Quantum teleportation using Ising anyons

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Anyons have been extensively investigated as information carriers in topological quantum computation. However, how to characterize the information flow in quantum networks composed by anyons is less understood, which motivates us to study quantum communication protocols in anyonic systems. Here we propose a general topologically protected protocol for quantum teleportation based on the Ising anyon model, and prove that with our protocol an unknown anyonic state of any number of Ising anyons can be teleported from Alice to Bob. Our protocol naturally generalizes quantum state teleportation from systems of locally distinguishable particles to systems of Ising anyons, which may promote our understandings of anyonic quantum entanglement as a quantum resource. In addition, our protocol is expected to be realized with the Majorana zero modes, one of possible physical realizations for the Ising anyon in experiments.

I. INTRODUCTION

Anyon \cite{1, 2}, as a kind of excitations different from boson and fermion living in two-dimensional system, has attracted the attention of theorists and experimentalists for their potential applications in fault-tolerant topological quantum computation due to its non-Abelian braid and topologically robustness \cite{3–6}. In theory, anyon is described by the anyon model \cite{7}, which is known as modular tensor category mathematically \cite{8, 9}. One of the most famous is the Ising anyon model. It is predicted that Majorana zero mode (MZM) as a physical realization of Ising anyon can exist on some physical platforms, such as fractional quantum Hall systems \cite{10, 11} and semiconductor nanowires \cite{12–14}.

Compared with the quantum information in conventional quantum states of distinguishable particles, which has a well-established quantum resource theory \cite{15}, the quantum information in anyonic states is less known to us. Since it is expected that the quantum resource theory of anyonic states could not only promote the development of topological quantum computation, but also guide the classification of topological phases in the condensed matter \cite{16, 17}. It urges us to study quantum information theory in anyonic systems. Some effort has been made to investigate novel quantum resources in anyonic system, such as anyonic entanglement \cite{18–20}, the entropy of anyonic charge entanglement \cite{20} and anyonic quantum mutual information \cite{21}.

Quantum teleportation \cite{22, 23} as the milestone of quantum information and quantum communication, which utilizes quantum entanglement and classical communication to teleport an unknown quantum state from one location to another, is a good starting point to investigate quantum entanglement of anyonic states. As we know the fragility of quantum states limits the application of quantum teleportation. However, anyonic system can provide a platform in which quantum states are very robust to resist environmental interference. Recently, Huang et al. \cite{22} have simulated quantum teleportation of a two-MZM state of the Kitaev chain \cite{12} using superconducting qubits, and given a modified teleportation for teleporting a qubit. It motivates us to consider the question of whether there is a general protocol for quantum teleportation to teleport an anyonic state with any number of anyons. However, the answer to this question is not straightforward. The Hilbert space of anyonic systems doesn’t equip with the tensor product structure like conventional quantum systems of distinguishable particles \cite{24}. Specifically, the Hilbert space of an anyonic system with total charge \(c\) containing two local subsystems \(A\) and \(B\) is

\[
\mathcal{H}^c_{AB} = \bigoplus_{a,b} \mathcal{H}^a_A \otimes \mathcal{H}^b_B \otimes V^c_{ab},
\]

where \(a\) (\(b\)) is the total charge of subsystem \(A\) (\(B\)), and \(V^c_{ab}\) is the fusion Hilbert space associated with the process that two anyons with charges \(a\) and \(b\) fuse into an anyon with charge \(c\). Thus, traditional methods such as quantum compiling \cite{25–27}, which aims to efficiently simulate a unitary gate by a series of elementary braidings, cannot be applied directly.

In this paper, we give a general topological protocol for quantum teleportation using Ising anyons. Specifically, \(2N + 1\) copies of the Bell states of two Ising anyons,

\[
|\sigma, \sigma; 0\rangle^{(2N+1)},
\]

which is the maximally entangled state of \(4N + 2\) Ising anyons \cite{21}, are distributed equally to Alice and Bob. Alice can teleport an anyonic state of \(M\) \((M \leq 2N + 1)\) Ising anyons to Bob by their respective local operation and one-way classical communication. As we all know...
that the technical core of the protocol for quantum teleportation is to construct a basis transformation between the computational basis and the maximally entangled basis. In the standard quantum teleportation [28, 29], Hadamard and CNOT gates are used to play a role of this basis transformation. In our anyonic teleportation, however, the key is how to distribute these anyons in an orderly fashion, which seems trivial to the standard quantum teleportation on the system of distinguishable particles due to the fact that braiding the distinguishable particles has no effect on the state. We find a systematic braiding, which equally distributes each pair of the Bell state of two Ising anyons to Alice and Bob. Based on this braiding, we give an equation which guides Bob to do the corresponding local braidings based on Alice’s measurement outcomes. This protocol can be viewed as an extension of quantum teleportation from systems of distinguishable particles to systems of Ising anyons.

II. RESULTS

In this section, we will give a brief review of the Ising anyon model [7, 30], and present the main result.

The Ising anyon model contains three types of particles, labeled by their topological charges \( I = \{0, \sigma, 1\} \), where particles 0, \( \sigma \), 1 are called vacuum, Ising anyon, and fermion respectively. The fusion rules for the model are given by

\[
\begin{align*}
\sigma \times \sigma &= 0 + 1, \\
1 \times \sigma &= \sigma, \\
1 \times 1 &= 0.
\end{align*}
\]

For example, Eq. 3 means that when two \( \sigma \)s are fused there are two possible fusion results 0 and 1.

Utilizing the fusion rules, we can define quantum states of an Ising anyonic system. For a system with two \( \sigma \)’s, the Hilbert space is spanned by two orthonormal anyonic states \( |\sigma, \sigma; 0\rangle \) and \( |\sigma, \sigma; 1\rangle \), which describe these two \( \sigma \)’s coming from vacuum 0 and a fermion 1, respectively. When there are more than two \( \sigma \)’s, we need to specify the order of fusion. Different orders will give different bases for the Hilbert space. The transformation between these bases is achieved through the \( F \) matrix (see details in App. A).

In addition to the fusion rules, the Ising anyon model also meets the rules of braiding. Specifically, exchanging \( \sigma \)’s in two-dimension space gives a unitary transformation named the \( R \) matrix (see details in App. A).

Unfortunately, based on these two unitary matrices \( F \) and \( R \), the Ising anyon model is known to be unable to realize universal topological quantum computing [31]. Further more, it is shown that not all Clifford gates can be realized by braiding Ising anyons [32]. However, multiqubit Pauli gates, which belong to the Clifford gates, are enough for our teleportation protocol.

Now we are ready to present our protocol for quantum teleportation using Ising anyons, where a player Alice wants to teleport an unknown state \( |\phi\rangle \) of \( M \) Ising anyons to a remote player Bob. A non-trivial example with \( N = 2 \) and \( M = 4 \) is illustrated in Fig. 1. Our protocol, as a generalization of the traditional quantum teleportation, is given as four steps:

**Step 1** : Preparation of Ising anyonic Bell state shared by Alice and Bob. The Bell state of \((4N + 2)\) Ising anyons [see Eq. (8) for the definition] prepared from \( 2N + 1 \) copies of the Bell states of two Ising anyons \( |\sigma, \sigma; 0\rangle \odot |2(2N + 1)\rangle \), through braiding the \( k \)-th copy to the center of the Bell state of \((2k - 2)\) Ising anyons in turn [see Eq. (7)], is shared between Alice and Bob, where \( M \leq 2N + 1 \).

**Step 2** : Ising anyonic Bell state measurement by Alice. Alice successively braids the middle two Ising anyons of remaining Ising anyons to the left side, to entangle the state \( |\phi\rangle \) of \( M \) Ising anyons with her \( M \) Ising anyons of the Bell state of \((4N + 2)\) Ising anyons. Then, she performs quantum measurements on these \( M \) Ising anyons to obtain fusion results \( a_k \) of the \((2k - 1)\)-th and the \(2k\)-th Ising anyons.

**Step 3** : Classical communication from Alice to Bob. Alice sends the measurement result \( \{a_1, a_2, \ldots , a_M\} \) to Bob by one-way classical communication.

**Step 4** : Conditional operation by Bob. Bob does braiding \( G_{a_1, \ldots , a_M} \) [see Eqs. (12) and (13)] on his \( M \) Ising anyons based on Alice’s measurement outcomes.

We have used the diagrammatic representation for the Ising anyon model in Fig. 1, where solid lines denote Ising anyons, dotted lines denote the Abelian anyons \( \{0, 1\} \), and the arrow of time goes up from the bottom. From the figure, we see that the information of the state \( |\phi\rangle \), encoded by 4 Ising anyonic lines, is accessible by Bob. Here the braiding performed by Alice and used for the preparation plays a role in building the path of information flow from the state \( |\phi\rangle \) to Bob’s output state.

In the following, we will prove the validity of this general protocol. To this end, we will define the braiding mentioned above named as the tangled braiding, and give an equation, which guides Bob to perform braiding based on Alice’s measurement outcomes. Finally, the quantum teleportation using Ising anyons will be built.
implies that there is a natural mapping between the Hilbert space of \(2N\) Ising anyons and that of \(N\) qubits: \(|a_1 a_2 \cdots a_N; c\rangle_{2N} \leftrightarrow |a_1 a_2 \cdots a_N\rangle\), where \(|a_1 a_2 \cdots a_N\rangle\) is the basis state of \(N\) qubits.

Here we are ready to give the definition of the tangled braiding:

*Definition* 1.– The tangled braiding \(T\) is a series of braiding acting on \(2N\) anyons, which can be written as \(T = T_N T_{N-1} \cdots T_2\), where braiding \(T_k\) \((k \in \{2, 3, \ldots, N\})\) is defined as

\[
T_k \equiv \begin{array}{c}
\vdots \\
\vdots \\
(k-1) \sigma’s \\
\vdots \\
(k-1) \sigma’s \\
\end{array}
\]

This tangled braiding \(T\) can be viewed as moving the \((2k-1)\)-th and \(2k\)-th Ising anyons from the right to the \(k\)-th and \((k+1)\)-th positions respectively, in turn from \(k = 2\) to \(k = N\).

The Bell state of 10 Ising anyons mentioned in the previous sections is just the state denoted as \(|B_{10}(00000)\rangle\) that can be obtained through the tangled braiding \(T\),

\[
|B_{10}(00000)\rangle \equiv T |00000\rangle_{10}
\]

This is the maximally entangled state that maximizes the anyonic von Neumann entropy \(S(\tilde{\rho}) = \tilde{\text{Tr}} \tilde{\rho} \log \tilde{\rho}\) of the anyonic reduced state \(\tilde{\rho}\) of the left (or right) 5 Ising anyons, where \(\tilde{\text{Tr}}\) is the quantum trace [20]. In addition to this state, another state that we will use in the following is

\[
|B_{10}(10000)\rangle \equiv T |10000\rangle_{10}
\]

To verify the validity of our quantum teleportation protocol, we prove a useful equality for the tangled braiding.

*Lemma*– For the state \(|a_1 a_2; a_1 + a_2\rangle\) of 4 Ising anyons, applying the tangled braiding \(T\) and then braiding the first two anyons twice successively, equals applying the braiding \(T\) acting on the state
\[(a_1 + 1)(a_2 + 1); a_1 + a_2\rangle_4 \text{ up to a global phase:} \]

\[
\begin{array}{c}
\vdots \\
|a_1\rangle \\
\vdots \\
\vdots \\
|a_2\rangle \\
\vdots \\
|a_1 + a_2 + 1\rangle \\
\end{array}
\]

Using these two gates, we can check Eq. (10) directly.

Proof. – We check this equality by mapping these braidings to the corresponding unitary operators. By using the \( R \) and \( F \) matrices of the Ising anyon model, we find that all \( n \)-qubit Pauli gates can be obtained by braiding 2\( n + 1 \) Ising anyons in the anyonic computational basis for 2\( n + 2 \) Ising anyons (see details in App. B):

\[
\begin{align*}
(b_{2j-1}^{(2n+2)})^2 &= \tau_3^{(j)}, \\
(b_{2j}^{(2n+2)})^2 &= \tau_1^{(j)} \otimes \tau_1^{(j+1)}, \\
(b_{2n+1}^{(2n+2)})^2 (b_{2n}^{(2n+2)})^2 &= i,
\end{align*}
\]

where 1 \( \leq j \leq n \), \( \tau_i^{(j)} \) is Pauli matrix \( \tau_i \) acting on the \( j \)-th qubit \( a_j \) in the anyonic computational basis, and \( b_j^{(2n+2)} \) is the generator of braid group \( B_{2n+2} \). Thus, braiding the first two Ising anyons gives the Pauli gate \( \tau_3 \) acting on the first qubit \( a_1 \). And the tangled braiding \( T \) gives two-qubit entangled gate:

\[
\frac{1}{\sqrt{2}}\begin{pmatrix}
1 & 0 & 0 & -i \\
0 & i & 1 & 0 \\
0 & -i & 1 & 0 \\
1 & 0 & 0 & i
\end{pmatrix}.
\]

Using these two gates, we can check Eq. (10) directly. \( \blacksquare \)

The physical significance of the tangled braiding \( T \) is apparent, which distributes the entanglement in \( N \) copies of the Bell states of two Ising anyons from left to right between Alice and Bob. Symmetrically, we can also define the dual tangled braiding \( \bar{T} \), which distributes the entanglement in \( N \) copies from right to left. This dual tangled braiding \( \bar{T} \) is just the tangled braiding \( T \) seen from the other side. Therefore, we will not distinguish between these two braidings, and use the symbol \( T \) uniformly.

IV. ISING ANYONIC TELEPORTATION

Now we will show that, by utilizing the tangled braiding \( T \) given above, we can teleport an anyonic state of Ising anyons from Alice to Bob, which can be seen as an anyonic version of quantum teleportation referred as Ising anyonic teleportation.

Theorem. – In the Ising anyonic teleportation, Alice and Bob share the state \( |B_{2(N+2)}(0\cdots0);0\rangle \), which can be prepared from the state \( |0\cdots0;0\rangle_{4N+2} \) by the tangled braiding \( T \), and each takes 2\( N + 1 \) Ising anyons. Alice has another unknown state \( |\phi\rangle \) of \( M \) (\( M \leq 2N + 1 \)) Ising anyons, which is prepared to be teleported to Bob. On Alice’s side, Alice does the operation \( \bar{T}^{-1} \) by braiding her 2\( M \) Ising anyons, then she measures every neighboring pair of these 2\( M \) Ising anyons to obtain \( M \) fusion results \( \{a_1 \cdots a_M\} \). She sends these results to Bob through one-way communication, who performs corresponding local operation \( G_{a_1 \cdots a_M} \) that depends on the results of Alice’s measurement by braiding his Ising anyons. When \( M \) is even,

\[
G_{a_1 \cdots a_M} = (b_0)^{2(c-a_1)} \cdots (b_j)^{2\sum_{i=j+1}^{M} a_i} \cdots (b_{M-2})^{2(a_M+a_{M-1})}(b_{M-1})^{2a_M}.
\]

When \( M \) is odd,

\[
G_{a_1 \cdots a_M} = (b_1)^{2(c-a_1)} \cdots (b_j)^{2\sum_{i=j+1}^{M} a_i} \cdots (b_{M-2})^{2(a_M+a_{M-1})}(b_{M-1})^{2a_M}.
\]

Here \( c = \sum_{i=1}^{M} a_i \), modulo 2 is the parity, \( b_j \) is the generator of Braid group, which braids the \( j \)-th and \((j+1)\)-th anyons (see Fig. 2 with \( M = 4 \)). Finally, the original anyonic state \( |\phi\rangle \) owned by Alice will appear on Bob’s side.

Specifically, in the Theorem, the inverse of the tangled braiding \( T^{-1} \) performed by Alice on 2\( M \) Ising anyons can be broken down into a series of fairly simple braidings:

\[
T^{-1} = T_2^{-1} T_{3}^{-1} \cdots T_M^{-1}.
\]

From Eq. (7), we know that \( T_k \) is the operation that moves the \((2k-1)\)-th and \((2k)\)-th Ising anyons to the \( k \)-th and \((k+1)\)-th positions respectively. The inverse of \( T_k \) is the opposite operation that moves the \((k+1)\)-th and \( k \)-th Ising anyons to the \( 2k \)-th and \((2k-1)\)-th positions respectively. An example with \( M = 4 \) is illustrated in Fig. 1. We can see that Alice first performs \( T_1^{-1} \) (black lines) that moves the 5-th and 6-th Ising anyons to the 8-th and 7-th positions respectively, and then performs \( T_3^{-1} \) (blue lines) and \( T_2^{-1} \) (green lines). The operation \( T^{-1} \) pairs the Ising anyons of the state \( |\phi\rangle \) with the Ising anyons of the shared Bell state in an orderly fashion.

Proof. – The main idea of the proof is to show that, for Bob, the case that Alice’s measurement outcomes are not \( \{0\cdots0\} \) can be equivalent to the case with outcomes \( \{0\cdots0\} \) by his operation.

First we note that, when Alice’s measurement outcomes are \( \{0\cdots0\} \), Bob will obtain the \( |\phi\rangle \) by doing nothing. Since we have

\[
\begin{array}{c}
\vdots \\
|\phi\rangle \\
\vdots \\
|\phi\rangle \\
\vdots \\
\vdots \\
|\phi\rangle
\end{array}
\]

which follows the fact that the zigzag gives identity \([8]\). In other words, these two diagrams are topological equivalent (isotopy) by continuous deformations as long as open endpoints are fixed \([7]\).
Second, for other measurement outcomes \( \{a_1 \cdots a_M\} \), the diagram will be complicated and can not become the diagram on left-hand side of Eq. (15) directly. Bob’s target is to deform the diagram to become the diagram on left-hand side of Eq. (15) by his local operation. Indeed, Bob can realize it by taking advantage of the Bell state \( |B_{2N+2}(0 \cdots 0); 0\rangle \) shared between Alice and him. To see this, without loss of generality, we consider two adjacent anyons on Bob’s side, namely, the \( j \)-th and \( (j + 1) \)-th anyons, which are connected to the measurement outcomes \( a_j \) and \( a_{j+1} \) on Alice’s side as shown in Fig. 2. It is reasonable to discuss only the \( j \)-th and \( (j + 1) \)-th anyons here and ignore the other, since braiding on any adjacent anyons will not affect other anyons which can be seen from Fig. 1. We suppose that Bob braids these two anyons twice illustrated in Fig. 2(a). Topologically, we can move this braiding performed by Bob to Alice’ side through the Bell state of 4 Ising anyons shared between them, whereby we obtain the diagram shown in Fig. 2(b). We notice that the diagram on Alice’s side in Fig. 2(b) is exactly the left-hand side of Eq. (10). Taking advantage of the Lemma in the previous section, the case that Bob braids the \( j \)-th and \( (j + 1) \)-th Ising anyons twice for Alice’s measurement outcomes \( \{a_j, a_{j+1}\} \) is equivalent to the case that Bob does nothing for Alice’s measurement outcomes \( \{a_j + 1, a_{j+1} + 1\} \). We conclude that when Alice’s measurement outcome \( a_{j+1} = 1 \), Bob only needs to braid the \( (j + 1) \)-th and \( j \)-th Ising anyons twice on his side, which is equivalent to the case with \( a_{j+1} = 0 \).

Take \( M = 4 \) as an example illustrated in TABLE 1. When Alice’s measurement outcomes are \( \{0011\} \), Bob only needs to braid the 3-rd and 4-th Ising anyons twice, which is equivalent to the case with measurement outcomes \( \{0000\} \). However, it should be noted that when the parity \( c \) of measurement outcomes is odd, Bob needs to use auxiliary Ising anyon labeled as 0-th illustrated in Fig. 2. Thus, Bob can take the local braiding \( G_{a_1 \cdots a_M} \) presented in Eq. (12) to hit the mark.

Third, we find that there is no need to perform the last step, \( b_0 \), to change \( a_1 \) if \( M \) is odd. To see it, we give the anyonic state of \( M \) (odd) Ising anyons when Alice’s measurement outcomes are \( \{0 \cdots 0\} \):

\[
\begin{align*}
\left| \psi \rightangle &= \left| \psi \right\rangle \\
&= \text{where } \left| \psi \right\rangle \text{ is the state of } M - 1 \text{ (even) Ising anyons with the parity } \sigma. \text{ On the right-hand side of Eq.}(16), \text{ charge } c \text{ denotes the total charged obtained by Alice’s measurement. By quantum tracing the charge } c \text{ (Alice’s side), we can obtain the state } \left| \phi \right\rangle \text{ of } M \text{ Ising anyons (see App. C), which Alice wants to teleport. }
\end{align*}
\]

In conclusion, we have shown that this Ising anyonic teleportation works. ■

In the above proof, we have used the Bell state of \( 4N + 2 \) Ising anyons to move the braiding performed by Bob to Alice side. In quantum information, we have a similar situation where we can move a unitary gate from one party to the other by using the general Bell state \( (1/\sqrt{d}) \sum_i \left| i \right\rangle_A \otimes \left| i \right\rangle_B \). That’s why we call the state in Eq. (8) the Bell state.

V. SUMMARY AND DISCUSSION

In summary, we have extended the quantum teleportation using quantum states of distinguishable particles to that using anyonic states of Ising anyons. We have found a systematic braiding \( T \) that distributes each pair of the Bell state of two Ising anyons to Alice and Bob, whereby proposed a general protocol for quantum teleportation based on the Ising anyon model. We have seen that there is a difference between the protocol for teleporting the unknown state \( \left| \phi \right\rangle \) of odd Ising anyons and that of even Ising anyons. The protocol of the latter is the same as the former except that we require Bob to perform the braiding \( b_0^2 \), which changes the parity \( c = 1 \) to \( c = 0 \). This is due to the fact that the structures of Hilbert spaces of odd number and even number of Ising anyons are different. The total charge of \( 2N + 1 \) Ising anyons is always \( \sigma \) while that of \( 2N \) could be 0 or 1, although the dimension of these two Hilbert spaces are the same. The superselection rules divide the Hilbert space of \( 2N \) Ising anyons into two subspaces, which cannot be transformed into each other by braiding these \( 2N \) Ising anyons. Thus, the parity \( c \) of \( 4N \) Ising anyons depends on the parities of \( 2N \) Ising anyons on two sides while the parity \( c \) of \( 4N + 2 \) Ising anyons isn’t up to the parities of \( 2N + 1 \) Ising anyons on two sides.

This Ising anyonic teleportation is protected by topol-
ogy. All that is needed in the protocol is braiding and measurement. It has been shown \([33, 34]\) that the projective measurements of two Ising anyons can be realized using the interferometry measurements, in which the target’s charge can be inferred from the effect on the interference of probe anyons braiding around the target’s charge.

Although quantum entanglement is well known as a resource consumed in quantum teleportation, there has been no definitive definition of quantum entanglement of quantum states of anyons. Our protocol confirms that the Bell state of Ising anyons \([\text{see Eq. (8)}]\) is one of the maximally entangled states. To further define the quantum resource theory, we should clarify the definitions of free operations and free states \([15]\) in anyonic systems. Along this direction, our protocol may promote our understandings of anyonic quantum entanglement as a quantum resource.

In the end, MZM as a reasonable candidate for Ising anyon, has been broadly searched in the experiments. We hope that our theoretical protocol might be further studied experimentally in future.

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**Appendix A: The Ising anyon model**

Here we will use the Ising anyon model \([7, 20]\), a kind of modular tensor categories \([8, 9]\), to derive that all \(n\)-qubit Pauli gates can be obtained by braiding \(2n + 2\) or \(2n + 1\) Ising anyons under the anyonic computational basis.

First, we review the Ising anyon model, which contains three types of topological charges \(I = \{0, \sigma, 1\}\) satisfying non-trivial fusion rules:

\[
\begin{align*}
\sigma \times \sigma &= 0 + 1, \\
1 \times \sigma &= \sigma, \\
1 \times 1 &= 0,
\end{align*}
\]

where \(\sigma\) denotes Ising anyon, \(1\) denotes fermion, and \(0\) is vacuum.

Based on the above fusion rules, we can define the Hilbert space called the fusion space, which is spanned by the different fusion paths. For example, the fusion space of two \(\sigma\)'s fusing into the vacuum is given by \(V_{\sigma^2}^0 = \text{span} \{\langle \sigma, \sigma; 0 \rangle\}\). Similarly the fusion space of two \(\sigma\)'s fusing into \(1\) is given by \(V_{\sigma^2}^1 = \text{span} \{\langle \sigma, \sigma; 1 \rangle\}\). In particular, each of these two spaces has only one basis vector since each of them has only one fusion path. It is useful to employ a diagrammatic representation for anyon models, where each anyon is associated with an oriented (we will omit the orientation here) line that can be understood as the anyon’s world line. In the diagrammatic representation, the two basis vectors above can be represented as

\[
\langle \sigma, \sigma; 0 \rangle = \left( \frac{1}{d_\sigma} \right)^{\frac{1}{2}} \begin{array}{c} 0 \\ \sigma \\ \sigma \end{array},
\]

\[
\langle \sigma, \sigma; 1 \rangle = \left( \frac{1}{d_\sigma} \right)^{\frac{1}{2}} \begin{array}{c} 1 \\ \sigma \\ \sigma \end{array},
\]

where \(d_\sigma = \sqrt{2}\) is the quantum dimension of anyon \(\sigma\), \((1/d_\sigma)^{\frac{1}{2}}\) is the normalized coefficient, the solid line denotes \(\sigma\), and the dotted line denotes the vacuum \(0\) and the fermion \(1\).

For a system with more \(\sigma\)'s, the Hilbert space is constructed by taking the tensor product of its composite parts. For example, the fusion space \(V_{\sigma^3}^\sigma\) of three \(\sigma\)'s with total charge \(\sigma\) can be constructed as

\[
V_{\sigma^3}^\sigma \cong \bigoplus_b V_{\sigma^b}^\sigma \otimes V_{b, \sigma},
\]

where \(b \in \{0, 1\}\). It should be noted that fusion order is not unique. In the example above you can choose to start with fusion of the two \(\sigma\)'s on the left or the two \(\sigma\)'s on the right. These two different methods of fusion are related
by $F$ matrix:

$$
\sigma \downarrow_{\sigma} \sum_d (F_{\sigma\sigma\sigma}^\dagger)^b \downarrow_{\sigma} \sigma \downarrow_{\sigma} = d \downarrow_{d},
$$

(A6)

where $b, d \in \{0, 1\}$ and

$$
F_{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.
$$

(A7)

A linear anyonic operator can be defined using the basis vectors in fusion space and splitting space as we do in quantum mechanics. For example, the identity operator for two Ising anyons is

$$
1_{\sigma\sigma} = |\sigma, \sigma; 0 \rangle \langle \sigma, \sigma; 0 | + |\sigma, \sigma; 1 \rangle \langle \sigma, \sigma; 1 | = \frac{1}{d_{\sigma}} \downarrow_{\sigma} + \frac{1}{d_{\sigma}} \downarrow_{\sigma}.
$$

(A8)

The quantum trace, which joins the outgoing anyon lines of the anyonic operator back onto the corresponding incoming lines, e.g.,

$$
\text{Tr} \left[ \frac{1}{d_{\sigma}} \downarrow_{\sigma} \right] = \frac{1}{d_{\sigma}} \uparrow_{\sigma} = 1.
$$

(A9)

By using the quantum trace, we can define an operator $\tilde{\rho}$, called an anyonic density operator satisfying the normalization condition $\text{Tr} [\tilde{\rho}] = 1$ and the positive semi-definite condition; that is, for any anyonic state $|\phi\rangle$, we have $\text{Tr} [\langle \phi | \tilde{\rho} | \phi \rangle] \geq 0$.

In addition to fusion rules, the Ising anyon model also needs to meet the rules of braiding. Specifically, exchanging neighboring $\sigma$'s gives to the anyonic state a unitary evolution named the $R$ matrix:

$$
\sigma \downarrow_{\sigma} = \sum_d (R_{\sigma\sigma})_{bd} \downarrow_{d} \uparrow_{\sigma},
$$

(A10)

where

$$
R_{\sigma\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

(A11)

**Appendix B: Pauli gates**

Suppose there are $2n + 1$ Ising anyons in a row. In the fusion order from left to right, the standard basis $|\bar{p}\rangle$ of this anyonic system can be denoted as

$$
|\bar{p}\rangle \equiv \bar{a}_1 \bar{a}_2 \cdots \bar{a}_{n}; \sigma_{2n+1} = \left( \frac{1}{d_{\sigma}} \right)^{n/2} \bar{a}_1 \bar{a}_2 \cdots \bar{a}_{n} \sigma_{2n} \sigma_{2n+1},
$$

(B1)
where the solid line denotes the Ising anyon $\sigma$, and the dotted line denotes two possible fusion results $\pi_i = 0, 1$, $i = 1, \ldots, n$, encoded as a binary code. Thus, we see this Hilbert space has dimension $2^n$. However, these binary variables are not independent of each other. To make this basis $|\overline{p}\rangle$ more like a $n$-qubit state, we adopt another fusion order, where $\sigma_{2j-1}$ and $\sigma_{2j}$ are fused first. Thus, we give another basis $|p\rangle$ called anyonic computational basis of the same Hilbert space:

$$|p\rangle \equiv |a_1 a_1 \cdots a_n; \sigma\rangle_{2n+1}$$

$$= \left( \frac{1}{d_\sigma} \right)^{n/2} \sigma_1 \sigma_2 \sigma_3 \sigma_4 \cdots \sigma_{2n-1} \sigma_2n \sigma_{2n+1}$$

Fusion path of $a_1 \times \cdots \times a_n \times \sigma = \sigma$

where $a_i = 0, 1$, $i = 1, \ldots, n$. The fusion path in the rectangle box above is not showed, since it can be uniquely determined due to abelian fusion rules of 0 and 1.

These two bases span the same Hilbert space are related by trivial $F$ moves in the Ising anyon model, for example, when $n = 2$, we have

$$|\overline{00}; \sigma\rangle_5 = |00; \sigma\rangle_5, \quad |\overline{01}; \sigma\rangle_5 = |01; \sigma\rangle_5, \quad |\overline{10}; \sigma\rangle_5 = |11; \sigma\rangle_5, \quad |\overline{11}; \sigma\rangle_5 = |10; \sigma\rangle_5. \quad (B3)$$

We can view these $F$ moves as a CNOT gate $U_{\text{CN}}^{1, 2}$ with $\overline{\pi}_1$ as the control qubit and $\overline{\pi}_2$ as the target qubit. Thus, we can get the basis $|p\rangle$ from the basis $|\overline{p}\rangle$ using a series of CNOT gates:

$$|a_1, a_1, \cdots, a_n; \sigma\rangle_{2n+1} = U_{\text{CN}}^{1, 2} U_{\text{CN}}^{2, 3} \cdots U_{\text{CN}}^{n-1, n} |\overline{\pi}_1,\overline{\pi}_2, \cdots, \overline{\pi}_n; \sigma\rangle_{2n+1}$$

$$\equiv U_{\text{CNOT}} |\overline{\pi}_1, \overline{\pi}_2, \cdots, \overline{\pi}_n; \sigma\rangle_{2n+1} \quad (B4)$$

Now we consider the states $|p\rangle$ as the basis for our representation space. We will use the $F$ move and the $R$ matrix to give a representation to braid group. Explicitly, we have the generators $b^{(2n+1)}_j$ of braid group $B_{2n+1}$, which denotes the exchange of the $j$-th and $(j + 1)$-th Ising anyons in the defined positive direction,

$$b^{(2n+1)}_{2j-1} = I_2 \otimes \cdots \otimes I_2 \otimes I_2 \otimes I_{n-j}^{-1} \otimes I_{n-j}^{1} \otimes I_{n-j}^{2} \otimes I_{n-j}^{3} \otimes \cdots \otimes I_{n-j}^{n-1}$$

$$= I_2 \otimes \cdots \otimes I_2 \otimes \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 1 \end{pmatrix} \otimes I_2 \otimes \cdots \otimes I_2, \quad 1 \leq j \leq n - 1,$n}

$$b^{(2n+1)}_{2j} = U_{\text{CN}}^{1, j+1} (P^{\sigma \sigma} \otimes I_2) (R_{\sigma \sigma} \otimes I_2) (P^{\sigma \sigma} \otimes I_2) U_{\text{CN}}^{1, j+1}$$

$$= I_2 \otimes \cdots \otimes I_2 \otimes \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \-i & 1 \end{pmatrix}, \quad 1 \leq j \leq n - 1,$n}

where $b^{(2n+1)}_j$, $1 \leq j \leq n - 1$, can be proved by noticing that

$$\sigma^{2j-1} \sigma^{2j} \sigma^{2j+1} \sigma^{2j+2} \quad \sigma^{2j-1} \sigma^{2j} \sigma^{2j+1} \sigma^{2j+2} \quad \sigma^{2j-1} \sigma^{2j} \sigma^{2j+1} \sigma^{2j+2}$$

$\overline{a}_j \quad \overline{a}_{j+1} \quad \overline{a}_j$
Now we have the generators $b_j^{(2n+1)}$ of braid group $B_{2n+1}$. It’s straightforward to extend to the case of $B_{2n+2}$:

$$b_{2j-1}^{(2n+2)} = I_2 \otimes \cdots \otimes I_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \otimes I_2 \otimes \cdots \otimes I_2, \quad 1 \leq j \leq n+1,$$

$$b_{2j}^{(2n+2)} = I_2 \otimes \cdots \otimes I_2 \otimes \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} \otimes I_2 \otimes \cdots \otimes I_2, \quad 1 \leq j \leq n. \quad (B7)$$

These matrices with dimension $2^{n+1} \times 2^{n+1}$ are reducible due to the superselection rule. In other words, the parity $\mathcal{P}$ of the state $|p\rangle = |a_1 \cdots a_{n+1}\rangle$, which is defined as $\mathcal{P} = \sum a_i \mod 2$, gives two representations for braid group $B_{2n+2}$. Indeed, we see that $\tau_3^{(n+1)}$ commutes with all generators of $B_{2n+2}$, where $\tau_i$, $i = 0, 1, 2, 3$, are Pauli matrices. Since the eigenvalues of $\tau_3^{(n+1)}$ have two different values $\pm$, this reducible representation can be reduced into two irreducible representations by projectors

$$p_{\pm}^{(2n+2)} = I \pm \tau_3^{(n+1)} \frac{1}{2}, \quad (B8)$$

where the superscript $\pm$ denotes two irreducible representations. The generators of $B_{2n+2}$ derived here are the same as the results in the Ref. [35].

Finally, all $n$-qubit Pauli gates can be obtained by braiding $2n+1$ Ising anyons in the computational basis with $2n+1$ Ising anyons:

$$\tau_3^{(j)} = \left(b_{2j-1}^{(2n+1)}\right)^2,$$

$$\tau_1^{(j)} = \left(b_{2j-1}^{(2n+1)}\right)^2 \left(b_{2j}^{(2n+1)}\right)^2 \cdots \left(b_{2n}^{(2n+1)}\right)^2,$$

$$i = b_{2n-1}^{(2n+1)} \left(b_{2n}^{(2n+1)}\right)^2 \left(b_{2n-1}^{(2n+1)}\right)^2, \quad (B9)$$

or by braiding $2n+1$ Ising anyons in the computational basis with $2n+2$ Ising anyons:

$$\tau_3^{(j)} = \left(b_{2j-1}^{(2n+2)}\right)^2,$$

$$\tau_1^{(j)} \otimes \tau_1^{(n+1)} = \left(b_{2j}^{(2n+2)}\right)^2 \left(b_{2j+2}^{(2n+2)}\right)^2 \cdots \left(b_{2n}^{(2n+2)}\right)^2,$$

$$i = b_{2n-1}^{(2n+2)} \left(b_{2n}^{(2n+2)}\right)^2 \left(b_{2n-1}^{(2n+2)}\right)^2, \quad (B9)$$

where $1 \leq j \leq n$, and $\tau_i^{(j)}$ is Pauli matrix $\tau_i$ acting on the $j$-th qubit $a_j$. It is worth noting that, in simulating $n$-qubit Pauli gates in the computational basis with $2n+1$ and $2n+2$ Ising anyons, we use the same braiding operations on $2n+1$ Ising anyons. However, for braiding $2n+2$ Ising anyons, if we want to apply $\tau_1^{(j)}$ or $\tau_2^{(j)}$ on qubit $a_j$, it will not only change the $a_j$ but will change the assistant qubit $a_{n+1}$ to maintain the parity.

**Appendix C: Quantum tracing Alice’s side**

In this section, we will give details of how to obtain the state $|\phi\rangle$ of $M$ (odd) Ising anyons from the state in Eq. (16) by quantum tracing charge $c \in \{0, 1\}$, which is the parity of Alice’s measurement outcomes, i.e., the total charge of the state obtained by Alice’s measurement:

$$\tilde{\text{Tr}}_c \left[ \begin{array}{c|c} c & \vdots \\ \hline \end{array} \right] = |\phi\rangle \langle \phi|, \quad (C1)$$
where $|\psi\rangle$ denotes the state of $M-1$ (even) Ising anyons, and $\tilde{\text{Tr}}_c$ denotes partial quantum trace over charge $c$. We use anyonic density matrices for anyonic states in order to apply quantum trace directly. From the left-hand side of Eq. (C1), we have:

$$
|\psi\rangle_i c c_i |\psi\rangle = |\psi\rangle_i - c c_i |\psi\rangle = |\phi\rangle \langle \phi |.
$$

(C2)

In the first step we have used $F$-move and the $F$ matrix $F_{\sigma i}^{c\sigma}$ is a number. In the second step we have used $F$-move with two lower and two upper legs and the fact that the tadpole diagram gives zero. In the last step we have used the fact that an unknotted loop carrying charge $c$ gives to its quantum dimension $d_c$.

From Eqs. (16) and (C1), we see that, when Alice’s measurement outcomes are $\{10 \cdots 0\}$, the state of $M$ (odd) Ising anyons on Bob’s side is $|\phi\rangle$ just like the case where the measurement outcomes $\{0 \cdots 0\}$. However, this does not apply when $M$ is even:

$$
|\phi\rangle = |\psi\rangle_i - c c_i |\psi\rangle = |\psi\rangle_i - c c_i |\psi\rangle,
$$

(C3)

where $|\psi\rangle$ denotes the state of $M-1$ (odd) Ising anyons. Similarly, by quantum tracing the charge $c$, we obtain the state of $M$ Ising anyons with parity $i-c$. When $c=0$, this state becomes the state $|\phi\rangle$ Alice wants to teleport. When $c=1$, this state has a different total charge than $|\phi\rangle$, and we need auxiliary anyon to change the parity.

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