Invariance of the Noether charge

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Abstract

Surprisingly, an interesting property of the Noether charge that it is by itself invariant under the corresponding symmetry transformation is never discussed in quantum field theory or classical mechanics textbooks we have checked. This property is also almost never mentioned in articles devoted to Noether’s theorem. Nevertheless, to prove this property in the context of Lagrangian formalism is not quite trivial and the proof, outlined in this article, can constitute an useful and interesting exercise for students.

Keywords: Noether theorem, classical mechanics, classical field theory

1. Introduction

Noether’s theorem [1–3] is a fundamental result which establishes a connection between continuous symmetries and conservation laws. Both concepts play a central role in modern physics. It is not surprising, therefore, that it is discussed in many quantum and classical field theory textbooks [4–16], as well as in some classical mechanics textbooks of various levels of sophistication [17–28]. It is surprising, however, that it is hard to find an answer in the quoted literature to the natural question of how these conserved Noether charges are affected by the corresponding symmetry transformations. Moreover, neither Hill’s well-known review [29] nor various pedagogical expositions of Noether’s theorem [30–36] discuss this question.

Our intuitive understanding is that symmetry is a property of the system to remain unchanged under some kind of transformation. Noether charges are among important characteristics of the system which determine its physical state. Therefore a natural expectation is that Noether charges should not be changed under the corresponding symmetry transformations. This is indeed the case. However the invariance property of the Noether charge is ‘rather hard to prove’ in Lagrangian formalism [37]. In the context of classical mechanics, the proofs were given by Lutzky, for the case of a system with one degree of freedom [38], and by Sarlet and Cantrijn for the general case [37] (see also [39, 40]).
In the field theory context, the invariance of the Noether charge follows from a more general mathematical result first proved by Khamitova [41] (after it was conjectured by Nail Ibragimov). Later Khamitova’s result describing the action of symmetries on conservation laws was reformulated in somewhat different language as proposition 5.64 in Olver’s book [42].

The aim of this note is to give a pedagogical exposition of this interesting property of the Noether charge in the frameworks of both classical mechanics and field theory.

2. Noether theorem in classical mechanics

Let us consider a classical mechanical system whose dynamics is determined by Hamilton’s variational principle

$$\delta \int_{t_1}^{t_2} L(t, q, \dot{q}) \, dt = 0$$

yielding the Euler–Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0.$$  

Here $q$ and $\dot{q}$ are shorthand notations for generalized coordinates $q = (q^1(t), q^2(t), \ldots, q^n(t))$ and the corresponding velocities $\dot{q} = (\dot{q}^1(t), \dot{q}^2(t), \ldots, \dot{q}^n(t))$. An infinitesimal transformation

$$t' = t + \epsilon \tau(t, q), \quad q^i'(t') = q^i(t) + \epsilon \xi^i(t, q)$$

is said to be a symmetry of the system considered if it leaves invariant the Euler–Lagrange equations of motion. A sufficient condition that the transformation (3) is a symmetry is provided by the existence of such function $K(t, q)$ that up to the first order in the transformation parameter $\epsilon \ll 1$ the following identity holds true:

$$L(t', q', \dot{q}'), \frac{dq^i(t')}{dt'}(t) \frac{d\dot{q}^i(t)}{dt} = L(t, q(t), \dot{q}(t)) + \epsilon \frac{dK(t, q)}{dt},$$

where (we use Einstein summation convention that repeated indexes are implicitly summed over)

$$\frac{dK(t, q)}{dt} = \frac{\partial K(t, q)}{\partial t} + \dot{q}^i \frac{\partial K(t, q)}{\partial q^i}.$$  

Indeed, in this case the new action integral

$$S' = \int_{t_1}^{t_2} L(t', q', \dot{q}'), \frac{dq^i(t')}{dt'}(t) \frac{d\dot{q}^i(t)}{dt} \, dt'$$

remains quasi-invariant:

$$S' = \int_{t_1}^{t_2} L(t', q', \dot{q}'), \frac{dq^i(t')}{dt'}(t) \frac{d\dot{q}^i(t)}{dt} \, dt' = S + \epsilon \left[ K(t_2, q(t_2)) - K(t_1, q(t_1)) \right].$$
and we will have \( \delta S' = \delta S + \epsilon \delta [K(t_2, q(t_2)) - K(t_1, q(t_1))] = 0 \), if \( \delta S = 0 \), because it is assumed in the Hamilton’s variational principle that variations of the generalized coordinates vanish at the initial and final points (at \( t = t_1 \) and \( t = t_2 \) respectively).

The velocity transformation law under (3) is the following

\[
\frac{dq'^i}{dt'} = \frac{dq^i}{dt} + \epsilon \frac{d\xi^i}{d\tau} \approx \dot{q}^i + \epsilon \left( \dot{\xi}^i - \dot{q}^i \right),
\]

(7)

where a dot denotes total derivative with respect to time \( t \). For example,

\[
\dot{\tau} = \frac{\partial \tau}{\partial t} + \dot{q}^i \frac{\partial \tau}{\partial q^i}.
\]

Therefore, we can introduce the generator of the transformation (3),

\[
\hat{G} = \tau(t, q) \frac{\partial}{\partial t} + \xi(t, q) \frac{\partial}{\partial q^i} + \eta(t, q, \dot{q}) \frac{\partial}{\partial \dot{q}^i},
\]

(8)

with

\[
\eta(t, q, \dot{q}) = \dot{\xi}^i - \dot{q}^i \dot{\tau},
\]

(9)

so that for any function \( f(t, q, \dot{q}) \) its variation under the transformation (3) is

\[
\delta f = f'(t', q', \dot{q}') \frac{dq'^i}{dt'} - f(t, q(t), \dot{q}(t)) = \epsilon \hat{G}(\dot{f}).
\]

(10)

Sometimes it is necessary to extend (8) by including higher derivatives. For example, in light of (7) we have

\[
\frac{d^2q'^i}{dt'^2} = \frac{d}{dt'} \left( \frac{dq'^i}{dt'} \right) \approx \frac{dq^i}{dt} + \epsilon \frac{d\eta^i}{d\tau} \approx \ddot{q}^i(t) + \epsilon \dddot{\xi}^i(t, q, \dot{q}, \ddot{q}).
\]

(11)

where

\[
\dddot{\xi}^i(t, q, \dot{q}, \ddot{q}) = \dot{\eta}^i - \dot{q}^i \dot{\tau}.
\]

(12)

Therefore the prolongation of the operator (8) on the space \((t, q, \dot{q}, \ddot{q})\) has the form (the same symbol will be used both for the transformation operator and any of its prolongations)

\[
\hat{G} = \tau(t, q) \frac{\partial}{\partial t} + \xi(t, q) \frac{\partial}{\partial q^i} + \eta(t, q, \dot{q}) \frac{\partial}{\partial \dot{q}^i} + \zeta(t, q, \dot{q}, \ddot{q}) \frac{\partial}{\partial \ddot{q}^i},
\]

(13)

Introducing the Lie characteristic function

\[
\sigma^i(t, q, \dot{q}) = \xi^i(t, q) - \dot{q}^i \tau(t, q),
\]

(14)

and using

\[
\eta^i - \tau \dot{q}^i = \sigma^i, \quad \zeta^i - \tau \ddot{q}^i = \ddot{\sigma}^i,
\]

(15)

along with

\[
\frac{\partial}{\partial t} = \frac{d}{dt} - \dot{q}^i \frac{\partial}{\partial q^i} - \ddot{q}^i \frac{\partial}{\partial \ddot{q}^i} - \dddot{q}^i \frac{\partial}{\partial \dddot{q}^i},
\]

(16)

the generator (13) can be rewritten in the form

\[
\hat{G} = \tau \frac{d}{dt} + \sigma^i \frac{\partial}{\partial q^i} + \sigma^i \frac{\partial}{\partial \dot{q}^i} + \sigma^i \frac{\partial}{\partial \ddot{q}^i} = \tau \frac{d}{dt} + \hat{L}_\beta,
\]

(17)

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Here we have introduced the canonical Lie–Bäcklund operator \[ L_B = \sigma^i \frac{\partial}{\partial q^i} + \sigma^i \frac{\partial}{\partial q^i} + \sigma^i \frac{\partial}{\partial q^i}. \] (18)

Although we shall not particularly need this fact here, the same simple pattern continues to hold for prolongations to higher jet spaces (by including higher derivatives of \( q^i \)) [43] and sometimes it is technically more convenient to work with completely prolonged operators. For example, let us show that the total time derivative operator commutes with the canonical Lie–Bäcklund operator \[ L_B \]. Assuming that \( k \) and \( l \) indexes run from zero to infinity, we write

\[
\left[ \frac{d}{dt}, L_B \right] = \left[ \frac{d}{dt}, \sigma^{ij} \frac{\partial}{\partial q^{ij}} \right] = \sigma^{ij} \left[ \frac{d}{dt}, \frac{\partial}{\partial q^{ij}} \right] = \sigma^{ij} \left( \frac{d}{dt} \right) \frac{\partial}{\partial q^{ij}} + \sigma^{ij} \left[ \frac{d}{dt}, \frac{\partial}{\partial q^{ij}} \right] = 0. \] (19)

On the other hand,

\[
\left[ \frac{d}{dt}, L_B \right] = \left[ \frac{d}{dt}, \frac{\partial}{\partial q^{ij}} \right] = \frac{\partial}{\partial t} + q^{(k+1)} \frac{\partial}{\partial q^{(k)}} \right] = -\delta^{k+1} \frac{\partial}{\partial q^{(k)}}, \] (20)

where \( \delta^{k+1} \) denotes the Kronecker delta function. Substituting this into (19), we get

\[
\left[ \frac{d}{dt}, L_B \right] = \sigma^{ij} \delta^{k+1} \frac{\partial}{\partial q^{ij}} = \sigma^{ij} \frac{\partial}{\partial q^{ij}} = 0. \] (21)

The canonical Lie–Bäcklund operator determines the so called vertical variation

\[
\tilde{f} = f(t, q^i(t), q^i(t)) - f(t, q^i(t), q^i(t)) = \epsilon \right L_B (f), \] (22)

which is caused solely by the changes in functional forms of generalized coordinates and their derivatives. In particular

\[
\tilde{q}^i = q^i(t) - q^i(t) = q^i(t) - q^i(t) \approx \epsilon (\xi^i - q^i(t)). \] (23)

Using

\[
\frac{d t'}{d t} = 1 + \epsilon \hat{\tau} (t, q, \dot{q}), \] (24)

and

\[
\epsilon \hat{\tau} (t, q, \dot{q}) L \left( t'(t), q^i(t'), \frac{d q^i(t')}{d t'} \right) \approx \epsilon \hat{\tau} (t, q, \dot{q}) L(t, q, \dot{q}), \] (25)

we get from (4)

\[
\hat{G}(L) = \tau L + \hat{L}_B (L) = \hat{K} - \tau L, \] (26)

which implies

\[
\hat{L}_B (L) = \frac{d}{d t} (K - \tau L). \] (27)
On the other hand,

\[ L_B(L) = \sigma^i \frac{\partial L}{\partial \dot{q}^i} + \dot{\sigma}^i \frac{\partial L}{\partial \dot{q}^i} = \sigma^i \left( \frac{\partial L}{\partial \dot{q}^i} - \frac{d}{dr} \frac{\partial L}{\partial \ddot{q}^i} \right) + \frac{d}{dr} \left( \sigma^i \frac{\partial L}{\partial \dot{q}^i} \right) = \sigma^i \frac{\delta L}{\delta q^i} + \frac{d}{dr} \left( \sigma^i \frac{\partial L}{\partial \dot{q}^i} \right), \]

where we have introduced the Euler–Lagrange operator (variational derivative)

\[ \frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - \frac{d}{dr} \frac{\partial}{\partial \dot{q}^i}. \]

Its prolongations to higher jet spaces can be read from the expression [40]

\[ \frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \sum_{l \geq 1} (-1)^l \frac{d^l}{dr^l} \frac{\partial}{\partial q^i^{(l)}}. \]

Equations (27) and (28) imply the validity of the so-called Rund–Trautman identity [44, 45]

\[ \frac{d}{dr} \left( K - \tau L - \sigma^i \frac{\partial L}{\partial \dot{q}^i} \right) = \sigma^i \frac{\delta L}{\delta q^i}, \]

from which the Noether theorem (in fact Noether’s first theorem) readily follows: for every continues symmetry transformation (3) there exits a conserved Noether charge

\[ Q = K - \tau L - \sigma^i \frac{\partial L}{\partial \dot{q}^i}. \]

Indeed, (31) and the Euler–Lagrange equation (2) guarantee that \( \dot{Q} = 0 \).

Sometimes \( K(t, q) \) is called the Bessel-Hagen function (see, for example, [35]), because Noether in her celebrated paper considered only \( K = 0 \) case and more general case of symmetries up to divergence were introduced later by Erich Bessel-Hagen [46]. However the problem was suggested to Bessel-Hagen by Noether herself [46, 47].

3. Invariance of the Noether charge in classical mechanics

The Noether charge (32) can be rewritten in the following way

\[ Q = K - \hat{N}(L), \]

where

\[ \hat{N} = \tau + \sigma^i \frac{\partial}{\partial q^i} + \left( \dot{\sigma}^i - \sigma^i \frac{d}{dr} \frac{\partial}{\partial \dot{q}^i} \right), \]

is the Ibragimov operator (in the more general form, it was introduced by Ibragimov [40, 43] under the name Noether operator. We find it more appropriate to call it Ibragimov operator).

The last term in (34) has no effect when working in the first jet space \((t, q, \dot{q})\) and that’s why (32) and (33) are equivalent on the \((t, q, \dot{q})\) space. So, at first sight, its introduction is superfluous. However this extra term will prove to be very useful as we are going now to show. Using (20), we get

\[ \frac{\partial}{\partial \dot{q}^i} \frac{d}{dr} = \left[ \frac{\partial}{\partial \dot{q}^i}, \frac{d}{dr} \right] + \frac{d}{dr} \frac{\partial}{\partial \dot{q}^i} = \frac{\partial}{\partial \dot{q}^i} + \frac{d}{dr} \frac{\partial}{\partial \dot{q}^i}, \]

(35)
and analogously
\[
\frac{\partial}{\partial \dot{q}^i} \frac{d}{dt} = \frac{\partial}{\partial q^i} + \frac{d}{dt} \frac{\partial}{\partial \dot{q}^i}.
\] (36)

Therefore
\[
\hat{N} \frac{d}{dt} = \tau \frac{d}{dt} + \sigma^i \left( \frac{\partial}{\partial q^i} + \frac{d}{dt} \frac{\partial}{\partial \dot{q}^i} \right) + \left( \dot{\sigma}^i - \sigma^i \frac{d}{dt} \right) \left( \frac{\partial}{\partial \dot{q}^i} + \frac{d}{dt} \frac{\partial}{\partial \ddot{q}^i} \right).
\] (37)

which simplifies to
\[
\hat{N} \frac{d}{dt} = \dot{\hat{G}} + \left( \dot{\sigma}^i - \sigma^i \frac{d}{dt} \right) \frac{d}{dt} \frac{\partial}{\partial \dot{q}^i}.
\] (38)

The last term can be neglected in the \((t, q, \dot{q})\) space and we get the following very useful identity (with above mentioned more general definition of \(\hat{N}\) it can be made strictly valid in all jet spaces [40])
\[
\hat{G} = \hat{N} \frac{d}{dt}.
\] (39)

Let us calculate the commutator
\[
\left[ \frac{d}{dt}, \hat{N} \right] = \left[ \frac{d}{dt}, \tau + \sigma^i \frac{\partial}{\partial q^i} + \left( \dot{\sigma}^i - \sigma^i \frac{d}{dt} \right) \frac{\partial}{\partial \dot{q}^i} \right].
\]

Neglecting the terms which are irrelevant in the first jet space \((t, q, \dot{q})\), we get
\[
\left[ \frac{d}{dt}, \hat{N} \right] = \tau + \dot{\sigma}^i \frac{\partial}{\partial \dot{q}^i} - \sigma^i \frac{\partial}{\partial \dot{q}^i} = \left( \dot{\sigma}^i - \sigma^i \frac{d}{dt} \right) \frac{\partial}{\partial \dot{q}^i} = \tau - \sigma^i \frac{d}{dt} \frac{\partial}{\partial \dot{q}^i}.
\] (40)

Therefore
\[
\left[ \hat{G}, \hat{N} \right] = \left[ \hat{N}, \hat{N} \right] = \hat{N} \left[ \frac{d}{dt}, \hat{N} \right] = \hat{N} \left( \tau - \sigma^i \frac{d}{dt} \frac{\partial}{\partial \dot{q}^i} \right).
\] (41)

Now we are well equipped to prove the invariance of the Noether charge. Indeed we have
\[
\hat{G}(Q) = \hat{G} \left( K - \hat{N}(L) \right) = \hat{G}(K) - \hat{G} \hat{N}(L).
\] (42)

But
\[
\hat{G} \hat{N}(L) = \left[ \hat{G}, \hat{N} \right](L) + \hat{N} \hat{G}(L),
\] (43)

which after using (26) and (41) and the Euler–Lagrange equations becomes
\[
\hat{G} \hat{N}(L) = \hat{N}(\tau L) + \hat{N}(K - \tau L) = \hat{N}(K).
\] (44)

Substituting this result into (42) and using (39), we get finally
\[
\hat{G}(Q) = \hat{G}(K) - \hat{N}(K) = \hat{N} \frac{d}{dt}(K) - \hat{N}(K) = 0.
\] (45)

As we see the Noether charge is indeed invariant under the corresponding symmetry transformation (3), as it should be according to our intuitive understanding of symmetry.
4. Noether theorem in classical field theory

Next we consider $n$-component classical field $u_a(x), a = 1, \ldots, n$ in the Minkowski space-time with coordinates $x^\mu$. It is assumed that the classical dynamics of the field is governed by the action principle

$$\delta S = \delta \int_\Omega \mathcal{L}(x, u, u_{\mu}) = 0.$$  \hfill (46)

Here $\Omega = [t_1, t_2] \times \mathbb{R}^3$ is the space–time domain and comma indicates differentiation with respect to $x$:

$$u_{a,\mu} = \frac{du_a(x)}{dx^{\mu}}. \hfill (47)$$

We shall proceed as much as possible in analogy with the classical mechanical case. In particular, the transformation

$$x'^{\mu} = x^{\mu} + \epsilon \tau^{\mu}(x, u), \quad u'_a(x') = u_a(x) + \epsilon \xi_a(x, u)$$  \hfill (48)

is a symmetry if the following holds true

$$\mathcal{L}(x'(x), u'(x'), \frac{du'(x')}{dx'}(x)) J(x) = \mathcal{L}(x, u(x), u_{\mu}(x)) + \epsilon K_{\mu}^{\nu}$$  \hfill (49)

for some functions $K^{\nu}(x, u)$. To avoid a confusion, for such functions comma denotes total differentiation with respect to the indicated component of $x$:

$$K_{\mu,\nu} = \frac{\partial K^{\mu}}{\partial x^{\nu}} = \frac{\partial K^{\mu}}{\partial x^{\nu}} + u_{a,\nu} \frac{\partial K^{\mu}}{\partial u_{a}}. \hfill (50)$$

At last, $J = \det[\partial x'^{\mu}/\partial x^{\nu}]$ is the Jacobian corresponding to the transformation $x \rightarrow x'$. Introducing the generator of the transformation (48), $\hat{G}$, and taking into account that

$$J \approx 1 + \epsilon \tau_{\mu}, \hfill (51)$$

the symmetry condition (49) can be rewritten in the form

$$\hat{G}(\mathcal{L}) = K_{\mu,\nu}^{\nu} - \tau_{\mu}^{\nu} \mathcal{L}. \hfill (52)$$

Under (48), the field derivatives transform as follows

$$\frac{du'_a(x')}{dx'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{du_a(x)}{dx^{\nu}} \approx \left( \delta_{\mu}^{\nu} - \epsilon \tau_{\mu}^{\nu} \right) \left( u_{\mu,\nu} + \epsilon \xi_{\mu,\nu} \right) \approx u_{a,\nu} + \epsilon \left( \xi_{a,\nu} - \tau_{\mu}^{\nu} u_{a,\mu} \right).$$ \hfill (53)

Therefore the generator $\hat{G}$ has the form

$$\hat{G} = \tau^{\mu} \frac{\partial}{\partial x^{\mu}} + \xi_a \frac{\partial}{\partial u_a} + \eta_{a,\mu} \frac{\partial}{\partial u_{a,\mu}}, \hfill (54)$$

where

$$\eta_{a,\mu} = \xi_{a,\mu} - \tau_{\mu}^{\nu} u_{a,\nu}. \hfill (55)$$
In complete analogy with (14) and (17), it is easy to rewrite the generator \( \hat{G} \) in the form
\[
\hat{G} = \tau^\mu \frac{d}{dx^\mu} + \sigma_a \frac{\partial}{\partial u_a} + \sigma_{a,\mu} \frac{\partial}{\partial u_{a,\mu}} = \tau^\mu \frac{d}{dx^\mu} + \hat{L}_B,
\]
with the Lie characteristic function
\[
\sigma_a = \xi_a - \tau^\mu u_{a,\mu}.
\]
Now we have
\[
\hat{L}_B(\mathcal{L}) = \sigma_a \left( \frac{\partial \mathcal{L}}{\partial u_a} - \frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial u_{a,\mu}} \right) + \frac{d}{dx^\mu} \left( \sigma_a \frac{\partial \mathcal{L}}{\partial u_{a,\mu}} \right) = \sigma_a \frac{\delta \mathcal{L}}{\delta u_a} + \frac{d}{dx^\mu} \left( \sigma_a \frac{\partial \mathcal{L}}{\partial u_{a,\mu}} \right)
\]
and, in combination with (52) and (56), (58) implies the validity of the field theoretical version of the Rund–Trautman identity
\[
\frac{d}{dx^\mu} \left( K^\mu - \tau^\mu \mathcal{L} - \sigma_a \frac{\partial \mathcal{L}}{\partial u_{a,\mu}} \right) = \sigma_a \frac{\delta \mathcal{L}}{\delta u_a}.
\]
Then the Euler–Lagrange equations
\[
\frac{\delta \mathcal{L}}{\delta u_a} = \frac{d \mathcal{L}}{d u_a} - \frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial u_{a,\mu}} = 0
\]
imply the existence of the conserved (divergence-free) current
\[
J^\mu = K^\mu - \tau^\mu \mathcal{L} - \sigma_a \frac{\partial \mathcal{L}}{\partial u_{a,\mu}}, \quad \frac{d}{dx^\mu} J^\mu = 0.
\]
The corresponding conserved Noether charge, associated with the symmetry transformation (48), is
\[
Q = \int J^\mu \, dx^\nu.
\]
So far, so good. However, unfortunately, here the simple analogy with the classical mechanical case ends and we need some extra labor to extend the proof of invariance of the Noether charge to the field theory case also.

5. Invariance of the Noether charge in classical field theory

Let us introduce again the Ibragimov operator
\[
\hat{N}^\mu = \tau^\mu + \sigma_a \frac{\partial}{\partial u_{a,\mu}},
\]
so that
\[
J^\mu = K^\mu - \hat{N}^\mu(\mathcal{L}).
\]
It is shown in the appendix that, in the first jet space \((x^\nu, u_a, u_{a,\nu})\), the following commutation relation, which will play an important role in our arguments below, holds true:
\[
\left[ \hat{G} + \tau^\nu, \hat{N}^\nu \right] = \tau^\nu \hat{N}^\nu.
\]
In fact, for suitably defined \( \hat{G} \) and \( \hat{N}^\nu \), (65) is valid in all jet spaces \([40]\). Now we use this commutation relation in the following way. We have
\[ \hat{G}\hat{N}^\mu (\mathcal{L}) = \left[ \hat{G} + \tau^\mu_\nu \hat{N}^\nu \right](\mathcal{L}) + \hat{N}^\mu \left( \hat{G} + \tau^\mu_\nu \right)(\mathcal{L}) - \tau^\mu_\nu \hat{N}^\mu (\mathcal{L}), \]  

which after using (52) and (65) becomes

\[ \hat{G}\hat{N}^\mu (\mathcal{L}) = \tau^\mu_\nu \hat{N}^\nu (\mathcal{L}) + \hat{N}^\mu \left( K^\nu_\mu \right) - \tau^\mu_\nu \hat{N}^\mu (\mathcal{L}). \]  

Let us substitute here \( \hat{N}^\mu (\mathcal{L}) = K^\mu - J^\mu \) from (64) and rearrange the terms. As a result we get [40]

\[ \hat{G} \left( J^\mu \right) + \tau^\mu_\nu J^\mu - \tau^\mu_\nu K^\nu = \hat{G} \left( K^\mu \right) + \tau^\mu_\nu K^\mu - \tau^\mu_\nu K^\nu - \hat{N}^\mu \left( K^\nu_\mu \right). \]  

Of course, this is far more complicated result than (45) and it is not immediately obvious how it can lead to invariance of the corresponding Noether charge. Nevertheless (68) indeed imply this invariance, as we now will show.

First of all it is necessary to understand what the invariance of the Noether charge does mean in the context of field theory. Let \( \mathbf{x} = (x^0 - \epsilon \tau^0(x, u), \mathbf{x}) \), so that after the transformation (48) \( \mathbf{x}' = (x^0, \mathbf{x}) \). The Noether charge \( Q \) does not depend on time. Therefore

\[ Q = \int J^0(x, u(x), u_\mu(x)) d\mathbf{x} = \int J^0(\mathbf{x}, u(\mathbf{x}), u_\mu(\mathbf{x})) d\mathbf{x}, \]  

and after the transformation (48) it becomes

\[ Q' = \int J^0(\mathbf{x}', u'(\mathbf{x}'), u'_\mu(\mathbf{x}')) d\mathbf{x}' = \int J^0(x, u'(x), u'_\mu(x)) d\mathbf{x}'. \]  

The last equality follows from the fact that \( \mathbf{x}' \) is a dummy variable in (70). Therefore, the invariance of the Noether charge, \( Q' = Q \), means that

\[ \int \left[ J^0(x, u'(x), u'_\mu(x)) - J^0(x, u(x), u_\mu(x)) \right] d\mathbf{x} \approx \epsilon \int \hat{L}_B(J^0) d\mathbf{x} = 0 \]  

and we come to the following condition

\[ \int \hat{L}_B(J^0) d\mathbf{x} = 0. \]  

Now let us return to (68) and substitute

\[ \hat{G} = \tau^\nu \frac{d}{dx^\nu} + \hat{L}_B. \]  

As a result we get

\[ \hat{L}_B(J^\mu) = \left[ \tau^\nu \left( K^\nu - J^\nu \right) \right]_\mu + \tau^\nu_\nu J^\nu - \tau^\nu_\nu K^\nu + \hat{L}_B(K^\nu) - \hat{N}^\mu \left( K^\nu_\mu \right), \]  

where we have taken into account that, for example

\[ \tau^\nu_\nu J^\mu + \tau^\nu_\nu J^\mu = (\tau^\nu J^\mu)_\nu. \]  

Next we have

\[ \hat{L}_B(K^\nu) - \hat{N}^\mu \left( K^\nu_\mu \right) = \sigma_a \frac{\partial K^\mu}{\partial u_a} - \tau^\mu K^\nu_\nu = \sigma_a \frac{\partial K^\nu}{\partial u_{a, \mu}}. \]  

But

\[ K^\nu_\nu = \frac{\partial K^\nu}{\partial x^\nu} + u_b, i \frac{\partial K^\nu}{\partial u_{ib}}. \]
and
\[
\frac{\partial K_{\nu}^{\mu}}{\partial u_{a,\mu}} = \delta_{b}^{a} \delta_{b}^{\mu} \frac{\partial K_{\nu}^{\mu}}{\partial u_{b}} = \frac{\partial K_{\nu}^{\mu}}{\partial u_{a}}. \tag{77}
\]
Therefore
\[
\hat{L}_{B}(K_{\nu}^{\mu}) - \hat{N}_{\nu}^{\mu}(K_{\nu}^{\mu}) = -\tau_{\nu}^{\mu} K_{\nu}^{\mu}, \tag{78}
\]
and (73) takes the form
\[
\hat{L}_{B}(J^{\mu}) = \left[ \tau^{\nu}(K^{\mu} - J^{\mu}) \right]_{\nu} + \tau_{\nu}^{\mu} J^{\nu} - (\tau_{\nu}^{\nu} K^{\nu})_{\nu}. \tag{79}
\]
But \( J_{\mu}^{\nu} = 0 \), as \( J^{\nu} \) is a conserved current. Therefore
\[
\tau_{\nu}^{\mu} J^{\nu} = \tau_{\nu}^{\nu} J^{\nu} + \tau^{\mu} J^{\nu} = (\tau^{\nu} J^{\nu})_{\nu}, \tag{80}
\]
and substituting this into (79) leads to a little miracle:
\[
\hat{L}_{B}(J^{\mu}) = \frac{d}{dx^{\nu}} \left[ \tau^{\nu}(K^{\mu} - J^{\mu}) - \tau^{\mu}(K^{\nu} - J^{\nu}) \right] = \frac{d G_{\mu}^{\nu}}{dx^{\nu}}. \tag{81}
\]
The fact that
\[
G_{\mu}^{\nu} = \tau^{\mu}(K^{\nu} - J^{\nu}) - \tau^{\nu}(K^{\mu} - J^{\mu}) = \tau^{\mu}(J^{\nu} - K^{\nu}) - \tau^{\nu}(J^{\mu} - K^{\mu}) \tag{82}
\]
is an antisymmetric tensor plays the crucial role, because then
\[
\hat{L}_{B}(J^{\mu}) = \frac{d G_{\mu}^{\nu}}{dx^{\nu}}, \quad i = 1, 2, 3, \tag{83}
\]
is the total three-dimensional divergence and the validity of (72) then follows from the Gauss theorem, provided our system is closed, so that fields fall sufficiently rapidly at spatial infinity to render the limit of the resulting surface integral zero.

6. Concluding remarks

Noether charge is invariant with respect to the corresponding symmetry transformation, as expected. In the context of classical mechanics, the initial rather brute-force proof by Lutzky [38] and by Sarlet and Cantrijn [37] can be significantly simplified by using ideas from [40].

In classical field theory, our presentation of this interesting property of the Noether charge is also based on the results of Ibragimov, Kara and Mahomed [40], in particular on the commutation relation (65). The crucial relation (81), from which the invariance of the Noether charge follows, is a particular case of a more general result of Khamitova [41]. However, the paper [41] is not an easy reading due to omission of many calculational details and to our knowledge it has not been used in the context of invariance of the Noether charge in the classical field theory.

One more point is worth to be mentioned. In Hamiltonian framework for classical mechanics both Noether theorem [19, 37, 48] and invariance of the Noether charge [37] are almost trivial results. Hamiltonian proof of the Noether theorem is so simple that it even ‘makes one question why the statement should be considered an important result’ [19]. Besides, the description of symmetry and conserved quantities in Hamiltonian framework is more straightforward and powerful than in the Lagrangian framework [48]. Why should we then bother about Lagrangian version of the Noether theorem at all? The answer is simple: because of its important applications in the field theories of modern physics. Since
Lagrangian is a relativistic invariant in field theory while Hamiltonian is not, the Lagrangian formalism has a special advantage over the Hamiltonian formalism in relativistic quantum mechanics' [49]. The quantization of modern gauge theories is also most straightforwardly formulated via Lagrangian path integral which is very convenient in practical calculations because it gives manifestly Lorentz invariant expressions (in covariant gauges) and easily leads to the Feynman rules.

Hamiltonian point of view is important as it underlines the rich geometrical ideas behind the Noether’s theorem. In such a framework the proof of the invariance of the Noether charge becomes trivial only after the corresponding mathematical machinery is fully developed [37]. However, all this does not make the Lagrangian viewpoint on symmetries obsolete, especially in field theory where it proved to be very useful being ‘one of the basic building blocks of modern field theories’ [2].

‘Judging by the number of papers devoted to it, Noether’s theorem must be one of the most popular propositions of all time’ [50]. Of course the simplicity of the Hamiltonian proof of the Noether theorem, without appreciating the rich underlying geometrical structures, cannot explain this popularity of the Noether theorem and if naively presented can only obscure its real significance. In fact to call its Hamiltonian counterpart ‘the Noether theorem’ is not quite correct because ‘the Hamiltonian point of view appears nowhere in Noether’s work, and it is therefore inappropriate to give her name to this important, yet easily proved result’ [47].

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Appendix. Calculation of the commutator \( [\hat{G} + \tau\nu, \tilde{N}\mu] \)

We have

\[
[\hat{G} + \tau\nu, \tilde{N}\mu] = \hat{G}(\tau\nu) + \hat{G}(\alpha_\mu \frac{\partial}{\partial u_{a,\mu}}) + \alpha_\nu \left[ \hat{G}, \frac{\partial}{\partial u_{a,\mu}} \right] + \alpha_\mu \left[ \tau\nu, \frac{\partial}{\partial u_{a,\mu}} \right],
\]

(A.1)

But

\[
\left[ \tau\nu, \frac{\partial}{\partial u_{a,\mu}} \right] = -\frac{\partial \tau\nu}{\partial u_{a,\mu}} = -\frac{\partial \tau\nu}{\partial u_a},
\]

(A.2)

because

\[
\tau\nu = \frac{\partial \tau\nu}{\partial x^\nu} + u_{b,\nu} \frac{\partial \tau}{\partial t_b},
\]

(A.3)

On the other hand, as \( \tau\mu \) does not depend on field derivatives,

\[
\hat{G}(\tau\nu) = \tau\nu \tau\nu + \alpha_\nu \frac{\partial \tau\nu}{\partial u_a}.
\]

(A.4)
Further we have
\[
\hat{G}(\sigma_a) = \tau^\nu \sigma_{a, \nu} + \sigma_b \frac{\partial \sigma_a}{\partial u_b} + \sigma_b \frac{\partial \sigma_a}{\partial u_{b, \nu}}.
\]
(A.5)

But
\[
\sigma_a = \xi(x, u) - \tau^\mu(x, u) u_{a, \mu},
\]
and, therefore,
\[
\frac{\partial \sigma_a}{\partial u_{b, \nu}} = -\delta^b_a \tau^\nu.
\]
(A.6)

Substituting this into (A.5), we get
\[
\hat{G}(\sigma_a) = \sigma_b \frac{\partial \sigma_a}{\partial u_b}.
\]
(A.7)

It remains to calculate the commutator
\[
\left[ \hat{G}, \frac{\partial}{\partial u_{a, \mu}} \right] = \left[ \tau^\nu \frac{\partial}{\partial x^\nu} + \xi_b \frac{\partial}{\partial u_b} + \eta_{b, \nu} \frac{\partial}{\partial u_{b, \nu}} \right] = \frac{\partial \eta_{b, \nu}}{\partial u_{a, \mu}} \frac{\partial}{\partial u_{b, \nu}},
\]
(A.8)

where we have used the fact that \(\tau^\nu\) and \(\xi_a\) do not depend on field derivatives. Using
\[
\eta_{b, \nu} = \xi_{b, \nu} - \tau_{b, \nu} u_{b, \alpha},
\]
along with
\[
\frac{\partial \xi_{b, \nu}}{\partial u_{a, \mu}} = \delta^\mu_b \frac{\partial \xi_b}{\partial u_a}, \quad \frac{\partial \tau^\alpha}{\partial u_{a, \mu}} = \delta^\mu_b \frac{\partial \tau^\alpha}{\partial u_a},
\]
(A.9)

we get
\[
\frac{\partial \eta_{b, \nu}}{\partial u_{a, \mu}} = \delta^\mu_b \left( \frac{\partial \xi_b}{\partial u_a} - u_{b, \alpha} \frac{\partial \tau^\alpha}{\partial u_a} \right) - \delta^b_a \tau^\nu = \delta^\mu_b \frac{\partial \sigma_b}{\partial u_a} - \delta^b_a \tau^\nu.
\]
(A.10)

Therefore
\[
\left[ \hat{G}, \frac{\partial}{\partial u_{a, \mu}} \right] = \tau^\nu \frac{\partial}{\partial u_{a, \mu}} - \frac{\partial \sigma_b}{\partial u_a} \frac{\partial}{\partial u_{b, \mu}}.
\]
(A.11)

Now (A.2), (A.4), (A.7) and (A.11), in combination with (A.1), imply the desired result (65):
\[
\left[ \hat{G} + \tau^\nu, \hat{N}^\mu \right] = \tau^\nu \left( \frac{\partial}{\partial u_{a, \mu}} + \sigma_a \frac{\partial}{\partial u_{a, \nu}} \right) = \tau^\nu \hat{N}^\mu.
\]

References

[1] Noether E 1918 Invariante variationsprobleme Gott. Nachr. 1918 235–57
Noether E 1971 Invariant variation problems Transp. Theory Stat. Phys. 1 186–207 (Engl. Transl.)
[2] Logan J D 1997 Invariant Variational Principles (New York: Academic)
[3] Lovelock D and Rund H 1989 Tensors, Differential Forms and Variational Principles (New York: Dover)
[4] Bogoliubov N N and Shirkov D V 1980 Introduction to the Theory of Quantized, Fields 3rd edn (New York: Wiley)
[5] Peskin M E and Schroeder D V 1995 An Introduction to Quantum Field Theory (Reading, MA: Addison-Wesley)
[6] Weinberg S 1995 The Quantum Theory of Fields: Foundations vol 1 (Cambridge: Cambridge University Press)
[7] Álvarez-Gaumé L and Vázquez-Mozo M Á 2012 An Invitation to Quantum Field Theory, Lecture Notes in Physics vol 839 (Berlin: Springer)
[8] Zee A 2010 Quantum Field Theory in a Nutshell 2nd edn (Princeton, NJ: Princeton University Press)
[9] Mandl F and Shaw G 2010 Quantum Field Theory 2nd edn (New York: Wiley)
[10] Srednicki M 2007 Quantum Field Theory (Cambridge: Cambridge University Press)
[11] Schwartz M D 2014 Quantum Field Theory and the Standard Model (Cambridge: Cambridge University Press)
[12] Lancaster T and Blundell S J 2014 Quantum Field Theory for the Gifted Amateur (Oxford: Oxford University Press)
[13] Hatfield B 1992 Quantum Field Theory of Point Particles and Strings (Cambridge: Perseus Books)
[14] Low F E 2004 Classical Field Theory: Electromagnetism and Gravitation (Weinheim: Wiley)
[15] Thirring W 1997 Classical Mathematical Physics: Dynamical Systems and Field Theories 3rd edn (New York: Springer)
[16] Arnold V I and Hadasz L 2010 Lectures on Classical and Quantum Field Theories (Berlin: Springer)
[17] Arnold V I 1989 Mathematical Methods of Classical Mechanics 2nd edn (New York: Springer)
[18] Lanczos C 1970 The Variational Principles of Mechanics 4th edn (Toronto: University of Toronto Press)
[19] Spirovski M 2010 Physics for Mathematicians: Mechanics 1 (Houston, TX: Publish or Perish)
[20] Goldstein H, Poole C and Safko J 2002 Classical Mechanics 3rd edn (Reading, MA: Addison-Wesley)
[21] Gregory R D 2006 Classical Mechanics (Cambridge: Cambridge University Press)
[22] Brizard A J 2008 An Introduction to Lagrangian Mechanics (Singapore: World Scientific)
[23] Hand L N and Finch J D 1998 Analytical Mechanics (Cambridge: Cambridge University Press)
[24] José J V and Saletan E J 1998 Classical Dynamics: A Contemporary Approach (Cambridge: Cambridge University Press)
[25] Mann R A 1974 The Classical Dynamics of Particles: Galilean and Lorentz; Relativity (New York: Academic)
[26] Marsden J E and Ratiu T S 1999 Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems 2nd edn (New York: Springer)
[27] Abraham R and Marsden J E 1978 Foundations of Mechanics 2nd edn (Reading: Benjamin-Cummings)
[28] Souriau J-M 1997 Structure of Dynamical Systems: A Symplectic View of Physics (Boston, MA: Birkhäuser)
[29] Hill E L 1951 Hamilton’s principle and the conservation theorems of mathematical physics Rev. Mod. Phys. 23 253–60
[30] Lévy-Leblond J 1971 Conservation laws for gauge-invariant Lagrangians in classical mechanics Am. J. Phys. 39 502–6
[31] Marinho R M Jr 2007 Noether’s theorem in classical mechanics revisited Eur. J. Phys. 28 37–43
[32] Desloge E A and Karch R I 1977 Noether’s theorem in classical mechanics Am. J. Phys. 45 336–40
[33] Boyer T H 1966 Derivation of conserved quantities from symmetries of the Lagrangian in field theory Am. J. Phys. 34 475–8
[34] Fleming H 1987 Noether’s theorem in classical field theories and gravitation Rev. Bras. Fis. 17 236–52
[35] Gorni G and Zampieri G 2014 Revisiting Noether’s theorem on constants of motion J. Nonlin. Math. Phys. 21 43–73
[36] Neuenschwander D E 2011 Emmy Noether’s Wonderful Theorem (Baltimore, MD: Johns Hopkins University Press)
[37] Sarlet W and Cantrijn F 1981 Generalizations of Noether’s theorem in classical mechanics SIAM Rev. 23 467–94
[38] Lutzky M 1978 Symmetry groups and conserved quantities for the Harmonic oscillator J. Phys. A: Math. Gen. 11 249–58
[39] Lutzky M 1979 Dynamical symmetries and conserved quantities J. Phys. A: Math. Gen. 12 973
[40] Ibragimov N H, Kara A H and Mahomed F M 1998 Lie–Bäcklund and Noether symmetries with applications Nonlinear Dyn. 15 115–36
[41] Khamitova R 1982 Group structure and a basis of conservation laws Theor. Math. Phys. 52 777–81
[42] Olver P J 1993 Applications of Lie Groups to Differential Equations 2nd edn (New York: Springer)
[43] Ibragimov N H 1985 Transformation Groups Applied to Mathematical Physics (Dordrecht: Reidel)
[44] Rund H 1972 A direct approach to Noether’s theorem in the calculus of variations Util. Math. 2 205–14
[45] Trautman A 1967 Noether equations and conservation laws Commun. Math. Phys. 6 248–61
[46] Bessel-Hagen E 1921 Über die Erhaltungssätze der Elektrodynamik Math. Ann. 84 258–76
[47] Kosmann-Schwarzbach Y 2011 The Noether Theorems: Invariance and Conservation Laws in the Twentieth Century (New York: Springer)
[48] Butterfield J 2006 On symmetry and conserved quantities in classical mechanics Physical Theory and its Interpretation: Essays in Honor of Jeffrey Bub ed W Demopoulos and I Pitowsky (Dordrecht: Springer) pp 43–99
[49] Nambu Y and Lagrangian On 1952 On Lagrangian and Hamiltonian formalism Prog. Theor. Phys. 7 131–70
[50] Chernoff P R and Marsden J E 1974 Lecture notes in mathematics Properties of Infinite Dimensional Hamiltonian Systems vol 425 (Berlin: Springer) p 149