Stability of charged scalar–tensor black holes coupled to Born–Infeld nonlinear electrodynamics

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Abstract
We study the linear stability against spherically symmetric perturbations of scalar–tensor black holes coupled to the Born–Infeld nonlinear electrodynamics. For this purpose we first derived the master equation governing the spherically symmetric perturbations of the black holes. By showing numerically the positiveness of the effective potential of the perturbations we prove the stability of the scalar–tensor black holes in the restricted case of spherically symmetric perturbations for the numerically studied region of their parameter space and for a large class of scalar–tensor theories. Thermodynamical arguments about the full stability of the studied solutions are also given in this paper.

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1. Introduction

One of the most natural generalizations of general relativity (GR) is the scalar–tensor theories (STT) of gravity which arise naturally in string theory and higher dimensional gravity theories [1]. STT satisfy the constraints imposed by the known experiments but also predict deviations from GR [1]. Following the general trend in the study of black holes in the STT we would like to find the effects that the scalar partner of the graviton may have on their properties. In [2] we obtained numerical solutions describing asymptotically flat, spherically symmetric, static, charged black holes coupled to nonlinear electrodynamics (NLED) in a class of STT with massless scalar field.
There are several theorems considering the possibility of existence of black holes in the case that a regular scalar field is present in the theory (see, for example, [3–5]). The conclusions which can be drawn from them are in the spirit that a non-trivial scalar field is not compatible with the presence of a regular event horizon in asymptotically flat spacetimes. The only case in STT for which asymptotically flat black holes exist is when the scalar field is a constant everywhere in the domain of outer communications so the solutions can be classified by the values of mass $M$ and the electric $Q$ (magnetic $P$) charge of the black hole and value of the scalar field at infinity $\phi_\infty$. As a result, the solutions in the STT coincide with the corresponding solutions in GR if we require that the scalar field be vanishing at infinity ($\phi_\infty = 0$). These theorems treat neutral black holes and charged black holes coupled to linear electrodynamics. When a nonlinear Lagrangian of the electrodynamics is considered, however, the energy–momentum tensor of the electromagnetic field has a non-vanishing trace and it serves as a source of the scalar field. This fact allowed us to find asymptotically flat black-hole solutions with non-trivial scalar field. So in this case even if we impose vanishing boundary conditions at infinity we can still have solutions with excited scalar field. Hence, the solutions we have obtained differ considerably from the corresponding solutions in GR even if they are uniquely determined by the values of their mass $M$, magnetic $P$ or electric $Q$ charge and the value of the scalar field at infinity $\phi_\infty$. In other STT there are also black-hole solutions that are not uniquely determined by the values of these parameters [6].

In this paper we have considered the Born–Infeld NLED. As a result of the presence of a non-zero scalar field, the causal structure of the solutions is simpler than the corresponding cases in GR [2].

For our solutions to describe physically realistic objects first they should be stable. In this paper, we study the linear stability against spherically symmetric perturbations and the thermodynamical stability of the numerical solutions obtained in [2]. The study of the linear stability of black holes started with the works of Regge and Wheeler [7] who studied the stability of the Schwarzschild black hole. The stability of the Schwarzschild black hole was studied in numerous papers (for example, [8–11]). Some mathematical aspects of that problem were clarified by Wald [12, 13]. Stability of charged black holes in the GR with NLED was discussed in [14–16].

In this paper we apply the standard scheme, make small, time-dependent perturbations of the static solutions and study their evolution with time. The time dependence of the perturbations can be separated in the form $e^{i\omega t}$ since the background solutions are independent of the time. The linearized field equations together with the proper boundary conditions give a spectral problem for the spacial part of the perturbations which can be cast in the form of standard quantum mechanical problem, through a proper change of the variables, with $\omega^2$ corresponding to the energy. We will consider perturbations whose spacial parts are square integrable functions. The presence of even one mode with negative $\omega^2$ (corresponding to bound states in quantum mechanics) satisfying the imposed boundary conditions would mean that the background solution is unstable since initially well-behaved perturbations (such that they are regular in the region between the event horizon and the spacial infinity) would grow unboundedly with time.

A thorough study of the stability of the system including the perturbations of the zero components of the background metric would require the solving of a singular Sturm–Liouville spectral problem for a system of coupled equations (the system is too complex and decoupling of the equations is not expected). This, however, is a highly non-trivial numerical task in the case the effective potential is not positively definite (a positively definite potential does not admit bound states so conclusion about the stability could easily be made) and is not in the scope of this paper. The problem gets even more complicated since our aim is to study
the stability of solutions not in a single theory but in a class of theories all of which are in agreement with current observational data. The effective potential cannot be expected to be positively definite for all theories in the considered class. In order to simplify the problem we should restrict our study to spherically symmetric perturbations. They are the simplest non-trivial perturbations in the system under consideration since in the case of spherical symmetry the gravitational field has no dynamical degrees of freedom and the perturbations of the metric functions are determined by the perturbations of the matter fields. The system of equations decouples and the problem is reduced to a single Schrödinger-like equation for the perturbations of the scalar field. In many other papers due to the complexity of the systems the authors were compelled to restrict their study of stability to the spherically symmetric case (see, for example, [17–21]).

Additional information about the full stability of the solutions can be obtained through thermodynamical considerations. Arguments from standard thermodynamics, however, are not always applicable to black holes since the black-hole entropy is not an additive quantity unlike the case of standard thermodynamics (we refer the reader to [22] for a detailed discussion on that subject). For the thermodynamical stability of black holes, however, the so-called turning point method can be used. Here we present it briefly and apply as an additional argument for the stability of the considered solutions.

The outline of this paper is as follows. In section 2, we present the numerical solutions obtained in [2] and in addition we obtain the electrically charged solutions through duality transformations. The equations for the small perturbations and figures representing the general behavior of the effective potential for the Schrödinger equation for the numerically studied values of the parameters of the solutions are given in section 3. In this section we also study the stability of the solutions from a thermodynamical point of view. Conclusions are made in the last section.

2. Charged scalar–tensor black holes coupled to Born–Infeld nonlinear electrodynamics

In [2] a four-parametric class of asymptotically flat, spherically symmetric black-hole solutions of the following field equations:

\[ R_{\mu\nu} = 2\partial_\mu \phi \partial_\nu \phi + 2V(\phi) g_{\mu\nu} - 2\partial_X L(X, Y) \left( F_{\mu\beta} F^\beta_\nu - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \]

\[ \nabla_\mu [\partial_X L(X, Y) F^{\mu\nu} + \partial_Y L(X, Y)(\ast F)^{\mu\nu}] = 0 \]

\[ \nabla_\mu \nabla^\mu \phi = \frac{dV(\phi)}{d\phi} - 4\alpha(\phi) A^4(\phi) [L(X, Y) - X \partial_X L(X, Y) - Y \partial_Y L(X, Y)], \]

where \( \alpha(\phi) = \frac{d\ln A(\phi)}{d\phi} \), is reported.

Here, \( L \) denotes the Lagrangian of the electromagnetic field. We also have that

\[ X = \frac{A^{-4}(\phi)}{4} F_{\mu\nu} F^{\mu\nu} F_{\alpha\beta}^{\ast}, \quad Y = \frac{A^{-4}(\phi)}{4} F_{\mu\nu} (\ast F)^{\mu\nu}, \]

where ‘\( \ast \)’ stands for the Hodge dual with respect to the metric \( g_{\mu\nu} \). In [2], the truncated Born–Infeld Lagrangian\(^6\) of the nonlinear electrodynamics

\[ L_{\text{BI}}(X) = 2b \left( 1 - \sqrt{1 + \frac{X}{b}} \right) \]

was considered. Here the potential of the scalar field \( V(\phi) \) is equal to zero.

\(^6\) Here we consider the pure magnetic or pure electric cases for which \( Y = 0 \).
Different types of STT are characterized by the particular choice of the coupling function which for the STT of Brans–Dicke has the following explicit form:

\[ A(\phi) = e^{\alpha \phi}, \]  

(4)

where \( \alpha \) is a positive constant. The general properties of the obtained class of solutions are presented below by several illustrative cases for which \( \alpha = 0.01 \) and \( \beta(\phi = 0) > -4.5 \), where \( \beta(\phi) = \frac{3\alpha(\phi)}{\alpha} \) (see [1]).

With the following ansatz for the metric of a static, spherically symmetric spacetime

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f(r) e^{-2\delta(r)} dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  

(5)

where

\[ f(r) = 1 - \frac{2m(r)}{r}, \]

the system (1) reduces to a system of coupled ordinary differential equations:

\[ \frac{d\delta}{dr} = -r \left( \frac{d\phi}{dr} \right)^2, \]  

(6)

\[ \frac{dm}{dr} = r^2 \left[ \frac{1}{2} f \left( \frac{d\phi}{dr} \right)^2 - A^4(\phi) L(X) \right], \]  

(7)

\[ \frac{d}{dr} \left( r^2 f \frac{d\phi}{dr} \right) = r^2 \left\{ -4\alpha(\phi) A^4(\phi) [\frac{X}{f} L(X) - X \partial_X L(X)] - rf \left( \frac{d\phi}{dr} \right)^3 \right\}. \]  

(8)

We impose the following boundary conditions:

\[ \lim_{r \to \infty} m(r) = M \quad (M \text{ is the mass of the black hole in the Einstein frame}), \]

\[ \lim_{r \to \infty} \delta(r) = \lim_{r \to \infty} \phi(r) = 0, \]

\[ f(r_H) = 0 \]

and use the regularization condition

\[ \left( \frac{df}{dr} \cdot \frac{d\phi}{dr} \right) \bigg|_{r=r_H} = \left. [4\alpha(\phi) A^4(\phi) [\frac{X}{f} L(X) - L(X)]] \right|_{r=r_H} \]

to determine the position of the event horizon \( r_H \) which is a priori unknown boundary (see [2] for more details).

2.1. Magnetically charged black holes

In the magnetically charged case

\[ F = P \sin \theta \, d\theta \wedge d\phi \]  

(9)

and \( X \) reduces to

\[ X = \frac{A^4(\phi) P^2}{2 r^2}. \]  

(10)

The behavior of the metric functions \( f \) and \( \delta \), and of the scalar field \( \phi \) in the theory of Brans–Dicke is given in figures 1 and 2, for \( b = 0.01 \), magnetic charge \( P = 3.55 \) and for different values of the black-hole mass \( M \). The abscissa in the left panel of figure 1 is given in the logarithmic scale in order to clarify the behavior of the function \( f \) better. As was proved by our previous analytical investigations, these figures represent the general behavior of the functions [2]. \( f \) has a single zero, namely the event horizon, \( \delta \) and \( \phi \) are monotonous and have no zeros.
Figure 1. The metric functions $f$ and $\delta$ in the theory of Brans–Dicke for $P = 3.55$. For the class of STT with $\alpha(\varphi) > 0$, $\alpha(\varphi = 0) = 0.01$ and $\beta(\varphi = 0) > -4.5$ the qualitative behavior of $f$ and $\delta$ is the same. The abscissa of the figure in the left panel is given in the logarithmic scale.

Figure 2. The behavior of the scalar field in the theory of Brans–Dicke for the same values of the parameters as in figure 1. For the class of STT with $\alpha(\varphi) > 0$, $\alpha(\varphi = 0) = 0.01$ and $\beta(\varphi = 0) > -4.5$ the qualitative behavior of $\varphi$ is the same.

2.2. Electrically charged black holes

The Born–Infeld Lagrangian (3) belongs to the class of Lagrangians that admit electric–magnetic duality rotations (see [23] for details concerning the invariance under electric–magnetic duality rotations and also [24] and references therein for more general duality transformations which, however, are not a symmetry of the system but rather relate different theories of NLED). So, in order to obtain the electrically charged solutions we can use the invariance of the system (1) under the following transformations:

$$
\{g_{\mu\nu}, \varphi, F_{\mu\nu}, P, X, L(X)\} \longleftrightarrow \{\bar{g}_{\mu\nu}, \varphi, \bar{G}_{\mu\nu}, \bar{Q}, \bar{X}, \bar{L}(\bar{X})\}.
$$

The barred quantities are related to the dual system. Here

$$
G_{\mu\nu} = -2 \frac{\partial(A^{4}(\varphi)L)}{\partial F^{\mu\nu}},
$$

(12)
'⋆' stands for the Hodge duality operator, and
\[ \hat{X} = -(\partial_X L(X))^2 X. \]  
(13)
The electromagnetic field of the electrically charged, dual system is given by
\[ \bar{F} = \frac{\bar{Q}}{e^{-r^2 \partial_X L}} \, dt \wedge dr. \]  
(14)
In order to obtain (14) through duality rotation we have also used the following relation:
\[ \partial_X L(X) \partial_{\hat{X}} L(\hat{X}) = 1, \]  
(15)
which holds for the Lagrangian (3). The energy–momentum tensor of the electromagnetic field is also preserved under electric–magnetic duality rotations so also in the electrically charged case the metric functions \( f \) and \( \delta \), and the scalar field \( \phi \) are those presented in figures 1 and 2.

3. Stability of the solutions

3.1. Linear stability of the magnetically charged black holes

In the present work we will study time-dependent, spherically symmetric perturbations of the black holes presented in the previous section. For this purpose, we will use the following ansatz for the metric of a spherically symmetric, time-dependent spacetime:
\[ ds^2 = -e^{\gamma} \, dt^2 + e^{\chi} \, dr^2 + e^{\beta} (d\theta^2 + \sin^2 \theta \, d\phi^2), \]  
(16)
where \( \gamma, \chi \) and \( \beta \) are the functions of \( r \) and \( t \). We take the function of the metric and the scalar field in the following form:
\[ \gamma(r, t) = [-2\delta(r) + \ln f(r)] + \triangle \gamma(r, t), \]  
\[ \chi(r, t) = -\ln f(r) + \triangle \chi(r, t), \]  
\[ \beta(r, t) = 2\ln r + \triangle \beta(r, t), \]  
\[ \phi(r, t) = \phi(r) + \triangle \phi(r, t), \]  
(17)
where \( \delta(r), f(r) \) and \( \phi(r) \) give the static background solution, and \( \triangle \gamma(r, t), \triangle \chi(r, t), \triangle \beta(r, t) \) and \( \triangle \phi(r, t) \) are small time-dependent perturbations. The convenient gauge is \( \triangle \beta(r, t) = 0 \) so in that case \( e^{\beta} = r^2 \). From the linearized (up to a first order in the small perturbations) \((tr)\) and \((\theta \theta)\) components of the generalized Einstein equations (1) we obtain
\[ \frac{1}{2} \partial_r \beta(r) \partial_t \triangle \chi = 2\partial_r \phi(r) \partial_t \triangle \phi, \]  
(18)
\[ \left[ 1 - R_{(0)}^{(0)} \right] \triangle \chi - \frac{1}{2} r^2 f(r) \partial_r \beta(r) (\partial_r \triangle \gamma - \partial_t \triangle \chi) \right. \]  
[ \left. = -8r^2 a(\phi) A^4(\phi) \left[ X \partial_X L(X) - L(X) \right] \right] \Delta \phi, \]  
(19)
The equation for the perturbations of the scalar field is
\[ \nabla_{\mu}^{(0)} \nabla_{\nu}^{(0)} \Delta \phi = -\left[ \nabla_{\mu}^{(0)} \nabla_{\nu}^{(0)} \phi \right] \Delta \chi + \frac{1}{2} f(r) \partial_r \phi(r) (\partial_t \Delta \gamma - \partial_r \Delta \chi) \]  
\[ = 4A^4(\phi) \left[ \frac{d\alpha(\phi)}{d\phi} [X \partial_X L(X) - L(X)] \right] \Delta \phi, \]  
(20)
where \( \nabla_{\mu}^{(0)} \) is the co-derivative operator with respect to the static background.

From the equations we can see that in the magnetically charged case the variations of the electromagnetic field \( \Delta F^{\mu \nu} \) vanish. The \((t \theta), (r \theta), (t \phi)\) and \((r \phi)\) components of the generalized Einstein equations (1) give
\[ \Delta F^{t \theta} = \Delta F^{r \theta} = \Delta F^{t \phi} = \Delta F^{r \phi} = 0. \]  
(21)
From the generalized Maxwell equations and the Bianchi equations for the other components of the variations of the electromagnetic field we obtain
\[ \Delta F^{\mu} = \Delta F^{\mu \phi} = 0. \] (22)

Integrating (18) we obtain
\[ \Delta \chi = 4 \frac{\partial \varphi(r)}{\partial r} \Delta \varphi = 2r \partial r \varphi(r) \Delta \varphi, \]
which together with equation (19) allows us to relate the perturbations of the metric functions to the perturbations of the scalar field. As a result, the equation for \( \Delta \varphi \) can be separated and takes the form
\[ \nabla^{(0)}_{\mu} \nabla^{(0)}{\mu} \Delta \varphi - U(r) \Delta \varphi = 0, \] (24)

where
\[ U(r) = -2[1 + 2r^2 A^4(\varphi)L(X)][\partial_r \varphi(r)^2 + A^4(\varphi)[16r \partial_r \varphi(r) \alpha(\varphi) + 4 \partial_r \alpha(\varphi)] \times [X \partial X L(X) - L(X)] - 16 \alpha^2(\varphi) A^4(\varphi) [X^2 \partial X^2 L(X) - X \partial X L(X) + L(X)]. \] (25)

Equation (24) admits a separation of the variables, which is a result of the fact that the background solution is static. So with the following substitution,
\[ \Delta \varphi(r, t) = \psi(r) e^{i \omega t}, \] (26)

from (24) we obtain an equation for the spatial part of the perturbations
\[ f(r) e^{-\delta(r)} \frac{d}{dr} \left[ f(r) e^{-\delta(r)} \frac{d \psi}{dr} \right] + \frac{2}{r} f^2(r) e^{-2\delta(r)} \frac{d \psi}{dr} + \omega^2 \psi = f(r) e^{-2\delta(r)} U(r) \psi, \] (27)

where \( \omega^2 \) acts as a spectral parameter. Introducing the tortoise radial coordinate \( r_* \) and making a proper substitution for \( \psi \)
\[ \psi(r) = \frac{u(r)}{r}, \quad dr_* = \frac{dr}{f(r) e^{-\delta(r)}}, \] (28)

we can cast (27) in the form of the Schrödinger equation
\[ \frac{d^2 u(r_*)}{dr_*^2} + \omega^2 u(r_*) = U_{\text{eff}}(r_*) u(r_*), \] (29)

where the effective potential is given by
\[ U_{\text{eff}}(r_*) = f(r_*) e^{-2\delta(r_*)} \left\{ U(r_*) + 2 A^4(\varphi) L(X) + \frac{1}{r^2(r_*)} [1 - f(r_*)] \right\}. \] (30)

For \( r \) varying in the interval \([r_H, \infty)\), where \( r_H \) is the radius of the event horizon, the tortoise radial coordinate \( r_* \in (-\infty, \infty) \). From this point we can use the techniques from the standard quantum mechanics to study the properties of the small perturbations.

We consider a class of STT, i.e., different coupling functions \( A(\varphi) \) which makes the potential (25), (30) difficult for analytical studying so we study it numerically. The qualitative behavior of the effective potential remains the same for different STT belonging to the class discussed in the previous section. The values of \( U_{\text{eff}} \) corresponding to different, small enough \( \alpha(\varphi = 0) \sim 10^{-3} - 10^{-2} \) and \( \beta(\varphi = 0) > -4.5 \) (such values are in agreement with the observations, for details see [1]) and to different STT are also quantitatively very close if the remaining parameters are kept fixed. Some illustrative cases from the theory of Brans–Dicke for \( \alpha = 0.01; b = 0.01 \) and \( P = 3.55 \) are given in figure 3. As can be seen from the figures, the effective potential is non-negative for the numerically studied range of values of the parameters. In all of the figures the effective potential is presented in terms of the radial coordinate \( r \). Obviously, in terms of the tortoise coordinate \( r_* \), \( U_{\text{eff}} \) would be more stretched but non-negative.
3.2. Linear stability of the electrically charged black holes

The functions of the metric and the scalar field are preserved under the electric–magnetic duality transformation. In the electrically charged case, however, there will be non-vanishing electro-magnetic perturbations. We can use the electric–magnetic duality rotation (11) to relate them to the perturbations of the magnetically charged solution

\[ \Delta \bar{F}_{\mu
u} = \Delta [(\ast G)_{\mu
u}] \]
\[ \Delta \bar{F}_{\mu
u} = (\Delta \ast)G_{\mu
u} + (\ast \Delta G)_{\mu
u} \]
\[ \Delta \bar{F}_{\mu
u} = \frac{1}{2}(\ast G)_{\mu
u}(\Delta \gamma + \Delta \chi) + 4\alpha(\phi)X \partial_X L(X)(\ast F)_{\mu
u} \Delta \phi - \partial_X L(X)(\ast \Delta F)_{\mu
u}. \]

This relation (31) allows us to express the perturbations of the electromagnetic field in terms of the functions of the background, static solution and the perturbations of the magnetically charged solution. So the perturbations of the electrically charged solution will remain bounded with time as long as the perturbations of the magnetically charged solution are.

3.3. Thermodynamical arguments for full stability of the solutions

Since the restricted case of spherically symmetric perturbations does not give us full information about the stability of the studied black holes here we will adduce additional thermodynamical considerations. Black holes have long been known to be thermodynamical systems [25]. The first law (FL) of black-hole thermodynamics for the present black holes is given and discussed in [2].

Information about the stability of black holes can be obtained through the so-called Poincaré method of stability [26] or turning point method. It was applied to gravitating systems by Katz [27, 28] and can be used to show the existence of unstable modes without having to solve any eigenvalue equation, only on the bases of the structure of the equilibrium sequence diagrams. A formal proof of the method is given by Katz [27] and Sorkin [29, 30]. For a nice discussion of the method we refer the reader to [31], where the method was applied.
Figure 4. The $M-T^{-1}$ relation, for $b = 0.01$ and $P = 1.0, 2.0, 3.0$. Its qualitative structure is the same for a large class of STT with $\alpha(\varphi) > 0$, $\alpha(\varphi = 0) = 0.01$ and $\beta(\varphi = 0) > -4.5$.

Concerning the application of the ‘turning point’ method to our case we shall consider scalar–tensor black holes in the micro-canonical ensemble. In this case, according to the method a change of stability can only occur at turning points or bifurcations in the equilibrium sequences on the conjugate diagram $M-T^{-1}$. Turning points are those where two equilibrium branches merge with a vertical tangent. By bifurcation points here we mean points where branching of equilibrium sequences occurs.

More details on the application of the ‘turning point’ method to study the stability of charged black holes in STT can be found in [6].

In figure 4 we present some illustrative cases of the conjugate diagram $M-T^{-1}$ from the theory of Brans–Dicke for $\alpha = 0.01; b = 0.01$ and several values of the magnetic charge $P = 1.0, 2.0, 3.0$. The qualitative structure of the conjugate diagram is the same for a large class of STT with $\alpha(\varphi) > 0$, $\alpha(\varphi = 0) = 0.01$ and $\beta(\varphi = 0) > -4.5$. For large masses $M$ the solutions approach the Schwarzschild black hole whose stability in the frame of the theory of Brans–Dicke has already been ascertained [32]. Hence, this justifies the expectation that the present solutions are stable for large (compared to the values of the magnetic charge $P$) values of $M$. As can be seen, there are no turning or bifurcation points in the diagram so no change of stability occurs with the decrease of $M$ and the entire equilibrium branches are stable.

4. Conclusion

Solutions of (29) with $\omega^2 < 0$, satisfying vanishing boundary conditions on the event horizon and at the spatial infinity (we consider perturbations which are square integrable functions, i.e. elements of the Hilbert space $L_2[\kappa^*, d\kappa^*]$) give the unstable modes and correspond to bound states in quantum mechanics. From quantum mechanics we know that when the potential is non-negative no bound states exist. The rigorous, formal mathematical proof is given by Wald (see papers [12, 13]). Thus we can conclude that the scalar–tensor black holes coupled
to Born–Infeld NLED are stable in the restricted case of spherically symmetric perturbations, for the numerically studied region of their parameter space and for a large class of STT. The conclusions obtained from the thermodynamical analysis are also in favor of the full stability of the presented black holes.

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