A DIAGRAMMATIC APPROACH TO CATEGORIZATION OF QUANTUM GROUPS II

MIKHAIL KHOVANOV AND AARON D. LAUDA

ABSTRACT. We categorify one-half of the quantum group associated to an arbitrary Cartan datum.

Cartan data and algebras. A Cartan datum $(I,\cdot)$ consists of a finite set $I$ and a symmetric bilinear form on $\mathbb{Z}[I]$ taking values in $\mathbb{Z}$, subject to the conditions

- $i \cdot i \in \{2,4,6,\ldots\}$ for any $i \in I$,
- $2 \delta_{ij} \in \{0, -1, -2, \ldots\}$ for any $i \neq j$ in $I$.

We set $d_{ij} = -2 \delta_{ij} \in \mathbb{N}$. To a Cartan datum assign a graph $\Gamma$ with the set of vertices $I$ and an edge between $i$ and $j$ if and only if $i \neq j$. We recall the definition of the negative half of the quantum group associated to an arbitrary Cartan datum.

We categorify one-half of the quantum group associated to an arbitrary Cartan datum.

The quantum version of the Gabber-Kac theorem says the following.

**Proposition 1.** Algebra $\mathcal{F}$ carries a unique $\mathbb{Q}(q)$-bilinear form such that $(1,1) = 1$ and

- $(\theta_i, \theta_j) = \delta_{ij}(1 - q^2)^{-1}$ for all $i, j \in I$,
- $(x, yy') = (r(x), y \otimes y')$ for $x, y, y' \in \mathcal{F}$,
- $(xx', y) = (x \otimes x', r(y))$ for $x, x', y \in \mathcal{F}$.

This bilinear form is symmetric.

The radical $\mathcal{J}$ of $(,),$ is a two-sided ideal of $\mathcal{F}$. The bilinear form descends to a non-degenerate bilinear form on the associated $\mathbb{Q}(q)$-algebra $\mathcal{F} = \mathcal{F}/\mathcal{J}$. The $\mathbb{N}[I]$-grading also descends:

$$\mathcal{F} = \bigoplus_{\nu \in \mathbb{N}[I]} \mathcal{F}_\nu.$$

The quantum version of the Gabber-Kac theorem says the following.
Proposition 2. The ideal $\mathcal{I}$ is generated by the elements

$$\sum_{a+b=d_{ij}+1} (-1)^a \theta_i^{(a)} \theta_j^{(b)}$$

over all $i, j \in I, i \neq j$.

Thus, $f$ is the quotient of $\mathcal{F}$ by the so-called quantum Serre relations

(1) $$\sum_{a+b=d_{ij}+1} (-1)^a \theta_i^{(a)} \theta_j^{(b)} = 0.$$

Denote by $\mathcal{A}f$ the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $f$ generated by the divided powers $\theta_i^{(a)}$, over all $i \in I$ and $a \in \mathbb{N}$.

Algebras $R(\nu)$. As in [4], we consider braid-like planar diagrams, each strand labelled by an element of $I$, and impose the following relations:

\begin{align*}
(2) & \quad \begin{array}{c}
\begin{array}{ccc}
\text{i} & \text{i} & \text{j} \\
\text{j} & \text{i} & \text{j}
\end{array}
\end{array} = \begin{cases}
0 & \text{if } i = j, \\
\begin{array}{c}
\text{i} \\
\text{j}
\end{array} & \text{if } i \cdot j = 0, \\
\begin{array}{c}
d_{ij} \\
d_{ji}
\end{array} & \text{if } i \cdot j \neq 0,
\end{cases} \\
(3) & \quad \begin{array}{cccc}
\text{i} & \text{j} & \text{i} & \text{j} \\
\text{i} & \text{j} & \text{i} & \text{j}
\end{array} = \begin{array}{c}
\text{i} \\
\text{j}
\end{array}, \quad \text{for } i \neq j, \\
(4) & \quad \begin{array}{cccc}
\text{i} & \text{i} & \text{i} & \text{i} \\
\text{i} & \text{i} & \text{i} & \text{i}
\end{array} = \begin{array}{c}
\text{i} \\
\text{i}
\end{array}, \\
(5) & \quad \begin{array}{cccc}
\text{i} & \text{i} & \text{i} & \text{i} \\
\text{i} & \text{i} & \text{i} & \text{i}
\end{array} = \begin{array}{c}
\text{i} \\
\text{i}
\end{array}, \\
(6) & \quad \begin{array}{cccc}
\text{i} & \text{j} & \text{k} & \text{i} \\
\text{i} & \text{j} & \text{k} & \text{i}
\end{array} = \begin{array}{c}
\text{i} \\
\text{i}
\end{array} \quad \text{unless } i = k \text{ and } i \cdot j \neq 0,
\end{align*}
\(d_{ij} = \sum_{a=0}^{d_{ij} - 1} a d_{ij} - 1 \) \(d_{ji} - a\) if \(i \cdot j \neq 0\).

**Example 3.** For the Cartan datum \(B_2 = \{i \cdot i = 2, j \cdot j = 4, i \cdot j = -2\}\) we have \(d_{ij} = 2, d_{ji} = 1\), and the relations involving \(d_{ij}, d_{ji}\) are

\[
\begin{align*}
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (0,1) -- (1,0);
\draw (2,0) -- (3,1);
\draw (2,1) -- (3,0);
\draw (0,2) -- (1,3);
\draw (0,3) -- (1,2);
\draw (2,2) -- (3,3);
\draw (2,3) -- (3,2);
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (0,1) -- (1,0);
\draw (2,0) -- (3,1);
\draw (2,1) -- (3,0);
\draw (0,2) -- (1,3);
\draw (0,3) -- (1,2);
\draw (2,2) -- (3,3);
\draw (2,3) -- (3,2);
\end{tikzpicture}
\end{align*}
\]

For each \(\nu \in \mathbb{N}[I]\) define the graded ring

\[
R(\nu) \overset{\text{def}}{=} \bigoplus_{i,j \in \text{Seq}(\nu)} j R(\nu)_i,
\]

where \(j R(\nu)_i\) is the abelian group of all linear combinations of diagrams with \(\text{bot}(D) = i\) and \(\text{top}(D) = j\) modulo the relations (2)-(7) and \(\text{Seq}(\nu)\) is the set
of weight \( \nu \) sequences of elements of \( I \). The multiplication is given by concatenation. Degrees of the generators are:

\[
\begin{align*}
\text{deg } \left( \begin{array}{c} i \\ i \end{array} \right) &= i \cdot i, \\
\text{deg } \left( \begin{array}{c} i \\ j \end{array} \right) &= -i \cdot j.
\end{align*}
\]

The rest of [4, Section 2.1] generalizes without difficulty to an arbitrary Cartan datum. To define the analogue of the module \( P_\nu^\ell \) over \( R(\nu) \), we choose an orientation of each edge of \( \Gamma \), then faithfully follow the exposition in Section 2.3 of [4], only changing the action of \( \delta_{k,i} \) in the last of the four cases to

\[
f \mapsto (x_k(s_k i)^d + x_{k+1}(s_k i)^{d'}) (s_k f) \text{ if } i_k \rightarrow i_{k+1},
\]

where \( d = d_{i_k i_{k+1}} \) and \( d' = d_{i_k i_{k+1}} \). Here notation \( i_k \rightarrow i_{k+1} \) means that \( i_k \cdot i_{k+1} \neq 0 \) and this edge of \( \Gamma \) is oriented from \( i_k \) to \( i_{k+1} \). Proposition 2.3 in [4] holds for an arbitrary \((I, \cdot)\). As in [4, Section 2.3], we define \( jB_i \), which might depend on minimal presentations of permutations in \( jS_i \) and gives a basis in \( jR(\nu) \). Corollary 2.6 in [4], showing that \( P_\nu^\ell \) is a faithful graded module over \( R(\nu) \), holds for an arbitrary Cartan datum and the properties of \( R(\nu) \) established in [4, Section 2.4] generalize without difficulty.

**Computations in the nil-Hecke ring.** In this section we slightly enhance the graphical calculus for computations in the nil-Hecke ring and record several lemmas to be used in the proof of categorified quantum Serre relations below. We use notation from Section 2.2 of [4].

A box with \( n \) incoming and \( n \) outgoing edges and \( \partial(n) \) written inside denotes the longest divided difference \( \partial_{w_0} \), the non-zero product of \( \frac{n(n-1)}{2} \) divided differences from \( \{ \partial_1, \ldots, \partial_{n-1} \} \):

![Diagram](https://example.com/diagram.png)

When this box is part of a diagram for an element of \( R(\nu) \), it denotes the corresponding element of \( R(ni) \subset R(\nu) \). A box labelled \( e_n \) denotes the idempotent \( e_n = x_1^{n-1}x_2^{n-2} \ldots x_{n-1} \partial_{w_0} \).

![Diagram](https://example.com/diagram.png)
A box labelled $e_{i,n}$ denotes the corresponding idempotent in $R(\nu)$:

$$
e_{i,n} = \begin{array}{c} \vdots \vline \hline \i \i \i \i \i \vline \end{array} = \begin{array}{c} n-1 \quad n-2 \quad 2 \\ i \i \i \i \i \end{array} \begin{array}{c} \vdots \vdots \vdots \vdots \vdots \end{array}
$$

**Remark 4.** Similar diagrams are used in the graphical calculus of Jones-Wenzl projectors (see [3]), but the latter has no direct relation to the graphical calculus in our paper.

**Lemma 5.** We have

$$\partial(n) \quad e_{i,n} = \partial(n) \quad \partial(n)$$

Proof is by induction on $n$:

The first equality uses that $x_1x_2\ldots x_{n-1}$ is central in the nil-Hecke ring $NH_{n-1}$, allowing us to move these dots across $\partial(n-1)$. The second equality is the induction hypothesis. \qed
The lemma implies the following graphical identities:

\[(10)\]

(11)

The following also hold:

\[(12)\]

\[(13)\]

For each \(i \in I\) the ring \(R(mi)\) is isomorphic to the nil-Hecke ring. The grading of a dot is now \(i \cdot i\), while that of a crossing is \(-i \cdot i\). For this reason one needs to generalize the grading convention described in [4, Section 2.2] and define \(i,mP\) to be the right graded projective module \(e_{i,m}R(mi)\{-\frac{m(m-1)i}{4}\}\), so that the grading starts in the degree \(-\frac{m(m-1)i}{4}\). Likewise, \(P_{i,m}\) is the left graded projective module \(R(mi)\psi(e_{i,m})\{-\frac{m(m-1)i}{4}\}\).

**The Grothendieck group, bilinear form and projectives.** We retain all notation and assumptions from [4], working over a field \(k\), denoting by \(K_0(R(\nu))\) the Grothendieck group of the category \(R(\nu)-\text{pmod}\) of graded finitely-generated projective left \(R(\nu)\)-modules and forming the direct sum

\[
R = \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu), \quad K_0(R) = \bigoplus_{\nu \in \mathbb{N}[I]} K_0(R(\nu)).
\]
Consider symmetric $\mathbb{Z}[q, q^{-1}]$-bilinear form
\begin{equation}
(,): K_0(R(\nu)) \times K_0(R(\nu)) \rightarrow \mathbb{Z}[q, q^{-1}], \ (\nu)_q
\end{equation}
where
\begin{equation}
(\nu)_q = \text{gdim}(\text{Sym}(\nu)) = \prod_{i \in \Gamma} \left( \prod_{a=1}^{\nu_i} \frac{1}{1 - q^{ai}} \right),
\end{equation}
and
\begin{equation}
([P], [Q]) = \text{gdim}_k(P^\psi \otimes_{R(\nu)} Q).
\end{equation}
The character $\text{ch}(M)$ of an $\mathbb{R}(\nu)$-module $M$, the divided power sequences $\text{Seqd}(\nu)$, and idempotents $1_i$ for $\nu = i_1^{(n_1)} \cdots i_r^{(n_r)} \in \text{Seqd}(\nu)$ are defined as in [4, Section 2.5]. Let $\nu = [n_1]_1! \cdots [n_r]_r!$ and
\begin{equation}
(i) = \sum_{k=1}^{r} \frac{n_k(n_k - 1)}{2} \cdot i_k \cdot i_k.
\end{equation}
Define graded left, respectively right, projective module
\begin{equation}
P_i = R(\nu)\psi(1_i)^{-\langle i \rangle}, \quad iP = 1_i R(\nu)^{-\langle i \rangle}.
\end{equation}
Quantum Serre relations. Let
\begin{equation}
\alpha^+_{a,b}(i,j) = \begin{array}{|c|}
\hline
a + 1 & \hline
\end{array} e_{i,a+1} \begin{array}{|c|}
\hline
j & \hline
\end{array},
\end{equation}
and also write $\alpha^+_{a,b}$ when $i$ and $j$ are fixed. To prove the categorified quantum Serre relations, we assume that $a + b = d + 1$, where $d = d_{ij}$. The element $\alpha^+_{a,b}$ belongs to
\begin{equation}
i^{a+1} j^{b-1} R(j + (a + b)i)^{\nu_{ij}}.
\end{equation}
By adding vertical lines on the left and on the right of the diagram, $\alpha^+_{a,b}$ can be viewed, more generally, as an element of
\begin{equation}
i^{i'} \nu_{i'}^{a+1} j^{i''} R(\nu)^{\nu_{i'}^{a} j^{i''} j^{i'}}
\end{equation}
for any sequences $i'$, $i''$ and the corresponding $\nu$. We can replace sequences $i'$ and $i''$ by dots to simplify notation.

Left multiplication by $\alpha^+_{a,b}$ is a homomorphism of projective modules
\begin{equation}
\ldots i^{a+1} j^{b-1} \rightarrow \ldots i^{a+1} j^{b-1} \rightarrow P.
\end{equation}
The top part of the diagram of $\alpha^+_{a,b}$ contains idempotents $e_{i,a+1}$ and $e_{i,b-1}$. Therefore, $\alpha^+_{a,b}$ induces a homomorphism of projective modules
\begin{equation}
\ldots i^{(a)} j^{(b)} \rightarrow \ldots i^{(a+1)} j^{(b-1)} \rightarrow P,
\end{equation}
denoted \( \alpha^+_{(a,b)} \) and given by the composition
\[
...i^a j^b... P \xrightarrow{\alpha^+_{a,b}} ...i^{a+1} j^{b-1}... P.
\]

It is easy to check that \( \alpha^+_{(a,b)} \) is a grading-preserving homomorphism. Likewise, let
\[
\alpha^-_{a,b}(i,j) = \begin{array}{c}
e_i, a-1 \\
e_i, b+1 \\
a-1 \\
b+1 \\
i \\
j \\
\end{array}
\]
and write \( \alpha^-_{a,b} \) instead of \( \alpha^-_{a,b}(i,j) \) when \( i \) and \( j \) are fixed. This element of
\[
...i^{a-1} j^{b-1}... R(\nu) ...i^a j^b...
\]
gives rise to a grading-preserving homomorphism of projectives
\[
\alpha^-_{(a,b)}: ...i^a j^b... P \rightarrow ...i^{a-1} j^{b+1}... P.
\]

For the next few pages, denote \( ...i^a j^b... P \) by \( (a,b)P \) (recall that \( a + b = d + 1, \ d = d_{ij} \)). We have a diagram of projective modules and grading-preserving homomorphisms
\[
\begin{array}{c}
\rightarrow (a-1, b+1) \ P \xrightarrow{\alpha^-_{a-1, b+1}} (a,b) \ P \xrightarrow{\alpha^+_{a,b}} (a+1, b-1) \ P \xrightarrow{\alpha^-_{a+1, b-1}} (a+1, b+1) \ P \xrightarrow{\alpha^+_{a, b}} (a,b) \ P \xrightarrow{\alpha^-_{a-1, b+1}} (a-1, b+1) \ P
\end{array}
\]
terminating on the left at
\[
(0, d+1) P \xrightarrow{\alpha^-_{0, d+1}} (1, d) P \xrightarrow{\alpha^+_{1, d}} (1, d) P \xrightarrow{\alpha^-_{1, d}} ...
\]
and on the right at
\[
\cdots \xrightarrow{\alpha^-_{d, 1}} (d, 1) P \xrightarrow{\alpha^+_{d, 1}} (d, 1) P \xrightarrow{\alpha^-_{d+1, 0}} (d+1, 0) P.
\]
Relations (10), (11) imply

\[ \alpha_{a+1,b-1}^+ \alpha_{a,b}^+ = \]

Furthermore,

\[ \alpha_{a-1,b+1}^+ \alpha_{a,b}^- = \]

Therefore,

\[ \alpha_{a-1,b+1}^+ \alpha_{a,b}^- - \alpha_{a+1,b-1}^- \alpha_{a,b}^+ = (-1)^{a-1} e_{i,a} \otimes 1_j \otimes e_{i,b}, \]
as elements of $R(\nu)$; see below:

\[
\alpha_{a-1,b+1}^+ \alpha_{a,b}^- - \alpha_{a+1,b-1}^- \alpha_{a,b}^+ = (\begin{array}{c}
\sum_{c=0}^{d-1} \\
\end{array}) e_{i,a} \otimes 1_j \otimes e_{i,b} - (-1)^{a-1} \otimes 1_j \otimes e_{i,a} \otimes 1_j \otimes e_{i,b},
\]

where $d = a + b - 1$. Consequently,

\[
\alpha_{a-1,b+1}^+ \alpha_{a,b}^- - \alpha_{a+1,b-1}^- \alpha_{a,b}^+ = (-1)^{a-1} \cdot \text{Id},
\]

as endomorphisms of the projective module $(a,b)_P$, since $e_{i,a} \otimes 1_j \otimes e_{i,b}$ acts by the identity on $(a,b)_P$. 

\[
\alpha_{a-1,b+1}^+ \alpha_{a,b}^- - \alpha_{a+1,b-1}^- \alpha_{a,b}^+ = (\begin{array}{c}
\sum_{c=0}^{d-1} \\
\end{array}) e_{i,a} \otimes 1_j \otimes e_{i,b} - (-1)^{a-1} \otimes 1_j \otimes e_{i,a} \otimes 1_j \otimes e_{i,b},
\]
Likewise,

\[ \alpha_0 d^+ \alpha_0^{-1} = \]

\[ e_{i,d+1} \]

\[ e_{i,d} \]

\[ e_{i,d+1} \]

\[ d' \]

\[ d \]

\[ e_{i,d+1} \]

\[ e_{i,d+1} \]

\[ e_{i,d+1} \]

\[ = \]

\[ 1_j \otimes e_{i,d+1} \]

where \( d' = d_{ji} \), and

\[ \alpha_0 d^+ \alpha_0^{-1} = \text{Id}, \]

as endomorphisms of \((0,d+1)P\). A similar computation shows that

\[ \alpha_+ d \alpha_0^{-1} = (-1)^d e_{i,d+1} \otimes 1_j, \]

as elements of \(R(\nu)\), and

\[ \alpha_0 d^+ \alpha_0^{-1} = (-1)^d \cdot \text{Id}, \]

as endomorphisms of \((d+1,0)P\). Moreover,

\[ \alpha_+ a_{a+1,b-1} \alpha_0^{-1} = 0 \quad \text{and} \quad \alpha_0^{-1} a_{a-1,b+1} \alpha_0^{-1} = 0. \]

**Proposition 6.** For each \( i,j \in I, i \neq j \) there are isomorphisms of graded right projective modules

\[
\bigoplus_{a=0}^{d+1} \cdots_{i(2a)} j_{j(i+1,2a)} \cdots P \cong \bigoplus_{a=0}^{d} \cdots_{i(2a+1)} j_{j(i,2a)} \cdots P.
\]

**Proof.** When \( i \cdot j = 0 \), the isomorphism reads

\[ \cdots_{i} P \cong \cdots_{ij} P \]

and is given by left multiplication by the \( ij \) intersection. When \( i \cdot j < 0 \), earlier computations show that the maps

\[
\bigoplus_{a=0}^{d+1} \cdots_{(2a,d+1-2a)} P \xrightarrow{\alpha'} \bigoplus_{a=0}^{d} \cdots_{(2a+1,d-2a)} P
\]
given by
\[
\alpha' = \sum_{a=0}^{\lfloor d/2 \rfloor} \alpha_{(2a,d+1-2a)}^+ + \sum_{a=0}^{\lfloor d+1/2 \rfloor} \alpha_{(2a,d+1-2a)}^+,
\]
\[
\alpha'' = \sum_{a=0}^{\lfloor d+1/2 \rfloor} \alpha_{(2a+1,d-2a)}^- - \sum_{a=0}^{\lfloor d/2 \rfloor} \alpha_{(2a+1,d-2a)}^-,
\]
are mutually-inverse isomorphisms, implying the proposition. Maps \(\alpha', \alpha''\) together are given by summing over all arrows in the diagram (24), with every fourth arrow appearing with the minus sign.

Corollary 7. For each \(i, j \in I, i \neq j\) there are isomorphisms of graded left projective modules
\[
\bigoplus_{a=0}^{\lfloor d/2 \rfloor} P_{...i^{(2a)}j^{(d+1-2a)}...} \cong \bigoplus_{a=0}^{\lfloor d+1/2 \rfloor} P_{...i^{(2a+1)}j^{(d-2a)}...}.
\]

Proposition 6 and Corollary 7 generalize Proposition 2.13 in [4] and can be considered a categorification of the quantum Serre relations. Corollaries 2.14 and 2.15 of [4], establishing quantum Serre relations for the characters of any \(M \in R(\nu)\mod\), generalize to an arbitrary Cartan datum in the same way.

**Grothendieck group as the quantum group.** Induction and restriction functors for inclusions \(R(\nu) \otimes R(\nu') \subset R(\nu + \nu')\) turn \(K_0(R)\) into a twisted bialgebra, and all results of [4, Section 2.6] remain valid for an arbitrary Cartan datum. As in [4, Section 3.1] we define a homomorphism of twisted bialgebras
\[
\gamma : \mathcal{A}f \rightarrow K_0(R)
\]
which takes the product of divided powers \(\theta_i = \theta_{i_1}^{(n_1)} \cdots \theta_{i_r}^{(n_r)}\) to \([P_i]\), where \(i = i_1^{(n_1)} \cdots i_r^{(n_r)}\). Homomorphism \(\gamma\) intertwines the bilinear forms on \(\mathcal{A}f\) and \(K_0(R)\),
\[
(x, y) = (\gamma(x), \gamma(y)), \quad x, y \in \mathcal{A}f.
\]
Due to the quantum Gabber-Kac theorem, this homomorphism is injective. Surjectivity of \(\gamma\) follows from the arguments identical to those given in [4, Section 3.2], which, in turn, were adopted from [5, Section 5]. Alternatively, the arguments could be adopted from [2] and [9]; we settled on using a single source. We obtain

**Theorem 8.** \(\gamma : \mathcal{A}f \rightarrow K_0(R)\) is an isomorphism of \(\mathbb{N}[I]\)-graded twisted bialgebras.

This theorem holds without any restrictions on the Cartan datum and on the ground field \(k\) over which \(R(\nu)\) is defined. All other results and observations of Sections 3.2 and 3.3 of [4] extend to the general case as well. The cyclotomic quotients of \(R(\nu)\), described in [4, Section 3.4], generalize to an arbitrary Cartan datum.
It would be interesting to relate our construction to Lusztig’s geometric realization of $U^-$ in the non-simply laced case [6] and to Brundan-Kleshchev’s categorification [1, 5] of $U_{q=1}$ in the affine Dynkin case $A^{(2)}_n$.

A multi-grading. For every pair $(i, j)$ of distinct vertices of $\Gamma$, algebras $R(\nu)$ can be equipped with an additional grading, by assigning degrees $-1$ and $1$ to the $ij$ and $ji$ crossings, respectively,

$$\text{deg} \begin{pmatrix} i & j \\ \end{pmatrix} = -1, \quad \text{deg} \begin{pmatrix} j & i \\ \end{pmatrix} = 1,$$

and degree 0 to all other diagrammatic generators of $R(\nu)$. These gradings are independent, and together with the principal grading, introduced above, make $R(\nu)$ into a multi-graded ring (with $n(n+1)/2 + 1$ independent gradings where $n = |\text{Supp}(\nu)|$). The direct sum of the categories of multi-graded finitely-generated projective left $R(\nu)$-modules, over all $\nu \in \mathbb{N}[I]$, categorifies a multi-parameter deformation [7, 8] of the quantum universal enveloping algebra $U^-$, the quotient of the free associative algebra on $\theta_i$, $i \in I$, by the relations

\begin{equation}
\sum_{a+b=d_{ij}+1} (-1)^a q_{ij}^a \theta_i^{(a)} \theta_j \theta_i^{(b)} = 0,
\end{equation}

where $q_{ij}$ are formal variables subject to conditions $q_{ij} q_{ji} = 1$.

Modifications in the simply-laced case. This section explains how to deform algebras $R(\nu)$ in the simply-laced case so that the main results of [4] will hold for the modified algebras. These deformations can be non-trivial only when the graph has cycles. As in [4], we start with an unoriented graph $\Gamma$ without loops and multiple edges. Next, fix an orientation of each edge of $\Gamma$, work over a base field $k$, and, for each oriented edge $i \rightarrow j$, choose two invertible elements $\tau_{ij}$ and $\tau_{ji}$ in $k$.

Denote such a datum \{orientations, invertible elements\} by $\tau$.

For each $\nu \in \mathbb{N}[I]$ consider $k$-vector space $\mathcal{P}_{\mathcal{O}_\nu}$ defined as in [4]. This space is the sum of polynomial rings in $|\nu|$ variables, over all sequences in $\text{Seq}(\nu)$. Define $R_\tau(\nu)$ to be the endomorphism algebra of $\mathcal{P}_{\mathcal{O}_\nu}$ generated by the endomorphisms $1_i$, $x_{k,i}$, $\delta_{k,i}$, over all possible $k$ and $i$, with the action as in [4 Section 2.3], with the only difference being the action of $\delta_{k,i}$ in the last of the four cases:

$$f \mapsto (\tau_{ik} x_{k+1}(s_k \delta) - \tau_{ik} x_{k}(s_k \delta))(s_k f) \quad \text{if} \quad i_k \rightarrow i_{k+1},$$

instead of

$$f \mapsto (x_{k}(s_k \delta) + x_{k+1}(s_k \delta))(s_k f) \quad \text{if} \quad i_k \rightarrow i_{k+1}.$$
The algebra $R_{\tau}(\nu)$ has a diagrammatic description similar to that of $R(\nu)$, with the following defining relations:

\[
\begin{array}{c}
\begin{array}{ccc}
\tau_{ij} & - & \tau_{ji} \\
i & j & i
\end{array}
\end{array}
\]

if $i \rightarrow j$,

\[
\begin{array}{c}
\begin{array}{ccc}
\tau_{ji} & - & \tau_{ij} \\
\tau_{ij} & - & \tau_{ji} \\
i & j & i
\end{array}
\end{array}
\]

if $i \leftarrow j$,

\[
\begin{array}{c}
\begin{array}{ccc}
i & j & i \\
i & j & i
\end{array}
\end{array}
\]

for $i \neq j$,

\[
\begin{array}{c}
\begin{array}{ccc}
i & j & i \\
i & i & i
\end{array}
\end{array}
\]

(28)

\[
\begin{array}{c}
\begin{array}{ccc}
i & j & i \\
i & i & i
\end{array}
\end{array}
\]

(29)

\[
\begin{array}{c}
\begin{array}{ccc}
i & j & i \\
i & i & i
\end{array}
\end{array}
\]

(30)

\[
\begin{array}{c}
\begin{array}{ccc}
i & j & i \\
i & i & i
\end{array}
\end{array}
\]

(31)

\[
\begin{array}{c}
\begin{array}{ccc}
i & j & i \\
i & i & i
\end{array}
\end{array}
\]

(32)

\[
\begin{array}{c}
\begin{array}{ccc}
i & j & i \\
i & i & i
\end{array}
\end{array}
\]

(33)

\[
\begin{array}{c}
\begin{array}{ccc}
i & j & i \\
i & i & i
\end{array}
\end{array}
\]

(34)

\[
\begin{array}{c}
\begin{array}{ccc}
i & j & i \\
i & i & i
\end{array}
\end{array}
\]

Reverse the orientation of a single edge $i \rightarrow j$ and change $\tau_{ij}$ to $-\tau_{ij}$ and $\tau_{ji}$ to $-\tau_{ji}$. Denote the new datum by $\tau'$. Algebras $R_{\tau}(\nu)$ and $R_{\tau'}(\nu)$ are isomorphic via a map which is the identity on diagrams. This way, the study of $R_{\tau}(\nu)$ reduces to the case of any preferred orientation of $\Gamma$. Rescaling one of the two possible types of the
ij crossing by \( \lambda \in k \) changes \( \tau_{ij} \) to \( \lambda \tau_{ij} \) and \( \tau_{ji} \) to \( \lambda \tau_{ji}^{-1} \) over all edges of \( \Gamma \), via non-canonical isomorphisms. Rescalings of \( ii \) crossings and dots further reduce the number of parameters to the rank of the first homology group of \( \Gamma \). When graph \( \Gamma \) is a forest (has no cycles), algebras \( R(\nu) \) are all isomorphic to \( R(\nu) \) via rescaling of generators. When \( \Gamma \) has a single cycle, rescaling of generators reduces this family of algebras to a one-parameter family, with the parameter taking values in \( k^* \). It is likely that \( R(\nu) \) has a description via equivariant convolution algebras in Lusztig’s geometrization \( \mathfrak{H} \) of \( U^- \) when all \( \tau_{ij} = 1 \) (compare with Conjecture 1.2 in \( \mathfrak{H} \)).

Form

\[
R(\tau) = \bigoplus_{\nu \in \mathbb{N}[t]} R(\nu).
\]

The Grothendieck group \( K_0(R(\tau)) \) of the category of finitely-generated graded left projective modules can be naturally identified with the integral version \( A_f \) of \( U^- \). All other essential constructions and results of \( \mathfrak{H} \) generalize from \( R(\nu) \) to algebras \( R(\nu) \) in a straightforward fashion.

Modifications in the general case. Rings \( R(\nu) \) associated to an arbitrary Cartan datum admit similar modifications that depend on choosing an orientation of \( \Gamma \) and invertible elements \( \tau_{ij}, \tau_{ji} \) of the ground field \( k \) for each oriented edge \( i \to j \).

The key point is the change in the definition of the endomorphism algebra, making \( \delta_{k,i} \) act by

\[
f \mapsto (\tau_{ik} x_{k+1} (s_k)^{d'} - \tau_{k+1,ik} x_k (s_k)^d) (s_k f) \quad \text{if} \quad i_k \to i_{k+1}
\]

in the last of the four cases, with \( d = d_{ik+1k} \) and \( d' = d_{ik,1k+1} \). Our proof of categorified quantum Serre relations for \( R(\nu) \) requires only minor changes in the general case of \( R(\nu) \). Everything else generalizes as well.

References

[1] J. Brundan and A. Kleshchev. Hecke-Clifford superalgebras, crystals of type \( A^{(2)}_{2l} \) and modular branching rules for \( S_n \). Represent. Theory, 5:317–403 (electronic), 2001, math.RT/0103060. MR1870595 (2002i:17024)

[2] I. Grojnowski and M. Vazirani. Strong multiplicity one theorems for affine Hecke algebras of type \( A \). Transform. Groups, 6(2):143–155, 2001. MR1835669 (2002c:20008)

[3] L. Kauffman and S. Lins. Temperley-Lieb recoupling theory and invariants of 3-manifolds, volume 134 of Annals of Mathematics Studies. Princeton University Press, 1994. MR1280463 (95c:57027)

[4] M. Khovanov and A. Lauda. A diagrammatic approach to categorification of quantum groups I. Representation Theory, 13:309–347 (electronic), 2009. MR2525917 (2010i:17023)

[5] A. Kleshchev. Linear and projective representations of symmetric groups, volume 163 of Cambridge Tracts in Mathematics. Cambridge Univ. Press, 2005. MR2165457 (2007b:20022)

[6] G. Lusztig. Introduction to quantum groups, volume 110 of Progress in Mathematics. Birkhäuser Boston Inc., Boston, MA, 1993. MR1227098 (94m:17016)

[7] M. Okado and H. Yamane. \( R \)-matrices with gauge parameters and multi-parameter quantized enveloping algebras. In Special functions (Okayama, 1990), ICM-90 Satell. Conf. Proc., pages 289–293. Springer, Tokyo, 1991. MR1166822 (93i:17025)
[8] N. Reshetikhin. Multiparameter quantum groups and twisted quasitriangular Hopf algebras. Lett. Math. Phys., 20(4):331–335, 1990. MR1077966 [91k:17012]

[9] M. Vazirani. Irreducible modules over the affine Hecke algebra: A strong multiplicity one result. Ph.D. thesis, UC Berkeley, 1999, math.RT/0107052.

Department of Mathematics, Columbia University, New York, New York 10027
E-mail address: khovanov@math.columbia.edu

Department of Mathematics, Columbia University, New York, New York 10027
E-mail address: lauda@math.columbia.edu