Observational constraints on DBI constant-roll inflation

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Abstract

DBI inflationary scenario using constant-roll approach will be studied and it seeks to compare the result with observational data. By considering the cosmological perturbations of the model the extra terms of the scalar spectral index with respect to the slow-roll inflationary scenario are specified, which they also appear in the amplitude of scalar perturbations and tensor-to-scalar ratio. In order to compare the model with observational data, some specific functions of scalar field are assumed for the $f(\phi)$ function. For power-law and exponential function, a constant slow-roll parameter $\epsilon$ is obtained which produce difficulties for the graceful exit from inflation. Then, a product of linear and exponential function, and also a hyperbolic function of scalar field are selected for $f(\phi)$, that result in a $\epsilon(\phi)$ with an end for the accelerated expansion phase. Considering the scalar spectral index, amplitude of scalar perturbations, and tensor-to-scalar ratio shows that for some values of the constant $\eta = \beta$ there could be a good consistency between the model prediction and observational data.

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I. INTRODUCTION

The idea that the universe undergoes an extreme accelerated expansion phase in a short period of time in the first era of its evolution is well known. The idea was put forth by Alan Guth for the first time in 1981 [1] to solve the problem of hot big bang model. From that time until today, many inflationary scenarios have been introduced such as new inflation [2, 3], chaotic inflation [4], k-essence inflation [5–10], brane inflation [11, 12], gauge inflation [13–17], warm inflation [18–24] and so on, however a huge class of this inflationary scenarios are based on the idea of slow-rolling, proposed by A. Linde [4], stating that the inflaton slowly rolls down from the tope of its potential. So far the scenario have received a huge support from WMAP and Planck observational data [25–28].

Inflation usually is driven by a scalar field which in the simplest case has a potential and a canonical kinetic term. This scenario stands on slow-roll approximation so that the kinetic part is negligible in comparison to the potential part of the scalar field energy density. Then, the scalar field slowly rolls down from the top of its potential toward the minimum of it. k-inflation models are a generalized models of canonical scalar field which includes a non-canonical kinetic term. The model first introduced in [5] and its cosmological perturbations was studied in [6]. k-essence inflation is a huge class of inflation in which many inflationary models could be classified in this class such as Tachyon inflation [29–32], DBI inflation [33–39] so that many works could be found in this topic [29–42]. In contrast to the canonical scalar field where the scalar perturbations propagate with speed of light, the scalar perturbation in k-essence inflation travels with sound speed which in general could changes [34, 36, 43, 44].

Investigating inflationary scenario in string theory, as a theory providing a consistent formulation of quantization of gravity involving extra dimension, have received a huge interest where inflaton might be an open string [33, 34]. This reliable theory is assumed to be our chance for understanding the fundamental characters and concepts of inflation which are missing in the standard picture [34]. By compacting the extra dimensions, string theory is able to anticipate a wide range of scalar fields which turns to some phenomenologically applicable inflation models such as DBI model [34]. In this case, the D-brane inflation, including a non-standard kinetic term, is clarified as moving of the D-brane through higher dimensions [33, 34]. Besides the non-standard kinetic term and scalar field potential,
the DBI effective action includes another function of scalar field, \( f(\phi) \), which contains information about the geometry of the compact manifold traversed by the D-brane \[33\]. An interesting feature of DBI model is that the sound speed could vary between zero and one. Then the field perturbations propagate at speed less than the speed of light which leads to this consequence that the Fourier modes freeze in at sound horizon \[35\]. The sound speed is given as inverse of the \( \gamma \) parameter which plays the same role of Lorentz factor in special relativity \[35\].

Slow-roll approximations are usually satisfied by almost flat part of the potential of scalar field. However, when the potential is exactly flat an strange situation occurs. From the scalar field equation of motion it could be concluded that the second slow-roll parameter becomes \( \eta = -3 \) that clearly breaks down the slow-roll approximation which states that the slow-roll parameters during the inflation should be smaller than unity. The situation was first studied in \[45\] where a flat potential was taken into account. The non-Gaussianity of the model was considered in \[46\], and they found that it is not necessary small. In \[47\], the situation was considered in more general case and the second slow-roll parameter was taken as a constant, and an approximate solution was obtained. Considering the cosmological perturbations comes to an amplitude of scalar perturbations which even vary on superhorizon scale. Taking a constant second slow-roll parameter and applying the Hamilton-Jacobi formalism \[48–55\] lead to an exact solution \[56\]. Also it was realized that for specific choices of \( \eta \) the amplitude of scalar perturbations could be frozen on superhorizon scale. The name ”constant-roll” was first used in \[56\]. The constant-roll inflationary scenario also has been investigated in modified gravity \[57–63\]. The scenario was generalized in \[64\], in which the second slow-roll parameter was taken as a function of scalar field and the scenario was named as ”smooth-roll inflationary scenario” \[64–66\].

The strong and interesting background of DBI scalar field motivates us to study the inflationary scenario with DBI scalar field as inflaton. In this order, the constant roll approach will be utilized where the second slow-roll parameter is taken as a constant that might not be small. The scenario of constant-roll DBI inflation was studied in \[67\], in which the work was limited to the constant sound speed. Taking \( \eta = \dot{\epsilon}/H\epsilon \), an analytical solutions for the Hubble parameter were obtained and some note about its consistency with observational were given. During the present work, we follow \[56, 68–70\], and defined the second slow-roll parameter as \( \eta = \dot{\epsilon}/H\dot{\phi} \) (where \( \epsilon = \pm 1 \)) which turns to a different
form of non-linear differential equation for the Hubble parameter. The constancy of the sound speed is released and working in ultra-relativistic regime is replaced \cite{33,71}. The main aim of the work is considering the model prediction with the latest observational data which will be performed in great detail. Constancy of $\eta$ enforces us to reinvestigate the cosmological perturbations of the model so that a generalized form of the amplitude of scalar perturbations is obtained. This modification also appears in the scalar spectral index and tensor-to-scalar ratio. Comparison with data is carried out by introducing some specific function of the scalar field for $f(\phi)$ function, and the amplitude of scalar perturbations, scalar spectral index and tensor-to-scalar ratio are estimated and they are depicted in terms of the number of e-fold for various choice of the constant $\beta$. The results shows that for some chases of $f(\phi)$ there could be a good consonance with observational data.

This paper is organized as follows: In Sec.II, the genera evolution equation of the model is presented. Then by assuming the DBI scalar field as inflaton, the equation is rewritten for DBI constant-roll inflation. Applying the assumption of constant-roll formalism, a non-linear differential equation is derived which gaining an analytical solution comes to difficulties. Then the work is limited to the ultra-relativistic regime. The cosmological perturbation of the model is studied in Sec.III where a generalized form of amplitude of scalar perturbations is obtained that only for specific choice of the constant $\eta = \beta$ is scale invariant. This modifications also appear in the scalar spectral index and tensor-to-scalar ratio so that with respect to the slow-roll inflationary scenario there are some modified terms. Computing the results of the model and comparing them with observational data for some choices of the $f(\phi)$ function are performed in Sec.IV and it is realized that the model have a good agreement with observational data.

\section{DBI MODEL}

In this section, we are going to introduce the main equations of motion of the model. The DBI model could be categorized as a subclass of a more general one addressed as K-essence whose action is indicated by

$$ S = \int \sqrt{-g} \left( \frac{1}{2} R + P(\phi, X) \right), $$

(1)
where \( g \) is the determinant of the metric tensor \( g_{\mu\nu} \), \( R \) is the scalar Ricci, the inflaton field is denoted by \( \phi \), and \( X \) is defined as \( X = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \). The term \( P(\phi, X) \) is the inflaton Lagrangian which is in general a function of \( \phi \) and \( X \). For DBI model, \( P(\phi, X) \) is given as

\[
P(\phi, X) = -f^{-1}(\phi) \left[ \sqrt{1 - 2f(\phi) X} - 1 \right] - V(\phi),
\]

so that \( V(\phi) \) is the potential of scalar field and \( f(\phi) \) is the inverse brane tension that is expressed as a function of scalar field \( \phi \).

Obtaining the field equation of the model by taking variation of above action with respect to the metric, and applying the FLRW spatially flat metric, \( ds^2 = dt^2 - a^2 \delta_{ij} dx^i dy^j \), the Friedmann equations are

\[
3H^2 = \rho,
\]

\[
2\dot{H} = \rho + p,
\]

in which \( \rho \) and \( p \) are respectively the energy density and pressure DBI scalar field which are read by

\[
\rho = \frac{\gamma - 1}{f} + V(\phi),
\]

\[
p = \frac{\gamma - 1}{f\gamma} - V(\phi),
\]

and the parameter \( \gamma \) is defined as \( \gamma = 1/\sqrt{1 - f(\phi) \dot{\phi}^2} \) known as Lorentz factor. The sound speed, which express the propagation speed of perturbations of scalar field through the homogeneous and isotropic background speceime, is obtained as

\[
c_s = \sqrt{dp/d\rho} = \frac{1}{\gamma}.
\]

From the second Friedmann equation, one has

\[
\gamma \dot{\phi} = -2H'(\phi), \quad \Rightarrow \quad \dot{\phi}^2 = \frac{4H'^2(\phi)}{1 + 4f(\phi)H^2(\phi)}
\]

Then, by substituting this in the definition of \( \gamma \), the coefficient is rewritten as

\[
\gamma = \sqrt{1 + 4f(\phi)H^2(\phi)}
\]
A. DBI constant-roll inflation

Inflation is known as a short period of accelerated expansion while the Universe undergoes an extreme expansion. The acceleration equation of the universe is usually given by $\ddot{a}/a = H^2(1 - \epsilon)$ where $\epsilon = -\dot{H}/H^2$. Therefore, to have an acceleration phase, there should be $\epsilon < 1$, and the condition $\epsilon = 1$ is usually taken as the end of inflation. This parameter is known as the first slow-roll parameter. Following [56, 68-70], the second slow-roll parameter is defined as the rate of variation of $\dot{\phi}$ during a Hubble time

$$
\eta = \frac{\varepsilon \ddot{\phi}}{H \dot{\phi}}
$$

(10)

where $\varepsilon = \pm 1$, mostly because there is an arbitrariness in putting the negative sign. In constant-roll approach of studying inflationary scenario, the second slow-roll parameter is taken as a constant, i.e. $\eta = \beta$. Then, one could found out a second order non-linear differential equation for the Hubble parameter as

$$
H''(\phi) - 2f'(\phi)H'^2(\phi) - \frac{\beta}{2} H \sqrt{1 + 4f(\phi)H^2(\phi)} = 0.
$$

(11)

Solving above equation and finding an analytical equation seems unlikely. So, the equation will be investigated in the ultra-relativistic regime. In this regime the quantity $\gamma$ is large [33, 71], then we have

$$
\gamma \simeq 2H'(\phi)\sqrt{f(\phi)}, \quad \text{and} \quad \dot{\phi} \simeq -\frac{1}{\sqrt{f(\phi)}}.
$$

(12)

Using the assumption of constant $\eta$, it is realized that the Hubble parameter for every specific function of $f(\phi)$ is derived as

$$
\eta = \beta = \frac{\varepsilon f'(\phi)}{2H f^{3/2}(\phi)}, \quad \Rightarrow \quad H(\phi) = \frac{\varepsilon}{2\beta} \frac{f'(\phi)}{f^{3/2}(\phi)}
$$

(13)

To solve the standard big bang theory problems, there should be enough expansion for the universe. The universe expansion is measured by the number of e-fold, given by

$$
N = \int_{t_e}^{t_e} H dt = \frac{-\varepsilon}{2\beta} \int_{\phi_e}^{\phi} \frac{f'(\phi)}{f^2(\phi)} d\phi
$$

(14)

where the subscript "e" indicates the quantity at the end of inflation, and "\star" denotes the time of horizon crossing.
III. COSMOLOGICAL PERTURBATION

Cosmological perturbations are an interesting prediction of inflationary scenario, which generally are divided to three types as: scalar, vector and tensor perturbations. Up to the first order of perturbations parameters, these types of perturbation are evolved independently. In this section, the cosmological perturbations of presented model will be considered by imposing this assumption that the second slow-roll parameter is constant and might not be small. Following [6] and applying a small inhomogeneous perturbation for the scalar field \( \phi(t, \mathbf{x}) = \phi_0(t) + \delta \phi(t, \mathbf{x}) \), the metric tensor will be disturbed as well, where in Newtonian gauge it is read by

\[
ds^2 = (1 + 2\Phi(t, \mathbf{x}))dt^2 - a^2(t)(1 - 2\Phi(t, \mathbf{x}))\gamma_{ij}dx^i dx^j
\]  

(15)

Note that it is presumed that the perturbed energy-momentum tensor is diagonal, \( \delta T_{ij} \propto \delta \delta^i \).

Substituting the above metric in the field equations, the \((0, 0)\) and \((0, i)\) component of perturbed equations are obtained as [6]

\[
\dot{\xi} = \frac{a(\rho + p)}{H^2} \zeta,
\]

\[
\dot{\zeta} = \frac{c_s^2 H^2}{a^3(\rho_T + p_t)} \nabla^2 \xi,
\]

(16)

where the variables \( \xi \) and \( \zeta \) are defined as

\[
\xi \equiv \frac{a}{4\pi GH} \Phi,
\]

\[
\zeta \equiv \frac{4\pi GH}{a} \xi + H \frac{\delta \phi}{\Phi} = \Phi + H \frac{\delta \phi}{\Phi}.
\]

The corresponding action for the equations (16) is

\[
S = \frac{1}{2} \int z^2 \left( \zeta'^2 + c_s^2 \zeta (\nabla \zeta)^2 \right) d\tau d^3 \mathbf{x},
\]

(17)

the prime denotes derivative with respect to the conformal time \( \tau \), and \( z \) is defined as \( z = a\sqrt{\rho + p}/c_s H \). By defining a canonical quantization variable \( v = z \zeta \), the above action is given as follows [6]

\[
S = \frac{1}{2} \int \left( v'^2 + c_s^2 v(\nabla v)^2 + \frac{z''}{z} v \right) d\tau d^3 \mathbf{x}.
\]

(18)

Then, the dynamical equation for the variable \( v \) is obtained as

\[
v''(t, \mathbf{x}) - c_s^2 \nabla^2 v(t, \mathbf{x}) - \frac{z''}{z} v(t, \mathbf{x}) = 0
\]

(19)
and by using the Fourier mode, one has

$$v''_k(\tau) + \left(c_s^2 k^2 - \frac{z''}{z}\right) v_k(\tau) = 0 \quad (20)$$

The term $z''/z$ up to the first order of the slow-roll parameters $\epsilon$ and $s$ could be expressed as follows

$$z = \frac{a\sqrt{\rho + \rho}}{c_s H} = \frac{a\sqrt{\gamma \dot{\phi}^2}}{c_s H} \quad (21)$$

$$z' = z(aH) \left(1 + \epsilon + s\eta - \frac{3}{2}s\right) \quad (22)$$

$$z'' = z(aH)^2 \left(2 + 6\epsilon - 3\eta - 3s - 9\epsilon\eta + 3s\eta + \eta^2 + 2\epsilon\eta^2\right) \quad (23)$$

Utilizing the variable changes $x = -c_s k\tau$ and $v_k = \sqrt{-\tau} f_k(\tau)$, the Eq. (20) could be transformed to the Bessel differential equation

$$\frac{d^2 f_k}{dx^2} + \frac{1}{x} \frac{df_k}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) f_k = 0, \quad (24)$$

in which the definition $z''/z = \frac{\nu^2 - 1}{\tau^2}$ has been used. The general solution is a combination of first and second type of Hankel function

$$f_k(\tau) = \alpha_1(k) H^{(1)}_{\nu}(-k\tau) + \alpha_2(k) H^{(2)}_{\nu}(-k\tau). \quad (25)$$

where $\alpha_1(k)$ and $\alpha_2(k)$ are constant that could be determined by considering the asymptotical feature of the equation.

In subhorizon scale, where $c_s^2 k^2 \gg a^2 H^2$ or in another word $c_s^2 k^2 \gg z''/z$, the differential equation (20), could be stated as

$$v''_k(\tau) + c_s^2 k^2 v_k(\tau) = 0 \quad (26)$$

and the solution is obtained as

$$v_k(\tau) = \frac{1}{\sqrt{2c_s k^2}} e^{-i c_s k \tau}. \quad (27)$$

It could be concluded that the general solution (25) should return to the subhorizon solution (27) for scale $c_s^2 k^2 \gg a^2 H^2$. Then, by studying the asymptotical behavior of Hankel function, one could found out that the constant $\alpha_2(k)$ should be eliminated, $\alpha_2(k) = 0$, and for $\alpha_1(k)$ there is $\alpha_1(k) = \frac{\sqrt{\pi}}{2} e^{\frac{\pi}{2} (\nu + \frac{1}{2})}$. On the other hand, at superhorizon limit, the solution is read as

$$\lim_{-k\tau \to 0} = \frac{2^{\nu - \frac{3}{2}} \Gamma(\nu)}{\sqrt{2c_s k} \Gamma(3/2)} e^{\frac{\pi}{2} (\nu - \frac{1}{2})} \left(-c_s k \tau\right)^{\frac{1}{2} - \nu}. \quad (28)$$
The amplitude of scalar perturbation is defined as

\[ P_s^{1/2} = \sqrt{\frac{k^3}{2\pi^2}} |\zeta| = \sqrt{\frac{k^3}{2\pi^2}} |\frac{v_k}{z}|. \]  

(29)

Then, one could arrive at

\[ P_s \bigg|_{\text{superhorizon}} = A_s = \left( \frac{2^{\nu-\frac{3}{2}} \Gamma(\nu)}{\Gamma(3/2)} \right)^2 \frac{H^4}{16\pi^2 f H^2 c_s^2} \left( \frac{c_s k}{aH} \right)^{3-2\nu}. \]  

(30)

The scalar spectral index, as an important observational parameter is defined through the relation of amplitude of scalar perturbation so that \( P_s = A_s \left( \frac{c_s k}{aH} \right)^{n_s-1} \), in which \( A_s \) is the amplitude of scalar perturbation at horizon crossing. Then, there is

\[ n_s - 1 = 3 - 2\nu, \quad \text{and} \quad \nu^2 = \frac{9}{4} + 6\epsilon + 3\varepsilon\eta - 3s + 9\varepsilon\eta\eta - 3\varepsilon s\eta + \eta^2 + 2\epsilon\eta^2. \]  

(31)

Tensor perturbations are same as the slow-roll scenario, since there is no contribution from the slow-roll parameter \( \eta \) in the amplitude of tensor perturbations. Therefore, we have the same relation for \( P_t \). Tensor perturbations are measured indirectly through the parameter \( r \) which is stated as the ratio of tensor perturbations to the scalar perturbations as

\[ r = \frac{P_t}{P_s} \quad \text{and at horizon crossing} \quad r = \left( \frac{\Gamma(3/2)}{2^{\nu-\frac{3}{2}} \Gamma(\nu)} \right)^2 \frac{32 f H^2 c_s^2}{H^2}. \]  

(32)

IV. OBSERVATIONAL CONSTRAINT

Considering the main evolution equations of the model, and discussing the cosmological perturbations of the model, we are set up to compare the model prediction with observational data. In this regards, it is required to specify an specific function of scalar field for \( f(\phi) \). Then, in the following lines, four typical example of \( f(\phi) \) will be introduced and for each one the consequences shall be discussed.

A. Power-law function

A power-law function of scalar field is taken as the first case, in which \( f(\phi) = f_0 \phi^n \). By substituting it in Eq. (13), the Hubble parameter is found in terms of the scalar field

\[ H(\phi) = \frac{n\varepsilon}{2\beta\sqrt{f_0}} \frac{1}{\phi^{\frac{n}{2}+1}}. \]  

(33)
Then, the first slow-roll parameter could be obtained using Eq. (12) and (33)

\[ \epsilon = -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}H'}{H^2} = \epsilon \beta \left(1 + \frac{1}{2n}\right) \]  

(34)

which is a constant. This constant might be smaller than unity and give an accelerated expansion phase however, graceful exit from inflation encounters difficulties. Due to this fact, this choice of the function \( f(\phi) \) might not be appropriate choice.

B. Exponential function

For the second case, an exponential function of the scalar field is selected for \( f(\phi) \) so that \( f(\phi) = f_0 e^{\lambda \phi} \). The Hubble parameter is achieved from Eq. (13)

\[ H(\phi) = \frac{\epsilon \lambda}{2 \beta \sqrt{f_0}} e^{-\frac{\lambda \phi}{2}}. \]  

(35)

The time derivative of scalar field has the same behavior as the Hubble parameter, i.e. \( \dot{\phi} = -e^{-\frac{\lambda \phi}{2}} / \sqrt{f_0} \). Then, utilizing \( \dot{H} = \dot{\phi}H' \), the first slow-roll parameter is given by

\[ \epsilon = -2 \epsilon \beta. \]  

(36)

A proper choice of \( \beta \) leads to an accelerated expansion phase, but there is the same problem that we have for the previous case and exiting from inflation seems unlikely.

C. Combination function

In this case, we combine two previous choice and take the function \( f(\phi) \) as \( f(\phi) = f_0 \phi e^{\lambda \phi} \). From Eq. (13), the Hubble parameter is derived in terms of the scalar field

\[ H(\phi) = \frac{\epsilon}{2 \beta} \frac{(1 + \lambda \phi) e^{-\lambda \phi/2}}{\phi \sqrt{f_0 \phi}}. \]  

(37)

Using above relation and applying the time derivative of scalar field, the first slow-roll parameter for this case is given by

\[ \epsilon = 2 \epsilon \beta \left[ \frac{\lambda \phi(1 + \lambda \phi)}{(1 + \lambda \phi)^2} - \frac{3}{2} \right]. \]  

(38)

which is a function of scalar field.

Inflation ends as \( \epsilon \) reaches unity, which occurs for scalar field \( \phi_e \)

\[ \lambda \phi_e = -1 \pm \frac{\sqrt{1 - \sigma}}{1 - \sigma}, \quad \sigma = \frac{\epsilon}{2 \beta} + \frac{3}{2}. \]  

(39)
Then, to have a physical answer, the condition $\sigma < 1$ should be satisfied which states that $\epsilon$ and $\beta$ must have a different sign. In order to solve the horizon and flatness problem, about $N = 60$ number of e-fold is required. To have this amount of expansion, reading from Eq. (14), the scalar field at the horizon crossing is obtained as

$$\lambda \phi_\ast = W \left[ \lambda \phi_\ast e^{\lambda \phi_\ast \epsilon^2 \beta N} \right],$$

where $W$ indicates the Lambert function. Fig. 1 displays the behavior of the slow-roll parameter $\epsilon$ versus scalar field for different choices of $\epsilon$, $\beta$ and $\lambda$. The two left hand figures are described $\epsilon$ for $\lambda > 0$ which indicates that the inflation occurs for the negative values of the scalar field. In addition the decreasing behavior of the scalar field, imposed by Eq. (12) is properly satisfied as well. However, the two right hand side figures show the parameter $\epsilon$ for $\lambda < 0$ where the inflation happens for positive values of the scalar field. But, it is seen that the scalar field increases during the inflationary times which opposes Eq. (12). Therefore it is resulted that for this case the inflation happens for negative values of the scalar field.

![Graphs showing the behavior of the slow-roll parameter $\epsilon$ versus scalar field.](image)

**FIG. 1:** The behavior of the slow-roll parameter $\epsilon$ versus the scalar field during the inflationary times.

The other slow-roll parameter which will be needed in estimating the scalar spectra index is
From the definition of the parameter, it could be acquired in terms of the scalar field as

\[ s = \frac{\dot{c}_s}{Hc_s} = -\dot{\phi} \left( \frac{H''}{H'} + \frac{f'}{2f} \right) \]  

(41)

and by using definition of \( f(\phi) \) and the Hubble parameter (37), one arrives at

\[ s = 2\varepsilon\beta \left\{ \frac{\lambda\phi \left[ \left( \frac{1}{1+\lambda\phi} - \frac{1}{2} - \frac{3}{2\lambda\phi} \right)^2 + \left( \frac{3}{2\lambda^2\phi^2} - \frac{1}{(1+\lambda\phi)^2} \right) \right]}{\left( \frac{1}{1+\lambda\phi} - \frac{1}{2} - \frac{3}{2\lambda\phi} \right)^2} + \frac{1 + \lambda\phi}{2} \right\} \]  

(42)

The second slow-roll parameter is \( \eta = \beta \) which is a constant. These slow-roll parameters are very important when the model predictions at horizon crossing are compared with observational data, which will be discussed next.

One of the most important observational parameter is the scalar spectral index, which has been estimated to be about \( n_s = 0.9649 \pm 0.0042 \) based on Planck-2018 (TT,TE,EE+lowE+lensing) [28]. Eq.(31) exhibits this parameters for the model. Then, using Eqs.(38), (40) and (42), the scalar spectral index could be calculated at the horizon crossing. The behavior of \( n_s \) versus the constant number of e-folds has been illustrated in Fig.2 for different values of \( \beta \). It is clear that for these selected values of \( \beta \) all three curves come close to each other. The scalar spectral index is low for small values of \( N \), and it come close to unity by increasing \( N \), so that for \( \beta = -0.00005 \) and \( N = 70 \) the scalar spectral index is about \( n_s = 0.9694 \) which is in good agreement with Planck data.

![FIG. 2: Diagram of \( n_s \) versus the constant \( \beta \).](image)

The latest observational data states that the amplitude of scalar perturbations should be about \( \ln(10^{10}A_s) = 3.044 \pm 0.014 \) [28]. The parameter was obtained in Sec.1111 for our model,
and by substituting the scalar field of the horizon crossing it could be computed, so that Fig. 3 shows the parameter versus number of e-folds for different values $\beta$. It states that for every specific values of $N$, the amplitude of scalar perturbation receives a bigger values for bigger choice of $\beta$, so that for $\beta = -0.00003$ the amplitude of scalar perturbations arrives at $2.17 \times 10^{-9}$ for $N = 43$ which is not proper results we are looking for. On the other hand, $A_s$ gets a consistent value for $\beta = -0.00005$ and $N = 70$ which is a better result.

![Diagram of $P_s$ versus the constant $\beta$.]

Measuring the tensor perturbations is performed through studying $r$ that is ratio of tensor-to-scalar perturbations that by combining the data of Planck and BICEP2/Keck Array BK14 gets a tighter bound as $r < 0.064$ [28]. From Eq. (32), and by inserting the scalar field (40), $r$ could be obtained at the horizon crossing. Dependence of $r$ on number of e-folds is determined in Fig. 4 for different values of $\beta$ where one realizes that for small values of number of e-folds the parameter $r$ gets larger for smaller values of $\beta$, however all curves approach to each other. The parameter $r$ is very small, of order $10^{-14}$, and for our interested range of number of e-fold, i.e. $N = 65 - 70$, it becomes even smaller, of order $10^{-15}$. Although this value perfectly stands in observational range it is very close to zero.

To end up this case, one of the most important diagram of inflationary studies, namely $r - n_s$ diagram, is presented in Fig. 5. By only considering the scalar spectral index and tensor-to-scalar ratio, it seems that all choices of $\beta$ is perfect and the results for $n_s$ and $r$ for appropriate number of e-folds, $N = 65 - 70$, are in good agreement with observational data. However, by considering the amplitude of scalar perturbations it is revealed that only $\beta = -0.00005$ gives a $A_s$ consistent
with observational data. Then, it might be concluded that the best choice of $\beta$, considered here, is $\beta = -0.00005$ which results in suitable values of $n_s$, $r$ and $A_s$ for $N = 70$ number of e-folds.

D. Hyperbolic function

For the final case, the function $f(\phi)$ is taken as a hyperbolic function, $f(\phi) = f_0 \cosh(\lambda \phi)$. Plugging it into Eq.(13), the Hubble parameter is extracted as a function of scalar field as

$$H(\phi) = \frac{\varepsilon \lambda}{2\beta \sqrt{f_0}} \frac{\sinh(\lambda \phi)}{\cosh^{3/2}(\lambda \phi)}.$$  \hspace{1cm}(43)

Using above result and Eq.(12), the slow-roll parameter $\epsilon$ is read as a function of scalar field

$$\epsilon = 2 \varepsilon \beta \left( \coth^2(\lambda \phi) - \frac{3}{2} \right).$$  \hspace{1cm}(44)

which make it possible to have a graceful exit from inflation. Fig.6 shows the behavior of $\epsilon$ versus scalar field for different choices of $\beta$ which clearly portrays that it could reach unity.
FIG. 6: The slow-roll parameter $\epsilon$ is plotted in terms of the scalar field during the inflationary times.

and ends the accelerated expansion phase of inflation.

Let us consider the situation in more analytical viewpoint. As the slow-roll parameters $\epsilon$ reaches unity inflation ends, thus the scalar field at the end of inflation is read as

$$\cosh(\lambda \phi_e) = \left[ \frac{3}{2 \beta} + \frac{3}{2} \right]^{1/2}. \quad (45)$$

On the other hand, from the equation of number of e-folds, the scalar field at the horizon crossing could be expressed as

$$\cosh(\lambda \phi_\ast) = \cosh(\lambda \phi_e) e^{2\epsilon \beta N} \quad (46)$$

It is demonstrated in Fig[6] where it is clearly seen that $\epsilon$ reaches unity by decreasing of the scalar field, and $\epsilon$ gets smaller values for bigger values of the scalar field. Also, unlike the previous case the scalar field could be positive which might be more favorable case.

The slow-roll parameter $s$ which appears in the scalar spectral index is obtained in terms of scalar field

$$s = 2\epsilon \beta \left[ \frac{1}{2} - \frac{9}{2 \cosh^2(\lambda \phi)} + \frac{1}{2} \right]. \quad (47)$$

Substituting $\phi_\ast$ in Eqs.(44) and (47), and using Eq.(31), the scalar spectral index is achieved at the horizon crossing, and Fig[7] displays its behavior versus the number of e-folds for different choices of the constant $\beta$. It is realized that the scalar spectral index increases by enhancement of the number of e-folds, and it asymptotically approaches unity, so that for $N = 60$, the scalar spectral index is obtained as $n_s = 0.9670$ which is consistent with the
recent observational data. Also, varying of the constant $\beta$ seems not to have great affect on the $n_s$ in which the three curves of $n_s$ corresponding to the three selected values of $\beta$ completely overlap.

Fig. 7 presents the variation of the amplitude of scalar perturbation versus the number of e-folds for different values of $\beta$. In contrast to the Fig. 7 where changing of $\beta$ might not be important, it results in great difference in magnitude of $A_s$ so that for every specific choice of number of e-fold, a larger values of $\beta$ gives a bigger $A_s$. For $\beta = -0.000003$ and $\beta = -0.000007$, the amplitude of scalar perturbations arrives at the observationally predicted values for respectively $N = 43$ and $N = 90$ which are not in our interest range of number of e-fold. However, by choosing $\beta = -0.000005$, the amplitude of scalar perturbations reaches $A_s = 2.17 \times 10^{-9}$ for $N = 60$ that properly satisfies our expectations.

The ratio of tensor-to-scalar perturbations is illustrated in Fig. 8 so that variation of $r$ in terms of number of e-fold for different choices of $\beta$ is made clear. After a fast enhancement for small values of $N$, the parameter $r$ decreases, and all three curves come close to each other for $60 - 70$ range of number of e-folds. It also should be noted that for $N = 60$, they satisfy the observationally predicted bound for $r$ as $r < 0.065$.

As the final step, the $r - n_s$ diagram for the case is also depicted in Fig. 9 where one realized that all choices of $\beta$ cross the observational area. However, the best choice of the constant $\beta$ seems to be $\beta = -0.000005$ which also results in a suitable value of $A_s$. 

FIG. 7: Diagram of $n_s$ versus the constant $\beta$. 
V. CONCLUSION

The constant-roll inflationary scenario in the framework of Einstein gravity was studied in which the inflaton was assumed to be a DBI scalar field. By acquiring the main dynamical equations of the model and applying Hamilton-Jacobi formalism, where the scalar field plays the role of time and the Hubble parameter is assumed to be a function of scalar field, a non-linear differential equation for the Hubble parameter was obtained. Gaining an analytical solution for this differential equation encountered with difficulties, then the work was restricted to the ultra-relativistic regime so that the Lorentz factor $\gamma$ is large. In this regime, the time derivative of scalar field was realized to be negative, consequently a decreasing behavior for the scalar field was expected during the inflationary times.

Comparing the results with the observational data was the main purpose of the presented work. In this regard, considering the cosmological perturbations of the model was required which was investigated in Sec.III. Since the second slow-roll parameter is assumed to be
FIG. 10: Diagram of $r - n_s$ versus the constant $\beta$.

a constant, which might not be small, some extra terms in the expression of the scalar spectra index appear that in turn affect the amplitude of scalar perturbations. On the other side, since perturbations of the energy-momentum tensor has no contribution in tensor perturbations, and due to this fact that only the slow-roll parameter $\epsilon$ contribute in the expression of the tensor perturbations, there was no modification with respect to the slow-roll scenario of inflation. Deriving the main perturbation parameters for the model, the consistency of the model predictions with observational data was performed in Sec. [IV] for specific choices of $f(\phi)$ function.

Attributing a power-law and exponential functions to $f(\phi)$ leads to a constant slow-roll parameter $\epsilon$. This result could not give a graceful exit from inflation, although might leads to an accelerated expansion phase by properly selecting the model parameters. As the third case, $f(\phi)$ function was taken as a product of linear and exponential function of scalar field. This choice results in a varying slow-roll parameter $\epsilon$ which also could give a graceful exit, i.e. $\epsilon = 1$. However, to have decreasing behavior of the scalar field during inflation, as it is required by Eq. [3], the computation indicates that the scalar field during inflation should be negative, plotted in Fig.[1]. The scalar spectral index, the amplitude of scalar perturbations and the ratio of tensor-to-scalar were estimated and depicted in terms of the number of e-folds for different choices of the constant $\beta$. The results express that the scalar spectral index is about 0.9695 which is in good agreement with Planck-2018 data, and the ratio of tensor-to-scalar was obtained of order $10^{-15}$ for $N = 70$. Although $r$ stands in the observationally estimated range $r < 0.062$, it is very close to zero. Besides that, it seems that varying the constant $\beta$ does not have a dramatic effect on $n_s$ and $r$ for $N = 65 - 70$ and they properly stand in the range predicted by Planck. However it could lead to the different
values of $A_s$ for our interest number of e-folds so that by enhancement of $\beta$ the amplitude of scalar perturbations grows and one arrives at $A_s = 2.17 \times 10^{-9}$ for $\beta = -0.00005$ and $N = 70$. A hyperbolic function of scalar field was investigated as the last case of $f(\phi)$. This choice also gives a varying $\epsilon$ which could produce a graceful exit from inflation after enough expansion. In contrast to the third case, to satisfy decreasing behavior of scalar field during inflation, the scalar field could be positive. Same as the previous case, the varying of $\beta$ does not have a great effect on $\beta$ and $r$ in our interest range of number of e-folds and they are consistent with observational data. Note to mention the fact that in contrast to the previous case, the perturbations parameter $r$ becomes larger than the its corresponding values in the third case. The effect of constant $\beta$ on $A_s$ was considered, plotted in Fig. that it was concluded enhancement of $\beta$ the amplitude of scalar perturbation increases and for $N = 60 - 70$ put it out of observationally predicted range. The best choice considered here was $A_s = 2.17 \times 10^{-9}$ for $\beta = -0.000005$ and $N = 60$.

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