Alpha-particle condensation in $^{16}$O via a full four-body OCM calculation

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In order to explore the 4α-particle condensate state in $^{16}$O, we solve a full four-body equation of motion based on the 4α OCM (Orthogonality Condition Model) in a large 4α model space spanned by Gaussian basis functions. A full spectrum up to the 0$^+_0$ state is reproduced consistently with the lowest six 0$^+$ states of the experimental spectrum. The 0$^+_0$ state is obtained at about 2 MeV above the 4α breakup threshold and has a dilute density structure, with a radius of about 5 fm. The state has an appreciably large α condensate fraction of 61%, and a large component of $\alpha+^{12}$C(0$^+_+\)$ configuration, both features being reliable evidence for this state to be of 4α condensate nature.

It is well established that α-clustering plays a very important role for the structure of lighter nuclei [1, 2]. The importance of α-cluster formation also has been discussed in infinite nuclear matter, where α-particle type condensation is expected at low density [3], quite in analogy to the recently realised Bose-Einstein condensation of bosonic atoms in magneto-optical traps [4]. On the other hand, for trapped fermions, quartet condensation also is an emerging subject, discussed, so far, only theoretically [5]. In nuclei the bosonic constituents always are only very few in number, nevertheless possibly giving rise to clear condensation characteristics, as is well known from nuclear pairing [6]. Concerning α-particle condensation, only the Hoyle state, i.e. the 0$^+_2$ state in $^{12}$C has clearly been established, so far. Several papers of the past [7, 8, 9, 10] and also more recently [11, 12, 13] have by now established beyond any doubt that the Hoyle state, only having about one third of saturation density, can be described, to good approximation, as a product state of three α-particles, condensed, with their c.o.m. motion, into the lowest mean field 0S-orbit [14, 15].

The establishment of this novel aspect of the Hoyle state naturally leads us to the speculation about α-particle type condensation in $^{16}$O, which is the focus of this work. The 0$^+$ spectrum of $^{16}$O has, in the past, very well been reproduced up to about 13 MeV excitation energy, including the ground state, with a semi-microscopic cluster model, i.e. the α+$^{12}$C OCM (Orthogonality Condition Model) [16]. In particular, this model calculation, as well as that of an α+$^{12}$C Generator-Coordinate-Method one [17], demonstrates that the 0$^+_2$ state at 6.05 MeV and the 0$^+_3$ state at 12.05 MeV have α+$^{12}$C structures [18] where the α-particle orbits around the $^{12}$C(0$^+_2$)-core in an S-wave and around the $^{12}$C(2$^+_1$)-core in a D-wave, respectively. Consistent results were later obtained by the 4α OCM calculation within the harmonic oscillator basis [14]. However, the model space adopted in Refs. [16, 17, 19] is not sufficient to account simultaneously for the α+$^{12}$C and the 4α gas-like configurations. On the other hand, the 4α-particle condensate state was first investigated in Ref. [11] and its existence was predicted around the 4α threshold with a new type of microscopic wave function of α-particle condensate character. While that wave function can well describe the dilute α cluster states as well as shell-model-like states, other structures such as α+$^{12}$C clustering are smeared out and only incorporated in an average way. Since there exists no calculation, so far, which reproduces both the 4α gas and α+$^{12}$C cluster structures simultaneously, it is crucial to perform an extended calculation for the simultaneous reproduction of both kinds of structures, which will give a decisive benchmark for the existence of the 4α-particle condensate state from a theoretical point of view.

The purpose of this Letter is to explore the 4α condensate state by solving a full OCM four-body equation of motion without any assumption with respect to the structure of the 4α system. Here we take the 4α OCM with Gaussian basis functions, the model space of which is large enough to cover the 4α gas, the α+$^{12}$C cluster, as well as the shell-model configurations. The OCM is extensively described in Ref. [20]. Many successful applications of OCM are reported in Ref. [2]. The 4α OCM Hamiltonian is given as follows:

$$\mathcal{H} = \sum_i T_i - T_{cm} + \sum_{i<j}^4 \left[ V_{2\alpha}^{(N)}(i,j) + V_{2\alpha}^{(C)}(i,j) + V_{2\alpha}^{(P)}(i,j) \right] + \sum_{i<j<k}^4 V_{3\alpha}(i,j,k) + V_{4\alpha}(1,2,3,4),$$

where $T_i$, $V_{2\alpha}^{(N)}(i,j)$, $V_{2\alpha}^{(C)}(i,j)$, $V_{3\alpha}(i,j,k)$ and $V_{4\alpha}(1,2,3,4)$ stand for the operators of kinetic energy for the $i$-th α particle, two-body, Coulomb, three-body and four-body forces between α particles,
respectively. The center-of-mass kinetic energy $T_{cm}$ is subtracted from the Hamiltonian. $V_{2\alpha}(i,j)$ is the Pauli exclusion operator \[21\], by which the Pauli forbidden states between two $\alpha$-particles in $0S$, $0D$ and $1S$ states are eliminated, so that the ground state with the shell-model-like configuration can be described correctly. The effective $\alpha-\alpha$ potentials reproduce the $\alpha-\alpha$ scattering phase shifts and energies of the $^8\text{Be}$ ground state and of the Hoyle state. The three-body force is phenomenologically introduced so as to fit the ground state energy of the $^{12}\text{C}$. The same force parameter set as used in Refs. \[15\] or \[24\] is adopted in the present calculation. We also mention that the ground state energy of $^{16}\text{O}$. The three-body and four-body forces are short-range, and, hence, they only act in compact configurations.

Utilizing the Gaussian expansion method \[25\] for the choice of variational basis functions, the total wave function $\Psi$ of the $4\alpha$ system is expanded in terms of Gaussian basis functions as follows:

$$
\Psi(0_i^n) = \sum_{c,\nu} A_c^\nu(\nu) \Phi_c(\nu),
$$

$$
\Phi_c(\nu) = \bar{S}[[\varphi_1(r_1, \nu_1)\varphi_2(r_2, \nu_2)]_{l_1, l_2} \varphi_3(r_3, \nu_3)]_c
$$

where $r_1$, $r_2$ and $r_3$ are the Jacobi coordinates describing internal motions of the $4\alpha$ system. $\bar{S}$ stands for the symmetrization operator acting on all $\alpha$ particles obeying Bose statistics. $\nu$ denotes the set of size parameters $\nu_1, \nu_2$ and $\nu_3$ of the normalized Gaussian function, $\varphi_i(r, \nu_i) = N_{\nu_i} r^{\nu_i} \exp\left(-\nu_i r^2\right) Y_{l_i m_i}(\hat{r})$, and $c$ the set of relative orbital angular momentum channels $[[l_1, l_2][l_3]]_c$ depending on either of the coordinate type of $K$ or $H$ \[22\], where $l_1$, $l_2$ and $l_3$ are the orbital angular momenta with respect to the corresponding Jacobi coordinates. The coefficients $A_c^\nu(\nu)$ are determined according to the Rayleigh-Ritz variational principle.

Figure 1 shows the energy spectrum with $J^\pi = 0^+$, which is obtained by diagonalizing the Hamiltonian, Eq. (1). It is confirmed that all levels are well converged. With the above mentioned effective $\alpha-\alpha$ forces, we can reproduce the full spectrum of $0^+$ states, and tentatively make a one-to-one correspondence of those states with the six lowest $0^+$ states of the experimental spectrum. In view of the complexity of the situation, the agreement is considered to be very satisfactory. We show in TABLE I the calculated rms radii and monopole transition matrix elements to the ground state $M(E0)$ in units of fm and fm$^2$, respectively. $R_{exp}$ and $M(E0)_{exp}$ are the corresponding experimental data.

![FIG. 1: Comparison of energy spectra between experiment and the present calculation. Two kinds of effective two-body nucleon-nucleon forces MHN and SW are adopted (see text). Dotted and dash-dotted lines denote the $\alpha+^{12}\text{C}$ and $4\alpha$ thresholds, respectively. Experimental data are taken from Ref. \[26\], and from Ref. \[27\] for the $0_1^+$ state. The assignments with experiment are tentative, see, however, detailed discussion in the text.]

### TABLE I: The rms radii $R$ and monopole transition matrix elements to the ground state $M(E0)$ in units of fm and fm$^2$, respectively. $R_{exp}$ and $M(E0)_{exp}$ are the corresponding experimental data.

|       | $R$    | $M(E0)$ | $R_{exp}$ | $M(E0)_{exp}$ |
|-------|--------|---------|-----------|--------------|
|       | SW MHN | SW MHN  |           |              |
| $0_1^+$ | 2.7    | 2.7     | 2.71 ± 0.02 |              |
| $0_2^+$ | 3.0    | 3.0     | 4.1 ± 3.9  | 3.55 ± 0.21  |
| $0_3^+$ | 2.9    | 2.9     | 2.6 ± 2.4  | 4.03 ± 0.09  |
| $0_4^+$ | 4.0    | 4.0     | 3.0 ± 2.4  | no data      |
| $0_5^+$ | 3.1    | 3.1     | 3.0 ± 2.6  | 3.3 ± 0.7    |
| $0_6^+$ | 5.0    | 5.6     | 0.5 ± 1.0  | no data      |

The $M(E0)$ values for the $0_2^+$, $0_3^+$, and $0_5^+$ states are consistent with the corresponding experimental values. As mentioned above, the structures of the $0_2^+$ and $0_3^+$ states are well established as having the $\alpha+^{12}\text{C}(0_1^+)$ and $\alpha+^{12}\text{C}(2_1^+)$ cluster structures, respectively. These structures of the $0_2^+$ and $0_5^+$ states are confirmed in the present calculation. We also mention that the ground state is described as having a shell-model configuration within the present framework, the calculated rms value.
agreeing with the observed one (2.71 fm).

On the contrary, the structures of the observed $0^+_1$, $0^+_3$ and $0^+_5$ states in Fig. 1 have, in the past, not clearly been understood, since they have never been discussed with the previous cluster model calculations [16, 17, 19]. Although Ref. [11] predicts the $4^+_1$ condensate state around the $4\alpha$ threshold, it is not clear which of those states corresponds to the condensate state. For example, the $0^+_1$ state has been considered as one of the candidates [27], and also the $0^+_5$ state with large $M(E0)$ value [28, 29], since the strong monopole transition implies a developed cluster structure [31]. This ambiguous situation stems from the fact that the $\alpha$-condensate-type wave function given in Ref. [11] can only reproduce the ground state and the $4\alpha$ condensate state, but not the $\alpha+^{12}\text{C}$ configurations which make up a large part of the $^{16}\text{O}$-spectrum up to the $0^+_6$ state.

As shown in Fig. 1, the present calculation succeeded, for the first time, to reproduce the $0^+_1$, $0^+_3$ and $0^+_5$ states, together with the $0^+_7$, $0^+_9$ and $0^+_12$ states. This puts us in a favorable position to discuss the $4\alpha$ condensate state, expected to exist around the $4\alpha$ threshold.

In Table I the largest rms value of about 5 fm is found for the $0^+_6$ state. Compared with the relatively smaller rms radii of the $0^+_1$ and $0^+_5$ states, this large size suggests that the $0^+_6$ state may be composed of a weakly interacting gas of $\alpha$ particles [31] of the condensate type.

While a large size is generally necessary for forming an $\alpha$ condensate, the best way for its identification is to investigate the single-$\alpha$ orbit and its occupation probability, which can be obtained by diagonalizing the one-body ($\alpha$) density matrix [14, 12, 32] in the following way:

$$\int dr' \rho(r, r') \phi^L(r') = \mu^L \phi^L(r),$$  \hspace{1cm} (4)

where $\phi^L(r)$ is the single-$\alpha$ natural orbit with orbital angular momentum $L$, and $\mu^L$ is its occupation probability. The one-body density matrix elements $\rho(r, r')$ have been calculated as in [14, 12]. As a result of the calculation of the $L = 0$ case, a large occupation probability of 61% of the lowest $0S$-orbit is found for the $0^+_6$ state, whereas the other five $0^+$ states all have appreciably smaller values, at most 25% ($0^+_5$). The corresponding single-$\alpha$ $S$ orbit is shown in Fig. 2. It has a strong spatially extended behaviour without any node ($0S$). This indicates that $\alpha$ particles are condensed into the very dilute $0S$ single-$\alpha$ orbit, see also Ref. [33]. Thus, the $0^+_6$ state clearly has $4\alpha$ condensate character. We should note that the orbit is very similar to the single-$\alpha$ orbit of the Hoyle state [14, 12]. We also show in Fig. 2 the single-$\alpha$ orbit for the ground state. It has maximum amplitude at around 3 fm and oscillations in the interior with two nodal ($2S$) behaviour, due to the Pauli principle and reflecting the shell-model configuration.

![FIG. 2: (Colors online) The radial parts of single-$\alpha$ orbits with $L = 0$ belonging to the largest occupation number, for the ground and $0^+_6$ states with MHN force.](image)

In order to further analyze the obtained wave functions, we calculate an overlap amplitude, which is defined as follows:

$$\mathcal{Y}(r) = \langle \Psi(0^+_6) \rvert \frac{\delta(r - r')}{r'^2} Y_L(\hat{r}') \Phi_L(^{12}\text{C}) \rangle_0.$$  \hspace{1cm} (5)

Here, $\Phi_L(^{12}\text{C})$ is the wave function of $^{12}\text{C}$, given by the $3\alpha$ OCM calculation [12], and $r$ is the relative distance between the center-of-mass of $^{12}\text{C}$ and the $\alpha$ particle. From this quantity we can see how large is the component in a certain $\alpha+^{12}\text{C}$ channel which is contained in our wave function (2) for $0^+_6$. The amplitudes for the $0^+_6$ state are shown in Fig. 3. It only has a large amplitude in the $\alpha+^{12}\text{C}(2^+_1)$ channel, whereas the amplitudes in other channels are much suppressed. The amplitude in the Hoyle-state channel has no oscillations and a long tail stretches out to $\sim 20$ fm. This behaviour is very similar.

![FIG. 3: (Colors online) $\mathcal{Y}(r)$ defined by Eq. (5) for the $0^+_6$ state with the MHN force.](image)
to that of the single-α orbit of the $0^+_6$ state discussed above.

The α decay width constitutes a very important information to identify the $0^+_6$ state from the experimental point of view. It can be estimated, based on the R-matrix theory, with the overlap amplitude Eq. (5) [34]. We find that the total α decay width of the $0^+_6$ state is as small as 50 keV (experimental value: 166 keV). This means that the state can be observed as a quasi-stable state. Thus, the width, as well as the excitation energy, are consistent with the observed data. All the characteristics found from our OCM calculation, therefore, indicate that the 6th $0^+_1$ state at 15.1 MeV and that it is the α-condensate state of $^{16}$O.

Finally we discuss the structures of the $0^+_1$ and $0^+_2$ states. Our present calculations show that the $0^+_1$ and $0^+_2$ states mainly have $\alpha+^{12}$C($0^+_1$) structure with higher nodal behaviour and $\alpha+^{12}$C($1^-$) structure, respectively. Further details will be given in a forthcoming extended paper. The calculated width of the $0^+_2$ state is $\sim 150$ keV, which is much larger than that found for the $0^+_3$ state $\sim 50$ keV. Both are qualitatively consistent with the corresponding experimental data, 600 keV and 185 keV, respectively. The reason why the width of the $0^+_2$ state is larger than that of the $0^+_3$ state, though the $0^+_2$ state has lower excitation energy, is due to the fact that the former has a much larger component of the $\alpha+^{12}$C($0^+_1$) decay channel, reflecting the characteristic structure of the $0^+_1$ state. The 4α condensate state, thus, should not be assigned to the $0^+_1$ or $0^+_2$ state [35] but very likely to the $0^+_6$ state.

In conclusion, the present 4α OCM calculation, for the first time, succeeded in describing the structure of the full observed 0+ spectrum up to the $0^+_6$ state in $^{16}$O. The 0+ spectrum of $^{16}$O up to about 15 MeV is now essentially understood, including the 4α condensate state. This is remarkable improvement concerning our knowledge of the structure of $^{16}$O. We found that the $0^+_6$ state above the 4α threshold has a very large rms radius of about 5 fm and a rather strong concentration of α particles 61% on a spatially extended single-α 0S orbit. The wave function was shown to have a larger $\alpha+^{12}$C amplitude only for $^{12}$C (Hoyle state). These results are strong evidence of the newly found $0^+_6$ state as being the 4α condensate state, i.e. the analog to the Hoyle state in $^{12}$C. Further experimental information is very much requested to confirm the existence of this novel state. Also independent theoretical calculations are strongly needed for confirmation of our results.

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