Kaluza-Klein Formalism of General Spacetimes

J.H. Yoon*
Department of Physics
Konkuk University, Seoul 143-701, Korea

I describe the Kaluza-Klein approach to general relativity of 4-dimensional spacetimes. This approach is based on the (2,2)-fibration of a generic 4-dimensional spacetime, which is viewed as a local product of a (1+1)-dimensional base manifold and a 2-dimensional fibre space. It is shown that the metric coefficients can be decomposed into sets of fields, which transform as a tensor field, gauge fields, and scalar fields with respect to the infinite dimensional group of the diffeomorphisms of the 2-dimensional fibre space. I discuss a few applications of this formalism.

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I. INTRODUCTION

It has been known for some time that there is a curious correspondence between (self-dual) Yang-Mills equations and the (self-dual) Einstein’s equations, when the Yang-Mills gauge symmetry is extended to an infinite dimensional symmetry of (volume-preserving) diffeomorphisms of some auxiliary manifold [1]. It is also well-known that the equations of motion of 2-dimensional non-linear sigma models with the target space as the area-preserving diffeomorphism of an auxiliary 2-surface [2–6] are identical to the the self-dual Einstein’s equations written in the Plebański form [7].

These correspondences are most striking for self-dual cases, and indicate an intriguing possibility that we may be able to reconstruct the full Einstein’s general relativity from suitable gauge field theories by replacing the usual finite dimensional gauge symmetry with an infinite dimensional group of the diffeomorphisms of some manifold. If we recall that the gauge symmetry of general relativity is the group of the diffeomorphisms of a 4-dimensional spacetime, this seemingly wild speculation is not totally unreasonable. Recently we have shown that such a description is indeed possible, by rewriting the Einstein-Hilbert action of general relativity of generic 4-dimensional spacetimes in the (2,2)-decomposition [8–13]. In this approach, the 4-dimensional spacetime is viewed, at least for a finite range of the spacetime, as a locally fibred manifold that consists of a (1+1)-dimensional base manifold \( M_{1+1} \) and a 2-dimensional fibre space \( N_2 \).

The Yang-Mills gauge fields, which naturally appear in this Kaluza-Klein setting [14], are defined on the (1+1)-dimensional base manifold \( M_{1+1} \), and turn out to be valued in the Lie algebra of an infinite dimensional group of the diffeomorphisms \( \text{diff}\ N_2 \). This feature is expected to simplify considerably certain issues concerned with the constraints of general relativity. Namely, in Yang-Mills gauge theories, it is well-known that the Gauss-law constraints associated with the Yang-Mills gauge invariance can be made “trivial”, if we consider gauge invariant quantities only. Thus, in principle, one might expect that the problem of solving the constraints of general relativity could be made “trivial”, at least for some of them, if such a gauge theory description is possible. The purpose of this paper is to show explicitly that our variables transform as a tensor field, gauge fields, and scalar fields with respect to the \( \text{diff}\ N_2 \) transformations, and discuss a general spacetime from the 4-dimensional fibre bundle point of view.

This paper is organized as follows. In section II, we shall outline the kinematics of the (2,2)-decomposition of a generic 4-dimensional spacetime, and introduce the Kaluza-Klein (KK) variables without assuming any spacetime isometries. In section III, we shall find the transformation properties of the KK variables with respect to the \( \text{diff}\ N_2 \) transformations, and introduce the notion of the \( \text{diff}\ N_2 \)-covariant derivatives. In section IV, we shall write down the Einstein-Hilbert action, and finally, we discuss possible applications of this formalism.

II. KINEMATICS

Let us decompose a generic 4-dimensional spacetime of the Lorentzian signature from the KK perspective, in which the spacetime under consideration is viewed as a 4-dimensional fibre bundle, consisting of a (1+1)-dimensional base

*Electronic address: yoonjh@cosmic.konkuk.ac.kr
manifold \( M_{1+1} \) and a 2-dimensional fibre space \( N_2 \). Let the basis vector fields of \( M_{1+1} \) and \( N_2 \) be \( \partial/\partial x^\mu (= \partial_\mu) \) and \( \partial/\partial y^a (= \partial_a) \), respectively, where \( \mu = 0,1 \) and \( a = 2,3 \). The horizontal vector fields \( \hat{\partial}_\mu \), which are defined to be orthogonal to \( N_2 \), can be expressed as linear combinations of \( \partial_\mu \) and \( \partial_a \),

\[
\hat{\partial}_\mu = \partial_\mu - A_\mu^a \partial_a,
\]

where the fields \( A_\mu^a \) are functions of \( (x^\mu, y^a) \). Let us denote by \( \gamma^{\mu\nu} \) the inverse metric of the horizontal space spanned by \( \hat{\partial}_\mu \), and by \( \phi^{ab} \) the inverse metric of \( N_2 \), respectively. In the horizontal lift basis which consists of \( \{ \hat{\partial}_\mu, \partial_a \} \), the metric of the 4-dimensional spacetime can then be written as

\[
\left( \frac{\partial}{\partial s} \right)^2 = \gamma^{\mu\nu} \left( \partial_\mu - A_\mu^a \partial_a \right) \otimes \left( \partial_\nu - A_\nu^b \partial_b \right) + \phi^{ab} \partial_a \otimes \partial_b.
\]

In the corresponding dual basis \( \{ dx^\mu, dy^a + A_\mu^a dx^\mu \} \), the metric becomes

\[
ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu + \phi_{ab} \left( dy^a + A_\mu^a dx^\mu \right) \left( dy^b + A_\nu^b dx^\nu \right).
\]

Formally the above metric looks similar to the “dimensionally reduced” metric in standard KK theories, but in fact it is quite different. In the standard KK reduction certain isometries are usually assumed, and dimensional reduction is made by projection along the directions generated by these isometries \([13]\). The fields \( A_\mu^a \) are identified as the KK gauge fields associated with the finite dimensional isometry group. In this paper, we do not assume such isometries; nevertheless, it turns out that the KK idea still works, and as we shall show shortly, the fields \( A_\mu^a \) can be identified as the gauge fields valued in the infinite dimensional Lie algebra of the diff \( N_2 \) transformations. Moreover, the fields \( \phi_{ab} \) and \( \gamma_{\mu\nu} \) transform as a tensor field and scalar fields with respect to the diff \( N_2 \) transformations.

### III. DIFFEOMORPHISMS AS A LOCAL GAUGE SYMMETRY

#### A. Finite transformations

Let us find the transformation properties of the fields \( \phi_{ab}, A_\mu^a, \) and \( \gamma_{\mu\nu} \) with respect to the diff \( N_2 \) transformations, which are the following coordinate transformations of \( N_2 \), while keeping \( x^\mu \) constant \([10]\),

\[
y^a = y^a(x,y), \quad x^\mu = x^\mu.
\]

Thus we have

\[
dy^a = \frac{\partial y^a}{\partial y^c} \left( dy^c - \left( \frac{\partial y^c}{\partial x^\mu} \right) dx^\mu \right), \quad dx^\mu = dx^\mu.
\]

In the new coordinates the term proportional to \( dx^\mu dy^a \) in \((2.3)\) becomes, keeping the \( (x^\mu, y^a) \) dependence explicit,

\[
2 \phi_{ab}(x,y) A_\mu^a(x,y) dx^\mu dy^b
\]

\[
= 2 \left( \frac{\partial y^a}{\partial y^c} \right) \left( \frac{\partial y^b}{\partial y^d} \right) \phi_{ab}(x,y) \left( \frac{\partial y^c}{\partial x^\mu} \right) A_\mu^c(x,y) dx^\mu \left( dy^c - \left( \frac{\partial y^c}{\partial x^\mu} \right) dx^\mu \right),
\]

where the identity

\[
\left( \frac{\partial y^a}{\partial y^c} \right) \left( \frac{\partial y^d}{\partial y^c} \right) = \delta^a_d
\]

was used. Also the term proportional to \( dy^a dy^b \) becomes

\[
\phi_{ab}(x,y) dy^a dy^b
\]

\[
= \left( \frac{\partial y^a}{\partial y^c} \right) \left( \frac{\partial y^b}{\partial y^d} \right) \phi_{ab}(x,y) \left\{ dy^c dx^\mu - 2 \left( \frac{\partial y^c}{\partial x^\mu} \right) dy^c dx^\mu + \left( \frac{\partial y^d}{\partial x^\mu} \right) dx^\mu dx^\nu \right\}.
\]

After rearranging terms, the metric \((2.3)\) can be written as, in the new coordinates,
\[ ds^2 = \gamma_{\mu\nu}(x, y)dx^\mu dx^\nu + \frac{\partial y^a}{\partial y^c}(\frac{\partial y^b}{\partial y^d})\phi_{ab}(x, y)dy^c dy^d \]

\[ + 2\left(\frac{\partial y^a}{\partial y^c}\right)\left(\frac{\partial y^b}{\partial y^d}\right)\phi_{ab}(x, y)\left\{\left(\frac{\partial y^d}{\partial y^c}\right)A^e_{\nu}(x, y) - \frac{\partial y^d}{\partial x^\mu}\right\}dx^\mu dy^c \]

\[ + \phi_{ab}(x, y)\left\{A^{a}_{\nu}(x, y)A^{b}_{\nu}(x, y) - 2\left(\frac{\partial y^a}{\partial y^c}\right)\left(\frac{\partial y^d}{\partial y^c}\right)A^e_{\nu}(x, y)\left(\frac{\partial y^c}{\partial x^\mu}\right) + \left(\frac{\partial y^a}{\partial y^c}\right)\left(\frac{\partial y^b}{\partial y^d}\right)\left(\frac{\partial y^d}{\partial x^\mu}\right)\right\}dx^\mu dx^\nu, \] (3.6)

which must be equal to

\[ ds^2 = \gamma'_{\mu\nu}(x', y')dx'^\mu dx'^\nu + \phi'_{ab}(x', y')\left\{dy^a + A^{a}_{\nu}(x', y')dx^\nu\right\}\left\{dy^b + A^{b}_{\nu}(x', y')dx^\nu\right\}, \] (3.7)

since the line element is invariant under the diff\(N_2\) transformations. If we compare terms containing \(dy^a dy^b\), we find that \(\phi_{ab}(x, y)\) transform as

\[ \phi'_{ab}(x', y') = \left(\frac{\partial x}{\partial y^a}\right)\left(\frac{\partial x}{\partial y^b}\right)\phi_{cd}(x, y). \] (3.8)

This shows that \(\phi_{ab}(x, y)\) is a tensor field with respect to the diff\(N_2\) transformations. If we use the equation (3.8) in (3.9), the metric becomes

\[ ds^2 = \gamma_{\mu\nu}(x, y)dx^\mu dx^\nu + \phi'_{cd}(x', y')\left\{dy^c + A^{c}_{a}(x', y')dx^a\right\}\left\{dy^d + A^{d}_{b}(x', y')dx^b\right\}, \]

\[ + \phi'_{cd}(x', y')\left\{\left(\frac{\partial y^c}{\partial y^a}\right)A^{a}_{\nu}(x, y) - \frac{\partial y^c}{\partial x^\mu}\right\}\left\{\left(\frac{\partial y^d}{\partial y^b}\right)A^{b}_{\nu}(x, y) - \frac{\partial y^d}{\partial x^\mu}\right\}dx^\mu dx^\nu, \] (3.9)

from which we deduce the following transformation properties of \(A_{\mu}^{\alpha}(x, y)\) and \(\gamma_{\mu\nu}(x, y)\)

\[ A'_{\mu}^{\alpha}(x', y') = \left(\frac{\partial x}{\partial y^a}\right)A^{a}_{\nu}(x, y) - \frac{\partial x}{\partial x^\mu} \] (3.10)

\[ \gamma'_{\mu\nu}(x', y') = \gamma_{\mu\nu}(x, y), \] (3.11)

under the diff\(N_2\) transformations.

**B. Infinitesimal transformations**

It will be instructive to examine the infinitesimal transformations corresponding to the above finite diff\(N_2\) transformations. The infinitesimal diff\(N_2\) transformations consist of the following transformations

\[ y^{'\alpha} = y^\alpha + \xi^a(x, y), \quad x^{'\mu} = x^\mu \quad (O(\xi^2) \ll 1), \] (3.12)

where \(\xi^a(x, y)\) is an arbitrary, infinitesimal, function of \((x^\mu, y^a)\). From this it follows that

\[ \frac{\partial y^c}{\partial y^a} = \delta^c_a - \frac{\partial \xi^c}{\partial y^a} + \cdots, \] (3.13)

where \(\cdots\) means terms of \(O(\xi^2)\). If we expand the l.h.s. of the equation (3.8) in \(\xi^a\), it becomes

\[ \phi'_{ab}(x', y + \xi) = \phi_{ab}(x, y) + \xi^a \frac{\partial}{\partial y^a} \phi_{ab}(x, y) + \cdots, \] (3.14)

whereas the r.h.s. becomes

\[ \left(\frac{\partial y^c}{\partial y^a}\right)\left(\frac{\partial y^d}{\partial y^b}\right)\phi_{cd}(x, y) = \phi_{ab}(x, y) - \frac{\partial \xi^c}{\partial y^a} \phi_{ab}(x, y) - \frac{\partial \xi^c}{\partial y^b} \phi_{ac}(x, y) + \cdots. \] (3.15)
Thus we have
\[
\delta \phi_{ab}(x, y) = \phi'_{ab}(x, y) - \phi_{ab}(x, y)
= -\xi^c \partial_c \phi_{ab}(x, y) - (\partial_a \xi^c) \phi_{cb}(x, y) - (\partial_b \xi^c) \phi_{ac}(x, y)
= -[\xi, \phi]_{lab},
\]
where the subscript \( L \) denotes the Lie derivative along the vector field \( \xi = \xi^a \partial_a \), i.e.
\[
[\xi, \phi]_{lab} = \xi^c \partial_c \phi_{ab} + (\partial_a \xi^c) \phi_{cb} + (\partial_b \xi^c) \phi_{ac}.
\]
It is a straightforward exercise to derive the infinitesimal transformation properties \( A^a_\mu \) and \( \gamma_{\mu\nu} \) from (3.10) and (3.11). They are found to be
\[
\delta A^a_\mu(x, y) = -\partial_\mu \xi^a + [A^a_\mu, \xi]_L^a
= -\partial_\mu \xi^a + A^c_\mu \partial_c \xi^a - \xi^c \partial_c A^a_\mu,
\]
\[
\delta \gamma_{\mu\nu}(x, y) = -[\xi, \gamma_{\mu\nu}]_L
= -\xi^c \partial_c \gamma_{\mu\nu},
\]
where \([A^a_\mu, \xi]_L^a\) and \([\xi, \gamma_{\mu\nu}]_L\) are the Lie derivatives of \( \xi^a \) and \( \gamma_{\mu\nu} \) along the vector fields \( A^a_\mu = A^a_\mu \partial_a \) and \( \xi = \xi^c \partial_c \), respectively. Notice that the Lie derivative acts on the fibre space index \( a \) only. The equations (3.16), (3.18), and (3.19) clearly show that the metric components \( \{ \phi_{ab}, A^a_\mu, \gamma_{\mu\nu} \} \) transform as a tensor field, gauge fields, and scalar fields under the \( \text{diff} N_2 \) transformations, respectively.

C. \( \text{diff} N_2 \)-covariant derivative

Using the Lie derivative along the \( \text{diff} N_2 \)-valued gauge fields, the \( \text{diff} N_2 \)-covariant derivative \( D_\mu \) can be naturally defined as
\[
D_\mu = \partial_\mu - [A_\mu, _L].
\]
With this definition, the equation (3.18) can be written as
\[
\delta A^a_\mu = -D_\mu \xi^a,
\]
which suggests that the \( \text{diff} N_2 \)-valued field strength \( F^a_{\mu\nu} \) be defined as
\[
[D_\mu, D_\nu] \eta = -F^a_{\mu\nu} \partial_a \eta
\]
for an arbitrary scalar function \( \eta \), where \( F^a_{\mu\nu} \) is given by
\[
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - [A^a_\mu, A^a_\nu]_L
= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + A^c_\mu \partial_c A^a_\nu + A^c_\nu \partial_c A^a_\mu.
\]
Similarly, the \( \text{diff} N_2 \)-covariant derivative of \( \phi_{ab} \) is defined as
\[
D_\mu \phi_{ab} = \partial_\mu \phi_{ab} - [A_\mu, \phi]_{lab}
= \partial_\mu \phi_{ab} - A^c_\mu \partial_c \phi_{ab} - (\partial_a A^c_\mu) \phi_{bc} - (\partial_b A^c_\mu) \phi_{ac}.
\]
It remains to show that \( F^a_{\mu\nu} \) and \( D_\mu \phi_{ab} \) transform covariantly under the infinitesimal \( \text{diff} N_2 \) transformations (3.12). Let us consider \( D_\mu \phi_{ab} \) first. The infinitesimal transformation of \( D_\mu \phi_{ab} \) becomes
\[
\delta (D_\mu \phi_{ab}) = -\partial_\mu \left( [\xi, \phi]_{lab} + [A_\mu, [\xi, \phi]_{lab}] + [D_\mu \xi, \phi]_{lab} \right),
\]
where we used the equations (3.16) and (3.18), and the Lie brackets are
\[
[A_\mu, [\xi, \phi]_{lab}] = A^c_\mu \partial_c (\xi, \phi)_{lab} + (\partial_a A^c_\mu) \phi_{bc} - (\partial_b A^c_\mu) \phi_{ac},
\]
\[
[D_\mu \xi, \phi]_{lab} = (D_\mu \xi^c) (\partial_c \phi_{ab}) + \partial_a (D_\mu \xi^c) \phi_{bc} + \partial_b (D_\mu \xi^c) \phi_{ac}.
\]
Using the Leibniz rule of the derivative $\partial_\mu$

$$\partial_\mu (\xi, \phi)_{Lab} = [\partial_\mu \xi, \phi]_{Lab} + [\xi, \partial_\mu \phi]_{Lab},$$  \hspace{1cm} (3.28)

and the properties of the Lie bracket

$$[D_\mu \xi, \phi]_{Lab} = [\partial_\mu \xi, \phi]_{Lab} - [[A_\mu, \xi]_L, \phi]_{Lab},$$ \hspace{1cm} (3.29)

$$[A_\mu, [\xi, \phi]_{Lab}]_{L} = -[\xi, [\phi, A_\mu]_{L}]_{Lab} - [\phi, [A_\mu, \xi]_{L}]_{Lab},$$ \hspace{1cm} (3.30)

we find that the equation (3.25) becomes

$$\delta (D_\mu \phi)_{ab} = -[\xi, \partial_\mu \phi]_{Lab} + [\xi, [A_\mu, \phi]_{L}]_{Lab}$$

$$= -[\xi, D_\mu \phi]_{Lab},$$ \hspace{1cm} (3.31)

which shows that $D_\mu \phi_{ab}$ transforms covariantly under the $\text{diffN}_2$ transformation.

Similarly, the infinitesimal transformation $\delta F_\mu^a$ becomes

$$\delta F_\mu^a = \partial_\mu \left( [A_\nu, \xi]^a_{\nu L} \right) + [D_\mu \xi, A_\nu]^a_\nu - (\mu \leftrightarrow \nu).$$ \hspace{1cm} (3.32)

Using the following identities

$$\partial_\mu \left( [A_\nu, \xi]^a_{\nu L} \right) = [\partial_\mu A_\nu, \xi]^a_{\nu L} + [A_\nu, \partial_\mu \xi]^a_{\nu},$$ \hspace{1cm} (3.33)

$$[D_\mu \xi, A_\nu]^a_{\nu L} = -[A_\nu, D_\mu \xi]^a_{\nu L} = -[A_\nu, \partial_\mu \xi]^a_{\nu} + [A_\nu, [A_\mu, \xi]_{L}]^a_{\nu},$$ \hspace{1cm} (3.34)

we find that

$$\delta F_\mu^a = [\partial_\mu A_\nu - \partial_\nu A_\mu, \xi]^a_{\nu L} + [A_\nu, [A_\mu, \xi]_{L}]^a_{\nu} - [A_\mu, [A_\nu, \xi]_{L}]^a_{\nu},$$

$$= -[\xi, F_\mu^a]_{L},$$ \hspace{1cm} (3.35)

where we used the Jacobi identity

$$[A_\nu, [A_\mu, \xi]_{L}]^a_{\nu} = -[A_\mu, [A_\nu, \xi]_{L}]^a_{\nu} - [\xi, [A_\nu, A_\mu]_{L}]^a_{\nu}.$$ \hspace{1cm} (3.36)

Therefore it follows that

$$\delta F_\mu^a = -[\xi, F_\mu^a]_{L},$$ \hspace{1cm} (3.37)

which shows that $F_\mu^a$ is indeed the $\text{diffN}_2$-valued field strength.

It must be marked here that, in the $(2,2)$-KK formalism, the Lie derivative, rather than the covariant derivative, appears naturally. The appearance of an infinite dimensional symmetry such as $\text{diffN}_2$ is not surprising, since in general relativity the underlying gauge symmetry is the infinite dimensional group of the diffeomorphisms of a 4-dimensional spacetime. The point is that it is the $\text{diffN}_2$ symmetry, the subgroup of the diffeomorphisms of a 4-dimensional spacetime, that shows up as a local gauge symmetry of the Yang-Mills type. This implies that the $(2,2)$-KK formalism can be made a viable method of studying general relativity from the standpoint of the $(1+1)$-dimensional Yang-Mills gauge theory with the $\text{diffN}_2$ symmetry as a local gauge symmetry.

**IV. THE ACTION**

The Einstein-Hilbert action in this KK formalism is given by

$$I = \int d^2 x d^2 y \sqrt{-\gamma} \sqrt{\phi} \left[ g^{\mu\nu} R_{\mu\nu} + \phi^{\alpha\beta} R_{\alpha\beta} + \frac{1}{4} \gamma^{\mu\nu\gamma\alpha\beta} \phi_{ab} F_{\mu\alpha}^a F_{\nu\beta}^b \right. $$

$$+ \frac{1}{4} \gamma^{\mu\nu} \phi^{ab} \phi^{cd} \left\{ (D_\mu \phi_{ac})(D_\nu \phi_{bd}) - (D_\mu \phi_{ab})(D_\nu \phi_{cd}) \right\} $$

$$+ \frac{1}{4} \phi^{ab} \gamma^{\mu\nu\gamma\alpha\beta} \left\{ (\partial_\alpha \gamma_{\mu\alpha})(\partial_\beta \gamma_{\nu\beta}) - (\partial_\alpha \gamma_{\mu\nu})(\partial_\beta \gamma_{\alpha\beta}) \right\}$$

$$+ \int d^2 x d^2 y (\partial A S^A).$$ \hspace{1cm} (4.1)
Let us summarize the notations:
1. The curvature tensors $R_{\mu\nu}$ and $R_{ac}$ are defined as

$$R_{\mu\nu} = \hat{\partial}_\mu \hat{\Gamma}^\alpha_{\nu\alpha} - \hat{\partial}_\nu \hat{\Gamma}^\alpha_{\mu\alpha} + \hat{\Gamma}^\alpha_{\mu\beta} \hat{\Gamma}^\beta_{\nu\alpha} - \hat{\Gamma}^\beta_{\nu\gamma} \hat{\Gamma}^\gamma_{\mu\alpha},$$

$$R_{ac} = \partial_a \Gamma_{bc}^b - \partial_b \Gamma_{ac}^b + \Gamma_{ad}^d \Gamma_{bc}^d - \Gamma_{db}^d \Gamma_{ac}^b,$$

$$\hat{\Gamma}^\alpha_{\mu\nu} = \frac{1}{2} \gamma^{\alpha\beta} \left( \partial_\mu \gamma_{\nu\beta} + \partial_\nu \gamma_{\mu\beta} - \partial_\beta \gamma_{\mu\nu} \right),$$

$$\Gamma^c_{ab} = \frac{1}{2} \phi^{cd} \left( \partial_a \phi_{bd} + \partial_b \phi_{ad} - \partial_d \phi_{ab} \right).$$

2. The last term in (4.1) is a surface integral, where $S^A = (S^\mu, S^a)$ is given by

$$S^\mu = \sqrt{-\gamma} \sqrt{\phi} j^\mu,$$

$$S^a = \sqrt{-\gamma} \sqrt{\phi} \left( - A^a_\mu j^\mu + j^a \right),$$

$$j^\mu = \gamma^{\mu\nu} \phi^{ab} D_\nu \phi_{ab},$$

$$j^a = \phi^{ab} j^\mu \phi_b \gamma_{\mu\nu}.$$
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