**Theorem 4.5.** Let \( r \geq 3 \) be an integer, and let \( D := D_1 \ast \cdots \ast D_r \) be the join of weighted oriented graphs \( D_1, \ldots, D_r \) as in Lemma 4.4. Then

(1) \( \text{depth} (S/I(D)) = 1 \);
(2) \( \text{reg} (S/I(D)) = \sum_{x \in V} w(x) - |V| + 1 \), where \( V = V_1 \sqcup \cdots \sqcup V_r \).

**Proof.** We apply induction on \( r \) with the \( r = 2 \) case verified in Theorem 4.2. Now, assume that \( r \geq 3 \). By Theorem 3.5 in the case of all \( I(D_i) = 0 \), one has \( I(D) = \bigcap_{i=1}^{r} \hat{I}_i = K \cap L \), where \( K = \hat{I}_1 = \bigcup_{j=2}^{r} I_j^{w_j} \) and \( L = \bigcap_{i=2}^{r} \hat{I}_i = I_1 + I(D_2 \ast \cdots \ast D_r) \) by Lemma 4.4 (2). Note that \( \supp(I_1) \cap \supp(I(D_2 \ast \cdots \ast D_r)) = \emptyset \), thus, by Lemma
\[2.5\] (1), (2) and the induction, we have

\[
depth (S/L) = \text{depth} (S_1/I_1) + \text{depth} (S_2/I(D_2 \ast \cdots \ast D_r)) = 0 + 1 = 1,
\]

\[
\text{reg} (S/L) = \text{reg} (S_1/I_1) + \text{reg} (S_2/I(D_2 \ast \cdots \ast D_r)) = \sum_{x \in V \setminus V_1} w(x) - |V \setminus V_1| + 1 = \sum_{x \in V \setminus V_1} w(x) - |V| + 1
\]

where the last equality holds because the weight of each vertex in \(V_1\) is one, and \(S_1 = k[x_{11}, \ldots, x_{1n_1}]\) and \(S_2 = k[x_{21}, \ldots, x_{2n_2}, \ldots, x_{r_1}, \ldots, x_{rn_r}]\). Moreover, we also have

\[
depth (S/K) = n_1 + \text{depth} (S_2/\sum_{j=2}^{r} I_j^{w_j}) = n_1,
\]

\[
\text{reg} (S/K) = \text{reg} (K) - 1 = \sum_{i=2}^{r} \text{reg} (I_i^{w_i}) - (r - 1) = \sum_{i=2}^{r} (\sum_{j=1}^{n_i} w_{ij} - |V_i| + 1) - (r - 1) = \sum_{x \in V \setminus V_1} w(x) - |V \setminus V_1| = \sum_{x \in V} w(x) - |V|.
\]

Notice that \(K + L = \sum_{j=2}^{r} I_j^{w_j} + I_1 + I(D_2 \ast \cdots \ast D_r) = I_1 + \sum_{j=2}^{r} I_j^{w_j}\), it follows that \(\text{depth} (S/(K, L)) = 0\) and \(\text{reg} (S/(K, L)) = \sum_{x \in V} w(x) - |V|\). The desired result holds by Lemma 2.4 (2), (4) and the following exact sequence

\[0 \rightarrow S/K \cap L \rightarrow S/K \oplus S/L \rightarrow S/(K, L) \rightarrow 0. \]

An immediate consequence of the above theorem is the following corollary.

**Corollary 4.6.** Let \(G = K_{a_1, a_2, \ldots, a_r}\) (resp. \(G = K_n\)) be a complete \(r\)-partite graph (resp. a complete graph), and let \(I(G)\) the edge ideal of \(G\), then

\[
\text{depth} (S/I(G)) = \text{reg} (S/I(G)) = 1.
\]

In particular, \(I(G)\) has a linear resolution.

**Proof.** This is a direct consequence of the above theorem in the case of all vertices of \(D_i\) having trivial weights. \(\square\)
In the following, we provide a technical lemma, which is useful to prove our main result of this section.

**Lemma 4.7.** Let $\mathbb{K}$ be a field of char $(\mathbb{K}) = 0$, and let $S = \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be a polynomial ring over $\mathbb{K}$ and $I \subset (x_1, \ldots, x_m)^2$ a monomial ideal in $\mathbb{K}[x_1, \ldots, x_m]$. Then, for all $t \geq 1$, we have

$$\text{reg} \left((I + (y_1^{w(y_1)}, \ldots, y_n^{w(y_n)}))^t \right) + (x_1, \ldots, x_m)^{(t)} \leq \sum_{i=1}^{n} w(y_i) - n + 1 + (t - 1)(w + 1)$$

where $w = \max\{w(y_i) \mid i \in [n]\}$.

**Proof.** Let $K = (I + (y_1^{w(y_1)}, \ldots, y_n^{w(y_n)}))^t + (x_1, \ldots, x_m)^{(t)}$. Since $x_i^t, y_j^{tw(y_j)} \in K$ for all $i \in [m]$ and $j \in [n]$, and $K$ is m-primary, where $m = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a maximal graded ideal in $S$. Therefore,

$$\text{reg} \left(K \right) = \max\{j + 1 : (S/K)_j \neq 0\}.$$ 

Let $A = \sum_{i=1}^{n} w(y_i) - n + 1 + (t - 1)(w + 1)$. Without loss of generality, we can assume that $w = w(y_n)$. In order to get the desired assertions, it is sufficient to prove that $(S/K)_j = 0$ for all $j \geq A$. That is, it is enough to prove that $f \in K$ for any monomial $f$ with $\deg(f) \geq A$.

Indeed, let $f = \left(\prod_{i=1}^{m} x_i^{a_i} \right) \left(\prod_{j=1}^{n} y_j^{b_j} \right)$ with $\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j \geq A$. We consider the following two cases:

1. If $\sum_{i=1}^{m} a_i \geq t$, then $f \in (x_1, \ldots, x_m)^{t} = (x_1, \ldots, x_m)^{(t)} \subseteq K$.
2. If $\sum_{i=1}^{m} a_i \leq t - 1$, then $\sum_{j=1}^{n} b_j \geq A - t + 1 = \sum_{i=1}^{n} w(y_i) - n + 1 + (t - 1)w$. In this case, let $f_1 = \prod_{j=1}^{n} y_j^{b_j}$, then $\deg(f_1) \geq \sum_{i=1}^{n} w(y_i) - n + 1 + (t - 1)w$.

Next, we prove that $f_1 \in (y_1^{w(y_1)}, \ldots, y_n^{w(y_n)})^t$ by induction on $t$ and $n$, thus $f_1 \in K$, since $(y_1^{w(y_1)}, \ldots, y_n^{w(y_n)})^t \subseteq (I + (y_1^{w(y_1)}, \ldots, y_n^{w(y_n)}))^t$.

The case $n = 1$ is trivial. If $t = 1$, then $\deg(f_1) = \sum_{j=1}^{n} b_j \geq \sum_{i=1}^{n} w(y_i) - n + 1$.

In this case, there exists some $i \in [n]$ such that $b_i \geq w(y_i)$, which implies that $f_1 \in (y_1^{w(y_1)}, \ldots, y_n^{w(y_n)})$, as wished. On the contrary, if $b_j \leq w(y_j) - 1$ for any $j \in [n]$, then $\deg(f_1) \leq \sum_{i=1}^{n} (w(y_i) - 1) = \sum_{i=1}^{n} w(y_i) - n$, contradicting with the assumption that $\deg(f_1) \geq \sum_{i=1}^{n} w(y_i) - n + 1$.

Now, we assume that $n, t \geq 2$. By similar arguments as the case $t = 1$, we can obtain that there exists some $i \in [n]$ such that $b_i \geq w(y_i)$. In this case, we set $b_i = pw(y_i) + q$ with $p \geq 1, 0 \leq q \leq w(y_i) - 1$. We divide into the following two cases:

1. If $p \geq t$, then $f_1 \in (y_1^{w(y_1)}, \ldots, y_n^{w(y_n)})^t.$
Lemma 4.1 and 4.3, one has

\[ \text{then, for all } i, \text{notations of Lemma 4.4, we have} \]

\[ w \]

\[ = \sum_{i=1}^{n} w(y_i) - n + 2 + (t - 1)w - (p + 1)w(y_i) \]

\[ \geq \sum_{i=1}^{n} w(y_i) - (n - 1) + 1 + (t - 1 - p)w \]

where the last inequality holds because of \( w = \max\{ w(y_i) | i \in [n] \} \) and \( w = w(y_n) \).

Hence, by induction, we have \( f_1/y_i^{b_i} \in (y_1^{w(y_1)}, \ldots, y_i^{w(y_i)}, \ldots, y_n^{w(y_n)})^{t-p} \), where \( y_i^{w(y_i)} \) denotes the element \( y_i^{w(y_i)} \) being omitted from \( (y_1^{w(y_1)}, \ldots, y_i^{w(y_i)}, \ldots, y_n^{w(y_n)})^{t-p} \). It follows that \( f_1 \in (y_1^{w(y_1)}, \ldots, y_i^{w(y_i)}, \ldots, y_n^{w(y_n)})^{t-p} \) \( y_i^{w(y_i)} \) \( \subseteq (y_1^{w(y_1)}, \ldots, y_n^{w(y_n)})^t \). And this concludes the proof. \( \square \)

**Theorem 4.8.** Let \( D := D_1 \ast D_2 \) be the join of two weighted oriented graphs \( D_1 \) and \( D_2 \), where \( D_i \) consists of isolated vertices with \( V_i = \{ x_{ij} | j \in [n_i] \} \) for \( i = 1, 2 \). Then, for all \( t \geq 1 \), we have

1. \( \text{depth}(S/I(D)^{(t)}) = 1; \)
2. \( \text{reg}(S/I(D)^{(t)}) = \sum_{x \in V} w(x) - |V| + 1 + (t - 1)(w + 1) \)

where \( w = \max\{ w(x) | x \in V \} \) and \( V = V_1 \cup V_2 \).

**Proof.** (1) follows from Theorem 4.2 since \( \text{depth}(S/I(D)^{(t)}) = \text{depth}(S/I(D)^{t}) \) in this case.

(2) It is clear for the \( t = 1 \) by Theorem 4.2 (2). Now, assume that \( t \geq 2 \). Using notations of Lemma 4.4, we have \( I(D)^{(t)} = I_1^{(t)} \cap (I_2^{w_2})^{(t)} \) by Lemma 2.6. Thus, by Lemmas 4.1 and 4.3, one has

\[ \text{reg}(S/I_1^{(t)}) = \text{reg}(S/I_1^t) = t - 1, \]

\[ \text{reg}(S/(I_2^{w_2})^{(t)}) = \text{reg}(S/(I_2^{w_2})^t) = \sum_{x \in V} w(x) - |V| + (t - 1)w, \]

\[ \text{reg}(S/(I_1^{(t)} + (I_2^{w_2})^{(t)})) = \text{reg}(S/(I_1^t + (I_2^{w_2})^t)) \]

\[ = \sum_{x \in V} w(x) - |V| + (t - 1)w + t - 1 \]

\[ = \sum_{x \in V} w(x) - |V| + (t - 1)(w + 1). \]
Hence, the desired results hold by using Lemma 2.7, 4.3 and the induction, we have
\[ 0 \rightarrow S/I(D)^{(t)} \rightarrow S/I_1^{(t)} \oplus S/(I_2^{w_2})^{(t)} \rightarrow S/(I_1^{(t)} + (I_2^{w_2})^{(t)}) \rightarrow 0. \ 
\]

Now we present our main theorem of this section.

**Theorem 4.9.** Let \( r \geq 2 \) be an integer and let \( D := D_1 \ast \cdots \ast D_r \) be the join of weighted oriented graphs \( D_1, \ldots, D_r \), where \( D_i \) consists of isolated vertices with \( V_i = \{ x_{ij} \mid j \in [n_i] \} \) for any \( i \in [r] \). Then, for any \( t \geq 1 \), we have
\[
(1) \ \text{depth}(S/I(D)^{(t)}) = 1;
\]
\[
(2) \ \text{reg}(S/I(D)^{(t)}) \leq \sum_{x \in V} w(x) - |V| + 1 + (t - 1)(w + 1).
\]
And these equalities hold if \( w = \max\{w(x) \mid x \in V_2\} \), where \( w = \max\{w(x) \mid x \in V\} \) and \( V = V_1 \sqcup \cdots \sqcup V_r \).

**Proof.** We prove the assertion by induction on \( r \) and \( t \). The cases \( t = 1 \), or \( t \geq 2 \) and \( r = 2 \) follows from Theorem 4.5 or Theorem 4.8 respectively.

Now, assume that \( t \geq 2 \) and \( r \geq 3 \). Using notations of Lemma 4.4, we have \( I(D) = \bigcap_{i=1}^r \tilde{I}_i \). By similar arguments as the case \( r = 2 \) in Theorem 4.8, we obtain
\[
I(D)^{(t)} = \left( \bigcap_{i=1}^r \tilde{I}_i \right)^{(t)} = \left( \bigcap_{i=1}^{r-1} \tilde{I}_i \right)^{(t)} \cap \left( \tilde{I}_r \right)^{(t)} = (J + I_r^{w_r})^{(t)} \cap \left( \tilde{I}_r \right)^{(t)},
\]
where \( J = I(D_1 \ast \cdots \ast D_{r-1}) \).

Note that \( \text{reg}(S/\tilde{I}_r^{(t)}) = t - 1 \) and \( \text{reg}(S/((J + I_r^{w_r})^{(t)} + \tilde{I}_r^{(t)})) \leq \sum_{x \in V_r} w(x) - |V_r| + 1 + (t - 1)(w' + 1) \) by Lemma 4.7, where \( w' = \max\{w(x) \mid x \in V_r\} \). Meanwhile, we also have \( \text{depth}(S/\tilde{I}_r^{(t)}) = n_r \) and \( \text{depth}(S/((J + I_r^{w_r})^{(t)} + \tilde{I}_r^{(t)})) = 0 \), since \((J + I_r^{w_r})^{(t)} + \tilde{I}_r^{(t)} \) is \( m \)-primary in \( S \), which is shown in Lemma 4.7. Therefore, by Lemmas 2.7, 4.3 and the induction, we have
\[
\text{reg}
\left( \frac{S}{(J + I_r^{w_r})^{(t)}} \right) = \max_{i \in [t-1]} \left\{ \text{reg} \left( \frac{S}{J^{(t-i)}} \right) + \text{reg} \left( \frac{S}{I_r^{w_r}} \right) + 1, \text{reg} \left( \frac{S}{J^{(t-j+1)}} \right) \right\}
\]
\[
+ \text{reg} \left( \frac{S}{I_r^{w_r}} \right)
\]
\[
= \max_{j \in [t]} \left\{ \text{reg} \left( \frac{S}{J^{(t-j+1)}} \right) + \text{reg} \left( \frac{S}{I_r^{w_r}} \right) \right\}
\]
\[
\leq \max_{j \in [t]} \left\{ \sum_{x \in V \setminus V_r} w(x) - |V \setminus V_r| + 1 + (t - j)(w'' + 1) + \sum_{x \in V_r} w(x) - |V_r| + (j - 1)w' \right\}
\]
\[
= \max_{j \in [t]} \left\{ \sum_{x \in V} w(x) - |V| + 1 + (t - j)w'' + (j - 1)w' + t - j \right\}
\]
\[
\leq \sum_{x \in V} w(x) - |V| + 1 + (t - 1)(w + 1),
\]
where \( w'' = \max \{ w(x) \mid x \in V \setminus V_r \} \) and \( V = V_1 \sqcup \cdots \sqcup V_r \). And the second equality holds because of \( \text{reg} \left( S/(I^{w_r}_{r})^{(i+1)} \right) \geq \text{reg} \left( S/(I^{w_r}_{r})^{(i)} \right) + 1 \) for any \( i \in [t-1] \) by Lemma 4.3.

If \( w = \max \{ w(x) \mid x \in V_2 \} \), then the first inequality in the expression of \( \text{reg} \left( S/(I^{w_r}_{r})^{(t)} \right) \) becomes equality by the induction, and the last inequality becomes equality because the function \( g(j) := \sum_{x \in V} w(x) - |V| + 1 + (t - j)w + (j - 1)w' + (t - j) \) is strictly monotonic decreasing with respect to \( j \).

Hence, the desired results hold by using Lemma 2.4 (1) to the following exact sequence
\[
0 \to S/I(D)^{(t)} \to S/(J + I^{w_r}_{r})^{(t)} + S/\hat{I}^{(t)} \to S/((J + I^{w_r}_{r})^{(t)} + \hat{I}^{(t)}) \to 0.
\]

An immediate consequence of the above theorem and Theorem 4.5 is the following corollary.

**Corollary 4.10.** Let \( D := D_1 \ast \cdots \ast D_r \) be the join of weighted oriented graphs \( D_1, \ldots, D_r \) as in Theorem 4.9, then, for any \( t \geq 1 \), we have
\[ \text{reg} \left( S/I(D)^{(t)} \right) \leq \text{reg} \left( S/I(D) \right) + (t - 1)(w + 1). \]

And these equalities hold if \( w = \max \{ w(x) \mid x \in V_2 \} \), where \( w = \max \{ w(x) \mid x \in V(D) \} \).

The following examples show that the upper bounds in Theorems 4.9 can be obtained, but may be strict.

**Example 4.11.** Let \( I(D) = (x_1 x_3^2, x_1 x_4^2, x_1 x_5^3, x_2 x_3^2, x_2 x_4^2, x_2 x_5^3, x_3 x_4^2, x_3 x_5^3) \) be the edge ideal of the join of weighted oriented graphs with the partition \( V_1 = \{ x_1, x_2 \}, V_2 = \{ x_3 \}, V_3 = \{ x_4, x_5 \} \). The weight function is \( w_1 = w_2 = 1, w_3 = w_4 = 2 \) and \( w_5 = 3 \). By using CoCoA, we obtain \( \text{reg} \left( S/I(D)^{(2)} \right) = 8 \). But we have \( \sum_{i=1}^{5} w_i - 5 + 1 + (3+1) = 9 \).

**Example 4.12.** Let \( I(D) = (x_1 x_3^2, x_1 x_4^2, x_1 x_5^3, x_2 x_3^2, x_2 x_4^2, x_2 x_5^3, x_3 x_4^2, x_3 x_5^3) \) be the edge ideal of the join of weighted oriented graphs with the partition \( V_1 = \{ x_1, x_2 \}, V_2 = \{ x_3 \}, V_3 = \{ x_4, x_5 \} \). The weight function is \( w_1 = w_2 = 1, w_3 = 5, w_4 = 2 \) and \( w_5 = 3 \). Thus \( w = \max \{ w_i \mid i \in [5] \} = 5 \). By using CoCoA, we have \( \text{reg} \left( S/I(D)^{(2)} \right) = \sum_{i=1}^{5} w_i - 5 + 1 + (5+1) = 14 \).

5. Ordinary powers of the edge ideal of the join of graphs with at least an oriented edge

In this section, we study depth and regularity of ordinary powers of the edge ideal of the join of weighted oriented graphs with at least an oriented edge for per graph. We give the exact formulas for depth and regularity of the edge ideal of the join of two weighted oriented graphs, and also provide the upper bounds of regularity of ordinary powers when the second graph is a weighted oriented complete graph.
Lemma 5.1. Let $D := D_1 \ast D_2$ be the join of two weighted oriented graphs $D_1$ and $D_2$, where the vertex set of $D_i$ is $V_i = \{x_{ij} \mid j \in [n_i]\}$ and $n_i = |V_i|$ for $i \in [2]$. If each $D_i$ contains at least an oriented edge. Then

1. $\text{depth}(S/I(D)) = 1$;
2. $\text{reg}(S/I(D)) = \max \{ \sum_{x \in V_2} w(x) - |V_2| + \text{reg}(S_1/I(D_1)), \text{reg}(S_2/I(D_2)) \}$, where $S_1 = k[x_{11}, \ldots, x_{1n_1}]$ and $S_2 = k[x_{21}, \ldots, x_{2n_2}]$.

Proof. We use notations of Lemma 4.4. Obviously, we have $I(D) = (\widehat{I}_1 + I(D_1)) \cap (\widehat{I}_2 + I(D_2))$ by Theorem 3.5. Set $J_i = \widehat{I}_i + I(D_i)$ for $i \in [2]$, then $I(D) = J_1 \cap J_2$ and $\text{supp} (\widehat{I}_i) \cap \text{supp} (I(D_i)) = \emptyset$. Thus, by Lemma 2.5 (2), one has

$$\text{depth}(S/J_1) = \text{depth}(S_1/I(D_1)) + \text{depth}(\widehat{I}_1) = \text{depth}(S_1/I(D_1)), \quad \text{depth}(S/J_2) = \text{depth}(S_2/I(D_2)) + \text{depth}(\widehat{I}_2) = \text{depth}(S_2/I(D_2)).$$

By Lemmas 2.2 (1) and 2.5 (1), we have

$$\text{reg}(S/J_1) = \text{reg}(\widehat{I}_1) + \text{reg}(S_1/I(D_1)) = \sum_{x \in V_2} w(x) - |V_2| + \text{reg}(S_1/I(D_1)), \quad \text{reg}(S/J_2) = \text{reg}(\widehat{I}_2) + \text{reg}(S_2/I(D_2)) = \text{reg}(S_2/I(D_2)).$$

Note that $(J_1, J_2) = (\widehat{I}_1 + I(D_1), \widehat{I}_2 + I(D_2)) = (x_{11}, \ldots, x_{1n_1}, x_{21}^{w_{21}}, \ldots, x_{2n_2}^{w_{2n_2}})$. Hence one has

$$\text{depth}(S/(J_1, J_2)) = 0 \quad \text{and} \quad \text{reg}(S/(J_1, J_2)) = \sum_{x \in V_2} w(x) - |V_2|.$$

Applying lemma 2.4 (1), (4) to the following exact sequence

$$0 \rightarrow S/J_1 \cap J_2 \rightarrow S/J_1 \oplus S/J_2 \rightarrow S/(J_1, J_2) \rightarrow 0,$$

we obtain the desired results.

Theorem 5.2. Let $r \geq 3$ be an integer, and let $D := D_1 \ast \cdots \ast D_r$ be the join of weighted oriented graphs $D_1, \ldots, D_r$, where the vertex set of $D_i$ is $V_i = \{x_{ij} \mid j \in [n_i]\}$ and $n_i = |V_i|$ for $i \in [r]$. If each $D_i$ contains at least an oriented edge. Then

1. $\text{depth}(S/I(D)) = 1$;
2. $\text{reg}(S/I(D)) = \max \{ \text{reg}(S_i/I(D_i)) + \sum_{x \in T_i} w(x) - |T_i| : i \in [r] \}$, where $T_i = V(D) \setminus \bigcup_{j=1}^{i} V_j$ for any $i \in [r]$.

Proof. We apply induction on $r$. The case $r = 2$ follows from Lemma 5.1. Now, assume that $r \geq 3$. In this case, we have $D = (D_1 \ast \cdots \ast D_{r-1}) \ast D_r$ and $I(D) = I((D_1 \ast \cdots \ast D_{r-1}) \ast D_r)$. Thus, by Lemma 5.1 and the inductive hypothesis, we
obtain that depth $(S/I(D)) = 1$ and
\[
\begin{align*}
\text{reg}\left( \frac{S}{I(D)} \right) &= \text{reg}\left( \frac{S}{I((D_1 \cdots \cdots D_{r-1})*D_r)} \right) \\
&= \max \left\{ \sum_{x \in V_r} w(x) - |V_r| + \text{reg}\left( \frac{S'}{I(D_1*D_2*\cdots*D_{r-1})} \right), \text{reg}\left( \frac{S_r}{I(D_r)} \right) \right\} \\
&= \max \left\{ \sum_{x \in V_r} w(x) - |V_r| + \max \left\{ \sum_{x \in T_i'} w(x) - |T_i'| + \text{reg}\left( \frac{S_i}{I(D_i)} \right) : i \in [r-1] \right\}, \text{reg}\left( \frac{S_r}{I(D_r)} \right) \right\} \\
&= \max \left\{ \text{reg}\left( \frac{S_i}{I(D_i)} \right) + \sum_{x \in T_i} w(x) - |T_i| : i \in [r] \right\}
\end{align*}
\]
where $S' = S_1 \otimes_K S_2 \otimes_K \cdots \otimes_K S_{r-1}$ and $T_i' = V(D) \setminus (V_r \cup (\bigcup_{j=1}^{i} V_j))$. \hfill \Box

**Remark 5.3.** Let $r \geq 2$ be an integer, and let $D := D_1 \ast \cdots \ast D_r$ be the join of weighted oriented graphs $D_1, \ldots, D_r$ with $V_i = \{ x_{ij} \mid j \in [n_i] \}$ and $n_i = |V_i|$ for $i \in [r]$. If there exist some $D_i$ consisting of isolated vertices and some $D_j$ containing at least an oriented edge. From the proof of Lemma 5.1 and Theorem 5.2, we still have depth $(S/I(D)) = 1$. But we can’t guarantee the regularity of $S/I(D)$ to obtain the equality, that is, we have
\[
\text{reg}(S/I(D)) \leq \max \left\{ \sum_{x \in T_i} w(x) - |T_i| + 1, \text{reg}\left( \frac{S_i}{I(D_i)} \right) + \sum_{x \in T_i} w(x) - |T_i| : i \in [r] \right\}.
\]

An immediate consequence of Theorem 5.2 is the generalizations of [17, Corollary 3.10, Proposition 3.12].

**Corollary 5.4.** Let $G := G_1 \ast \cdots \ast G_r$ be the join of graphs $G_1, \ldots, G_r$. If each $G_i$ contains at least an edge. Then
\begin{enumerate}
\item depth $(S/I(G)) = 1$;
\item $\text{reg}(S/I(G)) = \max \{ \text{reg}(S_i/I(G_i)) \mid i \in [r] \}$.
\end{enumerate}

**Proof.** This is a direct consequence of the above theorem in the case of all vertices of $D_i$ having trivial weights. \hfill \Box

**Theorem 5.5.** Let $K_n$ be a weighted oriented complete graph with edge ideal $I(K_n) = (x_i x_j^{w_{ij}} : i, j \in [n] \text{ and } i < j)$. Then, for any $t \geq 1$, we have
\[
\text{reg}\left( \frac{S}{I(K_n)^t} \right) \leq \sum_{i=1}^{n} w_i - n + 1 + (t-1)(w+1)
\]
where $w = \max \{ w_i \mid i \in [n] \}$. These equalities hold here when $w = w_2$. \hfill \Box
Proof. We apply induction on $n$ and $t$. The case $n = 2$ is trivial and the case $t = 1$ follows from Theorem 4.5 (2). Now, assume that $n \geq 3$ and $t \geq 2$. Let $J = (x_1, \ldots, x_{n-1})$, then $I(K_n)^t = (I(K_n \setminus x_n) + J x_n)^t = \sum_{i=1}^{t} I(K_n \setminus x_n)^{t-i}(J x_n)^i + I(K_n \setminus x_n)^t$. Let $P_0 = I(K_n)^t$, $P_j = I(K_n)^t : (x_n^{w_n})^j$ and $Q_{j-1} = P_{j-1} + (x_n^{w_n})$ for any $j \in [t]$. Since $I(K_n \setminus x_n) \subseteq J$, we have $P_j = \sum_{i=j+1}^{t} (J^i x_n^{w_n(i-j)} I(K_n \setminus x_n)^{t-i}) + J^j I(K_n \setminus x_n)^{t-j}$ with the convention $P_t = J^t$, and $Q_{j-1} = J^{j-1} I(K_n \setminus x_n)^{t-j+1} + (x_n^{w_n})$. Thus, by Lemma 2.5 (1) and the inductive hypothesis, we have

\[
\text{reg} \left( \frac{S}{Q_0} \right) = \text{reg} \left( Q_0 \right) - 1 = \text{reg} \left( I(K_n \setminus x_n)^t + (x_n^{w_n}) \right) - 1 \\
= \text{reg} \left( I(K_n \setminus x_n)^t \right) + w_n - 2 \\
\leq \sum_{i=1}^{n-1} w_i - (n - 1) + 2 + (t - 1)(w' + 1) + w_n - 2 \\
= \sum_{i=1}^{n} w_i - n + 1 + (t - 1)(w' + 1) \\
\leq \sum_{i=1}^{n} w_i - n + 1 + (t - 1)(w' + 1),
\]

where $w' = \max\{w_i \mid i \in [n - 1]\}$. By Lemma 4.1 we have

\[
\text{reg} \left( \frac{S}{P_t}(-tw_n) \right) = \text{reg} \left( S/J^t \right) + tw_n = (t - 1)(w_n + 1) + w_n \\
\leq \sum_{i=1}^{n} w_i - n + 1 + (t - 1)(w + 1).
\]

Note that $\dim \left( \frac{S}{J^j} \right) = 1$ for any $j \in [t - 1]$. Then, by Lemmas 2.3 2.2 (1), 2.5 (1) and the inductive hypothesis, we have

\[
\text{reg} \left( \frac{S}{Q_j}(-jw_n) \right) = \text{reg} \left( J^j I(K_n \setminus x_n)^{t-j} + (x_n^{w_n}) \right) - 1 + jw_n \\
\leq \text{reg} \left( J^j \right) + \text{reg} \left( I(K_n \setminus x_n)^{t-j} \right) + w_n - 2 + jw_n \\
\leq \sum_{i=1}^{n} w_i - n + 1 + (t - j - 1)(w' + 1) + j(w_n + 1) \\
\leq \sum_{i=1}^{n} w_i - n + 1 + (t - 1)(w + 1).
\]
where the last inequality holds since \( w_n, w' \leq w \). Using Lemma 2.4 (1) to the following exact sequences by shifting

\[
\begin{align*}
0 & \rightarrow \frac{S}{P_1}(-w_n) \xrightarrow{-w_n} I(K_n) \rightarrow \frac{S}{Q_0} \rightarrow 0 \\
0 & \rightarrow \frac{S}{P_2}(-w_n) \xrightarrow{-w_n} \frac{S}{P_1} \rightarrow \frac{S}{Q_1} \rightarrow 0 \\
& \vdots \vdots \vdots \\
0 & \rightarrow \frac{S}{P_{t-1}}(-w_n) \xrightarrow{-w_n} \frac{S}{P_{t-2}} \rightarrow \frac{S}{Q_{t-2}} \rightarrow 0 \\
0 & \rightarrow \frac{S}{P_t}(-w_n) \xrightarrow{-w_n} \frac{S}{P_{t-1}} \rightarrow \frac{S}{Q_{t-1}} \rightarrow 0,
\end{align*}
\]

we get the desired results.

If \( w_2 = w \), then, by the same technique, we can get \( \text{reg} (S/Q_0) = \sum_{i=1}^{n} w_i - n + 1 + (t - 1)(w + 1) \). Hence, the expected equality also holds in this case. \( \square \)

The following examples show that the upper bounds in Theorems 5.5 can be obtained, but may be strict.

**Example 5.6.** Let \( I(K_4) = (x_1x_2^7, x_1x_3^3, x_1x_4^6, x_2x_3^3, x_2x_4^6, x_3x_4^6) \) be the edge ideal of a weighted oriented complete graph \( K_4 \), its weight function is \( w_1 = 1, w_2 = 7, w_3 = 3 \) and \( w_4 = 6 \). By using CoCoA, we obtain \( \text{reg} (S/I(K_4)^2) = \sum_{i=1}^{4} w_i - 4 + 1 + (7 + 1) = 22 \), where \( w = 7 \).

**Example 5.7.** Let \( I(K_4) = (x_1x_2^2, x_1x_3^3, x_1x_4^6, x_2x_3^3, x_2x_4^6, x_3x_4^6) \) be the edge ideal of a weighted oriented complete graph \( K_4 \), its weight function is \( w_1 = 1, w_2 = 2, w_3 = 3 \) and \( w_4 = 6 \). By using CoCoA, we obtain \( \text{reg} (S/I(K_4)^2) = 15 \). However, \( \sum_{i=1}^{4} w_i - 4 + 1 + (2 - 1)(w + 1) = 16 \), where \( w = 6 \).

**Lemma 5.8.** Let \( D = D_1 \star D_2 \) be the join of two weighted oriented graphs \( D_1 \) and \( D_2 \), where \( V(D_i) = \{x_{ij} \mid j \in [n_i]\} \) and \( n_i = |V(D_i)| \) for \( i \in [2] \). If \( n_2 = 1 \), then

\[
\text{reg} (S/I(D)) \geq \text{reg} (S/I(D_1)) + w_{21} - 1.
\]

where \( S_1 = \mathbb{K}[x_{11}, \ldots, x_{1n_1}] \). The equality holds when \( D_1 \) contains at least an oriented edge.

**Proof.** The case \( D_1 \) contains of isolated vertices is obvious by Theorem 1.2 (2). We can assume that \( D_1 \) contains at least an oriented edge. Since \( I(D) : x_{21}^{w_{21}} = (x_{1i} \mid i \in [n_1]) \) and \( I(D) + (x_{21}^{w_{21}}) = I(D_1) + (x_{21}^{w_{21}}) \), we have \( \text{reg} ((S/I(D) : x_{21}^{w_{21}})(-w_{21})) = w_{21} \) and \( \text{reg} (S/I(D) + (x_{21}^{w_{21}})) = \text{reg} (S/I(D_1)) + w_{21} - 1 \). We can obtain the desired result by a fact that \( \text{reg} (S/I(D_1)) \geq 1 \) and by using Lemma 2.4 (1) to the following exact sequence

\[
0 \rightarrow (S/I(D) : x_{21}^{w_{21}})(-w_{21}) \xrightarrow{-w_{21}} S/I(D) \rightarrow S/I(D_1, x_{21}^{w_{21}}) \rightarrow 0. \quad \square
\]

Now we are ready to present the main result of this section.
Theorem 5.9. Let $D = D_1 \ast D_2$ be the join of two weighted oriented graphs $D_1$ and $D_2$, where $V(D_1) = \{x_{ij} \mid j \in [n_i]\}$ and $n_i = |V(D_i)|$ for $i \in [2]$. If $D_2$ is a weighted oriented complete graph as Theorem 5.4 and $\text{reg}(S/I(D_1)') \leq \text{reg}(S/I(D_1)) + (t - 1)(w' + 1)$, where $S_1 = \mathbb{K}[x_{ij} : i \in V(D_1)]$ and $w' = \max\{w(x) \mid x \in V(D_1)\}$. Then, for any integer $t \geq 1$, we have

$$\text{reg}(S/I(D)'^t) \leq \text{reg}(S/I(D)) + (t - 1)(w + 1)$$

where $w = \max\{w(x) \mid x \in V(D_1) \cup V(D_2)\}$. The equality holds when $w' = w$ and $\text{reg}(S/I(D)'^t)) = \text{reg}(S/I(D_1)) + (t - 1)(w' + 1)$.

Proof. We apply induction on $t$ and $n_2$. The case $t = 1$ is trivial and the $n_2 = 1$ case follows from Lemma 5.10. Now, suppose $t \geq 2$ and $n_2 \geq 2$. Let $P_0 = I(D)'$, $P_j = I(D)' : (x_{2n_2}^{w_{2n_2}})^j$ and $Q_{j-1} = P_{j-1} + (x_{2n_2}^{w_{2n_2}})$ for any $j \in [t]$, and set $J = \{x_{1i} \mid i \in [n_1]\} + \{x_{2i} \mid i \in [n_2 - 1]\}$. By some simple calculations, we can get

$$P_j = \sum_{i=j+1}^{2n_2} J_i x_{2n_2}^{(i-j)w_{2n_2}} I(D \setminus x_{2n_2})^{t-i} + J_i I(D \setminus x_{2n_2})^{t-i}$$

with the convention $P_t = J_t$ and $Q_{j-1} = J^{j-1} I(D \setminus x_{2n_2})^{t-j+1} + (x_{2n_2}^{w_{2n_2}})$.

By Lemmas 2.3 (1), 5.8 and the inductive hypothesis, we have

$$\text{reg}(S/Q_0) = \text{reg}(S/I(D \setminus x_{2n_2})^t) + w_{2n_2} - 1$$

$$\leq \text{reg}(S/I(D \setminus x_{2n_2})) + (t - 1)(w'' + 1) + w_{2n_2} - 1$$

$$= \text{reg}(S/I(D)) + (t - 1)(w'' + 1)$$

$$\leq \text{reg}(S/I(D)) + (t - 1)(w + 1),$$

where $w'' = \max\{w(x) \mid x \in V(D \setminus x_{2n_2})\}$. By Lemma 1.1, we obtain that

$$\text{reg}((S/P_t)(-tw_{2n_2})) = \text{reg}(S/J^t) + tw_{2n_2} = (t - 1)(w_{2n_2} + 1) + w_{2n_2}$$

$$\leq \text{reg}(S/I(D)) + (t - 1)(w + 1).$$

where the inequality holds since $w \geq w_{2n_2}$ and $\text{reg}(S/I(D)) \geq w_{2n_2}$.

Note that $\dim(S/J^j) = 1$ for any $j \in [t - 1]$. Then, by Lemmas 2.3, 2.2 (1), 2.5 (1) and the inductive hypothesis, we have

$$\text{reg}((S/Q_j)(-jw_{2n_2})) = \text{reg}(J^j I(D \setminus x_{2n_2})^{t-j} + (x_{2n_2}^{w_{2n_2}})) - 1 + jw_{2n_2}$$

$$= \text{reg}(J^j I(D \setminus x_{2n_2})^{t-j}) + w_{2n_2} - 2 + jw_{2n_2}$$

$$\leq \text{reg}(J^j) + \text{reg}(I(D \setminus x_{2n_2})^{t-j}) + w_{2n_2} - 2 + jw_{2n_2}$$

$$= j + \text{reg}(S/I(D \setminus x_{2n_2})^{t-j}) + w_{2n_2} - 1 + jw_{2n_2}$$

$$\leq \text{reg}(S/I(D \setminus x_{2n_2})) + (t - j - 1)(w'' + 1)$$

$$+ w_{2n_2} - 1 + j(w_{2n_2} + 1)$$

$$= \text{reg}(S/I(D)) + (t - j - 1)(w'' + 1) + j(w_{2n_2} + 1)$$

$$\leq \text{reg}(S/I(D)) + (t - 1)(w + 1)$$

where the last inequality holds since $w_{2n_2}, w'' \leq w$, and the penultimate equality holds because $D \setminus x_{2n_2}$ contains at least an oriented edge.
In particular, if \( w' = w \) and \( \text{reg} \left( S_1/I(D_1)^t \right) = \text{reg} \left( S_1/I(D_1) \right) + (t-1)(w'+1) \), it follows from Lemmas 2.3 (1), 5.8 and the inductive hypothesis that \( \text{reg}(S/Q_0) = \text{reg}(S/I(D)) + (t-1)(w+1) \).

We can confirm the assertion by applying Lemma 2.4 (1) to the following exact sequences

\[
\begin{align*}
0 \rightarrow & \quad \frac{S}{P_1}(-w_{2n_2}) \xrightarrow{\cdot x_{2n_2}} \frac{S}{P_1}(-w_{2n_2}) \rightarrow \frac{S}{Q_0} \rightarrow 0 \\
0 \rightarrow & \quad \frac{S}{P_2}(-w_{2n_2}) \xrightarrow{\cdot x_{2n_2}} \frac{S}{P_2}(-w_{2n_2}) \rightarrow \frac{S}{Q_0} \rightarrow 0 \\
& \quad \vdots \quad \vdots \\
0 \rightarrow & \quad \frac{S}{P_{t-1}}(-w_{2n_2}) \xrightarrow{\cdot x_{2n_2}} \frac{S}{P_{t-1}}(-w_{2n_2}) \rightarrow \frac{S}{Q_{t-1}} \rightarrow 0.
\end{align*}
\]

(1)

\[\square\]

**Lemma 5.10.** With the assumptions and notation of Theorem 5.9, if \( n_2 = 1 \), then

\[\text{reg} \left( S/I(D)^t \right) \leq \text{reg} \left( S/I(D) \right) + (t-1)(w+1)\]

where \( w = \max\{w(x) \mid x \in V_1 \cup V_2\} \). These equalities hold when \( w' = w \) and \( \text{reg} \left( S_1/I(D_1)^t \right) = \text{reg} \left( S_1/I(D_1) \right) + (t-1)(w'+1) \) for all \( t \geq 1 \).

**Proof.** The proof will be essentially the same as that for Theorem 5.9. We apply induction on \( t \) with the \( t = 1 \) case being trivial. Now, suppose \( t \geq 2 \). Let \( J = \{x_{1i} \mid i \in [n_1]\}, P_0 = I(D)^t, P_j = I(D)^t : (x_{21}^{w_{21}})^j \) and \( Q_{j-1} = P_{j-1} + (x_{21}^{w_{21}}) \) for any \( j \in [t] \).

By Lemmas 2.3 (1) and 5.8, we have

\[
\text{reg} \left( S/Q_0 \right) = \text{reg} \left( S_1/I(D \setminus x_{21})^t \right) + w_{21} - 1
= \text{reg} \left( S_1/I(D_1)^t \right) + w_{21} - 1
\leq \text{reg} \left( S_1/I(D_1) \right) + (t-1)(w' + 1) + w_{21} - 1
\leq \text{reg} \left( S/I(D) \right) + (t-1)(w' + 1)
\leq \text{reg} \left( S/I(D) \right) + (t-1)(w + 1)
\]

where the first inequality holds by the assumption that \( \text{reg} \left( S_1/I(D_1)^t \right) + w_{21} - 1 \leq \text{reg} \left( S_1/I(D_1) \right) + (t-1)(w' + 1) + w_{21} - 1 \).

By some similar arguments as the proof of Theorem 5.9, we can get

\[
\text{reg} \left( S/P_1(-tw_{2n_2}) \right) \leq \text{reg} \left( S/I(D) \right) + (t-1)(w + 1),
\]

\[
\text{reg} \left( S/Q_j(-jw_{21}) \right) \leq \text{reg} \left( S/I(D) \right) + (t-1)(w + 1) \quad \text{for } j = 1, \ldots, t-1.
\]

In particular, if \( w' = w \) and \( \text{reg} \left( S_1/I(D_1)^t \right) = \text{reg} \left( S_1/I(D_1) \right) + (t-1)(w' + 1) \) for all \( t \geq 1 \), then \( D_1 \) contains at least an oriented edge, because on the contrary, we have \( \text{reg} \left( S_1/I(D_1)^t \right) = 0 \). It follows from Lemmas 2.3 (1) and 5.8 that \( \text{reg}(S/Q_0) = \text{reg}(S/I(D)) + (t-1)(w+1) \). Thus we can obtain the assertion by applying Lemma 2.4 (1) to the exact sequences (1). \[\square\]
Remark 5.11. The conditions that \( \operatorname{reg} (S/I(D_1)^t) \leq \operatorname{reg} (S/I(D_1)) + (t-1)(w^t + 1) \) in Theorems 5.9 can be obtained for many weighted oriented graphs, such as weighted rooted forests, naturally weighted oriented cycles, the disjoint union of some weighted oriented gapfree bipartite graph and so on (see [21] Theorem 4.1, [20] Theorem 4.5, [23] Theorem 4.9).

As a consequence of Lemma 5.10, we have the following:

Corollary 5.12. Let \( D_1 \) be a naturally weighted oriented cycle with \( w(x) \geq 2 \) for all \( x \in V(D_1) \) and let \( D_2 \) be a graph composed of a isolated vertex. If \( D = D_1 \ast D_2 \), i.e., \( D \) is a weighted oriented wheel graph, and \( w := \max \{w(x) \mid x \in V(D)\} = \max \{w(x) \mid x \in V(D_1)\} \). Then
\[
\operatorname{reg} (S/I(D)^t) = \sum_{x \in V(D)} w(x) - |V(D)| + 1 + (t-1)(w + 1).
\]

Proof. By [20] Theorem 4.5, we have \( \operatorname{reg} (S_1/(I(D_1)^t)) = \operatorname{reg} (S_1/I(D_1)) + (t-1)(w + 1) \) for all \( t \geq 1 \). The expected formulas follow from Theorem 5.9.

The following examples show that the upper bounds in Theorems 5.9 can be obtained, but may be strict.

Example 5.13. Let \( I(D) = (x_1x_2^2, x_2x_3^3, x_3x_1^4, x_1y^3, x_2y^3, x_3y^3) \) be the edge ideal of a weighted oriented wheel graph \( D \), its weight function is \( w(x_1) = 4, w(x_2) = w(x_3) = 2 \) and \( w(y) = 3 \). By using CoCoA, we have \( \operatorname{reg} (S/I(D)) = 7, \operatorname{reg} (S/I(D)^2) = 12 \). Thus \( \operatorname{reg} (S/I(D)^2) = \operatorname{reg} (S/I(D)) + (w + 1) \), where \( w = 4 \).

Example 5.14. Let \( I(D) = (x_1x_2^2, x_2x_3^3, x_3x_1^4, x_1y^3, x_2y^3, x_3y^3) \) be the edge ideal of a weighted oriented wheel graph \( D \), its weight function is \( w(x_1) = w(x_2) = w(x_3) = 2 \) and \( w(y) = 3 \). By using CoCoA, we have \( \operatorname{reg} (S/I(D)) = 5, \operatorname{reg} (S/I(D)^2) = 8 \). Thus \( \operatorname{reg} (S/I(D)^2) < \operatorname{reg} (S/I(D)) + (w + 1) \).

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