A Note on modified Veselov-Novikov Hierarchy

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ABSTRACT

Because of its relevance to lower-dimensional conformal geometry, known as a generalized Weierstrass inducing, the modified Veselov-Novikov (mVN) hierarchy attracts renewed interest recently. It has been shown explicitly in the literature that an extrinsic string action à la Polyakov (Willmore functional) is invariant under deformations associated to the first member of the mVN hierarchy. In this note we go one step further and show the explicit invariance of the functional under deformations associated to all higher members of the hierarchy.

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1 Introduction

In a last decade we have witnessed a tremendous flow of applications of 1(+1)-dimensional exactly soluble models in 2-d CFT, 2-dimensional gravity — both continuum and matrix model approach —, (super) string theories, etc. Specifically, KdV, KP hierarchy and their modified cousins played important and remarkable roles in various occasions. We owe this success mainly to the existence of fascinating mathematical structures underlying such exactly soluble models, i.e. an infinite dimensional symmetry, known as \( \mathcal{W} \)-algebras, including Virasoro algebra.

In the case of higher dimensional exactly soluble models (see e.g. \cite{1, 2}), however, even though their importance was pointed out some time ago \cite{3}, due to their mathematical complexity, not much application has been explored until recently.

Veselov-Novikov (VN) hierarchy \cite{4} and its modified cousin (mVN) \cite{5} are demonstrated as another type of 2(+1)-dimensional extension of KdV and mKdV, as compared to the well-known KP-hierarchy. One interesting feature of this higher dimensional generalization is that in the mVN case one deals with a deformation problem of Dirac operators in 2 dimensions \cite{1}, rather than that of quadratic differential operators as in ordinary cases. In addition, thanks to the contributions of the authors of the recent literature \cite{6, 7, 8} we now know the important relevance of mVN to conformally Euclidean immersion of 2-surfaces into 3 (or higher) dimensional Euclidean (or Minkowski) manifold \cite{9, 10, 11}. There the potential term in the mVN equation is interpreted as mean curvature of the immersed surface (times \( \sqrt{g} \)), and the first integral of the mVN equation (i.e. 1-st member of the hierarchy) is shown \cite{12, 7} to be in agreement with Polyakov’s extrinsic string action \cite{13} (Willmore functional \cite{14}) in Euclidean signature at classical level.

Purpose of this note is to show explicitly that the first integral is also invariant under the deformations associated to the rest of the members of the hierarchy. This confirms the statement suggested in the literature \cite{12} obtained from general argument.
of the soluble system. The derived transformation laws for the potential will also be useful, following similar methods to ordinary (intrinsic) string cases, to pin down the algebraic structure of the infinite symmetries in the extrinsic strings.† (For the current status of the extrinsic strings in connection with QCD, see e.g. [15] and references therein.)

2 The mVN and generalized Weierstrass inducing

We first review generalized Weierstrass inducing and see how the mVN hierarchy is involved there. The Weierstrass representation is the construction of the conformally Euclidean minimal surface (i.e. that with vanishing mean curvature) in Euclidean 3-space. (See also [16].) The generalized Weierstrass inducing considered here is the extension of this construction to non-minimal surfaces. To explain this, here we follow the notations of Kenmotsu [9].

Let \( x_j : \Sigma \to \mathbb{R}^3 \) (\( j = 1, \ldots, 3 \)) be a conformally Euclidean immersion of an oriented 2-surface \( \Sigma \) (coordinatized locally by \( z, \bar{z} \in \mathbb{C} \)) into \( \mathbb{R}^3 \). Then following [9] we have:

\[
\partial \bar{\partial} x_j = \frac{\lambda^2}{4} (h_{11} + h_{22}) e_{3j},
\]

\[
\overline{\partial}^2 x_j = \bar{\partial} \lambda \cdot (e_1 + i e_2)_j + \frac{\lambda^2}{4} (h_{11} - h_{22} + 2i h_{12}) e_{3j},
\]

\[
\partial^2 x_j = \partial \lambda \cdot (e_1 - i e_2)_j + \frac{\lambda^2}{4} (h_{11} - h_{22} - 2i h_{12}) e_{3j}.
\]

Here \( \lambda(>0) \) is related to the conformal factor of the induced metric

\[
ds^2 \equiv \sum_{j=1}^{3} (dx_j)^2 = \lambda^2 dz \, d\bar{z}, \quad \text{or} \quad \lambda^2 = 2 \sum_j \left| \frac{\partial x_j}{\partial z} \right|^2 = 2 \sqrt{|g|},
\]

† As a decade-old subject, there are a huge number of works/contributions to the subject. The references cited in this Letter may not reflect all of them.
and $h_{ij}$ are related to the mean $H$ and Gaussian $K(\sim$ Ricci scalar) curvatures

\begin{align}
H &= \frac{1}{2}(h_{11} + h_{22}) \quad \text{(5)} \\
K &= H^2 - |\phi|^2, \quad \phi \equiv \frac{1}{2}(h_{11} - h_{22}) - i h_{12}, \quad \text{(6)}
\end{align}

respectively. The quantities $e_\alpha (\alpha = 1, \ldots, 3)$ are normalized tangent and normal vectors of the immersed surface $\Sigma$, whose components are defined as follows:

\begin{align}
e_{1j} &= \frac{1}{\lambda} \left( \frac{\partial x_j}{\partial z} + \frac{\partial x_j}{\partial \bar{z}} \right), \quad e_{2j} = i \frac{1}{\lambda} \left( \frac{\partial x_j}{\partial z} - \frac{\partial x_j}{\partial \bar{z}} \right), \quad e_{3i} = \epsilon_{ijk} e_{1j} e_{2k}. \quad \text{(7)}
\end{align}

Next we introduce $\psi_1, \psi_2$ [8]

\begin{align}
\psi_1 &\equiv \left[ \bar{\partial}(x_2 + i x_1) \right]^{1/2}, \quad \text{(8)} \\
\psi_2 &\equiv \left[ -\partial(x_2 + i x_1) \right]^{1/2}. \quad \text{(9)}
\end{align}

Then by direct calculation, using above formulas, we find that $\psi_i$’s have to satisfy

\begin{align}
\partial \psi_1 &= \frac{\lambda H}{2} \psi_2 \quad \text{(10)} \\
\bar{\partial} \psi_2 &= -\frac{\lambda H}{2} \psi_1 \quad \text{(11)}
\end{align}

and

\begin{align}
\bar{\partial}(\psi_1/\lambda) &= \frac{\bar{\partial}}{2} \psi_2 \quad \text{(12)} \\
\partial(\psi_2/\lambda) &= -\frac{\partial}{2} \psi_1. \quad \text{(13)}
\end{align}

It is also easy to show

\begin{equation}
\lambda = |\psi_1|^2 + |\psi_2|^2. \quad \text{(14)}
\end{equation}

It is remarkable that eqs.(10),(11) guarantee the integrability of forms $\Omega_{\pm}, \Omega_3$ defined by

\begin{align}
\Omega_+ &= \psi_1^2 dz - \psi_2^2 d\bar{z}, \quad \Omega_- = \overline{(\Omega_+)} , \quad \Omega_3 = -(\psi_2 \overline{\psi_1} dz + \psi_1 \overline{\psi_2} d\bar{z}), \quad \text{(15)}
\end{align}

namely, we have $d\Omega_\bullet = 0$. 

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Now we can consider a converse problem, given a solution to eqs. (10), (11) as a system of differential equations with respect to $\psi_i$’s. Since forms $\Omega_\bullet$ are all integrable, comparing with (8), (9), we observe that eqs. (10), (11) induces an immersion $X_j(z, \bar{z})$ ($j = 1, \ldots, 3$) of conformally Euclidean 2d surface in $\mathbb{R}^3$ via relations

$$X_2 = i X_1 = \int_\Gamma \Omega_\pm, \quad X_3 = \int_\Gamma \Omega_3.$$  

Here $\Gamma$ is an appropriate integration contour ending at $(z, \bar{z})$. The last relation comes from imposed conformally Euclidean property $g_{zz} = (\partial X_3)^2 + (\partial X_1)^2 + (\partial X_2)^2 = 0$, and $g_{z\bar{z}} = 0$. The original Weierstrass inducing corresponds to the special case $H = 0$, i.e. induction for minimal surfaces.

An important observation made in the recent literature [6, 7, 8, 17] is the relevance of eqs. (10), (11) to the 2(+1) dimensional exactly soluble mVN system. The mVN hierarchy is defined as a deformation problem associated to 2-dimensional Dirac operator (times some $\gamma$-matrix) $\mathcal{L}$ with a potential $p = p(z, \bar{z})$:

$$\mathcal{L} = \begin{pmatrix} \partial & -p \\ p & \bar{\partial} \end{pmatrix}, \quad \mathcal{L} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0.$$  

We note that eqs. (10), (11) correspond to taking special potential $p = \lambda H/2$. The $n$-th deformation in the hierarchy is defined via

$$\frac{\delta}{\delta t_n} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = A_n \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$  

where the deformation operator $A_n$ takes the form

$$A_n = \partial^{2n+1} + \sum_{i=0}^{2n-1} X^{(i)} \partial^i + \mathbf{\bar{\partial}}^{2n+1} + \sum_{i=0}^{2n-1} \mathbf{\bar{X}}^{(i)} \mathbf{\bar{\partial}}^i.$$  

\[\dag\text{We would rather use following form (17) of the operator for computational simplicity. We could have used ordinary form for the Dirac operator. Only $B_n$ in equation (20) get changed under such redefinition. In any event essential point is unaffected.}\]
with $X^{(i)}$, $\tilde{X}^{(j)}$ ($i, j = 0, \ldots, 2n - 1$) being $2 \times 2$ matrices. These matrices are completely determined, together with the other matrix-valued differential operator $B_n$ in eq. (20), from the compatibility condition:

$$\left[ \frac{\delta}{\delta t_n} - A_n, L \right] = B_n L.$$  \hspace{1cm} (20)

The operator $B_n$ has a similar expression as in the case $A_n$:

$$B_n = \sum_{i=0}^{2n-1} S^{(i)} \partial^i + \sum_{i=0}^{2n-1} \tilde{S}^{(i)} \partial^i$$  \hspace{1cm} (21)

with $2 \times 2$ matrices $S^{(i)}$, $\tilde{S}^{(j)}$ ($i, j = 0, \ldots, 2n - 1$). The compatibility condition (21) also gives a deformation equation for the potential $p$ in the form

$$\frac{\delta p}{\delta t_n} = \partial^{2n+1} p + \tilde{\partial}^{2n+1} p + \cdots.$$  

The first case $n = 1$ is known to yield modified Veselov-Novikov equation. Here we just write down the result. We will provide more technical details for higher mVN case in later sections.

$$\frac{\delta p}{\delta t_1} = \partial^3 p + 3\omega \partial p + \frac{3}{2} p \partial \omega + \tilde{\partial}^3 p + 3\bar{\omega} \tilde{\partial} p + \frac{3}{2} \bar{p} \tilde{\partial} \bar{\omega}, \ \ \bar{\omega} \equiv \partial p^2.$$  \hspace{1cm} (22)

It is remarkable that we have a simple first integral of this deformation, which is obtained from the relation

$$\frac{\delta p^2}{\delta t_1} = \partial \left( \partial^2 p^2 - 3(\partial p)^2 + 3p^2 \omega \right) + \tilde{\partial} \left( \tilde{\partial}^2 p^2 - 3(\tilde{\partial} p)^2 + 3\bar{p}^2 \bar{\omega} \right).$$  \hspace{1cm} (23)

Namely, the integral $S$

$$S = 2 \int p^2 dz d\bar{z}$$  \hspace{1cm} (24)

does not change its value under the first deformation $\delta/\delta t_1$ (if $p$ is localized). This conserved quantity has special meaning in the generalized Weierstrass inducing discussed before. Substituting $p = \lambda H/2$, we find that $S$ is nothing but Polyakov’s extrinsic string action

$$S = \int \sqrt{|g|} \ H^2 \ d^2 x.$$
(That is known as a Willmore functional in the mathematics literature.)

To sum, Polyakov’s extrinsic string action is invariant under the deformation associated to the (1st) mVN equation.

3 Second mVN deformation

We write deformation operators for the 2nd mVN as
\[
\frac{\delta}{\delta t_2} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = A_2 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad A_2 = A_2^{(+)} + A_2^{(-)},
\]
(25)

where \(2 \times 2\) matrix-valued operators \(A_2^{(\pm)}\) are defined as
\[
A_2^{(+)} = \partial^5 + V\partial^3 + W\partial^2 + X\partial + Z, \quad A_2^{(-)} = \bar{\partial}^5 + \tilde{V}\bar{\partial}^3 + \tilde{W}\bar{\partial}^2 + \bar{X}\bar{\partial} + \bar{Z}.
\]
(26)

Then all we have to do is to work out the compatibility condition (20) for \(B_2 = B_2^{(+)} + B_2^{(-)}\), here represented as
\[
B_2^{(+)} = Q\partial^3 + R\partial^2 + S\partial + T, \quad B_2^{(-)} = \tilde{Q}\bar{\partial}^3 + \tilde{R}\bar{\partial}^2 + \tilde{S}\bar{\partial} + \tilde{T}.
\]

We will perform this independently on (+) and (−) parts of the compatibility conditions
\[
\left[ \frac{\delta}{\delta t_2^{(\pm)}} - A_2^{(\pm)}, \mathcal{L} \right] = B_2^{(\pm)} \mathcal{L}, \quad \frac{\delta}{\delta t_2} = \frac{\delta}{\delta t_2^{(+)}} + \frac{\delta}{\delta t_2^{(-)}}.
\]
(27)

Some cares must be taken with regards to matrix components \(V_{11}, W_{11}, X_{11}, \) and \(\tilde{V}_{22}, \tilde{W}_{22}, \tilde{X}_{22}\). For instance, \(X_{11}\) and \(\tilde{X}_{22}\) give rise to a term in the deformation
\[
\frac{\delta}{\delta t_2} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \sim \begin{pmatrix} X_{11}(\partial \psi_1 - p\psi_2) \\ \tilde{X}_{22}(\bar{\partial} \psi_2 + p\psi_1) \end{pmatrix},
\]
(28)

which vanishes on-shell, so we set \(X_{11} = \tilde{X}_{22} = 0\) by hand. Similarly we set \(V_{11} = W_{11} = \tilde{V}_{22} = \tilde{W}_{22} = 0\). In the course of calculation we also get \(Z_{11} = \tilde{Z}_{11} = \text{const}\), and \(Z_{22} = \tilde{Z}_{22} + \cdots\). This type of deformation
\[
\frac{\delta}{\delta t_2} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \sim \tilde{Z}_{11} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]

\(7\)
is merely an overall constant scalar transformation, so we set $\tilde{Z}_{11} = 0$ as well. Same is true for $\tilde{Z}_{22}$.

After all these done, the deformation matrices are uniquely determined. For the operator $A_2^{(+)}$ we obtain

$$V = \begin{pmatrix} 0 & -5\partial p \\ 0 & 5\omega \end{pmatrix}, \quad W = \begin{pmatrix} 0 & -5\partial^2 p + 5p\omega \\ 0 & 15\partial\omega/2 \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & \frac{5}{2}(p\partial\omega - 2\omega\partial p - 2\partial^2 p) \\ 0 & \frac{5}{2}(2\omega^2 + 3\partial^2\omega + 2\zeta) \end{pmatrix}, \quad \overline{\partial\zeta} \equiv \partial(p^2\omega - (\partial p)^2),$$

$$Z = \begin{pmatrix} 0 & 5(p(\omega^2 + \zeta + \partial^2\omega) + \omega\partial^2 p + \frac{1}{2}\partial p \partial\omega) \\ 0 & \frac{5}{2}\partial(\omega^2 + \zeta + \partial^2\omega) \end{pmatrix}. \quad (29)$$

For the operator $B_2^{(+)}$ the result is

$$Q = \begin{pmatrix} 0 & -5\partial p \\ 5\partial p & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & ((W_{12})) \\ 10\partial^2 p + 5p\omega & 0 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & ((X_{12})) \\ \frac{5}{2}(3p\partial\omega + 6\omega\partial p + 4\partial^2 p) & 0 \end{pmatrix},$$

$$T = \begin{pmatrix} 0 & ((Z_{12})) \\ \frac{5}{2}p(2\omega^2 + 2\zeta + 3\partial^2\omega) + 15\omega\partial^2 p + 15\partial p \partial\omega + 5\partial^4 p & 0 \end{pmatrix}. \quad (30)$$

The ‘12*-components should be taken from those in eq.(29). The results for operators $A_2^{(-)}$ and $B_2^{(-)}$ are given by the general rule

$$A_2^{(-)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{A_2^{(+)}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_2^{(-)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{B_2^{(+)}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\overline{A_2^{(+)}}$ and $\overline{B_2^{(+)}}$ imply taking complex conjugate of respective components of the matrix operators.
Finally the deformation equation for the potential \( p \) is obtained as

\[
\frac{\delta p}{\delta t^\pm_2} = \partial^4p + 5\omega \partial^3p + \frac{15}{2} \partial \omega \partial^2p + \frac{5}{2} \partial p (2 \omega^2 + 3 \partial^2 \omega + 2 \zeta) + \frac{5}{2} \partial (\omega^2 + \zeta + \partial^2 \omega),
\]

and \( \frac{\delta p/\delta t^2_-}{\delta t^2_-} = \frac{\delta p/\delta t^2_+}{\delta t^2_+} \), respectively.

At this stage we can check explicit invariance of our first integral (24). From our deformation equation (32) we immediately have

\[
\frac{\delta p^2}{\delta t^2_+} = \partial \left[ \partial^4p^2 - 5\partial^2(\partial p)^2 + 5(\partial^2 p)^2 + 5\omega \partial^2 p^2 - 15\omega(\partial p)^2 + \frac{5}{2} \partial \omega \partial p^2 + 5p^2 (\omega^2 + \zeta + \partial^2 \omega) \right],
\]

and similarly for \( \frac{\delta p^2}{\delta t^2_-} = \frac{\delta p^2}{\delta t^2_+} \). Thus we find that for localized \( p \) the integral (24) remains invariant under deformations associated to our 2nd member of mVN hierarchy.

Actually, if the purpose is only to show \( \frac{\delta p^2/\delta t^+_2}{\delta t^+_2} = (\text{total divergence}) \), the following is more straightforward. We just inspect ‘21’- and ‘12’-components of every terms from the compatibility conditions (27), which provide us

\[
\begin{align*}
5\partial p + V_{12} &= 0 \\
10\partial^2 p + \partial V_{12} - p V_{22} + W_{12} &= 0 \\
10\partial^3 p + \partial W_{12} - p W_{22} + X_{12} &= 0 \\
5\partial^4 p + \partial X_{12} - p X_{22} + Z_{12} &= 0 \\
2\partial^5 p + \partial Z_{12} + V_{22} \partial^3 p + W_{22} \partial^2 p + X_{22} \partial p &= 2\delta p/\delta t^+_2.
\end{align*}
\]

Eventually we end up with a simpler expression

\[
\frac{\delta p^2}{\delta t^2_+} = \partial \left( \partial^4p^2 + V_{12} \partial^3p + W_{12} \partial^2p + X_{12} \partial p + Z_{12}p \right).
\]

This agrees with the previous result (33) after substitutions of the matrix-components from eqs.(29). The argument for the other deformation \( \frac{\delta p^2/\delta t^-_2}{\delta t^-_2} \) goes completely in parallel.
From these calculations we can naturally infer that the similar structure persists in higher mVN deformations, and we can state quite safely that we have generally

\[
\frac{\delta p^2}{\delta t_n} = \partial \left( \partial^{2n} p^2 + \sum_{i=0}^{2n-1} X^{(i)}_{12} \partial^i p \right) + \overline{\partial} \left( \overline{\partial}^{2n} p^2 + \sum_{i=0}^{2n-1} \overline{X}^{(i)}_{12} \overline{\partial}^i p \right)
\]

(36)
in our original notation (19). On the physics side this indicates the invariance of the extrinsic action à la Polyakov under all deformations associated to mVN hierarchy.

### 4 Discussions

We have derived 2nd member of the mVN hierarchy, and checked its consistency to the first integral (one of the so-called Kruskal integrals) derived from the 1st member of the hierarchy. Even though the result is not unexpected from the general argument of exactly soluble models of this sort, writing down a correct form of explicit deformation equation is very important in various respects. First of all, we would like to know the complete symmetry structure of the Polyakov’s extrinsic string. In order to pin down such an infinite symmetry we have to be aware of the algebraic structure of the Poisson algebra associated to this system. In the case of KdV, KP we have successfully identified their Poisson structures [18] to be \(W_\infty\)-algebra family [3]. The latter has played an important role in other string analyses. Higher Kruskal integrals are also important there. The derived deformation equations here are crucial to deduce such Poisson structures.

Secondly, though the immediate relevance found is to Polyakov’s extrinsic string, we have pointed out several years ago [3] the importance of 2- (or higher) dimensional exactly soluble system (that reduces to KdV in one dimension) in its application to 4-dimensional self-dual gravity. The (m)VN is one of the simplest extension of (m)KdV, other than KP. Clarifying its symmetry structure through associated Poisson algebra is an important step towards that goal as well. We would like to know the possible relevance of mVN to that problem, before proceeding to more complex...
Davey-Stewartson hierarchy. Though calculations become more involved in higher dimensional exactly soluble system, we wish to report on our analyses of these issues in future publications.

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