REMARKS ON LOCAL THEORY FOR
SCHRÖDINGER MAPS NEAR HARMONIC
MAPS ∗†‡

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Abstract
We consider the initial-value problem for the equivariant Schrödinger maps near a family of harmonic maps. We provide some supplemental arguments for the proof of local well-posedness result by Gustafson, Kang and Tsai (Duke Math. J. 145(3) 537–583, 2008). We also prove that the solution near harmonic maps is unique in $C(I; \dot{H}^1(\mathbb{R}^2) \cap \dot{H}^2(\mathbb{R}^2))$ for time interval $I$. In the proof, we give a justification of the derivation of the modified Schrödinger map equation in low regularity settings without smallness of energy.

1 Introduction
We consider the initial value problem for the Schrödinger map equation (or Schrödinger flow) from $\mathbb{R}^n$ to a sphere $S^2$, which is given by
\[ \partial_t u = u \times \Delta u, \quad u(x, 0) = u_0(x), \quad (1.1) \]
where $u = u(x, t)$ is unknown function from $\mathbb{R}^n \times \mathbb{R}$ to a sphere
\[ S^2 = \{ y \in \mathbb{R}^3 : |y| = 1 \} \subset \mathbb{R}^3, \quad (1.2) \]
and $\times$ denotes the vector product of vectors in $\mathbb{R}^3$. This equation arises in various ways in physics; we refer, for example, to [6], [13] for details.

The equation $(1.1)$ admits the following conserved energy
\[ \mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx, \quad (1.3) \]
and $(1.1)$ has the scale invariance
\[ u(x, t) \mapsto u(\frac{x}{\lambda}, \frac{t}{\lambda^2}) \text{ for } \lambda > 0. \quad (1.4) \]

In this work, we restrict ourselves to the case $n = 2$. Our aim of the present paper is to supplement arguments concerning the regularity, which is used without proof in the paper by Gustafson, Kang and Tsai. ∗

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We first recall the background of the problem. For $m \in \mathbb{N}$, a map $u : \mathbb{R}^2 \to \mathbb{S}^2$ is said to be $m$-equivariant if $u$ has the form
\[ u(x) = e^{m\theta R} v(r), \]
where $(r, \theta)$ is the polar coordinates of $x$, $v = i(v_1, v_2, v_3)$ is a function from $(0, \infty)$ to $\mathbb{R}^3$, and $R$ is the matrix $R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that $e^{\alpha R}$ represents a rotation in angle $\alpha$ around the $u_3$-axis for $\alpha \in \mathbb{R}$. In this paper, we focus on the latter case.

Each choice corresponds to different homotopy type of maps. When $\delta$ is the rotational symmetry of (1.1), we may assume $v(0) = \vec{k}$ without loss of generality. Here, we have two choices of $v(\infty)$:
\[ v(\infty) = -\vec{k} \quad \text{or} \quad v(\infty) = \vec{k}. \]

If an $m$-equivariant map $u = e^{m\theta R} v(r)$ satisfies $\mathcal{E}(u) < \infty$, the following hold: $v : (0, \infty) \to \mathbb{R}^3$ is continuous, and both limits $v(0) := \lim_{r \to 0} v(r)$ and $v(\infty) := \lim_{r \to \infty} v(r)$ exist and are equal to either of $\pm \vec{k}$, where $\vec{k} = i(0, 0, 1)$. (For the proof, see Section 4 below.) By the rotational symmetry of (1.1), we may assume $v(0) = -\vec{k}$ without loss of generality. Here, $u$ is a stationary solution to (1.1): $u = \Delta u = 0$.

Our function space is
\[ \Sigma_m := \left\{ u = e^{m\theta R} v(r) \mid u \in \dot{H}^1, v(0) = -\vec{k}, v(\infty) = \vec{k} \right\}. \]

Then $\Sigma_m$ is complete metric space with metric $d(u, \bar{u}) = \|u - \bar{u}\|_{\dot{H}^1}$. For $u \in \Sigma_m$, we have
\[ \mathcal{E}(u) = \pi \int_0^\infty \left| \partial_r u_r^2 + m^2 u_2^2 + u_2^2 \right| r \, dr. \]

Then we can write (1.7) as
\[ \mathcal{E}(u) = \pi \int_0^\infty \left| v_r - \frac{m}{r} J^* R v \right|^2 r \, dr + 4 \pi m \]
for $u \in \Sigma_m$, where we define $J^* := v \times \cdot$. (See [8] for details.) Thus, we have $\mathcal{E}(u) \geq 4 \pi m$ for all $u \in \Sigma_m$, and $u$ minimizes the energy if and only if $v_r - \frac{m}{r} J^* R v = 0$ for almost all $r \in (0, \infty)$. By solving this ODE, it turns out that the minimizing set can explicitly be written as
\[ \mathcal{O}_m = \left\{ e^{\alpha R} Q \left( \frac{\vec{k}}{s} \right) \mid s > 0, \alpha \in \mathbb{R} \right\}, \]
where $Q := e^{m\theta R} h(r)$, $h(r) = i(h_1(r), 0, h_3(r))$, $h_1(r) = \frac{2 e^m}{r^{m+1}}$, $h_3(r) = \frac{2 e^m}{r^{m+1}}$. We call $\mathcal{O}_m$ the family of harmonic maps. Note that any element in $\mathcal{O}_m$ is a stationary solution to (1.1): $u \times \Delta u = 0$.

We recall a geometric description of $\mathcal{O}_m$ obtained in [8].
Proposition 1.1. There exist \( \delta_0 > 0 \) and \( C_0, C_1 > 0 \) such that for \( u \in \Sigma_m \) with \( \mathcal{E}(u) < 4\pi m + \delta_0^2 \), the following hold:

(i) There exist unique \( s_* = s_*(u) \in (0, \infty) \) and \( \alpha_* = \alpha_*(u) \in T^1 \) such that
\[
\text{dist}_{H^1}(u, \Omega_m) = \left\| u - e^{\alpha_* R} Q\left(\frac{\cdot}{\alpha_*}\right) \right\|_{H^1}.
\]

(ii) The map \( u \mapsto (s_*(u), \alpha_*(u)) \) is continuous.

(iii) \( C_0 \text{dist}_{H^1}(u, \Omega_m) \leq \sqrt{\mathcal{E}(u) - 4\pi m} \leq C_1 \text{dist}_{H^1}(u, \Omega_m) \).

The above proposition ensures the unique existence of \( \dot{H}^1 \)-closest harmonic map for each \( u \in \Sigma_m \) with \( \mathcal{E}(u) - 4\pi m \ll 1 \). By the scaling pair \((s_*(u), \alpha_*(u))\), we can get precise information on the position of \( u \) relative to \( Q \) along the harmonic map family.

2 The Paper by Gustafson et al. and Our Main Result

With the aim of studying the stability of \( \Omega_m \), Gustafson, Kang, and Tsai \cite{O} considers local problems for \( (1.1) \) near the family \( \Omega_m \) in the class \( \Sigma_m \).

To present the statement of their results, we introduce the notion of weak solution of \( (1.1) \). We first note that the equation \( (1.1) \) can be written by divergence form as follows:
\[
\partial_t u = \sum_{j=1}^2 \partial_{x_j} (u \times (\partial_{x_j} u))
\]
where \( x_j \) is the \( j \)-th spatial coordinate. Considering \( (2.1) \), we define the weak solution to \( (1.1) \) in the following way.

Definition 2.1. For interval \( I \), \( u(t) \in L^\infty_{loc}(I; \Sigma_m) \) is said to be a weak solution if \( u(t) \) satisfies
\[
\int_{I \times \mathbb{R}^2} u \partial_t \phi \ dx dt = \int_{I \times \mathbb{R}^2} \sum_{j=1}^2 (u \times \partial_{x_j} u) \partial_{x_j} \phi \ dx dt
\]
for all \( \phi \in C^0_0(I \times \mathbb{R}^2) \).

The pioneering paper \cite{O} by Gustafson et al. contains the following results (Theorem 1.4, page 543):

(LP1) (Existence) There exist \( \delta_0 > 0 \), \( \sigma > 0 \), and \( C > 0 \) such that the following holds: If \( u_0 \in \Sigma_m \) satisfies \( \delta := \sqrt{\mathcal{E}(u_0) - 4\pi m} < \delta_0 \), then \( (1.1) \) has a weak solution \( u(t) \in C(I; \Sigma_m) \), \( I = [0, \sigma s_0^2] \), where \( s_0 := s_*(u_0) \).

(LP2) (Uniqueness) The above solution is unique in \( C(I; \Sigma_m) \); i.e., if \( \tilde{u}(t) \in C(I'; \Sigma_m) \) satisfies \( (1.1) \) for \( I' = [0, T] \) with \( T > 0 \), then \( \tilde{u}(t) = u(t) \) for all \( t \in I \cap I' \).

(LP3) (Energy conservation) The above solution conserves the energy, that is, \( \mathcal{E}(u(t)) = \mathcal{E}(u_0) \) for all \( t \in I \).

(LP4) (Regularity) If we further assume that \( u_0 \in \dot{H}^2 \), then the above solution \( u(t) \) is in \( C(I; \Sigma_m \cap \dot{H}^2) \).

(LP5) (Continuous dependence) The map \( \{ u \in \Sigma_m : \mathcal{E}(u) < 4\pi m + \delta_0^2 \} \ni u_0 \mapsto u(t) \in C(I; \Sigma_m) \) is continuous.
These assertions play an important role in the investigation of global behavior of the solution to (1.1) near the harmonic map family $\mathcal{O}_m$. Indeed, the ensured existence time $\sigma_s^0$ in (LP1) implies that the possible finite time blow-up scenario for $(u(t))$ is $s^*(u(t)) \to 0$. See [9] for more details. (See also [11], [16] and [17].)

In the present paper, we mainly focus on the following three points which are not explicitly mentioned in their paper. The first one is concerned with the limiting argument in their proof. Their way to show (LP1) is to reduce the problem to a PDE-ODE system (3.2) and (3.10) defined below. For the construction of weak solution, they first construct a solution $(q(t), s(t), \alpha(t))$ of (3.2) and (3.10), then reconstruct the original map $u(t)$ from it. Then, they claim that this $u(t)$ is actually a weak solution. To show that, they approximate $u(t)$ by smooth solutions. In the argument, they implicitly use the fact that the maximal existence time of each element of approximating sequence of solutions $\{u_k(t)\}_{k=1}^\infty$ is bounded from below uniformly in $k$. Our first aim is to provide a proof of this fact.

The second point is related to the regularity persistence stated in (LP4). In their argument, there is no explicit mention of how we ensure the continuity of the map $u : I \to \dot{H}^2$. Hence, we give a proof of this fact in the present paper.

The third one is concerned with the uniqueness of solutions. They implicitly reduce the problem to the modified system (3.2) and (3.10), and then show the uniqueness for these equations. For sufficiently smooth solutions (more precisely, when $u(t) \in C(I; \dot{H}^1 \cap \dot{H}^3)$), the reduction is justified since it is known that the corresponding $(q(t), s(t), \alpha(t))$ actually satisfies (3.2) and (3.10). However, such kind of justification is not given for weak solutions. Hence, we attempt to give a new justification of the derivation of (3.2) and (3.10) in a larger class of solutions.

In this paper, we reproduce the proof of (LP1), (LP3), and (LP4). Moreover, we show the restated propositions (LP2)' and (LP5)' as follows.

**Theorem 2.1.** The propositions (LP1), (LP3), and (LP4) hold. Moreover, the restated propositions (LP2)' and (LP5)' hold.

(LP2)' For $u_0 \in \{u \in \Sigma_m : E(u) < 4\pi m + \delta_0^2\} \cap \dot{H}^2$, the solutions to (1.1) are unique in $C(I; \Sigma_m \cap \dot{H}^2)$.

(LP5)' The map $\{u \in \Sigma_m : E(u) < 4\pi m + \delta_0^2\} \cap \dot{H}^2 \ni u_0 \mapsto u(t) \in C(I; \Sigma_m)$ is continuous with $\dot{H}^1$-topology. Moreover, the above map can be uniquely extended to $\{u \in \Sigma_m : E(u) < 4\pi m + \delta_0^2\}$ as a limit of $C(I; \Sigma_m \cap \dot{H}^2)$-solutions, and coincide with the solutions constructed in (LP1).

**Remark 2.1.** The uniqueness stated in (LP2)' is more restricted than that in (LP2), but stronger than that ensured in [15]. The statement in (LP5)' is essentially unchanged from that in (LP5), since we just make the definition of solution map clearer according to (LP2)'.

We briefly explain how we supplement the points mentioned above. For the first one, we establish a priori estimate for second derivative of $q(t)$ (see (3.19)), which leads to a priori bound for third order derivative of $u(t)$. Hence, by McGahagan [15], we can ensure that $u^3(t)$ continues to exist as long as the corresponding solution to (3.2) and (3.10) exists.
The essential step for the second point is the continuity of reconstruction stated in Lemma 7.1, in which we claim that the map $H^1 \times \mathbb{R}^+ \times T^1 \ni (q,s,\alpha) \mapsto u \in \Sigma_m \cap \dot{H}^2$ is continuous.

To prove (LP2)', which is concerned with the third point, we make a new justification of the derivation of (3.2) and (3.10) for the solutions $u(t) \in C(I; \dot{H}^1) \cap L^\infty(I; \dot{H}^2)$. This class is the lowest regularity ever. This immediately leads to (LP2)' by the uniqueness of the system (3.2) and (3.10) established in [9]. The main difficulty is that the calculation needs to be performed in the distributional class, while we have to use the polar coordinates essentially. This is why we introduce a new function class $H_{\alpha}^{-1}$ defined below. In this space, several kinds of calculation related to polar coordinates are justified in a larger class than $L^2$.

The rest of the proof is essentially a reproduction of the argument of [9], while we make small modifications.

Here, we make a few remarks on the preceding results concerning the well-posedness for Schrödinger maps. In [15], the existence of global weak solution $u(t) \in L^\infty(\mathbb{R}; \dot{H}^1)$ is established, which, however, says nothing about the singularities or uniqueness. The local well-posedness for large data has been studied, for example, by [15], [7], [14], and [15]. The lowest regularity is the work by McGahagan [15], in which the existence and uniqueness of solutions is established in $L^\infty(I; \dot{H}^1 \cap \dot{H}^3)$. (We essentially use this result in the present paper.) For small data, the local and global well-posedness have been extensively studied, and the cutting-edge result is [1]. The propositions (LP1)–(LP5) by Gustafson, Kang and Tsai [9], which is the main subject of the present work, is the first local-in-time result for rough data near harmonic maps under the restriction to equivariance.

The organization of this paper is as follows. In Section 3, we provide a proof of Theorem 2.1 reproducing the argument in [9]. The subsequent sections is devoted to the proof of technical lemmas. In detail, we provide a justification of derivation of the modified system (3.2) and (3.10) in Section 4. In Section 5, we derive the a priori estimates (3.18) and (3.19). In Section 6, a detailed proof of the properties of scaling pair $(s,\alpha)$ is given. In Section 7, the continuity of the map $H^1 \times \mathbb{R}^+ \times T^1 \ni (q,s,\alpha) \mapsto u \in \Sigma_m \cap \dot{H}^2$ is shown. In the same section, we provide a proof of a lemma concerning approximation by smooth maps.

We close this section with introducing notations used in the present paper. We set $\mathbb{N} := \{n \in \mathbb{Z} : n \geq 1\}$ and $\mathbb{R}^+ := \{s \in \mathbb{R} : s > 0\}$. We use the letter $C$ in many times to indicate a constant, and the representing quantity varies from each situation, if there is no risk of mathematical validity. For $p,q \in [1,\infty]$ and for interval $I \subset \mathbb{R}$, we sometimes abbreviate $L^p(I; L^q(\mathbb{R}))$ as $L^p_1 L^q_2(I)$, or $L^p_1 L^q_2$. We define $L^2_{\text{rad}}(\mathbb{R}^2) := \{f \in L^2(\mathbb{R}^2) : f \text{ is radially symmetric}\}$. For a Banach space $X$, $\langle \cdot, \cdot \rangle_X$ denotes the coupling of the elements in $X^*$ and in $X$. And for a Hilbert space $H$, $\langle \cdot, \cdot \rangle_H$ denotes the inner product of $H$.

3 Proof of Theorem 2.1

3.1 Coulomb Gauge and Modified Schrödinger Map

We begin with a proposition concerning the choice of frame, which we show in more general setting in Section 4 (see Lemma 4.1).
Proposition 3.1. \((\text{i})\) Let \(u = e^{mR}v(r) \in \Sigma_m\). Then, there exists \(\hat{e}(r) : (0, \infty) \to \mathbb{R}^3\) such that

(i) \(\hat{e}\) is absolutely continuous on any closed subinterval of \((0, \infty)\).

(ii) \(\lim_{r\to\infty} v(r) = \hat{e}(1, 0, 0)\).

(iii) \(\partial_r \hat{e}(r) = - (\hat{e}(r) \cdot v_r)(r) v(r)\) for \(r \in (0, \infty)\).

From the properties of \(\hat{e}\), it follows that
\[
|\hat{e}(r)| \equiv 1, \quad \hat{e}(r) \cdot v(r) = 0 \text{ for all } r \in (0, \infty).
\]

Therefore, \(\{e^{mR}\hat{e}, J^*e^{mR}\hat{e}\}\) forms an orthonormal frame of \(T_p\mathbb{S}^2\), where \(T_p\mathbb{S}^2\) denotes the tangent space of \(\mathbb{S}^2\) at \(p \in \mathbb{S}^2\). (In other words, this is an orthonormal frame of the tangent bundle \(u^{-1}\mathbb{S}^2\)). This choice of frame is called the Coulomb gauge (or Coulomb frame).

For \(u \in \Sigma_m\), define \(q = q(u) \in L^2_{rad}\) and \(\nu = \nu(u) \in L^\infty(\mathbb{R}^2)\) by
\[
q := (v_r - \frac{m}{r} J^* \nu) \cdot (\hat{e} + i J^* \hat{e}), \quad \nu := J^* \nu \cdot (\hat{e} + i J^* \hat{e}).
\]

Let \(I \subset \mathbb{R}\) be an interval. For \(u(t) \in C(I; \Sigma_m)\), we write \(q(t) := q(u(t))\), and set \(\tilde{q}(t) = e^{i(m+1)\theta} q(t)\). Then, the following holds.

Proposition 3.2. Let \(u(t) \in C(I; \Sigma_m) \cap C(I; \mathcal{H}^2)\) be a solution to \((\text{EA})\). Then, \(\tilde{q}(t) \in C(I; H^1) \cap C^1(I; H^{-1})\), and \(\tilde{q}(t)\) satisfies
\[
\ddot{q} + \Delta q = \frac{m(1 + v_\nu)}{r^2} \tilde{q} + \frac{mv_\nu}{r} \tilde{q} + \tilde{q} N(q) \quad (3.2)
\]
where
\[
N(q) := \text{Re} \int_0^\infty \left( \tilde{q} + \frac{m}{r} \nu \right) \left( q_r + \frac{1 - mv_\nu}{r} \tilde{q} \right) dr. \quad (3.3)
\]

Moreover, if a solution \(u(t)\) is in \(C(I; \Sigma_m) \cap L^\infty(I; \mathcal{H}^2)\), then \(\tilde{q}(t) \in C(I; L^2) \cap L^\infty(I; H^1) \cap W^{1, \infty}(I; H^{-1})\), and \((3.2)\) holds.

The equation \((3.2)\) is called the modified Schrödinger map equation. The equation \((3.2)\) is first obtained by Chang et al. \([5]\) by formal calculations. Recently, Bejenaru and Tataru \([3]\) shows that the calculation is rigorous when \(u(t) \in C(I; \Sigma_m \cap H^3)\). Our claim is that the calculation can also be justified when \(u(t) \in C(I; \Sigma_m) \cap L^\infty(I; H^2)\). Moreover, our proof of the proposition does not need any condition of smallness of energy, although this improvement brings no benefit on our main argument. We prove this proposition in Section 4.

3.2 Scaling

The quantity \(q\) does not have sufficient information about the original map \(u\). Indeed, we have
\[
q = 0 \iff u \in \mathcal{O}_m,
\]
which indicates the scale indefiniteness of original map. (Note that the situation is different in the case when \(v(0) = v(\infty) = -k\), where \(u\) can be completely reconstructed from \(q\). See \([2]\) for more details.) From this observation, we need to consider the information about the position of \(u\) along \(\mathcal{O}_m\). To this end, it seems to be natural to see \((s_u(u), \alpha_u(u))\) defined in Proposition \((\text{I})\). However, we make a different choice of scaling by following \([3]\), instead of \((s_u(u), \alpha_u(u))\).
We make here two preparations. First, we introduce Hilbert space $H^1_e$ as follows:

$$H^1_e := \{ f : (0, \infty) \to \mathbb{C} \mid \|f\|_{H^1_e} < \infty \},$$

(3.4)

$$\langle f, g \rangle_{H^1_e} := \int_0^\infty \left( f_+^2 + m^2 f^-_+^2 \right) r \, dr.$$  

(3.5)

Some properties of this space are observed in Section 4. The notation $H^1$ is adopted from [2], while [8] and [9] use $X$ instead.

Next, we set

$$j := \tilde{t}(0, 1, 0), \quad J^h j := \tilde{t}(-h_3, 0, h_1).$$

(3.6)

Then, $\{j, J^h j, h\}$ forms an orthonormal basis of $\mathbb{R}^3$. Using (3.6), for given $s > 0$, $\alpha \in \mathbb{T}^1$, and for a map $u \in \Sigma_m$, we can decompose

$$e^{-\alpha R} v(s) = \alpha_j + \beta J^h j + (1 + \gamma)h,$$

(3.7)

and we set $z = \alpha_j + iz_2$.

**Proposition 3.3.** ([2]) There exist $\delta_0 > 0$, $C > 0$ such that the followings hold:

(i) For $u \in \Sigma_m$ satisfying $\delta := \sqrt{2}(u) - 4\pi m < \delta_0$, there exist $(s, \alpha) = (s(u), \alpha(u)) \in \mathbb{R}_{>0} \times \mathbb{T}^1$ such that

$$\langle s, h \rangle_{H^1_e} = 0 \quad \text{and} \quad \left| \frac{s}{s^*} - 1 \right| + |\alpha - \alpha^*(u)| \leq C\delta$$

(3.8)

(ii) For the $u$ above, if $(\tilde{s}, \tilde{\alpha})$ satisfies (3.3) and $|s(u)^{-1}s - 1| + |\alpha - \alpha^*(s)| \leq C\delta_0$, then $(\tilde{s}, \tilde{\alpha}) = (s, \alpha)$.

(iii) If $u(t) \in C(I; \Sigma_m) \cap C^1(I; L^2(\mathbb{R}^2))$ for some open interval $I \subset \mathbb{R}$, then $s(u(t)), \alpha(u(t))$ are $C^1$. If $u(t) \in C(I; \Sigma_m) \cap W^{1, \infty}(I; L^2)$ for some open interval $I \subset \mathbb{R}$, then $s(u(t)), \alpha(u(t)) \in W^{1, \infty}(I; \mathbb{R})$.

A proof of Proposition 3.3 is given in Section 6.

For $u \in \Sigma_m$, we have extracted three quantities $q(u) \in L^2_{rad}, s(u) > 0,$ and $\alpha \in \mathbb{T}^1$. Conversely, we can show that $(q, s, \alpha) \in L^2_{rad} \times \mathbb{R}^+ \times \mathbb{T}^1$ possesses enough information to reconstruct the original map $u \in \Sigma_m$ completely, which is stated in the following proposition.

**Proposition 3.4.** ([2]) There exists $\delta_0 > 0$ such that for $(q, s, \alpha) \in L^2_{rad} \times \mathbb{R}^+ \times \mathbb{T}^1$ with $\delta := \|q\|_{L^2} < \delta_0$, there is a unique $u \in \Sigma_m$ which satisfies $(q, s, \alpha) = (q(u), s(u), \alpha(u))$. Moreover, the map $L^2_{rad} \times \mathbb{R}^+ \times \mathbb{T}^1 \ni (q, s, \alpha) \mapsto u \in \Sigma_m$ is continuous.

The proof of Proposition 3.4 can be found in [2], Lemma A.2.

Now, let $u(t) \in C(I; \Sigma_m) \cap L^\infty(I; H^2)$ be a solution to (1.1), and set $s(t) := s(u(t)), \alpha(t) := \alpha(u(t))$. Direct calculations yield the equation which $s(t)$ and $\alpha(t)$ satisfy.

**Proposition 3.5.** ([2]) There exists $\delta_0 > 0$ such that the following holds: If $u(t) \in C(I; \Sigma_m) \cap L^\infty(I; H^2)$ is a solution to (1.1) which satisfies $\delta := \sqrt{2}(u) - 4\pi m < \delta_0$, then

$$\begin{bmatrix} s(t) \\ \alpha(t) \end{bmatrix} = \begin{bmatrix} s(0) \\ \alpha(0) \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & -(ms)^{-1} \\ s^{-2} & 0 \end{bmatrix} \begin{bmatrix} ||h_1||_{H^2_e}^2 & I + A^{-1}G_2 \end{bmatrix} (r) \, dr,$$

(3.10)
The following holds: For there exists a solution to the system (3.2) and (3.10) as in [9].

Indeed, the quantities such as Proposition 3.6.

The following proposition is established.

For the proof of this proposition, see [9], Section A.2.

3.3 Local well-posedness of the PDE-ODE System

We have seen that if \( u(t) \in C(I; \Sigma_m) \cap L^\infty(I; \mathbb{H}^2) \) is a solution to (1.13), then \((\tilde{q}(t), s(t), \alpha(t))\) must satisfy the system of equations (3.12) and (3.10). Note that this is a closed system. Indeed, the quantities such as \( q, s, \alpha \) can be reconstructed from Proposition 3.4. In the converse direction, it is reasonable to expect that if \((\tilde{q}(t), s(t), \alpha(t))\) is a solution to the system (3.2) and (3.10), then the reconstructed map \( u(t) \) is a weak solution to (1.13). Hence, we now consider the local-wellposedness of the PDE-ODE system (3.2) and (3.10) as in [9].

For time interval \( I \), we set \( \text{Str}(I) := L^\infty_t L^2_x \cap L^4_t L^4_x \cap L_t^{8/3} L^6_x(I) \). In [9], the following proposition is established.

**Proposition 3.6.** ([9]) (i) There exist \( \delta_0 > 0, \sigma > 0, \) and \( C > 0 \) such that the following holds: For \((q_0, s_0, \alpha_0) \in L^2_x(\mathbb{R}^+ \times \mathbb{R}) \) satisfying \( \|q_0\|_{L^2} \leq \delta_0 \), there exists a solution to the system (3.2) and (3.10): \((q(t), s(t), \alpha(t))\) on the interval \( I = [0, \sigma \delta_0^2] \) which satisfies the following properties:

- \((q(t), s(t), \alpha(t)) \in C(I; L^2_{rad} \times \mathbb{R}^+ \times \mathbb{R})\),
- \((q(0), s(0), \alpha(0)) = (q_0, s_0, \alpha_0)\),
- \( \tilde{q} \in \text{Str}(I) \) and \( \|\tilde{q}\|_{\text{Str}(I)} \leq C \|q_0\|_{L^2} \),
- \( s_0^2 s(t) \in [0.5, 1.5] \) for all \( t \in I \).

(ii) The solution to (3.2) and (3.10) is unique in \( C(I; L^2_{rad}) \cap \text{Str}(I) \times C(\mathbb{R}^+ \times \mathbb{R}) \).

(iii) There exists \( C > 0 \) such that if \((\tilde{q}^i(t), s^i(t), \alpha^i(t)) \in C(I; L^2(\mathbb{R}^2) \times \mathbb{R}^+ \times \mathbb{R}) \) are solutions with initial data \((q_0^i, s_0^i, \alpha_0^i)\) as in (i) for \( i = 1, 2 \) for \( I = [0, \bar{T}] \subset [0, \min_{i=1,2} \sigma(s_0^i)^2] \), then the following difference estimate holds:

\[
\|q^1 - q^2\|_{\text{Str}(I)} + \|s^1 - s^2\|_{L^\infty(I)} + \|\alpha^1 - \alpha^2\|_{L^\infty(I)} \leq C \left( \|q_0^1 - q_0^2\|_{L^2} + |s_0^1 - s_0^2| + |\alpha_0^1 - \alpha_0^2| \right)
\]
Proposition 3.4, we reconstruct $q$. By Proposition 3.6, for initial data $(q, V)$, solution $(\tilde{q}, \tilde{V})$ makes sense for $q \in \mathcal{S}(I)$ via Duhamel formula.

Remark 3.1. The nonlocal term $N(q) \tilde{q}$ can be written as

$$N(q) \tilde{q} = \left( -V(r) + \int_{r}^{\infty} \frac{\partial}{\partial r} V(r') dr' \right) \tilde{q}, \quad V(r) = \frac{|q|^3}{2} + \Re \frac{m}{r} q$$

Thus (3.16) makes sense for $q \in \mathcal{S}(I)$ via Duhamel formula.

3.4 The Proof of (LP1)

Let us see the proof of (LP1). We may assume that $s(u_0) = 1$ by rescaling. By Proposition 3.6 for initial data $(q(u_0), 1, \alpha(u_0))$, there is a unique solution $(q(t), s(t), \alpha(t))$ to (2.2) and (3.10) for $I = [0, \sigma]$ if $\delta < \delta_0$. By Proposition 3.2 we reconstruct $u(t) \in \Sigma_m$ from $(q(t), s(t), \alpha(t))$ for each $t \in I$. By the continuity of reconstruction, $u(t) \in C(I; \Sigma_m)$. It suffices to show:

Claim 1. $u(t)$ is a weak solution to (1.4).

We prove this claim by approximation by smooth solutions. First, by Lemma 7.2 we take the following sequence:

$$\{u^k\}_{k=1}^{\infty} \subset \Sigma_m \cap H^3, \quad \lim_{k \to \infty} u^k = u_0 \text{ in } H^1.$$ Since $s(u_0^k) \to s(u_0) = 1$, we may assume that $s(u_0^k) = 1$ for all $k \in \mathbb{N}$ by rescaling. By McGahagan’s theorem [15], there exists a unique solution $u^k(t) \in L^\infty([0, T_k]; \Sigma_m \cap H^3)$ for each $k \in \mathbb{N}$, and $\lim_{k \to \infty} \|u^k\|_{H^2} = \infty$ if $T_k < \infty$.

For each $k \in \mathbb{N}$ and $t \in [0, T_k)$, we denote $(q^k(t), s^k(t), \alpha^k(t)) = (q(u^k(t)), s(u^k(t)), \alpha(u^k(t)))$. Then Propositions 3.2 and 3.4 imply that $(q^k(t), s^k(t), \alpha^k(t))$ satisfies (3.2) and (3.10). On the other hand, applying Proposition 3.2 to the initial data $(q^k(0), s^k(0), \alpha^k(0)) = (q(u_0^k), 1, \alpha(u_0^k))$, we have another solution $(q^k_W(t), s^k_W(t), \alpha^k_W(t))$. Since both of these belong to the space as in Proposition 3.6 (ii), we have $(q^k(t), s^k(t), \alpha^k(t)) = (q^k_W(t), s^k_W(t), \alpha^k_W(t))$ on $t \in [0, \min\{T_k, \sigma\}]$. Here, we claim that

Claim 2. $T_k \geq \sigma$.

We first prove Claim 1 provided that Claim 2 holds. By Claim 2, $(q^k(t), s^k(t), \alpha^k(t))$ satisfies all the properties in Proposition 3.6. Therefore, by the difference estimate in Proposition 3.6 (iii), we have

$$\|q^k - q\|_{\mathcal{S}(I)} + \|s^k - s\|_{L^\infty(I)} + \|\alpha^k - \alpha\|_{L^\infty(I)} \leq \|q^k(0) - q_0\|_{L^2} + |s^k(0) - s(0)| + |\alpha^k(0) - \alpha(0)| \to 0 \quad (3.17)$$

as $k \to \infty$. Hence, by the continuity of reconstruction (see Proposition 3.3, $\|u^k(t) - u(t)\|_{H^1} \to 0$ as $k \to \infty$ for all $t \in I = [0, \sigma]$. Since all $a^k(t)$ satisfy (1.1), $u(t)$ is a weak solution to (1.1), which is the desired conclusion.

We now return to Claim 2. While the above argument is established by [9], the proof of Claim 2 is not explicitly described in [9]. It is essential to ensure that $T_k$ is bounded from below uniformly in $k$ so that the approximation works. Hence, we provide a proof of this claim here.

Our main ingredients are a priori estimates for modified Schrödinger map equation (3.2). More precisely,
Proposition 3.7. There exists $\delta_0 > 0$ and $C > 0$ such that the following holds: If $u(t) \in C(I; \Sigma_m) \cap L^\infty(I; \dot{H}^2)$ is a solution to (1.1) on interval $I = (\tau, \tau + \sigma)$ for $\tau, \sigma > 0$ which satisfies $\delta := \sqrt{u} - 4\pi m < \delta_0$, then

\[
\|\nabla \tilde{q}\|_{\text{Str}(t)} \leq C(\|\nabla \tilde{q}(\tau)\|_{L^2_t} + (\sigma^{-1} + \sigma^{-3}/2) \|\nabla \tilde{q}\|_{L^2_t}; \Sigma) \leq \|\tilde{q}\|_{L^2_t} \cap L^{5/3}_t L^{5/4}_x, \]

(3.18)

where $\sigma = \inf_{t \in I} s(t)$. Furthermore, if $u(t) \in L^\infty(I; \Sigma_m \cap \dot{H}^3)$, then

\[
\|\Delta \tilde{q}\|_{\text{Str}(t)} \leq C(\|\Delta \tilde{q}(\tau)\|_{L^2_t} + (\sigma^{-3/2}) \|\tilde{q}\|_{L^2_t} \cap L^{5/3}_t L^{5/4}_x + (\sigma^{1/2} \|\nabla \tilde{q}\|_{L^2_t} \cap L^{5/3}_t L^{5/4}_x) \|\Delta \tilde{q}\|_{L^\infty_t L^2_x}).
\]

(3.19)

The former estimate is obtained in [8], Lemma 3.1. What is new is the estimate (3.19), which is concerned with the second derivative of $\tilde{q}$. This provides an a priori bound if we use the following estimate:

Proposition 3.8. There exists $\delta_0 > 0$ and $C > 0$ such that for $u \in \Sigma_m \cap \dot{H}^4$ with $\delta := \sqrt{\varepsilon} - 4\pi m < \delta_0$, we have

\[
\|u\|_{\dot{H}^3} \leq C(\|\Delta \tilde{q}\|_{L^2_t} + \|\tilde{q}\|_{L^6_t} \cap L^{5/3}_t L^{5/4}_x). \quad (3.20)
\]

The $\|u\|_{\dot{H}^2}$ counterpart to (3.20) is obtained in [8], Lemma 4.8. A proof of Propositions 3.7 and 3.8 is given in Section 5.

Proof of Claim 2. We omit the index $k$ for simplicity. Suppose $T < \tau$. We take two numbers $0 < \epsilon < \theta < 1$ and set the interval $J = [T - \theta, T - \epsilon]$. Then, by (3.15) in Proposition 3.7, we have

\[
\|\nabla \tilde{q}\|_{\text{Str}(J)} \leq C \left(\|\nabla \tilde{q}(T - \tau)\|_{L^2_t} + (\tau^{1/2} + \epsilon^{3/4} + \|\tilde{q}\|_{L^2_t} \cap L^{5/3}_t L^{5/4}_x(\tau)) \|\nabla \tilde{q}\|_{\text{Str}(J)} \right). \quad (3.21)
\]

By absolute continuity of integral, there exists $\tau_0$ such that for $\tau \leq \tau_0$,

\[
C(\tau^{1/2} + \epsilon^{3/4} + \|\tilde{q}\|_{L^2_t} \cap L^{5/3}_t L^{5/4}_x(\tau)) < \frac{1}{2}. \quad (3.22)
\]

Therefore, by (3.21) and (3.22), for $\tau \leq \tau_0$, we obtain

\[
\|\nabla \tilde{q}\|_{\text{Str}(J)} \leq 2C \|\nabla \tilde{q}(T - \tau)\|_{L^2_t}. \quad (3.23)
\]

Next, by (3.19) in Proposition 3.7 for $\tau \leq \tau_0$,

\[
\|\Delta \tilde{q}\|_{L^\infty_t \cap L^2_x(\tau)} \leq C(\|\Delta \tilde{q}(T - \tau)\|_{L^2_t} + (\tau^{3/2} + \|\tilde{q}\|_{L^2_t} \cap L^{5/3}_t L^{5/4}_x(\tau)) \|\nabla \tilde{q}\|_{L^2_t}) \|\Delta \tilde{q}\|_{L^\infty_t \cap L^2_x(J)} \|\Delta \tilde{q}\|_{L^2_t} \cap L^{5/3}_t L^{5/4}_x(J)). \quad (3.24)
\]

(Here, we have applied (3.23) with $\tau = \tau_0$.) As in (3.23), there exists $\tau_1 \leq \tau_0$ such that for $\tau \leq \tau_1$,

\[
C(\tau^{3/2} + \|\tilde{q}\|_{L^2_t} \cap L^{5/3}_t L^{5/4}_x(\tau)) \|\Delta \tilde{q}\|_{L^\infty_t \cap L^2_x(J)} + 2C \tau^{3/2} \|\nabla \tilde{q}(T - \tau_0)\|_{L^2_t} \|\Delta \tilde{q}\|_{L^\infty_t \cap L^2_x(J)) < \frac{1}{2}. \quad (3.25)
\]

By (3.22) and (3.23), for $\tau \leq \tau_1$, we obtain

\[
\|\Delta \tilde{q}\|_{L^\infty_t \cap L^2_x(J)} \leq 2C \|\Delta \tilde{q}(T - \tau)\|_{L^2_t} \quad (3.26)
\]

(3.26)
In particular, from (3.29) and (3.30),
\[
\|\nabla \tilde{q}(T - \epsilon)\|_{L^2(\mathbb{R}^2)} \leq 2C \|\nabla \tilde{q}(T - \tau_1)\|_{L^2(\mathbb{R}^2)},
\]
(3.27)
\[
\|\Delta \tilde{q}(T - \epsilon)\|_{L^2(\mathbb{R}^2)} \leq 2C \|\Delta \tilde{q}(T - \tau_1)\|_{L^2(\mathbb{R}^2)}.
\]
(3.28)
Hence, taking limit \(\epsilon \to 0\), we have
\[
\limsup_{t \to T} \|\nabla \tilde{q}(t)\|_{L^2(\mathbb{R}^2)} \leq 2C \|\nabla \tilde{q}(T - \tau_1)\|_{L^2(\mathbb{R}^2)} < \infty,
\]
(3.29)
\[
\limsup_{t \to T} \|\Delta \tilde{q}(t)\|_{L^2(\mathbb{R}^2)} \leq 2C \|\Delta \tilde{q}(T - \tau_1)\|_{L^2(\mathbb{R}^2)} < \infty.
\]
(3.30)
Here, Proposition 3.6 and Sobolev embedding imply
\[
\|u(t)\|_{H^3} \leq C(\|\Delta \tilde{q}(t)\|_{L^2} + \|\nabla \tilde{q}(t)\|_{L^2}^2 + 1),
\]
(3.31)
hence
\[
\limsup_{t \to T} \|u(t)\|_{H^3} \lesssim \limsup_{t \to T} \|\Delta \tilde{q}(t)\|_{L^2} + \limsup_{t \to T} \|\nabla \tilde{q}(t)\|_{H^1} + 1 < \infty,
\]
(3.32)
which leads to a contradiction. Therefore, we obtain \(T \geq \sigma\).
\[\square\]

### 3.5 Energy Conservation, Regularity, Uniqueness and Continuous Dependence

We first note that if the solution \(u(t)\) is in \(C(I; \Sigma_m) \cap L^\infty(I; H^2)\), then (LP3) can be obtained directly by differentiation with respect to \(t\). Thus, the energy conservation (LP3) follows from approximation by smooth solution as in the previous subsection.

Next, we prove regularity propagation (LP4). Let \(u_0 \in \Sigma_m \cap \dot{H}^2\) with \(\delta < \delta_0\). Without loss of generality, we may assume \(s(u_0) = 1\). Lemma 4.5 below implies \(\tilde{q}(u_0) \in H^1(\mathbb{R}^2)\). As in the previous subsection, we take a sequence \(\{u^k\}_{k=1}^\infty \subset \Sigma_m \cap \dot{H}^3\) such that \(u^k \to u_0\) in \(\Sigma_m\). We may assume \(s(u^k) = 1\). As shown above, there exists a unique solution \(u^k(t) \in \Sigma_m \cap \dot{H}^3\) with \(u^k(0) = u^k_0\), where \(\sigma\) is independent of \(k\). If we denote \((\tilde{q}^k(t), s^k(t), \alpha^k(t)) := (\tilde{q}(u^k(t)), s(u^k(t)), \alpha(u^k(t)))\), Proposition 3.2 implies \(\tilde{q}^k(t) \in L^\infty(I; H^2)\) and \(\tilde{q}^k\) satisfies (3.2). Furthermore, Proposition 3.6 implies \(\|\tilde{q}^k\|_{\mathcal{G}([0,\sigma], L^2)} \leq C \delta_0\) and \(s^k(t) \in [0.5, 1.5]\) for all \(k\) (by restricting the sequence to sufficiently large \(k\), if necessary). Combining these with the estimate (3.13), it follows that \(\|\tilde{q}^k\|_{L^\infty([0,\sigma], H^1)}\) is bounded uniformly in \(k\). On the other hand, Proposition 3.6 (iii) implies that \(\|\tilde{q}^k - \tilde{q}\|_{L^\infty([0,\sigma], L^2)} \to 0\) as \(k \to \infty\). Thus, it follows that \(\tilde{q} \in L^\infty([0,\sigma]; H^2)\) (see Proposition 1.4.24 in [1]). By direct differentiation of the integral form of (3.2), we have
\[
\nabla \tilde{q}(t) = e^{-(t-\tau)\Delta} \nabla \tilde{q}_0 - i \int_0^t e^{-i(t-\tau)\Delta} \nabla \Gamma(\tilde{q}) dt,
\]
where
\[
\Gamma(\tilde{q}) = \frac{m(1 + m^2)mv_\gamma - m^2 - 1}{\tau^3} \tilde{q} + \frac{m^2v_\gamma - m}{\tau} \tilde{q} + \tilde{q} N(\tilde{q}).
\]
Since \(\nabla \tilde{q}_0 \in L^2\) and \(\nabla \Gamma(\tilde{q}) \in L^{1/2} L^{5/3}\), which is the consequence of the proof of [3]. Lemma 3.1, the Strichartz estimates provides \(\tilde{q}(t) \in C([0,\sigma]; H^1)\). Hence Lemma 3.1 which is shown later, implies that \(u(t) \in C(I; \Sigma_m \cap \dot{H}^2)\). This is the desired conclusion.

(LP2)' is now immediate from Propositions 3.2, 3.4, 3.5 and 3.6 (LP6)' also follows immediately from Proposition 3.6 (iii), and from Propositions 3.4.
4 Derivation of Modified Schrödinger Map

In this section, we prove Proposition 4.2 by following the argument in Bejenaru and Tataru [3]. Chapter 3. We only prove in the case where \( u(t) \in C(I; \Sigma_m \cap H^2) \), while the case where \( u(t) \in C(I; \Sigma_m) \cap L^\infty(I; H^2) \) is almost parallel and achieved by small modifications.

We introduce some notations here. For a subset \( I \subset (0, \infty) \), define

\[
L^2_e(I) := \left\{ f : (0, \infty) \rightarrow C \mid \| f \|_{L^2_e(I)} := \left( \int_0^\infty |f|^2 r dr \right)^{1/2} < \infty \right\},
\]

and \( L^2_e := L^2((0, \infty)) \). Note that we can write \( \| f \|_{H^1_e}^2 = \| \partial_r f \|_{L^2_e}^2 + m^2 \| \hat{f} \|_{L^2_e}^2 \), where \( \hat{f} \) is defined in Section 3.

Next, we make a few fundamental observations. For \( m \)-equivariant maps \( u = e^{mBR_v(r)} \), we have the equivalence

\[
\| u \|_{H^1} \sim \| v_1 \|_{H^1} + \| v_2 \|_{H^1} + \| \partial_r v_3 \|_{L^2_e}.
\] (4.1)

We can show that the norm \( \hat{H}_1 \) bounds the \( L^\infty \)-norm, i.e.,

\[
\| f \|_{L^\infty} \leq C \| f \|_{\hat{H}_1}
\] (4.2)

for some constant \( C > 0 \). Indeed, for \( f \in \hat{H}_1 \) and \( 0 < r_1 < r_2 < \infty \),

\[
|f^2(r_1) - f^2(r_2)| \leq \frac{1}{2} \int_{r_1}^{r_2} |f(r)| \cdot |\partial_r f(r)| dr \leq \frac{1}{2} \| \partial_r f \|_{L^2_e((r_1, r_2))} \| f \|_{L^2_e((r_1, r_2))},
\] (4.3)

which implies \( \lim_{r \to \infty} f(r) \) exists, and it must be \( 0 \) since \( \hat{f} \in L^2_e \). Taking the limit as \( r_2 \to \infty \), we have (4.2) with \( C = 2^{-1/2} \). (4.2) also implies that for any \( m \)-equivariant map \( u = e^{mBR_v(r)} \in \hat{H}_1 \), \( \lim_{r \to 0} v(r) \) and \( \lim_{r \to \infty} v(r) \) exist and in \( \{ \pm \hat{k} \} \), as we remarked before.

**Lemma 4.1.** (i) Let \( u(x) = e^{mBR_v(r)} \in \hat{H}^1(\mathbb{R}^2; \mathbb{S}^2) \) and \( \hat{v}(\infty) = \hat{k} \). Then there exists a unique function \( \hat{v}(r) \in C((0, \infty); \mathbb{R}^3) \) such that

- \( \hat{v}(r) \) is absolutely continuous on any closed subinterval of \( (0, \infty) \).
- \( \lim_{r \to \infty} \hat{v}(r) = \hat{v}(0, 0, 0) \).
- \( D_r \hat{v} \equiv \partial_r \hat{v} + (\hat{v} \cdot \partial_r \hat{v}) \hat{v} = 0 \) for almost every \( r \in (0, \infty) \).

(ii) Let \( u^{(i)}(x) = e^{mBR_v^{(i)}(r)}(r) \in \hat{H}^1(\mathbb{R}^2; \mathbb{S}^2) \), and suppose \( v^{(i)}(\infty) = \hat{k} \) for \( i = 1, 2 \). And let \( M > 0 \) satisfying \( \| u^{(i)} \|_{\hat{H}_1} \leq M \) for \( i = 1, 2 \). Then, there exists \( C = C(M) \) such that

\[
\| \hat{v}^{(1)} - \hat{v}^{(2)} \|_{\hat{H}_1} \leq C(M) \| u^{(1)} - u^{(2)} \|_{\hat{H}_1},
\] (4.4)

where for \( f = (f_1, f_2, f_3) : (0, \infty) \to \mathbb{R}^3 \),

\[
\| f \|_{\hat{H}_1} := \| \partial_r f \|_{L^2_e} + \| f \|_{L^\infty} + \| \hat{f} \|_{L^2_e}.
\] (4.5)

(iii) Let \( I \subset \mathbb{R} \) be an open interval, and let \( u(t, r) = e^{mBR_v(t, r)} \in C(I; \hat{H}^1(\mathbb{R}^2; \mathbb{S}^2)) \) with \( v(t_0, \infty) = \hat{k} \) for some \( t_0 \in I \). If \( \hat{v}(t) \) is the function as in (i) corresponding to \( u(t) \), then we have \( \hat{v}(t) \in C(I; \hat{H}_1) \).
Remark 4.1. We can obtain the same result by replacing \( \tilde{k} \) by \( -\tilde{k} \). \( D_c \) corresponds to the covariant derivative along the curve \( v(\cdot) : (0, \infty) \to S^2 \).

Proof of Lemma 4.1. The proof of (i) is the same as that in [3], Chapter 3. Since (iii) is immediate from (ii), we only prove (ii).

Proof of Lemma 4.1. The proof of (i) is the same as that in [3], Chapter 3. Since (iii) is immediate from (ii), we only prove (ii).

In the proof, we use the notation \( \delta \cdot \) to describe the difference between \( j = 1 \) and \( j = 2 \) (for example, \( \delta v = v^{(1)} - v^{(2)} \)). We take a partition \( 0 = b_0 < b_1 < \cdots < b_{n-1} < b_n = \infty, n = n(M) \) satisfying

\[
\max_{t=1,2} \left\{ \left\| \partial_r v^{(i)} \right\|_{L_2^2(I_k)} + \left\| \frac{\partial_r v^{(i)}}{r} \right\|_{L_2^2(I_k)} + \left\| \frac{\partial_r v^{(i)}}{r} \right\|_{L_2^2(I_k)} \right\} \leq \frac{1}{8} \tag{4.6}
\]

for \( i = 1, 2 \) and \( k = 1, \ldots, n \) where \( I_k = [b_{k-1}, b_k] \). Then for \( I = I_k \) and \( j = 1, 2 \), we have

\[
\| \partial_r \delta \hat{e}_j \|_{L^{1}(I, dr)} \leq \| -[\delta \hat{e} \cdot \partial_r v^{(1)}]e_j^{(1)} - [\delta \hat{e} \cdot \partial_r v^{(2)}] \delta v_j^{(2)} \|_{L^{1}(I, dr)} \leq \| \delta \hat{e} \|_{L^{\infty}(I)} \left\| \partial_r v^{(i)} \right\|_{L_2^2(I)} \left\| e_j^{(1)} \right\|_{r I} \right\|_{L_2^2(I)} \left\| \partial_r v^{(2)} \right\|_{\delta v_j^{(2)}} \left\| \delta v_j \right\|_{L_2^2(I, dr)} \leq \frac{1}{16} \| \delta \hat{e} \|_{L^{\infty}(I)} + R(I) \tag{4.7}
\]

where we set

\[
R(I') = \max_{j=1,2} \left\{ \left\| \partial_r v^{(i)} \cdot e_j^{(1)} \right\|_{L^{1}(I', dr)} + \left\| \partial_r v^{(2)} \cdot e_j^{(1)} \right\|_{L^{1}(I', dr)} \right\} \tag{4.8}
\]

for interval \( I' \subset \mathbb{R} \). Next, let \( \hat{f} := J \hat{e} \). Then \( \hat{f} \) satisfies \( \partial_r \hat{f} = -\hat{f} \cdot \partial_r v \), and \( \lim_{r \to \infty} \hat{f}(r) = (0, 1, 0) \). Therefore, \( \hat{f} \) possesses the same properties as \( \hat{e} \) (except boundary condition). Thus, (4.7) also holds when we replace \( \hat{e} \) by \( \hat{f} \). Here, we consider the interval \( I = I_n \). From (4.7),

\[
|\delta \hat{e}_j(r)| \leq \int_{r}^{\infty} |\partial_r \delta \hat{e}_j(r')| dr' \leq \| \partial_r \delta \hat{e}_j \|_{L^{1}(I_n, dr')} \leq \frac{1}{16} \| \delta \hat{e} \|_{L^{\infty}(I_n)} + R(I_n) \tag{4.9}
\]

for \( r \in I_n \) and \( j = 1, 2 \). Hence we have

\[
\| \delta \hat{e}_j \|_{L^{\infty}(I_n)} \leq \frac{1}{16} \| \delta \hat{e} \|_{L^{\infty}(I_n)} + R(I_n) \tag{4.10}
\]

for \( j = 1, 2 \), and (4.10) also holds when \( \hat{e} \) is replaced by \( \hat{f} \). To bound the third components, we use the simple relations

\[
\hat{e}_3 = \hat{f}_1 v_2 - \hat{f}_2 v_1, \quad \hat{f}_3 = v \times \hat{e} \tag{4.11}
\]

In particular, we have \( \hat{e}_3 = \hat{f}_1 v_2 - \hat{f}_2 v_1 \). Thus, from (4.10),

\[
\| \delta \hat{e}_3 \|_{L^{\infty}(I_n)} = \left\| \delta \hat{f}_1 v_2^{(1)} + \delta \hat{f}_2 v_1^{(1)} - \delta \hat{f}_1 v_2^{(2)} + \delta \hat{f}_2 v_1^{(2)} \right\|_{L^{\infty}(I_n)} \leq \sum_{j=1}^{2} \| \delta \hat{f}_j \|_{L^{\infty}(I_n)} + \sum_{j=1}^{2} \| \delta v_j \|_{L^{\infty}(I_n)} \leq \frac{1}{8} \| \delta \hat{f} \|_{L^{\infty}(I_n)} + 2R(I_n) + 2C_0 \| \delta v \|_{H^1(\mathbb{R}^2)} \tag{4.12}
\]
where \( C_0 = \pi C \) and \( C \) is the constant in (4.22). Similarly, we have
\[
\|\delta f_3\|_{L^\infty(I_{n-1})} \leq \frac{1}{8} \|\delta \tilde{e}\|_{L^\infty(I_{n-1})} + 2R(I_n) + 2C_0 \|\delta u\|_{\dot{H}^1(\mathbb{R}^2)}.
\] (4.13)

Hence,
\[
\|\delta \tilde{e}\|_{L^\infty(I_{n-1})} + \|\delta f\|_{L^\infty(I_{n-1})} \leq \sum_{j=1}^{3} \left( \|\delta \tilde{e}_j\|_{L^\infty(I_{n-1})} + \|\delta f_j\|_{L^\infty(I_{n-1})} \right)
\leq \frac{1}{4} \left( \|\delta \tilde{e}\|_{L^\infty(I_{n-1})} + \|\delta f\|_{L^\infty(I_{n-1})} \right) + 8R(I_n) + 4C_0 \|\delta u\|_{\dot{H}^1},
\] (4.14)
which implies
\[
\|\delta \tilde{e}\|_{L^\infty(I_{n-1})} + \|\delta f\|_{L^\infty(I_{n-1})} \leq \frac{32}{3} R(I_n) + \frac{16}{3} C_0 \|\delta u\|_{\dot{H}^1}.
\] (4.15)

Next, we consider \( I = I_{n-1} \). For \( r \in I_{n-1} \) and for \( j = 1, 2 \),
\[
|\delta \tilde{e}_j(r)| \leq |\delta \tilde{e}_j(b_{n-1})| + \int_{r'}^{b_{n-1}} |\delta \tilde{e}_j'(r')| \, dr'
\leq \|\delta \tilde{e}_j\|_{L^\infty(I_{n-1})} + \|\delta \tilde{e}_j\|_{L^1(I_{n-1}; dr)}
\leq \frac{32}{3} R(I_n) + \frac{16}{3} C_0 \|\delta u\|_{\dot{H}^1} + \frac{1}{16} \|\delta \tilde{e}\|_{L^\infty(I_{n-1})} + R(I_{n-1}),
\] (4.16)
where the last inequality comes from (4.14) and (4.15). Taking the supremum, we have
\[
\|\delta \tilde{e}_j\|_{L^\infty(I_{n-1})} \leq \frac{1}{16} \|\delta \tilde{e}\|_{L^\infty(I_{n-1})} + \frac{32}{3} R(I_{n-1} \cup I_n) + \frac{16}{3} C_0 \|\delta u\|_{\dot{H}^1}
\] (4.17)
for \( j = 1, 2 \), and (4.17) also holds when we replace \( \tilde{e} \) by \( \hat{f} \). By using (4.11),
\[
\|\delta \tilde{e}_3\|_{L^\infty(I_{n-1})}
\leq \sum_{j=1}^{2} \|\delta \hat{f}_j\|_{L^\infty(I_{n-1})} + \|\delta \tilde{e}_j\|_{L^\infty(I_{n-1})}
\leq \frac{1}{8} \|\delta \hat{f}\|_{L^\infty(I_{n-1})} + \frac{64}{3} R(I_{n-1} \cup I_n) + \left( \frac{32}{3} + 2 \right) C_0 \|\delta u\|_{\dot{H}^1(\mathbb{R}^2)},
\] (4.18)
and
\[
\|\delta \hat{f}_3\|_{L^\infty(I_{n-1})} \leq \frac{1}{8} \|\delta \tilde{e}\|_{L^\infty(I_{n-1})} + \frac{64}{3} R(I_{n-1} \cup I_n)
+ \left( \frac{32}{3} + 2 \right) C_0 \|\delta u\|_{\dot{H}^1(\mathbb{R}^2)},
\] (4.19)
Thus,
\[
\|\delta \tilde{e}\|_{L^\infty(I_{n-1})} + \|\delta f\|_{L^\infty(I_{n-1})} \leq \sum_{j=1}^{3} \left( \|\delta \tilde{e}_j\|_{L^\infty(I_{n-1})} + \|\delta f_j\|_{L^\infty(I_{n-1})} \right)
\leq \frac{1}{4} \left( \|\delta \tilde{e}\|_{L^\infty(I_{n-1})} + \|\delta f\|_{L^\infty(I_{n-1})} \right)
+ \frac{32}{3} R(I_{n-1} \cup I_n) + \frac{9}{3} C_0 \|\delta u\|_{\dot{H}^1},
\] (4.20)
which implies
\[ \| \delta \hat{e} \|_{L^\infty(t_{n-1})} + \| \delta \hat{f} \|_{L^\infty(t_{n-1})} \leq \left( \frac{32}{3} \right)^2 + 12 \cdot \frac{16}{3} C_0 \| \delta u \|_{H^1}. \tag{4.21} \]

We repeat the above argument, then for \( k = 1, \ldots, n \), we have
\[ \| \delta \hat{e} \|_{L^\infty(t_k)} + \| \delta \hat{f} \|_{L^\infty(t_k)} \leq \left( \frac{32}{3} \right)^{n+k-1} R((0, \infty)) + \frac{16}{3} \cdot (12)^{n-k} C_0 \| \delta u \|_{H^1}. \tag{4.22} \]

Therefore,
\[ \| \delta \hat{e} \|_{L^\infty(0, \infty)} = \max_{k=1, \ldots, n} \| \delta \hat{e} \|_{L^\infty(t_k)} \leq \left( \frac{32}{3} \right)^n R((0, \infty)) + \frac{16}{3} \cdot (12)^n C_0 \| \delta u \|_{H^1} \leq C(M) \| \delta u \|_{H^1}. \tag{4.23} \]

Also holds when \( \hat{e} \) is replaced by \( \hat{f} \). As a consequence of the \( L^\infty \) bound, using the relation \( v = \hat{e} \times f \), we obtain
\[ \| \delta v_3 \|_{L^\infty} \leq C(M) \| \delta u \|_{H^1}. \tag{4.24} \]

which implies
\[ \| \delta u \|_{L^\infty} \leq C(M) \| \delta u \|_{H^1}. \tag{4.25} \]

Thus, we have
\[ \| \partial_r \delta v \|_{L^2} \leq \left\| \left[ \delta \hat{e} \cdot \partial_r v^{(1)} \right] v^{(1)} - \left[ \delta \hat{e} \cdot \partial_r \delta v \right] v^{(1)} - \left[ \delta \hat{e} \cdot \partial_r v^{(2)} \right] \delta v_j \right\|_{L^2} \leq \left\| \partial \delta \hat{e} \right\|_{L^\infty} \left\| \partial_r v^{(1)} \right\|_{L^2} + \left\| \partial_r \delta v \right\|_{L^2} + \left\| \partial_r v^{(2)} \right\|_{L^2} \left\| \delta v_j \right\|_{L^\infty} \leq C(M) \| \delta u \|_{H^1}. \tag{4.26} \]

On the other hand, since we have \( \hat{v} = \hat{f} \hat{v} - \hat{f}_1 \hat{v} \) by \( 11 \), similar argument yields
\[ \left\| \frac{\delta \hat{e}}{r} \right\|_{L^2} \leq C(M) \| \delta u \|_{H^1}, \tag{4.27} \]

which is the desired conclusion. \( \Box \)

Here, let us introduce a new function space. Set Hilbert space \( H_{e}^1 := L^2 \cap H^1 \) with inner product
\[ \left\langle f, g \right\rangle_{H_{e}^1} := (f, g)_{L^2} + (\partial_r f, \partial_r g)_{L^2} + \left( \frac{f}{r}, \frac{g}{r} \right)_{L^2}. \tag{4.28} \]

Then, we define
\[ H_{e}^{-1} := (H_{e}^1)^* \tag{4.29} \]
\[ \| f \|_{H_{e}^{-1}} := \sup_{g \in H_{e}^1} \left\langle f, g \right\rangle_{H_{e}^{-1}, H_{e}^1}. \tag{4.30} \]

This space is introduced so that \( \partial_r f \) and \( \frac{1}{r} \) are accommodated for \( f \in L^2 \). Such kind of space is also considered in [2], which, however, uses \( H_{e}^{-1} \) defined as the dual space of \( H_{e}^1 \). The reason why we take account of the \( L^2 \)-finiteness in our definition is the compatibility with the space \( H^{-1}(\mathbb{R}^3) \), which will be seen in Lemma L7.

Although \( H_{e}^{-1} \) penetrates the space of distribution, some calculations such as differentiation and multiplication are permitted under certain restrictions.
Lemma 4.2. (i) For \( f \in L^2 \), we can define \( \partial_r f \in H^{-1}_r \) and \( \frac{f}{r} \in H^{-1}_r \) by
\[
⟨\partial_r f, φ⟩_{H^{-1}_r, H^1_r} := -⟨f, \partial_r φ + \frac{φ}{r}⟩_{L^2}, \quad (4.31)
\]
\[
⟨\frac{f}{r}, φ⟩_{H^{-1}_r, H^1_r} := -⟨f, \frac{φ}{r}⟩_{L^2}. \quad (4.32)
\]
Moreover, we have
\[
∥\partial_r f∥_{H^{-1}_r} ≤ C ∥f∥_{L^2} , \quad ∥\frac{f}{r}∥_{H^{-1}_r} ≤ ∥f∥_{L^2}. \quad (4.33)
\]
(ii) Let \( g \in L^∞((0, ∞)) \) with \( \partial_r g \in L^2 \). Then, for \( f \in H^{-1}_r \), we can define \( gf \in H^{-1}_r \) by
\[
⟨gf, φ⟩_{H^{-1}_r, H^1_r} := ⟨f, gφ⟩_{H^{-1}_r, H^1_r}. \quad (4.34)
\]
Moreover, we have
\[
∥gf∥_{H^{-1}_r} ≤ C ∥f∥_{H^{-1}_r} \left(∥g∥_{L^∞} + ∥\partial_r g∥_{L^2}\right). \quad (4.35)
\]
(iii) For \( f \in L^2 \) and \( g \in L^∞((0, ∞)) \) with \( \partial_r g \in L^2 \), the Leibniz rule holds for \( fg \). Namely, \( \partial_r (fg) = \partial_r f \cdot g + f \cdot \partial_r g \).

Proof. (i) is immediate. For \( g \in L^∞ \) with \( \partial_r g \in L^2 \) and \( φ \in H^{-1}_r \),
\[
∥gφ∥_{H^1_r} ≤ C \left(∥g∥_{L^∞} + ∥\partial_r g∥_{L^2}\right) ∥φ∥_{H^1_r}, \quad (4.36)
\]
which leads to (ii). (iii) follows easily from (i) and (ii).

Lemma 4.3. In addition to the conditions in Lemma 4.1 (iii), we further assume that \( u(t, x) \) is weakly differentiable with respect to \( t \) (i.e., \( \partial_t u \in L^1_{loc}(I × \mathbb{R}^2) \), and that \( \partial_r u \in C(I; L^2(\mathbb{R}^2)) \). Then,
\[
(i) \ u(t, x) − u(t_0, x) \in C^1(I; L^2(\mathbb{R}^2)) \text{ for any fixed } t_0 \in I.
\]
\[
(ii) \ ∂_t u(t) \in C^1(I; H^{-1}_r) \text{ and }
\]
\[
∂_r ∂_t u(t) = ∂_t ∂_r u(t). \quad (4.37)
\]
\[
(iii) \ ∂_r \tilde{e}(t) − \tilde{e}(t_0) \in C^1(I; L^2) \text{ for any fixed } t_0 \in I. \text{ Furthermore, if } \sup_{t \in I} ∥u∥_{H^1} < ∞, \text{ then }
\]
\[
∥∂_r \tilde{e}∥_{L^∞(I; L^2)} ≤ C ∥∂_r u∥_{L^∞(I; L^2)} \quad (4.38)
\]
for some \( C = C(\sup_{t \in I} ∥u∥_{H^1}) > 0 \).

Proof. (i) By assumptions, the map \( t \mapsto u(t, x) \) is absolutely continuous for almost every \( x \in \mathbb{R}^2 \), and its differentiation \( \frac{du}{dt}(t, x) \) coincides with the weak derivative of \( u \) with respect to \( t \) (See [12], Problem 7.8, for example). Therefore, for \( s, t \in I \),
\[
u(t, x) − u(s, x) = \int_s^t ∂_t u(t', x)dt' \quad \text{for almost every } x \in \mathbb{R}^2, \quad (4.39)
\]
which implies \( u(t) − u(s) \in C^1(I; L^2(\mathbb{R}^2)) \).

(ii) By pointwise relation (4.39), we have
\[
v(t, r) − v(s, r) = \int_s^t ∂_r v(t', r)dt' \quad \text{for almost every } r \in (0, ∞). \quad (4.40)
\]
We now interpret (4.40) as a relation in $L^2$. (The right hand side is considered as the Bochner integral.) Since $\partial_t : L^2 \to H^{-1}$ is bounded, we have

$$\partial_t v(t) - \partial_t v(s) = \int_s^t \partial_t \partial v(t') dt',$$  \hfill (4.41)

which implies (ii).

(iii) We may replace $I$ by an interval contained compactly in $I$, and thus maybe assume $M := \sup_{t \in I} \|u\|_{\tilde{H}^1} < \infty$. To take care of integrability, we perform the proof via approximations. Let $I' \subseteq I$ be arbitrary intervals. Applying Lemma 7.2, we take a sequence $(u_n(t))_{n=1}^{\infty} \subseteq C^\infty(I' \times \mathbb{R}^2)$ satisfying (A), (B), and (C) in Lemma 7.2. And then let $\hat{e}_n(t)$ be the function in Lemma 4.1 (i) corresponding to $u_n(t)$. The elementary ODE theory yields that $\hat{e}_n(t,r) \in C^\infty(I' \times (0,\infty))$, and that there exists $R_n > 0$ such that if $r \geq R_n$, then $\hat{e}_n(t,r) = \psi(h_1(r),0,-h_1(r))$ for all $t \in I'$.

Furthermore, (4.3) implies

$$\sup_{t \in I'} \|\hat{e}_n(t)\|_{\tilde{H}_B^1} \leq C(M) \sup_{t \in I'} \|u(t) - u_n(t)\|_{\tilde{H}^1} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \hfill (4.42)$$

In particular, there exists $C_1(M) > 0$ such that $\sup_{t \in I'} \|\partial_t u_n(t)\|_{L^2} + \sup_{t \in I'} \|\partial_t \hat{e}_n(t)\|_{L^2} \leq C_1(M)$ for sufficiently large $n$.

Here, we use the following inequality:

**Lemma 4.4.** For $R_0 > 0$ and $f : (0,\infty) \to \mathbb{C}$, the following holds:

$$\left\| \int_0^\infty f(r') dr' \right\|_{L^2_{\tilde{H}((0,R_0) \cup (R_0,\infty))}} \leq \|f\|_{L^2_{\tilde{H}}((R_0,\infty))}. \hfill (4.43)$$

For the proof, see Lemma 4.1 in [10].

To the moment, we write $\hat{e}, u$ instead of $\hat{e}_n, u_n$ for abbreviation, respectively. We take $t, s \in I'$ and use the notation $\delta \cdot$ as the difference between the values at $t$ and at $s$ ($\delta \hat{e} = \hat{e}(t) - \hat{e}(s)$, for example). Then

$$\delta \hat{e} = -\int_r^\infty (\delta \hat{e} \cdot \partial_t v(t)) v(t) + (\hat{e}(s) \cdot \delta \partial_t v(t) + (\hat{e}(s) \cdot \partial_s v(s)) \delta v) dr'$$

$$= \left( (\hat{e}(s) \cdot \partial_t v(t)) (r) + \int_r^\infty (\delta \hat{e} \cdot \partial_s v(t)) v(t) + (\partial_t \hat{e}(s) \cdot \delta v) v(t) \right)$$

$$\quad + (\hat{e}(s) \cdot \delta v) \partial_s v(t) - (\hat{e}(s) \cdot \partial_s v(s)) \delta v \right) dr' \hfill (4.44)$$

where we have used integration by part. By Lemma 4.3 for $R > 0$, we have

$$\|\delta \hat{e}\|_{L^2_{\tilde{H}}(R,\infty)} \leq \|\delta \hat{e} \cdot \partial_t v(t)\|_{L^1_{\tilde{H}}(R,\infty)}$$

$$+ \left( 1 + \|\partial_s \hat{e}(s)\|_{L^2} + \|\partial_s v(s)\|_{L^2} + \|\partial_t v(t)\|_{L^2} \right) \|\delta v\|_{L^2} \cdot \hfill (4.45)$$

Here, we take a partition $0 = b_0 < b_1 < \cdots < b_{k-1} < b_k = \infty$, $k = k(M)$, such that $\|\partial_t v(t)\|_{L^2((b_k,b_{k+1}))} \leq \frac{1}{2}$. (Note that $k$ is independent of $n$.)

When $R = b_{k-1}$, (4.45) and Hölder’s inequality give

$$\|\delta \hat{e}\|_{L^2((b_k))} \leq \frac{1}{2} \|\delta \hat{e}\|_{L^2((b_k))} + (1 + 3C_1(M)) \|\delta v\|_{L^2} \hfill (4.46)$$

which implies

$$\|\delta \hat{e}\|_{L^2((b_k))} \leq 2(1 + 3C_1(M)) \|\delta v\|_{L^2} \hfill (4.47)$$
When \( R = b_{k-2} \), (4.45) implies
\[
\|\delta\hat{t}\|_{L^2_{\xi}(t_{k-1}\cup t_k)} \leq \sum_{j=k-1}^k \frac{1}{2} \|\delta\hat{t}\|_{L^2_{\eta}(t_j)} + (1 + 3C_1(M)) \|\delta v\|_{L^2},
\]  
(4.48)
which implies
\[
\|\delta\hat{t}\|_{L^2_{\xi}(t_{k-1})} \leq 4(1 + 3C_1(M)) \|\delta v\|_{L^2}. 
\]  
(4.49)
Repeating this argument and undoing the abbreviation, we obtain
\[
\|\hat{e}_n(t) - \hat{e}_n(s)\|_{L^2} \leq C_2(M) \|\delta v_n\|_{L^2} \leq C_2(M) \int_s^t \|\partial_t v_n(t')\|_{L^2} \, dt',
\]  
(4.50)
for some constant \( C_2(M) \). It follows from (4.42) and (4.50) that \( \hat{e}(t) - \hat{e}(s) \) in \( L^2 \) and \( \hat{e}_n(t) - \hat{e}_n(s) \to \hat{e}(t) - \hat{e}(s) \) in \( L^2 \). Moreover, (4.50) gives
\[
\|\hat{e}(t) - \hat{e}(s)\|_{L^2} \leq \liminf_{n\to\infty} \|\hat{e}_n(t) - \hat{e}_n(s)\|_{L^2} 
\leq C_2(M) \liminf_{n\to\infty} \int_s^t \|\partial_t v_n(t')\|_{L^2} \, dt' 
= C_2(M) \int_s^t \|\partial_t v(t')\|_{L^2} \, dt'.
\]  
(4.51)
Since \( I' \subseteq I \) is arbitrary, this holds for all \( t, s \in I \). This implies \( \hat{e}(t) - \hat{e}(t_0) \in W^{1,\infty}(I, L^2) \) for any fixed \( t_0 \in I \) (See Theorem 1.4.40 in [1], for example), and \( \|\partial_t \hat{e}\|_{L^{1,\infty}(I, L^2)} \leq C(M) \|\partial_t v\|_{L^{1,\infty}(I, L^2)} \).

Next, we check the continuity of \( \partial_t \hat{e} \). Since it has turned out that \( \hat{e}(t) - \hat{e}(s) \ma \in L^2 \) for all \( t, s \in I \), it follows that \( v \) and \( \hat{e} \) satisfy (4.44). Dividing (4.44) by \( t - s \) and taking the limit as \( s \to t \), we obtain
\[
\partial_t \hat{e}(t) = \left[\left[(\hat{e}(t) \cdot \partial_t v(t))v(t)\right]\right](r) + \int_r^\infty \left[-(\partial_t \hat{e}(t) \cdot \partial_t v(t))v(t) + (\partial_t \hat{e}(t) \cdot \partial_t v(t))v(t) \right. 
\]  
+ (\hat{e}(t) - \partial_t v(t))\partial_t v(t) - (\hat{e}(t) - \partial_t v(t))\partial_t v(t) \right] \, dr'
\]  
(4.52)
for almost all \( t \in I \) and for all \( n \in \mathbb{N} \). It also holds that \( v_n \) and \( \hat{e}_n \) satisfy (4.52) when \( v, \hat{e} \) are replaced by \( v_n, \hat{e}_n \), respectively. By taking the difference, the same argument provides
\[
\|\partial_t \hat{e}_n(t) - \partial_t \hat{e}_n(t)\|_{L^2} \leq C(M) \|\partial_t v_n(t) - \partial_t v_n(t)\|_{L^2},
\]  
(4.53)
\[
\|\partial_t \hat{e}(t) - \partial_t \hat{e}(t)\|_{L^2} \leq C(M) \|\partial_t v(t) - \partial_t v(t)\|_{L^2},
\]  
(4.54)
for almost all \( t \) and for sufficiently large \( n, n' \). (Here, \( C(M) \) is a constant independent of \( t \) and \( n \).) Hence by the property (C)’ in Lemma 1.7.2 we have \( \hat{e}_t \in C(I; L^2) \).

We now turn to consider a solution to (4.41); \( u(t) \in C(I; \Sigma_m \cap \bar{H}^2) \), where \( I \subseteq R \) is an open interval. Since (4.22) implies \( u(t) \in C(I; L^\infty(\mathbb{R}^2)) \), it follows from (4.41) that \( \partial_t u(t) \in C(I; L^2(\mathbb{R}^2)) \), which enables us to apply Lemma 1.3.1.

Here, note that a map \( u \in \Sigma_m \) belongs to \( \bar{H}^2 \) if and only if \( \Delta u \in L^2 \), hence
\[
H_m v_j, \quad H_0 v_3 \in L^2, \quad (j = 1, 2)
\]  
(4.55)
where

\[ H_k := \partial_r + \frac{1}{r} \partial_r - \frac{k^2}{r^2} \quad \text{for } k \in \mathbb{N}. \]  

(4.56)

It is known that there are equivalences

\[ \|H_s f\|_{L^2_k} \sim \|\partial_r f\|_{L^2_k} + \left\| \frac{\partial r f}{r} \right\|_{L^2_k} + \left\| f \right\|_{L^2_k} \quad \text{for } k \geq 2, \]  

(4.57)

\[ \|H_s f\|_{L^2_k} \sim \left\| \partial_r f\right\|_{L^2_k} + \left\| \frac{\partial r f}{r} - \frac{f}{r^2} \right\|_{L^2_k}. \]  

(4.58)

We can show these either by direct calculations or by using Hankel transform, for which we refer to [2]. Note that (4.57) does not hold for \( k = 1 \). Thus, we have to be careful in the 1-equivariant case. Denoting \( J = J^{(r,t)} = v(r,t) \times \cdot \), we define

\[ W(r,t) := \partial_r v - \frac{m}{r} J_r v \]  

(4.59)

\[ q(r,t) \equiv q_1(r,t) + iq_2(r,t) := W \cdot (\dot{e} + i \dot{\tilde{e}}), \]  

(4.60)

\[ p(r,t) \equiv p_1(r,t) + ip_2(r,t) := \partial_r v \cdot (\dot{e} + i \dot{\tilde{e}}), \]  

(4.61)

\[ \nu(r,t) \equiv \nu_1(r,t) + i \nu_2(r,t) := J_r v \cdot (\dot{e} + i \dot{\tilde{e}}). \]  

(4.62)

Besides these notations, we define

\[ \alpha(r,t) := \partial_t \dot{e} \cdot \dot{J} \tilde{e} = D_t \dot{e} \cdot J \tilde{e}, \]  

(4.63)

or equivalently \( D_t \dot{e} = \alpha \dot{J} \tilde{e} \).

The following lemma is concerned with the regularity of each quantity.

**Lemma 4.5.**

(i) \( W \in C(I; H^1_r) \cap C^1(I; H^{-1}_r). \)

(ii) \( q \in C(I; H^1_r) \cap C^1(I; H^{-1}_r). \)

(iii) \( p \in C(I; L^2). \)

(iv) \( \nu \in C(I; \hat{H}^1_r). \)

(v) \( \alpha \in C(I; L^1). \)

**Proof.** (iii), (iv), and (v) are immediate from Lemma 4.3 and (4.22). Now we prove (i) by following [2] partially. \( W \in C(I; L^2) \) follows from the definition. When we write

\[ W = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} \partial_r v_1 + \frac{m}{r} v_3 v_1 \\ \partial_r v_2 + \frac{m}{r} v_3 v_2 \\ \partial_r v_3 - \frac{m}{r} (v_1^2 + v_2^2) \end{pmatrix}, \]  

(4.64)

we have

\[ \partial_r W_j = v_{j \theta} + m v_3 \left( \frac{v_{1 \theta}}{r} \right) + m v_3 v_j \quad \text{for } j = 1, 2, \]  

(4.65)

\[ \partial_r W_3 = v_{3 \theta} + m \left( \frac{v_1^2}{r} + \frac{v_2^2}{r} \right) + 2m v_3 v_{3 \theta}. \]  

(4.66)

Thus, to derive \( \partial_r W \in C(I; L^2), \) it suffices to show that \( v_{1 \theta}^2 + v_{2 \theta}^2 \in C(I; L^2). \)

We first note that \( \frac{v_{1 \theta}^2 + v_{2 \theta}^2}{r^2} = \frac{v_{1 \theta} v_{1 \theta} + v_{2 \theta} v_{2 \theta}}{r^2} \), it suffices to show that \( \frac{v_{1 \theta}}{r} \) and \( \frac{v_{2 \theta} v_{3 \theta}}{r} \). Hence,

\[ \frac{v_{1 \theta}^2 + v_{2 \theta}^2}{r^2} = \frac{v_{1 \theta} v_{1 \theta} + v_{2 \theta} v_{2 \theta}}{r} - v_1 \left( \frac{v_{1 \theta}}{r} \right) - v_2 \left( \frac{v_{2 \theta}}{r} \right). \]  

(4.67)
which is in $C(I; L^2_r)$. This also implies $\frac{W_j}{r} \in C(I; L^2_r)$. For $j = 1, 2$, we can write
\begin{equation}
W_j = \frac{\partial_x v_j}{r} - \frac{v_j}{r^2} + \frac{1 + v_3}{r^2} v_j + \frac{m - 1}{r^2} v_3 v_j, \tag{4.68}
\end{equation}
In order to derive $\frac{W_j}{r} \in C(I; L^2_r)$, it suffices to show that $\frac{\partial_x v_j}{r} \in C(I; L^2_r)$. Since $v_3(0, t) = -1$ for all $t \in I$, there exists $R_0$ such that $v_3(r, t) < 0$ for all $r \in (0, R_0]$ and for all $t \in I$. Then,
\begin{equation}
\frac{1 + v_3}{r^2} = \frac{1 + v_3}{r^2} (\chi_{(0, R_0)} + \chi_{(R_0, \infty)}) \\
\leq \frac{1 - v_3^2}{r^2} + \frac{1}{r^2} \chi_{(R_0, \infty)} = \frac{v_1^2 + v_3^2}{r^2} + \frac{1}{r} \chi_{(R_0, \infty)}, \tag{4.69}
\end{equation}
which implies $\frac{\partial_x v_j}{r} \in L^2_r$ for all $t \in I$. (Here $\chi_A$ represents the characteristic function for some set $A$.) The continuity with respect to $t$ also follows easily from the expression in (4.69). Hence, $W \in C(I; H^{-1}_r)$.

To derive $W \in C^4(I; H^{-1}_r)$, it suffices to show that $\frac{\partial_x v_j}{r}$ and $\frac{v_1^2 + v_3^2}{r^2}$ are in $C^4(I; H^{-1}_r)$ for $j = 1, 2$. By Lemma [4.2] it is reduced to showing that $v_j(t)v_k(t) - v_j(t)v_k(t) \in C^4(I; L^2_r)$ for some fixed $t_0 \in I$ and $j, k = 1, 2, 3$, which follows immediately from Lemma [4.3]. Hence (ii) is achieved.

We now prove (ii). $q \in C(I; H^2_r)$ is easily derived from (i) and Lemma [4.4]. Let $t \in I$ and $h > 0$ with $t + h \in I$. By definition, we have
\begin{equation}
\frac{q_1(t + h) - q_1(t)}{h} = \frac{W(t + h) - W(t)}{h} \cdot \frac{\partial_t \tilde{q}(t)}{h}, \tag{4.70}
\end{equation}
From (i) and Lemmas [4.1] and [4.2], the first term tends to $\partial_t W(t) \cdot \partial_t \tilde{q}(t)$. The second term converges to $W(t) \cdot \partial_t \tilde{q}(t)$ by (i) and Lemma [4.5]. The argument for $q_2$ is similar. Hence $q \in C^4(I; H^{-1}_r)$. 

Let us move on to the derivation of Lemma [4.4]. The outline is based on [5]. The original equation [4.1] gives, in $L^2_r$ relation,
\begin{equation}
\partial_t v = \nu \left( \partial_r v + \frac{1}{r} \partial_r v + \frac{m^2}{r^2} R^2 v \right) \\
= J \left( D_r W + \frac{1}{r} W - \frac{m}{r} v_3 W \right). \tag{4.71}
\end{equation}
This implies
\begin{equation}
p = i \left( q_r + \frac{q}{r} - \frac{m}{r} v_3 q \right), \tag{4.72}
\end{equation}
in $L^2_r$ relation. Next, by Lemmas [4.2] and [4.3], we have
\begin{equation}
\partial_t W = \partial_r \partial_t v + \frac{m}{r} (\partial_t v_3 + v_3 \partial_t v), \tag{4.73}
\end{equation}
in $H^{-1}_r$ relation. Hence, considering Lemma [4.2] (ii) and (iii), we have
\begin{equation}
\partial_t W \cdot (\tilde{e} + iJ \tilde{e}) = \left( \partial_r \partial_t v + \frac{m}{r} v_3 \partial_t v \right) \cdot (\tilde{e} + iJ \tilde{e}) = \partial_r p + \frac{m}{r} v_3 p. \tag{4.74}
\end{equation}
On the other hand, we obtain
\begin{equation}
\partial_t q = \partial_t W \cdot (\tilde{e} + iJ \tilde{e}) + W \cdot D_r (\tilde{e} + iJ \tilde{e}) = \partial_t W \cdot (\tilde{e} + iJ \tilde{e}) - i\alpha q. \tag{4.75}
\end{equation}
Therefore,
\begin{equation}
\partial_t q + i\alpha q = \partial_t p + \frac{m}{r} v_3 p. \tag{4.76}
\end{equation}
in $H^{-1}_e$ relation. In turn, direct calculations give

\[ D_tD_t \hat{e} = \partial_t D_t \hat{e} + (D_t \hat{e} \cdot \partial_r) v \]
\[ = \partial_t \partial_r \hat{e} + (\partial_r \hat{e} \cdot \partial_r v) + (\hat{e} \cdot \partial_r \partial_r v) + (\partial_r \hat{e} \cdot \partial_r v) v \]

(4.77)
in $H^{-1}_e$ relation. Similarly, we have

\[ D_tD_t \hat{e} = (\partial_t \hat{e} \cdot \partial_r v) + (\hat{e} \cdot \partial_r \partial_r v) + (\partial_r \hat{e} \cdot \partial_r v) v \]

(4.78)
in $H^{-1}_e$ relation. Substituting (4.77) from (4.78), we obtain

\[ D_tD_t \hat{e} - D_tD_e \hat{e} = (\hat{e} \cdot \partial_r v) \partial_r v - (\hat{e} \cdot \partial_r v) \partial_r v, \]

(4.79)
where we use the $H^{-1}_e$ identity \( \partial_t \partial_r \hat{e} = \partial_t \partial_r \hat{e} \), which is shown in the same manner as Lemma 4.3 (ii). On the other hand, we have \( D_t \hat{e} = 0 \) and \( D_t D_t \hat{e} = D_t (aJ \hat{e}) = \alpha_r J \hat{e} \) in $H^{-1}_e$. Substituting these for (4.79), we have

\[ \alpha_r J \hat{e} = (\hat{e} \cdot \partial_r v) \partial_r v - (\hat{e} \cdot \partial_r v) \partial_r v. \]

(4.80)
Note that both sides of (4.80) belong to the class $L^1$, while we have carried out the calculation in $H^{-1}_e$. In particular, it makes sense to consider the inner product of (4.80) and $J \hat{e}$ in $\mathbb{R}^3$ pointwise, which leads to

\[ \alpha_r = (\hat{e} \cdot \partial_r v) (\partial_r J \hat{e}) - (\hat{e} \cdot \partial_r v) (\partial_r J \hat{e}) = \text{Im} \left[ \left( q + \frac{m}{r} \right) \right]. \]

(4.81)
By (4.72),

\[ \alpha_r = -\text{Re} \left[ \left( q + \frac{m}{r} \right) \left( q_r + \frac{1 - mv^2}{r} q \right) \right]. \]

(4.82)
Keeping in mind that (4.82) is a relation in $L^1$, we integrate (4.81) with respect to $r$ and obtain

\[ \alpha(r) = \text{Re} \int_r^\infty \left( q + \frac{m}{r} \right) \left( q_r + \frac{1 - mv^2}{r} q \right) dr, \]

(4.83)
which coincides with $N(q)$ defined in (4.3). On the other hand, by substituting (4.72) for (4.70), we have

\[ \partial_r q + i\alpha q = i \left\{ q_r + \frac{q_r - \frac{m}{r} \left( 1 - \frac{mv^2}{r} \right) q - \frac{mv^2}{r} q} {\frac{1 - mv^2}{r} q} \right\}. \]

(4.84)
in $H^{-1}_e$ relation. This is the prototype of (8.2), which represents the relation of the radial components of $\tilde{q}$.

Here, we need to justify the multiplication of elements in $H^{-1}_e$ by $e^{i(m+1)\theta R}$ in order to derive the relation of $\tilde{q} = e^{i(m+1)\theta} q$. For $f \in L^2_2$, we define $S(f) \in L^2_2(\mathbb{R}^2)$ by $S(f)(x) := e^{i(m+1)\theta} f(r)$, where $(r, \theta)$ is the polar coordinates of $x$. It is trivial that $\|S(f)\|_{L^2_2(\mathbb{R}^2)} = \sqrt{2\pi} \|f\|_{L^2_2}$. Note that $H_{m+1}^1 : H^1_e \rightarrow H^{-1}_e$ is bounded.

**Lemma 4.6.** (i) $f \in H^1_e$ if and only if $S(f) \in H^1(\mathbb{R}^2)$, and we have

\[ \|f\|_{H^1_1} \sim \|S(f)\|_{H^1(\mathbb{R}^2)}. \]

(ii) $H_{m+1} - 1 : H^1_e \rightarrow H^{-1}_e$ is invertible.

(iii) If $f \in L^2_2$, then $S((H_{m+1} - 1)^{-1} f) \in H^2(\mathbb{R}^2)$.

(iv) For $f \in L^2_2$, we have $(\Delta - 1)S((H_{m+1} - 1)^{-1} f) = S(f)$.
Then, \( F = F_{\theta_0, \gamma}^{\alpha, r_1} := \{(r \cos \theta, r \sin \theta) | r \in (r_0, r_1), \theta - \theta_0 \in (-\gamma, \gamma)\}. \) (4.85)

For a fan-shaped set \( F \), the polar coordinates transformation \( \psi : (x, y) \mapsto (r, \theta) \) is a \( C^\infty \)-diffeomorphism. Since \( S(f) \circ \psi^{-1} \) is weakly differentiable on \( \psi^{-1}(F) = (r_0, r_1) \times (\theta_0 - \gamma, \theta_0 + \gamma) \), we have \( S(f) \in W^1(F) \) and

\[
\partial_r S(f) = \frac{x}{\sqrt{x^2 + y^2}} S(\partial_r f) - i(m + 1) \frac{y}{x^2 + y^2} S(f),
\]

(4.86)
\[
\partial_\theta S(f) = \frac{y}{\sqrt{x^2 + y^2}} S(\partial_\theta f) + i(m + 1) \frac{x}{x^2 + y^2} S(f)
\]

(4.87)
on \( F \) (see [12] for example). Note that this expression does not depend on the choice of \( F \). We define \( g_x, g_y : \mathbb{R}^2 \to \mathbb{C} \) by the right hand sides of (4.80) and (4.87) for \( (x, y) \in \mathbb{R}^2 \setminus \{0\} \), respectively, and \( g_x(0) = g_y(0) = 0 \). Obviously, \( g_x, g_y \in L^2(\mathbb{R}^2) \), and thus \( S(f), g_x \) and \( g_y \) belong to the space of tempered distributions \( S' = S'(\mathbb{R}^2) \). It suffices to show that \( \partial_r S(f) = g_x \) and \( \partial_\theta S(f) = g_y \). We only observe the former equality. From the above argument, for \( \varphi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}) \), we have

\[
\langle S(f), \partial_r \varphi \rangle_{S', S} = -\langle g_x, \varphi \rangle_{S', S}, \quad \langle S(f), \partial_\theta \varphi \rangle_{S', S} = -\langle g_y, \varphi \rangle_{S', S}.
\]

(4.88)

Indeed, if we take a finite cover of \( \text{supp} \varphi \) which consists of fan-shaped sets, and take a partition of unity subordinate to it, then (4.88) follows from (4.80) and (4.87). It further follows that (4.88) holds for \( \varphi \in \mathcal{S}(\mathbb{R}^2) \) with \( \mathcal{S}(\text{supp} \varphi) \) containing a neighborhood of origin. Indeed, let \( \eta_0 \) be a \( C_0^\infty(\mathbb{R}^2) \)-function which is 1 for \( |x| \leq 1 \), 0 for \( |x| \geq 2 \), and \( 0 \leq \eta(x) \leq 1 \) for all \( x \in \mathbb{R}^2 \). And let \( \eta_j := \eta_0(\cdot/2^j) - \eta_0(\cdot/2^{j+1}) \) for \( j \in \mathbb{N} \). Then, for the above \( \varphi \), we have \( \left( \sum_{j=0}^{\infty} \eta_j \right) \varphi \to \varphi \) in \( S \) as \( J \to \infty \), which implies (4.88).

Hence, it follows that \( \mathcal{F}[\partial_r S(f) - g_x] \) is a polynomial, where \( \mathcal{F} \) is the Fourier transform. However, since \( (1 + |\xi|)^{-1} \mathcal{F}[\partial_r S(f) - g_x](\xi) \in L^2(\mathbb{R}^2) \), it must be 0, which is the desired conclusion.

(ii) We define bilinear form \( F : H^1_0 \times H^1_0 \to \mathbb{R} \) by

\[
F(f, g) := -\langle (H_{m+1} - 1)f, g \rangle_{H^{-1, 1}}.
\]

Then, \( F \) is bounded and coercive, and thus \( H_{m+1} - 1 \) is invertible by Lax-Milgram’s theorem.

(iii), (iv) Let \( \psi := (H_{m+1} - 1)^{-1} f \). Then we have

\[
\psi_{rr} + \frac{1}{r} \psi_r - \frac{(m + 1)^2}{r^2} \psi = f + \psi \quad \text{in} \quad D'((0, \infty)).
\]

(4.89)

Since \( f + \psi \in L^2_{loc}((0, \infty)) \), it holds that \( \psi \in H^2_{loc}((0, \infty)) \) (see Theorem 8.8 in [12] for example). This implies that \( S(\psi) \in W^1(F) \) for every fan-shaped set \( F \), and

\[
\Delta S(\psi) = S(\psi_{rr} + \frac{1}{r} \psi_r - \frac{(m + 1)^2}{r^2} \psi) = S(f) + S(\psi)
\]

(4.90)
in \( F \). By the same argument as (i), it follows that \( \mathcal{F}[\Delta S(\psi) - S(f) - S(\psi)] \) is a polynomial. From (i), we have \( S(\psi) \in H^1(\mathbb{R}^2) \). Thus, \( (1 + |\xi|)^{-1} \mathcal{F}[\Delta S(\psi) - S(f) - S(\psi)](\xi) \in L^2(\mathbb{R}^2) \), which implies \( \Delta S(\psi) = S(f) + S(\psi) \) in \( S' \). In particular, \( \Delta S(\psi) \in L^2(\mathbb{R}^2) \), thus (iii) and (iv) follows.

Considering Lemma [7.6] for \( f \in H^{-1} \), we define

\[
S(f) := (\Delta - 1) S((H_{m+1} - 1)^{-1} f).
\]

(4.91)
Lemma 4.7. (i) $S$ is a bounded linear operator from $H^{-1}_e$ to $H^{-1}(\mathbb{R}^2)$.
(ii) For $f \in H^1_e$, we have $\Delta S(f) = S(H_{m+1}f)$.

Proof of Proposition 3.7

In this section, we provide a proof of (3.19) in Proposition 3.7. The proof is an extension of the work in [3], where the a priori estimates for (3.2) are established for up to first spatial derivatives.

5 Derivation of Estimates

5.1 Proof of Proposition 3.7

In this section, we provide a proof of (3.19) in Proposition 3.7. The proof is an extension of the work in [3], where the a priori estimates for (3.2) are established for up to first spatial derivatives.

Let $u(t) \in L^\infty(I; \Sigma_m \cap \mathcal{H}^3)$ be a solution to (4.1), where $I = (\tau, \tau + \sigma)$ for some $\tau > 0$ and $\sigma > 0$. Note that $u(t)$ is automatically in $C(I; \Sigma_m)$ from the identity (1.1). We have checked the regularity of $\tilde{q}(t)$ in the previous section when $u(t) \in C(I; \Sigma_m) \cap L^\infty(I; \mathcal{H}^2)$. In the present case, $\tilde{q}(t)$ has additional regularity as follows.

Lemma 5.1. $\tilde{q}(t) \in L^\infty(I; H^2) \cap W^{1,\infty}(I; L^2)$. Proof. We first note that if $u \in \Sigma_m \cap \mathcal{H}^3$, then

$$\partial_r H_m v_j, \quad \frac{1}{r} H_m v_j, \quad \partial_r H_0 v_3 \in L^2_e$$

for $j = 1, 2$, and

$$\|u\|_{H^3} \sim \sum_{j=1}^2 \left( \|\partial_r H_m v_j\|_{L^2_e} + \left\| \frac{1}{r} H_m v_j \right\|_{L^2_e} \right) + \|\partial_r H_0 v_3\|_{L^2_e}. \quad (5.2)$$

This is immediate from the equivalence $\|u\|_{H^3} \sim \|\Delta u\|_{H^1}$. For $W$ in (4.63) and for $j = 1, 2$, direct calculations yield

$$H_{m+1} W_j = \partial_r H_m W_j + m v_3 \frac{1}{r} H_m v_j - 2m \frac{1 + v_3}{r^2} v_j - 2m \frac{1 + v_3}{r^2} v_j + 2mv_3 r \left( \frac{v_j}{r} - \frac{v_j}{r^2} \right) + m v_j H_0 v_3 \quad (5.3)$$
Then, the above representation implies \( H_{m+1}W_j \in L^\infty(I; L^2_t) \). Indeed, if we regard \( v_{jr}, \frac{\partial}{\partial r}v_r \), and \( H_0v_3 \) as radial symmetric functions in \( L^2(\mathbb{R}^2) \), Sobolev embedding implies all of these three quantities are in \( L^\infty(I; L^p_t) \) for \( p \in [2, \infty) \). Thus it follows from (1.20) that \( \frac{1}{2r^2}v_{rr} \in L^\infty(I; L^p_t) \) for \( p \in [2, \infty) \). Summarizing these up, we obtain \( H_{m+1}W_j \in C(I; L^2_t) \), and hence \( (H_{m+1}W_j) \cdot \tilde{e} \in L^\infty(I; L^2_t) \).

We next show \( (H_{m+1}W_3) \cdot \tilde{e} \in L^\infty(I; L^2_t) \). Direct calculations yield
\[
(H_{m+1}F_3) \cdot \tilde{e}_3 = \partial_r H_0v_3 \cdot \tilde{e}_3 + m(m^2 + 2m)\frac{1 - v_r^2}{r^2} \frac{\tilde{e}_3}{r} - 2m v_{3r} \frac{\tilde{e}_3}{r} + 2mv_3v_{3r} \frac{\tilde{e}_3}{r} - (m^2 + 2m) \frac{v_{3r} \tilde{e}_3}{r}.
\]

Thus, it suffices to show that \( \frac{\partial}{\partial r} \) and \( \frac{\partial}{\partial r} \) are in \( L^\infty(I; L^p_t) \) for \( p \in [2, \infty) \). The former follows from (1.11). Since \( v_{3rr} + \frac{v_{3r}}{r} \in L^\infty(I; L^p_t) \) by the Sobolev inequality, it suffices to show \( v_{3r} \in L^\infty(I; L^p_t) \). However, this is immediate from change of coordinates; \( u_{rr} = \frac{\partial^2}{\partial x_3 \partial y} u_{xy} \). Hence, \( (H_{m+1}W) \cdot \tilde{e} \in L^\infty(I; L^2_t) \). Note that \( H_{m+1}q = (H_{m+1}W) \cdot \tilde{e} + 2W_r \cdot \tilde{e}_r \). Since both \( W_r \) and \( \tilde{e}_r \) is in \( L^\infty(I; L^2_t) \), we have \( q \in L^\infty(I; H^2) \) from Lemma 7 (iii).

\( \bar{q}(t) \in W^{1,\infty}(I; L^2) \) follows from the fact that \( \bar{q} \) satisfies (5.2) and by Hardy’s inequality (see (5.17) below).

We move on to the derivation of the estimate (5.19). In the proof, we sometimes write the spaces of radial component like \( L_r^p \) as \( L^p_r \) for abbreviation if there is no ambiguity. Let \( x_i \) indicate the \( i \)-th spatial coordinate of \( \mathbb{R}^2 \) for \( i = 1, 2 \). By operating \( \partial_{x_i}, \partial_{x_j} \) on the equation (4.72) for \( i, j = 1, 2 \), we obtain
\[
iU_t + \Delta U = \sum_{k=1}^9 A_k,
\]
where
\[
U = \partial_{x_i} \partial_{x_j} \bar{q},
\]
\[
A_1 = m(1 + v_3)(mv_3 - m - 2)U,
\]
\[
A_2 = \left( \frac{m(1 + v_3)(mv_3 - m - 2)}{x_i} \right) \bar{q}_{x_j} + \left( \frac{m(1 + v_3)(mv_3 - m - 2)}{x_j} \right) \bar{q}_{x_i},
\]
\[
A_3 = \left( \frac{m(1 + v_3)(mv_3 - m - 2)}{x_i x_j} \right) \bar{q},
\]
\[
A_4 = \frac{mv_3}{r} \bar{q},
\]
\[
A_5 = \left( \frac{mv_3}{r} \right) \bar{q}_{x_j} + \left( \frac{mv_3}{r} \right) \bar{q}_{x_i},
\]
\[
A_6 = \left( \frac{mv_3}{r} \right) \bar{q},
\]
\[
A_7 = N(q)U,
\]
\[
A_k = (N(q))_{x_i} \bar{q}_{x_j} + (N(q))_{x_j} \bar{q}_{x_i},
\]
\[
A_9 = (N(q))_{x_i x_j} \bar{q}.
\]

By Strichartz estimates, we have
\[
\|U\|_{\text{Str}(t)} \leq C \left( \|U(\tau)\|_{L^2_t} + \sum_{k=1}^9 \|A_k\|_{L^\infty_t L^2_x} \right).
\]

To derive the bound for each \( A_k \), we make here some preparations based on the papers [5], [9].
If \( \delta \) is sufficiently small, then
\[
\|z\|_{H^1_*^\delta} \sim \sqrt{\mathcal{L}(u) - 4\pi m}.
\] (5.7)
In particular, \( \|z\|_{L^\infty} \) is sufficiently small, and hence we have
\[
|\gamma| \lesssim |z|^2, \quad |\gamma_r| \lesssim |z||z_r|.
\] (5.8)

For simplicity, we sometimes write \( a_1\hat{e} + a_3J^\delta\hat{e}, a_1j + a_3J^\delta j \) as \( a\hat{e}, a_j \) for \( a = a_1 + ia_2 \in \mathbb{C} \), respectively. Under this convention, \( q \) and \( z \) satisfy the relation
\[
s e^{-\alpha R}(q\hat{e})(sr) = (L_0 z)j + (\gamma h), \quad + \frac{2m}{r}h_3\gamma h + \frac{m}{r} \xi e, \quad \text{ (5.9)}
\]
where \( L_0z := z_r + \frac{2}{r}h_3z \), and \( \xi(r) := e^{-\alpha R v}(sr) - h(r) = zj + \gamma h \).

To obtain bounds of quantities related to \( z \) by those of \( q \), the following lemma effectively works:

**Lemma 5.2.** Let \( g(r) : (0, \infty) \to \mathbb{C} \) be a measurable function satisfying \( g \in L^\infty((0, \infty)) \) and \( g_r \in L^1_{\text{loc}}((0, \infty)) \). And assume that \( m \in \mathbb{N}, p \in [2, \infty) \), and \( a \in \mathbb{R} \) satisfy \( m - a > 1 \) and \( pa > 2 \). Then, if \( \frac{\partial}{\partial r} - \frac{m}{r^p}g \in L_*^p \), we have \( \frac{\partial}{\partial r} - \frac{m}{r^p}g \in L_*^p \) and
\[
\left\| \frac{g_r}{r^{p-1}} \right\|_{L_*^p} + \left\| \frac{g}{r^p} \right\|_{L_*^p} \leq C \left\| \frac{g_r}{r^{p-1}} \right\|_{L_*^p} + \left\| \frac{m}{r^p}g \right\|_{L_*^p}
\] (5.10)
for some constant \( C = C(m, p, a) \).

This is a simple extension of Lemma 3.6 in [5] and Lemma 4.2 in [8]. The proof of this lemma follows that of Lemma 4.2 in [5], replacing \( \partial_r, \tau \) by \( \frac{\partial}{\partial r} \), \( \frac{1}{r^{p-1}} \), respectively.

The following estimate for \( z \) is obtained in [8], Lemma 4.8. Namely, there exists \( \delta_0 > 0 \) such that for \( u \in \Sigma_m \), with \( \delta = \sqrt{\mathcal{L}(u) - 4\pi m} < \delta_0 \) and for \( p \in [2, \infty) \), we have
\[
\left\| z_r \right\|_{L_*^p} + \left\| \frac{z}{r} \right\|_{L_*^p} \lesssim s^{-2/p} \left\| q \right\|_{L_*^p} + \left\| q \right\|_{L_*^a} \quad (5.11)
\]
In [8], the estimate for \( \left\| z_r \right\|_{L_*^2} \) is also obtained and used to find the bound for \( \left\| z_r \right\|_{L_*^2} \). However, we slightly modify their method, and we only need (5.11) to estimate the terms which \( z \) concerns.

We observe here several simple estimates which are also seen in [8]. Let \( u \in \Sigma_m \) with \( \delta = \sqrt{\mathcal{L}(u) - 4\pi m} < \delta_0 \). Since \( \left\| q \right\|_{L_*^a} = \pi^{-1}\delta \lesssim 1 \), (5.8) and (5.11) provide
\[
\left\| \frac{1 + v}{r^2} \right\|_{L_*^4} = s^{-3/2} \left\| \frac{1 + h_3 + 2z h_1 + \gamma h_3}{r^2} \right\|_{L_*^4}
\leq s^{-3/2} \left( \left\| \frac{1 + h_3}{r^2} \right\|_{L_*^4} + \left\| \frac{2z h_1}{r^2} \right\|_{L_*^4} + \left\| \frac{\gamma h_3}{r^2} \right\|_{L_*^4} \right)
\lesssim s^{-3/2} \left( 1 + \left\| \frac{z}{r} \right\|_{L_*^4} + \left\| \frac{z r}{L_*^4} \right\|_{L_*^4} \right)
\lesssim s^{-3/2} + s^{-1} \left\| q \right\|_{L_*^4} + \left\| q \right\|_{L_*^a}^2.
\] (5.12)
Similarly, we have
\[
\| \frac{v_{rr}}{r} \|_{L^4} = \left\| \frac{1}{8r} \left[ \frac{m\beta_2^2}{r} \right] + h_1 zr - \frac{m h_3 zr_2}{r} + \gamma_r h_3 + \frac{m h_2^2}{r} \right\|_{L^4} \\
\lesssim s^{-3/2} \left( 1 + \|zr\|_{L^4} + \left\| \frac{x}{r} \right\|_{L^4} + \|zr\|_{L^8} + \left\| \frac{x}{r} \right\|_{L^8} \right) \\
\lesssim s^{-3/2} + s^{-1} \|q\|_{L^4} + \|q\|_{L^8}^2.
\] (5.13)

\[
\| v_{3r} \|_{L^4} \lesssim s^{-1/2} + \|q\|_{L^4}, \quad \| v_{3r} \|_{L^8} \lesssim s^{-3/4} + \|q\|_{L^8}.
\] (5.14)

Here, the notation in the first line of (5.13) means the composite of the functions in the square brackets and \( \frac{x}{r} \). Since
\[
v_{3rr} = \partial_r W_3 - \frac{1 - \nu_3^2}{r^2} - 2m v_{3r},
\] (5.15)
we have
\[
\| v_{3rr} \|_{L^4} \lesssim \| \partial_r (q\hat{e}) \|_{L^4} + \left\| \frac{1 + \nu_3}{r^2} \right\|_{L^4} + \left\| \frac{v_{3r}}{r} \right\|_{L^4} \\
\lesssim \| q_r \|_{L^4} + \left\| \frac{q}{r} \right\|_{L^4} + \| q \|_{L^8}^2 + s^{-1} \| q \|_{L^4} + s^{-3/2}.
\] (5.16)

- The nonlocal terms are treated in the following manner. By Hardy’s inequality, for \( p \in [1, \infty) \) and for \( f \in L^p_v \), we have
\[
\left\| \int_{r}^{\infty} \frac{f(v\gamma)}{r'} \, dr' \right\|_{L^p_v} \lesssim \| f \|_{L^p_v}.
\] (5.17)

Since \( |v| = \sqrt{1 - \frac{v_3^2}{v^2}} \), (5.16) and (5.17) implies
\[
\| N(q) \|_{L^4} \lesssim \| q \|_{L^8}^2 + \left\| \frac{q}{r} \right\|_{L^4} \lesssim \left\| \frac{1 - \nu_3^2}{r^2} \right\|_{L^4} + \| q \|_{L^8}^2 \\
\lesssim s^{-3/2} + s^{-1} \| q \|_{L^4} + \| q \|_{L^8}^2.
\] (5.18)

Now, we derive the bound for each \( A_k \).

\[
\| A_1 \|_{L^{4/3}_t L^{6/3}_x} \lesssim \left\| \frac{1 + \nu_3}{r^2} \right\|_{L^{4/3}_t L^{6/3}_x} \| U \|_{L^{\infty}_t L^2_x} \\
\lesssim \left( \frac{s^{-3/2}}{\sigma^{3/4} + s^{-1}} \right) \| q \|_{L^{6/3}_t L^{6/3}_x} + \| q \|_{L^{6/3}_t L^{6/3}_x} \| U \|_{L^{\infty}_t L^2_x} \\
\lesssim \left( \frac{s^{-3/2}}{\sigma^{3/4} + \| q \|_{L^{6/3}_t L^{6/3}_x}^2} \right) \| q_r \|_{L^{6/3}_t L^{6/3}_x} + \left\| \frac{q}{r} \right\|_{L^{6/3}_t L^{6/3}_x}.
\] (5.19)

\[
\| A_2 \|_{L^{4/3}_t L^{6/3}_x} \lesssim \left\| \left( \frac{1 + \nu_3}{r^2} \right) \left( \frac{q_{rr}}{r^2} + \left\| \frac{q_{3r}}{r^2} \right\|_{L^{6/3}_t L^{6/3}_x} \right) \right\|_{L^{4/3}_t L^{6/3}_x} \\
\lesssim \left( \frac{s^{-3/2} \sigma^{3/4} + \| q \|_{L^{6/3}_t L^{6/3}_x}^2} \right) \| q_r \|_{L^{6/3}_t L^{6/3}_x} + \left\| \frac{q}{r} \right\|_{L^{6/3}_t L^{6/3}_x}.
\] (5.20)

\[
\| A_3 \|_{L^{4/3}_t L^{6/3}_x} \lesssim \left\| \left( \frac{1 + \nu_3}{r^2} \right) + \left\| \frac{q_{3r}}{r^2} \right\|_{L^{6/3}_t L^{6/3}_x} + \| q_{3r} \|_{L^{6/3}_t L^{6/3}_x} \right\|_{L^{\infty}_t L^2_x} \\
\lesssim \left( \frac{s^{-3/2} \sigma^{3/4} + \| q \|_{L^{6/3}_t L^{6/3}_x}^2} \right) \| q_r \|_{L^{6/3}_t L^{6/3}_x} + \left\| \frac{q}{r} \right\|_{L^{6/3}_t L^{6/3}_x} + \| q_{3r} \|_{L^{6/3}_t L^{6/3}_x}.
\] (5.21)
\[ \|A_4\|_{L^3_{\sigma}L^{3\alpha}_{L^3}} \lesssim \frac{1}{r^2} \|U\|_{L^3_{\sigma}L^3} \] (5.22)

\[ \|A_5\|_{L^3_{\sigma}L^{3\alpha}_{L^3}} \lesssim \left( \|q\|_{L^3_{\sigma}L^3} + \|q\|_{L^3_{\sigma}L^3} \right) \|U\|_{L^3_{\sigma}L^3} \] (5.23)

\[ \|A_7\|_{L^3_{\sigma}L^{3\alpha}_{L^3}} \lesssim \|N(q)\|_{L^3_{\sigma}L^3} \|U\|_{L^3_{\sigma}L^3} \] (5.24)

\[ \|A_8\|_{L^3_{\sigma}L^{3\alpha}_{L^3}} \lesssim \frac{1}{r} \|N(q)\|_{L^3_{\sigma}L^3} + \frac{1}{r^2} \|N(q)\|_{L^3_{\sigma}L^3} \] (5.25)

The remaining term is \( A_6 \). First, we have

\[ \|A_6\|_{L^3_{\sigma}L^{3\alpha}_{L^3}} \lesssim \|q_{3rr}\|_{L^3_{\sigma}L^{3\alpha}_{L^3}} + \frac{\|q_{3rr}\|_{L^3_{\sigma}L^{3\alpha}_{L^3}}}{r^2} + \frac{\|q_{3rr}\|_{L^3_{\sigma}L^{3\alpha}_{L^3}}}{r^3} \] (5.26)

The second and third terms are estimated in the same manner above. Hence, it suffices to find the bound of the first term. By direct calculations,

\[ v_{3rr} = \partial_3 W_3 - \frac{2m}{r^2} v_3 \partial_3 W_3 - \frac{2m}{r^2} W_3^2 + \frac{4m}{r^2} (2mv_3^2 + v_3 - m) W_3 + \frac{m}{r^3} (1 - v_3^2)(6mv_3^2 + 6mv_3 + 2 - 2m^2). \] (5.27)

Hence

\[ \|q_{3rr}\|_{L^3_{\sigma}L^{3\alpha}_{L^3}} \lesssim \frac{1}{r^2} \|q\|_{L^3_{\sigma}L^{3\alpha}_{L^3}} + \frac{1}{r^3} \|q\|_{L^3_{\sigma}L^{3\alpha}_{L^3}} \] (5.28)

Each term is estimated as follows:
Therefore, we have
\[ \| \frac{1 + v_3}{r^2} \|_{L_t^{5/2} \mathcal{L}_{x}^{1/3}} \lesssim \left( \frac{2^{3/2} \sigma^{3/4} + \| q \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} + \| q_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}}} {r^2} \right) \| q_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}}. \] (5.30)

The third and forth terms are bounded by
\[ \left( \| q \|_{L_t^{s/3} \mathcal{L}_{x}^{s/3}} + \sigma^{1/2} \| q \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} \right) \| q_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}}. \] (5.31)

For the second term,
\[ \| (q \dot{e})_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} \leq \left( \| q_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} + \| q_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} + \| \dot{q_r} \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} \right) \| q_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} \leq \left( \| q \|_{L_t^{s/3} \mathcal{L}_{x}^{s/3}} + \sigma^{1/2} \left( \| q_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} + \| \dot{q_r} \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} \right) \right) \| q_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}}. \] (5.32)

We note that
\[ (q \dot{e})_{rr} = q_r \dot{e} - 2 (q_r \dot{e}_r + q_{rr} (J \dot{e})_r) - (q_l (v_r \cdot \dot{e}) + q_{ll} (v_r \cdot \dot{e})) v_r + (q_l (v_{rr} \cdot \dot{e}) + q_{lll} (v_{rr} \cdot \dot{e})) v_r. \] (5.33)

Hence, the first term is bounded by
\[ \left( \| q \|_{L_t^{s/3} \mathcal{L}_{x}^{s/3}} + \sigma^{1/2} \left( \| q_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} + \| \dot{q_r} \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} \right) \right) \times \left( \| q_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} + \| \dot{q_r} \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} \right). \] (5.34)

Therefore, we have
\[ \| A_b \|_{L_t^{4/3} \mathcal{L}_{x}^{4/3}} \lesssim \left( \| q \|_{L_t^{s/3} \mathcal{L}_{x}^{s/3}} + \sigma^{1/2} \left( \| q_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} + \| \dot{q_r} \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} \right) \right) \times \left( \| q_r \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} + \| \dot{q_r} \|_{L_t^{1/3} \mathcal{L}_{x}^{1/3}} \right). \] (5.35)

Applying the equivalence (1.17) to \( \tilde{q} \), we obtain (3.14).

5.2 Proof of Proposition 3.8

In this section, we prove Proposition 3.8. Let \( u \in \Sigma_\delta \cap \dot{H}^3 \) with \( \delta = \sqrt{2}(u) - 4\pi m < \delta_0 \). We first note that since we have the equivalence (5.2), it suffices to find the bounds for \( \partial_t H_m v_1, \frac{1}{r} H_m v_1, \partial_t H_m v_2 \) and \( \partial_t H_m v_3 \) for \( j = 1, 2 \). By straightforward computations, we have
\[ \partial_t H_m v_1 = \partial_t W_1 + (1 - m \nu_3) \frac{1}{r^2} W_1 - 2 m \frac{1}{r} W_1 W_3 + 2 m^2 \nu_3 \frac{1}{r^2} W_3 \]
\[ + (-3m^2 (1 - \nu_3^2) + m \nu_3 - 1) \frac{1}{r^2} W_1 - m \frac{1}{r} \partial_t W_3 \]
\[ + (m + 3m^2 \nu_3) \frac{1}{r^2} W_3 + \frac{4m^2}{r^3} v_1 (1 - \nu_3^2) + \frac{6m^3}{r^3} \nu_3 v_1 (1 - \nu_3^2), \] (5.36)
\[
\frac{1}{r} H_m v_1 = \frac{1}{r} \partial_r W_1 + (1 - mv_3) \frac{1}{r^2} W_1 - mv_1 \frac{1}{r^2} W_3 - \frac{2m^2}{r^2} v_1 (1 - v_3^2),
\]
(5.37)
\[
\partial_r H_0 v_3 = \partial_r W_3 + (-2mv_3 + 1) \frac{1}{r} \partial_r W_3 - 2m \frac{1}{r} W_3^2
\]
\[+ (8m^2 v_3 + 2mv_3 - 4m^2 - 1) \frac{1}{r^2} W_3 \]
\[+ \frac{1 - v_3^2}{r^3} (6mv_3^2 + 4m^2 v_3 - 2m^3).
\]

Hence, we obtain
\[
\|u\|_{H^3} \lesssim \|\partial_r W\|_{L^2} + \left\| \frac{1}{r} \partial_r W \right\|_{L^2} + \left\| \frac{1}{r^2} W \right\|_{L^2} + \left\| \frac{1}{r^2} W^2 \right\|_{L^2}
\]
\[+ \sum_{j=1}^2 \left\| \frac{1}{r^2} (1 - v_3^3) \right\|_{L^2} + \frac{1 - v_3^2}{r^3} (6mv_3^2 + 4m^2 v_3 - 2m^3) \right\|_{L^2}
\]
\[= : 6 B_k. \quad (5.39)
\]
By (5.38), we have
\[
B_1 \lesssim \|q v_r\|_{L^2} + \|q v_r\|_{L^2} + \|q v_r\|_{L^2} + \|q v_r\|_{L^2}.
\]
(5.40)
Direct calculations yield
\[
v_{1rr} = \partial_r W_1 - mv_1 \frac{1}{r} W_1 - mv_1 \frac{1}{r^2} W_3 + \frac{1}{r^2} v_1 (-m^2 + 2m^2 v_3^2 + mv_3),
\]
\[v_{3rr} = \partial_r W_3 - 2mv_3 \frac{1}{r} W_3 - \frac{1 - v_3^2}{r^2} (m + 2m^2 v_3),
\]
(5.42)
hence
\[
\|q v_{rr}\|_{L^2} \lesssim \|q \partial_r W\|_{L^2} + \left\| \frac{1}{r} \partial_r W \right\|_{L^2} + \left\| \frac{q}{r} \right\|_{L^2}
\]
\[\lesssim \|qq_r\|_{L^2} + \|q^2 v_r\|_{L^2} + \left\| \frac{q^2}{r} \right\|_{L^2} + \frac{q}{r} \right\|_{L^2}.
\]
(5.43)
Each term is estimated by
\[
\|qq_r\|_{L^2} \leq \|q\|_{L^2} \|q_r\|_{L^\infty} \lesssim \|q v_r\|_{L^2} + \frac{q}{r} \right\|_{L^2}.
\]
(5.44)
\[
\left\| \frac{q^2}{r} \right\|_{L^2} \leq \|q\|_{L^\infty} \lesssim \left\| \frac{q}{r} \right\|_{L^2} + \frac{q}{r} \right\|_{L^2}.
\]
(5.45)
\[
\|q^2 v_r\|_{L^2} \lesssim \|q^2 \|_{L^2} + \left\| \frac{q}{r} \right\|_{L^2} \lesssim \|q q_r\|_{L^2} + \left\| \frac{q}{r} \right\|_{L^2}.
\]
(5.46)
Therefore, (5.43) implies
\[
\|q v_{rr}\|_{L^2} \lesssim \|q v_r\|_{L^2} + \left\| \frac{q}{r} \right\|_{L^2} + \left\| \frac{q}{r} \right\|_{L^2} + \|q\|_{L^2}.
\]
(5.47)
Moreover, we obtain
\[
\|q v_r\|_{L^2} \lesssim \|q v_r - \frac{m}{r} J R v_0\|_{L^2} + \left\| \frac{q}{r} \right\|_{L^2} \lesssim \|q v_r\|_{L^2} + \|q v_r\|_{L^2},
\]
(5.48)
\[
\|q v_r^2\|_{L^2} \lesssim \|q v_r - \frac{m}{r} J R v_0\|_{L^2}^2 + \left\| \frac{q}{r} \right\|_{L^2} \lesssim \|q v_r\|_{L^2} + \|q v_r\|_{L^2}.
\]
(5.49)
Hence, (5.40) provides

\[ B_1 \lesssim \| q_r \|_{L^2} + \left\| \frac{q_r}{r} \right\|_{L^2} + \left\| \frac{q}{r^2} \right\|_{L^2} + \| q \|_{L^2} \, . \]  

(5.50)

We can similarly obtain

\[ B_2 \lesssim \left\| \frac{q}{r^3} \right\|_{L^2} + \left\| \frac{q v_r}{r^3} \right\|_{L^2} \lesssim \left\| \frac{q}{r^3} \right\|_{L^2} + \left\| \frac{q}{r^2} \right\|_{L^2} \, . \]  

(5.51)

\[ B_3 \leq \left\| \frac{q}{r^2} \right\|_{L^2} \, , \quad B_4 \leq \left\| \frac{q}{r^2} \right\|_{L^2} + \left\| \frac{q}{r^2} \right\|_{L^2} \, . \]  

(5.52)

In order to estimate \( B_5 \), we first observe that

\[ v_1^2 + v_2^2 = \left[ (1 + \gamma)h_1 - z_2 h_3 \right]^2 + z_1^2 \right] \left( \frac{r}{s} \right) \lesssim \left[ |z|^2 + h_3^2 \right] \left( \frac{r}{s} \right) \]  

(5.53)

where we use (5.5). Hence

\[ B_5 \lesssim \left\| \frac{(v_1^2 + v_2^2)^{3/2}}{r^3} \right\|_{L^2} \lesssim s^{-2} \left\| \frac{|z|^3 + h_3^3}{r^3} \right\|_{L^2} \lesssim \| q \|^3_{L^2} + s^{-2} \]  

(5.54)

where we use (5.11) in the last inequality.

It remains to control \( B_6 \). To this end, we require dividing the case into \( m = 1 \) and \( m = 2 \). When \( m = 1 \), we have the factorization

\[ \frac{1 - v_3^2}{r^3} (6v_3^2 + 4v_3 - 2) = \frac{2}{r^3}(1 - v_3^2)(1 + v_3)(3v_3 - 1). \]  

(5.55)

Moreover, we have

\[ 1 + v_3 \lesssim |(1 + h_3) + |z|| \left( \frac{r}{s} \right) \]  

(5.56)

Hence

\[ B_6 \lesssim s^{-2} \left\| \frac{|z|^3 + (1 + h_3)^3 + h_3^3}{r^3} \right\|_{L^2} \lesssim \| q \|^3_{L^2} + s^{-2}. \]  

(5.57)

In the case when \( m \geq 2 \), we no longer have such a factorization as (5.60).

Instead, we are able to use Lemma 5.2 in this case. First, we have

\[ B_6 \lesssim \| v_1^2 + v_2^2 \|_{L^2} \lesssim s^{-2} \left\| \frac{|z|^2 + h_3^2}{r^3} \right\|_{L^2} \lesssim \| q \|^2_{L^2} + s^{-2}. \]  

(5.58)

Here, we apply Lemma 5.2 with \( a = \frac{3}{2} \) and \( p = 4 \). Then we obtain

\[ \left\| \frac{s^2}{r^3} \right\|_{L^2} = \left\| \frac{z_r - m z_r}{r} \right\|_{L^2}^2 \lesssim \left\| \frac{z_r - m z_r}{r^2} \right\|_{L^2} \]  

\[ \quad \lesssim \left\| \frac{z_r}{r} \right\|_{L^2}^2 + \left\| \frac{m}{r} \right\|_{L^2} \]  

\[ =: E_1 + E_2. \]  

(5.59)

If we use the relation (5.5), then we have

\[ E_1 = \left\| \frac{1}{r} se^{-\alpha R} \right\|_{L^2}^2 \lesssim s^2 \left\| \left[ q \right]_{(sr)}^2 \right\|_{L^2} + \left\| \frac{1}{r} \right\|_{L^2} (\gamma h)_v^2 \right\|_{L^2} + \left\| \frac{2m}{r} \right\|_{L^2} \]  

\[ \lesssim \left\| \left[ q \right]_{(sr)}^2 \right\|_{L^2} + \left\| \frac{1}{r} \right\|_{L^2} (\gamma h)_v^2 \right\|_{L^2} + \left\| \frac{2m}{r} \right\|_{L^2} \]  

\[ \lesssim \| q \|^2_{L^2} \]  

(5.60)
Thus, by the implicit function theorem, there exist neighborhoods of $Q$ with the norm $(s, \alpha)$ coincides with $(s_0(u), \alpha_0(u))$. Moreover, if $(s, \alpha)$ satisfies $(h_1, z)$ $H^1_0 = 0$ and $|s - 1| + |\alpha| \leq C_1 \delta_1$, then $(s, \alpha)$ coincides with $(s_0(u), \alpha_0(u))$.

First, we introduce the function space

$$Y := \left\{ e^{m \theta R} v(r) : (0, \infty) \rightarrow \mathbb{R}^3, v_1, v_2, v_3 \in L^\infty, \partial_r v_3 \in L^2 \right\}$$

with the norm

$$\|u\|_Y := \|v_1\|_{H^1_0} + \|v_2\|_{H^1_0} + \|v_3\|_{L^\infty} + \|\partial_r v_3\|_{L^2}.$$  (6.1)

Then, we can easily check that $(Y, \|\cdot\|_Y)$ is a Banach space. Define $F = (F_1, F_2) : Y \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^2$ by

$$
\begin{align*}
F_1(u, s, \alpha) &:= \left( h_1(1), \frac{1}{h_1^2} \right) \quad \text{and} \\
F_2(u, s, \alpha) &:= \left( \frac{1}{h_1^2} \right)
\end{align*}
$$

where $H_m$ is defined in (5.60). Then, $F$ is $C^1$, and we have

$$F(Q, 1, 0) = 0, \quad \begin{pmatrix} \partial_uf_1(Q, 1, 0) \\ \partial_uf_2(Q, 1, 0) \end{pmatrix} = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right).$$  (6.3)

Thus, by the implicit function theorem, there exist neighborhoods $V \subset Y$ of $Q$, and $W \subset \mathbb{R}^+ \times \mathbb{R}$ of $(1, 0)$, respectively, and a function $(s_0(u), \alpha_0(u)) : V \rightarrow W$ such that

- $(s_0(u), \alpha_0(u))$ is $C^1$ on $V$.
- $(s_0(Q), \alpha_0(Q)) = (1, 0)$.
- For $(u, s, \alpha) \in V \times W$,

$$F(u, s, \alpha) = 0 \Longleftrightarrow (s, \alpha) = (s(u), \alpha(u)).$$  (6.4)

- For any $u \in V$, we have

$$\det \begin{pmatrix} \partial_uf_1(u, s_0(u), \alpha_0(u)) & \partial_uf_1(u, s_0(u), \alpha_0(u)) \\ \partial_uf_2(u, s_0(u), \alpha_0(u)) & \partial_uf_2(u, s_0(u), \alpha_0(u)) \end{pmatrix} \neq 0.$$  (6.5)

6 Proof of Proposition 3.3

Step 1. We begin with showing the following claim:

There exists $\delta_1 > 0$ and $C_1 > 0$ such that for $u \in \Sigma_m$ with $\|u - Q\|_{H^1}$ $< \delta_1$, there is a pair $(s, \alpha) = (s_0(u), \alpha_0(u)) \in \mathbb{R}^+ \times \mathbb{T}$ such that

- $(h_1, z)_{H^1} = 0,$
- $|s_0(u) - 1| + |\alpha_0(u)| \leq C_1 \|u - Q\|_{H^1}.$

Hence we complete the proof.

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and
\[
\begin{bmatrix}
(d_u s_0(u), \delta u)_{Y^\cdot, Y} \\
(d_u a_0(u), \delta u)_{Y^\cdot, Y}
\end{bmatrix} = \left( \begin{array}{cc}
\partial_x F_1 & \partial_x F_2 \\
\partial_t F_1 & \partial_t F_2
\end{array} \right)^{-1}\begin{bmatrix}
(d_u F_1, \delta u)_{Y^\cdot, Y} \\
(d_u F_2, \delta u)_{Y^\cdot, Y}
\end{bmatrix}
\]
for \( \delta u \in Y \).

From (6.7), there is a constant \( C \) such that
\[
\|d_u s(u)\|_{Y^\cdot} \leq C, \quad \|d_u a(u)\|_{Y^\cdot} \leq C
\]
for all \( u \in V \). (If necessary, we replace \( V \) by a smaller neighborhood.)

Now, let \( u \in \Sigma_m \). Then \( u \in Y \), and from Lemma 1.1 we have
\[
\|u - Q\|_{Y^\cdot} \leq C(\|u\|_{H^1}) \|u - Q\|_{H^2}.
\]
In particular, there exists \( \delta_1 > 0 \) such that \( u \in V \) if \( \|u - Q\|_{Y^\cdot} < \delta_1 \). Then, \( Q + t(u - Q) \in V \) for all \( t \in [0, 1] \), and
\[
|s(u) - 1| = |s(u) - s(Q)| = \left| \int_0^1 \frac{d}{dt}((Q + t(u - Q))dt) \right|
= \left| \int_0^1 \langle d_u s(Q + t(u - Q)), u - Q \rangle_{Y^\cdot, Y} dt \right|
\leq C \|u - Q\|_{Y^\cdot} \leq \frac{1}{2} C_1 \|u - Q\|_{H^1},
\]
for some \( C_1 > 0 \), and similarly
\[
|\alpha(u)| = |\alpha(u) - \alpha(Q)| \leq \frac{1}{2} C_1 \|u - Q\|_{H^2}.
\]
Moreover, when \( \delta_1 \) is sufficiently small, we have \((s, \alpha) \in W \) if \(|s - 1| + |\alpha| \leq C_1 \delta_1 \). Hence, by (6.9), the claim of Step 1 follows.

Step 2. We now prove the existence of \( s(u), \alpha(u) \) as in Proposition 3.2 (i) and (ii). Suppose \( \delta_0 > 0 \) and \( u \in \Sigma_m \) with \( \delta < \delta_0 \). To avoid the ambiguity, we change the notation \( \delta_0 \) in Proposition 1.1 into \( \delta_2 \). Then, there exists \( C_2 > 0 \) such that if \( \delta < \delta_2 \), then
\[
\left\| u - \exp^{\alpha_*(u)\tilde{Q}} (s_*(u) \tilde{x}) \right\|_{H^1} \leq C_2 \delta,
\]
Here we set \( \tilde{u}(x) := \exp^{-\alpha_*(u)\tilde{Q}}(s_*(u) \tilde{x}) \). By the scale invariance, we have \( \|\tilde{u} - Q\|_{H^1} \leq C_2 \delta \). Thus, if we choose \( \delta_0 = \min\{C_2^{-1} \delta, \delta_2 \} \), we can apply the result in Step 1 to \( \tilde{u} \). Namely, there exists \((s_0(\tilde{u}), a_0(\tilde{u}))\) such that \( F(u, s_0(\tilde{u}), a_0(\tilde{u})) = 0 \) and that \(|s_0(\tilde{u}) - 1| + |a_0(\tilde{u})| \leq C_1 \|\tilde{u} - Q\|_{H^1} \leq C_1 C_2 \delta \). Here we set \( C_3 := C_1 C_2 \) and
\[
s(u) := s_*(u) s_0(\tilde{u}), \quad \alpha(u) := \alpha_*(u) + a_0(\tilde{u}).
\]
Then we have
\[
F(u, s(u), \alpha(u)) = F(\tilde{u}, s_0(\tilde{u}), a_0(\tilde{u})) = 0,
\]
\[
\left\| \frac{s(u)}{s_*(u)} - 1 \right\| + |\alpha(u) - \alpha_*(u)| \leq C_3 \delta.
\]
Moreover, if \((s, \alpha)\) satisfies (6.15) in which \((s(u), \alpha(u))\) is replaced by \((s, u)\), then we obtain \((\frac{s}{s_*}, \alpha - \alpha_*) = (s_0(\tilde{u}), a_0(\tilde{u}))\) since \( F(\tilde{u}, \frac{s}{s_*}, \alpha - \alpha_*) = 0 \). Therefore, we achieve (i) and (ii) in Proposition 4.
Step 3. Finally, we show the regularity property of \((s(u), \alpha(u))\) as in Proposition 3.3 (iii). We only consider the case \(u(t) \in C(I; \Sigma_m) \cap C^1(I; L^2(\mathbb{R}^2))\). (The case \(u(t) \in C(I; \Sigma_m) \cap \dot{W}^{1,\infty}(I; L^2(\mathbb{R}^2))\) can be derived by small modifications.)

Fix \(t_0 \in I\). It suffices to show that \(s(u(t)), \alpha(u(t))\) are \(C^1\) in some neighborhood of \(t = t_0\). Define linear transform \(T\) on \(Y\) by \(Tu := e^{-\alpha(s(t_0))} \cdot u(s(t_0))\) for \(u \in Y\). Then, if \(t - t_0\) is sufficiently small, we have

\[
s(u(t)) = s_0(Tu(t))s_0(u(t_0)), \quad \alpha(u(t)) = \alpha(Tu(t)) + \alpha(s(t_0)) \quad (6.16)
\]

by the uniqueness proved in Step 1. Hence, we may assume \(s_0(u(t_0)) = 1, \alpha(s(t_0)) = 0\), and thus \(\|u(t_0) - Q\|_{\dot{H}^1} < \delta_1\). Under this simplification, we can use the fact that \(s(u)\) and \(\alpha(u)\) is \(C^1\)-function on \(Y\) by Step 1. By continuity, there exists \(\delta_0 > 0\) such that \(\|u(t) - Q\|_{\dot{H}^1} < \delta_1\) if \(|t - t_0| \leq \delta_0\).

Note that for fixed \(s\) and \(\alpha\), \(F(\cdot, s, \alpha)\) is bounded linear functional on \(L^2_y\).

It suffices to show that \(s(u(t))\) and \(\alpha(u(t))\) is \(C^1\) on \((t_0 - \delta_0, t_0 + \delta_0)\) and

\[
\frac{\partial}{\partial t}(s(u(t))) = -B(u(t)) \begin{pmatrix} F_1(\partial_t u(t), s(u(t)), \alpha(u(t))) \\ F_2(\partial_t u(t), s(u(t)), \alpha(u(t))) \end{pmatrix}, \quad (6.17)
\]

where

\[
B(u(t)) = \begin{pmatrix} B_{11}(u) & B_{12}(u) \\ B_{21}(u) & B_{22}(u) \end{pmatrix}
\]

\[
:= \begin{pmatrix} \partial_t F_1(u, s(u), \alpha(u)) & \partial_s F_1(u, s(u), \alpha(u)) \\ \partial_t F_2(u, s(u), \alpha(u)) & \partial_s F_2(u, s(u), \alpha(u)) \end{pmatrix}^{-1} \quad (6.18)
\]

for \(t \in (t_0 - \delta_0, t_0 + \delta_0)\). Let \(\sigma \in \mathbb{R}\) with \(|t + \sigma - t_0| < \delta_0\). Then,

\[
\frac{1}{\sigma} [s(u(t + \sigma)) - s(u(t))] + \sum_{j=1}^2 B_{1j} F_j(\partial_t u, s(u), \alpha(u))
\]

\[
= \frac{1}{\sigma} \int_0^1 \langle d_u s(u(t) + \xi(u(t + \sigma) - u(t))), u(t + \sigma) - u(t) \rangle_{\dot{H}^1_y, \dot{H}^1_y} d\xi 
\]

\[
+ \sum_{j=1}^2 B_{1j} F_j(\partial_t u, s(u), \alpha(u))
\]

\[
=: K_1 + K_2,
\]

where

\[
K_1 := \int_0^1 (d_u s(c(\xi)) - d_u s(c(0)), \Delta_\sigma u)_{\dot{H}^1_y, \dot{H}^1_y} d\xi \quad (6.20)
\]

\[
K_2 := (d_u s(u), \Delta_\sigma u)_{\dot{H}^1_y, \dot{H}^1_y} + \sum_{j=1}^2 B_{1j} F_j(\partial_t u, s(u), \alpha(u)) \quad (6.21)
\]

\[
c(\xi) := u(t) + \xi (u(t + \sigma) - u(t)), \quad \Delta_\sigma u = \frac{1}{\sigma} (u(t + \sigma) - u(t)) \quad (6.22)
\]

Using \(6.17\), we have

\[
K_2 = -\sum_{j=1}^2 B_{1j} [F_j(\Delta_\sigma u - \partial_t u(t), s(u(t)), \alpha(u(t)))] \to 0 \quad (6.23)
\]

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as $\sigma \to 0$, since $u(t) \in C^1(I; L^2)$. On the other hand, we can write
\[
\langle d_\alpha s(c(\xi)) - d_\alpha s(c(0)), \Delta_\sigma u \rangle_{Y^0, Y}
\]
\[
= - \sum_{j=1}^{2} \{ B_{ij}(c(\xi)) F_j(\Delta_\sigma u, s(c(\xi)), \alpha(c(\xi))) \}
\]
\[
= -(L_1 + L_2 + L_3),
\]
where
\[
L_1 := \sum_{j=1}^{2} \{ B_{ij}(c(\xi)) - B_{ij}(c(0)) \} F_j(\Delta_\sigma u, s(c(\xi)), \alpha(c(\xi))) \tag{6.25}
\]
\[
L_2 := \sum_{j=1}^{2} B_{ij}(c(0)) [ F_j(\Delta_\sigma u, s(c(\xi)), \alpha(c(\xi))) - F_j(\Delta_\sigma u, s(c(0)), \alpha(c(\xi))) ] \tag{6.26}
\]
\[
L_3 := \sum_{j=1}^{2} B_{ij}(c(0)) [ F_j(\Delta_\sigma u, s(c(0)), \alpha(c(\xi))) - F_j(\Delta_\sigma u, s(c(0)), \alpha(c(0))) ] \tag{6.27}
\]
Each $|L_k|$ is bounded by some constants independent of $\sigma$ and $\xi$, and converges to 0 as $\sigma \to 0$. Hence, by Lebesgue’s dominant convergence theorem, we obtain $K_1 \to 0$ as $\sigma \to 0$. The same argument can be applied to $\alpha$, and hence we achieve (6.17).

The continuity of (6.17) can be shown in the same manner as above. Hence, $s(u(t)), \alpha(u(t)) \in C^1$.

7 Some Technical Lemmas

7.1 Continuity of Reconstruction in Higher Regularity

In this section, we show some technical lemmas used in the previous sections. We first make a further observation concerned with the continuous dependence of the map $(q, s, \alpha) \mapsto u$ in Proposition 3.4.

**Lemma 7.1.** The reconstruction map $(q, s, \alpha) \mapsto u$ in Proposition 3.4 is continuous from $H^1_1 \times \mathbb{R}^+ \times \mathbb{T}^1$ to $\Sigma_m \cap H^2$.

**Proof.** We first fix $(q, s, \alpha) \in H^1_1 \times \mathbb{R}^+ \times \mathbb{T}^1$. It suffices to show that $\|u(q', s', \alpha') - u(q, s, \alpha)\|_{H^2} \to 0$ as $(q', s', \alpha') \to (q, s, \alpha)$ in $H^1_1 \times \mathbb{R}^+ \times \mathbb{T}^1$.

(Here $u(q, s, \alpha) \in \Sigma_m$ denotes the map reconstructed from $(q, s, \alpha)$.) In the proof, we write the difference $q' - q$ as $\delta q$, and also adopt this convention to other quantities.

We write $u = e^{m^R\bar{v}}(r) := u(q, s, \alpha), \ u' = e^{m^R\bar{v}'}(r) := u(q', s', \alpha')$. Taking account of the equivalence
\[
\|\delta u\|_{H^2} \sim \sum_{j=1}^{2} \|\delta H^m v_j\|_{L^2} + \|\delta H^0 v_3\|_{L^2} \tag{7.1}
\]
we only need to see the right hand side of (7.1). Direct computations yield
\[
\delta H_m,v_j = \delta \partial_t W_j - \delta \left( \frac{m}{r} v_j W_j \right) + \delta \left( \frac{1 - mv_j}{r} W_j \right) - \delta \left( \frac{2m^2}{r^2} v_j (1 - v_j^2) \right),
\]
for \( j = 1, 2 \). Here, we have
\[
\| \delta \partial_t W \|_{L^2} \leq \| \delta (q \dot{e} + q \dot{e}_r) \|_{L^2} \\
\leq \| \delta q \|_{L^2} + \| q \|_{L^2} \| \delta \dot{e} \|_{L^2} + \| q \|_{L^2} \| \delta \dot{e}_r \|_{L^2} \\
\leq \| \delta q \|_{H^1} + \| q \|_{H^1} \| \delta u \|_{H^1} + \| \delta q \|_{H^2}.
\]
Therefore, in both (7.2) and (7.3), all terms but the last term converge to 0 by (5.12). Since
\[
|1 - \frac{1}{r^2}| \leq \frac{1}{r^2} (2 \gamma + 1) (1 + \gamma) z_2 h_1 h_3 - z_2^2 h_1^2,
\]
the second term in (7.6) can be written as
\[
\left\| \frac{1}{s} \left( \frac{h_1^2 - (2 \gamma + 1) h_1 h_3 - 2(1 + \gamma) z_2 h_1 h_3 - z_2^2 h_1^2}{r^2} \right) \right\|_{L^2} \\
\leq \frac{1}{s} \left( \left\| \delta \gamma \right\|_{L^2} + \left\| (\delta \gamma)^2 \right\|_{L^2} + \left\| (\delta \gamma) z_2 + (1 + \gamma) \delta z_2 \right\|_{L^2} + \left\| \delta (z_2^2) \right\|_{L^2} \right). \tag{7.8}
\]
Since \( |\delta \gamma| \leq (|z| + |z'|)|\delta z| \), (7.8) is bounded by
\[
\frac{C}{s} \left( \left\| \frac{z}{r} \right\|_{L^2} + \left\| \frac{z'}{r} \right\|_{L^2} + \left\| \frac{\delta \gamma}{r} \right\|_{L^2} + \left\| \frac{\delta z_2}{r} \right\|_{L^2} \right), \tag{7.9}
\]
for some constant \( C \). Since \( z = (e^{-\alpha_R v}(sv)) \cdot (j + iJ^\phi \dot{J}) \), we can easily show that the second term in (7.8) converges to 0 as \((q', s', \alpha') \rightarrow (q, s, \alpha)\). Hence, it suffices to prove \( \| \frac{z}{r} \|_{L^4} \rightarrow 0 \).
Using the relation (5.9), we obtain
\[
\left\| \delta z_r - \frac{m}{r} \delta z \right\|_{L^4} \leq \left\| \delta(L_w z) \right\|_{L^4} + \left\| \frac{\alpha}{r^\delta} \delta z \right\|_{L^4} \\
\lesssim \left\| \delta \left( se^{-\alpha R} [q \tilde{e}](sr) \right) \right\|_{L^4} + \left\| \delta G_0(z) \right\|_{L^4} + \left\| \delta z \right\|_{L^\infty}
\]
where \( G_0(z) := (\gamma h)_r + \frac{2m}{r^\delta} \gamma h + \frac{m}{r} \xi \delta z \). Similarly, we have
\[
\left\| \delta G_0(z) \right\|_{L^4} \lesssim (\left\| z \right\|_{L^\infty} + \left\| z' \right\|_{L^\infty}) \left( \left\| \delta z_r \right\|_{L^4} + \left\| \delta z_s \right\|_{L^4} \right) \\
\lesssim (\left\| z \right\|_{L^\infty} + \left\| z' \right\|_{L^\infty}) \left\| \delta z_r - \frac{m}{r} \delta z \right\|_{L^4},
\]
where we apply Lemma 5.2 in the last inequality. By the smallness of \( \left\| z \right\|_{L^\infty} \) and \( \left\| z' \right\|_{L^\infty} \); (7.10) yields
\[
\left\| \delta z_r - \frac{m}{r} \delta z \right\|_{L^4} \lesssim \left\| \delta \left( se^{-\alpha R} [q \tilde{e}](sr) \right) \right\|_{L^4} + \left\| \delta z \right\|_{L^\infty},
\]
and the right hand side tends to 0 as \((q', s', \alpha') \rightarrow (q, s, \alpha)\). By using Lemma 5.2 again, we obtain \( \left\| \delta x \right\|_{L^4} \rightarrow 0 \). Hence the proof is accomplished.

### 7.2 Approximation

Finally, we show that each function \( u(t) = e^{\theta R} v(t, r) \in C(I; \dot{H}^1(\mathbb{R}^2)) \) can be approximated by smooth functions as follows.

**Lemma 7.2.** (i) Let \( I \) be an open interval, and suppose that \( u(t) = e^{\theta R} v(t, r) \in C(I; \dot{H}^1(\mathbb{R}^2)) \) with \( v(\infty) = \tilde{v} \). Then, for every \( I' \subset I \), there exist \( u_n(t, x) = e^{\theta R} v_n(t, r) \), \( n \in \mathbb{N} \) such that

(A) \( u_n(t, x) \in C^\infty(I' \times \mathbb{R}^2) \).

(B) For all \( n \in \mathbb{N} \), there exists \( R_n > 0 \) such that \( v_n(t, r) = h(r) \) for all \( t \in I' \) and \( r \geq R_n \).

(C) \( \sup_{t \in I'} \left\| u_n(t) - u(t) \right\|_{H^1} \rightarrow 0 \) as \( n \rightarrow \infty \).

(ii) Moreover, if \( \partial_t u \in C(I; L^2) \), then there exist \( u_n(t, x) = e^{\theta R} v_n(t, r), n \in \mathbb{N} \) satisfying (A), (B), and the following (C):

(C) \( \sup_{t \in I'} \left\| u_n(t) - u(t) \right\|_{H^1} + \sup_{t \in I'} \left\| \partial_t u_n(t) - \partial_t u(t) \right\|_{L^2(\mathbb{R}^2)} \rightarrow 0 \) as \( n \rightarrow \infty \).

**Remark 7.1.** Such kind of approximation is originally considered in \( C \) in an implicit way.

**Proof of Lemma 7.2** The proof consists of two steps.

**Step 1.** We first show the existence of sequence \( \{u_n\}_{n=1}^\infty \) satisfying (A) and (C) in the case (i), or (A) and (C)’ in the case (ii).

We take an interval \( I' \subset I \) and set \( \hat{u}(t, x) := u(t, x) - Q(x) \), where \( Q(x) = e^{\theta R} h(r) \). We also take radially symmetric mollifiers \( \eta_1 \in C^\infty_0(\mathbb{R}) \) and \( \eta_2 \in C^\infty_0(\mathbb{R}^2) \), and then define \( \eta(t, x) := \eta_1(t) \eta_2(x) \in C^\infty_0(\mathbb{R} \times \mathbb{R}^2) \).

Here, for \( (t, x) \in I' \times \mathbb{R}^2 \) and \( \varepsilon > 0 \), we define
\[
\eta_\varepsilon(t, x) := Q(x) + \eta_1 \ast \hat{u}(t, x) \\
= Q(x) + \int_{\mathbb{R}^2} \eta_\varepsilon(t - s, x - y) \hat{u}(s, y) \, ds \, dy.
\]
where \( \eta_\epsilon(t, x) := \epsilon^{-3} \eta(\epsilon^{-1} t, \epsilon^{-2} x) \). (Note that \( \eta_\epsilon \) is well-defined if \( \epsilon \) is sufficiently small.) Obviously, \( u_\epsilon \in C^\infty(I' \times \mathbb{R}^2) \). Since \( \tilde{u}(t, x) \to 0 \) as \( |x| \to \infty \) uniformly on any subinterval of \( I \), it follows that \( \eta_\epsilon \ast \tilde{u} \to \tilde{u} \) as \( \epsilon \to 0 \) in \( L^\infty(I' \times \mathbb{R}^2) \). In particular, the above claim implies that for sufficiently small \( \epsilon \), we can define

\[
U_n := |u_n|^{-1} u_n \quad (7.14)
\]

for sufficiently large \( n \in \mathbb{N} \). Then, \( U_n(t, \cdot) \) is \( m \)-equivariant for each \( t \), since we can check that

\[
\tilde{u}_n(t, R(\theta)x) = e^{in\theta R} \tilde{u}_n(t, x), \quad R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]

for all \( \theta \in \mathbb{T}^1 \). We can verify that \( \{U_n\}_n \) satisfies the desired properties, hence Step 1 is established.

Step 2. We show that \( u_\epsilon \) obtained in Step 1 can be chosen with satisfying (B). By Step 1, we may assume \( u_\epsilon(t) \in C(I; \dot{H}^1) \cap C^\infty(I \times \mathbb{R}^2) \). We take a cut-off function \( \phi_\epsilon \in C^\infty(\mathbb{R}) \) which satisfies \( 0 \leq \phi(s) \leq 1 \) for all \( s \in \mathbb{R} \), \( \phi(s) = 1 \) if \( |s| \leq 1 \), and \( \phi(s) = 0 \) when \( |s| \geq 2 \). And then we set \( \phi_n(s) := \phi(n^{-1}s) \) for \( n \in \mathbb{N} \). Define

\[
u_n(t, x) := Q(x) + \phi_n(r)\tilde{u}(t, x). \quad (7.16)
\]

Since \( \tilde{u}(t, x) \to 0 \) as \( |x| \to \infty \) uniformly on \( I' \), we can define

\[
U_n := |u_n|^{-1} u_n \quad (7.17)
\]

for sufficiently large \( n \). It is clear that \( U_n \) is \( m \)-equivariant and satisfies (A) and (B). Thus, it suffices to show (C) or (C)'. We can easily check that sup \( \sup_{t \in I'} \|u_n - u\|_{L^\infty} \to 0 \) as \( n \to \infty \). Moreover, denoting one of \( \partial_{x_1} \), \( \partial_{x_2} \), and \( \partial_t \) by \( \partial \), we have

\[
\|\partial u_n - \partial u\|_{L^2} \leq \frac{1}{n} \|\phi\|_{L^\infty} \|\tilde{u}(t)\chi_{n \leq |x| \leq 2n}\|_{L^2} + \|\tilde{u}\chi_{|x| \geq n}\|_{L^2} \quad (7.18)
\]

for each \( t \). Thus

\[
\sup_{t \in I'} \|\partial u_n - \partial u\|_{L^2} \lesssim \sup_{t \in I'} \|\tilde{u}(t)\chi_{n \leq |x| \leq 2n}\|_{L^\infty} + \sup_{t \in I'} \|\tilde{u}(t)\chi_{|x| \geq n}\|_{L^2} \to 0 \quad (7.19)
\]

as \( n \to \infty \). Hence, we have \( \sup_{t \in I'} \|\partial U_n - \partial u\|_{L^2} \to 0 \) as \( n \to \infty \) in the same manner as Step 1. Hence, we complete the proof.

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