FOURIER-LIKE MULTIPLIERS AND APPLICATIONS FOR INTEGRAL OPERATORS

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Abstract. Timelimited functions and bandlimited functions play a fundamental role in signal and image processing. But by the uncertainty principles, a signal cannot be simultaneously time and bandlimited. A natural assumption is thus that a signal is almost time and almost bandlimited. The aim of this paper is to prove that the set of almost time and almost bandlimited signals is not excluded from the uncertainty principles. The transforms under consideration are integral operators with bounded kernels for which there is a Parseval Theorem. Then we define the wavelet multipliers for this class of operators, and study their boundedness and Schatten class properties. We show that the wavelet multiplier is unitary equivalent to a scalar multiple of the phase space restriction operator. Moreover we prove that a signal which is almost time and almost bandlimited can be approximated by its projection on the span of the first eigenfunctions of the phase space restriction operator, corresponding to the largest eigenvalues which are close to one.

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1. Introduction

Timelimited functions and bandlimited functions are basic tools of signal and image processing. Unfortunately, the simplest form of the uncertainty principle tells us that a
signal cannot be simultaneously time and bandlimited. This leads to the investigation of the set of almost time and almost bandlimited functions, which has been initially carried through Landau, Pollak \[17, 18\] and then by Donoho, Stark \[9\]. In the current paper, we review the uncertainty principles on this set and present and compare different measure of localization. We made use of compositions of time and bandlimiting operators and considered the eigenvalue problem associated with these operators. The resulting operators yield an orthonormal set of eigenfunctions (well-known as prolate spheroidal functions) which satisfy some optimality in concentration in a region in the time-frequency domain. We prove a characterization of functions that are approximately time and bandlimited in the region of interest, and we obtain approximation inequalities for such functions using a finite linear combination of eigenfunctions.

The aim of this paper is to continue the study of the uncertainty principle to a very general class of integral operators, which has been started in \[11, 12\]. The transforms under consideration are integral operators \(T\) with bounded kernels \(K\) and for which there is a Parseval Theorem. This class includes the usual Fourier transform, the Fourier-Bessel (or Hankel) transform, the Dunkl transform and the deformed Fourier transform as particular cases. A version of Hardy’s and Donoho-Stark’s uncertainty principles for integral operators has been proved in \[7, 8\]. In this paper, we consider results of a different nature on the subspaces of functions that are essentially timelimited on \(S\) and bandlimited on \(\Sigma\), or functions that are essentially concentrated on \(S\) and bandlimited on \(\Sigma\), where \(S\) and \(\Sigma\) are general subsets of finite measure.

Let us now be more precise. Let \(\Omega\) be a convex cones in \(\mathbb{R}^d\) (i.e. \(\lambda x \in \Omega\) if \(\lambda > 0\) and \(x \in \Omega\)) with non-empty interior, endowed with the Borel measure \(\mu\). The Lebesgue spaces \(L^p(\Omega, \mu)\), \(1 \leq p \leq \infty\), are then defined in the usual way, where \(\| \cdot \|_{\infty}\) is the usual essential supremum norm and form \(1 \leq p < \infty\),

\[
\|f\|_{p,\mu}^p = \int_{\Omega} |f(x)|^p \, d\mu(x).
\]

We assume that the measure \(\mu\) is absolutely continuous with respect to the Lebesgue measure and has a polar decomposition of the form \(d\mu(r\zeta) = r^{2\alpha - 1} \, d\sigma(Q(\zeta)) \, d\sigma(\zeta)\) where \(d\sigma\) is the Lebesgue measure on the unit sphere \(S^{d-1}\) of \(\mathbb{R}^d\) and \(Q \in L^1(S^{d-1}, d\sigma), Q \neq 0\). Then \(\mu\) is homogeneous of degree \(2\alpha\) in the following sense: for every continuous function \(f\) with compact support in \(\Omega\) and every \(\lambda > 0\),

\[
\int_{\Omega} f\left(\frac{x}{\lambda}\right) \, d\mu(x) = \lambda^{2\alpha} \int_{\Omega} f(x) \, d\mu(x). \tag{1.1}
\]

One can then define the integral operator \(T\) on \(S(\Omega)\) by

\[
Tf(\xi) = \int_{\Omega} f(x)K(x, \xi) \, d\mu(x), \quad \xi \in \Omega,
\]

where \(K : \Omega \times \Omega \rightarrow \mathbb{C}\) is a kernel such that:

(1) \(K\) is continuous,

(2) \(K\) is bounded: \(|K(x, \xi)| \leq c_K\),

(3) \(K\) is homogeneous: \(K(\lambda x, \xi) = K(x, \lambda \xi)\).

Then \(T\) extends into a continuous operator from \(L^1(\Omega, \mu)\) to the space of bounded continuous functions \(C(\Omega)\), with

\[
\|Tf\|_{\infty} \leq c_K \|f\|_{1,\mu}. \tag{1.3}
\]
Further, if we introduce the dilation operator $D_\lambda$, $\lambda > 0$ by:

$$D_\lambda f(x) = \frac{1}{\lambda^s} f\left(\frac{x}{\lambda}\right).$$

Then the homogeneity of $\mathcal{K}$ implies

$$T D_\lambda = D_{\lambda^s} T. \quad (1.4)$$

The integral operators under consideration will be assumed to satisfy some of the following properties that are common for Fourier-like transforms:

(1) $\mathcal{T}$ has an Inversion Formula: When both $f \in L^1(\Omega, \mu)$ and $\mathcal{T} f \in L^1(\Omega, \mu)$ we have $f \in \mathcal{C}(\Omega)$ and for almost every $x \in \Omega$:

$$f(x) = \int_{\Omega} \mathcal{T} f(\xi) \overline{\mathcal{K}(x, \xi)} \, d\mu(\xi). \quad (1.5)$$

(2) $\mathcal{T}$ satisfies Parseval’s Theorem: for every $f, g \in \mathcal{S}(\Omega)$,

$$\langle \mathcal{T} f, \mathcal{T} g \rangle_\mu = \langle f, g \rangle_\mu, \quad (1.6)$$

where $\langle \cdot, \cdot \rangle_\mu$ is the inner product defined on the Hilbert spaces $L^2(\Omega, \mu)$ by

$$\langle f, g \rangle_\mu = \int_{\Omega} f(x) \overline{g(x)} \, d\mu(x).$$

In particular, $\mathcal{T}$ extends to an unitary transform from $L^2(\Omega, \mu)$ onto $L^2(\Omega, \mu)$, such that

$$\|\mathcal{T} f\|_{2, \mu} = \|f\|_{2, \mu}, \quad (1.7)$$

and for all $f \in L^2(\Omega, \mu)$,

$$\mathcal{T}^{-1} f(\xi) = \overline{\mathcal{T} f(\xi)}, \quad \xi \in \Omega. \quad (1.8)$$

This family of transforms includes for instance the Fourier transform (see e.g. [20]), the Hankel transform (see e.g. [13]), the the Dunkl transform (see e.g. [14]), the $G$-transform (see e.g. [29]), the deformed Fourier transform (see e.g. [5]), etc.

The inversion formula gives us back a signal $f$ via (1.5), and this is the basis for pseudo-differential operators on $\Omega$. Indeed if $\sigma$ be a suitable function on $\Omega$, then we define the pseudo-differential operator $F_\sigma$ by

$$F_\sigma f(x) = \int_{\Omega} \sigma(\xi) \mathcal{T} f(\xi) \overline{\mathcal{K}(x, \xi)} \, d\mu(\xi).$$

Pseudo-differential operators $F_\sigma$ are known as the multiplier, and in the case where $\sigma = \chi_A$ is a characteristic function, the operator $F_\sigma$ is known as the frequency limiting operator on $\Omega$, we simply denote it by $F_A$. Now if $\sigma$ is identically equal to 1, then $F_\sigma : L^2(\Omega, \mu) \to L^2(\Omega, \mu)$ is the identity operator in view of (1.8).

Our starting point is the following general form of Heisenberg-type uncertainty inequality (see [11, 12]).

**Theorem 1.1.** Let $s, \beta > 0$. Then

1. there exists a constant $C = C(s, a, \beta)$ such that for all $f \in L^2(\Omega, \mu)$,

$$\|\|x|^s f\|_{2, \mu}^{s+\beta} \|\xi|^\beta \mathcal{T} f\|_{2, \mu}^{s+\beta} \geq C \|f\|_{2, \mu}^{s+\beta}, \quad (1.9)$$

2. there exists a constant $C = C(s, a, \beta)$ such that for all $f \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$,

$$\|\|x|^s f\|_{1, \mu}^{a+s} \|\xi|^\beta \mathcal{T} f\|_{2, \mu}^{a+s} \geq C \|f\|_{1, \mu}^{a+s} \|f\|_{2, \mu}^{a+s}. \quad (1.10)$$
The proof of Inequality (1.10) can be obtained by combining a Nash-type inequality [12, Proposition 2.2] and a Carlson-type inequality [12, Proposition 2.3], while the proof of Inequality (1.9) can be obtained from either the Faris-type local uncertainty inequalities [11, Theorem A], or from the fact that the Benedicks-Amrein-Berthier uncertainty inequality [11, Theorem B].

Theorem 1.1 can be refined for orthonormal sequence in $L^2(\Omega, \mu)$. In particular an orthonormal sequence in $L^2(\Omega, \mu)$ cannot have uniform time-frequency localization. This is a consequence of the following Shapiro-type uncertainty principles.

**Theorem A.** Let $s > 0$ and let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal sequence in $L^2(\Omega, \mu)$.

1. There exists a positive constant $C$ such that, for every $N \geq 1$,
\[
\sum_{n=1}^{N} \left( \|x|^s f_n\|_{2, \mu}^2 + \|\xi|^s \mathcal{T} f_n\|_{2, \mu}^2 \right) \geq C N^{1+\frac{s}{2}}. \tag{1.11}
\]

2. If $\{f_n\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(\Omega, \mu)$, then
\[
\sup_n \left( \|x|^s f_n\|_{2, \mu} \|\xi|^s \mathcal{T} f_n\|_{2, \mu} \right) = \infty. \tag{1.12}
\]

Notice that the homogeneity of the kernel $K$ plays a key role only in the proof of Inequality (1.12). Moreover the proof of Theorem A is inspired from the classical result for the Fourier transform in [20], where the author prove that Inequality (1.11) is sharp and the equality cases are attained for the sequence of Hermite functions, (see also [19]). For the Hankel transform [13] this inequality is also optimal and the optimizers are the sequence of Laguerre functions.

One would like to find nonzero functions $f \in L^2(\Omega, \mu)$, which are timelimited on a subset $S \subset \Omega$ (i.e. supp$f \subset S$) and bandlimited on a subset $\Sigma \subset \Omega$ (i.e. supp$\mathcal{T} f \subset \Sigma$). Unfortunately, such functions do not exist, because if $f$ is time and bandlimited on subsets of finite measure, then $f = 0$ (see [11]). As a result, it is natural to replaced the exact support by the essential support, and to focus on functions that are essentially time and bandlimited to a bounded region like $S \times \Sigma$ in the time-frequency plane. To do this, we introduce the time limiting operator
\[
E_S f = \chi_S f, \quad f \in L^1(\Omega, \mu) \cup L^2(\Omega, \mu),
\]
and from [1, 9, 12] we recall the following definition.

**Definition 1.2.** Let $0 \leq \varepsilon < 1$ and let $S, \Sigma \subset \Omega$. Then

1. a nonzero function $f \in L^2(\Omega, \mu)$ is $\varepsilon$-concentrated on $S$ if $\|E_{S^c} f\|_{2, \mu} \leq \varepsilon \|f\|_{2, \mu}$,
2. a nonzero function $f \in L^1(\Omega, \mu)$ is $\varepsilon$-timelimited on $S$ if $\|E_{S^c} f\|_{1, \mu} \leq \varepsilon \|f\|_{1, \mu}$,
3. a nonzero function $f \in L^2(\Omega, \mu)$ is $\varepsilon$-bandlimited on $\Sigma$ if $\|F_{\Sigma^c} f\|_{2, \mu} \leq \varepsilon \|f\|_{2, \mu}$,
4. a nonzero function $f \in L^2(\Omega, \mu)$ is $\varepsilon$-localized with respect to an operator $L : L^2(\Omega, \mu) \to L^2(\Omega, \mu)$ if $\|Lf - f\|_{2, \mu} \leq \varepsilon \|f\|_{2, \mu}$.

Here $A^c = \Omega \setminus A$ is the complement of $A$ in $\Omega$. It is clear that, if $f$ is $\varepsilon$-bandlimited on $\Sigma$ then by Inequality (1.7), $\mathcal{T} f$ is $\varepsilon$-concentrated on $\Sigma$. Notice also that, the $\varepsilon$-concentration measure was introduced in [17, 18, 9], and the idea of $\varepsilon$-localization has been recently introduced in [1], which arises from the concept of pseudospectra of linear operators.

If $\varepsilon = 0$ in the $\varepsilon$-concentration measures, then $S$ and $\Sigma$ are respectively the exact support of $f$ and $\mathcal{T} f$, moreover when $\varepsilon \in (0, 1)$, $S$ and $\Sigma$ may be considered as the essential support of $f$ and $\mathcal{T} f$ respectively. On the other hand, a function $f \in L^2(\Omega, \mu)$
is $\varepsilon$-localized with respect to an operator $L$ is an eigenfunction of $L$ corresponding to the eigenvalue 1, if $\varepsilon = 0$, otherwise $f$ is called an $\varepsilon$-approximated eigenfunction (or $\varepsilon$-pseudoeigenfunction) of $L$ with pseudoeigenvalue 1, see [17, 24]. For example, since on $L^2(\Omega, \mu)$, we have $F_{\Sigma} + F_{\Sigma^c} = E_{\Sigma} + E_{\Sigma^c} = I$, then a nonzero function $f \in L^2(\Omega, \mu)$ is $\varepsilon$-concentrated on $S$ (resp. $\varepsilon$-bandlimited on $\Sigma$), if and only if, $f$ is $\varepsilon$-localized with respect to $E_{\Sigma}$ (resp. $\varepsilon$-localized with respect to $F_{\Sigma}$).

Let $\varepsilon_1, \varepsilon_2 \in (0, 1)$ and let $S, \Sigma$ two measurable subsets of $\Omega$ such that $0 < \mu(S), \mu(\Sigma) < \infty$. We denote by $L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$ the subspace of $L^2(\Omega, \mu)$ consisting of functions that are $\varepsilon_1$-concentrated on $S$ and $\varepsilon_2$-bandlimited on $\Sigma$ (clearly $L^2(0, 0, S, \Sigma) = \emptyset$). We denote also by $L^1 \cap L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$ the subspace of $L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$ consisting of functions that are $\varepsilon_1$-timelimited on $S$ and $\varepsilon_2$-bandlimited on $\Sigma$.

As a first result, we can remark that the essential supports $S$ and $\Sigma$ cannot be too small, and this is a simple consequence of the following Donoho-Stark type uncertainty principle (see [8, 11, 12]):

1. If $f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$ such that $\varepsilon_1^2 + \varepsilon_2^2 < 1$, then
   \[
   \mu(S)\mu(\Sigma) \geq C^{-2}\left(1 - \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right)^2.
   \] (1.13)

2. If $f \in L^1 \cap L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$, then
   \[
   \mu(S)\mu(\Sigma) \geq C^{-2}(1 - \varepsilon_1)^2(1 - \varepsilon_2^2).
   \] (1.14)

The second Inequality (1.14) is stronger than (1.13), since it is true for all $\varepsilon_1, \varepsilon_2 \in (0, 1)$, and since in (1.14) we can give separately a lower bound for $\mu(S)$ and $\mu(\Sigma)$, (see Inequality (3.22)).

It is natural to ask if there is a Heisenberg-type uncertainty inequalities for functions in the subspaces $L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$ and $L^1 \cap L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$, with constant which depends on $\varepsilon_1, \varepsilon_2, S, \Sigma$. In Section 3, we use the local uncertainty principles for functions either in $L^2(\Omega, \mu)$ or in $L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$ to obtain an uncertainty inequalities comparing the support and the essential support with the time dispersion or the frequency dispersion.

**Theorem B.** Let $\varepsilon_1, \varepsilon_2 \in [0, 1]$.

1. If $s, \beta > a$, then
   [(a)] there exists a constant $C$ such that for all function $f \in L^2(\Omega, \mu)$ which is $\varepsilon_1$-concentrated on $S$,
   \[
   \mu(S)^{\frac{\beta}{2}} \| |\!| |\!|^{\beta} \mathcal{T} f \|_{2, \mu}^2 \geq C (1 - \varepsilon_1^2)^{\frac{\beta}{2}} \| f \|_{2, \mu}^2.
   \] (1.15)
   [(b)] there exists a constant $C$ such that for all function $f \in L^2(\Omega, \mu)$ which is $\varepsilon_2$-bandlimited on $\Sigma$,
   \[
   \mu(\Sigma)^{\frac{\beta}{2}} \| |\!| |\!|^{\beta} \mathcal{T} f \|_{2, \mu}^2 \geq C (1 - \varepsilon_2^2)^{\frac{\beta}{2}} \| f \|_{2, \mu}^2.
   \] (1.16)

2. If $s, \beta > 0$, then
   [(a)] there exists a constant $C$ such that for all function $f \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$, which is $\varepsilon_1$-timelimited on $S$,
   \[
   \mu(S)^{\frac{\alpha + \beta}{2\alpha}} \| |\!| |\!|^{\beta} \mathcal{T} f \|_{2, \mu} \geq C (1 - \varepsilon_1)^{\frac{\alpha + \beta}{2\alpha}} \| f \|_{1, \mu}.
   \] (1.17)
   [(b)] there exists a constant $C$ such that for all function $f \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$, which is $\varepsilon_2$-bandlimited on $\Sigma$,
   \[
   \mu(\Sigma)^{\frac{\alpha + \beta}{2\alpha}} \| |\!| |\!|^{\beta} \mathcal{T} f \|_{1, \mu} \geq C (1 - \varepsilon_2^2)^{\frac{\alpha + \beta}{2\alpha}} \| f \|_{2, \mu}.
   \] (1.18)
Of course, if \( \varepsilon_1 = \varepsilon_2 = 0 \), then \( S = \text{supp} f \) and \( \Sigma = \text{supp} \mathcal{T} f \). Combining the inequalities in Theorem A, we obtain the following Heisenberg-type uncertainty principles, which can be viewed as the \( \varepsilon \)-concentration version of Theorem 1.1:

1. If \( s, \beta > a \), then there exists a constant \( C \) such that for all \( f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma) \),

\[
\| x^s f \|_{2, \mu}^{\beta} \| \xi^\beta \mathcal{T} f \|_{2, \mu}^{\varepsilon} \geq C \left( \frac{1 - \varepsilon_2^2}{\mu(S) \mu(\Sigma)} \right)^{\frac{2d}{2n}} \| f \|_{2, \mu}^{s + \beta}. \tag{1.19}
\]

2. If \( s, \beta > 0 \), then there exists a constant \( C \) such that for all \( f \in L^1 \cap L^2(\varepsilon_1, \varepsilon_2, S, \Sigma) \),

\[
\| x^s f \|_{1, \mu}^{\alpha + \beta} \| \xi^\beta \mathcal{T} f \|_{2, \mu}^{\varepsilon} \geq C \left( \frac{1 - \varepsilon_2^2}{\mu(S) \mu(\Sigma)} \right)^{\frac{(a + s)(\alpha + \beta)}{2n}} \| f \|_{1, \mu}^{\alpha + s} \| f \|_{2, \mu}^{\varepsilon + \beta}. \tag{1.20}
\]

Notice that Inequalities (1.15), (1.16) and (1.19) hold also for \( 0 < s, \beta \leq a \), but not necessarily with the same constants. Notice also that results in Theorem B are stronger than Inequalities (1.19) and (3.35), since in Theorem B we have a lower bounds for the measures of the time and frequency dispersions separately, this give more information than a lower bound of the product between them.

Now let \( \phi \) and \( \psi \) two bounded functions in \( L^2(\Omega, \mu) \) such that \( \| \phi \|_{2, \mu} = \| \psi \|_{2, \mu} \) and \( \| \phi \|_{\infty} \| \psi \|_{\infty} = 1 \). The first aim of Section 4 is to make precise the definition of the pseudo-differential operator (known as the wavelet multiplier) \( \psi \mathcal{F}_\sigma \phi : L^2(\Omega, \mu) \to L^2(\Omega, \mu) \), where \( \sigma \) is a symbol in \( L^p(\Omega, \mu), 1 \leq p \leq \infty \), and to prove that the resulting bounded linear operator is in the Schatten-von Neumann class \( S_p \). On the other hand, we use the \( \varepsilon \)-localization measure introduced in [1] to state a new uncertainty inequality involving the wavelet multiplier. More precisely we establish the following results.

**Theorem C.** Let \( \sigma \in L^p(\Omega, \mu), 1 \leq p \leq \infty \), and let \( \varepsilon_1, \varepsilon_2 \in (0, 1) \) such that \( \varepsilon_1 + \varepsilon_2 < 1 \).

1. The wavelet multiplier \( \psi \mathcal{F}_\sigma \phi : L^2(\Omega, \mu) \to L^2(\Omega, \mu) \) is in \( S_p \) and

\[
\| \psi \mathcal{F}_\sigma \phi \|_{S_p} \leq c_{\sigma}^{\frac{2}{p}} \| \sigma \|_{p, \mu}. \tag{1.21}
\]

2. If a nonzero function \( f \in L^2(\Omega, \mu) \) is \( \varepsilon_1 \)-localized with respect to \( (\psi \mathcal{F}_\sigma \phi) \) and \( \varepsilon_2 \)-localized with respect to \( (\psi \mathcal{F}_\sigma \phi) \), then

\[
\mu(S) \mu(\Sigma) \geq c_{\sigma}^4 (1 - \varepsilon_1 - \varepsilon_2). \tag{1.22}
\]

Here we denote by \( S_\infty \) for the space of bounded operators from \( L^2(\Omega, \mu) \) into itself. Notice also that the related results of Theorem B and Theorem C for the Hankel transform has been studied by the author in [15].

In Section 5, we will prove that the wavelet multiplier is unitary equivalent to a scalar multiple of the phase space restriction operator \( L_{S, \Sigma} = E_S F_{\Sigma} S_S \) on \( L^2(\Omega, \mu) \) arising from the Landau-Pollak theory in signal analysis [17, 18]. This leads to a compact self-adjoint operator with spectral representation:

\[
L_{S, \Sigma} f = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n \rangle_{\mu} \varphi_n. \tag{1.23}
\]

In the classical setting, these eigenfunctions are known as the prolate spheroidal wave functions. In particular,

\[
\| L_{S, \Sigma} \varphi_n - \varphi_n \|_{2, \mu} = \| \varphi_n - L_{S, \Sigma} \varphi_n, \varphi_n \rangle_{\mu} = 1 - \lambda_n < 1, \tag{1.24}
\]

then each eigenfunction \( \varphi_n \) is \((1 - \lambda_n)\)-localized with respect to \( L_{S, \Sigma} \), and a simple computation shows also that each function \( f \) in \( L^2(\varepsilon_1, \varepsilon_2, S, \Sigma) \) is \((2\varepsilon_1 + \varepsilon_2)\)-localized.
with respect to \( L_{S,\Sigma} \). Moreover, if a function \( f \in L^2(\Omega, \mu) \) is \( \varepsilon \)-localized with respect to \( L_{S,\Sigma} \), then it satisfies

\[
\langle f - L_{S,\Sigma}f, f \rangle_\mu \leq 2\varepsilon \| f \|_{2,\mu}^2.
\]  

(1.25)

Conversely, if we denote by

\[
L^2(\varepsilon, S, \Sigma) = \left\{ f \in L^2(\Omega, \mu) : \langle f - L_{S,\Sigma}f, f \rangle_\mu \leq \varepsilon \| f \|_{2,\mu}^2 \right\},
\]

(1.26)

then, each function \( f \in L^2(\varepsilon, S, \Sigma) \) is \( \sqrt{\varepsilon} \)-localized with respect to \( L_{S,\Sigma} \), and each \( f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma) \) is in \( L^2(2\varepsilon_1 + \varepsilon_2, S, \Sigma) \) (see Proposition 5.3 for more details).

Now let \( n(\varepsilon, S, \Sigma) = \text{card}\{ n : \lambda_n \geq 1 - \varepsilon \} \) the number of eigenvalues which are close to one. In [18], Landau and Pollak gave an asymptotic estimate for \( n(\varepsilon, S, \Sigma) \), when \( T \) is the Fourier transform and \( S, \Sigma \) are real intervals. This result can be interpreted as follows: there exist, up to a small error, \( \frac{|S||\Sigma|}{2\pi} \) independent functions that are \( \varepsilon_1 \)-concentrated on \( S \) and \( \varepsilon_2 \)-bandlimited on \( \Sigma \), these functions are the so-called prolate spheroidal wave functions. The last estimate has been recently refined in [1], where the authors instead of counting the number of eigenfunctions with eigenvalue close to one, they count the maximum number of orthogonal functions that are \( \varepsilon \)-localized with respect to \( L_{S,\Sigma} \). In [2], Abreu and Pereira noted that the sharp asymptotic number of these orthogonal functions is \( \approx (1 - \varepsilon)^{-1}\frac{|S||\Sigma|}{2\pi} \). On the other hand, we establish the following results, which characterize functions that are in \( L^2(\varepsilon, S, \Sigma) \), and approximate almost time and bandlimited functions.

**Theorem D.** Let \( f_{\ker} \) denote the orthogonal projection of \( f \) onto the kernel of \( L_{S,\Sigma} \).

1. A function \( f \) is in \( L^2(\varepsilon, S, \Sigma) \) if and only if,

\[
\sum_{n=1}^{n(\varepsilon, S, \Sigma)} (\lambda_n + \varepsilon - 1) \left| \langle f, \varphi_n \rangle_\mu \right|^2 \geq (1 - \varepsilon) \| f_{\ker} \|^2_{2,\mu} + \sum_{n=1+n(\varepsilon, S, \Sigma)}^{\infty} (1 - \varepsilon - \lambda_n) \left| \langle f, \varphi_n \rangle_\mu \right|^2.
\]

(1.27)

2. If \( f \) is in \( L^2(\varepsilon, S, \Sigma) \), then

\[
\left\| f - \sum_{n=1}^{n(\varepsilon_0, S, \Sigma)} \langle f, \varphi_n \rangle_\mu \varphi_n \right\|_{2,\mu} \leq \sqrt{\frac{\varepsilon}{\varepsilon_0}} \| f \|_{2,\mu}.
\]

(1.28)

Using the above comparison, it follows that, if \( f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma) \), then

\[
\left\| f - \sum_{n=1}^{n(\varepsilon_0, S, \Sigma)} \langle f, \varphi_n \rangle_\mu \varphi_n \right\|_{2,\mu} \leq \sqrt{\frac{2\varepsilon_1 + \varepsilon_2}{\varepsilon_0}} \| f \|_{2,\mu},
\]

(1.29)

\[ \text{and if } f \in L^2(\Omega, \mu) \text{ is } \varepsilon \text{-localized with respect to } L_{S,\Sigma}, \text{ then} \]

\[
\left\| f - \sum_{n=1}^{n(\varepsilon_0, S, \Sigma)} \langle f, \varphi_n \rangle_\mu \varphi_n \right\|_{2,\mu} \leq \sqrt{\frac{2\varepsilon}{\varepsilon_0}} \| f \|_{2,\mu}.
\]

2. Preliminaries

2.1. **Notation.** Throughout this paper we denote by \( \langle \cdot, \cdot \rangle \) the usual Euclidean inner product in \( \mathbb{R}^d \), we write for \( x \in \mathbb{R}^d \), \( |x| = \sqrt{\langle x, x \rangle} \) and if \( A \) is a measurable subset in \( \mathbb{R}^d \), we will write \( A^c \) for its complement in \( \Omega \).

For \( \xi \in \Omega \), we denote by \( \mathcal{K}_\xi : \Omega \to \Omega \) the kernel defined by \( \mathcal{K}_\xi(x) = \mathcal{K}(x, \xi) \), and for \( r > 0 \), we denote by \( B_r \) the closed ball in \( \Omega \) centred at \( 0 \) and of radius \( r \).
We will write along this paper $C$ for a constant that depends on the parameters $a$, $s$ and $c_K$ defined above (and may be depends also on some other parameter $\beta$, $\varepsilon$, ...). This constant may changes from line to line.

2.2. Generalities. Let $X$ be a separable and complex Hilbert space (of infinite dimension) in which the inner product and the norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let $A : X \to X$ be a compact operator for which we denote by $A^* : X \to X$ its adjoint. Then the linear operator $|A| = \sqrt{A^*A} : X \to X$ is positive and compact. The singular values of $A$ are the eigenvalues of the self-adjoint operator $|A|$. For $1 \leq p < \infty$, the Schatten class $S_p$ is the space of all compact operators whose singular values lie in $\ell_p$. In particular, $S_2$ is the space of Hilbert-Schmidt operators, and $S_1$ is the space of trace class operators. Moreover from [22, Section VI.6] and [26, Proposition 2.6], we have the following criterion for a bounded linear operator to be in the trace class.

**Proposition 2.1.** Let $A : X \to X$ be a bounded linear operator such that, for all orthonormal bases $\{\varphi_n\}_{n=1}^\infty$ for $X$,

$$\sum_{n=1}^\infty |\langle A\varphi_n, \varphi_n \rangle| < \infty,$$

(2.1)

then $A : X \to X$ is in the trace class $S_1$ with,

$$\operatorname{tr}(A) = \sum_{n=1}^\infty \langle A\varphi_n, \varphi_n \rangle,$$

(2.2)

where $\{\varphi_n\}_{n=1}^\infty$ is any orthonormal basis for $X$.

If, in addition $A$ is positive, then (see [26, Proposition 2.7]),

$$\|A\|_{S_1} = \operatorname{tr}(A).$$

(2.3)

Moreover from [26, Proposition 2.8], we have the following criterion for a bounded linear operator $A : X \to X$ to be in the Hilbert-Schmidt class $S_2$.

**Proposition 2.2.** Let $A : X \to X$ be a bounded linear operator such that, for all orthonormal bases $\{\varphi_n\}_{n=1}^\infty$ for $X$,

$$\sum_{n=1}^\infty \|A\varphi_n\|^2 < \infty,$$

(2.4)

then $A : X \to X$ is in the Hilbert-Schmidt class $S_2$ with,

$$\|A\|^2_{S_2} = \sum_{n=1}^\infty \|A\varphi_n\|^2,$$

(2.5)

where $\{\varphi_n\}_{n=1}^\infty$ is any orthonormal basis for $X$.

Finally, if the compact operator $A : X \to X$ is Hilbert-Schmidt, then the positive operator $A^*A$ is in the space of trace class $S_1$ and

$$\|A\|^2_{HS} = \|A\|^2_{S_2} = \|A^*A\|_{S_1} = \operatorname{tr}(A^*A) = \sum_{n=1}^\infty \|A\varphi_n\|^2,$$

(2.6)

for any orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ for $X$. 
For consistency, we define \( S_\infty := B(X) \) to be the space of bounded operators from \( X \) into \( X \), equipped with norm,

\[
\|A\|_{S_\infty} = \sup_{f : \|f\| \leq 1} \|Af\|.
\]

(2.7)

It is obvious that \( S_p \subseteq S_q \), \( 1 \leq p \leq q \leq \infty \).

2.3. Fourier-like Multipliers. For \( \sigma \in L^\infty(\Omega, \mu) \), we define the linear operator \( F_\sigma : L^2(\Omega, \mu) \to L^2(\Omega, \mu) \) by

\[
F_\sigma f = \mathcal{T}^{-1} [\sigma \mathcal{T} f] .
\]

(2.8)

In the case of the Fourier transform, this operator is known as the Fourier multiplier. Clearly, if \( \sigma = 1 \), then \( F_\sigma = I \), where \( I \) is the identity operator. Moreover, from the formula (1.7), it is clear that \( F_\sigma \) is bounded with

\[
\|F_\sigma\|_{S_\infty} \leq \|\sigma\|_\infty .
\]

(2.9)

**Definition 2.3.** Let \( \sigma \in L^1(\Omega, \mu) \cup L^\infty(\Omega, \mu) \) and let \( \phi, \psi \in L^\infty(\Omega, \mu) \cap L^2(\Omega, \mu) \) such that \( \|\phi\|_{2,\mu} = \|\psi\|_{2,\mu} = 1 \). We define the linear operator \( P_{\sigma,\phi,\psi} : L^2(\Omega, \mu) \to L^2(\Omega, \mu) \) by

\[
\langle P_{\sigma,\phi,\psi} f, g \rangle_\mu = \langle \sigma \mathcal{T}(\phi f), \mathcal{T}(\psi g) \rangle_\mu .
\]

(2.10)

In the case of the Fourier transform, this operator is known as the wavelet Fourier multiplier, which can be viewed as a variant of a localization operator with respect to the symbol \( \sigma \) and the admissible wavelets \( \phi \) and \( \psi \), see the book [26] and the reference therein. Notice that, if \( \sigma = \chi_A \) is the characteristic function on the subset \( A \subset \Omega \), then we write \( F_\sigma \) as \( F_A \). In this case, we also write \( P_{\sigma,\phi,\psi} \) as \( P_{A,\phi,\psi} \) if \( \phi \neq \psi \) and \( P_{A,\phi} \), if \( \phi = \psi \). The linear operator \( F_A : L^2(\Omega, \mu) \to L^2(\Omega, \mu) \) is a self-adjoint projection, it is known as the frequency limiting operator on \( L^2(\Omega, \mu) \) and has many applications in time-frequency analysis. Moreover we will prove in the last section that \( P_{A,\phi} \) can be viewed as the phase space (or time frequency) limiting operator.

The next proposition shows that \( P_{\sigma,\phi,\psi} : L^2(\Omega, \mu) \to L^2(\Omega, \mu) \) and \( \bar{\psi} F_\sigma \phi : L^2(\Omega, \mu) \to L^2(\Omega, \mu) \) are unitary equivalent.

**Proposition 2.4.** Let \( \sigma \in L^1(\Omega, \mu) \cup L^\infty(\Omega, \mu) \) and let \( \phi, \psi \in L^\infty(\Omega, \mu) \cap L^2(\Omega, \mu) \) such that \( \|\phi\|_{2,\mu} = \|\psi\|_{2,\mu} = 1 \). Then

\[
\langle P_{\sigma,\phi,\psi} f, g \rangle_\mu = \langle \bar{\psi} F_\sigma \phi f, g \rangle_\mu .
\]

(2.11)

**Proof.** From (2.8) and Parseval’s formula (1.6), we have

\[
\langle P_{\sigma,\phi,\psi} f, g \rangle_\mu = \langle \sigma \mathcal{T}(\phi f), \mathcal{T}(\psi g) \rangle_\mu
\]

\[
= \langle T F_\sigma(\phi f), T(\psi g) \rangle_\mu
\]

\[
= \langle F_\sigma(\phi f), \psi g \rangle_\mu
\]

\[
= \langle (\bar{\psi} F_\sigma \phi) f, g \rangle_\mu .
\]

The proof is complete.

\( \Box \)

3. Uncertainty Principles by Means of the Frequency Limiting Operator

We will need to introduce the following time limiting operator, defined by

\[
E_S f = \chi_S f, \quad f \in L^1(\Omega, \mu) \cup L^2(\Omega, \mu),
\]

where \( S \subset \Omega \). Clearly \( E_S : L^2(\Omega, \mu) \to L^2(\Omega, \mu) \) is a self-adjoint projection.
3.1. Uncertainty principles on the space $L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$. The first known result for functions in $L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$ is the following Donoho-Stark type uncertainty inequality, see [11, Inequality (3.4)].

**Theorem 3.1.** Let $\varepsilon_1, \varepsilon_2 \in (0, 1)$ such that $\varepsilon_1^2 + \varepsilon_2^2 < 1$. Then if $f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$ we have

$$
\mu(S)\mu(\Sigma) \geq \varepsilon_1^{-2} \left( 1 - \sqrt{\varepsilon_1^2 + \varepsilon_2^2} \right)^2.
$$

In the case of the Fourier transform, it goes back to Donoho and Stark [9]. This inequality implies that the essential support of $f$ and $\mathcal{T}f$ cannot be too small. Moreover, we recall the following local uncertainty principle, see [11].

**Theorem 3.2.**

1. If $0 < s < a$, then there exists a constant $C$ such that for all $f \in L^2(\Omega, \mu)$ and all measurable subset $\Sigma \subset \Omega$ of finite measure $0 < \mu(\Sigma) < \infty$,

$$
\|F_{\Sigma}f\|_{2,\mu}^2 \leq C \mu(\Sigma)^{\frac{s}{2}} \|\hat{f}\|_{2,\mu}^2.
$$

2. If $s > a$, then there exists a constant $C$ such that for all $f \in L^2(\Omega, \mu)$ and all measurable subset $\Sigma \subset \Omega$ of finite measure $0 < \mu(\Sigma) < \infty$,

$$
\|F_{\Sigma}f\|_{2,\mu}^2 \leq C \mu(\Sigma)^{2-2a} \|\hat{f}\|_{2,\mu}^{2-2a}.
$$

Next, take $s = a$. Then, if we apply the first inequality (3.2) with $a(1 - \varepsilon)$, $\varepsilon \in (0, 1)$, replacing $s$ and then apply the following classical inequality

$$
\|\|x\|^{a - \varepsilon} f\|_{2,\mu} \leq C \|\hat{f}\|_{2,\mu} \|x\|^{a} f\|_{2,\mu}^{1-\varepsilon},
$$

we obtain for all $\varepsilon \in (0, 1)$,

$$
\|F_{\Sigma}f\|_{2,\mu}^2 \leq C \mu(\Sigma)^{1-\varepsilon} \|\hat{f}\|_{2,\mu}^{2\varepsilon} \|x\|^{a} f\|_{2,\mu}^{2-2\varepsilon}.
$$

Consequently we conclude the following first corollary comparing the support of $\mathcal{T}f$ and the generalized time dispersion $\|\|x\|^{\delta} f\|_{2,\mu}^2$ for function in the range of $F_{\Sigma}$:

$$
\text{Im}(F_{\Sigma}) = \{ f \in L^2(\Omega, \mu) : \text{supp}\mathcal{T}f \subset \Sigma \}.
$$

**Corollary 3.3.** Let $s > 0$. Then there exists a constant $C$ such that for all $f \in \text{Im}(F_{\Sigma})$,

$$
\mu(\text{supp}\mathcal{T}f) \|\|x\|^{\delta} f\|_{2,\mu}^{2\alpha} \geq C \|\hat{f}\|_{2,\mu}^{2\alpha}.
$$

**Proof.** Let $s > 0$ and $f \in \text{Im}(F_{\Sigma})$. Then $f = F_{\Sigma}f$, and we apply (3.2), (3.3), (3.5) to obtain the desired result.

Notice that, if $\mu(\text{supp}\mathcal{T}f)$ is finite, then $\mu(\text{supp}f)$ is infinite, because $f$ and $\mathcal{T}f$ cannot be simultaneously supported on subsets of finite measure, see [11, Corollary 3.7]. This result is known as the Benedicks-Amrein-Berthier uncertainty principle.

Moreover, we can also obtain an inequality comparing the essential support of $\mathcal{T}f$ and the generalized time dispersion $\|\|x\|^{\delta} f\|_{2,\mu}^2$ for functions that are $\varepsilon_2$-bandlimited on $\Sigma$.

**Corollary 3.4.** Let $s > 0$.

1. If $0 < s < a$, then there exists a constant $C$ such that for all function $f$ which is $\varepsilon_2$-bandlimited on $\Sigma$,

$$
\mu(\Sigma)^{\frac{s}{2}} \|\|x\|^{\delta} f\|_{2,\mu}^{\alpha} \geq C \left( 1 - \varepsilon_2^2 \right) \|\hat{f}\|_{2,\mu}^{2\alpha}.
$$
(2) If \( s > a \), then there exists a constant \( C \) such that for all function \( f \) which is \( \varepsilon_2 \)-bandlimited on \( \Sigma \),
\[
\mu(\Sigma) \frac{2}{\beta} \| |x|^s f\|_{2, \mu}^2 \geq C \left( 1 - \varepsilon_2^2 \right)^{\frac{1}{2}} \| f \|_{2, \mu}^2.
\] (3.8)

(3) For all \( \varepsilon \in (0,1) \), there exists a constant \( C \) such that for all function \( f \) which is \( \varepsilon_2 \)-bandlimited on \( \Sigma \),
\[
\mu(\Sigma) \| |x|^\varepsilon f\|_{2, \mu}^2 \geq C \left( 1 - \varepsilon_2^2 \right)^{\frac{1}{1-\varepsilon}} \| f \|_{2, \mu}^2.
\] (3.9)

Proof. Since \( f \in L^2(\Omega, \mu) \) is \( \varepsilon_2 \)-bandlimited on \( \Sigma \), then
\[
\| F_{\Sigma} f \|_{2, \mu}^2 = \| f \|_{2, \mu}^2 - \| F_{\Sigma} f \|_{2, \mu}^2 \geq \left( 1 - \varepsilon_2^2 \right) \| f \|_{2, \mu}^2.
\]
For the first result, we use the local inequalities (3.2). Analogously, for the second inequality, we use (3.3), and finally, for the third inequality, we use (3.5).

Now, since \( \| F_{\Sigma} f \|_{2, \mu} = \| E_{\Sigma} tf \|_{2, \mu} \), then by interchanging the roles of \( f \) and \( T f \) in Theorem 3.2, Corollary 3.3 and Corollary 5.8, we obtain the following results involving the time limiting operator instead of the frequency limiting operator, and the frequency dispersion instead of the time dispersion.

Theorem 3.5. Let \( \beta > 0 \).

(1) If \( 0 < \beta < a \), then

(a) there exists a constant \( C \) such that for all \( f \in L^2(\Omega, \mu) \) and all measurable subset \( S \subset \Omega \) of finite measure \( 0 < \mu(S) < \infty \),
\[
\| E_S f \|_{2, \mu}^2 \leq C \mu(S) \frac{2}{\beta} \| |x|^\beta T f\|_{2, \mu}^2,
\]
(3.10) 

(b) there exists a constant \( C \) such that for all function \( f \) which is \( \varepsilon_1 \)-concentrated on \( S \),
\[
\mu(S) \frac{2}{\beta} \| |x|^\beta T f\|_{2, \mu}^2 \geq C \left( 1 - \varepsilon_1^2 \right) \| f \|_{2, \mu}^2.
\] (3.11)

(2) If \( \beta > a \), then

(a) there exists a constant \( C \) such that for all \( f \in L^2(\Omega, \mu) \) and all measurable subset \( S \subset \Omega \) of finite measure \( 0 < \mu(S) < \infty \),
\[
\| E_S f \|_{2, \mu}^2 \leq C \mu(S) \| f \|_{2, \mu}^2 - \frac{2 \beta}{\beta - a} \| |x|^\beta T f\|_{2, \mu}^{\frac{2 \beta}{\beta - a}},
\]
(3.12) 

(b) there exists a constant \( C \) such that for all function \( f \) which is \( \varepsilon_1 \)-concentrated on \( S \),
\[
\mu(S) \frac{2}{\beta} \| |x|^\beta T f\|_{2, \mu}^2 \geq C \left( 1 - \varepsilon_1^2 \right) \| f \|_{2, \mu}^2.
\] (3.13)

(3) For all \( \varepsilon \in (0,1) \),

(a) there exists a constant \( C \) such that for all \( f \in L^2(\Omega, \mu) \) and all measurable subset \( S \subset \Omega \) of finite measure \( 0 < \mu(S) < \infty \),
\[
\| E_S f \|_{2, \mu}^2 \leq C \mu(S)^{1-\varepsilon} \| f \|_{2, \mu}^2 \| |x|^\alpha T f\|_{2, \mu}^{2-2\varepsilon},
\]
(3.14) 

(b) there exists a constant \( C \) such that for all function \( f \) which is \( \varepsilon_1 \)-concentrated on \( S \),
\[
\mu(S) \| |x|^\alpha T f\|_{2, \mu}^2 \geq C \left( 1 - \varepsilon_1^2 \right) \| f \|_{2, \mu}^2.
\] (3.15)

(4) There exists a constant \( C \) such that for all \( f \in \text{Im}(E_S) = \{ f \in L^2(\Omega, \mu) : \text{supp} f \subset S \} \),
\[
\mu(\text{supp} f) \| |x|^\beta T f\|_{2, \mu}^{\frac{2 \beta}{\beta - a}} \geq C \| f \|_{2, \mu}^{\frac{2 \beta}{\beta - a}}.
\] (3.16)
Finally we can formulate our new Heisenberg-type uncertainty inequalities for functions in $L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$, with constants that depend on $\varepsilon_1$, $\varepsilon_2$, $S$ and $\Sigma$.

**Theorem 3.6.** Let $s, \beta > 0$. Then for all $f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$:

1. If $0 < s, \beta < a$,

$$\||x^s f\|_{\mu}^\beta \||x^s \mathcal{T} f\|_{\mu}^\beta \geq C \frac{(1 - \varepsilon_1^2)^{s/2}(1 - \varepsilon_2^2)^{\beta/2}}{(\mu(S)\mu(\Sigma))^{\beta/2}} \|f\|_{2,\mu}^{s+\beta},$$

(3.17)

2. If $s, \beta > a$,

$$\||x^s f\|_{\mu}^\beta \||x^s \mathcal{T} f\|_{\mu}^\beta \geq C \left(\frac{(1 - \varepsilon_1^2)^{s/2}(1 - \varepsilon_2^2)^{\beta/2}}{\mu(S)\mu(\Sigma)}\right)^{\frac{s}{2s+\beta}} \|f\|_{2,\mu}^{s+\beta},$$

(3.18)

3. For all $e \in (0, 1)$,

$$\||x^a f\|_{\mu}^\beta \||x^a \mathcal{T} f\|_{\mu}^\beta \geq C \left(\frac{(1 - \varepsilon_1^2)^{s/2}(1 - \varepsilon_2^2)^{\beta/2}}{\sqrt{\mu(S)\mu(\Sigma)}}\right)^{\frac{s}{2s+\beta}} \|f\|_{2,\mu}^{s+\beta}.$$  

(3.19)

**Remark 3.7.**

1. Notice that Corollary 5.8 and Inequalities (3.11), (3.13) and (3.15) give separately a lower bounds for the measures of the time dispersion $\||x^s f\|_{\mu}$ and the frequency dispersion $\||x^s \mathcal{T} f\|_{\mu}$, which give more information than a lower bound of the product between them in Theorem 3.6.

2. On the other hand, from Corollary 5.8 and Inequalities (3.11), (3.13) and (3.15), we can obtain separately a lower bounds, that depend of the signal $f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$, for the measures of $\mu(S)$ and $\mu(\Sigma)$, from which we deduce, in the spirit of [6], the following lower bounds for the product between them:

$$\mu(S)\mu(\Sigma) \geq \begin{cases} 
C.C.f(s, a, \beta)\left((1 - \varepsilon_1^2)^{\frac{1}{\beta}}(1 - \varepsilon_2^2)^{\frac{1}{\beta}}\right)^a, & 0 < s, \beta < a, \\
C.C.f(s, a, \beta)\left(1 - \varepsilon_1^2\right)\left(1 - \varepsilon_2^2\right), & s, \beta > a, \\
C.C.f(a, a, a)\left((1 - \varepsilon_1^2)(1 - \varepsilon_2^2)\right)^{\frac{1}{1-\varepsilon}}, & \text{otherwise,}
\end{cases}$$

(3.20)

where $C$ is a constant that depend only on $s, a, c_\kappa, \beta, \varepsilon$, and

$$C.f(s, a, \beta) = \left(\frac{\|f\|_{2,\mu}^{s+\beta}}{\||x^s f\|_{\mu}^\beta \||x^s \mathcal{T} f\|_{\mu}^\beta}\right)^{\frac{2a}{2s+\beta}}.$$  

(3.21)

3.2. **Uncertainty principles on the space $L^1 \cap L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$.** The first known result for functions in $L^1 \cap L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$ is the following Donoho-Stark type uncertainty inequality, see [12, Proposition 2.6].

**Theorem 3.8.** Let $\varepsilon_1, \varepsilon_2 \in (0, 1)$. Then if $f \in L^1 \cap L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)$ we have

$$\mu(S) \geq \frac{\|f\|_{2,\mu}^2(1 - \varepsilon_1)^2}{\|f\|_{\mu}^4}, \quad \mu(\Sigma) \geq \frac{c_\kappa^{-2}\|f\|_{2,\mu}^2(1 - \varepsilon_2)^2}{\|f\|_{\mu}^4},$$

(3.22)

and then

$$\mu(S)\mu(\Sigma) \geq c_\kappa^{-2}(1 - \varepsilon_1)^2(1 - \varepsilon_2)^2.$$  

(3.23)
Theorem 3.8 is stronger than Theorem 3.1, in the sense that the previous theorem give a lower bound of $\mu(S)$ and $\mu(\Sigma)$ separately, which is not possible in Theorem 3.1.

Now we will recall the following Carlson-type and Nash-type inequalities, see [12, Proposition 2.2, Proposition 2.3].

**Theorem 3.9.** Let $s, \beta > 0$. Then we have:

1. A Carlson-type inequality: there exists a constant $C = C(s, a)$ such that for all $f \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$,
   \[
   \|f\|_{1,\mu}^{1+\frac{s}{2}} \leq C \|f\|_{2,\mu} \|x|^s f\|_{1,\mu}.
   \] (3.24)

2. A Nash-type inequality: there exists a constant $C = C(\beta, a)$ such that for all $f \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$,
   \[
   \|f\|_{2,\mu}^{1+\frac{s}{2}} \leq C \|f\|_{2,\mu} \|\xi|^\beta \mathcal{T} f\|_{2,\mu}.
   \] (3.25)

Consequently we obtain a lower bounds for the time and frequency dispersions:

\[
\|x|^s f\|_{1,\mu} \geq C \left( \frac{\|f\|_{1,\mu}}{\|f\|_{2,\mu}} \right)^{\frac{s}{\beta}} \|f\|_{1,\mu} \quad \text{and} \quad \|\xi|^\beta \mathcal{T} f\|_{2,\mu} \geq C \left( \frac{\|f\|_{2,\mu}}{\|f\|_{1,\mu}} \right)^{\frac{\beta}{s}} \|f\|_{2,\mu}.
\] (3.26)

**Corollary 3.10.** Let $s, \beta > 0$. Then

1. there exists a constant $C = C(a, \beta, s)$ such that for all $f \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$,
   \[
   \|x|^s f\|_{1,\mu} \|\xi|^\beta \mathcal{T} f\|_{2,\mu} \geq C \|f\|_{2,\mu} \|f\|_{1,\mu}^s.
   \] (3.30)

2. there exists a constant $C$ such that for all $f \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$ and all measurable subset of $\Sigma$ of finite measure,
   \[
   \|F_{\Sigma} f\|_{2,\mu} \leq C \mu(\Sigma) \|f\|_{2,\mu} \|\xi|^\beta \mathcal{T} f\|_{2,\mu}.
   \] (3.27)

3. there exists a constant $C$ such that for all $f \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$ and all measurable subset $S$ of finite measure,
   \[
   \|E_S f\|_{1,\mu} \leq C \mu(S) \|f\|_{1,\mu} \|\xi|^\beta \mathcal{T} f\|_{2,\mu}.
   \] (3.28)

4. there exists a constant $C$ such that for all $f \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$ with $\text{supp}\mathcal{T} f \subset \Sigma$,
   \[
   \mu(\text{supp} \mathcal{T} f) \|x|^s f\|_{1,\mu} \|\xi|^\beta \mathcal{T} f\|_{2,\mu} \geq C \|f\|_{2,\mu}^{\frac{2s}{s+\beta}}.
   \] (3.29)

5. there exists a constant $C$ such that for all $f \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$ with $\text{supp} f \subset S$,
   \[
   \mu(\text{supp} f) \|\xi|^\beta \mathcal{T} f\|_{2,\mu} \|x|^s f\|_{1,\mu} \geq C \|f\|_{1,\mu}^{\frac{2s}{s+\beta}}.
   \] (3.30)

**Proof.** The first inequality follows by combining the Carlson inequality (3.24) and the Nash inequality (3.25). Next by (1.7) and (1.3),

\[
\|F_{\Sigma} f\|_{2,\mu}^2 = \|\chi_{\Sigma} \mathcal{T} f\|_{2,\mu}^2 \leq \mu(\Sigma) \|\mathcal{T} f\|_{2,\mu}^2 \leq c^2 \mu(\Sigma) \|f\|_{1,\mu}^2,
\]

and by the Carlson inequality (3.24) we obtain (3.28). Now by the Cauchy-Schwartz inequality we have,

\[
\|E_S f\|_{1,\mu}^2 \leq \mu(S) \|f\|_{2,\mu}^2,
\]

and by the Nash type inequality (3.25) we deduce (3.29). Finally (3.30) follows directly from (3.28) by taking $\Sigma = \text{supp} \mathcal{T} f$ and if we take $S = \text{supp} f$ in (3.29) we obtain (3.31).
Remark 3.11. Clearly, Inequality (3.26) implies also that, for all \( f \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu) \),
\[
|||x|^s f||^a+\beta ||\xi|^\beta Tf||^{a+\beta}_{2,\mu} \geq C ||f||^{a+\beta}_{1,\mu} ||f||^{a+\beta}_{2,\mu}.
\] (3.32)

Corollary 3.12. Let \( s, \beta > 0 \). Then

(1) there exists a constant \( C \) such that for all function \( f \), which is \( \varepsilon_1 \)-timelimited on \( S \),
\[
\mu(S) \frac{a+\beta}{a} \frac{\varepsilon_1}{a} |||x|^s f||^{a+\beta}_{1,\mu} \geq C (1 - \varepsilon_1) \frac{a+\beta}{a} ||f||^{a+\beta}_{1,\mu},
\] (3.33)

(2) there exists a constant \( C \) such that for all function \( f \), which is \( \varepsilon_2 \)-bandlimited on \( \Sigma \),
\[
\mu(\Sigma) \frac{a+\beta}{a} \frac{\varepsilon_2}{a} |||\xi|^\beta Tf||^{a+\beta}_{2,\mu} \geq C (1 - \varepsilon_2) \frac{a+\beta}{a} ||f||^{a+\beta}_{2,\mu},
\] (3.34)

(3) there exists a constant \( C \) such that for all \( f \in L^1 \cap L^2(\varepsilon_1, \varepsilon_2, S, \Sigma) \),
\[
|||x|^s f||^{a+\beta}_{1,\mu} ||\xi|^\beta Tf||^{a+\beta}_{2,\mu} \geq C \left( \frac{(1 - \varepsilon_1)^2 (1 - \varepsilon_2)^2}{\mu(S)\mu(\Sigma)} \right)^\frac{(a+\beta)^2}{2a} ||f||^{a+\beta}_{1,\mu} ||f||^{a+\beta}_{2,\mu}.
\] (3.35)

Proof. If \( f \) is \( \varepsilon_1 \)-timelimited, then
\[
||E_{\Sigma} f||^{a+\beta}_{1,\mu} \geq ||f||^{a+\beta}_{1,\mu} - ||E_{\Sigma^c} f||^{a+\beta}_{1,\mu} \geq (1 - \varepsilon_1) ||f||^{a+\beta}_{1,\mu},
\]
and if \( f \) is \( \varepsilon_2 \)-bandlimited, then
\[
||F_{\Sigma} f||^2_{2,\mu} = ||f||^2_{2,\mu} - ||F_{\Sigma^c} f||^2_{2,\mu} \geq (1 - \varepsilon_2^2) ||f||^2_{2,\mu}.
\]
Hence the desired result follows from (3.28) and (3.29). \( \square \)

Remark 3.13. Let \( s, \beta > 0 \) and let \( f \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu) \).

(1) If \( f \) is \( \varepsilon_1 \)-timelimited on \( S \), then
\[
\mu(S) \geq C \left( \frac{||f||^{a+\beta}_{1,\mu}}{|||x|^s f||^{a+\beta}_{1,\mu}} \right)^\frac{2a}{a+\beta} (1 - \varepsilon_1)^2.
\] (3.36)

(2) If \( f \) is \( \varepsilon_2 \)-bandlimited on \( \Sigma \), then
\[
\mu(\Sigma) \geq C \left( \frac{||f||^{a+\beta}_{2,\mu}}{|||x|^s f||^{a+\beta}_{1,\mu}} \right)^\frac{2a}{a+\beta} (1 - \varepsilon_2^2).
\] (3.37)

(3) If \( f \in L^1 \cap L^2(\varepsilon_1, \varepsilon_2, S, \Sigma) \), then
\[
\mu(S)\mu(\Sigma) \geq C \tilde{C}_f(a, s, \beta)(1 - \varepsilon_1)^2 (1 - \varepsilon_2^2),
\] (3.38)

where
\[
\tilde{C}_f(a, s, \beta) = \left( \frac{||f||^{a+\beta}_{1,\mu} ||f||^{a+\beta}_{2,\mu}}{|||x|^s f||^{a+\beta}_{1,\mu} ||\xi|^\beta Tf||^{a+\beta}_{2,\mu}} \right)^\frac{2a}{(a+\theta)(a+s)}.
\] (3.39)

4. The wavelet multiplier

Our motivation here came from the classical setting stated in [27, 28]. In this section let \( \phi \) and \( \psi \) will be two functions in \( L^\infty(\Omega, \mu) \cap L^2(\Omega, \mu) \) such that \( ||\phi||_{2,\mu} = ||\psi||_{2,\mu} = 1 \).
4.1. **Boundedness.** The aim of this section is to prove that we can also define \( P_{\sigma,\phi,\psi} \) for symbol \( \sigma \in L^p(\Omega, \mu), \) \( 1 < p < \infty \). First, if \( \sigma \in L^\infty(\Omega, \mu) \), we have the following result.

**Proposition 4.1.** Let \( \sigma \in L^\infty(\Omega, \mu) \). Then \( P_{\sigma,\phi,\psi} \) is in \( S_\infty \) and
\[
\| P_{\sigma,\phi,\psi} \|_{S_\infty} \leq \| \phi \|_\infty \| \psi \|_\infty \| \sigma \|_\infty. 
\] (4.1)

**Proof.** By the Cauchy-Schwartz inequality,
\[
|\langle P_{\sigma,\phi,\psi} f, g \rangle_\mu| \leq \| \sigma \|_\infty \| \mathcal{T}(\phi f) \|_{2,\mu} \| \mathcal{T}(\psi g) \|_{2,\mu}.
\]
Then by Plancherel’s formula (1.7), we obtain
\[
|\langle P_{\sigma,\phi,\psi} f, g \rangle_\mu| \leq \| \sigma \|_\infty \| \phi f \|_{2,\mu} \| \psi g \|_{2,\mu}.
\]
This completes the proof. \( \square \)

Now, if we consider \( \sigma \in L^1(\Omega, \mu) \), then we obtain the following result.

**Proposition 4.2.** Let \( \sigma \in L^1(\Omega, \mu) \). Then \( P_{\sigma,\phi,\psi} \) is in \( S_\infty \) and
\[
\| P_{\sigma,\phi,\psi} \|_{S_\infty} \leq cK \| \sigma \|_{1,\mu}. 
\] (4.2)

**Proof.** Since \( \mathcal{T}(\phi f)(\xi) = \langle f, \overline{\phi(\xi)} \rangle_\mu \), then by the Cauchy-Schwartz inequality,
\[
\| \mathcal{T}(\phi f) \|_\infty \leq cK \| f \|_{2,\mu} \| \phi \|_{2,\mu}.
\]
Therefore, since \( \| \phi \|_{2,\mu} = \| \psi \|_{2,\mu} = 1 \), we obtain
\[
|\langle P_{\sigma,\phi,\psi} f, g \rangle_\mu| \leq \| \sigma \|_{1,\mu} \| \mathcal{T}(\phi f) \|_\infty \| \mathcal{T}(\psi g) \|_\infty \leq cK^2 \| \sigma \|_{1,\mu} \| f \|_{2,\mu} \| g \|_{2,\mu}.
\]
This completes the proof. \( \square \)

Thus, by (4.1), (4.2) and the Riesz-Thorin interpolation argument [23, Theorem 2] (see also [26, Theorem 12.4]) we obtain the following theorem.

**Theorem 4.3.** Let \( \sigma \in L^p(\Omega, \mu), \) \( 1 < p < \infty \). Then the linear operator \( P_{\sigma,\phi,\psi} : L^2(\Omega, \mu) \to L^2(\Omega, \mu) \) is bounded and
\[
\| P_{\sigma,\phi,\psi} \|_{S_\infty} \leq cK^2 \| \sigma \|_{p,\mu}. 
\] (4.3)

Hence we can define the operator \( \tilde{\psi} F_\sigma \phi : L^2(\Omega, \mu) \to L^2(\Omega, \mu) \), where \( \sigma \in L^p(\Omega, \mu), \) \( 1 \leq p \leq \infty \) by
\[
\langle P_{\sigma,\phi,\psi} f, g \rangle_\mu = \langle \tilde{\psi} F_\sigma \phi f, g \rangle_\mu. 
\] (4.4)

4.2. **Schatten class properties.** Let us begin with the following theorem.

**Theorem 4.4.** Let \( \sigma \) be symbol in \( L^1(\Omega, \mu) \). Then \( P_{\sigma,\phi,\psi} \) is Hilbert Schmidt and
\[
\| P_{\sigma,\phi,\psi} \|_{S_2}^2 = \int_{\Omega} \sigma(\xi) \langle P_{\sigma,\phi,\psi} \tilde{\psi}, \phi \rangle \| K_\xi \|_\mu^2 \, d\mu(\xi) \leq \| \sigma \|_{L^1(\mu)}^2. 
\] (4.5)
Proof. First by (2.10) it follows immediately that the adjoint of $P_{\sigma, \psi, \phi}$ is $P_{\sigma, \phi, \psi} : L^2(\Omega, \mu) \to L^2(\Omega, \mu)$. Now, Let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis for $L^2(\Omega, \mu)$. Then by (2.10) and by Fubini's theorem, we obtain,

\[
\sum_{n=1}^\infty \|P_{\sigma, \phi, \psi}\varphi_n\|_{2, \mu}^2 = \sum_{n=1}^\infty \langle P_{\sigma, \phi, \psi}\varphi_n, P_{\sigma, \phi, \psi}\varphi_n \rangle_{\mu}
\]

\[
= \sum_{n=1}^\infty \langle \sigma T(\phi \varphi_n), T(\psi P_{\sigma, \phi, \psi}\varphi_n) \rangle_{\mu}
\]

\[
= \sum_{n=1}^\infty \int_{\Omega} \sigma(\xi)\langle \varphi_n, \phi K_\xi \rangle_{\mu} \langle P_{\sigma, \phi, \psi}\varphi_n, \psi K_\xi \rangle_{\mu} d\mu(\xi)
\]

\[
= \int_{\Omega} \sigma(\xi) \sum_{n=1}^\infty \langle P_{\sigma, \phi, \psi}\varphi_n, \psi K_\xi \rangle_{\mu} \langle \varphi_n, \phi K_\xi \rangle_{\mu} d\mu(\xi)
\]

\[
= \int_{\Omega} \sigma(\xi) \langle P_{\sigma, \psi, \phi}\psi K_\xi, \phi K_\xi \rangle_{\mu} d\mu(\xi),
\]

where we have used Parseval's identity in the last line. Therefore from Proposition 4.2 and since $|K_\xi| \leq c_K$,

\[
\sum_{n=1}^\infty \|P_{\sigma, \phi, \psi}\varphi_n\|_{2, \mu}^2 \leq \|P_{\sigma, \phi, \psi}\|_{S_\infty} \|\phi\|_{2, \mu} \|\psi\|_{2, \mu} \|\sigma\|_{1, \mu}
\]

\[
\leq c_K^2 \|\sigma\|_{1, \mu}.
\]

Thus from Proposition 2.2, the operator $P_{\sigma, \phi, \psi}$ is in $S_2$ and $\|P_{\sigma, \phi, \psi}\|_{S_2} \leq c_K^2 \|\sigma\|_{1, \mu}$. □

Consequently the operator $P_{\sigma, \phi, \psi}$ is also compact for symbols in $L^p(\Omega, \mu)$.

Corollary 4.5. Let $\sigma$ be symbol in $L^p(\Omega, \mu)$, $1 \leq p < \infty$. Then the operator $P_{\sigma, \phi, \psi}$ is compact.

Proof. Let $\{\sigma_n\}_{n=1}^\infty$ be a sequence of functions in $L^1(\Omega, \mu) \cap L^\infty(\Omega, \mu)$ such that $\sigma_n \to \sigma$ in $L^p(\Omega, \mu)$ as $n \to \infty$. Then by Theorem 4.3,

\[
\|P_{\sigma_n, \phi, \psi} - P_{\sigma, \phi, \psi}\|_{S_\infty} \leq \|\phi\|_{L_\infty} \|\psi\|_{L_\infty} \|\sigma_n - \sigma\|_{p, \mu}.
\]

(4.6)

Therefore $P_{\sigma_n, \phi, \psi} \to P_{\sigma, \phi, \psi}$ in $S_\infty$ as $n \to \infty$. Now, since by Theorem 4.4, the operators $P_{\sigma_n, \phi, \psi}$ are in $S_2$ and hence compact, and since the set of compact operators is a closed subspace of $S_\infty$, then the operator $P_{\sigma, \phi, \psi}$ is also compact. □

More precisely we will prove that the operator $P_{\sigma, \phi, \psi}$ is in fact in the Schatten class $S_p$, $1 \leq p < \infty$. Of particular interest is the Schatten-von Neumann class $S_1$ (see [27, 28]).

Theorem 4.6. Let $\sigma \in L^1(\Omega, \mu)$. Then $P_{\sigma, \phi, \psi} : L^2(\Omega, \mu) \to L^2(\Omega, \mu)$ is trace class with

\[
\|P_{\sigma, \phi, \psi}\|_{S_1} \leq c_K^2 \|\sigma\|_{1, \mu},
\]

and we have the following trace formula

\[
\text{tr} (P_{\sigma, \phi, \psi}) = \int_{\Omega} \sigma(\xi) \langle \psi K_\xi, \phi K_\xi \rangle_{\mu} d\mu(\xi).
\]
Proof. Let \( \{ \varphi_n \}_{n=1}^{\infty} \) be an orthonormal basis for \( L^2(\Omega, \mu) \). Then
\[
\sum_{n=1}^{\infty} \langle P_{\sigma,\phi,\psi} \varphi_n, \varphi_n \rangle_{\mu} = \sum_{n=1}^{\infty} \int_{\Omega} \sigma(\xi) T(\phi \varphi_n)(\xi) \overline{T(\psi \varphi_n)(\xi)} \, d\mu(\xi) = \sum_{n=1}^{\infty} \int_{\Omega} \sigma(\xi) \langle \overline{\psi \xi}, \varphi_n \rangle_{\mu} \langle \varphi_n, \overline{\phi \xi} \rangle_{\mu} \, d\mu(\xi).
\]
Thus by Fubini’s theorem,
\[
\sum_{n=1}^{\infty} \langle P_{\sigma,\phi,\psi} \varphi_n, \varphi_n \rangle_{\mu} = \int_{\Omega} \sigma(\xi) \sum_{n=1}^{\infty} \langle \overline{\psi \xi}, \varphi_n \rangle_{\mu} \langle \varphi_n, \overline{\phi \xi} \rangle_{\mu} \, d\mu(\xi). \tag{4.9}
\]
Therefore by Parseval’s identity, and the fact that \( \|\phi\|_{2,\mu} = \|\psi\|_{2,\mu} = 1 \),
\[
\sum_{n=1}^{\infty} \| P_{\sigma,\phi,\psi} \varphi_n \|^2_{\mu} \leq \frac{1}{2} \int_{\Omega} |\sigma(\xi)| \sum_{n=1}^{\infty} \left( |\langle \overline{\phi \xi}, \varphi_n \rangle_{\mu}|^2 + \left| \langle \overline{\psi \xi}, \varphi_n \rangle_{\mu} \right|^2 \right) \, d\mu(\xi)
= \frac{1}{2} \int_{\Omega} |\sigma(\xi)| \left( \|\phi \xi\|_{2,\mu}^2 + \|\psi \xi\|_{2,\mu}^2 \right) \, d\mu(\xi)
\leq \frac{C^2}{2} \|\sigma\|_{1,\mu}.
\]
By Proposition 2.1, the operator \( P_{\sigma,\phi,\psi} \) is in \( S_1 \) and with (4.9) and Parseval’s identity,
\[
\text{tr}(P_{\sigma,\phi,\psi}) = \sum_{n=1}^{\infty} \langle P_{\sigma,\phi,\psi} \varphi_n, \varphi_n \rangle_{\mu} = \int_{\Omega} \sigma(\xi) \langle \overline{\psi \xi}, \overline{\phi \xi} \rangle_{\mu} \, d\mu(\xi).
\]
This allows to conclude. \( \square \)

Moreover by (4.1), (4.7) and by interpolation argument we deduce the following result.

Corollary 4.7. Let \( \sigma \in L^p(\Omega, \mu), 1 \leq p < \infty \). Then \( P_{\sigma,\phi,\psi} : L^2(\Omega, \mu) \to L^2(\Omega, \mu) \) is in \( S_p \) and
\[
\| P_{\sigma,\phi,\psi} \|_{S_p} \leq \frac{C}{p} \|\psi\|_{\infty} \|\phi\|_{\infty} \|\sigma\|_{p,\mu}. \tag{4.10}
\]

4.3. An uncertainty relation. In this subsection, we will assume that \( \phi \) and \( \psi \) satisfy \( \|\phi\|_{\infty} \|\psi\|_{\infty} = 1 \). Now let \( \sigma_1 = \chi_S \) and \( \sigma_2 = \chi_\Sigma \) and let \( L_1 = P_{\sigma_1,\phi,\psi} \) and \( L_2 = P_{\sigma_2,\phi,\psi} \).

Theorem 4.8. Let \( \varepsilon_1, \varepsilon_2 \in (0, 1) \) such that \( \varepsilon_1 + \varepsilon_2 < 1 \). If \( f \in L^2(\Omega, \mu) \) is \( \varepsilon_1 \)-localized with respect to \( P_{\sigma_1,\phi,\psi} \) and \( \varepsilon_2 \)-localized with respect to \( P_{\sigma_2,\phi,\psi} \) then,
\[
\mu(S) \mu(\Sigma) \geq C \varepsilon_1^4 (1 - \varepsilon_1 - \varepsilon_2). \tag{4.11}
\]

Proof. From Proposition 4.1,
\[
\| f - L_2 L_1 f \|_{2,\mu} \leq \| f - L_2 f \|_{2,\mu} + \| L_2 f - L_2 L_1 f \|_{2,\mu} \leq \| L_2 f - f \|_{2,\mu} + \| L_2 \|_{S_\infty} \| L_1 f - f \|_{2,\mu} \leq (\varepsilon_2 + \varepsilon_1) \| f \|_{2,\mu}.
\]
Therefore
\[
\| L_2 L_1 f \|_{2,\mu} \geq \| f \|_{2,\mu} - \| f - L_2 L_1 f \|_{2,\mu} \geq (1 - \varepsilon_1 - \varepsilon_2) \| f \|_{2,\mu}.
\]
Thus from Proposition 4.2 it follows that
\[
1 - \varepsilon_1 - \varepsilon_2 \leq \|L_2 L_1\|_{S_\infty} \leq \|L_1\|_{S_\infty} \|L_2\|_{S_\infty} \leq c_k^4 \mu(S) \mu(\Sigma).
\]

This proves the desired result. \(\square\)

Notice that, when \(\varepsilon_1 = \varepsilon_2 = 0\) in the classical Donoho-Stark uncertainty inequality (3.1), we have \(\mu(S) \mu(\Sigma) \geq c_k^{-2}\). This is a trivial assertion since in this case \(S = \text{supp} f, \Sigma = \text{supp} T f\) and then from [11], either \(\mu(\text{supp} f) = \infty\) or \(\mu(\text{supp} T f) = \infty\). The case \(\varepsilon_1 = \varepsilon_2 = 0\) in Theorem 4.8 is not trivial and gives the following result.

**Corollary 4.9.** If \(f \in L^2(\Omega, \mu)\) is an eigenfunction of \(P_{S,\phi,\psi}\) and \(P_{\Sigma,\phi,\psi}\) corresponding to the same eigenvalue 1, then
\[
\mu(S) \mu(\Sigma) \geq c_k^{-2}. \tag{4.12}
\]

5. **Uncertainty principles for orthonormal sequences**

5.1. **The phase space restriction operator.** We define the phase space restriction operator by

\[
L_{S,\Sigma} = E_{S} F_{\Sigma} E_{S} = (F_{\Sigma} E_{S})^* F_{\Sigma} E_{S}.
\]

We know that the operator \(F_{\Sigma} E_{S}\) is Hilbert-Schmidt (see [11, Inequality (3.2)]), and since the pair \((S, \Sigma)\) is strongly annihilating, then from [11], we have

\[
\|L_{S,\Sigma}\|_{S_\infty} = \|E_{S} F_{\Sigma}\|_{S_\infty}^2 = \|F_{\Sigma} E_{S}\|_{S_\infty}^2 < 1. \tag{5.1}
\]

Moreover, the operator \(L_{S,\Sigma}\) is self-adjoint, positive and from (2.6) it is compact and even trace class with

\[
\|L_{S,\Sigma}\|_{S_1} = \|F_{\Sigma} E_{S}\|_{S_2}^2. \tag{5.2}
\]

In the fundamental paper [18], Landau and Pollak have considered the eigenvalue problem associated with the positive self-adjoint operator \(E_{S} F_{\Sigma} E_{S} : L^2(\mathbb{R}) \to L^2(\mathbb{R})\), where \(S, \Sigma\) are real intervals, for which they proved an asymptotic estimate for the number of eigenvalues.

Motivated by the process in [16], we will show that the phase space restriction operator \(L_{S,\Sigma}\) can be viewed as a wavelet multiplier, and then we will deduce a trace formula.

**Theorem 5.1.** Let \(\phi = \psi\) be the function on \(\Omega\) defined by \(\phi = \frac{1}{\sqrt{\mu(S)}} \chi_S\) and let \(\sigma = \chi_{\Sigma}\). Then

\[
L_{S,\Sigma} = \mu(S) P_{\Sigma,\phi}. \tag{5.3}
\]

**Proof.** Clearly, the function \(\phi\) belongs to \(L^2(\Omega, \mu) \cap L^\infty(\Omega, \mu)\), with \(\|\phi\|_{L^2,\mu} = 1\). Then, since \(E_{S}\) is self-adjoint and by Parseval’s equality (1.6), we have for all \(f, g \in L^2(\Omega, \mu)\),

\[
\langle L_{S,\Sigma} f, g \rangle_{\mu} = \langle F_{\Sigma} E_{S} f, \chi_S g \rangle_{\mu} = \sqrt{\mu(S)} \langle F_{\Sigma} E_{S} f, \phi g \rangle_{\mu} = \sqrt{\mu(S)} \langle T F_{\Sigma} E_{S} f, T(\phi g) \rangle_{\mu} = \sqrt{\mu(S)} \langle \chi_{\Sigma} T \chi_S f, T(\phi g) \rangle_{\mu} = \mu(S) \langle \sigma T(\phi f), \chi_S g \rangle_{\mu} = \mu(S) \langle P_{\Sigma,\phi}, g \rangle_{\mu}.
\]
This completes the proof. \(\square\)

From Theorem 4.6 and Theorem 5.1, we deduce the following trace formula.

**Corollary 5.2.** The phase space operator \(L_{S,\Sigma}\) is trace class with
\[
\text{tr}(L_{S,\Sigma}) = \mu(S)\text{tr}(P_{\Sigma,\phi}) = \int_{S} \int_{\Sigma} |K(x, \xi)|^2 \, d\mu(x) \, d\mu(\xi).
\] (5.4)

The compact operator \(L_{S,\Sigma} : L^2(\Omega, \mu) \to L^2(\Omega, \mu)\) is self-adjoint and then can be diagonalized as
\[
L_{S,\Sigma}f = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n \rangle \mu \varphi_n,
\] (5.5)
where \(\{\lambda_n = \lambda_n(S, \Sigma)\}_{n=1}^{\infty}\) are the positive eigenvalues associated to the arranged in a non-increasing manner
\[
\lambda_n \leq \cdots \leq \lambda_1 < 1,
\] (5.6)
and \(\{\varphi_n = \varphi_n(S, \Sigma)\}_{n=1}^{\infty}\) is the corresponding orthonormal set of eigenfunctions. In particular
\[
\|L_{S,\Sigma}\|_{S,\infty} = \lambda_1,
\] (5.7)
where \(\lambda_1\) is the first eigenvalue corresponding to the first eigenfunction \(\varphi_1\) of the compact operator \(L_{S,\Sigma}\). This eigenfunction realizes the maximum of concentration on the set \(S \times \Sigma\). On the other hand, since \(\varphi_n\) is an eigenfunction of \(L_{S,\Sigma}\) with eigenvalue \(\lambda_n\), then
\[
\|L_{S,\Sigma}\varphi_n - \varphi_n\|_{2,\mu} = \langle \varphi_n - L_{S,\Sigma}\varphi_n, \varphi_n \rangle_{2,\mu} = 1 - \lambda_n,
\] (5.8)
and
\[
\|L_{S,\Sigma}(L_{S,\Sigma}\varphi_n) - L_{S,\Sigma}\varphi_n\|_{2,\mu} = \lambda_n^{-1} \langle L_{S,\Sigma}\varphi_n - L_{S,\Sigma}(L_{S,\Sigma}\varphi_n), L_{S,\Sigma}\varphi_n \rangle_{2,\mu}
\]
\[
= \lambda_n(1 - \lambda_n) = (1 - \lambda_n)\|L_{S,\Sigma}\varphi_n\|_{2,\mu}.
\] (5.9)
Thus, for all \(n\), the functions \(\varphi_n\) and \(L_{S,\Sigma}\varphi_n\) are \((1 - \lambda_n)\)-localized with respect to \(L_{S,\Sigma}\).

More generally, we have the following comparisons of the measures of localization.

**Proposition 5.3.** Let \(\varepsilon, \varepsilon_1, \varepsilon_2 \in (0, 1)\).

1. If \(f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)\), then \(f\) is \((\varepsilon_1 + \varepsilon_2)\)-localized with respect to \(F_{\Sigma}E_S\) and \((2\varepsilon_1 + \varepsilon_2)\)-localized with respect to \(L_{S,\Sigma}\).
2. If \(f \in L^2(\Omega, \mu)\) is \(\varepsilon\)-localized with respect to \(L_{S,\Sigma}\), then
\[
\langle f - L_{S,\Sigma}f, f \rangle_{\mu} \leq (\varepsilon^2 + \varepsilon)\|f\|_{2,\mu}^2.
\] (5.10)
3. If \(f \in L^2(\Omega, \mu)\) satisfies
\[
\langle f - L_{S,\Sigma}f, f \rangle_{\mu} \leq \varepsilon\|f\|_{2,\mu}^2,
\] (5.11)
then \(f\) is \(\sqrt{\varepsilon}\)-localized with respect to \(L_{S,\Sigma}\).
4. If \(f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)\), then
\[
\langle f - L_{S,\Sigma}f, f \rangle_{\mu} < (2\varepsilon_1 + \varepsilon_2)\|f\|_{2,\mu}^2.
\] (5.12)

**Proof.** Recall that \(\|E_S\|_{S,\infty} = \|F_{\Sigma}\|_{S,\infty} = 1\). First we have
\[
\|F_{\Sigma}E_Sf - f\|_{2,\mu} \leq \|F_{\Sigma}f - f\|_{2,\mu} + \|F_{\Sigma}E_Sf - F_{\Sigma}f\|_{2,\mu}
\]
\[
\leq \|F_{\Sigma}f\|_{2,\mu} + \|F_{\Sigma}\|_{S,\infty}\|E_Sf\|_{2,\mu}
\]
\[
\leq (\varepsilon_1 + \varepsilon_2)\|f\|_{2,\mu}.
\]
Moreover, \[
\|L_{S,\Sigma}f - f\|_{2,\mu} \leq \|ESF\Sigma Ef - ESf\|_{2,\mu} + \|ESf - f\|_{2,\mu}
\]
\[
\leq \|ES\|_{S,\infty}\|F\Sigma Ef - f\|_{2,\mu} + \|ESf - f\|_{2,\mu}
\]
\[
\leq (2\varepsilon_1 + \varepsilon_2)\|f\|_{2,\mu}.
\]
Now since
\[
2\langle f - L_{S,\Sigma}f, f \rangle_{\mu} = \|L_{S,\Sigma}f - f\|_{2,\mu}^2 + \|f\|_{2,\mu}^2 - \|L_{S,\Sigma}f\|_{2,\mu}^2
\]
\[
\leq \|L_{S,\Sigma}f - f\|_{2,\mu}^2 + \left(\|L_{S,\Sigma}f - f\|_{2,\mu} + \|L_{S,\Sigma}f\|_{2,\mu}\right)^2 - \|L_{S,\Sigma}f\|_{2,\mu}^2
\]
\[
= 2\|L_{S,\Sigma}f - f\|_{2,\mu}^2 + 2\|L_{S,\Sigma}f - f\|_{2,\mu}\|L_{S,\Sigma}f\|_{2,\mu},
\]
and since \(\|L_{S,\Sigma}\|_{S,\infty} \leq 1\), then
\[
\langle f - L_{S,\Sigma}f, f \rangle_{\mu} \leq \|L_{S,\Sigma}f - f\|_{2,\mu}^2 + \|L_{S,\Sigma}f - f\|_{2,\mu}\|f\|_{2,\mu} \leq (\varepsilon^2 + \varepsilon)\|f\|_{2,\mu}^2,
\]
and the second result follows.

On the other hand, since
\[
\langle (L_{S,\Sigma})^2 f, f \rangle_{\mu} \leq \langle L_{S,\Sigma} f, f \rangle_{\mu},
\]
and since \(L_{S,\Sigma}\) is self-adjoint, then
\[
\|L_{S,\Sigma}f - f\|_{2,\mu}^2 = \langle (I - L_{S,\Sigma})^2 f, f \rangle_{\mu} \leq \|(I - L_{S,\Sigma})f, f\|_{\mu} \leq \varepsilon\|f\|_{2,\mu}^2.
\]
Finally, since
\[
\langle f - L_{S,\Sigma}f, f \rangle_{\mu} = \langle E\Sigma f, f \rangle_{\mu} + \langle E\Sigma f, F\Sigma f \rangle_{\mu} + \langle F\Sigma Ef, E\Sigma f \rangle_{\mu},
\]
then we obtain the last result.

The definition (5.11) is equivalent to
\[
\langle L_{S,\Sigma} f, f \rangle_{\mu} \geq (1 - \varepsilon)\|f\|_{2,\mu}^2,
\]
and we denote by \(L^2(\varepsilon, S, \Sigma)\) the subspace of \(L^2(\Omega, \mu)\) consisting of functions \(f \in L^2(\Omega, \mu)\) satisfying (5.16). Hence from (5.8) and (5.9) we have,
\[
\forall n \geq 1, \quad \varphi_n, L_{S,\Sigma} \varphi_n \in L^2(1 - \lambda_n, S, \Sigma).
\]
Moreover from Proposition 5.3, if \(f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma)\), then \(f \in L^2(2\varepsilon_1 + \varepsilon_2, S, \Sigma)\), and if \(f\) is \(\varepsilon\)-localized with respect to \(L_{S,\Sigma}\), then \(f \in L^2(2\varepsilon, S, \Sigma)\). Therefore we are interested to study the following optimization problem
\[
\text{Maximize} \quad \langle L_{S,\Sigma} f, f \rangle_{\mu}, \quad \|f\|_{2,\mu} = 1,
\]
which aims to look for orthonormal functions in \(L^2(\Omega, \mu)\), which are approximately time and band-limited to a bounded region like \(S \times \Sigma\). It follows that the number of eigenfunctions of \(L_{S,\Sigma}\) whose eigenvalues are very close to one, are an optimal solutions to the problem (5.18), since if \(\varphi_n\) is an eigenfunction of \(L_{S,\Sigma}\) with eigenvalue \(\lambda_n \geq (1 - \varepsilon)\), we have from the spectral representation,
\[
\langle L_{S,\Sigma} \varphi_n, \varphi_n \rangle_{\mu} = \lambda_n \geq (1 - \varepsilon).
\]
We denote by \(n(\varepsilon, S, \Sigma)\) for the number of eigenvalues \(\lambda_n\) of \(L_{S,\Sigma}\) which are close to one, in the sense that
\[
\lambda_1 \geq \cdots \geq \lambda_{n(\varepsilon, S, \Sigma)} \geq 1 - \varepsilon > \lambda_{1+n(\varepsilon, S, \Sigma)} \geq \cdots,
\]

and we denote by $V_{n(\varepsilon,S,\Sigma)} = \text{span}\{\varphi_n\}_{n=1}^{N(\varepsilon,S,\Sigma)}$ the span of the first eigenfunctions of $L_{S,\Sigma}$ corresponding to the largest eigenvalues $\{\lambda_n\}_{n=1}^{n(\varepsilon,S,\Sigma)}$. Therefore, by (5.19) and (5.17), each eigenfunction $\varphi_n$ and its resulting function $L_{S,\Sigma}\varphi_n$ are in $L^2(\varepsilon,S,\Sigma)$, if and only if $1 \leq n \leq n(\varepsilon,S,\Sigma)$. Now, if $f \in V_{n(\varepsilon,S,\Sigma)}$, then
\[
\left\langle L_{\Sigma}^\psi f, f \right\rangle_\mu = \sum_{n=1}^{n(\varepsilon,S,\Sigma)} \lambda_n \left| \left\langle f, \varphi_n \right\rangle_\mu \right|^2 \geq \lambda_{n(\varepsilon,S,\Sigma)} \sum_{n=1}^{n(\varepsilon,S,\Sigma)} \left| \left\langle f, \varphi_n \right\rangle_\mu \right|^2 \geq (1 - \varepsilon)\|f\|^2_{2,\mu}.
\]
Thus $V_{n(\varepsilon,S,\Sigma)}$ determines the subspace of $L^2(\Omega,\mu)$ with maximum dimension that is in $L^2(\varepsilon,S,\Sigma)$. Motivated by the recent paper [25] in the Gabor setting, we obtain the following theorem that characterizes functions that are in $L^2(\varepsilon,S,\Sigma)$.

**Theorem 5.4.** Let $f_{\ker}$ denote the orthogonal projection of $f$ onto the kernel $\text{Ker}(L_{S,\Sigma})$ of $L_{S,\Sigma}$. Then a function $f$ is in $L^2(\varepsilon,S,\Sigma)$ if and only if,
\[
\sum_{n=1}^{n(\varepsilon,S,\Sigma)} (\lambda_n + \varepsilon - 1)\left| \left\langle f, \varphi_n \right\rangle_\mu \right|^2 \geq (1 - \varepsilon)\|f_{\ker}\|^2_{2,\mu} + \sum_{n=n(\varepsilon,S,\Sigma)+1}^{\infty} (1 - \varepsilon - \lambda_n)\left| \left\langle f, \varphi_n \right\rangle_\mu \right|^2.
\]

**Proof.** A given function $f \in L^2(\Omega,\mu)$, write
\[
f = \sum_{n=1}^{\infty} \left\langle f, \varphi_n \right\rangle_\mu \varphi_n + f_{\ker},
\]
where $f_{\ker} \in \text{Ker}(L_{S,\Sigma})$. Then
\[
\left\langle L_{S,\Sigma} f, f \right\rangle_\mu = \sum_{n=1}^{\infty} \lambda_n \left| \left\langle f, \varphi_n \right\rangle_\mu \right|^2.
\]
So the function $f$ is in $L^2(\varepsilon,S,\Sigma)$ if and only if
\[
\sum_{n=1}^{\infty} \lambda_n \left| \left\langle f, \varphi_n \right\rangle_\mu \right|^2 \geq (1 - \varepsilon)\left(\|f_{\ker}\|^2_{2,\mu} + \sum_{n=1}^{\infty} \left| \left\langle f, \varphi_n \right\rangle_\mu \right|^2\right),
\]
and the conclusion follows. \hfill \Box

While a function $f$ that is in $L^2(\varepsilon,S,\Sigma)$ does not necessarily lies in some subspace $V_N = \text{span}\{\varphi_n\}_{n=1}^{N}$, it can be approximated using a finite number of such eigenfunctions. Let $\varepsilon_0 \in (0,1)$ be a fixed real number and let $\mathcal{P}$ the orthogonal projection onto the subspace $V_{n(\varepsilon_0,S,\Sigma)}$.

**Theorem 5.5.** Let $f$ be a function in $L^2(\varepsilon,S,\Sigma)$. Then
\[
\left\| f - \sum_{n=1}^{n(\varepsilon_0,S,\Sigma)} \left\langle f, \varphi_n \right\rangle_\mu \varphi_n \right\|_{2,\mu} \leq \sqrt{\frac{\varepsilon}{\varepsilon_0}} \|f\|_{2,\mu}.
\]

**Proof.** An easy adaptation of the proof of Proposition 3.3 in [25], we can conclude that
\[
\|\mathcal{P} f\|_{2,\mu}^2 \geq (1 - \varepsilon/\varepsilon_0)\|f\|_{2,\mu}^2.
\]
It follows then,
\[
\|f\|_{2,\mu}^2 = \|\mathcal{P} f + (f - \mathcal{P} f)\|_{2,\mu}^2 = \|\mathcal{P} f\|_{2,\mu}^2 + \|f - \mathcal{P} f\|_{2,\mu}^2.
\]
Thus
\[
\|f - \mathcal{P} f\|_{2,\mu}^2 = \|f\|_{2,\mu}^2 - \|\mathcal{P} f\|_{2,\mu}^2 \leq \|f\|_{2,\mu}^2 - (1 - \varepsilon/\varepsilon_0)\|f\|_{2,\mu}^2 = \varepsilon/\varepsilon_0\|f\|_{2,\mu}^2.
\]
This completes the proof of the theorem.

Consequently and from Proposition 5.3, we immediately deduce the following approximating results.

**Corollary 5.6.** Let \( \varepsilon, \varepsilon_1, \varepsilon_2 \in (0, 1) \).

1. If \( f \in L^2(\varepsilon_1, \varepsilon_2, S, \Sigma) \), then
\[
\left\| f - \sum_{n=1}^{n(\varepsilon_0, S, \Sigma)} \langle f, \varphi_n \rangle \mu \varphi_n \right\|_{2, \mu} \leq \sqrt{\frac{2\varepsilon_1 + \varepsilon_2}{\varepsilon_0}} \| f \|_{2, \mu}.
\]

2. If \( f \in L^2(\Omega, \mu) \) is \( \varepsilon \)-localized with respect to \( L_{S, \Sigma} \), then
\[
\left\| f - \sum_{n=1}^{n(\varepsilon_0, S, \Sigma)} \langle f, \varphi_n \rangle \mu \varphi_n \right\|_{2, \mu} \leq \sqrt{\frac{2\varepsilon}{\varepsilon_0}} \| f \|_{2, \mu}.
\]

5.2. **Shapiro–type uncertainty principles.** Based on Malinnikova’s ideas [20], we will prove in this section a quantitative dispersion inequality for orthonormal sequences and a strong uncertainty principle for orthonormal bases. Notice also that the homogeneity of the kernel \( K \) plays a key role in this section, especially in Lemma 5.12.

5.2.1. **Localization theorem.** Our starting point is the following theorem which states that any orthonormal system in \( L^2(\varepsilon_1, \varepsilon_2, S, \Sigma) \) cannot be infinite.

**Theorem 5.7.** Let \( \varepsilon_1, \varepsilon_2 \in (0, 1) \) such that \( 2\varepsilon_1 + \varepsilon_2 < 1 \) and let \( \{f_n\}_{n=1}^N \) be an orthonormal system in \( L^2(\varepsilon_1, \varepsilon_2, S, \Sigma) \). Then
\[
N < c_K^2 \frac{\mu(S)\mu(\Sigma)}{1 - 2\varepsilon_1 - \varepsilon_2}.
\]

**Proof.** The result follows immediately from Inequalities (5.12) and (5.4).

Consequently, if the generalized dispersions of each element of an orthonormal sequence are uniformly bounded, then this sequence is also finite.

**Corollary 5.8.** Fix \( A_1, A_2 > 0 \). Let \( s > 0 \) and let \( \{f_n\}_{n=1}^N \) be an orthonormal sequence in \( L^2(\Omega, \mu) \) such that \( \| |x|^s f_n \|_{2, \mu} \leq A_1 \) and \( \| |\xi|^s \mathcal{T} f_n \|_{2, \mu} \leq A_2 \). Then each \( f_n \) is in \( L^2 \left( \frac{1}{2}, \frac{1}{4}, B_{4^2 A_1}, B_{4^2 A_2} \right) \), and
\[
N \leq C (A_1 A_2)^2 a.
\]

**Proof.** By assumption we have, for all \( n \geq 1 \),
\[
\int_{|x| > 4^2 A_1} |f_n(x)|^2 \, d\mu(x) = \int_{|x| > 4^2 A_1} |x|^{-2s} |x|^{2s} |f_n(x)|^2 \, d\mu(x) \leq \frac{1}{16 A_1^{2s}} \| |x|^s f_n \|_{2, \mu}^2 \leq \frac{1}{16}.
\]
In the same way we obtain
\[
\int_{|\xi| > 4^2 A_2} |\mathcal{T} f_n(\xi)|^2 \, d\mu(\xi) \leq \frac{1}{16}.
\]
Thus \( f_n \in L^2 \left( \frac{1}{2}, \frac{1}{4}, B_{4^2 A_1}, B_{4^2 A_2} \right) \), and from (5.28) we conclude the desired result.
5.2.2. Quantitative dispersion inequality for orthonormal sequences. From Inequality (1.9), there exists a constant $C$ such that for all $f \in L^2(\Omega, \mu)$,
\[ \|x\|^s f \|_{2, \mu} \|\xi|^s \mathcal{T} f \|_{2, \mu} \geq C\|f\|_{2, \mu}^2, \]  
(5.30)
and the dilation argument (1.4) shows that (5.30) is equivalent to
\[ \|x\|^s f \|_{2, \mu}^2 + \|\xi|^s \mathcal{T} f \|_{2, \mu}^2 \geq 2C\|f\|_{2, \mu}^2. \]  
(5.31)
Consequently we obtain immediately the following result.

**Corollary 5.9.** Let $s > 0$ and let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal sequence in $L^2(\Omega, \mu)$. Then there exists $j_0 \in \mathbb{Z}$ such that
\[ \forall n \geq 1, \quad \max \left( \|x\|^s f_n \|_{2, \mu}, \|\xi|^s \mathcal{T} f_n \|_{2, \mu} \right) \geq 2^{s(j_0 - 1)}. \]  
(5.32)
This corollary with Corollary 5.8 allows as to prove the following quantitative dispersion inequality.

**Theorem 5.10.** Let $s > 0$ and let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal sequence in $L^2(\Omega, \mu)$. Then for every $N \geq 1$,
\[ \sum_{n=1}^{N} \left( \|x\|^s f_n \|_{2, \mu}^2 + \|\xi|^s \mathcal{T} f_n \|_{2, \mu}^2 \right) \geq C N^{1 + \frac{1}{2s}}. \]  
(5.33)

**Proof.** For each $j \in \mathbb{Z}$ we define
\[ P_j = \left\{ n : \max \left( \|x\|^s f_n \|_{2, \mu}^{\frac{1}{2}}, \|\xi|^s \mathcal{T} f_n \|_{2, \mu}^{\frac{1}{2}} \right) \in \left[ 2^{j - 1}, 2^j \right) \right\}. \]
First, by Inequality (5.32), we see that $P_j$ is empty for all $j < j_0$. Moreover, since for each $n \in P_j, j \geq j_0$,
\[ \|x\|^s f_n \|_{2, \mu}^{\frac{1}{2}} \leq 2^j \quad \text{and} \quad \|\xi|^s \mathcal{T} f_n \|_{2, \mu}^{\frac{1}{2}} \leq 2^j, \]  
(5.34)
then by Corollary 5.8, $P_j$ is finite, for all $j \geq j_0$, and if we denote by $N_j$ the number of elements in $P_j$ then
\[ N_j \leq C 4^{2a_j}. \]
Therefore, for every $m \geq j_0$, the number of elements in $\bigcup_{j=j_0}^{m} P_j$ is less than $C 4^{2am}$, where $C$ is a constant that does not depend on $m$.

Now, if $N > 2C 4^{2a_{j_0}}$, then we can choose an integer $m > j_0$ such that
\[ 2C 4^{(m-1)2a} < N \leq 2C 4^{2am}. \]
Therefore at least half of $\{1, \ldots, N\}$ does not belong to $\bigcup_{j=j_0}^{m-1} P_j$ and we obtain
\[ \sum_{n=1}^{N} \left( \|x\|^s f_n \|_{2, \mu}^2 + \|\xi|^s \mathcal{T} f_n \|_{2, \mu}^2 \right) \geq \sum_{n=1}^{N} \max \left( \|x\|^s f_n \|_{2, \mu}^2, \|\xi|^s \mathcal{T} f_n \|_{2, \mu}^2 \right) \geq \frac{N}{2} \frac{1}{4^s} \left( \frac{N}{2C} \right)^{\frac{1}{2s}}. \]
Finally, if $N \leq 2C 4^{2a_{j_0}}$, then from Corollary 5.9 we have
\[ \sum_{n=1}^{N} \left( \|x\|^s f_n \|_{2, \mu}^2 + \|\xi|^s \mathcal{T} f_n \|_{2, \mu}^2 \right) \geq \sum_{n=1}^{N} \max \left( \|x\|^s f_n \|_{2, \mu}^2, \|\xi|^s \mathcal{T} f_n \|_{2, \mu}^2 \right) \geq N 4^{s(j_0 - 1)} \geq \frac{N}{4^s} \left( \frac{N}{2C} \right)^{\frac{1}{2s}}. \]
This completes the proof. □

The last Dispersion inequality implies in particular that, there does not exist an infinite sequence \( \{f_n\}_{n=1}^{\infty} \) in \( L^2(\Omega, \mu) \) such that the two sequences \( \{\|\|x|^s f_n\|_{2, \mu}\}_{n=1}^{\infty} \) and \( \{\|\|\xi|^s \mathcal{T} f_n\|_{2, \mu}\}_{n=1}^{\infty} \) are bounded. More precisely:

**Corollary 5.11.** Let \( s > 0 \) and let \( \{f_n\}_{n=1}^{\infty} \) be an orthonormal sequence in \( L^2(\Omega, \mu) \). Then for every \( N \geq 1 \),

\[
\sup_{1 \leq n \leq N} \left\{ \|\|x|^s f_n\|_{2, \mu}^2, \|\|\xi|^s \mathcal{T} f_n\|_{2, \mu}^2 \right\} \geq C N^{\frac{1}{2s}}. \tag{5.35}
\]

In particular

\[
\sup_n \left( \|\|x|^s f_n\|_{2, \mu}^2 + \|\|\xi|^s \mathcal{T} f_n\|_{2, \mu}^2 \right) = \infty. \tag{5.36}
\]

**5.2.3. Strong uncertainty principle for orthonormal bases.** One wonders if (5.36) still valid for the product instead of the sum. We will show that this statement is not true in general for orthonormal sequences, but still valid for orthonormal bases. Indeed, there is an infinite orthonormal sequence \( \{f_n\}_{n=1}^{\infty} \) in \( L^2(\Omega, \mu) \), with bounded product of dispersions. Fix \( f : \Omega \rightarrow \mathbb{R} \) a radial, real-valued Schwartz function supported in \( B(0, 2) \setminus B(0, 1) \), with \( \|f\|_{2, \mu} = 1 \), and consider \( f_n(x) = 2^{na} f(2^n x) \). Then

\[
\|f_n\|_{2, \mu} = \|f\|_{2, \mu}, \quad \text{supp} f_n \subset B(0, 2^{-n+1}) \setminus B(0, 2^{-n}) \quad \text{and} \quad \mathcal{T} f_n(\xi) = 2^{-na} \mathcal{T}(f) (2^{-n} \xi).
\]

Therefore \( \{f_n\}_{n=1}^{\infty} \) form an orthonormal sequence in \( L^2(\Omega, \mu) \) and for every \( s > 0 \),

\[
\|\|x|^s f_n\|_{2, \mu} = 2^{-ns} \|\|x|^s f\|_{2, \mu}, \quad \|\|\xi|^s \mathcal{T} f_n\|_{2, \mu} = 2^{ns} \|\|\xi|^s \mathcal{T} f\|_{2, \mu}.
\]

Hence for all \( n \),

\[
\|\|x|^s f_n\|_{2, \mu} \|\|\xi|^s \mathcal{T} f_n\|_{2, \mu} = \|\|x|^s f\|_{2, \mu} \|\|\xi|^s \mathcal{T} f\|_{2, \mu} < \infty.
\]

To prove the main result of this subsection, we will need the following special form of the uncertainty principle for sets of finite measure, see e.g. [3, 4, 10, 11, 21].

**Lemma 5.12.** Let \( S \) and \( \Sigma \) be measurable subsets of finite measure \( 0 < \mu(S), \mu(\Sigma) < \infty \). Then there exists a nonzero function \( f \in L^2(\Omega, \mu) \) such that \( \text{supp} f \subset S^c \) and \( \text{supp} \mathcal{T} f \subset \Sigma^c \).

**Proof.** From [11, Corollary 3.7], there exist a positive constant \( C(S, \Sigma) \) such that for all functions \( f \in \text{Im}(F_{S^c}) \),

\[
\|f\|_{2, \mu} \leq C(S, \Sigma) \|E_{S^c} f\|_{2, \mu}.
\]

Therefore the trace space \( \Lambda = \{f|_{S^c} : f \in \text{Im}(F_{S^c})\} \) form a closed subspace in \( L^2(S^c, \mu) \) which is obviously not the whole space. Let \( g \) be a nonzero function in \( \Lambda^c = L^2(S^c, \mu) \setminus \Lambda \). Since \( g = F_{S^c} g + F_{\Sigma^c} g \), then \( f = F_{S^c} g \) is a nonzero function in \( L^2(\Omega, \mu) \) such that \( f \) is supported on \( S^c \) and \( \mathcal{T} f \) is supported on \( \Sigma^c \). We extend \( f \) by zero on \( S \) in order to get the required function. □

**Theorem 5.13.** Let \( s > 0 \) and let \( \{f_n\}_{n=1}^{\infty} \) be an orthonormal basis for \( L^2(\Omega, \mu) \). Then

\[
\sup_n \left( \|\|x|^s f_n\|_{2, \mu} \|\|\xi|^s \mathcal{T} f_n\|_{2, \mu} \right) = \infty. \tag{5.37}
\]

**Proof.** Assume that there exists an orthonormal basis \( \{f_n\}_{n=1}^{\infty} \) such that

\[
\|\|x|^s f_n\|_{2, \mu} \|\|\xi|^s \mathcal{T} f_n\|_{2, \mu} \leq A^2.
\]


Let $k \in \mathbb{Z}$ and let

$$A_k = \left\{ f_n : \| |x|^s f_n |^{1/s} \|_{2, \mu} \leq \left( 2^{-k} A, 2^{-k+1} A \right) \right\}.$$  

Clearly, $\{f_n\}_{n=1}^{\infty} = \bigcup_{k} A_k$, and for each $f_n \in A_k$, we have

$$\| |x|^s f_n |^{1/s} \|_{2, \mu} \leq 2^{-k+1} A \quad \text{and} \quad \| \xi^s T f_n |^{1/s} \|_{2, \mu} \leq A 2^k.$$  

Then by Corollary 5.8, $A_k$ is finite, and if $N_k$ is the number of elements in $A_k$ then $N_k$ is bounded by a constant $C$ that does not depend on $k$.

Let $R > 0$, then by using Lemma 5.12, we take a nonzero function $f \in L^2(\Omega, \mu)$ with $\| f \|_{2, \mu} = 1$, such that

$$\text{supp} f, \text{supp} T f \subset B_R^-.$$  

Then for $k \geq 0$ and $f_n \in A_k$ we obtain by the Cauchy-Schwartz inequality that

$$\left| \langle f, f_n \rangle_{\mu} \right|^2 \leq R^{-2s} \| f \|_{2, \mu}^2 \| |x|^s f_n |^{1/s} \|_{2, \mu} \leq (2AR^{-1})^{2s} 4^{-sk}. \quad (5.38)$$  

Similarly, for $k < 0$ and $f_n \in A_k$ we obtain by Parseval theorem (1.6),

$$\left| \langle f, f_n \rangle_{\mu} \right|^2 = \left| \langle T f, T f_n \rangle_{\mu} \right|^2 \leq R^{-2s} \| f \|_{2, \mu} \| \xi^s T f_n \|_{2, \mu} \leq (AR^{-1})^{2s} 4^{sk}. \quad (5.39)$$  

Now, since $\{f_n\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(\Omega, \mu)$, then

$$1 = \| f \|_{2, \mu}^2 = \sum_{k} \sum_{f_n \in A_k} \left| \langle f, f_n \rangle_{\mu} \right|^2,$$

and by combining Inequalities (5.38) and (5.39), we obtain

$$1 \leq (2AR^{-1})^{2s} \sum_{k=0}^{\infty} 4^{-sk} N_k + (AR^{-1})^{2s} \sum_{k=1}^{\infty} 4^{-sk} N_{-k} \leq C (2AR^{-1})^{2s} \sum_{k=0}^{\infty} 4^{-sk} + C (AR^{-1})^{2s} \sum_{k=1}^{\infty} 4^{-sk} \leq \frac{C}{R^{2s}}.$$  

Choosing $R$ large enough, we get a contradiction. The theorem is proved. \qed

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