LIE ALGEBROID FOLIATIONS AND 
$\mathcal{E}^1(M)$-DIRAC STRUCTURES

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Abstract

We prove some general results about the relation between the 1-cocycles of an arbitrary Lie algebroid $A$ over $M$ and the leaves of the Lie algebroid foliation on $M$ associated with $A$. Using these results, we show that a $\mathcal{E}^1(M)$-Dirac structure $L$ induces on every leaf $F$ of its characteristic foliation a $\mathcal{E}^1(F)$-Dirac structure $L_F$, which comes from a precontact structure or from a locally conformal presymplectic structure on $F$. In addition, we prove that a Dirac structure $\tilde{L}$ on $M \times \mathbb{R}$ can be obtained from $L$ and we discuss the relation between the leaves of the characteristic foliations of $L$ and $\tilde{L}$.

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1 Introduction

It is well-known the fundamental role that Poisson algebras play in Dirac’s theory of constrained Hamiltonian systems [1]. Two natural ways for Poisson algebras to arise from a manifold $M$ are through Poisson or presymplectic structures on $M$. Both structures are examples of Dirac structures in the sense of Courant-Weinstein [2, 3]. A Dirac structure on a manifold $M$ is a vector sub-bundle $\tilde{L}$ of $TM \oplus T^*M$ that is maximally isotropic under the natural symmetric pairing on $TM \oplus T^*M$ and such that the space of sections of $\tilde{L}$, $\Gamma(\tilde{L})$, is closed under the Courant bracket $[\cdot, \cdot]^\imath$ on $\Gamma(TM \oplus T^*M)$ (see Section 2.3, Example 1). If $\tilde{L}$ is a Dirac structure on $M$, then $\tilde{L}$ is endowed with a Lie algebroid structure over $M$ and the leaves of the induced Lie algebroid foliation $\mathcal{F}_{\tilde{L}}$ on $M$ are presymplectic manifolds.
reasonable to introduce a proper definition of a Dirac structure on the vector bundle $E$ then the pair $(TM, \Gamma(L))$ is closed under a suitable bracket $[\cdot, \cdot]$ and such that $\phi_L(e) = f$, for $e = (X, f) + (\alpha, g) \in \Gamma(L)$, is a 1-cocycle in the theory (see [11, 12, 13]).

Several aspects related with the geometry of $\mathcal{E}^1(M)$-Dirac structures were investigated (see [2]). In the particular case when the Dirac structure $\tilde{L}$ comes from a Poisson structure $\Pi$ on $M$, then $\tilde{L}$ is isomorphic to the cotangent Lie algebroid associated with $\Pi$ and $\mathcal{F}_L$ is just the symplectic foliation of $M$ (see [2]).

An algebraic treatment of Dirac structures was developed by Dorfman in [3] using the notion of a complex over a Lie algebra. This treatment was applied to the study of general Hamiltonian structures and their role in integrability. More recently, the properties of the Courant bracket $[\cdot, \cdot]$ have been systematized by Liu, Weinstein and Xu [19] in the definition of a Courant algebroid structure on a vector bundle $E \to M$ (see also [20, 25]). The natural example of a Courant algebroid is the Whitney sum $E = A \oplus A^*$, where the pair $(A, A^*)$ is a Lie bialgebroid over $M$ in the sense of Mackenzie-Xu [22].

On the other hand, a Jacobi structure on a manifold $M$ is a local Lie algebra structure, in the sense of Kirillov [13], on the space $C^\infty(M, \mathbb{R})$ (see [4, 14]). Apart from Poisson manifolds, interesting examples of Jacobi manifolds are contact and locally conformal symplectic manifolds [1]. Moreover, the 1-jet bundle $T^*M \times \mathbb{R} \to M$ is a Lie algebroid and the corresponding Lie algebroid foliation is just the characteristic foliation of $M$ (see [14]). However, for a Jacobi manifold $M$ the vector bundle $T^*M$ is not, in general, a Lie algebroid and, in addition, if one considers the usual Lie algebroid structure on $TM \times \mathbb{R}$ then the pair $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ is not a Lie bialgebroid (see [12, 29]). Thus, it seems reasonable to define a proper definition of a Dirac structure on the vector bundle $\mathcal{C}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ (a $\mathcal{C}^1(M)$-Dirac structure in our terminology). This job was done by A. Wade in [30]. A $\mathcal{C}^1(M)$-Dirac structure is a vector sub-bundle $L$ of $\mathcal{C}^1(M)$ that is maximally isotropic under the natural symmetric pairing of $\mathcal{C}^1(M)$ and such that the space $\Gamma(L)$ is closed under a suitable bracket $[\cdot, \cdot]$ on $\Gamma(\mathcal{C}^1(M))$ (this bracket may be defined using the general algebraic constructions of Dorfman [3]). Apart from $\mathcal{C}^1(M)$-Dirac structures which come from Dirac structures on $M$ or from Jacobi structures on $M$, other interesting examples can be obtained from a homogeneous Poisson structure on $M$, from a 1-form on $M$ (a precontact structure in our terminology) or from a locally conformal presymplectic (l.c.p.) structure, that is, a pair $(\Omega, \omega)$, where $\Omega$ is a 2-form on $M$, $\omega$ is a closed 1-form and $d\Omega = \omega \wedge \Omega$ (see [30]).

If $L$ is a $\mathcal{C}^1(M)$-Dirac structure, $[\cdot, \cdot]_L$ is the restriction to $\Gamma(L) \times \Gamma(L)$ of the extended Courant bracket $[\cdot, \cdot]$ and $\rho_L$ is the restriction to $L$ of the canonical projection $\rho : \mathcal{C}^1(M) \to TM$, then the triple $(L, [\cdot, \cdot]_L, \rho_L)$ is a Lie algebroid over $M$ (see [30]). Using the same terminology as in the Jacobi case, the Lie algebroid foliation $\mathcal{F}_L$ on $M$ associated with $L$ is called the characteristic foliation of $L$. An important remark is that the section $\phi_L$ of the dual bundle $L^*$ defined by $\phi_L(e) = f$, for $e = (X, f) + (\alpha, g) \in \Gamma(L)$, is a 1-cocycle of the Lie algebroid $(L, [\cdot, \cdot]_L, \rho_L)$. Anyway, since $\mathcal{C}^1(M)$-Dirac structures are closely related with Jacobi structures, it is not very surprising the presence of a Lie algebroid and a 1-cocycle in the theory (see [11, 12, 13]).
by Wade in [30]. However, the nature of the induced structure on the leaves of the characteristic foliation of a $\mathcal{E}^1(M)$-Dirac structure $L$ was not discussed in [30]. So, the aim of our paper is to describe such a nature. In addition, we will show that one may obtain, from $L$, a Dirac structure $\tilde{L}$ on $M \times \mathbb{R}$ in the sense of Courant-Weinstein and we will discuss the relation between the induced structures on the leaves of the characteristic foliations of $L$ and $\tilde{L}$. For the above purposes, we will prove some general results about the relation between the 1-cocycles of an arbitrary Lie algebroid $A$ over $M$ and the leaves of the Lie algebroid foliation on $M$ associated with $A$. In our opinion, these last results could be of independent interest.

The paper is organized as follows. In Section 2, we recall several definitions and results about $\mathcal{E}^1(M)$-Dirac structures which will be used in the sequel. We also present some examples that were obtained in [30]. In Section 3, we prove that if $(A, [[ , ]], \rho)$ is a Lie algebroid over $M$, $\phi$ is a 1-cocycle of $A$ and $F$ is a leaf of the Lie algebroid foliation $\mathcal{F}_A$ on $M$ then $S^\phi_F = \emptyset$ or $S^\phi_F = F$, where $S^\phi_F$ is the subset of $F$ defined by $S^\phi_F = \{x \in F/\ker (\rho|_{A_x}) \subseteq <\phi(x) >^\circ \}$ (see Theorem 3.2). Here, $A_x$ is the fiber of $A$ over $x$ and $<\phi(x) >^\circ$ is the annihilator of the subspace of $A_x^*$ generated by $\phi(x)$. On the other hand, the Lie algebroid structure $([ , ]), \rho)$ and the 1-cocycle $\phi$ induce a Lie algebroid structure $([ , ]^\phi, \rho^\phi)$ on the vector bundle $\tilde{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ (see (3.1)). Then, if $F$ and $\tilde{F}$ are the leaves of the Lie algebroid foliations $\mathcal{F}_A$ and $\mathcal{F}_{\tilde{A}}$ passing through $x_0 \in M$ and $(x_0, t_0) \in M \times \mathbb{R}$, we obtain, in the two possible cases ($S^\phi_F = \emptyset$ or $S^\phi_F = F$), the relation between $F$ and $\tilde{F}$ (see Theorem 3.3). Now, assume that $L$ is a $\mathcal{E}^1(M)$-Dirac structure and that $F$ is a leaf of the characteristic foliation $\mathcal{F}_L$. Then, in Section 4, we prove that $L$ induces, in a natural way, a $\mathcal{E}^1(F)$-Dirac structure $L_F$ and, moreover (using the results of Section 3), we describe the nature of $L_F$. In fact, we obtain that in the case when $S^\phi_L = \emptyset$, $L_F$ comes from a precontact structure on $F$ and in the case when $S^\phi_F = F$, $L_F$ comes from a l.c.p. structure on $F$ (see Theorem 4.2). Using this Theorem, we directly deduce the results of Courant [2] about the leaves of the characteristic foliation of a Dirac structure and the results of Kirillov [15] and Guedira-Lichnerowicz [9] about the leaves of the characteristic foliation of a Jacobi structure. We also apply the theorem to the particular case when $L$ comes from a homogeneous Poisson structure and some interesting consequences are derived. Finally, in Section 5, we prove that a Dirac structure $\tilde{L}$ on $M \times \mathbb{R}$ can be obtained from a $\mathcal{E}^1(M)$-Dirac structure $L$ in such a way that the Lie algebroid associated with $\tilde{L}$ is isomorphic to the Lie algebroid over $M \times \mathbb{R}$, $(\tilde{L} = L \times \mathbb{R}, [ , ]^\phi_L, \rho^\phi_L)$. Thus, if $(x_0, t_0)$ is a point of $M \times \mathbb{R}$, one may consider the leaves $F$ and $\tilde{F}$ of the characteristic foliations of $L$ and $\tilde{L}$ passing through $x_0$ and $(x_0, t_0)$. Then, using the results of Section 3, we obtain the relation between $F$ and $\tilde{F}$ and, in addition, we describe the presymplectic 2-form on $\tilde{F}$ in terms of the precontact structure on $F$, when $S^\phi_F = \emptyset$, or in terms of the 1-c.p. structure on $F$, when $S^\phi_F = F$ (see Theorem 5.3). As an application, we directly deduce some results of Guedira-Lichnerowicz [9] about the relation between the leaves of the characteristic foliation of a Jacobi structure on $M$ and the leaves of the symplectic foliation of the Poisson structure on $M \times \mathbb{R}$ induced by the Jacobi structure.
2 $\mathcal{E}^1(M)$-Dirac structures

All the manifolds considered in this paper are assumed to be connected and of class $C^\infty$. Moreover, if $M$ is a differentiable manifold, we will denote by $\mathcal{E}^1(M)$ the vector bundle $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}) \to M$. Note that the space of global sections $\Gamma(\mathcal{E}^1(M))$ of $\mathcal{E}^1(M)$ can be identified with the direct sum $(\mathfrak{x}(M) \times C^\infty(M, \mathbb{R})) \oplus (\Omega^1(M) \times C^\infty(M, \mathbb{R}))$.

2.1 Definition and characterization of $\mathcal{E}^1(M)$-Dirac structures

Along this Section, we will recall the definition of a $\mathcal{E}^1(M)$-Dirac structure, which was introduced by A. Wade in [30]. We will also give several results related to this notion.

The natural symmetric and skew-symmetric pairings $\langle , \rangle_+$ and $\langle , \rangle_-$ on $V \oplus V^*$, $V$ being a real vector space of finite dimension, can be extended, in a natural way, to the Whitney sum $A \oplus A^*$, where $A \to M$ is a real vector bundle over a manifold $M$. We also denote by $\langle , \rangle_+$ and $\langle , \rangle_-$ the resultant pairings on $\Gamma(A \oplus A^*) \cong \Gamma(A) \oplus \Gamma(A^*)$. In the particular case when $A = TM \times \mathbb{R}$, the explicit expressions of $\langle , \rangle_+$ and $\langle , \rangle_-$ on $\Gamma(\mathcal{E}^1(M))$ are

$$\langle (X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2) \rangle_+ = \frac{1}{2} \left( i_{X_2} \alpha_1 + f_2 g_1 + i_{X_1} \alpha_2 + f_1 g_2 \right),$$

$$\langle (X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2) \rangle_- = \frac{1}{2} \left( i_{X_2} \alpha_1 + f_2 g_1 - i_{X_1} \alpha_2 - f_1 g_2 \right),$$

for $(X_i, f_i) + (\alpha_i, g_i) \in \Gamma(\mathcal{E}^1(M))$, $i \in \{1, 2\}$. One may also consider the homomorphism of $C^\infty(M, \mathbb{R})$-modules $\rho : \Gamma(\mathcal{E}^1(M)) \to \mathfrak{x}(M)$ defined by

$$\rho((X, f) + (\alpha, g)) = X.$$

On the other hand, in [30] A. Wade introduced a suitable $\mathbb{R}$-bilinear bracket $[,] : \Gamma(\mathcal{E}^1(M)) \times \Gamma(\mathcal{E}^1(M)) \to \Gamma(\mathcal{E}^1(M))$ on the space $\Gamma(\mathcal{E}^1(M))$. This approach is based on an idea that can be found in [3], where the author generalizes the Courant bracket on $\Gamma(TM \oplus T^*M)$ to the case of complexes over Lie algebras. The bracket $[,]$ is given by

$$[(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)] = \left( [X_1, X_2], X_1(f_2) - X_2(f_1) \right) + \left( \mathcal{L}_{X_1} \alpha_2 - \mathcal{L}_{X_2} \alpha_1 + \frac{1}{2} d(i_{X_2} \alpha_1 - i_{X_1} \alpha_2) \right) + f_1 \alpha_2 - f_2 \alpha_1 + \frac{1}{2} (g_2 df_1 - g_1 df_2 - f_1 dg_2 + f_2 dg_1),$$

$$X_1(g_2) - X_2(g_1) + \frac{1}{2} (i_{X_2} \alpha_1 - i_{X_1} \alpha_2 - f_2 g_1 + f_1 g_2),$$

for $(X_i, f_i) + (\alpha_i, g_i) \in \Gamma(\mathcal{E}^1(M))$, $i \in \{1, 2\}$, where $[,]$ is the usual Lie bracket of vector fields and $\mathcal{L}$ is the Lie derivative operator on $M$. This bracket is skew-symmetric and, moreover, we have that

$$[e_1, f e_2] = f[e_1, e_2] + \rho(e_1)(f)e_2 - \langle e_1, e_2 \rangle_+ \left( (0, 0) + (df, 0) \right).$$

(2.4)
for \(e_1, e_2 \in \Gamma(\mathcal{E}^1(M))\) and \(f \in C^\infty(M, \mathbb{R})\). We note that \([, ]\) is not, in general, a Lie bracket, since the Jacobi identity does not hold (see [30]).

Now, let \(L\) be a vector sub-bundle of \(\mathcal{E}^1(M)\) which is isotropic under the symmetric pairing \(<, >_+\). We may consider the map \(T_L : \Gamma(L) \times \Gamma(L) \times \Gamma(L) \to C^\infty(M, \mathbb{R})\) given by

\[
T_L(e_1, e_2, e_3) = < [e_1, e_2], e_3 >_+, \quad \text{for } e_1, e_2, e_3 \in \Gamma(L).
\]

If \(e_i = (X_i, f_i) + (\alpha_i, g_i)\) with \(i \in \{1, 2, 3\}\) then, using (2.7), (2.3), (2.5) and the fact that \(< e_i, e_j >_+ = 0\), for \(i, j \in \{1, 2, 3\}\), we deduce that

\[
T_L(e_1, e_2, e_3) = \frac{1}{2} \sum_{\text{Cycl.}(e_1, e_2, e_3)} \left( i_{[X_1, X_2]} \alpha_3 + g_3(X_1(f_2) - X_2(f_1)) \right. \\
+ X_3(i_{X_2} \alpha_1 + f_2 g_1) + f_3(i_{X_2} \alpha_1 + f_2 g_1) \right).
\]

Thus, from (2.4), (2.3) and (2.6), it follows that \(T_L\) is a skew-symmetric \(C^\infty(M, \mathbb{R})\)-trilinear map, that is, \(T_L \in \Gamma(\wedge^3 L^*)\). Furthermore, using Proposition 3.3 in [30], we obtain that

\[
[[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2] = (0, 0) + (dT_L(e_1, e_2, e_3), T_L(e_1, e_2, e_3)),
\]

for \(e_1, e_2, e_3 \in \Gamma(L)\).

**Definition 2.1** [30] A \(\mathcal{E}^1(M)\)-Dirac structure on \(M\) is a sub-bundle \(L\) of \(\mathcal{E}^1(M)\) which is maximally isotropic under the symmetric pairing \(<, >_+\) and such that \(\Gamma(L)\) is closed under \([, ]\).

It is clear that if \(L\) is a \(\mathcal{E}^1(M)\)-Dirac structure on \(M\) then the section \(T_L \in \Gamma(\wedge^3 L^*)\) vanishes. In fact, we have the following result.

**Proposition 2.2** [30] Let \(L\) be a sub-bundle of \(\mathcal{E}^1(M)\) which is maximally isotropic under the symmetric pairing \(<, >_+\). Then, \(L\) is a \(\mathcal{E}^1(M)\)-Dirac structure if and only if the section \(T_L \in \Gamma(\wedge^3 L^*)\) given by (2.3) vanishes.

From (2.7) and Proposition 2.2, we conclude that the restriction of \([, ]\) to \(\Gamma(L)\) satisfies the Jacobi identity.

### 2.2 Lie algebroids and the characteristic foliation of a \(\mathcal{E}^1(M)\)-Dirac structure

A **Lie algebroid** \(A\) over a manifold \(M\) is a vector bundle \(A\) over \(M\) together with a Lie algebra structure \([, ]\) on the space \(\Gamma(A)\) and a bundle map \(\rho : A \to TM\), called the **anchor map**, such that if we also denote by \(\rho : \Gamma(A) \to \mathfrak{X}(M)\) the homomorphism of \(C^\infty(M, \mathbb{R})\)-modules induced by the anchor map then:
i) $\rho : (\Gamma(A), [\cdot, \cdot]) \to (\mathfrak{x}(M), [\cdot, \cdot])$ is a Lie algebra homomorphism and

ii) for all $f \in C^\infty(M, \mathbb{R})$ and for all $X, Y \in \Gamma(A)$, one has

$$[X, fY] = f[X, Y] + (\rho(X)(f))Y.$$ 

The triple $(A, [\cdot, \cdot], \rho)$ is called a Lie algebroid over $M$ (see [21]).

Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid over $M$. We consider the generalized distribution $\mathcal{F}_A$ on $M$ whose characteristic space at a point $x \in M$ is given by

$$F_A(x) = \rho(A_x),$$

(2.8)

where $A_x$ is the fiber of $A$ over $x$. Since $\Gamma(A)$ is a finitely generated $C^\infty(M, \mathbb{R})$-module (see [3]), the distribution $\mathcal{F}_A$ is also finitely generated. Moreover, it is clear that it is involutive. Thus, $\mathcal{F}_A$ defines a generalized foliation on $M$ in the sense of Sussman [26]. $\mathcal{F}_A$ is the Lie algebroid foliation on $M$ associated with $A$.

**Remark 2.3** If $F$ is the leaf of $\mathcal{F}_A$ passing through $x \in M$, $\dim F = r$ and $y \in M$ then $y \in F$ if and only if there exists a continuous piecewise smooth path $\gamma : I \to M$ from $x$ to $y$, which is tangent to $\mathcal{F}_A$ and such that $\dim \mathcal{F}_A(\gamma(t)) = r$, for all $t \in I$ (see [14, 28]).

If $(A, [\cdot, \cdot], \rho)$ is a Lie algebroid over $M$, one can introduce the *Lie algebroid cohomology complex with trivial coefficients* (for the explicit definition of this complex we remit to [21]). The space of 1-cochains is $\Gamma(A^*)$, where $A^*$ is the dual bundle to $A$, and a 1-cochain $\phi \in \Gamma(A^*)$ is a 1-cocycle if and only if $\phi[X, Y] = \rho(X)(\phi(Y)) - \rho(Y)(\phi(X))$, for all $X, Y \in \Gamma(A)$.

Now, suppose that $M$ is a differentiable manifold, that $[\cdot, \cdot]$ is the bracket on $\Gamma(\mathcal{E}^1(M))$ given by (2.3) and that $\rho : \Gamma(\mathcal{E}^1(M)) \to \mathfrak{x}(M)$ is the homomorphism of $C^\infty(M, \mathbb{R})$-modules defined by (2.2). Assume also that $L$ is a $\mathcal{E}^1(M)$-Dirac structure and that $\rho_L$ (respectively, $[\cdot, \cdot]_L$) is the restriction of $\rho$ (respectively, $[\cdot, \cdot]$) to $\Gamma(L)$ (respectively, $\Gamma(L) \times \Gamma(L)$). Then, it is clear that the triple $(L, [\cdot, \cdot]_L, \rho_L)$ is a Lie algebroid over $M$ (see [30] and Section 2.1). Thus, one can consider the Lie algebroid foliation $\mathcal{F}_L$ on $M$ associated with $L$. $\mathcal{F}_L$ is called the *characteristic foliation of the $\mathcal{E}^1(M)$-Dirac structure*.

On the other hand, we may introduce a section $\phi_L$ of the dual bundle $L^*$ as follows:

$$\phi_L(e) = f, \text{ for } e = (X, f) + (\alpha, g) \in \Gamma(L).$$

(2.9)

A direct computation, using (2.2), (2.3) and (2.4), proves that $\phi_L$ is a 1-cocycle.

### 2.3 Examples of $\mathcal{E}^1(M)$-Dirac structures

Next, we will present some examples of $\mathcal{E}^1(M)$-Dirac structures which were obtained in [30]. In addition, we will describe the Lie algebroids, the characteristic foliations and the 1-cocycles associated with these structures.
1.- Dirac structures

Let \( \tilde{L} \) be a vector sub-bundle of \( TM \oplus T^*M \) and consider the vector sub-bundle \( L \) of \( \mathcal{E}^1(M) \) whose sections are

\[
\Gamma(L) = \{(X, 0) + (\alpha, f)/X + \alpha \in \Gamma(\tilde{L}), f \in C^\infty(M, \mathbb{R})\}.
\]

Then, \( \tilde{L} \) is a Dirac structure on \( M \) in the sense of Courant-Weinstein \([2, 3]\) if and only if \( L \) is a \( \mathcal{E}^1(M) \)-Dirac structure (see \([30]\)). We recall that a vector sub-bundle \( L \) of \( TM \oplus T^*M \) is a Dirac structure on \( M \) if \( \tilde{L} \) is maximally isotropic under the natural symmetric pairing \( \langle \cdot, \cdot \rangle \) on \( TM \oplus T^*M \) and, in addition, the space of sections of \( L, \Gamma(L) \), is closed under the Courant bracket \([\cdot, \cdot]\) which is defined by

\[
[X_1 + \alpha_1, X_2 + \alpha_2] = [X_1, X_2] + (\mathcal{L}_{X_1} \alpha_2 - \mathcal{L}_{X_2} \alpha_1 + \frac{1}{2} d(i_{X_2} \alpha_1 - i_{X_1} \alpha_2)),
\]

for \( X_1 + \alpha_1, X_2 + \alpha_2 \in \mathfrak{x}(M) \oplus \Omega^1(M) \cong \Gamma(TM \oplus T^*M) \).

If \( \tilde{L} \subseteq TM \oplus T^*M \) is a Dirac structure on \( M \) then the triple \((\tilde{L}, [\cdot, \cdot]_{\tilde{L}}, \tilde{\rho}_{\tilde{L}})\) is a Lie algebroid over \( M \), where \([\cdot, \cdot]_{\tilde{L}}\) is the restriction to \( \Gamma(\tilde{L}) \times \Gamma(\tilde{L}) \) of the Courant bracket given by (2.11) and \( \tilde{\rho}_{\tilde{L}} \) is the restriction to \( \Gamma(\tilde{L}) \) of the map \( \tilde{\rho} : \Gamma(TM \oplus T^*M) \rightarrow \mathfrak{x}(M) \) defined by

\[
\tilde{\rho}(X + \alpha) = X,
\]

for all \( X + \alpha \in \Gamma(TM \oplus T^*M) \) (see \([2]\)). The characteristic foliation associated with \( \tilde{L} \) is the Lie algebroid foliation \( \mathcal{F}_{\tilde{L}} \). It is clear that \( \mathcal{F}_{\tilde{L}}(x) = \mathcal{F}_L(x) \), for all \( x \in M \). In addition, from (2.9) and (2.10), it follows that the 1-cocycle \( \phi_L \) identically vanishes.

2.- Locally conformal presymplectic structures

A locally conformal presymplectic (l.c.p.) structure on a manifold \( M \) is a pair \((\Omega, \omega)\), where \( \Omega \) is a 2-form on \( M \), \( \omega \) is a closed 1-form and \( d\Omega = \omega \wedge \Omega \). If \((\Omega, \omega)\) is a l.c.p. structure on \( M \), one may define the vector sub-bundle \( L_{(\Omega, \omega)} \) of \( \mathcal{E}^1(M) \) whose sections are

\[
\Gamma(L_{(\Omega, \omega)}) = \{(X, -i_X \omega) + (i_X \Omega + f \omega, f)/(X, f) \in \mathfrak{x}(M) \times C^\infty(M, \mathbb{R})\}.
\]

It is clear that the vector bundles \( L_{(\Omega, \omega)} \) and \( TM \times \mathbb{R} \) are isomorphic. In addition, \( L_{(\Omega, \omega)} \) is a \( \mathcal{E}^1(M) \)-Dirac structure \([30]\). Note that if \( \omega = 0 \) then \( \Omega \) is a presymplectic form on \( M \). Furthermore, if \((\Omega, \omega)\) is a l.c.p. structure on a manifold \( M \) of even dimension and \( \Omega \) is a nondegenerate 2-form then \((\Omega, \omega)\) is a locally conformal symplectic structure (see \([1, 15, 27]\)).

Let \((\Omega, \omega)\) be a l.c.p. structure on a manifold \( M \) and \( L_{(\Omega, \omega)} \) be the associated \( \mathcal{E}^1(M) \)-Dirac structure. Then, using (2.2), (2.3) and (2.13), we deduce that the Lie algebroids \((L_{(\Omega, \omega)}, [\cdot, \cdot]_{L_{(\Omega, \omega)}}, \rho_{L_{(\Omega, \omega)}})\) and \((TM \times \mathbb{R}, [\cdot, \cdot]_{(\Omega, \omega)}, \rho_{(\Omega, \omega)})\) are isomorphic, where \([\cdot, \cdot]_{(\Omega, \omega)} \) and \( \rho_{(\Omega, \omega)} \) are given by

\[
[(X, f), (Y, g)]_{(\Omega, \omega)} = [(X, Y), \Omega(X, Y) + (X(g) - g\omega(X)) - (Y(f) - f\omega(Y))],
\]

\[
\rho_{(\Omega, \omega)}(X, f) = X,
\]
for \((X, f), (Y, g) \in \mathfrak{x}(M) \times C^\infty(M, \mathbb{R})\). We remark that the map \(\nabla : \mathfrak{x}(M) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})\) defined by \(\nabla_X f = X(f) - f_\omega(X)\), for \(X \in \mathfrak{x}(M)\) and \(f \in C^\infty(M, \mathbb{R})\), induces a representation of the Lie algebroid \((TM, [, ], Id)\) on the trivial vector bundle \(M \times \mathbb{R} \to M\) and that \(\Omega\) is a 2-cocycle of \((TM, [, ], Id)\) with respect to this representation. In addition, the Lie algebroid \((TM \times \mathbb{R}, [, ], \rho_{\Omega, \omega})\) is the extension of \((TM, [, ], Id)\) via \(\nabla\) and \(\Omega\) (for the definition of the extension of a Lie algebroid \(A\) with respect to a 2-cocycle and a representation of \(A\) on a vector bundle, see \([2]\)). On the other hand, it is clear that \(\mathcal{F}_{L(\Omega, \omega)}(x) = T_x M\), for all \(x \in M\), and that, under the isomorphism between \(L(\Omega, \omega)\) and \(TM \times \mathbb{R}\), the 1-cocycle \(\phi_{L(\Omega, \omega)}\) is the pair \((-\omega, 0) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^* M \times \mathbb{R})\) (see \((2.9)\) and \((2.13)\)).

### 3.- Precontact structures

A precontact structure on a manifold \(M\) is a 1-form \(\eta\) on \(M\). A precontact structure \(\eta\) on \(M\) induces an \(\mathcal{E}^1(M)\)-Dirac structure \(L_\eta\). More precisely, suppose that \(\Phi\) is a 2-form on \(M\), that \(\eta\) is a 1-form and consider the vector sub-bundle \(L\) of \(\mathcal{E}^1(M)\) whose sections are

\[
\Gamma(L) = \{(X, f) + (i_X \Phi + f \eta, -i_X \eta)/(X, f) \in \mathfrak{x}(M) \times C^\infty(M, \mathbb{R})\}. \tag{2.14}
\]

The vector bundles \(L\) and \(TM \times \mathbb{R}\) are isomorphic. Moreover, \(L\) is a \(\mathcal{E}^1(M)\)-Dirac structure if and only if \(\Phi = d\eta\) (see \([30]\)). Thus,

\[
\Gamma(L_\eta) = \{(X, f) + (i_X d\eta + f \eta, -i_X \eta)/(X, f) \in \mathfrak{x}(M) \times C^\infty(M, \mathbb{R})\}. \tag{2.15}
\]

Note that a precontact structure \(\eta\) on a manifold \(M\) of odd dimension \(2n + 1\) such that \(\eta \wedge (d\eta)^n\) is a volume form is a contact structure (see \([1, 13, 14, 18]\)).

Let \(\eta\) be a precontact structure on a manifold \(M\) and \(L_\eta\) be the associated \(\mathcal{E}^1(M)\)-Dirac structure. Then, the Lie algebroids \((L_\eta, [, ], \rho_{L_\eta})\) and \((TM \times \mathbb{R}, [, ], \pi)\) are isomorphic, where \(\pi : TM \times \mathbb{R} \to TM\) is the canonical projection over the first factor and \([\ , ]\) is the usual Lie bracket on \(\Gamma(TM \times \mathbb{R}) \cong \mathfrak{x}(M) \times C^\infty(M, \mathbb{R})\) given by

\[
[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f)),
\]

for \((X, f), (Y, g) \in \mathfrak{x}(M) \times C^\infty(M, \mathbb{R})\). We also have that \(\mathcal{F}_{L_\eta}(x) = T_x M\), for all \(x \in M\). Moreover, under the isomorphism between \(L_\eta\) and \(TM \times \mathbb{R}\), the 1-cocycle \(\phi_{L_\eta}\) is the pair \((0, 1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^* M \times \mathbb{R})\) (see \((2.9)\) and \((2.13)\)).

### 4.- Jacobi structures

A Jacobi structure on a manifold \(M\) is a pair \((\Lambda, E)\), where \(\Lambda\) is a 2-vector and \(E\) is a vector field, such that \([\Lambda, \Lambda] = 2E \wedge \Lambda\) and \([E, \Lambda] = 0\) being the Schouten-Nijenhuis bracket. If the vector field \(E\) identically vanishes then \((M, \Lambda)\) is a Poisson manifold. Jacobi and Poisson manifolds were introduced by Lichnerowicz \([17, 18]\) (see also \([4, 5, 9, 15, 16, 28, 31]\)).

Now, given a 2-vector \(\Lambda\) and a vector field \(E\) on a manifold \(M\), we can consider the vector sub-bundle \(L_{(\Lambda, E)}\) of \(\mathcal{E}^1(M)\) whose sections are

\[
\Gamma(L_{(\Lambda, E)}) = \{(#_\Lambda(\alpha) + f E, -i_E \alpha + (\alpha, f)/(\alpha, f)) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})\}, \tag{2.16}
\]
where \( \#_\Lambda : \Omega^1(M) \to \mathfrak{x}(M) \) is the homomorphism of \( C^\infty(M, \mathbb{R}) \)-modules defined by \( \beta(\#_\Lambda(\alpha)) = \Lambda(\alpha, \beta) \), for \( \alpha, \beta \in \Omega^1(M) \). Note that the vector bundles \( L_{(\Lambda, E)} \) and \( T^*M \times \mathbb{R} \) are isomorphic. Moreover, we have that \( L_{(\Lambda, E)} \) is a \( \mathcal{E}^1(M) \)-Dirac structure if and only if \( (\Lambda, E) \) is a Jacobi structure (see \([30]\)).

If \( (\Lambda, E) \) is a Jacobi structure on a manifold \( M \) and \( L_{(\Lambda, E)} \) is the associated \( \mathcal{E}^1(M) \)-Dirac structure then the Lie algebroids \( (L_{(\Lambda, E)}, [, ]_{L_{(\Lambda, E)}}, \rho_{L_{(\Lambda, E)}}) \) and \( (T^*M \times \mathbb{R}, [, ]_{(\Lambda, E)}, \#_{(\Lambda, E)}) \) are isomorphic, where \([, ]_{(\Lambda, E)} \) and \( \#_{(\Lambda, E)} \) are defined by

\[
[(\alpha, f), (\beta, g)]_{(\Lambda, E)} = (\mathcal{L}_{\#_\Lambda(\alpha)}(\beta) - \mathcal{L}_{\#_\Lambda(\beta)}(\alpha) - d(\Lambda(\alpha, \beta))) + f \mathcal{L}_E \beta - g \mathcal{L}_E \alpha - i_E(\alpha \wedge \beta),
\]

\[
\#_{(\Lambda, E)}(\alpha, f) = \#_\Lambda(\alpha) + f E,
\]

for \( (\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R}) \) (see \([30]\)). The Lie algebroid structure \( ([, ]_{(\Lambda, E)}, \#_{(\Lambda, E)}) \) on \( T^*M \times \mathbb{R} \) was introduced in \([14]\). In this case, the characteristic foliation of the \( \mathcal{E}^1(M) \)-Dirac structure \( L_{(\Lambda, E)} \) is just the characteristic foliation on \( M \) associated with the Jacobi structure \( (\Lambda, E) \) (see \([4, 3, 13]\)). In addition, under the isomorphism between \( L_{(\Lambda, E)} \) and \( T^*M \times \mathbb{R} \), the 1-cocycle \( \phi_{L_{(\Lambda, E)}} \) is the pair \((E, 0) \in \mathfrak{x}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R}) \) (see \((2.9)\) and \((2.10)\)).

5.- Homogeneous Poisson structures

A homogeneous Poisson manifold \((M, \Pi, Z)\) is a Poisson manifold \((M, \Pi)\) with a vector field \( Z \) such that \([Z, \Pi]\) = \(-\Pi\) (see \([4]\)). Given a 2-vector \( \Pi \) and a vector field \( Z \) on a manifold \( M \), we can define the vector sub-bundle \( L_{(\Pi, Z)} \) of \( \mathcal{E}^1(M) \) whose sections are

\[
\Gamma(L_{(\Pi, Z)}) = \{((\#_\Pi(\alpha) - f Z, f) + (\alpha, i_Z \alpha)/(\alpha, f) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})\}.
\]

(2.17)

The vector bundles \( L_{(\Pi, Z)} \) and \( T^*M \times \mathbb{R} \) are isomorphic. Furthermore, \((M, \Pi, Z)\) is a homogeneous Poisson manifold if and only if \( L_{(\Pi, Z)} \) is a \( \mathcal{E}^1(M) \)-Dirac structure (see \([30]\)).

Let \((M, \Pi, Z)\) be a homogeneous Poisson manifold and \( L_{(\Pi, Z)} \) be the associated \( \mathcal{E}^1(M) \)-Dirac structure. Then, from \((2.2)\), \((2.3)\) and \((2.17)\), it follows that the Lie algebroids \((L_{(\Pi, Z)}, [, ]_{L_{(\Pi, Z)}}, \rho_{L_{(\Pi, Z)}})\) and \((T^*M \times \mathbb{R}, [, ]_{(\Pi, Z)}, \#_{(\Pi, Z)})\) are isomorphic, where \([, ]_{(\Pi, Z)} \) and \( \#_{(\Pi, Z)} \) are defined by

\[
[(\alpha, f), (\beta, g)]_{(\Pi, Z)} = \left(\mathcal{L}_{\#_\Pi(\alpha)}(\beta) - \mathcal{L}_{\#_\Pi(\beta)}(\alpha) - d(\Pi(\alpha, \beta))) - f(\mathcal{L}_Z \beta - \beta) + g(\mathcal{L}_Z \alpha - \alpha),
\]

\[
\#_{(\Pi, Z)}(\alpha, f) = \#_\Pi(\alpha) + f Z,
\]

(2.18)

for \( (\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R}) \). Furthermore, it is clear that \( \mathcal{F}_{L_{(\Pi, Z)}}(x) = \#_\Pi(T_x^*M) + <Z(x)> \), for all \( x \in M \). In other words, if \( \mathcal{F}_{\Pi} \) is the symplectic foliation of the Poisson manifold \((M, \Pi)\) then

\[
\mathcal{F}_{L_{(\Pi, Z)}}(x) = \mathcal{F}_{\Pi}(x) + <Z(x)>, \text{ for all } x \in M.
\]

(2.19)
In addition, under the isomorphism between $L_{(\Pi,Z)}$ and $T^*M \times \mathbb{R}$, the 1-cocycle $\phi_{L_{(\Pi,Z)}}$ is the pair $(0, 1) \in \mathfrak{x}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$ (see (2.3) and (2.17)).

On the other hand, we may consider the Lie algebroid structure $(\mathbb{L}, \#_{\Pi})$ on the vector bundle $T^*M \to M$ induced by the Poisson structure $\Pi$ and the Lie algebroid structure $(\mathbb{L}, \rho_Z)$ on the vector bundle $M \times \mathbb{R} \to M$ induced by the vector field $Z$. The explicit definitions of $\mathbb{L}$, $\#_{\Pi}$, $\#_Z$ and $\rho_Z$ are

\[
[\alpha, \beta]_{\Pi} = \mathcal{L}_{\#_{\Pi}(\alpha)} \beta - \mathcal{L}_{\#_{\Pi}(\beta)} \alpha - d(\Pi(\alpha, \beta)),
\]

\[
[f, g]_Z = g Z(f) - f Z(g), \quad \rho_Z(f) = -f Z,
\]

for $\alpha, \beta \in \Omega^1(M)$ and $f, g \in C^\infty(M, \mathbb{R})$. Then, using (2.18), we conclude that $(T^*M, [\cdot], \#_{\Pi})$ and $(M \times \mathbb{R}, [,], \#_Z, \rho_Z)$ are Lie subalgebroids of $(T^*M \times \mathbb{R}, [\cdot], \#_{(\Pi,Z)})$. This implies that $(T^*M, [\cdot], \#_{\Pi})$ and $(M \times \mathbb{R}, [,], \#_Z, \rho_Z)$ form a matched pair of Lie algebroids in the sense of Mokri [23].

In Section 4, we will prove that if $F$ is a leaf of the characteristic foliation of a $\mathcal{C}^1(M)$-Dirac structure $L$ then $L$ induces a $\mathcal{C}^1(F)$-Dirac structure $L_F$ and, in addition, we will describe the nature of $L_F$. First, in the next Section, we will show two general results about the relation between the 1-cocycles of an arbitrary Lie algebroid $A$ and the leaves of the Lie algebroid foliation $\mathcal{F}_A$.

### 3 1-cocycles of a Lie algebroid and the leaves of the Lie algebroid foliation

Let $(A, [\cdot], \rho)$ be a Lie algebroid over $M$, $\phi \in \Gamma(A^*)$ be a 1-cocycle and $\pi_1 : M \times \mathbb{R} \to M$ be the canonical projection onto the first factor. We consider the map $\cdot : \Gamma(A) \times C^\infty(M \times \mathbb{R}, \mathbb{R}) \to C^\infty(M \times \mathbb{R}, \mathbb{R})$ given by

\[
X \cdot \bar{f} = \rho(X)(\bar{f}) + \phi(X) \frac{\partial \bar{f}}{\partial t}.
\]

It is easy to prove that $\cdot$ is an action of $A$ on $M \times \mathbb{R}$ in the sense of [10] (see Definition 2.3 in [10]). Thus, if $\pi_1^* A$ is the pull-back of $A$ over $\pi_1$ then the vector bundle $\pi_1^* A \to M \times \mathbb{R}$ admits a Lie algebroid structure $(\mathbb{L}, [\cdot], \rho^\phi)$ (see Theorem 2.4 in [10]). For the sake of simplicity, when the 1-cocycle $\phi$ is zero, we will denote by $(\mathbb{L}, [,], \rho)$ the resultant Lie algebroid structure on $\pi_1^* A \to M \times \mathbb{R}$. On the other hand, it is clear that the vector bundles $\pi_1^* A \to M \times \mathbb{R}$ and $\tilde{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ are isomorphic and that the space of sections $\Gamma(\tilde{A})$ of $\tilde{A} \to M \times \mathbb{R}$ can be identified with the set of time-dependent sections of $A \to M$. Under this identification, we have that $[\tilde{X}, \tilde{Y}] = [\tilde{X}_t, \tilde{Y}_t](x)$ and that $\bar{\rho}(\tilde{X})(x, t) = \rho(X_t)(x)$, for $X, Y \in \Gamma(\tilde{A})$ and $(x, t) \in M \times \mathbb{R}$ (see [10]). In addition,

\[
[X, Y] = [\tilde{X}, \tilde{Y}]^\phi + \phi(\tilde{X}) \frac{\partial \tilde{Y}}{\partial t} - \phi(\tilde{Y}) \frac{\partial \tilde{X}}{\partial t}, \quad \bar{\rho}^\phi(\tilde{X}) = \bar{\rho}(\tilde{X}) + \phi(\tilde{X}) \frac{\partial}{\partial t},
\] (3.1)
where $\frac{\partial X}{\partial t} \in \Gamma(A)$ denotes the derivative of $X$ with respect to the time.

Now, if $F_A$ is the Lie algebroid foliation of $(\bar{A}, [\cdot, \cdot]^{\phi}, \bar{\rho}^{\phi})$ then, from (3.1), it follows that

$$F_A(x, t) = \{\rho(e_x) + \phi(x)(e_x)\frac{\partial}{\partial t} \mid (M \times \mathbb{R}) / e_x \in A_x\},$$

(3.2)

for all $(x, t) \in M \times \mathbb{R}$. Moreover, a direct computation shows that,

$$\text{dim } F_A(x) \leq \text{dim } F_A(x, t) \leq \text{dim } F_A(x) + 1,$$

(3.3)

$$\text{dim } F_A(x, t) = \text{dim } F_A(x) \iff \ker (\rho|_{A_x}) \subseteq <\phi(x)>^o,$$

(3.4)

where $F_A$ is the Lie algebroid foliation of $A$ and $<\phi(x)>^o$ is the annihilator of the subspace of $A_x^*$ generated by $\phi(x)$, that is,

$$<\phi(x)>^o = \{e_x \in A_x / \phi(x)(e_x) = 0\}.$$

**Remark 3.1** Note that the vector field $\frac{\partial}{\partial t}$ on $M \times \mathbb{R}$ is an infinitesimal automorphism of the foliation $F_A$. Therefore, if $(x, t_0)$ and $(x, t_0')$ are points of $M \times \mathbb{R}$ and $F, \bar{F}'$ are the leaves of $F_A$ passing through $(x, t_0)$ and $(x, t_0')$, then the map $(y, s) \mapsto (y, s + (t_0' - t_0))$ is a diffeomorphism from $F$ to $\bar{F}'$.

Next, we will discuss some relations between the leaves of $F_A$ and the 1-cocycle $\phi$ and between the leaves of $F_A$ and $F_A$. More precisely, the aim of this Section is to prove the following two results.

**Theorem 3.2** Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid and $\phi \in \Gamma(A^*)$ be a 1-cocycle. If $F$ is a leaf of the Lie algebroid foliation $F_A$ and $S^\phi_F$ is the subset of $F$ defined by

$$S^\phi_F = \{x \in F / \ker (\rho|_{A_x}) \subseteq <\phi(x)>^o\},$$

then $S^\phi_F = \emptyset$ or $S^\phi_F = F$. Furthermore, in the second case ($S^\phi_F = F$), the 1-cocycle $\phi$ induces a closed 1-form $\omega_F$ on $F$ which is characterized by the condition

$$\omega_F(\rho(e)|_F) = -\phi(e)|_F, \text{ for all } e \in \Gamma(A).$$

(3.5)

**Theorem 3.3** Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid, $\phi \in \Gamma(A^*)$ be a 1-cocycle and consider on the vector bundle $\bar{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ the Lie algebroid structure $([\cdot, \cdot]^{\phi}, \bar{\rho}^{\phi})$ given by (3.1). Suppose that $(x_0, t_0) \in M \times \mathbb{R}$ and that $F$ and $\bar{F}$ are the leaves of the Lie algebroid foliations $F_A$ and $F_A$ passing through $x_0 \in M$ and $(x_0, t_0) \in M \times \mathbb{R}$, respectively. Then:

i) If $\ker (\rho|_{A_{x_0}}) \not\subseteq <\phi(x_0)>^o$ (or, equivalently, $S^\phi_F = \emptyset$) we have that $\bar{F} = F \times \mathbb{R}$.

ii) If $\ker (\rho|_{A_{x_0}}) \not\subseteq <\phi(x_0)>^o$ (or, equivalently, $S^\phi_F = F$) and $\pi_1 : M \times \mathbb{R} \to M$ is the canonical projection onto the first factor, we have that $\pi_1(\bar{F}) = F$ and that the map $\pi_1|_F$:
$\tilde{F} \rightarrow F$ is a covering map. In addition, if $\omega_F$ is the closed 1-form on $F$ characterized by the condition (2.2) and $i_F : \tilde{F} \rightarrow M \times \mathbb{R}$ is the canonical inclusion then $\tilde{F}$ is diffeomorphic to a Galois covering of $F$ associated with $\omega_F$ and

$$(\pi_{1\tilde{F}})^*(\omega_F) = -d((i_F)^*(t)).$$

In order to prove the above results, we will use the following three lemmas.

**Lemma 3.4**

i) If $\tilde{F}$ is a leaf of the Lie algebroid foliation $\mathcal{F}_A$ and $S^\phi_F$ is the subset of $\tilde{F}$ defined by

$$S^\phi_F = \{(x, t) \in F / \ker(\rho_{Ax}) \subseteq \phi(x) > 0\},$$

then $S^\phi_F$ is an open subset of $\tilde{F}$.

ii) If $F$ is a leaf of the Lie algebroid foliation $\mathcal{F}_A$ then $S^\phi_F$ is an open subset of $F$.

**Proof:** i) Assume that $\dim \tilde{F} = r$ and let $(x_0, t_0)$ be a point of $S^\phi_F$. We will show that there exists a connected open subset $\tilde{W}_{(x_0, t_0)}$ of $\tilde{F}$ such that $(x_0, t_0) \in \tilde{W}_{(x_0, t_0)}$ and $\tilde{W}_{(x_0, t_0)} \subseteq S^\phi_F$ or, equivalently (see (3.4)),

$$\dim \mathcal{F}_A(x) = \dim \mathcal{F}_A(x, t) = r, \text{ for all } (x, t) \in \tilde{W}_{(x_0, t_0)}. \quad (3.6)$$

Note that

$$\dim \mathcal{F}_A(x, t) = r, \text{ for all } (x, t) \in \tilde{F}. \quad (3.7)$$

Therefore, we can choose a global generator system $\{e_1, \ldots, e_m\}$ of $\Gamma(A)$ in such a way that the set of vectors $\{\rho(e_i(x_0)) + \phi(x_0)(e_i(x_0)) \frac{\partial}{\partial t}|_{t_0}\}_{1 \leq i \leq r}$ are a basis of the vector space $\mathcal{F}_A(x_0, t_0)$. Then, using that $(x_0, t_0) \in S^\phi_F$, we deduce that the vectors $\{\rho(e_i(x_0))\}_{1 \leq i \leq r}$ are linearly independent in $T_{x_0}M$. This implies that $\dim \mathcal{F}_A(x_0) \geq r$ and, since the rank of a differentiable generalized distribution is a lower semicontinuous function (see [28]), there exists an open subset $V'_{x_0}$ of $M$, $x_0 \in V'_{x_0}$, such that

$$\dim \mathcal{F}_A(x) \geq \dim \mathcal{F}_A(x_0) \geq r, \text{ for all } x \in V'_{x_0}. \quad (3.8)$$

Thus, if $\tilde{W}_{(x_0, t_0)}$ is the connected component of the point $(x_0, t_0)$ in the open subset (of $\tilde{F}$) $\tilde{F} \cap (V'_{x_0} \times \mathbb{R})$ then, from (3.3), (3.7) and (3.8), it follows that (3.4) holds.

ii) Let $x_0$ be a point of $S^\phi_F$. We will prove that there exists an open neighbourhood $V_{x_0}$ of $x_0$ in $F$ which is contained in $S^\phi_F$.

Suppose that $t_0 \in \mathbb{R}$, that $F$ is the leaf of $\mathcal{F}_A$ passing through $(x_0, t_0)$ and that the dimension of $\tilde{F}$ is $r$. Then, the point $(x_0, t_0) \in S^\phi_F$ and thus there exists a connected open subset $\tilde{W}_{(x_0, t_0)}$ of $\tilde{F}$ such that $(x_0, t_0) \in \tilde{W}_{(x_0, t_0)}$ and (3.6) holds.

Now, if $\pi_1 : M \times \mathbb{R} \rightarrow M$ is the canonical projection onto the first factor, (3.2), (3.6) and Remark 2.3 imply that $\pi_1(\tilde{W}_{(x_0, t_0)}) \subseteq F$ and that $\pi_1 : \tilde{W}_{(x_0, t_0)} \rightarrow F$ is a local diffeomorphism. Therefore, $V_{x_0} = \pi_1(\tilde{W}_{(x_0, t_0)})$ is an open subset of $F$, $x_0 \in V_{x_0}$ and it is clear that $V_{x_0} \subseteq S^\phi_F$ (see (3.4) and (3.6)). QED
Lemma 3.5 If $F$ is a leaf of the Lie algebroid foliation $\mathcal{F}_A$ then $S^\phi_F$ is a closed subset of $F$.

Proof: We must prove that $F - S^\phi_F$ is an open subset of $F$.

Let $x_0$ be a point of $F - S^\phi_F$.

Assume that $\{s_1, \ldots, s_n\}$ is a local basis of $\Gamma(A)$ in an open neighbourhood of $x_0$ in $M$. If the dimension of $F$ is $r$ we can suppose, without loss of the generality, that the vectors $\{\rho(s_i(x_0))\}_{1 \leq i \leq r}$ are linearly independent in $T_{x_0}M$. Then, there exists an open subset $U'_{x_0}$ of $M$, $x_0 \in U'_{x_0}$, such that for every $x \in U'_{x_0}$ the set of vectors $\{\rho(s_i(x))\}_{1 \leq i \leq r}$ are linearly independent in $T_xM$. Thus, this set is a basis of $T_xF$, for all $x \in U_{x_0} = U'_{x_0} \cap F$. This implies that

$$\rho(s_{r+k})|_{U_{x_0}} = \sum_{i=1}^{r} g_k^i \rho(s_i)|_{U_{x_0}},$$

for all $k \in \{1, \ldots, n-r\}$, with $g_k^i \in C^\infty(U_{x_0}, \mathbb{R})$. Therefore, if $k \in \{1, \ldots, n-r\}$ and $e_{r+k}: U_{x_0} \to A$ is the map defined by

$$e_{r+k}(x) = s_{r+k}(x) - \sum_{i=1}^{r} g_k^i(x)s_i(x), \text{ for all } x \in U_{x_0},$$

we deduce that the set $\{e_{r+k}(x)\}_{1 \leq k \leq n-r}$ is a basis of the vector space $\ker(\rho|_{A_x})$, for all $x \in U_{x_0}$. Consequently, since $x_0 \in F - S^\phi_F$, it follows that $\phi(x_0)(e_{r+k_0}(x_0)) \neq 0$, for some $k_0 \in \{1, \ldots, n-r\}$. Hence, we conclude that there exists an open subset $V_{x_0}$ of $F$, $x_0 \in V_{x_0} \subseteq U_{x_0}$, such that $\phi(x)(e_{r+k_0}(x)) \neq 0$, for all $x \in V_{x_0}$. Then, it is clear that $V_{x_0} \subseteq F - S^\phi_F$. \(\Box\)

Lemma 3.6 If $\tilde{F}$ is a leaf of the Lie algebroid foliation $\mathcal{F}_A$ and $S^\phi_{\tilde{F}}$ is the subset of $\tilde{F}$ defined by

$$S^\phi_{\tilde{F}} = \{(x,t) \in \tilde{F} / \ker(\rho|_{A_x}) \subseteq <\phi(x)>^\circ\},$$

then $S^\phi_{\tilde{F}}$ is a closed subset of $\tilde{F}$.

Proof: Assume that the dimension of $\tilde{F}$ is $r$. We will prove that $\tilde{F} - S^\phi_{\tilde{F}}$ is an open subset of $\tilde{F}$.

Let $(x_0,t_0)$ be a point of $\tilde{F} - S^\phi_{\tilde{F}}$ and suppose that $F$ is the leaf of $\mathcal{F}_A$ passing through $x_0$. We have that $x_0 \in F - S^\phi_F$. Thus, using Lemmas 3.4 and 3.5 and the fact that $F$ is connected, we deduce that

$$S^\phi_F = \emptyset. \tag{3.9}$$

Therefore, from (3.2) and (3.9), it follows that

$$\mathcal{F}_{\tilde{A}}(x,t) = \mathcal{F}_A(x) \oplus \left< \frac{\partial}{\partial t} \big|_{t} \right>, \text{ for all } (x,t) \in F \times \mathbb{R}.$$
Consequently, $F \times \mathbb{R}$ is a connected integral submanifold of $\mathcal{F}_{\bar{A}}$ and its dimension is $r$. This implies that $F \times \mathbb{R}$ is an open subset of $\bar{F}$. Finally, from (3.9), we conclude that $F \times \mathbb{R} \subseteq \bar{F} - S^\phi_F$.

**Proof of Theorem 3.2:** Using Lemmas 3.4 and 3.5 and the fact that $F$ is connected it follows that $S^\phi_F = \emptyset$ or $S^\phi_F = F$.

Now, if $S^\phi_F = F$ we may introduce a 1-form $\omega_F$ on $F$ given by

$$\omega_F(x)(\rho(e_x)) = -\phi(x)(e_x),$$

for all $x \in F$ and $e_x \in A_x$. Note that the condition

$$\ker (\rho|_{A_x}) \subset \phi(x)^{\circ},$$

implies that $\omega_F(x): T_x F \to \mathbb{R}$ is well defined. Moreover, it is clear that $\omega_F$ satisfies (3.5) and, since $\phi$ is a 1-cocycle, we deduce that $\omega_F$ is closed.

**Proof of Theorem 3.3:** i) If $\ker (\rho|_{A_{x_0}}) \not\subset \phi(x_0)^{\circ}$ then $(x_0, t_0) \in \bar{F} - S^\phi_F$ and thus, using Lemmas 3.4 and 3.6 and the fact that $\bar{F}$ is connected, we obtain that

$S^\phi_F = \emptyset.$ (3.10)

Now, proceeding as in the proof of Lemma 3.4, we deduce that $F \times \mathbb{R}$ is an open subset of $\bar{F}$. On the other hand, from (3.3), (3.4) and (3.10), it follows that

$$\dim \mathcal{F}_A(x) = \dim \mathcal{F}_{\bar{A}}(x, t) - 1 = \dim \bar{F} - 1,$$

for all $(x, t) \in \bar{F}$.

Using this fact, (3.2) and Remark 2.3, we have that $\pi_1(\bar{F}) \subseteq F$. Therefore, we have proved that $\bar{F} = F \times \mathbb{R}$.

ii) Assume that $\ker (\rho|_{A_{x_0}}) \subset \phi(x_0)^{\circ}$. Then, $(x_0, t_0) \in S^\phi_F$ which implies that $\bar{F} = S^\phi_F$, that is,

$$\dim \mathcal{F}_A(x) = \dim \mathcal{F}_{\bar{A}}(x, t) = \dim \bar{F},$$

for all $(x, t) \in \bar{F}$. (3.11)

Using (3.2), (3.11) and Remark 2.3, we obtain that $\pi_1(\bar{F}) \subseteq F$ and that $\pi_{1|\bar{F}}: \bar{F} \to F$ is a local diffeomorphism. Consequently, $\pi_1(\bar{F})$ is an open subset of $F$.

In addition, from (3.2) and (3.3), it follows that

$$(\pi_{1|\bar{F}})^* (\omega_F) = -d((i_F)^* t).$$ (3.12)

Next, we will show that $\pi_1(\bar{F})$ is a closed subset of $F$ and that $\pi_{1|\bar{F}}: \bar{F} \to F$ is a covering map.

Let $x$ be a point of $F$. Since $\omega_F$ is a closed 1-form, there exists a connected open subset $U$ in $F$ and a real $C^\infty$-differentiable function $f_F$ on $U$ such that $x \in U$ and

$$\omega_F = df_F \quad \text{on } U.$$ (3.13)
Then, using (3.2), (3.3), (3.13), Remark 2.3 and the fact that \( \pi_1(\bar{F}) \subseteq F \), we deduce the following result

\[
(y, s) \in (\pi_1|F)^{-1}(U) = \pi_1^{-1}(U) \cap \bar{F} \implies \\
\{(z, s + f_F(y) - f_F(z)) \in M \times \mathbb{R} / z \in U\} \subseteq \bar{F}.
\] (3.14)

Thus, if \( x \in F - \pi_1(\bar{F}) \), we have that \( U \subseteq F - \pi_1(\bar{F}) \). This proves that \( \pi_1(\bar{F}) \) is a closed subset of \( F \) which implies that \( \pi_1(\bar{F}) = F \).

Now, suppose that \((x, t)\) is a point of \( \bar{F} \) and let \( U \) be a connected open subset of \( F \) and \( f_F \) be a real \( C^\infty \)-differentiable on \( F \) such that \( x \in U \) and (3.13) holds. If \( C_{(y,s)} \) is the connected component of a point \((y, s) \in (\pi_1|F)^{-1}(U)\) then, using (3.12) and (3.13), it follows that the function \((\pi_1|F)^*(f_F) + (i_F)^*t\) is constant on \( C_{(y,s)} \). Therefore, from (3.14), we obtain that

\[
C_{(y,s)} = \{(z, s + f_F(y) - f_F(z)) \in M \times \mathbb{R} / z \in U\}.
\]

Consequently, the map \( \pi_1|C_{(y,s)} : C_{(y,s)} \to U \) is a diffeomorphism. This proves that \( \pi_1|F : \bar{F} \to F \) is a covering map.

Finally, let \( E \) be the covering of \( F \) associated with \( \omega_F \), that is, \( E \) is the sheaf of germs of \( C^\infty \) functions \( g_F \) on \( F \) such that \( dg_F = \omega_F \) (see Section 2 of Chapter XIV in [7]). Denote by \( (f^0_F)^{[x_0]} \) the germ of \( f^0_F \) at \( x_0 \), where \( f^0_F \) is a \( C^\infty \) function on a connected open subset \( U_0 \) of \( F \) such that \( x_0 \in U_0 \), \( (f^0_F)(x_0) = t_0 \) and \( \omega_F|U_0 = df^0_F \). Then, using the above description of the leaf \( \bar{F} \) and the results in [3], we deduce that \( \bar{F} \) is diffeomorphic to the connected component of \( (f^0_F)^{[x_0]} \) in \( E \). In other words, \( \bar{F} \) is diffeomorphic to a Galois covering of \( F \) associated with \( \omega_F \).  

\[Q.E.D.\]

4 \( \mathcal{E}^1(M) \)-Dirac structures, submanifolds of the base space and the leaves of the characteristic foliation

4.1 \( \mathcal{E}^1(M) \)-Dirac structures and submanifolds of the base space

In this Section, we will prove that if \( S \) is a submanifold of \( M \) then, under certain regularity conditions, a \( \mathcal{E}^1(M) \)-Dirac structure induces a \( \mathcal{E}^1(S) \)-Dirac structure. This result will be used in Section 4.2.

Let \( L \) be a vector sub-bundle of \( \mathcal{E}^1(M) \) which is maximally isotropic under the symmetric pairing \( <, >_+ \) and \( S \) be a submanifold of \( M \). If \( x \) is a point of \( S \), we may define the vector space \((L_S)_x\) by

\[
(L_S)_x = \frac{L_x \cap ((T_xS \times \mathbb{R}) \oplus (T^*_xM \times \mathbb{R}))}{L_x \cap ((\{0\} \oplus ((T_xS)^c \times \{0\}))},
\] (4.1)
where \((T_xS)^\circ\) is the annihilator of \(T_xS\), that is, \((T_xS)^\circ = \{ \alpha \in T^*_xM \mid \alpha|_{T_xS} = 0 \}\). We have that the linear map \((L_S)_x \to (T_xS \times \mathbb{R}) \oplus (T^*_xS \times \mathbb{R})\) given by

\[
[(u, \lambda) + (\alpha, \mu)] \mapsto (u, \lambda) + (\alpha|_{T_xS}, \mu),
\]

is a monomorphism and thus \((L_S)_x\) can be identified with a subspace of \((T_xS \times \mathbb{R}) \oplus (T^*_xS \times \mathbb{R})\). Moreover, using the results of Section 1.4 in \([2]\), we deduce that \((L_S)_x\) is a maximally isotropic subspace of \((T_xS \times \mathbb{R}) \oplus (T^*_xS \times \mathbb{R})\) under the symmetric pairing \(<, >\). In particular, this implies that \(\dim (L_S)_x = \dim S + 1\), for all \(x \in S\). In addition, we may prove the following proposition.

**Proposition 4.1** Let \(L\) be a \(\mathcal{E}^1(M)\)-Dirac structure and \(S\) be a submanifold of \(M\). If the dimension of \(L_x \cap \{(T_xS \times \mathbb{R}) \oplus (T^*_xM \times \mathbb{R})\}\) keeps constant for all \(x \in S\) (or, equivalently, the dimension of \(L_x \cap \{0\} \oplus ((T_xS)^\circ \times \{0\})\) keeps constant for all \(x \in S\)) then \(L_S = \bigcup_{x \in S} (L_S)_x\) is a vector sub-bundle of \(\mathcal{E}^1(S)\) and, furthermore, \(L_S\) is a \(\mathcal{E}^1(S)\)-Dirac structure.

**Proof:** It is clear that \(L_S\) is a maximally isotropic vector sub-bundle of \(\mathcal{E}^1(S)\) under the symmetric pairing \(<, >\).

Now, we consider the vector bundle \(\hat{L}_S\) over \(S\) such that the fiber \((\hat{L}_S)_x\) of \(\hat{L}_S\) over \(x \in S\) is given by

\[
(\hat{L}_S)_x = L_x \cap \{(T_xS \times \mathbb{R}) \oplus (T^*_xM \times \mathbb{R})\}.
\]

Denote by \(i_S : \hat{L}_S \rightarrow L\) the inclusion map, by \(\pi_S : \hat{L}_S \rightarrow L_S\) the canonical projection and by \(T_L\) (respectively, \(T_{L_S}\)) the section of \(\wedge^3 L^*\) (respectively, \(\wedge^3 L^*_S\)) associated with the isotropic vector sub-bundle \(L\) (respectively, \(L_S\)). The map \(i_S\) (respectively, \(\pi_S\)) is a monomorphism (respectively, epimorphism) of vector bundles. Furthermore, if \(e_i = (X_i, f_i) + (\alpha_i, g_i) \in \Gamma(\hat{L}_S)\), with \(i \in \{1, 2, 3\}\), then, from (2.3) and Proposition 2.2, we get that

\[
(\pi^*_S T_{L_S})(e_1, e_2, e_3) = \frac{1}{2} \sum_{\text{cycl.}(e_1, e_2, e_3)} \left( i_{[X_1, X_2]} \alpha_3 + g_3(X_1(f_2) - X_2(f_1)) + X_3(i_{X_2} \alpha_1 + f_2 g_1) + f_3(i_{X_2} \alpha_1 + f_2 g_1) \right)
\]

\[
= (i^*_S T_L)(e_1, e_2, e_3) = 0.
\]

Therefore, \(\pi^*_S T_{L_S} = 0\) and, since \(\pi_S\) is an epimorphism of vector bundles, we conclude that \(T_{L_S} = 0\). This implies that \(L_S\) is a \(\mathcal{E}^1(S)\)-Dirac structure. \(\Box\)

### 4.2 The induced structure on the leaves of the characteristic foliation of a \(\mathcal{E}^1(M)\)-Dirac structure

Let \(L\) be a \(\mathcal{E}^1(M)\)-Dirac structure. Denote by \((L, [\cdot, \cdot]_L, \rho_L)\) the associated Lie algebroid and by \(\phi_L \in \Gamma(L^*)\) the 1-cocycle defined by (2.9).
We consider the bundle map \((\rho_L, \phi_L) : L \to TM \times \mathbb{R}\) given by
\[
(\rho_L, \phi_L)(e_x) = (\rho_L(e_x), \phi_L(x)(e_x)),
\]
for \(e_x \in L_x\) and \(x \in M\). Then, we may define the 2-form \(\Psi_L(x)\) on the vector space \((\rho_L, \phi_L)(L_x)\) by
\[
\Psi_L(x)((\rho_L, \phi_L)((e_1)_x), (\rho_L, \phi_L)((e_2)_x)) = \langle (e_1)_x, (e_2)_x \rangle, \]
for \((e_1)_x, (e_2)_x \in L_x, \langle , \rangle_\ast\) being the natural skew-symmetric pairing on \((T_xM \times \mathbb{R}) \oplus (T^*_xM \times \mathbb{R})\). Since \(L\) is an isotropic vector sub-bundle of \(\mathcal{E}^1(M)\) under the symmetric pairing \(\langle , \rangle_+\), we deduce that the 2-form \(\Psi_L(x)\) is well defined. Note that if \(e_1, e_2 \in \Gamma(L)\) then one may consider the function \(\Psi_L((\rho_L, \phi_L)(e_1), (\rho_L, \phi_L)(e_2)) \in C^\infty(M, \mathbb{R})\) given by
\[
\Psi_L((\rho_L, \phi_L)(e_1), (\rho_L, \phi_L)(e_2))(x) = \Psi_L(x)((\rho_L, \phi_L)((e_1)_x), (\rho_L, \phi_L)((e_2)_x)), \text{ for all } x \in M.
\]
In fact, if \(e_i = (X_i, f_i) + (\alpha_i, g_i), \text{ with } i \in \{1, 2\}\), we have that
\[
\Psi_L((X_1, f_1), (X_2, f_2)) = i_{X_1} \alpha_1 + f_2 g_1.
\]
Now, let \(F_L\) be the characteristic foliation of the \(\mathcal{E}^1(M)\)-Dirac structure \(L\) and \(F\) be a leaf of \(F_L\). If \(x\) is a point of \(F\), we will denote by \((L_F)_x\) the vector subspace of \((T_xF \times \mathbb{R}) \oplus (T^*_xF \times \mathbb{R})\) given by (see Section \[4.1\])
\[
(L_F)_x = \frac{L_x \cap ((T_xF \times \mathbb{R}) \oplus (T^*_xM \times \mathbb{R}))}{L_x \cap \{(0) \oplus ((T_xF)^0 \times \{0\})\}}.
\]
Then, we will prove that \(L_F = \bigcup_{x \in F}(L_F)_x\) defines a \(\mathcal{E}^1(F)\)-Dirac structure and we will describe the nature of \(L_F\).

**Theorem 4.2** Let \(L\) be a \(\mathcal{E}^1(M)\)-Dirac structure and \(F\) be the leaf of the characteristic foliation \(F_L\) passing through \(x_0 \in M\). Then, \(L_F = \bigcup_{x \in F}(L_F)_x\) defines a \(\mathcal{E}^1(F)\)-Dirac structure and we have two possibilities:

\[
\text{i) If ker} \ (\rho_{L|L_{x_0}}) \not\subseteq \phi_L(x_0)^0, \text{ the} \ \mathcal{E}^1(F)\text{-Dirac structure} \ L_F \text{ comes from a precontact structure} \ \eta_F \text{ on} \ F, \text{ that is,} \ L_F = L_{\eta_F}. \text{ In this case,} \ F \text{ is said to be a precontact leaf.}
\]
\[
\text{ii) If ker} \ (\rho_{L|L_{x_0}}) \subseteq \phi_L(x_0)^0, \text{ the} \ \mathcal{E}^1(F)\text{-Dirac structure} \ L_F \text{ comes from a locally conformal presymplectic structure} \ (\Omega_F, \omega_F) \text{ on} \ F, \text{ that is,} \ L_F = L(\Omega_F, \omega_F). \text{ In this case,} \ F \text{ is said to be a locally conformal presymplectic leaf.}
\]

**Proof:** From the definition of \(F_L\), it follows that
\[
(L_F)_x = L_x \cap ((T_xF \times \mathbb{R}) \oplus (T^*_xM \times \mathbb{R})) = L_x, \text{ for all } x \in F.
\]
Thus, since $L$ is a maximally isotropic vector sub-bundle of $\mathcal{E}^1(M)$ under the symmetric pairing $<, >_+$, we obtain that $\dim(\mathcal{L}_F)_x = \dim M + 1$, for all $x \in F$. Therefore, using Proposition \[1.3\], we deduce that $L_F$ defines a $\mathcal{E}^1(F)$-Dirac structure.

Next, we will distinguish the two cases:

i) Assume that $\ker(\rho_{L|\mathcal{L}_x}) \not\subseteq \phi_L(x_0) >^0$. Then, from Theorem \[3.2\], we have that $\ker(\rho_{L|\mathcal{L}_x}) \not\subseteq \phi_L(x) >^0$, for all $x \in F$. This implies that the map $(\rho_L, \phi_L)|_{\mathcal{L}_x} : \mathcal{L}_x \to T_x F \times \mathbb{R}$ is a linear epimorphism, for all $x \in F$ (see \[1.3\]). Consequently, the restriction of $\Psi_L$ to $F$ defines a section of the vector bundle $\wedge^2(T^* F \times \mathbb{R}) \to F$, i.e., a pair $(\Phi_F, \eta_F) \in \Omega^2(F) \times \Omega^1(F)$. The relation between $\Psi_L$ and $(\Phi_F, \eta_F)$ is given by

$$
\Psi_L(x)((u_1, \lambda_1), (u_2, \lambda_2)) = \Phi_F(x)(u_1, u_2) + \lambda_1 \eta_F(x)(u_2) - \lambda_2 \eta_F(x)(u_1),
$$

(4.6)

for all $x \in F$ and $(u_1, \lambda_1), (u_2, \lambda_2) \in T_x F \times \mathbb{R}$.

Now, suppose that $(u, \lambda) + (\alpha, \mu) \in (\mathcal{L}_F)_x = \mathcal{L}_x$, with $x \in F$. From (2.1), (4.4) and (4.6), it follows that

$$(i_u \Phi_F(x) + \lambda \eta_F(x))(v) + \nu(-\eta_F(x)(u)) = \alpha(v) + \nu \mu,$$

for all $(v, \nu) \in T_x F \times \mathbb{R}$, that is, $\alpha|T_x F = i_u \Phi_F(x) + \lambda \eta_F(x)$ and $\mu = -\eta_F(x)(u)$. In other words, if we consider $(\mathcal{L}_F)_x$ to be a subspace of $(T_x F \times \mathbb{R}) \oplus (T^*_x F \times \mathbb{R})$, we have that

$$(\mathcal{L}_F)_x \subseteq \{(u, \lambda) + (i_u \Phi_F(x) + \lambda \eta_F(x), -\eta_F(x)(u)) / (u, \lambda) \in T_x F \times \mathbb{R})\}.$$

But, since $\dim(\mathcal{L}_F)_x = \dim F + 1$, we deduce that

$$(\mathcal{L}_F)_x = \{(u, \lambda) + (i_u \Phi_F(x) + \lambda \eta_F(x), -\eta_F(x)(u)) / (u, \lambda) \in T_x F \times \mathbb{R})\}.$$

Thus,

$$
\Gamma(\mathcal{L}_F) = \{(X, f) + (i_X \Phi_F + f \eta_F, -i_X \eta_F) / (X, f) \in \mathfrak{x}(F) \times C^\infty(F, \mathbb{R})\}.
$$

(4.7)

Finally, using (4.7) and the fact that $L_F$ is a $\mathcal{E}^1(F)$-Dirac structure, we conclude that (see Section 2.3, Example 3),

$$
\Phi_F = d\eta_F
$$

(4.8)

and $L_F = L_{\eta_F}$.

ii) Assume that $\ker(\rho_{L|\mathcal{L}_x}) \subseteq \phi_L(x_0) >^0$. Then, from Theorem \[3.2\], we obtain that $\ker(\rho_{L|\mathcal{L}_x}) \subseteq \phi_L(x) >^0$, for all $x \in F$. This implies that the map

$$
\varphi_x = pr_1|_{(\rho_L, \phi_L)(L_x)} : (\rho_L, \phi_L)(L_x) \to \rho_L(L_x) = T_x F
$$

is a linear isomorphism, for all $x \in F$, where $pr_1 : T_x F \times \mathbb{R} \to T_x F$ is the projection onto the first factor.
Therefore, $\Psi_L$ induces a 2-form $\Omega_F$ on $F$ which is characterized by the condition
\[
\Omega_F(x)(\rho_L((e_1)_x), \rho_L((e_2)_x)) = \Psi_L(x)((\rho_L, \phi_L)((e_1)_x), (\rho_L, \phi_L)((e_2)_x)), \tag{4.9}
\]
for $x \in F$ and $(e_1)_x, (e_2)_x \in L_x$.

Moreover, since $S^\phi_L = F$, Theorem 3.2 allows us to introduce the closed 1-form $\omega_F$ on $F$ characterized by (3.5).

Now, suppose that $(u, \lambda) + (\alpha, \mu) \in \hat{(L_F)}_x = L_x$, with $x \in F$. From (2.1), (2.9), (3.5), (4.4) and (4.9), it follows that $\lambda = -\omega_F(x)(u)$ and that $\alpha_{|T_xF} = i_u \Omega_F(x) + \mu \omega_F(x)$. In other words, if we consider $(L_F)_x$ to be a subspace of $(T_xF \times \mathbb{R}) \oplus (T^*_xF \times \mathbb{R})$ then, since $\dim (L_F)_x = \dim F + 1$, we deduce that
\[
(L_F)_x = \{(u, -\omega_F(x)(u)) + (i_u \Omega_F(x) + \mu \omega_F(x), \mu) / (u, \mu) \in T_xF \times \mathbb{R}\}.
\]
Thus,
\[
\Gamma(L_F) = \{(X, -\omega_F(X)) + (i_X \Omega_F + f \omega_F, f) / (X, f) \in \mathfrak{x}(F) \times C^\infty(F, \mathbb{R})\}. \tag{4.10}
\]
Finally, using that $\omega_F$ is closed, (4.10) and the fact that $L_F$ is a $\mathcal{E}(F)$-Dirac structure, we conclude that the pair $(\Omega_F, \omega_F)$ is a l.c.p. structure on $F$ (see Section 2.3, Example 2) and that $L_F = L_{(\Omega_F, \omega_F)}$. \(\square\)

Examples 4.3 1.- Dirac structures

Let $\tilde{L} \subseteq TM \oplus T^*M$ be a Dirac structure on $M$ and $L$ be the $\mathcal{E}(M)$-Dirac structure associated with $\tilde{L}$ (see (2.10)). We know that the characteristic foliations $\mathcal{F}_{\tilde{L}}$ and $\mathcal{F}_L$ associated with $\tilde{L}$ and $L$, respectively, coincide (see Section 2.3, Example 1). Thus, if $\tilde{F}$ is a leaf of $\mathcal{F}_{\tilde{L}}$ then, using Theorem 4.2 and the fact that the 1-cocycle $\phi_L$ identically vanishes, it follows that $\tilde{F}$ carries an induced l.c.p. structure $(\Omega_{\tilde{F}}, \omega_{\tilde{F}})$. Moreover, from the definition of $\omega_{\tilde{F}}$ (see (3.3)), we obtain that $\omega_{\tilde{F}} = 0$, that is, $\Omega_{\tilde{F}}$ is a presymplectic form on $\tilde{F}$. Therefore, we deduce a well-known result (see [2]): the leaves of the characteristic foliation $\mathcal{F}_{\tilde{L}}$ are presymplectic manifolds.

2.- Locally conformal presymplectic structures

Let $(\Omega, \omega)$ be a l.c.p. structure on a manifold $M$ and $L_{(\Omega, \omega)}$ be the corresponding $\mathcal{E}(M)$-Dirac structure (see (2.13)). It is clear that $\mathcal{F}_{L_{(\Omega, \omega)}}(x) = T_xM$, for all $x \in M$, and thus there is only one leaf of the foliation $\mathcal{F}_{L_{(\Omega, \omega)}}$, namely, $M$. Besides, since $\ker (\rho_{L_{(\Omega, \omega)}}(\xi_{L_{(\Omega, \omega)}})) \subseteq \phi_{L_{(\Omega, \omega)}}(x)^\circ$, for all $x \in M$ (see Section 2.3, Example 2), $M$ carries an induced l.c.p. structure which is just $(\Omega, \omega)$.

3.- Precontact structures

Let $\eta$ be a precontact structure on a manifold $M$ and denote by $L_{\eta}$ the corresponding $\mathcal{E}(M)$-Dirac structure (see (2.13)). As in the case of a l.c.p. structure, there is only one leaf of the characteristic foliation $\mathcal{F}_{L_{\eta}}$: the manifold $M$. In addition, since
Let \( \ker (\rho_{L_\mathcal{E}}(L_\mathcal{E})_{\mathcal{E}_0}) \not\subseteq \phi_{L_\mathcal{E}}(x) >^o \), for all \( x \in M \) (see Section 2.3, Example 3), \( M \) carries an induced precontact structure. Such a structure is defined by the 1-form \( \eta \).

4.- Jacobi structures

Suppose that \( (\Lambda, E) \) is a Jacobi structure on a manifold \( M \) and let \( L_{(\Lambda, E)} \) be the corresponding \( \mathcal{E}^1(M) \)-Dirac structure. We know that the characteristic foliation \( \mathcal{F}_{L_{(\Lambda, E)}} \) of \( L_{(\Lambda, E)} \) is just the characteristic foliation associated with the Jacobi structure \( (\Lambda, E) \) (see Section 2.3, Example 4). Moreover, using that the Lie algebroid \( (L_{(\Lambda, E)}, [\cdot, \cdot]_{L_{(\Lambda, E)}}, \rho_{L_{(\Lambda, E)}}) \) can be identified with the Lie algebroid \( (T^*M \times \mathbb{R}, [\cdot, \cdot]_{(\Lambda, E)}, \#_{(\Lambda, E)}) \) and that, under this identification, the 1-cocycle \( \phi_{L_{(\Lambda, E)}} \) is the pair \((-E, 0)\), we obtain that if \( x_0 \) is a point of \( M \) then

\[
\ker (\rho_{L_{(\Lambda, E)}}(L_{(\Lambda, E)})_{\mathcal{E}_0}) \not\subseteq \phi_{L_{(\Lambda, E)}}(x_0) >^o \iff E(x_0) \in \#_{\Lambda}(T^*_xM). \tag{4.11}
\]

Thus, if \( F \) is the leaf of \( \mathcal{F}_{L_{(\Lambda, E)}} \) passing through \( x_0 \in M \) and \( E(x_0) \not\in \#_{\Lambda}(T^*_xM) \), from (4.11) and Theorem 4.2, it follows that \( F \) carries an induced l.c.p. structure \((\Omega_F, \omega_F)\). In fact, using (2.1), (4.4) and (4.6), we get that

\[
\Omega_F(y)(\#_{\Lambda}(\alpha_1), \#_{\Lambda}(\alpha_2)) = \alpha_1(\#_{\Lambda}(\alpha_2)), \quad \omega_F(y)(\#_{\Lambda}(\alpha_1)) = \alpha_1(E(y)),
\]

for all \( y \in F \) and \( \alpha_1, \alpha_2 \in T^*_yM \). Therefore, the pair \((-\Omega_F, \omega_F)\) is the locally conformal symplectic structure on \( F \) induced by the Jacobi structure \((\Lambda, E)\).

On the other hand, if \( F \) is the leaf of \( \mathcal{F}_{L_{(\Lambda, E)}} \) passing through \( x_0 \in M \) and \( E(x_0) \not\in \#_{\Lambda}(T^*_xM) \) then, from (4.11) and Theorem 4.2, we obtain that the \( \mathcal{E}^1(F) \)-Dirac structure comes from a precontact structure \( \eta_F \) on \( F \). In addition, \( E(y) \not\in \#_{\Lambda}(T^*_yM) \) and \( T_yF = \#_{\Lambda}(T^*_yM) \oplus <E(y)> \), for all \( y \in F \). Moreover, using (2.7), (4.4) and (4.6), we get that

\[
\eta_F(y)(\#_{\Lambda}(\alpha) + \lambda E(y)) = -\lambda,
\]

for all \( y \in F \), \( \alpha \in T^*_yM \) and \( \lambda \in \mathbb{R} \). Consequently, \(-\eta_F\) is the contact structure on \( F \) induced by the Jacobi structure \((\Lambda, E)\).

In conclusion, we deduce a well-known result (see [3, 14]): the leaves of the characteristic foliation of a Jacobi manifold are contact or locally conformal symplectic manifolds.

5.- Homogeneous Poisson structures

Let \( (M, \Pi, Z) \) be a homogeneous Poisson manifold and \( L_{(\Pi, Z)} \) be the corresponding \( \mathcal{E}^1(M) \)-Dirac structure (see (2.17)). Using that the Lie algebroid \( (L_{(\Pi, Z)}, [\cdot, \cdot]_{L_{(\Pi, Z)}}, \rho_{L_{(\Pi, Z)}}) \) can be identified with the Lie algebroid \( (T^*M \times \mathbb{R}, [\cdot, \cdot]_{(\Pi, Z)}, \#_{(\Pi, Z)}) \) and that, under this identification, the 1-cocycle \( \phi_{L_{(\Pi, Z)}} \) is the pair \((0,1)\) (see Section 2.3, Example 5), we obtain that if \( x_0 \) is a point of \( M \) then

\[
\ker (\rho_{L_{(\Pi, Z)}}(L_{(\Pi, Z)})_{\mathcal{E}_0}) \not\subseteq \phi_{L_{(\Pi, Z)}}(x_0) >^o \iff Z(x_0) \not\in \#_{\Pi}(T^*_xM). \tag{4.12}
\]

Thus, if \( x_0 \) is a point of \( M \) and \( F \) is the leaf of the characteristic foliation \( \mathcal{F}_{L_{(\Pi, Z)}} \) passing through \( x_0 \), we will distinguish two cases:
a) $Z(x_0) \in \#_\Pi(T^*_x M)$. In such a case, from (2.19), (4.12) and Theorem 3.2, it follows that $T_y F = \mathcal{F}_{L_y}(y) = \mathcal{F}_\Pi(y)$, for all $y \in F$, where $\mathcal{F}_\Pi$ is the symplectic foliation of the Poisson manifold $(\Pi, M)$. Therefore, $F$ is the leaf of $\mathcal{F}_\Pi$ passing through $x_0$. In addition, using Theorem 1.2, we deduce that the induced $\mathcal{E}^1(F)$-Dirac structure comes from a precontact structure $\eta_F$ on $F$. Moreover, from (2.1), (4.12) and (4.5), we have that

$$\eta_F(y)(\#_\Pi(\alpha_1)) = -\alpha_1(Z(y)), \quad \alpha_1 = \#_\Pi(\alpha_2) = \alpha_1(\#_\Pi(\alpha_2)),$$

for all $y \in F$ and $\alpha_1, \alpha_2 \in T^*_y M$. This implies that $d\eta_F$ is, up to sign, the symplectic 2-form of $F$.

b) $Z(x_0) \notin \#_\Pi(T^*_x M)$. In such a case, from (2.19) and (4.12), we get that $T_y F = \mathcal{F}_{L_y}(y) = \mathcal{F}_\Pi(y) \oplus <Z(y)>$, for all $y \in F$. Consequently, the dimension of $F$ is odd and the leaf $F_\Pi$ of the foliation $\mathcal{F}_\Pi$ passing through $x_0$ is a submanifold of $F$ of codimension one. Furthermore, the induced $\mathcal{E}^1(F)$-Dirac structure comes from a l.c.p. structure $(\Omega_F, \omega_F)$ on $F$ and, using (2.1), (3.3), (4.4) and (4.9), it follows that

$$\Omega_F(y)(\#_\Pi(\alpha_1) + \lambda_1 Z(y), \#_\Pi(\alpha_2) + \lambda_2 Z(y)) = \alpha_1(\#_\Pi(\alpha_2)),$$

$$\omega_F(y)(\#_\Pi(\alpha_1) + \lambda_1 Z(y)) = \lambda_1,$$

for all $y \in F$, $\alpha_1, \alpha_2 \in T^*_y M$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Note that if $i : F_\Pi \to F$ is the canonical inclusion, we deduce that $i^* \omega_F = 0$ and that $-i^* \Omega_F$ is the symplectic 2-form on $F_\Pi$. Thus, if the dimension of $F$ is $2n + 1$, we obtain that $\omega_F \wedge \Omega_F^n = \omega_F \wedge \Omega_F \wedge \Omega_F^{n-1}$ and $\Omega_F$ is a volume form on $F$.

5 Dirac structure associated with a $\mathcal{E}^1(M)$-Dirac structure and characteristic foliations

Let $M$ be a differentiable manifold and $L$ be a vector sub-bundle of $\mathcal{E}^1(M)$.

We consider the vector sub-bundle $\tilde{L}$ of $T(M \times \mathbb{R}) \oplus T^*(M \times \mathbb{R})$ such that the fiber $\tilde{L}_{(x,t)}$ of $\tilde{L}$ over $(x,t) \in M \times \mathbb{R}$ is given by

$$\tilde{L}_{(x,t)} = \left\{(u + \lambda \frac{\partial}{\partial t}) + e^t(\alpha + \mu dt) \mid (u, \lambda) + (\alpha, \mu) \in L_x \right\}, \quad (5.1)$$

where $L_x$ is the fiber of $L$ over $x$. Note that the linear map $\psi_{(x,t)} : L_x \to \tilde{L}_{(x,t)}$ given by

$$\psi_{(x,t)}((u, \lambda) + (\alpha, \mu)) = (u + \lambda \frac{\partial}{\partial t}) + e^t(\alpha + \mu dt), \quad (5.2)$$

is an isomorphism of vector spaces, for all $(x,t) \in M \times \mathbb{R}$. Using this fact, (2.8) and (2.10), we deduce the following result.
Proposition 5.1 \( L \) is a \( \mathcal{E}^1(M) \)-Dirac structure if and only if \( \bar{L} \) is a Dirac structure on \( M \times \mathbb{R} \).

Now, suppose that \( L \) is a \( \mathcal{E}^1(M) \)-Dirac structure and denote by \((L,[\cdot,\cdot]_L,\rho_L)\) the associated Lie algebroid and by \( \phi_L \) the 1-cocycle of \((L,[\cdot,\cdot]_L,\rho_L)\) given by (2.9). Then, we may consider the Lie algebroid structure \([\cdot,\cdot]_{\bar{L}}^\phi,\bar{\rho}_L^\phi\) defined by (3.1) on the vector bundle \( \bar{L} = L \times \mathbb{R} \to M \times \mathbb{R} \).

On the other hand, let \( \bar{L} \) be the Dirac structure on \( M \times \mathbb{R} \) associated with \( L \), \((\bar{L},[\cdot,\cdot]_{\bar{L}},\bar{\rho}_L)\) be the corresponding Lie algebroid over \( M \times \mathbb{R} \) and \( \mathcal{F}_L \) be the characteristic foliation of \( \bar{L} \) (see Section 2.3, Example 1).

It is clear that the linear maps \( \psi_{(x,t)} \), \((x,t) \in M \times \mathbb{R} \), induce an isomorphism of vector bundles \( \tilde{\psi} \) between \( \bar{L} \) and \( \bar{L} \). Moreover, we have

Lemma 5.2 The map \( \tilde{\psi} \) is an isomorphism of Lie algebroids over the identity, that is,

\[
\bar{\rho}_L(\tilde{\psi}(\bar{e}_1)) = \bar{\rho}_L^\phi(\bar{e}_1), \quad \bar{\psi}[\bar{e}_1,\bar{e}_2]_{\bar{L}}^\phi = [\bar{\psi}(\bar{e}_1),\bar{\psi}(\bar{e}_2)]_{\bar{L}},
\]

for \( \bar{e}_1,\bar{e}_2 \in \Gamma(\bar{L}) \). Thus, the characteristic foliation \( \mathcal{F}_{\bar{L}} \) of the Dirac structure \( \bar{L} \) coincides with the Lie algebroid foliation \( \mathcal{F}_L \).

Proof: Using (2.2), (2.3), (2.9), (2.11), (2.12), (3.1) and (5.2), we deduce that (5.3) hold. In addition, from (5.3), it follows that \( \mathcal{F}_{\bar{L}}(x,t) = \mathcal{F}_L(x,t) \), for all \((x,t) \in M \times \mathbb{R} \). \( \square \)

Now, assume that \( \bar{F} \) is a leaf of the foliation \( \mathcal{F}_L = \mathcal{F}_\bar{L} \). Then, we know that \( \bar{F} \) is a presymplectic manifold with presymplectic 2-form \( \Omega_{\bar{F}} \) characterized by the condition

\[
\Omega_{\bar{F}}(x,t)(\bar{\rho}_L((\bar{e}_1)(x,t)),\bar{\rho}_L((\bar{e}_2)(x,t))) = <(\bar{e}_1)(x,t), (\bar{e}_2)(x,t)>,
\]

for all \((x,t) \in M \times \mathbb{R} \) and \((\bar{e}_1)(x,t), (\bar{e}_2)(x,t) \in \bar{L}(x,t) \), where \(<,> \) is the natural skew-symmetric pairing on \( T(x,t)(M \times \mathbb{R}) \oplus T^*(x,t)(M \times \mathbb{R}) \) (see [2] and Examples 4.3).

Next, we will discuss the relation between the leaves of \( \mathcal{F}_{\bar{L}} \) and the leaves of the characteristic foliation \( \mathcal{F}_L \) associated with \( L \). In addition, we will describe the relation between the induced structures on them.

Theorem 5.3 Let \( L \) be a \( \mathcal{E}^1(M) \)-Dirac structure and \( \bar{L} \) be the Dirac structure on \( M \times \mathbb{R} \) associated with \( L \). Suppose that \((x_0,t_0) \in M \times \mathbb{R} \) and that \( F \) and \( \bar{F} \) are the leaves of \( \mathcal{F}_L \) and \( \mathcal{F}_{\bar{L}} \) passing through \( x_0 \) and \((x_0,t_0) \), respectively. Then:

i) If \( F \) is a precontact leaf we have that \( \bar{F} = F \times \mathbb{R} \). Moreover, if \( \eta_F \) is the precontact structure on \( F \),

\[
\Omega_{\bar{F}} = e^t((\pi_{1|\bar{F}})^*(dT) + dt \wedge (\pi_{1|\bar{F}})^*(\eta_F)),
\]

where \( \pi_{1|\bar{F}} : \bar{F} \to F \) is the restriction to \( \bar{F} \) of the canonical projection \( \pi_1 : M \times \mathbb{R} \to M \).
ii) If $F$ is a l.c.p. leaf and $(\Omega_F, \omega_F)$ is the l.c.p. structure on $F$ then $\pi_1(\tilde{F}) = F$, $\pi_{1|\tilde{F}} : \tilde{F} \to F$ is a covering map and $\tilde{F}$ is diffeomorphic to a Galois covering of $F$ associated with $\omega_F$. Furthermore, if $i_{\tilde{F}} : \tilde{F} \to M \times \mathbb{R}$ is the canonical inclusion and $\tilde{\sigma} \in C^\infty(\tilde{F}, \mathbb{R})$ is the function given by $\tilde{\sigma} = -(i_{\tilde{F}})^*(t)$, we have that

$$d\tilde{\sigma} = (\pi_{1|\tilde{F}})^*(\omega_F), \quad \Omega_{\tilde{F}} = e^{-\tilde{\sigma}}(\pi_{1|\tilde{F}})^*(\Omega_F).$$

Proof: i) Since $F$ is a precontact leaf, it follows that $\ker(\rho_{L_x}|_{L_{x_0}}) \not\subset \phi_L(x_0) >^0$ (see Theorem 4.2). Thus, from Theorem 3.3 and Lemma 5.2, we deduce that $\tilde{F} = F \times \mathbb{R}$.

On the other hand, if $(x, t) \in \tilde{F}$, $(\tilde{e}_i)(x,t) \in \tilde{L}(x,t), i \in \{1, 2\}$, and $(\tilde{e}_i)(x,t) = (u_i + \lambda_i \frac{\partial}{\partial t}) + e^t(\alpha_i + \mu_i dt)$, with $(u_i, \lambda_i) + (\alpha_i, \mu_i) \in L_x$ then, using (4.4), (4.6), (4.8) and (5.4), we get

$$\Omega_{\tilde{F}}(x,t)(\tilde{\rho}_L((\tilde{e}_1)(x,t)), \tilde{\rho}_L((\tilde{e}_2)(x,t))) = \frac{1}{2} e^t(\alpha_1(u_2) + \lambda_2\mu_1 - \alpha_2(u_1) - \mu_2\lambda_1)$$

This implies that $\Omega_{\tilde{F}} = e^t((\pi_{1|\tilde{F}})^*(\eta_F) + dt \wedge (\pi_{1|\tilde{F}})^*(\sigma_F)).$

ii) If $F$ is a l.c.p. leaf then $\ker(\rho_{L_x}|_{L_{x_0}}) \not\subset \phi_L(x_0) >^0$ (see Theorem 4.2). Therefore, from Theorem 3.3 and Lemma 5.2, we obtain that $\pi_1(F) = F$, that $\pi_{1|\tilde{F}} : \tilde{F} \to F$ is a covering map, that $\tilde{F}$ is diffeomorphic to a Galois covering of $F$ associated with $\omega_F$ and that $d\tilde{\sigma} = (\pi_{1|\tilde{F}})^*(\omega_F)$.

Finally, if $(x, t) \in \tilde{F}$, $(\tilde{e}_i)(x,t) \in \tilde{L}(x,t), i \in \{1, 2\}$, and $(\tilde{e}_i)(x,t) = (u_i + \lambda_i \frac{\partial}{\partial t}) + e^t(\alpha_i + \mu_i dt)$, with $(u_i, \lambda_i) + (\alpha_i, \mu_i) \in L_x$ then, using (4.4), (4.9), (5.4) and the definition of $\tilde{\sigma}$, we deduce

$$\Omega_{\tilde{F}}(x,t)(\tilde{\rho}_L((\tilde{e}_1)(x,t)), \tilde{\rho}_L((\tilde{e}_2)(x,t))) = \frac{1}{2} e^t(\alpha_1(u_2) + \lambda_2\mu_1 - \alpha_2(u_1) - \mu_2\lambda_1)$$

This implies that $\Omega_{\tilde{F}} = e^{-\tilde{\sigma}}(\pi_{1|\tilde{F}})^*(\Omega_F).$ \hfill \textit{Q.E.D.}

Examples 5.4

1.- Dirac structures

Let $L$ be a $\mathcal{E}^1(M)$-Dirac structure which comes from a Dirac structure on $M$ and $\tilde{L}$ be the associated Dirac structure on $M \times \mathbb{R}$. If $x_0$ is a point of $M$ and $F$ is the leaf of the characteristic foliation $\mathcal{F}_L$ passing through $x_0$, then $F$ is a presymplectic manifold with presymplectic 2-form $\Omega_F$ (see Examples 4.3). Moreover, since $\mathcal{F}_L(x) = \tilde{\mathcal{F}}_L(x,t)$, for all $(x, t) \in M \times \mathbb{R}$, we deduce that the leaf $\tilde{F}$ of $\tilde{\mathcal{F}}_L$ passing through $(x_0, t_0) \in M \times \mathbb{R}$ is $\tilde{F} = F \times \{t_0\}$. In addition, from Theorem 5.3, it follows that $\Omega_{\tilde{F}} = e^{\alpha F} \Omega_F$.

2.- Precontact structures

Let $\eta$ be a precontact structure on a manifold $M$ and $L_{\eta}$ be the associated $\mathcal{E}^1(M)$-Dirac structure. Then, the characteristic foliation $\mathcal{F}_{L_{\eta}}$ has a unique leaf, the manifold $M$ (see Examples 4.3). Furthermore, if $\tilde{L}_{\eta}$ is the Dirac structure on $M \times \mathbb{R}$ associated with $L_{\eta}$, we obtain that $\tilde{L}_{\eta}$ is the graph of the presymplectic 2-form $\tilde{\Omega}$ on $M \times \mathbb{R}$ given by

$$\tilde{\Omega} = e^t((\pi_1)^*(\eta) + dt \wedge (\pi_1)^*(\eta)).$$
In other words,

\[ \Gamma(\tilde{L}_\eta) = \{ \tilde{X} + i_\tilde{X} \tilde{\Omega} / \tilde{X} \in \mathfrak{x}(M \times \mathbb{R}) \} \subseteq \mathfrak{x}(M \times \mathbb{R}) \oplus \Omega^1(M \times \mathbb{R}). \]

On the other hand, using Theorem 5.3, we deduce a well-known result (see [2]): the characteristic foliation \( \mathcal{F}_{\tilde{L}_\eta} \) of \( \tilde{L}_\eta \) also has a unique leaf \( \tilde{F} \) (the manifold \( M \times \mathbb{R} \)) and the presymplectic 2-form \( \Omega_{\tilde{F}} \) on \( \tilde{F} \) is just \( \tilde{\Omega} \).

3.- Jacobi structures

Suppose that \( (\Lambda, E) \) is a Jacobi structure on a manifold \( M \). Then, it is well-known that the 2-vector \( \tilde{\Lambda} \) on \( M \times \mathbb{R} \) given by \( \tilde{\Lambda} = e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E) \) defines a Poisson structure on \( M \times \mathbb{R} \) (see [18]). Thus, one may consider the Dirac structure \( \tilde{L}_{\tilde{\Lambda}} \) on \( M \times \mathbb{R} \) associated with \( \tilde{\Lambda} \) (see [2]). In fact, we have that

\[ \Gamma(\tilde{L}_{\tilde{\Lambda}}) = \{ \#_{\tilde{\Lambda}}(\tilde{\alpha}) + \tilde{\alpha} / \tilde{\alpha} \in \Omega^1(M \times \mathbb{R}) \}. \]

Moreover, if \( L_{(\Lambda, E)} \) is the \( \mathcal{E}^1(M) \)-Dirac structure induced by the Jacobi structure \( (\Lambda, E) \), it is easy to prove that the Dirac structure \( \tilde{L}_{(\Lambda, E)} \) on \( M \times \mathbb{R} \) associated with \( L_{(\Lambda, E)} \) is isomorphic to \( \tilde{L}_{\tilde{\Lambda}} \). Therefore, using Theorem 5.3 (see also Examples 13), we directly deduce the results of Guedira-Lichnerowicz (see Section III.16 in [9]) about the relation between the leaves of the characteristic foliation of the Jacobi manifold \( (M, \Lambda, E) \) and the leaves of the symplectic foliation of the Poisson manifold \( (M \times \mathbb{R}, \tilde{\Lambda}) \).

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