A three-parametric deformation of $GL(1/1)$

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A three-parametric $R$-matrix satisfying a graded Yang-Baxter equation is introduced. This $R$-matrix allows us to construct new quantum supergroups which are deformations of the supergroup $GL(1/1)$ and the universal enveloping algebra $U[gl(1/1)]$.

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Using the $R$-matrix formalism is one of the approaches to quantum groups which can be interpreted as a kind of (quantum) deformations of ordinary (classical) groups or algebras. It has proved to be a powerful method in investigating quantum groups and related topics. A physical meaning of this approach is the so-called (universal) $R$-matrix associated to a quantum group satisfying the famous Yang-Baxter equation (YBE) representing an integrability condition of a physical system. The mathematical advantage of this approach is both the algebraic and co-algebraic structure of the corresponding quantum group can be expressed in a few compact (matrix) relations. Quantum groups as symmetry groups of quantum spaces or as deformations of universal enveloping algebras can be also derived in an elegant way in the framework of the $R$-matrix formalism. Combined with the supersymmetry idea, the quantum deformations lead to the concept of quantum supergroups. In this case, an $R$-matrix becomes graded and satisfies a graded YBE.

By construction, a quantum (super) group depends on one or more, complex in general, parameters. For about two decades quantum groups have been investigated in great detail in many aspects. These investigations were carried out first and mainly on the one-parametric case and they were extended later to on the multi-parametric deformations. Having in principle richer structures, multi-parametric quantum groups are also a subject of interest of a number of authors (see and references therein) and have been applied to considering some physics models (see for example, some recent works, Refs. but in comparison with the one-parametric quantum groups, they are considerably less understood (even, in some cases they can be proved to be equivalent to one-parametric deformations). Moreover, most of the multi-parametric deformations considered so far are two-parametric ones including those of supergroups (it is clear that two-parametric deformations of supergroups cannot be always reduced to one-parametric ones). In particular, a two-parametric deformation of the supergroup $GL(1/1)$ was considered in Refs. The authors obtained a two-parametric quantum deformation of $GL(1/1)$ but the corresponding deformation of the universal enveloping algebra $U[gl(1/1)]$ can be made to look like an one-parametric deformation by re-scaling its generators appropriately. Indeed, starting from the defining relations of the deformation of $U[gl(1/1)]$ given in Ref. 

\[ [K, H] = 0, \ [K, \chi_\pm] = 0, \ [H, \chi_\pm] = \pm 2\chi_\pm, \]

\[ \{\chi_+, \chi_-\}_{q/p} = \left(\frac{q}{p}\right)^{H/2} [K]_{qp}, \]

where

\[ \{\chi_+, \chi_-\}_{q/p} = \left(\frac{q}{p}\right)^{H/2} \chi_+ \chi_- \left(\frac{q}{p}\right)^{-H/2} \chi_+ \chi_-^{-1}, \]

\[ [K]_{qp} = \frac{(qp)^{K/2} - (qp)^{-K/2}}{(qp)^{1/2} - (qp)^{-1/2}} \]

and making re-scaling $\chi_\pm \rightarrow \chi'_\pm = \left(\frac{q}{p}\right)^{-1/4} \chi_\pm$, we get

\[ [K, H] = 0, [K, \chi'_\pm] = 0, [H, \chi'_\pm] = \pm 2\chi'_\pm, \{\chi'_+, \chi'_-\} = [K]_{qp}. \]

The latter relations are (conventional) defining relations of an one-parametric deformation of $U[gl(1/1)]$ with parameter $\sqrt{qp}$. In the present paper we suggest an $R$-matrix allowing us to construct a three-parametric deformation of $GL(1/1)$. This suggestion, however, is two-fold, as it leads us to a true two-parametric deformation of $U[gl(1/1)]$. 

\[ [K, H] = 0, [K, \chi_{1/2}] = 0, [H, \chi_{1/2}] = \pm 2\chi_{1/2}, \{\chi'_+, \chi'_-\} = [K]_{qp}. \]
Let us start with the operator

\[ R = q(e^1_j \otimes e^1_i) + r(e^1_i \otimes e^2_j) + s(e^2_i \otimes e^1_j) + \lambda(e^1_i \otimes e^1_j) + p(e^2_i \otimes e^2_j), \]

where \( p, q, r, s, \lambda \) are complex deformation parameters \((p, q, r, s, \lambda \in \mathbb{C})\), while \( e^1_i, e^2_i \), \( i, j = 1, 2 \), are Weyl generators of \( GL(1|1) \) with a \( \mathbb{Z}_2 \)-grading given as follows:

\[ [e^1_j] = [i] + [j] \mod 2, \quad [i] = \delta_{2i}. \]

We call the latter operator an \( R \)-matrix although it has a (finite) matrix form only in a finite-dimensional representation. In the fundamental representation \( e^1_j \) are super-Weyl matrices, \( (e^1_j)^h = \delta^h_i \delta^h_j \), and \( R \) is a \( 4 \times 4 \) matrix. Three of the five parameters, say, \( p, q, r, \) can be chosen to be independent, while the remaining parameters, \( s \) and \( \lambda \), are subject to the constraints

\[ rs = pq, \quad \lambda = q - p. \]

By this choice of the parameters, the \( R \)-matrix \( (1) \) satisfies the graded YBE

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \quad (3) \]

with

\[
\begin{align*}
R_{12} &= R \otimes I \equiv R \otimes e^1_i, \quad i = 1, 2, \\
R_{13} &= q(e^1_i \otimes e^1_i \otimes e^1_j) + r(e^1_i \otimes e^1_i \otimes e^2_j) + s(e^2_i \otimes e^1_i \otimes e^1_j) + ( -1)^{[i]} \lambda (e^2_i \otimes e^1_i \otimes e^2_j) + p(e^2_i \otimes e^1_i \otimes e^2_j), \\
R_{23} &= I \otimes R \equiv e^1_i \otimes R, \quad (4)
\end{align*}
\]

where repeated indices are summation indices, \( I \) is the identity operator and the \( \mathbb{Z}_2 \)-grading is given in \((2)\).

Now suppose the operator subject

\[ T = a e^1_1 + \beta e^1_2 + \gamma e^2_1 + d e^2_2 \equiv t^1_i e^1_j, \]

which in the fundamental representation is a \( 2 \times 2 \) matrix,

\[ T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}, \quad (6) \]

obeys the so-called \( RTT \) equation

\[ RT_1 T_2 = T_2 T_1 R, \]

where

\[
\begin{align*}
T_1 &= T \otimes I \equiv (ae^1_1 + \beta e^1_2 + \gamma e^2_1 + de^2_2) \otimes e^1_j, \\
T_2 &= I \otimes T \\
&\equiv e^1_j \otimes [ae^1_1 + ( -1)^{[i]} \beta e^1_2 + ( -1)^{[i]} \gamma e^2_1 + de^2_2].
\end{align*}
\]

The Eq. \((7)\) leads to the commutation relations between the elements of \( T \):

\[
\begin{align*}
a \beta &= \frac{r}{p} \beta a, \quad a \gamma = \frac{q}{r} \gamma a, \quad ad = da + \frac{\lambda}{r} \gamma \beta, \quad \beta^2 = 0 = \gamma^2, \\
\beta \gamma &= - \frac{s}{r} \gamma \beta, \quad \beta d = \frac{p}{r} \gamma \beta d, \quad \gamma d = \frac{r}{q} d \gamma. \quad (9)
\end{align*}
\]

Let us denote \( G \) a set of all operators \((a)\) satisfying \((4)\) and let \( T \) and \( T' \) be two independent copies of \((5)\) in the sense that all elements \( t^1_j \) of \( T \) commute with all those of \( T' \). The fact that the multiplication \( T.T' \) preserves the relation \((7)\), that is, the relations \((9)\), reflects the group nature of \( G \). Next, since the quantity

\[ D(T) \equiv (a - \beta d^{-1} \gamma)d^{-1} = d^{-1}(a - \beta d^{-1} \gamma) = a(d - \gamma a^{-1} \beta)^{-1} \]

commutes with \( T \) and has the “multiplicative” property \( D(T.T') = D(T).D(T') \) it can be identified with a representation of a quantum superdeterminant. Thus we can take \( G \) with \( D(T) \neq 0, \forall T \in G \), as a three-parametric deformation, denoted by \( GL_{p,q,r}(1/1) \), of a representation of \( GL(1/1) \). In the fundamental representation, \( D(T) \) is a quantum superdeterminant of matrix \( T \) in \((10)\) and the corresponding \( GL_{p,q,r}(1/1) \) is a three-parametric deformation of \( GL(1/1) \). When we set \( D(T) = 1 \) we get a three-parametric deformation of \( SL(1/1) \). We note that the form of \( D(T) \) is the same as in \((14)\), that is, it remains non-deformed and belongs to the center of \( GL_{p,q,r}(1/1) \). The Hopf structure is straightforward and given by the following maps:

- the co-product:

\[ \Delta(T) = T \otimes T, \quad (11) \]

- the antipode:

\[ S(T).T = I, \quad (12) \]

- the counit:

\[ \varepsilon(T) = I. \quad (13) \]

In components they read

\[ \Delta(t^1_j) = t^1_j \otimes t^1_k, \]

\[ S(t^1_j t^1_k) = S(t^1_j) t^1_k = a^{-1}(1 + \beta d^{-1} \gamma a^{-1}) t^1_j - (a^{-1} \beta d^{-1}) t^1_k \\
- (d^{-1} \gamma a^{-1}) t^1_j + d^{-1}(1 - \beta a^{-1} d^{-1}) t^1_k. \quad (15) \]
\[ \varepsilon(t^i_j) = \delta^i_j. \] (16)

A quantum superplane with symmetry (automorphism) group \( GL_{p,q,r}(1/1) \) is given by the coordinates
\[ \left( \begin{array}{c} x \\ \theta \\ \eta \\ y \end{array} \right) \] (17)
subject to the commutation relations
\[ x\theta = \frac{q}{p} \theta x, \quad \theta^2 = 0 \quad \text{or} \quad \eta^2 = 0, \quad \eta y = \frac{p}{r} y \eta, \] (18)
respectively. Note that these quantum superplanes (which are "two-dimensional") are still two-parametric (of course, we cannot make relations between two coordinates to depend on more than two parameters). Finally, in order to complete our program we must construct a deformation, denoted below as \( U_{p,q,r}[gl(1/1)] \), of the universal enveloping algebra \( U[gl(1/1)] \) corresponding to the \( R \)-matrix [1].

First, we introduce two auxiliary operators
\[ L^+ = H_1^+ e_1^1 + H_2^+ e_2^2 + \lambda X^+ e_3^1, \]
\[ L^- = H_1^- e_1^1 + H_2^- e_2^2 + \lambda X^- e_3^1, \] (19)
with \( H_i^\pm \) and \( X^\pm \) belonging to \( U_{p,q,r}[gl(1/1)] \) to be constructed. Then, demanding
\[ L_i^\pm = L_i^\pm \otimes e_i^1, \]
\[ L_2^+ = e_1^1 \otimes [H_1^+ e_1^1 + H_2^+ e_2^2 + (-1)^{[i]} \lambda X^+ e_3^1], \]
\[ L_2^- = e_1^1 \otimes [H_1^- e_1^1 + H_2^- e_2^2 + (-1)^{[i]} \lambda X^- e_3^1], \] (20)
to obey the equations
\[ RL_i^1 L_2^2 = L_2^2 L_i^1 R, \] (21)
where \((e_1,e_2) = (+,+)\), \((-,-)\), \((+,-)\), we get the following commutation relations between \( H_i^\pm \) and \( X^\pm \):
\[ H_i^+ H_j^2 = H_j^H H_i^+ \]
\[ p H_i^+ X^+ = r X^+ H_i^+, \quad q H_i^- X^+ = r X^+ H_i^-, \]
\[ r H_i^+ X^- = p X^+ H_i^+, \quad r H_i^- X^- = q X^- H_i^-, \]
\[ r X^+ X^- + s X^- X^+ = \lambda^{-1}(H_2^- H_1^+ - H_2^+ H_1^-), \] (22)
which are taken to be the defining relations of \( U_{p,q,r}[gl(1/1)] \). Its Hopf structure is given by
\[ \Delta(L^\pm) = L^\pm \otimes L^\pm, \] (23)
\[ S(L^\pm) = (L^\pm)^{-1}, \] (24)
\[ \varepsilon(L^\pm) = I, \] (25)
or equivalently (no summation on \( i = 1,2 \)),
\[ \Delta(H_i^\pm) = H_i^\pm \otimes H_i^\pm, \]
\[ \Delta(X^+) = H_1^+ \otimes X^+ + X^+ \otimes H_2^+, \]
\[ \Delta(X^-) = H_2^- \otimes X^- + X^- \otimes H_1^-, \] (26)
\[ S(H_i^+) = (H_i^\pm)^{-1}, \]
\[ S(X^+) = -(H_i^\pm)^{-1} X^+ (H_i^\pm)^{-1}, \]
\[ S(X^-) = -(H_i^\pm)^{-1} X^- (H_i^\pm)^{-1}, \] (27)
\[ \varepsilon(H_i^\pm) = 1, \quad \varepsilon(X^\pm) = 0. \] (28)

At first sight \( U_{p,q,r}[gl(1/1)] \) given in (22) is a three-parametric quantum supergroup depending on three parameters \( p, q \) and \( r \) (or \( s \)). However, making the substitution
\[ H_1^+ = \left( \frac{r}{p} \right)^{E_{11}}, \quad H_2^+ = \left( \frac{p}{r} \right)^{E_{22}}, \]
\[ H_1^- = \left( \frac{r}{q} \right)^{E_{11}}, \quad H_2^- = \left( \frac{q}{r} \right)^{E_{22}}, \]
\[ E_{12} = X^+ r^{E_{22}}, \quad E_{21} = X^- s^{E_{11}}, \] (29)
we obtain a two-parametric deformation of \( U[gl(1/1)] \), namely,
\[ [E_{ii}, E_{jj}] = 0, \]
\[ [E_{ii}, E_{jj} \pm 1] = (\delta_{ij} - \delta_{ij} \pm 1) E_{jj} \pm 1, \]
\[ \{E_{12}, E_{21}\} = [K]_{q,p}, \] (30)
where \( 1 \leq i,j \leq 1 \) and \( 1 \leq i,j \leq 2 \) and
\[ [K]_{q,p} = \frac{q^K p^K}{q - p}, \quad K = E_{11} + E_{22}. \] (31)
The latter deformation is a true two-parametric deformation of \( U[gl(1/1)] \) as it cannot be made to become one-parametric by rescaling its generators. Of course, (20) is not the only realization of the generators of
up,q,r \[ gl(1/1) \] in terms of the deformed Weyl generators \( E_{ij} \).

We have suggested in the present paper an \( R \)-matrix which satisfies a three-parametric graded YBE (and modified Hecke conditions which are not exposed here). Using this \( R \)-matrix we obtained new deformations of \( GL(1/1) \) and \( U[gl(1/1)] \). For conclusion, let us emphasize that the deformation \( GL_{p,q,r}(1/1) \) of \( GL(1/1) \) obtained is a three-parametric quantum group despite the fact that the corresponding deformation \( U_{p,q,r}[gl(1/1)] \) is equivalent to a two-parametric deformation of \( U[gl(1/1)] \). On the other side, however, the introduction of the latter solves a small problem of [11] mentioned above.

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