Group Actions, Cyclic Coverings and Families of K3-Surfaces

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Abstract. In this paper we describe six pencils of K3-surfaces which have large Picard number ($\rho = 19, 20$) and each contains precisely five special fibers: four have A-D-E singularities and one is non-reduced. In particular, we characterize these surfaces as cyclic coverings of some K3-surfaces described in a recent paper by Barth and the author. In many cases, using 3-divisible sets, resp., 2-divisible sets, of rational curves and lattice theory, we describe explicitly the Picard lattices.

Introduction

Recently, using various methods (toric geometry, mirror symmetry, etc.), many families of K3-surfaces with large Picard number and small number of special fibers have been constructed and studied (see [D, VY, Be]). Here we use group actions and cyclic coverings to describe six new families where the generic surface has Picard number 19 and we identify four surfaces with Picard number 20. These six pencils are related to three families of K3-surfaces studied by Barth and the author [BS]: the generic surface has Picard number 19 and the pencils contain four surfaces with singularities of A − D − E type and $\rho = 20$ and one non-reduced fiber. The families arise as minimal resolutions of quotients $X^n_\lambda / G_n$ where $G_n$ is a special finite subgroup of $SO(4, \mathbb{R})$ containing the Heisenberg group and $\{X^n_\lambda\}_{\lambda \in \mathbb{P}_1}$ is a $G_n$-invariant pencil of surfaces in $\mathbb{P}_3$, the latter are described in [S1] (we recall some facts in Section 1). In Sections 1 and 2 we describe six normal subgroups $H$ of $G_n$ which contain the Heisenberg group, describe the fixed points of $H$ on $X^n_\lambda$, and show that the minimal resolutions are pencils of K3-surfaces which contain five special surfaces. In Section 3 we show that the new families are certain cyclic coverings of the surfaces of [BS]. Then, by a classical result of Inose, [I, Cor. 1.2], they have the same Picard number, hence the general surface in each of the six pencils has Picard number 19 and we have four surfaces with Picard number 20. In Section 4 by using the rational curves on the minimal resolutions and 2-divisible and 3-divisible sets of rational curves, we describe completely the Picard lattice of many of the surfaces.

1 Notations and Preliminaries

There are two classical 2:1 coverings:

$$SU(2) \rightarrow SO(3, \mathbb{R}) \quad \text{and} \quad \sigma: SU(2) \times SU(2) \rightarrow SO(4, \mathbb{R}).$$
Denote by $T, O \subset SO(3, \mathbb{R})$, the rotation groups of the tetrahedron and octahedron, by $\tilde{T}, \tilde{O}$ the corresponding binary subgroups of $SU(2)$, and let $G_6 := \sigma(\tilde{T} \times \tilde{T})$, $G_8 := \sigma(\tilde{O} \times \tilde{O})$. We denote an element of $SU(2) \times SU(2)$ and its image in $SO(4, \mathbb{R})$ by $(p_1, p_2)$. Let $X_6^s = s_6 + \lambda q^4$ and $X_8^s = s_8 + \lambda q^4$ denote the pencils of $G_6$- and $G_8$-invariant surfaces in $\mathbb{P}^3$, which are described in [S1]; $s_6$ denotes a $G_6$-invariant homogeneous polynomial of degree six and $s_8$ denotes a $G_8$-invariant homogeneous polynomial of degree eight; and $q := x_0^2 + x_1^2 + x_2^2 + x_3^2$ is the equation of the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ in $\mathbb{P}^3$. The base locus of the pencils $X_n^s$ are $2n$ lines on the quadric with $n$ in each ruling, and each pencil contains exactly four nodal surfaces, cf. [S1].

Now recall the matrix:

$$C := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \in O(4, \mathbb{R}),$$

which operates on an element $(p_1, p_2) \in G_1 \times G_2$ by:

$$C^{-1}(p_1, p_2)C = (p_2, p_1).$$

Moreover we specify the following matrices of $SO(4, \mathbb{R})$:

$$\begin{pmatrix}
q_1, 1
\end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} q_2, 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix}
q_3, 1
\end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} q_4, 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Using these matrices the groups have the following generators:

| Group | Generators |
|-------|------------|
| $G_6$ | $(q_2, 1), (1, q_2), (p_3, 1), (1, p_3)$ |
| $G_8$ | $(q_2, 1), (1, q_2), (p_3, 1), (1, p_3), (p_4, 1), (1, p_4)$ |

Denote by $PG$ the image of a subgroup $G \subset SO(4, \mathbb{R})$ in $\mathbb{P} GL(4, \mathbb{R})$. We define the types of lines in $\mathbb{P}^3$ which are fixed by elements $(p_1, p_2) \in PG$ of order 2, 3 or 4 in the following way:

| order | type |
|-------|------|
| 2     | $M$  |
| 3     | $N$  |
| 4     | $R$  |
1.1 Normal Subgroups

In [S2] the author classifies all the subgroups of $SO(4, \mathbb{R})$ which contain the Heisenberg group $V \times V$. Here we consider all the normal subgroups of $G_6$ and of $G_8$ which contain the subgroup $V \times V$, resp. $G_6$. We denote by $H$ such a normal subgroup, by $o(H)$ its order and by $i(H) = [G_n:H]$ the index of $H$ in $G_n$. We list below all the groups $H$ and their generators, following the notation of [S2]. Moreover, we do not consider separately the groups $H$ and $C^{-1}HC$ or, in general, groups which are conjugate in $O(4, \mathbb{R})$. The group $T \times T$ is in fact the same as $G_6$, but to avoid confusion we use this notation when we consider it as subgroup of $G_8$.

![Table]

1.2 Fixed Points

We analyze the different kind of fixed points for elements of the subgroups $PH \subset PG$ in the same way as in [BS]. Recall that the elements of the form $(p, 1)$ or $(1, p)$ have each two disjoint lines of fixed points contained in one ruling, respectively, in the other ruling of the quadric (cf. [S1, 5.4, p. 439]).

**Fixed Points on the Quadric:** The subgroups $G_1 \times 1$ and $1 \times G_2$ of $PH$ operate on the two rulings of the quadric and determine orbits of lines. We give the lengths of the orbits in the following tables. In the first row we write the order of the element which fixes two lines of the orbit:

![Table]
Fixed Points Off the Quadric: We denote by $F_L$ the fixed group of a line $L$ of $\mathbb{P}^3$ in $PH$ and by $H_L$ the stabilizer group of $L$ in $PH$, i.e.,

$$F_L := \{ h \in PH \text{ s.t. } hx = x \text{ for all } x \in L \}$$

$$H_L := \{ h \in PH \text{ s.t. } hL = L \}.$$  

Moreover, denote by $\ell(L)$ the length of the $H$-orbit of the line $L$ and by $g$ a representative of a conjugacy class in $H$:

| Group | $T \times V$ | $(TT)'$ | $V \times V$ |
|-------|--------------|----------|-------------|
| $F_L$ | $(q_1, q_1)$ | $(q_1, q_1)$ | $(q_1, q_1)$ |
| type | $Z_2$ | $Z_2$ | $Z_2$ |
| $\ell(L)$ | 6 | 6 | 6 |
| $|H_L|/|F_L|$ | 4 | 4 | 4 |

Here we denote by $q_3 \in SU(2)$ the product of $q_1$ and $q_2$. In the last column of the table the sum runs over $i, j = 1, 2, 3$. In this case we have nine distinct conjugacy classes with just one element.

| Group | $O \times T$ | $(OO)'$ |
|-------|--------------|----------|
| $F_L$ | $(q_1, q_1)$ | $(p_3, p_3)$ |
| type | $Z_2$ | $Z_3$ |
| $\ell(L)$ | 18 | 32 |
| $|H_L|/|F_L|$ | 8 | 3 |

| Group | $T \times T$ |
|-------|--------------|
| $F_L$ | $(q_2, q_2)$ |
| Type | $Z_2$ |
| $\ell(L)$ | 18 |
| $|H_L|/|F_L|$ | 4 |

Remark 1.1 By taking the generator $(p_3, p_3)$ for $(TT)'$ instead of $(p_3, p_3)$ we find a group $(TT)''$ which is conjugate in $O(4, \mathbb{R})$ to $(TT)'$. The description of the fixed points is similar to that in the case of $(TT)'$.

2 Quotient Surfaces

2.1 Quotient Singularities

We now consider the projections:

$$\pi_H: X^6_\lambda \rightarrow X^6_\lambda / H, \quad \pi_{H'}: X^8_\lambda \rightarrow X^8_\lambda / H'$$
with \( H = T \times V \), \((TT)\)' or \( V \times V \); \( H' = O \times T \), \((OO)''\) or \( T \times T \). In this section we run the same program as in [BS, §3] and describe the singularities of the quotients (for the details cf. [BS]).

**Fixed Lines on \( q \):** The image in the quotient of the lines of the base locus of the pencils \( X_6^\lambda \) and \( X_8^\lambda \) and of the intersection points of the lines of the base locus are smooth. Observe that the points of intersection of the lines of the base locus of the pencils form one orbit under the action of \( T \times V \), \( V \times V \), \( O \times T \) and \( T \times T \). In the case of the groups \((TT)'\) we have three orbits and in the case of the group \((OO)''\) we have two orbits, as described in §1.2, this means that the lines in the quotient will meet three times and two times. Now we consider the points of intersection of the lines of the base locus with the other fixed lines on \( q \). In the table below we do not write the groups \((TT)'\) and \( V \times V \) because they do not have fixed points on \( q \) other than the lines of the base locus. We denote by \( \text{Fix}(P) \) the fix-group in \( PG \) of a point \( P \). In the next table we write the length and the number of orbits of fixed points, and we describe which kind of singularities we have in the quotient:

| Group | \( T \times V \) | \( O \times T \) | \( (TT)' \) | \( V \times V \) | \( (OO)'' \) | \( T \times T \) |
|-------|------------------|----------------|-------------|----------------|-------------|-------------|
| Fix(\( P \)) | \( Z_3 \times Z_2 \) | \( Z_3 \times Z_2 \) | \( Z_4 \times Z_3 \) | \( Z_2 \times Z_3 \) | \( Z_3 \times Z_2 \) | \( Z_3 \times Z_3 \) |
| Length | 8 | 24 | 48 | 48 | 24 | 24 |
| Number | 6 | 2 | 1 | 2 | 2 | 1 |
| Sing. | 6A_2 | 1A_1 | 2A_3 | 2A_1 | 1A_1 | 1A_1 |

**Fixed Lines Off \( q \):** Denote by \( o(L) \) the order of the fix-group \( F_L \) of \( L \). The number of points not on \( q \) cut out on \( X_7^\lambda \) by \( L \) is:

| Group | \( T \times V \) | \( (TT)' \) | \( V \times V \) | \( O \times T \) | \( (OO)'' \) | \( T \times T \) |
|-------|------------------|-------------|----------------|-------------|-------------|-------------|
| \( o(L) \) | 2 | 2 | 2 | 2 | 2 | 2 |
| Number | 4 | 4 | 6 | 4 | 8 | 6 |

In the next table we show in each case length and number of \( H_L \)-orbits, the number and type(s) of the quotient singularity(ies):

| Group | \( O \times T \) | \( (OO)'' \) | \( T \times T \) |
|-------|------------------|-------------|-------------|
| \( o(L) \) | 2 | 2 | 2 |
| Type | \( M \) | \( M' \) | \( M' \) |
| Length | 8 | 3 | 3 |
| Number | 1 | 2 | 2 |
| Singularities | \( A_1 \) | 2A_2 | 2A_1 |

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The Singular Surfaces: We denote by $ns$ the number of nodes on the surfaces and by $F$ the fix-group of a node in $H$. In the table below, we give the number of orbits of nodes and their fix-groups in $PH, PH'$ and we describe the singularities in the quotient. We recall [BS, Proposition 3.1]:

**Proposition 2.1** Let $X$ be a nodal surface with $F \subset SO(3)$ the fix-group of the node. Then the image of this node on $X/H$ is a quotient singularity locally isomorphic with $C^2/F$, where $F \subset SU(2)$ is the binary group which corresponds to $F$.

| Group | $T \times V$ | $(TT)'$ | $V \times V$ |
|-------|-------------|---------|-------------|
| $\lambda$ | $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ | $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ | $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ |
| $ns$ | 12 48 48 12 | 12 48 48 12 | 12 48 48 12 |
| Orbit | 1 1 1 1 | 3 3 1 1 | 3 3 3 3 |
| $F$ | $Z_2 \times Z_2 \ id \ id \ Z_2 \times Z_2$ | $T \ Z_3 \ id \ Z_2 \times Z_2$ | $Z_2 \times Z_2 \ id \ id \ Z_2 \times Z_2$ |
| Lines | $1M_1$ | $1M_1$ | $3M_{ij}$ |
| Meeting | $1M_2$ | $1M_2$ | $4N_1$ |
| Sing. | $D_4$ | $A_1 \ A_1 \ D_4$ | $3E_6 \ 3A_5 \ A_1 \ D_4$ |

| Group | $O \times T$ | $(OO)'$ | $T \times T$ |
|-------|-------------|---------|-------------|
| $\lambda$ | $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ | $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ | $\lambda_1 \lambda_2 \lambda_3 \lambda_4$ |
| $ns$ | 24 72 144 96 | 24 72 144 96 | 24 72 144 96 |
| Orbit | 1 1 1 1 | 2 2 3 2 | 2 1 1 2 |
| $F$ | $T \ Z_2 \times Z_2 \ Z_2 \ Z_3$ | $O \ Z_4 \ Z_2 \ D_3$ | $T \ Z_2 \ id \ Z_3$ |
| Lines | $3N \ 1M \ 1M' \ 1N$ | $3R \ 1R \ 1M \ 1N(4') \ 3M \ 1M \ 1N(4')$ |
| Meeting | $4N_1 \ 2M'$ | $4N(4') \ 3M$ | $4N(4')$ |
| Sing. | $E_6 \ D_4 \ A_3 \ A_5$ | $2E_7 \ A_7 \ A_3 \ 2D_5$ | $2E_6 \ A_3 \ A_1 \ 2A_5$ |

### 2.2 Rational Curves

Let 

$$\mu: Y_{\lambda, H} \rightarrow X^n_{\lambda}/H$$

be the minimal resolution of the singularities of $X^n_{\lambda}/H$. In the following table we give the number of rational curves coming from the curves of the base locus of $X^n_{\lambda}$ (denote it by $\nu_1$) and from the resolution of the singularities. The latter are of three kinds: those coming from the intersection points of the lines of the base locus with other fixed lines on $q$, those coming from fixed points which are off $q$ and do not come from nodes of $X^n_{\lambda}$, and those coming from the nodes. We denote their numbers by $\nu_2, \nu_3$ and $\nu_4$; then the total number of rational curves is $\nu := \nu_1 + \nu_2 + \nu_3 + \nu_4$. The configurations of some of the curves are then given in the figures in Section 5. In the table we write the discriminant $d$ of the intersection matrix, too. This is easy to compute because we know the configurations of the rational curves. Since in each case $d \neq 0$, the classes of the curves are independent in $NS(Y_{\lambda, H})$. 

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The Smooth $X^n_\lambda$:

| Group | $T \times V$ | $(TT)'$ | $V \times V$ | $O \times T$ | $(OO)'''$ | $T \times T$ |
|-------|--------------|----------|-------------|-------------|------------|--------------|
| $\nu_1$ | 4 | 2 | 6 | 3 | 2 | 4 |
| $\nu_2$ | 12 | — | — | 9 | 2 | 4 |
| $\nu_3$ | 3 | 15 | 9 | 7 | 14 | 10 |
| $\nu$ | 19 | 17 | 15 | 19 | 18 | 18 |
| $d$ | $2^5 \cdot 3^3 \cdot 5$ | $2^3 \cdot 3^6 \cdot 7$ | $2^1 \cdot 3^5 \cdot 7$ | $2^4 \cdot 3^2 \cdot 7$ | $2^4 \cdot 3^6 \cdot 7$ | $2^2 \cdot 3^6 \cdot 7$ |

The Singular $X^\mu_\lambda$:

In this case the surfaces $X^\mu_\lambda$ do not have extra singularities on $q_\lambda$, hence the number $\nu_1$ and $\nu_2$ remain the same as above and we do not write them again.

| Group | $T \times V$ | $(TT)'$ |
|-------|--------------|----------|
| $\lambda$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
| $\nu_3$ | — | 3 | 3 | — | — | 3 | 15 | 12 |
| $\nu_4$ | 4 | 1 | 1 | 4 | 18 | 15 | 1 | 4 |
| $\nu$ | 20 | 20 | 20 | 20 | 20 | 20 | 18 | 18 |
| $d$ | $-2^5 \cdot 3^3 \cdot 5$ | $-2^6 \cdot 3^6 \cdot 7$ | $-2^6 \cdot 3^3 \cdot 5$ | $-2^4 \cdot 3^1 \cdot 5$ | $-3^3 \cdot 5$ | $-2^6 \cdot 3^1 \cdot 5$ | $-2^4 \cdot 3^6 \cdot 5$ | $-2^2 \cdot 3^6 \cdot 5$ |

| Group | $V \times V$ | $O \times T$ |
|-------|--------------|-------------|
| $\lambda$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
| $\nu_3$ | — | 9 | 9 | — | 2 | 4 | 5 | 3 |
| $\nu_4$ | 12 | 3 | 3 | 12 | 6 | 4 | 3 | 5 |
| $\nu$ | 18 | 29 | 18 | 18 | 20 | 20 | 20 | 20 |
| $d$ | $-2^{10} \cdot 5$ | $-2^{10} \cdot 5$ | $-2^{10} \cdot 5$ | $-2^{10} \cdot 5$ | $-2^4 \cdot 3^2 \cdot 7$ | $-2^4 \cdot 3^1 \cdot 7$ | $-2^5 \cdot 3^1 \cdot 7$ | $-2^6 \cdot 3^2 \cdot 7$ |

| Group | $(OO)'''$ | $T \times T$ |
|-------|------------|--------------|
| $\lambda$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
| $\nu_3$ | — | 8 | 12 | 6 | — | 8 | 10 | 2 |
| $\nu_4$ | 16 | 7 | 3 | 10 | 12 | 3 | 1 | 10 |
| $\nu$ | 20 | 19 | 19 | 20 | 20 | 19 | 19 | 20 |
| $d$ | $-2^4 \cdot 7$ | $2^2 \cdot 3^2 \cdot 7$ | $2^8 \cdot 3^2 \cdot 7$ | $-2^8 \cdot 7$ | $-3^4 \cdot 7$ | $2^4 \cdot 3^6 \cdot 7$ | $2^3 \cdot 3^6 \cdot 7$ | $-2^4 \cdot 3^4 \cdot 7$ |

2.3 K3-Surfaces

Since the groups $H$ and $H'$ contain the subgroups $V \times V$ of $G_6$, resp., $T \times T$ of $G_8$, the projections $\pi_H$ and $\pi_H'$ are ramified on the lines of the base locus of the families $X^\mu_\lambda$ with ramification index two and three. By using Hurwitz’ formula and the fact that in each case the previous rational curves are independent in the Neron–Severi group, the same computation as in [BS, section 5] shows that the minimal resolutions of the quotients are K3-surfaces, a direct proof of this fact is given in the next section.
3 Cyclic Coverings

We give another description of the pencils of K3-surfaces by using cyclic coverings. We consider the pairs $G_n$ and $H$ so that $G_n/H$ is cyclic, in our cases either $|G_n/H| = 3$ or $|G_n/H| = 2$, and we consider the map:

$$\pi: X^n_0/H \longrightarrow X^n_0/G_n$$

3.1 The General Case

For the moment assume that $X^n_0$ is smooth. The group $G_n/H$ acts on the points of the fiber $\pi^{-1}(P)$. If the point $P$ is not fixed by $G_n/H$, then the map is $3:1$ or $2:1$ there. If $P$ is fixed by $G_n/H$ then we have a singularity on $X^n_0/H$, more precisely an $A_2$ or an $A_1$, now the fiber $\pi^{-1}(P)$ is one point and the map has multiplicity 2 or 3 there (cf. [M2, Lemma 3.6 p. 80]). We have a rational map between the minimal resolutions of $X^n_0/H$ and $X^n_0/G_n$:

$$\gamma: Y_{\lambda,H} \longrightarrow Y_{\lambda,G_n}$$

which is $3:1$ or $2:1$. Observe that this map is not defined over the $(-2)$-rational curves in the blow up of singular points of $X^n_0/G_n$ that come from fixed points of $G_n/H$ on $X^n_0/H$. The surfaces $Y_{\lambda,H}$ are K3-surfaces as well and by [I, Cor. 1.2] these have the same Picard number $\rho(Y_{\lambda,H}) = \rho(Y_{\lambda,G_n}) = 19$.

In this section we describe the map $\gamma$ by using cyclic coverings. For the general theory about 2-cyclic coverings and 3-cyclic coverings we refer to the article by Nikulin [N] and to articles by Miranda [M1] and Tan [T]. For the convenience of the reader, in Section 5, Figure 1, we recall the configurations of $(-2)$-rational curves on the smooth surfaces $Y_{\lambda,G_n}$ and on $Y_{\lambda,G_n}$ given in [BS]. By [BS, Proposition 6.1] the following classes are 3-divisible in $\text{NS}(Y_{\lambda,G_n})$:

$$\mathcal{L} := L_1 - L_2 + L_4 - L_5 + N_1 - N_2 + N_3 - N_4 + N_5 - N_6 + N_7 - N_8,$$
$$\mathcal{L}' := L_1' - L_2' + L_4' - L_5' + N_1 - N_2 + N_3 - N_4 - N_5 + N_6 - N_7 + N_8,$$

and also:

$$\mathcal{L} - \mathcal{L}' = L_1 - L_2 + L_4 - L_5 - L_1' + L_2' - L_4' + L_5' + 2(N_5 - N_6 + N_7 - N_8),$$
$$\mathcal{L} + \mathcal{L}' = L_1 - L_2 + L_4 - L_5 + L_1' - L_2' + L_4' - L_5' + 2(N_1 - N_2 + N_3 - N_4).$$

Making reduction modulo three we find the classes:

$$\mathcal{M} := L_1 - L_2 + L_4 - L_5 - L_1' + L_2' - L_4' + L_5' - (N_5 - N_6 + N_7 - N_8),$$
$$\mathcal{M}' := L_1 - L_2 + L_4 - L_5 + L_1' - L_2' + L_4' - L_5' - (N_1 - N_2 + N_3 - N_4).$$

In $\text{NS}(Y_{\lambda,G_n})$ the following classes are 2-divisible:

$$\mathcal{L} := L_1 + L_3 + L_5 + M_1 + M_3 + M_4 + R_1 + R_3,$$
$$\mathcal{L}' := L_1' + L_3' + L_5' + M_2 + M_3 + M_4 + R_1 + R_3.$$
Consider also the classes $\mathcal{L} + \mathcal{L}'$ and $\mathcal{L} - \mathcal{L}'$, which after reduction modulo two are the same as:

$$\mathcal{M} := L_1 + L_3 + L_5 + L'_1 + L'_3 + M_1 + M_2.$$ 

These classes consist of six disjoint $A_2$-configurations of curves and of eight disjoint $A_1$-configurations of curves (according to [T, N]). These are the resolutions of $A_2$ and $A_1$ singularities of $X^n_\lambda / G_n$ which arise by doing the quotient of $X^n_\lambda / H$ by $G_n / H$. We construct the 3-cyclic coverings and the 2-cyclic coverings by using the divisors $\mathcal{L}, \mathcal{L}', \mathcal{M}, \mathcal{M}'$. We recall briefly the construction in the case of 3-cyclic coverings. Then in the case of 2-cyclic coverings it is similar. First to avoid producing singularities, we blow up the meeting points of the $A_2$-configurations. Call $Y^0_{\lambda, G_n}$ the surface which we obtain after these blow-ups. The meeting points are replaced by $(-1)$-curves and the two $(-2)$-curves become now $(-3)$-curves. Denote by $\phi: Y^0_{\lambda, G_n} \to Y^0_{\lambda, G_n}$ the 3-cyclic covering with branching divisor $\mathcal{L}, \mathcal{L}'$ or $\mathcal{M}, \mathcal{M}'$ then

**Proposition 3.1** A configuration of curves on $Y^0_{\lambda, G_n}$:

$$\begin{align*}
-3 & \quad -1 & \quad -3 \\
-3 & \quad -1 & \quad -1
\end{align*}$$

becomes a configuration:

$$\begin{align*}
-1 & \quad -3 & \quad -1
\end{align*}$$

on $Y^1_{\lambda, G_n}$.

**Proof** We do the computation for one configuration of curves $L_1 - L_2$; this is the same in the other cases. Denote again by $L_1$ and $L_2$ the curves on $Y^0_{\lambda, G_n}$ which now are $(-3)$-curves and denote by $E$ the exceptional $(-1)$-curve. By the properties of cyclic coverings we have $\phi^*L_1 = 3\tilde{L}_1$, where $\tilde{L}_1$ is the strict transform of $L_1$. Then:

$$9(\tilde{L}_1)^2 = (\phi^*L_1)^2 = (\deg \phi)L_1^2 = -9.$$ 

Hence $(\tilde{L}_1)^2 = -1$. Since $E \cdot (L_1 - L_2) = 0$ the map $\phi$ is not ramified on $E$ and the restriction $\phi|_E$ is 3:1 onto $E$. Hence we have $\phi^*E = \bar{E}$ and $E^2 = (\phi^*E)^2 = (\deg \phi)E^2 = 3E^2 = -3$. 

Our surface $Y^1_{\lambda, G_n}$ is now no longer minimal. By blowing down the $(-1)$-curves, the curve $E$ also becomes a $(-1)$-curve, so we blow it down, too. By construction, the surfaces which we obtain are minimal $K3$-surfaces and are exactly the surfaces $Y_{\lambda, H}$.
which are obtained as the minimal resolutions of $X^6_\lambda/H$, in fact we have a commutative diagram:

\[
\begin{array}{ccccccccc}
Y^1_{\lambda,G_6} & \phi & Y^0_{\lambda,G_6} \\
Y_{\lambda,H} & \gamma & Y_{\lambda,G_6} \\
X^6_\lambda/H & \pi & X^6_\lambda/G_6
\end{array}
\]

The construction is similar in the case of 2-cyclic coverings of the surfaces $Y_{\lambda,G_6}$. This gives another description of the families of K3-surfaces $Y_{\lambda,T\times V}, Y_{\lambda,(TT)'}$ and $Y_{\lambda,O\times T}, Y_{\lambda,(OO)''}$ as finite coverings of the families $Y_{\lambda,G_6}$ and $Y_{\lambda,G_6}$.

**Remark** By using the divisors $L'$ and $M'$ on $Y_{\lambda,G_6}$ for the coverings we obtain the surfaces $Y_{\lambda,V\times T}$ and $Y_{\lambda,(TT)''}$ and by taking the divisor $L'$ on $Y_{\lambda,G_6}$ we obtain the surface $Y_{\lambda,T\times O}$. We do not discuss these surfaces separately in the sequel.

### 3.2 The Special Cases

In these cases, the situation is a little more complicated. Now in the counterimage $\pi^{-1}(P)$ of some singular point $P$ of $X^6_\lambda/G_6$, coming from the $A_1$-singularities of $X^6_\lambda$ we have singularities on $X^6_\lambda/H$, too. In the following table we give the singularities in the quotient $X^6_\lambda/G_n, n = 6, 8$ and the type and the number of singularities in the counterimage on $X^6_\lambda/H$. As in [BS] we denote by 6, 1, . . . , 8, 4 the special surfaces in the families.

| $G_6$ | $G_8$ | $\pi^{-1}(P)$ |
|-------|-------|----------------|
| $T\times V$ | $E_6$ | $A_1$ | $A_3$ | $A_5$ | $E_6$ | $D_4$ | $A_3$ | $A_5$ |
| $(TT)'$ | $3E_6$ | $3A_5$ | $A_1$ | $D_4$ | $(OO)'$ | $2E_7$ | $A_3$ | $D_5$ |
| $V\times V$ | $3D_4$ | $3A_1$ | $3A_3$ | $3D_4$ | $T\times T$ | $2E_6$ | $A_3$ | $A_1$ | $2A_5$ |

By resolving the quotients we get a map like $\gamma$ as before and so again by the result of Inose the minimal resolutions of the special K3-surfaces are K3-surfaces too with Picard number 20. We can describe this map as before by using cyclic coverings. In the case of the special surfaces in the family $Y_{\lambda,G_6}$ we construct 3-cyclic covering as in the general case by using the divisors $L, L', M, M'$, which are in the case of the special surfaces 3-divisible, too, cf. [BS, 6.2]. By taking $L$ and $L'$ we obtain the special K3-surfaces in the families $Y_{\lambda,T\times V}$, resp., $Y_{\lambda,V\times T}$, by taking $M$ and $M'$ we obtain the special K3-surfaces in the covering $Y_{\lambda,(TT)'}$ and $Y_{\lambda,(TT)''}$. In the case of the
special surfaces in the family \( Y_{\lambda,G_6} \) we take the divisors \( \mathcal{L}, \mathcal{L}', \mathcal{M} \) and we do 2-cyclic coverings. By taking \( \mathcal{L} \) or \( \mathcal{L}' \) we find the singular surfaces in the family \( Y_{\lambda,O \times T} \), resp., \( Y_{\lambda,T \times O} \), and by taking \( \mathcal{M} \) we find the singular surfaces in the family \( Y_{\lambda,(00)''} \).

4 Picard Lattices

We compute the Picard lattices of the general K3-surface in the families \( Y_{\lambda,T \times V} \), \( Y_{\lambda,O \times V} \) and of the special surfaces with \( \rho = 20 \) in each pencil. First we recall some facts. Denote by \( W \) the lattice spanned by the curves of Section 2, 2.2. If \( W \) is not the total Picard lattice, which we call \( NS \), there is an integral lattice \( W' \) such that \( W \subset W' \subset NS \) with \( \rho := [W':W] \), a prime number. Denote by \( d(W), d(W') \) the discriminant of the lattices \( W, W' \). Since \( [W':W]^2 = d(W) \cdot d(W')^{-1} \) (cf. [BPV, Lemma 2.1, p. 12]), we find that \( p^2 \) divides the discriminant of \( W \). Denote by \( (W'/W)^p \) the \( p \)-subgroup of \( (W'/W) \subset NS'/NS \) and denote by \( T \) the transcendental lattice orthogonal to the Picard lattice. Since the discriminant groups \( T'/T \) and \( NS'/NS \) are isomorphic (cf. [BPV, p. 13, Lemma 2.5]), they have the same rank which is \( \leq \text{rk}(T) \). It follows that also \( \text{rk}(W'/W)^p \leq \text{rk}(T) \).

**Proposition 4.1** The Picard lattices of the generic surface \( Y_{\lambda,T \times V} \) and \( Y_{\lambda,O \times T} \) are generated by the 19 rational curves of Section 2.2 and the classes:

\[
\begin{align*}
\tilde{L}'/3 & := L_1 - L_2 + L_4 - L_5 + L'_1 - L'_2 + L'_3 - L'_4 + L''_1 - L''_2 + L''_3, \\
\tilde{h}_1/2 & := L_1 + L_3 + L_5 + L'_1 + L'_3 + L'_5 + M_1 + M_3, \\
\tilde{h}_2/2 & := L_1 + L_3 + L_5 + L''_1 + L''_3 + L''_5 + M_1 + M_3,
\end{align*}
\]

of \( NS(Y_{\lambda,T \times V}) \), then the lattice has discriminant \( 2 \cdot 3 \cdot 5 \), resp., the classes

\[
\begin{align*}
\tilde{L}''/2 & := L_1 + L_3 + L_5 + L'_1 + L'_3 + L'_5 + M_1 + M_2, \\
k_1/3 & := L_1 - L_2 + L_4 - L_5 - L'_1 + L'_3 - L'_4 + L''_1 + N_1 - N_2 + N_3 - N_4,
\end{align*}
\]

of \( NS(Y_{\lambda,O \times T}) \), then the lattice has discriminant \( 2^3 \cdot 3 \cdot 7 \).

**Proof**

**Step 1.** The discriminant of the lattice generated by the 19 curves is \( 2^5 \cdot 3^3 \cdot 5 \) hence we can have 2-divisible classes or 3-divisible classes. The divisor \( \tilde{L}' \) is 3-divisible since it is the pull back of the divisor \( \mathcal{L}' \) on \( Y_{\lambda,G_6} \) which is 3-divisible, too. And we cannot have more 3-divisible classes. If there are no 2-divisible classes, then the group
would contain the classes $M_1/2, M_2/2, M_3/2, (L_1 + L_3 + N_1 + L_1' + L_2')/2,
(L_1 + L_3 + L_5 + L_1'' + L_2'' + L_3')/2, (L_1' + L_3' + L_5' + L_1'' + L_2'' + L_3')/2$ which are independent
classes with respect to the intersection form. Since the rank of $(W'\setminus W)^2$ is less than or equal to the rank of $T'\setminus T$ which is at most three, it cannot happen that we find five classes as before. Hence some combination of them must be contained in the Neron–Severi group. So we have

$$
\frac{1}{2} \left( \lambda (L_1 + L_3 + L_5) + \lambda' (L_1' + L_3' + L_5') + \lambda'' (L_1'' + L_3'' + L_5'') + \mu_1 M_1 + \mu_2 M_2 + \mu_3 M_3 \right) \in NS
$$

for some parameters $\lambda, \lambda', \lambda'', \mu_1, \mu_2, \mu_3 \in \mathbb{Z}_2$.

By Nikulin $[N]$ such a 2-divisible set contains 8 curves. So putting $\lambda'' = 0$ and $\mu_3 = 0$ we get the divisor $h_1/2$, putting $\lambda' = 0$ and $\mu_2 = 0$ we get the divisor $h_2/2$. The discriminant of the lattice $W$ together with these three classes now become $2 \cdot 3 \cdot 5$, hence we cannot have more torsion classes.

**Step 2.** Again the class $L''$ is the pull back of the class $M'$ on $Y_{L, C_6}$ hence it is 2-divisible. If there are no 3-divisible classes then the group $(W'\setminus W)^3$ would contain the classes $N_1 - N_2/3, N_3 - N_4/3$ and $(L_1 - L_2 + L_4 - L_5 + L_1' - L_3' + L_5')/3$ which are independent. By specializing to the surfaces $Y_{L, O \times O}^{(8,2)}$ and $Y_{L, O \times O}^{(6,3)}$, we also find here these three independent classes and so $rk(W'\setminus W)^3 \geq 3$. This is not possible. In fact, on these surfaces we have $rk(W) = 20$ which implies $rk(W'\setminus W)^3 \leq 2$. This means that the three classes fit together giving a 3-divisible class in $NS(Y_{L, O \times O}^{(8,2)})$ and $NS(Y_{L, O \times O}^{(6,3)})$ and so in $NS(Y_{L, O \times O})$ (cf. [vGT, Lemma 2.3]).

In the same way as before we can compute the Picard lattices of the special surfaces in the families. We state the results leaving the proofs to the reader.

**Proposition 4.2**

(i) The Picard lattice of the special surfaces in $Y_{L, T \times Y}$ and $Y_{L, O \times T}$ is generated in all the cases but $Y_{L, O \times T}^{(8,4)}$ by the curves of Section 2.2 and by the classes $L'/3$, $h_1/2$, $h_2/2$, resp. $L''/2$, $k_1/3$ of proposition 4.1. In the case of $Y_{L, O \times T}^{(8,4)}$ the class

$$
\frac{L_1 + L_3 + L_5 + N_1 + C + N_4 + R_2 + M_1}{2}
$$

is a generator too, and they span the 20-dimensional Picard lattice.

(ii) In the case of $Y_{L, O \times (L')}$, and $Y_{L, (L')}$, the class

$$
\frac{L}{3} := \frac{N_1 - N_2 + N_3 - N_4 + N_5 - N_6 + N_7 - N_8 + N_9 - N_{10} + N_{11} - N_{12}}{3}
$$

is a generator too.
is in the Neron–Severi group and in the case of \(Y_{\lambda,(TT)}^{(6,2)}\), the classes
\[
\frac{N_1 + C_1 + N_4 + N_5 + C_2 + N_9 + M_1 + M_2}{2},
\frac{N_1 + C_1 + N_4 + N_9 + C_3 + N_{12} + M_1 + M_3}{2}
\]
are in the Neron–Severi group, too. These together with the 20 curves of Section 2.2 span the 20-dimensional Picard lattice.

(iii) In the case of \(Y_{\lambda,(OO)}^{(8,1)}\), \(Y_{\lambda,(OO)}^{(8,4)}\), the class
\[
\frac{\bar{L}}{2} := M_1 + M_2 + M_3 + M_4 + R_1 + R_3 + R_1' + R_3'
\]
is in the Neron–Severi group and in the case of \(Y_{\lambda,(OO)}^{(8,4)}\), the class
\[
\frac{W}{4} := \frac{R_1 + 2R_2 + 3R_3 + R_1' + 2R_2' + 3R_3'}{4} + \frac{2N_1 + 2C_1 + 3M_1 + M_2 + 2N_3 + 2C_2 + 3M_3 + M_4}{4}
\]
is in the Neron–Severi group, too. Again these classes together with the 20 curves of Section 2.2 span the 20-dimensional Picard lattice. The discriminants of the Picard lattices then are:

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
 & Y_{\lambda,T \times V} & & Y_{\lambda,(TT)'} & & \\
\hline
 d & 6,1 & 6,2 & 6,3 & 6,4 & 6,1 & 6,2 \\
\hline
 & -3 \cdot 5 & -2^2 \cdot 3 \cdot 5 & -2^2 \cdot 3 \cdot 5 & -3 \cdot 5 & -3 \cdot 5 & -2^2 \cdot 3 \cdot 5 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
 & Y_{\lambda,O \times T} & & Y_{\lambda,(OO)''} & & \\
\hline
 d & 8,1 & 8,2 & 8,3 & 8,4 & 8,1 & 8,4 \\
\hline
 & -2^2 \cdot 7 & -2^2 \cdot 3 \cdot 7 & -2^3 \cdot 3 \cdot 7 & -2^2 \cdot 7 & -2^2 \cdot 7 & -2^4 \cdot 7 \\
\hline
\end{array}
\]

### 4.1 More Cyclic Coverings

Now we can construct the 3-cyclic covering of \(Y_{\lambda,T \times V}\), by using the 3-divisible classes \(\bar{T}^7\) and the 2-cyclic coverings of \(Y_{\lambda,O \times T}\), by using the 2-divisible class \(\bar{T}^7\). We can do this for the general surface in the pencil and for the special surfaces, too. In this case we obtain another description of the families \(Y_{\lambda,V \times V}\) and \(Y_{\lambda,T \times T}\). In particular, in these cases the general surface in the family also has Picard number 19 and we have four surfaces with Picard number 20. The description of the Picard lattices of the surfaces with \(\rho = 20\) is given in the following proposition (again, we leave the proof to the reader).
Proposition 4.3  The classes
\[
\begin{align*}
L_2 - L_4 + L'_3 - L'_1 &+ N_1 - N_2 + N_3 - N_4 + N_5 - N_6 + N_7 - N_8 \\
N_1 - C_1 + N_4 + N_5 + C_2 + N_8 + M_1 + M_2 \\
N_3 - L'_4 - L_3 - L_4 + N_1 - N_2 - N_3 + N_4 + N_5 - N_6 - N_7 + N_8
\end{align*}
\]
are in \(NS(Y^{(8,1)}_{\lambda,T\times T})\) and in \(NS(Y^{(8,4)}_{\lambda,T\times T})\). Moreover, the class
\[
\frac{N_1 + C_1 + N_4 + N_5 + C_2 + N_8 + M_1 + M_2}{2}
\]
is in \(NS(Y^{(8,4)}_{\lambda,T\times T})\) too. These classes together with the rational curves of Section 2, 2.2 span the 20-dimensional Picard lattices of the surfaces \(Y^{(8,1)}_{\lambda,T\times T}\) and \(Y^{(8,4)}_{\lambda,T\times T}\). Then the lattices have discriminant \(-7\), resp., \(-2^1 \cdot 7\).

5 Configurations of Rational Curves

In Figures 1, 2, 3, and 4 we give the configurations of rational curves on the surfaces with Picard number 19 and 20. In the case of the singular surfaces of the families \(Y_{\lambda,T\times V}\) and \(Y_{\lambda,O\times T}\) also the curves \(L_i, L'_i, L''_i\) are contained on the surfaces, but we do not draw the picture again. Moreover the configurations of curves on the surfaces \(Y^{(6,4)}_{\lambda,T\times V}\) and \(Y^{(6,3)}_{\lambda,T\times V}\) are the same as on the surfaces \(Y^{(6,1)}_{\lambda,T\times V}\), resp., \(Y^{(6,2)}_{\lambda,T\times V}\), so again we draw only one picture.

6 Final Remarks

In Section 4 we identify explicitly the Picard lattice of some K3-surfaces. It is our next aim to compute the transcendental lattices orthogonal to the Picard lattices to classify the K3-surfaces. In particular by a result of Shioda and Inose [SI] K3-surfaces with \(\rho = 20\) are classified by means of their transcendental lattice. By a result of Morrison [Mo], each K3-surface with \(\rho = 19\) or 20 admits a so called Shioda–Inose structure. This means that there is a Nikulin involution, an involution with eight isolated fixed points and the quotient is birational to a Kummer surface. It would be desirable to have an explicit description of this structure for our surfaces.

We do not describe the quotients 3-folds \(\mathbb{P}_3/G_n, \mathbb{P}_3/H\), \(H\) a normal subgroup of \(G_n\). It would be interesting to have a global resolution of these spaces and to see our K3-surfaces as smooth pencils on the smooth 3-folds.

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Figure 1

Figure 2

Figure 3
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