An exactly solvable model of two three-dimensional harmonic oscillators interacting with the quantum electromagnetic field: the Casimir-Polder potential

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Abstract

We consider two three-dimensional isotropic harmonic oscillators interacting with the quantum electromagnetic field in the Coulomb gauge and within dipole approximation. Using a Bogoliubov-like transformation, we can obtain transformed operators such that the Hamiltonian of the system, when expressed in terms of these operators, assumes a diagonal form. We are also able to obtain an expression for the energy shift of the ground state, which is valid at all orders in the coupling constant. From this energy shift the nonperturbative Casimir-Polder potential energy between the two oscillators can be obtained. When approximated to the fourth order in the electric charge, the well-known expression of the Casimir-Polder potential in terms of the polarizabilities of the oscillators is recovered.

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I. INTRODUCTION

The Casimir-Polder potential is a long range interaction between neutral atoms/molecules in vacuo. This potential arises from the interaction of the two atoms with the common radiation field. It is also a result of the quantum field fluctuations in the vacuum state, and it is one of the few unambiguous manifestations of the quantum nature of the electromagnetic field.

The Casimir-Polder potential is usually obtained by fourth-order perturbation theory. Higher-order corrections exist, as well as non-additive components in the case of three or more atoms. More recently Casimir-Polder forces for excited atoms have been considered, even if the precise meaning of a dressed excited state is not clear, as well as time-dependent Casimir-Polder energies for partially dressed atoms or atoms in an excited-state.

The Casimir-Polder potential can be expressed in terms of the dynamical polarizabilities of the two atoms/molecules, and this makes clear that a similar potential is expected whenever we consider two neutral systems with a discrete spectrum interacting with a quantum field (not necessarily in the framework of quantum electrodynamics, but also in condensed matter physics, for example).

In this paper we consider two harmonic oscillators interacting with the electromagnetic radiation field. This system has the advantage that exact solutions can be obtained by a Bogoliubov-like transformation, as we will show in this paper. It is also a good representation for atomic systems; approximating atoms by harmonic oscillators is a common procedure, due to the resulting simplification of calculations (for a few examples, see [11, 12, 13]). An harmonic oscillator interacting with a one-dimensional scalar field has been recently considered in order to study phenomena relevant to quantum optics, and exactly solved by a Bogoliubov transformation [8, 14]; similar methods have been used also for the description of unstable states in relativistic field theory [15]. An exact solution for two two-level atoms interacting with a one-dimensional scalar field, in the one excitation sector, has been recently obtained for the description of decaying states in the rotating wave approximation [16]. The rotating wave approximation, however, prevents application of these results to the Casimir-Polder potential, where the virtual processes play an essential role.

We are mainly interested in the study of the Casimir-Polder potential. We therefore con-
Consider two spatially separated isotropic three-dimensional harmonic oscillators, (thus having a spherical symmetry), interacting with the quantum electromagnetic field. Counterrotating terms and the related virtual processes are included in the Hamiltonian. By a Bogoliubov-like transformation we can obtain an exact diagonalization of the Hamiltonian describing this system. The Hamiltonian of the system, in terms of the transformed operators, has the form of a free field Hamiltonian. Moreover, using the inverse transformations we are also able to obtain the exact energy shift of the ground state of the system, due to the interaction between the oscillators and the radiation field. This result is exact within the model considered and valid at all orders in the coupling constant. The part of the energy shift depending on the distance between the two oscillators yields the Casimir-Polder potential between the ground-state oscillators, both in the so-called near and far zones. Known results are reproduced by approximation to the fourth-order in the electric charge. Due to the generality of our model, and the fact that the validity of our results is not limited by perturbation theory, the results obtained in this paper can be also used to obtain Casimir-Polder-like potentials in all cases in which the coupling constant is not small. We also expect that our nonperturbative approach can be useful in dealing with Casimir-Polder forces for atoms in excited states, being able to overcome the conceptual difficulties related to the vanishing energy denominators occurring in the perturbative expansions when excited states are considered [6, 17].

In Section II we introduce the Hamiltonian describing our system and the transformation used to obtain the new operators; in terms of the new operators, the Hamiltonian assumes a diagonal form. The explicit relation between the new and old operators is also obtained, as well as the inverse relations. In section III we obtain the energy shift of the ground state of the interacting system, from which the Casimir-Polder potential between the two isotropic harmonic oscillators is obtained. The physical meaning of our results is discussed, as well as possible applications to Casimir-Polder forces for excited atoms.

II. HAMILTONIAN AND BOGOLIUBOV TRANSFORMATION

We consider two isotropic three-dimensional harmonic oscillators with frequency $\omega_0 = ck_0$ interacting with the electromagnetic field in the Coulomb gauge. In the multipolar coupling
scheme the Hamiltonian of this system is

\[ H = \sum_{i=1,2} \sum_{\ell=x,y,z} k_0 a_{i\ell}^\dagger a_{i\ell} + \sum_{kj} k B_{kj}^\dagger B_{kj} + H_{int}^1 + H_{int}^2 \]  

(1)

(we use units such that \( \hbar = 1 \) and \( c = 1 \)). The index \( i \) labels the two oscillators (\( i = 1, 2 \)), the index \( \ell \) the space direction of the oscillator (\( \ell = x, y, z \)), \( a_{i\ell}^\dagger, a_{i\ell} \) are the creations and annihilation operators of the two three-dimensional oscillators and \( B_{kj}^\dagger, B_{kj} \) for the field (\( j \) is the polarization index), \( k = |k| \). The creation and annihilation operators in (1) satisfy the usual bosonic commutation rules

\[ [a_{i\ell}, a_{i'\ell'}^\dagger] = \delta_{ii'}\delta_{\ell\ell'}, \quad [B_{kj}, B_{kj'}^\dagger] = \delta_{kk'}\delta_{jj'} . \]  

(2)

\( H_{int}^1 \) and \( H_{int}^2 \) are the interaction Hamiltonians of the oscillators 1 and 2 with the field, respectively. Within the dipole approximation, they are given by

\[ H_{int}^1 = \sum_{\ell=x,y,z} \sum_{kj} f_{kj}^{1\ell} (a_{1\ell} + a_{1\ell}^\dagger) (B_{kj} - B_{kj}^\dagger) \]  

(3)

\[ H_{int}^2 = \sum_{\ell=x,y,z} \sum_{kj} f_{kj}^{2\ell} (a_{2\ell} + a_{2\ell}^\dagger) (B_{kj}e^{ikr} - B_{kj}^\dagger e^{-ikr}) \]  

(4)

where the atom 1 is supposed to be located at the origin and the atom 2 at \( r \), and the coupling constants are

\[ f_{kj}^{\ell} = -i\sqrt{\frac{2\pi k}{V}} \mu_{i\ell} \cdot \hat{e}_{kj} \]  

(5)

\( \mu_{i\ell} \) is the transition dipole moment of the atom \( i \) in the \( \ell \) direction and \( V \) is the quantization volume; the dipole moments are supposed real.

In order to diagonalize the Hamiltonian (1), we look for a linear transformation of the operators, similarly to the case of just one one-dimensional oscillator interacting with a scalar field [14]. Thus we write the new operators \( b_{kj}, b_{kj}^\dagger \) in terms of the old operators in the following form

\[ b_{kj}^\dagger = \sum_{i=1,2} \sum_{\ell=x,y,z} (t_{kj}^{i\ell} a_{i\ell}^\dagger + r_{kj}^{i\ell} a_{i\ell}) + \sum_{k'j'} (T_{kj}^{k'j'} B_{k'j'}^\dagger + R_{kj}^{k'j'} B_{k'j'}) \]  

(6)

and its Hermitian conjugate for the annihilation operators \( b_{kj} \). The coefficients \( t_{kj}^{i\ell}, r_{kj}^{i\ell}, T_{kj}^{k'j'}, R_{kj}^{k'j'} \) are chosen in such a way that the “new” operators satisfy the free-field
and the r-dependent function

\[ [H, b^\dagger_{kj}] = k^2 b^\dagger_{kj} \]
\[ [H, b_{kj}] = -k b_{kj} \]  

(7)

Substitution of (8) into (7) yields the following set of coupled equations for the coefficients

\[(k - k_0)t_{kj}^{1\ell} = \sum_{k'j'} f_{k'j'}^{1\ell} (T_{k'j'}^{k'j'} + R_{k'j'}^{k'j'}) \]
\[(k + k_0)r_{kj}^{1\ell} = \sum_{k'j'} f_{k'j'}^{1\ell} (T_{k'j'}^{k'j'} + R_{k'j'}^{k'j'}) \]
\[(k - k_0)t_{kj}^{2\ell} = \sum_{k'j'} f_{k'j'}^{2\ell} (T_{k'j'}^{k'j'} e^{ik'r} + R_{k'j'}^{k'j'} e^{-ik'r}) \]
\[(k + k_0)r_{kj}^{2\ell} = \sum_{k'j'} f_{k'j'}^{2\ell} (T_{k'j'}^{k'j'} e^{ik'r} + R_{k'j'}^{k'j'} e^{-ik'r}) \]
\[(k - k')T_{k'j'}^{k'j'} = \sum_{k'j'} f_{k'j'}^{1\ell} (r_{k'j'}^{1\ell} - t_{k'j'}^{1\ell}) + \sum_{k'j'} f_{k'j'}^{2\ell} e^{-ik'r} (r_{k'j'}^{2\ell} - t_{k'j'}^{2\ell}) \]
\[ -(k + k')R_{k'j'}^{k'j'} = \sum_{k'j'} f_{k'j'}^{1\ell} (r_{k'j'}^{1\ell} - t_{k'j'}^{1\ell}) + \sum_{k'j'} f_{k'j'}^{2\ell} e^{ik'r} (r_{k'j'}^{2\ell} - t_{k'j'}^{2\ell}) \]

(12)

(13)

This set of coupled equations can be solved exactly. The solution can be expressed in terms of the resolvent

\[(G_i(k))^{-1} = \frac{1}{k_0^2 - k^2} - 2k_0 \sum_{k'j'} \frac{2k'}{k^2 - k^2} f_{k'j'}^{1\ell} f_{k'j'}^{2\ell} \]

(14)

and the r-dependent function

\[
\sigma_\ell(k, r) = -2k_0 \sum_{k'j'} \left( e^{ik'r} \frac{k + k'}{k - k'} - e^{-ik'r} \frac{k - k'}{k + k'} \right) f_{k'j'}^{1\ell} f_{k'j'}^{2\ell} \]

(15)

For isotropic oscillators, as we assume, the resolvent (14) does not depend on \( \ell \). Explicit expressions of (14) and (15) in the continuous limit are given in the Appendix A. In this limit, the two functions (14) and (15) have poles for \( k = k' \), and must be extended to the complex plane

\[
G_i^{(\pm)}(k) = G_i(k \pm i\epsilon) \]
\[ \sigma_\ell^{(\pm)}(k, r) = \sigma_\ell(k \pm i\epsilon, r) \]

(16)

(17)

However, whenever unnecessary, for simplicity of notations we shall omit the apex relative to the choice of the analytic continuation of these functions. For convenience of notations the two functions defined in (14) and (15) can be expressed as the elements of a symmetric
2x2 matrix, with \((G_1)^{-1}\) and \((G_2)^{-1}\) as diagonal elements and \(\sigma_\ell\) as off-diagonal elements, apart from a proportional factor. So we define the matrix

\[
M_\ell(k, r) = \frac{G_1 G_2}{1 - \sigma_\ell^2 G_1 G_2} \begin{pmatrix} G_2^{-1} & -\sigma_\ell \\ -\sigma_\ell & G_1^{-1} \end{pmatrix}
\]

We also define the following two-dimensional column vectors

\[
f^\ell_{kj} = \begin{pmatrix} f_{k_j}^\ell \\ f_{k_j' e^{i k \cdot r}}^\ell \end{pmatrix} \quad t^\ell_{kj} = \begin{pmatrix} t_{k_j}^\ell \\ t_{k_j'}^\ell \end{pmatrix} \quad r^\ell_{kj} = \begin{pmatrix} r_{k_j}^\ell \\ r_{k_j'}^\ell \end{pmatrix}
\]

The following relations exist among \(r_{kj}^\ell\) and \(t_{kj}^\ell\), and among \(R_{kj'}^{k' j'}\) and \(T_{kj}^{k' j'}\)

\[
r_{kj}^\ell = \frac{k - k_0}{k + k_0} t_{kj}^\ell \\
R_{kj}^{k' j'} = \frac{k' - k}{k' + k} T_{kj}^{-k' j'}
\]

The solution for the coefficients defined in (6) can be expressed in the compact form

\[
t_{kj}^\ell = \frac{k + k_0}{k + k_0} f_{kj}^\ell \\
r_{kj}^\ell = - (k - k_0) M_\ell f_{kj}^\ell \\
T_{kj}^{k' j'} = \delta_{jj'} \delta_{kk'} + \frac{2k_0}{k - k'} (f_{k' j'}^\ell)^\dagger M_\ell f_{kj}^\ell \\
R_{kj}^{k' j'} = - \frac{2k_0}{k' + k} (f_{k' j'}^\ell)^\dagger M_\ell f_{kj}^\ell
\]

The correct commutation relation for the transformed operators

\[
[b_{kj}, b_{k' j'}^\dagger] = \delta_{jj'} \delta_{kk'}
\]

can be proved by expressing \(b_{kj}\) and \(b_{k' j'}^\dagger\) in terms of the “old” operators by means of eq. (6) and eqs. (23, 24, 25, 26). Equations (A7) and (A8) of Appendix A have been used in order to obtain the commutation relations (27).

The inverse relations, expressing old operators in terms of new operators, can be also obtained in the form

\[
d_{ij}^\dagger = \sum_{k_j} \left( \lambda_{ij}^{k_j} b_{kj}^\dagger + \tau_{ij}^{k_j} b_{kj} \right)
\]

\[
B_{kj}^\dagger = \sum_{k' j'} \left( \mu_{k_j}^{k' j'} b_{k' j'}^\dagger + \eta_{k_j}^{k' j'} b_{k' j'} \right)
\]
and the Hermitian conjugate relations for $a_{i\ell}$ and $B_{kj}$. The coefficients appearing in eqs. (28,29) can be obtained with the same procedure used for the “direct” relations. The result for the $\lambda$ coefficients is

$$
\lambda_{1\ell}^{kj} = \frac{(k + k_0) G_1(k)}{1 - \sigma^2_\ell(k, r) G_1(k) G_2(k)} \left( f_{k_j}^{1\ell} - f_{k_j}^{2\ell} e^{-i k r} \sigma_\ell(k, r) G_2(k) \right)
$$

(30)

$$
\lambda_{2\ell}^{kj} = \frac{(k + k_0) G_2(k)}{1 - \sigma^2_\ell(k, r) G_1(k) G_2(k)} \left( f_{k_j}^{2\ell} e^{-i k r} - f_{k_j}^{1\ell} \sigma_\ell(k, r) G_1(k) \right)
$$

(31)

The other coefficients may be expressed in terms of the $\lambda$ coefficients as

$$
\rho_{i\ell}^{kj} = \frac{k - k_0}{k + k_0} \left( \lambda_{i\ell}^{kj} \right)^*
$$

(32)

$$
\mu_{kj}^{k'j'} = \delta_{jj'} \delta_{kk'} + \frac{2k_0}{(k + k_0)(k' - k)} \left( \sum \lambda_{1\ell}^{kj} \lambda_{1\ell}^{k'j'} + \sum \lambda_{2\ell}^{kj} \lambda_{2\ell}^{k'j'} \right)
$$

(33)

$$
\eta_{kj}^{k'j'} = \frac{k' - k}{k' + k} \left( \mu_{kj}^{k'j'} \right)^*
$$

(34)

III. THE GROUND-STATE ENERGY

We now evaluate the energy shift of the ground-state energy due to the interaction of the two harmonic oscillators with the electromagnetic field. The ground-state energy has terms which depend on the distance $r$ between the two oscillators; this yields a Casimir-Polder potential between the two oscillators.

In order to evaluate the energy of the ground state we substitute the inverse relations (30,31,32,33,34) in the Hamiltonian (1). After this substitution, the Hamiltonian in terms of the new operators assumes a diagonal form

$$
H = \sum_{kj} k^i b_{kj}^i b_{kj} + E_0
$$

(35)

where $E_0$ is the ground-state energy. The part of $E_0$ depending from the interatomic distance $r$ gives the Casimir-Polder potential energy between the oscillators. After extensive algebraic calculations, and use of eqs. (A9) and (A10) of Appendix A we obtain

$$
E_0 = k_0 \sum_{kj} \left( 3k^2 - 2 k_0 k - k_0^2 \right) \sum_{i\ell} \lambda_{i\ell}^{kj} \left( \lambda_{i\ell}^{kj} \right)^* - 4 \omega_0^2 \sum_{k'j'} \frac{k}{(k + k')^2} \left( \lambda_{2\ell}^{k'j'} \left( \lambda_{2\ell}^{k'j'} \right)^* \right.

\sum_{\ell} \left[ \left( f_{k_j}^{1\ell} \right)^2 \frac{\lambda_{1\ell}^{k'j'} \left( \lambda_{1\ell}^{k'j'} \right)^*}{(k' + k_0)^2} + \left( f_{k_j}^{2\ell} \right)^2 \frac{\lambda_{2\ell}^{k'j'} \left( \lambda_{2\ell}^{k'j'} \right)^*}{(k' + k_0)^2}

+ \left( f_{k_j}^{1\ell} f_{k_j}^{2\ell} e^{-i k r} \lambda_{1\ell}^{k'j'} \left( \lambda_{2\ell}^{k'j'} \right)^* \right) \frac{1}{(k' + k_0)^2} + c.c. \right]

(36)
(it should be noted that the $\lambda$ functions in (36) depend on $r$). This expression is valid at any order in the electric charge. In the continuous limit, using the results of Appendix A, we can obtain an explicit expression of the shift of the ground state energy $E_0$. By retaining only the $r$ dependent parts of eq. (36) up to the $4^{th}$ order in the charge, approximating to the far zone $(k, k' \ll k_0)$ [2], the following expression for the interatomic potential $V_{CP}(r)$ is obtained

$$V_{CP}(r) = \frac{16}{\pi^2 k_0^2} \left(2I_1 + I_2\right) \frac{1}{r^7}$$

(37)

where $I_1$ and $I_2$ stand for the following integrals

$$I_1 = \int_0^\infty dx \int_0^\infty dy \frac{x^2 y^4}{x^2 - y^2} \left[x j_0(x) \left(j_0(y) - j_1(y)\right) + j_1(x) \left(3j_1(y) - j_0(y)\right)\right]$$

(38)

$$I_2 = \int_0^\infty dx \int_0^\infty dy \frac{x^2 y^4}{(x + y)^2} \left[x j_0(x) \left(j_0(y) - j_1(y)\right) + j_1(x) \left(3j_1(y) - j_0(y)\right)\right]$$

(39)

Once the above integrals are computed by standard methods as $I_1 = -\frac{23}{16\pi}$ and $I_2 = \frac{23}{16\pi}$, the final result is

$$V_{CP}(R) = -\frac{23}{16\pi} \frac{\alpha_1 \alpha_2}{r^7}$$

(40)

where $\alpha_i (i = 1, 2)$ is the static polarizability of the $i$-th three-dimensional isotropic harmonic oscillator, that is

$$\alpha_i = \frac{2}{3} \sum_{\ell=x,y,z} \frac{\mu_i^2}{k_0} = \frac{2 \mu_i^2}{k_0}$$

(41)

The quantity (40) is the well-known expression of the Casimir-Polder potential in the far zone and for isotropic molecules [2, 3]. Therefore, in the appropriate limits, the exact result (36) of the ground-state energy $E_0$ yields the usual Casimir-Polder potential energy and this confirms the validity of our approach.

IV. CONCLUSIONS

We have considered a system of two isotropic three-dimensional harmonic oscillators interacting with the electromagnetic radiation field, described in the Coulomb gauge and in the dipole approximation. Through an appropriate transformation we are able to diagonalize the Hamiltonian of this system and obtain exact solutions. The diagonalized Hamiltonian

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is an infinite set of harmonic oscillators with the same spectrum of the free electromagnetic field, plus an energy shift. We have obtained an expression of this energy shift valid at all orders in the coupling constant, which contains terms depending from the distance $r$ between the two three-dimensional oscillators. These $r$-dependent terms yield a potential energy between the two oscillators in the ground state of the interacting system, i.e. the Casimir-Polder interaction. Our results allow to obtain this energy exactly, at any order in the coupling constant. Approximation to the fourth order in the coupling constant reproduces known results of the Casimir-Polder long-range interaction in terms of the static polarizabilities of the oscillators (in the far zone). We plan to discuss in a future publication the extension of the results obtained in this paper to the case of the Casimir-Polder potential for excited states. In this case, we should be able to obtain the potential without the well-known difficulties related to vanishing energy denominators that appear when the stationary perturbation theory is applied to excited states.

Finally, we wish to stress that our method is quite general and not limited to systems interacting with the electromagnetic field. Our results can be easily extended to other matter-field interacting systems, for example the interaction with phonons. Thus the method presented in this paper could be particularly useful for various physical systems with a strong coupling constant.

**APPENDIX A: THE FUNCTIONS $G_i(k)$ AND $\sigma(k,r)$ IN THE CONTINUOUS LIMIT**

In this Appendix we derive explicit expressions for the functions defined in (14) and (15), in the case of isotropic oscillators, as well as quantities useful for the explicit evaluation of the commutation relations (27) and of energy shift $E_0$ given in Sec. III. The hypotheses of isotropic oscillators allows significant simplifications of the final equations. In equation (14) the summation over the polarization index $j'$ can be done using $\sum_{j'}(\hat{e}_{k'j'})_m(\hat{e}_{k'j'})_n = \delta_{mn} - \hat{k}'_m \hat{k}'_n$. Also, in the continuous limit

$$\sum_{k'} \rightarrow \frac{V}{(2\pi)^3} \int dk' k'^2 \int d\Omega'$$  \hspace{1cm} (A1)

Thus, after polarization sum and angular integration we obtain

$$\frac{V}{(2\pi)^3} \int d\Omega' \sum_{j'} f^{\text{el}}_{k'j'} f^{\text{el}}_{k'j'} = -\frac{2}{3\pi} k' \mu_1^2 \delta_{\ell\ell'}$$  \hspace{1cm} (A2)
where we have used that for symmetrical oscillators the strength of the dipole moment in the three spatial directions is the same, here indicated with $\mu_i$ ($i = 1, 2$).

We can proceed similarly for the quantity appearing in (15). After summation over polarizations and angular integration, we obtain

$$V (2\pi)^3 \int d\Omega' e^{\pm ik' \cdot r} \sum_{j'} f^1_{k'j'} f^{2\ell}_{k'j'} = -\frac{k'}{\pi} h_{\ell}(k' r) \mu_1 \mu_2 \delta_{\ell\ell}$$  \hfill (A3)

The function $h_{\ell}(k' r)$ is defined by

$$h_{\ell}(k' r) = \begin{cases} j_0(k' r) - \frac{1}{k' r} j_1(k' r) & \text{for } \ell = x, y \\ \frac{2}{k' r} j_1(k' r) & \text{for } \ell = z \end{cases}$$  \hfill (A4)

where the axis $z$ has been taken along the direction of $r$, and $j_0, j_1$ are spherical Bessel's functions.

Thus, in the case of isotropic oscillators the functions (14) and (15) take the form

$$(G_i(k))^{-1} = k_0^2 - k^2 - \frac{4}{3\pi} k_0 \mu_i^2 \int dk' \frac{2k'^4}{k'^2 - k^2}$$  \hfill (A5)

$$\sigma_\ell(k, r) = -\frac{2}{\pi} k_0 \mu_1 \mu_2 \int dk' \frac{2k'^4}{k'^2 - k^2} h_{\ell}(k' r)$$  \hfill (A6)

Equation (A5) contains an ultraviolet divergence that, however, is inessential for our purposes. (A6) is convergent due to the oscillating behaviour of $h_{\ell}(k' r)$.

By using the definitions (14) and (15) of $(G_i(k))^{-1}$ and $\sigma_\ell(k, r)$, the following identities, which have been used to obtain the commutation relations (27) and to evaluate the energy shift $E_0$, hold

$$\sum_{k j} (f^{(1)}_{k j})^2 F^{(1)}_{k', k}(k) = \frac{1}{2k_0} \left[ \frac{(G_i(k'))^{-1} - (G_i(\tilde{k}))^{-1}}{k' - \tilde{k}} + (k' + \tilde{k}) \right]$$  \hfill (A7)

$$\sum_{k j} f^{(1)}_{k j} f^{(2)}_{k j} e^{\pm ik \cdot r} F^{(1)}_{k', k}(k) = \frac{\sigma_\ell(k', r) - \sigma_\ell(\tilde{k}, r)}{2k_0(\tilde{k} - k')}$$  \hfill (A8)

$$\sum_{k j} k (f^{(2)}_{k j})^2 F^{(2)}_{k', k}(k) = \frac{k_0}{2} - \frac{k^3 + \tilde{k}^3 + k' (G_i(k'))^{-1} + \tilde{k} (G_i(\tilde{k}))^{-1}}{2k_0(k' + \tilde{k})}$$  \hfill (A9)

$$\sum_{k j} k f^{(1)}_{k j} f^{(2)}_{k j} e^{\pm ik \cdot r} F^{(2)}_{k', k}(k) = \frac{1}{2k_0(k' + \tilde{k})} \left[ k' \sigma_\ell(k', r) + \tilde{k} \sigma_\ell(\tilde{k}, r) \right]$$  \hfill (A10)
where $F_{k',k}(k)$ and $F_{k',\tilde{k}}(k)$ are given by

\begin{align}
F_{k',k}(k) &= \frac{1}{(k'-k)(\tilde{k}-k)} - \frac{1}{(k+k)(k'+k)} = \frac{2k}{k-k'} \left( \frac{1}{k^2-k'^2} - \frac{1}{\tilde{k}^2-k'^2} \right) \quad (A11) \\
F_{k',\tilde{k}}(k) &= \frac{1}{(k'-k)(\tilde{k}+k)} + \frac{1}{(k-k)(k'+k)} = -\frac{2}{k+k'} \left( \frac{k'}{k^2-k'^2} + \frac{\tilde{k}}{k^2-\tilde{k}^2} \right) \quad (A12)
\end{align}

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