Improved Heinz inequality and its application
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Abstract
We obtain an improved Heinz inequality for scalars and we use it to establish an inequality for the Hilbert-Schmidt norm of matrices, which is a refinement of a result due to Kittaneh.

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1. Introduction
Let $M_n$ be the space of $n \times n$ complex matrices and $||\cdot||$ stand for any unitarily invariant norm on $M_n$. So, $||UAV|| = ||A||$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. If $A = [a_{ij}] \in M_n$, then

$$
||A||_2 = \left( \sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2}
$$

is the Hilbert-Schmidt norm of matrix $A$. It is known that the Hilbert-Schmidt norm is unitarily invariant.

The classical Young’s inequality for nonnegative real numbers says that if $a, b \geq 0$ and $0 \leq v \leq 1$, then

$$
a^v b^{1-v} \leq va + (1-v) b
$$

with equality if and only if $a = b$. Young’s inequality for scalars is not only interesting in itself but also very useful. If $v = \frac{1}{2}$, by (1.1), we obtain the arithmetic-geometric mean inequality

$$
2\sqrt{ab} \leq a + b.
$$

Kittaneh and Manasrah [1] obtained a refinement of Young’s inequality as follows:

$$
a^v b^{1-v} + r_0 \left( \sqrt{a} - \sqrt{b} \right)^2 \leq va + (1-v) b,
$$

where $r_0 = \min \{v, 1-v\}$.

Let $a, b \geq 0$ and $0 \leq v \leq 1$. The Heinz means are defined as follows:

$$
H_v(a, b) = \frac{a^v b^{1-v} + a^{1-v} b^v}{2}.
$$
It follows from the inequalities (1.1) and (1.2) that the Heinz means interpolate between the geometric mean and the arithmetic mean:

\[
\sqrt{ab} \leq H_v(a, b) \leq \frac{a + b}{2}.
\] (1.4)

The second inequality of (1.4) is known as Heinz inequality for nonnegative real numbers.

As a direct consequence of the inequality (1.3), Kittaneh and Manasrah [1] obtained a refinement of the inequality as follows:

\[
H_v(a, b) + r_0 \left( \sqrt{a} - \sqrt{b} \right)^2 \leq \frac{a + b}{2},
\] (1.5)

where \( r_0 = \min \{v, 1 - v\} \).

Bhatia and Davis [2] proved that if \( A, B, X \in M_n \) such that \( A \) and \( B \) are positive semidefinite and if \( 0 \leq v \leq 1 \), then

\[
2 \left\| A^{1/2}XB^{1/2} \right\| \leq \left\| A^vXB^{1-v} + A^{1-v}XB^v \right\| \leq \|AX + XB\|. \tag{1.6}
\]

This is a matrix version of the inequality (1.4). Kittaneh [3] proved that if \( A, B, X \in M_n \) such that \( A \) and \( B \) are positive semidefinite and if \( 0 \leq v \leq 1 \), then

\[
\left\| A^vXB^{1-v} + A^{1-v}XB^v \right\| \leq 4r_0 \left\| A^{1/2}XB^{1/2} \right\| + (1 - 2r_0) \| AX + XB \|, \tag{1.7}
\]

where \( r_0 = \min \{v, 1 - v\} \). This is a refinement of the second inequality in (1.6).

In this article, we first present a refinement of the inequality (1.5). After that, we use it to establish a refinement of the inequality (1.7) for the Hilbert-Schmidt norm.

2. A refinement of the inequality (1.5)

In this section, we give a refinement of the inequality (1.5). To do this, we need the following lemma.

**Lemma 2.1.** [4,5] Let \( f(x) \) be a real valued convex function on an interval \( [a, b] \). For any \( x_1, x_2 \in [a, b] \), we have

\[
f(x) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2 - x_1}, \quad x \in (x_1, x_2).
\]

**Theorem 2.1.** Let \( a, b \geq 0 \) and \( 0 \leq v \leq 1 \). If \( r_0 = \min \{v, 1 - v\} \), then

\[
2H_v(a, b) \leq \begin{cases} (1 - 4r_0)(a + b) + 4r_0(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}), & v \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1], \\ 2(4r_0 - 1)\sqrt{ab} + 2(1 - 2r_0)(a^{1/4}b^{3/4} + a^{3/4}b^{1/4}), & v \in [\frac{1}{4}, \frac{3}{4}]. \end{cases}
\] (2.1)

**Proof.** It is known that as a function of \( v \), \( H_v(a, b) \) is convex and attains its minimum at \( v = \frac{1}{2} \). Let

\[
f(v) = 2H_v(a, b) = a^{1-v}b^v + a^v b^{1-v}, \quad 0 \leq v \leq 1.
\]

Obviously, \( f(v) \) is convex. For \( 0 \leq v \leq \frac{1}{4} \), since \( f(v) \) is convex on \( 0[1] \), by Lemma 2.1, we have
\[ f(v) \leq \frac{f\left(\frac{1}{4}\right) - f(0)}{\frac{1}{4} - 0} v - \frac{0f\left(\frac{1}{4}\right) - \frac{1}{4}f(0)}{\frac{1}{4} - 0}, \]

which is equivalent to

\[ f(v) \leq 4 \left( f\left(\frac{1}{4}\right) - f(0) \right) v + f(0). \]

That is,

\[ f(v) \leq (1 - 4v) f(0) + 4vf\left(\frac{1}{4}\right). \]

So,

\[ a^v b^{1-v} + a^{1-v} b^v \leq (1 - 4r_0) (a + b) + 4r_0 \left( a^{1/4} b^{3/4} + a^{3/4} b^{1/4} \right). \]

For \( \frac{1}{4} \leq v \leq 1 \), similarly, we have

\[ f(v) \leq \frac{f(1) - f\left(\frac{3}{4}\right)}{1 - \frac{3}{4}} v - \frac{\frac{3}{4}f(1) - f\left(\frac{3}{4}\right)}{1 - \frac{3}{4}}, \]

which is equivalent to

\[ f(v) \leq 4 \left( f(1) - f\left(\frac{3}{4}\right) \right) v - 3f(1) + 4f\left(\frac{3}{4}\right). \]

That is,

\[ f(v) \leq (4v - 3) f(1) + 4(1 - v) f\left(\frac{3}{4}\right). \]

So,

\[ a^v b^{1-v} + a^{1-v} b^v \leq (1 - 4r_0) (a + b) + 4r_0 \left( a^{1/4} b^{3/4} + a^{3/4} b^{1/4} \right). \]

If \( \frac{1}{4} \leq v \leq \frac{1}{2} \), then by Lemma 2.1, we have

\[ f(v) \leq \frac{f\left(\frac{1}{2}\right) - f\left(\frac{3}{4}\right)}{\frac{1}{2} - \frac{3}{4}} v - \frac{\frac{3}{4}f\left(\frac{1}{2}\right) - f\left(\frac{3}{4}\right)}{\frac{1}{2} - \frac{3}{4}}, \]

and so

\[ f(v) \leq (4v - 1) f\left(\frac{1}{2}\right) + 2(1 - 2v) f\left(\frac{1}{4}\right), \]

which is equivalent to

\[ a^v b^{1-v} + a^{1-v} b^v \leq 2(4r_0 - 1) \sqrt{ab} + 2(1 - 2r_0) \left( a^{1/4} b^{3/4} + a^{3/4} b^{1/4} \right). \]
If \( \frac{1}{2} \leq v \leq \frac{3}{4} \), similarly, we have
\[
f(v) \leq \frac{f \left( \frac{4}{3} \right) - f \left( \frac{3}{4} \right)}{\frac{1}{2} - \frac{3}{4}}v - \frac{f \left( \frac{4}{3} \right) - f \left( \frac{3}{4} \right)}{\frac{4}{3} - \frac{3}{4}},
\]
and so
\[
f(v) \leq (3 - 4v) f \left( \frac{1}{2} \right) + 2(2v - 1) f \left( \frac{3}{4} \right),
\]
which is equivalent to
\[
f(v) \leq (4r_0 - 1) f \left( \frac{1}{2} \right) + 2(1 - 2r_0)f \left( \frac{3}{4} \right).
\]
That is,
\[
a'^b + a'^{-b} \leq 2(4r_0 - 1)\sqrt{ab} + 2(1 - 2r_0) \left( a^{3/4}b^{3/4} + a^{3/4}b^{1/4} \right).
\]
This completes the proof. \( \square \)

Now, we give a simple comparison between the upper bound for \( a^v b^{1-v} + a^{1-v} b^v \) in (1.5) and (2.1). If \( v \in [0, \frac{1}{2}] \cup \left[ \frac{3}{4}, 1 \right] \), then
\[
a + b - 2r_0 \left( \sqrt{a} - \sqrt{b} \right)^2 \leq (1 - 4r_0)(a + b) - 4r_0(a^{3/4}b^{3/4} + a^{3/4}b^{1/4})
\]
\[= 2r_0 \left( a + b + 2\sqrt{ab} - 2 \left( a^{3/4}b^{3/4} + a^{3/4}b^{1/4} \right) \right) \geq 0.
\]
If \( v \in \left[ \frac{1}{2}, \frac{3}{4} \right] \), then
\[
a + b - 2r_0 \left( \sqrt{a} - \sqrt{b} \right)^2 \leq 2(4r_0 - 1)\sqrt{ab} - 2(1 - 2r_0) \left( a^{3/4}b^{3/4} + a^{3/4}b^{1/4} \right)
\]
\[= (1 - 2r_0) \left( a + b + 2\sqrt{ab} - 2 \left( a^{3/4}b^{3/4} + a^{3/4}b^{1/4} \right) \right) \geq 0.
\]
So, the inequality (2.1) is a refinement of the inequality (1.5).

3. An application

In this section, we give a refinement of the inequality (1.7) for the Hilbert-Schmidt norm based on the inequality (2.1).

**Theorem 3.1.** Let \( A, B, X \in M_n \) such that \( A \) and \( B \) are positive semidefinite and suppose that
\[
\phi(v) = ||A^v XB^{1-v} + A^{1-v}XB^v||_2, \quad 0 \leq v \leq 1.
\]

Then
\[
\phi(v) \leq \begin{cases} 
(1 - 4r_0)\phi(0) + 4r_0\phi \left( \frac{1}{2} \right), & v \in [0, \frac{1}{4}] \cup \left[ \frac{3}{4}, 1 \right] \\
(4r_0 - 1)\phi \left( \frac{1}{2} \right) + 2(1 - 2r_0)\phi \left( \frac{3}{4} \right), & v \in \left[ \frac{1}{4}, \frac{3}{4} \right]
\end{cases}
\]

where \( r_0 = \min \{ v, 1 - v \} \).
Proof. Since every positive semidefinite matrix is unitarily diagonalizable, it follows that there exist unitary matrices $U, V \in M_n$ such that $A = U\Lambda_1 U^*$ and $B = V\Lambda_2 V^*$, where $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $\Lambda_2 = \text{diag}(\mu_1, \ldots, \mu_n)$ and $\lambda_i, \mu_i \geq 0, i = 1, \ldots, n$. Let

$$Y = U^*XV = [y_{ij}].$$

If $v \in [0, \frac{1}{4}] \cup \left[\frac{1}{2}, 1\right]$, then by (2.1) and the Cauchy-Schwarz inequality, we have

$$\|A^*XB^{1-v} + A^{1-v}XB^v\|_2^2 = \sum_{i,j=1}^{n} (\lambda_i^{1/4} \mu_j^{3/4} + \lambda_i^{3/4} \mu_j^{1/4})^2 |y_{ij}|^2$$

$$\leq \sum_{i,j=1}^{n} \left( (1 - 4r_0)(\lambda_i + \mu_j) + 4r_0(\lambda_i^{1/4} \mu_j^{3/4} + \lambda_i^{3/4} \mu_j^{1/4}) \right)^2 |y_{ij}|^2$$

$$= (1 - 4r_0)^2 \sum_{i,j=1}^{n} (\lambda_i + \mu_j)^2 |y_{ij}|^2$$

$$+ 16r_0^2 \sum_{i,j=1}^{n} \left( \lambda_i^{1/4} \mu_j^{3/4} + \lambda_i^{3/4} \mu_j^{1/4} \right)^2 |y_{ij}|^2$$

$$+ 8r_0(1 - 4r_0) \sum_{i,j=1}^{n} (\lambda_i + \mu_j) \left( \lambda_i^{1/4} \mu_j^{3/4} + \lambda_i^{3/4} \mu_j^{1/4} \right) |y_{ij}|^2$$

$$\leq (1 - 4r_0)^2 \phi^{(0)} + 16r_0^2 \phi^2 \left( \frac{1}{4} \right) + 8r_0(1 - 4r_0) \phi^{(0)} \phi \left( \frac{1}{4} \right)$$

$$= \left( (1 - 4r_0) \phi^{(0)} + 4r_0 \phi \left( \frac{1}{4} \right) \right)^2.$$ 

If $v \in \left[\frac{1}{4}, \frac{3}{4}\right]$, the result follows from the inequality (2.1) and the same method above. This completes the proof. □

Remark. For the Hilbert-Schmidt norm, by the inequality (1.7), we have

$$\phi (v) \leq 2r_0 \phi \left( \frac{1}{2} \right) + (1 - 2r_0) \phi (0).$$

So, for $v \in [0, \frac{1}{4}] \cup \left[\frac{3}{4}, 1\right]$, we have

$$2r_0 \phi \left( \frac{1}{2} \right) + (1 - 2r_0) \phi (0) - (1 - 4r_0) \phi (0) - 4r_0 \phi \left( \frac{1}{4} \right)$$

$$= 2r_0 \left( \phi \left( \frac{1}{2} \right) + \phi (0) - 2\phi \left( \frac{1}{4} \right) \right) \geq 0.$$ 

If $v \in \left[\frac{1}{4}, \frac{3}{4}\right]$, then

$$2r_0 \phi \left( \frac{1}{2} \right) + (1 - 2r_0) \phi (0) - (4r_0 - 1) \phi \left( \frac{1}{2} \right) - 2(1 - 2r_0) \phi \left( \frac{1}{4} \right)$$

$$= (1 - 2r_0) \left( \phi \left( \frac{1}{2} \right) + \phi (0) - 2\phi \left( \frac{1}{4} \right) \right) \geq 0.$$ 

So, the inequality (3.1) is a refinement of the inequality (1.7) for the Hilbert-Schmidt norm.

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Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.
Competing interests
The authors declare that they have no competing interests.

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