Supersymmetric Gauge Theories, Vortices and Equivariant Cohomology

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Abstract
We construct actions for $(p, 0)$- and $(p, 1)$- supersymmetric, $1 \leq p \leq 4$, two-dimensional gauge theories coupled to non-linear sigma model matter with a Wess-Zumino term. We derive the scalar potential for a large class of these models. We then show that the Euclidean actions of the $(2, 0)$ and $(4, 0)$-supersymmetric models without Wess-Zumino terms are bounded by topological charges which involve the equivariant extensions of the Kähler forms of the sigma model target spaces evaluated on the two-dimensional spacetime. We give similar bounds for Euclidean actions of appropriate gauge theories coupled to non-linear sigma model matter in higher spacetime dimensions which now involve the equivariant extensions of the Kähler forms of the sigma model target spaces and the second Chern character of gauge fields. The BPS configurations are generalisations of abelian and non-abelian vortices.
1 Introduction

Two-dimensional field theories have found many applications in string theory. For example, two-dimensional sigma models and some two-dimensional gauge theories have been used to model the dynamics of fundamental and D-strings, respectively. The small fluctuations of strings which arise as intersections in various brane configurations are described by two-dimensional gauge theories coupled to scalars. Because of this, many of the properties and the various objects that arise in gauge theories coupled to scalars have a brane interpretation. Supersymmetric gauge theories coupled to linear sigma models have been constructed and they have been used to illuminate the relation between Landau-Ginzburg models and Calabi-Yau spaces. Recently a two-dimensional gauged theory coupled to a linear sigma model was used to investigate aspects of the dynamics of vortices using branes.

In two dimensions, the Wess-Zumino term has the same mass dimension as the kinetic term of sigma model scalars. Therefore two-dimensional supersymmetric gauged theories can couple to non-linear sigma model matter which also has a non-vanishing Wess-Zumino coupling. Such a theory is renormalizable. The gauging of supersymmetric two-dimensional non-linear sigma models with a Wess-Zumino term has been considered in [7, 8]. However in these papers the part of the action which involves the gauge field kinetic terms has not been given. It has been found in [7] that the Wess-Zumino term of a non-linear sigma model cannot always be gauged. The conditions for gauging a Wess-Zumino term have been identified as the obstructions to the extension of the closed form associated with the Wess-Zumino term to an element of the equivariant cohomology [9] of the sigma model target space [10]. Scalar potentials for supersymmetric two-dimensional sigma models with Wess-Zumino term have been investigated in [11, 12, 13, 14].

In this paper we shall construct the actions of (p,0)- and (p,1)-supersymmetric, \(1 \leq p \leq 4\), two-dimensional gauge theories coupled to non-linear sigma model matter and with non-vanishing Wess-Zumino term. In addition we shall also consider the scalar potentials that arise in these theories. This will generalise various partial results that have already appeared in the literature. To simplify the description of the results from here on, we shall use the term ‘sigma models’ instead of the term ‘non-linear sigma models’ unless otherwise explicitly stated. The method we shall use to construct the various actions of supersymmetric two-dimensional gauged theories coupled to sigma models is based on the superfields found in the context of supersymmetric sigma models [15, 16] and later used in the context supersymmetric gauged sigma models [17]. One advantage of this method is that it keeps manifest the various geometric properties of the couplings that appear in these theories. This will be used in the second part of the paper to construct of various bounds for vortices. Since the parts of the actions that we shall describe involving the kinetic term of the sigma model scalars, the Wess-Zumino term and their couplings to gauge fields are known, we shall focus on the kinetic term of the gauge fields and the scalar potentials of these theories. We shall allow the gauge couplings to depend on the sigma...
model scalars and we shall derive the various conditions on these couplings required by gauge invariance and supersymmetry. We shall find that the scalar potential of gauge theories coupled to sigma models in two dimensions, even in the presence of Wess-Zumino term, is the sum of a ‘F’ term or a ‘D’ term or both. The presence of a D-term, or Fayet-Iliopoulos term, may come as a surprise. This is because in the presence of a Wess-Zumino term the geometry of the target space, say, of (2,0)-supersymmetric models where such a term is expected, is not Kähler but Kähler with torsion (KT). Thus the Kähler form is not closed and there are no obvious moment maps. However, it has been shown in [17] that KT geometries under certain conditions admit moment maps and are those that appear in the Fayet-Iliopoulos term of these models. Similar results hold for the (4,0)-supersymmetric models. We shall observe that the gauged (p,1), p = 1, 2, 4, multiplets are associated with scalar superfields. For the gauge theories with (2,1) and (4,1) supersymmetry, these scalar multiplets satisfy the same supersymmetry constraints as the associated sigma model multiplets. Therefore these gauge theories can be thought of as sigma models with target spaces LG ⊗ R^p, p = 1, 2, 4. This will allow us to combine the (p,1) gauge multiplet and the standard sigma model (p,1) multiplet to a new sigma model multiplet. As a result, sigma model type of actions can be written for these gauge theories coupled to matter for which the associated couplings depend on the scalar fields of both the sigma model and gauge multiplets. This generalizes the results of [8].

In the second part of this paper, we shall show that the Euclidean actions of (2,0)- and (4,0)-supersymmetric two-dimensional gauge theories coupled to sigma models with a Fayet-Iliopoulos term but with vanishing Wess-Zumino term admit bounds. In particular we shall find that the Euclidean action SE of the (2,0)-supersymmetric theory is bounded by the absolute value of a topological charge Q which is the integral over the two-dimensional spacetime of the equivariant extension of the Kähler form of its sigma model target space, SE ≥ |Q|. The sigma model manifold in the (4,0)-supersymmetric theory is hyper-Kähler and so there are three Kähler forms each having an equivariant extension. The Euclidean action SE of the (4,0)-supersymmetric theory is bounded by the length of the three topological charges Q₁, Q₂, Q₃ each associated with the integral over the two-dimensional spacetime of the equivariant extensions of the three Kähler forms, SE ≥ √Q₁² + Q₂² + Q₃². This is another application of the equivariant cohomology in the context of two-dimensional gauged sigma models which is distinct from that found in [10] and we have mentioned above. (For many other applications see for example [18].) The configurations that saturate these bounds are vortices and include the Nielsen-Olesen type of vortices associated with gauge theories coupled to linear sigma models. In particular the bounds above generalise that found by Bogomol’nyi in [20] for the abelian vortices of a gauge theory coupled to a single linear complex scalar field.

We also find that similar bounds exist in higher dimensions for action type of functionals that involve maps between Kähler manifolds of any dimension coupled to gauge fields or maps from a Kähler manifold into a hyper-Kähler manifold again coupled to gauge fields. The structure of these functionals is
such that it includes the Euclidean actions of some supersymmetric gauge theories in higher dimensions coupled to sigma models with Fayet-Iliopoulos terms. In particular the first case, which involves maps between two-Kähler spaces, includes the Euclidean action of a four-dimensional $N = 1$ supersymmetric gauge theory coupled to a sigma model. The latter case, which involved maps from a Kähler manifold into a hyper-Kähler one, can be associated with the Euclidean action of a four-dimensional $N = 2$ supersymmetric gauge theory coupled to a sigma model. Note that in $N = 1$ theories in four dimensions the sigma model target space is Kähler while in the $N = 2$ theories in four dimensions the sigma model target space is hyper-Kähler. In all these cases the action functionals are bounded by topological charges which involve the equivariant extensions of the Kähler forms of the sigma model target space as well as the second Chern character of the gauge fields. Our results are different from those of [21, 22] for non-abelian vortices which involve non-abelian gauge theories coupled to linear sigma model matter. Note that in the bound constructed in [21], the topological term involves the first class and second Chern character of the gauge fields instead of the equivariant extension of the Kähler form and the second Chern character of the gauge field that we find. It turns out that in the case of gauge theories coupled to linear sigma models of [1] the two different topological charges can be related, see also [23]. However this involves a partial integration procedure in which various surface terms are taken to vanish. We remark that in the construction of the bounds that involve the equivariant extensions of the Kähler forms, and also in [20], the topological terms are identified with what remains after writing the Euclidean actions of the theories as a sum of squares without the use of partial integrations. It is clear that our results can also be used to construct bounds for solitons in appropriate gauge theories coupled to sigma model matter in odd-spacetime dimensions. This is the usual situation where instantons in a n-dimensional theory can be thought of as static solitons of a $(n+1)$-dimensional theory. In particular, there is a bound for the energy of static configurations of a three-dimensional $N = 2$ supersymmetric gauge theory coupled to sigma model matter. This new bound is an extension to gauged theories of the of bounds found in [24] and generalizes that of [22].

This paper is organised as follows: In section two, we shall describe non-supersymmetric gauge theories coupled to sigma model matter to set up our notation and give some universal conditions which appear in all such models. In section three, we shall describe the (1,0)-supersymmetric gauge theories coupled to sigma model matter with a Wess-Zumino term. We shall also give the scalar potential of the system. In sections four and five, we shall describe the (2,0)- and (4,0)-supersymmetric gauge theories coupled to sigma model matter and give the scalar potentials of the systems. We shall find that these systems have Fayet-Iliopoulos terms constructed from the moment maps of KT and HKT geometries, respectively. In section six, we shall describe the (1,1)-supersymmetric gauge theory coupled to sigma model matter. We shall find that both the gauge multiplet and the sigma model multiplet are scalar superfields allowing for non-polynomial interactions between them. In sections seven and eight, we shall describe the (2,1)- and (4,1)-supersymmetric gauge theories
coupled to sigma model matter and give the scalar potentials of the systems. In both these theories the gauge multiplets and the sigma model multiplets are scalar superfields and therefore admit non-polynomial interactions. In section nine, we describe a bound for the Euclidean action of a (2,0)-supersymmetric gauge theory without Wess-Zumino term. We show that the equivariant extension of the Kähler class enters in the bound. The BPS configurations are vortices and the scalar fields take values in a curved manifold $M$. In section ten, we describe a bound for the Euclidean action of a (4,0)-supersymmetric gauge theory without Wess-Zumino term. We show that the topological term in the bound is a linear combination equivariant extensions of the Kähler classes of the hyper-Kähler manifold which is the sigma model target space. In sections eleven and twelve, we present a generalisation of the bounds we have described for (2,0)- and (4,0)-supersymmetric models to a class of gauge theories coupled to scalars which are maps between Kähler manifolds or from a Kähler manifold to a hyper-Kähler, respectively. The topological term this time involves, apart from the equivariant extensions of Kähler forms, the second Chern character of the gauge connection. Finally in section thirteen, we give our conclusions.

2 Two-dimensional gauged sigma models with Wess-Zumino term

2.1 Geometric Data and Action

To describe two-dimensional supersymmetric gauge theories coupled to sigma model matter with a Wess-Zumino term, it is instructive to begin with a non-supersymmetric system as a toy example. Let $\Xi$ be the two-dimensional Minkowski spacetime with light-cone coordinates $(x^+, x^-)$. The fields of the model that we shall consider here are the following: a gauge potential $A$ with gauge group $G$, sigma model matter fields $\phi$ which are locally maps from $\Xi$ into a sigma model manifold or target space $M$ and real fermions $\psi^-$ and $\lambda^+$ on $\Xi$ of opposite chirality.

The couplings of two-dimensional gauged sigma model are described by a Riemannian metric $g$ on $M$ and the Wess-Zumino term which is a locally-defined two-form $b$ on $M$; $H = db$ is a globally defined closed three-form on $M$. In addition, the gauge group $G$ acts on $M$ leaving invariant both the metric and the Wess-Zumino term, ie

$$L_\alpha g = 0 \quad L_\alpha H = 0$$

where $L_\alpha$ is the Lie-derivative with respect to the vector fields $\{\xi_\alpha : a = 1, \ldots, \text{dim} L G\}$ generated by the action of the gauge group $G$ on $M$; $L G$ is the Lie algebra of $G$. Therefore, we have

$$[\xi_\alpha, \xi_\beta]^i = -f_{\alpha \beta}^\gamma \xi_\gamma^i,$$
where $f$ are the structure constants of $G$; $a, b, c = 1, \ldots, \dim L G$ are gauge indices. The first condition in (2.1) implies that $\xi_a$ are Killing vectors, ie

$$
\nabla_i \xi_{aj} + \nabla_j \xi_{ai} = 0
$$

(2.3)

where $\nabla$ is the Levi-Civita connection of $g$ and $i, j = 1, \ldots, \dim M$. The second condition in (2.1) together with $dH = 0$ imply that $i_a H$ is closed and so

$$
\xi_a^i H_{ijk} = 2 \partial_j w_{ka}
$$

(2.4)

for some locally defined one-form $w_a$.

It is useful to give a geometric interpretation for the gauge potential $A$ of the sigma model field $\phi$. Let $P$ be a principal bundle over the spacetime $\Xi$ with fibre the gauge group $G$. The gauge potential $A$ is locally the pull-back of a connection one-form $A$ of $P$ onto a open set of the spacetime $\Xi$. The sigma model maps $\phi$ are sections of the bundle $P \times_G M$. Locally they can be represented as maps from the spacetime $\Xi$ into $M$.

To describe the couplings of the fermions, we consider two vector bundles $E$ and $F$ over $M$ equipped with connections $B$ and $C$ and with fibre metrics $h$ and $k$, respectively. The fermions $\psi_-$ and $\lambda_+$ can be thought of as sections of $S_- \otimes E$ and $S_+ \otimes F$, respectively, where $S_- \text{ and } S_+$ are spin bundles over $\Xi$ associated to the two inequivalent real representation of $Spin(1,1)$. Note that in two dimensions there are Majorana-Weyl fermions and so two inequivalent one-dimensional real spinor representations of $Spin(1,1)$. In addition we shall assume that the connections $B$ and $C$ as well as the fibre metrics $h$ and $k$ are invariant under the action of the gauge group on $M$. These conditions imply that

$$
\mathcal{L}_a B^A_B = - \nabla_i U_a^A_B
$$

(2.5)

$$
\mathcal{L}_a h_{AB} = - U_a^C A h_{CB} - U_a^C B h_{AC},
$$

(2.6)

where

$$
\nabla_i U_a^A_B = \partial_i U_a^A_B + B_i^C U_a^C_B - U_a^A_C B_i^C_B,
$$

(2.7)

and $U_a$ are infinitesimal gauge transformations, $A, B, C = 1, \ldots, \text{rank } E$, and similarly for the connection $C$ and fibre metric $k$. The above conditions on the connection $B$ have appeared in (2.3). An action for the fields $A, \phi, \psi_-$ and $\lambda_+$ is

$$
S = \int d^2 x \left( u_{ab} F_{aB}^b F_{bA}^a + g_i U_a^A_B \nabla_+^i \phi^j \nabla_-^j \phi^k - w_{ia} \partial_i \phi^j F_{aB}^j \right)
$$

$$
+ \int d^2 x dt \left( H_{ijk} \partial_i \phi^j \nabla_+^k \nabla_-^k \phi^k - w_{ia} \partial_i \phi^j F_{aB}^j \right)
$$

$$
+ \int d^2 x \left( i h_{AB} \psi_+^A \nabla_+ \psi_-^B - i k_{A'B'} \lambda_+^{A'} \nabla_+ \lambda_-^{B'} \right)
$$

(2.8)

where $u_{ab} = u_{ab}(\phi)$ are the gauge couplings which in general depend on $\phi$, $V$ is a scalar potential and

$$
F_{aB} = [\nabla_+^a, \nabla_-^B],
$$

(2.9)
where we have suppressed gauged indices. The covariant derivatives in the action above are defined as follows,

\[ \nabla_\mu \phi^i = \partial_\mu \phi^i + A^a_\mu \xi^i_a \]  

(2.10)

and

\[ \tilde{\nabla}_\mu \psi^A = \partial_\mu \psi^A + \nabla_\mu \phi^i B^A_B \psi^B + A^a_\mu U^A_a B \psi^B, \]  

(2.11)

\[ \mu = \pm, =, \text{and similarly for } \tilde{\nabla} \lambda^A_+. \]  

The latter can be rewritten as

\[ \tilde{\nabla}_\mu \psi^A = \partial_\mu \psi^A + \partial_\mu \phi^i B^A_B \psi^B + A^a_\mu \psi^A_B \psi^B. \]  

(2.12)

where

\[ \mu^A_a = U^A_a B + \xi^i_a B^A_B. \]  

(2.13)

Observe that the part of the action involving the Wess-Zumino term has been written as an integral over a three-dimensional space. The conditions for this term to be written in a two-dimensional form as well as the conditions for the gauge invariance of the action will be investigated in the next section, see also [7].

2.2 Conditions for gauge invariance

The gauge transformations of the fields are

\[ \delta_\epsilon A^a_\mu = -\nabla_\mu \epsilon^a \]  

\[ \delta_\epsilon \phi^i = \epsilon^a \xi^i_a \]  

\[ \delta_\epsilon \psi^A_+ = \epsilon^a U^A_a B \psi^B_+ \]  

\[ \delta_\epsilon \lambda^A_+ = \epsilon^a V^A_a B \lambda^B_+ \]  

(2.14)

where \( \epsilon \) is the parameter of infinitesimal gauge transformations. Some of the conditions required for the invariance of the action (2.8) have been incorporated as part of the geometric data of the sigma model in the previous section. In particular, in addition to the conditions (2.1) and (2.5), we require that (i) \( w_a \) is a globally defined one-form on \( M \) which (ii) satisfies

\[ \mathcal{L}_a w_b = -f_{ab}^c w_c. \]  

(2.15)

To write the Wess-Zumino part of the action in a two-dimensional form, it is necessary for the relevant three form to be closed. This in addition requires that

\[ \xi^i_a w_{ab} + \xi^i_b w_{ia} = 0. \]  

(2.16)
Then the three-dimensional part of the action (2.8) can be written locally [7] as

\[ S' = \int d^2 x \left( b_{ij} \partial_i \phi^j \partial_\phi = \partial_\phi \phi^i - A^a_{\phi} w_{ia} \nabla_\phi \phi^i ight. 
\left. + A^a_{\phi} w_{ia} \nabla_\phi \phi^i + A^a_{\phi} A^b_{\phi} \xi_{ab}^i w_{ai} \right). \] (2.17)

The conditions above that require (i) \( H \) to be invariant under the group action, (ii) \( w_a \) to be globally defined on \( \mathcal{M} \), (iii) (2.15) and (iv) (2.16) are those for the closed form \( H \) to have an extension, i.e. equivariant extension [9], as a closed form in \( \mathcal{M} \times_G \mathcal{E}G \) [10].

Next let us consider the conditions for gauge invariance of part of the action (2.8) involving the fermions. We find that this requires that

\[ L_c u_{ab} + u_{db} f_{ca}^d + u_{ad} f_{cb}^d = 0 \] (2.18)

for the gauge couplings \( u_{ab} \),

\[ L_a U_b - L_b U_a = [U_a, U_b] - f_{ab} c^c U_c \] (2.19)

and

\[ L_a V = 0. \] (2.20)

We have not assumed that \( \nabla_i h = 0 \). However given a connection on a vector bundle with a fibre metric \( h \), there always exist another connection \( \nabla' \) such that \( \nabla'_i h = 0 \). Suppose that the \( \nabla' \) is used for the fermionic couplings. If this is the case, the gauge group is \( O(N) \) and therefore the right-hand-side of (2.6) vanishes.

Observe that the equation (2.5) can be written in a more covariant form as

\[ \xi_{a}^{j} G_{ij}^{\mu A} = \nabla_{i} \mu_{a} A_{\mu B}. \] (2.21)

In what follows we shall assume that the conditions stated in this section by requiring gauge invariance of the non-supersymmetric model described by the action (2.8) hold. We shall see that for supersymmetric sigma models more conditions are necessary.

### 3 (1, 0) supersymmetric gauged models

The (1,0)-supersymmetric gauged sigma model involves the coupling of three different (1,0)-multiplets. To simplify the construction of this model we shall describe each multiplet and the conditions for supersymmetry and gauge invariance separately. There are different ways of approaching this problem. Here we shall use ‘standard’ (1,0)-superfields. The action will be constructed using (1,0)-superspace methods.
3.1 The gauge multiplet

The \((1,0)\)-superspace \(\mathcal{X}^{1,0}\) has coordinates \((x^+, x^-, \theta^+)\), where \((x^+, x^-)\) are bosonic light-cone coordinates and \(\theta^+\) is a Grassmann odd-coordinate. The \((1,0)\)-supersymmetric Yang-Mills multiplet with gauge group \(G\) is described by a connection \(A\) in superspace which has components \((A^+, A_-, A^-)\). In addition, it is required that these satisfy the supersymmetry constraints \[ \begin{align*} [\nabla_+, \nabla_] &= 2i \nabla_+ , \\ [\nabla_+, \nabla_-] &= F_+ , \\ [\nabla_- , \nabla_-] &= W_- , \end{align*} \] (3.1)

where \(F_+, W_-\) are the components of the curvature of the superspace connection \(A\). (The gauge indices have been suppressed.) Jacobi identities imply that

\[ \nabla_+ W_- = i F_+ \] (3.2)

Therefore the independent components of the gauge multiplet are

\[ \chi^a = W_- | \]
\[ F^a_+ = -i \nabla_+ W^-_a \] (3.3)

where \(\chi^a\) is the gaugino and \(F^a_+\) is the two-form gauge field strength and the vertical line denotes evaluation of the associated superfield at \(\theta^+ = 0\). This notation for identifying the components of a superfield will also be used later for other theories.

3.2 Sigma model multiplets

To described the sigma model multiplet that couples to the above gauge field, we introduce a Riemannian manifold \(M\) with metric \(g\) and a locally defined two form \(b\). In addition we assume that \(M\) admits a vector bundle \(E\) with fibre metric \(h\), connection \(B\) and a section \(s\). The data required for the description of the sigma model multiplet are the same as those given in section 2.1. In addition we take the section \(s\) to satisfy

\[ \mathcal{L}_a s_A = -U_a B^A s_B . \] (3.4)

The \((1,0)\)-supersymmetric sigma model multiplet is described by a real scalar superfield \(\phi\) and a fermionic superfield \(\psi_-\). The superfield \(\phi\) is a map from the superspace \(\mathcal{X}^{1,0}\) into a sigma model manifold \(M\) and the fermionic superfield \(\psi_-\) is a section of the bundle \(\phi^* E \otimes S_-; S_-\) is a spin bundle over \(\mathcal{X}^{1,0}\).

The components of the superfields \(\phi, \psi_-\) are

\[ \phi^i = \phi^i | \quad \lambda^i_+ = \nabla_+ \phi^i | \]
\[ \psi_-^A = \psi_-^A | \quad \ell^A = \nabla_+ \psi_-^A | , \] (3.5)

where the covariant derivatives are defined as

\[ \nabla_+ \phi^i = D_+ \phi^i + A_+ \xi^i_a \]
\[ \nabla_+ \psi_-^A = D_+ \psi_-^A + \nabla_+ \phi^i B^A_i \psi_-^B + A_+ U_a A^B \psi_-^B , \] (3.6)

where \(D_+\) is the usual flat superspace derivative, \(D_+^2 = i \partial_+\).
3.3 Supersymmetric action

It is straightforward to couple the gauge multiplet to \((1,0)\)-supersymmetric sigma model matter. The full action is

\[
S = S_g + S_\sigma + S_f + S_p
\]

where

\[
S_g = -\int d^2 x d\theta^+ \left( u_{ab} W_a^+ \nabla^+ W_b^+ - i z_a W_a^- \right),
\]

where \(u_{ab} = u_{ab}(\phi)\), \(u_{ab}\) is not necessarily symmetric in the gauge indices and \(z_a\) is a theta type of term which may depend on the scalar field \(\phi\). \(z_a = z_a(\phi)\).

The gauge covariant supersymmetric action for the fields \(\phi\) is

\[
S_\sigma = -i \int d^2 x d\theta^+ \left( g_{ij} \nabla^+ \phi^i \nabla^+ \phi^j - w_{ia} \partial_t \phi^i W_a^- \right),
\]

which as in section 2.1 can also be written as an integral over \(\Xi^{1,0}\) superspace provided that \(H\) admits an equivariant extension. In particular we have

\[
S_\sigma = -i \int d^2 x d\theta^+ \left( g_{ij} \nabla^+ \phi^i \nabla^- \phi^j + b_{ij} D_+ \phi^i \partial_- \phi^j \right. \\
- \left. A_a^n w_{ia} \partial_- \phi^i + A_a^b w_{ia} D_+ \phi^i + A^a A_b \psi_{[i} w_{a]i} \right).
\]

The action of the gauged fermionic multiplet is

\[
S_f = \int d^2 x d\theta^+ h_{AB} \psi_+^A \bar{\nabla}^+ \bar{\psi}_-^B.
\]

The definition of the covariant derivative \(\bar{\nabla}_+\) is similar to the one given in section 2.1 for the covariant derivative \(\bar{\nabla}_-\).

The action for the potential term is

\[
S_p = \int d^2 x d\theta^+ m_A \psi_+^A
\]

which is similar to that of the ungauged model in \[1\].

The superfields transform under the gauge group \(G\) as

\[
\delta A_\mu = -\nabla_\mu \epsilon^a, \quad \delta \phi^i = \epsilon^a \xi_a^i(\phi), \quad \delta \psi_+^A = \epsilon^a U_a^A \psi_+^B.
\]

where \(\mu = \pm, =, +\) is a \(\Xi^{1,0}\) superspace index and \(\epsilon^a\) is an infinitesimal gauge transformation parameter. Gauge invariance of the action (3.7) requires, in addition to the conditions given in section 2.2, the condition (3.4) and

\[
L_a z_b = -f_{ab}^c z_c.
\]
3.4 The action of (1,0)-model in components and scalar potential

The action of (1,0)-supersymmetric two-dimensional gauged sigma model described by the action (3.7) can be easily written in components by performing the \( \theta^+ \) integration and using the definition of the various component fields of the (1,0)-multiplets which we have described in the previous sections. In particular we find for the part of the action involving the kinetic term of the gauge multiplet that

\[
S_g = \int d^2 x \left( u_{ab} F_{+}^a = F_{+}^b + z_a F_{+}^a 
- i u_{ab} \chi_a^{\pm} \nabla_{\pm} \chi^b_{\pm} + i \partial_{\lambda} u_{ab} \lambda^i_+ \chi^a_{\mp} F_{+}^b + i \partial_{\lambda} z_a \lambda^i_+ \chi^a_{\mp} \right) .
\]

(3.15)

Next we find that

\[
S_\sigma = \int d^2 x \left( g_{ij} \nabla_{\pm} \phi^i \nabla_{\pm} \phi^j + b_{ij} \partial_+ \phi^i \partial_- \phi^j + i \lambda^i_+ \nabla^{(\pm)}_\pm \lambda^j_+ + i w_{ia} \lambda^i_+ \chi^a_{-} - A^a_+ w_{ia} \partial_- \phi^i + A^a_- w_{ia} \partial_+ \phi^i \right) \]

(3.16)

where \( \nabla^{(\pm)}_\pm \) are the usual metric connections with torsion \( \pm H \) and \( \nabla^{(\pm)}_\pm \) are the associated connections involving also the gauge connection \( A \). For the fermionic multiplet we have

\[
S_f = \int d^2 x \left( - i h A B \psi_+^{A_+} \nabla_+ \psi_+^{B} + h A B \ell^A \ell^B - \frac{1}{2} G_{ijab} \psi_+^{A_+} \psi_+^{B} \lambda^i_+ \lambda^j_+ \right)
\]

(3.17)

and

\[
S_p = \int d^2 x \left( \nabla_i s_A \lambda^i_+ + s_A \ell^A \right) .
\]

(3.18)

The scalar potential in these models is precisely that of the ungauged (1,0) sigma models in [11], i.e.

\[
V = \frac{1}{4} m^2 h A B s_A s_B .
\]

(3.19)

So we find that only ‘F-terms’ contribute to the potential. This is because the gauge multiplet does not have an auxiliary field. Therefore the classical vacua of the theory are the points of the sigma model manifold \( M \) for which the section \( s \) vanishes modulo gauge transformations. Therefore the vacua of the theory are those orbits of the gauge group \( G \) in \( M \) for which the section \( s \) vanishes.

4 (2, 0) supersymmetry
4.1 The gauge multiplet

The (2,0) superspace $\Xi^{2,0}$ has coordinates $(x^+, x^-, \theta_0^+, \theta_1^+)$ where $(x^+, x^-)$ are the usual light-cone coordinates and $\{\theta_p^+: p = 0, 1\}$ are anticommuting coordinates. The (2,0)-supersymmetric Yang-Mills multiplet is described by a connection $A$ in $\Xi^{2,0}$ superspace with components $(A^+, A_-, A_{p+})$, $p = 0, 1$. In addition it is required that these satisfy the supersymmetry constraints

$$\left[\nabla_{p+}, \nabla_{q+}\right] = 2i\delta_{pq}\nabla_+ \quad \left[\nabla_+, \nabla_-=\right] = F_+ \quad \left[\nabla_{p+}, \nabla_-=\right] = W_{p-} , \quad (4.1)$$

where $p, q = 0, 1$. Jacobi identities imply that

$$\nabla_{p+}W_{q-} + \nabla_{q+}W_{p-} = 2i\delta_{pq}F_+ + \nabla_+ (4.2)$$

The components of the gauge multiplet are

$$\chi_0- = W_0- \quad \chi_1- = W_1- \quad iF_+ = \nabla_+W_0- \quad f = \nabla_+W_1- . \quad (4.3)$$

The components, $(\chi_0-, \chi_1-)$, are the gaugini which are real chiral fermions in two dimensions, $F_+$ is the field strength and $f$ is an auxiliary field. (We have suppressed the gauge indices.)

4.2 The sigma model multiplet

It is well known the target manifold $M$ of (2,0)-supersymmetric ungauged sigma model is Kähler manifold with torsion (KT). Therefore $M$ is a hermitian manifold with metric $g$ and equipped with a complex structure $J$ which is parallel with respect to the $\nabla^{(+)}$ connection. This connection is a metric connection with torsion $H$, ie $\nabla^{(+)} = \nabla + \frac{1}{2}H$ where $\nabla$ is the Levi-Civita connection. (For the definition of these geometries see [27, 28]). To gauge the model, we assume as in section 2.1 that the gauge group $G$ acts on $M$ and leaving invariant the metric $g$ and the Wess-Zumino term $H$. In addition we require that the action of the group $G$ is holomorphic. This means that

$$\mathcal{L}_aJ = 0 , \quad (4.4)$$

where the Lie derivative is along vector fields $\xi_a$ generated by the group action of $G$. The sigma model fields are maps $\phi : \Xi^{2,0} \rightarrow M$ into a complex manifold $M$ which in addition satisfy

$$\nabla_1^+\phi^i = J^i_j\nabla_0^+\phi^j , \quad (4.5)$$

where $\nabla_0^+\phi = D_p^+\phi^i + A_{p+}^a\xi_a^i$, $p = 0, 1$. Note that the requirement for $M$ to be a complex manifold can be derived from the above condition.

The components of the sigma model multiplet $\phi$ are as follows:

$$\phi^i = \phi|^i \quad \lambda^i_+ = \nabla_0^+\phi^i . \quad (4.6)$$
4.3 The fermionic multiplet

Let $E$ be a vector bundle over $M$ equipped with a connection $B$ and a fibre (almost) complex structure $I$. The fermionic multiplet $\psi_-$ is a section of $\phi^* E \otimes S_-$ over the $\Xi^{2,0}$, where $S_-$ is a spin bundle over $\Xi^{2,0}$. In addition we require that the fermionic multiplet $\psi_-$ satisfies

$$\tilde{\nabla}_{1+} \psi_-^A = I^A_B \tilde{\nabla}_{0+} \psi_-^B + \frac{1}{2} mL^A,$$  \hspace{1cm} (4.7)

where $L$ is a section of $E$ and

$$\tilde{\nabla}_{p+} \psi_-^A = D_{p+} \psi_-^A + \nabla_{p+} \phi^i B_i \psi_-^A + A_p^a U_a^A B,$$ \hspace{1cm} (4.8)

for $p = 0, 1$.

Compatibility of the condition (4.7) with gauge transformations requires that

$$\mathcal{L}_a I^A_B = U_a^A_C I^C_B - I^A_C U_a^C_B$$
$$\mathcal{L}_a L^A = U_a^A_B L^B.$$

(4.9)

These are the conditions for the gauge transformations and the $(2,0)$-supersymmetry transformations to commute.

The compatibility of the condition (4.7) with the algebra of covariant derivatives $\nabla$ implies the following conditions:

$$G_{kl}^A B J^1_k J^j_l = G_{ij}^A B$$
$$J^k \nabla_k L^A - I^A_B \nabla_i L^B = 0$$
$$J^k \nabla_k I^A_B - I^A_C \nabla_i I^C_B = 0.$$  \hspace{1cm} (4.10)

These are precisely the conditions required for the off-shell closure of $(2,0)$ supersymmetry algebra.

It is always possible to find a connection $B$ on the bundle $E$ such that $\nabla I = 0$. In such case the last condition in (4.10) is satisfied. Decomposing $E \otimes \mathbb{C}$ as $E \otimes \mathbb{C} = \mathcal{E} \oplus \bar{\mathcal{E}}$ using $I$, the first condition implies that $\mathcal{E}$ is a holomorphic vector bundle. Then the second condition in (4.10) implies that the section $L$ is the real part of a holomorphic section of $\bar{\mathcal{E}}$.

The components of the fermionic multiplet are as follows:

$$\psi_-^A = \psi_-^A|$$
$$\ell^A = \nabla_{0+} \psi_-^A|,$$  \hspace{1cm} (4.11)

where $\psi_-$ is a two-dimensional real chiral fermion and $\ell$ is an auxiliary field.

4.4 Action

The action of the $(2,0)$-supersymmetric gauged sigma model can be written as

$$S = S_g + S_\sigma + S_f,$$  \hspace{1cm} (4.12)

where $S_g$ is the action of the gauge multiplet, $S_\sigma$ is the action of the sigma-model multiplet and $S_f$ is the action of the fermionic multiplet. We shall describe each term separately.
4.5 The gauge multiplet action

The most general action for the (2,0)-supersymmetric gauge multiplet up to terms quadratic in the field strength is

\[ S_g = \int d^2 x d^2 \theta^+ \left( -u^0_{ab} \delta^{pq} W^a_p \nabla^0_{q-} W^b_q + u^1_{ab} \delta^{pq} W^a_p \nabla^1_{q-} W^b_q + i z^a_{p} W^a_p \right), \]

(4.13)

where \( u^0 \) and \( u^1 \) are the gauge coupling constants which in general depend on the superfield \( \phi \) and similarly for the ‘theta’ terms \( z^p \). Both \( u^0 \) and \( u^1 \) are not necessarily symmetric in the gauge indices. The above action can be written in different ways. However there are always field and coupling constant redefinitions which can bring the action to the above form.

Observe that this action is not an integral over the full \( \Xi^{2,0} \) superspace. Therefore it is not manifestly (2,0)-supersymmetric. The requirement of invariance under (2,0) supersymmetry imposes the conditions

\[ J^j_i \partial_j u^0 = - \partial_i u^1 \]
\[ J^j_i \partial_j z^1 = - \partial_i z^0. \]

(4.14)

This is most easily seen by verifying that the Lagrangian density is independent of \( \theta^+ \) up to \( \theta^\pm, x^\pm, x^\mp \)-surface terms. The conditions (4.14) are the Cauchy-Riemann equations which imply that \( u^0 + i u^1 \) and \( z^1 + i z^0 \) are holomorphic.

Indeed provided that the holomorphicity conditions (4.14) hold, the action (4.13), apart from the theta terms, can be written as an integral over the \( \Xi^{2,0} \) superspace as

\[ S_g = \int d^2 x d^2 \theta^+ d^2 \theta^+ \left( \alpha u^0_{ab} W^a_0 \nabla^0_{q-} W^b_q + (\alpha - 1) u^1_{ab} W^a_0 \nabla^1_{q-} W^b_q - \alpha u^1_{ab} W^a_0 \nabla^1_{q-} W^b_q + (\alpha - 1) u^0_{ab} W^a_0 \nabla^0_{q-} W^b_q \right) \]

(4.15)

for any constant \( \alpha \). After integrating over the odd coordinate \( \theta^+ \) we recover the action (4.13). Observe that the action (4.15) simplifies if one takes \( u^0, u^1 \) to be symmetric matrices. In particular one finds that

\[ S_g = \int d^2 x d^2 \theta^+ d^2 \theta^+ \left( u^0_{ab} W^a_0 \nabla^0_{q-} W^b_q + (\alpha - 1) u^1_{ab} W^a_0 \nabla^1_{q-} W^b_q \right). \]

(4.16)

Invariance of the action (4.13) under gauge transformations requires that the couplings \( u^0, u^1 \) and \( z^p \) satisfy

\[ \mathcal{L}_a u^0_{bc} = - f^e_{ac} u^0_{be} - f^e_{ae} u^0_{ce} \]
\[ \mathcal{L}_a u^1_{bc} = - f^e_{ac} u^1_{be} - f^e_{ae} u^1_{ce} \]
\[ \mathcal{L}_a z^p_{bc} = - f^p_{ae} z^p_{be} \]

(4.17)

The gauge transformations of the gauge multiplet and the sigma model multiplet that are required to derive the above result are as in the (1,0)-supersymmetric models studied in the previous sections.
4.6 The sigma model action

The part of the action which describes the coupling of the sigma model (2,0)-multiplet to the gauge multiplet has already been given in \[8\]. This action is

\[
S_\sigma = -i \int d^2x \theta_0^+ \left( g_{ij} \nabla_0^+ \phi^i \nabla^- \phi^j + \nu_a W_1^a \right) - i \int d^2x dt \theta_0^+ \left( H_{ijk} \partial_t \phi^i \nabla_0^+ \phi^j \nabla^- \phi^k - w_{ia} \partial_t \phi^i W_0^a \right)
\]

(4.18)

where \(\nu_a\) is a function on \(M\), possibly locally defined, given by

\[
I^j_i (\xi_a + w_{aj}) = -\partial_i \nu_a
\]

(4.19)

Under certain conditions the maps \(\nu\) can be thought of as the moment maps of KT geometry \[17\].

Gauge invariance of the above part of action requires in addition to the conditions on \(w\), which we have already mentioned in section 2.2, that \(\nu_a\) is globally defined on \(M\) and that

\[
\mathcal{L}_a \nu_b = -f_{ab}^c \nu_c.
\]

(4.20)

4.7 The action of the fermionic multiplet

This part of the action is

\[
S_f = \int d^2x \theta_0^+ \left( h_{AB} \psi^A \nabla_0^+ \psi^B + ms_A \psi^A \right)
\]

(4.21)

Gauge invariance of this part of the action requires the same conditions as those appearing for the couplings of (1,0)-multiplet in (2.19).

The conditions required by (2,0)-supersymmetry on the couplings of the above action are the same as those of the ungauged model and have been given in \[29\]. These can be easily derived by requiring that the Lagrangian density is independent from \(\theta_0^+\) up to \(x^+, x^-, \theta_0^+\) surface terms. In particular, we find that

\[
\begin{align*}
&h_{CB} h^{IA} + h_{CA} h^{IB} = 0 \\
h^{ij} \nabla_j h_{AB} + \nabla_i h_{AC} h^{IB} = 0 \\
h^{ij} \nabla_j s_A - \nabla_i (s_B h^{IB}) - \frac{1}{2} \nabla_i h_{AB} L^B = 0 \\
& s_A L^A = \text{const}.
\end{align*}
\]

(4.22)

The first condition implies that the fibre metric is hermitian is respect to the fibre complex structure. It is always possible to choose such a fibre metric given a fibre complex structure on a bundle vector bundle \(E\). In the context of sigma models this has been explained in \[14\]. The rest of the conditions can be considerably simplified if the connection \(B\) is chosen such that \(\nabla I = \nabla h = 0\). Such connection always exists on a hermitian vector bundle \(E\). In such case, the third equation in (4.22) implies that \(s\) is the real part of a holomorphic section of \(E^*\).
4.8 The action in components

It is straightforward to write the action $S$ of the $(2,0)$-supersymmetric gauge sigma model in components. In particular we find that the component action of the gauge multiplet (4.13) is

$$S_g = \int d^2 x \left( u_{ab}^0 F_a^b = F_a^b = z_0^a F_a^b = - u_{ab} f^a f^b + iz_1^a f^a ight.$$ \[+ iu_{ab}^1 \nabla_+ \chi_0^a - \nabla_+ \chi_0^b + iu_{ab}^0 \nabla_- \chi_0^a - \nabla_- \chi_0^b - 2iu_{[ab]}^1 \nabla_+ \chi_{[0]}^a - \nabla_+ \chi_{[0]}^b \\+ i\partial x_0^1 \nabla_+ \chi_0^a + i\partial x_0^0 \nabla_- \chi_0^a - \lambda_i^a \partial u_{ab} (i\chi_{[0]}^a - F_{[0]}^a + \chi_{[1]}^a - F_{[1]}^a) \\
+ \chi_{[1]}^i \partial u_{ab} (-\chi_{[0]}^a - F_{[0]}^a + i\chi_{[1]}^a - F_{[1]}^a) \right) . \] \hspace{1cm} (4.23)

The component action of the sigma model part is

$$S_\sigma = \int d^2 x \left( b_{ij} \nabla_+ \phi_i \nabla_- \phi_j + b_{ij} \partial_+ \phi_i \partial_- \phi_j + i\lambda_i^a \nabla_+ \phi_j \lambda_i^a - i\partial_+ \lambda_i^a \nu_a \lambda_i^a - i\partial_- \lambda_i^a \nu_a \lambda_i^a - i\partial_+ \lambda_i^a \nu_a \lambda_i^a - i\partial_- \lambda_i^a \nu_a \lambda_i^a \right.$$ \[+ g_{ij} \lambda_i^a \nu_a \lambda_j^a + iw_{ia} \lambda_i^a \xi_i^a + iw_{ia} \lambda_i^a \xi_i^a - A_{ij}^a w_{ia} \partial_+ \phi_i + A_{ij}^a w_{ia} \partial_- \phi_i - A_{ij}^a A_{il}^a \xi_i^a \xi_l^a \right) \hspace{1cm} (4.24)

and the component action of the fermionic multiplet is

$$S_f = \int d^2 x \left( -ih_{AB} \psi_+ A_+^B \psi_+^B + h_{AB} \ell^A \ell^B - \frac{1}{2} h_{AB} \psi_+^B \psi_+^B \lambda_i^A \lambda_i^A G_{ij} + ms_A \lambda_i^A \psi_+^A + ms_A \ell^A \right) . \hspace{1cm} (4.25)
Eliminating the auxiliary fields of the gauge and fermionic multiplets, we find that

\[ S = \int d^2 x \left( g_{ij} \nabla_i \phi^j \nabla = \phi^j + b_{ij} \partial_\phi \phi^j \partial_\phi^j + u_{ab}^0 F_{\pm}^a F_{\pm}^b \right. \]
\[ \left. + i u_{ab}^0 \lambda_0^a \nabla + \lambda_0^b \nabla - i u_{ab}^0 \lambda_1^a \nabla + \lambda_1^b \nabla \right) + i h_{AB} \psi^A \nabla_\psi B + z_a^0 F_{\pm}^a - \frac{1}{4} m^2 h_{AB} s_A s_B - \frac{1}{4} u_{ab}^0 (\nu_a - z_a^1) (\nu_b - z_b^1) \]
\[ - \frac{1}{2} h_{AB} \psi^A \psi^B \lambda_0^A \lambda_0^B G_{ijAB} + m \nabla_i s_A \lambda_0^A \nabla_i \]
\[ - A_a^+ w_{ia} \partial_\psi^i + A_a^0 w_{ia} \partial_\psi^i - A_a^0 A_b^\dagger \xi_{ij} w_{ai} \]
\[ - u_{ab}^0 (\nu_a - z_a^1) u_{[ab]}^1 F_{\pm}^c = - u_{ab}^0 u_{[ac]}^1 u_{[bd]}^1 F_{\pm}^d = (4.26) \]
\[ + i \partial_\psi^i z_a^0 \lambda_i^a \nabla - i \partial_\psi^i z_a^1 \lambda_i^a \nabla - i \partial_\psi^i u_{ab}^1 \lambda_i^a \nabla + i \partial_\psi^i u_{ab}^1 \lambda_i^a \nabla \]
\[ - i \partial_\psi^i u_{ab}^1 \lambda_1^i \nabla^a - i \partial_\psi^i u_{ab}^1 \lambda_0^i \nabla^a - i \partial_\psi^i u_{ab}^1 \lambda_1^i \nabla^a - i \partial_\psi^i u_{ab}^1 \lambda_0^i \nabla^a \]
\[ + \frac{1}{2} i u_{ab}^0 (\nu_a - z_a^1) (\partial_\psi_0^0 \lambda_1^a - \partial_\psi_0^0 \lambda_0^a) \]
\[ + \frac{1}{4} u_{ab}^0 (\partial_\psi_0^0 \lambda_1^a - \partial_\psi_0^0 \lambda_0^a - \partial_\psi_0^0 \lambda_1^a + \partial_\psi_0^0 \lambda_0^a) \]
\[ - i u_{ab}^0 u_{[ac]}^1 \nabla_c^a \nabla_\psi^d \]
\[ + 2 i u_{ab}^0 u_{[bc]}^0 \nabla_\psi^d \]

where \( u_{ab}^0 \) is the matrix inverse of \( u_{(ab)}^0 \),

\[ u_{ab}^0 u_{(bc)}^0 = \delta_{a}^{b} . \] (4.27)

Note that we have assumed that \( u^0 \) is invertible.

### 4.9 Scalar potential and classical vacua

The scalar potential of the (2,0)-supersymmetric gauge theories coupled to sigma model matter is

\[ V = \frac{1}{4} u_{ab}^0 (\nu_a - z_a^1) (\nu_b - z_b^1) + \frac{1}{2} m h_{AB} s_A s_B . \] (4.28)

The scalar potential in these models is written as a sum of a ‘\( D \)’ and an ‘\( F \)’ term. The classical supersymmetric vacua of the theory are those for which

\[ \nu_a - z_a^1 = 0 \quad s_A = 0 . \] (4.29)

The inequivalent classical vacua are the space of orbits of the gauge group on the zero set of the section \( s \) and \( \nu - z^1 \). If the section \( s \) and \( z^1 \) vanish, then
the space of inequivalent vacua is the KT reduction $M//G$ of the sigma model target space $M$. It has been shown in [17] that the space of vacua inherits the KT structure of the sigma model manifold $M$ and under certain assumptions is a smooth manifold. However the three-form of the Wess-Zumino term on $M//G$ is not necessarily closed.

5 (4,0) supersymmetry

5.1 The gauge multiplet

The (4,0) superspace $\Xi^{4,0}$ has coordinates $(x^+, x^-, \theta^+_p)$, where $\theta^+_p$, $p = 0, 1, 2, 3$, are the odd coordinates. The (4,0)-supersymmetric Yang-Mills multiplet is described by a connection $A$ in superspace with components $(A^+, A_-, A^+_p)$, $p = 0, 1, 2, 3$. In addition it is required that these satisfy the supersymmetry constraints

\[
\begin{align*}
[\nabla_{p+}, \nabla_{q+}] &= 2i\delta_{pq} \nabla_+ \\
[\nabla_+, \nabla_-] &= F_+ \\
[\nabla_{p+}, \nabla_-] &= W_p \\
\nabla_{p^+}W_{q^-} &= \frac{1}{2} \epsilon_{pq} q^r 
abla_{r^+}W_{q^-} 
\end{align*}
\] (5.1)

where $p, q, p'q' = 0, \ldots, 3$ and $p \neq q$ in the last condition. (We have suppressed gauge indices.) Jacobi identities imply that

\[
\nabla_{p^+}W_{q^-} + \nabla_{q^+}W_{p^-} = 2i\delta_{pq}F_+ 
\] (5.2)

The components of the gauge multiplet are

\[
\begin{align*}
\chi_p &= W_{p^-} \\
iF_+ &= \nabla_{0^+}W_{0^-} \\
f_r &= \nabla_{0^+}W_{r^-} (r = 1, 2, 3) 
\end{align*}
\] (5.3)

The first four fields, $(\chi_p : p = 0, 1, 2, 3)$, are the gaugini which are real chiral fermions in two dimensions, $F_+^+$ is the field strength and $\{f_r : r = 1, 2, 3\}$ are the auxiliary fields.

5.2 The sigma model multiplet

Let $M$ be a hyper-Kähler manifold with torsion (HKT). This implies that $M$ admits a hypercomplex structure $\{J_r : r = 1, 2, 3\}$ the metric $g$ on $M$ is trihermitian and the hypercomplex structure is parallel with respect to a metric connection with torsion the three-form $H$, $\nabla^{(+)}J_r = 0$; (see [27, 28] for more details). In addition we assume that the gauge group $G$ acts on $M$ preserving the metric, three-form $H$ and the hypercomplex structure. The latter condition implies that

\[
L_aJ_r = 0 
\] (5.4)
where the Lie derivative $L_a$ is along vector field $\xi_a$ generated by the action of $G$ on $M$. The sigma model fields are maps $\phi : \Xi^{k,0} \to M$ into the HKT manifold $M$ which in addition satisfy
\begin{equation}
\nabla_{r+} \phi^i = J_{r^i}^j \nabla_{0+} \phi^j ,
\end{equation}
where $\nabla_{r+} \phi = D_{r+} \phi^i + A^a_{r+} \xi^i_a$ and $r = 1, 2, 3$. We remark that the algebra of $(4,0)$ supersymmetry transformations closes as a consequence of the HKT condition we imposed on $M$.

The components of the sigma model multiplet $\phi$ are as follows,

\begin{equation}
\phi^i = \phi^i | \quad \lambda^i_+ = \nabla_{0+} \phi^i .
\end{equation}

### 5.3 The fermionic multiplet

Let $E$ be a vector bundle over $M$ equipped with a connection $B$ and a fibre (almost) hypercomplex structure $\{I_r : r = 1, 2, 3\}$. The fermionic multiplet $\psi_-$ is a section of $\phi^* E \otimes S_- \Sigma^{k,0}$, where $S_-$ is a spin bundle over $\Sigma^{k,0}$. In addition the fermionic multiplet $\psi_-$ satisfies
\begin{equation}
\tilde{\nabla}_{r+} \psi_- = D_{r+} \phi^i + \nabla_{r+} \phi^i + A^a_{r+} \xi^i_a + \frac{1}{2} m L_a^A \nabla_{0+} \psi_- ^A,
\end{equation}
where $\{L_r : r = 1, 2, 3\}$ are sections of $E$ and
\begin{equation}
\tilde{\nabla}_{r+} \psi_- ^A = D_{r+} \psi_- ^A + \nabla_{r+} \phi^i B^A_i \psi_- ^B + A^a_{r+} U_a A B ,
\end{equation}
p = 0, 1, 2, 3.

Compatibility of the condition (5.7) with gauge transformations requires that
\begin{equation}
L_a I_r^A = U_a A C I_r^C - I_r^A C U_a C B
\end{equation}
\begin{equation}
L_a L_r^A = U_a A B L_r^B .
\end{equation}

These are the conditions for the gauge transformations and the $(2,0)$-supersymmetry transformations to commute.

The conditions required for the closure of the $(4,0)$ supersymmetry algebra are similar to those found for the ungauged $(4,0)$ model in \cite{11, 15}. Here to simplify the analysis, we shall in addition assume that the fibre hypercomplex structure is parallel with respect to $\nabla$, ie $\nabla I_r = 0$. (See \cite{13} for a discussion on the conditions required for the existence of such a connection $\nabla$ on the vector bundle $E$.) The more general case can be easily derived but we shall not use these results later. In this special case, we find that
\begin{equation}
G_{kl}^A B J_r^k j I_s^l j + G_{kl}^A B J_s^k i J_r^l i = 2 \delta_{rs} G_{ij} A B
\end{equation}
\begin{equation}
J_r^j \nabla_j L_s^A + J_s^A \nabla_r L_r^A - I_r^A B \nabla_s L_s^B - I_s^B B \nabla_r L_r^B = 0 .
\end{equation}
The first condition implies that the curvature $G$ of the vector bundle $E$ is an $(1,1)$-form with respect to all three complex structures $J_r$. Observe that the
diagonal relation \( r = s \) implies all the rest. The diagonal part of the second condition implies that each section \( L_r \) is a holomorphic section with respect to the pair \((J_r, I_r)\).

The components of the fermionic multiplet are as follows:

\[
\psi_r^A = \psi^A |_{Z^r} \quad \ell^A = \nabla_{0+} \psi^A, \tag{5.11}
\]

where \( \psi_+ \) is a two-dimensional real chiral fermion and \( \ell \) is an auxiliary field.

### 5.4 Action

The action of the (4,0)-supersymmetric gauged sigma model can be written as

\[
S = S_g + S_\sigma + S_f, \tag{5.12}
\]

where \( S_g \) is the action of the gauge multiplet, \( S_\sigma \) is the action of the sigma-model multiplet and \( S_f \) is the action of the fermionic multiplet. We shall describe each term separately.

### 5.5 The gauge multiplet action

The most general action for the (4,0)-supersymmetric gauge multiplet up to terms quadratic in the field strength is

\[
S_g = \int d^2x d\theta^+ \left(-u^0_{ab} \delta^{pq} W^a_p - \nabla_{0+} W^b_q + \sum_{r=1}^3 u^r_{ab} \delta^{pq} W^a_p - \nabla_r W^b_q + i z^p W^a - p \right) \tag{5.13}
\]

where \( \{u^p : p = 0, 1, 2, 3\} \) and \( \{z^p : p = 0, 1, 2, 3\} \) are the gauge coupling constants which in general depend on the superfield \( \phi \) and \( u^p \) are not necessarily symmetric in the gauge indices. The above action can be written in different ways. However there are always field and coupling constant redefinitions which bring the action to the above form.

Observe that this action is not an integral over the full \( \Xi^{4,0} \) superspace. Therefore it is not manifestly (4,0)-supersymmetry. Define \( J^a_{ij} = \delta^a_j \). The requirement of invariance the action (5.13) under (4,0) supersymmetry imposes the conditions

\[
J^a_{ij} \partial_j u_q = \frac{1}{2} \epsilon_{p'q'pq} J^{p'q'}_{ij} \partial_j u_{q'} \quad (p \neq q) \tag{5.14}
\]

\[
\partial_0 u_0 = J^1_{ij} \partial_j u_1 = J^2_{ij} \partial_j u_2 = J^3_{ij} \partial_j u_3 ,
\]

and \( \{u^r : r = 1, 2, 3\} \) are symmetric in the gauge indices. In addition, we have

\[
J^j_r \partial_j z^r = -\partial_0 z^0 \tag{5.15}
\]

\[
J^j_p \partial_j z_q = -\frac{1}{2} \epsilon_{p'q'pq} J^{p'q'}_{k} \partial_k z_{q'}.
\]
The conditions (5.14) and (5.15) imply that in fact \( \{ u^p : p = 0, \ldots, 3 \} \) and \( \{ z^p : p = 0, \ldots, 3 \} \) are constant, i.e. independent from the sigma model superfield \( \phi \).

The above conditions are most easily derived by verifying that the Lagrangian density is independent of \( \theta + r \) up to surface terms in \( x^+, x^- \) and \( \theta_0^+ \). In addition, gauge invariance requires that the coupling constants \( u^p \) and \( z^p \) satisfy the condition

\[
f^d_{\ ab} u^p_{dc} + f^d_{\ ac} u^p_{bd} = 0 \\
f^c_{\ ab} z^p_{c} = 0.
\]

(5.16)

In particular \( u^r \) should be proportional to an invariant quadratic form on the Lie algebra of the gauge group \( G \). If \( G \) semi-simple, the condition on \( z^p \) implies that \( z^p = 0 \). If \( G \) is abelian, then the above conditions are satisfied for any constants \( u^p \) and \( z^p \).

5.6 The sigma model multiplet action

This part of the action has already been described in [8]. Here we shall summarise some of results relevant to this paper. The action of this multiplet is

\[
S_\sigma = -i \int d^2x d\theta_0^+ \left( g_{ij} \nabla_{0+} \phi^i \nabla_{0-} \phi^j + \sum_{r=1}^3 \nu_{ra} W^a_{r-} \right) \\
- i \int d^2x dt d\theta_0^+ \left( H_{ijk} \partial_t \phi^i \nabla_{0+} \phi^j \nabla_{0-} \phi^k - w_{ia} \partial_t \phi^i W^a_{0-} \right),
\]

(5.17)

where \( \nu_a \) is a function on \( M \), possibly locally defined, given by

\[
I_{r i} (\xi_{aj} + w_{aj}) = -\partial_i \nu_{ra}.
\]

(5.18)

It has been shown in [7] that under certain conditions \( \nu \) is a moment map of HKT geometry.

The gauge transformations of the superfield \( \phi \) are

\[
\delta \phi^i = \lambda^a \xi^i (\phi)
\]

(5.19)

Gauge invariance of the above action requires that \( w \) should satisfy the conditions mentioned in section 2.2; the moment maps should be globally defined on the sigma model target space \( M \) and

\[
\mathcal{L}_a \nu_{ra} = -f_{ab} c \nu_{rc}.
\]

(5.20)

5.7 The action of the fermionic multiplet

This part of the action is

\[
S_f = \int d^2x d\theta_0^+ \left( h_{AB} \psi^A \nabla_{0+} \psi^B + ms_A \psi^A \right)
\]

(5.21)
Gauge invariance of this part of the action requires the same conditions as those appearing for the couplings of (1,0)-multiplet in (5.19).

The conditions required by (4,0)-supersymmetry on the couplings of the above action are the same as those of the ungauged model and have been given in (5.13). These can be easily derived by requiring that the Lagrangian density is independent from \( \theta^+ \) up to \( x^+ \), \( x^- \), \( \theta^+_0 \) surface terms. In particular, we find that

\[
\begin{align*}
\frac{1}{2} \left( \delta \phi \right)_+^a \chi_p^a + \frac{1}{2} \chi_+^a \delta \phi^a & = 0 \\
\frac{1}{2} \left( \delta \phi \right)_-^a \chi_q^a + \frac{1}{2} \chi_-^a \delta \phi^a & = 0 \\
\frac{1}{2} \left( \delta \phi \right)_0^a \chi_{\alpha}^a + \frac{1}{2} \chi_0^a \delta \phi^a & = 0 \\
\end{align*}
\]

To derive this we have used that \( \nabla I_r = 0 \) as we have assumed in the construction of the fermionic multiplet. The first condition implies that the fibre metric is tri-hermitian. These conditions can be further simplified if the connection \( B \) is chosen such that \( \nabla h = 0 \). In such case, the third equation in (5.22) implies that \( s \) is the real part of three holomorphic sections of \( \mathcal{E}^* \) each with respect to the three doublets \( (J_r, I_r) \) of complex structures, i.e. \( s \) is triholomorphic.

### 5.8 The action in components and scalar potential

The part of the action of the theory involving the kinetic term of the gauge multiplets (5.13) can be easily expanded in components as follows:

\[
S_g = \int d^2 x \left( u^0_{ab} F^a_+ = F^b_+ + i u^0_{ab} \delta \phi^a \chi_p^a - \chi_q^b \right)
\]

\[
+ \frac{1}{2} \left( \delta \phi \right)_+^a \chi_p^a + \frac{1}{2} \chi_+^a \delta \phi^a = 0
\]

\[
+ \frac{1}{2} \left( \delta \phi \right)_-^a \chi_q^a + \frac{1}{2} \chi_-^a \delta \phi^a = 0
\]

\[
+ \frac{1}{2} \left( \delta \phi \right)_0^a \chi_{\alpha}^a + \frac{1}{2} \chi_0^a \delta \phi^a = 0
\]

To derive this we have used that \( \{ w^p : 0, \ldots, 3 \} \) are constant and \( \{ u^r : 1, 2, 3 \} \) symmetric in the gauge indices.

The part of the action that contains the kinetic term and the Wess-Zumino term of the sigma model fields in components is as follows,

\[
S_o = \int d^2 x \left( g_{ij} \nabla_i \phi^i \nabla_j \phi^j \right) + b_{ij} \partial_+ \phi^i \partial_+ \phi^j + i \lambda_+^a \nabla_+^{(i)} \lambda_+^j
\]

\[
- i \sum_r \left( \nu_{ra} f^a_r + \partial_r \nu_{ra} \lambda_+^a \lambda_r^a \right)
\]

\[
+ i g_{ij} \lambda_+^a \lambda_0^b - \xi_{\alpha}^a \lambda_+^a \lambda_0^a + i w_{ia} \lambda_+^a \lambda_0^a
\]

\[
- A_+^a w_{ia} \partial_+ \phi^i + A_+^a w_{ia} \partial_+ \phi^j - A_+^a A_+^b \xi_{[i} w_{a]}^j
\]

(5.24)
The part of the action that contains the kinetic term the fermionic multiplet in components is as follows:

\[
S_f = \int d^2x \left( -ih_{AB}\psi^A\nabla^+\psi_B - h_{AB}\ell^A\ell^B - \frac{1}{2}h_{AB}\psi^A\psi_B^B\lambda^i_+\lambda^j_+G_{i,jAB} \right)
+ m\nabla_i\lambda^i_+\psi^A - m_sA\ell^A )
\] (5.25)

After eliminating the auxiliary fields of both the gauge multiplet and the fermionic multiplet, we find that the action of (4,0)-supersymmetric gauge theories coupled to sigma models is

\[
S_g + S_f = \int d^2x \left( u^0_{ab}F^a_+F^b_+ + i\frac{1}{4}u^a_{ab}\lambda^i_+\lambda^j_+G_{i,jAB} - \frac{1}{4}m^2h^{AB}s_A\lambda^i_+\psi^A \right)
- \frac{1}{4}u^a_{ab}\nabla_{\mu}\nabla^\mu\psi^A - \frac{1}{4}z^0_aF^a_+ + \frac{1}{4}u^0_c\lambda^i_+\chi^a_c
\] (5.26)

It is straightforward to write the action \(S\) of the (2,0)-supersymmetric gauge sigma model in components. In particular we find the following:

\[
S = \int d^2x \left( g_{ij}\nabla^+_\phi^i\nabla^-\phi^j + b_{ij}\partial^+\phi^i\partial^-\phi^j + u^0_{ab}\ell^a_{\pm} + F^a_+ \right)
+ \frac{1}{4}u^a_{ab}\lambda^i_+\lambda^j_+G_{i,jAB}
+ \frac{1}{4}u^0_{ab}\lambda^i_+\psi^A
\] (5.27)
5.9 Scalar Potential and Classical Vacua

The scalar potential of the (4,0)-supersymmetric gauged sigma models is

\[ V = \frac{1}{4} \sum_{r=1}^{3} u_0^{ab} \nu_{ra} \nu_{rb} + \frac{1}{4} m h^{AB} s_A s_B , \]  

(5.28)

where we have absorbed the constants \( z_r^a \) into the definition of the moment maps \( \nu_{ra} \). The scalar potential in these models is written as a sum of a ‘\( D \)’ and an ‘\( F \)’ term. The classical supersymmetric vacua of the theory are those for which \( \nu_{ra} = 0 \) and \( s_A = 0 \).

\[ (5.29) \]

The inequivalent classical vacua are the space of orbits of the gauge group on the zero set of the section \( s \) and the HKT moment maps \( \nu_{ra} \). If the section \( s \) vanishes, then the space of inequivalent vacua is the theory is the HKT reduction of the sigma model target space \( M//G \). It has been shown in \([17]\) that under certain assumptions the space of vacua inherits the HKT structure of the sigma model manifold \( M \) and it is a smooth space. However the three-form of the Wess-Zumino term on \( M//G \) is not necessarily closed.

6 (1,1) supersymmetry

6.1 The gauge multiplet

The (1,1) superspace \( \Xi^{1,1} \) has coordinates \((x^+, x^-, \theta^+, \theta^-)\), where \( \theta^\pm \) are Grassman valued odd coordinates. The (1,1)-supersymmetric Yang-Mills multiplet is described by a connection \( A \) in superspace with components \((A^+, A_- , A_+, A_-)\). In addition it is required that these satisfy the supersymmetry constraints \[ \begin{align*}
[\nabla_+ , \nabla_-] &= W \\
[\nabla_+ , \nabla_+] &= 2i \nabla_+ \\
[\nabla_-, \nabla_-] &= 2i \nabla_-
\end{align*} \]

(6.1)

(We have suppressed the gauge indices.) The Jacobi identities imply that

\[ [\nabla_+, \nabla_-] = i \nabla_- W \quad [\nabla_-, \nabla_+] = i \nabla_+ W \]

\[ F_{\pm\pm} = \nabla_+ \nabla_- W \]

(6.2)

It is worth mentioning that the (1,1)-supersymmetric gauge multiplet can be constructed from a scalar superfield. This allows for the possibility of non-polynomial couplings between the sigma model multiplet and the gauge multiplet. The components of the gauge multiplet are

\[ \begin{align*}
W &= W| \\
\chi_+ &= \nabla_+ W| \\
F_{\pm\pm} &= \nabla_+ \nabla_- W|
\end{align*} \]

(6.3)

The field, \( W \), is a scalar, \( \chi_+ , \chi_- \) are gaugini which are real chiral fermions in two dimensions, and \( F_{\pm\pm} \) is the (gauge) field strength.
6.2 The sigma model multiplet

Let $M$ be a Riemannian manifold with metric $g$ and locally defined two-form $b$. We take the gauge group $G$ to act on $M$ with isometries preserving the Wess-Zumino three-form $H = db$ and generating the vector fields $\xi_a$ as in section 2.1. In addition we assume that the one-form $w$ satisfies the conditions of section 2.2.

The sigma model $(1,1)$ multiplet $\phi$ is a map from the $(1,1)$ superspace $\Xi^{1,1}$ into the Riemannian manifold $M$. The components of $\phi$ are as follows,

$$
\phi^i = \phi^i|_{\ell^i} = \nabla_+ \phi^i|_{\ell^i} = \nabla_- \phi^i|_{\ell^i},
$$

where $\nabla_+ \phi^i = D_+ \phi^i + A_a^i \xi_a^{i}$ and similarly for $\nabla_-$. Observe that the first two components of $(1,0)$ superfield $\phi$ can be identified with the two components of a $(1,0)$ superfield $\phi$ while the latter two components can be identified with those of a $(1,0)$ fermionic superfield $\psi_\nu$. The vector bundle associated with this fermionic multiplet is the tangent bundle of $M$.

6.3 Action

The action of the $(1,1)$-supersymmetric gauged sigma model can be written as sum of three terms,

$$
S = S_g + S_\sigma + S_p,
$$

where $S_g$ is the action of the gauge multiplet, $S_\sigma$ is the action of the sigma-model multiplet and $S_p$ is a potential term. We shall describe each term separately.

6.4 The gauge multiplet action

An action of the $(1,1)$-supersymmetric gauge multiplet is most easily written in $(1,1)$ superspace. In particular we have

$$
S_g = \int d^2 x d\theta^+ d\theta^- \left( -u_{ab} \nabla_+ W^a \nabla_- W^b + \frac{1}{2} v_{ab} W^a W^b + z_a W^a \right),
$$

where $u_{ab} = u_{ab}(\phi)$, $u$ is not necessarily symmetric in the gauge indices, $v_{ab} = v_{ab}(\phi)$ and the theta term $z_a = z_a(\phi)$. Of course this action is manifestly $(1,1)$-supersymmetric because it is an integral over full superspace. Gauge invariance imposes the additional conditions

$$
\mathcal{L}_a u_{bc} = -f^d_{ab} u_{dc} - f^d_{ac} u_{bd}, \\
\mathcal{L}_a v_{bc} = -f^d_{ab} v_{dc} - f^d_{ac} v_{bd}, \\
\mathcal{L}_a z_b = -f^d_{ab} z_d
$$

on the couplings $u,v$ and $z$. 25
The action (6.6) can be easily expanded in components to find

\[ S_g = \int d^2x \left( + u_{ab} F^a_{\pm} F^b_{\pm} - u_{ab} \nabla^a \nabla^b W^a \nabla^b W^b + i u_{ab} \nabla^a \chi^b_{\pm} - i u_{ab} \chi^a_{\pm} \nabla^b \right) \]

\[ + \partial_t u_{ab} F^a_{\pm} F^b_{\pm} - \partial_t u_{ab} \chi^a_{\pm} \chi^b_{\pm} + i \partial_t u_{ab} \lambda^i_{\pm} \nabla^a \nabla^b W^a \nabla^b W^b + i \partial_t u_{ab} \lambda^i_{\pm} \nabla^a \nabla^b W^a \nabla^b W^b + u_{ab} f_{cd} \lambda^c \nabla^d W^a W^b - \nabla^i \partial_j u_{ab} \lambda^i_{\pm} \lambda^a_{\pm} \chi^b_{\pm} - \partial_t u_{ab} \ell^i \lambda^a_{\pm} \chi^b_{\pm} + v_{ab} \chi^a_{\pm} W^b + \ell^i \partial_t z^a W^a + \lambda^i_{\pm} \lambda^a_{\pm} \nabla^i \partial_j z^a W^a \right) . \] 

\[ (6.8) \]

6.5 The sigma model multiplet action and potential term

A (1,1)-supersymmetric gauged sigma model action has been given in [8]. This action can be written as

\[ S_\sigma = \int d^2x d\theta^+ d\theta^- g_{ij} \nabla^i \nabla^j - g_{ij} \nabla^i \phi^j \]

\[ + \int d^2x d\theta^+ d\theta^- (H_{ijk} \partial^i \phi^j \nabla^k - w_{ia} \partial^i \phi^j W^a) \] 

\[ (6.9) \]

This can be rewritten without the \( t \) integration as

\[ S_\sigma = \int d^2x d\theta^+ d\theta^- \left( g_{ij} \nabla^i \phi^j + b_{ij} D_+ \phi^i D_- \phi^j - A^a_{\pm} w_{ia} D_- \phi^i - A^a_{\pm} w_{ia} D_+ \phi^i + A^a_{\pm} A^b_{\pm} \xi^i w_{ai} \right) . \] 

\[ (6.10) \]

It is straightforward to add a potential term to the above actions as

\[ S_p = \int d^2x d\theta^+ d\theta^- h \] 

\[ (6.11) \]

where \( h = h(\phi) \) is a function of the superfield \( \phi \). Gauge invariance of the above action requires that \( w \) should satisfy the conditions stated in section 2.2.

6.6 A generalisation of the action

The action of (1,1)-supersymmetric gauge theory presented above can be generalized by allowing the various couplings of the theory to depend on the scalar component of the gauge multiplet superfield. Supersymmetry then requires
additional fermionic couplings. The new theory can be organised as a (1,1)-supersymmetric sigma model which has target space \( L = M \times \mathcal{L}(G) \), where \( \mathcal{L}(G) \) is the Lie algebra of the gauge group \( G \). The various allowed couplings are restricted by two-dimensional Lorentz invariance, supersymmetry and gauge invariance. The superfields of the (1,1)-supersymmetric gauge theory coupled to sigma model matter are maps \( Z = (\phi, W) \) from the (1,1) superspace \( \Xi^{1,1} \) into \( L \), where \( \phi \) is the usual (1,1) sigma model superfield and \( W \) is the (1,1) gauge theory multiplet. We again allow the gauge group \( G \) to act on \( L \) with a group action on \( M \) and the adjoint action on \( \mathcal{L}(G) \). The vector fields generated by such a group action are

\[
\xi_a = \xi^A_a \partial_A - f^c_{\ ab} \partial_c \phi + W^b f^c_{\ ab} \partial_c \phi + \ldots
\]  

where \( \xi_a \) denotes a vector field generated by the group action. The partial derivative with the gauge index denotes differential with respect to \( W \).

Next we introduce a metric \( g \) and a Wess-Zumino term \( H \) on \( L \) and assume that the gauge group \( G \) acts on \( L \) with isometries leaving the Wess-Zumino term \( H \) invariant. We also define \( w \) as \( \iota_{\xi} H = dw \), where \( \xi_a \) is the new Killing vector field \( (6.12) \). Then an action can be written for this new sigma model as

\[
S_\sigma = \int d^2 x d\theta^+ d\theta^- \left( g_{AB} \nabla^+ Z^A \nabla^- Z^B + h \right) + \int d^2 x dt d\theta^+ d\theta^- \left( H_{ABC} \partial_t Z^A \nabla^+ Z^B \nabla^- Z^C + w_{Ba} \partial_t Z^B W^a \right)
\]

where \( h \) is a function which depends on \( Z \). This action is clearly supersymmetric because it is a full (1,1) superspace integral. Gauge invariance requires that \( w \) above satisfies all the conditions stated in section 2.2 but for the group action with associated vector fields \( (6.12) \) and a Wess-Zumino term in \( L \). In addition, gauge invariance requires that

\[
\mathcal{L}_\xi h = 0
\]

where the Lie derivative is with respect to the vector field \( (6.12) \).

### 6.7 Scalar Potential

To compute the scalar potential we express the action in components and eliminate the auxiliary field of the sigma model superfield \( \phi \) from the action using the field equations. The scalar potential is

\[
V(W, \phi) = \frac{1}{4} g^{ij} \partial_i h \partial_j h ,
\]

where \( g^{ij} \) is the inverse of the restriction of the metric of \( L \) on \( M \). Observe that \( V \) depends on both the sigma model scalar \( \phi \) and the gauge multiplet scalar \( W \). The classical supersymmetric vacua of the theory are those values of \( (\phi, W) \) for which \( \partial_i h = 0 \). For example, for the special (1,1)-supersymmetric model investigated in the beginning of the section, \( V = \frac{1}{4} g^{ij} (\partial_i h + \partial_j z_0 W^a) (\partial_j h + \partial_i z_b W^b) \), where in this case \( h = h(\phi) \).
7 (2, 1) supersymmetry

7.1 The gauge multiplet

The (2, 1) superspace $\Xi^{2,1}$ has coordinates $\{x^+, x^-, \theta^+ \, \theta^-\}$, where $(x^+, x^-)$ are the even and $(\theta^+ \, \theta^-)$ are odd coordinates. The (2,1)-supersymmetric Yang-Mills multiplet is described by a connection $A$ in superspace with components $(A^+, A^-, A^0, A^1)$, $p=0,1$. In addition it is required that these satisfy the supersymmetry constraints

\[ \nabla_+ W = F_+ \]
\[ \nabla_- = 2i\delta pq \nabla_p \]  

(7.1)

We have suppressed the gauge indices. We remark that $W_p$ are scalar superfields. The Jacobi identities imply that

\[ \nabla_+ W = i\nabla_- W \]
\[ F^a_+ = \nabla_+ \nabla_- W_a \]
\[ \nabla_1 W_0 + \nabla_0 W_1 = 0 \]  

(7.2)

The two scalar superfields $(W_0, W_1)$ can be viewed as a map $W$ from the (2,1) superspace $\Xi^{2,1}$ into $LG \otimes \mathbb{R}^2$, where $LG$ is the Lie algebra of the group $G$. Next introduce a complex structure $I = \text{Id} \otimes \epsilon$ in $LG \otimes \mathbb{R}^2$ where $\epsilon$ is the constant complex structure in $\mathbb{R}^2$ with $\epsilon_{01} = -1$. The last two conditions in (7.1) can be expressed as

\[ \nabla_1 W_0 + \nabla_0 W_1 = 0 \]  

(7.3)

In fact this implies that $W$ is a covariantly chiral superfield, $(\nabla_1 + i\nabla_0)(W_1 + iW_0) = 0$.

The components of the gauge superfields $W_p$ are

\[ W_p = W_p \]
\[ F^a_+ = \nabla_+ \nabla_- W^a_0 \]
\[ f = \nabla_0 \nabla_- W_1 \]
\[ \chi_{0+} = \nabla_0 W_0 \]
\[ \chi_{1+} = \nabla_0 W_1 \]
\[ \chi_{p-} = \nabla_- W_p \]  

(7.4)

where $W_p$ are scalars, $\chi_{p+}$ and $\chi_{p-}$ are the gaugini which are real chiral fermions in two dimensions, $F^a_+$ is the field strength and $f$ is a real auxiliary field. As in the (1,1)-supersymmetric gauge theory, the gauge multiplet is determined by scalar superfields. This will lead again to non-polynomial interactions between the gauge and sigma multiplets of the theory.

7.2 The sigma model multiplet

Let $M$ be a KT manifold with metric $g$ and complex structure $J$. We in addition assume that the gauge group $G$ acts on $M$ with isometries which furthermore preserve the complex structure $J$ and the Wess-Zumino term $H$. These conditions are the same as those in the case of (2,0)-supersymmetric theory. The
(2,1) sigma model superfield $\phi$ is a map from the (2,1) superspace $\mathbb{E}^{2,1}$ into the sigma model manifold $M$. In addition it is required that

$$\nabla_{1^+}\phi^i = J^i_j \nabla_{0^+}\phi^j ,$$  \hspace{1cm} (7.5)$$

where $\nabla_{p^+}\phi^i = D_{p^+}\phi^i + A_{p^+}^a\xi_a^i$ and $\xi_a^i$ are the vector fields on $M$ generated by the group action. As we have seen, the (2,1) gauge multiplet satisfies the condition (7.3) similar to (7.5). The superfield $\phi$ is also covariantly chiral, as can be seen by choosing complex coordinates on the sigma model manifold $M$. These results will be used later for the construction of actions of (2,1)-supersymmetric gauge theories coupled to sigma models.

The components of the sigma model multiplet $\phi$ are as follows:

$$\phi^i = \phi^i |$$
$$\ell^i = \nabla_{0^+} \nabla_{-}\phi^i |$$
$$\lambda^i_+ = \nabla_{0^+}\phi^i |$$
$$\lambda^i_- = \nabla_{-}\phi^i |,$$

(7.6) where $\phi$ is a scalar, $\lambda_+$ and $\lambda_-$ are real fermions, and $\ell$ is an auxiliary field.

### 7.3 Action

An action of a (2,1)-supersymmetric gauge theory coupled to sigma model matter can be written as

$$S = S_g + S_{\sigma} + S_p ,$$  \hspace{1cm} (7.7)$$

where $S_g$ is the action of the gauge multiplet, $S_{\sigma}$ is the action of the sigma-model multiplet and $S_p$ contains the potential term. We shall describe each term separately.

### 7.4 The gauge multiplet action

An action for the (2,1)-supersymmetric gauge multiplet is

$$S_g = \int d^2x d\theta_0^+ d\theta^- ( - u^0_{ab} \delta^{pq}_0 \nabla_{0^+} W^a_p \nabla_{-} W^b_q + u^1_{ab} \delta^{pq}_0 \nabla_{1^+} W^a_p \nabla_{-} W^b_q + z^a W^a )$$  \hspace{1cm} (7.8)$$

where $u^0, u^1$ are the gauge coupling constants and $z^a$ are theta term type of couplings. All the couplings are allowed to depend on the superfield $\phi$. We shall assume that both $u^0, u^1$ are symmetric in the gauge indices but this restriction can be lifted.

Observe that the action (7.3) is not an integral over the full $\mathbb{E}^{2,1}$ superspace. Therefore it is not manifestly (2,1)-supersymmetric. The requirement of invariance of the action under (2,1) supersymmetry imposes the conditions

$$J^j_i \partial_j u^0_{ab} = - \partial_i u^1_{ab}$$
$$J^j_i \partial_j z^a = - \partial_i z^a .$$  \hspace{1cm} (7.9)$$
Therefore the couplings $u^0 + iu^1$ and $z^1 + iz^0$ are holomorphic.

In addition gauge invariance of the action (7.8) implies that

$$L_a u^p_{bc} = - f_{ab} u^p_{dc} - f_{ac} u^p_{bd}$$
$$L_a z^p_b = - f_{ab} z^p_d .$$

(7.10)

7.5 The sigma model multiplet action and potential

The action of the (2,1)-supersymmetric gauged sigma model with Wess-Zumino term has been given in [8]. Here we shall summarise some of results relevant to this paper. The action of this multiplet is

$$S = \int d^2 x d\theta^+ d\theta^- \left( g_{ij} \nabla_0^+ \phi^i \nabla_- \phi^j + \nu_a W^a_i \right)$$
$$+ \int d^2 x dt d\theta^+ d\theta^- \left( H_{ijk} \partial_t \phi^i \nabla_0^+ \phi^j \nabla_- \phi^k - w_{ia} \partial_t \phi^i W^a_i \right)$$

(7.11)

This action can be written without the $t$ integration as

$$S = \int d^2 x d\theta^+ d\theta^- \left( g_{ij} \nabla_+ \phi^i \nabla_- \phi^j + b_{ij} D_+ \phi^j D_- \phi^j \right.$$  
$$- A^a_+ w_{ia} D_- \phi^j - A^a_- w_{ia} D_+ \phi^j + A^a_+ A^b_+ \xi^i_{[b} w_{a]i} \bigg)$$

(7.12)

Gauge invariance of the above action requires that $w$ should satisfy the conditions described in section 2.2. As in the case of (2,0)-supersymmetric gauged sigma model, it is also required that $\nu$ is globally defined and $L_a \nu_b = - f_{ab} \nu_c$. In fact $\nu$ is a moment map associated with the action of the gauge group on the KT manifold $M$.

The part of the action involving the potential is

$$S_p = \int d^2 x d\theta^+ d\theta^- h ,$$

(7.13)

where $h = h(\phi)$ Invariance under (2,1) supersymmetry requires that

$$\partial_i h = J^k_i \partial_k h^1$$

(7.14)

where $h^1 = h^1(\phi)$. This implies that $h$ is the real part of a holomorphic function on $M$.

The scalar potential of (2,1)-supersymmetric gauge theories coupled to sigma models described above is

$$V = \frac{1}{4} w^a_0 (\nu_a + z^1_a)(\nu_b + z^1_b) + \frac{1}{4} \phi^{ij} (\partial_i h + \partial_i z^p_a W^a_p)(\partial_j h + + \partial_j z^p_a W^a_p) .$$

(7.15)
7.6 A generalisation

As we have shown both the (2,1) gauge multiplet and the (2,1) sigma model multiplet are constructed from covariantly chiral scalar superfields, ie both satisfy the conditions (7.5) and (7.3). Because of this, these two superfields can be combined to a single superfield \( Z = (W_0, W_1, \phi) \) which is a map from the (2,1) superspace \( \Sigma^{2,1} \) into \( (LG \otimes \mathbb{R}^2) \times M \). In addition we can take \( Z \) to satisfy a chirality condition which is the combination of (7.3) and (7.5). Next we can take the gauge group \( G \) to act on \( (LG \otimes \mathbb{R}^2) \times M \) with the adjoint action in the first factor and a group action on \( M \). The vector fields associated by such a group action are

\[
\xi_a = \sum_p f^c_{ab} W_p^b \frac{\partial}{\partial W^c_p} + \xi^i \partial_i .
\]  
(7.16)

Treating the (2,1)-supersymmetric gauge theory coupled to sigma model matter as a sigma model with superfield \( Z \), which satisfies (7.3) and (7.5), we can write the action

\[
S_\sigma = \int d^2x d\theta^- (g_{AB} \nabla_{\theta^0} Z^A \nabla_- Z^B + \nu_\phi W^\phi_i + h) \\
+ \int d^2x dt d\theta^+ d\theta^- (H_{ABC} \partial_t Z^A \nabla_{\theta^0} Z^B \nabla_- Z^C - w_{Ba} \partial_t Z^B W^a) 
\]  
(7.17)

where now all the couplings are defined using the geometry of \((LG \otimes \mathbb{R}^2) \times M\). Of course (2,1)-supersymmetric requires that \((LG \otimes \mathbb{R}^2) \times M\) is a KT manifold with respect to the complex structure \((J, id \otimes \epsilon)\). In particular the metric and the rest of the couplings depend on the coordinates of \((LG \otimes \mathbb{R}^2) \times M\). The conditions for gauge invariance are easily determined from those of the (2,1)-supersymmetric gauged sigma model. We remark that the couplings of (7.17) can be arranged such that the \( SO(2) \) R-symmetry of the (2,1)-supersymmetry algebra is broken. In particular the \( SO(2) \) rotation that rotate the \( W_p \) scalar components is not a symmetry of the action. However if one insists in preserving the R-symmetry, then the KT manifold \((LG \otimes \mathbb{R}^2) \times M\) should admit a \( SO(2) \) action preserving all the geometric data.

8 (4,1) supersymmetry

8.1 The gauge multiplet

The (4,1) superspace \( \Sigma^{4,1} \) has coordinates \((x^+, x^-, \theta^+_{p}, \theta^-)\), where \((x^+, x^-)\) are the even and \(\{\theta^-, \theta^+_{p}, p = 0, \ldots, 3\}\) are the odd coordinates. The (4,1)-supersymmetric Yang-Mills multiplet is described by a connection \( A \) in \( \Sigma^{4,1} \) superspace with components \((A_+, A_-, A_{p+}, A_-)\) with \( p = 0, \ldots, 3 \). In addition...
it is required that these satisfy the supersymmetry constraints
\[ ([\nabla_p^+, \nabla_-] = W_p) \quad \text{and} \quad ([\nabla_+, \nabla_-] = F_\#) \]
\[ ([\nabla_p^+, \nabla_q^+] = 2i\delta_{pq}\nabla_\#) \quad \text{and} \quad ([\nabla_-, \nabla_-] = 2i\nabla_-) \]  \hspace{1cm} (8.1)
\[ \nabla_{p^+} W_q = \epsilon_{pq} v^q \nabla_{p^+} W_{q^+}. \]

(We have suppressed all the gauge indices.) The Jacobi identities imply that
\[ ([\nabla_p^+, \nabla_+] = i\nabla_- W_p) \]
\[ ([\nabla_-, \nabla_-] = i\nabla_{0+} W_0) \]
\[ F_\# = \nabla_{0+} \nabla_- W_0^a \]  \hspace{1cm} (8.2)
\[ \nabla_{p^+} W_q + \nabla_{q^+} W_p = 0 \quad (p \neq q) \]
\[ \nabla_{0+} W_0 = \nabla_{1+} W_1 = \nabla_{2+} W_2 = \nabla_{3+} W_3 \]

The (4,1) gauge multiplet is determined by four scalar superfields. Some of the conditions on these superfields given in (8.1), like in the (2,1) model previously, can be expressed as conditions of a (4,1) sigma model multiplet. For this, view the four-scalar superfields \( \{ W_p : p = 0, 1, 2, 3 \} \) as maps from the superspace \( \Xi^{4,1} \) into \( LG \otimes \mathbb{R}^4 \), where \( LG \) is the Lie algebra of the gauge group \( G \). Then introduce three constant complex structures \( \{ I_r \} \) in \( \mathbb{R}^4 \) such that \( (I_r)^0_s = \delta_{rs} \) and \( (I_r)^i_t = -\epsilon_{rst} \) where \( r, s, t = 1, 2, 3 \). The conditions on \( W_p \) in (8.1) and (8.2) can be expressed as
\[ \nabla_{r^+} W_p^a = I_r^q \nabla_{0^+} W_q^a. \]  \hspace{1cm} (8.3)

The components of the gauge multiplet are
\[ W_p = W_p^| \quad F_\# = \nabla_{0^+} \nabla_- W_0^| \]
\[ \chi_{p^+} = \nabla_- W_p^| \quad \chi_{p^-} = \nabla_{0^+} W_p^| \]  \hspace{1cm} (8.4)
\[ f_r = \nabla_{0^+} \nabla_- W_r^| \quad r = 1, 2, 3 \]

where \( W_p \) are scalars, \( \chi_{p^+}, \chi_{p^-} \) are the gaugini which are real chiral fermions in two dimensions, \( F_\# \) is the field strength and \( \{ f_r : r = 1, 2, 3 \} \) are auxiliary fields. The \( SO(4) \) R-symmetry of the (4,1)-supersymmetric gauge theory rotates both the scalars and the fermions of the gauge multiplet.

### 8.2 The sigma model multiplet

Let \( M \) be a HKT manifold with metric \( g \) and hypercomplex structure \( \{ J_r : r = 1, 2, 3 \} \). We in addition assume that the gauge group \( G \) acts on \( M \) with isometries which in addition preserve the hypercomplex structure \( J_r \) and the Wess-Zumino term \( H \). These conditions are the same as in the case of (4,0)-supersymmetric model. The (4,1) sigma model superfield \( \phi \) is a map from the (4,1) superspace \( \Xi^{4,1} \) into the sigma model manifold \( M \). In addition it is required that
\[ \nabla_{r^+} \phi^j = J_r^{i} j \nabla_{0^+} \phi^j, \]  \hspace{1cm} (8.5)
\[ \nabla_{p+} \phi^i = D_{p+} \phi^i + A_{p+}^a \xi^i_a \]
and \( \xi \) are the vector fields on \( M \) generated by the group action. As we have seen the \((4,1)\) gauge multiplet satisfies the condition \((8.3)\) similar to \((8.3)\). These results will be used later for the construction of actions of \((4,1)\)-supersymmetric gauge theories coupled to sigma models.

The components of the sigma model \((4,1)\) multiplet \( \phi \) are as follows,

\[ \begin{align*}
\phi^i &= \phi^i |_{\ell^i} = \nabla_0+ \phi^i |_{\lambda^i_+} = \nabla_{\lambda^i_+} \phi^i |_{\lambda^i_-} = \nabla_{\lambda^i_-} .
\end{align*} \tag{8.6} \]

where \( \phi \) is a scalar, \( \lambda_+ \) and \( \lambda_- \) are real fermions, and \( \ell \) is an auxiliary field.

### 8.3 Action

The action of a \((4,1)\)-supersymmetric gauged theory coupled to sigma model matter can be written as

\[ S = S_g + S_\sigma + S_p , \tag{8.7} \]

where \( S_g \) is the action of the gauge multiplet, \( S_\sigma \) is the action of the sigma-model multiplet and \( S_p \) is the potential. We shall describe each term separately.

### 8.4 The gauge multiplet action

An action for the \((4,1)\)-supersymmetric gauge multiplet is

\[ S_g = \int d^2 x d\theta^+_0 \left( - w^0_{ab} \delta_{pq} \nabla_{0+} W^a_p \nabla_{-} W^b_q + w^r_{ab} \delta_{pq} \nabla_{r+} W^a_p \nabla_{-} W^b_q + z^p_{a} W^a_p \right) \tag{8.8} \]

where \( \{ w^p \} = \{ w^0, w^r \} \) and \( z^p \) are the gauge coupling constants and theta type of terms, respectively, which in general depend on the superfield \( \phi \). We shall assume that both \( w^p \) are symmetric in the gauge indices but this restriction can be lifted.

Observe that this action is not an integral over the full \( \Xi^{4,1} \) superspace. Therefore it is not manifestly \((4,1)\)-supersymmetric. The requirement of invariance under \((4,1)\) supersymmetry imposes the condition that \( w^p \) and \( z^p \) are constant. This is similar to the condition that arises in \((4,0)\) supersymmetric gauge theories. In addition, gauge invariance of the action \((8.8)\) requires that

\[ f^d_{ab} w^p_{dc} + f^d_{ac} w^p_{bd} = 0 , \quad f^d_{ab} z^p_d = 0 . \tag{8.9} \]

Thus \( w^p \) must be invariant quadratic forms on the Lie algebra of the group \( G \) and \( z^p \) must be invariant elements of the Lie algebra. Of course \( z^p = 0 \), if \( G \) is semi-simple.
8.5 The sigma model multiplet action and the potential

An action for the (4,1) sigma model multiplet coupled to gauge fields has been given in \[8\]. Here we shall summarise some of results relevant to this paper.

The action of the (4,1)-supersymmetric gauged sigma model is

\[
S_{\sigma} = \int d^2x \theta^+ d\theta^- (g_{ij} \nabla^+ \phi^i \nabla^j - \phi^i + \sum_r \nu^a_r W^a_r) + \int d^2x dt d\theta^+ d\theta^- (H_{ijk} \partial_t \phi^i \nabla^j - \phi^k - \omega^a_i \partial_t \phi^i W^a_i).
\] (8.10)

The gauge transformations of \(\phi\) are \(\delta \phi^i = \lambda^a \xi^a_i(\phi)\). Gauge invariance of the above action requires that \(w\) should satisfy the conditions described in section 2.2. As in the case of (4,0)-supersymmetric gauged sigma model, \(\nu^r\) should satisfy \(\mathcal{L}_a \nu^r_b = -f_{ab}^c \nu^r_c\). In fact \(\nu^r\) is a moment map associated with the action of the gauge group \(G\) on the HKT manifold \(M\).

The part of the action involving the potential is

\[
S_p = \int d^2x d\theta^+ d\theta^- h, \tag{8.11}
\]

where \(h = h(\phi)\). Invariance under (4,1) supersymmetry requires that

\[
\partial_\theta h = J^k_i \partial_i h^r
\] (8.12)

where \(h^r = h^r(\phi)\). This implies that \(h\) is the real part of three holomorphic functions on \(M\), i.e., \(h\) is tri-holomorphic.

The scalar potential of (4,1)-supersymmetric gauge theories coupled to sigma models is

\[
V = \frac{1}{4} \nu^a_b \sum_{r=1}^3 \nu^r_a b^r + \frac{1}{4} g^{ij} \partial_i h \partial_j h, \tag{8.13}
\]

where we have shifted the moment maps \(\nu^r\) by a constant \(z^r\).

8.6 A generalisation

As we have shown both the (4,1) gauge multiplet and the (4,1) sigma model multiplet are constructed from scalar superfields which satisfy the similar constraints \(8(3)\) and \(8(5)\), respectively. Because of this, these two superfields can be combined to a single superfield \(Z = (W, \phi)\) which is a map from the (4,1) superspace \(\mathbb{H}^{1,1}\) into \((\mathcal{L}G \otimes \mathbb{R}^4) \times M\). In addition we can take \(Z\) to satisfy a condition which is the combination of \(8(3)\) and \(8(3)\). Next we can take the gauge group \(G\) to act on \((\mathcal{L}G \otimes \mathbb{R}^4) \times M\) with the adjoint action in the first factor and a group action on \(M\). The vector fields associated by such a group action are

\[
\xi_a = \sum_p f^e_{ab} W^b_p \frac{\partial}{\partial W^e_p} + \xi^i \partial_i. \tag{8.14}
\]
Treating the (4,1)-supersymmetric gauge theory coupled to sigma model matter as a sigma model with superfield $Z$, which satisfies (8.3) and (8.5), we can write the action

$$S = \int d^2x d\theta^+ d\theta^- (g_{AB} \nabla_{0+} Z^A \nabla_{-} Z^B + \sum_r \nu^r W^a_r + h)$$

$$+ \int d^2x dt d\theta^+ d\theta^- (H_{ABC} \partial_t Z^A \nabla_{0+} Z^B \nabla_{-} Z^C - w_B \partial_t Z^B W^a)$$

(8.15)

where now all the couplings are defined using the geometry of $(LG \otimes \mathbb{R}^4) \times M$. Of course (4,1)-supersymmetry requires that $(LG \otimes \mathbb{R}^4) \times M$ is a HKT manifold with respect to the hypercomplex structure $(J_r, I_r)$. In particular the metric and the rest of the couplings depend on the coordinates of $(LG \otimes \mathbb{R}^4) \times M$. The conditions for gauge invariance are easily determined from those of the (4,1)-supersymmetric gauged sigma model. We remark that the couplings of (8.15) can be arranged such that the $SO(4)$ R-symmetry of the (4,1)-supersymmetry algebra is broken. In particular the $SO(4)$ rotation that rotates the $W_p$ scalar components is not a symmetry of the action. However if one insists in preserving the R-symmetry, then the HKT manifold $(LG \otimes \mathbb{R}^4) \times M$ should admit a $SO(4)$ action preserving all the geometric data.

9 A bound for vortices in the (2,0) model

Vortices are the instantons of two-dimensional gauge theories coupled to sigma models. Bogomol’nyi type of bounds for both abelian [20] and non-abelian vortices [21, 22] have been investigated in the context of linear sigma models. Here we shall establish bounds for vortices for non-linear sigma models. For this we shall consider the Euclidean action of the (2,0)-supersymmetric gauge sigma model without Wess-Zumino term. The sigma model target space $M$ is Kähler with metric $g$, complex structure $J$ and associated Kähler form $\Omega_J((\Omega_J)_{ij} = g_{ik} J_{kj})$. After a Wick rotation the two-dimensional spacetime is $\mathbb{R}^2$ with the standard Euclidean metric. The relevant part of the bosonic Euclidean action of a (2,0)-supersymmetric gauge theory coupled to a sigma model is

$$S_E = \int_{\mathbb{R}^2} d^2x \left( \frac{1}{4} g_{ij} \delta^{\mu\nu} \nabla_\mu \phi^i \nabla_\nu \phi^j + \frac{1}{2} u_{ab} F^a_{\mu\nu} F^b_{\lambda\rho} \delta^{\mu\lambda} \delta^{\nu\rho} + \frac{1}{4} u^{ab} \nu_a \nu_b \right).$$

(9.1)

Next we introduce $I$ a constant complex structure on $\mathbb{R}^2$ such that $\mathbb{R}^2$ is a Kähler manifold. The associated Kähler form $\Omega_I$ is the volume form of $\mathbb{R}^2$. In such a case the Euclidean action (9.1) can be rewritten as

$$S_E = \int_{\mathbb{R}^2} d^2x \left[ \frac{1}{4} u_{ab} ((\Omega_I \cdot F^a + \frac{1}{2} \nu^a)(\Omega_I \cdot F^b - \frac{1}{2} \nu^b)

+ \frac{1}{4} g_{ij} \delta^{\mu\nu} (I^\mu \nabla_\nu \phi^i \equiv \nabla_\mu \phi^k J^i_k)(I^\nu \nabla_\sigma \phi^j \equiv \nabla_\nu \phi^\ell J^j_\ell) \right]$$

(9.2)
where $\Omega I \cdot F = (\Omega I)^{\mu \nu} F_{\mu \nu}$, $\nu^a = u^{ab} \nu_b$, $u_{ab} = u_{ab}^0$ and $u_{ac} u_{cb} = \delta^a_b$ ($u_{ab} = u_{(ab)}$). We remark that the above expression for the Euclidean action has been constructed from (9.1) by completing squares and collecting all the remaining terms which organise themselves in the last term of (9.2).

The last term in (9.2),

$$Q = \int_{\mathbb{R}^2} \omega_J,$$

(9.3)
is a topological charge, where the form

$$\omega_J = (\Omega J)_{ij} \nabla \phi^i \wedge \nabla \phi^j + \nu_a F^a$$

(9.4)
is the equivariant extension of the Kähler form $\Omega J$ of the sigma model target space $M$. The form $\omega_J$ is closed. Viewing $\omega_J$ as form on $\mathbb{R}^2$, it is apparent. In fact $\omega_J$ is closed as a two-form on any manifold $N$ for any map $\phi$ from $N$ into the sigma model manifold $M$ and for any choice of connection $A$. This can be easily seen and we shall not demonstrate it here.

The Euclidean action of the (2,0)-supersymmetric two-dimensional gauge theory coupled to a sigma model is bounded by the absolute value of the topological charge $Q$, $S_E \geq |Q|$. This is because it is always possible to choose the signs in the Bogomol’nyi bound above such that the topological term is positive. If the topological charge is positive, then the bound is attained whenever

$$\Omega I \cdot F^a - \nu^a = 0$$

$$J^i_j \nabla_{\mu} \phi^j - \nabla_{\nu} \phi^i F_{\mu} = 0.$$  

(9.5)

In two-dimensions, the curvature $F$ is a (1,1)-form. Choosing complex coordinates $(z, \bar{z})$ on $\mathbb{R}^2$ with respect to the complex structure $I$, it is always possible to arrange using a (complex) gauge transformation that $A_{\bar{z}} = 0$. Choosing complex coordinates in the sigma model target space $M$ as well, it is easy to see that the second BPS condition implies that the map $\phi$ is holomorphic from the spacetime $\mathbb{R}^2$ into the sigma model manifold $M$.

A special case of this bound arises for gauge theories coupled to linear sigma models for which the sigma model manifold $M = \mathbb{R}^{2n}$ with the Euclidean metric and equipped with a constant compatible complex structure $J$. This case includes the Nielsen-Olesen vortices [19]. (For these, existence of a solution was shown in [23] and the moduli were studied in [30], [21] and more recently in [31], see also [32]). The case with a single complex scalar has been analysed in [20]. Choosing complex coordinates $\{q^\alpha; \alpha = 1, \ldots, n\}$ in $\mathbb{R}^{2n}$, we write

$$ds^2 = \sum_{\alpha} dq^\alpha dq^{\bar{\alpha}}$$

$$\Omega_J = -i \sum_{\alpha} dq^\alpha \wedge dq^{\bar{\alpha}}.$$  

(9.6)
Next consider the abelian group $U(1)$-action $q^\alpha \to e^{iQ^\alpha}q^\alpha$ which generates the holomorphic Killing vector fields

$$\xi = i \sum_{\alpha} Q_\alpha (q^\alpha \frac{\partial}{\partial q^\alpha} - q^{\bar{\alpha}} \frac{\partial}{\partial q^{\bar{\alpha}}}).$$

(9.7)

The moment map is

$$\nu = -\sum_{\alpha} (Q_\alpha q^\alpha q^{\bar{\alpha}}) - \Lambda,$$

(9.8)

where $\Lambda$ is a (cosmological) constant. This is an example of a $(2,0)$-supersymmetric gauged linear sigma model with gauge group $U(1)$ of the type considered in [5].

The topological charge is

$$Q = \int_{R^2} d^2z \left( \sum_{\alpha} (\nabla_z q^\alpha \nabla_{\bar{z}} q^{\bar{\alpha}} - \nabla_{\bar{z}} q^\alpha \nabla_z q^{\bar{\alpha}}) + \nu F_{z\bar{z}} \right)$$

(9.9)

where $\nabla_z q^\alpha = \partial_z q^\alpha + iA_z q^\alpha$, $\nabla_{\bar{z}} q^{\bar{\alpha}} = \partial_{\bar{z}} q^{\bar{\alpha}} - iA_{\bar{z}} q^{\bar{\alpha}}$, $\nabla_z q^{\bar{\alpha}} = (\nabla_z q^\alpha)^*$, and $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z$. To compare the bound above (9.12) with that of vortices in [4], we observe that after some integration by parts we have

$$Q = \int_{R^2} d^2z \left( \sum_{\alpha} (\partial_z q^\alpha \partial_{\bar{z}} q^{\bar{\alpha}} - \partial_{\bar{z}} q^\alpha \partial_z q^{\bar{\alpha}}) - \Lambda F_{z\bar{z}} \right) + \text{surfaces}$$

(9.10)

The first term in the above expression is the topological charge expected for the vortices (instantons) of ungauged two-dimensional sigma models. The same topological charge also appears in the kink solitons of three-dimensional nonlinear sigma models. The last part in the above expression involving the cosmological constant and the Maxwell field is the usual degree of an abelian vortex. The relation between the topological charge $Q$ in (9.9) and the degree of an abelian vortex involves integration by parts. Under certain boundary conditions the two topological charges are the same. However as we have shown, the bound that involves the equivariant extension of the Kähler form generalizes in the context of gauge theories coupled to non-linear sigma models.

10 A bound for vortices in the $(4,0)$ model

A bound similar to the one we have described in the previous section for the Euclidean action of $(2,0)$-supersymmetric gauge theory coupled to sigma model matter can also be found for the Euclidean action of $(4,0)$-supersymmetric gauge theory. The Euclidean action of a $(4,0)$-supersymmetric gauge theory coupled to sigma model matter with vanishing Wess-Zumino term is

$$S_E = \int_{R^2} d^2x \left( \frac{1}{2} g_{ij} \delta^{\mu\nu} \nabla_\mu \phi^i \nabla_\nu \phi^j + \frac{1}{2} u_{ab} F_{\mu\nu}^a F_{\lambda\rho}^b \delta^{\mu\lambda} \delta^{\nu\rho} + \frac{1}{4} \sum_{r=1}^3 u^{ab} \sum_r \nu_r^a \nu_r^b \right)$$

(10.1)
The sigma model target space is hyper-Kähler with metric $g$, hypercomplex structure \{ $J_r : r = 1, 2, 3$ \} and associated Kähler forms $\Omega_J$. After the Wick rotation, the two-dimensional spacetime is $\mathbb{R}^2$ with the standard Euclidean metric. Let $I$ be a compatible constant complex structure such that $\mathbb{R}^2$ is a Kähler manifold with associated Kähler form $\Omega_I$. In such a case the Euclidean action can be written as

$$S_E = \int d^2 x \left[ \frac{1}{4} \sum_{r=1}^{3} u_{ab}(a_r \Omega_I \cdot F^a) + u_{r}^a (a_r \Omega_I \cdot F^b) \right]$$

(10.2)

where \{ $a_r : r = 1, 2, 3$ \} is a constant vector with length one, \( \sum_{r=1}^{3} (a_r)^2 = 1 \), $u_{ab} = u_{ba}^0$, $u_{r}^a = u_{a0r}^0$, $u^{ac}u_{cb} = \delta^a_b$, and

$$\omega_J_r = (\Omega_J_r)_{ij} \nabla \phi^i \wedge \nabla \phi^j + \nu_a F^a$$

(10.3)

is the equivariant extension of the Kähler form $\Omega_J$.

The strictest bound is attained whenever the unit vector \{ $a_r : r = 1, 2, 3$ \} is parallel to the vector of the topological charges $Q_r : r = 1, 2, 3$, where

$$Q_r = \int_{\mathbb{R}^2} \omega_J_r$$

(10.4)

and the sign is chosen such that the topological term in the bound is positive. If the inner product of \{ $a_r : r = 1, 2, 3$ \} and \{ $Q_r : r = 1, 2, 3$ \} in (10.2) is positive, we have that

$$S_E \geq \sqrt{Q_1^2 + Q_2^2 + Q_3^2}.$$  

(10.5)

This bound is attained whenever

$$a_r \Omega_I \cdot F^a - \nu_r^a = 0$$

$$J_r^i \nabla \phi^j - a_r \nabla \phi^i I^\nu \mu = 0.$$  

(10.6)

Using a rotation in the space of three complex structures, we can arrange such that $a_1 = 1$ and $a_2 = a_3 = 0$. In such case, the last equation in (10.6) implies that

$$\nabla \phi^i = 0.$$  

(10.7)

This in turn gives

$$F^a_{\mu \nu} \xi^i_{\mu} = 0.$$  

(10.8)
Therefore either $\phi$ takes values in the fixed point set $M_f$ of the group action of $G$ in $M$ or the curvature $F$ of the connection $A$ vanishes. In the latter case, the first equation in (10.6) implies that $\nu = 0$ and these are the vacua of the theory. If these are no non-trivial flat connections and $M_f \cap \nu_1^{-1}(0) \cap \nu_2^{-1}(0) \cap \nu_3^{-1}(0)$ is empty, then the space of solutions if the hyper-Kähler reduction $M//G$ of $M$ and it is a hyper-Kähler manifold. On the other hand if $\phi$ take values in $M_f$, then the second equation in (10.6) implies that $\phi$ are constant. Substituting this in the first equation in (10.6) implies that $\phi$ are in $M_f \cap \nu_2^{-1}(0) \cap \nu_3^{-1}(0)$.

In addition we have that

$$\Omega_I \cdot F^a - \nu^a = 0 .$$

(10.9)

This is the Hermitian-Einstein equation in two dimensions.

## 11 Kähler manifolds and non-abelian vortices

The bounds that we have described in the previous sections can be easily generalized as follows. Consider two Kähler manifolds $(N, h, I)$ and $(M, g, J)$ of dimensions $2k$ and $2n$, and Kähler forms $\Omega_I$ and $\Omega_J$, respectively. Next allow $M$ to admit a holomorphic $G$-action with associated killing vector fields $\xi$ and moment map $\nu$. In our conventions $i_\xi \Omega_J = -d\nu$. Next consider the functional

$$S_E = \int_N d\text{vol}(N) \left( \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} |F|^2 + \frac{1}{4} |\nu|^2 \right) ,$$

(11.1)

where $|\nabla|^2 = g_{ij} h^{\mu \nu} \nabla_\mu \phi^i \nabla_\nu \phi^j$, $\nabla_\mu \phi^j = \partial_\mu \phi^j + A^a_\mu \xi^a_j$, $|F|^2 = u_{ab} F^a_{\mu \nu} h^{\mu \rho} h^{\nu \sigma}$, $|\nu|^2 = u_{ab} \nu^a \nu^b$ and $u$ is a fibre inner product on the gauge bundle which we can set $u_{ab} = \delta_{ab}$.

The functional $S_E$ can be rewritten as follows:

$$S_E = \int_N d\text{vol}(N) \left[ \frac{1}{4} |\Omega_I \cdot F + \nu|^2 + |F^{2,0}|^2 + \frac{1}{4} |J \nabla \phi + J \nabla \phi|^2 \right]$$

$$\pm \frac{1}{(k-1)!} \int_N \omega_J \wedge \Omega_I^{k-1} - \frac{1}{(k-2)!} \int_N u_{ab} F^a \wedge F^b \wedge \Omega_I^{k-2}$$

(11.2)

where we have chosen the normalisation $d\text{vol}(N) = \frac{1}{k!} \Omega_I^k$, $\Omega_I \cdot F = \Omega_I^{\mu \nu} F_{\mu \nu}$, $F^{2,0}$ is the $(2,0)$ part of the curvature $F$ and

$$\omega_J = (\Omega_J)_{ij} \nabla_\phi^i \wedge \nabla_\phi^j + \nu_a F^a$$

(11.3)

is the equivariant extension of the Kähler form $\Omega_J$. (The inner products are taken with respect to the Riemannian metrics $h$ and $g$.) The rest of the notation is self-explanatory. We remark that if $\Omega_J$ represents the first Chern class of a line bundle, i.e. the Kähler manifold is Hodge, then $\omega_J$ can be thought of as the equivariant extension of the first Chern class (see [3]).

If $u_{ab}$ is a constant invariant quadratic form on the Lie algebra of the gauge group $G$, it is clear that the functional $S_E$ is bounded by a topological term
Q which involves the equivariant extension of the Kähler form and the second Chern character of the bundle \( P \times_G \mathcal{L}(G) \), where \( P \) is a principal bundle of the gauge group \( G \) and \( G \) acts on \( \mathcal{L}(G) \) with the adjoint representation. In particular we can write

\[
S_E = \int_N d\text{vol}(N) \left[ \frac{1}{4} \Omega_I \cdot F + |F|^2 + \frac{1}{4} |J \nabla \phi + J \nabla \phi|^2 \right] + \frac{1}{(k-1)!} \int_N \omega_J \wedge \Omega_{k-1}^I - \frac{8\pi^2}{\lambda(k-2)!} \int_N \text{ch}_2 \wedge \Omega_{k-2}^I ,
\]

where \( \lambda \) is an appropriate normalisation factor involving the ratio between the fibre inner product on \( P \times_G \mathcal{L}(G) \) and \( u \); where \( G \) is simple. It is worth pointing out that the term involving the second Chern character is not affected by the choice of sign in writing (11.2). Therefore there are three cases to consider the following: (i) there is no choice of sign such that the topological charge \( Q \) is positive. In such a case the bound cannot be attained. (ii) There is a critical case in which for one choice of sign the topological charge is negative while for the other choice is zero. This case implies that the Euclidean action vanishes and so every term should vanish. Solutions exist for \( F = \nabla \phi = \nu = 0 \). (iii) For one of the choice of signs the topological charge is positive. Suppose that \( Q \) is positive in (11.2) for the first choice of sign. In such case the bound is attained provided that the equations

\[
\begin{align*}
F^{2,0} &= 0 \\
\Omega_I \cdot F^a - \nu^a &= 0 \\
\nu^\mu \nabla_\mu \phi^i - J^i_j \nabla_\mu \phi^j &= 0
\end{align*}
\]

(11.5)

hold. The first equation implies that \( F \) is a (1,1)-form. The last equation in (11.5) implies that the maps \( \phi \) are holomorphic. Finally the middle equations are a generalization of non-abelian vortex equations. If the term involving the moment map is constant, then the resulting equation is the Hermitian-Einstein equation.

12 Hyper-Kähler manifolds and non-abelian vortices

Let \((N, h, I)\) be a Kähler manifold of dimension \(2k\) with associated Kähler form \( \Omega_I \) and \((M, g, J_r)\) be a hyper-Kähler manifold of dimension \(4n\) with associated Kähler forms \( \Omega_{J_r} \). Next allow \( M \) to admit a tri-holomorphic \( G \)-action with associated killing vector fields \( \xi \) and moment maps \( \nu_r \). In our conventions \( i_\xi \Omega_{J_r} = -d\nu_r \). Next consider the functional

\[
S_E = \int_N d\text{vol}(N) \left[ \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} |F|^2 + \frac{1}{4} \sum_{r=1}^3 |\nu_r|^2 \right] ,
\]

(12.1)
where $|\nabla|^2 = g_{ij}h^{\mu\nu}\nabla_\mu\phi^i \nabla_\nu\phi^j$, $\nabla_\mu\phi^i = \partial_\mu\phi^i + A^a\xi^i_a$, $|F|^2 = u_{ab}F^a_\mu F^b_\nu h^{\mu\nu}$, $|\nu_r|^2 = u_{ab}\nu_{ra}\nu_{rb}$ and $u$ is a fibre inner product on the gauge bundle which we can set $u_{ab} = \delta_{ab}$.

The functional $S_E$ can be rewritten as follows:

$$S_E = \int_N d\text{vol}(N) \left[ \frac{1}{4} \sum_{r=1}^3 [a_r \Omega_I \cdot F + \nu_r]^2 + \frac{1}{4} \sum_{r=1}^3 [a_r I^\nu \nabla_\mu \phi^i + J_r \nabla_\mu \phi^i]^2 \right]$$

$$\pm \frac{1}{(k-1)!} \int_N \sum_{r=1}^3 a_r \omega_{J_r} \wedge \Omega^{k-1}_I - \frac{1}{(k-2)!} \int_N u_{ab} F^a \wedge F^b \wedge \Omega^{k-2}_I$$

(12.2)

where $d\text{vol}(N) = \frac{1}{k!} \Omega^k_I$, $\{a_r : r = 1, 2, 3\}$ is a constant vector of length one, $\sum_{r=1}^3 (a_r)^2 = 1$, $\Omega_I \cdot F = \Omega^{\mu\nu}_I F_{\mu\nu}$, $F^{2,0}$ is the $(2,0)$ part of the curvature $F$ and

$$\omega_{J_r} = (\Omega_{J_r})_{ij} \nabla_\mu \phi^i \wedge \nabla_\nu \phi^j + \nu_{ra} F^a$$

(12.3)

is the equivariant extension of the Kähler form $\Omega_{J_r}$. (The inner products are taken with respect to the Riemannian metrics $h$ and $g$. The rest of the notation is self-explanatory.

It is clear that the functional $S_E$ is bounded by a topological charge $Q$ which involves the equivariant extensions of the Kähler forms $\Omega_{J_r}$ and, if $u$ is a constant invariant quadratic form on $L(G)$, the second Chern character of the gauge bundle $P \times_G L(G)$. It is worth pointing out that the term involving the second Chern character is not affected by the choice of sign in writing (12.2).

Therefore as in the Kähler case, there are several cases to consider but we shall not repeat the analysis again. Suppose that both $Q$ and that the inner product of the vector $\{a_r : r = 1, 2, 3\}$ with $\{\tilde{Q}_r : r = 1, 2, 3\}$ are positive in (12.2), where

$$\tilde{Q}_r = \frac{1}{(k-1)!} \int_N \omega_{J_r} \wedge \Omega^{k-1}_I.$$  

(12.4)

Then the bound is attained provided that the equations

$$F^{2,0} = 0$$

$$a_r \Omega_I F^a - \nu_r^a = 0$$

$$a_r I^\nu \nabla_\mu \phi^i - J_r^a J \nabla_\mu \phi^i = 0$$

(12.5)

hold. The first equation implies that $F$ is a $(1,1)$-form. It is always possible with a rotation in the space of complex structures of the hyper-Kähler manifold $M$ to set $a_1 = 1$ and $a_2 = a_3 = 0$. Then last equation in (12.5) implies that

$$\nabla_\mu \phi^i = 0.$$  

(12.6)

This in turn implies that

$$F^a_\mu \xi^i_a = 0.$$  

(12.7)
Therefore either the connection $A$ is flat or the maps $\phi$ take values in the fixed set $M_f$ of the $G$-group action on $M$. In the former case, in the absence of non-trivial flat connections, the moduli space of solutions to these equations is the hyper-Kähler reduction $M//G$ of $G$ and it is a smooth manifold provided that the level set does not intersect $M_f$. In the latter case, the maps $\phi$ are constant and the two remaining equations in (12.5) are the Hermitian-Einstein equations for the connection $A$.

One can also consider the case where $(N, h, I_r)$ is a hyper-Kähler manifold while $(M, g, J)$ is a Kähler manifold which admits a $G$-holomorphic action of isometries. This case can be treated as that considered in the previous section involving only Kähler manifolds. A Kähler structure on $N$ can chosen with respect to any complex structure which lies in the two-sphere of complex structures of $N$.

13 Concluding Remarks

We have constructed the actions of two-dimensional $(p,0)$- and $(p,1)$-supersymmetric gauge theories coupled to sigma model matter with Wess-Zumino term. We have also given the scalar potentials of these theories. Our method of constructing these theories relies on a superfield method. Then we have shown that the Euclidean actions these theories admit vortex type of bounds which generalise to higher dimensions.

The gauge theories that we have constructed are not the most general ones. It is known for example that the $(1,1)$-supersymmetric sigma model admits a scalar potential which is the length of a killing vector field [33]. Our superfield method cannot describe such a term. There are also other possibilities, for example the sigma models with almost complex manifolds as a target space as well as those associated with $(p,0)$ fermionic multiplets for which the supersymmetry algebra closes on-shell [14, 25]. Other models of interest that we have not described here are those with $(p,2)$, $p = 2, 4$, and $(4,4)$ supersymmetry. All the above models can be described using $(1,0)$ superfields. This method has been used before, see [14, 13, 14]. This means that the action of such models can be written in terms of $(1,0)$ superfields and the additional supersymmetries can be implemented by requiring invariance of the action under additional suitable transformations. The $(2,2)$ and $(4,4)$ supersymmetric gauge theories have been described using other methods in [5] and [34].

The gauge theories coupled to sigma models which we have described with $(p, 1)$ supersymmetry have soliton type of bounds in addition to the vortex type of bounds that we have described. For the former bounds the energy of these models can be written as a sum of squares and a topological term. This is very similar to bounds of (ungauged) sigma models [24] and so we have not described them here. It would be of interest to investigate the solutions of the vortex equations we have presented for different types of moment maps. It may be that for a suitable choice, the vortex equations can be solved exactly.
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