THE HERMITE-KRICHEVER ANSATZ FOR FUCHSIAN EQUATIONS WITH APPLICATIONS TO THE SIXTH PAINLEVÉ EQUATION AND TO FINITE-GAP POTENTIALS

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Abstract. Several results including integral representation of solutions and Hermite-Krichever Ansatz on Heun’s equation are generalized to a certain class of Fuchsian differential equations, and they are applied to equations which are related with physics. We investigate linear differential equations that produce Painlevé equation by monodromy preserving deformation and obtain solutions of the sixth Painlevé equation which include Hitchin’s solution. The relationship with finite-gap potential is also discussed. We find new finite-gap potentials. Namely, we show that the potential which is written as the sum of the Treibich-Verdier potential and additional apparent singularities of exponents $-1$ and $2$ is finite-gap, which extends the result obtained previously by Treibich. We also investigate the eigenfunctions and their monodromy of the Schrödinger operator on our potential.

1. Introduction

It is well known that a Fuchsian differential equation with three singularities is transformed to a Gauss hypergeometric equation, and plays important roles in substantial fields in mathematics and physics. Several properties of solutions to the hypergeometric equation have been explained in various textbooks.

A canonical form of a Fuchsian equation with four singularities is written as

$$\left( \left( \frac{d}{dw} \right)^2 + \left( \frac{\gamma}{w} + \frac{\delta}{w-1} + \frac{\epsilon}{w-t} \right) \frac{d}{dw} + \frac{\alpha \beta w - q}{w(w-1)(w-t)} \right) \tilde{f}(w) = 0$$

with the condition

$$\gamma + \delta + \epsilon = \alpha + \beta + 1,$$

and is called Heun’s equation. Heun’s equation frequently appears in physics, i.e., general relativity \[21\], fluid mechanics \[3\] and so on. Despite that Heun’s equation was resolved in the 19th century; several results of solutions have only been recently revealed. Namely, integral representations of solutions, global monodromy in terms of hyperelliptic integrals, relationships with the theory of finite-gap potential and the Hermite-Krichever Ansatz for the case $\gamma, \delta, \epsilon, \alpha - \beta \in \mathbb{Z} + 1/2$ are contemporary (see \[1\] \[6\] \[19\] \[23\] \[24\] \[25\] \[26\] \[28\] etc.), though they are not written in a textbook on Heun’s equation \[17\].
In this paper, we consider differential equations which have additional apparent
singularities to Heun’s equation. More precisely, we consider the equation

\[
\left\{ \frac{d^2}{dw^2} + \left( \frac{1}{2} - l_1 \right) \frac{1}{w} + \frac{1}{2} - l_2 \frac{1}{w - 1} + \frac{1}{2} - l_3 \frac{1}{w - t} + \sum_{i'=1}^{M} \frac{-r_{i'}}{w - b_{i'}} \right\} \frac{d}{dw} \\
\frac{(\sum_{i=0}^{3} l_i + \sum_{i'=1}^{M} r_{i'})(-1 - l_0 + \sum_{i=1}^{3} l_i + \sum_{i'=1}^{M} r_{i'})w + \tilde{p} + \sum_{i'=1}^{M} \frac{s_{i'}}{w - b_{i'}}}{4w(w - 1)(w - t)} \right\} \tilde{f}(w) = 0,
\]

for the case \( l_i \in \mathbb{Z}_{\geq 0} (0 \leq i \leq 3) \), \( r_{i'} \in \mathbb{Z}_{>0} (1 \leq i' \leq M) \) and the regular singular
points \( b_{i'} (1 \leq i' \leq M) \) are apparent.

By a certain transformation, Eq. (1.3) is rewritten in terms of elliptic functions such as

\[
\left\{ -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i (l_i + 1) \wp(x + \omega_i) \\
+ \sum_{i'=1}^{M} \left( \frac{r_{i'}}{2} \left( \frac{r_{i'}}{2} + 1 \right) \left( \wp(x - \delta_{i'}) + \wp(x + \delta_{i'}) \right) + \frac{s_{i'}}{\wp(x) - \wp(\delta_{i'})} \right) - E \right\} f(x) = 0,
\]

with the condition that logarithmic solutions around the singularities \( x = \pm \delta_{i'} \) (\( i' = 1, \ldots, M \)) disappear. We then establish that solutions to Eq. (1.4) have an
integral representation and they are also written as a form of the Hermite-Krichever
Ansatz. For details see Proposition 2.3 and Theorem 2.5. Note that the results on the
Hermite-Krichever Ansatz are related to Picard’s theorem on differential equations
with coefficients of elliptic functions \([11, \S 15.6]\). By the Hermite-Krichever Ansatz,
we can obtain information on the monodromy of solutions to differential equations.

Results on the integral representation and the Hermite-Krichever Ansatz are applied for particular
cases. One example is Painlevé equation. For the case \( M = 1 \) and \( r_1 = 1 \), it is known that Eq. (1.3) produces the sixth Painlevé equation by monodromy
preserving deformation (see \([12]\)). On the other hand, solutions to Eq. (1.4) are expressed as a form of the Hermite-Krichever Ansatz for the case \( l_i \in \mathbb{Z}_{\geq 0} (i = 0, 1, 2, 3) \),
and we obtain an expression of monodromy. Fixing monodromy corresponds to the
monodromy preserving deformation; thus, we obtain solutions to the sixth Painlevé
equation by fixing monodromy (see section 3). For the case \( l_0 = l_1 = l_2 = l_3 = 0 \), we
recover Hitchin’s solution \([9]\). Note that the sixth Painlevé equation and the Hitchin’s
solution appear in topological field theory \([16]\) and Einstein metrics \([9]\).

Another example for application of the integral representation and the Hermite-
Krichever Ansatz is finite-gap potential. On solid-state physics, band structure of
spectral is essential, and examples and properties of finite-gap (finite-band) potential
could be applicable (e.g. see \([2]\)).

Recently several authors have been active in producing a variety of studies of finite-
gap potential, and several results have been applied to the analysis of Schrödinger-
type operators and so on. Here we briefly review these results. Let \( q(x) \) be a periodic,
smooth, real function, \( H \) be the operator \(-d^2/dx^2 + q(x)\), and \( \sigma_b(H) \) be the set such
that

\[ E \in \sigma_b(H) \iff \text{Every solution to } (H - E)f(x) = 0 \text{ is bounded on } x \in \mathbb{R}. \]

If the closure of the set \( \sigma_b(H) \) can be written as

\[ \overline{\sigma_b(H)} = [E_0, E_1] \cup [E_2, E_3] \cup \cdots \cup [E_{2g}, \infty), \]

where \( E_0 < E_1 < \cdots < E_{2g} \), then \( q(x) \) is called the finite-gap potential.

Let \( \wp(x) \) be the Weierstrass \( \wp \)-function with periods \((2\omega_1, 2\omega_3)\). Ince [10] established in 1940 that if \( n \in \mathbb{Z}_{\geq 1}, \omega_1 \in \mathbb{R} \) and \( \omega_3 \in \sqrt{-1}\mathbb{R} \), then the potential of the Lamé’s operator,

\[ -\frac{d^2}{dx^2} + n(n+1)\wp(x), \]

is finite-gap. From the 1960s, relationships among finite-gap potentials, odd-order commuting operators and soliton equations were investigated. If there exists an odd-order differential operator \( A = (d/dx)^{2g+1} + \sum_{j=0}^{2g-1} b_j(x) (d/dx)^{2g-1-j} \) such that \([A, -d^2/dx^2 + q(x)] = 0\), then \( q(x) \) is called the algebro-geometric finite-gap potential. Under the condition that \( q(x) \) is real-valued, smooth and periodic, it is known that \( q(x) \) is a finite-gap potential if and only if \( q(x) \) is an algebro-geometric finite-gap potential. For a detailed historical review, see [7] and the references therein.

In the late 1980s, Treibich and Verdier invented the theory of elliptic solitons, which is based on an algebro-geometric approach to soliton equations developed by Krichever [14] among others, and found a new algebro-geometric finite-gap potential, which is now called the Treibich-Verdier potential (see [28]). This potential may be written in the form

\[ v(x) = \sum_{i=0}^{3} l_i(l_i + 1)\wp(x + \omega_i) \]

for the Schrödinger operator \(-d^2/dx^2 + v(x)\), where \( l_i \ (i = 0, 1, 2, 3) \) are integers and \( \omega_1, \omega_3, \omega_0 = 0, \omega_2 = \omega_1 - \omega_3 \) are half-periods. Subsequently several studies [6, 30, 19, 23, 24, 25, 26] have further added to understanding of this subject. Note that the function in Eq. (1.7) corresponds to the potential of the Schrödinger operator as Eq. (1.4) for the case \( M = 0 \), and it is closely related to Heun’s equation.

Later, by following his joint work with Verdier, Treibich [27] established that, if \( l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0} \) and \( \delta \) satisfy

\[ \sum_{i=0}^{3} (l_i + 1/2)^2\wp'((\delta + \omega_i) = 0, \]

then the potential

\[ v(x) = 2(\wp(x - \delta) + \wp(x + \delta)) + \sum_{i=0}^{3} l_i(l_i + 1)\wp(x + \omega_i) \]

for the Schrödinger operator \(-d^2/dx^2 + v(x)\) is algebro-geometric finite-gap. In [20] Smirnov presented further results.

In this paper, we generalize the results of Treibich and Smirnov. In particular, we will find that, if \( l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}, \delta_j \neq \omega_i \mod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z} \ (0 \leq i \leq 3, 1 \leq j \leq M) \)
and $\delta_j \pm \delta_{j'} \not\equiv 0 \mod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z}$ ($1 \leq j < j' \leq M$), and $\delta_1, \ldots, \delta_M$ satisfy the equation

\[(1.10) \quad 2 \sum_{j' \neq j} (\wp'((\delta_j - \delta_{j'})) + \wp'((\delta_j + \delta_{j'})) + \sum_{i=0}^{3} (l_i + 1/2)^2 \wp'((\delta_j + \omega_i)) = 0 \quad (j = 1, \ldots, M),\]

then the potential

\[(1.11) \quad v(x) = \sum_{i=0}^{3} l_i(l_i + 1)\wp(x + \omega_i) + 2 \sum_{i'=1}^{M} (\wp(x - \delta_{i'}) + \wp(x + \delta_{i'}))\]

for the Schrödinger operator $-d^2/dx^2 + v(x)$ is algebro-geometric finite-gap. Note that the potential in Eq. (1.11) corresponds to Eq. (1.4) with conditions $r_{i'} = 2$ and $s_{i'} = 0$ ($i' = 1, \ldots, M$). For the special case $M = 1$, we recover the Treibich’s result [27].

Our approach differs from that of Treibich and Verdier and is elementary; we do not use knowledge of sophisticated algebraic geometry. The approach is based on writing the product of two specific eigenfunctions of the Schrödinger operator in the form of a doubly-periodic function for all eigenvalues $E$, which follows from the appearance of regular singularities of the Schrödinger operator (see section 2). Using the doubly-periodic function, an odd-order commuting operator is constructed, and it follows that the potential is algebro-geometric finite-gap. As a consequence, we obtain results concerning integral representations of solutions, monodromy formulae in terms of a hyperelliptic integral, the Bethe-Ansatz and the Hermite-Krichever Ansatz, as is shown in [25, 26] for Heun’s equation. We can also obtain two expression of monodromy. By comparing the two expressions, we obtain hyperelliptic-to-elliptic integral reduction formulae. Note that our approach can be related to the theory of Picard’s potential, which is developed by Gesztesy and Weikard [7].

This paper is organized as follows. In section 2 we obtain integral representations of solutions to the differential equation of the class mentioned above and rewrite them to the form of the Hermite-Krichever Ansatz. To obtain an integral representation, we introduce doubly-periodic functions that satisfy a differential equation of order three. Some properties related with this doubly-periodic function are investigated, and we obtain another expression of solutions that looks like the form of the Bethe Ansatz. In section 3 we consider the relationship with the sixth Painlevé equation. We show that solutions of the sixth Painlevé equation are obtained from solutions expressed in the form of the Hermite-Krichever Ansatz of linear differential equations considered in section 2 by fixing monodromy. Some explicit solutions that include Hitchin’s solution are displayed. In section 4 we discuss the relationship with the results on finite-gap potential. In subsection 4.1 we show that the potential $v(x)$ in Eq. (1.11) is algebro-geometric finite-gap under the conditions of Eq. (1.10). In subsection 4.2 we express global monodromy of eigenfunctions of the Schrödinger operator in terms of a hyperelliptic integral. In subsection 4.3 we investigate the eigenfunctions and monodromy by the Bethe Ansatz and the Hermite-Krichever Ansatz. As a consequence, we are able to derive another monodromy formula. In subsection 4.4 we obtain hyperelliptic-to-elliptic integral reduction formulae by comparing two expressions of monodromy. In section 5 we consider several examples on finite-gap potential. In section 6 we give concluding remarks and present an open problem. In the appendix, we note definitions and formulae for elliptic functions.
2. Fuchsian differential equation and Hermite-Krichever Ansatz

2.1. Fuchsian differential equation. To begin with, we introduce the following differential equation;

\[ \begin{align*}
&\left\{ \frac{d^2}{dz^2} + \left( \sum_{i=1}^{3} \frac{1}{2} \left( l_i - e_i \right) + \sum_{i'=1}^{M} \frac{-r_{i'}}{z-b_{i'}} \right) \frac{d}{dz} + \frac{N(N-2l_0-1)z + p + \sum_{i'=1}^{M} \frac{o_{i'}}{z-b_{i'}}}{4(z-e_1)(z-e_2)(z-e_3)} \right\} \tilde{f}(z) = 0,
\end{align*} \]

where \( N = \sum_{i=0}^{3} l_i + \sum_{i'=1}^{M} r_{i'} \). This equation is Fuchsian, i.e., all singularities \( \{e_i\}_{i=1,2,3}, \{b_{i'}\}_{i'=1,...,M} \) and \( \infty \) are regular. The exponents at \( z = e_i \) (\( i = 1, 2, 3 \)) (resp. \( z = b_{i'} \) (\( i' = 1, \ldots, M \))) are 0 and \( l_i + 1/2 \) (resp. 0 and \( r_{i'} + 1 \)), and the exponents at \( z = \infty \) are \( N/2 \) and \( (N-2l_0-1)/2 \). Conversely, any Fuchsian differential equation that has regular singularities at \( \{e_i\}_{i=1,2,3}, \{b_{i'}\}_{i'=1,...,M} \) and \( \infty \) such that \( e_i \) and \( b_{i'} \) for all \( i \in \{1, 2, 3\} \) and \( i' \in \{1, \ldots, M\} \) are zero is written as Eq. (2.1). By the transformation \( z \rightarrow z + \alpha \), we can change to the case \( e_1 + e_2 + e_3 = 0 \). In this paper we restrict discussion to the case \( e_1 + e_2 + e_3 = 0 \).

We remark that any Fuchsian equation with \( M + 4 \) singularities is transformed to Eq. (2.1) with the condition \( e_1 + e_2 + e_3 = 0 \).

It is known that, if \( e_1 + e_2 + e_3 = 0 \) and \( e_1 \neq e_2 \neq e_3 \neq e_1 \), then there exists some periods \( (2\omega_1, 2\omega_3) \) such that \( \wp(\omega_1) = e_1 \) and \( \wp(\omega_3) = e_3 \), where \( \wp(x) \) is the Weierstrass \( \wp \)-function with periods \( (2\omega_1, 2\omega_3) \). We set \( \omega_0 = 0 \) and \( \omega_2 = -\omega_1 - \omega_3 \). Then we have \( \wp(\omega_2) = e_2 \).

Now we rewrite Eq. (2.1) in an elliptic form. We set

\[ \Phi(z) = \prod_{i=1}^{3} (z - e_i)^{-l_i/2} \prod_{i'=1}^{M} (z - b_{i'})^{-r_{i'}/2}, \quad z = \wp(x), \]

and \( \tilde{f}(z)\Phi(z) = f(x) \). Then we have

\[ (H - E)f(x) = 0, \]

where \( H \) is a differential operator defined by

\[ H = -\frac{d^2}{dx^2} + v(x), \]

\[ v(x) = \sum_{i=0}^{3} l_i(l_i + 1)\wp(x + \omega_i) \]

\[ + \sum_{i'=1}^{M} \frac{r_{i'}}{2} \left( \frac{r_{i'}}{2} + 1 \right) \left( \wp(x - \delta_{i'}) + \wp(x + \delta_{i'}) \right) + \frac{s_{i'}}{\wp(x) - \wp(\delta_{i'})}. \]
\[ \varphi(\delta_{i'}) = b_{i'}, \quad (i' = 1, \ldots, M), \]

(2.6) \[ a_{i'} = -s_{i'} + r_{i'} \left( \frac{1}{8} r_{i'} (12b_{i'}^2 - g_2) + \frac{1}{2} (4b_{i'}^3 - g_2b_{i'} - g_3) \left( \sum_{i'' \neq i'} \frac{r_{i''}}{(b_{i'} - b_{i''})} \right) \right. \]
\[ + 2(l_1(b_{i'} - e_2)(b_{i'} - e_3) + l_2(b_{i'} - e_1)(b_{i'} - e_3) + l_3(b_{i'} - e_1)(b_{i'} - e_2)) \right) \],

(2.7) \[ p = E + (e_1l_1^2 + e_2l_2^2 + e_3l_3^2) - 2(l_1l_2c_3 + l_2l_3c_1 + l_3l_1c_2) - \frac{1}{2} \sum_{i' = 1}^{M} b_{i'} r_{i'}^2 \]
\[ + 2 \sum_{i' = 1}^{M} \sum_{i = 1}^{3} l_i r_{i'} (e_i + b_{i'}) + 2 \left( \sum_{i' = 1}^{M} b_{i'} r_{i'} \right) \left( \sum_{i' = 1}^{M} r_{i'} \right), \]

(2.8) \[ g_2 = -4(e_1 e_2 + e_2 e_3 + e_3 e_1), \quad g_3 = 4e_1 e_2 e_3. \]

Conversely, Eq. (2.11) is obtained from Eq. (2.3) by the transformation above.

We consider another expression. Set

(2.10) \[ H_g = -\frac{d^2}{dx^2} + \sum_{i' = 1}^{M} \frac{r_{i'} \varphi'(x)}{\varphi(x) - \varphi(\delta_{i'})} \frac{d}{dx} + \left( l_0 + \sum_{i' = 1}^{M} r_{i'} \right) \left( l_0 + 1 - \sum_{i' = 1}^{M} r_{i'} \right) \varphi(x) \]
\[ + \sum_{i = 1}^{3} l_i (l_i + 1) \varphi(x + \omega_i) + \sum_{i' = 1}^{M} \frac{\tilde{s}_{i'}}{\varphi(x) - \varphi(\delta_{i'})}, \]

(2.11) \[ f_g(x) = f(x) \Psi_g(x), \quad \Psi_g(x) = \prod_{i' = 1}^{M} (\varphi(x) - \varphi(\delta_{i'}))^{r_{i'}/2}. \]

Then Eq. (2.3) is also equivalent to

(2.12) \[ (H_g - E - C_g) f_g(x) = 0, \]
where

(2.13) \[ \tilde{s}_{i'} = s_{i'} - r_{i'} \left( \frac{1}{8} r_{i'} (12b_{i'}^2 - g_2) + \frac{1}{2} (4b_{i'}^3 - g_2b_{i'} - g_3) \left( \sum_{i'' \neq i'} \frac{r_{i''}}{(b_{i'} - b_{i''})} \right) \right. \]
\[ \left. + 2(l_1(b_{i'} - e_2)(b_{i'} - e_3) + l_2(b_{i'} - e_1)(b_{i'} - e_3) + l_3(b_{i'} - e_1)(b_{i'} - e_2)) \right) \),

(2.14) \[ C_g = -\frac{1}{2} \sum_{i' = 1}^{M} b_{i'} r_{i'}^2 + 2 \left( \sum_{i' = 1}^{M} b_{i'} r_{i'} \right) \left( \sum_{i' = 1}^{M} r_{i'} \right). \]

Note that the exponents at \( x = \pm \delta_{i'} (i' = 1, \ldots, M) \) are 0 and \( r_{i'} + 1 \).

In this paper, we consider solutions to Eq. (2.11), which is equivalent to Eq. (2.3) or Eq. (2.12) for the case \( l_i \in \mathbb{Z} \), and the regular singular point \( z = b_{i'} \) is apparent for all \( i' \). Here, a regular singular point \( x = a \) of a linear differential equation of order two is said to be apparent, if and only if the differential equation does not have a logarithmic solution at \( x = a \). It is known that the regular singular point \( x = a \) is apparent, if and only if the monodromy matrix around \( x = a \) is a unit matrix. Note that Smirnov investigated solutions to Eq. (2.3) in \([20]\) with the assumptions that \( s_{i'} = 0, r_{i'} \in 2\mathbb{Z} \) for all \( i' \).

Now we study the condition that the regular singular point \( x = a \) is apparent. More precisely, we describe the condition that a differential equation of order two does not have logarithmic solutions at a regular singular point \( x = a \) (\( a \neq \infty \)) for
the case $\alpha_2 - \alpha_1 \in \mathbb{Z}$, where $\alpha_1$ and $\alpha_2$ are exponents at $x = a$. If $\alpha_1 = \alpha_2$, then the differential equation has logarithmic solutions at $x = a$. We assume that the exponents satisfy $\alpha_2 - \alpha_1 = n \in \mathbb{Z}_{\geq 1}$. Since the point $x = a$ is a regular singular, the differential equation is written as

$$\left\{ \frac{d^2}{dx^2} + \sum_{j=0}^{\infty} p_j(x-a)^{j-1} \frac{d}{dx} + \sum_{j=0}^{\infty} q_j(x-a)^{j-2} \right\} f(x) = 0,$$

for some $p_j, q_j \in \mathbb{C} (j = 0, 1, \ldots)$. Let $F(t)$ be the characteristic polynomial at the regular singular point $x = a$. Since exponents at $x = a$ are $\alpha_1$ and $\alpha_2$, $F(t)$ is written as $F(t) = t^2 + (p_0 - 1)t + q_0 = (t - \alpha_1)(t - \alpha_2)$. We now calculate solutions to Eq.\((2.15)\) in the form

$$f(x) = \sum_{j=0}^{\infty} c_j(x-a)^{\alpha_1+j},$$

where $f(x)$ is normalized to satisfy $c_0 = 1$. By substituting it into Eq.\((2.15)\) and comparing the coefficients of $(x-a)^{\alpha_1+j-2}$, we obtain the relations

$$F(\alpha_1 + j)c_j + \sum_{j'=0}^{j-1} \{(\alpha_1 + j')p_j-j' + q_{j-j'}\} c_{j'} = 0.$$

If the positive integer $j$ satisfies $F(\alpha_1 + j) \neq 0$ (i.e. $j \neq 0, n)\), then the coefficient $c_j$ is determined recursively. For the case $j = n$, we have $F(\alpha_1 + n) = 0$ and

$$\sum_{j'=0}^{n-1} \{(\alpha_1 + j')p_n-j' + q_{n-j'}\} c_{j'} = 0.$$

Eq.\((2.18)\) with recursive relations \((2.17)\) for $j = 1, \ldots, n - 1$ is a necessary and sufficient condition that Eq.\((2.15)\) does not have a logarithmic solution for the case $\alpha_2 - \alpha_1 = n \in \mathbb{Z}_{\geq 1}$. In fact, if $p_0, q_0, \ldots, p_n, q_n$ satisfy Eq.\((2.18)\), then there exist solutions to Eq.\((2.15)\) that include two parameters $c_0$ and $c_n$. Thus any solutions are not logarithmic at $x = a$. Conversely, if Eq.\((2.18)\) is not satisfied, there exists a logarithmic solution written as $f(x) = \sum_{j=0}^{\infty} c_j(x-a)^{\alpha_1+j} + \log(x-a) \sum_{j=n}^{\infty} \tilde{c}_j(x-a)^{\alpha_1+j}$.

It follows from $\wp(\delta_{\nu}) = b_{\nu}, \wp'(\delta_{\nu}) \neq 0$ and holomorphy of $\prod_{i=1}^{3}(\wp(x) - \epsilon_i)^{-l_i/2}$ at $x = \pm \delta_{\nu}$, that, the monodromy matrix to Eq.\((2.1)\) around a regular singular point $z = b_{\nu}$ is a unit matrix, if and only if the monodromy matrix to Eq.\((2.12)\) around a regular singular point $x = \pm \delta_{\nu}$ is a unit matrix. It is obvious that, if the monodromy matrix to Eq.\((2.1)\) around a regular singular point $z = b_{\nu}$ is a unit matrix, then we have $r_{\nu} \in \mathbb{Z}_{\neq 0}$ for all $\nu$.

2.2. Integral representation and the Hermite-Krichever Ansatz. We introduce doubly-periodic functions to obtain an integral expression of solutions to Eq.\((2.13)\) (or Eq.\((2.12)\)) for the case $l_i \in \mathbb{Z}_{\geq 0}$ ($i = 0, 1, 2, 3$), $r_{\nu} \in \mathbb{Z}_{\geq 0}$ ($\nu = 1, \ldots, M$) and the regular singular points $z = b_{\nu}$ ($\nu = 1, \ldots, M$) of Eq.\((2.1)\) are apparent.

**Proposition 2.1.** Let $v(x)$ be the function defined in Eq.\((2.2)\). If $l_i \in \mathbb{Z}_{\geq 0}$ ($i = 0, 1, 2, 3$), $r_{\nu} \in \mathbb{Z}_{\geq 0}$ ($\nu = 1, \ldots, M$) and regular singular points $z = b_{\nu}$ ($\nu = 1, \ldots, M$)
of Eq. (2.1) are apparent, then the equation

\begin{equation}
\left\{ \frac{d^3}{dx^3} - 4(v(x) - E) \frac{d}{dx} - 2 \frac{dv(x)}{dx} \right\} \Xi(x) = 0,
\end{equation}

has an even nonzero doubly-periodic solution that has the expansion

\begin{equation}
\Xi(x) = c_0 + \sum_{i=0}^{3} \sum_{j=0}^{l_i-1} b_j^{(i)} \varphi(x + \omega_i)^{l_i-j} + \sum_{i'=1}^{3} \sum_{j'=0}^{M} \sum_{r'=1}^{r_{i'}-1} \frac{\partial^{(r')}}{\partial \varphi(x)} (\varphi(x) - \varphi(\delta_{i'}))^{r_{i'}-j}.
\end{equation}

**Proof.** First, we show a lemma that is related to the monodromy of solutions to Eq. (2.12).

**Lemma 2.2.** If $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{>0}$, then the monodromy matrix of Eq. (2.12) around a point $x = n_1 \omega_1 + n_3 \omega_3$ $(n_1, n_3 \in \mathbb{Z})$ is a unit matrix.

**Proof.** Due to periodicity, it is sufficient to consider the case $x = \omega_i$ $(i = 0, 1, 2, 3)$. We first deal with the case $x = \omega_i$ $(i = 1, 2, 3)$. The exponents at the singular point $x = \omega_i$ $(i = 1, 2, 3)$ are $-l_i$ and $l_i+1$. Because Eq. (2.12) is invariant under the transformation $x - \omega_i \rightarrow -(x - \omega_i)$ and the gap of the exponents at $x = \omega_i$ (i.e. $l_i + 1 - (-l_i)$) is odd, there exist solutions in the form $f_{i,1}(x) = (x - \omega_i)^{-l_i} (1 + \sum_{j=1}^{\infty} a_j (x - \omega_i)^{2j})$ and $f_{i,2}(x) = (x - \omega_i)^{l_i+1} (1 + \sum_{j=1}^{\infty} a'_j (x - \omega_i)^{2j})$. Since the functions $f_{i,1}(x)$ and $f_{i,2}(x)$ form a basis for solutions to Eq. (2.12) and they are non-branching around the point $x = \omega_i$, the monodromy matrix around $x = \omega_i$ is a unit matrix. For the case $i = 0$, the exponents at $x = 0$ are $-l_0 - \sum_{i'=1}^{3} r_{i'}$ and $l_0 + 1 - \sum_{i'=1}^{3} r_{i'}$, the gap of the exponents is odd, and similarly it is shown that the monodromy matrix around the point $x = 0$ is a unit matrix. Hence we obtain the lemma.

We continue the proof of Proposition 2.1. Let $M_j$ $(j = 1, 3)$ be the transformations obtained by the analytic continuation $x \rightarrow x + 2\omega_j$. It follows from double-periodicity of Eq. (2.12) that, if $f_j(x)$ is a solutions to Eq. (2.12), then $M_j f_j(x)$ is also a solution to Eq. (2.12). From the assumption that regular singular points $z = b_{i'}$ are apparent for all $i'$, the monodromy matrix to Eq. (2.12) around a regular singular point $x = \pm \delta_{i'}$ is a unit matrix for all $i'$. By combining with Lemma 2.2 it follows that all local monodromy matrices around any singular points are units. Hence the transformations $M_j$ do not depend on the choice of paths. From the fact that the fundamental group of the torus is commutative, we have $M_1 M_3 = M_3 M_1$. Recall that the operators $M_j$ act on the space of solutions to Eq. (2.12) for each $E$, which is two dimensional. By the commutativity $M_1 M_3 = M_3 M_1$, there exists a joint eigenvector $\hat{A}_g(x)$ for the operators $M_1$ and $M_3$. It follows from Proposition 2.2 and the appearance of singular points that the function $\hat{A}_g(x)$ is single-valued and satisfies equations $(H_g - E - C_g) \hat{A}_g(x) = 0$, $M_1 \hat{A}_g(x) = \hat{m}_1 \hat{A}_g(x)$ and $M_3 \hat{A}_g(x) = \hat{m}_3 \hat{A}_g(x)$ for some $\hat{m}_1, \hat{m}_3 \in \mathbb{C} \setminus \{0\}$. By changing parity $x \leftrightarrow -x$, it follows immediately that $(H_g - E - C_g) \hat{A}_g(-x) = 0$, $M_1 \hat{A}_g(-x) = \hat{m}_1^{-1} \hat{A}_g(-x)$ and $M_3 \hat{A}_g(-x) = \hat{m}_3^{-1} \hat{A}_g(-x)$. Then the function $\hat{A}_g(x) \hat{A}_g(-x)$ is single-valued, even and doubly-periodic. We set $\tilde{A}(x) = \hat{A}_g(x) / \Psi_g(x)$. Then $\tilde{A}(x)$ and $\tilde{A}(-x)$ are solutions to Eq. (2.23).

Now consider the function $\Xi(x) = \hat{A}_g(x) \hat{A}_g(-x) / \Psi_g(x)^2$. Since the function $\Psi_g(x)^2$ is single-valued, even and doubly-periodic, the function $\Xi(x)$ is single-valued, even (i.e. $\Xi(x) = \Xi(-x)$), doubly-periodic (i.e. $\Xi(x + 2\omega_1) = \Xi(x + 2\omega_3) = \Xi(x)$), and
satisfies the equation
\[
\left\{ \frac{d^3}{dx^3} - 4 (v(x) - E) \frac{d}{dx} - 2 \frac{dv(x)}{dx} \right\} \Xi(x) = 0
\]
that the products of any pair of solutions to Eq. (2.3) satisfy.

Since the function \( \Xi(x) \) is an even doubly-periodic function that satisfies the differential equation (2.19) and the exponents of Eq. (2.19) at \( x = \omega_i \) (\( i = 0, \ldots, 3 \)) (resp. \( x = \pm \delta_i' \) (\( i' = 1, \ldots, M \)) are \(-2l_i, 1, 2l_i + 2\) (resp. \( -r_i', 1, r_i' + 2\)), it is written as a rational function of variable \( \wp(x) \), and it admits the expansion as Eq. (2.20) by considering exponents. □

The function \( \Xi(x) \) is calculated by substituting Eq. (2.20) into the differential equation (2.19) and solving simultaneous equations for the coefficients. We introduce an integral formula for a solution to the differential equation Eq. (2.3) in use of the function \( \Xi(x) \). Set
\[
Q = \Xi(x)^2 (E-v(x)) + \frac{1}{2} \Xi(x) \frac{d^2 \Xi(x)}{dx^2} - \frac{1}{4} \left( \frac{d \Xi(x)}{dx} \right)^2.
\]
It follows from Eq. (2.19) that
\[
\frac{dQ}{dx} = \frac{1}{2} \Xi(x) \left( 4 \frac{d \Xi(x)}{dx} (E-v(x)) - 2 \Xi(x) \frac{dv(x)}{dx} + \frac{d^3 \Xi(x)}{dx^3} \right) = 0.
\]
Hence the value \( Q \) is independent of \( x \).

**Proposition 2.3.** Let \( \Xi(x) \) be the doubly-periodic function defined in Proposition 2.1 and \( Q \) be the value defined in Eq. (2.21). Then the function
\[
\Lambda(x) = \Xi(x) \exp \int \frac{\sqrt{-Q} dx}{\Xi(x)},
\]
is a solution to the differential equation (2.3), and the function
\[
\Lambda_g(x) = \Psi_g(x) \Xi(x) \exp \int \frac{\sqrt{-Q} dx}{\Xi(x)},
\]
is a solution to the differential equation (2.12).

**Proof.** From Eqs. (2.23, 2.21) we have
\[
\frac{\Lambda'(x)}{\Lambda(x)} = \frac{1}{2} \frac{\Xi'(x)}{\Xi(x)} + \frac{\sqrt{-Q}}{\Xi(x)},
\]
\[
\frac{\Lambda''(x)}{\Lambda(x)} = \frac{1}{2} \frac{\Xi''(x)}{\Xi(x)} - \frac{1}{4} \left( \frac{\Xi'(x)}{\Xi(x)} \right)^2 - \frac{Q}{\Xi(x)^2} = v(x) - E.
\]
Hence we have \(-\frac{d}{dx} \Lambda(x) + v(x) \Lambda(x) = E \Lambda(x)\). It follows from the equivalence of Eq. (2.3) and Eq. (2.12) that the function \( \Lambda_g(x) \) is a solution to Eq. (2.12). □

**Proposition 2.4.** If \( Q \neq 0 \), then the functions \( \Lambda(x) \) and \( \Lambda(-x) \) are linearly independent and any solution to Eq. (2.3) is written as a linear combination of \( \Lambda(x) \) and \( \Lambda(-x) \).
Proof. It follows from Eq. (2.25) and the evenness of the function $\Xi(x)$ that

\begin{equation}
\frac{d}{dx} \Lambda(-x) = \frac{1}{2} \frac{\Xi'(x)}{\Xi(x)} - \frac{\sqrt{-Q}}{\Xi(x)}.
\end{equation}

Hence we have

\begin{equation}
\Lambda(-x) \frac{d}{dx} \Lambda(x) - \Lambda(x) \frac{d}{dx} \Lambda(-x) = \Lambda(x) \Lambda(-x) 2\sqrt{-Q} \Xi(x).
\end{equation}

If $\Lambda(x)$ and $\Lambda(-x)$ are linearly dependent, then the l.h.s. of Eq. (2.28) must be zero; however, this is impossible because $Q \neq 0$. Hence the functions $\Lambda(x)$ and $\Lambda(-x)$ are linearly independent. It follows from the invariance of Eq. (2.3) with respect to the transformation $x \leftrightarrow -x$ that $\Lambda(-x)$ is also a solution to Eq. (2.3).

Since solutions to Eq. (2.3) form a two-dimensional vector space and the functions $\Lambda(x)$ and $\Lambda(-x)$ are linearly independent, the functions $\Lambda(x)$ and $\Lambda(-x)$ form a basis of the space of solutions to Eq. (2.3), and any solution to Eq. (2.3) is written as a linear combination of $\Lambda(x)$ and $\Lambda(-x)$.

It follows from Proposition 2.4 that, if $Q \neq 0$, then the functions $\Lambda_g(x)$ and $\Lambda_g(-x)$ are linearly independent, and any solution to Eq. (2.12) is written as a linear combination of $\Lambda_g(x)$ and $\Lambda_g(-x)$.

From the formulae (2.23, 2.24) and the doubly-periodicity of the functions $\Xi(x)$ and $\Psi_g(x)^2$, we have

\begin{equation}
\Lambda(x + 2\omega_j) = \pm \Lambda(x) \exp \int_{0+\epsilon}^{2\omega_j+\epsilon} \frac{\sqrt{-Q}dx}{\Xi(x)}, \quad (j = 1, 3),
\end{equation}

\begin{equation}
\Lambda_g(x + 2\omega_j) = \pm \Lambda_g(x) \exp \int_{0+\epsilon}^{2\omega_j+\epsilon} \frac{\sqrt{-Q}dx}{\Xi(x)}, \quad (j = 1, 3),
\end{equation}

with $\epsilon$ a constant determined so as to avoid passing through the poles while integrating. The sign $\pm$ is determined by the analytic continuation of the function $\sqrt{\Xi(x)}$, and the integrations in Eqs. (2.29, 2.30) may depend on the choice of the path. The function $\Lambda(x)$ may have branching points, although the function $\Lambda_g(x)$ does not have branching points and is meromorphic on the complex plane, because $\Lambda_g(x)$ is a solution to Eq. (2.12) and any singularity of Eq. (2.12) is apparent. It follows from Eq. (2.31) that there exists $m_1, m_3 \in \mathbb{C}$ such that

\begin{equation}
\Lambda_g(x + 2\omega_j) = \exp(\pi \sqrt{-1} m_j) \Lambda_g(x), \quad (j = 1, 3).
\end{equation}

We now show that a solution to Eq. (2.12) can be expressed in the form of the Hermite-Krichever Ansatz. We set

\begin{equation}
\Phi_i(x, \alpha) = \frac{\sigma(x + \omega_i - \alpha)}{\sigma(x + \omega_i)} \exp(\zeta(\alpha)x), \quad (i = 0, 1, 2, 3),
\end{equation}

where $\sigma(x)$ (resp. $\zeta(x)$) is the Weierstrass sigma (resp. zeta) function. Then we have

\begin{equation}
\left(\frac{d}{dx}\right)^k \Phi_i(x + 2\omega_j, \alpha) = \exp(-2\eta_j \alpha + 2\omega_j \zeta(\alpha)) \left(\frac{d}{dx}\right)^k \Phi_i(x, \alpha)
\end{equation}

for $i = 0, 1, 2, 3$, $j = 1, 2, 3$ and $k \in \mathbb{Z}_{\geq 0}$, where $\eta_j = \zeta(\omega_j)$ ($j = 1, 2, 3$).
Theorem 2.5. Set \( \bar{l}_0 = l_0 + \sum_{i'=1}^{M} r_{i'} \) and \( \bar{l}_i = l_i \) \((i = 1, 2, 3)\). The function \( \Lambda_g(x) \) in Eq. (2.24) is expressed as

\[
\Lambda_g(x) = \exp (\kappa x) \left( \sum_{i=0}^{3} \sum_{j=0}^{\bar{l}_i-1} \bar{b}_j^{(i)} \left( \frac{d}{dx} \right)^j \Phi_i(x, \alpha) \right)
\]

for some values \( \alpha, \kappa \) and \( \bar{b}_j^{(i)} \) \((i = 0, \ldots, 3, j = 0, \ldots, \bar{l}_i - 1)\), or

\[
\Lambda_g(x) = \exp (\bar{\kappa} x) \left( c + \sum_{i=0}^{3} \sum_{j=0}^{\tilde{l}_i-2} \tilde{b}_j^{(i)} \left( \frac{d}{dx} \right)^j \Phi(x + \omega_i) + \sum_{i=1}^{3} \eta_i \frac{\partial}{\partial x} \right)
\]

for some values \( \bar{\kappa}, c, \eta_i \) \((i = 1, 2, 3)\) and \( \tilde{b}_j^{(i)} \) \((i = 0, \ldots, 3, j = 0, \ldots, \tilde{l}_i - 2)\).

If the function \( \Lambda_g(x) \) is expressed as Eq. (2.34), then

\[
\Lambda_g(x + 2\omega_j) = \exp(-2\eta_j \alpha + 2\omega_j \zeta(\alpha) + 2\kappa \omega_j) \Lambda_g(x), \quad (j = 1, 3),
\]

else

\[
\Lambda_g(x + 2\omega_j) = \exp(2\bar{\kappa} \omega_j) \Lambda_g(x), \quad (j = 1, 3).
\]

Proof. Set

\[
\alpha = -m_1 \omega_3 + m_3 \omega_1,
\]

where \( m_1 \) and \( m_3 \) are determined in Eq. (2.31).

If \( \alpha \equiv 0 \pmod{2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z}} \), then we set

\[
\kappa = \zeta(m_1 \omega_3 - m_3 \omega_1) - m_1 \eta_3 + m_3 \eta_1.
\]

It follows from Legendre’s relation \( \eta_1 \omega_3 - \eta_3 \omega_1 = \pi \sqrt{-1}/2 \) and the relation \( \zeta(\alpha) = -\zeta(\alpha) \) that

\[
\exp(\kappa(x + 2\omega_j)) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha)
\]

\[
= \exp(-2\eta_j \alpha + 2\omega_j(\zeta(\alpha) + \kappa)) \exp(\kappa x) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha)
\]

\[
= \exp(2m_1(\eta_1 \omega_3 - \eta_3 \omega_j) + 2m_3(\eta_1 \omega_j - \eta_j \omega_1)) \exp(\kappa x) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha)
\]

\[
= \exp(\pi \sqrt{-1}m_j) \exp(\kappa x) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha)
\]

for \( i = 0, 1, 2, 3, j = 1, 3 \) and \( k \in \mathbb{Z}_{\geq 0} \). Hence the function \( \Lambda_g(x) \) and the functions \( \exp(\kappa x) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha) \) have the same periodicity with respect to periods \((2\omega_1, 2\omega_3)\). Since the meromorphic function \( \Lambda_g(x) \) satisfies Eq. (2.12), the regular singular point \( x = \pm \delta_{i'} \) \((i' = 1, \ldots, M)\) is apparent, and the exponents at \( x = \pm \delta_{i'} \) are 0 and \( r_{i'} + 1 \), it is holomorphic except for \( \mathbb{Z} \omega_1 \oplus \mathbb{Z} \omega_3 \) and has a pole of degree \( \bar{l}_i \) or zero of degree \( \bar{l}_i + 1 \) at \( x = \omega_i \) \((i = 0, 1, 2, 3)\). The function \( \exp(\kappa x) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha) \) has a pole of degree \( k + 1 \) at \( x = \omega_i \). By subtracting the functions \( \exp(\kappa x) \left( \frac{d}{dx} \right)^k \Phi_i(x, \alpha) \) from the function \( \Lambda_g(x) \) to erase the poles, we obtain a holomorphic function that has the same periods as \( \Phi_0(x, \alpha) \), and must be zero, because if we denote the holomorphic function by \( f(x) \), then \( f(x)/(\exp(\kappa x)\Phi_0(x)) \) is doubly-periodic and have only one
pole of degree one in a fundamental domain, and \( f(x) \) must be zero. Hence we obtain the expression (2.31). The periodicity (see Eq. (2.36)) follows from Eq. (2.40).

If \( \alpha \equiv 0 \pmod{2\omega_1 \mathbb{Z} + 2\omega_3 \mathbb{Z}} \) (i.e. \( m_1\omega_3 \equiv m_3\omega_1 \pmod{2\omega_1 \mathbb{Z} + 2\omega_3 \mathbb{Z}} \)), then we set

\[
\bar{\kappa} = -m_1\eta_3 + m_3\eta_1.
\]

The function \( \Lambda_g(x) \) and the function \( \exp(\bar{\kappa}x) \) have the same periodicity with respect to periods \((2\omega_1, 2\omega_3)\). Hence the function \( \Lambda_g(x) \exp(-\bar{\kappa}x) \) is doubly periodic, and we obtain the expression (2.35) by considering the poles. Periodicity (see Eq. (2.37)) follows immediately.

We investigate the situation that Eq. (2.12) has a non-zero solution of an elliptic function. Let \( \mathcal{F}_{\epsilon_1,\epsilon_3} \) and \( \tilde{\mathcal{F}}_{\epsilon_1,\epsilon_3} \) \((\epsilon_1, \epsilon_3 \in \{\pm 1\})\) be the spaces defined by

\[
\begin{align*}
\mathcal{F}_{\epsilon_1,\epsilon_3} &= \{ f(x); \text{meromorphic} \mid f(x + 2\omega_1) = \epsilon_1 f(x), \ f(x + 2\omega_3) = \epsilon_3 f(x) \}, \\
\tilde{\mathcal{F}}_{\epsilon_1,\epsilon_3} &= \left\{ f(x) \middle| \begin{array}{c}
\Psi_g(x) \in \mathcal{F}_{\epsilon_1,\epsilon_3},
\ f(x)\Psi_g(x) \text{ is holomorphic}
\end{array} \right. \\
&\quad \text{except for } \mathbb{Z}\omega_1 + \mathbb{Z}\omega_3, \text{ and the degree of the pole at } x = \omega_i \text{ is no more than } \begin{cases}
\ l_i, & i = 1, 2, 3, \\
\ l_0 + \sum_{j=1}^M r_j, & i = 0.
\end{cases}
\end{align*}
\]

where \((2\omega_1, 2\omega_3)\) are basic periods of elliptic functions. Then \( \tilde{\mathcal{F}}_{\epsilon_1,\epsilon_3} \) is a finite-dimensional vector space. Note that, if a solution \( f(x) \) to Eq. (2.23) satisfies the condition \( f(x + 2\omega_1)\Psi_g(x + 2\omega_1) = \epsilon_1 f(x)\Psi_g(x) \) and \( f(x + 2\omega_3)\Psi_g(x + 2\omega_3) = \epsilon_3 f(x)\Psi_g(x) \) for some \( \epsilon_1, \epsilon_3 \in \{\pm 1\} \), then we have \( f(x) \in \mathcal{F}_{\epsilon_1,\epsilon_3} \), because the position of the poles and their degree are restricted by the differential equation.

Proposition 2.6. Assume that Eq. (2.23) has a non-zero solution in the space \( \mathcal{F}_{\epsilon_1,\epsilon_3} \) for some \( \epsilon_1, \epsilon_3 \in \{\pm 1\} \). Then the signs \((\epsilon_1, \epsilon_3)\) are determined uniquely for each \( E, \tilde{s}_\nu \) (\( \nu = 1, \ldots, M \)) etc.

Proof. Assume that Eq. (2.23) has a non-zero solution in both the spaces \( \tilde{\mathcal{F}}_{\epsilon_1,\epsilon_3} \) and \( \tilde{\mathcal{F}}_{\epsilon_1',\epsilon_3'} \). Let \( f_1(x) \) (resp. \( f_2(x) \)) be the solution to the differential equation (2.23) in the space \( \tilde{\mathcal{F}}_{\epsilon_1,\epsilon_3} \) (resp. the space \( \tilde{\mathcal{F}}_{\epsilon_1',\epsilon_3'} \)). Then periodicity of the function \( f_1(x)\Psi_g(x)\) and \( f_2(x)\Psi_g(x)\) is different, more precisely there exists \( j \in \{1, 3\} \) such that

\[
\begin{align*}
\begin{cases}
\ f_1(x + 2\omega_j)\Psi_g(x + 2\omega_j) = \pm f_1(x)\Psi_g(x), \\
\ f_2(x + 2\omega_j)\Psi_g(x + 2\omega_j) = \mp f_2(x)\Psi_g(x).
\end{cases}
\end{align*}
\]

Then the functions \( f_1(x) \) and \( f_2(x) \) are linearly independent. Since the functions \( f_1(x) \) and \( f_2(x) \) satisfy Eq. (2.23), we have \( \frac{d}{dx}(f_2(x)f_1'(x) - f_1(x)f_2'(x)) = f_2(x)f_1''(x) - f_1(x)f_2''(x) = 0 \). Therefore \( f_2(x)f_1'(x) - f_1(x)f_2'(x) = C \) for constants \( C \), and \( C \) is non-zero, which follows from linear independence. By Eq. (2.44), the function \( (f_2(x)f_1'(x) - f_1(x)f_2'(x))\Psi_g(x)^2 \) is anti-periodic with respect to the period \( 2\omega_j \), but it contradicts to \( C \neq 0 \). Hence, we proved that Eq. (2.23) does not have a non-zero solution in both the spaces \( \tilde{\mathcal{F}}_{\epsilon_1,\epsilon_3} \) and \( \tilde{\mathcal{F}}_{\epsilon_1',\epsilon_3'} \).

Proposition 2.7. If \( Q = 0 \), then we have \( \Lambda(x) \in \tilde{\mathcal{F}}_{\epsilon_1,\epsilon_3} \) for some \( \epsilon_1, \epsilon_3 \in \{\pm 1\} \).

Proof. It follows from Eq. (2.23) and the double-periodicity of the function \( \Xi(x)\Psi_g(x)^2 \) that

\[
(\Lambda(x + 2\omega_j)\Psi_g(x + 2\omega_j))^2 = \Xi(x + 2\omega_j)\Psi_g(x + 2\omega_j)^2 = \Xi(x)\Psi_g(x)^2 = (\Lambda(x)\Psi_g(x))^2,
\]

The function \( \Lambda_g(x) \) and the function \( \exp(\bar{\kappa}x) \) have the same periodicity with respect to periods \((2\omega_1, 2\omega_3)\). Hence the function \( \Lambda_g(x) \exp(-\bar{\kappa}x) \) is doubly periodic, and we obtain the expression (2.35) by considering the poles. Periodicity (see Eq. (2.37)) follows immediately.
for \( j = 1, 3 \). Hence \( \Lambda(x + 2\omega_j)\Psi_g(x + 2\omega_j) = \pm \Lambda(x)\Psi_g(x) \) \((j = 1, 3)\) and we have \( \Lambda(x) \in \tilde{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) for some \( \epsilon_1, \epsilon_3 \in \{\pm 1\} \).

It follows from Proposition 2.7 that the dimension of the space of solutions to Eq.\((2.19)\), which are even doubly-periodic, is no less than one. Since the exponents of Eq.\((2.19)\) at \( x = 0 \) are \(-2l_0\), \(1\) and \(2l_0 + 2\), the dimension of the space of even solutions to Eq.\((2.19)\) is at most two. Hence, the dimension of the space of solutions to Eq.\((2.19)\), which are even doubly-periodic, is one or two.

**Proposition 2.8.** Assume that the dimension of the space of solutions to Eq.\((2.19)\), which are even doubly-periodic, is two for some eigenvalue \( E \). Then all solutions to Eq.\((2.3)\) for the eigenvalue \( E \) are contained in the space \( \tilde{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) for some \( \epsilon_1, \epsilon_3 \in \{\pm 1\} \).

**Proof.** Since the differential equation \((2.3)\) is invariant under the change of parity \( x \leftrightarrow -x \) and exponents at \( x = 0 \) are even one and odd one, a basis of the solutions to Eq.\((2.3)\) is taken as \( f_\epsilon(x) \) and \( f_o(x) \) such that \( f_\epsilon(x) \) (resp. \( f_o(x) \)) satisfies \( f_\epsilon(-x) = f_\epsilon(x) \) (resp. \( f_o(-x) = -f_o(x) \)). Then the functions \( f_\epsilon(x)^2 \) and \( f_o(x)^2 \) are even and they are solutions to Eq.\((2.19)\). Since the dimension of the space of even solutions to Eq.\((2.19)\) is at most two, and the dimension of the space of solutions to Eq.\((2.19)\), which are even doubly-periodic, is two, the even functions \( f_\epsilon(x)^2 \) and \( f_o(x)^2 \) must be doubly-periodic. Hence \( (f_\epsilon(x + 2\omega_j)\Psi_g(x + 2\omega_j))^2 = (f_\epsilon(x)\Psi_g(x))^2 \) \((j = 1, 3)\) and it follows that \( f_\epsilon(x + 2\omega_j)\Psi_g(x + 2\omega_j) = \pm f_\epsilon(x)\Psi_g(x) \) \((j = 1, 3)\). Therefore we have \( f_\epsilon(x) \in \tilde{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) for some \( \epsilon_1, \epsilon_3 \in \{\pm 1\} \). Similarly we have \( f_o(x) \in \tilde{\mathcal{F}}_{\epsilon_1', \epsilon_3'} \) for some \( \epsilon_1', \epsilon_3' \in \{\pm 1\} \), and it follows from Proposition 2.6 that \( \epsilon_1' = \epsilon_3' = \epsilon_1, \epsilon_3 \) (resp. \( j = 1, 3 \)). Since \( f_\epsilon(x) \) and \( f_o(x) \) are a basis of solutions to Eq.\((2.3)\), all solutions to Eq.\((2.3)\) are contained in the space \( \tilde{\mathcal{F}}_{\epsilon_1, \epsilon_3} \).

**Proposition 2.9.** If \( M = 0 \) or \( M = 1 \) and \( r_1 = 1 \), then the dimension of the space of solutions to Eq.\((2.19)\), which are even doubly-periodic, is one for all \( E \).

**Proof.** Assume that the dimension of the space of solutions to Eq.\((2.19)\), which are even doubly-periodic, is two. From Proposition 2.8, all solutions to Eq.\((2.3)\) are contained in the space \( \tilde{\mathcal{F}}_{\epsilon_1, \epsilon_3} \) for some \( \epsilon_1, \epsilon_3 \in \{\pm 1\} \). Since the differential equation \((2.12)\) is invariant under the change of parity \( x \leftrightarrow -x \) and exponents at \( x = 0 \) are even one and odd one, a basis of the solutions to Eq.\((2.12)\) can be taken as \( f_\epsilon(x) \) and \( f_o(x) \) such that \( f_\epsilon(x) \) (resp. \( f_o(x) \)) is even (resp. odd) function. From the assumption that \( l_i \in \mathbb{Z} \) \((i = 0, 1, 2, 3)\) and that regular singular points \( b_i \) are apparent \((i' = 1, \ldots, M)\), the functions \( f_\epsilon(x) \) and \( f_o(x) \) are meromorphic. Since the function \( f_\epsilon(x) \) (resp. \( f_o(x) \)) satisfies Eq.\((2.12)\), it does not have poles except for \( \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_3 \). Hence the function \( f_\epsilon(x) \) admits the expression \( f_\epsilon(x) = \varphi_1(x)^{\beta_1}\varphi_2(x)^{\beta_2}\varphi_3(x)^{\beta_3}(P^{(1)}(\varphi(x)) + \varphi'(x)P^{(2)}(\varphi(x))) \), where \( \varphi_i(x) \) \((i = 1, 2, 3)\) are co-\( \varphi \) functions and \( P^{(1)}(z), P^{(2)}(z) \) are polynomials in \( z \). Since the function \( f_\epsilon(x) \) is even, we have \( P^{(1)}(z) = 0 \) or \( P^{(2)}(z) = 0 \). By combining with the relation \( \varphi'(z) = -2\varphi_1(z)\varphi_2(z)\varphi_3(z) \), the function \( f_\epsilon(x) \) is expressed as

\[
(2.46) \quad f_\epsilon(x) = \varphi_1(x)^{\beta_1}\varphi_2(x)^{\beta_2}\varphi_3(x)^{\beta_3}P_\epsilon(\varphi(x)),
\]

where \( P_\epsilon(z) \) is a polynomial in \( z \). Because the exponents of Eq.\((2.12)\) at \( x = \omega_i \) \((i = 1, 2, 3)\) are \(-l_i \) and \( l_i + 1 \), we have \( \beta_i \in \{-l_i, l_i + 1\} \) \((i = 1, 2, 3)\). Similarly the function \( f_o(x) \) is expressed as

\[
(2.47) \quad f_o(x) = \varphi_1(x)^{\beta_1'}\varphi_2(x)^{\beta_2'}\varphi_3(x)^{\beta_3'}P_o(\varphi(x)),
\]
where \( P_0(z) \) is a polynomial in \( z \) and \( \beta'_i \in \{-l_i, l_i + 1\} \).

Since \( \varphi(-x) = \varphi(x) \), \( \varphi_i(-x) = -\varphi_i(x) \) \( (i = 1, 2, 3) \) and the parity of functions \( f_e(x) \) and \( f_o(x) \) is different, we have \( \beta_1 + \beta_2 + \beta_3 \equiv \beta'_1 + \beta'_2 + \beta'_3 \pmod{2} \). Since \( f_e(x + 2\omega_1) = (1)^{\beta_2 + \beta_3} f_e(x) \), \( f_o(x + 2\omega_1) = (1)^{\beta'_2 + \beta'_3} f_o(x) \), \( f_e(x + 2\omega_3) = (1)^{\beta_2 + \beta_3} f_e(x) \), \( f_o(x + 2\omega_3) = (1)^{\beta'_2 + \beta'_3} f_o(x) \), we have \( \beta_2 + \beta_3 \equiv \beta'_2 + \beta'_3 \pmod{2} \) and \( \beta_1 + \beta_2 \equiv \beta'_1 + \beta'_2 \pmod{2} \). Hence we have \( \beta_i \equiv \beta'_i \pmod{2} \) for \( i = 1, 2, 3 \). Therefore \( \beta_i, \beta'_i = (l_i, l_i + 1) \) or \( (\beta_i, \beta'_i) = (l_i + 1, -l_i) \) for each \( i \in \{1, 2, 3\} \). Let \( \beta_0 \) (resp. \( \beta_0' \)) be the exponent of the function \( f_e(x) \) (resp. \( f_o(x) \)) at \( x = 0 \). Since the parity of functions \( f_e(x) \) and \( f_o(x) \) is different and the exponents of Eq. (2.12) at \( x = 0 \) are \( -l_0 - \sum_{i=1}^{M} r_i \) and \( l_0 + 1 - \sum_{i=1}^{M} r_i \), we have \( (\beta_0, \beta_0') = (-l_0 - \sum_{i=1}^{M} r_i, l_0 + 1 - \sum_{i=1}^{M} r_i) \) or \( (\beta_0, \beta_0') = (l_0 + 1 - \sum_{i=1}^{M} r_i, -l_0 - \sum_{i=1}^{M} r_i) \).

Since the function \( f_e(x) \) is doubly-periodic with periods \((4\omega_1, 4\omega_3)\), the sum of degrees of zeros of \( f_e(x) \) on the basic domain is equal to the sum of degrees of poles of \( f_e(x) \). Since the function \( f_e(x) \) does not have poles except for \( \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_3 \), we have \( \sum_{i=0}^{3} \beta_i \leq 0 \). Similarly we have \( \sum_{i=0}^{3} \beta'_i \leq 0 \). Hence \( 0 \geq \sum_{i=0}^{3} (\beta_i + \beta'_i) = 4 - 2 \sum_{i=1}^{M} r_i \). Therefore we have \( \sum_{i=1}^{M} r_i \geq 2 \).

Thus we obtain that, if \( M = 0 \) or \( (M = 1 \text{ and } r_1 = 1) \), then the dimension of the space of solutions to Eq. (2.19), which are even doubly-periodic, is one.

Note that the case \( M = 0 \) corresponds to Heun’s equation, and the case \( M = 1 \) and \( r_1 = 1 \) is related with the sixth Painlevé equation.

**Example 1.** Let us consider the following differential equation:

\[
(2.48) \quad \left\{ - \left( \frac{d}{dx} \right)^2 + \left( \frac{\varphi'(x)}{\varphi(x) + \sqrt{\frac{12}{3}}} + \frac{\varphi'(x)}{\varphi(x) - \sqrt{\frac{12}{3}}} \right) \frac{d}{dx} \right\} f(x) = 0.
\]

This equation corresponds to the case \( l_0 = 1, l_1 = l_2 = l_3 = 0, M = 2 \) and \( r_1 = r_2 = 1 \), if \( g_2 \neq 0 \). From the relation

\[
(2.49) \quad \frac{\varphi'(x)}{\varphi(x) + \sqrt{\frac{12}{3}}} + \frac{\varphi'(x)}{\varphi(x) - \sqrt{\frac{12}{3}}} = \frac{\varphi''(x)}{\varphi'(x)},
\]

a basis of the solutions to Eq. (2.48) is 1, \( \varphi'(x) \). The dimension of the solutions to Eq. (2.19), which are even doubly-periodic, is two, and a basis of the solutions to Eq. (2.19) is written as \( 1/\varphi''(x), \varphi'(x)^2/\varphi'''(x), \varphi'(x)/\varphi''(x) \).

**Proposition 2.10.** Assume that the dimension of the space of the solutions to Eq. (2.19), which are even doubly-periodic, is one. Let \( c_0 \) and \( b^{(i)}_j \) be constants defined in Eq. (2.20).

(i) If there exists a non-zero solution to Eq. (2.3) in the space \( \mathcal{F}_{e_1 e_3} \) for some \( e_1, e_3 \in \{\pm 1\} \), then we have \( Q = 0 \).

(ii) If \( Q \neq 0 \) and \( l_i \neq 0 \), then the function \( \Lambda(x) \) has a pole of degree \( l_i \) and \( b^{(i)}_0 \neq 0 \).

(iii) If \( Q \neq 0 \) and \( l_i = 0 \), then \( \Lambda(\omega_i) \neq 0 \) and \( \Xi(\omega_i) \neq 0 \). If \( Q \neq 0 \) and \( l_0 = l_1 = l_2 = l_3 = 0 \), then \( c_0 \neq 0 \).

**Proof.** First we prove (i). Suppose that there exists a non-zero solution to Eq. (2.3) in the space \( \mathcal{F}_{e_1 e_3} \) and \( Q \neq 0 \). From the condition \( Q \neq 0 \), the functions \( \Lambda(x) \) and \( \Lambda(-x) \) form the basis of the space of the solutions to the differential equation (2.3). Since there is a non-zero solution to Eq. (2.3) in the space \( \mathcal{F}_{e_1 e_3} \), there exist constants...
(C₁, C₂) \neq (0, 0) such that C₁Λ(x) + C₂Λ(−x) ∈ \tilde{\mathcal{F}}_{ε₁, ε₃}. By shifting \( x \to x + 2ω_j \) \((j = 1, 3)\), it follows from Eq.\((2.31)\) that

\[
\begin{align*}
(2.50) & \quad (C₁Λ(x + 2ω_j) + C₂Λ(-(x + 2ω_j)))Ψ_g(x + 2ω_j) \\
& = C₁Λ(x + 2ω_j)Ψ_g(x + 2ω_j) ± C₂Λ(-x - 2ω_j)Ψ_g(-x - 2ω_j) \\
& = C₁\exp(π\sqrt{-1m_j})Λ(x)Ψ_g(x) ± C₂\exp(-π\sqrt{-1m_j})Λ(-x)Ψ_g(-x) \\
& = (C₁\exp(π\sqrt{-1m_j})Λ(x) + C₂\exp(-π\sqrt{-1m_j})Λ(-x))Ψ_g(x),
\end{align*}
\]

where the sign \( \pm \) is determined by the branching of the function \( Ψ_g(x) \), and the function \( C₁\exp(π\sqrt{-1m_j})Λ(x) + C₂\exp(-π\sqrt{-1m_j})Λ(-x) \) also satisfies Eq.\((2.3)\). On the other hand, it follows from the definition of the space \( \tilde{\mathcal{F}}_{ε₁, ε₃} \) that \( (C₁Λ(x + 2ω_j) + C₂Λ(-(x + 2ω_j)))Ψ_g(x + 2ω_j) = (C₁Λ(x) + C₂Λ(-x))Ψ_g(x) \) or \( (C₁Λ(x + 2ω_j) + C₂Λ(-(x + 2ω_j)))Ψ_g(x + 2ω_j) = -(C₁Λ(x) + C₂Λ(-x))Ψ_g(x) \). By comparing two expressions, we have \( \exp(π\sqrt{-1m_j}) ∈ \{±1\} \ (j = 1, 3) \) and the periodicities of the functions \( Λ(x)Ψ_g(x) \) and \( (C₁Λ(x) + C₂Λ(-x))Ψ_g(x) \) coincide. Thus \( Λ(x), Λ(-x) ∈ \tilde{\mathcal{F}}_{ε₁, ε₃} \). The functions \( Λ(x)^2 \) and \( Λ(-x)^2 \) are even doubly-periodic functions and satisfy Eq.\((2.19)\), because they are the products of a pair of solutions to Eq.\((2.3)\). Since the functions \( Λ(x) \) and \( Λ(-x) \) are linearly independent, the functions \( Λ(x)^2 \) and \( Λ(-x)^2 \) are linearly independent. Hence the dimension of the space of solutions to Eq.\((2.3)\), which are even doubly-periodic, is no less than two, and contradict the assumption of the proposition. Therefore the supposition \( Q \neq 0 \) is false, and we obtain (i).

Next we show (ii). Assume that \( l_i ≠ 0 \). Since the exponents of Eq.\((2.3)\) at \( x = ω_i \) are \(-l_i\) or \( l_i + 1\), the function \( Λ(x) \) has a pole of degree \( l_i \) or a zero of degree \( l_i + 1 \) at \( x = ω_i \). It follows from the periodicity (see Eq.\((2.31)\)) that, if the function \( Λ(x) \) has a zero at \( x = ω_i \), then \( Λ(x) \) has also a zero at \( x = -ω_i \). Hence the function \( Λ(-x) \) has a zero at \( x = ω_i \). From the assumption \( Q ≠ 0 \), any solution to Eq.\((2.3)\) is written as a linear combination of functions \( Λ(x) \) and \( Λ(-x) \). But it contradicts that one of the exponents at \( x = ω_i \) is \(-l_i\). Hence the function \( Λ(x) \) has a pole of degree \( l_i \). Since the dimension of the space of the solutions to Eq.\((2.3)\), which are even doubly-periodic, is one, we have \( Ξ(x) = CΛ(x)Λ(-x) \) for some non-zero constant \( C \) and \( b_0^{(i)} ≠ 0 \).

(iii) is proved similarly by showing that the function \( Λ(x) \) does not have zero at \( x = ω_i \).

By combining Propositions \(2.7\) and \(2.10\) (i) we obtain the following proposition:

**Proposition 2.11.** Assume that the dimension of the space of solutions to Eq.\((2.19)\), which are even doubly-periodic, is one. Then the condition \( Q = 0 \) is equivalent to that there exists a non-zero solution to Eq.\((2.3)\) in the space \( \tilde{\mathcal{F}}_{ε₁, ε₃} \) for some \( ε₁, ε₃ ∈ \{±1\} \).

We show that the function \( Λ(x) \) admits an expression of the Bethe Ansatz type.

**Proposition 2.12.** Set \( l = \sum_{i=0}^{3} l_i + \sum_{i' = 1}^{M} r_{i'} \), \( \tilde{l}_0 = l_0 + \sum_{i' = 1}^{M} r_{i'} \) and \( \tilde{l}_i = l_i \ (i = 1, 2, 3) \). Assume that \( Q ≠ 0 \) and the dimension of the space of the solutions to Eq.\((2.19)\), which are even doubly-periodic, is one.

(i) The function \( Λ_g(x) \) in Eq.\((2.24)\) is expressed as

\[
Λ_g(x) = \frac{C_0\prod_{j=1}^{3} σ(x - t_j)}{σ(x)^{l_0}σ_1(x)^{l_1}σ_2(x)^{l_2}σ_3(x)^{l_3}} \exp(cx),
\]

(2.51)
for some $t_1, \ldots, t_l, c$ and $C_0(\neq 0)$ such that $t_j \equiv 0 \pmod{\omega_1 \mathbb{Z} + \omega_3 \mathbb{Z}}$ for all $j$, where $\sigma_i(x) \ (i = 1, 2, 3)$ are co-sigma functions.

(ii) $t_j + t_{j'} \equiv 0 \pmod{2 \omega_1 \mathbb{Z} + 2 \omega_3 \mathbb{Z}}$ for $1 \leq j < j' \leq l$.

(iii) If $t_j \neq \pm \delta_{i'} \pmod{2 \omega_1 \mathbb{Z} + 2 \omega_3 \mathbb{Z}}$ for all $i' \in \{1, \ldots, M\}$, then we have $t_j \neq t_{j'} \pmod{2 \omega_1 \mathbb{Z} + 2 \omega_3 \mathbb{Z}}$ for all $j' \neq j$.

(iv) If $t_j \equiv \pm \delta_{i'} \pmod{2 \omega_1 \mathbb{Z} + 2 \omega_3 \mathbb{Z}}$, then $\# \{j' \mid t_j \equiv t_{j'} \pmod{2 \omega_1 \mathbb{Z} + 2 \omega_3 \mathbb{Z}}\} = r_{i'} + 1$.

(v) If $l_0 \neq 0 \ (\text{resp.} \ l_0 = 0)$, then we have $c = \sum_{i=1}^{l'} \zeta(t_j)$ \ (resp. $c = \sum_{i=1}^{l'} \zeta(t_j) + \sqrt{-Q}/\Xi(0)$). (Note that it follows from Proposition 2.10 (iii) that $\sqrt{-Q}/\Xi(0)$ is finite.)

(vi) Set $z = \wp(x)$ and $z_j = \wp(t_j)$. Then

$$
(2.52) \quad \frac{d\Xi(x)}{dz} \bigg|_{z=z_j} = \frac{2\sqrt{-Q}}{\wp'(t_j)}.
$$

**Proof.** Let $\alpha$ be the value defined in Eq. (2.35). First, we consider the case $\alpha \neq 0 \pmod{2 \omega_1 \mathbb{Z} + 2 \omega_3 \mathbb{Z}}$. Let $\kappa$ be the value defined in Eq. (2.39). Then the function $\Lambda_g(x)/ (\exp(\kappa x) \Phi_0(x, \alpha))$ is meromorphic and doubly-periodic. Hence there exists $a_1, \ldots, a_{r'}, b_1, \ldots, b_{r'}$ such that $a_1 + \cdots + a_{r'} = b_1 + \cdots + b_{r'}$ and

$$
\Lambda_g(x)/ (\exp(\kappa x) \Phi_0(x, \alpha)) = \frac{\prod_{j=1}^{r'} \sigma(x - a_j)}{\prod_{j=1}^{r'} \sigma(x - b_j)}.
$$

For the case $\alpha \equiv 0 \pmod{2 \omega_1 \mathbb{Z} + 2 \omega_3 \mathbb{Z}}$ the function $\Lambda_g(x)/ \exp(\tilde{x}x)$ is similarly expressed as

$$
\Lambda_g(x)/ \exp(\tilde{x}x) = \frac{\prod_{j=1}^{r'} \sigma(x - a_j)}{\prod_{j=1}^{r'} \sigma(x - b_j)}.
$$

Since the function $\Lambda_g(x)$ satisfies Eq. (2.12), it does not have poles except for $\omega_1 \mathbb{Z} + \omega_3 \mathbb{Z}$ and we have

$$
(2.53) \quad \Lambda_g(x) = \frac{C_0 \prod_{j=1}^{r'} \sigma(x - t_j)}{\sigma(x)^{r_0} \sigma_1(x)^{r_1} \sigma_2(x)^{r_2} \sigma_3(x)^{r_3}} \exp(cx),
$$

for some $t_1, \ldots, t_l, c$ and $C_0(\neq 0)$. It follows from Proposition 2.10 (ii) and (iii) that $t_j \not\equiv 0 \pmod{\omega_1 \mathbb{Z} + \omega_3 \mathbb{Z}}$. Therefore we obtain (i).

Suppose that $t_j + t_{j'} \equiv 0 \pmod{2 \omega_1 \mathbb{Z} + 2 \omega_3 \mathbb{Z}}$ for some $j$ and $j'(\neq j)$. From Eq. (2.51) and $-t_j \equiv t_{j'} \pmod{2 \omega_1 \mathbb{Z} + 2 \omega_3 \mathbb{Z}}$, we have $\Lambda_g(t_j) = \Lambda_g(-t_j) = 0$. Since $Q \neq 0$, all solutions to Eq. (2.12) are written as linear combinations of $\Lambda_g(x)$ and $\Lambda_g(-x)$. Hence $t_j$ is a zero for all solutions to Eq. (2.12), but they contradict that one of the exponents at $x = t_j$ is zero. Therefore we obtain (ii).

If $t_j \not\equiv \pm \delta_{i'} \omega_i \pmod{2 \omega_1 \mathbb{Z} + 2 \omega_3 \mathbb{Z}}$ for all $i$ and $i'$, then the exponents of Eq. (2.12) at $x = t_j$ are 0 and 1, and $x = t_j$ is a zero of $\Lambda_g(x)$ of degree one. Incidentally, the exponents of Eq. (2.12) at $x = \pm \delta_{i'}$ are 0 and $r_{i'} + 1$. Hence, if $t_j \equiv \pm \delta_{i'} \pmod{2 \omega_1 \mathbb{Z} + 2 \omega_3 \mathbb{Z}}$, then $x = t_j$ is a zero of $\Lambda_g(x)$ of degree $r_{i'} + 1$. Thus we obtain (iii) and (iv).

It follows from Eq. (2.51) and $\Lambda_g(x) = \Psi_g(x) \Lambda(x)$ that

$$
(2.54) \quad \frac{\Lambda'(x)}{\Lambda(x)} = c - \tilde{l}_0 \frac{\sigma'(x)}{\sigma(x)} - \sum_{i=1}^{3} l_i' \frac{\sigma_i'(x)}{\sigma_i(x)} + \sum_{j=1}^{l} \frac{\sigma'(x - t_j)}{\sigma(x - t_j)} - \sum_{r'=1}^{M} \frac{r_{i'} \varphi'(x)}{2 \varphi(x) - \varphi(\delta_{i'})}.
$$
By expanding Eq. (2.25) at $x = 0$ and observing coefficient of $x^0$, we obtain

\[(2.55)\]

\[
c - \sum_{j=1}^{l} \zeta(t_j) = \left. \frac{\sqrt{-Q}}{\Xi(x)} \right|_{x=0},
\]

because the functions $\sigma'(x)/\sigma(x)$, $\sigma'_i(x)/\sigma_i(x)$, $\varphi'(x)/(\varphi(x) - \varphi(\delta_i'))$ and $\Xi'(x)/\Xi(x)$ are odd and $\sigma'(-t)/\sigma(-t) = -\zeta(t)$. It follows from $Q \neq 0$ and Proposition 2.10 that, if $l_0 \neq 0$, then $\left. \sqrt{-Q}/\Xi(x) \right|_{x=0}$ is finite. Thus we obtain (v).

We show (vi). The function $\Lambda(x)\Lambda(-x)$ is even doubly-periodic and satisfies Eq. (2.19), because it is a product of the solutions to Eq. (2.3). Since the dimension of the space of the solutions to Eq. (2.3), which are even doubly-periodic, is one, we have $\Xi(x) = C\Lambda(x)\Lambda(-x)$ for some non-zero constant $C$. Hence we have $\Xi(t_j) = \Xi(-t_j) = 0$. On the other hand, we have $\Lambda(-t_j) \neq 0$ from (ii). At $x = -t_j$, the l.h.s. of Eq. (2.25) is finite, and the denominator of the r.h.s. is zero. Therefore we have

\[(2.56)\]

\[
\left. \Xi'(x) \right|_{x=-t_j} + 2\sqrt{-Q} = 0.
\]

By changing the variable $z = \varphi(x)$ and the oddness of the function $\varphi'(x)$, we obtain (vi).

Note that Gesztesy and Weikard [8] obtained a similar expression to Eq. (2.51) in the framework of Picard’s potential.

3. The Case $M = 1$, $r_1 = 1$ and Painlevé Equation

3.1. We consider Eq. (2.12) for the case $M = 1$, $r_1 = 1$. For this case, Eq. (2.12) is written as

\[(3.1)\]

\[
(H_g - \tilde{E})f_g(x) = 0,
\]

where

\[(3.2)\]

\[
H_g = -\frac{d^2}{dx^2} + \frac{\varphi'(x)}{\varphi(x) - \varphi(\delta_1)} \frac{d}{dx} + \frac{\tilde{s}_1}{\varphi(x) - \varphi(\delta_1)} + \sum_{i=0}^{3} l_i(l_i + 1)\varphi(x + \omega_i).
\]

We set

\[(3.3)\]

\[
\Psi_g(x) = \sqrt{\varphi(x) - \varphi(\delta_1)}, \quad b_1 = \varphi(\delta_1),
\]

\[(3.4)\]

\[
\mu_1 = \frac{-\tilde{s}_1}{4b_1^3 - 2g_2b_1 - g_3} + \sum_{i=1}^{3} \frac{l_i}{2(b_1 - e_i)},
\]

\[(3.5)\]

\[
p = \tilde{E} - 2(l_1l_2e_3 + l_2l_3e_1 + l_3l_1e_2) + \sum_{i=1}^{3} l_i(l_i + 2(e_i + b_1)).
\]

The condition that, the regular singular points $x = \pm \delta_1$ is apparent, is written as

\[(3.6)\]

\[
p = (4b_1^3 - 2g_2b_1 - g_3) \left\{ -\mu_1^2 + \sum_{i=1}^{3} \frac{l_i + 1/2}{b_1 - e_i} \mu_1 \right\} - b_1(l_1 + l_2 + l_3 + l_0)(l_1 + l_2 + l_3 + l_0 + 1).
\]
From now on we assume that $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ and the eigenvalue $\bar{E}$ satisfies Eqs. (3.6). Then the assumption in Proposition 2.1 is true, and propositions and theorem in the previous section are valid. The function $\Xi(x)$ in Proposition 2.1 is written as

\begin{equation}
\Xi(x) = c_0 + \frac{d_0}{(x - \varphi(\delta_1))} + \sum_{i=0}^{3} \sum_{j=0}^{l_i-1} b^{(i)}_j \varphi(x + \omega_i)^{l_i-j}.
\end{equation}

It follows from Proposition 2.9 that the function $\Xi(x)$ is determined uniquely up to multiplicative constant. Ratios of the coefficients $c_0/d_0$ and $b^{(i)}_j/d_0$ $(i = 0, 1, 2, 3, \ j = 0, \ldots, l_i - 1)$ are written as rational functions in variables $b_1$ and $\mu_1$, because the coefficients $b^{(i)}_j$, $c_0$ and $d_0$ satisfy linear equations whose coefficients are rational functions in $b_1$ and $\mu_1$, which are obtained by substituting Eq. (3.7) into Eq. (2.19). The value $Q$ is calculated by Eq. (2.21) and it is expressed as a rational function in $b_1$ and $\mu_1$ multiplied by $d_0$. It is shown by observing asymptotic $\mu_1 \to \infty$ that $Q$ is not identically zero. By an appropriate choice of $d_0$, $Q$ is expressed as a polynomial in $b_1$ and $\mu_1$. We set

\begin{equation}
\Lambda_g(x) = \Psi_g(x) \sqrt{\Xi(x)} \exp \int \frac{\sqrt{-Q} \ dx}{\Xi(x)}.
\end{equation}

Due to Proposition 2.3 the function $\Lambda_g(x)$ is a solution to the differential equation (3.1). By Theorem 2.5 the eigenfunction $\Lambda_g(x)$ is also expressed in the form of the Hermite-Krichever Ansatz. Namely, it is expressed as

\begin{equation}
\Lambda_g(x) = \exp(\kappa x) \left( \sum_{i=0}^{3} \sum_{j=0}^{l_i-1} b^{(i)}_j \left( \frac{d}{dx} \right)^j \Phi_i(x, \alpha) \right)
\end{equation}

or

\begin{equation}
\Lambda_g(x) = \exp(\bar{\kappa} x) \left( \bar{c} + \sum_{i=0}^{3} \sum_{j=0}^{l_i-2} b^{(i)}_j \left( \frac{d}{dx} \right)^j \varphi(x + \omega_i) + \sum_{i=1}^{3} \bar{c}_i \varphi'(x) \varphi(x - e_i) \right)
\end{equation}

where $\bar{l}_0 = l_0 + 1$ and $\bar{l}_i = l_i$ ($i = 1, 2, 3$). Now we investigate the values $\alpha$ and $\kappa$ in Eq. (3.9). Note that, if $\alpha \neq 0 \ (\text{mod} \ 2\omega_1 \mathbb{Z} \oplus 2\omega_2 \mathbb{Z})$, then the function $\Lambda_g(x)$ is expressed as Eq. (3.9) and we have

\begin{equation}
\Lambda_g(x + 2\omega_j) = \exp(-2\eta_j \alpha + 2\omega_j \zeta(\alpha) + 2\kappa \omega_j) \Lambda_g(x), \quad (j = 1, 3).
\end{equation}

Proposition 3.1. Assume that $M = 1$, $r_1 = 1$, $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$ and the value $p$ satisfies Eq. (3.6). Let $\alpha$ and $\kappa$ be the values determined by the Hermite-Krichever Ansatz (see Eq. (3.9)). Then $\varphi(\alpha)$ is expressed as a rational function in variables $b_1$ and $\mu_1$, $\varphi'(\alpha)$ is expressed as a product of $\sqrt{-Q}$ and a rational function in variables $b_1$ and $\mu_1$, and $\kappa$ is expressed as a product of $\sqrt{-Q}$ and a rational function in variables $b_1$ and $\mu_1$.

Proof. It follows from Eqs. (2.5) and (3.9) that

\begin{equation}
\Lambda_g(x + 2\omega_j) = \exp \left( 2\eta_j \left( -\frac{t_j}{j' + 1} + \frac{l_i \omega_i}{3} \right) + 2\omega_j \left( c - \frac{3}{i=1} l_i \eta_k \right) \right) \Lambda_g(x),
\end{equation}
for $j = 1, 3$, where $l = l_0 + l_1 + l_2 + l_3 + 1$. By comparing with Eq. (3.11), we have

(3.13)

$$-2\eta_1\alpha + 2\omega_1(\zeta(\alpha) + \kappa) = -2\eta_1\left(\sum_{j'=1}^{l} t_{j'} - \sum_{i=1}^{3} l_i\omega_i\right) + 2\omega_1\left(c - \sum_{i=1}^{3} l_i\eta_i\right) + 2\pi\sqrt{-1}n_1,$$

(3.14)

$$-2\eta_3\alpha + 2\omega_3(\zeta(\alpha) + \kappa) = -2\eta_3\left(\sum_{j'=1}^{l} t_{j'} - \sum_{i=1}^{3} l_i\omega_i\right) + 2\omega_3\left(c - \sum_{i=1}^{3} l_i\eta_i\right) + 2\pi\sqrt{-1}n_3,$$

for integers $n_1, n_3$. It follows that

(3.15) $$(\alpha - \left(\sum_{j'=1}^{l} t_{j'} - \sum_{i=1}^{3} l_i\omega_i\right))(\omega_1 + 2\eta_3\omega_1) = 2\pi\sqrt{-1}(n_1\omega_3 - n_3\omega_1),$$

(3.16) $$(\zeta(\alpha) + \kappa - c + \sum_{i=1}^{3} l_i\eta_i)(2\eta_1\omega_1 - 2\eta_3\omega_3) = 2\pi\sqrt{-1}(n_1\eta_3 - n_3\eta_1).$$

From Legendre’s relation $\eta_1\omega_3 - \eta_3\omega_1 = \pi\sqrt{-1}/2$, we have

(3.17) $$\alpha \equiv \sum_{j'=1}^{l} t_{j'} - \sum_{i=1}^{3} l_i\omega_i \pmod{2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z}}.$$ Combining Eqs. (3.15, 3.16) with Proposition 2.12 (v) and relations $\zeta(\alpha + 2\omega_j) = \zeta(\alpha) + 2\eta_j$ ($j = 1, 3$), we have

(3.18) $$\kappa = -\zeta\left(\sum_{j'=1}^{l} t_{j'} - \sum_{i=1}^{3} l_i\omega_i\right) + \sum_{j'=1}^{l} \zeta(t_{j'}) - \sum_{i=1}^{3} l_i\eta_i + \delta_{b,0}\sqrt{-Q}(0).$$

Next, we investigate values $\phi(\alpha), \psi'(\alpha)$ and $\kappa$. The functions $\phi(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i\omega_i)$, $\psi'(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i\omega_i)$ and $\zeta(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i\omega_i) - \sum_{j=1}^{l} \zeta(t_j) + \sum_{i=1}^{3} l_i\eta_i$ are doubly-periodic in variables $t_1, \ldots, t_l$. Hence by applying addition formulae of elliptic functions and considering the parity of functions $\phi(x)$, $\psi'(x)$ and $\zeta(x)$, we obtain the expressions

(3.19) $$\phi\left(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i\omega_i\right) = \sum_{j_1 < j_2 < \cdots < j_m \text{ even}} f^{(1)}_{j_1, \ldots, j_m}(\phi(t_1), \ldots, \phi(t_l))\psi'(t_{j_1}) \cdots \psi'(t_{j_j}),$$

$$\psi'\left(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i\omega_i\right) = \sum_{j_1 < j_2 < \cdots < j_m \text{ even}} f^{(2)}_{j_1, \ldots, j_m}(\phi(t_1), \ldots, \phi(t_l))\psi'(t_{j_1}) \cdots \psi'(t_{j_j}),$$

$$\zeta\left(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i\omega_i\right) - \sum_{j=1}^{l} \zeta(t_j) + \sum_{i=1}^{3} l_i\eta_i = \sum_{j_1 < j_2 < \cdots < j_m \text{ odd}} f^{(3)}_{j_1, \ldots, j_m}(\phi(t_1), \ldots, \phi(t_l))\psi'(t_{j_1}) \cdots \psi'(t_{j_j}),$$

where $f^{(i)}_{j_1, \ldots, j_m}(x_1, \ldots, x_l)$ ($i = 1, 2, 3$) are rational functions in $x_1, \ldots, x_l$. From Eq. (2.52), the function $\psi'(t_j)/\sqrt{-Q}$ is expressed as a rational function in $b_1, \mu_1$ and
\[ \varphi(t_j) \text{. Hence, } \varphi(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i), \varphi'(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i)/\sqrt{-Q} \text{ and } (\zeta(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i) + \sum_{j=1}^{l} \zeta(t_j) + \sum_{i=1}^{3} \eta_i)/\sqrt{-Q} \text{ are expressed as rational functions in the variable } \varphi(t_1), \ldots, \varphi(t_l) \text{.}

Since the dimension of the space of the solutions to Eq. (2.3), which are even doubly-periodic, is one, we have \( \Xi(x) = C \Lambda(x) \Lambda(-x) \) for some non-zero scalar \( C \). Hence, we have the following expression:

\[
\Xi(x) \Psi_g(x)^2 = \frac{D \prod_{j=1}^{l}(\varphi(x) - \varphi(t_j))}{(\varphi(x) - e_1)^{l_1}(\varphi(x) - e_2)^{l_2}(\varphi(x) - e_3)^{l_3}}
\]

for some value \( D \neq 0 \). Thus

\[
\prod_{j=1}^{l}(\varphi(x) - \varphi(t_j)) = \Xi(x) \Psi_g(x)^2(\varphi(x) - e_1)^{l_1}(\varphi(x) - e_2)^{l_2}(\varphi(x) - e_3)^{l_3}/D.
\]

Hence, the elementary symmetric functions \( \sum_{j_1<\ldots<j_{l'}} \varphi(t_{j_1}) \ldots \varphi(t_{j_{l'}}) \) \((l' = 1, \ldots, l)\) are expressed as rational functions in \( b_1 \) and \( \mu_1 \). By substituting elementary symmetric functions into the symmetric expressions of \( \varphi(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i), \varphi'(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i) \) and \((\zeta(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i) + \sum_{j=1}^{l} \zeta(t_j) + \sum_{i=1}^{3} \eta_i)/\sqrt{-Q} \), it follows that \( \varphi(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i), \varphi'(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i)/\sqrt{-Q} \) and \((\zeta(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i) + \sum_{j=1}^{l} \zeta(t_j) + \sum_{i=1}^{3} \eta_i)/\sqrt{-Q} \) are expressed as rational functions in \( b_1 \) and \( \mu_1 \). Hence, \( \varphi(\alpha), \varphi'(\alpha)/\sqrt{-Q} \) and \( \kappa/\sqrt{-Q} \) are expressed as rational functions in variables \( b_1 \) and \( \mu_1 \).

We now discuss the relationship between the monodromy preserving deformation of Fuchsian equations and the sixth Painlevé equation. For this purpose we recall some definitions and results of Painlevé equation.

The sixth Painlevé equation is a non-linear ordinary differential equation written as

\[
\frac{d^2 \lambda}{dt^2} = \frac{1}{2} \left( \frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t} \right) \left( \frac{d \lambda}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t} \right) \frac{d \lambda}{dt} + \frac{\lambda(\lambda - 1)(\lambda - t)}{t^2(t - 1)^2} \left\{ \frac{\kappa_0}{2} t^2 - \frac{\kappa_0^2}{2} \frac{t}{\lambda^2} + \frac{\kappa_1}{2} \frac{(t - 1)}{\lambda - 1} + \frac{1}{2} \frac{(1 - \kappa_0^2)}{\lambda - t} \right\}.
\]

A remarkable property of this differential equation is that its solutions do not have movable singularities other than poles. This equation is also written in terms of a Hamiltonian system by adding the variable \( \mu \), which is called the sixth Painlevé system:

\[
\frac{d \lambda}{dt} = \frac{\partial H_{VI}}{\partial \mu}, \quad \frac{d \mu}{dt} = \frac{\partial H_{VI}}{\partial \lambda},
\]

with the Hamiltonian

\[
H_{VI} = \frac{1}{t(t - 1)} \left\{ \lambda(\lambda - 1)(\lambda - t) \mu^2 - \{ \kappa_0(\lambda - 1)(\lambda - t) + \kappa_1 \lambda(\lambda - t) + (\kappa_t - 1) \lambda(\lambda - 1) \} \mu + \kappa(\lambda - t) \right\},
\]
where \( \kappa = ((\kappa_0 + \kappa_1 + \kappa_t - 1)^2 - \kappa_\infty^2)/4 \). The sixth Painlevé equation for \( \lambda \) is obtained by eliminating \( \mu \) in Eq. (3.23). Set \( \omega_1 = 1/2, \omega_3 = \tau/2 \) and write

\[
(3.25) \quad t = \frac{e_3 - e_1}{e_2 - e_1}, \quad \lambda = \frac{\varphi(\delta) - e_1}{e_2 - e_1}.
\]

Then the sixth Painlevé equation is equivalent to the following equation (see [16, 22]):

\[
(3.26) \quad \frac{d^2 \delta}{d\tau^2} = -\frac{1}{4\pi^2} \left\{ \frac{\kappa_\infty^2}{2} \varphi'(\delta) + \frac{\kappa_0^2}{2} \varphi'(\delta + 1/2) + \frac{\kappa_1^2}{2} \varphi'(\delta + \tau/2) + \frac{\kappa_\infty^2}{2} \varphi'(\delta + \tau/2) \right\},
\]

where \( \varphi'(z) = (\partial\varphi/\partial z)(\varphi(z)) \).

It is widely known that the sixth Painlevé equation is obtained by the monodromy preserving deformation of a certain linear differential equation. Let us introduce the following Fuchsian differential equation:

\[
(3.27) \quad \frac{d^2 y}{d w^2} + p_1(w) \frac{d y}{d w} + p_2(w) y = 0,
\]

where

\[
(3.28) \quad p_1(w) = \frac{1 - \kappa_0}{w} + \frac{1 - \kappa_1}{w - 1} + \frac{1 - \kappa_t}{w - t} - \frac{1}{w - \lambda},
\]

\[
(3.29) \quad p_2(w) = \frac{\kappa}{w(w - 1)} - \frac{t(t - 1)H_{VI}}{w(w - 1)(w - t)} + \frac{\lambda(\lambda - 1)\mu}{w(w - 1)(w - \lambda)}.
\]

This equation has five regular singular points \( \{0, 1, \infty, \lambda\} \) and the exponents at \( w = \lambda \) are 0 and 2. It follows from Eq. (3.21) that the regular singular point \( w = \lambda \) is apparent. Then the sixth Painlevé equation is obtained by the monodromy preserving deformation of Eq. (3.23), i.e., the condition that the monodromy of Eq. (3.27) is preserved as deforming the variable \( t \) is equivalent to that \( \mu \) and \( \lambda \) satisfy the Painlevé system (see Eq. (3.23)), provided \( \kappa_0, \kappa_1, \kappa_t, \kappa_\infty \notin \mathbb{Z} \). For details, see [12].

Now we transform Eq. (3.27) into the form of Eq. (3.2). We set

\[
(3.30) \quad w = \frac{\varphi(x) - e_1}{e_2 - e_1}, \quad y = f_g(x) \prod_{i=1}^3 (\varphi(x) - e_i)^{y_i/2},
\]

\[
(3.31) \quad t = \frac{e_3 - e_1}{e_2 - e_1}, \quad \lambda = \frac{b_1 - e_1}{e_2 - e_1}, \quad \varphi(\delta_1) = b_1.
\]

Then we obtain Eq. (3.2) by setting

\[
(3.32) \quad \kappa_0 = l_1 + 1/2, \quad \kappa_1 = l_2 + 1/2, \quad \kappa_t = l_3 + 1/2, \quad \kappa_\infty = l_0 + 1/2,
\]

\[
(3.33) \quad \mu = (e_2 - e_1)\mu_1, \quad \kappa = (l_1 + l_2 + l_3 + l_0 + 1)(l_1 + l_2 + l_3 - l_0),
\]

\[
(3.34) \quad H_{VI} = \frac{1}{t(1 - t)} \left\{ \frac{p + \kappa e_3}{e_2 - e_1} + \lambda(1 - \lambda)\mu \right\},
\]

(see Eqs. (3.3, 3.5)), and Eq. (3.24) is equivalent to Eq. (3.6), that means that the appearance of regular singularity is inherited. Mapping from the variable \( x \) to the variable \( w \) (see Eq. (3.30)) is a double covering from the punctured torus \((\mathbb{C}/(2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z})) \setminus \{0, \omega_1, \omega_2, \omega_3\} \) to the punctured Riemann sphere \( \mathbb{P}^1 \setminus \{0, 1, t, \infty\} \). A solution \( y(w) \) to Eq. (3.27) corresponds to a solution \( f_g(x) \) to Eq. (3.2) by \( y(w) = f_g(x) \prod_{i=1}^3 (\varphi(x) - e_i)^{y_i/2} \). Hence the monodromy preserving deformation of Eq. (3.27) in \( t \) corresponds to the monodromy preserving deformation of Eq. (3.2) in \( \tau \).
Now we consider monodromy preserving deformation in the variable $\tau$ ($\omega_1 = 1/2, \omega_3 = \tau/2$) by applying solutions obtained by the Hermite-Krichever Ansatz for the case $l_i \in \mathbb{Z}_{\geq 0}$ ($i = 0, 1, 2, 3$). Let $\alpha$ and $\kappa$ be values determined by the Hermite-Krichever Ansatz (see Eq. (3.39)). We consider the case $Q \neq 0$. Then a basis for solutions to Eq. (2.12) is given by $\Lambda_\phi(x)$ and $\Lambda_\phi(-x)$, and the monodromy matrix with respect to the cycle $x \to x + 2\omega_j$ ($j = 1, 3$) is diagonal. The elements of the matrix are obtained from Eq. (3.11). Hence, the eigenvalues $\exp(\pm (-2\eta_i \alpha + 2\omega_j \zeta(\alpha) + 2\kappa \omega_j))$ ($j = 1, 3$) of the monodromy matrices are preserved by the monodromy preserving deformation. We set

\begin{align*}
(3.35) & \quad -2\eta_1 \alpha + 2\omega_1 \zeta(\alpha) + 2\kappa \omega_1 = \pi \sqrt{-1} C_1, \\
(3.36) & \quad -2\eta_3 \alpha + 2\omega_3 \zeta(\alpha) + 2\kappa \omega_3 = \pi \sqrt{-1} C_3,
\end{align*}

for constants $C_1$ and $C_3$. By Legendre’s relation, we have

\begin{align*}
(3.37) & \quad \alpha = C_3 \omega_1 - C_1 \omega_3, \\
(3.38) & \quad \kappa = \zeta(C_1 \omega_3 - C_3 \omega_1) + C_3 \eta_1 - C_1 \eta_3,
\end{align*}

(see Eqs. (2.38, 2.39)). From Proposition 3.1, the value $\varphi(\alpha)(= \varphi(C_3 \omega_1 - C_1 \omega_3))$ is expressed as a rational function in variables $b_1$ and $\mu_1$, the value $\varphi'(\alpha)(= \varphi'(C_3 \omega_1 - C_1 \omega_3))$ is expressed as a product of $\sqrt{-Q}$ and a rational function in variables $b_1$ and $\mu_1$, and the value $\kappa(= \zeta(C_1 \omega_3 - C_3 \omega_1) + C_3 \eta_1 - C_1 \eta_3)$ is expressed as a product of $\sqrt{-Q}$ and rational function in variables $b_1$ and $\mu_1$. By solving these equations for $b_1$ and $\mu_1$ and evaluating them into Eq. (3.2), the monodromy of the solutions on the cycles $x \to x + 2\omega_j$ ($j = 1, 3$) is preserved for the fixed values $C_1$ and $C_3$. Let $\gamma_0$ be the path in the $x$-plane which is obtained by the pullback of the cycle turning the origin around anti-clockwise in the $w$-plane, where $x$ and $w$ are related with $w = (\varphi(x) - e_1)/(e_2 - e_1)$. Then the monodromy matrix on $\gamma_0$ with respect to the basis $(\Lambda_\phi(x), \Lambda_\phi(-x))$ is written as

\begin{equation}
(3.39) \quad (\Lambda_\phi(x), \Lambda_\phi(-x)) \to (\Lambda_\phi(-x), \Lambda_\phi(x)) = (\Lambda_\phi(x), \Lambda_\phi(-x)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{equation}

and does not depend on $\tau$. Since the fundamental group on the punctured Riemann sphere $\mathbb{P}^1 \setminus \{0, 1, t, \infty\}$ is generated by the images of $\gamma_0$ and the cycles $x \to x + 2\omega_j$ ($j = 1, 3$), Eqs. (3.37, 3.38) describe the condition for the monodromy preserving deformation on the punctured Riemann sphere by rewriting the variable $\tau$ to $t$. Summarizing, we have the following proposition.

**Proposition 3.2.** We set $\omega_1 = 1/2, \omega_3 = \tau/2$ and assume that $l_i \in \mathbb{Z}_{\geq 0}$ ($i = 0, 1, 2, 3$) and $Q \neq 0$. By solving the equations in Proposition 3.1 in variable $b_1 = \varphi(\delta_1)$ and $\mu_1$, we express $\varphi(\delta_1)$ and $\mu_1$ in terms of $\varphi(\alpha)$, $\varphi'(\alpha)$ and $\kappa$, and we replace $\varphi(\alpha)$, $\varphi'(\alpha)$ and $\kappa$ with $\varphi(C_3 \omega_1 - C_1 \omega_3)$, $\varphi'(C_3 \omega_1 - C_1 \omega_3)$ and $\zeta(C_1 \omega_3 - C_3 \omega_1) + C_3 \eta_1 - C_1 \eta_3$. Then $\delta_1$ satisfies the sixth Painlevé equation in the elliptic form

\begin{equation}
(3.40) \quad \frac{d^2 \delta_1}{d\tau^2} = -\frac{1}{8\pi^2} \left\{ \sum_{i=0}^{3} (l_i + 1/2)^2 \varphi'(\delta_1 + \omega_i) \right\}.
\end{equation}

We observe the expressions of $b_1$ and $\mu_1$ in detail for the cases $l_0 = l_1 = l_2 = l_3 = 0$ and $l_0 = 1, l_1 = l_2 = l_3 = 0$. 
3.2. The case $M = 1$, $r_1 = 1$, $l_0 = l_1 = l_2 = l_3 = 0$. We investigate the case $M = 1$, $r_1 = 1$, $l_0 = l_1 = l_2 = l_3 = 0$ in detail. The differential equation (3.41) is written as

$$\left\{-\frac{d^2}{dx^2} + \frac{\varphi'(x)}{\varphi(x) - b_1} \frac{d}{dx} - \frac{\mu_1(4b_1^3 - g_2b_1 - g_3)}{\varphi(x) - b_1} - p\right\} f_g(x) = 0. \tag{3.41}$$

We assume that $b_1 \neq e_1, e_2, e_3$. The condition that the regular singular points $x = \pm \delta_1$ ($\varphi(\delta_1) = b_1$) are apparent is written as

$$p = -(4b_1^3 - g_2b_1 - g_3)\mu_1^2 + (6b_1^2 - g_2/2)\mu_1, \tag{3.42}$$

(see Eq.(3.6)). The doubly-periodic function $\Xi(x)$ (see Eq.(3.7)) which satisfies Eq.(2.19) is calculated as

$$\Xi(x) = 2\mu_1 + \frac{1}{\varphi(x) - b_1}. \tag{3.43}$$

The value $Q$ (see Eq.(2.21)) is calculated as

$$Q = 2\mu_1(2\mu_1(e_1 - b_1) + 1)(2(e_2 - b_1)\mu_1 + 1)(2\mu_1(e_3 - b_1) + 1). \tag{3.44}$$

We set

$$\Lambda_g(x) = \sqrt{\Xi(x)(\varphi(x) - b_1)} \exp \int \frac{\sqrt{-Q}dx}{\Xi(x)}, \tag{3.45}$$

(see Eq.(3.8)). Then a solution to Eq.(3.41) is written as $\Lambda_g(x)$, and is expressed in the form of the Hermite-Krichever Ansatz as

$$\Lambda_g(x) = b_0^{(0)} \exp(\kappa x) \Phi_0(x, \alpha) \tag{3.46}$$

for generic ($\mu_1, b_1$). The values $\alpha$ and $\kappa$ are determined as

$$\varphi(\alpha) = b_1 - \frac{1}{2\mu_1}, \quad \varphi'(\alpha) = -\frac{\sqrt{-Q}}{2\mu_1^2}, \quad \kappa = \frac{\sqrt{-Q}}{2\mu_1}. \tag{3.47}$$

Hence we have

$$\mu_1 = -\frac{\kappa}{\varphi'(\alpha)}, \quad b_1 = \varphi(\alpha) - \frac{\varphi'(\alpha)}{2\kappa}. \tag{3.48}$$

From Proposition 3.2 the function $\delta_1$ determined by

$$\varphi(\delta_1) = b_1 = \varphi(C_3\omega_1 - C_1\omega_3) - \frac{\varphi'(C_3\omega_1 - C_1\omega_3)}{2(\zeta(C_1\omega_3 - C_3\omega_1) - C_1\eta_3 + C_3\eta_1)} = \varphi(C_1\omega_3 - C_3\omega_1) + \frac{\varphi'(C_1\omega_3 - C_3\omega_1)}{2(\zeta(C_1\omega_3 - C_3\omega_1) - (C_1\eta_3 - C_3\eta_1))} \tag{3.49}$$

is a solution to the sixth Painlevé equation in the elliptic form (see Eq.(3.40)). This solution coincides with the one found by Hitchin [9] when he studied Einstein metrics and isomonodromy deformations.

Note that in [4, 13], solutions in terms of theta functions are obtained.

Now we consider the case $Q = 0$. If $Q = 0$, then $\mu_1 = 0$ or $\mu_1 = 1/(2(b_1 - e_i))$ for some $i \in \{1, 2, 3\}$.

If $\mu_1 = 0$, then a solution to Eq.(3.41) is $1(= \Lambda_g(x))$ and another solution is written as

$$\zeta(x) + b_1x = \int -(\varphi(x) - b_1)dx. \tag{3.50}$$

We investigate the monodromy preserving deformation on the basis $s_1(x) = B(\tau)$ and $s_2(x) = \zeta(x) + b_1x$, where $B(\tau)$ is a constant that is independent of $x$. The
monodromy matrix with respect to the path $\gamma_0$ is written as diag$(1, -1)$. Since $s_2(x + 2\omega_j) = s_2(x) + 2(\eta_j + \omega_j b_1) \ (j = 1, 3)$, the monodromy matrix with respect to the basis $(s_1(x), s_2(x))$ on the cycle $x \to x + 2\omega_j \ (j = 1, 3)$ is written as

$$
\begin{pmatrix}
1 & 2(\eta_j + \omega_j b_1)/B(\tau) \\
0 & 1
\end{pmatrix}.
$$

To preserve monodromy, the matrix elements should be constants of the variable $\tau(= \omega_3/\omega_1)$ up to simultaneous change of basis. Hence we obtain

$$
\begin{align*}
2(\eta_1 + \omega_j b_1) &= D_1 B(\tau), \\
2(\eta_3 + \omega_j b_1) &= D_3 B(\tau),
\end{align*}
$$

for some constants $D_1$ and $D_3$. By using Legendre’s relation, we obtain that $B(\tau) = \pi\sqrt{-1}/(D_1\omega_3 - D_3\omega_1)$ and

$$
\varphi(\delta_1) = b_1 = -\frac{D_1\eta_3 - D_3\eta_1}{D_1\omega_3 - D_3\omega_1}.
$$

Since Eq. (3.53) is obtained by monodromy preserving deformation, the function $\delta_1$ satisfies the sixth Painlevé equation.

If $\mu_1 = 1/(2(b_i - e_i))$ for some $i \in \{1, 2, 3\}$, then $\varphi_i(x)(= \Lambda(y(x)))$ is a solution to Eq. (3.41), and another solution is written as

$$
\varphi_i(x) \left\{ \frac{e_i - b_1}{(e_i - e_i')(e_i - e_i'')} \zeta(x + \omega_i) + (1 - \frac{e_i - b_1}{(e_i - e_i')(e_i - e_i''})x \right\} = \varphi_i(x) \int \frac{\varphi(x) - b_i}{\varphi(x) - e_i} dx,
$$

where $i'$ and $i''$ are elements in $\{1, 2, 3\}$ such that $i' \neq i, i'' \neq i$ and $i' < i''$. By calculating similarly to the case $\mu_1 = 0$, we obtain that the function $\delta_1$, which is determined by

$$
\varphi(\delta_1) = b_1 = \frac{(g_2/2 - 2e_i^3)(D_1\omega_3 - D_3\omega_1) + e_i(D_1\eta_3 - D_3\eta_1)}{e_i(D_1\omega_3 - D_3\omega_1) + (D_1\eta_3 - D_3\eta_1)},
$$

is a solution to the sixth Painlevé equation for constants $D_1$ and $D_3$.

We now show that Eqs. (3.53) (3.55) are obtained by suitable limits from Eq. (3.49). Set $(C_1, C_3) = (CD_1, CD_3)$ in Eq. (3.49) and consider the limit $C \to 0$, then we recover Eq. (3.53). Similarly, set $(C_1, C_3) = (CD_1, -1 + CD_3)$ (resp. $(C_1, C_3) = (-1 + CD_1, 1 + CD_3)$, $(C_1, C_3) = (1 + CD_1, CD_3)$) and consider the limit $C \to 0$, then we recover Eq. (3.55) for the case $i = 1$ (resp. $i = 2, i = 3$). Hence the space of the parameters of the solutions to the sixth Painlevé equation (i.e. the space of initial conditions) for the case $l_0 = l_1 = l_2 = l_3 = 0$ is obtained by blowing up four points on the surface $\mathbb{C}/2\mathbb{Z} \times \mathbb{C}/2\mathbb{Z}$, and this reflects the $A_1 \times A_1 \times A_1 \times A_1$ structure of Riccati solutions by Saito and Terajima [18].

3.3. The case $M = 1$, $r_1 = 1$, $l_0 = 1$, $l_1 = l_2 = l_3 = 0$. The differential equation (3.1) for this case is written as

$$
\begin{align*}
&-\frac{d^2}{dx^2} + \frac{\varphi'(x)}{\varphi(x) - b_1} \frac{d}{dx} - \frac{\mu_1(4b_1^3 - 2g_2b_1 - g_3)}{\varphi(x) - b_1} + 2\varphi(x) - p \left\{ f_\gamma(x) = 0,
\end{align*}
$$

We assume that $b_1 \neq e_1, e_2, e_3$. The condition that the regular singular points $x = \pm \delta_1$ ($\varphi(\delta_1) = b_1$) are apparent is written as

$$
p = -(4b_1^3 - g_2b_1 - g_3)\mu_1^2 + (6b_1^2 - g_2/2)\mu_1 + 2b_1,
$$

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(see Eq. (3.6)). The doubly-periodic function \( \Xi(x) \) (see Eq. (3.7)), which satisfies Eq. (2.19), is calculated as
\[
\Xi(x) = \varphi(x) + ((-4b_1^3 + b_1g_2 + g_3)\mu_1^2 + (6b_1^2 - g_2/2)\mu_1 - b_1)
+ ((-4b_1^3 + b_1g_2 + g_3)\mu_1/2 + 3b_1^2 - g_2/4)/(\varphi(x) - b_1).
\]

The value \( Q \) (see Eq. (2.21)) is calculated as
\[
Q = -((2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4)(2(b_1^2 + e_1b_1 + e_2e_3)\mu_1 - 2b_1 - e_1)
(2(b_1^2 + e_2b_1 + e_1e_2)\mu_1 - 2b_1 - e_2)(2(b_1^2 + e_3b_1 + e_1e_3)\mu_1 - 2b_1 - e_3).
\]

We set
\[
\Lambda_g(x) = \sqrt{\Xi(x)(\varphi(x) - b_1)} \exp \int \frac{\sqrt{-Q} \, dx}{\Xi(x)}.
\]

Then a solution to Eq. (3.56) is written as \( \Lambda_g(x) \), and it is expressed in the form of the Hermite-Krichever Ansatz as
\[
\Lambda_g(x) = \exp(\kappa x) \left\{ \tilde{b}_0^{(0)} \Phi_0(x, \alpha) + \tilde{b}_1^{(0)} \frac{d}{dx} \Phi_0(x, \alpha) \right\}
\]
for generic \( (\mu_1, b_1) \). The values \( \alpha \) and \( \kappa \) are determined as
\[
\varphi(\alpha) = \frac{2(4b_1^3 - b_1g_2 - g_3)b_1\mu_1^3 + (-24b_1^3 + 4g_2b_1 + 3g_3)\mu_1^2 + (24b_1^2 - 2g_2)\mu_1 - 8b_1}{2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4},
\]
\[
\varphi'(\alpha) = \frac{-4((4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 12b_1\mu_1 - 4)}{(2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4)^2} \sqrt{-Q},
\]
\[
\kappa = \frac{2\mu_1}{2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4} \sqrt{-Q}.
\]

Hence we have
\[
b_1 = \frac{2\varphi(\alpha)\kappa^3 - 3\varphi'(\alpha)\kappa^2 + (6\varphi(\alpha)^2 - g_2)\kappa - \varphi(\alpha)\varphi'(\alpha)}{2(\kappa^3 - 3\varphi(\alpha)\kappa + \varphi'(\alpha))},
\]
\[
\mu_1 = \frac{-2\varphi'(\alpha)\kappa^3 + (12\varphi(\alpha)^2 - g_2)\kappa^2 - 6\varphi(\alpha)\varphi'(\alpha)\kappa + \varphi'(\alpha)^2}{2(\kappa^3 - 3\varphi(\alpha)\kappa + \varphi'(\alpha))\kappa}.
\]

From Proposition 3.2 the function \( \delta_1 \) determined by
\[
\varphi(\delta_1) = b_1 = \frac{2\varphi(\omega)\zeta(\omega) - \eta)^3 + 3\varphi'(\omega)(\zeta(\omega) - \eta)^2 + (6\varphi(\omega)^2 - g_2)(\zeta(\omega) - \eta) + \varphi(\omega)\varphi'(\omega)}{2((\zeta(\omega) - \eta)^3 - 3\varphi(\omega)(\zeta(\omega) - \eta) - \varphi'(\omega))},
\]
\[
(\omega = C_1\omega_3 - C_3\omega_1, \quad \eta = C_1\eta_3 - C_3\eta_1),
\]
is a solution to the sixth Painlevé equation in the elliptic form (see Eq. (3.40)). In the sixth Painlevé equation, it is known that the case \( (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (1/2, 1/2, 1/2, 3/2) \) is linked to the case \( (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (1/2, 1/2, 1/2, 1/2) \) by Bäcklund transformation. For a table of Bäcklund transformation of the sixth Painlevé equation, see...
By transforming the solution in Eq. (3.49) of the case \((\kappa_0, \kappa_1, \kappa_2, \kappa_3) = (1/2, 1/2, 1/2, 1/2)\) to the one of the case \((\kappa_0, \kappa_1, \kappa_2, \kappa_3) = (1/2, 1/2, 1/2, 3/2)\), we recover the solution in Eq. (3.67).

Now we consider the case \(Q = 0\). If \(Q = 0\), then \(\mu_1\) is a solution to the equation \(2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4 = 0\) or \(\mu_1 = (2b_1 + e_i)/(2(b_1^2 + e_i b_1 + e_1^2 - g_2/4))\) for some \(i \in \{1, 2, 3\}\). We set \(\omega = D_1\omega_3 - D_3\omega_1\) and \(\eta = D_1\eta_3 - D_3\eta_1\), where \(D_1\) and \(D_3\) are constants. For the case that \(\mu_1\) is a solution to the equation \(2(4b_1^3 - b_1g_2 - g_3)\mu_1^3 - (12b_1^2 - g_2)\mu_1^2 + 4 = 0\), the corresponding solutions to the sixth Painlevé equation are written as the function \(\delta_1\), where

\[
\varphi(\delta_1) = b_1 = \frac{4\eta^3 + g_2\omega^2 \eta - 2g_3\omega^3}{\omega(2g_2\omega^2 - 12\eta^2)}.
\]

For the case \(\mu_1 = (2b_1 + e_i)/(2(b_1^2 + e_i b_1 + e_1^2 - g_2/4))\) \((i \in \{1, 2, 3\})\), we have

\[
\varphi(\delta_1) = b_1 = \frac{-g_2 e_i \omega^2 + (6e_1^2 - g_2)\eta}{(6e_1^2 - g_2)\omega - 6e_i \eta}.
\]

Note that these solutions are also obtained by suitable limits from Eq. (3.67), and Eq. (3.68) (resp. Eq. (3.69)) is transformed by Bäcklund transformation from Eq. (3.55).

4. Relationship with finite-gap potential

4.1. Finite-gap property. We investigate the condition that the potential in Eq. (2.5) is finite-gap.

If \(M = 0\) \((M\) is the number of additional apparent singularities) and \(l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}\), then the potential is called the Treibich-Verdier potential, and it is algebra-geometric finite-gap.

Next we consider the case \(M = 1\) and \(r_1 = 2\). Set \(b_1 = \varphi(\delta_1)\). The condition that the regular singularity \(x = \pm \delta_1\) of Eq. (2.3) is apparent (which is equivalent to that the regular singularity \(z = b_1\) of Eq. (2.4) is apparent) is written as

\[
s_1^3 + (12b_1^2 - g_2)s_1^2 + (4(4b_1^3 - g_2b_1 - g_3)E + f_1(b_1))s_1 + f_0(b_1) = 0.
\]

where \(f_1(b_1)\) and \(f_0(b_1)\) are given by

\[
f_1(b_1) = -2(2l_0^2 + 2l_0 + 5)b_1(4b_1^3 - g_2b_1 - g_3) + (6b_1^2 - g_2/2)^2
- 8(2l_1^2 + 2l_1 + 1)(b_1 - e_2)(b_1 - e_3)(e_1b_1 + e_1^2 + e_2 e_3)
- 8(2l_2^2 + 2l_2 + 1)(b_1 - e_1)(b_1 - e_3)(e_2b_1 + e_2^2 + e_1 e_3)
- 8(2l_3^2 + 2l_3 + 1)(b_1 - e_1)(b_1 - e_2)(e_3b_1 + e_3^2 + e_1 e_2),
\]

\[
f_0(b_1) = (2l_0 + 1)^2(4b_1^3 - g_2b_1 - g_3)^2
- 16(2l_1 + 1)^2(e_1 - e_2)(e_1 - e_3)(b_1 - e_2)^2(b_1 - e_3)^2
- 16(2l_2 + 1)^2(e_2 - e_1)(e_2 - e_3)(b_1 - e_1)^2(b_1 - e_3)^2
- 16(2l_3 + 1)^2(e_3 - e_1)(e_3 - e_2)(b_1 - e_1)^2(b_1 - e_2)^2.
\]

If \(s_1 = 0\), then we obtain an equation

\[
f_0(b_1) = 0.
\]

Remarkably, the value \(b_1\) determined by this equation does not depend on the value \(E\). It is shown by Treibich [27] that, if \(M = 1\), \(r_1 = 2\), \(s_1 = 0\), \(l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}\) and
Proposition 4.1. Assume that \( b_1(= \phi(\delta_1)) \) satisfies Eq.(4.3) (which is equivalent to Eq.(1.8) by setting \( \delta_1 = \delta \)), then the potential is algebro-geometric finite-gap.

In this section, we investigate the differential equation

\[
(4.5) \quad \left(-\frac{d^2}{dx^2} + v(x)\right) f(x) = Ef(x),
\]

for the case when the regular singular points \( x = \pm \delta_{i'} \) (\( i' = 1, \ldots , M \)) of Eq.(1.5) are apparent, \( \delta_j \not\equiv \omega_i \mod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z} \) (\( 0 \leq i \leq 3, 1 \leq j \leq M \)) and \( \delta_j \pm \delta_{j'} \not\equiv 0 \mod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z} \) (\( 1 \leq j < j' \leq M \)). Note that the potential in Eq.(4.5) corresponds to the one in Eq.(2.5) with conditions \( r_{i'} = 2 \) and \( s_{i'} = 0 \) (\( i' = 1, \ldots , M \)). The exponents of Eq.(4.5) at the regular singularity \( x = \pm \delta_{i'} \) (\( i' = 1, \ldots , M \)) are \(-1\) and \(2\).

**Proposition 4.1.** Assume that \( l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}, \delta_j \not\equiv \omega_i \mod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z} \) (\( 0 \leq i \leq 3, 1 \leq j \leq M \)) and \( \delta_j \pm \delta_{j'} \not\equiv 0 \mod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z} \) (\( 1 \leq j < j' \leq M \)). If the values \( \delta_1, \ldots , \delta_M \) satisfy the equation

\[
(4.6) \quad 2 \sum_{j' \neq j} (\phi'(\delta_j - \delta_{j'}) + \phi'(\delta_j + \delta_{j'})) + 3 \sum_{i=0} l_i (l_i + 1/2)^2 \phi' \omega_i = 0 \quad (j = 1, \ldots , M),
\]

then the regular singular points \( x = \pm \delta_{i'} \) (\( i' = 1, \ldots , M \)) of Eq.(4.5) are apparent.

**Proof.** We show that, if \( \delta_1, \ldots , \delta_M \) satisfy Eq.(4.6), then the regular singular points \( x = \delta_j \) (\( j = 1, \ldots , M \)) are apparent. The coefficients of expansion around \( x = \delta_j \) of Eq.(4.5) written in the form of Eq.(2.15) \((a = \delta_j)\) are given by

\[
(4.7) \quad p_k = 0 \quad (k \in \mathbb{Z}_{\geq 0}), \quad q_0 = -2, \quad q_1 = 0,
\]

\[
q_2 = E - \left(2\phi(2\delta_j) + 2 \sum_{j' \neq j} (\phi(\delta_j - \delta_{j'}) + \phi(\delta_j + \delta_{j'})) + 3 \sum_{i=0} l_i (l_i + 1) \phi(\delta_j + \omega_i)\right),
\]

\[
q_3 = - \left(2\phi'(2\delta_j) + 2 \sum_{j' \neq j} (\phi'(\delta_j - \delta_{j'}) + \phi'(\delta_j + \delta_{j'})) + 3 \sum_{i=0} l_i (l_i + 1) \phi'(\delta_j + \omega_i)\right),
\]

and the characteristic polynomial \( F(t) \) at \( x = \delta_j \) is written as \( F(t) = (t - 2)(t + 1) \).

In section 2.1, we obtained a condition for apparentness of a regular singular point. On the case \( x = \delta_j \), it is written as

\[
(4.8) \quad (-p_3 + q_3)c_0 + q_2c_1 + (p_1 + q_1)c_2 = 0,
\]

with \( c_0 = 1, -2c_1 + (-p_1 + q_1)c_0 = 0, -2c_2 + (-p_2 + q_2)c_0 + q_1c_1 = 0 \) (see Eqs.(2.18, 2.17)). Hence \( c_1 = 0 \) and the condition that \( x = \delta_j \) is apparent is written as \( q_3 = 0 \), i.e.,

\[
(4.9) \quad 2\phi'(2\delta_j) + 2 \sum_{j' \neq j} (\phi'(\delta_j - \delta_{j'}) + \phi'(\delta_j + \delta_{j'})) + 3 \sum_{i=0} l_i (l_i + 1) \phi'(\delta_j + \omega_i) = 0.
\]

From the identity

\[
(4.10) \quad 8\phi'(2x) = \phi'(x) + \phi'(x + \omega_1) + \phi'(x + \omega_2) + \phi'(x + \omega_3),
\]

we have

\[
(4.11) \quad 2\phi'(2\delta_j) + 2 \sum_{j' \neq j} (\phi'(\delta_j - \delta_{j'}) + \phi'(\delta_j + \delta_{j'})) + 3 \sum_{i=0} l_i (l_i + 1) \phi'(\delta_j + \omega_i) = 0.
\]
Eq. (4.9) is equivalent to Eq. (4.6). The condition that \( x = -\delta_j \) is apparent is given by

\[
2\varphi'(-2\delta_j) + 2 \sum_{j' \neq j} (\varphi'(-\delta_j - \delta_{j'}) + \varphi'(-\delta_j + \delta_{j'})) + \sum_{i=0}^{3} l_i (l_i + 1) \varphi'(-\delta_j + \omega_i) = 0,
\]

and it is equivalent to Eq. (4.6) by the oddness and the double-periodicity of the function \( \varphi'(x) \). Therefore, if Eq. (4.6) is satisfied, then the points \( x = \pm\delta_j \) \( (j = 1, \ldots, M) \) are apparent.

It is remarkable that Eq. (4.6) does not contain the variable \( E \). We examine this equation with the introduction of

\[
\Phi(\delta_1, \ldots, \delta_M) = 2 \sum_{1 \leq j_1 < j_2 \leq M} (\varphi(\delta_{j_1} - \delta_{j_2}) + \varphi(\delta_{j_1} + \delta_{j_2})) + \sum_{j=1}^{M} \sum_{i=0}^{3} (l_i + 1/2)^2 \varphi(\delta_j + \omega_i),
\]

in which case Eq. (4.6) is equivalent to the equations

\[
\frac{\partial}{\partial \delta_j} \Phi(\delta_1, \ldots, \delta_M) = 0 \quad (j = 1, \ldots, M).
\]

We will now show that Eq. (4.6) has a good solution.

**Proposition 4.2.** Assume that \( l_0, l_1 \in \mathbb{R} \setminus \{-1/2\}, l_2, l_3 \in \mathbb{R}, \omega_1 \in \mathbb{R}_{>0} \) and \( \omega_3 \in \sqrt{-1} \mathbb{R}_{>0} \). Then Eq. (4.6) has a solution such that \( \delta_j \in \mathbb{R}, \delta_j \not\equiv \omega_i \mod 2\omega_1 Z + 2\omega_3 Z \) \( (0 \leq i \leq 3, 1 \leq j \leq M) \) and \( \delta_j \pm \delta_j' \not\equiv 0 \mod 2\omega_1 Z + 2\omega_3 Z \) \( (1 \leq j < j' \leq M) \).

**Proof.** From the assumption \( \omega_1 \in \mathbb{R}_{>0} \) and \( \omega_3 \in \sqrt{-1} \mathbb{R}_{>0} \), the functions \( \varphi(x + \omega_i) \) \( (i = 0, 1, 2, 3) \) are real-valued for \( x \in \mathbb{R} \setminus \omega_1 Z \) and \( \lim_{x \to 0, x \in \mathbb{R}} \varphi(x) = \lim_{x \to \omega_1, x \in \mathbb{R}} \varphi(x + \omega_1) = +\infty \). Now we consider the function \( \Phi(\delta_1, \ldots, \delta_M) \) on the real domain \( D = \{(\delta_1, \ldots, \delta_M) \in \mathbb{R}^M | 0 < \delta_1 < \cdots < \delta_M < \omega_1, \delta_j + \delta_j' < \omega_1 (\forall j, j' \text{ s.t. } j < j')\} \). Then \( \Phi(\delta_1, \ldots, \delta_M) \) is real-valued and continuous on the domain \( D \). As \( (\delta_1, \ldots, \delta_M) \) tends to the boundary of the domain \( D \), the value \( \Phi(\delta_1, \ldots, \delta_M) \) tends to \( +\infty \) by the assumption \( l_0 \neq -1/2, l_1 \neq -1/2 \). Since \( \varphi(x + \omega_i) \geq \min(e_1, e_2, e_3) = e_3 \) for \( x \in \mathbb{R} \) and \( i = 0, 1, 2, 3 \), we have \( \Phi(\delta_1, \ldots, \delta_M) \geq (2M - 2 + \sum_{i=0}^{3} (l_i + 1/2)^2)Me_3 \). Therefore the function \( \Phi(\delta_1, \ldots, \delta_M) \) has a minimum value at \( \exists (\delta_1^0, \ldots, \delta_M^0) \in D \). Since \( (\delta_1^0, \ldots, \delta_M^0) \) is an extremal point of the function \( \Phi(\delta_1, \ldots, \delta_M) \), it satisfies Eq. (4.13). Hence \( (\delta_1^0, \ldots, \delta_M^0) \) is a solution to Eq. (4.6).

Because \( (\delta_1^0, \ldots, \delta_M^0) \in D \), it satisfies \( \delta_j^0 \not\equiv \omega_i \mod 2\omega_1 Z + 2\omega_3 Z \) \( (0 \leq i \leq 3, 1 \leq j \leq M) \) and \( \delta_j^0 \pm \delta_j^0' \not\equiv 0 \mod 2\omega_1 Z + 2\omega_3 Z \) \( (1 \leq j < j' \leq M) \).

Upon introducing \( b_j = \varphi(\delta_j) \) \( (j = 1, \ldots, M) \), it follows from the relations

\[
\varphi'(x + y) + \varphi'(x - y) = \frac{\varphi'(x)\varphi''(y)}{(\varphi(x) - \varphi(y))^2} - \frac{2\varphi'(x)\varphi'(y)^2}{(\varphi(x) - \varphi(y))^3},
\]

\[
\varphi'(x + \omega_i) = -\frac{3e_i^2 - g_2/4}{(\varphi(x) - e_i)^2} \varphi'(x), \quad (i = 1, 2, 3),
\]
that Eq. (4.6) may be expressed in the algebraic form

$$\sum_{j' \neq j} \left\{ \frac{12b_{j'}^2 - g_2}{(b_j - b_{j'})^2} + \frac{4(4b_{j'}^3 - g_2b_{j'} - g_3)}{(b_j - b_{j'})^3} \right\}$$

$$= (l_0 + 1/2)^2 - \sum_{i=1}^{M} (l_i + 1/2)^2 \frac{3\epsilon_i^2 - g_2/4}{(b_j - e_i)^2}, \quad (j = 1, \ldots, M),$$

under the condition $b_j \neq e_1, e_2, e_3 \ (j = 1, \ldots, M)$. For the case $M = 1$, it is written as Eq. (4.4).

If $\delta_1, \ldots, \delta_M$ satisfy Eq. (4.6), then the regular singular points $x = \pm \delta_i$ ($i' = 1, \ldots, M$) of Eq. (4.5) are apparent for all $E$, and it follows from Proposition 2.1 that there exists a non-zero solution to the third-order differential equation satisfied by products of two solutions to Eq. (4.5). Namely, if $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$, $\delta_1, \ldots, \delta_M$ satisfy Eq. (4.6), $\delta_j \neq \omega_i$ mod $2\omega_1Z \oplus 2\omega_3Z$ ($0 \leq i \leq 3, 1 \leq j \leq M$) and $\delta_j \pm \delta_{j'} \neq 0$ mod $2\omega_1Z \oplus 2\omega_3Z$ ($1 \leq j < j' \leq M$), then Eq. (2.19) has an even non-zero doubly-periodic solution that has the expansion

$$\Xi(x) = c_0 + \sum_{i=0}^{3} \sum_{j=0}^{l_i-1} b_{j}^{(i)} \varphi(x + \omega_i)^{l_i-j} + \sum_{i'=1}^{M} \left( \frac{d_{0}^{(i')}}{(\varphi(x) - \varphi(\delta_{i'}))^2} + \frac{d_{1}^{(i')}}{(\varphi(x) - \varphi(\delta_{i'}))} \right),$$

for all $E$. On the present situation, we can improve Proposition 2.1.

**Proposition 4.3.**

(i) For each $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$, periods $(2\omega_1, 2\omega_3)$ and values $\delta_1, \ldots, \delta_M$, the number of eigenvalues $E$, such that the dimension of the space of even doubly-periodic solutions to Eq. (2.19) is no less than two, is finite.

(ii) If $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$, $\delta_1, \ldots, \delta_M$ satisfy Eq. (4.6), $\delta_j \neq \omega_i$ mod $2\omega_1Z \oplus 2\omega_3Z$ ($0 \leq i \leq 3, 1 \leq j \leq M$) and $\delta_j \pm \delta_{j'} \neq 0$ mod $2\omega_1Z \oplus 2\omega_3Z$ ($1 \leq j < j' \leq M$), then Eq. (2.19) has a unique non-zero doubly-periodic solution $\Xi(x, E)$, which has the expansion

$$\Xi(x, E) = c_0(E) + \sum_{i=0}^{3} \sum_{j=0}^{l_i-1} b_{j}^{(i)}(E) \varphi(x + \omega_i)^{l_i-j} + \sum_{i'=1}^{M} d_{0}^{(i')}(E)(\varphi(x + \delta_{i'}) + \varphi(x - \delta_{i'})),$$

where the coefficients $c_0(E), b_{j}^{(i)}(E)$ and $d_{0}^{(i')}(E)$ are polynomials in $E$ such that these polynomials do not share any common divisors and the polynomial $c_0(E)$ is monic.

We set $g = \deg_E c_0(E).$ Then the coefficients satisfy $\deg_E b_{j}^{(i)}(E) < g$ for all $i$ and $j$, and $\deg_E d_{0}^{(i')}(E) < g$ for all $i'$.

**Proof.** By substituting Eq. (4.17) into Eq. (2.19), we derive linear equations in coefficients $c_0, b_{j}^{(i)}, d_{0}^{(i')}$ and $d_{1}^{(i')}$ to satisfy Eq. (2.19). We replace $c_0, b_{j}^{(i)}, d_{0}^{(i')}, d_{1}^{(i')}$ with $\tilde{c}_1, \ldots, \tilde{c}_M$ ($M' = 1 + l_0 + l_1 + l_2 + l_3 + 2M$). Then the linear equations are written as

$$\sum_{i=1}^{M'} (m_{k,i}E + n_{k,i})\tilde{c}_i = 0, \quad (k = 1, \ldots, M''),$$

where $M''$ is the number of equations. It follows from Proposition 2.1 that there exists a non-zero solution to Eq. (2.19). Hence all minors of the matrix $(m_{k,i}E + n_{k,i})_{k,i}$ of rank $M'$ are identically zero.
Now we assume that there exists infinitely-many values $E$ such that the dimension of the space of even doubly-periodic solutions to Eq. (2.19) is no less than two. Since any minors of the matrix $(m_{k,i}E + n_{k,i})_{k,i}$ of rank $M' - 1$ are written as polynomials in $E$, and they must be zero at infinitely-many values of $E$ by the assumption, they are identically zero. Hence the dimension of the space of even doubly-periodic solutions to Eq. (2.19) is no less than two for all $E$. Because the coefficients of Eq. (4.19) are written as polynomials in $E$, there exist linearly independent functions $\Xi^{(1)}(x, E)$ and $\Xi^{(2)}(x, E)$ which solve Eq. (2.19) and may be expressed in the form

$$\Xi^{(k)}(x, E) = \sum_{j=0}^{g_k} a_j^{(k)}(x) E^{g_k-j}, \quad a_0^{(k)}(x) \neq 0, \quad (k = 1, 2).$$

**Lemma 4.4.** If $\tilde{\Xi}(x)$ is a solution of Eq. (2.14) written in the form $\tilde{\Xi}(x) = \sum\limits_{i=0}^{\tilde{g}} \tilde{a}_i(x) E^{\tilde{g}-i}$ ($\tilde{a}_0(x) \neq 0$), then $\tilde{a}_0(x)$ is independent of $x$.

**Proof.** By substituting $\tilde{\Xi}(x)$ into Eq. (2.19) and considering the coefficients of $E^{\tilde{g}+1}$, we obtain that $\tilde{a}_0'(x) = 0$. Hence $\tilde{a}_0(x)$ is independent of $x$. \hfill \Box

By the lemma, $a_0^{(0)}(x)$ and $a_0^{(1)}(x)$ are independent of $x$, and so we may denote them by $a_0^{(0)}$ and $a_0^{(1)}$. If $g_1 \geq g_2$ (resp. $g_1 < g_2$), then we set $\tilde{\Xi}^{(1)}(x, E) = \Xi^{(1)}(x, E) - (a_0^{(1)}/a_0^{(2)})\Xi^{(2)}(x, E)E^{g_1-g_2}$, $\tilde{\Xi}^{(2)}(x, E) = \Xi^{(2)}(x, E)$ (resp. $\tilde{\Xi}^{(1)}(x, E) = \Xi^{(1)}(x, E) - (a_0^{(2)}/a_0^{(1)})\Xi^{(1)}(x, E)E^{g_2-g_1}$). Then the degree of either $\tilde{\Xi}^{(1)}(x, E)$ or $\tilde{\Xi}^{(2)}(x, E)$ in $E$ decreases from the one of $\Xi^{(1)}(x, E)$ or $\Xi^{(2)}(x, E)$, and so $\tilde{\Xi}^{(1)}(x, E)$ and $\tilde{\Xi}^{(2)}(x, E)$ are linearly-independent solutions to Eq. (2.19). From Lemma 4.4, the top terms of $\tilde{\Xi}^{(1)}(x, E)$ and $\tilde{\Xi}^{(2)}(x, E)$ in $E$ are non-zero constants. By the same procedure, we can construct functions $\tilde{\Xi}^{(1)}(x, E)$ and $\tilde{\Xi}^{(2)}(x, E)$ which are linearly independent solutions to Eq. (2.19) and the degree of either $\tilde{\Xi}^{(1)}(x, E)$ or $\tilde{\Xi}^{(2)}(x, E)$ in $E$ decreases from the one of $\tilde{\Xi}^{(1)}(x, E)$ or $\tilde{\Xi}^{(2)}(x, E)$. By repeating this decreasing procedure, we find that there exist linearly-independent solutions to Eq. (2.19) such that their degrees in $E$ are zero. This is a contradiction, because if a solution, $f(x)$, to Eq. (2.19) is independent of $E$, then $f'(x) = v'(x) = 0$. Therefore we have that the number of eigenvalues $E$, such that the dimension of the space of even doubly-periodic solutions to Eq. (2.19) is no less than two, is finite at most.

From Proposition 2.1, there exist non-zero solutions to Eq. (2.19). Thus Eq. (4.19) has a non-zero solution for all $E$. Because the coefficients in Eq. (4.19) are polynomial in $E$, a solution to Eq. (4.19) is written in terms of rational functions in $E$. By multiplying by an appropriate term, a solution to Eq. (4.19) (i.e., $c_0, b_j^{(i)}, d_0^{(i)}, d_1^{(i)}$) may be expressed by polynomials in $E$ which do not share a common divisor, and they are determined uniquely up to scalar multiplication, because the dimension of solutions is one. We denote the doubly-periodic function uniquely determined in this way by $\Xi(x, E)$. By combining with the relation

$$\varphi(x + \delta) + \varphi(x - \delta) = 2\varphi(\delta) + \frac{\varphi''(\delta)}{\varphi(x) - \varphi(\delta)} + \frac{\varphi'(\delta)^2}{(\varphi(x) - \varphi(\delta))^2};$$

(4.21)
the function $\Xi(x, E)$ is expressed as
\begin{equation}
\Xi(x, E) = c_0(E) + \sum_{i=0}^{3} \sum_{j=0}^{l_i-1} b_j^{(i)}(E) \phi(x + \omega_i)^{l_i-j}
\end{equation}
\begin{equation}
\quad + \sum_{i'=1}^{M} \left( d^{(i')}(E) (\phi(x + \delta_{i'}) + \phi(x - \delta_{i'})) + \frac{d^{(i')}_1(E)}{(\phi(x) - \phi(\delta_{i'}))} \right),
\end{equation}
where $c_0(E)$, $b_j^{(i)}(E)$, $d^{(i')}(E)$ and $d^{(i')}_1(E)$ are polynomials in $E$ which do not share a common divisor. At $x = \delta_{i'}$ we have the expansion
\begin{equation}
\Xi(x, E) = \frac{d^{(i')}(E)}{(x - \delta_{i'})^2} + \frac{d^{(i')}_1(E)}{\phi'(\delta_{i'})(x - \delta_{i'})} + \text{(holomorphic at } x = \delta_{i'}).\end{equation}
By substituting this expansion into Eq.(2.19), we obtain the equality $d^{(i')}_1(E) = 0$ upon observing the coefficient of $1/(x - \delta_{i'})^4$. Hence we obtain the expression (4.18).

We express the function $\Xi(x, E)$ in descending order of powers of $E$. From Lemma 4.4, the top term is constant, hence the degrees of the coefficients in Eq.(4.18), other than $c_0(E)$, are strictly less than the degree of the function $\Xi(x, E)$ in $E$. Therefore $c_0(E) \neq 0$, $\deg_E b_j^{(i)}(E) < \deg_E c_0(E)$ and $\deg_E d^{(i')}(E) < \deg_E c_0(E)$ for all $i$, $i'$, $j$. By multiplying by a constant, $c_0(E)$ is normalized to be monic. Thus we obtain (ii).

If there exists an odd-order differential operator $A = (d/dx)^{2g+1} + \sum_{j=0}^{2g-1} b_j(x) (d/dx)^{2g-1-j}$ such that $[A, -d^2/dx^2 + q(x)] = 0$, then $q(x)$ is called the algebro-geometric finite-gap potential. Note that the equation $[A, -d^2/dx^2 + q(x)] = 0$ is equivalent to the function $q(x)$ being a solution to a stationary higher-order KdV equation.

Now we construct the commuting operator, $A$, for the operator $-d^2/dx^2 + v(x)$ ($v(x)$ defined in Eq.(4.5)) by using an expansion of the function $\Xi(x, E)$ in $E$. Write
\begin{equation}
\Xi(x, E) = \sum_{i=0}^{g} a_{g-i}(x) E^i.
\end{equation}
It follows from Proposition 4.3 that $a_0(x) = 1$. Since the function $\Xi(x, E)$ in Eq.(4.24) satisfies the differential equation (2.19), we obtain the following relations by equating the coefficients of $E^{g-j}$:
\begin{equation}
A_j''(x) - 4v(x) a_j'(x) - 2v'(x) a_j(x) + 4a_j'_{j+1}(x) = 0.
\end{equation}

**Theorem 4.5.** Assume that $l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$, values $\delta_1, \ldots, \delta_M$ satisfy Eq.(4.6), $\delta_j \not\equiv \omega_i \bmod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z}$ ($0 \leq i \leq 3$, $1 \leq j \leq M$) and $\delta_j \pm \delta_{j'} \not\equiv 0 \bmod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z}$ ($1 \leq j < j' \leq M$). Let $v(x)$ be the function defined in Eq.(4.5). Define the $(2g+1)$st-order differential operator $A$ by
\begin{equation}
A = \sum_{j=0}^{g} \left\{ a_j(x) \frac{d}{dx} - \frac{1}{2} \left( \frac{d}{dx} a_j(x) \right) \right\} \left( -\frac{d^2}{dx^2} + v(x) \right)^{g-j},
\end{equation}
where the $a_j(x)$ are defined in Eq.(4.24). Then the operator $A$ commutes with the operator $H = -d^2/dx^2 + v(x)$. In other words, the function $v(x)$ is an algebro-geometric finite-gap potential.
Proof. The commutativity of the operators $A$ and $H$ follow from Eq. (4.25). See also [25, Theorem 3.1]. □

Upon setting

$$Q(E) = Ξ(x, E)^2 (E - v(x)) + \frac{1}{2} Ξ(x, E) \frac{d^2 Ξ(x, E)}{dx^2} - \frac{1}{4} \left( \frac{dΞ(x, E)}{dx} \right)^2,$$

it is shown similarly to Eq. (2.22) that $Q(E)$ is independent of $x$. By definitions of $Ξ(x, E)$ and $Q(E)$, $Q(E)$ is a monic polynomial in $E$ of degree $2g + 1$. The following proposition is proved by reviewing [25, Proposition 3.2]:

**Proposition 4.6.** Let $H$ be the operator $-d^2/dx^2 + v(x)$, $A$ be the operator defined by Eq. (4.26) and $Q(E)$ be the polynomial defined in Eq. (4.27). Then

$$A^2 + Q(H) = 0.$$  

(4.28)

We now relate the present work to Picard’s potential. Let $q(x)$ be an elliptic function. If the differential equation $(-d^2/dx^2 + q(x))f(x) = Ef(x)$ has a meromorphic fundamental system of solutions with respect to $x$ for all values of $E$, then $q(x)$ is called a Picard potential (see [5]). It is known that, under the condition that $q(x)$ is an elliptic function, $v(x)$ is a Picard potential if and only if $q(x)$ is an algebro-geometric potential (see [7] and the references therein). Hence the function $v(x)$ defined in Eq. (4.15) with Eq. (4.16) is a Picard potential. It is possible to prove directly that $v(x)$ is a Picard potential by combining Lemma 2.2 and the apparent of singularities at $x = ±δ_i$ (i.e. $i = 1, ..., M$) ensured by Eq. (4.6).

**4.2. Monodromy and hyperelliptic integral.** We obtain an integral representation of solutions to the differential equation (4.5), and express the monodromy in terms of a hyperelliptic integral. Throughout this subsection, we assume that $l_0, l_1, l_2, l_3 ∈ \mathbb{Z}_{>0}$, $δ_1, ..., δ_M$ satisfy Eq. (1.6), $δ_j ≠ ω_i \mod 2ω_1\mathbb{Z} ⊕ 2ω_2\mathbb{Z}$ ($0 ≤ i ≤ 3$, $1 ≤ j ≤ M$) and $δ_j ± δ_{j'} ≠ 0 \mod 2ω_1\mathbb{Z} ⊕ 2ω_2\mathbb{Z}$ ($1 ≤ j < j' ≤ M$).

An integral representation of solutions is obtained in Proposition 2.3. Namely, the function

$$Λ(x, E) = \sqrt{Ξ(x, E)} \exp \left( \int \frac{-Q(E) dx}{Ξ(x, E)} \right),$$

is a solution to the differential equation (1.5).

Assume that the value $E_0$ satisfies $Q(E_0) = 0$. Then it follows from Proposition 2.7 that the function $Λ(x, E_0)$ is doubly-periodic up to signs, i.e., $Λ(x + 2ω_k, E_0)/Λ(x, E_0) ∈ \{±1\}$ ($k = 1, 3$). In [25, Theorem 3.7] the monodromy of solutions to Heun’s equation for the case $l_0, l_1, l_2, l_3 ∈ \mathbb{Z}$ is calculated in terms of a hyperelliptic integral. Similarly, we can calculate the monodromy of solutions to Eq. (4.5) in terms of a hyperelliptic integral.

**Proposition 4.7.** (c.f. [25, Theorem 3.7]) Assume that $E_0$ satisfies $Q(E_0) = 0$. Then there exist $q_1, q_3 ∈ \{0, 1\}$ such that $Λ(x + 2ω_k, E_0) = (-1)^{q_k} Λ(x, E_0)$ and

$$Λ(x + 2ω_k, E) = (-1)^{q_k} Λ(x, E) \exp \left( -\frac{1}{2} \int_{E_0}^{E} \int_{0+\varepsilon}^{2ω_k+\varepsilon} \frac{Ξ(x, E') dx}{\sqrt{-Q(E')}} dE' \right)$$

for $k = 1, 3$ with $\varepsilon$ denoting a constant chosen in order to avoid passing through the poles in the integration.
Proof. This proposition is proved by analogous argument to the proof of [25, Theorem 3.7]. □

We express Eq. (4.30) more explicitly. Since the function \( \varphi(x)^n \) is written as a linear combination of the functions \( \left( \frac{d}{dx} \right)^{2j} \varphi(x) \) \( (j = 0, \ldots, n) \), the function \( \Xi(x, E) \) can be expressed as

\[
\Xi(x, E) = c(E) + \sum_{i=0}^{3} \sum_{j=0}^{l_i-1} a_j^{(i)}(E) \left( \frac{d}{dx} \right)^{2j} \varphi(x + \omega_i)
+ \sum_{i' = 1}^{M} d^{(i')}(E) (\varphi(x + \delta_{i'}) + \varphi(x - \delta_{i'})).
\]

Set

\[
a(E) = \sum_{i=0}^{3} a_j^{(i)}(E) + 2 \sum_{i' = 1}^{M} d^{(i')}(E).
\]

From Proposition 4.7 we have

\[
\Lambda(x + 2\omega_k, E) = (-1)^{\eta_k} \Lambda(x, E) \exp \left( -\frac{1}{2} \int_{E_0}^{E} \frac{-2\eta_k a(E) + 2\omega_k c(E)}{\sqrt{-Q(E)}} dE \right),
\]

for \( k = 1, 3 \), where \( \eta_k = \zeta(\omega_k) \) \( (k = 1, 3) \). If \( Q(E') \neq 0 \), then the functions \( \Lambda(x, E') \) and \( \Lambda(-x, E') \) are a basis of the space of solutions to Eq. (4.29) (see Proposition 2.4). Thus, if \( Q(E') \neq 0 \), then the monodromy matrix of solutions to Eq. (4.29) on the basis \( (\Lambda(x, E'), \Lambda(-x, E')) \) with respect to the cycle \( x \to x + 2\omega_k \) \( (k = 1, 3) \), is diagonal and described by hyperelliptic integrals as Eq. (4.33).

4.3. Bethe Ansatz and Hermite-Krichever Ansatz. In this subsection we express a solution to Eq. (4.29) in the form of the Bethe Ansatz and also in the form of the Hermite-Krichever Ansatz. The monodromy is described by the data of the Hermite-Krichever Ansatz (or the Bethe Ansatz). Throughout this subsection, we will also assume that \( l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}, \delta_1, \ldots, \delta_M \) satisfy Eq. (4.6), \( \delta_j \neq \omega_i \) mod \( 2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z} \) \( (0 \leq i \leq 3, 1 \leq j \leq M) \) and \( \delta_j \pm \delta_{j'} \neq 0 \) mod \( 2\omega_1 \mathbb{Z} \oplus 2\omega_3 \mathbb{Z} \) \( (1 \leq j < j' \leq M) \).

Set \( l = 2M + \sum_{i=0}^{3} l_i, \bar{l}_0 = 2M + l_0, \bar{l}_i = l_i \) \( (i = 1, 2, 3) \) and

\[
\Psi_g(x) = \prod_{i' = 1}^{M} (\varphi(x) - \varphi(\delta_{i'})).
\]

Assume that \( Q(E') \neq 0 \) and the dimension of the space of even doubly-periodic solutions to Eq. (2.14) is one. By Proposition 2.12 (i), the function \( \Lambda(x, E') \) in Eq. (4.29) is expressed in the form of the Bethe Ansatz. Namely,

\[
\Lambda(x, E') = \frac{C_0 \prod_{j=1}^{t} \sigma(x - t_j)}{\Psi_g(x) \sigma(x)^{i_0} \sigma_1(x)^{i_1} \sigma_2(x)^{i_2} \sigma_3(x)^{i_3}} \exp(cx),
\]

for some \( t_1, \ldots, t_l, c \) and \( C_0 \neq 0 \). It follows from Proposition 2.12 (vi) that

\[
\left. \frac{d\Xi(x, E')}{dz} \right|_{z = z_j} = \frac{2\sqrt{-Q(E')}}{\varphi'(t_j)},
\]

where \( z = \varphi(x) \) and \( z_j = \varphi(t_j) \).
The function $\Lambda(x, E')$ is also expressed in the form of the Hermite-Krichever Ansatz. Recall that the function $\Phi_i(x, \alpha)$ ($i = 0, 1, 2, 3$) defined in Eq. (2.32) has periodicity described as Eq. (2.33).

**Proposition 4.8.** (i) The function $\Lambda(x, E)$ in Eq. (4.29) is expressed as

$$
\Lambda(x, E) = \frac{\exp(\kappa x)}{\Psi_g(x)} \left( \sum_{i=0}^{3} \sum_{j=0}^{\bar{l}_i-1} \bar{b}_j^{(i)} \left( \frac{d}{dx} \right)^j \Phi_i(x, \alpha) \right)
$$

for $\alpha, \kappa$ and $\bar{b}_j^{(i)}$ ($i = 0, \ldots, 3$, $j = 0, \ldots, \bar{l}_i - 1$), or

$$
\Lambda(x, E) = \frac{\exp(\bar{\kappa} x)}{\Psi_g(x)} \left( \bar{c} + \sum_{i=0}^{3} \sum_{j=0}^{\bar{l}_i-2} \bar{b}_j^{(i)} \left( \frac{d}{dx} \right)^j \varphi(x + \omega_i) + \sum_{i=1}^{3} \bar{c}_i \frac{\varphi'(x)}{\varphi(x) - \bar{c}_i} \right)
$$

for $\bar{\kappa}, \bar{c}, \bar{c}_i$ ($i = 1, 2, 3$) and $\bar{b}_j^{(i)}$ ($i = 0, \ldots, 3$, $j = 0, \ldots, \bar{l}_i - 2$). If $\Lambda(x, E)$ is expressed as Eq. (4.37), then

$$
\Lambda(x + 2\omega_k, E) = \exp(-2\eta_k \alpha + 2\omega_k \zeta(\alpha) + 2\kappa \omega_k) \Lambda(x, E), \quad (k = 1, 3).
$$

(ii) There exist polynomials $P_1(E), \ldots, P_6(E)$ such that, if $P_2(E') \neq 0$, then the function $\Lambda(x, E')$ in Eq. (4.29) is written in the form of Eq. (4.37), and the values $\alpha$ and $\kappa$ are expressed as

$$
\varphi(\alpha) = \frac{P_1(E')}{P_2(E')}, \quad \varphi'(\alpha) = \frac{P_3(E')}{P_4(E')} \sqrt{-Q(E')}, \quad \kappa = \frac{P_5(E')}{P_6(E')} \sqrt{-Q(E')}
$$

If $P_2(E') = 0$, then the function $\Lambda(x, E')$ in Eq. (4.29) is expressed in the form of Eq. (4.38).

**Proof.** (i) follows from Theorem 2.5. Note that $\Lambda(x, E)\Psi_g(x)$ is a solution to Eq. (2.12).

(ii) is proved by quite a similar argument to that of the proof of Proposition 3.1.

We provide a sketch of the proof of (ii).

We assume that $Q(E') \neq 0$ and the dimension of the space of solutions to Eq. (2.19), which are even doubly-periodic for fixed $E'$, is one. For the case $Q(E') = 0$, or the case when the dimension of the space of solutions to Eq. (2.19), which, for fixed $E'$, are even and doubly-periodic, is more than one, (ii) is shown by considering a continuation on parameter $E$.

It follows from Eq. (4.35) that

$$
\Lambda(x + 2\omega_k, E') = \exp \left( 2\eta_k \left( -\sum_{j=1}^{l} t_j + \sum_{i=1}^{3} l_i \omega_i \right) + 2\omega_k \left( c - \sum_{i=1}^{3} l_i \eta_i \right) \right) \Lambda(x, E')
$$

for $k = 1, 3$. By comparing with Eq. (4.39), we have

$$
\alpha \equiv \sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i \quad (\text{mod } 2\omega_1 Z \oplus 2\omega_3 Z),
$$

$$
\kappa = -\zeta \left( \sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i \right) + \sum_{j=1}^{l} \zeta(t_j) - \sum_{i=1}^{3} l_i \eta_i + \delta_{0,0} \sqrt{-Q(E')} \Xi(0, E').
$$
It follows from expressing \( \varphi(\sum_{j=1}^{l} t_j - \sum_{i=1}^{3} l_i \omega_i) \) as a combination of \( \varphi(t_j) \) and \( \varphi'(t_j) \) \((j = 1, \ldots, l)\), and applying Eq. (4.36) together with the expression

\[
\Xi(x, E)\Psi_g(x)^2 = \frac{D \prod_{j=1}^{l}(\varphi(x) - \varphi(t_j))}{(\varphi(x) - e_1)^{l_1}(\varphi(x) - e_2)^{l_2}(\varphi(x) - e_3)^{l_3}}
\]

for \( D \neq 0 \) that \( \varphi(\alpha) \) is expressed as a rational function in \( E \). We can similarly obtain expressions for \( \varphi'(\alpha) \) and \( \kappa \) in the form of Eq. (4.40).

The condition \( P_2(E') = 0 \) is equivalent to the condition \( \alpha \equiv 0 \mod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z} \). If \( \alpha \equiv 0 \) (resp. \( \alpha \not\equiv 0 \)), then the function \( \Lambda(x, E') \) is expressed as Eq. (4.37) (resp. Eq. (4.38)). Thus we obtain (ii). \( \square \)

4.4. Hyperelliptic-elliptic reduction formulae. We obtain hyperelliptic-elliptic reduction formulae by comparing two expressions of monodromies. The following argument is analogous to the one in \[26, \S 3\].

By comparing Eq. (4.33) and Eq. (4.39), we have

\[
-\eta_k \left( 2\alpha + \int_{E_0}^{E} \frac{a(\tilde{E})}{\sqrt{-Q(\tilde{E})}} d\tilde{E} \right) + \omega_k \left( 2(\zeta(\alpha) + \kappa) + \int_{E_0}^{E} \frac{c(\tilde{E})}{\sqrt{-Q(\tilde{E})}} d\tilde{E} \right)
= \pi\sqrt{-1}(q_k + 2n_k),
\]

for \( k = 1, 3 \) and integers \( n_1 \) and \( n_3 \). By Legendre’s relation \( \eta_1\omega_3 - \eta_3\omega_1 = \pi\sqrt{-1}/2 \), it follows that

\[
\alpha + \frac{1}{2} \int_{E_0}^{E} \frac{a(\tilde{E})}{\sqrt{-Q(\tilde{E})}} d\tilde{E} = -(q_1 + 2n_1)\omega_3 + (q_3 + 2n_3)\omega_1,
\]

(4.46)

\[
\zeta(\alpha) + \kappa + \frac{1}{2} \int_{E_0}^{E} \frac{c(\tilde{E})}{\sqrt{-Q(\tilde{E})}} d\tilde{E} = -(q_1 + 2n_1)\eta_3 + (q_3 + 2n_3)\eta_1.
\]

(4.47)

We set \( \xi = \varphi(\alpha) \). By a similar argument to that of \[26\], Proposition 2.4, it may be proved that \( \alpha \to 0 \) \((\mod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z})\) as \( E \to \infty \). Combining with the relation \( \int \left( 1/\varphi'(\alpha) \right) d\xi = \int d\alpha \), we have

\[
\int_{\infty}^{\xi} \frac{d\tilde{\xi}}{\sqrt{4\tilde{\xi}^3 - g_2\tilde{\xi} - g_3}} = \alpha = -\frac{1}{2} \int_{\infty}^{E} \frac{a(\tilde{E})}{\sqrt{-Q(\tilde{E})}} d\tilde{E}.
\]

(4.48)

Note that \( Q(E) \) is a polynomial of degree \( 2g + 1 \), while \( a(E) \) is a polynomial of degree \( g \). Hence Eq. (4.48) represents a formula which reduces a hyperelliptic integral of the first kind to an elliptic integral of the first kind. The transformation of variables is given by \( \xi = P_1(E)/P_2(E) \) for polynomials \( P_1(E) \) and \( P_2(E) \) \((\text{see Eq. (4.40)})\). Let \( \alpha_0 \) denote the value of \( \alpha \) at \( E = E_0 \), where \( E_0 \) is the value satisfying \( Q(E_0) = 0 \). It follows from Eq. (4.46) that \( \alpha_0 = -(q_1 + 2n_1)\omega_3 + (q_3 + 2n_3)\omega_1 \) and

\[
\alpha - \alpha_0 + \frac{1}{2} \int_{E_0}^{E} \frac{a(\tilde{E})}{\sqrt{-Q(\tilde{E})}} d\tilde{E} = 0.
\]

(4.49)
If \( \alpha_0 \equiv 0 \pmod{2\omega_1\mathbb{Z} + 2\omega_3\mathbb{Z}} \), then \( \zeta(\alpha - \alpha_0) = \zeta(\alpha) + (q_1 + 2n_1)\eta_3 - (q_3 + 2n_3)\eta_1 \).

Combining with Eqs. (4.47, 4.49), we have

\[
\kappa = -\frac{1}{2} \int_{E_0}^{E} \frac{c(E)}{-Q(E)} dE + \zeta \left( \frac{1}{2} \int_{E_0}^{E} \frac{a(E)}{-Q(E)} dE \right).
\]

If \( \alpha_0 \not\equiv 0 \pmod{2\omega_1\mathbb{Z} + 2\omega_3\mathbb{Z}} \), then \( \zeta(\alpha_0) = -(q_1 + 2n_1)\eta_3 + (q_3 + 2n_3)\eta_1 \) and

\[
\kappa = -\frac{1}{2} \int_{E_0}^{E} \frac{c(E)}{-Q(E)} dE + \int_{\wp(\alpha_0)}^{\tilde{\xi}} \frac{\tilde{\xi} d\tilde{\xi}}{\sqrt{4\tilde{\xi}^3 - g_2\tilde{\xi} - g_3}}.
\]

Note that \( Q(E) \) is a polynomial of degree \( 2g + 1 \), \( c(E) \) is a polynomial of degree \( g + 1 \) and \( \kappa \) is expressed as \( \kappa = \sqrt{-Q(E)}P_5(E)/P_6(E) \) for polynomials \( P_5(E) \) and \( P_6(E) \) (see Eq. (4.40)). Hence Eq. (4.51) represents a formula which reduces a hyper-elliptic integral of the second kind to an elliptic integral of the second kind, and the transformation of variables is also given by \( \xi = P_1(E)/P_2(E) \).

The following proposition describes the asymptotic behavior of \( \wp(\alpha) \) and \( \kappa \) as \( E \to \infty \), which is proved in a similar manner to [26, Proposition 3.2].

**Proposition 4.9.** (c.f. [26, Proposition 3.2]) As \( E \to \infty \), we have \( \alpha \sim \frac{1}{2\sqrt{-E}}(4M + \sum_{i=0}^{3} l_i(l_i + 1)) \), \( \wp(\alpha) \sim -4E/(4M + \sum_{i=0}^{3} l_i(l_i + 1))^2 \) and \( \kappa \sim \sqrt{-E}(1 - 2/(4M + \sum_{i=0}^{3} l_i(l_i + 1))) \).

In [26], following Maier [15], twisted Heun polynomials and theta-twisted Heun polynomials are introduced. We can extend the notions of twisted Heun polynomials and theta-twisted Heun polynomials to our potential to express the transformation of variables \( \xi = P_1(E)/P_2(E) \) and the value \( \kappa = \sqrt{-Q(E)}P_5(E)/P_6(E) \).

## 5. Examples on finite-gap potential

We here consider in detail several examples on finite-gap potential discussed in section 4. The results below partially overlap with those of Smirnov [20].

### 5.1. The case \( M = 1 \), \( l_0 = l_1 = l_2 = l_3 = 0 \).

The differential equation is written as

\[
\left( -\frac{d^2}{dx^2} + 2(\wp(x - \delta_1) + \wp(x + \delta_1)) \right) f(x) = Ef(x).
\]

Set \( b_1 = \wp(\delta_1) \). Then the condition that the regular singular points \( x = \pm \delta_1 \) of Eq. (5.1) are apparent is given by

\[
\prod_{i=1}^{3} (b_1^2 - 2c_1b_1 - 2c_1^2 + g_2/4) = 0
\]

(see Eq. (4.4)). This equation is equivalent to \( \wp(2\delta_1) = e_i \) for some \( i \in \{1, 2, 3\} \), which is solved by \( \delta_1 \equiv \omega_i/2 \pmod{\omega_1\mathbb{Z} + \omega_3\mathbb{Z}} \). By the shift \( x \to x + \delta_1 \), Eq. (5.1) is written as

\[
\left( -\frac{d^2}{dx^2} + 2(\wp(x) + \wp(x + \omega_i)) \right) f(x) = Ef(x),
\]

whose potential is the Treibich-Verdier potential for the case \( (l_0, l_1, l_2, l_3) = (1, 1, 0, 0) \), \( (1, 0, 1, 0) \) or \( (1, 0, 0, 1) \).
We derive the functions that have appeared in section 4 for the case \((b_0^2 - 2e_ib_1 - 2e_i^2 + g_2/4) = 0\) \((i \in \{1, 2, 3\})\). The functions \(\Xi(x, E)\) and \(Q(E)\) are given by

\[
\Xi(x, E) = E - 3e_i + \varphi(x - \delta_1) + \varphi(x + \delta_1),
\]

\[
Q(E) = (E - 4e_i)(E^2 - 2e_iE + g_2 - 11e_i^2).
\]

Hence the genus of the associated curve \(\nu^2 = -Q(E)\) is one, and a third-order commuting operator is constructed from \(\Xi(x, E)\) (see Theorem 4.5). The function \(\Lambda(x, E)\) defined by Eq. (4.29) is a solution to Eq. (5.1), and the monodromy formula corresponding to Eq. (4.33) is given by

\[
\Lambda(x + 2\omega_k, E) = \Lambda(x, E) \exp \left( -\frac{1}{2} \int_{4e_i}^{E} \frac{-4\eta_k + 2\omega_k(\tilde{E} - 3e_i)}{\sqrt{-Q(\tilde{E})}} d\tilde{E} \right),
\]

for \(k = 1, 3\). The function \(\Lambda(x, E)\) admits an expression in the form of the Hermite-Krichever Ansatz as

\[
\Lambda(x, E) = \frac{\exp (\kappa x)}{\varphi(x) - \varphi(\delta_1)} \left( \tilde{b}_0 \Phi_0(x, \alpha) + \tilde{b}_1 \frac{d}{dx} \Phi_0(x, \alpha) \right)
\]

for generic \(E\), and the values \(\alpha\) and \(\kappa\) satisfy

\[
\varphi(\alpha) = e_i - \frac{E^2 - 2e_iE + g_2 - 11e_i^2}{4(E - 4e_i)}, \quad \kappa = \frac{1}{2} \sqrt{\frac{-(E^2 - 2e_iE + g_2 - 11e_i^2)}{E - 4e_i}}.
\]

The monodromy is written by using the values \(\alpha\) and \(\kappa\) (see Eq. (4.39)). By comparing the two expressions of monodromy, we obtain

\[
\int_{\infty}^{\xi} \frac{d\tilde{\xi}}{\sqrt{4\tilde{\xi}^3 - g_2\tilde{\xi} - g_3}} = -\int_{\infty}^{E} \frac{d\tilde{E}}{\sqrt{-Q(\tilde{E})}},
\]

\[
\kappa = -\frac{1}{2} \int_{E_0}^{E} \frac{\tilde{E} - 3e_i}{\sqrt{-Q(\tilde{E})}} d\tilde{E} + \int_{e_i}^{\xi} \frac{\tilde{\xi} d\tilde{\xi}}{\sqrt{4\tilde{\xi}^3 - g_2\tilde{\xi} - g_3}},
\]

for the transformation

\[
\xi = e_i - \frac{E^2 - 2e_iE + g_2 - 11e_i^2}{4(E - 4e_i)},
\]

where \(E_0\) satisfies \(E_0^2 - 2e_iE_0 + g_2 - 11e_i^2 = 0\), and these formulae are related to the Landen transformation. Note that our results are compatible with the one of the Treibich-Verdier potential for the case \((l_0, l_1, l_2, l_3) = (1, 1, 0, 0)\) (see [26]).

5.2. The case \(M = 1, l_0 = 1, l_1 = l_2 = l_3 = 0\). The differential equation is written as

\[
\left( -\frac{d^2}{dx^2} + 2(\varphi(x) + \varphi(x - \delta_1) + \varphi(x + \delta_1)) \right) f(x) = Ef(x).
\]

Set \(b_1 = \varphi(\delta_1)\). Then the condition that the regular singular points \(x = \pm \delta_1\) of Eq. (5.12) are apparent is given by

\[
(b_1 - g_2b_1^2/2 - g_3b_1 - g_2^2/48)(b_1^2 - g_2/12) = 0.
\]

We will first obtain the functions of the present study for the case \(b_1^2 - g_2b_1^2/2 - g_3b_1 - g_2^2/48 = 0\).
The functions $\Xi(x, E)$ and $Q(E)$ are given by

\begin{align}
\Xi(x, E) &= E - 6b_1 + \varphi(x) + \varphi(x - \delta_1) + \varphi(x + \delta_1), \\
Q(E) &= E^3 - 12b_1 E^2 + 9(2b_1^2 + g_2/4)E + 126b_1^3 - 39g_2b_1/2 - 27g_3/4, \\
\end{align}

and $Q(E)$ is factorized as $Q(E) = (E - \alpha_i)(E - \alpha_2)(E - \alpha_3)$, where $\alpha_i = 3(8b_1^3 + 8e_i b_1^2 - 2(s_e^2 + g_2)b_1 + 3e_i g_2 - 12e_1^3)/(g_2 - 12e_1^3) \ (i = 1, 2, 3)$. The genus of the associated curve $\nu^2 = -Q(E)$ is one, and a third-order commuting operator is constructed from $\Xi(x, E)$ (see Theorem 4.5). The function $\Lambda(x, E)$ defined by Eq. (4.29) is a solution to Eq. (4.12), and the monodromy formula corresponding to Eq. (4.33) may be expressed in the form

\begin{equation}
\Lambda(x + 2\omega_k, E) = -\Lambda(x, E) \exp \left( -\frac{1}{2} \int_{\alpha_2}^E -6\eta_k + 2\omega_k(\tilde{E} - 6b_1) \right),
\end{equation}

for $k = 1, 3$. The function $\Lambda(x, E)$ admits an expression in the form of the Hermite-Krichever Ansatz as

\begin{equation}
\Lambda(x, E) = \frac{\exp(\kappa x)}{\varphi(x) - \varphi(\delta_1)} \left( \tilde{b}_0 \Phi_0(x, \alpha) + \tilde{b}_1 \frac{d}{dx} \Phi_0(x, \alpha) + \tilde{b}_2 \left( \frac{d}{dx} \right)^2 \Phi_0(x, \alpha) \right)
\end{equation}

for generic $E$, and the values $\alpha$ and $\kappa$ satisfy

\begin{align}
\varphi(\alpha) &= e_i - \frac{(E - \alpha_i)(E - 9b_1 + 9e_i/2 + \alpha_i/2)^2}{9(E - 7b_1)^2} \ (i = 1, 2, 3) \\
&= -\frac{E^3 - 18b_1 E^2 + 99b_1^2 E - 126b_1^3 + (-93b_1/2 + 9E/2)g_2 - 27g_3}{9(E - 7b_1)^2}, \\
\kappa &= \frac{2}{3(E - 7b_1)} \sqrt{-Q(E)}.
\end{align}

The monodromy is written by using the values $\alpha$ and $\kappa$ (see Eq. (4.39)). By comparing the two expressions of monodromy, we obtain

\begin{align}
\int_{\infty}^{\xi} \frac{d\bar{\xi}}{\sqrt{4\bar{\xi}^3 - g_2 \bar{\xi} - g_3}} &= -\frac{3}{2} \int_{\infty}^E \frac{d\tilde{E}}{\sqrt{-Q(\tilde{E})}}, \\
\kappa &= -\frac{1}{2} \int_{\epsilon_i}^E \frac{\tilde{E} - 6b_1}{\sqrt{-Q(\tilde{E})}} d\tilde{E} + \int_{\alpha_i}^{\xi} \frac{\bar{\xi} d\bar{\xi}}{\sqrt{4\bar{\xi}^3 - g_2 \bar{\xi} - g_3}}, \quad (i = 1, 2, 3),
\end{align}

for the transformation

\begin{equation}
\xi = -\frac{E^3 - 18b_1 E^2 + 99b_1^2 E - 126b_1^3 + (-93b_1/2 + 9E/2)g_2 - 27g_3}{9(E - 7b_1)^2}.
\end{equation}

We now consider the case $b_2^2 - g_2/12 = 0$. The functions $\Xi(x, E)$ and $Q(E)$ are given by

\begin{align}
\Xi(x, E) &= \left( E^2 - 3g_2/2 \right) + \frac{(2b_1 + 3g_3)E + g_2^2 + 18b_1 g_3}{2b_1 g_2 + 3g_3} \varphi(x) \\
&\quad + \frac{2(2b_1 g_2 + 3g_3)E - (g_2 + 18b_1 g_3)g_2}{2(2b_1 g_2 + 3g_3)} (\varphi(x - \delta_1) + \varphi(x + \delta_1)), \\
Q(E) &= (E - 6b_1)(E + 6b_1)(E - 3e_1)(E - 3e_2)(E - 3e_3).
\end{align}
Hence the genus of the associated curve $\nu^2 = -Q(E)$ is two, and a fifth-order commuting operator is constructed from $\Xi(x, E)$ (see Theorem 4.5). The function $\Lambda(x, E)$ defined by Eq. (4.29) is a solution to Eq. (5.12), and the monodromy formula corresponding to Eq. (4.33) is given by

$$
(5.25) \quad \Lambda(x + 2\omega_k, E) = \Lambda(x, E) \exp \left( -\frac{1}{2} \int_{\omega_1}^{E} \frac{-6\eta_k \bar{E} + 2\omega_k (\bar{E}^2 - 3g_2/2)}{\sqrt{-Q(E)}} \, d\bar{E} \right),
$$

for $k = 1, 3$. The function $\Lambda(x, E)$ admits an expression in the form of the Hermite-Krichever Ansatz as Eq. (5.17) for generic $E$, and the values $\alpha$ and $\kappa$ satisfy

$$
(5.26) \quad \phi(\alpha) = -\frac{E^3 - 27g_3}{9(E^2 - 3g_2)}, \quad \kappa = \frac{2}{3} \sqrt{\frac{-(E^3 - 9g_2E/4 - 27g_3/4)}{(E^2 - 3g_2)}}.
$$

The monodromy may be written upon using the values $\alpha$ and $\kappa$ (see Eq. (4.39)). By comparing the two expressions of monodromy, we obtain

$$
(5.27) \quad \int_{\infty}^{E} \frac{d\bar{\xi}}{\sqrt{4\bar{\xi}^3 - g_2\bar{\xi} - g_3}} = -\frac{3}{2} \int_{\infty}^{E} \frac{\tilde{E} \, d\tilde{E}}{\sqrt{-Q(E)}},
$$

$$
(5.28) \quad \kappa = -\frac{1}{2} \int_{E_1}^{E} \frac{\tilde{E}^2 - 3g_2/2}{\sqrt{-Q(E)}} \, d\tilde{E} + \int_{3E_1}^{\xi} \frac{\tilde{\xi} \, d\tilde{\xi}}{\sqrt{4\bar{\xi}^3 - g_2\bar{\xi} - g_3}},
$$

for the transformation

$$
(5.29) \quad \xi = -\frac{E^3 - 27g_3}{9(E^2 - 3g_2)}.
$$

These formulae reduce hyperelliptic integrals of genus two to elliptic integrals. Note that our results are similar to that of the Treibich-Verdier potential for the case $(l_0, l_1, l_2, l_3) = (2, 0, 0, 0)$ (see [20]), and these two potentials may be related by an isospectral deformation (see [20]).

5.3. The case $M = 1, l_0 = 2, l_1 = l_2 = l_3 = 0$. The differential equation is written as

$$
(5.30) \quad \left( -\frac{d^2}{dx^2} + 6\varphi(x) + 2(\varphi(x - \delta_1) + \varphi(x + \delta_1)) \right) f(x) = Ef(x).
$$

Set $b_1 = \varphi(\delta_1)$. Then the condition that the regular singular points $x = \pm \delta_1$ of Eq. (5.30) are apparent is given by

$$
(5.31) \quad b_1^6 - 53g_2b_1^4/100 - 17g_3b_1^3/25 + 19g_2^2b_1^2/400 + 11g_2g_3b_1/100 + g_3^2/1600 + g_2^2/25 = 0.
$$

Upon setting

$$
(5.32) \quad H^{(0)}(E) = E^2 - \frac{160b_1^4 - 24g_2b_1 - 16g_3}{12b_1^2 - g_2} \cdot E - \frac{800b_1^4 - 124g_2b_1^2 - 32g_3b_1 + 3g_2^2}{12b_1^2 - g_2},
$$

$$
(5.33) \quad H^{(1)}(E) = E - \left\{ 800b_1^5 + 800c_1b_1^4 + (-384g_2 + 320c_2^2)b_1^3 + (60g_2 - 656c_2^2)c_1b_1^2 
+ (30g_2^2 + 144c_2^2g_2 - 512c_1^2)b_1 + (-23g_2^2 + 12g_2c_1^2 + 256c_2^2)c_1 \right\}/\{(12c_1^2 - g_2)(12b_1^2 - g_2)\},
$$

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for \( i = 1, 2, 3 \), the functions \( \Xi(x, E) \) and \( Q(E) \) are given by
\[
\Xi(x, E) = E^2 - 10b_1E - \frac{3800b_1^4 - 124b_1^2g_2 - 32b_1g_3 + 3g_2^2}{12b_1^2 - g_2} + 9\varphi(x)^2
\]
\[
+ 3(E - 4b_1)\varphi(x) + \left( E - \frac{-72b_1^2 + 14b_1g_2 + 12g_3}{12b_1^2 - g_2} \right) (\varphi(x - \delta_1) + \varphi(x + \delta_1)),
\]
\[
Q(E) = H^{(0)}(E)H^{(1)}(E)H^{(2)}(E)H^{(3)}(E).
\]

The genus of the associated curve \( \nu^2 = -Q(E) \) is two, and a fifth-order commuting operator is constructed from \( \Xi(x, E) \) (see Theorem 4.5). The function \( \Lambda(x, E) \) defined by Eq. (4.33) is a solution to Eq. (5.30), and the monodromy formula corresponding to Eq. (4.33) is written as
\[
\Lambda(x + 2\omega_k, E) = -\Lambda(x, E).
\]

\[
\exp \left( \int_{\alpha_2}^{E} \eta_k \left( \frac{5\tilde{E} - \frac{8(2b_1g_2 + 3g_1)}{12b_1^2 - g_2}}{\sqrt{-Q(\tilde{E})}} \right) \phi_k \right) \left( \sum_{j=0}^{3} b_j \left( \frac{d}{dx} \right)^j \Phi_0(x, \alpha) \right)
\]

for \( k = 1, 3 \), where \( \alpha_i \) satisfies \( H^{(i)}(\alpha_i) = 0 \) \( (i = 1, 2, 3) \). The function \( \Lambda(x, E) \) admits an expression in the form of the Hermite-Krichever Ansatz as
\[
\Lambda(x, E) = \frac{\exp (\kappa x)}{\varphi(x) - \varphi(b)} \left( \sum_{j=0}^{3} b_j \left( \frac{d}{dx} \right)^j \Phi_0(x, \alpha) \right)
\]

for generic \( E \), and the values \( \alpha \) and \( \kappa \) satisfy
\[
\varphi(\alpha) = e_i - \frac{H^{(i)}(E)H^{(i)}(E^2)}{25H^{(0)}(E)H^{(0)}(E^2)}, \quad (i = 1, 2, 3),
\]
\[
\kappa = \frac{4H^{(1)}(E)H^{(2)}(E)H^{(3)}(E)}{5H^{(0)}(E)H^{(0)}(E)},
\]

where
\[
H^{(0)}(E) = E + \frac{-420b_1^4 + 59g_2b_1 + 36g_3}{2(12b_1^2 - g_2)}, \quad H^{(i)}(E) = E + \frac{-900b_1^4 + 83g_2b_1 + 12g_3}{5(12b_1^2 - g_2)},
\]
\[
H^{(i)}(E) = E^2 + E\{400b_1^5 + 400e_i b_1^4 + (108g_2 - 3440e_i^2)b_1^3 + (-120g_2 + 1472e_i^2)e_i b_1^2 + (-18g_2^2 + 468e_i^2 g_2 - 256e_i^4)b_1 + (13g_2^3 - 336g_2 e_i^2 + 704e_i^4)e_i} / \{(12e_i^2 - g_2)(12b_1^2 - g_2)\}
\]
\[
+ \{1600e_i b_1^5 + (-2800g_2 + 35200e_i^2)b_1^4 + (6352g_2 - 84800e_i^2)e_i b_1^3
\]
\[
+ (-24g_2^2 + 2688e_i^2 g_2 - 28672e_i^4)b_1^2 + (-1152g_2^2 + 16240g_2 e_i^2 - 17920e_i^4)e_i b_1
\]
\[
+ 41g_2^3 - 328e_i^2 g_2 - 4096e_i^4 g_2 + 19712e_i^6} / \{(2(12e_i^2 - g_2)(12b_1^2 - g_2)\}, \quad (i = 1, 2, 3).
The monodromy may be written using the values \( \alpha \) and \( \kappa \) (see Eq. (4.39)). By comparing the two expressions of monodromy, we obtain

\[
\int_\infty^\xi \frac{d\tilde{\xi}}{\sqrt{4\tilde{\xi}^3 - g_2\tilde{\xi} - g_3}} = -\frac{1}{2} \int_\infty^E \frac{5\bar{E} - \frac{8(2b_1g_2 + 3g_3)}{12b_1^2 - g_2}}{\sqrt{-Q(E)}}d\bar{E},
\]

\[
\kappa = -\frac{1}{2} \int_\infty^E \bar{E}^2 - 10b_1\bar{E} - \frac{3(2006\bar{E}^2 - 4\bar{E}^2g_2 - 2b_1g_3 + g_2^2)}{12b_1^2 - g_2}\sqrt{-Q(E)}d\bar{E} + \int_\alpha^\xi \frac{\tilde{\xi}d\tilde{\xi}}{\sqrt{4\tilde{\xi}^3 - g_2\tilde{\xi} - g_3}},
\]

for the transformation

\[
\xi = e_i - \frac{H^{(i)}(E)H_t^{(i)}(E)^2}{25H^{(0)}(E)H_t^{(0)}(E)^2}, \quad (i = 1, 2, 3).
\]

These formulae reduce hyperelliptic integrals of genus two to elliptic integrals, which may not be reduced to the case of the Treibich-Verdier potential.

5.4. The case \( M = 2, l_0 = l_1 = l_2 = l_3 = 0 \). The differential equation is written as

\[
-\frac{d^2}{dx^2} + 2\sum_{l'=1}^2 (\varphi(x - \delta_{l'}) + \varphi(x + \delta_{l'})) f(x) = E f(x).
\]

The equation for the apparenity of singularity is written as

\[
\varphi'(2\delta_1) + \varphi'(\delta_1 + \delta_2) + \varphi'(\delta_1 - \delta_2) = 0, \quad \varphi'(2\delta_2) + \varphi'(\delta_2 + \delta_1) + \varphi'(\delta_2 - \delta_1) = 0.
\]

We set \( \alpha = \delta_1 + \delta_2 \) and \( \beta = \delta_1 - \delta_2 \). Then Eq. (5.46) are equivalent to

\[
\varphi'(\alpha + \beta) + \varphi'(\alpha - \beta) + 2\varphi'(\alpha) = 0, \quad \varphi'(\alpha + \beta) + \varphi'(\beta - \alpha) + 2\varphi'(\beta) = 0.
\]

From the relation

\[
\varphi'(x + y) + \varphi'(x - y) = -\frac{\varphi''(y)\varphi'(x)}{(\varphi(x) - \varphi(y))^2} - 2\frac{\varphi'(y)^2\varphi'(x)}{(\varphi(x) - \varphi(y))^3},
\]

we have that Eq. (5.47) is equivalent to

\[
(A(\alpha, \beta) + 2)\varphi'(\alpha) = 0, \quad (A(\beta, \alpha) + 2)\varphi'(\beta) = 0,
\]

\[
A(x, y) = -\frac{\varphi''(y)}{(\varphi(x) - \varphi(y))^2} - 2\frac{\varphi'(y)^2}{(\varphi(x) - \varphi(y))^3}.
\]

Thus solutions to Eq. (5.49) are divided into four cases; the case \( \varphi'(\alpha) = \varphi'(\beta) = 0 \), the case \( \varphi'(\alpha) = A(\beta, \alpha) + 2 = 0 \), the case \( \varphi'(\beta) = A(\alpha, \beta) + 2 = 0 \), and the case \( A(\alpha, \beta) + 2 = A(\beta, \alpha) + 2 = 0 \).

We first consider the case \( \varphi'(\alpha) = \varphi'(\beta) = 0 \). Since \( \varphi'(x) = 0 \) is equivalent to \( x \equiv \omega_1, \omega_2, \omega_3 \mod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z} \), we have \( \alpha, \beta \equiv \omega_1, \omega_2, \omega_3 \mod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z} \). By considering the condition \( \delta_1 \neq 0 \neq \delta_2 \), we have \( (\delta_1, \delta_2) \equiv \pm((\omega_i + \omega_j)/2, (\omega_i - \omega_j)/2) \mod 2\omega_1\mathbb{Z} \oplus 2\omega_3\mathbb{Z} \) for \( 1 \leq i \neq j \leq 3 \). For this case, we have \( \varphi(2\delta_1) = \varphi(2\delta_2) \) and

\[
\Xi(x, E) = E - 3(\varphi(2\delta_1) + \varphi(\delta_1 + \delta_2) + \varphi(\delta_1 - \delta_2))
+ \varphi(x + \delta_1) + \varphi(x - \delta_1) + \varphi(x + \delta_2) + \varphi(x - \delta_2).
\]

The degree of the polynomial \( Q(E) \) is three, and the genus of the associated curve \( \nu^2 = -Q(E) \) is one.
Secondly, we consider the case \( \varphi'(\alpha) = A(\beta, \alpha) + 2 = 0 \). It follows from \( \varphi'(\alpha) = 0 \) that \( \alpha \equiv \omega_i \mod 2\omega_i \mathbb{Z} \oplus 2\omega_j \mathbb{Z} \) for some \( i \in \{1, 2, 3\} \). Then we have \( A(\beta, \omega_i) + 2 = 0 \), which may be written as \( 6\epsilon_i^2 - g_2/2 = 2(e_i - \varphi(\beta))^2 \). The solutions of this equation are given by \( 2\beta \equiv \omega_i \mod 2\omega_i \mathbb{Z} \oplus 2\omega_j \mathbb{Z} \). Hence \( (\delta_1, \delta_2) \equiv (\omega_j/2 \pm \omega_i/4, \omega_j/2 \mp \omega_i/4) \), \(- (\omega_j/2 \pm \omega_i/4, \omega_j/2 \mp \omega_i/4) \mod 2\omega_i \mathbb{Z} \oplus 2\omega_j \mathbb{Z} \) for \( 1 \leq i \leq 3, \, 0 \leq j \leq 3 \). For this case, we have \( \varphi(2\delta_1) = \varphi(2\delta_2) \) and

\[
(5.52) \quad \Xi(x, E) = E - 3(\varphi(2\delta_1) + \varphi(\delta_1 + \delta_2) + \varphi(\delta_1 - \delta_2)) + \varphi(x + \delta_1) + \varphi(x - \delta_1) + \varphi(x + \delta_2) + \varphi(x - \delta_2).
\]

The degree of the polynomial \( Q(E) \) is three, and the genus of the associated curve \( \nu^2 = -Q(E) \) is one. The case \( \varphi'(\beta) = A(\alpha, \beta) + 2 = 0 \) can be treated similarly.

We now consider the case \( A(\alpha, \beta) + 2 = A(\beta, \alpha) + 2 = 0 \). It follows from a direct derivation that \( \varphi(\alpha) + \varphi(\beta) = 2e_i \) and \( \varphi(\alpha)\varphi(\beta) = 2e_i^2 - g_2/12 \) for some \( i \in \{1, 2, 3\} \). The function \( \Xi(x, E) \) is given by

\[
(5.53) \quad \Xi(x, E) = c(E) + \sum_{j=1}^{2} d^{(j)}(E)(\varphi(x + \delta_j) + \varphi(x - \delta_j)),
\]

where

\[
(5.54) \quad d^{(1)}(E) = E - 2e_i + 3\varphi(2\delta_2), \quad d^{(2)}(E) = E - 2e_i + 3\varphi(2\delta_1),
\]

\[
(5.55) \quad c(E) = (d^{(1)}(E) + d^{(2)}(E))(\frac{E}{2} - 3e_i) - \frac{3}{2}(d^{(1)}(E)\phi(2\delta_1) + d^{(2)}(E)\phi(2\delta_2)).
\]

The degree of the polynomial \( Q(E) \) is five, and the genus of the associated curve \( \nu^2 = -Q(E) \) is two.

6. Concluding Remarks

We have shown in sections 2 and 3 that solutions of the linear differential equation that produces the sixth Painlevé equation have integral representations and that they are expressed in the form of the Hermite-Krichever Ansatz. Furthermore we got a procedure for obtaining solutions of the sixth Painlevé equation (see Eq. (3.26) for the cases \( \kappa_0, \kappa_1, \kappa_t, \kappa_\infty \in \mathbb{Z} + 1/2 \) by fixing the monodromy, and we presented explicit solutions for the cases \( (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (1/2, 1/2, 1/2, 1/2) \) and \( (1/2, 1/2, 1/2, 3/2) \).

By Bäcklund transformation of the sixth Painlevé equation (see 29 etc.), Hitchin’s solution (i.e., solutions for the case \( (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (1/2, 1/2, 1/2, 1/2) \)) is transformed to the solutions for the case \( (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in O_1 \cup O_2 \), where

\[
(6.1) \quad O_1 = \left\{ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) | \kappa_0, \kappa_1, \kappa_t, \kappa_\infty \in \mathbb{Z} + 1/2 \right\},
\]

\[
(6.2) \quad O_2 = \left\{ (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \mid \begin{array}{c} \kappa_0, \kappa_1, \kappa_t, \kappa_\infty \in \mathbb{Z} \\
\kappa_0 + \kappa_1 + \kappa_t + \kappa_\infty \in 2\mathbb{Z} \end{array} \right\}.
\]

Note that solutions for the case \( (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (0, 0, 0, 0) \in O_2 \) are already known and are called Picard’s solution.

For the case \( (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in O_1 \), solutions of the linear differential equation are investigated by our method, and solutions of the sixth Painlevé equation follow from them. On the other hand, for the case \( (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in O_2 \), we cannot obtain results on integral representation and the Hermite-Krichever Ansatz by our method, although solutions of the sixth Painlevé equation are obtained in principle by Bäcklund transformation from the case \( (\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) = (1/2, 1/2, 1/2, 1/2) \). Note that the
condition \((\kappa_0, \kappa_1, \kappa_t, \kappa_\infty) \in O_2\) corresponds to the condition \(l_0, \ldots, l_3 \in \mathbb{Z} + 1/2, l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z}\) (see Eq. (3.32)).

Now we propose a problem to investigate solutions and their monodromy of the linear differential equation (Eq. (3.1) with the condition (3.6)) for the cases \(l_0, \ldots, l_3 \in \mathbb{Z} + 1/2, l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z}\). In particular, how can we investigate solutions and their monodromy of the linear differential equation for the case \(\kappa_0 = \kappa_1 = \kappa_t = \kappa_\infty = 0\) (i.e. \(l_0 = l_1 = l_2 = l_3 = -1/2\))?\(^1\)

In section 4, we produced new examples of finite-gap potential and investigated properties of them. More properties should be clarified in near future. For instance, it is not immediate to calculate the genus of the associated curve. As is seen from examples in sections 5.2 and 5.4, the genus depends on the solution of the equations which determine the position of apparent singularities (i.e. Eq. (4.6)). Related results were obtained by Treibich [27] for the case \(M = 1\), and they are to be simplified and generalized.

To find finite-gap potential, we considered only the case \(r'_i = 2\) and \(s'_i = 0\) (see Eq. (2.5)) in section 4. We propose a problem for a study of finite-gap potential for the cases \(r'_i \neq 2\) for some \(i'\).

**Appendix A. Elliptic functions**

This appendix presents the definitions of and the formulas for the elliptic functions. The Weierstrass \(\wp\)-function, the Weierstrass sigma-function and the Weierstrass zeta-function with periods \((2\omega_1, 2\omega_3)\) are defined as follows:

\[
\wp(z) = \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}} \left( \frac{1}{(z - 2m\omega_1 - 2n\omega_3)^2} - \frac{1}{(2m\omega_1 + 2n\omega_3)^2} \right),
\]

\[
\sigma(z) = z \prod_{(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}} \left( 1 - \frac{z}{2m\omega_1 + 2n\omega_3} \right)
\cdot \exp\left( \frac{z}{2m\omega_1 + 2n\omega_3} + \frac{z^2}{2(2m\omega_1 + 2n\omega_3)^2} \right),
\]

\[
\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}.
\]

Setting \(\omega_2 = -\omega_1 - \omega_3\) and

\[
e_i = \wp(\omega_i), \quad \eta_i = \zeta(\omega_i), \quad (i = 1, 2, 3)
\]

yields the relations

\[
e_1 + e_2 + e_3 = \eta_1 + \eta_2 + \eta_3 = 0,
\]

\[
\eta_1 \omega_3 - \eta_3 \omega_1 = \eta_3 \omega_2 - \eta_2 \omega_3 = \eta_2 \omega_1 - \eta_1 \omega_2 = \pi \sqrt{-1}/2,
\]

\[
\wp(z) = -\zeta'(z), \quad (\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).
\]

\(^1\)Note added in 2008: Results on this subject were obtained by Takemura K., Integral representation of solutions to Fuchsian system and Heun’s equation, *J. Math. Anal. Appl.* 342 (2008), 52–69.
The periodicity of functions $\varphi(z)$, $\zeta(z)$ and $\sigma(z)$ are as follows:
\begin{equation}
\varphi(z + 2\omega_i) = \varphi(z), \quad \zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i, \quad (i = 1, 2, 3),
\end{equation}
\begin{equation}
\sigma(z + 2\omega_i) = -\sigma(z) \exp(2\eta_i(z + \omega_i)), \quad \frac{\sigma(z + t + 2\omega_i)}{\sigma(z + 2\omega_i)} = \exp(2\eta_i t) \frac{\sigma(z + t)}{\sigma(z)}.
\end{equation}

The constants $g_2$ and $g_3$ are defined by
\begin{equation}
g_2 = -4(e_1 e_2 + e_2 e_3 + e_3 e_1), \quad g_3 = 4e_1 e_2 e_3.
\end{equation}

The co-sigma functions $\sigma_i(z)$ ($i = 1, 2, 3$) and co-$\varphi$ functions $\varphi_i(z)$ ($i = 1, 2, 3$) are defined by
\begin{equation}
\sigma_i(z) = \exp(-\eta_i z) \frac{\sigma(z + \omega_i)}{\sigma(\omega_i)}, \quad \varphi_i(z) = \frac{\sigma_i(z)}{\sigma(z)}.
\end{equation}

and satisfy
\begin{equation}
\varphi_i(z)^2 = \varphi(z) - e_i, \quad (i, i' = 1, 2, 3)
\end{equation}
\begin{equation}
\varphi_i(z + 2\omega_{i'}) = \exp(2(\eta_{i'} \omega_i - \eta_i \omega_{i'})) \varphi_i(z) = (-1)^{\delta_{i,i'}} \varphi_i(z).
\end{equation}

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