A specialization of definitions in common knowledge logic

Aurimas Paulius Girčys, Regimantas Pliuškevičius

Institute of Mathematics and Informatics, Vilnius University
Akademijos 4, LT-08663 Vilnius
E-mail: aurimas.gircys@gmail.com; regimantas.pliuskevicius@mii.vu.lt

Abstract. It is known that one of main aims of specializations of derivations in non-classical logics is the various tools which allow us to simplify the searching of termination of derivations. The traditional techniques used to ensure termination of derivations in various non-classical logics are based on loop-checking. In this paper the reflexive common knowledge logic based on modal logic $K_2$ is considered. For considered logic a sequent calculus with specialized non-logical (loop-type) axioms is presented.

Keywords: sequent calculus, common knowledge logic, termination.

1 Introduction

The common knowledge logics are important class of non-classical logics and play a significant role in several areas of computer science, artificial intelligence, game theory, economics and etc.

Common knowledge logics are based on multi-modal logics extended with common knowledge operator. In this paper, a reflexive common knowledge logic (RCKL in short) based on multi-modal logic $K_2$ and reflexive common knowledge operator [3] is considered.

Common knowledge operator satisfies some induction like axioms. In derivation this induction-like tool is realized using loop-type axioms. Determination of these loop-type axioms involves creating a new “good loop” in contrast to “bad loops” and the new loop checking along with ordinary non-induction-type loop checking. Based on history method a method of determination of “good loops” for common knowledge logic is described in [1].

In this paper for reflexive common knowledge logic (based on multi-modal logic $K_2$), a sequent calculus with specialized non-logical (loop-type) axioms is presented. The specialization is achieved using some splitting rules.

2 Initial calculi for considered logic

The language of considered RCKL contains:

1. A set of propositional symbols $P_1, P_2, \ldots, Q_1, Q_2, \ldots$;
2. A finite set of agent constants $i_1, i_2, \ldots, i_k$ ($I_i, i \in \{1, \ldots, n\}$, $1 \leq l \leq k$ for simplicity we assume than $n = 2$);
(3) Multi-agent knowledge operator $K_i$, where $i \in \{1, \ldots, n\}$ is an agent constant;

(4) Reflexive common knowledge operator $C$;

(5) Logical operators $\supset, \land, \lor, \lnot$.

Formulas of RCKL are defined as follows: every propositional symbol is a formula; if $A$, $B$ are formulas then $A \supset B$, $A \land B$, $A \lor B$, $\lnot(A)$ are formulas; if $i$ is an agent constant, $A$ is a formula, then $K_i(A)$ is a formula; if $A$ is a formula, then $C(A)$ is a formula. The formula $K_i(A)$ means “Agent $i$ knows $A$” $(l \in \{1, 2\})$. The expression $E(A)$ means “every agent knows $A$” and is used as abbreviation of formula $K_1(A) \land K_2(A)$. The formula $C(A)$ means “$A$ is common knowledge of all agents”. The formula $C(A)$ has the same meaning as the infinite conjunction $A \land E(A) \land E(E(A)) \land \cdots \land E^K(A) \land \cdots$ where $E^0(A) = A$. Formal semantics of the formulas $K_i(A)$ and $C(A)$ is defined as RCKL [3]. Hilbert-type calculus HRC is obtained from Hilbert-type calculus for propositional logic by adding the following postulates:

1. Infinitary calculus $G_w RC$ is obtained from classical propositional calculus (with invertible logical rules) by adding the following rules:

$$\Gamma \rightarrow A$$

$$\Pi_1, K_i \Gamma \rightarrow \Pi_2, K_i(A)(K_i)$$

where $K_i \Gamma$ is empty or consists of formulas of the shape $K_i(B)$.

$$\Gamma \rightarrow \Delta, K_i(A) \land K_2(A)(\rightarrow E),$$

$$\rightarrow E, \Gamma \rightarrow \Delta, K_i(A) \rightarrow \Delta, E(A),$$

$$\rightarrow E, \Gamma \rightarrow \Delta, K_i(A),$$

$$\rightarrow E, \Gamma \rightarrow \Delta, E(A),$$

$$\Gamma \rightarrow \Delta, A, E(C(A)),$$

$$\Gamma \rightarrow \Delta, C(A),$$
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\[
\begin{align*}
\Gamma \to \Delta, A; & \Gamma \to A, E(A); \ldots; \Gamma \to \Delta; E^k(A) \ldots (\to \negw), \\
\end{align*}
\]

where \( E^k(A) = E^{k-1}(E(A)) \) and \( E^0(A) = A \).

Analogously as in [2] one can prove that \( G_wRC \) is sound and complete.

2. Loop-type calculus \( G_LRC \) is obtained from the calculus \( G_wRC \) by replacing the infinity rule \((\rightarrow \Box w)\) by the following rule

\[
\begin{align*}
\Gamma \to \Delta, A; & \Gamma \to \Delta, E(C(A)), \\
\end{align*}
\]

and adding loop-type axioms as follows: a sequent \( S \) is a loop-type axiom if \( S \) is above a sequent \( S' \) and on the same branch of a derivation, such that \( S' \) can be obtained from \( S \) using structural rules of weakening and contraction and there is the right premise of the rule \((\rightarrow C)\) between \( S \) and \( S' \).

Analogously as in [1] one can prove that \( G_LRC \) is sound and complete. Therefore the calculi \( G_wRC \) and \( G_LRC \) are equivalent.

Let us introduce a canonical form of sequents. A sequent \( S \) is a primary sequent, if \( S \) is of the following shape \( \Sigma_1, K\Gamma_1, C\Delta_1 \to \Sigma_2, K\Gamma_2, C\Delta_2 \), where for every \( i (i \in \{1, 2\}) \) \( \Sigma_i \) is empty or consists of propositional symbols; \( K\Gamma_i \) is empty or consists of formulas of the shape \( K_l(A) (l \in \{1, 2\}) \); \( C\Delta_i \) is empty or consists of formulas of the shape \( CA \).

It is easy to see that bottom-up applying logical rules each sequent can be reduced to a set of primary sequents.

The primary sequent \( \Sigma_1, K\Gamma_1, C\Delta_1 \to \Sigma_2, K\Gamma_2, C\Delta_2 \) is a \( K \)-primary one; the primary sequent \( \Sigma_1, K\Gamma_1, C\Delta_1 \to \Sigma_2, C\Delta_2 \) is a \( C \)-primary one. The primary sequent \( S \) is a non-splittable primary one if \( S \) is either a \( K \)-primary or \( C \)-primary sequent. Otherwise, the primary sequent \( S \) is a splittable primary one.

Let \( G^S_LRC \) be a calculus obtained from the calculus \( G_LRC \) by following transformations:

1. Adding the following splitting rule

\[
\begin{align*}
\Sigma_1, K\Gamma_1, C\Delta_1 \to \Sigma_2, K\Gamma_2, C\Delta_2 (Sp) \\
\end{align*}
\]

where the conclusion of the rule \((Sp)\) is splittable primary sequent; \( \Sigma_1 \cap \Sigma_2 = \emptyset \) (i.e. the sequent \( \Sigma_1 \to \Sigma_2 \) is not an axiom); \( S_1 \) is \( K \)-primary sequent \( \Sigma_1, K\Gamma_1, C\Delta_1 \to \Sigma_2, K\Gamma_2; S_2 \) is \( C \)-primary sequent;

2. Replacing a loop-type axiom by specialized loop-type axiom. A specialized loop-type axiom is a loop-type axiom, which is a \( C \)-primary sequent;

3. Using that the rule \((Sp)\) is admissible in the calculus \( I \in \{G_wRC, G_LRC\} \) we can prove that the calculi \( G^S_LRC \) and \( I \in \{G_wRC, G_LRC\} \) are equivalent, therefore the calculus \( G^S_LRC \) is sound and complete.
In construction of derivation it is convenient to use the following (admissible in $G^S_LRC$) rule:

$$
\frac{\Gamma \rightarrow A}{\Sigma_1, \Sigma_2 \rightarrow \Sigma_1, \Sigma_2 \rightarrow E(A), \Sigma_2 (E)}
$$

where $\Sigma_1 \cap \Sigma_2 = \emptyset$.

**Example 1.** Let $S = P, C(P \supset E(P)) \rightarrow C(P), K_1(Q)$. Let’s construct a derivation of $S$ in $G^S_LRC$. Since $S$ is splittable primary sequent let us backward apply to $S$ the rule $(Sp)$ and let us try to construct a derivation of $C$-primary sequent. $S_1 = P_1 C(P \supset E(P)) \rightarrow C(P)$.

$$
\begin{align*}
S_1^* &= P_1 C(P \supset E(P)) \rightarrow C(P) \\
\frac{P \rightarrow P; P_1 E(P), E(C(P \supset E(P)) \rightarrow E(C(P))}{\frac{P_1 (P \supset E(P), E(C(P \supset E(P)) \rightarrow E(C(P))}{(\supset \rightarrow)}}{S_1 = P_1 C(P \supset E(P)) \rightarrow C(P)}
\end{align*}
$$

Since $S_1 = S_1^*$, $S_1^*$ is a loop type axiom. Therefore $G^S_LRC \vdash S$.

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**REZIUMĖ**

**Išvedimų specializacija bendrojo žinojimo logikai**

A.P. Girčys, R. Pliuškevičius

Straipsnyje pateikiama ciklinių aksiomų specializacija refleksyviai bendro žinojimo logikai.

**Raktiniai žodžiai:** sekvencinis skaičiavimas, bendrojo žinojimo logika, ciklinės aksiomos.