Abstract. Assuming the Generalized Riemann Hypothesis, we establish explicit bounds in the \( q \)-aspect for the logarithmic derivative \((L'/L)(\sigma, \chi)\) of Dirichlet \( L \)-functions, where \( \chi \) is a primitive character modulo \( q \geq 10^{30} \) and \( 1/2 + 1/\log \log q \leq \sigma \leq 1 - 1/\log \log q \). In addition, for \( \sigma = 1 \) we improve upon the result by Ihara, Murty and Shimura (2009). Similar results for the logarithmic derivative of the Riemann zeta-function are given.

1. Introduction

Let \( \zeta(s) \) be the Riemann zeta-function and \( s = \sigma + it \), where \( \sigma \) and \( t \) are real numbers. It is well-known that for \( \sigma > 1 \) the logarithmic derivative of the zeta-function admits an expansion into the Dirichlet series

\[
\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s},
\]

where \( \Lambda(n) \) is the von Mangoldt function. A classical result due to Littlewood (1924) asserts that the Riemann Hypothesis (RH) implies

\[
\frac{\zeta'(s)}{\zeta(s)} \ll ((\log t)2^{-2\sigma} + 1) \min \left\{ \frac{1}{|\sigma - 1|}, \log \log t \right\}
\]

for \( 1/2 + 1/\log \log t \leq \sigma \leq 3/2 \) and \( t \) large, see [13, Corollary 13.14]. In particular, \( (\zeta'/\zeta)(s) \ll (\log t)^{2-2\sigma} \) for \( 1/2 + \delta \leq \sigma \leq 1 - \delta \) and any fixed \( \delta \in (0, 1/4) \), and \( (\zeta'/\zeta)(1 + it) \ll \log \log t \). The shape of these bounds has never been improved, and efforts have been placed into obtaining explicit constants for the main terms. Recently, an explicit bound for [14] has been given by Gonçalves and the first author in [7, Theorem 1].

Similar results hold for \( L \)-functions. Let \( \chi \) be a primitive character modulo \( q \), and let \( L(s, \chi) \) be the associated Dirichlet \( L \)-function. From now on we will assume that the Generalized Riemann Hypothesis (GRH) holds, i.e., all non-trivial zeros of \( L(s, \chi) \) have the form \( \rho_\chi = 1/2 + i\gamma_\chi \) for \( \gamma_\chi \in \mathbb{R} \).

In 2009, Ihara, Murty and Shimura [18, Corollary 3.3.2] proved under GRH that

\[
\left| \frac{L'}{L}(1, \chi) \right| \leq 2 \log \log q + 2 (1 - \log 2) + O \left( \frac{\log \log q}{\log q} \right).
\]

The implicit constant in the error term is stated explicitly. Note that \( 2 (1 - \log 2) = 0.613 \ldots \). In the present paper we improve the latter result to the following.

Theorem 1. Assume the Generalized Riemann Hypothesis. Let \( \chi \) be a primitive character modulo \( q \geq 10^{30} \). Then

\[
\left| \frac{L'}{L}(1, \chi) \right| \leq 2 \log \log q - 0.4989 + 5.91 \frac{(\log \log q)^2}{\log q}.
\]

Also, for \( q \geq 10^{153} \) we have \( \left| (L'/L)(1, \chi) \right| \leq 2 \log \log q \).

Numerical considerations concerning the extremal values of \( M_q = \max_{\chi \neq \chi_0} |(L'/L)(1, \chi)| \) for odd prime numbers \( q \leq 10^7 \) can be found in [16], see also [13] and [15]. Theorem 1 may be used to obtain conditional and effective results in the distribution of prime numbers in arithmetic progressions. Here, one of the necessary ingredients is also an estimate on \( b(\chi) \), i.e., the constant term in the Laurent expansion of \( (L'/L)(s, \chi) \) at
s = 0. Bennett et al. [11 Proposition 1.12] proved unconditionally that $|b(\chi)| \leq 0.2515q \log q$ for a Dirichlet character $\chi$ modulo $q \geq 10^5$. From [18, Equation 10.35] we see that
\[ b(\chi) = -\frac{L'(1, \chi)}{L(1, \chi)} - \log \frac{q}{2\pi} + \gamma, \]
where $\chi$ is the conjugate Dirichlet character and $\gamma$ is the Euler–Mascheroni constant.

**Corollary 2.** Assume the Generalized Riemann Hypothesis. Let $\chi$ be a primitive character modulo $q \geq 10^{30}$. Then $|b(\chi)| \leq \log q + 2 \log \log q - 0.224$.

This improves the conditional estimate in [8 Lemma 17] for $q \geq 10^{30}$. Our next theorem provides effective and conditional estimates of the form [11] for Dirichlet $L$-functions of primitive characters modulo $q$ and
\[ s = \sigma \in \mathbb{R} \text{ in the range } 1/2 + 1/\log \log q \leq \sigma \leq 1 - 1/\log \log q. \]

**Theorem 3.** Assume the Generalized Riemann Hypothesis. Let $\chi$ be a primitive character modulo $q \geq 10^{30}$. For the range
\[ \frac{1}{2} + \frac{1}{\log \log q} \leq \sigma \leq 1 - \frac{1}{\log \log q}, \]
we have
\[ \left| \frac{L'(\sigma, \chi)}{L(\sigma, \chi)} \right| \leq A_\sigma (\log q)^{2-2\sigma} - \frac{\sigma^{1-\sigma}}{1-\sigma} + \frac{5.561 (\log q)^{3-4\sigma}}{1-\sigma} + \frac{0.306 (\log \log q)^2}{2\sigma - 1}, \]
where
\[ A_\sigma := \frac{2(2\sigma - 1) \left( 1 - \exp\left( -\frac{3(1-\sigma)}{2(2\sigma - 1)} \right) \right)}{3(1-\sigma)^2} + 2.079. \]

Recently, explicit and conditional results on the logarithmic derivative of $L$-functions in the Selberg class of functions with a polynomial Euler product were published in [24]. However, they are worse than [11]. See also [3, 14, 17, 25, 26] for other similar results, and the recent work of N. Palojärvi and the third author in [19].

It was established in [6, Theorem 1] by means of bandlimited functions that for
\[ \frac{1}{2} + \frac{1}{\log \log q} \leq \sigma \leq 1 - \frac{1}{\sqrt{\log \log q}} \]
and sufficiently large $q$ one has
\[ \left| \Re \left\{ \frac{L'(\sigma, \chi)}{L(\sigma, \chi)} \right\} \right| \leq \left( -\sigma^2 + 3\sigma - 1 \right) \left( \log q \right)^{2-2\sigma} + O \left( \frac{(\log q)^{2-2\sigma}}{(\sigma - \frac{1}{2}) (1-\sigma)^2 \log \log q} \right). \]

Following [7] we provide a similar estimate for the imaginary part.

**Theorem 4.** Assume the Generalized Riemann Hypothesis. Let $\chi$ be a primitive character modulo $q$. For sufficiently large $q$ in the range [1,4] we have
\[ \left| \Im \left\{ \frac{L'(\sigma, \chi)}{L(\sigma, \chi)} \right\} \right| \leq \sqrt{\frac{2(-\sigma^2 + 3\sigma - 1)^2(-\sigma^2 + \sigma + 1)}{\sigma^2(1-\sigma)^2(2-\sigma)} (\log q)^{2-2\sigma}} + O \left( \frac{\log (\sigma - \frac{1}{2}) (\log q)^{2-2\sigma}}{\left( \sigma - \frac{1}{2} \right) (1-\sigma)^2 \log \log q} \right). \]

In comparison to Theorem [11] we are able to provide a similar conditional estimate also for the Riemann zeta-function on the 1-line.

**Theorem 5.** Assume the Riemann Hypothesis. For $t \geq 10^{30}$, we have
\[ \left| \frac{\zeta'}{\zeta} (1 + it) \right| \leq 2 \log \log t - 0.4989 + 5.35 \frac{(\log \log t)^2}{\log t}. \]

Also, for $t \geq 10^{137}$, we have $|\zeta'(1 + it)| \leq 2 \log t$.

\footnote{Following the same approach it is possible to get $(L'/L) (\sigma, \chi) \ll \log \log q$ in the range $1 - 1/\log \log q \leq \sigma \leq 1$.}
In the proof of Theorems 1 and 3 we are using a slightly modified version of Selberg’s moment formula [24], and the sum over the non-trivial zeros is estimated with a help of bandlimited majorants. By taking the same approach, we can recover under RH that

\[ \frac{\zeta'(\sigma + it)}{\zeta(\sigma)} \leq A_\sigma (\log t)^{2-2\sigma} + O \left( (\log t)^{3-4\sigma} + \frac{1}{(2\sigma - 1)^2} \right) \]

in the range

\[ \frac{1}{2} + \frac{1}{\log \log t} \leq \sigma \leq 1 - \frac{1}{\log \log t}, \]

with \( t \) sufficiently large and \( A_\sigma \) defined by (1.3). Note that (1.5) improves [7] Theorem 1 in the range (1.6) and [7] Theorem 2 for \( \sigma \geq 0.51 \). It would be interesting to prove (1.5) for a larger family of \( L \)-functions (see [19]).

The outline of this paper is as follows. In Section 2 we revise Selberg’s moment formula for Dirichlet \( L \)-functions and in Sections 3 and 4 we derive general estimates for the corresponding sums over prime numbers and non-trivial zeros, respectively. In Section 5 we use these bounds to prove Theorems 1 and 3. The proof of Theorems 4 and 5 is provided in Sections 6 and 7, respectively.

2. The Selberg moment formula

Selberg [23, Lemma 2] discovered an interesting connection between the logarithmic derivative of the Riemann zeta-function and a special truncated Dirichlet series, which is also known as the Selberg moment formula. We apply this formula in the context of Dirichlet \( L \)-functions. Let \( \chi \) be a primitive character modulo \( q \) and let \( L(s, \chi) \) be the associated Dirichlet \( L \)-function. We write \( a = (1 - \chi(-1))/2 \in \{0, 1\} \), depending on whether the character \( \chi \) is even or odd. A variation of Selberg’s formula [18, Equation 13.35] (see also [12, Chapter 4, Theorem 1.7]) asserts for \( x \geq 2 \) and \( y \geq 2 \) that

\[ \frac{L'(s, \chi)}{L(s, \chi)} = - \sum_{n \leq xy} \Lambda_{x,y}(n) \chi(n) n^s + \frac{1}{\log y} \sum_{\rho_x} x^{\rho_x - s} - (xy)^{\rho_x - s} \quad \text{for } s \neq \rho_x \chi, \]

for \( s \neq \{-2n - a: n \in \mathbb{N}_0\} \) and \( s \neq \rho_x \chi \), where

\[ \Lambda_{x,y}(n) := \begin{cases} \Lambda(n), & 1 \leq n \leq x, \\ \Lambda(n) \frac{\log \frac{2n+a}{n}}{\log x}, & x < n \leq xy, \end{cases} \]

and the second sum runs over the non-trivial zeros \( \rho_x \chi \) of \( L(s, \chi) \). Let \( q \geq 10^{30} \) and consider the range

\[ \frac{1}{2} + \frac{1}{\log \log q} \leq \sigma \leq 1. \]

In (2.1) we take the parameters

\[ y = \exp \left( \frac{\lambda}{\sigma - \frac{1}{2}} \right), \quad x = y^{-1} \log^2 q, \]

where \( \lambda > 0 \) is chosen such that \( x \geq 2 \) and \( y \geq 2 \). Let us bound each term on the right-hand side of (2.1). The first term is estimated easily by

\[ \left| \sum_{n \leq xy} \Lambda_{x,y}(n) \chi(n) n^s \right| \leq \sum_{n \leq x} \Lambda(n) n^s + \frac{1}{\log y} \sum_{x < n \leq xy} \frac{\Lambda(n) \log \frac{2n+a}{n}}{n^\sigma} =: S_{x,y}(\sigma). \]

Since GRH holds, we estimate the second term in (2.1) as

\[ \frac{1}{\log y} \sum_{\rho_x} x^{\rho_x - s} - (xy)^{\rho_x - s} \quad \text{for } \rho_x \chi, \]

\[ \leq e^{\lambda \frac{\sigma - 1}{2}} (\log q)^{1-2\sigma} \sum_{\gamma_x} \frac{\sigma - 1/2}{(\sigma - 1/2)^2 + \gamma_x^2}. \]
where the last sum runs over the ordinates of the non-trivial zeros of \( L(s, \chi) \). Finally, using \(^2\)** we bound the last term as

\[
\frac{1}{\log y} \sum_{n=0}^{\infty} \frac{x^{-2n-a-\sigma} - (xy)^{-2n-a-\sigma}}{(2n+a+\sigma)^2} \leq \left( \frac{y^\sigma + 1}{\log y} \right) \sum_{n=0}^{\infty} \frac{1}{(2n+\frac{1}{2})^2} \leq \frac{4.3}{\lambda(\log q)^{2\sigma}}.
\]

The main difficulty remains to estimate two sums over primes in \(^3\)**, and to estimate the sum over zeros \(^4\)**. We are going to do this in the following two sections.

3. THE SUM OVER PRIME NUMBERS

In this section we bound the sum over the primes in \(^2\)**. Firstly, we will provide an estimate when \( \sigma = 1 \). We use the following lemma.

**Lemma 6.** Let \( \psi(x) = \sum_{n \leq x} \Lambda(n) \) and assume the Riemann Hypothesis. Then, for \( x \geq 60 \)

\[
\sum_{n \leq x} \frac{\Lambda(n)}{n} \leq \log x - \gamma + \frac{\psi(x) - x}{x} + 0.24 \frac{\log^2 x}{\sqrt{x}}.
\]

In particular, for \( x \geq 32 \) we obtain

\[
\sum_{n \leq x} \frac{\Lambda(n)}{n} \leq \log x - \gamma + 0.04 \frac{\log^2 x}{\sqrt{x}}.
\]

**Proof.** We follow partially the proof in \(^5\)** Lemma 2.2]. Using integration by parts and the fact\(^6\)** that \( \int_1^\infty (\psi(u) - u)/u^2 \, du = -\gamma - 1 \), for \( x \geq 2 \) we have

\[
\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x - \gamma + \frac{\psi(x) - x}{x} - \int_x^\infty \frac{\psi(u) - u}{u^2} \, du.
\]

Using Weil’s explicit formula \( \psi(u) = u - \sum_{\rho} \frac{u^\rho}{\rho} - \log 2\pi - \frac{1}{2} \log(1 - u^{-2}) \) when \( u \) is not a prime power, we arrive at

\[
\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x - \gamma + \frac{\psi(x) - x}{x} - \sum_{\rho} \frac{x^{\rho - 1}}{\rho(\rho - 1)} + \int_x^\infty \frac{\log 2\pi + \frac{1}{2} \log(1 - u^{-2})}{u^2} \, du.
\]

Since RH holds, using\(^7\)** \( \sum_{\rho} \frac{1}{\rho(\rho - 1)} = 2 + \gamma - \log 4\pi \) and discarding the second part of the integral in the above expression, we arrive at

\[
\sum_{n \leq x} \frac{\Lambda(n)}{n} \leq \log x - \gamma + \frac{\psi(x) - x}{x} + 2 + \gamma - \log 4\pi + 2 \frac{\log 2\pi}{\sqrt{x}}.
\]

When \( x \geq 90 \), using \(^8\)** we arrive at \(^8\)**. We check by computer that it also holds for \( 60 \leq x \leq 90 \). Now, using the explicit conditional bound\(^9\)** \( |\psi(x) - x| \leq \frac{\sqrt{x}}{4\pi} \log^2 x \), we conclude that \(^8\)** is true for \( x \geq 4 \cdot 10^6 \). We check by computer that it also holds for \( 32 \leq x \leq 4 \cdot 10^6 \).

**Remark 7.** We remark that in \(^10\)** Lemma 2.1 and Lemma 2.2] there are some minor typos. For instance, the sign in front of \( \log 2\pi \). In \(^10\)** p. 81] this typo is mentioned. We claim that Lemma 2.2 in \(^10\)** should be replaced by \(^11\)** (after integration). This is valid unconditionally for all \( x \geq 2 \). Furthermore we claim that \( \sum_{n \leq x} \Lambda(n)/n = \log x - \gamma + O^*(0.0067/\log x) \) for \( x \geq 23 \) in \(^10\)** Corollary on p. 114] is wrong, and has many counterexamples even for \( x \geq 10^4 \).

---

2. See \( **11**\) Proposition 3.4.4.
3. See \( **19**\) Equation 10.30.
4. See \( **22**\) Equation 6.22]. Computer verification shows that it actually holds for \( x \geq 59 \). Moreover, we have that \( \psi(x) - x \leq 42 \sqrt{x} \log^2 x \) for \( x \geq 2 \).
Now, assume that \( x \geq 60 \) and \( \frac{1}{2} < \sigma \leq 1 \). Using integration by parts one can see that
\[
\frac{1}{\log y} \sum_{x < n \leq xy} \frac{\Lambda(n) \log \frac{xy}{n}}{n^2} = \frac{(xy)^{1-\sigma}}{\log y} \int_1^y \frac{\log u}{u^{2-\sigma}} \, du - \frac{\psi(x) - x}{x^\sigma} - \frac{1}{\log y} \int_x^{xy} \left( \psi(u) - u \right) \left( \frac{1}{u^{\sigma} \log u} \right) \, du
\]
\[
\leq \frac{(xy)^{1-\sigma}}{\log y} \int_1^y \frac{\log u}{u^{2-\sigma}} \, du - \frac{\psi(x) - x}{x^\sigma} + \left( \sigma + \frac{1}{\log y} \right) \left( \frac{x^{\frac{1}{2} - \sigma} - (xy)^{\frac{1}{2} - \sigma}}{8\pi (\sigma - \frac{1}{2})} \right)^2 \log^2(xy).
\]
(3.5)

Then, for \( \sigma = 1 \), combining (3.1), (3.5) and recalling (2.3) we have
\[
S_{x,y}(1) \leq \log x - \gamma + \frac{0.24}{\sqrt{x}} + \frac{\log y}{2} + \left( \sigma + \frac{1}{\log y} \right) \left( \frac{x^{\frac{1}{2}} - (xy)^{\frac{1}{2}}}{4\pi} \right)^2 \log^2(xy)
\]
\[
= 2 \log \log q - \gamma - \lambda + \left( \frac{e^\lambda - 1}{2(\lambda + 1)} \right) \left( \frac{\log \log q}{\log q} \right)^2 + \frac{0.24 e^\lambda}{\log q}.
\]
(3.6)

When \( 1/2 < \sigma < 1 \), by integration by parts we see that
\[
\sum_{n \leq x} \frac{\Lambda(n)}{n^\sigma} = \frac{\psi(x) - x}{x^\sigma} + \frac{x^{1-\sigma} - \sigma 2^{1-\sigma}}{1 - \sigma} + \sigma \int_2^x \frac{\psi(u) - u}{u^{\sigma + 1}} \, du
\]
\[
\leq \frac{\psi(x) - x}{x^\sigma} + \frac{x^{1-\sigma} - \sigma 2^{1-\sigma}}{1 - \sigma} + \frac{\sigma}{8\pi} \int_2^x \frac{\log^2 u}{u^{\sigma + 1}} \, du
\]
\[
\leq \frac{\psi(x) - x}{x^\sigma} + \frac{x^{1-\sigma} - \sigma 2^{1-\sigma}}{1 - \sigma} + \frac{\sigma \log^2 x}{2^{\sigma + 1} \pi (\sigma - \frac{1}{2})}.
\]
(3.7)

We directly combine (3.5) and (3.7), with
\[
\frac{(xy)^{1-\sigma}}{\log y} \int_1^y \frac{\log u}{u^{2-\sigma}} \, du = \frac{(xy)^{1-\sigma} - x^{1-\sigma}}{(1 - \sigma)^2 \log y} - \frac{x^{1-\sigma}}{1 - \sigma}
\]
to get
\[
S_{x,y}(\sigma) \leq \frac{(xy)^{1-\sigma} - x^{1-\sigma}}{(1 - \sigma)^2 \log y} - \frac{\sigma 2^{1-\sigma}}{1 - \sigma} + \frac{\sigma \log^2 x}{2^{\sigma + 1} \pi (\sigma - \frac{1}{2})} + \left( \sigma + \frac{1}{\log y} \right) \left( \frac{x^{\frac{1}{2} - \sigma} - (xy)^{\frac{1}{2} - \sigma}}{8\pi (\sigma - \frac{1}{2})} \right)^2 \log^2(xy)
\]
\[
\leq B_{\sigma,\lambda}(\log q)^{2-2\sigma} - \frac{\sigma 2^{1-\sigma}}{1 - \sigma} + \frac{\sigma (\log \log q)^2}{2^{\sigma + 1} \pi (2\sigma - 1)} + \frac{2\sigma}{2\sigma - 1} + \frac{1}{\lambda} \left( e^\lambda - 1 \right) \left( \frac{\log \log q}{\log q} \right)^2 \log^2(xy).
\]
(3.8)

Here \( B_{\sigma,\lambda} \) is given by
\[
B_{\sigma,\lambda} = \frac{(2\sigma - 1) - \left( 1 - \exp \left( -\frac{2\lambda(1-\sigma)}{2\sigma - 1} \right) \right)}{2\lambda (1-\sigma)^2}.
\]
(3.9)

4. The sum over the non-trivial zeros

In this section we obtain an explicit upper bound for the sum in (2.5) over the non-trivial zeros of \( L(s, \chi) \).

Firstly, we are going to derive an estimate when \( \sigma = 1 \). To do that, we will use the known constant \( B(\chi) \) since \( \Re \{ B(\chi) \} = -\sum_{\rho_\chi} \Re \{ 1/\rho_\chi \} \), and assuming GRH we have
\[
\sum_{\gamma_\chi} \frac{1}{\gamma_\chi + \gamma_\chi^2} = |\Re \{ B(\chi) \}|.
\]

---

5See [13] Equation 10.38.
Using [14] Lemmas 2.3 and 2.4, one can deduce that
\[ |\Re\{B(\chi)\}| \leq \left( 1 - \frac{1}{\sqrt{z}} \right)^{-2} \left( \frac{1}{2} \left( 1 - \frac{1}{z} \right) \log \frac{q}{\pi} + \log z \right) \]
for \( z \geq 4 \). Choosing \( z = \frac{1}{4} \log^2 q \) and recalling that \( q \geq 10^{30} \) we obtain
\[
\sum_{\gamma_x} \frac{1}{4 + \gamma_x^2} \leq \frac{1}{2} \left( \frac{2}{\log q} - 2 \right) \log \frac{q}{\pi} + 2 \left( 1 - \frac{2}{\log q} \right)^{-2} \log \log q \leq \frac{1}{2} \log \frac{q}{\pi} + 2.6 \log \log q. \tag{4.1}
\]
For the range (1.2) we proceed in a different way. Let \( a = \sigma - \frac{1}{4} \) and let \( f_a : \mathbb{R} \to \mathbb{R} \) be the function
\[
f_a(x) = \frac{a}{a^2 + x^2}. \tag{4.2}
\]
We want to estimate \( \sum \gamma_x f_a(\gamma_x) \) and the classical machinery to bound this sum is the Guinand–Weil explicit formula for Dirichlet L-functions [3] Lemma 4] and for the Riemann zeta-function [3] Lemma 8].

**Lemma 8.** Let \( q \) be a positive integer and \( \chi \) be a primitive character modulo \( q \). Let \( h(s) \) be analytic in the strip \( \Re\{s\} \leq \frac{1}{2} + \varepsilon \) for some \( \varepsilon > 0 \), and assume that \( |h(s)| \ll (1 + |s|)^{-3/2} \) as \( |\Re\{s\}| \to \infty \), for some \( \delta > 0 \). Assume GRH and that \( h(s) \) is a real-valued function. Then\(^6\)
\[
\sum_{\gamma} h(\gamma) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} h(u)\Re \left\{ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{1}{4} - iu \right) \right\} du - \frac{1}{\pi} \sum_{n=1}^{\infty} \Lambda(n) \Re \left\{ \chi(n) \hat{h} \left( \frac{\log n}{2\pi} \right) \right\}
\]
and
\[
\sum_{\gamma} h(\gamma) = 2 \Re \left\{ h \left( \frac{1}{2} \right) \right\} - \frac{\log \pi \hat{h}(0) + 1}{2\pi} \int_{-\infty}^{\infty} h(u)\Re \left\{ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{1}{4} + iu \right) \right\} du - \frac{1}{\pi} \sum_{n=2}^{\infty} \Lambda(n) \Re \left\{ \hat{h} \left( \frac{\log n}{2\pi} \right) \right\},
\]
where the sums on the left-hand sides run over the imaginary parts of the non-trivial zeros of \( L(s, \chi) \) and \( \zeta(s) \), respectively.

The function \( f_a \) does not satisfy the conditions in Lemma 8 and the key idea is to replace \( f_a \) with certain explicit bandlimited majorants\(^7\) which are admissible for the classical Guinand–Weil explicit formula.

In [6] Lemma 9], it is proved that for any \( \Delta > 0 \) the function
\[
h(s) = h_{a, \Delta}(s) = \left( \frac{a}{a^2 + s^2} \right) \left( \frac{e^{2\pi a \Delta} + e^{-2\pi a \Delta} - 2 \cos(2\pi s)}{(e^{\pi a \Delta} - e^{-\pi a \Delta})^2} \right) \tag{4.3}
\]
is a real entire function of exponential type \( 2\pi \Delta \) such that \( f_a(s) \leq h(s) \) for all \( s \in \mathbb{R} \), and its Fourier transform satisfies \( \hat{h}(\xi) \geq 0 \) for all \( |\xi| \leq \Delta \), \( \hat{h}(\xi) = 0 \) for all \( |\xi| > \Delta \) and \( \hat{h}(0) = \pi \coth(\pi a \Delta) \). Now, we follow the idea in \[7]. Let \( \Delta > 0 \) such that \( \pi a \Delta \geq 1 \). Because \( h(u) \geq 0 \) for all \( u \in \mathbb{R} \), we have
\[
\sum_{\gamma} f_a(\gamma) \leq \sum_{\gamma} h(\gamma) + \sum_{\gamma} h(\gamma), \tag{4.4}
\]
where the last sum runs over the ordinates \( \gamma \) of the non-trivial zeros of \( \zeta(s) \). From Lemma 8 we see that
\[
\sum_{\gamma} h(\gamma) = \frac{\log q}{2\pi} \hat{h}(0) - \frac{\log \pi}{\pi} \hat{h}(0) + 2 \Re \left\{ h \left( \frac{1}{2} \right) \right\}
\]
\[
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u)\Re \left\{ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{1}{2} + iu \right) \right\} du + \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u)\Re \left\{ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + iu \right) \right\} du + \frac{1}{\pi} \sum_{n=2}^{\infty} \Lambda(n) \Re \left\{ \chi(n) \hat{h} \left( \frac{\log n}{2\pi} \right) \right\}. \tag{4.5}
\]

\(^6\)The functions \( E_a(z) \) defined in [14] Lemmas 2.3] satisfy \( E_a(z) \leq 0 \) for \( z \geq 4 \).
\(^7\)Here \( \hat{h} \) denotes the Fourier transform of \( h \), i.e., \( \hat{h}(\xi) = \int_{-\infty}^{\infty} h(u)e^{-2\pi i u \xi} du \).
\(^8\)To get the better bounds, we seek majorants that are extremal in the sense that they solve the Beurling–Selberg problem associated to \( f_a \). This idea has been employed to estimates objects in the theory of the Riemann zeta-function and L-functions. See, for instance, [8][9][10][11][12].
Since $\Re \{\chi(n)\} + 1 \geq 0$ and $\tilde{h}(\xi) \geq 0$, we discard the last sum in (4.3). Using the bound $\Re \{(\Gamma'/\Gamma)(s)\} \leq |s|$ for $\Re\{s\} \geq \frac{1}{2}$ (see Lemma 2.3) and the estimate

$$0 \leq h(u) \leq \frac{a}{2\pi} \cdot \left( e^{\pi a} + e^{-\pi a} \right)^2 \leq \frac{1.725 a}{a^2 + u^2}, \quad (4.6)$$

one can bound the terms involving the gamma function as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \Re \left\{ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{a}{2} + iu \right) \right\} du \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \log \left( \frac{3}{4} + \frac{iu}{2} \right) du \leq \frac{1}{2\pi} \int_{|u| \geq 2} h(u) \log \left( \frac{3}{4} + \frac{iu}{2} \right) du \leq 0.298.$$

Therefore, the contribution of these terms is at most 0.596. Since $\tilde{h}(0) \geq \pi$, combining (4.4), (4.5) and using $\tilde{h}(0) = \pi \coth(\pi a \Delta)$ we get

$$\sum_{\gamma_\chi} f_\sigma(\gamma_\chi) \leq \frac{\log q}{2\pi} \tilde{h}(0) + 2 \Re \left\{ h \left( \frac{i}{2} \right) \right\} = \frac{\coth(\pi a \Delta) \log q}{2} + \left( \frac{2a}{1 - a^2} \right) \frac{e^{\pi a} + e^{-\pi a} - e^{2\pi a} - e^{-2\pi a} \left( e^{\pi a} - e^{-\pi a} \right)^2}{1 - e^{-2}}.$$

While discarding the negative term on the right-hand side and using $\pi a \Delta \geq 1$, it follows that

$$\sum_{\gamma_\chi} f_\sigma(\gamma_\chi) \leq \frac{\log q}{2} + \frac{e^{-2\pi a} \log q}{1 - e^{-2}} + \left( \frac{2a}{1 - a^2} \right) \frac{e^{1-2a} \pi a \left( 1 + e^{-2\pi a} \right)}{1 - e^{-2}}.$$

We choose $\pi a = \log q$ and recall that $a = \sigma - \frac{1}{2}$. Letting $\alpha = (1 - e^{-2})^{-1}$ and $\beta = 1 + (\log 10^{10})^{-2}$, we obtain

$$\sum_{\gamma_\chi} \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + \gamma_\chi^2} \leq \frac{\log q}{2} + \alpha \left( 1 + 2a\beta - \frac{(\sigma + a\beta \sigma^{-1})}{1 - \sigma} \right) (\log q)^{2-2\sigma}.$$

Clearly, in the range $\frac{1}{2} < \sigma < 1$ we have that $\sigma + a\beta \sigma^{-1} \geq 1 + a\beta$. Therefore,

$$\sum_{\gamma_\chi} \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + \gamma_\chi^2} \leq \frac{\log q}{2} + \frac{1.338}{1 - \sigma} (\log q)^{2-2\sigma}. \quad (4.7)$$

5. Proof of Theorems 2 and 3

Having derived estimates for the terms in the Selberg moment formula (2.1) in the previous sections, we are now ready to prove Theorem 2 and Theorem 3.

Proof of Theorem 2 Letting $s = 1$ in (2.1), and combining (2.5), (2.6), (3.6) and (4.11) it follows that

$$\left| \frac{L'(1, \chi)}{L(1, \chi)} \right| \leq 2 \log q - \gamma - \lambda + \frac{e^\lambda + 1}{2\lambda} + \left( \frac{e^\lambda - 1)(2\lambda + 1)}{2\pi \lambda} \right) \frac{(\log \log q)^2}{\log q} + \left( \frac{2.6}{\lambda} \right) \frac{\log \log q}{\log q} - \left( \frac{(e^\lambda + 1) \log \pi}{2\lambda} - 0.24 e^\lambda \right) \frac{1}{\log q} + \left( \frac{2.15}{\lambda} \right) \frac{1}{(\log q)^2}.$$  

We choose $\lambda = 2.1862$ in order to minimize the constant term in the latter inequality. Note that this also implies $y = e^{2\lambda} \geq 2$, and that $q \geq 10^{30}$ implies $x = e^{-2\lambda} \log^2 q \geq 60$. We arrive at

$$\left| \frac{L'(1, \chi)}{L(1, \chi)} \right| \leq 2 \log q - 0.4989 + \left( \frac{11.776}{\log \log q} - \frac{0.455}{(\log \log q)^2} + \frac{78.906}{\log \log q} \right) \frac{(\log \log q)^2}{\log q}.$$ 

This implies our desired results.
Proof of Theorem 4. We choose $\lambda = \frac{4}{3}$. Recalling that $q \geq 10^{30}$ and using (2.9), we have $x \geq 60$ and $y \geq 2$ and the estimates obtained in the previous sections hold. Combining (2.5), (2.6), (6.1) and (4.7) it follows that

$$|L'(\sigma, \chi)| \leq A_\sigma (\log q)^{2-2\sigma} - \sigma^2 \frac{1}{1-\sigma} + \sigma (\log \log q)^2 + \left( \frac{(14\sigma - 4)(2\sigma - 1)}{6\pi(2\sigma - 1)} \right) (\log \log q)^2 (\log q)^{1-2\sigma}$$

$$+ \frac{4(\pi^2 + 1)1.338}{3(1-\sigma)} (\log q)^{3-4\sigma} + \frac{8.6(2\sigma - 1)(e^{\pi(2\sigma - 1)} + 1)}{3(\log q)^{2\sigma}}$$

with $A_\sigma = B_\sigma + \frac{2}{3} (e^{3/4} + 1)$, where $B_\sigma$ is defined in (3.2). Finally, we bounded each term conveniently using (12). We remark that the factor $14\sigma - 4$ is bounded by $10 - 14(\log \log q)^{-1}$ and this negative part cancels the last summand in the right-hand side of (6.1). Also, we use the fact that the function $\sigma^2 - \sigma$ is increasing in $\sigma \in (\frac{1}{2}, 1)$.

6. Proof of Theorem 4

Since the proof closely follows [4] (see also Section 4), we will highlight only the main differences.

6.1. Bounds for $\Re \{(L'/L)'(s, \chi)\}$. In this section we are going to establish a lower and an upper bound for $\Re \{(L'/L)'(s, \chi)\}$, where $s = \sigma + it$ with $\frac{1}{2} < \sigma < 1$ and $|t| \leq \frac{1}{2}$. Taking real part of the derivative of the partial fraction decomposition of $L'/L$, see [18] Equation 10.37, using a classical estimate for $(\Gamma'/\Gamma)'$ and using GRH, we arrive at

$$\Re \left\{ \left( \frac{L'}{L} \right)'(s, \chi) \right\} = \sum_{\gamma} g_a(t - \gamma) + \sum_{\gamma} g_a(\gamma) + O(1),$$

(6.1)

where $a = \sigma - \frac{1}{2}$, the function $g_a : \mathbb{R} \to \mathbb{R}$ is defined by $g_a(x) = \frac{e^{2x} - 2}{(e^{x} + a)^2}$, and the second sum runs over the imaginary part of the zeros of $\zeta(s)$. Note that we can add this sum here since $\sum_{\gamma} 1/\gamma^2 < \infty$.

Now, we replace the function $g_a$ by the bandlimited majorants and minorants described in [4] Lemmas 6, 7, 8. Let $\Delta \geq 1$ be a parameter such that $\pi a \Delta \geq 1$. The minorant function $m := m_{a, \Delta}$ help us to derive the desired lower bound. We have that $m(u) \leq g_u(u)$ for all $u \in \mathbb{R}$, $m(u) = O((u^2 + a^2)^{-1})$ and $m \left( \frac{1}{2} \right) = 2\pi \Delta a e^{(1-2a)\pi \Delta} \left( a^2 - \frac{1}{4} \right)^{-1} + O \left( e^{(1-2a)\pi \Delta} \left( a - \frac{1}{2} \right)^{-2} \right)$. Moreover $\hat{m}(\xi) \leq 0$ for all $\xi \in \mathbb{R}$ and $\hat{m}(0) = -4\pi^2 \Delta e^{-2a\pi \Delta} + O \left( \Delta e^{-4a\pi \Delta} \right)$. Then, we apply Lemma 3 as in (1.5). Since $|t| \leq \frac{1}{2}$, by Stirling’s formula the terms with $\Gamma'/\Gamma$ are $O(1/a^2)$. Choosing $\pi \Delta = \log \log q$ we conclude that

$$\Re \left\{ \left( \frac{L'}{L} \right)'(s, \chi) \right\} \geq \left( \frac{-2a^2 + 6a - 2}{\sigma(1-\sigma)} \right) \log \log q (\log q)^{2-2\sigma} + O \left( \frac{(\log q)^{2-2\sigma}}{(\sigma - \frac{1}{2}) (1-\sigma)^2} \right)$$

for $(\sigma - \frac{1}{2}) \log \log q \geq 1$. Similarly, the majorant one can get in the range $1.4$ the upper bound

$$\Re \left\{ \left( \frac{L'}{L} \right)'(s, \chi) \right\} \leq \left( \frac{-2a^2 + 2a + 2}{\sigma(1-\sigma)} \right) \log \log q (\log q)^{2-2\sigma} + O \left( \frac{(\log q)^{2-2\sigma}}{(\sigma - \frac{1}{2}) (1-\sigma)^2} \right).$$

(6.3)

Proof of Theorem 4. Define the function $\varphi(t) = -\log |L(s, \chi)|$. Note that $\varphi'(t) = \Re \{(L'/L)'(s, \chi)\}$, and $\varphi''(t) = \Re \{(L'/L)^2(s, \chi)\}$. Let $|t| \leq 1/2$ and $q$ be sufficiently large. Denoting by $-\beta$ and $\alpha$ the right-hand sides in (6.2) and (6.3) respectively, we write $-\beta \leq \varphi''(t) \leq \alpha$. Also, from [6] Theorem 1 one can get the bounds $-\gamma \leq \varphi(t) \leq \gamma$, where

$$\gamma = \left( \frac{-\sigma^2 + 3\sigma - 1}{2\sigma(1-\sigma)} \right) \frac{(\log q)^{2-2\sigma}}{\log \log q} + c \frac{\log (\sigma - \frac{1}{2}) (\log q)^{2-2\sigma}}{(\sigma - \frac{1}{2}) (1-\sigma)^2 (\log \log q)^2}.$$

Let $|h| \leq 1/4$. An application of the mean value theorem gives that

$$\varphi'(0) - \varphi'(-h) = \varphi''(h) h \leq \max\{h, 0\} \alpha + \max\{-h, 0\} \beta.$$
Here \( h^* \in [-\pi, 0] \) or \([0, -\pi]\). Averaging in \( h \) in the interval \([-\nu(1-A), \nu A]\), we obtain \( \varphi(0) \leq \frac{\pi}{\nu} \) for \( 0 < \nu < \frac{1}{2\pi} \) and \( 0 < A < 1 \). We minimize the right-hand side of the above expression by choosing \( \nu = 2\sqrt{(\alpha^{-1} + \beta^{-1})}\gamma \) and \( A = \beta(\alpha + \beta)^{-1} \). Note that we have \( \nu < \frac{1}{2\pi} \) as \( q \to \infty \). We conclude that \( \varphi(0) \leq 2\sqrt{\alpha \beta(\alpha + \beta)^{-1}} \gamma \). The proof of the lower bound for \( \varphi(0) \) is similar. This implies the desired result.

\[ \tag*{\text{Remark 9.}} \]

By using \(|\Im\{L'(s, \chi)\}| \leq |\Im\{L'(L)/2\} \leq |\Im\{L'(s, \chi)\}| \leq \frac{|\Im\{L'(s, \chi)\}|}{\log\log t} \), one can observe that the bound for the imaginary part in Theorem 4 actually is better than Theorem 3 only when \( \sigma \) is very close to 0.5, about 0.536. We believe that to improve Theorem 4 we should estimate \( \Im\{L'(L)/2\} \) directly, without using the interpolation argument mentioned above. In fact, it is possible to obtain a representation for \( \Im\{L'(s, \chi)\} \) as in \( \text{Lemma 9} \), and then one gets a function for which the Beurling–Selberg problem needs to be solved. However, for this specific function, the Beurling–Selberg problem is a hard problem and it is still open.

### 7. Proof of Theorem 5

**Proof of Theorem 5.** We use Selberg’s moment formula for the Riemann zeta-function \([13, Equation (13.35)]\) with \( s = 1 + it \) and \( y = e^{2\lambda} \) and \( x = y^{-1}\log^2 t \):

\[
\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n \leq x} \frac{\Lambda(n)}{n^s} + \frac{1}{\log y} \sum_{\rho} \frac{x^{\rho} - (xy)^{\rho - s}}{(\rho - s)^2} + \frac{1}{\log y} \sum_{n=1}^{\infty} \frac{x^{-2n-s} - (xy)^{-2n-s}}{(2n+s)^2} \frac{x^{1-s} - (xy)^{1-s}}{\log y (1-y)}.
\]

(7.1)

Note that the first sum is bounded exactly as in \( \text{Lemma 10} \) replacing \( q \) by \( t \). Bounding as in \( (2.5) \), the sum over the zeros in the right-hand side of \( (7.2) \) is bounded by \( (e^\lambda + 1)^{-1} \lambda^{-1} \log(t)^{-1} \sum f_\Delta(t-\gamma) \), where the function \( f_\Delta \) is defined in \( \text{Lemma 2} \). For \( \Delta \geq 1 \) and \( a = \frac{1}{2} \) we apply Lemma 8 for the function \( s \mapsto h(t-s) \) to get

\[
\sum_{\gamma} f_\Delta(t-\gamma) \leq \sum_{\gamma} h(t-\gamma) \leq 2 \left| h \left( \frac{t-1}{2} \right) \right| - \log \pi \frac{\hat{h}(0)}{2\pi} + \frac{1}{\pi} \int_{-\infty}^{\infty} h(u) \Re \left\{ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{i(t-u)}{2} \right) \right\} du.
\]

(7.2)

where \( h(s) \) is the majorant function defined in \( \text{Lemma 3} \). Since \( t \geq 1630 \), for all \( u \in \mathbb{R} \) we have

\[
\Re \left\{ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{i(t-u)}{2} \right) \right\} \leq \log \left| \frac{1}{4} + \frac{i(t-u)}{2} \right| = \log t + \log \left( 1 + \frac{1}{16t^2} \right) \leq \log t + \log(1 + |u|).
\]

Thus, by \( \text{Lemma 10} \) with \( a = \frac{1}{2} \), the third term in the right-hand side of \( (7.2) \) is bounded by \( \frac{\hat{h}(0)}{2\pi} \log t + 0.541 \). We bound the first term in the right-hand side of \( (7.2) \) using directly \( (1.3) \), and since \( \hat{h}(0) \geq \pi \) we arrive at

\[
\sum_{\gamma} f_\Delta(t-\gamma) \leq \frac{\hat{h}(0)}{2\pi} \log t + \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left| \hat{h} \left( \frac{\log n}{2\pi} \right) \right|.
\]

To bound the above sum over primes, we remark from \( \text{Lemma 9} \) that \( \hat{h}(\xi) = 0 \) for all \( |\xi| \geq \Delta \) and

\[
\hat{h}(\xi) = \pi \left( \frac{e^{\pi(\Delta-|\xi|)} - e^{-\pi(\Delta-|\xi|)}}{e^{\pi\Delta} (1 - e^{-\pi\Delta})^2} \right)
\]

for all \( |\xi| \leq \Delta \).

Choosing \( \pi \Delta = \log t \) and using \( (3.1) \) it follows that

\[
\frac{1}{\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left| \hat{h} \left( \frac{\log n}{2\pi} \right) \right| = \frac{1}{(1 - \log t)^{1/2}} \sum_{n \leq (\log t)^2} \frac{\Lambda(n)}{n} = \frac{1}{(1 - \log t)^{1/2}} \sum_{n \leq (\log t)^2} \frac{\Lambda(n)}{n} \leq 2 \frac{\log t - 1 + 0.24(\log t)^{-1}}{(1 - \log t)^{1/2}}.
\]
Therefore, \( \sum_{\gamma} f_y(t - \gamma) \leq 0.5 \log t + 2 \log \log t \) (compare this estimate with (4.1)). We conclude that
\[
\left| \frac{1}{\log y} \sum_{\rho} x^{\rho - s} - (xy)^{\rho - s} \right| \leq \frac{e^\lambda + 1}{2\lambda} + \frac{2(e^\lambda + 1) \log t}{\lambda(\log t)}.
\]
Finally, we bound the last two terms in (7.1) trivially. Therefore, taking \( \lambda = 2.1862 \) and considering that \( t \geq 10^{30} \) we obtain
\[
\left| \frac{\zeta'}{\zeta}(1 + it) \right| \leq 2 \log \log t - 0.4989 + \left( 3.091 + \frac{9.06}{\log \log t} + \frac{2.137}{(\log \log t)^2} \right) \left( \frac{\log t}{\log \log t} \right)^2.
\]
Now the final result easily follows.

Acknowledgements

The authors would like to thank Olivier Ramaré for useful remarks concerning the proof of Lemma 10 and Alessandro Languasco for pointing out a few references, as well as Tim Trudgian for taking time to read the manuscript. The project started when A. C. was a postdoctoral fellow at NTNU. A. C. was supported by Grant 275113 of the Research Council of Norway, and M. V. H. was supported by the Olav Thon Foundation through the StudForsk program. The authors also thankful to the anonymous referee for the valuable comments and suggestions.

References

[1] M. A. Bennett, G. Martin, K. O’Bryant, and A. Rechnitzer, Explicit bounds for primes in arithmetic progressions, Illinois J. Math. 62 (2018), no. 1-4, 427–532.
[2] E. Carneiro, V. Chandee and M. B. Milinovich, A note on the zeros of zeta and \( L \)-functions, Math. Z. 281 (2015), 315–332.
[3] E. Carneiro, A. Chirre and M. B. Milinovich, Bandlimited approximations and estimates for the Riemann zeta-function, Publ. Mat. 63 (2019), no. 2, 601–661.
[4] V. Chandee, Explicit upper bounds for \( L \)-functions on the critical line, Proc. Amer. Math. Soc. 137 (2009), no. 12, 4049–4063.
[5] V. Chandee and K. Soundararajan, Bounding \(|\zeta(\frac{1}{2} + it)|\) on the Riemann hypothesis, Bull. London Math. Soc. 43 (2011), no. 2, 243–250.
[6] A. Chirre, A note on entire \( L \)-functions, Bull. Braz. Math. Soc. (N.S.) 50 (2019), no. 1, 67–93.
[7] A. Chirre and F. Gonçalves, Bounding the log-derivative of the zeta-function, Math. Z. 300 (2022), no. 1, 1041–1053.
[8] A-M. Ernvall-Hytonen and N. Palojärvi, Explicit bound for the number of primes in arithmetic progressions assuming the generalized Riemann hypothesis, Math. Comp. 91 (2022), no. 335, 1317–1365.
[9] D. A. Goldston and S. M. Gonek, A note on \( \sum_{\rho} x^{\rho - s} \) from the one of \( \Lambda(n) \), Acta Arith. 137 (2009), no. 3, 253–276.
[10] Y. Ihara, V. K. Murty, and M. Shimura, On the logarithmic derivatives of Dirichlet \( L \)-functions on the critical line, Proc. Amer. Math. Soc. 137 (2009), no. 12, 4279–4291.
[11] G. J. O. Jameson, The prime number theorem, London Mathematical Society Student Texts 53, Cambridge University Press, Cambridge, 2003.
[12] D. Joyner, Distribution theorems of \( L \)-functions, Pitman Research Notes in Mathematics Series 142, Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1986.
[13] Y. Lamzouri and A. Languasco, Small values of \( |L'/L(1, \chi)| \), Exp. Math. 32 (2023), no.2, 362–377.
[14] Y. Lamzouri, X. Li, and K. Soundararajan, Conditional bounds for the least quadratic non-residue and related problems, Math. Comp. 84 (2015), no. 295, 2391–2412.
[15] A. Languasco, Efficient computation of the Euler-Kronecker constants of prime cyclotomic fields, Res. Number Theory 7 (2021), no. 1, Paper No. 2, 22.
[16] A. Languasco and L. Righi, A fast algorithm to compute the Ramanujan-Deninger gamma function and some number-theoretic applications, Math. Comp. 90 (2021), no. 332, 2899–2921.
[17] A. Languasco and T. S. Trudgian, Uniform effective estimates for \( |L(1, \chi)| \), J. Number Theory 236 (2022), 245–260.
[18] H. L. Montgomery and R. C. Vaughan, Multiplicative Number Theory: I. Classical Theory, Cambridge Studies in Advanced Mathematics 97, Cambridge University Press, 2006.
[19] N. Palojärvi and A. Simonić, Conditional estimates for \( L \)-functions in the Selberg class, preprint available at arXiv:2111.01121 (2022).
[20] O. Ramaré, Explicit estimates for the summatory function of \( \Lambda(n)/n \) from the one of \( \Lambda(n) \), Acta Arith. 159 (2013), no. 2, 113–122.
[21] O. Ramaré and D. J. Platt, Explicit estimates: from \( \Lambda(n) \) in arithmetic progressions to \( \Lambda(n)/n \), Exp. Math. 26 (2017), no. 1, 77–92.
[22] L. Schoenfeld, Sharper bounds for the Chebyshev functions \( \theta(x) \) and \( \psi(x) \), II, Math. Comp. 30 (1976), no. 134, 337–360.
[23] A. Selberg, On the normal density of primes in small intervals, and the difference between consecutive primes, Arch. Math. Naturvid. 47 (1943), no. 6, 87–105.

[24] A. Simonić, Estimates for $L$-functions in the critical strip under GRH with effective applications, Mediterr. J. Math. 20 (2023), no. 2, paper no. 87, 24pp.

[25] A. Simonić, Explicit estimates for $\zeta(s)$ in the critical strip under the Riemann Hypothesis, Q. J. Math. 73 (2022), no. 3, 1055–1087.

[26] A. Simonić, On explicit estimates for $S(t)$, $S_1(t)$, and $\zeta(1/2 + it)$ under the Riemann Hypothesis, J. Number Theory 231 (2022), 464–491.

Departamento de Ciencias - Sección Matemáticas, Pontificia Universidad Católica del Perú, Lima, Perú
Email address: cchirre@pucp.edu.pe

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway
Email address: markus.v.hagen@ntnu.no

School of Science, The University of New South Wales (Canberra), ACT, Australia
Email address: a.simonic@adfa.edu.au