CONSTRUCTIONS OF KUMMER STRUCTURES ON GENERALIZED KUMMER SURFACES

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Abstract. We study generalized Kummer surfaces $Km_3(A)$, by which we mean the K3 surfaces obtained by desingularization of the quotient of an abelian surface $A$ by an order 3 symplectic automorphism group. Such a surface carries 9 disjoint configurations of two smooth rational curves $C, C'$ with $CC' = 1$. This $9A_2$-configuration plays a role similar to the Nikulin configuration of 16 disjoint smooth rational curves on (classical) Kummer surfaces. We study the (generalized) question of T. Shioda: suppose that $Km_3(A)$ is isomorphic to $Km_3(B)$, does that imply that $A$ and $B$ are isomorphic? We answer by the negative in general, by two methods: by a link between that problem and Fourier–Mukai partners of $A$, and by construction of $9A_2$-configurations on $Km_3(A)$ which cannot be exchanged under the automorphism group.

1. Introduction

A Kummer surface $Km(A)$ is the minimal desingularization of the quotient of an abelian surface $A$ by the standard involution $[-1]$. It is a K3 surface containing 16 disjoint $(-2)$-curves, which lie over the 16 singularities of $A/\langle [-1] \rangle$. Such set of curves is called a Kummer (or $16A_1$) configuration. A well-known result of Nikulin [25] gives the converse: if a K3 surface contains a $16A_1$-configuration, then it is the Kummer surface of an abelian surface $A$, such that the 16 $(-2)$-curves lie over the singularities of $A/\langle [-1] \rangle$.

In 1977 Shioda [33] asked the following question: if two abelian surfaces $A$ and $B$ satisfy $Km(A) \simeq Km(B)$, is it true that $A \simeq B$?

Gritsenko and Hulek [15] gave a negative answer to that question in general. In [29, 30], we studied and constructed examples of two $16A_1$-configurations on the same Kummer surface such that their associated abelian surfaces are not isomorphic.

Kummer surfaces have natural generalizations to quotients of an abelian surface $A$ by other symplectic groups $G \subseteq \text{Aut}(A)$. If $G \simeq \mathbb{Z}/3\mathbb{Z}$, then the quotient surface $A/G$ for the action of $G$ on $A$ has 9 cusp singularities, in bijection with the fixed points of $G$. Its minimal desingularization, denoted by $Km_3(A)$, is a K3 surface which contains what we call a generalized Kummer configuration (or $9A_2$-configuration), which means that the surface contains 9 disjoint $A_2$-configurations, i.e. pairs $(C, C')$ of $(-2)$-curves.
such that $CC' = 1$. Barth [2] proved that if a K3 surface contains a $9\mathbf{A}_2$-configuration, then there exists an abelian surface $A$ and a symplectic order 3 automorphism group such that $X = \text{Km}_3(A)$. It is then natural to ask the \textit{generalized Shioda’s question}: does an isomorphism $\text{Km}_3(A) \simeq \text{Km}_3(B)$ between two generalized Kummer surfaces implies that the abelian surfaces $A$ and $B$ are isomorphic?

A generalized Kummer structure on a K3 surface $X$ is an isomorphism class of pairs $(A, G)$ of abelian surfaces equipped with an order 3 symplectic automorphism subgroup $G \subseteq \text{Aut}(A)$, such that $X \simeq \text{Km}_3(A)$, where $\text{Km}_3(A)$ is the minimal desingularization of $A/G$. Thus Shioda’s question is if there is only one generalized Kummer structure on $X$. In [17], we proved that the re is a one-to-one correspondence between Kummer structures on $X$ and $\text{Aut}(X)$-orbits of $9\mathbf{A}_2$-configurations. In the present paper, we obtain the first explicit examples of generalized Kummer surfaces which possess two distinct generalized Kummer structures. For that aim, we construct two $9\mathbf{A}_2$-configurations $C, C'$ on the Kummer surface, and prove that there is no automorphism sending one configuration to the other. A generalized Kummer surface $X = \text{Km}_3(A)$ has a natural $9\mathbf{A}_2$-configuration

$$C = \{A_1, B_1, \ldots, A_9, B_9\}.$$ 

We suppose that $X$ is generic projective, so that its Picard number is 19. That hypothesis is assumed in all the paper. Let $L$ be the big and nef generator of the orthogonal complement of the curves in the Néron-Severi group. By a result of Barth [2], one has either $L^2 = 6k + 2$ or $L^2 = 6k$, for $k$ an integer. We suppose that $6L^2$ is not a square, so that the two Pell-Fermat equations

$$x^2 - 12(3k + 1)y^2 = 1 \quad \text{and} \quad x^2 - 4ky^2 = 1$$

have non-trivial solutions. Let us denote by $(x_0, y_0)$ the fundamental solution according to these cases and let us define accordingly:

$$\begin{align*}
B_1' &= 3y_0L - \left(\frac{1}{7}(x_0 + 1)A_1 + x_0B_1\right) \quad \text{if} \quad L^2 = 6k + 2, \\
B_1' &= y_0L - \left(\frac{1}{9}(x_0 + 1)A_1 + x_0B_1\right) \quad \text{if} \quad L^2 = 6k.
\end{align*}$$

The class $B_1'$ is a $(-2)$-class in the Néron-Severi group of $X$ (i.e. $B_1'^2 = -2$) such that $B_1'A_1 = B_1A_1 = 1$. Our main result is

**Theorem 1.** Suppose that $L^2 = 2t$ is such that either $L^2 = 2 \mod 6$ or $L^2 \neq 0 \mod 18$ or $3|y_0$. Then $B_1'$ is the class of a $(-2)$-curve such that $B_1'A_1 = 1$ and the 18 $(-2)$-curves

$$C' = \{A_1, B_1', A_2, B_2, \ldots, A_9, B_9\}$$

form a $9\mathbf{A}_2$-configuration.

Suppose moreover that $L^2 = 2 \mod 6$ and $x_0 \neq \pm 1 \mod 2t$, or $L^2 = 6$ or $12 \mod 18$ and $x_0 \neq \pm 1 \mod 2k$. There are no automorphisms sending $C$ to $C'$. For these cases, there are (at least) two generalized Kummer structures on the generalized Kummer surface $X$. 
As pointed out in Example 4.1, the first values of $L^2$ for which our theorem produces new generalized Kummer structures are:

$$20, 44, 68, 84, 92, 104, 110, 116, 120, 126, 132, 140, 164, 168, 176, 188.$$ 

Theorem 22 shows that the hypothesis $x_0 \not\equiv \pm 1 \mod 2k$ is sharp, since if $x_0 = \pm 1 \mod 2k$ and $L^2 \leq 200$, there exists an automorphism sending $C$ to $C'$. We expect that one can remove the somehow technical hypothesis $L^2 \not\equiv 0 \mod 18$ or $3\mid y_0$ of Theorem 1.

We observe that our construction can be applied to other configurations $A_j, B_j$ than $A_1, B_1$. For $L^2 \leq 200$, one can check that repeating twice the construction, we go back (up to automorphisms) to the original Kummer configuration.

The paper is structured as follows: in Section 2, we give a precise description of the Néron-Severi group of a generalized Kummer surface and how a divisor can be written in the $\mathbb{Q}$-basis $L, A_1, B_1, \ldots, A_9, B_9$. In the case that $L^2 = 2 \mod 6$, we also obtain that $\text{Km}_3(A) \simeq \text{Km}_3(B)$ if and only if $A$ and $B$ are Fourier–Mukai partners. In Section 3, according to the cases $L^2 = 0$ or $2 \mod 6$, we construct two generalized Kummer configurations. In Section 4, we give a sufficient condition on which these Kummer configurations give rise to two generalized Kummer structures. In the last section we describe the projective model of the K3 surface determined by $L$ and we recall some known constructions in the literature. We describe more in detail the case $L^2 = 20$, which is the first case for which our Theorem gives two non-equivalent generalized Kummer structures. In particular we obtain a model of $X$ as a double cover of the plane and show that the automorphism group preserving the double cover is isomorphic to $\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes S_3)$.

We aim to study more projective models in a forthcoming paper.

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2. The Néron–Severi lattice and its properties, Fourier–Mukai partners

2.1. Construction of the Néron-Severi lattice of $X$. Let $A$ be an abelian surface with an action of a group $G := \mathbb{Z}/3\mathbb{Z}$ that leaves invariant each element of the space $H^0(A, \Omega^2_A)$ (we call this action symplectic). It is well known that the quotient $A/G$ has $9A_2$ singularities. The minimal resolution denoted by $X := \text{Km}(A, G)$ is a K3 surface, called a generalized Kummer surface, which carries a configuration of rational curves with Dynkin diagram $9A_2$. Observe that the abelian surface $A$ has Picard number at least 3, see [2] Proposition on p. 10 and the K3 surface $X$ has generically Picard number 19. Let $K_3$ denotes the minimal primitive sub-lattice of the K3 lattice, $A_{K3}$, that contains the 9 configurations $A_2$. This is a rank 18 negative definite even lattice of discriminant 3^3, which is described
as follows. Denote by $A_j, B_j, j = 1, \ldots, 9$ the nine couples of $(-2)$-curves generating the nine $\mathbf{A}_2$. Then by [6] Proof of Proposition 1.3] the lattice $K_3$ is generated by the classes $A_1, B_1, \ldots, A_9, B_9$ and the three classes

$$t_1 = \frac{1}{3}(\sum_{i=1}^{9}(A_i - B_i)),$$
$$t_2 = \frac{1}{3}((A_2 - B_2) + 2(A_3 - B_3) + A_6 - B_6 + 2(A_7 - B_7) + A_8 - B_8 + 2(A_9 - B_9)),$$
$$t_3 = \frac{1}{3}((A_4 - B_4) + 2(A_5 - B_5) + A_6 - B_6 + 2(A_7 - B_7) + 2(A_8 - B_8) + A_9 - B_9),$$

with intersection matrix:

$$\begin{pmatrix}
-6 & -6 & -6 \\
-6 & -10 & -6 \\
-6 & -6 & -10
\end{pmatrix}.$$  

The discriminant group $K_3'/K_3$ is generated by the classes

$$w_1 = \frac{1}{3}(A_5 - B_5 + A_7 - B_7 + A_8 - B_8),$$
$$w_2 = \frac{1}{3}(2(A_4 - B_4) + A_6 - B_6 + 2(A_7 - B_7) + A_8 - B_8),$$
$$w_3 = \frac{1}{3}(A_3 - B_3 + A_5 - B_5 + A_6 - B_6),$$

with intersection matrix:

$$\begin{pmatrix}
-2 & -2 & -2 \\
-2 & -\frac{20}{3} & -2 \\
-\frac{2}{3} & -\frac{2}{3} & -2
\end{pmatrix}.$$  

**Theorem 2.** Assume $\rho(\text{Km}(A,G)) = 19$ and let $L$ be a generator of $K_3'/K_3 \subset \text{NS}(\text{Km}(A,G))$ such that $L$ is big and nef. Then $L^2 \equiv 0 \text{ mod } 6$ or $L^2 \equiv 2 \text{ mod } 6$. For an integer $k$, let us denote by $K_{6k}$ (respectively $K_{6k+2}$) the lattice $\mathbb{Z}L \oplus K_3$ when $L^2 = 6k$ (respectively $L^2 = 6k + 2$). Then

1. If $L^2 \equiv 0 \text{ mod } 6$ then

$$\text{NS}(\text{Km}(A,G)) = K_{6k}',$$

where $K_{6k}'$ is generated by $K_{6k}$ and by a class $(L + v_{6k})/3$ where $v_{6k}/3 \in K_3'/K_3$ with $L^2 = -v_{6k}^2$ mod 18, moreover $K_{6k}$ is the unique even lattice, up to isometry, such that $[K_{6k}' : K_{6k}] = 3$ and $K_3$ is a primitive sublattice of $K_{6k}'$, so that we can assume that

(a) If $L^2 \equiv 0 \text{ mod } 18$ then $v_{6k}^2 \equiv 0 \text{ mod } 18$.
(b) If $L^2 \equiv 12 \text{ mod } 18$ then $v_{6k}^2 \equiv -12 \text{ mod } 18$.
(c) If $L^2 \equiv 6 \text{ mod } 18$ then $v_{6k}^2 \equiv -6 \text{ mod } 18$.

2. If $L^2 \equiv 2 \text{ mod } 6$ then

$$\text{NS}(\text{Km}(A,G)) = K_{6k+2}.$$  

**Proof.** The fact that $L^2 \equiv 0 \text{ mod } 6$ or $L^2 \equiv 2 \text{ mod } 6$ follows from [2] Section 2.2]. In the first case if $L^2 \equiv 0 \text{ mod } 6$ then the discriminant group of the lattice $K_{6k}$ contains the generators $w_1, w_2, w_3$ and $L^2/6k$. Recall that for a K3 surface the discriminant group of the Néron-Severi group is the same as the discriminant group of the transcendental lattice, with the quadratic form which changes the sign. Since here the rank of the transcendental lattice is three, the number of independent generators of the discriminant group does
not exceed three, hence a class of the discriminant group of the lattice $\mathcal{K}_{6k}$ is contained in the Néron-Severi group. This class is of the form $\frac{L + v_{6k}}{3}$, where $\frac{v_{6k}}{n}$ belongs to the discriminant group of $\mathcal{K}_3$. By the previous description we necessarily have that $n = 3$. Moreover since the class $\frac{L + v_{6k}}{3}$ is now in the Néron-Severi group, then $(\frac{L + v_{6k}}{3})^2 \in 2\mathbb{Z}$ so that $L^2 + v_{6k}^2 \in 18\mathbb{Z}$ since $L^2 = 6k$ for some integer $k$, thus we get the cases for $L$ and $v_{6k}$ listed in the statement. Moreover if $\frac{L + v_{6k}'}{3}$ is another class contained in the Néron-Severi group as before then $\frac{v_{6k} - v_{6k}'}{3}$ belongs to the Néron-Severi group and so to $\mathcal{K}_3$ but $\frac{v_{6k} - v_{6k}'}{3} \in \mathcal{K}_3'$ so it must be a zero class.

For the unicity statement, we use a similar argument as in [20 Proposition 2.2], which is as follows: First, one computes some generators of the isometry group $O(\mathcal{K}_3)$ of the negative definite lattice $\mathcal{K}_3$. These elements act on the discriminant group $\mathcal{K}_3'/\mathcal{K}_3 \simeq (\mathbb{Z}/3\mathbb{Z})^3$ and one obtains that the image of $O(\mathcal{K}_3)$ in $O(\mathcal{K}_3'/\mathcal{K}_3)$ is a group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times S_4$, which has exactly 4 orbits: $O_z = \{0\}, O_0, O_1, O_2$, where the elements in $O_i$ ($i \in \{0, 1, 2\}$) have square $\frac{2i}{3}$ mod 2. Therefore the isometry class of the gluing $\mathcal{K}_3'$ does not depend on the choice of the element $v_{6k}$ and is unique for each of the respective cases (a), (b) or (c).

In case that $L^2 = 6k + 2$ with $k$ an integer, observe that a class of the form $\frac{L + v_{6k+2}}{3}$ must satisfy $(L \cdot \frac{L + v_{6k+2}}{3}) = \frac{6k + 2}{3} \in \mathbb{Z}$ which is impossible, so this class does not exist and we get that

$$\text{NS}(\text{Km}(A, G)) = \mathcal{K}_{6k+2},$$

which finishes the proof.

Remark 3. The classes $v_{6k}$ can be chosen equal to be $3w_1$ or $3w_3$ if $L^2 \equiv 0 \text{ mod } 18$; equal to $3w_2$ if $L^2 \equiv 6 \text{ mod } 18$ and equal to $3(w_3 - w_2)$ if $L^2 \equiv 12 \text{ mod } 18$.

The following result is due to Barth:

**Theorem 4.** ([2, Section 2.2]). There exists a K3 surface $X$ such that $\text{NS}(X) = \mathcal{K}_{6k}$ (respectively $\text{NS}(X) = \mathcal{K}_{6k+2}$) for an integer $k > 0$ (respectively $k \geq 0$) and such a surface $X$ is a generalized Kummer surface.

Remark 5. i) When the Picard number of $X$ is 19, the discriminant group of $\text{NS}(X)$ determines uniquely $L^2$ and $\text{NS}(X)$.

ii) Theorems 3 and 4 and their proofs (with $L \neq 0$) are also valid when $A$ is a complex non-algebraic torus, equivalently, when $L^2 \leq 0$. If $L = 0$, then $\text{NS}(X) = \mathcal{K}_3$ is negative definite of rank 18, and the K3 surface $X$ is also non-algebraic.

2.2. Classes and polarizations in the Néron-Severi lattice of $X$.

2.2.1. Notations and divisibility of the classes. Let us denote the $(-2)$-curves forming the generalized Kummer configuration by $A_1, B_1, \ldots, A_9, B_9$, where $A_jB_j = 1$ and denote by $L$ the orthogonal complement of these 18 curves
in the rank 19 lattice $\text{NS}(X)$, we recall that $L^2 = 2t$, for $t \in \mathbb{N}^*$. Let $a, \alpha_j, \beta_j \in \frac{1}{3}\mathbb{Z}$ be such that the class

$$
\Gamma = aL - \sum_{j=1}^{9} (\alpha_j A_j + \beta_j B_j),
$$

is in the Néron-Severi lattice $\text{NS}(X)$.

**Corollary 6.** 0) For $j \in \{1, \ldots, 9\}$, one has $\alpha_j \in \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z} \iff \beta_j \in \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$.

1) Suppose $L^2 = 2 \mod 6$. Then $a \in \mathbb{Z}$ and if one coefficient $\alpha_j$ or $\beta_j$ is in $\frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$, then there are 12 or 18 coefficients that are in $\frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$.

2) Suppose $L^2 = 0 \mod 6$, and $a \in \mathbb{Z}$. If one coefficient $\alpha_j$ or $\beta_j$ is in $\frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$, then there are 12 or 18 coefficients that are in $\frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$.

3) Suppose $L^2 = 0 \mod 18$ (respectively $L^2 = 6 \mod 18$ and $L^2 = 12 \mod 18$), and $a \in \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$. Then there are at least 6 (respectively 8 and 10) coefficients $\alpha_j$ or $\beta_j$ that are in $\frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$.

4) The group $L^2 / \langle A_1, \ldots, B_9 \rangle$ (where the orthogonal of $L$ is taken in $\text{NS}(X)$) is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$. The 27 elements of that group are: 24 elements which are supported on 6 $A_2$ blocs, an element $S$ supported on the 9 blocs $A_2$, the element $2S$, and the zero element.

**Proof.** It is a consequence of Theorem 2 and Remark 3. But we may also use the more conceptual result of Barth, that a 3-divisible set of $A_2$-configuration on a K3 surface has support on 6 or 9 $A_2$-configurations. For example, in case 2), since $L^2 = 2 \mod 6$, by Theorem 2 the Néron-Severi lattice is $\text{NS}(X) = \mathbb{Z}L \oplus \mathcal{K}_3$, therefore the coefficient of $L$ is an integer. Let $S = \{r_1, \ldots, r_k\}$ be a set of $k$ elements in $\{1, \ldots, 9\}$ and consider the singular surface $X_S$ obtained by contracting curves $A_j, B_j$ to cusps for $j \in \{r_1, \ldots, r_k\}$. By classical results on cyclic covers (see e.g. [5, Chapter I, 17], see also [3] for the particular case of cusps), there exists an order 3 cyclic cover branched exactly over the cusps of $X_S$ if and only if $\sum_{j \in R} \frac{1}{3}(A_j - B_j) \in \text{NS}(X)$. In [3, Lemma 1], Barth obtains that when such a cover exists, one has necessarily $k = 6$ or 9 (this is the analogous result of a well-known result of Nikulin for the classical Kummer surfaces). That implies that there are 12 or 18 coefficients of $\Gamma$ that are in $\frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$. \hfill $\square$

Let $\Gamma \in \text{NS}(X)$ be the class of a divisor and let us write

$$
\Gamma = aL - \frac{1}{3} \sum_{j=1}^{9} (a_j A_j + b_j B_j),
$$

with $a \in \frac{1}{3}\mathbb{Z}$, $a_j, b_j \in \mathbb{Z}$. The intersection numbers $\Gamma A_j = \frac{1}{3}(2a_j - b_j)$ and $\Gamma B_j = \frac{1}{3}(2b_j - a_j)$ are integers. Since $2a_j - b_j$ and $2b_j - a_j$ are divisible by 3, there exist integers $u_j, v_j$ such that

$$
\begin{align*}
& a_j = u_j + 2v_j \\
& b_j = 2u_j + v_j,
\end{align*}
$$

with $u_j, v_j \in \mathbb{Z}$. By Remark 2, the coefficients of $\Gamma$ are respectively divisible by 18, 12, and 6, according to the parity of $a_j$. By the argument of Section 7.3, we conclude that the 27 elements of $\text{NS}(X)$ with $\frac{1}{3}\mathbb{Z}$-coefficients are supported on 6 $A_2$-blocs, an element $S$ supported on the 9 blocs $A_2$, the element $2S$, and the zero element.
so that we can write
\[ \Gamma = aL - \frac{1}{3} \sum_{j=1}^{9} ((u_j + 2v_j)A_j + (2u_j + v_j)B_j), \]
with \( u_j, v_j \in \mathbb{Z} \), which is also
\[ \Gamma = aL - \frac{1}{3} \sum_{j=1}^{9} (u_j F_j + v_j G_j) \]
for
\[ F_j = A_j + 2B_j, \quad G_j = 2A_j + B_j. \]
We have \( F_j^2 = G_j^2 = -6 \), \( F_j G_j = -3 \), so that
\[ \Gamma^2 = 2ta^2 - \frac{2}{3} \sum_{j=1}^{9} \left( u_j^2 + u_j v_j + v_j^2 \right). \]

Let us suppose moreover that \( \Gamma \) is the class of an irreducible curve which is not among the 18 curves \( A_1, \ldots, B_9 \). Then \( a \in \frac{1}{3} \mathbb{N}^* \) and the intersection numbers \( \Gamma A_j, \Gamma B_j \) are positive or zero:
\[ \begin{cases} 2a_j \geq b_j, \\ 2b_j \geq a_j, \end{cases} \]
which inequalities are equivalent to
\[ (2.1) \quad u_j \geq 0, \text{ and } v_j \geq 0. \]

2.2.2. Polarizations. Let be \( u \in \mathbb{Z} \) and define
\[ D = uL - \sum_{j=1}^{9} (A_j + B_j). \]
The 18 curves \( A_1, B_1, \ldots, A_9, B_9 \) have degree 1 for \( D \): \( DA_k = DB_k = 1 \). With the same notations, we show:

**Proposition 7.** The minimal integer \( u_0 \) such that for \( u \geq u_0 \) the divisor \( D \) is ample is given in the following table (according to the cases of \( L^2 \)):

| \( L^2 \) | \( L^2 \geq 20 \) |
|----------|------------------|
| \( u_0 \) | \( u_0 \) | \( u_0 \) |
| \( L^2 = 2 \text{ mod } 6 \) | \( L^2 = 2 \) | \( L^2 = 8 \) or 14 | \( L^2 \geq 30 \) |
| 4 | 2 | 1 |
| \( L^2 = 0 \text{ mod } 18 \) | \( L^2 = 18 \) | \( L^2 \geq 36 \) |
| 2 | 1 |
| \( L^2 = 6 \text{ mod } 18 \) | \( L^2 = 6 \) | \( L^2 \geq 24 \) |
| 3 | 1 |
| \( L^2 = 12 \text{ mod } 18 \) | \( L^2 = 12 \) | \( L^2 \geq 30 \) |
| 2 | 1 |
Proof. We start by defining $D = u_0 L - \sum_{j=1}^{9} (A_j + B_j)$ with $u_0$ as defined in the Table. We check that $D^2 > 0$ and $D^\perp$ contains no vector with square $-2$ (for example when the coefficients on the diagonal of a Gram matrix obtained from a base of $D^\perp$ are multiples of $-4$). Using that the ample cone is a fundamental domain for the Weyl group (the reflection group generated by reflection by vectors of square $-2$), we can choose $D$ as an ample class. We have $DA_j = 1$; if $A_j = C + C'$ with effective divisors $C, C'$, then $DC$ or $DC'$ is 0 thus, since $D$ is ample, $C$ or $C'$ is 0, which proves that $A_j$ is a $(-2)$-curve. Using that fact, one easily check that $L$ is nef. Then for $u \geq u_0$, the divisor $(u - u_0)L + D$ is also ample, which proves the result. □

2.3. Fourier–Mukai partners and generalized Kummer structures. Recall that for a K3 surface (resp. an abelian surface) $X$, a Fourier–Mukai partner of $X$ is a K3 surface (resp. an abelian surface) $Y$ such that there is an isomorphism of Hodge structures $(T(Y), C\omega_Y) \simeq (T(X), C\omega_X)$, where $\omega_X$ is a generator of $H^0(X, \Omega^2_X)$, and $T(X)$ is the transcendental lattice. The set of isomorphism classes of Fourier–Mukai partners of $X$ is denoted by $\text{FM}(X)$.

Let $A$ be an abelian surface and $G_A$ be an order 3 automorphism group of $A$ acting symplectically. Let $X = \text{Km}_3(A)$ (or sometimes $\text{Km}_3(A, G_A)$ or $\text{Km}_3(A, J_A)$, where $\langle J_A \rangle = G_A$) be the minimal resolution of the quotient surface $A/G_A$; since $G_A$ is symplectic, $X$ is a generalized Kummer surface.

An isomorphism class of pairs $(B, G_B)$ where $B$ is an abelian surface and $G_B$ is an order 3 symplectic automorphism group such that $\text{Km}_3(B) \simeq X$ is called a generalized Kummer structure on $X$. Let us denote by $\mathcal{K}(X)$ the set of these isomorphism classes.

In order to state our results on a link between $\mathcal{K}(X)$ and Fourier–Mukai partners of $A$, let us recall some results and notations in Barth’s article [2]. According to [2, Section 1.3], the $G_A$-invariant part of $H_2(A, \mathbb{Z})$ is a rank 4 lattice $L_A$, for which there is a basis $g_1, \ldots, g_4$ (in the notions of [2], these are $g_1 = \gamma_1, g_2 = \gamma_2, g_3 = \gamma_3, g_4 = \gamma_3 + \gamma_4$) with Gram matrix

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 2 & 3 \\
0 & 0 & 3 & 6
\end{pmatrix}.
$$

For $X = \text{Km}_3(A)$, the lattice $L_X \subset H^2(X, \mathbb{Z})$ which is orthogonal to the 18 $(-2)$-curves in $X$ has rank 4; it is generated by elements $\zeta_1, \ldots, \zeta_4$ with intersection matrix

$$
\begin{pmatrix}
0 & 3 & 0 & 0 \\
3 & 0 & 0 & 0 \\
0 & 0 & 6 & 3 \\
0 & 0 & 3 & 2
\end{pmatrix}.
$$
The canonical rational map
\[ \pi_A : A \to \text{Km}_3(A) = X \]
induces the morphisms
\[ \pi_{A*} : L_A \to L_X, \quad \pi_A^* : L_X \to L_A \]
which are such that
\[ \pi_{A*}(g_i) = \zeta_i \text{ for } i \leq 3, \quad \pi_A^*(g_4) = 3\zeta_4 \]
and
\[ \pi_A^*(\zeta_i) = 3g_i \text{ for } i \leq 3, \quad \pi_A^*(\zeta_4) = g_4, \]
so that \( \pi_{A*}\pi_A^* : L_X \to L_X \) and \( \pi_A\pi_{A*} : L_A \to L_A \) are the multiplication by 3 maps. The lattice \( \pi_A(L_A) \) has index 3 in \( L_X \), and it is easy to check that \( \pi_A(L_A) \) is isometric to \( L_A(3) \). Here, for a lattice \( L := (L, (, )) \) and a non-zero integer \( m \), we define the lattice \( L(m) := (L, m(, )) \).

The Picard number of \( A \) is either 3 or 4; we suppose that we are in the generic case so that the Picard number is 3. Then the \( G_A \)-invariant part of \( \text{NS}(A) \) is generated by a divisor \( L_A \) and the transcendental lattice of \( A \) is \( T(A) = L_A^1 \subset L_A \) (see [2]). The orthogonal complement in \( \text{NS}(X) \) of the 18 (-2)-curves on \( X \) (which are the components of the exceptional loci of the resolution \( X \to A/G_A \)) is generated by a divisor \( L_X = \sum_{j=1}^4 n_j\zeta_j \in L_X \) (for some coprime integers \( n_j \in \mathbb{Z} \)) which is such that \( L_X^2 = 0 \) or \( 2 \) mod 6, moreover:

- if \( L_X^2 = 2 \) mod 6, then \( \gcd(n_4, 3) = 1 \) and \( L_A = \pi_A^*(L_X) = 3n_1g_1 + 3n_2g_2 + 3n_3g_3 + n_4g_4 \)
- if \( L_X^2 = 0 \) mod 6, then \( n_4 = 3n'_4 \) for some \( n'_4 \in \mathbb{Z} \), \( \gcd(n_1, n_2, n_3, 3) = 1 \) and \( L_A = \frac{1}{3}\pi_A^*(L_X) = n_1g_1 + n_2g_2 + n_3g_3 + n'_4g_4 \).

The orthogonal complement of \( L_X \) and the 18 (-2)-curves on \( X \) is the transcendental lattice \( T(X) \) of \( X = \text{Km}_3(A) \); it is contained in \( L_X \). One has \( \pi_A^*L_X = \nu L_A \) with \( \nu \in \{1, 3\} \).

**Theorem 8.** Let \( (A, G_A) \) and \( (B, G_B) \) be two abelian surfaces with an order 3 symplectic automorphism group and let \( X = \text{Km}_3(A) \). Suppose that \( X \) has Picard number 19 and \( L_X^2 = 2 \) mod 6. We have \( \text{Km}_3(B) \simeq \text{Km}_3(A) = X \) (i.e. \( \{(B, G_B)\} \in K(X) \)) if and only if \( B \) is a Fourier–Mukai partner of \( A \).

The proof is similar to [13 Theorem 0.1] for the classical Kummer surfaces. Let us prove the following result

**Lemma 9.** Suppose that \( L_X^2 = 2 \) mod 6. The canonical rational map
\[ \pi_A : A \to X = \text{Km}_3(A) \]
induces a Hodge isometry
\[ \pi_{A*} : (T(A)(3), C\omega_A) \to (T_{\text{Km}_3(A)}, C\omega_{\text{Km}_3(A)}). \]
For the proof of Lemma \[9\] we will use the following elementary result:

**Lemma 10.** Let $N$ be a lattice and let $N' \subset N$ be a sub-lattice of finite index $k = [N : N']$, let $M \subset N$ be a sub-lattice and let $M' = M \cap N'$. The index $[M : M']$ divides $[N : N']$.

*Proof.* The map $\phi : m + M' \in M/M' \to m + N' \in N/N'$ is a well defined morphism. For $m \in M$, one has $\phi(m + M') = N' \in N/N'$ if and only if $m \in N'$ thus if and only if $m \in N' \cap M = M'$: the kernel of $\phi$ is trivial. Thus the morphism $\phi$ is injective and identifying $M/M'$ with its image, by Lagrange Theorem the order of $M/M'$ divides the order of $N/N'$. \[\square\]

*Proof.* (Of Lemma \[9\].) Since $\pi_{A^*}$ is an isometry between $L_A(3)$ and $\pi_{A^*}(L_A)$, it restricts to an isometry from $T(A)(3)$ to $\pi_{A^*}(T(A))$ (let us remark that one may also prove that $\pi_{A^*}(T(A))$ is isometric to $T(A)(3)$ by using \[16\] Proposition 1.1 & Remark of Inose).

Define $N = L_X$, $N' = \pi_{A^*}(L_A)$, $v = \pi_{A^*}(L_A)$, $M = v^\perp N = L_X^\perp (= T(X))$ and $M' = \pi_{A^*}(L_A^\perp) = \pi_{A^*}(T(A)) = M \cap N'$.

Since $[N : N'] = 3$, by Lemma \[10\] we get that $[M : M'] \in \{1, 3\}$.

Suppose that $L_X^\perp = 2 \mod 6$ and write

$$L_A = \pi_A^*(L_X) = 3n_1g_1 + 3n_2g_2 + 3n_3g_3 + n_4g_4,$$

(we recall that $L_X = n_1\zeta_1 + n_2\zeta_2 + n_3\zeta_3 + n_4\zeta_4$ with coprime integers $n_1, \ldots, n_4$). Then $v = 3w$ for $w = L_X$. Again by Lemma \[10\] the lattice

$$M' \oplus Zv = (M \oplus Zv) \cap N'$$

has index 1 or 3 in $M \oplus Zv$. Since

$$[M \oplus Zv : M' \oplus Zv] = [M : M'][Zv : Zv] = 3[M : M'],$$

that forces $[M : M'] = 1$, which implies that

$$\pi_{A^*}(T(A)) = \pi_{A^*}(L_A^\perp) = L_X^\perp = T(X),$$

and therefore $T(X)$ is isometric to $T(A)(3)$, since $\pi_{A^*}(T(A)) \simeq T(A)(3)$. \[\square\]

**Remark 11.** For any $k \in \mathbb{Z}$, the polarization $L_X = \zeta_1 + k\zeta_2$ is such that $L_X^2 = 6k$ (as remarked by Barth in \[2\]). Using the Gram matrix of the $\zeta_k$’s, one finds that the transcendental lattice of $X$ is generated by elements $\zeta_1 - k\zeta_2, \zeta_3, \zeta_4$, whereas $T(A)$ is generated by $g_1 - kg_2, g_3, g_4$. Thus, for these examples one obtains that $\pi_{A^*}(T(A))$ (generated by $\zeta_1 - k\zeta_2, \zeta_3, 3\zeta_4$) has index 3 in $T(X)$. In fact, using Lemma \[10\] one can prove more generally that for any polarization $L_X$ such that $L_X^2 = 6k$, the lattice $\pi_{A^*}(T(A))$ has index 3 in $T(X)$.

Let us prove Theorem \[8\]
Proof. Suppose that $L_X^2 = 2 \mod 6$. From Lemma \[9\] there exists an isomorphism of Hodge structures

$$(T(B), C\omega_B) \cong (T(A), C\omega_A)$$

if and only if there is an isomorphism of Hodge structures

$$(T(Km_3(B)), C\omega_{Km_3(B)}) \cong (T(Km_3(A)), C\omega_{Km_3(A)}).$$

Therefore $B$ is a Fourier–Mukai partner of $A$ if and only if $Km_3(B)$ is a Fourier–Mukai partner of $Km_3(A)$. By \[14\] Corollary 2.6, since $X = Km_3(A)$ has Picard number $19 > 2 + \ell$ (here $\ell$ is the length of the discriminant group of $NS(X)$, which is also the length of $T(X)$, and therefore is $\leq 3$), one has $\{Km_3(A)\} = FM(Km_3(A))$ (the isomorphism class of $Km_3(A)$ is the unique Fourier–Mukai partner of $Km_3(A)$), therefore $B$ is a Fourier–Mukai partner of $A$ if and only if $Km_3(B) \simeq Km_3(A) = X$. In other words: $B$ is a Fourier–Mukai partner of $A$ if and only if $Km_3(B) \in K(X)$. \qed

By \[9\] Proposition 5.3, the number $|FM(A)|$ of Fourier–Mukai partners of $A$ is finite. So, in case $L_X^2 = 2 \mod 6$, to find the number of generalized Kummer structures on $X$ reduces to find the number of conjugacy classes of order 3 symplectic groups $G'$ on a finite number of abelian surfaces $B$ such that $Km_3(B, G') \simeq Km_3(A)$.

3. CONSTRUCTIONS OF KUMMER STRUCTURES

3.1. $(−2)$-classes with intersection one with $A_1$. Let $A_1, B_1, \ldots, A_9, B_9$ be a generalized Nikulin configuration on a generalized Kummer surface $X$. Let $L$ be the big and nef generator of the orthogonal complement of these 18 curves and let $t \in \mathbb{N}$ such that $L^2 = 2t$. According to the two possible cases, $L^2 = 2 \mod 6$ and $L^2 = 0 \mod 6$, we denote by $k \in \mathbb{N}$ the integer such that $t = 1 + 3k$ in the first case, and such that $t = 3k$ in the second case.

Our aim is to construct a $(−2)$-curve $B'_1 \neq B_1$ in the lattice $L$ generated by $L, A_1, B_1$ such that $A_1 B'_1 = 1$, so that $(A_1, B'_1)$ is another $A_2$-configuration. As we will see in the next Section, under some conditions on $t$, such a curve exists and is unique. In order to prove this result, we will study the properties of $L'$, the orthogonal complement of $A_1, B'_1$ in $L$.

By Theorem \[2\] an element in the lattice generated by $L, A_1, B_1$ has the form

$$B'_1 = aL - (a_1 A_1 + b_1 B_1),$$

for integers $a, a_1, b_1$. The class $B'_1$ satisfies $B'_1 A_1 = 1$ if and only if $2a_1 - b_1 = 1$, which gives

$$B'_1 = aL - (a_1 A_1 + (2a_1 - 1)B_1).$$

Moreover since we search for a $(−2)$-curve, we have

$$B'^2_1 = -2 = 2ta^2 - 6a^2_1 + 6a_1 - 2,$$

which is equivalent to

$$3a_1(a_1 - 1) - ta^2 = 0.$$
We can write that condition as
\[ 3 \left( (2a_1 - 1)^2 - 1 \right) - 4ta^2 = 0, \]
which is equivalent to
\[ (3.1) \quad 3(2a_1 - 1)^2 - 4ta^2 = 3. \]

3.1.1. Case \( t = 1 \mod 3 \).

Since we suppose that \( t = 1 \mod 3 \), the integer \( a \) in Equation (3.1) must be divisible by 3. Let us define the integers \( x_0, y_0 \) by
\[ x_0 = 2a_1 - 1, \quad a = 3y_0. \]

Equation (3.1) is then equivalent to the Pell-Fermat equation
\[ (3.2) \quad x_0^2 - 12ty_0^2 = 1. \]

Since \( t = 1 \mod 3 \), the integer \( 12t \) is never a square and there exist non-trivial solutions. Let us fix such a solution \((x_0, y_0)\) (we observe that \( x_0 \) is necessarily odd). Then
\[ B'_1 = 3y_0L - (a_1A_1 + (2a_1 - 1)B_1) \]
with \( 2a_1 - 1 = x_0 \) is such that \( B'_1^2 = -2 \) and \( B'_1A_1 = 1 \), and conversely, all (-2)-classes with these two properties are obtained in that way.

Let us search for the class of \( L' = \alpha L - (\beta_1A_1 + \beta'_1B_1) \), \( \alpha, \beta_1, \beta'_1 \in \mathbb{Z} \), such that \( L' \) generates the orthogonal complement of \( A_1, B'_1, A_2, B_2, \ldots, A_9, B_9 \).

From \( L'A_1 = 0 \), we obtain
\[ \beta'_1 = 2\beta_1. \]

Since also \( L'B'_1 = 0 \), we get
\[ 2\alpha t = 3\beta_1(2a_1 - 1), \]
in other words:
\[ 2\alpha ty_0 = \beta_1 x_0. \]

Since by equation (3.2) the integers \( x_0, y_0 \) are co-primes and \( x_0 \) is co-prime to \( 2t \), we get:
\[ \beta_1 = 2ty_0, \quad \alpha = x_0, \]
and the class \( L' = x_0L - 2ty_0(A_1 + 2B_1) \) is primitive with \( L'L > 0 \), and such that
\[ L'^2 = 2tx_0^2 - 24t^2y_0^2 = 2t. \]

For any solution \((x_0, y_0)\) of the Pell-Fermat equation (3.2), the classes
\[ B'_1 = 3y_0L + (\frac{1}{2}(x_0 + 1)A_1 + x_0B_1) \],
\[ L' = x_0L - 2ty_0(A_1 + 2B_1) \]
in \( L \) have the properties \( B'_1^2 = -2 \), \( B'_1A_1 = 1 \), \( L'A_1 = L'B'_1 = 0 \), \( L'^2 = 2t \), \( L'L > 0 \) and all the classes \( B, L_0 \) with these properties are obtained in that way.
3.1.2. Case \( t = 0 \mod 3 \). Let us suppose that \( t = 0 \mod 3 \), and let \( k \in \mathbb{N}^* \) such that \( t = 3k \). Then the equation
\[
3(2a_1 - 1)^2 - 4ta^2 = 3
\]
is equivalent to
\[
(2a_1 - 1)^2 - 4ka^2 = 1.
\]
By defining \( x_0 = 2a_1 - 1, \ y_0 = a \), we are reduced to the Pell-Fermat equation \((3.2)\):
\[
x_0^2 - 4ky_0^2 = 1,
\]
\( (x_0 \) is necessarily odd). Let us fix such a solution \((x_0, y_0) \). Then
\[
B'_1 = y_0L - (a_1A_1 + (2a_1 - 1)B_1),
\]
with \( 2a_1 - 1 = x_0 \) satisfies \( B'_1 = -2, \ B'_1A_1 = 1 \).

Let us search for the class of \( L' = \alpha L - (\beta_1A_1 + \beta'_1B_1) \), \( \alpha, \beta_1, \beta'_1 \in \mathbb{Z} \), such that \( L' \) generates the orthogonal complement of \( A_1, B'_1, A_2, B_2, \ldots, A_9, B_9 \). From \( L'A_1 = 0 \), we obtain
\[
\beta'_1 = 2\beta_1,
\]
and since \( L'B'_1 = 0 \), we get
\[
2a_1t = 3\beta_1(2a_1 - 1),
\]
in other words:
\[
2\alpha ky_0 = \beta_1 x_0.
\]
Since \( x_0, y_0 \) are co-primes, and \( L' \) is primitive, we get:
\[
\beta_1 = 2k y_0, \ \alpha = x_0
\]
and \( L' = x_0L - 2k y_0(A_1 + 2B_1) \), thus
\[
L'^2 = 2tx_0^2 - 24k^2y_0^2 = 2t(x_0^2 - 4ky_0^2) = 2t.
\]
For any solution \((x_0, y_0) \) of the Pell-Fermat equation \((3.3)\), the classes
\[
B'_1 = y_0L + \left(\frac{1}{2}(x_0 + 1)A_1 + x_0B_1\right),
\]
\[
L' = x_0L - 2k y_0(A_1 + 2B_1)
\]
have the properties we required: \( B'_1 = -2, \ B'_1A_1 = 1, \ L'A_1 = L'B'_1 = 0 \) and all classes with these properties are obtained in that way.

3.2. Existence and unicity of \( B'_1 \).

3.2.1. Case \( t = 1 \mod 3 \). Let \((x_0, y_0) \) be a non-trivial solution of the Pell-Fermat equation \((3.2)\). In Section 3.1.1, we defined the classes
\[
L' = x_0L - 2ty_0(A_1 + 2B_1),
\]
\[
B'_1 = 3y_0L - \left(\frac{1}{2}(x_0 + 1)A_1 + x_0B_1\right).
\]
Let us prove the following result:

**Proposition 12.** Suppose that \((x_0, y_0) \) is the fundamental solution of the Pell-Fermat equation \((3.2)\). The class \( L' \) is nef. Suppose that \( L'\Gamma = 0 \) for the class \( \Gamma \) of an irreducible \((-2)\)-curve. Then \( \Gamma \) is in \( \{A_1, B'_1, A_2, \ldots, A_9, B_9\} \).
Proof. Let $\Gamma$ be a $(-2)$-curve, we write it as (see Section 2.2.1)
\[
\Gamma = aL - \frac{1}{3} \sum_{j=1}^{9} ((u_j + 2v_j)A_j + (2u_j + v_j)B_j),
\]
with $u_j \geq 0$, $v_j \geq 0$ (we can apply inequality (2.1) since we suppose that $\Gamma$ is irreducible) and $u_j, v_j \in \mathbb{Z}$, $a \in \mathbb{Z}$, so that
\[
(3.4) \quad \Gamma^2 = 2ta^2 - \frac{2}{3} \sum_{j=1}^{9} (u_j^2 + u_jv_j + v_j^2) = -2.
\]
Suppose that $\Gamma L' \leq 0$. This is equivalent to:
\[
\left( aL - \frac{1}{3} ((u_1 + 2v_1)A_1 + (2u_1 + v_1)B_1) \right) (x_0L - 2ty_0(A_1 + 2B_1)) \leq 0
\]
which is equivalent to
\[
ax_0 \leq (2u_1 + v_1)y_0.
\]
By taking the square (recall that $x_0 > 0, y_0 > 0$ and $a \geq 0$), we get
\[
x_0^2a^2 \leq (4(u_1^2 + u_1v_1 + v_1^2) - 3v_1^2)y_0^2,
\]
which is equivalent to
\[
-4(u_1^2 + u_1v_1 + v_1^2)y_0^2 \leq -x_0^2a^2 - 3v_1^2y_0^2,
\]
thus
\[
-\frac{2}{3}(u_1^2 + u_1v_1 + v_1^2) \leq -\frac{1}{6y_0^2}(x_0^2a^2 + 3v_1^2y_0^2)
\]
and using equality $\Gamma^2 = -2$ in (3.4), we get
\[
-2 \leq 2ta^2 - \frac{1}{6y_0^2}(x_0^2a^2 + 3v_1^2y_0^2) - \frac{2}{3}S,
\]
here and hereafter, we denote
\[
S = \sum_{j=2}^{9} (u_j^2 + u_jv_j + v_j^2) \geq 0.
\]
Thus we obtain
\[
\frac{1}{2}v_1^2 + \frac{1}{6}a^2\left(\frac{x_0^2}{y_0^2} - 12t\right) + \frac{2}{3}S \leq 2,
\]
which is equivalent to
\[
(3.5) \quad \frac{1}{2}v_1^2 + \frac{1}{6}a^2\left(\frac{x_0^2}{y_0^2} - 2\right) + \frac{2}{3}S \leq 2.
\]
If $a = 0$, then $\Gamma = A_1$ or $A_j, B_j$ with $j \geq 2$. We therefore suppose that $a \neq 0$, and then $a > 0$ since $\Gamma$ is effective. Let us suppose that one of the coefficients $a_j = \frac{1}{3}(u_j + 2v_j)$, $\beta_j = \frac{1}{3}(2u_j + v_j)$ of $\Gamma$ is in $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Then by Corollary 6 at least 12 of the coefficients $\alpha_j, \beta_j$ are non-zero, and therefore at least 5 of the $u_j$ or $v_j$ (which are $\geq 0$) with the condition $j \geq 2$ are non-zero. That implies that $S \geq 5$, thus $\frac{2}{3}S \geq \frac{10}{3} > 2$, which is a contradiction.
So all the coefficients of $\Gamma$ are integers. Suppose that $S > 0$. Since $S < 3$, there exist one or two indices $j \geq 2$ such that $u_j$ or $v_j$ is equal to 1 (and the other coefficients with index $k \geq 2$ are 0). But then the coefficients of $\Gamma$ are not integral: the only possibility is $S = 0$. Since $a \neq 0$, we have $v_1 \in \{0, 1\}$. Since $\Gamma^2 = -2$, the integers $a, u_1, v_1, t$ satisfies

$$2ta^2 - \frac{2}{3}(u_1^2 + u_1v_1 + v_1^2) = -2,$$

which is equivalent to

$$u_1^2 + u_1v_1 + v_1^2 = 3(ta^2 + 1).$$

If $v_1 = 0$, since $t = 1 \mod 3$, $ta^2 + 1 = 1$ or $2 \mod 3$. But then $3(ta^2 + 1)$ is not a square, and there is no such a solution. Therefore $v_1 = 1$, and from inequality (3.5), the integer $a$ is in the range

$$1 \leq a \leq 3y_0.$$

The equality

$$u_1^2 + u_1 + 1 = 3(ta^2 + 1),$$

is equivalent to

$$(2u_1 + 1)^2 + 3 = 12(ta^2 + 1),$$

thus to

$$(2u_1 + 1)^2 = 3(4ta^2 + 3).$$

Then $2u_1 + 1$ must be divisible by 3: let $w$ be such that $3w = 2u_1 + 1$. The above equation is equivalent to

$$3w^2 = 4ta^2 + 3,$$

which in turn, since $t = 1 \mod 3$, implies that there exists an integer $A$ such that $a = 3A$, then the equation is equivalent to the Pell-Fermat equation

$$w^2 - 12tA^2 = 1.$$

Let $w_0, A_0$ be a solution of that Pell-Fermat equation, then $a = 3A_0, u_1 = \frac{1}{2}(3w_0 - 1), v_1 = 1$ are such that $\Gamma^2 = -2$. Since $a \leq 3y_0$, and we now use that $(x_0, y_0)$ is the fundamental solution of the Pell-Fermat equation to conclude that $a = 3y_0$ and $u_1 = \frac{1}{2}(3x_0 - 1)$, for which integers one has $\Gamma = B_1'$, and $L'B_1' = 0$. That concludes the proof. □

We obtain:

**Corollary 13.** The divisor $B_1' = 3y_0L - (\frac{1}{2}(x_0 + 1)A_1 + x_0B_1)$ is the class of a $(-2)$-curve.

The curves $B_1$ and $B_1'$ are the unique $(-2)$-curves in the lattice generated by $L, A_1, B_1$ which have intersection 1 with $A_1$.

**Proof.** A suitable multiple of the big and nef divisor $L'$ defines a map which is generically one to one onto a projective model. It contracts the irreducible curves subjacent to $B_1'$ and the curves $A_1, A_2, B_2, \ldots, A_9, B_9$ to ADE singularities. By the genericity assumption on the K3 surface $X$, the surface has
Picard number 19, which forces the image of $A_1 + B'_1$ to be a $A_2$-singularity, therefore the curve $B'_1$ is irreducible.

Let us prove the unicity claim. By Section 3.1 if $\tilde{B}_1$ is the class of a $(-2)$-curve such that $\tilde{B}_1A_1 = 1$, there exists a solution $(x, y)$ of the Pell-Fermat equation (3.3) such that

$$\tilde{B}_1 = 3yL - \left(\frac{1}{2}(x + 1)A_1 +xB_1\right).$$

Suppose $\tilde{B}_1 \neq B_1$ ie $(x, y) \neq (-1, 0)$. Then since $\tilde{B}_1$ is effective, one has $x > 0, y > 0$ and there exists $k \in \mathbb{N}^*$ such that $x = x_k, y = y_k$ for

$$x_k + \sqrt{12ty_k} = (x_0 + \sqrt{12ty_0})^k = (x_0 + \sqrt{12ty_0})(x_{k-1} + \sqrt{12ty_{k-1}}),$$

where $(x_0, y_0)$ is the fundamental solution of the Pell-Fermat equation (3.3) (see e.g. [1], Section 4.2). One has

$$\tilde{B}_1B'_1 = (3yL - \left(\frac{1}{2}(x + 1)A_1 +xB_1\right))(3y_0L -(\frac{1}{2}(x_0 + 1)A_1 +x_0B_1)) = 18y_0t - \frac{3}{2}xx_0 - \frac{1}{2},$$

therefore $\tilde{B}_1B'_1 < 0$ if and only if $xx_0 + \frac{1}{2} > 12y_0$. Using an induction on $k$, one can check that this is the case for all $k \geq 1$. Thus if $k > 1$, the $(-2)$-class $\tilde{B}_1$ cannot be the class of an irreducible curve. We observe moreover that if $k = 1$, then $\tilde{B}_1 = B'_1$, and that concludes the proof. \qed

3.2.2. Case $t = 0 \mod 3$. Suppose that $L^2 = 2t = 6k$. Let $(x_0, y_0)$ be the fundamental solution of the Pell-Fermat equation (3.3). Let us search when the $(-2)$-class

$$B'_1 = y_0L - \left(\frac{1}{2}(x_0 + 1)A_1 +x_0B_1\right),$$

is the class of a $(-2)$-curve. Recall that

$$L' = x_0L - 2ky_0(A_1 +2B_1),$$

generates in $\text{NS}(X)$ the orthogonal complement of $A_1, B'_1, A_2, B_2, \ldots, A_9, B_9$.

**Proposition 14.** a) Suppose that $L^2 = 6$ or $12 \mod 18$ or $L^2 = 0 \mod 18$ and $3 \mid y_0$. Then the class $L'$ is nef. Moreover suppose that $\Gamma$ is the class of an irreducible $(-2)$-curve such that $L'T = 0$, then $\Gamma \in \{A_1, B'_1, A_2, \ldots, A_9, B_9\}$. b) Suppose that $L^2 = 0 \mod 18$ and $3 \not| y_0$. Up to exchanging the curves $A_1$ and $B_1$, the same result holds true.

**Remark 15.** i) If $\Gamma$ is in $\{A_1, B'_1, A_2, \ldots, A_9, B_9\}$, one has $L'T = 0$. The classes $D = A_j + B_j$ (for $j \geq 2$) or $D = A_1 + B'_1$ are also a $(-2)$-classes such that $L'D = 0$.

ii) The class $B'_1$ is constructed such that $A_1B'_1 = 1$. Using the same solutions of the Pell-Fermat equation, we can also construct a class $A'_1$ such that $A'_1B_1 = 1$, this is what we mean in b), when we say that the result holds true up to exchanging the role of the curves $A_1$ and $B_1$. 

Proof (of Proposition 12). As we will see the proof of Proposition 12 is easy for the cases \( L^2 = 6, 12 \mod 18 \), the main difficulty is for \( L^2 = 0 \mod 18 \).

By definition, \( L \) is nef if and only if for any \((-2)\)-curve \( \Gamma \), one has \( \Gamma L \geq 0 \). Let therefore

\[
\Gamma = aL - \frac{1}{3} \sum_{j=1}^{9} ((u_j + 2v_j)A_j + (2u_j + v_j)B_j),
\]

be an effective \((-2)\)-class and let us suppose moreover that this is the class of an irreducible \((-2)\)-curve not in \( \{A_1, \ldots, B_9\} \). Then by inequality \((2.1)\), necessarily one has \( u_j \geq 0, v_j \geq 0, u_j, v_j \in \mathbb{Z} \), and \( a > 0 \), in \( \frac{1}{3}\mathbb{Z} \) (since \( t = 0 \mod 3 \)). As in Section 3.2.1, let us study if \( L' \) is nef. After computations similar to those of Section 3.2.1, we obtain that \( \Gamma L' \leq 0 \) if and only if

\[
\frac{1}{2}v_1^2 + \frac{3a^2}{2y_0} + \frac{2}{3}S \leq 2,
\]

where \( S = \sum_{j=2}^{9} \left( u_j^2 + u_jv_j + v_j^2 \right) \in \mathbb{N} \). We observe that \( S < 3 \) since \( a \neq 0 \).

**Case 1: one of the coefficients of \( \Gamma \) is in \( \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z} \).** Suppose that one of the coefficients of \( \Gamma \) is in \( \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z} \). By Corollary 6 there are at least 6, 8 or 10 coefficients \( \alpha_j = \frac{1}{3}(u_j + 2v_j) \), \( \beta_j = \frac{1}{3}(2u_j + v_j) \) that are in \( \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z} \) according if \( L^2 = 0, 6 \) or 12 mod 18. Thus respectively, at least 3, 4 or 5 integers \( u_j, v_j \) are non-zero, and therefore the sum \( S \) (which is over the indices \( j \geq 2 \) is larger or equal to 2, 3 or 4 respectively. Since \( S < 3 \), Corollary 6 implies that \( L^2 = 0 \mod 18, S = 2 \), moreover \( a \in \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z} \) (still by Corollary 6) and Equation \((3.6)\) is then equivalent to

\[
\frac{1}{2}v_1^2 + \frac{3a^2}{2y_0} \leq \frac{2}{3}.
\]

**Sub-case 1) a) \( v_1 = 1 \) (and \( L^2 = 0 \mod 18 \).** Let us suppose that \( v_1 = 1 \) and define \( a' \in \mathbb{Z} \) such that \( a = \frac{a'}{3} \) (since \( a \in \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z} \), \( a' \) is coprime to 3). Inequality \((3.7)\) implies that \( a' \leq y_0 \). Since \( S = 2 \) and \( v_1 = 1 \), the \((-2)\)-class \( \Gamma \) has the form:

\[
\Gamma = \frac{1}{3}a'L - \frac{1}{3}(u_1F_1 + G_1 + H + H')
\]

with classes \( H, H' \) in the set \( \{F_j, G_j \mid j \geq 2\} \) and such that \( HH' = 0 \) (we recall that \( F_j = A_j + 2B_j, G_j = 2A_j + B_j \)). Equality \( \Gamma^2 = -2 \) is equivalent to

\[
\frac{2}{3}ka'^2 - \frac{2}{3}(u_1^2 + u_1 + 1) - \frac{2}{3} - \frac{2}{3} = -2,
\]

which is equivalent to

\[
(u_1^2 + u_1 + 1) - ka'^2 = 1,
\]

(observe here that since \( L^2 = 0 \mod 18 \), one has \( k = 0 \mod 3 \), thus \( u_1^2 + u_1 = 0 \mod 3 \)) and finally, to

\[
(2u_1 + 1)^2 - 4ka'^2 = 1.
\]
Since \( a' \leq y_0 \) and \((x_0, y_0)\) is the fundamental solution of the Pell-Fermat equation (3.3), we have \( a' = y_0 \), (in other words \( a = \frac{1}{3}y_0 \)) and \( u_1 = \frac{1}{2}(x_0 - 1) \).

We obtain that the class \( \Gamma \) is

\[
(3.8) \quad \Gamma_0 = \frac{1}{3}y_0L - \frac{1}{3}(\frac{1}{2}(x_0 - 1)F_1 + G_1 + H + H') .
\]

By Corollary 6, this class can be in the Néron-Severi group only if \( y_0 \) is coprime to 3. Now, let us suppose that \( y_0 \) is coprime to 3 (recall that we are in the case \( L^2 = 0 \mod 18 \)). We will see that up to exchanging the role of \( A_1 \) and \( B_1 \), one can also suppose that such a \( \Gamma_0 \) is not in the Néron-Severi group.

If the pair \( \{H, H'\} \) in the definition of \( \Gamma_0 \) of equation (3.8) exists, it is unique, otherwise the difference between the two obtained \((-2)\)-classes \( \Gamma_0 \) would have at least 2 and at most 8 coefficients in \( \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z} \), with the coefficient of \( L \) in \( \mathbb{Z} \), but this is impossible by part 2) of Corollary 6.

For some classes \( H_1, H'_1 \in \{ F_j, G_j \mid j \geq 2 \} \) with \( H_1H'_1 = 0 \), let us consider the \((-2)\)-class

\[
\Gamma'_0 = \frac{1}{3}y_0L - \frac{1}{3}(F_1 + \frac{1}{2}(x_0 - 1)G_1 + H_1 + H'_1)
\]

The number of coefficients in \( \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z} \) of the class

\[
\Gamma_0 - \Gamma'_0 = \frac{1}{3}(G_1 - F_1) + H + H' - H_1 - H'_1
\]

is at least 2 (because \( u_1 \neq 1 \mod 3 \)) and at most 10 (because each class \( H, H', H_1, H'_1 \) can give at most two coefficients in \( \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z} \)). By part 2) of Corollary 6, this is impossible. We thus obtain that either \( \Gamma_0 \) or \( \Gamma'_0 \) is not in the Néron-Severi group. Thus up to exchanging \( A_1 \) and \( B_1 \), and working with \( B_1 \) instead of \( A_1 \), one can suppose that \( \Gamma_0 \) is not in the Néron-Severi group.

**Remark 16.** If \( \Gamma_0 \in \text{NS}(X) \), then

\[ B'_1 = 3\Gamma_0 + B_1 + H + H', \]

and the curve \( B'_1 \) is not irreducible.

**Sub-case 1) b)** \( v_1 = 0 \) (and \( L^2 = 0 \mod 18 \)). We are still in the case \( L^2 = 0 \mod 18, S = 2, a \in \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z} \) and we suppose now that \( v_1 = 0 \). Equation (3.7) is equivalent to \( a' \leq 2y_0 \), where \( a = \frac{a'}{3} \). The \((-2)\)-class \( \Gamma \) has the form:

\[
\Gamma = \frac{1}{3}a'L - \frac{1}{3}(u_1F_1 + H + H'),
\]

and equality \( \Gamma^2 = -2 \) implies that

\[
(3.9) \quad u_1^2 - ka'^2 = 1.
\]

Since \( a' \leq 2y_0 \) and \((x_0, y_0)\) is a fundamental solution of (3.3) one has \( a' = 2y_0 \), \( u_1 = x_0 \) and \( \Gamma \) has the form

\[
\Gamma = \frac{2}{3}y_0L - \frac{1}{3}(x_0(A_1 + 2B_1) + H + H'),
\]
with $H, \, H'$ in $\{ F_j, G_j \mid j \geq 2 \}$ such that $HH' = 0$. Since cases $v_1 = 0$ and $v_1 = 1$ are disjoint, again, by exchanging the role of $A_1$ and $B_1$ and by considering
\[ \Gamma' = \frac{2}{3} y_0 L - \frac{1}{3} \left( x_0 (2A_1 + B_1) + H_1 + H'_1 \right), \]
and the difference $\Gamma - \Gamma'$, we see that $\Gamma$ or $\Gamma'$ is not in the Néron-Severi lattice.

Case 2) All the coefficients are integers. From the previous discussion, we can suppose that $a$ is not in $\frac{1}{3} \mathbb{Z} \setminus \mathbb{Z}$ (here $L^2 = 0 \mod 6$). Then by Corollary 6 all the coefficients of $\Gamma$ are integers, which implies that $S$ is divisible by 3, and since $S < 3$, we obtain that $S = 0$, moreover, using Equation 3.6 one has $v_1 \in \{0, 1\}$. We thus obtain that
\[ \Gamma = aL - \frac{1}{3} (u_1 F_1 + v_1 G_1) \]

with
\[ (3.10) \quad 6ka^2 - \frac{2}{3} (u_1^2 + u_1 v_1 + v_1^2) = -2, \]
where we recall that $L^2 = 6k$.

Sub-case 2) a) Suppose $v_1 = 0$, then Equation (3.10) is equivalent to
\[ u_1^2 = 3(1 + 3ka^2), \]
which has no solutions.

Sub-case 2) b) Suppose that $v_1 = 1$. By defining $U = \frac{1}{3} (2u_1 + 1)$, Equation (3.10) is equivalent to
\[ U^2 - 4ka^2 = 1. \]
Since by (3.6), $a \leq y_0$ and $(x_0, y_0)$ is the fundamental solution of the above Pell-Fermat equation, we obtain that $a = y_0, \, u_1 = \frac{1}{3} (3x_0 - 1)$ and
\[ \Gamma = y_0 L - \left( \frac{1}{2} (x_0 + 1) A_1 + x_0 B_1 \right) = B'_1. \]
That finishes the proof of Proposition 14.

As for the case $L^2 = 2 \mod 6$, we get:

Corollary 17. Suppose that $(x_0, y_0)$ is the fundamental solution of the Pell-Fermat equation 3.3 and that $L^2 = 6$ or $12 \mod 18$ or $L^2 = 0 \mod 18$ and $3 | y_0$. The divisor $B'_1 = y_0 L - \left( \frac{1}{2} (x_0 + 1) A_1 + x_0 B_1 \right)$ is the class of a $(-2)$-curve.

The curves $B_1$ and $B'_1$ are the unique $(-2)$-curves in the lattice generated by $L, \, A_1, \, B_1$ which have intersection one with $A_1$.

Up to exchanging the role of the curves $A_1, \, B_1$, the same result holds true for $L^2 = 0 \mod 18$ and $3 \nmid y_0$.

Proof. For proving that $B'_1$ is a $(-2)$-curve we use the same argument as in the proof of Corollary 13. Let $B_1$ be a $(-2)$-curve such that $B_1 A_1 = 1$.

Suppose $B_1 \neq B_1$, there exists $(x, y)$ with $x > 0, \, y > 0$ solution of the
Pell-Fermat equation \[ B_1 = yL - \left(\frac{1}{2}(x + 1)A_1 + xB_1\right) \]

One has $\tilde{B}_1B'_1 < 0$ if and only if

$$4kyy_0 < xx_0 + \frac{1}{3}.$$ 

We proceed as in the proof of Corollary 13 and obtain that this inequality holds for any such $(x, y)$, therefore $(x, y) = (x_0, y_0)$. \(\square\)

4. Existence of two generalized Kummer structures

4.1. A theoretical approach. Let $X$ be a generalized Kummer surface, we keep the notations as before, in particular the polarization $L$ generates the orthogonal complement to the 18 curves $A_1, \ldots, B_9$. Through this section we suppose that $6L^2$ is not a square (when $L^2 = 2 \mod 6$, that assumption is always satisfied). For $L^2 = 2 \mod 6$ (respectively for $L^2 = 0 \mod 6$), let $(x_0, y_0)$ be the fundamental solution of the Pell-Fermat equation $x^2 - 12ty^2 = 1$, for $t$ such that $L^2 = 2t$ (respectively $x^2 - 4ky^2 = 1$, for $k$ such that $L^2 = 6k$). We remark that $x_0^2 = 1 \mod 12t$ (respectively $x_0^2 = 1 \mod 4k$). Let $B'_1$ be the $(-2)$-class $B'_1 = 3y_0L - \left(\frac{1}{2}(x_0 + 1)A_1 + x_0B_1\right)$, (respectively $B'_1 = y_0L - \left(\frac{1}{2}(x_0 + 1)A_1 + x_0B_1\right)$).

Let us suppose that $B'_1$ is the class of a $(-2)$-curve. This is guarantied by Corollaries 13 and 17 if $L^2 \neq 0 \mod 18$, or if $L^2 = 0 \mod 18$ and $3 | y_0$, or up to exchanging the role of $A_1$ and $B_1$ if $L^2 = 0 \mod 18$ and $3 \not{|} y_0$. Then, we know two generalized Nikulin configurations $C = \{A_1, B_1, \ldots, A_9, B_9\}$, $C' = \{A_1, B'_1, A_2, B_2, \ldots, A_9, B_9\}$.

Let us prove the following:

**Theorem 18.** Suppose that $x_0 \neq \pm 1 \mod 2t$ (respectively $x_0 \neq \pm 1 \mod 2k$). There is no automorphism sending the configuration $C$ to the configuration $C'$. As a consequence, there are (at least) two generalized Kummer structures on the generalized Kummer surface $X$.

**Proof.** Let us suppose that such an automorphism $g$ sending $C$ to $C'$ exists. The automorphism $g$ induces an isometry on $\text{NS}(X)$, it therefore sends the orthogonal complement of $C$ to the orthogonal complement of $C'$. Since $L$, $L'$ are the positive generators of these complements, it maps $L$ to $L'$. Suppose
that $L$ is such that $L^2 = 2 \mod 6$. We recall that $L' = x_0 L - 2ty_0(A_1 + 2B_1)$ and that by Section 2.1, the Néron-Severi lattice is

$$NS(X) = \mathbb{Z}L \oplus \mathcal{K}_3,$$

therefore $\frac{1}{2t}L$ is in the dual of $NS(X)$ (we recall that $L^2 = 2t$). Since we know that $g(L) = L'$, the action of $g$ on the class of $\frac{1}{2t}L$ in the discriminant group is

$$g^*(\frac{1}{2t}L) = \frac{x_0}{2t}L - y_0(A_1 + 2B_1) = \frac{x_0}{2t}L \in NS(X)^\vee/NS(X).$$

However, the action of an automorphism on the discriminant group must be $\pm$ identity (see e.g. [32, Section 8.1]). Since the hypothesis is that $x_0 \neq \pm 1 \mod 2t$, such a $g$ does not exist.

Suppose that $L^2 = 0 \mod 6$, i.e. $L^2 = 6k$ ($k \in \mathbb{N}^*$), then $L' = x_0 L - 2ky_0(A_1 + 2B_1)$. The Néron-Severi lattice is generated by the lattice $\mathbb{Z}L \oplus \mathcal{K}_3$ (see section 2.1) and by a vector $\frac{1}{3}(L + v_{6k})$, $v_{6k} \in \mathcal{K}_3$. The class $\frac{1}{2t}L$ is thus in the dual lattice of $NS(X)$. The action of $g$ on the class of $\frac{1}{2t}L$ in the discriminant group is

$$g^*(\frac{1}{2k}L) = \frac{x_0}{2k}L - y_0(A_1 + 2B_1) = \frac{x_0}{2k}L.$$

Again, since we supposed that $x_0 \neq \pm 1 \mod 2k$, this is impossible. \hfill \Box

**Remark 19.** Suppose for simplicity that $L^2 = 2 \mod 6$. One could play again the same game: pick-up a $\mathbb{A}_2$ configuration $C_1, D_1$ in $C'$, then Corollary [13] implies that $D'_1 = 3y_0 L - (\frac{1}{3}(x_0 + 1)C_1 + x_0 D_1)$ is irreducible and we obtain in that way a new $9\mathbb{A}_2$-configuration $C''$, with orthogonal complement $L''$. Again there is no automorphism sending $C'$ to $C''$. However, the coefficient on $L$ of $L''$ will be $x_0^2$, which is congruent to 1 modulo $2t$, therefore there could be an automorphism sending $C'$ to $C''$, and in fact, in all the tested cases, computations show that this always happens.

**Example 20.** The first $L^2$ for which Theorem [18] provides two generalized Kummer structures on the generalized Kummer surface are:

$$\begin{align*}
(4.1) & \quad 20, 44, 68, 84, 92, 104, 110, 116, 120, 126, 132, 140, 164, 168, 176, 188. 
\end{align*}$$

The following infinite series of examples was given to us by Olivier Ramaré:

**Example 21.** Let $k$ be an integer, and let $a = 8 + 12k$. Then $t = 6 + 17k + 12k^2$ is such that $a^2 + a = 12t$. The pair $(2a + 1, 2)$ is the fundamental solution of the Pell-Fermat equation $x^2 - 12ty^2 = 1$. One can check that moreover $2a + 1 \neq \pm 1 \mod 2t$ and therefore we can apply Theorem [18] for such $t$’s.

The next Section suggests that the criteria in Theorem [18] for having two generalized Kummer structures is quite sharp.
4.2. A computational approach. Suppose that the polarization $L$ on the
generalized Kummer surface $X$ is such that $6L^2$ is not a square and $L^2 \neq$
$0 \mod 18$ so that the curve $B'_1$ obtained in Corollaries 13 and 17 is irreducible.
Then, we know two generalized Nikulin configurations
\[ C = \{A_1, B_1, \ldots, A_9, B_9\}, \quad C' = \{A_1, B'_1, A_2, B_2, \ldots, A_9, B_9\}. \]
Let $L$ and $L'$ be the big and nef generators of the orthogonal complements
of $C$ and $C'$ respectively. Suppose that there is an automorphism $g$ sending
$C$ to $C'$. Then it induces an isometry $g^*$ of the lattice $\text{NS}(X)$, in particular
it sends $L$ to $L'$. Using the Torelli Theorem for K3 surfaces, one can test all
linear maps
\[ \psi : \text{NS}(X) \otimes \mathbb{Q} \to \text{NS}(X) \otimes \mathbb{Q} \]
which satisfies to the following conditions:

i) it sends the $\mathbb{Q}$-base $\{L\} \cup C$ to the $\mathbb{Q}$-base $\{L'\} \cup C'$ and sends $A_2$
-configurations in $C$ to $A_2$-configurations in $C'$,

ii) it preserves the Néron-Severi lattice

iii) it acts on the discriminant group of $\text{NS}(X)$ by $\pm Id$,

iv) it sends an ample class to an ample class.

About that last point, we remark that it is always satisfied by such a $\psi$.
Indeed, the nef divisor $L'$ generates the orthogonal complement of a $9A_2$
configuration $C'$, thus by Section 2.2.2 for $u_0 \geq 4$, the divisor
\[ D' = u_0L' - \sum_{C \in C'} C \]
is ample, but $D'$ is also the image by $\psi$ of the ample class $D = u_0L - \sum_{C \in C} C$.

Let us change the notations and define $C_1 = A_1, D_1 = B'_1, C_j = A_j, D_j = B_j$ for $j \geq 2$. There are
\[ 9!9^9 = 185794560 \]
maps $\psi$ satisfying condition i): $9!$ is for the number of maps $\{A_k, B_k\} \to$
$\{C_{\sigma(k)}, D_{\sigma(k)}\}$, where $\sigma \in S_9$ is a permutation, and this is times $2^9$, because
one must choose to send $A_k$ either to $C_{\sigma(k)}$ or to $D_{\sigma(k)}$. It can be quite long
to sort among them the maps that satisfy ii) and iii), because one must deal
with rank 19 matrices.

However, recall that from Corollary 4 the divisors supported only on the
$A_2$-blocs $A_j, B_j$ ($j \in \{1, \ldots, 9\}$) have restrictions: they form in $\text{NS}(X) \otimes$
$\mathbb{Z}/3\mathbb{Z}$ a group isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$, each with support on either 6 blocs of
$A_2$ (12 such words) or on 9 blocs (2 words) or 0 (thus a total of $2 \times 12 + 2 \times$
$1 + 1 = 3^3$ classes which are $3$-divisible). With a computer, it is not difficult
to obtain the set $\text{BL}_{12}$ (respectively $\text{BL}'_{12}$) of 6 blocs which are $3$-divisible
for $C$ (respectively for $C'$). Since $\psi$ is an isometry it must send $3$-divisible
sets with support on $C$ to $3$-divisible sets with support on $C'$.

Using a computer, one can find in a few seconds the set of permutations $\sigma$
such that if $\sum a_i(A_i + 2B_i) \in \text{BL}_{12}$ is $3$-divisible, then $\sum a_i(C_{\sigma(i)} + 2D_{\sigma(i)})$
is $3$-divisible. That leaves a set of 432 permutation, instead of $9!$. Therefore
the number of possibilities is divided by 800, which makes the computation for checking conditions ii) to iv) above last a few minutes only.

By these computations, we obtain the following result which gives a more precise result (with an independent proof) than Theorem 18, but only for polarizations \( L \) such that \( L^2 \) is below some bound:

**Theorem 22.** Let \( X \) be a generalized K3 surface polarized with \( L \), such that \( L^2 < 200 \) and \( L^2 \neq 0 \mod 18 \).

There is an automorphism \( \sigma \) sending the generalized Nikulin configuration \( C \) to \( C' \) if and only if \( L^2 \) is not in the list (4.1).

Note that one can often choose \( \sigma \) of order 2, but for cases \( L^2 = 42, 48 \), all the automorphisms sending \( C \) to \( C' \) have infinite order. For the cases \( L^2 = 36 \) or \( 180 \), provided that the curve \( B'_{1} \) constructed in Section 3.2.2 is irreducible, there are also two generalized Kummer structures.

5. **Examples**

5.1. **A birational models of \( X \) with \( 9A_2 \) singularities.** We keep the notations: \( X \) is a generalized Kummer surface with Picard number 19, \( A_1, \ldots, B_9 \) is a \( 9A_2 \)-configuration, and \( L \) is the nef divisor that generates the orthogonal complement of these 18 curves (see proof of Proposition 7).

**Proposition 23.** Suppose \( L^2 > 2 \). The linear system \( |L| \) induces a morphism \( \varphi_L : X \to \mathbb{P}^{\frac{L^2}{2}}+1 \) which is an embedding outside the \((-2)\)-curves \( A_1, B_1, \ldots, A_9, B_9 \) and maps these curves to 9 cusps.

**Proof.** We know that \( L \) is nef and big. If \( |L| \) has base points, then

\[
L = uF + \Gamma,
\]

where \( F \) is an elliptic curve and \( \Gamma \) is an irreducible \((-2)\)-curve such that \( F\Gamma = 1 \) (see [26, Section 3.8]). Moreover since \( L \) is big and nef we have

\[
0 \leq L\Gamma = u - 2
\]

so that \( u > 0 \). If \( \Gamma \neq A_k, B_k \) we have \( FA_k \geq 0, \Gamma A_k \geq 0 \) and we compute

\[
0 = LA_k = uFA_k + \Gamma A_k
\]

Since \( u > 0 \) we obtain \( FA_k = \Gamma A_k = 0 \) similarly \( FB_k = \Gamma B_k = 0 \) so that \( \Gamma \) is in the orthogonal complement of \( A_1, B_1, \ldots, A_9, B_9 \), which is clearly impossible. If \( \Gamma = A_k \) we have \( L = uF + A_k \) hence

\[
0 = LA_k = u - 2
\]

which gives \( u = 2 \), now \( L = 2F + A_k \) which has \( L^2 = 2 \), which contradicts the assumption, similarly for \( B_k \). In conclusion \( |L| \) is base-point-free. Suppose that \( |L| \) is hyperelliptic (see [31]). Since \( L \) is primitive, one cannot have \( L = 2D_2 \) with \( D_2^2 = 2 \). Suppose there is an elliptic curve \( F \) such that \( FL = 2 \). Write

\[
F = aL - \sum \alpha_i A_i + \beta_i B_i,
\]
with $a \geq 0$, $a, \alpha_i, \beta_i \in \frac{1}{3} \mathbb{Z}$, then
\[
2 = FL = aL^2.
\]

Since $L^2 > 2$, this is possible only if $L^2 = 6$. But in that case J. Bertin and P. Vanhaecke [7] proved that $\varphi_L$ is an embedding outside the $(-2)$-curves $A_1, B_1, \ldots, A_9, B_9$ (see the description below).

5.2. Case $L^2 = 2$. In the case $L^2 = 2$, the generalized Kummer surface $X = \text{Km}(A, G)$ is the double cover of the plane ramified on a sextic curve with $9A_2$ singularities. This double cover were first studied by Ch. Birkenhake and H. Lange in [18]. In [17], the two authors of the present paper and D. Kohel determined several $9A_2$-configurations on $X$ related to a special configuration of conics in the plane. If $L^2 = 2$ by Section 3.2 we have $B_1' = 6L - (4A_1 + 7B_1)$. Since $LB_1' = 12$, the curve $B_1'$ is sent to a singular curve of degree 6 in the plane, which passes through the cusp obtained by the contraction of $A_1, B_1$. As shown in [28] (see also Corollary 22), up to automorphism of the K3 surface, there is only one $9A_2$-configuration, so that there exists an automorphism sending the configuration $\mathcal{C}$ to the configuration $\mathcal{C}'$; observe that clearly this automorphism is not the covering involution.

5.3. Case $L^2 = 6$. The polarization $L^2 = 6$ exhibits the K3 surface as a complete intersection of a quadric and a cubic in $\mathbb{P}^4$ with $9A_2$ singularities, see the paper by J. Bertin and P. Vanhaecke [7] for more details and the equations. Observe that in this case the Pell–Fermat equation $x^2 - 4y^2 = 1$ has no solution so that we can not apply our construction. By [10] in this case there is only one generalized Nikulin configuration.

5.4. Case $L^2 = 20$. This is the first example for which our construction gives two non–equivalent Kummer structures (see Example 4.1), so we study it more in detail. As before let $L$ be the class such that $L^2 = 20$, let $A_1, B_1, \ldots, A_9, B_9$ be the $9A_2$-configuration orthogonal to $L$. In this case the class $B_1'$ is
\[
B_1' = 3L - (6A_1 + 11B_1)
\]
and we have shown that $\mathcal{C} = \{A_1, B_1, \ldots, A_9, B_9\}$ and $\mathcal{C}' = \{A_1, B_1', \ldots, A_9, B_9\}$ are not equivalent. The projective model determined by $L$ is a surface in $\mathbb{P}^{11}$, we describe here another projective model as a double plane ramified on a special sextic curve. By Proposition 7, the class
\[
D_2 = L - \sum_{k=1}^{9} (A_k + B_k)
\]
is ample with $D_2^2 = 2$ and $D_2 A_k = D_2 B_k = 1$, $k \in \{1, \ldots, 9\}$. The $(-2)$-classes
\[
E_k = D_2 - A_k, \quad F_k = D_2 - B_k, \quad k \in \{1, \ldots, 9\}
\]
are also of degree 1 for $D_2$, hence are classes of $(-2)$-curves. Moreover one can check easily that:
Proposition 24. The 18 \((-2)\)-curves \(E_1, F_1, \ldots, E_9, F_9\) form a \(9A_2\)-configuration.

Suppose that \(D_2\) has base points. Then there exist an elliptic curve \(F\) and an irreducible \((-2)\)-curve \(\Gamma\), \([26] \text{ Section 3.8}\) such that \(D_2 = 2F + \Gamma\) and \(\Gamma D_2 = 1\). But then \(\Gamma D_2 = 0\), which is a contradiction since \(D_2\) is ample. Therefore, with the previous notations:

Proposition 25. The generalized Kummer surface is a double cover of \(\mathbb{P}^2\) branched over a smooth sextic curve \(C_6\) which has 18 tritangent lines, which are the images of the 18 couples of curves \((E_k, A_k)\) and \((F_k, B_k)\) for \(k \in \{1, \ldots, 9\}\).

Plane sextic curves with several tritangents were studied by A. Degtyarev in \([12]\). Using the Néron-Severi lattice and Vinberg’s algorithm \([34]\), one can compute moreover, that there are 1728 \(6\)-tangent conics to \(C_6\) and 67212 rational cuspidal curves which are tangent to \(C_6\), with a cusp on \(C_6\). Also we have:

Theorem 26. The automorphism group \(G_{36}\) preserving the polarization \(D_2\) is isomorphic to

\[ \mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes S_3), \]

it has order 36. It is generated by the involution \(\sigma\) of the double cover and a symplectic group of automorphisms \(G_{18}\) of order 18; \(\sigma\) generates the center of \(G_{36}\).

Proof. This is obtained by a computation as explained in Section 4.2, using the divisibility relations among the \((-2)\)-curves. The algorithm in Section 4.2 is also able to construct the automorphisms if such exists, one just have to check that the hypothesis of the Torelli Theorem are satisfied. \(\square\)

The involution \(\sigma\) is such that \(\sigma(A_k) = E_k, \sigma(B_k) = F_k\), so that the two \(9A_2\)-configurations \(E_1, F_1, \ldots, E_9, F_9\) and \(C = \{A_1, B_1, \ldots, A_9, B_9\}\) are equivalent.

The orbit of \(A_1\) under \(G_{36}\) is \(\{A_k, E_k \mid 1 \leq k \leq 9\}\) and the orbit of \(B_1\) is \(\{B_k, F_k \mid 1 \leq k \leq 9\}\). Let \(L_k\) (respectively \(L'_k\)) be the image of \(A_k\) (respectively \(B_k\)) by the double cover map \(X \to \mathbb{P}^2\). The group \(\mathbb{Z}_3 \rtimes S_3\) acts on the plane and the orbit of \(L_1\) (resp. \(L'_1\)) is \(\{L_k \mid 1 \leq k \leq 9\}\) (resp. \(\{L'_k \mid 1 \leq k \leq 9\}\)).

The general abelian surfaces \(A\) such that \(X = \text{Km}_3(A)\) are simple \([28]\).

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