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Entanglement and Quantum Computation

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Introduction

The phenomenon of quantum entanglement is perhaps the most enigmatic feature of the formalism of quantum theory. It underlies many of the most curious and controversial aspects of the quantum mechanical description of the world. In [1] Penrose gives a delightful and accessible account of entanglement illustrated by some of its extraordinary manifestations. Many of the best known features depend on issues of non-locality. These include the seminal work of Einstein, Podolsky and Rosen [18], Bell’s work [19] on the EPR singlet state, properties of the GHZ state [20, 21] and Penrose’s dodecahedra [1]. In this paper we describe a new feature of entanglement which is entirely independent of the auxiliary notion of non-locality.

We will argue that the phenomenon of entanglement is responsible for an essential difference in the complexity (as quantified below) of physical evolution allowed by the laws of quantum physics in contrast to that allowed by the laws of classical physics. This distinction was perhaps first explicitly realised by Feynman [17] in 1982 when he noted that the simulation of a quantum evolution on a classical computer appears to involve an unavoidable exponential slowdown in running time. Subsequently in the development of the subject of quantum computation – which represents a hybrid of quantum physics and theoretical computer science – it was realised that quantum systems could be harnessed to perform useful computations more efficiently than any classical device. Below we will examine some of the basic ingredients of quantum computations and relate their singular efficacy to the existence of entanglement.

The perspective of information theory also provides further new insights into the relationship between entanglement and non-locality, beyond the well studied mediation of non-local correlations between local measurement outcomes. To outline some of these effects we first introduce the notion of “quantum information”.

A fundamental difference between quantum and classical physics is that the state of an unknown quantum system is in principle unmeasureable e.g. given an unknown state $|\psi\rangle$ of a 2-level system it is not possible to identify it. In fact any measurement on $|\psi\rangle$ will reveal at most one bit of information about its identity whereas the full description of $|\psi\rangle$ requires the specification of two real numbers. In terms of binary expansions this corresponds to a double infinity of bits of information. We refer to
the full (largely inaccessible) information represented by a quantum state as quantum information in contrast to the more familiar notion of classical information such as the outcome of a measurement which is in principle fully accessible.

The inaccessibility of quantum information is closely related to the possibility of non-local influences which do not violate classical causality, as necessitated by Bell’s analysis of local measurements on an EPR pair – the non-local influences are simply restricted to a level which is inaccessible to any local observations. However it has recently been shown that entanglement plays a more subtle role here than just mediating correlations between the classical information of local measurement outcomes. According to the process known as quantum teleportation entanglement acts as a channel for the transmission of quantum information: an unknown quantum state of a 2-level system may be transferred intact from one location to another by sending only two bits of classical information, if the locations are also linked by entanglement in the form of a shared EPR pair. The remaining quantum information can be interpreted as having flown across the entanglement which is destroyed by the process.

Another novel application of entanglement and non-locality inspired by concepts from theoretical computer science, is the existence and construction of quantum error correcting codes, first introduced by Shor in 1995. Entanglement provides a way of delocalising quantum information in a system composed of several subsystems. For example if $|0\rangle$ and $|1\rangle$ are orthogonal states then the entangled states $\frac{1}{\sqrt{2}}(|0\rangle |0\rangle + |1\rangle |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle |1\rangle + |1\rangle |0\rangle)$ are identical in terms of local quantum information (each subsystem being in the maximally mixed state in each case) whereas they differ in terms of their global quantum information content. By an ingenious extension of this idea one may encode an unknown state of a 2-level system into an entangled state of several 2-level systems in such a way that if any (unknown) one of the subsystems is arbitrarily corrupted then the original state may still be perfectly reconstructed i.e. none of the information of the original state resides locally in the encoding. In this way a state may be protected against the effects of spurious environmental interactions if (as is generally a good approximation) these interactions act by local means.

Thus quantum computation and quantum information theory provide a rich variety of new applications of entanglement and we now turn to a more detailed discussion of issues in quantum computation in particular.

**Quantum Computation and Complexity**

The theory of computation and computational complexity is normally presented as an entirely mathematical theory with no reference to considerations of physics. However any actual computation is a physical process involving the physical evolution of selected properties of a physical system. Consequently the issues of “what is computable” and “what is the complexity of a computation” must depend essentially
on the laws of physics and cannot be characterised by mathematics alone. This fundamental point was emphasised by Deutsch [8] and it is dramatically confirmed by the recent discoveries [2, 3, 4, 5, 6, 10] that the formalism of quantum physics allows one to transgress some of the boundaries of the classical theory of computational complexity, whose formulation was based on classical intuitions. In [1] Penrose proposes the possible introduction of non-computable elements into physics (i.e. non-computable within the standard existing theory of computability). This however requires going beyond the existing formalism of quantum theory since the latter lies entirely within the bounds of classical computability. Our considerations here lie entirely within the standard framework so that, for us, quantum processes cannot result in any computation which is not already possible by classical means. This notwithstanding, there does appear to be a significant difference in the efficiency of computation as noted in Feynman’s remark.

A fundamental notion of the theory of computational complexity is the distinction between polynomial and exponential use of resources in a computation. This will provide a quantitative measure of our essential distinction between quantum and classical computation. Consider a computational task such as the following: given an integer $N$, decide whether $N$ is a prime number or not. We wish to assess the resources required for this task as a function of the size of the input which is measured by $n = \log_2 N$, the number of bits needed to store $N$. If $T(n)$ denotes the number of steps (on a standard universal computer) needed to solve the problem, we ask whether $T(n)$ can be bounded by some polynomial function in $n$ or whether $T(n)$ grows faster than any polynomial. More generally we may consider any language $\mathcal{L}$ – a language being a subset of the set of all finite strings of 0’s and 1’s – and consider the computational task of recognising the language i.e. given a string $\sigma$ of length $n$ the computation outputs 0 if $\sigma \in \mathcal{L}$ and outputs 1 if $\sigma \notin \mathcal{L}$. (In our example above $\mathcal{L}$ is the set of all prime numbers written in binary). The language $\mathcal{L}$ is said to be in complexity class $\mathcal{P}$ (“polynomial time”) if there exists an algorithm which recognises $\mathcal{L}$ and runs in time $T(n)$ bounded by a polynomial function. Otherwise the recognition of $\mathcal{L}$ is said to require exponential time. More generally it is useful to consider algorithms which involve probabilistic choices (“coin tosses”) [7, 9]. $\mathcal{L}$ is said to be in the class $\mathcal{BPP}$ (“bounded-error probabilistic polynomial time”) if there is a polynomial time algorithm which correctly classifies the input string $\sigma$ with probability at least $2/3$ (or equivalently, any other value strictly between $1/2$ and 1). Thus a $\mathcal{BPP}$ algorithm may give the wrong answer but by simple repetition and taking the majority answer, we can amplify the probability of success as close to 1 as desired while retaining a polynomial time for the whole process ([7, 9]). The class $\mathcal{BPP}$ of algorithms which run in polynomial time but allow for “small imperfections”, is often regarded as the class of computational tasks which are “feasible in practice” (or at least a first mathematical approximation to this notion). We also use the term “efficient computation” for a computation which runs in polynomial time.
In the above considerations the exact number of steps $T(n)$ will generally depend on the choice of underlying computer and the model of computation adopted. However if we stay within the confines of classical physics, the distinction between polynomial and exponential time i.e. between efficient and non-efficient computation, appears to be robust, being independent of these choices. It is thus a distinction with physical significance. In the discussion above we have illustrated it in its most familiar form – in terms of the resource of time. From the physical point of view it is natural to extend the notion of efficient computation to require the efficient use of all possible physical resources. Indeed in the following discussion we will be led to consider other resources such as energy. The absolute significance of the distinction between efficient and non-efficient computation provides an extension of the classical Church-Turing thesis \cite{8,5} which refers to a similar distinction between computability and non-computability. The fundamental raison d’être of quantum computation is the fact that quantum physics appears to allow one to transgress this classical boundary between polynomial and exponential computations.

The concept of quantum computation may be rigorously formalised as a natural extension of classical mathematical models of computation \cite{8,3,11,9} in which the computational steps are allowed to be quantum processes restricted by a suitable notion of locality. For our purposes it will suffice to envisage a quantum computer as a standard universal computer in which the memory bits are 2–level quantum systems instead of 2–state classical systems. The quantum systems are each endowed with a preferred basis $\{|0\rangle, |1\rangle\}$ corresponding to the classical bit values of 0 and 1. We refer to these 2–level systems as qubits \cite{12}.

The computer is able to support arbitrary superpositions of the values 0 and 1 within each qubit and also entanglements of many qubits. Furthermore the computer may be programmed to perform unitary transformations of any number of qubits. It is important however that large unitary transformations be constructed or “programmed” from a finite set of fixed given unitary transformations. In this way we can assess the complexity of unitary transformations by the length of their programs. There are many known examples of small finite sets of transformations, out of which one can program any unitary transformation of any number of qubits (to arbitrary precision) \cite{13,14,15}. Indeed it is known \cite{14} that almost any single transformation of two qubits by itself suffices. The distinction between polynomial and exponential time does not depend on the choice of these basic transformations as any one such set can first be used to build the members of any other set leading to only an overall constant expansion in the number of elementary steps in any computation.

**Superposition and Entanglement in Quantum Computation**

There are several quantum algorithms known \cite{2,3,4,5} which strongly support the view that a quantum computer can perform some computational tasks exponentially faster than any classical device. The most significant of these is Shor’s polynomial
time algorithm for integer factorisation \cite{5,9}, a problem which is believed to lie outside the class $BPP$ of classical computation. We then ask: what is the essential quantum effect that gives rise to this exponential increase in computing power?

All of the quantum algorithms utilise the process of computation by quantum parallelism \cite{8}. This refers to a quantum computer’s capability to carry out many computations simultaneously in superposition if the input is set up in a suitable superposition of classically distinct inputs. Thus one might conclude that it is superposition that is at the root of the quantum computational speedup. However closer examination will show that entanglement is the essential feature rather than just superposition itself. Note that entanglement may be viewed as a special kind of superposition – superposition in the presence of a product structure on the state space – which arises from the system being made up of several subsystems. In our considerations these are the qubits comprising the computer. An entangled state is then a superposition of product states which cannot be expressed as a single product state.

To see that superposition itself is not the essential feature we need only note that classical waves also exhibit superposition. Any effect depending on quantum interference alone can be readily mimicked by classical waves. However in other respects quantum states and classical waves differ considerably e.g. the measurement theories are very different (being far more favourable for computation with classical waves than with quantum states!) and there is no classical analogue of the phenomenon of entanglement.

To illustrate the above remarks and highlight the role of entanglement consider the following example of computation by quantum parallelism. Let $B = \{0, 1\}$ and consider any (non-trivial) function $f : B^n \to B$. Suppose that we have a quantum computer programmed to compute $f$ in polynomial time. The computer has $n$ input qubits and one output qubit and its operation corresponds to a unitary transformation $U_f$ on $n + 1$ qubits which effects the evolution:

$$U_f : |i_1\rangle |i_2\rangle \ldots |i_n\rangle |0\rangle \longrightarrow |i_1\rangle |i_2\rangle \ldots |i_n\rangle |f(i_1, \ldots, i_n)\rangle$$

Here each $i_k$ is either 0 or 1. The output register is initially in state $|0\rangle$ and at the end of the computation it contains the basis state corresponding to the value of the function. Consider the one-qubit operation:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

If the input qubits are all initially in state $|0\rangle$ then applying $H$ to each one successively gives an equal superposition of all $2^n$ values in $B^n$:

$$H \otimes \cdots \otimes H \cdot |0\rangle \cdots |0\rangle = \frac{1}{2^{n/2}} (|0\rangle + |1\rangle)^n = \frac{1}{2^{n/2}} \sum_{i=0}^{2^n-1} |i\rangle$$

(1)
(where we have identified \((i_1, \ldots, i_n)\) with the binary number \(i_1 \ldots i_n < 2^n\)). Note that this state is prepared in polynomial (in fact linear) time. Running the computer with \((I)\) as input yields the final state

\[
|f\rangle = \sum_{i=0}^{2^n-1} |i\rangle |f(i)\rangle
\]

Thus by quantum parallelism we have computed exponentially many values of \(f\) in superposition with only polynomial computing effort.

Can we mimick the above with classical waves? We represent each qubit by a classical wave system and select two modes of vibration to represent the states \(|0\rangle\) and \(|1\rangle\) e.g. \(|0\rangle\) and \(|1\rangle\) may be the two lowest energy modes of a vibrating elastic string with fixed endpoints. It is then straightforward to construct the superposition corresponding to \(|0\rangle + |1\rangle\) and performing this separately on \(n\) pieces of string we obtain the product state \((I)\). However, regardless of how much the strings interact with each other in their subsequent (externally driven) vibrational evolution, their joint state is always a product state of \(n\) separate vibrations. The total state space of the total classical system is the Cartesian product of the individual state spaces of the subsystems whereas quantum-mechanically, it is the tensor product. This crucial distinction between Cartesian and tensor products is precisely the phenomenon of quantum entanglement. The state \((2)\) is generally entangled (for non-trivial \(f\)'s) so that the transition from \((1)\) to \((2)\) cannot be achieved in a classical scenario.

We may attempt to represent entanglement using classical waves in the following manner. The state of \(n\) qubits is a \(2^n\) dimensional space and can be isomorphically viewed as the state space of a single particle with \(2^n\) levels. Thus we simply interpret certain states of a single \(2^n\) level particle as “entangled” via their correspondence under a chosen isomorphism between \(\otimes^n H_2\) and \(H_{2^n}\) (where \(H_k\) denotes a Hilbert space of dimension \(k\).) In this way, \(2^n\) modes of a classical vibrating system can apparently be used mimick general entanglements of \(n\) qubits. However the physical implementation of this correspondence appears always to involve an exponential overhead in some physical resource so that the isomorphism is not a valid correspondence for considerations of complexity. For example suppose that the \(2^n\) levels of the one-particle quantum system are equally spaced energy levels. A general state of \(n\) qubits requires an amount of energy that grows linearly with \(n\) whereas a general state of this \(2^n\) level system (and also the corresponding classical wave system) requires an amount of energy that grows exponentially with \(n\). To physically realise a system in a general superposition of \(2^n\) modes we need exponential resources classically and linear resources quantum mechanically because of the existence of entanglement.

This comparison is reminiscent of the representation of whole numbers in unary (i.e. representing \(k\) as a string of \(k\) 1’s analogous to \(k\) equally spaced levels) versus the binary representation of \(k\) which requires a string of length \(\log_2 k\) and is therefore exponentially more efficient. Note that \(n\) classical bits can also accommodate \(2^n\)
possible alternatives but only one such alternative can be present at any time, even if probabilistically determined. In contrast $n$ qubits can accommodate $2^n$ possible alternatives which may all be simultaneously present in superposition. As a quantum computation proceeds a new qubit may be incorporated into the overall processed state at each step leading to an exponential growth in time of the quantum information, because of entanglement. If we were to compute this quantum evolution “by hand” from the laws of quantum mechanics then we would suffer an exponential slowdown in handling an exponentially growing amount of information. In contrast, Nature manages to process this growing information in linear time! This is an example of Feynman’s remark mentioned in the introduction. Note that if the state of the accumulating qubits were always a product state (i.e. no entanglement) then the quantum information would accumulate only linearly.

There exist physical systems with infinitely many discrete energy levels which accumulate below a finite bound $E_0$. Thus we may use these levels to represent general superpositions of exponentially many modes with only a constant cost in energy and apparently circumvent the above objections! However in this case the levels will crowd together exponentially closely and we will need to build our instruments with exponentially finer precision. This again will presumably require exponentially increasing physical resources.

It has been occasionally suggested that the interferences in Shor’s efficient quantum factoring algorithm and other quantum algorithms, can be readily represented by classical wave interferences but on closer inspection all such proposals involve an exponential overhead in some physical resource as illustrated in our discussion above.

The standard mathematical theory of computational complexity [7] assesses the complexity of a computation in terms of the resources of time (number of steps needed) and space (amount of memory required). In the above we have been led to consider the accounting of other physical resources such as energy and precision. This reinforces our earlier remark that the notion of computational complexity must rest on the laws of physics and consequently the proper assessment of complexity will need to take into account all possible varieties of physical resource. A theory of computational complexity based on such general physical foundations remains to be formulated.

**Entanglement and the Super-Fast Quantum Fourier Transform**

The efficiency of Shor’s factoring algorithm rests largely on the fact that the discrete Fourier transform\(^5\) $DFT_{2^n}$ in dimension $2^n$ (a particular unitary transformation in $2^n$ dimensions) may be implemented on a quantum computer in time polynomial in $n$ (in fact quadratic in $n$). Similarly the efficiency of Deutsch’s and Simon’s algorithms \(^2\) \(^4\) rest on the polynomial-time computability of the Fourier transform over the additive group $B^n$. The standard classical Fourier transform algorithm implements
Let $M$ be a unitary matrix of size $2^n$ and $v$ a column vector of length $2^n$. To compute $w = Mv$ by direct matrix multiplication requires $O((2^n)^2)$ operations, each entry of the result requiring $O(2^n)$ operations. Suppose now that the space of $v$ is the tensor product of $n$ two dimensional spaces $V^{(1)} \otimes \ldots \otimes V^{(n)}$ and that $M$ decomposes as a simple tensor product

$$M = S^{(1)} \otimes \ldots \otimes S^{(n)} \tag{3}$$

where each $S^{(i)}$ is a 2 by 2 matrix acting on the respective component space $V^{(i)}$. Thus we can label the components of $v$ by indices $i_1 \ldots i_n \in B^n$ and the computation of $w$ becomes

$$w_{j_1 \ldots j_n} = \sum_{i_1 \ldots i_n} S^{(1)}_{j_1 i_1} \ldots S^{(n)}_{j_n i_n} v_{i_1 \ldots i_n} \tag{4}$$

Now consider first $S^{(1)}$. Each application of this 2 by 2 matrix requires some fixed (i.e. independent of $n$) number of operations and $S^{(1)}$ needs to be applied $2^{n-1}$ times, once for each choice of the indices $i_2 \ldots i_n \in B^{n-1}$. The same accounting applies to each of the $n$ matrices $S^{(i)}$ giving a total number of operations $O(n2^{n-1}) = O(n2^n)$. Thus the tensor product factorisation (3) leads to an exponential speed-up compared to straightforward matrix multiplication for a general $M$. A similar argument will apply if $M$ decomposes more generally into the successive application of a polynomial number of matrices, each of which applies to a bounded number $b$ of the component spaces (not necessarily disjoint) and $b$ is independent of $n$. (3) is merely the simplest example of such a decomposition. The classical fast Fourier transform algorithm is based on the fact that the Fourier transform in dimension $2^n$ decomposes in just this way (although not as simply as (3) c.f. [9] for an explicit description of the decomposition).

Consider finally the implementation of $M$ as given in (3) on a quantum computer. The data given by the components of $v$ is represented by the amplitudes of a general state of $n$ qubits. Each of the $n$ operations $S^{(i)}$ is a one-qubit operation and needs to be applied only once to its respective qubit i.e. the $2^{n-1}$ repetition of the classical calculation is eliminated! This is due to the rules of formation of the tensor product (i.e. entanglement) requiring for example that $S^{(1)}$ applied to the first qubit automatically carries through to all possible values of the indices $i_2 \ldots i_n$ in (4) i.e. the global operation $S^{(1)} \otimes I \otimes \ldots \otimes I$ is implemented on the whole space. Thus $M$ is implemented in time $O(n)$. In a similar way, the more complicated decomposition of the Fourier transform can be seen [9] to lead to a time $O(n^2)$. This comparison of classical and quantum implementations of $M$ is yet another illustration of Feynman’s remark [17] that the simulation of a quantum process on a classical computer
generally involves an exponential slowdown.

The classical and quantum scenarios for the above computation of $Mv$ differ significantly in the following respect. After the classical computation we are able to read off all $2^n$ components of $w$ presented as classical information whereas the quantum computation results in the quantum information of one copy of the state $|w\rangle = \sum w_{i_1...i_n} |i_1\rangle \ldots |i_n\rangle$ from which we are unable to extract the individual values of $w_{i_1...i_n}$. This is the issue of inaccessibility of quantum information that was mentioned in the introduction. Nevertheless we are able to extract classical information from $|w\rangle$ that depends on exponentially many of the values and classically this information would require a preliminary exponential computational effort. This phenomenon also occurs in computation by quantum parallelism. From the quantum information $|f\rangle$ in (2) we are unable to extract all the individual values $f(i_1 \ldots i_n)$ but we can obtain certain global properties of the function. Indeed in Shor’s factoring algorithm [5, 9], the analogue of $f$ is a periodic function and after applying the Fourier transform we are able to extract the period.

**Concluding Remarks**

In summary the effects of quantum entanglement enable certain large unitary transformations to be implemented exponentially more efficiently on a quantum computer than on any classical computer. However after quantum computation the full results are coded in a largely inaccessible form. Remarkably this inaccessibility, dictated by the principles of quantum measurement theory, does not serve to cancel out the exponential gain in computing effort. Limited information may be obtained about the transformed data which, although small, would nevertheless require an exponential amount of computing effort to obtain by classical computation.

Another fundamental issue intimately related to entanglement is the so-called measurement problem of quantum mechanics. This refers to the reconciliation of the apparent “collapse of wave function” in a measurement with the unitary evolution of quantum mechanics and explaining why, after a measurement, we see merely one of the possible outcomes rather than experiencing some weird reality in entanglement with all possible outcomes. In [4] Penrose discusses several of the best known interpretations of quantum mechanics and it appears that none of them provides a resolution of this phenomenon. With extraordinarily innovative and broad-ranging arguments, Penrose suggests that the resolution might involve non-computational ingredients. The fact that quantum theory has resisted unification with the theory of gravitation suggests that the essentially linear concept of entanglement may not persist at a macroscopic level. Indeed it is difficult to imagine that the linearity of quantum theory could survive a unification with the nonlinear foundations of the general theory of relativity. The algorithms of quantum computation such as Shor’s algorithm depend critically for their efficiency and validity on effects of increasingly large scale entanglements with increasing input size. Thus efforts to experimentally
implement these algorithms may provide particularly acute opportunities to witness a possible breakdown of the current conventional quantum formalism.

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