SANDWICH CELLULARITY AND A VERSION OF CELL THEORY

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Abstract. We explain how the theory of sandwich cellular algebras can be seen as a version of cell theory for algebras. We apply this theory to many examples such as Hecke algebras, and various monoid and diagram algebras.

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1. Introduction

One of the main tools to study the representation theory of algebras is the notion of a cellular algebra due to Graham and Lehrer [GL96], and cellular algebras are nowadays ubiquitous in representation theory. Sandwich cellular algebras can be seen as a common and strict generalization of Graham–Lehrer cellular algebras and affine cellular algebras as in [KX12], as well as partial generalizations of Kazhdan–Lusztig bases, monoid and diagram algebras. All of these, in a precise sense, fit under the umbrella of sandwich cellularity.

These sandwich cellular algebras are certain algebras equipped with a sandwich cell datum which in turn gives rise to the notion of cells for these algebras. Cells partition sandwich cellular algebras in the same way as Green’s relations [Gre51] partition monoids, or more generally semigroups (we stay with monoids in this paper for simplicity), and Kazhdan–Lusztig cells [KL79] partition Hecke algebras. (The latter is the reason why we use the name cells.) Sandwich cellular structures were also successfully applied very early on, albeit in disguise, for example in the study of Brauer algebras as in [Bro55] and [FG95].

The analogy between sandwich cellular algebras and monoids respectively Hecke algebras goes even further. Parts of a sandwich cell datum are sandwich cellular bases and sandwiched algebras. For cellular algebras all sandwiched algebras are trivial, and the sandwiched algebras are the main new ingredient in the theory. These sandwiched algebras play the role of Green’s $H$-groups [Gre51] and asymptotic Hecke algebras associated to intersections of left and right cells as e.g. in [Lus87]. The sandwich cellular bases are versions of Kazhdan–Lusztig bases in the theory of sandwich cellularity.

The most important theorem regarding sandwich cellular algebras is $H$-reduction, see Theorem 2A.17. $H$-reduction classifies simple modules of sandwich cellular algebras by using their cells and the sandwiched algebras. In the theory of monoids $H$-reduction is known as the celebrated Clifford–Munn–Ponizovskii theorem that classifies simple modules of monoids by Green’s $J$-classes and the simple modules of the $H$-groups, see [GMS09] or [Ste16] for modern expositions. In the theory of Kazhdan–Lusztig cells the corresponding theorem does not have a name (as far as we know), and is weaker than the Clifford–Munn–Ponizovskii theorem, see Section 3 for details. A variant of the $H$-reduction for sandwich cellular algebras is the theorem with the same name in categorification, see e.g. [MMM20] and [MMM+21], where we got the nomenclature from. Note however that this categorical $H$-reduction is similar in spirit but different in nature, which becomes evident when comparing [MMM+23] with Section 3.

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As we will see, all algebras are sandwich cellular with potentially many different cell structures. Hence, the main point is the cell structure itself and not the property of being sandwich cellular. The important task is thus to find a useful cell structure. What useful means is difficult to gauge and example dependent. But what one should roughly have in mind here is a fine cell structure with many cells and well-understood sandwiched algebras.

Examples of sandwich cellular algebras with such cell structures include many algebras from various diverse fields of mathematics: all cellular algebras with useful cell datum, some Hecke algebras and their (p-)Kazhdan–Lusztig bases, many monoid algebras such as transformation monoids, many diagram algebras such as Brauer algebras, KLR(W) algebras, potentially weighted, seem to have natural sandwich cellular structures, see e.g. [Bow22], [MT24] and [MT23], diagram algebras that appear in the study of higher genus knot invariants [RT21], [TV23] have useful sandwich cell structures. Note that some, but not all, of the named algebras are cellular but the sandwich cell structure is often much easier to find and to work with, as we will see.

Sandwich cellular algebras have been around for a long time, however, in various disguises and often implicit in the literature. They originate from at least four perspectives as mentioned above. In historical order, they appeared via the Clifford–Munn–Ponizovskii theorem in monoid theory, in the study of the Brauer algebra, they are related to Kazhdan–Lusztig cells, and are generalizations of cellular algebras. In this work we will draw connections between these different fields by taking all of these perspectives at once.

This paper is organized as follows:

- **We give a concise summary and advance the theory of sandwich cellular algebras at the same time, see Section 2. In Section 2B we give a reformulation of the original definition that is useful in practice and explains our choice of using cell theory in the title. We also make the connection to cellular algebras and monoid theory precise.**
- **In Section 3 we apply the theory of sandwich cellular algebras to Hecke algebras of finite Coxeter type and their Kazhdan–Lusztig and p-Kazhdan–Lusztig bases.**
- **In Section 4 we study various diagram algebras from the viewpoint of sandwich cellular algebras. This includes diagram algebras without antiinvolution such as transformation and planar transformation monoids, as well as diagram algebras with an antiinvolution such as Brauer and Temperley–Lieb algebras.**
- **In all the examples we study, we classify simple modules using $H$-reduction and also compute some dimensions of simple modules, which is often doable with the general theory of sandwich cellular algebras at hand.**

Quite a few, but not all, results in this paper have been obtained before, sometimes a long time ago. However, our point is that all of them fit under the umbrella of sandwich cellularity. At the end of each section we, for convenience, systematically collect references and explain how the results in this paper compare to known theorems in the literature.

**Remark 1.1.** Parts of this paper are based on computer calculations. For the reader that wants to run these calculations themselves, we have collected all relevant material on GitHub [Tub23].

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## 2. Cell theory for algebras

We now recall the notion of a sandwich cellular algebra, and discuss a modification of sandwich cellularity which is useful in practice (in particular, for all algebras in this paper).

### 2A. Sandwich cellular algebras

**We start with notation:**
Notation 2A.1. We use the following conventions.

(a) We fix a commutative unital ring $\mathbb{K}$. This is our ground ring throughout, and e.g. ranks $\text{rk}_\mathbb{K}$ are with respect to $\mathbb{K}$ if not specified otherwise. We sometimes need $\mathbb{K}$ to be a field, e.g. for classification results. Hence the notation $\mathbb{K}$.

(b) Throughout this section let $\mathcal{A}$ denote an associative unital $\mathbb{K}$-algebra. Algebras in this paper are always assumed to be associative and unital.

(c) Whenever we define an order, say $<_p$, then we will also use $>_p$, $\leq_p$ or $\geq_p$, having the evident meanings. We also write e.g. $\mathcal{A}^{>_p \lambda}$ which stands for the $\mathbb{K}$-submodule of $\mathcal{A}$ spanned by $$\{c^n_{\ell,m,V} \mid \mu \in \mathcal{P}, \mu >_p \lambda, U \in \mathcal{T}(\mu), V \in \mathcal{B}(\mu), n \in B_\mu\}$$

(d) Unless otherwise specified, modules will always be left modules. In diagrammatic terms these are given by acting from the top, and we use

\[ ab = \begin{array}{c} a \\ b \end{array} \]

We sometimes use right modules and bimodules, and we will stress whenever that is the case.

Remark 2A.2. As in Notation 2A.1, we will use colors in this paper. The colors are a visual aid only and the paper is readable in black-and-white without restrictions.

The following definition is a modification of [MT23, Definition 2A.2], see also [TV23, Section 2] for the same definition using a slightly different formulation.

Definition 2A.3. A sandwich cell datum for $\mathcal{A}$ is a quadruple $(\mathcal{P}, (\mathcal{T}, B), (\mathcal{H}_\lambda, B_\lambda), C)$, where:

- $\mathcal{P} = (\mathcal{P}, <_p)$ is a poset (the middle poset with sandwich order $<_p$),
- $\mathcal{T} = \bigcup_{\lambda \in \mathcal{P}} \mathcal{T}(\lambda)$ and $\mathcal{B} = \bigcup_{\lambda \in \mathcal{P}} \mathcal{B}(\lambda)$ are collections of finite sets (the top/bottom sets),
- For $\lambda \in \mathcal{P}$ we have algebras $\mathcal{H}_\lambda$ (the sandwiched algebras) and bases $B_\lambda$ of $\mathcal{H}_\lambda$,
- $C$: $\prod_{\lambda \in \mathcal{P}} \mathcal{T}(\lambda) \times B_\lambda \times B(\lambda) \to \mathcal{A}$; $(T, m, B) \mapsto c^\lambda_{T,m,B}$ is an injective map, such that:

\[(AC_1) \text{ The set } B_\mathcal{A} = \{c^\lambda_{T,m,B} \mid \lambda \in \mathcal{P}, T \in \mathcal{T}(\lambda), B \in \mathcal{B}(\lambda), m \in B_\lambda\} \text{ is a basis of } \mathcal{A}. \text{(We call } B_\mathcal{A} \text{ a sandwich cellular basis.)} \]

\[(AC_2) \text{ For all } x \in \mathcal{A} \text{ there exist scalars } r^T_U \in \mathbb{K} \text{ that do not depend on } B \text{ or on } m, \text{ such that} \]

\[(2A.4) \quad xc^\lambda_{T,m,B} \equiv \sum_{U \in \mathcal{T}(\lambda)} r^T_U c^\lambda_{U,n,B} \quad (\text{mod } \mathcal{A}^{>_p \lambda}). \]

Similarly for right multiplication by $x$.

\[(AC_3) \text{ There exists a free } \mathcal{A}-\mathcal{H}_\lambda \text{-bimodule } \Delta(\lambda), \text{ a free } \mathcal{H}_\lambda-\mathcal{A} \text{-bimodule } \nabla(\lambda), \text{ and an } \mathcal{A} \text{-bimodule isomorphism} \]

\[(2A.5) \quad \mathcal{A}_\lambda = \mathcal{A}^{>_p \lambda}/\mathcal{A}^{>_p \lambda} \cong \Delta(\lambda) \otimes_{\mathcal{H}_\lambda} \nabla(\lambda). \]

We call $\mathcal{A}_\lambda$ the cell algebra, and $\Delta(\lambda)$ and $\nabla(\lambda)$ left and right cell modules.

The algebra $\mathcal{A}$ is a sandwich cellular algebra if it has a sandwich cell datum.

Definition 2A.6. In the setup of Definition 2A.3 assume that $\mathcal{T}(\lambda) = \mathcal{B}(\lambda)$ for all $\lambda \in \mathcal{P}$, and that there is an antiinvolution $(\_)^{*}: \mathcal{A} \to \mathcal{A}$ and order two bijections $(\_)^{*}: B_\lambda \to B_\lambda$ such that:

\[(AC_4) \text{ We have } (c^\lambda_{T,m,B})^* \equiv c^\lambda_{T,m,B} \quad (\text{mod } \mathcal{A}^{>_p \lambda}). \]

In this case we call the sandwich cell datum involutive and write $(\mathcal{P}, \mathcal{T}, (\mathcal{H}_\lambda, B_\lambda), C, (\_)^{*})$ for it.

Diagrammatically, although not quite accurate, we think of (2A.4) and (AC4) as

\[ \begin{array}{c} T \\ m \\ B \\ T' \\ m' \\ B' \end{array} \equiv \begin{array}{c} B \\ T' \\ m' \end{array} \quad (\text{mod } \mathcal{A}^{>_p \lambda}), \quad \begin{array}{c} T \\ m \\ B \\ T' \\ m' \end{array} \equiv \begin{array}{c} B \\ m' \end{array} \quad (\text{mod } \mathcal{A}^{>_p \lambda}). \]
Notation 2A.7. We will also say, for example, that \( \mathcal{A} \) itself is sandwich cellular, although this is a bit misleading since an algebra can be sandwich cellular with different sandwich cell data, see e.g. Example 2A.11.

Example 2A.8. Two crucial special cases of involutive sandwich cellular algebras are:

(a) If all sandwiched algebras are isomorphic to \( \mathbb{K} \), then \( (\mathcal{P}, \mathcal{T}, (\mathcal{H}_\lambda \cong \mathbb{K}, B_\lambda = \{1\}), C, (-)^* \) is a cell datum and \( \mathcal{A} \) is a cellular algebra in the sense of [GL96]. (Strictly speaking \( \mathcal{A} \) is cellular in the sense of [GG11] due to the weakened condition (AC4) on the antiinvolution. Throughout, we will always use this weaker version of cellularity.)

(b) If all sandwiched algebras are commutative, then \( (\mathcal{P}, \mathcal{T}, (\mathcal{H}_\lambda, B_\lambda), C, (-)^* \) is an affine cell datum and \( \mathcal{A} \) is an affine cellular algebra in the sense of [KK12].

The mnemonic (in particular for readers familiar with diagram algebras) is:

\[
\begin{align*}
\text{cellular:} & \quad c_{T,1,B}^\lambda \xrightarrow{\mathcal{T}} \mathcal{H}_\lambda \cong \mathbb{K}, & \text{affine cellular:} & \quad c_{T,m,B}^\lambda \xrightarrow{\mathcal{T}} \text{commutative } \mathcal{H}_\lambda, \\
\text{sandwich cellular:} & \quad c_{T,m,B}^\lambda \xrightarrow{\mathcal{T}} \text{general } \mathcal{H}_\lambda,
\end{align*}
\]

where we assume the existence of an antiinvolution for the top two pictures. \(\diamond\)

The comparison of sandwich cellular and cellular algebras is as follows.

**Proposition 2A.10.** An involutive \( (\mathcal{P}, \mathcal{T}, (\mathcal{H}_\lambda, B_\lambda), C, (-)^* \) sandwich cell datum is a cell datum if and only if \( \mathcal{H}_\lambda \cong \mathbb{K} \) for all \( \lambda \in \mathcal{P} \).

Moreover, an involutive sandwich cellular algebra \( \mathcal{A} \) such that all \( \mathcal{H}_\lambda \) are cellular (with the same antiinvolution) is cellular with a refined sandwich cell datum. Conversely, if at least one \( \mathcal{H}_\lambda \) is non-cellular, then \( \mathcal{A} \) is non-cellular.

**Proof.** The first claim is immediate, for the second see [TV23, Proposition 2.9]. \(\square\)

**Example 2A.11.** For a group \( G \) let \( \mathcal{A} = \mathbb{K}[G] \). Then the group element basis is a sandwich cellular basis. For \( G \) being the symmetric group or the dihedral group associated to a polygon with an odd number of edges we will see a very different sandwich cellular basis in Section 3.

We stress that the group element basis is not a cellular or affine cellular basis in general. For example, if \( G \) is any noncommutative group, then the group element basis is neither cellular nor affine cellular basis but still sandwich cellular.

In Proposition 2E.1 we give a condition (that can likely be weakened) on a monoid that ensures that the monoid element basis is a sandwich cellular basis.

We will discuss more examples, and also return to Example 2A.11, later on.

**Proposition 2A.12.** Any algebra has the structure of a sandwich cellular algebra. Moreover, if \( \mathbb{K} \) is an algebraically closed field, then there exists a sandwich cell datum with \( \mathcal{T}(\lambda) = \mathcal{B}(\lambda) \) and \( \mathcal{H}_\lambda \cong \mathbb{K} \) for all \( \lambda \in \mathcal{P} \).

**Proof.** For the first statement take all ingredients to be trivial, e.g. \( \mathcal{P} = \{0\}, \mathcal{T}(0) = \mathcal{B}(0) = \{0\}, \mathcal{H}_0 = \mathcal{A} \) etc. After comparison of definitions, the second claim is [CZ19, Corollary B]. \(\square\)

**Remark 2A.13.** By Proposition 2A.12, the main point is not to prove that an algebra is sandwich cellular, but rather to find a useful sandwich cell datum. Note also that (the proofs of) Proposition 2A.10 and Proposition 2A.12 imply that any algebra is cellular over an algebraically closed field if one does not assume the existence of an antiinvolution.

**Notation 2A.14.** The \( \mathcal{A} \)-bimodule \( \mathcal{A}_\lambda \) has an induced multiplication, but might not have a unit in this multiplication. If it does not have a unit, then we always adjoin a unit whenever we want to see it as an algebra.

An apex of an \( \mathcal{A} \)-module \( M \), if it exists, is a maximal \( \lambda \in \mathcal{P} \) such that

\[
\mathbb{K}\{ c_{T,m,B}^\lambda \mid T \in \mathcal{T}(\lambda), B \in \mathcal{B}(\lambda), m \in B_\lambda \}
\]

does not annihilate \( M \). The same notion is used for \( \mathcal{A}_\lambda \) instead of \( \mathcal{A} \).
Example 2A.15. The cell algebra $\mathcal{A}_\lambda$ has at most two apexes, one for the unit and $\lambda$.

Lemma 2A.16. Every simple $\mathcal{A}$-module has an apex $\lambda \in \mathcal{P}$, and similarly for simple $\mathcal{A}_\lambda$-modules.

Proof. See [TV23, Lemma 2.15].

For a set $X$ of modules we write $X / \cong$ for the set of isomorphism classes obtained from $X$ by identifying isomorphic modules. The main theorem about sandwich cellular algebras is the following version of the Clifford–Munn–Ponizovskii theorem:

Theorem 2A.17. (H-reduction) Let $\mathbb{K}$ be a field.

(a) If $\lambda \in \mathcal{P}$ is an apex and $\mathcal{A}_\lambda$ is Artinian, then we have bijections

\[
\text{Start: } \{\text{simple } \mathcal{A}\text{-modules with apex } \lambda\} / \cong \\
\downarrow \downarrow 1:1 \\
J\text{-reduction: } \{\text{simple } \mathcal{A}_\lambda\text{-modules with apex } \lambda\} / \cong \\
\downarrow \downarrow 1:1 \\
H\text{-reduction: } \{\text{simple } \mathcal{A}_\lambda\text{-modules}\} / \cong.
\]

(b) All $\mathcal{A}$-modules with apex $\lambda$ have composition factors of apex $\mu$ with $\mu \leq \mathcal{P} \lambda$.

Proof. See [TV23, Theorem 2.16].

Notation 2A.18. Whenever Theorem 2A.17 applies, we will write $\mathcal{P}^{\text{op}} \subset \mathcal{P}$ for the set of apexes, and $L(\lambda, K)$ for the simple $\mathcal{A}$-modules associated to $\lambda \in \mathcal{P}^{\text{op}}$ and a simple $\mathcal{A}_\lambda$-module $K$. If $\mathcal{A}_\lambda \cong \mathbb{K}$, then we write $L(\lambda)$ for the unique simple $\mathcal{A}$-module of apex $\lambda$.

Remark 2A.19. The bijections in Theorem 2A.17 can be made explicit and $L(\lambda, K)$ is the head of the induction of $K$ to $\mathcal{A}$, see [TV23, Theorem 2.16] for details.

Notation 2A.20. As already indicated by the Artinian condition in Theorem 2A.17, there are some technicalities when working with infinite dimensional algebras, see [TV23, Section 2] for a more detailed treatment. To simplify the exposition, we will from now on assume that our algebras are finite dimensional.

Remark 2A.21. Sandwich cellular algebras are inspired by the theory of cellular algebras on the one hand, see e.g. [GL96], [HM10], [KKX12], [AST18], [ET21], [GW15] or [TV23], and ideas coming from monoid representation theory on the other hand, see e.g. [Gre51], [GMS09] or [KST24]. There was also a huge influence from Kazhdan–Lusztig (KL) theory and based algebras [KL79], [Lus87], [KM16]. Sandwich cellular algebras have appeared in disguise in, for example, [Bro55], [FG95], [KKX99] and [KX01]. We learned the idea underlying sandwich cellular algebras from [GW15].

2B. Green’s theory of cells in algebras. We now discuss a special case of sandwich cellular algebras for which we can reformulate Definition 2A.3 to be closer to Green’s classical theory of cells a.k.a. Green’s relations, and KL theory.

Remark 2B.1. The main slogan of cell theory is the following. In contrast to groups, the multiplication in an algebra often destroys information. For example, if $b = ca$, then $b$ can be obtained from $a$ by left multiplication, and we can say that $b$ is left bigger than $a$. In a group we can go back by $c^{-1}b = a$ so $a$ is also left bigger than $b$, but this is not always possible in an algebra. Cells can then be thought of as keeping track of the information loss during multiplication. The reader is encouraged to keep this analogy in mind while reading the text below.

Recall that $\mathcal{A}$ denotes an algebra as in Notation 2A.1. Fix a basis $B_{\mathcal{A}}$ of $\mathcal{A}$. Everything below depends on the choice of this basis, and we will use the (based) pair $(\mathcal{A}, B_{\mathcal{A}})$.

For $a, b, c \in B_{\mathcal{A}}$ we write $b \in ca$ if, when $ca$ is expanded in terms of $B_{\mathcal{A}}$, $b$ appear with nonzero coefficient in $ca$. We define preorder on $B_{\mathcal{A}}$ by

\[
(a \leq_l b) \iff \exists c : b \in ca, \quad (a \leq_r b) \iff \exists d : b \in ad, \quad (a \leq_{lr} b) \iff \exists c, d : b \in cad.
\]

We call these left, right and two-sided cell orders.

Remark 2B.2. Our convention for the cell orders is the same as the most common convention used in the theory of cellular algebras but the opposite of the one usually used in monoid theory.
Definition 2B.3. We define equivalence relations by

\[(a \sim_l b) \iff (a \leq_l b \text{ and } b \leq_l a), \quad (a \sim_r b) \iff (a \leq_r b \text{ and } b \leq_r a), \quad (a \sim_{lr} b) \iff (a \leq_{lr} b \text{ and } b \leq_{lr} a).\]

The respective equivalence classes are called left, right respectively two-sided cells.

We also say \(J\)-cells instead of two-sided cells, following the notation in [Gre51].

Definition 2B.4. An \(H\)-cell \(H = H(L, R) = L \cap R\) is an intersection of a left cell \(L\) and a right cell \(R\).

The following is easy:

Lemma 2B.5. The left, right and \(J\)-cell orders induce preorders on the left, right and \(J\)-cells, respectively. Similarly, we can also compare elements and cells via the cell orders. \(\Box\)

We will use the same symbols for the various preorders.

Example 2B.6. Let \(\mathcal{A} = \mathbb{K}S\) for a finite monoid \(S\) (all monoids we use are finite, and we drop that adjective), and fix \(B_{\mathcal{A}} = S\), the monoid basis.

(a) The above recovers Green’s relations. That is, the preorders simplify to

\[(a \leq_l b) \iff \exists c : b = ca, \quad (a \leq_r b) \iff \exists d : b = ad, \quad (a \leq_{lr} b) \iff \exists c, d : b = cad.\]

In monoid theory the corresponding cells are called \(L\), \(R\) and \(H\)-classes.

(b) Let \(G \subseteq S\) be the group of invertible elements. We have \(G \leq_l a, G \leq_r a\) and \(G \leq_{lr} a\) for all \(a \in S\). This can be seen by \(a = (ag^{-1})g\) and similar calculations for the right and \(J\)-orders. In particular, if \(G = S\) (thus, \(S\) is a group), then there is only one cell, which is a left, right, \(J\) and \(H\)-cell at the same time.

If not specified otherwise, when referring to monoids or groups, then we will always fix \(B_{\mathcal{A}}\) to be the monoid basis. \(\Diamond\)

Remark 2B.7. We think of cells as a matrix decomposition of \(B_{\mathcal{A}}\):

\[H(\mathcal{L}, \mathcal{R}) = H_{33}\]

In the left picture we use matrix notation for the twelve \(H\)-cells in \(J\) so that \(H_{ij}\) is the intersection of the \(i\)th right with the \(j\)th left cell of \(J\). The \(J\)-cell \(J\) is then a matrix with entries from \(H\)-cells. Moreover, the left cells are column vectors and the right cells are row vectors of \(J\).

The right picture is a GAP output using the package Semigroups, see also [Tub23]. The illustration shows the cell structure of a given, not further specified, monoid in the matrix notion. The \(J\)-order is illustrated as well. The shaded \(H\)-cells are strictly idempotent in the sense of Definition 2B.13.

We denote the indexing set for the \(J\)-cells by \(P\), and denote the \(J\)-cells by \(J_\lambda\) for \(\lambda \in P\).

Notation 2B.9. We use a subscript \(\lambda\) to indicate that we are working in a fixed \(J\)-cell \(J_\lambda\).

The next lemma is immediate and will be used throughout.

Lemma 2B.10. \(\mathcal{A}^{>lr}_{\lambda} = \mathbb{K}\left\{ \cup_{\mu \in P, \mu >_{lr} \lambda} J_\mu \right\}\) is a two-sided ideal in \(\mathcal{A}\). \(\Box\)

Example 2B.11. If \(J_\lambda\) is the \(J\)-cell of size 3-3 on the right-hand side in (2B.8), then \(\mathcal{A}^{>lr}_{\lambda}\) is the two-sided ideal supported on the size 9-9 and the top \(J\)-cells. Note that we can ignore the \(J\)-cell of size 2-2 as the multiplication will never get us from \(J_\lambda\) into it. \(\Diamond\)

Notation 2B.12. An equation etc. holds up to higher order terms if it holds modulo \(\mathcal{A}^{>lr}_{\lambda}\).
Definition 2B.13. If the $\mathbb{K}$-linear span of a $J$-cell $\mathcal{J}$ contains a pseudo-idempotent up to higher order terms, that is, $e \in \mathbb{K}\mathcal{J}$ with $e^2 = s(e) \cdot e \pmod{\mathcal{A}^{\geq \nu \lambda}}$ for $s(e) \in \mathbb{K} \setminus \{0\}$, then we call $\mathcal{J}$ idempotent. We say $\mathcal{J}$ is strictly idempotent if $\mathcal{J}$ itself contains a pseudo-idempotent up to higher order terms. We use the same terminology for $H$-cells.

Note that pseudo-idempotents can be rescaled to idempotents if $s(e)$ is invertible, which is always true, for example, if $\mathbb{K}$ is a field. We write $\mathcal{J}_\lambda(e)$ and $\mathcal{H}_\lambda(e) \subset \mathcal{J}_\lambda(e)$ for idempotent cells.

Example 2B.14. All idempotent $H$ and $J$-cells are strictly idempotent for monoids. The notion then corresponds to having an idempotent and is called regular in monoid theory.

A bottom $J$-cell is a $J$-cell that is minimal in the $J$-order. Similarly, a top $J$-cell is a $J$-cell that is maximal in the $J$-order.

Proposition 2B.15. For the pair $(\mathcal{A}, B_{\mathcal{A}})$ we have:

(a) Every $H$-cell is contained in some $J$-cell, and every $J$-cell is a disjoint union of $H$-cells.

(b) If $s(e) \in \mathbb{K} \setminus \{0\}$ is invertible, then $\mathcal{H}_\lambda = \mathbb{K}\mathcal{H}_\lambda(e)/\mathcal{A}^{\geq \nu \lambda}$ is an algebra with identity $\frac{1}{s(e)} e$.

This algebra is a subalgebra of $\mathcal{A}^{\leq \nu \lambda} = \mathcal{A}/\mathcal{A}^{\geq \nu \lambda}$ and $\mathcal{A}^\lambda = \mathcal{A}^{\leq \nu \lambda} \cap \mathbb{K}\mathcal{J}_\lambda$.

(c) Assume that $1 \in B_{\mathcal{A}}$. The pair $(\mathcal{A}, B_{\mathcal{A}})$ has a unique bottom $J$-cell, and this $J$-cell is strictly idempotent.

(d) The pair $(\mathcal{A}, B_{\mathcal{A}})$ has a unique top $J$-cell, which is an $H$-cell at the same time.

Point (a) works analogously for left and right cells in a fixed $J$-cell.

If they exist, then we write $\mathcal{J}_b$ and $\mathcal{J}_t$ for the bottom and top $J$-cell, respectively.

Proof. (a)+(b). These follow by construction.

(c). The bottom $J$-cell is easy to find: it is the subset of invertible basis elements, and the unit is the idempotent in it.

(d). If $\mathcal{J}$ and $\mathcal{J}'$ are maximal $J$-cells, then $\mathcal{J} = \mathcal{J} \mathcal{J}' = \mathcal{J}'$ by maximality. Existence of a maximal $J$ cell follows from the finiteness of $\mathcal{A}$. □

Definition 2B.16. The pair $(\mathcal{A}, B_{\mathcal{A}})$ is called involutive if it admits an order two bijection $\_^* : B_{\mathcal{A}} \to B_{\mathcal{A}}$ that gives rise to an antiinvolution on $\mathcal{A}$.

Example 2B.17. Groups are involutive with the inversion operation being the antiinvolution. □

In the involutive setting all $J$-cells are square, meaning they have the same number of left and right cells:

Lemma 2B.18. In the involutive setting, $\_^*$ gives rise to mutually inverse bijections $\{\text{left cells}\} \leftrightarrow \{\text{right cells}\}$ that preserve containment in $J$-cells.

Proof. Because $(b = ca)^* \Rightarrow (b^* = a^*c^*)$. □

The connection to sandwich cellular algebras is given by:

Theorem 2B.19. For any sandwich cellular algebra $\mathcal{A}$ we get a pair $(\mathcal{A}, B_{\mathcal{A}})$ for which the above theory applies with the following cell structure:

(i) The basis $B_{\mathcal{A}}$ is the sandwich cellular basis.

(ii) The poset can be taken (potentially changing the order) to be $\mathcal{P} = (\mathcal{P}, <_{lr})$.

(iii) The set $T(\lambda)$ indexes the right cells within $\mathcal{J}_\lambda$, and $B(\lambda)$ indexes the left cells within $\mathcal{J}_\lambda$.

(iv) All left, right and $H$-cells within one $J$-cell are of the same size.

(v) If $\mathcal{J}_\lambda$ is idempotent, then the respective sandwiched algebra and cell algebra are isomorphic to $\mathcal{H}_\lambda$ and $\mathcal{A}^\lambda$.

Proof. The first three points are just reformulations of (AC1) and (AC2), while the fourth and fifth conditions follow from (AC3). □

Notation 2B.20. We will use Theorem 2B.19 to associate a pair $(\mathcal{A}, B_{\mathcal{A}})$ to any (involutive) sandwich cellular algebra. We call such a pair a(n involutive) sandwich pair.
If \((\mathcal{A}, B_{\mathcal{A}})\) is a sandwich pair, then \(H\)-reduction Theorem 2A.17 applies.

**Remark 2B.21.** Part (iv) of Theorem 2B.19 is crucial for the \(H\)-reduction Theorem 2A.17 to work. See Example 3A.6 for an explicit (counter)example.

We now collect a few numerical properties of cells. We denote by e.g. \(|J_\lambda|\) the number of elements in the cells and by \(\#\mathcal{L}_\lambda\) etc. the number of such cells, measured within one fixed \(J\)-cell.

**Lemma 2B.22.** For a sandwich pair \((\mathcal{A}, B_{\mathcal{A}})\) we have:

(a) The number of \(J\)-cells is \(|\mathcal{P}|\) and \(\text{rk}_K(\mathcal{A}) = |B_{\mathcal{A}}| = \sum_{\lambda \in \mathcal{P}} |J_\lambda|\).

(b) \(|J_\lambda| = \#\mathcal{L}_\lambda \cdot |\mathcal{H}_\lambda| \cdot \#\mathcal{R}_\lambda\) and \(|J_\lambda| \cdot |\mathcal{H}_\lambda| = |\mathcal{L}_\lambda| \cdot |\mathcal{R}_\lambda|\).

(c) \(|\mathcal{L}_\lambda| = |\mathcal{H}_\lambda| \cdot \#\mathcal{R}_\lambda|\).

(d) \(|\mathcal{R}_\lambda| = |\mathcal{H}_\lambda| \cdot \#\mathcal{L}_\lambda|\).

**Proof.** (a) is clear. Using the notation from (2A.9): as in [TV23, Section 2], that is, as free \(K\)-modules we have

\[
\mathcal{K}\mathcal{L}_\lambda \cong \mathcal{K}T(\lambda) \otimes_K \mathcal{H}_\lambda \quad \text{and} \quad \mathcal{K}\mathcal{R}_\lambda \cong \mathcal{H}_\lambda \otimes_K \mathcal{K}B(\lambda) \quad \frac{\mathcal{T}}{\mathcal{R}}.
\]

(2B.23)

\[
J_\lambda \cong \mathcal{K}T(\lambda) \otimes_K \mathcal{H}_\lambda \otimes_K \mathcal{K}B(\lambda) \quad \frac{\mathcal{T}}{\mathcal{R}}.
\]

We shaded the parts that one can think of as being fixed. This follows from (AC\(_2\)) and (AC\(_3\)) and implies (b), (c) and (d). 

**Proposition 2B.24.** Let \(K\) be a field and \((\mathcal{A}, B_{\mathcal{A}})\) be a sandwich pair. The following are equivalent.

(i) The algebra \(\mathcal{A}\) is semisimple.

(ii) All \(J\)-cells are idempotent and square, all \(\mathcal{H}_\lambda\) are semisimple and all \(L(\lambda, K)\) are isomorphic as \(\mathcal{A}^{\operatorname{ss}}\)-modules to \(\operatorname{Ind}_{\mathcal{H}_\lambda}^{\mathcal{A}} K\).

**Proof.** We will use Lemma 2B.22 in this proof.

Assume condition (ii) holds. Then we get \(|J_\lambda| = \#\mathcal{L}_\lambda \cdot |\mathcal{H}_\lambda| \cdot \#\mathcal{R}_\lambda = (\#\mathcal{L}_\lambda)^2 \cdot |\mathcal{H}_\lambda|\), because \(J_\lambda\) is square. Moreover, since \(\mathcal{H}_\lambda\) is semisimple we get \(\dim_K(\mathcal{H}_\lambda) = |\mathcal{H}_\lambda| = \sum_{\text{simples}} \dim_K (K)^2\), and combining this with the final assumption gives \(|J_\lambda| = \sum_{\text{simples}} \dim_K (L(\lambda, K))^2\). Since all \(J\)-cells are idempotent we get \(\dim_K(\mathcal{A}) = \sum_{\text{simples}} \dim_K (L(\lambda, K))^2\) which shows that \(\mathcal{A}\) is semisimple.

Assume that (i) holds. Then Remark 2A.19 shows that we can read the argument above backwards, showing that (i) implies (ii).

**Remark 2B.25.** The discussion above is new but of course strongly inspired by very similar construction known throughout the literature, see e.g. [Gre51], [KL79], [KM16].

2C. **\(J\)-reduction and \(H\)-reduction.** Only sandwich pairs will satisfy \(H\)-reduction in general, but (the much weaker) \(J\)-reduction works for all pairs:

**Proposition 2C.1.** Let \(K\) be a field and consider the pair \((\mathcal{A}, B_{\mathcal{A}})\).

(a) \(J\)-reduction as in Theorem 2A.17 holds for the pair \((\mathcal{A}, B_{\mathcal{A}})\).

(b) \(H\)-reduction as in Theorem 2A.17 does not hold in general for the pair \((\mathcal{A}, B_{\mathcal{A}})\).

**Proof.** (a). One can copy the arguments in [TV23, Lemma 2.15].

(b). See Example 3A.6 for a counterexample.
2D. **Sandwich and Gram matrices.** With reference to (2A.9) and (2B.23), consider the following, purely symbolic, equations:

\[
(2D.1) \quad \begin{array}{c}
\begin{array}{c}
T \quad m \\
B \quad m \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
T' \\
B' \\
\end{array}
\end{array} \equiv r_{TB} \cdot m'm \quad (\text{mod } \mathcal{A}^{>_{r,\lambda}}).
\]

The scalar \( r_{TB} \in \mathbb{K} \) is essentially given by (2A.4). This construction using half-diagrams and pairings is standard in quantum algebra and we will exploit it for our purposes. We omit the colored boxes from illustrations as they are fixed and do not play any crucial role.

**Remark 2D.2.** As already indicated in (2D.1), pairings in the sandwich setting take naturally place in \( \mathcal{H}_\lambda \) and not in \( \mathbb{K} \). The comparison to [GL96] is that all pairing therein still take values in \( \mathcal{H}_\lambda \) but we have \( \mathcal{H}_\lambda \cong \mathbb{K} \) in the cellular setting anyway.

As shown in [TV23, Lemma 2.12], the natural multiplication on the \( \mathcal{A} \mathcal{A} \)-bimodule \( \mathcal{A}_\lambda \) in (2A.5) is determined by a bilinear map \( \phi^\lambda : \nabla(\lambda) \otimes _{\mathcal{A}} \Delta(\lambda) \to \mathcal{H}_\lambda \). One can think of this map as being given by (2D.1). This map in turn gives rise to a bilinear form, denoted using the same symbol, \( \phi^\lambda : \Delta(\lambda) \to \text{Hom}_\mathbb{K}(\nabla(\lambda), \mathbb{K}) \) that we can extend via \( _- \otimes _{\mathcal{H}_\lambda} \), \( K \) is any \( \mathcal{H}_\lambda \)-module:

\[
\phi^\lambda \otimes _{\mathcal{H}_\lambda} K = \phi^\lambda \otimes _{\mathcal{H}_\lambda} \text{id}_K : \Delta(\lambda) \otimes _{\mathcal{H}_\lambda} K \to \text{Hom}_\mathbb{K}(\nabla(\lambda), \mathbb{K}) \otimes _{\mathcal{H}_\lambda} K.
\]

**Lemma 2D.3.** Let \( (\mathcal{A}, B_{\mathcal{A}}) \) be a sandwich pair. The form \( \phi^\lambda \) is determined by a \( (|\mathcal{H}_\lambda| \cdot |\mathcal{R}_\lambda|) \)-matrix with values in \( \mathcal{H}_\lambda \) given by (2D.1). Moreover, if \( \mathcal{H}_\lambda \) is semisimple, then the same is true for \( \phi^\lambda \otimes _{\mathcal{H}_\lambda} K \) with a modification (see in the proof).

**Proof.** Take \( |\mathcal{R}_\lambda| \) many representatives of the free right \( \mathcal{H}_\lambda \)-module \( \Delta(\lambda) \) and \( |\mathcal{L}_\lambda| \) many representatives of the free left \( \mathcal{H}_\lambda \)-module \( \nabla(\lambda) \). Then (2D.1) defines the element in \( \mathcal{H}_\lambda \) which determines the pairing. Note hereby that we still have \( m \) and \( m' \) in these pictures which increases the size of the matrix to \( (|\mathcal{H}_\lambda| \cdot |\mathcal{R}_\lambda|) \) times \( (|\mathcal{L}_\lambda| \cdot |\mathcal{H}_\lambda|) \).

With respect to (2D.1), the entries are then as follows:

\[
\begin{array}{c}
|B| \\
|T| \\
|m|
\end{array} = \begin{array}{c}
|B'|
\end{array} \begin{array}{c}
|T'|
\end{array} m'm \quad \text{if the product is in } \mathcal{H}_\lambda \text{ up to higher order terms},
\]

\[
0 \quad \text{else}.
\]

For general \( K \), semisimplicity ensures that every simple \( \mathcal{H}_\lambda \)-module \( K \) has at least one associated idempotent \( e_K \in \mathcal{H}_\lambda \) and the same argument as before works, but using the elements of the form

\[
(2D.4) \quad \begin{array}{c}
\begin{array}{c}
T \\
B \\
\end{array}
\end{array} \equiv r_{TB} \cdot m'm \quad (\text{mod } \mathcal{A}^{>_{r,\lambda}}).
\]

These are obtained from the previous ones by putting idempotents (denoted above as a box) at the bottom of the diagram used in the first part of the proof.

**Definition 2D.5.** Under the assumptions from Lemma 2D.3, the **sandwich matrix** \( S^\lambda,K \) (associated to \( \lambda \in \mathcal{P}^{op} \) and a simple \( \mathcal{H}_\lambda \)-module \( K \)) is the \( (|\mathcal{H}_\lambda| \cdot |\mathcal{R}_\lambda|) \times (|\mathcal{L}_\lambda| \cdot |\mathcal{H}_\lambda|) \)-matrix with values in \( \mathcal{H}_\lambda \) given by Lemma 2D.3.

For a sandwich pair \( (\mathcal{A}, B_{\mathcal{A}}) \), the **Gram matrix** \( G^\lambda \) (associated to \( \lambda \in \mathcal{P}^{op} \)) is the \( (|\mathcal{R}_\lambda|) \times (|\mathcal{L}_\lambda|) \)-matrix with values in \( \mathbb{K} \) defined by

\[
(G^\lambda)_{i,j} = \begin{cases} s(e) & \text{if } H_{ij} \text{ is idempotent with eigenvalue } s(e), \\ 0 & \text{else}. \end{cases}
\]

In other words, \( G^\lambda \) is the matrix with entries being the eigenvalues of pseudo-idempotents.
Example 2D.6. For admissible monoids the sandwich matrix is a variation of the matrix with the same name in monoid theory, see e.g. [Ste16, Section 5.4]. Precisely, the matrix given in [Ste16, Section 5.4] would be the analog of the matrix for \( \phi^{\lambda} \).

The Gram matrices for the monoid with cell structure as in (2B.8) are identity matrices of sizes 1-1 (twice), 3-3, 2-2 and 9-9.

Proposition 2D.7. Let \( K \) be a field and \((\mathcal{A}, B_{\mathcal{A}})\) be a sandwich pair.

(a) Under the assumptions from Lemma 2D.3, for \( \lambda \in P^{op} \) and a simple \( \mathcal{H}_{\lambda} \)-modules \( K \), we have

\[
\dim_K(L(\lambda, K)) = \text{rank}_K(S^{\lambda,K}).
\]

(b) For \( \mathcal{H}_{\lambda} \cong K \) and all \( \lambda \in P^{op} \) we have

\[
\dim_K(L(\lambda)) = \text{rank}_K(G^{\lambda}).
\]

Proof. (a). This follows from [TV23, Lemma 2.13] and the proof of Lemma 2D.3.

(b). For \( \mathcal{H}_{\lambda} \cong K \), the Gram matrix is the sandwich matrix, so (a) applies.

As we will see in Theorem 4A.17 and in contrast to cellular algebras, the Gram matrix is in general not enough to determine the simple dimensions and one needs to know the (in general much bigger) sandwich matrices.

Remark 2D.8. As we wrote above, the idea of using pairings and matrices to determine information about modules is everywhere in quantum algebra and related fields. Its a bit hard to track down, but see [CP61, Section 5.2] for an early reference. The above is the version in the theory of sandwich cellular algebras.

2E. Monoids and sandwich cellularity. We call a monoid \( S \) admissible if all \( J \)-cells are either idempotent with \( H \)-cells of arbitrary size, or have \( H \)-cells of size one.

Proposition 2E.1. For any admissible monoid \( S \) (with an antiinvolution) we have a(n involutive) sandwich pair \((K S, S)\).

Proof. The crucial fact about monoids we need is that left, right and \( H \)-cells in one \( J \)-cells are always of the same size, see [Gre51, Theorem 1]. In particular, we can let \( P \) be the poset coming from Green cells, we can let \( T \) and \( B \) be indexed by right and left cells, and \( B_{\lambda} = H \) to be any \( H \)-cell in \( J_{\lambda} \).

This choice satisfies \((AC_1)\) and \((AC_2)\), by construction.

We need to work a bit more for \((AC_3)\). As free \( K \)-modules we can take \( \Delta(\lambda) = K L, \nabla(\lambda) = K R \) and \( \mathcal{H}_{\lambda} = K H \) for any choice of cells. As a free \( K \)-module we thus get \( \mathcal{H}_{\lambda} \cong \Delta(\lambda) \otimes_K \mathcal{H}_{\lambda} \otimes_K \nabla(\lambda) \).

If the \( H \)-cells in \( J \) are of size one, then this construction satisfies \((AC_3)\).

The final case is when \( J_{\lambda}(e) \) is idempotent, but has arbitrary sized \( H \)-cells. In this case \( J_{\lambda}(e) \) is strictly idempotent by Example 2B.14. It then follows that \( H_{\lambda}(e) \) is a subgroup by [Gre51, Theorem 7]. We can take \( \mathcal{H}_{\lambda} = KH_{\lambda}(e) \) and \( \Delta(\lambda) = KL, \nabla(\lambda) = KR \) for the left and right cells defining \( H_{\lambda}(e) \) via their intersection. It is then easy to see (again with reference to [Gre51]) that \((AC_3)\) holds for these choices since \( H_{\lambda}(e) \) acts freely on its defining left and right cell.

Finally, having an antiinvolution on \( S \) clearly implies \((AC_4)\).

Example 2E.2. For admissible monoids the cell modules \( \Delta(\lambda) \) are the classical Schützenberger modules with their origin in [Sch58].

Proposition 2E.3. For any admissible monoid \( S \) with an antiinvolution the sandwich pair \((K S, S)\) can be refined into a cellular pair if and only if all \( K H(e) \) are cellular algebras with a compatible antiinvolution.

Proof. This follows directly from Proposition 2A.10 and the proof of Proposition 2E.1.

Remark 2E.4. The admissibility condition in Proposition 2E.1 can be removed if one works with algebras that are potentially not unital. In fact, the theory of sandwich cellular algebras that are potentially not unital runs in parallel to the one in this paper; only the technical details will change.

Remark 2E.5. The discussion in this section is new, but the question whether monoid algebras are cellular is well-studied. See e.g. [Eas06] which is very similar to Proposition 2E.3, but our point is that this follows from the general theory of sandwich cellular algebras. Our general theory can also be used to reprove the main results in [Wil07] or [GX09], which in turn generalize [Eas06].
3. Kazhdan–Lusztig bases and sandwich cellularity

The KL bases of Hecke algebras partially motivated the definition of sandwich cellularity. In this section we discuss the relation between these two notions.

3A. The classical story. Throughout this section, let $W = (W, S)$ be a (connected finite) Coxeter system. We often identify $W$ with its Coxeter diagram.

**Definition 3A.1.** For a parameter $v$ let $H_W = H_{(W, S)}$ denote the $\mathbb{Z}[v, v^{-1}]$-algebra generated by \{ $H_s | s \in S$ \} subject to the braid relations and $H_s^2 = 1 + (v^{-1} - v)H_s$.

**Notation 3A.2.** Throughout this section $\mathbb{K}$ will denote a specialization of $\mathbb{Z}[v, v^{-1}]$. Everything below is defined over $\mathbb{K}$ because the main ingredients are defined over $\mathbb{Z}[v, v^{-1}]$.

By [KL79, Theorem 1.1] there exists a distinguished basis of $H_W$ that we call the **KL basis** and that we denote by $B_W = B_{(W, S)} = \{ w | w \in W \}$. We will consider the pair $(H_W, B_W)$.

**Remark 3A.3.** The precise conventions for $H_W$ and its distinguished basis will only be of importance when we do explicit calculations in dihedral type. To not distract the reader from the main points, we specify our conventions in Section 3C after the main statements.

**Example 3A.4.** For the pair $(H_W, B_W)$ the cells are the KL cells from [KL79]. Explicitly, take $\mathbb{K} = \mathbb{C}(v)$ and consider the dihedral type Coxeter system $I_2(n)$ determined by $\circlearrowright$ with the left node called 1 and the right node called 2. Then we have the cell structure

\[
\begin{align*}
J_t \quad b_{1212} \\
J_m \quad b_{11, 121} \quad b_{12} \\
J_b \quad b_0 \\
\end{align*}
\]

$n = 4$ (type $B_2$),

\[
\begin{align*}
J_t \quad b_{1212} \\
J_m \quad b_{11, 121} \quad b_{12} \\
J_b \quad b_0 \\
\end{align*}
\]

$n = 5$ (type $H_2$): \[b_{21, 2121}, b_{12, 2121}, b_{2, 2122}\].

The pattern for general even and $n$ odd is the same; in the even case the diagonal $H$-cells have one more element than the off-diagonal $H$-cells, while they have the same size for $n$ odd.

As before, the shaded $H$-cells indicate strictly idempotent (diagonals) and idempotent but not strictly idempotent (off-diagonals) $H$-cells, respectively, in the sense of Definition 2B.13. This is where we use that $\mathbb{K} = \mathbb{C}(v)$ in this example, the rest works for any $\mathbb{K}$.

For the following theorem recall the **bar involution** of $H_W$, see [KL79, Introduction].

**Theorem 3A.5.** For a Coxeter system $W$ we have:

(a) The pair $(H_W, B_W)$ is a sandwich pair if and only if $W$ is of type $A$ or type $I_2(n)$ for odd $n$.

(b) If $W$ is of type $A$ or type $I_2(n)$ for odd $n$, then the sandwich pair $(H_W, B_W)$ is involutive with the bar involution.

(c) For type $I_2(n)$ the involutive sandwich pair $(H_W, B_W)$ comes neither from a cellular nor an affine cellular algebra, but can be refined into a cellular pair.

**Proof.** (a). If the Coxeter graph is not of type $A$ or type $I_2(n)$ for odd $n$, then one will always find $H$-cells of different sizes within one $J$-cell, see e.g. [Lus84] for Weyl types and [MMM+23, Section 8] for the other types. So these types cannot give a sandwich pair $(H_W, B_W)$ by Theorem 2B.19. Conversely, in type $A$ the KL basis is a cellular basis, as follows from [KL79], and for type $I_2(n)$ with $n$ odd sandwich cellularity can be easily verified by hand using the calculations in Section 3C.

(b). This can be proven using the results in [KL79].

(c). This follows by (a) and Proposition 2A.10 (and some care with the antiinvolution). \( \square \)

The following example shows that the assumption of $H$-cells being of the same size within one $J$-cell is crucial for Theorem 2A.17 to work.

**Example 3A.6.** In Example 3A.4 consider the non-quantum case over $\mathbb{K} = \mathbb{C}$, that is the group algebras of dihedral groups equipped with the KL basis $B_{1n}$ (we write $I_n$ for $I_2(n)$).

We consider $n = 4$ and $n = 5$, the dihedral groups $D_4$ and $D_5$ of orders 8 and 10. The character tables are (the conjugacy classes index the columns, the simple characters the rows and $\phi$ is the
golden ratio):\[
\begin{array}{c|cccc}
\chi_t & \chi^t & \chi^{m,1} & \chi^{m,2} \\
\hline
\chi_1 & 1 & 1 & 1 & 1 \\
\chi_b & 1 & 1 & -1 & 1 & -1 \\
\chi_n,1 & 1 & 1 & -1 & 1 & -1 \\
\chi_n,2 & 1 & 1 & -1 & -1 & 1 \\
\chi_n,3 & 2 & 2 & 0 & 0 & 0
\end{array}
\quad \text{D}_4;
\begin{array}{c|cccc}
\chi_t & \chi^t & \chi^{m,1} & \chi^{m,2} \\
\hline
\chi_1 & 1 & 1 & 1 & 1 \\
\chi_b & 1 & 1 & 1 & 1 \\
\chi_n,1 & 0 & 0 & 0 & 0 \\
\chi_n,2 & 2 & 0 & \sigma^t & 0
\end{array}
\quad \text{D}_5:
\] (3A.7)

Now, \((\mathbb{C}[\text{D}_5], B_{15})\) is a sandwich pair, with sandwiched algebras \(\mathcal{H}_b \cong \mathbb{C}, \mathcal{H}_m \cong \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]\) and \(\mathcal{H}_t \cong \mathbb{C}\). Thus, \(H\)-reduction Theorem 2A.17 gives the expected classification of simple \(D_5\)-modules as in (3A.7) where the subscripts indicate the associated cells.

In contrast, \((\mathbb{C}[\text{D}_4], B_{14})\) is not a sandwich pair since the algebra one gets are \(\mathcal{H}_b \cong \mathbb{C}, \mathcal{H}_m \cong \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]\) or \(\mathcal{H}_t \cong \mathbb{C}\). The \(H\)-reduction for the middle cell fails: neither \(\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]\) nor \(\mathbb{C}\) give the expected count of three simple \(D_4\)-modules.

For \(K\) let \(p \in \mathbb{Z}_{\geq 0}\) be minimal such that \(p \cdot 1 = 0 \in K\), and let \(p = \infty\) if no such \(p\) exists. That is, \(p = \text{char}(K)\) but we consider \(\text{char}(K) = 0 = p = \infty\). We will write \(\text{char}^{0=\infty}(K)\) if we want to see \(\text{char}(K) = 0 \) as \(p = \infty\).

We use the English convention for Young diagrams, i.e. we draw left-justified rows of boxes successively down the page. Let \(P(\lambda|p)\) the set of \(p\)-restricted Young diagrams with \(\lambda\) boxes. That is, each element of \(P(\lambda|p)\) is a Young diagram with \(\lambda\) boxes such that the difference between the length of two consecutive columns is \(< p\) (empty columns are of length zero). Note that the set \(P(\lambda|p)\) for \(p > \lambda\) is the set of all Young diagrams with \(\lambda\) boxes. The set \(P(\lambda|p) = \{\lambda\}\) is a poset when equipped with the dominance order \(<_d\) (with conventions specified by Example 3A.8).

Example 3A.8. For example, \(p > 3: P(3|p) = \{\square \square <_d \square <_d \square \}, p \in \{2,3\}: P(3|p) = \{\square \square <_d \square \square \},\)

are the sets of \(p\)-restricted partitions of 3 for all possible \(p\).

For \(a \in \mathbb{Z}_{\geq 0}\), let \([a] = \frac{v^{-a} - v^a}{v - v^{-1}} \in \mathbb{Z}[v, v^{-1}]\) denote the usual quantum numbers. Below we will need the condition that \(K = F(q)\) for \(F\) an algebraically closed field and \(K\) is such that quantum numbers \([a]\) evaluated at \(q\) are invertible. We call such fields admissible.

Example 3A.9. The fields \(K = \mathbb{C}(v)\) or \(K = \mathbb{C}\), where \(q = v\) and \(q = 1\) respectively, are examples of admissible fields.

Theorem 3A.10. Consider the case where \((H_W, B_W)\) is an involutive sandwich pair.

(a) Let \(W\) be of type A.
(i) The \(J\)-cells of \(H_A\) are given by \(P = \{P(\lambda|\infty), <_d\}\).
(ii) The left cells and right cells of \(H_A\) are determine by the Robinson–Schensted correspondence.
(iii) A \(J\)-cell of \(H_A\) is strictly idempotent if and only if its corresponding partition is \(p\)-restricted. For strictly idempotent \(J\)-cells and \(K\) a field we have \(\mathcal{H}_t \cong K\).
(iv) Let \(K\) be a field. The set of apexes for simple \(H_A\)-modules is \(P^{\text{op}} = \{P(\lambda|\text{char}^{0=\infty}(K)), <_d\}\), and there is precisely one simple \(H_A\)-module \(L(\lambda)\) for \(\lambda \in P^{\text{op}}\).

(b) Let \(W\) be odd dihedral type \(I_2(n)\).
(i) The \(J\)-cells of \(H_{I_n}\) are given by \(P = \{b <_{I_r} m <_{I_r} t\}\).
(ii) \(J_b\) and \(J_t\) are left and right cells, while the left cells in \(J_m\) are given by reduced expressions starting with either 1 or 2, and the right cells by reduced expressions that end with either 1 or 2.
(iii) \(J_b\) is always strictly idempotent, \(J_m\) is strictly idempotent if \([2]\) is invertible in \(K\), and \(J_m\) is strictly idempotent if \([2][n]\) is invertible in \(K\). Assume \(K\) is an admissible field. Then we have \(\mathcal{H}_b \cong K, \mathcal{H}_m \cong K[\mathbb{Z}/(\frac{n-1}{2})\mathbb{Z}], \mathcal{H}_t \cong K\).
(iv) Let $\mathbb{K}$ be an admissible field. The set of apexes for simple $H_{I^2}$-modules is $p^{op} = p$, and there is precisely one simple $H_{I^2}$-module for $\lambda \in \{b, t\} \subset p^{op}$, and there are precisely $\frac{n-1}{2}$ simple $H_{I^2}$-modules for $m \in p^{op}$. The dimensions of the simple $H_{I^2}$-modules are 1, 1 and 2, the latter $\frac{n-1}{2}$ times.

(c) Let $\mathbb{K}$ be an admissible field. We have that $H_{A}$ and $H_{I^2}$ for $n$ odd are semisimple.

Proof. (a). Proving how the cell structure looks like is not trivial, but still well-known and follows from [KL79]. Thus, it remains to verify part (iv). That $P^{op} = P(\lambda | \text{char}^{\geq 0}(\mathbb{K}))$ follows from the calculation of eigenvalues of distinguished involutions in the Hecke algebra, see [Lus03, Conjectures 14.2 and Chapter 15] (we mean the arXiv version of the book). The rest is then Theorem 3A.5 and $H$-reduction Theorem 2A.17.

(b). This follows from Section 3C below. That is, the hardest parts are proven in Lemma 3C.3 and Lemma 3C.4. The rest is easy to prove using the explicit description of the multiplication of KL basis elements given in Section 3C.

(c). By the above and Proposition 2B.24, it remains to show that the $\Delta(\lambda)$ are simple. However, we already know the dimensions of the simples by (a).(iv) and (b).(iv), and the proof completes. □

Example 3A.11. Back to Example 3A.6 for $n = 5$. The sandwich matrices for the bottom and top $J$-cell are $S^b = (b_9)$ and $S^t = (10 \cdot b_{12121})$, respectively. We compute the sandwich matrices $S^{m,1}$ and $S^{m,2}$ in the proof of Lemma 3C.4. These matrices are of rank 2. Thus, the bottom and top $J$-cells give one dimensional simple $\mathcal{A}$-modules, while the middle $J$-cell gives two simple $\mathcal{A}$-modules of dimension 2.

Remark 3A.12. Theorem 3A.5 also shows that the KL basis is cellular if and only if $W$ is of type $A$. That the KL basis of type $A$ is cellular follows from [KL79], as we wrote above. The converse is known, but hard to find in the literature. The case of general sandwich cellularity is new.

We point out that all finite type Hecke algebras are sandwich cellular, even cellular by [Gec07], and the statement in Theorem 3A.5 is that the KL basis is not a sandwich cellular basis.

Finally, our sandwich approach to classify simple $H_{I^2}$-modules from Theorem 3A.10 is new, but the results are certainly not new: In the semisimple case the classification follows directly from Tits’ deformation theorem (explicitly, [GP00, Theorem 8.1.7]) and classical theory. We also stress that the approach taken in Theorem 3A.10 with a bit more work, see Remark 3C.6 below, also classifies simple $H_{I^2}$-modules over arbitrary fields. This classification is again well-known, see e.g. [GP00, Chapter 8] for a concise treatment and bibliographical remarks, but our point is that it also follows as part of the general theory of sandwich cellular algebras.

Note that, by Proposition 2C.1, $J$-reduction Theorem 2A.17 works for all pairs $(H_W, B_W)$ in arbitrary type (not necessarily connected or finite). This recovers an result of e.g. [Lus84, Section 4].

3B. $p$-Kazhdan–Lusztig bases and sandwich cellularity. For a fixed prime $p$ e.g. [JW17] defines another distinguished basis $B^p_W = \{b^p_w \mid w \in W\}$ of $H_W$ that we call the $p$-KL basis. Not much is known about the (sandwich) cellularity of the $p$-KL basis, and all we can say is:

Theorem 3B.1.

(a) If $W$ is of type $A$, then $(H_W, B^p_W)$ is a sandwich pair for all primes, and the cell structure is the same as for $(H_W, B_W)$.

(b) If $W$ is of type $B_2, C_2, G_2, B_3, C_3, B_4, C_4$ or $D_4$, then $(H_W, B^p_W)$ is never a sandwich pair.

(c) Let $W$ be a Weyl group that is not of type $A$. Varying over all primes $p$, the pair $(H_W, B^p_W)$ is not a sandwich pair up to finitely many potential exceptions.

Note that Theorem 3A.5.(c) and Theorem 3B.1.(a) give the same classification of simple $H_W$-modules via $H$-reduction Theorem 2A.17.

Proof. (a). This is [Jen20b, Theorem 5.14].

(b). We will show that $(H_W, B^p_W)$ cannot be a sandwich pair by contradicting Theorem 2B.19.(d). For $p > 2$ in types $B_2$ and $C_2$ and for $p > 3$ in type $G_2$ the $p$-KL basis is the usual KL basis, so Theorem 3A.5 applies. For the remaining cases we simply list the $p$-KL cells, which have $H$-cells of
different sizes:

\[
\begin{array}{c|c|c}
\text{type } B_2: & b_{1212}^2 & b_{212}^2 \\
\hline
b_{21}^2 & b_{221}^2 & b_{212}^2, b_{21212}^2 \\
\end{array}
\quad
\begin{array}{c|c|c}
\text{type } C_2: & b_{21212}^2 & b_{212}^2 \\
\hline
b_{21}^2 & b_{221}^2 & b_{212}^2, b_{21212}^2 \\
\end{array}
\quad
\begin{array}{c|c|c}
\text{type } G_2: & b_{1212}^2 & b_{212}^2 \\
\hline
b_{212}^2 & b_{21212}^2 \\
\end{array}
\quad
\begin{array}{c|c|c}
\text{type } D_\infty: & b_{1212}^2 & b_{212}^2 \\
\hline
b_{21}^2 & b_{212}^2 & b_{21212}^2, b_{2121212}^2, b_{212121212}^2 \\
\end{array}
\]

Here we used \(\leftrightarrow\), \(\rightleftharpoons\) and \(\iff\) with the same labeling of vertices as in the dihedral case. Similar calculations verify the remaining cases. Note hereby that for \(p\) big enough we can use Theorem 3A.5 since the \(p\)-KL basis and the KL agree for big enough primes, and there are only finitely many cases left to check which we verified with computer help (one can verify this without computer, but we did it by computer).

\((c)\). This follows from [JW17, Proposition 4.2, part (7)] and Theorem 3A.5.(a). \(\square\)

**Remark 3B.2.** For finite Coxeter systems of Dynkin type calculations suggest that \((H_W^p, B_W^p)\) is a sandwich pair if and only if \(W\) is of type \(A\).

**Remark 3B.3.** The statements in Theorem 3B.1 are reformulations of the vast literature on \(p\)-KL cells, see e.g. [JW17], [Jen20a] or [Jen20b], into the theory of sandwich cellular algebras.

### 3C. Kazhdan–Lusztig structure constants in dihedral type

Consider \(\leftarrow\) with nodes labeled 1 and 2. The Coxeter group associated to this Coxeter diagram is the infinite dihedral group \(D_\infty = \langle 1, 2 | i^2 = 2^i = 1 \rangle\). Every element of \(D_\infty\) has a unique reduced expression and we write \(k21\) and \(k12\) for the reduced expressions \(\ldots 21\) and \(\ldots 12\) in \(k\) symbols. For example \(521 = 121212\).

The associated Hecke algebra \(H_{D_\infty}\) has a KL basis \(\{b_w | w \in D_\infty\}\) (whose precise definition does not matter) with identity \(b_\emptyset\). Set \(b_{0ab} = 0\). The nonidentity multiplication rules are given by the **(scaled) Clebsch–Gordan formula** where the steps size is two:

\[
b_{k12}b_{j12} = \begin{cases} 
\left[2\right]b_{(k+j)|12} + \ldots + \left[2\right]b_{(k+j)-1|12} & j12=21\ldots12, \\
\left[2\right]b_{k12j} + 2\left[2\right]b_{(k+j)+1|2} + \ldots + 2\left[2\right]b_{(k+j)-2|12} + b_{k+j|12} & j12=12\ldots12. 
\end{cases}
\]

There are also similar formulas with \(b_{j21}\) on the right and \(b_{2k1}\) on the left (in total four configurations).

Let \(D_n = \langle 1, 2 | i^2 = 2^i = \left(12\right)^n = 1 \rangle\) be the dihedral group of the \(n\) gon. The longest element is \(w_0 = n12 = n21\). Let \(\left[2\right]_i = (v^i + v^{-i})\). With respect to the KL basis and its multiplication rules, the only change compare to \(D_\infty\) is that expressions of the form \((\text{here } d > 0)\)

\[
b_{(n-d)|12} + b_{(n+d)|12} \mapsto \left[2\right]_d b_{w_0}, \quad b_{(n-d)|21} + b_{(n+d)|21} \mapsto \left[2\right]_d b_{w_0},
\]

are replaced as indicated. This is the **(scaled) truncated Clebsch–Gordan formula**.

**Example 3C.1.** For the infinite dihedral group we get

\[
b_{1212}b_{21212} = \left[2\right]b_{12} + \left[2\right]b_{1212} + \left[2\right]b_{121212} + \left[2\right]b_{12121212},
\]

\[
b_{1212}b_{121212} = b_{12} + 2b_{1212} + 2b_{121212} + b_{12121212} + b_{1212121212},
\]

from the Clebsch–Gordan formula. Moreover,

\[
b_{1212}b_{21212} = \left[2\right]b_{12} + \left[2\right]b_{1212} + \left[2\right]b_{121212} + \left[2\right]b_{12121212} = \left[2\right]b_{12} + \left(\left[2\right]_3 + \left[2\right]_2\right)b_{121212},
\]

\[
b_{1212}b_{121212} = b_{12} + 2b_{1212} + 2b_{121212} + 2b_{12121212} + b_{1212121212} = \left(\left[2\right]_4 + \left[2\right]_2 + \left[2\right]_0\right)b_{12121212},
\]

are calculations for \(n = 6\). Here we used \(\left[2\right]_2\left[2\right] + \left[2\right]_3 = \left[2\right]_3 + \left[2\right]_2\) and \(\left[2\right]_0 = 2\). \(\diamond\)
The next lemma can be easily proven using the truncated Clebsch–Gordan formulas.

**Lemma 3C.2.** For the pair \((H_W, B_W)\) the cell structure for \(H_{I_n}\) is as in Example 3A.4. \(\square\)

**Lemma 3C.3.** Let \(\mathcal{K}\) be an admissible field. For the involutive sandwich pair \((H_W, B_W)\) in odd dihedral type \(I_2(n)\) we have \(\mathcal{H}_n \cong \mathcal{K}[\mathbb{Z}/(\frac{n-1}{2})\mathbb{Z}]\).

**Proof.** We focus on the \(H\)-cell containing \(b_1\). Let \(c_{121} = \frac{1}{\sqrt{2}}b_{121}\). Note that \(n \geq 3\) and \(n\) odd.

Let \(P_0(X) = 1\), \(P_1(X) = X\) and \(P_{k+1}(X) = (X - 1)P_k(X) - P_{k-1}(X)\) for \(k > 1\). The polynomial \(P_{n-1}(X)\), known as multiplication by \([3]\), is the characteristic polynomial of the graph

\[
\begin{array}{ccccccc}
& & & & & & \\
| & | & | & | & | & |
\end{array}
\]

This graph is the fusion graph of \(SO_q(3)\), the semisimplification of quantum \(\mathfrak{so}_3(\mathbb{C})\) tilting modules for \(q = \exp(\pi i / n)\). The illustrated number below the vertices are the values of the Perron–Frobenius eigenvector of the graph.

The Clebsch–Gordan formulas show that this graph is also the action graph of \(c_{121}\) on the \(H\)-cell containing \(b_1\), up to rescaling of the KL basis, and we get \(P_{(n-1)/2}(c_{121}) = 0\) via the identification of \(c_{121}\) with the Grothendieck class of the defining representation of \(\mathfrak{so}_3(\mathbb{C})\).

Hence, we have \(\mathcal{H}_n \cong \mathcal{K}[X]/(P_{(n-1)/2}(X))\). It is known that the polynomial \(P_k(X)\) has \(k\) distinct roots. Thus, \(\mathcal{H}_n \cong \mathcal{K}[X]/(P_{(n-1)/2}(X))\) and the Chinese reminder theorem imply the lemma. \(\square\)

**Lemma 3C.4.** Let \(\mathcal{K}\) be an admissible field. For the involutive sandwich pair \((H_W, B_W)\) the ranks of the sandwich matrices are 1 for the bottom, 2 for all middle ones, and 1 for the top.

The involved calculations are a bit ugly and, for the sake of brevity, we only sketch the proof:

**Sketch of a proof of Lemma 3C.4.** The claim is immediate for the bottom and top \(J\)-cells. For \(J_m\) we first compute the general pairing matrix. We scale such that we do not need to worry about \([2]\) and write \(c_w = \frac{1}{\sqrt{m}}b_w\). Take \(C = (C_1, \ldots) = (c_1, c_{121}, \ldots, c_{121}, c_{1212}, \ldots)\) and \(R = (R_1, \ldots) = (c_1, c_{121}, \ldots, c_{21}, c_{2121}, \ldots)\) as index sets for columns and rows. Next, we write down a matrix \(M\) with the \(i\)-\(j\) entry being \(R_iC_j (\mod \mathcal{K}\{b_{w_0}\})\).

For \(n = 5\) we for example get

\[
M = \begin{bmatrix}
c_1 & c_{121} & c_{1212} \\
c_{121} & c_1 + c_{121} & c_{1212} + c_{1212} \\
c_{1212} & c_{21} + c_{2121} & c_2 + c_{2121}
\end{bmatrix}
\]

The isomorphism constructed in the proof of Lemma 3C.3 implies that the matrix \(M\), up to scaling, can be replaced by a matrix \(N\) with four blocks corresponding to \(\mathcal{K}[\mathbb{Z}/(\frac{n-1}{2})\mathbb{Z}]\). For example, for \(n = 5\) one gets

\[
N = \begin{bmatrix}
& d_1 & d_{121} & d_{12} & d_{1212} \\
& d_{121} & d_1 & d_{121} & d_{12} & d_{1212} \\
& d_{12} & d_{121} & d_1 & d_{121} & d_{12} & d_{1212} \\
& d_{1212} & d_{21} & d_{2121} & d_1 & d_{2121} & d_{212} & d_{2}
\end{bmatrix}
\]

In this example, the units in the four blocks corresponding to \(\mathcal{K}[\mathbb{Z}/2\mathbb{Z}]\) are \(d_1, d_{1212}, d_{2121}\) and \(d_2\).

In general the units are \(d_1, d_{(n-1)/2}, d_{(n-1)/2}d_2\) and \(d_2\).

The idempotents for the simple \(\mathcal{K}[\mathbb{Z}/(\frac{n-1}{2})\mathbb{Z}]\)-modules are, of course, easy to write down, and it is then also easy to show that the ranks of the associated sandwich matrices are rank\(\mathcal{K}(\mathbb{S}^{m,i}) = 2\). For example, for \(n = 5\) the idempotents for the northwest corner are \(e_1 = d_1 + d_{121}\) and \(e_2 = d_1 - d_{121}\). Let \(e_3 = d_{1212} + d_{12}\) be the idempotent for the southwest corner. For \(e_1\) we get the matrix

\[
N_{e_1} = \begin{pmatrix}
e_1 & e_1 & e_1 & e_1 \\
e_1 & e_1 & e_1 & e_1 \\
e_3 & e_3 & e_3 & e_3 \\
e_3 & e_3 & e_3 & e_3
\end{pmatrix}
\]

The matrix is clearly of rank two, which shows that the sandwich matrix for \(e_1\) is also of rank two. The rank of the sandwich matrix for \(e_2\) can be computed similarly. \(\square\)
Example 3C.5. With appropriate care one can use $H$-reduction Theorem 2A.17 for arbitrary fields to classify simple $D_n$-modules for $n$ odd as well as their quantum counterparts.

For example, take $n = 5$ and $v = 1$ and let us exclude char($\mathbb{K}$) = 2. The polynomial from the proof of Lemma 3C.3 defining the quotient is $P_2(X) = X^2 - X - 1$ whose roots are the golden ratio $\phi$, interpreted in $\mathbb{K}$, appearing in (3A.7) and its conjugate. (For the general case note that $\phi$ is specialized at $q = \exp(\pi i/n)$ with $n = 5$.) Thus, unless char($\mathbb{K}$) = 5, we can apply $H$-reduction for the middle $J$-cell and obtain two simple $D_5$-modules of that apex. For char($\mathbb{K}$) = 5 one needs to distinguish the cases when $\mathbb{K} = \mathbb{F}_{25}$ and $\mathbb{K} = \mathbb{F}_{5}$, but one still gets the correct number of simple $D_5$-modules with the middle apex.

In general, we cannot rescale the KL basis and the polynomial defining the quotient is given by $P'_0(X) = 1, P'_1(X) = X$ and $P'_{k+1}(X) = (X - [2])P'_k(X) - [2]P'_{k-1}(X)$ for $k > 1$.

Remark 3C.6. As in Example 3C.5, it is not hard to discuss the case for general $\mathbb{K}$, but there is some annoying rescaling involved so we decide not to include this into this paper.

Remark 3C.7. The multiplication rules for the KL basis of the dihedral group are well-known. The above, in particular, is a reformulation of [dC06, Section 4]. The application of sandwich cellularity to dihedral representation theory, e.g. Lemma 3C.2, Lemma 3C.3 and Lemma 3C.4, is new.

4. Diagram algebras and sandwich cellularity

We now discuss several examples of diagram algebras, all of which fit into the theory of sandwich cellular algebras.

4A. Some non-involutive diagram algebras. Let $S_n = \text{Aut}(\{1, \ldots, n\})$ be the symmetric group on the set $\{1, \ldots, n\}$, and in this paper the planar symmetric group is $S^n_p = 1$, independent of $n$. We now discuss two examples, which in some sense are analogs of the (planar) symmetric group in the theory of monoids.

Definition 4A.1. The transformation monoid $T_n$ on the set $\{1, \ldots, n\}$ is $\text{End}(\{1, \ldots, n\})$.

Note that the subgroup of invertible elements $G \subset T_n$ is isomorphic to $S_n$ and we will identify $S_n$ as a subgroup of $T_n$ in this way.

The elements of $T_n$ can be written in one-line notation with $(ijk\ldots)$ denoting the map $1 \mapsto i$, $2 \mapsto j$, $3 \mapsto k$ etc. Alternatively we can model $T_n$ using string diagrams as follows.

We consider isotopy classes of diagrams of $2n$ points in the rectangle $[0,1] \times [0,1]$, with $n$ equally spaced points at the bottom and top. For $(ijk\ldots)$, connect the first bottom point with the $i$th on the top, the second with the $j$th on the top, the third with the $k$th on the top and so on. Two diagrams represent the same element if and only if they represent the same map in $T_n$. For example:

\begin{equation}
\begin{array}{c}
(24138567) \sim \sim \sim \sim \sim, \\
(23153555) \sim \sim \sim \sim \sim.
\end{array}
\end{equation}

The dots on strings used in the pictures are reminders that there are top point not in the image of the displayed maps. The first two diagrams are string diagrams for elements of $S_8 \subset T_8$.

Notation 4A.3. We have crossings, merges and top dots, which are:

crossings: $\big\downarrow \big\uparrow$, merges: $\big\downarrow \big\uparrow \big\downarrow \big\uparrow \ldots$, top dots: $\big\cdot$.

We will use this terminology throughout this section.

The diagrammatic multiplication $\circ$ in the transformation monoid is given by vertical gluing (and rescaling), using the convention from Notation 2A.1. For example,

\[ a \circ b = \]
Definition 4A.4. The \textit{planar transformation monoid} $T^n_p$ on the set \{1, ..., $n$\} is the submonoid of $T_n$ for which we can draw planar string diagrams without leaving the defining rectangle $[0, 1] \times [0, 1]$.

In other words, $T^n_p$ does not have any crossings.

Remark 4A.5. In the semigroup literature $T_n$ is also called \textit{full transformation monoid} and $T^n_p$ is called \textit{order-preserving transformation monoid}.

Example 4A.6. From the four elements displayed in (4A.2) only the southeast is in $T^n_p$. Moreover, we have $T^n_p \cap S_n = \{id\}$, where $id$ is the unit in $T_n$.

Using the diagrammatic description we find:

Lemma 4A.7. For $a \in T_n$ there is a unique factorization of the form $a = \tau \circ \sigma \circ \beta$ such that $\beta$ has a minimal number of crossings, $\tau$ has no crossings, $\beta$ contains no top dots, $\tau$ contains no merges and $\sigma \lambda \in S_\lambda$ for minimal $\lambda$.

Similarly for $a \in T^n_p$ but with $\sigma \lambda \in S^n_p = 1$.

Proof. The rearrangement

\[ a = \begin{array}{c}
\Downarrow \\
\sigma_4 \\
\tau \\
\Downarrow \\
\beta
\end{array} \]

generalizes immediately. \hfill \Box

With respect to the factorization in Lemma 4A.7, we call $\lambda$ the number of \textit{through strands}, $\beta$ the \textit{bottom}, $\tau$ the \textit{top} and $\sigma \lambda \in S_\lambda$ the \textit{middle} of $a$. The middle in Lemma 4A.7 also motivates:

Definition 4A.8. We define the \textit{symmetric} $KT_n = \mathbb{K}[T_n]$ and the \textit{planar diagram algebra} $KT^n_p = \mathbb{K}[T^n_p]$ associated the (planar) transformation monoid.

Notation 4A.9. We write $D_n$ for either $T_n$ or $T^n_p$.

Let \{\binom{n}{\lambda}\} denote the $(n, \lambda)$th Stirling number (of the second kind), see also Remark 4A.11 below.

Proposition 4A.10. We have the following for the pair $(KD_n, D_n)$.

(a) The $J$-cells of $KD_n$ are given by diagrams with a fixed number of through strands $\lambda$. The $\leq_{lr}$-order is a total order and increases as the number of through strands decreases. That is,

\[ \mathcal{P} = \{ n <_{lr} n - 1 <_{lr} ... <_{lr} 1 \}. \]

(b) The left cells of $KD_n$ are given by diagrams where one fixes the bottom of the diagram, and similarly right cells are given by diagrams where one fixes the top of the diagram. The $\leq_{l}$ and the $\leq_{r}$-order increases as the number of through strands decreases. Within $\mathcal{J}_\lambda$ we have

\[ \mathbb{K}T^n_p : \# \mathcal{L}_\lambda = \binom{n}{\lambda}, \quad \mathbb{K}T^n_p : \# \mathcal{R}_\lambda = \binom{n-1}{\lambda-1}, \quad \mathbb{K}D_n : \# \mathcal{R}_\lambda = \binom{n}{\lambda}. \]

(c) Each $J$-cell of $KD_n$ is strictly idempotent, and

\[ \mathbb{K}T^n_p : \mathcal{H}_\lambda \cong \mathbb{K}[S_\lambda], \quad \mathbb{K}T^n_p : \mathcal{H}_\lambda \cong \mathbb{K}[S^n_p] \cong \mathbb{K}. \]

(d) The pair $(KD_n, D_n)$ is a sandwich pair, that comes neither from a cellular nor an affine cellular algebra.

(e) Let $\mathbb{K}$ be a field with char($\mathbb{K}$) = 0. For $\lambda \neq 1$ we have

\[ \mathbb{K}T^n_p : \text{rank}_\mathbb{K}(G^\lambda) = \binom{n}{\lambda}, \quad \mathbb{K}T^n_p : \text{rank}_\mathbb{K}(G^\lambda) = \binom{n-1}{\lambda-1}. \]

We also have \text{rank}_\mathbb{K}(G^1) = 1 for both.

Note that the numerical data given in Proposition 4A.10 is sufficient to determine the numbers and sizes of left, right, $J$ and $H$-cells by Lemma 2B.22.
Remark 4A.11. Very similar to binomials, the **Stirling numbers** \( \{ \lambda \} \) count certain combinatorial set partitions, and thus appear very often for diagram algebras. Explicitly, \( \{ \lambda \} \) counts the number of ways to partition a set of \( n \) labeled objects into \( \lambda \) nonempty unlabeled subsets. Not surprisingly, \( \{ \lambda \} \) also admit a triangle description:

\[
\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc...
Example 4A.13. For \( n = 3 \) we get

\[
\begin{align*}
&J_t & \mathcal{H}_t & \cong \mathbb{K}[S_1] \\
&J_m & \mathcal{H}_m & \cong \mathbb{K}[S_2] \\
&J_b & \mathcal{H}_b & \cong \mathbb{K}[S_3]
\end{align*}
\]

which display the cells of \( KT_3 \) in diagrammatic notation and in one-line notation. For \( KT_p^3 \) one gets

\[
\begin{align*}
&J_t & \mathcal{H}_t & \cong \mathbb{K}[S_1] \\
&J_m & \mathcal{H}_m & \cong \mathbb{K}[S_2] \\
&J_b & \mathcal{H}_b & \cong \mathbb{K}[S_3]
\end{align*}
\]

(We have not illustrated the cell structure in one-line notation.)

Note that there is redundant information in the cell pictures in Example 4A.13. For example, the \( H \)-cells of \( KT_n \) within \( J_\lambda \) are always of size \( \lambda ! \), so we actually only need to remember the idempotent \( H \)-cells. GAP’s package Semigroups does exactly that:

Example 4A.14. The cell structures of \( KT_n \) and \( KT_p^3 \) are of the form

where \( n = 3, n = 4 \) and \( n = 5 \), respectively. The Gram matrices, except the topmost, have rank given by the number of rows, e.g. the \( (5 \times 10) \)-matrix for \( KT_5 \) has rank 5.

Note also that we need \( \text{char}(\mathbb{K}) = 0 \), since, for example, the Gram matrix \( G^2 \) for \( KT_3 \) has determinant \(-2\). 

\( \diamond \)
Theorem 4A.15. Assume that $\mathbb{K}$ is a field, and consider the sandwich pair $(\mathbb{K}D_n,D_n)$. The set of apexes for simple $\mathbb{K}D_n$-modules is $\mathcal{P}^{op} = \{ n <_{tr} n-1 <_{tr} \ldots <_{tr} 1 \}$, and there are precisely $|P(\lambda|\text{char}^{\mathbb{K}}(\mathbb{K}))|$ simple $\mathbb{K}T_n$-modules and one simple $\mathbb{K}T'_n$-module for $\lambda \in \mathcal{P}^{op}$.

Proof. By Proposition 4A.10 and $H$-reduction Theorem 2A.17, it suffices to identify the simple $\mathcal{H}_\lambda$-modules. This is immediate for $\mathbb{K}T'_n$. For $\mathbb{K}T_n$ we use the classical result that one can index simple $S_\lambda$-modules by $P(\lambda|\text{char}^{\mathbb{K}}(\mathbb{K}))$, see for example [Mat99, Theorem 3.43]. □

Example 4A.16. Continuing Example 4A.14, and let $\mathbb{K}$ be a field with $\text{char}(\mathbb{K}) \nmid 5!$. Then the number of simple $\mathbb{K}T_n$-modules for $n = 5$ are given by $(7,5,3,2,1)$, where we ordered them by apex reading from the bottom to the top. For $\mathbb{K}T'_n$ the sequence is $(1,1,1,1,1)$. □

By convention, the trivial and the sign $S_1$-modules are the same. This is relevant for the top cell in the following theorem.

Theorem 4A.17. Assume that $\mathbb{K}$ is a field with $\text{char}(\mathbb{K}) = 0$ or $\text{char}(\mathbb{K}) \gg 0$ (meaning up to finitely many characteristics), and consider the sandwich pair $(\mathbb{K}D_n,D_n)$.

(a) For $\mathbb{K}T_n$, if $K$ is simple but not the sign $\mathcal{H}_\lambda$-module, then we have

$$\dim_{\mathbb{K}}(L(\lambda, K)) = \binom{n}{\lambda} \cdot \dim_{\mathbb{K}}(K)$$

for $\lambda \in \mathcal{P}^{op}$. If $K$ is the sign $\mathcal{H}_\lambda$-module, then

$$\dim_{\mathbb{K}}(L(\lambda, K)) = \binom{n-1}{\lambda-1}$$

for $\lambda \in \mathcal{P}^{op}$.

(b) For $\mathbb{K}T'_n$ and $\lambda \in \mathcal{P}^{op}$ we have

$$\dim_{\mathbb{K}}(L(\lambda)) = \binom{n-1}{\lambda-1}.$$

Remark 4A.18. Note that the dimensions of the simple $S_\lambda$-modules that appear in Theorem 4A.17 can be computed using well-known formulas since we assume that $\text{char}(\mathbb{K}) = 0$. Thus, Theorem 4A.17 gives an explicit description of the dimensions of the simple $\mathbb{K}D_n$-modules.

In contrast, the characteristic $p$ version of Theorem 4A.17 is much more difficult. In fact, to the best of our knowledge, the problem of determining the dimensions of simple $\mathbb{K}D_n$-modules in characteristic $p$ is still open and for $D_n = T_n$ that result would generalize the problem of finding the dimensions of the simple $\mathbb{K}S_\lambda$-modules.

Remark 4A.19. The dimensions of the simple $\mathbb{K}D_n$-modules computed in Theorem 4A.17 are fairly large. This could potentially be relevant for cryptographical purposes following [KST24].

Proof of Theorem 4A.17. The calculations below use invertibility of certain, finitely many, numbers. These numbers are always invertible if $\text{char}(\mathbb{K}) = 0$ but also for $\text{char}(\mathbb{K}) \gg 0$ since we only need to invert finitely many numbers.

(a). The proof splits into two parts. We will see why we get two different cases using the diagrammatic description of the primitive idempotents of $\mathbb{K}[S_\lambda]$ due to (Gyoja-)Aiston(-Morton) [AM98], see also [TVW17, Definition 2.26]. Fix $\mu \in P(\lambda|\infty)$, seen as a Young diagram, with $\mu_1',\ldots,\mu_k'$ rows and $\mu_1,\ldots,\mu_k$ columns. Note that $\mu_1' + \ldots + \mu_k' = n = \mu_1 + \ldots + \mu_k$. The full symmetrizer for $\mu_1'$ is the sum of all symmetric group elements on the respective strands, and the full antisymmetrizer for $\mu_1'$ is the signed sum of all symmetric group elements on the respective strands. Define $e_{\text{row}}(\mu)$ and $e_{\text{col}}(\mu)$ to be the tensor product (horizontal juxtaposition of strands) of the full symmetrizers and full antisymmetrizers associated to the rows respectively columns of $\mu$. Let $w$ be any shortest presentation (with respect to simple transpositions) of the permutation that permutes the row standard filling of $\mu$ to the column standard filling of $\mu$. Then, for some nonzero scalar $s \in \mathbb{K}$, $e_{\mu} = \frac{1}{s} \cdot e_{\text{row}}(\mu) \cdot (id \otimes w) \cdot e_{\text{col}}(\mu) \cdot (id \otimes w^{-1})$ is a primitive idempotent in the group algebra of $S_\lambda$ projecting to the simple $\mathbb{K}[S_\lambda]$-module $K = K(\mu)$ associated to $\mu$.

As not unusual in these types of diagrammatics, we draw symmetrizers as reddish shaded boxes, and antisymmetrizer as greenish shaded boxes labeled by their number of strands and an $s$ or an $a$.
to distinguish the boxes. Let $w$ be the permutation (23)(45)(34) in this presentation. Illustrating $w$ and $w^{-1}$ also as shaded boxes, we get

\[
\mu = \begin{array}{c}
\text{3s} \\
\text{w} \\
\text{2s}
\end{array}; \quad e_\mu = \frac{1}{s},
\]
as the primitive idempotent associated to $\mu$. We have the following relations:

\[
(4A.20) \quad \begin{array}{c}
\begin{array}{c}
\text{3s} \\
\text{w} \\
\text{2s}
\end{array} = \begin{array}{c}
\text{3} \\
\text{w} \\
\text{2}
\end{array}, \\
\begin{array}{c}
\text{3s} \\
\text{w} \\
\text{2s}
\end{array} = \begin{array}{c}
\text{3} \\
\text{w} \\
\text{2}
\end{array}, \\
\begin{array}{c}
\text{3s} \\
\text{w} \\
\text{2s}
\end{array} = \begin{array}{c}
\text{2} \\
\text{3} \\
\text{w}
\end{array}, \\
\begin{array}{c}
\text{3s} \\
\text{w} \\
\text{2s}
\end{array} = 0.
\end{array}
\]

That is, the (anti)symmetrizers eat crossings, while merges eat symmetrizers but annihilate antisymmetrizers. As we will see, the case where there is no symmetrizer box is special as all merges are annihilated.

**Case 1:** $K$ is not the sign $\mathcal{H}_\lambda$-module. Assume that $K$ is simple but not the sign $\mathcal{H}_\lambda$-module. We will show that $\Delta(\lambda)e_K$ is a simple $\Bbbk T_n$-module. Note that this implies $\dim_{\Bbbk}(L(\lambda, K)) = \binom{n}{\lambda} \cdot \dim_{\Bbbk}(K)$, which is what we wanted to prove.

To prove that $\Delta(\lambda)e_K$ is a simple $\Bbbk T_n$-module, we use induction on $n - \lambda$. If $n - \lambda = 0$, then we are in the bottom cell and the claim is clear. So let $n - \lambda = k$ and assume that we have proven the claim for $n - \lambda < k$. Define idempotents

\[
e_{i,j} = [\begin{array}{c}
i \\
j
\end{array}], \quad 1 \leq i < j \leq n,
\]

that merge the $i$th and $j$th strands and are the identity everywhere else. Note that the top of the diagrams for the $e_{i,j}$ have $n - 1$ strands.

For $x \in \Delta(\lambda)e_K$ suppose that $e_{i,j}x \neq 0$ for some $e_{i,j}$. Using the permutation $\sigma_{i,j}$ of the $i$th and $j$th strand, this implies that $y = e_{1,n}\sigma_{i,j}x \in e_{1,n}\Delta(\lambda)e_K$ is nonzero. Since $\Bbbk T_{n-1} \cong e_{1,n}\Bbbk T_n e_{1,n}$, we know by induction that $e_{1,n}\Delta(\lambda)e_K$ is simple as a $\Bbbk T_{n-1}$-module and we get that $\Delta(\lambda)e_K \subset e_{1,n}\Delta(\lambda)e_K \subset \Bbbk T_n x$, which shows that $\Delta(\lambda)e_K$ is a simple $\Bbbk T_n$-module.

From (4A.20) it follows that at least one $e_{i,j}$ does not annihilate $\Delta(\lambda)e_K$: in the case where $K = K(\mu)$ is not the sign $\mathcal{H}_\lambda$-module the idempotent $e_K$ has at least one nontrivial symmetrizer box at the top and hitting it will produce a nonzero diagram.

**Case 2:** $K$ is the sign $\mathcal{H}_\lambda$-module. The case where $K$ is the sign $\mathcal{H}_\lambda$-module is special and better to be analyzed by different means. Let $e_\lambda$ be the idempotent as in the proof of Proposition 4A.10. One easily sees that $e_\lambda T_n e_\lambda \cong S_\lambda$. Let $M = \{ (x_1, \ldots, x_n) \in \Bbbk^n \mid x_1 + \ldots + x_n = 0 \}$ denote the $\Bbbk T_n$-module with action induced from the natural action of $\Bbbk T_n$ on $\Bbbk^n$. The space $M$ is a $e_\lambda T_n e_\lambda \cong S_\lambda$-module by restriction and hence, we get a $e_\lambda \Bbbk T_n e_\lambda \cong \Bbbk[S_\lambda]$-module $\wedge^{\lambda-1} M$. It is known that $\wedge^{\lambda-1} M$ is a simple $\Bbbk[S_\lambda]$-module, and thus, is also a simple $\Bbbk T_n$-module. Its dimension is $\binom{n-1}{\lambda-1}$ and $\wedge^{\lambda-1} M$ has apex $\lambda$, by construction. Since we already know all the other simple $\Bbbk T_n$-module with apex $\lambda$, and they are of bigger dimensions, we conclude that $L(\lambda, K) \cong \wedge^{\lambda-1} M$.

(b). This follows directly from Proposition 2D.7 and Proposition 4A.10.

In the proof of Theorem 4A.17 we could have alternatively computed the sandwich matrices as the following example shows.

**Example 4A.21.** Continuing Example 4A.16, and let $\Bbbk$ be such that $\text{char}(\Bbbk) \nmid 3!$. Then the dimensions of simple $\Bbbk T_3$-modules are given by 1, 2, 1 for the bottom apex, 3, 2 for the middle apex and 1 for the top apex. The two sandwich matrices of ranks 3 and 2 for the middle apex are

\[
S_{\text{m, triv}} = \begin{pmatrix}
\varepsilon_{\text{triv}} & \varepsilon_{\text{triv}} & \varepsilon_{\text{triv}} & \varepsilon_{\text{triv}} \\
0 & 0 & 0 & 0 \\
\varepsilon_{\text{triv}} & \varepsilon_{\text{triv}} & \varepsilon_{\text{triv}} & \varepsilon_{\text{triv}} \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad S_{\text{m, sign}} = \begin{pmatrix}
\varepsilon_{\text{sign}} & -\varepsilon_{\text{sign}} & \varepsilon_{\text{sign}} & \varepsilon_{\text{sign}} \\
-\varepsilon_{\text{sign}} & \varepsilon_{\text{sign}} & \varepsilon_{\text{sign}} & \varepsilon_{\text{sign}} \\
-\varepsilon_{\text{sign}} & \varepsilon_{\text{sign}} & \varepsilon_{\text{sign}} & \varepsilon_{\text{sign}} \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

where $\varepsilon_{\text{triv}}$ and $\varepsilon_{\text{sign}}$ are the idempotents for the trivial and the sign $\mathcal{H}_2$-module, respectively.
For $\mathbb{KT}_3^p$ there is actually no restriction on the field since the Gram matrices are

$$G^b = (1), \quad G^m = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad G^t = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and we get 1, 2, 1 as the simple dimensions. □

**Remark 4A.22.** The quantum version of the above discussion simply replaces crossings by *over* and *undercrossings*:

$$\begin{array}{c} \begin{array}{c} \uparrow \downarrow \end{array} \end{array} \leftrightarrow \begin{array}{c} \begin{array}{c} \uparrow \downarrow \end{array} \end{array}$$

In this case the sandwiched algebras are the Hecke algebras of type $A$ as in Section 3. Otherwise there is no difference.

**Remark 4A.23.** The transformation monoid and its planar counterpart are around for donkey’s years, and so are their diagrammatic descriptions. The diagrammatic incarnations appear, for example, in the formulation of categories of (planar) transformations, see e.g. [ER23, Figure 8].

The representation theory of $\mathbb{KT}_n$ was studied since the early days of monoid representation theory. Often relying on versions of Theorem 2A.17, but in a less general setting with the focus on monoids. For example, the classification of simple $\mathbb{KT}_n$-modules is given in [Ste16, Section 5.3], but in a different language and not using sandwich cellularity. That section also lists a few original references, starting with [HZ57].

We do not know any reference for the representation theory of the planar transformation monoid. This discussion appears to be new, and is easy from the perspective of sandwich cellularity.

### 4B. Some involutive diagram algebras

In the previous section we found the cells and parameterized the simple $\mathbb{KD}_n$-modules, for $D_n$ being $T_n$ or $T^p_n$, using their diagrammatic incarnation. The same strategy works, by its very construction, for a wide range of diagram monoids and algebras.

We list now a few such algebras (the reader unfamiliar with these is referred to e.g. [HR05] for more details). These algebras have in common that their multiplication is defined via an underlying monoid, up to evaluations of closed components.

**Notation 4B.1.** As before, *planar* means the submonoids having only planar diagrams of the same type, while the other listed diagrammatic descriptions are *symmetric*. Below we will give one prototypical example of the diagrammatics for these monoids.

- **The partition monoid** $P_n$ of all diagrams of partitions of a 2$n$-element set. The **planar partition monoid** $P^p_n$ is the respective planar submonoid of $P_n$.

- **The rook-Brauer monoid** $RoBr_n$ consisting of all diagrams with components of size 1 or 2. The planar rook-Brauer monoid $RoBr^p_n = Mo_n$ is also called the **Motzkin monoid**.

- **The Brauer monoid** $Br_n$ consisting of all diagrams with components of size 2. The planar Brauer monoid $Br^p_n = TL_n$ is known as the **Temperley–Lieb monoid** (sometimes $TL_n$ is called the **Jones monoid** or **Kauffman monoid**).

- **The rook monoid or symmetric inverse semigroup** $Ro_n$ consisting of all diagrams with components of size 1 or 2, and all partitions have at most one component at the bottom and at most one at the top. The **planar rook monoid** $Ro^p_n$ is the corresponding submonoid.
• The **symmetric group** $S_n$ consisting of all matchings with components of size 1. The **planar symmetric group** is trivial $S^p_n \cong 1$.

$$\in S_n, \quad \in S^p_n.$$  

The monoids $S_n$ and $S^p_n$ are, of course, the same groups as in Section 4A. $S_n$ is isomorphic to a subgroup of any of the symmetric monoids, and $S^p_n$ is isomorphic to a subgroup of any of the planar monoids, both in the evident way. We will use this below.

**Notation 4B.2.** Below we write $D_n$ for any of the monoids listed above.

**Notation 4B.3.** Additionally to Notation 4A.3 we now also have caps: $\sqcup$, cups: $\sqcap$, splits: $\sqrt{\cdot}, \sqrt[\cdot]{\cdot}, \ldots$, bottom dots: $\cdot$.

which we, as indicated, give the names caps, cups, splits and bottom dots.

**Definition 4B.4.** Fix $\delta \in \mathbb{K}$. For $D_n$ we define its associated (symmetric or planar) **diagram algebra** $D_n(\delta)$ to be the algebra with basis $D_n$ and multiplication of basis elements given by the monoid multiplication except that all closed components are evaluated to $\delta$.

Note that $D_n(\delta)$ is $\mathbb{K}$-linear, but we decided to use $D_n(\delta)$ as the notation instead of e.g. $\mathbb{K}[D_n](\delta)$.

**Example 4B.5.** Typical examples how the multiplication in diagram algebras works is

which illustrate the multiplication of $M_6(\delta)$.

**Remark 4B.6.** For some of the $D_n$ one can define associated multiparameter diagram algebras. For example, for $M_n$ one could evaluate circles to $\delta_1$ and intervals to $\delta_2$. Our discussion below works mutatis mutandis for these multiparameter diagram algebras as well.

A main difference between $D_n$ and $T_n$, $T^p_n$ is that $D_n$ is involutive using the **diagrammatic antiainvolution** $-^*$, e.g.:

$$\left( \begin{array}{c} \cdot \\ \cdot \end{array} \right)^* = \begin{array}{c} \cdot \\ \cdot \end{array}.$$  

The $\mathbb{K}$-linear extensions of the diagrammatic antiainvolution endows $D_n(\delta)$ with the structure of an involutive sandwich cellular algebra, as we will see in Proposition 4B.10 below. (We will always use the diagrammatic antiainvolution in this section.) Hence, by Lemma 2B.18 the $J$-cells are squares and we can focus on describing left cells and right cells come for free.

**Example 4B.7.** Let us give two examples how the cells structure of $TL_n(\delta)$ looks like.

\begin{align*}
J_1 & \quad \mathcal{H}_1 \cong 1 \\
J_3 & \quad \mathcal{H}_3 \cong 1 \\
J_0 & \quad \mathcal{H}_0 \cong 1 \\
J_2 & \quad \mathcal{H}_2 \cong 1 \\
J_4 & \quad \mathcal{H}_4 \cong 1
\end{align*}

(4B.8)
These are the cells of \( \text{TL}_3(\delta) \) and \( \text{TL}_4(\delta) \) for invertible \( \delta \). If \( \delta \) is not invertible, then the picture is the same but with uncolored diagonal \( H \)-cells for \( J_1 \) and \( J_2 \), and \( J_0 \) having no idempotent at all.

The following analog of Lemma 4A.7 is easy to verify.

**Lemma 4B.9.** Let \( D_n \) be symmetric. For \( a \in D_n \) there is a unique factorization of the form \( a = \tau \circ \sigma_\lambda \circ \beta \) such that \( \beta \) and \( \tau \) have a minimal number of crossings, \( \beta \) contains no cups, splits or top dots, \( \tau \) contains no caps, merges or bottom dots and \( \sigma_\lambda \in S_\lambda \) for minimal \( \lambda \).

Similarly for \( a \in D_n \) when \( D_n \) is planar, but with \( \sigma_\lambda \in S_{\lambda n}^n = 1 \). \( \square \)

As before, we get the notions of through strands etc.

**Proposition 4B.10.** We have the following for the pair \( (D_n(\delta), D_n) \).

(a) The \( J \)-cells of \( D_n(\delta) \) are given by diagrams with a fixed number of through strands \( \lambda \). The \( \leq_{\text{tr}} \)-order is a total order and increases as the number of through strands decreases. See (4B.11) for a summary.

(b) The left cells of \( D_n(\delta) \) are given by diagrams where one fixes the bottom of the diagram, and similarly right cells are given by diagrams where one fixes the top of the diagram. The \( \leq_{\text{r}} \) and the \( \leq_{\text{tr}} \)-order increases as the number of through strands decreases. For \( \# L_\lambda \) see (4B.11).

(c) The idempotency of the \( J \)-cells are as follows. Assume \( \delta \) is invertible in \( \mathbb{K} \). Then all \( J \)-cells of \( D_n(\delta) \) are strictly idempotent. Second, assume \( \delta \) is not invertible in \( \mathbb{K} \). Then the following \( J \)-cells for \( D_n(\delta) \) are strictly idempotent, while all other \( J \)-cells are not idempotent (we list the various \( D_n \)):

(i) All \( J \)-cells except the top for \( P_{\lambda n} \), \( P_{\lambda n}^p \), \( R \circ B \text{r}_n \) and \( M_{\lambda n} \), and \( B \text{r}_n \), \( T \text{l}_n \) for even \( n \).

(ii) Only the bottom \( J \)-cell for \( R \text{o}_n \) and \( R \text{o}_n^p \).

(iii) All \( J \)-cells for \( S \text{n}_n \) and \( S_{\lambda n}^n \), and \( B \text{r}_n \), \( T \text{l}_n \) for \( n \) odd.

See also (4B.11). The sandwiched algebras are given in (4B.11).

(d) The pair \( (D_n(\delta), D_n) \) is an involutive sandwich pair, that for \( D_n \) symmetric comes neither from a cellular nor an affine cellular algebra, but can be refined to a cellular pair.

The following table summarizes the cell structure of these diagram monoids where \( \lambda \in \mathcal{P} \) (the column for \( \mathcal{H}_\lambda \) only applies if the parent \( J \)-cell is strictly idempotent):

| Monoid | \( \mathcal{P} \) for \( \delta^{-1} \in \mathbb{K} \) | \( \mathcal{P} \) for \( \delta \) not inv. | \( \# L_\lambda \) | \( \mathcal{H}_\lambda \) |
|--------|---------------------------------|---------------------------------|-----------------|-----------------|
| \( P_{\lambda n} \) | \{n, \ldots, n-1, \ldots, 0\} | \{n, \ldots, n-1, \ldots, 1\} | \frac{n(n+1)}{2}(2n-1) | \( S_\lambda \) |
| \( P_{\lambda n}^p \) | \{n, \ldots, n-1, \ldots, 0\} | \{n, \ldots, n-1, \ldots, 1\} | \frac{n(n-1)}{2}(2n-1) | \( S_\lambda \) |
| \( R \circ B \text{r}_n \) | \{n, \ldots, n-1, \ldots, 0\} | \{n, \ldots, n-1, \ldots, 1\} | \frac{n(n-1)}{2}(2n-1) | \( S_\lambda \) |
| \( M_{\lambda n} \) | \{n, \ldots, n-1, \ldots, 0\} | \{n, \ldots, n-1, \ldots, 1\} | \frac{n(n-1)}{2}(2n-1) | \( S_\lambda \) |
| \( B \text{r}_n \) | \{n, \ldots, n-2, \ldots, 0\} | \{n, \ldots, n-2, \ldots, 1\} | \frac{n(n-1)}{2}(2n-1) | \( S_\lambda \) |
| \( T \text{l}_n \) | \{n, \ldots, n-2, \ldots, 0\} | \{n, \ldots, n-2, \ldots, 1\} | \frac{n(n-1)}{2}(2n-1) | \( S_\lambda \) |
| \( R \text{o}_n \) | \{n, \ldots, n-1, \ldots, 0\} | \{n\} | \frac{n(n-1)}{2}(2n-1) | \( S_\lambda \) |
| \( R \text{o}_n^p \) | \{n, \ldots, n-1, \ldots, 0\} | \{n\} | \frac{n(n-1)}{2}(2n-1) | \( S_\lambda \) |
| \( S \text{n}_n \) | \{n\} | \{n\} | 1 | \( S_\lambda \) |
| \( S_{\lambda n}^n \) | \{n\} | \{n\} | 1 | \( S_\lambda \) |

For \( T \text{l}_n \) and \( B \text{r}_n \) the last entry of \( \mathcal{P} \) is either 0 or 1, depending on the parity of \( n \). Similarly, the last entry of \( \mathcal{P} \) for \( \delta \) not invertible is either 2 or 1, again depending on the parity of \( n \).

**Example 4B.12.** GAP produces the following outputs for the underlying monoids:
The monoids $S_n$ and $S_n^n$ are not illustrated as they just have one cell.

Proof of Proposition 4B.10. (a) and (b). Only the counts for $\#\mathcal{L}_\lambda$ are not immediate. Counting them is a combinatorial exercise that has been solved several times in the literature. The respective triangles, cf. Remark 4A.11, are A049020, A008313, A096713, A064189, A111062, and A007318 on OEIS.

(c). This is easy to see and omitted. The main trick is to use variations of $\bigcirc = 1$

to ensure that certain cells stay strictly idempotent even when $\delta$ is not invertible.

(d). Since $D_n$ is an admissible monoid by the above, Proposition 2E.1 applies for $(\mathbb{K}[D_n], D_n)$. This result can be pulled over to $(D_n(\delta), D_n)$. The second claim about cellularity follows then from Lemma 2B.18 and Proposition 2A.10 (and some care with the antiinvolution).

From now on we use that $(D_n(\delta), D_n)$ is an involutive sandwich pair.

Theorem 4B.13. Assume that $\mathbb{K}$ is a field, and consider the involutive sandwich pair $(D_n(\delta), D_n)$.

(a) Assume that $\delta \neq 0$. The set of apexes for simple $D_n(\delta)$-modules is $\mathcal{P}^{ap}$ as in (4B.11), and there are precisely $|P(\lambda|\text{char}^0=\infty(\mathbb{K}))|$ (symmetric) or one (planar) simple $D_n(\delta)$-modules for $\lambda \in \mathcal{P}^{ap}$.

(b) Assume that $\delta = 0$. Then the same statement holds for the restricted set of apexes as detailed in (4B.11).

Proof. This is clear by Proposition 4B.10 and $H$-reduction Theorem 2A.17.

Below we compute ranks of Gram matrices. To get started, here is an example:

Example 4B.14. For $TL_5(\delta)$ we have

$$G^1 = \begin{pmatrix} \delta^2 & \delta & 1 & \delta & 1 \\ \delta & \delta & \delta & 1 & \delta \\ 1 & \delta & \delta & \delta & 1 \\ \delta & 1 & \delta & \delta & \delta \\ 1 & \delta & 1 & \delta & \delta \end{pmatrix}, \quad \det(G^1) = (\delta - 1)^4(\delta + 1)^4(\delta^2 - 2),$$

$$G^3 = \begin{pmatrix} \delta & 1 & 0 & 0 & 0 \\ 1 & \delta & 1 & 0 & 0 \\ 0 & 1 & \delta & 1 & 0 \\ 0 & 0 & 1 & \delta & 1 \end{pmatrix}, \quad \det(G^3) = (\delta^2 + \delta - 1)(\delta^2 - \delta - 1),$$

and

$$G^5 = (1), \quad \det(G^3) = 1,$$

as cells, Gram matrices and their determinants. The ranks of $G^1$ and $G^3$ depend on $\delta$ and are easy to determine from the determinants. For example, if $\delta = 1$ or $\delta = \sqrt{2}$, then $\text{rank}_\mathbb{K}(G^1) = 1$ and $\text{rank}_\mathbb{K}(G^3) = 4$, respectively.

The following proposition computes ranks of Gram matrices for $D_n \in \{S_n, S_n^n, Ro_n, Ro_n^p, TL_n, Pa_n^p\}$ explicitly and implicitly for $D_n = M_n$ in $\text{char}(\mathbb{K}) = 0$ as well. We do not know nice formulas for the ranks of the remaining monoids, but we give some partial results by computing the ranks of the cells close to the bottom, including explicit formulas for $D_n = M_n$.
Proposition 4B.15. Assume that \( \mathbb{K} \) is a field, and consider the involutive sandwich pair \((D_n(\delta), D_n)\). The ranks of (some of the) Gram matrices are as follows.

(a) For \( D_n \in \{ S_n, S_n^p \} \) we have \( \text{rank}_\mathbb{K}(G^n) = 1 \) for \( n \in \mathcal{P} \).

(b) For \( D_n \in \{ R_n, R_n^p \} \) we have \( \text{rank}_\mathbb{K}(G^\lambda) = \binom{n}{\lambda} \) for \( \lambda \in \mathcal{P} \).

(c) For \( D_n = \text{TL}_n \) we have \( \text{rank}_\mathbb{K}(G^\lambda) = t_l^\lambda \) for \( \lambda \in \mathcal{P} \), with \( t_l^\lambda \) explicitly given in Remark 4B.16 below. For \( D_n = \text{Pa}_n^p \) we have \( \text{rank}_\mathbb{K}(G^\lambda) = t_{2n}^\lambda \) for \( \lambda \in \mathcal{P} \).

(d) For \( D_n = \text{Mo}_n \) and \( \text{char}(\mathbb{K}) = 0 \) the rank \( \text{rank}_\mathbb{K}(G^\lambda) \) depends on the multiplicity of \( \delta \) in an explicit multitset \( \text{Roots}_{n,\lambda} \) explained in the proof below.

(e) For \( D_n \in \{ B_n, \text{RoBr}_n, \text{Mo}_n, \text{Pa}_n \} \) we have \( \text{rank}_\mathbb{K}(G^n) = 1 \) for \( n \in \mathcal{P} \).

(f) For \( D_n = \text{Br}_n \), \( \text{char}(\mathbb{K}) = 0 \) and \( n-2 \in \mathcal{P} \) we have

\[
\text{rank}_\mathbb{K}(G^{n-2}) = \begin{cases} 
\frac{1}{2}n(n-1) & \text{if } \delta \not\in \{2,-n+4,-2n+4\}, \\
n & \text{if } \delta = 2, \\
\frac{1}{2}n(n-3)+1 & \text{if } \delta = -n+4, \\
\frac{1}{2}(n+1)(n-2) & \text{if } \delta = -2n+4.
\end{cases}
\]

(g) For \( D_n = \{ \text{RoBr}_n, \text{Mo}_n \} \) and \( n-1 \in \mathcal{P} \) we have \( \text{rank}_\mathbb{K}(G^{n-1}) = n-1 \).

(h) For \( D_n = \text{Mo}_n \), \( \text{char}(\mathbb{K}) = 0 \) and \( n-2 \in \mathcal{P} \) we have

\[
\text{rank}_\mathbb{K}(G^{n-2}) = \begin{cases} 
\frac{1}{2}(n^2+n-2) & \text{if } \delta \not\in \{0\} \cup \text{Roots}_{n-1}, \\
\frac{1}{2}(n^2+n-4) & \text{if } \delta \in \text{Roots}_{n-1} \setminus \{0\}, \\
n-1 & \text{if } \delta = 0, n \not\equiv 0 \mod 2, \\
n-2 & \text{if } \delta = 0, n \equiv 0 \mod 2.
\end{cases}
\]

The set \( \text{Roots}_{n-1} \) can be explicitly computed as explained in the proof below.

(i) For \( D_n = \text{RoBr}_n \), \( \text{char}(\mathbb{K}) = 0 \) and \( n-2 \in \mathcal{P} \) we have

\[
\text{rank}_\mathbb{K}(G^{n-2}) = \begin{cases} 
n(n-1) & \text{if } \delta \not\in \{0,3,5-n,5-2n\}, \\
\frac{1}{2}n(n-1) & \text{if } \delta = 0, \\
\frac{1}{2}n(n+1) & \text{if } \delta = 3, \\
(n-1)^2 & \text{if } \delta = 5-n, \\
n(n-1)-1 & \text{if } \delta = 5-2n.
\end{cases}
\]

Note that Proposition 4B.15 always assumes that \( \lambda \in \mathcal{P} \). Hence, depending on \( \delta \), some cases might not appear.

Remark 4B.16. The number \( t_l^\lambda \) is given as follows.

Let \( l \in \mathbb{Z}_{\geq 0} \) be minimal such that \( U_{l+1}(\delta) = 0 \) where \( U_k(X) \) is the (normalized) Chebyshev polynomial (of the second kind) defined by \( U_0(X) = 1, U_1(X) = X \) and \( U_k(X) = XU_{k-1}(X) - U_{k-2}(X) \) for \( k > 1 \). If no such \( l \in \mathbb{Z}_{\geq 0} \) exists we set \( l = \infty \). The number \( l \) is often called the quantum characteristic of \( (\mathbb{K}, \delta) \). Similarly, let further \( p \in \mathbb{Z}_{>0} \) be minimal such that \( p \cdot 1 = 0 \in \mathbb{K} \), and let \( p = \infty \) if no such \( p \) exists, so \( p = \text{char}^{0=\infty}(\mathbb{K}) \), see Section 3A.

Let \( \nu_p \) denote the \( p \)-adic valuation. Let \( \nu_{l,p}(x) = 0 \) if \( x \not\equiv 0 \mod l \), and \( \nu_{l,p}(x) = \nu_p(x) \) otherwise. Let further \( x = [...,x_1,x_0] \) denote the \((l,p)\)-adic expansion of \( x \) given by

\[
[... x_1, x_0] = \sum_{i=1}^{\infty} l^p-i x_i + x_0 = x, \quad x_i \geq 0 \in \{0,...,p-1\}, x_0 \in \{0,...,l-1\}.
\]

Write \( x < y \) if \([...,x_1,x_0]\) is digit-wise smaller or equal to \([...,y_1,y_0]\). We also use \( x < y \) if \( x < y \), \( \nu_{l,p}(x) = \nu_{l,p}(y) \) and \( \nu_{l,p}(x) \)th digit of \( x \) and \( y \) agree set

\[
e_{n,k} = \begin{cases} 
1 & \text{if } n \equiv k \mod 2, \nu_{l,p}(k) = \nu_{l,p}\left(\frac{n+\lambda}{2}\right), k < \frac{n+\lambda}{2}, \\
-1 & \text{if } n \equiv k \mod 2, \nu_{l,p}(k) < \nu_{l,p}\left(\frac{n+\lambda}{2}\right), k < \frac{n+\lambda}{2} - 1, \\
0 & \text{else}.
\end{cases}
\]

Then \( t_{l,n}^\lambda = \sum_{r=0}^{(n-\lambda)/2} e_{n-2\lambda+1,\lambda+1}(\frac{2\lambda+2}{n+\lambda+2}) \binom{n}{(n-\lambda)/2} \).
Proof of Proposition 4B.15. We compute the Gram matrices using a case-by-case argument. (The arguments for the various cases are similar but differ in details.) We will also use Proposition 4B.10 since we need the various numerical data computed therein.

Case Ro_n and Ro_l^i. In this case all Gram matrices are \( \delta^{n-\lambda} \) multiples of identity matrices. This can be seen as follows. We use the same order for left and right cells and then one calculates that the middle (where rows and columns are swapped) is of the form

\[
\begin{vmatrix}
\delta & 0 & 0 \\
0 & \delta^2 & 0 \\
0 & 0 & \delta^2
\end{vmatrix}.
\]

This calculation generalizes without problems, hence, \( G^\lambda \) is of full rank. Note hereby, as for the rest of the proof, that we only care about apexes in the theorem. In particular, for \( \delta = 0 \) there is only one apex, the bottom \( J \)-cell, and the relevant Gram matrix is the identity.

Case TL_n and Pa_n^p. The calculation of the Gram matrices for TL_n is known, but not easy, so we will not recall it here. See e.g. [Spe23] (general case using Temperley–Lieb combinatorics), [And19] (general case using tilting modules) or [RSA14] (characteristic zero) for details. The monoid isomorphism \( Pa_n^p \cong TL_{2n} \) from [HR05, Section 1] can then be used to prove the statement for the planar partition algebra from the Temperley–Lieb algebra case.

Case D_n \in \{ Br_n, Mo_n, RoBr_n, Pa_n \}, bottom cell. This is clear.

Case Br_n, \( \lambda = n - 2 \). In this case there is one cap and one cup. Thus, it suffices to remember the endpoints of the middle. We do this by using

\[
(4B.17)
\]

The numbers in these pictures are the endpoints of the strings, read from left to right. The Gram matrix becomes symbolically \( G_{n-2} = (b[i,j]t[k,l])_{i,j,k,l} \) for \( 1 \leq i < j \leq n \) and \( 1 \leq k < l \leq n \). The entries of the Gram matrix are determined by

\[
(4B.19)
\]

This can be seen as indicated above. The matrices are then easy to write down, for example

\[
(4B.18)
\]

Induction verifies that the determinant is

\[
\det(G_{n-2}) = (\delta - 2)^{\frac{1}{2}n(n-3)}(\delta + n - 4)^{n-1}(\delta + 2n - 4).
\]

Thus, unless \( \delta \in \{2, -n + 4, -2n + 4\} \), we get \( \text{rk}_\mathbb{F}(G_{n-2}) = \frac{1}{2}n(n-1) \). For the reaming cases we get \( \text{rk}_\mathbb{F}(G_{n-2}) = n \) if \( \delta = 2 \), \( \text{rk}_\mathbb{F}(G_{n-2}) = \frac{1}{2}n(n-3) + 1 \) if \( \delta = -n + 4 \) and \( \text{rk}_\mathbb{F}(G_{n-2}) = \frac{1}{2}(n+1)(n-2) \) if \( \delta = -2n + 4 \).

Case RoBr_n and Mo_n, \( \lambda = n - 1 \). The only way to reduce the number of through strands by one is to have a start and an top dot. Thus, this cell is exactly as in (4B.17), up to permutations, and we get the same Gram matrix as for Ro_n and Ro_l^i, namely \( \delta \) times identity.

Case Mo_n, \( \lambda = n - 2 \). In order to get \( n - 2 \) through strands one either needs to have one cap or two bottom dots at the bottom, and one cup or two top dots at the top of the diagram. Having one cap-cup pair is the Temperley–Lieb case, having two start and two top dots is the planar rook monoid case, and then there are the mixed cases. In the following illustration we again mark the
endpoints of strings by their positions read from left to right. As for \( \text{Br}_n \) above, that is \((4B.18)\), we use a translation from diagrams to symbols:

\[
\bigcup_{i=1}^n \rightsquigarrow b[i, i+1], 1 \leq i \leq n, \quad \bigcup_{i=1}^n \rightsquigarrow c[i, i+1], 1 \leq i \leq n, \quad \bigcup_{j=1}^k \rightsquigarrow d[j, k], 1 \leq j < k \leq n.
\]

We use the notation \( t[i, i+1], u[i, i+1] \) and \( v[j, k] \) for right cells. Additionally to \((4B.19)\) we have:

\[
\begin{cases}
  b[i, i+1]u[j, j+1] = \delta & \text{if } i = j, \\
  b[i, i+1]u[j, j+1] = 0 & \text{else,} \\
  b[i, i+1]v[j, k] = 0 & \text{always,} \\
  u[i, i+1]v[j, k] = \delta^2 & \text{if both endpoints match,} \\
  u[i, i+1]v[j, j+1] = \delta^2 & \text{if } i = j, \\
  u[i, i+1]v[j, j+1] = 0 & \text{else.}
\end{cases}
\]

Using these formulas and the corresponding symbolic Gram matrix, an analysis as for \( \text{Br}_n \) shows that the Gram matrix of square size \((n-1) + (n-1) + \left(\frac{1}{2}(n-1)(n-2)\right)\) is

\[(4B.20)\quad G^{n-2} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix},
\]

\[A_{11} = \begin{pmatrix} \delta & 1 & 0 & 0 & \ldots & 0 \\ 0 & \delta & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \delta \\ 0 & 0 & 0 & 0 & \ldots & \delta \\ 0 & 0 & 0 & 0 & \ldots & \delta \end{pmatrix}, \quad A_{12} = \delta id, \quad A_{22} = \delta^2 id, \quad A_{33} = \delta^2 id.
\]

The northwest corner is the same as for \( TL_n \). An example is

\[n = 4: G^{n-2} = \begin{pmatrix}
\delta & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta
\end{pmatrix}.
\]

Let \( U_k(X) \) again be the Chebyshev polynomial, see Remark \(4B.16\). It is not hard to see (and well-known) that \( \det(A_{11}) = U_{n-1}(\delta) \). Using this determinant formula and \((4B.20)\) we get

\[\det(G^{n-2}) = \delta^{n(n-1)}U_{n-1}(\delta - 1).
\]

The Chebyshev polynomial \( U_{n-1}(X - 1) \) is a polynomial of degree \( n - 1 \) and has distinct roots (given by explicit formulas) and we denote the set of these roots by \( \text{Roots}_{n-1} \). Thus, for \( \delta \notin \{0\} \cup \text{Roots}_{n-1} \), we get \( \text{rk}_G(G^{n-2}) = \frac{1}{2}(n^2 + n - 2) \). If we have \( \delta \neq 0 \) and \( \delta \in \text{Roots}_{n-1} \), then we get \( \text{rk}_G(G^{n-2}) = \frac{1}{2}(n^2 + n - 4) \). For the final case \( \delta = 0 \) we recall that \( 0 \in \text{Roots}_{n-1} \) holds if and only if \( n \equiv 0 \mod 2 \). Thus, we get \( \text{rk}_G(G^{n-2}) = n - 1 \) if \( n \equiv 0 \mod 2 \), and \( \text{rk}_G(G^{n-2}) = n - 2 \) if \( n \equiv 0 \mod 2 \).

**Case \( Mo_n, \text{ general } \lambda \)**: In general one gets a recursion for \( \det(G^\lambda) \). To explain it, let us use \( n \) together with \( \lambda \) in our notation. The recursion is

\[(4B.21) \quad \det(G^\lambda) = \det(G^{n-1, \lambda-1}) \left( \delta^{\#L_{n-1, \lambda}} \det(G^{n-1, \lambda}) \right) \left( U_{\lambda+1}(\delta - 1)/U_\lambda(\delta - 1) \right)^{\#L_{n-1, \lambda+1}} \det(G^{n, \lambda+1}),
\]

where one omits a factor when the respective Gram matrix is empty. Here \( \#L_{n, \lambda} \) is the number of left cells in \( J_\lambda \) given in \((4B.11)\). This is not hard to show using the same strategy as above, see for example [BH14, Section 5] (the paper [BH14] has different parameters then we do but the arguments given therein work mutatis mutandis). One can then inductively solve \((4B.21)\) and gets

\[(4B.22) \quad \det(G^\lambda) = \delta^{x(n, \lambda)} \prod_{t=1}^{[(n-\lambda)/2]} \left( U_{\lambda+t}(\delta - 1)/U_{t-1}(\delta - 1) \right)^{\#L_{\lambda+2t}},
\]

with \( x(n, \lambda) \) recursively determined by \( x(n, \lambda) = x(n - 1, \lambda - 1)\delta^{\#L_{n-1, \lambda}}x(n - 1, \lambda)x(n - 1, \lambda + 1) \) under the same conditions as in \((4B.21)\). From this one gets different ranks depending on the multiset (counting multiplicities as well) \( \text{Roots}_{n, \text{all}} = \{0\} \cup \text{Roots}_{n+\lambda-1}/2 \cup \ldots \cup \text{Roots}_{\lambda+1} \) for \( n + \lambda \) even respectively \( \text{Roots}_{n, \text{all}} = \{0\} \cup \text{Roots}_{(n+\lambda-1)/2} \cup \ldots \cup \text{Roots}_{\lambda+1} \) for \( n + \lambda \) odd.
Case $RoBr_n$, $\lambda = n - 2$. This case is very similar to $Br_n$ and $Mo_n$ for $\lambda = n - 2$ with some small differences. That is, we still use the shorthand notation from (4B.18) but now also

\[(4B.23) \quad \begin{array}{l}
\uparrow & \mid & \mid & \uparrow \quad \leftrightarrow c[i,j],
\hline
\downarrow & \mid & \mid & \downarrow \quad \leftrightarrow u[k,l].
\end{array}\]

As one easily checks by drawing the relevant pictures, we then get

\[
\begin{align*}
& b[i,j]t[k,l] = \delta \quad \text{if both endpoints match}, \\
& b[i,j]t[k,l] = 1 \quad \text{if one endpoint matches}, \\
& b[i,j]t[k,l] = 0 \quad \text{else}, \\
& c[i,j]t[k,l] = \delta^2 \quad \text{if both endpoints match}, \\
& c[i,j]t[k,l] = 0 \quad \text{else},
\end{align*}
\]

as entries of the Gram matrix. Thus, we have the same block decomposition as in (4B.20) but with northwest corner corresponding to $Br_n$ and $A_{33}$ being empty. For example,

\[
n = 4: G^{n-2} = \begin{pmatrix}
\delta & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \delta & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \delta & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & \delta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

From the previous formulas one easily gets that

\[
\det(G^{n-2}) = \delta^{\frac{3}{2}n(n-1)}(\delta - 3)^{\frac{1}{2}n(n-3)}(\delta + n - 5)^{n-1}(\delta + 2n - 5).
\]

Analyzing the rank using this determinant formula gives the claimed result. That is, unless $\delta \in \{0, 3, 5 - n, 5 - 2n\}$ we have $rk_K(G^{n-2}) = n(n - 1)$. Otherwise, reading $\{0, 3, 5 - n, 5 - 2n\}$ left to right, we get $rk_K(G^{n-2}) = \frac{1}{2}n(n - 1)$, $rk_K(G^{n-2}) = \frac{1}{2}n(n + 1)$, $rk_K(G^{n-2}) = (n - 1)^2$ and $rk_K(G^{n-2}) = n(n - 1) - 1$.

\[
\square
\]

Remark 4B.24. For a multiparameter version of the Motzkin diagram algebra the formula (4B.22) holds after replacing $\delta^{c(n,\lambda)}$ with the parameter for intervals, keeping the same exponent.

We now compute the dimensions of the simple $D_n$-modules for $D_n \in \{S_n, S_n^p, Ro_n, Ro_n^p, TL_n, Pa_n^p\}$ explicitly, and implicitly for $D_n = Mo_n$. We also give some partial results for the remaining ones.

Theorem 4B.25. Assume that $\mathbb{K}$ is a field with char($\mathbb{K}$) = 0 for the symmetric monoids, and consider the involutive sandwich pair $(D_n(\delta), D_n)$.

\[\begin{align*} \text{(a)} \quad & \text{For } D_n \in \{S_n, S_n^p, Ro_n, Ro_n^p, TL_n, Mo_n, Pa_n^p\} \text{ the dimension of the simple } D_n(\delta) \text{-modules for } \lambda \in \mathcal{P}^{np}, \text{ and additionally a simple } \mathcal{H}_\delta \text{-module } K \text{ for } S_n \text{ and } Ro_n, \text{ are} \\
& \dim_K(L(\lambda)) = \text{rank}_K(G^\lambda), \quad \dim_K(L(\lambda, K)) = \text{rank}_K(G^\lambda) \cdot \dim_K(K). \\
& (\text{The ranks of the Gram matrices are computed in Proposition 4B.15.})
\end{align*}\]

\[\begin{align*} \text{(b)} \quad & \text{Let char}(\mathbb{K}) = 0 \text{ for } D_n = Mo_n, \text{ and let } \mathbb{K} \text{ be an arbitrary field otherwise. For } D_n \in \{S_n, S_n^p, Ro_n, Ro_n^p, TL_n, Mo_n, Pa_n^p\} \text{ the algebra } D_n(\delta) \text{ is semisimple if and only if we are in the following cases:} \\
& \quad (i) \quad \text{All cases for } D_n \in \{S_n, S_n^p\}; \\
& \quad (ii) \quad \text{The parameter } \delta \text{ is nonzero for } D_n \in \{Ro_n, Ro_n^p\}; \\
& \quad (iii) \quad \text{The number } l \text{ from Remark 4B.16 is in } \mathbb{Z}_{\geq n+1} \cup \{\infty\} \text{ for } D_n \in \{TL_n, Pa_n^p\}; \\
& \quad (iv) \quad \text{We have } \delta \notin \text{Roots}_{n, \text{all}}, \text{ for all } \lambda \in \mathcal{P}, \text{ for } D_n = Mo_n.
\end{align*}\]

\[\begin{align*} \text{(c)} \quad & \text{For } \lambda \in \mathcal{P}^{np} \text{ we have the following lower bounds, where we always want to bound the left side by the right side:} \\
& \dim_K(L(K_{\text{triv}}, \lambda)_{Pa_n}) \geq tl_n^2, \quad \dim_K(L(K_{\text{triv}}, \lambda)_{Mo_n}) \geq \dim_K(L(\lambda)_{Mo_n}) \geq tl_n^3, \quad \dim_K(L(K_{\text{triv}}, \lambda)_{Br_n}) \geq tl_n^4.
\end{align*}\]
Here $K_{triv}$ always denotes the trivial $\mathcal{H}_\lambda$-module and we use subscripts to indicate what kind of simple $D_n(\delta)$-modules we consider.

Proof. (a). For the planar diagram algebras this is just Proposition 2D.7, and for $D_n = S_n$ the statement is clear. For $R_n(\delta)$ and $\lambda \in \mathcal{P}^0$ we define a pseudo-idempotent $e_\lambda$ that have a start-top dot pair for all positions 1 to $n - \lambda$ and is the identity otherwise. For example, for $n = 5$ and $\lambda = 3$ we get

$$e_3 = \begin{array}{c|c|c|c|c|c} & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \end{array}.$$  

Hence, we get idempotents $e'_{\sigma(\lambda)} = \frac{1}{\delta_{\sigma} - \sigma} e_{\lambda}$ in $\mathcal{J}_\lambda$. Note also that $\Delta(\lambda) \cong R_n(\delta)e_{\lambda}$ as $R_n(\delta)$-modules. Let $e_{\sigma(\lambda)}$ denote the projection to the component of $\mathcal{H}_\lambda^{\oplus k}$ determined by the top through strand points of $\sigma e_\lambda$, where we write $k = \binom{\lambda}{n}$. The $K$-linear map $f_\lambda: \Delta(\lambda) \to \mathcal{H}_\lambda^{\oplus k}$ given by $f_\lambda(\sigma e'_{\lambda}) = \sigma e_{\sigma(\lambda)}$ is an isomorphism of $\mathcal{H}_\lambda$-modules. Thus, $\Delta(\lambda, K) \cong K^{\oplus k}$ for all simple $\mathcal{H}_\lambda$-modules $K$. Using now the permutation action of $\mathcal{H}_\lambda$, which connects the various direct summands, one sees that $\Delta(\lambda, K)$ is a simple $R_n(\delta)$-module.

(b). This follows from (a) and Proposition 2B.24.

(c). Note that we have algebra embeddings $P_n^\delta(\delta) \hookrightarrow P_n(\delta)$, $T_n^\delta(\delta) \hookrightarrow M_n(\delta) \hookrightarrow R_n(\delta)$ and $T_n(\delta) \hookrightarrow Br_n(\delta)$ given by sending diagrams to themselves interpreted in the bigger monoids. Moreover, by Proposition 4B.10, these embeddings behave well with the cell order in the sense that diagrams in $\mathcal{J}_k^{source}$ (for the source algebra) are send to diagrams in $\mathcal{J}_k^{target}$ (for the target algebra). This follows then from the isomorphism to $\mathcal{J}_k^{source}$ with the one for $\mathcal{J}_k^{target}$. Recalling the picture that defines the sandwich matrices, see (2D.4), since the embedding sends diagrams to themselves, the computations for each entry remain the same. Thus, using an appropriate order, the sandwich matrices for the small monoids are submatrices of the sandwich matrices for the bigger monoids and Proposition 4B.15 completes the proof.

Remark 4B.26. Similarly as in Remark 4A.22, the quantum versions of the diagram algebras $D_n(\delta)$ (e.g. BMW algebras) can be studied verbatim, and the Hecke algebras of type $A$ appear again as the sandwiched algebras.

Remark 4B.27. The study of diagram algebras has a long history, which we cannot cover here in any satisfying way. Nevertheless, let us give a few references for some of the results above that have appeared in the literature, but not under the umbrella of sandwich cellularity, which is new in this paper.

The representation theories of $S_n$ and $S_n^p$ is, of course, well-studied, and we do not comment on these any further.

Statements such as Proposition 4B.10 and Proposition 4B.15 appear throughout the semigroup literature, see e.g. [HR05], [DEE+19], [HJ20], sometimes as disguised lemmas such as [EMRT18, Remark 9.21 and Lemma 9.19], and many more. They have very similar appearances for diagram algebras with the catch that the parameter $\delta$ is not necessarily 1 as for the monoids.

Computations of dimensions of simple modules of diagram algebras such as in Theorem 4B.25 have also appeared in many works (although some of the results above appear to be new). For example, for $R_n$ see [Mun57] and [Sol02], and for $R_n^p$ see [FHH09]. The formulas from Remark 4B.16 for $T_n$ go back to [RSA14], [And19] and [Spe23], and the ones for $P_n^\delta$ follows then from the isomorphism to $T_2 \mathbb{A}_2$ from [HR05]. We do not know any reference for the dimensions of simple modules for any of the remaining symmetric monoids, but the Gram determinants for $M_n$ appear in [BH14]. The bounds however have been studied in [KST24]. Let us also note that a lot of these diagram algebras and similar diagram algebra can be studied using Schur–Weyl–Brauer duality, see e.g. [AST17, Section 3] for a collection of these dualities.

The quantum versions, which also fit into the theory, cf. Remark 4B.26, have also been studied extensively, see e.g. [Xi00] or [Eny04] for the BMW case. Note also that Proposition 4B.10(d) shows that all the diagram algebras in this section are cellular. This is known, see e.g. [GL96], [Xi99] and [Eny04]. But again our point is that we get all of this by using the theory of sandwich cellular algebras.
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