STABILITY OF GEODESIC SPHERES IN $\mathbb{S}^{n+1}$ UNDER CONSTRAINED CURVATURE FLOWS

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ABSTRACT. In this paper we discuss the stability of geodesic spheres in $\mathbb{S}^{n+1}$ under constrained curvature flows. We prove that under some standard assumptions on the speed and weight functions, the spheres are stable under perturbations that preserve a volume type quantity. This extends results of [3] and [7] to a Riemannian manifold setting.

1. Introduction

We consider a family of hypersurfaces that are compact without boundary, $\{\Omega_t\}_{t \in [0, T]}$, inside a Riemannian manifold $(N^{n+1}, \bar{g})$ moving with a speed function $\hat{G}$ in the direction of the normal. If $\Omega_0$ is given by an embedding $\tilde{X}_0 : M^n \to N^{n+1}$ then the family is obtained by solving for an $X : M^n \times [0, T) \to N^{n+1}$ that satisfies

$$\frac{\partial X}{\partial t} = \hat{G}(X)\nu, \quad X(\cdot, 0) = \tilde{X}_0$$

where $\nu$ is the outer unit normal to $\Omega_t$, with $\Omega_t = X(M^n, t)$. We will consider speed functions, $\hat{G}$, of the form:

$$\hat{G}(X) = \int_{M^n} F(\kappa) \hat{\Xi}(X) d\mu - \int_{M^n} \hat{\Xi}(X) d\mu - F(\kappa),$$

where $d\mu$ and $\kappa = (\kappa_1, \ldots, \kappa_n)$ are, respectively, the induced volume form and principal curvatures (i.e. eigenvalues of the Weingarten map $W$) on $X(\Omega)$, $\hat{\Xi}$ is a weight function, and $F$ is a smooth, symmetric function on $\mathbb{R}^n$ satisfying $\frac{\partial F}{\partial \kappa} (\kappa(\tilde{X}_0)) > 0$.

We will consider a fairly general form of the weight function $\hat{\Xi}$, however of special interest are the cases when

$$\hat{\Xi}(X) = \sum_{a=0}^{n+1} c_a \hat{\Xi}_a(X)$$

for some $\{c_0, \ldots, c_{n+1}\} \in \mathbb{R}^{n+2}$, where

$$\hat{\Xi}_a(X) := \begin{cases} \frac{\partial^a}{\partial \kappa^a} [\bar{g}(\bar{R}(\nu, T_k)\nu, T_j) + \nabla_j \nabla_k (\frac{\partial E_{n+1-a}}{\partial \kappa^a})] + (n+1-a)^{-1} E_{n+1-a} & \text{if } a = 0, \ldots, n, \\ 1 & \text{if } a = n+1, \end{cases}$$

$g$ is the induced metric on $X(M^n)$, $\bar{R}(U,W)Z = \bar{\nabla}_W \bar{\nabla}_U Z - \bar{\nabla}_U \bar{\nabla}_W Z + \bar{\nabla}_{[U,W]} Z$ is the Riemann curvature tensor of $(N^{n+1}, \bar{g})$, $\bar{\nabla}$ and $\nabla$ are the Levi-Civita connections on $(N^{n+1}, \bar{g})$ and induced on $(X(M^n), g)$ respectively, and

$$E_a := \sum_{1 \leq h_1 < \ldots < h_a \leq n} \prod_{i=1}^{a} \kappa_{h_i}.$$
are the elementary symmetric functions of the Weingarten map. Note that for hypersurfaces in Euclidean space
\( g^{ik} \nabla_k \left( \frac{\partial E_n}{\partial h^i} \right) = 0 \); see the proof of Lemma 2.2.2. in [6]. With \( \hat{\Xi} \) as in (3), the flow (1) preserves the real valued quantity

\[
\hat{V}(\Omega) := \sum_{a=0}^{n+1} c_a \hat{V}_a(\Omega),
\]

where \( \hat{V}_a \) are the mixed volumes

\[
\hat{V}_a(\Omega) := \begin{cases} \frac{1}{(a+1)!} \int_{M^p} E_{n-a} \, d\mu & \text{if } a = 0, \ldots, n, \\ \operatorname{Vol}(\Omega) & \text{if } a = n + 1. \end{cases}
\]

The topic of intrinsic volumes is more complicated in spherical space than Euclidean space. For example, in [4] they consider three different definitions, however each is a linear combination of the mixed volumes defined above and hence can be preserved by choosing the \( c_a \) constants appropriately. See Appendix [X] for a proof that \( \hat{V} \) is preserved under the flow when \( \hat{\Xi} \) is given by (3).

This flow in Euclidean space, and with a weight function such that a mixed volume is preserved, has been studied previously by McCoy in [11]. There it was proved that under some additional conditions on \( F \), for example homogeneity of degree one and convexity or concavity, initially convex hypersurfaces admit a solution for all time and that the hypersurfaces converge to a sphere as \( t \to \infty \). This was an extension of a result by Huisken [9] who proved the result for the volume preserving mean curvature flow (VPMCF). The stability of spheres has previously been considered by Escher and Simonett in [3] for the case of the VPMCF in Euclidean space where it was proved that they were stable under small perturbations in the little Hölder space \( h^{1,\alpha} \), any \( \alpha \in (0, 1) \). This result was extended by the author, [8], to the case of mixed-volume preserving curvature flows, with the perturbations in the space \( h^{2,\alpha} \) to account for the fully nonlinear nature of the flows.

For flows of this nature in Riemannian manifolds Huisken noted in [9] that even the VPMCF in \( S^{n+1} \) will, in general, not preserve the convexity of a hypersurface, thus making the standard analysis more difficult. In [11] Alikakos and Freire prove that if the manifold \( N^{n+1} \) has a finite number of critical points of the scalar curvature that are all non-degenerate, then the geodesic spheres close to the critical points are stable under volume preserving perturbations. In the case when \( N^{n+1} \) is hyperbolic space, Cabezas-Rivas and Vicente in [2] prove that the VPMCF of hypersurfaces that satisfy a certain convexity property exist for all time and converge to geodesic spheres. They also prove that geodesic spheres in these manifolds are stable with respect to the VPMCF under \( h^{1,\alpha} \) perturbations.

In this paper we consider the stability of geodesic spheres in \( S^{n+1} \) under the flow (1).

The main theorem is:

**Theorem 1.1.** A geodesic sphere \( S \subset S^{n+1} \) is stable under perturbations in \( h^{2,\alpha} \), for any \( \alpha \in (0, 1) \), with respect to the flow (1), with \( \hat{G} \) as in (2), if the following hold:

- \( F \) is a smooth, symmetric function of the principal curvatures,
- \( \frac{\partial F}{\partial \kappa_i}(\kappa(S)) > 0 \), and
- \( \hat{\Xi}(S) = \text{const} \neq 0. \)

To be precise let \( \Omega_0 \) be a \( h^{2,\alpha} \)-close normal geodesic graph over \( S \), then the flow by (1) exists for all time and the hypersurfaces \( \Omega_t := X(M^p, t) \) converge in \( h^{2,\alpha} \) to a geodesic sphere close to \( S \).
Note that the little H"older spaces on a manifold, $h^{k,\alpha}(M^n)$ for $k \in \mathbb{N}_0$ and $\alpha \in (0,1)$, are defined as the completion of the smooth functions inside the standard H"older space $C^{k,\alpha}(M^n)$. They are useful in analysing stability properties as they obey a self interpolation property, \cite[Equation 19]{5}.

The paper is structured as follows. In Section \ref{normalgraphs} we consider the properties of hypersurfaces that are normal geodesic graphs over a base hypersurface. We also give an equation on the space of graph functions that is equivalent to \cite{1}. Section \ref{derivative} derives the linearisation of the speed function in the case where the hypersurfaces are graphs over a geodesic sphere in $\mathbb{S}^{n+1}$. The space of functions that define geodesic spheres close to the base sphere is then analysed in Section \ref{spaces} and finally Theorem \ref{mainresult} is proved in Section \ref{mainresult}. In Appendix \ref{appendix} we derive the formula for the weight function so that the quantity $\hat{V}$ is preserved under the flow.

2. Normal Geodesic Graphs and Equivalent Equations

We will consider the situation where the hypersurfaces are normal (geodesic) graphs over a base hypersurface $X_0(M^n)$ with outer unit normal $v_0$. These hypersurfaces can be written as $X_\alpha(p) = \gamma(p, u(p))$ where $u : M^n \rightarrow \mathbb{R}$, and $\gamma_p(s) := \gamma(p, s)$ is the unique unit speed geodesic satisfying $\gamma_p(0) = X_0(p)$ and $\dot{\gamma}_p(0) = v_0(p)$, we use a dot to denote a derivative with respect to the geodesic parameter, $s$. Note that we require $|u|_{C^0} < C_0$, where $C_0$ is the injectivity radius of $X_0(M^n) \subset N^{n+1}$.

Standard calculations give us the following formulas

\textbf{Lemma 2.1.} The tangent vectors for a normal graph $X_\alpha(M^n)$ are given by
\[ T_i(u) = \frac{\partial \gamma}{\partial p^i} + \nabla_i \mu \gamma \bigg|_{s=0}, \]
the induced metric components are given by
\[ g_{ij}(u) = \tilde{g} \left( \frac{\partial \gamma}{\partial p^i}, \frac{\partial \gamma}{\partial p^j} \right) + \nabla_i \nabla_j \mu, \]
and the unit normal by
\[ v(u) = \sqrt{1 - |\nabla u|^2_{g(u)}} \gamma - \frac{g^{ij}(u) \nabla_i \mu \frac{\partial \gamma}{\partial p^j}}{\sqrt{1 - |\nabla u|^2_{g(u)}}}, \]
where $g^{ij}(u)$ are the components of the inverse of $g(u)$, also note that $|\nabla u|_{g(u)} = \sqrt{g^{ij}(u) \nabla_i \mu \nabla_j \mu} < 1$.

\textbf{Proof.} The formula for the tangent inverse follows directly from $X_\alpha(p) = \gamma(p, u(p))$. The metric formula then follows from using the unit speed condition $\tilde{g}(\dot{\gamma}, \dot{\gamma}) = 1$ and the formula $\tilde{g} \left( \frac{\partial \gamma}{\partial p^i}, \dot{\gamma} \right) = 0$. In fact,
\[ \frac{\partial}{\partial s} \left( \tilde{g} \left( \frac{\partial \gamma}{\partial p^i}, \dot{\gamma} \right) \right) = \tilde{g} \left( \nabla_{\dot{\gamma}} \gamma, \dot{\gamma} \right) + \tilde{g} \left( \frac{\partial \gamma}{\partial p^j}, \nabla_j \dot{\gamma} \right) = 0, \]
where $\nabla_U V = V(U) + \Gamma^\nu_{\beta\gamma} V^\beta U^\gamma \partial_\alpha$ is the Levi-Civita connection on $(N^{n+1}, \tilde{g})$, and we have used the space derivative of the unit speed condition and the geodesic condition. Hence $\tilde{g} \left( \frac{\partial \gamma}{\partial p^i}, \dot{\gamma} \right) \bigg|_{s=0} = \tilde{g} \left( \frac{\partial \gamma}{\partial p^i}, v_0 \right) = 0$, so
\[ g_{ij}(u) = \tilde{g} \left( T_i(u), T_j(u) \right) = \tilde{g} \left( \frac{\partial \gamma}{\partial p^i}, \frac{\partial \gamma}{\partial p^j} \right) + \nabla_i \nabla_j \mu \tilde{g}(\dot{\gamma}, \dot{\gamma}) \bigg|_{s=0}. \]
The formula for the unit normal can be seen by taking its inner product with the tangent vectors and using that $\hat{g}\left(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^j}\right)|_{\varphi=\varphi_0} = g_{ij}(u) - \nabla_u \nabla_j u$. □

We now aim to show that the equation (1) is equivalent to an equation on $C^2(M^n)$. We define $L(u) := \hat{g}\left(\hat{\nu}\right)_{\varphi=\varphi_0} = \frac{1}{\sqrt{1 - |\nabla u|^2}}$ and $G(u) := L(u)\hat{G}(X_u)$, and consider the flow

$$\frac{\partial u}{\partial t} = G(u), \quad u(0) = u_0,$$

then we have

$$\frac{\partial X_u}{\partial t} = \frac{\partial u}{\partial t} \left(\hat{\nu}\right)_{\varphi=\varphi_0} = G(u)\left(\hat{\nu}\right)_{\varphi=\varphi_0},$$

and in particular

$$\left(\frac{\partial X_u}{\partial t}, \nu(u)\right) = \hat{G}(X_u).$$

Therefore there is a tangential diffeomorphism $\phi_t : M^n \to M^n$, with $\phi_0 = id$, such that $X_u(\phi_t(\varphi), t)$ satisfies (1), with initial embedding $X_u$. Likewise if $X$ satisfies (1) and $X = X_u$ for some $u : M^n \times [0, T) \to \mathbb{R}$ then from $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t} \left(\hat{\nu}\right)_{\varphi=\varphi_0}$ we obtain, via the inner product with $\nu(u)$, that $\hat{G}(X_u) = L(u)^{-1}\frac{\partial u}{\partial t}$ and hence $u$ satisfies (6). Therefore equations (6) and (1) are equivalent.

Note that because (6) is equivalent to (1), up to a tangential diffeomorphism, when $\hat{S}$ is given by (3) we have that $V(u) := \hat{V}(X_u(M^n))$ is also a preserved quantity for (6). This can also be seen by using equation (17) in Appendix A to calculate the linearisation of $V(u)$:

$$DV(u)[w] = D\hat{V}(X_u)[DX_u[w]]$$

$$= D\hat{V}(X_u)[\hat{\nu}]_{\varphi=\varphi_0} w$$

$$= \int_{M^n} \hat{g}(X_u)\hat{g}(\hat{\nu}w, \nu(u)) d\mu_u$$

$$= \int_{M^n} \hat{g}(X_u)L(u)^{-1}w d\mu_u.$$  \hspace{1cm} (7)

Now by setting $w = \frac{\partial u}{\partial t} = L(u)\hat{G}(X_u)$ and using the form of $\hat{G}$ in (2) we obtain $\frac{\partial X_u}{\partial t} = DV(u)[\frac{\partial u}{\partial t}] = 0$.

3. LINEARISATION ABOUT GEODESIC SPHERES IN $\mathbb{S}^{n+1}$

We start by giving some standard linearisation formulas, and we suppress that quantities are to be evaluated at $X$.

**Lemma 3.1.** The components of the metric of $X(M^n)$, $g_{ij}(X) := \hat{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$, have the linearisation

$$Dg_{ij}(X)[Y] = \hat{g}\left(\hat{\nabla}_{T_i} Y, T_j\right) + \hat{g}\left(\hat{\nabla}_{T_j} Y, T_i\right),$$

where $T_i(X) := \frac{\partial}{\partial x^i}$, and $\hat{\nabla}_{T_i} Y = \frac{\partial Y}{\partial x^i} + \hat{\Gamma}^{\alpha}_{ij} Y^\alpha \hat{\partial}_i$. The volume element $\mu(X) := \sqrt{\det(g(X))}$ has the linearisation

$$D\mu(X)[Y] = g^{ij}\hat{g}\left(\hat{\nabla}_{T_i} Y, T_j\right)\mu,$$

the unit normal of $X(M^n)$ satisfies

$$D\nu(X)[Y] + \hat{\Gamma}^\alpha_{ij} Y^\beta \hat{\partial}_i \nu = -g^{ij}\hat{g}\left(\hat{\nabla}_{T_i} Y, \nu\right) T_j,$$
the linearisation of the components of the second fundamental form, \( h_{ij}(X) := \tilde{g} \left( \nabla_{\tau_i}(X), T_j(X) \right) \) is
\[
Dh_{ij}(X)[Y] = -\tilde{g} \left( \nabla_{\tau_i}(\tilde{g} Y, T_j), \nabla_{\tau_j}(X) \right) - \tilde{g} \left( \nabla_{\nabla_{\tau_i}Y}T_j, T_j \right)
\]
and finally the linearisation of the elements of the Weingarten map, \( h^i_j(X) = g^{ik}(X)h_{kj}(X) \), are given by
\[
Dh^i_j(X)[Y] = -g^{ik} \left( h_j^l \left( \tilde{g} \left( \nabla_{\tau_k}Y, T_l \right) + \tilde{g} \left( \nabla_{\nabla_{\tau_k}Y}T_l \right) \right) + \tilde{g} \left( \nabla_{\tau_j}T_l, T_l \right) \right).
\]

**Proof.** The variations of the metric and volume element are straight from the definitions. By taking the linearisations of the formulas
\[
\tilde{g}(v, T_i) = 0, \text{ and } \tilde{g}(v, v) = 1,
\]
to obtain
\[
\tilde{g} \left( Dv(X)[Y] + \Gamma^a_{\phi\psi}(\phi^\rho Y^\sigma) \frac{\partial}{\partial a}, T_j \right) + \tilde{g}(v, \nabla_{T_i}Y) = 0, \text{ and } 2\tilde{g} \left( Dv(X)[Y] + \Gamma^a_{\phi\psi}(\phi^\rho Y^\sigma) \frac{\partial}{\partial a}, v \right) = 0,
\]
we are able to conclude the formula for the linearisation of the unit normal.

Now we consider the linearisation of the second fundamental form:
\[
Dh_{ij}(X)[Y] = \tilde{g} \left( D \left( \nabla_{\tau_i}Y \right)(X)[Y] + \Gamma^a_{\phi\psi}(\phi^\rho Y^\sigma) \frac{\partial}{\partial a}, T_j \right) + \tilde{g}(\nabla_{\tau_i}Y, \nabla_{\tau_j}Y)
\]
\[
= \tilde{g} \left( \nabla_{\tau_i} \left( Dv(X)[Y] + \Gamma^a_{\phi\psi}(\phi^\rho Y^\sigma) \frac{\partial}{\partial a} \right) + \tilde{g}(T_i, Y) \right) + \tilde{g} \left( \nabla_{\tau_i}Y, \nabla_{\tau_j}Y \right)
\]
\[
= \tilde{g} \left( \nabla_{\tau_i} \left( -g^{ik} \tilde{g} \left( \nabla_{\tau_k}Y, T_i \right) \right) \right) + \tilde{g} \left( \nabla_{\tau_i}Y, \nabla_{\tau_j}Y \right)
\]
\[
= -\tilde{g} \left( \nabla_{\tau_i}Y, \nabla_{\tau_j}Y \right) - \tilde{g} \left( \nabla_{\tau_i}Y, v \right) + \tilde{g} \left( \nabla_{\tau_j}Y, T_i \right).
\]

Finally we use that \( h^i_j = g^{ik}h_{kj} \) and that \( Dg^{ik}(X)[Y] = -g^{ip}g^{kq}Dg_{pq}(X)[Y] \) to obtain the linearisation of the Weingarten map components
\[
Dh^i_j(X)[Y] = -g^{ip}g^{kq} \left( \tilde{g} \left( \nabla_{\tau_k}Y, T_q \right) + \tilde{g} \left( \nabla_{\nabla_{\tau_k}Y}T_q \right) \right) h_{ij} - g^{ik} \left( \tilde{g} \left( \nabla_{\tau_k}Y, v \right) + \tilde{g} \left( \nabla_{\tau_k}Y, T_j \right) \right)
\]
\[
= -g^{ip} \left( h_j^l \left( \tilde{g} \left( \nabla_{\tau_k}Y, T_l \right) + \tilde{g} \left( \nabla_{\nabla_{\tau_k}Y}T_l \right) \right) + \tilde{g} \left( \nabla_{\tau_k}Y, T_j \right) \right) + \tilde{g} \left( \nabla_{\tau_k}Y, T_j \right).
\]

\[\square\]

**Lemma 3.2.**
\[
D\hat{G}(X)[Y] = \frac{\int_M \sum_{a=1}^n \frac{\partial F}{\partial \kappa_a}(\kappa(X))D\kappa_a(X)[Y] \hat{\Xi}(X) - \frac{\partial F}{\partial \kappa a}(\hat{\Xi})D(\hat{\Xi})}{\int_M \hat{\Xi}(X) d\mu} - \sum_{a=1}^n \frac{\partial F}{\partial \kappa_a}(\kappa(X))D\kappa_a(X)[Y],
\]
\[
D\kappa_a(X)[w_0 w_0] = -\tilde{\zeta}_a^i \tilde{\zeta}_a^j \left( \nabla_{\hat{T}_i} \nabla_{\hat{T}_j} w + \tilde{g} \left( \nabla_{\hat{T}_i} \nabla_{\hat{T}_j} w, \hat{T}_j \right) \right) - \tilde{\zeta}_a^w w,
\]
where \( \hat{T}_i \) are tangent vectors, \( \hat{\nabla} \) is the Levi-Civita connection, and \( \tilde{\zeta}_a \) is the principle direction (eigenvector of the Weingarten map \( W \)) corresponding to the principle curvature \( \hat{k} \) of \( X_0(M^n) \). In particular if \( X_0(M^n) \) is totally umbilic we have
\[
\sum_{a=1}^n \frac{\partial F}{\partial \kappa_a}(\kappa(X))D\kappa_a(X)[w_0 w_0] = -\frac{\partial F}{\partial \kappa_1}(\kappa) \left( \Delta_w w + [\hat{W}^i_0 w + \hat{Ric}(w_0, w_0)w \right)
\]
where \( \hat{\Xi} \) is the metric of \( X_0(M^n) \).

**Proof.** The first formula follows directly from the definition of \( \hat{G}(X) \), while the second formula follows from \( D\kappa_a(X)[Y] = \zeta^i(X)\zeta^j(X)g_{ij}Dh^i_j(X)[Y] \) and Lemma 3.1 \[\square\]
Now we consider $N^{n+1} = \mathbb{S}^{n+1}$ with coordinates $q_\alpha$, $\alpha = 1, \ldots, n+1$, with $q_1 \in [0, 2\pi)$ and $q_\alpha \in [0, \pi)$ for $\alpha = 2, \ldots, n+1$, such that the metric is
\[
\tilde{g} = g_{\mathbb{S}^{n+1}} = \sum_{\alpha=1}^{n+1} \prod_{\beta=\alpha+1}^{n+1} \sin(q_\beta)^2 \, dq^{\alpha 2},
\]
and $X_0(M^n)$ equal to the $n$-sphere $q_{n+1} = \theta$, for some fixed $\theta \in (0, \pi)$, which we denote $S_\theta$. With this set up normal graphs take the form
\[
(8) \quad X_\alpha(p) = (p_1, \ldots, p_n, \theta + u(p)),
\]
where $p = (p_1, \ldots, p_n) \in \mathbb{S}^n$ and $C_0 = \min(\theta, \pi - \theta)$. The tangent vectors, metric, and Weingarten map for $X_0(M^n)$ are then
\[
\hat{T}_i = \delta^\alpha_i \frac{\partial}{\partial q^{\alpha}},
\]
\[
\hat{g}_{ij} = \sin(\theta)^2 g_{\mathbb{S}^n} = \sin(\theta)^2 \sum_{\alpha=1}^{n} \prod_{\beta=\alpha+1}^{n} \sin(p_\beta)^2 \, dp^{\alpha 2},
\]
\[
\hat{\mathcal{W}} = \cot(\theta) Id.
\]

Lemma 3.3.
\[
\Xi_a(X_0) = \left\{ \begin{array}{cl}
\frac{\cot(\theta)^2 - n+a}{1} & \text{if } a = 0, \ldots, n, \\
0 & \text{if } a = n+1.
\end{array} \right.
\]

Proof. This follows from a straightforward calculation using (3). Firstly we note that since $\hat{\mathcal{H}}_j h_j^i = 0$ we have
\[
\hat{\mathcal{H}}_j \hat{\mathcal{H}}^i_k \left( \frac{\partial E_{n-a}(k)}{\partial h_j^{i}} \right) = \hat{\mathcal{H}}_j \left( \frac{\partial^2 E_{n-a}(k) \hat{\mathcal{H}}_j^i}{\partial h_j^{i} \partial h_j^{i}} \right) = 0.
\]
Next we use that $\hat{g} \left( \hat{\mathcal{R}}(v_0, \hat{T}_k)v_0, \hat{T}_j \right) = \hat{g}_{k,j}, E_a(k) = (\binom{n}{a}) \cot(\theta)^a$, and the formula (see Proposition B.0.2. in [8] for example)
\[
(9) \quad \frac{\partial E_a}{\partial h_j^i}(k) = \sum_{b=0}^{a-1} (-1)^b \binom{n}{a-b} E_{a-b}(k),
\]
to calculate the remaining terms:
\[
\Xi_a(X_0) = \frac{-\hat{g}_{ab}}{(n+1)(a+n)} \frac{\partial E_{n-a}}{\partial h_j^i}(k) \hat{g}_{kj} + \frac{E_{n-a+1}(k)}{(n+1)(a+n)}
\]
\[
= \frac{-\delta_j^i}{(n+1)(a+n)} \sum_{b=0}^{n-a-1} (-1)^b \binom{n}{a-b} E_{n-a-b}(k) + \frac{(n+1)(a+n) \cot(\theta)^{n+1-a}}{(n+1)(a+n)}
\]
\[
= \frac{-1}{(n+1)(a+n)} \sum_{b=0}^{n-a-1} (-1)^b \binom{n}{a-b} E_{n-a-b}(k) + \frac{a}{(n+1)(a+n)} \cot(\theta)^{a+1-n} + \frac{a}{(n+1)(a+n)} \cot(\theta)^{a+1-n}
\]
\[
= \frac{-n}{(n+1)(a+n)} \sum_{b=0}^{n-a-1} (-1)^b \binom{n}{a-b} E_{n-a-b}(k) + \frac{a}{(n+1)(a+n)} \cot(\theta)^{a+1-n} + \frac{a}{(n+1)(a+n)} \cot(\theta)^{a+1-n}
\]
\[
= \frac{a-n}{n+1} \cot(\theta)^{a+1-n} + \frac{a}{n+1} \cot(\theta)^{a+1-n}.
\]
\[\square\]
Lemma 3.5.

- For our purposes the important thing here is that each $\Xi_a(X_0)$ is a constant and, hence, the form $\Xi$ will be allowable in our theorem, provided $c_a \in \mathbb{R}, a = 0, \ldots, n + 1$, are such that $\Xi(X_0) \neq 0$.
- It should be noted that we have $\Xi_a(X_0) = 0$ if and only if $\theta \in \Theta(a, \pi, \pi - \theta_a)$, where $\theta_a = \arcsin \left( \frac{\sqrt{n}}{n} \right)$, except in the cases of $a = n - 1$ when it is if and only if $\theta \in \Theta(a, \pi - \theta_a)$ and $a = n + 1$ when it is never zero.
- As suggested in [4] a better intrinsic volume to use in spherical space may be

$$
\hat{U}_a(\Omega) = \frac{\Gamma(n + 2)}{2^{n+2}} \sum_{b=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{V_{a+2b}(\Omega)}{\Gamma \left( \frac{a+2b}{2} + 1 \right) \Gamma \left( \frac{a-2b}{2} + 1 \right)}.
$$

where $\Gamma$ is the usual Gamma function, for which the linearisation is

$$
D\hat{U}_a(X)[Y] = \int_{\Omega} \hat{Z}_a(X) \delta(Y, \nu) \, d\mu,
$$

where

$$
\hat{Z}_a(X) = \frac{\Gamma(n + 2)}{2^{n+2}} \sum_{b=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{\hat{\Xi}_{a+2b}(X)}{\Gamma \left( \frac{a+2b}{2} + 1 \right) \Gamma \left( \frac{a-2b}{2} + 1 \right)}.
$$

In this case the linearisation functions at $X_0$ have the simpler form

$$
\hat{Z}_a(X_0) = \frac{a\Gamma(n + 1) \cos(\theta)^{a+1-a}}{2^{n+2} \pi^2 \Gamma \left( \frac{n}{2} + 1 \right) \Gamma \left( \frac{n-a}{2} + 1 \right)}
$$

and is zero if and only if $\theta = \frac{\pi}{2}$ and $a \neq n + 1$.

Lemma 3.5. We assume that $\hat{\Xi}(X_0) = \text{const} \neq 0$, then

$$
DG(0)[w] = \frac{\partial F}{\partial k_1}(k) \sin(\theta)^{-2} \left( \Delta_{g^w} w + nw - \int_{\mathbb{S}^n} w \, d\mu_0 \right)
$$

Proof. To calculate $DG(0)$ we first note that $DL(0)[w] = 0$ and $DX_a|_{w=0}[w] = wv_0$, so that

$$
DG(0)[w] = DG(X_0)[wv_0]
$$

$$
= \int_{\mathbb{S}^n} \sum_{a=1}^{n} \frac{\partial F}{\partial k_1}(k) \Delta_{g^w} D(\hat{\Xi}_a)(X_0)[wv_0] \, d\mu_0 - \frac{\partial F}{\partial k_1}(k) \Delta_{g^w} \hat{\Xi}(X_0) \, d\mu_0 - \sum_{a=1}^{n} \frac{\partial F}{\partial k_1}(k) \Delta_{g^w} \hat{\Xi}_a(X_0)[wv_0]
$$

$$
= \frac{\partial F}{\partial k_1}(k) \left( \Delta_{g^w} + |W^2_\delta w + \hat{Ric}(v_0, v_0)| w - \int_{\mathbb{S}^n} \frac{\partial F}{\partial k_1}(k) \left( \Delta_{g^w} + |W^2_\delta w + \hat{Ric}(v_0, v_0)| w \right) \, d\mu_0 \right)
$$

$$
= \frac{\partial F}{\partial k_1}(k) \left( \Delta_{g^w} + |W^2_\delta w + \hat{Ric}(v_0, v_0)| w - \int_{\mathbb{S}^n} \Delta_{g^w} + |W^2_\delta w + \hat{Ric}(v_0, v_0)| w \, d\mu_0 \right)
$$

$$
= \frac{\partial F}{\partial k_1}(k) \left( \sin(\theta)^{-2} \Delta_{g^w} w + nw - \int_{\mathbb{S}^n} \frac{\partial F}{\partial k_1}(k) \sin(\theta)^{-2} \left( \Delta_{g^w} w + nw \right) \, d\mu_0 \right)
$$

$\square$

Corollary 3.6. Let $c_a \in \mathbb{R}, a = 0, \ldots, n + 1$, be such that $\hat{\Xi}(X_0) = \text{const} \neq 0$, and the smooth, symmetric function $F$ be such that $\frac{\partial F}{\partial k_1}(k) > 0$, then

$$
\sup \{ \lambda : \lambda \in \sigma(DG(0)) \neq \{0\} \} < 0,
$$
and 0 is an eigenvalue of $DG(0)$ with multiplicity $n + 2$ and eigenfunctions given by the first order spherical harmonics on $S^n$, $Y_1^{(n)}, \ldots, Y_{n+1}^{(n)}$, and the constant function.

**Remark 3.7.** Our coordinates give the following formula for the spherical harmonics

$$Y_{i}^{(n)}(p) = \prod_{j\neq i}^{n} \sin(p_j) \cos(p_{i-1}).$$

4. **Space of Geodesic Spheres in $S^{n+1}$**

In this section we consider how to parameterise the space of geodesic spheres as normal geodesic graphs over a particular sphere. That is we look to find the space of functions such that $X_u$ is a sphere near $X_0$ for any $u$ in the space, and that all spheres near $X_0$ are accounted for. To find the parametrisation we embed $S^{n+1}$ in $\mathbb{R}^{n+2}$ using the first order spherical harmonics on $S^{n+1}$:

$$Z(q) = \left( Y_1^{(n+1)}(q), \ldots, Y_{n+2}^{(n+1)}(q) \right).$$

The image of $X_0$ lies in the plane $x_{n+2} = \cos(\theta)$, so any non-perpendicular sphere is determined by the values $b_i$, $i = 1, \ldots, n + 2$, such that it lies in the plane

$$x_{n+2} + \sum_{i=1}^{n+1} b_i x_i = \sqrt{1 + |b|^2} (\cos(\theta) + b_{n+2}),$$

for $(b_1, \ldots, b_{n+2}) \in \mathbb{R}^{n+1} \times (-1 - \cos(\theta), 1 - \cos(\theta))$, where $|b| = \left( \sum_{i=1}^{n+1} b_i^2 \right)^{\frac{1}{2}}$. The radius of the sphere is given by $\bar{R} = \bar{R}(b_{n+2}) = \sqrt{\sin(\theta)^2 - 2 \cos(\theta) b_{n+2} - b_{n+2}^2}$. Using the form of the normal graph $\mathcal{K}$ we find that $u$ satisfies

$$\cos(\theta + u(p)) + \sin(\theta + u(p)) \sum_{i=1}^{n+1} b_i \prod_{j=1}^{n} \sin(p_j) \cos(p_{i-1}) = \sqrt{1 + |b|^2} (\cos(\theta) + b_{n+2}),$$

so by using the formula for the spherical harmonics on $S^n$ and solving for $u$ we obtain that the graph functions for the spheres are given by

$$u_b = \arctan \left( \frac{\sum_{i=1}^{n+1} b_i Y_i^{(n)} + \sqrt{1 + |b|^2} (\cos(\theta) + b_{n+2}) \sqrt{1 + \left( \sum_{i=1}^{n+1} b_i Y_i^{(n)} \right)^2 - (1 + |b|^2) \cos(\theta) + b_{n+2})^2}}{(1 + |b|^2) \cos(\theta) + b_{n+2})^2 - \left( \sum_{i=1}^{n+1} b_i Y_i^{(n)} \right)^2} \right) - \theta,$$

where $\arctan$ is defined such that $\arctan: \mathbb{R} \to [0, \pi)$ and we require $|b|^2 < \frac{1}{(\cos(\theta) + b_{n+2})^2} - 1$. This requirement means that the geodesic spheres considered divide the poles $q_{n+1} = 0$ and $q_{n+1} = \pi$.

Note $u_0 = 0$ so this gives the base geodesic sphere, and if we linearise, with respect to the parameters, at the base sphere we have

$$Du_{b|_{b=0}}[z] = \sum_{j=1}^{n+1} z_j Y_j^{(n)} + \frac{1}{\sin(\theta)} z_{n+2}.$$

We have thus proved the following
Lemma 4.1. The space of graph functions defining a sphere non-perpendicular to $X_0(M^n)$ and separating the poles is given by

$$\mathcal{S} := \{ u_b \in C^0(\mathbb{S}^n) : b \in \mathbb{R}^{n+1} \times (-1 - \cos(\theta), 1 - \cos(\theta)), |b|^2 = \frac{1}{(\cos(\theta) + b_{n+2})^2} - 1 \}$$

where $u_b$ is defined as in (10). Further, at $b = 0$ it has the tangent space $T_0 \mathcal{S} = \text{span} \{(Y_i^{(n)} \in C^0(\mathbb{S}^n) : i = 1, \ldots, n + 1) \cup \{1\}\}$ and $\mathcal{S}$ is locally a differentiable graph over it.

5. Proof of Main Theorem

The proof of Theorem 1 follows precisely as in [2] since $DG(0)$ is a positive multiple of the linear operator in that paper, see Lemma 3.1. of [2] for its definition, and the stationary solutions are again a graph over $\text{Null}(DG(0))$. Firstly, since $DG(0)[w] + \frac{\partial F}{\partial \nu}(k) \sin(\theta)^{-2} \int_{\mathbb{S}^n} w \, dm_0$ is the negative of an elliptic operator, it is sectorial as a map from $h^{2,\alpha}(\mathbb{S}^n)$ to $h^{0,\alpha}(\mathbb{S}^n)$ for any $\alpha \in (0, 1)$. Next, we use that $\frac{\partial F}{\partial \nu}(k) \sin(\theta)^{-2} \int_{\mathbb{S}^n} w \, dm_0$ is a bounded linear map from $h^{2,\alpha}(\mathbb{S}^n)$ to $h^{2,\alpha}(\mathbb{S}^n)$ to conclude that $DG(0) : h^{2,\alpha}(\mathbb{S}^n) \to h^{0,\alpha}(\mathbb{S}^n)$ is sectorial for any $\alpha \in (0, 1)$. Now, as being sectorial is a stable condition, this implies that $DG(w) : h^{2,\alpha}(\mathbb{S}^n) \to h^{0,\alpha}(\mathbb{S}^n)$ is also sectorial for any $w$ in a neighbourhood $0 \in O_0 \subset h^{2,\alpha}(\mathbb{S}^n)$ and $\alpha \in (0, 1)$. Short-time existence for (6) then follows directly from Theorem 8.4.1 in [10].

Theorem 5.1. For any $\alpha \in (0, 1)$ there are constants $\delta, r > 0$ such that if $\|u_0\|_{h^{2,\alpha}(\mathbb{S}^n)} \leq r$ then equation (6) has a unique maximal solution:

$$u \in C^0([0, \delta), h^{2,\alpha}(\mathbb{S}^n)) \cap C^1([0, \delta), h^{0,\alpha}(\mathbb{S}^n)).$$

Next we see the existence of a center manifold which attracts solutions, moreover, locally this is our space of stationary solution $\mathcal{S}$. Let $P$ be the spectral projection from $h^{2,\alpha}(\mathbb{S}^n)$ onto $T_{0,\mathcal{S}}$ associated with $DG(0)$, $\lambda_1$ be the first non-zero eigenvalue of $DG(0)$, and $\psi : U \subset T_{0,\mathcal{S}} \to (I - P)[h^{2,\alpha}(\mathbb{S}^n)]$ be the local graph function for $\mathcal{S}$.

Lemma 5.2. The space $\mathcal{S}$ is a local, invariant, exponentially attractive, center manifold for (6). In particular, there exists $r_1, r_2 > 0$ such that if $\|u_0\|_{h^{2,\alpha}(\mathbb{S}^n)} < r_2$ then there exists $z_0 \in T_{0,\mathcal{S}}$ such that

$$\|P[u(t)] - z(t)\|_{h^{2,\alpha}(\mathbb{S}^n)} + \|u(t) - \psi(z(t))\|_{h^{2,\alpha}(\mathbb{S}^n)} \leq C \exp(-\omega t)\|u_0 - \psi(z_0)\|_{h^{2,\alpha}(\mathbb{S}^n)},$$

for as long as $\|P[u(t)]\|_{h^{2,\alpha}(\mathbb{S}^n)} < r_1$, where $\omega \in (0, -\lambda_1)$, $C$ is a constant depending on $\omega$, and

$$\dot{z}(t) = P \left[ G \left( \frac{z(t)}{r_1} \right) z(t) + \psi(z(t)) \right], \quad z(0) = z_0,$$

where $\eta : T_{0,\mathcal{S}} \to \mathbb{R}$ is a smooth cut off function such that $0 \leq \eta(x) \leq 1$, $\eta(x) = 1$ if $\|x\|_{h^{2,\alpha}(\mathbb{S}^n)} \leq 1$ and $\eta(x) = 0$ if $\|x\|_{h^{2,\alpha}(\mathbb{S}^n)} \geq 2$.

Proof. The existence of a local center manifold, $M^c$, follows from Theorem 9.2.2 in [10], where it is also shown to be a local graph over the nullspace of $DG(0)$, i.e. $T_{0,\mathcal{S}}$. Theorem 2.3 in [12] states that $M^c$ contains all local stationary solutions, i.e. $\mathcal{S} \subset M^c$, so combining these two facts we see that $M^c = \mathcal{S}$. The exponential attractivity comes from Proposition 9.2.4 of [10].
Using (11) evaluated at $t = 0$ we obtain
\[
\|z_0\|_{P^{\alpha}(\mathbb{R}^n)} \leq \|P[u_0]\|_{P^{\alpha}(\mathbb{R}^n)} + \|P[u_0] - z_0\|_{P^{\alpha}(\mathbb{R}^n)}
\]
\[
\leq \|P[u_0]\|_{P^{\alpha}(\mathbb{R}^n)} + C\|I - P\|_{\mathcal{M}(\mathbb{R}^n)}\|\psi(u_0)\|_{P^{\alpha}(\mathbb{R}^n)},
\]
so since $\psi$ is Lipschitz and $P$ is bounded, this leads to a bound of the form $\|z_0\|_{P^{\alpha}(\mathbb{R}^n)} \leq C\|u_0\|_{P^{\alpha}(\mathbb{R}^n)}$. Therefore we can ensure that $\|z_0\|_{P^{\alpha}(\mathbb{R}^n)} < r_1$ by taking $\|u_0\|_{P^{\alpha}(\mathbb{R}^n)}$ small enough, and since $z_0 + \psi(z_0)$ defines a sphere, we see $G\left(\frac{\omega}{r_1}, z_0 + \psi(z_0)\right) = G\left(z_0 + \psi(z_0)\right) = 0$. Hence $z(t) = z_0$ is the solution to (12) and we can restate (11) as
\[
\|P[u(t)] - z_0\|_{P^{\alpha}(\mathbb{R}^n)} + \|I - P\|_{\mathcal{M}(\mathbb{R}^n)}\|\psi(u_0)\|_{P^{\alpha}(\mathbb{R}^n)} \leq C\exp(-\omega t)\|I - P\|_{\mathcal{M}(\mathbb{R}^n)}\|\psi(u_0)\|_{P^{\alpha}(\mathbb{R}^n)},
\]
for as long as $P[u(t)] \in B_{r_1}(0)$. However using this bound, and our bound for $z_0$, it follows that $\|P[u(t)]\|_{P^{\alpha}(\mathbb{R}^n)} < C\|u_0\|_{P^{\alpha}(\mathbb{R}^n)}$ as long as $\|P[u(t)]\|_{P^{\alpha}(\mathbb{R}^n)} < r_1$. By choosing $\|u_0\|_{P^{\alpha}(\mathbb{R}^n)}$ small enough we can therefore ensure $\|P[u(t)]\|_{P^{\alpha}(\mathbb{R}^n)} < \frac{\omega}{2}$ for all $t \geq 0$. Thus (13) is true for all $t \geq 0$ and this proves that $u(t)$ converges to $z_0 + \psi(z_0)$ as $t \to \infty$. This completes the proof of Theorem 1.1 since $z_0 + \psi(z_0)$ is the graph function of a sphere.

We also have the following corollary that follows by a simple continuity argument as in [5, Corollary 3.8] or [7, Corollary 4.4].

**Corollary 5.3.** Let $\Omega_0$ be a graph over a sphere with height function $u_0$ such that the solution, $u(t)$, to the flow (6) with initial condition $u_0$ exists for all time and converges to zero. Suppose further that $\nabla_i u_0(\bar{x}(X)) > 0$ for all $t \in [0, \infty)$ and $i = 1, \ldots, n$. Then there exists a neighbourhood, $O$, of $u_0$ in $H^{2-\alpha}(\mathbb{R}^n)$, $0 < \alpha < 1$, such that for every $w_0 \in O$ the solution to (6) with initial condition $w_0$ exists for all time and converges to a function near zero whose graph is a sphere.

### Appendix A. Form of the Weight Function

In this appendix we determine the form of the weight function must take in order to preserve the quantity $\tilde{V}$ in (5). We start by considering the linearisation of the mixed volumes. We will abuse notation and set $\hat{V}(X) = \tilde{V}(X(M^n))$ and $\hat{V}_a(X) = \tilde{V}_a(X(M^n))$.

**Lemma A.1.** The mixed volumes have the linearisation
\[
\hat{D}V_a(X)[Y] = \int_{M^n} \hat{\Sigma}_a(X)\hat{g}(Y, v) \, d\mu,
\]
for $a = 0, \ldots, n+1$, with $\hat{\Sigma}_a$ as defined in (4).

**Proof.** We first note that the formula for $\hat{D}V_{a+1}(X)[Y]$ is standard.

Now we consider the linearisation of the mixed volumes with $0 \leq a \leq n$, but before starting the calculation we state some useful relations for the elementary symmetric functions:
\[
\frac{\partial E_a}{\partial h_i} = g_{ik} \frac{\partial E_a}{\partial h_k},
\]
\[
\frac{\partial E_{a+1}}{\partial h_i} = E_a \delta_i^j - h_k \frac{\partial E_a}{\partial h_k},
\]
and
\[
h_{ij} \frac{\partial E_a}{\partial h_j} = h_{ik} \frac{\partial E_a}{\partial h_k},
\]
which are all easily obtained from \((9)\).

We can now calculate the linearisation for \(a = 0, \ldots, n\) using Lemma \([3,1]\).

\[
(n + 1) \frac{\partial}{\partial h_i} D\hat{V}_{n-a}(X)[Y] = \int_{M^p} E_a D \mu_{ij} h_j(X)[Y] + E_a h_i(X)[Y] D\mu(X)[Y] d\mu
\]

\[
= \int_{M^p} -2 g^a \frac{\partial E_a}{\partial h_j} \left( \tilde{g} \left( \tilde{\nabla}_T \tilde{\nabla}_T Y, v \right) + \tilde{g} \left( \tilde{\nabla}_T Y, T_j \right) \right) + \tilde{E}_a g^{\beta} \tilde{g} \left( \tilde{\nabla}_T Y, T_k \right) d\mu
\]

\[
= \int_{M^p} -2 g^a \frac{\partial E_a}{\partial h_j} \left( \tilde{g} \left( \tilde{\nabla}_T \tilde{\nabla}_T Y, v \right) + \tilde{g} \left( \tilde{\nabla}_T Y, T_j \right) \right) + \tilde{E}_a g^{\beta} \tilde{g} \left( \tilde{\nabla}_T Y, T_k \right) d\mu
\]

We now use Equation (14) to cancel the first two terms in the \(\tilde{g} \left( \tilde{\nabla}_T Y, T_i \right)\) factor, and Equation (16) to alter the third term of the factor:

\[
(n + 1) \frac{\partial}{\partial h_i} D\hat{V}_{n-a}(X)[Y] = \int_{M^p} \left( E_a g^{\alpha \gamma} - g^{\alpha \gamma} q_i h_j \frac{\partial E_a}{\partial h_j} \right) \tilde{g} \left( \tilde{\nabla}_T Y, T_i \right) - g_a \frac{\partial E_a}{\partial h_j} \tilde{g} \left( \tilde{\nabla}_T Y, T_i \right) d\mu
\]
Now we use the homogeneity $E_a$ and the Gauss-Codazzi equation:

$\left( n + 1 \right) D \hat{\mathcal{V}}_{n-d}(X)[Y] = \int_{M^v} -g^{ij} g^a \frac{\partial E_a}{\partial T_j} \bar{g} \left( \bar{R}(T_i, T_j) \bar{v}, \bar{v} \right) + (a + 1) E_{a+1} \bar{g}(Y, \nu) \left[ \begin{array}{c} \frac{\partial E_a}{\partial T_j} \end{array} \right] \bar{g} \left( \bar{R}(Y, T_j) \nu, \nu \right) d\mu$

and the Gauss-Codazzi equation:

$= \int_{M^v} -g^{ij} \frac{\partial E_a}{\partial T_j} \bar{g} \left( \bar{R} \left( \bar{g}(Y, \nu) \nu, \nu \right) \nu, \nu \right) + (a + 1) E_{a+1} \bar{g}(Y, \nu) - g^{ij} \nabla_j \nabla_k \left( \frac{\partial E_a}{\partial T_j} \right) \bar{g}(Y, \nu) d\mu$

$= \int_{M^v} -g^{ij} \frac{\partial E_a}{\partial T_j} \bar{g} \left( \bar{R} \left( \bar{g}(Y, \nu) \nu, \nu \right) \nu, \nu \right) + (a + 1) E_{a+1} \bar{g}(Y, \nu) - g^{ij} \nabla_j \nabla_k \left( \frac{\partial E_a}{\partial T_j} \right) \bar{g}(Y, \nu) d\mu.$

$\square$

**Corollary A.2.** If

$$\hat{\mathcal{V}}(X) = \sum_{a=0}^{n+1} c_a \hat{\mathcal{V}}_a(X)$$

for some constants $c_a \in \mathbb{R}$, $a = 0, \ldots, n + 1$, where $\hat{\mathcal{V}}_a(X)$ are defined in (1), then $\mathcal{V}(X)$ is preserved by the flow (1).

**Proof.** By Lemma [3,4] and linearity we have

$$D \hat{\mathcal{V}}(X)[Y] = \int_{M^v} \hat{\mathcal{V}}(X) \bar{g}(Y, \nu(X)) d\mu.$$  

It then follows from the form of $\hat{\mathcal{G}}(X)$ in (2) that under (1)

$$\frac{\partial \hat{\mathcal{V}}}{\partial t} = D \hat{\mathcal{V}}(X) \left[ \frac{\partial X}{\partial t} \right]$$

$$= D \hat{\mathcal{V}}(X) [\hat{\mathcal{G}}(X) \nu(X)]$$

$$= \int_{M^v} \hat{\mathcal{V}}(X) \hat{\mathcal{G}}(X) d\mu$$

$$= 0,$$

thus under this weight function we have $\hat{\mathcal{V}}(\Omega_t) = \hat{\mathcal{V}}(\Omega_0)$ as long as the flow exists.  

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