Invisibility carpet in a channel with a structured fluid

G. Dupont\textsuperscript{a} S. Guenneau\textsuperscript{a,1} S. Enoch\textsuperscript{a}

\textsuperscript{a}Institut Fresnel, CNRS, Aix-Marseille Universit\'e, Campus Universitaire de Saint-Jerome, 13013 Marseille, France

Received May 2011; accepted after revision +++++

Presented by

Abstract

We first note it is possible to construct two linear operators defined on two different domains, yet sharing the same spectrum using a geometric transform. However, one of these two operators will necessarily have spatially varying, matrix valued, coefficients. This mathematical property can be used in the design of metamaterials whereby two different domains behave in the same electromagnetic, acoustic, or hydrodynamic way (mimetism). To illustrate this property, we describe a feasible invisibility carpet for linear surface liquid waves in a channel. This structured metamaterial bends surface waves over a finite interval of Hertz frequencies.

To cite this article: G. dupont, S. Guenneau, S. Enoch, C. R. Mecanique xxx (2011).

Résumé

Tapis d’invisibilité dans un canal avec un fluide structuré Nous observons en premier lieu qu’il est possible de construire deux opérateurs linéaires définis sur deux domaines distincts mais qui possèdent le même spectre de valeurs propres, par le truchement d’une transformation géométrique. Néanmoins, un des deux opérateurs aura nécessairement des coefficients hétérogènes et non scalaires. Cette propriété mathématique peut néanmoins être utilisée dans le design de métamatériaux grâce auxquels deux objets distincts présentent les mêmes caractéristiques acoustique, optique ou hydrodynamique (mimétisme). Pour illustrer notre propos, nous proposons un modèle réaliste de cylindres rigides judicieusement disposés (un métamatériau appelé tapis d’invisibilité) qui fonctionne sur une plage de fréquences hertziennes pour des vagues de faible amplitude dans un canal.

Key words: Eigenvalue problem; Transformation acoustics; Cloak; Carpet; Metamaterials; Finite Elements

Mots-clés : Problème aux valeurs propres ; Acoustique de Transformation ; Cloque ; Tapis ; Métamatériaux ; Eléments Finis

Email address: sebastien.guenneau@fresnel.fr (S. Guenneau).

1 Corresponding author

Preprint submitted to Elsevier Science 3 octobre 2011
Version française abrégée

Nous considérons un problème spectral modélisé qui consiste à trouver les couples de valeurs propres $\lambda$ et vecteurs propres associés $\phi$ tels que :

$$A_1(\phi) = -\Delta \phi = \lambda \phi, \quad (1)$$

dans un domaine $\Omega_1$ borné dans $\mathbb{R}^2$. Nous nous intéressons plus particulièrement au cas de conditions de Neumann au bord du domaine, en vue d’une application à l’acoustique (voir équations 3-5). Il est bien connu que la résolvante de cet opérateur $A_1$ est compacte dans l’espace de Hilbert $H^1(\Omega_1)$ (par injection compacte de $H^1(\Omega_1)$ dans $L^2(\Omega_1)$), et donc que le spectre $\sigma(A_1)$ de l’opérateur $A_1$ est un ensemble discret de valeurs propres réelles positives tendant vers $+\infty$ qui peuvent être rangées par ordre croissant (Gram-Schmidt).

Il est bon de noter que ce problème mathématique modélise par exemple la recherche de modes susceptibles de se propager dans un guide (acoustique, électromagnétique, hydraulique...) en régime harmonique, auquel cas la racine carrée de la valeur propre $\lambda$ à la dimension physique d’une fréquence par une vitesse.

La question que l’on se pose est de savoir si l’on peut construire un autre opérateur $A_2$ agissant sur un domaine borné $\Omega_2$ distinct de $\Omega_1$ dont le spectre $\sigma(A_2)$ est identique au précédent. La réponse est affirmative dans la mesure où l’on procède à un changement de variables qui applique le domaine $\Omega_1$ sur le domaine $\Omega_2$, cf. Figure 1. En effet, les couples de valeurs propres $\beta$ et vecteurs propres associés $\psi$ tels que :

$$A_2(\psi) = -T^{-1}_{33} \nabla \cdot T^{-1}_T \nabla \psi = \beta \psi, \quad (2)$$

où les valeurs propres $\beta$ sont réelles positives ($T$ est symétrique), peuvent-être mises en correspondance (une à une) avec les valeurs propres $\lambda$.

Ce tour de passe-passe est bien connu des spécialistes de l’inversion [1,2] dans le cadre de l’étude du problème de Calderon (qui revient à connaître les propriétés de l’application Dirichlet-Neumann [3]). Néanmoins, nous n’avons trouvé nulle part dans la littérature physique un exposé mathématique élémentaire qui prend la mesure de ces implications pour les problèmes aux valeurs propres dans les résonances de cavités : deux opérateurs linéaires définis sur des domaines bornés distincts peuvent présenter des spectres identiques, pourvu que l’un au moins ait des coefficients hétérogènes anisotropes, ce qui renvoie à la question de la reconnaissance de forme d’un tambour à travers sa signature acoustique [4].

Les aspects mathématiques sous-jacents dépassent le cadre de cette étude, mais nous présentons dans la Figure 2 et le Tableau 1 des résultats numériques qui appuient notre propos : les graphes de gauche de la figure 2 et les trois premières colonnes du tableau 1 de gauche démontrent qu’un canal à vagues...
à bord droit à un canal à bord courbe avec un fluide transformé (i.e. hétérogène anisotrope déduit de la formule (7), voir figure 3). Les graphes de droite de la figure 2 et les trois premières colonnes du tableau 1 de droite démontrent que les normes dans $L^2$ des fonctions propres sont très similaires (aux erreurs numériques près) pour les canaux à vagues à bord droit et à bord courbe avec un fluide transformé.

La version anglaise de l'article est quant à elle dédiée à l'étude d'un problème de diffraction en hydrodynamique qui est le pendant des problèmes aux valeurs propres analysés ci-dessus. L'accent est mis sur une simulation numérique avec un design de tapis structuré qui doit faire l'objet d'une validation expérimentale ultérieure dans un canal à houle. Il est intéressant de noter que les valeurs propres du
1. Setup of the hydrodynamic problem

The transformation based solutions to the Maxwell equations in curvilinear coordinate systems reported by Pendry et al. in [6] bend electromagnetic waves around arbitrarily sized and shaped surfaces (see also [7] for a conformal optics approach). The electromagnetic carpet is a metamaterial which maps a concealment region onto a surrounding surface: as a result of the coordinate transformation the permittivity and permeability are strongly heterogeneous and anisotropic within what physicists call an invisibility carpet [8], yet fulfilling impedance matching with the surrounding vacuum. The carpet thus neither scatters waves nor induces a shadow in the reflecting field.

In the present paper, we build upon the recent proposal by Li and Pendry [8] to map a curved surface onto a flat surface in order to control the wave front of an electromagnetic wave scattered by a bump located on a flat mirror so that if we now dress this bump with a heterogeneous anisotropic material, the wave seems to be reflected by a flat mirror, thereby making the bump invisible [8,9]. We actually design a structured material in order to mimick the prerequisite material properties deduced from a geometric transformation. We focus here on linear surface water waves propagating within a channel, but we emphasize that our approach is generic and works for any wave governed by a Helmholtz equation...
subject to Neumann boundary conditions e.g. anti-plane shear waves in an elastic material with cracks, pressure water waves in a fluid with rigid inclusions, transverse electric waves in a dielectric medium with infinite conducting inclusions.

Let Ω denote the region of a channel occupied by a fluid. The conservation of momentum leads to the Navier-Stokes equations:

\[
\rho \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) u - \mu \nabla^2 u = -\nabla p + \rho g , \quad \text{in } \Omega ,
\]

where \( u \) denotes the velocity field, \( t \) the time variable, \( \rho \) the density of the fluid \( \mu \) its viscosity, and \( g \) the gravity.

If we assume that the fluid is incompressible and irrotational, we know that \( u \) derives from a potential which under the hypothesis of small perturbations of the free interface separating the fluid with ambient atmosphere, leads to the Helmholtz equation:

\[
\nabla^2 \phi - \kappa^2 \phi = 0 ,
\]

with \( \kappa \) the spectral parameter related to the frequency of the wave \( \omega \) via the dispersion relation:

\[
\omega = \sqrt{g \kappa \tanh (h \kappa) \left( 1 + \frac{\kappa^2 \sigma}{g \rho} \right)} .
\]

Here, \( h \) is the depth of water in the channel and \( \sigma \) the surface tension at the free surface.

The linearized problem (4-5) allows for straightforward analogies between transverse electromagnetic and acoustic waves propagating in structured cylindrical domains, see [10] for the design of an invisibility cloak for surface liquid waves, experimentally shown to work between 10 and 15 Hertz (broadband).

### 2. Design of a heterogeneous anisotropic fluid

As we already announced in the French abridged version, our aim here is to approximate the spectrum of the Laplace operator defined on a bounded region of a certain shape with a perturbed Laplace operator defined on another bounded region of a different shape. In both cases, we assume Neumann boundary conditions, so that the resolvents of both operators are compact and their spectra consist only of a countable set of discrete eigenvalues with a single accumulation point (0 or infinity depending upon whether we look at the operator or its inverse) [11]. This allows for a one-to-one correspondence between the spectra of the Laplace operator \( A_1 \) and perturbed Laplace operators \( A_\eta \) associated with the structured fluid. The underlying asymptotic mechanism is that one wants to approximate each eigenvalue of \( A_1 \) by an eigenvalue of \( A_\eta \), in the limit when \( \eta \) goes to zero: The smaller \( \eta \), the larger the number of rigid cylinders (of order \( \eta^{-1} \)) of decreasing diameter (\( \sim \eta \)), the finer the approximation (in the homogenization limit) of the transformed fluid obtained by mapping the first region on the seconde one. This means that the sequence of spectra \( \sigma(A_\eta) \) of operators \( A_\eta \) should tend (pointwise) to the spectrum \( \sigma(A_1) \) of the operator \( A_1 \) when \( \eta \) tends to zero. As a result, if the first region \( \Omega_1 \) is filled with an isotropic homogeneous medium (say a fluid), the second one \( \Omega_2 \) (associated with the domain of an operator \( A_2 \)) is now filled with the same fluid, however with a collection of small rigid cylinders approximating an anisotropic heterogeneous fluid: \( \sigma(A_1) = \lim_{\eta \to 0} \sigma(A_\eta) = \sigma(A_2) \). We now want to numerically validate this conjecture: one can design a *meta-fluid* by structuring the second region with rigid cylinders, in which case it can be simply filled with an ordinary fluid. However, such an asymptotic approach can only work to certain extent (within the framework of effective medium theory, hence for small enough frequencies).
2.1. Geometric transform

Let us first introduce a simple geometric transform mapping the first region to the second one. The bottom line is the bold proposal by Li and Pendry to conceal an object that is placed under a curved reflecting surface by imitating the reflection of a flat surface [8] in the context of electromagnetic waves in open space. In the present case, the domain is bounded and the geometric transform reads as follows:

\[
\begin{align*}
    x' &= x \\
y' &= \frac{y_2 - y_1}{y_2} y + y_1 \\
z' &= z
\end{align*}
\]

with the associated Jacobian matrix

\[
J_{xx'} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial y}{\partial x'} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

and \( \alpha = (y_2 - y_1)/y_1 \).

The metric tensor associated with the transformed coordinates takes the following form (and its effect on the Cartesian metric is shown in figure 3):

\[
T^{-1} = J^{-1}_{xx'} J_{xx'} det(J_{xx'}) = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ -\frac{\partial y}{\partial x'} \left( 1 + \left( \frac{\partial y}{\partial x'} \right)^2 \right)^2 \alpha & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix}.
\]

Figure 3. Left: Metrics associated with the Cartesian coordinate system (original domain, leftmost panel) and the transformed coordinate system (invisibility carpet, right panel) mapped onto one another via the transformation matrix \( T \) (note that the right angles are not preserved i.e. the transformation is not conformal); Right: Numerical simulations at frequency \( \nu = 1.99 \text{Hz} \); (a) Field inside a straight channel filled with a homogeneous isotropic fluid; (b) Field inside a curved channel filled with a homogeneous isotropic fluid; (c) Field inside a curved channel filled with a heterogeneous anisotropic fluid described by formula (7). The color scale is in arbitrary units. The strong similarity between fields in (a) and (c) is noted.

It is interesting to look at the expression of the eigenvalues of \( T^{-1} \) as these are the relevant quantities to design a structured channel:

\[
\lambda_1 = \frac{1}{\alpha}, \lambda_i = \frac{1}{2\alpha} \left( 1 + \alpha^2 + \left( \frac{\partial y}{\partial x'} \right)^2 \alpha^2 + (-1)^{i-1} \sqrt{-4\alpha^2 + \left( 1 + \alpha^2 + \left( \frac{\partial y}{\partial x'} \right)^2 \alpha^2 \right)^2} \right) .
\]

We note that \( \lambda_1 \) and \( \lambda_i, i = 2, 3, \) are strictly positive functions as obviously \( 1 + \alpha^2 + \left( \frac{\partial y}{\partial x'} \right)^2 \alpha^2 > \sqrt{-4\alpha^2 + \left( 1 + \alpha^2 + \left( \frac{\partial y}{\partial x'} \right)^2 \alpha^2 \right)^2} \) and also \( \alpha > 0 \). This establishes that \( T^{-1} \) is not a singular matrix for
a two-dimensional carpet, which is a big advantage over two-dimensional cloaks obtained by blowing up a point onto a disc [8]: the transformation matrix is then singular at the cloak’s inner boundary (one eigenvalue goes to infinity, while the other two go to zero [3]).

2.2. **Structured fluid**

Let us now mimic the heterogeneous anisotropic fluid using an effective medium approach whereby an assembly of rigid cylinders judiciously located is now fixed to the bottom of the channel. It is clear that such a design will only work to certain extent and moreover will be constrained by the working eigenfrequency (the larger the eigenvalue of the operator, the larger the discrepancy between the ideal and approximated cases). In Figures 4 and 5, we show some representative fields corresponding to given eigenfrequencies in the range $1.72 \, Hz < \nu < 2.33 \, Hz$ for a curved channel with a carpet (Figure 4) and without a carpet (Figure 5). We emphasize that the wavefront of the fields is nearly flat in Figure 4. We report in table 1 the $L^2$ norm of these eigenfields and compare them to the benchmark of a straight channel and a curved channel filled with a transformed fluid. These numerical results clearly show the positive effect of the structured carpet. However, some care needs be taken when commenting these results, as shown by the discrepancy between the eigenvalues for spectral problems set in the straight channel and the curved channel filled with the *structured fluid*: cloaking is only achieved for the control of the fields’s wavefront, not for the eigenfrequencies.

![Figure 4. Eigenfields for $1.72 \, Hz < \nu < 2.33 \, Hz$ in a curved channel with the structured carpet. The flat wavefronts of all eigenfields is noted.](image-url)
Figure 5. Eigenfields for $1.72 \, \text{Hz} < \nu < 2.23 \, \text{Hz}$ in the same curved channel as in figure 4 but without the structured carpet. The disturbed wavefronts for the eigenfields is noted (except for the first two leftmost eigenfields).

3. Conclusions

In this note, we have reported some preliminary results on a structured invisibility carpet for the control of linear surface water waves in a channel. Unlike for the structured invisibility cloak some of us designed earlier for linear surface water waves [10] (which avoids any backscattering of an incident wave), the carpet mimics the backscattering of a flat boundary (i.e. it flattens the wavefront of backscattered waves). The numerical illustrations demonstrate the high potential for a practical realization of a meta-fluid working over a large bandwidth. We hope this analysis will foster experimental efforts towards a new generation of dykes without overtopping phenomena. Similar ideas could be implemented in the design of structured fluids for an enhanced control of pressure waves [12]. It should be finally pointed out that recent theoretical and experimental work drawing analogies between water waves and cosmological physics [13,14] suggests new avenues for structured meta-fluids in the non-linear regime.

Acknowledgement

Mr Guillaume Dupont is thankful for a PhD funding from the University of Aix-Marseille III.

References

[1] A. Greenleaf, M. Lassas and G. Uhlmann, On nonuniqueness for Calderons inverse problem, Math. Res. Lett. 10, 685 (2003).

[2] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Isotropic transformation optics: approximate acoustic and quantum cloaking, New J. Phys. 10, 115024 (2008).
[3] R.V. Kohn, H. Shen, M.S. Vogelius, and M.I. Weinstein, Cloaking via change of variables in electric impedance tomography, Inverse Problems 24, 015016 (2008).

[4] M. Kac, Can one hear the shape of a drum? Am. Math. Mon. 73, 1-23 (1966).

[5] S. Guenneau, A.B. Movchan, F. Zolla, N.V. Movchan and A. Nicolet, Acoustic band gaps in arrays of neutral inclusions, J. Comput. Appl. Math. 234, 1962-1969 (2010).

[6] J.B. Pendry, D. Schurig, and D.R. Smith, Controlling electromagnetic fields, Science 312, 1780 (2006).

[7] U. Leonhardt, Optical conformal mapping, Science 312, 1777 (2006).

[8] J. Li and J.B. Pendry, Hiding under the Carpet: A New Strategy for Cloaking, Phys. Rev. Lett. 101, 203901, 2008.

[9] G. Dupont, S. Guenneau and S. Enoch, Electromagnetic analysis of arbitrarily shaped pinched carpets, Phys. Rev. A 82, 033840 (2010).

[10] M. Farhat, S. Enoch, S. Guenneau and A.B. Movchan, Broadband invisibility for linear surface water waves, Phys. Rev. Lett. 101, 134501 (2008).

[11] C. Conca, J. Planchard, M. Vanninathan, Fluids and periodic structures, John Wiley and Sons, 1997.

[12] D. Torrent and J. Sanchez-Dehesa, Acoustic cloaking in two dimensions: A feasible approach, New J. Phys. 10, 063015 (2008).

[13] G. Rousseaux, C. Mathis, P. Maissa, T. G. Philbin, and U. Leonhardt, Observation of negative-frequency waves in a water tank: a classical analogue to the Hawking effect?, New J. Phys. 10, 053015 (2008).

[14] G. Rousseaux, P. Maissa, C. Mathis, P. Coullet, T. G. Philbin, and U. Leonhardt, Horizon effects with surface waves on moving water, New J. Phys. 12, 095018 (2010).