An Alternative Method for Primary Decomposition of Zero-dimensional Ideals over Finite Fields

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Abstract. We present an alternative method for computing primary decomposition of zero-dimensional ideals over finite fields. Based upon the further decomposition of the invariant subspace of the Frobenius map acting on the quotient algebra in the algorithm given by S. Gao, D. Wan and M. Wang in 2008, we get an alternative approach to compute all the primary components at once. As one example of our method, an improvement of Berlekamp’s algorithm by theoretical considerations which computes the factorization of univariate polynomials over finite fields is also obtained.

1 Introduction

Let $k$ be a field and $k[x_1, \ldots, x_n]$ (or $k[x]$ for short) be the ring of polynomials in the variables $x_1, \ldots, x_n$ with coefficients in $k$. An ideal $I \subset k[x]$ is called zero-dimensional if the quotient algebra $k[x]/I$ is a finite-dimensional $k$–vector space. There are several well-know algorithms for computing primary ideal decomposition based on zero-dimensional decomposition, and we refer the readers to [1,3,8,9].

Gao, Wan and Wang in [7] present an interesting approach to compute primary decomposition of zero-dimensional ideals over finite fields. The method is based on the invariant subspace of the Frobenius map acting on the quotient algebra $k[x]/I$. Since the dimension of the invariant subspace just equals the number of primary components, a basis of the invariant subspace leads a complete primary decomposition of $I$ by computing Gröbner bases.

In the method in [7], if one chooses an element of the basis of the invariant subspace which is separable for $I$, then all the primary components can be computed at once. Otherwise, the further decomposition is necessary even though the probability of separable element in the invariant subspace is not low in most cases, see Proposition 3.2 in [7] for details. In this paper, we aim to find an approach to decompose the above invariant subspace into a direct product of several one-dimensional $k$–algebras in Proposition 1 in Section 3. Based upon
this theoretical work, we get an alternative approach to compute primary decomposition of zero-dimensional ideals over finite fields, which allows us to find all the primary components completely.

2 Preliminaries

In this section, we assume $R$ is a commutative ring. For basic notations in commutative algebra, we refer the readers to the monographs [5,6].

Definition 1.[5] Given a commutative ring $R$, let $I$ be an ideal in $R$, let $U$ be an $R$–module. Then the set \( \{ m \in U | m \cdot s = 0, \ \forall s \in I \} \) is an $R$–submodule of $U$. It is called the colon module of 0 by $I$ in $U$, denoted by $M(I)$.

Lemma 1. Let $R$ be a commutative ring with identity element, and $U$ be an $R$–module. If $I_1, \ldots, I_t$ are pairwise comaximal ideals in $R$, then

\[
M(I) = M(I_1) \oplus \cdots \oplus M(I_t),
\]

where $I = I_1 \cap \cdots \cap I_t$ or $I_1 \cdots I_t$.

Proof: We give a proof by induction on $t$. When $t = 2$, first we prove the following claim

\[
M(I) = M(I_1 \cap I_2) = M(I_1) + M(I_2).
\]

It is obvious that $M(I) = M(I_1 \cap I_2) \supseteq M(I_1) + M(I_2)$ by the definition of $M(I)$ and the fact that $I = I_1 \cap I_2 = I_1 I_2$.

Since $I_1$ and $I_2$ are pairwise comaximal ideal in $R$, there exist some $a_1 \in I_1$ and $a_2 \in I_2$ such that $a_1 + a_2 = 1$ where 1 is the identity element in $R$, which implies that $m = a_1 \cdot m + a_2 \cdot m$ for any $m \in M(I_1 \cap I_2)$. The other direction follows from the fact that $a_1 \cdot m \in M(I_2)$ and $a_2 \cdot m \in M(I_1)$.

Since $a_1 \cdot m = a_2 \cdot m = 0$ for any $m \in M(I_1) \cap M(I_2)$, which means $m = a_1 \cdot m + a_2 \cdot m = 0$. Thus,

\[
M(I) = M(I_1) \oplus M(I_2).
\]

Now let $t > 2$, consider that $I_1 \cap \cdots \cap I_{t-1}$ and $I_t$ are pairwise comaximal ideals in $R$. It follows from the induction hypothesis that

\[
M(I_1 \cap \cdots \cap I_{t-1} \cap I_t) = M(I_1 \cap \cdots \cap I_{t-1}) \oplus M(I_t) = M(I_1) \oplus \cdots \oplus M(I_{t-1}) \oplus M(I_t).
\]

This completes the proof. \( \square \)

We need the following ring-theoretic version of the Chinese Remainder Theorem in our discussion.

Lemma 2 (Chinese Remainder Theorem).[5] Let $R$ be a commutative ring with identity element. If $I_1, \ldots, I_t$ are pairwise comaximal ideals in $R$, then the canonical map is an isomorphism of $R$-modules

\[
R/I \cong R/I_1 \oplus \cdots \oplus R/I_t,
\]

where $I = I_1 \cap \cdots \cap I_t$. 
Let $k$ be any field, now we consider the above ring $R = k[x_1, \ldots, x_n]$ or $k[x]$.

**Theorem 1.** Assume that $I_1, \ldots, I_t$ are pairwise comaximal ideals in $k[x_1, \ldots, x_n]$, and let $I = I_1 \cap \cdots \cap I_t$ or $I_1 \cdots I_t$. Then the canonical map $\Phi$ is an isomorphism of $k[x]$–modules, i.e.,

$$k[x]/I = J_1/I \oplus \cdots \oplus J_t/I \cong k[x]/I_1 \oplus \cdots \oplus k[x]/I_t$$

where $J_i = I_1 \cap \cdots \cap I_{i-1} \cap I_{i+1} \cap \cdots \cap I_t$ for $i = 1, 2, \ldots, t$. Moreover, the restriction of $\Phi$ to $J_i/I$ is an isomorphism of $k[x]$–modules

$$J_i/I \cong k[x]/I_i,$$

for each $i = 1, 2, \ldots, t$.

**Proof:** Consider the $k[x]$–module $U = k[x]/I$. By Lemma 1, we have

$$M(I) = M(I_1) \oplus \cdots \oplus M(I_t).$$

Since $I_i$ and $J_i$ are comaximal for each $i$ and $M(I) = k[x]/I$ and $M(I_i) = J_i/I$, it is easy to see that $k[x]/I = J_1/I \oplus \cdots \oplus J_t/I$.

Applying Lemma 2, the canonical map $\Phi$ is an isomorphism of $k[x]$–modules

$$k[x]/I \cong k[x]/I_1 \oplus \cdots \oplus k[x]/I_t.$$

Furthermore, it implies that each restriction of $\Phi$ to $J_i/I$ is an isomorphism $J_i/I \cong k[x]/I_i$ by the proof of Chinese Remainder Theorem. For the details, please see the proof of Lemma 3.7.4 in [6]. This completes the proof. □

Let $k$ be any field containing a finite field $\mathbb{F}_q$ as a subfield. An ideal $I \subseteq k[x]$ is called primary if each non-zero zerodivisor of $k[x]/I$ is a non-zero nilpotent element. Further $I$ is called quasi-primary if $\sqrt{I}$ is a prime ideal, that is, if $I$ has only one minimal component and all other components are embedded.

**Definition 2.** Let $k$ be any field containing $\mathbb{F}_q$ as a subfield, and $I$ be an ideal in $k[x]$, we establish the following $\mathbb{F}_q$–linear transformation $\Psi_I$ from $\mathbb{F}_q$–vector space $k[x]/I$, to itself, defined by

$$\Psi_I(\bar{f}) = \bar{f}^q - \bar{f}$$

for each $\bar{f} \in k[x]$.

In fact $\text{Ker}(\Psi_I)$ is the invariant subspace of the Frobenius map acting on $k[x]/I$ which plays an essential role in [7] and our improvement. With the notation in Definition 2, Lemma 2.1 in [7] can be described as follows.

**Lemma 3.** Let $k$ be any field containing $\mathbb{F}_q$ as a subfield, and $I_0 \subseteq k[x]$ be a quasi-primary ideal. Then

$$\text{Ker}(\Psi_{I_0}) = \mathbb{F}_q.$$

Now consider an arbitrary ideal $I \subseteq k[x]$. Suppose $I$ has an irredundant primary decomposition

$$I = I_1 \cap I_2 \cap \cdots \cap I_t,$$  \hspace{1cm} (1)
where $I_i \in k[x]$ are primary ideals, and $I_i$’s are pairwise comaximal.

The following result is another version of Theorem 2.2 in [7]. Here we present an alternative proof.

**Theorem 2.** Let $I \subset \mathbb{F}_q[x]$ be a zero-dimensional ideal with $t$ irredundant primary components $I_1, I_2, \ldots, I_t$. If we consider $\mathbb{F}_q[x]/I$ as $\mathbb{F}_q$-vector space, then $\text{Ker}(\Psi_f) \cong \mathbb{F}_q^t$.

**Proof:** From Theorem 1, we know that there exists an isomorphism of $\mathbb{F}_q$-vector spaces

$$\mathbb{F}_q[x]/I = J_1/I_1 \oplus \cdots \oplus J_t/I_t \cong F_q[x]/I_1 \oplus \cdots \oplus F_q[x]/I_t$$

and $J_i/I \cong F_q[x]/I_i$ for each $i$. By Lemma 3, we have $\text{Ker}(\Psi_f) \cong \mathbb{F}_q^t$. This completes the proof. □

3 Main Results

Let $I$ be a zero-dimensional ideal in $\mathbb{F}_q[x]$. In the following, we assume that a Gröbner basis for $I$ is already known or computed for certain term order. Then one can easily find a linear basis for $\mathbb{F}_q[x]/I$ over $\mathbb{F}_q$ by Macaulays Basis Theorem in [56].

Based upon the result of Theorem 2.2 in [7] or Theorem 2 in this paper, we are ready to decompose $\text{Ker}(\Psi_f)$ into a direct product of some one-dimensional algebras using the following method. Furthermore, by applying Gröbner basis theory, an improving approach to compute all the primary components of $I$ is given.

**Lemma 4.** In the situation of Theorem 2, $\text{Ker}(\Psi_f)$ is a subring of $\mathbb{F}_q[x]/I$ and has no non-zero nilpotent elements.

**Proof:** It is easy to check that $\text{Ker}(\Psi_f)$ is a subring of $\mathbb{F}_q[x]/I$. We proceed to show that $\text{Ker}(\Psi_f)$ has no non-zero nilpotent elements.

Suppose there is some $\bar{f} \in \text{Ker}(\Psi_f)$ and a positive integer $m$ satisfying $\bar{f}^m = \bar{0}$. Consider that the greatest common divisor of polynomials $x^m$ and $x^q - x$ is $x$ in $\mathbb{F}_q[x]$, there exist some $u(x), v(x) \in \mathbb{F}_q[x]$ such that $x = u(x)x^m + v(x)(x^q - x)$. It implies that

$$\bar{f} = u(\bar{f})\bar{f}^m + v(\bar{f})(\bar{f}^q - \bar{f}) = \bar{0}$$

by $\bar{f}^q = \bar{f}$. This completes the proof. □

The proof of the following theorem presents an approach to decompose a finite dimensional $k_0$-algebra which has no non-zero nilpotent elements.

**Proposition 1.** Let $k_0$ be a field, and $V$ be a $k_0$-algebra with $\dim_{k_0}(V) = t$. If $V$ has no non-zero nilpotent elements, then there exist $\bar{g}_1, \ldots, \bar{g}_t \in V$ such that $V = \text{span}_{k_0}(\bar{g}_1, \ldots, \bar{g}_t)$ and $\bar{g}_i\bar{g}_j = \bar{0}$ for any $i \neq j$. Furthermore, $V$ can be written as a direct product of several one-dimensional $k_0$-algebras

$$V = \langle \bar{g}_1 \rangle \oplus \cdots \oplus \langle \bar{g}_t \rangle.$$

**Proof:** Let $\{\bar{f}_1, \ldots, \bar{f}_t\}$ be a basis of the $k_0$-vector space $V$. For a given no-zero zerodivisor $\bar{h} \in V$, we define that next $\mathbb{F}_q$-linear transformation

$$V \rightarrow V,$$
\[ \phi_k : \bar{g} \to \bar{h}g. \]

It is well known that \( \dim_{k_0}(\text{Ker}(\phi_h)) + \dim_{k_0}(\text{Im}(\phi_h)) = t. \)

We claim that
\[ V = \text{Ker}(\phi_h) \oplus \text{Im}(\phi_h). \]

To prove the claim, it suffices to show \( \text{Ker}(\phi_h) \cap \text{Im}(\phi_h) = \{0\} \). For any \( \bar{p} \in \text{Ker}(\phi_h) \cap \text{Im}(\phi_h) \), it implies that there exists some \( \bar{q} \in V \) such that \( \bar{p} = \bar{h} \bar{q} \).

Notice that \( \bar{h} \bar{p} = \bar{h}^2 \bar{q} = 0 \), hence \( \bar{p}^2 = 0 \). Since \( V \) has no non-zero nilpotent elements, we have \( \bar{p} = 0 \). This finishes the proof of our claim. In addition both \( \text{Ker}(\phi_h) \) and \( \text{Im}(\phi_h) \) are \( k_0 \)-algebras. Thus the claim implies a direct product of \( k_0 \)-algebras
\[ V = \text{Ker}(\phi_h) \oplus \text{Im}(\phi_h). \]

With an analogous operation \( \text{Ker}(\phi_h) \) and \( \text{Im}(\phi_h) \), respectively, we can decompose \( V \) into a direct product of one-dimensional \( k_0 \)-algebras
\[ V = \langle \bar{g}_1 \rangle \oplus \cdots \oplus \langle \bar{g}_t \rangle. \]

This completes the proof. \( \square \)

**Remark:** Since the number of regular elements of \( V \) is \( |k_0| \), most of elements are non-zero divisors.

Based upon the above results, we can decompose \( \text{Ker}(\Psi_f) \) in Theorem 2.

**Theorem 3.** Let \( F_q \) be a finite field. Suppose \( I \subset F_q[x] \) is a zero-dimensional ideal with \( t \) irredundant primary components \( I_1, I_2, \ldots, I_t \). Then there exists a direct product of one dimensional \( F_q \)-algebras
\[ \text{Ker}(\Psi_f) = \langle \bar{h}_1 \rangle \oplus \cdots \oplus \langle \bar{h}_t \rangle, \]

with each \( \bar{h}_i \in J_i \setminus I_i \) and \( \bar{h}_i^2 = \bar{h}_i \) where \( J_i = I_1 \cap \cdots \cap I_{i-1} \cap I_{i+1} \cap \cdots \cap I_t \) for \( i = 1, 2, \ldots, t \).

**Proof:** We regard \( F_q[x]/I \) as an \( F_q \)-algebra. With the notation in Definition 2, let \( V_0 = \text{Ker}(\Psi_f) \). It follows from Theorem 2 that \( V_0 \) is a finite dimensional \( F_q \)-algebra with \( \dim_{F_q}(V_0) = t \). Furthermore, it follows that \( V_0 \) has no non-zero nilpotent elements from Lemma 4.

Applying Proposition 1, one can get the next direct product of one dimensional \( F_q \)-algebras
\[ V_0 = \langle \bar{g}_1 \rangle \oplus \cdots \oplus \langle \bar{g}_t \rangle. \]

Taking some permutation of \( \bar{g}_1, \ldots, \bar{g}_t \), we can make sure that \( g_i \in J_i \) for \( i = 1, 2, \ldots, t \). Since there exists some \( k_i \in F_q \setminus \{0\} \) satisfying \( \bar{g}_i^2 = k_i \bar{g}_i \) for each \( i \), we can easily get some \( \bar{h}_i \in \langle \bar{g}_i \rangle \setminus \{0\} \) such that \( \bar{h}_i^2 = \bar{h}_i \) for \( i = 1, 2, \ldots, t \). This completes the proof. \( \square \)

The following example which is the same one given in [7] illustrates our decomposition.

**Example 1.** Consider the ideal
\[ I = \langle y^2 - xz, z^2 - x^2 y, x + y + z - 1 \rangle \subset F_5[x, y, z]. \]
Moreover, there is a direct product of two-dimensional algebras and have no non-zero nilpotent elements. Under the lex order with \( x \succ y \succ z \), \( I \) has a Gröbner basis

\[
G = [x + y + z - 1, y^2 + 3y - 2z^4 + z^3 + 2z^2 + z, yz + 2y + 2z^4 - z^3 - 2z - 2z^5 - z^4 + 3z^3 - z^2 + 2z],
\]

Let \( R = \mathbb{F}_5[x, y, z]/I \). By Macaulay’s Basis Theorem, we can get a \( \mathbb{F}_5 \)-basis \( B \) of \( R \) with \( B = \{z^4, z^3, z^2, z, y, 1\} \). Using the matrix of \( \mathbb{F}_5 \)-linear transformation \( \Psi_I \) on the basis \( B \), or referring to [7], one can easily compute that

\[
\mathbf{V} = \text{Ker}(\Psi_I) = \{ \bar{g} \in R \mid \bar{g}^5 = \bar{g} \} = \text{span}_{\mathbb{F}_5}(\bar{g}_1, \bar{g}_2, \bar{g}_3, \bar{g}_4)
\]

where

\[
\bar{g}_1 = 1, \quad \bar{g}_2 = z - z^2, \quad \bar{g}_3 = z^2 + z^3, \quad \bar{g}_4 = z^3 - 2z^4.
\]

We proceed to show how to decompose \( \mathbf{V} \) using the method given in the proof of Proposition 1.

As \( \bar{h} = \bar{g}_4 \in \mathbf{V} \) is a zerodivisor, we define the next \( \mathbb{F}_5 \)-linear transformation

\[
\phi_{\bar{g}_4} : \bar{g} \mapsto \bar{g}_4 \bar{g}.
\]

The matrix of \( \phi_{\bar{g}_4} \) under the basis \( \bar{g}_1, \bar{g}_2, \bar{g}_3, \bar{g}_4 \) of \( \mathbf{V} \) is

\[
A_{\bar{g}_4} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 2 & 2 & -1 \\
0 & 2 & 2 & -1 \\
1 & 3 & 2 & 1 \\
\end{bmatrix}
\]

such that \((\phi_{\bar{g}_4}(\bar{g}_1), \phi_{\bar{g}_4}(\bar{g}_2), \phi_{\bar{g}_4}(\bar{g}_3), \phi_{\bar{g}_4}(\bar{g}_4)) = (\bar{g}_1, \bar{g}_2, \bar{g}_3, \bar{g}_4)A_{\bar{g}_4}\). We can easily compute that

\[
\text{Ker}(\phi_{\bar{g}_4}) = \text{span}_{\mathbb{F}_5}(\bar{h}_1, \bar{h}_2) \quad \text{and} \quad \text{Im}(\phi_{\bar{g}_4}) = \text{span}_{\mathbb{F}_5}(\bar{h}_3, \bar{h}_4)
\]

where

\[
\bar{h}_1 = -4 - \bar{g}_2 + \bar{g}_3 = -4 + 2z^2 + z^3,
\]

\[
\bar{h}_2 = 3\bar{g}_2 + \bar{g}_4 = 3z - 3z^2 + z^3 - 2z^4,
\]

\[
\bar{h}_3 = \bar{g}_2 + \bar{g}_3 = z^3 + z,
\]

\[
\bar{h}_4 = \bar{g}_4 = z^3 - 2z^4.
\]

Let \( \mathbf{V}_1 = \text{Ker}(\phi_{\bar{g}_4}) \) and \( \mathbf{V}_2 = \text{Im}(\phi_{\bar{g}_4}) \). It is easy to check that both \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \) are two-dimensional \( \mathbb{F}_5 \)-algebras and have no non-zero nilpotent elements. Moreover, there is a direct product of two-dimensional \( \mathbb{F}_5 \)-algebras

\[
\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2.
\]
Continuing this way, we can compute analogously as the following direct product of one-dimensional $F_5$-algebras

$$V_1 = \langle \bar{h}_2 \rangle \oplus (\bar{h}_1 + \bar{h}_2), \quad V_1 = \langle \bar{h}_3 \rangle \oplus (\bar{h}_3 + \bar{h}_4).$$

Thus, $V = \langle \bar{h}_2 \rangle \oplus (\bar{h}_1 + \bar{h}_2) \oplus \langle \bar{h}_3 \rangle \oplus (\bar{h}_3 + \bar{h}_4)$ such that

$$(\bar{h}_2)^2 = 2\bar{h}_2, \quad (\bar{h}_1 + \bar{h}_2)^2 = \bar{h}_1 + \bar{h}_2,$$

$$(\bar{h}_3)^2 = \bar{h}_3, \quad (\bar{h}_3 + \bar{h}_4)^2 = \bar{h}_3 + \bar{h}_4.$$ 

Furthermore, since $(3\bar{h}_2)^2 = 3\bar{h}_2$, we have

$$V = \langle 3\bar{h}_2 \rangle \oplus (\bar{h}_1 + \bar{h}_2) \oplus \langle \bar{h}_3 \rangle \oplus (\bar{h}_3 + \bar{h}_4).$$

It remains to compute all the primary components of $I$ by the following results.

**Proposition 2.** Let $R$ be a commutative ring with identity element 1. If there exist $f_0 \in I_0$, $g_0 \in J_0$ such that $f_0 + g_0 = 1$, namely, $I_0, J_0$ are two comaximal ideals in $R$, then

$$I : \langle g_0 \rangle^\infty = I_0,$$

where $I = I_0 \cap J_0$.

**Proof:** We first show that $I : \langle g_0 \rangle^\infty \supseteq I_0$. Given any $h \in I_0$, we have $h = hf_0 + hg_0$. It implies that $hg_0 = h - hf_0 \in I_0$. Since $hg_0 \in J_0$, $hg_0 \in I_0 \cap J_0 = I$. Thus, $h \in I : \langle g_0 \rangle \subseteq I : \langle g_0 \rangle^\infty$.

In another direction, for any $h \in I : \langle g_0 \rangle^\infty$, it means that there exists some integer $k > 0$ such that

$$hg_0^k \in I.$$ 

According to $(f_0 + g_0)^k = 1$, we have that there is some $f_0^* \in I_0$ such that

$$f_0^* + g_0^k = 1.$$ 

Hence

$$h = hf_0^* + hg_0^k \in I_0 + I \subseteq I_0.$$ 

This completes the proof. $\square$

One can compute $I : \langle g_0 \rangle^\infty$ in $k[x]$ by the following Gröbner basis theory, see [1] for the details.

**Lemma 5.** Let $k$ be a field and $I$ an ideal generated by $\{f_1, \ldots, f_s \}$ in $k[x]$, and some $g_0 \in k[x]$. If $G^*$ is the Gröbner basis of $\{f_1, \ldots, f_s, 1 - ug_0 \}$ in $k[x, u]$ with respect to the purely lexicographical order determined with $x_i < u$. Then

$$I : \langle g_0 \rangle^\infty = (G^* \cap k[x]).$$ 

Next we proceed to present an alternative algorithm for computing primary decomposition of zero-dimensional ideals over finite fields.

**Primary Decomposition:** $I_1, \ldots, I_t \leftarrow I$. Given a zero-dimensional ideal in $F_q[x]$, this algorithm computes an irredundant primary decomposition $I = I_1 \cap \cdots \cap I_t$. 
D1. Computing a Gröbner basis $G$ of $I$, get the basis $B$ for $\mathbb{F}_q[x]/I$ by Macaulay’s Basis Theorem.

D2. Compute $\text{Ker}(\Psi_I)$ and $t$:

D2.1. Compute $\text{Ker}(\Psi_I)$ by the matrix of $\mathbb{F}_q$–linear transformation $\Psi_I$ on the basis $B$ and set $V \leftarrow \text{Ker}(\Psi_I)$.

D2.2. Set $t \leftarrow \text{dim}_{\mathbb{F}_q}(V)$.

D3. Decompose $V$ into a direct product of one-dimensional $\mathbb{F}_q$–algebras by the method given in the proof of Proposition 1,

$$V = \langle \bar{h}_1 \rangle \oplus \cdots \oplus \langle \bar{h}_t \rangle.$$ 

D4. Compute each Gröbner basis $G^*_i$ of $I \cup \{1 - u\bar{h}_i\}$ in $\mathbb{F}_q[x,u]$. Set $I_i \leftarrow \langle G^*_i \cap \mathbb{F}_q[x] \rangle$ for $i = 1, 2, \ldots, t$.

Example 2. Continued from Example 1, compute Gröbner basis $G^*_1$ of $\{y^2 - xz, z^2 - x^2y, x + y + z - 1, 1 - u\bar{h}_1\}$ in $\mathbb{F}_5[x,y,z,u]$ by Lemma 5. $G^*_1 = [z, y, x + 4, 4 + u]$, so

$$I_1 = \langle z, y, x + 4 \rangle.$$ 

Similarly, we can compute that

$$G^*_2 = [z + 2, y^2 + 3y + 1, x + y + 2, 2 + u],$$
$$G^*_3 = [z^2 + 4z + 2, y + 2z + 1, x + 4z + 3, 4 + u],$$
$$G^*_4 = [z + 3, y + 4, x + 2, 4 + u].$$

Therefore,

$$I_2 = \langle z + 2, y^2 + 3y + 1, x + y + 2 \rangle,$$
$$I_3 = \langle z^2 + 4z + 2, y + 2z + 1, x + 4z + 3 \rangle,$$
$$I_4 = \langle z + 3, y + 4, x + 2 \rangle.$$ 

From the above, we have the following irredundant primary decomposition

$$I = I_1 \cap I_2 \cap I_3 \cap I_4.$$ 

As a direct application of our approach, we easily give to an alternative method of Berlekamp’s algorithm which computes the factorization of univariate polynomials over finite fields, see [2] for the details.

Example 3. Consider the polynomial $f = x^6 + x^5 + x^4 + 2 \in \mathbb{F}_3[x]$. Let $B = \{x^5, x^4, x^3, x^2, x, 1\}$ which is an $\mathbb{F}_3$–basis of $\mathbb{F}_3[x]/\langle f \rangle$.

Then we have the following $\mathbb{F}_3$–linear transformation

$$\Psi(f) : \mathbb{F}_3[x]/\langle f \rangle \to \mathbb{F}_3[x]/\langle f \rangle$$

defined by

$$\Psi(f) (\bar{g}) = \bar{g}^3 - \bar{g}$$
for $\bar{g} \in \mathbb{F}_3[x]/\langle f \rangle$.

The matrix of $\mathbb{F}_3$–linear transformation $\Psi_{\langle f \rangle}$ on the basis $B$ is obtained as follows:

$$
\begin{bmatrix}
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
2 & 1 & 0 & 0 & 1 \\
0 & 2 & 2 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 \\
0 & 2 & 1 & 1 & 0
\end{bmatrix}.
$$

We can easily compute that

$$
\mathbf{V} = \text{Ker}(\Psi_{\langle f \rangle}) = \{ \bar{g} \in \mathbb{F}_3[x]/\langle f \rangle \mid \bar{g}^3 = \bar{g} \} = \text{span}_{\mathbb{F}_3}(\bar{g}_1, \bar{g}_2, \bar{g}_3)
$$

where

$$
\bar{g}_1 = 1, \quad \bar{g}_2 = -x^3 + x^2, \quad \bar{g}_3 = x^5 + x.
$$

We proceed to show how to decompose $\mathbf{V}$ by the method given in the proof of Proposition 1.

As $\bar{g}_3 \in \mathbf{V}$ is a zerodivisor, we suppose that the next $\mathbb{F}_3$–linear transformation

$$
\mathbf{V} \rightarrow \mathbf{V},
$$

$\phi_{\bar{g}_3} : \bar{g} \mapsto \bar{g} \bar{g}_3$.

The matrix of $\phi_{\bar{g}_3}$ under the basis $\{\bar{g}_1, \bar{g}_2, \bar{g}_3\}$ of $\mathbf{V}$ is

$$
A_{\bar{g}_3} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}
$$

such that $(\phi_{\bar{g}_3}(\bar{g}_1), \phi_{\bar{g}_3}(\bar{g}_2), \phi_{\bar{g}_3}(\bar{g}_3)) = (\bar{g}_1, \bar{g}_2, \bar{g}_3)A_{\bar{g}_3}$. We can readily compute that

$$
\text{Ker}(\phi_{\bar{g}_3}) = \text{span}_{\mathbb{F}_3}(\bar{s}_1) \text{ and } \text{Im}(\phi_{\bar{g}_3}) = \text{span}_{\mathbb{F}_3}(\bar{s}_2, \bar{s}_3)
$$

where

$$
\bar{s}_1 = \bar{g}_2 + \bar{g}_3 = x^5 + 2x^3 + x^2 + x,
$$

$$
\bar{s}_2 = \bar{g}_3 = x^5 + x,
$$

$$
\bar{s}_3 = \bar{g}_1 + \bar{g}_2 + \bar{g}_3 = x^5 + 2x^3 + x^2 + x + 1.
$$

Let $\mathbf{V}_1 = \text{Ker}(\phi_{\bar{g}_3})$ and $\mathbf{V}_2 = \text{Im}(\phi_{\bar{g}_3})$. It is easy to check that $\mathbf{V}_2$ is a two-dimensional $\mathbb{F}_3$–algebra and has no non-zero nilpotent elements. Moreover, there is a direct product of $\mathbb{F}_3$–algebras

$$
\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2.
$$
Similarly, we can compute analogously as the following direct product of one-dimensional $\mathbb{F}_3$-algebras

$$V_2 = \langle \bar{s}_2 - \bar{s}_3 \rangle \oplus \langle \bar{s}_2 + \bar{s}_3 \rangle.$$ 

Thus, $V = \langle \bar{s}_1 \rangle \oplus \langle \bar{s}_2 - \bar{s}_3 \rangle \oplus \langle \bar{s}_2 + \bar{s}_3 \rangle$. It is easy to check that 

$$(2\bar{s}_1)^2 = 2\bar{s}_1, \quad (\bar{s}_2 - \bar{s}_3)^2 = \bar{s}_2 - \bar{s}_3, \quad (2(\bar{s}_2 + \bar{s}_3))^2 = 2(\bar{s}_2 + \bar{s}_3).$$

Set $\bar{h}_1 = 2\bar{s}_1$, $\bar{h}_2 = \bar{s}_2 - \bar{s}_3$, $\bar{h}_3 = 2(\bar{s}_2 + \bar{s}_3)$. We have 

$$V = \langle \bar{h}_1 \rangle \oplus \langle \bar{h}_2 \rangle \oplus \langle \bar{h}_3 \rangle.$$ 

Compute Gröbner basis $G^*_1$ of $\{ f, 1 - u\bar{h}_1 \}$ in $\mathbb{F}_3[x, u]$ by Lemma 5. 

$$G^*_1 = \{ 2 + x^2 + x, u + 1 \},$$

so 

$$f_1 = 2 + x^2 + x.$$ 

Similarly, we can compute that 

$$G^*_2 = \{ x^3 + 2x^2 + 1, 2 + u \}, \quad G^*_3 = \{ x + 1, t + 1 \}.$$ 

Therefore, 

$$f_2 = x^3 + 2x^2 + 1, \quad f_3 = x + 1.$$ 

It yields the following factorization 

$$f = f_1 f_2 f_3.$$ 

Remark: The virtue of our approach to compute primary decomposition of zero-dimensional ideals over finite fields is that it allows us to find all the primary components completely. In particular, Proposition 1 contributes a new and simple method to decompose the invariant subspace of the Frobenius map on the quotient algebra $k[x]/I$ by theoretical and practical considerations. But the complexity of our approach is mainly depended on computing Gröbner bases. There is no detailed discussion of the complexity and implementation of our alternative method here.

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