Weak $q$-Deformed Coherent States Approximate Eigenfunctions and its Resolution of Unity

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Abstract. We use the weak deformed version of $q$-boson annihilation operator to solve the eigenvalues coherent states equation for the generalized $q$-deformed harmonic oscillator. We also describe the construction of their resolution of unity.

Key words: Coherent states; Weak $q$-deformed bosons; Riccati equation; Harmonic oscillator

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1 Introduction

Quantum groups (QG) have been the subject of intensive research and several standard textbooks have been devoted to this exciting field [1, 2, 3, 4, 5]. They are considered to be a generalization of the fundamental symmetry concepts of classical Lie groups and stand for a certain Hopf algebras [6] which are nontrivial deformation of the so-called universal enveloping algebra (UEA) of semi-simple Lie algebras. Quantum algebras are obtained by deforming UEA via a deformation parameter $q = e^{i\hbar}$ [7, 8] and UEA associated with the quantum groups are a Hopf algebras with a deformed Hopf structure (see e.g. [6]). In analogy with non-deformed case, quantum algebras have been constructed in such a way that we regain the standard Lie algebras as $q \rightarrow 1$, i.e. $\hbar \rightarrow 0$.

Since then QG have attracted great attention on behalf of physicists and found close applications in various fields of physics and chemistry such as quantum inverse scattering theory [9], nuclear physics [10], solvable statistical mechanics models [11] and molecular physics [12].

The idea to associate quantum groups to the classical groups lead to develop the concept of $q$-deformed quantum mechanics [13, 14, 15, 16, 17, 18, 19, 20, 21] and a special attention is devoted to the $q$-deformed Weyl-Heisenberg algebra ($q$-WH) of raising and lowering (creation and annihilation) operators [13, 14] referred in mathematical literature as $\mathfrak{osp}_q(2|1)$. As a result a plethora of a rich perspective is then open to study some models directly related to specific topics originating from phase space [22], uncertainty relation [23] and hydrogen atom [24], etc.

On the other hand the basic features of coherent states (CS) are now well understood and have been the subject of intensive investigations [25, 26, 27, 28]. It was Glauber [29] who showed that CS can be used to describe the electromagnetic correlation function in the context of quantum optics. They are specific superpositions of the eigenstates of the harmonic oscillator and may be constructed in three different ways: (i) they are eigenstates of the boson annihilation operator (Glauber’s approach), (ii) they are displayed version of the ground wave-function (Klauder’s approach) and (iii) they minimize the Heisenberg uncertainty relationship (Schrödinger’s approach).

As a natural implication the relation between $q$-WH algebra and CS is of great interest due to the numerous physical applications of the features of CS. In analogy with non-deformed CS, the
$q$-deformed coherent states ($q$-CS) are the eigenstates of the $q$-deformed boson annihilation operator and are superpositions of $q$-deformed harmonic oscillators ($q$-HO). The $q$-deformed CS have been well studied and widely applied to mathematical physics \cite{30,31,32,33,34,35,36,37,38,39}.

The $q$-deformed HO can be defined with the help of coordinate description of the creation and annihilation operators realized in a space of functions, $B_s^\dagger$ and $B_s$, introduced by Macfarlane \cite{13} ($\hbar = 1$):

$$B_s = \alpha \left( e^{-2i s x} - e^{-i s x} e^{i s \partial} \right),$$

$$B_s^\dagger = \alpha \left( e^{2i s x} - e^{i s \partial} e^{-i s x} \right),$$

where $\partial \equiv \frac{d}{dx}$ and obeying the $q$-commutation ($q$-mutator) rule

$$[B_s, B_s^\dagger]_q \equiv B_s B_s^\dagger - q^2 B_s^\dagger B_s = 1, \quad (1.3)$$

where $q$ is assumed to be real and related to the parameter $s$ via $q = e^{-s^2}$, and is taken to lie between zero and 1. Using (1.3) it is easy to be convinced that $\alpha \overline{\alpha} = (1 - q^2)^{-1}$.

However, as far as we know, the resolution of the eigenvalues equation for $q$-CS associated to generalized $q$-deformed CS has received much less attention. So, the main aim of this paper is to fill this gap. The purpose of the present paper, working in contrast with known $q$-deformed HO, is to apply a weak deformed approximation of $q$-deformed boson annihilation operator to the generalized $q$-deformed HO in order to solve the eigenvalues equation associated with the generalized $q$-deformed CS, using some lemmas directly related to the resolution of the Riccati differential equation. We prove that these states admit a resolution of unity with a specific measure coinciding with the elliptic Jacobi $\vartheta_3$-function and the space of functions is considered to be the unit circle, which seems here to be suited and more appropriate.

This paper is structured as follows. In section 2, we introduce a weak deformed version of $q$-deformed boson annihilation operator whose characteristics enables us to describe the eigenfunctions for the generalized $q$-deformed CS in terms of the Riccati equation. Section 3 is dedicated to some features integrability lemmas of the Riccati equation. Section 4 is devoted to the construction process of the generalized $q$-deformed CS in the sense described above and its alternative approach via the perturbation procedure is presented. In section 5, we prove that the constructed $q$-deformed CS admit a unity resolution relation and expressed through a positive-definite weight function coinciding with the elliptic Jacobi-function. Finally, last section contains the conclusion.

## 2 Weak deformed approximation and Riccati equation

Following a part of development made in the introduction, we begin by specify the coordinate representation for the creation and annihilation operators related to our object of study. We suggest taking it in the form

$$B_s \rightarrow \mathfrak{B}_s = \alpha \left( e^{-2i s \beta(x)} - e^{-i s \beta(x)} e^{i s \partial} \right),$$

$$B_s^\dagger \rightarrow \mathfrak{B}_s^\dagger = \overline{\alpha} \left( e^{2i s \beta(x)} - e^{i s \partial} e^{-i s \beta(x)} \right), \quad (2.1)$$

where $\beta(x)$ is the deformation function to be determined. This determination comes from the restriction that the ladder operators in (2.1) and (2.2) satisfy the $q$-mutator rule (1.3) under the same constraints imposed to $q$ and $\alpha$ in the introduction, i.e. $q = e^{-s^2}$ and $\alpha \overline{\alpha} = (1 - q^2)^{-1}$. 
Then taking into account (2.3) and applying the braiding identity related to the Campbell-Baker-Hausdorff relation

\[ e^X e^Y = \exp \left\{ Y + \sum_{n=1}^{\infty} \frac{1}{n!} [X,Y]_{(n)} \right\} e^X, \]

(2.3)

where

\[ [X,Y]_{(n)} = \left[ X, \left[ X, \left[ X, \ldots, \left[ X,Y \right], \ldots \right] \right] \right], \]

we obtain after some straightforward calculation the difference equation for \( \beta(x) \)

\[ \beta(x + is) = \beta(x) + is. \]

(2.4)

Note that considering the deformation function \( \beta(x) \) as an analytic function in the complex plan, one can approximate the left-hand side of (2.4) by its Taylor expansion \( \beta(x + is) = \beta(x) + is\beta'(x) + O(s^2) \) and comparing this with (2.4), we get \( \beta(x) = x \). However it is interesting to find a general solution to (2.4), so that \( \beta_0(x) = x \) will be a particular solution, by solving the difference equation whose its solution is \( \beta(x) = x + b(x) \), where \( b(x) \) is an arbitrary periodic function with period equal to \( is \). As a consequence it is worth mentioning that a \( is \)-periodic function \( b(x) \) of the variable \( x \) can be written as a power series by defining a new variable \( \frac{x}{s} \) as

\[ b(x) = \sum_{n=-\infty}^{\infty} c_n \exp \left( \frac{2n\pi x}{s} \right), \]

(2.5)

where the coefficients \( c_n \) are arbitrary. As \( b(x) \) has a period of \( is \), it is natural that we expand (2.5) in a series of functions with period \( 2\pi \) in order to give rise to the Fourier series. This restriction may be easily relaxed by substituting \( x \) by \( \frac{is}{2\pi}t \), we get

\[ b(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, \quad \text{with} \quad c_n = \frac{1}{2\pi i} \int_{(C)} \frac{b(\zeta)}{\zeta^{n+1}} d\zeta, \]

(2.6)

where \( \zeta = e^{it} \).

Equation (2.6) guarantees that our \( 2\pi \)-periodic function is now constructed sectionally on each interval \( I_n = [(2n - 1)\pi, (2n + 1)\pi] \), with the Fourier part-function \( b_n(t) = e^{int} \). Then it is obvious that \( c_n \) are complex Fourier coefficients and \( (C) \) is the contour of a unit circle.

Inspired by the properties of the difference equation of \( \beta(x) \) discussed previously and using the fact that \( 0 < s \ll 1 \) is the small parameter, we will, in what follows, express the \( q \)-deformed boson annihilation operator \( \mathfrak{B}_s \) in its weak deformed approximation scheme given by

\[ \mathfrak{B}_s \simeq \frac{s}{\sqrt{2}} \frac{d^2}{dx^2} - \frac{i}{\sqrt{2}} \left( 1 - is\beta(x) \right) \frac{d}{dx} - \frac{i}{\sqrt{2}} \left( \beta(x) - \frac{3is}{2} \beta^2(x) \right), \]

(2.7)

in order to solve the associated \( q \)-deformed CS eigenvalues equation in the sense of Barut-Girardello, i.e.

\[ \mathfrak{B}_s|\Lambda, s\rangle = \lambda_s |\Lambda, s\rangle, \]

(2.8)

and using the configuration space, \( \langle x|\Lambda, s \rangle \equiv \Lambda_s(x) \), the eigenvalues equation (2.8) becomes

\[ \mathcal{L}_s\Lambda_s(x) \equiv \left[ \frac{d^2}{dx^2} - \frac{2i}{s} \left( 1 - is\beta(x) \right) \frac{d}{dx} - \frac{2i}{s} \left( \beta(x) - \frac{3is}{2} \beta^2(x) \right) \right] \Lambda_s(x) = \frac{2\sqrt{\lambda_s}}{s} \Lambda_s(x). \]

(2.9)
However assuming that the parameter \( s \) is smaller that 1, we then expect that eigenfunctions \( \Lambda_s(x) \) are independent on \( s \), in contrast to \( \lambda_s \) as it will be shown in sections 4 and 5. But we will continue to label them through the subscript \( s \).

As a last step of our calculations, let us look for solutions of (2.9) in the form of

\[
\Lambda_s(x) = \xi_s(x) \exp \left[ \frac{i}{s} x + \delta \int x \beta(x') \, dx' \right],
\]

where \( \delta \) is some constant to be determined subsequently and different than one \( (\delta \neq 1) \). We may guess that (2.10) is, as a matter of fact, a good candidate in describing \( q \)-deformed CS eigenfunctions for the generalized \( q \)-deformed HO. If one substitutes (2.10) into (2.9) it is easy to show that the corresponding eigenvalues equation for \( \xi_s(x) \) reads as

\[
\frac{d^2 \xi_s(x)}{dx^2} + 2(\delta - 1)\beta(x) \frac{d\xi_s(x)}{dx} + \left( \frac{1 - 2\sqrt{s}\lambda_s}{s^2} - \frac{4i}{s} \beta(x) + (\delta + 1)(\delta - 3)\beta^2(x) + \delta \beta'(x) \right) \xi_s(x) = 0,
\]

and by means of change of function

\[
z_s(x) = -\frac{d \ln \xi_s(x)}{dx} \quad \Rightarrow \quad \xi_s(x) \sim \exp \left[ - \int z_s(x') \, dx' \right],
\]

the above equation is reduced to the Riccati equation given by

\[
\frac{dz_s(x)}{dx} = z_s^2(x) - 2(\delta - 1)\beta(x)z_s(x) + \frac{1 - 2\sqrt{s}\lambda_s}{s^2} - \frac{4i}{s} \beta(x) + (\delta + 1)(\delta - 3)\beta^2(x) + \delta \beta'(x).
\]

In order to obtain the solution of (2.13), we introduce in the next section some integrability lemmas satisfying some restrictions on the coefficients of the Riccati equation.

### 3 Some lemmas about Riccati equation

The Riccati equation

\[
z'(x) = p_2(x)z^2(x) + p_1(x)z(x) + p_0(x),
\]

plays a significant role in many fields of applied and fundamental science and is one of the most studied first order non-linear differential equations [10].

It is well established that the solutions are obtained by assuming certain relations among the coefficients \( p_i(x) \), \( (i = 0, 1, 2) \), of (3.1) which lead to involve some lemmas.

Let us briefly review two important lemmas about the properties of the solutions of Riccati equation and solved analytically. The proofs of both lemmas are discussed and given in the cited references.

**Lemma 3.1** ([10] [11]). Let \( p_2(x) = 1 \), \( p_0(x) \) and \( p_1(x) \) be polynomials. If the degree of the polynomial \( S(x) = p_2^2(x) - 2p_1'(x) - 4p_0(x) \) is odd, the Riccati equation can not possess a polynomial solution. If the degree of \( S(x) \) is even, the equation involved may possess only the following polynomial solutions:

\[
z_{\pm}(x) = -\frac{1}{2} \left( p_1(x) \pm \sqrt{S(x)} \right),
\]

where \( \lfloor \sqrt{S(x)} \rfloor \) denotes an integer rational part of the expansion of \( S(x) \) in decreasing powers of \( x \).
Lemma 3.2 ([40] [42]). The Riccati equation (3.1) is solvable by quadrature if a relationship

$$\omega_1^2 p_2(x) + \omega_1 \omega_2 p_1(x) + \omega_2^2 p_0(x) = 0,$$

exists with constant coefficients \(\omega_1\) and \(\omega_2\), not simultaneously zero, and satisfying the condition \(|\omega_1| + |\omega_2| > 0\).

4 Generalized \(q\)-deformed coherent states

In accordance with two lemmas exposed in last section, our key aim in this section is to test their validity and reliability by combining both lemmas in order to solve the Riccati equation (2.13).

Following lemma 3.1, the coefficients \(p_0(x)\) and \(p_1(x)\) are polynomials satisfying

$$p_0(x) = \frac{1 - 2\sqrt{2} s \lambda_s}{s^2} - \frac{4i}{s} \beta(x) + (\delta + 1)(\delta - 3)\beta^2(x) + \delta \beta'(x),$$

$$p_1(x) = -2(\delta - 1)\beta(x),$$

and as a consequence, these coefficients of the Riccati equation involve that the function \(\beta(x)\), i.e. \(b(x)\), is a polynomial, too. Then the polynomial \(S(x)\) is given by

$$S(x) = 16\beta^2(x) + \frac{16i}{s}\beta(x) - \frac{4}{s^2}(1 - 2\sqrt{2}s\lambda_s) - 4\beta'(x).$$

On the other hand, the lemma 3.2 give us the possibility to solve (2.13) through quadratures. Therefore (3.3) can be express as

$$\delta \beta'(x) + (\delta + 1)(\delta - 3)\beta^2(x) - \frac{2}{s}(2i + (\delta - 1)\Omega s)|\beta(x)| - \frac{1 - 2\sqrt{2}s\lambda_s + \Omega^2 s^2}{s^2} = 0,$$

where \(\Omega = \frac{\omega_2}{\omega_1}\) is the new parameter of quadrature \((\omega_2 \neq 0)\) and it is considered here to be a real parameter. The next step consists in eliminating the term \(\beta'(x)\) from (4.3) and (4.4) which lead us to express the function \(S(x)\) as a quadratic function in \(\beta(x)\)

$$S(x) = \frac{4}{s} (\delta - 1)(\delta + 3)\beta^2(x) + \frac{8(\delta - 1)}{s\delta}(2i - \Omega s)|\beta(x)| - \frac{4}{s^2}\Omega^2 s^2$$

$$\left(\Omega^2 s^2 + 2\sqrt{2}(\delta - 1) s \lambda_s - (\delta - 1)\right).$$

Unfortunately, it seems that lemma 3.1 as it is postulated is inconvenient for application because the polynomial \(S(x)\), taken into considerations, has its integer rational part in decreasing powers of \(x\) which is not necessarily the case here. Then the best thing we do can is to think about the problem otherwise. So we try to choose \(S(x)\) so that (3.2) will be both as simple as possible and convenient for application in the context of our subject. In this way since \(S(x)\) is a polynomial, the expression under the square root sign in (3.2) must be regarded as the square of a polynomial. This is possible only if the discriminant \(\Delta(S)\) of (4.5) is equal to zero, i.e.

$$\frac{64(\delta - 1)}{s^2\delta^2} \left( (\delta - 1)^2 - 2(\delta - 1)(2i \Omega + \sqrt{2}(\delta + 3)s \lambda_s)s + 4\Omega^2 s^2 \right) = 0,$$

with \(\delta \neq 1\) and hence we obtain the expression of eigenvalues \(\lambda_s\) related to the different parameters \(s\), \(\delta\) and \(\Omega\) as

$$\lambda_s(\delta, \Omega) = \frac{(\delta - 1 - 2i s \Omega)^2}{2\sqrt{2}s(\delta - 1)(\delta + 3)}.$$
Now since $S(x)$ is a quadratic function in $\beta(x)$ and its discriminant is equal to zero, then the expression under the square root has a quadratic form and determine completely the polynomial $S(x)$, which is given by

$$S(x) = \left( \beta(x) + \frac{1}{s} \cdot \frac{2i - \Omega s}{\delta + 3} \right)^2,$$

and then we obtain $z_s(x)$, $\xi_s(x)$ and the generalized $q$-deformed CS, $\Lambda_s(x)$, using (3.12), (2.10) and (2.10), respectively, up to normalization constant

$$z_s^{(\pm)}(x) = \mp \frac{1}{2s} \cdot \frac{2i - \Omega s}{\delta + 3} + \left( \delta - 1 \mp \frac{1}{2} \right) x + \left( \delta - 1 \mp \frac{1}{2} \right) b(x),$$

$$\xi_s^{(\pm)}(x) = \exp \left[ \frac{1}{2s} \cdot \frac{2i - \Omega s}{\delta + 3} - \left( \delta - 1 \mp \frac{1}{2} \right) \frac{x^2}{2} - \left( \delta - 1 \mp \frac{1}{2} \right) \int^x b(x') \, dx' \right],$$

$$\Lambda_s^{(\pm)}(x) \sim \exp \left[ \frac{i}{s} \gamma_s(\Omega, \delta) x + \left( \delta - 1 \mp \frac{1}{2} \right) \frac{x^2}{2} + \left( \delta - 1 \mp \frac{1}{2} \right) \int^x b(x') \, dx' \right],$$

where one can observe, due to the lemma 3.1, that the generalized $q$-deformed CS in (4.11) have two possibilities and are both solutions of (2.9) and the parameter $\gamma_s(\Omega, \delta)$ is defined by

$$\gamma_s(\Omega, \delta) = 1 \mp \frac{2 + i \Omega s}{2(\delta + 3)}.$$

At this stage one can ask whether there are any other alternative approaches which allows us to interpret (4.11). Here we try to answer this question by keeping in mind that choosing the right approach for our construction leads, very naturally, to a good approximation by assuming that the parameter $s$ is smaller than 1. Then, for this it is helpful to use a perturbation procedure by retaining just a small number of terms and/or a small number of $q$-deformed CS basis.

It is well-known that the perturbation theory of first-order is applicable and quantitatively sufficient, then we can choose the two-first basis of $q$-deformed CS to investigate the meaning of (4.11). Let us expand the eigenfunctions of interest in a first-order in $s$ as

$$\Lambda_s^{(\pm)}(x) \sim \Lambda_{0,s}^{(\pm)}(x) + s \Lambda_{1,s}^{(\pm)}(x) + \mathcal{O}(s^2),$$

where we restrict ourselves, in the remainder of the paper, just to the negative case.

Substituting the expansion (4.13) into (2.8), using (2.7) and (4.7), and equating terms with like powers of $s$ leads to a series of equations, shown here up to first-order

$$\Lambda_{0,s}^{(-)}(x) + (\beta(x) - i \sqrt{2} \lambda_0) \Lambda_{0,s}^{(-)}(x) = 0,$$

$$\Lambda_{1,s}^{(-)}(x) + (\beta(x) - i \sqrt{2} \lambda_0) \Lambda_{1,s}^{(-)}(x) = -i \Lambda_{0,s}^{(-)}(x) + i \beta(x) \Lambda_{0,s}^{(-)}(x) + i \left( \sqrt{2} \lambda_1 + \frac{3}{2} \beta^2(x) \right) \Lambda_{0,s}^{(-)}(x),$$

where $\lambda_0 = \frac{-i \Omega}{\sqrt{2(\delta + 3)}}$ and $\lambda_1 = \frac{-i \sqrt{2}s}{(\delta - 1)(\delta + 3)}$.

We can solve (4.14) and (4.15) to obtain the correction terms $\Lambda_{0,s}^{(-)}(x)$ and $\Lambda_{1,s}^{(-)}(x)$, then (4.13) has a form

$$\Lambda_s^{(-)}(x) \sim \left\{ 1 + is \left[ \kappa x + \frac{b(x)}{2} + \sqrt{2} \lambda_0 x^2 + 2i \sqrt{2} \lambda_0 \int^x b(x') \, dx' \right] + \mathcal{O}(s^2) \right\} \Lambda_{0,s}^{(-)}(x),$$

where

$$\Lambda_{0,s}^{(-)}(x) = \exp \left[ i \sqrt{2} \lambda_0 x - \frac{x^2}{2} - \int^x b(x') \, dx' \right],$$

(4.17)
and \( \kappa = \lambda_0^2 + \sqrt{2} \lambda_1 + \frac{1}{2} \). Therefore it is possible to interpret \((4.16)\) as an approximation of the first-order for the eigenstates \((4.11)\). Another important property to be discuss in the next section concerns the resolution of unity for a set \((4.11)\).

5 Resolution of unity and its consequence

It is well-known that the determination of a unity resolution relation for any set of CS is indeed a difficult task, because it imposes some severe constraints on CS. In this sense we are going to prove that the generalized \(q\)-deformed CS, \(\Lambda^{(-)}_s(x)\), are endowed with a resolution of unity and expressed in terms of a certain positive-definite weight function. Our proof follows basically the formal mathematical treatment sketched in \([15]\) but differs slightly in some points.

To demonstrate this specific identity, we first begin by defining

\[
1_E \equiv \int_{(\mathcal{I})} d\mu_s(x) \Lambda^{(-)}_s(x) \Lambda^{(-)}_s(x),
\]

(5.1)

where \(d\mu_s(x) = \sigma_s(x) dx\) serving as a measure in \(E\) and \(\sigma_s(x)\) is a real and positive-definite weight function to be determined. Here \(\mathcal{I}\) stands for the domain of integration which depends closely on the generalized \(q\)-deformed CS, \(\Lambda^{(-)}_s(x)\).

However the compactness of the physical configuration space, \(\mathcal{E}\), is unfortunately ill-defined. To solve the problem, we begin first by considering all \(\Lambda^{(-)}_s(x)\) defined in \(\mathcal{I}_\infty = (-\infty, \infty)\) and we will use the change of variable of the section 2, \(x = \frac{i t}{2\pi}\), which has an advantage to deduce the nature of the space of functions. In the other words this allows us to reduce \(\mathcal{E}\) to the unit circle which seems here to be more appropriate.

Then \((4.11)\) can be expressed as

\[
\Lambda^{(-)}_s(t) \sim \exp \left[- \frac{\gamma_s(\Omega, \delta)}{2\pi} t + \frac{1}{4} \left( \frac{is}{2\pi} \right)^2 + \frac{1}{2} \left( \frac{is}{2\pi} \right) \int t(t') dt' \right],
\]

(5.2)

where \(b(t)\) is \(2\pi\)-periodic function. Now \((5.2)\) are well-defined over a unit circle and obeying to the relation

\[
\Lambda^{(-)}_s(t + 2\pi) \sim \exp \left[- \frac{\gamma_s(\Omega, \delta)}{4\pi} t - \frac{s^2}{4\pi} t - \frac{s^2}{4} \right] \Lambda^{(-)}_s(t),
\]

(5.3)

and by mathematical induction, we get from \((5.3)\)

\[
\Lambda^{(-)}_s(t + 2\pi n) \sim \exp \left[- n \gamma_s(\Omega, \delta) - n \frac{s^2}{4\pi} t - n \frac{s^2}{4} \right] \Lambda^{(-)}_s(t),
\]

(5.4)

which is valid for all \(n \in \mathbb{Z}\). Exploiting this property, it is interesting – theoretically speaking – to express \((5.1)\) in the equivalent form

\[
1_E = \int_{-\infty}^{\infty} dx \Lambda^{(-)}_s(x) \sigma_s(x) \Lambda^{(-)}_s(x) \equiv \sum_{n=-\infty}^{+\infty} \int_{(2n-1)\pi}^{(2n+1)\pi} dx \Lambda^{(-)}_s(x) \Lambda^{(-)}_s(x),
\]

(5.5)

where the domain of weak oscillator coordinate is considered to be covered by the infinite sum of the finite interval \(\mathcal{I}_n = [(2n - 1)\pi, (2n + 1)\pi]\), deduced in section 2, as

\[
\mathcal{I}_\infty = \bigcup_{n=-\infty}^{\infty} \mathcal{I}_n,
\]

(5.6)
and what Sogami and Koizumi call a periodic structure in [13].

Let \( x = t + 2\pi n \), where \( t \) is confined to the partial interval \( [−\pi, \pi] \) and substituting (5.4) in the right-hand side of (5.5), we obtain after some straightforward calculations

\[
1_\varepsilon = \int_{−\pi}^{\pi} dt \, \Lambda_s^{(−)}(t)\sigma_s(t)\Lambda_s^{(−)}(t),
\]

where by adopting a particular parametrization \( s \to \sqrt{2}s \) for the parameter \( s \), the weight function \( \sigma_s(t) \) is found to be

\[
\sigma_s(t) = \sum_{n=-\infty}^{+\infty} \exp \left( -2n\gamma_{\Re} - 2n^2s^2 - 2n^2\pi t - n^2s^2 \right)
\]

\[
\equiv \sqrt{\frac{\pi}{s}} \exp \left[ \left( \frac{s}{2\pi} t + \frac{\gamma_{\Re}}{s} \right)^2 \right] \vartheta_3 \left( \frac{t}{2} + \frac{\pi\gamma_{\Re}}{s^2} \mid \tau \right),
\]

(5.8)

where \( \gamma_{\Re} = \frac{\gamma + \gamma}{2} \). As we can see the weight function (5.8) is positive-definite and coincides exactly with the well-known elliptic Jacobi \( \vartheta_3 \)-function, where by definition \( \tau = \frac{\gamma}{\pi} \) [14]. It is worth noting that the particular case for (5.8) can be obtained, in order to establish the value of \( \delta \), if we consider that the coefficient \( \gamma_{\Re} \) in (5.8) must be specified to be zero; i.e. \( \gamma_s(\Omega, \delta) \in i\mathbb{R} \), from which we can infer to the parameter \( \delta \) the value: \( \delta = -2 \).

This particular result allows us to decrease the number of parameters related to \( q \)-deformed CS to one, namely \( \Omega \), instead of two. The generalized \( q \)-deformed CS, \( \Lambda_s^{(−)}(x) \), is reduce to the expression

\[
\Lambda_s^{(−)}(x) \sim \exp \left[ \frac{\Omega x}{2} + \frac{x^2}{4} + \frac{1}{2} \int x b(x') dx' \right],
\]

(5.9)

which are effectively independent of the parameter \( s \) but, however, depend on a free parameter \( \Omega \) associated with the quadrature solution.

6 Conclusion

Although the \( q \)-deformed CS play an important role in theoretical and mathematical physics, the resolution of their eigenvalues equation, as far as we known, still remain practically unexplored and received much less attention, hence the purpose of this study.

In our treatment, we have taken a weak deformed version of \( q \)-boson annihilation operator in order to solve the eigenvalues equation for the generalized \( q \)-deformed CS. With the help of change of function, this equation was transformed into Riccati equation where under some integrability restrictions of its coefficients, the eigenfunctions \( \Lambda_s^{(−)}(x) \) as well as their associated eigenvalues \( \lambda_s \) are obtained from this construction. We have also studied the question of interpreting \( \Lambda_s^{(−)}(x) \) using the property that \( s \) is smaller than 1. It turned out that the perturbation theory of the first-order, with small number of \( q \)-deformed CS basis, can be considered as an alternative approach for describing \( \Lambda_s^{(−)}(x) \).

Finally, we have shown that the simple and explicit form of the deduced \( q \)-deformed CS has enabled us to establish the relation of resolution of unity by means of a particular measure, the elliptic Jacobi \( \vartheta_3 \)-function, on the unit circle which seems to be especially suited for this problem in order to bring out the nature of the space of functions.

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