Time evolution of the energy density inside a one-dimensional non-static cavity with a vacuum, thermal and a coherent state

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Abstract. We study the time evolution of the energy density for a real massless scalar field in a two-dimensional spacetime, inside an expanding (not oscillating) cavity, and also the force exerted by the field on the moving boundary. Our calculations are based on the exact numerical approach proposed by Cole and Schieve. Considering Neumann and Dirichlet boundary conditions, we investigate the following initial states of the field: vacuum, thermal and the coherent state.

1. Introduction

Moore, in the context of a real massless scalar field inside a non-static cavity, obtained an exact formula for the expected value of the energy-momentum tensor (in a two-dimensional spacetime), given in terms of a functional equation, usually called Moore’s equation [1]. For this equation, there is no general technique of analytical solution, but exact analytical solutions for particular movements of the boundary [2–4], and also approximate analytical solutions (see, for instance [5, 6]) have been obtained. The dynamical Casimir effect has been investigated for a thermal bath as the initial field state in a non-static cavity [5, 7–10], and also for a coherent state [9,11–13]. Squeezed states have also been considered [9].

In the context of the Dynamical Casimir effect, the problem of boundaries oscillating with small amplitudes is commonly related to the problem of photon creation in laboratory, and it is usually investigated via approximate analytical solutions of the Moore equation [5, 6], and also via other perturbative methods [14]. On the other hand, the investigation of non-oscillating movements, with large amplitudes, are frequently motivated by problems of particle creation in cosmological models and radiation emitted by collapsing black holes [2,15, 16], and the related energy density has been investigated just for particular laws of motion for which the Moore equation can be solved exactly [2,16]. For general non-oscillating trajectories, an approach based on the numerical method to solve exactly the Moore equation for a general law of motion, proposed by Cole and Schieve [17], can be useful. This method gives results in agreement with those obtained by perturbative approaches [17,18], and also can give results for models that are not solvable via perturbative techniques, as the problem of expanding cavities.

In the present paper, we apply a technique of calculation based on the Cole-Schieve approach [17,19], and examine the energy density in an expanding one-dimensional cavity, with a law of
motion for which there is no exact analytical solution for the correspondent Moore equation. Beyond the vacuum, we also consider thermal and coherent states as the initial field states.

The article is organized as follows. In section 2 we discuss the general field solution and the law of motion of the boundary. In section 3 we investigate the non-static cavity for a vacuum, thermal and a coherent state. In section 4 we summarize our results.

2. General field solution and law of motion

Let us start considering the field satisfying the Klein-Gordon equation (we assume throughout this paper \(\hbar = c = k_B = 1\)) and obeying conditions imposed at the static boundary located at \(x = 0\), and also at the moving boundary’s position at \(x = L(t)\), where \(x = L(t)\) is a prescribed law for the moving boundary and \(L(t < 0) = L_0\), with \(L_0\) being the length of the cavity in the static situation. As done in Ref. [20], we consider four types of boundary conditions. The Dirichlet-Neumann (DN) boundary condition imposes Dirichlet condition at the static boundary, whereas the space derivative of the field taken in the instantaneously co-moving Lorentz frame vanishes (Neumann condition) at the moving boundary’s position. We also consider: Dirichlet-Dirichlet (DD), Neumann-Neumann (NN) and Neumann-Dirichlet (ND) boundary conditions. A general solution of the wave equation can be written as [20]:

\[
\hat{\psi}(t, x) = \lambda(\hat{A} + \hat{B}\psi^{(0)}) + \sum_{n=1-2\beta}^{\infty} [\hat{a}_n \psi_n(t, x) + H.c.],
\]

(1)

where the field modes \(\psi_n(t, x)\) are given by

\[
\psi_n(t, x) = \frac{1}{\sqrt{4(n + \beta)\pi}} \left[ \gamma e^{-i(n+\beta)\pi R(v)} + \gamma^* e^{-i(n+\beta)\pi R(u)} \right],
\]

(2)

with \(\psi^{(0)} = [R(v) + R(u)]/2\) (see Ref. [21]), \(u = t - x, v = t + x\), and \(R\) satisfying the functional equation \(R[t + L(t)] - R[t - L(t)] = 2\), which is the Moore equation. The operators \(\hat{A}\) and \(\hat{B}\) satisfy the commutation rules \([\hat{A}, \hat{B}] = i\), \([\hat{A}, \hat{a}_n] = [\hat{B}, \hat{a}_n] = 0\). The NN solution is recovered for \(\lambda = \gamma = 1\) and \(\beta = 0\). The other three cases are recovered if \(\lambda = 0\) and: \(\beta = 0\) and \(\gamma = i\) for the DD case; \(\beta = 1/2\) and \(\gamma = i\) for the DN case; \(\beta = 1/2\) and \(\gamma = 1\) for the ND case.

In the present paper we investigate the following particular trajectory \(x = L(t)\), based on the one proposed by Walker and Davies [22]:

\[
t = L_0 - L + A \left( e^{-2(L_0 - L)/B} - 1 \right)^{1/2},
\]

(3)

valid for \(t \geq 0\), where \(A\) and \(B\) are positive constants, with \(A > B\) (so that \(|\dot{L}| < 1\)). For \(A = 2\) and \(B = 1\) (values also chosen in Ref. [22]) the trajectory is showed in Fig. 1. This trajectory has some interesting properties: it has a discontinuity of the boundary acceleration when \(t = 0\) \((L = 1)\) (as showed in Fig. 1(b)), but it is smooth and asymptotically static for \(t \to \infty\) (see Fig. 1 (a)). With this feature, the “in” and “out” states can be well defined and the average number of produced particles can be calculated. In addition, we have a discontinuity of the boundary acceleration at one spacetime point only, in contrast with other laws of motion investigated in the literature, which have discontinuities of the acceleration at the two points: in the beginning and also at the end of the time interval in which the boundary is accelerated. According to Walker and Davies, asymptotically static trajectories brought the advantage of avoiding certain pathologies related to the radiation emitted by a moving mirror with abrupt acceleration [22].
3. Vacuum, thermal and coherent states

As examples of initial field states such that the density matrix is diagonal in the Fock basis, let us consider the vacuum and the thermal state. It can be shown that the expected value of the energy density operator $T = \langle \hat{T}_{00}(t, x) \rangle$ can be split in $T = T_{\text{vac}} + T_{\text{non-vac}}$, where $T_{\text{vac}}$ is the contribution to the energy density due the vacuum part and $T_{\text{non-vac}}$ is the non-vacuum contribution due to the real particles in the initial state of the field. Hereafter we consider the averages $\langle ... \rangle$ taken over initial field states annihilated by $\hat{B}$. Let us start considering the vacuum as the initial field state ($T_{\text{non-vac}} = 0$). The vacuum contribution to the energy density inside the oscillating cavity can be written as $T_{\text{vac}} = -f(v) - f(u)$, where:

$$f = \frac{|\gamma|^2}{24\pi} \left\{ \frac{R'''}{R'} - \frac{3}{2} \left( \frac{R''}{R'} \right)^2 + \pi^2 \left[ \frac{1}{2} - 3(\beta - \beta^2) \right] R'^2 \right\}. \quad (4)$$

For the static situation the function $R$ is given by $R(z) = z/L_0$ and its first derivative is a constant $R'(z) = 1/L_0$. From this equation we get (see [1,2,23] the known static Casimir forces $F_{\text{vac}}^{(s)}$ acting on the right boundary:

$$F_{\text{vac}}^{(s)} \text{DD} = F_{\text{vac}}^{(s)} \text{NN} = -\pi/(24L_0^2), \quad F_{\text{vac}}^{(s)} \text{DN} = F_{\text{vac}}^{(s)} \text{ND} = \pi/(48L_0^2), \quad (5)$$

where the superscripts DD, NN, DN and ND mean the types of boundary conditions considered in the calculations. In Fig. 2(a) we plot, for both DD and NN cases and the vacuum as the initial field state, the time evolution of the actual force $F_{\text{vac}} = T_{\text{vac}}[t, L(t)]$ acting on the moving boundary (solid line) for each position $L$, whereas the dotted line shows the value of the static Casimir force $-\pi/[24L(t)^2]$ which would act on the boundary if it was static at the position $x = L$. In analogous manner, in Fig. 2(b) we plot, for the DN and ND cases, the time evolution of the force acting on the moving boundary (solid line), whereas the dotted line shows the repulsive static Casimir force $\pi/[48L(t)^2]$. We see in Fig. 2 that, for DD and NN, and also for DN and ND, the radiation force acting on the moving boundaries are the same:

$$F_{\text{vac}}^{\text{DD}} = F_{\text{vac}}^{\text{NN}}, \quad F_{\text{vac}}^{\text{DN}} = F_{\text{vac}}^{\text{ND}}. \quad (6)$$

Fig. 2 also shows the dynamical force approaching the static Casimir one in the limit $t \to \infty$. 

Figure 1. (a) Moving mirror trajectory defined by Eq. (3); (b) Moving mirror velocity (vertical axis) as function of the boundary position.
Figure 2. (a) The quantum force $F_{\text{vac}}$ acting on the moving boundary (solid line) and the attractive static Casimir force $F_{\text{vac}}^{(s)}$ (dotted line) for the DD or NN cases; (b) The quantum force $F_{\text{vac}}$ acting on the moving boundary (solid line) for the DN or ND cases, and the repulsive static Casimir force $F_{\text{vac}}^{(s)}$ (dotted line). In both cases the vacuum was considered as the initial field state. The discontinuities in the derivative occur at $t_1 \approx 2.67, t_2 \approx 6.78, t_3 \approx 12.20, t_4 \approx 18.56$.

In Fig. 2 we can see discontinuities of the derivatives. These discontinuities always occur when the front of the wave in the energy density meets the right boundary. When $t = 0$ the right boundary starts to move, interacting with the vacuum field, and generating a wave in the energy density, propagating leftward in the cavity. The wave will be reflected back by the left (static) boundary and propagates rightward until meeting the right boundary at time $t = t_1 \approx 2.67$. This value can be obtained by solving the equation $t_1 - L_0 = L(t_1)$. From this instant on, the right boundary will interact also with this reflected wave, what generates the mentioned discontinuity. In general, the wave in the energy density, after reflections, will meet the right boundary at the instants $t = t_i$, where $i = 1, 2, ..., $ which can be obtained by solving the equation

$$t_i - t_{i-1} - L(t_{i-1}) = L(t_i)$$

with $t_0 = 0$.

In Fig. 3(a) we show the energy density at the point $x = L_0/2$, as function of time, for DD or NN cases (solid line), and also for DN or ND cases (dotted line). From $t = 0$ to $t = 1/2$ we observe a constant value, correspondent to the static Casimir energy density. At $t = 1/2$ the wave in the energy density arrives at the point $x = L_0/2$. The jump observed at this point is related to abrupt motion started by the right boundary, in the sense that there is a discontinuity of the boundary acceleration as showed in Fig. 1(b). After reflected by the left boundary, this discontinuity point propagates and can be observed again at $t = 1.5$, an so on, after successive reflections. The energy density at the point $x = L_0/2$ goes to zero, since, for our law of motion, the length of the cavity goes to infinity and the motion is asymptotically static as $t \to \infty$. In Fig. 3(b) we show the energy for all points in the cavity for $t = 30L_0$.

Now, let us consider the thermal state as the initial field state. For this case we have $\langle \hat{a}_n^\dagger \hat{a}_{n'} \rangle = \delta_{nn'}\pi(n, \beta)$ and $\langle \hat{a}_n \hat{a}_{n'} \rangle = \langle \hat{a}_n^\dagger \hat{a}_{n'}^\dagger \rangle = 0$, where $\pi(n, \beta) = [\exp(\kappa(n + \beta)) - 1]^{-1}$ and $\kappa = 1/T$. In Fig. 4 we compare the behavior of the force $F_T = T_{\text{non-vac}}[t, L(t)]$ for $T = 1$ and $F_{\text{vac}}$, obtaining that the difference between $F_T^{(\text{DD})}$ and $F_T^{(\text{DN})}$ is a scale factor. We also see that

$$F_T^{(\text{DD})} = F_T^{(\text{NN})}, \quad F_T^{(\text{DN})} = F_T^{(\text{ND})}.$$
Figure 3. (a) The energy density for DD or NN cases (solid line) and also for DN or ND cases (dotted line), at the point \( x = L_0/2 \), as function of time. The dashed and dashed-point lines represents, respectively, the static Casimir energy densities for DD-NN and ND-DN cases. (b) The energy density for for DD or NN cases (solid line) and also for DN or ND cases (dotted line), as function of the normalized position \( x/L(t) \) in the cavity, for a time \( t = 30L_0 \). In both cases the vacuum was considered as the initial field state.

Note that the discontinuities of the derivative visualized in Fig. 4 occur for the same values \( t_i \) given in Eq. 7.

Figure 4. (a) The quantum forces \( F_T \) (solid line) and \(-30 \times F_{\text{vac}} \) (dotted line) for DD or NN cases, as function of time; (b) The quantum forces \( F_T \) (solid line) and \( 30 \times F_{\text{vac}} \) (dotted line) for DN or ND cases, as function of time. In both cases the thermal bath with temperature \( T = 1 \) was considered as the initial field state.

The coherent state, an example of a non-diagonal state \( \langle \hat{a}_n^\dagger \hat{a}_{n'} \rangle \neq 0 \), can be defined as an eigenstate of the annihilation operator: \( \hat{a}_n |\alpha\rangle = \alpha \delta_{nn_0} |\alpha\rangle \), where \( \alpha = |\alpha|e^{i\theta} \) and \( n_0 \) is related to the frequency of the excited mode \[24\]. In Fig. 5 we visualize the behavior of the force \( F_\alpha = F_{\text{non-vac}}[t, L(t)] \) for \( \alpha = 1 \) and \( \theta = 0 \). Note that, for the coherent state, the symmetry
between the DD and NN cases, and also between DN and ND cases, is broken in the sense that

\[
\mathcal{F}_{\alpha}^{\text{DD}} \neq \mathcal{F}_{\alpha}^{\text{NN}}, \quad \mathcal{F}_{\alpha}^{\text{DN}} \neq \mathcal{F}_{\alpha}^{\text{ND}}.
\]

(9)

Figure 5. The coherent force \( \mathcal{F}_{\alpha} \) acting on the moving boundary (vertical axis), as function of time, for a coherent state with \( |\alpha| = 1 \) and \( \theta = 0 \). (a) The DD case (solid line) and the NN case (dashed-point line); (b) The DN case (solid line) and the ND case (dashed-point line).

Even in the static case \( (t < 0) \), for a certain value of \( \alpha \) and \( \theta \), the energy density for the coherent state is spacetime dependent, and also different if we consider NN, DD, ND or DN cases [20]. In contrast, for vacuum and thermal initial states, the energy density is a constant, having a same value for NN and DD cases, and also for ND and DN cases [20]. These differences between coherent (non-diagonal) and thermal-vacuum (diagonal) states in the static situation propagate along the dynamical situation \( (t > 0) \), as viewed in Eqs. (6), (8) and (9).

4. Final comments

In summary, using a technique of calculation based on the Cole-Schieve approach, we obtained the behavior of the energy density inside an expanding cavity, and also the behavior of the dynamical force acting on the moving boundary. We also considered Dirichlet and Neumann boundary conditions We observed that the energy density and the force on the moving mirror are not affected if we change NN by DD (or DN by ND) boundary conditions, for vacuum or thermal initial field states (diagonal states). On the other hand, the same invariance is not observed for the coherent (non-diagonal) state.

The number of produced particles in this model will be calculated further.

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References

[1] Moore G T 1970 *J. Math. Phys.* **11** 2679
[2] Fulling S A and Davies P C W 1976 *Proc. R. Soc. London* A **348** 393
[3] Law C K 1994 *Phys. Rev. Lett.* **73** 1931
[4] Wu Y, Chan K W, Chu M C and Leung P T 1999 *Phys. Rev. A* **59** 662; Wegrzyn P 2007 *J. Phys.* B **40** 2621
[5] Dodonov V V, Klimov A B and Nikonov D E 1993 *J. Math. Phys.* **34** 2742
[6] Dalvit D A R and Mazzitelli F D 1998 *Phys. Rev. A* **57** 2113
[7] Jaekel M T and Reynaud S 1993 *J. Phys. I (France)* **3** 339; Jaekel M T and Reynaud S 1993 *Phys. Lett.* A **172** 319; Machado L A S, Maia Neto P A and Farina C 2002 *Phys. Rev. D* **66** 105016.
[8] Plunien G, Schutzhold R and Soff G 2000 *Phys. Rev. Lett.* **84** 1882
[9] Alves D T, Granhen E R and Lima M G 2008 *Phys. Rev. D* **77** 125001
[10] Hui J, Qing-Yun S and Jian-Sheng W 2000 *Phys. Lett.* A **268** 174; Schutzhold R, Plunien G and Soff G 2002 *Phys. Rev. A* **65** 043820; Schaller G, Schutzhold R, Plunien G and Soff G 2002 *Phys. Rev. A* **66** 023812
[11] Alves D T, Farina C and Maia Neto P A 2003 *J. Phys.* A **36** 11333
[12] Dodonov V V, Klimov A B and Man’ko V I 1990 *Phys. Lett.* A **149** 225
[13] Andreata M A and Dodonov V V 2000 *J. Phys.* A **33** 3209
[14] Razavy M and Terning J 1985 *Phys. Rev. D* **31** 307; Calucci G 1992 *J. Phys.* A **25** 3873; Law C K 1994 *Phys. Rev. A* **49** 433; Law C K 1995 *Phys. Rev. A* **51** 2537; Dodonov V V and Klimov A B 1996 *Phys. Rev. A* **53** 2664; Mundarain D F and Maia Neto P A 1998 *Phys. Rev. A* **57** 1379
[15] Haro J 2005 *J. Phys.* A **38** L307
[16] Castagnino M and Ferraro R 1984 *Ann. Phys.* **154** 1-23
[17] Cole C K and Schieve W C 1995 *Phys. Rev. A* **52** 4405
[18] Alves D T and Granhen E R 2008 *Phys. Rev. A* **77** 015808
[19] Cole C K and Schieve W C 2001 *Phys. Rev. A* **64** 023813-1
[20] Alves D T, Granhen E R, Lima M G and Silva H O *Exact solution for the energy density inside a non-static cavity with an arbitrary initial field state* (to be published)
[21] Dalvit D A R, Mazzitelli F D and Milin O 2006 *J. Phys.* A **39** 6261
[22] Walker W R and Davies P C W 1982 *J. Phys.* A **15** L477
[23] Boyer T H 2003 *Am. J. Phys.* **71** 990
[24] Glauber R J 1963 *Phys. Rev.* **131** 2766; Glauber R J 1963 *Phys. Rev. Lett.* **10** 84