A new model of black hole in Lovelock Unique Vacuum theory using the gravitational decoupling method

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Abstract

We show a new exact solution of black hole for the Lovelock theory with Unique Vacuum (LUV). For this, we provide an extension of the Gravitational Decoupling (GD) algorithm for LUV, which provide a simplest way to solve the equations of motion. As a consequence of the application of the GD, the total mass of black hole varies. Our solution can have up to three horizons: two inner horizons plus the black hole horizon. This horizons structure is a proper characteristic of our solution. The temperature is an increasing function of the black hole horizon and the heat capacity is always positive, i.e. the solution is stable. Both the horizons structure as the thermodynamics behavior differs from the other black hole solutions previously studied in the literature for LUV.
I. INTRODUCTION

Finding new solutions of physical interest to the Einstein field equations is not an easy task due to the highly nonlinear behavior of its equations of motion. In this connection, in (2017) it was proposed the Gravitational Decoupling method (GD) [1], which represents an easy algorithm to decouple gravitational sources in General Relativity. One interesting extension of this method in reference [2]. This algorithm involves a Minimal Geometric Deformation (MGD) to the metric tensor of one seed solution. Furthermore, the sources are decoupled in one seed and one extra source. The method was described in reference [3] as follows: “given two gravitational sources: a seed source A and an extra source B, standard Einstein’s equation are first solved for A, and then a simpler set of quasi-Einstein equations are solved for B. Finally, the two solutions can be combined in order to derive the complete solution for the total system”. Since its appearance, by applying the algorithm to previously known seed solutions, this method has served to find several new solutions of physical interest for GR, as for example stellar distributions [3–13] and black hole solutions [5, 14–18]. See other applications in references [19–34].

The recent detection of gravitational waves through the collision of two rotating black holes [35, 36], together with the also recent assignment of the Nobel Prize, have positioned to the black holes as one of the most interesting and intriguing objects in gravitation. In this connection, the fact that the black holes, due to quantum fluctuations, emit as black bodies, where its temperature is related to its surface gravity [37], shows that in these objects the geometry and thermodynamics are directly connected.

As mentioned in the previous paragraph, applying GD method to the Schwarzschild solution, new black hole solutions have been found. As for example, three interesting solutions, characterized by isotropic, conformal and barotropic equations of state, were found in reference [14]. Other interesting solutions with hair were found in reference [16]. Non-trivial extensions of the Kerr and Kerr-Newman black holes with primary hair were found in reference [15]. Thus, it is undoubtedly interesting the search of new black hole solutions by using the GD method and test its physical behaviors.

On the other hand, during the last years, several branches of theoretical physics have predicted the existence of extra dimensions. Now, considering higher dimension in gravity opens up a range of new possibilities that retain the core of the Einstein gravity in four
dimensions. Lovelock theory \[38\] is a one of these possibilities, as, although includes higher powers of curvature corrections in the action, its equations of motion are of second order on the metric and the energy momentum tensor is conserved. It is worth to mention that the higher curvature terms can modify the structure of the black hole solutions, showing different properties with respect to the solutions in General Relativity. So, the study of BH solutions with presence of higher curvature terms is undoubtedly interesting from the physical point of view. One interesting extension of the Gravitational Decoupling algorithm for Pure Lovelock gravity \[39\] was developed in reference \[23\], showing a regular black hole solution. Some examples of recent studies about black holes in Lovelock gravity in references \[40–44\].

One potential drawback of the generic Lovelock gravity is the existence of more than a single ground state, namely more than a single constant curvature spaces solution, or equivalently, more than a single potential effective cosmological constants \[45\]. The potential effective cosmological constants can be complex numbers, which makes those ground states unstable under dynamical evolution. One way to avoid this is by choosing the coupling constant \(\{\gamma_n\}\) such that the equations of motion have, roughly speaking, the form \((R - \Lambda)^n = 0\). In this case there is a single, but \(n\)-fold degenerated ground state of constant curvature. This case is known as Lovelock with unique vacuum (LUV) or Lovelock with \(n\) fold degenerated ground state. The vacuum and static solution with an AdS ground state was studied in references \[46, 47\] and with a dS ground state in reference \[48\]. Regular black hole solutions for this theory have been studied in references \[49, 50\].

In this work we will provide an extension of the Gravitational Decoupling algorithm for Lovelock with Unique Vacuum theory, in order to obtain a new analytical black hole solution. We will compare the horizons structure and the thermodynamics behavior of our solution with the solutions previously studied in the literature. Furthermore we will test the consequences of the application of the GD method on the computation of the conserved charges.
II. GENERIC LOVELOCK THEORY AND THE CASE WITH UNIQUE VACUUM

The generic Lovelock Lagrangian is:

\[ L = \sqrt{-g} \sum_{n=0}^{N} \gamma_n L_n, \]  

(1)

where \( N = \frac{d}{2} - 1 \) for \( d \) even and \( N = \frac{d-1}{2} \) for \( d \) odd and, \( \gamma_n \) are arbitrary coupling constants. \( L_n \) is a topological density defined as:

\[ L_n = \frac{1}{2n} \delta^{\mu_1 \nu_1 \ldots \mu_n \nu_n}_{\alpha_1 \beta_1 \ldots \alpha_n \beta_n} \prod_{r=1}^{n} R^{\alpha_r \beta_r}_{\mu_r \nu_r}, \]  

(2)

where \( R^{\alpha \beta}_{\mu \nu} \) is a \( n \) order generalization of the Riemann tensor for the Lovelock theory, and:

\[ \delta^{\mu_1 \nu_1 \ldots \mu_n \nu_n}_{\alpha_1 \beta_1 \ldots \alpha_n \beta_n} = \frac{1}{n!} \delta_{\alpha_1 \beta_1} \ldots \delta_{\alpha_n \beta_n} \]  

(3)

is the generalized Kronecker delta.

It is worth to stress that, the terms \( L_0, L_1 \) and \( L_2 \) are proportional to the cosmological constant, Ricci Scalar and the Gauss Bonnet Lagrangian, respectively. The corresponding equation of motion is given by:

\[ \sum_{n=0}^{N} \gamma_n G^{(n)}_{AB} = T_{AB}, \]  

(4)

where \( G^{(n)}_{AB} \) is a \( n \) order generalization of the Einstein tensor due to the topological density \( L_n \). As example \( G^{(1)}_{AB} \) is just the Einstein tensor, \( G_{AB} \), associated with the Ricci scalar and \( G^{(2)}_{AB} \) is the Lanczos tensor, \( H_{AB} \), associated with the Gauss Bonnet Lagrangian.

As for example, the Einstein Gauss Bonnet equations of motion up to \( n = 2 \), with cosmological constant are:

\[ \gamma_1 G^A_B + \gamma_0 \Lambda \delta^A_B + \gamma_2 H^A_B = T^A_B, \]  

(5)

where the Lanczos tensor is:

\[ H_{AB} = 2 \left( RR_{AB} - 2R_{AC}R^C_B - 2R^{CD}R_{ACBD} + R^CDE_{A}R_{BCDE} \right) - \frac{1}{2} g_{AB}L_2. \]  

(6)
A. Lovelock with unique vacuum

As mentioned in the introduction, Lovelock theory can be factorized in several effective cosmological constants \([45]\). In this connection, for \(\gamma_p = 0\) from \(p > I\), the vacuum equations of motion can be written as \([49]\):

\[
G^\mu_{(LL)} = \delta^{\alpha_1 \beta_1 ... \alpha_I \beta_I \mu}_{\alpha_1 \beta_1} (R_{\alpha_1 \beta_1} + \kappa_1 \delta_{\alpha_1 \beta_1} \mu \nu) \cdots (\nabla_{\alpha_I \beta_I} + \kappa_I \delta_{\alpha_I \beta_I} \nu) = 0.
\] (7)

This shows, as expected, that the Lovelock gravity can be factorized in several ground states of constant curvature. To analyze those backgrounds one can introduce the ansatz \(R_{\alpha_1 \beta_1} = x \delta_{\alpha_1 \beta_1} \). This maps (7) into \(G^\mu = \delta^\mu = P_l(x) \delta^\mu\) where

\[
P_l(x) = \sum_{p=0}^{I} \gamma_p x^p = (x + \kappa_1) \cdots (x + \kappa_I).
\] (8)

where \(\kappa_i\) must be a real number \(\forall \gamma_p \in \mathbb{R}\)

One way of avoiding equations of motion with several ground states is consider the following action:

\[
S_n = \int \sum_{p=0}^{p=n} \gamma_p^n L_n.
\] (9)

In order to obtain \(\kappa_1 = \kappa_2 = \cdots = \kappa_I\), i.e. a \(n\)-fold degenerated vacuum state, we choose the following coupling constants \([46]\)

\[
\gamma_p^n = \begin{cases} \frac{\gamma_0}{d-2p} \binom{n}{p} & \text{for } 0 \leq p \leq n \\ 0 & \text{for } n < p \leq N \end{cases}
\] (10)

So, in references \([46, 47]\) was showed that, using this coupling constants, the equations of motion adopt the following form:

\[
\frac{\delta}{\delta g_{\mu \nu}} L \sqrt{g} \sim ((R \pm l^{-2})^n)^{\mu \nu} = 0.
\] (11)

So, the Lovelock theory has a unique vacuum (A)dS, but \(n\)-fold degenerated.

III. LUV EQUATIONS OF MOTION FOR MULTIPLES SOURCES

In this work we study the static \(d\) dimensional spherically symmetric metric:
\[ ds^2 = -\mu(r) dt^2 + \frac{dr^2}{\mu(r)} + r^2 d\Omega^2_{d-2}, \] (12)

where \( d\Omega^2_{d-2} \) corresponds to the metric of a \((d - 2)\) unitary sphere. The energy momentum tensor corresponds to a neutral perfect fluid:

\[ T^A_B = \text{diag}(-\rho, p_r, p_\theta, p_\phi, ...), \] (13)

where, from the spherical symmetry, we have for all the \((d - 2)\) angular coordinates that \( p_\theta = p_\phi = ... \). It is worth to mention that this form of metric \((12)\) imposes that \( T^0_0 = T^1_1 \), i.e. \(-\rho = p_r\).

For the cosmological constant:

\[ \Lambda = \mp \frac{(d - 1)(d - 2)}{l^2}, \] (14)

the equations of motion are \([46, 49]\):

\[ \frac{d}{dr} \left( r^{d-1} \left[ \frac{1 - \mu(r)}{r^2} \pm \frac{1}{l^2} \right] \right) = r^{d-2} \rho(r) \] (15)

with \( d - 2n - 1 \geq 0 \). The conservation law \( T_{iB}^{;AB} = 0 \) gives:

\[ p'_r + \frac{d - 2}{r}(p_r - p_\theta) = 0. \] (16)

We start by decoupling the energy momentum tensor into a seed energy momentum tensor, \( T^A_B \), and \( n \) extra sources

\[ T^A_B = T^A_B + \alpha(\theta_1)_B^A + \alpha^2(\theta_2)_B^A + ... + \alpha^{n-1}(\theta_{n-1})_B^A + \alpha^n(\theta_n)_B^A \] (17)

therefore the number of sources is determined by the value of \( n \), i.e depend on the power of the Riemann tensor. The seed energy momentum tensor has the form \( T^A_B = \text{diag}(-\bar{\rho}, \bar{p}_r, \bar{p}_\theta, \bar{p}_\phi, ...). \) So, it is easy to check that:

\[ \rho = \bar{\rho} - \alpha(\theta_1)_0^0 - \alpha^2(\theta_2)_0^0 - ... - \alpha^{n-1}(\theta_{n-1})_0^0 - \alpha^n(\theta_n)_0^0 \] (18)

\[ p_r = \bar{p}_r + \alpha(\theta_1)_1^1 + \alpha^2(\theta_2)_1^1 + ... + \alpha^{n-1}(\theta_{n-1})_1^1 + \alpha^n(\theta_n)_1^1 \] (19)

\[ p_\theta = \bar{p}_\theta + \alpha(\theta_1)_2^2 + \alpha^2(\theta_2)_2^2 + ... + \alpha^{n-1}(\theta_{n-1})_2^2 + \alpha^n(\theta_n)_2^2 \] (20)
Due that the form of metric imposes that \(-\rho = p_r\), we impose arbitrarily that \(-\bar{\rho} = \bar{p}_r\) and \((\theta_i)_0^0 = (\theta_i)_1^1\). On the other hand, due that \(p_{\theta} = p_{\phi} = \ldots\), we impose arbitrarily that \(\bar{p}_{\theta} = \bar{p}_{\phi} = \ldots\) and \((\theta_i)_2^2 = (\theta_i)_3^3 = \ldots\).

Thus, replacing equation (18) into the equation (15), we obtain the \((t, t)\) and \((r, r)\) components of the equations of motion:

\[
\frac{d}{dr} \left( r^{d-1} \left[ 1 - \frac{\mu(r)}{r^2} \pm \frac{1}{l^2} \right]^n \right) = r^{d-2} \left( \bar{\rho} - \alpha (\theta_1)_0^0 - \alpha^2 (\theta_2)_0^0 - \ldots - \alpha^{n-1} (\theta_{n-1})_0^0 - \alpha^n (\theta_n)_0^0 \right)
\]

We solve the \((t, t)\) and \((r, r)\) components of the LUV equations together with the conservation equation. Using the Bianchi identities, we ignore the remaining \((\theta, \theta) = (\phi, \phi) = \ldots\) components (the suspense points indicate that all the tangential components of the LUV equations are similar).

By inserting equations (18), (19) and (20) into equation (16):

\[
\bar{p}_r' + \frac{d - 2}{r} (\bar{p}_r - \bar{p}_t) + \alpha \left( ((\theta_1)_1^1)' + \frac{d - 2}{r} ((\theta_1)_1^1 - (\theta_1)_2^2) \right) + \alpha^2 \left( ((\theta_2)_1^1)' + \frac{d - 2}{r} ((\theta_2)_1^1 - (\theta_2)_2^2) \right) + \ldots + \alpha^{n-1} \left( ((\theta_{n-1})_1^1)' + \frac{d - 2}{r} ((\theta_{n-1})_1^1 - (\theta_{n-1})_2^2) \right) + \alpha^n \left( ((\theta_n)_1^1)' + \frac{d - 2}{r} ((\theta_n)_1^1 - (\theta_n)_2^2) \right) = 0
\]

Thus, the system to solve corresponds to equations (21) and (22).

The LUV equations of motion for the seed energy momentum tensor are recovered for the limit \(\alpha \to 0\):

\[
\nabla_A T^A_B = 0,
\]

For \(n = 1\) both components of energy momentum tensor are directly conserved, i.e \(\nabla_A (\theta_1)_B^A = 0\). However, for \(n > 1\) one can notice that:

\[
\alpha \nabla_A (\theta_1)_B^A + \alpha^2 \nabla_A (\theta_2)_B^A + \ldots + \alpha^{n-1} \nabla_A (\theta_{n-1})_B^A + \alpha^n \nabla_A (\theta_n)_B^A = 0,
\]

where the covariant derivative is computed by using the line element (12). In this work we impose in arbitrarily way that:

\[
\alpha^i \nabla_A (\theta_i)_B^A = 0,
\]
with \( \alpha^i \neq 0 \). Thus, we will solve the system (21), (23) and (25). Under this assumption each source is separately conserved, and thus, there is no exchange of energy momentum between them. Therefore, our energy momentum tensor (17) is a specific way of decoupling the system inspired by the approach of reference [1].

IV. GRAVITATIONAL DECOUPLING BY MGD FOR LUV

We start with a solution to the equations (21), (23) and (25) with \( \alpha = 0 \), namely seed solution \( \{ \bar{\mu}, \bar{\rho}, \bar{p}_r, \bar{p}_\theta \} \), where \( \bar{\mu} \) is the corresponding metric functions:

\[
\begin{align*}
ds^2 &= -\mu(r)dt^2 + \bar{\mu}(r)^{-1}dr^2 + r^2d\Omega^2_{d-2}.
\end{align*}
\] (26)

Turning on the parameter \( \alpha \), the effects of the sources \( (\theta_i)_{AB} \) appear on the seed solution. These effects can be encoded in the geometric deformation undergone by the seed fluid geometry \( \{ \bar{\mu} \} \) in equation (26) as follows:

\[
\bar{\mu}(r) \rightarrow \mu(r) = \bar{\mu}(r) - \alpha g(r).
\] (27)

This is known as Minimal Geometric Deformation [1].

Replacing equation (27) into (21), we use the binomial theorem as follows in the left side of this equation (21):

\[
\left[ \left( \frac{1 - \bar{\mu}(r)}{r^2} \right)^\pm \frac{1}{l^2} + \left( \frac{\alpha g}{r^2} \right) \right]^n = \left( \frac{1 - \bar{\mu}(r)}{r^2} \right)^\pm \frac{1}{l^2} +
\]

\[
\left( \frac{1 - \bar{\mu}(r)}{r^2} \right)^{n-1} \left( \frac{g}{r^2} \right)^\pm \frac{1}{l^2} + \ldots + \left( \frac{1 - \bar{\mu}(r)}{r^2} \right)^{n-1} \left( \frac{\alpha g}{r^2} \right)^{n-1} \alpha^{n-1} +
\]

\[
\left( \frac{g}{r^2} \right)^n \alpha^n
\] (28)

Thus, the system splits into the following sets of equations:

- The standard LUV equations for a seed solution (with \( \alpha = 0 \)):

\[
\frac{d}{dr} \left( r^{d-1} \left[ \frac{1 - \bar{\mu}(r)}{r^2} \pm \frac{1}{l^2} \right]^n \right) = r^{d-2} \bar{\rho}
\] (29)

and the conservation equation given by:

\[
\bar{p}'_r + \frac{d-2}{r} (\bar{p}_r - \bar{p}_\theta) = 0
\] (30)
• The terms of order $\alpha$ give rise to the quasi-LUV equations of order $\alpha^1$,
\[
\frac{d}{dr} \left( n \cdot r^{d-1} \left( \frac{1 - \bar{\mu}(r)}{r^2} \pm \frac{1}{l^2} \right) \frac{g}{r^2} \right) = -r^{d-2}(\theta_1)_0^0
\]
and the conservation equation given by:
\[
((\theta_1)_1^1)' + \frac{d-2}{r}((\theta_1)_1^1 - (\theta_1)_2^1) = 0
\]
Thus, following the iteration, it is possible to obtain the quasi LUV equations of order $\alpha^2$, $\alpha^3$...$\alpha^{n-3}$, $\alpha^{n-2}$.

• The terms of order $\alpha^{n-1}$ give rise to the following quasi LUV equations of order $\alpha^{n-1}$:
\[
\frac{d}{dr} \left( n \cdot r^{d-1} \left( \frac{1 - \bar{\mu}(r)}{r^2} \pm \frac{1}{l^2} \right) \frac{g}{r^2} \right) = -(\theta_{n-1})_0^0
\]
and the conservation equation given by:
\[
((\theta_{n-1})_1^1)' + \frac{d-2}{r}((\theta_{n-1})_1^1 - (\theta_{n-1})_2^1) = 0
\]

• The terms of order $\alpha^n$ give rise to the quasi-LUV equations of order $\alpha^n$:
\[
\frac{d}{dr} \left( r^{d-1} \frac{g}{r^2} \right) = -r^{d-2}(\theta_n)_0^0
\]
and the conservation equation given by:
\[
((\theta_n)_1^1)' + \frac{d-2}{r}((\theta_n)_1^1 - (\theta_n)_2^1) = 0.
\]

It is worth stressing that each quasi LUV equation cannot be formally identified as the spherically symmetric LUV equations of motion for $n > 1$, because the left side of each quasi LUV equation do not have the standard form given by the equation (21). Furthermore, the Bianchi identities are not satisfied for each quasi LUV equation. For $n = 1$ the quasi Einstein equations can be transformed in the standard Einstein equations after a convenient redefinition of the energy momentum tensor [1], however, the method of the reference [1] has been widely used to find new solutions without using this mentioned redefinition in several works.

Despite the above mentioned, our imposed way for solving the original system (21), (23) and (25), based in the decoupling of sources by means of the standard and quasi LUV equations, ensures us to solve successfully this original system.
Furthermore, under our assumptions, each conservation equation (30), (32), ..., (34), (36) is separately conserved, and thus, there is no exchange of energy momentum between the seed fluid and each sector \((\theta_i)_{AB}\). So, in our gravitational decoupling method there is only purely gravitational interaction.

V. THE NEW SOLUTION

To find a new solution we propose the following strategy:

1. Pick up a seed solution \(\{\bar{\mu}, \bar{\rho}\}\) of the Standard LUV equation (29).

   In this connection we choose the following generalization vacuum and asymptotically AdS solution of reference [46]:
   \[
   \bar{\mu} = 1 + \frac{r^2}{l^2} - \left( \frac{2M \cdot H(r)}{\Omega_{d-2} \cdot r^{d-2n-1}} \right)^{1/n} \tag{37}
   \]
   where \(H(r)\) corresponds to the Heaviside function, the cosmological constant \([14]\) is negative, and \(\Omega_{d-2}\) is the area of a unitary \((d-2)\) sphere. So, the seed energy density is:
   \[
   \frac{2M}{\Omega_{d-2} r^{d-2}} \cdot \delta(r) = \bar{\rho} \tag{38}
   \]

2. We solve the quasi LUV equation of order \(\alpha^n\). For this, we write \(g(r)\) as:
   \[
   g(r) = \left( \frac{2m(r)}{\Omega_{d-2} \cdot r^{d-2n-1}} \right)^{1/n} \tag{39}
   \]
   So, from equation (35), we obtain that:
   \[
   m(r) = -\Omega_{d-2} \int r^{d-2}(\Theta_n)_0^0 dr. \tag{40}
   \]
   So, from equation (27), the solution is:
   \[
   \mu(r) = 1 + \frac{r^2}{l^2} - \left( \frac{2M \cdot H(r)}{\Omega_{d-2} \cdot r^{d-2n-1}} \right)^{1/n} - \alpha \left( \frac{2m(r)}{\Omega_{d-2} \cdot r^{d-2n-1}} \right)^{1/n}. \tag{41}
   \]
   The source \((\Theta_n)_2^2\) is computed by the Quasi LUV equation of order \(\alpha^n\), equation (36).

3. Once we know the functions \(\bar{\mu}(r)\) and \(g(r)\), we compute directly the sources \((\theta_i)_0^0\) in the quasi LUV equations of order \(\alpha^1...\alpha^{n-1}\). The sources \((\theta_i)_2^2\) are computed by the corresponding conservation equation of each quasi LUV equation.
Thus, the equation (41) represents a new type of black hole solution by gravitational decoupling for the LUV theory. It is worth to mention that this solution has a central singularity. So, this singularity must be protected by the black hole horizon. For the limit \(\alpha \to 0\) and for \(r > 0\) our solution coincide with reference [46]. The inclusion of the central Heaviside function will allow us to compute the mass of the seed solution as we will see bellow.

VI. CONSERVED CHARGE

Recently, in references [51, 52] has appeared a new definition of conserved charges. In [51] the energy and momentum can be computed by integrating a covariantly conserved current \(J^\mu = T^\mu_\nu \xi^\nu\) in a volume integral. It is worth to mention that the definition of reference [51] is reduced to the definition of conserved energy of reference [52] for a Killing vector \(\xi^\mu = -\delta^\mu_0\). Following [51], the energy is defined as:

\[
E = \int d^{d-1}x \sqrt{-g} J^0 = \int d^{d-1}x \sqrt{-g} T^0_\nu \xi^\nu,
\]

where \(\xi^\nu\) is a Killing vector and \(d\) is the number of dimensions.

It is worth mentioning that this definition is diffeo-invariant. Furthermore the current is covariantly conserved due that \(\nabla_\mu T^\mu_\nu = 0\) and \(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0\).

a. Energy of the seed solution: Following [51], we choose \(\xi^\mu = -\delta^\mu_0\). Evaluating the component \(\bar{T}^0_0\) and the energy density of the seed solution, equation (38), into the equation (42), we obtain the energy of the seed solution:

\[
\bar{E} = -\Omega_{d-2} \int_0^\infty r^{d-2} \bar{T}^0_0 dr = \Omega_{d-2} \int_0^\infty r^{d-2} \bar{\rho}(r) dr = M
\]

b. Energy of each extra source: The constituition of each source \((\Theta_i)_0\) is:

\[
E_i = -\Omega_{d-2} \int_0^\infty r^{d-2}(\Theta_i)_0 dr
\]

c. The total energy: The total energy is:

\[
E = \bar{E} + \alpha E_1 + \alpha^2 E_2 + \ldots + \alpha^{n-1} E_{n-1} + \alpha^n E_n
\]

So, one consequence of the application of the gravitational decoupling algorithm to the seed solution is that the total mass varies. If the total energy increases or decreases depends on the sign of the coupling constants \(\alpha^i\) and the value of each integral \(E_i\).
VII. TOY MODEL

Still, we have not proposed any form for \((\Theta_n)_0\). In this connection we will use the generalization of Hayward density of reference [49]

\[
(\Theta_n)_0 = \frac{d - 1}{\Omega_{d-2}} \frac{L\tilde{M}^2}{(LM + r^{d-1})^2}.
\] (46)

where \(\tilde{M}\) represents the mass in the generalization of the Hayward metric [49] and \(L\) is a constant of integration. Replacing into equation (40):

\[
m(r) = C - \frac{L\tilde{M}^2}{LM + r^{d-1}}.
\] (47)

Choosing arbitrarily the integration constant \(C = \tilde{M}\):

\[
m(r) = \frac{\tilde{M}r^{d-1}}{LM + r^{d-1}}.
\] (48)

Replacing in equation (41):

\[
\mu(r) = 1 + \frac{r^2}{l^2} - \left(\frac{2M \cdot H(r)}{\Omega_{d-2} \cdot r^{d-2n-1}}\right)^{1/n} - \alpha \left(\frac{\tilde{M}r^{2n}}{LM + r^{d-1}}\right)^{1/n}.
\] (49)

where \(d - 2n - 1 > 0\) to ensure the behavior asymptotically AdS.

VIII. THERMODYNAMICS

As example, in this work, we will analyze the solution (49) for \(n = 2\) with \(d = 7\) and for \(n = 3\) with \(d = 8\). It is direct to check that the behavior of the solutions showed is generic for other values of \(d\).

A. Mass parameter and horizon structure

The horizon structure is showed for \(n = 2\) with \(d = 7\) and for \(n = 3\) with \(d = 8\) in figures 1 and 2, respectively. It is direct to check that these behaviors are generic for other values of \(d\). For an asymptotically AdS space time, the black hole horizon corresponds to the largest root of the equation \(f(r = R) = 0\). This differs from the space time with positive cosmological constant, where the radial coordinate is time–like for large enough values of \(r\) and so it is formed a cosmological horizon [54].
We can see, in the first and second panel of the figures 1 and 2, that for a value of the mass parameter $M < M_{CRI}$, the black hole has three horizons: the inner horizon one, the inner horizon two and the black hole horizon. For $M = M_{CRI}$ both inner horizons coincide. For $M > M_{CRI}$ the solutions only has the black hole horizon. In the third panel we see the behavior of the function $f(r)$, which can have up to three horizons. The inner horizons are protected for the black hole horizon, so, a priori the inner horizons do not have physical interest.

The existence of two inner horizons is a proper characteristic of our solution. In the vacuum solution of LUV of references [46, 47] there are not inner horizons. In the regular black hole of LUV of reference [49] there is only one inner horizon.

On the other hand, it is well known that both the Pure Lovelock vacuum solution [39] and
FIG. 2: **first panel:** first part of Mass Parameter. **second panel:** second part of Mass Parameter. **Third Panel:** f(r).

the Einstein Gauss Bonnet solution \cite{55}, apart of the central singularity, have a singularity of curvature (in \cite{55} is called branch singularity) for negative cosmological constant. A point of singularity of curvature is where the Ricci scalar diverges but the function $f(r)$ does not diverge. Adding one extra source to the LUV vacuum solutions \cite{46}, without the gravitational decoupling method, also appears a singularity of curvature. Thus, one advantage of our extension of the GD method is that this latter leads to solutions free of singularities of curvature.
FIG. 3: first panel: Temperature for $n = 2$ and $d = 7$. second panel: Temperature for $n = 3$ and $d = 8$.

B. Temperature and stability

In the figure we show the behavior of the temperature for $n = 2$ with $d = 7$ and for $n = 3$ with $d = 8$, respectively. In both cases the temperature is an increasing function of the black hole horizon. This behavior differs from the vacuum solution LUV of reference where the temperature changes from a decreasing function to an increasing function, and thus, there is a phase transition. Our temperature also differs from the regular black hole solution of LUV of reference, where there are two phase transitions. In our case, due that $dT/dr_+$ is always positive, the specific heat $C = (dM/dr_+)(dT/dr_+)^{-1}$ is always positive. Thus our solution is always stable and does not have phase transitions.

IX. CONCLUSION AND DISCUSSION

We have showed a new exact solution of black hole for the Lovelock theory with Unique Vacuum (LUV). For this we have provide an extension of the Gravitational Decoupling algorithm for LUV. In this extension, the number of extra sources is determined by the value of $n$, i.e. depend on the power of the Riemann tensor in the action. In this extension, the equations of motion split in the standard LUV equations of motion and in the quasi LUV of order $\alpha^i$ equations.
Under the assumptions imposed, each conservation equation is separately conserved, and thus, there is no exchange of energy momentum between the seed fluid and each sector $(\theta_i)_{AB}$. So, in our gravitational decoupling method there is only purely gravitational interaction. It is worth to mention that the equation of motion for LUV is only possible for the case where $-g_{tt} = (g_{rr})^{-1}$.

In section V we have proposed the following strategy:

1. Pick up a seed solution $\{\bar{\mu}, \bar{\rho}\}$ of the Standard LUV equation.

2. Impose a form for the source $(\Theta_n)^0_0$ and solve the quasi LUV equation of order $\alpha^n$.

So, in this step we have a very simple system because the function $g$ is directly computed. Furthermore, the source $(\Theta_n)^2_2$ is also directly computed from the conservation law of the quasi LUV of order $\alpha^n$.

3. Once we know the functions $\bar{\mu}(r)$ and $g(r)$, we compute directly the sources $(\theta_i)^0_0$ in the quasi LUV equations of order $\alpha^1 ... \alpha^{n-1}$. The sources $(\theta_i)^2_2$ are also computed directly by the corresponding conservation equation of each quasi LUV equation.

So, the application of our extension provide a simplest way to solve the original system.

We have showed that, one consequence of the application of the gravitational decoupling algorithm is that the total mass varies. If the total energy increases or decreases depends on the sign of the coupling constants $\alpha^i$ and the value of each integral $E_i$.

We have showed that the obtained solution can have up to three horizons depending on the value of the critical mass parameter, namely $M_{CRI}$. For $M < M_{CRI}$, the black hole has three horizons: the inner horizon one, the inner horizon two and the black hole horizon. For $M = M_{CRI}$ both inner horizons coincide. For $M > M_{CRI}$ the solutions only has the black hole horizon.

The existence of two internal horizons is a proper characteristic of our solution. In the vacuum solution of LUV of references [46, 47] there are not inner horizon. In the regular black hole of LUV of reference [49] there is only one inner horizon.

In our solution, the temperature is an increasing function of the black hole horizon. This behavior differs from the vacuum solution LUV of reference [46] where the temperature changes from a decreasing function to a increasing function, and thus, there is a phase transition. Our temperature also differs from the regular black hole solution of LUV of
reference [49], where there are two phase transitions. In our case, due that \(\frac{dT}{dr_+}\) is always positive, the specific heat \(C = (dM/dr_+)(dT/dr_+)^{-1}\) is always positive. Thus our solution is always stable and does not have phase transitions.

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