Magnetic permeability of constrained scalar QED vacuum

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March 28, 2022

Abstract

We compute the influence of boundary conditions on the Euler-Heisenberg effective Lagrangian scalar QED scalar for the case of a pure magnetic field. The boundary conditions constrain the quantum scalar field to vanish on two parallel planes separated by a distance $a$ and the magnetic field is assumed to be constant, uniform and perpendicular to the planes. The effective Lagrangian is obtained using Schwinger’s proper-time representation and exhibits new contributions generated by the boundary condition much in the same way as a material pressed between two plates exhibits new magnetic properties. The confined bosonic vacuum presents the expected diamagnetic properties and besides the new non-linear $a$-dependent contributions to the susceptibility we show that there exists also a new $a$-dependent contribution for the vacuum permeability in the linear approximation.

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The classical electromagnetic field $F_{\mu\nu}$ in classical vacuum is described by the Maxwell Lagrangian $-F_{\mu\nu}F_{\mu\nu}/4$. We consider here the case in which there is only a magnetic field $B$ and so the Lagrangian is given by:

$$\mathcal{L}^{(0)}(B) = -\frac{1}{2}B^2.$$ (1)

If the field is in a medium it can be described by an effective Lagrangian $\mathcal{L}^{(0)}(B) + \mathcal{L}^{(1)}(B)$, where $\mathcal{L}^{(1)}(B)$ takes into account the net influence of the medium on the field by means of permeability constants and functions. The permeability constant appears in the quadratic term of $\mathcal{L}^{(1)}(B)$ while the higher order terms of $\mathcal{L}^{(1)}(B)$ describe non-linear susceptibility effects. In their seminal work Euler and Heisenberg noticed that usual QED vacuum behaves as a medium under an applied electromagnetic field and obtained for this vacuum the one-loop effective Lagrangian [1, 2]. An effective Lagrangian can be obtained also for the bosonic charged vacuum of scalar QED [2] which is the case that interests us here. The idea of effective Lagrangians is now an important tool in quantum field theory and in the original case of QED, which provides an interesting and not too complicated formalism, it has a privileged setting to test new methods and ideas [3]. There is also an ongoing experimental effort to measure non-linear effects under strong magnetic fields predicted by the Euler-Heisenberg Lagrangian [4, 5, 6] and recent experiments involving intense electron and laser beams have shown that the critical electrical field strength of the Euler-Heisenberg Lagrangian may be within reach [7, 8].

Vacuum fluctuations of any field are affected by the imposition of boundary conditions, a phenomenon generally known as Casimir effect [9, 10]. In the original Casimir effect [11] two large metallic parallel plates implement boundary conditions on the vacuum fluctuations of the quantum electromagnetic field. The influence of the boundary conditions appears as a force of attraction between the plates. This force was measured by Sparnaay [12] and more recently with high precision in experiments conducted by Lamoreaux [13] and by Mohideen and Roy [14]. In the case of a charged quantum field the vacuum fluctuations can be influenced by an applied external field and by the imposition of boundary conditions and it poses by itself the question of how those two agents together affect the charged vacuum fluctuations. There are two quite distinct points of view in which to pose the question and they lead to different physical phenomena to be investigated. From one point of
view we ask how the boundary conditions affects the Euler-Heisenberg effective Lagrangian and from the other how the external applied field affects the Casimir energy. In a previous work we have considered the influence of boundary conditions on the effective Lagrangian of a charged fermionic vacuum \[15\]. Here we wish to consider the influence of the boundary conditions on a charged bosonic vacuum. In both situations the obtained results are in accord with the general view that fermionic and bosonic vacua should exhibit paramagnetic and diamagnetic properties, respectively. Let us notice that we can think of the boundary conditions as providing an applied stress on the slab of vacuum between the plates and from this point of view it is quite natural to have a change in the vacuum constitutive relations when the separation of the plates \(a\) is changed. We should mention that the refractive index of a non-trivial vacuum can be related with the vacuum expectation value of the stress tensor in a general formalism in the framework of QED effective actions \[29\]. We have also considered the other point of view and studied the influence of an external magnetic field on the Casimir energy of bosonic \[10\] and fermionic \[17\] fields. Our results show that the external field enhances the fermionic Casimir effect and inhibits the bosonic Casimir effect.

The question of how boundary conditions affect the effective Lagrangian is an important one because vacuum fluctuations can be constrained by boundary conditions implemented by several types of cavities and in the case of QCD the confining boundary conditions on vacuum fluctuations cannot be avoided. Certainly the QCD problem is immensely complicated and by this very reason it renders useful a previous investigation in a simpler context, one in which we can fix our attention on the main features of the problem before facing more complicated gauge groups and boundary conditions. Therefore, let us consider here the simple situation of a massive charged scalar field between two parallel planes on which we assume Dirichlet boundary conditions for the field. We make our calculation considering the planes as large square plates with side \(\ell\) and separated by a distance \(a\) \((\ell \gg a)\); at the appropriated moment the limit \(\ell \to \infty\) may be taken. We consider the applied external magnetic field \(B\) as constant and uniform which is perpendicular to the plates and points in a direction that makes \(eB\) positive. With the charged scalar field we are free of kinematical complexities, the choice of a pure magnetic external field excludes the possibility of pair creation for any field strength, and the Dirichlet boundary conditions on the plates is a
rather simple constraint on the charged vacuum which changes its magnetic permeability properties. Let us observe that the permeability and permittivity properties of QED vacuum may also be changed by considering boundary conditions imposed on the quantum electromagnetic field while the quantum charged field is free of boundary conditions. In this case the change in the vacuum properties requires a two-loop process and it is known as Scharnhorst effect \[18, 19, 20\] (see also \[21, 22\]). On the other hand we study here boundary conditions on the charged quantum field and the resulting changes in the vacuum properties appear already at the one-loop level.

To obtain the effective Lagrangian of the charged scalar fluctuations in the presence of the magnetic field we will employ Schwinger’s proper time representation for the effective action \[2\] and a straightforward method of calculation also used by Schwinger to calculate the original Casimir energy \[23\] and later applied to several other problems \[15, 16, 17, 24\]. The proper time representation for the effective action \(\mathcal{W}^{(1)}\) is given by \[2\]:

\[
\mathcal{W}^{(1)} = -\frac{i}{2} \int_{s_0}^{\infty} \frac{ds}{s} \text{Tr} e^{-isH},
\]

where \(s_0\) is a cutoff in the proper-time \(s\), \(\text{Tr}\) stands for the total trace and \(H\) is the proper-time Hamiltonian, which in the present case of a scalar field is given by \((p - eA)^2 + m^2\), where \(p_\mu = -i\partial_\mu\), \(m\) is the mass of the scalar field, \(e\) is its charge and \(A\) is the potential of the external magnetic field \(B\), whose direction can be arbitrarily chosen to make \(eB\) positive. The spatial components of \(p\) parallel to the plates are constrained by the Landau levels created by \(B\) with multiplicity \(eB\ell^2/2\pi\) and the component of \(p\) perpendicular to the plates is discretized by Dirichlet boundary conditions into the values \(\pi n/a\), where \(n\) is a positive integer. The time component of \(p\) has as eigenvalue any real number \(\omega\) and the charge degrees of freedom of the complex scalar field contributes with a factor of 2 to the total trace. Therefore, we obtain for the trace in (2):

\[
\text{Tr} e^{-isH} = 2e^{-ism^2} \sum_{n'=0}^{\infty} \frac{eB\ell^2}{2\pi} e^{-iseB(2n'+1)} \sum_{n=1}^{\infty} e^{-is(n\pi/a)^2} \int \frac{dt d\omega}{2\pi} e^{i\omega t},
\]

where the range of the time integral is taken to be a large observation time \(T\). For the first sum in (3) we have

\[
\sum_{n'=0}^{\infty} \frac{eB\ell^2}{2\pi} e^{-iseB(2n'+1)} = \frac{\ell^2}{4\pi is} [1 + iseB \mathcal{M}(iseB)],
\]
where \( \mathcal{M} \) is the function defined by:

\[
\mathcal{M}(\xi) = cosech\xi - \xi^{-1}.
\]  

(5)

The function \( \mathcal{M} \) plays in the present bosonic formalism the same role as the Langevin function in the fermionic formalism \[15\]. For the second sum we make use of Poisson sum formula \[25\] in order to obtain:

\[
\sum_{n=1}^{\infty} e^{-i(n\pi/a)^2} = \frac{a}{\sqrt{i\pi s}} \sum_{n=1}^{\infty} e^{i(an)^2/s} + \frac{a}{2\sqrt{i\pi s}} - \frac{1}{2}.
\]  

(6)

Using (6) and (4) into (3), we obtain for the trace:

\[
Tr e^{-isH} = \frac{a\ell^2 T}{4\pi^2} e^{-sm^2} \frac{1}{is} [1 + iseB \mathcal{M}(iseB)] \times
\]

\[
\left[ \frac{1}{2\sqrt{i\pi s}} + \frac{1}{\sqrt{i\pi s}} \sum_{n=1}^{\infty} e^{i(an)^2/s} - \frac{1}{2a} \right] \sqrt{\frac{\pi}{-is}}.
\]  

(7)

Now we substitute equation (7) into the effective action (2) and following Schwinger \[2, 23\] we use Cauchy contour theorem to make a \( \pi/2 \) clockwise rotation in the integration \( s \)-axes in order to obtain the effective action as the following manifestly real quantity:

\[
\mathcal{W}^{(1)} = \frac{a\ell^2 T}{16\pi^2} \int_{s_0}^{\infty} \frac{ds}{s^3} e^{-sm^2} [1 + seB \mathcal{M}(seB)] +
\]

\[
\frac{a\ell^2 T}{8\pi^2} \int_{s_0}^{\infty} \frac{ds}{s^3} e^{-sm^2} \left[ \sum_{n=1}^{\infty} e^{-(an)^2/s} + \sqrt{\frac{\pi s}{2a}} \right] [1 + seB \mathcal{M}(seB)].
\]  

(8)

From this action we obtain the corresponding effective Lagrangian which we add to the Maxwell Lagrangian \[1\] to obtain the still unrenormalized cutoff dependent complete Lagrangian:

\[
\mathcal{L}(a, B) = -\frac{1}{2} B^2 + \frac{1}{16\pi^2} \int_{s_0}^{\infty} \frac{ds}{s^3} e^{-sm^2} [1 + seB \mathcal{M}(seB)] +
\]

\[
\frac{1}{8\pi^2} \int_{s_0}^{\infty} \frac{ds}{s^3} e^{-sm^2} \left[ \frac{\sqrt{\pi s}}{2a} + \sum_{n=1}^{\infty} e^{-(an)^2/s} \right] [1 + seB \mathcal{M}(seB)].
\]  

(9)

Now we follow the usual procedure for renormalizing this Lagrangian \[1, 2\] \[3\]. Consider in the first integral in equation (3) the first two terms in the
expansion of the function $1 + seB \mathcal{M}(seB)$ in powers of $seB$, to wit: 1 and $(seB)^3/6$. The first term gives for the first integral in (i) a contribution which in the limit $s_0 \to 0$ tends to the infinite constant $m^4 \Gamma(-2)/16\pi^2$, where $\Gamma$ is the Euler gamma function. This infinity can be eliminated by simply subtracting it from the Lagrangian $\mathcal{L}(a, B)$. The second term gives for the first integral in (ii) a contribution which is proportional to $B^2/2$, with a constant of proportionality which tends to the infinite constant $-e^2 \Gamma(0)/48\pi^2$ in the limit $s_0 \to 0$. We write this constant as $Z_3^{-1} - 1$ and absorb it in the definitions of renormalized charge, $e_R = e Z_3^{1/2}$, and renormalized field, $B_R = B Z_3^{-1/2}$. With this procedure the second infinity is also disposed off, in the first term in (i) the bare field $B$ is replaced by $B_R$ and in the rest of $\mathcal{L}(a, B)$ we have $eB$ replaced by $e_R B_R$. Notice that the rest of $\mathcal{L}(a, B)$ does not change its form under these replacements because $e_R B_R = eB$. After the introduction of these renormalization procedures all the infinities are removed from the Lagrangian $\mathcal{L}(a, B)$ and we can take the limit $s_0 \to 0$ in order to eliminate the cutoff. The first term in the expansion of $1 + seB \mathcal{M}(seB)$ gives for the second integral in (ii) a finite term which is actually the Casimir energy of the charged scalar field [10]. However, this energy is totally irrelevant for the effective Lagrangian because it is independent of the field $B$. Therefore we can for convenience also subtract this term from $\mathcal{L}(a, B)$ to obtain a final well-defined renormalized Lagrangian $\mathcal{L}_R$ written in terms of the renormalized quantities $e_R$ and $B_R$. With the proviso that from now on we will deal only with renormalized quantities we write the final renormalized form of the effective Lagrangian suppressing all subindexes $R$, thereby obtaining:

$$\mathcal{L}(a, B) = -\frac{1}{2}B^2 + \mathcal{L}_H^{(1)}(B) + \mathcal{L}_H^{(1)}(a, B) , \quad (10)$$

where

$$\mathcal{L}_H^{(1)}(B) = \frac{1}{16\pi^2} \int_0^{\infty} \frac{ds}{s^3} e^{-sm^2} [seB \mathcal{M}(seB) + \frac{1}{6}(seB)^2] \quad (11)$$

and

$$\mathcal{L}_H^{(1)}(a, B) = \frac{1}{8\pi^2} \int_0^{\infty} \frac{ds}{s^3} e^{-sm^2} \left[ \frac{\sqrt{\pi} s}{2a} + \sum_{n=1}^{\infty} e^{-(an)^2/s} \right] seB \mathcal{M}(seB) . \quad (12)$$

The result in (11) agrees with the effective Lagrangian first obtained by Schwinger [2] which is the version for the charged scalar field of the effective
Lagrangian obtained by Euler and Heisenberg for the charged Dirac field \[1\]. The expression for \( \mathcal{L}_{HE}^{(1)}(B) \) in (11) can be further worked out to express the integral in it in terms of a Riemann zeta function \[26\]. Let us consider the regime of \( B \) small compared to the critical field \( B_{cr} = m^2/e \). In this regime \( \mathcal{L}_{HE}^{(1)}(B) \) can be expanded in powers of \( B^2 \) and we obtain (cf. formula 1411,12 in \[27\]):

\[
\mathcal{L}_{HE}^{(1)}(B) = - \frac{m^4}{16\pi^2} \sum_{k=2}^{\infty} \frac{(2^{2k-1} - 1)B_{2k}}{k(2k-1)(2k-2)} \left( \frac{B}{B_{cr}} \right)^{2k},
\]

(13)

where \( B_{2k} \) is the \( 2k \)-th Bernoulli number. This expansion shows that the lowest order contribution from the Euler-Heisenberg effective Lagrangian is a term in \( B^4 \), which means a non-linear contribution to the magnetic susceptibility. If we want this contribution to be relevant the external field \( B \) should not be much smaller than the critical field \( B_{cr} \). The effective Lagrangian \( \mathcal{L}_{HEC}^{(1)}(a, B) \) in (12) is the result that we were looking for. It takes into account the effect of the boundary conditions on the charged vacuum fluctuations and can be called the Casimir-Euler-Heisenberg effective Lagrangian for the charged scalar field.

The result that we have obtained in (12) for the bosonic vacuum and the previous result for the fermionic vacuum \[15\] shows quantitatively how the magnetic permeability of both vacua changes with the parameter \( a \) of the boundary conditions. In the limit \( a \to \infty \) we have that \( \mathcal{L}_{HEC}^{(1)}(a, B) \to 0 \) and (14) reduces to the usual effective Lagrangian (11) of the unconstrained charged scalar vacuum. For finite \( a \) we expect from (12) significant contributions to the complete effective Lagrangian if the dimensionless parameter \( am \) is not large. To see more clearly the new contributions yielded by the Casimir-Euler-Heisenberg effective Lagrangian (12) to the permeability properties of vacuum let us rewrite it in the following form:

\[
\mathcal{L}_{HEC}^{(1)}(a, B) = - \frac{1}{2} \left[ \frac{1}{\mu(am)} - 1 \right] + \mathcal{L}_{HE}^{(1)'}(a, B),
\]

(14)

where

\[
\frac{1}{\mu(am)} = 1 + \frac{e^2}{48\pi am} + \frac{e^2}{12\pi^2} \sum_{n=1}^{\infty} K_0(2amn),
\]

(15)
and

\[ L^{(1)'}_{HE}(a, B) = \frac{1}{8 \pi^2} \int_0^\infty ds \frac{d}{s^3} e^{-sm^2} \left[ \frac{\sqrt{s}}{2a} - \sum_{n=1}^{\infty} \frac{e^{-(an)^2/s}}{n^2} \right] \left[ se B \mathcal{M}(seB) + \frac{1}{6} (seB)^2 \right]. \]  

(16)

The expression in (16) provides the new non-linear \( a \)-dependent contributions to the magnetic susceptibility stemming jointly from the external field \( B \) and the boundary conditions. It provides at each order in the expansion (13) a contribution due to the influence of the boundary conditions. Those contributions are given explicitly by the following expansion of \( L^{(1)'}_{HE}(a, B) \) in the regime of small \( B \) (cf. formula 1411,12 in [27]):

\[
L^{(1)'}_{HE}(B) = -\frac{m^4}{16\pi^2} \sum_{k=2}^{\infty} \frac{(2^{2k-1} - 1)B_{2k}}{k(2k - 1)(2k - 2)} \times \\
\times \left[ \frac{4\pi}{3am} + 2 \sum_{n=1}^{\infty} \frac{(2amn)^{2k-2}}{(4k - 5)!!} K_{2k-2}(2amn) \right] \left(\frac{B}{B_{cr}}\right)^{2k}. 
\]

(17)

In this expression we can read at each order of \( B/B_{cr} \) the corrections to be added to (13) if the boundary conditions are enforced.

In (15) we have from \( L^{(1)}_{HEC}(a, B) \) a type of contribution to the permeability constant which is not present in the Euler-Heisenberg Lagrangian (11). This novel contribution provides the charged scalar vacuum with an \( a \)-dependent permeability constant \( \mu(am) \). From the properties of the \( K_0 \) Bessel function [27] we see in (13) that \( \mu(am) \) is strictly less than 1, i.e., the charged scalar vacuum magnetic permeability is definitely diamagnetic. We also see that the expected limit of \( \mu(am) \to 1 \) is obtained when \( a \to \infty \) and that \( \mu(am) \to 0 \) when \( a \to 0 \). From those two limits the general behaviour of the permeability constant is quite predictable on physical grounds and is depicted in Figure 1. To highlight the importance of the permeability constant \( \mu(am) \) generated by the boundary conditions let us consider the weak field regime in which only quadratic terms in the Lagrangian are not negligible. In this situation the whole Euler-Heisenberg Lagrangian \( L^{(1)}_{HE}(B) \) (11) as well as the corrections \( L^{(1)'}_{HE}(a, B) \) in (14) make no contribution to the effective Lagrangian and (11) reduces to the expression:

\[ L(a, B) = -\frac{1}{2} \frac{B^2}{\mu(am)}, \]  

(18)
in which the charged vacuum manifests itself only through the permeability constant $\mu(\alpha m)$.

In this paper we have obtained the effect on the Euler-Heisenberg effective Lagrangian of the charged scalar field due to Dirichlet boundary conditions on two parallel plates. We have obtained corrections to the non-linear susceptibility effects of the Euler-Heisenberg Lagrangian and also a novel contribution providing the charged scalar vacuum with a permeability constant which depends on the boundary conditions. The permeability constant shows that the bosonic vacuum under consideration behaves as a diamagnetic medium, a result that should be compared with a previous one [15] which exhibits the paramagnetic properties of a fermionic vacuum. The obtained results provide in the simple context of a quantum charged scalar field some physical insights that can be useful in the treatment of more complicated gauge groups and boundary conditions.

Acknowledgements
We thank J. Rafelski for several insightful discussions on this subject. We also acknowledge A. A. Actor for useful conversations on the subject with one of us (A. C. T.). M. V. C.-P. and C. F. thanks CNPq (The National Research Council of Brazil) and M. R. Negrão thanks CAPES (Brazilian Council for Graduate Training) for partial financial support.

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FIGURE CAPTION

Figure 1. The permeability constant $\mu$ as a function of the confining distance $a$ in units of Compton wavelength $1/m$. 