Proposal to observe the strong Van der Waals force in
\[ e + \bar{e} \rightarrow 2\pi \]

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Abstract

Large discrepancy of the p-wave phase shift data \( \delta_1(\nu) \) of the \( \pi-\pi \) scattering from those of the dispersion calculation is pointed out. In order to determine which is correct, the pion form factor \( F_\pi(\nu) \), which is the second source of information of the phase shift \( \delta_1(\nu) \), is used. It is found that the phase shift obtained from the dispersion is not compatible with the data of the pion form factor. What is wrong with the dispersion calculation, is considered.

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1  p-wave phase shift $\delta_1(\nu)$ of the $\pi-\pi$ scattering

It is known that $\delta_1(\nu)$ is reproduced well by Wagner’s straight line fit

$$\frac{s_\rho}{s} \left( \frac{\nu}{\nu_\rho} \right)^{3/2} \cot \delta_1^{(1)}(\nu) = \frac{s_\rho - s}{\sqrt{s_\rho}^2}$$ \hspace{1cm} (1)

in the low energy region, where $\nu$ is the momentum squared in the center of mass system and $s = 4\nu + 4$ in which the unit $\mu^2 = 1$ is adopted. If we compare it with the effective range function $X_1(\nu)$, which is defined by

$$X_1(\nu) = \frac{\nu^{3/2}}{\sqrt{\nu + 1}} \cot \delta_1^{(1)}(\nu),$$ \hspace{1cm} (2)

we can rewrite Wagner’s fit in terms of the effective range function

$$\frac{X_1(\nu)}{\sqrt{\nu + 1}} = \tilde{c}(\nu - \nu_\rho).$$ \hspace{1cm} (3)

From the values of the mass and the width of the $\rho$-meson $m_\rho = 775.65\text{MeV.}$ and $\Gamma_\rho = 143.85\text{MeV.}$, the parameters of Eq.(3) are determined: $\nu_\rho = 6.721$ and $\tilde{c} = -1.576$ in the unit of $\mu = 1$. In figure 1, Wagner’s fit and the data points are shown.

![Figure 1: Wagner’s straight line fit of Eq.(3).](image1)

![Figure 2: Effective range functions $X_1^{zp}(\nu)$, $X_1^{disp}(\nu)$ and $X_1^{wag}(\nu)$.) The corridor is the result of the pion form factor.](image2)

Before the dispersion calculation, it is convenient to introduce the zero-potential amplitude $a_1^{zp}(\nu)$, which is characterized by vanishing of the left hand spectrum. Moreover it is expected to have the $\rho$-meson pole at right location which is specified by $m_\rho$ and $\Gamma_\rho$. The following effective range function $X_1^{zp}(\nu)$ will do the job:

$$X_1^{zp}(\nu) = c_0 + c_1\nu + \frac{2}{\pi} \frac{\nu^{3/2}}{\sqrt{\nu + 1}} \log(\sqrt{\nu} + \sqrt{\nu + 1})$$ \hspace{1cm} (4)

with $c_0 = 31.72$ and $c_1 = -5.719$. 

If we remember the relation between the amplitude $a_1(\nu)$ and $X_1(\nu)$
\[
\frac{a_1(\nu)}{\nu} = \frac{1}{X_1(\nu) - i\nu\sqrt{\nu/(\nu + 1)}} = \frac{1}{X_1(\nu) - \nu(-\nu/(\nu + 1))^{1/2}}
\] (5)
the necessity of the logarithmic term in Eq.(4) is evident, because the term $-\nu(-\nu/(\nu + 1))^{1/2}$ has cuts in $\nu < -1$ as well as in $\nu > 0$. We can numerically confirm the property that the amplitude does not have the left hand spectrum by computing
\[
\frac{K_1(\nu)}{\nu} \equiv \frac{a_1(\nu)}{\nu} - \frac{1}{\pi} \int_0^\infty d\nu' \frac{\text{Im} a_1(\nu')}{\nu'(\nu' - \nu)},
\] (6)
which is sometimes called Kantor amplitude. If we use the zero-potential amplitude $a_1^{zp}(\nu)$ in evaluating Eq.(6), it must become identically zero, namely $K_1^{zp}(\nu) = 0$. In general, Kantor amplitude can be computed in principle from the experimental data, and can be used to explore the left hand spectrum namely to examine the force acting between the scattering particles.

In the $\pi-\pi$ scattering the two-pion exchange spectrum is known to be computed

from the crossing symmetry. The explicit form of the contribution of the two-pion exchange spectrum to the p-wave amplitude $a_1^{(1)}(\nu)/\nu$ is[1]
\[
\left( \frac{K_1^{2\pi}(\nu)}{\nu} \right) = \frac{K_1^{2\pi}(\nu)}{\nu} = \frac{1}{\pi} \int_0^\infty d\nu'' \text{Im} a_1^{(1)}(\nu'') \left( \frac{1 + 2(\nu + 1)/\nu''}{\nu'^2} \right) 2Q_1(1 + 2(\nu'' + 1)/\nu) - \frac{1 + 2/\nu''}{6(\nu'' + 1)^2} \right) d\nu''
\] (7)
\[
+ \frac{1}{\pi} \int_0^\infty \left[ \frac{2}{3} \text{Im} a_1^{(0)}(\nu'') - \frac{5}{3} \text{Im} a_1^{(2)}(\nu'') \right] \left( \frac{2}{\nu'^2} Q_1(1 + 2(\nu'' + 1)/\nu) - \frac{1}{6(\nu'' + 1)^2} \right) d\nu''
\]
in which $\text{Im} a_1^{(I)}(\nu'')$ are the imaginary part of the $\ell$-th partial waves of isospin $I$. It turns out that $K_1^{2\pi}(\nu)/\nu$ is small.[2] The effective range function of the dispersion calculation $X_1^{\text{disp}}(\nu)$, which corresponds to the amplitude $(a_1^{zp}(\nu) + K_1^{2\pi}(\nu))/\nu$, stays close to $X_1^{zp}(\nu)$. In figure 2, the effective range curves $X_1^{zp}(\nu)$, $X_1^{\text{disp}}(\nu)$ and $X_1^{\text{zag}}(\nu)$, which is given in Eq.(3), are shown along with the data points. Although the locations of the $\rho$-meson and the slopes at $\nu = \nu_\rho$ are kept the same for three curves, $X_1^{\text{zag}}(\nu)$ deviates appreciably from other curves in the low energy region. The corridor just below Wagner’s curve is the effective range function obtained from the pion form factor in the next section.

2 Cross check of the p-wave phase shift $\delta_1(\nu)$

Because of the final state interaction, the phase of the pion form factor $F(\nu)$ coincides with the p-wave phase shift $\delta_1(\nu)$ at least in the elastic region of the corresponding $\pi-\pi$ scattering. Let us introduce the phase function $\Delta(\nu)$ by $F_\pi(\nu) = |F_\pi(\nu)| e^{i\Delta(\nu)}$, which is expected to be equal to $\delta_1(\nu)$ in the low energy region. It is convenient to define a function
\[
f(\nu) = \frac{1}{\nu + 1} \log |F_\pi(\nu)|.
\] (8)
If we remember \( F_\pi(\nu) \) is normalized at \( \nu = -1 \), the denominator \((\nu + 1)\) in Eq.(8) is necessary to remove zero at \( \nu = -1 \) and it also serve to make \( f(\nu) \) to decrease at large \(|\nu|\). The integral representation of \( f(\nu) \) has the form of the Hilbert transformation:

\[
\frac{\log |F_\pi(\nu)|}{\nu + 1} = \frac{P}{\pi} \int_{-\infty}^{\infty} d\nu' \frac{\theta(\nu') \Delta(\nu')}{(\nu' + 1)(\nu' - \nu)} ,
\]

and whose inversion is

\[
\frac{\theta(\nu) \Delta(\nu)}{\nu + 1} = -\frac{P}{\pi} \int_{-\infty}^{\infty} d\nu' \frac{\log |F_\pi(\nu')|}{(\nu' + 1)(\nu' - \nu)} ,
\]

because the square of the Hilbert transformation is equal to minus identity.\[1\]

Equations (9) and (10) enable us to extract more complete information on the phase shift or on the pion form factor by analyzing the phase shift and the form factor data jointly. When the precise data of \(|F_\pi(\nu)|\) were available in the space-like region \((\nu < -1)\) as well as in the time-like region \((\nu > 0)\), we could use Eq.(10) to evaluate the phase \(\Delta(\nu)\). However even in such situation, we have to interpolate the data to the narrow unphysical region \((-1 < \nu < 0)\), where experimental data are not available, although the interpolation function is strictly restricted by the condition that the integration of Eq.(10) must vanish in \(\nu < 0\). Since the precision of the data of the pion form factor in the threshold region is not sufficient, our program in this paper will become modest one to estimate the deviation of the phase shift in the sub-\(\rho\) region, namely to determine the deviation coefficient \(a\) introduced in

\[
\Delta(\nu) = a(\delta_1^{\text{wag}}(\nu) - \delta_1^{\text{zp}}(\nu)) + \delta_1^{\text{zp}}(\nu) \quad \text{in} \quad 0 < \nu < \nu_\rho
\]

by fitting the integration of Eq.(9) to the data of the pion form factor in the space-like region.

![Figure 3: \((\delta_1^{\text{wag}}(\nu) - \delta_1^{\text{zp}}(\nu))/(\nu' + 1)\) in sub-\(\rho\) region.](image1)

![Figure 4: \((f(\nu) - f_\text{zp}(\nu))\) in the \(\rho\)-resonance region for \(a = 0, 1, 2, 3\).](image2)

In figure 3, the spectral function \((\delta_1^{\text{wag}}(\nu) - \delta_1^{\text{zp}}(\nu))/(\nu' + 1)\) necessary to calculate \((f(\nu) - f_\text{zp}(\nu))\) is plotted against \(\nu\). It is interesting to examine qualitatively what is the results of the Hilbert transformation of this spectral function in Eq.(9). Firstly
$f(\nu)$ must shift upward in the space-like region, whereas in the $\rho$-meson and the higher energy region it must shift downward. Secondly because of the rapid raise of the curve of the spectral function at small $\nu$, $f(\nu)$ must have a narrow peak in the threshold region. Although the phase $\Delta(\nu)$ coincides with the $\pi$-$\pi$ phase shift $\delta_1(\nu)$ in the $\rho$-resonance region, they deviate each other in the higher energy region where the inelasticity is not negligible. Therefore before we evaluate $f(\nu)$ in the space-like region, we must determine $\Delta(\nu)$ in the higher energy region, for various values of the deviation coefficient ‘$a$’ appeared in Eq.(11), in such a way that it reproduces the form factor $f(\nu)$ well in the $\rho$-resonance region. In figure 4, curves $(f(\nu) - f^{zp}(\nu))$ are plotted against $\nu$ for $a = 0, 1, 2$ and 3, along with experimental data of CMD-2, in which the $\omega$-pole is removed. Although for $a = 0$ the curve continues monotonously to the space-like region, for $a > 0$ curves $(f(\nu) - f^{zp}(\nu))$ have narrow peaks in the threshold region. Our proposal is to observe such a narrow peak by measuring the cross section of $e^+ e^- \rightarrow 2\pi$ precisely in the low energy region, and which serves to confirm that the long range force such as the strong Van der Waals force is acting between pions.

In figure 5 and 6, the same curves are plotted in the space-like region ($\nu < -1$) and in the unphysical region ($-1 < \nu < 0$). The data points in fig.5 are those of Amendolia et.al.in the region of small momentum transfer, whereas the points in fig.6 are other data in the space-like region. The graphs indicate that the data points differ from the curve $a = 0$, which is obtained by choosing the zero-potential phase shift $\delta_1^{zp}(\nu')$ as $\Delta(\nu')$ in $0 < \nu' < \nu_\rho$ in the evaluation of $f(\nu)$. From the chi square search, $a = 1.1$ is the best fit for the data of low momentum transfer of fig.5. On the other hand, for the joint space-like data of fig.5 and 6, the minimum of chi square occurs at $a = 1.7$. Therefore the pion form factor data supports the Wagner’s fit rather than the dispersin calculation or the zero-potential curve as shown in fig.2.
3 Long range interaction in the $\pi$-$\pi$ scattering

In order to see what type of force is acting between pions, let us compute the contribution from the spectrum on the left hand cut, namely the Kantor amplitude $K_1^{\text{wag}}(\nu)/\nu$ given in Eq.(6) by substituting $\Delta(\nu')$ by the phase shift $\delta_1^{\text{wag}}(\nu')$ which is close to the experimental data. In figure 7, $K_1^{\text{wag}}(\nu)/\nu$ is plotted against $\nu$.

![Kantor amplitude from Wagner's fit](image)

Figure 7: Kantor amplitude $K_1^{\text{wag}}(\nu)/\nu$ of $\pi$-$\pi$ scattering. A cusp of the attractive sign appears at $\nu = 0$, which is characteristic to the long range force.

The curve $K_1^{\text{wag}}(\nu)/\nu$ is characterized by its large slope and very large curvature in the threshold region. On the other hand, since the spectrum of the short range force starts at far left, for example the 4-pion exchange spectrum starts at $\nu = -4$, $K_1(\nu)/\nu$ must be almost constant with small slope and extremely small curvature in the threshold region. Therefore the curve indicates that strong force whose range is longer than that of the pion exchanges must acting.

In general the long range potential, whose asymptotic form is $V(r) \sim -C/r^\alpha$, induces a left hand spectrum in $a_1(\nu)$ starting from $\nu = 0$ and the threshold behavior of the spectrum is $\text{Im} a_1(\nu') \approx C''(-\nu')^\gamma$. The powers $\alpha$ and $\gamma$ are related by $\alpha = 2\gamma + 3$, and the coefficient $C''$ is proportional to $C$ of the potential. In particular for $\alpha = 6$, which is the Van der Waals potential of the London type, the amplitude has the singular term $a_1(\nu) = -C''\nu^{3/2} + \cdots$, whereas for $\alpha = 7$, which is the Van der Waals potential of the Casimir-Polder type, the amplitude has the singular term $a_1(\nu') = C'''\nu^2 \log \nu + \cdots$. It is important that $C''$ and $C'''$ are positive for the attractive potential. Figure 7 indicates that the behavior of the curve $K_1(\nu)/\nu$ is close to $c_0'' - C''\sqrt{\nu}$, namely case of $\alpha = 6$, although possibility of $\alpha = 7$, namely $c_0''' + C'''\nu \log \nu$, is not excluded. We can conclude that the attractive Van der Waals force dominates the pion-pion interaction rather than the short range force.

Finally we shall consider why the strong Van der Waals force appears in the hadron physics. When the hadron was regarded as an elementary particle, the interaction...
between hadrons must occur by the exchanges of mesons, and therefore it was inevitably short range. However after the introduction of the composite model of hadron, whose basic constructive force is strong or superstrong Coulomb type, because of the quantum fluctuation, we cannot avoid the strong Van der Waals force between the composite particles, namely between hadrons. Although the appearance of the Van der Waals force is simply a logical consequence of such composite model, what is important is its strength, which dominates the pion-pion interaction. It is known that the order of magnitude of the strength $C$ of the Van der Waals potential $V(r) \sim -C/r^6$ is

$$C = \frac{2}{3}(\ast e^2)^2 a_1^2 a_2^2 / \Delta E_1,$$

where $\ast e^2$ is the “fine structure constant” of the basic Coulomb force whereas $a_1$ and $a_2$ are the radii of the composite particle 1 and 2 respectively. $\Delta E_1$ is the first excitation energy. From the size of the cusp of figure 7, we can estimate the strength $C$, and which indicates that the fundamental Coulomb interaction is superstrong. Therefore the magnetic monopole model of hadron must be the favorite model, because from the charge quantization condition of Dirac $\ast e^2$ is equal to $137/4$. If we remember that the Van der Waals interaction is universal, we can expect to observe the singular behavior also in other scatterings, whenever sufficiently precise data are available. In fact the attractive cusp is observed in the once subtracted S-wave amplitude $(a_0(\nu) - a_0(0))/\nu$ of the proton-proton scattering at $\nu = 0$, when the repulsive Coulomb singularity is properly removed.\textsuperscript{5}

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