AN EXISTENCE THEOREM, WITH ENERGY BOUNDS, OF FLOER'S PERTURBED CAUCHY-RIEMANN EQUATION WITH JUMPING DISCONTINUITY

YONG-GEUN OH

Abstract. This is a sequel to the paper [Oh5]. The main purpose of the present paper is to give the proof of an existence theorem with energy bounds of certain pseudo-holomorphic sections of the deformed mapping cylinder that is needed for the proof of nondegeneracy of the homological invariant pseudo-norm which the author has constructed on general symplectic manifolds [Oh4,5]. The existence theorem is also the crux of the author’s recent proof of an optimal energy-capacity inequality given in [Oh5]. In this paper, we give a more general existence result than needed in that we study Floer’s perturbed Cauchy-Riemann equations with discontinuous Hamiltonian perturbation terms and prove an existence theorem of certain piecewise smooth finite energy solutions of the equation. The proof relies on a careful study of the product structure on the Floer homology and a singular degeneration (“adiabatic degeneration”) of Floer’s perturbed Cauchy-Riemann equation. In the course of the proof, we also derive certain general energy identity of pseudo-holomorphic sections of the Hamiltonian fibration.

Contents

§1. Introduction and the main results
§2. Perturbed Cauchy-Riemann equations re-visited
§3. Energy identity on the deformed mapping cylinder
§4. Pants product and the Hamiltonian fibration
§5. Construction of the $W^{1,2}$-sections I: analysis of the thick part
§6. Construction of the $W^{1,2}$-sections II: analysis of the thin part
§7. The case with $H_3 = H_1 \# H_2$

Key words and phrases. Hamiltonian diffeomorphisms, perturbed Cauchy-Riemann, equation, $W^{1,2}$-section, minimal area metric, Hamiltonian fibrations, pseudo-holomorphic sections, Floer homology, pants product, spectral invariants.

1Partially supported by the NSF Grant # DMS-9971446 & DMS-0203593 and by # DMS-9729992 in the Institute for Advanced Study, Vilas Associate Award in the University of Wisconsin and by a grant of the Korean Young Scientist Prize

Typeset by $\LaTeX$
§1. Introduction

In [Oh3,4], on general (non-exact) compact symplectic manifolds, the present author defined spectral invariants

$$
\rho : \widehat{Ham}(M, \omega) \times QH^*(M) \setminus \{0\} \to \mathbb{R} \quad (1.1)
$$

by constructing a function

$$
\rho_a = \rho(\cdot; a) : C^0_{\mathfrak{m}}(S^1 \times M) \to \mathbb{R}
$$

for each \(a \in QH^*(M) \setminus \{0\}\) such that

$$
\rho(H; a) \in \text{Spec}(H)
$$

and

$$
\rho(H; a) = \rho(F; a)
$$

if \([H] = [F]\), when \(H, F\) are \(C^2\)-functions. The map \((1.1)\) is then defined by

$$
\rho(h; a) := \rho(H; a)
$$

for \(h \in \widehat{Ham}(M, \omega)\) and \([H] = h\). We also prove that \(\rho_a\) is \(C^0\)-continuous and the invariant \((1.1)\) satisfies certain axioms which we refer to [Oh4].

Among these invariants, the invariant \(\rho(h; 1)\) or \(\rho(H; 1)\) is of particular interest. Using them, we [Oh5] constructed an invariant pseudo-norm by considering the sum

$$
\tilde{\gamma}(h) = \rho(h; 1) + \rho(h^{-1}; 1) \quad (1.2)
$$

and taking the infimum

$$
\gamma(\phi) = \inf_{\pi(h) = \phi} \tilde{\gamma}(h). \quad (1.3)
$$

The inequality

$$
\rho(h; 1) \leq E^-(h) := \inf_{[K] = h} E^-(K)
$$

$$
\rho(h^{-1}; 1) \leq E^+(h) := \inf_{[K] = h} E^+(K) \quad (1.4)
$$

is also proved in [Oh5] in general. Here we denote

$$
E^+(K) = \int_0^1 \max K_t \, dt, \quad E^-(K) = \int_0^1 -\min K_t \, dt.
$$

In particular we have

$$
\rho(H; 1) \leq E^-(h), \quad \rho(\tilde{F}; 1) \leq E^+(f) \quad (1.5)
$$

for any \(H, F\) with \([H] = h, [F] = f\).

One crucial ingredient in the nondegeneracy proof of \(\gamma : \widehat{Ham}(M, \omega) \to \mathbb{R}_+\) given in [Oh5] is the usage of a geometric invariant \(A(\phi, J_0)\) of the pair \((\phi, J_0)\) and certain existence theorem of the perturbed Cauchy-Riemann equation for nondegenerate
Hamiltonian $H$. One of the purposes of the present paper is to provide a proof of this existence theorem.

We first recall the definition of the invariant $A(\phi, J_0)$ from [Oh5]. Denote by $J_0$ a compatible almost complex structure on $(M, \omega)$ and by $\mathcal{J}_\omega$ the set of compatible almost complex structures on $M$. For given $\phi$ and $J_0$, we consider the set of paths $J' : [0, 1] \to \mathcal{J}_\omega$ with

$$J'(0) = J_0, \quad J'(1) = \phi^* J_0$$

and denote the set of such paths by $\mathcal{j}(\phi, J_0)$.

For each given $J' \in \mathcal{j}(\phi, J_0)$, we define the constant

$$A_S(\phi, J_0; J') = \inf \{ \omega([u]) \mid u : S^2 \to M \text{ non-constant and}$$

$$\text{satisfying } \overline{\partial}_{J'_t} u = 0 \text{ for some } t \in [0, 1] \}$$

and then

$$A_S(\phi, J_0) = \sup_{J' \in \mathcal{j}(\phi, J_0)} A_S(\phi, J_0; J').$$

As usual, we set $A_S(\phi, J_0) = \infty$ if there is $J \in \mathcal{j}(\phi, J_0)$ for which there is no $J$-holomorphic sphere for any $t \in [0, 1]$ as in the weakly exact case. The positivity $A_S(\phi, J_0; J') > 0$ when it is not infinite and so $A_S(\phi, J_0) > 0$ is an immediate consequence of the one parameter version of the uniform $\epsilon$-regularity theorem (see [SU], [Oh1]).

Next for each given $J' \in \mathcal{j}(\phi, J_0)$, we consider the equation of $v : \mathbb{R} \times [0, 1] \to M$

$$\begin{cases} \frac{\partial v}{\partial \tau} + J'_t \frac{\partial v}{\partial t} = 0 \\ \phi(v(\tau, 1)) = v(\tau, 0), \quad \int |\frac{\partial v}{\partial \tau}|^2_{J'_t} < \infty. \end{cases}$$

This equation itself is analytically well-posed and (1.6) enables us to interpret solutions of (1.9) as pseudo-holomorphic sections of the mapping cylinder of $\phi$ with respect to a suitably chosen almost complex structure $\tilde{J}$ on the mapping cylinder.

Note that any such solution of (1.9) also has the limit $\lim_{\tau \to \pm \infty} v(\tau)$ and satisfies

$$\lim_{\tau \to \pm \infty} v(\tau) \in \text{Fix}\phi.$$ 

Now it is a crucial matter to produce non-constant solutions of (1.9), when $\phi$ is not the identity and in particular when $\phi$ is a nondegenerate Hamiltonian diffeomorphism.

Suppose that $\phi \neq id$ is nondegenerate and choose a symplectic ball $B(u)$ such that

$$\phi(B(u)) \cap B(u) = \emptyset$$

where $B(u)$ is the image of a symplectic embedding into $M$ of the standard Euclidean ball of radius $r$ with $u = \pi r^2$. We then study (1.9) together with the condition

$$v(0, 0) \in B(u).$$
Because of (1.10), it follows
\[ v(\pm \infty) \in \text{Fix } \phi \subset M \setminus B(u). \] (1.12)

Therefore such a solution cannot be constant because of (1.11) and (1.12).

We now define the constant
\[ A_D(\phi, J_0; J') := \inf \left\{ \int v^*\omega, | v \text{ non-constant solution of (1.9)} \right\} \] (1.13)
for each \( J' \in j(\phi, J_0) \). Again we have \( A_D(\phi, J_0; J') > 0 \). We set
\[ A(\phi, J_0; J') = \min\{ A_S(\phi, J_0; J'), A_D(\phi, J_0; J') \} \] (1.14)

The following existence theorem is an immediate consequence of Theorem II below.

**Theorem I.** For any nondegenerate Hamiltonian diffeomorphism \( \phi \neq id \), we have
\[ 0 < A(\phi, J_0; J') < \infty \] (1.15)
for any \( J_0 \in J_\omega \) and \( J' \in j(\phi, J_0) \). More precisely, the following alternative holds:
for any given point in \( B(u) \) with
\[ \phi(B(u)) \cap B(u) = \emptyset, \]

(1) either there exists a non-constant \( J'_t \)-holomorphic sphere for some \( t \in [0, 1] \) that pass through the point or
(2) (1.9) has a non-constant solution that pass through the point.

In fact, Theorem II was also the crux in the proof of the following optimal energy-capacity inequality which in particular gives rise to nondegeneracy of the pseudo-norm \( \gamma \). We refer to [Oh5] for its proof based on Theorem II.

**[Optimal Energy-Capacity inequality] [Oh5].** We denote by \( e_\gamma(A) \) the \( \gamma \)-displacement energy of closed subset \( A \) of \( M \), i.e.,
\[ e_\gamma(A) := \inf \{ \gamma(\phi) \mid \phi(A) \cap A = \emptyset \} \]
and by \( c(A) \) the Gromov capacity
\[ c(A) := \sup\{ u \mid \exists \text{ a symplectic embedding } B(u) \hookrightarrow \text{Int } A \}. \]

Then we have following inequalities:
\[ e_\gamma(A) \geq c(A) \]

As we pointed out in [Oh5], this gives rise to the optimal form of the inequality between the Hofer displacement energy and the Gromov capacity
\[ e(A) \geq c(A) \]
since we have \( e(A) \geq e_\gamma(A) \), where
\[
e(A) = \inf_\phi \{ \| \phi \| \mid \phi(A) \cap A = \emptyset \}.
\]

Finally we define
\[
A(\phi, J_0) := \sup_{J' \in j(\phi, J_0)} \min \{ A_S(\phi, J_0; J'), A_D(\phi, J_0; J') \}
\]
(1.16)

and
\[
A(\phi) = \sup_{J_0} A(\phi, J_0).
\]

Because of the assumption that \( \phi \) has only finite number of fixed points, it is clear that \( A(\phi, J_0) > 0 \) and so we have \( A(\phi) > 0 \). Note that when \((M, \omega)\) is weakly exact and so \( A_S(\phi, J_0; J') = \infty \), \( A(\phi, J_0) \) is reduced to
\[
A(\phi, J_0) = \sup_{J \in j(\phi, J_0)} \{ A_D(\phi, J_0; J') \}.
\]

In [Oh5] we have proved
\[
A(\phi) \leq \gamma(\phi) \leq \| \phi \|_{\text{mid}},
\]
postponing to the present paper the proof of some existence theorem of the following equation
\[
\begin{align*}
\frac{\partial u}{\partial t} + J_t \left( \frac{\partial u}{\partial t} - X_H\left( u \right) \right) &= 0 \\
u(-\infty) &= [z^-, w^-], \quad u(\infty) = [z^+, w^+] \\
w_- \# u &\sim w_+, \quad u(0, 0) = q \in B(u)
\end{align*}
\]
(1.18)

for the particular family
\[
J_t = (\phi_H')_t J_t', \quad J' \in j(\phi, J_0).
\]
(1.19)

Note that \( J_t \) in (1.19) is t-periodic by definition of \( j(\phi, J_0) \), i.e. \( J_1 = J_0 \) and hence (1.18) is well-posed.

Let \( k : M \to \mathbb{R} \) be a Morse function and \( \epsilon > 0 \) be any small positive number. We denote by \( \alpha, \beta \in CF_n(ek) \) Floer cycles of \( ek \) that realize the fundamental cycle \( 1^b = [M] \) and \( h_H(\alpha), h_H(\beta) \) be the cycles in \( CF_n(H) \) of \( H \) transferred via the linear homotopy
\[
\mathcal{H} : s \in [0, 1] \mapsto (1 - s)ek + sH.
\]

What we needed in the proofs of the optimal Energy-Capacity inequality and (1.17) in [Oh5] is the following existence theorem of non-stationary solutions of (1.18) with the upper bound on the action. We refer to [Oh4,5] for more explanations on the definitions and notations of various terms undefined in this theorem. The proof of this theorem (Theorem 7.1) will be finished in section 7.
Theorem II [Theorem 3.11, Oh5]. Let $H$ and $J_0$ be as before. And let $q \in \text{Int } B_u$ and $\delta > 0$ be given. Then for any $J' \in \mathfrak{J}(\phi, J_0)$, there exist some generators $[z, w] \in h_H(\alpha)$ and $[z', w'] \in h_H(\beta)$ with

$$
\mathcal{A}_H([z, w]) \leq \rho(H; 1) + \frac{\delta}{2}
$$

$$
\mathcal{A}_\tilde{H}([z', w']) \leq \rho(\tilde{H}; 1) + \frac{\delta}{2}
$$

such that the following alternative holds:

(1) The equation

$$
\begin{align*}
\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) &= 0 \\
u(\infty) &= [z', w'], \\u (-\infty) &= [z, w]
\end{align*}
$$

has a cusp-solution $u_1 \# u_2 \# \cdots \# u_N$ which is a connected union of Floer trajectories, possibly with a finite number of sphere bubbles, for $H$ that satisfies the conditions

$$
u_N(\infty) = [z', w'], u_1(-\infty) = [z, w], \quad u_j(0, 0) = q \in B_u.
$$

for some $1 \leq j \leq N$.

(2) or there is some $J'_t$-holomorphic sphere $v : S^2 \to M$ for some $t \in [0, 1]$ that passes through the point $q$.

This in particular implies

$$0 < A(\phi, J_0) \leq A(\phi) \leq \gamma(\phi) < \infty$$

for any $\phi$ and $J_0$.

We would like to emphasize that without the upper estimate (1.20) the existence of such pair $[z, w]$ and $[z', w']$ would have been a result much easier to prove from the nontriviality of the quantum product $1 \ast 1 = 1$

and its interpretation in terms of the pants product in the Floer complex. However obtaining the energy bound (1.20) requires careful control of the levels of the Floer cycles representing the Floer class $1^2$ under the pants product. We have carried out this analysis exploiting the general properties of spectral invariants $\rho(\cdot; 1)$ and the energy identity that we derive for the deformed mapping cylinder in section 3.

Using our techniques proving this existence result, we can in fact prove a more general existence theorem (see Theorem 5.1) on certain piecewise smooth finite energy solutions, i.e. solutions satisfying

$$
\int \int \left(\left|\frac{\partial u}{\partial \tau}\right|^2 + \left|\frac{\partial u}{\partial t} - X_{(H, F)}\right|^2\right)dt \, d\tau < \infty,
$$

for some $H$ and $F$. This is a result of higher-order energy bounds, and we refer to [Oh5] for details.
of the following perturbed Cauchy-Riemann equation with discontinuous Hamiltonian perturbation term

\[
\begin{align*}
\frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_{(H,F)}(u) \right) &= 0 \\
u(\infty) &= z^+ \in \text{Per}(F), \quad u(-\infty) = z^- \in \text{Per}(H)
\end{align*}
\]

(1.21)

where \( X_{(H,F)} \) is the discontinuous family of vector field

\[
X_{(H,F)}(\tau,t,x) = \begin{cases} 
X_H(t,x) & \text{for } \tau < 0 \\
X_F(t,x) & \text{for } \tau > 0
\end{cases}
\]

(1.22)

and \( J \) is the discontinuous family

\[
J(\tau,t,x) = \begin{cases} 
(\phi_H^t)^*J'_t & \text{for } \tau < 0 \\
(\phi_F^t)^*J'_t & \text{for } \tau > 0
\end{cases}
\]

(1.23)

for \( J' \in j(\phi,J_0) \). Note that if we define

\[
v(\tau,t) := \begin{cases} 
(\phi_H^t)^{-1}(u(\tau,t)) & \text{for } \tau < 0 \\
(\phi_F^t)^{-1}(u(\tau,t)) & \text{for } \tau > 0
\end{cases}
\]

then \( v \) satisfies (1.9) away from \( \tau = 0 \) and has finite energy i.e., lie in \( W^{1,2} \). A priori, \( v \) may be discontinuous but since \( J'_t \) is smooth everywhere, \( W^{1,2} \) condition implies that \( v \) must be smooth across \( \tau = 0 \) and becomes the classical solution of (1.9), if it is continuous.

We refer to \( \S \)4 and \( \S \)5 for more detailed study of (1.21), especially Theorem 5.1. Theorem 5.1 is the main existence result of the present paper, whose statement however is somewhat long and awkward to state in this introduction. We refer readers to section 5 for the precise statement. We would like to point out that Kaszturirangan and the author [KO] already demonstrated the necessity of studying the Cauchy-Riemann equation with non-smooth Lagrangian boundary condition. The appearance of the perturbed Cauchy-Riemann equation (1.20) with discontinuous Hamiltonian perturbation term is somehow reminiscent of the story from [KO]. Both appear in our study of the chain level operators in the Floer theory when we go to some limiting cases where the usual Floer’s perturbed Cauchy-Riemann equation is not analytically well-posed but allows smooth approximations which enable us to take the limit of the homology. The existence of this limiting homology (‘Fary functor’) is the source of this existence theorem of discontinuous \( W^{1,2} \) solution of the above equation.

One motivation of ours to study (1.21) is that this equation is naturally related to the small Hofer pseudo-norm: the small Hofer pseudo-norm, denoted by \( \| \cdot \|_{sm} \) is defined by

\[
\| \phi \|_{sm} := \rho^+(\phi) + \rho^-(\phi)
\]

where

\[
\rho^+(\phi) = \inf_{H \to \phi} \int_0^1 \max_{\pi(h)=\phi} H_t \, dt = \inf_{\pi(h)=\phi} E^+(h)
\]

\[
\rho^-(\phi) = \inf_{F \to \phi} \int_0^1 \min_{\pi(h)=\phi} F_t \, dt = \inf_{\pi(h)=\phi} E^-(h)
\]

The question whether this pseudo-norm is nondegenerate was first posed by Polterovich [Po] and proved by McDuff [Mc] for the case of \( \mathbb{C}P^n \) or for the weakly exact case. We refer to [Mc], [Po] for some more background materials on various related norms of Hamiltonian diffeomorphisms.
Theorem III. Let \( X_{(H,F)} \) and \( J \) as in (1.21) and (1.22) respectively. Suppose \( u : \mathbb{R} \times S^1 \to M \) is a piece-wise smooth map with finite energy that satisfies

\[
\begin{align*}
\frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_{(H,F)}(u) \right) &= 0 \\
u(\infty) = [z^+, w^+] \in \text{Crit}_F, \quad u(-\infty) = [z^-, w^-] \in \text{Crit}_H
\end{align*}
\]

and

\[
\int (w^-)^* \omega + \int (u_+)^* \omega + \int (u_-)^* \omega - \int (w^+)^* \omega = 0
\]

where \( u_\pm \) are the parts of \( u \) on \( \tau < 0 \) and \( \tau > 0 \) respectively. Then we have the following inequality

\[
\int \left| \frac{\partial u}{\partial \tau} \right|^2 J_t \leq -A_F([z^+, w^+]) + A_H([z^-, w^-]) + \int_0^1 \max H \, dt + \int_0^1 - \min F \, dt.
\]

This is the key inequality which relates all three basic quantities in the Floer theory (or more generally in the symplectic topology), the energy of the pseudo-holomorphic curves, the actions of periodic orbits and the Hofer type quantities of the corresponding Hamiltonians. We hope to pursue applications of Theorem 5.1 and Theorem III in the future.

Finally we would like to note that the kind of singular degeneration problem that we consider in section 5 and 6 will be the first step towards understanding the complete picture of adiabatic degeneration of Floer’s perturbed Cauchy-Riemann equation which will be studied jointly with Fukaya [FOh2].

We would like to thank the Institute for Advanced Study in Princeton for the excellent environment and hospitality during our participation of the year 2001-2002 program “Symplectic Geometry and Holomorphic Curves”. Much of this research has been carried out while we are visiting the Korea Institute for Advanced Study in Seoul. We thank KIAS for providing excellent atmosphere of research.

§2. Perturbed Cauchy-Riemann equations re-visited

In this section, we re-examine Floer’s perturbed Cauchy-Riemann equation that defines the chain map in the Floer homology theory and show how the Cauchy-Riemann equation with jumping discontinuity in its coefficients naturally appears in the chain level Floer theory.

The story goes in the following way. Let \( H, F \) be two \( t \)-periodic Hamiltonians and \( J^0, J^1 \) with \( J^i = \{ J^i_t \}_{0 \leq t \leq 1}, \; i = 0, 1 \) be a \( t \)-periodic family of almost complex structures. Without loss of any generality, we will always assume

\[
H_t \equiv 0, \quad J_t \equiv J_0 \quad \text{near} \; t = 0 \equiv 1
\]

When we are given a homotopy \( \mathcal{H} \) with \( \mathcal{H} = \{ H^s \}_{0 \leq s \leq 1} \) with \( H^0 = H, \quad H^1 = F \) and \( \mathcal{j} = \{ J^s \}_{0 \leq s \leq 1} \) connecting \( J^0 \) and \( J^1 \), the chain homomorphism

\[
h^*_{(\mathcal{H}, J)} : CF_*(H, J^0) \to CF_*(F, J^1)
\]
is defined by the non-autonomous equation
\[
\begin{cases}
\frac{\partial u}{\partial \tau} + J^{\rho_1(\tau)} \left( \frac{\partial u}{\partial \tau} - X_{H_{\rho_2(\tau)}}(u) \right) = 0 \\
\lim_{\tau \to -\infty} u(\tau) = z^-,
\lim_{\tau \to \infty} u(\tau) = z^+
\end{cases}
\]  \tag{2.2}

with finite energy condition
\[
\int \left| \frac{\partial u}{\partial \tau} \right|^2_{J^{\rho_1(\tau)}} < \infty. \tag{2.3}
\]

Here \( \rho_i \) are the cut-off functions of the type \( \rho : \mathbb{R} \to [0, 1] \),
\[
\rho(\tau) = \begin{cases}
0 & \text{for } \tau \leq R_1 \\
1 & \text{for } \tau \geq R_2
\end{cases}
\]
\[
\rho'(\tau) \geq 0
\]
for given arbitrary pair \( R_1 < R_2 \) in \( \mathbb{R} \). For the simplicity of exposition, we will consider mostly the case \( \rho_1 = \rho_2 = \rho \). (2.2) and (2.3) uniquely determine the asymptotic condition
\[
\lim_{\tau \to -\infty} u(\tau) = z^- \in \text{Per}(H), \quad \lim_{\tau \to \infty} u(\tau) = z^- \in \text{Per}(F) \tag{2.4}
\]

Since we consider (2.2) as an equation on the \( \Gamma \)-covering space \( \tilde{\Omega}_0(M) \), we also put the topological condition
\[
w^- \# u \sim w^+
\]  \tag{2.5}

where \( w^\pm \) are discs bounding \( z^\pm \) respectively. Here \( \Gamma \) is the covering group
\[
\Gamma = \frac{\pi_2(M)}{\ker(\omega|_{\pi_2(M)}) \cap \ker(c_1|_{\pi_2(M)})}.
\]

Then we lift the asymptotic condition (2.4) to the covering space as
\[
\lim_{\tau \to -\infty} u(\tau) = [z^-, w^-] \in \text{Crit}A_H, \quad \lim_{\tau \to \infty} u(\tau) = [z^+, w^+] \in \text{Crit}A_F. \tag{2.6}
\]

The following identity is the fundamental identity which has been used frequently in our previous works [Oh2,3,4]. We leave its proof to readers.

**Lemma 2.1.** For any solution of (2.2) with (2.5), we have
\[
A_F([z^+, w^+]) - A_H([z^-, w^-])
= -\int \left| \frac{\partial u}{\partial \tau} \right|^2_{J^{\rho_1(\tau)}} - \int_{-\infty}^\infty \rho'_2(\tau)(F(t, u(\tau, t)) - H(t, u(\tau, t))) \, dt \, d\tau
\leq -\int \left| \frac{\partial u}{\partial \tau} \right|^2_{J^{\rho_1(\tau)}} + \int_0^1 -\min F_t \, dt + \int_0^1 \max H_t \, dt.
\]  \tag{2.7}

In particular, we have
\[
\int \left| \frac{\partial u}{\partial \tau} \right|^2_{J^{\rho_1(\tau)}} \leq A_H([z^-, w^-]) - A_F([z^+, w^+]) + \int_0^1 -\min F_t \, dt + \int_0^1 \max H_t \, dt.
\]  \tag{2.8}
We recall that when $H = F$ and $\rho \equiv 1$, we have the improved identity
\[
\int J \left| \frac{\partial u}{\partial \tau} \right|^2 = A_H([z^-, w^-]) - A_H([z^+, w^+]).
\]
Note that the upper bound (2.8) is independent of the cut-off functions $\rho$ and also of $R_1, R_2$. One important fact which we are going to exploit is that the chain map itself
\[
h^\rho_{(H,j)} : CF_*(H, J^0) \to CF_*(F, J^1)
\]
depends on the choice of $\rho$ although two chain maps $h^\rho_{(H,j)}$ for different $\rho$'s are chain homotopic to each other and hence they induce the same homomorphism
\[
h_{HF} : HF_*(H) \to HF_*(F)
\]
in homology.

We now would like to study the equation (2.2) with the condition (2.5) as $R_1 \to 0_-$ and $R_2 \to 0_+$, i.e., when the cut-off function $\rho$ converges to the discontinuous Heaviside function
\[
\rho = \begin{cases} 
0 & \text{for } \tau < 0 \\
1 & \text{for } \tau > 0.
\end{cases}
\]
Note that the corresponding limit of the equation (2.2) is
\[
\begin{cases} 
\frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_{(H,F)}(u) \right) = 0 \\
\lim_{\tau \to -\infty} u(\tau) = z^-, \lim_{\tau \to \infty} u(\tau) = z^+
\end{cases}
\]
(2.10)
where $X_{(H,F)}$ are the discontinuous family of Hamiltonian vector fields satisfying (1.21) and
\[
J(\tau, t, x) = \begin{cases} 
J^0(t, x) & \text{for } \tau < 0 \\
J^1(t, x) & \text{for } \tau > 0.
\end{cases}
\]
We will be particularly interested in the family
\[
J^0_t = (\phi^t_H)^* J^0_t, \quad J^1_t = (\phi^t_F)^* J^1_t
\]
(2.11)
for $J' \in j_{(\phi, J^0)}$. We note that these are $t$-periodic due to the definition of $j_{(\phi, J^0)}$ and so (2.10) is a well-defined equation over $\mathbb{R} \times S^1$ in the classical sense, except at $\tau = 0$. Due to the condition (2.1), the equation has smooth coefficients and the Hamiltonian perturbation term is smooth in a neighborhood of $\mathbb{R} \times \{0\} \subset \mathbb{R} \times S^1$ including $(0, 0)$. Therefore any $W^{1,2}$ solution of (2.10) will be smooth near $(0, 0)$. More importantly, under the choice (2.11), (2.10) is transformed into
\[
\begin{cases} 
\frac{\partial v}{\partial \tau} + J_t^t \frac{\partial v}{\partial t} = 0 \\
\phi(v(\tau, 1)) = v(\tau, 0)
\end{cases}
\]
(2.12)
by the map
\[
v(\tau, t) = \begin{cases} 
(\phi^t_H)^{-1}(u(\tau, t)) & \text{for } \tau < 0 \\
(\phi^t_F)^{-1}(u(\tau, t)) & \text{for } \tau > 0.
\end{cases}
\]
(2.13)
If \( u \) is in \( W^{1,2} \), so is \( v \). Since (2.12) has smooth coefficients, such \( W^{1,2} \) solution \( v \) will be indeed smooth if \( v \) is continuous across \( \tau = 0 \).

The equation (2.10) now has bounded discontinuous coefficients and zero-order perturbations. Therefore we regard (2.10) as the first order quasi-linear PDE for the distributional maps. For this purpose, we will always embed \( M \) with a fixed metric \( g \) into \( \mathbb{R}^N \) by the Nash isometric embedding theorem and regard \( u \) as a vector valued distributional maps defined on \( \mathbb{R} \times S^1 \). As usual in the geometric PDE, we define a finite energy map into \( M \) by a \( \mathbb{R}^N \)-valued distribution \( u \) with \( u(z) \in M \subset \mathbb{R}^N \) almost everywhere

and

\[
\int |\frac{\partial u}{\partial \tau}|^2 + |\frac{\partial u}{\partial t} - X_{(H,F)}|^2 dt d\tau < \infty
\]  

(2.14)

which does not depend on the choice of metric \( g \). Note that since \( M \) is compact, (2.14) in particular implies that \( u \in L^\infty \) and so in \( L^p \) for every \( p > 0 \).

The following is the key property that will be important later in the nondegeneracy proof.

**Theorem 2.2.** Let \( u : \mathbb{R} \times S^1 \to M \) be a solution lying with finite energy which satisfies (2.10) and (2.6). Suppose that it satisfies in addition

\[
\int w^- \omega + \int u^- \omega + \int u^+ \omega - \int w^+ \omega = 0. 
\]  

(2.15)

Then we have the following identity

\[
\int |\frac{\partial u}{\partial \tau}|^2 J \leq -A_F([z^+, w^+]) + A_H([z^-, w^-])
\]

\[
+ \int_0^1 \max H_t dt + \int_0^1 -\min F_t dt. 
\]  

(2.16)

**Proof.** We first note that since \( u \) has finite energy, both \( \int u^- \omega \) and \( \int u^+ \omega \) are finite and so well-defined (see [Oh2] for the proof of this fact). It also implies

\[
\int |\frac{\partial u}{\partial \tau}|^2 J = \lim_{\epsilon \to 0} \int_{(\mathbb{R} \setminus [-\epsilon, \epsilon]) \times S^1} |\frac{\partial u}{\partial \tau}|^2 J 
\]

\[
= \lim_{\epsilon \to 0} \int_{-\infty}^{-\epsilon} \int_0^1 |\frac{\partial u}{\partial \tau}|^2 J \ dt d\tau + \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \int_0^1 |\frac{\partial u}{\partial \tau}|^2 J \ dt d\tau. 
\]

On the other hand, we have

\[
\int_{-\infty}^{-\epsilon} \int_0^1 |\frac{\partial u}{\partial \tau}|^2 J \ dt d\tau = A_H(u(-\epsilon)) - A_H(u(-\infty))
\]

\[
\int_{-\infty}^{-\epsilon} \int_0^1 |\frac{\partial u}{\partial \tau}|^2 J \ dt d\tau = A_F(u(\infty)) - A_F(u(\epsilon))
\]

and hence

\[
\int |\frac{\partial u}{\partial \tau}|^2 J = -A_F(u(\infty)) + A_H(u(-\infty)) + \lim_{\epsilon \to 0} (-A_H(u(-\epsilon)) + A_F(u(\epsilon)). 
\]  

(2.17)
It follows from (2.15) that the term of the integral of $\omega$ in the action functional vanishes and so
\[
\lim_{\varepsilon \to 0} \left( -\mathcal{A}_H(u(-\varepsilon)) + \mathcal{A}_F(u(\varepsilon)) \right) = \lim_{\varepsilon \to 0} \left( \int_0^1 -F(u(\varepsilon, t)) \, dt + \int H(u(-\varepsilon, t)) \, dt \right) \\
\leq \int_0^1 \max H_t \, dt + \int_0^1 -\min F_t \, dt.
\] (2.18)

Combining (2.17) and (2.18), we have finished the proof. \(\square\)

The essential matter now is to prove some existent result of the equation (2.2) with suitable asymptotic condition and (2.15). One way of obtaining a solution is by taking a limit of a sequence of solutions of (2.2) as $\rho$ converges to the step function (2.10). But this limit could be stationary, i.e., $\tau$-independent which is not suitable for our purpose because the solution we need is the one that is non-trivial as in [Oh5]. For this purpose, we need to generalize the above story to the maps from the arbitrary compact Riemann surface of genus zero with $k$ punctures. We will focus only on $k = 3$ in this paper because this is the case that is directly relevant to the proof of Theorem I.

§3. Energy identity on the deformed mapping cylinder

In this section, we recast Floer’s equation for the chain map in the point of the deformed mapping cylinder, and calculate the vertical energy of the pseudo-holomorphic sections for a suitably chosen almost complex structure $\tilde{J}$ in terms of the variation of the Hamiltonians and the curvature of the natural connection associated to the Hamiltonians. This will form the basis of the energy calculation of pseudo-holomorphic sections of the Hamiltonian fibration over the Riemann surface $\Sigma$ of genus zero with arbitrary number of punctures, when we realize the conformal structure of $\Sigma$ by the minimal area metric on $\Sigma$ [Z]. Our calculation of the energy is a crucial ingredient in the optimal Energy-Capacity inequality and nondegeneracy proof of $\gamma$-norm, and also in our construction of $W^{1,2}$-solution mentioned in Theorem II (or Theorem 5.1). This calculation is very much in the spirit of geometric calculations in the differential geometry.

Consider the two parameter family of Hamiltonian diffeomorphisms
\[
\phi : (s, t) \in [0, 1] \times [0, 1] \to \text{Ham}(M, \omega)
\]
with
\[
\phi(s, 0) = id
\]
for all $s \in [0, 1]$. We also denote $\phi_s^t = \phi(s, t)$ and
\[
\phi_s := \phi(s, 1).
\]
We will assume that
\[
\phi(s, t) = \begin{cases} 
  id & \text{for } t \text{ near } 0 \\
  \phi_s & \text{for } t \text{ near } 1
\end{cases}
\]
We define the vector fields
\[
X(s, t, x) = \frac{\partial \phi_s^t}{\partial t} \circ (\phi_s^t)^{-1}
\]
\[
Y(s, t, x) = \frac{\partial \phi_s^t}{\partial s} \circ (\phi_s^t)^{-1}
\] (3.1)
and denote by
\[
H : [0,1] \times [0,1] \times M \to \mathbb{R}
\]
\[
F : [0,1] \times [0,1] \times M \to \mathbb{R}
\]
the normalized Hamiltonians generating \(X\) and \(Y\) respectively. We will be particularly interested in the family generated by the linear homotopy
\[
H(s, t, x) = sH(t, x).
\]
Now we consider the cylinder \(\Sigma = \mathbb{R} \times S^1\) or the semi-cylinders
\[
\Sigma_+ = [0, \infty) \times S^1
\]
\[
\Sigma_- = (-\infty, 0] \times S^1
\]
equipped with the standard flat metric on them. We will focus on \(\Sigma = \Sigma_-\) but the parallel story goes for \(\Sigma\) or \(\Sigma_+\) with some obvious changes. For each given \(0 < R_1 < R_2\), we consider the cut-off functions
\[
\rho = \rho_{R_1, R_2} = \begin{cases} 0 & \text{for } -\infty < \tau \leq -R_2 \\ 1 & \text{for } -R_1 \leq \tau \leq 0 \end{cases}
\]
with \(\rho' \leq 0\) using the function \(\rho\). We reparameterize the \(s\)-variable in the family \(\phi\) and denote
\[
\phi_\rho(\tau, t) = \phi(\rho(\tau), t).
\]
We denote by \(X_\rho\) and \(Y_\rho\) the corresponding reparameterized family of vector fields. If we denote by \(H_\rho\) and \(F_\rho\) the corresponding Hamiltonians for the reparameterized family \(\phi_\rho\), the following is easy to check
\[
H_\rho(\tau, t, x) = \rho(\tau)H(t, x)
\]
\[
F_\rho(\tau, t, x) = \rho'(\tau)F(\rho(\tau), t, x). \tag{3.2}
\]
Considering the path
\[
f : \tau \mapsto f_\tau = \phi_\rho^1(\tau) : \mathbb{R}_- \to \text{Ham}(M, \omega)
\]
we define the deformed mapping cylinder by
\[
E =Ef := \mathbb{R}_- \times \mathbb{R} \times M/(\tau, t, x) \sim (\tau, t-1, f_\tau(x))
\]
which defines a symplectic fibration over the semi-cylinder \(\Sigma = \mathbb{R}_+ \times S^1\). We consider the closed two form
\[
\omega_E := \omega + d(H_\rho dt) = \omega + d(\rho H_1 dt) \tag{3.3}
\]
on \(\mathbb{R}_- \times \mathbb{R} \times M\) which projects down to a closed two form on \(E\). We denote the push down of the form again by \(\omega_E\). This form is nondegenerate in the fiber of \(E \to \mathbb{R} \times S^1\) and restricts to \(\omega\) and so defines a canonical connection \(\nabla\) [GLS] whose
The coupling form is exactly $\omega_E$. We need to describe this connection more explicitly to carry out precise calculation of the energy.

We trivialize the mapping cylinder $E \to \mathbb{R}_+ \times S^1 \times M$ by

$$[	au, t, v] \in E \mapsto (\tau, t, u(\tau, t)) \quad (3.3)$$

where

$$u(\tau, t) = (\phi^t_{\rho(\tau)})^{-1}(v). \quad (3.4)$$

Note that (3.4) is $t$-periodic because of the defining equivalence relation of the deformed mapping cylinder $E_f$ and so (3.4) provides a well-defined trivialization of $E_f$.

Under this trivialization, a straightforward calculation proves the formula for the horizontal lifts of $\frac{\partial}{\partial \tau}$ and $\frac{\partial}{\partial t}$ for the connection $\nabla$

$$(Du)^h\left(\frac{\partial}{\partial \tau}\right) = \frac{\partial}{\partial \tau}$$

$$=(Du)^h\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t} + X_\rho \quad (3.5)$$

for the arbitrary sections $v$. If we denote by

$$\Pi_{\tau, t, x} : T_{(\tau, t, x)}E \to (TE)_{(\tau, t, x)}^v \cong T_xM$$

the associated vertical projection, it has the formula

$$\Pi_{\tau, t, x}(\alpha, \beta, \xi) = \xi - \beta X_\rho \quad (3.6)$$

in the trivialization (3.3). We provide a symplectic form on $E$ by

$$\Omega_\lambda = \Omega_{E, \lambda} = \omega_E + \lambda \omega_\Sigma$$

for a sufficiently large $\lambda > 0$, where $\omega_\Sigma$ is an area form on the base $\mathbb{R}_+ \times S^1$. We normalize it so that

$$\int \omega_\Sigma = 1. \quad (3.7)$$

Now let $J^s_t$ for $(s, t) \in [0, 1]^2$ be two parameter family of $\omega$-compatible almost complex structures on $M$ such that $J$ is constant near $s = 0, 1$ and $J^s_0 = J^1_t$. We also assume that $J^s_t$ is constant near $t = 0 \equiv 1$. Using this, we define the almost complex structure $\tilde{J}$ on $E$ in the trivialization (3.3) by the formula

$$\tilde{J}(\tau, t, x)(\alpha, \beta, \xi) = (-\beta, \alpha, (\phi^t_{\rho(\tau)})^*J^s_t(\tau)(\xi - \beta X_\rho) + \alpha X_\rho). \quad (3.8)$$

One can easily check that $\tilde{J}$ is well-defined on $E$ and tame to $\Omega_E$ for a sufficiently large $\lambda$, but it is not compatible in the usual sense in that the bilinear form

$$\Omega(\cdot, \tilde{J}\cdot)$$

is not symmetric. However we can symmetrize this and define the associate metric $g_{\tilde{J}}$ by

$$\langle V, W \rangle = g_{\tilde{J}}(V, W) := \frac{1}{2}(\Omega(V, \tilde{J}W) + \Omega(W, \tilde{J}V)). \quad (3.9)$$

We call (3.9) the metric associated to $\tilde{J}$ and denote

$$|V|^2 = |V|^2_{\tilde{J}} = g_{\tilde{J}}(V, V).$$

With respect to this metric, we still have the following basic identity whose proof we omit.
Lemma 3.1. Let $v : \mathbb{R} \times S^1 \to E$ be any $\bar{J}$-holomorphic map $v$. Then we have

$$\frac{1}{2} \int |Dv|_{\bar{J}}^2 = \int v^* \Omega. \quad (3.11)$$

Next note that for any $\bar{J}$-holomorphic section $v : \Sigma \to E_f$, if we identify it with a map

$$v : \mathbb{R}_+ \times \mathbb{R} \to M$$

satisfying

$$f_\tau(v(\tau,1)) = v(\tau,0),$$

the map $u$ defined by

$$u(\tau,t) = (\phi_{\rho(\tau)}^t)^{-1}(v(\tau,t))$$

satisfies

$$u(\tau,0) = u(\tau,1)$$

and is smooth near $t = 0 \equiv 1$ because of the condition (2.1) for the Hamiltonian $H$, and hence defines a well-defined map $u : \mathbb{R}_+ \times S^1 \to M$. It also satisfies

$$\frac{\partial u}{\partial \tau} + J_{\rho(\tau)} \left( \frac{\partial u}{\partial t} - X_{H_{\rho(\tau)}}(u) \right) = 0. \quad (3.12)$$

With this preparation, we now compute the energy density $|Dv|^2$ for $\bar{J}$-holomorphic section $v : \Sigma \to E_f$. Decomposing $Dv$ into its vertical and horizontal components

$$Dv = (Dv)^v + (Dv)^h$$

we have

$$|Dv|^2 = |(Dv)^v|^2 + |(Dv)^h|^2 + 2\langle (Dv)^v, (Dv)^h \rangle. \quad (3.13)$$

If we denote the curvature of the connection $\nabla$ by

$$K(v)d\tau \wedge dt$$

then it is straightforward to compute

$$\frac{1}{2} \int |(Dv)^h|^2 = \int K(v)d\tau dt + \lambda \quad (3.14)$$

by integrating the identity

$$\sum_{i=1}^2 |(Dv)^h(e_i)|^2 = \sum_{i=1}^2 \Omega_{E,\lambda}((Dv)^h(e_i), \bar{J}(Dv)^h(e_i))$$

$$= \sum_{i=1}^2 (\omega_E + \lambda \omega_\Sigma)((Dv)^h(e_i), \bar{J}(Dv)^h(e_i))$$

$$= 2(\omega_E((Dv)^h(e_1), (Dv)^h(e_2)) + \lambda \omega_\Sigma(e_1, e_2))$$
for an orthonormal frame $\{e_1, e_2\}$ of $T\Sigma$, e.g., for the frame
$$\left\{ \frac{\partial}{\partial \tau}, \frac{\partial}{\partial t} \right\}$$
and then applying the curvature identity
$$d(\omega(e_1^#, e_2^#)) = -\iota_{[e_1^#, e_2^#]}\omega$$
where $e_i^#$ is the horizontal lift of $e_i$ (see [1.12], GLS] but with caution on the sign convention). Applying (3.5) and (3.8), we derive
\begin{equation}
\tilde{J}(Dv)^h\left( \frac{\partial}{\partial \tau} \right) = \frac{\partial}{\partial t} + X_\rho \\
\tilde{J}(Dv)^h\left( \frac{\partial}{\partial t} \right) = -\frac{\partial}{\partial \tau}.
\end{equation}
(3.5) immediately gives rise to
\begin{equation}
K(v)(\tau, t) = \omega_E \left( (Dv)^h\left( \frac{\partial}{\partial \tau} \right), (Dv)^h\left( \frac{\partial}{\partial t} \right) \right) = \rho'(\tau)H(t, u(\tau, t))
\end{equation}
which in particular proves that $K$ is compactly supported and so the curvature integral in (3.14) in finite. Combining (3.14) and (3.16), we have derived
\begin{equation}
\frac{1}{2} \int |(Dv)^h|^2 = \lambda + \int \rho'(\tau)H(t, u(\tau, t))
\end{equation}
Next we will compute $\langle (Dv)^v, (Dv)^h \rangle$
\begin{equation}
\langle (Dv)^v, (Dv)^h \rangle = \langle (Dv)^v\left( \frac{\partial}{\partial \tau} \right), (Dv)^h\left( \frac{\partial}{\partial t} \right) \rangle + \langle (Dv)^v\left( \frac{\partial}{\partial t} \right), (Dv)^h\left( \frac{\partial}{\partial \tau} \right) \rangle.
\end{equation}
We separately compute the two terms. First from the definition (3.9) of the associated metric, we have
\begin{equation}
\langle (Dv)^v\left( \frac{\partial}{\partial \tau} \right), (Dv)^h\left( \frac{\partial}{\partial \tau} \right) \rangle
\end{equation}
\begin{equation}
= \frac{1}{2} \left( \Omega((Dv)^v\left( \frac{\partial}{\partial \tau} \right)), \tilde{J}(Dv)^h\left( \frac{\partial}{\partial \tau} \right) \right) + \Omega((Dv)^h\left( \frac{\partial}{\partial \tau} \right), \tilde{J}(Dv)^v\left( \frac{\partial}{\partial \tau} \right)) \right).
\end{equation}
Since $\tilde{J}$ preserves the vertical space, it is easy to see that the second term vanishes. For the first term, we derive from (3.6) and (3.15)
\begin{equation}
2\Omega((Dv)^v\left( \frac{\partial}{\partial \tau} \right), \tilde{J}(Dv)^h\left( \frac{\partial}{\partial \tau} \right)) = 2\omega_E\left( \frac{\partial u}{\partial \tau}, \frac{\partial}{\partial t} + X_\rho \right) = 0
\end{equation}
and hence
\begin{equation}
2\langle (Dv)^v\left( \frac{\partial}{\partial \tau} \right), (Dv)^h\left( \frac{\partial}{\partial \tau} \right) \rangle = 0
\end{equation}
On the other hand, a straightforward calculation using (3.6) and (3.15) also show
\begin{equation}
2\langle (Dv)^v\left( \frac{\partial}{\partial t} \right), (Dv)^h\left( \frac{\partial}{\partial t} \right) \rangle
\end{equation}
\begin{equation}
= \left( \omega + d(\rho H dt) \right) \left( \frac{\partial u}{\partial t} - X_\rho, \frac{\partial}{\partial \tau} \right) = 0.
\end{equation}
Adding (3.18) and (3.19), we derive the identity
\begin{equation}
2\langle (Dv)^v, (Dv)^h \rangle = 0.
\end{equation}
We summarize the above calculations into
Lemma 3.3. Let $\tilde{J}$ be the above almost complex structure on $E$ and $\nu: \Sigma \rightarrow E$ be a $\tilde{J}$-holomorphic section. Then we have

$$\frac{1}{2} \int |(D\nu)^{\nu}|^2 = \lambda + \int \rho'(\tau)H(t, u(\tau, t))$$

Applying Lemma 3.3, we derive the following formula for the vertical energy

Proposition 3.4. Let $\rho = \rho_{R_1, R_2}$ be the cut-off functions as above. Then we have

$$\frac{1}{2} \int |(D\nu)^{\nu}|^2 = \int v^*\omega_E - \int \rho'(\tau)H(t, u(\tau, t)) dt d\tau$$

for any $\rho$ and $\tilde{J}$-holomorphic section $\nu: \Sigma = \mathbb{R} \times S^1 \rightarrow E$ of the deformed mapping cylinder. Similar formula holds on $\Sigma$ or $\Sigma = \mathbb{R} \times S^1$. In particular, we have the inequality

$$\frac{1}{2} \int |(D\nu)^{\nu}|^2 \leq \int v^*\omega_E + \int_0^1 \max H_t dt$$

for any finite energy $\tilde{J}$-holomorphic sections $\nu$ for any $0 < R_1 < R_2 < \infty$.

Proof. (3.22) immediately follows by substituting (3.16) and (3.20) into (3.21). (3.23) is a consequence of integration over $\tau$ noting that $\rho' \leq 0$ on $\Sigma = (\mathbb{R}, 0) \times S^1$. □

§4. Pants product and the Hamiltonian fibration

Let $h, f, g \in \tilde{H}am(M, \omega)$ be an arbitrary nondegenerate triple. The product structure

$$HF_*(h) \otimes HF_*(f) \rightarrow HF_*(g).$$

is defined by considering the “pants product” in the chain complex

$$CF_*(H) \otimes CF_*(F) \rightarrow CF_*(G)$$

where $[H] = h, [F] = f, [G] = g$. Here one can already see that the pants product in the chain level depend on the choice of the Hamiltonians representing the classes $h, f, g \in \tilde{H}am(M, \omega)$, not just on $h, f, g$ let alone their projections $\phi = \pi(h), \psi = \pi(f), \eta = \pi(g)$. The definition of the pants product (4.2) depend on many additional data, while the induced product in homology (4.1) is independent of the choice of such data. The best way of describing the pants product is using the Hamiltonian fibration with connection of fixed boundary monodromy as described by Entov [En]. We briefly recall Entov’s construction here with few notational changes and different convention on the grading, but fully exploiting the minimal area metric representation [Z] of the conformal structure on $\Sigma$. As in [Oh5], our grading will be provided by

$$k = \mu_H$$
where $\mu_H([z, w])$ is the Conley-Zehnder index of the critical point $[z, w] \in A_H$. This grading convention respects the grading under the pants product

$$CF_k(H) \otimes CF_\ell(F) \to CF_{k+\ell-n}(G).$$

Let $\Sigma$ be the compact Riemann surface of genus 0 with three punctures with the minimal area metric $[Z]$ which we now describe. We conformally identify $\Sigma$ with three half cylinders with which we denote by $\Sigma_1, \Sigma_2$ and $\Sigma_3$ in the following way: the conformal structure on $\Sigma \setminus \{z_1, z_2, z_3\}$ can be described in terms of the minimal area metric $[Z]$ which we denote by $g_\Sigma$. This metric makes $\Sigma$ as the union of three half cylinders $\Sigma_i$'s with flat metric with each meridian circle has length $2\pi$. The metric is singular only at two points $p, \overline{p} \in \Sigma$ which lie on the boundary cycles of $\Sigma_i$. Therefore the conformal structure induced from the minimal area metric naturally extends over the two points $p, \overline{p}$. The resulting conformal structure is nothing but the standard unique conformal structure on $\Sigma = S^2 \setminus \{z_1, z_2, z_3\}$. We will use this metric for the analytic estimates of pseudo-holomorphic curves implicit in the argument. One important property of this singular metric is that it is flat everywhere except at the two points $p, \overline{p}$ where the metric is singular but Lipschitz. Because the metric is Lipschitz at the two points and smooth otherwise, it follows that the corresponding Sobolev constants of the metric are all finite even though the metric is singular at the two points $p, \overline{p}$ (see §10, FOh] for a similar consideration of such metric for the open string version).

Following §10, FOh], §11, En], we identify $\Sigma$ as the union of $\Sigma_i$'s

$$\Sigma = \cup_{i=1}^3 \Sigma_i$$

in the following way: if we identify $\Sigma_i$ with $(-\infty, 0] \times S^1$, then there are 3 paths $\theta_i$ of length $\frac{1}{2}$ for $i = 1, 2, 3$ in $\Sigma$ connecting $p$ to $\overline{p}$ such that

$$\begin{align*}
\partial_1 \Sigma &= \theta_1 \circ \theta_3^{-1} \\
\partial_2 \Sigma &= \theta_2 \circ \theta_1^{-1} \\
\partial_3 \Sigma &= \theta_3 \circ \theta_2^{-1}
\end{align*}$$

We fix a holomorphic identification of each $\Sigma_i, i = 1, 2, 3$ with $(-\infty, 0] \times S^1$ and $\Sigma_3$ with $[0, \infty) \times S^1$ consider the decomposition (4.4).

We denote the identification by

$$\varphi_i^+: \Sigma_i \to (-\infty, 0] \times S^1$$

for the outgoing ends and

$$\varphi_3^- : \Sigma_3 \to [0, \infty) \times S^1$$

for the incoming end. We denote by $(\tau, t)$ the standard cylindrical coordinates on the cylinders with identification $S^1 = \mathbb{R}/\mathbb{Z}$. So in our case, the length of the meridian has length 1 instead of $2\pi$.

We consider a cut-off function $\rho^- : (-\infty, 0] \to [0, 1]$ of the type

$$\rho^- = \begin{cases} 
1 & -\infty < \tau \leq -R_2 \\
0 & -R_1 \leq \tau \leq 0
\end{cases}$$

(4.5)
and \( \rho^+ : [0, \infty) \to [0, 1] \) by \( \rho^+(\tau) = \rho^-(\tau) \) where \( 0 < R_1 < R_2 < \infty \) are arbitrary numbers. We will just denote by \( \rho \) these cut-off functions for both cases when there is no danger of confusion.

We now consider the (topologically) trivial bundle \( P \to \Sigma \) with fiber isomorphic to \((M, \omega)\) and fix a trivialization \( \Phi_i : P_i := P|_{\Sigma_i} \to \Sigma_i \times M \) on each \( \Sigma_i \). On each \( P_i \), we consider the closed two form of the type

\[
\omega_{P_i} := \Phi_i^* (\omega + d(\rho H t dt)) \tag{4.6}
\]

for a time periodic Hamiltonian \( H : [0, 1] \times M \to \mathbb{R} \). Note that this naturally extends to a closed two form \( \omega_P \) defined on the whole \( P \to \Sigma \) since the cut-off functions vanish in the center region of \( \Sigma \). Note that this form is nondegenerate in the fiber and restricts to \( \Phi_i^* \omega \).

Such \( \omega_P \) induces a canonical symplectic connection \( \nabla = \nabla_{\omega_P} \) [GLS], [En]. We consider the symplectic form on \( P \)

\[
\Omega_{P, \lambda} := \omega_P + \lambda \omega_{\Sigma} \tag{4.7}
\]

where \( \omega_{\Sigma} \) is an area form and \( \lambda > 0 \) is a sufficiently large constant. We will always normalize \( \omega_{\Sigma} \) so that \( \int_{\Sigma} \omega_{\Sigma} = 1 \) as before.

Next let \( \tilde{J} \) be the almost complex structure defined as in (3.5) on each \( \Sigma_i \) which naturally extends to the whole \( P \) due to the cut-off functions. This almost complex structure on \( P \) is \((H, J_0)\)-compatible in that it satisfies

1. \( \tilde{J} \) is \( \omega_P \)-compatible on each fiber in that it preserves the vertical tangent space
2. the projection \( \pi : P \to \Sigma \) is pseudo-holomorphic, i.e, \( d\pi \circ \tilde{J} = j \circ d\pi \).

When we are given three \( t \)-periodic Hamiltonian \( H = (H_1, H_2, H_3) \) and a periodic family \( J = (J^1, J^2, J^3) \), \( \tilde{J} \) additionally satisfies

3. For each \( i \), \( (\Phi_i)^* \tilde{J} = j \oplus J_{H_i} \) where

\[
J_{H_i}(\tau, t, x) = (\phi_{H_i}^t)^* J_i^1 \tag{4.8}
\]

on \( M \) over the cylinder \( \Sigma_i' \subset \Sigma_i \) in terms of the cylindrical coordinates. Here

\[
\Sigma_i' = \varphi_i^{-1}((\infty, -R_{i,2}] \times S^1), \quad i = 1, 2
\]

and

\[
\Sigma_3 = \varphi_3^{-1}([R_{3,2}, \infty) \times S^1)
\]

for some \( R_{i,2} > 0 \). \( \tilde{J} \) also satisfies

\[
\tilde{J} = j \oplus J_0
\]

on

\[
\Sigma_{\text{cn}} := \bigsqcup_{i=1}^3 C_i
\]
where \( C_i \subset \Sigma_i \) is the region corresponding to \([-R_i, 1, 0] \times S^1 \) for \( i = 1, 2 \) and the one corresponding to \([0, R_3, 1] \times S^1 \).

The \( \overline{J} \)-holomorphic sections \( v \) over \( \Sigma'_i \) are precisely the solutions of the equation

\[
\frac{\partial u}{\partial \tau} + J_i^{\rho(\tau)} \left( \frac{\partial u}{\partial t} - X_{\rho(\tau)}(u) \right) = 0
\]

(4.9)

if we write \( v(\tau, t) = (\tau, t, u(\tau, t)) \) in the trivialization with respect to the cylindrical coordinates \((\tau, t)\) on \( \Sigma'_i \) induced by \( \phi_1 \) above. In the center region \( \Sigma_{cn} \), they just become \( J_0 \) holomorphic curves

\( u : \Sigma_{cn} \to M \)

with respect to the given conformal structure \( j \) on \( \Sigma_{cn} \).

Now we define the moduli space which will be relevant to the definition of the pants product that we need to use. To simplify the notations, we denote \( \hat{z} = [z, w] \) in general and \( \hat{z} = (\hat{z}_1, \hat{z}_2, \hat{z}_3) \) where \( \hat{z}_i = [z_i, w_i] \in \text{Crit}A_H \) for \( i = 1, 2, 3 \). We denote by

\[
\tilde{\pi}_2(M) = \text{Im}(\pi_2(M) \to H_2(M, \mathbb{Z}))/\text{Tor}(H_2(M, \mathbb{Z}))
\]

the set of spherical homology classes (mod) torsion elements.

**Definition 4.2.** Consider the Hamiltonians \( H = \{H_i\}_{1 \leq i \leq 3} \) and the closed two form \( \omega_P \) on the trivial bundle \( P = \Sigma \times M \) and its associated connection \( \nabla \). Let \( \bar{J} \) be a \( H \)-compatible almost complex structure.

1. We denote by \( \mathcal{M}(H, \bar{J}; \hat{z}) \) the space of all \( \bar{J} \)-holomorphic sections \( v : \Sigma \to P \) such that the closed surface obtained by capping off \( u := pr_M \circ v \) with the discs \( w_i \) taken with the same orientation for \( i = 1, 2 \) and the opposite one for \( i = 3 \) represents zero in \( \tilde{\pi}_2(M) \):

\[
[u\#(\bigcup_{i=1}^{3} w_i) = 0 \quad \text{in} \quad \tilde{\pi}_2(M)].
\]

(4.10)

Note that \( \mathcal{M}(H, \bar{J}; \hat{z}) \) depends only on the equivalence class of \( \hat{z} \)'s:

2. we say that \( \hat{z}' \sim \hat{z} \) if they satisfy

\[
z'_i = z_i, \quad w'_i = w_i \# A_i
\]

for \( A_i \in \Gamma \) and \( \sum_{i=1}^{3} A_i = 0 \) in \( \tilde{\pi}_2(M) \).

The (virtual) dimension of \( \mathcal{M}(H, \bar{J}; \hat{z}) \) is given by

\[
\dim \mathcal{M}(H, \bar{J}; \hat{z}) = 2n - (\mu_{H_1}(\hat{z}_1) + n) - (\mu_{H_2}(\hat{z}_2) + n) - (\mu_{H_3}(\hat{z}_3) + n)
\]

\[
= n + (\mu_{H_1}(\hat{z}_3) - \mu_{H_1}(\hat{z}_1) - \mu_{H_2}(\hat{z}_2)).
\]
Note that when \( \dim \mathcal{M}(H, J; \tilde{\mathbf{z}}) = 0 \), we have

\[
n = -\mu_{H_3}(\tilde{z}_3) + \mu_{H_1}(\tilde{z}_1) + \mu_{H_2}(\tilde{z}_2)
\]

which is equivalent to

\[
k_3 = k_1 + k_2 - n
\]

if we write

\[
k_i = \mu_H(\tilde{z}_i).
\]

This is exactly the grading of the Floer complex we adopted in [Oh5]. Now the pair-of-pants product \(*\) for the chains is defined by

\[
\tilde{z}_1 * \tilde{z}_2 = \sum_{\tilde{z}_3} \#(\mathcal{M}(H, J; \tilde{z})) \tilde{z}_3
\]

(4.12)

for the generators \( \tilde{z}_i \) and then by linearly extending over the chains in \( CF_*(H_1) \otimes CF_*(H_2) \). Our grading convention makes this product is of degree zero.

The following uniform energy bound will be used in our adiabatic convergence argument in \( \S 6 \).

**Proposition 4.3.** Let \( \tilde{z}_i = [z_i, w_i] \in \text{Crit} A_H \) and \( v : \Sigma \to M \) be a \( J \)-holomorphic section of \( P \to \Sigma \) with the asymptotic condition \( \tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) \). Then we have the following inequality

\[
\frac{1}{2} \int_{\Sigma} |(Dv)^v|^2 = \int v^* \omega_P - \sum_{i=1}^{2} \left( \int_0^1 \rho_+^i(\tau) H_i(t, u(\tau, t)) dt - \int_0^1 \rho_-^i(\tau) H_3(t, u(\tau, t)) \right)
\]

\[
= -A_{H_3}([z_3, w_3]) + A_{H_1}([z_1, w_1]) + A_{H_2}([z_2, w_2])
\]

\[
- \sum_{i=1}^{2} \left( \int_0^1 \rho_+^i(\tau) H_i(t, u(\tau, t)) dt - \int_0^1 \rho_-^i(\tau) H_3(t, u(\tau, t)) \right)
\]

\[
\leq -A_{H_3}([z_3, w_3]) + A_{H_1}([z_1, w_1]) + A_{H_2}([z_2, w_2])
\]

\[
+ \sum_{i=1}^{2} \int_0^1 \max H_{i,t} dt + \int_0^1 -\min H_{3,t} dt
\]

(4.13)

(4.14)

(4.15)

In particular, we have the uniform upper bound for the vertical energy of \( v \in \mathcal{M}(H, J; \tilde{z}) \) which is independent of the cut-off functions \( \rho^\pm \) or of the choice of \( 0 < R_1 < R_2 < \infty \).

**Proof.** This is an immediate consequence of summing over \( i = 1, 2, 3 \) of (3.22) in Proposition 3.4. The only matter to be clarified is the question on the orientation about the embedding

\[
\Sigma_i \hookrightarrow \Sigma.
\]

Recall that the embedding of the outgoing end

\[
\Sigma_i \cong (-\infty, 0] \times S^1 \hookrightarrow \Sigma \quad \text{for } i = 1, 2
\]

is orientation preserving, but the embedding

\[
\Sigma_3 \cong [0, \infty) \times S^1 \hookrightarrow \Sigma
\]
is orientation reversing. The latter is responsible for the negative sign in front of $\int \rho'_s(\tau) H_3(t, u(\tau, t)) \, dt$. The calculation leading to (3.22) was carried out for the outgoing end $(-\infty, 0) \times S^1$. Taking the orientation change on the incoming end into consideration and summing (3.22) over $i$, we get (4.13).

For the proof of (4.14), we just apply the identity from [§5,En]

$$\int_{\Sigma} v^* \omega_P = -A_{H_3}([z_3, w_3]) + A_{H_1}([z_1, w_1]) + A_{H_2}([z_2, w_2]).$$

(See also [Sc] but with different conventions). This finishes the proof. □

This general inequality can be also used the other way around. More precisely, the following lower bound will be a crucial ingredient for the limiting arguments in the next section for the pseudo-holomorphic sections with some asymptotic conditions which are allowed to vary inside given Floer cycles.

**Corollary 4.4.** Let $\tilde{z}_i = [z_i, w_i]$ for $i = 1, 2, 3$. Suppose $A_{H_3}([z_3, w_3]) \geq c$ for some constant $c$. Then we have the lower bound

$$A_{H_1}([z_1, w_1]) + A_{H_2}([z_2, w_2]) \geq c - (E^+(H_1) + E^+(H_2)) + E^-(H_3). \quad (4.16)$$

**§5. Construction of the $W^{1,2}$-sections: analysis of the thick part**

In this section, we will start with the proof of the main existence result. We will treat the case $F = H$ in §8 in a more careful way and improve the existence statement to prove [Theorem 3.11, Oh5] which was postponed to the present paper. We refer to [Oh4,5] for more explanation on the Novikov cycles $h_{H_1}(\alpha)$ or $h_{H_2}(\beta)$ here.

**Theorem 5.1.** Let $H, F$ be as above and $J_0$ be any compatible almost complex structure on $M$ and let $\delta, \delta_1 > 0$ be given. Then for any $J' \in j_{(\phi, J_0)}$ there exist some $[z, w] \in h_{H_1}(\alpha)$ and $[z', w'] \in h_{H_4}(\beta)$ with

$$A_{H_1}([z, w]) \leq \rho(H; 1) + \frac{\delta}{2} \quad (5.1)$$

$$A_{F'}([z', w']) \leq \rho(F'; 1) + \frac{\delta}{2} \quad (5.2)$$

for which the following alternative holds:

1. There exists a cusp solution $u : \mathbb{R} \times S^1 \to M$

   $$u = u^- \# u^+ := (u_1^- \# u_2^- \# \cdots u_{N_1}^-) \# (u_1^+ \# \cdots u_{N_2}^+)$$

   where each $u_i^-$ is a cusp-curve consisting of a finite number of sphere bubbles and at most one principal component that satisfies

   $$\frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_F(u)) = 0, \quad J_i^- = (\phi^*_{H_1})_* J_i'$$

   and similarly for $u_j^+$'s whose principal components satisfy

   $$\frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_H(u)) = 0, \quad J_i^+ = (\phi^*_{F'})_* J_i'$$

   where $\phi^*$ is the pull-back by the diffeomorphism $\phi : M \to M$. (Here $\phi^*$ is defined by the flow of $H$).
and $u_{N_1}^− # u_1^+$ satisfies
\[
\begin{cases}
\frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_{(F,H)}(u) \right) = 0 \\
u(\infty) = [\tilde{z}', \tilde{w}'], \quad u(-\infty) = [z, w]
\end{cases}
\tag{5.3}
\]
on $(\mathbb{R} \times S^1) \setminus \{ \tau = 0 \}$ which lies in $W^{1,2}$ and satisfies
\[
\int (w^-)^* \omega + \int (u_-)^* \omega + \int (u_+)^* \omega - \int (w^+)^* \omega = 0
\tag{5.4}
\]
where $u^\pm$ are the parts on $\tau < 0$ and $\tau > 0$ respectively, and $u$ is smooth near $(0, 0)$ and satisfies
\[
u_{N_2}^+(\infty) = [\tilde{z}', \tilde{w}'], \quad u_1^-(\infty) = [z, w],
\]
\[
u_{N_1}^+(0, 0) = u_1^+(0, 0) = q \in B(u)
\tag{5.5}
\]
(2) or there is some $J_t'$-holomorphic sphere $v : S^2 \to M$ for some $t \in [0, 1]$ that passes through the point $q$ and in particular
\[
\mathcal{A}_H([z, w]) + \mathcal{A}_{\tilde{J}}([z', w']) + \int_0^1 \max H_t \, dt + \int_0^1 - \min F_t \, dt \geq A_S(\phi, J_0; J') - \delta_1.
\tag{5.6}
\]

In this section and the next, we give the proof of Theorem 5.1. Before launching the proof of Theorem 5.1, we would like to give some heuristic explanation of our construction. Morally we would like to apply the pants product to the case $H_1 = H$, $H_2 = \tilde{F}$, $H_3 = 0$ (5.7) and study the moduli space
\[
\mathcal{M}(H, \tilde{J}; z)
\]
of pseudo-holomorphic sections of an appropriate Hamiltonian bundle $P \to \Sigma$, for the $\tilde{J}$ chosen as in section 4, that has the boundary condition
\[
u|_{\partial_1 \Sigma} = [z, w] \in h_{H_1}(\alpha), \quad \nu|_{\partial_2 \Sigma} = [z', w'] \in h_{H_2}(\beta)
\]
Here we denote $\tilde{z} = ([z, w], [z', w']; [q, \tilde{q}])$. Note that because $H_3 = 0$ in the monodromy condition (5.8) and the outgoing end with monodromy $H_2 = F$ is equivalent to the incoming end with monodromy $H_2 = F$, we can fill-up the hole $z_3 \in \Sigma$ and consider the cylinder with one outgoing and one incoming end with the monodromy $H$ and $F$ respectively. In other words our Hamiltonian bundle $P \to \Sigma$ becomes just the deformed mapping cylinder $E_f$ for
\[
f : \tau \to \phi_1^{\rho(\tau)}
\]
as defined in §2 where $\phi_1^s$ is the two parameter family generated by the Hamiltonians
\[
s \in [0, 1] \mapsto (1 - s)H + sF
\]
for the limiting cut-off function
\[
\rho = \begin{cases} 
0 & \text{for } \tau < 0 \\
1 & \text{for } \tau > 1.
\end{cases}
\]
This limiting moduli space is precisely the space of solutions of (5.3). This heuristic reasoning is the motivation to the approach that we take in the proof below.
Remark 5.2. We note that when $F = H$ and $\rho \equiv 1$ the Hamiltonian bundle $P$ is nothing but the mapping cylinder $P = E_\phi$ with $\phi = \phi_H^1$ and the the almost complex structure $\tilde{J}$ is nothing but the pushforward of the complex structure

$$\tilde{J}(\tau, t, x) = j \oplus J'(t, x)$$

on $\mathbb{R} \times \mathbb{R} \times M$ under the covering projection

$$\pi : \mathbb{R} \times \mathbb{R} \times M \to E_\phi = (\mathbb{R} \times \mathbb{R} \times M)/(\tau, t+1, \phi(x)) \sim (\tau, t, x).$$

Furthermore, the natural connection induced by the Hamiltonian $H : [0, 1] \times \mathbb{R} \to M$ is flat. This remark will be important in section 7 for the proof of Theorem II.

We now suppose that we are given any $J' \in j(\phi, J_0)$, and one fixed $J' \in j(\phi^{1-\epsilon}_k, J_0)$. Let $(\phi, J_0)$ be as before and $k$ be a generic Morse function. For the sake of simplicity, we will also assume that $k$ has the unique minimum point.

Instead of the triple (5.7) which is not allowed in defining the pants product in the Floer theory because $H_3 = 0$ is degenerate, we consider the admissible triple

$$H_1 = H, H_2 = \epsilon k \# F, H_3 = \epsilon k$$

for the nondegenerate $H$ and $F$. We note that if $F$ is nondegenerate and $\epsilon$ is sufficiently small, $(\epsilon k) \# F$ is also nondegenerate and there is a natural one-one correspondence between $\text{Per}(H)$ and $\text{Per}((\epsilon k) \# F)$ and the critical points $\text{Crit}A_F$ and $\text{Crit}A_{(\epsilon k) \# F}$. We will again denote by $[z', w']$ the critical point of $A_{H_2}$ corresponding to $[z', w'] \in \text{Crit}A_F$.

Then we consider the triple of the families of almost complex structures $J = (J_1, J_2, J_3)$ on the three ends $\Sigma_i \cong (-\infty, R] \times S^1$ under the trivialization

$$\Phi_i : P|_{\Sigma_i} \to \Sigma_i \times M$$

which are defined by

$$J_1^t = (\phi_H')^* J'$$

$$J_2^t = (\phi_{\epsilon k \# F}^1)^* (\phi^{1-t})^* J'(1-t)$$

$$J_3^t = (\phi_{\epsilon k}^1)^* J_{3, t}$$

respectively. Note that the family $(\phi_{\epsilon k}^{1-t})^* J'(1-t) \in j((\phi_{\epsilon k \# F}^1)^{-1}, J_0)$ and $J_2$ is still a $t$-periodic family. And as $\epsilon \to 0$, $J_2$ converges to the time reversal family of $t \mapsto (\phi_F^t)^* J'_t$

$$t \mapsto (\phi_F^{1-t})^* J'_{(1-t)}.$$

We also choose $J_{3, t}$ so that as $\epsilon \to 0$, the path converges to the constant path $J_0$ which is possible as $\phi_{\epsilon k}^1$ converges to the identity map.

With these choices, we study the limit as $\epsilon \to 0$ of the moduli space

$$\mathcal{M}(H', J; z')$$
with the asymptotic boundary condition (5.4) for each $\epsilon$. The main difficulty in studying this limit problem in general is that the limit is a singular limit and it is possible that the image of pseudo-holomorphic curves can collapse to an object of the Hausdorff dimension one in the limit (see [FOh1] for the study of this kind of the singular limit problem). Our first non-trivial task is to prove that there is a suitable limiting procedure for any sequence $\nu^\epsilon \in \mathcal{M}(H^\epsilon, \tilde{J}^\epsilon; \bar{z})$ as $\epsilon \to 0$. We will show that there exists a sub-sequence of $\nu^\epsilon \in \mathcal{M}(H^\epsilon, \tilde{J}^\epsilon; \bar{z})$ that converges to the union

$$u \cup \chi$$

in the Hausdorff topology, where $\chi$ is a negative (cusp) gradient trajectory of $-k$ landing at the critical point $q \in \text{Crit } f$ and $u$ is a solution of (6.20) but satisfying

$$u(0,0) \in \text{Im } \chi$$

instead of $u(0,0) = q$. In particular, we will prove that the piece of Hausdorff dimension one is always a (negative) gradient trajectory of $-k$.

This proven, it is easy to see that the above mentioned heuristic discussion cannot produce a solution required in Theorem 5.1 unless the negative gradient trajectory of $-k$ landing at $q$ is trivial for some reason. At this stage, the condition

$$\mu_{\text{ Morse }}(q) = 2n$$

will play an essential role and enable us to conclude that the only gradient trajectories of $-f$ landing at $q$ are the constant map $q$ and so we are in a good position to start with.

Let $P \to \Sigma$, $\omega_P$ and $\Omega_{P, \lambda}$ be as in §4. We equip $P$ with an $H$-compatible almost complex structure $\tilde{J}$ such that $\tilde{J} = J \oplus J'$ on each $\Sigma_i$ where

$$J'_1 \in j_{(\phi,J_0)}, \quad J'_2 = \tilde{J}'_1 \in j_{(\phi^{-1},J_0)}, \quad J'_3 \in j_{(\phi^1,J_0)}.$$  \hspace{1cm} (5.9)

More explicitly we define $\tilde{J}$ by

$$\tilde{J}(\tau,t,x)(\alpha, \beta, \xi) = (-\beta, \alpha, (\phi_t^{\rho(\tau)})^*J_t^\rho(\tau)(\xi - \beta X_{\rho(\tau)H_i}) + \alpha X_{\rho(\tau)H_i})$$  \hspace{1cm} (5.10)

on each $\Sigma_i$, where $s \in [0,1] \to J_s^\rho$ is a fixed path from $J_i^1 = \{ (\phi_t^{\rho})_s J_t^i \}_{t \in [0,1]}$ to $J_0$ for each $i = 1, 2, 3$. Note that $\tilde{J}$ is $H$-compatible and naturally extends to the whole $P$ because of the cut-off function $\rho$ and $J_t^\rho$ was chosen to be $J_t^0 = J_0$ and the connection is trivial on $\Sigma_{cn}$.

The $\tilde{J}$-holomorphic sections $v$ over $\Sigma_i$ are precisely the solutions of the equation

$$\frac{\partial u}{\partial \tau} + J_t^{\rho(\tau)}(\frac{\partial u}{\partial t} - X_{\rho(\tau)H_i}(u)) = 0$$  \hspace{1cm} (5.11)

if we write $v(\tau,t) = (\tau,t, u(\tau,t))$ in the trivialization with respect to the cylindrical coordinates $(\tau,t)$ on $\Sigma_i'$ induced by $\phi_t^1$ above. In the center region $\Sigma_{cn}$, they just become $J_0$-holomorphic curves

$$v : \Sigma_{cn} \to M$$
with respect to the given conformal structure \( j \) on \( \Sigma_{cn} \).

We denote \( h = [\phi, H] \) and \( f = [\phi, F] \) and let \( H_1 \) and \( H_2' \) be the paths from \( \epsilon k \) to \( H \) and \( (\epsilon k)\#F \) respectively and \( H_3 \) the constant path \( \epsilon k \). Then by the construction we have the identity

\[
\lambda_{H_1}(\alpha) * h_{H_2'}(\beta) = \gamma + \partial_{\epsilon k}(\eta)
\]

in the homology. In the level of cycles, for any cycles \( \alpha \in CF_*(H) \), \( \beta \in CF_*(\epsilon k)#F \) and \( \gamma \in CF_*(\epsilon k) \) with

\[
[\alpha] = 1^p, \quad [\beta] = 1^p, \quad [\gamma] = 1^p,
\]

we have

\[
h_{H_1}(\alpha) * h_{H_2'}(\beta) = \gamma + \partial_{\epsilon k}(\eta)
\]

for some \( \eta \in CF_*(\epsilon k) \). We recall the following Non-pushing down lemma

**Lemma 5.3 [Lemma 6.8, Oh3].** Consider the cohomology class \( 1 \in QH^*(M) = HQ^*(-\epsilon k) \). Then there is the unique Novikov cycle \( \gamma \) of the form

\[
\gamma = \sum_j c_j[x_j, \hat{x}_j]
\]

with \( x_j \in \text{Crit}(-\epsilon k) \) that represents the class \( 1^p = [M] \). Furthermore for any Novikov cycle \( \beta \) with \( [\beta] = [M] \), i.e., \( \beta \) with

\[
\beta = \gamma + \partial_{\epsilon k}(\delta)
\]

with \( \delta \) a Novikov chain, we have

\[
\lambda_{\epsilon k}(\beta) \geq \lambda_{\epsilon k}(\gamma).
\]

It follows from Lemma 5.3 and (5.12) that

\[
\lambda_{\epsilon k}(h_{H_1}(\alpha) * h_{H_2'}(\beta)) \geq -\frac{\delta}{3}
\]

with \( \delta \) independent of \( \epsilon \) by choosing \( \epsilon \) sufficiently small. By the definition of \( \rho(\cdot; 1) \), we can find \( \alpha \) and \( \beta \) such that

\[
\rho(H; 1) - \frac{\delta}{2} \leq \lambda_{H}(h_{H_1}(\alpha)) \leq \rho(H; 1) + \frac{\delta}{2}
\]

\[
\rho(\epsilon k#F; 1) - \frac{\delta}{2} \leq \lambda_{\epsilon k#F}(h_{H_2'}(\beta)) \leq \rho(\epsilon k#F; 1) + \frac{\delta}{2}.
\]

Furthermore by the continuity of \( \rho(\cdot; 1) \), we also have

\[
\lim_{\epsilon \to 0} \rho(\epsilon k#F; 1) = \rho(F; 1).
\]

Now we need to compare the levels of \( \lambda_{H}(h_{H_1}(\alpha)) \), \( \lambda_{\epsilon k#F}(h_{H_2'}(\beta)) \) and \( \lambda_{\epsilon k}(h_{H_1}(\alpha) * h_{H_2'}(\beta)) \).
For each given $\epsilon > 0$, let $[z_\epsilon, w_\epsilon] \in h_{H_1}(\alpha)$ and $[z', w'] \in h_{H_2}(\beta)$ and $[q, \tilde{q}\# A] \in \gamma + \partial_k(\eta)$ such that the moduli space $\mathcal{M}(H^\epsilon, J^\epsilon; \bar{z}^\epsilon)$ is non-empty. We may choose $[q, \tilde{q}\# A] \in \gamma + \partial_k(\eta)$ so that
\[
\mathcal{A}_{ek}([q, \tilde{q}\# A]) \geq -\frac{\delta}{2}
\] (5.18)
using (5.15). Furthermore since $[h_{H_1}(\alpha) * h_{H_2}(\beta)] = 1$, and the unique maximum point of $-\epsilon k$ is homologically essential, we may also assume that $q$ is the unique maximum point and $A = 0$. For each given $\epsilon > 0$, let $v^\epsilon$ be any $\tilde{J}$-holomorphic section of $P \to \Sigma$ that has the asymptotic boundary condition
\[
v|_{\partial_1 \Sigma} = [z_\epsilon, w_\epsilon] \in h_{H_1}(\alpha), \quad v|_{\partial_2 \Sigma} = [z'_\epsilon, w'_\epsilon] \in h_{H_2}(\beta) \quad \text{and} \quad v|_{\partial_3 \Sigma} = [q, \tilde{q}] \in \gamma.
\] (5.19)
Then it follows from (4.16) of Corollary 4.4 that we have
\[
\mathcal{A}_H([z_\epsilon, w_\epsilon]) + \mathcal{A}_{\epsilon k \# F}([z'_\epsilon, w'_\epsilon]) \geq -\frac{\delta}{2} - (E^+(H) + E^+(\epsilon k \# F)) + E^-(\epsilon k).
\]
By choosing $\epsilon > 0$ sufficiently small, we may assume that
\[
\mathcal{A}_H([z_\epsilon, w_\epsilon]) + \mathcal{A}_{\epsilon k \# F}([z'_\epsilon, w'_\epsilon]) \geq -(E^+(H) + E^+(\epsilon k \# F)) - \delta_2 := C(H, \tilde{F})
\] (5.20)
where $\delta_2$ can be made arbitrarily small by letting $\epsilon \to 0$. In particular, combining (5.16) and (5.20), we derive the bounds
\[
C(H, \tilde{F}) - \rho(H; 1) \leq \mathcal{A}_H([z_\epsilon, w_\epsilon]) \leq \rho(H; 1) + \frac{\delta}{2}
\]
\[
C(H, \tilde{F}) - \rho(\epsilon k \# F; 1) \leq \mathcal{A}_{\epsilon k \# F}([z'_\epsilon, w'_\epsilon]) \leq \rho(\epsilon k \# F; 1) + \frac{\delta}{2}
\] (5.21)
for all $\epsilon > 0$. Therefore for each given $\epsilon > 0$, the lower bound of (5.21) implies that there are only finitely many possible asymptotic periodic orbits among the generators $[z_\epsilon, w_\epsilon] \in h_{H_1}(\alpha)$ and $[z'_\epsilon, w'_\epsilon] \in h_{H_2}(\beta)$ respectively such that the corresponding moduli space
\[
\mathcal{M}(H^\epsilon, \tilde{J}; \bar{z}^\epsilon)
\]
becomes non-empty for the asymptotic condition (5.19). On the other hand, since we assume that $F$ is generic nondegenerate $\epsilon k \# F$ are nondegenerate for all sufficiently small $\epsilon > 0$ and hence there are canonical one-one correspondences between $\text{Crit}_A F$ and $\text{Crit}_{\epsilon k \# F}$. Therefore as we let $\epsilon \to 0$, we may assume that the asymptotic orbits $[z'_\epsilon, w'_\epsilon] \in \text{Crit}_{\epsilon k \# F}$ converges to $[z', w'] \in \text{Crit}_A F$ and $[z', w'] \in \text{Crit}_A H$ to $[z, w] \in \text{Crit}_A H$ in $C^\infty$-topology. By taking the limit of (5.21) as $\epsilon \to 0$, we obtain (5.1) in Theorem 5.1.

Having this convergence statement made for the asymptotic orbits, for the simplicity of notations, we will omit the subscript $\epsilon$ from $[z_\epsilon, w_\epsilon]$ and $[z'_\epsilon, w'_\epsilon]$ and just denote them by $[z, w]$ and $[z', w']$ from now on. We also note that because the degree of 1 is zero, we have
\[
\mu_H([z, w]) = \mu_{\epsilon k \# F}([z', w']) = \mu_{\epsilon k}([q, \tilde{q}]) = n
\]
and so
\[ \mu_{\text{Morse}}(q) = n + \mu_k([q, \hat{q}]) = 2n. \]
In particular, any gradient trajectory \( \chi : (K, \infty) \to M \) of \(-\epsilon k\) satisfying
\[ \dot{\chi} - \epsilon \text{grad} k(\chi) = 0, \quad \lim_{\tau \to \infty} \chi(\tau) = q \]
must be the constant map \( \chi \equiv q \), which is exactly what we wanted to have in our heuristic discussion in the beginning of this section.

We recall the uniform energy bound in Proposition 3.3 for the vertical energy of the section \( v \). Since this bound is uniform for all sufficiently small \( \epsilon \) and \( R_{i,2} > R_{i,1} \), we can now carry out the adiabatic convergence argument for \( \Sigma_3 \). To carry out this adiabatic convergence argument, we will conformally change the metric on the base \( \Sigma \) of the fibration. We note that the vertical energy of the section \( v \) is invariant under the conformal change of the base metric. We will realize this conformal change by a conformal diffeomorphism
\[ \psi_\epsilon : C \setminus \{(0,0)\} \to (\Sigma, g_\epsilon) \]
where \( C \) is the standard flat cylinder \( \mathbb{R} \times S^1 \) and the metric \( g_\epsilon \) is constructed in a way similar to the minimal area metric, but we will change the lengths of \( \theta_i \)'s in (4.4) in the following way:
\begin{align*}
\text{length } \theta_1 &= 1 - \epsilon/2 \\
\text{length } \theta_2 &= \text{length } \theta_3 = \epsilon/2.
\end{align*}
(5.22)
It is easy to see that we can choose the conformal diffeomorphism \( \psi_\epsilon \) so that it restricts to a quasi-isometry
\[ \psi_\epsilon : C \setminus D(\delta(\epsilon)) \to (\Sigma_1 \cup \Sigma_2, g_\epsilon) \]
and to a conformal diffeomorphism
\[ \psi_\epsilon : D(\delta(\epsilon)) \to (\Sigma_3, g_\epsilon) \]
where \( D(\delta) \) is the disc around \((0,0)\) and \( \delta(\epsilon) \to 0 \) as \( \epsilon \to 0 \). We choose any sequence \( \epsilon_j \to 0 \) and \( v_j \in \mathcal{M}(H^{\epsilon_j}, \hat{J}; \hat{z}^{\epsilon_j}) \) that satisfy (5.19). Consider the compositions
\[ \tilde{v}_j = v_j \circ \psi_\epsilon ; C \setminus \{(0,0)\} \to P. \]
(5.23)
Since \( \psi_\epsilon \) is conformal, we have
\[ \int |(D\tilde{v}_j)^\nu|_j^2 = \int |(Dv_j)^\nu|_j^2 \]
and hence the uniform energy bound
\begin{align*}
\frac{1}{2} \int |(D\tilde{v}_j)^\nu|_j^2 \leq -A_{\nu k}([q, \hat{q}]) + A_{\nu k\#F}([z', w']) + A_H([z, w]) \\
&\quad + \left( \int_0^1 \max H \ dt + \int_0^1 \max \hat{k\#F} \ dt + \int_0^1 \min \epsilon k \ dt \right) \\
&\leq \rho(H; 1) + \frac{\delta}{2} + (\rho(\epsilon k\#F) + \frac{\delta}{2} + \frac{\delta}{2} \\
&\quad + \left( \int_0^1 \max H \ dt + \int_0^1 \max \hat{k\#F} \ dt + \int_0^1 \min \epsilon k \ dt \right) \\
&\leq \rho(H; 1) + \rho(\hat{F}; 1) + \int_0^1 \max H \ dt + \int_0^1 \max \hat{F} \ dt + 2\delta
\end{align*}
(5.25)
provided \( \epsilon \) is sufficiently small. Here we used (4.15) for the first inequality, (5.16)-(5.18) for the second and the third inequalities. There are two alternatives to consider:

1. there exists \( c > 0 \) such that

\[
\frac{1}{2} J^2(\delta(\epsilon_j)) |(Dv_j)''|_J^2 \geq c > 0
\]  

for all sufficiently large \( j \) or

2. we have

\[
\limsup_{j \to \infty} \frac{1}{2} J^2(\delta(\epsilon_j)) |(Dv_j)''|_J^2 = 0.
\]  

For the first alternative, \( \widetilde{v}_j \) bubbles-off as \( \epsilon \to 0 \). Since the bubble must be contained in a fiber by the maximum principle by the \( H \)-compatibility of \( \tilde{J} \), it is \( J''_t \)-holomorphic for some \( t \in [0, 1] \), we indeed have

\[
\limsup_{j \to \infty} \frac{1}{2} \int \Sigma_3 |(Dv_j)''|_J^2 \leq \delta_j \to 0
\]  

as \( j \to \infty \). On \([1, \infty) \times S^1\), (6.1-3) is equivalent to

\[
\frac{\partial u}{\partial \tau} + J''_t \left( \frac{\partial u}{\partial t} - X_{\epsilon f} \right) = 0
\]  

for the section \( v(\tau, t) = (\tau, t, u(\tau, t)) \) where \( J''_t = (\phi''_{\epsilon f})_t, J''_t \) with \( J''_t \in j(\phi''_{\epsilon f}, J_0). \)

We recall that \( J''_t \to J_0 \) as \( \epsilon \to 0 \). The energy bound (6.3) and the \( \epsilon \)-regularity theorem (see Corollary 3.4, Oh1) for the context of pseudo-holomorphic curves, we immediately derive the following uniform \( C^1 \)-estimate.
Lemma 6.1. Let $\delta_j > 0$ as in (6.1). Then there exists $j_0$ such that for any $j \geq j_0$ and $u$, we have
\[
|Du_j(\tau, t)| \leq C\delta_j
\]
on $\Sigma_3 = \mathbb{R}_+ \times S^1$ with $C > 0$ independent of $j$. In particular, we have
\[
\text{length}(t \mapsto u_j(\tau, t)) \leq C\delta_j. \quad (6.5)
\]
for any $\tau \in [0, \infty)$.

We reparameterize $u_j$ to define
\[
\pi_j(\tau, t) = u_j(\frac{\tau}{\epsilon_j}, \frac{t}{\epsilon_j})
\]
on $(\epsilon_j, \infty) \times \mathbb{R}/\epsilon_j\mathbb{Z}$. Then $\pi_j$ satisfies
\[
\frac{\partial \pi_j}{\partial \tau}(\tau, t) + J_t^\tau \frac{\partial \pi_j}{\partial t}(\tau, t) - \text{grad}_{g_{j_0}} f(\pi_j)(\tau, t) = 0. \quad (6.6)
\]
We will just denote grad for $\text{grad}_{g_{j_0}}$ from now on. We choose the ‘center of mass’ for each circle $t \mapsto u_j(\tau, t)$ which we denote by $\chi_j(\tau)$. More precisely, $\chi_j(\tau)$ is defined by the following standard lemma (see [K] for its proof).

Lemma 6.2. Suppose a circle $z : S^1 \to M$ has diameter less than $\delta$ with $\delta = \delta(M)$ sufficiently small and depending only on $M$. Then there exists a unique point $x_z$, which we call the center of mass of $z$, such that
\[
z(t) = \exp_{x_z} \xi(t), \quad \int_{S^1} \xi(t) \, dt = 0. \quad (6.7)
\]
Furthermore $x_z$ depends smoothly on $z$ but does not depend on its parameterization.

If $j_0$ is sufficiently large, (6.4) implies that the diameter of all the circles $t \mapsto u_j(\tau, t)$ is less than $\delta$ given in Lemma 6.2 for any $j$ and $\tau \in (0, \infty)$. Therefore we can write $\pi_j(\tau, t)$ as
\[
\pi_j(\tau, t) = \exp_{\chi_j(\tau)}(\bar{\xi}_j(\tau, t))
\]
with $\bar{\xi}_j(\tau, t) \in T_{\chi_j(\tau)}M$ for any $\tau \in [1, \infty)$ for all $j \geq j_0$, $j_0$ sufficiently large. Furthermore the $C^1$ estimate in Lemma 6.1 proves
\[
|\bar{\xi}(\tau, t)| \leq C\delta_j.
\]
We consider the exponential map
\[
\exp: \mathcal{U} \subset TM \to M; \quad \exp(x, \xi) := \exp_x(\xi)
\]
and denote
\[
D_1 \exp(x, \xi) : T_xM \to T_xM \quad \text{the (covariant) partial derivative with respect to $x$ and}
\]
\[
d_2 \exp(x, \xi) : T_xM \to T_xM
\]
the usual derivative \( d_2 \exp(x, \xi) := T_\xi \exp_x : T_x M \to T_x M \). We recall the property
\[
D_1 \exp(x, 0) = d_2 \exp(x, 0) = id
\]
which is easy to check. We now compute
\[
\frac{\partial \eta}{\partial \tau} = d_2 \exp(\chi(\tau), \xi(\tau, t)) \left( \frac{\partial \xi}{\partial \tau}(\tau, t) \right) + D_1 \exp(\chi(\tau), \xi(\tau, t))(\dot{\chi}(\tau))
\]
\[
\frac{\partial \eta}{\partial t} = d_2 \exp(\chi(\tau), \xi(\tau, t)) \left( \frac{\partial \xi}{\partial t}(\tau, t) \right)
\]
\[
\text{grad} f(\eta) = \text{grad} f(\exp(\chi(\tau), \xi(\tau, t))).
\]
Substituting these into (6.5) and multiplying by \( (d_2 \exp(\chi(\tau), \xi(\tau, t)))^{-1} \), we get
\[
\frac{D\xi_j}{D\tau}(\tau, t) + (d_2 \exp(\chi(\tau), \xi_j(\tau, t)))^{-1}(D_1 \exp(\chi(\tau), \xi_j(\tau, t))(\dot{\chi}_j(\tau))
\]
\[
+ (\exp(\chi(\tau)))^* J^*_t(\chi(\tau)) \left( \frac{\partial \xi_j}{\partial t}(\tau, t) \right) - (\exp(\chi(\tau)))^* (\text{grad} f)(\xi_j(\tau, t)) = 0
\]
on \( T_{\chi(\tau)}M \). Using the center of mass condition (6.7) and integrating over \( t \in \mathbb{R}/\epsilon_j \mathbb{Z} \), we obtain
\[
\int_{0}^{1} (d_2 \exp(\chi(\tau), \xi_j(\tau, t)))^{-1}(D_1 \exp(\chi(\tau), \xi_j(\tau, t))(\dot{\chi}_j(\tau)) dt
\]
\[
- \int_{0}^{1} (\exp(\chi(\tau)))^* (\text{grad} f)(\xi_j(\tau, t)) dt = 0.
\]
Here we used the identities
\[
\int_{S^1} \frac{\partial \xi_j}{\partial t}(\tau, t) dt = 0 \quad (6.9)
\]
\[
\int_{S^1} \xi_j(\tau, t) dt = 0 \equiv \int_{S^1} \frac{D\xi_j}{D\tau}(\tau, t) dt \quad (6.10)
\]
where the second identity of (6.10) is a consequence of the first. On the other hand, it follows that there exists sufficiently small \( \delta_j > 0 \) such that
\[
(1 - C\delta_j)|\dot{\chi}(\tau)| \leq |d_2 \exp(\chi(\tau), \xi(\tau, t)))^{-1}(D_1 \exp(\chi(\tau), \xi(\tau, t))(\dot{\chi}(\tau))| \leq (1 + C\delta_j)|\dot{\chi}(\tau)|
\]
and
\[
|(\exp(\chi(\tau)))^* (\text{grad} f)(\xi(\tau, t))| \leq (1 + C\delta_j) \| \text{grad} f(\xi(\tau, t)) \| \leq (1 + C\delta_j) \| \text{grad} f \|_{C^0}.
\]
These follow from (6.7) and the following standard estimates on the exponential map
\[
|d_2 \exp(x, \xi)(u)| \leq C |\xi||u|
\]
\[
|D_1 \exp(x, \xi)(u)| \leq C |\xi||u|
\]
for \( \xi, u \in T_xM \) where \( C \) is independent of \( \xi \), as long as \(|\xi|\) is sufficiently small, say smaller than injectivity radius of the metric on \( M \) (see \([K]\)).

Combining (6.8), (6.11) and (6.12), we have obtained

\[
|\dot{\chi}_j(\tau)| \leq C \|\text{grad } f\|_{C^0}.
\]

Therefore \( \chi_j \) is equi-continuous and so on any given fixed interval \([1, R] \subset [1, \infty)\), there exists a subsequence of \( \chi_j : [1, R] \rightarrow M \) uniformly converges to some \( \chi_\infty : [1, R] \rightarrow M \). Furthermore it easily follows from (6.12) and smoothness of the exponential map, we also have the estimates

\[
|d_exp(\chi_j(\tau), \vec{\xi}_j(\tau, t))^{-1}(D_1 exp(\chi_j(\tau), \vec{\xi}_j(\tau, t))(\dot{\chi}_j(\tau))| - \dot{\chi}_j(\tau)| \leq C|\vec{\xi}_j(\tau, t)||\dot{\chi}_j(\tau)|
\]  

\[\text{(6.13)}\]

and

\[
|exp^*_{\chi_j(\tau)}(\text{grad } f)(\vec{\xi}_j(\tau, t)) - \text{grad } f(\chi_j(\tau))| \leq C|\vec{\xi}_j(\tau, t)|.
\]  

\[\text{(6.14)}\]

where the constant \( C \) depends only on \( M \). Since \( \max |\xi_j|_{C^0} \leq C \delta_j \rightarrow 0 \) uniformly, the equation (6.8) converges uniformly to

\[
\dot{\chi}_\infty - \text{grad } f(\chi_\infty) = 0.
\]

Therefore \( \chi_\infty \) is a gradient trajectory of \( f \). Recalling the \( C^1 \)-estimate in Lemma 6.1 and

\[\pi_j(\tau, t) = \exp_{\chi_j(\tau)}(\vec{\xi}(\tau, t))\]

we have proven that \( \pi_j|_{[0, R] \times R/\epsilon, Z} \) uniformly converges to \( \chi_\infty \). By a boot-strap argument by differentiating (6.8), this convergence can be turned into a \( C^\infty \)-convergence.

Therefore by a standard argument of local convergence as in \([Fl1]\), the sequence \( u_j \) converges to a \textit{connected} finite union of gradient trajectories of \( -f \)

\[
\chi_0 \# \chi_1 \# \cdots \# \chi_N
\]

for some \( N \in \mathbb{Z}_+ \) such that

\[\chi_N(\infty) = q.\]  

\[\text{(6.15)}\]

However since \( \mu_{\text{Morse}}^M(q) = 2n \), i.e, \( q \) is a (local) maximum of \(-f\), \( \chi_N \) must be constant which in turn implies that all \( \chi_j \) must be constant map \( q \) for all \( j \). Recalling

\[u_j(\tau, t) = \pi_j(\epsilon_j \tau, \epsilon_j t),\]

we have proven that \( u_j|_{\Sigma_3} \) converges to the constant map \( q \) in the \( C^\infty \)-topology.

Now we analyze the sequence \( v_j \) over \( \Sigma_1 \cup \Sigma_2 \). Consider a sequence \( \epsilon_j \rightarrow 0 \) and \( R_{3,2} = R_j \rightarrow 0 \). Obviously \( R_{3,1} \) also goes to zero since \( R_{3,2} > R_{3,1} \). As we mentioned before the lower bound (5.21) implies by the definition of the Novikov chains that there are only finitely many elements \([z, w] \in h_{\mathcal{H}_1}(\alpha)\) and \([z', w'] \in h_{\mathcal{H}_2}(\beta)\) above the lower bound. Therefore we can now apply the Gromov-Floer type compactness arguments as \( \epsilon \rightarrow 0 \).
When \( v_j \mid_{\Sigma_1 \cup \Sigma_2} \) bubble-off, we are again in the second alternative in Theorem 5.1. Therefore we assume that there exists a constant \( C > 0 \) such that
\[
|Dv_j|_{\Sigma_1 \cup \Sigma_2} \leq C
\]  
(6.16)
by choosing a subsequence if necessary. Using this \( C^1 \) bound together with the matching condition (6.2), we can bootstrap to extract a (cusp)-limit \( v_\infty \) on \( \Sigma = \Sigma_1 \cup \Sigma_2 \) such that
\[
v_\infty = (v_1^- \# v_2^- \# \cdots \# v_N^-) \# (v_1^+ \# \cdots \# v_L^+)
\]
where \( v_i^- \) for \( 1 \leq i \leq N - 1 \) and \( v_j^+ \) for \( 2 \leq i \leq L \) are maps from \( \mathbb{R} \times [0, 1] \) to \( M \) but \( v_N^- \) one from \( (-\infty, 0] \times [0, 1] \) and \( v_1^+ \) from \([0, \infty) \times [0, 1] \). All of them satisfy
\[
\begin{cases}
\partial_{\tau} v + J_t^2 \partial_{\tau} v = 0 \\
\phi(v(\tau, 1)) = v(\tau, 0), \\
\int |\partial_{\tau} v|_J^2 < \infty
\end{cases}
\]  
(6.17)
In particular, the join \( v_N^- \# v_1^+ \) also satisfies (6.17) on \( \mathbb{R} \times [0, 1] \) with possible singularities along \( \{\tau = 0\} \). Here we would like to point out that by the definition of the data \( (P^e, \tilde{J}, H^e) \) and by the choice of minimal area metrics \( g_e \) in (5.22), the bundle data \( (P^e, \tilde{J}) \) on \( \mathbb{R} \times S^1 \setminus \{(0, 0)\} \) smoothly converges to \( (P, \tilde{J}) \) on compact sets of \( \mathbb{R} \times S^1 \setminus \{(0, 0)\} \). Note that the latter datum is smooth (see Remark 5.2). Now the matching condition (6.2) and the way how \( v_\infty \) arises, especially the uniform \( C^1 \)-bound (6.16) implies that \( v_N^- \# v_1^+ \) becomes indeed smooth even across \( \tau = 0 \) by the elliptic regularity. And then since \( v_N^- \# v_1^+ \) has finite energy on \( \mathbb{R} \times \{(0, 1)\} \) (as a \( J \)-holomorphic section of \( P \)), it can be smoothly extended across \( (0, 0) \) by the removal singularity theorem. Hence we have produced a smooth cusp-solution of the equation (6.17).

Now we consider \( u_\infty : \mathbb{R} \times S^1 \to M \) defined by
\[
u_\infty(\tau, t) = \begin{cases} (\phi_H^t)(v_\infty(\tau, t)) & \text{for } \tau < 0 \\
(\phi_F^t)(v_\infty(\tau, t)) & \text{for } \tau > 0
\end{cases}
\]  
(6.18)
The basic topological hypothesis
\[
[u \cup (\bigcup_{i=1}^{3} w_i)] = 0 \quad \text{in } \pi_2(M)
\]
in the definition of the pants product (see Definition 4.2) gives rise to (5.4). Furthermore since \( X_{(H,F)} \) smoothly matches near \((0, 0)\) by the basic assumption (2.1) and \( u_\infty \) lies in \( W^{1,2} \), \( u_\infty^- \) and \( u_\infty^+ \) smoothly matches around \((0, 0)\) and \( u(0, 0) = q \) Furthermore
\[
v(0, 0) = u(0, 0) = q \in B(u)
\]
while \( v(\pm \infty) \in \text{Fix } \phi \) and hence \( v \) cannot be constant. This takes care of the alternative (1) in Proposition 5.2. This finally finishes the proof of Proposition 5.2 and so the proof of Theorem 5.1.
§7. The case with $H_3 = H_1 \# H_2$

In this section, we specialize to the case $F = H$ in §6 and §6. In this case, we can improve the arguments therein to prove the following theorem, which is Theorem 3.11 in [Oh5] whose proof was postponed to the present paper.

**Theorem 7.1.** Let $H$ and $J_0$ be as before. And let $q \in \text{Int} B(u)$ and $\delta > 0$ be given. Then for any $J' \in j(\phi, J_0)$, there exist some generators $[z, w] \in h_H(\alpha)$ and $[z', w'] \in h_H(\beta)$ with

$$A_H([z, w]) \leq \rho(H; 1) + \frac{\delta}{2}$$

$$A_H([z', w']) \leq \rho(\tilde{H}; 1) + \frac{\delta}{2}$$

such that the following alternative holds:

1. The equation

$$\begin{cases}
\frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_H(u) \right) = 0 \\
u(\infty) = [z', \tilde{w'}], \quad u(-\infty) = [z, w]
\end{cases}$$

has a cusp-solution

$$u_1 \# u_2 \# \cdots \# u_N$$

which is a connected union of Floer trajectories, possibly with a finite number of sphere bubbles, for $H$ that satisfies the conditions

$$u_N(\infty) = [z', \tilde{w'}], \quad u_1(-\infty) = [z, w], \quad u_j(0, 0) = q \in B(u).$$

for some $1 \leq j \leq N$,

2. or there is some $J'_t$-holomorphic sphere $v : S^2 \to M$ for some $t \in [0, 1]$ that passes through the point $q$.

This in particular implies

$$0 < A(\phi, J_0) \leq A(\phi) \leq \gamma(\phi) < \infty$$

for any $\phi$ and $J_0$.

The proof of this theorem will be identical except that we exploit the fact that in the case where $F = H$, we can actually construct a ‘flat’ connection on $P \to \Sigma$ by modifying the two parameter family

$$\phi : [0, 1] \times [0, 1] \to \text{Ham}(M, \omega)$$

used in the construction of §3. We now explain this modification here.

We start with $i = 2$ where $H_2 = e \# H$. We will interpolate the Hamiltonian isotopy

$$\phi_2(0, t) = \phi_{e \# H}^t$$

and

$$\phi_2(1, t) = \begin{cases}
\phi_{H}^{2t} & \text{for } 0 \leq t \leq \frac{1}{2} \\
\phi_{H}^{1} \cdot \phi_{H}^{2t-1} & \text{for } \frac{1}{2} \leq t \leq 1
\end{cases}$$
by a family \( \phi_s \) satisfying \( \phi_s^1 = \phi_s^{-1} \cdot \phi_{\epsilon k}^{-1} \) for all \( s \in [0, 1] \). The corresponding family is nothing but

\[
\phi_2(s, t) = \begin{cases} 
\phi_{\epsilon k}^{2t} & \text{for } 0 \leq t \leq \frac{s}{2} \\
\phi_{\epsilon k}^1 \cdot \phi_{\epsilon k}^{2t-s} & \text{for } \frac{s}{2} \leq t \leq 1
\end{cases}
\] (7.4)

For the case \( i = 1 \), we consider the family

\[
\phi_1(s, t) = \begin{cases} 
\phi_{\epsilon k}^{2t} & \text{for } 0 \leq t \leq \frac{s}{2} \\
\phi_{\epsilon k}^1 & \text{for } \frac{s}{2} \leq t \leq 1
\end{cases}
\] (7.5)

For the case \( i = 3 \), we define the family

\[
\phi_3(s, t) = \begin{cases} 
\phi_{\epsilon k}^{2t} & \text{for } 0 \leq t \leq \frac{s}{2} \\
\phi_{\epsilon k}^1 \cdot \phi_{\epsilon k}^{2t-s} & \text{for } \frac{s}{2} \leq t \leq 1
\end{cases}
\] (7.6)

Note that this family smoothly matches under the identification of (4.4) except at the two points \( p, \overline{p} \) where it is continuous. These family define a flat connection on the Hamiltonian fibration \( P \to \Sigma \) which is smooth away from \( p, \overline{p} \). For this family, it is straightforward to compute the identity

\[
\frac{1}{2} \int_{\Sigma} |(Dv)^n|^2 = -A_{\epsilon k}([q, \overline{q}]) + A_H([z, w]) + A_{\epsilon k \# H}([z', w']).
\] (7.7)

Once we have this, a simple examination of the proof in §6, 7 gives rise to the proof of Theorem 7.1.

**Remark 7.2.** In fact, the above construction of the flat connection works for any triple \( H = (H_1, H_2, H_3) \) satisfying

\[
H_3 = H_1 \# H_2.
\] (7.8)

This fact was used by Schwarz [Sc] in his proof of the triangle inequality. Since the details are not given in [Sc], we provide the detail of this construction here. For each \( i \), we consider the family

\[
\phi_1(s, t) = \begin{cases} 
\phi_{H_i}^{2t} & \text{for } 0 \leq t \leq \frac{s}{2} \\
\phi_{H_i}^1 & \text{for } \frac{s}{2} \leq t \leq 1
\end{cases}
\] (7.9)

\[
\phi_2(s, t) = \begin{cases} 
\phi_{H_i}^1 & \text{for } 0 \leq t \leq \frac{s}{2} \\
(\phi_{H_i}^1) \cdot \phi_{H_2}^{2t-s} & \text{for } \frac{s}{2} \leq t \leq 1
\end{cases}
\] (7.10)

\[
\phi_3(s, t) = \begin{cases} 
\phi_{H_i}^{2t} & \text{for } 0 \leq t \leq \frac{s}{2} \\
\phi_{H_i}^1 \cdot \phi_{H_2}^{2t-s} & \text{for } \frac{s}{2} \leq t \leq 1
\end{cases}
\] (7.11)

on \( \Sigma_i \) for \( i = 1, 2, 3 \) respectively. It follows that this smoothly matches under the identification of (4.4) away from \( p, \overline{p} \).
References

[En] Entov, M., K-area, Hofer metric and geometry of conjugacy classes in Lie groups, Invent. Math. 146 (2001), 93-141.

[FOh1] Fukaya, K., Oh, Y.-G., Zero-loop open strings in the cotangent bundle and Morse homotopy, Asian J. Math. 1 (1997), 96-180.

[FOh2] Fukaya, K., Oh, Y.-G., in preparation.

[GLS] Guillemin, V., Lerman, E., Sternberg, S., Symplectic Fibrations and Multiplicity Diagrams, Cambridge University Press, 1996.

[K] Karcher, H., Riemannian center of mass and mollifier smoothing, Comm. Pure Appl. Math. 30 (1977), 509-541.

[KO] Kasturirangan, R., Oh, Y.-G., Floer homology for open subsets and a relative version of Arnold’s conjecture, Math. Z. 236 (2001), 151-189.

[Mc] McDuff, D., Geometric variants of the Hofer norm, preprint, 2001.

[Oh1] Oh, Y.-G., Removal of boundary singularities of pseudo-holomorphic curves with Lagrangian boundary conditions, Comm. Pure Appl. Math. 45 (1992), 121-139.

[Oh2] Oh, Y.-G., Gromov-Floer theory and disjunction energy of compact Lagrangian embeddings, Math. Res. Lett. 4 (1997), 895-905.

[Oh3] Oh, Y.-G., Chain level Floer theory and Hofer’s geometry of the Hamiltonian diffeomorphism group, Asian J. Math. 6 (2002), 579-624, math.SG/0104243.

[Oh4] Oh, Y.-G., Construction of spectral invariants of Hamiltonian diffeomorphisms, submitted.

[Oh5] Oh, Y.-G., Spectral invariants and geometry of the Hamiltonian diffeomorphism group, submitted.

[Po] Polterovich, L., The Geometry of the Group of Symplectic Diffeomorphisms, Birkhäuser, 2001.

[SU] Sacks, J., Uhlenbeck, K., The existence of minimal immersions of 2 spheres, Ann. Math. 113 (1981), 1-24.

[Sc] Schwarz, M., On the action spectrum for closed symplectically aspherical manifolds, Pacific J. Math. 193 (2000), 419-461.

[Z] Zwiebach, B., Closed string field theory: quantum action and the B-V master equation, Nucl. Phys. B 390 (1993), 33.