LIPSCHITZ NORMAL EMBEDDINGS AND POLAR EXPLORATION OF COMPLEX SURFACE GERMS

ANDRÉ BELOTTO DA SILVA, LORENZO FANTINI, AND ANNE PICHON

Abstract. We undertake a systematic study of Lipschitz Normally Embedded surface germs with isolated singularities. We prove in particular that the topological type of such a germ determines the combinatorics of its minimal resolution which factors through the blowup of its maximal ideal and through its Nash transform, as well as the polar curve and the discriminant curve of a generic plane projection, thus generalizing results of Spivakovsky and Bondil that were known for minimal surface singularities. This fits in the program of polar explorations, the quest to determine the generic polar variety of a singular surface germ, to which the final part of the paper is devoted.

A ranger walked from his tent 10 km southwards, turned east, walked straight eastwards 10 km more, met his bear friend, turned north and after another 10 km found himself by his tent. What colour was the bear and where did all this happen?

V. I. Arnold’s (Odessa, 12 June 1937 – Paris, 3 June 2010) selection of problems for children from 5 to 15

1. Introduction

A germ of a real or complex analytic space \((X, 0)\) embedded in \((\mathbb{R}^n, 0)\) or in \((\mathbb{C}^n, 0)\) is equipped with two natural metrics: its outer metric \(d_o\), induced by the standard metric of the ambient space, and its inner metric \(d_i\), which is the associated arc-length metric on the germ. The germ \((X, 0)\) is said to be Lipschitz normally embedded (LNE for short) if the identity map of \((X, 0)\) is a bilipschitz homeomorphism between the inner and the outer metric, that is if there exist a neighborhood \(U\) of 0 in \(X\) and a constant \(K \geq 1\) such that

\[ d_i(x, y) \leq K d_o(x, y) \]

for all \(x\) and \(y\) in \(U\). This property only depends on the analytic type of \((X, 0)\), and not on the choice of an embedding in some smooth ambient space \((\mathbb{R}^n, 0)\) or \((\mathbb{C}^n, 0)\).

The study of Lipschitz Normal Embedded singularities is a very active research area with many recent results, for example by Birbrair, Fernandes, Kerner, Mendes, Misev, Neumann, Nuño-Ballesteros, Pedersen, Pichon, Ruas, and Sam paio (see [BMNB18, FS19a, KPR18, NPP19a, NPP19b, BM18]), but despite the current progress it is still in its infancy. While an irreducible complex curve germ \((X, 0)\) is LNE if and only if it is smooth (see [PT69, Fer03, NP07]), the situation is far richer already for complex surface germs. Lipschitz Normal Embedded germs are fairly common in this context, including in particular all minimal surface singularities (as proven in [NPP19b] exploiting a characterization obtained in [NPP19a]), and the superisolated surface singularities with LNE tangent cone (see [MP18]).
this paper we prove several properties of a general complex LNE surface germ with isolated singularities, describing in particular its generic polar curves and the discriminant curves of its generic plane projections.

Among LNE surface singularities, the most widely studied are minimal singularities, which have been introduced in greater generality in [Kol85]. In dimension two they are the rational surface singularities with reduced fundamental cycle, and they have the remarkable property that the topological type of \((X, 0)\) determines the following data, which is a priori of analytic nature:

1. The dual graph of the minimal good resolution of \((X, 0)\) which factors through the blowup of the maximal ideal and through the Nash transform, decorated by two families of arrows corresponding to the strict transform of a generic hyperplane section and and the strict transform of the polar curve of a generic plane projection.

2. The topological type of the discriminant curve of a generic projection.

Moreover, this data can be computed explicitly from the dual graph of the minimal good resolution of \((X, 0)\).

The first property is a deep result of Spivakovsky [Spi90, III, Theorem 5.4], the second one was later proven by Bondil [Bon03, Theorem 4.1], [Bon16, Proposition 5.4].

Observe that by good resolution of \((X, 0)\) we mean a proper bimeromorphic morphism \(\pi: X_\pi \to X\) from a smooth surface \(X_\pi\) to \(X\) which is an isomorphism outside of a simple normal crossing divisor \(E = \pi^{-1}(0)\), and the vertices of dual graph \(\Gamma_\pi\) of \(E\) carry as weights the genera and self-intersections of the corresponding components of \(E\). The fact that the topological type of a surface germ determines the dual graph of its minimal resolution is a classical result of Neumann [Neu81].

The two main results of the present paper extend the theorems of Spivakovsky and Bondil to all LNE surface singularities. Furthermore, we strengthen Spivakovsky’s result by showing that another important datum is an invariant of the topological type of \((X, 0)\), namely the inner rates of \((X, 0)\), an infinite family of rational numbers which measures the local metric structure of the germ \((X, 0)\) with respect to its inner metric. If \(E_v\) is an exceptional component in a good resolution of \((X, 0)\), then its inner rate \(q_v\), introduced in [BNP14] and further studied in [BdSFP19], measures the shrinking rate of the piece of the link of \((X, 0)\) that corresponds to \(E_v\) (see [BdSFP19, §1, §3]).

In order to give a precise statement of our results we need to introduce some additional notation. Let \(\pi: X_\pi \to X\) denote a good resolution of \((X, 0)\), let \(\Gamma_\pi\) be the dual graph of \(\pi\), and denote by \(V(\Gamma_\pi)\) the set of vertices of \(\Gamma_\pi\), so that every element \(v\) of \(V(\Gamma_\pi)\) corresponds to an exceptional component \(E_v\) of the exceptional divisor \(E = \pi^{-1}(0)\) of \(\pi\). We denote by \(Z_{\max}(X, 0) = \sum_{v \in V(\Gamma_\pi)} m_v E_v\) the maximal ideal divisor of \((X, 0)\), that is the divisor of \(X_\pi\) supported on \(E\) and whose coefficient \(m_v\), called \emph{multiplicity of} \(v\), is the multiplicity along the component \(E_v\) of the pullback via \(\pi\) of a generic linear form \(h: (X, 0) \to (\mathbb{C}, 0)\) on \((X, 0)\). While in general the divisor \(Z_{\max}(X, 0)\) depends on the analytic type of \((X, 0)\), there is another divisor supported on \(E\), namely the fundamental cycle \(Z_{\min}\) of \(\Gamma_\pi\), defined as the unique minimal nonzero element of the Lipman cone of \(\Gamma_\pi\) (see Section 2 for the relevant definitions), which only depends on the graph \(\Gamma_\pi\). Finally, we denote by \(Z_{\Gamma_\pi} \cdot E_v = -E_v^2 + 2g(E_v) - 2\) for every vertex \(v\) of \(\Gamma_\pi\).
For each vertex $v$ of $\Gamma_\pi$, set $l_v = -Z_{\max}(X,0) \cdot E_v$, that is, $l_v$ is the intersection multiplicity of $E_v$ with the strict transform of a generic hyperplane section $h^{-1}(0)$ of $(X,0)$ via $\pi$. We call $L$-vector of $(X,0)$ the vector $L_\pi = (l_v)_{v \in V(\Gamma_\pi)} \in \mathbb{Z}_{\geq 0}^{V(\Gamma_\pi)}$. Recall that the blowup $Bl_0 X$ of the maximal ideal of $(X,0)$ is the minimal resolution of the base points of the family of generic hyperplane sections of $(X,0)$. Therefore, whenever $\pi: X_\pi \to X$ factors through $Bl_0 X$, the strict transform of such a generic hyperplane section via $\pi$ consists of a disjoint union of smooth curves that intersect transversely $E$ at smooth points of $E$, and $l_v$ is the number of such curves passing through the component $E_v$; we then call $L$-node of $\Gamma_\pi$ (or simply of $(X,0)$) any vertex $v$ such that $l_v > 0$. Similarly, we denote by $p_v$ the intersection multiplicity of the strict transform of the polar curve of a generic plane projection $\ell: (X,0) \to (\mathbb{C}^2,0)$ with $E_v$ and we call $P$-vector of $(X,0)$ the vector $P_\pi = (p_v)_{v \in V(\Gamma_\pi)} \in \mathbb{Z}_{\geq 0}^{V(\Gamma_\pi)}$. Since the Nash transform $\nu$ of $(X,0)$ is the minimal resolution of the base points of the family of generic polar curves of $(X,0)$, whenever $\pi: X_\pi \to X$ factors through $\nu$ and such a strict transform consists of smooth curves intersecting $E$ transversely at smooth points, and $p_v$ equals the number of such curves through $E_v$; we then call $P$-node of $\Gamma_\pi$ (or simply of $(X,0)$) any vertex $v$ such that $p_v > 0$. Finally, whenever $\pi: X_\pi \to X$ factors through the blowup of the maximal ideal, we define a natural distance $d$ on $\Gamma_\pi$ by declaring the length of an edge $e$ between two vertices $v$ and $v'$ of $\Gamma_\pi$ to be $1/\lcm\{m_v, m_{v'}\}$.

We can now state our first main theorem, which generalizes Spivakovsky’s result [Spi90, III, Theorem 5.4] to all LNE surface germs with isolated singularities.

**Theorem 1.1.** Let $(X,0)$ be an LNE surface germ with an isolated singularity, let $\pi: X_\pi \to X$ be the minimal good resolution of $(X,0)$, and let $\Gamma_\pi$ be the dual graph of $\pi$. Then the following properties hold.

(i) The resolution $\pi$ factors through the blowup of the maximal ideal of $(X,0)$ and all $L$-nodes have multiplicity one.

(ii) The maximal ideal divisor $Z_{\max}(X,0)$ of $(X,0)$ coincides with the fundamental cycle $Z_{\min} \Gamma_\pi$. In particular, $\Gamma_\pi$ determines the multiplicity $m_v$ associated with every vertex $v$ of $\Gamma_\pi$, and therefore also the set $V_L \Gamma_\pi$ of $L$-nodes of $\Gamma_\pi$, the $L$-vector $L_\pi$ of $(X,0)$, and the distance $d$ on $\Gamma_\pi$.

(iii) The inner rate $q_v$ of each vertex (or, more generally, of each divisorial point) of $\Gamma_\pi$ is given by

$$q_v = d(v, V_L) + 1.$$ 

(iv) The $P$-vector $P_\pi$ of $(X,0)$ is determined by

$$p_v = -E_v \cdot \left( \sum_{v'} ((m_{v'}q_{v'}) - 1)E_{v'} - (Z_{\Gamma_\pi} - Z_{\min}) \right)$$

for every vertex $v$ of $\Gamma_\pi$.

(v) Let $\pi'$ be the minimal good resolution of $(X,0)$ that factors through its Nash transform. A vertex $v$ of $\Gamma_{\pi'}$ is a $P$-node of $(X,0)$ if and only if either $l_v > 1$ or there exist two distinct vertices $v'$ and $v''$ of $\Gamma_{\pi'}$ adjacent to $v$ and such that $q_{v'} > q_{v''} < q_v$.

(vi) An edge $e = [v, v']$ of $\Gamma_\pi$ contains a $P$-node of $(X,0)$ in its interior if and only if $q_e - q_{v'} < d(v, v')$. When this is the case, $e$ contains exactly one $P$-node $w$, and its inner rate is $q_w = (d(v, v') + q_v + q_{v'})/2$. The resolution $\pi'$ is obtained by composing $\pi$ with a finite sequence of blowups of double
points of the respective exceptional divisor: at each step, a double point in \(E_v \cap E_{v'}\) has to be blown up if and only if \(|q_v - q_{v'}| < d(v, v')\).

In particular, we can build from \(\Gamma_{\pi}\) the resolution graph \(\Gamma_{\pi'}\) of \(\pi'\), decorated by arrows corresponding to the components of the polar curve of a generic plane projection \(\ell: (X, 0) \to (\mathbb{C}^2, 0)\) and by the inner rate of each vertex. While the first two parts of the Theorem are quite elementary, the remaining parts rely heavily on a careful study of the generic projections of LNE surfaces (see Lemma 4.1) that builds on results from \([NPP19a]\), and parts (iii) and (iv) also depend on the study of inner rates of \([BdSFP19]\), and in particular on the so-called Laplacian Formula of loc. cit.

We then move our attention to the study of the discriminant curve \(\Delta\) of a generic plane projection \(\ell: (X, 0) \to (\mathbb{C}^2, 0)\) of \((X, 0)\). Our second main result, which generalizes Bondil’s results \([Bon03, \text{Theorem 4.1}], [Bon16, \text{Proposition 5.4}]\), can be stated as follows.

**Theorem 1.2.** Let \((X, 0)\) be a LNE surface germ with isolated singularities and let \(\pi: X_{\pi} \to X\) be the minimal good resolution of \((X, 0)\). Then the dual graph \(\Gamma_{\pi}\) of \(\pi\) determines the embedded topological type of the discriminant curve of a generic plane projection \(\ell: (X, 0) \to (\mathbb{C}^2, 0)\) of \((X, 0)\).

To be more precise, the embedded topological type of a plane curve can be conveniently encoded in a combinatorial object, its Eggers–Wall tree, whose construction will be recalled in Section 7 (see also \([GBGPPP19, \text{Definition 3.9}]\)). We will give a more precise statement of Theorem 1.2 in Theorem 7.5, showing explicitly how to obtain the Eggers–Wall of the discriminant curve \(\Delta\) of a generic plane projection \(\ell: (X, 0) \to (\mathbb{C}^2, 0)\) of \((X, 0)\) as the quotient of the graph \(\Gamma_{\pi'}\) by a suitable equivalence relation.

Part (iv) of Theorem 1.1 can be thought of as the uniqueness of a solution, within the class of LNE surface singularities, to what we refer to as the problem of polar exploration of surface singularities, which asks to determine the possible configurations of arrows of a finite graph that can be realized as polar curves of a complex surface germ \((X, 0)\). Recall that surface singularities can be resolved either by a sequence of normalized point blowups, following seminal work of Zariski \([Zar39]\) from the late nineteen thirties, or by a sequence of normalized Nash transforms, as was done half a century later by Spivakovsky \([Spi90]\). The relationship between these two resolution algorithms, and therefore between hyperplane sections and polar curves of a surface singularity, is still quite mysterious, and they seem to be in some sense dual, as was observed by Lê \([L00, \text{§4.3}]\).

Recall that the incidence matrix of the dual graph \(\Gamma_{\pi}\) associated with a good resolution \(\pi: X_{\pi} \to X\) of \((X, 0)\) is negative definite by a classical result of Mumford \([Mum61, \text{§1}]\). Moreover, Grauert \([Gra62]\) proved that every weighted graph \(\Gamma\) without loops and with negative definite incidence matrix can be realized as dual graph \(\Gamma_{\pi}\) associated with a good resolution of some normal complex surface germ \((X, 0)\). It is well known that the weighted graph \(\Gamma_{\pi}\) determines the topology of \((X, 0)\), since \(\Gamma_{\pi}\) is a plumbing graph of the link of \((X, 0)\), and conversely, as we have already mentioned, Neumann \([Neu81]\) proved that the plumbing graph \(\Gamma_{\pi}\) is determined up to a natural equivalence relation by the topology of the surface germ. It is thus natural to consider the plumbing graph \(\Gamma_{\pi}\) endowed with an \(L\)- and a \(P\)-vector.
From this point of view, our result implies the following statement.

**Corollary 1.3.** Let \( \Gamma \) be a finite connected graph without loops weighted by attaching to each vertex \( v \) a genus \( g(v) \geq 0 \) and a self-intersection \( e(v) < 0 \). Then there exist finitely many pairs \( (L, P) \) of vectors \( L = (l_i), P = (p_i) \in (\mathbb{Z}_{\geq 0})^{V(\Gamma)} \) such that there exist a normal surface singularity \((X, 0)\) and a good resolution \( \pi \) of \((X, 0)\) satisfying
\[
(\Gamma, L, P) = (\Gamma_\pi, L_\pi, P_\pi).
\]
Moreover, at most one such pair \( (L, P) \) can be realized by a LNE germ \((X, 0)\).

Observe that, strictly speaking, only the last part of the statement is derived from Theorem 1.1. We deduce the rest of the statement, regarding the finiteness of potential solutions to the problem of polar exploration, from a result of Caubel–Nemethi–Popescu-Pampu [CNPP06], and from the classical Lê–Greuel–Teissier formula [LT81].

We conclude the paper by discussing two tools that can be used for polar exploration, that is to reduce the list of possibly realizable pairs of \( L \)- and \( P \)-vectors, either by using the Laplacian formula for the inner rates of [BdSFP19] to restrict the relative positions of polar curves and hyperplane sections on a resolution graph, or by studying carefully the topology of a Milnor–Lê fiber of our surface germ. In Example 8.2 we show that, by combining these tools, we can sometimes find a unique solution to the problem of polar explorations also for non LNE surface germs.

**Acknowledgments.** We thank Patrick Popescu-Pampu and Bernard Teissier for interesting discussions about quasi-ordinary singularities and generic plane projections. We would also like to thank Camille Le Van and Delphine Menard for fruitful conversations. This work has been partially supported by the project *Lipschitz geometry of singularities (LISA)* of the *Agence Nationale de la Recherche* (project ANR-17-CE40-0023) and by the *PEPS–JCJC Métriques singulières, valuations et géométrie Lipschitz des variétés* of the *Institut National des Sciences Mathématiques et de leurs Interactions* of the *Centre National de la Recherche Scientifique*. The second author has also been supported by a *Research Fellowship* of the *Alexander von Humboldt Foundation*.

2. **Surface germs with unique \( L \)-vector**

In this section we prove parts (i) and (ii) of Theorem 1.1. More generally, we are interested in finding a suitable geometric condition yielding a class of complex surfaces \((X, 0)\) whose \( L \)-vector is completely determined by the topology of a resolution. In order to achieve this, we recall the precise definitions of the divisors \( Z_{\text{max}}(X, 0) \) and \( Z_{\text{min}} \) that have been mentioned in the introduction, and determine a condition that guarantees their equality.

We begin by recalling the notion of Lipman cones. Let \( \Gamma \) be a finite connected graph without loops and such that each vertex \( v \in V(\Gamma) \) is weighted by two integers, its genus \( g(v) \geq 0 \) and its self-intersection \( e(v) < 0 \). We assume that the incidence matrix induced by the self-intersections of the vertices of \( \Gamma \), that is the matrix \( I_\Gamma \in \mathbb{Z}^{V(\Gamma)} \) whose \((v, v')\)-th entry is \( e(v) \) if \( v = v' \), and the number of edges of \( \Gamma \) connecting \( v \) to \( v' \) otherwise, is negative definite. Let \( E \) be a configuration of curves whose dual graph is \( \Gamma \), so that \( I_\Gamma = (E_i \cdot E_j) \), and consider the free additive group
\(G\) generated by the irreducible components of \(E\), that is
\[
G = \left\{ D = \sum_{v \in V(\Gamma)} m_v E_v \middle| m_i \in \mathbb{Z} \right\}.
\]

By a slight abuse of notation, we refer to the elements of \(G\) as \textit{divisors on} \(\Gamma\). On \(G\) there is a natural intersection pairing \(D \cdot D'\), described by the incidence matrix \(I_\Gamma\), and a natural partial ordering given by setting \(\sum m_v E_v \leq \sum m'_v E_v\) if an only if \(m_v \leq m'_v\) for every \(v \in V(\Gamma)\).

The \textit{Lipman cone} of \(\Gamma\) is the semi-group \(\mathcal{E}^+\) of \(G\) defined as
\[
\mathcal{E}^+ = \{ D \in G \mid D \cdot E_v \leq 0 \text{ for all } v \in V(\Gamma) \}.
\]

**Remark 2.1.** By looking at a divisor’s coefficients we can identify \(G\) with the additive group \(\mathbb{Z}^{V(\Gamma)}\). Then the Lipman cone \(\mathcal{E}^+\) of \(\Gamma\) is naturally identified with the cone \(\mathbb{Z}^{V(\Gamma)} \cap -I_\Gamma^{-1}(\mathbb{Z}^{V(\Gamma)})\), since by definition a divisor \(\sum m_v E_v\) belongs to \(\mathcal{E}^+\) if and only if the vector \(I_\Gamma \cdot (m_v)_{v \in V(\Gamma)}\) belongs to \(\mathbb{Z}^{V(\Gamma)}\).

A fundamental property of the Lipman cone \(\mathcal{E}^+\), proven in [Art66, Proposition 2], is that it has a unique nonzero minimal element \(Z_{\min}\), called the \textit{fundamental cycle} of \(\Gamma\), and that moreover \(Z_{\min} > 0\), that is the coefficients of \(Z_{\min}\) are all strictly positive. Observe that the existence of the fundamental cycle and the fact that \(Z_{\min} > 0\) are equivalent to the fact that \(D > 0\) for every nonzero divisor \(D\) in \(\mathcal{E}^+\).

Assume from now on that \(\Gamma\) is the dual graph of a good resolution of an isolated surface singularity \((X, 0)\). Notice that the Lipman cone, and therefore its fundamental cycle, only depend on the graph \(\Gamma\), that is on the topology of \((X, 0)\), and not on the complex geometry of \((X, 0)\); the fundamental cycle \(Z_{\min}\) can be explicitly computed from \(\Gamma\) by using Laufer’s algorithm from [Lau72, Proposition 4.1].

Consider now a germ of analytic function \(f : (X, 0) \rightarrow (\mathbb{C}, 0)\). The \textit{total transform} of \(f \circ \pi\) is the divisor \((f)_{\Gamma} = (f)_{\Gamma} + f^*\) on \(X_\pi\), where \(f^*\) is the strict transform of \(f\) and \((f)_{\Gamma} = \sum_{v \in V(\Gamma)} m_v(f) \cdot E_v\) is the divisor supported on \(E\) such that \(m_v(f)\) is the multiplicity of \(f \circ \pi\) along \(E_v\). By [Lau71, Theorem 2.6], we have
\[
(f)_{\Gamma} \cdot E_v = 0 \quad \text{for all } v \in V(\Gamma).
\]

In particular, \((f)_{\Gamma}\) belongs to the Lipman cone \(\mathcal{E}^+\) of \(\Gamma\), and therefore the semi-group \(\mathcal{A}_X^+ = \{(f)_{\Gamma} \mid f \in O_{(X, 0)}\}\) of \(G\) is contained in \(\mathcal{E}^+\); it has a unique nonzero minimal element \(Z_{\max}(X, 0)\), which is called the \textit{maximal ideal divisor} of \((X, 0)\). Observe that the divisor \(Z_{\max}(X, 0)\) coincides with the cycle \((h)_{\Gamma}\) of a generic linear form \(h : (X, 0) \rightarrow (\mathbb{C}, 0)\), and that by the definition of the fundamental cycle we have \(Z_{\min} \leq Z_{\max}(X, 0)\).

The following proposition is the main result of this section.

**Proposition 2.2.** Let \((X, 0)\) be a normal surface singularity and let \(\pi : (X_\pi, E) \rightarrow (X, 0)\) be the minimal good resolution of \((X, 0)\). If a generic hyperplane section of \((X, 0)\) is a union of smooth curves, then:

(i) \(\pi\) factors through the blowup of the maximal ideal of \((X, 0)\) and all \(\mathcal{L}\)-nodes have multiplicity one;

(ii) the maximal ideal divisor \(Z_{\max}(X, 0)\) of \((X, 0)\) coincides with the fundamental cycle \(Z_{\min}\) of \(\Gamma_\pi\).

**Proof.** Let \(\pi' : X_{\pi'} \rightarrow X\) be the minimal good resolution of \((X, 0)\) which factors through the blowup of its maximal ideal, and let \(E_v\) be the component of \((\pi')^{-1}(0)\)
associated with an \(\mathcal{L}\)-node \(v\) of \((X, 0)\). Let \(h: (X, 0) \to (\mathbb{C}, 0)\) be a generic linear form and let \(\gamma\) be an irreducible component of \(h^{-1}(0)\) whose strict transform \(\gamma^*\) by \(\pi'\) intersects \(E_v\). Then \(\gamma^*\) is a curvette of \(E_v\), that is a smooth curve intersecting transversally \(E_v\) at a smooth point of \((\pi')^{-1}(0)\). If \(F \subset (X, 0)\) is the Milnor–Lê fiber of a generic linear form \(h'\) on \((X, 0)\) whose strict transform by \(\pi'\) is disjoint from the one of \(h\), then the multiplicity \(m_v\) of \(E_v\) equals the intersection multiplicity of \(\gamma\) with \(F\), which is 1 since \(\gamma\) is smooth by hypothesis. This proves that \(m_v = 1\). Assume now that \(\pi\) does not factor through the blowup of the maximal ideal, so \(\pi' = \pi \circ \alpha\), where \(\alpha\) is a finite composition of point blowups. By minimality of \(\pi'\) there exists an \(\mathcal{L}\)-node \(v_0\) of \((X, 0)\) which is associated with the exceptional component of one of the point blowups in \(\alpha\). Let \(\alpha_1\) be the first blowup in the sequence \(\alpha\), that is, \(\alpha_1\) is the blowup of \(X_\pi\) at a point \(p\) of \(E_v \cap h^*\), where \(E_v\) is a component of \(\pi^{-1}(0)\), and let \(E_w\) be the exceptional curve of \(\alpha_1\). Since \(h^*\) passes through \(p\), we have \(m_v = m_w(h) > m_w(h) \geq 1\). As this argument can be repeated for every blowup forming \(\alpha\), we deduce that \(m_v(h) \geq 1\) as well, contradicting the first part of the proof. This implies that \(\alpha\) must be an isomorphism, proving (i).

To prove (ii), write \(Z_{\max} = Z_{\max}(X, 0) = \sum_{v \in V(\Gamma)} m_v E_v\) and \(Z_{\min} = \sum_{v \in V(\Gamma)} \tilde{m}_v E_v\), and for every \(v\) in \(V(\Gamma)\) consider the positive integers

\[
I_v = -Z_{\max} \cdot E_v \quad \text{and} \quad \tilde{I}_v = -Z_{\min} \cdot E_v.
\]

Since \(Z_{\min} \leq Z_{\max}\) by definition of \(Z_{\min}\), it is enough to prove that \(Z_{\min} - Z_{\max}\) is positive. Since \(I_{\Gamma}\) is negative definite, it is therefore sufficient to show that \((Z_{\min} - Z_{\max}) \cdot E_v = I_v - \tilde{I}_v\) is negative for every vertex \(v\) of \(\Gamma\). Whenever \(I_v = 0\), this follows immediately from the definition, so let us fix a vertex \(v\) such that \(I_v > 0\). From part (i), we know that \(m_v = 1\). It follows from the inequality \(0 < \tilde{m}_v \leq m_v = 1\) that \(\tilde{m}_v = 1\) as well. We therefore get:

\[
l_v - \tilde{l}_v = (Z_{\min} - Z_{\max}) \cdot E_v = \sum_{w \in V(\Gamma)} (\tilde{m}_w - m_w) E_w \cdot E_v = \sum_{w \neq v} (\tilde{m}_w - m_w) E_w \cdot E_v \leq 0,
\]

since \(E_w \cdot E_v \geq 0\) whenever \(w \neq v\) and \(\tilde{m}_w \leq m_w\) at all vertices. \(\square\)

The hypothesis of Proposition 2.2 is quite weak, as it is satisfied by every normal surface germ with reduced tangent cone (in which case the components of a generic hyperplane section are not only smooth but also transverse, see for example [GSLJ97, §1]), for example by every minimal surface singularity. More generally, the hypothesis holds for all LNE surface germs, as was proven in [FS19b, Theorem 3.10]. In particular, the proposition implies parts (i) and (ii) of Theorem 1.1.

Observe that it follows by equation (1) that the vector \(-I_{\Gamma} \cdot Z_{\max}(X, 0)\) of \(\mathbb{Z}^{V(\Gamma)}\) coincides with the \(\mathcal{L}\)-vector \(L_{\pi}\) of \((X, 0)\) considered in the introduction. Therefore, whenever \(Z_{\max}(X, 0)\) is determined by the topological type of \((X, 0)\), the same holds true for \(L_{\pi}\). We collect this result, which is the first step towards the proof of Corollary 1.3, in the following corollary.

**Corollary 2.3.** Let \(\Gamma\) be a weighted graph. Then there exists at most one vector \(L \in \mathbb{Z}^{V(\Gamma)}\) such that there exist a normal surface germ \((X, 0)\) whose generic hyperplane section is a union of smooth curves and a good resolution \(\pi: X_{\pi} \to (X, 0)\) satisfying \((\Gamma, L) = (\Gamma_{\pi}, L_{\pi})\).
3. A LEMMA ON GENERIC PROJECTIONS

In this section we will introduce three notions that will prove fundamental in the remaining part of the paper, namely generic projections, non-archimedean links, and local degrees. We will also prove an important result, Lemma 3.1, that shows the compatibility of generic projections with minimal resolutions.

We begin by discussing the notion of generic projection, which is based on seminal work of Teissier. Fix an embedding of \((X, 0)\) in a smooth germ \((\mathbb{C}^n, 0)\), and consider the morphism \(\ell_D: (X, 0) \to (\mathbb{C}^2, 0)\) obtained as the restriction to \(X\) of the projection along a \((n - 2)\)-dimensional linear subspace \(D\) of \(\mathbb{C}^n\). Recall that whenever \(\ell_D\) is finite, the associated polar curve \(\Pi_D\) is the closure in \((X, 0)\) of the ramification locus of the restriction of \(\ell_D\) to \(X \setminus \{0\}\), and the associated discriminant curve is the plane curve \(\Delta_C = \ell_D(\Pi_D)\). The Grassmannian variety \(\text{Gr}(n - 2, \mathbb{C}^n)\) of \((n - 2)\)-planes in \(\mathbb{C}^n\) contains an analytic dense open subset \(\Omega\) such that for every \(D\) in \(\Omega\) the projection \(\ell_D\) is finite and both the families \(\{\Pi_D\}_{D \in \Omega}\) and \(\{\Delta_D\}_{D \in \Omega}\) well behaved (for example, they are equisingular in a strong sense). We say that a morphisms \(\ell: (X, 0) \to (\mathbb{C}^2, 0)\) is a generic projection of \((X, 0)\) if \(\ell = \ell_D\) for some \(D\) in \(\Omega\). A discussion of the properties satisfied by a generic projection, leading to a precise definition of \(\Omega\), can be found in [NPP19a, §2], building on work of Teissier (see in particular [Tei82, Lemme-clé V 1.2.2]); we will come back to this matter later in this section.

We now recall the definition of the non-archimedean link \(\text{NL}(X, 0)\) of the germ \((X, 0)\). Indeed, our goal for this section is to study the map induced by a generic projection \(\ell: (X, 0) \to (\mathbb{C}^2, 0)\) on the dual graphs of a good resolution of \((X, 0)\). In principle, for this to make sense it is necessary to chose a suitable good resolution \(\pi: X_\pi \to X\) of \((X, 0)\), and a compatible sequence of blowups \(\sigma: Y_\sigma \to \mathbb{C}^2\) of \(\mathbb{C}^2\) above \(0\) in order for \(\ell\) to induce a map \([\Gamma_{\pi}] \to [\Gamma_{\sigma}]\) between the topological spaces underlying \(\Gamma_\pi\) and \(\Gamma_\sigma\). In this paper we will use \(\text{NL}(X, 0)\) as a convenient way of encoding intrinsically all the dual graphs of good resolutions of \((X, 0)\); for this purpose, we can adopt the following ad hoc definition. Recall that, if \(\pi: X_\pi \to X\) and \(\pi': X_{\pi'} \to X\) are two good resolution of \((X, 0)\) such that \(\pi'\) dominates \(\pi\) (that is, \(\pi'\) factors through \(\pi\)), then we have a natural inclusion \([\Gamma_{\pi}] \to [\Gamma_{\pi'}]\) between the topological spaces underlying the dual graphs \(\Gamma_\pi\) and \(\Gamma_{\pi'}\), and a retraction \([\Gamma_{\pi'}] \to [\Gamma_\pi]\) obtained by contracting the trees in \([\Gamma_{\pi'}] \setminus [\Gamma_\pi]\). The non-archimedean link can then be seen the inverse limit \(\text{NL}(X, 0) = \varprojlim [\Gamma_\pi]\) in the category of topological spaces and with respect to the various retraction morphisms, where the limit runs over the poset of good resolutions of \((X, 0)\), ordered by domination. In particular, \(\text{NL}(X, 0)\) contains a copy the dual graph of each good resolution of \((X, 0)\), and it can be seen as a compactification of the infinite union \(\bigcup_\pi [\Gamma_\pi]\) of all the dual graphs of the good resolutions of \((X, 0)\). As such, it can be thought of as a universal dual graph of the singularity \((X, 0)\). To unburden the notation, in the remaining part of the paper we will usually identify a dual graph \(\Gamma_\pi\) with its image \([\Gamma_\pi]\) in \(\text{NL}(X, 0)\).

Traditionally, the non-archimedean link \(\text{NL}(X, 0)\) can also be built as a space of normalized semivaluations on the complete local ring \(\widehat{O}_{X,0}\) of \(X\) at \(0\). In particular, if \(\pi: X_\pi \to X\) is a good resolution of \((X, 0)\) and \(E_\pi\) is a component of its exceptional divisor \(\pi^{-1}(0)\), the corresponding vertex of \(\Gamma_\pi\) is identified with the corresponding divisorial valuation \(v: \widehat{O}_{X,0} \to \mathbb{R}_+ \cup \{+\infty\}\) defined by \(v(f) = \text{ord}_{E_\pi}(\pi^*f)/m_v\),
where \( \text{ord}_{E_v}(\pi^*f) \) denotes the order of vanishing along \( E_v \) of the pullback of \( f \) via \( \pi \).

Throughout the paper, we will freely make use of this terminology, calling divisorial point of \( \text{NL}(X,0) \) (or of a given dual graph \( \Gamma_\pi \)) any point that can arise in this way, and denoting by \( E_v \) any exceptional curve corresponding to a divisorial point \( v \). Observe that the subset of \( \text{NL}(X,0) \) consisting of its divisorial points is dense in the non-archimedean link; this corresponds to the fact that any given dual graph \( \Gamma_\pi \) can be refined ad infinitum by passing to resolutions dominating \( \pi \), subdividing each edge \( e = [v,v'] \) into smaller edges by successively blowing up double points starting with the blowup of \( X_\pi \) at \( E_v \cap E'_v \). We refer the reader to [BdSFP19, §2.1] and [Fan18] for further details on this point of view.

The morphism \( \ell: (X,0) \rightarrow (C^2,0) \) induces a natural map \( \hat{\ell}: \text{NL}(X,0) \rightarrow \text{NL}(C^2,0) \). From the point of view of semivaluations, this is simply defined functorially by pre-composing a semivaluation on \( \mathcal{O}_{X,0} \) with the morphism of complete local rings \( \mathcal{O}_{C^2,0} \rightarrow \mathcal{O}_{X,0} \) induced by \( \ell \).

Concretely, \( \hat{\ell}(v) \) can also be computed explicitly on a divisorial point \( v \) of \( \text{NL}(X,0) \) as follows: we can find a sequence of blowups \( \sigma_{\ell,v}: Y_{\sigma_{\ell,v}} \rightarrow C^2 \) of \( C^2 \) above \( 0 \) and a good resolution \( \pi_{\ell,v}: X_{\pi_{\ell,v}} \rightarrow X \) of \( (X,0) \) such that \( v \) corresponds to an exceptional component \( E_v \) of \( \pi_{\ell,v} \), the composition \( \ell \circ \pi_{\ell,v}: X_{\pi_{\ell,v}} \rightarrow C^2 \) factors through a map \( \hat{\ell}: X_{\pi_{\ell,v}} \rightarrow Y_{\sigma_{\ell,v}} \) making the following diagram commute

\[
\begin{array}{ccc}
X_{\pi_{\ell,v}} & \xrightarrow{\pi_{\ell,v}} & X \\
\downarrow{\hat{\ell}} & & \downarrow{\ell} \\
Y_{\sigma_{\ell,v}} & \xrightarrow{\sigma_{\ell,v}} & C^2
\end{array}
\]

and such that \( E_v \) is mapped by \( \hat{\ell} \) surjectively onto an exceptional component \( E_w \) of \( \sigma_{\ell,v} \); we then have \( \hat{\ell}(v) = w \).

Let \( \pi: X_\pi \rightarrow X \) be a good resolution of \( (X,0) \). Then, since \( \ell \) is a finite map, ramified precisely over the associated polar curve, the induced map \( \hat{\ell}|_{\Gamma_\pi}: \Gamma_\pi \rightarrow \tilde{\ell}(\Gamma_\pi) \) is itself a finite cover, ramified precisely at the set of \( P \)-nodes of \( (X,0) \) contained in \( \Gamma_\pi \). In particular, while an edge of \( \Gamma_\pi \) can be folded by \( \ell \), it cannot be contracted by it.

Observe that the map \( \hat{\ell} \) clearly depends on the choice of \( \ell \). Indeed, if \( \ell': (X,0) \) is another generic projection obtained by composing \( \ell \) with an automorphism \( \phi \) of \( (C^2,0) \), then \( \phi \) induces a nontrivial automorphism \( \tilde{\phi} \) of \( \text{NL}(C^2,0) \), and we have \( \hat{\ell} = \tilde{\phi} \circ \hat{\ell}' \). While in general two generic projections of \( (X,0) \) do not differ by an automorphism of \( (C^2,0) \), it is possible to control this phenomenon if we restrict \( \hat{\ell} \) to the dual graph \( \Gamma_\pi \) of the minimal good resolution \( \pi: X_\pi \rightarrow X \) of \( (X,0) \) that factors through its Nash transform, as we explain in Lemma 3.1 below.

In order to do this, we need to dive deeper into the definition of generic projections, to be able to study the polar curves and the discriminant curves of \( (X,0) \) in families. Let us begin by recalling the precise notion of strong equiresolution of singularities given in [Tei80, 3.1.1 and 3.1.5]. Given a morphism \( \beta: M \rightarrow \Lambda \) with reduced fibers between smooth connected complex manifolds and a snc divisor \( E \) on \( M \), we say that \( \beta \) is simple (with respect to \( E \)) if \( \beta \) is smooth and its restriction \( \beta|_E: E \rightarrow \Lambda \) to \( E \) is proper and locally a trivial deformation along its fibers. If we have another morphism \( \sigma: M' \rightarrow M \), we say that \( \sigma \) is \( \beta \)-compatible if the composition \( \beta' = \beta \circ \sigma \)
is simple (with respect to \( E' = \sigma^{-1}(E) \)). Finally, given a (singular) subvariety \( X \) of \( M \), we say that an embedded resolution of singularities \( \pi: \tilde{M} \to M \) of \( X \) is a (strong) equiresolution (along \( \Lambda \)) of \( X \) if \( \pi \) is \( \beta \)-compatible and all of its restrictions \( \pi_{\lambda} \) over \( \lambda \in \Lambda \) are good embedded resolutions of \( X_{\lambda} \).

According to [Tei82, Lemme-clé V 1.2.2] (see [NPP19a, Proposition 2.3] for an English presentation), there exists an analytic open dense subset \( \Omega \) of the Grassmannian \( \text{Gr}(n - 2, \mathbb{C}^n) \) where the family \( \{ (\Delta_D, D) \}_{D \in \Omega} \) of discriminant curves, which can be seen as a surface in \( (\mathbb{C}^2, 0) \times \Omega \) fibered over \( \Omega \) via the projection \( \beta: (\mathbb{C}^2, 0) \to \Omega \) on the second factor, admits a strong embedded equiresolution

\[
(\mathcal{Y}, \mathcal{F}) \xrightarrow{\sigma} (\mathbb{C}^2, 0) \times \Omega \xrightarrow{\beta} \Omega
\]

with \( F = \sigma^{-1}(\{0\} \times \Omega) \) a smooth divisor on \( \mathcal{Y} \).

For each \( D \in \Omega \), denote by \( \sigma_D: (\mathcal{Y}_D, \mathcal{F}_D) \to (\mathbb{C}^2, 0) \) the restriction of \( \sigma \) to the fiber \( \beta_{\mathcal{Y}}^{-1}(D) \), which is a sequence of blowups of \( \mathbb{C}^2 \) above \( 0 \). This will allow us to define an isomorphism of graphs \( \eta_{D, D'}: \Gamma_{\sigma_D} \xrightarrow{\sim} \Gamma_{\sigma_{D'}} \), as follows. For each \( v \in V(\Gamma_{\sigma_D}) \), if we denote by \( F^D_v \) the corresponding irreducible component of \( F = \sigma_D^{-1}(0) \), there is a unique irreducible component \( F^D_v \) of \( \sigma^{-1}(\{0\} \times \Omega) \) such that \( F^D_v = F^D \cap \sigma^{-1}(0) \). We then set \( \eta_{D, D'}(v) = v' \), where \( v \) is the vertex of \( \Gamma_D \) such that \( F^D_v \cap \sigma_D^{-1}(0) = F^D_v \setminus (\mathbb{C}^2, 0) \). This yields a bijection \( V(\Gamma_{\sigma_D}) \to V(\Gamma_{\sigma_{D'}}) \) which extends to a natural homeomorphism

\[
\eta_{D, D'}: \Gamma_{\sigma_D} \to \Gamma_{\sigma_{D'}}
\]

defined on the divisorial points of \( \Gamma_{\sigma_D} \) as follows. Fix \( D \in \Omega \) and consider a divisorial point \( v \) on an edge \([v_1, v_2]\) of \( \Gamma_{\sigma_D} \). Then \( E^D_v \) is created by a finite sequence of blowups of double points of the previous exceptional divisor, starting with the blowup of the point \( F^D_{v_1} \cap F^D_{v_2} \). We can perform this blowups in family by blowing up along successive intersections of the form \( F^D_{w_1} \cap F^D_{w_2} \), starting with the blowup along \( F^D_{v_1} \cap F^D_{v_2} \). By composing this sequence of blowups with \( \sigma \), we obtain a (\( \beta \)-compatible) morphism \( \mathcal{Y}_{\sigma_v} \to (\mathbb{C}^2, 0) \times \Omega \). The last blowup creates an irreducible new component \( F^D_{v'} \) in the exceptional divisor, and as before we define \( v' = \eta_{D, D'}(v) \) by declaring that the corresponding irreducible component \( F^D_{v'} \) is the intersection \( F^D_{v'} \cap (\mathbb{C}^2, 0) \). Observe that, since multiplicities are constant along a smooth family, we have \( m_v = m_{\eta_{D, D'}(v)} \) for every divisorial point \( v \) of \( \Gamma_{\sigma_D} \).

The following lemma relating the graphs of \( \Gamma_{\sigma_D} \) with \( \Gamma_\pi \) plays a crucial role in several arguments in the rest of the paper.

**Lemma 3.1.** Let \( (X, 0) \) be a normal surface singularity, let \( \pi: X_\pi \to X \) be the minimal good resolution of \( (X, 0) \) that factors through the blowup of its maximal ideal and its Nash transform, and let \( \Gamma_\pi \subset \text{NL}(X, 0) \) be the dual graph of \( \pi \). Then for all \( D \) and \( D' \) in \( \Omega \) the diagram

\[
\begin{array}{ccc}
\Gamma_\pi & \xrightarrow{\tilde{\iota}_{D|\Gamma_\pi}} & \Gamma_{\sigma_D} \\
\eta_{D, D'} \downarrow & & \downarrow \eta_{D, D'} \\
\Gamma_{\sigma_{D'}} & \xrightarrow{\tilde{\iota}_{D'|\Gamma_{\sigma_{D'}}}} & \Gamma_{\sigma_{D'}}
\end{array}
\]
obtained by restricting to the graph $\Gamma_\sigma$ the two induced morphisms of non-archimedean links $\tilde{\ell}_D, \tilde{\ell}'_D : \text{NL}(X, 0) \to \text{NL}(\mathbb{C}^2, 0)$, is commutative.

Before moving to the proof of the lemma, which is rather technical, we observe that the automorphism $\eta_{D, \sigma'} : \Gamma_{\sigma'} \to \Gamma_{\sigma''}$ extends naturally to an automorphism $\eta_{D, \sigma''}$ of the dual graph $\Gamma_{\sigma''}$ of any good resolution $\pi' : X_{\sigma'} \to X$ of $(X, 0)$. However, the commutativity $\eta_{D, \sigma''} \circ \tilde{\ell}_D$ does not necessarily hold on the whole of $\Gamma_{\sigma''}$. We defer an illustration of this phenomenon to Example 4.4, since showing this now would require a lengthy local computation, while after proving Lemma 4.1 we can give a more conceptual explanation.

As our needs go slightly beyond what was done by Teissier, let us explain how to adapt his constructions accordingly. We start by proving a technical lemma about resolution in families of surfaces, much in the spirit of [Tei80, 4.1 and 4.2]:

**Lemma 3.2.** Let $M$ and $\Omega$ be connected complex manifold such that $\dim(M) = \dim(\Omega) + 2$, let $E$ be an snc divisor over $M$, and let $\beta : M \to \Omega$ be a simple morphism (with respect to $E$). Consider a finite sequence of (adapted) smooth blowups $\sigma : (M', E') \to (M, E)$ whose centers have codimension at least 2. Then, up to shrinking the size of the dense open $\Omega$ (and, therefore, of $M$ and $M'$), the composition $\beta' = \beta \circ \sigma$ is simple (with respect to $E'$).

**Proof.** It is enough to prove the claim in the case that $\sigma$ is a single blowup with center $C$. By Remmert Proper Map Theorem applied to $\beta|_E$, the image $\beta(C)$ is a closed analytic subset of $\Omega$. If $\dim(\beta(C)) < \dim(\Omega)$, set $Z = \beta(C)$ and note that, once we replace $\Omega$ by $\Omega \setminus Z$, the result easily follows from the fact that $\sigma : M' \to M$ is an isomorphism. We can therefore assume that $\dim(\beta(C)) = \dim(\Omega)$, so that $\beta(C) = \Omega$. Since $\dim(C) \leq \dim(\Omega)$ by hypothesis, we conclude that $\dim(C) = \dim(\Omega)$, and in particular the restriction $\beta|_C : C \to \Omega$ is generically a local isomorphism. Let $Y \subset C$ be the set of critical points of $\beta|_C$, which is a proper closed analytic subset of $C$. Again by Remmert Proper Map Theorem, the image $Z' = \beta(Y)$ is a closed analytic subset of $\text{Gr}(n - 2, \mathbb{C}^n)$, properly contained in $\text{Gr}(n - 2, \mathbb{C}^n)$ because $\dim(Y) < \dim(\text{Gr}(n - 2, \mathbb{C}^n))$. Now, after replacing $\Omega$ by $\Omega \setminus Z'$, we can assume that $\beta : C \to \Omega$ is everywhere a local isomorphism. We now claim that $\beta'$ is simple via direct computation. Indeed, since smoothness can be verified locally, let us fix a point $p \in C$, and denote by $f_1$ and $f_2 \in \mathcal{O}_p$ local generators of $C$. Since $\beta$ is simple at $p$, there exist an (analytic) local coordinate system $(\lambda, x_1, x_2)$ at $p$ such that $\beta(\lambda, x_1, x_2) = \lambda$ and $E$ is locally contained in $(x_1 x_2 = 0)$. Since $\pi : C \to \Omega$ is a local isomorphism around $p$ and $C$ is smooth and adapted to $E$, it follows that the map $(\lambda, x_1, x_2) \to (\lambda, f_1, f_2)$ is a local isomorphism and $E \subset (f_1 f_2 = 0)$. Therefore, up to a local change of variables, we can assume that $f_1 = x_1$ and $f_2 = x_2$, and we easily conclude that $\beta' : M' \to \Omega$ is simple. \[\square\]

Now, recall that we have an embedding of $(X, 0)$ in $(\mathbb{C}^n, 0)$ and let $\Phi : (X, 0) \times \Omega \to (\mathbb{C}^2, 0) \times \Omega$ be the morphism defined as $\Phi(x, D) = (\ell_D(x), D)$, which is generically of maximal rank. Let $\pi : (X, E) \to (X, 0)$ be a resolution of singularities of $(X, 0)$ which factors through the blowup of its maximal ideal and through its Nash transform. We note that, by using Grauert Proper Mapping Theorem, resolution of singularities, and the universal property of blowups, there exists a sequence of blowups $\alpha : (Z, \mathcal{G}) \to (X, E) \times \Omega$ and an analytic morphism $\Psi : (Z, \mathcal{G}) \to (Y, F)$
such that $\Psi^{-1}(\mathcal{F})_{\text{red}} = \mathcal{G}_{\text{red}}$ and the following diagram

$$
\begin{array}{ccc}
(Z, \mathcal{G}) & \xrightarrow{\alpha} & (X_\pi, E) \times \Omega \\
\downarrow \Psi & & \downarrow \pi \times \text{Id} \\
(Y, \mathcal{F}) & \xrightarrow{\sigma} & (\mathbb{C}^2, 0) \times \Omega \\
\end{array}
$$

is commutative, with $\beta_Y = \beta \circ \sigma$ simple. Thanks to Lemma 3.2, up to shrinking the size of the open $\Omega$ if necessary, the morphism $\beta_Z = \beta_Y \circ \Psi$ is simple as well. We are now ready to complete the proof of Lemma 3.1.

**Proof of Lemma 3.1.** The map $\tilde{\ell}|_{\Gamma_v}$ is determined by its restriction to the set of divisorial points of $\Gamma_\pi$, as those form a dense subset of $\Gamma_\pi$. Since $\beta_Z$ and $\beta_Y$ are simple, for every pair of elements $D, D'$ of $\Omega$, the following diagram commutes

$$
\begin{array}{ccc}
V(\Gamma_\pi) & \xrightarrow{\tilde{\ell}_D} & V(\Gamma_{\sigma_D}) \\
\downarrow \tilde{\ell}_{D'} & & \downarrow \tilde{\ell}_{D'} \\
V(\Gamma_{\sigma_{D'}}) & \xrightarrow{\tilde{\ell}_{D', D'}} & V(\Gamma_{\sigma_{D'}}) \\
\end{array}
$$

We now need to prove the result on the divisorial points of $\Gamma_\pi$ which are not vertexes of $\Gamma_\pi$. It is sufficient to consider the case where $v$ is the divisorial point associated with the exceptional curve of the blowup $\pi': (X'_\pi, E') \to (X_\pi, E)$ of center $E_{v_1} \cap E_{v_2}$, since the same argument can then be repeated verbatim for general sequence of point blowups. Observe that if $E_v \times \Omega$ is already a component of $\mathcal{G}$, then $\tilde{\ell}_D(v) \in \Gamma_{\sigma_D}$ for every $D \in \Omega$, and we conclude easily. If $G_v = E_v \times \Omega$ is not a component of $\mathcal{G}$, we note that $G_{v_1} \cap G_{v_2} = (E_{v_1} \cap E_{v_2}) \times \Omega$ is an admissible center in $(Z, \mathcal{G})$, since all blowups in $\alpha$ are admissible. We therefore may perform this extra blowup $\alpha': (Z', G') \to (Z, \mathcal{G})$, whose exceptional divisor $G_v = E_v \times \Omega$ is trivial with respect to the family structure. Fix $D \in \Omega$, set $w_1 = \ell_D(v_1)$ and $w_2 = \ell_D(v_2)$, and consider the associated components $F_{w_1}^D$ and $F_{w_2}^D$ of $\mathcal{F}$. Then, after performing a sequence of combinatorial blowups $\rho: (Y', \mathcal{F}') \to (Y, \mathcal{F})$, starting with blowing up the center $F_{w_1}^D \cap F_{w_2}^D$, the projection $\tilde{\ell}_{D'}(v)$ belongs to the graph of $\Gamma_{\sigma_{D'} \circ \sigma_{D'}}$ for every $D'$ in $\Omega$. We have obtained, without the need to shrink the size of $\Omega$, the following commutative diagram:

$$
\begin{array}{ccc}
(Z', G') & \xrightarrow{\alpha'} & (Z, \mathcal{G}) & \xrightarrow{\alpha} & (X_\pi, E) \times \Omega & \xrightarrow{\pi \times \text{Id}} & (X, 0) \times \Omega \\
\downarrow \Psi' & & \downarrow \Psi & & \downarrow \Phi \\
(Y', \mathcal{F}') & \xrightarrow{\rho} & (Y, \mathcal{F}) & \xrightarrow{\sigma} & (\mathbb{C}^2, 0) \times \Omega & \xrightarrow{\beta} & \Omega \\
\end{array}
$$

where $\beta_{Y'} = \beta \circ \sigma \circ \rho$ and $\beta_{Z'} = \beta_Y \circ \Psi'$ are simple morphisms. We conclude easily. \qed

**Remark 3.3.** If $(X, 0)$ is an hypersurface in $(\mathbb{C}^3, 0)$, shrinking the open set $\Omega$ is not necessary when applying Lemma 3.2, since a resolution of the family can be constructed everywhere by performing a Hirzebruch–Jung process in family, exploiting the fact that, thanks to [PP02, Corollary 3.4] (or, more generally, to [PP04, Theorem 5.1]), the combinatorial data of the quasi-ordinary singularities that appear during the process are constant in the family.
We conclude the section by recalling the definition of the local degree of a divisorial point \( v \) of \( \text{NL}(X,0) \), as it will be very important in the remaining part of the paper. Let \( \ell : (X,0) \to (\mathbb{C}^2,0) \) be a generic projection of \( (X,0) \) and consider the diagram (2) (page 9). For each component \( E_v \) of \( \sigma_{v,-1}(0) \) (respectively \( E_{v'} \) of \( \sigma_{v',-1}(0) \)), let us choose a tubular neighborhood disc bundle \( \mathcal{N}(E_v) \) (resp. \( \mathcal{N}(E_{v'}) \)), and consider the two sets

\[
\mathcal{N}(E_v) = N(E_v) \setminus \bigcup_{E_{v'} \neq E_v} N(E_{v'}) \quad \text{and} \quad \mathcal{N}(E_{v'}) = N(E_{v'}) \setminus \bigcup_{E_v \neq E_{v'}} N(E_v)
\]

in \( X_{\sigma_v} \) and \( Y_{\sigma_{v'}} \) respectively. We can then adjust the disc bundles \( \mathcal{N}(E_v) \) and \( \mathcal{N}(E_{v'}) \) in such a way that the cover \( \ell \) restricts to a cover

\[
\ell_v : \pi_{\ell,v}(\mathcal{N}(E_v)) \to \sigma_{\ell,v}(\mathcal{N}(E_{v'}))
\]

branched precisely on the polar curve of \( \ell \) (if \( v \) is not a \( \mathcal{P} \)-node, the branching locus is just the origin). Using a resolution in family over \( \Omega \) as in the proof of the Lemma 3.1, it is easy to deduce the following result.

**Lemma 3.4.** For every divisorial point \( v \) of \( \text{NL}(X,0) \), the map \( \Omega : X \to \mathbb{N} \) that sends a generic projection \( \ell : (X,0) \to (\mathbb{C}^2,0) \) to the degree \( \deg(\ell_v) \) of the cover \( \ell_v \) is constant.

Therefore, we can set \( \deg(v) = \deg(\ell_v) \). We call this integer the local degree of a generic projection of \( (X,0) \) at \( v \), or simply the local degree of \( (X,0) \) at \( v \).

4. **Generic projections of LNE surfaces**

In this section we study LNE surface germs by establishing some properties related to their generic projections.

We begin by proving the invariance of multiplicities under generic projections, and showing a characterization of the \( \mathcal{P} \)-nodes of a LNE surface in terms of their local degrees. More precisely, we prove the following result:

**Lemma 4.1.** Let \( (X,0) \) be an LNE surface germ, let \( \ell : (X,0) \to (\mathbb{C}^2,0) \) be a generic projection, let \( \pi : X_\pi \to X \) be the minimal good resolution of \( (X,0) \) which factors through its Nash transform and the blowup of its maximal ideal, and let \( \ell : X_\pi \to X \) be a divisorial point of \( \Gamma_{\pi} \subset \text{NL}(X,0) \). Then:

(i) \( m_v = m_{\ell(v)} \);

(ii) \( v \) is a \( \mathcal{P} \)-node of \( (X,0) \) if and only if \( \deg v > 1 \).

**Proof.** We begin by proving (i). We use again the notations introduced in the proof of Lemma 3.1. Write \( \ell = \ell_D \) and set \( \pi_{v,D} = \pi_D \circ \alpha_{v,D} : X_{v,D} \to X \) and \( \sigma_{v,D} = \sigma_D \circ \alpha_{v,D} : Y_{v,D} \to \mathbb{C}^2 \). Let \( E_v \) be the irreducible component of \( (\pi_{v,D})^{-1}(0) \) corresponding to \( v \). Set \( w = \tilde{\ell}_D(v) \) and denote by \( E_w \) be the corresponding irreducible component of \( (\sigma_{v,D})^{-1}(0) \).

Take a curvette \( \gamma^* \) of \( E_w \) which does not intersect a component of the strict transform of the discriminant curve \( \Delta_D \) and let \( (\gamma,0) \subset (\mathbb{C}^2,0) \) be the irreducible curve germ defined by \( \gamma = \sigma_{v,D}(\gamma^*) \), so that we have \( m_w = \text{mult}(\gamma) \). Among the components of \( (\ell_D)^{-1}(\gamma) \) we can find an irreducible curve germ \( \hat{\gamma} \) on \( (X,0) \) whose strict transform by \( \pi_{v,D} \) is a curvette of \( E_{\gamma} \), so that we have \( m_{\gamma} = \text{mult}(\hat{\gamma}) \). We then have \( \text{mult}(\hat{\gamma}) = k \cdot \text{mult}(\gamma) \), where \( k \) is the degree of the covering \( \hat{\gamma} \to \gamma \) induced by \( \ell \).
We will argue by contradiction. Assume that \( \text{mult}(\tilde{\gamma}) \neq \text{mult}(\gamma) \), that is that \( k > 1 \). Our goal will be to construct two \( \delta_1 \) and \( \delta_2 \) inside \( \tilde{\gamma} \) whose inner and outer contact do not coincide; this will then imply that \( (X, 0) \) is not LNE, contradicting our hypothesis. In order to do so, we consider another generic projection \( \ell_D : (X, 0) \to (\mathbb{C}^2, 0) \), chosen to be generic with respect to the curve \( \tilde{\gamma} \) as well, and set \( \gamma' = \ell_D(\tilde{\gamma}) \). Then the cover \( \tilde{\gamma} \to \gamma' \) induced by \( \ell_D \) has degree 1, and thus \( \tilde{\gamma} \) and \( \gamma' \) have the same multiplicity since \( \text{mult}(\tilde{\gamma}) = \text{degree}(\ell_D|_{\tilde{\gamma}}) \cdot \text{mult}(\gamma') = \text{mult}(\gamma') \). Set \( w' = \ell_D(v) \). By Lemma 3.1, we have \( w' = \eta_D(w) \), and thus \( m_w = m_w \) and \( q_w = q_w \). Moreover, by definition of \( \eta_D \), the strict transform of \( \gamma' \) by \( \sigma_{v, D} \) intersects \( \left( \sigma_{v, D} \right)^{-1}(0) \) in a smooth point \( p \) of \( E_w' \).

Observe that, since the plane curve germ \( \gamma \) is the image through \( \sigma_{v, D} \) of a curvette of \( E_w \), it has no characteristic Puiseux exponent strictly greater than the inner rate \( q_w \) of \( w \). On the other hand, the strict transform of \( \gamma' \) by \( \sigma_{v, D} \) cannot be a curvette of \( E_w' \) since \( \text{mult}(\gamma') = \text{mult}(\tilde{\gamma}) = km_w > m_w \). Therefore, the minimal good embedded resolution of \( \gamma' \) is obtained by composing \( \sigma_{v, D} \) with a nontrivial sequence of point blowups, starting with the blowup of \( Y_{\sigma_{v, D}} \) at \( p \). Let \( E_{w'} \) be the last irreducible curve created by this sequence, so that the strict transform of \( \gamma' \) is a curvette of \( E_{w'} \). Then the inner rate \( q_{w'} \) of \( E_{w'} \), which is strictly greater than \( q_w = q_w \), is a characteristic Puiseux exponent of \( \gamma' \).

Let us choose an embedding \( (X, 0) \subset (\mathbb{C}^n, 0) \) and coordinates \( (x_1, \ldots, x_n) \) of \( \mathbb{C}^n \) such that \( \ell_D(x) = (x_1, x_2) \) and \( \gamma' \) is not tangent to the line \( x_1 = 0 \). Therefore, since \( q_{w'} \) is a characteristic Puiseux exponent of \( \gamma' \), we can find a pair of real arcs \( \delta_1 \) and \( \delta_2 \) among the components of the intersection \( \gamma' \cap \{ x_1 = t \} \in \mathbb{R} \) such that their contact \( q(\delta_1, \delta_2) \) is equal to \( q_{w'} \) (we refer to [NP14, §3] for details on this classical result about Puiseux expansions). Let \( \delta_1 \) and \( \delta_2 \) be two liftings of \( \delta_1 \) and \( \delta_2 \) via \( \ell' \). Since the projection \( \ell_D \) is generic with respect to \( \tilde{\gamma} \), it induces by [Tei82, pp. 352-354] a bilipschitz homeomorphism for the outer metric from \( \tilde{\gamma} \) onto \( \gamma' \), and therefore the outer contacts \( q_{\text{out}}(\delta_1, \delta_2) \) and \( q(\delta_1', \delta_2') \) coincide, so that in particular \( q_{\text{out}}(\delta_1, \delta_2) = q_{w'} \) is greater than \( q_w \).

We will now show that the inner contact \( q_{\text{inn}}(\delta_1, \delta_2) \) between \( \delta_1 \) and \( \delta_2 \) is at most \( q_w \), which will yield the contradiction we were after.

Observe that the inner contact \( q = q_{\text{inn}}(\delta_1, \delta_2) \) between \( \delta_1 \) and \( \delta_2 \), which is defined by \( d_{\text{inn}}(\delta_1(t), \delta_2(t)) = \Theta(t^q) \), can also be computed as \( d_{\text{inn}}(\delta_1(t), \delta_2(t)) = \Theta(t^q) \), where \( d_{\text{inn}}(\delta_1(t), \delta_2(t)) \) denotes the inner distance between \( \delta_1(t) \) and \( \delta_2(t) \) inside the Milnor fiber \( F_t = X \{ x_1 = t \} \), that is the distance measured by taking the infimum of the inner lengths of the paths joining \( \delta_1(t) \) to \( \delta_2(t) \) inside \( F_t \). This is a consequence of the fact that, by [BNP14] and in the language therein, the subset \( \pi(N(E_w)) \) of \( (X, 0) \) is a \( B(q_w) \)-piece fibered by the restriction of the generic linear form \( x_1 \) whenever \( q_w > 1 \), and it is a conical piece if \( q_v = 1 \).

In order to conclude, consider a small disc \( D \) contained in the divisor \( E_v \) and centered at the point \( \tilde{\gamma} \cap E_v \) and let \( N \cong D \times D' \) be a trivialization of the normal disc-bundle to \( E_v \) over \( D \) such that \( \tilde{\gamma}^* = \{ 0 \} \times D' \). The intersection \( F_t \cap \pi(N) \) consists of \( m_v \) disjoint discs each centered at one of the \( m_v \) distinct points of \( \tilde{\gamma} \cap F_t \). Since \( \delta_1(t) \) and \( \delta_2(t) \) are two of these points, then they are the centers of two of these discs, \( D_1 \) and \( D_2 \) respectively. Since these two discs have diameters \( \Theta(t^{q_k}) \), any path from \( \delta_1(t) \) to \( \delta_2(t) \) inside \( F_t \) will have intersections with \( D_1 \) and \( D_2 \) of length...
Let us show with an example that the hypothesis that deg(v) > 1, because the cover ℓ is ramified in a neighborhood of the polar curve. Assume that v is not a P-node and that deg(v) > 1. We use again the plane curve γ introduced in the proof of (i).

By the definition of deg(v), the curve ℓ⁻¹(γ) has k_v irreducible components whose strict transforms by π_v,Δ are curvettes of E_v, where k_v divides deg(v), and we have m_v = m_ℓ(v)deg(v)/k_v. Since m_v = m_ℓ(v) by (i), then deg(v) = k_v, so k_v > 1. Let 1 and 2 be two components of ℓ⁻¹(γ) whose strict transforms by π_v,Δ are curvettes of E_v, and let us consider two real arcs 1 ⊂ 1 and 2 ⊂ 2 such that ℓ_D(1) = ℓ_D(2).

By definition of q_v, we have q_∞(1,2) = q_v and then q_∞(1,2) = q_v. Since v is not a P-node, the lifted Gauss map λ (see [NPP19a, Definition 6.11]) is constant along E_v, and we then have 1(1) = 1(2), see page 19 in loc. cit. By [NPP19a, Lemma 9.1], this implies q_∞(1,2) < q_∞(1,2), therefore (X,0) is not LNE, contradicting the hypothesis.

Remark 4.2. Whenever v is not a P-node, then part (i) of the theorem is an immediate consequence of (ii). Indeed, consider the degree deg(v) cover ℓ_v: π_ℓ,v(üler(E_v)) → σ_ℓ,v(üler(E_ℓ(v))) of equation (4) (page 13), choose coordinates of C^2 so that ℓ = (z_1, z_2), with h = z_1 a generic linear form on (X,0). Then ℓ_v restricts to a degree deg(v) cover from the intersection F_v = oder(E_v) ∩ {h = t} to its image ℓ(F_v), implying that m_v = km_ℓ(v), where the integer k divides deg(v).

As a simple consequence of Lemma 4.1(ii) we deduce the following result.

Corollary 4.3. Let (X,0) be a LNE surface germ, let v be a divisorial point of NL(X,0), and assume that v is associated with an exceptional component E_v of genus g > 0 in some good resolution of (X,0). Then v is a P-node of (X,0).

Proof. Consider again the finite cover ℓ_v: π_ℓ,v(üler(E_v)) → σ_ℓ,v(üler(E_ℓ(v))) of equation (4) (page 13), and assume that v is not a P-node. Then ℓ_v is an homeomorphism by Lemma 4.1, and so is its restriction ℓ_v: E_v ∩ E_v → E_ℓ(v) ∩ E_ℓ(v). Observe that N(E_v) ∩ E_v (respectively N(E_ℓ(v)) ∩ E_ℓ(v)) is the complex curve E_v (resp. E_ℓ(v)) with a finite union of discs removed. Since E_ℓ(v) has genus zero, this implies that E_v also has genus zero.

Example 4.4. Let us show with an example that the hypothesis that π is minimal is required in Lemma 4.1. Let (X,0) be the standard singularity A_2, which is the hypersurface singularity in (C^3,0) defined by the equation x^2 + y^2 + z^3 = 0. A good resolution π: X_0 → X of (X,0) can be obtained by the method described in [Lau71, Chapter II]. It considers the generic projection ℓ = ℓ_D = (y, z): (X,0) → (C^2,0) and, given a suitable embedded resolution σ_Δ: Y_σ_Δ → C^2 of the associated discriminant curve Δ: y^2 + z^3 = 0, gives a simple algorithm to compute a resolution of (X,0) as a cover of Y. In this example, Δ is a cusp and the dual graph Γ_σ_Δ of its minimal embedded resolution σ_Δ is depicted on the left of Figure 1. Its vertices are labeled as w_0, w_1, and w_2, in their order of appearance as exceptional divisors of blowups in the resolution process, the negative number attached to each vertex denotes the self-intersection of the corresponding exceptional curve, while the positive numbers in parentheses denote the multiplicities and the arrow denotes the strict transform
of $\Delta$. In this case Laufer’s algorithm gives us the dual graph of a good resolution $\pi_\ell$ of $(X,0)$ such that $\ell \circ \pi_\ell$ factors through $\sigma_\Delta$, appearing as the second graph from the left in Figure 1. Again, all exceptional components are rational, each vertex is decorated by the self-intersection of the corresponding exceptional curve, the arrow denotes the strict transform of $\Delta$, and the vertices as labeled in a way that $\ell(v_0) = \ell(v'_0) = w_0$, $\ell(v_1) = w_1$, and $\ell(v_2) = w_2$. Observe that the vertex $v_1$ has multiplicity 2, but it is sent by $\ell$ to the vertex $w_1$, which has multiplicity 1. However, the rational curve $E_{v_1}$ associated with the vertex $v_1$ has self-intersection $-1$ and can thus be contracted. The resulting map $\pi: X_\pi \to X$, which no longer factors through $\sigma_\Delta$, is the minimal resolution of $(X,0)$ factoring through its Nash transform. Observe that its $P$-node $v_2$ can also be contracted, yielding the minimal good resolution of $(X,0)$, which in this case does not factor through the Nash transform of $(X,0)$.

![Diagram](image)

Figure 1. Dual resolution graphs for the plane curve $\Delta$ (left) and for the surface singularity $X = A_2$ (middle and right).

In the proof of Lemma 4.1, the minimality of $\pi$ is only required in order to apply Lemma 3.1. Therefore, this examples also shows how the commutativity of the diagram of Lemma 3.1 may fail to hold on a larger dual graph such as $\Gamma_\pi$.

We can now move our focus to the morphism $\tilde{\ell}$ induced by a generic projection $\ell: (X,0) \to (\mathbb{C}^2,0)$, and more precisely to its restriction to the dual graph $\Gamma_\pi$ of some good resolution of $(X,0)$. Recall that, given a graph $\Gamma$, we denote by $V(\Gamma)$ the set of its vertices. In general, even whenever $\pi$ factors through the Nash transform of $(X,0)$, it is not possible to find a suitable sequence of blowups $\sigma: Y \to \mathbb{C}^2$ above 0 such that $\tilde{\ell}$ induces a morphism of graphs $\tilde{\ell}|_{\Gamma_{\pi}}: \Gamma_\pi \to \Gamma_\sigma$, since in order to make the elements of $\tilde{\ell}(V(\Gamma_\pi))$ appear among the vertices of $\Gamma_\sigma$, one usually introduces too many additional vertices, so that the image $\tilde{\ell}(e)$ of some edge $e$ of $\Gamma_\sigma$ is not an edge of $\Gamma_\pi$, but only a string of several edges. Remarkably, thanks to Lemma 4.1.(ii), in the case of LNE surfaces we can control this phenomenon completely. Indeed, the following proposition explains that in this case we do get a morphism of graphs, provided that we restrict our attention to a subgraph of $\Gamma_\pi$ that does not contain a $P$-node of $(X,0)$ in its interior.

**Proposition 4.5.** Let $(X,0)$ be an LNE surface germ, let $\pi: X_\pi \to X$ be the minimal good resolution of $(X,0)$ that factors through its Nash blowup, let $\ell: (X,0) \to (\mathbb{C}^2,0)$ be a generic projection, and let $\tilde{\ell}: \NL(X,0) \to \NL(\mathbb{C}^2,0)$ be the map induced by $\ell$. Let $\Gamma_\pi$ be a subgraph of $\Gamma_\pi$ and assume that there exists a connected component $W$ of $\Gamma_\pi \setminus \{P \text{- nodes} \}$ such that $\Gamma_0$ is contained in the topological closure of $W$ in $\Gamma_\pi$. Let $\sigma_{\Gamma_0}, Y_{\sigma_{\Gamma_0}} \to \mathbb{C}^2$ be the minimal sequence of point blowups such that $\tilde{\ell}(V(\Gamma_0)) \subset V(\Gamma_{\sigma_{\Gamma_0}})$. Then the restriction $\tilde{\ell}|_{\Gamma_0}: \Gamma_0 \to \NL(\mathbb{C}^2,0)$ induces an isomorphism of
graphs between $\Gamma_0$ and a subgraph of $\Gamma_{\sigma_{\Gamma_0}}$. Moreover, this isomorphism respects the weights of all vertices of $\Gamma_0$ that are not $P$-nodes.

In particular, this means that not only vertices and edges of $\Gamma_0$ are sent respectively to vertices and edges of $\Gamma_{\sigma_{\Gamma_0}}$, but also that if $v$ is a vertex of $\Gamma_0$ which is not a $P$-node, then we have $g(E_v) = g(E_{\tilde{v}(v)}) = 0$ and $E_v^2 = E_{\tilde{v}(v)}^2$ (the two self-intersection being computed in $X_\pi$ and in $Y_{\sigma_{\Gamma_0}}$ respectively).

**Proof.** Let us assume for the time being that $\Gamma_0$ is connected. If $W$ contains at least one vertex of $\Gamma_0$, that is if $\Gamma_0$ has at least a vertex which is not a $P$-node of $(X,0)$, denote by $\Gamma'_0$ the maximal subgraph of $\Gamma_0$ contained in $W$ and set $V(\Gamma'_0) = \{v_1, \ldots, v_r\}$. Let $U$ be a tubular neighborhood of the curve $C = E_{v_1} \cup \ldots \cup E_{v_r}$ in $X_\pi$. Since the incidence matrix of $\Gamma'_0$ is negative definite, the analytic contraction $\eta: U \rightarrow (S,p)$ of the curve $C$ onto a point $p$ defines a normal surface singularity $(S,p)$. Observe that, since $\pi$ is the minimal resolution of $(X,0)$ which factors through its Nash transform, the only exceptional components of $\pi$ that could be contracted while retaining smoothness of the ambient surface are associated with $P$-nodes of $(X,0)$; since $\Gamma_0$ contains no $P$-node, this implies that $\eta: U \rightarrow (S,p)$ is the minimal good resolution of the surface germ $(S,p)$. If $W$ contains no vertex of $\Gamma_0$, then either $\Gamma_0$ consists of a single $P$-node, in which case the lemma is clearly true (we can thus disregard this case in the rest of the proof), or $\Gamma_0$ consists of two $P$-nodes $v$ and $v'$ and a single edge corresponding to an intersection point $p = E_v \cap E_{v'}$, in which case we set $S = U = (X_\pi,p)$ and $\eta = \text{id}_U$.

Let $\pi': X_{\pi'} \rightarrow X$ be the minimal resolution of $(X,0)$ that factors through its Nash transform and through $Y_{\Gamma_0}$. Then $\pi'$ factors through $\pi$ by minimality of the latter, so that we obtain a commutative diagram as follows:

$$
\begin{array}{ccc}
X_\pi' & \xrightarrow{\beta} & X_\pi \\
\downarrow{\ell} & & \downarrow{\pi} \\
Y_{\Gamma_0} & \xrightarrow{\sigma_{\Gamma_0}} & \mathbb{C}^2
\end{array}
$$

Set $\tilde{U} = \beta^{-1}(U)$ and $U' = \tilde{\ell}(\tilde{U})$. By minimality of $\sigma_{\Gamma_0}$, $\beta$ restricts to an isomorphism $\beta|_{\tilde{U}}: \tilde{U} \rightarrow U$. Write $\tilde{\ell}(W \cap \Gamma_0) \cap V(\Gamma_{\sigma_{\Gamma_0}}) = \{w_1, \ldots, w_s\}$, so that by construction $U'$ is a tubular neighborhood of $E_{w_1} \cup \ldots \cup E_{w_s}$ in $Y_{\Gamma_0}$. Similarly as above, the contraction of the union of curves $C' = E_{w_1} \cup \ldots \cup E_{w_s}$ in $U'$ defines a normal surface singularity $(S,p')$ and an analytic map $\eta': U' \rightarrow (S',p')$ which is a good resolution of $(S',p')$. Moreover, the restriction $\tilde{\ell}|_U$ induces a finite analytic map $\tilde{\ell}: (S,p) \rightarrow (S',p')$.

Since $W$ contains no $P$-node, then by Lemma 4.1.(ii) we have $\deg(w) = 1$ for every divisorial point $w$ of $W \cap \Gamma_0$. This implies that all fibers of $\tilde{\ell}$ have cardinality 1 and therefore $\tilde{\ell}$ is an isomorphism. It follows that $\tilde{\ell}^{-1} \circ \eta'$ is a good resolution of $(S,p)$, and therefore by minimality of the resolution $\eta$ the map $\tilde{\ell}^{-1} \circ \eta'$ factors through $\eta$ via an analytic map $\alpha: (U', C') \rightarrow (U, C)$, so that we obtain the following
commutative diagram:

\[
\begin{array}{ccc}
(U, C) & \xrightarrow{(\beta|_U)^{-1}} & (S, p) \\
\downarrow{\ell} & \alpha & \downarrow{\ell^{-1}} \\
(U', C') & \xrightarrow{\eta'} & (S', q)
\end{array}
\]

The morphism \(\alpha\) is therefore an analytic isomorphism with inverse \(\ell \circ (\beta|_U)^{-1}\), which completes the proof of the lemma for \(\Gamma_0\) connected. Whenever \(\Gamma_0\) is not connected, we can repeat the proof above for each one of its connected components, now that we know that \(\ell\) is injective on the closure of \(W\), and therefore on \(\Gamma_0\). \(\square\)

What might prevent Proposition 4.5 from holding globally on \(\Gamma\) is that, for example, there might exist an edge \(e\) of \(\Gamma\) such that \(\ell(e)\) contains in its interior one (and, for the sake of the example, exactly one) point of the form \(\ell(v)\) for some vertex \(v\) elsewhere in \(\Gamma\). However, whenever this happens it is always possible to refine the graph \(\Gamma\), performing a blowup of the double point of the exceptional divisor of \(X\) that corresponds to \(e\) and thus subdividing the edge \(e\) by adding a new vertex \(w\), and this vertex satisfies \(\ell(w) = \ell(v)\). Observe that, if \((X, 0)\) were arbitrary, this may still fail to give a morphism of graphs since the vertex associated with the blowup would not necessarily be sent to \(\ell(v)\) by \(\ell\). The fact that this does not occur in the case of LNE surfaces, and that therefore we can refine \(\Gamma\) to obtain a morphism of graphs, is the the content of the following corollary.

**Corollary 4.6.** Let \((X, 0)\) be an LNE surface germ, let \(\pi: X_\pi \to X\) be the minimal good resolution of \((X, 0)\) that factors through its Nash blowup, let \(\ell: (X, 0) \to (\mathbb{C}^2, 0)\) be a generic projection, let \(\tilde{\ell}: \mathrm{NL}(X, 0) \to \mathrm{NL}(\mathbb{C}^2, 0)\) be the map induced by \(\ell\), and let \(\sigma: Y_{\sigma} \to \mathbb{C}^2\) be the minimal sequence of blowups of \(\mathbb{C}^2\) above \(0\) such that \(\tilde{\ell}(V(\Gamma_\pi)) \subset \tilde{\ell}(V(\Gamma_\pi)).\) Then there exists a good resolution \(\pi': X_{\pi'} \to X\) of \((X, 0)\), obtained by composing \(\pi\) with a finite sequence of blowups of double points of the successive exceptional divisors, such that \(\tilde{\ell}\) induces a morphism of graphs \(\tilde{\ell}|_{\Gamma_{\pi'}}: \Gamma_{\pi'} \to \Gamma_{\sigma}\).

**Proof.** Let \(e\) be an edge of \(\Gamma\) and let \(\Gamma_0\) be the subgraph of \(\Gamma\) that consists of \(e\) and of the two vertices \(v\) and \(v'\) to which the latter is adjacent. Let \(\sigma_{\Gamma_0}\) be the minimal sequence of blowups above \(0\) such that \(\tilde{\ell}(v), \tilde{\ell}(v') \in \tilde{\ell}(V(\Gamma_{\Gamma_0})).\) By Proposition 4.5, \(\tilde{\ell}(e)\) is an edge of \(\Gamma_{\Gamma_0}\), and in particular \(\tilde{\ell}\) induces an isomorphism of smooth germs \(\alpha: (X_{\pi}, E_v \cap E_{v'}) \xrightarrow{\sim} (Y_{\sigma_{\Gamma_0}}, E_{\tilde{\ell}(v)} \cap E_{\tilde{\ell}(v')})).\) Now, \(\sigma_\ell\) factors through \(\sigma_{\Gamma_0}\) by minimality of the latter. In particular, a finite sequence of blowups above \(E_{\tilde{\ell}(v)} \cap E_{\tilde{\ell}(v')}\) occur in this factorization. By performing the same sequence of blowups on \((X_{\pi}, E_v \cap E_{v'})\) via the isomorphism \(\alpha\), we subdivide the edge \(e\) in a chain of edges that is sent isomorphically to a subgraph of \(\Gamma_{\sigma}\) via \(\tilde{\ell}\). Repeating this procedure for every edge \(e\) of \(\Gamma\), we obtain the resolution \(\pi'\) that we were after. \(\square\)

Observe that the resulting morphism of graphs \(\tilde{\ell}|_{\Gamma_{\pi'}}: \Gamma_{\pi'} \to \Gamma_{\sigma}\) is not surjective, as is clear from Example 4.4. This issue be discussed further in Section 7.
5. INNER RATES ON LNE SURFACE GERMS

In this section we move to the study of the inner rates on LNE surface germs and prove parts (iii) and (iv) of Theorem 1.1.

Recall that the inner rate $q_v$ of a divisorial point $E_v$ of $NL(X, 0)$ is defined as the inner contact $q_{inn}(\gamma_1, \gamma_2)$, where $\gamma_1, \gamma_2 \subset (X, 0)$ are two curve germs that pullback to two curvettes through distinct point of the divisor $E_v$ associated with $v$ via any good resolution $\pi: X_\pi \to X$ of $(X, 0)$ that makes the divisor $E_v$ appear. This definition only depends on the divisorial valuation $v$ (see [BdSFP19, Lemma 3.2]).

We begin by endowing the dual graph of a good resolution of $(X, 0)$ with a natural metric. Let $\pi: X_\pi \to X$ be a good resolution of $(X, 0)$ factoring through the blowup of its maximal ideal of $(X, 0)$, and denote by $|\Gamma_\pi|$ the topological space underlying the graph $\Gamma_\pi$. We endow $|\Gamma_\pi|$ with the metric defined by declaring that the length of an edge connecting two vertices $v$ and $w$ is equal to $1/\lcm(m_v, m_w)$, and denote by $d$ the associated distance function. Observe that, since the exceptional component of the blowup of an intersection point between the two components associated with $v$ and $w$ is $m_v + m_w$, and $1/\lcm(m_v, m_w) = 1/\lcm(m_v, m_v + m_w) + 1/\lcm(m_v + m_w, m_w)$, the metric on $|\Gamma_\pi|$ is compatible with subdividing the edges of the graph $\Gamma_\pi$ by blowing up $X_\pi$ at double points of $\pi^{-1}(0)$, and thus induces a metric on $NL(X, 0)$. The reader should be warned that this metric on $\Gamma_\pi$ is not the same as the one defined in [BdSFP19, §2.1], albeit it is strictly related to the latter and was already briefly used in Lemma 5.5 of loc. cit.

The following proposition is strictly stronger than part (iii) of Theorem 1.1, as it computes inner rates on the whole $NL(X, 0)$ rather than on a specific resolution graph.

**Proposition 5.1.** Let $(X, 0)$ be an LNE surface germ. Then, for every divisorial point $v$ of $NL(X, 0)$, the inner rate $q_v$ of $v$ equals $d(v, V_\gamma) + 1$, where $d(v, V_\gamma)$ denotes the distance of $v$ from the set $V_\gamma$ of all $\mathcal{L}$-nodes of $(X, 0)$.

**Proof.** Let $\pi: X_\pi \to X$ be the minimal good resolution of $(X, 0)$ which factors through its Nash transform and let $\Gamma_{\pi'}$ be a refinement of $\Gamma_\pi$ as in Corollary 4.6. We will begin by proving the wanted equality for divisorial points contained in $\Gamma_{\pi'}$. Denote by $w_0$ be the unique $\mathcal{L}$-node of $(\mathbb{C}^2, 0)$. For every divisorial point $w$ of $NL(\mathbb{C}^2, 0)$ the inner rate of $v$ is $d(w, w_0) + 1$ by [BdSFP19, Lemma 5.5] (or by a simple computation using Lemma 3.6 of loc. cit.). Since the inner rates on $(X, 0)$ and $(\mathbb{C}^2, 0)$ commute with the map $\ell$ (see [BdSFP19, Lemma 3.2]), we need to show that $d(v, V_\gamma) = d(\ell(v), w_0)$. We claim that, if $\gamma$ is an injective path in $\Gamma_{\pi'}$ connecting two divisorial points $v_1$ and $v_2$, then the length of $\gamma$ is greater or equal to the length of its image $\ell(\gamma)$ in $NL(\mathbb{C}^2, 0)$, with equality holding as long as $\ell$ maps $\gamma$ injectively onto its image. Indeed, any edge $e$ in $\gamma$ is sent via $\ell$ to an edge $\ell(e)$ of the dual graph of some sequence of blowups of $(\mathbb{C}^2, 0)$ thanks to Corollary 4.6. It then follows from Lemma 4.1.(i) that the edges $e$ and $\ell(e)$ have the same length, which implies our claim. In particular, since $V_\gamma = \ell^{-1}(w_0)$, we deduce that $d(v, V_\gamma) = d(\ell(v), w_0)$. To obtain the converse inequality it is sufficient to prove that there exists a path $\gamma$ from $v$ to an element of $V_\gamma$ where $\ell$ is injective. This follows from the fact that there exists such a path along which the inner rate function is strictly decreasing (and hence injective), which was proven in [BdSFP19, Proposition 3.9]. The fact that the equality holds on the whole of $NL(X, 0)$ is a consequence of [BdSFP19,
Lemma 5.5] (which is itself based on the same computations from Lemma 3.6 of loc. cit. that we have already used above. □

**Remark 5.2.** Proposition 5.1 shows that the inner rate function generalizes the function $s$ used by Spivakovsky in [Spi90, Definition 5.1] to study minimal and sandwiched surface singularities.

In order to prove part (iv) of Theorem 1.1, we need to rely on a deeper result, the so-called *Laplacian formula* for the inner rate function that we obtained in [BdSFP19] and that we will briefly recall now. In order to state this formula we will introduce two additional vectors indexed on the vertices of the dual graph $\Gamma_\pi$ of a good resolution $\pi : X_\pi \to (X, 0)$ of $(X, 0)$. Assume that $\pi$ factors through the blowup of the maximal ideal of $(X, 0)$ and let $L_\pi$ and $P_\pi$ be respectively the $\mathcal{L}$- and the $\mathcal{P}$-vector of $(X, 0)$ as before. For every vertex $v$ of $\Gamma_\pi$, set $k_v = \text{val}_{\Gamma_\pi}(v) + 2g(v) - 2$ and $a_v = m_v q_v$, and consider the vectors $K_\pi = (k_v)_{v \in V(\Gamma_\pi)}$ and $A_\pi = (a_v)_{v \in V(\Gamma_\pi)}$. Denote by $I_{\Gamma_\pi}$ the incidence matrix of the exceptional divisor of $\pi$. Then the following equality holds:

$$I_{\Gamma_\pi} \cdot A_\pi = K_\pi + L_\pi - P_\pi.$$  

(5)

This equality is an effective version (see [BdSFP19, Proposition 5.3]) of the main result of loc. cit.

**Proof of part (iv) of Theorem 1.1.** For every vertex $v$ of $\Gamma_\pi$, equation (5) yields

$$m_v q_v E_v^2 + \sum_{v' \in \Gamma_\pi} m_v q_v = \text{val}_{\Gamma_\pi}(v) + 2g(E_v) - 2 + l_v - p_v,$$

where the sum runs over the vertices $v'$ of $\Gamma_\pi$ adjacent to $v$. Then the equality we want follows from the fact that $E_v \cdot \sum_{v \in V(\Gamma_\pi)} E_v = E_v^2 + \text{val}_{\Gamma_\pi}(v)$, that $l_v = E_v \cdot Z_{\max}(X, 0)$ by definition of $l_v$, that $Z_{\max}(X, 0) = Z_{\min}$ by part (ii) of the theorem, and that $E_v \cdot Z_{\Gamma_\pi} = -E_v^2 + 2g(E_v) - 2$ by definition of $Z_{\Gamma_\pi}$.

6. End of the proof of Theorem 1.1

In this section we conclude the proof of Theorem 1.1, showing parts (v) and (vi), which means that we are interested in determining the $\mathcal{P}$-nodes of the LNE surface germ $(X, 0)$.

We begin with two definitions. Let $\pi$ denote a good resolution of $(X, 0)$ that factors through the blowup of its maximal ideal and through its Nash transform, let $v$ be a vertex of $\Gamma_\pi$, and let $e = [v, v']$ be an edge of $\Gamma_\pi$ adjacent to $v$. We say that $e$ is *incoming* at $v$ if we have $q_v > q_{v'}$. Following [Spi90, Definition 5.3], whenever $v$ has at least two incoming edges, we say that it is a *central node* of $\Gamma_\pi$.

Observe that the $\mathcal{L}$-nodes of $\Gamma_\pi$ have no incoming edges, and that the number of incoming edges at a vertex $v$ does not depend on the choice of a resolution such that $v$ is a vertex of the associated graph, since the inner rate increases along any new edge introduced by blowing up a smooth point. In the LNE case, we can prove the following more precise result, building on the local degree formula [BdSFP19, Lemma 4.18].

**Lemma 6.1.** Let $(X, 0)$ be a LNE surface germ with isolated singularities, let $\pi$ be a good resolution of $(X, 0)$ that factors through its Nash transform, and let $v$ be a vertex of $\Gamma_\pi$. Then the local degree $\deg(v)$ at $v$ equals $l_v$ if $v$ is an $\mathcal{L}$-node of $\Gamma_\pi$, or the number of incoming edges of $\Gamma_\pi$ at $v$ otherwise.
Proof. Denote by \( \ell : (X, 0) \to (\mathbb{C}^2, 0) \) a generic projection. Assume first that \( v \) is an \( \mathcal{L} \)-node. In this case, we can compute the degree directly via the definition and a generic projection \( h : (X, 0) \to (\mathbb{C}, 0) \) that factors through \( \ell \) (that is, there exists a projection \( \ell_h : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) such that \( h = \ell_h \circ \ell \)). More precisely, let \( \gamma \) be the curve \( h^{-1}(0) \cap X \). Since \( h \) factors through \( \ell \), we know that \( \ell(\gamma) \) is a line in \((\mathbb{C}^2, 0)\). Now, \( l_v \) corresponds to the number of distinct irreducible components of the strict transform of \( \gamma \) by \( \pi \) that intersect \( E_v \). Since each of those components is smooth, we conclude that \( \deg(v) = l_v \) by computing the degree via \( \gamma \) using the definition. Assume now that \( v \) is not an \( \mathcal{L} \)-node. By Corollary 4.6, there exist a refinement \( \Gamma_{\pi'} \) of \( \Gamma_{\pi} \) and a sequence of blowups \( \sigma : Y \to \mathbb{C}^2 \) such that \( \tilde{\ell} \) induces a morphism of graphs \( \Gamma_{\pi'} \to \Gamma_{\pi} \). In particular, since \( \tilde{\ell} \) respects inner rates (see again [BdSFP19, Lemma 3.2]), all edges of \( \Gamma_{\pi'} \) that are incoming at \( v \) are sent to the unique edge of \( \Gamma_{\pi} \) that is incoming at \( \tilde{\ell}(v) \) (its uniqueness can for example be seen as a consequence of Proposition 5.1). The proof then follows from applying the formula of [BdSFP19, Lemma 4.18], observing that the local degree \( \deg(e) \) along every edge adjacent to \( v \) equals 1 by Lemma 4.1.(ii) and that, even if further blowups are needed to pass from \( \pi' \) to a resolution adapted to \( \ell \), no new edge can be incoming at \( v \). \( \square \)

We can now complete the proof of our main theorem.

End of proof of Theorem 1.1. Let \((X, 0)\) be an LNE surface germ with an isolated singularity, let \( \pi : X_\pi \to X \) be the minimal good resolution of \((X, 0)\), let \( \pi' : X_{\pi'} \to X \) be the minimal one that factors through the Nash transform, and let \( v \) be a vertex of \( \Gamma_{\pi'} \). By combining the lemmas 4.1.(ii) and 6.1, we obtain the following:

\[ v \text{ is a } \mathcal{P} \text{-node of } \Gamma_{\pi'} \text{ if and only if either } l_v > 1 \text{ or } v \text{ is a central node of } \Gamma_{\pi'} \quad (*) \]

which establishes part (v) of the theorem. We claim that this also implies that the \( \mathcal{P} \)-nodes of \( \Gamma_{\pi'} \) are already on the graph \( \Gamma_{\pi} \subseteq \Gamma_{\pi'} \) (possibly in the interior of some edge). Indeed, if \( v \) is a vertex of \( \Gamma_{\pi'} \setminus \Gamma_{\pi} \), since \( \pi' \) is obtained from \( \pi \) by a sequence of point blowups, we deduce from Proposition 5.1 that there is only one incoming edge at \( v \) (observe that all \( \mathcal{L} \)-nodes of \((X, 0)\) are contained in \( \Gamma_{\pi} \) thanks to Proposition 2.2), so that the claim follows from (\( * \)). Therefore we obtain \( \pi' \) by successive blowups of double points on the exceptional divisor of \( \pi \). Now, let \( e \) be an edge of \( \Gamma_{\pi} \). If \( e = \{v, v'\} \) contains no \( \mathcal{P} \)-node, then it is also an edge of \( \Sigma_{\pi'} \), and by applying Proposition 4.5 to its closure we deduce that its image through the map induced by a generic projection \( \ell : (X, 0) \to (\mathbb{C}^2, 0) \) is an edge \( \tilde{\ell}(e) = [\tilde{\ell}(v), \tilde{\ell}(v')] \) of \( \Gamma_{\sigma} \). Therefore we have \( |q_{\tilde{\ell}(v)} - q_{\tilde{\ell}(v')}| = d(\tilde{\ell}(v), \tilde{\ell}(v')) \), as it can for example be seen by Proposition 5.1, and since the inner rate map commutes with \( \tilde{\ell} \) and \( d(v, v') = d(\tilde{\ell}(v), \tilde{\ell}(v')) \) by Lemma 4.1.(i), we deduce that if \( |q_v - q_{v'}| = d(v, v') \). This shows that if \( |q_v - q_{v'}| < d(v, v') \) then \( e \) must contain a \( \mathcal{P} \)-node. Conversely, if \( e \) contains a \( \mathcal{P} \)-node \( w \) then, as it can only contain one \( \mathcal{P} \)-node, it is folded in two by the projection \( \tilde{\ell} \). It follows that, with respect to the distance \( d \), the inner rate grows linearly with slope 1 from \( v \) to \( w \), and then decreases linearly with slope 1 from \( w \) to \( v' \), so that \( |q_v - q_{v'}| < d(v, v') \). We also deduce that \( d(v, v') = d(v, w) + d(w, v') = (q_w - q_v) + (q_w - q_{v'}) \), and therefore \( q_w = (d(v, v') + q_v + q_{v'})/2 \). This reasoning can be repeated after blowing up the double point of \( \pi^{-1}(0) \) corresponding to \( e \), and is therefore sufficient to establish part (vi) of Theorem 1.1 and thus conclude its proof. \( \square \)
7. Discriminant curves

We will now move our focus to the discriminant curve of a generic plane projection of a LNE surface germ. We will describe those curves completely, proving in Theorem 7.5 a more precise version of Theorem 1.2 from the introduction. In order to do so, we need to pursue in greater depth the study of the properties of the map \( \ell \) already undertaken in Section 4

Let \((X, 0)\) be an LNE normal surface singularity, let \(\pi : X_\pi \to X\) be the minimal good resolution which factors through its Nash transform, let \(\ell : (X, 0) \to (\mathbb{C}^2, 0)\) be a generic projection, let \(\tilde{\ell} : \text{NL}(X, 0) \to (\mathbb{C}^2, 0)\) be the induced morphism, and, as in Section 4, let \(\sigma_\ell : Y_{\sigma_\ell} \to \mathbb{C}^2\) be the minimal sequence of blowups above 0 such that \(\tilde{\ell}(V(\Gamma_\pi)) \subset V(\Gamma_{\sigma_\ell})\). We call \(\Delta\)-node of \(\Gamma_{\sigma_\ell}\) any vertex \(v\) which is the image by \(\tilde{\ell}\) of a \(\mathcal{P}\)-node of \(\Gamma_\pi\), and we call root vertex of \(\Gamma_{\sigma_\ell}\) the image by \(\tilde{\ell}\) of the \(\mathcal{L}\)-nodes of \(\Gamma_\pi\). Observe that the root vertex of \(\Gamma_{\sigma_\ell}\) is the divisorial point associated with the exceptional divisor of blowup of \(\mathbb{C}^2\) at 0, which can also be seen as the unique \(\mathcal{L}\)-node of \((\mathbb{C}^2, 0)\), and that a vertex \(v\) of \(\Gamma_{\sigma_\ell}\) is a \(\Delta\)-node if and only if \(E_v\) intersects the strict transform \(\Delta^*\) of the discriminant curve \(\Delta\) of \(\ell\).

The following proposition explains that for LNE surfaces the morphism \(\sigma_\ell\) coincides with the minimal good embedded resolution of the discriminant curve \(\Delta\).

**Proposition 7.1.** Let \((X, 0)\) be an LNE normal surface germ, let \(\ell : (X, 0) \to (\mathbb{C}^2, 0)\) be a generic projection of \((X, 0)\), and let \(\pi : X_\pi \to X\) be the minimal good resolution of \((X, 0)\) that factors through its Nash transform. Consider the three finite sequences of point blowups of \(\mathbb{C}^2\) above 0 defined as follows:

- \(\sigma_\Delta : Y_{\sigma_\Delta} \to \mathbb{C}^2\) is the minimal good embedded resolution of the discriminant curve \(\Delta\) associated with \(\ell\);
- \(\sigma_\Omega : Y_{\sigma_\Omega} \to \mathbb{C}^2\) is the minimal sequence which resolves the base points of the family of projected polar curves \(\{\ell(P_\Delta)\}_{P_\Delta \in \Omega}\);
- \(\sigma_\ell : Y_{\sigma_\ell} \to \mathbb{C}^2\) is the minimal sequence such that \(V(\Gamma_{\sigma_\ell})\) contains \(\tilde{\ell}(V(\Gamma_\pi))\).

Then \(\sigma_\Delta, \sigma_\Omega\) and \(\sigma_\ell\) coincide.

**Proof.** We begin by showing that \(\sigma_\ell\) and \(\sigma_\Omega\) coincide. First observe that \(\sigma_\ell\) factors through \(\sigma_\Omega\), since the latter is the minimal sequence of blowups of \(\mathbb{C}^2\) over 0 such that \(V(\Gamma_{\sigma_\Omega})\) contains all the \(\Delta\)-nodes. This is the case because the base points of the family of polar curves \(\{P_\Delta\}_{P_\Delta \in \Omega}\) are resolved by the Nash transform of \((X, 0)\), whose exceptional components correspond to the \(\mathcal{P}\)-nodes of \((X, 0)\), and the \(\Delta\)-nodes are by definition the image through \(\ell\) of the \(\mathcal{P}\)-nodes of \((X, 0)\).

Now, to show that \(\sigma_\ell\) equals \(\sigma_\Omega\) it is enough to show that if \(w\) is a vertex of \(\sigma_\ell\) which is not a \(\Delta\)-node (nor the root), then \(w\) can not be contracted. Now, we can pick a vertex \(v\) of \(\Gamma_\pi\) which is not a \(\mathcal{P}\)-node (nor a \(\mathcal{L}\)-node) and such that \(\tilde{\ell}(v) = w\). Let \(\Gamma_\ell\) be the star of \(v\) in \(\Gamma_\pi\), which by definition is the subgraph consisting of \(v\), all adjacent vertices and the edges connecting those to \(v\). By applying 4.5 we see that \(g(E_v) = g(E_w) = 0\) and \(E_v^2 = E_w^2\); note that the first intersection is computed in \(X_\pi\), the second one in \(Y_{\Gamma_\ell}\). Since \(\sigma_\ell\) factors through \(\sigma_\Gamma_\ell\), the self-intersection of \(E_w\) in \(Y_{\sigma_\ell}\) is smaller than or equal to \(E_w^2\). For \(E_w\) to be contractible in \(Y_{\sigma_\ell}\) we would therefore need to have \(E_v^2 = -1\). Since \(E_v^2\) is not contractible in \(X_\pi\) (because it is not a \(\mathcal{P}\)- nor a \(\mathcal{L}\)-node and \(\pi\) is minimal) and \(g(E_v) = 0\), this would imply that \(E_v\) has valence at least 3 in \(\Gamma_\pi\). However, this is not possible because Proposition 4.5
applied to $Γ_0$ would tell us that $w$ has to have valence at least 3 in $Γ_{σ_Ω}$, which can not occur for the last divisor appearing in a sequence of blowups of $ℂ^2$.

Let us now prove that $σ_Ω$ factors through $σ_Δ$, that is that it is a good embedded resolution of the curve $Δ$. Assume by contradiction that this is not the case, so that there exist a $Δ$-node $w$ and a component $Δ_0$ of $Δ$ whose strict transform by $σ_Ω$, while intersecting $E_w$ at a smooth point $p$ of an exceptional component $E_w$, is not a curvette of $E_w$. This implies that the multiplicity of $Δ_0$ is strictly greater than $m_w$. Let $Π_0$ be the component of the polar curve $Π$ of $ℓ$ such that $Δ_0 = ℓ(Π_0)$. Since $π$ is minimal, the strict transform of $Π_0$ by $π$ is a curvette on an exceptional component $E_v$ such that $ℓ(v) = w$, so that the multiplicity of $Π_0$ equals $m_v$. Since $ℓ(v) = w$, by Lemma 4.1.(ii) we have $m_v = m_w$, and therefore $\text{mult}(Δ_0) > \text{mult}(Π_0)$. However, the restriction $ℓ|_{Π_0} : Π_0 → Δ_0$ is a bilipschitz homeomorphism with respect to the outer metric by [Tei82, pp. 352-354], so that in particular we have $\text{mult}(Δ_0) = \text{mult}(Π_0)$, yielding a contradiction. This proves that $σ_Ω$ is a good embedded resolution of $Δ$.

To prove the proposition, it is now sufficient to show that $σ_Δ$ factors through $σ_Ω$, that is that it also resolves the base points of the family $\{ℓ(Π_Δ)\} _{Π∈Ω}$. Assume by contradiction that this is not the case, so that there exists a component $Δ_q$ of $Δ$ whose strict transform by $σ_Δ$ meets the exceptional divisor $σ_Δ^{-1}(0)$ at a (smooth) point $p$ which is a base point of the family $\{ℓ(Π_Δ)\} _{Π∈Ω}$. Let $w$ be the vertex of $Γ_σ$ such that $E_w$ is the irreducible component of $σ_Ω^{-1}(0)$ that contains $p$. The base point $p$ is resolved by a sequence of point blowups $δ$ which creates a bamboo (that is, a chain of two-valent vertices ending with a univalent vertex) $B$ living inside $Γ_σ\setminus Γ_σ$, stemming from the vertex $w$ and having the corresponding $Δ$-node $w'$ at its extremity. Since $σ_Ω$ is a resolution of $Δ$, we can perform the Hirzebruch–Jung resolution of $(X, 0)$ with the morphism $ℓ$ and the morphism $σ_Ω$. One takes the strict transform of $(X, 0)$ by the fiber product of $ℓ$ and $σ_Ω$, then normalize it. Since $ℓ$ is a cover branched over the discriminant curve $Δ$, we then get a normal surface $Z$ and a finite cover $ℓ' : Z → Y_σ_Ω$ which is ramified over the total transform $σ_Ω^{-1}(Δ)$ of the discriminant curve $Δ$. Resolving the singularities, we obtain a resolution $π' : X_π' → X$ of $(X, 0)$. Since $Γ_σ$ contains all $Δ$-nodes, then $Γ_π'$ contains all $P$-nodes of $(X, 0)$, and therefore $π'$ factors through $π$. Since by the previous part $σ_Ω$ factors through $σ_Δ$, the total transform $σ_Ω^{-1}(Δ)$ has normal crossings in $Y_σ_Ω$, each singularity of $Z$ is a quasi-ordinary singularity branched over a double point of $σ_Ω^{-1}(Δ)$. The resolution of each one of those quasi-ordinary singularities of $Z$ has as exceptional divisor a string of rational curves, and the strict transform of the branching locus consists of the union of two curvettes, one at each extremity of the string. This implies that the bamboo $B$ lifts by $ℓ$ to a bamboo $B'$ in the resolution graph $Γ_π'$ with a $P$-node at its extremity. This gives a $P$-node with a unique inward edge in $Γ_π'$, and therefore a unique inward edge in $Γ_π$, contradicting the statement (⋆) appearing on page 21. □

We denote by $V_N(Γ_π)$ the set of nodes of $Γ_π$, that is the subset of $V(Γ_π)$ consisting of the $P$-nodes, the $L$-nodes, and of all the vertices of valency at least three in $Γ_π$ (that is, those with at least three adjacent edges). Similarly, we call node of $Γ_σ$, a vertex which is either the root vertex, a $Δ$-node, or a vertex of valency three in $Γ_σ$, and we denote by $V_N(Γ_σ)$ the set of nodes of $Γ_σ$. The following proposition relates the nodes of $Γ_π$ to the ones of $Γ_σ$.
Proposition 7.2. Let $(X,0)$, $\pi_0$, $\ell$, and $\sigma_\ell$ be as above. Then we have:

(i) $\ell(V(\Gamma_\pi)) = V(\Gamma_{\sigma_\ell});$

(ii) $\ell(V_N(\Gamma_\pi)) = V_N(\Gamma_{\sigma_\ell}).$

Proof. Part (i) is easy, since $\ell(V(\Gamma_\pi)) \subset V(\Gamma_{\sigma_\ell})$ by definition of $\sigma_\ell$, while the other inclusion is a direct consequence of Proposition 4.5.

Let us now prove part (ii). By definition, a vertex $w$ of $\Gamma_{\sigma_\ell}$ is the root vertex (respectively a $\Delta$-node) of $\Gamma_{\sigma_\ell}$ if and only if $\ell^{-1}(w)$ contains a $\mathcal{L}$-node (respectively a $\mathcal{P}$-node) of $(X,0)$. To prove the proposition, it is then enough to prove the following claim: a given vertex $w$ of $\Gamma_{\sigma_\ell}$ that is not the root vertex nor a $\Delta$-node has valency at least three if and only if $\ell^{-1}(w)$ contains at least a vertex of valency at least three in $\Gamma_\pi$.

The “if” part of the claim can be easily obtained by taking a vertex $v$ in $\ell^{-1}(w)$ having valency at least three and applying Proposition 4.5 to the subgraph of $\Gamma_\pi$ consisting of $v$ and its adjacent vertices.

Let us prove the “only if” part of the claim. Assume that $v$ is a vertex of $\Gamma_\pi$ whose valency is at most 2 and such that $\ell(v) = w$. Since $w$ has valency at least three in $\Gamma_{\sigma_\ell}$, there are at least two outgoing (that is, not incoming in the terminology of Section 6) edges $e_w$ and $e'_w$ of $\Gamma_{\sigma_\ell}$ at $v$. Since $v$ is not a $\mathcal{L}$-node it must have one incoming edge, and therefore at most one of the two edges $e_w$ and $e'_w$ (say it is $e_w$) is contained in the image through $\ell$ of an edge $e_v$ adjacent to $v$. Therefore there must be another vertex $v'$ of $\Gamma_\pi$ such that $\ell(v') = w$, and an edge $e_{v'}$ adjacent to $v'$ such that $\ell(e_{v'})$ contains $e'_w$. Since if $v'$ is at least trivalent there is nothing to prove, with the goal of deriving a contradiction we can assume that both $v$ and $v'$ have valency two. It follows that any path $\tau$ connecting $v$ and $v'$ in $\Gamma_\pi$ must pass through at least one of the incoming edges at $v$ and $v'$, or otherwise it would have to become a loop in $\Gamma_{\sigma_\ell}$, which is a tree, so that in particular $\tau$ must contain a point of inner rate strictly smaller than the one of $v$ and $v'$. Now let $\gamma$ be a curve in $(\mathbb{C}^2,0)$ which is the image through $\sigma_\ell$ of a curvette of $E_w$, and let $\hat{\gamma}$ (respectively $\hat{\gamma}'$) be a component of $\ell^{-1}(\gamma)$ which is the image of a curvette of $E_w$ (respectively $E_{v'}$) through $\pi$. By [NPP19a, Proposition 15.3], we deduce that the inner contact $q_{\text{inn}}(\hat{\gamma},\hat{\gamma}')$ between $\hat{\gamma}$ and $\hat{\gamma}'$ is strictly smaller than $q_v$. However, by repeating the argument used in the proof of Lemma 4.1, we can see that $q_{\text{inn}}(\hat{\gamma},\hat{\gamma}') \geq q_v$, which contradicts the fact that $(X,0)$ is LNE. This concludes the proof of the claim, and therefore of the proposition. 

\[\square\]

Remark 7.3. Observe that if $w$ is the root vertex of $\Gamma_{\sigma_\ell}$ then $\ell^{-1}(w)$ is exactly the set of $\mathcal{L}$-nodes of $(X,0)$, so that $\bar{\ell^{-1}}(w) \subset V_N(\Gamma_\pi)$. However, if $w$ is a $\Delta$-node of $\Gamma_{\sigma_\ell}$, not all vertices in $\bar{\ell^{-1}}(w)$ need to be $\mathcal{P}$-nodes of $(X,0)$ (nor, more generally, nodes of $\Gamma_\pi$), as [NPP19b, Example 3.13] shows. If $w$ is a node of $\Gamma_{\sigma_\ell}$ which is not a $\Delta$-node and which has valency at least three $\Gamma_{\sigma_\ell}$, we do not know whether $\bar{\ell^{-1}}(w)$ may contain vertices having valency less than three in $\Gamma_\pi$.

Let $\Gamma$ be either of the two graphs $\Gamma_\pi$ or $\Gamma_{\sigma_\ell}$. We call principal part of $\Gamma$ the subgraph $\Gamma'$ of $\Gamma$ generated by the set $V_N(\Gamma)$ of nodes of $\Gamma$, that is the subgraph defined as the union of all injective paths connecting pairs of points of $V_N(\Gamma)$. The closure of each component of $\Gamma \setminus \Gamma'$ is a bamboo (that is, a chain of valency 2 vertices ending with a valency 1 vertex) stemming from a node of $\Gamma$. 

As showed by Example 4.4, the map \( \tilde{\ell} : \Gamma_\pi \to \Gamma_{\sigma_\ell} \) may fail to be surjective. However, as a consequence of Proposition 7.2.(ii), \( \tilde{\ell} \) restricts to a surjective map \( \tilde{\ell}|_{\Gamma'_\pi} : \Gamma_\pi \to \Gamma'_\sigma \) between the principal parts of \( \Gamma_\pi \) and \( \Gamma_\sigma \). Recall that the inner rates extend uniquely to a continuous map \( \Gamma_\pi \to \mathbb{R}_{\geq 1} \) (see [BdSFP19, Lemma 3.8]). We exploit this fact to define an equivalence relation on the topological space underlying \( \Gamma'_\sigma \).

**Proposition 7.4.** Let \( \sim \) be the equivalence relation on \( \Gamma'_\pi \) defined by declaring that two points \( v \) and \( v' \) are equivalent if the two following conditions hold:

(i) \( q_v = q_{v'} \);

(ii) there exists a path \( \tau \) in \( \Gamma'_\pi \) from \( v \) to \( v' \) such that the inner rate \( q_w \) of any point \( w \) in \( \tau \) is greater than or equal to \( q_v \).

Then the map \( \tilde{\ell}|_{\Gamma'_\pi} : \Gamma_\pi \to \Gamma'_\sigma \) identifies \( \Gamma'_\sigma \) with the quotient \( \Gamma'_\sigma / \sim \).

**Proof.** The fact that \( \sim \) is an equivalence relation follows immediately from the definition. We have to prove that \( v \sim v' \) if and only if \( \tilde{\ell}(v) = \tilde{\ell}(v') \). Assume that \( \tilde{\ell}(v) = \tilde{\ell}(v') \). Then \( q_v = q_{v'} \) since both inner rates are equal to \( q_{\tilde{\ell}(v)} \) by [BdSFP19, Lemma 3.2]. Assume then by contradiction that the condition (ii) is not satisfied, and let \( \gamma \) be a curve in \( (\mathbb{C}^2, 0) \) which is the image through \( \sigma_\ell \) of a curvette of \( E_{\tilde{\ell}(v)} \).

Arguing again in a similar way as in the proof of 7.2, let \( \tilde{\gamma} \) (respectively \( \tilde{\gamma}' \)) be a component of \( \ell^{-1}(\gamma) \) which is the image of a curvette of \( E_v \) (respectively \( E_{v'} \)) via a suitable resolution factoring through \( \pi \). Since \( \Gamma'_\pi \) is path connected but (ii) is not satisfied, we have \( q_{\text{ann}}(\tilde{\gamma}, \tilde{\gamma}') < q_v \) by [NPP19a, Proposition 15.3], but \( q_{\text{ann}}(\tilde{\gamma}, \tilde{\gamma}') \geq q_v \) by repeating the argument used in the proof of Lemma 4.1, contradicting the fact that \( (X, 0) \) is LNE.

To prove the converse implication, observe that if \( w \) and \( w' \) are two points of \( \Gamma'_{\sigma_\ell} \) then there exists a unique injective path \( \tau_{w,w'} \) between \( w \) and \( w' \) in \( \Gamma'_{\sigma_\ell} \), since the latter is a connected tree. Moreover, for each point \( w'' \) in the interior of \( \tau_{w,w'} \) we have \( q_{w''} < \max\{q_w, q_{w'}\} \) (for example, this can be derived from Proposition 5.1). Now assume that \( v \) and \( v' \) are two points of \( \Gamma'_\pi \) that satisfy the conditions (i) and (ii) and let \( \tau \) be any path in \( \Gamma'_\pi \) between \( v \) and \( v' \). By continuity of the projection, \( \tilde{\ell}(\tau) \) must contain \( \tau|_{\tilde{\ell}(v)}, \tilde{\ell}(v') \), and the latter has nonempty interior as soon as \( \tilde{\ell}(v) \neq \tilde{\ell}(v') \), therefore we deduce that when this is the case then \( \tau \) contains a point \( w \) mapping to the interior of \( \tau|_{\tilde{\ell}(v)}, \tilde{\ell}(v') \), so that \( q_w = q_{\tilde{\ell}(w)} < q_{\tilde{\ell}(v)} = q_v \). As this would contradict condition (ii), we must have \( \tilde{\ell}(v) = \tilde{\ell}(v') \). \( \square \)

We have now collected all the results we need to move to the study of the embedded topological type of the discriminant curve \((\Delta, 0) \subset (\mathbb{C}^2, 0)\). Fix once and for all a set of coordinates \((x_1, x_2)\) on \((\mathbb{C}^2, 0)\) such that \( x_1 = 0 \) is transverse to \( \Delta \). The topological type we are interested in is then completely determined by the characteristic exponents of the Newton–Puiseux expansion with respect to \( x_1 \) of each branch of \( \Delta \) and by the coincident exponents between each pair of branches, a data which is encoded by another combinatorial object, the so-called Eggers–Wall tree \( \Theta(\Delta) = \Theta_{x_1}(\Delta) \) of \( \Delta \). We refer the reader to [GBGPP19, §3] for a thorough introduction to this object, and in particular to Definition 3.8 and Remark 3.14 of loc. cit. for a formal definition starting from Newton–Puiseux expansion and an interesting historical remark. From our point of view, it is more convenient to describe the Eggers–Wall tree \( \Theta(C) \) of a plane curve germ \((C, 0) \subset (\mathbb{C}^2, 0)\) starting from the dual graph of a good embedded resolution of \( \Delta \) and from the invariants we
Then, it remains to show how to combine results we proved in Sections 4, 6, and 7 to the exceptional components intersecting the strict transform of $C$, and its vertices of valency at least three. The Eggers–Wall tree, denoted by $\text{Algorithm B}$.

Algorithm A. Denote by $\sigma_C : Y_{\sigma_C} \to C^2$ the minimal good embedded resolution of the curve $C$. The set of nodes $V_N(\Gamma_{\sigma_C})$ of the dual graph $\Gamma_{\sigma_C}$ of $\sigma_C$ is by definition the set consisting of its root, its $C$-nodes, which are the vertices corresponding to the exceptional components intersecting the strict transform of $C$, and its vertices of valency at least three. The Eggers–Wall tree $\Theta(C)$ is obtained from the set of nodes $V_N(\Gamma_{\sigma_C})$ of the tree $\Gamma_{\sigma_C}$, from its principal part $\Gamma'_{\sigma_C}$, and from the multiplicities and the inner rates of the vertices of $\Gamma'_{\sigma_C}$, as follows:

- From $\Gamma'_{\sigma_C}$, attach one extra edge to the root and one to each $C$-node $w$ for every branch of $C$ passing through $E_w$.
- Decorate each node $w \in V_N(\Gamma_{\sigma_C})$ (this includes vertices that have valency larger than three in $\Gamma_{\sigma_C}$ but less than three in $\Gamma'_{\sigma_C}$) with the rational number $e_C(w) = q_w$.
- If $e = [w, w']$ is an edge of $\Gamma'_{\sigma_C}$, decorate it with the integer $i(e) = \text{lcm}(m_w, m_{w'})$.
- If $e$ is one of the new edges of $\Theta(C)$ adjacent to a vertex $w$, decorate it with the integer $i(e) = m_w$.

The rational numbers $e_C(w)$ on the nodes $w$ on the path connecting the root to a $C$-node $w'$ are then precisely the characteristic exponents of any branch of $C$ passing through $E_w$, while the coincident exponent between two branches can be computed from the functions $e_C$ and $i_C$, as explained in [GBGPPP19, Theorem 3.25].

In order to describe the embedded topological type of the discriminant curve $\Delta$, it remains to show how to combine results we proved in Sections 4, 6, and 7 to determine the input of Algorithm A from the minimal good resolution of $(X, 0)$.

Algorithm B. Denote by $\pi_0 : X_{\pi_0} \to X$ the minimal good resolution of $(X, 0)$. Then:

- The multiplicities and the inner rates of the vertices of $\Gamma_{\pi_0}$ are uniquely determined by parts (ii) and (iii) of Theorem 1.1.
- The minimal resolution $\pi : X_{\pi} \to X$ of $(X, 0)$ factoring through its Nash transform, decorated with its multiplicities and inner rates, is obtained from $\pi_0$ applying the algorithm of part (vi) of Theorem 1.1. This also determines the set of nodes $V_N(\Gamma_{\pi})$ of $\Gamma_{\pi}$ and its principal part $\Gamma'_{\pi}$.
- Recall that we have $\sigma_\ell = \sigma_\Delta$ by Proposition 7.1. Therefore, combining Propositions 7.2 and 7.4 we obtain the principal part $\Gamma'_{\sigma_\Delta}$ of $\Gamma_{\sigma_\Delta}$ and the subset $V_N(\Gamma_{\sigma_C})$ consisting of the nodes of $\Gamma_{\sigma_\Delta}$.
- The multiplicities of the vertices of $\Gamma'_{\sigma_\Delta}$ are determined by the ones of the vertices of $\Gamma_{\pi}$ thanks to Lemma 4.1.(i).
- The inner rates of the vertices of $\Gamma'_{\sigma_\Delta}$ are determined by the ones of the vertices of $\Gamma_{\pi}$ because inner rates commute with $\ell$ thanks to [BdSFP19, Lemma 3.2].

We have proven the following result, which is a more precise version of Theorem 1.2 from the introduction.
Theorem 7.5. Let \((X, 0)\) be an LNE normal surface germ and let \(\ell: (X, 0) \to (\mathbb{C}^2, 0)\) be a generic projection. Then the embedded topology of the discriminant curve \(\Delta\) of \(\ell\) is completely determined by the topology of \((X, 0)\). More precisely, the Eggers–Wall tree of \(\Delta\) can obtained by applying Algorithm B followed by Algorithm A to the dual graph of the minimal resolution of \((X, 0)\).

8. Polar explorations on surface germs

In this section we prove the finiteness of realizable pairs of \(L\)- and \(P\)-vectors on a weighted graph \(\Gamma\). This suffices to establish Corollary 1.3, since the uniqueness in the LNE case is an immediate consequence of Theorem 1.1. We then provide a recipe to greatly bound the number of possibly realizable \(P\)-vectors based on the Laplacian formula for the inner rates called in Section 5.

The finiteness is established in the the following proposition, where we make use of a result of [CNPP06] to bound the number of realizable \(L\)-vectors, and of the classical Lê–Greuel–Teissier formula of [LT81] (see also [BdSFP19, Proposition 5.1] for a proof using the Laplacian formula) to relate \(L\)- and \(P\)-vectors to each other.

Proposition 8.1. Let \(\Gamma\) be a weighted graph. Then there exist finitely many pairs \((L, P)\) of vectors \(L = (l_i), P = (p_i) \in (\mathbb{Z}_{\geq 0})^{V(\Gamma)}\) such that there exists a normal surface singularity \((X, 0)\) and a good resolution \(\pi\) of \((X, 0)\) satisfying

\[(\Gamma, L, P) = (\Gamma_{\pi}, L_{\pi}, P_{\pi}).\]

Proof. Let \(E = \cup_{v \in V(\Gamma)} E_v\) be a configuration of curves dual to \(\Gamma\). Consider a divisor \(D_0 = \sum_{i=1}^n d_i E_i\) on \(\Gamma\) with \(d_i \in \mathbb{Z}\) such that \(D_0 \cdot E_v + \text{val}_v(v) + 2g_v \geq 0\) for all \(v \in V(\Gamma)\), where \(\text{val}_v(\Gamma)\) denotes the valency of \(v\) in \(\Gamma\). Consider the finite subset \(\mathcal{K}^+ = \{D \in \mathcal{E}^+; D \leq D_0\}\) of the Lipman cone \(\mathcal{E}^+\) of \(\Gamma\). Now, let \((X, 0)\) be any normal surface singularity whose link realizes \(\Gamma\). By [CNPP06, Theorem 4.1], there exists a ger of analytic function \(f: (X, 0) \to (\mathbb{C}, 0)\) such that \((f)_\Gamma = D_0\), which implies that \(D_0\) belongs to the subset \(\mathcal{A}_X\) of \(\mathcal{E}^+\) defined on page 6. We conclude that \(Z_{\text{max}}(X, 0) \leq D_0\), so that \(Z_{\text{max}}(X, 0) \in \mathcal{K}^+\). As \(L_\pi = (l_1, \ldots, l_n)\) is defined by the property that \(Z_{\text{max}}(X, 0) \cdot E_i + l_i = 0\) for every \(i = 1, \ldots, n\), this proves that there are only a finite number of possibilities for \(L_\pi\), since \(K^+\) is a finite set. Since the multiplicity \(m(X, 0)\) of the singularity \((X, 0)\) is equal to the sum of the products \(m_v E_v\) over the \(L\)-nodes of \((X, 0)\), we deduce an explicit bound for \(m(X, 0)\).

By the Lê–Greuel–Teissier formula [BdSFP19, Proposition 5.1], the multiplicity \(m_{\Pi}(X, 0)\) of the polar curve \(\Pi\) of a generic projection \(\ell: (X, 0) \to (\mathbb{C}^2, 0)\), is equal to \(m(X, 0) - \chi(F_i)\), where \(F_i\) denotes the Milnor–Lê fiber of a generic linear form on \((X, 0)\). Observe that by additivity of the Euler characteristic, and recalling the notation \(\mathcal{N}(E_v)\) of page 13, we can compute \(\chi(F_i)\) as

\[\chi(F_i) = \sum_{v \in V(\Gamma)} m_v \chi(\mathcal{N}(E_v) \cap E_v) = \sum_{v \in V(\Gamma)} m_v (g(E_v) - \text{val}_v(v) + 2 - l_v).\]

This shows that \(\chi(F_i)\) only depends on the weighted graph \(\Gamma\) and on the \(L\)-vector \(L_i\), and so that \(m_{\Pi}(X, 0)\) is itself bounded by an interger we can compute. Since \(m_{\Pi}(X, 0)\) is equal to the sum of the products \(m_v p_v\) over the \(P\)-nodes of \((X, 0)\), the value \(p_v\) for any vertex \(v\) of \(\Gamma\) is bounded as well. This implies that only finitely many vectors \(P\) are realizable, concluding the proof of the proposition. \(\square\)
While Proposition 8.1 provides a finite list of possible realizable pairs of $L$- and $P$-vectors, the list outputted by following its proof could still be fairly long. In order to get an effective bound on the number of realizable $L$-vectors, it is useful in the course of the proof of Proposition 8.1 to take the vector $D$ to be $\sum_{v \in V(\Gamma)} [d_v] E_v$, where the $d_v$ are determined by $-I_\gamma \cdot (d_v)_{v \in V(\Gamma)} = (\text{val}_s(v) + 2g_v)_{v \in V(\Gamma)}$, as this is the smallest vector which allows us to apply Caubel–Némethi–Popescu-Pampu’s result. On the other hand, the bound on generic polar curves, being solely based on the polar multiplicities given by the Lê–Grenel–Teissier formula, gives no information on the position of polar curves relative to the hyperplane sections.

We now discuss restrictions on these relative positions, thus providing a sharper bound to the number of realizable pair of vectors and a better understanding on the polar geometry of $\Gamma$. For this, we can shift our focus to the following situation: an $L$-vector $L$ is fixed, and we give geometric conditions on a $P$-vector $P$ such that the pair $(L, P)$ may be realizable. Assume that $(X, 0)$ is a normal surface germ realizing $\Gamma$ and $L$. The Laplacian formula, recalled in equation (5) at page 20, yields

$$A_\pi = I_{\Gamma_\pi}^{-1} \cdot (K_\pi + L - P_\pi).$$

Observe that $K_\pi + L$ and $P_\pi$ are known, while $A_\pi$ and $P_\pi$ are not, but either of them is determined by the other one thanks to the formula above. What we are going to prove is that there is only a very limited number of possible values of $A_\pi$. Since the vector $P_\pi$ has positive coordinates, this implies that $-I_{\Gamma_\pi}^{-1} \cdot P_\pi$ belongs to the Lipman cone $E^+$ of $\Gamma$, and so $A_\pi$ belongs to the translate $I_{\Gamma_\pi}^{-1} \cdot (K_\pi + L) + E^+$ of $E^+$. Moreover, if a vertex $v_0$ of $\Gamma_\pi$ is an $L$-node of $(X, 0)$, that is if $t_{v_0} \neq 0$, we know that its inner rate $q_{v_0}$ must be equal to 1. This implies that $A_\pi$ belongs to the intersection of the hyperplanes of $\mathbb{Z}^V(\Gamma_\pi)$ of $v_0$-th coordinate equal to $m_{v_0}$, for every $L$-node $v_0$ of $(X, 0)$, has inner rate $q_v$ equal to 1. This intersection is finite, and in fact rather small, since $D > 0$ for every nonzero element $D$ of the Lipman cone $E^+$. The construction is illustrated in Figure 2 below.

Additional restrictions may be derived from the topological properties of the germ $(X, 0)$. In particular, let $\pi: X_\pi \to X$ be a good resolution of $(X, 0)$, let $\ell: (X, 0) \to (\mathbb{C}^2, 0)$ be a generic projection, let $\sigma_\ell: Y_\ell \to \mathbb{C}^2$ be a sequence of blowups of $\mathbb{C}^2$ above 0 such that $V(\Gamma_{\sigma_\ell})$ contains $\ell(V(\Gamma_\pi))$, and let $\pi_\ell: X_{\pi_\ell} \to X$ be a good resolution of $(X, 0)$ such that $\pi_\ell \circ \ell$ factors through $\sigma_\ell$. Let $v$ be a vertex of $\Gamma_{\pi_\ell}$, and let $v_1, \ldots, v_r$ be the vertices of $\Gamma_{\pi_\ell}$ that are adjacent to $v$ and contained in $\Gamma_\pi$. Let $\Gamma_v$ be the subgraph of $\Gamma_{\pi_\ell}$ defined as the closure in $\Gamma_{\pi_\ell}$ of the connected component of $\Gamma_{\pi_\ell} \setminus \ell^{-1}(\ell((v_1, \ldots, v_r)))$ containing $v$, and consider the subgraph of $\Gamma_{\pi_\ell}$ defined as $T_v = \ell(\Gamma_v)$. Set

$$\mathcal{N}(\Gamma_v) = \bigcup_{w \in V(\Gamma_v)} N(E_w) \setminus \bigcup_{w' \in V(\Gamma_{\pi_\ell}) \setminus V(\Gamma_v)} N(E_{w'})$$

and

$$\mathcal{N}(T_v) = \bigcup_{w \in V(T_v)} N(E_w) \setminus \bigcup_{w' \in V(\Gamma_{\pi_\ell}) \setminus V(T_v)} N(E_{w'}).$$

Adjusting the fiber bundles $N(E_w)$ if necessary, by restricting $\ell$ to $\pi_\ell(\mathcal{N}(\Gamma_v))$ we obtain a degree $\deg(v)$ cover $\ell|_{\pi_\ell(\mathcal{N}(\Gamma_v))}: \pi_\ell(\mathcal{N}(\Gamma_v)) \to \pi_\ell(\mathcal{N}(T_v))$. Observe that $\ell|_{\pi_\ell(\mathcal{N}(\Gamma_v))}$ is an extension of the cover $\ell_v$ introduced in equation (4) (page 13) to define the local degree $\deg(v)$ of $v$. Now, pick a generic linear form $h: (X, 0) \to \mathbb{C}$.
Figure 2. Observe that, since \( Z_{\min} > 0 \), then the Lipman cone \( E^+ \) (in blue), and thus \( I^{-1}_{\pi} (K_\pi + L) + E^+ \) (in red), contain no horizontal line. Only six values of \( A_\pi \) are possible in this example.

(C, 0) on \( (X, 0) \), denote by \( \hat{F}_v \) the intersection of \( \pi_{\ell}(N(\Gamma_v)) \) with the Milnor fiber \( X \cap \{ h = t \} \) of \( x \), and set \( \hat{F}'_v = \ell_v(\hat{F}_v) \). Restricting again \( \ell \), we get a \( \deg(v) \)-cover \( \ell|_{\hat{F}_v} : \hat{F}_v \to \hat{F}'_v \). The Hurwitz formula applied to this cover yields the following equality:

\[
\chi(\hat{F}_v) + m_v p_v = \deg(v) \chi(\hat{F}'_v).
\]

(6)

Let us illustrate how this can be used with the help of an example.

**Example 8.2.** We will discuss in detail Example 3 from the paper [MM20], showing that we can determine its \( P \)-vector completely. Consider the hypersurface \( (X, 0) \) in \( (\mathbb{C}^3, 0) \) defined by the equation \( z^2 = (y + x^3)(y + x^2)(x^3 + y^3) \). The dual graph of the minimal resolution of \( (X, 0) \) which factors through the blowup of the maximal ideal is given in Figure 3. The arrow represents the strict transform of a generic linear form, the negative numbers are the self intersections \( E_\pi^2 \) in \( X \) of the irreducible components \( E_v \) of \( \pi^{-1}(0) \), and all curves \( E_v \) are rational. The multiplicities \( m_v \), which are computed from this data using the equalities (1) (page 6), are within parentheses in the figure.

Figure 3. The graph \( \Gamma_\pi \), decorated with the self-intersections and multiplicities.
Applying \[\text{BdSFP19, Proposition 3.9}\] to the vertex \(v\), therefore the corresponding to \(p\) obtained in \(\text{equation(5)}\) (page 20) to try to eliminate some of these possibilities for \(P\) and to compute the inner rates. Writing the formula for every vertex \(v\) in \(\{v_1, v_2, v_3, v_4, w_3, w_4\}\), for which we know \(p_v\), and using the fact that \(q_v = 1\), we obtain the corresponding inner rates \(q\) and the inner rate \(q_v\), which are as follows: \(q_v = 2\), \(q_v = q_w = \frac{5}{2}\), \(q_v = q_w = \frac{13}{5}\), and \(q_v = \frac{34}{13}\). Therefore \(q_v = 9\) and \(q_v = 5\), and \(q_v = 1\). We know from the Hurwitz arguments above that \(p_v = 1\) or 2. Therefore, the only possibility is \(q_v = \frac{5}{2}\) and \(p_v = 1\).

The Laplacian formula for \(v_5\) gives \(-15q_v + 13q_v + 2q_v = -p_v\). Then \(2q_v + p_v = 11\) with \(q_v = \frac{1}{2}\), \(q_v = \frac{5}{3}\), and \(p_v = 1\). The unique possibility is \(q_v = 3\) and \(p_v = 1\) or 2. Therefore, the only possibility is \(q_v = \frac{5}{2}\) and \(p_v = 1\).

The Laplacian formula for \(v_7\) gives \(q_v + p_v = 4q_v - 3q_v = 4\), with \(q_v \geq 0\). \(q_v > 3\), and \(p_v = 1\). The unique possibility is \(q_v = 4\) and \(p_v < 4\).

Among the four possibilities for \((p_v, q_v, p_v, p_v)\) identified above, the unique possibility is then \((1, 0, 0, 1)\), so \(p_v = 1\). This is indeed compatible with the Laplacian formula for \(v_8\).
We have obtained a unique possibility for $P$ and the inner rates, as shown in Figure 4. Observe that since there are no edges joining two vertices with nonzero $p_v$, then the strict transform $\Pi^*$ of the polar curve $\Pi$ by $\pi$ meets $E$ at smooth points of the exceptional divisor $\pi^{-1}(0)$. Moreover, since each $p_v$ equals either zero or one, then $\pi$ is a good resolution of $\Pi$, that is $\pi^{-1}(0)$ is a simple normal crossing divisor. The arrows in Figure 4 represent the strict transform of a generic polar curve.

![Figure 4](image_url)

**Figure 4.** The graph $\Gamma_{\pi}$, decorated with the inner rates of its vertices and arrows corresponding to the components of a generic polar curve.

### References

[Art66] Michael Artin. On isolated rational singularities of surfaces. *Amer. J. Math.*, 88:129–136, 1966.

[BdSFP19] André Belotto da Silva, Lorenzo Fantini, and Anne Pichon. Inner geometry of complex surfaces: a valuative approach. *arXiv preprint arXiv:1905.01677*, 2019.

[BM18] Lev Birbrair and Rodrigo Mendes. Arc criterion of normal embedding. In *Singularities and foliations. geometry, topology and applications*, volume 222 of *Springer Proc. Math. Stat.*, pages 549–553. Springer, Cham, 2018.

[BMNB18] Lev Birbrair, Rodrigo Mendes, and Juan J Nuño-Ballesteros. Metrically un-knotted corank 1 singularities of surfaces in $\mathbb{R}^4$. *J. Geom. Anal.*, 28(4):3708–3717, 2018.

[BNP14] Lev Birbrair, Walter D. Neumann, and Anne Pichon. The thick-thin decomposition and the bilipschitz classification of normal surface singularities. *Acta Math.*, 212(2):199–256, 2014.

[Bon03] Romain Bondil. Discriminant of a generic projection of a minimal normal surface singularity. *C. R. Math. Acad. Sci. Paris*, 337(3):195–200, 2003.

[Bon16] Romain Bondil. Fine polar invariants of minimal singularities of surfaces. *J. Singul.*, 14:91–112, 2016.

[CNPP06] Clément Caubel, András Némethi, and Patrick Popescu-Pampu. Milnor open books and Milnor fillable contact 3-manifolds. *Topology*, 45(3):673–689, 2006.

[Fan18] Lorenzo Fantini. Normalized Berkovich spaces and surface singularities. *Trans. Amer. Math. Soc.*, 370(11):7815–7859, 2018.

[Fer03] Alexandre Fernandes. Topological equivalence of complex curves and bi-Lipschitz homeomorphisms. *Michigan Math. J.*, 51(3):593–606, 2003.

[FS19a] Alexandre Fernandes and J Edson Sampaio. Tangent cones of Lipschitz normally embedded sets are Lipschitz normally embedded. Appendix by Anne Pichon and Walter D. Neumann. *Int. Math. Res. Not. IMRN*, (15):4880–4897, 2019.

[FS19b] Alexandre Fernandes and J. Edson Sampaio. Tangent cones of Lipschitz normally embedded sets are Lipschitz normally embedded. Appendix by Anne Pichon and Walter D. Neumann. *Int. Math. Res. Not. IMRN*, (15):4880–4897, 2019.

[GBGPPP19] Evelia R. García Barroso, Pedro D. González Pérez, and Patrick Popescu-Pampu. The valuative tree is the projective limit of eggers-wall trees. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 113(4):4051–4105, Mar 2019.
32 ANDRÉ BELOTTO DA SILVA, LORENZO FANTINI, AND ANNE PICHON

[Gra62] Hans Grauert. Über Modifikationen und exceptionelle analytische Mengen. *Math. Ann.*, 146:331–368, 1962.

[GSLJ97] Gérard Gonzalez-Sprinberg and Monique Lejeune-Jalabert. Families of smooth curves on surface singularities and wedges. *Ann. Polon. Math.*, 67(2):179–190, 1997.

[Kol85] János Kollár. Toward moduli of singular varieties. *Compositio Math.*, 56(3):369–398, 1985.

[KPR18] Dmitry Kerner, Helge Møller Pedersen, and Maria A S Ruas. Lipschitz normal embeddings in the space of matrices. *Math. Z.*, 290(1-2):485–507, 2018.

[Lê97] Dung Tráng Lê. Geometry of complex surface singularities. In *Singularities—Sapporo 1998*, volume 29 of *Adv. Stud. Pure Math.*, pages 163–180. Kinokuniya, Tokyo, 2000.

[Lau71] Henry B. Laufer. *Normal two-dimensional singularities*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971. Annals of Mathematics Studies, No. 71.

[Lau72] Henry B. Laufer. On rational singularities. *Amer. J. Math.*, 94:597–608, 1972.

[LT81] Dung Tráng Lê and Bernard Teissier. Variétés polaires locales et classes de Chern des variétés singulières. *Ann. of Math. (2)*, 114(3):457–491, 1981.

[MM20] Hélène Maugendre and Françoise Michel. On the growth behaviour of Hironaka quotients. *J. Singul.*, 20:31–53, 2020.

[MP18] Filip Misev and Anne Pichon. Lipschitz normal embedding among superisolated singularities. *arXiv preprint arXiv:1810.10179*, 2018. To appear in *Int. Math. Res. Not. IMRN*.

[Mum61] David Mumford. The topology of normal singularities of an algebraic surface and a criterion for simplicity. *Inst. Hautes Études Sci. Publ. Math.*, (9):5–22, 1961.

[Neu81] Walter D. Neumann. A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves. *Trans. Amer. Math. Soc.*, 268(2):299–344, 1981.

[NP07] Walter D. Neumann and Anne Pichon. Complex analytic realization of links. In *Intelligence of low dimensional topology 2006*, volume 40 of *Ser. Knots Everything*, pages 231–238. World Sci. Publ., Hackensack, NJ, 2007.

[NP14] Walter D. Neumann and Anne Pichon. Lipschitz geometry of complex curves. *J. Singul.*, 10:225–234, 2014.

[NPP19a] Walter D. Neumann, Helge Møller Pedersen, and Anne Pichon. A characterization of lipschitz normally embedded surface singularities. *Journal of the London Mathematical Society*, 2019.

[NPP19b] Walter D. Neumann, Helge Møller Pedersen, and Anne Pichon. Minimal singularities are lipschitz normally embedded. *Journal of the London Mathematical Society*, 2019.

[PP02] Patrick Popescu-Pampu. On the invariance of the semigroup of a quasi-ordinary surface singularity. *C. R. Math. Acad. Sci. Paris*, 334(12):1001–1106, 2002.

[PP04] Patrick Popescu-Pampu. On the analytical invariance of the semigroups of a quasi-ordinary hypersurface singularity. *Duke Math. J.*, 124(1):67–104, 2004.

[PT69] Frédéric Pham and Bernard Teissier. Fractions lipschitziennes d’une algèbre analytique complexe et saturation de Zariski. *Prépublications Ecole Polytechnique*, No. M17.0669, 1969.

[Spi90] Mark Spivakovsky. Sandwiched singularities and desingularization of surfaces by normalized Nash transformations. *Ann. of Math. (2)*, 131(3):411–491, 1990.

[Tei80] Bernard Teissier. Résolution simultanée, ii. *Séminaire sur les Singularités des Surfaces Lecture Notes in Mathematics*, 777, 1980.

[Tei82] Bernard Teissier. Variétés polaires. II. *Multiplicités polaires, sections planes, et conditions de Whitney*. In *Algebraic geometry (La Rábida, 1981)*, volume 961 of *Lecture Notes in Math.*, pages 314–491. Springer, Berlin, 1982.

[Zar39] Oscar Zariski. The reduction of the singularities of an algebraic surface. *Ann. of Math. (2)*, 40:639–689, 1939.
Aix-Marseille Université, CNRS, Centrale Marseille, I2M, Marseille, France
E-mail address: andre-ricardo.belotto-da-silva@univ-amu.fr
URL: https://andrebelotto.com

Goethe-Universität Frankfurt, Institut für Mathematik, Frankfurt am Main, Germany
E-mail address: fantini@math.uni-frankfurt.de
URL: https://lorenzofantini.eu/

Aix-Marseille Université, CNRS, Centrale Marseille, I2M, Marseille, France
E-mail address: anne.pichon@univ-amu.fr
URL: http://liml.univ-mrs.fr/~pichon/