Some Observations on the 3x+1 Problem

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Abstract
We present some interesting observations on the 3x+1 problem. We propose a new algorithm which eliminates certain steps while we check the action of 3x+1 procedure on a number. Also, we propose a reason why many numbers follow a similar pattern during execution of 3x+1 algorithm. We advocate towards the end (a heuristic argument) that the 3x+1 conjecture is more likely to be true than false.

1. Introduction: Perhaps the most fascinating problem in number theory is the so called 3x+1 problem. This problem is also known as Collatz’s problem, Kakutani’s problem, Syracuse problem, Ulam’s problem, and Hasse’s algorithm [1]. The problem can be stated as follows:

Let \( T : \mathbb{Z} \rightarrow \mathbb{Z} \) be defined by

\[
T(x) = \frac{x}{2^k} \quad \text{if} \quad x \equiv 0 \mod(2), \quad \text{and} \quad k \text{ is the largest power of 2 that divides } x.
\]

\[
T(x) = \frac{(3x + 1)}{2^k} \quad \text{if} \quad x \equiv 1 \mod(2), \quad \text{and} \quad k \text{ is the largest power of 2 that divides } (3x + 1).
\]

It is conjectured that if \( x \in \mathbb{N} \) then the trajectory \( x, T(x), T^2(x), \ldots \) eventually reaches (converges to) 1 in finitely many steps.

We now proceed to discuss

2. Some Interesting Observations:

Observation (1): (a) For an odd number \( x \) of type \((4k + 1)\) we have \( T(x) < x \).

(b) For an odd number \( x \) of type \((4k + 3)\) we have \( T(x) > x \).

Proof of (a): \( T(x) = \frac{3x + 1}{2^u} = \frac{12k + 4}{2^u} = \frac{4(3k + 1)}{2^u} \), hence divisible at least by 4 (i.e. \( u \) can be at least equal to 2).
**Proof of (b):** \( T(x) = \frac{3x + 1}{2^u} = \frac{12k + 10}{2^u} = \frac{2(6k + 5)}{2^u} \), hence divisible at most by 2 (i.e. \( u \) can be at most equal to 1).

**Observation (2):** Every odd integer \( y \) can be uniquely expressed as 
\[ y = 2^n x + 2^{(n-1)} - 1, \]
i.e. for every odd integer \( y \) there exist unique \( x, n \) such that the above equation holds.

**Proof:** Find the unique largest power of 2 that divides \((y+1)\), say \((n-1)\), and let \( \frac{(y+1)}{2^{(n-1)}} = 2x + 1 \). Hence etc.

**Observation 3:** There are numbers which rise \( n \) number of times where \( n \) is any arbitrary positive integer without a single fall under the action of \( 3x + 1 \) algorithm.

**Proof:** Consider the following number 
\[ y = 2^{(n+2)} x + 2^{(n+1)} - 1, \]
then 
\[ T(y) = 2^{(n+1)}(3x + 1) + 2^n - 1, \]
\[ T^2(y) = 2^n(3^2 x + 3^1 + 3^0) + 2^{(n-1)} - 1, \]
\[ \vdots \]
\[ T^n(y) = 2^2(3^n x + \frac{(3^n - 1)}{2}) + 2^1 - 1 = y_1. \]

After this stage we get 
\[ T^{(n+1)}(y) = T(y_1). \]

Note that \( T^j(y), j = 1, 2, (n - 1) \) is an odd number of type \((4k+3)\) while \( T^n(y) \) is an odd number of type \((4k+1)\). Therefore, it is clear that 
\[ y < T^1(y) < T^2(y) < \cdots < T^n(y), \]
while 
\[ T^{(n+1)}(y) < T^n(y). \]

This observation offers us a faster algorithm by which we can omit certain steps while we process a positive odd integer by \( 3x + 1 \) algorithm as follows:

**A Faster 3x + 1 Algorithm:**
(1) Express the given positive odd integer \( y \) uniquely as
\[ y = 2^n x + 2^{(n-1)} - 1 \]

(2) Find directly
\[ T^{(n-2)}(y) = 2^2 \left( 3^{(n-2)} x + \frac{3^{(n-2)} - 1}{2} \right) + 2^1 - 1 = y_1, \text{ say} \]

(3) Find \( T(y_1) = y_2 \), say and if \( y_2 \neq 1 \) then set \( y \leftarrow y_2 \) and go to step (1).

**Example 1:** Let \( y = 1023 \), then \( T^8(y) = y_1 = 39365 \) and \( T(y_1) = y_2 = 7381 \) (= a new \( y \) to start with). Thus, 8 executions are avoided!!

**Observation 4:** If \( y \rightarrow 1 \) then \( 2^k y \rightarrow 1 \) for all \( k \geq 0 \).

**Proof:** Obvious.

**Observation 5:** If \( y \rightarrow 1 \) then \( z = 4^n y + \left( \frac{4^n - 1}{3} \right) \rightarrow 1 \)

for all \( n = 1, 2, \ldots \). More precisely, if \( T(y) = y_0 \) then \( T(z) = y_0 \).

**Proof:** We proceed by induction on \( n \).

**Step (1):** Let \( n = 1 \). Therefore, we have \( z = 4y + 1 \).

Now, \( T(y) = y_0 = \frac{(3y + 1)}{2^j} \) where \( j \) is the largest power of 2 that divides \( (3y + 1) \). Also, \( 3z + 1 = 3(4y + 1) + 1 = 4(3y + 1) \), therefore the largest power that divides \( (3z + 1) \) will be \( 2^{(j+2)} \) and
\[ T(z) = y_0 = \frac{(3z + 1)}{2^{(j+2)}}. \]

**Step (2):** We assume by induction the result holds for \( n = k \) and proceed to show it for \( n = (k + 1) \). Let \( z = 4^k y + \left( \frac{4^k - 1}{3} \right) \) and
\[ T(z) = y_0. \] We have to see that when \( z' = 4^{(k+1)} y + \left( \frac{4^{(k+1)} - 1}{3} \right) \) we still have \( T(z') = y_0 \). Note that \( z' = 4z + 1 \) and when \( T(z) = y_0 \) then also \( T(z') = y_0 \) by step (1).

We now proceed to see why many numbers follow a similar pattern during execution of \( 3x+1 \) algorithm.
**Numbers with Similar Pattern:** From observations 4 and 5 suppose a number $x_0$ has the following sequence under the action of $3x + 1$ algorithm:

$$x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_j \rightarrow \cdots \rightarrow x_n \rightarrow 1,$$

then

1. The numbers $2^k x_j$ will have same pattern for all $k, j$.

2. The numbers $4^k x_j + \left(\frac{4^k - 1}{3}\right)$ will have same pattern for all $k, j$.

**Example 2:** Count the numbers $\leq 1000$ which have maximum value 9232.

**Solution:** Using (1) and (2) we count the numbers as follows:

1. Find the smallest number $x_0^1$ say such that $x_0^1 \leq 1000$.

   Let $x_0^1 \rightarrow x_1^1 \rightarrow \cdots \rightarrow x_j^1 \rightarrow \cdots \rightarrow x_k^1 \rightarrow 9232$.

2. Consider the set of numbers $\{2^m (4^n x_j^1 + \left(\frac{4^n - 1}{3}\right))\}$,

   for all nonnegative integers $m, n$ and $0 \leq j \leq k$, such that all these numbers are $\leq 1000$. It is clear that all these numbers when processed under $3x+1$ algorithm will have 9232 as maximum.

3. Find all the numbers $z \leq 1000$ which fall to $x_j^1$, $0 \leq j \leq k$, under the $3x + 1$ algorithm. Consider all distinct numbers $z_i$ on the paths from $z$ to $x_j^1$ and find as above their multiples $\{2^m (4^n z_i + \left(\frac{4^n - 1}{3}\right))\}$,

   for all nonnegative integers $m, n$ and $0 \leq i \leq k$, such that all these numbers are $\leq 1000$. Again, all these numbers when processed under $3x+1$ algorithm will have 9232 as maximum.

4. Consider next smallest number other than the numbers considered above having maximum 9232, say $x_0^2$, etc. etc. and go to (1).

5. Continue till all the numbers $\leq 1000$ that go to 9232 are considered in some set.

One can easily check that there are in all 350 numbers $\leq 1000$ having maximum 9232 under the $3x+1$ algorithm.
Observation 6: Solving the 3x+1 conjecture is equivalent to showing that every positive odd integer \( x \) has a representation as:

\[
x = \frac{2^{n(k+1)} - \left\{ 3^k 2^0 + 3^{(k-1)} 2^{n_1} + \cdots + 3^0 2^{n_k} \right\}}{3^{(k+1)}} \rightarrow (1)
\]

where the integral indices satisfy \( 0 < n_1 < n_2 < \cdots < n_{(k+1)} \),

\( n_{(k+1)} \neq n_k + 2 \).

Proof: Simple.

Example 3: \( 7 = \frac{2^{11} - \{3^4 2^0 + 3^3 2^1 + 3^2 2^2 + 3^1 2^4 + 3^0 2^7\}}{3^5} \)

Observation 7: If we consider all the solutions (integral as well as rational) of equation (1) we get all possible \( x \) that go to 1 in \((k+1)\) steps. Consider following sets of integral and rational solutions:

Let \( U_1 = \left\{ \frac{2^k - 3^0 2^0}{3^1}, k = 1, 2, \cdots \right\} \)

\( U_2 = \left\{ \frac{2^{k_2} - \{3^1 2^0 + 3^0 2^{k_1}\}}{3^2}, 0 < k_1 < k_2, k_1, k_2 = 1, 2, \cdots \right\}, \text{ etc.} \)

Observation 8: Solving 3x+1 conjecture is equivalent to showing that

\[
Z_{Odd} \subset \bigcup_{j=1}^{\infty} U_j
\]

Why 3x+1 conjecture is more likely to be true than false? Suppose a number \( x_0 \) requires \( k \) steps to become 1. One can see (by numerically tackling some examples) that if one finds closest possible \( x_1^i, x_2^i \) belonging to \( U_i, 1 \leq i \leq (k-1) \) such that \( x_1^i < x_0 < x_2^i \) then one observes that one moves closer and closer to \( x_0 \) as \( i \) is increased from 1 to \((k-1)\) (as one gets more parameters for maneuvering) and finally at \( k \)-th step one gets a solution in \( U_k \) which is equal to \( x_0 \).

References

1. Jeffrey. C. Lagarias, The 3x+1 Problem and its Generalizations, The American Mathematical Monthly, Vol. 92, No.1, pp. 3-23, 1985.