An Energy Independent Scaling of Transverse Momentum Spectra of Direct (Prompt) Photons from Two-Body Processes in High-Energy Proton–Proton Collisions

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1. Introduction

Quantum chromodynamics (QCD) and similar strong interaction theories predict that Quark–Gluon plasma (QGP) may be generated in a high-temperature and high-density environment within a few microseconds after the little Big Bang in heavy ion collisions at high energy.[1–5] Understanding the dynamic evolution of quarks and gluons in nuclei is an important issue for modern nuclear physics.[2–5] The main reason for studying high-energy heavy ion collisions is to understand the formation and properties of QGP.[6–9] High-energy proton–proton (p+p), proton–nucleus, and nucleus–nucleus collisions can provide a high-temperature and/or high-density environment, which is similar to the early days of the Big Bang in the universe.[10–13] In the evolution process of high-energy collisions,[14–16] the chemical and kinetic freeze-outs are two very important stages. In the chemical freeze-out stage, the types and ratios of particles are fixed, and the system is in chemical equilibrium. In the kinetic freeze-out stage, the elastic collisions of particles disappear, and, hence, the kinematic properties of particles remain unchanged afterward.[15,17]

During the whole process of high-energy collisions, the particles are continuously emitted outward. The types and ratios of different particles are changeable before the kinetic free-out stage. In the collisions, some final products or particles are produced from initial and/or intermediate stages. When some primary particles produced in collisions are difficult to be detected and measured directly, measuring the final particles that carry a large amount information on system evolution becomes an important choice and tool for researchers. One can extract the information on collision process by analyzing the spectra of final particles. The information about evolution and various properties of the system is deduced by studying the properties of final particles. This study is useful to understand the evolution of early universe and structure of tight stars, the deconfinement of quarks and formation of QGP, as well as the mechanisms of particle production.[17–9] In abundant experimental data, the transverse momentum ($p_T$) spectra of particles are particularly important[18] due to the fact that they reflect the excitation degree and transverse structure of the system. Indeed, this information is related
to the QGP and particle production \cite{19-26} because QGP eventually decomposes into final particles.

Direct photons are defined as all photons generated directly in the scattering process, not from hadron decay. Direct photons provide a powerful tool for studying QGP. As a part of direct photons, prompt photons are defined as those coming from hard scattering of two partons from the incident hadrons, or those from the collinear fragmentation of high-\(p_T\) final state partons.\cite{27,28} Prompt photons provide a method and approach for the measurement of perturbation QCD (pQCD), and are also a probe for the study of the initial state of protons or nuclei. At the lowest order of pQCD, parton produced photons come from two hard scattering sub-processes: \(qg \rightarrow q\gamma\) (main), where \(q, g, \) and \(\gamma\) denote quark, gluon, and photon, respectively; \(q\bar{q} \rightarrow g\gamma\) (primary), where \(\bar{q}\) denotes antiquark. In addition, prompt photons can also be produced by higher-order processes such as bremsstrahlung or fragmentation. The collinear part of this process has proved to contribute at the lowest order. We do not need to select the prompt photons from the direct photons particularly. The study of \(p_T\) spectra of direct (prompt) photons will help to study the QCD and pQCD processes, and understand some properties of the initial state of protons or nuclei.\cite{27}

The invariant or differential cross-sections of direct (prompt) photons have been extensively studied in high-energy collisions. Several collaborations at the Large Hadron Collider of the European Organization for Nuclear Research (CERN, abbreviated from the previous French name of “Conseil Européen pour la Recherche Nucléaire”), the Relativistic Heavy Ion Collider of the Brookhaven National Laboratory, and other colliders have measured the \(p_T\) spectra of generated direct (prompt) photons in various collisions. Compared to nuclear collisions, \(p+p\) collisions are the basic and cleaner process with absence of the influence of nucleon-nucleon correlation and cold spectator nucleons. Understanding the characteristics of particle productions is very necessary for researchers to study the evolution of collision system and interactions among various particles.\cite{10-37}

The \(p_T\) spectrum of direct (prompt) photons is very wide, which distributes in a range from 0 to above 1000 GeV/c in some cases. In different \(p_T\) regions, the forms of spectrum may be different.\cite{38} Generally speaking, the spectrum in low-\(p_T\) (intermediate-\(p_T\)) region is related to the soft (intermediate) scattering process, and the spectrum in high-\(p_T\) region is related to the hard scattering process in which the direct photons are also called the prompt photons. However, there are not clear demarcation points to distinguish various \(p_T\) regions. In the production of multiple particles,\cite{39} in the context of considering a given collision energy and the constant particle species, we usually choose a fitting function with the best effect for \(p_T\) spectrum to extract some quantities such as the rapidity (\(y\)) or pseudorapidity (\(\eta\)) density (\(dN/dy\) or \(dN/d\eta\)), kinetic freeze-out temperature (\(T_{\text{kin}}\) or \(T_0\)), average transverse velocity (\(v_{\perp}\) or \(\beta_T\)), and etc., where \(N\) denotes the number of particles. In the two-body processes such as \(gg \rightarrow q\gamma\) and \(q\bar{q} \rightarrow g\gamma\), some parameters such as \(T_0\) and \(\beta_T\) may be unavailable. Instead, we may use the effective temperature (\(T\)) and others.

To fit the \(p_T\) spectrum of direct (prompt) photons, different models or functions can be used. These models or functions include, but are not limited to, the inverse power-law or Hagedorn function,\cite{40} Tsallis–Lévy\cite{41,42} or Tsallis–Pareto-type function,\cite{42-45} Bose–Einstein or Fermi–Dirac distribution, Boltzmann distribution,\cite{46,47} and so on. After constant exploration, we know that it is difficult to fit well the spectrum in both the low- and high-\(p_T\) regions with only a simple probability density function.\cite{19} Therefore, for a wider \(p_T\) spectrum, a multi-component (at least a two-component) function is generally used in the fit.\cite{48} In the two-component function, the first component represents the low-\(p_T\) region, while the second component represents the high-\(p_T\) region. However, it is not ideal if the two-component function is superimposed in terms of usual function (in which the parameters are correlated) or step function (in which the connection point is unsmooth).\cite{49,50} We hope to find a new function which can fit well the spectrum of whole \(p_T\) region.

In view of the problem of two-component function, in this paper, we used a few one-component functions to describe the \(p_T\) spectrum of direct (prompt) photons and compare their fitting qualities. We collected the \(p_T\) spectra of direct (prompt) photons from two-body processes in \(p+p\) collisions over a center-of-mass energy (\(\sqrt{s}\)) range from 24.3 GeV to 13 TeV, which are measured by a few international collaborations. The values of parameters are obtained and the trends of parameters are analyzed. Thus, an energy independent scaling, the \(x_T\) scaling\cite{11,52} of \(p_T\) spectra are obtained and analyzed.

The theoretical formalism used in this paper is based on the statistical models (nonextensive statistics). Several models were tested, like the revised Tsallis–Pareto-type (TP-like) function at the particle level and the convolution of two TP-like functions at the parton level as well as a revised Tsallis-like function at the particle level and the root-sum-of-squares of two revised Tsallis-like functions at the parton level. This paper provides useful information for high energy physics phenomenologists and researches working on nonextensive statistics applied to scattering processes for addressing direct (prompt) photon production. The formalism considered in this paper is widely used to describe the charged particle production in high energy collisions, and for the first time it is used for the direct (prompt) photon production by us.

The rest of this paper is structured as follows. The formalism and method are briefly introduced in Section 2. The results and discussion are given in Section 3. In Section 4, we summarize our main observations and conclusions.

2. Formalism and Method

2.1. The TP-Like Function and the Convolution of Two Functions

According to refs. \cite{42-45}, we know that the Tsallis–Pareto-type function can be written as

\[
\mathcal{f}(p_T) = \frac{\mathcal{A}(n_0 - 1)(n_0 - 2)}{n_0 T[n_0 T + m_0(n_0 - 2)]} p_T^n m_T \times \left(1 + \frac{m_T - m_0}{n_0 T}\right)^{-n_0} \tag{1}
\]

where the effective temperature \(T\) and power index \(n_0\) are free parameters, \(A\) is the normalization constant, \(m_0\) is the rest mass, and \(m_T = \sqrt{p_T^2 + m_0^2}\) is the transverse mass. Equation (1) can be
simplified and rewritten to the following form\textsuperscript{[53–56]}

\[ f(p_T) = C p_T m_T (1 + \frac{m_T - m_0}{n_0 T})^{-a_0} \]  

(2)

where \( C \) is the normalization constant related to \( T, n_0, \) and \( m_0. \)

The above probability density function is expected to fit simultaneously the spectrum in low- and high-\( p_T \) regions. However, our studies show that the spectrum in very-low-\( p_T \) region cannot be fitted well by the equation. Empirically, a revised form

\[ f(p_T) = C(p_T, m_T) a_0 (1 + \frac{m_T - m_0}{n_0 T})^{-a_0} \]  

(3)

is much better in the fitting treatment, where the power exponent \( a_0 \) is a correction factor. The normalization constants in Equations (2) and (3) are different, though both of them are denoted by the same symbol \( C. \) Meanwhile, although \( a_0 \) is dimensionless, \( C \) has a dimension so that the units in the two sides of the equation are the same \([\text{GeV}/c]^2\) in general. In addition, we have used the same symbols, \( f(p_T) \), in Equations (2) and (3), though the functions are different.

Equation (3) is in fact a revised Tsallis–Pareto-type function. For convenience, we call Equation (3) the TP-like function. The parameter \( T \) used above is not a real temperature, but the effective temperature. The parameter \( n_0 \) reflects the degree of equilibrium of the system in terms of the grand canonical ensemble for a large amount of events which are indeed true in experiments. A larger \( n_0 \) corresponds to a more equilibrium of the system. As a function that describes the \( p_T \) spectrum at the particle level, Equation (3) is still not ideal in some cases. Empirically, we need to consider the convolution of two TP-like functions\textsuperscript{[39]} if we assume that two participant quarks or partons contribute to the \( p_T \) spectrum in terms of two energy sources.

The first and second energy sources contribute \( p_1 \) and \( p_2 \) to \( p_T \) respectively, that is, \( p_T = p_{1T} + p_{2T} \), where \( p_{1T} \) and \( p_{2T} \) obey the TP-like function. Due to Equation (3), we have the probability density function obeyed by the \( i \)-th energy source to be

\[ f_i(p_T) = C_i(p_{iT}, m_{iT}) a_0 (1 + \frac{m_{iT} - m_{0T}}{n_{0T} T})^{-a_0} \]  

(4)

where \( C_i \) is the normalization constant related to the parameters, \( m_{iT} = \sqrt{p_{iT}^2 + m_T^2} \) is the transverse mass, and \( m_{0T} \) is the constituent mass for the \( i \)-th quark or parton. In the two-body process, the lightest constituent mass (0.31 GeV/c\(^2\)) of quark or \( d \) is used for each participant partons in \( p+\bar{p} \) collisions.

The \( p_T \) distribution contributed by two participant partons is the convolution of two TP-like functions. That is \textsuperscript{[57]}

\[ f(p_T) = \int_0^{p_T} f_1(p_{1T}) f_2(p_{2T} - p_{1T}) dp_{1T} \]

\[ = \int_0^{p_T} f_2(p_{2T} - p_{1T}) f_1(p_{1T}) dp_{2T} \]  

(5)

If Equation (3) fits the \( p_T \) spectrum at the particle level in which the rest mass of particle is used, Equation (5) fits the \( p_T \) spectrum at the quark or parton level in which the constituent mass of quark is used. Although Equation (3) can fit the \( p_T \) spectrum at the particle level satisfactorily, Equation (5) can fit the \( p_T \) spectrum at the quark level more significantly. Since Equation (5) describes the contributions of participant quarks or partons, we may call it the result of the participant quark model. In fact, Equation (5), that is, the convolution, is also a result in the framework of the multi-source thermal model\textsuperscript{[58,59]}

### 2.2. The Revised Tsallis-Like Function and the Root-Sum-of-Squares of Two Functions

According to refs. [6, 42, 43, 60–62], we know that, at mid-rapidity, the Tsallis-like distribution obeyed by \( p_T \) can be written as

\[ f(p_T) = C p_T m_T \times \left[ 1 + \frac{(q - 1)(m_T - \mu - m_0)}{T} \right]^{-q/(q-1)} \]  

(6)

where \( \mu \) is the chemical potential, \( q \) is the entropy index that describes the degree of equilibrium, and \( C \) is the normalization constant which is different from those in Equations (2) and (3). Equation (6) is very similar to Equation (2), if not equal. Comparing with Equation (2), Equation (6) is more in line with the requirement of thermodynamic consistency due to the fact that \( q/(q - 1) \) in Equation (6) replaced \( n_0 = 1/(q - 1) \) in Equation (2), regardless of the value of \( \mu \). The meanings of the indexes \( q \) and \( n_0 \) are consistent. The closer the index \( q \) to 1 is (the larger the index \( n_0 \) is), the larger the degree of equilibrium of the system is.

Similar to Equation (2), in fitting the \( p_T \) spectrum, Equation (6) is not ideal in very-low-\( p_T \) region, too. Empirically, we may revise Equation (6) to the following form

\[ f(p_T) = C(p_T, m_T) a_0 \times \left[ 1 + \frac{(q - 1)(m_T - \mu - m_0)}{T} \right]^{-q/(q-1)} \]  

(7)

It is similar to Equation (3). Equation (7) is called the revised Tsallis-like function by us, to distinguish it from the TP-like function. For the \( i \)-th parton, we have

\[ f_i(p_T) = C(p_{iT}, m_{iT}) a_0 \times \left[ 1 + \frac{(q - 1)(m_{iT} - \mu - m_0)}{T} \right]^{-q/(q-1)} \]  

(8)

where \( \mu \) denotes the chemical potential of the \( i \)-th parton.

Of course we can do the convolution of two Equations (8). The result is similar to the convolution of two Equations (4), regardless of the value of \( \mu \). Although \( \mu \) causes \( m_0 \) to increase and different parameters can be obtained, we may combine \( \mu \) with \( m_0 \) and use smaller \( m_0 \) to reduce the influence of \( \mu \). For other possibilities on the synthesis of \( p_{1T} \) and \( p_{2T} \), we give up to use the convolution of two Equations (8). Instead, we consider the root-sum-of-squares of two Equations (8) in which the vectors \( p_{1T} \) and \( p_{2T} \) are perpendicular, and they are the two components of the vector \( p_T \). That is \( p_T = \sqrt{p_{1T}^2 + p_{2T}^2} \). This relation is different from \( p_T = p_{1T} + p_{2T} \) discussed in the previous subsection in which the vectors \( p_{1T} \) and \( p_{2T} \) are parallel.

Let \( \phi \) denote the azimuthal angle of the vector \( p_T \) relative to the vector \( p_{1T} \). According to refs. [57, 63], \( p_{1T} = p_T \cos \phi, p_{2T} = p_T \sin \phi \). Therefore, we can obtain the unified probability density function.
of $p_T$ and $\phi$ as

$$f_{p_T,\phi}(p_T, \phi) = p_T f_1(p_T, p_{T_1})$$

$$= p_T f_1(p_T, p_{T_1})$$

$$= p_T f_1(p_T, \cos \phi) f_2(p_T, \sin \phi)$$

(9)

The probability density function of $p_T$ is

$$f(p_T) = \int_0^{2\pi} f_{p_T,\phi}(p_T, \phi) d\phi$$

$$= p_T \int_0^{2\pi} f_1(p_T, \cos \phi) f_2(p_T, \sin \phi) d\phi$$

(10)

We now discuss the chemical potential $\mu_i$ in Equation (8) that is used in Equation (10). It is known that the chemical potential has a great influence on $p_T$ spectrum at low energy, but has a little influence at high energy.\cite{64–70} According to refs. \cite{71–73}, the chemical potential $\mu_B$ of baryons in central nucleus-nucleus collisions is related to the center-of-mass energy $\sqrt{s_{NN}}$ per nucleon pair. Empirically, one has

$$\mu_B = \frac{1.303 \text{ (GeV)}}{1 + 0.286 \sqrt{s_{NN}/\text{GeV}}}$$

(11)

It is known from ref. \cite{64} that $\mu_B = \mu_b/3$ because a baryon consists of three quarks. In $p+\bar{p}$ collisions, we have approximately the same $\mu_B$ as given is Equation (11). In this paper, we consider two cases, $\mu = \mu_B/3$ and $\mu = 0$, for checking the influence of nonzero chemical potential on $p_T$ spectrum.

In the above discussions, the case of mid-rapidity is used. This means that we have used $y \approx 0$ for a narrow rapidity range which covers the mid-rapidity. If the narrow rapidity range does not cover the mid-rapidity, one may simply shift it to cover the mid-rapidity by adding or deducting a value, based on the additivity of rapidity. This treatment is used to exclude the kinetic energy of directed motion of the source, which should not contribute to the temperature. If the rapidity range is not too narrow, for example, it is larger than three rapidity units, one may consider an integral for $y$. Simply, one may replace the transverse mass by the energy obtained by the product of the transverse mass and $\cosh y$. For the narrow rapidity range, considering the integral for $y$ may cause mainly the variation of normalization and does not cause the free parameters to change significantly.

In this work, the main modification for the TP-like function and the Tsallis-like distribution is the introduction of correction index $a_0$. This introduction seems to cause the variation of thermodynamic consistency.\cite{74} However, we may think that the thermodynamic consistency is mainly determined by the remainder included the added item $(p_T m_T)^{a_0-1}$ or $(p_T m_{B_\pi})^{a_0-1}$ in Equations (3), (4), (7), and (8). The influence of $(p_T m_T)^{a_0-1}$ or $(p_T m_{B_\pi})^{a_0-1}$ can be combined into the normalization. This combination also includes the dimension. As a result, the dimensions for the two sides of the equations are finally the same. As mentioned in the last subsection, both the units in the two sides of the equations are (GeV/c)$^{-1}$, if the unit of $p_T$ is GeV/c.

In addition, as free parameters, the values of $q$ and $a_0$ are determined by the $p_T$ spectra. If the $p_T$ spectra can be approximately fitted by the Boltzmann–Gibbs distribution, we have naturally $q \rightarrow 1$ and $a_0 \rightarrow 1$. Then, the Boltzmann–Gibbs distribution is recovered. If the $p_T$ spectra cannot be fitted by the Boltzmann–Gibbs distribution, we have $q \neq 1$ and any $a_0$, or $q \rightarrow 1$ and $a_0 \neq 1$. It does not matter whether the Boltzmann–Gibbs distribution is recovered or not. It depends on the values of parameters themselves. In the particular case where $q \rightarrow 1$ and $a_0 \neq 1$, the Boltzmann–Gibbs distribution is modified. In the particular case where $q \neq 1$ and $a_0 \rightarrow 1$, the TP-like function and the Tsallis-like distribution are recovered.
2.3. An Energy Independent Scaling—the $x_T$ Scaling

We know that the prompt photons refer to all photons directly generated by hard processes or bremsstrahlung. From the $p_T$ spectra of prompt photons in p+p collisions, it can be seen that the fragmentation of hard scattering partons is the main mechanism of high-$p_T$ prompt photons. It is predicted that this production mechanism may produce a behavior called $x_T$ scaling, where $x_T$ is defined as $x_T = 2p_T / \sqrt{s}$. This scaling behavior was first discovered in the preliminary data from the intersecting storage rings (ISR) of CERN.\(^{[73]}\) Up to now, this scaling behavior has been used to analyze the transformation and other properties of the soft and hard processes which produce particles.\(^{[74]}\)

According to ref. \(^{[76]}\), in the corresponding $x_T$ range and for the case of mid-rapidity, the invariant cross-section can be expressed as

$$E \frac{d^3 \sigma}{dp^3} = \frac{1}{p_T} F(x_T) = \frac{1}{\sqrt{s}} G(x_T)$$

where $\sigma$ is the cross-section and $F(x_T)$ and $G(x_T)$ are general scaling functions which will be determined from the quoted data. Focus in the index $n$, especially noting that this index is not a constant. In addition, it is different from the power index in the TP-like function introduced in the beginning of this section. The value of $n$ depends on the energy $\sqrt{s}$ of the constituent parton and the range of $x_T$. At least, $n$ is related to $\sqrt{s}$.

To know the value of $n$, one may use two methods based on Equation \((12)^{[76]}\)
i) Using the ratio of logarithmic yield ratio to logarithmic energy ratio

\[ n = -\frac{\log[\text{yield}(x_T, \sqrt{s_1})/\text{yield}(x_T, \sqrt{s_2})]}{\log(\sqrt{s_1}/\sqrt{s_2})} \]  

(13)

ii) Fitting the \( x_T \) distribution in terms of invariant cross-section at different energies

\[ E \frac{d^3\sigma}{dp^3} = \left( \frac{B}{\sqrt{s/\text{GeV}}} \right)^n x_T^{-m} \]  

(14)

where \( B \) and \( m \) are parameters related to \( \sqrt{s} \) and \( x_T \). In a real treatment, the form of

\[ (\sqrt{s/\text{GeV}})^n E \frac{d^3\sigma}{dp^3} = G(x_T) \]  

(15)

is used to fit the spectra of \( x_T \) scaling, where the general scaling function \( G(x_T) \) is pending to be determined.

### 3. Results and Discussion

#### 3.1. Comparison with Data

In the above section, we have used the probability density function, \( f(x_T) = (1/N)dN/dx_T \). The experimental data quoted in this paper are in the form of invariant cross-section, \( E d^3\sigma/dp^3 = (1/2\pi p_T) d^3\sigma/dp_T dy \). Thus, one has the relation \( (1/2\pi p_T) d\sigma_f(x_T)/dy = E d^3\sigma/dp^3 \). Here, \( f(x_T)/dy \) only means that \( f(x_T) \) is divided by a concrete amount \( dy \) and \( d\sigma_f \) is the normalization constant which is used in the calculations to compare with the data. If the experimental data are in other forms, one may adjust the relation conveniently. For examples, one has at least the following relations: \( (1/2\pi p_T) N_d f(x_T)/dy \)
\[ \frac{dy}{dN} = \frac{Ed^3N}{dp^3}, \quad \sigma_\text{eff}(p_T)/dy = d^3\sigma/dp_T dy, \quad N_{\text{tot}}(p_T)/dy = d^3N/dp_T dy, \quad \sigma_\text{eff}(p_T) = d\sigma/dp_T, \quad \text{and} \quad N_{\text{tot}}(p_T) = dN/dp_T, \] where the representations in the right side of these equations are various possible forms used in experiments, and \( N_0 \) is the normalization constant used to compare with the data.

As mentioned above, the function form of \( f(p_T) = (1/N) dN/dp_T \) is used in the calculations. This renders that \( f(p_T) \) normalized to 1. The normalization constant in \( f(p_T) \) is not related to \( \sigma_0 \) and \( N_0 \). The latter two are related to the data in the comparison. In addition, in the distributions based on nonextensive statistics satisfying thermodynamic consistency relations, the normalization which is proportional to the volume \( V \) and the degeneracy factor \( g \) is renormalized in \( f(p_T) \) because the probability density function is needed for us. After the renormalization, although the volume no longer appears, its meaning can be also reflected by \( N_0 \). The larger \( N_0 \) is, the larger the volume is.

**Figure 1a** shows the invariant cross-section of direct photons produced in two-body processes in p+p collisions at different collision energies with different mid-\( \eta \) ranges. The points (symbols) in different colors in the picture represent different rapidities, energies, and collaborations. For clarity, the results (symbols) corresponding to the UA6 Collaboration at 24.3 GeV with rapidity \( -0.1 < |\eta| < 0.9 \) [7], the R806 Collaboration at 63 GeV with rapidity \( |\eta| < 0.2 \) [7], and the R110 Collaboration at 63 GeV with rapidity \( |\eta| < 0.8 \) [7] are multiplied by the factors \( 10^1, 10^2, \) and \( 10^4 \), respectively. The result corresponding to the CCOR Collaboration at \( 1.4 \) GeV with rapidity \( |\eta| < 0.45 \) [7] is not rescaled (i.e., the rescaling factor is \( 10^0 \) in this case). The solid, dashed, dot-dashed, and dotted curves in Figure 1a represent the results obtained by Equations (3), (5), (10) with \( \mu_1 = \mu_2 / 3 \), and (10) with \( \mu_1 = 0 \), respectively. Because Equations (3) and (7) are similar to each other, only the result from Equation (3) is presented.
In the fits, we have used $m_i = 0$ for direct photon and $m_{i\gamma} = 0.31 \text{ GeV}/c^2$ for participant parton in the two-body process, where the heavier partons are not considered due to their very small production probability. The least square method is used to determine the values of free parameters and normalization constant, and the statistical simulation method is used to determine the errors. The free parameters ($T$, $\eta$, and $a_0$), the normalization constant ($\sigma_0$), $\chi^2$, and the number of degrees of freedom (ndof) obtained from the fits by Equations (3) and (5) are listed in Tables 1 and 2 respectively. The values of $T$, $q$, $a_0$, $\sigma_0$, $\chi^2$, and ndof obtained from the fits by Equation (10) with $\mu_1 = \mu_0/3$ and $\mu_2 = 0$ are listed in Tables 3 and 4 respectively.

In order to compare the degree of deviation between the fitting results and the data, Figure 1b–e show the ratios of data to fit obtained from four cases: Equations (3), (5), (10) with $\mu_1 = \mu_0/3$, and (10) with $\mu_1 = 0$, orderly. One can see that the fit results reproduce satisfactorily the data in the whole $p_T$ range. As seen from Figure 1a and corresponding results of Tables 1–4, the fits in four cases are in satisfactory agreement with the experimental data. Although the data of direct photons generated in $p$–$p$ collisions measured by the UA6, R108, R086, and R110 Collaborations at 24.3, 62.4, and 63 GeV[77–80] can be fitted by Equations (3), (5), and (10), the mid-$\eta$ ranges used in different experiments are not the same exactly. These differences do not affect significantly the results due to the fact that we have used the results per $\eta$ unit.

It seems that we do not need to use so many fitting functions, because Equation (3) fits well enough the experimental data. However, the fits using Equations (5) and (10) can reveal more quantities due to different physical ideas used. In particular, the results fitted by various functions can be compared with each other. So it is useful to use different functions to fit the data.
Figure 2. Same as in Figure 1, but showing the results at 200 GeV, 300 GeV, and 2.76 TeV. a) The symbols represent the data measured by the PHENIX, NA24, CMS, and ALICE Collaborations.\cite{81–84}

Similar to Figure 1a, Figures 2a–8a show the invariant cross-section of direct photons produced in two-body processes in p+p collisions at higher energies. The curves have the same meanings as Figure 1a, and the parameters are listed in Tables 1–4.

In Figure 2a, the points (symbols) in different colors represent orderly the data measured by the PHENIX Collaboration at 200 GeV with $|\eta| < 0.35$;\cite{81} the NA24 Collaboration at 300 GeV with $-0.65 < \eta < 0.52$;\cite{82} the CMS Collaboration at 2.76 TeV with $|\eta| < 0.9$;\cite{83} and the ALICE Collaboration at 2.76 TeV with $|\eta| < 1.44$ \cite{84} with different rescaling factors ($10^6$, $10^5$, $10^4$, and $10^3$) marked in the panel.

In Figure 3a, the symbols represent orderly the data measured by the CMS Collaboration at 5.02 TeV with $|\eta| < 1.44$;\cite{85} and at 7 TeV with $|\eta| < 1.45$;\cite{86} as well as the results measured by the ATLAS Collaboration at 7 TeV with $|\eta| < 1.37$;\cite{87} and by the ALICE Collaboration at 7 TeV with $|\eta| < 0.27$;\cite{88} with different rescaling factors ($10^6$, $10^5$, $10^4$, and $10^3$) marked in the panel, where the rescaling factor $10^6$ is not marked.

The symbols in Figure 4a represent orderly the data measured by the ATLAS Collaboration at 7 TeV with $|\eta| < 0.6$, $0.6 < |\eta| < 1.37$, $1.52 < |\eta| < 1.81$, and $1.81 < |\eta| < 2.37$;\cite{90,91}

The symbols in Figure 5a represent orderly the data measured by the CMS Collaboration at 7 TeV with $|\eta| < 0.9$, $0.9 < |\eta| < 1.44$, $1.57 < |\eta| < 2.1$, and $2.1 < |\eta| < 2.5$;\cite{91}

Different rescaling factors ($10^6$, $10^5$, $10^4$, and $10^3$) are used for clarity.

In Figure 6a, the symbols represent orderly the data measured by the ATLAS Collaboration at 8 TeV with $|\eta| < 0.6$, $0.6 < |\eta| < 1.37$, $1.56 < |\eta| < 1.81$, and $1.81 < |\eta| < 2.37$;\cite{92}

In Figure 7a, the symbols represent orderly the data measured by the CMS Collaboration at 13 TeV with $|\eta| < 0.8$, $0.8 < |\eta| < 1.44$, $1.57 < |\eta| < 2.1$, and $2.1 < |\eta| < 2.5$;\cite{93}

In Figure 8a, the symbols represent orderly the data measured by the ATLAS Collaboration at 13 TeV with $|\eta| < 0.6$, $0.6 < |\eta| < 1.37$, $1.56 < |\eta| < 1.81$, and
1.81 < |\eta| < 2.37.\cite{94} In the three figures, the rescaling factors are orderly $10^4$, $10^2$, $10^0$, and $10^{-2}$.

The ratios of data to fit in Figures 2–8 are the same as in Figure 1. In some cases, the deviations between the fits and data are significant, which means that the fits are approximate. In most cases, the fits are satisfactory and acceptable. At least, the trends of data can be revealed from the fits by Equations (3), (5), and (10). From the values of $\chi^2$ listed in Tables 1–4, one can see that the fitting qualities of the three equations in four cases are nearly the same. It is hard to judge which one is better. In our opinion, the method using the convolution of two TP-like functions is more convenient to be extended to the convolution of three or more functions and to be explained by the underlying multi-parton process, even performing the convolution by other functions.

Concerning the fit quality, from Tables 1–4 one can see that the values for $\chi^2$/ndof are systematically higher than one. On the other hand, fits using Tsallis-like distribution on particle level present consistently $\chi^2$/ndof < 1 in case of charged particle production in p+p collisions.\cite{43} The reason is that the present work deals with the data sample for direct (prompt) photons which have lower statistics and wider $p_T$ range than those for charged particles in experiments.

3.2. Trends of Parameters and Discussion on Methodology

We now study the variation trend of the extracted free parameters. Figure 9a–d show the dependences of effective temperature $T$, power index $n_0$, entropy index $q$, and correction index $a_0$ on collision energy $\sqrt{s}$, respectively. The symbols represent the values of parameters listed in Tables 1–4 and extracted from Figures 1–8 by using Equations (3), (5), and (10). One can see that with the increase of $\sqrt{s}$, the parameters $T$, $q$, and $a_0$ increase, and the parameter $n_0$ decreases.

The results from Figure 9 is understandable. At higher energy, the collision system stays at higher excitation state with
lesser probability of equilibrium. The parameters $T$ and $q$ ($n_0$) are naturally larger (smaller). In addition, the spectra of $p_T$ are wider at higher energy. This causes a valley in the spectra at very low $p_T$, which needs a larger $a_0$ to fit it. In some references (for instance[43,95]) it is proposed a QCD inspired model for the energy dependence of parameters $T$ and $q$ concerning charged particle production. In general, a $\ln \sqrt{s}$ dependence is presented. From Figure 9, it seems this is not the case (it is steeper) for direct (prompt) photons. The reason is that, compared with charged particles, direct (prompt) photons are emitted earlier in the source with higher excitation degree. In addition, resonance decay and medium effect affect charged particle production more significantly. These factors cause different behaviors for two types of particles.

The effective temperature is fitted in particle level and in parton level. Both of them contain the contributions of thermal motion and transverse flow effect (if available in $p+p$ collisions). The latter should be excluded from the contributions to extract the real temperature. As a nonreal temperature, the effective temperature is model-dependent. So, the effective temperatures obtained from different functions may be different. In addition, direct (prompt) photons are emitted earlier than light charged particles in general. This is also why the effective temperature obtained from this work is quite higher than ones typically obtained in Tsallis-like fits to light charged particle production.[61,96]

In most cases, Figure 9 shows that the parameters from different equations are similar to each other. However, in Figure 9d, although the same symbol $a_0$ is used, the values may be different in Equations (3) and (5) due to Equation (3) having the item $(p_T \sqrt{p_T^2 + m_0^2})^{a_0}$ and Equation (5) having the product of $(p_T \sqrt{p_T^2 + m_0^2})^{a_0}$ and $(p_T - p_{T_1}) \sqrt{(p_T - p_{T_1})^2 + m_0^2}$. The influence of $a_0$ in Equation (5) is double. This renders the difference in $a_0$ from Equations (3) and (5). This difference does not happen in Equation (10) due to another calculation.
can see that Equation (16) is approximately an anti-correlation in double logarithmic representations.

The form of the function \( (\sqrt{s}/\text{GeV})^n \sigma / dp^3 = (\sqrt{s}/\text{GeV})^n (1/2\pi p_T) d^2\sigma / dp_T dy \) in the dependence on the variable \( x_T = 2p_T/\sqrt{s} \) can be empirically constructed. The concrete form obtained in the present work is a simple expression with three parameters. This scaling law renders that there is common similarity and universality in many aspects in high energy collisions. The underlying physics reason is that participant quarks or partons from the collision system take part in the considered scattering process, which is related to the parton saturation and higher-order QCD contribution from \( gg \) and \( q\bar{q} \) channels.

The functions which are used to fit the spectra in the present work are not sole ones. Although we have not presented the related results, Equation (7) is also suitable to fit the spectra due to small difference from Equation (3). As mentioned in the second section, \( n_0 = 1/(q-1) \) and \( q \) is close to 1. This causes Equation (7) to be similar to Equation (3). However, the difference between Equations (5) and (10) is large. Their results are only similar in the available data region in the \( p_T \) spectra.

We would like to point out that Equations (5) and (10) are two special cases from two participant partons. If the vectors \( p_1 \) and \( p_2 \) are parallel, we have \( p_T = p_1 + p_2 \) and Equation (5). If the vectors \( p_{1z} \) and \( p_{2z} \) are perpendicular, we have \( p_T = \sqrt{p_{1z}^2 + p_{2z}^2} \) and Equation (10). If the vectors \( p_1 \) and \( p_2 \) form an arbitrary angle, the analytical expression of \( p_T \) is not available for us. Instead, we may use the Monte Carlo calculation to give many concrete \( p_T \) by using the relation \( p_T = \sqrt{p_{1z}^2 + p_{2z}^2} \) and Equation (10). The values of \( p_{1z} \) and \( p_{2z} \) can be evenly distributed in \([0, 2\pi] \), and the values of \( p_1 \) and \( p_2 \) obey Equation (4) or (8). After repeated calculations for many times, the distribution of \( p_T \) can be obtained by statistics. Our exploratory research shows that the Monte Carlo calculation is suitable.

If the participants are three or more partons, one may extend Equations (5) and (10) in principle. In fact, one may consider the contributions of the first two partons, and combine the contributions with the third one. Then, one may combine the contributions of the first three partons with the fourth one, and so on. In the case of the number of participants being large, the analytical expression becomes complex and inconvenient. Instead, one may use the Monte Carlo calculation for the two special cases and the general case. What one does is just adding items in the expressions of \( p_T \) for the three or more participant partons.

Before summary and conclusions, we would like to collect a few other physical approaches describing the spectra of direct (prompt) photons. For examples, i) QCD color dipole picture in which the low-\( p_T \) region is described by the parton saturation effects and pQCD evolution is reproduced accurately. The \( x_T \) scaling can be also addressed in this formalism, ii) High energy or \( k_t \)-factorization approach which is based on off-shell matrix elements to compute the sub-process cross section. Building blocks include unintegrated parton distribution functions (uiPDFs), iii) Next-to-leading-order (NLO), next-to-next-to-leading-order (NNLO), and
Figure 9. Dependences of a) $T$, b) $n_0$, c) $q$, and d) $a_0$ on $\sqrt{s}$. As marked in the panels, $T$ and $a_0$ are obtained from four cases, and $n_0$ and $q$ are obtained from two cases.

next-to-next-to-leading-order (NNNLO) QCD calculations$^{[109–111]}$ in which the well known pQCD approach usually implemented in the Monte Carlo simulations. iv) Color glass condensate effective field theory (CGC EFT)$^{[112–117]}$ in which the dipole forward scattering amplitude is given in terms of fundamental lightlike Wilson lines. It is known at NLO accuracy level. The present paper is consistent with these approaches.

In addition, we would like to emphasize the significance of the present work. We have used two types of parameterization which come from the statistical method to fit the direct (prompt) photon spectra as measured in $p+p$ collisions in a wide range of colliding energy and found a scaling form to describe the spectra. Although the direct (prompt) photon spectra with large $p_T$ at collider energies can be described by pQCD and similar theoretical treatments, the present work proposes an alternative method, that is, the statistical method that was developed for charged particle production in high-energy heavy ion collisions, to describe the direct (prompt) photon spectra in $p+p$ collisions. These fits conform that the underlying participants in $p+p$ collisions are partons, but not protons. Participants are partons, which is the source of the common similarity and universality.$^{[97–104]}$

4. Summary and Conclusions

We summarize here our main observations and conclusions.

(a) In this paper, we have analyzed the transverse momentum spectrum of direct photons generated at mid-rapidity in $p+p$ collisions over an energy range from 24.3 GeV to 13 TeV. We have conducted the research through two methods: i) The TP-like function at the particle level and the convolution of two TP-like functions at the parton level; ii) The revised Tsallis-like function at the particle level and the root-sum-of-squares of two revised Tsallis-like functions at the parton level. The calculated results can be used to fit the experimental data measured by international collaborations.

(b) In the first method, the $p_T$ value of direct photon is directly regarded as both the subjecting to the TP-like function and the equaling to the sum of transverse momenta of two participant partons. The contribution of each participant parton to the $p_T$ spectrum of particles is also the TP-like function. In the second method, we regard $p_T$ of direct photon as both the subjecting to the revised Tsallis-like function and the equaling to the root-sum-of-squares of transverse momenta.
of two participant partons. The transverse momentum of partons also obeys the revised Tsallis-like function. In the two methods, we describe the $p_T$ spectrum not only at the particle level, but also at the parton level in the framework of multi-source thermal model. The fitting results of the two methods are basically consistent with the experimental data. Although it is hard to judge which case is more suitable due to the incomplete $p_T$ range in data, the idea at the parton level in the first method is more convenient to be extended to the scene of multiple participant partons.

(c) The two methods correspond to two special cases of the relation between the transverse momenta of two partons. In the first method, the transverse momenta of two partons are parallel. In the second method, the transverse momenta of two partons are perpendicular. In the general case, the transverse momenta of two partons may have any azimuthal angles in

Figure 10. a) $n$ as the function of $x_T$. The symbols are obtained from Equation (13) due to the fittings. b–f) $(\sqrt{s}/GeV)^n E^3 d^3 \sigma/dp^3$ as the functions of $x_T$ at different energies with various $|\eta|$ marked in the panels. The symbols are obtained from Equation (15) due to the fittings, where various $n$ are used as those listed in Table 5. The curves are the result of fits by $G(x_T)$ presented in Equation (16).
Figure 10b UA6 24.3 GeV |0.1 < η < 0.9 | 6.0 ± 0.2
Figure 10b CCOR 62.4 GeV |0.1 < η < 0.9 | 6.4 ± 0.2
Figure 10b R806 63 GeV |0.1 < η < 0.9 | 6.5 ± 0.2
Figure 10b PHENIX 100 GeV |0.1 < η < 0.9 | 6.5 ± 0.2

Table 5. Summary of the parameter n for the spectra with the form of x_T scaling in p+p collisions. These n are used for the symbols in Figure 10b–f in which the curves are presented by Equation (16).

(d) Based on the analyses of the p_T spectra of direct photons, an energy independent scaling, that is, the x_T scaling, is obtained to show approximately anti-correlation in double logarithmic representations. Not only for the spectra in low-p_T region, but also for the spectra in high-p_T region, the data follow approximately the function (\sqrt{s}/GeV)\eta Ed\sigma/dp = a x_T^{2\eta/\sqrt{s}}. Suitable n may show an energy independent scaling which is expressed by (\sqrt{s}/GeV)\eta Ed\sigma/dp = a x_T^{2\eta/\sqrt{s}}. This scaling law renders that there are many similarities and universality in some distributions in various collisions at high energies. The underlying physics reason is that the participant partons, but not nucleons, from the collision system take part in the two-body scattering process, which involves the parton saturation and higher-order QCD contribution from gg and gq channels.

Acknowledgements

The work of Q.Z. and F.H.L. was supported by the National Natural Science Foundation of China under Grant Nos. 12147215, 12047571, 11575103, and 11947418, the Shanxi Provincial Natural Science Foundation under Grant No. 201901D111043, and the Fund for Shanxi “1331 Project” Key Subjects Construction. The work of Y.Q. G. was supported by the Scientific and Technological Innovation Programs of Higher Education Institutions in Shansi (STIP) under Grant No. 201901D111043. The work of K.K.O. was supported by the Ministry of Innovative Development of the Republic of Uzbekistan within the fundamental project No. F3-20200929146 on analysis of open data on heavy-ion collisions at Relativistic Heavy Ion Collider and Large Hadron Collider.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Keywords
direct photons, revised Tsallis-like functions, TP-like functions, transverse momentum spectra, x_T scaling

Received: December 4, 2021
Revised: December 26, 2021
Published online: January 15, 2022

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