HYPERBOLIC RELAXATION OF THE 2D NAVIER-STOKES EQUATIONS IN A BOUNDED DOMAIN

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Abstract. A hyperbolic relaxation of the classical Navier-Stokes problem in 2D bounded domain with Dirichlet boundary conditions is considered. It is proved that this relaxed problem possesses a global strong solution if the relaxation parameter is small and the appropriate norm of the initial data is not very large. Moreover, the dissipativity of such solutions is established and the singular limit as the relaxation parameter tends to zero is studied.

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2000 Mathematics Subject Classification. 35B40, 35B45.

Key words and phrases. Navier-Stokes equations, hyperbolic relaxations, singular perturbations, attractors.

A.I. and Yu. G. acknowledge financial support from the Russian Science Foundation (grant no. 14-21-00025) and S.Z.’s research is supported by the Russian Science Foundation (grant no. 14-41-00044) and the RFBR grant 15-01-03587. The authors would also like to thank Varga Kalantarov for many stimulating discussions.

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1. Introduction

Various versions of hyperbolic Navier-Stokes equations are of increasing current interest. For instance, such equations may appear from the relaxation approximations of the Euler equations in the diffusive scaling limit:

\[
\begin{cases}
\partial_t u + \text{div}(U) + \nabla_x p = 0, & \text{div } u = g \\
\varepsilon \partial_t U + \nabla u + U = u \otimes u,
\end{cases}
\]  

(1.1)

where \( \varepsilon > 0 \) and \( U \) is a supplementary matrix valued variable. Excluding this variable, we end up with the following version of hyperbolic Navier-Stokes equations

\[
\varepsilon \partial_t^2 u + \partial_t u + \text{div}(u \otimes u) + \nabla_x p = \Delta_x u + g,
\]  

(1.2)

see [3, 7, 13, 17] for the details.

Another source of such equations is the theory of viscoelastic fluids. In particular, the equations

\[
\varepsilon \partial_t^2 u + \varepsilon \partial_t \text{div}(u \otimes u) + \partial_t u + \text{div}(u \otimes u) + \nabla_x p = \Delta_x u + g
\]  

(1.3)

with the additional term \( \varepsilon \partial_t \text{div}(u \otimes u) \) naturally arise in the theory of Jeffrey flows, see [5, 6, 15, 16], see also references therein.

We also report here on one more motivation for the hyperbolic relaxation of Navier-Stokes equations related with the computational aspects. From this point of view, the usefulness of hyperbolization of the Navier-Stokes equations can be described as follows. The usage of the explicit schemes, which can turn out to be the most adequate for flows with complex structure and very convenient for parallel computations, requires the time step \( \tau \sim h^2 \), where \( h \) is the typical size of the spatial grid. In the hyperbolized version of Navier–Stokes system with small parameter \( \varepsilon \) the time step is \( \tau \sim h \sqrt{\varepsilon} \) because of the hyperbolic nature of modified system. If we take \( \varepsilon \) of the order of \( h \), then there is a significant gain in the computation time. At the same time many physical systems has limited level of detailing, see, for example, [4]. This fact leads to natural limitation for the space discretization in practical problems analogously to the situation with multi-phase models. The estimates of possible influence of scales on the quality of numerical algorithms and the proximity estimates in the linear case can be found, for example, in [10, 12, 14].

The mathematical study of problem (1.2) in the case where \( x \in \mathbb{R}^d \), \( d = 2 \) or 3 as well as for periodic boundary conditions are presented in [3, 13], see also [13, 16].

The main aim of the present paper is to study problem (1.2) in a bounded domain \( \Omega \subset \mathbb{R}^2 \) with Dirichlet boundary conditions. Note
that, in contrast to the cases mentioned above, we cannot use the vorticity equation as it done in [3] or Strichartz estimates for wave operators which have been essentially used in [13] (to the best of our knowledge nothing is known concerning the validity of Strichartz estimates for hyperbolic Stokes equations in bounded domains). By this reason, the have to work on the level of energy type estimates only and cannot use the approaches developed in [3] and [13] at least in a direct way.

One more principal difficulty related with these equations is that they do not possess any reasonable energy inequality for $\varepsilon \neq 0$, so to obtain the global existence of solutions we need to use the energy equality for $\varepsilon = 0$ and perturbation arguments. By this reason, our results are restricted to the case of small $\varepsilon$ and cannot be extended to the case of arbitrarily large $\varepsilon$. We expect that this restriction is not technical but is related with the nature of the considered problem. In order to support this point of view, we consider the simplified model of 1D hyperbolic Burgers equation

$$
\varepsilon \partial_t^2 u + \partial_t u + u \partial_x u = \partial_x^2 u
$$

and prove that the solutions may blow up in finite time if the initial energy or and $\varepsilon > 0$ is large enough, see Section 5.2 for the details.

We will study the strong solutions $\xi_u(t) := \{u(t), \partial_t u(t)\}$ of problem (1.2) which belong to the phase space

$$
E_1^\varepsilon := \left\{ \{u, v\} \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega), \mbox{ div } u = \mbox{ div } v = 0 \right\}
$$

endowed by the following norm:

$$
\|\xi_u\|_{E_1^\varepsilon}^2 := \varepsilon \|\partial_t u\|_{H^1}^2 + \|\partial_t u\|_{L^2}^2 + \|u\|_{H^2}^2.
$$

The main result of the paper is a global existence and dissipativity of strong solutions of problem (1.2) if $\varepsilon$ is small enough and the initial data satisfies

$$
\|\xi_u(0)\|_{E_1^\varepsilon} \leq R(\varepsilon),
$$

where the monotone decreasing function $R$ satisfies $\lim_{\varepsilon \to 0} R(\varepsilon) = \infty$. As usual, the dissipativity means that

$$
\|\xi_u(t)\|_{E_1^\varepsilon} \leq Q(\|\xi_u(t)\|_{E_1^\varepsilon}) e^{-\alpha t} + Q(\|g\|_{L^2}),
$$

where the monotone function $Q$ and positive constant $\alpha$ are independent of $u$, $t$ and $\varepsilon$, see Theorem 3.1.

The paper is organized as follows.
In Section 2, we introduce the necessary spaces and notations which will be used throughout of the paper. Section 3 is devoted to the proof of the main result stated above.

The singular limit $\varepsilon \to 0$ is studied in Section 4. In particular, we give there the results concerning the convergence of individual trajectories on finite time interval as well as the convergence of the corresponding global attractors.

Finally, in Section 5, we discuss possible extensions of proved results to the 3D case as well as to Jeffrey flows. In addition, the result concerning blow up in the hyperbolic Burgers equations is given there.

2. Preliminaries

In a bounded smooth domain $\Omega \subset \mathbb{R}^2$, we study the following problem:

\[
\begin{aligned}
\varepsilon \partial_t^2 u + \partial_t u + (u, \nabla_x)u + \nabla_x p &= \Delta_x u + g, \\
\text{div } u &= 0, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u'_0, \\
&\quad u|_{\partial \Omega} = 0.
\end{aligned}
\]

Here $u = (u^1, u^2)$ and $p$ are an unknown velocity vector field and pressure, respectively, and $g \in L^2(\Omega)$ is the given external force and $\varepsilon > 0$ is a given parameter which is assumed to be small enough.

As usual, we introduce the spaces $V$ and $H$ as follows:

\[
\begin{aligned}
V := \{ u \in [H^1_0(\Omega)]^2, \quad \text{div } u = 0 \}, \\
H := \{ u \in [L^2(\Omega)]^2, \quad \text{div } u = 0, \quad u \cdot n|_{\partial \Omega} = 0 \}.
\end{aligned}
\]

and denote by $P : [L^2(\Omega)]^2 \to H$ the Leray–Helmholtz orthogonal projection onto solenoidal vector fields, see [19]. We also denote by $A := -P\Delta$ the Stokes operator in $H$ with Dirichlet boundary conditions. Then $A$ is a positive-definite self-adjoint operator in $H$ with compact inverse with domain

\[
D(A) = [H^2(\Omega) \cap H^1_0(\Omega)]^2 \cap \{ \text{div } u = 0 \}.
\]

In addition, $D(A^{1/2}) = V$ (with equality of norms), and $D(A^{-1/2}) = D(A^{1/2})^* = V^* = H^{-1}(\Omega)$ (with equality of norms), and we also set

\[
(u, v)_s := (u, A^sv), \quad \text{for } |s| \leq 1.
\]

We now introduce the natural energy spaces related with the hyperbolic relaxation (2.1) of the Navier-Stokes equations as follows:

\[
E^s := D(A^{(s+1)/2}) \times D(A^{s/2}), \quad s \in \mathbb{R},
\]

\[
(2.3)
\]
although we will use below only the cases where \( s = -1, 0, 1 \). The norms in these spaces are given by
\[
\|\xi u\|_{E^s_{\varepsilon}}^2 := \varepsilon \|\partial_t u\|_{D(A^{s/2})}^2 + \|u\|_{D(A^{(s-1)/2})}^2 + \|u\|_{D(A^{(s+1)/2})}^2, \tag{2.4}
\]
where
\[
\xi u := \{u, \partial_t u\}.
\]
Note that these norms are equivalent for different positive values of the parameter \( \varepsilon \), but the dependence on \( \varepsilon \) is included to the definition of this norms in order to capture the right dependence of solutions on \( \varepsilon \) as \( \varepsilon \to 0 \).

The truncated norms
\[
\|\xi u\|_{E^s_{\varepsilon}}^2 := \varepsilon \|\partial_t u\|_{D(A^{s/2})}^2 + \|u\|_{D(A^{(s+1)/2})}^2 \tag{2.5}
\]
in these energy spaces will also be useful in what follows. Furthermore, all the technical estimates in Section 3 below will be carried out in terms of norm (2.5) up to the last step when the full norm (2.4) is appended to the final estimate.

Finally, in order to exclude the pressure, we apply the Leray operator \( P \) to both sides of equation (2.1) and get the equation for the velocity field \( u \) only:
\[
\varepsilon \partial_t^2 u + \partial_t u + P((u, \nabla_x)u) = -Au + g, \quad \xi u\big|_{t=0} = \{u_0, u_0'\}, \tag{2.6}
\]
where we assume for simplicity that \( g = Pg \). Thus, by definition, a vector field \( u = u(t, x) \) is a strong solution of the Navier-Stokes problem (2.1) on the interval \( t \in [0, T] \) if
\[
u \in C(0, T; D(A)), \quad \partial_t u \in C(0, T; V), \quad \varepsilon \partial_t^2 u \in C(0, T; \mathcal{H}) \tag{2.7}
\]
and \( u \) satisfies (2.6) as an equality in \( \mathcal{H} \).

For \( \varepsilon = 0 \) the limiting equation is the classical Navier-Stokes system
\[
\begin{cases}
\partial_t v + (u, \nabla_x)v + \nabla_x p = \Delta_x v + g, \\
\text{div } v = 0, \quad v\big|_{t=0} = v_0,
\end{cases} \tag{2.8}
\]
which has a unique strong solution \( v \) and this solution possesses the dissipative estimate in \( H^2 \):
\[
\|v(t)\|_{H^2} \leq Q(\|v(0)\|_{H^2})e^{-\alpha t} + Q(\|g\|_{L^2}), \tag{2.9}
\]
where the monotone function \( Q \) and positive constant \( \alpha \) are independent of \( t, g \) and \( v_0 \), see [1], [20] and the references therein. Moreover, differentiating equation (2.1) in time and arguing in a standard way,
we may obtain the corresponding estimates for the time derivatives of $v$, namely,

$$
\| \partial_t v(t) \|_{L^2} + \| \partial_t v \|_{L^2(t,t+1;V)} + \| \partial_t^2 v \|_{L^2(t,t+1;V^*)} \leq Q(\|v(0)\|_{H^2}) e^{-\alpha t} + Q(\|g\|_{L^2}).
$$

(2.10)

Since the derivation of this estimate from estimate (2.9) is straightforward, we leave it for the reader.

**Remark 2.1.** In what follows we shall be using the same notation for different monotone increasing functions in dissipative estimates like (2.9), (2.10).

### 3. Key dissipative estimate

The main aim of this section is to obtain the analogue of the dissipative estimate (2.9) for the case $\varepsilon > 0$. The main difference here is that, in contrast to the classical Navier-Stokes equations, we do not have basic energy identity if $\varepsilon > 0$. Moreover, we expect that the solutions of the perturbed problem may blow up in finite time if the initial energy is large enough, see Section 5 for more details. Thus, we may expect only that (2.9) remains true if $\varepsilon > 0$ is small or the initial energy is not very large. Namely, the following estimate can be considered as a main result of the paper.

**Theorem 3.1.** Let the external forces $g \in H$. Then, for every $R > 0$, there exists $\varepsilon_0 = \varepsilon_0(R) > 0$ such that, for every $\varepsilon \leq \varepsilon_0$ and every initial data $\xi_u(0) \in E^1$ satisfying $\|\xi_u(0)\|_{E^1} \leq R$, problem (2.1) possesses a unique global strong solution $\xi_u(t) \in E^1$ and the following dissipative estimate holds:

$$
\|\xi_u(t)\|_{E^1} \leq Q(\|\xi_u(0)\|_{E^1}) e^{-\alpha t} + Q(\|g\|_{L^2}),
$$

(3.1)

where the positive constant $\alpha$ and monotone function $Q$ are independent of $R$, $\varepsilon \leq \varepsilon_0$, $t \geq 0$ and the initial data $\xi_u(0)$ satisfying $\|\xi_u(0)\|_{E^1} \leq R$.

**Proof.** We give below only the formal derivation of the dissipative estimate (3.1). The existence a solution can then be obtained in a standard way, for instance, by the Galerkin approximations, see [1], [19]. The uniqueness of strong solutions is also straightforward and we do not discuss it here.

The derivation of estimate (3.1) is based on the fact that a solution $u(t)$ of the perturbed equation (with small $\varepsilon > 0$), if it exists, should be close (on a finite time interval $t \in [0,T]$) to the corresponding limit Navier-Stokes system (2.8), which corresponds to $\varepsilon = 0$. Thus, we take the strong solution $v = v(t)$ of the Navier-Stokes problem (2.8) as an
approximation of the corresponding solution \( u = u(t) \) of the perturbed system. Important for us is that
\[
|v|_{t=0} = |u|_{t=0}.
\]
Then, on the one hand, (2.8) is globally solvable and we have (2.9) and (2.10). On the other hand, we expect that the difference
\[
w(t) = u(t) - v(t)
\]
is small in the appropriate norm which in turn would allow us to verify the desired estimate using the perturbation arguments. However, the realization of this scheme is not straightforward for two reasons. The first one is that the passage from \( \varepsilon = 0 \) to \( \varepsilon > 0 \) is a singular perturbation and, in particular, a boundary layer appears at \( t = 0 \). The second, more technical one is that the nonlinearity \((u, \nabla_x)u\) is critical from the point of view of a hyperbolic equation even in the phase space \( E_1^\varepsilon \). So, the interplay between these two difficulties makes the derivation of the desired estimate rather delicate and rather technical. For the convenience of the reader, we will split it in several steps.

**Step 1. Estimates for** \( w \): **weak norms.** The function \( w = u - v \) solves the equation
\[
\begin{cases}
\varepsilon \partial_t^2 w + \partial_t w + Aw = \\
\quad -P \left[ ((w, \nabla_x)w) + (w, \nabla_x) + ((v, \nabla_x)w) \right] - \varepsilon \partial_t^2 v, \\
|w|_{t=0} = 0, \quad \partial_t w|_{t=0} = u_0' - \partial_t v(0), \\
|w|_{\partial\Omega} = 0.
\end{cases}
\]
We multiply this equation by \( A^{-1}(\partial_t w + \alpha w) \) where \( \alpha > 0 \) and first consider the linear left-hand side. Integrating by parts and taking into account (2.5) we obtain
\[
\left( \varepsilon \partial_t^2 w + \partial_t w + Aw, A^{-1}(\partial_t w + \alpha w) \right) =
\frac{1}{2} \frac{d}{dt} E_\varepsilon^{-1}(\xi_w) + (1 - \varepsilon \alpha) \| \partial_t w \|_{H^{-1}}^2 + \alpha \| w \|_{H}^2,
\]
where
\[
E_\varepsilon^{-1}(w) := \| \xi_w \|_{E_\varepsilon^{-1}}^2 + \alpha \| w \|_{H^{-1}}^2 + 2\varepsilon \alpha (w, \partial_t w)_{-1}.
\]
Next, in view of the Poincaré inequality
\[
\| w \|_{H^{-1}}^2 \leq \lambda_1^{-1} \| w \|_{H}^2,
\]
taking \( \alpha \) sufficiently small (and independent of \( \varepsilon \) as \( \varepsilon \to 0 \)), we have
\[
c_1 \| \xi_w \|_{E_\varepsilon}^2 \leq E_\varepsilon^{-1}(\xi_w) \leq c_2 \| \xi_w \|_{E_\varepsilon},
\]
where positive constants \( c_1 \) and \( c_2 \) are independent of \( \varepsilon \) and \( \alpha \).
Thus, the scalar product of (3.3) and $A^{-1}(\partial_t w + \alpha w)$ gives

$$\frac{d}{dt} E_{\varepsilon}^{-1}(\xi_w) + (1 - \alpha \varepsilon)\|\partial_t w\|_{H^{-1}}^2 + \alpha \|w\|_H^2 \leq$$

$$\leq ||(w, \nabla_x)w, A^{-1}(\partial_t w + \alpha w)|| + ||(w, \nabla_x)\nu, A^{-1}(\partial_t \nu + \alpha \nu)|| +$$

$$+ ||(v, \nabla_x)w, A^{-1}(\partial_t w + \alpha w)|| + \varepsilon|\partial_t^2 v, A^{-1}(\partial_t w + \alpha w)|. \tag{3.6}$$

Moreover, under this choice of $\alpha$, there exists $\beta > 0$, which is also independent of $\varepsilon \to 0$, such that (3.6) takes the following form (here and below we write $E_{\varepsilon}^{-1}(w)$ instead of $E_{\varepsilon}^{-1}(\xi_w)$):

$$\frac{d}{dt} E_{\varepsilon}^{-1}(w) + \beta E_{\varepsilon}^{-1}(w) + \beta \|\partial_t w\|_{H^{-1}}^2 \leq ||(\text{div}(w \otimes w), A^{-1}(\partial_t w + \alpha w)|| +$$

$$+ ||(\text{div}(w \otimes \nu), A^{-1}(\partial_t \nu + \alpha \nu)|| +$$

$$+ ||(\text{div}(v \otimes w), A^{-1}(\partial_t w + \alpha w)|| + \varepsilon|\partial_t^2 v, A^{-1}(\partial_t w + \alpha w)|, \tag{3.7}$$

where

$$\text{div}(u \otimes v) := \sum_{i,j=1}^2 \partial_i(u^i v^j) = (u, \nabla_x) v - v \text{ div } u = (u, \nabla_x) v$$

on divergent free vector fields. So, we only need to estimate the terms on the right-hand side of this inequality. The most difficult term here is the first one, and we are actually unable to estimate in a closed form. Instead, using the regularity of the Stokes operator and the interpolation inequality $\|w\|_L^4 \leq C\|w\|_{H^2}\|w\|_{L^2}^3$, we can estimate it as follows:

$$||(\text{div}(w \otimes w), A^{-1}(\partial_t w + \alpha w)|| \leq$$

$$\leq C\|w \otimes w\|_{L^2}\|A^{-1}(\partial_t w + \alpha w)\|_{H^1} \leq C\|w\|_{L^4}(\|\partial_t w\|_{H^{-1}} + \|w\|_{H^{-1}}) \leq$$

$$\leq C\|w\|_{H^2}\|w\|_{L^2}^3 + \frac{\beta}{3}(\|\partial_t w\|_{H^{-1}}^2 + \|w\|_{H^{-1}}^2).$$

The second and third terms are estimated in the same way, but using the fact that $\|v\|_{L^\infty}$ is under control. Using also (3.4) we obtain

$$||(\text{div}(w \otimes \nu), A^{-1}(\partial_t \nu + \alpha \nu)|| + ||(\text{div}(v \otimes w), A^{-1}(\partial_t w + \alpha w)|| \leq$$

$$\leq C||w| \cdot |v||_{L^2}(\|\partial_t w\|_{H^{-1}} + \|w\|_{H^{-1}}) \leq C_R\|w\|_{H^2}^2 + \frac{\beta}{3}\|w\|_{H^{-1}}^2.$$
and collecting the above estimates we finally obtain
\[
\frac{d}{dt} E^{-1}_\varepsilon(w) \leq (C_R + C\|w\|_{H^2} E^{-1}_\varepsilon(w)^{1/2}) E^{-1}_\varepsilon(w) + C\varepsilon^2 \|\partial^2 v\|_{H^{-1}}^2, \tag{3.8}
\]
where the constant \(C_R\) depends only on \(R\) (recall that we assume that \(\|\xi_u(0)\|_{E^1} \leq R\)). It is also important that by the equivalence (3.5)
\[
E^{-1}_\varepsilon(w(0)) \leq C\varepsilon^2 \|\partial_t u(0)\|_{H^{-1}}^2 \leq C_R\varepsilon. \tag{3.9}
\]
Unfortunately, estimate (3.8) is not enough to obtain an estimate for \(w\) due to the presence of the \(H^2\)-norm of \(w\), so we need more estimates to control it.

**Step 2. \(H^2\)-estimates for \(u\).** Recall that \(u = v + w\) and the \(H^2\)-norm of \(v\) is under control, so estimating the \(H^2\)-norm of \(u\) is equivalent to estimating the \(H^2\)-norm of \(w\). However, it is more convenient to work with \(u\) on this stage (due to the presence of the term \(\varepsilon \partial^2_w v\) in the right-hand side of the equation for \(w\) which requires too much regularity of the initial data to be properly estimated). Multiplying the initial equation (2.1) by \(A(\partial_t u + \alpha u)\) and arguing as before, we end up with
\[
\frac{d}{dt} E^1_\varepsilon(u) + \beta E^1_\varepsilon(u) + \beta \|\partial_t u\|_{H^1}^2 \leq
\]
\[
\leq |((u, \nabla_x)u, A(\partial_t u + \alpha u))| + |(g, A(\partial_t u + \alpha u))|. \tag{3.10}
\]
The key problem is again to estimate the first term on the right-hand side. We write it as follows
\[
(u, \nabla_x)u = (w, \nabla_x)w + (v, \nabla_x v) + (v, \nabla_x w) + (w, \nabla_x v)
\]
and estimate each term separately. For simplicity we will estimate only the terms with multiplication by \(A\partial_t u\) (the multiplication by \(Au\) is analogous, but easier). Integrating by parts we have
\[
|((w, \nabla_x)w, A\partial_t u)| \leq C(\|w\|_{L^\infty}^2 \|w\|_{H^2}^2 + \|w\|_{W^{1,4}}^4) + \frac{\beta}{4} \|\partial_t u\|_{H^1}^2 \leq
\]
\[
\leq C E^{-1}_\varepsilon(w)^{1/2}[E^1_\varepsilon(u)^{3/2} + \|v\|_{H^2}^3] + \frac{\beta}{4} \|\partial_t u\|_{H^1}^2, \tag{3.11}
\]
where we have used the interpolation inequalities
\[
\|w\|_{L^\infty}^2 \leq C\|w\|_{L^2}^2 \|w\|_{H^2}, \quad \|w\|_{W^{1,4}}^4 \leq C\|w\|_{L^2}^2 \|w\|_{H^2}^3
\]
and (3.2). Since we have (2.9), the second term is straightforward
\[
|((v, \nabla_x)w, A\partial_t u)| \leq C\|v\|_{H^2}^2 \|\partial_t u\|_{H^1} \leq
\]
\[
\leq Q(\|g\|_{L^2}^2) + Q(\|u(0)\|_{H^2}) e^{-\alpha t} + \frac{\beta}{4} \|\partial_t u\|_{H^1}^2.
\]
The rest two terms can be estimated analogously to (3.11):

\[ |((v, \nabla_x)w, A\partial_t u) + (w, \nabla_x)\partial_t u) | \leq C\|v\|_{H^2}\|w\|_{H^2}^2 + \frac{\beta}{4}\|\partial_t u\|_{H^1}^2 \leq \frac{\beta}{4}\|\partial_t u\|_{H^1}^2 + \frac{\beta}{4}\|\partial_t u\|_{H^1}^2 + \left(Q(\|g\|_{L^2}) + Q(\|u(0)\|_{H^2})e^{-\alpha t}\right)\|w\|_{H^2}^2 \leq \frac{\beta}{4}\|\partial_t u\|_{H^1}^2 + \left(Q(\|g\|_{L^2}) + Q(\|u(0)\|_{H^2})e^{-\alpha t}\right)(1 + \|u\|_{H^2}^2)
\]

with a different function \(Q'\) with the same properties. Inserting these estimates into (3.10) and omitting the primes, we get

\[
\frac{d}{dt}E_\varepsilon^1(u) + \beta E_\varepsilon^1(u) + \frac{\beta}{4}\|\partial_t u\|_{H^1}^2 \leq CE_\varepsilon^{-1}(w)^{1/2}[E_\varepsilon^1(u)^{3/2} + \|v\|_{H^2}^3] + \left(Q(\|g\|_{L^2}) + Q(\|u(0)\|_{H^2})e^{-\alpha t}\right)(1 + \|u\|_{H^2}^2). \tag{3.12}
\]

Inequalities (3.8), (3.9) and (3.12) are enough to verify the global existence on any finite interval \(t \in [0, T]\) if \(\varepsilon \leq \varepsilon(T)\) is small enough, but still not sufficient to iterate the estimates and get the global existence and dissipativity (due to the presence of the big term \(Q(\|g\|_{L^2})\|u\|_{H^2}^2\) which can not be absorbed by the term \(\beta E_\varepsilon^1(u)\) in the left-hand side). To overcome this we need one more step involving the “parabolic” type estimates for \(Au\).

**Step 3. Parabolic estimates for \(Au\).** We multiply equation (2.1) by \(Au\) to get

\[
\frac{d}{dt}\left(\varepsilon(\partial_t u, Au) + \frac{1}{2}\|u\|_{H^1}^2\right) + \|u\|_{H^2}^2 - \varepsilon\|\partial_t u\|_{H^1}^2 \leq |(u, \nabla_x)u, Au)| + |(g, Au)|. \tag{3.13}
\]

Estimating the non-linear term analogously to Step 2, we get

\[
\frac{1}{2}\|u\|_{H^2}^2 + \frac{d}{dt}\left(\varepsilon(\partial_t u, Au) + \frac{1}{2}\|u\|_{H^1}^2\right) - \varepsilon\|\partial_t u\|_{H^1}^2 \leq CE_\varepsilon^{-1}(w)^{1/2}[E_\varepsilon^1(u)^{3/2} + \|v\|_{H^2}^3] + \left(Q(\|g\|_{L^2}) + Q(\|u(0)\|_{H^2})e^{-\alpha t}\right)(1 + \|u\|_{H^2}^2). \tag{3.14}
\]
We now multiply inequality (3.14) by $2Q(\|g\|_{L^2})$ and add the result to inequality (3.12). This gives the following estimate
\[
\frac{d}{dt} E^1_\varepsilon(u) + \beta E^1_\varepsilon(u) + \left(\frac{\beta}{4} - 2\varepsilon Q(\|g\|_{L^2})\right) \|\partial_t u\|_{H^1}^2 \leq C(1 + 2Q(\|g\|_{L^2})) E^{-1}_\varepsilon(w)^{1/2}[E^1_\varepsilon(u)^{3/2} + \|v\|_{H^2}^2] + 2Q(\|g\|_{L^2})^2 + 2Q(\|g\|_{L^2})Q(\|u(0)\|_{H^2})e^{-\alpha t} + Q(\|u(0)\|_{H^2}) \|u\|_{H^2}^2 e^{-\alpha t},
\] (3.15)
where
\[
\tilde{E}^1_\varepsilon(u) := E^1_\varepsilon(u) + Q(\|g\|_{L^2})\left(2(\partial_t u, Au) + \|u\|_{H^1}^2\right).
\]
Thus, for sufficiently small $\varepsilon$ we have
\[
C^{-1} E^1_\varepsilon(u) \leq \tilde{E}^1_\varepsilon(u) \leq C E^1_\varepsilon(u),
\]
where the constant $C = C_R$ is independent of $\varepsilon$. Therefore, we may replace $E^1_\varepsilon(u)$ by $\tilde{E}^1_\varepsilon(u)$. Using the dissipative estimate (2.9) for the term $\|v\|_{H^2}^3$ in (3.15) and dropping the bar-sign we obtain the final estimate:
\[
\frac{d}{dt} E^1_\varepsilon(u) + (\beta - Q(\|u(0)\|_{H^2})e^{-\alpha t} - C[E^{-1}_\varepsilon(w)]^{1/2}[E^1_\varepsilon(u)]^{1/2}) E^1_\varepsilon(u) \leq C(1 + E^{-1}_\varepsilon(w)^{1/2})(Q(\|g\|_{L^2}) + Q(\|u(0)\|_{H^2})e^{-\alpha t}),
\] (3.16)
where all of the constants are positive and are independent of $\varepsilon \to 0$. As we will see below, this estimate together with inequalities (3.8) and (3.9) is sufficient to obtain the desired dissipative estimate for $u$.

**Step 4. Completion of the proof.** Following [22] (see also [21] and [8]), we first establish the dissipative estimate on a finite time interval $t \in [0, T]$ where $T$ is large enough. Namely, we claim that inequalities (3.8), (3.9) and (3.16) imply the following intermediate result.

**Lemma 3.2.** For every $R > 0$ and every $T \in \mathbb{R}_+$ there exists $\varepsilon_0 = \varepsilon_0(T, R)$ such that for every $\varepsilon < \varepsilon_0$ and every initial data $\xi_u(0)$ satisfying $\|\xi_u(0)\|_{E^1} \leq R$, there exists a unique solution $\xi_u(t), t \in [0, T]$ of problem (2.1) satisfying the dissipative estimate
\[
\varepsilon \|\partial_t u(t)\|_{H^1}^2 + \|u(t)\|_{H^2}^2 \leq Q_R(\|\xi_u(0)\|_{E^1}) e^{-\alpha_R t} + Q(\|g\|_{L^2}),
\] (3.17)
where the function $Q$ is independent of $T$, $R$ and $\varepsilon$ and the functions $Q_R$ and $\alpha_R$ are independent of $T$ and $\varepsilon$.

**Proof.** Indeed, assume for the first that
\[
C[E^{-1}_\varepsilon(w)]^{1/2}[E^1_\varepsilon(u)]^{1/2} \leq \frac{1}{2} \beta, \quad E^{-1}_\varepsilon(w) \leq 1, \quad t \in [0, T].
\] (3.18)
Then the Gronwall inequality applied to (3.16) gives us the desired estimate
\[ E^1_\varepsilon(u(t)) \leq \tilde{Q}(\|g\|_{L^2}) + \tilde{Q}(\|\xi_u(0)\|_{E^1_\varepsilon})e^{-\beta t}, \]  
for some monotone function $\tilde{Q}$ and positive constant $\bar{\beta}$ which are independent of $t$ (but may depend on $R$). Recall that this estimate is obtained under the assumption (3.18) and we still need to justify it. To this end, we note that estimate (3.19) gives us the following control
\[ \|w(t)\|_{H^2} \leq C(E^1_\varepsilon(u(t)) + C\|v\|_{H^2}^2)^{1/2} \leq Q_R \]  
and, therefore (3.8) reads
\[ \frac{d}{dt} E^{-1}_\varepsilon(w) \leq (C_R + Q_R E^{-1}_\varepsilon(w)^{1/2}) E^{-1}_\varepsilon(w) + C\varepsilon^2\|\partial^2_t v\|_{H^{-1}}, \]  
This estimate, together with the fact that $E^{-1}_\varepsilon(w(0)) \leq C_R\varepsilon$ implies that, for sufficiently small $\varepsilon \leq \varepsilon_0(T, R)$, the quantity $E^{-1}_\varepsilon(w(t))$ remains small and can be estimated as follows:
\[ E^{-1}_\varepsilon(w(t)) \leq Q_R\varepsilon e^{(C_R + Q_R)T}, \quad t \in [0, T]. \]  
Thus, we end up with the control
\[ C[E^{-1}_\varepsilon(w(t))]^{1/2} E^1_\varepsilon(u(t))]^{1/2} \leq \tilde{Q}_R e^{Q_R T}, \]  
where the constant $Q_R$ is independent of $\varepsilon \to 0$. Finally, since the bounds in (3.18) are satisfied for $t = 0$ for sufficiently small $\varepsilon$, if we assume in addition that $Q_R\varepsilon e^{Q_R T} \leq \beta/2$, we will get (3.18) by continuity arguments. This finishes the proof of the lemma.

Note that (3.17) still not enough to verify the global existence and dissipativity since the left-hand side does not contain does the $L^2$-norm of $\partial_t u$ without $\varepsilon$, see definition (2.4). To overcome this problem and append $\|\partial_t u\|_{L^2}$ to our estimate obtained in the norm $E^1_\varepsilon$, we recall the boundary layer estimate for the second order ODE
\[ \varepsilon\partial^2_t u + \partial_t u = h(t), \quad \|h(t)\|_{L^2} \leq C(\|u(t)\|_{H^2}^2 + \|g\|_{L^2} + 1), \]  
Namely,
\[ \|\partial_t u(t)\|_{L^2} \leq \|\partial_t u(0)\|_{L^2} e^{-t/\varepsilon} + \frac{1}{\varepsilon} \int_0^t e^{-(t-s)/\varepsilon} \|h(s)\|_{L^2} ds \leq \|\partial_t u(0)\|_{L^2} e^{-t/\varepsilon} + C\left(1 + \|g\|_{L^2} + \max_{s \in [0, t]} \|u(s)\|_{H^2}^2\right). \]  
Using this estimate on the interval $t \in [T - 1, T]$ and assuming that $T \geq 1$, we derive from (3.17) that
\[ \|\xi_u(t)\|_{E^1_\varepsilon} \leq Q_R(\|\xi_u(0)\|_{E^1_\varepsilon})e^{-\alpha_\varepsilon T} + Q(\|g\|_{L^2}), \]  
(3.24)
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for \( \|\xi_u(0)\|_{E^1} \leq R \) and \( 1 \leq t \leq T \). This estimate is already enough to get the desired dissipative estimate and finish the proof of the theorem. Indeed, if we take \( T = T(R) \) large enough, the \( E^1_\varepsilon \)-norm of the solution \( u(t) \) at point \( t = T \) will be less than this norm at point \( t = 0 \) according to (3.17) if \( R \) is large enough (say, if \( R \geq 2Q(\|g\|_{L^2}) \)). This allows to iterate the procedure and obtain the global existence and estimate (3.1). Thus, Theorem 3.1 is proved. \( \Box \)

4. ATTRACTORS AND THEIR SINGULAR LIMIT AS \( \varepsilon \to 0 \)

In this section we verify the existence of global attractors associated with the Navier-Stokes problem (2.1) and their convergence as \( \varepsilon \to 0 \) to the limit attractor associated with the classical Navier-Stokes problem. We first recall that the global well-posedness and dissipativity of problem (2.1) is established not for all initial data \( \xi_u(0) \in E^1_\varepsilon \) and not for all \( \varepsilon > 0 \), but only for relatively small \( \varepsilon > 0 \) and initial data satisfying the assumption

\[
\xi_u(0) \in B(0, R, E^1_\varepsilon) := \{ \xi_0 \in E^1_\varepsilon, \|\xi_0\|_{E^1_\varepsilon} \leq R \}, \quad (4.1)
\]

where \( R = R(\varepsilon) \) is monotone increasing and satisfying

\[
\lim_{\varepsilon \to 0} R(\varepsilon) = \infty, \quad (4.2)
\]

see Theorem 3.1. By this reason, it looks natural to consider equation (2.1) in the phase space

\[
\Phi_\varepsilon := \cup_{t \geq 0} \{ \xi_u(t), \xi_u(0) \in B(0, R, E^1_\varepsilon) \}. \quad (4.3)
\]

Then, according to Theorem 3.1, the solution semigroups

\[
S_\varepsilon(t)\xi_u(0) := \xi_u(t), \quad S_\varepsilon(t)\Phi_\varepsilon \subset \Phi_\varepsilon \quad (4.4)
\]

is well-defined and dissipative on \( \Phi_\varepsilon \) for \( \varepsilon > 0 \) being small enough. Moreover, these semigroups are continuous on \( \Phi_\varepsilon \) with respect to the initial data (in the topology of the space \( E^1 \)) for every fixed \( t \) and \( \varepsilon \) and, in particular, the set \( \Phi_\varepsilon \) is closed in \( E^1 \). Thus, we may speak about global attractors of semigroups \( S_\varepsilon(t) \) on \( \Phi_\varepsilon \). For the convenience of the reader, we remind the definition of a global attractor, see [1] [20] for more details.

**Definition 4.1.** A set \( \mathcal{A}_\varepsilon \) is a global attractor of the semigroup \( S_\varepsilon(t) : \Phi_\varepsilon \to \Phi_\varepsilon \) if

1) The set \( \mathcal{A}_\varepsilon \) is compact in \( \Phi_\varepsilon \);
2) The set \( \mathcal{A}_\varepsilon \) is strictly invariant: \( S_\varepsilon(t)\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon \);
The set $A_\varepsilon$ attracts the images of all bounded subsets of $\Phi_\varepsilon$ as $t \to \infty$, i.e., for every bounded subset $B \subset \Phi_\varepsilon$ and every neighbourhood $\mathcal{O}(A_\varepsilon)$ of the attractor $A_\varepsilon$, there exists $T = T(B, \mathcal{O})$ such that

$$S_\varepsilon(t)B \subset \mathcal{O}(A_\varepsilon)$$

if $t \geq T$. In the case $\varepsilon > 0$ the whole phase space $\Phi_\varepsilon$ is bounded, so we may state and check the attraction property for $B = \Phi_\varepsilon$ only.

In the case where the phase space $\Phi_\varepsilon$ is endowed by the weak or strong topology of the space $E^1$, we will refer to $A_\varepsilon$ as weak or strong attractor respectively.

The most straightforward is the existence of a weak attractor, so we will start with establishing this fact.

**Proposition 4.2.** Let $\varepsilon > 0$ be small enough. Then the solution semigroup $S_\varepsilon(t)$ acting on the phase space $\Phi_\varepsilon$ defined above possesses a weak attractor $A_\varepsilon$ which satisfies the estimate

$$\|A_\varepsilon\|_{E^1} \leq \bar{R}, \quad (4.5)$$

where $\bar{R}$ is independent of $\varepsilon \to 0$. As usual this attractor is generated by all bounded solutions of problem (2.1) defined for all $t$

$$A_\varepsilon = K_\varepsilon|_{t=0}, \quad (4.6)$$

where

$$K_\varepsilon := \{ \xi_u, \|\xi_u(t)\|_{E^1} \leq \bar{R},$$

$$S_\varepsilon(h)\xi_u(t) = \xi_u(t + h), \quad h \geq 0, \quad t \in \mathbb{R} \subset C(\mathbb{R}, \Phi_\varepsilon). \quad (4.7)$$

**Proof.** In order to prove the proposition, it is sufficient to verify two facts. Namely, that there exists a compact absorbing set for the semigroup $S_\varepsilon(t)$ and that the semigroup is weakly continuous on it, see [1] for details. The first fact follows from the dissipative estimate proved in Theorem 3.1. Indeed, estimate (3.1) guarantees that the ball $B(0, \bar{R}, E^1)$ will be an absorbing ball for this semigroup if, say, $\bar{R} = 2Q(\|g\|_{L^2})$. This ball is weakly compact by Alaoglu theorem. The weak continuity is straightforward and standard, so we left its rigorous verification to the reader. Thus, the existence of a weak attractor $A_\varepsilon$ is verified and the representation formula (4.6) also followed from the abstract attractor’s existence theorem and the proposition is proved. \[\square\]

We are now ready to verify that the constructed weak attractor is actually a strong one.

**Proposition 4.3.** Let $\varepsilon > 0$ be small enough. Then the solution semigroup $S_\varepsilon(t) : \Phi_\varepsilon \to \Phi_\varepsilon$ associated with the hyperbolic Navier-Stokes
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system (2.1) possesses a strong global attractor which coincides with the weak attractor constructed above.

Proof. We will check the asymptotic compactness of the semigroup $S_{\varepsilon}(t)$ using the so-called energy method. To this end, we take an arbitrary sequence of the initial data $\xi_n \in \Phi_{\varepsilon}$ and arbitrary sequence $t_n \to \infty$ and need to verify that the sequence $\{S_{\varepsilon}(t_n)\xi_n\}_{n=1}^{\infty}$ is precompact. Let $\xi_{u_n}(t) := S_{\varepsilon}(t + t_n)\xi_n$, $t \geq -t_n$ be a sequence of solutions of problem (2.1) associated with this sequence. Extending it by zero for $t \leq -t_n$, from the key dissipative estimate (3.1), we infer that $\xi_{u_n}$ is uniformly bounded in the space $L^\infty(\mathbb{R}, E^1)$. Thus, without loss of generality, we may assume that $\xi_{u_n} \to \xi_u$ weakly-star in $L^\infty_{\text{loc}}(\mathbb{R}, E^1)$.

Moreover, utilizing the fact that $u_n$ solves (2.1), we also get that $\xi_{u_n}(t) \to \xi_u(t)$ for every fixed $t \in \mathbb{R}$ as well as the fact that the limit function $\xi_u \in \mathcal{K}_{\varepsilon}$. In particular,

$$\xi_{u_n}(0) \to \xi_u(0)$$

and to verify the desired asymptotic compactness we only need to check that this convergence is in fact strong. We will utilize the so-called energy method, see [2, 11] for more details. Namely, multiplying equation (2.1) by $\partial_t Au + \alpha Au$, where $\alpha > 0$ is a big number which will be determined later, we get

$$\frac{d}{dt} \left( \frac{1}{2} \left( \|\xi_u\|_{E^1}^2 + \alpha \|u\|_{H^1}^2 + 2\alpha \varepsilon \langle \partial_t u, Au \rangle \right) \right) +$$

$$+ \beta \left( \|\xi_u\|_{E^2}^2 + \alpha \|u\|_{H^1}^2 + 2\alpha \varepsilon \langle \partial_t u, Au \rangle \right) + (1 - (\alpha + \beta)\varepsilon)\|\partial_t u\|_{H^1}^2 +$$

$$+ (\alpha - \beta)\|u\|_{H^2}^2 - \alpha \beta \|u\|_{H^1}^2 + 2\varepsilon \alpha \beta \langle \partial_t u, Au \rangle +$$

$$+ ((u, \nabla_x)u, \partial_t Au + \alpha Au) = (g, \partial_t Au + \alpha Au),$$

where $\beta > 0$ is also a positive number. The validity of the energy identity for strong solutions can be verified in a standard way, say, by approximating the solution $u$ by $P_N u$, see e.g. [11]. Denoting

$$E_{\varepsilon}(\xi_u) := \|\xi_u\|_{E^1}^2 + \alpha \|u\|_{H^1}^2 + 2\alpha \varepsilon \langle \partial_t u, Au \rangle,$$

$$N(\xi_u) := \alpha ((u, \nabla_x)u, Au) - ((\partial_t u, \nabla_x)u, Au) -$$

$$- ((u, \nabla_x)\partial_t u, Au) - \alpha (g, Au)$$

and

$$L(\xi_u) := (1 - (\alpha + \beta)\varepsilon)\|\partial_t u\|_{H^1}^2 + (\alpha - \beta)\|u\|_{H^2}^2 - \alpha \beta \|u\|_{H^1}^2 - 2\varepsilon \alpha \beta \langle \partial_t u, Au \rangle,$$
we transform the identity as follows:

$$\frac{1}{2} \frac{d}{dt} E^1_\varepsilon(\xi_u) + \beta E^1_\varepsilon(\xi_u) + L(\xi_u) + N(\xi_u) = \frac{d}{dt} I(u), \quad (4.12)$$

where $I(u) := (g, Au) - ((u, \nabla_x)u, Au)$.

Using the last identity for solutions $u_n$ instead of $u$, after integration in time we get

$$E^1_\varepsilon(\xi_{u_n}(0)) + 2 \int_{-t_n}^{0} e^{2\beta s} (L(\xi_{u_n}(s)) + N(\xi_{u_n}(s)) + \beta I(u_n(s))) \, ds =$$

$$= 2I(u_n(0)) + 2e^{-\beta t_n} (E^1_\varepsilon(\xi_{u_n}(-t_n)) - I(u_n(-t_n))). \quad (4.13)$$

Our task now is to pass to the limit $n \to \infty$ in this identity. To this end, we first note that due to the boundedness of the sequence $\xi_n$ and the fact that $t_n \to \infty$, the terms containing $u_n(-t_n)$ vanish in the limit. Next, due to already proved weak convergence in $E^1$, the passage to the limit in terms containing the functional $I$ is also immediate. Let us now pass to the limit in terms containing $L(\xi)$ and $N(\xi)$. The only non-trivial term in $N(\xi_{u_n})$ is $((u_n, \nabla_x) \partial_t u_n, Au_n)$ (for other terms of $N$ the proved weak convergence is enough to pass to the limit). We write this terms as follows:

$$((u_n, \nabla_x) \partial_t u_n, Au_n) = ((u, \nabla_x) \partial_t u_n, Au_n) + ((u_n - u, \nabla_x) \partial_t u_n, Au_n).$$

Since the sequence $\xi_{u_n}$ is bounded in $E^1$ and $u_n \to u$ strongly in the space $C^1_{loc}(\mathbb{R} \times \overline{\Omega})$, the second term tends to zero and we only need to study the first one. Let us introduce the quadratic form

$$Q(\xi_{u_n}) := L(\xi_{u_n}) - ((u, \nabla_x) \partial_t u_n, Au_n).$$

Then, since the $C$-norm of $u(t)$ is uniformly bounded with respect to $\varepsilon \to 0$ and $t \in \mathbb{R}$, for sufficiently small $\varepsilon > 0$, we may fix $\alpha > 0$ large enough and $\beta > 0$ small enough (both independent of $\varepsilon \to 0$) such that both quadratic forms $E^1_\varepsilon(\xi_{u_n})$ and $Q(\xi_{u_n})$ will be positive definite. Then, passage to the weak limit gives us that

$$\int_{-\infty}^{0} e^{\beta s} (L(\xi_u(s)) + N(\xi_u(s)) + \beta I(u(s))) \, ds \leq$$

$$\leq \liminf_{n \to \infty} \int_{-t_n}^{0} e^{\beta s} (L(\xi_{u_n}(s)) + N(\xi_{u_n}(s)) + \beta I(u_n(s))) \, ds \quad (4.14)$$
and, therefore
\[
\limsup_{n \to \infty} E_1^\varepsilon(\xi u_n(0)) + 2 \int_{-\infty}^{0} e^{\beta s} (L(\xi u(s)) + N(\xi u(s)) + \beta I(u(s))) \, ds \leq 2I(u(0)). \tag{4.15}
\]
Comparing this with the energy identity for the limit solution \(u(t)\), we infer
\[
\limsup_{n \to \infty} E_1^\varepsilon(\xi u_n(0)) \leq E_1^\varepsilon(\xi u(0)) \leq \liminf_{n \to \infty} E_1^\varepsilon(\xi u_n(0)),
\]
where the second inequality follows from the weak convergence \(4.9\) and the fact that \(E_1^\varepsilon(\xi)\) is positive definite. Therefore,
\[
\lim_{n \to \infty} E_1^\varepsilon(\xi u_n(0)) = E_1^\varepsilon(\xi u(0))
\]
and, indeed, \(\xi u_n(0) \to \xi u(0)\) strongly in \(E^1\). Thus, the asymptotic compactness is verified and the proposition is proved. \(\square\)

We now turn to study the singular limit \(\varepsilon \to 0\). We start with the convergence of individual trajectories. Let \(u_\varepsilon(t)\) and \(u_0(t)\) be the solutions of hyperbolic Navier-Stokes equation \(2.1\) and the limit classical Navier-Stokes system \(2.8\). Let also
\[
w(t) = w_\varepsilon(t) := u_\varepsilon(t) - u_0(t).
\]
and \(w(t) = w_\varepsilon(t) := u_\varepsilon(t) - u_0(t)\). Then, as actually established in the proof of Theorem \(3.1\)
\[
\|w_\varepsilon(t)\|_{H}^2 \leq C\varepsilon\|\xi u_\varepsilon(0)\|_{E^1}^2 e^{Kt}, \tag{4.17}
\]
where the constants \(C\) and \(K\) are independent of \(\varepsilon \to 0\), see also \(7\) for similar estimates. However, this estimate is far from being optimal. In order to improve it, we add the first boundary layer term at \(t = 0\) and write
\[
u_\varepsilon(t) = u_0(t) + \varepsilon(\partial_t u_\varepsilon(0) - \partial_t u_0(0))(1 - e^{-\frac{t}{\varepsilon}}) + \tilde{w}_\varepsilon(t). \tag{4.18}
\]
where the value of \(\partial_t u_0(0)\) is determined by \(u_0(0)\) via equation \(2.8\). Then, the following result holds.

**Proposition 4.4.** Let the initial data \(\xi u_\varepsilon(0)\) satisfy the assumptions of Theorem \(3.1\). Then the remainder \(\tilde{w}(t) = \tilde{w}_\varepsilon(t)\) satisfies the following estimate:
\[
\|\xi \tilde{w}(t)\|_{E^{-1}} \leq \varepsilon Q(\|\xi u_\varepsilon(0)\|_{E^1}) e^{Kt}, \tag{4.19}
\]
where the constant \(K\) and the function \(Q\) are independent of \(\varepsilon \to 0\).
Proof. Indeed, the remainder \( \tilde{w} \) solves the equation
\[
\varepsilon \partial_t^2 \tilde{w} + \partial_t \tilde{w} + A \tilde{w} = -P[\text{div}(u_\varepsilon \otimes \tilde{w}) + \text{div}(\tilde{w} \otimes u_0)] - \\
- \varepsilon P[\text{div}(u_\varepsilon \otimes w_1) + \text{div}(w_1 \otimes u_0) - Aw_1 + \partial_t^2 u_0], \quad \tilde{w}|_{t=0} = 0,
\]
where \( w_1(t) := (\partial_t u_\varepsilon(0) - \partial_t u_0(0))(1 - e^{-\frac{t}{\varepsilon}}) \). Multiplying this equation by \( \partial_t A^{-1} \tilde{w} \) and using the obvious estimate
\[
\left| \langle P \text{div}(u_1 \otimes u_2), \partial_t A^{-1} \tilde{w} \rangle \right| \leq \\
\leq C \|u_1 \otimes u_2\|_{L^2} \|\partial_t \tilde{w}\|_{H^{-1}} \leq C \|u_1\|_{L^\infty} \|u_2\|_{L^2} \|\xi\|_{E_\varepsilon^{-1}}^{1/2} \quad (4.21)
\]
together with the facts that \( u_\varepsilon \) and \( u_0 \) are uniformly bounded in \( C \) as well as \( w_1 \) is uniformly bounded in \( H \), we end up with the following inequality
\[
\frac{d}{dt} \|\xi\|_{E_\varepsilon^{-1}}^2 \leq K \|\xi\|_{E_\varepsilon^{-1}}^2 + \varepsilon \|\xi\|_{E_\varepsilon^{-1}} + \varepsilon^2 \|\partial_t^2 u_0\|_{H^{-1}}^2
\]
for some positive constant \( K \) depending on the initial data, but being independent of \( \varepsilon \to 0 \). So, we only need to estimate the second term in the right-hand side. To this end, we integrate by parts and arrive at
\[
\frac{d}{dt} \left( \|\xi\|_{E_\varepsilon^{-1}}^2 - \varepsilon \|w_1, \tilde{w}\|_H \right) - K \left( \|\xi\|_{E_\varepsilon^{-1}}^2 - \varepsilon \|w_1, \tilde{w}\|_H \right) \leq \\
\leq \varepsilon \|\partial_t w_1\|_H \|\tilde{w}\|_H + K \varepsilon \|w_1\|_H \|\tilde{w}\|_H + \varepsilon^2 \|\partial_t^2 u_0\|_{H^{-1}}^2 \quad (4.22)
\]
which in turn gives
\[
\frac{d}{dt} \left( \|\xi\|_{E_\varepsilon^{-1}}^2 - \varepsilon \|w_1, \tilde{w}\|_H \right) - (K + C \|\partial_t w_1\|_H) \left( \|\xi\|_{E_\varepsilon^{-1}}^2 - \varepsilon \|w_1, \tilde{w}\|_H \right) \leq \\
\leq C \varepsilon^2 \left( \|\partial_t u_0\|_{H^{-1}}^2 + \|w_1\|_H^2 + \|\partial_t w_1\|_H \right). \quad (4.23)
\]
Integrating this inequality and using the facts that we have the uniform control of the \( L^2(V^*) \)-norm of \( \partial_t^2 u_0 \) and the \( L^1(H) \)-norm of \( \partial_t w_1 \), we get
\[
\|\xi\|_{E_\varepsilon^{-1}} \leq \varepsilon Q\|\xi_{u_\varepsilon}(0)\|_{E_\varepsilon} e^{Kt}. \quad (4.24)
\]
Thus, it only remains to estimate the \( H^{-2} \)-norm of \( \partial_t \tilde{w}(t) \). This can be done exactly as at the end of the proof of Theorem 3.1 using the fact that \( \partial_t^2 u_0 \) is bounded in \( H^{-2} \) and the proposition is proved. \( \Box \)

Our next task is to compare the attractors \( \mathcal{A}_\varepsilon \) of hyperbolic Navier-Stokes system (2.1) with the global attractor of the limit equation (2.8). To do this, we note that the solution operator of the limit equation is defined on a different space (since the initial data \( \partial_t u|_{t=0} \) is not required for solving the limit parabolic equation). To overcome this difficulty,
we introduce following the standard scheme the phase space of the limit problem as follows:

\[ \Phi_0 := \{(u_0, u_1) \in \mathcal{E}_0^1, \quad u_1 = -Au_0 - P(u_0, \nabla x)u_0 + g \} \quad (4.25) \]

where \( \mathcal{E}_0^1 = D(A) \times \mathcal{H} \), and introduce the solution semigroup \( S_0(t) : \Phi_0 \to \Phi_0 \) by the natural expression

\[ S_0(t)(u_0(0), \partial_t u_0(0)) := (u_0(t), \partial_t u_0(t)) \]

where \( u_0(t) \) is the solution of the Navier-Stokes problem (2.8).

It is well-known that the semigroup \( S_0(t) \) associated with the classical Navier-Stokes equation possesses a global attractor \( A_0 \) in \( \Phi_0 \) which is related with the usual attractor \( \bar{A}_0 \) of the Navier-Stokes problem in the phase space \( H^2 \) via the following expression:

\[ A_0 = \{(u_0, u_1), \quad u_0 \in \bar{A}_0, \quad u_1 = -Au_0 - P(u_0, \nabla x)u_0 + g \}. \quad (4.26) \]

The next proposition gives the strong convergence of the attractors \( A_\varepsilon \) to the limit attractor \( A_0 \).

**Proposition 4.5.** The family of attractors \( A_\varepsilon \) is upper semi continuous at \( \varepsilon = 0 \) in the topology of the space \( E_0^1 \), i.e., for every neighbourhood \( \mathcal{O}(A_0) \) of the limit attractor \( A_0 \), there exists \( \varepsilon_0 = \varepsilon_0(\mathcal{O}) > 0 \) such that

\[ A_\varepsilon \subset \mathcal{O}(A_0) \quad (4.27) \]

for all \( \varepsilon < \varepsilon_0 \).

**Proof.** Indeed, according to the general theory, see e.g. [1], it is sufficient to show that for every sequence \( \varepsilon_n \to 0 \) and every sequence \( \xi_{u_n} \in \mathcal{K}_{\varepsilon_n} \) there is a subsequence \( \xi_{u_{n_k}} \) which is convergent in \( C_{\text{loc}}(\mathbb{R}, E_0^1) \) to some \( \xi_{u_0} \in \mathcal{K}_0 \). Since according to the dissipative estimate, the sequence \( \xi_{u_n} \) is uniformly bounded in \( C_b(\mathbb{R}, E_0^1) \), we may assume without loss of generality that

\[ \xi_{u_n} \to \xi_{u_0} \quad (4.28) \]

in the space \( L_{\text{loc}}^\infty(\mathbb{R}, E_0^1) \) and passing to the weak limit \( n \to \infty \) in the equations for \( u_n(t) \), we see that \( \xi_{u_0} \in \mathcal{K}_0 \). It remains to note that the strong convergence can be derived from the weak convergence using the energy method described in the proof of Proposition 4.3. Thus, the proposition is proved. \( \square \)

### 5. Generalizations and concluding remarks

In this section, we briefly discuss possible generalizations of the obtained results and related topics. We start with the comments concerning the 3D case.
5.1. Hyperbolic relaxation of the NS equations in 3D. Note that our estimates related with the closeness of the solutions of the relaxed problem and the initial parabolic one are based on the embedding $H^2(\Omega) \subset C(\Omega)$ and for this reason work in 3D case as well. The principal difference is related with the initial equation itself. Indeed, in contrast to the 2D case, we cannot solve globally the NS problem in 3D in the class of strong solutions, so we need to postulate it. This does not allow us to iterate estimate (3.24) and get the global existence of a solution for the perturbed system, so we may guarantee only the local result stated in the next proposition.

Proposition 5.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary and let $u_0 \in H^2$ be such that there exists a global strong solution $u_0(t) \in H^2$, $t \geq 0$, of the classical Navier-Stokes problem (2.8). Then, for every $T > 0$ and $R > 0$, there exists $\varepsilon_0 = \varepsilon_0(T, R) > 0$ such that for all $\varepsilon \leq \varepsilon_0$ and all initial data $u_0' \in H^1$ such that $\|\{u_0, u_0'\}\|_{E_1} \leq R$ there exist a unique strong solution $u_\varepsilon(t)$, $t \in [0, T]$ of problem (2.1) satisfying the initial condition

$$
\xi_{u_\varepsilon}|_{t=0} = \{u_0, u_0'\}.
$$

Moreover, the analogue of Proposition 4.4 holds on the time interval $t \in [0, T]$.

The proof actually repeats the one given above for the 2D case with non-essential minor corrections, so we left it to the reader.

Remark 5.2. We also mention that there is an exceptional case where we expect the global existence of solutions near $u_0(t)$ for the relaxed problem. Namely, this is the case where this limit solution is asymptotically stable (in a proper sense). Indeed, in this case we may establish the global existence of strong solutions for 3D Navier-Stokes equations in a neighbourhood of the solution $u_0(t)$ (using, say, the standard arguments related with the implicit function theorem). After that it should be possible to iterate the analogue of estimate (3.24) and get the global existence of solutions of the relaxed equations in the neighbourhood of $u_0(t)$ if $\varepsilon > 0$ is small enough. We return to this problem somewhere else.

5.2. The case of 1D Burgers equation. We now discuss a possible blow up of solutions of the hyperbolic Navier-Stokes equations for $\varepsilon > 0$ and arbitrarily large initial data. We start with more simple hyperbolic relaxation of 1D Burgers equation

$$
\varepsilon \partial_t^2 u + \partial_t u + \partial_x (u^2) = \partial_x^2 u + g, \quad x \in [0, L], \quad \xi_u|_{t=0} = \xi_0
$$

(5.1)
endowed by the Dirichlet boundary conditions. Then, on the one hand the arguments given in Section 3 we see that the analogue of Theorem 3.1 holds. Namely, for every $R > 0$, there exists $\varepsilon_0 = \varepsilon_0(R)$ such that a unique global solution $\xi_u(t) \in \mathcal{E}^1_\varepsilon$ of problem (5.1) exists for all $\varepsilon \leq \varepsilon_0$ and the initial data $\xi_u(0)$ such that $\|\xi_u(0)\|_{\mathcal{E}^1_\varepsilon} \leq R$. Moreover, this solution satisfies the dissipative estimate (3.1).

On the other hand, this model is essentially simpler than the original Navier-Stokes problem and we are able to prove here that the solutions may blow up if the initial energy is large enough. To see this, we fix $\varepsilon = 1$ (which we may always assume due to scaling) and consider the following hyperbolic equation on the whole line $x \in \mathbb{R}$:

$$
\partial^2_t u + \partial_t u + \partial_x (u^2) = \partial^2_x u, \quad x \in \mathbb{R}, \quad \xi_u \big|_{t=0} = \xi_0.
$$

(5.2)

Actually this equation possesses the finite propagation speed property, so the boundary conditions are also not essential since the blowing up solution which we construct will be localized in space. Then, the following result can be proved.

**Proposition 5.3.** There exist smooth finitely supported initial data $\xi_u(0)$ such that the corresponding solution of (5.2) blows up in finite time.

**Proof.** Also this result is well-known (see e.g., [18]), for the convenience of the reader we reproduce key points of the proof here. We start with the finite propagation property which is the key technical tool for verifying the blow up.

Namely, if the support of the initial data $\xi_u(0)$ satisfies

$$
\text{supp } \xi_u(0) \subset [-R, R],
$$

then

$$
\text{supp } \xi_u(t) \subset [-R - t, R + t].
$$

(5.3)

To verify this, following [7] (see also references therein), we consider the cone $K$ in the $(x, t)$-plane with base $-R \leq x \leq R$ and vertex $t = R$, $x = 0$. Let, for $0 \leq t \leq R$,

$$
e(t) := \int_{-(R-t)}^{R-t} \left( (\partial_t u(t, x))^2 + (\partial_x u(t, x))^2 \right) dx
$$

be the energy at the section of the cone $K$ at time $t$. Then
\[
\frac{d}{dt} e(t) = 2 \int_{-(R-t)}^{R-t} \left( \partial_t u \partial_u u + \partial_x u \partial_x u \right) dx - \\
\left[ ((\partial_t u)^2 + (\partial_x u)^2)|_{(t,R-t)} + ((\partial_t u)^2 + (\partial_x u)^2)|_{(t,-(R-t))} \right] = \\
2 \int_{-(R-t)}^{R-t} \left( \partial_t u (\partial_t u - \partial_x u) \right) dx - \\
\left[ ((\partial_t u)^2 + 2\partial_t u \partial_x u + u(\partial_x u)^2)|_{(t,R-t)} - \\
- [((\partial_t u)^2 + 2\partial_t u \partial_x u + (\partial_x u)^2)|_{(t,-(R-t))} \right] \leq \\
2 \int_{-(R-t)}^{R-t} \left( \partial_t u (\partial_t u - \partial_x u) \right) dx = 2 \int_{-(R-t)}^{R-t} \left( \partial_t u (-\partial_t u - 2\partial_x u) \right) dx \leq \\
4 \int_{-(R-t)}^{R-t} u \partial_t u \partial_x u dx \leq 2 \|u\|_{L^\infty} e(t).
\]

Now, if \( \text{supp} \xi u(0) \cap [-R, R] = \emptyset \), then \( e(0) = 0 \) and, hence, \( e(t) = 0 \) for \( 0 \leq t \leq R \). Therefore the solution vanishes in the cone \( K \), which is equivalent to (5.3).

We are now ready to verify the blow up. Following [18] (see also [7]), multiply equation (5.2) by \( e^{-x} \) and integrate over \( x \in \mathbb{R} \). Then, denoting
\[
y(t) := \int_{\mathbb{R}} e^{-x} u(t, x) \, dx,
\]
we get
\[
y''(t) + y'(t) = y(t) + \int_{\mathbb{R}} e^{-x} u^2(t, x) \, dx. \tag{5.4}
\]
All of the terms in this equation will be finite if we start from the initial data \( \xi u(0) \) with finite support due to the property (5.3). Let us fix an arbitrary \( T > 0 \) and assume that \( \text{supp} \xi u(0) \subset [-1, 1] \). Then, due to (5.3) and Jensen inequality, we can estimate the nonlinear term in (5.4)
\[
\int_{\mathbb{R}} e^{-x} u^2(t, x) \, dt \geq e^{-T-1} \int_{\mathbb{R}} (e^{-x} u(t, x))^2 \, dx \geq e^{-T-1} (2(T+1)^{-1} y^2(t) \tag{5.5}
\]
which gives
\[
y''(t) + y'(t) \geq y(t) + e^{-T-1} (2(T+1)^{-1} y^2(t), \quad t \in [0, T] \tag{5.6}
\]
which guarantees the blow up of solutions of (5.2) if the localized initial data is large enough, see [18] for details. Thus, the proposition is proved. \( \square \)
5.3. **Finite propagation approximation of the Navier-Stokes problem.** We emphasize once more that the finite propagation speed property is crucial for this method. Since for the initial hyperbolic Navier-Stokes system this property clearly fails due to the presence of the non-local pressure term, the method is not applicable to these equation and one should find an alternative way to establish the finite time blow up. However, if we modify slightly the approximation scheme for the original Navier-Stokes problem, we may get the finite propagation speed property. Namely, let us consider the following problem:

\[
\varepsilon \partial_t^2 u + \partial_t u + \text{div}(u \otimes u) = \Delta_x u + \frac{1}{\alpha} \nabla_x \text{div} u + g, \quad u|_{\partial \Omega} = 0, \tag{5.7}
\]

where \(\alpha > 0\) is one more small parameter. Then, on the one hand, as not difficult to see, the solutions of (5.7) converge as \(\alpha \to 0\) to the corresponding solutions of the hyperbolic Navier-Stokes problem (2.1). On the other hand, equation (5.7) possesses the finite propagation speed property, see [7] for the details. When the finite propagation speed property is established, the blow up of smooth solutions for problem (5.7) can be verified exactly as for the case of Burgers equation, see [17] for the details.

5.4. **Connection with viscoelastic fluids.** Note that the existence of blow up solutions discussed is related mainly with the fact that the hyperbolic relaxation (2.1) of the Navier-Stokes equations does not possess a reasonable energy functional for \(\varepsilon > 0\). By this reason, the model (2.1) looks a bit non-physical. Fortunately, this drawback can be easily corrected by adding an extra small term to (2.1) which does not destroy the hyperbolic structure of the equations and the estimates obtained above, but restores the energy identity. Indeed, let us consider the following particular case of the so-called Jeffrey model for viscoelastic fluids:

\[
\begin{align*}
\partial_t u + (u, \nabla_x)u + \nabla_x p &= \text{div} \sigma + g, \quad \text{div} u = 0, \\
\varepsilon \partial_t \sigma + \sigma &= \gamma,
\end{align*}
\tag{5.8}
\]

where \(\gamma := \frac{1}{2} (\nabla_x u + \nabla^*_x u)\) is a strain rate tensor, see [5] [6] for more details. Then, integrating the second equation in time, we may write (5.8) as an Euler equation with memory term

\[
\partial_t u + (u, \nabla_x)u + \nabla_x p = \int_{-\infty}^{t} \beta(t - s) \Delta_x u(s) \, ds + g, \quad \beta(s) = \frac{1}{\varepsilon} e^{-\frac{s}{\varepsilon}}.
\]
Alternatively, excluding $\sigma$ from the first equation by differentiating it in time and using the second one gives

$$
\varepsilon \partial_t^2 u + \varepsilon \partial_t[(u, \nabla_x)u] + \partial_t u + (u, \nabla_x)u + \nabla_x p = \Delta_x u + g
$$

(5.9)

which coincides with (2.11) up to the desired extra term $\varepsilon \partial_t[(u, \nabla_x)u]$. This extra term allows to restore the energy identity. Namely, multiplying the first equation of (5.8) by $u$ and integrating over $x$, we get the energy identity of the form

$$
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{L^2}^2 + \varepsilon \|\text{div} \sigma\|_{L^2}^2 \right) + \|\text{div} \sigma\|_{L^2}^2 = (u, g).
$$

(5.10)

This identity guarantees at least the global existence of weak solutions and destroys the blow up mechanism described above. We return to this problem somewhere else.

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