Kahler geometry on toric manifolds, and some other manifolds with large symmetry

S. K. Donaldson

April 14, 2008

Contents

1 Background
   1.1 Gauge theory and holomorphic bundles. . . . . . . . . . . . . . . 2
   1.2 Symplectic and complex structures . . . . . . . . . . . . . . . 3
   1.3 The equations . . . . . . . . . . . . . . . . . . . . . . . . . . . 4

2 Toric manifolds
   2.1 Local differential geometry . . . . . . . . . . . . . . . . . . . . 5
      2.1.1 Complex coordinates . . . . . . . . . . . . . . . . . . . . 5
      2.1.2 Symplectic coordinates . . . . . . . . . . . . . . . . . . . 6
   2.2 The global structure . . . . . . . . . . . . . . . . . . . . . . . . 9
      2.2.1 Complex charts . . . . . . . . . . . . . . . . . . . . . . . 10
      2.2.2 Symplectic construction . . . . . . . . . . . . . . . . . . 12
      2.2.3 Algebraic construction . . . . . . . . . . . . . . . . . . . 13
      2.2.4 Real forms . . . . . . . . . . . . . . . . . . . . . . . . . 14
   2.3 Algebraic metrics and asymptotics . . . . . . . . . . . . . . . . 15
      2.3.1 Asymptotics of $L^2$-metrics . . . . . . . . . . . . . . . . 15
      2.3.2 The Veronese embedding and the Central Limit theorem 16
   2.4 Extremal metrics on toric varieties . . . . . . . . . . . . . . . . 18

3 Toric Fano manifolds
   3.1 The Kahler-Ricci soliton equation . . . . . . . . . . . . . . . . 20
   3.2 Continuity method, convexity and a fundamental inequality . . . 21
   3.3 A priori estimate . . . . . . . . . . . . . . . . . . . . . . . . . 23
   3.4 The method of Wang and Zhu . . . . . . . . . . . . . . . . . . 27

4 Variants of toric differential geometry
   4.1 Multiplicity-free manifolds . . . . . . . . . . . . . . . . . . . . 30
   4.2 Manifolds with a dense orbit . . . . . . . . . . . . . . . . . . 34
In this article we discuss some aspects of existence problems in Kahler geometry; a field which owes so much to Yau. Kahler manifolds, in general, are rather sophisticated mathematical objects and—the author feels—it is often hard to acquire an intuition to back up the more abstract ideas. Thus the main point of this article is to discuss cases where the manifolds in question can be, to some extent, visualised and the existence problems stand out more clearly. Our central topic is the class of “toric varieties” and we will begin by reviewing the differential-geometric theory of these. Then we move on to consider two variants of the toric condition—both involving manifolds with large symmetry groups—and make a special study of a Fano 3-fold found by Mukai and Umemura. In the companion article (with R.S. Bunch), immediately following this in the volume, we take the “visualisation” theme in a different direction, with numerical results for toric surfaces.

The author is grateful to Gabor Szekelyhidi, Rosa Sena-Dias and Yanir Rubinstein for pointing out a substantial mistake in an earlier version of this paper.

1 Background

We begin by reviewing some basic notions in Kahler geometry. The author’s view of this subject is coloured by an analogy with gauge theory so, while it is only indirectly relevant, we will begin with that.

1.1 Gauge theory and holomorphic bundles.

Here we consider a complex vector bundle $E$ over a complex manifold $X$. We want to study the interaction between two structures

- A hermitian metric on $E$;
- A holomorphic structure on $E$, which can be defined by a $\bar{\partial}$-operator

  \[ \bar{\partial} : \Omega^0(E) \to \Omega^{0,1}(E). \]

A basic fact is that given both of these structures there is a unique compatible unitary connection, in the sense that the $\bar{\partial}$-operator is the $(0,1)$-component of
the covariant derivative. Now there are two ways of setting up the theory. In the first—the traditional point of view in complex geometry, as is [14] for example—we fix a holomorphic structure and consider the various Hermitian metrics. Then we have, for example, the formula

\[ F_h = -\partial(h^{-1} \partial h) \]  

(1)

for the curvature tensor in a local holomorphic trivialisation, where the metric is defined by a matrix-valued function \( h \). In the second point of view—closer to what one does in general Yang-Mills theory—we fix the Hermitian metric and consider various \( \bar{\partial} \)-operators. We can identify the set of these operators with the space \( \mathcal{A} \) of unitary connections on \( E \). This point of view brings in two infinite dimensional groups. First, the group \( U(E) \) of unitary automorphisms of \( E \) and second the group \( GL(E) \) of general linear automorphisms. Then \( GL(E) \) acts on the space of \( \bar{\partial} \)-operators by conjugation, and hence on the set \( \mathcal{A} \) of connections. The \( \bar{\partial} \)-operators which define equivalent holomorphic structures are exactly those which are in the same orbit of the \( GL(E) \)-action.

The advantage of this second point of view comes when studying the “jumping” of holomorphic structures. This arises from the fact that the \( GL(E) \) orbits are not usually closed in \( \mathcal{A} \). Fix a Kahler metric on the base space \( X \) and use this to define the Yang-Mills functional: the \( L^2 \) norm of the curvature. When one seeks Yang-Mills connections compatible with a given holomorphic structure \( \mathcal{E} \) one attempts to minimise this functional over a \( GL(E) \) orbit in \( \mathcal{A} \). But it may happen that there is no minimum, in the simplest case because the infimum is achieved at a point in \( \mathcal{A} \) in the closure but not in the orbit itself. Then one finds a Yang-Mills connection not on the original holomorphic bundle \( \mathcal{E} \), but on another one \( \mathcal{E}' \), such that there are arbitrarily small deformations of \( \mathcal{E}' \) which are isomorphic to \( \mathcal{E} \). This lies at the root of the link between Yang-Mills theory and the stability of holomorphic bundles expressed by the Kobayashi-Hitchin conjecture [3], [33], [7].

### 1.2 Symplectic and complex structures

Now we pass on to Kahler geometry. We study the interaction between two structures on an underlying manifold \( M \): a complex structure and a symplectic form. We require these to be algebraically compatible in the sense that the symplectic form is the imaginary part of a hermitian metric. As before there are two points of view we can take. In the first—the conventional point of view in complex differential geometry—we fix the complex structure and vary the Kahler form. If we choose a reference form \( \omega_0 \) and vary in the fixed cohomology class then (at least when \( M \) is compact) any other form can be represented by a Kahler potential, in the shape

\[ \omega_\psi = \omega_0 + i \partial \bar{\partial} \psi. \]

For the alternative point of view we fix a symplectic form \( \omega \) and consider the space \( \mathcal{J} \) of algebraically-compatible almost-complex structures on \( M \). Then
the group SDiff of symplectomorphisms of \((M,\omega)\) acts on \(\mathcal{J}\), and this is the analogue of the unitary gauge group \(U(E)\) in the previous case. We consider the subset \(\mathcal{J}_{\text{int}}\) of integrable almost complex structures, which is preserved by SDiff. This is partitioned into equivalence classes under the relation \(J_1 \sim J_2\) if \((M, J_1), (M, J_2)\) are isomorphic as complex manifolds. Although the group SDiff does not have a true complexification one can argue that the equivalence classes in \(\mathcal{J}_{\text{int}}\) are formally the orbits of such a (mythical) complexified group, in the sense that they behave that way at the level of tangent spaces and Lie algebras [8].

1.3 The equations

The focus of this article is on the existence question for four different kinds of special Kahler metrics, working within a fixed Kahler class on a compact manifold.

1. **Extremal Kahler metrics** The definition is due to Calabi [6]. They are critical points (and in fact local minima) of the Calabi functional

\[
\int_M |\text{Riem}(\omega)|^2 d\mu_\omega,
\]

where \(\omega\) varies over the Kahler metrics in a fixed Kahler class and \(\text{Riem}\) is the Riemann curvature tensor. The Euler-Lagrange equation is

\[
\overline{\partial} (\text{grad} S_\omega) = 0,
\]

where \(\text{grad}\) is the gradient operator defined by \(\omega\) and \(S(\omega)\) is the scalar curvature. In other words, the vector field \(\text{grad} S_\omega\) should be a holomorphic vector field. On the face of it, this is a sixth order partial differential equation for the Kahler potential \(\psi\).

2. **Constant scalar curvature Kahler metrics** These are just those with \(S_\omega\) constant. Certainly they are extremal metrics (since the gradient vanishes), and if it happens that \(M\) has no non-trivial holomorphic vector fields then an extremal metric must have constant scalar curvature.

3. **Kahler-Einstein metrics** By definition these are those where the Ricci tensor is a multiple \(\lambda \omega\). We will only consider the case when \(\lambda\) is positive (the zero and negative cases being completely understood through the results of Yau and Aubin). By rescaling there is no loss in supposing that \(\lambda = 1\). Solutions can only exist when \(M\) is a “Fano” manifold and the class \([\omega]\) is \(\sim c_1(M)\).

4. **Kahler-Ricci solitons** These again occur only in the Fano case. They are metrics for which

\[
\text{Ric} - \omega = L_v \omega,
\]

where \(L_v\) is the Lie derivative along a holomorphic vector field \(v\).
Obviously a Kahler-Einstein metric has constant scalar curvature. There is no simple relation between the other two classes—extremal metrics and Kahler-Ricci solitons—but they can each be thought of as variants of the theory which take account of the possible holomorphic vector fields on the manifold. All this is elucidated by the theory of the *Futaki invariant*. We will not go in to this in detail here, since we will see later how the theory works in explicit examples. Suffice it to say that in either situation the relevant holomorphic vector field which can be determined *a priori* from standard topological data. More precisely, the vector field it determined once we fix a maximal compact connected subgroup of the group of holomorphic automorphisms. In either situation, an extremal metric or Kahler-Ricci soliton will necessarily be Einstein/constant scalar curvature if the Futaki invariant vanishes.

There is, of course, as yet no general existence theory for these structures but at the conjectural level one can see a detailed analogy with the Yang-Mills case. We do not want to go into this further here—partly because there is a comprehensive recent survey article [24]—but proceed with our study of special classes of manifolds.

2 Toric manifolds

We say that a compact Kahler manifold $X$ of complex dimension $n$ is *toric* if the compact torus $T^n$ acts by isometries on $X$ and the extension of the action to the complex torus $T^n \cong (\mathbb{C}^*)^n$ acts holomorphically with a free, open, dense orbit $X_0 \subset X$.

2.1 Local differential geometry

2.1.1 Complex coordinates

Here we work in the neighbourhood of a point in the free orbit $X_0$. We can use the group action to define local co-ordinates. So we have complex co-ordinates

$$
\tau_a = \frac{1}{2} (t_a + i \theta_a)
$$

say. The factor 2 here will simplify the formulae later. Locally the isometry group acts by translations in the $\theta_a$ directions. (Later, when we work globally, the $\theta_a$ will become “angular” co-ordinates, with period $4\pi$.) Locally, a Kahler metric is given by $i \partial \bar{\partial} \phi$ for a function $\phi$ of the complex variables $\tau_a$. If this function only depends on the real parts $t_a$ then the metric will obviously be invariant under translations in the $\theta_a$ directions and it is not hard to see that any metric of the kind we are considering arises in this way. Now if we write $\phi = \phi(t_a)$ then the tensor $i \partial \bar{\partial} \phi$ is just

$$
\sum_{ab} \frac{\partial \phi}{\partial t_a \partial t_b} d\tau_a d\tau_b,
$$

5
and this defines a positive Hermitian form if and only if the Hessian matrix of \( \phi \) is positive definite; or in other words \( \phi \) is a convex function of the real variables \( \tau_a \). Thus the theory of convex functions on Euclidean spaces is embedded, as this translationally invariant case, in the theory of Kähler geometry. We write \( \nabla^2 \phi \) for the Hessian of \( \phi \) and also use index notation \( \nabla^2 \phi = (\phi^{ab}) \). The placing of the indices is unconventional but will be convenient later. We write \( (\phi_{ab}) \) for the inverse matrix. Explicitly the symplectic form \( \omega \) is

\[
\frac{1}{2} \sum \phi^{ab} dt_a \wedge d\theta_b,
\]

and the Riemannian metric is

\[
\frac{1}{2} \left( \sum \phi^{ab} dt^a dt^b + \sum \phi^{ab} d\theta^a d\theta^b \right).
\]

We regard the curvature tensor of this metric as an element of \( \Lambda^2 \otimes \Lambda^2 \). Then the curvature tensor is

\[
\sum R^{abcd} d\tau_d d\tau_c \otimes d\tau_b d\tau_a,
\]

where

\[
R^{abcd} = \phi^{abcd} - \phi^{ac} \phi^{bd} \mu \phi^{\lambda \mu}.
\] (2)

(Here we use the summation convention over the repeated indices. The third and fourth order derivatives of \( \phi \) are written as \( \phi^{abc} \), \( \phi^{abcd} \) in the obvious way.) This formula for the curvature tensor is just the formula (1), expressed in our current notation.

### 2.1.2 Symplectic coordinates

We now take a different point of view, following Guillemin [15] and Abreu [1], and also the general scheme outlined in the previous section. Thus we consider an open set in \( \mathbb{R}^n \times \mathbb{R}^n \) with linear coordinates \( x^a, \theta_a \). More invariantly, we should write the ambient space as \( V \times V^* \) where \( V = \mathbb{R}^n \), with coordinates \( x^a \). We assume the open set has the form \( Q \times V^* \) where \( Q \subset V \) is convex. On this open set we consider the standard symplectic form

\[
\Omega = \frac{1}{2} \sum dx^a d\theta_a.
\]

This is preserved by the translations in the \( \theta \) variables. More precisely we have a Hamiltonian action of the group \( G = V^* \) on the symplectic manifold \( Q \times V^* \) and the moment map is just the projection to \( Q \), with components the coordinates \( x^a \). We consider \( G \)-invariant almost-complex structures on \( Q \times V^* \), algebraically compatible with \( \Omega \). Now at each point such a structure is specified by a subspace of the complexified cotangent bundle which has a unique basis of the form

\[
\epsilon_a = d\theta_a + Z_{ab} dx^b,
\]

\[
\epsilon_a = d\theta_a + Z^{ab} dx_b.
\]
where \((Z_{ab})\) is a symmetric complex matrix with positive definite imaginary part. (This is just the standard description of the Siegel upper half-space \(Sp(n, \mathbb{R})/U(n)\).) So our almost-complex structure is represented by a matrix-valued function \((Z_{ab})\) and \(G\)-invariance specifies that \(Z\) is a function of the variables \(x^a\). Following our general scheme we should now determine when such an almost-complex structure is integrable. By definition this means that the 2-forms
\[
de\epsilon_a = \frac{\partial Z_{ab}}{\partial x^c} dx^c dx^b
\]
can be expressed as \(\sum \alpha_{ab} \wedge \epsilon_b\) and this only happens when all the \(de_a\) are zero (since \(de_a\) does not contain any terms involving \(d\theta_i\)). So the integrability condition is
\[
\frac{\partial Z_{ab}}{\partial x^c} = \frac{\partial Z_{ac}}{\partial x^b}.
\] (3)

Now consider the action of the infinite-dimensional symplectomorphism group. In this situation we need to consider the symplectic diffeomorphisms that commute with the \(G\)-action. More precisely we want to take the Hamiltonian diffeomorphisms generated by functions that Poisson-commute with the generators of the \(G\)-action; but these are just the functions of the \(x^i\) variables. The corresponding group \(\mathcal{G}\) of diffeomorphisms can be identified with smooth functions on \(Q\), where a function \(f\) acts by taking a point \((x, \theta)\) to \((x, \theta + Df)\). This gives an action on the space of almost-complex structures which simply takes \(Z_{ab}\) to \(Z_{ab} + f_{ab}\), where \(f_{ab}\) is the Hessian of \(f\).

Now consider the action of \(\mathcal{G}\) on the integrable structures. The condition (3) implies, by the elementary “criterion for an exact differential”, that there are complex-valued functions \(it_a\) such that
\[
Z_{ab} = i\frac{\partial t_a}{\partial x^b}.
\]

The fact that \(Z_{ab}\) is symmetric implies, by the same criterion, that there is a single complex valued function \(F\) such that \(t_a = \frac{\partial F}{\partial x^a}\), in other words
\[
Z_{ab} = \frac{\partial^2 F}{\partial x^a \partial x^b}.
\]

If we let \(f\) be minus the real part of \(F\) then the action of \(f \in \mathcal{G}\) takes the structure \((Z_{ab})\) to a new structure with zero real part. So, taking account of this diffeomorphism group, we can reduce to considering \(Z = iY\), with \(Y\) real and positive definite. Now the functions \(t_a\) are real and \(\epsilon_a = d(t_a + i\theta_a)\) so \(t_a + i\theta_a\) are local complex co-ordinates. (Thus we confirm the Newlander-Nirenberg integrability theorem in this special case.). Write \(u\) for the imaginary part of the function \(F\) above, so
\[
Y_{ab} = \frac{\partial^2 u}{\partial x^a \partial x^b} = u_{ab}.
\]
Some linear algebra shows that the metric defined by the almost complex structure and the fixed form $\Omega$ is

$$
\frac{1}{2} \sum u_{ij} dx^i dx^j + u^{ij} d\theta_i d\theta_j,
$$

where $(u^{ij})$ is the matrix inverse of the Hessian $(u_{ij})$.

The conclusion of this is that we have another description of the local differential geometry, defined by a convex function $u$ of the variables $x^a$. The relation between this picture and that in complex co-ordinates discussed above is just the Legendre transform for convex functions. That is, given a convex function $u$ on $Q \subset V$ we define a function $\phi$ on an open set $Q^* \subset V^*$ by decreeing that

$$
\phi(t) = \sum x^a t_a - u(x),
$$

where the point $x \in V$ is the unique point where $Du = t$. As is well-known, this transform expresses a symmetric relation between $u$ and $\phi$, so $u$ is the Legendre transform of $\phi$. Further, the Hessian $\phi^{ab} = \frac{\partial^2 \phi}{\partial t_a \partial t_b}$ is the inverse of the Hessian $u^{ab}$ of $u$ at the corresponding point. It is easy to see using this that the Legendre transform does give a Kähler potential for the same metric expressed in the complex co-ordinates. Conversely if we start with the complex description and a convex function $\phi$ then the Legendre transform gives the symplectic picture. More invariantly, the map $x$ is characterised as the moment map for the action of the group of translations.

Thus we have two natural coordinate systems to use when discussing this local differential geometry, and of course we can transform any formulae from one set-up to the other. Working in the symplectic picture we set

$$
F_{ijkl} = u_{ia} u_{jb} \frac{\partial^2 u^{ab}}{\partial x^k \partial x^l}. 
$$

Then one finds that the Riemann curvature tensor is

$$
F_{ijkl} \eta^i \wedge \eta^j \otimes \eta^k \wedge \eta^l, 
$$

where $\eta^a = dx^a + i u^{ab} d\theta_b$. So the four-index tensor $F$ is essentially the same as the curvature tensor. For example the norm if the Riemann curvature tensor is the same as the natural norm of $F$ i.e.

$$
|F|^2 = \sum F_{ijkl} F_{abcd} u^i u^j u^k u^l.
$$

The Ricci tensor is in the same fashion, equivalent to the tensor

$$
G_{ij} = F_{ijkl} u^{kl},
$$

which can also be expressed as

$$
G_{ij} = \frac{\partial^2 L}{\partial x^i \partial x^j}.
$$
where \( L = \log \det(u_{ij}) \). The scalar curvature is given by another contraction yielding Abreu’s formula

\[
S = G_{ij}u^{ij} = \sum_{ij} \frac{\partial^2 u_{ij}}{\partial x^i \partial x^j}.
\]

(6)

We mentioned in the previous section that in the general case the group of symplectomorphisms does not have a complexification, and this limits the practicality of the symplectic approach to Kahler geometry. But in this special situation there is a complexification of \( G \): simply the complex valued functions on \( Q \) under addition. Further, in it is nearly true that this complexified group \( G^c \) acts on the set of almost complex structures, represented as matrix-valued functions \( Z_{ab} \). The “action” is simply to map \( Z \) to \( Z + \frac{\partial^2 F}{\partial x^a \partial x^b} \). It is only a local action because the condition that the imaginary part of \( X \) is positive definite could be violated. Our discussion above asserts that all the integrable structures are in a single orbit of this complexified action and the parametrisation by the function \( u \) is the parametrisation by an open set in the quotient \( G^c/G \). Further, it is easy to verify in this framework that the scalar curvature given by the formula (6) is a moment map for the action of \( G \) with respect to the natural symplectic structure on the space of almost-complex structures (which is derived from the invariant symplectic form on the Siegel upper half space), see [9].

2.2 The global structure

In the previous section we discussed the local differential geometry of a toric manifold in the dense open set where the torus action is free. We now go on to the global picture. There are at least three different points of view we can take but the essential thing is that this structure is encoded by a bounded polytope \( P \subset \mathbb{R}^n \), or more invariantly \( P \subset V \) in the notation of the previous section. This polytope is defined by a finite collection of linear inequalities \( \lambda_r(x) > c_r \) corresponding to the codimension-1 faces. So \( \lambda_r \) are vectors in the dual space \( V^* \). We suppose that there is an integer lattice in \( V \), which we can take to be the standard \( \mathbb{Z}^n \) in \( \mathbb{R}^n \). Then there is a dual lattice in \( V^* \) and we suppose that the \( \lambda_r \) lie in this dual lattice. We can rescale so that the \( \lambda_r \) are primitive vectors with respect to this lattice. Further, we suppose that each vertex of \( P \) is contained in exactly \( n \) codimension faces and that the corresponding \( \lambda_r \) form an integer basis for the dual lattice. Such a polytope is called a Delzant polytope. Another way of expressing the condition is via the group \( \Gamma \) of maps

\[
\tilde{x} \mapsto Ax + b
\]

from \( \mathbb{R}^n \) to itself, where \( A \) is restricted to lie in \( GL(n, \mathbb{Z}) \). Up to the action of \( \Gamma \), a neighbourhood of any vertex of \( P \) is equivalent to a neighbourhood of 0 in the infinite polytope \( \{ x_i > 0 \} \subset \mathbb{R}^n \). If the vertices of the polytope are integral we call it an integral Delzant polytope.
Example The standard simplex in $\mathbb{R}^n$, given by the inequalities
\[ x^1 > 0, x^2 > 0, \ldots, x^n > 0, x^1 + x^2 + \ldots + x^n \leq 1 \]
is a Delzant polytope.

2.2.1 Complex charts

Start with a Delzant polytope $P$. Let $S$ be the finite set of pairs of

- a vertex $p$ of $P$;
- an ordering $\lambda_{r(i)}$ of the faces containing $p$.

For any two $\sigma = (p, r(\ ))$ and $\sigma' = (p', r'(\ ))$ in $S$ there is a unique element $\gamma_{\sigma,\sigma'}$ of $\Gamma$ which maps $p$ to $p'$ and matches up the corresponding faces. Obviously we have
\[ \gamma_{\sigma,\sigma'} = 1 ; \gamma_{\sigma,\sigma'}^{-1} = \gamma_{\sigma',\sigma} ; \gamma_{\sigma,\sigma''} = \gamma_{\sigma,\sigma'} \circ \gamma_{\sigma',\sigma''} . \]

Now suppose we have any space $M^*$ on which $\Gamma$ acts and $M^*$ is a subset of a larger space $M$. We take the product $S \times M$ and define a relation
\[ (\sigma, m) \sim (\sigma', \gamma_{\sigma,\sigma'}(m)) , \]
for $m \in M^*$. The properties above tell us that this is an equivalence relation, so we can take the quotient $S \times M/\sim$. In our case we take $M$ to be $\mathbb{C}^n$ and $M^* = (\mathbb{C}^*)^n \subset \mathbb{C}^n$. Then $GL(n, \mathbb{Z})$ acts on $M^*$. This is clear if we identify $\mathbb{C}^*$ with $\mathbb{C}/\mathbb{Z}$ and hence $M^*$ with $\mathbb{C}^n/\mathbb{Z}^n$. In terms of the original description, with co-ordinates $z_i$ on $\mathbb{C}^n$, we make a matrix $(a_{ij})$ act on $(\mathbb{C}^*)^n$ by
\[ z_i' = \prod z_j^{a_{ij}} , \]
which is well-defined since the $a_{ij}$ are integers. There is a natural homomorphism from $\Gamma$ to $GL(n, \mathbb{Z})$ so $\Gamma$ acts on $M^*$ via this. Then it is clear from the construction that the quotient $X_{\text{cx}}$ is a complex manifold covered by charts $M_{\sigma}$ labelled by elements of $\Sigma$, each chart being a copy of $M = \mathbb{C}^n$. The charts for the $n!$ different elements of $\Sigma$ belonging to the same vertex of $P$ have the same image so it suffices just to take one of them. There is an action of the complex torus $T^m_{\mathbb{C}}$ with a dense orbit, which is the image of any $\{\sigma\} \times M^*$. The construction behaves well with respect to restriction to faces, so for each $m$-dimensional face $\Pi$ of $P$ there is a submanifold $X^\Pi_{\text{cx}} \subset X_{\text{cx}}$ which is an $m$-dimensional complex submanifold with an action of $T^m_{\mathbb{C}}$ induced from the action on $X_{\text{cx}}$. Indeed the orbits of the $T^m_{\mathbb{C}}$ action on $X_{\text{cx}}$ correspond to these faces. In particular the vertices of $P$ correspond to points of $X_{\text{cx}}$; the fixed points under the $T^m_{\mathbb{C}}$ action.

Example When $P$ is the $n$-simplex, as above, the manifold $X_{\text{cx}}$, we construct is $\mathbb{C}P^n$. 

10
So far we have not used the full strength of the data we began with. For example, we could simply have omitted some vertices of $P$ and run the same construction. We have also thrown away some of the data, through the homomorphism from $\Gamma$ to $GL(n, \mathbb{Z})$. First, the fact that the vertices come from a bounded polytope yields the compactness of the space $X_{\text{cx}}$ we have defined. We leave this as an exercise for the reader. For the second point, it is indeed the case that if we vary the constants $c_r$ slightly (so that we do not introduce or remove any vertices) we get the same complex manifold $X_{\text{cx}}$. The extra structure of the specific polytope corresponds to fixing a distinguished cohomology class in $H^2(X_{\text{cx}}; \mathbb{R})$. This is easiest to see in the case when the polytope is integral. Then the $\gamma_{\sigma\sigma'}$ lie in a smaller group $\Gamma_Z \subset \Gamma$ which is an extension

$$\mathbb{Z}^n \to \Gamma_Z \to GL(n, \mathbb{Z}).$$

We take the trivial complex line bundle $\mathbb{C}$ over $M = \mathbb{C}^n$. Then $\Gamma_Z$ acts on the restriction of $\mathbb{C}$ to $M^*$ and the same construction gives a complex line bundle $L \to X_{\text{cx}}$. Furthermore this is an equivariant line bundle for the $T^*_n$ action. The distinguished cohomology class is just the first Chern class of $L$. In general, when the vertices are not integral we consider the sheaf $Z^1$ of closed 1-forms over $X_{\text{cx}}$. We can use the $\gamma_{\sigma\sigma'}$ to define a closed 1-form on $M_\sigma \cap M_{\sigma'}$ and this yields a Cech cocycle with values in this sheaf. Then the short exact sequence of sheaves

$$0 \to \mathbb{R} \to C^\infty(X_{\text{cx}}) \to Z^1 \to 0$$

gives a boundary map from $H^1(X_{\text{cx}}; Z^1)$ to $H^2(X_{\text{cx}}, \mathbb{R})$ which defines the distinguished cohomology class. (In fact this cohomology class is not changed if we translate $P$. A more precise statement is that the Delzant polytope $P$ can be recovered from the complex manifold $X$ with a suitable distinguished $T^*_n$-equivariant cohomology class.)

**Example.** Consider a vertex $p$ of a Delzant polytope $P$. There is no loss of generality in supposing that $p$ is the origin and that near the origin $P$ agrees with the standard model $\{x^i > 0\}$. Then, for $\delta > 0$, we define $P_{\delta}$ to be the subset of $P$ defined by the additional inequality $\sum x_i > \delta$. For small enough $\delta$ this is again a Delzant polytope and the complex manifold $X_\delta$ is the blow-up of $X$ at the fixed point corresponding to $P$. The exceptional divisor $E$ is a copy of projective space, associated to the “new” $n - 1$-simplex in the boundary of $P_\delta$. The manifold does not vary with $\delta$ but the evaluation of the distinguished cohomology class on the standard generator of $H_2(E) \subset H_2(X_\delta)$ is $\delta$.

Now we go back to differential geometry. If we have a Kahler metric on $X_{\text{cx}}$, its restriction to the open orbit is described by a Kahler potential; a convex function $\phi$ on $\mathbb{R}^n$, as above. Conversely we can define an “admissible” convex function $\phi$ to be one which defines a Kahler metric over the orbit which extends smoothly to the compact manifold. This is a condition on the asymptotic behaviour of $\phi$ at infinity in $\mathbb{R}^n$. The essence of the condition is that $\phi$ is
asymptotic to the piecewise linear function
\[ \Phi(t) = \max_p p \cdot t, \]
where \( p \) runs over the vertices of the polytope. Thus if we let \( \phi_\lambda \) be the rescaling \( \phi_\lambda(t) = \lambda^{-1} \phi(\lambda t) \) for \( \lambda \in \mathbb{R} \) then \( \phi_\lambda \to \Phi \) (in \( C^0 \)) as \( \lambda \) tends to infinity. In the model case when 0 is a vertex and \( P \) agrees locally with \( \{ x^i > 0 \} \) the local complex co-ordinates are \( z_a = \log \tau_a \) and so \( |z_a|^2 = e^{t_a} \). The admissible condition is that \( \phi \) extends to a smooth function of the complex co-ordinates \( z_a \).

**Example** The round metric on the 2-sphere with area \( 2\pi \) is given by the Kahler potential
\[ \phi(t) = \log(1 + e^t). \]
In terms of a local complex co-ordinate \( z \) this is \( \log(1 + |z|^2) \).

### 2.2.2 Symplectic construction

Here we start with the product \( P \times T^n \) with standard co-ordinates \( x^a, \theta_a \) as before, except of course that now the \( \theta_a \) are taken to be “angular” co-ordinates with period \( 4\pi \). This is a noncompact symplectic manifold with the standard symplectic form \( \Omega = \sum dx^a d\theta_a \) and with Hamiltonian \( T^n \) action whose moment map is the projection to \( P \). The essential point is that this can be compactified to a compact symplectic manifold \( X_{\text{symp}} \) and the moment map extends to a map with image the closure \( \overline{P} \). This works in a similar fashion to the complex picture. For example, consider the neighbourhood of a vertex of \( P \) which as usual we can take to be the origin, with \( P \) locally modelled on \( \{ x^i > 0 \} \). Then \( \Omega \) is the pull-back of the standard form on \( \mathbb{C}^n \) under the map
\[ (x^a, \theta_a) \mapsto (|x_a|^{1/2} e^{i\theta_a}), \]
We adjoin a neighbourhood of 0 in \( \mathbb{C}^n \) to \( P \times T^n \) using this map and repeat the construction, modified in the obvious way, for all other boundary points of \( P \).

Now of course this symplectic construction describes the same object as the complex construction in the previous section. We return to the discussion of the local differential geometry taking now \( Q = P \). We can start with an admissible Kahler potential \( \phi \) on \( \mathbb{R}^n = V^* \). Then its Legendre transform is a function on \( P \). Around a vertex, as above, this has the form
\[ u = \sum x^i \log x^i + v, \]
where \( v \) is a smooth function (on the manifold with corners). We say that a symplectic potential \( u \) is admissible if it is the Legendre transform of an admissible Kahler potential \( \phi \). Stated explicitly in terms of \( u \) this the requirement of “Guillemin boundary conditions”, which are

1. \( u \) is a continuous function on \( \overline{P} \), smooth in the interior.
2. The restriction of $u$ to each face is smooth and strictly convex.

3. Let $q$ a boundary point which lies on a codimension $r$ face of $P$, so without loss of generality $q = 0$ and $P$ is locally defined by equations $x^1 > 0, \ldots, x^r > 0$. Then near $q$

$$u = \sum_{i=1}^{r} x_i \log x_i + v$$

where $v$ is smooth.

It is easy to see that such functions exist. For example we can take the Guillemin function

$$u = \sum_{r} (\lambda_r - c_r) \log(\lambda_r - c_r).$$

Either way, we get a map from the complex manifold $X_{cx}$ to the symplectic manifold $X_{symp}$ which matches up the structures involved.

**Example** The round metric on $S^2$, of area $2\pi$, is defined by the symplectic potential, on the interval $[0, 1]$,

$$u(x) = (x \log x + (1 - x) \log(1 - x)).$$

### 2.2.3 Algebraic construction

Here we suppose that the Delzant polytope $P$ is integral. We consider all the multiples $kP$ for integers $k \geq 0$ and let $B_k$ be the set of lattice points

$$B_k = kP \cap \mathbb{Z}^n.$$

Let the number of points in $B_k$ be $N_k + 1$. We can put all these sets together by considering the cone over $P$

$$cone(P) = \{(x, y) \in \mathbb{R}^{n+1} : y \geq 0, x \in yP\}.$$

The disjoint union of the sets $B_k$ can be identified with the set $B = cone(P) \cap \mathbb{Z}^{n+1}$. Now $B$ is an abelian semi-group under addition and we have a corresponding ring $R$ over $\mathbb{C}$ with one generator $s_b$ for each point of $b \in B$ and relations $s_b s_{b'} = s_{b+b'}$. This is a graded ring, $R = \bigoplus R_k$, where $R_k$ has a basis $s_{\nu}$ corresponding to the points $\nu$ of $B_k$. Further, there is an obvious action of the torus $T^n$ on $R$.

All of these definitions make sense for any convex set $P$. The crucial fact is that when the $P$ is an integral polytope the ring is finitely generated. Thus there is a corresponding projective variety $X_{alg} = \text{Proj}(R)$, and the group action on $R$ defines an action on $X_{alg}$. Second, if $P$ is Delzant, then $X_{alg}$ is smooth and of course this recovers the same complex manifold $X_{cx}$. The vector spaces $R_k$ are the sections

$$R_k = H^0(X_{cx}, L^k)$$
and it is not hard to see that for any \( k \geq 1 \) the sections give an embedding \( X_{cx} \rightarrow P(R_k^*) \). From this algebro-geometric point of view the integer \( \lambda_r(\nu) - c_r \), for lattice points \( \nu \in P \), is the order of vanishing of the section \( s_\nu \) along the corresponding divisor in \( X_{cx} \).

**Example** Let \( P \) be the square \((0,1)^2 \subset \mathbb{R}^2 \). The corresponding manifold is the product \( S^2 \times S^2 \). The points in \( B_1 \) are the four vertices \( p_0 = (0,0), p_1 = (0,1), p_2 = (1,0), p_3 = (1,1) \) so \( R_1 \) has a corresponding basis \( s_0 s_1, s_2, s_3 \) say. The equation \( p_0 + p_3 = p_1 + p_2 \) goes over to the relation \( s_0 s_3 = s_1 s_2 \). The embedding of \( X_{cx} \) in \( P^3 \) has image the quadric hypersurface cut out by the equation \( Z_0 Z_1 - Z_2 Z_3 = 0 \).

When the polytope \( P \) is integral but not Delzant the variety \( X_{alg} \) we construct is singular. If each vertex lies on exactly \( n \) codimension-1 faces then \( X_{alg} \) is an orbifold. Much of the theory, including the differential-geometric constructions, extends easily to this case.

To sum up we have three ways—complex, symplectic and algebraic— of constructing a compact manifold associated to an integral Delzant polytope. From now on we will just denote this by \( X \).

### 2.2.4 Real forms

A toric manifold \( X \) contains a submanifold \( X_R \) of one half the dimension which is a “real form” in the complex picture and Lagrangian in the symplectic picture. To define this from the first point of view we just observe that the action of \( \Gamma \) on \( M^* = (\mathbb{C}^*)^n \) preserves the subset \( M^*_R \) of real points. Then we run the same construction. From the symplectic point of view we let \( A \) be the subgroup of the real torus \( T^n \) given by the elements of order 2, so \( A \) is isomorphic to \( (\mathbb{Z}/2)^n \). Then we consider the subset \( A \times P \subset T^n \times P \) and check that the closure of this in \( X \) is a smooth \( n \)-dimensional manifold. From the algebro-geometric point of view we simply observe that all our relations are real, so complex conjugation acts on everything and we get a real form of our complex algebraic variety.

This construction is particularly vivid in the symplectic picture [?]. The composite

\[
X_R \rightarrow X \rightarrow \bar{P},
\]

is a \( 2^n \)-fold covering map over the interior \( P \subset \bar{P} \) so we can construct \( X_R \) by taking \( 2^n \) copies of \( \bar{P} \) and gluing the boundary components appropriately. The Riemannian metric on \( P \) given by the Hessian \( u_{ij} \) of an admissible symplectic potential extends to a smooth Riemannian metric on \( X_R \). In particular we get a conformal structure on \( X_R \) and when \( n = 2 \) a Riemann surface structure on the oriented cover of \( X_R \). (The surface \( X_R \) is only itself orientable in the case when \( P \) is a rectangle.) For example, if \( P \) is the standard triangle in \( \mathbb{R}^2 \) then \( X_R \) is a real projective plane in \( X = \mathbb{C}P^2 \) and can be constructed.
by gluing four triangles. The oriented cover is $S^2$, constructed by gluing eight triangles. In general we get a class of Riemann surfaces obtained by gluing eight polygons. Given a symplectic potential $u$, the induced conformal structure on $\mathcal{P}$ is equivalent to the standard disc. So if $P$ has $s$ vertices we get an invariant of $u$ in the moduli space $\mathcal{M}_s$ of configurations of $s$ distinct points on $S^1 = \mathbb{RP}^1$ modulo the action of $PSL(2,\mathbb{R})$. This determines the conformal structure of $X_R$, and is an interesting global invariant of a toric Kahler surface.

2.3 Algebraic metrics and asymptotics

If $X$ is any compact complex manifold and $L \to X$ a very ample line bundle we can generate Kahler metrics on $X$ by the following procedure. Choose a Hermitian metric on the complex vector space $H^0(X; L)$. This induces a metric on the dual space and hence a standard Fubini-Study metric on the complex projective space $\mathbb{P}(H^0(X; L)^*)$. Now we use the embedding $\iota : X \to \mathbb{P}(H^0(X; L)^*)$ to induce a Kahler metric on $X$. We call metrics of this kind “algebraic Kahler metrics”.

This construction becomes very simple and explicit in the toric case. We consider metrics on $H^0(L)$ which are invariant under the torus action, hence are diagonal in the standard basis $s_\nu$. A collection of positive numbers $a_\nu$, for each lattice point $\nu$ in $\mathbb{P}$, defines an invariant metric with $\|s_\nu\|^2 = a_\nu^{-1}$. Given this data $\{a_\nu\}$ we have a Kahler potential on $\mathbb{R}^n$:

$$\phi(t) = \log \left( \sum_\nu a_\nu e^{\nu^* t} \right),$$

(7)

where $\nu^* t$ denotes the dual pairing between the copy of $\mathbb{R}^n$ on which $\phi$ defined and the copy of $\mathbb{R}^n$ containing $P$. This is the potential which defines the algebraic metric via the projective embedding.

We will not discuss this topic at length here, but we want to make the point that the data $-\log a_\nu$—a real-valued function on the lattice points in $\mathbb{P}$—can be thought of as a “discrete approximation” to the symplectic potential $u$—a real-valued function on $\mathbb{P}$. This only makes sense as an asymptotic statement, when we replace the bundle $L$ by $L^k$ and $P$ by $kP$ for large $k$. Rescaling, we can equivalently fix $P$ and replace the integer lattice by $k^{-1}\mathbb{Z}^n$. We discuss two simple precise statements which illustrate this general idea but for many further developments in a similar vein we refer to the recent works of Zelditch [36].

2.3.1 Asymptotics of $L^2$-metrics

Suppose we start with some symplectic potential $u$ and corresponding Kahler potential $\phi$. Then $\phi$ can be regarded as a Hermitian metric on the line bundle $L$ over the toric variety. Thus we have a natural $L^2$-metric on $H^0(X; L)$

$$\|s\|^2 = \int_X |s|^2 d\mu_\phi,$$
where the pointwise norm $|s|$ is defined by $\phi$ and $d\mu_\phi$ is the volume form of the Kahler metric. Thus, starting with $u$ we get a collection of numbers $a_\nu = \|s_\nu\|^{-1}$. Now replace $L$ by $L^k$, as above. The same symplectic potential $u$ defines a metric on $L^k$ and we get a collection of numbers $a_\nu^{(k)}$ say, for $\nu \in P \cap k^{-1}\mathbb{Z}^n$. One precise statement expressing the general idea above is that for each $\epsilon > 0$ and compact subset $K \subset P$ there is a $k_0$ such that

$$|u(\nu) - k^{-1}\log a_\nu^{(k)}| < \epsilon,$$

once $k \geq k_0$, for all $\nu \in K \cap k^{-1}\mathbb{Z}^n$.

The proof of this is very simple. Go back to the case $k = 1$ for the moment. Unravelling the definitions, the coefficients $a_\nu$ are given by

$$a_\nu^{-1} = \int_{\mathbb{R}^n} e^{-\phi} e^{\nu \cdot \mathbf{t}} \det(\nabla^2 \phi) \, d\mathbf{t},$$

where $\phi$ is the given Kahler potential. (Notice, by the way, that Holder’s inequality shows that $\nu \mapsto -\log a_\nu$ is a convex function, in the obvious sense.) Rescaling, we get $a_{\nu,k} = I_{\nu}(k)$ say, where

$$I_{\nu}(k) = \int_{\mathbb{R}^n} e^{-k(\phi - \nu \cdot \mathbf{t})} \det(\nabla^2 \phi) d\mathbf{t}.$$  \hspace{1cm} (8)

(Notice that these formulae make sense for any $\nu \in P$ and the restriction to the lattice $k^{-1}\mathbb{Z}^n$ is not really relevant here.) So we see that our question reduces to the standard discussion of the asymptotic behaviour of the integral $*$ as $k \to \infty$. The dominant contribution comes from the a neighbourhood of the point $\mathbf{t}_0$ where $\phi - \mathbf{t} \nu$ is minimal and the standard Laplace approximation is

$$I_{\nu}(k) \sim (2\pi k)^{-n/2} \exp(-k(\phi(t_0) - t_0 \nu)) \det \nabla^2 \phi(\mathbf{t}_0).$$

But $\mathbf{t}_0$ is just the point which corresponds to $\nu$ under the Legendre transform, and $\phi(\mathbf{t}_0) - \mathbf{t}_0 \nu$ is $-u(\nu)$. So

$$k^{-1}\log I_{\nu}(k) = u(\nu) + O(k^{-1}\log k),$$

and our result follows since $k^{-1}\log k \to 0$ as $k \to \infty$.

Following on this line, it is easy to derive a special case of Tian’s Theorem from [29]. If we start with any Kahler metric with potential $\phi$, then use the $a_\nu^{(k)}$ as above to define an algebraic metric with potential $\phi^{(k)}$ then, after suitable normalisation the $\phi^{(k)}$ converge to $\phi$ as $k \to \infty$. In particular the algebraic metrics are dense in the space of all metrics.

### 2.3.2 The Veronese embedding and the Central Limit theorem

Suppose, in the general situation, that the sections of $L$ generate the sections of $L^k$ so that we have a surjective linear map

$$s^k(H^0(L)) \to H^0(L^k).$$
A metric on $H^0(L)$ defines a metric on the symmetric power $s^k(H^0(L))$ in a standard way. Then we can define a metric on $H^0(L^k)$ by identifying it with the orthogonal complement of the kernel of the map above. Then we can use this to define an algebraic Kahler metric on $X$ by the embedding $\iota_k : X \to \mathbf{P}(H^0(L^k)^*)$. Now, up to a scale factor, these Kahler metrics are independent of $k$. One way of seeing this is that the embedding $\iota_k$ is the composite of $\iota_1$ and the Veronese embedding

$$j : \mathbf{P}(\mathbb{C}^N) \to \mathbf{P}(s^k\mathbb{C}^N),$$

and, up to scale, $j$ is an isometry of the two Fubini-Study metrics. (This is forced by $U(N)$-invariance.) So the same Kahler metric has a whole series of algebraic representations.

Let us see how this works in the toric case. We start with data $a_\nu$ on $\overline{P} \cap \mathbb{Z}^n$. Then we can write

$$k\phi = \log \left( \sum a_\nu e^{i\nu \cdot \xi} \right)^k = 2 \log \sum B_\mu e^{i\mu \cdot \xi},$$

where the coefficients $B_\mu$ are

$$B_\mu = \sum_{\nu_1 + \cdots + \nu_k = \mu} a_{\nu_1}a_{\nu_2}\cdots a_{\nu_k}.$$

So if we regard $(a_\mu)$ as a measure supported on the lattice points in $\overline{P}$ then the $(B_\mu)$ represent the $k$-fold convolution $A \ast \cdots \ast A$, supported on the lattice points in $kP$. Now rescale back to the fixed polytope $P$, so we write $b_\nu^{(k)} = B_{k\nu}$, for $\nu \in \overline{P} \cap k^{-1}\mathbb{Z}^n$. These define an admissible Kahler potential with Legendre transform $ku$, where $u$ is the Legendre transform of $\phi$. Then on compact subsets of $P$ we claim that

$$k^{-1} \log b_\nu^{(k)} = u + O(k^{-1} \log k). \quad (9)$$

This is essentially the Central Limit theorem, for the convolutions of the discrete measure $A$. By applying a translation we can reduce to calculating at the point $\nu = 0 \in P$. Changing the coefficients $a_\nu$ to $a_\nu e^{iz \nu}$, for any fixed $z \in \mathbb{R}^n$, does not change either side of $(9)$, when $\nu = 0$, so we can reduce to the case when $\sum a_\nu = 0$. That is to say, that $\phi$ attains its minimum at the point $\xi = 0$. Now we consider the function

$$f(\theta) = \sum a_\nu e^{i\nu \cdot \theta}.$$  

This is a finite trigonometric polynomial which can be regarded as a function on our compact torus $T$. Then

$$b_0^{(k)} = \int_T f^k d\theta,$$

and our assertion follows from the stationary phase approximation, since the maximum value of $|f|$ is $\sum a_\nu = u(0)$. 

17
Of course $f$ is just the analytic continuation of $e^\phi$, for our Kahler potential $\phi$. This makes one wonder if there may be other contexts when it is useful to consider such analytic continuations.

**Example** For each $k$, the round metric on $S^2$ is described as an algebraic metric with the coefficients $a_{\nu} = \left( \begin{array}{c} k \\ \nu \end{array} \right)$.

Notice that the asymptotics approximations we have discussed hold uniformly over compact subsets of the open polytope $P$. The discussion near the boundary of $P$ is more delicate, because one gets different asymptotic models. A prototype is the different approximations—normal or Poisson—for the binomial distribution in different regimes.

### 2.4 Extremal metrics on toric varieties

The author has written at length on this topic in other papers, so we shall be rather brief here. Expressed in terms of a symplectic potential $u$ the condition for an extremal metric is that the scalar curvature

$$S(u) = -u_{ij}^{ij},$$

is an affine-linear function on $P$. More generally, it is natural in this context to consider the prescribed scalar curvature equation $S(u) = A$ for some given function $A$ on $P$. This can be expressed as a variational problem. Recall that our polytope $P$ comes with preferred defining inequalities $\lambda_r(\underline{x}) \geq c_r$. These linear functions $\lambda_r$ define a measure $d\sigma$ on the boundary of $P$ (just a multiple of standard Lebesgue measure on each codimension-1 face). Then, given a function $A$ on $P$ we define a linear functional

$$L_A(f) = \int_{\partial P} f d\sigma - \int_P A f \, d\underline{x}.$$  

Now define a nonlinear functional by

$$F_A(u) = L_A(u) - \int_P \log \det \nabla^2 u \, d\underline{x}.$$  

Then an admissible symplectic potential $u$ which satisfies the equation $u_{ij}^{ij} = -A$ is an absolute minimiser of the functional $F_A$.

The functional $F_A$ is a variant of the **Mabuchi functional**, which is defined in the general Kahler context. It is a convex functional on the space of convex functions on the polytope $P$. The equation $u_{ij}^{ij} = -A$, together with the Guillemin boundary conditions asserts that the functional $L_A$ is represented by the inverse of the Hessian of $u$ in the sense that

$$L_A(f) = \int_P u_{ij}^{ij} f_{ij},$$  

(10)
for all test functions $f$. We see immediately from this that if a solution $u$ is to exist then $L_A$ must vanish on the affine linear functions $f$. This is set of $n+1$ linear constraints on the function $A$. If we take $A$ to be the constant

$$\frac{\text{Vol}(\partial P, d\sigma)}{\text{Vol}(P, d\xi)},$$

then $L_A$ vanishes on the constant functions $f$. The restriction of this functional $L_A$ to the linear functions $f$ is the Futaki invariant, in this special setting. Otherwise said, this is essentially the difference between the centre of mass of $(\partial P, d\sigma)$ in $\mathbb{R}^n$ and the centre of mass of $(P, d\xi)$. If this Futaki invariant does not vanish then we cannot have a constant scalar curvature metric, but there is a unique affine-linear function $A$ satisfying the constraint above, and we seek an extremal metric with this prescribed scalar curvature.

It is not true that any toric variety admits an extremal metric. To see this observe that if a solution exists then the weak formulation (10) implies that $L_A(f) \geq 0$ for convex functions $f$ (with strict inequality if $f$ is, say, smooth and not affine linear). But one can construct examples of toric surfaces where $L_A$ does not satisfy this condition, for the affine-linear $A$ above. To fit this in with the discussion of Section 1, imagine following a minimising sequence $u^{(\alpha)}$ for the functional $F_A$, in the case when no solution exists (there would be a similar discussion for the Calabi functional). Then the typical phenomenon (which one can see explicitly in some simple examples, and probably holds in general) is that $u^{(\alpha)}$ behaves like

$$u^{(\alpha)} \sim C_\alpha v,$$

where $C_\alpha$ are real, $C_\alpha \to \infty$ and $v$ is a piecewise-linear convex function on $\mathbb{P}$. Differential geometrically this corresponds to the collapsing of some directions in the torus fibration over the parts of $P$ where the derivative of $v$ is discontinuous. Algebro-geometrically, the data $v$ describes a toric degeneration of $X$ into a singular toric variety $X_0$ (at least, this is the case if $v$ is defined by “rational data”). In other words we have a picture much like that sketched in 1.1, except that rather than “jumping” to a different complex structure on the same underlying smooth manifold we have to allow singularities. (In fact a similar thing happens in the Yang-Mills case in higher dimensions, where the limiting structures may be sheaves rather than holomorphic bundles.)

In this way, one has a good understanding of one mechanism by which existence can fail. The more formidable problem is to see if this is the only way. More precisely, it is natural to make the

\textbf{Conjecture 1} If $P \subset \mathbb{R}^n$ is a Delzant polytope and $A$ is a smooth function on $\mathbb{P}$ with the property that $L_A(f)$ vanishes if $f$ is affine linear and $L_A(f) > 0$ if $f$ is a convex function which is not affine linear, then there is an admissible symplectic potential satisfying the equation $u_{ij}^{(1)} = -A$.

We refer to [9], [10], [11] for more information about this, particularly in the case when $n = 2$. 

19
3 Toric Fano manifolds

3.1 The Kahler-Ricci soliton equation

The condition that a toric manifold $X$ be Fano, with $L = K_X^{-1}$, is easily stated in terms of the polytope $P$. There is a preferred “centre” $\nu_0 \in P$ such that for each face $\lambda_r(\nu_0) - c_r = 1$. This follows because the wedge product of the vector fields generating the action is a meromorphic $n$-form on $X$ with a simple pole along each of the divisors corresponding to the faces. Then the inverse is a section of $K_X^{-1}$ and is a multiple of the standard basis element $s_{\nu_0}$. This centre is also the centre of mass of $(\partial P, d\sigma)$.

In this Section we discuss a Theorem of Wang and Zhu [34].

**Theorem 1** Any toric Fano manifold has a Kahler-Ricci soliton metric, unique up to holomorphic automorphisms.

We will begin by giving a proof which is somewhat different to that of Wang and Zhu (although it borrows ideas from that paper and from [32]), working largely with the symplectic description. We can assume that the centre $\nu_0$ is the origin. Given a symplectic potential $u$ we write

$$h = x^i u_i - u,$$

and

$$L = \log \det \nabla^2 u.$$

These are smooth functions on $P$ but both tend to infinity at the boundary. Note that $h$ depends on a choice of origin in $\mathbb{R}^n$. Of course $h$ is just the composite of the Kahler potential $\phi$ with the derivative of $u$, mapping $P$ to $\mathbb{R}^n$. The assumption that the toric manifold $X$ be Fano is equivalent to the fact that, for any admissible $u$, the difference $L - h$ is a smooth function on $P$. The condition that $u$ describe a Kahler-Ricci soliton is that

$$L - h = \sum c_i x^i,$$  \hspace{1cm} (11)

for constants $c_i$ (which of course specify the relevant holomorphic vector field on the Kahler manifold). Just as in our discussion of extremal metrics, it is natural in this context to consider more generally an equation $L - h = A$ for some prescribed smooth function $A$ on $P$. Again, much as for the extremal case, there are elementary constraints that we need to impose on $A$. For any symplectic potential $u$ we consider the integrals

$$\int_P x^i e^{L-h} d\mu,$$

for $i = 1, \ldots, n$. Transforming the integral to the dual space, it becomes

$$\int_{\mathbb{R}^n} \frac{\partial \phi}{\partial t_i} e^{-\phi} dt_i = - \int_{\mathbb{R}^n} \frac{\partial e^{-\phi}}{\partial t_i} dt_i = 0.$$
So a necessary condition that the equation $L - h = A$ has a solution is that, for each $i$,
\[ \int_P x_i e^A d\mu = 0. \] (12)
This fixes the constants $c_i$ in (11). To see this, consider the function of $c \in \mathbb{R}^n$:
\[ F(c) = \int_P e^{\sum c_i x^i} d\mu \]
This is convex and proper (since the origin lies in $P$) and so has a unique critical point. But the derivative of $F$ with respect to $c_i$ is
\[ \int_P x^i e^{\sum c_i x^i} d\mu. \]
So the unique critical point of $F$ gives exactly the constants $c_i$ required to satisfy the constraint.

In sum, the theorem of Wang and Zhu follows from

**Theorem 2** For any smooth function $A$ on $\overline{P}$ which satisfies the constraint (12) there is a solution $u$ to the equation $L - h = A$, which is unique up to the addition of a linear function.

An equivalent statement is

For any smooth function $A$ on $\overline{P}$ there are constants $\gamma_i$ and an admissible potential $u$ such that $L - h = A + \sum \gamma_i x^i$. The $\gamma_i$ are unique and $u$ is unique up to the addition of a linear function.

The equivalence of the statements follows from the same argument as above.

### 3.2 Continuity method, convexity and a fundamental inequality

For any symplectic potential $u$ on our Fano polytope, centred at the origin, we write $\rho = L - h$. Now we define the following weighted norms, for functions $f, g$ on $P$:

\[ \langle f, g \rangle_u = \int_P f g e^\rho \, d\mu; \]
\[ \langle \nabla f, \nabla g \rangle_u = \int_P f_i g_a u^{ia} e^\rho \, d\mu; \]
\[ \langle \nabla^2 f, \nabla^2 g \rangle_u = \int_P f_{ij} g_{ab} u^{ia} u^{jb} e^\rho \, d\mu. \]

The first variation of $\rho$ with respect to an infinitesimal variation $f$ in $u$ is $\delta \rho = \Box f$, where $\Box$ is the differential operator
\[ \Box f = u^{ij} f_{ij} - x^i f_i + f. \] (13)
Since
\[ \rho_j = -u^{ja}u^a \rho_j - x^a u_{ja}, \]
this can also be written as
\[ \Box f = (u^{ij} f_i)_j - u^j \rho_j f_i + f, \] (14)
from which it follows that
\[ \langle \Box f, g \rangle_u = -\langle \nabla f, \nabla g \rangle_u + \langle f, g \rangle_u. \] (15)
In particular, \( \Box \) is self-adjoint with respect to the weighted norm.

Now define a functional by
\[ F(u) = \int_P e^\rho \, dx. \] (16)
Then the first variation is
\[ \delta F = \int_P \Box e^\rho \, dx = \langle \Box f, 1 \rangle_u. \] (17)
By the self-adjoint property we can also write this as
\[ \delta F = \langle f, \Box 1 \rangle_u = \langle f, 1 \rangle_u = \int_P f e^\rho \, dx. \] (18)
This leads to two different expressions for the second variation of \( F \). If we put
\( u_t = u + tf, \rho_t = \rho(u_t) \) and write \( \Box_t \) for the operator defined by \( u_t \) then
\[ \frac{d}{dt} \Box_t f = -u^{ia} u^{jb} f_{ij} f_{ab}. \]
So,
\[ \frac{d^2}{dt^2} F(u_t) = \frac{d}{dt} \int_P \Box_t e^\rho \, dx = \int_P (\Box_t f \Box_t e^\rho - u^{ia} u^{jb} f_{ij} f_{ab}) e^\rho \, dx, \]
which is equal to
\[ \langle \Box_t f, \Box_t f \rangle u_t - \langle \nabla^2 f, \nabla^2 f \rangle u_t. \]
On the other hand
\[ \frac{d^2}{dt^2} F(u_t) = \frac{d}{dt} \int_P f e^\rho \, dx = \int_P f \Box_t e^\rho \, dx. \]
So, evaluating at \( t = 0 \) and dropping \( t \) from the notation, we have the identity
\[ \langle \Box f, \Box f \rangle_u - \langle \nabla^2 f, \nabla^2 f \rangle_u = \langle f, \Box f \rangle_u. \] (19)
Applying (15), with \( g = \Box f \), this gives,
\[ \langle \nabla f, \nabla \Box f \rangle_u = -\langle \nabla^2 f, \nabla^2 f \rangle_u. \] (20)
It is obvious from the definition that \( \Box \) vanishes on the linear functions and \( \Box 1 = 1 \). If \( f \) is any eigenfunction of \( \Box \), with eigenvalue \( \lambda \), which is orthogonal to the linear functions and then constants, then \( \nabla^2 f \) is non-zero and the identity gives

\[
\lambda \langle \nabla f, \nabla f \rangle_u = -\langle \nabla^2 f, \nabla^2 f \rangle_u,
\]

so \( \lambda < 0 \). (This is a variant of the standard lower bound on the eigenvalues of the Laplacian on a manifold with positive Ricci curvature, the identity can of course be verified more directly, but the argument above avoids some laborious manipulation.) In sum, we have derived an inequality

\[
\langle 1, f \rangle_u \langle 1, 1 \rangle_u - \langle f, \Box f \rangle_u \geq 0,
\]

with equality if and only if \( f \) is a linear function.

Now to apply this to our problem. First, we can use the continuity method for the equation \( L - h = A + \sum \gamma_i x^i \), with respect to variations in \( A \). The linearised equation is \( \Box u = \delta A + \sum \delta \gamma_i x^i \). Since the cokernel of \( \Box_u \) is identified with the linear functions this linearised equation has a solution and we can apply the implicit function theorem in the usual way.

Second, we obtain the uniqueness of solutions. Consider the functional \(-\log \mathcal{F}\). Along a line \( u_t = u + tf \) we have

\[
\frac{d^2}{dt^2} (-\log \mathcal{F}) = \frac{1}{\mathcal{F}^2} (\mathcal{F}' \mathcal{F}' - \mathcal{F}'')
\]

where \( \mathcal{F}' \), \( \mathcal{F}'' \) denote the derivatives of \( \mathcal{F} \). Evaluating at \( t = 0 \) we have

\[
\mathcal{F} = \langle 1, 1 \rangle_u, \mathcal{F}' = \langle 1, f \rangle_u, \mathcal{F}'' = \langle f, \Box u f \rangle_u,
\]

so our inequality (21) asserts that the second derivative of \(-\log \mathcal{F}\) is positive, and strictly positive unless \( f \) is affine-linear. Thus \(-\log \mathcal{F}\) is a convex function. Now if \( \rho = A + \sum \gamma_i x^i \) the \( \gamma_i \) are determined by \( A \), using the same argument as in the previous subsection. So we may as well suppose that \( \gamma_i = 0 \). Then \( \mathcal{F}(u) = C \) where \( C \) is the integral of \( e^A \). The equation \( \rho = A \) is the Euler-Lagrange equation for critical points of the linear function

\[
u \mapsto \int \rho \, e^A \, d\mu
\]

subject to the constraint \(-\log \mathcal{F} = -\log C\). The convexity gives uniqueness, modulo linear functions.

### 3.3 A priori estimate

To prove Theorem 2 we need to establish appropriate a priori bounds on a solution to our equation. We proceed in four steps.
Step 1: Preliminaries

We want to appeal to some of the standard body of theory for compact Kähler manifolds, that is, where we consider a fixed reference metric $\omega_0$ on a compact manifold and another metric $\omega = \omega_0 + i\partial\bar{\partial}\psi$. Our problem differs a little from that usually considered in the literature. To fit into a general setting we could consider a fixed smooth function $G$ of $p$-variables, a compact Kähler manifold $X$ with $p$ fixed holomorphic vector fields $v_\alpha$ and a function $\psi$ which satisfies an equation

$$(\omega_0 + i\partial\bar{\partial}\psi)^n = \exp(\psi + G(\nabla_1\psi, \ldots, \nabla_p\psi))$$

where $\nabla_\alpha\psi$ denotes the derivative of $\psi$ along the vector field $v_\alpha$. Then the modification by Tian and Zhu ([32], Section 5, especially Prop. 5.1) of the standard argument of Yau, shows that in this situation an $L^\infty$ bound on $\psi$ leads to bounds on all higher derivatives. (Apart from this the proof we give is self-contained.)

In our toric setting, we choose some fixed admissible Kähler potential $\phi_0$ on $\mathbb{R}^n$ with Legendre transform $u_0$. Then we consider some general Kähler potential $\phi$, with Legendre transform $u$ and set $\psi = \phi - \phi_0$. So an $L^\infty$ bound on $\psi$ on the compact toric manifold is identical to an $L^\infty$ bound on $\phi - \phi_0$ on $\mathbb{R}^n$. Now a general property of the Legendre transform is that it is an isometry with respect to the $L^\infty$ distance: that is to say

$$\sup_{\ell \in \mathbb{R}^n} |\phi(\ell) - \phi_0(\ell)| = \sup_{x \in P} |u(x) - u_0(x)|.$$

This is an elementary exercise.

In our situation, $u_0$ is a fixed continuous function on $P$ so an $L^\infty$ bound on the function $\psi$ on the compact Kähler manifold is equivalent to an $L^\infty$ bound on the “unknown” symplectic potential $u$.

In sum, we see that to prove our proposition it suffices to establish an a priori $L^\infty$ bound on symplectic potentials $u$ satisfying a differential inequality

$$|L - h| \leq C,$$

for fixed $C$. Of course for this to make sense we have to normalise the non-uniqueness under the addition of linear functions, but we can do this very simply by restricting to functions $u$ whose derivative vanishes at the origin. i.e are minimised at the origin. We write $m = -u(0)$, so our problem comes down to obtaining upper and lower bounds on $m$ and an upper bound on $\max_P u - u(0)$. By the Sobolev inequality, the latter will follow if we establish an a priori bound on $\|\nabla u\|_{L^p}$, for any $p > n$.

Step 2

Here we get a lower bound on $m = -u(0)$. By definition $h(0) = m$. Let the polytope $P$ be contained in the $R_1$-ball about 0 in $\mathbb{R}^n$ and let $\Omega \subset P$ be the set
where $|\nabla u| \leq 1$. Since the derivative of the Legendre transform $\phi$ is bounded in size by $R_1$ we have $|h(x) - m| \leq R_1$ for all $x \in \Omega$ and the basic assumption (22) gives $L \leq m + R_1 + C$ so

$$\text{det}(\nabla^2 u) \leq \exp(m + R_1 + C).$$

But the integral of $\text{det}(\nabla^2 u)$ over $\Omega$ gives the volume $\omega_n$ of the unit ball in $\mathbb{R}^n$ so

$$\exp(m + R_1 + C) \text{Vol}(\Omega) \geq \omega_n.$$ 

Since the volume of $\Omega$ cannot exceed the volume of $P$ this gives a lower bound on $m$.

**Step 3**

Here we bring in a crucial identity.

**Lemma 1** For any admissible symplectic potential $u$ and each index $i$ we have

$$\int_P e^{\rho} u_i x^i dx = \int_P e^{\rho} dx,$$

where $\rho = L - h$.

To see this, transform the integral to $\mathbb{R}^n$, giving

$$\int_P e^{\rho} u_i x^i dx = \int_{\mathbb{R}^n} e^{-\phi} \frac{\partial \phi}{\partial t_i} t_i dt.$$

Now integrate by parts to write this as

$$\int_{\mathbb{R}^n} e^{-\phi} dt,$$

which transforms back to

$$\int_P e^{\rho}.$$

In our situation we are given $|\rho| \leq C$, so summing over the index $i$, we have

$$\int_P \sum u_i x^i dx \leq ne^{2C} \text{Vol}(P).$$

In terms of “generalised polar co-ordinates” $(r, \theta)$ we can write this as

$$\int_P r^n \frac{\partial u}{\partial r} dr d\theta \leq ne^{2C} \text{Vol}(P).$$

By hypothesis, the derivative of $u$ vanishes at the origin and the radial derivative $\frac{\partial u}{\partial r}$ is increasing along each ray. It is clear then that if we fix some $R_0$ such that
the $3R_0$-ball $B(3R_0)$ about the origin is contained in $P$ then we get from Lemma 1 an a priori bound on the integral of $u - u(0)$ over the boundary of $B(2R_0)$. In turn this gives, by an easy elementary argument, a bound on the derivative over an interior ball, say

$$|\nabla u| \leq C'$$ on $B(R_0)$,

(23)

where $C'$ can be computed explicitly in terms of $n, C, R_0, \text{Vol}(P)$.

**Step 4**

We can now complete our task, using some elementary convexity arguments. First, throughout $P$ we have $h \geq m$ so $\det \nabla^2 u \geq e^{-C} e^m$. Thus the volume of the image of the ball $B(R_0)$ under the map $Du$ is at least $e^{-C} e^m \text{Vol}(B(R_0))$. But by (23) this image is contained in the ball of radius $C'_0$, so

$$e^{-C} e^m \leq \left( \frac{C'_0}{R_0} \right)^n.$$  

This gives our upper bound on $m$.

Next we claim that at any point $x_0 \in P$ where $h(x_0) = m + 1$ we have $|\nabla u| \leq \kappa$, where $\kappa = C' + R_0^{-1}$. To see this, consider the affine-linear function $\pi$ defining the supporting hyperplane of $u$ at $x_0$. By definition, this has the form $\pi(x) = -h(x_0) + D(x) = -(m + 1) + D(x)$ where $D$ is a linear function, equal to the derivative of $u$ at $x_0$. Thus there is a point $y$ on the boundary of $B(R_0)$ where $\pi(y) = -(m + 1) + R_0 |D|$. By convexity we have $u(y) \geq \pi(y)$, so $u(y) \geq -(m + 1) + R_0 |D|$. On the other hand (23) implies that $u(y) \leq -m + C'R_0$, so we have

$$-(m + 1) + R_0 |D| \leq -m + C'R_0,$$

and $|D| \leq C' + R_0^{-1}$.

Now consider the situation under the Legendre transform. The statement above becomes that for any point $\xi_0$ where $\phi(\xi_0) = \phi(0) + 1 = m + 1$ we have $|\xi_0| \leq \kappa$. It follows easily from the convexity of $\phi$ that for any $\xi$ we have

$$\phi(\xi) \geq \kappa^{-1} |\xi| + m - \kappa^{-1}.$$  

(24)

Recall from the conclusion of Step 1 that we seek a bound on $\|\nabla u\|_{L^p}$. By the definition of the Legendre transform we have

$$\int_P |\nabla u|^p \, dx = \int_{\mathbb{R}^n} |\xi|^p \det(\nabla^2 \phi) \, d\xi.$$  

Using our hypothesis (22) we have then

$$\|\nabla u\|_{L^p}^p \leq e^C \int_{\mathbb{R}^n} |\xi|^p e^{-\phi} \, d\xi.$$  

26
Now write
\[ \int_{\mathbb{R}^n} |t|^p e^{-\phi} \, dt \leq e^{m-\kappa-1} \int_{\mathbb{R}^n} |t|^p e^{-\kappa-1} \, dt, \]
to finish the argument.

**Remark**

The identity in Lemma 1 is related to the positivity condition for the linear functional \( L_A \) discussed in Section 2. Suppose we have a Fano polytope with vanishing Futaki invariant, so we seek a Kahler-Einstein metric which *a fortiori* has constant scalar curvature. Thus, in the setting of Section 2, the function \( A \) is a constant. By an argument of Zhou and Zhu [37] the linear functional \( L_A \) satisfies the positivity condition in this case. On the other hand, in the setting of Section 2, anytime \( L_A \) satisfies the positivity condition we can deduce an *a priori* interior bound on the derivative of a normalised solution, see [9], just as in (23) above. Thus the identity of Lemma 1 can be seen as an extension of this discussion to the case of general Ricci soliton metrics.

### 3.4 The method of Wang and Zhu

We will now discuss briefly the original approach of Wang and Zhu (again with some modifications). For simplicity we will just consider the case when the Futaki invariant vanishes, so we seek a Kahler-Einstein metric. Recall from the above that the vanishing Futaki invariant is equivalent to fact that the centre of mass of the polytope \( P \) is the preferred centre, which we are taking as \( 0 \in \mathbb{R}^n \).

Wang and Zhu use the continuity method with respect to the family of equations
\[
\det(\nabla^2 \phi) = \exp(-(s\phi + (1-s)f)), \tag{25}
\]
where \( f \) is a fixed admissible Kahler potential and \( 0 \leq s < 1 \). We discuss first the case when \( s = 0 \). Then the equation in question is just the toric case of the “prescribed volume form” equation, solved, for general Kahler manifolds, by Yau. But let us see how to give a simple proof in this special situation. The equation (25) with \( s = 0 \) is degenerate, in that we can obviously change \( \phi \) by the addition of a constant, so we may normalise \( u \) to be zero at some point. Thus, as before, all we need to do is bound the \( L^p \) norm of \( \nabla u \). But for this we simply write
\[
\int_P |\nabla u|^p \, d\lambda = \int_{\mathbb{R}^n} |t|^p \det \nabla^2 \phi \, dt = \int_{\mathbb{R}^n} |t|^p e^{-f} \, dt < \infty.
\]
This concludes the proof of the \( L^\infty \) estimate for the case \( s = 0 \). (Here we have not used the fact that \( f \) is convex, so by deforming \( f \) one can prove the toric case of Yau’s Theorem: the existence of a solution for any \( f \).)
Now we go on to the main case, when $s > 0$. It suffices to obtain estimates for $s \geq s_0$ for some fixed $s_0 > 0$.

Set $w = s\phi + (1 - s)f$. Then $w$ is another admissible function and

$$\det(\nabla^2 w) \geq s_0^n \det(\nabla^2 \phi) = s_0^n e^{-w}.$$

Let the minimal value of $w$ be $m$, attained at a point $\zeta \in \mathbb{R}^n$. The first main step in the proof is

**Proposition 1** We have

$$w(t) \geq \epsilon |t - \zeta| - C$$

for known $\epsilon, C$.

The foundation of the approach of Wang and Zhu is the following fact.

**Proposition 2** Suppose that $v$ is a convex function on $\mathbb{R}^n$, attaining minimal value 0, and suppose $\det(\nabla^2 v) \geq \lambda$ when $v \leq 1$. Then if $K$ is the set where $v \leq 1$ we have $\text{Vol}(K) \leq C\lambda^{-1/2}$ for some constant $C$ depending only on the dimension $n$.

Wang and Zhu prove this using a comparison argument. It can also be shown using the elementary geometry of the derivative of $v$ (see [17] Prop. 3.2.3), but both approaches depend on the fact that after a unimodular affine transformation we can suppose that there are concentric balls

$$B(R_1) \subset K \subset B(R_2),$$

with the ratio $R_2/R_1$ of the radii bounded by a fixed constant depending on the dimension. Notice that a reverse inequality holds. If in the same situation $\det(\nabla^2 v) \leq \Lambda$ then $\text{Vol}(K) \geq C\Lambda^{-1/2}$ ([17], Cor. 3.2.4).

With this background in place we can proceed to explain the proof of Wang and Zhu. Let $m$ be the minimal value of the function $w$ and set $v = w - m$. Then $\det(\nabla^2 v) \geq \lambda = t_0^n e^{m+1}$ on the set $K$ where $v \leq 1$. So we deduce that

$$\text{Vol}(K) \leq C\lambda^{-1/2} = C'e^{m/2},$$

say. For each positive $\mu$ let $K_\mu$ be the set $\{v \leq \mu\}$ and $V(\mu) = \text{Vol}(K_\mu)$. Then convexity implies that $K_\mu$ is contained in the dilate of $K$ by factor $\mu$ about the minimum point of $v$. Thus

$$V(\mu) = \text{Vol}(K_\mu) \leq h^n \text{Vol}(K) \leq \mu^n C'e^{m/2}.$$

By the co-area formula

$$\int_{\mathbb{R}^n} e^{-w} \, dt = \int_0^\infty e^{-\mu} V(\mu) \, dh.$$
Now the volume form \( \det(\nabla^2 \phi) \) is at most \( e^{-m} e^{-v} \) and its integral is the volume of our manifold \( X \). So
\[
\text{Vol}(X) \leq e^{-m} \int_{0}^{\infty} C' e^{m/2} e^{-\mu} \mu^n \, d\mu = C'' e^{-m/2},
\]
say. We see that
\[
m \leq n_0 = 2 \log(I_0/C''),
\]
and then deduce from (26) that
\[
\text{Vol}(K) \leq C' e^{-n_0/2}.
\]
(28)

Now we use the fact that \( |\nabla w| \leq b \) say. This means that the distance from the boundary of \( K \) to the minimum point \( \zeta \) is at least \( b^{-1} \), so \( K \) contains a ball of this fixed radius about \( \zeta \). If \( K \) contains a point \( \zeta' \) with \( |\zeta - \zeta'| = R \) for large \( R \), then the volume of \( K \) would be large, contradicting the bound (28). So we conclude that \( K \) is contained in the ball \( \{ \zeta' : |\zeta' - \zeta| \leq R_0 \} \) for some fixed \( R_0 \). But then convexity implies that
\[
|\xi - \zeta| \leq R_0^{-1} v(\xi).
\]

This completes the proof of Proposition 1.

The second main step is to show that \( |\zeta| \) is not large. This is where the hypothesis that the the Futaki invariant vanishes is used. Consider the derivative \( Df \) of the fixed admissible function \( f \). This is a vector-valued function on \( \mathbb{R}^n \), which gives a proper map to the open polytope \( P \). The crucial thing is an identity
\[
\int_{\mathbb{R}^n} Df e^{-w} \, d\mu = 0.
\]
(29)

To see this, consider one component \( \frac{\partial f}{\partial t^a} = f^a \) of \( Df \), and observe first that
\[
\int_{\mathbb{R}^n} \left( (1 - s) \frac{\partial f}{\partial t^a} + s \frac{\partial \phi}{\partial t^a} \right) e^{-w} \, d\mu = \int_{\mathbb{R}^n} \frac{\partial w}{\partial t^a} e^{-w} \, d\mu = 0.
\]
So it is the same to show that
\[
\int_{\mathbb{R}^n} \frac{\partial \phi}{\partial t^a} e^{-w} \, d\mu = 0.
\]
But this integral is
\[
\int_{\mathbb{R}^n} \frac{\partial \phi}{\partial t^a} \det(\phi_{ab}) \, d\mu
\]
which is the same as
\[
\int_{\mathbb{P}} x^n \, d\nu
\]
and this vanishes by our hypothesis.
Consider a codimension-1 face of $P$ defined by an equation $\lambda_r(x) = c_r$. Let $g_r$ be the function

$$g_r(t) = \log(\lambda_r(Df(t)) - c_r).$$

It is easy to check that the derivative of $g_r$ is bounded on $\mathbb{R}^n$. Suppose $|\zeta|$ is large. This means that $Df(\zeta)$ is close to the boundary of $P$, so there is some $r$ for which $g_r(\zeta)$ is very negative $g_r(\zeta) \leq -M$ say, for $M$ large. Then the bound on the derivative of $g_r$ means that we can find a constant $\sigma$ such that on the ball $B$ of radius $\sigma M$ about $\zeta$ we have $g_r \leq -M/2$. Thus $\lambda_r(Df) \geq c_r/2$ say, on $B$, if $M$ is large enough. Equally, it follows from Proposition 1 that when $M$ is large the integral of $e^{-w}$ over $\mathbb{R}^n \setminus B$ is small. This shows that

$$\int_{\mathbb{R}^n} \lambda_r(Df)e^{-w} > 0,$$

if $M$ is large, which is a contradiction to the identity (29) above.

It is now easy to complete the proof. Since $\zeta$ is bounded we have

$$w(t) \geq c|t| - c$$

and the bound on the $L^p$ norm of $\nabla u$ follows just as before. Then it is straightforward to get upper and lower bounds on $u$ at some point, for example the point corresponding to $\zeta$.

4 Variants of toric differential geometry

4.1 Multiplicity-free manifolds

The special features of toric differential geometry can be traced back to the fact that the group of Hamiltonian diffeomorphisms which commute with the action is abelian. In general, the action of a compact group $G$ on a symplectic manifold $(M, \omega)$ is called “multiplicity-free” if it has this property. This is equivalent to saying that all the $G$-invariant functions Poisson-commute. The theory has been developed by a number of authors. The analogous notion in algebraic geometry is that of a spherical variety. The theory of extremal metrics and the Mabuchi functional in this setting has been studied by Alexeev and Katzarkov [2] and by Raza [27] and Podesta and Spiro [25] have extended the theorem of Wang and Zhu for Fano manifolds in this direction. There is also related work of Bielwaski [3]. We will now outline some of these ideas.

There is a general classification of multiplicity-free manifolds ([33], [20]), but rather than attempting to discuss the most general situation we focus on a simple class of examples. Pick a maximal torus $T$ in the compact connected Lie group $G$ and let $V$ be the dual of the Lie algebra of $T$. There is a weight lattice $\Lambda \subset V$. Pick a positive Weyl chamber in $V$ and consider an integral Delzant polytope $P$ whose closure is contained in the interior of this chamber. We construct a
manifold from this data and as usual we can take either a symplectic or complex point of view.

*Complex*

The choice of a Weyl chamber defines a Borel subgroup $B$ of the complexified group $G^c$, containing the complexified torus $T_c$. For example if $G = U(n)$ the Borel subgroup is the group of complex matrices with zeros below the diagonal. Then we have a generalised flag manifold $Y = G^c/B$, which is a compact complex manifold. There is a homomorphism from $B$ to $T^c$ which is a left inverse to the inclusion. Now form the toric manifold $X$ associated to the polytope $P$. Then $T^c$ acts holomorphically on $X$ and so $B$ does also via the homomorphism above. So we get a complex manifold

$$Z = G^c \times_B X,$$

with a holomorphic fibration $\pi : Z \rightarrow Y$, having fibre $X$. The group $G^c$ acts on $Z$ and $\pi$ is a $K^c$-equivariant map. Further, we have a $T^c$-equivariant line bundle $L \rightarrow X$ so the same construction yields a $G^c$-equivariant line bundle $\mathcal{L} \rightarrow Z$ which restricts to $L$ on each fibre. We can identify $H^0(Z, \mathcal{L})$ with the sections of the vector bundle $\pi_*(\mathcal{L})$ over $F$. Recall that there is a standard basis for $H^0(X, L)$ labelled by the lattice points $\nu$ in $\overline{P}$. This yields an isomorphism between $\pi_*(\mathcal{L})$ and the direct sum of line bundles $\xi_\nu \rightarrow F$ associated to these weights. The Borel-Weil theorem asserts that the holomorphic sections of $\xi_\nu$ define the irreducible representation $W_\nu$ of $G^c$ with highest weight $\nu$. So we see that, as a representation of $G^c$,

$$H^0(Z, \mathcal{L}) = \bigoplus_{\nu \in \overline{P}} W_\nu.$$  

In particular the representation is “multiplicity-free”, in the sense that all irreducibles appear with multiplicity at most one. This is the same as saying that the algebra of $G^c$-equivariant endomorphisms of $H^0(Z, \mathcal{L})$ is commutative. The terminology “multiplicity free” in the symplectic setting is derived by analogy with this.

Notice that replacing $P$ by a multiple $kP$ yields the same complex manifold $Z$ but replaces $\mathcal{L}$ by $\mathcal{L}^k$. Translating $P$ by $\nu$ does not change $Z$ but changes the line bundle $\mathcal{L}$ to $\mathcal{L} \otimes \pi^*(\xi_\nu)$. In none of the above do we use the fact that $\overline{P}$ lies in the interior of the positive Weyl chamber. This is exactly the condition which implies that $\mathcal{L}$ is an *ample* line bundle over $Z$.

**Example** Take $G = SU(2)$, so $V$ can be identified with $\mathbb{R}$ and the positive Weyl chamber with the positive reals. Let $P$ be the interval $(p_1, p_2)$. Then $X = Y = \mathbb{CP}^1$ and $Z$ is the blow-up of the complex projective plane at one point. As $p_1, p_2$ vary we get all positive line bundles $\mathcal{L}$ over $Z$.

For the symplectic description we start by writing $Y = G/T$, and think of $G$ as a principal $T$-bundle over $Y$. As a manifold $Z$ is the associated bundle $G \times_T X$. Now $T$ has a Hamiltonian action on $X$. In general suppose a Lie
group $K$ has a Hamiltonian action on a symplectic manifold $(M, \Omega)$ and we have a principal $K$-bundle $E \to U$. Then there is a canonical closed 2-form $\tilde{\Omega}$ on the associated bundle $E \times_K M$ which restricts to $\Omega$ (in the obvious sense) on each fibre. Indeed this is true in the “universal” case when we take the group of all Hamiltonian diffeomorphisms of a symplectic manifold. This theory is explained in detail in [21], Sect. 6.1). It is easy to say explicitly how this works in the case at hand. Choose a basis of $V = \text{Lie}(T)^*$. The basis elements can be regarded as left-invariant 1-forms $\alpha_i$ on $G$ and also as the components of a connection form on the $T$-bundle $G \to Y$. The moment map $\mu : X \to V$ has components, relative to this basis, which we denote by $x^i$, in line with our previous notation. Since the moment map is equivariant we can also regard $\mu$ as a map from $Z$ to $V$ and the components $x^i$ as functions on $Z$. Restrict to the open set $Z_0 \subset Z$ corresponding to the open set $X_0 \subset X$ where $T$ acts freely. This can be identified with the product $P \times G$, so we can also regard $\alpha_i$ as 1-forms on $X_0$. Then we set

$$\tilde{\Omega} = d\left(\sum x^i \alpha_i\right)$$

on $Z_0$. On each fibre the 1-forms $\alpha_i$ can be identified with the $d\theta_i$ and we recover the form $\sum dx^i d\theta_i$. The point is that, although the 1-forms $\alpha_i$ do not extend over $Z$, the closed 2-form $\tilde{\Omega}$ does. This is fairly clear from the corresponding discussion on the fibres. The condition that $P$ lies inside an open Weyl chamber is exactly the condition that the form $\tilde{\Omega}$ is symplectic. The $G$-invariant functions on $Z$ are just the composite of $\mu$ with functions on $P$ and these all Poisson-commute.

An important object in this theory is the “Duistermaat-Heckmann” function $W$ on $V = \text{Lie}(T)^*$. It is a polynomial function which, on the open Weyl chamber, gives the symplectic volume of the corresponding coadjoint orbit. Algebraically it is the product of the positive roots, where the roots are viewed as linear functions on $V$. The push-forward $\mu_*(\tilde{\Omega}^N)$ of the symplectic measure on $Z$ is the restriction to $P$ of $(2\pi)^n W$ times the Lebesgue measure on $V$. Thus if we identify functions on $P$ with $G$-invariant functions on $Z$ the operation of integration over $Z$ corresponds to the weighted integral

$$\int_P f W dx$$

(31)

Raza extended the symplectic point of view on toric differential geometry, as outlined (2.1.2) above, to this setting [27]. The orthogonal complement with respect to $\Omega$ defines a field of horizontal subspaces in $Z$, transverse to the fibres. Any $G$-invariant almost-complex structure on $Z$, compatible with $\tilde{\Omega}$, must respect this decomposition and agree with the standard complex structure, induced from $Y$, in the horizontal subspace. So such almost-complex structures correspond to the same $T$-invariant almost-complex structures on $X$ which we studied before, and the integrable structures are determined by an admissible symplectic potential $u$ on $\overline{P}$, as before. The whole difference in the theory resides in the weight function $W$. Raza shows that the scalar curvature of the
metric on $Z$ defined by a symplectic potential $u$ is

$$\frac{1}{W} \frac{\partial^2 W u^i}{\partial x^i \partial x^j} + f_G,$$

where $f_G$ is function determined by the group $G$. In fact if we let $\sigma \in \text{Lie}(T)^*$ be the sum of the positive roots of $G$ then

$$f_G = W^{-1} (W \sigma^i) :$$

the derivative of $\log W$ in the direction $\sigma$. This extends Abreu’s formula in the toric case, and also a formula of Calabi, for the case when $K = SU(2)$ ([6], [18]). There there seems to be considerable scope for extending the analytical theory developed in the toric case to this more general setting, similar to the work of Szekelyhidi in [28].

Now we consider the Fano case, where the line bundle $\mathcal{L}$ is $K_Z^{-1}$. This requires, first, that the fibre $X$ be Fano. Recall that there is a preferred centre $\nu_0$ in $P$ (the centre of mass of the boundary). The second requirement, to identify $\mathcal{L}$ with $K_Z^{-1}$, is that $\nu_0$ is equal to $\sigma$, the sum of the positive roots. (To see this, observe that the line bundle over $Y$ associated to the weight $\sigma$ is the $K_Y^{-1}$.) In Section 3 we took this centre to be the origin, but here that would conflict with the Weyl chamber structure. So, given a polytope $P$ satisfying these two conditions above, and an admissible symplectic potential $u$, we define

$$h = (x^i - \sigma^i) u_i - u.$$

Then $L - h$ is smooth on $\overline{P}$. The Ricci soliton condition is

$$L - h = G + \sum c_i x^i,$$

for suitable constants $c_i$. This falls into the class of equations we considered in 3.2, and the existence theorem of Podesta and Spiro is another illustration of our result there.

What we have discussed is the simplest class of multiplicity-free manifolds. One gets other examples in at least two ways.

- One can allow the boundary of $\overline{P}$ to touch the boundary of the Weyl chamber.

- One can consider polytopes contained in proper affine subspaces of $\text{Lie}(T)^*$.

There seems to be considerable scope for developing this theory, both in the Fano case and for extremal metrics. In the latter case one could hope to extend the results proved for toric varieties, along the lines of the work of Szekelyhidi [28] in the case when $G = SU(2)$. 

33
4.2 Manifolds with a dense orbit

Now we consider another generalisation of toric geometry. Let $G$ be a compact Lie group and $G^c$ its complexification. Suppose $G^c$ acts holomorphically on a compact complex manifold $V$ and that there is a point $x_0 \in V$ whose $G^c$ orbit is dense. We also want to suppose that the stabiliser $\Gamma \subset G^c$ is finite. Then the orbit is a copy of $G^c/\Gamma$ in $V$ and the complement is an analytic subvariety (which must contain a divisor if $X$ is Kahler). Of course the case of a toric manifold fits into this picture, except that in that case we can assume $\Gamma$ is trivial (but see the further discussion below). In the next section we will study a particular example of this set-up: the Mukai-Umemura manifold.

Now there is no loss of generality in supposing that $\Gamma$ lies in the compact group $G$ and we can study $G$-invariant Kahler metrics on $V$. Over the dense orbit these can be represented by Kahler potentials $\Phi$ on $G^c$ which are invariant under the two groups $G$ (acting by left multiplication) and $\Gamma$ (acting by right multiplication). In other words, $\Phi$ can be regarded as a function on the symmetric space $M = G^c/G$ which is invariant under the action of the finite group $\Gamma$ on $M$. We will denote the corresponding function on $M$ by $\phi$.

A finite group $\Gamma$ can enter in the toric case in slightly different way, but leading to the same conclusion. Suppose $\Gamma$ is a finite subgroup of $GL(n, \mathbb{Z})$ which preserves the polytope $P$ of a toric manifold $X$. (For example if $X$ is $\mathbb{C}P^n$, so $P$ is the standard simplex, we can take $\Gamma$ to be the permutations of the $n$ coordinates.) Then there is a group $\hat{T}$ which fits into a split exact sequence

$$1 \to T \to \hat{T} \to \Gamma \to 1$$

and which acts on $X$. As a toric manifold, we know that we can represent $T$-invariant Kahler metrics on $X$ by potentials $\phi$ on $\mathbb{R}^n$, but now we can further restrict to $\hat{T}$-invariant metrics and these correspond to $\Gamma$-invariant functions $\phi$, for the natural action of $\Gamma$ on $\mathbb{R}^n$ (of course, this copy of $\mathbb{R}^n$ is really the dual of that containing $P$).

We now develop the local Kahler differential geometry in this situation, working in terms of a function $\phi$ on the symmetric space $M$. This has a standard connection on its tangent bundle, which is the Levi-Civita connection for any $G^c$-invariant metric. Thus we have a Hessian operator $\nabla^2$ taking functions on $M$ to sections of $s^2(T^*M)$. The tangent space of $V$ at a point $gx_0$ can be identified with the complexification of the tangent space of $M$ at the point $Gg$. Thus we have an identification with the symmetric tensors $s^2(T^*M)$ at $Gg$ with a subspace of $\Lambda^{1,1}TG^c$ at $g$. This just corresponds to embedding the real symmetric matrices in the complex Hermitian matrices.

**Lemma 2** Under this identification for any function $\phi$ on $M$ and corresponding function $\Phi$ on $G^c$ the form $i\partial \bar{\partial} \Phi$ corresponds to $\nabla^2 \phi$.

We can see this as follows. First note that in the toric case this is just what we have seen when we identify the Kahler metric with the Hessian $\phi^{ab}$. For the
general case, there is no loss in working at the point $g = 1$. To evaluate $\nabla^2 \phi$ on a tangent vector $v$ we take the geodesic $\gamma(t)$ in $M$ starting with initial velocity $v$. Then

$$\nabla^2 \phi(v) = \frac{d^2}{dt^2} \phi(\gamma),$$

evaluated at 0. Now geodesics in $G^c/G$ through the identity coset correspond to 1-parameter subgroups in $G^c$ so we have a homomorphism $\tilde{\gamma} : \mathbb{C} \to G^c$, such that $\gamma(t) = K\tilde{\gamma}(it) \in M$. Then we are essentially reduced to the toric case, restricting to this 1-parameter subgroup.

Thus the local Kahler geometry in this situation reduces to the study of convex functions on $M$ which, by definition, are those functions $\phi$ with $\nabla^2 \phi > 0$ at each point. Equivalently, they are functions which are convex along geodesics in $M$. Of course this is a generalisation of the case when $M = \mathbb{R}^n = T^c/T^n$. We can go on to write out the equations we want to solve explicitly in this framework. The Kahler-Einstein equation, in the Fano case, is

$$\det \nabla^2 \phi = e^{-\phi}.$$

For the scalar curvature; given a convex function $\phi$, we define an operator

$$\Delta_\phi(f) = (\nabla^2 \phi)^{-1} \nabla^2 f,$$

where $(\nabla^2 \phi)^{-1}$ is the quadratic form on $T^*M$ induced by the nondegenerate quadratic form $\nabla^2 \phi$ on $TM$, in the usual way, and the dot denotes the contraction between $s^2 TM$ and $s^2 T^*M$. Then the scalar curvature of the Kahler metric defined by $\Phi$ is

$$S = \Delta_\phi(\log \det \nabla^2 \phi).$$

Notice that these local constructions make sense on any manifold equipped with a connection and volume form.

There are some important differences between this theory in the case of a semi-simple group $G$ and that in the abelian, toric, case.

- When we go beyond the local differential geometry we need to consider a class of “admissible” functions $\phi$ which define metrics which extend smoothly to $V$. This imposes some asymptotic growth conditions on $\phi$ (as in the toric case) but these can be more complicated, since they encode the structure of the compactification.

- In the toric case the local equations are affine invariant, but there is no substitute for the affine group in the semi-simple case. In the semi-simple case we have a preferred metric which changes the character of the theory.

- The geometry of $M$ in the semi-simple case has negative curvature, reflecting the non-abelian nature of $G$. This makes a radical difference to arguments involving volumes of balls etc.
Again, there seems to the author to be a lot of scope for development of this theory. For example one could consider a function \( w \) on a Riemannian manifold of negative curvature which satisfies a differential inequality

\[
\det \nabla^2 w \geq e^{-w},
\]

and try to establish analogs of the results proved by Wang and Zhu in the toric case.

5 The Mukai-Umemura manifold and its deformations

The first part of this section gives an account, not aimed at algebraic geometry specialists, of a very interesting family of Fano 3-folds, following Mukai. The basic references are [22], [23], but there are also many other relevant papers in the algebraic geometry literature. Then we go on to discuss the existence of Kahler-Einstein metrics on some manifolds in this family.

5.1 Mukai’s construction

We start with a 7-dimensional complex vector space \( V \) and write \( Gr_3(V) \) for the Grassmann manifold of 3-dimensional subspaces of \( V \). So \( Gr_3(V) \) has dimension \( 3(7-3) = 12 \). A form \( \Omega \in \Lambda^2(V^*) \) defines a subset \( Z_{\Omega} \subset Gr_3(V) \) consisting of the 3-planes \( P \) such that \( \Omega|_P \) vanishes. In other language we consider the tautological rank 3 vector bundle \( U \to Gr_3(V) \); the form \( \Omega \) defines a section \( s_{\Omega} \) of \( \Lambda^2 U^* \) with zero set \( Z_{\Omega} \). For generic \( \Omega \) this zero set is a smooth subvariety of codimension 3. Now let \( \Omega_1, \Omega_2, \Omega_3 \) be three such forms and consider

\[
X = Z_{\Omega_1} \cap Z_{\Omega_2} \cap Z_{\Omega_3} \subset Gr_3(V).
\]

Of course this only depends on the 3-plane \( \Pi \in \Lambda^2 V^* \) spanned by the \( \Omega_i \), so we may sometimes write \( X_{\Pi} \). Obviously there is a Zariski-open subset \( U \) in the Grassmannian \( Gr_3(\Lambda^2 V^*) \) of 3-planes \( \Pi \) such that \( X_{\Pi} \) is a smooth subvariety of dimension \( 12 - 3 \cdot 3 = 3 \). This set \( U \) is non-empty, as we will see later. The group \( SL(V) \) acts on the whole construction and obviously different subspaces \( \Pi \) which lie in the same \( SL(V) \) orbit define isomorphic manifolds \( X_{\Pi} \), so we get a set of equivalence classes of manifolds constructed in this way, parametrised by the quotient \( U/SL(V) \). (Mukai shows further that this parametrisation is effective: i.e. \( X_{\Pi_1} \) is isomorphic to \( X_{\Pi_2} \) if and only if \( \Pi_1, \Pi_2 \) lie in the same \( SL(V) \) orbit. Moreover, he shows that all “prime Fano 3-folds of genus 12” arise in this way.)

We compute the canonical bundle \( K_X \) of the variety \( X = X_{\Pi} \) for some \( \Pi \in U \). We have

\[
\Lambda^2 U^* = U \otimes H
\]
where $H$ is the ample line bundle $\Lambda^3 U^*$. So, writing $\det$ for the the top exterior power of a vector bundle, we have

$$\det \Lambda^2 U^* = H^{\otimes 2}.$$  

The tangent bundle of the Grassmannian at a 3-plane $P \subset V$ can be identified with $P^* \otimes V/P$. So

$$\det TGr_3 = H^{\otimes 7}.$$  

Now since the tangent bundle of $X$ is the kernel of a surjective map from $TGr_3(V)$ to $\Lambda^2 U^* \oplus \Lambda^2 U^* \oplus \Lambda^2 U^*$ we have

$$K_X^{-1} = \det TX = H^{\otimes (7-3.2)} = H.$$  

Thus $X$ is a Fano manifold. The sections of $H$ over $Gr_3$ give the Plucker embedding

$$Gr_3(V) \to \mathbf{P}(\Lambda^3 V) = \mathbf{P}^{34}.$$  

For any 3-form $A \in \Lambda^3 V^*$ we get a hyperplane section $Y_A \subset Gr_3(V)$ which just consists of the 3-planes $P$ such $A|_P = 0$. By definition this occurs if $P$ is in $X$ and $A$ is in the image of the wedge product map $\Pi \otimes V^* \to \Lambda^3 V^*$. We expect this map to have an image of dimension $7.3 = 21$ in which case the image of the composite

$$X \to Gr_3(V) \to \mathbf{P}^{34}$$  

lies in a linear subspace $\mathbf{P}^{34-21} = \mathbf{P}^{13}$. Certainly this map is defined by sections of $K_X^{-1}$, we will see later that $H^0(X, K_X^{-1})$ has dimension 14 and that this embedding is that given by the anticanonical system.

To make this more concrete we show now that $X$ is a rational variety; that is, we construct an explicit parametrisation of a dense open set in $X$. Suppose we have a pair of 3-dimensional subspaces $P_0, Q_0 \subset V$ with $P_0 \cap Q_0 = 0$. We ask what 3-planes $P$ in the 6-dimensional subspace $P_0 \oplus Q_0$ lie in $X$. In matrix notation, we can write the restriction of a form $\Omega$ to $P_0 \oplus Q_0$ as

$$\begin{pmatrix} \sigma & A \\ -A^T & \tau \end{pmatrix}$$

Now consider the 3-dimensional subspaces $P$ which arise as the graphs of linear maps $M : P_0 \to Q_0$. The condition becomes

$$\sigma + M^T \tau M + (AM - (AM)^T) = 0.$$  \hspace{1cm} (33)$$

So our three forms $\Omega_i$ give us three triples $A_i, \sigma_i, \tau_i$ and we have three equations of the form (33) to solve to find a point of $X$. We have 9 unknowns: the entries of the matrix $M$. The left hand side of (33) takes values in the 3-dimensional space of skew symmetric $3 \times 3$ matrices so we obtain a total of $3.3 = 9$ equations in these 9 unknowns and we expect a finite number of solutions. These equations are quadratic and one can solve them explicitly, to see that there are generically two solutions. However it is easier to suppose that we are in the case when
$P_0$ itself lies in $X$. This means that all the $\tau_i$ are zero, so the equations (33) become linear. Generically this system of 9 linear equations in 9 unknowns is nondegenerate and there is a unique solution. Now suppose we have found one point $P_0$ in $X$ and consider the space of 6-planes in $V$ which contain $P_0$. This is a copy of projective 3-space $\mathbf{P}^3$. Given a point in $\mathbf{P}^3$, that is to say a 6-dimensional subspace $E$ of $V$, we choose a complementary subspace to write is $E = P_0 \oplus Q_0$. Then we can proceed as above and, by solving linear equations, find the points of $X \cap \text{Gr}_3(E)$. Generically there is just one, $P_E$ say, different from the original $P_0$. Conversely for any $P' \in X$ the sum $P \oplus P'$ lies in a 6-dimensional subspace. Of course there will be various exceptional cases, but the upshot is that we get a birational map from $\mathbf{P}^3$ to $X$ which takes a subspace $E$ containing $P_0$ to $P_E$.

We now consider a special manifold in this family. Take the vector space $V$ to be the sixth symmetric power $s^6$ of the fundamental representation of $SL(2, \mathbb{C})$. Then $\Lambda^2 V^* = \Lambda^2 s^6$ decomposes into distinct irreducible representations

$$\Lambda^2 s^6 = s^{10} \oplus s^6 \oplus s^2.$$ 

The $s^2$ summand is a 3-plane $\Pi_0$ invariant under $SL(2, \mathbb{C})$, so there is a natural $SL(2, \mathbb{C})$ action on the corresponding variety, the Mukai-Umemura manifold, $X_0 = X_{\Pi_0}$. We will see below that $X_0$ admits a Kahler-Einstein metric. The representation $s^6$ has a standard invariant symmetric form $(\cdot, \cdot)$ and the inclusion $s^2 \to \Lambda^2 s^6$ is just the map from the Lie algebra of $SL(2, \mathbb{C})$ given by the action on $s^6$. This comes down to saying that a 3-plane $P$ is in $X_0$ if and only if

$$\langle \delta p, q \rangle = 0$$

(34)

for all $p, q \in P$ and $\delta \in \text{sl}_2$. Notice that the action of $SL(2, \mathbb{C})$ on all the spaces involved actually factors through $PSL(2, \mathbb{C})$.

Identify the projectivisation of the fundamental representation $s^1 = \mathbb{C}^2$ with the standard round sphere and fix an icosahedron, which can be regarded as a set of 12 vertices in in this sphere. Thus we get a symmetry group $\Gamma \subset SO(3) \subset PSL(2, \mathbb{C})$ of order 60. There is a simple way to see that the 7-dimensional representation $s^6$ of $PSL(2, \mathbb{C})$ becomes reducible when restricted to $\Gamma$. There are 6 pairs of antipodal vertices and for each such pair $p, \overline{p}$ we have a 1-dimensional subspace consisting of polynomials which vanish to order 3 at $p, \overline{p}$. The sum of these 6 subspaces is obviously invariant under $\Gamma$ and is a proper subspace of $s^6$ since it has codimension at least 1. A little calculation shows that this invariant subspace is of dimension 3 and satisfies the criterion (34). So this subspace gives a point $P_0$ in $X_0$ fixed by $\Gamma$. On the other hand the stabiliser of $P_0$ is obviously not the whole of $SO(3)$ and, since there is no finite subgroup of $SO(3)$ strictly larger than $\Gamma$, the stabiliser must be exactly $\Gamma$.

Now go back to the wedge product $P_0 \wedge V^* \to \Lambda^3 V^*$. In terms of representations this is an $SL(2, \mathbb{C})$-map

$$s^2 \otimes s^6 \to \Lambda^3 s^6.$$ 

38
It is an exercise in representation theory to show that
\[ \Lambda^3 s^6 = s^{12} \oplus s^8 \oplus s^6 \oplus s^4 \oplus s^2 \oplus s^0. \]
So comparing with
\[ s^2 \otimes s^6 = s^8 \oplus s^6 \oplus s^4 \oplus s^2 \oplus s^0 \]
we see that the embedding \( X_0 \subset Gr_3(V) \subset P(\Lambda^3 V) \) gives rise to an \( SL(2, \mathbb{C}) \)-equivariant embedding
\[ X_0 \hookrightarrow P(s^0 \oplus s^{12}). \]
In other words, by our identification of the anticanonical bundle \( K^{-1} \) we have
\[ H^0(X_0, K^{-1}) = s^0 \oplus s^{12}, \]
as a representation of \( SL(2, \mathbb{C}) \). In particular, there is an \( SL(2, \mathbb{C}) \)-invariant section \( \sigma \) of \( K^{-1} \). Explicitly, if we identify \( \Lambda^3 s^6 \) with \( \Lambda^4 s^6 \) then \( \sigma \) corresponds to the 4-form on \( V = s^5 \) defined as follows. We choose any orthonormal basis \( \Omega_1, \Omega_2, \Omega_3 \) of \( P_0 \) and write down the 4-form
\[ \ast \sigma = \Omega_1^2 + \Omega_2^2 + \Omega_3^2. \]
In this way, we get another description of the manifold \( X_0 \). Our point \( P_0 \in X_0 \) cannot lie in the zero set of \( \sigma \) (since its orbit is 3-dimensional). So, in the embedding (35), we have
\[ P_0 = [1, v_0] \in P(C \oplus s^{12}). \]
Thus \( v_0 \) is an element of \( s^{12} \) whose stabiliser in \( PSL(2, \mathbb{C}) \) is exactly \( \Gamma \). Now there is an obvious element of the projective space \( P(s^{12}) \) with stabiliser \( \Gamma \), just the configuration of vertices of the icosahedron, regarded as an element of the symmetric product. Since \( \Gamma \) is a perfect group it must act trivially on the corresponding line in \( s^{12} \), so we get a vector in \( s^{12} \) with stabiliser \( \Gamma \). It is easy to see that, up to a multiple, this is the only element of \( s^{12} \) with stabiliser \( \Gamma \), and thus we have identified \( v_0 \). Then we can simply define \( X_0 \) to be the closure in \( P(C \oplus s^{12}) \) of the \( PSL(2, \mathbb{C}) \)-orbit of \( v_0 \) in \( s^{12} \). (Here we are regarding the vector space \( s^{12} \) as being a subset of the projective space \( P(C \oplus s^{12}) \) in the familiar way.)
In this description, the intersection of \( X_0 \) with the hyperplane at infinity \( D = P(s^{12}) \subset P(C \oplus s^{12}) \), is, by definition, the zero set of the invariant section \( \sigma \) of \( K^{-1} \). Consider a 1-parameter subgroup \( \lambda_t \) in \( PSL(2, \mathbb{C}) \). Thus we have a pair of distinct point \( z_+, z_- \) such that when \( t \) is large positive the map \( \lambda_t \) contracts most of the sphere to a small neighbourhood of \( z_+ \), and when \( t \) is large negative to a small neighbourhood of \( z_- \). If \( y_1, \ldots, y_{12} \) is any configuration of distinct points it is not hard to see that the limit as \( t \to \infty \) of
\[ \lambda_t(y) = (\lambda_t(y_1), \lambda_t(y_2, \ldots, \lambda_t(y_{12})) \]
39
in the symmetric product $\mathbf{P}(s^{12})$ is either $12z_+ = (z_+, z_+, \ldots, z_+)$ (in the generic case) or $11z_+ + z_- = (z_+, \ldots, z_+, z_-)$ (in the case when one of the $y_i$ is $z_-)$.

Using this, Mukai and Umemura show that the divisor at infinity $D$ consists precisely of the union of points of the form $12z_+$ or $11z_+ + z_-$ in $\mathbf{P}(s^{12})$. It is easy to identify this geometrically. The points of the form $12z_+$ make up the rational normal curve in $\mathbf{P}(s^{12})$. Our divisor $D$ is the surface swept out by the lines in $\mathbf{P}(s^{12})$ tangent to the rational normal curve. As a set we can identify $D$ with $\mathbf{P}^1 \times \mathbf{P}^1$: we just map $(z_+, z_-)$ to $11z_+ + z_- \in D$. But the surface $D$ is singular and a more precise statement is that the map above is a holomorphic map $\nu : \mathbf{P}^1 \times \mathbf{P}^1 \to D$ which is the normalisation of $D$. The singular set of $D$ is the image of the diagonal in $\mathbf{P}^1 \times \mathbf{P}^1$, and it is easy to check that the singularity has the form of a cusp transverse to the diagonal. That is to say, we can choose local co-ordinates $z_1 z_2, z_3$ in $X_0$ around a singular point of $D$ such that $D$ is defined by the equation $z_1^2 = z_2^3$.

We now have a rather explicit description of $X_0$, as the compactification of $PSL(2, \mathbb{C})/T$ formed by adjoining the divisor $D$. We can use this to compute the action of $PSL(2, \mathbb{C})$ on all of the spaces of sections $H^0(X_0, K^{-p})$. For the pull back $\nu^*(K^{-1})$ is isomorphic to the line bundle $\mathcal{O}(11, 1)$ over $\mathbf{P}^1 \times \mathbf{P}^1$. We can regard the structure sheaf of $D$ as a subsheaf of that of $\mathbf{P}^1 \times \mathbf{P}^1$. From the local model of the singularity along the diagonal one sees that the quotient can be identified with sections of $\mathcal{O}(2)$ along the diagonal. This means that $H^0(D, K^{-p}|_D)$ is the kernel of a map $H^0(\mathbf{P}^1 \times \mathbf{P}^1; \mathcal{O}(11p, p)) \to H^0(\mathbf{P}^1; \mathcal{O}(12p - 2))$. As representations of $PSL(2, \mathbb{C})$ this is a map

$$s^{11p} \otimes s^p \to s^{12p - 2}.$$

Now

$$s^{11p} \otimes s^p = s^{12p} + s^{12p - 2} \ldots + s^{10p}$$

and the map above is just the projection to the second factor. So

$$H^0(D; K^{-p}|_D) = s^{12p} \oplus s^{12p - 4} \oplus s^{12p - 6} \ldots \oplus s^{10p + 2} \oplus s^{10p}.$$

Then the exact cohomology sequence of

$$0 \to K^{-p-1} \to K^{-p} \to K^{-p}|_D \to 0$$

together with Kodaira vanishing on $X_0$ gives

$$H^0(X_0, K^{-p}) = H^0(X_0, K^{-p-1}) \oplus s^{12p} \oplus s^{12p - 4} \ldots \oplus s^{10p},$$

and inductively we get a description of each $H^0(X_0, K^{-p})$. Thus

$$H^0(X_0, K^{-1}) = s^0 \oplus s^{12},$$

$$H^0(X_0, K^{-2}) = s^0 \oplus s^{12} \oplus s^{24} \oplus s^{20}.$$
5.2 Topological and symplectic picture

We will now get another explicit picture of $X_0$, taking the point of view of symplectic geometry. Recall that all Kahler metrics in the cohomology class $c_1(X_0)$ define equivalent symplectic structures, so we have a well-defined symplectic manifold $(X_0, \omega)$ with an $SO(3)$-action. Thus we have an equivariant moment map

$$\mu : X_0 \rightarrow \mathbb{R}^3 = \text{Lie}(SO(3))^*.$$ 

whose image is clearly a ball in $\mathbb{R}^3$. We can understand the structure of this moment map by restricting to a subgroup $S^1 \subset SO(3)$, say that corresponding to the $x_1$-axis in $\mathbb{R}^3$. Then the Hamiltonian $H$ for this circle action on $X_0$ is the composite of $\mu$ with projection to the $x_1$-axis. The critical points of $H$ are the fixed points of the circle action and we can find these explicitly. We can suppose that our circle subgroups corresponds to the standard action of

$$\left( \begin{array}{cc} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{array} \right),$$

acting on $\mathbb{C}^7$ with weights $\lambda^3, \ldots, \lambda^{-3}$. We write $e_i$ for the basis vector belonging to the weight $\lambda^i$. This induces an action on the Grassmannian $Gr_3(V)$ whose fixed points are just invariant 3-dimensional subspaces of $\mathbb{C}^7$ and these are just the spans $P_{ijk} = \langle e_i, e_j, e_k \rangle$ for distinct $i, j, k$. By checking the 35 different cases, or otherwise, one finds that the only $P_{ijk}$ which satisfy the criterion (34) to lie in $X_0$ are $P_{123}, P_{023}, P_{0-2-3}, P_{-1-2-3}$. There is an action of the Weyl group $\{ \pm 1 \}$ on the whole situation which commutes, up to sign, with the circle action, takes $H$ to $-H$ and takes $P_{ijk}$ to $P_{-i-j-k}$. So there are four fixed points of the circle action but to analyse the local structure around them it suffices to consider the two cases $P_{123}, P_{023}$. Notice that, by considering $H$ as a Morse function we immediately see that $X_0$ has the same additive homology as $\mathbb{C}P^3$. Notice also that the value of $H$ at a critical point is just given by the weight of the action on the fibre of $K^{-1}$ over this point, which is just $i + j + k$ at $P_{ijk}$.

We next compute the weights of the circle action on the tangent spaces at the fixed points. This is similar to the calculation of the canonical bundle. At a fixed point the tangent space $TX_0$, viewed as a representation of $S^1$, can be written as the formal difference

$$TGr_3(V) - (\Lambda^2 U^* \otimes \text{Lie}(SO(3))).$$

Computing the weights of these two terms and subtracting we find that the weights of the action on the tangent space at $P_{123}$ are $(1,2,3)$ and on the tangent space at $P_{023}$ are $(1,-1,5)$. In either case the orbit of the fixed point is a copy of $SO(3)/S^1 = S^2$ in $X_0$ and the weight 1 in the action on $TX_0$ just corresponds to the tangent space of this orbit. The weights normal to the orbit are $(2,3)$ in the case of $P_{123}$ and $(-1,5)$ in the case of $P_{023}$.

With these calculations we can get a good picture of the map $\mu$. Write $\Sigma, \Sigma'$ for the orbits of $P_{123}$ and $P_{023}$ respectively. Then $\mu$ restricts to an $SO(3)$-equivariant equivalence between $\Sigma$ and the sphere of radius $1 + 2 + 3 = 6$ in $\mathbb{R}^3$. 

41
and between $\Sigma'$ and the sphere of radius $0 + 2 + 3 = 5$. The image of $\mu$ is the ball of radius 6 and the critical values of $\mu$ are precisely these two spheres. So $\mu$ is a fibration away from these spheres. For $x \in \mathbb{R}^4$, write $F_x$ for the preimage $\mu^{-1}(x)$. If $|x| \neq 5, 6$ the fibre $F_x$ is a 3-manifold. If also $|x| > 0$ then this 3-manifold has a natural circle action defined by the circle subgroup of $SO(3)$ fixing $x$. When $x = 0$ the fibre has an $SO(3)$ action. If also $|x| > 0$ then this 3-manifold has a natural circle action defined by the circle subgroup of $SO(3)$ fixing $x$. When $x = 0$ the fibre has an $SO(3)$ action. As $x$ varies in $\mathbb{R}^3$ the fibre $F_x$ is a 3-manifold. If also $|x| > 0$ then this 3-manifold has a natural circle action defined by the circle subgroup of $SO(3)$ fixing $x$. When $x = 0$ the fibre has an $SO(3)$ action. As $x$ moves along the positive $x_1$-axis, the preimage by $\mu$ of the positive $x_1$-axis is a smooth 4-manifold, with a circle action, and the fibres $F_x$, for $x$ on the axis, are the level sets of the Hamiltonian $H$, restricted to $V$. Then we have the usual Morse-theory description of these changes, from the Hessian of $H$ on $V$, which is determined by the weights of the circle action. As $x$ moves across the point $(6, 0, 0)$ the situation is modelled by the level sets $2|z_1|^2 + 3|z_3|^2 = \epsilon$, for $(z_1, z_2) \in \mathbb{C}^2$, with the circle action of weight $(2, 3)$. Thus the fibre changes from the empty set to a 3-sphere with an action given by these weights. As $x$ moves across the point $(5, 0, 0)$ the situation is modelled, locally, by the level sets $-|z_1|^2 + 5|z_2|^2 = \epsilon$, with the circle action of weight $(-1, 5)$. The effect on the fibres is to perform a “Dehn surgery” on an $S^1$-orbit. Thus the fibres $F_x$ for $|x| < 5$ are obtained by performing this surgery on a knot $\Gamma$ in $S^3$. Now $\Gamma$ is a free orbit of the $(2, 3)$ action so it is the $(2, 3)$ “torus knot” which is just a trefoil. To nail down the Dehn surgery completely we need to specify a framing of the knot but this is determined by the fact that the linking number of a nearby orbit with $\Gamma$ is the weight 5, from which one concludes that the framing is $+1$. This is a well-known description of the Poincaré homology sphere (the result of $+1$-surgery on a trefoil), and ties in with our previous discussion since the fibre $F_0$ is the $SO(3)$-orbit $SO(3)/\Gamma$. (Another way of expressing this is that the fibres $F_x$ are Seifert-fibred 3-manifolds: for $5 < |x| < 6$ we have two multiple fibres with multiplicity $(2, 3)$ and the surgery across $|x| = 5$ introduces another multiple fibre with multiplicity 5, so for $|x| < 5$ we get the Seifert manifold with multiplicities $(2, 3, 5)$, which is another well-known description of the Poincaré manifold.)

It is interesting to match this picture up with the algebro-geometric description. This illustrates the general theory of Kirwan [19]. The 2-sphere $\Sigma$ at which $|\mu|$ attains its maximal value 6 is a holomorphic sphere in $X_0$: it is just the rational normal curve in our divisor $D \subset \mathbb{P}(s^{12})$. The other sphere $\Sigma'$ is not holomorphic. It is a critical manifold for the function $|\mu|^2$ on $X_0$ and the divisor $D$ appears as the associated “ascending set”: the closure of the set of points which flow to $\Sigma'$ under the decreasing gradient flow of $|\mu|^2$. In our description of $D$ as $S^2 \times S^2$ the holomorphic curve $\Sigma$ is the diagonal and $\Sigma'$ is the
“anti-diagonal” consisting of pairs of antipodal points. One can also see the cusp singularity in \( D \), transverse to \( \Sigma \), from the weights \((2, 3)\) of the circle action on the normal bundle.

Notice that if we write \( \mathbb{CP}^3 = \mathbb{P}(s^3) \), for the 4-dimensional representation \( s^3 \) of \( SU(2) \), the moment map \( \mu : \mathbb{CP}^3 \to \mathbb{R}^3 \) for the action gives a description of \( \mathbb{CP}^3 \) very similar to that above. In this case \( \mu^{-1}(0) \) is \( SO(3)/H \) where \( H \subset SO(3) \) is the group of symmetries of an equilateral triangle, and we see this 3-manifold described as the Seifert fibration with multiple fibres \((2, 2, 3)\).

### 5.3 Deformations

Here we study the deformations of Mukai’s construction about the special solution \( X_0 \). Recall that a manifold in this family is specified by a 3-plane in \( \Lambda^2 \mathbb{C}^7 \).

We start with the 3-plane \( s^2 \subset \Lambda^2 s^6 = s^{10} \oplus s^6 \oplus s^2 \). The tangent space of the Grassmannian at this point is given by the linear maps from \( s^2 \) to the complementary subspace \( s^{10} \oplus s^6 \), that is (using the fact that all these representations are isomorphic to their duals)

\[
TGr_3(\Lambda^2 \mathbb{C}^7) = (s^{10} \oplus s^6) \otimes s^2 = s^{12} \oplus 2s^8 \oplus s^6 \oplus s^4.
\]

The action of the group \( SL(\mathbb{C}^7) = SL(s^6) \) gives a linear map

\[
\mathfrak{sl}(7) \to TGr_3(\Lambda^2 \mathbb{C}^7),
\]

which we know has kernel the Lie algebra \( \mathfrak{sl}(2) \) of the stabiliser. Now the Lie algebra of \( GL(\mathbb{C}^7) = GL(s^6) \) is

\[
s^6 \otimes s^6 = s^{12} \oplus s^{10} \oplus s^8 \oplus s^6 \oplus s^4 \oplus s^2 \oplus s^0
\]

so the Lie algebra of \( SL(s^6) \) is \( s^{12} \oplus \ldots s^2 \). It is clear then that the quotient of the tangent space by the tangent space to the orbit is just \( s^8 \), as a representation of \( PSL(2, \mathbb{C}) \). By general theory there is an equivariant slice: a \( PSL(2, \mathbb{C}) \) equivariant embedding \( j \) from a neighbourhood of 0 in \( s^8 \) into \( Gr_3(\Lambda^2) \), mapping 0 to our fixed subspace \( s^2 \), such that two points \( j(p), j(q) \) in the same \( SL(7) \) orbit if and only if \( p, q \) are in the same \( PSL(2) \) orbit. In fact, although we do not really need this, what we are describing is the versal deformation of \( X_0 \), so \( H^1(TX_0) = s^8 \), as a representation of \( PSL(2, \mathbb{C}) \).

One can gain a lot of insight from this simple calculation. The structure of the orbits of \( PSL(2, \mathbb{C}) \) on \( s^8 \) (or any \( s^p \)) is a standard example in Geometric Invariant Theory. There are five cases

1. The trivial orbit \{0\}.

2. The orbits of polynomials having no zero of multiplicity \( \geq 4 \). These are closed in \( s^8 \).

3. The orbit of polynomials having two distinct zeros, each of multiplicity four. This orbit is closed and each point in it has stabiliser \( \mathbb{C}^* \subset PSL(2, \mathbb{C}) \).
4. The orbits of polynomials having a zero of multiplicity four and other zeros each of multiplicity less than four. These orbits are not closed but there closure contains the orbit of type (3).

5. The orbits of polynomials having a zero of multiplicity \( \geq 5 \). These are not closed and contain 0 in their closure.

This is the source of the famous example of Tian of Fano manifolds without Kahler-Einstein (or Ricci soliton) metrics \[31\]. Tian shows that the manifolds corresponding to any \( PSL(2, \mathbb{C}) \) orbit of type (5) cannot have such metrics. Tian’s general results also show the same for the manifolds corresponding to orbits of type (4). Tian’s results are of course deep and difficult but we note now that a weaker statement is rather obviously true. For this we need to recall some background.

In general, the linearisation of the Kahler-Einstein equations on a complex manifold \( Z \) at a solution \( \omega_0 \) is given by the self-adjoint operator \( \Delta + 1 \) and, much as we have seen in Section 3, the kernel of this can be identified with the Lie algebra of the isometry group \( G \) of \( \omega_0 \). Suppose we have a \( G \)-equivariant deformation of \( Z_0 \); i.e. a complex manifold \( Z \) with a \( G \)-action, an action of \( G \) on a ball \( B \subset \mathbb{C}^m \) and a \( G \)-equivariant submersion \( \pi: Z \rightarrow B \). In this situation we automatically get “local actions” of the complexified group \( G^c \) on \( Z \) and \( B \), compatible with \( \pi \). The standard “Kuranishi method”, which depends only on the formal properties of the situation, yields the following structure (after possibly restricting to a smaller ball \( B \)).

- A \( G \)-invariant family of Kahler metrics \( \omega_t \) on the fibres \( Z_t = \pi^{-1}(t) \) such that \( \omega_t \) is isometric to \( \omega_{t'} \) if and only if \( t \) and \( t' \) are in the same \( G \)-orbit.

- A smooth map \( \nu: B \rightarrow \mathfrak{g}^* \), equivariant for the action of \( G \) on \( B \) and the co-adjoint action on \( \mathfrak{g}^* \), such that \( \omega_t \) is Kahler-Einstein if and only if \( \nu(t) = 0 \).

Now in this general situation we can see that, if the \( G \)-action on \( B \) is non-trivial the map \( \nu \) cannot be identically zero. For if \( t, t' \) are in the same orbit of the local \( G^c \) action on \( B \) then \( Z_t \) and \( Z_{t'} \) are isomorphic complex manifolds. But if \( \nu(t) \) and \( \nu(t') \) both vanish then \( \omega_t \) and \( \omega_{t'} \) are Kahler-Einstein and, by the uniqueness of the Kahler-Einstein solution, they must be isometric and this only happens if \( t, t' \) are in the same \( G \)-orbit. Thus what we see from this elementary argument is that as we deform \( Z_0 \) in the smooth family \( Z_t \) we cannot deform the metric \( \omega_0 \) in a smooth family of Kahler-Einstein metrics, for all small \( t \). Tian’s much stronger result is that if the Futaki invariant of \( Z_0 \) vanishes (say), and if 0 lies in the closure of the the \( G^c \)-orbit of a point \( t \in B \) then \( Z_t \) does not admit any Kahler-Einstein metric at all. This is an example of the “jumping of structures” phenomenon discussed in Section 1: there are arbitrarily small deformations of \( Z_0 \) which are equivalent to a different structure \( Z_t \).

Returning to our special case of the Mukai-Umemura manifold, we can see conversely that there are some deformations of \( X_0 \) which do admit Kahler-Einstein metrics. The general theory of these “obstruction maps” \( \nu \) is being
developed by T. Brömle, in his Ph.D thesis, but in this special case we can make some simple deductions from symmetry arguments. Let $p$ be a point in $s^8$ which is fixed by a subgroup $J \subset SO(3)$. Then $J$ acts on $\mathbb{R}^3 = su(2)$ and if $\nu$ is any equivariant map from $s^8$ to $\mathbb{R}^3$ then $J$ must fix $\nu(p)$. So if the origin is the only point in $\mathbb{R}^3$ fixed by $J$ then we must have $\nu(p) = 0$. Consider, for example,

$$p = C(z^4 - \alpha w^4)(w^4 - \alpha z^4),$$

with any $\alpha, C \in \mathbb{C}$. This is fixed by a dihedral group $J$ of order 8 which has the desired property, so we see that the deformations corresponding such elements of $s^8$ admit Kahler-Einstein metrics, for small $C$. For $\alpha, C \neq 0$ the element $p$ has a discrete stabiliser in $SO(3)$ and it follows that the corresponding metrics have discrete isometry groups. But then the deformation theory implies that all small deformations of these manifolds admit Kahler-Einstein metrics. So we conclude that there is a non-empty open set in $\mathcal{U}$ where the manifolds admit Kahler-Einstein metrics.

Taking $\alpha = 0$ above we get a special family of deformations, admitting Kahler-Einstein metrics, where we can take $J = O(2) \subset SO(3)$. It follows that the corresponding manifolds have a $\mathbb{C}^*$-action. We can see this family of manifolds explicitly as follows. Fix the action on $\mathbb{C}^7$ with weights $\lambda_3, \ldots, \lambda^{-3}$ as usual. Then we want to look at 3-dimensional subspaces $\Pi$ of $\Lambda^2\mathbb{C}^7$ preserved by the action and we just consider those on which the action has weights 1, 0, −1. Now the weight 1-subspace of $\lambda^2$ has a basis $e_3 \wedge e_{-2}, e_2 \wedge e_{-1}, e_1 \wedge e_0$ and our space $\Pi$ must contain a vector $u = u_{3,-2} e_3 \wedge e_{-2} + u_{2,-1} e_2 \wedge e_{-1} + u_{1,0} e_1 \wedge e_0$, for scalars $u_{3,-2}$ etc. Similarly $\Pi$ must contain a vector $v = v_{1,-1} e_1 \wedge e_{-1} + v_{2,-2} e_2 \wedge e_{-2} + v_{3,-3} e_3 \wedge e_{-3}$ and a vector $w = w_{-3,2} e_{-3} \wedge e_2 + w_{-2,1} e_{-2} \wedge e_1 + w_{-1,0} e_{-1} \wedge e_0$.

The vector space $\Pi$ is determined by these three vectors $u, v, w$. The coefficients are not unique. We could change $u, v, w$ to $\mu_1 u, \mu_2 v, \mu_3 w$. Also we could change our basis vectors $e_i$ to $\lambda_i e_i$ to give an equivalent 3-plane. This would change the coefficients, for example $u_{3,-2}$ would change to $\lambda_3 \lambda_{-2} u_{3,-2}$. However the expression

$$\tau = \frac{u_{3,-2} w_{-3,2} v_{1,-1}}{u_{2,-1} w_{-2,1} v_{3,-3}}$$

is invariant under all these changes and gives a “modulus” for this family. The Mukai-Umemura manifold has $\tau = 1$. When $\tau$ is close to 1 we have seen that the corresponding manifold admits a Kahler-Einstein metric. It seems likely that this true for all $\tau$ but, as far the author is aware, this is not known. It seems an interesting test case for future developments in the existence theory.
5.4 The $\alpha$-invariant

In this subsection we establish the fact used above, that the Mukai-Umemura manifold has a Kahler-Einstein metric\(^1\), which is . For this we appeal to the theory of the $\alpha$-invariant, developed by Tian \cite{30}. We begin by recalling the definition. Let $Z$ be a Fano manifold on which a compact group $G$ acts by holomorphic automorphisms and fix a $G$-invariant Kahler metric $\omega_0$ in the cohomology class $-c_1(K_Z)$. Let $\mathcal{P}$ be the set of $G$-invariant Kahler potentials $\psi$ on $X$ such that $\omega_\psi = \omega_0 + i\partial\bar{\partial}\psi > 0$ and $\max_Z \psi = 0$. Thus $\mathcal{P}$ can be identified with the set of all $G$-invariant Kahler metrics in the given Kahler class. Let $A \subset \mathbb{R}$ be the set defined by the condition that $\beta \in A$ if there exists a $C_\beta \in \mathbb{R}$ such that

$$\int_Z e^{-\beta \psi} \, d\mu_0 \leq C_\beta,$$

for all $\psi \in \mathcal{P}$. Here $d\mu_0$ is the volume form defined by the fixed metric $\omega_0$. Then Tian sets

$$\alpha_G(Z) = \sup \{ \beta : \beta \in A \},$$

and shows that this does not depend on the choice of $\omega_0$. He shows that $\alpha_G(Z)$ is always strictly positive and that if $\alpha_G(Z) > \frac{2}{n+1}$ then $Z$ has a Kahler-Einstein metric. What we really show in this subsection is that if we take the Mukai-Umemura manifold $X$ with the action of $SO(3)$ then,

**Theorem 3** The $\alpha$-invariant $\alpha_{SO(3)}(X_0)$ is $5/6$.

So, since $5/6 > 3/4$, Tian’s theory proves the existence of a Kahler-Einstein metric. We should say straightaway that this is not really a new result. Alessio Corti has explained to the author that, given the facts above, it can be obtained from the more general theories of \cite{13}. But our argument is extremely simple and fits well into the general framework of this article.

We will only write down the proof that $\alpha \geq 5/6$, which is what is relevant to Corollary 1. The proof that $\alpha = 5/6$ is an easy extension of this.

**Lemma 3** There is an $M \in \mathbb{R}$ such that

$$\int_Z \psi \, d\mu_0 \geq -M$$

for all $\psi \in \mathcal{P}$.

This is a step in Tian’s proof that $\alpha > 0$ and we repeat his argument. If $\psi \in \mathcal{P}$ we have

$$\Delta_0 \psi = 2\Lambda(i\partial\bar{\partial}\psi) \geq -2n.$$

\(^1\)This material appeared in the preprint A note on the $\alpha$-invariant of the Mukai-Umemura 3-fold arxiv DG 07114357.
Let $K$ be the Green’s function for $\Delta_0$, so that for all functions $f$ on $Z$

$$f(x) = -\int_Z K(x,y)(\Delta_0 f)(y) d\mu_0(y) + \frac{1}{V} \int f(y) d\mu_0(y),$$

where $V$ is the volume of the manifold. With our sign conventions, $K$ is bounded below and, since we can change $K$ by the addition of a constant without affecting the identity, we may suppose that $K \geq 0$. While $K$ is singular along the diagonal it is integrable in each variable. Let $x$ be the point where $\psi$ vanishes. Then applying the Green’s identity to $\psi$ we have

$$\int_Z \psi(y) d\mu_0(y) = V \int_Z K(x,y) \Delta_0 \psi d\mu_0(y) \geq -2nV \int_Z K(x,y) d\mu_0(y).$$

So we can take

$$M = 2nV \max_x \int_Z K(x,y) d\mu_0(y).$$

For the rest of this section we work with the Mukai-Umemura manifold, which we denote by $X$. Let $\sigma$ be the $SO(3)$-invariant section of the anticanonical bundle $K^{-1}$ cutting out the divisor $D$. There is a Hermitian metric on this line bundle such that the curvature of the associated unitary connection is $-i\omega_0$. Set

$$f_0 = -\log (|\sigma|^2).$$

This is a smooth function on $X \setminus D$ and $i\partial \bar{\partial} f_0 = \omega_0$.

**Lemma 4** For any $\beta < \frac{5}{6}$ the function $\exp(\beta f_0)$ is integrable.

This is also standard. The integral in question is

$$\int_Z |\sigma|^{-2\beta} d\mu_0.$$

By what we know about the singularities of $D$, we can reduce to considering the integrals

$$\int_B |z^2 - w^3|^{-2\beta},$$

where $B$ is the unit ball in $\mathbb{C}^2$ and $z, w$ are complex co-ordinates. Let $T$ be the linear map $T(z, w) = (z/8, w/4)$ and for $r \geq 1$ set

$$\Omega_r = T^r(B) \setminus T^{r-1}(B).$$

Set

$$I_r = \int_{\Omega_r} |z^2 - w^3|^{-2\beta}.$$

The substitution $(z', w') = T(z, w)$ shows that

$$I_{r+1} = 2^{(12\beta - 10)} I_r.$$
Thus $\sum_r I_r$ is finite if $\beta < 5/6$ and the union of the $\Omega_r$ cover $B^4 \setminus \{0\}$.

Now we give the main proof. Let $x_0 \in X$ be the point with stabiliser $\Gamma$. We identify $SO(3)$-invariant functions on $X \setminus D$ with $\Gamma$-invariant functions on $M = PSL(2, \mathbb{C})/SO(3)$ as in (4.2). The function $f_0$ on $x \setminus D$ corresponds to a convex function $\phi_0$ on $M = PSL(2, \mathbb{C})/SO(3)$ which is an “admissible potential” in the language of (4.2). For any other admissible potential $\phi$ the difference $\phi - \phi_0$ corresponds to $\psi$, restricted to $X \setminus D$. The normalisation that $\max \psi = 0$ becomes the condition that $\sup \phi - \phi_0 = 0$, and in particular $\phi \leq \phi_0$.

Let $P_0 \in M$ be the identity coset. It is the unique point fixed by the action of $\Gamma$. Any admissible potential function $\phi$ on $M$ is proper and bounded below so achieves a minimum in $M$. By the convexity and $\Gamma$-invariance this minimum must occur at $P_0$. Set $\phi(P_0) = -b$. Then the inequality $\phi_0 \geq -b$ translates back into the statement that $\psi \geq f_0 - b$. So

$$\int_Z e^{-\beta \psi} \, d\mu_0 \leq e^{b\beta} \int_Z f_0^{-\beta} \, d\mu_0.$$  

By Lemma 4, it suffices to obtain an upper bound on $b$. Let $B$ be the geodesic ball in $M$ centred on $P_0$, of radius 1 say, and let $\bar{\pi}$ be the maximum value of $\phi_0$ on $B$, so for any $\phi$ we have $\phi \leq \bar{\pi}$ on $B$. Convexity along geodesics emanating from $P_0$ implies that $\phi(Q) \leq -b + (\bar{\pi} + b) \text{dist}(Q, P_0)$, for any point $Q$ in $B$. In particular, on the ball $\frac{1}{2}B$ of radius $1/2$ about $P_0$ we have $\phi \leq (\bar{\pi} - b)/2$.

Take the inverse image in $PSL(2, \mathbb{C})$ of the ball $\frac{1}{2}B$ and map this to $X$ by $g \rightarrow g(x_0)$. The image obviously contains a neighbourhood $N$ of $x_0$ and on $N$ we have $\psi \leq \frac{\bar{\pi}}{2} - b + f_0$. Then Lemma 3 implies that $b$ cannot be very large. In fact, if the minimum of $f_0$ on $N$ is $\underline{a}$, we have $\psi \leq (\bar{\pi} - \underline{a}) - \frac{b}{2}$ on $N$, so

$$-M \leq \int_N \psi \, d\mu_0 \leq (\bar{\pi} - \underline{a}) - \frac{b}{2} \text{Vol}(N),$$

hence

$$b \leq (\bar{\pi} - 2\underline{a}) + \frac{2M}{\text{Vol}(N)}$$

where $M$ is as in Lemma 3. This completes the proof of Theorem 3.

Notice that the same argument can be applied in the toric case, when the polytope $P$ has a group $\Gamma$ of symmetries, as discussed in (4.2). We should suppose that $\Gamma$ has a unique fixed point in $P$: then the proof proceeds exactly as before. The analogue of Lemma 4 holds with $\beta < 1$ since the local models for the zeros of $s$ are $f_p(z_1, \ldots, z_n) = 0$ where $f_p(z_1, \ldots, z_n) = z_1 \ldots z_p$ and $|f_p|^{-\beta}$ is locally integrable for $\beta < 1$. the conclusion is that the $\alpha$-invariant in this case is 1. which is a theorem of Batyrev and Selinova [3]. (Song gave another proof in [24], and showed conversely that for polytopes which do not have such a symmetry group the $\alpha$-invariant never exceeds $n/n + 1$.)
References

[1] Abreu, M. Kahler geometry of toric varieties and extremal metrics Int. Jour. Math. 9 1198 641-651

[2] Alexeev, V. and Katzarkov,L. On K-stability of reductive varieties Geometric and Functional Analysis 15 2005 (297-310)

[3] M.F. Atiyah and R. Bott The Yang-Mills equations over Riemann surfaces Philosophical Transactions of the Royal Society of London, Series A 308 (1982) 523-615

[4] Batyrev, V. and Selivanova, E. Einstein-Kahler metrics on symmetric toric Fano manifolds J. reine angew. Math. 512 (1999) 225-236

[5] Bielawski, R. Kahler metrics on GC Jour. Reine Angew. Math. 559 2003(123-136)

[6] E. Calabi Extremal Kahler metrics In:Seminar in Differential geometry (ed. Yau) Princeton U.P. (1983)

[7] S. K. Donaldson Infinite determinants, stable bundles and curvature Duke Math. Jour. 54 (1987) 231-47

[8] Donaldson, S. K. Remarks on gauge theory, complex geometry and four-manifold topology Fields Medallists' Lectures (Atiyah, Iagolnitzer eds.) World Scientific 1997 (384-403)

[9] Donaldson, S. K. Scalar curvature and stability of toric varieties Jour. Differential Geometry 62 (2002) 289-349

[10] Donaldson, S. K. Extremal metrics on toric surfaces: a continuity method To appear in Jour. Diff. Geom.

[11] Donaldson,S.K. Constant scalar curvature metrics on toric surfaces In preparation.

[12] Donaldson, S. and Kronheimer, P. The geometry of four-manifolds Oxford U.P. 1990

[13] Demailly, J-P., and Kollar, J. Semi-continuity of complex singularity exponents and Kahler-Einstein metrics on Fano orbifolds Ann. Sci. Ecole Normale Superiore 34 (2001) 525-556

[14] Griffiths, P. and Harris, J. Principles of Algebraic Geometry Wiley 1978

[15] Guillemin, V. Kahler structures on toric varieties Jour. Differential Geom. 40 (1994) 285-309

[16] Guillemin,V. Moment maps and combinatorial invariants of Hamiltonian T^n-spaces Birkhauser 1994
[17] Gutiérrez, C. E. The Monge-Ampère equation Birkhauser 2001

[18] Hwang, A. and Singer, M. A momentum construction for circle-invariant Kahler metrics Trans. Amer. Math. Soc. 354 2002 (2285-2325)

[19] Kirwan, F.C. Cohomology of quotients in symplectic and algebraic geometry Princeton U.P. 1984

[20] Losev, I.V. Proof of the Knop conjecture Arxiv SG/0612561

[21] McDuff, D and Salamon, D Introduction to symplectic topology Oxford U.P. 1995

[22] Mukai, S. Fano 3-folds in “Complex projective geometry” 255-263 London math. Soc. Lecture Notes 179 Cambridge UP (1992)

[23] Mukai, S. and Umemura, H. Minimal rational threefolds In: Algebraic Geometry (Tokyo/Kyoto 1982) 490-518 Lecture Notes in Math. 1016 Springer 1983

[24] Phong, D.H. and Sturm, J. Lectures on stability and constant scalar curvature Kahler metrics Arxiv DG/08014179

[25] Podesta and Spiro, Kahler-Ricci solitons on homogeneous toric bundles I, II Arxiv DG/0604070/0604071

[26] Song, J. The $\alpha$-invariant on toric Fano manifolds Amer. Jour. Math. 127 (2005) No. 6 1247-1259

[27] Raza, A.A. Scalar curvature and multiplicity-free actions Ph. D. Thesis, Imperial College London (2005)

[28] Szekelyhidi, G. The Calabi functional on a ruled surface Arxiv DG/0703562

[29] Tian, G. On a set of polarised Kahler metrics on algebraic manifolds J. Differential Geometry 32 (1990) 99-130

[30] Tian, G. On Kahler-Einstein metrics on certain Kahler manifolds with $c_1(M) > 0$ Inventiones Math. 89 (1987) 225-246.

[31] Tian, G. Kahler-Einstein metrics with positive scalar curvature Inventiones Math. 130 (1987) no. 1 1-37

[32] Tian, G. and Zhu, X.H. Uniqueness of Kahler-Ricci solitons Acta Math. 184 (2000) 271-305

[33] K. K. Uhlenbeck and S-T. Yau On the existence of hermitian Yang-Mills connections on stable bundles over compact Kahler manifolds Commun. Pure Applied Math. 39 (1986) 257-93

[34] Wang, X-J and Zhu, X.H. Kahler-Ricci solitons on toric manifolds with positive first Chern class Advances in Math. 188 (2004) 87-103
[35] Woodward, C.T. *Spherical varieties and the existence of invariant Kahler structures* Duke Math.Jour. 93 1998 (345-377)

[36] Zelditch, S. *Bernstein polynomials, Bergmann kernels and toric Kahler varieties* Arxiv DG/07052879

[37] B. Zhou and X. Zhu *Relative K-stability and modified K-energy on toric manifolds* arxiv:math.DG/06032337