THE ASYMPTOTIC NUMBER OF PLANAR, SLIM, SEMIMODULAR LATTICE DIAGRAMS

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Abstract. A lattice $L$ is slim if it is finite and the set of its join-irreducible elements contains no three-element antichain. We prove that there exists a positive constant $C$ such that, up to similarity, the number of planar diagrams of these lattices of size $n$ is asymptotically $C \cdot 2^n$.

1. Introduction and the result

A finite lattice $L$ is slim if $\text{Ji}(L)$, the set of join-irreducible elements of $L$, contains no three-element antichain. Equivalently, $L$ is slim if $\text{Ji}(L)$ is the union of two chains.

Slim, semimodular lattices were heavily used while proving a recent generalization of the classical Jordan-Hölder theorem for groups in [4]. These lattices are planar, that is, they have planar diagrams, see [4]. Hence it is reasonable to study their planar diagrams, which are called slim, semimodular (lattice) diagrams for short.

The size of a diagram is the number of elements of the lattice it represents. Let $D_1$ and $D_2$ be two planar lattice diagrams. A bijection $\varphi : D_1 \to D_2$ is a similarity map if it is a lattice isomorphism preserving the left-right order of (upper) covers and that of lower covers of each element of $D_1$. If there is a similarity map $D_1 \to D_2$, then these two diagrams are similar, and we will treat them as equal ones. Let $N_{ssl}(n)$ denote the number of slim, semimodular diagrams of size $n$, counting them up to similarity.

Our target is to prove the following result.

Theorem 1.1. There exists a constant $C$ such that $0 < C < 1$ and $N_{ssl}(n)$ is asymptotically $C \cdot 2^n$, that is, $\lim_{n \to \infty} (N_{ssl}(n) / 2^n) = C$.

Note that there are two different methods to deal with $N_{ssl}(n)$. The present one yields the asymptotic statement above, while the method of [1] gives the exact values of $N_{ssl}(n)$ up to $n = 50$ (with the help of a usual personal computer). Also, [1] determines the number $N_{ssl}(n)$ of slim, semimodular lattices of size $n$ up to $n = 50$ while we do not even know $\lim_{n \to \infty} (N_{ssl}(n) / N_{ssl}(n - 1))$, and it is only a conjecture that this limit exists.

Note also that, besides [1] and [2], there are several papers on counting lattices; see, for example, M. Erné, J. Heitzig, and J. Reinhold [7], M. M. Pawar and B. N. Waphare [11], and J. Heitzig and J. Reinhold [9].
2. Lattice theoretic lemmas

A minimal non-chain region of a planar lattice diagram \( D \) is called a cell, a four-element cell is a 4-cell; it is also a covering square, that is, cover-preserving four-element Boolean sublattice. We say that \( D \) is a 4-cell diagram if all of its cells are 4-cells. We shall heavily rely on the following result of G. Grätzer and E. Knapp [8, Lemmas 4 and 5].

**Lemma 2.1.** Let \( D \) be a finite, planar lattice diagram.

(i) If \( D \) is semimodular, then it is a 4-cell diagram. If \( A \) and \( B \) are 4-cells of \( D \) with the same bottom, then these 4-cells have the same top.

(ii) If \( D \) is a 4-cell diagram in which no two 4-cells with the same bottom have distinct tops, then \( D \) is semimodular.

In what follows, we always assume that \( 4 \leq n \in \mathbb{N} = \{1, 2, \ldots \} \), and that \( D \) is a slim, semimodular diagram of size \( n \). Let \( w^f_D \) be the smallest doubly irreducible element of the left boundary chain \( BC^f(D) \) of \( D \), and let \( \text{rank}^l(D) \) be the height of \( w^f_D \). The left-right duals of these concepts are denoted by \( w^r_D \) and \( \text{rank}^r(D) \). See Figure 1 for an illustration, where \( w^f_D \) and \( w^r_D \) are the black-filled elements. By D. Kelly and I. Rival [10, Proposition 2.2], each planar lattice diagram with at least three elements contains a doubly irreducible element \( \neq 0,1 \) on its left boundary. This implies the following statement, on which we will rely implicitly.

**Lemma 2.2.** Either \( \text{rank}^l(D) = \text{rank}^r(D) = 0 \) and \( w^f_D = w^r_D = 0 \), or \( \text{rank}^l(D) > 0 \) and \( \text{rank}^r(D) > 0 \).

For a \( a \in D \), the ideal \( \{x \in D : x \leq a\} \) is denoted by \( \downarrow a \).

**Lemma 2.3.** \( BC^f(D) \cap \downarrow w^f_D \subseteq \text{Ji}D \).

**Proof.** Suppose, for a contradiction, that the lemma fails, and let \( u \) be the smallest join-reducible element belonging to \( BC^f(D) \cap \downarrow w^f_D \). By D. Kelly and I. Rival [10 Proposition 2.2], there is a doubly irreducible element \( v \) of the ideal \( \downarrow u = \{x \in D : x \leq u\} \) such that \( v \in BC^f(\downarrow u) \); notice that \( v \) also belongs to \( BC^f(D) \). Clearly, \( v < u \) and \( v \) is join-irreducible in \( D \). Therefore, since \( v < u \leq w^f_D \) and \( w^f_D \) is the least doubly irreducible element of \( BC^f(D) \), \( v \) is meet-reducible in \( D \). Hence there exist a \( p \in D \) such that \( v \prec p \) and \( p \notin \downarrow u \). Denote by \( u_0 \) the unique lower cover of \( u \) in \( BC^f(D) \). Since \( v < u \), we have that \( v \leq u_0 \). By semimodularity and \( p \nleq u_0 \), we obtain that \( u_0 = u_0 \lor v \prec u_0 \lor p \neq u \). Hence \( u_0 \) has two covers, \( u \) and \( u_0 \lor p \).

**Figure 1.** Left and right ranks
Thus \( u_0, u \in BC_\ell(D), u_0 \prec u, u \) is join-reducible, and \( u_0 \) is meet-reducible. This contradicts [5, Lemma 4]. \( \square \)

Next, we prove the following lemma.

**Lemma 2.4.** For \( 4 \leq n \in \mathbb{N} \), we have that

\[
N_{\text{ssd}}(n-1) + N_{\text{ssd}}(n-3) \leq N_{\text{ssd}}(n), \tag{2.1}
\]

\[
N_{\text{ssd}}(n) \leq 2 \cdot N_{\text{ssd}}(n-1). \tag{2.2}
\]

**Proof.** The set of slim, semimodular diagrams of size \( n \) is denoted by \( \text{SSD}(n) \). Let

\[
\text{SSD}_0(n) = \{ D \in \text{SSD}(n) : \text{rank}_\ell(D) = \text{rank}_r(D) = 0 \},
\]

\[
\text{SSD}_1(n) = \{ D \in \text{SSD}(n) : \text{rank}_\ell(D) = \text{rank}_r(D) = 1 \}, \text{ and}
\]

\[
\text{SSD}_+(n) = \text{SSD}(n) - \text{SSD}_0(n).
\]

Since we can omit the least element and the least three elements, respectively, and the remaining diagram is still slim and semimodular by Lemma 2.1, we conclude that \( |\text{SSD}_0(n)| = N_{\text{ssd}}(n-1) \) and \( |\text{SSD}_1(n)| = N_{\text{ssd}}(n-3) \). This implies (2.1). For \( D \in \text{SSD}_+(n) \), we define

\[
D^* = D - \{ w_D^\ell \}.
\]

We know from By D. Kelly and I. Rival [10, Proposition 2.2], mentioned earlier, that

\[
w_D^\ell \notin \{0, 1\}, \quad \text{provided } D \in \text{SSD}_+(n). \tag{2.3}
\]

This together with the fact that \( D \in \text{SSD}_+(n) \) is not a chain yields that

\[
\text{length } D^* = \text{length } D. \tag{2.4}
\]

Let \( w_D^- \) denote the unique lower cover of \( w_D^\ell \) in \( D \). Since each meet-reducible element has exactly two covers by [5, Lemma 2], we conclude from Lemma 2.3 that

\[
w_D^\ell = w_D^- \tag{2.5}
\]

It follows from Lemma 2.1 that \( D^* \in \text{SSD}(n-1) \). From (2.5) we obtain that

\[
D^* \in \text{SSD}(n-1) \text{ determines } D. \tag{2.6}
\]

Hence \( |\text{SSD}_+(n)| \leq |\text{SSD}(n-1)| = N_{\text{ssd}}(n-1) \). Combining this with \( |\text{SSD}_0(n)| = N_{\text{ssd}}(n-1) \) and \( \text{SSD}(n) = \text{SSD}_0(n) \cup \text{SSD}_+(n) \), where \( \cup \) stands for disjoint union, we obtain (2.2). \( \square \)

Next, let

\[
W(n) = \text{SSD}(n-1) - \{ D^* : D \in \text{SSD}_+(n) \}.
\]

This is the “wrong” set from our perspective since \( W(n) = \emptyset \), which is far from reality, would turn inequality (2.2) into an equality. Fortunately, this set is relatively small by the following lemma. The upper integer part of a real number \( r \) is denoted \( \lceil x \rceil \), for example, \( \lceil \sqrt{2} \rceil = 2 \).

**Lemma 2.5.** If \( 4 \leq n \), then \( |W(n)| \leq \sum_{j=2}^{n+1 - \lceil \sqrt{n-1} \rceil} N_{\text{ssd}}(j) \).
Using semimodularity, we obtain that
\[(2.7)\]
\[W(n) = \{ E \in SSD(n-1) : w^E_E \text{ is a coatom of } E \} \]
The \(\subseteq\) inclusion is clear from (2.3), (2.4), and (2.5). These facts together with Lemma 2.1 imply the reverse inclusion since by adding a new cover to \(w^E_E\), to be positioned to the left of \(BC_E(E)\), we obtain a slim, semimodular diagram \(D\) such that \(D^* = E\).

It follows from Lemma 2.3 that no down-going chain starting at \(BC_E(E)\) can branch out. Thus
\[(2.8)\]
\[\downarrow w^E_E \subseteq BC_E(E) \text{ and } \downarrow w^E_E \text{ is a chain.} \]
Since \(w^E_E\) is a coatom, we have that
\[(2.9)\]
\[\text{with the notation } E^\bullet = E \setminus \downarrow w^E_E, \quad |E^\bullet| = |E| - \text{length } E. \]
Clearly, \(E^\bullet\) is a join-subsemilattice of \(E\) since it is an order-filter. To prove that
\[(2.10)\]
\[E^\bullet \text{ is a slim, semimodular diagram,} \]
assume that \(x, y \in E^\bullet - \{1\}\). We want to show that \(x \land y\), taken in \(E\), belongs to \(E^\bullet\). Let \(x_0 \) and \(y_0\) be the smallest element of \(BC_y(E) \cap \downarrow x\) and \(BC_x(E) \cap \downarrow y\), respectively. Since \(x_0, y_0 \in BC_y(E) \cap \{w^E_E \setminus \{w^E_E\}\}\), (2.10) implies that \(x_0\) and \(y_0\) are meet-reducible. Hence they have exactly two covers by [5 Lemma 2]. Let \(x_1\) and \(y_1\) denote the cover of \(x_0\) and \(y_0\), respectively, that do not belong to \(BC_y(E)\), and let \(x^+\) and \(y^+\) be the respective covers belonging to \(BC_y(E)\). By the choice of \(x_0\), we have that \(x^+ \not\leq x\), whence \(x_1 \leq x\). Similarly, \(y_1 \leq y\). Since \(BC_y(E)\) is a chain and the case \(x_0 = y_0\) will turn out to be trivial, we can assume that \(x_0 < y_0\). We know that \(x_1 \not\leq y_0\) since otherwise \(x_1\) would belong to \(BC_y(E)\) by (2.8). Using semimodularity, we obtain that \(x_1 \lor y_0 > y_0\). Since \(y_0\) has only two covers by [5 Lemma 2] and \(x_1 \leq y^+\) would imply \(x_1 \in BC_y(E)\) by (2.8), it follows that \(x_1 \lor y_0 = y_1\). Hence \(x_1 \leq y, x_1 \leq x,\) and \(x_1 \in E^\bullet\) implies that \(x \land y\) belong to (the order filter) \(E^\bullet\). Thus \(E^\bullet\) is (to be more precise, determines) a sublattice of (the lattice determined by) \(E\). The semimodularity of \(E^\bullet\) follows from Lemma 2.1.

This proves (2.10).

By (2.9), (2.10), by a trivial argument,
\[(2.11)\]
\[E^\bullet \in SSD(n-\text{length } E) \text{ and } E^\bullet \text{ determines } E. \]

Next, we have to determine what values \(h = \text{length } E\) can take. Clearly, \(h \leq |E| - 1 = n - 2\). There are various ways to check that \(|E| \leq (1+\text{length } E)^2 = (1+h)^2\); this follows from the main theorem of [6], and follows also from the proof of [3 Corollary 2]. Since now \(|E| = n - 1\), we obtain that \(\lceil \sqrt{n - 1} \rceil - 1 \leq h\). Therefore, combining (2.10), (2.11), we obtain that
\[W(n) \leq \sum_{h=\lceil \sqrt{n - 1} \rceil - 1}^{n-2} N_{\text{sead}}(n-h). \]

Substituting \(j\) for \(n-h\) we obtain our statement.

We conclude this section by the following lemma.

**Lemma 2.6.**
\[2 \cdot N_{\text{sead}}(n - 1) - \sum_{j=2}^{n+1-\lceil \sqrt{n - 1} \rceil} N_{\text{sead}}(j) \leq N_{\text{sead}}(n) \leq 2 \cdot N_{\text{sead}}(n - 1). \]
Figure 2. An illustration to Lemma 3.1

Proof. By (2.6) and the definition of $W(n)$, we have that

$$N_{ssd}(n) = |SSD_{00}(n)| + |SSD_{++}(n)| = N_{ssd}(n-1) + |SSD(n-1) - W(n)|$$

$$= N_{ssd}(n-1) + N_{ssd}(n-1) - |W(n)|,$$

and the statement follows from Lemma 2.5 and (2.2). □

3. Tools from Analysis at work

For $k \geq 2$, define $\kappa_k = N_{ssd}(k)/N_{ssd}(k-1)$. Since $N_{ssd}(n-3)/N_{ssd}(n-1) = 1/(\gamma_{n-1}\gamma_{n-2})$, dividing the inequalities of Lemma 2.4 by $N_{ssd}(n-1)$ we obtain that $1 + 1/(\kappa_{n-1}\kappa_{n-2}) \leq \kappa_n \leq n$, for $n \geq 4$. Therefore, since $\kappa_k \leq 2$ also holds for $k \in \{2, 3\}$ and $1 + 1/(2 \cdot 2) = 5/4$, we conclude that

$$\frac{5}{4} \leq \kappa_n \leq 2, \quad \text{for } n \geq 4.$$

Clearly, $N_{ssd}(k-1) = N_{ssd}(k)/\kappa_n \leq \frac{1}{2}N_{ssd}(k)$ if $k \geq 4$. Thus, by iteration, we obtain that

$$N_{ssd}(k-j) \leq (4/5)^j \cdot N_{ssd}(k), \quad \text{for } j \in \mathbb{N}_0 \text{ and } k \geq j + 4.$$

If $k \geq 5$, then using $N_{ssd}(k) \geq N_{ssd}(5) \geq 3$ (actually, $N_{ssd}(5) = 3$), we obtain that

$$N_{ssd}(1) + \cdots + N_{ssd}(k) = 1 + 1 + 1 + N_{ssd}(4) + \cdots + N_{ssd}(k)$$

$$\leq 3 + N_{ssd}(k) \cdot ((4/5)^{k-4} + (4/5)^{k-5} + \cdots + (4/5)^0)$$

$$\leq N_{ssd}(k) + N_{ssd}(k) \cdot 1/(1 - 4/5) = 6N_{ssd}(k).$$

Combining Lemma 2.6 with 3.3 and 3.2 we obtain that

$$2N_{ssd}(n-1) - 6 \cdot (4/5)^{\lceil \sqrt{n-1} \rceil - 2} \cdot N_{ssd}(n-1) \leq$$

$$2N_{ssd}(n-1) - 6N_{ssd}(n + 1 - \lceil \sqrt{n-1} \rceil)$$

$$\leq N_{ssd}(n) \leq 2N_{ssd}(n-1).$$

Dividing the formula above by $2N_{ssd}(n-1)$ and (3.1) by 2, we obtain that

$$\max\left(\frac{5}{8}, 1 - 3 \cdot (4/5)^{\lceil \sqrt{n-1} \rceil - 2}\right) \leq \kappa_n/2 \leq 1, \quad \text{for } n \geq 5.$$

Next, let us choose an integer $m \geq 5$, and define

$$z_0 = z_0(m) = \min\left(3/8, 3 \cdot (4/5)^{\lceil \sqrt{m-1} \rceil - 2}\right).$$
Lemma 3.1. If $0 < z \leq z_0$, then
\[- \ln(1 - z) \leq z/(1 - z) \leq z/(1 - z_0).\]

Proof. The statement follows from $\ln'(1 - z) = 1/(1 - z)$ and the similarity of the triangle $ABT$ to the triangle $A'B'T$, see Figure 2.

With the auxiliary steps made so far, we are ready to start the final argument.

Proof of Theorem 1.1. For $n > m$, let
\[ p_n = \prod_{j=m+1}^{n} (\kappa_j/2). \]

Clearly,
\[ N_{\text{ssd}}(n)/2^m = p_n \cdot N_{\text{ssd}}(m)/2^m. \]

Hence it suffices to prove that the sequence $\{p_n\}$, that is $\{p_n\}_{n=m+1}^\infty$, is convergent. Let $s_n = -\ln p_n$, $\mu = 3(1 - z_0)^{-1}$, $\alpha = 4/5$, and $\nu = 5\mu/4 = \mu/\alpha$. Then, using (3.4) together with Lemma 3.1, we obtain that $\leq^*$, and using that the function $f(x) = \alpha \sqrt{x} - 1$ is decreasing, we obtain that
\[ 0 < s_n = \sum_{j=m+1}^{n} (-\ln(\kappa_j/2)) \leq^* \sum_{j=m+1}^{n} (1 - \kappa_j/2)/(1 - z_0) \]
\[ \leq^* \mu \cdot \sum_{j=m+1}^{n} \alpha \sqrt{(j - 1) - 2} \leq \mu \cdot \sum_{j=m+1}^{n} \alpha \sqrt{j - 1} \leq \mu \cdot \sum_{k=m}^{n-1} \alpha \sqrt{k - 1} \]
\[ = \nu \cdot \sum_{k=m}^{n-1} \alpha \sqrt{k} \leq \nu \cdot \int_{x=m-1}^{n-1} \alpha \sqrt{x} \, dx \leq \nu \cdot (F(\infty) - F(m - 1)), \]

where $F(x)$ is a primitive function of $f(x)$. Let $\delta = -\ln \alpha = \ln (5/4)$. It is routine to check (by hand or by computer algebra) that, up to a constant summand,
\[ F(x) = -2 \cdot \delta^{-2} \cdot (1 + \delta \sqrt{x}) \cdot \alpha \sqrt{x}. \]

Clearly, $F(\infty) = \lim_{x \to \infty} F(x) = 0$. This proves that the sequence $\{s_n\}$ converges; and so does $\{p_n\} = \{e^{-s_n}\}$ by the continuity of the exponential function. Therefore, since $N_{\text{ssd}}(m)/2^m$ in (3.5) does not depend on $m$, we conclude Theorem 1.1.

Remark 3.2. We can approximate the constant in Theorem 1.1 as follows. Since $e^{-\nu \cdot (F(\infty) - F(m))} \leq e^{-s_n} = p_n \leq 1$ and, by (3.5), $C = \lim_{n \to \infty} (p_n N_{\text{ssd}}(m)/2^m)$, we obtain that
\[ e^{\nu F(m)} \cdot N_{\text{ssd}}(m)/2^m = e^{-\nu \cdot (F(\infty) - F(m))} \cdot N_{\text{ssd}}(m)/2^m \leq C \leq N_{\text{ssd}}(m)/2^m. \]

Unfortunately, our computing power yields only a very rough estimation. The largest $m$ such that $N_{\text{ssd}}(50)$ is known is $m = 50$, see [4]. With $m = 50$ and $N_{\text{ssd}}(50) = N_{\text{ssd}}(50) = 81 287 566 224 125$, it is a routine task to turn (3.6) into
\[ 0.42 \cdot 10^{-57} \leq C \leq 0.073. \]

We have reasons (but no proof) to believe that $0.023 \leq C \leq 0.073$, see the Maple worksheet (version V) available from the authors’s home page.

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