IP*-sets in function field and mixing properties

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Abstract

The ring of polynomial over a finite field $F_q[x]$ has received much attention, both from a combinatorial viewpoint as in regards to its action on measurable dynamical systems. In the case of $(\mathbb{Z}, +)$ we know that the ideal generated by any nonzero element is an IP*-set. In the present article we first establish that the analogous result is true for $F_q[x]$. We further use this result to establish some mixing properties of the action of $(F_q[x], +)$. We shall also discuss on Khintchine’s recurrence for the action of $(F_q[x] \setminus \{0\}, \cdot)$.

Keywords: IP*-set, Central*-set, \triangle-set, Strong mixing, Finite Field

2010 MSC: primary 54D35, 22A15, secondary 05D10, 54D80

1. Introduction

By a measurable dynamical system (MDS), we mean $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$, where $(X, \mathcal{B}, \mu)$ is a probability space and for each $g \in G$, $T_g : X \to X$ is an invertible and measure preserving transformation. For an MDS $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$, let $\mathcal{B}^+$ be the set of all positive measured sets, and $N(A, B) = \{g \in G : \mu(A \cap T_g B) \neq 0\}$. The classical results in ergodic theory state that a transformation $(T_g)_{g \in G}$ is ergodic iff $N(A, B) \neq \emptyset$ for each pair of $A, B \in \mathcal{B}^+$, weakly mixing iff $\{g \in G : |\mu(A \cap T_g B) - \mu(A)\mu(B)| < \epsilon\}$ is a central* set for each pair $A, B$. 

\textsuperscript{1}This author is partially supported by DST-PURSE programme.
\textsuperscript{2}The work of this article was a part of this author’s Ph.D. dissertation which was supported by a CSIR Research Fellowship.
of $A, B \in \mathcal{B}^+$, mildly mixing iff $\{g \in G : |\mu(A \cap T_g B) - \mu(A)\mu(B)| < \epsilon\}$ is an IP*-set, and strongly mixing iff $\{g \in G : |\mu(A \cap T_g B) - \mu(A)\mu(B)| < \epsilon\}$ is a cofinite set. See for example [BD, KY2]. In [KY2] authors described the notions in terms of families. We are here interested with the family of central*-sets, IP*-sets, cofinite sets and difference sets which we shall denote by $C^*$, $\mathcal{IP}^*$, $C_f$ and $\triangle$ respectively. The notions of $C^*$, $\mathcal{IP}^*$ will be defined latter. In these terms $C^*$-mixing implies weak mixing, $\mathcal{IP}^*$-mixing implies mild mixing and $C_f$-mixing implies strong mixing.

**Definition 1.** Let $A$ be a subset of a semigroup $S$.

(1) $A$ is called an IP-set if there is a subsequence $\langle x_n \rangle_{n=1}^{\infty}$ such that all finite subset $F \in \mathcal{P}_f(\mathbb{N})$ sums of forms $\sum_{n \in F} x_n$ are in $A$. A subset $A \subset S$ is said to be an IP*-set, if it meets every IP-set in $S$. The collection of all IP-sets is denoted by $\mathcal{IP}$ and the collection of all IP*-sets will be denoted by $\mathcal{IP}^*$.

(2) $D$ is called a $\triangle$-set if it contains an infinite difference set, i.e. there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ of $S$ such that $D \supset \triangle(\langle x_n \rangle_{n=1}^{\infty}) = \{x_n \cdot x_m^{-1} : m, n \in \mathbb{N}\}$. A subset $D \subset S$ is said to be an $\triangle^*$-set, if it meets every $\triangle$-set in $S$. The collection of $\triangle$-sets is denoted by $\triangle$ and the collection of all $\triangle^*$-sets will be denoted by $\triangle^*$.

In recent years $(\mathbb{F}_q[x], +)$, where $\mathbb{F}_q[x]$ denotes the ring of all polynomials over the finite field of characteristic $q$, has received much attention both from a combinatorial viewpoint as in regards to its action on measurable dynamical systems. In [Le] the author has proved a version of celebrated Green-Tao Theorem for $(\mathbb{F}_q[x], +)$. In [BTZ] the authors proved some higher order versions of Khintchine’s recurrence theorem for the action of $(\mathbb{F}_q[x], +)$. In the present article we shall present some combinatorial properties of $(\mathbb{F}_q[x], +)$ and $(\mathbb{F}_q[x], \cdot)$. Further we will apply these combinatorial properties to find some interesting properties of their actions on measure space.

In order to discuss combinatorial properties of $(\mathbb{F}_q[x], +)$ and $(\mathbb{F}_q[x], \cdot)$ we shall need algebraic properties of its Stone-Čech compactification of $\mathbb{F}_q[x]$. For
this purpose we need to discuss the algebra of the Stone–Čech compactification of a discrete semigroup $S$. For a discrete semigroup $S$, the Stone–Čech compactification of $S$ will be denoted by $\beta S$. We take the points of $\beta S$ to be the ultrafilters on $S$, identifying the principal ultrafilters with the points of $S$ and thus pretending that $S \subseteq \beta S$. Given $A \subseteq S$, we denote

$$A = \{ p \in \beta S : A \in p \}.$$ 

The set $\{A : A \subseteq S\}$ is a basis for the closed sets of $\beta S$. The operation ‘·’ on $S$ can be extended to the Stone–Čech compactification $\beta S$ of $S$ so that $(\beta S, \cdot)$ is a compact right topological semigroup (meaning that for any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with $S$ contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). A nonempty subset $I$ of a semigroup $T$ is called a left ideal of $S$ if $TI \subseteq I$, a right ideal if $IT \subseteq I$, and a two sided ideal (or simply an ideal) if it is both a left and right ideal. A minimal left ideal is the left ideal that does not contain any proper left ideal. Similarly, we can define minimal right ideal and smallest ideal.

Any compact Hausdorff right topological semigroup $T$ has the smallest two sided ideal

$$K(T) = \bigcup \{ L : L \text{ is a minimal left ideal of } T \} = \bigcup \{ R : R \text{ is a minimal right ideal of } T \}.$$ 

Given a minimal left ideal $L$ and a minimal right ideal $R$, $L \cap R$ is a group, and in particular contains an idempotent. If $p$ and $q$ are idempotents in $T$ we write $p \leq q$ if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal $K(T)$ of $T$. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if the set $\{ x \in S : x^{-1}A \in q \} \in p$, where $x^{-1}A = \{ y \in S : x \cdot y \in A \}$. See [HS] for an elementary introduction to the algebra of $\beta S$ and for any unfamiliar details.

**Definition 2.** Let $C$ be a subset of a semigroup $S$. Then $C$ is called a central set if there exists a minimal idempotent $p \in K(\beta S)$ such that $C \in p$. A subset of
S, which meets every central set, called central set. We shall denote the class of all central sets as $C$ and that of all central* sets as $C^*$.

The notion of central set was first introduced by Furstenberg in [F] using topological dynamics and proved to be equivalent with definition in [BH]. The basic fact that we need about central sets is given by the Central Sets Theorem, which is due to Furstenberg [F, Proposition 8.21] for the case $S = \mathbb{Z}$.

**Theorem 3** (Central Sets Theorem). Let $S$ be a semigroup. Let $T$ be the set of sequences $\langle y_n \rangle_{n=1}^{\infty}$ in $S$. Let $C$ be a subset of $S$ which is central and let $F \in \mathcal{P}_f(T)$. Then there exist a sequence $\langle a_n \rangle_{n=1}^{\infty}$ in $S$ and a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and for each $L \in \mathcal{P}_f(\mathbb{N})$ and each $f \in F$, $\sum_{n \in L} (a_n + \sum_{t \in H_n} f(t)) \in C$.

However, the most general version of Central Sets Theorem is presented in [DHS], where all the sequences have been handled simultaneously.

To end this preliminary discussions let us recall Khintchine’s Theorem, which states that for any measure preserving system $(X, \mathcal{B}, \mu, T)$, and for any $\epsilon > 0$ the set $\{ n \in \mathbb{Z} : \mu(A \cap T^{-n}A) > \mu(A)^2 - \epsilon \}$ is an IP*-set and in particular syndetic. In [BHK] the authors proved that for any ergodic system $(X, \mathcal{B}, \mu, T)$ the sets $\{ n \in \mathbb{Z} : \mu(A \cap T^{-n}A \cap T^{-2n}A) > \mu(A)^3 - \epsilon \}$ and $\{ n \in \mathbb{Z} : \mu(A \cap T^{-n}A \cap T^{-2n}A \cap T^{-3n}A) > \mu(A)^4 - \epsilon \}$ are syndetic subsets of $\mathbb{Z}$. On the other hand they proved that for $n \geq 4$ the above result does not hold in general.

In [BTZ] the authors proved result analogous to Khintchine’s Theorem. In fact they proved that for $q > 2$ if $c_0, c_1, c_2$ are distinct elements of $F_q[x]$ and $(X, \mathcal{B}, \mu, T_{f \in F_q[x]})$ is an ergodic system, then for any $A \in \mathcal{B}^+$, and $\epsilon > 0$ the set

$$\{ f \in F_q[x] : \mu(T_{c_0 f}A \cap T_{c_1 f}A \cap T_{c_2 f}A) > \mu(A)^3 - \epsilon \}$$

is syndetic.

The authors also proved that for $q > 3$ and $c_0, c_1, c_2, c_3 \in F_q[x]$ the above conclusion is true provided that $c_1 + c_i = c_k + c_l$ for some permutation $\{i, j, k, l\}$ of $\{1, 2, 3, 4\}$. Further analogous to [BHK] the authors proved that for any $k \geq 3$
there exists \((c_0, c_1, \ldots, c_k) \in F_q[x]^{k+1}\) for which Khintchine’s Theorem does not hold in general.

At the end of this article we shall present some observation on Khintchine’s Theorem for the action of \((F_q[x], \cdot)\).

Acknowledgement: We would like to thank Professor Neil Hindman for his helpful suggestions. We also thank the referee for her/his comments that have resulted in substantial improvements to this paper.

2. Combinatorial Properties of \(F_q[x]\)

In the terminology of Furstenberg \cite{F}, an IP\(^*-\) set \(A\) in \(Z\) is a set, which meets FS\((x_n)_{n=1}^{\infty}\) for any sequence \((x_n)_{n=1}^{\infty}\) in \(Z\). This in turn implies that \(A\) is an IP\(^*-\) set iff it belongs to every idempotent of \(\beta Z\). IP\(^*-\) sets are known to have rich combinatorial structures. For example IP\(^*-\) sets are always syndetic. Given any IP\(^*-\) set \(A\) which is a subset of the set of integers \(Z\) and a sequence \((x_n)_{n=1}^{\infty}\) in \(Z\) there exists a sum subsystem \((y_n)_{n=1}^{\infty}\) of \((x_n)_{n=1}^{\infty}\) such that

\[
\text{FS}\((y_n)_{n=1}^{\infty}\) \cup \text{FP}\((y_n)_{n=1}^{\infty}\) \subseteq A,
\]

where for any sequence \((x_n)_{n=1}^{\infty}\) in \(Z\), \(\text{FS}\((x_n)_{n=1}^{\infty}\) is defined to be the set \(\{\sum_{n \in F} x_n : F \text{ is a finite subset of } \mathbb{N}\}\). \(\text{FP}\((x_n)_{n=1}^{\infty}\) can be defined analogously.

It is well known that in the ring \((\mathbb{Z}, +, \cdot)\) the non trivial principal ideals are IP\(^*-\) sets. So the natural question is, whether this result is true for arbitrary rings. The answer is no. In fact in the ring \(\mathbb{Z}[x]\) the ideal generated by \(x\) has empty intersection with \(\mathbb{N}\), whereas \(\mathbb{N}\) is an IP-set in \(\mathbb{Z}[x]\). We will prove that in the ring \((F_q[X], +, \cdot)\) every principal ideal is an IP\(^*-\) set. In fact we will also prove that in \((F_q[X_1, X_2, \ldots, X_k], +, \cdot)\) every ideal of the form \(\langle f_1(X_1), f_2(X_2), \ldots, f_k(X_k) \rangle\) is an IP\(^*-\) set.

**Theorem 4.** In the polynomial ring \((F_q[X_1, X_2, \ldots, X_k], +, \cdot)\) over the finite field \(F_q\), the ideal \(\langle f_1(X_1), f_2(X_2), \ldots, f_k(X_k) \rangle\) generated by \(f_1(X_1), f_2(X_2), \ldots,\)
$f_k(X_k)$ (at most one of which is non constant), is an IP*-set in the corresponding additive group.

**Proof.** For simplicity we work with $k = 2$. Let $(g_n(X_1, X_2))_{n=1}^{\infty}$ be a sequence in $F_q[X_1, X_2]$.

Let $g(X_1, X_2)$ be a polynomial in $F_q[X_1, X_2]$. Then

$$g(X_1, X_2) = \sum_{i \leq n, j \leq m} a_{i,j} X_1^i X_2^j,$$

where $a_{i,j} \in F_q$.

Since $X_1^i, f_1(X_1) \in F_q[X_1]$ and $X_2^j, f_2(X_2) \in F_q[X_2]$ by applying a division algorithm we have

$$X_1^i = f_1(X_1)q_{1,i}(X_1) + r_{1,i}(X_1), \text{ where } \deg(r_{1,i}(X_1)) < \deg f_1(X_1)$$

$$X_2^j = f_2(X_2)q_{2,j}(X_2) + r_{2,j}(X_2), \text{ where } \deg(r_{2,j}(X_2)) < \deg f_2(X_2).$$

Then $g(X_1, X_2)$ can be expressed as

$$g(X_1, X_2) = f_1(X_1)h_1(X_1, X_2) + f_2(X_2)h_2(X_1, X_2)$$

$$+ \sum_{i \leq n, j \leq m} a_{i,j} r_{1,i}(X_1)r_{2,j}(X_2).$$

$$\deg(r_{1,i}(X_1)) < \deg f_1(X_1)$$

$$\deg(r_{2,j}(X_2)) < \deg f_2(X_2)$$

Therefore we can write

$$g(X_1, X_2) = h(X_1, X_2) + r(X_1, X_2),$$

where

$$h(X_1, X_2) \in (f_1(X_1), f_2(X_2))$$

and $r(X_1, X_2)$ is a polynomial such that $\deg r(X_1, X_2) < \deg f_1(X_1) + \deg f_2(X_2)$.

This implies that

$$g_n(X_1, X_2) = h_n(X_1, X_2) + r_n(X_1, X_2)$$
where

\(h_n(X_1, X_2) \in \langle f_1(X_1), f_2(X_2) \rangle\)

and \(r_n(X_1, X_2)\) is a polynomial such that \(\deg r_n(X_1, X_2) < \deg f_1(X_1) + \deg f_2(X_2)\).

But the set \(\{r_n(X_1, X_2) : n \in \mathbb{N}\}\) is finite. Since \(\{g_n(X_1, X_2) : n \in \mathbb{N}\}\) is infinite there exists \(q\) many polynomials \(g_{n_i}(X_1, X_2)\) : \(i = 1, 2, \ldots, q\) such that the corresponding \(r_{n_i}(X_1, X_2)\) for \(i = 1, 2, \ldots, q\) are equals. Now adding we get

\[
\sum_{i=1}^{q} g_{n_i}(X_1, X_2) = \sum_{i=1}^{q} h_{n_i}(X_1, X_2) + \sum_{i=1}^{q} r_{n_i}(X_1, X_2).
\]

This implies that \(\sum_{i=1}^{q} g_{n_i}(X_1, X_2) \in \langle f_1(X_1), f_2(X_2) \rangle\) as \(\sum_{i=1}^{q} r_{n_i}(X_1, X_2) = 0\). Therefore, \(\langle f_1(X_1), f_2(X_2) \rangle\) is an IP*-set.

In case of \(\mathbb{Z}\) we know that iterated spectra of an IP* set are also IP* but may not contain any ideal [BHK96]. But for \((F_q[X], +)\) any IP* set contains an ideal up to finitely many terms.

**Theorem 5.** Any IP* set in \((F_q[X], +)\) contains an ideal of the form \(\langle X^m \rangle\), for some \(m \in \mathbb{N}\), up to finitely many terms.

**Proof.** Let us claim that any syndetic IP set \(A\) in \((F_q[X], +)\) contains \(\langle X^m \rangle\) for some \(m \in \mathbb{N}\). Now \(A\) being a syndetic set will be of the form

\[
A = \bigcup_{i=1}^{k} (f_i(X) + \langle X^m \rangle) \text{ (up to finitely many terms)}
\]

for some \(m, k \in \mathbb{N}\) with \(m > \deg f_i(X)\). Again, since \(A\) is an IP-set, one of \(f_i(X)\) must be zero. In fact \(A\) being an IP set, there exist a sequence \(\langle g_i(x) \rangle_{i=1}^{\infty}\) such that \(FS(g_i(x)) \subseteq A\). This implies that for each \(i \in \mathbb{N}\), there exists \(j \in \{1, 2, \cdots, k\}\) and some \(h_i(X) \in F_q[X]\) such that \(g_i(X) = f_j(X) + h_i(X)X^m\).

Since \(\{g_i(X) : i \in \mathbb{N}\}\) is infinite, there exist \(q\) many polynomials \(g_{n_i}(X) : i = 1, 2, \ldots, q\) such that the corresponding \(f_{j_i}(X)\) are equal for \(i = 1, 2, \ldots, q\), and such sum of \(q\) many polynomials is equal to zero. Hence some \(f_j(X)\) is equal to zero. \(\square\)
We end this section with the following observation. We know that in the case of $\mathbb{Z}$, the intersection of thick set and an IP syndetic set may not be central, but in case of $F_q[X]$, such sets will be always central set. In fact in $F_q[X]$, any IP syndetic set is an IP$^+$ set. Again since the thick sets are always central set, intersection of a thick set and an IP syndetic set is central set.

3. Mixing Properties of the action of $(F_q[x], +)$

In this section we shall show that all the mixing properties are equivalent under the action of $(F_q[x], +)$. First let us recall the following definitions.

**Definition 6.** A measure preserving system $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ is said to be ergodic if for any set $A \in \mathcal{B}$ which satisfies $\mu(A \triangle T_gA) = 0$ for any $g \in G$ has either measure 0 or 1.

**Definition 7.** Let $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ be a measure preserving dynamical system. Then

1. $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ is called strong mixing if for any $\epsilon > 0$ and any $A, B \in \mathcal{B}$ with positive measure, the set $\{g \in G : |\mu(A \cap T_gB) - \mu(A)\mu(B)| < \epsilon\}$ is a cofinite set.
2. $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ is called mild mixing if for any $\epsilon > 0$ and any $A, B \in \mathcal{B}$ with positive measure, the set $\{g \in G : |\mu(A \cap T_gB) - \mu(A)\mu(B)| < \epsilon\}$ is an IP$^+$-set.
3. $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$ is called weak mixing if for any $\epsilon > 0$ and any $A, B \in \mathcal{B}$ with positive measure, the set $\{g \in G : |\mu(A \cap T_gB) - \mu(A)\mu(B)| < \epsilon\}$ is a central$^*$-set.

**Lemma 8.** Let $(X, \mathcal{B}, \mu, (T_f)_{f \in F_q[x]})$ be a measure preserving system. Then it is strong mixing iff for each $B \in \mathcal{B}$ with $\mu(B) > 0$ and an infinite set $F$, there exists a sequence of polynomials $\langle f_n \rangle_{n=1}^{\infty}$ in $F$ such that $\chi_B \circ T_{f_n} = U_{T_{f_n}} \chi_B \rightarrow f_B = \mu(B)$.

**Proof.** Let $(X, \mathcal{B}, \mu, (T_f)_{f \in F_q[x]})$ be a measure preserving system. Let $\{A_i\}_{i=1}^{\infty}$ be a countable basis of $\mathcal{B}$ i.e. $\{A_i\}_{i=1}^{\infty}$ is dense in $\mathcal{B}$ with the metric $d(A, B) =$
\( \mu(A \triangle B) \). Let \( B \in \mathcal{B} \), with \( \mu(B) > 0 \) and \( F \) be an infinite set. Let us set for each \( i \in \mathbb{N} \) and \( \epsilon > 0 \),

\[
F(i, \epsilon) = \{ f \in F_q[x] : |\mu(A_i \cap T_f B) - \mu(A_i)\mu(B)| < \epsilon \}.
\]

Then for each \( i \in \mathbb{N} \), \( F(i, \epsilon) \) is a cofinite set. Let us choose one \( f_1(x) \in F \cap F(1,1) \). Next we choose another \( f_2(x) \in F \cap F(1,\frac{1}{2}) \cap F(2,\frac{1}{2}) \) such that \( \deg f_2 > \deg f_1 \). Inductively we choose a sequence \( \langle f_n \rangle_{n=1}^{\infty} \) with \( \deg f_{n+1} > \deg f_n \) such that

\[
f_{n+1} \in F \cap F(1, \frac{1}{n+1}) \cap \ldots \cap F(i+1, \frac{1}{n+1}).
\]

So we get a subsequence \( \langle f_n \rangle_{n=1}^{\infty} \) of \( F \). By choosing again a subsequence from it as our requirements we can assume \( \chi_B \circ T_{f_n} = U_{T_{f_n}} \chi_B \rightarrow f_B \) (weakly). It is clear that for each \( i \)

\[
\int \chi_{A_i} (f_B - \mu(B)) d\mu = 0.
\]

This implies that \( f_B = \mu(B) \).

Conversely given that, for each \( B \in \mathcal{B} \) with \( \mu(B) > 0 \) and an infinite set \( F \), there exists a sequence of polynomials \( \langle f_n \rangle_{n=1}^{\infty} \) in \( F \) such that \( \chi_B \circ T_{f_n} = U_{T_{f_n}} \chi_B \rightarrow f_B = \mu(B) \). Now if \( (X, \mathcal{B}, \mu, (T_f)_{f \in F_q[x]}) \) is not strong mixing then there exist \( A, B \in \mathcal{B} \) with positive measure and \( \epsilon > 0 \) such that \( \{ f \in F_q[x] : |\mu(A \cap T_f B) - \mu(A)\mu(B)| \geq \epsilon \} \) is an infinite set. For this \( B \in \mathcal{B} \), and the infinite subset \( F \subset F_q[x] \) we consider the set \( \text{cl}_w \{ U_{T_f}(\chi_B) : f \in F \} \). Then by the given hypothesis there is a constant function \( f_B \in \text{cl}_w \{ U_{T_f}(\chi_B) : f \in F \} \). Clearly the set \( \{ f \in F_q[x] : \mu(A \cap T_f B) \geq \mu(A)\mu(B) + \epsilon \} \) is infinite. Therefore each \( f \in \text{cl}_w \{ U_{T_f}(\chi_B) : f \in F \} \) satisfies that \( \int \chi_A \cdot f d\mu \geq \mu(A)\mu(B) + \epsilon \). This contradicts the assumption \( \mu(B) \in \text{cl}_w \{ U_{T_f}(\chi_B) : f \in F \} \).

**Theorem 9.** Let \( (X, \mathcal{B}, \mu, (T_f)_{f \in F_q[x]}) \) be a measure preserving action. Then \( T \) is strongly mixing iff for any \( \epsilon > 0 \) and \( A \in \mathcal{B} \) with \( \mu(A) > 0 \)

\[
\{ f \in F_q[x] : |\mu(A \cap T_f A) - \mu(A)^2| < \epsilon \} \in \Delta^*.
\]
Proof. Strong mixing clearly implies the given condition.

For the converse, by Lemma 8 it is sufficient to show that for each \( B \in \mathcal{B} \) with \( \mu(B) > 0 \) and an infinite set \( F \), there exists a sequence of polynomials \( \langle f_n \rangle_{n=1}^{\infty} \) in \( F \) such that

\[
\chi_B \circ T_{f_n} = U_{T_{f_n}} \chi_B \longrightarrow f_B = \mu(B) \quad \text{in the weak topology.}
\]

With out loss of generality we can assume that \( F \) has a sequence \( \langle f_{n_i} \rangle_{n=1}^{\infty} \) such that \( \deg f_{i} < \deg f_{i+1} \). Since \( \triangle \)-sets always have the Ramsey property, there exists some \( F_1 \subseteq F \) such that

\[
F_1 - F_1 \subseteq (F - F) \cap \{ f \in F_q[x] : |\mu(B \cap T_f B) - \mu(B)|^2 < \frac{1}{2} \}.
\]

Choosing \( F_1 \supseteq F_2 \supseteq \ldots \supseteq F_k \), we can inductively choose \( F_{k+1} \subseteq F_k \) such that

\[
F_{k+1} - F_{k+1} \subseteq (F_{k} - F_{k}) \cap \{ f \in F_q[x] : |\mu(B \cap T_{f} B) - \mu(B)|^2 < \frac{1}{2^{k+1}} \}.
\]

Therefore for any \( f \neq g \in F_k \), we have \( |\mu(T_f B \cap T_g B) - \mu(B)|^2 < \frac{1}{2^k} \).

Now let us consider a sequence \( \langle f_{n_i} \rangle_{n=1}^{\infty} \) such that \( f_{n_i} \in F_{n_i} \). Then clearly \( \chi_B \circ T_{f_{n_i}} = U_{T_{f_{n_i}}} \longrightarrow f_B \) in the weak topology. Thus

\[
\langle f_B, f_B \rangle = \lim_i \lim_j \langle \chi_B \circ T_{f_{n_i}}, \chi_B \circ T_{f_{n_j}} \rangle \leq \mu(B)^2 + \lim_i \frac{1}{2^i} = \mu(B)^2 = \left( \int f_B d\mu \right)^2.
\]

This shows that \( f_B = \mu(B) \) due to the Cauchy-Schwarz inequality. \( \square \)

So the above theorem shows that like \((\mathbb{N},+)\) action, in case of \((F_q[x],+)\) action also \(\triangle^*\)-mixing and strong mixing are equivalent. But the authors believe that this is not true for the action of arbitrary group.

4. Action of \((F_q[x],\cdot)\)

In this section we shall show that \((F_q[x],\cdot)\) behaves quite similarly like \((\mathbb{N},+)\) and using some established examples for the action of \((\mathbb{N},+)\), we shall produce some examples of \((F_q[x],\cdot)\). First, we require the following lemmas.

**Lemma 10.** Let \( \varphi : (F_q[x],\cdot) \rightarrow (\mathbb{N},+) \) be a map defined by \( \varphi(f) = \deg f \). Then \( C \) is an IP-set in \((\mathbb{N},+)\) iff \( \varphi^{-1}(C) \) is an IP-set in \((F_q[x],\cdot)\).
Corollary 11. Let \( \varphi : (F_q[x], \cdot) \to (\mathbb{N}, +) \) be a map defined by \( \varphi(f) = \deg f \).
Then \( C \) is an IP\(^*\)-set in \( (\mathbb{N}, +) \) iff \( \varphi^{-1}(C) \) is an IP\(^*\)-set in \( (F_q[x], \cdot) \).

The following lemma follows from [BG, Corollary 4.22]. We still include a proof suggested by Prof. Neil Hindman, for the sake of completeness.

Lemma 12. Let \( \varphi : (F_q[x], \cdot) \to (\mathbb{N}, +) \) be a map defined by \( \varphi(f) = \deg f \). Then \( C \) is a central set in \( (\mathbb{N}, +) \) iff \( \varphi^{-1}(C) \) is a central set in \( (F_q[x], \cdot) \).

Proof. Clearly \( \varphi \) is a homomorphism so that it has continuous extension \( \hat{\varphi} \) over \( (\beta F_q[x], \cdot) \) onto \( (\beta \mathbb{N}, +) \) [HS, Corollary 4.22 and Exercise 3.4.1].

Necessity. Let \( C \) be a central set in \( (\mathbb{N}, +) \) and let \( p \) be a minimal idempotent containing \( C \). Let \( M = \hat{\varphi}^{-1}(\{p\}) \). Then \( M \) is a compact Hausdorff right topological semigroup, so pick an idempotent \( q \in K(M) \). We claim that \( q \) is minimal in \( (\beta F_q[x], \cdot) \). So let \( r \) be an idempotent in \( (\beta F_q[x], \cdot) \) with \( r \leq q \). Then \( \varphi(r) \leq \varphi(q) = p \). Hence \( r \in M \) and thus \( r = q \).

Sufficiency. Assume that \( \varphi^{-1}(C) \) is central in \( (\beta F_q[x], \cdot) \). Pick an idempotent \( p \in K(\beta F_q[x]) \) such that \( \varphi^{-1}(C) \in p \). Then \( \hat{\varphi}(p) \in \hat{\varphi}(K(\beta F_q[x])) \) and \( \hat{\varphi}(K(\beta F_q[x])) = K(\beta \mathbb{N}) \) by [HS, Exercise 1.7.3]. Thus by [HS, Lemma 3.30] \( \varphi(\varphi^{-1}(C)) \in \hat{\varphi}(p) \) and \( \varphi(\varphi^{-1}(C)) = C \).


Corollary 13. Let \( \varphi : (F_q[x], \cdot) \to (\mathbb{N}, +) \) be a map defined by \( \varphi(f) = \deg f \). Then \( C \) is a central* set in \( (\mathbb{N}, +) \) iff \( \varphi^{-1}(C) \) is a central* set in \( (F_q[x], \cdot) \).

We know that strong mixing \( \Rightarrow \) mild mixing \( \Rightarrow \) weak mixing; but the converses are not true in general. In case of the action of \( (F_q[x], +) \) on a measure space \( (X, \mathcal{B}, \mu) \) all the mixings are equivalent. In contrast to the action of \( (F_q[x], \cdot) \) we see that there are weak mixing systems that are not mild mixing and there are mild mixing systems that are not strong mixing. Let \( (X, \mathcal{B}, \mu, T) \) be a weak mixing system which is not mild mixing. We define an action of \( (F_q[x], \cdot) \) by the formula
\[
T_f(x) = T^{\deg f}(x).
\]
Let $\epsilon > 0$ and any $A, B \in \mathcal{B}$ with positive measure. Since $(X, \mathcal{B}, \mu, T)$ is weak mixing

$$\{n \in \mathbb{Z} : |\mu(A \cap T^{-n}B) - \mu(A)\mu(B)| < \epsilon\}$$

is a central*-set. This further implies that the set

$$\{f : |\mu(A \cap T^f B) - \mu(A)\mu(B)| < \epsilon\} = \{f : |\mu(A \cap T^{\deg f} B) - \mu(A)\mu(B)| < \epsilon\}$$

is a central*-set so that $(X, \mathcal{B}, \mu, \{T_f : f \in \mathbb{F}_q[x], \cdot\})$ is weak mixing. Using quite similar technique and the fact that a set $A$ in $\mathbb{N}$ is an IP*-set iff $\varphi^{-1}(A)$ is an IP*-set, we can show that $(X, \mathcal{B}, \mu, \{T_f : f \in \mathbb{F}_q[x], \cdot\})$ is not mild mixing. Similarly we can show that there exists $(X, \mathcal{B}, \mu, \{T_f : f \in \mathbb{F}_q[x], \cdot\})$ which is a mild mixing system but not strong mixing. In [BH] the authors introduced the notion of dynamical IP*-set.

**Definition 14.** A set $A$ in a semigroup $S$ is called a dynamical IP*-set if there exists a measure preserving dynamical system $(X, \mathcal{B}, \mu, (T_s)_{s \in S})$, $A \in \mathcal{B}$ with $\mu(A) > 0$, such that $\{s \in S : \mu(A \cap T_s^{-1}A) > 0\} \subseteq A$.

By [HS, Theorem 16.32], there is an IP* set $B$ in $(\mathbb{N}, +)$ such that for each $n \in \mathbb{N}$, neither $n + B$ nor $-n + B$ is an IP* set. Consequently, the following theorem shows that not every IP*-set is a dynamical IP*-set.

**Theorem 15.** Let $B$ be a dynamical IP* set in $(\mathbb{N}, +)$. There is a dynamical IP*-set $C \subset B$ such that for each $n \in C$, $-n + C$ is a dynamical IP* set (and hence not every IP* set is a dynamical IP*-set).

**Proof.** [Theorem 19.35 HS].

Now, since there is an IP* set $B$ in $(\mathbb{N}, +)$ such that for each $n \in \mathbb{N}$, neither $n + B$ nor $-n + B$ is an IP* set, we can show that $\varphi^{-1}B$ is an IP*-set in $(\mathbb{F}_q[x], \cdot)$, but neither $\varphi^{-1}B/f$ nor $(\varphi^{-1}B)f$ is an IP*-set in $(\mathbb{F}_q[x], \cdot)$ for any $f \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$. Using an analogous method used in [Theorem 19.35 HS], we can prove the following.

**Theorem 16.** Let $B$ be a dynamical IP* set in $(\mathbb{F}_q[x], \cdot)$. There is a dynamical IP*-set $C \subset B$ such that for each $f \in C$, $C/f$ is a dynamical IP* set (and hence not every IP* set is a dynamical IP* set in $(\mathbb{F}_q[x], \cdot)$).
It can be proved that if $B$ be a dynamical IP$^*$ set in $(\mathbb{N}, +)$ then $\varphi^{-1}(C)$ is also a dynamical IP$^*$ set in $(F_q[x], \cdot)$ by the transformation $T_f(x) = T^{\deg f}(x)$. But we do not know whether the converse is true or false. In the discussion of the action of $(F_q[x], \cdot)$, we have noticed that it behaves quite similar to the action of $(\mathbb{N}, +)$. This motivates us to raise the following question.

**Question 1.** Given any ergodic system $(X, B, \mu, T_f \in (F_q[x], \cdot))$, are the sets,

$$\{ f \in F_q[x] : \mu(A \cap T_f A \cap T_{f^2} A) > \mu(A)^3 - \epsilon \}$$

and

$$\{ f \in F_q[x] : \mu(A \cap T_f A \cap T_{f^2} A \cap T_{f^3} A > \mu(A)^4 - \epsilon \}$$

syndetic subsets of $(F_q[x], \cdot)$? In contrast, it can be shown that for $k > 3$, $\{ f \in F_q[x] : \mu(A \cap T_f A \cap T_{f^2} A \cap T_{f^3} A \cap \ldots \cap T_{f^k} A > \mu(A)^{k+1} - \epsilon \}$ are not syndetic subsets of $(F_q[x], \cdot)$ by the transformation $T_f(x) = T^{\deg f}(x)$.

**References**

[B] V. Bergelson, Minimal idempotents and ergodic Ramsey theory. Topics in Dynamics and Ergodic Theory, editors: S. Bezuglyi and S. Kolyada, 8–39, London Math. Soc. Lecture Note Ser. 310, Cambridge Univ. Press, Cambridge, 2003.

[BD] V. Bergelson, T. Downarowicz, Large sets of integers and hierarchy of mixing properties of measure preserving systems. Colloq Math, 2008, 110: 117–150

[BG] V. Bergelson, and D. Glasscock, Interplay between notions of additive and multiplicative largeness, arXiv:1610.09771v1.

[BH] V. Bergelson and N. Hindman, IP$^*$-sets and central sets, Combinatorica 14 (1994), 269–277.
[BHK] V. Bergelson, B. Host and B. Kra, Multiple recurrence and nilsequences. Invent Math, 2005, 160: 261–303

[BHK96] V. Bergelson, N. Hindman and B. Kra, Iterated spectra of numbers - elementary, dynamical, and algebraic approaches, Trans. Amer. Math. Soc. 348 (1996), 893-912.

[BTZ] V. Bergelson, T. Tao, and Z. Tamar, Multiple recurrence and convergence results associated to $F_p^\infty$-actions, J. Anal. Math. 127 (2015), 329–378.

[DHS] D. De, N. Hindman and D. Strauss, A new and stronger central sets theorem, Fundamenta Mathematicae 199 (2008), 155-175.

[F] H. Furstenberg, Recurrence in Ergodic Theory and Combinatorial Number Theory, Princeton University Press, 1981.

[HS] N. Hindman and D. Strauss, Algebra in the Stone-Čech compactification: theory and applications, de Gruyter, Berlin, 1998.

[Le] T. H. Le, Green-Tao theorem in function fields, Acta Arith. 147 (2011), no. 2, 129–152.

[KY] R. Kuang and X. Ye, The return times set and mixing for measure preserving transformations. Discrete Contin Dyn Syst, 2007, 18: 817–827.

[KY2] R. Kuang and X. Ye, Mixing via families for measure preserving transformations. Colloq Math, 2008, 110: 151–165.

[R14] R. Kuang, Mixing via the extended family, Sci. China Math. 57 (2014), no. 2, 367–376.