Travelling waves in expanding spatially homogeneous space–times

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Abstract

Some classes of the so-called ‘travelling wave’ solutions of Einstein and Einstein–Maxwell equations in general relativity and of dynamical equations for massless bosonic fields in string gravity in four and higher dimensions are presented. Similarly to the well known plane-fronted waves with parallel rays (pp-waves), these travelling wave solutions may depend on arbitrary functions of a null coordinate which determine the arbitrary profiles and polarizations of the waves. However, in contrast with pp-waves, these waves do not admit the null Killing vector fields and can exist in some curved (expanding and spatially homogeneous) background space–times, where these waves propagate in certain directions without any scattering. Mathematically, some of these classes of solutions arise as the fixed points of Kramer–Neugebauer transformations for hyperbolic integrable reductions of the above mentioned field equations or, in other cases, after imposing the ansatz that these waves do not change the part of the spatial metric transverse to the direction of wave propagation. It is worth noting that the strikingly simple forms of all the solutions presented prospectively make possible the consideration of the nonlinear interaction of these waves with the background curvature and singularities, as well as the collision of such wave pulses with solitons or with each other in the backgrounds where such travelling waves may exist.

Keywords: exact solutions, gravity, string gravity, travelling bosonic waves, hyperbolic (matrix) Ernst equations, Kasner universe background

Introduction

Among the many exact solutions that have been discovered during the almost 100 year history of general relativity (the majority of which can be found in the books [1]–[4]), there are very few classes of explicitly known solutions of Einstein’s field equations in
general relativity or in other gravity models that are able to shed some light on various nonlinear wave phenomena which characterize strong gravitational, electromagnetic or other types of waves propagating through (and interacting with) some non-trivial background space–time geometries and external fields.

From the physical point of view, the consideration of any nonlinear wave phenomenon becomes more clear if we can distinguish the waves which take part in the interactions. This applies, for example, to the case of the collision of waves with distinct wavefronts in which certain kinds of waves approach each other before their collision, to the case of the interaction of incident waves of infinite duration but rapidly decreasing amplitudes in which the parameters of the initial waves can only be determined asymptotically, as well as to the case of the decay of some initial field configuration producing some outgoing waves. In such cases, we can distinguish some simple waves, which are here called 'travelling waves' and which propagate in some space–time region in a certain direction without any scattering and without caustics or focusing singularities.

In the literature of the last few decades, a lot of attention has been concentrated on the study of one type of travelling wave—plane gravitational and electromagnetic waves, which represent a subclass of the class of plane-fronted waves with parallel rays or ‘pp-waves’. These solutions (discovered long ago for a vacuum by Brinkman [5] and then considered for the Einstein–Maxwell case by Baldwin and Jeffery [6]) are algebraically special and the corresponding space–time geometries admit shear-free, twist-free and expansion-free null congruences and possess rather large groups of isometries including the null Killing vector field. Differing from other pp-waves, the plane-wave solutions are characterized by the supplemental condition that for these waves various physical and geometrical parameters are constant along the wavefronts. Later, the plane-wave solutions were presented by Rosen [7] in a very convenient general form which shows explicitly the transversal character of these waves. In this form, all components of the solutions depend only on a null coordinate, which labels the wavefronts. It is remarkable that in general relativity the class of plane-wave solutions for gravitational waves in a vacuum as well as for gravitational and electromagnetic waves in the electrovacuum case depend on arbitrary functions of a null coordinate, which determine the amplitudes and polarizations of the waves. Despite the very simple explicit form of plane-wave solutions, these may have rather interesting and non-trivial geometrical structures (see [2, 4] for details and references).

Because of the existence of a null Killing vector field and other symmetries of plane waves, these solutions can be matched easily with the Minkowski background on the wavefronts. This leads to the construction of more realistic solutions for wave pulses, which propagate through the Minkowski background with distinct wavefronts and, in particular, for sandwich waves. However, for a description of waves that propagate in the Minkowski background and which possess some non-planar wavefront geometry, or of the waves propagating in some non-trivial curved backgrounds in which the curvature characteristics inevitably evolve along the wavefronts, we need to consider other types of travelling waves.

Many particular solutions for travelling waves (different from pp-waves) are known within the class of vacuum space–times, the metrics of which are diagonal and depend on time and one spatial coordinate. These metrics describe the propagation of gravitational waves with constant linear polarization. As is well known, the Einstein equations for these metrics reduce to a linear equation for one real unknown function—the Euler–Poisson–Darboux equation, which admits many explicit solutions for travelling waves. In particular, examples of solutions for such waves with distinct wavefronts and with different levels of smoothness of the metrics on these fronts, propagating in different cosmological backgrounds, can be found in [8]. Waves propagating through the Minkowski background with different forms of wavefronts (plane, cylindrical,
spherical, toroidal, etc) were considered in [9, 10]. More general solutions for such ‘linear’ travelling gravitational waves with constant linear polarization propagating in some non-trivial backgrounds can arise as specific solutions of the corresponding characteristic initial value problem formulated for the Euler–Poisson–Darboux equation. The general solution of the characteristic initial problem for this equation was described in terms of the Abel transform in [11, 12] and was later discussed in [2, 13].

Many fewer solutions are known for travelling waves which do not possess a constant linear polarization and/or include, in addition to the gravitational part, the electromagnetic or some other field components. The majority of the known examples of such solutions belong to the classes of fields for which the components depend on only two coordinates—time and one spatial coordinate or, more precisely, to the classes of space–times for which the metrics admit a two-dimensional (in four-dimensional space–times) or, in general, a \((D-2)\)-dimensional (in \(D\)-dimensional space–time) Abelian isometry group with space-like orbits, and all matter fields and their potentials share this symmetry. With this space–time symmetry ansatz, Einstein’s field equations remain nonlinear, but in many physically important cases these symmetry reduced equations (similarly to elliptic Ernst equations for stationary axisymmetric vacuum and electrovacuum fields [14, 15]) can be expressed conveniently in the form of hyperbolic Ernst equations (Einstein equations for vacuum fields and Einstein–Maxwell equations for electrovacuum fields and some others) or generalized (matrix) hyperbolic Ernst equations (the equations for bosonic dynamics of some string gravity models in four and higher dimensions). In these cases, the field equations are completely integrable. The integrability of these equations allows for the construction of solutions the use of various symmetry transformations, soliton generating techniques, the monodromy transform approach and various integral equation methods (see the review [16] for more details and references). These provide us with effective methods for the construction of large families of particular solutions (such as solitons and some others) and some principle algorithms for the solution of various (symmetry reduced) initial, characteristic and boundary value problems for these equations. However, the explicit realization of these algorithms is very difficult for more or less generic ‘input data’ for the solutions and these can be realized only for very particular choices of data. This leads to the explicit construction of families of solutions which include only a finite number of new parameters, but no arbitrary functions.

On the other hand, some solutions for travelling waves (different from pp-waves) are known. These are the solutions for waves in curved space–times which depend also on arbitrary functions of a null coordinate. First, we have to mention here a class of vacuum and stiff matter fluid solutions for gravitational wave pulses which has been found and investigated in detail in [17]. The solutions of this class describe the waves which have no linear polarization and propagate in special kinds of spatially homogeneous Kasner universes. These solutions depend explicitly on one arbitrary function of a null coordinate which determines the profiles of the waves.

Another known example of solutions for travelling waves (different from pp-waves) is the class of solutions for cylindrical pure electromagnetic waves which is found in [18] and is also mentioned in the equation (22.59) in the ‘Cylindrical waves’ chapter of [1]. These solutions depend on two arbitrary functions of a null coordinate which determine the arbitrary amplitudes of each of two states of wave polarization. It is necessary to note here, however, that the cylindrical wave interpretation of these solutions does not seem appropriate because of the singular behaviour of these metrics on the axis \( \rho = 0 \).

The purpose of this paper is a construction of solutions for travelling waves (different from pp-waves) depending on some arbitrary functions of a null coordinate which describe, in general relativity and in some string gravity models in four and higher dimensions, the
propagation in some curved space–times of gravitational, electromagnetic and other massless bosonic fields with arbitrary profiles and polarizations. For the construction of solutions for such waves, we suggest two methods based on two different ansatzes which can be used to search for the travelling wave solutions as the solutions of hyperbolic Ernst equations in general relativity and generalized matrix hyperbolic Ernst equations which arise in some (symmetry reduced) string gravity models.

One way in which the travelling wave solutions may arise is the consideration of fixed points of the (generalized) Kramer–Neugebauer transformation of the solution spaces of hyperbolic integrable reductions of Einstein’s field equations. This transformation of vacuum solutions to vacuum solutions for stationary axisymmetric fields was discovered by Kramer and Neugebauer—see [1] for further details and references. It is easy to see that similar transformations exist in the space of four-dimensional vacuum solutions admitting the two-dimensional Abelian isometry group with 2-surface orthogonal space-like orbits (or, equivalently, in the space of solutions of the hyperbolic vacuum Ernst equation). It is surprising, perhaps, but rather simple calculation of the fixed points of this transformation lead to the class of solutions which depend on an arbitrary function of a null coordinate and which admit unambiguous interpretation as travelling waves propagating on some special case of Kasner background. Also, this class of solutions coincides with the class of Wainwright vacuum metrics mentioned above. Thus, we do not obtain new solutions for this simplest case, but this suggests the method for finding similar solutions for travelling waves on some spatially homogeneous backgrounds for other integrable hyperbolic symmetry reductions of Einstein’s field equations. This is confirmed below by the construction of travelling wave solutions for hyperbolic symmetry reductions of dynamical equations for massless bosonic fields in string gravity.

Another ansatz also leads to the construction of some classes of travelling wave solutions for not gravitational but pure electromagnetic or other massless bosonic fields in four and higher dimensions which depend on a set of arbitrary functions of a null coordinate. This ansatz restricts the consideration by the solutions to plane-fronted waves which propagate on the expanding spatially homogeneous background and for which the part of the spatial metric transverse to the direction of wave propagation remains unperturbed. In the case of Einstein–Maxwell fields in four dimensions this ansatz leads unambiguously to plane-fronted pure electromagnetic travelling waves propagating in the symmetric Kasner space–time along its axis of symmetry. The solutions of this class depend on two arbitrary functions of a null coordinate which determine the arbitrary amplitudes, forms and polarizations of these waves. We note here that formally the class of waves derived in this way is a ‘twin’ of the above mentioned class of solutions for cylindrical electromagnetic waves propagating on some static background found in [18], but its interpretation as cylindrical waves is not appropriate because of the singular behaviour of the solutions on the axis of symmetry. In our case, we have plane-fronted waves propagating on the non-static, expanding Kasner background and the existence of the Kasner initial singularity in these solutions is justified from the physical point of view.

In this paper we also use an ansatz similar to that mentioned just above for the construction of travelling wave solutions for some massless bosonic fields in string gravity which also propagate in the expanding spatially homogeneous backgrounds in four and higher dimensions and possess arbitrary profiles and polarizations.

In the following section of this paper, we first recall the hyperbolic vacuum Ernst equation and the corresponding real hyperbolic form of the Kramer–Neugebauer transformation of its solution space. Then we describe the derivation of solutions which arise as the fixed points of this transformation. After that, in another section we recall the hyperbolic
electrovacuum Ernst equations in Einstein–Maxwell theory in four dimensions and apply the second ansatz to these equations. Then we describe a derivation of the corresponding class of pure electromagnetic travelling waves propagating in the expanding Kasner background.

In the subsequent sections, we use the same methods for the construction of solutions for massless bosonic gauge fields in string gravity in four and higher dimensions. These solutions describe the travelling waves with arbitrary profiles and polarizations propagating in some expanding spatially homogeneous (Kasner) backgrounds. In the conclusion we summarize the results and outline the related interesting questions.

The hyperbolic vacuum Ernst equation and the Kramer–Neugebauer transformation of its solution space

The space–time symmetry ansatz

It is well known that in four-dimensional space–time which admits the two-dimensional Abelian orthogonally transitive isometry group with space-like orbits, the coordinates can be chosen in such a way that all metric components depend only on time and one spatial coordinate, say $x^0 = t$ and $x^1 = x$, or, equivalently, on null cone coordinates $u = t - x$ and $v = t + x$ and the metric can be presented in the form

$$\text{d} s^2 = - f \text{d} u \text{d} v + g_{a b} \text{d} x^a \text{d} x^b,$$

where $a, b, \ldots = 2, 3$ and $\{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}$; the conformal factor $f$ and the metric components $g_{a b}$ are functions of $u$ and $v$ only. It is convenient to parameterize the metric components $g_{a b}$ by three real functions $\alpha$, $H$ and $\Omega$ so that

$$g_{a b} \delta^{a b} = H (\text{d} y + \Omega \, \text{d} z)^2 + \frac{\alpha^2}{H} \text{d} z^2$$

where $H > 0$, $\alpha > 0$ and $\det \| g_{a b} \| = \alpha^2$.

The Ernst potential and hyperbolic Ernst equation for vacuum metrics

Similarly to stationary axisymmetric vacuum metrics for which the Ernst potential was introduced in [14], the metrics (1) can also be characterized by a complex scalar potential $\mathcal{E}$ defined up to an imaginary constant by the relations

$$\text{Re} \, \mathcal{E} = -H, \quad \text{d}(\text{Im} \, \mathcal{E}) = \alpha^{-1} H^2 \ast d \Omega,$$

where $\ast d$ is the Hodge star operator on the plane $(u, v)$. In accordance with the vacuum Einstein equations for metrics (1), the functions $\mathcal{E}(u, v)$ and $\alpha(u, v)$ should satisfy the nonlinear hyperbolic Ernst equation and the d’Alembert equation, respectively:

$$\begin{cases}
\text{Re} \, \mathcal{E} \left( 2\mathcal{E}_{u u} + \frac{\alpha_u}{\alpha} \mathcal{E}_v + \frac{\alpha_v}{\alpha} \mathcal{E}_u \right) - 2 \mathcal{E}_u \mathcal{E}_v = 0 \\
\alpha_{u v} = 0
\end{cases}$$

where all suffixes represent the derivatives. Given a solution $(\alpha, \mathcal{E})$ of (4), the functions $H$ and $\Omega$ can be determined from (3), while the function $f$ is determined by the relations:
For different choices of the solution for $\alpha(u, v)$, the gauge freedom remaining in (1) allows one to choose $\alpha$ as a new time-like or spatial coordinate $t$ or $x$, respectively. However, in some contexts, such as the consideration of colliding plane gravitational waves with distinct wavefronts propagating through the Minkowski background, one may need to consider more complicated solutions for $\alpha(u, v)$ determined by the profiles of waves before their collision (see [2, 19] and references therein).

The alternative vacuum Ernst potential and Kramer–Neugebauer transformation

As can be checked by a direct calculation, the following combination of vacuum metric functions (2) should also satisfy the hyperbolic Ernst equation (4):

$$\mathcal{E} = -\frac{\alpha}{H} + i\Omega.$$  (5)

Therefore, given a solution $(\alpha, \mathcal{E})$ of (4), one can construct two vacuum metrics corresponding to different ‘interpretations’ (3) and (5) of the vacuum Ernst potential:

$$\begin{cases}
H_{(1)} = -\text{Re} \mathcal{E} \\
\Omega_{(1)} = \alpha(\text{Re} \mathcal{E})^{-2} \partial_t(\text{Im} \mathcal{E}),
\end{cases} \quad \begin{cases}
H_{(2)} = -\frac{\alpha}{\text{Re} \mathcal{E}}, \\
\Omega_{(2)} = \text{Im} \mathcal{E},
\end{cases} \quad f_{(2)} = \frac{H_{(1)}}{\sqrt{\alpha}} f_{(1)}.$$  (6)

Accordingly, one can consider the transformation acting on the space of solutions of the above described symmetry reduced vacuum Einstein equations which take the form

$$\{ H_{(1)}, \Omega_{(1)} \} \leftrightarrow \{ H_{(2)}, \Omega_{(2)} \}.$$  (7)

(Such transformations of vacuum solutions for stationary axisymmetric fields were discovered by Neugebauer and Kramer—see [1] for further details and references.)

Travelling vacuum gravitational waves that are fixed points of Kramer–Neugebauer transformation

Consider now the question of whether the Kramer–Neugebauer transformation possess fixed points, i.e. whether solutions exist which are invariant under this transformation of the solution space. It is clear that these solutions should satisfy

$$H_{(1)} = H_{(2)}, \quad \Omega_{(1)} = \Omega_{(2)}.$$  (7)

These conditions lead, in accordance with (6), to the relations:

$$\text{Re} \mathcal{E} = -\sqrt{\alpha}, \quad \text{d}(\text{Im} \mathcal{E}) = *\text{d}(\text{Im} \mathcal{E}).$$

The second relation here (due to the definition of $^*\text{d}$ in (3)) reduces to $\partial_t(\text{Im} \mathcal{E}) = 0$ and the class of invariant solution is determined by the Ernst potential

$$\mathcal{E} = -\sqrt{\alpha} + i\nu_{\alpha}(u), \quad \alpha_{\nu\nu} = 0$$

where $\alpha$ is an arbitrary solution of the last of the above equations and $\nu_{\alpha}(u)$ is an arbitrary function of $u$. The corresponding class of vacuum metrics takes the form
In the form (1) this family of solutions is described by the expression

\[
\frac{\psi''(u)}{2a^2} \alpha_{uu} du \, dv + \sqrt{\alpha} \left[ dy + \psi'(u) dz \right]^2 + \alpha^{1/2} dz^2.
\]

In addition to this, instead of (7), slightly different invariance conditions \( H(1) = H(2) \), \( \Omega(1) = -\Omega(2) \) lead to another class of vacuum metrics which possess the form

\[
\frac{\psi''(v)}{2a^2} \alpha_{vv} dv \, du + \sqrt{\alpha} \left[ dx + \psi'(v) dy \right]^2 + \alpha^{1/2} dy^2.
\]

One can check easily that the metrics (8) and (9) satisfy the vacuum Einstein equations. For different choices of \( \alpha \) as a solution of the equation \( a_{uv} = 0 \), for which \( \alpha = c \) is a space-like or time-like surface, the appropriate transformations \( u \to h(u), v \to g(v) \) allow one to choose \( t = \alpha \) or \( x = \alpha \), respectively. The case \( \alpha = t \) is the most interesting because (8) with \( \psi_{u}(u) = 0 \) as well as (9) with \( \psi_{v}(u) = 0 \) represent a cosmological Kasner solution with a special set of exponents \( (p_0, p_1, p_2) = (-\frac{1}{3}, \frac{4}{13}, \frac{12}{13}) \) and therefore, for \( \psi_{u}(u) \neq 0 \) or \( \psi_{v}(v) \neq 0 \) these solutions describe the plane-fronted travelling gravitational waves the profiles of which are determined by the arbitrary functions \( \psi_{u}(u) \) in (8) and \( \psi_{v}(v) \) in (9) and which propagate on this specific Kasner background in the positive and negative directions of the \( r \)-axis, respectively.

In this case \( \alpha = t \), the classes of solutions (8) and (9) are already known. The physical and geometrical properties of these solutions as the solutions for the plane-fronted waves on a specific Kasner background were considered in detail in [17]. In particular, for a specially chosen null tetrad, the components of the Weyl tensor calculated in [17] possess the structure \( \Psi' = \Psi'' = 0 \), and from the expressions found there for \( \Psi_0, \Psi'_0 \) and \( \Psi_1 \), one can conclude easily that the solutions (8) and (9) are of algebraically general type (type I in Petrov classification). However, as was also shown in [17], any possible weak discontinuities in the function \( \psi_{u}(u) \) in this solution give rise to Weyl tensor discontinuities of algebraic type N, which justifies the interpretation of this vacuum solution as a gravitational wave pulse.

The solutions (8) and (9) can certainly be used in further investigations of the propagation of waves with arbitrary amplitudes in a curved space–time as well as their collisions and nonlinear interactions. In addition to this, it is also important for us here that the classes of solutions (8) and (9) in the above considerations arose from the ansatz of invariance of the solution under the Kramer–Neugebauer transformation. This suggests the opportunity of using a similar ansatz for the construction of classes of travelling wave solutions for integrable reductions of some other gravity models in four and higher dimensions.

Hyperbolic Ernst equations for electrovacuum fields

The space–time symmetry ansatz

For four-dimensional electrovacuum space–times for which the metric satisfies the same space–time symmetry ansatz as the vacuum metrics discussed above and the electromagnetic

\footnote{The usual form of the family of vacuum cosmological Kasner solutions is \( ds^2 = -dt^2 + \kappa_{01} dx^2 + \kappa_{12} dy^2 + \kappa_{02} dz^2 \), where the exponents should satisfy \( \kappa_{01} + \kappa_{12} + \kappa_{02} = 1 \). However, in the form (1) this family of solutions is described by the expression \( ds^2 = \kappa_{01}( -dt^2 + dx^2 ) + \kappa_{12}(dy^2 + dz^2) \), where \( \kappa_{01} = \kappa_{01}'/(1 - \kappa_{01}) \), \( \kappa_{12} = \kappa_{12}'/(1 - \kappa_{12}) \), \( \kappa_{02} = \kappa_{02}'/(1 - \kappa_{02}) \).}

\footnote{The Newman–Penrose formalism notations used here and below correspond to those described in [1].}
potential of which shares the same symmetry, the metric and electromagnetic potential can be taken in the form

\[ dx^2 = -f \, dv + g_{ab} \, dx^a dx^b, \quad A = \{ 0, 0, A_\alpha \} \] (10)

where \( x^a = \{ y, z \} \) and \( g_{ab}, f \) and \( A_\alpha \) are functions of \( u \) and \( v \) only. We use here the null coordinates \( u, v = (t - x, t + x) \) and the same parametrization (2) for \( g_{ab} \) in terms of the functions \( \alpha > 0 \) and \( \Omega \) with \( \det [g_{ab}] = \alpha^2 \).

**Electrovacuum Ernst potentials and hyperbolic Ernst equations**

Similarly to stationary axisymmetric fields [15], the electrovacuum Einstein–Maxwell fields, which satisfy the symmetry ansatz mentioned just above, admit a complete description in terms of two complex Ernst potentials \( \mathcal{E}(u, v) \) and \( \Phi(u, v) \). In the hyperbolic case, these potentials are determined by the relations:

\[
\begin{align*}
\text{Re } \mathcal{E} &= -H - \Phi \bar{\Phi} \\
\text{Im } \mathcal{E} &= \alpha^{-1} H^2 \ast \text{d} \Omega + i (\Phi \partial \Phi - \Phi \bar{\Phi}) \\
\text{Re } \Phi &= A_y \\
\text{Im } \Phi &= \alpha^{-1} H (\Omega \ast \text{d} A_y - \ast \text{d} A_z) \\
\alpha_{uv} &= 0.
\end{align*}
\] (11)

where the star operator \( \ast \text{d} \equiv du \, \partial_u - dv \, \partial_v \), and \( A_y, A_z \) are the non-vanishing components of a real electromagnetic vector potential. In accordance with the electrovacuum Einstein–Maxwell equations for the fields (10), the functions \( \mathcal{E}(u, v), \Phi(u, v) \) and \( \alpha(u, v) \) should satisfy the nonlinear Ernst equations and the d’Alembert equation, respectively:

\[
\begin{align*}
\left( \text{Re } \mathcal{E} + \Phi \bar{\Phi} \right) \left( 2 \mathcal{E}_{uv} + \frac{\alpha_u}{\alpha} \mathcal{E}_y + \frac{\alpha_y}{\alpha} \mathcal{E}_u \right) - (\mathcal{E}_u + 2 \Phi \partial_u \Phi) \mathcal{E}_v &= 0 \\
\left( \text{Re } \mathcal{E} + \Phi \bar{\Phi} \right) \left( 2 \Phi_{uv} + \frac{\alpha_u}{\alpha} \Phi_y + \frac{\alpha_y}{\alpha} \Phi_u \right) - (\mathcal{E}_u + 2 \Phi \partial_u \Phi) \Phi_v &= 0 \\
\alpha_{uv} &= 0.
\end{align*}
\] (12)

Given a solution \( (\alpha, \mathcal{E}, \Phi) \) of (12), the functions \( H, \Omega \) and \( A_y, A_z \) can be determined from (11), while the function \( f \) is determined by the relations

\[
\begin{align*}
f_u &\quad \left( f_u = \frac{\alpha_{uv}}{\alpha_u} - \frac{H_u}{H} + \frac{\alpha}{2\alpha_u} \left[ \frac{\mathcal{E}_u + 2 \Phi \partial_u \Phi}{H^2} \right]^2 \right) + \frac{4}{H} |\Phi_u|^2 \\
f_v &\quad \left( f_v = \frac{\alpha_{uv}}{\alpha_v} - \frac{H_v}{H} + \frac{\alpha}{2\alpha_v} \left[ \frac{\mathcal{E}_v + 2 \Phi \partial_v \Phi}{H^2} \right]^2 \right) + \frac{4}{H} |\Phi_v|^2.
\end{align*}
\] (13)

In the presence of electromagnetic fields we have no transformations similar to the Kramer–Neugebauer vacuum-to-vacuum transformations and, therefore, for the construction of solutions for travelling electromagnetic waves in some Kasner backgrounds we need to find some other transformations or ansatzes.

**Travelling electromagnetic waves propagating in Kasner space–time**

*On the Bonnor transformation*

The simplest way to construct the solutions for travelling electromagnetic waves in some Kasner backgrounds could be the application to solutions for travelling vacuum gravitational
waves of the Bonnor transformation [1, 20] which maps (in the hyperbolic case) a class of vacuum metrics admitting the Abelian two-dimensional orthogonally transitive isometry group with space-like orbits onto a class of electrovacuum solutions with the same symmetry but with diagonal metrics and an electromagnetic potential which arise from a non-diagonal part of the original vacuum metric.

The application of the Bonnor transformation to the solutions (8) and (9) leads to the classes of solutions for waves with linear polarizations each dependent only on one arbitrary real function determining the profile of the electromagnetic wave. It is interesting to note here that this Bonnor transformation also changes the parameters of the background Kasner solution so that from the gravitational waves (8) and (9) on the Kasner background with the exponents $(p_1, p_2, p_3) = (-3/13, 4/13, 12/13)$ we obtain pure electromagnetic waves propagating through the symmetric Kasner background with the exponents $(p_1, p_2, p_3) = (-1/7, 2/7, 2/7)$ along its axis of symmetry.

However, we do not follow this method. Instead we use some ansatz (obviously inspired by the application of the Bonnor transformation to (8) or (9)) which leads to a class of solutions depending on two arbitrary functions which determine the amplitudes of the electromagnetic waves of both polarizations.

**Electromagnetic waves propagating through a symmetric Kasner background**

Let us consider a class of electrovacuum fields for which the metric on the orbits of the isometry group possess a plane symmetry and the non-zero components of electromagnetic potential depend only on the null coordinate $u = t - x$:

$$H = \alpha(u, v), \quad \Omega = 0, \quad A_\gamma = A_\gamma(u), \quad A_z = A_z(u).$$

This means that for the Ernst potentials for this class of fields we have

$$\begin{cases}
\text{Re } \mathcal{E} = -\alpha(u, v) - \phi_\gamma(u)\phi_\gamma^*(u), \\
\partial_u \text{Im } \mathcal{E} = i[\phi_\gamma^*(u)\phi_\gamma'(u) - \phi_\gamma(u)\phi_\gamma'(u)], \quad \Phi = \phi_\gamma(u) \equiv A_\gamma(u) - iA_z(u).
\end{cases}$$

Substitution of these expressions for the Ernst potentials into the Ernst equation (12) shows that these equations are satisfied for the arbitrary complex function $\phi_\gamma(u)$ or, equivalently, for the arbitrary real functions $A_\gamma(u)$ and $A_z(u)$. Then the conformal factor $f$ can be calculated from (13) and the full metric takes the form

$$ds^2 = -\frac{1}{\sqrt{\alpha}} \exp \left[ 2 \int_{u_0}^{u} \left| \frac{\phi_\gamma'(u)}{\alpha_u} \right|^2 du \right] \alpha_u \alpha_v du dv + \alpha \left( dv^2 + dz^2 \right)$$

where the prime denotes a derivative. If the electromagnetic field depends only on the other null coordinate $v = t + x$, i.e. in the case $\Phi = \phi_\gamma(v)$, we use the ansatz

$$H = \alpha(u, v), \quad \Omega = 0, \quad A_\gamma = A_\gamma(v), \quad A_z = A_z(v).$$

This leads to another class of solutions similar to (15):

$$ds^2 = -\frac{1}{\sqrt{\alpha}} \exp \left[ 2 \int_{v_0}^{v} \left| \frac{\phi_\gamma'(v)}{\alpha_v} \right|^2 dv \right] \alpha_u \alpha_v du dv + \alpha \left( dv^2 + dz^2 \right)$$

where $\Phi = \phi_\gamma(v) \equiv A_\gamma(v) + iA_z(v)$ is an arbitrary complex function of $v$. Similarly to vacuum solutions (8) and (9), the function $\alpha(u, v)$ (as the solution of the last equation in (12))
can be chosen in (15) and (17) so that the surfaces $\alpha = \text{const}$ can be space-like or time-like and, therefore, we can choose for simplicity $\alpha = t$ or $\alpha = x$, respectively.

For the case of time-like $\alpha = \text{const}$, the ‘twins’ of solutions (15) and (17) representing some cylindrical waves with $\alpha = \rho$ propagating in some static axisymmetric background have been found in [18]. However, as was mentioned in the introduction, the cylindrical wave interpretation of these solutions is not completely appropriate because of the singular behaviour of these metrics on the axis $\rho = 0$.

In the case $\alpha = t$, in the absence of electromagnetic fields, i.e. for $\phi_+(u) = 0$ and $\phi_-(v) = 0$, each of the solutions (15) and (17) coincides with the symmetric Kasner solution. Therefore, if $\phi_+(u) \neq 0$ and $\phi_-(v) \neq 0$, these solutions describe travelling electromagnetic waves propagating through (and interacting with) the symmetric Kasner background along its axis of symmetry ($x$-axis) in its positive and negative directions, respectively.

It is interesting to note that in the metrics (15) and (17), the electromagnetic field does not affect at all the part of metric which is transverse to the direction of wave propagation and therefore these are pure electromagnetic waves without any co-moving gravitational wave component. This can also be confirmed by calculation of the Newman–Penrose scalars for this solution. Namely, choosing the null tetrad for the metric (15) in the appropriate form, in the notations of [1] we obtain the following expressions for the null tetrad components of the Weyl and Maxwell tensors:

\[
\begin{align*}
&k^i dx^i = \frac{1}{\sqrt{2}} \frac{1}{\nu^{1/4}} e^{2I(u)} du, \quad \nu_0 = \nu_1 = 0, \quad \phi_0 = 0, \quad I(u) = \int_{u_0}^{u} \left| \phi'_+(u) \right|^{1/2} du, \\
&l^i dx^i = \frac{1}{\sqrt{2}} \frac{1}{\nu^{1/4}} e^{2I(u)} dv, \quad \nu_2 = \nu_3 = 0, \quad \phi_1 = 0, \\
&m^i dx^i = -\sqrt{\epsilon} (dy + idz), \quad \nu_2 = \nu_3 = 0, \quad \phi_2 = -\frac{\phi'_+(u)}{\nu^{1/4}} e^{-2I(u)}. \\
\end{align*}
\]

However, as one can see here, the metric (15) is of algebraic type D and the electromagnetic field in this solution is null. Both principal null directions are geodesic, twist-free and shear-free, but both of them are divergent. Formally this means that the solution (15) (and similarly, the solution (17)) belongs to the corresponding subclass of the Robinson–Trautman class of Einstein–Maxwell fields and, in accordance with [1], this class of fields is also known formally. In this case, it may be expected that the solution (15) coincides with the solution (28.43) of [1], but these solutions seem to be essentially different. In any case, the expressions for the Newman–Penrose scalars given above as well as the pure radiation structure of the energy–momentum tensor confirm our interpretation of the solutions (15) and (17) with $\alpha = t$ as the solutions for plane-fronted electromagnetic waves propagating on the expanding (Kasner) background and the existence of the Kasner initial singularity in these solutions is obviously inevitable from the physical point of view.

The solutions (15) and (17) can be useful in the further investigation of the propagation and nonlinear interaction of travelling electromagnetic waves with a curved background as well as of the collision of these waves with gravitational and electromagnetic solitons as well as with each other. In addition to this, the purpose of considering these waves in the context of this paper is that a very simple ansatz (14) or (16), which leads, respectively, to the solutions (15) and (17), can be generalized appropriately and used for the construction of solutions for travelling massless bosonic waves propagating through some expanding cosmological backgrounds in the other (symmetry reduced) gravity models in four and higher dimensions.
Matrix hyperbolic Ernst equations for bosonic fields in string gravity

Massless bosonic dynamics in string gravity

We consider now the string gravity models in space–time with $D \geq 4$ dimensions for which the dynamics of massless bosonic fields is determined by the effective action

$$S = \int e^{-\Phi} \left\{ \tilde{R}^{(D)} + \nabla_M \tilde{\Phi} \nabla^M \tilde{\Phi} - \frac{1}{12} H_{MNP} H^{MNP} - \frac{1}{2} \sum_{p=1}^{n} F_{MN}^{(p)} F^{MN(p)} \right\} \sqrt{-G} \, d^D x \tag{18}$$

where $M, N, \ldots = 1, 2, \ldots, D$ and $p = 1, \ldots, n$; $G_{MN}$ possesses the ‘most positive’ Lorentz signature. The string frame metric $\tilde{G}_{MN}$ and dilaton field $\tilde{\Phi}$ are related to the metric $G_{MN}$ and dilaton $\Phi$ in the Einstein frame as

$$\tilde{G}_{MN} = e^{2\Phi} G_{MN}, \quad \tilde{\Phi} = (D - 2) \Phi.$$ 

The components of a three-form $H$ and two-form $F^{(p)}$ are determined in terms of the antisymmetric tensor field $B_{MN}$ and Abelian gauge field potentials $A_{M}^{(p)}$ as

$$H_{MNP} = 3 \left( \partial_M B_{NP} - \sum_{p=1}^{n} A_{[M}^{(p)} F_{NP]}^{(p)} \right),$$

$$F_{MN}^{(p)} = 2 \partial_M A_N^{(p)}, \quad B_{MN} = -B_{NM}.$$ 

The space–time symmetry ansatz

The integrable hyperbolic reductions of the dynamical equations of the model (18) arise from the assumption that the space–time admits $d = D - 2$ commuting space-like Killing vector fields. All field components and potentials are assumed to be functions of time $t$ and one spatial coordinate $x$ (or, equivalently, of the null cone coordinates $u = t - x$ and $v = t + x$). It is assumed also that the metric components possess the structure

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & G_{ab} \end{pmatrix} \quad \mu, \nu = 1, 2, \ a, b = 3, 4, \ldots, D$$

while the components of the gauge field potentials take the forms

$$B_{MN} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix}, \quad A_{M}^{(p)} = \begin{pmatrix} 0 \\ A_a^{(p)} \end{pmatrix}. $$

The coordinates $x^1, x^2$ can be chosen such that the metric $g_{\mu\nu}$ takes a conformally flat form $g_{\mu\nu} dx^\mu dx^\nu = f(-dt^2 + dx^2) = -f \, du dv$ where the conformal factor $f(u, v) > 0$. The other metric and gauge field components can be combined into three matrix functions

$$G = e^{2\Phi} \| G_{ab} \|, \quad B = \| B_{ab} \|, \quad A = \| A_a^{(p)} \| \tag{19}$$

where $G$ is a symmetric $d \times d$-matrix of string frame metric components, $B$ is the antisymmetric $d \times d$-matrix of the components of gauge potential for the three-form $H$ and $A$ is a rectangular $d \times n$-matrix of components of $n$ Abelian vector gauge potentials.
Matrix Ernst potentials and hyperbolic matrix Ernst equations

The space–time symmetry ansatz described above reduces the dynamical equations for the model (18) to a completely integrable hyperbolic system which can be presented in different convenient forms. One of the forms of these equations represents a matrix analogue of the hyperbolic Ernst equations (see [21, 22]). In this form, any solution is determined by the $d \times d$-matrix of the Ernst-like potential $\mathcal{E}$, the rectangular $d \times n$-matrix $\mathcal{A}$ and a scalar function $\alpha$.

The matrix Ernst potential and the function $\alpha$ are defined in terms of matrix variables (19) and dilaton $\Phi$ by the following relations

\[
\begin{align*}
\mathcal{E} &= \mathcal{G} + B + A \mathcal{A}^T, \\
\det \mathcal{G} &\equiv e^{2\mathcal{G}} \alpha^2, \\
\partial_u \partial_v \alpha &= 0, \\
\end{align*}
\]

where the superscript $T$ represents a matrix transposition, the second equation in (20) is a definition of the function $\alpha(u, v) > 0$ and the final linear equation for $\alpha$ follows immediately from the symmetry reduced field equations for (18). In terms of the matrix potentials $\mathcal{E}$ and $\mathcal{A}$ and the function $\alpha$ the dynamical equations take the form

\[
\begin{align*}
2\mathcal{E}_{uv} + \frac{\alpha_u}{\alpha} \mathcal{E}_v + \frac{\alpha_v}{\alpha} \mathcal{E}_u &= \left( \mathcal{E}_u - 2\mathcal{A}_u \mathcal{A}^T \right) \mathcal{G}^{-1} \mathcal{E}_v - \left( \mathcal{E}_v - 2\mathcal{A}_v \mathcal{A}^T \right) \mathcal{G}^{-1} \mathcal{E}_u = 0, \\
2\mathcal{A}_{uv} + \frac{\alpha_u}{\alpha} \mathcal{A}_v + \frac{\alpha_v}{\alpha} \mathcal{A}_u &= \left( \mathcal{E}_u - 2\mathcal{A}_u \mathcal{A}^T \right) \mathcal{G}^{-1} \mathcal{A}_v - \left( \mathcal{E}_v - 2\mathcal{A}_v \mathcal{A}^T \right) \mathcal{G}^{-1} \mathcal{A}_u = 0, \\
\alpha_{uv} &= 0,
\end{align*}
\]

(21)

where the suffixes represent the derivatives and the matrix $\mathcal{G} = \frac{1}{2}(\mathcal{E} + \mathcal{E}^T) - A \mathcal{A}^T$.

Given a solution $(\alpha, \mathcal{E}, \mathcal{A})$ of (21), the matrix functions $\mathcal{G}$ and $B$ can be determined, respectively, from the symmetric and antisymmetric parts of the matrix Ernst potential, while the string frame conformal factor $\tilde{f}$ is determined from the relations

\[
\begin{align*}
\partial_u \log \left( \frac{\alpha^T}{\alpha_u \alpha_v} \right) &= \frac{\alpha}{\alpha_u} \text{tr} \left[ \frac{1}{4} \left( \mathcal{E}_u - 2\mathcal{A}_u \mathcal{A}^T \right) \mathcal{G}^{-1} \left( \mathcal{E}_v^T - 2\mathcal{A} \mathcal{A}^T \right) \mathcal{G}^{-1} + \mathcal{A}_u \mathcal{G}^{-1} \mathcal{A}_v \right], \\
\partial_v \log \left( \frac{\alpha^T}{\alpha_u \alpha_v} \right) &= \frac{\alpha}{\alpha_v} \text{tr} \left[ \frac{1}{4} \left( \mathcal{E}_v - 2\mathcal{A}_v \mathcal{A}^T \right) \mathcal{G}^{-1} \left( \mathcal{E}_u^T - 2\mathcal{A} \mathcal{A}^T \right) \mathcal{G}^{-1} + \mathcal{A}_v \mathcal{G}^{-1} \mathcal{A}_u \right],
\end{align*}
\]

(22)

and the conformal factor in the Einstein frame is $f = e^{-2\tilde{f}}$.

It is useful to note that for any solution $(\alpha, \mathcal{E}, \mathcal{A})$ of the matrix Ernst equation (21) and corresponding $\mathcal{G}$ and $B$, there exist dual matrix potentials $\tilde{B}$ and $\tilde{A}$ such that

\[
\begin{align*}
\tilde{B}_u &= -\alpha \mathcal{G}^{-1} \left( B_u - \mathcal{A}_u \mathcal{A}^T + A \mathcal{A}^T \right) \mathcal{G}^{-1}, \\
\tilde{B}_v &= \alpha \mathcal{G}^{-1} \left( B_v - \mathcal{A}_v \mathcal{A}^T + A \mathcal{A}^T \right) \mathcal{G}^{-1}, \\
\tilde{A}_u &= -\alpha \mathcal{G}^{-1} \mathcal{A}_u + \tilde{B} \mathcal{A}_u, \\
\tilde{A}_v &= \alpha \mathcal{G}^{-1} \mathcal{A}_v + \tilde{B} \mathcal{A}_v,
\end{align*}
\]

(23)

These dual potentials will be used in our construction of the alternative Ernst potential and matrix generalization of the Kramer–Neugebauer transformation.

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3 In particular, in [21, 22], a special $(2d + n) \times (2d + n)$-matrix form of these dynamical equations was found and used for the proof and exploration of the integrable structure of these equations.
The hyperbolic ‘vacuum’ matrix Ernst equation and the Kramer–Neugebauer transformation of its solution space

The matrix analogue of the vacuum Ernst equation

From the mathematical point of view, the matrix analogue of the vacuum Ernst equation of general relativity arises from the matrix Ernst equation (21) in the corresponding ‘vacuum’ case for which \( \mathcal{A} = 0 \), i.e. if all Abelian vector gauge fields vanish. In this ‘vacuum’ case, however, the set of dynamical variables consists of the string frame metric \( \mathcal{G} \) and the antisymmetric matrix potential \( \mathcal{B} \) which can be combined (as in the general case (20)), into a matrix Ernst potential. Then the dynamical equations take the form

\[
\begin{align*}
2E_{uv} + \frac{\alpha_u}{\alpha} E_v + \frac{\alpha_v}{\alpha} E_u - \mathcal{G}_{uv}^{-1} E_v E_u - \mathcal{G}_{uv}^{-1} E_u = 0, & \quad \mathcal{E} = \mathcal{G} + \mathcal{B} \\
\alpha_{uv} = 0. & \quad \det \mathcal{G} \equiv e^{2\bar{\delta} \alpha^2}.
\end{align*}
\]

(24)

The dual matrix potential \( \mathcal{B} \) defined above is determined in terms of these variables as

\[
\mathcal{B}_u = -\alpha \mathcal{G}^{-1} \mathcal{B}_u \mathcal{G}^{-1}, \quad \mathcal{B}_v = \alpha \mathcal{G}^{-1} \mathcal{B}_v \mathcal{G}^{-1}.
\]

(25)

The alternative vacuum Ernst potential and Kramer–Neugebauer transformation

As can be checked by direct calculation, for any solution \( (\alpha, \mathcal{E}) \) of the matrix Ernst equation (24) the following combination of \( \mathcal{G} \) and \( \mathcal{B} \) should also satisfy (24):

\[
\mathcal{E} = \alpha \mathcal{G}^{-1} + \mathcal{B}.
\]

(26)

Therefore, given a solution \( (\alpha, \mathcal{E}) \) of (24), we can construct two ‘vacuum’ solutions corresponding to different ‘interpretations’ (24) and (26) of the matrix Ernst potential:

\[
\begin{align*}
\mathcal{G}(1) = \frac{1}{2} \left( \mathcal{E} + \mathcal{E}^{\mathcal{T}} \right), & \quad \mathcal{G}(2) = \alpha \mathcal{G}(1)^{-1}, \\
\mathcal{B}(1) = \frac{1}{2} \left( \mathcal{E} - \mathcal{E}^{\mathcal{T}} \right), & \quad \mathcal{B}(2) = -\alpha \mathcal{G}(1)^{-1} \mathcal{B}(1) \mathcal{G}(1)^{-1}, \\
\mathcal{E} \mathcal{B}(1), & \quad \mathcal{E} \mathcal{B}(2), \quad \mathcal{E} \mathcal{G}(1).& \quad \mathcal{E} \mathcal{G}(2), \quad \mathcal{E} \mathcal{G}(1), \quad \mathcal{E} \mathcal{B}(1), \quad \mathcal{E} \mathcal{B}(2), \quad \mathcal{E} \mathcal{G}(1). \quad (27)
\end{align*}
\]

Thus, we have the transformation acting on the space of solutions of the above described symmetry reduced ‘vacuum’ \( (\mathcal{A} = 0) \) string gravity model (18) which takes the form

\[
\begin{align*}
\mathcal{G}(1), \mathcal{B}(1), \mathcal{E}(1) \quad & \leftrightarrow \quad \mathcal{G}(2), \mathcal{B}(2), \mathcal{E}(2) \\
\end{align*}
\]

This transformation generalizes to the case of hyperbolic symmetry reductions of the ‘vacuum’ string gravity model the transformation discovered by Kramer and Neugebauer for stationary axisymmetric solutions of the vacuum Einstein equations in general relativity (see [1] for further details and references).

Travelling ‘vacuum’ massless bosonic waves that are fixed points of the generalized Kramer–Neugebauer transformation

To construct the solutions for the travelling ‘vacuum’ \( (\mathcal{A} = 0) \) bosonic waves, we use the same method as used in one of the previous sections of this paper for the construction of pure vacuum travelling gravitational waves in general relativity—the solutions (8) and (9). Namely, we consider the solutions of the matrix Ernst equation (24) which are fixed points of the transformation (27). These solutions should satisfy the conditions
\[ G = aG^{-1}, \quad \tilde{B} = B. \]  

The first of these equations, together with the condition of positive signature of the matrix \( G \), means that this matrix is proportional to a unit matrix and thus, we have

\[ \det G = \alpha^{d/2} \quad \Rightarrow \quad e^{\tilde{\Phi}} = \alpha^{\frac{d}{2}}. \]

Then, in accordance with the definition \( \Phi \) of \( \tilde{B} \), the second of the invariance conditions \( \alpha = 0 \) means that \( B = B_+(v) \) where \( B_+(v) \) is an arbitrary antisymmetric \( d \times d \)-matrix function of \( v = t + x \). A direct substitution of this solution into the matrix Ernst equation \( \text{(24)} \) shows that this equation is satisfied. Calculating the conformal factor from \( \text{(22)} \) we obtain the solution in the string frame in the form

\[
\begin{align*}
\alpha_\alpha &= \int \alpha_\alpha, \\
&= \left[ \frac{d - 4}{4} \log \alpha \right], \\
&= \alpha(u, v), \\
&= \alpha = 0.
\end{align*}
\]

Similarly to the solutions constructed in previous sections, for different choices of \( \alpha \) (as a solution of \( \alpha_\alpha = 0 \)) for which \( \alpha = \text{const} \) are space-like or time-like surfaces, the transformations \( u \rightarrow h(u), v \rightarrow g(v) \) allow one to choose \( t = \alpha \) or \( x = \alpha \), respectively.

It is easy to see that in the case \( \alpha = t \), for \( B_+(v) = 0 \) and \( B_+(u) = 0 \) the solutions \( \alpha_\alpha = 0 \) and \( \alpha = 0 \) describe some travelling waves of a three-form field \( H \) from the bosonic sector of the string gravity model \( \text{(18)} \). The arbitrary antisymmetric matrix functions \( B_+(v) \) and \( B_+(u) \) describe arbitrary amplitudes of these waves and arbitrary states of their polarizations. It is also interesting to note that in these travelling wave solutions the part of the metric transverse to the direction of wave propagation (the \( x \)-axis) remains unperturbed and therefore these waves do not comprise pure gravitational and dilaton wave components.

**Travelling waves of bosonic gauge fields in the Kasner-like background**

To construct the solutions of \( \text{(21)} \) which describe the travelling waves for vector gauge fields, we follow the same method we used in previous sections for the construction of pure electromagnetic waves propagating on the symmetric Kasner background in the Einstein–Maxwell theory. Namely, we assume that in these waves the part of the metric transverse to the direction of wave propagation is not perturbed, i.e. the gravitational component of these waves is absent. As before, we assume also for the string frame spatial metric a Kasner-like form with specially chosen (equal to each other) exponents and restrict our consideration to the case in which the dual potential \( \tilde{B} \) vanishes:

\[ \alpha_\alpha = 0. \]
With these two assumptions we obtain immediately from (23) the relations

\[
\begin{align*}
B_u &= A_u A^T - A A^T_u, \\
\overline{A}_u &= -\overline{A}_u, \\
B_v &= A_v A^T - A A^T_v, \\
\overline{A}_v &= A_v.
\end{align*}
\]  

(32)

These relations imply the condition \(A_{uv} = 0\), but for travelling waves we should restrict ourselves to one of the cases \(A_v = 0\) or \(A_u = 0\) which imply, respectively, the conditions \(B_v = 0\) or \(B_u = 0\). In the first case, we obtain

\[
\begin{align*}
A &= A_\alpha u(u), \\
B &= B_\alpha u(u), \\
B'_\alpha(u) &= A'_\alpha u(u) A^T_\alpha u(u) - A_\alpha u(u) A'_\alpha u^T(u),
\end{align*}
\]

while in the second case we have

\[
\begin{align*}
A &= A_\alpha v(v), \\
B &= B_\alpha v(v), \\
B'_\alpha(v) &= A'_\alpha v(v) A^T_\alpha v(v) - A_\alpha v(v) A'_\alpha v^T(v).
\end{align*}
\]

Hence, given the matrix \(A_\alpha(u)\) (in the first case) or the matrix \(A_\alpha(v)\) (in the second case), one can calculate from the above expressions the corresponding three-form potentials \(B_\alpha(u)\) or \(B_\alpha(v)\), respectively. Direct substitution of all these expressions into the matrix Ernst equation (21) shows that these equations are satisfied for arbitrary matrix functions \(A_\alpha(u)\) or \(A_\alpha(v)\). After calculation of the conformal factors from the expressions (22), we obtain a class of solutions with string frame metrics

\[
\begin{align*}
ds^2_D &= -\alpha^{d-1} \exp \left[ \int_{u_0}^u \frac{\text{tr}(A'_\alpha(u) \cdot A^T_\alpha(u))}{\alpha_u} du \right] \alpha_u \alpha_v dv \cdot dv + \alpha \left[ dx^2 + \ldots + dx^{D^2} \right] \]
\end{align*}
\]

(33)

where \(A_\alpha(u)\) is an arbitrary \(d \times n\)-matrix function; \(\alpha = \alpha(u, v)\) and \(\alpha_{uv} = 0\). The other class of solutions is

\[
\begin{align*}
ds^2_D &= -\alpha^{d-1} \exp \left[ \int_{v_0}^v \frac{\text{tr}(A'_\alpha(v) \cdot A^T_\alpha(v))}{\alpha_v} dv \right] \alpha_u \alpha_v du \cdot du + \alpha \left[ dx^2 + \ldots + dx^{D^2} \right] \]
\end{align*}
\]

(34)

where \(A_\alpha(v)\) is an arbitrary \(d \times n\)-matrix function; \(\alpha = \alpha(u, v)\) and \(\alpha_{uv} = 0\).

Similarly to our comments on previous solutions, we note that for different choices of \(\alpha\) as a solution of \(\alpha_{uv} = 0\) for which \(\alpha = \text{const}\) are space-like or time-like surfaces, the transformations \(u \rightarrow h(u), v \rightarrow g(v)\) allow one to choose \(t = \alpha\) or \(x = \alpha\), respectively. In the case \(\alpha = t\), each of the solutions (33) and (34) for \(A_\alpha(u) = 0\) and \(A_\alpha(v) = 0\), respectively, reduces to a \(D\)-dimensional Kasner-like background and therefore, for \(A_\alpha(u) \neq 0\) and \(A_\alpha(v) \neq 0\) these solutions describe traversing waves of an arbitrary number \(n\) of Abelian vector gauge fields from the bosonic sector of string gravity (18). The arbitrary \(d \times n\)-matrix functions \(A_\alpha(u)\) and \(A_\alpha(v)\) describe arbitrary amplitudes of these waves (propagating, respectively, in the positive and negative directions of the \(x\)-axis) and arbitrary states of their polarizations. It can be noted here that the space–time geometries in the solutions (33) and (34), in the four-dimensional case (in which the dilaton field vanishes), are similar to those in the solutions (15) and (17), respectively, i.e. these metrics belong to type D of the Petrov classification, and the differences between these solutions are in their different sets of non-gravitational fields and in their influence on the rate of space–time expansion along the direction of wave.
propagation. For these waves, the part of the metric transverse to the direction of propagation (the $x$-axis) also remains unperturbed and therefore these waves also do not comprise pure gravitational and dilaton wave components, but some ‘induced’ wave components of the three-form field with the potentials $B_3(u)$ or $B_3(v)$ (depending on $A_+(u)$ and $A_-(v)$, respectively) are present in these solutions.

Summary of results and conclusions

In this paper, we constructed several classes of exact travelling wave solutions (i.e. the solutions for waves which propagate in a given space–time region in certain direction without any scattering, caustics or singularities) for gravitational and electromagnetic waves in general relativity as well as for massless bosonic fields in some string gravity models in four and higher dimensions. Each of these classes of solutions are different from pp-waves and depend on a set of arbitrary functions of a null coordinate which allow one to consider the waves with arbitrary profiles and states of polarizations. The solutions of these classes describe the propagation of these travelling waves through some expanding spatially homogeneous Kasner-like space–times.

From the mathematical point of view, it is interesting that these classes of solutions were obtained as fixed points of (generalized) Kramer–Neugebauer transformations of the solution spaces of hyperbolic integrable reductions of vacuum Einstein equations in general relativity and certain ‘vacuum-like’ hyperbolic integrable symmetry-reduced equations for massless bosonic fields in string gravity in four and higher dimensions. The ‘source’ of the other travelling wave solutions found in this paper is the ansatz which restricts the consideration of solutions of the hyperbolic Ernst equations in general relativity and the matrix Ernst equations in string gravity to the solutions for which the part of the spatial metric transverse to the direction of wave propagation coincides with that for the background solution.

From the physical point of view, it is worth noting that the integrability of the hyperbolic Ernst equations and generalized (matrix) hyperbolic Ernst equations (which govern the dynamics of the waves considered in this paper) as well as the very simple form of all the solutions constructed here, make possible the application of some effective methods (based on the modern theory of integrable systems) for the consideration of physically interesting questions concerning various aspects of the nonlinear interactions of these waves propagating through curved space–times. In particular,

- The application to these solutions of various global symmetry transformations such as Harrison and Bonnor transformations allows one to construct various solutions that also include some arbitrary functions of the null variable and which describe travelling gravitation and electromagnetic waves propagating on curved cosmological backgrounds not only of Kasner-type but also some other types.
- The application to these solutions of vacuum [23] and electrovacuum [24] soliton generating techniques allows one to consider the interaction of these waves (with arbitrary profiles) with, respectively, gravitational or gravitational and electromagnetic solitons on the same Kasner background.
- The collision of such travelling wave pulses also has some physical interest. In particular, it appears interesting to analyse the gravitational waves which may be created as a result of the collision of pure electromagnetic or other massless bosonic travelling wave pulses with arbitrary amplitudes on the expanding spatially homogeneous background. The
general approach to the solution of the corresponding characteristic initial value problems was developed in [19, 25, 26].

However, the questions of the collision and nonlinear interaction of waves are not in the scope of this paper, and it is expected that these will be the subject of future studies.

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