Dynamics of a Diffusive Predator–Prey Model: The Effect of Conversion Rate

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Abstract A general diffusive predator–prey model is investigated in this paper. We prove the global attractivity of constant equilibria when the conversion rate is small, and the non-existence of non-constant positive steady states when the conversion rate is large. The results are applied to several predator–prey models and give some ranges of parameters where complex pattern formation cannot occur.

Keywords Reaction–diffusion · Non-existence · Steady state · Global attractivity

1 Introduction

During the past decades predator–prey interaction has been investigated extensively, and there are several reaction–diffusion equations modelling the predator–prey interaction, see [4–6,9,11,12,32] and references therein. The spatiotemporal patterns induced by diffusion, such as Turing pattern, can be used to explain the complex phenomenon in ecology. A prototypical one is the following diffusive predator–prey system with Holling type-II functional response

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from models (1.1) and (1.2) that the predator can survive without the specific prey. For Yi et al. [32] investigated the Hopf and steady state bifurcations near the unique positive equilibrium of system (1.1). Peng and Shi [21] proved that the global bifurcating branches of steady state solutions are bounded loops containing at least two bifurcation points, which improved the result in [32]. Ko and Ryu [11] investigated the dynamics of system (1.1) with a prey refuge. For the case of the homogeneous Dirichlet boundary conditions, Zhou and Mu [33] showed the existence of positive steady states through bifurcation theory and fixed point index theory. Recently, Wang et al. [28] studied a diffusive predator–prey model in the following general form

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + au \left(1 - \frac{u}{k}\right) - \frac{buv}{1 + mu}, & x \in \Omega, & t > 0, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v - \theta v + \frac{evu}{1 + mu}, & x \in \Omega, & t > 0, \\
\partial_\nu u = \partial_\nu v &= 0, & x \in \partial \Omega, & t > 0, \\
u(x, 0) = u_0(x) &\geq (\neq)0, & v(x, 0) = v_0(x) &\geq (\neq)0, & x \in \Omega.
\end{align*}
\] (1.1)

where predator functional response \(g(u)\) is increasing. They investigated the Hopf and steady state bifurcations near the unique positive equilibrium of system (1.2), and the existence and non-existence of non-constant positive steady states were also addressed with respect to diffusion coefficients \(d_1\) and \(d_2\). Similar results on the Hopf and steady state bifurcations near the positive equilibrium can be found in [10, 23, 25, 26]. Moreover, a non-monotonic functional response was proposed to model the prey’s group defense, see [8, 29]. That is, the predator functional response in model (1.2) is non-monotonic and can be chosen as follows:

\[(\text{Holling type-IV}) \quad g(u) = \frac{bu}{1 + nu + mu^2}, \quad \text{where} \quad b, n, m > 0. \] (1.3)

Related to the work on model (1.2) with non-monotonic functional response, see [16, 24, 30, 34] and references therein. Another prototypical predator–prey model has the following form

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &= u(a - u) - \frac{buv}{1 + nu + mu^2}, & x \in \Omega, & t > 0, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &= v(d - v) + \frac{evu}{1 + mu}, & x \in \Omega, & t > 0, \\
\partial_\nu u = \partial_\nu v &= 0, & x \in \partial \Omega, & t > 0, \\
u(x, 0) = u_0(x) &\geq (\neq)0, & v(x, 0) = v_0(x) &\geq (\neq)0, & x \in \Omega.
\end{align*}
\] (1.4)

Here the growth rate of the predator is logistic type in the absence of prey, and it is different from models (1.1) and (1.2) that the predator can survive without the specific prey. For \(m = 0\), Leung [12] proved the global attractivity of constant equilibria, which still holds when saturation \(m\) is small [1, 3, 6]. Du and Lou [6] investigated the existence and non-existence of non-constant steady states when saturation \(m\) is large, and see also [5] for the case of the homogeneous Dirichlet boundary conditions. Peng and Shi [21] proved the non-existence of non-constant positive steady states. Moreover, Yang et al. [31] considered a diffusive predator–prey model under the homogeneous Dirichlet boundary conditions, where
the growth rate of the predator is like a Beverton–Holt function, and see [2] for the case of the homogeneous Neumann boundary conditions.

Motivated by the above work of [6] and [28], we analyze a diffusive predator–prey model in the following general form

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + g(u) (f(u) - v), & x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v + v (h(v) + cg(u)), & x \in \Omega, \ t > 0, \\
\partial_\nu u = \partial_\nu v &= 0, & x \in \partial \Omega, \ t > 0, \\
u(x, 0) &= u_0(x) \geq (\neq)0, \ v(x, 0) = v_0(x) \geq (\neq)0, & x \in \Omega.
\end{align*}
\]

Here initial values \(u_0(x), v_0(x) \in C^{2,\alpha}(\Omega)\) for some \(\alpha \in (0, 1)\), \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) \((N \leq 3)\) with a smooth boundary \(\partial \Omega\), \(f(u)g(u)\) is the growth rate of prey without predator, \(g(u)\) is the predator functional response, \(c > 0\) is the conversion rate, and \(h(v)\) is the growth rate per capita of the predator in the absence of prey. For system (1.5), we see that \((u, v)\) is a constant positive equilibrium if and only if \(u \in (0, a)\) is a solution of the following equation

\[H(u) := h(f(u)) + cg(u) = 0.\]  

However, it is hard to determine the exact numbers and expressions of constant positive equilibria of system (1.5), not to mention the bifurcations near constant positive equilibria. In this paper, we first assume that \(f\) and \(g\) satisfy the following assumptions:

(A1) \(f \in C^1([0, \infty))\) and there exists a unique \(a > 0\) such that \(f(u)\) is positive for \(u \in [0, a)\) and negative for \(u > a\). Moreover, \(f'(u) < 0\) on \((0, a)\) or there exists \(\lambda \in (0, a)\) such that \(f'(u) > 0\) on \((0, \lambda)\) and \(f'(u) < 0\) on \((\lambda, a)\),

(A2) \(g \in C^1([0, \infty))\), \(g(0) = 0\), and \(g'(u) > 0\) for \(u \geq 0\),

and investigate the effect of small conversion rate on the positive steady states of system (1.5). There are several examples of \(f\) and \(g\) satisfying Assumptions (A1) and (A2). For example,

1. Richards growth rate for prey and Holling type-II functional response:

\[f(u) = \frac{\gamma (1 + mu)(a^p - u^p)}{b} \quad \text{and} \quad g(u) = \frac{bu}{1 + mu},\]

where \(a, b, m, \gamma > 0\) and \(p \geq 1\);

2. weak Allee effect in prey and Holling type-II functional response:

\[f(u) = \frac{\gamma (1 + mu)(a - u)(u + p)}{b} \quad \text{and} \quad g(u) = \frac{bu}{1 + mu},\]

where \(a, b, p, m, \gamma > 0\) and \(a > p\);

3. logistic growth rate for prey and Ivlev type functional response:

\[f(u) = \begin{cases} 
\gamma u(a - u) / \alpha (1 - e^{-\beta u}), & \text{for } u \neq 0, \\
\alpha \gamma / \alpha \beta, & \text{for } u = 0,
\end{cases} \quad \text{and} \quad g(u) = \alpha \left(1 - e^{-\beta u}\right),\]

where \(a, \alpha, \beta, \gamma > 0\).

Then, we consider the case that \(f\) and \(g\) satisfy the following assumptions:
(A'₀) \( f \in C^1(\mathbb{R}^+_{\geq 0}) \) and there exists a unique \( a > 0 \) such that \( f(u) \) is positive for \( u \in [0, a) \) and negative for \( u > a \),
(A'₁) \( g \in C^2(\mathbb{R}^+) \), \( g(0) = 0 \), \( g'(0) > 0 \) and \( g(u) > 0 \) for \( u > 0 \),
and investigate the effect of large conversion rate on the positive steady states of system (1.5). Here we remark that (A'₀) is more general that (A₁), \( g \) may be nonmonotonic for (A'₁), and the assumption that \( g \in C^2(\mathbb{R}^+) \) is needed to guarantee the regularity of the positive steady states. There are also several examples of \( f \) and \( g \) satisfying Assumptions (A'₀) and (A'₁).

For example,

(1) logistic growth rate for prey and Holling type-IV functional response:

\[
\begin{align*}
  f(u) &= \gamma(1 + nu + mu^2)(a - u) \quad \text{and} \quad g(u) = \frac{bu}{1 + nu + mu^2},
\end{align*}
\]

where \( a, b, m, n, \gamma > 0 \).

The rest of the paper is organized as follows. In Sect. 2, we show the global attractivity of constant equilibria of system (1.5) when the conversion rate is small, which also implies the non-existence of non-constant positive steady states. In Sect. 3, we prove the non-existence of non-constant positive steady states of system (1.5) when the conversion rate is large. In Sect. 3, we apply the obtained theoretical results to some concrete examples.

### 2 The Case of Small Conversion Rate

In this section, we investigate the positive steady states of system (1.5) when conversion rate \( c \) is small. Throughout this section, we assume that \( h \) satisfies the following assumption:

(A₃) \( h \in C^1(\mathbb{R}^+_{\geq 0}) \), and there exists a unique \( d > 0 \) such that \( h(v) \) is positive for \( v \in [0, d) \) and negative for \( v > d \). Moreover, \( h'(v) < 0 \) for \( v \geq d \).

Then \( h(v) \) has a inverse function, denoted by \( h^{-1} \), when \( v \in [d, \infty) \). We first recall the following well-known result for later application.

**Lemma 2.1** Assume that \( H : (0, \infty) \to \mathbb{R} \) is a smooth function satisfying \( H(w)(w - w₀) < 0 \) for any \( w > 0 \) and \( w \neq w₀ \). If \( w(x, t) \) satisfies the following problem

\[
\begin{align*}
  \frac{\partial w}{\partial t} &= d \Delta w + H(w), \quad x \in \Omega, \quad t > t₀, \\
  \frac{\partial w(x, t)}{\partial v} &= 0, \quad x \in \partial \Omega, \quad t > t₀, \\
  w(x, t₀) &= (\neq)0, \quad x \in \Omega,
\end{align*}
\]

where \( d > 0, t₀ \in \mathbb{R}^+ \), then \( w(x, t) \) exists for all \( t > t₀ \), and \( w(x, t) \to w₀ \) uniformly for \( x \in \overline{\Omega} \) as \( t \to \infty \).

From Lemma 2.1, we give the exact asymptotic bounds of the solutions for system (1.5).

**Lemma 2.2** Assume that \( f, g \) and \( h \) satisfy Assumptions (A₁), (A₂) and (A₃). If \( h(f(0)) < -cg(a) \), then there exist \( (\ddot{u}, \ddot{v}), (\overline{u}, \overline{v}) > (0, 0) \) satisfying

\[
\begin{align*}
  f(\ddot{u}) - \ddot{v} &\leq 0, \quad h(\ddot{v}) + cg(\ddot{u}) \leq 0, \\
  f(\overline{u}) - \overline{v} &\geq 0, \quad h(\overline{v}) + cg(\overline{u}) \geq 0.
\end{align*}
\]
Moreover, for any initial value \( \phi = (u_0(x), v_0(x)) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega}) \), where \( u_0(x) \geq (\neq)0, \ v_0(x) \geq (\neq)0 \) for all \( x \in \overline{\Omega} \), there exists \( t_0(\phi) > 0 \) such that the corresponding solution \((u(x,t), v(x,t))\) of system (1.5) satisfies
\[
(u, v) \leq (u(x,t), v(x,t)) \leq (\overline{u}, \overline{v}) \quad (2.2)
\]
for any \( t > t_0(\phi) \).

**Proof** Since \( h(f(0)) < -cg(a) \), it follows from assumption (A3) that there exists \( \epsilon > 0 \) such that \( h^{-1}(-cg(a + \epsilon)) \) exists and
\[
\epsilon < d, \quad h^{-1}(-cg(a + \epsilon)) > d, \quad f(0) - [h^{-1}(-cg(a + \epsilon)) + \epsilon] > 0, \quad (2.3)
\]
where \( h^{-1} \) is the inverse function of \( h(v) \) for \( v \in [d, \infty) \). It follows from
\[
\frac{\partial u}{\partial t} \leq d_1 \Delta u + g(u)f(u)
\]
and Lemma 2.1 that, for any initial value \( \phi \), there exists \( t_1(\phi) > 0 \) such that \( u(x,t) \leq a + \epsilon \) for \( t > t_1(\phi) \), where \( a > 0 \) is the unique zero of \( f(u) \). Since
\[
\frac{\partial v}{\partial t} \geq d_2 \Delta v + v h(v),
\]
there exists \( t_2(\phi) > t_1(\phi) \) such that \( v(x,t) \geq d - \epsilon > 0 \) for \( t > t_2(\phi) \). Consequently, we have
\[
\frac{\partial v}{\partial t} \leq d_2 \Delta v + v [h(v) + cg(a + \epsilon)] \quad \text{for} \quad t > t_2(\phi),
\]
and then there exists \( t_3(\phi) > t_2(\phi) \) such that \( v(x,t) \leq h^{-1}(-cg(a + \epsilon)) + \epsilon \) for \( t > t_3(\phi) \). Since \( f, g \) satisfy (A1), (A2) and Eq. (2.3), there exists \( \overline{a} \in (0, a) \), depending on \( c \), such that
\[
f(u) - [h^{-1}(-cg(a + \epsilon)) + \epsilon] > 0 \quad \text{for} \quad u \in [0, \overline{a}],
\]
and
\[
f(u) - [h^{-1}(-cg(a + \epsilon)) + \epsilon] < 0 \quad \text{for} \quad u \in (\overline{a}, a].
\]
It follows from Lemma 2.1 that there exists \( t_4(\phi) > t_3(\phi) \) such that \( u(x,t) \geq \frac{\overline{a}}{2} > 0 \) for \( t > t_4(\phi) \). Choose
\[
\overline{u} = a + \epsilon, \quad u = \frac{\overline{a}}{2}, \quad (2.4)
\]
\[
\overline{v} = d - \epsilon, \quad v = h^{-1}(-cg(a + \epsilon)) + \epsilon.
\]
Then \((u, v)\) and \((\overline{u}, \overline{v})\) satisfy Eq. (2.1), and there exists \( t_0(\phi) > t_4(\phi) \) such that the corresponding solution \((u(x,t), v(x,t))\) of system (1.5) satisfies Eq. (2.2) for any \( t > t_0(\phi) \).

Then, through the upper and lower solution method [17–20], we have the following results on the global attractivity of the positive equilibrium.

**Theorem 2.3** Assume that \( f, g \) and \( h \) satisfy Assumptions (A1), (A2) and (A3). If \( f(0) > d \), then there exists \( c_0 > 0 \), depending on \( f, g, h \), such that system (1.5) has a unique constant positive steady state \((u_*, v_*)\) for any \( c \in (0, c_0) \). Furthermore, for any initial value \( \phi = (u_0(x), v_0(x)) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega}) \), where \( u_0(x) \geq (\neq)0, \ v_0(x) \geq (\neq)0 \) for all \( x \in \overline{\Omega} \), the corresponding solution \((u(x,t), v(x,t))\) of system (1.5) converges uniformly to \((u_*, v_*)\) as \( t \to \infty \).
Proof Since \( f(0) > d \) (or equivalently, \( h(f(0)) < 0 \)), there exists \( c_1 > 0 \) such that \( h(f(0)) < -cg(a) \) for \( c \in (0, c_1] \). In the following, we always assume that \( c < c_1 \). From Lemma 2.2, we see that there exist \((u, v), (\bar{u}, \bar{v}) > (0, 0)\), which is a pair of coupled upper and lower solution of system (1.5). Since \( f \) and \( g \) are smooth, there exists \( K > 0 \) such that, for any \((u_1, v_1)\) and \((u_2, v_2)\) satisfying \((u, v) \leq (u_1, v_1), (u_2, v_2) \leq (\bar{u}, \bar{v})\), we have
\[
|g(u_1)(f(u_1) - v_1) - g(u_2)(f(u_2) - v_2)| \leq K(|u_1 - u_2| + |v_1 - v_2|),
\]
\[
|v_1h(v_1) + cg(u_1)v_1 - v_2h(v_2) - cg(u_2)v_2| \leq K(|u_1 - u_2| + |v_1 - v_2|).
\]

Define two iteration sequences \((\tilde{u}^{(m)}, \tilde{v}^{(m)})\) and \((\bar{u}^{(m)}, \bar{v}^{(m)})\) as follows: for \( m \geq 0 \),
\[
\tilde{u}^{(m+1)} = \tilde{u}^{(m)} + \frac{g(\tilde{u}^{(m)})}{K} \left[ f(\tilde{v}^{(m)}) - \tilde{v}^{(m)} \right],
\]
\[
\bar{u}^{(m+1)} = \bar{u}^{(m)} + \frac{g(\bar{u}^{(m)})}{K} \left[ f(\bar{v}^{(m)}) - \bar{v}^{(m)} \right],
\]
\[
\tilde{v}^{(m+1)} = \tilde{v}^{(m)} + \frac{h(\tilde{v}^{(m)}) + cg(\tilde{u}^{(m)})}{K} \left[ \frac{\bar{u}^{(m)}}{K} \right],
\]
\[
\bar{v}^{(m+1)} = \bar{v}^{(m)} + \frac{h(\bar{v}^{(m)}) + cg(\bar{u}^{(m)})}{K} \left[ \frac{\bar{u}^{(m)}}{K} \right],
\]
where \((\tilde{u}^{(0)}, \tilde{v}^{(0)}) = (\bar{u}, \bar{v})\) and \((\bar{u}^{(0)}, \bar{v}^{(0)}) = (u, v)\). Then there exist \((\tilde{u}, \tilde{v})\) and \((\bar{u}, \bar{v})\) such that \((u, v) \leq (\tilde{u}, \tilde{v}), (\tilde{u}, \tilde{v}) \leq (\bar{u}, \bar{v})\), \(\lim_{m \to \infty} \tilde{u}^{(m)} = \tilde{u}, \lim_{m \to \infty} \tilde{v}^{(m)} = \tilde{v}, \lim_{m \to \infty} \bar{u}^{(m)} = \bar{u}, \lim_{m \to \infty} \bar{v}^{(m)} = \bar{v}\), and
\[
0 = f(\tilde{u}) - \tilde{v}, 0 = h(\tilde{v}) + cg(\tilde{u}),
\]
\[
0 = f(\bar{u}) - \bar{v}, 0 = h(\bar{v}) + cg(\bar{u}).
\]
(2.5)

It follows from Eq. (2.5) that
\[
h(f(\tilde{u})) - cg(\tilde{u}) = h(f(\tilde{u})) - cg(\tilde{u}).
\]
(2.6)

The following proof is given in two cases.

Case I. If \( f'(u) < 0 \) on \((0, a)\), where \( a \) is the unique zero of \( f(u) \), then it follows from Eq. (2.5) and Assumptions \((A_1)-(A_3)\) that
\[
f(\tilde{u}) = \tilde{v} = h^{-1}(-cg(\tilde{u})) < h^{-1}(-c_1g(a)) < f(0) \quad \text{for any} \quad c \in (0, c_1).
\]
(2.7)

Then there exists \( \lambda \in (0, a) \) such that \( f(\lambda) = h^{-1}(-c_1g(a)) \) and \( \bar{u} \geq \lambda \) for any \( c \in (0, c_1) \). Hence
\[
0 = h(f(\tilde{u})) - cg(\tilde{u}) = h(f(\tilde{u})) + cg(\bar{u}) \geq \left[ \min_{u \in [\lambda, a]} [h(f(u))]' - c \max_{u \in [\lambda, a]} g'(u) \right](\bar{u} - \tilde{u})
\]
for any \( c \in (0, c_1) \). Since \( \min_{u \in [\lambda, a]} [h(f(u))]' > 0 \), there exists \( c_0 \in (0, c_1) \), depending on \( f, g, h \), such that \( \bar{u} = \tilde{u} \) for any \( c \in (0, c_0) \).

Case II. If there exists \( \lambda \in (0, a) \) such that \( f'(u) > 0 \) on \((0, \lambda)\) and \( f'(u) < 0 \) on \((\lambda, a)\), then there exist \( \lambda \in (\lambda, a) \) such that \( f(\lambda) = f(0) \). It follows from Eq. (2.7) that \( \bar{u} \geq \lambda \) for any \( c \in (0, c_1) \). Hence
\[
0 = h(f(\tilde{u})) - cg(\tilde{u}) = h(f(\tilde{u})) + cg(\bar{u}) \geq \left[ \min_{u \in [\lambda, a]} [h(f(u))]' - c \max_{u \in [\lambda, a]} g'(u) \right](\bar{u} - \tilde{u})
\]
for any \( c \in (0, c_1) \). Since \( \min_{u \in [\lambda, a]} [h(f(u))]' > 0 \), there exists \( c_0 \in (0, c_1) \), depending on \( f, g, h \), such that \( \bar{u} = \tilde{u} \) for \( 0 < c < c_0 \).
It follows from the upper and lower solution method \([17–20]\) that, for any \(c \in (0, c_0)\), system (1.5) has a unique constant positive equilibrium \((u_*, v_*)\), which is globally attractive.

From Lemma 2.1 and Theorem 2.3, we have the following results on the global attractivity of constant equilibria.

**Corollary 2.4** Assume that \(f\), \(g\) and \(h\) satisfy Assumptions \((A_1)\), \((A_2)\) and \((A_3)\). Then the following two statements are true.

1. If \(f(0) > d\) (or equivalently, \(h(\max_{u \in [0,a]} f(u)) < 0\)), then there exists \(c_0 > 0\), depending on \(f\), \(g\) and \(h\), such that, for any \(c \in (0, c_0)\), system (1.5) has a unique constant positive steady state, which is globally attractive. Hence system (1.5) has no non-constant positive steady states for \(c \in (0, c_0)\) if \(f(0) > d\).

2. If \(\max_{u \in [0,a]} f(u) < d\) (or equivalently \(h(\max_{u \in [0,a]} f(u)) > 0\)), then steady state \((0, d)\) of system (1.5) is globally attractive. Hence system (1.5) has no positive steady states if \(\max_{u \in [0,a]} f(u) < d\).

**Proof** If \(f(0) > d\), then it follows from Theorem 2.3 that there exists \(c_0 > 0\) such that, for any \(c \in (0, c_0)\), system (1.5) has a unique constant positive steady state, which is globally attractive. Hence system (1.5) has no non-constant positive steady states for \(c \in (0, c_0)\).

Then, we consider the case that \(\max_{u \in [0,a]} f(u) < d\). Since

\[
\frac{\partial v}{\partial t} \geq d_2 \Delta v + vh(v),
\]

it follows from Lemma 2.1 that for any initial value and

\[
\epsilon \in \left(0, \frac{1}{2} \left(d - \max_{u \in [0,a]} f(u)\right)\right),
\]

there exists \(t_2 > 0\) such that \(v(x, t) \geq d - \epsilon > 0\) for \(t > t_2\), and consequently,

\[
\frac{\partial u}{\partial t} \leq d_1 \Delta u + g(u)(f(u) - v) \leq d_1 \Delta u + g(u) \left(\max_{u \in [0,a]} f(u) - d + \epsilon\right)
\]

for \(t > t_2\), which implies that \(u(x, t)\) converges uniformly to 0 as \(t \to \infty\). Therefore, the steady state \((0, d)\) of system (1.5) is globally attractive.

We remark that the above results do not mention the case that

\[
h\left(\max_{u \in [0,a]} f(u)\right) < 0 < h(f(0)),
\]

that is,

\[
\max_{u \in [0,a]} f(u) > d > f(0).
\]

For this case, the dynamics is complex even when \(c = 0\). The steady states of Eq. (1.5) for \(c = 0\) satisfy

\[
\begin{cases}
-d_1 \Delta u = g(u)(f(u) - v), & x \in \Omega, \\
-d_2 \Delta v = vh(v), & x \in \Omega, \\
\partial_n u = \partial_n v = 0, & x \in \partial \Omega.
\end{cases}
\]
Clearly, if \((u(x), v(x))\) is positive and satisfies Eq. (2.8), then \(v(x) \equiv d\) and \(u(x)\) satisfies
\[
\begin{align*}
-d_1 \Delta u &= g(u)(f(u) - d), \quad x \in \Omega, \\
\partial_v u &= 0, \quad x \in \partial\Omega.
\end{align*}
\] (2.9)

Equation (2.9) exhibits a strong Allee effect if \(\max_{u \in [0, a]} f(u) > d > f(0)\), which has a stable constant positive equilibrium and an unstable constant positive equilibrium, see [7,26,27] for related work on strong Allee effect.

### 3 The Case of Large Conversion Rate

In this section, we prove the non-existence of non-constant positive steady states of system (1.5) when conversion rate \(c\) is large, and the method used here is motivated by [21]. Throughout this section, we assume that \(f\) and \(g\) satisfy (A′1) and (A′2). Define
\[
q(u) = \begin{cases} g(u)/u, & \text{if } u > 0, \\ g'(0), & \text{if } u = 0, \end{cases}
\]
and then \(q(u) \in C^1(\mathbb{R}^+)\). Let \(w = cu, \rho = 1/c\), and then \(w\) and \(v\) satisfy
\[
\begin{align*}
-d_1 \Delta w &= wq(\rho w)(f(\rho w) - v), \quad x \in \Omega, \\
-d_2 \Delta v &= v[h(v) + q(\rho w)w], \quad x \in \Omega, \\
\partial_v u &= \partial_v v = 0, \quad x \in \partial\Omega.
\end{align*}
\] (3.2)

Hence the existence/non-existence of positive steady states of system (1.5) for large \(c\) is equivalent to that of solutions of system (3.2) for small \(\rho\). In the following, we first cite the maximum principle for weak solutions from [13,15,22] and the Harnack inequality for weak solutions from [14,21,22] for later application.

**Lemma 3.1** Assume that \(\Omega\) is a bounded Lipschitz domain in \(\mathbb{R}^N\), and \(g \in C(\overline{\Omega} \times \mathbb{R})\). If \(z \in H^1(\Omega)\) is a weak solution of the inequalities
\[
\begin{align*}
\Delta z + g(x, z) &\geq 0, \quad x \in \Omega, \\
\partial_v z &\leq 0, \quad x \in \partial\Omega,
\end{align*}
\]
and there exists a constant \(K\) such that \(g(x, z) < 0\) for \(z > K\), then \(z \leq K\) a.e. in \(\Omega\).

**Lemma 3.2** Assume that \(\Omega\) is a bounded Lipschitz domain in \(\mathbb{R}^N\), and \(c(x) \in L^q(\Omega)\) for some \(q > N/2\). If \(u \in H^1(\Omega)\) is a non-negative weak solution of the following problem
\[
\begin{align*}
\Delta u + c(x)u &= 0, \quad x \in \Omega, \\
\partial_v u &= 0, \quad x \in \partial\Omega,
\end{align*}
\]
then there is a positive constant \(C\), which is determined only by \(\|c(x)\|_q\), \(q\) and \(\Omega\), such that
\[
\sup_{x \in \Omega} u \leq C \inf_{x \in \Omega} u.
\]

Then, we consider the positive solutions of system (3.2) when \(\rho = 0\).
Lemma 3.3 Assume that \( f \) and \( g \) satisfy Assumptions (A\( _1' \)) and (A\( _2' \)), \( h \) satisfies
\[
(A_4) \quad [h(v) - h(f(0))] (v - f(0)) < 0 \text{ for any } v > 0 (v \neq f(0)),
\]
and \( h(f(0)) < 0 \). Then, for \( \rho = 0 \), system (3.2) has a unique positive steady state \(( -\frac{h(f(0))}{g(0)}, f(0)) \). \hfill \( \Box \)

Proof Since \( h(f(0)) < 0 \), system (3.2) has a constant positive steady state \((w_\ast, v_\ast) = (-\frac{h(f(0))}{g(0)}, f(0)) \) for \( \rho = 0 \). We construct the following function
\[
G(w, v) := \int_\Omega \left\{ \frac{w - w_\ast}{w} \left[ d_1\Delta w + wg'(0)(f(0) - v) \right] \right\} dx + \int_\Omega \left\{ \frac{v - v_\ast}{v} \left[ d_2\Delta v + v \left( h(v) + g'(0)w \right) \right] \right\} dx = -\int_\Omega \left[ d_1\frac{w_\ast|\nabla w|^2}{w^2} + d_2\frac{v_\ast|\nabla v|^2}{v^2} \right] dx + \int_\Omega (v - v_\ast)[h(v) - h(v_\ast)] dx.
\]
Therefore, if \((w(x), v(x))\) is a positive solution of system (3.2) for \( \rho = 0 \), then
\[
G(w(x), v(x)) = 0.
\]
Since \( h \) satisfies (A\( _4 \)), it follows that \((w(x), v(x)) \equiv (w_\ast, v_\ast) \). \hfill \( \Box \)

Based on Lemmas 3.1 and 3.2, we will give a priori estimates for positive solutions of system (3.2) under the following assumption (A\( _5 \)).

(A\( _5 \)) \( h \in C^1(\overline{\mathbb{R}^+}) \) and there exist \( n \in \mathbb{N}^+ \), \( \{q_i\}_{i=0}^n \), \( \{k_i\}_{i=0}^n \) and \( \{\bar{k}_i\}_{i=0}^n \) such that
\[
\sum_{i=0}^n k_i v^{q_i} \leq -h(v) \leq \sum_{i=0}^n \bar{k}_i v^{q_i} \text{ for any } v \geq 0,
\]
where \( 0 = q_0 < q_1 < q_2 < \cdots < q_n, q_n > \frac{1}{2} \text{ and } k_n, \bar{k}_n > 0 \).

The above mentioned Assumptions (A\( _4 \)) and (A\( _5 \)) are not strong, and we will remark that many common used growth rate per capita functions satisfy (A\( _4 \)) and (A\( _5 \)) at the end of this section.

Theorem 3.4 Assume that \( f \), \( g \) and \( h \) satisfy Assumptions (A\( _1' \)), (A\( _2' \)) and (A\( _5 \)), \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N \leq 3) \) with a smooth boundary \( \partial \Omega \), and \((w_\rho, v_\rho)\) is a positive solution of system (3.2). Denote
\[
v_0 = \begin{cases} 
\max\{v \geq 0 : h(v) = 0\}, & \text{if } \{v \geq 0 : h(v) = 0\} \neq \emptyset, \\
0, & \text{if } \{v \geq 0 : h(v) = 0\} = \emptyset.
\end{cases} \tag{3.3}
\]
Then the following two statements are true.

1. There exists \( \overline{C} > 0 \) such that \( \sup_{x \in \Omega} w_\rho, \sup_{x \in \Omega} v_\rho < \overline{C} \) for all \( \rho > 0 \).
2. If \( f(0) > v_0 \), then there exists \( M > 0 \) such that
\[
\inf_{0 \leq \rho \leq M} w_\rho > 0 \text{ and } \inf_{0 \leq \rho \leq M} v_\rho > 0.
\]
Proof Since \( h \) satisfies assumption \((A_5)\), we see that \( \lim_{v \to -\infty} h(v) = -\infty \), which implies that \( v_0 \) is well defined. From Eq. (3.2), we obtain

\[
- \int_\Omega v_\rho h(v_\rho) \, dx = \int_\Omega w_\rho q(\rho w_\rho) f(\rho w_\rho) \, dx,
\]

\[
\int_\Omega [h(v_\rho) + w_\rho q(\rho w_\rho)] \, dx = -d_2 \int_\Omega \frac{|\nabla v_\rho|^2}{v_\rho^2} \, dx \leq 0.
\]

Since \( \rho w_\rho \) satisfies

\[
-d_1 \rho \Delta w_\rho \leq \rho w_\rho f(\rho w_\rho) q(\rho w_\rho),
\]

it follows from Lemma 3.1 (see also \([13,15,22]\)) that \( \rho w_\rho \leq a \), where \( a > 0 \) is the unique zero of \( f(u) \). Noticing that

\[
\max_{u \in [0,a]} f(u), \max_{u \in [0,a]} q(u), \min_{u \in [0,a]} q(u) > 0,
\]

from Eq. (3.4), we have

\[
- \int_\Omega v_\rho h(v_\rho) \, dx \leq - \max_{u \in [0,a]} f(u) \max_{u \in [0,a]} q(u) \int_\Omega h(v_\rho) \, dx.
\]

This relation and assumption \((A_5)\) imply

\[
- \int_\Omega v_\rho h(v_\rho) \, dx \leq \sum_{i=0}^n \sigma_i \| v_\rho \|_{q_{n+1}}^{q_i},
\]

\[
k_n \| v_\rho \|_{q_{n+1}}^{q_{n+1}} \leq \sum_{i=0}^{n-1} \sigma_i \| v_\rho \|_{q_{n+1}}^{q_i} + \sum_{i=0}^n \sigma_i \| v_\rho \|_{q_{n+1}}^{q_i},
\]

where \( \sigma_i \) and \( \sigma_i \) are positive and depend only on \( f, g, q_i, \) and \( k_i \) and \( \Omega \). Therefore, there exists a constant \( C_1 > 0 \) such that \( \| v_\rho \|_{q_{n+1}} \leq C_1 \) for all \( \rho \geq 0 \), which implies

\[
\|q(\rho w) [f(\rho w) - v] \|_{q_{n+1}} \leq \max_{u \in [0,a]} q(u) \left( \max_{u \in [0,a]} f(u) \right) \left( \frac{1}{q_{n+1}} + C_1 \right).
\]

Since \( N \leq 3 \) and \( q_n + 1 > \frac{3}{2} \geq \frac{N}{2} \), from Lemma 3.2 we see that there exists \( C_2 > 0 \) such that

\[
\sup_{x \in \Omega} w_\rho \leq C_2 \inf_{x \in \Omega} w_\rho \text{ for all } \rho \geq 0.
\]

It follows from Eqs. (3.4) and (3.6) that

\[
\inf_{x \in \Omega} w_\rho < \frac{\sum_{i=0}^n \sigma_i C_1^{q_i}}{\min_{u \in [0,a]} q(u) \|\Omega\|} \text{ for all } \rho \geq 0.
\]

Therefore, from Eqs. (3.7) and (3.8), we see that there exists a constant \( C_3 > 0 \) such that

\[
\sup_{x \in \Omega} w_\rho \leq C_3 \text{ for all } \rho \geq 0.
\]

Consequently,

\[
- d_2 \Delta v_\rho = v_\rho [h(v_\rho) + q(\rho w_\rho) w_\rho] \leq v_\rho \left[ h(v_\rho) + \max_{u \in [0,a]} q(u) C_3 \right].
\]
It follows from assumption (A₅) that \( \lim_{v \to -\infty} h(v) = -\infty \), which implies that there exists \( C₄ > 0 \) such that
\[
h(v) + \max_{u \in [0,a]} q(u)C₃ < 0 \quad \text{for} \quad v > C₄.  \tag{3.11}
\]
Therefore, from Eqs. (3.10) and (3.11) and Lemma 3.1, we obtain
\[
\sup_{x \in \Omega} v_\rho \leq C₄ \quad \text{for all} \quad \rho \geq 0.  \tag{3.12}
\]
Letting \( \overline{C} = \max\{C₃, C₄\} \), we have
\[
\sup_{x \in \Omega} w_\rho, \sup_{x \in \Omega} v_\rho < \overline{C} \quad \text{for all} \quad \rho \geq 0.  \tag{3.13}
\]
In the following, we find the lower bound of \( w_\rho \) and \( v_\rho \). We first claim that there exists \( M₁ > 0 \) such that
\[
\inf_{0 \leq \rho \leq M₁} \sup_{x \in \Omega} w_\rho > 0.  \tag{3.14}
\]
By way of contradiction, there exists a sequence \( \{\rho_j\}_{j=1}^{\infty} \) such that \( \lim_{j \to \infty} \rho_j = 0 \) and \( \lim_{j \to \infty} \inf_{x \in \Omega} w_{\rho_j} = 0 \), which implies that \( \lim_{j \to \infty} \sup_{x \in \Omega} w_{\rho_j} = 0 \) from Eq. (3.7). Then we only need to consider two cases.

**Case I.** \( \{v \geq 0 : h(v) = 0\} = \emptyset \). Then \( v_0 = 0 < f(0) \). Noticing that \( \lim_{v \to -\infty} h(v) = -\infty \) from assumption (A₅), we obtain \( h(v) < 0 \) for \( v \geq 0 \) and \( \max_{v \geq 0} h(v) < 0 \). Since \( \lim_{j \to \infty} \sup_{x \in \Omega} w_{\rho_j} = 0 \), we have
\[
\int_{\Omega} v_{\rho_j} \left[-h(v_{\rho_j}) - q(\rho_j w_{\rho_j})w_{\rho_j}\right] dx \geq \int_{\Omega} v_{\rho_j} \left[-\max_{v \geq 0} h(v) - q(\rho_j w_{\rho_j})w_{\rho_j}\right] dx > 0
\]
for sufficiently large \( j \), which contradicts with the fact that
\[
\int_{\Omega} v_{\rho_j} \left[-h(v_{\rho_j}) - q(\rho_j w_{\rho_j})w_{\rho_j}\right] dx = 0.
\]

**Case II.** \( \{v \geq 0 : h(v) = 0\} \neq \emptyset \) and \( v_0 < f(0) \). Since \( \lim_{j \to \infty} \sup_{x \in \Omega} w_{\rho_j} = 0 \), we see that, for any \( \epsilon > 0 \), there exists \( j₀(\epsilon) > 0 \) such that \( \sup_{x \in \Omega} |w_{\rho_j}| \max_{u \in [0,a]} q(u) < \epsilon \) for any \( j > j₀(\epsilon) \), which implies that
\[
-d\Delta v_{\rho_j} \leq v_{\rho_j} \left[h\left(v_{\rho_j}\right) + \epsilon\right] \quad \text{for} \quad j > j₀(\epsilon).
\]
This relation and Lemma 3.1 lead to
\[
v_{\rho_j} \leq v_\epsilon := \max\{v \geq 0 : h(v) + \epsilon = 0\} \quad \text{for} \quad j > j₀(\epsilon).
\]
It follows from \( \lim_{\epsilon \to 0} v_\epsilon = v_0 < f(0) \) that \( v_\epsilon < f(0) \) for sufficiently small \( \epsilon \), and without loss of generality, we assume \( v_\epsilon < f(0) \). Since
\[
-d₁\rho_j \Delta w_{\rho_j} \geq \rho_j w_{\rho_j} q(\rho_j w_{\rho_j}) \left[f\left(\rho_j w_{\rho_j}\right) - v_{\rho_j}\right] \geq \rho_j w_{\rho_j} q(\rho_j w_{\rho_j}) \left[f\left(\rho_j w_{\rho_j}\right) - v_{\rho_j}\right] \tag{3.15}
\]
for \( j > j₀(\epsilon) \), it follows from Lemma 3.1 and \( v_\epsilon < f(0) \) that there exists \( w_0 > 0 \) such that \( \rho_j w_{\rho_j} \geq w_0 > 0 \) for \( j > j₀(\epsilon) \), which contradicts with the fact that \( \lim_{j \to \infty} \rho_j = 0 \) and \( \lim_{j \to \infty} \sup_{x \in \Omega} w_{\rho_j} = 0 \).

Therefore, the claim is proved and Eq. (3.14) hold. Then, we prove that there exists \( M₂ > 0 \) such that
\[
\inf_{0 \leq \rho \leq M₂} \sup_{x \in \Omega} v_\rho > 0.  \tag{3.16}
\]
Assuming the contrary, we see that there exists a sequence \( \{ \rho_i \} \) such that \( \lim_{i \to \infty} \rho_i = 0 \) and \( \lim_{i \to \infty} \inf_{x \in \Omega} v_{\rho_i} = 0 \). Since

\[
\| h(v_{\rho}) + q(\rho w_{\rho})w_{\rho} \|_\infty \leq \max_{v \in [0, \overline{C}]} |h(v)| + \max_{u \in [0, \overline{C}]} q(u) \overline{C},
\]

it follows from Lemma 3.2 that \( \lim_{i \to \infty} \sup_{x \in \Omega} v_{\rho_i} = 0 \), and consequently,

\[
\int_{x \in \Omega} w_{\rho_i}q(\rho_i w_{\rho_i}) \left[ f(\rho_i w_{\rho_i}) - v_{\rho_i} \right] dx > 0
\]

for sufficiently large \( i \), which is a contradiction. Letting \( M = \min \{ M_1, M_2 \} \), we have

\[
\inf_{0 \leq \rho \leq M} \inf_{x \in \Omega} w_{\rho} > 0 \quad \text{and} \quad \inf_{0 \leq \rho \leq M} \inf_{x \in \Omega} v_{\rho} > 0.
\]

This completes the proof. \( \square \)

From Theorem 3.4, we obtain the non-existence of non-constant positive solutions of system (3.2) for small \( \rho \).

**Theorem 3.5**  Assume that \( f, g \) and \( h \) satisfy Assumptions (A1'), (A2') and (A3), and \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N \leq 3) \) with a smooth boundary \( \partial \Omega \). Then the following two statements are true.

1. If \( f(0) > v_0 \) and \( h \) satisfies (A4), where \( v_0 \) is defined as in Eq. (3.3), then there exists \( c_0 > 0 \), depending on \( f, g, h, d_1, d_2 \) and \( \Omega \), such that system (1.5) has a unique constant positive steady state and no non-constant positive steady states for any \( c > c_0 \).
2. If \( f(0) < v_0 \) and \( h(v) > 0 \) for \( v < v_0 \), then there exists \( c_0 > 0 \), depending on \( f, g, h, d_1, d_2 \) and \( \Omega \), such that system (1.5) has no positive steady states for any \( c > c_0 \).

**Proof**  Since \( f(0) > v_0 \), it follows from Eq. (3.3) that \( h(f(0)) < 0 \). This relation and Lemma 3.3 imply that system (3.2) has a unique positive solution \( (w_*, v_*) = \left( -\frac{h(f(0))}{g(0)}, f(0) \right) \) for \( \rho = 0 \). Since \( h \) satisfies (A4), we have \( h'(f(0)) \leq 0 \), which implies that \( (w_*, v_*) \) is non-degenerate in the sense that zero is not the eigenvalue of the linearized problem with respect to \( (w_*, v_*) \). Then it follows from the implicit function theorem that there exists \( \rho_0 > 0 \) such that system (3.2) has a constant positive solution \( (w_{\rho}, v_{\rho}) \) for \( 0 < \rho < \rho_0 \). Therefore, the existence is proved, and the uniqueness is proved in the following. By the implicit function theorem, we only need to show that if \( (w^\rho, v^\rho) \) is a positive solution of system (3.2), then

\[
(w^\rho, v^\rho) \to (w_*, v_*) \quad \text{in} \quad C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \quad \text{as} \quad \rho \to 0.
\]

It follows from Theorem 3.4 that

\[
C \leq w^\rho, v^\rho \leq \overline{C} \quad \text{for} \quad x \in \overline{\Omega} \quad \text{and} \quad \rho \in [0, M],
\]

where \( M \) is defined as in Theorem 3.4 and

\[
\overline{C} = \min \{ \inf_{0 \leq \rho \leq M} \inf_{x \in \Omega} w^\rho, \inf_{0 \leq \rho \leq M} \inf_{x \in \Omega} v^\rho \} > 0.
\]

This, combined with the \( L^p \) theory, implies that \( w^\rho \) and \( v^\rho \) are bounded in \( W^{2,p}(\Omega) \) for any \( p > N \). It follows from the embedding theorem that \( w^\rho \) and \( v^\rho \) are precompact in \( C^1(\overline{\Omega}) \). This implies that, for any sequence \( \{ \rho_i \} \) satisfying \( \lim_{i \to \infty} \rho_i = 0 \), there exist a subsequence \( \{ \rho_{i_k} \} \) and \( (w^*(x), v^*(x)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \) such that

\[
(w^{\rho_{i_k}}, v^{\rho_{i_k}}) \to (w^*(x), v^*(x)) \quad \text{in} \quad C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \quad \text{as} \quad k \to \infty.
\]
where \( w^*(x) \) and \( v^*(x) \) is positive from Eq. (3.17). Note that

\[
\begin{align*}
  w^{\rho_k} & = -d_1 \Delta + I \rightleftharpoons \frac{1}{\rho_k} \left\{ w^{\rho_k} + w^{\rho_k} q(\rho_k w^{\rho_k}) \left[ f(\rho_k w^{\rho_k}) - v^{\rho_k} \right] \right\}, \\
v^{\rho_k} & = -d_2 \Delta + I \rightleftharpoons \frac{1}{\rho_k} \left\{ v^{\rho_k} + v^{\rho_k} \left[ h(v^{\rho_k}) + q(\rho_k w^{\rho_k}) w^{\rho_k} \right] \right\},
\end{align*}
\]

(3.18)

and \( \lim_{k \to \infty} \rho_k w^{\rho_k} = 0 \) in \( C^1(\overline{\Omega}) \). Then, taking the limit of Eq. (3.18) as \( k \to \infty \) and by the Schauder theorem, we see that \( (w^*(x), v^*(x)) \) is a positive solution of system (3.2) for \( \rho = 0 \), and

\[
(w^{\rho_k}, v^{\rho_k}) \to (w^*(x), v^*(x)) \text{ in } C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \text{ as } k \to \infty.
\]

This completes the proof of part (1).

Then, we prove the part (2). It follows from \( f(0) < v_0 \) and Eq. (3.3) that \( v_0 > 0 \) and \( v_0 = \max \{ v > 0 : h(v) = 0 \} \). Then

\[
h(v) > 0 \text{ for } v < v_0, \quad \text{and } h(v) < 0 \text{ for } v > v_0.
\]

(3.19)

Assuming the contrary, we see that there exists a sequence \( \{ \rho_j \}_{j=1}^{\infty} \) such that \( \lim_{j \to \infty} \rho_j = 0 \) and system (3.2) has a positive solution \( (w_{\rho_j}, v_{\rho_j}) \) for \( \rho = \rho_j \). Since

\[
-d_2 \Delta v_{\rho_j} = v_{\rho_j} \left[ h(v_{\rho_j}) + q(\rho_j w_{\rho_j}) w_{\rho_j} \right] \geq v_{\rho_j} h(v_{\rho_j}),
\]

it follows from (3.19) that \( v_{\rho_j} \geq v_0 \), which implies that

\[
-d_1 \Delta w_{\rho_j} \leq w_{\rho_j} q(\rho_j w_{\rho_j}) \left[ f(\rho_j w_{\rho_j}) - v_0 \right].
\]

From Eq. (3.13), we see that \( \lim_{j \to \infty} \rho_j w_{\rho_j} = 0 \) uniformly on \( \overline{\Omega} \). This, together with the fact that \( f(0) < v_0 \), implies that there exists \( j_0 > 0 \) such that \( f(\rho_j w_{\rho_j}) - v_0 < 0 \) for any \( j > j_0 \). Hence \( w_{\rho_j} \leq 0 \) for \( j > j_0 \). This contradicts with the fact that \( w_{\rho_j} \) is positive. \( \square \)

Remark 1 We remark that Assumptions (A4) and (A5) are not strong. Some examples of \( h \) satisfying (A5) are

(1) logistic:

\[
h(v) = \beta(d - v), \quad \beta, d > 0,
\]

(3.20)

(2) weak Allee effect:

\[
h(v) = \beta(d - v)(v + p), \quad d > p > 0, \beta > 0,
\]

(3.21)

(3) strong Allee effect:

\[
h(v) = \beta(d - v)(v - p), \quad d, p, \beta > 0,
\]

(3.22)

(4) strong Allee effect:

\[
h(v) = \beta \frac{(d - v)(v - p)}{v + r}, \quad d, p, \beta, r > 0.
\]

(3.23)

Moreover, Eq. (3.20) always satisfies (A4), Eq. (3.21) satisfies (A4) for \( f(0) > d - p \), Eq. (3.22) satisfies (A4) for \( f(0) > d + p \), and Eq. (3.23) satisfies (A4) for

\[
f(0) > \frac{dp + dr + pr}{r}.
\]
4 Applications

In this section, we apply the previously obtained results to some concrete predator–prey models.

4.1 A Predator–Prey Model with Holling-II Functional Response

In this subsection, we consider model (1.4), which is a predator–prey model with logistic growth rate for predator and Holling type-II functional response. Here $\Omega_1$ is a bounded domain in $\mathbb{R}^N (N \leq 3)$ with a smooth boundary $\partial \Omega_1$, parameters $a, b, e, m, d_1$ and $d_2$ are all positive constants, and $d$ may be positive constant, negative constant or zero. Letting

$$f(u) = \frac{(1 + mu)(a - u)}{b}, \quad g(u) = \frac{bu}{1 + mu}, \quad h(v) = d - v, \quad c = \frac{e}{b},$$

system (1.5) is transformed to system (1.4), and parameter $c$ in system (1.5) is equivalent to $e$ in system (1.4). In this case, $f, g$ and $h$ satisfy Assumptions (A1), (A2), (A′1), (A′2), (A4) and (A5), and

$$v_0 = \begin{cases} d, & d \geq 0, \\ 0, & d < 0, \end{cases}$$

where $v_0$ is defined as in Eq. (3.3). Moreover, $h$ satisfies (A3) if $d > 0$. Then, from Theorem 3.5, we have the following results on the non-existence of non-constant steady states when conversion rate $e$ is large. This results supplements the result in [21], which consider the case that $e$ equals to $b$ and is sufficiently large.

**Proposition 4.1** (a1) If $a > bd$, then there exists $e_0 > 0$, depending on $a, b, d, m, d_1, d_2$ and $\Omega_1$, such that system (1.4) has a unique constant positive steady state and no non-constant positive steady states for any $e > e_0$.

(a2) If $a < bd$, then there exists $e_0 > 0$, depending on $a, b, d, m, d_1, d_2$ and $\Omega_1$, such that system (1.4) has no positive steady states for any $e > e_0$.

From Corollary 2.4, we have the following results on the global attractivity of constant equilibria for small conversion rate, which also imply the non-existence of non-constant steady states.

**Proposition 4.2** (b1) If $d < 0$, then there exists $e_0 > 0$, depending on $a, b, m$ and $d$, such that the steady state $(a, 0)$ of system (1.4) is globally attractive for $e \in (0, e_0)$.

(b2) If $0 < d < \frac{a}{b}$, then there exists $e_0 > 0$, depending on $a, b, m$ and $d$, such that, for $e \in (0, e_0)$, system (1.4) has a unique constant positive steady state, which is globally attractive.

(b3) If $d > \frac{(am+1)^2}{4mb}$, then the steady state $(0, d)$ of system (1.4) is globally attractive.

**Proposition 4.3** If $d > 0$, then there exists $a_0 > 0$, depending on $b, d, e$ and $m$, such that, for any $a > a_0$, system (1.4) has a unique constant positive steady state, which is globally attractive.

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Proof Letting \( a_1 := b \left( d + \frac{e}{m} \right) \), we have

\[
h(f(0)) = d - \frac{a}{b} < -cg(a) = -\frac{ea}{1+ma}
\]

for any \( a > a_1 \). This, together with \( h(0) = d > 0 \), imply that the assumption of Lemma 2.2 hold. Then there exist \((u, v), (\tilde{u}, \tilde{v}) > (0, 0)\), which is a pair of coupled upper and lower solution of system (1.4) for \( a > a_1 \). Using the similar arguments as Theorem 2.3, there exist \((\tilde{u}, \tilde{v})\) and \((\tilde{\tilde{u}}, \tilde{\tilde{v}})\) such that \((u, v) \leq (\tilde{u}, \tilde{v}) \leq (\tilde{\tilde{u}}, \tilde{\tilde{v}}), \lim_{m \to \infty} \tilde{u}^{(m)} = \tilde{u}, \lim_{m \to \infty} \tilde{v}^{(m)} = \tilde{v}, \lim_{m \to \infty} u^{(m)} = \tilde{u}, \lim_{m \to \infty} v^{(m)} = \tilde{v}\), and Eqs. (2.5) and (2.6) hold. If \( a > a_2 := \max\{b(d + \frac{e}{m}), 1\} \), then

\[
f(\tilde{u}) = \tilde{v} < \left( d + \frac{e}{m} \right) < \frac{a}{b} = f(0),
\]

which implies that \( \tilde{u} > \tilde{\tilde{u}} > \tilde{\lambda} = \frac{am - 1}{m} > 0 \), where \( f(\tilde{\lambda}) = f(0) \). It follows from Eq. (2.6) that

\[
0 = h(f(\tilde{u})) - cg(\tilde{u}) - h(f(\tilde{\tilde{u}})) + cg(\tilde{\tilde{u}})
= \frac{(1+m\tilde{u})(a - \tilde{u})}{b} - \frac{(1+m\tilde{\tilde{u}})(a - \tilde{\tilde{u}})}{b} + \frac{e\tilde{u}}{1+m\tilde{u}} - \frac{e\tilde{\tilde{u}}}{1+m\tilde{\tilde{u}}}
\geq \left[ \frac{1}{b} \left( 2m\tilde{\lambda} - am + 1 \right) - e \right] (\tilde{u} - \tilde{\tilde{u}})
\geq \left[ \frac{1}{b} (ma - 1) - e \right] (\tilde{u} - \tilde{\tilde{u}}),
\]

which implies that \( \tilde{u} = \tilde{\tilde{u}} \) for \( a > a_3 := \frac{be + 1}{m} \). Therefore, for any \( a > a_0 = \max\{a_1, a_2, a_3\} \), system (1.4) has a unique constant positive steady state, which is globally attractive.

Similarly, we have the global attractivity of the positive equilibrium with respect to saturation \( m \).

Proposition 4.4 If \( 0 < d < \frac{a}{b} \), then there exists \( m_0 > 0 \), depending on \( a, b, d \) and \( e \), such that, for any \( m > m_0 \), system (1.4) has a unique constant positive steady state, which is globally attractive.

Propositions 4.3 and 4.4 supplement Theorem 2.3 in [6], which prove the global attractivity of the positive equilibrium for \( ma \leq 1 \).
4.2 A Predator–Prey Model with Weak Allee Effect in Predator and Holling-II Functional Response

In this subsection, we consider the following predator–prey model

\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &= u(a - u) - \frac{buv}{1 + mu}, & x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &= \beta v(d - v)(v + p) + \frac{euv}{1 + mu}, & x \in \Omega, \ t > 0, \\
\partial_n u &= \partial_n v = 0, & x \in \partial \Omega, \ t > 0, \\
u(x, 0) &= u_0(x) \geq (\neq)0, \ v(x, 0) = v_0(x) \geq (\neq)0, \ x \in \Omega.
\end{align*}

where \(a, b, d, p, e, m, \beta, d_1 \) and \(d_2 \) are positive constants, \(d > p\), and \(\Omega\) is a bounded domain in \(\mathbb{R}^N (N \leq 3)\) with a smooth boundary \(\partial \Omega\). Letting

\begin{equation}
\begin{aligned}
f(u) &= \frac{(1 + mu)(a - u)}{b}, & g(u) = \frac{bu}{1 + mu}, & h(v) = \beta(d - v)(v + p), & c = \frac{e}{b},
\end{aligned}
\end{equation}

system (1.5) is transformed to system (4.4), and parameter \(c\) in system (1.5) is equivalent to \(e\) in system (4.4). In this case, \(f, g\) and \(h\) satisfy \((A_1), (A_2), (A_1'), (A_2'), (A_3)\) and \((A_5)\), \(v_0 = d\), and \(h(v) > 0\) for \(v < v_0\). It follows from Remark 1 that \(h(v)\) satisfies \((A_4)\) if \(\frac{a}{b} = f(0) > d - p\). Then, from Theorem 3.5 we have the following results on the non-existence of non-constant steady states when conversion rate \(e\) is large.

**Proposition 4.5**  
(a1) If \(a > bd\), then there exists \(e_0 > 0\), depending on \(a, b, d, p, \beta, m, d_1, d_2\) and \(\Omega\), such that system (4.4) has a unique constant positive steady state and no non-constant positive steady states for \(e > e_0\).

(a2) If \(a < bd\), then there exists \(e_0 > 0\), depending on \(a, b, d, p, \beta, m, d_1, d_2\) and \(\Omega\), such that system (4.4) has no positive steady states for \(e > e_0\).

From Corollary 2.4, we have the following results on the global attractivity of constant equilibria for small conversion rate, which also imply the non-existence of non-constant steady states.

**Proposition 4.6**  
(b1) If \(d < \frac{a}{b}\), then there exists \(e_0 > 0\), depending on \(a, b, d, p, \beta\) and \(m\), such that, for \(e \in (0, e_0)\), system (4.4) has a unique constant positive steady state, which is globally attractive.

(b2) If \(d > \frac{(am + 1)^2}{4mb}\), then the steady state \((0, d)\) of system (4.4) is globally attractive.

Here we remark that assumption \(d > p\) throughout this subsection means that the growth rate of predator is weak Allee type in the absence of prey. The results of this subsection can also be applied to the case of \(d < p\), since Assumptions \((A_1), (A_2), (A_1'), (A_2'), (A_3), (A_4)\) and \((A_5)\) are also satisfied.
4.3 A Predator–Prey Model with Strong Allee Effect in Predator and Holling-IV Functional Response

In this subsection, we consider the following model,

$$
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \Delta u &= u(a - u) - \frac{buv}{1 + nu + mu^2}, \quad x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} - d_2 \Delta v &= \beta v(d - v)(v - p) + \frac{euv}{1 + nu + mu^2}, \quad x \in \Omega, \ t > 0, \\
\partial_n u &= \partial_n v = 0, \quad x \in \partial \Omega, \ t > 0,
\end{align*}
$$

which is a diffusive predator–prey model with strong Allee effect in predator. Here $a, b, d, p, e, m, n, \beta, d_1$ and $d_2$ are positive constants. Letting

$$
\begin{align*}
f(u) &= \frac{(1 + nu + mu^2)(a - u)}{b}, \quad c = \frac{e}{b}, \\
g(u) &= \frac{buv}{1 + nu + mu^2}, \quad h(v) = \beta(d - v)(v - p).
\end{align*}
$$

Then system (1.5) is transformed to system (4.6), and parameter $c$ in system (1.5) is equivalent to $e$ in system (4.6). In this case, $f, g$ and $h$ satisfy $(A'_1), (A'_2)$ and $(A_5)$, and $v_0 = \max\{d, p\}$. It follows from Remark 1 that $h(v)$ satisfies Assumptions $(A_4)$ if $\frac{a}{b} = f(0) > d + p$. Then from Theorem 3.5, we have the following results on the non-existence of non-constant steady states when conversion rate $e$ is large.

**Proposition 4.7** If $a > b(d + p)$, then there exists $e_0 > 0$, depending on $a, b, d, p, m, n, \beta, d_1, d_2$ and $\Omega$, such that system (4.6) has a unique constant positive steady state and no non-constant positive steady states for any $e > e_0$.

5 Conclusions

In this paper, we consider a general diffusive predator–prey system. It covers a wide range of predator–prey models which include some well-known ones but also some less studied ones. We find that the conversion rate is a key parameter to affect the dynamics of a general predator–prey model, and there are almost no complex patterns for large and small conversion rate. Hence, this phenomenon can occur commonly for predator–prey models, which was found in [2] for a special model with a nonlinear growth rate for the predator.

For the case of small conversion rate, we show the global attractivity of the unique constant positive steady state even when $h(v)$ is nonmonotonic, and hence it can be applied to model (4.4) with weak Allee effect in predator. A special case where $h(v)$ is monotonic was analyzed in [2]. We remark that our result is not a direct conclusion from [17–20], and needs some detailed analysis for the relations between the limits of upper solutions sequence and lower solutions sequence. Moreover, our result supplements some existing ones. For example, Propositions 4.3 and 4.4 supplement Theorem 2.3 in [6], which show the global attractivity of the positive equilibrium for $ma \leq 1$.

For the case of large conversion rate, we show the nonexistence of the positive steady states. Our method is motivated by [21], but we need to modify many of their arguments to derive our result. We find that, even with a nonmonotonic functional response, there exist no nonconstant positive steady states for large conversion rate. Moreover, our result in Theorem
4.1 also supplements Theorem 1.2 of [21], which show the nonexistence when \( e \) equals to \( b \) and is sufficiently large.

Finally, we should mention that the dimension \( N \) of the domain \( \Omega \) we studied is less than three, and this is meaningful in biology. For a general dimension \( N \), we can also obtain the similar results, if \( h \) satisfies the following assumption:

\[
(A'_5) \quad h \in C^1(\mathbb{R}^+) \text{ and there exist } n \in \mathbb{N}^+, \{q_i\}_{i=0}^n, \{k_i\}_{i=0}^n \text{ and } \{\overline{k}_i\}_{i=0}^n \text{ such that }
\]

\[
\sum_{i=0}^n k_i v^{q_i} \leq -h(v) \leq \sum_{i=0}^n \overline{k}_i v^{q_i} \quad \text{for any } v \geq 0,
\]

where \( 0 = q_0 < q_1 < q_2 < \cdots < q_n, q_n > \frac{N}{2} - 1 \) and \( k_n, \overline{k}_n > 0 \).

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