Prediction of Neutrino Mixing based on $C_2 \times D_3$

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The lepton mixing angles of the PMNS matrix are predicted based on the lepton flavor symmetry of a finite group $C_2 \times D_3$, where the cyclic group $C_2$ acts on the charged lepton mass terms and the dihedral group $D_3$ on the neutrino ones. All three mixing angles of the PMNS matrix are given in terms of just one parameter, the charged lepton mixing angle, and fit extremely well to the observed values. In particular, the smallness of $\theta_{13}$ is explained in terms of the smallness of the muon-to-tau mass ratio.

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1. Introduction

The success of the Standard Model (SM) of the Electroweak (EW) theory and Quantum Chromodynamics (QCD) is quite impressive as the last loophole is finally closed with the long-waited discovery of the Higgs at LHC. Nevertheless, it is equally well known that the SM is not our ultimate theory to describe the Nature. The observations of neutrino flavor violations\[1\][2] unambiguously supports the existence of physics beyond the SM. These observations are commonly interpreted to indicate that neutrinos are massive and the flavor-violations are due to the difference between the flavor and mass eigenstates of the leptons\[3\][4]. For massive neutrinos, right-handed neutrinos are inevitably required and the mystery of the neutrino physics originates from the fact that the right-handed neutrinos do not carry any charges of the SM, but we have to rely on their visible companions of their left-handed partners to investigate their properties. The only connection between the left-handed and right-handed neutrinos are via their Dirac mass terms (i.e. Yukawa interactions involving the SM Higgs or something similar), unless there is a new physics in which the right-handed neutrinos also carry some new gauge charges even below the EW scale\[5\][6]. So, to go beyond the SM it is important to understand the structure of these neutrino masses and mixings. If there is any nontrivial structure which can be traced back to some fundamental principle, it will be a clue to the physics beyond SM. Partly motivated by this, in this paper we will present mass matrices based on new symmetry constraints, which explain the observed mixing angles.\(^1\)

The relevant terms for the interactions between neutrinos and charged leptons in the SM Lagrangian are given by

\[
\mathcal{L}_{\ell\nu} = \frac{g}{\sqrt{2}} W^\mu_{\mu} \ell^\dagger L \gamma^\mu \nu^\prime_L + \text{h.c.} = \frac{g}{\sqrt{2}} W^\mu_{\mu} \ell^\dagger_L U^\dagger_{\ell} \gamma^\mu U_{\nu} \nu_L + \text{h.c.},
\]

where \(\ell^\prime\) and \(\nu^\prime\) denote the Weak flavor eigenstates of the charged leptons and neutrinos, respectively, \(\ell\) and \(\nu\) denote their mass eigenstates, and \(U_{\ell}\) is the theoretical (unitary) mixing matrix of the charged leptons and \(U_{\nu}\) is that of the left-handed neutrinos. Since the existence of the neutrinos is only indirectly inferred by observing the companion charged leptons, the observed neutrino (to be precise, lepton) mixing is given by the celebrated PMNS matrix\[3\][4]

\[
U_{\text{PMNS}}(\theta_{23}, \theta_{13}, \theta_{12}) = U_{\ell}^\dagger(\theta_{\ell 23}, \theta_{\ell 13}, \theta_{\ell 12}) U_{\nu}(\theta_{\nu 23}, \theta_{\nu 13}, \theta_{\nu 12}).
\]

Since three dimensional rotations satisfy the SO(3) group symmetry, no matter how many angles

\[^1\text{Explaining the neutrino mixing based on a discrete symmetry was initiated in [7].}\]
are involved on the r.h.s., only three Euler angles are needed to express the final rotation for the PMNS matrix, and these three angles are experimentally measured.

The latest best-fit numbers from the pdgLive[8] are $^2$: $\sin^2(2\theta_{12}) = 0.857 \pm 0.023 - 0.025$, $\sin^2(2\theta_{23}) > 0.95$, and $\sin^2(2\theta_{13}) = 0.095 \pm 0.010$, which can also be expressed as

$$\sin^2 \theta_{12} \simeq 0.311 \pm 0.016,$$
$$\sin^2 \theta_{23} \simeq (0.39 \sim 0.61),$$
$$\sin^2 \theta_{13} \simeq 0.024 \pm 0.003.$$  \hfill (3)

The masses are not completely determined, but two constraints are known:

$$\Delta m_{21}^2 \simeq 7.50^{+0.19}_{-0.20} \times 10^{-5} \text{ eV}^2,$$
$$\Delta m_{32}^2 \simeq 2.32^{+0.12}_{-0.08} \times 10^{-3} \text{ eV}^2.$$  \hfill (4)

In principle, there can be three different CP-violating phases for three generations of leptons, but in the Dirac case a $3 \times 3$ unitary matrix allows only one phase to be independent and the rest can be eliminated by chiral phase transformations. However, in this paper we will ignore the CP-violation for simplicity of the argument. The generalization should be straightforward, and we will comment on it in the discussion section later.

We emphasize that there is no a priori reason to demand $U_\ell = 1$. In fact, allowing more general $U_\ell$ in this paper, we can easily justify the measured PMNS matrix based on the new symmetry argument. We may even call it a prediction since no free parameter is involved once the right symmetry is imposed. The new symmetry we introduce here is based on the cyclic groups $C_2$, and enhanced to the dihedral group $D_3[9]$. The case of pure cyclic groups is presented in [10] and in this paper we will show what happens if the symmetry is enhanced to the dihedral group and, at the same time, some technical details will be explained. The symmetry we have introduced have not been previously considered, although many other discrete symmetries are investigated for the neutrino masses and mixing (see, for example, a recent review [11] and references therein; also see [12]; some mathematical details can be found in [13]).

This paper is organized as follows. In section 2, we introduce a new discrete symmetry based on finite groups. In section 3, the details of the $C_2$ symmetry for the charged lepton mass matrix is explained, and the charged lepton mixing matrix $U_\ell$ is constructed. In section 4, the details of the $C_2 \times C_2$ symmetry for the neutrinos are explained and the condition toward the enhanced $D_3$ symmetry is given. Also the theoretical neutrino mixing matrix $U_\nu$ is constructed.

$^2$No error range for $\sin^2(2\theta_{23})$ is posted yet, but it is not going to be really crucial for us since our predicted numbers are almost 0.95.
Then in section 5, the PMNS matrix is given numerically and compared to the best-fit values. Finally, the conclusions and discussions are given in the final section.

2. The Symmetry

The mixing matrices are dictated by the structure of the mass matrices. So the symmetry we need is not only the symmetry of the Lagrangian but also what constrains the mass matrices to the desired form at the same time. Thus, first, we can demand that the mass matrices are invariant as

\[ T^\daggerMT' = M, \]  
\[ T^\dagger m_D S = m_D, \]  
\[ S^\dagger m_R S^* = m_R, \]

where \( T' \), \( T \), and \( S \) are (nontrivial) operations acting on the charged leptons, left-handed neutrinos \( \nu_L \), right-handed neutrinos \( \nu_R \), respectively. For \( T' \), \( T \), and \( S \) to be interpreted as symmetry transformations, they must form a group. Since they can be all independent, the simplest group we can consider is a direct product of three cyclic groups \( C_2 \) of order two\[9\]. \( C_2 \) is a group of just one nontrivial element in addition to the identity, a simplest possible nontrivial group. So, if \( T'^2 = 1 \), \( T^2 = 1 \), and \( S^2 = 1 \), then we have a symmetry group \( C_2(T') \times C_2(T) \times C_2(S) \). For them to be a symmetry of the Lagrangian, \( T' \), \( T \), and \( S \) must be also unitary, and that they are also hermitian. Then the discrete symmetry under \( C_2(T') \times C_2(T) \times C_2(S) \) is a global (lepton flavor) symmetry of the Lagrangian.

In principle, we could demand different symmetry transformations for the left-handed and right-handed charged leptons, but as we will see later, even if we impose eq.(5a) differently as \( T^\dagger MT'_R = M, \) \( T'_R^\dagger = T'_R \) for the desired \( M \), eq.(9a), will dictate that \( T'_R = T' \) has to be satisfied. One may also wonder if \( T' \) and \( T \) should be the same because left-handed leptons form SU(2) doublets, but it is not necessary because their transformations are independent from the SU(2) isospin rotations, which are always compensated by those of the Higgs doublet. In some sense, this is also why neutrino mixing and charged lepton mixing can be different even though they form SU(2) doublets. Since \( T' \), \( T \), and \( S \) have eigenvalues \( \pm 1 \), they can be regarded as generalized parity operators analogous to \( \gamma_5 \). \( T' \), \( T \), and \( S \) also fix mass matrices to the desired forms, as we will see later.

Furthermore, when \( C_2(T) \times C_2(S) \) is enhanced to the dihedral group \( D_3(T, S) \), the entire
mixing matrices will be fixed by the symmetry without an additional free parameter. The
dihedral group $D_3(T, S)$ can be presented by

$$D_3(T, S) = \langle T, S | T^2 = 1, S^2 = 1, (TS)^3 = 1 \rangle.$$  \hspace{1cm} (6)

Geometrically, $D_3$ is also known as a symmetry of a regular triangle in the two-dimensions, but
we consider $D_3(T, S)$ in the three-dimensions.

For notational convenience, we first define three basic real symmetric operators

$$\Gamma_1(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & s_1 & -c_1 \end{pmatrix}, \quad \Gamma_2(\theta_2) = \begin{pmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & -c_2 \end{pmatrix}, \quad \Gamma_3(\theta_3) = \begin{pmatrix} c_3 & s_3 & 0 \\ s_3 & -c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$  \hspace{1cm} (7)

where we call $c_i \equiv \cos \theta_i$ or $s_i \equiv \sin \theta_i$ the symmetry parameters, and $\Gamma_i$ acts about $i$-axis. $\Gamma_i$'s
are combinations of a rotation and an inversion, satisfying $\Gamma_i^2 = 1$, and $\det \Gamma_i = -1$. Because
of the latter condition, $\Gamma_i$'s are not group elements of $SO(3)$ or $SU(3)$, hence the finite groups
we consider here are not their subgroups. In the followings, all $T', T$, and $S$ can be expressed
in terms of these, so these are the building blocks of our cyclic groups and the dihedral group.

As was first observed in [6], the current best-fit mixing angles can be extremely well
reproduced if we parametrize $U_\ell(\theta_{\ell 13})$ and $U_\nu(\theta_{\nu 23}, \theta_{\nu 12})$ such that

$$U_{PMNS}(\theta_{23}, \theta_{13}, \theta_{12}) = U_{\ell}^\dagger(\theta_{\ell 13}) U_\nu(\theta_{\nu 23}, \theta_{\nu 12})$$  \hspace{1cm} (8)

with respect to the mass matrices (in the Weak flavor basis) of the form

$$\mathbf{M} = \begin{pmatrix} M_{11} & 0 & M_{13} \\ 0 & M_{22} & 0 \\ M_{13} & 0 & M_{33} \end{pmatrix},$$  \hspace{1cm} (9a)

$$\mathbf{m}_D = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & 0 \\ m_{13} & 0 & m_{22} \end{pmatrix},$$  \hspace{1cm} (9b)

where $M_{22} = M_\mu$ is the muon mass for the charged lepton mass matrix $\mathbf{M}$ and $\mathbf{m}_D$ is the Dirac
neutrino mass matrix with $m_{33} = m_{22}$. So, motivated by [6], even without the extra $U(1)_\lambda$
symmetry, we demand these mass matrices to be hermitian. The specific form of the Majorana
mass matrix for the right-handed neutrinos are not crucial in our argument, so we will only
invoke later when we need it. Now, our mission is to show the existence of proper $T'$, $T$, and
$S$, which constrain the mass matrices to be the desired forms given here.
3. Charged Leptons

Our plan is as follows: We will first construct the symmetry, then show that the symmetry indeed constrains the mass matrix uniquely to the desired form. Next, we will diagonalize the desired mass matrix to obtain the mixing matrix with the mixing angle constrained by the symmetry parameter.

The symmetry $C_2(T')$ we need is generated explicitly by

$$
T' = \begin{pmatrix}
\frac{\bar{M}_{33} - \bar{M}_0}{\bar{M}_{33} + \bar{M}_0} & 0 & \frac{2M_{13}}{\bar{M}_{33} + \bar{M}_0} \\
0 & 1 & 0 \\
\frac{2M_{13}}{\bar{M}_{33} + \bar{M}_0} & 0 & -\frac{\bar{M}_{33} - \bar{M}_0}{\bar{M}_{33} + \bar{M}_0}
\end{pmatrix},
$$

(10)

where we have introduced a shorthand notation

$$
\bar{M} \equiv M - M_e,
$$

(11)

such that the required cyclic condition $T'^2 = 1$ is satisfied provided

$$
M_{13}^2 = \bar{M}_0 \bar{M}_{33}.
$$

(12)

In terms of the notations introduced in eq.(7) this $T'$ can be expressed as

$$
T' = \Gamma_2(\theta_\ell),
$$

(13)

where the symmetry parameters are given by

$$
c_\ell = \frac{\bar{M}_{33} - \bar{M}_0}{\bar{M}_{33} + \bar{M}_0},
$$

(14a)

$$
s_\ell = -\frac{2M_{13}}{\bar{M}_{33} + \bar{M}_0},
$$

(14b)

and $c_\ell^2 + s_\ell^2 = 1$ because of eq.(12).

Uniqueness of $M$

We can now show that eq.(5a) indeed leads $M$ to be the desired form of eq.(9a) with $M_{11} = M_0$. For this purpose, we will first consider a more general $M = (M_{\ell\ell'})$, then eq.(5a) reads

$$
\begin{pmatrix}
c_\ell^2 M_{11} + 2c_\ell s_\ell M_{13} + s_\ell^2 M_{33} & c_\ell M_{12} + s_\ell M_{23} & c_\ell s_\ell (M_{11} - M_{33}) + (s_\ell^2 - c_\ell^2) M_{13} \\
c_\ell M_{12} + s_\ell M_{23} & M_{22} & s_\ell M_{12} - c_\ell M_{23} \\
c_\ell s_\ell (M_{11} - M_{33}) + (s_\ell^2 - c_\ell^2) M_{13} & s_\ell M_{12} - c_\ell M_{23} & s_\ell^2 M_{11} - 2c_\ell s_\ell M_{13} + c_\ell^2 M_{33}
\end{pmatrix} = (M_{\ell\ell'}),
$$

(15)
which consists of six linear equations for six mass parameters $M_{\ell\ell'} = M_{\ell'\ell}$. The (13)-components lead to

$$t_\ell \equiv \tan \theta_\ell = - \frac{2M_{13}}{M_{33} - M_{11}},$$

which (11)- and (33)-components also satisfy. Comparing this to eqs.(14a)(14b), $M_{11}$ can be fixed as

$$M_{11} = M_0.$$  \hspace{1cm} (17)

(12)-components (also (23)-components) lead to

$$M_{23} = \frac{1 - c_\ell}{s_\ell} M_{12} = - \frac{\tilde{M}_0}{M_{13}} M_{12}.$$  \hspace{1cm} (18)

Since the three eigenvalues of $\mathbf{M}$ are the physical charged lepton masses $M_e$, $M_\mu$, and $M_\tau$,

$$\text{Det}(\mathbf{M} - \lambda \mathbf{1}) = (M_e - \lambda)(M_\mu - \lambda)(M_\tau - \lambda).$$  \hspace{1cm} (19)

With eq.(17), for $\lambda = M_e$ this becomes

$$0 = \tilde{M}_{22}\left(\tilde{M}_0 M_{33} - M_{13}^2\right) - M_{12}^2 \frac{(\tilde{M}_{33} + \tilde{M}_0)^2}{M_{33}}.$$  \hspace{1cm} (20)

The first term vanishes because of eq.(12), then the remaining term implies

$$M_{12} = 0$$  \hspace{1cm} (21)

and that $M_{23} = 0$ can be shown from eq.(18). Similarly, for $\lambda = M_\mu$, we can easily obtain $M_{22} = M_\mu$, then $\lambda = M_\tau$ is also satisfied with eqs.(12)(21). Thus, the symmetry $C_2(T')$ uniquely fixes the charged lepton mass matrix to the desired form given in eq.(9a) with $M_{11} = M_0$.

**Proof of $T_R' = T'$**

Next, we will show $T_R' = T'$ for $T'^{\dagger} \mathbf{M} T_R' = \mathbf{M}$, if $\mathbf{M}$ is given by eq.(9a). Let us start with

$$T_R' = \begin{pmatrix} T_{11}' & 0 & T_{13}' \\ 0 & 1 & 0 \\ T_{31}' & 0 & -T_{11}' \end{pmatrix}$$  \hspace{1cm} (22)

which is not necessarily symmetric and satisfies

$$T_R'^{\dagger} = \mathbf{M}^{-1} T'R \mathbf{M}.$$  \hspace{1cm} (23)

Then we can easily see $T_R'^2 = 1$, Since $T'^2 = 1$.  \hspace{1cm} 6
Next we need to make sure $T'_R$ is also hermitian or real-symmetric. The components of $T'_R$ can be read off explicitly from the above as

\begin{align*}
T'_{11} &= \frac{M_\mu}{\text{Det}\,M} \left( c_\ell (M_{11}M_{33} + M_{13}^2) + s_\ell M_{13} (M_{33} - M_{11}) \right), \quad (24a) \\
T'_{13} &= \frac{M_\mu}{\text{Det}\,M} \left( 2c_\ell M_{13}M_{33} + s_\ell (M_{33}^2 - M_{13}^2) \right), \quad (24b) \\
T'_{31} &= \frac{M_\mu}{\text{Det}\,M} \left( -2c_\ell M_{11}M_{13} + s_\ell (M_{11}^2 - M_{13}^2) \right). \quad (24c)
\end{align*}

To obtain symmetric $T'_R$, we demand that $T'_{13} = T'_{31}$, which leads to eq.(16), and that $T'_{11} = c_\ell$ and $T'_{13} = s_\ell$. Thus we obtain $T'_R = T'$. So the left- and right-handed charged leptons transform in the same way and eq.(5a) is good enough for our purpose.

\underline{Obtaining $U_\ell$}

Having established that the charged lepton mass matrix $M$ given in eq.(9a) indeed is what we get under the symmetry $C_2(T')$, we can now diagonalize it as

$$U_\ell^{-1} M U_\ell = M_4 = \text{diag}(M_e, M_\mu, M_\tau)$$

(25)

to obtain the charged lepton mixing matrix

$$U_\ell(\theta_{\ell 13}) = \begin{pmatrix} c_{\ell 13} & 0 & -s_{\ell 13} \\ 0 & 1 & 0 \\ s_{\ell 13} & 0 & c_{\ell 13} \end{pmatrix},$$

(26)

where the mixing angle satisfies

\begin{align*}
s_{\ell 13} &= -\frac{M_{13}}{\sqrt{(M_{33} - M_e)^2 + M_{13}^2}}, \quad (27a) \\
c_{\ell 13} &= \frac{M_{33} - M_e}{\sqrt{(M_{33} - M_e)^2 + M_{13}^2}}. \quad (27b)
\end{align*}

In addition to $M_\mu = M_{22}$, the other eigenvalues are now given by

\begin{align*}
M_e &= \frac{1}{2} \left( M_{33} + M_{11} - \sqrt{(M_{33} - M_{11})^2 + 4M_{13}^2} \right), \quad (28a) \\
M_\tau &= \frac{1}{2} \left( M_{33} + M_{11} + \sqrt{(M_{33} - M_{11})^2 + 4M_{13}^2} \right). \quad (28b)
\end{align*}

Note that eq.(28a) is actually equivalent to eq.(12). Since $M_e$ and $M_\tau$ are given by three parameters $M_{11}$, $M_{33}$ and $M_{13}$, one of which is a free parameter. This free parameter is constrained by $c_\ell$ so that fixing $c_\ell$ in terms of $M_0$ can fix $M_{11}$, and vice versa.
Together with $M_\tau$, eq.(27a) can be expressed as

$$s_{\ell 13} = \sqrt{\frac{M_{11} - M_e}{M_\tau - M_e}}.$$  

(29)

This is interesting because it is almost given in terms of physical charged lepton masses except $M_{11}$. So, it certainly motivates us to choose $M_{11} = M_\mu$ so that it can be entirely given in terms of all three physical charged lepton masses. This can be achieved by choosing the symmetry of the generator $T'$ given in terms of $M_0 = M_\mu$, which will fix $M_{11} = M_\mu$. Therefore, a good $C_2(T)$ symmetry motivated by phenomenology is to let $M_0 = M_\mu$, as in [6], so that $s_{\ell 13}$ is given by the muon-to-tau mass ratio to a good approximation. As we will see later, this leads to neutrino mixing angles very close to the best-fit values.

There is also an interesting relationship between the symmetry parameter $\theta_\ell$ and the mixing angle $\theta_{\ell 13}$. Comparing eq.(27a) to eq.(14a), we can show that

$$1 - 2s_{\ell 13}^2 = \frac{\tilde{M}_{33}^2 - M_{13}^2}{\tilde{M}_{33}^2 + M_{13}^2} = c_\ell,$$  

(30)

where eq.(12) is used for the last equality. This leads to an interesting identity

$$\theta_\ell = 2\theta_{\ell 13}.$$  

(31)

Thus the symmetry angle of $C(T')$ is actually twice of the mixing angle so that the charged lepton mixing angle $\theta_{\ell 13}$ can play an important role in our model.

4. Neutrinos

For neutrinos, we will proceed similarly as the charged lepton case but with some alterations. First, we construct the simpler symmetry based on the cyclic groups, and show the uniqueness of the form of the mass matrix. However, since $m_{11} = m_{22}$ is a special case, we will treat $m_{11} \neq m_{22}$ and $m_{11} = m_{22}$ cases separately. Next, we will look for a condition to enhance the symmetry group to the dihedral group, in which the symmetry parameters of the cyclic groups will be further constrained. Then we will construct the mixing matrix, in which now mixing angles will be constrained by the symmetry parameters.

For the symmetry $C_2(T)$ of the left-handed neutrinos, we choose, using the notation given in eq.(7),

$$T = \Gamma_2(\theta_\nu),$$  

(32)
where \( \theta_\nu \) should be properly chosen to satisfy our need. In principle, we can choose any value for \( c_\nu \) because \( T^2 = 1 \) for any \( c_\nu \). However, if we want to relate it to the values in the charged lepton case, an interesting choice is

\[
c_\nu = t_{\ell 13} = - \frac{M_{11} - M_e}{M_{13}}.
\]

(33)

Although this is not necessarily a unique choice, but certainly a good choice so that \( \theta_\nu \) can be related to \( \theta_{\ell 13} = \theta_\ell / 2 \). It will be extremely interesting if there exists a symmetry argument to force this identity, which we will leave as future work.

Once \( T \) is chosen, \( S \) for the right-handed neutrinos should be what can constrain the Dirac neutrino mass matrix to be the desired form given in eq.(9b). To find such \( S \), let us start with

\[
S = \Gamma_1(\theta_\alpha)\Gamma_3(\theta_\nu)\Gamma_1(\theta_\alpha) = \begin{pmatrix}
c_\nu & s_\nu c_\alpha & s_\nu s_\alpha \\
s_\nu c_\alpha & s_\alpha^2 - c_\nu c_\alpha^2 & c_\alpha s_\alpha (1 + c_\nu) \\
s_\nu s_\alpha & -c_\alpha s_\alpha (1 + c_\nu) & c_\alpha^2 - c_\nu s_\alpha^2
\end{pmatrix}.
\]

(34)

Note that \( S^2 = 1 \) and \( S^\dagger = S \) which we can easily see from eq.(7) so that \( S \) certainly forms the cyclic group \( C_2(S) \) which satisfies our criteria to be a symmetry of the Lagrangian. Then, \( \theta_\alpha \) can be determined specifically to satisfy eq.(5b), which we will show next.

\[m_{11} \neq m_{22}\] Case

Let us first consider a more general Dirac neutrino mass matrix, eq.(9b), with \( m_{11} \neq m_{22} \). With our candidate representation of \( S \) given in eq.(34), the symmetry invariance constraint eq.(5b) in components reads

\[
m_{11} = c_\nu (c_\nu m_{11} + s_\nu m_{13}) + c_\nu s_\nu c_\alpha m_{12} + s_\nu s_\alpha (c_\nu m_{13} + s_\nu m_{22}),
\]

(35a)

\[
m_{12} = s_\nu c_\alpha (c_\nu m_{11} + s_\nu m_{13}) + c_\nu (s_\alpha^2 - c_\nu c_\alpha^2) m_{12} - c_\alpha s_\alpha (1 + c_\nu) (c_\nu m_{13} + s_\nu m_{22}),
\]

(35b)

\[
m_{12} = c_\nu m_{12} + s_\nu c_\alpha m_{22},
\]

(35c)

\[
m_{22} = s_\nu c_\alpha m_{12} + (s_\alpha^2 - c_\nu c_\alpha^2) m_{22},
\]

(35d)

\[
m_{13} = s_\nu s_\alpha (c_\nu m_{11} + s_\nu m_{13}) - c_\alpha s_\alpha (1 + c_\nu) c_\nu m_{12} + (c_\alpha^2 - c_\nu s_\alpha^2) (c_\nu m_{13} + s_\nu m_{22}),
\]

(35e)

\[
m_{13} = c_\nu (s_\nu m_{11} - c_\nu m_{13}) + s_\nu c_\alpha m_{22} + s_\nu s_\alpha (s_\nu m_{13} - c_\nu m_{22}),
\]

(35f)

\[
0 = m_{23} = s_\nu s_\alpha m_{12} - c_\alpha s_\alpha (1 + c_\nu) m_{22},
\]

(35g)

\[
0 = m_{23} = s_\nu c_\alpha (s_\nu m_{11} - c_\nu m_{13}) + s_\nu (s_\alpha^2 - c_\nu c_\alpha^2) m_{12} - c_\alpha s_\alpha (1 + c_\nu) (s_\nu m_{13} - c_\nu m_{22}),
\]

(35h)

\[
m_{22} = s_\nu s_\alpha (s_\nu m_{11} - c_\nu m_{13}) - c_\alpha s_\alpha (1 + c_\nu) s_\nu m_{12} + (c_\alpha^2 - c_\nu s_\alpha^2) (s_\nu m_{13} - c_\nu m_{22}).
\]

(35i)

Eqs.(35c)(35d)(35g) lead to (for \( c_\alpha \neq 0 \neq s_\alpha \))

\[
\frac{m_{12}}{m_{22}} = \frac{s_\nu c_\alpha}{1 - c_\nu} = \frac{(1 + c_\nu) c_\alpha}{s_\nu}.
\]

(36)
Eq. (35a) × (c_ν / s_ν) + eq. (35f) leads to

\[ \frac{m_{13}}{m_{12}} = \frac{c_\alpha}{1 - s_\alpha}. \]  \hspace{1cm} (37)

With eqs. (36)(37), eq. (35a) × \( s_\nu - c_\nu \times \text{eq. (35f)} \) leads to

\[ \frac{m_{12}}{m_{11}} = \frac{s_\nu c_\alpha}{2c_\nu + s_\alpha(1 + c_\nu)}. \]  \hspace{1cm} (38)

The rest of equations equivalently reproduce these, so eqs. (36)-(38) are complete constraints obtained from the symmetry constraint eq. (5b) for \( C_2(T) \times C_2(S) \). So the mass parameters are constrained by the symmetry parameters \( \theta_\nu \) and \( \theta_\alpha \).

The above relations between the mass parameters and the symmetry parameters can be rearranged to show more useful relations. From eqs. (36)(38), eliminating \( m_{12} \), we can obtain a relation between \( m_{11} \) and \( m_{22} \) as

\[ m_{11} - m_{22} = \left( \frac{1 + c_\nu}{1 - c_\nu} s_\alpha - \frac{1 - 3c_\nu}{1 - c_\nu} \right) m_{22}. \]  \hspace{1cm} (39)

This in particular shows why \( m_{11} = m_{22} \) is a special case, for which \( s_\alpha \) and \( c_\nu \) become no longer independent. If \( m_{11} \neq m_{22} \), \( s_\alpha \) and \( c_\nu \) are independent.

Eq. (37) also enables us to express \( \theta_{\nu 23} \), one of the two neutrino mixing angles, in terms of a symmetry parameter as

\[ s_{\nu 23}^2 = \frac{m_{13}^2}{m_{12}^2 + m_{13}^2} = \frac{1 + s_\alpha}{2}, \]  \hspace{1cm} (40)

where the former identity is from eq. (70b). Then together with eqs. (36)(39), we can obtain the other neutrino mixing angle \( \theta_{\nu 12} \), depending on the choice of \( m_1 \), as

\[ \frac{1 - c_\nu}{1 + c_\nu} \frac{2}{1 + s_\alpha} = \frac{m_{12}^2 + m_{13}^2}{(m_+ - m_{22})^2} = \left\{ \begin{array}{ll} t_{\nu 12}^2, & \text{for } m_1 = m_-, \\ \frac{1}{t_{\nu 12}}, & \text{for } m_1 = m_+, \end{array} \right. \]  \hspace{1cm} (41a)

where the latter identity is from eqs. (70c)(70d) for \( m_1 = m_- \) or eqs. (71c)(71d) for \( m_1 = m_+ \) and

\[ m_\pm \equiv \frac{1}{2} \left( m_{11} + m_{22} \pm \sqrt{(m_{11} - m_{22})^2 + 4(m_{12}^2 + m_{13}^2)} \right). \]  \hspace{1cm} (42)

So both mixing angles are now expressed in terms of the symmetry parameters, \( c_\nu \) and \( s_\alpha \). If we eliminate \( s_\alpha \) from eqs. (41a)(41b) with the help of eq. (40), we can obtain interesting relationships between two mixing angles in terms of \( c_\nu \) as

\[ \frac{1 - c_\nu}{1 + c_\nu} = \left\{ \begin{array}{ll} s_{\nu 23}^2 t_{\nu 12}^2, & \text{for } m_1 = m_-, \\ s_{\nu 23}^2 \frac{t_{\nu 23}}{t_{\nu 12}^2}, & \text{for } m_1 = m_+. \end{array} \right. \]  \hspace{1cm} (43a)
\(m_{11} = m_{22}\) Case

As we have noticed in eq.(39), \(m_{11} = m_{22}\) can simplify the symmetry parameters by relating each other. This is also the case in which the Dirac neutrino mass matrix has identical diagonal components, i.e. diagonal components have maximal symmetry[10].

From eq.(39), if \(m_{11} = m_{22}\), we can eliminate one symmetry parameter using the other as

\[
s_\alpha = \frac{1 - 3c_\nu}{1 + c_\nu},
\]
then eq.(38) simplifies as

\[
c_\alpha = \frac{\sqrt{8c_\nu(1 - c_\nu)}}{1 + c_\nu} = \frac{m_{12}}{m_{11}}s_{\nu 23},
\]

In particular, eq.(40) leads to

\[
\frac{1 - c_\nu}{1 + c_\nu} = s_{\nu 23}^2,
\]

hence eqs.(41a)(41b) simplify to

\[
t_{\nu 12}^2 = 1.
\]
such that both eqs.(43a)(43b) now reduce to eq.(46). So, if we choose the symmetry parameter \(s_\alpha\) as eq.(44) in the beginning, we can force \(m_{11} = m_{22}\) so that the symmetry can lead to \(\theta_{\nu 12} = \pi/4\).

Uniqueness of \(m_D\)

Once we have figured out how the symmetry angles \(\theta_\alpha\) and \(\theta_\nu\) satisfy the symmetry constraint eq.(5b), it is straightforward to show how the symmetry eq.(5b) constrains the Dirac neutrino mass matrix \(m_D\) to be the desired form given in eq.(9b). For this purpose, we only need to show that there are \(s_\alpha\) and \(c_\nu\) which set \(m_{23} = 0\) and \(m_{33} = m_{22}\).

As in the charged lepton case, let us first start with a general hermitian \(m_D = (m_{\ell\ell})\), then the symmetry invariance condition eq.(5b) becomes

\[
m_D = T m_D S
\]

\[
= \begin{pmatrix}
c_\nu m_{11} + s_\nu m_{13} & c_\nu m_{12} + s_\nu m_{23} & c_\nu m_{13} + s_\nu m_{33} \\
m_{12} & m_{22} & m_{23} \\
s_\nu m_{11} - c_\nu m_{13} & s_\nu m_{12} - c_\nu m_{23} & s_\nu m_{13} - c_\nu m_{33}
\end{pmatrix}
\times
\begin{pmatrix}
c_\nu & s_\nu s_\alpha & s_\nu s_\alpha \\
s_\nu c_\alpha & s_\nu^2 - c_\nu c_\alpha^2 & -c_\alpha s_\alpha (1 + c_\nu) \\
s_\nu s_\alpha & s_\alpha s_\alpha (1 + c_\nu) & c_\alpha^2 - c_\nu s_\alpha^2
\end{pmatrix}
\]

The (21)-components read

\[
m_{12} = c_\nu m_{12} + s_\nu c_\alpha m_{22} + s_\nu s_\alpha m_{23}.
\]

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Now we choose the symmetry parameter for $S$ to satisfy (see eq.(36))
\[
c_\alpha = \left(\frac{1 - c_\nu}{s_\nu}\right) \frac{m_{12}}{m_{22}},
\]
then (21)-components satisfy
\[
m_{23} = \frac{1}{s_\alpha} \left(\frac{1 - c_\nu}{s_\nu} m_{12} - c_\alpha m_{22}\right) = 0.
\]

In fact, eq.(50) is the only condition we need to constrain even $m_{33} = m_{22}$. The (11)- and (31)-components lead to eq.(37) after eliminating $m_{33}$ terms, then together with eq.(50), we can obtain
\[
m_{13} = \frac{(1 + c_\nu)c_\alpha^2}{s_\nu(1 - s_\alpha)}.
\]
On the other hand, the (13)- and (33)-components independently lead to
\[
m_{13} = \frac{(1 + c_\nu)c_\alpha^2}{s_\nu(1 - s_\alpha)}.
\]
Comparing these two, we can obtain
\[
m_{33} = m_{22}.
\]
So, with the choice of eq.(50), the symmetry based on $C_2(T) \times C_2(S)$ indeed constrains the Dirac neutrino mass matrix $m_D$ to be the desired form uniquely.

**$D_3$ Symmetry for $m_{11} \neq m_{22}$**

Under $C_2(T) \times C_2(S)$, $\theta_\alpha$ and $\theta_\nu$ are not directly related but in terms of some mass parameters. (For example, see eq.(36).) So inevitably it will raise the question if the symmetry can be enlarged so that they can be more directly related. This is indeed the case under certain conditions and the larger symmetry, as we will show in the following.

Using the definitions of $T$ and $S$ given in eqs.(32)(34), we can compute
\[
T S = \begin{pmatrix}
c_\nu^2 + s_\alpha s_\nu^2 & s_\nu c_\alpha \{c_\nu - s_\alpha(1 + c_\nu)\} & s_\nu(c_\nu s_\alpha + c_\alpha^2 - c_\nu s_\alpha^2) \\
\frac{s_\nu c_\alpha}{s_\nu} & s_\alpha^2 - c_\nu c_\alpha^2 & -c_\alpha s_\alpha(1 + c_\nu) \\
c_\nu s_\nu(1 - s_\alpha) & s_\nu c_\alpha + c_\alpha s_\alpha c_\nu(1 + c_\nu) & s_\nu^2 s_\alpha - c_\nu(c_\alpha^2 - c_\nu s_\alpha^2)
\end{pmatrix}.
\]

Now we need to check if
\[
(TS)^3 = 1, \text{ i.e. } (TS)^2 = ST.
\]
From the latter equation, with $ST = (TS)^\dagger$, the (11)-component reads
\[
0 = (c_\nu^2 + s_\alpha s_\nu^2)^2 + s_\nu^2 c_\alpha^2 \{c_\nu - s_\alpha(1 + c_\nu)\} + c_\nu s_\nu^2 (1 - s_\alpha)(c_\nu s_\alpha + c_\alpha^2 - c_\nu s_\alpha^2) - (c_\nu^2 + s_\alpha s_\nu^2)
\]
\[
= s_\nu^2(s_\alpha - 1) \{(c_\nu + 1)s_\alpha - c_\nu\} \{(c_\nu + 1)s_\alpha + 2 - c_\nu\}.
\]
Note that \( s_\alpha = \frac{c_\nu - 2}{1 + c_\nu} \) leads to undesirable \( s_{\nu 23}^2 \) in eq.(40), so not interesting to our purpose. So, for \( s_\nu \neq 0 \) and \( s_\alpha \neq 1 \), as our solution, we take

\[
s_\alpha = \frac{c_\nu}{1 + c_\nu}.
\] (58)

Then, the following useful identities can be worked out:

\[
c_\nu^2 + s_\alpha s_\nu^2 = \frac{c_\nu^2}{1 + c_\nu} = c_\nu,
\] (59a)

\[
c_\alpha^2 - c_\nu s_\alpha^2 = 1 - \frac{c_\nu^2}{1 + c_\nu},
\] (59b)

\[
s_\alpha^2 - c_\nu c_\alpha^2 = -s_\alpha.
\] (59c)

With these identities, eq.(55) can be simplified as

\[
(TS)^2 = \begin{pmatrix}
    c_\nu & 0 & s_\nu \\
    s_\nu c_\alpha & -s_\alpha & -c_\nu c_\alpha \\
    s_\nu s_\alpha & c_\alpha & -c_\nu s_\alpha
\end{pmatrix}.
\] (60)

Now we can easily show that

\[
(TS)^2 = \begin{pmatrix}
    c_\nu & s_\nu c_\alpha & s_\nu s_\alpha \\
    0 & -s_\alpha & c_\alpha \\
    s_\nu & -c_\nu c_\alpha & -c_\nu s_\alpha
\end{pmatrix} = (TS)^\dagger = ST, \ i.e. \ (TS)^3 = 1.
\] (61)

Therefore, with eq.(58), the symmetry \( C_2(T) \times C_2(S) \) now gets enhanced to the dihedral group \( D_3(T, S) \) with a presentation given by eq.(6), under which \( s_\alpha \) is fixed in terms of \( c_\nu \). So, as the symmetry gets enhanced, we can eliminate another independent symmetry parameter, leaving the entire symmetry of the neutrino mass matrix controlled solely by \( \theta_\nu \) now.

The mixing angles can now be expressed only in terms of \( c_\nu \). Substituting \( s_\alpha \) using eq.(58), eq.(40) becomes

\[
s_{\nu 23}^2 = \frac{2c_\nu + 1}{2(1 + c_\nu)},
\] (62)

and, from eqs.(41)(41a), we obtain

\[
\nu_{12}^2 = \begin{cases}
    \frac{2(1 - c_\nu)}{2c_\nu + 1}, & \text{for } m_1 = m_-, \\
    \frac{2c_\nu + 1}{2(1 - c_\nu)}, & \text{for } m_1 = m_+.
\end{cases}
\] (63a)

So the neutrino mixing angles are entirely given in terms of the symmetry parameter \( \theta_\nu \) now. Furthermore, eq.(39) can be also given in terms of \( c_\nu \) as

\[
m_{11} - m_{22} = \frac{4c_\nu - 1}{1 - c_\nu} m_{22}.
\] (64)
$D_3$ Symmetry for $c_\nu = \frac{1}{4}$ and $m_{11} = m_{22}$

From eq.(64), if $m_{11} = m_{22}$, we get a simple fraction $c_\nu = 1/4$. In fact, if we set eq.(63a) and eq.(63b) to be the same, we also obtain $c_\nu = 1/4$ and that $t_{\nu 12}^2 = 1$. So, the value $c_\nu = 1/4$ as well as $m_{11} = m_{22}$ are, in this sense, special, calling for extra attention.

We can also explicitly show in numbers that the dihedral symmetry condition, $(TS)^3 = 1$, is satisfied as follows. The group generators are given explicitly as

$$T = \begin{pmatrix}
\frac{1}{4} & 0 & \sqrt{\frac{15}{4}} \\
0 & 1 & 0 \\
\sqrt{\frac{15}{4}} & 0 & -\frac{1}{4}
\end{pmatrix},$$

$$S = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{\sqrt{24}}{5} & \frac{1}{5} \\
0 & \frac{1}{5} & -\frac{\sqrt{24}}{5}
\end{pmatrix}\begin{pmatrix}
\frac{1}{4} & \sqrt{15} & 0 \\
\sqrt{15} & -\frac{1}{4} & 0 \\
0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{\sqrt{24}}{5} & \frac{1}{5} \\
0 & \frac{1}{5} & -\frac{\sqrt{24}}{5}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{4} & 3\sqrt{\frac{10}{5}} & 0 \\
\frac{3\sqrt{10}}{10} & -\frac{1}{5} & -\frac{\sqrt{6}}{10} \\
\sqrt{\frac{15}{4}} & -\sqrt{\frac{6}{10}} & \frac{19}{20}
\end{pmatrix},$$

then

$$TS = \begin{pmatrix}
\frac{1}{4} & 0 & \sqrt{\frac{15}{4}} \\
\frac{3\sqrt{10}}{10} & -\frac{1}{5} & -\frac{\sqrt{6}}{10} \\
\frac{\sqrt{15}}{20} & \frac{2\sqrt{5}}{5} & -\frac{1}{20}
\end{pmatrix},$$

$$(TS)^2 = \begin{pmatrix}
\frac{1}{4} & \frac{3\sqrt{10}}{10} & \sqrt{\frac{15}{4}} \\
0 & -\frac{1}{5} & \frac{2\sqrt{5}}{5} \\
\frac{\sqrt{15}}{4} & -\sqrt{\frac{6}{10}} & -\frac{1}{20}
\end{pmatrix} = (TS)^T = ST. \quad (66b)$$

So, under this $D_3(T, S)$, since five (out of total six) mass parameters of $m_D$ except the overall mass scale given by $m_{22}$ are fixed by the symmetry, the neutrino mixing angles are to be completely fixed by the symmetry without any additional free parameter left over.

**Diagonalizing $m_D$**

Having established the fact that the symmetry indeed constrains the Dirac neutrino mass matrix $m_D$ to be the desired form given in eq.(9b), we can now diagonalize $m_D$ as

$$U_\nu^{-1}m_D U_\nu = m_{Dd} = \text{diag}(m_1, m_2, m_3) \quad (67)$$

with the theoretical neutrino mixing matrix

$$U_\nu(\theta_{\nu 23}, \theta_{\nu 12}) = \begin{pmatrix}
c_{\nu 12} & s_{\nu 12} & 0 \\
-s_{\nu 23}s_{\nu 12} & c_{\nu 23}c_{\nu 12} & s_{\nu 23} \\
s_{\nu 23}s_{\nu 12} & -s_{\nu 23}c_{\nu 12} & c_{\nu 23}
\end{pmatrix}, \quad (68)$$
where $\theta_{\nu23}$ and $\theta_{\nu12}$ depend on the choice of $m_1$ and $m_2$ from $m_\pm$ given by

$$m_\pm \equiv \frac{1}{2} \left( m_{11} + m_{22} \pm \sqrt{(m_{11} - m_{22})^2 + 4(m_{12}^2 + m_{13}^2)} \right). \quad (69)$$

If $m_1 = m_-$, $\theta_{\nu12}$ and $\theta_{\nu23}$ are given by

$$c_{\nu23} = \frac{m_{12}}{\sqrt{m_{12}^2 + m_{13}^2}}, \quad (70a)$$

$$s_{\nu23} = -\frac{m_{13}}{\sqrt{m_{12}^2 + m_{13}^2}}, \quad (70b)$$

$$c_{\nu12} = -\frac{m_- - m_{22}}{\sqrt{(m_- - m_{22})^2 + m_{12}^2 + m_{13}^2}} = \frac{\sqrt{m_{12}^2 + m_{13}^2}}{\sqrt{(m_+ - m_{22})^2 + m_{12}^2 + m_{13}^2}}, \quad (70c)$$

$$s_{\nu12} = \frac{m_+ - m_{22}}{\sqrt{(m_- - m_{22})^2 + m_{12}^2 + m_{13}^2}} = \frac{m_{12}}{\sqrt{(m_+ - m_{22})^2 + m_{12}^2 + m_{13}^2}}, \quad (70d)$$

where we have used an identity $(m_+ - m_{22})(m_- - m_{22}) = -(m_{12}^2 + m_{13}^2)$ for the second equality in the latter two equations. But, if $m_1 = m_+$, we get the opposite for $\theta_{\nu12}$ as

$$c_{\nu23} = -\frac{m_{12}}{\sqrt{m_{12}^2 + m_{13}^2}}, \quad (71a)$$

$$s_{\nu23} = \frac{m_{13}}{\sqrt{m_{12}^2 + m_{13}^2}}, \quad (71b)$$

$$s_{\nu12} = -\frac{m_- - m_{22}}{\sqrt{(m_- - m_{22})^2 + m_{12}^2 + m_{13}^2}} = \frac{\sqrt{m_{12}^2 + m_{13}^2}}{\sqrt{(m_+ - m_{22})^2 + m_{12}^2 + m_{13}^2}}, \quad (71c)$$

$$c_{\nu12} = \frac{m_+ - m_{22}}{\sqrt{(m_- - m_{22})^2 + m_{12}^2 + m_{13}^2}} = \frac{m_{12}}{\sqrt{(m_+ - m_{22})^2 + m_{12}^2 + m_{13}^2}}. \quad (71d)$$

Then mass eigenvalues, if we demand $m_2^2 > m_1^2$, are given by

$$m_{\text{Dd}} = \begin{cases} 
\text{diag}(m_1 = m_-, m_2 = m_+, m_3 = m_{22}), & \text{for } m_{11} + m_{22} > 0, \\
\text{diag}(m_1 = m_+, m_2 = m_-, m_3 = m_{22}), & \text{for } m_{11} + m_{22} < 0.
\end{cases} \quad (72a)$$

Note that for $s_{\nu23} > 0$ and $c_{\nu23} > 0$, we should choose the sign of $m_{12}$ and $m_{13}$ accordingly. In either case, $m_{12}$ and $m_{13}$ should have opposite signs such that $m_{12}m_{13} < 0$.

This possibility of having two different choices is the main reason behind the two different identities between the symmetry parameters and $t_{\nu12}$ as in eqs.(41a)(41b) and subsequent equations involving $t_{\nu12}$. This is also eventually related to the fact that there is no constraint currently to tell whether $\theta_{23}$ of the PMNS matrix is smaller or larger than $\pi/4$, even if we restrict the other angle to satisfy $\theta_{12} < \pi/4$. 

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The Physical Dirac Case

If the observed neutrinos are Dirac types, only two symmetry constraints eqs.(5a)(5b) are sufficient. As we have seen, the symmetry constrains not only the form of the mass matrix but also its components. In the pure cyclic case of $C_2(T') \times C_2(T) \times C_2(S)$, eq.(36) implies

$$m_{12} = \frac{s_\nu c_\alpha}{1 - c_\nu} m_{22},$$

and that, with eq.(37), we get

$$m_{13} = \frac{s_\nu (1 + s_\alpha)}{1 - c_\nu} m_{22},$$

while, with eq.(38), we get

$$m_{11} = \frac{2c_\nu + s_\alpha (1 + c_\nu)}{1 - c_\nu} m_{22}.$$ (75)

Then, from eq.(72a), we can express the mass eigenvalues in terms of the symmetry parameters as

$$m_1 = -m_{22}, \quad m_2 = \frac{2 + s_\alpha (1 + c_\nu)}{1 - c_\nu} m_{22}, \quad m_3 = m_{22},$$ (76)

where $m_{22}$ provides over-all mass scale. Under the enhanced symmetry of $C_2(T') \times D_3(T, S)$, there is an additional constraint, eq.(58), which relates $s_\alpha$ to $c_\nu$, then the masses are more strictly constrained as

$$m_1 = -m_{22}, \quad m_2 = \frac{2 + c_\nu}{1 - c_\nu} m_{22}, \quad m_3 = m_{22}.$$ (77)

Note that in either cases, $m_1$ and $m_3$ are degenerate, which contradicts with the measurement of $\Delta m_{31}^2 \neq 0$. This situation does not change even with eq.(72b) because it only swaps the values of $m_1$ and $m_2$.

Therefore, the symmetry we have imposed disfavors the physical Dirac neutrinos unless there is a mechanism to break this global symmetry to lift the degeneracy. However, in the following Majorana case, this degeneracy can be lifted.

The Majorana Case

In the Majorana case, the physical left-handed neutrino masses are due to the see-saw mechanism w.r.t. the right-handed Majorana neutrino masses. The right-handed Majorana neutrino mass matrix must satisfy eq.(5c), which implies that

$$[m_R, S] = 0.$$ (78)

Since $S$ is a real symmetric (or hermitian) matrix, it can be easily diagonalized as

$$S_d = V^{-1} S V$$ (79)
by an orthogonal (or unitary) matrix $V$. Then we can easily generate $m_R$ with three different
eigenvalues, such that $m_{Rd} = \text{diag}(m_{R1}, m_{R2}, m_{R3})$, using the similarity transformation that
diagonalizes symmetric $S$, as

$$m_R = V m_{Rd} V^{-1}$$

such that eq.(78) is satisfied by construction. So, after the see-saw mechanism, we can have
non-degenerate left-handed Majorana neutrinos masses given by

$$m_{L1} = \frac{m^2_1}{m_{R1}}, \quad m_{L2} = \frac{m^2_2}{m_{R2}}, \quad m_{L3} = \frac{m^2_3}{m_{R3}},$$

where $m_i$ are from either eq.(76) or eq.(77) depending on the symmetry.

In most of cases we consider in this paper, the Dirac neutrino mass eigenvalues satisfy the
ratio $m^2_1 : m^2_2 : m^2_3 \sim 1 : 3^2 : 1$, hence the left-handed Majorana mass eigenvalues satisfy

$$m^2_{L1} : m^2_{L2} : m^2_{L3} \sim \frac{1}{m^2_{R1}} : \frac{81}{m^2_{R2}} : \frac{1}{m^2_{R3}}.$$  \hspace{1cm} (82)

For the normal hierarchy, i.e. $m^2_{R3} \gg m^2_{R2} \gg m^2_{R1}$, the ratio of the right-handed neutrino masses are

$$m^2_{L1} : m^2_{L2} : m^2_{L3} \sim 1 : 2 : 32 \sim \frac{1}{m^2_{R1}} : \frac{81}{m^2_{R2}} : \frac{1}{m^2_{R3}},$$

or

$$m_{R1} : m_{R2} : m_{R3} \sim 5.7 : 36 : 1 \sim 6 : 36 : 1,$$ \hspace{1cm} (83)

or

$$m_{R1} : m_{R2} : m_{R3} \sim 1 : 8.9 : 5.6 \sim 1 : 9 : 6$$ \hspace{1cm} (84)

to meet the observed mass relations, eq.(4). For the inverted hierarchy, i.e. $m^2_{L2} \gg m^2_{L1} \gg m^2_{L3}$, they are now

$$m^2_{L1} : m^2_{L2} : m^2_{L3} \sim 31 : 32 : 1 \sim \frac{1}{m^2_{R1}} : \frac{81}{m^2_{R2}} : \frac{1}{m^2_{R3}}$$ \hspace{1cm} (85)

or

$$m_{R1} : m_{R2} : m_{R3} \sim 1 : 8.9 : 5.6 \sim 1 : 9 : 6$$ \hspace{1cm} (86)

So the symmetry can constrain the relative ratios of the right-handed Majorana mass eigenvalues. From the naturalness point of view, the inverted hierarchy may be slightly favored, but the difference is not significant.

5. The PMNS Matrix

Now we have all ingredients and ready to compute the PMNS matrix. For this purpose, let us recall and summarize the relations between the mixing angles and the symmetry parameters.
For $C_2(T')$, $\theta_{\ell 13}$ is given by eq.(29) with $M_{11} = M_\mu$, i.e. the ratio in terms of of physical charged lepton masses as

$$s_{\ell 13}^2 = \frac{M_\mu - M_e}{M_\tau - M_e} \approx \frac{M_\mu}{M_\tau}, \quad (87)$$

The neutrino part of the mixing angles are such that, under pure cyclic groups $C_2(T) \times C_2(S)$, from eqs.(40),

$$s_{\nu 23}^2 = \frac{1 + s_\alpha}{2}, \quad (88)$$

and from eqs.(41a)(41b) we have

$$t_{\nu 12}^2 = \begin{cases} \frac{1 - c_\nu}{1 + c_\nu}, & \text{for } m_1 = m_-, \\ \frac{1 + c_\nu}{2}, & \text{for } m_1 = m_. \end{cases} \quad (89a)$$

Under the enhanced symmetry with the dihedral group $D_3(T, S)$, $s_\alpha$ is no longer independent from $c_\nu$ such that, with eq.(62), we have

$$s_{\nu 23}^2 = \frac{2c_\nu + 1}{2(1 + c_\nu)}, \quad (90)$$

and, from eqs.(63a)(63b)

$$t_{\nu 12}^2 = \begin{cases} \frac{2(1 - c_\nu)}{2c_\nu + 1}, & \text{for } m_1 = m_-, \\ \frac{2c_\nu + 1}{2(1 - c_\nu)}, & \text{for } m_1 = m_. \end{cases} \quad (91a)$$

So the mixing angles are entirely given in terms of the symmetry parameters and, as the symmetry gets enhanced, they becomes more restrictive.

If we further choose eq.(33) such that symmetry parameters of $T$ and $T'$ are related as

$$c_\nu = t_{\ell 13}, \quad (92)$$

even the symmetry parameters are no longer free at all in the case of $C_2(T') \times D_3(T, S)$. So it is much desirable to look for an enhanced symmetry in which $C_2(T')$ and $D_3(T, S)$ are combined into a single group satisfying $c_\nu = t_{\ell 13}$.

The PMNS matrix in our model is given in eq.(8) as

$$U_{\text{PMNS}} = U_{\ell}^\dagger(\theta_{\ell 13})U_{\nu}(\theta_{\nu 23}, \theta_{\nu 12}), \quad (93)$$

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which eventually will only depend on the symmetry parameters. Using eq.(26) and eq.(68), we can compare the r.h.s. to the PMNS matrix in the standard (PDG) parametrization as

\[
U_{\text{PMNS}} = \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13} \\
-(s_{12}c_{23} + c_{12}s_{23}s_{13}) & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{13} \\
 s_{12}s_{23} - c_{12}c_{23}s_{13} & -(c_{12}s_{23} + s_{12}c_{23}s_{13}) & c_{23}s_{13}
\end{pmatrix}
\]

Then we can obtain the mixing angles of the PMNS matrix in terms of the our theoretical mixing angles of charged lepton and Dirac neutrino mass matrices as

\[
s_{13} = s_{\ell 13}c_{\nu 23}, \quad t_{23} = \frac{t_{\nu 23}}{c_{\ell 13}}, \quad t_{12} = \frac{t_{\nu 12} - t_{\ell 13}s_{\nu 23}}{1 + t_{\ell 13}s_{\nu 23}t_{\nu 12}}.
\]

Eq.(95a) implies that the smallness of \( s_{13} \) is related to the smallness of the muon-to-tau mass ratio.

There are three symmetry parameters: \( \theta_\ell = \theta_{\ell 13}/2 \) for \( C_2(T') \), \( \theta_\nu \) for \( C_2(T) \), and additional \( \theta_\alpha \) for \( C_2(S) \). Although our main interest is the most restrictive case (Case IV below), but for comparison purpose we will consider the following four different cases:

- **Case I:** \( C_2 \times C_2 \times C_2 \) with all independent \( \theta_\ell, \theta_\nu, \) and \( \theta_\alpha \).
- **Case II:** \( C_2 \times C_2 \times C_2 \) with \( c_\nu = t_{\ell 13} \), but independent \( \theta_\alpha \).
- **Case III:** \( C_2 \times D_3 \) with only \( \theta_\ell \) and \( \theta_\nu \) independent.
- **Case IV:** \( C_2 \times D_3 \) with \( c_\nu = t_{\ell 13} \) such that all symmetry parameters are related.

Note that, in the following, unless specifically mentioned, we use eq.(91a) for \( t_{\nu 12} \).

**Case I**

With simple choices \( s_{\ell 13}^2 = \widetilde{M}_\mu/\widetilde{M}_\nu \simeq M_\mu/M_\nu \simeq 0.05919, \ c_\nu = 1/4 \) and \( s_\alpha = 1/5 \) such that \( s_{\ell 23}^2 = 3/5 \) and \( t_{\nu 12}^2 = 1 \), we can obtain \( s_{12}^2 \simeq 0.313, \ s_{13}^2 \simeq 0.0237 \) and \( s_{23}^2 \simeq 0.615 \). These agree with the best-fit values given in eq.(3) impressively well, albeit with \( \theta_{23} > \pi/4 \).

Note that the current observed data do not provide any information whether \( \theta_{23} \) is smaller or larger than \( \pi/4 \), so \( s_{23}^2 \simeq 0.61 \) is equally acceptable as \( s_{23}^2 \simeq 0.39 \). Nevertheless, we can obtain \( s_{23}^2 \simeq 0.4 \), too. For example, if we choose \( c_\nu \simeq 0.55, \ s_\alpha \simeq -0.24 \) and \( s_{\ell 13}^2 \simeq 0.04 \) such
that $t_{\nu 12}^2 \simeq 0.76$ and $s_{\nu 23}^2 \simeq 0.38$, then we obtain $s_{\nu 23}^2 \simeq 0.39$, $s_{\tau 12}^2 \simeq 0.312$ and $s_{\tau 13}^2 \simeq 0.0248$. However, $s_{\ell 13}^2$ needs to be much smaller to obtain the needed $s_{\ell 13}^2$, and no longer related to the muon-to-tau mass ratio.

If we assume $c_{\nu} = 1/3$ and $s_{\alpha} = 0$ to get $s_{\nu 23}^2 = 1/2$ and $t_{\nu 12}^2 = 1$, i.e. BM (bimaximal)[14]. For $s_{\ell 13}^2 \simeq M_{\mu}/M_{\tau} \simeq 0.05919$, we obtain $s_{\tau 12}^2 \simeq 0.328$, $s_{\tau 13}^2 \simeq 0.0296$ and $s_{\nu 23}^2 \simeq 0.515$, but $s_{\ell 13}^2$ about $2\sigma$ away and $s_{\nu 23}^2$ is more than $\sigma$ away. Even if we relax $s_{\ell 13}^2$ to different values, we cannot get better results. So, although it may not be ruled out, it is not as good as other cases we consider.

**Case II**

In this case, $c_{\nu}$ is no longer independent but given by eq.(33), $c_{\nu} = t_{\ell 13}$. Then, for $s_{\ell 13}^2 \simeq M_{\mu}/M_{\tau} \simeq 0.05919$, we have $c_{\nu} \simeq 0.2508$. Now we choose $s_{\alpha} = 1/5$ such that $s_{\nu 23}^2 \simeq 3/5$ and $t_{\nu 12}^2 = 0.998$. Then we obtain $s_{\tau 12}^2 \simeq 0.312$, $s_{\nu 23}^2 \simeq 0.615$ and $s_{\ell 13}^2 \simeq 0.0237$.

If we relax $M_{11}$ so that $M_{11} \neq M_{\mu}$ is allowed, then we can choose $c_{\nu} = 1/4 = t_{\ell 13}$, which leads to $s_{\ell 13} = 0.2425$ Then, with $s_{\alpha} = 1/5$, we can obtain $s_{\nu 23}^2 = 3/5$, and that $s_{\tau 12}^2 \simeq 0.313$, $s_{\nu 23}^2 \simeq 0.614$ and $s_{\ell 13}^2 \simeq 0.0235$.

Note that, compared to the $s_{\nu 23}^2 \simeq 0.6$ case, a larger $c_{\nu}$ is needed to obtain $s_{\tau 12}^2 \simeq 0.31$ if $s_{\nu 23}^2$ and $s_{\nu 23}^2$ are smaller. But, $c_{\nu} = t_{\ell 13}$ forbids larger $c_{\nu}$ without making $s_{\ell 13}^2$ become larger, so Case II favors $s_{\nu 23}^2 \simeq 0.6$.

**Case III**

Once the symmetry $C_2(T) \times C_2(S)$ is enhanced to $D_3(T, S)$, $s_{\alpha}$ is no longer independent so that $s_{\ell 13}$ and $c_{\nu}$ can control all mixing angles.

For $s_{\ell 13}^2 \simeq M_{\mu}/M_{\tau} \simeq 0.05919$, if we choose $c_{\nu} = 1/4$, then $s_{\nu 23}^2 = 3/5$ and $t_{\nu 12}^2 = 1$ so that we can obtain $s_{\tau 12}^2 \simeq 0.313$, $s_{\nu 23}^2 \simeq 0.615$ and $s_{\ell 13}^2 \simeq 0.0237$.

One interesting case which may have more geometrical connection is to choose $\theta_{\nu} = 75^\circ$ such that $c_{\nu} = (\sqrt{3} - 1)/2\sqrt{2} \simeq 0.2588$, then $s_{\nu 23}^2 \simeq 0.603$ and $t_{\nu 12}^2 \simeq 0.977$ so that we can obtain $s_{\tau 12}^2 \simeq 0.307$, $s_{\nu 23}^2 \simeq 0.617$ and $s_{\ell 13}^2 \simeq 0.0235$. $75^\circ$ happens to be an half of the inner angle of a dodecagon, but at this moment we do not know if this has any significance from the symmetry point of view.

**Case IV**

Now even $c_{\nu}$ is no longer independent but given by $c_{\nu} = t_{\ell 13} \simeq 0.2508$ for $s_{\ell 13}^2 \simeq M_{\mu}/M_{\tau} \simeq$
Case I is with independent $c_\nu$ and $s_\alpha$, Case II is with $c_\nu = t_{\ell 13}$. For $C_2 \times D_3$, Case III is with independent $c_\nu$, Case IV is with $c_\nu = t_{\ell 13}$. The numbers in brackets for $s_{\ell 13}, c_\nu$ and $s_\alpha$, are dependent numbers.

0.05919. Then $s^2_{\nu 23} \simeq 0.6$ and $t^2_{\nu 12} \simeq 0.998$ such that we obtain $s^2_{12} \simeq 0.312$, $s^2_{23} \simeq 0.615$ and $s^2_{13} \simeq 0.0237$. Physically, this is the most plausible case because $\tilde{\theta}_{13}$ is related to the muon-to-tau mass ratio. However, it is hard to see if the symmetry we impose has any connection to a geometry, which often enables us to justify the origin of the symmetry.

For a close simple fraction, if we choose $c_\nu = 1/4$, which leads to $s_{\ell 13} = 0.2425$, $s^2_{\nu 23} = 3/5$ and $t^2_{\nu 12} = 1$, then we obtain $s^2_{12} \simeq 0.313$, $s^2_{23} \simeq 0.614$ and $s^2_{13} \simeq 0.0235$.

If we choose $\theta_\nu = 75^\circ$, then $c_\nu = (\sqrt{3} - 1)/2\sqrt{2} \simeq 0.2588$ and $s_{\ell 13} \simeq 0.2506$. We get $s^2_{\nu 23} \simeq 0.603$ and $t^2_{\nu 12} \simeq 0.977$. This leads to $s^2_{12} \simeq 0.301$, $s^2_{23} \simeq 0.618$ and $s^2_{13} \simeq 0.0249$.

So far, all the examples in Case IV favor $s^2_{23} \simeq 0.6$, and we have already noticed in Case I that we have to make quite difference choices to obtain $s^2_{23} \simeq 0.4$. In Case IV we can also find out what kind of choice we have to make to obtain $\theta_{23} < \pi/4$ and at what expense. From eq.(91b) if we let $s_{\ell 13} = -\sqrt{0.036} \simeq -0.190$, i.e. $c_\nu = t_{\ell 13} \simeq -0.193$, then $s^2_{\nu 23} \simeq 0.38$ and $t^2_{\nu 12} \simeq 0.257$ so that we can obtain $s^2_{23} \simeq 0.389$, $s^2_{13} \simeq 0.022$ and $s^2_{12} \simeq 0.308$. Again, $\theta_{\ell 13}$ is no longer related to the muon-to-tau mass ratio.

Some of these are summarized in the Table 1 for a quick view.

### 6. Discussions

We have shown that there is a new discrete symmetry which leads closely to the observed PMNS matrix with the small $s^2_{13}$ being related to the small muon-to-tau mass ratio (to be precise, if $\theta_{23} > \pi/4$). In the absence of the CP-violation, the combined number of parameters in both (hermitian) mass matrices are twelve, and all of them are fixed by, at most, three symmetry

| Case | $s^2_{\ell 13}$ | $c_\nu$ | $s_\alpha$ | $\sin^2(2\theta_{12})$ | $\sin^2(2\theta_{23})$ | $\sin^2(2\theta_{13})$ |
|------|----------------|---------|-----------|----------------------|----------------------|----------------------|
| I    | $M_\mu/M_\tau$ | 1/4     | 1/5       | 0.860                | 0.948                | 0.0925               |
| II   | $M_\mu/M_\tau$ | (0.2508) | 1/5       | 0.859                | 0.948                | 0.0925               |
| III  | $M_\mu/M_\tau$ | 1/4     | (1/5)     | 0.860                | 0.948                | 0.0925               |
| IV   | $M_\mu/M_\tau$ | (0.2508) | (0.201)   | 0.859                | 0.947                | 0.0924               |
| IV   | (0.0588)       | 1/4     | (1/5)     | 0.861                | 0.948                | 0.0919               |

Table 1: Comparison between our prediction and the best fit given by [8]. For $C_2 \times C_2 \times C_2$, Case I is with independent $c_\nu$ and $s_\alpha$, Case II is with $c_\nu = t_{\ell 13}$. For $C_2 \times D_3$, Case III is with independent $c_\nu$, Case IV is with $c_\nu = t_{\ell 13}$. The numbers in brackets for $s_{\ell 13}, c_\nu$ and $s_\alpha$, are dependent numbers.
parameters, depending on the symmetry. In the most restrictive case of $C_2(T') \times D_3(T, S)$ with $c_\nu = t_{\ell 13}$, just one symmetry parameter $\theta_\ell = 2\theta_{\ell 13}$ ends up controlling all twelve parameters of mass matrices and three mixing angles of the PMNS matrix. In other words, one charged lepton mixing angle $\theta_{\ell 13}$, which itself is given in terms of the physical charged lepton masses as $\theta_{\ell 13} = \sin^{-1}\sqrt{(M_\mu - M_e)/(M_\tau - M_e)} \simeq \sin^{-1}\sqrt{M_\mu/M_\tau}$, actually controls all parameters. So there is no free parameter and no fine-tuning involved, hence it should be sufficient for us to call our production of mixing angles as a prediction, which happens to match the currently observed data extremely well.

In this paper, the choice $c_\nu = t_{\ell 13}$ is not based on any symmetry relations, so it will be very interesting to find out if there is a larger symmetry enhanced from $C_2(T') \times D_3(T, S)$, under which this shows up as an identity. We believe such a symmetry exists.

Based on the success of the symmetry we have used in this paper, understanding the origin of this $C_2(T') \times D_3(T, S)$ from more fundamental point of view could be the key to find the correct model to go beyond the SM. Since our symmetry generating group elements have negative determinants, our discrete group cannot be a subgroup of compact gauge symmetry groups. However, this type of finite groups might rise from a lattice compactification of string theory. So it will be interesting to investigate such a possibility.

There is an interesting aspect of our model on charged lepton masses because our model also relates charged lepton masses to the symmetry. The charged lepton mass matrix given in eq. (9a) with $M_{11} = M_\mu$ has largest mass ratio only about 16, but it reproduces the large mass hierarchy $M_\tau/M_e \sim 10^3$. So, in some sense, the lepton mixing under $C_2(T')$ justifies the lepton mass hierarchy. Furthermore, if the tau mass had not been known, we could have actually predicted it based on the symmetry we impose here. For example, let $c_\nu = t_{\ell 13} = 1/4$ and $M_{11} = M_\mu$, then we can estimate $M_\tau$ in terms of $M_e$ and $M_\mu$ as $M_\tau = 17M_\mu - 16M_e \simeq 1788$ MeV. This is less than 1 % off from the measured $M_\tau \simeq 1776.82 \pm 0.16$ MeV. Even though it is still $70 \sigma$ away, the outcome is quite intriguing. It will be interesting to find out if there is any deeper meaning to this. For example, $\theta_{\ell 13} = \tan^{-1}(1/4)$ could be satisfied with the tree level values of charged leptons, or at some energy scale.

Another interesting aspect of our model is that the symmetries we impose favor Majorana neutrinos because they symmetries force Dirac neutrino masses be degenerate. As is well known, the Majorana neutrinos are also favored because we do not have to assume unnaturally small Yukawa coupling constants unlike the Dirac case. The symmetry we impose also puts a
constraint on the right-handed neutrino mass matrix such that $[S, m_R] = 0$.

In this paper, we have not considered the CP-violation, but it should be straightforward to include it. For example, in the case for $s_{13} = -\sqrt{0.036} < 0$ to obtain $\theta_{23} < \pi/4$, we actually get $s_{13} < 0$ so that this can be interpreted as $s_{13} > 0$ with $\delta = \pi$. Furthermore, since our model favors the Majorana neutrinos, there should be three CP-violating phases, which can be easily introduced because our symmetry generating transformation matrices can be all hermitian. We will leave details of this as a future work.

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**References**

[1] K. Nakamura and S.T. Petcov, in Particle Data Group, Phys. Rev. D 86, 010001 (2012) (http://pdg.lbl.gov/2012/reviews/rpp2012-rev-neutrino-mixing.pdf) and references therein.

[2] A. de Gouvea et al. [Intensity Frontier Neutrino Working Group Collaboration], arXiv:1310.4340 [hep-ex].

[3] B. Pontecorvo, Sov. Phys. JETP 7, 172 (1958) [Zh. Eksp. Teor. Fiz. 34, 247 (1957)].

[4] B. Pontecorvo, Sov. Phys. JETP 6, (1957) 429; Zh. Eksp. Teor. Fiz. 33 (1957) 549; Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. 28 (1962) 870; B. Pontecorvo, Sov. Phys. JETP 26 (1968) 984; Zh. Eksp. Teor. Fiz. 53 (1968) 1717; V. N.Gribov and B. Pontecorvo, Phys. Lett. B 28 (1969) 493.

[5] H. La, “Hidden Neutrino Gauge Symmetry,” arXiv:1209.1377 [hep-ph].

[6] H. La, “Flavor Violating Lepton Family U(1)$_\lambda$,” arXiv:1308.0088 [hep-ph].

[7] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530, 167 (2002) [hep-ph/0202074].

[8] Particle Data Group, http://pdg.lbl.gov/rpp2013v2/pdgLive/

[9] M. Hamermesh, *Group Theory and Its Application to Physical Problems*, (Dover, New York, 1962).

[10] H. La and T.J. Weiler, “A Simple Ansatz for Neutrino Mixing,” to appear.
[11] S. F. King and C. Luhn, “Neutrino Mass and Mixing with Discrete Symmetry,” Rept. Prog. Phys. 76, 056201 (2013) [arXiv:1301.1340 [hep-ph]].

[12] G. Altarelli and F. Feruglio, Rev. Mod. Phys. 82, 2701 (2010) [arXiv:1002.0211 [hep-ph]].

[13] W. Grimus and P. O. Ludl, J. Phys. A 45, 233001 (2012) [arXiv:1110.6376 [hep-ph]].

[14] V. D. Barger, S. Pakvasa, T. J. Weiler and K. Whisnant, Phys. Lett. B 437, 107 (1998) [hep-ph/9806387].