Dirac Spin Precession in Kerr Spacetime by the parallelism description

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Abstract
In the framework of parallelism general relativity (PGR), the Dirac particle spin precession in the rotational gravitational field is studied. In terms of the equivalent tetrad of Kerr frame, we investigate the torsion axial-vector spin coupling in PGR. In the case of the weak field and slow rotation approximation, we obtain that the torsion axial-vector has the dipole-like structure, but different from the gravitomagnetic field, which indicates that the choice of the Kerr tetrad will influence on the physics interpretation of the axial-vector spin coupling.

key words: torsion, parallelism, Kerr spacetime, spin

1 Introduction
As an extension of Einstein’s general relativity, the metric-affine theory is one of the choices [1, 2, 3], which incorporates the geometric structure of spacetime into curvature and torsion. In order for the fundamental exploration of spacetime, the tetrad theory of gravitation has been paid more attention by many people [4, 5, 6, 7, 8, 9, 10], where the spacetime is characterized by the torsion tensor and the vanishing curvature, the relevant spacetime
is the Weitzenböck spacetime [4], which is a special case of the Riemann-Cartan spacetime, such as the metric-affine theory of gravitation [1, 2, 3]. The tetrad theory of gravitation will be equivalent to general relativity when the convenient choice of the parameters of the Lagrangian [4].

We will use the greek alphabet \((\mu, \nu, \rho, \ldots = 1, 2, 3, 4)\) to denote tensor indices, that is, indices related to spacetime. The latin alphabet \((a, b, c, \ldots = 1, 2, 3, 4)\) will be used to denote local Lorentz (or tangent space) indices. Of course, being of the same kind, tensor and local Lorentz indices can be changed into each other with the use of the tetrad \(h^a_\mu\), which satisfy

\[
h^a_\mu \ h_a^\nu = \delta^\nu_\mu \ ; \ h^\sigma_\mu \ h^\mu_b = \delta^b_\nu . \quad (1)
\]

A nontrivial tetrad field can be used to define the linear Cartan connection[4, 8]

\[
\Gamma^\sigma_\mu\nu = h_a^\sigma \partial_\nu h^a_\mu , \quad (2)
\]

with respect to which the tetrad is parallel:

\[
\nabla_\nu \ h^a_\mu = \partial_\nu h^a_\mu - \Gamma^\rho_\mu\nu h^a_\rho = 0 . \quad (3)
\]

The Cartan connection can be decomposed according to

\[
\Gamma^\sigma_\mu\nu = \check{\Gamma}^\sigma_\mu\nu + K^\sigma_\mu\nu , \quad (4)
\]

where

\[
\check{\Gamma}^\sigma_\mu\nu = \frac{1}{2} g^{\sigma\rho} [\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}] \quad (5)
\]

is the Levi-Civita connection of the metric

\[
g_{\mu\nu} = \eta_{ab} \ h^a_\mu \ h^b_\nu , \quad (6)
\]

where \(\eta^{ab}\) is the metric in flat space with the line element

\[
d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu , \quad (7)
\]

and

\[
K^\sigma_\mu\nu = \frac{1}{2} [T^\sigma_\mu\nu + T^\sigma_\nu\mu - T^\sigma_\mu\nu] \quad (8)
\]
is the contorsion tensor, with

$$T^\sigma_{\mu \nu} = \Gamma^\sigma_{\mu \nu} - \Gamma^\sigma_{\nu \mu}$$

(9)

the torsion of the Cartan connection [4, 8]. The irreducible torsion vectors, i.e., the torsion vector and the torsion axial-vector, can then be constructed as [4, 8]

$$V_\mu = T^\nu_{\nu \mu},$$

(10)

$$A_\mu = \frac{1}{6} \epsilon_{\mu \nu \rho \sigma} T^\nu_{\rho \sigma},$$

(11)

with $\epsilon_{\mu \nu \rho \sigma}$ being the completely antisymmetric tensor normalized as $\epsilon_{0123} = \sqrt{-g}$ and $\epsilon^{0123} = \frac{1}{\sqrt{-g}}$, where $g$ is the determinant of metric.

The spacetime dynamic effects on the spin is incorporated into Dirac equation through the “spin connection” appearing in the Dirac equation in gravitation [4]. In Weitzenböck spacetime, as well as the general version of torsion gravity, it has been shown by many authors [4, 5, 11, 12, 13, 14, 15, 16, 17] that the spin precession of a Dirac particle is intimately related to the torsion axial-vector, and it is interesting to note that the torsion axial-vector represents the deviation of the axial symmetry from the spherical symmetry [5].

$$\frac{dS}{dt} = -\frac{3}{2} A \times S,$$

(12)

where $S$ is the semiclassical spin vector of a Dirac particle, and $A$ is the spacelike part of the torsion axial-vector. Therefore, the corresponding extra Hamiltonian energy is of the form,

$$\delta H = -\frac{3}{2} A \cdot S.$$  

(13)

The purpose of the paper is to derive the torsion axial vector spin coupling in the Kerr spacetime with the given tetrad, which is performed in section 2. In the weak field and slow rotation approximation, the analytical expression of the torsion axial-vector is obtained in section 3. Discussions and conclusions are given in the last section. Throughout this paper we use the unit with $c = 1$.
2 The torsion axial-vector in Kerr spacetime

The gravitational field of a rotating mass is described by the axially symmetric stationary Kerr metric [18],

\[ d\tau^2 = g_{00} dt^2 + g_{11} dr^2 + g_{22} d\theta^2 + g_{33} d\phi^2 + 2 g_{03} d\phi \, dt, \]

where

\[ g_{00} = 1 - \frac{r_s}{\Sigma}; \quad g_{11} = -\frac{\Sigma}{\Delta}; \quad g_{22} = -\Sigma \]

\[ g_{33} = -\left( r^2 + a^2 + \frac{r_s a^2}{\Sigma} \sin^2 \theta \right) \sin^2 \theta \]

\[ g_{03} = g_{30} = \frac{r_s a}{\Sigma} \sin^2 \theta \]

with

\[ \Delta = r^2 - r_s r + a^2 \quad \text{and} \quad \Sigma = r^2 + a^2 \cos^2 \theta. \]

In these expressions, \( r_s \) is Schwarzschild radius and \( a \) is the angular momentum of a gravitational unit mass source. If \( a = 0 \), the Kerr metric becomes the Schwarzschild metric in the standard form. In Kerr spacetime, the tetrad can be expressed by the dual basis of the differential one-form [19] through choosing a coframe of the coordinate system,

\[ d\vartheta^\mu = \begin{cases} A [dt - a \sin^2 \theta d\phi], \\ A^{-1} dr, \\ \sqrt{\Sigma} d\theta, \\ B [-(adt + (r^2 + a^2) d\phi)], \end{cases} \]

where \( A = \sqrt{\Delta/\Sigma}, \ B = \sin \theta/\sqrt{\Sigma}. \) Therefore, the tetrad can be obtained with the subscript \( \mu \) denoting the column index (c.f. [?]),

\[ h^\alpha_{\ \mu} = \begin{pmatrix} A & 0 & 0 & -aA \sin^2 \theta \\ 0 & 1/A & 0 & 0 \\ 0 & 0 & \sqrt{\Sigma} & 0 \\ -aB & 0 & 0 & (r^2 + a^2)B \end{pmatrix}, \]
where \( h = \det(h^a_\mu) = \Sigma \sin \theta = \sqrt{-g} \) with \( g = \det(g_{\mu\nu}) \). We can inspect that Eqs. (23) and (24) satisfy the conditions in Eq. (1). From Eqs. (23) and (24), we can now construct the Cartan connection, whose nonvanishing components are:

\[
\begin{align*}
    h\Gamma^0_{01} &= (A'/A)(r^2 + a^2) \sin \theta - B' a^2 \sqrt{\Sigma} \sin^2 \theta, \\
    h\Gamma^0_{31} &= -(A'/A)a(r^2 + a^2) \sin^3 \theta + [B(r^2 + a^2)]' \sqrt{\Sigma} \sin^2 \theta, \\
    h\Gamma^0_{02} &= (A'/A)(r^2 + a^2) \sin \theta - a^2 \sin^2 \theta \cos \theta = a^4 \sin^4 \theta \cos \theta / \Sigma, \\
    h\Gamma^0_{32} &= -[A \sin^2 \theta]'a \sin \theta (r^2 + a^2)/A + B'(r^2 + a^2)a \sqrt{\Sigma} \sin^2 \theta, \\
    h\Gamma^1_{12} &= -a^2 \sin^2 \theta \cos \theta, \\
    h\Gamma^2_{21} &= r \sin \theta,
\end{align*}
\]

\[
\begin{align*}
    h\Gamma^3_{01} &= (A'/A) a \sin \theta - B' a \sqrt{\Sigma}, \\
    h\Gamma^3_{31} &= -(A'/A)a^2 \sin^2 \theta + [B(r^2 + a^2)]' \sqrt{\Sigma}, \\
    h\Gamma^3_{02} &= (A'/A)a \sin \theta - B' \sqrt{\Sigma} a = (a^2 / \Sigma - 1)a \cos \theta, \\
    h\Gamma^3_{32} &= -(A \sin^2 \theta)'a^2 \sin \theta / A + B'(a^2 + r^2) \sqrt{\Sigma},
\end{align*}
\]

where

\[
\begin{align*}
    A' &= \frac{\partial A}{\partial r} = (r - r_s/2)/(A \Sigma) - Ar / \Sigma, \\
    A'_\theta &= \frac{\partial A}{\partial \theta} = \frac{a^2 A}{\Sigma} \sin \theta \cos \theta, \\
    B' &= \frac{\partial B}{\partial r} = -Br / \Sigma, \\
    B'_\theta &= \frac{\partial B}{\partial \theta} = \cos \theta / \sqrt{\Sigma} + \frac{a^2 B}{\Sigma} \sin \theta \cos \theta.
\end{align*}
\]

The nonzero torsion axial–vector components are

\[
\begin{align*}
    A^{(1)} \times (6h) &= -2(g_{00}T^0_{23} + g_{03}T^3_{23} + g_{30}T^0_{02} + g_{33}T^3_{02}) \\
    &= -2(g_{00}T^0_{23} + g_{03}T^3_{23} + g_{30}T^0_{02} + g_{33}T^3_{02}), \\
    A^{(2)} \times (6h) &= 2(g_{00}T^0_{13} + g_{03}T^3_{13} + g_{30}T^0_{01} + g_{33}T^3_{01}) \\
    &= 2[g_{00}T^0_{31} + g_{03}T^3_{31} - g_{30}T^0_{01} - g_{33}T^3_{01}].
\end{align*}
\]
3 Slow Rotation and Weak Field Approximations

In the case of slow rotation and weak field, we keep the terms up to first order in the angular momentum $a$ and in $r_s/r$. The related quantities are simplified as follows:

$$\Delta = r^2 - r_s r; \quad \Sigma = r^2$$

(38)

$$g_{00} = (-g_{11})^{-1} = 1 - \frac{r_s}{r}; \quad g_{22} = -r^2$$

(39)

$$g_{33} = -r^2 \sin^2 \theta; \quad g_{03} = \frac{r_s a}{r} \sin^2 \theta$$

(40)

$$h = r^2 \sin \theta; \quad A = \sqrt{1 - r_s/r}; \quad B = \sin \theta/r.$$  

(41)

In this approximation, all terms reduce to the values of the Schwarzschild solution except $g_{03}$. On the other hand, in the weak field limit, characterized by keeping terms up to first order in $r_s/r$, the nonzero components of the axial–vector torsion become

$$h \Gamma^0_{32} = -ar^2 \sin^2 \theta \cos \theta, \quad h \Gamma^3_{32} = r^2 \cos \theta,$$

$$h \Gamma^0_{02} = 0, \quad h \Gamma^3_{02} = -a \cos \theta.$$  

(42)

and

$$h \Gamma^0_{31} = ar \sin^3 \theta(1 - r_s/2r), \quad h \Gamma^3_{31} = r \sin \theta,$$

$$h \Gamma^0_{01} = (r_s/2) \sin \theta, \quad h \Gamma^3_{01} = (a \sin \theta/r)(1 + r_s/2r).$$

(43)

Substituting Eqs.(38), (39), (40) and (41) into (36) and (37), we obtain

$$A^{(1)} = \frac{2}{3} (1 - r_s/r) \frac{a}{r^2} \cos \theta,$$

(44)

and

$$A^{(2)} = \frac{2}{3} \frac{a}{r^3} \sin \theta.$$  

(45)

In spacelike vector form, the axial–vector becomes,

$$-\mathbf{A} = A^{(1)} \sqrt{-g_{11}} \mathbf{e}_r + A^{(2)} \sqrt{-g_{22}} \mathbf{e}_\theta,$$

(46)
where $e_r = \sqrt{-g_{11}} \, dr$ and $e_{\theta} = \sqrt{-g_{22}} \, d\theta$ are unit vectors in $(r, \theta)$ directions, and then we have,

$$-\mathbf{A} = \frac{2a}{3r^2} \left[\sqrt{1 - r_s/r} \, \cos \theta \, e_r + \sin \theta \, e_{\theta}\right]. \tag{47}$$

It has been shown by many authors [4, 12, 13, 14] that the spin precession of a Dirac particle in torsion gravity is intimately related to the axial-vector

$$\frac{ds}{dt} = \mathbf{b} \times \mathbf{s}, \tag{48}$$

where $\mathbf{s}$ is the spin vector, and $\mathbf{b} = -3\mathbf{A}/2$. Therefore,

$$\mathbf{b} = \frac{J}{Mr^2} \left[\sqrt{1 - r_s/r} \, \cos \theta \, e_r + \sin \theta \, e_{\theta}\right]. \tag{49}$$

with $J = Ma$ the angular momentum.

4 Discussions and conclusions

The torsion axial-vector Dirac spin coupling by the special choice of the Kerr tetrad in the framework of the torsion spacetime without curvature has been derived. We employ the given Kerr tetrad to derive the torsion axial-vector, as one of the three irreducible quantities in Weitzenböck spacetime, which will couple with the Dirac spin. Unlike the previous work where another Kerr tetrad is used [9], we have not obtained the gravitomagnetic spin coupling here, which indicates that the choice of the tetrad incurs the preferred reference frame where the physics measurement is performed. It is worth noting the implications of some special cases from the analysis of the $\mathbf{b}$ field in Eq.(49), and we find that $\mathbf{b}$ field is still a dipole-like field although it is not a standard dipole gravitomagnetic field as obtained before [9], which is on account of the axisymmetric property of Kerr spacetime. If we set the gravitational constant $G=0$ or $r_s = 0$, say the null gravitational field, then we find that $\mathbf{b}$ field is not vanished. This fact shows that $G=0$ will arise the spacetime curvature to be zero, but the torsion would not be cancelled automatically. The similar phenomenon has also been found when we deal with the rotation spin coupling in the rotational system, where the nonzero Cartan connections still survive in the Minkowski spacetime [20]. Of course,
if there is no rotation, say $a=0$, then the $b$ field disappears because the axial-vector represents the measurement of the axial symmetry deviated from the spherical symmetry [5].

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is the Weitzenböck spacetime [4], which is a special case of the Riemann-Cartan spacetime, such as the metric-affine theory of gravitation [1, 2, 3]. The tetrad theory of gravitation will be equivalent to general relativity when the convenient choice of the parameters of the Lagrangian [4].

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$$h^a_\mu h^a_\nu = \delta^a_\mu ; \quad h^a_\mu h^a_\rho = \delta^a_\rho . \quad (1)$$

A nontrivial tetrad field can be used to define the linear Cartan connection[4, 8]

$$\Gamma^a_\mu \nu = h^a_\sigma \partial_\nu h^\sigma_\mu ; \quad (2)$$

with respect to which the tetrad is parallel:

$$\nabla_\nu h^a_\mu \equiv \partial_\nu h^a_\mu - \Gamma^a_\mu \rho h^a_\rho = 0 . \quad (3)$$

The Cartan connection can be decomposed according to

$$\Gamma^a_\mu \nu = \overset{o}{\Gamma}^a_\mu \nu + K^a_\mu \nu ; \quad (4)$$

where

$$\overset{o}{\Gamma}^a_\mu \nu = \frac{1}{2} g^a_\rho [\partial_\mu g_{\nu \rho} + \partial_\nu g_{\rho \mu} - \partial_\rho g_{\mu \nu}] \quad (5)$$

is the Levi-Civita connection of the metric

$$g_{\mu \nu} = \eta_{ab} h^a_\mu h^b_\nu , \quad (6)$$

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$$d\tau^2 = g_{\mu \nu} dx^\mu dx^\nu , \quad (7)$$

and

$$K^a_\mu \nu = \frac{1}{2} \left[ T^a_\mu \nu + T^a_\nu \mu - T^a_\sigma \mu \right] \quad (8)$$
is the contorsion tensor, with

\[ T^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu} \]  

(9)

the torsion of the Cartan connection [4, 8]. The irreducible torsion vectors, i.e., the torsion vector and the torsion axial-vector, can then be constructed as [4, 8]

\[ V_\mu = T^\nu_{\nu\mu}, \]  

(10)

\[ A_\mu = \frac{1}{6} \epsilon_{\mu\nu\rho\sigma} T^\nu_{\nu\mu}, \]  

(11)

with \( \epsilon_{\mu\nu\rho\sigma} \) being the completely antisymmetric tensor normalized as \( \epsilon_{0123} = \sqrt{-g} \) and \( \epsilon^{0123} = \frac{1}{\sqrt{-g}} \), where \( g \) is the determinant of metric.

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(12)

where \( S \) is the semiclassical spin vector of a Dirac particle, and \( A \) is the spacelike part of the torsion axial-vector. Therefore, the corresponding extra Hamiltonian energy is of the form,

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(13)

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2 The torsion axial-vector in Kerr spacetime

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(14)

where

\[ g_{00} = 1 - \frac{r_s r}{\Delta}; \quad g_{11} = -\frac{\Delta}{r_s r}; \quad g_{22} = -\Sigma \]

(15)

\[ g_{33} = -\left( r^2 + a^2 + \frac{r_s r a^2}{\Sigma} \sin^2 \theta \right) \sin^2 \theta \]

(16)

\[ g_{03} = g_{30} = \frac{r_s r a}{\Sigma} \sin^2 \theta \]

(17)

with

\[ \Delta = r^2 - r_s r + a^2 \quad \text{and} \quad \Sigma = r^2 + a^2 \cos^2 \theta. \]

(18)

In these expressions, \( r_s \) is Schwarzschild radius and \( a \) is the angular momentum of a gravitational unit mass source. If \( a = 0 \), the Kerr metric becomes the Schwarzschild metric in the standard form. In Kerr spacetime, the tetrad can be expressed by the dual basis of the differential one-form [19] through choosing a coframe of the coordinate system,

\[ d\varphi^0 = A [dt - a \sin^2 \theta d\phi], \]

(19)

\[ d\varphi^1 = A^{-1} dr, \]

(20)

\[ d\varphi^2 = \sqrt{\Sigma} d\theta, \]

(21)

\[ d\varphi^3 = B [(-ad\theta + (r^2 + a^2) d\phi], \]

(22)

where \( A = \sqrt{\Delta/\Sigma} \), \( B = \sin \theta / \sqrt{\Sigma} \). Therefore, the tetrad can be obtained with the subscript \( \mu \) denoting the column index (c.f. [?]).

\[ h^a_{\mu} = \begin{pmatrix}
A & 0 & 0 & -aA \sin^2 \theta \\
0 & 1/A & 0 & 0 \\
0 & 0 & \sqrt{\Sigma} & 0 \\
-aB & 0 & 0 & (r^2 + a^2)B
\end{pmatrix}, \]

(23)
with the inverse
\[ hh^a_{\mu} = \begin{pmatrix} (r^2 + a^2) \sin \theta / A & 0 & 0 & a \sin \theta / A \\ 0 & A \Sigma \sin \theta & 0 & 0 \\ 0 & 0 & B \Sigma & 0 \\ a \sqrt{\Sigma \sin^2 \theta} & 0 & 0 & \sqrt{\Sigma} \end{pmatrix}, \tag{24} \]
where \( h = \det(h^a_{\mu}) = \Sigma \sin \theta = \sqrt{-g} \) with \( g = \det(g_{\mu \nu}) \). We can inspect that Eqs. (23) and (24) satisfy the conditions in Eq. (1). From Eqs. (23) and (24), we can now construct the Cartan connection, whose nonvanishing components are:
\[ h \Gamma^0_{01} = (A'/A)(r^2 + a^2) \sin \theta - B' a^2 \sqrt{\Sigma} \sin^2 \theta, \tag{25} \]
\[ h \Gamma^0_{31} = -(A'/A) a(r^2 + a^2) \sin^3 \theta + [B(r^2 + a^2)]' a \sqrt{\Sigma} \sin^2 \theta, \tag{26} \]
\[ h \Gamma^0_{02} = (A'_\theta / A)(r^2 + a^2) \sin \theta - a^2 \sin^2 \theta \cos \theta = a^4 \sin^4 \theta \cos \theta / \Sigma, \tag{27} \]
\[ h \Gamma^0_{32} = -[A \sin^2 \theta]' a \sin \theta (r^2 + a^2) / A + B'(r^2 + a^2)a \sqrt{\Sigma} \sin^2 \theta, \tag{28} \]
\[ h \Gamma^1_{12} = -a^2 \sin^2 \theta \cos \theta, \quad h \Gamma^2_{21} = r \sin \theta, \tag{29} \]
\[ h \Gamma^3_{01} = (A'/A) a \sin \theta - B' a \sqrt{\Sigma}, \tag{30} \]
\[ h \Gamma^3_{31} = -(A'/A) a^2 \sin^3 \theta + [B(r^2 + a^2)]' \sqrt{\Sigma}, \tag{31} \]
\[ h \Gamma^3_{02} = (A'_\theta / A) a \sin \theta - B'_\theta \sqrt{\Sigma} a = (a^2 / \Sigma - 1) a \cos \theta, \tag{32} \]
\[ h \Gamma^3_{32} = -(A \sin^2 \theta)' a^2 \sin \theta / A + B'(a^2 + r^2) \sqrt{\Sigma}, \tag{33} \]
where
\[ A' = \frac{\partial A}{\partial r} = (r - r_s / 2) / (A \Sigma) - A r / \Sigma, \quad A'_\theta = \frac{\partial A}{\partial \theta} = \frac{a^2 A}{\Sigma} \sin \theta \cos \theta, \tag{34} \]
\[ B' = \frac{\partial B}{\partial r} = -B r / \Sigma, \quad B'_\theta = \frac{\partial B}{\partial \theta} = \cos \theta / \sqrt{\Sigma} + \frac{a^2 B}{\Sigma} \sin \theta \cos \theta. \tag{35} \]

The nonzero torsion axial–vector components are
\[ A^{(1)} \times (6h) = -2(9_0 T_{02}^0 + 9_0 T_{23}^0 + 9_0 T_{02}^0 + 9_3 T_{02}^0) \tag{36} \]
\[ = -2(9_0 \Gamma_{32}^0 + 9_3 \Gamma_{32}^0 - 9_3 \Gamma_{02}^0 - 9_3 \Gamma_{02}^0), \]
\[ A^{(2)} \times (6h) = 2[9_0 T_{13}^0 + 9_0 T_{13}^0 + 9_3 T_{01}^0 + 9_3 T_{01}^0] \tag{37} \]
\[ = 2[9_0 \Gamma_{31}^0 + 9_3 \Gamma_{31}^0 - 9_3 \Gamma_{01}^0 - 9_3 \Gamma_{01}^0]. \]
3 Slow Rotation and Weak Field Approximations

In the case of slow rotation and weak field, we keep the terms up to first order in the angular momentum $a$ and in $r_s/r$. The related quantities are simplified as follows:

$$\Delta = r^2 - r_s r; \quad \Sigma = r^2$$

(38)

$$g_{00} = (-g_{11})^{-1} = 1 - \frac{r_s}{r}; \quad g_{22} = -r^2$$

(39)

$$g_{33} = -r^2 \sin^2 \theta; \quad g_{03} = \frac{r_s a}{r} \sin^2 \theta$$

(40)

$$h = r^2 \sin \theta; \quad A = \sqrt{1 - r_s/r}; \quad B = \sin \theta/r.$$  

(41)

In this approximation, all terms reduce to the values of the Schwarzschild solution except $g_{03}$. On the other hand, in the weak field limit, characterized by keeping terms up to first order in $r_s/r$, the nonzero components of the axial–vector torsion become

$$h \Gamma^0_{32} = -ar^2 \sin^2 \theta \cos \theta, \quad h \Gamma^3_{32} = r^2 \cos \theta,$$

$$h \Gamma^0_{02} = 0, \quad h \Gamma^3_{02} = -a \cos \theta,$$

(42)

and

$$h \Gamma^0_{31} = ar \sin^3 \theta (1 - r_s/2r), \quad h \Gamma^3_{31} = r \sin \theta,$$

$$h \Gamma^0_{01} = (r_s/2) \sin \theta, \quad h \Gamma^3_{01} = (a \sin \theta/r)(1 + r_s/2r).$$

(43)

Substituting Eqs. (38), (39), (40) and (41) into (36) and (37), we obtain

$$A^{(1)} = \frac{2}{3}(1 - r_s/r) \frac{a}{r^2} \cos \theta,$$

(44)

and

$$A^{(2)} = \frac{2a}{3} \frac{1}{r^3} \sin \theta.$$

(45)

In spacelike vector form, the axial–vector becomes,

$$-\mathbf{A} = A^{(1)} \sqrt{-g_{11}} \mathbf{e}_r + A^{(2)} \sqrt{-g_{22}} \mathbf{e}_\theta,$$

(46)
where \( \mathbf{e}_r = \sqrt{-g_{rr}} \, dr \) and \( \mathbf{e}_\theta = \sqrt{-g_{\theta\theta}} \, d\theta \) are unit vectors in \((r, \theta)\) directions, and then we have,

\[
-A = \frac{2a}{3r^2} \left[ \sqrt{1 - r_s/r} \cos \theta \, \mathbf{e}_r + \sin \theta \, \mathbf{e}_\theta \right].
\]

(47)

It has been shown by many authors \([4, 12, 13, 14]\) that the spin precession of a Dirac particle in torsion gravity is intimately related to the axial-vector

\[
\frac{ds}{dt} = \mathbf{b} \times \mathbf{s},
\]

(48)

where \( \mathbf{s} \) is the spin vector, and \( \mathbf{b} = -3A/2 \). Therefore,

\[
\mathbf{b} = \frac{J}{Mr^2} \left[ \sqrt{1 - r_s/r} \cos \theta \, \mathbf{e}_r + \sin \theta \, \mathbf{e}_\theta \right].
\]

(49)

with \( J = Ma \) the angular momentum.

4 Discussions and conclusions

The torsion axial-vector Dirac spin coupling by the special choice of the Kerr tetrad in the framework of the torsion spacetime without curvature has been derived. We employ the given Kerr tetrad to derive the torsion axial-vector, as one of the three irreducible quantities in Weitzenböck spacetime, which will couple with the Dirac spin. Unlike the previous work where another Kerr tetrad is used \([9]\), we have not obtained the gravitomagnetic spin coupling here, which indicates that the choice of the tetrad incurs the preferred reference frame where the physics measurement is performed. It is worth noting the implications of some special cases from the analysis of the \( \mathbf{b} \) field in Eq.(49), and we find that \( \mathbf{b} \) field is still a dipole-like field although it is not a standard dipole gravitomagnetic field as obtained before \([9]\), which is on account of the axisymmetric property of Kerr spacetime. If we set the gravitational constant \( G=0 \) or \( r_s = 0 \), say the null gravitational field, then we find that \( \mathbf{b} \) field is not vanished. This fact shows that \( G=0 \) will arise the spacetime curvature to be zero, but the torsion would not be cancelled automatically. The similar phenomenon has also been found when we deal with the rotation spin coupling in the rotational system, where the nonzero Cartan connections still survive in the Minkowski spacetime \([20]\). Of course,
if there is no rotation, say a=0, then the b field disappears because the axial-vector represents the measurement of the axial symmetry deviated from the spherical symmetry [5].

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