The total disjoint irregularity strength of some certain graphs

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Abstract

Under a totally irregular total \( k \)-labeling of a graph \( G = (V, E) \), we found that for some certain graphs, the edge-weight set \( W(E) \) and the vertex-weight set \( W(V) \) of \( G \) which are induced by \( k = ts(G) \), \( W(E) \cap W(V) \) is a nonempty set. For which \( k \), a graph \( G \) has a totally irregular total labeling if \( W(E) \cap W(V) = \emptyset \)? We introduce the total disjoint irregularity strength, denoted by \( ds(G) \), as the minimum value \( k \) where this condition satisfied. We provide the lower bound of \( ds(G) \) and determine the total disjoint irregularity strength of cycles, paths, stars, and complete graphs.

Keywords: total disjoint irregularity strength, total irregularity strength, irregular total labeling

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1. Introduction

Let \( G \) be a finite, simple, and undirected graph with the vertex set \( V \) and the edge set \( E \). Let \( f : V \cup E \rightarrow \{1, 2, \cdots, k\} \) be a total \( k \)-labeling. Under \( f \), the weight of a vertex \( v \in V \) is \( w(v) = f(v) + \sum_{uv \in E} f(uv) \) and the weight of an edge \( uv \in E \) is \( w(uv) = f(u) + f(uv) + f(v) \). If all the vertex (or edge)-weights are distinct then \( f \) is called a vertex (or edge) irregular total
For instance, Fig. 1 shows a totally disjoint irregular total labeling of $P_5$, $C_3$, $K_4$, and $K_5$.

$k$-labeling and the minimum value $k$ such that $G$ has a vertex (or edge) irregular total $k$-labeling is called the total vertex (or edge) irregularity strength, denoted by $\text{tvs}(G)$ (or $\text{tes}(G)$), respectively. This parameter was introduced by Baca et al. [2]. They gave the boundary of $\text{tes}(G)$ and $\text{tvs}(G)$ and determined that for $n$ vertices, $\text{tes}(C_n) = \text{tes}(P_n) = \left\lceil \frac{n+2}{3} \right\rceil$, $\text{tes}(P_n) = \left\lceil \frac{n+1}{3} \right\rceil$, $\text{tvs}(S_n) = \text{tes}(S_n) = \left\lceil \frac{n+1}{2} \right\rceil$, and $\text{tvs}(K_n) = 2$.

Later, Jendrol et al. [7] provided a better lower bound of $\text{tes}(G)$ and determined that $\text{tes}(K_5) = 5$ and $\text{tes}(K_n) = \left\lceil \frac{n^2-n+4}{6} \right\rceil$, for $n \neq 5$. For any tree $T$, Ivanco and Jendrol [6] proved that $\text{tes}(T)$ is equal to the lower bound. Nurdin et al. [9] gave the lower bound for $\text{tvs}$ for any graph $G$.

Recently, Marzuki et al. [8] combined the properties of $\text{tes}(G)$ and $\text{tvs}(G)$ and gave new parameter called the total irregularity strength, denoted by $\text{ts}(G)$. It is the minimum value $k$ for which $G$ has a totally irregular total $k$-labeling. They proved that the lower bound $\text{ts}(G) \geq \max\{\text{tes}(G), \text{tvs}(G)\}$ is sharp for $C_n$ and $P_n$ except for $P_2$ or $P_3$. In [14], we proved that for $n \neq 2$, $\text{tes}(K_n) = \text{tes}(K_n)$. In [5], Indriati et al. proved that for $n \geq 3$, $\text{tes}(S_n) = \text{tvs}(S_n)$. For further reading, one can see [1], [3], [4], [5], and [10] - [13]. All these results showed that the lower bound is sharp.

Observing $\text{tes}(G)$, for the vertex weight-set $W(V)$ and the edge weight-set $W(E)$ under a totally irregular total labeling on $G$, $W(V) \cap W(E) \neq \emptyset$. Considering this condition, we define a new parameter called the total disjoint irregularity strength. A totally disjoint irregular total $k$-labeling of a graph $G$ as a total labeling $f : V \cup E \rightarrow \{1, 2, \cdots, k\}$ which satisfies: (i) for any two vertices $x \neq y \in V$, $w(x) \neq w(y)$; (ii) for any two edges $x_1y_1 \neq x_2y_2 \in E$, $w(x_1y_1) \neq w(x_2y_2)$; (iii) $W(V) \cap W(E) = \emptyset$; where $W(V)$ (and $W(E)$) is the vertex (and edge) weight-set, respectively. The minimum value $k$ such that a graph $G$ has a totally disjoint irregular total labeling is called the total disjoint irregularity strength of a graph $G$, denoted by $\text{ds}(G)$. Thus, for any graph $G$,

$$\text{ds}(G) \geq \text{ts}(G).$$

For instance, Fig. 1 shows a totally disjoint irregular total labeling of $P_3$, $C_3$, $K_4$, and $K_5$.

In this paper, we determine $\text{ds}$ of cycles, paths, stars, and complete graphs.

2. Main Results

Let $G = (V, E)$ be a connected graph. For $G$ has a totally irregular total $k$-labeling $f : V \cup E \rightarrow \{1, 2, \cdots, k\}$, we need $|V| + |E|$ distinct weights. Let $\delta = \delta(G)$ (or $\Delta = \Delta(G)$) be the minimum (or maximum) degree of vertex in $G$, respectively. Let $n_i$ be the number of vertices of degree $i$, where $i = \delta, \delta + 1, \cdots, \Delta$. Then $|V| = \sum_{i=\delta}^{\Delta} n_i$. Now, assume that $\delta = 1$. Let $f$ be a optimal
labeling with respect to $ds(G)$. Then the maximum weight has to be at least $|E| + |V| + 1$. The maximum vertex weight is the sum of $\Delta + 1$ labels and every edge weight is the sum of three labels imply that $k \geq \lceil \frac{|E| + |V| + 1}{\Delta + 1} \rceil$. Moreover, when $n_1 \leq \Delta$, $|V| + |E|$ distinct weights are only exist if $\lceil \frac{|E| + |V| + 1}{\Delta + 1} \rceil \geq n_1$. In the other hand, when $n_1 \geq \Delta$, we have $2n_1$ distinct weights depend on 2 labels, such that $|V| + |E|$ distinct weights are only exist if $\lceil \frac{|E| + |V| + 1}{\Delta + 1} \rceil \geq n_1$. Hence, the minimum value $k \geq \max \{ n_1, \lceil \frac{|E| + n_1 + n_2 + 3}{3} \rceil \}$. If $v \notin V$, then $\delta \neq 1$. For $f$ is optimal then the minimum weight is at least 3. Then, $k \geq \lceil \frac{|E| + n_1 + n_2 + 3}{3} \rceil$. Thus we have the lower bound of $ds(G)$.

**Theorem 2.1.** Let $G = (V, E)$ be a connected graph. Let $v$ be a pendant vertex and $n_i (i = 1, 2)$ be the number of vertices of degree $i$. Then

$$ds(G) \geq \begin{cases} \max \left\{ n_1, \left\lceil \frac{|E| + n_1 + n_2 + 3}{3} \right\rceil \right\}, & \text{if } v \in V; \\ \left\lceil \frac{|E| + n_1 + n_2 + 3}{3} \right\rceil, & \text{if } v \notin V. \end{cases}$$

Our next results show that this lower bound is sharp.

**Theorem 2.2.** Let $m_1 \geq 3$ and $m_2 \in \mathbb{N}$. Let $C_{m_1}$ be a cycle with $m_1$ vertices and $P_{m_2}$ be a path with $m_2$ vertices. Then

$$ds(C_{m_1}) = \left\lceil \frac{2m_1 + 2}{3} \right\rceil;$$
$$ds(P_{m_2}) = \begin{cases} 3, & \text{for } m_2 = 3; \\ \left\lceil \frac{2m_2}{3} \right\rceil, & \text{otherwise}. \end{cases}$$

To prove Theorem 2.2, we need this lemma.

**Lemma 2.1.** For any integers $y$ and $x_i, 1 \leq i \leq 2n$, let $\{x_i\}$ be a sequence. If the sum of 3 consecutive integers in $\{x_i\}$ is

$$x_i + x_{i+1} + x_{i+2} = \begin{cases} y + 2i - 2, & \text{for } 1 \leq i \leq n - 1; \\ y + 4n - 2i - 3, & \text{for } n \leq i \leq 2n - 2; \end{cases}$$

then

$$x_i = \begin{cases} x_{2n-1}, & \text{for } i = n - 3j \text{ and } 1 \leq j \leq \left\lceil \frac{2n}{3} \right\rceil; \\ x_{2n-i+2}, & \text{for } i = n - 3j + 1 \text{ and } 1 \leq j \leq \left\lceil \frac{2n}{3} \right\rceil. \end{cases}$$

**Proof.** Set all equations above as a linear system leads to the solution which is required. \qed

Now, we are able to prove Theorem 2.2.
Case 1. $m_2 = 3$

Suppose that $ds(P_3) = 2$. Since we need 5 distinct weight from 2 to 6, one endpoint (and its incidence edge) can be labeled by 1 to have smallest weight. In the other hand, maximum weight 6 only can occur when the rest vertices and edge are labeled by 2. Hence, there is one vertex and one edge of the same weight. Contrary to hypothesis. Thus, $ds(P_3) \geq 3$. By label $P_3$ as in Fig.1, we have $ds(P_3) = 3$.

Case 2. $m_1 \geq 3$ and $m_2 \neq 3$

It is trivial for $n = 2$. For $n_1 \geq 3$ and $n_2 \geq 2$, by Theorem 2.1, $ds(C_{m_1}) \geq t(m_1)$ and $ds(P_{m_2}) \geq t(m_2)$. For the reverse inequality, we construct $f_i : V \cup E \rightarrow \{1, 2, \ldots, t(m_i)\}$, for $i = 1, 2$, as follows: Let $f_1^{m_1} = \{v_1^{m_1}, v_2^{m_1}, \ldots, v_{m_1}^{m_1}\}$ and $f_2^{m_2} = \{v_1^{m_2}, v_2^{m_2}, \ldots, v_{m_2}^{m_2}\}$ be the alternating vertex (and edge) label-sets, where $f_1(v_i) = v_i^{m_1}$, $f_2(v_i) = v_i^{m_2}$, $f_1(e_i) = e_i^{m_1}$, $f_2(e_i) = e_i^{m_2}$, and $W(C_{m_1}) = \{w(v_i), w(e_i)\}$ for $1 \leq i \leq m_1$ and $W(P_{m_2}) = \{w(v_1), w(e_1), w(v_2), w(e_2), \ldots, w(v_{m_2})\}$ be the alternating vertex (and edge) weight-sets of $C_{m_1}$ and $P_{m_2}$, respectively. Let

$$d(m_i) = \begin{cases} t(m_i) - 1, & \text{for } m_1 \equiv j \mod 3, j = 0, 1, m_2 \equiv j \mod 3, j = 0, 2; \\ t(m_i), & \text{for } m_1 \equiv 2 \mod 3, m_2 \equiv 1 \mod 3. \end{cases}$$

We prove by induction on $m_i$. For the base step, it is true that for $f_1^3 = \{1, 1, 2, 3, 1\}$, $f_2^3 = \{1, 2, 3, 4, 4, 1\}$, and $f_1^5 = \{1, 1, 2, 3, 4, 4, 1\}$, we have $W(C_3) = \{3, 4, 6, 8, 7, 5\}$, $W(C_4) = \{4, 5, 7, 8, 10, 11, 9, 6\}$, and $W(C_5) = \{3, 4, 5, 7, 8, 10, 11, 12, 9, 6\}$ and for $f_2^3 = \{1, 1, 2\}$, $f_2^4 = \{1, 1, 2, 3, 4\}$, and $f_2^5 = \{1, 1, 2, 3, 4, 4, 2, 3, 1, 3\}$, we have $W(P_2) = \{2, 4, 3\}$, $W(P_4) = \{2, 3, 4, 5, 7, 8, 6\}$, and $W(P_6) = \{2, 3, 5, 8, 11, 12, 10, 9, 6, 7, 4\}$ such that for $i = 1, 2$, $f_1$ is a totally disjoint irregular total $t(m_i)$ labeling, $ds(C_{m_1}) = t(m_1)$ for $m_1 \in \{3, 4, 5\}$ and $ds(P_{m_2}) = t(m_2)$ for $m_2 \in \{2, 4, 6\}$, where the maximum weight is $w(e_{d(m_i)})$.

For the inductive step, we assume that for all $k_1$ and $k_2$, $f_1$ is a totally disjoint irregular total $t_1$-labeling of $C_{k_1}$ and $f_2$ is a totally disjoint irregular total $t_2$-labeling of $P_{k_2}$, where

$$e_{d_1(k_1)} = \begin{cases} t_1 - 1, & \text{for } i = 2, k_2 \equiv j \mod 9, j \in \{1, 2, 8\}; \\ t_1, & \text{for } i \in \{1, 2\}, k_2 \equiv j \mod 9, j \in \{0, 3, 4, 5, 6, 7\}; \end{cases}$$

$$v_{d_1(k_1)+1} = \begin{cases} t_1 - 1, & \text{for } i \in \{1, 2\}, k_1 = 6, k_2 \equiv j \mod 9, j \in \{5, 7, 8\}; \\ t_1, & \text{for } i \in \{1, 2\}, k_1 \neq 6, k_2 \equiv j \mod 9, j \in \{0, 1, 2, 3, 4, 6\}; \end{cases}$$

and the maximum weight is $w(e_{d_1(k_1)})$.

Let $G_{k_1} \cong C_{k_1}$ and $G_{k_2} \cong P_{k_2}$. To prove that $ds(G_{(k_1)+3}) = t(k_1 + 3) = ds(G_{k_1}) + 2$, we construct $G_{(k_1)+3}$ from $G_{k_1}$ by subdivide $e_{d_1(k_1)}$ as described in Fig.2. Define $f_i^{(k_1)+3} = \{(v_1^{(k_1)+3}, e_{d_1(k_1)+1}^{(k_1)+3}), (v_2^{(k_1)+3}, e_{d_1(k_1)+3})\}$ and $f_i^{(k_1)+3} = f_i^{(k_1)+3}$. Setting $w(e_{d_1(k_1)}) = w(e_{d_2(k_1)+3})$ and $w(v_{d_1(k_1)+1}) = w(v_{d_2(k_1)+1})$, we have $a \neq b$ for $a, b \in W(G_{(k_1)+3}) \{v_1^{(k_1)+1}, e_{d_1(k_1)+1}^{(k_1)+1}, v_2^{(k_1)+2}, e_{d_1(k_1)+3}, e_{d_1(k_1)+1}\}$. Moreover, $e_{d_1(k_1)+3} = e_{d_1(k_1)+3}$ and $v_{d_1(k_1)+1} = e_{d_1(k_1)+1}$. This is sufficient to apply Lemma 2.1. Let \( x_i \mid 1 \leq i \leq 8 \) = \( e_{d_1(k_1)+3}, v_{d_1(k_1)+1}, e_{d_1(k_1)+1}, v_{d_1(k_1)+2}, e_{d_1(k_1)+2}, e_{d_1(k_1)+3}, v_{d_1(k_1)+3}, v_{d_1(k_1)+1} \) and \( y = w(e_{d_1(k_1)}) + 1 \). Then, we have $v_{d_1(k_1)+3} = e_{d_1(k_1)+2}$, $e_{d_1(k_1)+3} = v_{d_1(k_1)+1} + 2$, and $v_{d_1(k_1)+3} - 1 = e_{d_1(k_1)+3}$.
The total disjoint irregularity strength of some certain graphs  |  M.I. Tilukay and A.N.M. Salman

Figure 2. The construction of $P_{k+3}$ from $P_k$

where

$$e^{(k_i)+3}_{(k_i)+1} = \begin{cases} 
3, & \text{for } i = 1, k_i = 3; \\
2k_i + t(k_i), & \text{for } i = 1, k_i \neq 3; \\
2k_i - 2t(k_i) + 3, & \text{for } i = 2, k_i \equiv 8 \text{ mod } 9; \\
2k_i - 2t(k_i) + 2, & \text{for } i = 2, k_i \equiv j \text{ mod } 9, j \in \{1, 2, 5, 7\}; \\
2k_i - 2t(k_i) + 1, & \text{for } i = 2, k_i \equiv j \text{ mod } 9, j \in \{3, 4, 6\}.
\end{cases}$$

Then, it can be checked that the maximum label is $ds(G_{(k_i)+3}) + 2 = ds(G_{(k_i)+3})$. We have completed the labeling $f_i$ on $G_{(k_i)+3}$ and have proved that $f_i$ is a totally disjoint irregular total $t(k_i)$-labeling. Thus, for any positive integer $m_1 \geq 3$ and $m_2 \in \mathbb{N}$, $ds(C_{m_1}) = \left\lfloor \frac{2m_1+2}{3} \right\rfloor$, $ds(P_{m_2}) = \left\lceil \frac{2m_2}{3} \right\rceil$, for $m_2 \neq 3$ and $ds(P_3) = 3$.

**Theorem 2.3.** Let $n \in \mathbb{N}$, $n \geq 3$ and $S_n$ be a star with $n+1$ vertices, then $ds(S_n) = n$.

*Proof.* Let $V(S_n) = \{v_i | 1 \leq i \leq n+1\}$ where $v_{n+1}$ is the vertex of degree $n$. By Theorem 2.1, $ds(S_n) \geq n$. To prove the reverse inequality, we construct an irregular total labeling $f : V \cup E \rightarrow \{1, 2, \cdots , t\}$ by define $f(v_i) = i$ for $1 \leq i \leq n$ and $f(v_n) = n-2$, $f(v_{n+1}) = n$, $f(v_iv_{n+1}) = 1$ for $1 \leq i \leq n-1$, and $f(v_nv_{n+1}) = 3$. Hence, we have $w(v_i) = i+1$ for $1 \leq i \leq n-1$, $w(v_n) = n+1$, $w(v_{n+1}) = 2n+2$, $w(v_iv_{n+1}) = n + i + 1$ for $1 \leq i \leq n-1$, and $w(v_nv_{n+1}) = 2n+1$. See that $W(V) \cap W(E) = \emptyset$. Thus, $f$ is a totally disjoint irregular total labeling and $ds(S_n) = n$ for $n \geq 3$. \hfill $\Box$

Next, by using our previous result in [14], we determine the exact value of $ds(K_n)$. For the convenience of reader, we provide the construction of totally irregular total labeling of $K_n$ for $n \neq 5, 10, 12$ given in [14]. Let $\left\lfloor \frac{n^2-n+4}{6} \right\rfloor = t$ and $\left\lfloor \frac{n+1}{3} \right\rfloor = s$. We divide the vertex-set into 3 mutually disjoint subsets, say $A$, $B$, and $C$, where $A = \{a_i | 1 \leq i \leq s\}$, $B = \{b_i | 1 \leq i \leq n-2s\}$, and $C = \{c_i | 1 \leq i \leq s\}$. Let $f : V \cup E \rightarrow \{1, 2, \cdots , t\}$ defined by:

\begin{align*}
  f(a_i) &= 1, \quad \text{for } 1 \leq i \leq s; \\
  f(b_i) &= \binom{s}{2} + s(i-1) + 1, \quad \text{for } 1 \leq i \leq n-2s; \\
  f(c_i) &= t, \quad \text{for } 1 \leq i \leq s; \\
  f(a_i a_j) &= \binom{j-1}{2} + i, \quad \text{for } 1 \leq i < j \leq s; \\
  f(a_i b_j) &= i, \quad \text{for } 1 \leq i \leq s, 1 \leq j \leq n-2s; \\
  f(a_i c_j) &= s(i-1) + j, \quad \text{for } 1 \leq i, j \leq s; \\
  f(b_i b_j) &= s(n-s-i-j+2) - \left(\binom{s}{2} + \binom{j-1}{2}\right) + i, \quad \text{for } 1 \leq i < j \leq n-2s; \\
  f(b_i c_j) &= \binom{n-2s}{2} + s(n-s) - t + j + 1, \quad \text{for } 1 \leq i \leq n-2s, 1 \leq j \leq s; \\
  f(c_i c_j) &= \binom{n}{2} - 2(t-1) - \binom{s-1+i}{2} + j - i, \quad \text{for } 1 \leq i < j \leq s.
\end{align*}

(2)
Theorem 2.4. Let \( n \in \mathbb{N} \), \( n \notin \{ i \mid 6 \leq i \leq 59 \} \cup \{ 61, 62, 65, 68, 71, 74 \} \) and \( K_n \) be a complete graph with \( n \) vertices. Then

\[
ds(K_n) = \begin{cases} 
n, & \text{for } n \leq 5; \\
\left\lceil \frac{n^2 - n + 4}{6} \right\rceil, & \text{otherwise.}
\end{cases}
\]

Proof. By 1 and Theorem 2.1, \( ds(K_n) \geq ts(K_n) \). Let \( t = \left\lceil \frac{n^2 - n + 4}{6} \right\rceil \). For the reverse inequality, we divide the proof into three cases as follows:

Case 1. \( n \leq 5 \)

It is obvious for \( n \leq 3 \). Now, suppose that \( ds(K_4) = 3 \). We need 10 distinct weight with minimum weight 3. We can label 2 vertices and one edge by label 1. In the other hand, the maximum weight should be 12. Labeling 3 edges and one vertex by label 3 implies that there are 2 edges with the same weight 7. Contrary to hypothesis. Thus, \( ds(K_4) \geq 4 \). To prove the upper bound for \( n = 4 \) or 5, we define \( f \) as in Fig. 1. Therefore, we have the exact value of \( ds(K_n) \) for \( n \leq 5 \).

Case 2. \( n = 77 \) or \( n \geq 80 \)

Consider that under the totally irregular total \( t \)-labeling of \( K_n \) in (2), the maximum edge weight is \( w(c_{s-1}c_s) = \left(\begin{array}{c} n \\ 2 \end{array}\right) + 2 \) and minimum vertex-weight is \( w(a_1) = \frac{s(s^2 - 1)}{6} + n \). It follow \( w(c_{s-1}c_s) < w(a_1) \) implies vertex-weight set and edge weight set are disjoint. Thus, \( ds(K_n) = t \) for \( n = 77 \) or \( n \geq 80 \).

Case 3. \( n \in \{ 60, 63, 64, 66, 67, 69, 70, 72, 73, 75, 76, 78, 79 \} \)

Consider that under the totally irregular total \( t \)-labeling of \( K_n \) provided in (2), we met condition where the minimum vertex-weight \( w(a_1) \) is equal to the weight of an edge connecting vertices in \((B, C)\) or \((C, C)\). Then, we modify \( f \). Let \( E(K_n) = \{ e_i \mid 1 \leq i \leq n(n - 1)/2 \} \). Let \( e_p \in E(K_n) \) be an edge where \( w(a_1) = w(e_p) \). Since \( t \equiv 2 \mod 3 \), then \( f(e_{n(n-1)/2}) = f(c_{s-1}c_s) = t - 1 \). It implies that we can change \( f(e_i) \) by \( f(e_i) + 1 \), for \( p \leq i \leq n(n - 1)/2 \) without changing the maximum label such that \( W(V(K_n)) \cap W(E(K_n)) = \emptyset \). It complete the proof.

\( \square \)

Open Problem

1. For \( n \in \mathbb{N}, n \in \{ i \mid 6 \leq i \leq 59 \} \cup \{ 61, 62, 65, 68, 71, 74 \} \), find the exact value of \( ds(K_n) \).
2. For any graph \( G \), find \( ds(G) \).

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