EXTREME POINTS IN LIMIT SETS

DANNY CALEGARI AND ALDEN WALKER

Abstract. Given an iterated function system of affine dilations with fixed points the vertices of a regular polygon, we characterize which points in the limit set lie on the boundary of its convex hull.

CONTENTS

1. Introduction 1
2. The limit set 2
3. Angles and hyperplanes 2
4. Extreme point alternatives 5
References 8

1. Introduction

1.1. Background. Fix $n \geq 2$ and $c \in \mathbb{C}$ with $|c| < 1$. Let $F_{n,c}$ be the iterated function system generated by $\{f_j\}_{j=0}^{n-1}$ with $f_j : \mathbb{C} \to \mathbb{C}$ defined

$$f_j(z) = cz + \xi^j,$$

where $\xi = e^{2\pi i/n}$ is the standard primitive $n$th root of unity. Let $\Xi_n = \{\xi^j\}_{j=0}^{n-1}$ be the set of all $n$th roots of unity. The iterated function system $F_{n,c}$, and thus its limit set $\Lambda_{n,c}$, has rotational symmetry of order $n$ around 0. Indeed, it is simple to check that conjugation by the multiplication-by-$\xi$ map takes $f_j$ to $f_{j+1 \mod n}$. Other definitions of iterated function systems with fixed points on the vertices of a regular $n$-gon are conjugate to $F_{n,c}$; we have chosen this one to simplify our arguments. We will only be interested in $n \geq 2$. This construction is a simple generalization of the one for $n = 2$ initially studied by [1].

Our objective in this note is to characterize the extreme points in $\Lambda_{n,c}$; that is, those which lie on the boundary of its convex hull. As a consequence, we provide an updated, more thorough version of Lemma 7.2.3 in [3], and we re-prove [4], Proposition 2.1. The material here is an extracted, expanded piece of [2].

1.2. Acknowledgements. Danny Calegari was supported by NSF grant DMS 1405466. Alden Walker was partially supported by NSF grant DMS 1203888.

Date: November 20, 2018.
2. The limit set

For convenience, we will use $F_{n,c}$ to denote the set of all finite words in the symbols $\{0, \ldots, n-1\}$, which will be a notational convenience obviously in bijection with the set of finite words in the generators $f_j$. Note that the set of finite words in the $f_j$ is slightly different from the set of finite compositions of the $f_j$ because different words may produce the same function. We define a map $\pi : F_{n,c} \times \mathbb{C} \rightarrow \mathbb{C}$ by

$$\pi((j_0, j_1, \cdots j_m), z) = (f_{j_0} \circ \cdots \circ f_{j_m})(z).$$

Let $F_{n,c}^\infty$ denote the set of infinite words in $\{0, \ldots, n-1\}$. Given an infinite word $f = (j_0, j_1, \cdots) \in F_{n,c}^\infty$, the limit $\lim_{m \rightarrow \infty} (f_{j_0} \circ \cdots \circ f_{j_m})(z)$ does not depend on $z$ because $|c| < 1$. Thus we can extend $\pi$ to a map $\pi : F_{n,c}^\infty \rightarrow \mathbb{C}$.

**Lemma 2.1.** $\pi(F_{n,c}^\infty) = \Lambda_{n,c}$.

**Proof.** The set $F_{n,c}^\infty$ is compact in the standard Cantor topology, and the map $\pi$ is continuous for this topology. The set of infinite words is evidently invariant in the sense of an iterated function system, so the image $\pi(F_{n,c}^\infty)$ is a compact set invariant under $F_{n,c}$, so it is $\Lambda_{n,c}$. \qed

This parameterization of $\Lambda_{n,c}$ as infinite words in $F_{n,c}^\infty$ will be our main tool. Let us see what the elements of $\Lambda_{n,c}$ can be. Suppose we have an infinite word $f = (j_0, j_1, \ldots)$. We can compute the associated point $\pi(f) \in \Lambda_{n,c}$ as

$$\pi(f) = f_{j_0}(\pi(j_1, \ldots)) = c\pi(j_1, \ldots) + \xi^{j_0}.$$

By induction, we see that

$$\pi(f) = \sum_{k=0}^{\infty} c^k \xi^{j_k}.$$

That is, the exponents of the $\xi$ are exactly the letters in the infinite word $f$. There is a clear geometric picture for such sums: at every step $k$, we select which power $j_k$ of $\xi$ we would like to multiply by $c^k$ and accumulate into the sum. Note that the set $\{c^k \xi^{j_k}\}_{j_k=0}^{j_k=n-1}$ is a rotated, scaled copy of the roots of unity. We imagine a constellation of potential vectors at each step, and we can choose to add in any one of them. See Figure 1.

3. Angles and hyperplanes

Any point in $\mathbb{C}$ can be thought of as a vector in $\mathbb{R}^2$, and any vector $v \in \mathbb{R}^2$ induces a (real) linear function on $\mathbb{C}$ given by taking the standard inner product with $v$. We will be interested in the linear functions coming from the unit circle, so we denote by $v_\theta : \mathbb{C} \rightarrow \mathbb{R}$ the linear function given by taking the real inner product with the point $e^{2\pi i \theta}$.

**Remark 3.1.** For convenience, we will normalize all angles to lie in $[0, 1]$. Thus, as above, we refer to the argument of $e^{2\pi i \theta}$ as “the angle $\theta$”.

The level sets of $v_\theta$ are (oriented) affine hyperplanes, and we say that they lie at angle $\theta$. Note that it is the normal vector that points in the direction $\theta$. Such a hyperplane $H$ supports a closed set $X \subseteq C$ if $H$ intersects $X$ and $X$ is contained in the closure of one of the two complements of $H$ in $\mathbb{C}$. Equivalently, $H$ supports $X$ if the value of $v_\theta$ on $X \cap H$ (which must be constant) is maximal over all points
in $X$. The points which lie in the intersection of $X$ and a hyperplane at angle $\theta$ which supports it are extreme points at angle $\theta$.

Turning to our problem of interest, we now consider the extreme points in $\Lambda_{n,c}$ in terms of the parameterization of $\Lambda_{n,c}$ from Section 2 and Lemma 2.1. The extreme points at angle $\theta$ are those points $p$ such that $v_\theta(p)$ is maximal over $\Lambda_{n,c}$. Denote the set of extreme points in $\Lambda_{n,c}$ at angle $\theta$ by $E_{n,c,\theta}$. Let us be given an infinite word $f = (j_0, \ldots) \in F_{\infty}^{n,c}$. We say that $f$ is an extreme word at angle $\theta$ if $\pi(f)$ is an extreme point at angle $\theta$. Denote the set of extreme words in $F_{n,c}$ at angle $\theta$ by $W_{n,c,\theta}$. It is cleaner to characterize the extreme words $W_{n,c,\theta}$ in $F_{\infty}^{n,c}$ and then translate that understanding to the set of extreme points $E_{n,c,\theta}$ in $\Lambda_{n,c}$.

We have

$$\pi(f) = \sum_{k=0}^{\infty} c^k \xi^{j_k},$$

so

$$v_\theta(\pi(f)) = \sum_{k=0}^{\infty} v_\theta(c^k \xi^{j_k}).$$

Now $v_\theta$ is real linear but not complex linear, so we cannot pull the coefficient $c^k$ out. It is difficult to understand exactly what this sum is equal to, but it is not difficult to maximize it: we just need to maximize each summand. For each $k$, the value of $v_\theta(c^k \xi^{j_k})$ is maximized when the vector $c^k \xi^{j_k}$ is the closest to angle $\theta$ among the vectors $\{c^k \xi^j\}_{j=0}^{n-1}$. So to construct an extreme point $p$ at angle $\theta$, we choose, for each $k$, the letter $j_k$ such that $c^k \xi^{j_k}$ is closest to $\theta$. It is possible that there will be multiple choices which are equidistant from $\theta$, in which case there are exactly two options. This produces a very clean geometric picture of each extreme point, which is illustrated in Figure 2.

Everything which follows is basically a direct observation from Figure 2. As mentioned above, for each given $k$, there are two possibilities for the set of function values $\{v_\theta(c^k \xi^j)\}_{j=0}^{n-1}$. Either all of these values are distinct, in which case the choice of coordinate $j_k$ is forced, or there is some $j_k$ such that $v_\theta(c^k \xi^{j_k}) = v_\theta(c^k \xi^{j_k+1})$. This happens exactly when the angle $\theta$ is equidistant from the arguments of $c^k \xi^{j_k}$.
To produce an extreme point at angle $\theta$, we consider the linear function $v_\theta$, where $e^{2\pi i \theta}$ is the arrow indicated in blue. For each $k$, we are forced to choose $j_k$ such that the $v_\theta(c^k \xi^{j_k})$ is maximal for the vectors $\{c^k \xi^{j_k}\}_{j=0}^{n-1}$. The partial sum after 4 steps is shown on right.

and $c^k \xi^{j_k + 1}$. Here and in what follows, we take all powers of $\xi$ modulo $n$. The two situations are shown in Figure 3.

When $\theta$ is equidistant from the arguments of $c^k \xi^{j_k}$ and $c^k \xi^{j_k + 1}$, as on the right, both $j_k$ and $j_k + 1$ are allowed as coordinate $k$ in an extreme point at angle $\theta$.

We now formalize the observations from Figures 2 and 3. Let $n \in \mathbb{N}$, $c = r e^{2\pi i \phi}$ and $\theta \in [0,1]$ be given. For each $k \geq 0$, define a set $J_{n,c,\theta,k} \subset \{0,\ldots,n-1\}$ as follows. If there exists an integer $m \in [0,n-1]$ such that $k \phi + \frac{m}{n} + \frac{1}{2n} \equiv \theta \mod 1$, then $J_{n,c,\theta,k} = \{m, m + 1\}$ (recall all indices are taken modulo $n$). Otherwise, $J_{n,c,\theta,k} = \{m\}$ where $k \phi + m/n$ is closest to $\theta$.

**Lemma 3.2.** In the above notation,

$$W_{n,c,\theta} = \prod_{k=0}^{\infty} J_{n,c,\theta,k}.$$  

That is, the set of extreme words in $F_{n,c}^\infty$ at angle $\theta$ is the infinite cartesian product of the sets $J_{n,c,\theta,k}$. Or, more simply, to enumerate all the extreme words,
we can go one letter at a time. When we need to decide on letter $k$, we consult $J_{n,c,\theta,k}$ to see what the allowed letters are, and any choice is allowed (all constraints on the letters are local).

**Proof of Lemma 3.2.** The proof is contained in the preceding discussion: if we are given an extreme word $f = (j_0, \ldots)$, and we write the linear function distributed over the sum:

$$v_\theta(\pi(f)) = \sum_{k=0}^\infty v_\theta(c^k j_k),$$

then is suffices to optimize each term independently. The definition of $J_{n,c,\theta,k}$ is the condition to check that there are two equidistant options as in Figure 3. □

### 4. Extreme Point Alternatives

#### 4.1. Main theorem.

We now follow the line of reasoning begun in Lemma 3.2. We first state our main theorem and then spend the rest of the section proving and explaining it.

**Theorem 4.1.** As above, let $c = r e^{2\pi i \phi}$.

1. If $\phi \notin \mathbb{Q}$, then
   
   (a) For all $\theta$, we have $|W_{n,c,\theta}| = |E_{n,c,\theta}| \in \{1, 2\}$.
   
   (b) The set of $\theta$ such that $|W_{n,c,\theta}| = 1$ is dense in $[0, 1]$.
   
   (c) The set of $\theta$ such that $|W_{n,c,\theta}| = 2$ is dense in $[0, 1]$.
   
   (d) $\Lambda_{n,c}$ is not convex.

2. If $\phi \in \mathbb{Q}$ with $\phi = p/q$ in lowest terms, then let $b$ be such that $bn = \text{lcm}(n,q)$ (e.g. if gcd$(n,q) = 1$, then $b = q$). Let

   $$\Theta = \left\{ \ell \phi + \frac{m}{n} + \frac{1}{2n} \right\}_{\ell=0,m=0}^{b-1,n-1}.$$ 

   Then

   (a) For $\theta \in \Theta$, we have that $E_{n,c,\theta}$ is itself the limit set of an iterated function system with two generators conjugate to $F_2,|c|$. That is, a Cantor set with dilation factor $|c|^b$ (or an interval if $|c|^b \geq 1/2$).
   
   (b) For $\theta \notin \Theta$, we have $|W_{n,c,\theta}| = |E_{n,c,\theta}| = 1$.
   
   (c) The convex hull of $\Lambda_{n,c}$ is a polygon with $nb$ sides at the angles in $\Theta$.
   
   (d) If $\Lambda_{n,c}$ is convex, then $|c| \geq 2^{-1/b}$.

#### 4.2. The irrational case.

Before the formal proof of the rational case of Theorem 4.1, we give the picture of the proof, which is actually much more convincing. If we construct an infinite word $f \in F_{n,c}^\infty$ by building each summand in the infinite sum $\pi(f)$ as in Figure 1, then for each $k$ we have $n$ options of which power $j_k$ to use in the term $c^k j_k \xi_k$ to accumulate. As we have seen in Figures 2 and 3 if we are building an extreme word for angle $\theta$, our choice is dictated by which $j_k$ makes $c^k j_k \xi_k$ as close to $\theta$ as possible. Let us consider what the set of options $\{c^k \xi_k\}_{k=0}^{n-1}$ looks like when $\phi \notin \mathbb{Q}$. The set $\mathbb{Z}\phi$ is dense in the interval $[0, 1]$, and there are no distinct integers $k,k'$ such that $k\phi \equiv k'\phi \mod 1$. That is, the constellation of choices never repeats. Thus, if we ever happen to stumble upon an index $k$ with $|J_{n,c,\theta,k}| = 2$, it will never happen again. Figure 4 gives a picture.
When $\phi \notin \mathbb{Q}$, the constellations $\{c^{\ell} \xi^j\}_{j=0}^{n-1}$ are dense (and not periodic). Thus, if we ever find a $k$ as in Figure 3 with two equidistant choices for some given $\theta$, that is the only $k$ for which it occurs. The figure shows the constellations for $k = 0, \ldots, 15$.

Proof of Theorem 4.1(1). To prove (a), it suffices to show that for all $\theta$, there is at most one $k$ such that $|J_{n,c,\theta,k}| = 2$. Towards a contradiction, assume there are two distinct such values $k,k'$. Then there exist $m,m'$ integers with

$$k\phi + \frac{m}{n} + \frac{1}{2n} \equiv \theta \equiv k'\phi + \frac{m'}{n} + \frac{1}{2n} \mod 1$$

So $(k-k')\phi \equiv (m-m')/n \mod 1$. Since $k \neq k'$, this implies that $\phi$ is rational, which is a contradiction.

To prove (b) and (c), we exhibit sets of $\theta$ with the desired properties. First, for any integer $\ell$, set $\theta = \ell \phi$. Then suppose there is any integer $k$ with

$$k\phi + \frac{m}{n} + \frac{1}{2n} \equiv \theta \equiv \ell \phi \mod 1.$$

We conclude that $(k-\ell)\phi \equiv m/n + 1/2n \mod 1$. The expression on the right is never 0, which implies that $\phi \in \mathbb{Q}$, a contradiction. Thus for all $k$ we have $|J_{n,c,\theta,k}| = 1$, meaning $|W_{n,c,\theta}| = 1$ and hence $|E_{n,c,\theta}| = 1$.

Next, we set $\theta = \ell \phi + 1/2n$ for any integer $\ell$. Now we have

$$k\phi + \frac{m}{n} + \frac{1}{2n} \equiv \ell \phi + \frac{1}{2n} \mod 1,$$

so $(k-\ell)\phi \equiv m/n \mod 1$. This does have exactly one solution, where $k = \ell$ and $m = 0$. Hence there is exactly one $k$ such that $|J_{n,c,\theta,k}| = 2$, so $|W_{n,c,\theta,k}| = 2$. In general, it is difficult to know the size of $E_{n,c,\theta}$ from knowing $W_{n,c,\theta}$. However, in this case the two extreme words differ in exactly one letter, so their images under $\pi$ differ in exactly one (nonzero) summand, so there are exactly two extreme points at angle $\theta$.

We have proved that for all $\theta \in \phi \mathbb{Z}$, we have $|W_{n,c,\theta}| = |E_{n,c,\theta}| = 1$ and for all $\theta \in \phi \mathbb{Z} + 1/2n$, we have $|W_{n,c,\theta}| = |E_{n,c,\theta}| = 1$. These sets are dense in the circle (represented here by the interval $[0,1]$) because $\phi \in \mathbb{Q}$, and we have proved (b) and (c).

Claim (d) is an immediate consequence of (c): if we exhibit any single $\theta$ with $|E_{n,c,\theta}| = 2$, then every point on the line segment between these two points lies in the convex hull of $\Lambda_{n,c}$ but not in $\Lambda_{n,c}$ itself, meaning that $\Lambda_{n,c}$ is not convex. □
4.3. The rational case. We now turn to understanding the extreme points in \( \Lambda_{n,c} \) when \( \phi \in \mathbb{Q} \). This case is simpler in some ways but more technical and interesting in others. In particular, we will now find sets of extreme words at certain angles which are infinite. In order to simplify the reasoning, we will first restrict \( c = r e^{2\pi i \phi} \) such that \( \phi \in \mathbb{Q} \) behaves nicely with respect to \( n \), meaning that \( \phi = p/q \) with \( \gcd(n,q) = 1 \). The general case will follow by proving a technical lemma (Lemma 4.2) and observing that with this lemma in hand, the general argument is essentially the same. The truly helpful picture of the situation, analogous to Figure 4, is shown in Figure 5. When \( \phi \in \mathbb{Q} \), one can see that the constellations \( \{c^k \xi_j \}_{j=0}^{n-1} \) are periodic in \( k \). The restriction that \( \gcd(n,q) = 1 \) ensures that this period is easy to compute (it is \( nq \)).

**Proof of Theorem 4.1(2) when \( \gcd(n,q) = 1 \).** Because \( \phi = p/q \), we have \( c^q = r^q \), i.e., the angle of \( c^q \xi^j \) is the same as the angle of \( c^{k+q} \xi^j \). This implies that the angles of the constellations \( \{c^k \xi_j \}_{j=0}^{n-1} \) are periodic in \( k \), and for all \( k \) and \( \theta \), we have \( J_{n,c,\theta,k} = J_{n,c,\theta,k+q} \). The fact that \( \gcd(q,n) = 1 \) means that we do not have any additional equalities — that is, there are no \( k,k',j,j' \) with \( 0 \leq k,k' < q-1 \) and \( 0 \leq j,j' < n-1 \) with \( \arg(c^k \xi^j) = \arg(c^{k'} \xi^{j'}) \).

The combination of these two facts implies that for exactly the angles listed in \( \Theta \), we have \( |E_{n,c,\theta}| > 1 \). Now consider a specific \( \theta \in \Theta \). The product \( \prod_{k=0}^{q-1} J_{n,c,\theta,k} \) contains exactly two words, since there will be exactly one \( k \) for which we have two options. Call these words \( w_0 \) and \( w_1 \). Now if \( f \in W_{n,c,\theta} \) is any extreme word for \( \theta \), the discussion above shows that both words \( w_0 + f \) and \( w_1 + f \) (where + means word concatenation) are also extreme words for \( \theta \). Pushing forward under \( \pi \), this means that the limit set of the iterated function system generated by the functions

\[
\begin{align*}
z &\mapsto c^q z + w_0 \\
z &\mapsto c^q z + w_1
\end{align*}
\]

is exactly the set of extreme points \( E_{n,c,\theta} \). Since \( c^q = r^q \) is real, this is conjugate to the function system \( F_{2,|c|^q} \), which has limit set as described.
If we consider any \( \theta_0, \theta_1 \) consecutive elements of \( \Theta \), then note that because of the discreteness of the constellations, for any two \( \theta, \theta' \) with \( \theta_0 < \theta, \theta' < \theta_1 \), we have \( J_{n,c,\theta,k} = J_{n,c,\theta',k} \) for all \( k \). Therefore, all these angles share one extreme point.

The previous two paragraphs imply facts (a), (b), and (c). To see (d), note that if we are to have \( \Lambda_{n,c} \) convex, then for \( \theta \in \Theta \), the set of extreme points \( E_{n,c,\theta} \) must be an interval. Thus \( |c|^n \geq 1/2 \), or \( |c| \geq 2^{-1/q} \).

For the general case when we do not necessarily have \( \gcd(n,q) = 1 \), the only thing we need to determine is the periodicity of \( \{c^k\xi^j\}_{j=0}^{n-1} \) as \( k \) varies. For simplicity, denote by \( C_k \) the set of angles \( \{\arg(c^k\xi^j)\}_{j=0}^{n-1} \). We are careful to say a set, because while assuming, as above, that \( \phi = p/q \) in lowest terms, we certainly have that \( C_k = C_{k+q} \), but we might have \( C_k = C_{k+b} \) for some \( b < q \) where the elements are not in the same order. As an example, if \( \phi = 1/6 \) and \( n = 3 \), then \( C_k = C_{k+2} \).

**Lemma 4.2.** In the above notation, let \( b \) be the smallest positive integer such that \( b(1/q) = a(1/n) \) for some integer \( a \). Equivalently, let \( bn = \lcm(n,q) \). Then for all \( k \), we have \( C_k = C_{k+b} \) and \( C_k \cap C_{k+i} = \emptyset \) for \( 0 < i < b \).

**Proof.** It is immediate that \( C_k = C_{k+b} \) when \( b(1/q) = a(1/n) \) because

\[
\arg(c^{k+b}\xi^j) = \phi(k+b)+j/n = \phi k + (p/q)b + j/n = \phi k + (pa + j)/n = \arg(c^k\xi^{pa+j})
\]

If we supposed towards a contradiction that for \( i < b \) we have \( C_k \cap C_{k+i} \neq \emptyset \), then an analogous chain of equalities gives that there are \( j_1, j_2 \) with \( (p/q)(k+i) + j_1/n = (p/q)k + j_2/n \), so \( (p/q)i = (j_1-j_2)n \). But \( b \) is the smallest positive integer such that this can hold, which contradicts that \( i < b \).

**Proof of Theorem 4.1(2) in the general case.** This is essentially a corollary of the above Lemma 4.2; the proof of Theorem 4.1(2) in the case when \( \gcd(n,q) = 1 \) depends only on characterizing the periodicity of the angles \( C_k \), which is done by Lemma 4.2.

4.4. **Final remarks.** A consequence of Theorem 4.1(2d) is that for any \( n \) and \( c \), if \( \Lambda_{n,c} \) is convex, then \( |c| \geq 1/2 \) (see [4], Proposition 2.1). Furthermore, this bound is sharp from the perspective of the set of extreme points: there are \( c \) (in particular, \( c = 1/2 + 0a \)) with \( |c| = 1/2 \) such that every \( E_{n,c,\theta} \) is an interval. Note that this does not imply that \( \Lambda_{n,c} \) is convex: consider the Sierpinski triangle, where the boundary of the limit set is a finite sided polygon, but the limit set itself is complicated. Figure 6 shows some more examples of limit sets and their supporting hyperplanes.

**References**

[1] M. Barnsley and A. Harrington, *A Mandelbrot set for pairs of linear maps*, Phys. D. **15** (1985), no. 3, 421–432.

[2] D. Calegari and A. Walker *Circle actions on the boundary of Schottky space*, in preparation.

[3] D. Calegari, S. Koch and A. Walker, *Roots, Schottky semigroups, and a proof of Bandt’s Conjecture*, Ergodic Theory and Dynamical Systems **37** (2017) no. 8, 2487–2555. doi:10.1017/etds.2016.17.

[4] Himeki, Y. and Ishii, Y. *M4 is regular-closed*, Ergodic Theory and Dynamical Systems, 1-8 doi:10.1017/etds.2018.27.
Figure 6. A variety of limit sets with $c = 0.48 + 0i$ and supporting hyperplanes, showing how the limit set may or may not be convex, even when the sets of extreme points are intervals (the dilation amounts here are just shy of 0.5 to show the detail). A small twist of $\phi = 1/100$ in the lower right produces a much more interesting collection of extreme points, and the boundary of the convex hull is now a polygon with 300 sides.