HOW MANY $n$-VERTEX TRIANGULATIONS DOES THE 3-SPHERE HAVE?

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Abstract. It is known that the 3-sphere has at most $2^{O(n^2 \log n)}$ combinatorially distinct triangulations with $n$ vertices. Here we construct at least $2^{\Omega(n^2)}$ such triangulations.

1. Introduction

For $d \geq 3$ fixed and $n$ large, Kalai \cite{Kalai} constructed $2^{\Omega(n^{\lfloor d/2 \rfloor})}$ combinatorially distinct $n$-vertex triangulations of the $d$-sphere (the squeezed spheres), and concluded from Stanley’s upper bound theorem for simplicial spheres \cite{Stanley} an upper bound of $2^{O(n^{\lfloor d/2 \rfloor} \log n)}$ for the number of such triangulations. In fact, this upper bound readily follows from the Dehn-Sommerville relations, as they imply that the number of $d$-dimensional faces is a linear combination of the number of faces of dimension $\leq \lceil d/2 \rceil - 1$, and hence is at most $O(n^{\lfloor d/2 \rfloor})$. Thus, as already argued in \cite{Kalai}, the number of different triangulations is at most

$$\binom{n}{\lfloor d/2 \rfloor}/n!,$$

namely at most $2^{O(n^{\lfloor d/2 \rfloor} \log n)}$.

For $d$ odd this leaves a big gap between the upper and lower bounds; most striking for $d = 3$. Pfeifle and Ziegler \cite{Pfeifle} constructed $2^{\Omega(n^{5/4})}$ combinatorially different $n$-vertex triangulations of the 3-sphere. Combined with the $2^{O(n \log n)}$ upper bound for the number of combinatorial types of $n$-vertex simplicial 4-polytopes \cite{Deza} (see also \cite{Bjorner}) it shows that most triangulations of the 3-sphere are not combinatorially isomorphic to boundary complexes of simplicial polytopes.

The bound in \cite{Pfeifle} is obtained by constructing a polyhedral 3-sphere with $\Omega(n^{5/4})$ combinatorial octahedra among its facets. Our bound will follow from constructing a polyhedral 3-sphere with $\Omega(n^2)$ combinatorial bipyramids among its facets. (A bipyramid is the unique simplicial 3-polytope with 5 vertices.) The idea for the construction is as follows: consider the boundary complex $C$ of the cyclic 4-polytope with $n$ vertices.

- Find particular $\Theta(n)$ simplicial 3-balls contained in $C$, with disjoint interiors.
- On the boundary of each such 3-ball find particular $\Theta(n)$ pairs of adjacent triangles (each pair forms a square), such that these squares have disjoint interiors, and the missing edge in each such pair is an interior edge of its 3-ball.
- Replace the interior of each such 3-ball with the cone from a new vertex over each boundary square (forming a bipyramid) and over each remaining boundary triangle (forming a tetrahedron).
- Show that the particular 3-balls and squares chosen have the property that the above construction results in a polyhedral 3-sphere.

Specifically, we prove the following.

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Theorem 1.1 For each $n \geq 1$, there exists a 3-dimensional polyhedral sphere with $5n + 4$ vertices, such that $n^2$ of its facets are combinatorially equivalent to a bipyramid.

Erickson [2, Sec.8] asked whether there exist 4-polytopes on $n$ vertices with $\Omega(n^2)$ non-simplicial facets, conjectured there are none, and further conjectured that there are no such polyhedral 3-spheres. The latter is refuted by Theorem 1.1. We leave open the question of whether the polyhedral 3-spheres constructed in Theorem 1.1 are combinatorially equivalent to the boundary complexes of some 4-polytopes.

Note that each of the bipyramids in the above theorem can be triangulated independently in two ways — either into 2 tetrahedra by inserting its missing triangle or into 3 tetrahedra by inserting its missing edge — to obtain a triangulation of the 3-sphere. Thus, for $v$ the number of vertices and $m$ the number of bipyramids, we obtain at least $\frac{2m}{v}$ combinatorially distinct triangulations. This gives the following result.

Corollary 1.2 The 3-sphere admits $2\Omega(n^2)$ combinatorially distinct triangulations on $n$ vertices. □

2. Preliminaries

For background on polytopes, the reader can consult [9] or [4], and for background on simplicial complexes see e.g. [3].

Let $X$ denote a complex, simplicial or polyhedral. In what follows, always assume $X$ to be pure, namely all its maximal faces with respect to inclusion have the same dimension. We write $\mathcal{V}(X)$ for the set of all vertices of $X$. By a facet of $X$ we mean a face $F \in X$ of maximal dimension. We use the term $k$-face as a shorthand for $k$-dimensional face, as usual. Similarly, we call a complex $X$ a $k$-complex if all of its facets are $k$-faces.

For a polyhedral ball $X$ we denote by $\partial X$ the boundary complex of $X$. That is, $\partial X$ is the subcomplex of $X$ whose facets are precisely the faces of $X$ that are contained in exactly one facet of $X$. In particular, if $X$ is a $k$-complex homeomorphic to the $k$-ball, then $\partial X$ is $(k-1)$-dimensional, homeomorphic to the $(k-1)$-sphere. We say that a face $F \in X$ is interior to a polyhedral ball $X$ if $F \notin \partial X$. For a $X$ simplicial complex, the link of a face $F \in X$ is the subcomplex $\{T \in X : F \cap T = \emptyset, F \cup T \in X\}$, and its (closed) star is the subcomplex generated by the faces $\{T \in X : F \subseteq T\}$ under taking subsets. If $P$ is a simplex, then we will identify $P$ with its set of vertices $\mathcal{V}(P)$, or with the simplicial complex $2^{\mathcal{V}(P)}$, when convenient.

We will make use of the following arithmetic notation. For any integer $n \geq 1$, we will use $[n]$ as a shorthand for $[1,n] \cap \mathbb{Z}$, the set of all integers from 1 to $n$. For a real number $r$, we will denote by $\lfloor r \rfloor$ the floor of $r$, and by $\lceil r \rceil$ the ceiling of $r$. We will also use the notation $\sigma \in \{+,-\}$, with the convention that $-\sigma \in \{+,\}$ is the sign different from $\sigma$.

The foundation for our construction is the cyclic polytope, so we restate its definition here. The moment curve in $\mathbb{R}^d$ is the curve $\alpha_d : \mathbb{R} \to \mathbb{R}^d$ defined by

$$\alpha_d(t) = (t, t^2, t^3, \ldots, t^d).$$

The convex hull of the image of $[n]$ under $\alpha_d$, which we will denote by $C(n,d)$, is the cyclic $d$-polytope with the $n$ vertices $\alpha_d(1), \alpha_d(2), \ldots, \alpha_d(n)$. The faces of the cyclic polytope admit a simple combinatorial description, called Gale’s evenness condition (see e.g. [4], p. 62, and [9], p. 14). We restate this property here as a lemma.
Lemma 2.1 All facets of \( C(n,d) \) are \((d-1)\)-simplices. Furthermore, for any set of \( d \) integers \( I \subset [n] \), the convex hull \( \text{conv}(\alpha_d(I)) \) is a facet of \( C(n,d) \) if and only if for every \( x,y \in [n] \setminus I \), there are an even number of elements \( z \in I \) satisfying \( x < z < y \).

3. Construction of the polyhedral sphere

Consider the cyclic \( 4 \)-polytope
\[
C(4n+4,4) = \text{conv}(\alpha_4([4n+4])).
\]
Let \( P(n) \) be a polyhedral complex that is combinatorially isomorphic to the boundary complex of \( C(4n+4,4) \). We label the set of vertices of \( P(n) \) by \([4n+4]\), ordered so that each vertex \( i \) of \( P(n) \) corresponds to the vertex \( \alpha_4(i) \) of \( C(4n+4,4) \), under this isomorphism. Note that \( P(n) \) is homeomorphic to the 3-sphere. By Lemma 2.1 all the facets of \( P(n) \) are tetrahedra. That is, \( P(n) \) is a simplicial complex.

In what follows, we will describe certain faces and subcomplexes of \( P(n) \), which we will ultimately use to construct the polyhedral 3-sphere of Theorem 1.1.

We define a set of integers
\[
A(n) = \{ m \in [n+2, 3n+1] \mid m = 2k, \ k \in \mathbb{Z} \}.
\]

Therefore \( |A(n)| = n \). For \( a \in A(n) \) and \( u \in [n] \) we define the collections of vertices
\[
I(a,u,1) = \{a-u-1, \ a-u, \ a+u, \ a+u+1\},
I(a,u,2) = \{a-u-1, \ a-u, \ a+u+1, \ a+u+2\},
I(a,u,3) = \{a-u, \ a-u+1, \ a+u+1, \ a+u+2\}.
\]

Then every set \( I(a,u,i) \) is a facet of \( P(n) \). This is because \( I(a,u,i) \) satisfies the criteria of Lemma 2.1 in the case \( d = 4 \), hence \( \alpha_4(I(a,u,i)) \) is the set of vertices of a facet of \( C(4n+4,4) \). For later use, let \( I_-(a,u,i) \) (resp. \( I_+(a,u,i) \)) denote the smallest (resp. largest) two elements in \( I(a,u,i) \).

We need the following auxiliary lemma.

Lemma 3.1 For all \( a \in A(n) \), \( u, u' \in [n] \) and \( i, j \in [3] \), if \( u' \leq u-1 \), then
\[
I(a,u,i) \cap I(a,u',j) \subseteq \begin{cases} 
\{a-u, \ a+u, \ a+u+1\} & i = 1 \\
\{a-u, \ a+u+1\} & i = 2 \\
\{a-u, \ a-u+1, \ a+u+1\} & i = 3 
\end{cases}
\]

Proof. Let \( m \in I(a,u',j) \). Then
\[
a-u' - 1 \leq m \leq a+u' + 2.
\]
Since \( u' \leq u-1 \), we have \( a-u \leq a-u' - 1 \) and \( a+u'+2 \leq a+u+1 \). Therefore
\[
a-u \leq m \leq a+u+1.
\]
The lemma now follows immediately from the definition of the sets \( I(a,u,i) \). \( \square \)

For each fixed \( a \in A(n) \), we consider the collection
\[
B_0(a) = \{ I(a,u,i) \mid u \in [n], i \in [3] \}
\]
of \( 3n \) facets of \( P(n) \). Let \( B(a) \) denote the simplicial complex obtained by taking the closure of \( B_0(a) \) under subsets. We chose the simplicial complex \( B(a) \) for two main reasons, which we establish in the following lemmas. The first is that \( B(a) \) is a 3-ball (see Lemma 3.2). The second is that any two such balls \( B(a), B(a') \) intersect “minimally” (see Lemma 3.3 and
Lemma 3.4. These two facts will be crucial to our construction of the polyhedral 3-sphere of Theorem 1.1.

Lemma 3.2 For each \( a \in A(n) \), the simplicial complex \( B(a) \) is a shellable simplicial 3-ball.

Before giving a formal proof, let us describe a way to “visualize” \( B(a) \). For fixed \( i = 1, 2, 3 \) let \( L(i) \) denote the “chain” of tetrahedra \( L(i) = \{ I(a, u, i) : u \in [n] \} \). In \( L(i) \) a tetrahedron \( I(a, u, i) \) intersects only the tetrahedra right before and after it in the chain; specifically, it intersects \( I(a, u + 1, i) \) in one edge and \( I(a, u - 1, i) \) in its opposite edge. From this description it is easy to see that putting \( L(2) \) “on top of” \( L(1) \) forms a simplicial 3-ball; see Figure 1. However, this ball has no interior edges, a property needed later (see Lemma 3.5). To fix this, we put \( L(3) \) on top of \( L(2) \), which gives the simplicial ball \( B(a) \).

Proof of Lemma 3.2. We exhibit a shelling order of the facets of \( B(a) \). In particular, we define an ordering \( F_1, F_2, \ldots, F_{3n} \) of the facets of \( B(a) \), such that, if \( G_k \) denotes the closure under subsets of \( \{ F_1, F_2, \ldots, F_k \} \), then the simplicial complex \( G_k \) is a 3-ball for all \( k \in [3n] \).

This order of the facets is easy to describe. For \( i \in \mathbb{Z} \), let \( r(i) \) denote the unique element of \( \{1, 2, 3\} \) for which \( i \equiv r(i) \) (mod 3). We also write \( u_i = \lceil \frac{i}{3} \rceil \). Then we define

\[
F_i = I(a, u_i, r(i)), \quad i \in [3n].
\]

We check that this is indeed a shelling order for \( B(a) \). The facet \( F_1 \) is a tetrahedron, so \( G_1 \) forms a 3-ball. Now assume inductively that \( G_k \) forms a 3-ball, for some \( k \in [3n - 1] \).

Note first of all that since \( P(n) \) is a 3-sphere, every triangle \( T \in P(n) \) is a face of exactly two tetrahedra of \( P(n) \). Therefore, for any triangle \( T \in F_{k+1} \cap G_k \), we must have

\[
T \in \partial G_k
\]
as otherwise, since \( G_k \) is a 3-ball, the triangle \( T \) would be a face of three tetrahedra in \( P(n) \), namely two tetrahedra of \( G_k \), and the tetrahedron \( F_{k+1} \).

We now show that \( G_{k+1} \) is a 3-ball. We consider separately the cases \( r(k) = 1, 2, 3 \).

When \( r(k) = 1 \), we have \( u_k = \lceil \frac{k}{3} \rceil = \lceil \frac{k+1}{3} \rceil = u_{k+1} \). Therefore \( F_k = I(a, u_k, 1) \) and \( F_{k+1} = I(a, u_k, 2) \). Thus \( F_{k+1} \cap F_k \) is the triangle

\[
T_{k,1} = \{a - u_k - 1, a - u_k, a + u_k + 1\}.
\]

If \( i \leq k - 1 \) then \( u_i \leq u_k - 1 \), so by Lemma 3.1

\[
F_{k+1} \cap F_i = I(a, u_i, r(i)) \cap I(a, u_k, 2) \subseteq \{a - u_k, a + u_k + 1\} \subset T_{k,1}.
\]
So we have $F_{k+1} \cap F_i \subseteq T_{k,1}$ for all $i \leq k$, and $F_{k+1} \cap F_k = T_{k,1}$. Thus $F_{k+1} \cap G_k = T_{k,1}$.

By the inductive hypothesis $G_k$ is a 3-ball. Therefore $G_{k+1}$ is the union of the two 3-balls $G_k$ and $F_{k+1}$, the intersection of which is the 2-ball $T_{k,1}$. By (1), this 2-ball is contained in the boundary of both $G_k$ and $F_{k+1}$. Hence $G_{k+1}$ is a 3-ball in the case $r(k) = 1$.

When $r(k) = 2$, we have $u_{k-1} = u_k = u_{k+1}$, so $F_{k-1} = I(a, u_k, 1)$, $F_k = I(a, u_k, 2)$, and $F_{k+1} = I(a, u_k, 3)$. Thus $F_{k+1} \cap F_k$ is the triangle
\[ T_{k,2} = \{a - u_k, a + u_k + 1, a + u_k + 2\}, \]
and
\[ F_{k+1} \cap F_{k-1} = \{a - u_k, a + u_k + 1\} \subseteq T_{k,2}, \]
finishing the proof if $k = 2$. Furthermore, if $k > 2$, then $F_{k-3} = I(a, u_k - 1, 2)$, so $F_{k+1} \cap F_{k-3}$ is the triangle
\[ T'_{k,2} = \{a - u_k, a - u_k + 1, a + u_k + 1\}. \]

If $i \leq k - 2$ then $u_i \leq u_k - 1$, so by Lemma 3.1
\[ F_{k+1} \cap F_i = I(a, u_i, r(i)) \cap I(a, u_k, 3) \subseteq \{a - u_k, a - u_k + 1, a + u_k + 1\} = T'_{k,2}. \]

Therefore $F_{k+1} \cap F_i$ is contained in the complex formed by $T_{k,2}$ and $T'_{k,2}$ for all $i \leq k$, and $F_{k+1} \cap F_k = T_{k,2}$ and $F_{k+1} \cap F_{k-3} = T'_{k,2}$. We conclude that $F_{k+1} \cap G_k$ has the two facets $T_{k,2}$ and $T'_{k,2}$, which form a 2-ball. Together with (1) it follows that $G_{k+1}$ is a 3-ball in the case $r(k) = 2$.

Finally, when $r(k) = 3$, we have $r(k+1) = 1$, and $u_{k+1} = u_k + 1$ and $u_{k-1} = u_k$. Therefore $F_{k+1} = I(a, u_k + 1, 1)$ and $F_{k-1} = I(a, u_k, 2)$, so $F_{k+1} \cap F_{k-1}$ is the triangle
\[ T_{k,3} = \{a - u_k - 1, a + u_k + 1, a + u_k + 2\}. \]

Also, if $i \leq k$, then $u_i \leq u_{k+1} - 1$, so by Lemma 3.1
\[ F_{k+1} \cap F_i = I(a, u_i, r(i)) \cap I(a, u_k + 1, 1) \subseteq \{a - u_k - 1, a + u_k + 1, a + u_k + 2\} = T_{k,3}. \]

Therefore, $F_{k+1} \cap F_i \subseteq T_{k,3}$ for all $i \leq k$, and $F_{k+1} \cap F_{k-1} = T_{k,3}$. We conclude that $F_{k+1} \cap G_k = T_{k,3}$. This and (1) imply that that $G_{k+1}$ is a 3-ball in the case $r(k) = 3$.

We now show that the intersection of two different balls $B(a)$ and $B(a')$ does not contain a triangle. As an immediate consequence, the balls $B(a)$ and $B(a')$ intersect only in their boundaries, which we state as a separate lemma.

**Lemma 3.3** For distinct $a, a' \in A(n)$, the intersection $B(a) \cap B(a')$ does not contain a 2-face of $P(n)$.

**Proof.** Let $a, a' \in A(n)$, with $a' < a$. Suppose there is a triangle $T \subseteq B(a) \cap B(a')$. As $T$ belongs to some tetrahedron $I(a, u, i)$ with $u \in [n]$ and $i \in [3]$, there is a unique way to write $T = \{k_1, k_2, k_3\}$ such that $|k_1 - k_2| = 1$, and then it satisfies $4a - 1 \leq k_1 + k_2 + 2k_3 \leq 4a + 5$. As $T$ also belongs to some tetrahedron $I(a', u', i')$ with $u' \in [n]$ and $i' \in [3]$, it follows that $4a' - 1 \leq k_1 + k_2 + 2k_3 \leq 4a' + 5$. However, by definition of $A(n)$, we have $a' \leq a - 2$, hence the intervals $[4a' - 1, 4a' + 5]$ and $[4a - 1, 4a + 5]$ are disjoint, a contradiction.

**Lemma 3.4** For distinct $a, a' \in A(n)$, we have $B(a) \cap B(a') \subseteq \partial B(a) \cap \partial B(a')$.

**Proof.** Let $a, a' \in A(n)$, with $a' \neq a$. Let $F \in B(a) \cap B(a')$ and suppose by contradiction that $F$ is interior to one of the 3-balls, say $B(a)$. Since $F \in B(a')$, and $P(n)$ is a 3-sphere, this implies that the closed star of $F$ in $B(a')$ is a subcomplex of $B(a)$. In particular, $B(a) \cap B(a')$ contains a tetrahedron, hence it contains a triangle. This contradicts Lemma 3.3. □
To understand the boundary complex $\partial B(a)$ of each ball $B(a)$, we introduce the following notation. For $a \in A(n)$, $u \in [n]$ and $i \in [3]$, let

$$
x_-(a, u, 1) = a - u - 1, \quad x_+(a, u, 1) = a + u,
$$

$$
x_-(a, u, 2) = a - u - 1, \quad x_+(a, u, 2) = a + u + 2,
$$

$$
x_-(a, u, 3) = a - u + 1, \quad x_+(a, u, 3) = a + u + 2.
$$

Our next result characterizes the boundary complex of each ball $B(a)$.

**Lemma 3.5** For every $a \in A(n)$, the 2-faces of the boundary complex $\partial B(a)$ are exactly the triangles

$$I_\sigma(a, u, i) \cup \{x_\sigma(a, u, i)\},$$

for $u \in [n]$, $i \in [3]$, and $\sigma \in \{-, +\}$.

**Proof.** Let $T \in B(a)$ be a triangle, so $T$ is in $I(a, u, i)$ for some $u \in [n]$ and $i \in [3]$, and can be written uniquely as $T = \{k_1, k_2, k_3\}$, where $|k_1 - k_2| = 1$. Together with $T$, one of $k_3 \pm 1$ forms the tetrahedron $I(a, u, i)$, and the other forms a tetrahedron $I' \in P(n)$. Then $T \in \partial B(a)$ if and only if $I' \neq I(a, u', i')$ for all $u' \in [n]$ and $i' \in [3]$. Essentially, fixing $a$ and $u$, there are 12 triangles for which we need to check this (though some work can be saved). We exhibit here a sample of these computations. A key invariant to compute is the label average $e(F)$ of the vertices of a tetrahedron $F$. For $I(a, u, i)$ this equals

$$e(I(a, u, i)) = \frac{1}{4} \sum_{k \in I(a, u, i)} k = a + \frac{i - 1}{2}.$$

In particular, $e(I(a, u, i)) = e(a, i)$ does not depend on $u$.

Let $T = I(a, u, 1) \setminus \{a + u + 1\}$, so $k_3 = a + u$. For $I' = T \cup \{k_3 - 1\}$, we obtain $e(I') = a - 1/2 < a$, thus $e(I')$ is smaller then all $e(I(a, u', i'))$, implying that $T \in \partial B(a)$.

Let $T = I(a, u, 1) \setminus \{a + u\}$, so $k_3 = a + u + 1$. For $I' = T \cup \{k_3 + 1\}$, we obtain $I' = I(a, u, 2) \in B(a)$, implying that $T \notin \partial B(a)$.

We leave the other 10 checks to the reader. \qed

The classification of Lemma 3.5 yields an important fact about the edges of $B(a)$, as follows. Define the edge $E(a, u)$ of $P(n)$ by

$$E(a, u) = \{a - u, a + u + 1\}.$$

Clearly $E(a, u)$ is an edge of $I(a, u, 1)$, hence an edge of $B(a)$. As it turns out, the interior edges of $B(a)$ are exactly the edges $E(a, u)$, a consequence of Lemma 3.5. This is the content of the next lemma.

**Lemma 3.6** The interior edges of $B(a)$ are exactly the edges $\{E(a, u) : u \in [n]\}$.

**Proof.** Fix $u \in [n]$. By Lemma 3.5 we see that, among the edges of the 3 tetrahedra $I(a, u, i)$, with $i \in [3]$, the only one not belonging to a boundary triangle of $B(a)$ is $E(a, u)$. Thus, all interior edges of $B(a)$ must be of the form $E(a, u)$ for some $u \in [n]$. To verify that each edge $E(a, u)$ is indeed interior in $B(a)$, we check that the link of $E(a, u)$ in $B(a)$ is a cycle. Indeed, this link contains the 4-cycle $(a - u - 1, a + u, a - u + 1, a + u + 2)$, and hence is equal to it (as the link in the entire complex $P(n)$ is a cycle). \qed

We now consider the two triangles of $I(a, u, 1)$ not having $E(a, u)$ as an edge, namely

$$T_\sigma(a, u) = I_\sigma(a, u, 1) \cup \{x_\sigma(a, u, 1)\}$$
Since $R$ take the average of the vertices of $D(a,u)$ if it did, the fact that $E(a,u)$ is a boundary triangle of $B(a)$. That is, $D(a,u)$ is a subcomplex of $\partial B(a)$. We define $R(a,u)$ to be the intersection of the two triangles of $D(a,u)$. That is,

$$R(a,u) = T_-(a,u) \cap T_+(a,u) = \{a - u - 1, a - u, a + u\}.$$ 

We define $D(a,u)$ to be the 2-dimensional simplicial complex obtained as the closure of $\{T_-(a,u), T_+(a,u)\}$ under subsets. From Lemma 3.7, we see that each $T_{\sigma'}(a,u)$ is a boundary triangle of $B(a)$. That is, $D(a,u)$ is a subcomplex of $\partial B(a)$. We define $R(a,u)$ to be the intersection of the two triangles of $D(a,u)$. That is,

$$R(a,u) = T_-(a,u) \cap T_+(a,u) = \{a - u - 1, a + u\}.$$ 

Since $R(a,u)$ is an edge, it follows that $D(a,u)$ is a 2-ball. Note also that $R(a,u)$ is the unique interior edge of $D(a,u)$.

Understanding the intersection of distinct disks $D(a,u)$ and $D(a',u')$ is crucial for constructing the polyhedral 3-sphere of Theorem 1.1. The relevant properties of this intersection are stated in the following lemma.

**Lemma 3.7** For $(a,u) \neq (a',u')$, the disks $D(a,u)$ and $D(a',u')$ intersect in a single face. When $a = a'$, this intersection lies on the boundary of both disks.

**Proof.** Let $a, a' \in A(n)$ and $u, u' \in [n]$ such that $(a,u) \neq (a',u')$.

First we consider the case $a = a'$ and $u' < u$. Then $u' \leq u - 1$, so by Lemma 3.4, we have

$$T_{\sigma'}(a,u') \cap T_{\sigma'}(a,u) \subseteq I(a,u',1) \cap I(a,u,1) \subseteq \{a - u, a + u, a + u + 1\},$$

for every choice of $\sigma, \sigma' \in \{-, +\}$. Thus

$$V(D(a,u) \cap D(a,u')) \subseteq \{a - u, a + u, a + u + 1\}. \quad (2)$$

By definition,

$$T_-(a,u') = \{a - u' - 1, a - u', a + u'\},$$

$$T_+(a,u') = \{a - u' - 1, a + u', a + u' + 1\}.$$ 

Since $u' \leq u - 1$, it follows that

$$T_-(a,u') \cap \{a - u, a + u, a + u + 1\} \subseteq \{a - u\},$$

$$T_+(a,u') \cap \{a - u, a + u, a + u + 1\} \subseteq \{a - u, a + u\}.$$ 

This and (2) imply that

$$V(D(a,u) \cap D(a,u')) \subseteq \{a - u, a + u\}.$$ 

As $\{a - u, a + u\}$ is an edge of $T_-(a,u)$ we conclude that $D(a,u) \cap D(a,u')$ is either an edge or a vertex or empty. Finally, observe that $\{a - u, a + u\}$ is not one of the edges $R(a,u)$, $R(a,u')$. Therefore

$$D(a,u) \cap D(a,u') \subseteq \partial D(a,u) \cap \partial D(a,u').$$

Now consider the case $a' < a$. In this case, $E(a,u) \neq E(a',u')$. For otherwise, we may take the average of the vertices of $E(a,u) = E(a',u')$, to obtain

$$a + \frac{1}{2} = \frac{1}{2} \sum_{k \in E(a,u)} k = a' + \frac{1}{2},$$

from which we conclude $a = a'$.

From the definitions, we see that $E(a,u)$ is the only missing edge of $D(a,u)$ (namely, a non-edge of $D(a,u)$ whose vertices are in $D(a,u)$), and $E(a',u')$ is the only missing edge of $D(a',u')$. It follows that $D(a,u) \cap D(a',u')$ does not contain both vertices of $E(a,u)$, as if it did, the fact that $E(a,u) \neq E(a',u')$ implies that $E(a,u)$ is an edge of $D(a',u')$. But
$E(a, u)$ is interior to $B(a)$, contradicting Lemma 3.4. Similarly, $D(a, u) \cap D(a', u')$ does not contain both vertices of $E(a', u')$. That is,

(3) \[ E(a, u), E(a', u') \not\subset V(D(a, u) \cap D(a', u')). \]

Now suppose (for a contradiction) that $D(a, u) \cap D(a', u')$ has at least 3 vertices. Then (3) implies that $D(a, u) \cap D(a', u')$ must contain one of the triangles $T_{\sigma}(a, u) = T_{\sigma'}(a', u')$, contradicting Lemma 3.3. It follows that $D(a, u) \cap D(a', u')$ is either an edge or a vertex or empty.

We are now ready to construct the polyhedral 3-sphere that proves Theorem 1.1.

**Proof of Theorem 1.1.** For each $a \in A(n)$, we do the following to $P(n)$. Remove all interior faces of $B(a)$. Add a new vertex $q(a)$ to $P(n)$, and cone from this vertex to $\partial B(a)$, to obtain a new simplicial 3-ball, call it $B'(a)$.

By Lemma 3.4, the collection of all tetrahedra that results from the above process (both new tetrahedra and tetrahedra which were not removed from $P(n)$) is a simplicial complex, call it $P'(n)$. Since each $B'(a)$ is a 3-ball with the same boundary as $B(a)$, the complexes $P'(n)$ and $P(n)$ are homeomorphic. That is, $P'(n)$ is also a 3-sphere.

Now, for each $a \in A(n)$ and $u \in [n]$, remove from $P'(n)$ the triangle $T'(a, u)$ defined by

\[ T'(a, u) = \{q(a)\} \cup R(a, u), \]

merging the two tetrahedra it borders into the same facet. Call the resulting collection of faces $Q(n)$. Recall that the edge $R(a, u)$ of $T'(a, u)$ is the unique interior edge of the disk $D(a, u)$. Therefore, it follows from Lemma 3.7 that removing a triangle $T'(a, u)$ creates exactly one non-simplicial facet of $Q(n)$, which we will denote by $F(a, u)$. That is, $F(a, u)$ is the combinatorial bipyramid obtained from the two tetrahedra

\[
T'(a, u) \cup \{a - u\}, \\
T'(a, u) \cup \{a + u + 1\},
\]

of $P'(n)$, by removing their intersection $T'(a, u)$. The 5 vertices of $F(a, u)$ are

\[ \mathcal{V}(F(a, u)) = T'(a, u) \cup E(a, u) = \{q(a)\} \cup I(a, u, 1). \]

We must show that $Q(n)$ is in fact a polyhedral complex—that is, that every two faces of $Q(n)$ intersect in a single face. As $P'(n)$ is a simplicial complex, we only need to show that each $F(a, u)$ intersects the other faces of $Q(n)$ properly. This follows from the fact that

\[ F(a, u) \cap \partial B'(a) = D(a, u), \]

together with Lemma 3.7.

Clearly the polyhedral complex $Q(n)$ is homeomorphic to $P'(n)$, hence is a 3-sphere. Finally, $Q(n)$ is obtained from $P(n)$ by adding $|A(n)|$ vertices. Therefore $Q(n)$ has

\[ 4n + 4 + |A(n)| = 5n + 4 \]

vertices. Furthermore, as noted above, we have exactly one non-simplicial facet $F(a, u)$ of $Q(n)$ for each $(a, u) \in A(n) \times [n]$. That is, $Q(n)$ has

\[ |A(n) \times [n]| = n^2 \]

non-simplicial facets. \[ \square \]

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