Under the premise that the current observations of the cosmic background radiation set a very stringent limit to the anisotropy of the universe, we present a proposal where the dark side of the universe is represented by one parameter, \(m_{\phi}\), with the aim of having a time-varying cosmological term \(\Lambda(t)\) in the dust epoch within an anisotropic cosmology and from there obtaining a scalar field potential that gives the inflationary behavior and isotropy to this day, we introduce the fluctuation deceleration parameter \(\Delta q(t)\) obtaining a negative value, where we consider two epoch in our universe, stiff and dust scenarios, which indicate that the universe has growing expansion in its average function volume. The main idea arises by the hypothesis that the cosmological term \(\Lambda\) is identified with the scalar potential as \(V(\phi(t)) = 2\Lambda(t)\). As a consequence of scaling solutions between the energy density of the scalar field and ordinary matter, exact solutions of the field equations are obtained by a special ansatz to solve the Einstein-Klein-Gordon (EKG) equation and the particular potential obtained by this approach. We use Misner’s variables considering a decomposition in an isotropic and an anisotropic part. We employ the Lagrangian formalism for a scalar field \(\phi\) with standard kinetic energy and arbitrary scalar potential \(V(\phi)\).

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I. INTRODUCTION

The present phase of an accelerated expansion of the universe stands as one of the most challenging open problems in modern cosmology and astrophysics considering the isotropic cosmological model, Friedmann-Robertson-Walker (FRW), as the model for excellency. This acceleration is characterized by which is popularly known as dark energy. Any theoretical model should be corroborated with observations for understanding the viability of that model. A number of cosmological observations (Knop et al. 2003; Riess et al. 2004; Tegmark et al. 2004; Spergel et al. 2006) indicate altogether that our universe is presently under a phase of accelerated expansion.

Among many possible alternatives, the simplest candidate for dark energy is the vacuum energy which is mathematically equivalent to the cosmological constant. Models with different decay laws for the variation of the cosmological term was investigated during the last two decades in a non covariant way, (Chen & Wu 1990); (Abdel 1990); (Pavon 1991); (Carvalho et al 1992); (Kalligas et al 1992); (Lima & Maia 1994); (Lima & Carvalho 1994); (Lima & Trodden 1996); (Arbab & Abdel 1994); (Birkel & Sarkar 1997); (Silveira & Waga 1997); (Starobinsky 1998); (Overduin & Cooperstock 1998); (Vishwakarma 2000,2001); (Arbab 2001,2003,2004); (Cunha & Santos 2004); (Carneiro & Lima 2005); (Fomin et al 2005); (Sola & Stefancic 2005,2006); (Pradhan et al 2007); (Jamil & Debnath 2011) and (Mukhopadhyay 2011); in particular, in Fomin et al (2005) there are several evolution relations for \(\Lambda\) which many author have used, also in Ref. Overduin & Cooperstock (1998) appears a table with these relations and the corresponding references.
where they were considered.

The observable high homogeneity of the Universe and its isotropy do not guarantee the specific properties today. On the other hand, the current observations of the cosmic background radiation set a very stringent limit to the anisotropy of the Universe (Martinez & Saez 1995), therefore it is important to consider the anisotropy cosmological model of the Universe which became isotropic during evolution (Belinskii & Khalatnikov 1972, Folomeev & Gurovich 2000).

Therefore, there is a natural desire to build an anisotropic cosmological model with a scalar field that has the advantages mentioned above within the Friedmann model with a scalar field and to analyze a possibility of its approach to the isotropic variant with the accuracy required by the observations. One of the objectives for our work is that the anisotropic parameter, defined using a Misner’s parametrization variant, becomes a constant when the cosmic time \( t \) tends to \( \infty \) (until the dust scenario) due that in the stiff matter (previous scenario) this isotropic behavior does not emerge yet. In addition, anisotropic cosmological models have been treated in this formalism from different points of view. (Aroonkumar 1993, 1994); (Arbab 1997); (Singh et al 1998); (Pradhan & Kumar 2001); (Pradhan 2003, 2007, 2009); (Pradhan & Pandey 2003, 2006); (Saha 2006) (Pradhan et al 2007, 2008, 2012, 2015); (Carneiro 2005); (Esposito et al 2007); (Bal & Singh 2008); (Belinchón 2008); (Singh et al 2008), (Singh et al 2013); (Shen 2013); (Tripathy 2013) and (Rahman & Ansary 2013).

In this paper, we introduce a combination of results using two different approaches. The first one has to do with the ideas of scaling behavior, where a set of variables is used in the transformation formalism (in connection with Misner’s transformations) so that the EKG equation is simplified and we can obtain an *isotropized cosmological model in the dust stage*; the second one, uses the ideas to obtain a cosmological term in an anisotropic universe in such a way that a scalar field potential that mimics the cosmological constant to this day may emerge, in the sense that the anisotropic cosmological model has the isotropization. In this order of ideas, and following closely a previous work of one the authors (Socorro et al 2015), the solutions for the stiff matter case are obtained, showing that the isotropization of the model is not achieved because the scalar potential is zero. In this way, we present an analysis in a covariant way using an anisotropic cosmological model *employing a Misner’s parametrization variant*, using the Lagrangian density of standard scalar field. The main idea arises by proposing that as in the cosmological constant case, the scalar potential is identified as \( V(\phi) = 2\Lambda \), with \( \Lambda \) a constant. So, we extend this idea and suggest that this correspondence is valid even when this cosmological term has a temporal dependence, i.e., \( V(\phi(t)) = 2\Lambda(t) \).

We include as a toy model a barotropic equation state between the pressure and energy density of the scalar field, \( p_\phi = \omega_\phi \rho_\phi \), quantities that we shall define in the following lines. In order to built up the analysis presented here, initially we solve the Klein-Gordon equations, whose solution implies that the energy density of a scalar field has a wide range of scaling behavior, \( \rho_\phi \sim A^{-m} \) with \( A \) the scale factor of the FRW model, (Ferreira & Joyce 1998); (Liddle & Sharrer 1998) and (Copeland et al 1998), that emerges as a proportionality law between the energy density of the scalar field and the energy density of the barotropic perfect fluid, relation that is known as an “attractor solution”, with the proportionality constant \( m_\phi \) (Liddle & Sharrer 1998), that is, \( \rho_\phi = m_\phi \rho \). In the reference by Singh (Singh H.P, 2008) to do the mention that the transition from a decelerated phase to accelerated stage of evolution can be due to the domination of dark energy over other kinds of matter elds, however this indication is indeed reassuring as the formation of structure in the universe is better supported by a decelerating model. To give a precise form for this reasoning, the model requires both decelerated and an accelerated phase of expansion. For this purpose, we need to have some form of elds that governs the dynamics of the universe in a such a way that deceleration parameter \( q \), being positive in the early epoch, becomes negative at late times. We introduce the fluctuation in deceleration parameter \( \Delta q(t) \), because in our toy model, both scenarios studied (stiff and dust epoch) gives a positive deceleration
parameter, this calculation we shall present it. In this sense, the parametrization of the dark matter-energy content of the universe will be given by the parameter \( m_\phi \) as follows: \( m_\phi = 5 \) for dark matter and \( m_\phi = 15 \) for dark energy. We expect that the evolution of the isotropic volume is different for each of these values, having a more significant growth for \( m_\phi = 15 \) than \( m_\phi = 5 \). However, the nature of the composition of the scalar field is unknown since this type of matter cannot be detected by the usual methods. Recently, one of the authors presented an analysis in covariant way (Socorro et al. 2015), where the cosmological term of the FRW model is used, obtaining the corresponding scalar potential in different scenarios as well as its relationship with the time-dependent cosmological term.

The non covariant treatment begins by giving the cosmological constant (usually taken as a constant) a geometric interpretation. To explain what we mean, consider Einstein’s field equations written as

\[
G_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu},
\]

where \( G_{\mu\nu} \) is the usual Einstein tensor and \( T_{\mu\nu} \) is the energy-momentum tensor of matter. When we take the covariant divergence of equation (1), the vanishing of the Einstein tensor is guaranteed by the Bianchi identities, then it is assumed that the energy-momentum tensor satisfies the corresponding conservation law \( \nabla^\nu T_{\mu\nu} = 0 \), and that the covariant divergence of the cosmological term must vanish, this implies that \( \Lambda = \text{constant} \). Usually, this argument situates this cosmological constant on the left-hand side of the field equations, given a geometrical interpretation of the cosmological term. However, if we consider putting the cosmological term on the right side of the field equations, the interpretation of \( \Lambda \) changes due to the fact that it can be considered as part of the content of matter. Taking this into account, the field equations now are written as

\[
G_{\mu\nu} = -8\pi G \tilde{T}_{\mu\nu}, \quad \tilde{T}_{\mu\nu} = T_{\mu\nu} + \frac{\Lambda}{8\pi G} g_{\mu\nu}.
\]

Once this is done, there is no a priori reason why this cosmological term should not vary, considering that it is the effective energy-momentum tensor \( \tilde{T}_{\mu\nu} \) that satisfies the conservation law

\[
\nabla^\nu \tilde{T}_{\mu\nu} = 0,
\]

the set of equations (2) and (3), together with a single state equation, are the tools needed to carry out this research, since we do not have a Lagrangian density that reproduces field equations (2) and (3).

In the present treatment we take into account the corresponding Lagrangian density with a scalar field

\[
\mathcal{L}[g, \phi] = \sqrt{-g} \left( R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi) \right) + \sqrt{-g} \mathcal{L}_{\text{matter}}
\]

where \( R \) is the Ricci scalar, \( \mathcal{L}_{\text{matter}} \) corresponds to a barotropic perfect fluid, \( p = \omega \rho \), \( \rho \) is the energy density, \( p \) is the pressure of the fluid in the co-moving frame and \( \omega \) is the barotropic constant. The corresponding variation of (4), with respect to the metric and the scalar field, results in the EKG field equations

\[
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\frac{1}{2} \left( \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right) + \frac{1}{2} g_{\alpha\beta} V(\phi) - 8\pi G T_{\alpha\beta},
\]

\[
\Box \phi - \frac{\partial V}{\partial \phi} = 0.
\]

From (5) it can be deduced that the energy-momentum tensor associated with the scalar field is

\[
8\pi G T^{(\phi)}_{\alpha\beta} = \frac{1}{2} \left( \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right) - \frac{1}{2} g_{\alpha\beta} V(\phi),
\]
and the corresponding tensor for a barotropic perfect fluid becomes

\[ T_{\alpha\beta} = (p + \rho) \, u_\alpha u_\beta + g_{\alpha\beta} \rho, \]

(8)

here \( u_\alpha \) is the four-velocity in the co-moving frame, and the barotropic equation of state is \( p = \gamma \rho \).

A. Misner’s parametrization variant

The line element for the anisotropic cosmological Bianchi type I model in the Misner’s parametrization

\[
\begin{align*}
\text{ds}^2 &= -N^2 dt^2 + a_1^2 dx^2 + a_2^2 dy^2 + a_3^2 dz^2, \\
&= -N^2 dt^2 + e^{2\Omega} \left[ e^{2\beta_+ + 2\sqrt{3}\beta_-} dx^2 + e^{2\beta_+ - 2\sqrt{3}\beta_-} dy^2 + e^{-4\beta_+} dz^2 \right]
\end{align*}
\]

(9)

where \( a_i \) (i = 1, 2, 3) are the scale factor on directions (x, y, z), respectively, and N is the lapse function. For convenience, and in order to carry out the analytical calculations, we consider the following representation for the line element (9)

\[
\begin{align*}
\text{ds}^2 &= -N^2 dt^2 + \eta^2 \left[ m_1^2 dx^2 + m_2^2 dy^2 + m_3^2 dz^2 \right],
\end{align*}
\]

(10)

where the relations between both representations are given by

\[
\begin{align*}
\eta &= e^\Omega, \\
m_1 &= e^{\beta_+ + \sqrt{3}\beta_-}, \quad \frac{\dot{m}_1}{m_1} = \dot{\beta}_+ + \sqrt{3}\dot{\beta}_-, \\
m_2 &= e^{\beta_+ - \sqrt{3}\beta_-}, \quad \frac{\dot{m}_2}{m_2} = \dot{\beta}_+ - \sqrt{3}\dot{\beta}_-, \\
m_3 &= e^{-2\beta_+}, \quad \frac{\dot{m}_3}{m_3} = -2\dot{\beta}_+,
\end{align*}
\]

where \( \eta \) is a function that has information regarding the isotropic scenario and the \( m_i \) is a function that has information about the anisotropic behavior of the universe, such that

\[
\begin{align*}
\Pi_{i=1}^3 m_i &= 1, \\
\Pi_{i=1}^3 a_i &= \eta^3, \\
\sum_{i=1}^3 \frac{\dot{m}_i}{m_i} &= 0,
\end{align*}
\]

(11)

act as constraint equations for the model.

II. FIELD EQUATIONS OF THE BIANCHI TYPE I COSMOLOGICAL MODEL

In this section we present the solutions of the field equations for the anisotropic cosmological model, considering the temporal evolution of the scale factors with barotropic fluid and standard matter. The solutions obtained already consider the particular choice of the Misner’s like transformation discussed lines above. Using the metric (10) and a
co-moving fluid, equations (5) take the following form

\[
\begin{align*}
(0) & \quad \frac{\dot{m}_1}{N_m} \frac{\dot{m}_2}{N_m} + \frac{\dot{m}_2}{N_m} \frac{\dot{m}_3}{N_m} + \frac{\dot{m}_1}{N_m} \frac{\dot{m}_3}{N_m} + 3 \left( \frac{\dot{\eta}}{N_\eta} \right)^2 - 8\pi G \rho - \frac{1}{2} \left( \frac{\dot{\phi}^2}{2 N^2} + V(\phi) \right) = 0, \\
(1) & \quad -\frac{\dot{N}}{N^2} \left[ \frac{\dot{m}_2}{N_m} + \frac{\dot{m}_3}{N_m} + 2 \left( \frac{\dot{\eta}}{N_\eta} \right)^2 + \frac{\dot{m}_2}{N_m} \frac{\dot{m}_3}{N_m} \right] + \frac{\dot{m}_1}{N_m} \frac{\dot{m}_3}{N_m} + \frac{\dot{m}_2}{N_m} \frac{\dot{m}_3}{N_m} + 3 \left( \frac{\ddot{\eta}}{N_\eta} \right)^2 \left[ \frac{\dot{m}_1}{N_m} + \frac{\dot{m}_3}{N_m} \right] \\
& \quad + 2 \left( \frac{\dot{\eta}}{N_\eta} \right)^2 + \left( \frac{\dot{\eta}}{N_\eta} \right)^2 + \frac{1}{2} \left( \frac{\dot{\phi}^2}{2 N^2} - V(\phi) \right) + 8\pi G \rho = 0, \\
(2) & \quad -\frac{\dot{N}}{N^2} \left[ \frac{\dot{m}_2}{N_m} + \frac{\dot{m}_3}{N_m} + 2 \left( \frac{\dot{\eta}}{N_\eta} \right)^2 + \frac{\dot{m}_1}{N_m} \frac{\dot{m}_3}{N_m} \right] + \frac{\dot{m}_1}{N_m} \frac{\dot{m}_3}{N_m} + \frac{\dot{m}_2}{N_m} \frac{\dot{m}_3}{N_m} + 3 \left( \frac{\ddot{\eta}}{N_\eta} \right)^2 \left[ \frac{\dot{m}_1}{N_m} + \frac{\dot{m}_3}{N_m} \right] \\
& \quad + 2 \left( \frac{\dot{\eta}}{N_\eta} \right)^2 + \left( \frac{\dot{\eta}}{N_\eta} \right)^2 + \frac{1}{2} \left( \frac{\dot{\phi}^2}{2 N^2} - V(\phi) \right) + 8\pi G \rho = 0, \\
(3) & \quad -\frac{\dot{N}}{N^2} \left[ \frac{\dot{m}_1}{N_m} + \frac{\dot{m}_2}{N_m} + 2 \left( \frac{\dot{\eta}}{N_\eta} \right)^2 + \frac{\dot{m}_1}{N_m} \frac{\dot{m}_2}{N_m} \right] + \frac{\dot{m}_1}{N_m} \frac{\dot{m}_2}{N_m} + \frac{\dot{m}_1}{N_m} \frac{\dot{m}_2}{N_m} + 3 \left( \frac{\ddot{\eta}}{N_\eta} \right)^2 \left[ \frac{\dot{m}_1}{N_m} + \frac{\dot{m}_2}{N_m} \right] \\
& \quad + 2 \left( \frac{\dot{\eta}}{N_\eta} \right)^2 + \left( \frac{\dot{\eta}}{N_\eta} \right)^2 + \frac{1}{2} \left( \frac{\dot{\phi}^2}{2 N^2} - V(\phi) \right) + 8\pi G \rho = 0, 
\end{align*}
\]

here a dot ( ‘ ) represents a time derivative. The corresponding Klein-Gordon (KG) equation is given by

\[
\frac{\ddot{N}}{N} \frac{\phi^2}{N^2} - \frac{\dot{\phi}^2}{N^2} - 3 \left( \frac{\dot{\eta}}{N_\eta} \right)^2 - \dot{V} = 0, 
\]

and the conservation law for the energy-momentum tensor reads

\[
3(\gamma + 1) \frac{\dot{\eta}}{N_\eta} + \frac{\dot{\rho}}{\rho} = 0, 
\]

where \( \rho = \rho_\gamma \eta^{-3(1+\gamma)} \) and \( \rho_\gamma \) is an integration constant that depends on the scenario that is being considered. By setting \( 16\pi G \rho_\phi = \frac{1}{2} \frac{\dot{\phi}^2}{N^2} - V(\phi) \) and \( 16\pi G \rho_\phi = \frac{1}{2} \frac{\dot{\phi}^2}{N^2} + V(\phi) \) as the energy density and pressure of the scalar field, respectively, and a barotropic equation of state for the scalar field of the form \( p_\phi = \omega_\phi \rho_\phi \), we can obtain the kinetic energy \( K = \frac{1+\omega_\phi}{1-\omega_\phi} V(\phi) \), therefore, equation (16) can be written as

\[
\frac{d}{d\tau} \left[ \ln \left( \eta^\phi V^{\frac{2}{1+\omega_\phi}} \right) \right] = 0, \quad \rightarrow \quad V = c_\omega \eta^{-3(1+\omega_\phi)}, 
\]

hence, the scalar field has solutions in quadrature form

\[
\Delta \phi = \alpha_\omega \int \frac{d\tau}{\eta^{\frac{2}{3(1+\omega_\phi)}},} 
\]

where \( c_\omega \) and \( \alpha_\omega \) are appropriate constants for the scenario considered. The relation \( p_\phi = \omega_\phi \rho_\phi \) implies that \( \rho_\phi = \frac{2c_\omega}{\alpha_\omega} \Lambda^{-3(1+\omega_\phi)} \sim \Lambda^{-m} \) where \( m = 3(1+\omega_\phi) \). On the other hand, the solution of the energy-momentum tensor for the perfect fluid \( (\nabla \nu \Gamma^{\mu \nu} = 0) \), gives \( \rho = \rho_\gamma \Lambda^{-3(1+\gamma)} \sim \Lambda^{-n} \), where \( n = 3(1 + \gamma) \), where in principle, the two barotropic indexes \( \omega_\phi \) and \( \gamma \) are different. It is well documented in the literature that in the FRW cosmological model, for the case \( m = n \) the solutions obtained are “attractor solutions” and corresponds when the potential of the scalar field \( \phi \) has an exponential behavior. This case has been studied by several authors, using different methods and where the potential is put by hand (Lucchin & Matarrese1985); (Halliwell 1985); (Burd & Barrow 1988); (Wetterich1998);(Wand et al 1993); (Ferreira & Joyce 1997) and (Copeland et al 1998), in order to understand the evolution of the universe. It turns out that for the anisotropic cosmological model there is no available literature regarding this topic.

In order to solve the set of equations (12)-(17), we consider the case \( m = n \), thus, we have that \( \gamma = \omega_\phi \), and found that the energy density of the scalar field and the energy density of ordinary matter must satisfy the relation
\[ \rho_{\phi} = m_\phi \rho \], where \( m_\phi \) is a positive constant that gives the proportionality between the dark matter-energy and ordinary matter, and as a consequence we find the corresponding potential of the scalar field for a wide range of values of the barotropic index \( \omega_\phi \), where the temporal solution for standard matter is found.

From here on, we introduce the parameter \( \alpha_\phi = 1 + m_\phi \) (this term is not taken into account when we obtain the exact solution with standard matter), a necessary condition to apply our approach with scalar field. To obtain the time dependent solutions of the scale factors we write the EKG equations as

\begin{align}
(0) & \quad \frac{\dot{m}_1}{N m_1} \frac{\ddot{m}_2}{N m_2} + \frac{\dot{m}_2}{N m_2} \frac{\ddot{m}_3}{N m_3} + \frac{\dot{m}_1}{N m_1} \frac{\ddot{m}_3}{N m_3} + 3 \left( \frac{\dot{\eta}}{N \eta} \right)^2 - 8\pi G \alpha_\phi \rho = 0, \\
(1) & \quad - \frac{\dot{N}}{N^2} \left[ \frac{\ddot{m}_1}{N m_1} + \frac{\ddot{m}_3}{N m_3} + 2 \frac{\dot{\eta}}{N \eta} \right] + \frac{\ddot{m}_2}{N^2 m_2} + \frac{\ddot{m}_3}{N^2 m_3} + 3 \frac{\dot{\eta}}{N \eta} \left[ \frac{\dot{m}_2}{N m_2} + \frac{\dot{m}_3}{N m_3} \right] \\
& \quad + 2 \frac{\ddot{\eta}}{N^2 \eta} + \left( \frac{\dot{\eta}}{N \eta} \right)^2 + 8\pi G \gamma_\phi \alpha_\phi P = 0,
\end{align}

\begin{align}
(2) & \quad - \frac{\dot{N}}{N^2} \left[ \frac{\ddot{m}_1}{N m_1} + \frac{\ddot{m}_3}{N m_3} + 2 \frac{\dot{\eta}}{N \eta} \right] + \frac{\ddot{m}_1}{N^2 m_1} + \frac{\ddot{m}_3}{N^2 m_3} + 3 \frac{\dot{\eta}}{N \eta} \left[ \frac{\dot{m}_1}{N m_1} + \frac{\dot{m}_3}{N m_3} \right] \\
& \quad + 2 \frac{\ddot{\eta}}{N^2 \eta} + \left( \frac{\dot{\eta}}{N \eta} \right)^2 + 8\pi G \gamma_\phi \alpha_\phi P = 0,
\end{align}

\begin{align}
(3) & \quad - \frac{\dot{N}}{N^2} \left[ \frac{\ddot{m}_1}{N m_1} + \frac{\ddot{m}_3}{N m_3} + 2 \frac{\dot{\eta}}{N \eta} \right] + \frac{\ddot{m}_1}{N^2 m_1} + \frac{\ddot{m}_2}{N^2 m_2} + \frac{\ddot{m}_3}{N m_3} + 3 \frac{\dot{\eta}}{N \eta} \left[ \frac{\dot{m}_1}{N m_1} + \frac{\dot{m}_2}{N m_2} \right] \\
& \quad + 2 \frac{\ddot{\eta}}{N^2 \eta} + \left( \frac{\dot{\eta}}{N \eta} \right)^2 + 8\pi G \gamma_\phi \alpha_\phi P = 0,
\end{align}

and the KG equation for this particularly case is the same as before (equation (16))

\[ \frac{\dot{N}}{N^2} \phi^2 - \frac{\phi \ddot{\phi}}{N^2} - 3 \frac{\dot{\eta}}{N \eta} \phi^2 - \ddot{V} = 0. \]

Now, subtracting (21) from the component (22) we obtain

\[ \frac{\dot{N}}{N^3} \left[ \frac{\ddot{m}_1}{N m_1} - \frac{\ddot{m}_2}{N m_2} \right] - \frac{\dot{m}_1}{N m_1} \frac{\ddot{m}_3}{N m_3} + \frac{1}{N^2} \left[ \frac{\ddot{m}_2}{N m_2} - \frac{\ddot{m}_1}{N m_1} \right] + \frac{\ddot{m}_2}{N m_2} \frac{\ddot{m}_3}{N m_3} + 3 \frac{\dot{\eta}}{N \eta} \left[ \frac{\ddot{m}_2}{N m_2} - \frac{\ddot{m}_1}{N m_1} \right] = 0, \]

noticing that

\[ \frac{1}{N} \left[ \frac{\ddot{m}_2}{N m_2} - \frac{\ddot{m}_1}{N m_1} \right] ^* = \frac{1}{N^2} \left[ \frac{\ddot{m}_2}{N m_2} - \frac{\ddot{m}_1}{N m_1} \right] - \frac{1}{N^2} \left[ \left( \frac{\ddot{m}_2}{N m_2} \right)^2 - \left( \frac{\ddot{m}_1}{N m_1} \right)^2 \right] + \frac{\dot{N}}{N^3} \left[ \frac{\ddot{m}_2}{N m_2} - \frac{\ddot{m}_1}{N m_1} \right], \]

equation (25) can be rearranged and written as (where \( ^* \) also denotes a time derivative)

\[ \frac{1}{N} \left[ \frac{\ddot{m}_2}{N m_2} - \frac{\ddot{m}_1}{N m_1} \right] ^* + 3 \frac{\dot{\eta}}{N \eta} \left[ \frac{\ddot{m}_2}{N m_2} - \frac{\ddot{m}_1}{N m_1} \right] = 0, \]

finally, defining \( R_{21} = \frac{\ddot{m}_2}{N m_2} - \frac{\ddot{m}_1}{N m_1} \), the last equation can be casted as

\[ \frac{\ddot{R}_{21}}{R_{21}} + 3 \frac{\dot{\eta}}{\eta} = 0, \]

whose solution is given by

\[ R_{21} = \frac{\ell_{21}}{\eta^3}, \]

where \( \ell_{21} \) is an integration constant.
When we perform the same procedure with the other pair of equations, namely, subtracting \( 22 \) from the component \( 23 \) one obtains
\[
\frac{\dot{N}}{N^3} \left[ \frac{\dot{m}_2}{m_2} - \frac{\dot{m}_3}{m_3} \right] - \frac{\dot{m}_2}{Nm_2} \frac{\dot{m}_1}{Nm_1} + \frac{1}{N^2} \left[ \frac{\dot{m}_3}{m_3} - \frac{\dot{m}_2}{m_2} \right] + \frac{\dot{m}_1}{Nm_1} \frac{\dot{m}_3}{Nm_3} + 3 \frac{\dot{\eta}}{N^2 \eta} \left[ \frac{\dot{m}_3}{m_3} - \frac{\dot{m}_2}{m_2} \right] = 0, \tag{30}
\]
which has the same structure as equation \( 25 \). Proceeding in the same manner as we did above, we define \( R_{32} = \frac{\dot{m}_3}{Nm_3} - \frac{\dot{m}_2}{Nm_2} \), obtaining a differential equation analogous to \( 28 \), whose solutions are given by
\[
R_{32} = \frac{\ell_{32}}{\eta^3}, \tag{31}
\]
where \( \ell_{32} \) is an integration constant. And lastly, subtracting \( 21 \) from \( 23 \) we get
\[
R_{13} = \frac{\ell_{13}}{\eta^3}, \tag{32}
\]
\( \ell_{13} \) is also a constant that comes from integration, these three constants satisfy \( \ell_{21} + \ell_{32} + \ell_{13} = 0 \).

From equation \( 29 \) we have that
\[
2 \frac{\dot{m}_2}{Nm_2} - \frac{\dot{m}_1}{Nm_1} - \frac{\dot{m}_2}{Nm_2} = \frac{\ell_{21}}{\eta^3}, \tag{33}
\]
and if we use the constraints from \( 11 \), the last equation reduces to
\[
2 \frac{\dot{m}_2}{Nm_2} + \frac{\dot{m}_3}{Nm_3} = \frac{\ell_{21}}{\eta^3}, \tag{34}
\]
finally, using equation \( 31 \) as a last step we get
\[
3 \frac{\dot{m}_2}{Nm_2} + \frac{\ell_{32}}{\eta^3} = \frac{\ell_{21}}{\eta^3}. \tag{35}
\]
In order to investigate the solution for the last equation we cast it in the following form
\[
\frac{\dot{m}_2}{Nm_2} = \frac{\ell_{21} - \ell_{32}}{3 \eta^3} = \frac{\ell_2}{\eta^3}, \tag{36}
\]
where \( \ell_2 = (\ell_{21} - \ell_{32})/3 \). The other components can be obtain in a similar fashion, which read
\[
\frac{\dot{m}_3}{Nm_3} = \frac{\ell_3}{\eta^3}, \tag{37}
\]
\[
\frac{\dot{m}_1}{Nm_1} = \frac{\ell_1}{\eta^3}, \tag{38}
\]
the constants being \( \ell_3 = (\ell_{32} - \ell_{13})/3 \) and \( \ell_1 = (\ell_{13} - \ell_{21})/3 \), also, these constants satisfy \( \sum_{j=1}^3 \ell_j = 0 \). Now that equations \( 36-38 \) are written in a more manageable way, obtaining the solutions is straightforward, which are given by
\[
m_i(t) = \alpha_i \exp \left[ \ell_i \int \frac{N \mathrm{d}t}{\eta^3} \right], \tag{39}
\]
where \( \Pi_{j=1}^3 \alpha_j = 1 \).

III. EXACT SOLUTIONS WITHOUT SCALAR FIELD IN THE GAUGE N=1.

First, we consider the case when the scalar field is null, this implies that \( \alpha_\phi = 1 \), (for convenience we left this in the following treatment) which can be identified as the shifting in ordinary matter when we promote the scalar field as a fundamental ingredient in the matter-energy density of the universe.
Using equation (20) along with the relations \((\dot{m}_i/m_i) = (\ell_i/\eta^3)\) and the particular case where the scalar field is null, we get

\[
\frac{\dot{m}_1}{m_1} + \frac{\dot{m}_2}{m_2} + \frac{\dot{m}_3}{m_3} + 3 \left( \frac{\dot{\eta}}{\eta} \right)^2 - 8\pi G \alpha \rho = \frac{(\ell_1 \ell_2 + \ell_1 \ell_3 + \ell_2 \ell_3)}{\eta^6} + 3 \left( \frac{\dot{\eta}}{\eta} \right)^2 - 8\pi G \alpha \rho \eta^{-3(1+\gamma)} = 0, \tag{40}
\]

however, we can obtain a master equation for \(\eta\), which is given by

\[
-\frac{\ell^2}{\eta^6} + 3 \left( \frac{\dot{\eta}}{\eta} \right)^2 - 8\pi G \alpha \rho \eta^{-3(1+\gamma)} = 0, \tag{41}
\]

where we have identify \((\ell_1 \ell_2 + \ell_1 \ell_3 + \ell_2 \ell_3) = -(1/2) \sum_{j=1}^3 \ell_j^2 = -\ell^2\) using the constraint \(\sum_{j=1}^3 \ell_j = 0\). Because \(|\gamma| < 1\), the previous equation can be expressed as

\[
\frac{\eta^2 d\eta}{\sqrt{8\pi G \alpha \rho \eta^{3(1-\gamma)} + \ell^2}} = dt, \tag{42}
\]

whose general solutions are given in terms of the hypergeometric function as

\[
3\ell \Delta t = \eta^3 \ _2F_1 \left[ \frac{1}{2}, \frac{1}{1-\gamma}, \frac{1}{1-\gamma} - \frac{8\pi G \alpha \rho \eta^{3(1-\gamma)}}{\ell^2} \right]. \tag{43}
\]

Although the general solutions of the master equation are given by hypergeometric equations, we can solve the master equation (42) for particular cases of parameter \(\gamma\), including our case: the dust era, as we will show below.

### A. Exact solutions for dust scenario: \(\gamma = 0\).

For the dust scenario, \(\gamma = 0\), the master equation (42) takes the following form

\[
\frac{\eta^2 d\eta}{\sqrt{8\pi G \alpha \rho_0 \eta^{3+\ell^2}}} = dt, \tag{44}
\]

where the solution is given by

\[
\frac{4\sqrt{3} \pi G \alpha \rho_0}{\ell} \Delta t = \sqrt{\frac{8\pi G \alpha \rho_0 \eta^{3+1-b_2}}{\ell^2}}, \tag{45}
\]

here \(b_2\) is a constant related to the initial conditions for the dust era. From the last equation we can obtain \(\eta^3\), which reads

\[
\eta^3 = b_0 \left[ (b_1 \Delta t + b_2)^2 - 1 \right], \tag{46}
\]

where \(b_0 = \ell^2/(8\pi G \alpha \rho_0)\) and \(b_1 = (4\sqrt{3} \pi G \alpha \rho_0)/\ell\), which in turn enables us to find that the expression for \(m_i(t)\) will be given by

\[
m_i(t) = \alpha_i \left( 1 - \frac{2}{b_{1t} + b_2 + 1} \right)^{q_1}, \tag{47}
\]

where \(q_1 = \sqrt{\frac{2\ell}{3\pi}}\) with the constraint between them \(q_2^2 + q_3^2 + q_2q_3 = \frac{1}{3}\). It is important to emphasize that the anisotropic parameter acquires a constant value in the limit when \(t \to \infty\), signaling that for an extended period of time this anisotropic model transits to the flat FRW model, which is the model that we “see” today.
B. Stiff matter: \( \gamma = 1 \).

The master equation (42) when the scenario of stiff matter, \( \gamma = 1 \), is considered becomes

\[
\frac{\eta^2 d\eta}{\sqrt{\frac{8\pi G \rho_0}{3} + \frac{\ell^2}{3}}} = dt \tag{48}
\]

which has a solution of the form

\[
\eta^3 = \left[ \eta_0 + 3b_0 \Delta t \right], \tag{49}
\]

where \( b_0 = \sqrt{\frac{8\pi G \rho_0}{3} + \frac{\ell^2}{3}} \), and we can see that the isotropic volume function depends linearly with respect to time. For the stiff matter case the functions \( m_i(t) \) will be given by

\[
m_i(t) = \alpha_i \left[ \eta_0 + 3b_0 \Delta t \right] \frac{\ell}{m_\phi}, \tag{50}
\]

where we can clearly notice that dynamically we do not obtain isotropization. In our approach, the scalar potential becomes null, property of the K-essence cosmological models (Socorro et al 2010,2104), (Espinoza García, Abraham., et al 2014)

C. Exact solutions including the scalar field

Lastly, if we consider the case when the scalar field is taken into account, it turns out that the solutions have the same form as for the dust scenario case. With the difference that the parameter \( m_\phi \) will appear in the solutions (as will be shown below) and which contains the proportionality between the energy density of the scalar field and the energy density of ordinary matter. Also, for this case, \( m_\phi \) gives the proportionality between the ordinary matter and the dark energy scenario.

In this line of thought the scalar field can be calculated as follows

\[
\Delta \phi = \int \frac{dt}{\sqrt{b_0 \sqrt{(b_1 \Delta t + b_2)^2 - 1}}}, \tag{51}
\]

making the following change of variables \( u = b_1 t + b_2 \) we get \( \Delta \phi = (1/b_1 \sqrt{b_0}) \ln [u + \sqrt{u^2 - 1}] \), to finally obtain \( u = \cosh \left[ \frac{1}{2} \sqrt{\frac{3}{2}} (1 + \frac{1}{m_\phi}) \Delta \phi \right] \), such that when is substituted into the scalar field potential (18) we find that

\[
V(t) = 8\pi G \rho_0 m_\phi \frac{1}{(b_1 \Delta t + b_2)^2 - 1}, \leftrightarrow \ V(\phi) = 8\pi G \rho_0 m_\phi \text{Csch}^2 \left( \frac{1}{2} \sqrt{\frac{3}{2}} (1 + \frac{1}{m_\phi}) \Delta \phi \right), \tag{52}
\]

thus, the cosmological term for this scenario becomes

\[
\Lambda(t) = \frac{4\pi G \rho_0 m_\phi}{(b_1 \Delta t + b_2)^2 - 1}, \tag{53}
\]

which is a behavior that anisotropic universes present. From the potential that has a temporal dependence, we can estimate that it will have very pronounced negative slope, causing the universe to evolve faster when the parameter \( m_\phi = 15 \), giving approximately a 78\% in the total matter density for dust scenario. The behavior of these functions can be seen in Fig. (1)

The time evolution for the volume function in this scenario will be given by

\[
\eta^3 = b_0 \left[ (b_1 \Delta t + b_2)^2 - 1 \right], \tag{54}
\]
Figure 1: In the dust scenario, the isotropic parameter (volume function) given by equation (54), has a fast growing for big values to the $m_\phi$ parameter, in the plot we choose for the constants the values $b_1 = 10$, $b_2 = 10$, $b_0 = 0.001$. $m_\phi = 5$ corresponds to dark matter in the down plot, and $m_\phi = 15$ is for dark energy in the up plot.

Figure 2: The scalar potential given by equation (52) in term of the scalar field. $m_\phi = 5$ corresponds to dark matter in the down plot, and $m_\phi = 15$ is for dark energy in the up plot.

where $b_0 = \ell^2/(8\pi G\alpha \rho_0)$ and $b_1 = (4\sqrt{3}\pi G\alpha \rho_0)/\ell$, that in terms of the scalar field can be written as

$$\eta^3 = b_0\sinh^2\left[\frac{1}{2} \sqrt{\frac{3}{2}} \left(1 + \frac{1}{m_\phi^2}\right) \Delta \phi\right],$$

(55)

where we have used $u = b_1 t + b_2 = \cosh \left[\frac{1}{2} \sqrt{\frac{3}{2}} \left(1 + \frac{1}{m_\phi^2}\right) \Delta \phi\right]$. From the previous equation we can infer that the volume function has a stronger dependence on the dark scenario. We can see that for early times the value of $m_\phi$ is
small having a growing behavior; but when $m_\phi$ has large values the volume function has decelerate behavior.

D. Fluctuation deceleration parameter $\Delta q(t)$

One realistic model for our universe when we employ an anisotropic cosmological model requires both decelerated and an accelerated phase of expansion. For this purpose, in the literature, see (Singh H.P, 2008), to do mention the condition over to have some form of fields which governs the dynamics of the universe in a such a way that fluctuation deceleration parameter $\Delta q$ become negative, where we employ as initial deceleration parameter as the stiff matter $\gamma = 1$ and the final deceleration parameter to this corresponding to dust scenario $\gamma = 0$. The Hubble function in anisotropic cosmology in analogy with the FRW universe, we define a generalized Hubble parameter $H$ and the generalized deceleration parameter $q$ as $H = \frac{\dot{a}}{a} = \frac{\dot{\eta}}{\eta}$, that is equivalent to $H = \frac{1}{3} (H_x + H_y + H_z)$ where the directional Hubble parameters in directions (x,y,z) are defined in the usual way, $H_x = \frac{\dot{a}_x}{a}, H_y = \frac{\dot{a}_y}{a},$ and $H_z = \frac{\dot{a}_z}{a}$, then, the deceleration parameter become

$$q = -\frac{\dot{H} + H^2}{H^2},$$

(56)

can be calculated for our model, and are depending of the gauge shift function $N$ (Socorro et al 2015), which should be important employing the observational data of Supernova type Ia, see references (Riess et al 1998) and (Perlmutter et al 1999).

Taking in account the corresponding solutions found previously, we have that $q_{\text{stiff}} = 2$ and $q_{\text{dust}} = \frac{1}{2} + \frac{3}{2u^2}$, where the function $u = b_1t + b_2$, thus

$$\Delta q(t) = q_{\text{dust}} - q_{\text{stiff}} = \frac{1}{2} + \frac{3}{2u^2} - 2 = -\frac{3}{2} + \frac{3}{2u^2} = -\frac{3}{2} + \lim_{t \to \infty} \frac{3}{2u^2} = -\frac{3}{2},$$

(57)

The time behavior for this fluctuation deceleration parameter is presented in the figure 3. This fluctuation deceleration parameter take in account that the previous scenario have a big value in its deceleration parameter, and the dust scenario become minus positive, having a growing expansion in its overage volume function.

IV. CONCLUSIONS AND REMARKS

In this work we have characterized the current stage of the universe introducing a combination of results using two different approaches in an anisotropic cosmological model. Employing a Misner’s like transformation we can consider a decomposition in an isotropic and an anisotropic part, where the properties of the latter are preserved, appearing as constraint equations (see (11)). This proposition arises due to the fact that we can identify $V(\phi) = 2\Lambda$, where $\Lambda$ is a constant (as in the cosmological constant case), and we argue that this identification should be kept even when the cosmological term has temporal dependence $V(t) = 2\Lambda(t)$. The other idea was to consider a law between the energy density of the scalar field and the energy density of ordinary matter as follows: $\rho_\phi = m_\phi \rho$; the equations of state for both the scalar field and for ordinary matter were considered as bartropic, $P_\phi = \omega_\phi \rho_\phi$ and $p = \gamma \rho$, respectively. We found that for the solutions to the EKG equations to be consistent, both cosmological parameters must have the same value.

We were able to find analytical solutions for the dust scenario when the content of matter was ordinary matter, and then, using the scaling solutions (previously found) between the energy density of the scalar field and the energy density of standard matter, for the scalar field. We found that the solutions had the same structures as in the case of ordinary matter, the only difference being that the parameter $\rho$ changed to $\rho_0 \to (1 + m_\phi)\rho_0$ wherever it appears.
This last parameter is the responsible of the parametrization of the dark side of the universe, where $m_\phi = 5$ is for dark matter and $m_\phi = 15$ for dark energy. Considering this two values for the evolution of the universe, it turns out that the behavior is quite different: for $m_\phi = 15$ the growth is faster than for $m_\phi = 5$, as is expected. Also we found that the anisotropic functions become isotropic in the limit $t \to \infty$, which means that for the dust stage the anisotropic Bianchi I type cosmological model evolves into the isotropic flat FRW cosmological model. Finally, if we want the universe to have an accelerated expansion today, the scalar potential must have a behavior like that of a hyperbolic cosecant and the dynamic cosmological constant as a decreasing function over time, that is, $\Lambda(t) \sim 1/t^2$.

We would like to conclude by highlighting that when one studies the universe from the point of view of scalar fields, this type of potential is not found in the literature.

We introduce the fluctuation deceleration parameter $\Delta q(t)$ in order to have a negative value, light that the actual scenario in our universe have a growing expansion in its overage function volume $\eta^3$, obtaining $\Delta q = -\frac{3}{2}$. We do not introducing new ansatz in the Hubble parameter in order to obtain the corresponding fluctuation deceleration parameter, only we use the standard definitions that appear in the literature.

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