A MOMENT INEQUALITY AND POSITIVITY FOR SIGNED GRAPH LAPLACIANS

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Abstract. A number of recent papers have considered signed graph Laplacians, a generalization of the classical graph Laplacian, where the edge weights are allowed to take either sign. In the classical case, where the edge weights are all positive, the Laplacian is positive semi-definite with the dimension of the kernel representing the number of connected components of the graph. In many applications one is interested in establishing conditions which guarantee the positive semi-definiteness of the matrix. In this paper we present an inequality on the eigenvalues of a weighted graph Laplacian (where the weights need not have any particular sign) in terms of the first two moments of the edge weights. This bound involves the eigenvalues of the equally weighted Laplacian on the graph as well as the eigenvalues of the adjacency matrix of the line graph (the edge-to-vertex dual graph). For a regular graph the bound can be expressed entirely in terms of the second eigenvalue of the equally weighted Laplacian, an object that has been extensively studied in connection with expander graphs and spectral measures of graph connectivity. We present several examples including Erdős–Rényi random graphs in the critical and subcritical regimes, random large d-regular graphs, and the complete graph, for which the inequalities here are tight.

Key words. Signed Laplacian, Eigenvalue Inequality

1. Introduction. There are a number of problems in applied mathematics where one is led to consider the eigenvalues of a signed (combinatorial) graph Laplacian: given a graph $G$ with $N$ vertices and $E$ edges the signed combinatorial Laplacian is an $N \times N$ matrix of the form

$$L_{ij}(\gamma) = \begin{cases} 
\sum_{k \neq i} \gamma_{ik} & i = j \\
-\gamma_{ij} & i \neq j, \ i \sim j \\
0 & i \neq j, \ i \not\sim j.
\end{cases}$$

(1.1)

Here $i \sim j$ denotes the relation that distinct vertices $i$ and $j$ are connected by an edge, $\gamma_{ij}$ denotes the weight of edge $ij$ and $\gamma \in \mathbb{R}^E$ is the vector of all edge weights. In this note Laplacian matrices are always symmetric, that is, $\gamma_{ij} = \gamma_{ji}$ for each $i$ and $j$. Note that the vector $1_N = (1, 1, 1, \ldots, 1)^t$ is always in the kernel of $L(\gamma)$. In the classical case where the edge weights are positive, $\gamma_{ij} \geq 0$, the matrix is positive semi-definite but in this paper the edge weights $\gamma_{ij}$ are not assumed to have any particular sign. An incomplete list of the applications of such matrices includes:

- Data mining in social networks [1, 2].
- Hypergraph clustering algorithms [3].
- Models for the evolution of multi-agent networks [4, 5].
- Finding the fastest mixing linear consensus model [6, 7].
- The stability of phase-locked solutions to the Kuramoto and related models [8, 9, 10, 11, 12, 13].

In many of these applications one is interested in establishing the semi-definiteness of a Laplacian matrix, which typically implies stability of the associated fixed point or a consensus state. For this reason a number of papers have considered the problem of establishing semi-definiteness of the Laplacian matrix, as in [14, 15, 16, 17, 18].

The purpose of this note is to present an inequality (Theorem 1.2) on the eigenvalues of a Laplacian matrix in terms of the first two moments — the mean and variance — of the edge weights. Following this we give a proof of the inequality and applications to both deterministic and random graphs. We first define:

$$Q = \frac{1}{E} \sum_{i \gg j} \gamma_{ij} \quad \text{and} \quad P = \frac{1}{E} \sum_{i > j} \gamma_{ij}^2.$$

(1.2)

Note that the Cauchy–Schwartz inequality implies that $P - Q^2 \geq 0$.

In what follows we will denote the $i$th eigenvalue of a symmetric $N \times N$ matrix $L$ by $\lambda_i(L)$, numbered in increasing order of absolute values $|\lambda_1(L)| \leq |\lambda_2(L)| \leq \cdots \leq |\lambda_N(L)|$. 

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DEFINITION 1.1. For a connected graph let $L(1_E)$ denote the equally weighted Laplacian—the (combinatorial) Laplacian on the graph $G$ with all edge weights taken to be unity: $\gamma_{ij} = 1$.

$$L_{ij}(1_E) = \begin{cases} \deg(v_i) & i = j \\ -1 & i \neq j, i \sim j \\ 0 & i \neq j, i \not\sim j, \end{cases}$$

where $\deg(v_i)$ is the degree of vertex $i$. In particular, let $\lambda_i^G = \lambda_i(L(1_E))$ denote the $i$th eigenvalue of $L(1_E)$, numbered in increasing order.

$$0 = \lambda_1^G < \lambda_2^G \leq \lambda_3^G \leq \cdots \leq \lambda_N^G. \tag{1.3}$$

To state our main result let $A^{L(G)}$ denote the adjacency matrix of the line graph of $G$ and define the quantity

$$\mu = \max_{\gamma \in \mathbb{R}^E : \gamma \perp 1_E} \frac{\langle \gamma, (A + A^{L(G)})\gamma \rangle}{\|\gamma\|^2}.$$ 

One has the inequalities

$$4 + \lambda_{E-1}(A^{L(G)}) \leq \mu \leq 4 + \lambda_E(A^{L(G)}) \leq 2d_{\max} + 2, \tag{1.4}$$

where $d_{\max}$ is the maximum degree of the vertices in the graph $G$. In the case of a $d$-regular graph we have the equality $\mu = 2d + 2 - \lambda_2^G = 4 + \lambda_{E-1}(A^{L(G)})$ (see the proof of Theorem 1.2).

THEOREM 1.2. Consider a weighted Laplacian matrix $L(\gamma)$ defined as in (1.1) on a connected graph $G$ with $\lambda_2^G$ and $\lambda_N^G$ as in (1.3) and $N \geq 3$. If $P$ and $Q$ are defined as in (1.2) then the $N - 1$ eigenvalues of $L(\gamma)$ corresponding to eigenvectors orthogonal to $1_N$ satisfy the inequality

$$Q\lambda_2^G - \sqrt{E(P - Q^2)\mu \frac{N - 2}{N - 1}} \leq \lambda_i(L(\gamma)) \leq Q\lambda_N^G + \sqrt{E(P - Q^2)\mu \frac{N - 2}{N - 1}}. \tag{4.1}$$

Further this inequality is tight: for the complete graph there are choices of edge weights realizing the upper and lower bounds.

In particular, if $G$ is a $d$-regular graph then

$$Q\lambda_2^G - \sqrt{E(P - Q^2)(2d + 2 - \lambda_2^G) \frac{N - 2}{N - 1}} \leq \lambda_i(L(\gamma)) \leq Q\lambda_N^G + \sqrt{E(P - Q^2)(2d + 2 - \lambda_2^G) \frac{N - 2}{N - 1}}. \tag{4.2}$$

It is notable that, at least in the case of a regular graph, the lower bound depends only on $\lambda_2^G$, the second largest eigenvalue of the graph Laplacian. The second eigenvalue is, of course, a well-studied object that encodes important geometric information on the connectivity of the graph, dating back at least to the work of Feidler [19], and is closely connected with the theory of expander graphs. See, for instance, the review article of Hoory, Linial, and Wigderson [20] for an overview of this area.

The lower bound in Theorem 1.2 is most important when considering the question of the positivity of $L(\gamma)$. It shows that $L(\gamma)$ is positive definite if the variance $P - Q^2 \geq 0$ is sufficiently small compared to the mean squared. Specifically, $L(\gamma)$ is positive definite if

$$\frac{(\lambda_2^G)^2}{\mu} E^{-1}Q^2 > P - Q^2. \tag{1.5}$$

Of course if the variance is small enough then each edge weight is necessarily positive, and hence the Laplacian is necessarily positive semi-definite. A computation shows that all the edge weights are positive if

$$E^{-1}Q^2 > P - Q^2. \tag{1.6}$$

Thus, when $(\lambda_2^G)^2/\mu > 1$ inequality (1.5) gives an improvement on the naive condition (1.6) in the sense that it allows for a larger variance. In Section 2 we present several examples where $(\lambda_2^G)^2/\mu > 1$ for graphs
with a large number of vertices. We also give an example of a graph where \((\lambda_2^G)^2/\mu \leq 1\). A simple example is the extreme case where \(G\) is disconnected so that \(\lambda_2^G = 0\).

The inequalities in Theorem 1.2 are also reminiscent of the Samuelson inequality for a finite set of real numbers. The original Samuelson inequality states that a finite set of real numbers is contained in a ball with center equal to the mean and radius proportional to the standard deviation of its elements (see [21] for example).

**Proof of Theorem 1.2.** Recall that \(1_E \in \mathbb{R}^E\) represents the vector \(1_E = (1, 1, 1, \ldots, 1)^t\) so we have the orthogonal decomposition

\[
\gamma = Q1_E + \hat{\gamma},
\]

where \(\hat{\gamma}\) satisfies

\[
\langle \hat{\gamma}, 1_E \rangle = 0 \quad \text{and} \quad \|\hat{\gamma}\|^2 = E(P - Q^2),
\]

and \(\|\cdot\|\) is the Euclidean norm.

This gives a decomposition of the graph Laplacian into a “mean” and “fluctuation” as follows

\[
L(\gamma) = L(Q1_E + \hat{\gamma}) = QL(1_E) + L(\hat{\gamma}).
\]

Recall that for symmetric matrices \(A\) and \(B\) we have the inequalities

\[
\lambda_{\min}(A + B) \geq \lambda_{\min}(A) + \lambda_{\min}(B) \quad \text{and} \quad \lambda_{\max}(A + B) \leq \lambda_{\max}(A) + \lambda_{\max}(B),
\]

where \(\lambda_{\min}(A) = \min_i \lambda_i(A)\) and \(\lambda_{\max}(A) = \max_i \lambda_i(A)\) are the smallest and largest eigenvalues of \(A\). Thus, to prove the theorem it is enough to bound the spectral radius of the fluctuation \(L(\hat{\gamma})\) and apply the above inequalities with \(A = QL(1_E)\) and \(B = L(\hat{\gamma})\).

To bound the fluctuation recall that the square of the Hilbert–Schmidt norm \(\|\cdot\|^2_{HS}\) of a matrix is the sum of the squares of its eigenvalues, that is

\[
\|L(\hat{\gamma})\|^2_{HS} = \sum_{i=1}^N \lambda_i^2.
\]

Also note that

\[
\lambda_1 = 0 \quad \text{and} \quad \text{Tr}(L(\hat{\gamma})) = 2 \sum_{i>j} \hat{\gamma}_{ij} = \sum_{i=1}^N \lambda_i = 0.
\]

Maximizing \(|\lambda_i|\) subject to the constraints (1.8) and (1.9) we have

\[
\max_i |\lambda_i| \leq \sqrt{\frac{N-2}{N-1}} \|L(\hat{\gamma})\|_{HS}.
\]

Next we express the Hilbert–Schmidt norm as a quadratic form in the edge-weights \(\hat{\gamma}_{ij}\):

\[
\|L(\hat{\gamma})\|^2_{HS} = \sum_i \lambda_i \left( L(\hat{\gamma}) \right)^2 = 2 \sum_{i<j} \hat{\gamma}_{ij}^2 + \sum_i \left( \sum_{j \neq i} \hat{\gamma}_{ij} \right)^2.
\]

To prove the estimate, we consider the Hilbert–Schmidt norm as a quadratic form on \(\gamma \in \mathbb{R}^E\) and maximize it subject to the constraints (1.7)

The quadratic form on \(\mathbb{R}^E\) in (1.10) can be expressed in graph-theoretic terms as \(\langle \gamma, (4I_E \times E + A^{LG(G)})\gamma \rangle\), where \(A^{LG(G)}\) is the adjacency matrix of the line graph of the graph \(G\). The line graph \(LG(G)\) has a vertex set given by the edge set of the original graph \(G\). Two vertices in the line graph are adjacent if the corresponding edges in \(G\) share a vertex. Thus we have that

\[
\|L(\hat{\gamma})\|^2_{HS} \leq E(P - Q^2) \max_{\gamma \in \mathbb{R}^E : \gamma \perp 1_E} \frac{\langle \gamma, (4I + A^{LG(G)})\gamma \rangle}{\|\gamma\|^2},
\]

where \(\perp\) denotes orthogonality to \(1_E\).
from which it follows that

\[ \max_i |\lambda_i(L(\tilde{\gamma}))| \leq \sqrt{E(P - Q^2)\frac{N-2}{N-1}} \mu. \]  

(1.11)

When \( G \) is \( d \)-regular we will prove that \( \mu = 4 + \lambda_{E-1}(A^{LG(G)}) = 2d + 2 - \lambda_2^G \) by relating the eigenvalues of the \( A^{LG(G)} \) to the eigenvalues of \( L(\mathbf{1}_E) \) — the Laplacian of the original graph. In this case, the line graph \( LG(G) \) is also \( d \)-regular, the vector \( \mathbf{1}_E \) is the eigenvector corresponding to the largest eigenvalue, and thus the maximum of the Rayleigh quotient in (1.11) is equal to the second largest eigenvalue of \( 4I_{E\times E} + A^{LG(G)} \). It is well-known, and easy to see, that the adjacency matrix of the line graph is related to the (unoriented) incidence matrix \( C \) of the graph \( G \) by

\[ A^{LG(G)} = C^T C - 2I_{E\times E}, \]

and that the adjacency matrix \( A^G \) of the original graph is related to the (unoriented) incidence matrix by

\[ A^G = CC^T - D, \]

where \( D \) is the degree matrix — the diagonal \( N \times N \) matrix with the vertex degrees along the diagonal. Since \( G \) is \( d \)-regular we have that \( D = dI_{N \times N} \) so that

\[
A^{LG(G)} + 4I_{E \times E} = C^T C + 2I_{E \times E} = CC^T = 2dI_{N \times N} - L(\mathbf{1}_E).
\]

To conclude the proof recall that the non-zero eigenvalues of \( C^T C \) are equal (counting by multiplicity) to the non-zero eigenvalues of \( CCT \). Thus, the second-largest eigenvalue of \( A^{LG(G)} + 4I_{E \times E} \), and therefore \( \mu \), is equal to \( 2d + 2 - \lambda_2^G \), where \( \lambda_2^G \) is the second-smallest eigenvalue of graph Laplacian.

2. Examples. In this section we present examples of Theorem 1.2 applied to the complete graph on \( N \) vertices, the Erdős–Rényi random graph in the critical and supercritical scaling regime, the cyclic graph, and random \( d \)-regular graphs. Recall that \( L(\gamma) \) is positive semi-definite whenever \( P \) and \( Q \) satisfy (1.5) or (1.6). In each example except the cyclic graph, the quantity \( (\lambda_2^G)^2/\mu \) is not regular and thus the upper and lower bounds are actually attained — while it is clear that each inequality in the derivation of Theorem 1.2 is tight it is not immediately clear that there is a single example for which all of the inequalities are extremized.

For the complete graph the mean, \( QL(\mathbf{1}_E) \) is a constant multiple of the orthogonal projection onto the \( N - 1 \) dimensional subspace \( (1,1,1,\ldots,1)^\perp \). It is easy to see that the eigenvalues of \( QL(\mathbf{1}_E) \) are given by 0, with multiplicity 1, and \( NQ \), with multiplicity \( N - 1 \). It is also noteworthy that in the case where the underlying topology is the complete graph the mean \( QL(\mathbf{1}_E) \) commutes with every combinatorial Laplacian, and thus with the fluctuation \( L(\tilde{\gamma}) \).

The line graph of the complete graph \( K_N \) is the Johnson \( J_{N,2} \). Using the fact that \( K_N \) is regular of degree \( N - 1 \) or known results about the spectrum of the adjacency matrix of the Johnson graph it follows that

\[
QN - \sqrt{E(P - Q^2)(2(N - 1) + 2 - N)}\frac{N-2}{N-1} \leq \lambda_i(L(\gamma)) \leq QN + \sqrt{E(P - Q^2)(2(N - 1) + 2 - N)}\frac{N-2}{N-1},
\]

or equivalently,

\[
N \left( Q - \sqrt{\frac{N-2}{2}} \sqrt{P - Q^2} \right) \leq \lambda_i(L(\gamma)) \leq N \left( Q + \sqrt{\frac{N-2}{2}} \sqrt{P - Q^2} \right).
\]
This is the sharp version of a simple inequality for the complete graph case that was proven in [22] via the Hilbert–Schmidt equality in order to establish the existence of a spectral gap. In particular it was shown there that if $L(\gamma)$ is a graph Laplacian with weights given by $\gamma \in \mathbb{R}^E$, then

\begin{equation}
N \left( Q - \sqrt{N-1} \sqrt{P - Q^2} \right) \leq \lambda_i(L(\gamma)) \leq N \left( Q + \sqrt{N-1} \sqrt{P - Q^2} \right)
\end{equation}

so the current inequality improves on the elementary estimate by roughly a factor of $\sqrt{2}$ for large $N$. Furthermore an example in the paper of Agbanusi and Bronski [22] shows that the current inequality is sharp: there exist explicit Laplace matrices for which the upper and lower limits are achieved.

2.2. Erdős–Rényi critical scaling. Consider an Erdős–Rényi random graph in the critical regime, where the edge probability is $p = \frac{\mu_0 \log N}{N}$ with $\mu_0 > 1$ to ensure connectivity of the graph. It has been shown by Kolokolnikov, Osting, and Von Brecht [23] that in the critical scaling regime one has that

$$\lambda_2 \sim a(p_0) p_0 \log N + O(\sqrt{\log N}), \quad \text{as } N \to \infty,$$

where $a(p_0) \in (0, 1)$ is defined to be the solution to $p_0 - 1 = a p_0 (1 - \log(a))$. The inequality holds in the sense that

$$\left| \frac{\lambda_2}{Np} - a(p_0) \right| \leq \frac{C_1}{Np},$$

is true with probability at least $1 - C_2 \exp\{-C_3 \sqrt{Np} \}$ for some constants $C_1, C_2, C_3$.

We are not aware of any precise results for $\mu$, but it is fairly easy to get (probabilistic) upper bounds since (1.4) says that $\mu \leq 2 d_{\max} + 2$. It follows a union bound argument (see Appendix A) that there is a constant $C_4 > 0$ such that

$$\mathbb{P} \left( \max_i \deg(v_i) \leq C p_0 \log N \right) \geq 1 - C_4 N^{\beta(C)}$$

where

$$\beta(C) = 2 - p_0 - C p_0 \log C + C p_0.$$

We can choose any $C$ such that $\beta(C) < 0$. For simplicity if we take $C = 4$ we have that $\beta \approx 1 - 2.55 p_0$.

Since this is a random graph we also need an estimate of $E$, the total number of edges. Since the edges are independent this essentially follows from the central limit theorem, and we have that $E = \frac{\mu_0 (N-1)}{2} \log N + o(N^{1/2 + \epsilon})$ for each $\epsilon > 0$ with high probability. Combining the above we have that the non-zero eigenvalues satisfy the lower bound

$$\lambda_i \geq p_0 \log(N) \left( a(p_0) Q (1 + o(1)) - \sqrt{4(N-2)(1+o(1)) \sqrt{P - Q^2}} \right)$$

with probability tending to 1 as $N \to \infty$.

The upper bound follows similarly — we are not aware of any result on the precise distribution of the largest eigenvalue of the Laplacian of an Erdős–Rényi graph, but the largest eigenvalue is obviously less than twice the largest degree of the graph, giving

$$\lambda_i \leq p_0 \log(N) \left( 8 Q (1 + o(1)) + \sqrt{4(N-2)(1+o(1)) \sqrt{P - Q^2}} \right).$$

Thus for an Erdős–Rényi graph in the critical scaling regime the Laplacian is (with probability tending to 1) positive definite if the inequality

$$Q^2 > \frac{4(N-1)}{a^2(p_0)} (P - Q^2).$$

Note that with high probability the number of edges will be $p_0 N \log N$ so the above estimate is asymptotically better than the naive estimate (1.6) by a factor of $\log N$. The constant in the above is obviously not sharp, as we have used a crude estimate on the largest eigenvalue of the adjacency matrix, and moreover no use has been made of the constraint that $\bar{\gamma}$ is mean zero. We do, however, expect that the scaling with $N$ is tight.
2.3. Erdős–Rényi supercritical scaling. Now we consider the Erdős–Rényi graphs in the supercritical regime with fixed edge probability $p \in (0, 1)$. Observe that since $p \geq p_0 \log(N)/N$ for large $N$ we have that the graph is connected almost surely. Moreover, in this regime the average degree of a vertex is $pN$. In fact, a similar calculation to the one in Appendix A shows that

$$\mathbb{P}\left( \max_i \deg(v_i) \leq (1 + N^{-1/2+\epsilon})pN \right) \to 1, \quad \text{as} \quad N \to \infty,$$

for $\epsilon \in (0, 1/2)$.

In particular, this shows that

$$\mu \leq 2(1 + N^{-1/2+\epsilon})pN + 2 \quad \text{and} \quad \lambda_N^G \leq 2(1 + N^{-1/2+\epsilon})pN,$$

with probability tending to 1 as $N \to \infty$. By Theorem 2 in [24], for each $\epsilon > 0$ we have

$$\lambda_2^G = pN + o(N^{2+\epsilon}), \quad \text{as} \quad N \to \infty.$$

The number of edges is $E = pN(N - 1)/2 + o(N^{1+\epsilon})$ as $N \to \infty$. Applying these bounds in the non-regular case of Theorem 1.2 we have

$$pN \left( Q(1+o(1)) - \sqrt{(N - 2)(1 + o(1)) \sqrt{P - Q^2}} \right) \leq \lambda_i \leq pN \left( 2Q(1+o(1)) + \sqrt{(N - 2)(1 + o(1)) \sqrt{P - Q^2}} \right)$$

with probability tending to 1 as $N \to \infty$, implying positivity when

$$Q \geq N(P - Q^2).$$

Notice that when we take $p = 1$ we recover the non-sharp bounds for the complete graph topology in (2.1) for large $N$. This is again due to the fact that we do not employ the constraint that $\gamma$ has mean zero. The same comments that were made for the critical case apply here as well — the constants can be improved but we believe the scaling to be optimal.

2.4. Cyclic graph. For the Cyclic graph on $N$ vertices, the graph Laplacian with all edge weights equal to 1 is twice the identity plus the circulant matrix generated by the vector $c = (0, -1, 0, \ldots, 0, -1)$. The eigenvalues of the circulant matrix, and hence those of the Laplacian, can be computed explicitly. The smallest nonzero and largest eigenvalues are $\lambda_2^G = 2(1 - \cos(2\pi/N))$ and $\lambda_N^G = 2$, respectively. Since the Cyclic graph has degree 2 we have the bounds $\mu = 6 - \lambda_2^G$. Putting all this together we find that

$$2 \left( Q(1 - \cos(2\pi/N)) - \sqrt{\frac{N}{2}} \frac{N - 2}{N - 1} \sqrt{P - Q^2} \right) \leq \lambda_i \leq 2 \left( Q + \sqrt{\frac{N}{2}} \frac{N - 2}{N - 1} \sqrt{P - Q^2} \right).$$

In this example the naive inequality (1.6) on $P$ and $Q$ is actually the stronger one since $(\lambda_2^G)^2/\mu \leq 2^2(1 - \cos(2\pi/N))^2/2 < 1$ for all large enough $N$. This is in contrast to each of the other examples where $(\lambda_2^G)^2/\mu > 1$ for large $N$.

2.5. Random $d$-regular graphs. Consider the probability space consisting of $d$-regular graphs ($d \geq 3$) on $N$ vertices with the uniform probability measure. Work of Freidman [25] implies that in this setting

$$\lambda_2^G \geq d - 2\sqrt{d - 1} + o(1), \quad \text{as} \quad N \to \infty,$$

with high probability. Applying the above inequality and that $\lambda_N^G \leq 2d$ to the bound for $d$-regular graphs in Theorem 1.2 we have that

$$\lambda_i \geq d \left( Q(1 - 2d^{-1/2} + o(1)) - \sqrt{\frac{N}{2}} \left( 1 + 4d^{-1/2} + o(1) \right) \sqrt{P - Q^2} \right),$$

and

$$\lambda_i \leq d \left( 2Q + \sqrt{\frac{N}{2}} \left( 1 + 4d^{-1/2} + o(1) \right) \sqrt{P - Q^2} \right).$$
Since $\mu = 2d + 2 - \lambda_G^2$ for $d$-regular graphs we have

$$\frac{(\lambda_G^2)^2}{\mu} \geq \frac{d^2 - 4d\sqrt{d-1} + 4(d-1) + o(1)}{d + 2\sqrt{d-1} + 2 + o(1)}, \quad \text{as } N \to \infty,$$

with high probability. Thus, for large $d$ the right side of (2.2) is roughly of size $d$, and in particular, $(\lambda_G^2)^2/\mu > 1$ almost surely as $N \to \infty$.

3. Concluding Remarks. In this paper we have derived bounds on the largest and smallest eigenvalues of a graph Laplacian in terms of the mean and variance of the edge weights and the second largest eigenvalue of the equally weighted graph Laplacian. These inequalities are tight in the case of the complete graph topology — there exist edge weightings which attain both the upper and the lower bounds.

There are a couple of ways in which it might be interesting to extend these results. Firstly while the bounds are tight for the complete graph topology it is unlikely that this is the case for most graph topologies. In the course of the proof we use the inequality

$$\lambda_{\min}(A + B) \geq \lambda_{\min}(A) + \lambda_{\min}(B),$$

where $A$ is the equally weighted Laplacian and $B$ is the fluctuation. In the case of the complete graph topology the equally weighted Laplacian is the identity on mean zero vectors, $A$ and $B$ commute, and this inequality is actually an equality. This is not true for other underlying graph topologies. It would be interesting to explore the extent to which this inequality fails to be tight for topologies other than the complete graph topology.

A second question concerns the quantity $\mu$, which is related to the maximum of a Rayleigh quotient for the adjacency of the line graph over mean zero vectors in $\mathbb{R}^E$. In the regular case we can, via a duality argument, compute $\mu$ in terms of the second largest eigenvalue of the graph Laplacian. For the non-regular case, we only bound $\mu$ in terms of the maximum degree, which does not exploit the mean zero condition at all. It would be interesting to develop a bound on $\mu$ in the non-regular case that exploits the mean zero condition.

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Appendix A. Upper bound on the maximum degree.

Proposition A.1. Suppose that $G$ is an Erdős–Rényi random graph where each possible edge is present with probability $p = \frac{p_0 \log N}{\sqrt{N}}$ with $p_0 > 1$. The probability that

$$\max_i \deg(v_i) \leq C p_0 \log(N)$$

tends to 1 algebraically as $N \to \infty$ for $C$ large enough. Choosing $C \geq 4$ is sufficient.

Proof. Observe that since

$$\mathbb{P}(\max_i \deg(v_i) \leq C p_0 \log(N)) = 1 - \mathbb{P}(\deg(v_i) > C p_0 \log(N) \text{ for some } i),$$

by a union bound it suffices to show that

$$N \mathbb{P}(\deg(v_i) > C p_0 \log(N)) \to 0, \quad \text{as } N \to \infty.$$
The inequality follows since the probability that an event occurs in \( N \) trials is larger than the probability that it occurs in \( N - 1 \) trials. Now suppose that \( K \geq K_{\max} \) — the \( K \) for which the maximum of \( \frac{N}{K} p^K (1 - p)^{N-K} \) is achieved. In this case

\[ P(\deg(v_i) > K) \leq N \left( \frac{N}{K} \right) p^K (1 - p)^{N-K}. \]

Now we apply Sterling's approximation to estimate \( \binom{N}{K} \) to find that

\[ P(\deg(v_i) > K) \leq N \cdot \frac{N^N e^{-N} \sqrt{2\pi N}}{K^K e^{-K} \sqrt{2\pi (N-K)^{N-K}} e^{-(N-K)}} p^K (1 - p)^{N-K}. \]

After regrouping and some elementary estimates we have

\[ P(\deg(v_i) > K) \leq N \sqrt{\frac{N}{(N-K)}} \left( \frac{Np}{K} \right)^K \left( \frac{N(1-p)}{N-K} \right)^{N-K}. \]

Now we choose \( K = CNp \) for \( C > 1 \) so that \( K \geq K_{\max} \) for large \( N \) since \( K_{\max} \leq \lfloor (N+1)p \rfloor \). It follows that

\[ P(\deg(v_i) > K) \leq N \left( \frac{1}{C} \right)^{Cp_0 \log(N)} \left( \frac{1 - p_o \log(N)}{N} \right)^{N-K} \leq N \left( \frac{1}{C} \right)^{Cp_0 \log(N)} \left( \frac{1 - p_o \log(N)}{N} \right)^N. \]

Using that \((1 - p_o \log(N)/N)^N = e^{N \log(1 - p_o \log(N)/N)}\) and Taylor expanding the outer logarithm we have

\[ P(\deg(v_i) > K) \leq N \left( \frac{1}{C} \right)^{Cp_0 \log(N)} \frac{e^{-p_o \log(N)}}{e^{-Cp_0 \log(N)}} = N N^{-Cp_0 \log(C)} N^{p_0} N^{Cp_0}. \]

Combining the exponents shows that

\[ NP \left( \deg(v_i) > C_{\max} \right) \leq N^{2 - C \log(C)p_0 + C_{p_0} - p_0}. \]

Since \( p_0 > 1 \) choosing \( C > 3.6 \) gives algebraic decay. In particular, choosing \( C = 4 \) gives

\[ P \left( \max_i \deg(v_i) > 4p_0 \log(N) \right) \leq N^{1 - 2.54p_0}. \]

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