An epiperimetric inequality approach to the regularity of the free boundary in the Signorini problem with variable coefficients

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Abstract

In this paper we establish the $C^{1,\beta}$ regularity of the regular part of the free boundary in the Signorini problem for elliptic operators with variable Lipschitz coefficients. This work is a continuation of the recent paper [GSVG14], where two of us established the interior optimal regularity of the solution. Two of the central results of the present work are a new monotonicity formula and a new epiperimetric inequality.

Résumé

Dans cet article nous établissons la régularité $C^{1,\beta}$ de la partie régulière de la frontière libre dans le problème de Signorini pour des opérateurs elliptiques à coefficients variables Lipschitz. Ce travail est une continuation de l’étude récente [GSVG14], dans laquelle deux d’entre nous ont établi la régularité inférieure optimale de la solution. Une nouvelle formule de monotonie et une nouvelle inégalité épipérimétrique présentent deux résultats centraux de cet article.

Keywords: Signorini problem, Regularity of free boundaries, Weiss-type monotonicity formula, Epiperimetric inequality

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1. Introduction

1.1. Statement of the problem and main assumptions

The purpose of the present paper is to establish the $C^{1,\beta}$ regularity of the free boundary near so-called regular points in the Signorini problem for elliptic operators with variable Lipschitz coefficients. Although this work represents a continuation of the recent paper [GSVG14], where two of us established the interior optimal regularity of the solution, proving the regularity of the free boundary has posed some major new challenges. Two of the central results of the present work are a new monotonicity formula (Theorem 4.3) and a new epiperimetric inequality (Theorem 6.3). Both of these results have been inspired by those originally obtained by Weiss in [Wei99] for the classical obstacle problem, but the adaptation to the Signorini problem has required a substantial amount of new ideas.

The lower-dimensional (or thin) obstacle problem consists of minimizing the (generalized) Dirichlet energy

$$\min_{u \in K} \int_{\Omega} \langle A(x) \nabla u, \nabla u \rangle \, dx,$$

(1.1)

where $u$ ranges in the closed convex set

$$K = K_{g,\varphi} = \{ u \in W^{1,2}(\Omega) \mid u = g \text{ on } \partial\Omega, \: u \geq \varphi \text{ on } M \cap \Omega \}.$$  

Here, $\Omega \subset \mathbb{R}^n$ is a given bounded open set, $M$ is a codimension one manifold which separates $\Omega$ into two parts, $g$ is a boundary datum and the function $\varphi : M \to \mathbb{R}$ represents the lower-dimensional, or thin, obstacle. The functions $g$ and $\varphi$ are required to satisfy the standard compatibility condition $g \geq \varphi$ on $\partial\Omega \cap M$. This problem is known also as (scalar) Signorini problem, as the minimizers satisfy Signorini conditions on $M$ (see (1.7)–(1.9) below in the case of flat $M$).

Our assumptions on the matrix-valued function $x \mapsto A(x) = [a_{ij}(x)]$ in (1.1) are that $A(x)$ is symmetric, uniformly elliptic, and Lipschitz continuous (in short $A \in C^{0,1}$). Namely:

$$a_{ij}(x) = a_{ji}(x) \quad \text{for } i, j = 1, \ldots, n, \text{ and every } x \in \Omega;$$

(1.2)

there exists $\lambda > 0$ such that for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$, one has

$$\lambda |\xi|^2 \leq \langle A(x) \xi, \xi \rangle \leq \lambda^{-1} |\xi|^2;$$

(1.3)

there exists $Q \geq 0$ such that

$$|a_{ij}(x) - a_{ij}(y)| \leq Q |x - y|, \quad x, y \in \Omega.$$

(1.4)

By standard methods in the calculus of variations it is known that, under appropriate assumptions on the data, the minimization problem (1.1) admits a unique solution $u \in K$, see e.g. [Fri88], or also [Tro87]. The set

$$\Lambda^{\varphi}(u) = \{ x \in M \cap \Omega \mid u(x) = \varphi(x) \}$$

is known as the coincidence set, and its boundary (in the relative topology of $M$)

$$\Gamma^{\varphi}(u) = \partial_M \Lambda^{\varphi}(u)$$

is known as the free boundary. In this paper we are interested in the local regularity properties of $\Gamma^{\varphi}(u)$. When $\varphi = 0$, we will write $\Lambda(u)$ and $\Gamma(u)$, instead of $\Lambda^0(u)$ and $\Gamma^0(u)$. 


We also note that since we work with Lipschitz coefficients, it is not restrictive to consider the situation in which the thin manifold is flat, which we take to be \( M = \{ x_n = 0 \} \). We thus consider the Signorini problem (1.1) when the thin obstacle \( \varphi \) is defined on \( B'_1 = M \cap B_1 = \{ (x',0) \in B_1 \mid |x'| < 1 \} \), which we call the thin ball in \( B_1 \). In this case we will also impose the following conditions on the coefficients
\[
a_{in}(x',0) = 0 \quad \text{in } B'_1, \quad \text{for } i < n, \tag{1.5}\]
which essentially means that the conormal directions \( A(x',0)\nu_\pm \) are the same as normal directions \( \nu_\pm = \mp e_n \). We stress here that the condition (1.5) is not restrictive as it can be satisfied by means of a \( C^{1,1} \) transformation of variables, as proved in Appendix B of [GSVG14].

We assume that \( \varphi \in C^{1,1}(B'_1) \) and denote by \( u \) the unique solution to the minimization problem (1.1). (Notice that, by letting \( \tilde{\varphi} \equiv \varphi(x') \), we can think of \( \varphi \in C^{1,1}(B_1) \), although we will not make such distinction explicitly.) Such \( u \) satisfies
\[
Lu = \text{div}(A\nabla u) = 0 \quad \text{in } B_1^+ \cup B_1^-, \tag{1.6}
\]
\[
u \geq 0 \quad \text{in } B'_1, \tag{1.7}
\]
\[
(A\nabla u, \nu_+) + (A\nabla u, \nu_-) \geq 0 \quad \text{in } B'_1, \tag{1.8}
\]
\[
(u - \varphi)(A\nabla u, \nu_+) + (A\nabla u, \nu_-) = 0 \quad \text{in } B'_1. \tag{1.9}
\]
The conditions (1.7)–(1.9) are known as Signorini or complementarity conditions. It has been recently shown in [GSVG14] that, under the assumptions above, the unique solution of (1.6)–(1.9) is in \( C^{1,1/2}(B_1^+ \cup B'_1) \). This regularity is optimal since the function \( u(x) = \Re(x_1 + i|x_n|)^{3/2} \) solves the Signorini problem for the Laplacian and with thin obstacle \( \varphi \equiv 0 \).

Henceforth, we assume that \( 0 \) is a free boundary point, i.e., \( 0 \in \Gamma^\varphi(u) \), and we suppose without restriction that \( A(0) = I \). Under these hypothesis, we consider the following normalization of \( u \)
\[
v(x) = u(x) - \varphi(x') + bx_n, \quad b = \partial_{x_n}, u(0). \tag{1.10}
\]
Note that \( \partial_{x_n} u(0) + \partial_{x_n} u(0) = 0 \), which follows from (1.9), by taking a limit from inside the set \( \{ (x',0) \in B'_1 \mid u(x',0) > \varphi(x') \} \), and from the hypothesis \( A(0) = I \). This implies that
\[
v(0) = |\nabla v(0)| = 0. \tag{1.11}
\]
Next, note that, in view of (1.4) above and of the assumption \( \varphi \in C^{1,1}(B_1) \), we have
\[
-L(\varphi(x') - bx_n) \overset{\text{def}}{=} f \in L^\infty(B_1). \tag{1.12}
\]
Hence, we can rewrite (1.6)–(1.9) in terms of \( v \) as follows:
\[
Lv = \text{div}(A\nabla v) = f \quad \text{in } B_1^+ \cup B_1^-, \quad f \in L^\infty(B_1), \tag{1.12}
\]
\[
v \geq 0 \quad \text{in } B'_1, \tag{1.13}
\]
\[
(A\nabla v, \nu_+) + (A\nabla v, \nu_-) \geq 0 \quad \text{in } B'_1, \tag{1.14}
\]
\[
v(A\nabla v, \nu_+) + (A\nabla v, \nu_-) = 0 \quad \text{in } B'_1. \tag{1.15}
\]
Note that from (1.12) we have
\[
\int_{B_1} ((A\nabla v, \nabla \eta) + f \eta) = \int_{B'_1} ((A\nabla v, \nu_+) + (A\nabla v, \nu_-)) \eta, \quad \eta \in C_0^\infty(B_1). \tag{1.16}
\]
and thus the conditions (1.13)–(1.15) imply that $v$ satisfies the variational inequality

$$\int_{B_1} (\langle A \nabla v, \nabla (w-v) \rangle + f(w-v)) \geq 0, \quad \text{for any } w \in \mathcal{K}_{v,0},$$

where $\mathcal{K}_{v,0} = \{ w \in W^{1,2}(B_1) \mid w = v \text{ on } \partial B_1, w \geq 0 \text{ on } B_1^c \}$. Since $\mathcal{K}_{v,0}$ is convex, and so is the energy functional

$$\int_{B_1} (\langle A \nabla w, \nabla w \rangle + 2fw),$$

this is equivalent to saying that $v$ minimizes (1.17) among all functions $w \in \mathcal{K}_{v,0}$. Notice as well that

$$\Lambda^v(u) = \{ u(\cdot,0) = \varphi \} = \{ v(\cdot,0) = 0 \} = \Lambda^0(v) = \Lambda(v).$$

We will write $\Gamma^v(u) = \Gamma^0(v) = \Gamma(v)$, and thus $0 \in \Gamma(v)$ now and (1.11) holds.

1.2. Main result

To state the main result of this paper, we need to further classify the free boundary points. This is achieved by means of the truncated frequency function

$$N(r) = N_L(v, r) \overset{\text{def}}{=} \frac{\sigma(r)}{2} e^{K' \frac{r}{\mu}} \frac{d}{dr} \log \max \left\{ \frac{1}{\sigma(r)r^{n-2}} \int_{S_r} v^2 \mu, r^{3+\delta} \right\}.$$

Here, $\mu(x) = \langle A(x)x, x \rangle / |x|^2$ is a conformal factor, $\sigma(r)$ is an auxiliary function with the property that $\sigma(r)/r \to \alpha > 0$ as $r \to 0$, $0 < \delta < 1$, and $K'$ is a universal constant (see Section 2 for exact definitions and properties). This function was introduced in [GSVG14], and represents a version of Almgren’s celebrated frequency function (see [Alm79]), adjusted for the solutions of (1.12)–(1.15). By Theorem 2.4 in [GSVG14], $N(r)$ is monotone increasing and hence the limit

$$\hat{N}(0+) = \lim_{r \to 0} \tilde{N}(r), \quad \text{where } \tilde{N}(r) = \frac{r}{\sigma(r)} N(r)$$

exists. The remarkable fact is that either $\hat{N}(0+) = 3/2$, or $\hat{N}(0+) \geq (3 + \delta)/2$, see Lemma 2.5 below. This leads to the following definition.

**Definition 1.1.** We say that $0 \in \Gamma(v)$ is a regular point if $\hat{N}(0+) = 3/2$. Shifting the origin to $x_0 \in \Gamma(v)$, and denoting the corresponding frequency function by $N_{x_0}$, we define

$$\Gamma_{3/2}(v) = \{ x_0 \in \Gamma(v) \mid \hat{N}_{x_0}(0+) = 3/2 \},$$

the set of all regular free boundary points, also known as the regular set.

The remaining part of the free boundary is divided into the sets $\Gamma_\kappa(v)$, according to the corresponding value of $\hat{N}(0+) \overset{\text{def}}{=} \kappa$. We note that the range of possible values for $\kappa$ can be further refined, provided more regularity is known for the coefficients $A(x)$. This can be achieved by replacing the truncation function $r^{3+\delta}$ in the formula for $N(r)$ with higher powers of $r$, similarly to what was done in [GP09] in the case of the Laplacian. This will provide more information on the set of possible values of $\hat{N}(0+)$ which will serve as a classification parameter.

The following theorem is the central result of this paper.

**Theorem 1.2.** Let $v$ be a solution of (1.12)–(1.15) with $x_0 \in \Gamma_{3/2}(v)$. Then, there exists $\eta_0 > 0$, depending on $x_0$, such that, after a possible rotation of coordinate axes in $\mathbb{R}^{n-1}$, one has $B'_{\eta_0}(x_0) \cap \Gamma(v) \subset \Gamma_{3/2}(v)$, and

$$B'_{\eta_0} \cap \Lambda(v) = B'_{\eta_0} \cap \{ x_{n-1} \leq g(x_1, \ldots, x_{n-2}) \}$$

for $g \in C^{1,\beta}(\mathbb{R}^{n-2})$ with a universal exponent $\beta \in (0,1)$. 


This result is known and well-understood in the case $L = \Delta$, see [ACS08] or Chapter 9 in [PSU12]. However, the existing proofs are based on differentiating the equation for $v$ in tangential directions $e \in \mathbb{R}^{n-1}$ and establishing the nonnegativity of $\partial_x v$ in a cone of directions, near regular free boundary points (directional monotonicity). This implies the Lipschitz regularity of $\Gamma_{3/2}(v)$, which can be pushed to $C^{1,\beta}$ with the help of the boundary Harnack principle. The idea of the directional monotonicity goes back at least to the paper [Alt77], while the application of the boundary Harnack principle originated in [AC85]; see also [Ca98] and the book [PSU12]. In the case $L = \Delta$, we also want to mention two recent papers that prove the smoothness of the regular set: Koch, the second author, and Shi [KPS14] establish the real analyticity of $\Gamma_{3/2}$ by using hodograph-type transformation and subelliptic estimates, and De Silva and Savin [DSS14b] prove $C^\infty$ regularity of $\Gamma_{3/2}$ by higher-order boundary Harnack principle in slit domains.

Taking directional derivatives, however, does not work well for the problem studied in this paper, particularly so since we are working with solutions of the non-homogeneous equation (1.12), which corresponds to nonzero thin obstacle $\phi$. In contrast, the methods in this paper are purely energy based, and they are new even in the case of the thin obstacle problem for the Laplacian. They are inspired by the homogeneity improvement approach of Weiss [Wei99] in the classical obstacle problem, which consists of a combination of a monotonicity formula and an epiperimetric inequality. In this connection we mention that recently in [FGS13], Focardi, Gelli, and Spadaro extended Weiss’ method to the classical obstacle problem for operators of a monotonicity formula and an epiperimetric inequality. In this connection we mention that recently in [FGS13], Focardi, Gelli, and Spadaro extended Weiss’ method to the classical obstacle problem for operators with Lipschitz coefficients. We also mention a recent preprint by Koch, Rüland, and Shi [KRS15a] in which they use Carleman estimates to establish the almost optimal interior regularity of the solution in the variable coefficient Signorini problem, when the coefficient matrix is $W$, with Lipschitz coefficients. We mention that the geometric meaning of the functional above is that it measures the closeness of the solution $v$ to the prototypical homogeneous solutions of degree $3/2$, i.e., the functions

$$a\mathcal{R}(\langle x', \nu \rangle + i|x_n|)^{3/2}, \ a \geq 0, \ \nu \in S_1.$$ 

The second central ingredient in the proof is the epiperimetric inequality for the functional $W(v) = W_{\Delta}(v, 1) = \int_{B_1} |\nabla v|^2 - \frac{3}{2} \int_{S_1} v^2$, which states that if $a (3/2)$-homogeneous function $w$, nonnegative on $B_1'$, is close to the solution $h(x) = \mathcal{R}(x_1 + i|x_n|)^{3/2}$ in $W^{1,2}(B_1)$-norm, then there exists $\zeta$ in $B_1$ with $\zeta = w$ on $\partial B_1$ such that $W(\zeta) \leq (1 - \kappa)W(w)$, for a universal $0 < \kappa < 1$, see Theorem 6.3 below.

The combination of Theorems 4.3 and 6.3 provides us with a powerful tool for establishing the following geometric rate of decay for the Weiss functional:

$$W_L(v, r) \leq Cr^\gamma,$$
for a universal $\gamma > 0$. In turn, this ultimately implies that

$$\int_{S_1'} |v_{x,0} - v_{y,0}| \leq C|\bar{x} - \bar{y}|^\beta$$

for properly defined homogeneous blowups $v_{x,0}$ and $v_{y,0}$ at $\bar{x}, \bar{y} \in \Gamma_{3/2}(v)$. This finally implies the $C^{1,\beta}$ regularity of $\Gamma_{3/2}(v)$ in a more or less standard fashion.

1.3. Structure of the paper

The paper is organized as follows.

- In Section 2 we recall those definitions and results from [GSVG14] which constitute the background results of this paper.
- In Section 3 we give a more in depth look at regular free boundary points and prove some preliminary but important properties such as the relative openness of the regular set $\Gamma_{3/2}(v)$ in $\Gamma(v)$ and the local uniform convergence of the truncated frequency function $\tilde{N}(r) \to 3/2$ on $\Gamma_{3/2}(v)$. We also introduce Almgren type scalings for the solutions, see (3.2), which play an important role in Section 7.
- In Section 4 we establish the first main technical tool of this paper, the Weiss-type monotonicity formula discussed above (Theorem 4.3). This result is instrumental to studying the homogeneous blowups of our function, which we do in Section 5.
- Section 6 is devoted to proving the second main technical result of this paper, the epiperimetric inequality (Theorem 6.3) which we have discussed above.
- Finally, in Section 7 we combine the monotonicity formula and the epiperimetric inequality to prove the main result of this paper, the $C^{1,\beta}$-regularity of the regular set $\Gamma_{3/2}(v)$ (Theorem 1.2).

2. Notation and preliminaries

2.1. Basic notation

Throughout the paper we use following notation. We work in the Euclidean space $\mathbb{R}^n$, $n \geq 2$. We write the points of $\mathbb{R}^n$ as $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Very often, we identify the points $(x', 0)$ with $x'$, thus identifying the “thin” space $\mathbb{R}^{n-1} \times \{0\}$ with $\mathbb{R}^{n-1}$.

For $x \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}$ and $r > 0$, we define the “solid” and “thin” balls

$$B_r(x) = \{y \in \mathbb{R}^n \mid |x - y| < r\}, \quad B_r'(x') = \{y' \in \mathbb{R}^{n-1} \mid |x' - y'| < r\}$$

as well as the corresponding spheres

$$S_r(x) = \{y \in \mathbb{R}^n \mid |x - y| = r\}, \quad S_r'(x') = \{y' \in \mathbb{R}^{n-1} \mid |x' - y'| = r\}.$$

We typically do not indicate the center, if it is the origin. Thus, $B_r = B_r(0)$, $S_r = S_r(0)$, etc. We also denote

$$B_+^r(x', 0) = B_r(x', 0) \cap \{\pm x_n > 0\}, \quad \mathbb{R}_+^n = \mathbb{R}^n \cap \{\pm x_n > 0\}.$$

For a given direction $e$, we denote the corresponding directional derivative by

$$\partial_e u = (\nabla u, e),$$

whenever it makes sense. For the standard coordinate directions $e = e_i$, $i = 1, \ldots, n$, we also abbreviate $\partial_i u = \partial_{e_i} u$. 

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In the situation when a domain $\Omega \subset \mathbb{R}^n$ is divided by a manifold $M$ into two subdomains $\Omega_+$ and $\Omega_-$, $\nu_+$ and $\nu_-$ stand for the exterior unit normal for $\Omega_+$ and $\Omega_-$ on $M$. Moreover, we always understand $\partial_{\nu_+} u$ ($\partial_{\nu_-} u$) on $M$ as the limit from within $\Omega_+$ ($\Omega_-$). Thus, when $\Omega_\pm = B_r^\pm$, we have
\[
\nu_\pm = \mp e_n, \quad \partial_{\nu_+} u(x', 0) = \lim_{y \to (x', 0), \ y \in \partial B_r^+} \partial_n u \overset{def}{=} \partial_n^+ u(x', 0), \quad \partial_{\nu_-} u(x', 0) = \lim_{y \to (x', 0), \ y \in \partial B_r^-} \partial_n u \overset{def}{=} \partial_n^- u(x', 0)
\]

In integrals, we often do not indicate the measure of integration if it is the Lebesgue measure on subdomains of $\mathbb{R}^n$, or the Hausdorff $\mathcal{H}^k$ measure on manifolds of dimension $k$.

Hereafter, when we say that a constant is universal, we mean that it depends exclusively on $n$, on the ellipticity bound $\lambda$, and on the Lipschitz bound $Q$ on the coefficients $a_{ij}(x)$. Likewise, we will say that $O(1)$, $O(r)$, etc, are universal if $|O(1)| \leq C$, $|O(r)| \leq Cr$, etc, with $C \geq 0$ universal.

### 2.2. Summary of known results

For the convenience of the reader, in this section we briefly recall the definitions and results proved in [GSVG14] which will be used in this paper.

As stated in Section 1, we work under the nonrestrictive situation in which the thin manifold $M$ is flat. More specifically, we consider the Signorini problem (1.1) when the thin obstacle $\varphi$ is defined on
\[
B_1' = M \cap B_1 = \{(x', 0) \in B_1 \mid |x'| < 1\}.
\]

We assume that $\varphi \in C^{1,1}(B_1')$ and denote by $u$ the unique solution to the minimization problem (1.1), which then satisfies (1.6)–(1.9). We assume that $0$ is a free boundary point, i.e., $0 \in \Gamma^\varphi(u)$, and that $A(0) = I$, and we consider the normalization of $u$ as in (1.10), i.e.,
\[
v(x) = u(x) - \varphi(x') + bx_n, \quad \text{with } b = \partial_{\nu_+} u(0).
\]

As remarked on Section 1, $v$ satisfies (1.12)–(1.15) with $f \overset{def}{=} -L(\varphi(x') - bx_n) \in L^\infty(B_1)$, and has the additional property that $v(0) = |\nabla v(0)| = 0$, see (1.11) above.

We then recall the following definitions from [GSVG14]. The Dirichlet integral of $v$ in $B_r$ is defined by
\[
D(r) = D_{L}(v, r) = \int_{B_r} \langle A(x) \nabla v, \nabla v \rangle,
\]
and the height function of $v$ in $S_r$ is given by
\[
H(r) = H_L(v, r) = \int_{S_r} v^2 \mu,
\]
where $\mu$ is the conformal factor
\[
\mu(x) = \mu_L(x) = \frac{|A(x)x|}{|x|^2}.
\]

We notice that when $A(x) \equiv I$, then $\mu \equiv 1$. We also define the generalized energy of $v$ in $B_r$
\[
I(r) = I_L(v, r) = \int_{S_r} v \langle A \nabla v, \nu \rangle = D_L(v, r) + \int_{B_r} vf
\]
where $\nu$ indicates the outer unit normal to $S_r$. The following result is Lemma 4.4 in [GSVG14].

**Lemma 2.1.** The function $H(r)$ is absolutely continuous, and for a.e. $r \in (0, 1)$ one has
\[
H'(r) = 2I(r) + \int_{S_r} v^2 |x|.
\]
As it was explained in [GSVG14], the second term in the right-hand side of (2.3) above represents a serious difficulty to overcome if one wants to establish the monotonicity of the generalized frequency. To bypass this obstacle, one of the main ideas in [GSVG14] was the introduction of the following auxiliary functions, defined for $v$ satisfying (1.12)–(1.15) and $0 < r < 1$:

$$
\psi(r) = e\int_0^r G(s) \, ds, \quad \sigma(r) = \frac{\psi(r)}{r^{n-2}}, \quad \text{where} \quad G(r) = \begin{cases} 
\frac{\int_{S_r} v^2 |x|}{\int_{S_r} v^2 \mu}, & \text{if } H(r) \neq 0, \\
\frac{n-1}{r}, & \text{if } H(r) = 0.
\end{cases}
$$

(2.4)

When $L = \Delta$, it is easy to see that $\psi(r) = r^{n-1}$ and that $\sigma(r) = r$. We have the following simple and useful lemma which summarizes the most relevant properties of $\psi(r)$ and $\sigma(r)$.

**Lemma 2.2.** There exists a universal constant $\beta \geq 0$ such that

$$
\frac{n-1}{r} - \beta \leq \frac{d}{dr} \log \psi(r) \leq \frac{n-1}{r} + \beta, \quad 0 < r < 1,
$$

(2.5)

and one has

$$
e^{-\beta(1-r)} r^{n-1} \leq \psi(r) \leq e^{\beta(1-r)} r^{n-1}, \quad 0 < r < 1.
$$

(2.6)

This implies, in particular, $\psi(0+) = 0$. In terms of the function $\sigma(r) = \psi(r)/r^{n-2}$ we have

$$
\left| \frac{d}{dr} \log \frac{\sigma(r)}{r} \right| \leq \beta, \quad 0 < r < 1
$$

(2.7)

and

$$
e^{-\beta(1-r)} \leq \frac{\sigma(r)}{r} \leq e^{\beta(1-r)}, \quad 0 < r < 1.
$$

(2.8)

In particular, $\sigma(0+) = 0$.

The next result is essentially Lemma 5.6 from [GSVG14].

**Lemma 2.3.** There exist $\alpha > 0$ such that

$$
\left| \frac{\sigma(r)}{r} - \alpha \right| \leq \beta e^\beta r, \quad r \in (0,1),
$$

(2.9)

for $\beta$ as in Lemma 2.2. In particular,

$$
\alpha = \lim_{r \to 0^+} \frac{\sigma(r)}{r}.
$$

(2.10)

Moreover, we also have that $e^{-\beta} \leq \alpha \leq e^\beta$.

With $\psi$ as in (2.4), we now define

$$
M_L(v, r) = \frac{1}{\psi(r)} H_L(v, r), \quad J_L(v, r) = \frac{1}{\psi(r)} I_L(v, r).
$$

(2.11)

The next relevant formulas are those of $J'(r) = \frac{d}{dr} J_L(v, r)$ and $M'(r) = \frac{d}{dr} M_L(v, r)$. Using formula (5.28) in [GSVG14], we have

$$
J'(r) = \left(-\frac{\psi'(r)}{\psi(r)} + \frac{n-2}{r} + O(1)\right) J(r) + \frac{1}{\psi(r)} \left\{ 2 \int_{S_r} (A \nabla v, v)^2 \mu + \frac{2}{r} \int_{B_r} (Z, \nabla v) f - \left(\frac{n-2}{r} + O(1)\right) \int_{B_r} v f + \int_{S_r} v f \right\},
$$

(2.12)
where the vector field $Z$ is given by
\[ Z = \frac{rA\nabla r}{\mu} = \frac{A(x)x}{\mu}. \] (2.13)

We also recall that (5.26) in [GSVG14] gives
\[ N'(r) = 2J(r). \] (2.14)

The central result in [GSVG14] is the following monotonicity formula.

**Theorem 2.4 (Monotonicity of the truncated frequency).** Let $v$ satisfy (1.11)–(1.15). Given $\delta \in (0, 1)$ there exist universal numbers $r_0, K_0 > 0$, depending also on $\delta$ and $\|f\|_{L^\infty}$, such that the function
\[ N(r) = N_L(v, r) \overset{\text{def}}{=} \frac{\sigma(r)}{2} e^{K_0 r^{\frac{1-\delta}{2}}} \frac{d}{dr} \log \max \{ M_L(v, r), r^{3+\delta} \}. \] (2.15)
is monotone non-decreasing on $(0, r_0]$.

We call $N(r)$ the truncated frequency function, by analogy with Almgren’s frequency function [Alm79] (see [GSVG14, GP09, CSS08] for more insights on this kind of formulas).

We then define a modification of $N$ as follows:
\[ \tilde{N}(r) = \tilde{N}_L(v, r) \overset{\text{def}}{=} \frac{r}{\sigma(r)} N_L(v, r) = \frac{r}{2} e^{K_0 r^{\frac{1-\delta}{2}}} \frac{d}{dr} \log \max \{ M_L(v, r), r^{3+\delta} \}. \]

We notice that by Theorem 2.4 the limit $\tilde{N}(0+)$ exists. Combining that with Lemma 2.3 above, which states that $\lim_{r \to 0^+} \frac{\sigma(r)}{r} = \alpha > 0$, we see that $\tilde{N}(0)$ also exists.

The following lemma provides a summary of estimates which are crucial for our further study. The lower bound on $\tilde{N}(0+)$ is proved in Lemma 6.3 in [GSVG14] (whose proof contains also that of the gap on the possible values of $\tilde{N}(0)$), the bound on $|v(x)|$ is Lemma 6.6 and the bound on $|\nabla v(x)|$ is proved in Theorem 6.7 there.

**Lemma 2.5.** Let $v$ satisfy (1.11)–(1.15) and $r_0 \in (0, 1/2]$ be as in Theorem 2.4. Then, $\tilde{N}(0+) \geq \frac{3}{2}$, and actually $\tilde{N}(0+) = \frac{3}{2}$ or $\tilde{N}(0+) \geq \frac{3+\delta}{2}$.

Moreover, there exists a universal $C$ depending also on $\delta$, $H(r_0)$ and $\|f\|_{L^\infty(B_1)}$ such that
\[ |v(x)| \leq C|x|^{3/2}, \quad |\nabla v(x)| \leq C|x|^{1/2}, \quad |x| \leq r_0. \]
(2.16)

**Corollary 2.6.** With $r_0$ as in Theorem 2.4, one has
\[ H(r) \leq Cr^{n+2}, \quad I(r) \leq Cr^{n+1}, \quad r \leq r_0. \] (2.17)

**Proof.** It is enough to use (2.16) in definitions (2.1) and (2.2) above. \qed

The results of this section have been stated when the free boundary point in question is the origin. However, given any $x_0 \in \Gamma(v)$, we can move $x_0$ to the origin by letting
\[ v_{x_0}(x) = v(x_0 + A^{1/2}(x_0)x) - b_{x_0}x_n, \quad \text{where } b_{x_0} = \langle A^{1/2}(x_0)\nabla v(x_0), e_n \rangle, \]
\[ A_{x_0}(x) = A^{-1/2}(x_0)A(x_0 + A^{1/2}(x_0)x)A^{-1/2}(x_0), \]
\[ \mu_{x_0}(x) = \langle A_{x_0}(x)\nu(x), \nu(x) \rangle, \]
\[ L_{x_0} = \text{div}(A_{x_0} \nabla \cdot). \]
(Note that, by the $C^{1, \frac{1}{2}}$-regularity of $v$ established in [GSVG14], the mapping $x_0 \mapsto b_{x_0}$ is $C^2$ on $\Gamma(v)$.)

Then, by construction, we have the normalizations $A_{x_0}(0) = I_n, \mu_{x_0}(0) = 1$. We also know that $0 \in \Gamma(v_{x_0})$, and that
\[ v_{x_0}(0) = v(x_0) = 0, \quad |\nabla v_{x_0}(0)| = 0. \]
Besides, $v_{x_0}$ satisfies (1.12)–(1.15) for the operator $L_{x_0}$. Thus, all results stated above for $v$ are also applicable to $v_{x_0}$.

We thus also have the versions of the quantities defined in this sections, such as $M_L$, $N_L$, etc, centered at $x_0$ (if we replace $L$ with $L_{x_0}$). Instead of using the overly bulky notations $M_{L_{x_0}}$, $N_{L_{x_0}}$, etc, we will use $M_{x_0}$, $N_{x_0}$, etc.

3. Regular free boundary points

Using Theorem 2.4, in this section we explore in more detail the notion of regular free boundary points and establish some preliminary properties of the regular set. We begin by recalling the following definition from Section 1.

Definition 3.1. We say that $x_0 \in \Gamma(v)$ is regular iff $\tilde{N}_{x_0}(v_{x_0}, 0+) = \frac{3}{2}$ and let $\Gamma_{3/2}(v)$ be the set of all regular free boundary points. $\Gamma_{3/2}(v)$ is also called the regular set.

In Lemma 3.3 below we prove that $\Gamma_{3/2}(v)$ is a relatively open subset of the free boundary $\Gamma(v)$. To accomplish this we prove that $\tilde{N}_{\bar{x}}(0+) = \frac{3}{2}$ for $\bar{x}$ in a small neighborhood of $x_0 \in \Gamma_{3/2}(v)$. Since the definition of $\tilde{N}(r)$ involves a truncation of $M(r)$, we first need to establish the following auxiliary result.

Lemma 3.2. Let $v$ satisfy (1.11)–(1.15) with $0 \in \Gamma_{3/2}(v)$. Then

$$\frac{r M'(r)}{2 M(r)} \to \frac{3}{2} \quad \text{as } r \to 0^+.\]$$

In particular, for every $\varepsilon > 0$ there exists $r_\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$\frac{r M'(r)}{2 M(r)} \leq \frac{3 + \varepsilon}{2}, \quad M(r) \geq C_\varepsilon r^{3+\varepsilon}, \quad \text{for } 0 < r \leq r_\varepsilon.$$

Proof. We first claim that since $0 \in \Gamma_{3/2}(v)$, then $M(r) \geq r^{3+\delta}$ for $r > 0$ small. Indeed, if there was a sequence $s_j \to 0$ such that $M(s_j) < s_j^{3+\delta}$, then

$$\tilde{N}(s_j) = \frac{3 + \delta}{2} e^{K' s_j^{1-\delta}} \to \frac{3 + \delta}{2} \neq \frac{3}{2} = \tilde{N}(0+),$$

which is a contradiction. Hence, for $r$ small we have

$$\tilde{N}(r) = \frac{r e^{K' r^{1-\delta}} M'(r)}{2 M(r)}.$$

Since $e^{K' r^{1-\delta}} \to 1$ as $r \to 0$ and $\tilde{N}(0+) = \frac{3}{2}$, we obtain the first part of the lemma. Hence, for every $\varepsilon > 0$ there exists a small $r_\varepsilon > 0$ such that

$$\frac{r M'(r)}{2 M(r)} \leq 3 + \varepsilon, \quad r < r_\varepsilon.$$

Integrating from $r$ to $r_\varepsilon$, this gives

$$M(r_\varepsilon) \leq \frac{r_\varepsilon}{r} M(r_\varepsilon) \leq \left( \frac{r_\varepsilon}{r} \right)^{3+\varepsilon},$$

from which we conclude, with $C_\varepsilon = M(r_\varepsilon)/r_\varepsilon^{3+\varepsilon}$, that $M(r) \geq C_\varepsilon r^{3+\varepsilon}$. [\square]
Lemma 3.3. Let $v$ satisfy (1.12)–(1.15) with $x_0 \in \Gamma_{3/2}(v)$. Then, there exists $\eta_0 = \eta_0(x_0) > 0$ such that $\Gamma(v) \cap B'_{\eta_0}(x_0) \subset \Gamma_{3/2}(v)$ and, moreover, the convergence

$$\hat{N}_\bar{x}(r) \to 3/2 \quad \text{as} \quad r \to 0+$$

is uniform for $\bar{x} \in \Gamma(v) \cap B'_{\eta_0}(x_0)$.

**Proof.** Fix $0 < \varepsilon < \delta/8$, where $\delta > 0$ is fixed as in the definition (2.15) of the frequency $N_{x_0}(r)$. Then, by Lemma 3.2, there exist $C_\varepsilon = C_\varepsilon(x_0) > 0$ and $r_\varepsilon = r_\varepsilon(x_0) > 0$ such that

$$\frac{r \, M'_{x_0}(r)}{2 \, M_{x_0}(r)} \leq \frac{3 + \varepsilon}{2}, \quad M_{x_0}(r) \geq C_\varepsilon r^{3+\varepsilon}, \quad r < r_\varepsilon.$$

We then want to show that similar inequalities will hold if we replace $x_0$ with $\bar{x} \in B'_{\eta_0}(x_0)$ for a sufficiently small $\eta_\varepsilon$. We will write $L_\bar{x} = \text{div}(A_\bar{x}\nabla \cdot)$. To track the dependence on $\bar{x}$, we write, using the differentiation formulas in [GSVG14], that

$$\frac{M'_{\bar{x}}(r)}{M_{\bar{x}}(r)} = \frac{H'_{\bar{x}}(r)}{H_{\bar{x}}(r)} - \frac{\psi'_{\bar{x}}(r)}{\psi_{\bar{x}}(r)} = \frac{H'_{\bar{x}}(r)}{H_{\bar{x}}(r)} - \frac{\int_{S_r} v_\bar{x}(x) 2L_\bar{x}|x|}{\int_{S_r} v_\bar{x}(x)^2 \mu_\bar{x}(x)}$$

$$= \frac{2H_{\bar{x}}(r)}{H_{\bar{x}}(r)} - \frac{2 \int_{S_r} v_\bar{x}(x) \langle A_\bar{x}(x)\nabla v_\bar{x}(x), \nu \rangle}{\int_{S_r} v_\bar{x}(x)^2 \mu_\bar{x}(x)}$$

$$= \frac{2r \int_{S_r} v_\bar{x}(x) \langle A_\bar{x}(x)\nabla v_\bar{x}(x), x \rangle}{\int_{S_r} v_\bar{x}(x)^2 \langle A_\bar{x}(x) x, x \rangle}.$$

This implies that for fixed $r > 0$, the mapping $\bar{x} \mapsto M'_{\bar{x}}(r)/M_{\bar{x}}(r)$ is continuous. Thus, if $\rho_\varepsilon < r_\varepsilon$, we can find a small $\eta_\varepsilon > 0$ such that

$$\frac{\rho_\varepsilon \, M'_{\bar{x}}(\rho_\varepsilon)}{2 \, M_{\bar{x}}(\rho_\varepsilon)} \leq \frac{3 + 2\varepsilon}{2}, \quad \text{for} \quad \bar{x} \in B'_{\eta_\varepsilon}(x_0).$$

On the other hand, since

$$M_{\bar{x}}(r) = \frac{r}{\sigma_{\bar{x}}(r)^{r+1}} \int_{S_r} v_\bar{x}(x)^2 \langle A_\bar{x}(x) x, x \rangle,$$

the continuity of the mapping $\bar{x} \mapsto M_{\bar{x}}(r)$ is not so clear. However, having that $c_0 \leq \frac{\sigma_{\bar{x}}(r)}{r} \leq c_0^{-1}$ for a universal constant $c_0 > 0$, we can write that

$$M_{\bar{x}}(\rho_\varepsilon) \geq \frac{1}{2} \, \sigma_{\bar{x}}(r)^{3+\varepsilon} \rho_\varepsilon^{3+\delta}, \quad \bar{x} \in B'_{\eta_\varepsilon}(x_0),$$

if we take $\eta_\varepsilon$ and $\rho_\varepsilon$ sufficiently small. The latter inequality implies that we can explicitly compute $\hat{N}_\bar{x}(\rho_\varepsilon)$ by

$$\hat{N}_\bar{x}(\rho_\varepsilon) = \frac{\rho_\varepsilon}{2} \, e^{K' \rho_\varepsilon^{3+\delta}/2} \frac{M'_{\bar{x}}(\rho_\varepsilon)}{M_{\bar{x}}(\rho_\varepsilon)} \leq e^{K' \rho_\varepsilon^{3+\delta}/2} \left( \frac{3 + 3\varepsilon}{2} \right) \geq \frac{3 + 3\varepsilon}{2},$$

again if $\rho_\varepsilon$ is small enough. Hence, by the monotonicity of $N_{\bar{x}}(r) = (\sigma_{\bar{x}}(r)/r)\hat{N}_{\bar{x}}(r)$, we obtain

$$\hat{N}_{\bar{x}}(0+) \leq \frac{1}{\alpha_{\bar{x}}} \frac{\sigma_{\bar{x}}(r) \, 3 + 3\varepsilon}{\rho_\varepsilon},$$

where

$$\alpha_{\bar{x}} \overset{\text{def}}{=} \lim_{r \to 0+} \frac{\sigma_{\bar{x}}(r)}{r}.$$

(3.1)
Using now the estimates in Lemma 2.3, we have
\[ \left| \frac{\sigma_x(r)}{r} - \alpha \right| \leq C_0 r \quad \text{and} \quad \alpha \geq c_0, \]
therefore we can guarantee that
\[ \tilde{N}_x(0+) \leq (1 + \varepsilon) \frac{3 + 3\varepsilon}{2} \leq \frac{3 + \delta}{2} < \frac{3 + \delta}{2}, \quad \bar{x} \in B'_{\eta_0}(x_0). \]
But then, by the gap of values of \( \tilde{N}_x(0+) \) between \( 3/2 \) and \( (3 + \delta)/2 \), we conclude that
\[ \tilde{N}_x(0+) = 3/2, \quad \bar{x} \in B'_{\eta_0}(x_0). \]

To prove the second part of the lemma, we note that for any fixed \( \bar{x} \in B'_{\eta_0}(x_0) \), the mapping
\[ r \mapsto e^{\beta r} \tilde{N}_x(r) \quad 0 < r < \rho_\varepsilon \]
is monotone increasing for a universal constant \( \beta > 0 \), which follows from the inequality
\[ \frac{d}{dr} \log \frac{\sigma_x(r)}{r} \leq \beta, \quad \text{(see Lemma 2.3)}, \]
the monotonicity of \( r \mapsto N_\varepsilon(r) = \frac{\sigma_x(r)}{r} \tilde{N}_x(r) \), as well as the nonnegativity of \( \tilde{N}_x(r) \).

Now, for each fixed \( 0 < r < \rho_\varepsilon \), the mapping
\[ \bar{x} \mapsto e^{\beta r} \tilde{N}_\varepsilon(r) = e^{\beta r + K'(1 - i/2) r^2} \frac{\int_{S_{r}} v_\varepsilon(x) (A_\varepsilon(x) \nabla v_\varepsilon(x), x)}{\int_{S_{r}} v_\varepsilon(x)^2 (A_\varepsilon(x) x, x)} \]
is continuous on \( B'_{\eta_0}(x_0) \). Since the limit
\[ \lim_{r \to \rho_0} e^{\beta r} \tilde{N}_\varepsilon(r) = \tilde{N}_\varepsilon(0+) = 3/2, \quad \text{for all } \bar{x} \in B'_{\eta_0}(x_0), \]
by the classical theorem of Dini, we have that the convergence \( e^{\beta r} \tilde{N}_\varepsilon(r) \to 3/2 \) as \( r \to 0^+ \) will be uniform on \( \Gamma(\varepsilon) \cap B'_{\eta_0/2}(x_0) \), implying also the uniform convergence \( \tilde{N}_\varepsilon(r) \to 3/2 \). \( \square \)

In the remaining part of this section we study Almgren type scalings
\[ \bar{v}_{\bar{x},r}(x) = \frac{v_{\bar{x},r}(x)}{d_{\bar{x},r}}, \quad d_{\bar{x},r} = \left( \frac{1}{r^{n-1}} H_{\varepsilon}(r) \right)^{1/2}. \quad (3.2) \]
This is slightly different from what was done in [GSVG14], but more suited for the study of the free boundary. Notice that we have the following normalization:
\[ \int_{S_1} \bar{v}_{\bar{x},r}^2 r \mu_{\bar{x},r} = 1, \quad \mu_{\bar{x},r} = \frac{\langle A_\varepsilon(x) x, x \rangle}{|x|^2}. \]

Now, if \( \bar{x} \in \Gamma_{3/2}(\varepsilon) \), then the results of [GSVG14] imply that over subsequences \( r = r_j \to 0 \), we have the convergence
\[ \bar{v}_{\bar{x},r}(x) \left( \frac{\sigma_\varepsilon(r)}{r} \right)^{1/2} \to a R(h', e' + i|x_0|)^{3/2} \quad \text{in } C^{1,\alpha}_{\text{loc}}(\mathbb{R}^{n} \cup \mathbb{R}^{n-1}) \]
for some \( e' \in \mathbb{R}^{n-1} \) with \( |e'| = 1 \) and \( a > 0 \). Since we also have the convergence
\[ \frac{\sigma_\varepsilon(r)}{r} \to \alpha_\varepsilon > 0, \]
see Lemma 2.3, we obtain that
\[ \tilde{v}_{x,r}(x) \to \frac{a}{\alpha_x^{1/2}} R((x', e') + i|x_n|)^{3/2} \text{ in } C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n \cup \mathbb{R}^{n-1}). \]

The normalization \( \int_{S^1} \tilde{v}_x^2 \mu_{x,r} = 1 \) then implies that there exists a dimensional constant \( c_n > 0 \) (independent of \( \bar{x} \)), such that, on a subsequence,
\[ \tilde{v}_{x,r}(x) \to c_n R((x', e') + i|x_n|)^{3/2}. \]

Moreover, we can actually prove the following result.

Lemma 3.4. Let \( v \) satisfy (1.12)–(1.15) with \( x_0 \in \Gamma_{3/2}(v) \). Given \( \theta > 0 \), there exists \( r_0 = r_0(x_0) > 0 \) and \( \eta_0 = \eta_0(x_0) > 0 \) such that
\[ \inf_{e' \in \mathbb{R}^{n-1}, |e'| = 1} \| \tilde{v}_{x,r}(x) - c_n R((x', e') + i|x_n|)^{3/2} \|_{C^{1,\alpha}(B^+_{\xi_{\eta_0}}(0) \cup B_{r_0}(x_0))} < \theta \]

for any \( x \in \Gamma_{3/2}(v) \cap \overline{B_{r_0}(x_0)} \) and \( r < r_0 \).

Proof. Suppose the contrary. Then, there exists a sequence \( x_j \to x_0 \) and \( r_j \to 0 \) such that
\[ \| \tilde{v}_{x_j, r_j} - c_n R((x', e') + i|x_n|)^{3/2} \|_{C^{1,\alpha}(B^+_{\xi_{\eta_0}}(0) \cup B_{r_0}(x_0))} \geq \theta \]

for any unit vector \( e' \in \mathbb{R}^{n-1} \). Observe now that
\[ e^{-K'(pr)^{(1-s)/2}} \tilde{N}_e(pr) = \frac{(pr)^2 \int_{S^1} v_x(x) A_x(x) \nabla v_x(x), x}{\int_{S^1} v_x(x)^2 A_x(x), x} \]
\[ = \frac{\rho^2 \int_{S^1} \tilde{v}_{x,r}(x) A_x(x) \nabla \tilde{v}_{x,r}(x), x}{\int_{S^1} \tilde{v}_{x,r}(x)^2 A_x(x) \nabla \tilde{v}_{x,r}(x), x} \]
\[ = \frac{\rho^2 \int_{S^1} \tilde{v}_{x,r}(x) A_x(x) \nabla \tilde{v}_{x,r}(x), x}{\int_{S^1} \tilde{v}_{x,r}(x)^2 A_x(x) \nabla \tilde{v}_{x,r}(x), x} \]
\[ \text{(3.3)} \]

Now, we claim that the scalings \( \tilde{v}_{x_j, r_j} \) are uniformly bounded in \( C^{1,1/2}(B^+_{R} \cup B'_{R}) \) for any \( R > 0 \). Indeed, by Lemma 3.3, given \( \varepsilon > 0 \) small, we have that
\[ \frac{t}{2} M_{\tilde{\varepsilon}}(t) \leq \frac{3}{2}, \quad t < \rho_\varepsilon, \quad \bar{x} \in B'_{\eta_0}(x_0). \]

Let \( R \geq 1 \) and \( Rr < \rho_\varepsilon \). Integrating the above inequality from \( t = r \) to \( Rr \), we obtain that
\[ M_{\tilde{\varepsilon}}(Rr) \leq M_{\tilde{\varepsilon}}(r) R^{3+\varepsilon}. \]

Changing \( M_{\tilde{\varepsilon}} \) to \( H_{\tilde{\varepsilon}} \) we therefore have, using (2.6), that
\[ H_{\tilde{\varepsilon}}(Rr) \leq C_0 H_{\tilde{\varepsilon}}(r) R^{n+2+\varepsilon}. \]

The latter can be written in the form
\[ \int_{S^1} \tilde{v}_{x,r}^2 \mu_{x,r} \leq C_0 R^{n+2+\varepsilon}. \]

Thus, the uniform boundedness of \( \tilde{v}_{x_j, r_j} \) in \( L^2(B_{R}) \), and consequently in \( C^{1,1/2}(B_{R}^+ \cup B'_{R}) \), follows. Hence, we can assume without loss of generality that \( \tilde{v}_{x_j, r_j} \to v_0 \) in \( C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n \cup \mathbb{R}^{n-1}) \). It is immediate to see that
$v_0$ will solve the Signorini problem for the Laplacian in the entire $\mathbb{R}^n$. Besides, by Lemma 3.3, we will have that $\bar{N}_{x_j}(\rho r_j) \to 3/2$. On the other hand, passing to the limit in (3.3), we obtain that
\[
\frac{3}{2} = \frac{\int_{S_{\rho}} v_0(x)\langle \nabla v_0(x), x \rangle}{\int_{S_{\rho}} v_0(x)^2} = \frac{\rho \int_{B_{\rho}} |\nabla v_0|^2}{\int_{S_{\rho}} v_0(x)^2}, \quad \text{for any } \rho > 0.
\]
Therefore, $v_0$ is $3/2$ homogeneous global solution of the Signorini problem and thus has the form $v_0(x) = c_n \Re \langle x', e_0' \rangle + i|x_n|^{3/2}$. Hence, for large $j$ we will have
\[
\|\bar{v}_{x_j, r_j} - c_n \Re \langle x', e_0' \rangle + i|x_n|^{3/2}\|_{C^1(B_{2}^{+} \cup B_{1}^{-})} < \theta,
\]
contradictory to our assumption. The proof is complete. □

4. A Weiss type monotonicity formula

In this section we establish a monotonicity formula which is reminiscent of that established by Weiss in [Wei99] for the classical obstacle problem, and which is one of the two main ingredients in our proof of the $C^{1,\beta}$ regularity of the regular set. We consider the solution $u$ to the Signorini problem (1.6)–(1.9) above, and we set $v$ as in (1.10).

**Definition 4.1.** Let $r_0 > 0$ be as in Theorem 2.4. For $r \in (0, r_0)$ we define the $\frac{3}{2}$-th generalized Weiss-type functional as follows
\[
W_L(v, r) = \frac{\sigma(r)}{r^3} \left\{ J_L(v, r) - \frac{3/2}{r} M_L(v, r) \right\}
= \frac{1}{r^{n+1}} I_L(v, r) - \frac{3/2}{r^{n+2}} H_L(v, r)
= \frac{1}{r^{n+1}} \int_{B_r} [(A(x)\nabla v, \nabla v) + vf] - \frac{3/2}{r^{n+2}} \int_{S_r} v^2 \mu,
\]
where $I_L$, $J_L$, $H_L$, and $M_L(v, r)$ are as in Section 2. Whenever $L = \Delta$, we write $W(v, r) = W_\Delta(v, r)$, and unless we want to stress the dependence on $v$, we will write $W_L(r) = W_L(v, r)$.

In this section we will show that there exists $C > 0$ such that $r \mapsto W_L(v, r) + Cr^{1/2}$ is monotone nondecreasing, that the limit $\lim_{r \to 0} W_L(v, r)$ exists and is zero, and that $W_L(v, r) \geq -Cr^{1/2}$. We start by proving that $W_L(v, \cdot)$ is bounded.

**Lemma 4.2.** The functional $W_L(v, \cdot)$ is bounded on the interval $(0, r_0)$.

**Proof.** Indeed, from Corollary 2.6 we have
\[
|W_L(v, r)| \leq \frac{1}{r^{n+1}} \left\{ |I(r)| + \frac{3}{2r} H(r) \right\} \leq C.
\]

The functional $W_L(r)$ in (4.1) is tailor-made to the study of regular free boundary points of solutions of the Signorini problem (1.11)–(1.15). The following “almost monotonicity” property of $W_L$ plays a crucial role in our further study.
Theorem 4.3 (Weiss type monotonicity formula). Let \( v \) satisfy (1.11)–(1.15) and let \( 0 < r < r_0 \), depending also on \( \|f\|_{L^\infty(B_1)} \), such that for every \( 0 < r < r_0 \) one has

\[
\frac{d}{dr} \left( W_L(v, r) + Cr^{1/2} \right) \geq 2 \frac{r}{n+1} \int_{S_r} \left( \frac{\langle A \nabla v, \nu \rangle}{\sqrt{\mu}} - \frac{(3/2)\sqrt{\mu}}{r} \right)^2.
\]  

(4.3)

In particular, there exists \( C > 0 \) such that function \( r \mapsto W_L(v, r) + Cr^{1/2} \) is monotone nondecreasing, and therefore the limit \( W_L(v, 0^+) \overset{\text{def}}{=} \lim_{r \to 0} W_L(v, r) \) exists.

Proof. We have from Definition 4.1 above,

\[
\frac{d}{dr} W_L(v, r) = \frac{\sigma(r)}{r^3} \left\{ \left( \frac{\psi'(r)}{\psi(r)} - \frac{n-2}{r} - \frac{3}{r} \right) J(r) - \frac{3}{r^2} M(r) \right\} + \left( \frac{\psi'(r)}{\psi(r)} - \frac{n-2}{r} + O(1) \right) J(r) + \frac{1}{\psi(r)} \left\{ 2 \int_{S_r} \frac{\langle A \nabla v, \nu \rangle^2}{\mu} \right\} - \frac{2}{r} \int_{B_r} \langle Z, \nabla v \rangle f - \left( \frac{n-2}{r} + O(1) \right) \int_{B_r} vf + \frac{1}{r} J(r) + \frac{3}{r^2} M(r) \right\}.
\]

Using (2.12) and (2.14) above, we thus find

\[
\frac{d}{dr} W_L(v, r) = \frac{\sigma(r)}{r^3} \left\{ \left( -\frac{6}{r} + O(1) \right) J(r) + \frac{2}{\psi(r)} \int_{S_r} \frac{\langle A \nabla v, \nu \rangle^2}{\mu} \right\} \]

\[
+ \frac{3}{r^2} \left( 1 - \left( \frac{r\psi'(r)}{\psi(r)} - n + 2 \right) \right) M(r)
\]

\[
+ \frac{1}{\psi(r)} \left\{ -2 \int_{B_r} \langle Z, \nabla v \rangle f - \left( \frac{n-2}{r} + O(1) \right) \int_{B_r} vf + \int_{S_r} vf \right\} \right\}.
\]

By (2.5) in Lemma 2.2 above, we see that \( \psi'(r) = \frac{n-1}{r} + O(1) \). We thus obtain from the latter chain of equalities

\[
\frac{d}{dr} W_L(v, r) = \frac{2\sigma(r)}{r^3 \psi(r)} \left\{ \left( -\frac{3}{r} + O(1) \right) I(r) + \int_{S_r} \frac{\langle A \nabla v, \nu \rangle^2}{\mu} + \frac{9}{r^2} (1 + O(r)) H(r) \right\} \]

\[
+ \frac{\sigma(r)}{r^3 \psi(r)} \left\{ -2 \int_{B_r} \langle Z, \nabla v \rangle f - \left( \frac{n-2}{r} + O(1) \right) \int_{B_r} vf + \int_{S_r} vf \right\}.
\]

By the definitions (2.1) and (2.2) of \( H(r) \) and \( I(r) \) we have

\[
\int_{S_r} \left( \frac{\langle A \nabla v, \nu \rangle}{\sqrt{\mu}} - \frac{(3/2)\sqrt{\mu}}{r} \right)^2 = \int_{S_r} \frac{\langle A \nabla v, \nu \rangle^2}{\mu} - \frac{3}{r} I(r) + \frac{9}{r^2} H(r).
\]

Since \( \sigma(r) = r^{2-n} \psi(r) \), we conclude that

\[
\frac{d}{dr} W_L(v, r) = 2 \frac{r}{n+1} \left\{ \int_{S_r} \left( \frac{\langle A \nabla v, \nu \rangle}{\sqrt{\mu}} - \frac{(3/2)\sqrt{\mu}}{r} \right)^2 + O(1) \right\} \]

\[
+ \frac{1}{r^{n+1}} \left\{ -2 \int_{B_r} \langle Z, \nabla v \rangle f - \left( \frac{n-2}{r} + O(1) \right) \int_{B_r} vf + \int_{S_r} vf \right\}.
\]

(4.4)
Returning to (4.4) and making use of (2.17), we conclude that
\[
\frac{d}{dr} W_L(v, r) = \frac{2}{r^{n+1}} \int_{S_r} \left( \frac{\langle A \nabla v, v \rangle}{\sqrt{\mu}} - \frac{(3/2)\sqrt{\mu}}{r} v \right)^2 + O(1)
+ \frac{1}{r^{n+1}} \left( - \frac{2}{r} \int_{B_r} \langle Z, \nabla v \rangle f - \left( n - \frac{2}{r} + O(1) \right) \int_{B_r} vf + \int_{S_r} vf \right).
\]

The proof of the estimate (4.3) will be completed if we can show that there exists a universal constant \( C > 0 \) such that
\[
\left| - \frac{2}{r} \int_{B_r} \langle Z, \nabla v \rangle f - \left( n - \frac{2}{r} + O(1) \right) \int_{B_r} vf + \int_{S_r} vf \right| \leq Cr^{n+\frac{1}{2}}.
\]

From the expression of the vector field \( Z = \frac{A(x)\mu}{\mu} \), see (2.13) above, we have \( |Z| \leq C r \) for \( |x| \leq r \). Since \( f \in L^\infty \), we obtain from the second inequality in (2.16)
\[
\left| - \frac{2}{r} \int_{B_r} \langle Z, \nabla v \rangle f - \left( n - \frac{2}{r} + O(1) \right) \int_{B_r} vf + \int_{S_r} vf \right| \leq Cr^{n+\frac{1}{2}},
\]
for a universal \( C > 0 \) which also depends on \( \|f\|_{L^\infty(B_1)} \). The first inequality in (2.16) gives instead
\[
\left| \left( n - \frac{2}{r} + O(1) \right) \int_{B_r} vf + \int_{S_r} vf \right| \leq Cr^{n+\frac{1}{2}},
\]
thus completing the proof of (4.3). The existence of \( W_L(v, 0+) \) now follows from the monotonicity and boundedness of \( W_L(v, r) + Cr^{1/2} \), see Lemma 4.2.

From Theorem 4.3 we obtain that \( W_L(v, 0+) = \lim_{r \to 0} W_L(v, r) \) exists. In the next lemma we prove that this limit must actually be zero.

**Lemma 4.4.** Let \( v \) satisfy (1.11)-(1.15) and \( 0 \in \Gamma_{3/2} (v) \). Then, \( W_L(v, 0+) = 0 \).

**Proof.** Recall from (4.1) that one has
\[
W_L(v, r) = W_L(r) = \sigma(r) \left\{ \frac{2J(r)}{2r^3} - \frac{3}{r} M(r) \right\} = H(r) \left\{ \frac{r}{r^{n+2}} \frac{M'(r)}{M(r)} - \frac{3}{2} \right\},
\]
where in the last equality we have used (2.14) and (2.11) above. The proof now follows from the boundedness of \( H(r) \) by (2.17) and the convergence \( \frac{r}{r^{n+2}} \frac{M'(r)}{M(r)} \to \frac{3}{2} \) by Lemma 3.2.

**Corollary 4.5.** Let \( C \) and \( r_0 \) be as in Theorem 4.3. Then, for every \( 0 < r < r_0 \) one has
\[
W_L(v, r) \geq -Cr^{1/2}.
\]

**Proof.** This follows directly by combining Theorem 4.3 with Lemma 4.4.

5. Homogeneous blowups

In this section we analyze the uniform limits of some appropriate scalings of a solution \( v \) to the Signorini problem (1.11)-(1.15) by making essential use of the monotonicity formula in Theorem 4.3 above. These scalings, together with the Almgren type ones defined in (3.2), will be instrumental in Section 7.

Let \( v \) satisfy (1.11)-(1.15) and \( 0 \in \Gamma_{3/2} (v) \). We consider the following homogeneous scalings of \( v \nabla v(x) = \frac{v(rx)}{r^{3/2}}. \)
**Lemma 5.1.** Define $A_r(x) = A(rx)$ and $f_r(x) = r^{1/2} f(rx)$. Then, $v_r$ solves the thin obstacle problem (1.11)–(1.15) in $B_{\frac{r}{2}}$ with zero thin obstacle, operator $L_r = \text{div}(A_r \nabla \cdot)$ and right-hand side $f_r$ in (1.12).

**Proof.** We only need to verify (1.16). Indeed, given $\eta \in C^\infty_0(B_{\frac{r}{2}})$, define $\rho \in C^\infty_0(B_{\frac{r}{2}})$ by letting $\rho(y) = r^{1/2} \eta \left( \frac{y}{r} \right)$. By a change of variable one easily verifies that

$$\int_{B_{\frac{r}{2}}} (A_r \nabla v_r, \nabla \eta) = \int_{B'_{\frac{r}{2}}} \left[ (A_r(x) \nabla v_r(x), \nu_r^+ + (A_r(x) \nabla v_r(x), \nu_r^-) \right] \eta(x) - \int_{B_{\frac{r}{2}}} f_r(x) \eta(x). \quad (5.2)$$

□

**Lemma 5.2.** Let $v$ satisfy (1.11)–(1.15) and let $0 \in \Gamma_{3/2}(v)$. Given $r_j \to 0$, there exists a subsequence (which we will still denote by $r_j$) and a function $v_0 \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_+ \cup \mathbb{R}^{n-1})$ for any $\alpha \in (0,1/2)$, such that $v_{r_j} \to v_0$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_+ \cup \mathbb{R}^{n-1})$. Such $v_0$ is a global solution of the Signorini problem (1.11)–(1.15) in $\mathbb{R}^n$ with zero thin obstacle and zero right-hand side $f$.

**Proof.** By Lemma 2.5, there exist universal constants $C, r_0 > 0$ such that

$$|v(x)| \leq C|x|^{3/2} \quad \text{and} \quad |
abla v(x)| \leq C|x|^{1/2}, \quad |x| < r_0.$$  

Moreover, as proved in [GSVG14], $v \in C^{1,\frac{1}{2}}_{\text{loc}}(B_{\frac{1}{2}}^+ \cup B_1')$ with

$$\|v\|_{C^{1,\frac{1}{2}}(B_{\frac{1}{2}}^+ \cup B_1')} \leq C(n, \lambda, Q, \|v\|_{W^{1,2}(B_1)}, \|f\|_{L^\infty}).$$

Given $r_j \searrow 0$, by a standard diagonal process we obtain convergence in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_+ \cup \mathbb{R}^{n-1})$, for any $\alpha \in (0,1/2)$, of a subsequence of the functions $v_{r_j}$ to a function $v_0$. Passing to the limit in (5.2) we conclude that $v_0$ is a global solution to the Signorini problem with zero thin obstacle. The fact (important for our later purposes) that the right-hand side $f$ is also zero, follows again from (5.2) since we obtain

$$\left| \int_{B_{\frac{r}{2}}} f_r(x) \eta(x) \right| = \sqrt{r} \left| \int_{B_{\frac{r}{2}}} f(x) \eta(x) \right| \leq \|f\|_{L^\infty(B_1)} \|\eta\|_{L^1(\mathbb{R}^n)} \sqrt{r} \to 0.$$  

□

**Remark 5.3.** Notice that we have not yet ruled out the possibility that $v_0 \equiv 0$. This crucial aspect will be dealt with later, in Proposition 7.3 below. The next result establishes an important homogeneity property of the global solution $v_0$.

**Proposition 5.4.** Let $v_0$ be a function as in Lemma 5.2. Then, $v_0$ is homogeneous of degree $3/2$.

**Proof.** In what follows we denote with $\mu_r(x) = \mu(rx)$. Furthermore, recall that the scaled functions in (5.1) are given by $v_{r}(x) = r^{-3/2} v(rx)$. Let $r_j \searrow 0$ and denote by $v_{r_j}$ a corresponding blowup as in Lemma 5.2.
Let $0 < r < R$. We integrate (4.3) over the interval $(r, R)$ obtaining

\[
W_L(v, r, R) - W_L(v, r, r) - C \sqrt{r} (\sqrt{R} - \sqrt{r}) \geq \int_{r, r}^{r, R} \frac{2}{r^{n+3}} \int_{S_t} \left( \frac{A(\nabla v, x)}{\sqrt{\mu}} - \frac{3}{2} \sqrt{\mu} v \right)^2 dt
\]

\[
= \int_r^R \frac{1}{(r s)^{n+3}} \int_{S_r} \left( \frac{A(\nabla v, x)}{\sqrt{\mu}} - \frac{3}{2} \sqrt{\mu} v \right)^2 r_s ds
\]

\[
= \int_r^R \frac{1}{(r s)^{n+3}} \int_{S_r} \left( \frac{A(r_j, y) \nabla v(x, y), r_j y)}{\sqrt{\mu(r_j, y)}} - \frac{3}{2} \sqrt{\mu(r_j, y)} v(r_j, y) \right)^2 r_j ds
\]

\[
= \frac{1}{r^2} \int_r^R \frac{1}{s^{n+3}} \int_{S_r} \left( \frac{A(r_j, y) \nabla v(x, y), y) r_j}{\sqrt{\mu(r_j, y)}} - \frac{3}{2} \sqrt{\mu(r_j, y)} v(r_j, y) \right)^2 ds
\]

\[
= \frac{1}{r^2} \int_r^R \frac{1}{s^{n+3}} \int_{S_r} \left( \frac{A(r_j, y) \nabla v(x, y), y) r_j}{\sqrt{\mu(r_j, y)}} - \frac{3}{2} \sqrt{\mu(r_j, y)} v(r_j, y) \right)^2 ds
\]

Since $W_L(v, 0, 1)$ exists, the left-hand side goes to zero as $j \to \infty$. Since $A(0) = I, \mu(0) = 1$ and we have $C^{1,\alpha}_{\text{loc}}(\mathbb{R}_+ \cup \mathbb{R}^{n-1})$ convergence of $v_{r_j}$ to $v_0$, passing to the limit as $j \to \infty$ in the above, we conclude that

\[
0 \geq \int_r^R \frac{1}{s^{n+3}} \int_{S_r} \left( \nabla v_0(y, x), y) - \frac{3}{2} v_0(y) \right)^2 ds.
\]

This inequality, and the arbitrariness of $0 < r < R$, imply that $v_0$ is homogeneous of degree 3/2.

**Definition 5.5.** We call such $v_0$ a homogeneous blowup.

### 6. An epiperimetric inequality for the Signorini problem

In this section we establish in the context of the Signorini problem a basic generalization of the epiperimetric inequality obtained by Weiss for the classical obstacle problem. Our main result, which is Theorem 6.3 below, is tailor made for analyzing regular free boundary points in the Signorini problem, being the second main tool we use to reach our goal – the $C^{1,\beta}$ regularity of the regular set.

**Definition 6.1.** Given $v \in W^{1,2}(B_1)$, we define the boundary adjusted energy as the Weiss type functional defined in (4.1) for the Laplacian operator with $r = 1$ and zero thin obstacle, i.e.,

\[
W(v) \overset{\text{def}}{=} W_\Delta(v, 1) = \int_{B_1} |\nabla v|^2 - \frac{3}{2} \int_{S_1} v^2.
\]

**Remark 6.2.** We observe explicitly that if $\int_{S_1} v^2 \neq 0$, then we can write

\[
W(v) = \left( \int_{S_1} v^2 \right) \left[ N(v, 1) - \frac{3}{2} \right].
\]

It follows that if $v$ is a solution to the Signorini for the Laplacian in $\mathbb{R}^n$, with zero thin obstacle, and which is homogeneous of degree $\frac{3}{2}$, then by [ACS08] we have $N(v, r) \equiv \frac{3}{2}$, and therefore $W(v) = 0$.  

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We now consider the function $h(x) = \Re(x_1 + i|x_n|^{3/2})$, which is a $\frac{3}{2}$-homogeneous global solution of the Signorini problem for the Laplacian with zero thin obstacle, and introduce the set

$$H = \{ a\Re((x', \nu) + i|x_n|^{3/2} | \nu \in S_1, a \geq 0 \}$$

of all multiples and rotations of the function $h$. The following is the central result of this section.

**Theorem 6.3 (Epiperimetric inequality).** There exists $\kappa \in (0, 1)$ and $\theta \in (0, 1)$ such that if $w \in W^{1,2}(B_1)$ is a homogeneous function of degree $\frac{3}{2}$ such that $w \geq 0$ on $B_1'$ and $\|w - h\|_{W^{1,2}(B_1)} \leq \theta$, then there exists $\zeta \in W^{1,2}(B_1)$ such that $\zeta = w$ on $S_1$, $\zeta \geq 0$ on $B_1'$ and

$$W(\zeta) \leq (1 - \kappa)W(w).$$

**Proof.** We argue by contradiction and assume that the result does not hold. Then, there exist sequences of real numbers $\kappa_m \to 0$ and $\theta_m \to 0$, and functions $w_m \in W^{1,2}(B_1)$, homogeneous of degree $\frac{3}{2}$, such that $w_m \geq 0$ on $B_1'$ and

$$\|w_m - h\|_{W^{1,2}(B_1)} \leq \theta_m,$$

but such that for every $\zeta \in W^{1,2}(B_1)$ with $\zeta \geq 0$ on $B_1'$, and for which $\zeta = w_m$ on $S_1$, one has

$$W(\zeta) > (1 - \kappa_m)W(w_m).$$

With such assumption in place we start by observing that there exists $g_m = a_m\Re((x', \nu_m) + i|x_n|^{3/2}) \in H$ which achieves the minimum distance from $w_m$ to $H$:

$$\|w_m - g_m\|_{W^{1,2}(B_1)} = \inf_{g \in H} \|w_m - g\|_{W^{1,2}(B_1)}.$$ 

Indeed, this follows from the simple fact that the set $H$ is locally compact. Combining this inequality with (6.1) we deduce that $\|g_m - h\|_{W^{1,2}(B_1)} \leq 2\theta_m$. As a consequence, we must have that $\nu_m \to e_1$ and $a_m \to 1$. Hence,

$$\left| \frac{w_m}{a_m} - \Re((x', \nu_m) + i|x_n|^{3/2}) \right|_{W^{1,2}(B_1)} \leq \frac{\theta_m}{a_m} \to 0.$$

If we rename $\frac{w_m}{a_m} \sim w_m$ and $\frac{\theta_m}{a_m} \sim \theta_m$, and rotate $\mathbb{R}^{n-1}$ to send $\nu_m$ to $e_1$, the renamed functions $w_m$ will be homogeneous of degree $\frac{3}{2}$, nonnegative on $B_1'$, and will satisfy

$$\inf_{g \in H} \|w_m - g\|_{W^{1,2}(B_1)} = \|w_m - h\|_{W^{1,2}(B_1)} \leq \theta_m.$$  

Moreover, (6.2) will still hold for the renamed $w_m$, because of the scaling property $W(tw) = t^2W(w)$ and the invariance of $W(w)$ under rotations in $\mathbb{R}^{n-1}$.

We note explicitly that (6.2) implies in particular that $w_m \neq h$ for every $m \in \mathbb{N}$, as $W(h) = 0$ (see Remark 6.2 above). Thus we may also set

$$\theta_m = \|w_m - h\|_{W^{1,2}(B_1)} > 0$$

for the rest of the proof.

We next want to rewrite (6.2) in a slightly different way, using the properties of function $h$. Given $\phi \in W^{1,2}(B_1)$, consider the first variation of $W$ at $h$ in the direction of $\phi$

$$\delta W(h)(\phi) \overset{\text{def}}{=} \int_{B_1} 2
abla h \cdot \nabla \phi - \frac{3}{2} \int_{S_1} 2h \phi,$$ 

(6.5)
where the boundary integrals in (6.5) and thereafter must be interpreted in the sense of traces. To compute \( \delta W(h)(\phi) \) we write the first integral in the right-hand side of (6.5) as \( \int_{B_1^+} 2(\nabla h, \nabla \phi) = \int_{B_1^+} 2(\nabla h, \nabla \phi) + \int_{B_1^-} 2(\nabla h, \nabla \phi) \). Now,

\[
\int_{B_1^+} 2(\nabla h, \nabla \phi) = -\int_{B_1^+} 2(\Delta h)\phi + \int_{\partial B_1^+} 2\phi(\nabla h, \nu) + \int_{B_1'^+} 2\phi(\nabla h, \nu_+) = \int_{\partial B_1^+} 2\phi(\nabla h, \nu) + \int_{B_1'^+} 2\phi(\nabla h, \nu_+),
\]

where we have used the fact that \( \Delta h = 0 \) in \( B_1^\pm \). Since \( h \) is homogeneous of degree \( \frac{3}{2} \), by Euler’s formula we have \( \langle \nabla h, \nu \rangle = \frac{3}{2}h \) in \( S_1^\pm \). Keeping in mind that on \( B_1' \) we have \( \nu_\pm = \mp e_n \), we find

\[
\int_{B_1^+} 2(\nabla h, \nabla \phi) = \frac{3}{2} \int_{S_1^+} 2\phi h + \int_{B_1'} 2\phi(\nabla h, \nu_\pm) = \frac{3}{2} \int_{S_1^+} 2\phi h + \int_{B_1'} 2\phi \partial_n^\pm h.
\]

Since \( h \) is even in \( x_n \), so that \( \partial_n^- h = -\partial_n^+ h \) in \( B_1' \), we conclude

\[
\delta W(h)(\phi) = -4 \int_{B_1'} \phi \partial_n^+ h. \tag{6.6}
\]

If now \( \zeta \in W^{1,2}(B_1) \) is a function with \( \zeta \geq 0 \) on \( B_1' \) and such that \( \zeta = w_\infty \) on \( S_1 \), by plugging in \( \phi = \zeta - h \) into (6.5) and (6.6), we obtain that

\[
W(\zeta) = W(\zeta) - W(h) - \delta W(h)(\zeta - h) - 4 \int_{B_1'} (\zeta - h) \partial_n^+ h
= \int_{B_1} |\nabla(\zeta - h)|^2 - \frac{3}{2} \int_{S_1} (\zeta - h)^2 - 4 \int_{B_1'} \zeta \partial_n^+ h,
\]

where we have used that \( W(h) = 0 \) and \( h \partial_n^+ h = 0 \) on \( B_1' \). By using a similar identity for \( W(w_\infty) \), we can rewrite (6.2) as

\[
(1 - \kappa_m) \left[ \int_{B_1} |\nabla(w_\infty - h)|^2 - \frac{3}{2} \int_{S_1} (w_\infty - h)^2 - 4 \int_{B_1'} w_\infty \partial_n^+ h \right]
< \int_{B_1} |\nabla(\zeta - h)|^2 - \frac{3}{2} \int_{S_1} (\zeta - h)^2 - 4 \int_{B_1'} \zeta \partial_n^+ h. \tag{6.7}
\]

Inequality (6.7) will play a key role in the completion of the proof and will be used repeatedly.

Let us introduce the normalized functions

\[
\hat{w}_m = \frac{w_\infty - h}{\theta_m}.
\]

By (6.4) we have

\[
\|\hat{w}_m\|_{W^{1,2}(B_1)} = 1 \quad \text{for every } m \in \mathbb{N}.
\]

By the weak compactness of the unit sphere in \( W^{1,2}(B_1) \) we may assume that

\[
\hat{w}_m \to \hat{w} \quad \text{weakly in } W^{1,2}(B_1).
\]

Besides, by the compactness of the Sobolev embedding and traces operator from \( W^{1,2}(B_1) \) into \( L^2(B_1) \), \( L^2(B_1') \), \( L^2(S_1) \) (see e.g. Theorem 6.3 in [Nec12]), we may assume

\[
\hat{w}_m \to \hat{w} \quad \text{strongly in } L^2(B_1), L^2(B_1'), \text{ and } L^2(S_1).
\]

We then make the following
Claim 1.

(i) $\hat{w} \equiv 0$;

(ii) $\hat{w}_m \to 0$ strongly in $W^{1,2}(B_1)$.

Note that (ii) will give us a contradiction since by construction $\|\hat{w}_m\|_{W^{1,2}(B_1)} = 1$. Hence, the theorem will follow once we prove the claim.

In what follows we will denote $\Lambda = \Lambda(h)$, the coincidence set of $h$.

Step 1. We start by showing that there is a constant $C > 0$ such that

$$\left\| \frac{w_m}{\theta_m^2} \partial_n^+ h \right\|_{L^1(B'_1)} \leq C, \quad \text{for every } m \in \mathbb{N}. \quad (6.8)$$

To this end, we pick a function $\eta \in W^{1,\infty}_0(B_1)$ such that $0 < \eta \leq 1$, and define $\zeta = (1 - \eta)w_m + \eta h$. Then, $\zeta = w_m$ on $S_1$ and $\zeta \geq 0$ on $B'_1$. Furthermore, $\zeta - h = (1 - \eta)(w_m - h)$. We can thus apply (6.7) to such a $\zeta$, obtaining

$$(1 - \kappa_m) \left( \int_{B_1} |\nabla (w_m - h)|^2 - \frac{3}{2} \int_{S_1} (w_m - h)^2 - 4 \int_{B'_1} w_m \partial_n^+ h \right)$$

$$< \int_{B_1} |\nabla ((1 - \eta)(w_m - h))|^2 - \frac{3}{2} \int_{S_1} (1 - \eta)^2 (h - w_m)^2 - 4 \int_{B'_1} ((1 - \eta)w_m + \eta h) \partial_n^+ h$$

$$= \int_{B_1} [(1 - \eta)^2 |\nabla (w_m - h)|^2 + |\nabla \eta|^2 (w_m - h)^2 - 2(1 - \eta)(w_m - h) \langle \nabla \eta, \nabla (w_m - h) \rangle]$$

$$- \frac{3}{2} \int_{S_1} (1 - \eta)^2 (h - w_m)^2 - 4 \int_{B'_1} (1 - \eta)w_m \partial_n^+ h.$$ 

Dividing by $\theta_m^2$, rearranging terms and using the fact that $\|\hat{w}_m\|_{W^{1,2}(B_1)} = 1$ and that $\partial_n^+ h \leq 0$ on $\Lambda$, we obtain

$$4 \int_{B'_1} (\eta - \kappa_m) \frac{w_m}{\theta_m^2} \partial_n^+ h \leq -(1 - \kappa_m) \left( \int_{B_1} |\nabla \hat{w}_m|^2 - \frac{3}{2} \int_{S_1} \hat{w}_m^2 \right)$$

$$+ \int_{B_1} [(1 - \eta)^2 |\nabla \hat{w}_m|^2 + |\nabla \eta|^2 \hat{w}_m^2 - 2(1 - \eta) \hat{w}_m \langle \nabla \eta, \nabla \hat{w}_m \rangle] - \frac{3}{2} \int_{S_1} (1 - \eta)^2 \hat{w}_m^2 \leq C,$$

where $C > 0$ is independent of $m \in \mathbb{N}$. At this point we choose $\eta(x) = \check{\eta}(|x|)$, and let

$$0 < \varepsilon \overset{\text{def}}{=} \int_0^1 \check{\eta}(r) r^n dr.$$

Since $\kappa_m \to 0$ as $m \to \infty$, possibly passing to a subsequence we can assume that $\kappa_m \leq \frac{\varepsilon}{2}(n + 1)$ for every $m \in \mathbb{N}$. With such choice we have

$$\int_0^1 (\check{\eta}(r) - \kappa_m) r^n dr \geq \frac{\varepsilon}{2}, \quad m \in \mathbb{N}.$$

Using the fact that $w_m$ and $h$ are homogeneous of degree $\frac{n}{2}$, we thus obtain

$$C \geq 4 \int_{B'_1} (\eta - \kappa_m) \frac{w_m}{\theta_m^2} \partial_n^+ h = 4 \left( \int_0^1 (\check{\eta}(r) - \kappa_m) r^n dr \right) \int_{S'_1} \frac{w_m}{\theta_m^2} \partial_n^+ h + 2\varepsilon \int_{S'_1} \frac{w_m}{\theta_m^2} \partial_n^+ h.$$

which, again by the homogeneity of $w_m$ and $h$, and the fact that $w_m \geq 0$ on $B'_1$, proves (6.8).
Step 2. We next show that
\[ \Delta \hat{w} = 0, \quad \text{in } B_1 \setminus \Lambda. \] 

To establish (6.9) it will suffice to show that for any ball \( B \), such that its concentric double \( 2B \subset B_1 \setminus \Lambda \), and for any function \( \phi \in W^{1,2}(B) \) such that \( \phi - \hat{w} \in W^{1,2}_0(B) \), one has
\[
\int_B |\nabla \hat{w}|^2 \leq \int_B |\nabla \phi|^2.
\]

To begin, we fix a function \( \phi \in L^\infty(B_1) \cap W^{1,2}(B) \), and we consider
\[ \zeta = \eta(h + \theta_m \phi) + (1 - \eta) w_m, \]
where \( \eta \in C_0^\infty(B_1 \setminus \Lambda), 0 \leq \eta \leq 1 \). Notice that on \( S_1 \), \( \zeta = w_m \), and, since \( \phi \in L^\infty(B_1) \) and \( \eta \in C_0^\infty(B_1 \setminus \Lambda) \), for \( m \) large enough we have \( \zeta \geq 0 \) on \( B'_1 \). For such sufficiently large \( m \)'s, we can thus use the function \( \zeta \) in (6.7), obtaining
\[
(1 - \kappa_m) \left( \int_{B_1} |\nabla (w_m - h)|^2 - \frac{3}{2} \int_{S_1} (w_m - h)^2 - 4 \int_{B'_1} w_m \partial_n^+ h \right)
< \int_{B_1} |\nabla ((1 - \eta)(w_m - h) + \eta \theta_m \phi)|^2 - \frac{3}{2} \int_{S_1} ((1 - \eta)(w_m - h) + \eta \theta_m \phi)^2 
- 4 \int_{B'_1} (\eta(h + \theta_m \phi) + (1 - \eta) w_m) \partial_n^+ h.
\]

Dividing by \( \theta_m^2 \) and recalling that \( h \partial_n^+ h = 0 \) in \( B'_1 \), we obtain
\[
(1 - \kappa_m) \left( \int_{B_1} |\nabla \hat{w}_m|^2 - \frac{3}{2} \int_{S_1} \hat{w}_m^2 - 4 \int_{B'_1} w_m \theta_m^+ \partial_n^+ h \right)
< \int_{B_1} \left[ |\nabla(\eta \phi)|^2 + |\nabla((1 - \eta) \hat{w}_m)|^2 + 2|\nabla(\eta \phi), \nabla((1 - \eta) \hat{w}_m)| \right]
- 4 \int_{B'_1} \left[ \frac{\eta \phi}{\theta_m} \partial_n^+ h + (1 - \eta) \frac{w_m \theta_m^+}{\theta_m^2} \partial_n^+ h \right] 
- \frac{3}{2} \int_{S_1} ((1 - \eta) \hat{w}_m + \eta \phi)^2 
= \int_{B_1} \left[ |\nabla(\eta \phi)|^2 + |\nabla((1 - \eta) \hat{w}_m)|^2 + 2|\nabla(\eta \phi), \nabla((1 - \eta) \hat{w}_m)| \right] 
- 4 \int_{B'_1} \frac{w_m \theta_m^+ \partial_n^+ h}{\theta_m^2} 
- \frac{3}{2} \int_{S_1} \hat{w}_m^2,
\]

since \( \eta \in C_0^\infty(B_1 \setminus \Lambda) \). Hence
\[
\int_{B_1} |\nabla \hat{w}_m|^2 < \kappa_m \int_{B_1} |\nabla \hat{w}_m|^2 + \frac{3}{2} (1 - \kappa_m) \int_{S_1} \hat{w}_m^2 + 4(1 - \kappa_m) \int_{B'_1} \frac{w_m \theta_m^+ \partial_n^+ h}{\theta_m^2} 
+ \frac{3}{2} \kappa_m \int_{S_1} \hat{w}_m^2 
= \kappa_m \int_{B_1} |\nabla \hat{w}_m|^2 - \frac{3}{2} \kappa_m \int_{S_1} \hat{w}_m^2 
- 4 \int_{B'_1} \frac{w_m \theta_m^+ \partial_n^+ h}{\theta_m^2} 
+ \frac{3}{2} \kappa_m \int_{S_1} \hat{w}_m^2 
\leq C \kappa_m + \int_{B_1} \left[ |\nabla(\eta \phi)|^2 + |\nabla((1 - \eta) \hat{w}_m)|^2 + 2|\nabla(\eta \phi), \nabla((1 - \eta) \hat{w}_m)| \right].
\]
Therefore

\[
\int_{B_1} (1 - (1 - \eta)^2) |\nabla \hat{w}_m|^2 \leq C \kappa_m + \int_{B_1} \left[ |\nabla (\eta \phi)|^2 + \hat{w}_m^2 |\nabla \eta|^2 - 2(1 - \eta) \hat{w}_m (\nabla \eta, \nabla \hat{w}_m) + 2(\nabla (\eta \phi), \nabla (1 - \eta) \hat{w}_m) \right].
\]

Passing to the limit \( m \to \infty \) we obtain

\[
\int_{B_1} (1 - (1 - \eta)^2) |\nabla \hat{w}|^2 \leq \int_{B_1} \left[ |\nabla (\eta \phi)|^2 + \hat{w}^2 |\nabla \eta|^2 - 2(1 - \eta) \hat{w} (\nabla \eta, \nabla \hat{w}) + 2(\nabla (\eta \phi), \nabla (1 - \eta) \hat{w}) \right].
\]

(6.10)

Notice that

\[
\int_{B_1} |\nabla (\eta \phi + (1 - \eta) \hat{w})|^2 = \int_{B_1} \left[ |\nabla (\eta \phi)|^2 + |\nabla ((1 - \eta) \hat{w})|^2 + 2(\nabla (\eta \phi), \nabla ((1 - \eta) \hat{w}) \right]
\]

\[
= \int_{B_1} \left[ |\nabla (\eta \phi)|^2 + \hat{w}^2 |\nabla \eta|^2 + (1 - \eta)^2 |\nabla \hat{w}|^2 - 2 \hat{w}(1 - \eta) \langle \nabla \hat{w}, \nabla \eta \rangle \right.
\]

\[
+ 2(\nabla (\eta \phi), \nabla (1 - \eta) \hat{w}) \right]
\]

hence (6.10) gives us,

\[
\int_{B_1} |\nabla \hat{w}|^2 \leq \int_{B_1} |\nabla (\eta \phi + (1 - \eta) \hat{w})|^2.
\]

By approximation we can drop the condition \( \phi \in L^\infty(B_1) \) and by considering open balls \( B \supset B_1 \setminus \Lambda \) we may choose \( \eta = 1 \) in \( B \) and \( \phi = \hat{w} \) outside \( B \). This will give

\[
\int_{B_1} |\nabla \hat{w}|^2 \leq \int_B |\nabla \phi|^2 + \int_{B \setminus B_1} |\nabla \hat{w}|^2,
\]

hence

\[
\int_B |\nabla \hat{w}|^2 \leq \int_B |\nabla \phi|^2,
\]

which proves the harmonicity of \( \hat{w} \) in \( B \).

**Step 3.** We next want to prove that

\[
\hat{w} = 0 \quad \mathfrak{H}^{n-1}\text{-a.e. in } \Lambda.
\]

(6.11)

We note that for the function \( h \) we have \( \partial_\alpha^+ h(x',0) \neq 0 \) for every \( (x',0) \in \Lambda^o \) (interior of \( \Lambda \) in \( \mathbb{R}^{n-1} \)). Therefore, given \( \omega \subset \Lambda^o \), there exists a constant \( C_\omega > 0 \) such that \( |\partial_\alpha^+ h(x',0)| \geq C_\omega \) for every \( (x',0) \in \omega \).

At points \( (x',0) \in \Lambda^o \), we can thus write

\[
\hat{w}_m = \frac{w_m - h}{\theta_m} = \frac{w_m}{\theta_m} \partial_\alpha^+ h \frac{\theta_m}{\partial_\alpha^+ h}.
\]

This gives

\[
\int_\omega |\hat{w}_m| \leq \frac{\theta_m}{C_\omega} \int_\omega \frac{w_m}{\theta_m} |\partial_\alpha^+ h| \leq \frac{C \theta_m}{C_\omega},
\]

where in the last inequality we have used (6.8) in Step 1 above. Since \( \theta_m \to 0 \), we conclude that \( \|\hat{w}_m\|_{L^1(\omega)} \to 0 \) as \( m \to \infty \). By the arbitrariness of \( \omega \subset \Lambda^o \) we infer that, in particular, we must have

\[
\hat{w}_m(x',0) \to 0, \quad \mathfrak{H}^{n-1}\text{-a.e. } (x',0) \in \Lambda
\]

(6.12)
which proves (6.11).

Step 4 (Proof of (i)). We next show that

\[ \hat{w}_m \to 0, \quad \text{weakly in } W^{1,2}(B_1), \]  

(6.13)
or, equivalently, \( \hat{w} = 0 \). We begin by observing that, since the \( \hat{w}_m \)'s are homogeneous of degree \( \frac{3}{2} \), their weak limit \( \hat{w} \) is also homogeneous of degree \( \frac{3}{2} \). Combining this observation with Steps 2 and 3 above, we then have the following properties for \( \hat{w} \):

(i) \( \Delta \hat{w} = 0 \) in \( B_1 \setminus \Lambda \);
(ii) \( \hat{w} = 0 \) \( H^{n-1} \) a.e. on \( \Lambda \);
(iii) \( \hat{w} \) is homogeneous of degree \( \frac{3}{2} \).

We next obtain an explicit representation for \( \hat{w} \). First, we note that \( \hat{w} \) is Hölder continuous up to the coincidence set \( \Lambda \) of \( h \). Indeed, this can be seen by making a bi-Lipschitz transformation \( T : B_1 \setminus \Lambda \to B_1^+ \) as in (b) on p. 501 of [AC85]. The function \( \tilde{w} = \hat{w} \circ T^{-1} \in W^{1,2}(B_1^+) \) solves a uniformly elliptic equation in divergence form

\[ \text{div}(b(x) \nabla \tilde{w}) = 0 \quad \text{in } B_1^+ \]

with bounded measurable coefficients \( b(x) = [b_{ij}(x)] \). We will also have

\( \tilde{w} = 0 \) on \( B_1^+ \)
in the sense of traces. Then, by the boundary version of De Giorgi-Nash-Moser regularity theorem (see, e.g., Theorem 8.29 in [GT01]) we have that \( \tilde{w} \) is \( C^\gamma \) up to \( B_1^+ \) for some \( \gamma > 0 \). Since \( \hat{w} = \tilde{w} \circ T(x) \), this implies that \( \hat{w} \in C^\gamma(B_1) \).

Once we know that \( \hat{w} \in C^\gamma(B_1) \), together with (i)–(iii) above, we can apply a theorem of De Silva and Savin on an expansion of harmonic functions in slit domains, see Theorem 3.3 in [DSS14a] (and also Theorem 4.5 in the same paper) which implies that there are constants \( a_1, a_2, \ldots, a_{n-1}, b, \) and \( c \) such that, for some \( \alpha \in (0, 1) \),

\[ |\hat{w}(x) - P_0(x)U_0(x) - cx_n| = O(|x|^{3/2 + \alpha}), \]

where

\[ U_0(x) = \frac{1}{\sqrt{2}} \sqrt{x_1 + \sqrt{x_1^2 + x_n^2}} = \Re(x_1 + i|x_n|)^{1/2}, \]

and

\[ P_0(x) = \sum_{k=1}^{n-1} a_k x_k + b \sqrt{x_1^2 + x_n^2}. \]

Since \( \hat{w} \) is \( 3/2 \)-homogeneous, we must have \( c = 0 \) and thus

\[ \hat{w}(x) = P_0(x)U_0(x). \]

Now, a direct computation shows that such \( \hat{w} \) will be harmonic in \( B_1 \setminus \Lambda \) only if

\[ a_1 + 2b = 0, \]

which implies the representation

\[ \hat{w}(x) = \frac{a_1}{2} \Re(x_1 + i|x_n|)^{3/2} + \sum_{j=2}^{n-1} a_j x_j \Re(x_1 + i|x_n|)^{1/2}, \]

\[ = \frac{a_1}{2} h(x) + \sum_{j=2}^{n-1} a_j x_j U_0(x). \]
We next show that all constants $a_j = 0$, $j = 1, \ldots, n - 1$. To simplify the notation we will write $\| \cdot \| = \| \cdot \|_{W^{1,2}(B_1)}$. We then use the fact that
\[
\|w_m - g\|^2 \geq \|w_m - h\|^2 \quad \text{for all } g \in H.
\]
Recalling that $\hat{w}_m = \frac{w_m - h}{\theta_m}$, we can write this as
\[
\|\theta_m \hat{w}_m + h - g\|^2 \geq \|\theta_m w_m\|^2,
\]
or
\[
2\theta_m \langle \hat{w}_m, h - g \rangle + \|h - g\|^2 \geq 0.
\]
Therefore,
\[
\langle \hat{w}_m, g - h \rangle \leq \frac{\|h - g\|^2}{2\theta_m}.
\]
(6.14)
Applying this to $g = (1 + \theta^2_m)h$, we obtain
\[
\langle \hat{w}_m, h \rangle \leq \frac{\theta_m}{2} \|h\|^2.
\]
Letting $m \to \infty$ we arrive at
\[
\langle \hat{w}, h \rangle = \frac{a_1}{2} \|h\|^2 \leq 0.
\]
This implies that $a_1 \leq 0$. Using the same argument for $g = (1 - \theta^2_m)h$ allows us to conclude that also $-a_1 \leq 0$, and therefore $a_1 = 0$. Further, rewriting (6.14) as
\[
\left\langle \frac{\hat{w}_m}{\theta^2_m}, g - h \right\rangle \leq \frac{\theta_m}{2} \left\| \frac{g - h}{\theta^2_m} \right\|^2,
\]
and taking for $j = 2, \ldots, n - 1$
\[
g = \Re(x_1 \cos(\theta^2_m) + \sin(\theta^2_m)x_j + i|x_n|)^{3/2},
\]
in such inequality, by letting $m \to \infty$ we obtain that
\[
\frac{3}{2} \langle \hat{w}, x_j U_0 \rangle = \frac{3}{2} a_j \|x_j U_0\|^2 \leq 0.
\]
(We note here that $\langle x_i U_0, x_j U_0 \rangle = 0$ for $i, j = 2, \ldots, n - 1$, $i \neq j$ and that
\[
\frac{\Re(x_1 \cos(\theta) + \sin(\theta)x_j + i|x_n|)^{3/2} - \Re(x_1 + i|x_n|)^{3/2}}{\theta} \to \frac{3}{2} x_j U_0(x)
\]
as $\theta \to 0$, strongly in $W^{1,2}(B_1)$.) Hence $a_1 \leq 0$. Replacing $x_j$ with $-x_j$ in the above argument, we also obtain $-a_1 \leq 0$. Thus, $a_j = 0$ for all $j = 1, \ldots, n - 1$, which implies $\hat{w} = 0$ and completes the proof of (6.13).

Step 5 (Proof of (ii)): Finally, we claim that, on a subsequence,
\[
\hat{w}_m \to 0 \quad \text{strongly in } W^{1,2}(B_1).
\]
(6.15)
Since we already have the strong convergence $\hat{w}_m \to \hat{w} = 0$ in $L^2(B_1)$, we are left with proving
\[
\nabla \hat{w}_m \to 0 \quad \text{strongly in } L^2(B_1).
\]
(6.16)
To this end, we pick \( \eta \in C^0(B_1) \), \( 0 \leq \eta \leq 1 \), and consider \( \zeta = (1 - \eta)w_m + \eta h \). Clearly, \( \zeta = w_m \) on \( S_1 \), \( \zeta \geq 0 \) on \( B'_1 \), and \( \zeta - h = (1 - \eta)(w_m - h) \). Applying (6.7) with this choice of \( \zeta \) we obtain

\[
(1 - \kappa_m) \left[ \int_{B_1} |\nabla (w_m - h)|^2 - \frac{3}{2} \int_{S_1} (w_m - h)^2 - 4 \int_{B'_1} w_m \partial_n^+ h \right]
\]

\[
< \int_{B_1} \| (1 - \eta) \nabla (w_m - h) - \nabla \eta (w_m - h) \|^2
\]

\[
- \frac{3}{2} \int_{S_1} (1 - \eta)^2 (w_m - h)^2 - 4 \int_{B'_1} (1 - \eta)w_m \partial_n^+ h.
\]

Dividing by \( \theta_m^2 \), and recalling that \( \hat{w}_m = \frac{w_m - h}{\theta_m} \), we obtain

\[
(1 - \kappa_m) \left( \int_{B_1} |\nabla \hat{w}_m|^2 - \frac{3}{2} \int_{S_1} \hat{w}_m^2 - 4 \int_{B'_1} \frac{w_m \partial_n^+ h}{\theta_m^2} \right)
\]

\[
< \int_{B_1} \left[ (1 - \eta)^2 |\nabla \hat{w}_m|^2 + \hat{w}_m^2 |\nabla \eta|^2 - 2(1 - \eta)\hat{w}_m \langle \nabla \hat{w}_m, \nabla \eta \rangle \right]
\]

\[
- \frac{3}{2} \int_{S_1} (1 - \eta)^2 \hat{w}_m^2 - 4 \int_{B'_1} (1 - \eta) \frac{w_m \partial_n^+ h}{\theta_m^2}.
\]

This gives

\[
\int_{B_1} |\nabla \hat{w}_m|^2 - 4 \int_{B'_1} \frac{w_m \partial_n^+ h}{\theta_m^2}
\]

\[
\leq \int_{B_1} \left[ (1 - \eta)^2 |\nabla \hat{w}_m|^2 + |\nabla \eta|^2 \hat{w}_m^2 - 2(1 - \eta)\hat{w}_m \langle \nabla \hat{w}_m, \nabla \eta \rangle \right]
\]

\[
- \frac{3}{2} \int_{S_1} (1 - \eta)^2 \hat{w}_m^2 - 4 \int_{B'_1} \frac{w_m \partial_n^+ h}{\theta_m^2} + (1 - \kappa_m) \frac{3}{2} \int_{S_1} \hat{w}_m^2
\]

\[
+ \kappa_m \left( \int_{B_1} |\nabla \hat{w}_m|^2 - 4 \int_{B'_1} \frac{w_m \partial_n^+ h}{\theta_m^2} \right)
\]

\[
= \int_{B_1} \left[ (1 - \eta)^2 |\nabla \hat{w}_m|^2 + |\nabla \eta|^2 \hat{w}_m^2 - 2(1 - \eta)\hat{w}_m \langle \nabla \hat{w}_m, \nabla \eta \rangle \right]
\]

\[
+ \kappa_m \left( \int_{B_1} |\nabla \hat{w}_m|^2 - 4 \int_{B'_1} \frac{w_m \partial_n^+ h}{\theta_m^2} \right)
\]

\[
+ \frac{3}{2} \int_{S_1} \hat{w}_m^2 - 4 \int_{B'_1} (1 - \eta) \frac{w_m \partial_n^+ h}{\theta_m^2}.
\]

If in this inequality we use the fact that \( \|\nabla \hat{w}_m\|_{L^2(B_1)} \leq \|\hat{w}_m\|_{W^{1,2}(B_1)} = 1 \), and that \( \frac{w_m \partial_n^+ h}{\theta_m^2} \) is uniformly bounded in \( L^1(B'_1) \), a fact which we have proved in (6.8) of Step 1, we obtain

\[
\int_{B_1} |\nabla \hat{w}_m|^2 - 4 \int_{B'_1} \frac{w_m \partial_n^+ h}{\theta_m^2}
\]

\[
\leq \int_{B_1} (1 - \eta)^2 |\nabla \hat{w}_m|^2 + |\nabla \eta|^2 \hat{w}_m^2 - 2(1 - \eta)\hat{w}_m \langle \nabla \hat{w}_m, \nabla \eta \rangle
\]

\[
+ C \kappa_m + \frac{3}{2} \int_{S_1} \hat{w}_m^2 - 4 \int_{B'_1} (1 - \eta) \frac{w_m \partial_n^+ h}{\theta_m^2}.
\]
We now make the choice in (6.17) of
\[ \eta(x) = \begin{cases} 
1, & \text{if } |x| \leq \frac{1}{2}, \\
2(1 - |x|), & \text{if } \frac{1}{2} < |x| < 1, \\
0, & \text{if } |x| \geq 1,
\end{cases} \]
we obtain
\[ \int_{B_{\frac{1}{2}}} |\nabla \hat{w}_m|^2 \leq \int_{B_1} [\nabla \eta]^2 \hat{w}_m^2 - 2(1 - \eta)\hat{w}_m \langle \nabla \hat{w}_m, \nabla \eta \rangle + \frac{3}{2} \int_{S_1} \hat{w}_m^2 + C\kappa_m + 4 \int_{B_r} \eta \hat{w}_m \frac{\partial}{\partial \eta} h^{+1} \]
\[ \leq \int_{B_1} [\nabla \eta]^2 \hat{w}_m^2 - 2(1 - \eta)\hat{w}_m \langle \nabla \hat{w}_m, \nabla \eta \rangle + \frac{3}{2} \int_{S_1} \hat{w}_m^2 + C\kappa_m, \]
since \( \eta, w_m \geq 0 \) and \( \partial^+ h \leq 0 \). We thus conclude that
\[ \int_{B_{\frac{1}{2}}} |\nabla \hat{w}_m|^2 \leq \int_{B_1} [\nabla \eta]^2 \hat{w}_m^2 - 2(1 - \eta)\hat{w}_m \langle \nabla \hat{w}_m, \nabla \eta \rangle + \frac{3}{2} \int_{S_1} \hat{w}_m^2 + C\kappa_m. \tag{6.18} \]
We now observe that, since \( \hat{w}_m \) is homogeneous of degree 3/2, and thus \( \nabla \hat{w}_m \) is homogeneous of degree 1/2, we have
\[ \int_{B_1} |\nabla \hat{w}_m|^2 = 2^{n+1} \int_{B_{\frac{1}{2}}} |\nabla \hat{w}_m|^2. \]
Using this identity in (6.18) we conclude that
\[ \int_{B_1} |\nabla \hat{w}_m|^2 \leq 2^{n+1} \left( \int_{B_1} [\nabla \eta]^2 \hat{w}_m^2 - 2(1 - \eta)\hat{w}_m \langle \nabla \hat{w}_m, \nabla \eta \rangle + \frac{3}{2} \int_{S_1} \hat{w}_m^2 + C\kappa_m \right). \]
To complete the proof of (6.15), and consequently of Theorem 6.3, all we need to do at this point is to observe that, on a subsequence, the right-hand side of the latter inequality converges to 0 as \( m \to \infty \). This follows from the facts that \( \kappa_m \to 0, \|\hat{w}_m\|_{L^1(B_1)} \to 0, \|\hat{w}_m\|_{L^1(S_1)} \to 0, \) and \( \|\nabla \hat{w}_m\|_{L^2(B_1)} \leq 1. \)

This completes the proof of the claim and that of the theorem. \( \square \)

7. \( C^{1,\beta} \) regularity of the regular part of the free boundary

In this final section we combine Theorems 4.3 and 6.3 to establish the \( C^{1,\beta} \) regularity of the regular part of the free boundary. We will consider two types of scalings: the Almgren one, defined in (3.2), and the homogeneous scalings defined in (5.1), which are suited for the study of regular free boundary points, i.e.,
\[ v_r(x) = \frac{v(rx)}{r^{3/2}}. \tag{7.1} \]
Throughout this section we continue to use the notation \( h(x) = \Re(x_1 + i|x_n|)^{2} \) adopted in Section 6. The symbol \( \theta > 0 \) will be used to exclusively denote the constant in the epiperimetric inequality of Theorem 6.3 above.

In Lemma 4.2 we showed that our Weiss type functional \( W_L(v, r) \) is bounded, when \( v \) is the solution to the problem (1.11)–(1.15). In the course of the proof of the next lemma we establish the much more precise statement that \( W_L(v, r) \leq Cr^\gamma \), for appropriate constants \( C, \gamma > 0 \). This gain is possible because of the assumption, in Lemma 7.1 below, that the scalings \( v_r \) have the epiperimetric property, i.e., the conclusion of the epiperimetric inequality holds for their extensions as 3/2-homogeneous functions in \( B_1. \)
Lemma 7.1. Let \( v \) be the solution of the thin obstacle problem (1.11)-(1.15), and suppose that \( 0 \in \Gamma_{3/2}(v) \). Assume the existence of radii \( 0 \leq s_0 < r_0 < 1 \) such that for every \( s_0 \leq r \leq r_0 \), if we extend \( v_r \big|_{S_r} \) as a 3/2-homogeneous function in \( B_1 \), call it \( w_r \), then there exists a function \( \zeta_r \in W^{1,2}(B_1) \) such that \( \zeta_r \geq 0 \) in \( B_1 \), \( \zeta_r = v_r \) on \( S_1 \) and

\[
W(\zeta_r) \leq (1 - \kappa)W(w_r),
\]

where \( \kappa \) is the constant in the epiperimetric inequality. Then, there exist universal constants \( C, \gamma > 0 \) such that for every \( s_0 \leq s \leq t \leq r_0 \) one has

\[
\int_{S_t} |v_t - v_s| \leq Ct^\gamma.
\]  

(7.2)

Proof. As before, we let \( L = \text{div}(A\nabla \cdot) \). The main idea of the proof of (7.2) is to relate \( \int_{S_t} |v_t - v_s| \) with the Weiss type functional \( W_L(v, t) \) defined in (4.1) above, and then control the latter in the following way:

\[
W_L(v, t) \leq Ct^\gamma, \quad 0 < t < r_0.
\]  

(7.3)

More specifically, combining equations (7.11) and (7.12) proved below, we obtain the following

\[
\int_{S_t} |v_t - v_s| \leq C(n) \int_s^t r^{-1/2} \left( \frac{d}{dr} W_L(v, r) + Cr^{-1/2} \right)^{1/2} dr.
\]  

(7.4)

After using Hölder’s inequality in the right-hand side of (7.4) we obtain

\[
\int_{S_t} |v_t - v_s| \leq C \left( \int_s^t r^{-1/2} dr \right)^{1/2} \left( \int_s^t \frac{d}{dr} W_L(v, r) + Cr^{-1/2} dr \right)^{1/2}
\]

\[
\leq C \left( \log \frac{t}{s} \right)^{1/2} \left( W_L(v, t) - W_L(v, s) + C(t^{1/2} - s^{1/2}) \right)^{1/2}
\]

\[
\leq C \left( \log \frac{t}{s} \right)^{1/2} \left( Ct^\gamma + Cs^{1/2} + C(t^{1/2} - s^{1/2}) \right)^{1/2}
\]

\[
\leq C \left( \log \frac{t}{s} \right)^{1/2} t^{\tilde{z}},
\]

where we have used (7.3), and we have estimated \(-W_L(v, s) \leq Cs^{1/2}\) using Corollary 4.5. With this estimate in hands we now use a dyadic argument. Assume that \( s \in [2^{-\ell}, 2^{-(\ell+1)}), t \in [2^{-h}, 2^{-(h+1)}) \) with \( h \leq \ell \) and apply the estimate above iteratively. We obtain

\[
\int_{S_t} |v_s - v_t| \leq \int_{S_t} |v_s - v_{2^{-(\ell+1)}}| + \cdots + \int_{S_t} |v_{2^{-h}} - v_t|
\]

\[
\leq C \log^{1/2} \left( \frac{2^{-h}}{s} \right) \left( 2^{-\ell+1} \right)^{1/2} + \cdots + C \log^{1/2} \left( \frac{t}{2^{-h}} \right) t^{\tilde{z}}
\]

\[
\leq C(\log 2)^{1/2} \left( 2^{\tilde{z}} \right)^{-\ell+1} + \cdots + C(\log 2)^{1/2} t^{\tilde{z}}
\]

\[
\leq C(\log 2)^{1/2} \sum_{j=h}^{\ell-1} \left( 2^{\tilde{z}} \right)^{-j} + C(\log 2)^{1/2} t^{\tilde{z}}
\]

\[
\leq C(\log 2)^{1/2} \left( 2^{\tilde{z}} \right)^{-h} + C(\log 2)^{1/2} t^{\tilde{z}} \leq Ct^{\tilde{z}},
\]

which yields the sought for conclusion (7.2).
Similarly, in the computations that follow we will estimate we obtain the following estimate for $H$.

where using (2.16) we have estimated $I$ as well as the identity $L$ with the observation that see (4.2). To simplify the notation we write $I_L$ and $H = H_L$. We start by observing that combining Lemma 2.1 with the observation that $L|x| = \text{div}(A(x)\nabla r) = \frac{n-1}{|x|}(1 + O(|x|))$ (see Lemma 4.1 in [GSVG14]), we obtain the following estimate for $H'(r)$:

$$H'(r) - \left(\frac{n-1}{r} + O(1)\right)H(r) = 2I(r). \quad (7.5)$$

In the computations that follow we will estimate $\frac{d}{dr}W_L(v, r)$ using formula (7.5), estimates (2.16), (2.17), as well as the identity $I(r) = D(r) + \int_{B_r} v f$, which gives $I'(r) = \int_{S_r} \langle A\nabla v, \nabla v \rangle + \int_{S_r} v f$. We thus have

$$\frac{d}{dr}W_L(v, r) \geq \frac{I'(r)}{r^{n+1}} - \frac{n+1}{r^{n+2}}I(r) - \frac{3}{2r^{n+2}}H'(r) + \frac{3(n+2)}{2r^{n+3}}H(r)$$

$$\geq \frac{1}{r^{n+1}} \int_{S_r} \langle A\nabla v, \nabla v \rangle - \frac{n+1}{r^{n+2}}D(r) + \frac{3(n+2)}{2r^{n+3}}H(r)$$

$$- \frac{3}{2r^{n+2}} \left(\frac{n-1}{r} H(r) + 2 \int_{S_r} v(A\nu, \nabla v) + CH(r)\right)$$

$$+ \frac{1}{r^{n+1}} \int_{S_r} v f - \frac{n+1}{r^{n+2}} \int_{B_r} v f$$

$$\geq \frac{1}{r^{n+1}} \int_{S_r} \langle A\nabla v, \nabla v \rangle - \frac{n+1}{r^{n+2}}D(r) + \frac{9}{2r^{n+3}}H(r) - \frac{3}{r^{n+2}} \int_{S_r} v(A\nu, \nabla v) - C r^{-1/2}$$

$$= \frac{1}{r^{n+1}} \int_{S_r} \langle A\nabla v, \nabla v \rangle - \frac{n+1}{r} W_L(v, r) - \frac{3(n-2)}{2r^{n+3}}H(r) - \frac{3}{r^{n+2}} \int_{S_r} v(A\nu, \nabla v) - C r^{-1/2},$$

where using (2.16) we have estimated $CH(r) \leq Cr^{n+2}$, $|\int_{S_r} v f| \leq C r^{n+\frac{1}{2}}$. Now

$$H(r) = \int_{S_r} \mu v^2 = \int_{S_r} v^2 + \int_{S_r} (\mu - 1)v^2$$

$$\leq \int_{S_r} v^2 + r \int_{S_r} v^2 \leq \int_{S_r} v^2 + Cr^{n-1+3} = \int_{S_r} v^2 + Cr^{n+3}.$$

Similarly,

$$\int_{S_r} \langle A\nabla v, \nabla v \rangle = \int_{S_r} |\nabla v|^2 + \int_{S_r} ((A(x) - A(0))\nabla v, \nabla v)$$

$$\leq \int_{S_r} |\nabla v|^2 + r \int_{S_r} |\nabla v|^2 \leq \int_{S_r} |\nabla v|^2 + Cr^{n+1}.$$
Finally,
\[
\int_{S_r} v(A\nu, \nabla v) = \int_{S_r} v(\nu, \nabla v) + \int_{S_r} v((A(x) - A(0))\nu, \nabla v)
\]
\[
\leq \int_{S_r} v(\nu, \nabla v) + Cr \int_{S_r} |v||\nabla v|
\]
\[
\leq \int_{S_r} v(\nu, \nabla v) + Cr \left( \int_{S_r} v^2 \int_{S_r} |\nabla v|^2 \right)^{\frac{1}{2}}
\]
\[
\begin{align*}
(2.16) & \leq \int_{S_r} v(\nu, \nabla v) + C r^{\frac{n+3}{2}} r^{\frac{n-3}{2}}
\end{align*}
\]
\[
= \int_{S_r} v(\nu, \nabla v) + C r^{n+2}.
\]

This implies
\[
\frac{d}{dr} W_L(v, r) \geq \frac{1}{r^{n+1}} \int_{S_r} |\nabla v|^2 - \frac{n + 1}{r} W_L(v, r) - \frac{3(n-2)}{2r^{n+3}} \int_{S_r} v^2 - \frac{3}{r^{n+2}} \int_{S_r} v(\nu, \nabla v) - C r^{-1/2}
\]
\[
= -\frac{n + 1}{r} W_L(v, r) + \frac{1}{r} \int_{S_1} |\nabla v_r|^2 - \frac{3(n-2)}{2r} \int_{S_1} v_r^2 - 3 \int_{S_1} v_r(\nu, \nabla v_r) - C r^{-1/2}
\]
\[
= -\frac{n + 1}{r} W_L(v, r) + \frac{1}{r} \int_{S_1} \left( (\nu, \nabla v_r)^2 - \frac{3}{2} v_r^2 + |\partial_r v_r|^2 - \frac{3}{2} \left( n - \frac{1}{2} \right) w_r^2 \right) - C r^{-1/2}.
\]

where $\partial_r v_r$ is the tangential derivative of $v_r$ along $S_1$. Let $w_r$ be the $\frac{3}{2}$-homogeneous extension of $v_r|_{S_1}$, then
\[
\int_{S_1} \left( |\partial_r v_r|^2 - \frac{3}{2} \left( n - \frac{1}{2} \right) w_r^2 \right) = \int_{S_1} \left( |\partial_r w_r|^2 - \frac{3}{2} \left( n - \frac{1}{2} \right) w_r^2 \right).
\]

Recalling that $w_r$ is homogeneous of degree $3/2$, we have on $S_1$ that $(\nabla w_r, \nu) = \frac{3}{2} w_r$. This gives
\[
\int_{S_1} \left( |\partial_r w_r|^2 - \frac{3}{2} \left( n - \frac{1}{2} \right) w_r^2 \right) = \int_{S_1} \left( |\nabla w_r|^2 - (\nabla w_r, \nu)^2 - \frac{3}{2} \left( n - \frac{1}{2} \right) w_r^2 \right)
\]
\[
= \int_{S_1} \left( |\nabla w_r|^2 - \frac{9}{4} w_r^2 - \frac{3}{2} \left( n - \frac{1}{2} \right) w_r^2 \right) = \int_{S_1} \left( |\nabla w_r|^2 - \frac{3}{2} (n + 1) w_r^2 \right).
\]

Now, since again by the homogeneity of $w_r$,
\[
\int_{S_1} |\nabla w_r|^2 = (n + 1) \int_{B_1} |\nabla w_r|^2,
\]
we conclude
\[
\int_{S_1} \left( |\partial_r w_r|^2 - \frac{3}{2} \left( n - \frac{1}{2} \right) w_r^2 \right) = (n + 1) W(w_r, 1),
\]
where we recall that, by definition (4.1) above,
\[
W(w, s) = W_{\Delta}(w, s) = \frac{1}{s^{n+1}} \int_{B_s} |\nabla w|^2 - \frac{3}{2s^{n+2}} \int_{S_s} w^2.
\]

Hence
\[
\frac{d}{dr} W_L(v, r) \geq \frac{n + 1}{r} (W(w_r, 1) - W_L(v, r)) + \frac{1}{r} \int_{S_1} (\nu, \nabla v_r) - \frac{3}{2} v_r^2 - C r^{-1/2}.
\]

(7.6)
By the hypothesis, for every \( s_0 \leq r \leq r_0 \) there exists a function \( \zeta_r \in W^{1,2}(B_1) \) such that \( \zeta_r \geq 0 \) in \( B'_1 \), \( \zeta_r = v_r \) on \( S_1 \) and
\[
W(\zeta_r, 1) \leq (1 - \kappa)W(v_r, 1). \tag{7.7}
\]
We note that this inequality continues to hold as \( \zeta_r \) we take the minimizer of the functional \( W(\cdot, 1) \) among all functions \( \zeta \in W^{1,2}(B_1) \) with \( \zeta|_{S_1} = v_r|_{S_1} \) and \( \zeta \geq 0 \) in \( B'_1 \). Taking such minimizer is equivalent to saying that \( \zeta_r \) is the solution of the thin obstacle problem in \( B_1 \) for the Laplacian with zero thin obstacle on \( B'_1 \) and boundary values \( v_r \) on \( S_1 \). In particular, with this choice of \( \zeta_r \) we will have \( W(\zeta_r, 1) \leq W(v_r, 1) \). Next,
\[
W(\zeta_r, 1) = \int_{B_1} |\nabla \zeta_r|^2 - \frac{3}{2} \int_{S_1} \zeta_r^2
\geq \int_{B_1} \langle A(rx) \nabla \zeta_r, \nabla \zeta_r \rangle - \frac{3}{2} \int_{S_1} \mu(rx) \zeta_r^2 - Cr \int_{B_1} |\nabla \zeta_r|^2 - Cr \int_{S_1} \zeta_r^2.
\]
If we now let \( \hat{\zeta} = \zeta_r(x/r)r^{3/2} \), then on \( S_r \) we have that \( \hat{\zeta} = v_r(x/r)r^{3/2} = v(x) \), and
\[
\int_{B_1} \langle A(rx) \nabla \zeta_r, \nabla \zeta_r \rangle - \frac{3}{2} \int_{S_1} \mu(rx) \zeta_r^2 = r^{-n-1} \int_{B_r} \langle A \nabla \hat{\zeta}, \nabla \hat{\zeta} \rangle - \frac{3}{2} \int_{S_r} \mu \hat{\zeta}^2
\]
Since \( \hat{\zeta} = v \) on \( S_r \), \( \hat{\zeta} \geq 0 \) on \( B'_r \) and \( v \) minimizes the energy \( (1.17) \) over \( B_r \) among all such functions, we obtain
\[
\int_{B_r} \langle A \nabla \hat{\zeta}, \nabla \hat{\zeta} \rangle - \frac{3}{2} \int_{S_r} \mu \hat{\zeta}^2 \geq \int_{B_r} \langle A \nabla v, \nabla v \rangle - 2f v.
\]
Next, by \( (2.16) \) we have \( |v(x)| \leq C|x|^{3/2} \leq Cr^{3/2} \) in \( B_r \). Besides, noting that \( \hat{\zeta} \) solves the thin obstacle problem in \( B_r \) with boundary values \( v \) on \( S_r \), by subharmonicity of \( \hat{\zeta}^\pm \), we will have that
\[
\sup_{B_r} |\hat{\zeta}| \leq \sup_{S_r} v^+ + \sup_{S_r} v^- \leq Cr^{3/2}.
\]
Hence, we obtain
\[
\int_{B_r} \langle A \nabla \hat{\zeta}, \nabla \hat{\zeta} \rangle \geq \int_{B_r} \langle A \nabla v, \nabla v \rangle + 2f v - 2f \hat{\zeta}
\geq \int_{B_r} \langle A \nabla v, \nabla v \rangle - Cr^{n+(3/2)}.
\]
Combining the inequalities above, we have
\[
W(\zeta, 1) \geq r^{-n-1} \int_{B_r} \langle A \nabla v, \nabla v \rangle - \frac{3}{2} r^{-n-2} \int_{S_r} \mu v^2 - Cr \int_{B_1} |\nabla \zeta_r|^2 - Cr \int_{S_1} \zeta_r^2 - Cr^{1/2}
= W_L(v, r) - Cr \left( W(\zeta, 1) + \frac{3}{2} \int_{S_1} \zeta_r^2 \right) - Cr \int_{S_1} \zeta_r^2 - Cr^{1/2}
\geq W_L(v, r) - Cr \left( W(v_r, 1) + \frac{3}{2} \int_{S_1} \zeta_r^2 \right) - Cr \int_{S_1} v_r^2 - Cr^{1/2}
= W_L(v, r) - Cr \left( \int_{B_1} |\nabla v_r|^2 - \frac{3}{2} \int_{S_1} v_r^2 + \frac{3}{2} \int_{S_1} \zeta_r^2 \right) - Cr \int_{S_1} v_r^2 - Cr^{1/2}
= W_L(v, r) - Cr \int_{B_1} |\nabla v_r|^2 - Cr \int_{S_1} v_r^2 - Cr^{1/2}
\geq W_L(v, r) - Cr^{1/2}. \tag{2.16}
\]

Combining (7.7) and (7.8) we obtain

\[
W(w_r, 1) - W_L(v, r) \geq \frac{W(\zeta_r, 1)}{1 - \kappa} - W_L(v, r) \geq \frac{W_L(v, r) - C r^{1/2}}{1 - \kappa} - W_L(v, r)
\]

\[
= \frac{\kappa}{1 - \kappa} W_L(v, r) - C r^{1/2}.
\]

(7.9)

Therefore, from (7.6) and (7.9) we conclude that

\[
\frac{d}{dr} W_L(v, r) \geq \frac{n + 1}{r} (W(w_r, 1) - W_L(v, r)) - C r^{-1/2}
\]

\[
\geq \frac{n + 1}{r} \left( \frac{\kappa}{1 - \kappa} W_L(v, r) - C r^{1/2} \right) - C r^{-1/2}
\]

\[
\geq \frac{n + 1}{r} \frac{\kappa}{1 - \kappa} W_L(v, r) - C r^{-1/2}.
\]

(7.10)

Taking \( \gamma \in (0, \frac{1}{2} \wedge (n + 1) \frac{\kappa}{1 - \kappa}) \), if we use that \( W_L(v, r) \geq -C r^{1/2} \), see Corollary 4.5, then from the above inequality we will have

\[
(W_L(v, r) r^{-\gamma})' \geq -C r^{-\gamma - 1/2}.
\]

Integrating in \((t, r_0)\), for \( t \geq s_0 \), we obtain

\[
W_L(v, t) r^{-\gamma} \leq W_L(v, r_0) r_0^{-\gamma} + C r_0^{-\gamma + 1/2} - C t^{-\gamma + 1/2}.
\]

This implies, in particular,

\[
W_L(v, t) \leq C t^\gamma,
\]

which establishes (7.3).

We now prove (7.4). For a fixed \( x \), define \( g(r) = \frac{v(rx)}{r^{3/2}} \), so that \( v_t(x) - v_s(x) = g(t) - g(s) \). Then

\[
\int_{S_1} |v_t - v_s| = \int_{S_1} \left| \int_s^t g'(r) dr \right|
\]

\[
\leq \int_s^t \left( \int_{S_1} r^{-1/2} \left( \nabla v(rx), \nu \right) - \frac{3}{2} \frac{v(rx)}{r} \right) dr
\]

\[
= \int_s^t \left( r^{-1} \int_{S_1} \left( \nabla v_r, \nu \right) - \frac{3}{2} v_r \right) dr
\]

\[
\leq C_n \int_s^t r^{-1/2} \left( r^{-1} \int_{S_1} \left( \nabla v_r, \nu \right) - \frac{3}{2} v_r \right)^2 \frac{1}{2} dr.
\]

By (7.6) and (7.9),

\[
\frac{1}{r} \int_{S_1} \left( \nabla v_r, \nu \right) - \frac{3}{2} v_r \right)^2 \leq \frac{d}{dr} W_L(v, r) + C r^{-1/2} - \frac{n + 1}{r} (W(w_r, 1) - W_L(v, r))
\]

\[
\leq \frac{d}{dr} W_L(v, r) - \frac{n + 1}{r} \left( \frac{k}{1 - k} \right) W_L(v, r) + C r^{-1/2}
\]

\[
\leq \frac{d}{dr} W_L(v, r) + C r^{-1/2},
\]

(7.12)

where we have used again Corollary 4.5, that gives \( W_L(v, r) \geq -C r^{1/2} \). This proves (7.4) and completes the proof. \( \square \)
The next important step after Lemma 7.1 is contained in Proposition 7.2 below. It proves that the regular set is a relatively open set of the free boundary, and that if \( x_0 \in \Gamma_{3/2}(v) \), then for \( r \) small enough and \( \bar{x} \in \Gamma(v) \) in a small neighborhood of \( x_0 \) we can apply Lemma 7.1 to the homogeneous scalings
\[
v_{\bar{x},r}(x) = \frac{v_{\bar{x}}(rx)}{r^{3/2}} = \frac{v(\bar{x} + A^{1/2}(\bar{x})rx) - b_\bar{x}rx_n}{r^{3/2}},
\]
which in turn proves the uniqueness of the blowup limits.

**Proposition 7.2.** Let \( v \) solve (1.12)–(1.15) and \( x_0 \in \Gamma_{3/2}(v) \). Then, there exist constants \( r_0 = r_0(x_0) \), \( \eta_0 = \eta_0(x_0) > 0 \) such that \( \Gamma(v) \cap B_{\eta_0}(x_0) \subset \Gamma_{3/2}(v) \). Moreover, if \( v_{\bar{x},0} \) is any blowup of \( v \) at \( \bar{x} \in \Gamma(\bar{u}) \cap B_{\eta_0}(x_0) \), as in Definition 5.5, then
\[
\int_{S_1} |v_{\bar{x},r} - v_{\bar{x},0}| \leq Cr^\gamma, \quad \text{for all } r \in (0, r_0),
\]
where \( C \) and \( \gamma > 0 \) are universal constants. In particular, the blowup limit \( v_{\bar{x},0} \) is unique.

**Proof.** Let \( r_0 \) and \( \eta_0 \) be as in Lemma 3.4. Then, for \( \bar{x} \in B_{\eta_0}(x_0) \cap \Gamma(v) \subset \Gamma_{3/2}(v) \) and \( 0 < r < r_0 \) consider two scalings, the homogeneous and Almgren types:
\[
v_{\bar{x},r}(x) = \frac{v_{\bar{x}}(rx)}{r^{3/2}}, \quad \tilde{v}_{\bar{x},r}(x) = \frac{v_{\bar{x}}(rx)}{\tilde{d}_{\bar{x},r}}
\]
By Lemma 3.4, the Almgren scaling \( \frac{1}{c_n} \tilde{v}_{\bar{x},r} \) on \( S_1 \) has the epiperimetric property in the sense that if we extend it as a \( 3/2 \)-homogeneous function in \( B_1 \), call it \( \tilde{w}_r \), then Lemma 3.4 allows us to apply the epiperimetric inequality to conclude that there exists \( \zeta_r \in W^{1,2}(B_1) \) such that \( \zeta_r = \frac{1}{c_n} \tilde{v}_{\bar{x},r} \) on \( S_1 \), \( \zeta_r \geq 0 \) in \( B_1^c \) and
\[
W(\zeta_r) \leq (1 - \kappa)W(\tilde{w}_r).
\]
We next observe that if a certain function on \( S_1 \) has the epiperimetric property then so does any of its multiples. In particular, \( v_{\bar{x},r} \leq \frac{\tilde{c}}{\tilde{d}_{\bar{x},r}} \tilde{v}_{\bar{x},r} \) has the epiperimetric property for any \( \bar{x} \in B_{\eta_0}(x_0) \cap \Gamma(v) \) and \( r < r_0 \). Thus, we can apply Lemma 7.1 to conclude that
\[
\int_{S_1} |v_{\bar{x},r} - v_{\bar{x},s}| \leq Cr^\gamma, \quad \text{for } 0 < s \leq r < r_0,
\]
for universal \( C \) and \( \gamma > 0 \). Now, if over some sequence \( v_{\bar{x},s_j} \to v_{\bar{x},0} \) (see Lemma 5.2), we will obtain that
\[
\int_{S_1} |v_{\bar{x},r} - v_{\bar{x},0}| \leq Cr^\gamma, \quad \text{for } 0 < r < r_0.
\]
\( \square \)

We notice explicitly that up to this point we have not excluded the possibility that the blowup \( v_{\bar{x},0} \equiv 0 \). This is done in the following proposition.

**Proposition 7.3.** The unique blowup \( v_{\bar{x},0} \) in Proposition 7.2 is nonzero.

**Proof.** Assume to the contrary that \( v_{\bar{x},0} = 0 \). Then, from Proposition 7.2 we have that
\[
\int_{S_1} |v_{\bar{x},r}| \leq Cr^\gamma, \quad \text{for } 0 < r < r_0.
\]
But then,
\[
\int_{S_1} |\tilde{v}_{\bar{x},r}| = \int_{S_1} |v_{\bar{x},r}| \frac{r^{3/2}}{\tilde{d}_{\bar{x},r}} \leq C \frac{r^{3/2+\gamma}}{\tilde{d}_{\bar{x},r}}.
\]
On the other hand, by Lemma 3.2 for every \( \varepsilon > 0 \) there exists \( r_\varepsilon > 0 \) such that

\[
d_{\xi,r} = \left( \frac{1}{r^{n-1}} H_{\xi}(r) \right) \geq c M_{\xi}(r)^{1/2} \geq cr^{(3+\varepsilon)/2}, \quad \text{for } 0 < r < r_\varepsilon
\]

If we choose \( \varepsilon < 2\gamma \), we then obtain as \( r \to 0^+ \)

\[
\int_{S_1} \left| \tilde{v}_{\xi,r} \right| \leq Cr^{\gamma-\varepsilon/2} \to 0.
\]

Since there exists \( e' \in \mathbb{R}^{n-1} \), and a subsequence \( r = r_j \to 0^+ \) such that \( \tilde{v}_{\xi,r} \to c_n \Re(\langle x', e' \rangle + i|x_n|)^{3/2} \), this is clearly a contradiction. \( \square \)

In what follows we denote by

\[
v_{\xi,0}(x) = a_{\xi} \Re(\langle x', \nu_{\xi} \rangle + i|x_n|)^{3/2}, \quad a_{\xi} > 0, \quad \nu_{\xi} \in S_1'
\]

the unique homogeneous blowup at \( \tilde{\xi} \).

**Proposition 7.4.** Let \( v \) be a solution of (1.12)–(1.15) with \( x_0 \in \Gamma_{3/2}(v) \). Then, there exists \( \eta_0 > 0 \) depending on \( x_0 \) such that

\[
\int_{S_1} |v_{\xi,0} - v_{\bar{y},0}| \leq C|\bar{x} - \bar{y}|^{\beta} \quad \text{for } \bar{x}, \bar{y} \in B_{\eta_0}(x_0) \cap \Gamma(v),
\]

where \( C \) and \( \beta > 0 \) are universal constants.

**Proof.** Let \( \eta_0 \) and \( r_0 \) be as in Proposition 7.2. Then, we will have

\[
\int_{S_1} |v_{\xi,0} - v_{\bar{y},0}| \leq C r^{\gamma} + \int_{S_1} |v_{\xi,r} - v_{\bar{y},r}|
\]

for any \( r < r_0 \) and \( \bar{x}, \bar{y} \in B_{\eta_0} \cap \Gamma(v) \). In this inequality we will chose \( r = |\bar{x} - \bar{y}|^{\sigma} \) with \( 0 < \sigma < 1 \) to be specified below. We then estimate the integral on the right-hand side of the above inequality. First we notice that

\[
v_{\xi}(z) = v(\bar{x} + A^{1/2}(\bar{x})z) - \langle A^{1/2}(\bar{x}) \nabla v(\bar{x}), e_n \rangle z_n
\]

\[
= v(\bar{x} + A^{1/2}(\bar{x})z) - \partial_n v(\bar{x}) \langle \bar{x} + A^{1/2}(\bar{x})z, e_n \rangle
\]

by using property (1.5) of the coefficient matrix \( A \). Therefore, denoting

\[
\xi(s) = [s\bar{x} + (1-s)\bar{y}] + [sA^{1/2}(\bar{x}) + (1-s)A^{1/2}(\bar{y})]z
\]

\[
p(s) = s\partial_n v(\bar{x}) + (1-s)\partial_n v(\bar{y})
\]

we will have

\[
v_{\xi}(z) - v_{\bar{y}}(z) = \int_0^1 \frac{d}{ds} \left( v(\xi(s)) - p(s)\langle \xi(s), e_n \rangle \right) ds
\]

\[
= \int_0^1 \left( \langle \nabla v(\xi(s)) - p(s)e_n, [\bar{x} - \bar{y}] + [A^{1/2}(\bar{x}) - A^{1/2}(\bar{y})]z \right)
\]

\[
- \left[ \partial_n v(\bar{x}) - \partial_n v(\bar{y}) \right] \left( [sA^{1/2}(\bar{x}) + (1-s)A^{1/2}(\bar{y})]z, e_n \right) ds
\]

where for the last term we have used the orthogonality \( \langle [s\bar{x} + (1-s)\bar{y}], e_n \rangle = 0 \). This gives

\[
|v_{\xi}(z) - v_{\bar{y}}(z)| \leq C(|\bar{x} - \bar{y}| + |z|)^{1/2}(|\bar{x} - \bar{y}| + |z||\bar{x} - \bar{y}|) + C|\bar{x} - \bar{y}|^{1/2}|z|.
\]
Using the above estimate we then obtain
\[
\int_{S_1} |v_{\bar{x}, r} - v_{\bar{y}, r}| = \int_{S_1} \frac{|v_{\bar{x}}(rz) - v_{\bar{y}}(rz)|}{r^{3/2}} \\
\leq C \left( (|\bar{x} - \bar{y}| + r)^{1/2} |\bar{x} - \bar{y}|(1 + r) + |\bar{x} - \bar{y}|^{1/2} r \right) \\
\leq C|\bar{x} - \bar{y}|^{1-\sigma} + C|\bar{x} - \bar{y}|^{(1-\sigma)/2} \leq C|\bar{x} - \bar{y}|^{(1-\sigma)/2},
\]
if we choose \( r = |\bar{x} - \bar{y}|^\sigma \) with \( 0 < \sigma < 1 \).

Going back to the beginning of the proof, we conclude
\[
\int_{S_1} |v_{\bar{x},0} - v_{\bar{y},0}| \leq C(|\bar{x} - \bar{y}|^{\sigma \gamma} + |\bar{x} - \bar{y}|^{(1-\sigma)/2}) = C|\bar{x} - \bar{y}|^{2\beta},
\]
with \( 2\beta = \gamma/(1 + 2\gamma) \) if we choose \( \sigma = 1/(1 + 2\gamma) \).

It remains to show that we can change the integration over the \((n-1)\)-dimensional sphere \( S_1 \) to the \((n-2)\)-dimensional \( S_0 \). To this end, we note that both \( v_{\bar{x},0} \) and \( v_{\bar{y},0} \) are solutions of the Signorini problem for the Laplacian with zero thin obstacle and therefore
\[
\Delta(v_{\bar{x},0} - v_{\bar{y},0})_\pm \geq 0.5
\]
Thus by the energy inequality we obtain that
\[
\int_{B_1} |\nabla(v_{\bar{x},0} - \nabla v_{\bar{y},0})|^2 \leq C \int_{B_1} |(v_{\bar{x},0} - v_{\bar{y},0})|^2 \leq C|\bar{x} - \bar{y}|^{2\beta}.
\]
(Recall that \( v_{\bar{x},0} \) are \( 3/2 \)-homogeneous functions with uniform \( C^{1,1/2} \) estimates). Then, using the trace inequality, we obtain that
\[
\int_{B_{1/2}} |v_{\bar{x},0} - v_{\bar{y},0}|^2 \leq C|\bar{x} - \bar{y}|^{2\beta}
\]
This is equivalent to
\[
\int_{S_1} |v_{\bar{x},0} - v_{\bar{y},0}|^2 \leq C|\bar{x} - \bar{y}|^{2\beta},
\]
and using Hölder’s inequality
\[
\int_{S_1} |v_{\bar{x},0} - v_{\bar{y},0}| \leq C|\bar{x} - \bar{y}|^\beta,
\]
as claimed. \( \square \)

**Lemma 7.5.** Let \( \nu \) be as in Proposition 7.4 and \( v_{\bar{x},0} = a_{\bar{x}} \Re(\langle x', v_{\bar{x}} \rangle + i |x_0|^3)^{3/2} \) be the unique homogeneous blowup at \( \bar{x} \in B_{\nu_0}'(x_0) \cap \Gamma(\nu) \). Then, for a constant \( C_0 \) depending on \( x_0 \) we have
\[
|a_{\bar{x}} - a_{\bar{y}}| \leq C_0 |\bar{x} - \bar{y}|^\beta \tag{7.14}
\]
\[
|v_{\bar{x}} - v_{\bar{y}}| \leq C_0 |\bar{x} - \bar{y}|^\beta, \tag{7.15}
\]
for \( \bar{x}, \bar{y} \in B_{\nu_0}'(x_0) \cap \Gamma(\nu) \).

\textsuperscript{5} Short proof: the only nontrivial place to verify the subharmonicity of \((v_{\bar{x},0} - v_{\bar{y},0})_+\) is at points \( z \in \mathbb{R}^{n-1} \) with \( v_{\bar{x},0}(z) > 0 \) and \( v_{\bar{y},0}(z) = 0 \); but near such \( z \), \( \Delta v_{\bar{x},0} = 0 \) and \( \Delta v_{\bar{y},0} \leq 0 \) by the Signorini conditions on \( \mathbb{R}^{n-1} \).
Proof. The first inequality follows from the observation that
\[ C_n a_{\bar{x}} = \|v_{\bar{x},0}\|_{L^1(S_1^c)}, \quad C_n a_{\bar{y}} = \|v_{\bar{y},0}\|_{L^1(S_1^c)} \]
with the same dimensional constant \( C_n \) and therefore
\[ C_n |a_{\bar{x}} - a_{\bar{y}}| \leq \|v_{\bar{x},0} - v_{\bar{y},0}\|_{L^1(S_1^c)} \leq C|\bar{x} - \bar{y}|^\beta, \]
which establishes (7.14). To prove (7.15), we first note that by (7.14)
\[ \|a_{\bar{x}}(z, \nu_{\bar{x}})^{3/2} - a_{\bar{y}}(z, \nu_{\bar{y}})^{3/2}\|_{L^1(S_1^c)} \leq C|\bar{x} - \bar{y}|^\beta \]
and therefore we obtain
\[ \|(z, \nu_{\bar{x}})^{3/2} - (z, \nu_{\bar{y}})^{3/2}\|_{L^1(S_1^c)} \leq C_0|\bar{x} - \bar{y}|^\beta. \]
(Here we have used that by (7.14) we may assume that \( a_{\bar{y}} > a_{x_0}/2 \) if \( \eta_0 \) is small).
On the other hand, it is easy to see from geometric considerations\(^6\) that
\[ \|(z, \nu_{\bar{x}})^{3/2} - (z, \nu_{\bar{y}})^{3/2}\|_{L^1(S_1^c)} \geq c_n|\nu_{\bar{x}} - \nu_{\bar{y}}| \]
implies that
\[ c_n|\nu_{\bar{x}} - \nu_{\bar{y}}| \leq C_0|\bar{x} - \bar{y}|^\beta. \]

We are now ready to prove the main result of this paper.

Proof of Theorem 1.2. We subdivide the proof into several steps.

Step 1. Let \( \eta_0 \) be as Proposition 7.2. We then claim that for any \( \varepsilon > 0 \) there is \( r_\varepsilon > 0 \) such that
\[ \|v_{\bar{x}, r} - v_{\bar{x}, 0}\|_{C^1(B_{r_\varepsilon}^c \cup B_1^c)} < \varepsilon, \quad \text{for } \bar{x} \in B_{\eta_0/2}(x_0) \cap \Gamma(v), \ r < r_\varepsilon. \]
Indeed, arguing by contradiction, we will have a sequence of points \( \bar{x}_j \in B_{\eta_0/2}(x_0) \cap \Gamma(v) \) and radii \( r_j \to 0 \) such that
\[ \|v_{\bar{x}_j, r_j} - v_{\bar{x}_j, 0}\|_{C^1(B_{r_j}^c \cup B_1^c)} \geq \varepsilon_0 \]
for some \( \varepsilon_0 > 0 \). Clearly, we may assume that \( \bar{x}_j \to \bar{x}_0 \in B_{\eta_0/2}(x_0) \cap \Gamma(v) \). Now, the scalings \( v_{\bar{x}_j, r_j} \) are uniformly bounded in \( C^{1,1/2}(B_{R}^\pm \cup B'_R) \) for any \( R > 0 \) and thus we may assume that
\[ v_{\bar{x}_j, r_j} \to w \quad \text{in } C^1_{loc}((\mathbb{R}^n)^c \cup \mathbb{R}^{n-1}). \]
We claim that actually \( w = v_{x_0,0} \). Indeed, by integrating the inequality in Proposition 7.2, we will have
\[ \|v_{\bar{x}, r} - v_{\bar{x}, 0}\|_{L^1(B_R)} \leq C_R r^\gamma, \quad \text{for } \bar{x} \in B_{\eta_0}(x_0) \cap \Gamma(v), \ r < r_0/R, \]
which will immediately imply that \( v_{\bar{x}, 0} \to w \) in \( L^1(B_R) \). On the other hand, Lemma 7.5 implies that \( v_{\bar{x}, 0} \to v_{x_0, 0} \) in \( C^1(B_{R}^\pm \cup B'_R) \). Hence \( w = v_{x_0,0} \). Moreover, we get that both \( v_{\bar{x}, r_j} \) and \( v_{\bar{x}, 0} \) converge in \( C^1(B_{r_j}^c \cup B_1^c) \) to the same function \( v_{x_0,0} \) and therefore
\[ \|v_{\bar{x}, r_j} - v_{\bar{x}, 0}\|_{C^1(B_{r_j}^c \cup B_1^c)} \to 0 \]
\(^6\)just notice that \( \partial_{\theta} \int_{\{z, e_1 \geq 1/2\} \cup \{z, e_2 \geq 1/2\}} (z, \cos \theta e_1 + \sin \theta e_2)^{3/2} |\theta = 0 = \frac{3}{2} \int_{\{z, e_1 \geq 1/2\} \cup \{z, e_2 \geq 1/2\}} (z, e_1)^{1/2}(z, e_2) > 0 \)
Step 2. For a given \( \varepsilon > 0 \) and a unit vector \( \nu \in \mathbb{R}^{n-1} \) define the cone 

\[
\mathcal{C}_\varepsilon(\nu) = \{ x' \in \mathbb{R}^{n-1} \mid \langle x', \nu \rangle \geq \varepsilon |x'| \}
\]

We then claim that for any \( \varepsilon > 0 \) there exists \( r_\varepsilon \) such that for any \( \bar{x} \in B_{\eta_0/2}(x_0) \cap \Gamma(\nu) \) we have 

\[
\mathcal{C}_\varepsilon(\nu) \cap B_{r_\varepsilon} \subset \{ v_{\bar{x}}(\cdot, 0) > 0 \}.
\]

Indeed, consider a cutout from the sphere \( S_{\nu}^{1/2} \) by the cone \( \mathcal{C}_\varepsilon(\nu) \)

\[
K_\varepsilon(\nu) = \mathcal{C}_\varepsilon(\nu) \cap S_{\nu}^{1/2}.
\]

Note that 

\[
K_\varepsilon(\nu) \in \{ v_{\bar{x},0}(\cdot, 0) > 0 \} \cap B^1 \quad \text{and} \quad v_{\bar{x},0}(\cdot, 0) \geq a_{\bar{x}}c_\varepsilon \quad \text{on} \quad K_\varepsilon(\nu_\varepsilon)
\]

for some universal \( c_\varepsilon > 0 \). Without loss of generality, by Lemma 7.5, we may assume that \( a_{\bar{x}} \geq a_0/2 \) for \( \bar{x} \in B_{\eta_0}(x_0) \cap \Gamma(\nu) \). Then, applying Step 1 above, we can find \( r_\varepsilon > 0 \) such that 

\[
v_{\bar{x},r}(\cdot, 0) > 0 \quad \text{on} \quad K_\varepsilon(\nu_\varepsilon), \quad \text{for all} \quad r < r_\varepsilon
\]

Scaling back by \( r \), we have 

\[
v_{\bar{x}}(\cdot, 0) > 0 \quad \text{on} \quad rK_\varepsilon(\nu_\varepsilon) = \mathcal{C}_\varepsilon(\nu) \cap S_{r/2}^1, \quad r < r_\varepsilon
\]

Taking the union over all \( r < r_\varepsilon \), we obtain 

\[
\mathcal{C}_\varepsilon(\nu) \cap B_{r_\varepsilon/2} \subset \{ v_{\bar{x}}(\cdot, 0) > 0 \}.
\]

Step 3. We next claim that for any \( \varepsilon > 0 \) there exists \( r_\varepsilon \) such that for any \( \bar{x} \in B_{\eta_0/2}(x_0) \cap \Gamma(\nu) \) we have 

\[
-\mathcal{C}_\varepsilon(\nu) \cap B_{r_\varepsilon} \subset \{ v_{\bar{x}}(\cdot, 0) = 0 \}.
\]

To prove that we note that 

\[
-K_\varepsilon(\nu_\varepsilon) \in \{ v_{\bar{x},0}(\cdot, 0) = 0 \} \cap B^1 \quad \text{and} \quad \langle \partial_\nu - \partial^+ \rangle v_{\bar{x},0}(\cdot, 0) \geq a_{\bar{x}}c_\varepsilon > (a_0/2)c_\varepsilon \quad \text{on} \quad -K_\varepsilon(\nu_\varepsilon)
\]

for a universal \( c_\varepsilon > 0 \). Then, from Step 1, we will have the existence of \( r_\varepsilon > 0 \) such that 

\[
\langle A_{\bar{x}}(rx)\nabla v_{\bar{x},r}(x), e^-_n \rangle + \langle A_{\bar{x}}(rx)\nabla v_{\bar{x},r}(x), e^+_n \rangle > 0 \quad \text{on} \quad -K_\varepsilon(\nu_\varepsilon), \quad \text{for all} \quad r < r_\varepsilon.
\]

Thus, from the complementarity conditions, we will have 

\[
v_{\bar{x},r}(\cdot, 0) = 0 \quad \text{on} \quad -K_\varepsilon(\nu_\varepsilon), \quad \text{for all} \quad r < r_\varepsilon.
\]

Arguing as in the end of Step 2, we conclude that 

\[
-\mathcal{C}_\varepsilon(\nu) \cap B_{r_\varepsilon/2} \subset \{ v_{\bar{x}}(\cdot, 0) = 0 \}.
\]

Step 4. Here without loss of generality we will assume that \( A(x_0) = I \) and \( \nu_{x_0} = e_{n-1} \). Changing from function \( v_{\bar{x}} \) to \( v \), we may rewrite the result of Steps 2 and 3 as 

\[
\bar{x} + A(\bar{x})^{1/2}(\mathcal{C}_\varepsilon(\nu_\varepsilon) \cap B_{r_\varepsilon/2}) \subset \{ v(\cdot, 0) > 0 \}.
\]

\[
\bar{x} - A(\bar{x})^{1/2}(\mathcal{C}_\varepsilon(\nu_\varepsilon) \cap B_{r_\varepsilon/2}) \subset \{ v(\cdot, 0) = 0 \}.
\]
for any $\bar{x} \in B'_{\eta_0}(x_0) \cap \Gamma(v)$. Taking $\bar{x}$ sufficiently close to $x_0$ (and using Lemma 7.5) we can guarantee that

$$A(\bar{x})^{1/2}(\mathcal{E}_x(v_\mathcal{E}) \cap B'_{r/4}) \supset \mathcal{E}_x(e_{n-1}) \cap B'_{r/4}.$$ 

Hence, there exists $\eta_\epsilon > 0$ such that

$$\bar{x} + (\mathcal{E}_x(e_{n-1}) \cap B'_{r/4}) \subset \{v(\cdot, 0) > 0\},$$

$$\bar{x} - (\mathcal{E}_x(e_{n-1}) \cap B'_{r/4}) \subset \{v(\cdot, 0) = 0\}$$

for any $\bar{x} \in B'_{\eta_0}(x_0) \cap \Gamma(v)$. Now, fixing $\epsilon = \epsilon_0$, by the standard arguments, we can conclude that there exists a Lipschitz function $g : \mathbb{R}^{n-2} \to \mathbb{R}$ with $|\nabla g| \leq C_n/\epsilon_0$ such that

$$B'_{\eta_0}(x_0) \cap \{v(\cdot, 0) = 0\} = B'_{\eta_0}(x_0) \cap \{x_{n-1} \leq g(x'')\}$$

$$B'_{\eta_0}(x_0) \cap \{v(\cdot, 0) > 0\} = B'_{\eta_0}(x_0) \cap \{x_{n-1} > g(x'')\}$$

Step 5. Using the normalization $A(x_0) = I$ and $\nu_{x_0} = e_{n-1}$ as in Step 4 and letting $\epsilon \to 0$ we see that $\Gamma(v)$ is differentiable at $x_0$ with normal $\nu_0$. Recentering at any $\bar{x} \in B'_{\eta_0}(x_0) \cap \Gamma(v)$, we see that $\Gamma(v)$ has a normal $A(\bar{x})^{-1/2} \nu_{\bar{x}}$

at $\bar{x}$. Finally noting that by Lemma 7.5 the mapping $\bar{x} \mapsto A(\bar{x})^{-1/2} \nu_{\bar{x}}$ is $C^\beta$, we obtain that the function $g$ in Step 4 is $C^{1,\beta}$. The proof is complete. \qed

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