REPETITIVE HIGHER CLUSTER CATEGORIES OF TYPE $A_n$

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Abstract. We show that the repetitive higher cluster category of type $A_n$, defined as the orbit category $\mathcal{D}^b(\text{mod}kA_n)/((\tau^{-1}[m])^p$, is equivalent to a category defined on a subset of diagonals in a regular polygon. This generalizes the construction of Caldero-Chapoton-Schiffler, [CCS06], which we recover when $p = m = 1$, and the work of Baur-Marsh, [BM08], treating the case $p = 1, m > 1$. Our approach also leads to a geometric model of the bounded derived category in type $A$.

1. Introduction

In this paper we give a geometrical-combinatorial model, in the spirit of Caldero-Chapoton-Schiffler, [CCS06], of repetitive higher cluster categories of type $A_n$. These are orbit categories of the bounded derived category of $\text{mod}kA_n$ under the action of the cyclic group generated by the auto-equivalence $(\tau^{-1}[m])^p$, where $kA_n$ is the path algebra associated to a Dynkin quiver of type $A_n$, $\tau$ is the AR-translation and $[m]$ the composition of the shift functor [1] on $\mathcal{D}^b(\text{mod}kA_n)$ with itself $m$-times and $1 \leq p \in \mathbb{N}$. We write $C_{n,p}^m := \mathcal{D}^b(\text{mod}kA_n)/((\tau^{-1}[m])^p)$. The class of objects of $C_{n,p}^m$ is the same as for $\mathcal{D}^b(\text{mod}kA_n)$ and the space of morphisms is as follows

$$\text{Hom}_{C_{n,p}^m}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\text{mod}kA_n)}(X, (\tau^{-p}[pm])^i Y).$$

When the index $p$ or $m$ is equal to one we omit it in the writing of $C_{n,p}^m$.

When $p = m = 1$ one recovers the usual cluster categories, which we denote simply by $C_n$, defined independently in [CCS06] for the case $A_n$ and in the general case in [BMR+06]. If $p = 1$ and $m > 1$ one regains the higher cluster categories $C_m^m$ defined in [Kel08], also called $m$-cluster categories. For $C_{n,p}^m$ Baur-Marsh gave a geometric model in [BM08], and in [BM07] for type $D_n$. In the case $p > 1$ and $m = 1$, the category is simply called repetitive cluster category $C_{n,p}$, studied also by Zhu in [Zhu11] from a purely algebraic point of view.

The main results of the paper are the following. On one side we are able to give an equivalence of categories between $C_{n,p}^m$ and a category defined on a subset of all the diagonals in a regular $p((n+1)m+1)$-gon $\Pi^p$. Then, the model we propose here also leads to a geometric interpretation of cluster tilting objects in $C_{n,p}^m$. Furthermore, for $m = 1$ we are able to prove an equivalence of triangulated categories between $C_{n,p}$ and a quotient of a cluster category of a certain rank. On the other hand as an application of the results obtained for $C_{n,p}^m$, a geometric model for $\mathcal{D}^b(\text{mod}kA_{n-1})$ will be given.

The association of geometric models to algebraic categories has been studied and developed by many authors, among others we mention: [BM08], [BM07], [BZ10], [BT09], [CCS06], [Tor11], [Sch08], . . . . This approach is not only beautiful but also fruitful as it gives new ways to understand the intrinsic combinatorics of the category.
The particularity of the repetitive higher cluster categories is that they are fractionally Calabi-Yau of dimension \( \frac{p(m+1)}{p} \), this means that \( \tau[1] \cong [p(m+1)] \) as triangle functors, and the fraction cannot be simplified. And it is precisely in this point that the category \( C_{n,p} \) differs from \( C_m \) and \( C_{m,n} \).

This generalization of the notion of a Calabi-Yau category is interesting because many categories happen to be of this type. Consider for example the bounded derived category of \( \text{mod} kQ \), for a quiver \( Q \) with underlying Dynkin diagram, or of the category of coherent sheaves on an elliptic curve or a weighted projective line of tubular type. A.-C. van Roosmalen was recently able to give a classification up to derived equivalence of abelian hereditary categories whose bounded derived category are fractionally Calabi-Yau, see [van10].

The structure of the paper is as follows. The next section is dedicated to a review of useful definitions on fractionally Calabi-Yau categories and repetitive cluster categories will be carefully defined. In Section 3 we define the repetitive polygon \( \Pi^p \) and we choose a subset of diagonals in \( \Pi^p \) together with a rotation rule between them. This will lead to the modelling of \( C_{n,p} \). In Section 4 we study the relation between repetitive cluster categories and cluster categories. Section 5 is dedicated to the study of cluster tilting objects in \( C_{n,p} \). The content of Section 6 is the geometric modelling of \( C_{n,p} \), the construction we give here generalizes the one of Section 3. Finally, in Section 7 we extend the construction of \( \Pi^p \) to a geometric figure with an infinite number of sides, \( \Pi^{\pm\infty} \). Applying the results of Section 7 to the category generated by the set of diagonals complementary to 2-diagonals in \( \Pi^{\pm\infty} \), we obtain a model for \( D^b(\text{mod} kA_{n+1}) \).

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2. Repetitive cluster categories of type \( A_n \)

2.1. Serre duality and Calabi-Yau categories. Let \( k \) be a field (assume it to be algebraically closed, even though in this section it is not needed) and let \( \mathcal{K} \) be a \( k \)-linear triangulated category which is Hom-finite, i.e. for any two objects in \( \mathcal{T} \) the space of morphisms is a finite dimensional vector space.

**Notation 2.1.** Throughout the paper we denote by \( D \) the category \( D^b(\text{mod} kA_n) \) or \( D^b(\text{mod} kA_{n-1}) \), and we believe that it will be clear from the context if it is one or the other.

**Definition 2.2.** A \( k \)-triangulated category \( \mathcal{K} \) has a Serre functor if it is equipped with an auto-equivalence \( \nu : \mathcal{K} \to \mathcal{K} \) together with bifunctorial isomorphisms

\[
D\text{Hom}_\mathcal{K}(X,Y) \cong \text{Hom}_\mathcal{K}(Y,\nu X),
\]

for each \( X, Y \in \mathcal{K} \). \( D \) indicates the vector space duality \( \text{Hom}_k(?,k) \).

We will say that \( \mathcal{K} \) has Serre duality if \( \mathcal{K} \) admits a Serre functor. In the case \( \mathcal{K} = D \) a Serre functor exists ([Kel10] p. 24), it is unique up to isomorphism and \( \nu \cong \tau[1] \), where \( \tau \) is the Auslander-Reiten translate and \([1]\) the shift functor of \( D \).

**Definition 2.3.** A triangle functor between two triangulated categories \( \mathcal{J} \) and \( \mathcal{K} \) is a pair \((F,\sigma)\) where \( F : \mathcal{J} \to \mathcal{K} \) is a \( k \)-linear functor and \( \sigma : F[1] \to [1]F \) an isomorphism of functors such that the image of a triangle in \( \mathcal{J} \) under \( F \) is a triangle in \( \mathcal{K} \).
Suppose \((F, \sigma)\) and \((G, \gamma)\) are triangle functors, then a morphism of triangle functors is a morphism of functors \(\alpha : F \to G\) such that the square

\[
\begin{array}{ccc}
F[1] & \xrightarrow{\sigma} & [1]F \\
\downarrow{\alpha[1]} & & \downarrow{[1]\alpha} \\
G[1] & \xrightarrow{\gamma} & [1]G
\end{array}
\]

commutes.

**Definition 2.4.** One says that a category \(K\) with Serre functor \(\nu\) is a fractionally Calabi-Yau category of dimension \(\frac{m}{n}\) or \(\frac{m}{n}\)-Calabi-Yau if there is an isomorphism of triangle functors:

\[
\nu^m \cong [m],
\]

for \(n, m > 0\), and where \([m]\) indicates the composition of the shift functor with itself \(m\) times.

**Remark 2.5.** Notice that a category of fractional CY dimension \(\frac{m}{n}\) is also of fractional CY dimension \(\frac{mt}{nt}\), \(t \in \mathbb{Z}\). However the converse is not always true.

### 2.2. Repetitive cluster categories of type \(A_n\)

In the following we give the algebraic description of the repetitive cluster category of type \(A_n\). This is the orbit category of the bounded derived category of \(\text{mod}(kA_n)\) under the action of the cyclic group generated by the auto-equivalence \((\tau^{-1})^p = \tau^{-p}[p]\) for \(p > 0\), where \(\tau\) is the AR-translation in \(D\) and \([1]\) the shift functor. Repetitive cluster categories were defined in \([Zhu 11]\) as orbit categories of \(D^b(H)\), for \(H\) a hereditary abelian category with tilting objects.

**Definition 2.6.** The repetitive cluster category \(C_{n,p} := D/\langle \tau^{-p}[p] \rangle\) of type \(A_n\), has as class of objects the same as in \(D\). The class of morphism is given by:

\[
\text{Hom}_{C_{n,p}}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_D(X, (\tau^{-p}[p])^i Y)
\]

Observe that when \(p = 1\), one gets back the usual cluster category which we simply denote by \(C_n\). Furthermore, one can define the projection functor \(\eta_p : C_{n,p} \to C_n\) which sends an object \(X\) in \(C_{n,p}\) to an object \(X\) in \(C_n\), and \(\phi : X \to Y\) in \(C_{n,p}\) to the morphism \(\phi : X \to Y\) in \(C_n\), \([Zhu 11]\). Then one has that \(\pi_1 = \pi_p \circ \eta_p\), where \(\pi_p : D \to C_{n,p}\).

In the following let \(F = F_1\) be the fundamental domain for the functor \(F := \tau^{-1}[1]\) in \(D\) given by the isoclasses of indecomposable objects in \(\text{mod}(kA_n)\) together with the \([1]\)-shift of the projective indecomposable modules. After \([BMR + 06, Proposition: 1.6]\) one can identify the subcategory of isomorphism classes of indecomposable objects of \(C_n\), denoted by \(\text{ind}(C_n)\), with the objects in \(F\). Let \(F^k := F \circ \cdots \circ F\), \(k\)-times, then denote by \(F_k\) the \(F^k\)-shift of \(F\) and we can draw the fundamental domain for the functor \(\tau^{-p}[p]\) as in Figure 1.

![Figure 1. Partition of the fundamental domain of \(\tau^{-p}[p]\) for an odd value of \(p\)](image)
As next we summarize some basic properties of $C_{n,p}$ proven in [Zhu11, Proposition: 3.3].

**Lemma 2.7.**
- $C_{n,p}$ is a triangulated category with AR-triangles and Serre functor $\nu := \tau[1]$.
- The projections $\pi_p : D \to C_{n,p}$ and $\eta_p : C_{n,p} \to C_n$ are triangle functors.
- $C_{n,p}$ is fractionally CY of dimension $\frac{2p}{p}$.
- $C_{n,p}$ is a Krull-Schmidt category.
- $\text{ind}(C_{n,p}) = \bigcup_{i=1}^{p} \text{ind}(F_i)$.

3. Geometric model of $C_{n,p}$

The geometric model for $C_{n,p}$ we present in this paper can be viewed as a $p$-covering of the polygon (which we view as a disc) modelling $C_n$, [CCS06]. However, the cover we propose here differs from the usual understanding of a covering space. In fact, the nature of $C_{n,p}$ suggests a connection between the first and last layer of the covering. This is the reason why the different “layers” in our model arise inside a disc as shown in 2.

3.1. The repetitive polygon $\Pi^p$.

Let $p > 1$. For the purpose of the geometric characterization of $C_{n,p}$, let $\Pi$ be a regular $N := n + 3$-gon and let $\Pi^p$ be a regular $p(n + 2)$-gon. Number the vertices of $\Pi^p$ clockwise repeating $p$-times the $N-$tuple $1, 2, ..., N - 1, N$ and letting correspond $N \equiv 1$. Then we denote by $\Pi_1$ a region homotopic to $\Pi$ inside $\Pi^p$ delimited by the segments $(1, 2), (2, 3), ..., (N - 1, N)$ and the inner arc $(1, N)$.

Denote by $\rho : \Pi^p \to \Pi^p$ the clockwise rotation through $\frac{2\pi}{p}$ around the center of $\Pi^p$, and set $\Pi_k := \rho^{k-1}(\Pi_1)$ for $1 \leq k \leq p$. In this way we divide $\Pi^p$ into $p$ regions. See Figure 2 where we illustrate this construction.

**Definition 3.1.** We call diagonals of $\Pi^p$ the union of all diagonals of $\Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_p$.

Denote the diagonals of $\Pi^p$ by the triple $(i, j, k)$, where $1 \leq k \leq p$ specifies a region $\Pi_k$ inside $\Pi^p$, and the tuple $(i, j)$ defines the diagonal in $\Pi_k$.

Notice that for us the set of diagonals of $\Pi^p$ consists of a subset of all the straight inner lines (drawn as arcs) joining the vertices of $\Pi^p$. Furthermore, the arcs $(1, N, k)$ for $1 \leq k \leq p$ are not diagonals of $\Pi^p$ as they correspond to boundary segments of $\Pi_k$.

**Notation 3.2.** Observe once and for all that in the writing $(i, j, k)$ we understand that the index $k$ has to be taken modulo $p$, and the indices $i, j$ modulo $N$. Furthermore, we always assume that $i < j$.

![Figure 2. The polygon $\Pi^3$ for $n = 3$](image-url)
3.2. Quiver of diagonals of $\Pi^p$. As next we associate a translation quiver to the diagonals of $\Pi^p$ with the intention of modelling the AR-quiver of the category $C_{n,p}$. The first part of the next definition goes back to [CCS06], the second part is new and essential for the modelling of the repetitive cluster category.

In the following we call a rotation of a diagonal around a fixed vertex irreducible whenever the other vertex of the diagonal moves to the preceding or successive vertex.

**Definition 3.3.** Let $\Gamma_{n,p}$ be the quiver whose vertices are the diagonals $(i,j,k)$ of $\Pi^p$, and whose arrow are defined as follows.

1.: For $1 \leq i < j < N$:

$$(i,j,k) \rightarrow (i,j+1,k)$$

That is, we draw an arrow if there is an irreducible clockwise rotation around the vertex $j$ or $i$ inside $\Pi_k$.

2.: For $1 \leq i < j = N$ we link $(i,N,k) \rightarrow (i+1,N,k)$ whenever there is an irreducible clockwise rotation around the vertex $N$ inside $\Pi_k$.

3.: For $1 \leq i < j = N$ we link $(i,N,k) \rightarrow (1,i,k+1)$. That is, we draw an arrow to the diagonal we can reach with a clockwise rotation around vertex $i$ inside $\Pi_k$ composed with $\rho$. I.e. $(i,N,k)$ is linked to $\rho(1,i,k) = (1,i,k+1)$.

The set of the operations 1. and 2. are denoted by $\text{IrrRot}$ and the operations 3. by $\text{Irr}\rho\text{Rot}$.

Notice that $\Gamma_{n,p}$ lies on a Möbius strip when $p$ is odd, and on a cylinder when the value of $p$ is even.

As next we equip $\Gamma_{n,p}$ with a translation $\tau : \Gamma_{n,p} \rightarrow \Gamma_{n,p}$ such that $(\Gamma_{n,p}, \tau)$ becomes a stable translation quiver in the sense of Riedtmann, [Rie80].

For the reader familiar with the article of [CCS06] we point out that the first part of the following definition agrees with the one given there, however the second is new and can be thought of as a connecting map between the different regions $\Pi_k$ in $\Pi^p$.

**Definition 3.4.** The translation $\tau$ on $\Gamma_{n,p}$ maps

- $(i,j,k)$ to $(i-1,j-1,k)$ for $1 < i,j \leq N$ and $1 \leq k \leq p$, i.e. $\tau$ is an anticlockwise rotation around the center of $\Pi^p$ by $\frac{2\pi}{\left( \frac{2p+1}{2} \right)}$.
- $(1,j,k)$ to $(j-1,N,k-1)$, i.e. the effect of $\tau$ on $(1,j,k)$ is induced by $\frac{1}{N}$th anticlockwise rotation in the regular $N$-gon $\Pi$ homotopic to $\Pi_k$, composed with $\rho^{-1}$. That is
  $$\tau(1,j,k) = (j-1,N,k-1) = \rho^{-1}(1,j,N,k).$$

**Lemma 3.5.** The pair $(\Gamma_{n,p}, \tau)$ is a stable translation quiver.

**Proof.** It is clear that the map $\tau$ is bijective. As $\Gamma_{n,p}$ is finite, we only need to persuade us that the number of arrows from a diagonal $D$ to $D'$ is equal to the number of arrows from $\tau D'$ to $D$. As there is at most one arrow between any two diagonals, we only have to check that there is an arrow from $D$ to $D'$ if and only if there is an arrow from $\tau D'$ to $D$.

For a given diagonal $D = (i,j,k)$ we distinguish two possible cases. Either there is an arrow to a diagonal $D'$ of $\Pi_k$ to a diagonal of the neighbouring region $\Pi_{k+1}$. 


• In the first case we distinguish depending on whether $i = 1$, or $i \neq 1$. If $i \neq 1$ the result follows from Proposition 2.2 in [BM08]. If $i = 1$, then

$$(1, j + 1, k) \xrightarrow{\imath} (1, j, k) \xrightarrow{\tau} (2, j, k) \xrightarrow{\imath} (1, j + 1, k)$$

Then $\tau(1, j + 1, k) = (j, N, k - 1)$, and $(j, N, k - 1) \rightarrow (1, j, k)$. Similarly $\tau(2, j, k) = (1, j - 1, k)$ and $(1, j - 1, k) \rightarrow (1, j, k)$.

• If there is an arrow $(i, j, k) \rightarrow (i', j', k + 1)$, we deduce that $j = N, i' = 1, j' = i \neq 1$.

Then $\tau(1, i, k + 1) = (i - 1, N, k)$, and we deduce that $(i - 1, N, k) \rightarrow (i, N, k)$.

3.3. Category of diagonals in $\Pi^p$. We now associate an additive category $\mathcal{C}(\Pi^p)$ to the diagonals of $\Pi^p$. The new feature arising here is that we only allow a subset of all possible diagonals of $\Pi^p$, and that we deal with an additional type of rotation.

As next we remind the reader the definition of mesh category. This will be used repeatedly in the next sections.

Let $\alpha$ denote the arrow $x \rightarrow y$, then $\sigma(\alpha)$ denotes the arrow $\tau(y) \rightarrow x$.

**Definition 3.6.** The mesh category of a translation quiver $(Q, \tau)$ is the factor category of the path category of $Q$ modulo the ideal generated by the mesh relations

$$r_v := \sum_{\alpha: u \rightarrow v} \alpha \cdot \sigma(\alpha),$$

where the sum is over all arrows ending in $v$, and $v$ runs through the vertices of $Q$.

Then the category of diagonals $\mathcal{C}(\Pi^p)$ in $\Pi^p$ arises as the mesh category of $(\Gamma^p, \tau)$. More specifically, the class of objects of $\mathcal{C}(\Pi^p)$ is given by formal direct sums of the diagonals in $\Pi^p$, i.e. diagonals in the regions $\Pi_k$ for $1 \leq k \leq p$. The class of morphisms is generated by the two rotations $\text{IrrRot}_{\Pi^p}$ and $\text{IrrRot}_{\Pi^p}$, carefully defined in Definition 3.3, modulo the mesh relations which can be read off from $(\Gamma^p, \tau)$.

Note that with our approach a new geometric type of mesh relation appears. It arises between diagonals of two consecutive regions: $\Pi_k$ and $\Pi_{k+1}$.

3.4. Equivalence of categories. The next results will show that the category $\mathcal{C}(\Pi^p)$ of diagonals in $\Pi^p$ is equivalent to the repetitive cluster category of type $A_n$. The notation is as specified after Lemma 2.7.

**Proposition 3.7.** There is a bijection

$$\varphi: X \mapsto D_X,$$

from $\text{ind}(\mathcal{C}_{n,p})$ to the diagonals of $\Pi^p$ such that:

• $\text{Irr}_{\mathcal{C}_{n,p}}(X, Y) \leftrightarrow \text{IrrRot}_{\Pi^p}(D_X, D_Y)$ if $X, Y$ are in $\mathcal{F}_i$, for some $i$.

• $\text{Irr}_{\mathcal{C}_{n,p}}(X, Y) \leftrightarrow \text{IrrRot}_{\Pi^p}(D_X, D_Y)$ if $X \in \mathcal{F}_i$ and $Y \in \mathcal{F}_j$, $i \neq j$.

**Proof.** We saw in Lemma 2.7 that

$$\text{ind}(\mathcal{C}_{n,p}) = \bigcup_{i=1}^{p} \text{ind}(\mathcal{F}_i).$$
Then one easily sees that \( \varphi \) is a bijection on the level of objects. In fact, we can apply \cite[Corollary 4.7]{CCS06} to every subcategory \( \text{ind}(F_i) \) of \( \mathcal{C}_{n,p} \) and to the corresponding region \( \Pi_i \) in \( \Pi^p \), \( 1 \leq i \leq p \).

Then for each \( 1 \leq i \leq p \), the irreducible morphisms between objects \( X, Y \in \mathcal{F}_i \) where \( X \not\cong (\tau^{-1}[1])^{i-1}P[1] \) for \( P \) some projective, or where \( X \cong (\tau^{-1}[1])^{i-1}P[1] \) and \( Y \cong (\tau^{-1}[1])^{i-1}Q[1] \), for projectives \( P \) and \( Q \), agree by \cite[Theorem 5.1]{CCS06} with the rotations \( \text{IrrRot}_{\Pi_i}(D_X, D_Y) \) described in part 1. and 2. of Definition \ref{def:irrrot}.

Therefore, we only need to study the correspondence between \( \text{IrrC}_{n,p}(X, Y) \) and the rotations in \( \Pi^p \) described in part 3. of Definition \ref{def:irrrot} for \( X \cong ((\tau^{-1}[1])^{i-1})\tilde{P}[1] \) and \( Y \not\cong (\tau^{-1}[1])^{i-1}\tilde{Q}[1], 1 \leq i \leq p \). Thus, assume \( \text{IrrC}_{n,p}(X, Y) \neq 0 \), in this case it follows from the shape of the AR-quiver of \( \mathcal{C}_{n,p} \) that \( Y \cong (\tau^{-1}[1])^iR \) for some projective \( R \). Under the bijection \( \varphi \) applied to the objects \( X \) and \( Y \) one then has:

\[
X \mapsto (i, N, k), \quad Y \mapsto (1, i, k + 1).
\]

From Definition \ref{def:irrrot} it then follows that there is an arrow \((i, N, k) \rightarrow (1, i, k + 1)\) in \( \Gamma_{n,p} \) defining a corresponding operation in \( \text{IrrpRot}_{\Pi^p}(D_X, D_Y) \). Similarly one shows the other direction. This proves that the mapping \( \varphi \) is a bijection also on the level of morphisms.

It only remains to check that the mesh relations in the two categories agree. But by the above we have precisely showed that the AR-quiver of \( \mathcal{C}_{n,p} \) and \( \Gamma_{n,p} \) are isomorphic, hence the mesh relations in both categories agree.

It follows from the previous result that the projection functor \( \eta_p : \mathcal{C}_{n,p} \rightarrow \mathcal{C}_n \) corresponds to the projection \( \mu_p : \Pi^p \rightarrow \Pi \).

4. \( \mathcal{C}_{n,p} \) and the link to the cluster category

The desire of comparing the category \( \mathcal{C}_{n,p} \) with the cluster category of type \( A_t \), for a certain \( t \) arose while studying the geometrical model of \( \mathcal{C}_{n,p} \). We will see in this section that the connection is slightly different than expected, as \( t \) cannot simply be deduced from the size of \( \Pi^p \). This is because the class of morphisms in \( \mathcal{C}_{n,p} \) is too particular.

However, starting with a bigger polygon modelling the cluster category, the category \( \mathcal{C}_{n,p} \) appears whenever we assume that \( p \neq 2 \). In fact, otherwise the diagonals of the two models overlap hence the description degenerates.

In the following denote by \((\Gamma_{n,p}, \tau_{n,p})\) the AR-quiver of \( \mathcal{C}_{n,p} \) and by \((\Gamma_t, \tau_t)\) the AR-quiver of \( \mathcal{C}_t \).

**Lemma 4.1.** Let \( p > 2 \). Then \((\Gamma_{n,p}, \tau_{n,p})\) is a subquiver of \((\Gamma_t, \tau_t)\) for a suitable value of \( t \), namely

- \( t := (n + 3)(\frac{p}{2}) - 3 \), if \( p \) is even,
- \( t := (n + 3)p - 3 \), if \( p \) is odd.

**Proof.** To prove the claim we establish an isomorphism of stable translation quivers between \((\Gamma_{n,p}, \tau_{n,p})\) and a stable translation subquiver of \((\Gamma_t, \tau_t)\) for the different cases.

Let us first study the case where \( p \) is even. We observe that the union of the \( \tau_t \)-orbits of the top and bottom \( n \) rows of the quiver \( \Gamma_t \), illustrated in the gray strip in Figure \ref{fig:cluster} defines a subquiver \( \tilde{\Gamma}_t \) of \( \Gamma_t \). Since \( p > 2 \), the top and the bottom \( n \) rows do not overlap.

As \( \tilde{\Gamma}_t \) is a stable translation quiver, the same remains true for \( \tilde{\Gamma}_t \). And it turns out that when \( t \) is as given in the claim, \( \tilde{\Gamma}_t \) and \( \Gamma_{n,p} \) are isomorphic as stable translation quivers. First: by construction, the two quivers have the same number of rows (namely \( n \)). To see the isomorphism just on the level of quivers we compare the induced action of the auto equivalence \((\tau_{n,p}^{-1}[1])^p\) on \( \Gamma_{n,p} \) with the action of
(\tau^{-1}[1])|_{\Gamma_t}$ on $\tilde{\Gamma}_t$. It is easy to check that the actions coincide. Furthermore, because the meshes of the quivers $\tilde{\Gamma}_t$ and $\Gamma_{n,p}$ coincide we deduce that this gives an isomorphism of stable translation quivers.

When $p$ is odd, the embedding of $\Gamma_{n,p}$ into $\Gamma_t$ can be shown in a similar way. However, in this case we need to embed $\Gamma_{n,p}$ into the central band of $\Gamma_t$ to preserve the induced action of the autoequivalence, i.e. to preserve the identifications of vertices in $\Gamma_{n,p}$. More precisely, this time we have to consider the horizontal band $n$ vertices wide at the high of $\frac{(p-1)(n+3)}{2} + 1$ vertices, counting from the bottom of $\Gamma_t$, see Figure 3. Then we concludes as in the previous case. □

4.1. Triangulated equivalence for $C_{n,p}$. Since the inclusion of the AR-quiver of $C_{n,p}$ in the AR-quiver of the cluster category $C_t$ for $t$ as in Lemma 4.1 does not give rise to an inclusion at the level of full subcategories, $C_{n,p}$ is in particular not a triangulated subcategory of $C_t$. However, it is possible to prove that it is triangulated equivalent to a quotient category of $C_t$.

Quotient categories of the type we are going to consider here have been studied in [Jør10], et al. In particular, Jørgensen showed that higher cluster categories are triangulated equivalent to quotients of the cluster category. We will now recall the definition of quotient categories and key properties, which will be needed in the following.

Let $\mathcal{C}$ be an additive category and $\mathcal{X}$ a class of objects of $\mathcal{C}$. Then the quotient category $\mathcal{C}/\mathcal{X}$ has by definition the same objects as $\mathcal{C}$, but the morphism spaces are taken modulo all the morphisms factoring through an object of $\mathcal{X}$.

Observe that if $\mathcal{C}$ is a triangulated category, then $\mathcal{C}/\mathcal{X}$ needs not to be triangulated for all choices of $\mathcal{X}$. However, it was shown in [Jør10, Theorem 2.2] that $\mathcal{C}/\mathcal{X}$ is always pre-triangulated. This means that $\mathcal{C}/\mathcal{X}$ is equipped with a pair $(\sigma, \omega)$ of adjoint endofunctors satisfying a number of properties. However none of them is necessarily an auto-equivalence. Nevertheless, taking a particular choice of the class $\mathcal{X}$, Jørgensen proved in [Jør10, Theorem 3.3] that the endofunctors $\sigma$ and $\omega$, can be turned into auto-equivalences, so that $(\mathcal{C}/\mathcal{X}, \sigma)$ becomes a triangulated category.

We use these facts to link the triangulated structure of $C_{n,p}$ to a quotient category of the cluster category of type $A_t$.

**Figure 3.** Inclusions $\Gamma_{n,4} \subset \Gamma_t$ and $\Gamma_{n,3} \subset \Gamma_t$, and $a = \frac{(p-1)(n+3)}{2} + 1$.

**Proposition 4.2.** Assume $p \neq 2$ and define $t$ as in Lemma 4.1. Then the category $C_{n,p}$ is triangulated equivalent to a quotient of a cluster category of type $A_t$.

**Proof.** For $p$ even, denote by $\mathcal{X}$ the additive full subcategory generated by the indecomposable objects in the band (union of $\tau$-orbits) of $t-2n$ vertices in the middle of $\Gamma_t$, for $t = (n+3)\frac{p}{2} - 3$. For $p$ odd i.e. $t = (n+3)p - 3$ one takes as $\mathcal{X}$ the additive full subcategory generated by the indecomposable objects in the band of $\frac{p-1}{2}$ vertices at the top and bottom of $\Gamma_t$. Then, in both cases we have that $\tau^{2n} \mathcal{X} = \mathcal{X}$. So the AR-quiver of the quotient category $(\mathcal{C}_t)/\mathcal{X}$ is obtained by deleting the vertices corresponding to the objects of $\mathcal{X}$ and the arrows linked with them ([Jør10, Theorem 4.2]). Furthermore, $(\mathcal{C}_t)/\mathcal{X}$ is connected, and has finitely many
indecomposable objects up to isomorphism. Furthermore, again by \[\text{Jør}10\] Theorem 4.2 (\(C_t\)) is standard and of algebraic origin. Proceeding as in \[\text{Jør}10\] Theorem 5.2 we conclude that (\(C_t\)) is triangulated equivalent to a quotient of a cluster category of type \(A_n\).

It remains to see that \((C_t)\) and \(C_{n,p}\) are equivalent as triangulated categories. For this we observe that \(C_{n,p}\) is of algebraic origin by results of \[\text{Kel}10\] Section 9.3 and standard by \[\text{Ami}07\] Proposition 6.1.1. Furthermore, it is straightforward to see that the AR-quivers of \((C_t)\) and \(C_{n,p}\) are isomorphic as translation quivers. Thus we are in the conditions of Amiot’s Theorem \[\text{Jør}10\] Theorem 5.1 applied to \((C_t)\) and \(C_{n,p}\). Hence we deduce that these categories are equivalent as triangulated categories, and so the claim follows.

5. Cluster tilting theory for \(C_{n,p}\)

It is well known that cluster tilting objects in cluster categories are strongly linked to clusters in cluster algebras discovered by Fomin-Zelevinski in \[\text{FZ}02\]. This link has been established in \[\text{BMR}+06\]. Studying the endomorphism algebra of these objects provides a way to recover the exchange matrix, indispensable in the mutation process in a cluster algebra.

In this section we are interested in understanding the cluster tilting objects of \(C_{n,p}\), and compare them with configurations of diagonals in the polygon \(\Pi^p\). When \(p = 1\), it is known that cluster tilting objects of \(C_t\) correspond to a triangulation of a regular \(n + 3\)-gon, that is a maximal collection of pairwise non crossing diagonals.

Cluster tilting objects in \(C_{n,p}\) have also been studied from an algebraic point of view in \[\text{Zhu}11\]. In fact, Zhu proves that the endomorphism algebra of a cluster tilting object of \(C_{n,p}\) gives a cover of the endomorphism algebra of the same object, seen now as a cluster tilting object of the cluster category.

In the following definition let \(\text{add}(T)\) be the full subcategory consisting of direct summands of direct sums of finitely many copies of \(T\).

**Definition 5.1.** An object \(T \in C_{n,p}\) is called a cluster tilting object if for any object \(X \in C_{n,p}\) we have that \(\text{Ext}^1_{C_{n,p}}(T, X) = 0\) if and only if \(X \in \text{add}(T)\), and \(\text{Ext}^1_{C_{n,p}}(X, T) = 0\) if and only if \(X \in \text{add}(T)\).

A cluster tilting object is called basic if all its direct summands are pairwise non isomorphic. In this paper all the cluster tilting objects we consider will be basic, so we omit the term basic. The next result follows from \[\text{CCS}06\] Section 5.

**Lemma 5.2.** \(T\) is a cluster tilting objects in \(C_t\) if and only if \(X_T\) is a triangulation of a regular \(n + 3\)-gon. The cardinality of \(X_T\) is \(n\).

As our goal is to link cluster tilting objects of \(C_{n,p}\) to diagonals in \(\Pi^p\) we first need to express what it means geometrically that two objects in \(C_{n,p}\) have no extension. We will see that it is not enough to study only diagonals in a single region \(\Pi_k\) in \(\Pi^p\), but also diagonals in the regions \(\Pi_{k-1}\) respectively \(\Pi_{k+1}\), depending on which entry of the bifunctor \(\text{Ext}^1_{C_{n,p}}(\cdot, \cdot)\) is fixed.

We remind the reader that diagonals of \(\Pi^p\) consist only of a subset of all the lines joining vertices of \(\Pi^p\). In the next result we assume that \(i < j\) and \(i' < j'\) in the writing of the diagonals \(D_X := (i, j, l)\) and \(D_Y := (i', j', l')\) of \(\Pi^p\).

**Lemma 5.3.** Let \(D_X := (i, j, l)\) be fixed and \(D_Y := (i', j', l')\). Then

\[
\dim(\text{Ext}^1_{C_{n,p}}(X, Y)) = 1 \iff \begin{cases} 
  l = l' & \text{and} \ 1 \leq i' < i < j < j' \leq N, \\
  l = l' + 1, \text{ and} \ 1 \leq i < i' < j < j' \leq N 
\end{cases}
\]
Dually, we have
\[
\dim(\text{Ext}^1_{C_{n,p}}(Y, X)) = 1 \Leftrightarrow \begin{cases} 
 l' = l, \text{ and } 1 \leq i < j < j' \leq N, \\
 l' = l + 1, \text{ and } 1 \leq i' < i < j < j' \leq N.
\end{cases}
\]
Otherwise \( \dim(\text{Ext}^1_{C_{n,p}}(-, -)) = 0. \)

**Proof.** When \( p = 1, \) it was remarked in [CCS06, Section 5] that
\[
\dim(\text{Ext}^1_{C_n}(X, Y)) = 1
\]
if and only if the corresponding diagonals cross. For all the other values of \( p \) we have to consider also diagonals in neighbouring regions. To compute the Ext-spaces we proceed using the fact that the triangulated category \( C_{n,p} \) has Serre duality by Lemma 2.7. Thus, in the first case we have
\[
D_{\text{Ext}}^1_{C_{n,p}}(X, -) = D_{\text{Hom}}_{C_{n,p}}(X, -[1]) \\
\cong \text{Hom}_{C_{n,p}}(-[1], \tau X[1]) \\
\cong \text{Hom}_{C_{n,p}}(-, \tau X).
\]
Here we used that \( C_{n,p} \) has finite dimensional Hom spaces, and therefore also finite dimensional Ext spaces. As next we compute the Hom(\( -\), \( \tau X \)) support using the mesh relations in the AR-quiver of \( C_{n,p} \). That gives a rectangular region, also known as the backwards hammock. Then one concludes using the bijection between indecomposable objects and diagonals from Proposition 3.7.

In the second case, we proceed dually. Then the study the forward hammocks leads to the desired claim. \( \square \)

With this result in mind we are able to deduce the desired geometric description of cluster tilting objects, of \( C_{n,p} \), and in fact this gives a geometric/combinatorial analogue to [Zhu11, Thm: 3.5] for the case \( H = \text{mod} kA_n \). Recall that the map \( \rho : \Pi^p \to \Pi^p \) is a clockwise rotation of \( \frac{2\pi}{p} \) around the center of \( \Pi^p \).

**Proposition 5.4.** \( T \) is a cluster tilting object of \( C_{n,p} \) if and only if
\[
X_T = X_T \cup \rho(X_T) \cup \cdots \cup \rho^{p-1}(X_T)
\]
where \( X_T \) is a triangulation of the region \( \Pi_1 \).

**Proof.** We saw in Lemma 5.3 that the non-vanishing of the Ext spaces can arise from crossing within the same \( \Pi_k \) or from crossings between diagonals in the neighbouring regions \( \Pi_{k+1} \) and \( \Pi_{k-1} \) of \( \Pi^p \). In particular, diagonals \( D_X \) in \( \Pi_k \) and \( D_Y \) in \( \Pi_{k+1} \) give rise to a non-vanishing Ext spaces if and only if \( \rho^{-1}(D_Y) \) crosses \( D_X \) inside \( \Pi_k \) or \( \rho(D_X) \) crosses \( D_Y \) inside \( \Pi_{k+1} \), and the vertices of the diagonals satisfy the conditions of Lemma 5.3. Thus, the only possible way to get a maximal configuration of diagonals which do not cross in the sense of both Lemma 5.3 is to take the same triangulation in each region \( \Pi_k \), for \( 1 \leq k \leq p \).

On the other hand, by Lemma 5.2 we have that a triangulation \( X_T \) of a region \( \Pi_k \) corresponds to a cluster tilting object in \( C_n \), then \( \rho^i(X_T) \), for \( 0 \leq i \leq p - 1 \), gives a cluster tilting object \( C_{n,p} \), by the previous argument. \( \square \)

**Corollary 5.5.** Any cluster tilting object in \( C_{n,p} \) contains \( pn \) pairwise non isomorphic summands.

**Proof.** Since each region \( \Pi_k \) is homotopic to a regular \( n+3 \)-gon, the set \( X_T \) consists of is \( pn \) diagonals of \( \Pi^p \). \( \square \)
Observe that in \( C_{n,p} \) there is exactly one complement to an object in a cluster tilting object \( T \), whereas there are exactly two in \( C_n \). This observation follows from the fact that whenever a diagonal \( D_X \) is removed from \( X_T \) as given in Proposition 5.3 and we replace \( D_X \) with a unique other diagonal \( D_X' \), completing again the triangulation, we obtain a new triangulation which is no longer stable under the rotation of \( \rho \). Thus, to get a new cluster tilting object in \( C_{n,p} \) one needs to replace \( p \) summands of \( T \) corresponding to a complete \( \rho \)-orbit of diagonals in \( IP^p \).

6. Repetitive higher cluster categories \( C^m_{n,p} \)

In this section we study orbit categories of the form

\[
C^m_{n,p} := D / (\langle \tau^{-1}[m] \rangle^p).
\]

We call them repetitive higher cluster categories of type \( A_n \). The class of objects is the same as the class of objects in \( D \) and the space of morphisms is given by

\[
\text{Hom}_{C^m_{n,p}}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_D(X, (\tau^{-p}[pm])^i Y).
\]

We will see below that these categories are interesting because their geometric models provide us with a geometric description of the bounded derived category \( D \).

Rpetitive higher cluster categories have similar properties to the repetitive cluster categories. In particular, identifying \( \text{ind} C^m \) with \( \mathcal{F} = \mathcal{F}_1 \), the fundamental domain for the functor \( \tau^{-1}[m] \) in \( D \), we can prove:

**Lemma 6.1.** For all \( m \in \mathbb{N} \).

1. The projection functors \( \pi^m_p : D \to C^m_{n,p} \) and \( \eta^m_p : C^m_{n,p} \to C^m_n \) are triangle functors.
2. The category \( C^m_{n,p} \) is triangulated, with Serre functor \( \tau[1] \) induced from \( D \).
3. \( C^m_{n,p} \) is fractionally Calabi-Yau of dimension \( \frac{n(p+1)}{p} \) and Krull-Schmidt.
4. \( \text{ind}(C^m_{n,p}) = \bigcup_{i=1}^{N} (\mathcal{F}_i) \).

**Proof.** The first two claims follow from [Kel08 Theorem 1] and [BMR+06 Proposition 1.3]. Concerning the fractionally Calabi-Yau dimension, one computes easily that \( (\tau[1])^p \cong [p(m+1)] \) in \( C^m_{n,p} \). The last claim can be deduced from a similar reasoning as in [BMR+06 Proposition 1.6]. \( \square \)

6.1. \( C^m_{n,p} \) via \( m \)-diagonals in \( IP^p \). In the following we present a geometric model for the category \( C^m_{n,p} \) which arises as a generalization of the one given for \( C_{n,p} \). Indecomposable objects are modelled by a subset of diagonals in \( IP^p \) and irreducible morphisms will be associated to rotations of such modulo the mesh relations.

Let \( IP^p \) be a regular \( p((n+1)m+1) \)-gon, \( m, n \in \mathbb{N} \). Then we divide \( IP^p \) into regions as we did in Section 5.3 when \( m = 1 \). Then each region \( \Pi_k \) in \( IP^p \) is homotopic to a regular \( ((n+1)m+2) \)-gon. See also Figure 2.

For the convenience of the reader we recall that an \( m \)-diagonal in a regular \( ((n+1)m+2) \)-gon, is a diagonal which divides the polygon into two parts, each having 2 vertices mod \( m \). Observe that \( m \)-diagonals have been studied by many authors, see [BM09], [BM07], [BT09], [FR06], [PS00], [Tho07], [Tor11], [Tza06]. . . . We will now adapt the notion of \( m \)-diagonals to the case at hand.

**Definition 6.2.** The \( m \)-diagonals of \( IP^p \) are given by the union of the \( m \)-diagonals in each region \( \Pi_k \), for \( 1 \leq k \leq p \).
Notation 6.3. In the writing of a diagonal \((i, j, k)\) in \(\Pi^p\), we consider the operations on the first two indices modulo \(m\), and the last one modulo \(p\). Remember that the last index specifies the region \(\Pi_k\) in \(\Pi^p\). As everywhere else in this paper we assume that \(i < j\).

We associate a quiver to the \(m\)-diagonals in \(\Pi^p\) as follows.

Definition 6.4. Let \(\Gamma_{n,p}^m\) be the quiver of \(m\)-diagonals of \(\Pi^p\). Its vertices are the \(m\)-diagonals, and the arrows are given as follows.

1.: If \(j \neq N - (m - 1)\),
\[
(i, j + m, k) \rightarrow (i, j, k) \rightarrow (i, j, k) \rightarrow (i - m, j, k)
\]

2.: If \(j = N - (m - 1)\) then \((i, N - (m - 1), k) \rightarrow (i - m, N - (m - 1), k)\).

3.: Furthermore, \((i, N - (m - 1), k) \rightarrow (1, i, k + 1)\).

Observe that the description 1. and 2. corresponds to an irreducible \(m\)-rotations in \(\Pi^p\) as defined in [BM08]. That is an irreducible clockwise rotations between \(m\)-diagonals around the vertex \(i\) or \(j\) inside a region \(\Pi_k\). We denote the set of these operations by \(\text{IrrRot}_m\). The third type describes the composition of an \(m\)-irreducible rotation inside \(\Pi_k\) around the vertex \(i\) with the clockwise rotation \(\rho(1, i, k) = (1, i, k + 1)\). We denote the second type of operation by \(\text{Irr}_m\rho\text{Rot}_m\).

Then we equip \(\Gamma_{n,p}^m\) with a translation map. The first part of the definition describes an anticlockwise rotation around the center of \(\Pi_k\) through \(\frac{2m\pi}{(n+1)m - 2}\), the second defines a composition of a rotation as in the first case, together with with an anticlockwise rotation around the center of \(\Pi^p\) through \(\frac{2\pi}{p}\).

Definition 6.5. The translation \(\tau_m\) maps

- \((i, j, k)\) to \((i - m, j - m, k)\) if \(i, j \neq 1\).
- If \(i = 1\), \(\tau_m(1, j, k) = \rho^{-1}(1 - m, j - m, k) = (1 - m, j - m, k - 1)\).

The proof of the next result is straightforward. We use the fact that the AR-quiver of \(C_{n,p}^m\) is isomorphic to the quiver of \(m\)-diagonals of \(\Pi\), as shown in [BM08, Proposition 5.4], combined with similar arguments as in the proof of Lemma 3.5.

Lemma 6.6. There is an isomorphism of stable translation quivers between the AR-quiver of \(C_{n,p}^m\) and \(\Gamma_{n,p}^m\).

6.2. Tilting theory for \(C_{n,p}^m\). In this section we determine the \(m\)-cluster tilting objects of \(C_{n,p}^m\) geometrically, we will see that they correspond to \(\rho\)-orbits of \(m\)-angulations of a region \(\Pi_k\) in \(\Pi^p\).

Definition 6.7. An object \(T \in C_{n,p}^m\) is called \(m\)-rigid if it is the direct sum of non isomorphic indecomposable objects \(T_1, \ldots, T_t\) such that \(\text{Ext}_{C_{n,p}^m}^i(T_j, T_k) = 0\) for all \(1 \leq i \leq m\), and \(1 \leq j, k \leq t\). It is called maximal \(m\)-rigid if it is \(m\)-rigid and maximal with respect to this property.

An \(m\)-cluster tilting object is an object \(T\) in \(C_{n,p}^m\) which is \(m\)-rigid and such that \(X \in \text{add}^* T\) if and only if \(\text{Ext}_{C_{n,p}^m}^i(T, X) = 0\) for \(1 \leq i \leq m\), and \(X \in \text{add} T\) if and only if \(\text{Ext}_{C_{n,p}^m}^i(X, T) = 0\) for \(1 \leq i \leq m\).

Observe that it was proven in [Wra09] that in \(C_n\), \(m\)-cluster tilting and maximal \(m\)-rigid objects coincide.

Definition 6.8. An \(m\)-angulation of \(\Pi^p\) consists of a maximal set of pairwise non crossing \(m\)-diagonals in \(\Pi^p\). We denote such a set by \(X^m\).
For higher cluster categories the following result is a consequence of \cite[Theorem 1]{ThH99} combined with \cite[Section 4]{BM08}.

**Lemma 6.9.** \(m\)-cluster tilting objects in \(\mathcal{C}_n^m\) correspond to \(m\)-angulations \(\mathcal{X}^m\) of a regular \((n+1)m+2\)-gon.

Thus, an \(m\)-cluster tilting object in \(\mathcal{C}_n^m\) has \(n\) non isomorphic summands, see also \cite{Zhu08} and \cite{Wra09}. The next useful result is due to A. Wrålsen, \cite{Wra09}.

**Lemma 6.10.** Any \(m\)-cluster tilting object in \(\mathcal{C}_n^m\) is induced by a maximal \(m\)-rigid object in the fundamental domain of \(\tau^{-1}[m]\) in \(\mathcal{D}\).

**Proposition 6.11.** \(T\) is an \(m\)-cluster tilting object in \(\mathcal{C}_{n,p}^m\) if and only if

\[
\mathcal{X}_T^m = \mathcal{X}_n^m \cup \rho(\mathcal{X}_n^m) \cup \cdots \cup \rho^{p-1}(\mathcal{X}_n^m)
\]

where \(\mathcal{X}_n^m\) is an \(m\)-angulation of the region \(\Pi_1\).

**Proof.** First let \(\mathcal{X}_n^m\) be an \(m\)-angulation of a given region \(\Pi_1\) in \(\mathcal{P}^n\). By Lemma 6.9 it corresponds to a cluster tilting object \(T_{\mathcal{C}_n^m}\) of \(\mathcal{C}_n^m\). By Lemma 6.10 it can be lifted to a maximal \(m\)-rigid object \(T_D\) in \(\mathcal{D}\) contained in a fundamental domain of \(\tau^{-1}[m]\). Consider its \((\tau^{-1}[m])\)-orbit \(\{\tau^{\pm i}[m] \mid i \in \mathbb{Z}\}\) in \(\mathcal{D}\) which we denote again by \(T_D\). Let \(T_{\mathcal{C}_n^m}\) be the projection of \(T_D\) under the functor \(\tau^p_{\mathcal{P}^m} : \mathcal{D} \to \mathcal{C}_{n,p}^m\). Then \(T_{\mathcal{C}_n^m}\) is a direct sum of pairwise non isomorphic indecomposable objects, which we denote by \(T_1, \ldots, T_t\) and we compute:

\[
\text{Ext}^i_{\mathcal{C}_{n,p}^m}(T_k, T_l) = \text{Hom}_{\mathcal{C}_{n,p}^m}(T_k, T_l[i]) = \bigoplus_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(T_k, (\tau^{-p}[mp])^j T_l[i]) = \bigoplus_{j \in \mathbb{Z}} \text{Ext}_{\mathcal{D}}(T_k, (\tau^{-p}[mp])^j T_l) = 0
\]

for all \(T_k, T_l, 1 \leq k, l \leq t\) and \(1 \leq i \leq m\). In fact, the first and third equality hold by definition. The second follows from the definition of morphisms in \(\mathcal{C}_{n,p}^m\). The last equality follows because if \(T_D\) is \(m\)-rigid then the same is true for \((\tau^{-1}[m])\langle T_D\rangle\), hence also for \((\tau^{-j}[m])\langle T_D\rangle\). And as \(T_D\) is maximal, it follows that in the fundamental domain of \((\tau^{-1}[m])\), \(\text{Ext}_{\mathcal{D}}(T_k, T_l) = 0\) holds. Hence \(T_{\mathcal{C}_n^m}\) is \(m\)-rigid.

Now we show that \(T_{\mathcal{C}_n^m}\) satisfies the second part of the definition of \(m\)-cluster tilting objects in \(\mathcal{C}_{n,p}^m\). For this we first remark that the Ext spaces can be determined by similar computations as in Lemma 5.8. Secondly, by construction and Lemma 6.9 the objects \(T_1, \ldots, T_t\) of \(\mathcal{C}_{n,p}^m\) correspond to a maximal set \(\mathcal{X}^m\) of pairwise non crossing \(m\)-diagonals in \(\mathcal{P}^n\). Furthermore, observe that this set is given by the \(\rho\)-orbit of \(\mathcal{X}_n^m\) in \(\mathcal{P}^n\). If \(T_{\mathcal{C}_n^m}\) was not \(m\)-cluster tilting, by the bijection given in Lemma 6.9 there must be a region \(\Pi_t\) in \(\mathcal{P}^n\) where the set of \(m\)-diagonals is not maximal, but this contradicts the assumption that \(\mathcal{X}_n^m\) was an \(m\)-angulation. This proves one direction of the claim.

For the other direction, assume we have an \(m\)-angulation \(\mathcal{X}_T^m\) of \(\mathcal{P}^n\), and that \(\mathcal{X}_T^m\) is not of the claimed form. Then \(\mathcal{X}_T^m\) is such that there are regions \(\Pi_k\) in \(\mathcal{P}^n\) with \(m\)-angulations that are not obtained one from the other under \(\rho\). Denote these two different \(m\)-angulations by \(\mathcal{X}_k^m\) and \(\mathcal{X}_l^m\). By Lemma 6.9 they correspond to \(m\)-cluster tilting objects in \(\mathcal{C}_n^m\).

Consider the projection functor \(\mu_p : \mathcal{P}^n \to \Pi\) sending a diagonal \(D_X\) of \(\mathcal{P}^n\) to the corresponding diagonal \(D_X\) in \(\Pi\), and a morphism between \(D_X\) and \(D_Y\) in \(\mathcal{P}^n\) to the corresponding morphism between \(D_X\) and \(D_Y\) in \(\Pi\). Then as \(\mu_p(\mathcal{X}_k^m) \neq \mu_p(\mathcal{X}_l^m)\)
µ_p(ξ^m_i) there are two m-diagonals, say D_V and D_W that cross in Π, because otherwise this would contradict the maximality of the m-angulation. Therefore, we conclude by Lemma 6.9 that the corresponding objects V and W in C^m_{n,p} are such that \( \text{dim}(\text{Ext}_{C^m_{n,p}}(V, W)) > 0 \) for an \( 1 \leq i \leq m \). Then, up to isomorphism, i.e. up to change the object W with an other indecomposable object in the same \( τ^{-1}[m] \)-orbit in D, it follows that also \( \text{dim}(\text{Ext}_{C^m_{n,p}}(V, W)) > 0 \). This proves that the object in C^m_{n,p} corresponding to the diagonals \( ξ^p_0 \) is not an m-cluster tilting object. □

With this result we deduce immediately the following.

**Corollary 6.12.** Any m-cluster tilting object in C^m_{n,p} contains pn pairwise non isomorphic summands.

### 7. Geometric model for \( D^b(\text{mod}kA_{n+1}) \)

Our strategy is to adapt the construction of C^m_{n,p} given in the previous section. For this we will cut the large polygon \( \Pi^p \) between two regions, and let \( p \to \infty \). In this way we obtain a figure with infinitely many sides in which the idea is to no longer study m-diagonals but their complements. More precisely, we consider the diagonals which are not 2-diagonals.

Complements to m-diagonals in a given polygon Π have been studied in [Lam11]. The result we need for the modelling of D is Lemma 7.1. It is based on the notion of the m-th power of a translation quiver introduced in [BM08].

#### 7.1. The m-th power of a translation quiver.

The m-th power of \((Γ, τ)\), is a translation quiver \((Γ^m, τ^m)\) which has the same vertices as Γ, and whose arrows are given by paths of length m in Γ: \((x = x_0 \to x_1 \to \cdots \to x_m = y)\), such that whenever \(τx_{i+1}\) is defined \(τx_{i+1} \neq x_i−1\) for \(i = 1, \ldots, m−1\). Furthermore, the translation is given as \(τ^m := τ \circ \cdots \circ τ\), m-times. If Γ is the AR-quiver of the cluster category of type \( A_{(n+1)m−1} \), the m-th power gives rise to \(Γ^m\), the AR-quiver of the m-cluster category of type \( A_n \), see [BM08 Theorem 7.2].

**Lemma 7.1.** Let Π be a regular \( (n + 1)m + 2 \)-gon and \( Γ_{A_{(n+1)m−1}} \) the quiver of its diagonals. Then we have

\[(Γ_{A_{(n+1)m−1}})^2 \cong Γ^2_n \sqcup Γ_1 \sqcup Γ_2,\]

where \( Γ_1 \cong Γ_2 \cong AR(D^b(\text{mod}kA_{n+1})/[1]) \), and [1] is the shift functor on D.

**Proof.** This result was obtained from Corollary 4.13 and Theorem 4.16 in [Lam11]. □

As a consequence of this result, one can say that the vertices of Γ_1 and Γ_2 are precisely the diagonals whose endpoints are vertices of Π of the same parity.

#### 7.2. The \( ∞ \)-gon Π^±∞.

In order to construct a polygon with infinitely many sides we first introduce a polygon Π^±p with \((2p + 1)(2n + 3)\) sides and then let \(p \to \infty\).

Let Π_1 be a N := \((2(n + 1) + 2)\)-gon and assume that its vertices all lie on a line (c.f. Figure 4). Let \( g \) be the map shifting the figure Π_1. \( N \) steps to the right so that the left most vertex of \( g(Π_1) \) coincides with the right most vertex of Π_1. Considering powers of \( g \) we obtain a figure with \( 2p + 1 \) regions homotopic to Π_1, each with vertices numbered from 1 up to N. Denote this figure by Π^±p. The diagonals in Π^±p we will consider are inner arcs connecting even numbered vertices in the same Π_k, \(-p \leq k \leq p\). One could also have chosen the odd numbered vertices. We call these diagonals 2^\text{-diagonals} . Observe that the 2^\text{-diagonals in any} Π_k give rise to a copy of Γ_1 in Lemma 7.1. Letting \( p \to \infty \) we obtain the \( ∞ \)-gon Π^±∞.
7.3. The quiver $\Gamma^{\pm\infty}$. We will define a translation quiver on the $2c$-diagonals of $\Pi^{\pm\infty}$, for this we start associating a quiver to the $2c$-diagonals of $\Pi^{\pm p}$. The arrows between $2c$-diagonals are defined in a similar way as in Section 6. Let $(i, j, k)$ be a $2c$-diagonal in a region $\Pi_k$ of $\Pi^{\pm p}$ with $j \neq N$, then

\[
(1, j + 2, k) \xrightarrow{\mu}(i, j, k) \xrightarrow{\mu}(i - 2, j, k)
\]

where the first two entries are understood modulo 2, and computations on the last entry have to be considered modulo $p$. Furthermore, for all $k \neq p$, we have arrows

\[
(2, i, k + 1) \xrightarrow{\rho}(i, N, k) \xrightarrow{\rho}(i, j - 2, N, k)
\]

where $2, i, k + 1 = \rho(2, i, k)$. The condition on $k$ is needed because we want to avoid that there is an arrow linking the diagonals of the region $\Pi_p$ to those of the region $\Pi_{-p}$. We will refer to the arrows $(i, N, k) \xrightarrow{\rho}(2, i, k + 1), k \neq p$ as connecting arrows.

Letting $p \to \infty$ we obtain infinitely many regions homotopic to $\Pi_1$, and we call the resulting figure $\Pi^{\pm \infty}$. We denote the quiver of $2c$-diagonals of $\Pi^{\pm \infty}$ by $\Gamma^{\pm \infty}$.

Let $\phi$ be a stable translation automorphism of $\Gamma^{\pm \infty}$.

\[
\phi^{-1} \xrightarrow{\phi} \phi
\]

\[
\ldots \quad \Pi_{-2} \quad \Pi_1 \quad \Pi_2 \quad \ldots
\]

Figure 4. $\Pi^{\pm \infty}$ as model for $D^b(modkA_3)$.

7.4. Translation map for $\Gamma^{\infty}$. We equip $\Gamma^{\pm \infty}$ with the translation $\tau_2$ from Definition 6.3 and let $p$ tend to infinity. More precisely, $\tau_2(i, j, k) = (i - 2, j - 2, k)$ for all $i, j$ different from 2, and $\tau_2(2, j, k) = \phi^{-1}(N, j - 2, k) = (j - 2, N, k - 1)$. It can be proven as in Lemma 6.5 that this defines a stable translation quiver, and taking the mesh category of it specifies morphisms for the $2c$-diagonals of $\Pi^{\pm \infty}$ arising from the mesh relations of $\Gamma^{\pm \infty}$.

7.5. Geometric model of $modkA_{n+1}$. Observe that when restricting to the region $\Pi_1$ of $\Pi^{\pm \infty}$ we obtain the AR-quiver of the module category of type $A_{n+1}$ equipped with translation $\tau$. In fact, write $\Gamma_{\Pi_1}$ for the full subquiver of $\Gamma^{\pm \infty}$, with translation given by $\tau_{2|\Gamma_{\Pi_1}}$, and assume that the path algebra $kA_{n+1}$ is taken over an equioriented Dynkin quiver of type $A_{n+1}$.

Lemma 7.2. There is an isomorphism of stable translation quivers

\[
\Gamma_{\Pi_1} \cong AR(modkA_{n+1}).
\]

Proof. By construction $\Pi_1$ is homotopic to a regular $2(n + 1) + 2$-gon. Hence, the number of $2c$-diagonals in $\Pi_1$ is $\frac{(n + 2)(n + 1)}{2}$. This number agrees with the isomorphism classes of indecomposable objects in $modkA_{n+1}$. Using Gabriel’s correspondence between indecomposable objects in $modkA_{n+1}$ and positive roots of the associated Dynkin diagram we associate positive roots to the $2c$-diagonals of $\Pi_1$ and indecomposable modules to the positive roots.
Given a 2c-diagonal \((i, j, 1)\) we let \(|i - j|\) be its size. Then we associate the 2c-diagonals of size 2 to the simple roots \(\alpha_i, 1 \leq i \leq n + 1\) of the associated Dynkin diagram. As the length of the segment augments, we increase the number of summands of the positive root. Remark that the longest 2c-diagonal of \(\Pi_1\) is \((2, 2(n + 1) + 2)\), and it corresponds to \(\alpha_1 + \cdots + \alpha_{n+1}\).

Due to the shape of \(\Gamma_{\Pi_1}^+\) and the well known shape of \(AR(\text{mod} kA_{n+1})\), it is clear that the map \(\varphi\) we just defined gives rise to an isomorphism of quivers, and it easy to check that it preserves the translation so that it becomes an isomorphism of stable translation quivers.

7.6. Geometric model of the bounded derived category \(D\). By taking the mesh category of \((\Gamma^\pm\infty, \tau_2)\) we obtain a geometric model of \(D\).

**Theorem 7.3.** There is an isomorphism of stable translation quivers between the \(AR\)-quiver of \(D\) and \((\Gamma^\pm\infty, \tau_2)\).

**Proof.** By Lemma 7.2 we know that

\[
(\Gamma_{\Pi_1}, \tau_2|\Gamma_{\Pi_1}) \cong (AR(\text{mod} kA_{n+1}), \tau).
\]

Furthermore, we observe that the translation \(\varphi\) reproduces copies of \(\Gamma_{\Pi_1}\) in \(\Gamma^\pm\infty\). Thus, it remains to study the connecting arrows. In \(\Gamma^\pm\infty\) these are the arrows between 2c-diagonals of the form \((i, N, k)\) and \((2, i - 2, k + 1)\) lying in the different regions \(\Pi_k\) and \(\Pi_{k+1}\) inside \(\Pi^\pm\infty\). By construction these arrows are exactly the connecting arrows between different copies of the \(AR\)-quiver of mod\( kA_{n+1}\) in \(AR(D)\).

**Corollary 7.4.** There is a morphism of functors between \(\varphi\) in the category of 2c-diagonals of \(\Pi^\pm\infty\) and the shift functor \([1]\) on \(D\).

**Proof.** This follows directly from the arguments given in the proof of the previous result.

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