Statistical Trends in the Obliquity Distribution of Exoplanet Systems

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Abstract

Important clues to the formation and evolution of planetary systems can be inferred from the stellar obliquity ψ. We study the distribution of obliquities using the California-Kepler Survey and the TEPCat Catalog of Rossiter–McLaughlin (RM) measurements, from which we extract, respectively, 275 and 118 targets. We infer a “best fit” obliquity distribution in ψ with a single parameter κ. Large values of κ imply that ψ is distributed narrowly around zero, while small values imply approximate isotropy. Our findings are as follows. (1) The distribution of ψ in Kepler systems is narrower than found by previous studies and consistent with κ ∼ 15 (mean ⟨ψ⟩ ∼ 19° and spread ψ ∼ 10°). (2) The value of κ in Kepler systems does not depend, at a statistically significant level, on planet multiplicity, stellar multiplicity, or stellar age; on the other hand, metal-rich hosts, small-planet hosts, and long-period planet hosts tend to be more oblique than the general sample (at a ∼ 2.5σ significance level). (3) The obliquities of Hot Jupiter (HJ) systems with RM measurements are consistent with κ ∼ 2, which corresponds to a broader distribution than for the general Kepler population. (4) A separation of the RM sample into cooler (T_{eff} ≤ 6250 K) and hotter (T_{eff} ≥ 6250 K) HJ hosts results in two distinct distributions, κ_{cooler} ∼ 4 and κ_{hotter} ∼ 1 (4σ significance), both more oblique than the Kepler sample. We hypothesize that the total mass in planets may be behind the increasing obliquity with metallicity and planet radius, and that the dependence on period could be due to primordial disk alignment rather than tidal realignment of stellar spin.

Key words: methods: statistical – planetary systems – planets and satellites: general – planet–star interactions – stars: rotation

1. Introduction

The alignment of planetary orbits with the spin axis of their host star is a fundamental feature of exoplanetary architectures, one that points directly to the physical mechanisms behind planetary formation. In the solar system, for example, the Sun’s stellar spin is tilted with respect to the ecliptic by only ∼7° (Carrington 1863; Beck & Giles 2005). Exoplanetary systems, on the other hand, may exhibit severe spin–orbit misalignments, including nearly polar orientations (e.g., Kepler-63b, Sanchis-Ojeda et al. 2013). A complete, predictive theory of planet formation must explain the origin of both large and small stellar obliquities, being able to discern whether these are inherited from the protoplanetary disk or rather a consequence of later dynamical interactions.

The true, three-dimensional obliquity ψ of a planet-hosting star is not a direct observable. This angle can be expressed as (e.g., Winn et al. 2007)

$$\cos \psi = \cos I_\kappa \cos i_\rho + \sin I_\kappa \sin i_\rho \cos \lambda$$

where $i_\rho$ is the planet’s orbital inclination, $I_\kappa$ is the stellar line-of-sight (LOS) inclination, and λ is the obliquity projected onto the plane of the sky. Typically, λ is measured via the Rossiter–McLaughlin (RM) effect (McLaughlin 1924; Rossiter 1924) (e.g., see Queloz et al. 2000; Ohta et al. 2005; Giménez 2006), or even estimated via the variability of stellar spots (Nutzman et al. 2011), and can be related to ψ via statistical arguments (Fabrycky & Winn 2009). The stellar LOS inclination $I_\kappa$ is more difficult to measure directly; it can be estimated from asteroseismology (Gizon & Solanki 2003; Campante et al. 2016) or inferred from a combination of projected velocity measurements $V \sin I_\kappa$, stellar radii $R_*$, and rotational periods $P_\text{rot}$ (e.g., Winn et al. 2007; Hirano et al. 2014; Morton & Winn 2014). Provided that stellar radii and rotational periods are available for a large number of stars (as is the case for the Kepler catalog; e.g., McQuillan et al. 2014), the $V \sin I_\kappa$ approach to inference of stellar obliquity is the most cost-effective, since $V \sin I_\kappa$ measurements are easier to make than RM ones: not only do they require less spectral resolution and sensitivity, but also they do not need to be taken during transit (e.g., Gaudi & Winn 2007).

An ensemble of either $I_\kappa$ or $V \sin I_\kappa$ measurements facilitates statistical tests that can constrain the degree to which planetary orbits tend to be aligned/misaligned with the spins of their host stars (Fabrycky & Winn 2009; Morton & Winn 2014; Campante et al. 2016). Spin–orbit statistics can also provide valuable tools for identifying distinct planet populations or for distinguishing between models of planet formation. Focusing on the origins of hot Jupiters, Morton & Johnson (2011) compared the obliquity outcomes of two such models, the Lidov–Kozai migration model of hot Jupiters by Fabrycky & Tremaine (2007) and the planet scattering scenario of Nagasawa et al. (2008), inferring that the scattering model is more likely given the observations of projected obliquity.

As the number of obliquity measurements increases further, astronomers are able to identify other emerging trends that relate the distribution of obliquity to different planetary and stellar properties. One such trend is that of hot Jupiters tending to have smaller values of λ when their host stars have effective temperatures below $T_{\text{eff}} \sim 6000–6300$ K (Winn et al. 2010; Albrecht et al. 2012, 2013; see also Winn & Fabrycky 2015). This temperature dependence is the most robust obliquity relation in the literature, and it has been reflected not only in λ,
but also in the distribution of photometric modulation amplitudes (Mazeh et al. 2015), which should depend on the orientation of the stellar spin axis (see also Li & Winn 2016). Another possible trend relates the obliquity of hot Jupiters to stellar age (Triaud 2011), and there is also evidence to suggest that hot Jupiters in general are more oblique than the overall planet population (Albrecht et al. 2013, and more recently Winn et al. 2017). One intriguing finding, perhaps pointing to the dynamical evolution of planetary systems, is that stars with multiple planets may be very closely aligned (Sanchis-Ojeda et al. 2012; Albrecht et al. 2013) although it is unclear whether the converse is true of single-transiting systems: Morton & Winn (2014) noted a modest trend suggestive of higher obliquity in systems with one transiting planet, but very recently Winn et al. (2017) reported that the trend has disappeared.

In the present work, we address the statistical trends mentioned above, and explore additional ones, making use of two publicly available catalogs: the California-Kepler Survey (CKS, Johnson et al. 2017; Petigura et al. 2017) and the TEPCat database (Southworth 2011).

2. Obliquity Distribution from Bayesian Inference

2.1. The Geometry of Stellar Obliquity

The stellar obliquity \( \psi \) is the angle between the stellar spin vector \( S_\kappa \) and the planetary orbit’s angular momentum vector \( L_p \). The three-dimensional orientation of \( S_\kappa \) in space is determined by a polar angle \( \theta \) and an azimuthal angle \( \phi \). Observationally, however, instead of \( \theta \) and \( \phi \), it is more convenient to work in terms of \( I_\kappa \), the angle between \( S_\kappa \) and the LOS, and \( \beta_\kappa \), the angle of \( S_\kappa \) projected onto the plane of the sky. These angles are related by

\[
\cos I_\kappa = \sin \theta \cos \phi, \quad (2a)
\sin I_\kappa \sin \beta_\kappa = \sin \theta \sin \phi, \quad (2b)
\sin I_\kappa \cos \beta_\kappa = \cos \theta. \quad (2c)
\]

Now, for transiting planets, one can assume that \( I_\kappa \approx 90^\circ \) (see Equation (1)), i.e., \( L_p \) is perpendicular to the LOS; this assumption allows us to set \( \theta = \psi \) and \( \beta_\kappa = \lambda \), where \( \lambda \) is the projected spin–orbit misalignment angle.

In the following, we describe how a collection of \( \lambda \) or \( \cos I_\kappa \) measurements can be used to constrain the statistical properties of the true obliquity \( \psi \).

2.2. The Fisher Distribution for \( \psi \) and the Concentration Parameter

We are interested in finding the distribution of obliquities for a sample of known exoplanetary systems. For this, it is convenient to have a model function, such as the Fisher distribution (Fisher 1953; Fisher et al. 1993), which was proposed for exoplanet obliquities by Fabrycky & Winn (2009) (see also Tremaine & Dong 2012) and has the form

\[
f_\psi(\psi|\kappa) = \frac{\kappa}{2 \sinh \kappa} \exp(\kappa \cos \psi) \sin \psi \quad (3)
\]

where \( \kappa \) is often referred to as the “concentration parameter.”

The Fisher distribution of Equation (3) does not have closed-form expressions for its moments. The mean \( \langle \psi \rangle \) and standard deviation \( \sigma_\psi \equiv \sqrt{\langle \psi^2 \rangle - \langle \psi \rangle^2} \) are shown in Figure 1 as a function of \( \kappa \). For large \( \kappa \), \( f_\psi(\psi|\kappa) \) reduces to the Rayleigh distribution with scale parameter \( \sigma \equiv \kappa^{-1/2} \), for which \( \langle \psi \rangle = \sqrt{\pi/2} \sigma \) and \( \sigma_\psi = \sqrt{2} \sigma \). Thus, the quantity \( \kappa^{-1/2} \) provides a scale for the mean and spread of the obliquity angle.

Given the actual observable angles, it is convenient to work with the probability distributions of \( \cos I_\kappa \) or \( \lambda \). Morton & Winn (2014, hereafter MW14) have shown that, by means of Equation (2a) and standard probability rules, one can derive \( f_{\cos I_\kappa}(z = \cos I_\kappa|\kappa) \) in closed form:

\[
f_{\cos I_\kappa}(z|\kappa) = \frac{2\kappa}{\sinh \kappa} \int_{\sqrt{1-(z/y)^2}}^{1} \frac{\exp(\kappa y)}{\sqrt{1-y^2}} \frac{1}{\sqrt{1-(z/y)^2}} dy \quad (4)
\]

which is normalized in such a way that \( \cos I_\kappa \in [0, 1] \). In this work (see Appendix A), we also show that a similar derivation can be carried out to compute the probability distribution function (PDF) of the projected obliquity \( \lambda \) given \( \kappa \):

\[
f_\lambda(\lambda|\kappa) = \frac{\kappa}{2\pi \sinh \kappa \cos^2 \lambda} \int_0^{\cos \lambda} \frac{\exp(\kappa y)}{\sqrt{1-y^2}} \times \frac{y}{\sqrt{1-(y^2/\tan^2 \lambda)^2}} dy \quad (5)
\]

normalized to be valid in the range \( \lambda \in [-\pi, \pi] \). To the extent of our knowledge, this expression for \( f_\lambda(\lambda|\kappa) \) has not been presented previously in the literature.

2.3. Hierarchical Bayesian Inference

For a given data set \( D \) containing \( N \) stars with \( N \) measurement posteriors of some quantity that depends on \( \psi \) given \( \kappa \), we write the total likelihood as

\[
\mathcal{L}_\kappa = \prod_{n=1}^{N} \mathcal{L}_{\kappa, n}. \quad (6)
\]

The contribution from each measurement \( \mathcal{L}_{\kappa, n} \) to the total likelihood \( \mathcal{L}_\kappa \) depends on the stellar quantity being measured. If we have posteriors for \( z = \cos I_\kappa \), then we have (e.g.,

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**Figure 1.** Moments of the Fisher distribution (Equation (3)) in degrees. The mean obliquity \( \langle \psi \rangle \) for a given value of \( \kappa \) is depicted in blue; the spread (standard deviation) of obliquity \( \sigma_\psi \) given \( \kappa \) is depicted in red. When \( \kappa = 0 \) (isotropy), \( \langle \psi \rangle = 90^\circ \) and \( \sigma_\psi = (\pi/4 - 2)^{1/2} \approx 39^\circ 17 \). When \( \kappa \gtrsim 30 \), \( f_\psi \) approximates a Rayleigh distribution with \( \langle \psi \rangle \approx \sqrt{\pi/2} \kappa^{-1/2} \) and \( \sigma_\psi = (4/\pi - 1)^{1/2} \sqrt{\pi/2} \kappa^{-1/2} \).

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\[ \mathcal{L}_{\kappa, n}(D|\kappa) \propto \int_0^1 dz \ p(z|D_n) \frac{f_{\cos I_0}(z|\kappa)}{\pi_{\cos I_0, 0}(z)} \]  
(7)

where \( \pi_{\cos I_0, 0}(z) = 1 \) for all \( z \) is an uninformative prior. Similarly, for another data set \( \tilde{D} \) from which \( N \) posteriors on \( \lambda \) can be obtained, we have

\[ \mathcal{L}_{\kappa, n}(\tilde{D}|\kappa) \propto \int_{-\pi}^\pi d\lambda p(\lambda|\tilde{D}_n) \frac{f_{\lambda}(\lambda|\kappa)}{\pi_{\lambda, 0}(z)} \]  
(8)

where the prior \( \pi_{\lambda, 0}(z) = 1/(2\pi) \) is also uninformative and thus only amounts to a normalization constant. These two likelihoods (Equations (7) and (8)) can be computed by direct numerical integration or by the method of \( K \)-samples (see Equation (27)).

The prior PDF of the meta-parameter \( \kappa \) that must multiply the total likelihood in Equation (6) is usually assumed to take the form (Fabrycky & Winn 2009)

\[ \pi_{\kappa}(\kappa) \propto (1 + \kappa^2)^{-3/4}. \]  
(9a)

This prior is chosen to be well behaved when \( \kappa = 0 \) and to become uniform in the Rayleigh scale parameter \( \sigma = \kappa^{-1/2} \) as \( \kappa \to \infty \). This prior may underestimate \( \kappa \) if it is large (\( \gtrsim 10 \)), and a flat prior might pick up the signal of a large concentration parameter if the data set is small (Campante et al. 2016). An alternative is to devise a prior that, for large \( \kappa \), becomes logarithmic uninformative in \( \sigma \) (i.e., a Jeffreys prior on a scale parameter) rather than uniform. This can be accomplished with a prior in \( \kappa \) of the form

\[ \pi'_{\kappa}(\kappa) \propto (1 + \kappa^2)^{-1/2}, \]  
(9b)

which satisfies \( \pi_{\kappa} \propto 1/\sigma \) for \( \kappa \gg 1 \). For the sake of consistency with previous studies, we will employ the prior function of Equation (9a) unless stated otherwise. As it turns out, the data set is large enough that the inference on \( \kappa \) is weakly sensitive to the choice of either prior. Further details of the Bayesian computation are provided in Appendix B.

We note that, in principle, measurements of \( \cos I_0 \) and \( \lambda \) could be used jointly to write a two-dimensional integral for \( \mathcal{L}_{\kappa, n} \) in place of Equations (7) and (8) and using a joint probability \( p(z, \lambda|\kappa) = f_{\cos I_0}(z|\kappa) \times f_{\lambda}(\lambda|\kappa) \). Unfortunately, the number of \( \text{Kepler} \) systems for which both \( \cos I_0 \) and \( \lambda \) have been derived/measured is small (we identify five such objects in Section 3.2). Thus, for the remainder of the paper, we will compute \( p(\kappa|D) \) for the different data sets independently, acknowledging that each data set might sample a different population of planetary systems and thus differences in the inferred values of \( \kappa \) are to be expected.

3. Obliquity Distribution from Observations

3.1. The California-Kepler Survey

Recently, Winn et al. (2017) used the CKS (Johnson et al. 2017; Petigura et al. 2017) to study the properties of LOS inclination \( I_a \) and projected rotational velocity \( V \sin I_a \) of numerous \( \text{Kepler} \) planet hosts. The extensive analysis of Winn et al. (2017) did not include inference on the concentration parameter \( \kappa \) (Equation (3)); thus, the analysis presented below is highly complementary to their work.

We are interested in objects for which \( V \sin I_a \), the stellar radius \( R_* \), and the rotational period \( P_\text{rot} \) are known. The CKS catalog contains 1305 \( \text{Kepler} \) Objects of Interest (KOIs) with \( V \sin I_a \) and \( R_* \) measurements, of which 773 have a “confirmed planet” disposition (Petigura et al. 2017). To assign rotational periods, we collect \( P_\text{rot} \) measurements from two main catalogs: Mazeh et al. (2015), which obtained them via autocorrelation function analysis, and Angus et al. (2018), who introduced a novel Bayesian parameter estimation of \( P_\text{rot} \) as part of a parametric model consisting of a quasi-periodic Gaussian random process (QPGP). We are able to find additional periods in the literature, recovering measurements from Bonomo & Lanza (2012), McQuillan et al. (2013, 2014), Hirano et al. (2014), García et al. (2014), Paz-Chinchón et al. (2015), and Buzasi et al. (2016), all of which were obtained using somewhat different, but still deeply related, methods. Most of these period identification techniques are based on Fourier analysis of time series (see discussion in Agrain et al. 2015), with one departure being the Morlet wavelet method of García et al. (2014), which is still inherently a spectral analysis method. We refer to all these strategies collectively as “spectral analysis” (SA) methods and group the corresponding periods along with the catalog of Mazeh et al. (2015), leaving the catalog of Angus et al. (2018) as the one truly distinct approach to period identification. Of the 773 entries in the CKS sample, we are able to assign SA periods for 734, and QPGP periods for 645; 614 targets have both SA and QPGP periods. Following Winn et al. (2017), we proceed with our analysis using only targets with “reliable” periods, meaning those for which SA and QPGP estimates coincide. Our method of period selection is analogous to although slightly more permissive than the one used by Winn et al. (2017).

First, we consider only period measurements with a signal-to-noise ratio of 3 and larger. Second, we consider that the two period estimates match if they are within 30% of the identity, provided that the uncertainties overlap (we use 3\( \sigma \) error bars). Period filtering removes 492 targets, and we are left with 291. Possibly blended sources within the \( \text{Kepler} \) photometric aperture (\( \sim 4^\circ \), Mullally et al. 2015; Furlan et al. 2017) mean that we cannot attribute the rotational period to the planet host; therefore, we remove 34 additional targets for which a nearby companion was detected by Furlan et al. (2017) within 3\( \text{mag} \) of the magnitude of the target \( \text{Kepler} \) Input Catalog (KIC) star (Winn et al. 2017). We are left with a database of 257 targets.

3.1.1. Inclination Posterior for Each Target

For each of the 257 CKS targets, we derive a posterior PDF of the LOS inclination. In principle, the value of \( I_a \) can be obtained from the straightforward operation

\[ \sin I_a = \frac{V \sin I_a}{V_{\text{eq}}} = \frac{V \sin I_a}{2\pi R_*/P_\text{rot}} \]  
(10)

(e.g., Borucki & Summers 1984; Doyle et al. 1984; Soderblom 1985; Winn et al. 2007; Albrecht et al. 2011). However, this approach has long been recognized as nontrivial because of the often large uncertainties in the different quantities involved.

As described in detail by MW14, if one has PDFs for the projected rotational velocity \( V \sin I_a \) and the equatorial rotational velocity \( V_\text{eq} = p_{V_\text{eq}} \), respectively—the posterior
PDF of \( \sin I_\ast \) given a prior \( \tau_{\sin I_\ast}(\zeta) = z/\sqrt{1-z^2} \) (i.e., uniform in \( \cos I_\ast \)) is given by

\[
p(\sin I_\ast | D) \propto \mathcal{L}(\sin I_\ast) \tau_{\sin I_\ast}(\sin I_\ast)
= \frac{\sin I_\ast}{\sqrt{1 - \sin^2 I_\ast}} \int_0^\infty v' p_{V\sin I}(v' \sin I_\ast) p_{V\sin I}(v') dv'.
\] (11)

It is convenient to work in terms of \( \cos I_\ast \). Since

\[p(\cos I_\ast | D) = p(\sin I_\ast | D) \left| \frac{d \sin I_\ast}{d \cos I_\ast} \right|, \]

the desired posterior is

\[
p(\cos I_\ast | D) = p(\sin I_\ast | D) \frac{\cos I_\ast}{\sqrt{1 - \cos^2 I_\ast}}
\times \int_0^\infty v' p_{V\sin I}(v' \sqrt{1 - \cos^2 I_\ast}) p_{V\sin I}(v') dv'.
\] (12)

MW14 assume \( p_\zeta(\cdot) \) is a Gaussian, but for \( p_{V\sin I}(\cdot) \) they derive an empirical PDF from the Monte Carlo sampling of \( P_{\text{rot}} \) and \( R_\ast \)—both assumed to be (double-sided) Gaussians—after incorporating the effects of differential rotation and the stochasticity of stellar spots. From the PDF \( p_\zeta(\cdot) \), we derive a most likely value \( V_{\text{eq}} \) with a confidence interval and plot it against the measured value of \( V \sin I_\ast \); this is shown in Figure 2 (top panel). All targets above the line \( V_{\text{eq}} = V \sin I_\ast \) (i.e., \( \sin I_\ast < 1 \)) are not edge-on (and potentially misaligned with respect to the planetary orbits). Targets for which \( \sin I_\ast < 1 \) with some statistical certainty are highlighted. Targets below the line \( V_{\text{eq}} = V \sin I_\ast \) (i.e., \( \sin I_\ast > 1 \)) are unphysical if the uncertainties cannot account for such a measurement (see figure caption for further details). Figure 2 (bottom panel) depicts the same comparison of velocities, but for the MW14 database. In our final database of 257 targets, two targets—KOIs 94 and 1848 (Kepler-89 and Kepler-978 respectively)—appear to be marginally unphysical, i.e., they are below the identity line at a distance greater than \( 3\sigma \) but smaller than \( 5\sigma \). We deem these two targets “anomalous,” but we do not remove them. Note that in the MW14 database, four targets are anomalous, but only one (KOI 244) is removed from their analysis. If not filtered out, anomalous targets will still have well behaved \( \cos I_\ast \) PDFs (Equation (12)) that strongly peak at \( \cos I_\ast = 0 \) (see Figure 3 below), and favor lower values of \( \kappa \) when entering Equation (7).

The 257 PDFs of \( \cos I_\ast \) are shown in Figure 3. Of all these targets, only 17 (highlighted curves) have \( \cos I_\ast \approx 90^\circ \) to a 95% confidence level (e.g., see MW14). This fraction (6.6%) is lower than in the sample of 70 targets of MW14, who concluded that the number of misaligned (not edge-on) stars was 12 (or 17%). We thus expect the overall population to have lower mean obliquities, or a higher value of \( \kappa \) in the distribution of Equation (3), than originally reported by MW14.

### 3.1.2. Obliquity Distribution

With the 257 \( \cos I_\ast \) posteriors depicted in Figure 3, we can carry out the hierarchical Bayesian inference method summarized in Section 2. The likelihood of the data given \( \kappa \) is computed from Equations (6) and (7), and the prior used is given by Equation (9a). The resulting posterior PDF of \( \kappa \) is
shown in Figure 4, with a (weighted) maximum a posteriori\(^5\) of 14.5 and a shortest 68%-probability interval of \(\kappa \in [8.5, 28.0];\) thus, we write \(\kappa_{\text{CKS}} = 14.5^{+13.5}_{-6}.\) This concentration parameter is significantly larger than the value reported by MW14, which we re-derived to be \(\kappa_{\text{MW}} = 6.2^{+1.6}_{-1.5}.\) A value of \(\kappa = 15\) in Equation (3) corresponds to a mean obliquity of \(\langle \psi \rangle = 19^\circ\) and a standard deviation of \(\sigma_\psi = 10^\circ\), while \(\kappa = 6\) results in \(\langle \psi \rangle = 30^\circ\) and \(\sigma_\psi = 16^\circ\). This “flattening” of the obliquity distribution might be a natural consequence of a larger number of systems with smaller planets being added to the list, revealing that the vast majority of systems have low obliquities. If we had removed KOIs 95 and 1848 (which are marginally unphysical and favor low obliquities; purple data points in Figures 2 and 3) we would have obtained \(\kappa_{\text{CKS}} = 13.8^{+12.2}_{-6.6}\) and thus their influence on \(\kappa\) is negligible. We have also checked the influence of the prior function \(\pi_\kappa\) of Equation (9a) versus the alternative prior \(\pi'_{\kappa}\) of Equation (9b). If \(\pi'_{\kappa}\) is used, we find that the posterior \(p(\kappa|D)\) produces \(\kappa_{\text{CKS}} = 15.4^{+23.8}_{-6.6}\) as expected from \(\pi'_{\kappa}\) being a more slowly declining function of \(\kappa\), but a minor change considering the uncertainties. Up to this point, we can conclude that the CKS test is more spin–orbit aligned than the data set used by MW14 and that the sample is consistent with obliquities in the range \(\psi = (19 \pm 10)^\circ\), provided that the Fisher distribution is an adequate model (see below).

A tentative trend discovered by MW14, which now seems to have disappeared (Winn et al. 2017), is that of \(\kappa\) having a dependence on planet multiplicity. In Figure 4, we show the separation of the CKS data set into a subset containing “singles” (systems with one transiting planet) and one containing “multis” (systems with multiple transiting planets). Indeed, the multiplicity trend no longer appears to be real, because we find \(\kappa_{\text{singles}} = 12.2^{+12.3}_{-5.6}\) and \(\kappa_{\text{multis}} = 13.3^{+7.5}_{-5.1}\) i.e., the two posterior PDFs are indistinguishable.

**Model selection.** Despite its useful functional form, the Fisher distribution (Equation (3)) is not fully justified on physical grounds; in principle, other obliquity distributions could fit the data better. Fabrycky & Winn (2009) considered alternative models, such as a mixture of Fisher distributions or a mixture of an isotropic distribution and a spin–orbit aligned one. A mixture model of two Fisher distributions (two concentration parameters \(\kappa_1\) and \(\kappa_2\) with relative weights \(f\) and \(1-f\)) has a joint posterior PDF given by

\[
p(\kappa_1, \kappa_2, f|D) \propto p(D|\kappa_1, \kappa_2, f) \pi(\kappa_1, \kappa_2, f) \approx \pi_\psi^\alpha \pi_\psi(\kappa_1) \pi_\psi(\kappa_2) \pi(f)
\]

(13)

where the contribution of the \(n\)th measurement to the total likelihood is

\[
\mathcal{L}_\alpha(D_n|\kappa_1, \kappa_2, f) = f \mathcal{L}_\psi(D_n|\kappa_1) + (1-f) \mathcal{L}_\psi(D_n|\kappa_2)
\]

(14)

where \(\mathcal{L}_\psi(D_n|\kappa)\) is given by Equation (7). In Equation (13) we have also assumed that the three-parameter prior \(p(\alpha) = p(\kappa_1, \kappa_2, f)\) is separable. Figure 5 shows the joint posterior of \(\kappa_1\) and \(\kappa_2\) and the marginalized posterior of \(f\). The marginalized “best fit” values are \(\kappa_1 = 0^{+11.4}_{-4}, \kappa_2 = 18.4^{+17.6}_{-7.2}\) and \(f = 0.02^{+0.02}_{-0.02}\). This roughly states that the data are consistent with a small fraction of the population (a few per cent, consistent with 17 of 257 targets being oblique; see Figure 3) being drawn from a high-obliquity distribution and a large fraction of the population being drawn from a low-obliquity distribution.

To compare this new model (which we call \(\mathcal{M}_0\)) with the previous, simpler one (\(\mathcal{M}_0\)), we need to compute

\[
p(D|\mathcal{M}) = \int p(D|\kappa_1, \kappa_2, f) \pi(\kappa_1, \kappa_2, f) \, d\kappa_1 d\kappa_2 df
\]

(15)

(sometimes called the “Bayesian evidence”) and then calculate the posterior odds ratio:

\[
p(\mathcal{M}_0|D) = \frac{p(D|\mathcal{M}_0) p(\mathcal{M}_0)}{p(D|\mathcal{M})} p(\mathcal{M}) \approx K
\]

(16)
where $\mathcal{K}$ is called the “Bayes factor” or “evidence ratio.” If we assume that $p(\mathcal{M}_0) = p(\mathcal{M}_1) = 1/2$, then the model choice is given by $\mathcal{K}$. We find that $\mathcal{K} = 2.25$, and thus the data support the null (simplest) model $\mathcal{M}_0$ in favor of the alternative model $\mathcal{M}_1$ (although not decisively, see Jeffreys 1961, Appendix B).

Alternatively, since one of the concentration parameters in $\mathcal{M}_1$ is consistent with zero, we can take $\kappa_1 = 0$ and create another mixture model ($\mathcal{M}_2$) consisting of an isotropic distribution and a Fisher distribution (e.g., see Campante et al. 2016). The likelihood function of the $n$th target becomes

$$L_{\alpha,n}(D_n | \kappa, f) = f + (1 - f) L_{\kappa,n}(D_n | \kappa).$$ (17)

The posterior distribution of this simpler model (of only two parameters) produces $\kappa = 18.4^{+17.4}_{-6.9}$ and $f = 0.02^{+0.04}_{-0.01}$. The evidence ratio, in this case, is $\mathcal{K} = p(D | \mathcal{M}_0) / p(D | \mathcal{M}_2) = 12$. Thus, the null model $\mathcal{M}_0$ is substantially favored by the data.

In what follows, we explore the obliquity properties of different subsets—say $D_A$ and $D_B$ where $\bigcup \{D_i\} = D$—within the CKS catalog. Our aim is to assess whether a given subset is “more oblique” or “less oblique” than its complement. Although we derive different values/posteriors of $\kappa_A$ and $\kappa_B$, our main goal is not to analyze the physical meaning of each concentration parameter, but to measure the statistical significance of the differences encountered.

### 3.1.3. Obliquity Trends: Stellar Properties

The larger size of the CKS data set with respect to previously published catalogs allows us to explore changes in $\kappa$ as a function of different physical variables. In the following, we focus on the properties of the stellar host.

**Effective temperature.** One intriguing stellar property that appears to affect the obliquity of planetary systems is the stellar effective temperature (Schlaufman 2010; Winn et al. 2010, 2017; Albrecht et al. 2012, 2013; Mazeh et al. 2015). This transition appears to coincide with the so-called “Kraft break” (Struve 1930; Schatzman 1962; Kraft 1967), identified as a sharp increase in the measured projected velocities in the field at around $T_{\text{eff}} = 6200$ K. This break is attributed to the transition that takes place when the width of the convective envelope of a solar-type star vanishes, giving rise to radiative envelopes at higher temperatures. We explore the temperature dependence in our compiled list of 257 *Kepler* targets. Placing a cut at 6200 K, we divide the data set into “hotter” systems ($T_{\text{eff}} \geq 6200$ K) and “cooler” systems ($T_{\text{eff}} < 6200$ K), with 20 and 237 targets respectively. We find no statistical difference: we derive $\kappa_{\text{hotter}} = 12.2^{+8.6}_{-4.2}$ and $\kappa_{\text{cooler}} = 12.2^{+8.6}_{-4.2}$. However, this inference is largely due to the small number of KOIs with effective temperatures above 6200 K. This lack of detected planets around “hotter” stars is at least in part due to a selection effect, since those hot stars for which planetary transits are detected tend to be the least variable ones, in turn preventing the measurement of their rotational periods from photometric variability (Mazeh et al. 2015; Winn et al. 2017). We return to this subject in Section 3.2 below.

**Metallicity.** The high precision provided by the CKS survey allows us to explore the impact of other fundamental stellar properties such as metallicity and age. In principle, these two variables are not entirely independent of each other, because we expect the older sample to have somewhat lower metallicity than the younger sample. However, the dynamical range in age is too narrow to reveal any correlation with metallicity; furthermore, both the lowest and highest metallicities in the sample, [Fe/H] = −0.4 ± 0.04 for KOI 623 and [Fe/H] = 0.396 ± 0.04 for KOI 941, have “old” ages of log $\Lambda = 9.95^{+0.13}_{-0.08}$ and log $\Lambda = 10.3^{+0.12}_{-0.25}$, respectively. Figure 6 (top panel) shows the concentration inference for a separation of the data set into a low-metallicity sample ($N = 166$) and a high-metallicity one ($N = 91$). The cut in metallicity is [Fe/H]$_{\text{cut}} = 0.135$, and it is chosen as the one that maximizes the difference in the obliquity properties of the two subsamples, i.e., the value of the cutoff is a free parameter, which is explored in a “sweeping” fashion. Inference results in a concentration parameter of $\kappa_{\text{CKS low [Fe/H]}} = 22.6^{+48.5}_{-12.4}$ for the lower metallicity subsample, and $\kappa_{\text{CKS high [Fe/H]}} = 7.6^{+4.1}_{-2.9}$ for the higher metallicity one. The two PDFs appear different, and thus a proper statistical assessment of their true distinctiveness is required. Following MW14, we use the squared Hellinger distance $\delta_{\text{HF}}$ as a difference metric between the two distributions. Then, we repeat the hierarchical inference 5000 times by Monte Carlo sampling the data set in such a way that two samples $D_1$ and $D_2$ of sizes $N_1 = 166$ and $N_2 = 91$ are generated for each try. For each of these tries, we compute $p(\kappa | D_1)$ and $p(\kappa | D_2)$, and measure the corresponding $\delta_{\text{HF}}$. We then count the fraction of realizations in which the synthetic $\delta_{\text{HF}}$ is equal to or larger than in the real sample. This test quantifies the likelihood of obtaining the observed difference between $\kappa_{\text{low [Fe/H]}}$ and $\kappa_{\text{high [Fe/H]}}$ by mere chance. We find that in ~2% of the random trials, the two resulting PDFs are more different than in the top panel of Figure 6 (by measure of $\delta_{\text{HF}}$), thus concluding that this difference is 98% significant (i.e., a 2.3σ detection) and thus suggestive, but not conclusive. Differences in the planet-bearing frequency as a function of metallicity have been reported in the literature (e.g., Mulders et al. 2016; Petigura et al. 2018), and thus the (moderate) trend of $\kappa$ with [Fe/H] might be an indirect reflection of a dependence of $\kappa$ on planet type (see Section 3.1.4 below). For example, one might expect, qualitatively, that larger and more numerous rocky cores are formed in high-metallicity protoplanetary disks (Petigura et al. 2018). The lower values of $\kappa$ at higher metallicities might be linked to the formation of more crowded/tightly packed systems, which will tend to be less stable (e.g., Pu & Wu 2015) and thus evolve toward excited mutual inclinations and obliquities.
Figure 6. Separation of the data set of 257 targets into two subsets according to stellar metallicity (Petigura et al. 2018) and stellar age (ages from Johnson et al. 2017) in the CKS survey. Top panel: separation into high-Z systems ([Fe/H] > 0.135, N = 91 objects) and low-Z systems ([Fe/H] < 0.135, N = 166 objects). The concentration parameter of the younger subset is 9.8 (i.e., \( \langle \psi \rangle \approx 23^\circ \) and \( \sigma_\psi \approx 12^\circ \)), and that of the older subset is 22.6 (i.e., \( \langle \psi \rangle \approx 15^\circ \) and \( \sigma_\psi \approx 8^\circ \)); the statistical significance of this difference is 97.9\%. Bottom panel: splitting of the data set into younger (yellow) and older (red) KOIs (ages from Johnson et al. 2017). The age cutoff chosen to separate the data set is \( \log A_{\text{sys}} = 9.63 \) (or \( A = 4.27 \) Gyr). The concentration parameter of the younger subset is 9.8 (i.e., \( \langle \psi \rangle \approx 23^\circ \) and \( \sigma_\psi \approx 12^\circ \)), and that of the older subset is 22.4 (i.e., \( \langle \psi \rangle \approx 15^\circ \) and \( \sigma_\psi \approx 8^\circ \)); the statistical significance of this difference is only 93.5\% (a 1.85\% detection). The overlap between the low-metallicity sample (\( N = 166 \)) and the older sample (\( N = 161 \)) is significant but not overwhelming; 103 objects satisfy both \([\text{Fe/H}] < 0.135\) and \(A > 6.6\).

Age. Stellar obliquity can be affected by stellar spin-down as well as by tidal coupling to close-in planets (e.g., Winn et al. 2010; Albrecht et al. 2013; Dawson 2014; Li & Winn 2016), both of which act over long periods of time. Thus, provided that accurate estimates and a wide enough range of stellar ages are available, one can in principle probe for changes in \( \kappa \) as a function of this quantity (e.g., see Triaud 2011). As a part of the CKS, Johnson et al. (2017) fitted evolutionary models to the observed spectroscopic parameters to obtain stellar masses and ages. We add these ages to the 257 target samples and split the data set into an “older” subset (\( \log A \geq \log A_{\text{sys}} = 9.63 \)) with \( N = 161 \) targets and a “younger” subset (\( \log A < \log A_{\text{sys}} \)) with \( N = 96 \) targets, where the cutoff value was deliberately chosen as the one that maximizes the statistical significance of the data splitting. We show the results of the concentration inference as a function of age in Figure 6 (bottom panel). We find that the youngest systems are consistent with a distribution of obliquities with \( \kappa_{\text{younger}} = 9.8^{+6.5}_{-2.4} \). The “older” subset produces \( \kappa_{\text{older}} = 22.4^{+62.1}_{-13.1} \). This difference has a statistical significance that is marginal at best (93.6\%), and further studies will help to decide whether this trend is real or not. As with the metallicity cutoff, the age cutoff is a free parameter, which we explore systematically, always requiring that both subsamples have more than 20 objects.

Stellar multiplicity. Another interesting property that can affect the obliquity of Kepler systems is stellar multiplicity. Different proposed channels for the origins of hot Jupiters (see Dawson & Johnson 2018, for a recent review) typically invoke the presence of a distant companion of stellar or planetary mass that triggers the Lidov–Kozai mechanism or some related secular process responsible for high eccentricities (e.g., Naoz 2016). Some of these studies have explicitly provided predictions of the distribution of obliquities generated (see, e.g., Fabrycky & Tremaine 2007; Naoz et al. 2012; Petrovich 2015a, 2015b; Anderson et al. 2016). However, moderate obliquities can be induced by a companion whether a system harbors a hot Jupiter or not, and these companions are not required to be in highly inclined orbits (e.g., Bailey et al. 2016; Lai 2016; Lai et al. 2018). To test this, we use the adaptive optics follow-up observations of Furlan et al. (2017). This campaign—part of the Kepler Follow-up Observation Program (KPOP, see Section 3.1.5 below)—observed 3357 KOI host stars, reporting on the detection of 2297 companions to 1903 stars. Of the 773 KIC stars in the CKS catalog with confirmed planets (Section 3.1), 766 were part of the observation log of Furlan et al. (2017), including all of the 257 targets used in the present work. Using the results of the high-resolution imaging campaign of Furlan et al. (2017), we assign companion numbers to our 257 CKS targets. We find that 65 KOIs have one or more candidates for stellar companions. Of these, nine have \( I_b \approx 90^\circ \) at a statistically significant level. As before, we split the data set according to the presence of companions and carry out the concentration inference for each subsample. In Figure 7 (top panel) we show the posterior PDFs corresponding to each subset, finding that \( \kappa_{\text{comp}} = 7.4^{+12.4}_{-4.6} \) and \( \kappa_{\text{no comp}} = 15.1^{+22.9}_{-6.6} \). Although the \( \kappa \) posteriors appear somewhat distinguishable, the statistical significance of this difference is negligible, because the synthetic distance metric \( d_H \) was larger than the measured one in \( \approx 35\% \) of the random data splittings attempted. The undetected effect of stellar multiplicity on obliquity might not be a surprise. The majority of these companions are detected beyond a separation of 1.5\,au, which in most cases (assuming a mean distance of \( \approx 500 \) pc) corresponds to 750\,au, a separation that might be too wide for a significant obliquity to accumulate due to differential precession over the age of the system (see, e.g., Boué & Fabrycky 2014, and the Discussion below). Note that we have removed 34 targets with close companions for being possible blends that can make the period detection ambiguous (Section 3.1). We have repeated the obliquity inference reinstating these KOIs to see whether they contain some tentative clues regarding the influence of companions. Again, the \( \kappa \) posteriors of the “companions” and “no companions” subsamples are statistically indistinguishable. Since the size of the sample allows us to split the data set according to more than one variable, we have also explored planet multiplicity in combination with stellar multiplicity. No convincing trends arise, although some multis seem misaligned at a statistically significant level. This is the case for KOI 377...
(three transiting planets) and KOI 1486 (two transiting planets), for which the 95% confidence upper limits on \( \cos I_p \) are 0.108 and 0.143, respectively (see caption in Figure 3). This might seem counterintuitive at first, since the presence of additional planets could protect the system against external perturbations (e.g., Boué & Fabrycky 2014; Lai & Pu 2017; Lai et al. 2018). However, multi-planet systems could still shield each other from the excitation of mutual inclinations, while being coherently inclined with respect to the stellar spin axis. The precise balance between external excitation and suppression of inclinations depends on all semimajor axes and masses of a given system, and thus any distinction between oblique and non-oblique populations might be much more subtle than what we have attempted here.

3.1.4. Obliquity Trends: Planetary Properties

As an initial test, in Section 3.1.2 we explored the role of planet multiplicity. However, we can explore other planetary properties such as orbital period and planet radius.

Orbital period. Under the tidal evolution hypothesis of Winn et al. (2010) and Albrecht et al. (2012), stellar obliquity should exhibit some dependence on planetary orbital period \( P_{\text{orb}} \). Using the photometric amplitude as an indicator of mean obliquity, Mazeh et al. (2015) concluded that low obliquities around cooler KOIs can extended out to \( \sim 50 \) days. However, using the same data set, Li & Winn (2016) argue that the evidence for spin–orbit alignment partially weakens for \( P_{\text{orb}} \gtrsim 10 \) days, and that it disappears when \( P_{\text{orb}} \gtrsim 30 \) days. In Figure 8 (top panel), we show that the splitting of the CKS data set by the period of the closest-in planet reveals a difference between short-period systems and long-period ones. For a period cutoff of \( P_{\text{orb,cut}} = 8.5 \) days, we find that short-period systems are more spin–orbit aligned than long-period ones, and that the statistical significance of this difference is 97.8% (a \( \sim 2.5\sigma \) detection). We remove the five targets that can be classified as hot Jupiters and repeat the statistical test (bottom panel of Figure 8), obtaining a mild increase in the significance of the trend. Despite the small number of targets removed, this additional test is not superfluous, because these five targets are the best candidates to test the tidal realignment hypothesis of Winn et al. (2010) (see also Albrecht et al. 2012; Dawson 2014). The fact that closer-in planets are, on average, more spin–orbit aligned with their host stars is in qualitative agreement with the tidal realignment hypothesis; however, as noted by Li & Winn (2016), it is suspect that this trend still applies for the small-mass planets in the CKS survey (see Discussion).

Planet radius. For each KOI, we define three statistics: the radius of the closest-in planet \( R_{p,c} \), the mean planet radius \( \bar{R}_p \), and the radius of the largest planet \( R_{p,\text{max}} \). As with the exploration of orbital period, we split the data set according to these three quantities, sweeping the cut value \( R_{p,\text{cut}} \) until we find a maximum in the statistical significance. The results of this exploration are presented in Figure 9, where the five hot Jupiter systems have been excluded in all three tests. All data splitting tests provide a consistent trend: systems with larger planets are more oblique than systems with smaller planets. For the metrics \( R_{p,c} \) and \( \bar{R}_p \) (top and middle panels of Figure 9), the statistical significance of this trend is 99.5% (\( \lesssim 3\sigma \)). The test suggests that systems containing planets larger than \( \sim 3 R_{\oplus} \) (i.e., Neptunes and sub-Saturns) have larger stellar obliquities. MW14 discussed the possibility of this trend being behind their reported dependence on planet multiplicity, because multiple transiting systems tend to have smaller mean radius (e.g., Latham et al. 2011). This idea is supported by the
radius appears to point toward a dependence on the total mass contained in planets, in turn a measure of the “dynamical temperature” of a system (e.g., Tremaine 2015). As we do not have accurate mass estimates for most of the planets in this sample, we cannot confidently define a statistic for the total mass contained in planets; however, in the Discussion, we speculate on ways of approximately assigning a total planetary mass to each KOI.

We briefly comment on the “sweeping data slicing” that we implemented above (Sections 3.1.3 and 3.1.4) in order to identify potential breaks in the obliquity distribution. This technique was necessary because, for the variables tested, we lack a hypothesis predicting a change in the distribution of obliquities at some critical value of the variable in question. In addition, these variables—such as metallicity, age, planet orbital period, and planet radius—are continuous, as opposed to categorical—such as multiplicity or planet type—which makes the a priori identification of an adequate cutoff ambiguous.

The sweeping of cutoff values means that we are carrying out $N$ different hypothesis tests, one for each time a new cutoff is tried and the data set is split. For each test, the “null hypothesis” $H_{0,n}$ ($n = 1, \ldots, N$) is that the two subsets resulting from the splitting are indistinguishable from each other; we look for the $n$th cutoff for which we can reject the null hypothesis at some confidence level. If the properties of the obliquity data set change abruptly at the $n$th cutoff, then the $n$th $p$-value will be much smaller than that of any of the other tests (as would be expected to happen, say, at the Kraft break). Alternatively, the obliquity data could depend smoothly and monotonically on some of these continuous variables, in which case the $p$-value could be small, not only for one, but for many of the $N$ tests.

When such multiple tests are attempted, it is important to quantify whether the null hypothesis in a given one of them could be rejected “by chance” due to “$p$-hacking.” This is sometimes referred to as the “multiple comparisons problem” (e.g., Miller 1981). Multiple strategies have been devised to correct for this effect, the most widely used being the Holm–Bonferroni method (Holm 1979), which adjusts the $p$-values of each test by a factor based on the number of tests carried out. An obstacle in implementing this adjustment is the clear correlation of our $N$ tests and of their resulting $p$-values (e.g., Conneely & Boehnke 2007). Given this difficulty, we stick to our Monte Carlo approach. This time, we compute statistical significance by comparing the distance metric $\delta_n$ resulting from the $n$th cutoff to the distance metrics obtained from all cutoffs attempted; thus, we compare to $5000 \times N$ distance metrics with varying cutoff, instead of 5000 metrics with a fixed cutoff. For example, in Figure 6 (top panel), the distance metric of the data splitting according to a metallicity cutoff of [Fe/H] $= 0.135$ is $\delta_{0.135} = 0.563$, which is larger than in 96.85% of all Monte Carlo data splittings across all $N$ tests: thus, the “global $p$-value” is $0.0315$ (up from a value of $1 - 0.979 = 0.021$ obtained with the previous test); this allows us to reject the null hypothesis (that there is no metallicity dependence) at an $\alpha = 0.04$ level. We repeat this analysis for orbital period (Figure 8, top panel), and find a global $p$-value of 0.0256 (up from $1 - 0.978 = 0.022$). Similarly, for the dependence on planet radius (Figure 9, top panel) we find a global $p$-value of 0.0093 (up from

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**Figure 9.** Similar to Figures 6–8, this time separating the CKS data set (minus five hot Jupiter systems) by three different metrics related to the size of the planets in each system. Top panel: separation of the data set by the radius of the closest-in planet $R_{p,c}$, where the radius cut is placed at $R_{p,c} = 2.75 R_\oplus$; the large-planet sample is more oblique on average with a concentration parameter $\kappa_{\text{large} R_{p,c}} = 5.0^{+0.8}_{-0.5}$ and the small-planet sample is less oblique with $\kappa_{\text{small} R_{p,c}} = 23.5^{+38.3}_{-12.5}$ ($N = 198$). Middle panel: separation of the data set by $R_p$, the mean planet radius, with $R_{p,\text{cut}} = 2.75 R_\oplus$ as before; the large-planet sample is more oblique than the small-planet sample: $\kappa_{\text{large} R_p} = 5.3^{+12.3}_{-1.3}$ and $\kappa_{\text{small} R_p} = 23.4^{+37.6}_{-12.4}$. Bottom panel: separation of the data set by maximum planet radius in each KOI $R_{p,\text{max}}$, with $R_{p,\text{cut}} = 3.4 R_\oplus$; the large-planet sample is more oblique than the small-planet sample: $\kappa_{\text{large} R_{p,\text{max}}} = 4.9^{+24.4}_{-3.4}$ and $\kappa_{\text{small} R_{p,\text{max}}} = 22.1^{+44.1}_{-11.1}$ ($N = 212$).
1 – 0.995 = 0.005). Therefore, after introducing the global p-value as a way of dealing with the multiple comparisons problem, the detection significance remains roughly unaltered.

3.1.5. Other Catalogs of V sin I

Spectroscopic studies in the literature can provide additional values of V sin I and \( R_\text{e} \) (e.g., see Buchhave et al. 2012; Hirano et al. 2012, 2014). However, the CKS has signified a major leap with respect to previous studies, not only because of the size of its sample, but because of the consistency and uniformity of its data collection and analysis. In general, we expect CKS to supersede any previously reported spectroscopic analyses. There are, however, significant amounts of KOI data that are publicly available via KFOP (Furlan et al. 2017), which has compiled follow-up imaging and spectroscopy of a large number KOIs. From the KFOP database, and as of 2018 February 20, we obtain 858 individual KOIs with values and uncertainties of V sin I and \( R_\text{e} \) reported by one or more users. Whenever more than one value of either V sin I or \( R_\text{e} \) is reported, we calculate a weighted mean of the values and their uncertainties. This database overlaps with the CKS database on 505 targets. Unfortunately, these reported values are inconsistent—uploaded by different users, using different fitting methods for rotational broadening, applied on spectra obtained from different telescopes—and often the secondary by-product of a different type of analysis. Although some consistency is found between KFOP and CKS (rotational velocities tend to agree for V sin I < 5 \( \text{km s}^{-1} \), albeit with significant scatter), the accuracy and uniformity needed for the statistical analysis in the present work make the CKS catalog the only source that can be used with confidence.

3.2. The TEPCat Catalog

The same kind of hierarchical Bayesian inference can be carried out for a database of projected obliquity measurements via use of Equation (5). Using an essentially equivalent method, Fabrycky & Winn (2009) inferred a value of \( \kappa \) from a list of 11 targets with RM observations. Using their sample, but implementing the formalism summarized in Section 2, we obtain \( \kappa_{\text{FW}} = 9.1^{+1.7}_{-1.1} \), in consistency with the results of that work.

We can extend the same analysis to a much larger RM database. We retrieve the data compiled in John Southworth’s TEPCat Catalog (Southworth 2011, http://www.astro.keele.ac.uk/jkt/tepcat/). This catalog contains 191 measurements for 118 unique systems. For multiple entries, we take the weighted mean of the observations, provided that there is some consistency between the reported measurements; if discrepancies are found, we take the latest/most accurate measurement. The measurement posterior of \( \lambda \) for each target, i.e., \( p(\lambda | D_\lambda) \) in Equation (8), is taken to be a Gaussian with mean and variance given by each measurement and its uncertainty, respectively. The likelihood of the data given \( \kappa \) is obtained via Equations (6) and (8), and the same prior (Equation (9a)) used in Section 3.1. The posterior PDF of \( \kappa \) is shown in Figure 10, and the 68% probability interval is given by \( \kappa_{\text{TEPCat}} = 2.2^{+0.2}_{-0.6} \). This value of the concentration parameter is much smaller than that obtained for the CKS sample, and corresponds to a much more oblique population (\( \kappa = 2 \) implies \( \langle \psi \rangle = 53^\circ \) and \( \sigma_\psi = 30^\circ \)). This discrepancy, however, is to be expected, because RM measurements are typically limited to hot Jupiter systems (Gaudi & Winn 2007), in contrast to the more diverse nature of the Kepler exoplanet systems (see, e.g., Albrecht et al. 2012; Winn et al. 2017).

Effective temperature. The size of the TEPCat catalog allows us to split the data set into subsamples. This is shown in Figure 10 (top panel), where concentration inference was carried out for a “hotter” (\( T_{\text{eff}} \geq 6250 \text{ K} \)) subset with \( N = 44 \) entries, and a “cooler” (\( T_{\text{eff}} < 6250 \text{ K} \)) subset with \( N = 74 \) entries. We find that \( \kappa_{\text{hotter}} = 1.2^{+0.3}_{-0.6} \) and \( \kappa_{\text{cooler}} = 3.5^{+1.0}_{-0.2} \) with a statistical significance of 99.7%. Although the TEPCat catalog is largely composed on hot Jupiters, not all of its entries qualify as such. Indeed, if we only select targets with \( R_\text{e} > 7R_\oplus \) and \( P_{\text{orb}} \leq 13 \text{ days} \), we remove 12 targets. Repeating the separation of the data set according to \( T_{\text{eff}} \) (Figure 10, bottom panel), we find \( \kappa_{\text{hotter, Hi}} = 0.9^{+0.3}_{-0.2} \) and \( \kappa_{\text{cooler, Hi}} = 4.2^{+0.8}_{-0.6} \) and that the hotter sample and the cooler sample are different with a 99.98% statistical significance (almost a 4\( \sigma \) detection).

![Figure 10](image-url)
This obliquity–temperature dependence is in accordance with the confirmed trend that hotter hot Jupiter hosts tend to be more oblique than cooler ones (Schlaufman et al. 2010; Winn et al. 2010; Hébrard et al. 2011; Albrecht et al. 2012; Dawson 2014; Mazeh et al. 2015). Nevertheless, these values of $\kappa$ indicate that hot Jupiter systems are in general more oblique ($\kappa_{\text{TEPCat}} \sim 4.2$) than the majority of configurations found in the $\text{Kepler}$ catalog ($\kappa_{\text{CKS}} \sim 14$), regardless of stellar effective temperature. In principle, one could focus solely on hot Jupiters in the CKS sample to explore the discrepancy between $\kappa_{\text{TEPCat}}$ and $\kappa_{\text{CKS}}$. Unfortunately, only 5/257 targets in the list compiled in Section 3.1 fall into the “hot Jupiter category” (see Section 4 below).

3.2.1. Three-dimensional Obliquities?

Having PDFs of both $\lambda$ and $\cos I_\text{J}$, one can, in principle, construct the three-dimensional obliquity $\psi$ (e.g., Benomar et al. 2014). The TEPCat and CKS databases overlap on five targets: KOI 377 (Kepler-9), KOI 203 (Kepler-17), KOI 806 (Kepler-30), KOI 63 (Kepler-63), and KOI 94 (Kepler-89). Of these five targets, KOI 63 ($T_{\text{eff}} = 5673$ K) is severely oblique in both $\lambda$ and $I_\text{J}$, as already reported by Sanchis-Ojeda et al. (2013). KOI 63 is also unusual because it is one of the fastest rotators in the sample with $P_{\text{rot}} = 5.5$ days (the mean in the data set is 18.8 days).

3.3. Obliquity Distribution of Hot Jupiters in the CKS and TEPCat Data Sets

The significance of the temperature separation of the TEPCat sample is clear (Figure 10). However, we lack the statistics to carry out an analogous test with the hot Jupiters in the CKS target list. The overall CKS sample used for our analysis ($N = 257$) contains 20 stars above 6200 K (7.8%). Of the 257 targets, only five are hot to hosts Jupiters, all below 6000 K. By contrast, 37% of targets in the TEPCat catalog (mostly hot Jupiter hosts) have $T_{\text{eff}} > 6200$ K. This severely limits our ability to find a consensus between what can be inferred from RM measurements and via the $V \sin I_\text{J}$ method. Winn et al. (2017) found a workaround to this limitation, and using a statistical analysis of $V \sin I_\text{J}$ alone (e.g., Schlaufman 2010), they identified six systems above the Kraft break that are likely to be oblique: KOI 2, KOI 18, KOI 98, KOI 167, KOI 1117, and KOI 1852. None of these systems made it to our original data set of 257 targets. KOI 98 was removed due to risk of blending and the other five have inconsistent SA and QPGP periods. The SA periods of KOIs 2, 18, 1117, and 1852 fall in a range from 60 to 90 days (very long for stars above the Kraft break, and likely to be wrong), while the corresponding QPGP periods are between 5 and 25 days, still somewhat long for these effective temperatures. If we use the QPGP periods for KOIs 2 (HAT-P-7), 18, and 1117—19.6 $\pm$ 4, 24$^{+1.2}_{-1.0}$, and 10 $\pm$ 1 days respectively—we find that these KOIs have equatorial velocities $V_\text{eq} \approx V \sin I_\text{J}$, and thus are consistent with $I_\text{J} \approx 90^\circ$. In this case, the low $V \sin I_\text{J}$ of these objects would result from them being anomalously slow rotators for their effective temperatures, and not from being highly oblique. On the other hand, the QPGP period of KOI 1852 (6.9 $\pm$ 0.2 days) does imply that this system could be severely oblique with $I_\text{J}(95%) = 50^\circ$, but it does not harbor a hot Jupiter. These estimates contradict some of the findings of Winn et al. (2017) in regard to the obliquity of hot Jupiter hosts above the Kraft break. It is interesting to note that KOI 2 (HAT-P-7) has already been reported to have either a polar or retrograde planetary orbit (Winn et al. 2009a). Thus, if indeed $\sin I_\text{J} \approx 1$ for this system, that could only be consistent with a fully retrograde configuration.

Whenever the SA and QPGP periods differ, it is typically when SA periods are extremely long. Some SA periods are longer than 100 days, while none of the QPGP periods is longer than 55 days. These extremely long periods could be an artifact of time series analysis or inherent to the $\text{Kepler}$ instrumental systematics, which were never optimized to capture such long variability timescales (see Montet et al. 2017). Thus, in the absence of matching, if one rotational period is to be chosen, we favor that of Angus et al. (2018). A further advantage of these estimates is that the confidence intervals in $P_{\text{rot}}$ of Angus et al. (2018) are the well-defined result of the Bayesian fitting of a parametric (albeit non-physical) model. If we use the QPGP for hot Jupiters, we increase the number of such objects in our database from five to 19. Under these circumstances, only four hot Jupiters are significantly oblique—KOIs 97, 127, 201, and 214—of which only KOI 97 has $T_{\text{eff}} > 6000$ K. The upside of this exercise is that this list of 19 hot Jupiters is now marginally large enough for $\kappa$-inference. Our analysis results in $\kappa_{97} = 0.01_{-0.01}^{+0.01}$, with a $\kappa$ posterior that is marginally distinguishable from the prior $\pi_\kappa$ (Equation (9a)). If we instead use the prior $\pi_\kappa$ in Equation (9b), which we may argue is a better representation of a Jeffreys prior at large $\kappa$, we find $\kappa_{97} = 4^{+3}_{-1.5}$. It is difficult to conclude something from these results alone, but these concentration parameters cannot rule out that $\kappa_{\text{TEPCat}} \approx 2.2$ provides an adequate representation of the underlying obliquity distribution of hot Jupiters.

4. Discussion

We have explored the influence of seven different variables on the obliquity concentration parameter using two publicly available catalogs of exoplanet systems: the CKS survey and the TEPCat catalog. The variables tested fall into the category “stellar properties”: (1) stellar effective temperature, (2) stellar age, (3) stellar metallicity, and (4) stellar multiplicity; or into the category of “planetary properties”: (5) planet multiplicity, (6) planet orbital period, and (7) planet radius. From the CKS survey, we have found that metallicity, planet orbital period, and planet radius are the variables to which obliquity is the most sensitive, while effective temperature is not testable using this catalog. Planet multiplicity, on the other hand, is found to have no significant correlation with stellar obliquity. The TEPCat catalog is used to find a very strong correlation with effective temperature, in agreement with similar previous claims in the literature.

In particular, exploration of three of those seven variables, namely $T_{\text{eff}}$, stellar multiplicity, and planet multiplicity, was motivated by previous observational and theoretical studies. All these three tests returned null results in the CKS catalog: those variables do not correlate with stellar obliquity.

The dependence of obliquity on stellar effective temperature for hot Jupiters is the most robust of the trends found in the literature. Unfortunately, the $V \sin I_\text{J}$ method is not very effective at probing this relation, since the sample of KOIs with both $V \sin I_\text{J}$ and $P_{\text{rot}}$ measurements is sparse above 6000 K. Hotter stars tend to be more variable and are also larger/brighter, and thus typically only large planets around the quietest of these stars can be detected (Mazeh et al. 2015; Winn et al. 2017). Nevertheless, this method may still be important for understanding the systematic effect of stellar variability on the motion of hot Jupiters.
In turn, stars with little variability cannot be used to infer rotational periods via photometric modulation (McQuillan et al. 2014; Angus et al. 2018). We have attempted several approaches to isolate the hot Jupiter sample and study its obliquity properties, but more data are needed to robustly compute a value of $\kappa$ that can be compared to the one derived from TEPCat data. An alternative use of Kepler data to infer $I_\phi$ is asteroseismology, as done by Campante et al. (2016). This method represents a powerful alternative and complement to the $V\sin I_\phi$ method when stellar activity is too low to enable $P_{\text{rot}}$ measurements, in turn permitting a reliable computation of the power spectrum of non-radial oscillations (e.g., Chaplin et al. 2011). Unfortunately, the list of 25 targets of Campante et al. (2016) contains only one hot Jupiter—KOI 2, which is oblique and above the Kraft break—not allowing us to address the temperature–obliquity relation reported by Winn et al. (2010). The fact that, by contrast, we find such a strong detection of temperature dependence in the TEPCat data set (a $4\sigma$ detection, see Figure 10) highlights the fact that the KS and TEPCat catalog are, in general, probing different planet populations.

A dependence of $\kappa$ on stellar multiplicity could be detectable in similar data sets, provided that we can identify companions in an adequate manner and that these are close enough to induce obliquity through cumulative differential precession (e.g., Boué & Fabrycky 2014). As we cannot know whether the visual companions of Furlan et al. (2017) are truly bound, and we have removed some of the closest to avoid confusion due to blending, the true effect of stellar multiplicity in the Kepler sample remains unknown. Perhaps future missions to observe nearby planet hosts such as the Transiting Exoplanet Survey Satellite (TESS), complemented with astrometric information from Gaia (e.g., see Quinn & White 2016), will not only improve our $\cos I_\phi$ estimates, but also help to identify truly bound multiple stellar systems. In our analysis, we find that some multi-planet systems with companions show significant spin–orbit misalignment (in particular KOIs 377 and 1486, which drive the measured low value of $\kappa$ for this subpopulation). It is possible that multi-planet systems are more susceptible to external torquing because they have a large collective quadrupole moment; provided that the multi-planet system is able to react nearly rigidly to external perturbations, the entire coplanar system can precess around the global angular momentum vector at a much faster rate than the host star would (e.g., Kaib et al. 2011; Boué & Fabrycky 2014). This process introduces an obliquity, which will depend on the mass, separation, and inclination of the stellar companion (e.g., Lai 2016).

The tentative dependence on planet multiplicity (MW14) is attractive because it fits into a picture of exoplanet statistics in which systems can be categorized according to their “dynamical temperature” (e.g., Tremaine 2015): multi-planet systems are “dynamically cold,” having low mutual inclinations and near-zero eccentricities; “dynamically hot” systems, on the other hand, contain fewer planets and have larger eccentricities and mutual inclinations (Xie et al. 2016; Zhu et al. 2018). However, it is known that single-transiting systems are not necessarily true single-planet systems (Tremaine & Dong 2012); indeed, Zhu et al. (2018) found that an important fraction of these observed singles exhibit signs of transit-timing variations (TTVs). Thus, even though a scrambled distribution of stellar obliquities would be consistent with dynamically hot systems, identifying which systems in our KOI sample are truly hot is difficult without measured eccentricities and/or TTVs. Unfortunately, we have not found any evidence to support this hypothesis by looking at planetary multiplicity alone.

Three variables appear to exhibit substantial to significant trends in the CKS catalog: stellar metallicity ($97.9\%$ significance), planet orbital period ($98\%$ significance), and planet radius ($99.5\%$ significance). The dependence on both stellar metallicity (Figure 6) and planet radius (Figure 9) could be pointing toward an underlying dependence on total mass contained in planets $M_{p,\text{tot}}$ or the total orbital angular momentum contained in planets $J_{p,\text{tot}}$. Probing the dynamical temperature of planetary systems in a more meaningful way than (observed) planet multiplicity alone. The total mass contained in planets $M_{p,\text{tot}}$ is difficult to estimate from the current data set. Even using the empirical mass–radius relation of Weiss & Marcy (2014), we still need to leave out 32 systems for which that relation is not valid (systems with one or more planets with $R_p < 4R_\oplus$). If we simply label those 32 systems as “heavy” planetary systems and the rest as “light” ones, we indeed get a difference in the concentration parameter that is significant with a $98\%$ confidence. In the future, better estimates of planet mass could shed light on the influence of $M_{p,\text{tot}}$ and $J_{p,\text{tot}}$ on the obliquity of the stellar host.

The dependence on planet orbital period is intriguing, especially because it seems to hold for planets of any mass. The work of Mazeh et al. (2015) suggests that some level of spin–orbit alignment can even extend out to orbital periods of $\sim 50$ days (our data set contains only 15 KOIs for which the shortest orbital period is greater than $50$ days). It is difficult to picture a scenario in which low-mass planets are able to modify the obliquity of their host star out to orbital periods of 10 days, let alone 50. Such planets cannot effectively torque the star because the total angular momentum contained in the planetary orbit is too small compared to that contained in stellar spin (e.g., Dawson 2014), and thus this finding presents a severe challenge to proposed mechanisms of tidal realignment (e.g., Li & Winn 2016). An alternative idea, proposed by Matsakos & Königl (2015), is that spin–orbit realignment can be achieved by planet ingestions, and that low obliquities are the result of the incorporation of orbital angular momentum into the star, bringing its spin closer to that of the (nearly coplanar) planetary system. Under this hypothesis, stars above the Kraft break are less susceptible to realignment because they are faster rotators. In principle, this hypothesis should produce a correlation between obliquity and stellar rotational period (for $T_{\text{eff}}$ above and below 6000 K). Since we use rotational periods to infer obliquities, we refrain from searching for such a correlation at this moment, because it would produce biased results. However, we encourage future observational work aiming at independently measuring obliquities and rotational periods of planet hosts.

Yet another novel hypothesis to explain the relation between obliquity and orbital period is that these planets are simply born within a gas disk that is closely aligned with the stellar spin. This would require the innermost regions of protoplanetary disks ($r \lesssim 0.1$ au) to be immune to any external mechanisms that act to induce misalignment with respect to the stellar equator. In principle, this can be achieved by a protostellar analog to the Bardeen–Petterson (BP) effect (Bardeen &
Petterson (1975), in which viscous accretion disks can reach a warped steady-state geometry that transitions into perfect spin–orbit alignment within some transition radius. The BP effect is the result of a competition between the nodal precession of test particle orbits at some frequency \( \Omega_p \) and the rate at which material and angular momentum are resupplied via advection (Kumar & Pringle 1985). To a very rough approximation, one can identify the BP transition radius \( r_{BP} \) with the distance at which precession and advection balance each other, i.e., when \( \Omega_p \sim 1/(\alpha^2 t_visc) \) (Papaloizou & Pringle 1983; Kumar & Pringle 1985), where \( \alpha \) is the viscosity parameter, \( t_visc \sim \alpha^{-1} h^{-2}(r)\rho_{eb} \) is the viscous time, and \( h \) is the disk aspect ratio. The precession rate due to a rapidly spinning star is proportional to the oblateness quadrupole moment \( J_2M_0R_*^2 \) and scales with orbital radius as \( \propto r^{7/2} \).

Alternatively, a magnetized protostar for which the magnetic moment \( \mu \) and spin vector are misaligned can also induce precessional torques, which act at a rate proportional to \( \mu^2 \) and scale with radius as \( \propto r^{-11/2} \) (Lai 1999; Pfeiffer & Lai 2004). Of these two effects, the magnetic torque seems to be the most efficient one, because Pfeiffer & Lai (2004) state that \( r_{BP} \sim 3r_{in} \sim 10R_* \), where \( r_{in} \propto \mu^2/7 \) is the magnetospheric truncation radius. On the other hand, for a rapidly spinning oblate star (say, rotating at a tenth of the breakup rate) the transition radius is smaller: \( r_{BP} \approx R_* \left( \frac{J_2 \alpha h^2}{2} \right)^{1/2} \sim \mathcal{O}(R_*) \). Thus, young stars can, in principle, enforce the spin–orbit alignment of the surrounding disk out to a distance of a few \( R_* \), while regions outside that radius would be subject to external torques or able to retain any primordial misalignments.

5. Summary

In this work, we have studied the obliquity distribution of exoplanet host stars, performing an estimation of the concentration parameter \( \kappa \), which serves as a measure of how narrow/wide the distribution of obliquities is around/away from \( \psi = 0^\circ \). We have carried out this parameter estimation using a hierarchical Bayesian inference method (Hogg et al. 2010; Foreman-Mackey et al. 2014, MW14), which we have applied to two publicly available data sets. The first one, the CKS (Johnson et al. 2017; Petigura et al. 2017), helps to constrain \( \kappa \) from measurements/inference of the LOS inclination of the stellar spin angle \( \lambda_* \). The second data set (TEPCat, Southworth 2011), provides measurements of the obliquity projected onto the plane of the sky \( \lambda \), which can also be used to estimate \( \kappa \) (Section 2). From these data sets, we can conclude:

1. The CKS data set of \( \sin \lambda_* \) and \( R_* \) provides us with a target list \((N=257)\) with LOS inclination angles that are consistent with a concentration parameter of \( \kappa_{\text{CKS}} = 14.5^{+13.5}_{-10} \), larger than previously measured for Kepler systems (MW14 found \( \kappa_{MW} = 6.2^{+1.6}_{-1.5} \) and consistent with a mean obliquity of \( \langle \psi \rangle = 19^\circ \) and a standard deviation of \( \sigma_\psi = 10^\circ \).

2. The TEPCat data set of \( \lambda \) measurements \((N=118)\) is consistent with \( \kappa_{\text{TEPCat}} = 2.2^{+0.6}_{-0.6} \), meaning that the class of systems probed by RM observations (typically hot Jupiters) follow a significantly wider distribution of obliquities than those probed by the Kepler telescope.

We have divided the CKS and TEPCat data sets into separate bins according to stellar and planetary properties. With the CKS catalog, we have explored stellar age, stellar metallicity, stellar multiplicity, planet multiplicity, planet orbital period, and planet radius. With the TEPCat catalog, we have explored stellar effective temperature. Our findings are:

1. Planet multiplicity does not affect stellar obliquity at a statistically significant level in Kepler systems, and the weak trend which originally pointed out by MW14 has vanished. This finding is in agreement the recent results of Winn et al. (2017). Although it is still true that compact multi-planet systems have very low obliquities (e.g., Sanchis-Ojeda et al. 2012; Albrecht et al. 2013), the converse is not necessarily true for single systems of any period and planet radius. Hot Jupiter systems, which do tend to lack nearby neighbors (Steffen et al. 2012), also tend to have higher obliquities on average (e.g., Hébrard et al. 2008; Winn et al. 2009b; Triaud et al. 2010; Albrecht et al. 2013), but this property does not extend to single-transiting systems with small planets.

2. We have looked for trends in \( \kappa \) as a function of stellar properties within the CKS data set, exploring the dependence on \( T_{\text{eff}} \), stellar multiplicity, stellar age, and stellar metallicity. The only obliquity trend that rises to a substantial statistical level \((\gtrsim 2.5\sigma)\) is that of metallicity. None of the other three stellar variables affects the inferred value of \( \kappa \) above a \( 2\sigma \) level.

3. A new well established trend is that hot Jupiter hosts with \( T_{\text{eff}} \gtrsim 6000 \) K tend to be more oblique. This trend is impossible to probe with the CKS data set (it lacks hot Jupiter systems with \( T_{\text{eff}} > 6000 \) K), but it is easily seen in the TEPCat catalog. By using concentration parameter inference on the TEPCat catalog, we have not only recovered a known temperature trend, but we have placed a quantitative measure of how much more oblique hotter hosts are. We find, when considering hot Jupiters alone (106 objects), \( \kappa_{\text{TEPCAT,HJ}}^{\text{hotter}} = 0.9 \pm 0.3 \) and \( \kappa_{\text{TEPCAT,HJ}}^{\text{cooler}} = 4.2^{+0.5}_{-0.6} \) with large statistical significance. Despite the greater obliquity of hot Jupiter host above the Kraft break, both concentration parameters \( \kappa_{\text{TEPCAT,HJ}}^{\text{hotter}} \) and \( \kappa_{\text{TEPCAT,HJ}}^{\text{cooler}} \) imply wide distributions of \( \psi \). Thus, all hot Jupiters hosts in the TEPCat catalog are on average more oblique than the overall Kepler systems, independently of the stellar effective temperature.

4. We have looked for trends in \( \kappa \) as a function of planetary properties within the CKS data set. We find a trend with planetary orbital period at a substantial level \((2.5\sigma)\) and a trend with planetary radius at a convincing level \((\sim 3\sigma)\). We propose the hypothesis that the correlation between obliquity and planet radius is indirectly probing an underlying trend with “dynamical temperature.” Under such a hypothesis, the more tightly packed systems or those with greater total mass contained in planets will exhibit larger stellar obliquities. This is in accordance with the trend found with stellar metallicity. On the other hand, we propose that spin–orbit alignment in short-period systems is due to primordial alignment of the innermost protoplanetary gas disk, and not due to tidal realignment of the host star at later times.
5. We have attempted to study the obliquity for hot Jupiters in the *Kepler* catalog. Given the small number of these objects for which all of $V\sin I_\ast$, $R_\ast$, and $P_{\text{rot}}$ can be compiled, it is difficult to provide a precise number. However, the data favor isotropic orientations over spin-orbit alignment, which is in rough agreement with the TEPCat results.

One of the remaining challenges when studying the statistical distribution of obliquity is being able to robustly measure the role of the Kraft break in stellar alignment. Although the $V \sin I_\ast$ method is observationally cheaper and less biased toward a specific type of planet than RM observations, detecting small planets around stars with $T_{\text{eff}} > 6000$ K is difficult, as is the unambiguous identification of rotational periods. Of these two difficulties, perhaps the most concerning is the frequent lack of reliable periods for the hotter stars with detected planets (see Mazeh et al. 2015; Winn et al. 2017). The QGP method of period identification seems promising when unveiling underlying rotational periods of active stars, although the possibility of significant differential rotation (e.g., Reiners 2006) might require extensions to this parametric model. The identification of stellar spin $I_\ast$ from future asteroseismological studies can be instrumental in providing an independent confirmation of the inference based on the $V \sin I_\ast$ approach.

We have provided open-source, documented software on how to carry out the hierarchical Bayesian inference of the concentration parameter $\kappa$. This software package has been made freely available online (github.com/djmunoz/obliquity_inference).

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**Appendix A**

**Projected Obliquity**

The three-dimensional orientation of the stellar spin vector can be written in terms of the polar and azimuthal angles $\theta$ and $\phi$ (Fabrycky & Winn 2009; Morton & Winn 2014) where the polar axis is $\vec{z}$ — the unit angular momentum vector of the planetary orbit — or in terms of the angles $I_\ast$ and $\lambda$ (LOS inclination and projected obliquity), in which the LOS vector $\vec{x}$ plays the role of the polar axis. These two sets of angles are related by (see Equation (2))

\[
\cos I_\ast = \sin \theta \cos \phi, \quad (18a)
\]
\[
\sin I_\ast \sin \lambda = \sin \theta \sin \phi, \quad (18b)
\]
\[
\sin I_\ast \cos \lambda = \cos \theta. \quad (18c)
\]

Given the PDFs $f_\theta(\theta|\kappa)$ and $f_\phi(\phi)$, one can obtain the PDF $f_{\text{cos} I_\ast}(\lambda|\kappa)$ using coordinate transformations (Morton & Winn 2014; their Equation (9)). In a similar fashion, since $\tan \lambda = \tan \theta \sin \phi$ (Equations (18b) and (18c)), one can obtain a PDF for the quantity $\tan \lambda$ after obtaining the PDFs $f_{\tan \theta}(\lambda|\kappa)$ and $f_{\tan \phi}(\lambda|\kappa)$:

\[
f_{\tan \theta}(\lambda|\kappa) = \frac{\kappa}{2 \sinh \kappa} \frac{|y|}{(1 + y^2)^{3/2}} \exp \left( \frac{\kappa}{\sqrt{1 + y^2}} |y| \right), \quad y \in (-\infty, \infty) \quad (19)
\]
\[
f_{\tan \phi}(x|\kappa) = \frac{2}{\pi} \frac{1}{\sqrt{1 - x^2}}, \quad x \in (0, 1) \quad (20)
\]

with $y = \tan \theta$ and $x = \sin \phi$. Combining these two expressions and using

\[
f_{\tan \lambda}(\lambda|\kappa) = \int_{-\infty}^{\infty} f_{\tan \theta}(\theta|\kappa) f_{\tan \phi}(\lambda|\kappa) \frac{1}{|y|} dy, \quad (21)
\]

we have, for $z = \tan \lambda$,

\[
f_{\tan \lambda}(\lambda|\kappa) = \frac{\kappa}{\pi \sinh \kappa} \int_{0}^{1} \frac{1}{\sqrt{1 + y^2}} \exp(\kappa \sqrt{1 + y^2}) \, dy \frac{\sqrt{1 + z^2}}{\sqrt{1 - z^2}} \sqrt{1 - \frac{y^2}{1 - z^2}}, \quad (22)
\]
This expression is valid in the range \((0, \infty)\), and thus not very practical for random sampling purposes. However, using \(f_{\tan \lambda}\) we can write

\[
f_{\lambda}(\lambda|\kappa) = \frac{\kappa/ \sinh \kappa}{\pi \cos^2 \lambda} \int_0^{\cos \lambda} dy \frac{\exp(\kappa y)}{\sqrt{1 - y^2}} \frac{y}{\sqrt{1 - 2 \cos \lambda \frac{y}{\sqrt{1 - y^2}}}},
\]

(23)

which we can confirm is correct by random sampling in \(\theta\) and \(\phi\) and computing the sampled values of \(\lambda\) using Equations (18) (see Figure 11).

**Appendix B**

**Background: Hierarchical Bayesian Inference of Meta-parameters**

Here, we briefly describe the hierarchical Bayesian method introduced by Hogg et al. (2010). In this method, one seeks to carry out the estimation of the parameter vector \(\alpha\), which controls the PDF \(f_{\lambda}(\omega|\alpha)\) of a vector of physical quantities \(\omega\), which in turn contains the parameters \(\{w_i\}\) of some parameterization of an individual object. With \(N\) such objects (in our case, exoplanet systems) the whole data set \(D\) can be split into subsets \(D_1, \ldots, D_N\), with \(D = \bigcup_{n=1}^{N} D_n\). By means of Bayes’ theorem, the posterior PDF of these “meta-parameters” \(\alpha\) for a given data set \(D\) is (Hogg et al. 2010)

\[
p(\alpha|D) \propto \mathcal{L}_\alpha(D) \prod_{n=1}^{N} \pi_{w,n}(\alpha)
\]

where \(\mathcal{L}_\alpha\) is the global likelihood, \(\mathcal{L}_\alpha(D_i)\) is the likelihood associated with data set \(D_n\) and \(\pi_{w,n}\) the prior information of the parameter vector \(\alpha\).

Instead of computing \(\mathcal{L}_\alpha(D_i|\alpha)\) directly from the global data set, this hierarchical method introduces a middle step, which is played by the vector of parameters per object \(\omega_n\), for which we already have previously computed posterior PDF obtained from its corresponding data set \(D_n:\)

\[
p(\omega|D_n) = \frac{1}{Z_n} \prod_{i=1}^{N} p(D_n|\omega_i)p(\omega_i|\omega_n)
\]

where \(Z_n\) is a normalization constant and \(\pi_{w,0}\) the (uninformative) prior for the vector of parameters \(\omega\). Then, one can write (Hogg et al. 2010; Foreman-Mackey et al. 2014)

\[
\mathcal{L}_{\alpha,n} = \int d\omega_n p(D_n|\omega_n)p(\omega_n|\omega_n)
\]

where \(Z_n\) is the normalization constant and \(\pi_{w,0}\) the (uninformative) prior for the vector of parameters \(\omega_n\). Then, one can write (Hogg et al. 2010; Foreman-Mackey et al. 2014)

\[
\mathcal{L}_{\alpha,n} = \int d\omega_n p(D_n|\omega_n)p(\omega_n|\omega_n)
\]

(26)

The PDFs in Equations (4) and (5) can replace \(p(\omega_n|\alpha)\) in Equation (26)—with the parameter vectors \(\omega_n\) and \(\alpha\) having only one element each—to carry out the hierarchical inference calculation on \(\kappa\).

Ignoring the normalization constant, \(\mathcal{L}_{\alpha,n}\) can be interpreted as the distribution-weighted average of the ratio \(p(\omega_n|\alpha)/\pi_{w,0}(\omega_n)\), which allows for a Monte Carlo approximation via \(K\)-sampling.
