Cartesian product and acyclic edge colouring

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Abstract

The acyclic chromatic index, denoted by $a'(G)$, of a graph $G$ is the minimum number of colours used in any proper edge colouring of $G$ such that the union of any two colour classes does not contain a cycle, that is, forms a forest. We show that $a'(G \square H) \leq a'(G) + a'(H)$ for any two graphs $G$ and $H$ such that $\max\{a'(G), a'(H)\} > 1$. Here, $G \square H$ denotes the cartesian product of $G$ and $H$. This extends a recent result of [15] where tight and constructive bounds on $a'(G)$ were obtained for a class of grid-like graphs which can be expressed as the cartesian product of a number of paths and cycles.

Keywords: cartesian product, acyclic edge colouring, acyclic chromatic index.

1 Introduction

All graphs we consider are simple and finite. Throughout the paper we use $\Delta = \Delta(G)$ to denote the maximum degree of a graph $G$. A colouring of the edges of a graph is proper if no pair of incident edges receive the same colour. A proper colouring $\mathcal{C}$ of the edges of a graph $G$ is acyclic if there is no two-coloured (bichromatic) cycle in $G$ with respect to $\mathcal{C}$. In other words, the subgraph induced by the union of any two colour classes in $\mathcal{C}$ is a forest. The minimum number of colours required to edge-colour a graph $G$ acyclically is termed the acyclic chromatic index of $G$ and is denoted by $a'(G)$. The notion of acyclic colouring was introduced by Grünbaum in [8]. The acyclic chromatic index and its vertex analogue are closely related to other parameters like oriented chromatic number and star chromatic number of a graph $G$ both of which have many practical applications [11, 3].

Determining $a'(G)$ is a hard problem both from a theoretical and from an algorithmic point of view. Even for the simple and highly structured class of complete graphs ($K_n$), the value of $a'(G)$ is still not determined exactly.
However, using probabilistic arguments, some loose upper bounds have been obtained. For example, see

(i) [12] for a bound of $a'(G) \leq 16\Delta$ which improves a previous bound of $a'(G) \leq 64\Delta$ due to [2].

(ii) [14] for a bound of $a'(G) \leq 4.52\Delta$ for graphs $G$ with girth (the length of the shortest cycle) at least 220.

It has been conjectured [1] that $a'(G) \leq \Delta + 2$ for any $G$ and this has been shown to be true for some special classes of graphs. However, the presently known bounds are far from the conjectured bound of $\Delta + 2$. It is still open whether the conjecture is true or if there are counterexamples.

Some tight upper bounds have also been obtained for some special classes of graphs. For example, see

(iii) [5] for a bound of $a'(G) \leq 5$ for 3-regular graphs,

(iv) [1] for a bound of $a'(G) \leq \Delta + 2$ for graphs with girth at least $c\Delta(\log \Delta)$,

(v) [17] for a bound of $a'(G) \leq d + 1$ for random $d$-regular ($d$ fixed) graphs.

Some constructive bounds which lead to an actual acyclic edge colouring have also been obtained. See

(vi) [20] for a constructive bound of $a'(G) \leq 5$ for graphs with $\Delta \leq 3$,

(vii) [21] for a constructive bound of $a'(G) \leq 6\Delta \log \Delta$ for any arbitrary graph,

(viii) [15] for a constructive bound of $\Delta + 1$ for grid-like graphs.

(ix) [16] for a constructive bound of $\Delta + 1$ for outerplanar graphs.

In this paper, we look at the cartesian product (defined in Section 2), denoted $G \boxtimes H$, of two arbitrary graphs $G$ and $H$ and show that the acyclic chromatic index of the product is at most the sum of acyclic chromatic indices of $G$ and $H$, provided at least one of these values exceeds 1. This is an extension of the work in [15], where it was shown that $a'(G \boxtimes H) \leq a'(G) + \Delta(H)$, whenever $H$ is a path or a cycle and $a'(G) \geq 2$, $a'(G) \geq 3$, respectively. While the bound we give here is slightly weaker, we remove the restriction that $H$ is a path or a cycle.

Section 2 contains definitions and our main result. Section 3 contains some concluding remarks. In the following subsection, in order to motivate the reader, we present a brief exposure to previous work on graph invariants in the context of graph products.

### 1.1 Previous work on graph invariants and cartesian product

Since it is well known (see [10] for details) that any connected graph can be expressed in a unique way (upto isomorphism) as the cartesian product of the smaller so-called "prime"
graphs, researchers have studied how various invariants of a graph can be expressed in terms of those of its factors. Specifically, it has been shown that

(a) \( \gamma(G \Box H) \leq \min\{\gamma(G)|V(G)|, \gamma(H)|V(H)|\} \) by Vizing [22]. Here, \( \gamma(G) \) denotes the minimum size of a dominating set of \( G \).

(b) Vizing [22] also studied the independence number \( \alpha(G) \) (the maximum size of an independent set of \( G \)) in the context of cartesian product of graphs and showed that

\[
\begin{align*}
\alpha(G \Box H) & \geq \alpha(G)\alpha(H) + \min\{|V(G)| - \alpha(G), |V(H)| - \alpha(H)\} \\
\alpha(G \Box H) & \leq \min\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}
\end{align*}
\]

(c) Let \( \chi(G) \) denote the chromatic number of \( G \), that is, the minimum number of colours required to properly colour the vertices of \( G \). It was first noticed by Sabidussi [18] and can also be easily verified that \( \chi(G \Box H) = \max\{\chi(G), \chi(H)\} \).

(d) The Hadwiger number of a graph \( G \), denoted by \( \eta(G) \), is the largest integer \( l \) such that \( G \) has a \( K_l \) minor where \( K_l \) is the complete graph on \( l \) vertices. Recently, Chandran and Raju [6] have obtained results relating \( \eta(G) \) and the cartesian product operation. In particular, it is proved in [6] that \( \eta(G \Box H) \geq (g - \sqrt{h})(\sqrt{h} - 2)/4 \) where \( g = \eta(G) \geq h = \eta(H) \).

### 2 Definitions and Results

For a comprehensive introduction and survey of results on various graph products, the reader is advised to refer to the book authored by Imrich and Klavzar [10].

**Definition 2.1** Given two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \), their cartesian product, denoted by \( G_1 \boxtimes G_2 \), is defined as the graph \( G = (V, E) \) where \( V = V_1 \times V_2 \) and \( E \) contains the edge joining \((u_1, u_2)\) and \((v_1, v_2)\) if and only if either \( u_1 = v_1 \) and \( \{u_2, v_2\} \) is an edge in \( E_2 \) or \( u_2 = v_2 \) and \( \{u_1, v_1\} \) is an edge in \( E_1 \).

Note that \( G = G_1 \Box G_2 \) can be thought of as being obtained as follows. Take \(|V_2|\) isomorphic copies of \( G_1 \) and label them with vertices from \( V_2 \). For each edge \((u, v)\) in \( E_2 \), introduce a perfect matching between \( G_u \) and \( G_v \) which joins each vertex in \( V(G_u) \) with its isomorphic image in \( V(G_v) \). Equivalently, one can also think of this as being obtained by taking \(|V_1|\) isomorphic copies of \( G_2 \) and introducing a perfect matching between corresponding copies of \( G_2 \) for each edge in \( E_1 \). The following facts are easy to verify.

**Fact 2.1** The cartesian product \( G_1 \Box G_2 \) is commutative in the sense that \( G_1 \Box G_2 \) is isomorphic to \( G_2 \Box G_1 \). Similarly, this operation is also associative. Hence the product \( G_1 \Box G_2 \Box \ldots \Box G_k \) is well-defined for each \( k \). For each \( G \) and \( k \geq 1 \), we define \( G^k \) as follows: \( G^1 = G \) and \( G^k = G^{k-1} \Box G \) for \( k > 1 \).
Fact 2.2 If $G = G_1 \square G_2 \square \ldots \square G_k$, then $G = (V, E)$ where $V$ is the set of all $k$-tuples of the form $(u_1, \ldots, u_k)$ with each $u_i \in V(G_i)$ and the edge joining $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ is in $E$ if and only if for some $i$, $1 \leq i \leq k$, (i) $u_j = v_j$ for all $j \neq i$ and (ii) the edge $\{u_i, v_i\}$ is in $E(G_i)$.

Fact 2.3 $G_1 \square G_2$ is connected if and only if both $G_1$ and $G_2$ are connected.

Remark 2.1 It is known (see [10] for further details and references) that any connected graph $G$ can be expressed as a product $G = G_1 \square \cdots \square G_k$ of prime factors $G_i$. Here, a graph is said to be prime with respect to the $\square$ operation if it is non-trivial and if it is not isomorphic to the product of two non-trivial graphs. A non-trivial graph is one having at least two vertices. Also, this factorisation is unique except for a re-ordering of the factors and is known as the Unique Prime Factorisation (UPF) of the graph. It is also known that the UPF of a graph can be computed in time polynomial in the size of $G$.

We now formally present our main result which relates acyclic chromatic index to the cartesian product of graphs. Without loss of generality, we can assume that the product graph is connected. In view of Fact 2.3, this implies that it suffices to consider only connected graphs as factors. Also, if $H$ is trivial (that is, $H$ is a graph on just one vertex), then $G \square H$ is isomorphic to $G$ for any $G$. Hence, we focus only on connected non-trivial graphs. We will often use the following easy-to-verify fact about acyclic edge colourings.

Fact 2.4 For any $\Delta > 1$, let $G$ be any $\Delta$-regular graph. Then, $\alpha'(G) \geq \Delta + 1$.

Theorem 2.1 Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two connected non-trivial graphs such that $\max\{\alpha'(G), \alpha'(H)\} > 1$. Then,

$$\alpha'(G \square H) \leq \alpha'(G) + \alpha'(H).$$

Note: If $G$ and $H$ are both connected and non-trivial with $\alpha'(G) = \alpha'(H) = 1$, then each of $G$ and $H$ is a $K_2$. In that case, $G \square H = C_4$ where $C_4$ is a cycle on 4 vertices. Only in this case, we have $\alpha'(G \square H) = 3$ whereas $\alpha'(G) + \alpha'(H) = 2$.

Proof: Let $\alpha'(G) = \eta$ and $\alpha'(H) = \beta$. Since $\square$ is commutative, without loss of generality, assume that $\eta \geq \beta$. Let $\Delta$ denote the maximum degree of $H$. Set $d$ to be $\Delta + 1$ if $H$ is either a complete graph on $\Delta + 1$ vertices or an odd cycle (in which case $\Delta = 2$). Otherwise, set $d$ to be $\Delta$. In any case, $H$ can be properly vertex coloured using colours from the set $[d] = \{0, \ldots, d - 1\}$.

We know that $\beta = \alpha'(H) \geq \Delta$ always. If $H = K_{\Delta+1}$, then (since $H$ is $\Delta$-regular) $\alpha'(H) \geq \Delta + 1$ (except when $H = K_2$). In both cases, $\eta \geq \beta \geq d$. If $H = K_2$, then $d = \Delta + 1 = 2$ and $\eta \geq 2$ by assumption. In any case, we have $\eta \geq d$. Let $X_G : E_G \to [\eta] =$
$\{0, \ldots, \eta - 1\}$ and $X_H : E_H \rightarrow [\beta'] = \{0', \ldots, (\beta - 1)\}'$ be two acyclic edge colourings of $G$ and $H$ respectively using disjoint sets of colours.

Each edge in $G \square H$ is either (i) an edge joining $(u_1, v)$ and $(u_2, v)$ for some $e = \{u_1, u_2\} \in E_G$ and $v \in V_H$ or (ii) an edge joining $(u, v_1)$ and $(u, v_2)$ for some $f = \{v_1, v_2\} \in E_H$ and $u \in V_G$. We denote the former edges by $e_v$ (where $e \in E_G, v \in V_H$) and the latter edges by $f_u$ (where $f \in E_H, u \in V_G$). Note that each edge of $G \square H$ lies either in some isomorphic copy $H_u$ of $H$ or in some isomorphic copy $G_v$ of $G$.

For each $i \in \{0, \ldots, d - 1\}$, let $\sigma_i : [\eta] \rightarrow [\eta]$ be a bijection defined by

$$\sigma_i(j) = (j + i) \mod \eta, \quad \forall j \in [\eta].$$

Since $\eta \geq d$, we notice that the bijections $\sigma_i(i \in [d])$ are mutually non-fixing, that is, for all $0 \leq i, k \leq d - 1$, $i \neq k$ and for each $j \in [\eta]$, $\sigma_i(j) \neq \sigma_k(j)$.

Let $Y_H : V_H \rightarrow \{0, \ldots, d - 1\}$ be a proper vertex colouring of $V_H$. We define a colouring of the edges of $G \square H$ based on the colourings $X_G, X_H$ and $Y_H$ as follows.

For each edge in $E$ of the form $f_u$, where $f \in E_H$ and $u \in V_G$, we colour $f_u$ using the colour $X_H(f)$. Now consider any arbitrary edge of the form $e_v$, where $e \in E_G$ and $v \in V_H$. Let $i = Y_H(v)$ be the colour used by $Y_H$ on $v$. Colour $e_v$ using the colour $\sigma_i(X_G(e))$.

In other words, edges $f_u$ in each isomorphic copy $H_u$ is coloured the same way as $f$ in $H$ is coloured by $X_H$. But edges $e_v$ in each isomorphic copy $G_v$ is coloured essentially (ignoring the labels of colours) the same way as $G$ is coloured but the colour labels are rotated by mutually non-fixing permutations. The permutation that is used for a $G_v$ is decided by the vertex colour assigned to $v$ by $Y_H$. As a result, for each edge $f = \{v_1, v_2\} \in E_H$ and for each edge $e = \{u_1, u_2\} \in E_G$, $e_v$ and $e_{v_1}$ get different colours but always from $[\eta]$.

Let $X : E(G \square H) \rightarrow \{0, \ldots, \eta - 1\} \cup \{0', \ldots, (\beta - 1)\}'$ be the colouring defined just above. We will show that $X$ is proper and acyclic.

Claim 2.1 $X$ is proper.

Proof: Consider any vertex $(u, v)$. The set of edges in $G \square H$ which are incident on $(u, v)$ can be partitioned into two subsets $A_u = \{f_u : v \in f \in E_H\}$ and $A_v = \{e_v : u \in e \in E_G\}$. Since edges in these two sets are coloured using colours from disjoint sets, namely from $[\eta]$ and $[\beta']$, there is no conflict between these two sets. Now, let us focus on edges in $A_u$. Since $f_u$’s are coloured in the same way as $f$’s are coloured in $H$, there is no conflict among edges in $A_u$. Similarly, the edges $e_v$’s in $A_v$ are coloured essentially in the same way (except for a rotation of the colour labels) as $e$’s are coloured in $G$ and hence coloured with distinct colours, there is no conflict among members of $A_v$ also. Hence $X$ is proper.

It is only left to prove acyclicity of $X$. We prove this by contradiction. Suppose there is a bichromatic (with respect to $X$) cycle $C$ in $G$. First, we note that
Claim 2.2 C cannot lie entirely within any isomorphic copy $G_v$ or $H_u$ of G or H respectively.

Proof: Note that X restricted to $H_u$ (or $G_v$) is basically either $X_H$ (or $X_G$ except for renaming of the colours). Hence if C lies within such an isomorphic copy, it implies that either $X_H$ or $X_G$ has a bichromatic cycle, which is a contradiction. ■

By the above claim, it follows that C should visit vertices in at least two different copies $G_v$ and $G_{v'}$. But different copies are only joined by edges of type $f_u$ for some $f \in E_H$ and $u \in V_G$. Thus, it follows that C has at least one edge each of the two types $e_v$ ($e \in E_G$, $v \in V_H$) and $f_u$ ($f \in E_H$, $u \in V_G$) which are coloured with respectively, say, $a \in [n]$ and $b \in [\beta']$.

Claim 2.3 Let $(u_1, v_1)$ be some arbitrary vertex in C. Let $(u_1, v_2)$ for some $v_2 \in V_H$ be the other end point of the unique b-coloured edge in C incident at $(u_1, v_1)$. C lies entirely within $G_{v_1}$ and $G_{v_2}$.

Proof: The proof is by induction on the distance $l$ in C from $(u_1, v_1)$ along the direction specified by the edge $\{v_1, v_2\}_{u_1}$. For $l = 0$, it is clearly true. Suppose it is true for vertices whose above-defined distance is at most $l'$. Let $(u_{l'}, v_{l'})$ be the vertex at distance $l'$. By inductive hypothesis, $v_{l'}$ is either $v_1$ or $v_2$. Let $c \in \{a, b\}$ be the colour of the edge joining $(u_{l'}, v_{l'})$ and $(u_{l'+1}, v_{l'+1})$. If $c = a$, then $v_{l'+1} = v_{l'}$ and hence the hypothesis is clearly true for $l = l' + 1$. If $c = b$ (hence $u_{l'+1} = u_{l'}$) and if $v_{l'} = v_1$, then $v_{l'+1} = v_2$. This follows from (i) the b-coloured edge incident at the copy of $u_1$ in $G_{v_1}$ joins it to the copy of $u_1$ in $G_{v_2}$ and hence (ii) all edges of the perfect matching joining isomorphic copies of vertices in $G_{v_1}$ and $G_{v_2}$ are coloured with $b$. In particular, the b-coloured edge incident at $(u_{l'}, v_1)$ joins it to $(u_{l'}, v_2)$. Similarly, one can argue that if $c = b$ and if $v_{l'} = v_2$, then $v_{l'+1} = v_1$. In any case, $v_{l'+1} \in \{v_1, v_2\}$, thereby proving that C lies entirely within $G_{v_1}$ and $G_{v_2}$. ■

Since the edges in $G_{v_1}$ and $G_{v_2}$ are coloured without using colour $b$ and since every alternate edge of C is coloured with $b$, we see that $b$ is used an even number of times in C. This implies $|C| = 0 \pmod{4}$. Thus, C looks like

$$C = \langle (u_1, v_1), (u_1, v_2), (u_2, v_2), (u_2, v_1), \ldots, (u_{2k-1}, v_2), (u_{2k}, v_2), (u_{2k-1}, v_1), (u_{2k-1}, v_1) \rangle.$$

For each of the a-coloured edges in $G_{v_2}$ joining $(u_{2l-1}, v_2)$ and $(u_{2l}, v_2)$, its isomorphic copy in $G_{v_1}$ joins $(u_{2l-1}, v_1)$ and $(u_{2l}, v_1)$ and is coloured with the colour $c = \sigma_i(\sigma_j^{-1}(a)) \neq a$ where $i = Y_H(v_1)$ and $j = Y_H(v_2)$. These isomorphic copies in $G_{v_1}$ of a-coloured edges of C in $G_{v_2}$ together with the a-coloured edges of C in $G_{v_1}$ constitute the following $\{a, c\}$-coloured bichromatic cycle in $G_{v_1}$:

$$D = \langle (u_1, v_1), (u_2, v_1), (u_3, v_1), \ldots, (u_{2k}, v_1), (u_{2k}, v_1) \rangle.$$

This is a contradiction to the fact that X restricted to $G_{v_1}$ is acyclic. This shows that X admits no bichromatic cycle and hence X is proper and acyclic. Since X uses only colours
from $[\eta] \cup [\beta']$, we get $a'(G \Box H) \leq a'(G) + a'(H)$.

Remark 2.2 Note that the above proof is constructive in the following sense: given two acyclic edge colourings $X_G$ and $X_H$ of $G$ and $H$ respectively, one can construct an acyclic edge colouring of $G \Box H$ in time polynomial in the size of $E(G \Box H)$.

Corollary 2.1 Let $G_1, \ldots, G_k$ be $k$ connected non-trivial graphs such that for each $i$, $1 \leq i \leq k$, $a'(G_i) = \Delta(G_i)$ and $\max\{a'(G_1), \ldots, a'(G_k)\} > 1$. Then,

$$a'(G_1 \square \ldots \square G_k) = \Delta(G_1 \square \ldots \square G_k).$$

Proof: Follows from (i) $a'(G) \geq \Delta(G)$ for any $G$, (ii) $\Delta(G_1 \square \ldots \square G_k) = \Delta(G_1) + \ldots + \Delta(G_k)$, (iii) Theorem 2.1.

Corollary 2.2 Let $G$ be a connected non-trivial graph such that $a'(G) = \Delta(G) > 1$. Then, for each $d \geq 1$,

$$a'(G^d) = d\Delta(G).$$

The following result first obtained in [15] now follows as a corollary of Theorem 2.1.

Corollary 2.3 ([15]) Let $G = K_2^d = K_2 \square \ldots \square K_2$ be the $d$-dimensional hypercube for some $d \geq 1$. Then,

$$a'(K_2) = 1 \quad \text{and} \quad a'(K_2^d) = d + 1 \quad \text{for} \quad d > 1.$$

Proof: Suppose $d > 1$. Since $G = K_2^d$ is $d$-regular, by Fact 2.4, we need at least $d+1$ colours in any acyclic edge colouring of $K_2^d$ and hence $a'(G) \geq d + 1$. Also, $a'(K_2^2) = a'(C_4) = 3$. Starting with $G = K_2^2$ and applying Theorem 2.1 repeatedly by setting $H = K_2$ each time, we get $a'(K_2^d) \leq a'(K_2^2) + (d-2) \leq d + 1$. Combining both the lower and upper bounds, we get the result.

3 Conclusions

It can be easily observed that $a'(G) \geq \Delta + 1$, for all regular graphs with $\Delta \geq 2$. It is conjectured in [1] that $a'(G) \leq \Delta + 2$ for all graphs. If we take the cartesian product of $t$ graphs each of whose $a'(G)$ value is known and each of which is regular with $\Delta \geq 2$, then the bound we would get (which is not better than $\Delta + t$) by applying the above result is very weak, assuming the conjecture is true. It would be interesting to make a statement like $a'(G \square H) \leq a'(G) + \Delta(H)$ for a wider class of graphs $H$, like the results obtained in [15] for grid-like graphs. At present, investigations are being carried out in this direction.
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