ON THE HÖLDER ESTIMATE OF KÄHLER-RICCI FLOW

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Abstract. In this work, we study the Hölder regularity of the Kähler-Ricci flow on compact Kähler manifolds with semi-ample canonical line bundle. By adapting the method in the work of Hein-Tosatti on collapsing Calabi-Yau metrics, we obtain a uniform spatial $C^\alpha$ estimate along the Kähler-Ricci flow as $t \to +\infty$.

1. Introduction

In this work, we study the normalized Kähler-Ricci flow which is a family of Kähler metrics satisfying

\begin{align}
\partial_t \omega(t) &= -\text{Ric}(\omega(t)) - \omega(t); \\
\omega(0) &= \omega_0
\end{align}

on a compact Kähler manifold $X$ with semi-ample canonical line bundle $K_X$, where $\omega_0$ is the initial Kähler metric. In this case, $X$ admits an Iitaka fibration structure given by a holomorphic map $f : X \to \Sigma \subset \mathbb{CP}^N$ with possibly singular fibers and possibly singular base manifold $\Sigma$. Let $S \subset \Sigma$ be the union of the set of singular values of $f$ and the singular set of $\Sigma$. The regular fibers $f^{-1}(z)$, where $z \in \Sigma \setminus S$, are Calabi-Yau manifolds. The complex dimension of $\Sigma$ is the Kodaira dimension of $X$. We focus on the case when $0 < \dim \Sigma < \dim X$, and we let $\dim_{\mathbb{C}} \Sigma = m$ and $\dim_{\mathbb{C}} X = m + n$, so that the Calabi-Yau fibers have complex dimension $n$.

Under semi-ample assumption, the canonical line bundle is nef and hence the flow exists on $X \times [0, +\infty)$ by the works of [12, 21, 26]. The Kähler-Ricci flow under semi-ample canonical line bundle has been extensively studied by various authors [4, 5, 6, 7, 11, 12, 17, 18, 19, 20, 22, 24, 25, 28, 29, 30, 31]. In [17, 18], Song-Tian proved that the flow converges to a generalized Kähler-Einstein metric in the sense of measure on the base manifold $\Sigma$ as $t \to +\infty$. The generalized Kähler-Einstein metric $\omega_\Sigma$ satisfies $\text{Ric}(\omega_\Sigma) = -\omega_\Sigma + \omega_{WP}$, where $\omega_{WP}$ is the Weil-Petersson form which measures the variation of complex structures of the fibers. It was conjectured that the regularity of convergence can be improved to $C^\infty_{\text{loc}}(f^{-1}(\Sigma \setminus S))$-convergence. This conjecture is still open.

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in general, although many progresses have been made. For instances, Tosatti-Weinkove-Yang proved in \cite{25} the $C_0^0(f^{-1}(\Sigma\setminus S))$-convergence of the metric to the generalized Kähler-Einstein metric on the base manifold $\Sigma$. In \cite{3}, the second named author and Fong considered the case when the generic fibres are biholomorphic to each other and developed a sharp parabolic Schauder estimate on cylinder using the idea of Hein-Tosatti in \cite{10}, and thus confirmed the above conjecture in the locally product case. More recently, Jian and Song \cite{13} considered the case when $m+n=3$ and proved that the Ricci curvature is uniformly locally bounded. For further discussions, we refer interested readers to \cite{1, 3, 4, 6, 7, 9, 13, 20, 22, 25} and the references therein.

The main goal of this article is to establish the $C_\alpha^\alpha(f^{-1}(\Sigma\setminus S))$-convergence of the flow in the general setting where the fibres are not necessarily biholomorphic to each other.

**Theorem 1.1.** Suppose that $(X, \omega_X)$ is a compact Kähler manifold with semi-ample canonical line bundle and $\omega(t)$ is a normalized Kähler-Ricci flow on $X$ defined by (1.1). Then for any $\alpha \in (0, 1)$ and any compact set $K \subset X \setminus f^{-1}(S)$, there is $C(\alpha, K)$ such that for all $t \in [0, +\infty)$,

$$\|\omega(t)\|_{C^\alpha(K, \omega_X)} \leq C.$$ (1.2)

The proof of Theorem 1.1 adapts the idea in Hein-Tosatti’s work \cite{10} on collapsing Calabi-Yau metrics, and some estimates for Kähler-Ricci flow in earlier works by Fong-Zhang \cite{4}, Song-Tian \cite{19} and Tosatti-Weinkove-Yang \cite{25}.

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2. Preliminary

In this section, we collect some known results which will be used in deriving the $C^\alpha$ estimate.

2.1. Local estimates for Kähler-Ricci flow. It is known that when the Kähler-Ricci flow $\omega(t)$ is uniformly equivalent to a fixed Kähler metric, then $\omega(t)$ is bounded in $C_k^k$ for all $k \in \mathbb{N}$. When the Kähler-Ricci flow is with respect to a mildly varying family of complex structures, we need the following parabolic regularization property, which is a slight modification of the elliptic case \cite{10} Proposition 2.3.

**Proposition 2.1.** For all $n, k, l_0 \in \mathbb{N}$, $\alpha \in (0, 1)$ and $A > 1$, there are $\kappa(n, \alpha)$ and $C(k, n, \alpha, A, l_0) > 0$ such that the following holds. Let $B_1(0)$ be the unit ball in $\mathbb{C}^n$ with the standard Euclidean metric $\omega_{\mathbb{C}^n}$ and $J$ be a complex structure on $B_1(0)$ such that

$$\|J - J_{\mathbb{C}^n}\|_{C^{1, \alpha}(B_1(0))} < \kappa, \quad \|J - J_{\mathbb{C}^n}\|_{C^{k, \alpha}(B_1(0))} \leq A.$$ (2.1)
If \( \omega(t) \) is a family of J-Kähler metrics satisfying
\[
\partial_{t} \omega(t) = -\text{Ric}(\omega(t)) - l \omega(t) \quad \text{on} \quad Q_{1}(0) = B_{1}(0) \times [-1, 0]
\]
for some \( |l| \leq l_{0} \) and
\[
A^{-1} \omega_{\mathbb{C}^{n}} \leq \omega(t) \leq A \omega_{\mathbb{C}^{n}} \quad \text{on} \quad Q_{1}(0),
\]
then we have
\[
\| \omega(t) \|_{C^{k, \alpha}(Q_{1/2}(0))} \leq C(k, n, \alpha, A, l_{0}).
\]

**Proof.** By re-parametrization of time, we may assume \( l_{0} = 0 \). Thanks to [10, Proposition 2.2], we can find J-holomorphic coordinates on \( B_{3/4}(0) \) which are close to the standard Euclidean coordinates in \( C^{2, \alpha} \) and differing from them by a bounded amount in \( C^{k+1, \alpha} \). In these J-holomorphic coordinates, applying the local estimates of Kähler-Ricci flow in [16], we obtain the required estimate. \( \square \)

### 2.2. Liouville theorems for Ricci-flat metric

The following Liouville theorems for Ricci-flat metrics will play an important role in analyzing the blow-up model of the Kähler-Ricci flow.

**Theorem 2.1** ([15]). Suppose that \( \omega \) is a Ricci-flat Kähler metric on \( \mathbb{C}^{n} \) such that
\[
A^{-1} \omega_{\mathbb{C}^{n}} \leq \omega(t) \leq A \omega_{\mathbb{C}^{n}} \quad \text{on} \quad \mathbb{C}^{n}
\]
for some \( A > 1 \), then \( \omega \) is constant.

When the underlying manifold is a product of \( \mathbb{C}^{n} \) and a compact Calabi-Yau Kähler manifold \( Y \), the following Liouville theorem was proved by Hein [8], see also [14] for an alternative proof using mean value inequality.

**Theorem 2.2** ([8][14]). Let \( Y \) be a compact Calabi-Yau Kähler manifold with Ricci-flat metric \( \omega_{Y} \). Suppose that \( \omega \) is a Ricci-flat Kähler metric on \( \mathbb{C}^{n} \times Y \) such that
\[
A^{-1}(\omega_{\mathbb{C}^{n}} + \omega_{Y}) \leq \omega \leq A(\omega_{\mathbb{C}^{n}} + \omega_{Y}) \quad \text{on} \quad \mathbb{C}^{n} \times Y
\]
for some \( A > 1 \) and \( \omega \) is d-cohomologous to \( \omega_{\mathbb{C}^{n}} + \omega_{Y} \), then \( \omega \) is parallel with respect to \( \omega_{\mathbb{C}^{n}} + \omega_{Y} \).

### 3. \( C^{\alpha} \) estimate of Kähler-Ricci flow

First, let us recall the setting: \( X \) is a compact Kähler manifolds with semiample canonical line bundle, \( f : X^{m+n} \rightarrow \Sigma^{m} \subset \mathbb{CP}^{N} \) is the corresponding Calabi-Yau fibration, and \( S \subset \Sigma \) is the union of the set of singular values of \( f \) and the singular set of \( \Sigma \). By [18], there exists a smooth Kähler metric \( \omega_{\Sigma} \) on \( \Sigma^{n} \setminus S \) satisfying the generalized Kähler-Einstein equation:
\[
\text{Ric}(\omega_{\Sigma}) = -\omega_{\Sigma} + \omega_{\text{WP}},
\]
where \( \omega_{\text{WP}} \) is the smooth semi-positive Weil-Petersson form. By rescaling, we may assume \( B_2(0) \subset \Sigma \setminus S \), where \( B_2(0) = B_{\mathbb{C}^m}(2) \) denotes the Euclidean ball. On \( B_2(0) \), we have \( \omega_{\mathbb{C}^m} = \omega_{\Sigma} + \sqrt{-1} \partial \bar{\partial} u \) for some \( u \in C^\infty(B_2(0)) \). For notational convenience, we still use \( u \) and \( \omega_{\Sigma} \) to denote their pull-backs to \( B_2(0) \times Y \).

The map \( f|_{f^{-1}(B_2(0))} : f^{-1}(B_2(0)) \to B_2(0) \) is a proper surjective holomorphic map with \( n \)-dimensional Calabi-Yau fibers. For each \( z \in B_2(0) \), we write \( X_z = f^{-1}(z) \). For Kähler metric \( \omega_0 \) on \( X \), using Yau’s theorem [27], there is a unique Kähler-Ricci flat metric \( \omega_{F,z} \) on each fibre \( X_z \) which is cohomologous to \( \omega_0|_{X_z} \). We may choose \( \rho \) locally smoothly such that \( \omega_F = \omega_0 + \sqrt{-1} \partial \bar{\partial} \rho \) and \( \omega_F|_{X_z} = \omega_{F,z} \). Following [10], we define the reference closed real \((1,1)\)-form on \( B_2(0) \) by

\[
\omega^s(t) = (1 - e^{-t})\omega_\infty + e^{-t}\omega_F,
\]

where \( \omega_\infty = f^*\omega_{\mathbb{C}^m} \). Note that \( \omega_F \) may be not positive definite along the base directions, and so \( \omega^s(t) \) is not necessarily positive definite for all \( t \geq 0 \). But the Cauchy-Schwarz inequality shows \( \omega^s(t) \) is positive definite for \( t \) sufficiently large, which is what we concern. For convenience, by translation of time, we always assume \( \omega^s(t) > 0 \) for all \( t \geq 0 \), and denote its associated Riemannian metric by \( g^s(t) \).

Suppose that \( \omega^s(t) \) is the normalized Kähler-Ricci flow [11] defined on \( X \times [0, +\infty) \). Let \( g^s(t) \) be the associated Riemannian metric. We use \( \varphi \) to denote the solution of the following ODE:

\[
\dot{\varphi} + \varphi = \log \frac{\omega^s(t)^{m+n}}{e^{-nt}C_n^{m+n}\omega_F^m \wedge \omega^m_{\Sigma}} - u, \quad \varphi(0) = \rho.
\]

Here we recall that \( u \) is the function such that \( \omega_\infty = \omega_{\Sigma} + \sqrt{-1} \partial \bar{\partial} u \). By taking \( \sqrt{-1} \partial \bar{\partial} \) on both sides of (3.3) and using \( \sqrt{-1} \partial \bar{\partial} \log(\omega_F^m \wedge \omega_{\Sigma}^n) = \omega_{\Sigma} \) (see e.g., [23] Section 5.7), we see that

\[
\omega^s(t) = \omega^s(t) + \sqrt{-1} \partial \bar{\partial} \varphi(t) \quad \text{on} \quad (B_2(0) \times Y) \times [0, \infty).
\]

This reduces back to the setting analogous to its elliptic counterpart in [10].

By Ehresmann’s theorem and shrinking \( B_2(0) \) slightly, \( f \) is a smooth fiber bundle. We may choose a trivialization \( \Phi : B_2(0) \times Y \to X \) where \( Y = f^{-1}(0) \) such that \( \Phi|_{\{0\} \times Y} : \{0\} \times Y \to f^{-1}(0) = Y \) is the identity map. On \( B_2(0) \times Y \), we use \( J^2 \) to denote the complex structure inherited from \( X \) via the trivialization \( \Phi \). Then the projection \( \text{pr}_{\mathbb{C}^m} \) is a \( J^2 \)-holomorphic submersion. Let \( J_{YZ} \) denote the restriction of \( J^2 \) to the \( J^2 \)-holomorphic fibre \( \{z\} \times Y \). Note that \( \Phi^*\omega_{F,z} \) is a Ricci-flat \( J_{YZ} \)-Kähler metric on \( \{z\} \times Y \). Let \( g_{YZ} \) be its associated Riemannian metric on \( \{z\} \times Y \). Extend it trivially to the product metric on \( \mathbb{C}^m \times Y \) and define the product shrinking metric \( g_z(t) = g_{\mathbb{C}^m} + e^{-t}g_{YZ} \), which is Kähler with respect to \( J_z = J_{\mathbb{C}^m} + J_{YZ} \). By the trivialization \( \Phi \), we may assume the above complex structures, metrics and the Kähler-Ricci
flow are defined on $B_2(0) \times Y$. We will omit the trivialization for notational convenience.

The main objective of this section is to prove the following $C^\alpha$ estimate of the Kähler-Ricci flow.

**Theorem 3.1.** For all $\alpha \in (0, 1)$, there is $C > 0$ such that for all $t \in [0, +\infty)$,

$$
\sup_{x=(z,y) \in B_{1/4} \times Y} \sup_{(x',t') \in B_{2\Gamma(t)}((x,t), \frac{1}{2})} |\eta(x, t) - P_{x'}^{g_z(t)}(\eta(x', t))|_{g_z(t)} \leq C,
$$

where $\eta = \sqrt{-1} \partial \bar{\partial} \varphi$ and $P_{x'}^{g_z(t)}$ denotes the $g_z(t)$-parallel transport along the $g_z(t)$-geodesic from $x'$ to $x$.

**Remark 3.1.** We note that the Ricci flow $g^*(t)$ is uniformly equivalent to the reference shrinking Riemannian metric $g_z(t)$ by the work of [4]. By standard parabolic regularity theory, the flow is $C^k$ regular in finite time. The major difficulties is the uniformity for all $t > 0$.

To prove Theorem 3.1, we first need the following lower order estimates along the Kähler-Ricci flow.

**Lemma 3.1.** Under the above setting, there are $C > 0$ and $T > 1$ such that for all $(x, t) \in B_2(0) \times [T, +\infty)$,

(i) $|R_{g^*(t)}| \leq C$;

(ii) $C^{-1}\omega^U(t) \leq \omega^*(t) \leq C\omega^U(t)$;

(iii) $|\nabla (\varphi + \varphi + u)|_{g^*(t)} \leq C$;

(iv) $e^{nt}\omega^m(t) \to C_n \omega_F^m \wedge \omega_{\Sigma}^m$ uniformly on $B_2(0)$ as $t \to +\infty$.

**Proof.** The uniform boundedness of scalar curvature follows directly from [19 Theorem 1.1]. By [4 Theorem 1.1], the flow $\omega^*(t)$ is uniformly equivalent to the shrinking reference metric $(1 - e^{-t})\omega_\Sigma + e^{-t}\omega_0$ and hence $\omega^U(t)$ on $B_2(0)$. These prove (i) and (ii).

By [19 Proposition 3.1], the function $|\nabla v|_{g^*(t)}$ is uniformly bounded where $v = \log\frac{\omega^m(t)^{m+n}}{e^{-nt\Omega}}$ and $\Omega$ is a smooth volume form on $X$. Since $\Omega$ and $\omega_F^m \wedge \omega_\Sigma^m$ are uniformly equivalent on $B_2(0)$, then (iii) follows. The volume convergence (iv) follows from [25 Lemma 3.1].

Next, let us include some important observation from [10]. Choosing a complex coordinate chart $(y^1, ..., y^n)$ on $Y$, together with complex coordinate chart $(z^1, ..., z^m)$ on $B_2(0)$, $(z,y)$ is a complex coordinate chart on $B_2(0) \times Y$. Since $\text{pr}_{C^m}$ is holomorphic with respect to $J^2$ and $J_{z_0}$, then

$$
(\text{pr}_{C^m})_* \circ (J^2 - J_{z_0}) = J_{C^m} \circ (\text{pr}_{C^m})_* - J_{C^m} \circ (\text{pr}_{C^m})_* = 0.
$$

Thus, ignoring the distinction between these complex coordinates and their complex conjugates, (3.6) shows

$$
(J^2 - J_{z_0})_{(z,y)} = A(z_0, z, y)dz \otimes \partial_y + B(z_0, z, y)dy \otimes \partial_y,
$$

where $A(z_0, z, y)$ and $B(z_0, z, y)$ are functions on $B_2(0)$.
where $A, B$ are smooth matrix-valued functions with $B(z_0, z_0, y) = 0$. Combining (3.7) with the definitions of $g_{z_0}(t)$ and $g^2(t)$, we see that
\begin{equation}
(g^2(t) - g_{z_0}(t)) \bigg|_{(z,y)}
= e^{-t} \left( C(z_0, z, y) dz \otimes dz + D(z_0, z, y) dz \otimes dy + E(z_0, z, y) dy \otimes dy \right),
\end{equation}
where $C, D, E$ are smooth matrix valued functions with $E(z_0, z_0, y) = 0$. Thanks to the factor $e^{-t}$ in (3.8) and the Cauchy-Schwarz inequality, we can find $C, T > 1$ such that for all $(z, t) \in B \times [T, +\infty)$,
\begin{equation}
C^{-1} g_z(t) \leq g^2(t) \leq C g_z(t).
\end{equation}
By translation of time, we will assume $T = 0$ since we only concern the behaviour of the Kähler-Ricci flow as $t \to +\infty$ by Remark 3.1.

Now, we are in a position to prove Theorem 3.1.

**Proof of Theorem 3.1.** We will follow closely the argument in [10] and adapt the argument in the parabolic setting. Let $B = B_1(0) \subset B_2(0) \Subset \Sigma \setminus S$. In the following, we say a constant is uniform if it is independent of $t$, and always use $C$ to denote a uniform constant.

For $x = (z, y) \in B \times Y$, we consider the function
\begin{equation}
\mu(x, t) = \left(d^{g_z(t)}(x, \partial(B \times Y))\right)^\alpha \times \sup_{x' \in B^{g_z(t)}(x, \frac{1}{2}d^{g_z(t)}(x, \partial(B \times Y)))} \left| \eta(x, t) - P_{x'x}^{g_z(t)}(\eta(x', t)) \right|_{g_z(x, t)}.
\end{equation}
where $P_{x'x}^{g_z(t)}$ denotes the $g_z(t)$-parallel transport along the $g_z(t)$-geodesic from $x'$ to $x$. To prove Theorem 3.1, it suffices to show that
\begin{equation}
\sup_{(B \times Y) \times [0, \infty)} \mu(x, t) \leq C
\end{equation}
for some $C > 0$. Suppose on the contrary, we can find sequences $t_i \to +\infty$ and $x_i \in B \times Y$ such that
\begin{equation}
\mu(x_i, t_i) = \sup_{(B \times Y) \times [0, t_i]} \mu(x, t) \to +\infty.
\end{equation}
Let $x_i = (z_i, y_i)$ and $x_i' \in B^{g_{z_i(t_i)}(x_i, \frac{1}{2}d^{g_{z_i(t_i)}}(x_i, \partial(B \times Y)))}$ be the point realizing the supremum in the definition of $\mu$. Define $\lambda_i$ by
\begin{equation}
\lambda_i^\alpha = \frac{|\eta(x_i, t_i) - P_{x'x_i}^{g_{z_i(t_i)}}(\eta(x_i', t_i))|_{g_z(x_i, t_i)}}{d^{g_{z_i(t_i)}(x_i, x_i')}^\alpha},
\end{equation}
and then
\begin{equation}
\mu(x_i, t_i) = d^{g_{z_i(t_i)}}(x_i, \partial(B \times Y))^\alpha \cdot \lambda_i^\alpha.
\end{equation}
By passing to subsequence, we may assume that
\begin{equation}
x_i \to x_\infty = (z_\infty, y_\infty) \in \overline{B} \times Y.
\end{equation}
Since \( g_z(t_i) = g_{\mathbb{C}^m} + e^{-t_i}g_{Y,z_i} \) and \( t_i \to +\infty \), the metric \( g_z(t_i) \) is shrinking, and so \( d^{g_z(t_i)}(x, \partial(B \times Y)) \) is uniformly bounded from above. Combining this with (3.12) and (3.14), we see that
\[
\lambda_i \to +\infty \quad \text{as} \quad i \to +\infty.
\]
Define the diffeomorphism \( \Psi_i : B_{\lambda_i} \times Y \to B \times Y \) by
\[
\Psi_i(z, y) = (\lambda_i^{-1}z, y),
\]
and pull back the complex structures, metrics, Kähler-Ricci flow and points to \( B_{\lambda_i} \times Y \) via \( \Psi_i \):
\[
\begin{align*}
\hat{J}_i &= \Psi_i^*J_{z_i}; \\
\hat{J}_i^2 &= \Psi_i^*J_i^2; \\
\hat{g}_i(t) &= \lambda_i^2\Psi_i^*g_z(t_i + \lambda_i^{-2}t); \\
\hat{\omega}_i^2(t) &= \lambda_i^2\Psi_i^*\omega^2(t_i + \lambda_i^{-2}t); \\
\hat{\eta}_i(t) &= \lambda_i^2\Psi_i^*\eta(t_i + \lambda_i^{-2}t); \\
\hat{\omega}_i^*(t) &= \hat{\omega}_i^2(t) + \hat{\eta}_i(t); \\
\hat{x}_i &= \Psi_i^{-1}(x_i); \\
\hat{x}_i' &= \Psi_i^{-1}(x_i').
\end{align*}
\]
Then \( \hat{g}_i(t) \) is a Ricci-flat \( \hat{J}_i \)-Kähler product metric, \( \hat{\omega}_i^2(t) \) is a semi-Ricci-flat \( \hat{J}_i^2 \)-Kähler metric and \( \hat{\omega}_i^*(t) \) is a Kähler-Ricci flow with respect to \( \hat{J}_i^2 \). Let \( \hat{g}_i^*(t) \) and \( \hat{g}_i^*(t) \) be the Riemannian metrics associated to \( \hat{\omega}_i^2(t) \) and \( \hat{\omega}_i^*(t) \), then we have the following properties which follow from Lemma 3.1, (3.9) and (3.18):
\[
\begin{align*}
\lambda_i &\geq \hat{g}_i(t) \leq \hat{g}_i^*(t) \leq C\hat{g}_i(t); \\
\lambda_i^{-1} &\geq \hat{g}_i(t) \leq \hat{g}_i^*(t) \leq C\hat{g}_i(t); \\
\hat{g}_i(t) &= g_{\mathbb{C}^m} + \lambda_i^2\hat{e}^{-\lambda_i^{-2}t})g_{Y,z_i}; \\
\hat{\omega}_i^2(t) &= (1 - e^{-\lambda_i^{-2}t})\omega_{\mathbb{C}^m} + \lambda_i^2e^{-\lambda_i^{-2}t}\Psi_i^*\omega_F.
\end{align*}
\]
Recalling the definition of \( \lambda_i \) (3.13) and using (3.18),
\[
\lambda_i^\alpha = \frac{|\eta(x_i, t_i) - \mathbf{P}_{g_z(t_i)}^{g_{x_i}(t_i)}(\eta(x_i', t_i))|_{g_z(x_i, t_i)}}{d^{g_z(t_i)}(x_i, x_i')^\alpha} = \frac{\lambda_i^\alpha}{\frac{d^{g_z(t_i)}(x_i, x_i')^\alpha}{d^{\hat{g}_i(0)}(\hat{x}_i, \hat{x}_i')}^\alpha} \cdot \lambda_i^\alpha,
\]
which implies
\[
\frac{|\hat{\eta}_i(\hat{x}_i, 0) - \mathbf{P}_{\hat{g}_i^*(\hat{x}_i')}^{\hat{g}_i^*(\hat{x}_i')}(\hat{\eta}_i(\hat{x}_i', 0))|_{\hat{g}_i(\hat{x}_i, 0)}}{d^{\hat{g}_i(0)}(\hat{x}_i, \hat{x}_i')^\alpha} = 1.
\]
By (3.19) and \( \hat{\omega}_i^*(t) = \hat{\omega}_i^2(t) + \hat{\eta}_i(t) \), the numerator of (3.21) is uniformly bounded and hence the distance between \( \hat{x}_i \) and \( \hat{x}_i' \) with respect to \( \hat{g}_i(0) \) is
uniformly bounded:
\begin{equation}
\tag{3.22}
d_{B^i(0)}(\hat{x}_i, \hat{x}'_i) \leq C.
\end{equation}

On the other hand, by the definition of \( \mu \) and (3.12),
\begin{equation}
\tag{3.23}
(\mu(\hat{B}_\lambda \times Y)) = \mu(x_i, t_i) \to +\infty.
\end{equation}
This implies that the pointed limit space of
\begin{equation}
\tag{3.24}
(B_\lambda \times Y, \hat{g}_i(0), \hat{x}_i)
\end{equation}
will be complete. By (3.22), we may assume \( \hat{x}_i \) modulo translations in the \( \mathbb{C}^m \) factor. Write \( \delta_i = \lambda_i e^{-t_i/2} \), then
\begin{equation}
\tag{3.25}
\hat{g}_i(0) = g_{\mathbb{C}^m} + \delta_i^2 g_{Y,z_i}.
\end{equation}
By passing to a subsequence, we may assume \( \delta_i \to \delta_{\infty} \in [0, +\infty] \). From the behaviour of \( \hat{g}_i(0) \), there are three cases to be considered:
\begin{enumerate}
\item[(a)] \( \delta_{\infty} = +\infty \);
\item[(b)] \( \delta_{\infty} \in (0, +\infty) \);
\item[(c)] \( \delta_{\infty} = 0 \).
\end{enumerate}
Before splitting into different cases, thanks to (3.7) and (3.15), we always have the following convergence of complex structures:
\begin{equation}
\tag{3.26}
\hat{J}_i, \hat{J}'_i \to J_{\mathbb{C}^m} + J_{Y,z_{\infty}} \text{ in } C^\infty_{\text{loc}}(\mathbb{C}^m \times Y).
\end{equation}

**Case (a):** \( \delta_{\infty} = +\infty \). In this case, \( (B_\lambda \times Y, \hat{g}_i(0), \hat{x}_i) \) converges to \( (\mathbb{C}^{m+n}, g_{\mathbb{C}^{m+n}}, 0) \) in the \( C^\infty \)-Cheeger-Gromov sense. More precisely, let \( \hat{y}_i = (\hat{y}_1^n) \) be a holomorphic chart of \( Y \) centred at \( y_{\infty} \) with respect to the complex structure \( J_{Y,z_{\infty}} \).

We may assume \( \hat{y}_i \in B_{\mathbb{C}^n}(1) \) and \( \hat{y}_i \to y_{\infty} = 0 \).

Consider the diffeomorphism \( \Lambda_i : B_\lambda \times B_{\delta_i} \to B_{\lambda_i} \times B_1 \) given by
\begin{equation}
\Lambda_i(z, y) = (z, \delta_i^{-1}y).
\end{equation}
Then (3.26) shows the convergence of the background complex structures:
\begin{equation}
\tag{3.27}
\Lambda_i^* \hat{J}_i, \Lambda_i^* \hat{J}'_i \to J_{\mathbb{C}^{m+n}} \text{ in } C^\infty_{\text{loc}}(\mathbb{C}^{m+n}),
\end{equation}
and (3.8), (3.19) show the convergence of the background metrics:
\begin{equation}
\tag{3.28}
\Lambda_i^* \hat{g}_i(t), \Lambda_i^* \hat{g}'_i(t) \to g_{\mathbb{C}^{m+n}} \text{ in } C^\infty_{\text{loc}}(\mathbb{C}^{m+n} \times (-\infty, 0])
\end{equation}

On the other hand, we write \( \tilde{\omega}_i^*(t) = \Lambda_i^* \tilde{\omega}_i^*(t) \), and then \( \tilde{\omega}_i^*(t) \) solves the approximated Kähler-Ricci flow with respect to the complex structure \( \Lambda_i^* \tilde{J}'_i \):
\begin{equation}
\tag{3.29}
\partial_t \tilde{\omega}_i^*(t) = -\text{Ric}(\tilde{\omega}_i^*(t)) - \lambda_i^{-2} \tilde{\omega}_i^*(t).
\end{equation}
By (3.19) and (3.28), we obtain
\begin{equation}
\tag{3.30}
C^{-1} \omega_{C^{m+n}} \leq \tilde{\omega}_i^*(t) \leq C \omega_{C^{m+n}}
\end{equation}
on any compact subset of \( C^{m+n} \times (-\infty, 0] \) if \( i \) is sufficiently large relative to the compact set. By (3.27), (3.28) and (3.30), we may apply Proposition 2.1 to obtain
\begin{equation}
C^\infty_{\text{loc}}(\mathbb{C}^{m+n} \times (-\infty, 0]) \text{ estimate of } \tilde{\omega}_i^*(t) \text{. Hence, } \tilde{\omega}_i^*(t) \text{ converges}
\end{equation}
to $\tilde{\omega}_\infty^\bullet(t)$ in $C^\infty_{\text{loc}}(\C^{m+n} \times (-\infty, 0])$ which remains uniformly equivalent to the Euclidean metric $\omega_{\C^{m+n}}$ for all $t \leq 0$.

Now, we follow similar argument in [3] to reduce the discussion back to the elliptic case. Using (3.29), $\tilde{\omega}_\infty^\bullet(t)$ solves the Kähler-Ricci flow:

$$\partial_t \tilde{\omega}_\infty^\bullet(t) = -\text{Ric}(\tilde{\omega}_\infty^\bullet(t)).$$

(3.31)

By Lemma 3.1, the scalar curvature of the original Kähler-Ricci flow is bounded, and hence the scalar curvature after parabolic rescaling converges to 0, i.e.,

$$R(\tilde{\omega}_\infty^\bullet(t)) \equiv 0 \text{ on } \C^{m+n} \times (-\infty, 0].$$

(3.32)

Recalling the evolution of scalar curvature along the Kähler Ricci flow:

$$(\partial_t - \Delta)R(\tilde{\omega}_\infty^\bullet(t)) = 2|\text{Ric}(\tilde{\omega}_\infty^\bullet(t))|^2,$$

(3.33)

we conclude that $\tilde{\omega}_\infty^\bullet(t)$ is Ricci-flat, and then $\tilde{\omega}_\infty^\bullet(t) \equiv \tilde{\omega}_\infty^\bullet(0)$ for all $t \leq 0$. Combining this with Theorem 2.1, $\tilde{\omega}_\infty^\bullet(t)$ is constant on $\C^{m+n} \times (-\infty, 0]$.

Define the pull-back geometric quantities by

$$\begin{cases}
\tilde{g}_t(t) = \Lambda_t^* \hat{g}_t(t); \\
\tilde{\eta}_t(t) = \Lambda_t^* \hat{\eta}_t(t); \\
\tilde{x}_i = \Lambda_t^{-1}(\hat{x}_i); \\
\tilde{x}_i' = \Lambda_t^{-1}(\hat{x}_i').
\end{cases}$$

(3.34)

By (3.21) and (3.22), we see that

$$\begin{cases}
d\tilde{g}_t^{(0)}(\tilde{x}_i, \tilde{x}_i') \leq C; \\
\frac{|\tilde{\eta}_t(\tilde{x}_i, 0) - \tilde{P}_{\tilde{x}_i'}(\tilde{\eta}_t(\tilde{x}_i', 0))|_{\tilde{g}_t(\tilde{x}_i, 0)}}{d\tilde{g}_t^{(0)}(\tilde{x}_i, \tilde{x}_i')^\alpha} = 1.
\end{cases}$$

(3.35)

Since $d\tilde{g}_t^{(0)}(\tilde{x}_i, \tilde{x}_i')$ is uniformly bounded, then there is a compact set $\Omega$ containing $\tilde{x}_i$ and $\tilde{x}_i'$. Note that $\Omega$ is independent of $t$. Using the convergence of reference complex structures (3.27) and metrics (3.28), the $C^1_{\text{loc}}$ estimate of $\tilde{\omega}_t^\bullet(t)$ from Proposition 2.1 implies

$$|\tilde{\eta}_t(\tilde{x}_i, 0) - \tilde{P}_{\tilde{x}_i'}(\tilde{\eta}_t(\tilde{x}_i', 0))|_{\tilde{g}_t(\tilde{x}_i, 0)} \leq C d\tilde{g}_t^{(0)}(\tilde{x}_i, \tilde{x}_i').$$

(3.36)

Combining this with (3.35),

$$C^{-1} \leq d\tilde{g}_t^{(0)}(\tilde{x}_i, \tilde{x}_i') \leq C.$$

(3.37)

However, $\tilde{\omega}_t^\bullet(t)$ and $\Lambda_t^* \hat{\omega}_t^\bullet(t)$ both converge to a constant real (1,1)-forms in $C^\infty_{\text{loc}}(\C^{m+n} \times (-\infty, 0])$. Using [10, Remark 3.7] and $\tilde{\eta}_t(t) = \tilde{\omega}_t^\bullet(t) - \Lambda_t^* \hat{\omega}_t^\bullet(t)$, (3.35) shows $d\tilde{g}_t^{(0)}(\tilde{x}_i, \tilde{x}_i') \to 0$, which contradicts with (3.37).

**Case (b): $\delta_\infty \in (0, +\infty)$.** In this case, the blowup limit is $\C^m \times \Omega$. We may assume $\delta_\infty = 1$ without loss of generality. This case is similar to **Case**
(a) except that we don’t need to apply an additional diffeomorphism. Indeed, similar to (3.26), (3.38) shows

\[ \hat{g}_i(t), \hat{g}_i^\perp(t) \to g_{C^m} + g_{Y,z_{\infty}} \ \text{in} \ C^\infty_{\text{loc}}(C^m \times Y \times (-\infty, 0]). \]

Thanks to (3.19), (3.26) and (3.38), we may apply Proposition 2.1 to obtain

\[ C^\infty_{\text{loc}}(C^m \times Y \times (-\infty, 0]) \]\n
for any \( \lambda > 0 \) estimate of \( \hat{\omega}_i^\bullet(t) \). Hence, \( \hat{\omega}_i^\bullet(t) \) converges to \( \hat{\omega}_i^\bullet(t) \) in \( C^\infty_{\text{loc}}(C^m \times Y \times (-\infty, 0]) \). Then for each \( t \leq 0 \), \( \hat{\omega}_i^\bullet(t) \) is uniformly equivalent to the product form \( \hat{\omega}_i^\bullet(t) = \hat{\omega}_i^\bullet(0) \) is Ricci-flat. Thanks to the uniform equivalence of metrics (3.19) and Theorem 2.2, \( \hat{\omega}_i^\bullet(t) \) is parallel to \( \hat{\omega}_i^\bullet(t) = \hat{\omega}_i^\bullet(0) \) is Ricci-flat. But this will contradict with (3.21) by the same argument of Case (a).

**Case (c):** \( \delta_{\infty} = 0 \). In this case, the blowup limit is \( C^m \). For \( \hat{x} = (\hat{z}, \hat{y}) \in B_{\lambda_i} \times Y \) and \( \lambda = \lambda_i^{-1} \hat{z} \), we write

\[ (\hat{g}_i(\hat{z}) = \lambda_i^2 \hat{y}^* g_{\hat{z}}(t_i + \lambda_i^{-2} t) = \hat{g}_{C^m} + \lambda_i^2 e^{-t_i \lambda_i^{-2} t} g_{Y,z_{\infty}}. \]

By the definition of \( \mu \) and (3.21), for all \( \hat{x} = (\hat{z}, \hat{y}) \in B_{\lambda_i} \times Y \) and \( \hat{x}' \in B^{(\hat{g}_i)_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \), we have

\[ (d^{(\hat{g}_i)_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \alpha \left| \hat{\eta}(\hat{x}, 0) - P^{(\hat{g}_i)_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \right| \]

\[ \leq \mu(x_i, t_i) = (d^{\hat{g}_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \alpha, \]

which implies

\[ \left| \hat{\eta}(\hat{x}, 0) - P^{(\hat{g}_i)_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \right| \]

\[ \leq C \left( \frac{d^{(\hat{g}_i)_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \alpha}{d^{\hat{g}_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \alpha} \right). \]

Recalling that metrics \( \hat{g}_i(0) \) and \( \hat{g}_i(0) \) are uniformly equivalent,

\[ \left| \hat{\eta}(\hat{x}, 0) - P^{(\hat{g}_i)_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \right| \]

\[ \leq C \left( \frac{d^{\hat{g}_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \alpha}{d^{\hat{g}_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \alpha} \right). \]

For any \( R > 1 \), \( \hat{x} \in B^{\hat{g}_i(\hat{x}, R)} \) and sufficiently large \( i \), (3.23) and (3.42) show

\[ \left| \hat{\eta}(\hat{x}, 0) - P^{(\hat{g}_i)_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \right| \]

\[ \leq C \left( \frac{d^{\hat{g}_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \alpha}{d^{\hat{g}_i(\hat{x}, \partial(B_{\lambda_i} \times Y))) \alpha} \right) \leq C R. \]

Combining this with [10], Lemma 3.6], \( \hat{\omega}_i^\bullet(0) \) have uniformly bounded \( C^\alpha \) norm with respect to any fixed non-collapsing reference metric when \( i \) is sufficiently large. Thus, we may assume \( \hat{\omega}_i^\bullet(0) \to \hat{\omega}_i^\bullet(0) \) in \( C^\beta_{\text{loc}}(C^m \times Y) \) for any \( \beta < \alpha \), where \( \hat{\omega}_i^\bullet(0) \in C^\alpha_{\text{loc}}(C^m \times Y) \) which satisfies the following:
(i) $\hat{\omega}_\infty^*(0)$ is a section of $\text{pr}^{*}_C(\Lambda^{1,1}C^m)$ uniformly equivalent to $\omega_{C^m}$;
(ii) $\hat{\omega}_\infty^*(0)$ is $g_{Y,\omega_{\infty}}$-parallel in the fiber directions;
(iii) $\hat{\omega}_\infty^*(0)$ is weakly closed.

The conclusions (i) and (ii) follows from (3.19) and (3.43). The conclusion (iii) is clear due to the uniform convergence and the fact that $\hat{\omega}_i^*(0)$ is closed. Hence, $\hat{\omega}_\infty^*(0)$ is the pull-back under $\text{pr}^*_C$ of a weakly closed $(1,1)$-form of class $C^\alpha_{\text{loc}}$ on $C^m$.

To derive contradiction in Case (c), we will prove three claims. First, we rule out $d\hat{g}^0_i(\hat{x}_i, \hat{x}'_i) \to 0$.

**Claim 3.1.** There is $\varepsilon > 0$ such that for all $i \in \mathbb{N}$,

$$d\hat{g}^0_i(\hat{x}_i, \hat{x}'_i) \geq \varepsilon.$$  

**Proof of Claim 3.1.** Suppose on the contrary, by passing to subsequence, we may assume

$$d_i := d\hat{g}^0_i(\hat{x}_i, \hat{x}'_i) \to 0.$$  

Consider the diffeomorphism $\Lambda_i : B_{d_i^{-1} \lambda_i} \times Y \to B_{\lambda_i} \times Y$ given by

$$\Lambda_i(z, y) = (d_i z, y),$$

and define the pull-back geometric quantities by

$$\begin{aligned}
\hat{J}_i & = \Lambda_i^* \hat{J}_i; \\
\tilde{J}_i & = \Lambda_i^* \tilde{J}_i; \\
\hat{g}_i(t) & = d_i^{-2} \Lambda_i^* \hat{g}_i(d_i^2 t); \\
\tilde{g}_i(t) & = d_i^{-2} \Lambda_i^* \tilde{g}_i(d_i^2 t); \\
\hat{\omega}_i^*(t) & = \Lambda_i^* \hat{\omega}_i^*(d_i^2 t); \\
\tilde{\omega}_i^*(t) & = \Lambda_i^* \tilde{\omega}_i^*(d_i^2 t); \\
\hat{x}_i & = \Lambda_i^{-1}(\hat{x}_i); \\
\tilde{x}_i & = \Lambda_i^{-1}(\tilde{x}_i).
\end{aligned}$$

Then

$$d\hat{g}^0_i(\hat{x}_i, \tilde{x}_i) = 1$$

and

$$\hat{\omega}_i^5(t) = (1 - e^{-s})\omega_{C^m} + e^{-s}d_i^{-2} \lambda_i^2 \Lambda_i^* \Psi_i^* \omega_F,$$

where $s = t_i + d_i^2 \lambda_i^{-2} t$. Hence, we can rewrite the equation (3.3) as

$$(\hat{\omega}_i^5 + \tilde{\eta}_i)^{m+n} = e^{\tilde{G}_i + \tilde{H}_i}(\hat{\omega}_i^5)^{m+n},$$

where

$$\begin{aligned}
\tilde{G}_i(t) & = \Lambda_i^* \Psi_i^*(\hat{\phi} + \varphi + u)(s); \\
\tilde{H}_i(t) & = \log \frac{C^m_n \omega_{\Sigma}^m \wedge (e^{-s}d_i^{-2} \lambda_i^2 \Lambda_i^* \Psi_i^* \omega_F)^n}{((1 - e^{-s})\omega_{C^m} + e^{-s}d_i^{-2} \lambda_i^2 \Lambda_i^* \Psi_i^* \omega_F)^{m+n}}.
\end{aligned}$$

(3.48)

(3.49)

(3.50)

(3.51)
Using (3.19) and (3.21),
\begin{align}
|\tilde{\eta}^i(0)|_{\tilde{g}^i(0)} & \leq C; \\
|\tilde{\eta}^i(\tilde{x}, 0) - P_{\tilde{x}, \tilde{x}'}(\tilde{\eta}^i(\tilde{x}', 0))|_{\tilde{g}^i(0)} & = d^i.
\end{align}
(3.52)

For $\tilde{x} = (\tilde{z}, \tilde{y}) \in B_{d_i^{-1}\lambda_i} \times Y$ and $z = d_i\lambda_i^{-1}\tilde{z}$, we write
\begin{align}
(\tilde{g}_i)_{\tilde{z}}(t) & = d_i^{-2}\lambda_i^2 \Lambda_i^* \Psi_i^* g_z(t_i + d_i^2\lambda_i^{-2}t).
\end{align}
(3.53)

Using (3.43), after pulling back via the diffeomorphism $\Lambda_i$, we conclude that if $i$ sufficiently large,
\begin{align}
\sup_{\tilde{x}, \tilde{x}' \in B_{\delta_i}(\tilde{x}_i, d_i^{-1})} \frac{|\tilde{\eta}^i(\tilde{x}, 0) - P_{\tilde{x}, \tilde{x}'}(\tilde{\eta}^i(\tilde{x}', 0))|_{(\tilde{g}_i)_{\tilde{z}}(0)}}{d_{\tilde{g}^i(0)}(\tilde{x}, \tilde{x}')} & \leq Cd_i^\alpha.
\end{align}
(3.54)

Decompose $\tilde{\eta}^i(0) = \tilde{\eta}_i^2 + \tilde{\eta}_i'$ where $\tilde{\eta}_i^2$ is the unique $\tilde{g}_i(0)$-parallel $(1, 1)$-form pulled back from $\mathbb{C}^m$ such that $\tilde{\eta}_i^2(\tilde{x}_i) = \tilde{g}_i(\tilde{x}_i, 0)$ is the $\tilde{g}_i(\tilde{x}_i, 0)$ orthogonal projection of $\tilde{\eta}_i(\tilde{x}_i, 0)$ onto $pr^*_i(\Lambda_{1,1}^{1,n})|_{\tilde{x}_i}$. Applying the proof of [10, (5.36)] by freezing the time at 0 and using (3.54), we can find a constant $C > 0$ such that for sufficiently large $i$,
\begin{align}
|d_i^{-\alpha}\tilde{\eta}_i'(\tilde{x}_i)|_{\tilde{g}_i(\tilde{x}_i, 0)} & \leq C(d_i^{-1}\delta_i)^\alpha.
\end{align}
(3.55)

Write $\varepsilon_i = d_i^{-1}\delta_i$. By passing to a subsequence, we may assume $\varepsilon_i \to \varepsilon_\infty \in [0, +\infty]$. By considering the behaviour of $\tilde{\eta}_i^2(0) = g_{\mathbb{C}^m} + \varepsilon_i^2 g_{Y, \tilde{z}_i}$, there are three distinct cases to be considered: $\varepsilon_\infty = +\infty$; $\varepsilon_\infty \in (0, +\infty)$; $\varepsilon_\infty = 0$. The motivation of the above discussion is to pass $d_i^{-\alpha}\tilde{\eta}_i'$ to a limiting $\sqrt{-1}\partial \bar{\partial}$ exact $(1, 1)$-form on $\mathbb{C}^{m+n}, \mathbb{C}^m \times Y$ or $\mathbb{C}^m$ that is $O(r^\alpha)$ as $r \to +\infty$ and not parallel, where $r$ denotes the corresponding distance function in different settings.

Before analyzing these three cases, we collect some useful estimates which hold in all cases. Using (3.54), (3.55) and $d_i \to 0$, for each $R > 0$ and $i$ sufficiently large, we have
\begin{align}
\sup_{\tilde{x} \in B_{\delta_i}(\tilde{x}_i, R)} |\tilde{\eta}_i'(\tilde{x})|_{\tilde{g}_i(\tilde{x}, 0)} & \leq C\delta_i^\alpha + Cd_i^\alpha R^\alpha.
\end{align}
(3.56)

Following the argument of [10, (5.41)], we define $\tilde{\omega}_i^\# = \tilde{\omega}_i^*(0) - \tilde{\eta}_i'$. Since $\delta_i, d_i \to 0$, then $\tilde{\omega}_i^\#$ is Kähler on $B_{\delta_i}(\tilde{x}_i, R)$ for sufficiently large $i$. Moreover, its associated Riemannian metric $\tilde{g}_i^\#$ is uniformly equivalent to $\tilde{g}_i(0)$. At time 0, we expand the Monge–Ampère equation (3.50) as
\begin{align}
\text{tr}_{\tilde{\omega}_i^\#} \tilde{\eta}_i' + \sum_{i=2}^{m+n} C_{i}^{m+n} \frac{(\tilde{\eta}_i')(m+n-i)}{(\tilde{\omega}_i^\#)^{m+n}} & = e^{G_i(0) + \tilde{H}_i(0)} \frac{(\tilde{\omega}_i^*(0))^{m+n}}{(\tilde{\omega}_i^\#)^{m+n}} - 1.
\end{align}
(3.57)

Write $e^{K_i} - 1$ for the right hand side.
Subclaim: There is $C > 0$ such that the following holds. For all $R > 0$, there is $N$ such that for all $i > N$ and $\tilde{x}' \in \tilde{B}_{\tilde{g}}(\tilde{x}, R)$,

(i) for $j \geq 2$, $d_i^{-\alpha}|(\tilde{\eta}_i^j(\tilde{x}))^2 - P_{\tilde{x}', \tilde{x}}^j((\tilde{\eta}_i^j(\tilde{x}))^2)|_{\tilde{g}_i(\tilde{x}, 0)} \leq C(\delta_i^\alpha + d_i^\alpha R^\alpha)R^\alpha$;

(ii) $d_i^{-\alpha}|\tilde{\omega}_{\tilde{x}, \tilde{x}}^\#(\tilde{x}_{i}) - P_{\tilde{x}', \tilde{x}}^j(\tilde{\omega}_{\tilde{x}, \tilde{x}}^\#(\tilde{x}'))(\tilde{g}_i(\tilde{x}, 0) \leq Cd_i^{-\alpha}\lambda_i^{-1}R$;

(iii) $d_i^{-\alpha}|e^{\tilde{K}_i(\tilde{x}, 0)} - e^{\tilde{K}_i(\tilde{x}', 0)}| \leq Cd_i^{-\alpha}\lambda_i^{-1}R$.

Proof of Subclaim. The proofs of (i) and (ii) are identical to that of [10, (5.42), (5.43)] by freezing the time at 0 and using (3.7), (3.8), (3.54) and (3.56).

For (iii), the contributions from $\tilde{H}_i(0)$ and $(\tilde{\omega}_i^\alpha(0))^{m+n}/(\tilde{\omega}_i^\#)^{m+n}$ are done in the proof of [10, (5.44)]. It suffices to handle the term $\tilde{G}_i(0)$. We denote $v = \varphi(0) + \varphi(0) + u$. For $\tilde{x}'$ such that $d_{\tilde{g}_i(0)}(\tilde{x}, \tilde{x}') < R$ and sufficiently large $i$, we have

$$d_i^{-\alpha}|\tilde{G}_i(\tilde{x}_i, 0) - \tilde{G}_i(\tilde{x}', 0)| = |v(\tilde{z}_i, \tilde{y}_i) - v(\tilde{z}_i, \tilde{y}_i)|$$

for some $\alpha$ and $c > 0$. Indeed, since $\tilde{\omega}_i^\#(0) \in C_{loc}^\alpha(\mathbb{C}^m)$ is closed, then it has potential in $C_{loc}^{2\alpha}(\mathbb{C}^m)$. Combining this with (3.61) and standard elliptic bootstrapping argument, $\tilde{\omega}_i^\#(0)$ is smooth. Recalling that $\tilde{\omega}_i^\#(0)$ is uniformly

Claim 3.2. There are two points $\tilde{z}, \tilde{z}' \in \mathbb{C}^m$ such that $\tilde{\omega}_i^\#(\tilde{z}, 0) \neq \tilde{\omega}_i^\#(\tilde{z}', 0)$.

Proof of Claim 3.2. The proof is identical to [10, 5.3.2 Claim 2] by freezing time at 0 and using Claim 3.1 and 3.2.1.

Claim 3.3. The $C^\alpha$ Kähler current $\tilde{\omega}_i^\#(0)$ on $\mathbb{C}^m$ is parallel with respect to Euclidean metric.

Proof of Claim 3.3. The proof is standard and very similar to that in [3, 10]. It suffices to show that

$$\tilde{\omega}_i^\#(0) = \tilde{\omega}_i^\#(0) \text{ weakly on } \mathbb{C}^m$$

for some constant $c > 0$. Indeed, since $\tilde{\omega}_i^\#(0) \in C_{loc}^\alpha(\mathbb{C}^m)$ is closed, then it has potential in $C_{loc}^{2\alpha}(\mathbb{C}^m)$. Combining this with (3.61) and standard elliptic bootstrapping argument, $\tilde{\omega}_i^\#(0)$ is smooth. Recalling that $\tilde{\omega}_i^\#(0)$ is uniformly
equivalent to $\omega_{cm}$, Theorem 2.1 implies $\hat{\omega}_i^0(0)$ is constant, which complete the proof of Claim 3.3.

Next, we prove (3.61). Write $s = t_i + \lambda_i^{-2} t$ and $\psi_i(t) = \lambda_i^2 \Psi_i^* \varphi(s)$. Using (3.3) and (3.18), we have

\[
\begin{align*}
\hat{\omega}_i^0(t) &= (1 - e^{-s})\omega_{cm} + e^{-\lambda_i^{-2} t} \delta_i^* \omega_F + \sqrt{-1} \partial \bar{\partial} \psi_i(t); \\
\hat{\omega}_i^0(t)^{m+n} &= C_{n}^{m+n} e^{\Psi_i^*(\varphi(s))} \delta_i^{2n} (\Psi_i^* \omega_F)^{n} \wedge \omega_{cm}^n.
\end{align*}
\]

where $\partial$ and $\bar{\partial}$ are with respect to $\hat{J}_i^m$. Using $z_i \to z_\infty \in \overline{B}$ and (3.7), we have the following convergence:

\[
\hat{J}_i^m \to J_{cm} + J_{Y,z_\infty}, \quad \Psi_i^* \omega_F \to \omega_{Y,z_\infty} \text{ in } C^\infty_{loc}(C^m \times Y).
\]

Then (3.61) follows from (3.62), (3.63) and same argument of [3] which is based on [10]. □

It is clear that Claim 3.2 contradicts with Claim 3.3. This completes the proof of Case (c), and hence Theorem 3.1. □

**Proof of Theorem 1.1.** The $C^0$ estimate of $\omega(t)$ follows from [4, Theorem 1.1]. Recalling $g_z(t) = g_{cm} + e^{-t} g_{Y,z_1}$, it is immediate that $P g_z(t) = P g_z(0)$ and $d g_z(t) \leq d g_z(0)$ and $|T| g_z(t) \geq |T| g_z(0)$ for all $t \geq 0$ and contravariant tensor $T$. Therefore, Theorem 3.1 implies a uniform bound on the $g_z(0)$-Hölder quotient of the Kähler-Ricci flow $\omega(t)$ for any $x, x' \in X$. Thanks to [10, Lemma 3.6], we obtain the required Hölder estimate. □

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