FORMULAS FOR COEFFICIENTS OF POLYNOMIALS
ASSIGNED TO ARITHMETIC FUNCTIONS

BERNHARD HEIM AND MARKUS NEUHAUSER

Abstract. We attach to normalized (non-vanishing) arithmetic functions $g$ and $h$ recursively defined polynomials. Let $P_{g,h}^0(x) := 1$. Then

\begin{equation}
P_{g,h}^n(x) := x \frac{h(n)}{h(n)} \sum_{k=1}^{n} g(k) P_{g,h}^{n-k}(x).
\end{equation}

For special $g$ and $h$, we obtain the D’Arcais polynomials, which are equal to the coefficients of the $-z$th powers of the Dedekind $\eta$-function and are also given by Nekrasov and Okounkov as a hook length formula. Examples are offered by Pochhammer polynomials, Chebyshev polynomials of the second kind, and associated Laguerre polynomials. We present explicit formulas and identities for the coefficients of $P_{g,h}^n(x)$ which separate the impact of $g$ and $h$. Finally, we provide several applications.

2010 Mathematics Subject Classification. Primary 05A10, 11B83, 11P84; Secondary 11F20, 33C45.

Key words and phrases. Arithmetic functions, Dedekind eta-function, partitions, polynomials.
1. Introduction

The investigation of formulas and properties for the $q$-expansion of $r^{th}$ powers of Euler products $\prod_{n=1}^{\infty} (1 - q^n)^r$ goes back to Euler 1748 ($r = \pm 1$), Jacobi 1828 ($r = 3$), and Ramanujan 1916 ($r = 24$) [Ra16]. This involves pentagonal numbers, partition numbers, triangle numbers, and the Ramanujan tau-function. We refer to [Ka78] for a historical introduction including Macdonald identities.

The topic interrelates several fields in mathematics, for example, combinatorics [AE04, Wi06], modular forms and number theory [Se85], representation theory of Lie algebras [Ko04, We06], and statistical mechanics [NO06].

Modular forms represented by powers of the Dedekind $\eta$-function have captivating properties. Some of them are still conjectural, like the Lehmer conjecture [Le47] on the non-vanishing of the coefficients of discriminant function $\Delta$. We refer to Balakrishnan, Craig and Ono [BKO20] for recent results. Let $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$, where $q := e^{2\pi i \tau}, \Im(\tau) > 0$. Varying $r$, Newman [Ne55] and Serre [Se85] investigated the $q$-expansion

$$\sum_{n=0}^{\infty} p_r(n) q^n = \prod_{n=1}^{\infty} (1 - q^n)^r.$$  

and obtained remarkable results. The starting point is the fact that the coefficients are polynomials in $r$. Newman utilized ([Ne55], formula (3)), a recursion formula for the coefficients of the polynomials

$$p_r(n) = \frac{(-1)^n}{n!} \sum_{k=0}^{n-1} (-1)^k A_k(n) r^{n-k},$$

$$A_k(n) = \sum_{s=1}^{k} s! \sigma(s+1) \sum_{\lambda=1}^{s} \binom{\lambda-1}{s} A_{k-s} (\lambda - 1 - s)$$

and recorded the first eleven polynomials. Recall that $\sigma(n) := \sum_{d|n} d$.

Let $r$ be an even, positive integer. Serre proved that $\eta^r$ is lacunary iff $r \in \{2, 4, 6, 8, 10, 14, 26\}$. Serre reduced his proof to specific properties of the first eleven polynomials ([Se85], Lemme 3). These polynomials belong to an interesting type of recursively defined polynomials $P_{g,h}^n(x)$ assigned to normalized arithmetic functions $g$ and $h$. Let $h$ be non-vanishing. Let $P_{g,h}^0(x) := 1$ and

$$P_{g,h}^n(x) = \frac{x}{h(n)} \sum_{k=1}^{n} g(k) P_{g,h}^{n-k}(x),$$

$$P_{g,h}^n(x) = \frac{1}{\prod_{k=1}^{n} h(k)} \left( A_{g,h}^{n,n} x^n + \ldots + A_{g,h}^{n,1} x \right).$$
Let \( \sigma_t(n) := \sum_{d|n} d^k \), \( \id(n) = n \) and \( 1(n) = 1 \). Then \( \sigma(n) = \sigma_1(n) \). We recover Newman’s approach \( \{2\} \):

\[
\sum_{n=0}^{\infty} P_n^{\sigma, \id}(z) q^n = (1 - q^n)^{-z}, \quad (z \in \mathbb{C}).
\]

for polynomials with \((g, h) = (\sigma, \id)\). The polynomials of degree \( n \leq 5 \) first appeared in work by Francesco D’Arcais \( \{DA13\} \). They are called D’Arcais polynomials \( \{We06, HN20B\} \).

Shortly after their discovery, Westbury \( \{We06\} \) and Han \( \{Ha10\} \) spotted and proved \( \mu \) both functions depend on \( g, h \). Let \( \sum \lambda \in \mathbb{P} \) be given. We denote by \( P_n^{\sigma, \id} \) the orbit of \( \mu \) of \( n \) of length \( m \) be normalized arithmetic functions. Let \( a_4(n) \) and \( a_6(n) \) be the coefficients of the \( q \) expansion of \( 1/E_4 \) and \( 1/E_6 \). Then \( \{HN20C\} \)

\[
\begin{align*}
a_4(n) &= P_n^{\sigma_3, 1}(-240),
a_6(n) &= P_n^{\sigma_5, 1}(504).
\end{align*}
\]

In 2003, Nekrasov and Okounkov \( \{NO03\} \) discovered a remarkable hook length formula, displaying the \( n \)th D’Arcais polynomial as a sum over all partitions \( \lambda \) of \( n \), \((\lambda + n)\), of products involving the multiset \( \mathcal{H}(\lambda) \) of hook lengths associated to \( \lambda \). Nekrasov and Okounkov’s result (finally published in 2006 \( \{NO06\} \)) is based on random partitions and the Seiberg–Witten theory:

\[
P_n^{\sigma, \id}(z) = \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \left( 1 + \frac{z + 1}{h^2} \right).
\]

Shortly after their discovery, Westbury \( \{We06\} \) and Han \( \{Ha10\} \) spotted and proved the formula in connection with Macdonald identities.

In this paper we present a formula for the coefficients \( A_{n,m}^{g,h} \) of the polynomials \( P_n^{g,h}(x) \) attached to normalized arithmetic functions \( g \) and \( h \), where \( h \) is non-vanishing. The main goal is to identify the impact of \( g \) and \( h \) separately. The formula makes it possible to study the polynomials and coefficients by varying \( g \) and \( h \). We expect to obtain an approximation of the coefficients with this method of the D’Arcais polynomials in the near future.

**Main Theorem.** Let \( g \) and \( h \) be normalized arithmetic functions. Let \( h \) be non-vanishing. Let \( 1 \leq m < n \). Let \( \mu = (\mu_1, \ldots, \mu_r) \) be a partition of \( n - m \). Let \( \mathcal{G}(\mu) := \prod_{k=1}^{r} g(\mu_k + 1) \) and \( \mathcal{H}(\mu, n) \) as provided by Definition \( \{1\} \) and Definition \( \{2\} \). Both functions depend on \( \mu \). Additionally, \( \mathcal{G} \) depends on \( g \) and \( \mathcal{H}(\mu, n) \) depends on \( h \) and \( n \). Then we have

\[
A_{n,m}^{g,h} = \sum_{\mu \vdash n-m} \mathcal{G}(\mu) \cdot \mathcal{H}(\mu, n).
\]

Let a partition \( \mu = (\mu_1, \ldots, \mu_r) \) of \( n \) be given. We denote by \( \ell(\mu) = r \) the length of \( \mu \) and \( |\mu| = n \) the size of \( \mu \). We denote \( \mathbf{m}_j = \mathbf{m}_j(\mu) \), the multiplicity of \( j \) in the partition \( \mu \). The symmetric group \( S_r \) acts on the set of all compositions of \( n \) of length \( r \). Let \( \text{Orb}(\mu) \) be the orbit of \( \mu \) \( \{St99\} \).
Theorem 1. Let \( g \) be a normalized arithmetic function. Let \( A_{n,m}^{g,1} \) be the \( m \)th coefficient of \( P_n^{g,1}(x) \). Let \( 1 \leq m < n \). Then
\[
A_{n,m}^{g,1} = \sum_{\mu \vdash n-m} G(\mu) \cdot \left( \frac{\ell(\mu)}{m_1, \ldots, m_m} \right) \binom{n - |\mu|}{\ell(\mu)}.
\]

Let \( 1 \leq m < n \). The coefficients for \( n - m = 1, 2, 3 \) are given by
\begin{itemize}
  \item \( A_{n,n-1}^{g,1} = g(2)(n-1) \),
  \item \( A_{n,n-2}^{g,1} = (g(2))^2 \binom{n-2}{2} + g(3)(n-2) \),
  \item \( A_{n,n-3}^{g,1} = (g(2))^3 \binom{n-3}{3} + 2g(2)g(3)\binom{n-3}{2} + g(4)(n-3) \).
\end{itemize}

Theorem 2. Let \( g \) be a normalized arithmetic function. Let \( A_{n,m}^{g,\text{id}} \) be the \( m \)th coefficient of \( P_n^{g,\text{id}}(x) \) for \( 1 \leq m < n \). Then
\[
A_{n,m}^{g,\text{id}} = \sum_{\mu \vdash n-m} G(\mu) \prod_{k=0}^{|\mu|+\ell(\mu)-1} (n-k) \sum_{\lambda \in \text{Orb}(\mu)} \prod_{k=1}^{k} \left( k + \sum_{i=1}^{\lambda_i} \right)^{-1}.
\]

Let \( 1 \leq m < n \). The coefficients for \( n - m = 1, 2, 3 \) are given by
\begin{itemize}
  \item \( A_{n,n-1}^{g,\text{id}} = g(2)\binom{2}{2} \),
  \item \( A_{n,n-2}^{g,\text{id}} = 3(g(2))^2 \binom{n}{2} + 2g(3)\binom{n}{3} \),
  \item \( A_{n,n-3}^{g,\text{id}} = 15(g(2))^3 \binom{n}{3} + 20g(2)g(3)\binom{n}{2} + 6g(4)\binom{n}{1} \).
\end{itemize}

Next, let us consider the most simple arithmetic function \( g = 1 \), which involves the Pochhammer polynomials. We have \( P_n^{1,1}(x) = x(x+1)^{n-1} \) and \( P_n^{1,\text{id}}(x) = \frac{1}{n!} \prod_{k=0}^{n-1}(x+k) \). This follows directly from Definition \( \{3\} \). Thus,
\begin{itemize}
  \item \( A_{n,m}^{1,1} = \binom{n-1}{m-1} \),
  \item \( A_{n,m}^{1,\text{id}} = |s(n,m)| \),
\end{itemize}

where \( s(n,m) \) is the Stirling number of the first kind. Note that Theorem \( \{1\} \) and Theorem \( \{2\} \) already provide non-trivial identities. This example is already quite interesting, since it shows the impact of \( h \) on the root distribution for \( g = 1 \). Let \( h \) be a normalized and non-vanishing arithmetic function. Throughout the paper we put \( h(0) := 0 \). Then
\[
P_n^{1,h}(x) = \frac{1}{\prod_{k=1}^{h} h(k)} \prod_{k=0}^{n-1}(x+h(k)).
\]

Remark. It would be beneficial to develop a deformation theory for the recursively defined polynomials \( P_n^{g,h}(x) \). This should lead to new results for the distribution of the roots, where \( h \) varies from \( h = 1 \) to \( h = \text{id} \).

Let \( g = 1 \). For \( h = 1 \) the set of roots is given by \( 0 \) and by \( -1 \) with multiplicity \( n-1 \). This set should move towards the set of simple roots given by \( 0, 1, \ldots, n-1 \) for \( h = \text{id} \). The coefficients of the involved polynomials can be described in terms of elementary symmetric polynomials.
Recently [HN20D], we discovered a conversion formula between the $m$th coefficients of the $n$th polynomials assigned to $g$ and to $\tilde{g}(n) = g(n)/n$ for $h = 1$ and $h = \text{id}$.

Let $g$ be of moderate growth, i.e. the generating series of $g$ is regular at 0. Let $G$ be the function attached to $g$ and $\tilde{G}$ be the function attached to $\tilde{g}$. Then

\begin{equation}
\frac{A_{n,m}}{n!} = \frac{A_{n,m}}{m!}.
\end{equation}

This implies

**Corollary 1.** Let $g$ be a normalized arithmetic function of moderate growth. Let $1 \leq m < n$. Let $G$ be assigned to $g$ and $\tilde{G}$ assigned to $\tilde{g}$:

\begin{equation}
\frac{1}{n!} \sum_{\mu \vdash n-m} G(\mu) \prod_{k=0}^{\ell(\mu)-1} (n-k) \sum_{\lambda \in \text{Orb}(\mu)} \prod_{k=1}^{\ell(\lambda)} \left( k + \sum_{i=1}^{k} \lambda_i \right) = \frac{1}{m!} \sum_{\mu \vdash n-m} \tilde{G}(\mu) \cdot \left( \begin{array}{c} \ell(\mu) \\ m_1, \ldots, m_m \end{array} \right) \left( \begin{array}{c} n - |\mu| \\ \ell(\mu) \end{array} \right)
\end{equation}

where $m_j$ is the multiplicity of $j$ in $\mu$.

2. **Deformation of the coefficients of D’Arcais polynomials**

2.1. **Notation.** Let $n \in \mathbb{N}$. A composition of $n$ is an ordered sum of integers. It is a sequence $\mu = (\mu_1, \ldots, \mu_r)$ of positive integers, which sum up to $n := |\mu|$, which is denoted by the size or weight of the partition. We denote by $\ell(\mu) = r$, the length of the composition and $C(n)$, the set of all compositions of $n$. Next we define the subset $P(n)$ of partitions of $n$. A partition of $n$ is any composition $\mu$ of $n$, where the sequence $\mu_1 \geq \ldots \geq \mu_r$ is non-increasing. If $r$ summands appear in a composition of $\mu$, we say that $\mu$ has $r$ parts. The number of all compositions and partitions of $n$ is denoted by $c(n)$ and $p(n)$, respectively. The number of compositions and partitions with $k$ parts is denoted by $c_k(n)$ and $p_k(n)$, respectively. Recall that $c_k(n) = \binom{n-1}{k-1}$.

The symmetric group $S_r$ acts on the set of all compositions of $n$ of length $r$ by permuting the parts: $\pi(\mu) := (\mu_{\pi^{-1}(1)}, \ldots, \mu_{\pi^{-1}(r)})$. Each orbit $\text{Orb}(\mu) := \{ \pi(\mu) : \pi \in S_{\ell(\mu)} \}$, where $\mu \in C(n)$ is a composition of $n$, contains exactly one partition. Let $\mu \in C(n)$. Then $m_j := m_j(\mu) = |\{ i : \mu_i = j \}|$ is the multiplicity of $j$ in the composition $\mu$ of $n$.

Let $\varepsilon$ be defined as the unique vector of length 0. We extend our notation by $\ell(\varepsilon) = 0$ and $|\varepsilon| = 0$. We define $C$ as the set of all ordered partitions, where we include the partition of length 0.

Let $\mu_1, \ldots, \mu_r$ be non-negative integers, which add up to $n$. Then we denote by

\begin{equation}
\binom{n}{\mu_1, \ldots, \mu_r} := \frac{n!}{\mu_1! \cdots \mu_r!}
\end{equation}
the multinomial coefficient, which has the property

\[(x_1 + \ldots + x_r)^n = \sum_{k_1, \ldots, k_r \geq 0, k_1 + \ldots + k_r = n} \binom{n}{k_1, \ldots, k_r} x_1^{k_1} \cdot \ldots \cdot x_r^{k_r}.\]

Consider a multiset of \(n\) objects, in which \(\mu_i\) objects are of type \(i\). Then the number of ways to linearly order these objects is equal to the multinomial coefficients in (16).

2.2. The function \(H(\mu, n)\). Throughout this section \(h\) will be a normalized non-vanishing function. Special cases are given by \(h(n) = \text{id}(n) = n\) or \(h(n) = 1(n) = 1\) for all \(n \in \mathbb{N}\). Our goal is to define a function \(H(\mu, n)\) for every composition \(\mu\) and every number \(n\). This function will play a significant role in our main formula for the coefficients \(A_{n,m}^{\mu, h}\). Let \(1 \leq m \leq n\) be integers. We put \(H(n) := \prod_{k=1}^{n} h(k)\) and \(H(0) = 1\). Further, let \(h_m(n) := \frac{H(n)}{H(n-m)} = \prod_{k=0}^{m-1} h(n-k)\).

**Definition 1.** Let \(h\) be a normalized non-vanishing arithmetic function. Let \(C\) be the set of all compositions including \(\varepsilon\). We define the function \(H(\mu, n)\) on \(C \times \mathbb{N}\) inductively by the length of \(\mu\). The initial values \(H(\varepsilon, n) := 1\) of length \(0\) for \(n \in \mathbb{N}_0\). Let further \(\mu \in C\) of length \(r + 1\), where \(r \geq 0\). For \(n \geq |\mu| + l(\mu)\), let

\[(18) \quad H(\mu, n) := \sum_{k=|\mu|+l(\mu)}^{n-1} h_{\mu+1}(k) H((\mu_1, \ldots, \mu_r), k - \mu_{r+1}),\]

If \(n < |\mu| + l(\mu)\) we put \(H(\mu, n) := 0\).

**Remarks.** a) Let the values of \(h\) be positive integers. Then function \(H(\mu, n)\) is an integer-valued function and has positive values for \(n \geq |\mu| + l(\mu)\). It is otherwise vanishing.
b) Let \(\mu \in \mathbb{N}\) and let \(n \geq \mu + 1\). Then

\[(19) \quad H(\mu, n) = \sum_{k=\mu}^{n-1} h_{\mu}(k).\]

For \(h(n) = 1\), or \(h(n) = \text{id}(n)\), and \(\mu\) a partition of length \(r\), we have a closed formula for \(H(\mu, n)\).

**Proposition 1.** Let \(h\) be a normalized non-vanishing arithmetic function. Let \(h(n) = 1\) or \(h(n) = n\), then we have the following closed formulas. Let \(n \in \mathbb{N}_0\) and \(n \geq 1\). Let \(\mu\) be a composition of length \(l(\mu)\) and size \(|\mu|\). Let \(h(n) = 1\) then

\[(20) \quad H(\mu, n) = |\mu| + l(\mu) - 1 \choose |\mu| \cdot l(\mu)\cdot (n - |\mu|) \frac{1}{\ell(\mu)} \cdot \left(\prod_{j=1}^{k} \mu_j \right).\]

Let \(h(n) = n\) then

\[(21) \quad H(\mu, n) = \prod_{k=0}^{\ell(\mu)} (n - k) \cdot \prod_{k=1}^{\ell(\mu)} \left(k + \sum_{j=1}^{k} \mu_j \right) - 1.\]
Proof. Let $h = 1$ be trivial. We prove the formula \((20)\) by induction over the length $r$. Let $n$ be arbitrary. We start with $r = 0$. Then the formula confirms that $H(\varepsilon, n) = 1$. Now let $r \geq 1$ and \((20)\) be true for all compositions with length smaller than $r$. Let a composition $\mu$ be given of length $r$. Then

$$H(\mu, n) = \sum_{k=|\mu|+\ell(\mu)-1}^{n-1} h_{\mu_r}(k) H((\mu_1, \ldots, \mu_{r-1}), k - \mu_r)$$

$$= \sum_{k=|\mu|+\ell(\mu)-1}^{n-1} \left( k - |\mu| \right) \left( \ell(\mu) - 1 \right) = \left( n - |\mu| \right) \left( \ell(\mu) - 1 \right).$$

The last step is given by the sum of the entries in the appropriate column in the Pascal triangle. Formula \((21)\) is slightly more complicated but proven in the same way. \qed

For example, for $h \in \{1, \id\}$ we have the following values, which illustrate some patterns and differences.

- Let $h(n) = 1$.
  Then we have $H(1, n) = n - 1$, $H((1, 1), n) = \binom{n-2}{2}$, $H(2, n) = n - 2$, and $H((1, 1), n) = \binom{n-3}{3}$. Further $H((1, 2), n) = H((2, 1), n) = \binom{n-2}{2}$.

- Let $h(n) = n$.
  Then we have $H(1, n) = \binom{n}{2}$, $H((1, 1), n) = 3 \binom{n}{4}$ and $H(2, n) = 2 \binom{n}{3}$.
  Further, $H((1, 2), n) = 12 \binom{n}{5}$ and $H((2, 1), n) = 8 \binom{n}{5}$.

Definition 2. Let $h$ be a normalized non-vanishing arithmetic function. Let $\mu \in C(n)$. Then

\[(22) \quad H(\mu, n) := \sum_{\lambda \in \text{Orb}(\mu)} H(\lambda, n).\]

2.3. Proof of Theorem 1 and Theorem 2.

Theorem 1 follows from the Main Theorem, formula \((20)\), and the following proposition.

Proposition 2. Let $h = 1$. Let $n$ and $m$ be positive integers and let $\mu \in \mathcal{P}(m)$. Let $m_j = m_j(\mu)$ be the multiplicity of $j$ in the partition $\mu$. Then we have

\[(23) \quad H(\mu, n) = \binom{\ell(\mu)}{m_1, \ldots, m_m} \binom{n - |\mu|}{\ell(\mu)}.\]

Proof. Let $r = \ell(\mu)$. In the case of $h(n) = 1$ for all $n$, $H(\pi(\mu), n) = H(\mu, n)$ for every $\pi \in S_r$, as $\ell(\pi(\mu)) = \ell(\mu)$ and $|\pi(\mu)| = |\mu|$. This implies that $H(\mu, n) = |\text{Orb}(\mu)| H(\mu, n)$. Let $F(\mu) = \{ \pi \in S_r : \pi(\mu) = \mu \}$ then $|\text{Orb}(\mu)| = \frac{\prod_{j=1}^{\ell(\mu)} S_{m_j}}{F(\mu)}$. For $\pi \in F(\mu)$ and $\nu = \pi(\mu)$ it must hold that $\nu_j = \mu_j$ for all $1 \leq j \leq r$ and any $\pi$ which satisfies this property is in $F(\mu)$. This means that $F(\mu) \cong \prod_{j=1}^{\ell(\mu)} S_{m_j}$, where $S_0$ is a trivial group consisting of only the neutral element. That means that $|F(\mu)| = \prod_{j=1}^{\ell(\mu)} m_j!$, which proves the claim. \qed

Theorem 2 follows from the Main Theorem, formula \((21)\), and the following proposition.
Proposition 3. Let $h = \text{id}$. Let $n$ and $m$ be positive integers. Let $\mu \in \mathcal{P}(m)$. Then we have

$$\mathcal{H}(\mu, n) = \prod_{k=0}^{|\mu|+\ell(\mu)-1} (n-k) \sum_{\lambda \in \text{Orb}(\mu)} \prod_{k=1}^{\ell(\lambda)} \left( k + \sum_{i=1}^{\lambda} \right)^{-1}$$

3. Main Theorem

Definition 3. Let $g$ be a normalized arithmetic function. Let $\mu \in \mathcal{P}(m)$. The function $G(\mu)$ is defined by the product of $g$ evaluated at $(\mu_k + 1)_k$.

$$G(\mu) := \prod_{k=1}^{\ell(\mu)} g(\mu_k + 1).$$

We would like to establish a formula for $A_{n,m}^{g,h}$, separating the contribution of the arithmetic functions $g$ and $h$. As a first step, we utilize the recursive definition of $P_{n}^{g,h}(x)$ given in (5). For simplification of the notation, we frequently put $A_{n,m} = A_{n,m}^{g,h}$. Recall that $A_{0,0} = 1$, $A_{n,0} = 0$ for $n \in \mathbb{N}$ and, $A_{n,n} = 1$.

Lemma 1. The coefficients of $A_{n,m}$ of $P_{n}^{g,h}(x)$ satisfy the recursion formula

$$A_{n,m} = \sum_{k=1}^{n-m+1} g(k) \frac{H(n-1)}{H(n-k)} A_{n-k,m-1}, \quad (1 \leq m \leq n).$$

Proof. We directly apply the definition of $P_{n}^{g,h}(x)$ by the recursion stated in (5). Let $1 \leq m \leq n$. Then $\sum_{m=1}^{n} A_{n,m} x^m = H(n) P_{n}(x)$ is equal to

$$H(n-1) x \sum_{k=1}^{n} g(k) P_{n-k}(x) = x \sum_{k=1}^{n} g(k) \frac{H(n-1)}{H(n-k)} \sum_{m=0}^{n-k} A_{n-k,m} x^m = \sum_{m=1}^{n} \sum_{k=1}^{n-m+1} g(k) \frac{H(n-1)}{H(n-k)} A_{n-k,m-1} x^m.$$

The value of $A_{n,m}$ is determined by the $n - m + 1$ values

$$A_{n-1,m-1}, A_{n-2,m-1}, \ldots, A_{m-1,m-1}.$$  

These are all $m - 1$st coefficients of all polynomials $P_d(x)$ of degree $0 \leq d < n$. Let $\delta := n - m$. From the computational point of view, to calculate $A_{n,m}$, we need $\delta + 1$ many previously given coefficients. As a special case we have

$$A_{n,n} = \frac{H(n-1)}{H(n-1)} A_{n-1,n-1} = A_{0,0} = 1.$$
3.1. Proof of the Main Theorem.

Proof. Let $\delta := n - m > 0$. The proof is given by induction on the degree $n$ of the polynomials $P_n(x)$. Let $n \in \mathbb{N}$. To formalize the induction, let $S(n)$ be the mathematical statement:

\begin{equation}
S(n) : A_{n,n-\delta} \text{ satisfies (9) for all } 0 < \delta \leq n.
\end{equation}

The statement $S(1)$ is true, since $A_{n,0} = 0$ for all $n \in \mathbb{N}$ and $A_{n,0} = 0$ for $\delta = n = 1$, evaluating the formula (9). This follows from $H(1,1) = 0$.

Let us now assume that the statements $S(1), \ldots, S(n-1)$ are true. We show that this implies, that $S(n)$ is also true. Our starting point is provided by the following formula, stated in Lemma 1:

\begin{equation}
A_{n,n-\delta} = A_{n-1,n-\delta-1} + \sum_{k=2}^{\delta+1} g(k) \frac{H(n-1)}{H(n-k)} A_{n-k,n-(\delta+1)}, \quad (0 \leq \delta < n).
\end{equation}

First, we replace $k$ by $k+1$ and utilize the fact that the last formula (29) is also true for $n$ substituted by $n-1, n-2, \ldots, \delta+1$:

\[ A_{n,n-\delta} = \sum_{N=\delta}^{n-1} \sum_{k=1}^{\delta} g(k+1) \frac{H(N)}{H(N-k)} A_{N-k,N-\delta}. \]

Next we insert the induction hypothesis. As $H(\varepsilon, 0) = 0$, we for $A_{n,n-\delta}$ we obtain the expression

\[
\sum_{N=\delta}^{n-1} \sum_{k=1}^{\delta} g(k+1) \prod_{j=1}^{\ell(\mu)} g(\mu_j + 1) h_k(N) H(\mu, N-k)
\]

\[ = \sum_{k=1}^{\delta} \sum_{\mu \in \mathcal{P}(\delta-k)} g(k+1) \prod_{j=1}^{\ell(\mu)} g(\mu_j + 1) \sum_{N=\delta}^{n-1} h_k(N) H(\mu, N-k). \]

Let $r = \ell(\mu)$ and $M = |\mu| + k$, then we use the one to one correspondence that $\nu = (\nu_1, \ldots, \nu_r, \nu_{r+1}) \in \mathbb{N}^{r+1}$ if and only if $\mu = (\nu_1, \ldots, \nu_r) \in \mathcal{C}(M - \nu_{r+1})$. Recall that in every orbit of a composition, there is always exactly one partition. This allows us finally to obtain for $A_{n,n-\delta}$ the expression

\[
\sum_{\nu \in \mathcal{P}(\delta)} g(\nu_{r+1} + 1) \prod_{j=1}^{\ell(\nu)-1} g(\nu_j + 1) \sum_{M=\nu_{r+1}}^{n-1} h_{\nu_{r+1}}(M) H((\nu_1, \ldots, \nu_r), M - \nu_{r+1})
\]

\[ = \sum_{\nu \in \mathcal{P}(\delta)} \prod_{j=1}^{\ell(\nu)} g(\nu_j + 1) H(\nu, M) \]

which shows the claim. \hfill \square

4. Applications and examples

We begin with some definitions and results. Let us first recall some properties of symmetric polynomials. Let $R$ be a commutative ring and let $t_1, \ldots, t_n$ be
algebraic independent elements of $R$. Let $x$ be a variable over $R[t_1, \ldots, t_n]$. We expand the polynomials

$$p(x) = \prod_{k=1}^{n} (x - t_k) = x^n - s_1 x^{n-1} + \ldots + (-1)^n s_n,$$

where each $s_k = s_k(t_1, \ldots, t_n)$ is a polynomial in $t_1, \ldots, t_n$. For instance,

$$(30) \quad s_1 = \sum_{k=1}^{n} t_k, \quad s_2 = \sum_{1 \leq k_1 < k_2 \leq n} t_{k_1} t_{k_2}, \ldots, \quad s_n = \prod_{k=1}^{n} t_k.$$ 

The polynomials $s_1, \ldots, s_n$ are the elementary symmetric polynomials of $t_1, \ldots, t_n$. The symmetric group $S_n$ operates on $R[t_1, \ldots, t_n]$ by

$$\pi(q(t_1, \ldots, t_n)) := q(t_{\pi^{-1}(1)}, \ldots, t_{\pi^{-1}(n)}), \quad \text{where } q \in R[t_1, \ldots, t_n], \pi \in S_n.$$ 

A polynomial $q$ is symmetric iff $\pi(q) = q$ for all $\pi \in S_n$. It is well-known that every symmetric polynomial $q(t_1, \ldots, t_n)$ is a polynomial in the elementary symmetric polynomials. We further recall the definitions of unimodal, log-concave, and ultra-log-concave.

**Definition 4.** Let $a_0, a_1, \ldots, a_n$ be a finite sequence of non-negative real numbers. This sequence is called:

- unimodal if $a_0 \leq a_1 \leq \ldots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \ldots \geq a_n$ for some $k$,
- log-concave if $a_j^2 \geq a_{j-1} a_{j+1}$ for all $j \geq 1$, and
- ultra-log-concave if the associated sequence $a_k / \binom{n}{k}$ is log-concave.

Note that ultra-log-concave implies log-concave and log-concave implies unimodal. The converses are in general not true, indicated by the following examples. The sequence of coefficients of the polynomials:

- $x^2 + 2x + 5$ is unimodal but not log-concave and
- $x^2 + 2x + 3$ is log-concave but not ultra-log-concave.

Let $P_n^1(x)$ be ultra-log-concave or log-concave, then $P_n^{\text{id}}(x)$ is ultra-log-concave or log-concave, respectively (see [HN20D]). This follows from relation (15) and

$$\frac{1}{m!^2} - \frac{1}{(m-1)! (m+1)!} = \frac{1}{(m-1)! m!} \left( \frac{1}{m} - \frac{1}{m+1} \right) > 0.$$

Let $\tilde{g}(n) = g(n) / n$. Then $(A_{n,m}^{\tilde{g}, \text{id}})^2 \geq A_{n,m-1}^{\tilde{g}, \text{id}} A_{n,m+1}^{\tilde{g}, \text{id}}$ implies

$$(A_{n,m}^{g, \text{id}})^2 = n!^2 \left( A_{n,m}^{\tilde{g}, \text{id}} \right)^2 \frac{1}{m!^2} \geq n!^2 A_{n,m-1}^{\tilde{g}, \text{id}} A_{n,m+1}^{\tilde{g}, \text{id}} \frac{1}{(m-1)! (m+1)!} = A_{n,m-1}^{g, \text{id}} A_{n,m+1}^{g, \text{id}}.$$

The proof for ultra-log-concave is exactly the same if the coefficients $A_{n,k}$ are replaced by $A_{n,k} / \binom{n}{k}$. 
4.1. The case $g = 1$.

4.1.1. Arbitrary $h$.
Let $h$ be a normalized non-vanishing arithmetic function. Then $P_n^{1,h}(x)$ is determined by

$$P_n^{1,h}(x) = \frac{1}{H(n)}(x + h(0)) \cdots (x + h(n - 1))$$

$$= \frac{1}{H(n)} \sum_{m=1}^{n} s_{n-m}(h(0), \ldots, h(n-1)) x^m.$$

4.1.2. Let $g = h = 1$. We have $P_n^{1,1}(x) = x(x + 1)^{n-1}$ for $n \geq 1$. This leads to $1 \leq m \leq n$:

$$A_{n,m}^{1,1} = \binom{n-1}{m-1} = s_{n-m}(0,1,\ldots,1).$$

Therefore, $P_n^{1,1}(x)$ is ultra-log-concave. Combining this formula with Theorem 1 leads to

**Corollary 2.** Let $1 \leq m < n$. Then

$$\sum_{\mu \in \mathcal{P}(n-m)} \binom{\ell(\mu)}{m_1, \ldots, m_n} \binom{n-|\mu|}{\ell(\mu)} = \binom{n-1}{m-1}.$$

4.2. The case $g = \text{id}$.

4.2.1. Let $g = \text{id}$ and $h = \text{id}$.
The explicit form of the coefficients and the ultra-log-concave properties immediately leads to

**Corollary 3.** The polynomials $P_n^{\text{id},\text{id}}(x)$ are ultra-log-concave and the coefficients are equal to the Lah numbers. These polynomials are associated Laguerre polynomials.

$$A_{n,m}^{\text{id},\text{id}} = \frac{1}{n!} = \frac{1}{n!} \binom{n-1}{m-1}.$$

4.2.2. Let $g = \text{id}$ and $h$ be arbitrary. We have $g(n + 2) - 2g(n + 1) + g(n) = 0$ for all $n \geq 1$, see [HNT20, Example 2.7 (1)]. From [HNT20, Theorem 2.1] we obtain

$$\frac{h(n)}{h(n + 2)} P_n^{g,h}(x) + \left(-2 \frac{h(n + 1)}{h(n + 2)} - \frac{x}{h(n + 2)}\right) P_{n+1}^{g,h}(x) + P_{n+2}^{g,h}(x) = 0.$$

We would like to mention that in the case $g = \text{id}$ and $h = 1$, we obtain the Chebyshev polynomials of the second kind. It is not a coincidence that $P_n^{1,\text{id}}(x)$ and $P_n^{\text{id},\text{id}}(x)$ both satisfy a 2nd order linear difference equation (3-term linear difference equation), since they are orthogonal polynomials.

**Remark.** Let $g$ fixed. Let the assumptions of Theorem 2.1 [HNT20] be fulfilled. Then $\{P_n^{g,h}(x)\}$ satisfies for all non-vanishing normalized arithmetic functions $h$ the same type of reduced recursion formula, with the same of amount of terms (see (34) for $g = \text{id}$).
4.3. Log-concavity.

**Proposition 4.** Let \( g \) and \( h \) be normalized arithmetic functions with positive values. Let

\[
a_{n,m} = \frac{A_{n,m}^{g,h}}{\prod_{k=1}^{n} h(k)}
\]

be the \( m \)th coefficient of the polynomial \( P_{n}^{g,h}(x) \). Then

\[
a_{n,n-1}^2 > a_{n,n-2} a_{n,n}
\]

for all \( n \geq 2 \) if \( g(2)^2 > g(3) \). There exists \( g \), independent of \( h \), such that (36) fails.

**Proof.** It is sufficient to consider \( \Delta(n) := (\prod_{k=1}^{n} h(k))^2 ((a_{n,n-1}^2 - a_{n,n-2} a_{n,n}) \).

Then

\[
\Delta(n) = (G(1)H(1, n-1))^2 - [G(2)H(2, n-2) + G(1)H((1,1), n-2)]
\]

\[
= g(2)^2 \sum_{k_1,k_2=1}^{n-2} h(k_1) h(k_2)
\]

\[
- g(3) \sum_{k=2}^{n-3} h(k) h(k-1) - g(2)^2 \sum_{k=3}^{n-3} h(k) \sum_{\ell=1}^{k-2} h(\ell)
\]

\[
\geq (g(2)^2 - g(3)) \sum_{k=2}^{n-3} h(k) h(k-1).
\]

This proves the first part of the claim in the Proposition. We also observe that we can chose \( g(3) \) sufficiently large that a \( n \) exists, such that \( \Delta(n) < 0 \). \(\square\)

The Nekrasov–Okounkov polynomials [NO06, HZ20, HN20D] defined by

\[
Q_n(x) := \sum_{\lambda\vdash n} \prod_{h\in H(\lambda)} \left( 1 + \frac{z}{h^2} \right) = \sum_{m=0}^{n} b_{n,m} x^m.
\]

are log-concave for \( n \leq 1500 \). Hong and Zhang [HZ20] have proven that for \( n \) sufficiently large, and \( m \gg \sqrt{n} \log n \): \( b_{n,m} \geq b_{n,m+1} \). From Proposition 4 we obtain

**Corollary 4.** Let \( n \geq 2 \) and \( Q_n(x) \) be the Nekrasov-Okounkov polynomial. Then

\[
b_{n,n-1}^2 > b_{n,n-2} b_{n,n}.
\]

4.4. Applications with the conversion formula. It is well known (see for example [Ko04]) that

\[
\prod_{n=1}^{\infty} (1 - q^n)^{-z} = 1 + \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n} \sum_{k_1,\ldots,k_m \in \mathbb{N}_{\geq 1}} \prod_{i=1}^{m} \frac{\sigma(k_i)}{k_i} x^m \right) q^n
\]

This can be easily generalized to a formula for the coefficients of \( P_{n}^{g,\mathrm{id}}(x) \). Utilizing the conversion formula (15) we also obtain a formula for the coefficients of \( P_{n}^{g,1}(x) \).
Corollary 5. Let $g$ be a normalized arithmetic function of moderate growth. Then we obtain for the coefficients of $P_n^{g,\text{id}}(x)$ and $P_n^{g,1}(x)$ the following formulas:

\begin{align*}
A_n^{g,\text{id}} &= \sum_{k_1,\ldots,k_m \in \mathbb{N}} \prod_{i=1}^m g(k_i), \\
A_n^{g,1} &= \frac{m!}{n!} \sum_{k_1,\ldots,k_m \in \mathbb{N}} \prod_{i=1}^m g(k_i).
\end{align*}

We utilized

$$\sum_{n=0}^{\infty} P_n^{g,\text{id}}(x) q^n = \exp \left( x \sum_{n=1}^{\infty} g(n) \frac{q^n}{n} \right)$$

and the conversion formula

\begin{equation}
\frac{A_n^{g,\text{id}}}{n!} = \frac{A_n^{g,1}}{m!}.
\end{equation}

It is a very challenging task to find similar formulas involving general $h$. In this paper we provide a formula (Main Theorem) for all pairs $g$ and $h$, which indicates already the complexity of this task.

Acknowledgments

To be entered later.

References

[AE04] G. E. Andrews, K. Eriksson: Integer Partitions. Cambridge University Press (2004).
[BKO20] J. Balakrishnan, W. Craig, K. Ono: Variations of Lehmer’s Conjecture for Ramanujan’s tau-function. To appear in: J. Number Theory (JNT Prime and Special Issue on Modular Forms and Function Fields) and arXiv: https://arxiv.org/abs/2005.10345.
[DA13] F. D’Arcais: Développement en série. Intermédiaire Math. 20 (1913), 233–234.
[Ha10] G. Han: The Nekrasov–Okounkov hook length formula: refinement, elementary proof and applications. Ann. Inst. Fourier (Grenoble) 60 no. 1 (2010), 1–29.
[HN20B] B. Heim, M. Neuhauser: The Dedekind eta function and D’Arcais-type polynomials. Res. Math. Sci. 7: 3 doi:10.1007/s40687-019-0201-5.
[HN20C] B. Heim, M. Neuhauser: On the reciprocals of Eisenstein series. International Journal of number theory (12.10.2020 online: https://doi.org/10.1142/S1793042120400199)
[HN20D] B. Heim, M. Neuhauser: Horizontal and vertical log-concavity. Submitted.
[HNT20] B. Heim, M. Neuhauser, R. Tröger: Zeros of recursively defined polynomials. J. Difference Equ. Appl. 26 no. 4 (2020), 510–531.
[HZ20] L. Hong, S. Zhang: Towards Heim and Neuhauser’s unimodality conjecture on the Nekrasov–Okounkov polynomials. arXiv:2008.10069.
[Ka78] V. G. Kac: Infinite-dimensional algebras, Dedekind’s $\eta$-function, classical Möbius function and the very strange formula. Advances in Mathematics 30 (1978), 85–136.
[Ko04] B. Kostant: Powers of the Euler product and commutative subalgebras of a complex simple Lie algebra. Invent. Math. 158 (2004), 181–226.
[Le47] D. Lehmer: The vanishing of Ramanujan’s $\tau(n)$. Duke Math. J. 14 (1947), 429–433.
[NO03] N. Nekrasov, A. Okounkov: Seiberg–Witten theory and random partitions. arXiv:hep-th/0306238v2.
[NO06] N. Nekrasov, A. Okounkov: Seiberg–Witten theory and random partitions. The unity of mathematics. Progr. Math. 244 Birkhäuser Boston (2006), 525–596.
M. Newman: *An identity for the coefficients of certain modular forms*. J. London Math. Soc. **30** (1955), 488–493.

K. Ono: *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series*. Conference Board of Mathematical Sciences **102** (2003).

S. Ramanujan: *On certain arithmetical functions*. Trans. Cambridge Philos. Soc. **22**, 159–184 (1916). In: Hardy, G. H., Seshu Aiyar, P. V., Wilson, B. M. (eds.) Collected Papers of Srinivasa Ramanujan, pp. 136–162. AMS Chelsea Publishing, American Mathematical Society, Providence, RI (2000)

J. Serre: *Sur la lacunarit´e des puissances de η*. Glasgow Math. J. **27** (1985), 203–221.

R. Stanley: *Enumerative Combinatorics, vol. 1*. Cambridge: Cambridge University Press, 1999.

B. Westbury: *Universal characters from the Macdonald identities*. Adv. Math. **202** no. 1 (2006), 50–63.

H. S. Wilf: *Generatingfunctionology*. A. K. Peters, Wellesley, Massachusetts, third edition (2006).

Faculty of Mathematics, Computer Science, and Natural Sciences, RWTH Aachen University, 52056 Aachen, Germany

*Email address: bernhard.heim@rwth-aachen.de*

Kutaisi International University (KIU), Youth Avenue, Turn 5/7 Kutaisi, 4600 Georgia

*Email address: markus.neuhauser@kiu.edu.ge*