Closed Bosonic String Partition Function in Time Independent Exact PP-Wave Background

Agapitos Hatzinikitas

University of Crete,
Department of Applied Mathematics,
L. Knosou-Ambelokipi, 71409 Iraklio Crete,
Greece,
Email: ahatzini@tem.uoc.gr

and

Ioannis Smyrnakis

University of Crete,
Department of Applied Mathematics,
L. Knosou-Ambelokipi, 71409 Iraklio Crete,
Greece,
Email: smyrnaki@tem.uoc.gr

Abstract

The modular invariance of the one-loop partition function of the closed bosonic string in four dimensions in the presence of certain homogeneous exact pp-wave backgrounds is studied. In the absence of an axion field the partition function is found to be modular invariant. In the presence of an axion field modular invariance is broken. This can be attributed to the light-cone gauge which breaks the symmetry in the $\sigma$, $t$-directions. Recovery of this broken modular invariance suggests the introduction of twists in the world-sheet directions. However, one needs to go beyond the light-cone gauge to introduce such twists.
1 Introduction

It has been known for some time [1] that there are certain solutions to the vacuum Einstein equations in four dimensions, with a covariantly constant null Killing vector, that can be interpreted as plane fronted gravitational waves, the so called pp-waves. These are in Brinkmann coordinates and in four dimensions of the form

\[ ds^2 = -dX^+dX^- + F(X^+, X_i)(dX^+)^2 + \sum_{i=1}^{2}(dX_i)^2 \] (1)

where \( \sum_{i=1}^{2}(dX_i)^2 \) denotes the standard metric in the Euclidean space \( E^2 \) and the metric is a solution to the vacuum Einstein field equations if and only if

\[ \partial^2 F(X^+, X_i) = 0. \] (2)

A particular solution of (2) is

\[ F(X^+, X_i) = C_{ij}(X^+)X^iX^j \]

with \( Tr(C_{ij}) = 0 \).

It was shown in [2] that pp-waves are exact solutions of the spacetime string equations to all orders of perturbation theory. So if this metric is used as a spacetime metric in string theory, the \( \beta \)-functions are zero, so conformal invariance is preserved [3]. This class of solutions can be extended by the introduction of an antisymmetric tensor and a dilaton field so that the \( \beta \)-functions remain zero [4]. The axion field strength corresponding to the antisymmetric tensor is given by

\[ H_{\mu\nu\rho} = A_{ij}(V + Z)l_\mu \nabla_\nu X_i \nabla_\rho X_j \] (3)

where \( l_\mu = \partial_\mu X^+ \) is the null Killing vector in the coordinates \( V = \frac{1}{2}(X^+ + X^-), Z = \frac{1}{2}(X^+ - X^-), X^i, i = 1, 2 \). The condition of Weyl invariance in two dimensions now becomes

\[ R_{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H_{\nu}^{\rho\sigma} - \nabla_\mu \nabla_\nu \Phi = 0. \] (4)

Equation (4) is rewritten as

\[ - \frac{1}{2}l_\mu l_\nu \partial_\ell \partial^\ell F(V + Z, X^i) - \frac{1}{36}l_\mu l_\nu A_{ij}(V + Z)A^{ij}(V + Z) - \partial_\mu \partial_\nu \Phi(V + Z) = 0 \] (5)

where only the \( \mu = \nu = (V, Z) \) components contribute. They both lead to the equation

\[ \partial^2 F + \frac{1}{18}A_{ij}A^{ij} + 2\Phi'' = 0 \] (6)

where \( \Phi'' \) refers to differentiation of the dilaton field w.r.t. \( X^+ \). The action corresponding to the above exact solution of the spacetime string equations is

\[ S = \frac{1}{2\pi} \int d\sigma dt \left( h^{ab} \partial_a X^+ \partial_b X^- - h^{ab} \partial_a X_i \partial_b X_i - h^{ab} F \partial_a X^+ \partial_b X^+ - \frac{1}{3} A_{ij}X_j \partial_a X^+ \partial_b X_i \epsilon^{ab} \right) \] (7)

where \( \alpha' = 1/2, \ h^{ab} = \sqrt{-g}g^{ab} and \epsilon^{\sigma} = -1 \). Note that \( h = deth_{ab} = -1 \). This action as it stands is not quadratic in the string coordinates so it is difficult to manipulate. For a particular class of pp-waves, the exact plane waves, the action turns out that by choosing the light-cone gauge it is possible to make this action quadratic and hence quantize it. For these exact plane waves \( F(X^+, X^i) = C_{ij}(X^+)X^iX^j \). Interestingly enough the transverse string coordinates become
massive bosonic fields. Nevertheless the conformal invariance is manifested in the partition function. The modular invariance of the partition function for the Nappi-Witten model \cite{5} has been investigated in \cite{6}.

Of particular interest are two families of exact plane waves that possess an extra Killing vector capable of generating translations in $X^+-$direction. These are called homogeneous exact plane waves. Their metrics in Brinkmann coordinates are given by \cite{7}

$$ds^2 = -dX^+dX^- + \left( e^{X^+f} A_0 e^{-X^+f} \right)_{ij} w_iw_j(dX^+)^2 + (d\bar{w})^2$$ \hspace{1cm} (8)

$$ds^2 = -dX^+dX^- + \left( e^{f\ln X^+} A_0 e^{-f\ln X^+} \right)_{ij} w_iw_j \frac{(dX^+)^2}{(X^+)^2} + (d\bar{w})^2$$ \hspace{1cm} (9)

where $A_0$ is a constant symmetric matrix and $f$ is a constant antisymmetric matrix. The first family consists of geodesically complete metrics having no singularities. The second family has a null singularity at $X^+ = 0$.

The second family of homogeneous exact plane waves has been investigated in \cite{8}. It has been shown that the closed string theory based on this special metric is an exactly solvable model meaning that it is possible to find explicitly the solution to the classical string equations, perform a canonical quantization, determine the spectrum of the Hamiltonian operator and compute some simple scattering amplitudes.

Here we will be concerned mostly with the first family although the covariant form of the action given at the end of Section 4 holds for any plane fronted wave. If we go to a rotating frame through the transformation

$$w_i = \left( e^{X^+f} \right)^i_k X^k$$ \hspace{1cm} (10)

the metric then becomes

$$ds^2 = -dX^+dX^- + \left( A_0 - f^2 \right)_{ij} X^i X^j(dX^+)^2 + (dX^i)^2 - 2f_{ij} X^i dX^j dX^+. \hspace{1cm} (11)$$

This is a more convenient frame for quantization since neither $A_0$ nor $f$ depend on $X^+$. Early studies of strings propagating on subclasses of these backgrounds have been made by \cite{9,10,11}.

Our aim in the present paper is to study the modular properties of the partition function for the closed bosonic string in four dimensions which propagates on the first family exact plane wave background with $f_{ij} = 0$, in the presence of the antisymmetric tensor and the dilaton field. To achieve this we organise the paper as follows.

In Section 2 the canonical momenta are used to write the phase-space action before fixing the light-cone gauge. From this we read off the phase-space Hamiltonian in the light-cone gauge. Next we specialize to time-independent exact plane waves. The transverse string coordinates and the canonical momenta are decomposed into oscillator modes. Finally each oscillator Hamiltonian is diagonalized through a canonical transformation.

In Section 3 the partition function for $A_{ij} = 0$ is computed in the path integral formalism by expanding the transverse fields, subject to free-theory boundary conditions, in modes and then integrating them out. Following a deformed zeta function regularization scheme one can prove that the partition function is explicitly modular invariant, although it does not split into a finite sum of holomorphic times antiholomorphic blocks in any obvious way.

In Section 4 the now non zero axion field $A_{ij}$ is interpreted as an $O(2)$ worldsheet gauge field in the $\sigma$-direction. This is shown to break the modular invariance due to inequivalence of
the $\sigma$- and $t$-directions in the worldsheet, for which the light-cone gauge is responsible. The necessity for modular invariance seems to indicate that we need the introduction of twists in both the $\sigma$- and $t$-directions. However, to justify this, one needs to go beyond the light-cone gauge.

### 2 Quantization

The canonical momenta corresponding to the action (7) are

$$P_+ = \frac{\partial L}{\partial X^-} = \frac{1}{2\pi} h^{a0} \partial_a X^+$$

$$P_- = \frac{\partial L}{\partial X^+} = \frac{1}{2\pi} \left( h^{a0} \partial_a X^- - 2F^+ + \frac{1}{3} A_{ij} X^i X^j \right)$$

$$P_i = \frac{\partial L}{\partial \dot{X}_i} = -\frac{1}{2\pi} \left( 2h^{a0} \partial_a X_i + \frac{1}{3} A_{ij} X^j X^j \right).$$

(12)

The canonical momenta (12) conjugate to the bosonic coordinates is the generalization of those used in [12] when an axion field is switched on. Substituting in the action we get

$$S = \frac{1}{2\pi} \int d\sigma dt \left[ 2\pi P_+ \dot{X}^- + 2\pi P_- \dot{X}^+ + 2\pi P_i \dot{X}_i + \frac{h_{01}}{h_{00}} \left( 2\pi P_+ X^- + 2\pi P_- X^+ + 2\pi P_i X^j \right) - \frac{1}{h_{00}} \left( 4\pi^2 P_+ P_- + 4\pi^2 F(P^+)^2 - \pi^2 (P_i)^2 + X^+ X^- - F(X^+)^2 - (X^i)^2 \right) + \frac{1}{h_{00}} \left( \frac{2}{3} A_{ij} P_j X_i X_i + \frac{2\pi}{6} A_{ij} P_i X_j X^+ + \left( \frac{1}{6} A_{ij} X^i X^j \right)^2 \right) \right].$$

(13)

In the conformal gauge $h_{00} = -1$ and $h^{10} = 0$ so the action becomes

$$S = \frac{1}{2\pi} \int d\sigma dt \left[ 2\pi P_+ \dot{X}^- + 2\pi P_- \dot{X}^+ + 2\pi P_i \dot{X}_i + \left( 4\pi^2 P_+ P_- + 4\pi^2 F(P^+)^2 - \pi^2 (P_i)^2 + X^+ X^- - F(X^+)^2 - (X^i)^2 \right) - \left( \frac{2}{3} A_{ij} P_j X_i X_i + \frac{2\pi}{6} A_{ij} P_i X_j X^+ + \left( \frac{1}{6} A_{ij} X^i X^j \right)^2 \right) \right].$$

(14)

From this we can read off the Hamiltonian in the light-cone gauge. Setting $X^+ = p_+ t$ we obtain

$$H = \int \frac{d\sigma}{2\pi} \left[ 2\pi p_+ P_- + \pi^2 (P_i)^2 + X^i X_i - F p_+^2 - \frac{p_+}{3} A_{ij} X_i X_j \right].$$

(15)

Next we specialize to the time independent exact plane waves. For these the polarisation tensor is given by $C_{ij} = \left( \begin{array}{cc} W_1 + \Phi & -W_2 \\ -W_2 & -W_1 + \Phi \end{array} \right)$ and we have

$$F(X^+, X_i) = C_{ij} X^i X^j = W_1 (X_1^2 - X_2^2) - 2W_2 X_1 X_2 + \Phi (X_1^2 + X_2^2).$$

(16)

Note that $\partial^2 F = 4\Phi$. Setting $A_{ij} = A \epsilon_{ij}$ the spacetime field equation (6) demands that

$$\ddot{\Phi} = -\frac{1}{36} A^2 - \frac{1}{2} \Phi''.$$ 

(17)
Substituting (16) into (15) we obtain

\[
H = \frac{1}{2\pi} \int d\sigma \left[ 2\pi p_+ P_- + \pi^2 (\mathcal{P}_i)^2 + X_i'^2 \right] \\
- \ p_+^2 \left( W_1(X_1'^2 - X_2'^2) - 2W_2X_1X_2 - \left( \frac{1}{36} A^2 + \frac{1}{2} \Phi'' \right) (X_1'^2 + X_2'^2) \right) \\
- \ \frac{p_+}{3} (X_1'X_2 - X_2'X_1) \right]. \tag{18}
\]

We proceed by expanding \( X^i \) and \( P^i \) in oscillator modes

\[
X^i(\sigma, t) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} X^i_n(t)e^{in\sigma} \\
P^i(\sigma, t) = \frac{1}{\pi \sqrt{2}} \sum_{n \in \mathbb{Z}} P^i_n(t)e^{-in\sigma}. \tag{19}
\]

Reality of \( X^i, P^i \) demands that \( X^i_{-n}(t) = \bar{X}^i_n(t), P^i_{-n}(t) = \bar{P}^i_n(t) \). This implies in particular that \( X^i_0(t), P^i_0(t) \) are real. The commutation relations of the \( \hat{X}^i_n, \hat{P}^j_m \) in the operator formalism are given by

\[
[\hat{X}^i_n, \hat{P}^j_m] = i \delta^{ij} \delta_{nm}. \tag{20}
\]

The Hamiltonian now becomes

\[
H = 2\pi p_+ p_- + H_0 + H_{\text{osc.}} \tag{21}
\]

where \( p_- \) is the average value of the momentum density \( P_- \),

\[
H_0 = \frac{1}{2} \left( P_0^i \right)^2 - \frac{1}{2} p_+^2 W_1((X_0^i)^2 - (X_0^i)^2) + p_+^2 W_2 X_0^1 X_0^2 + \frac{1}{2} p_+^2 \left( \frac{1}{36} A^2 + \frac{1}{2} \Phi'' \right) (X_0^i)^2 \tag{22}
\]

and

\[
H_{\text{osc.}} = \frac{1}{2} \sum_{n \neq 0} \left[ |P_n^i|^2 + n^2 |X_n^i|^2 - p_+^2 W_1(|X_n^1|^2 - |X_n^2|^2) + p_+^2 W_2 (X_n^1 X_n^2 + X_n^1 X_n^2) \right. \\
\left. + \ p_+^2 \left( \frac{1}{36} A^2 + \frac{1}{2} \Phi'' \right) |X_n^i|^2 - \frac{p_+}{3} A (X_n^1 X_{-n}^2 - X_n^1 X_{-n}^2) \right]. \tag{23}
\]

If we define as in [4]

\[
\phi_1 = p_+^2 \left( -W_1 + \frac{A^2}{36} + \frac{\Phi''}{2} \right) \\
\phi_2 = p_+^2 \left( W_1 + \frac{A^2}{36} + \frac{\Phi''}{2} \right) \\
\rho = W_2 p_+^2 \\
\lambda = \frac{1}{3} p_+ A \tag{24}
\]
we have

\[ H_0 = \frac{1}{2} (P^i_0)^2 + \frac{1}{2} (X^1_0 X^2_0) \begin{pmatrix} \phi_1 & \rho \\ \rho & \phi_2 \end{pmatrix} \begin{pmatrix} X^1_0 \\ X^2_0 \end{pmatrix} \]

\[ H_{osc.} = \frac{1}{2} \sum_{n \in \mathbb{Z} - \{0\}} \left[ |P^i_n|^2 + n^2 |X^i_n|^2 + (X^{1i}_n X^{2i}_n) \begin{pmatrix} \phi_1 & \rho + in\lambda \\ \rho - in\lambda & \phi_2 \end{pmatrix} \begin{pmatrix} X^1_n \\ X^2_n \end{pmatrix} \right]. \tag{25} \]

It is possible to transform canonically the phase-space variables so as to diagonalize the Hamiltonian. Let

\[ \tilde{X}^i_n = M_{ij}^i X^j_n \quad \tilde{P}^i_n = M_{ij}^i P^j_n \tag{26} \]

where

\[ (M_{ij}) = i \sqrt{\rho^2 + n^2 \lambda^2} \begin{pmatrix} \alpha_- e^{i\theta} & \alpha_+ \\ \alpha_+ & -\alpha_- e^{-i\theta} \end{pmatrix}, \tag{27} \]

\[ \alpha_\pm = \frac{1}{\sqrt{G_\pm^2 + \rho^2 + n^2 \lambda^2}} \]

\[ G_\pm = \pm \left( \frac{\phi_1 - \phi_2}{2} \right) + \sqrt{\left( \frac{\phi_1 - \phi_2}{2} \right)^2 + \rho^2 + n^2 \lambda^2} \tag{28} \]

and \( \theta = \text{arg}(\rho - in\lambda) \). The diagonalized Hamiltonians become

\[ H_0 = \frac{1}{2} (\tilde{P}^i_0)^2 + \frac{1}{2} \left( S_{0,+}^2 (\tilde{X}^1_0)^2 + S_{0,-}^2 (\tilde{X}^2_0)^2 \right) \]

\[ H_{osc.} = \frac{1}{2} \sum_{n \in \mathbb{Z} - \{0\}} \left[ |\tilde{P}^i_n|^2 + n^2 |\tilde{X}^i_n|^2 + S_{n,+}^2 |\tilde{X}^1_n|^2 + S_{n,-}^2 |\tilde{X}^2_n|^2 \right] \]

\[ = \sum_{n > 0} \sum_{i=1,2} H^i_n \tag{29} \]

where

\[ H^i_n = \tilde{P}^i_n \tilde{P}_{-n}^i + (\omega^i_n)^2 \tilde{X}^i_n \tilde{X}_{-n}^i, \tag{30} \]

\[ S_{n,\pm}^2 = \left( \frac{\phi_1 + \phi_2}{2} \right) \pm \sqrt{\left( \frac{\phi_1 - \phi_2}{2} \right)^2 + \rho^2 + n^2 \lambda^2} \tag{31} \]

and \((\omega^1_n)^2 = n^2 + S_{n,\pm}^2\). Here we have assumed that the gravitational wave amplitudes are small compared to \( \Phi'' \) so as to have positivity of \( S_{n,\pm}^2 \). In case the gravitational waves have large amplitudes then the average number of excitation modes of the string diverges exponentially and a string singularity appears \[4\].
3 Partition Function when $A = 0$

In the particular case of $A = 0$ we have $\phi_1 = p_+^2 (W_1 + \frac{\Phi''}{2})$, $\phi_2 = p_+^2 (W_1 + \frac{\Phi''}{2})$, $\rho = W_2 p_+^2$ and $\lambda = 0$, so $S^2_\pm = (\phi_1 + \phi_2) / 2 + (\phi_1 - \phi_2) / 2 + \rho^2$ becomes independent of $n$. Time independence of $\phi_1$, $\phi_2$, $\rho$ demands that the dilaton is at most quadratic in $X^+$, $\Phi(X^+) = c_1(X^+)^2 + c_2 X^+ + c_3$ and that $W_1$, $W_2$ are independent of $X^+$. So we have

$$S^2_\pm = p_+^2 \left( c_1 \pm \sqrt{W_1^2 + W_2^2} \right) = p_+^2 S^2_\pm. \quad (32)$$

The partition function now becomes

$$Z_{A=0} = \int_F \frac{d\tau d\bar{\tau}}{\tau_2} Z_{A=0} (\tau, \bar{\tau}) \quad (33)$$

where

$$Z_{A=0} (\tau, \bar{\tau}) = C \int dp_+ dp_- Tr \left( e^{-2 i \pi \tau \dot{\Pi} \cdot \hat{\Pi} e^{2 i \pi \tau \Pi} \Pi} \right), \quad (34)$$

$\dot{\Pi} = \sum_{i=1}^n \sum_{n=0}^\infty \dot{\Pi}^i_n$ is the momentum operator of the string, $\Pi = 2 \pi p_+ p_- + \sum_{i=1}^n \sum_{n=0}^\infty \Pi^i_n + \frac{1}{2} \sum_{n \in \mathbb{Z}} \sqrt{\frac{1}{2} + n^2 + S^2_\pm} + \frac{1}{2} \sum_{n \in \mathbb{Z}} \sqrt{\frac{1}{2} + S^2_\pm}$ is the normal ordered Hamiltonian and $F$ is the fundamental domain of the modular transformations. Substituting the above operators in (34) and using the results of appendix A we get, in Euclidean time, that

$$Z_{A=0} (\tau, \bar{\tau}) = C \int dp_+ dp_- e^{-2 \pi \tau \Pi \cdot \hat{\Pi} \Pi} \left( \prod_{i=1}^n \left( \prod_{n \geq 0} Z^i_n \right) \right)$$

$$= C \int dp_+ dp_- e^{-2 \pi \tau \Pi \cdot \hat{\Pi} \Pi} \left( \prod_{i=1}^n \left( \prod_{n \geq 0} \text{det}^{-\frac{1}{2}} (D^i_0) \prod_{n \geq 0} \text{det}^{-1} (D^i_n) \right) \right)$$

$$= C \int dp_+ dp_- e^{-2 \pi \tau \Pi \cdot \hat{\Pi} \Pi} \prod_{i=1}^n \prod_{n \in \mathbb{Z}} \text{det}^{-\frac{1}{2}} (D^i_n) \quad (35)$$

where $Z^i_n$ for $n \geq 0$ are given in appendix A. Using now the determinant formulae derived in appendix B we have

$$\prod_{n \in \mathbb{Z}} \text{det} (D^i_n) = \prod_{n \in \mathbb{Z}} e^{2 \pi \omega_n^i \tau_2 (1 - e^{-2 \pi \omega_n^i \tau_2 + 2 i \pi n \tau_1}) (1 - e^{-2 \pi \omega_n^i \tau_2 - 2 i \pi n \tau_1})} = e^{4 \pi \tau_2 \Delta \int \Delta^2 (\tau, \bar{\tau})} \quad (36)$$

$$\Delta (1,2) = \Delta_{\Pi_+ S^2 (-,+)} = \left( \sum_{n \in \mathbb{Z}} \omega_n^i \right)^2 = \left( \sum_{n \in \mathbb{Z}} \sqrt{n^2 + p_+^2 S^2_\pm} \right)^2 = \frac{1}{2 \pi^2} \sum_{n=1}^\infty \int_0^\infty ds e^{-n^2 s} \frac{p_+^2 S^2_\pm}{s} \quad (37)$$

and

$$f_i (\tau, \bar{\tau}) = \prod_{n \in \mathbb{Z}} (1 - e^{-2 \pi \omega_n^i \tau_2 + 2 i \pi n \tau_1}). \quad (38)$$

Note that in formula (37) we have regularized the sum by analytically continuing the formula (see appendix C)

$$\sum_{n=1}^\infty \frac{1}{(n^2 + c^2)^\nu} = \frac{1}{2 c^{2 \nu}} + \frac{\sqrt{\pi}}{2 c^{2 \nu - 1} \Gamma (\nu)} \left[ \Gamma (\nu - \frac{1}{2}) + 4 \sum_{n=1}^\infty (\pi n c)^{\nu - \frac{1}{2}} K_{\nu - 1/2} (2 \pi n c) \right] \quad (39)$$

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to $\nu = -1/2$ and dropping the infinite uniform vacuum energy $\frac{\sqrt{\pi}}{2^{z-1}\Gamma(\nu)}\Gamma(\nu - \frac{1}{2})$ [13]. We also make use of the fact that [14]

$$K_{-1}(z) = \frac{1}{z} \int_0^\infty e^{-t - \frac{t^2}{4z}} dt. \quad (40)$$

This regularized $\Delta_0$ is the Casimir energy of the theory. The regularization procedure we followed corresponds to a deformed zeta function regularization. Using the notation of deformed modular forms as defined in appendix D we have $\prod_{n\in\mathbb{Z}} det(D_n^{(1,2)}) = \hat{\eta}_c^2(\bar{\tau}, \tau)$ so

$$Z_{A=0}(\tau, \bar{\tau}) = C \int dp_+ dp_- e^{-2\pi i \tau p_+ \bar{p}_-} \left[ \hat{\eta}_{p_+ \bar{S}_+}(\tau, \bar{\tau}) \hat{\eta}_{p_- \bar{S}_-}(\tau, \bar{\tau}) \right]^{-1}. \quad (41)$$

The function $\hat{\eta}_c(\tau, \bar{\tau})$ has the modular properties [15]

$$\hat{\eta}_c(\tau + 1, \bar{\tau} + 1) = \hat{\eta}_c(\tau, \bar{\tau})$$
$$\hat{\eta}_c(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}) = \hat{\eta}_{c/|\tau|}(\tau, \bar{\tau}) \quad (42)$$

and in the limit $c \to 0$ it degenerates according to

$$\hat{\eta}^R_0(\tau, \bar{\tau}) = \eta(\tau)\bar{\eta}(\bar{\tau}). \quad (43)$$

Here the symbol $R$ means that we have regularized $\hat{\eta}_0(\tau, \bar{\tau})$ by dropping a zero factor in the limit $c \to 0$.

Because of the modular properties [12] and the fact we integrate over $p_+, p_-$ we have that the partition function transforms according to

$$Z_{A=0}(\tau + 1, \bar{\tau} + 1) = Z_{A=0}(\tau, \bar{\tau})$$
$$Z_{A=0}(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}) = |\tau|^2 Z_{A=0}(\tau, \bar{\tau}) \quad (44)$$

so $Z_{A=0}(\tau, \bar{\tau}) d\tau d\bar{\tau}/\tau_2$ is modular invariant. In order to avoid infinities coming from the momenta integration an IR regularization procedure is needed. A possible way to regularize the partition function without destroying the modular invariance is to use an infrared cut-off $\epsilon$ and take the limit $\epsilon \to 0$ after we divide by the singular term. Doing this we get

$$Z^{R}_{A=0}(\tau, \bar{\tau}) = \lim_{\epsilon \to 0} \frac{\int^{\epsilon}_0 \int^{\infty}_0 dp_+ dp_- e^{-2\pi i \tau p_+ \bar{p}_-} \left[ \hat{\eta}_{p_+ \bar{S}_+}(\tau, \bar{\tau}) \hat{\eta}_{p_- \bar{S}_-}(\tau, \bar{\tau}) \right]^{-1}}{\tau_2^2 \left( \sqrt{s} \eta(\tau)\bar{\eta}(\bar{\tau}) \right)^2}. \quad (45)$$

4 The partition function when $A \neq 0$

When $A \neq 0$ the above calculation does not lead to a modular invariant partition function. This is because $A$ assumes the role of a worldsheet $O(2)$ gauge field in the $\sigma$-direction. It is possible to write the light-cone gauge fixed Hamiltonian [18] in the following form

$$H = 2\pi p_+ p_- + \frac{1}{2\pi} \int d\sigma \left[ \pi^2 (P^2)^2 + (D\sigma X_i)^2 \right. - p_+^2 \left( W_1(X_1^2 - X_2^2) - 2W_2X_1X_2 - \frac{1}{2}\phi''(X_1^2 + X_2^2) \right) \right] \quad (46)$$
where \( D^R_{\alpha} X_i = \partial_\sigma X_i - \frac{p_{\pm A}}{6} \epsilon_{ij} X_l \). This suggests that \( \frac{p_{\pm A}}{6} \epsilon_{ij} \) is an \( O(2) \) connection along the \( \sigma \)-direction. It is more convenient to define a complex spacetime coordinate to turn the \( O(2) \) connection to a \( U(1) \) connection. If \( Z = X_1 + i X_2 \) then the Hamiltonian takes the form

\[
H = 2\pi p_+ p_- + \frac{1}{2\pi} \int d\sigma \left[ (2\pi)^2 P_x P_x - \frac{p^2}{2} W Z^2 - \frac{p^2}{2} W \bar{Z}^2 + \frac{p^2}{2} \Phi'' Z \bar{Z} + D_\sigma Z D_\sigma \bar{Z} \right] \tag{47}
\]

where now \( D_\sigma = \partial_\sigma + i \frac{p_{\pm A}}{6} \) and \( W = W_1 + i W_2 \).

If we make the gauge transformation

\[
Z = e^{-i \frac{p_{\pm A}}{6} \sigma} \tilde{Z}
\]

then the Hamiltonian becomes

\[
H = 2\pi p_+ p_- + \frac{1}{2\pi} \int d\sigma \left[ (2\pi)^2 P_x P_x - \frac{p^2}{2} W e^{-i \frac{p_{\pm A}}{6} \sigma} \tilde{Z}^2 - \frac{p^2}{2} W e^{i \frac{p_{\pm A}}{6} \sigma} \bar{\tilde{Z}}^2 + \frac{p^2}{2} \Phi'' \frac{\tilde{Z} \bar{\tilde{Z}}}{\tilde{\bar{Z}}} + D_\sigma \tilde{Z} D_\sigma \bar{\tilde{Z}} \right]. \tag{49}
\]

The boundary condition satisfied by the gauge transformed variable \( \tilde{Z} \) is

\[
\tilde{Z}(\sigma + 2\pi, t) = e^{i \frac{p_{\pm A}}{6} \sigma} \tilde{Z}(\sigma, t). \tag{50}
\]

Expanding the fields

\[
Z(\sigma, t) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} Z_n(t) e^{i n \sigma}
\]

\[
P\tilde{Z}(\sigma, t) = \frac{1}{2\pi \sqrt{2}} \sum_{n \in \mathbb{Z}} P_n(t) e^{-i n \sigma} \tag{51}
\]

we get that

\[
\tilde{Z}(\sigma, t) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} Z_n(t) e^{i (n + \frac{p_{\pm A}}{6}) \sigma} \tag{52}
\]

Now the Hamiltonian becomes

\[
H = 2\pi p_+ p_- + \frac{1}{2} \sum_{n \in \mathbb{Z}} \left[ P^2_n P^2_n - \frac{p^2}{2} W Z_n \bar{Z}_n - \frac{p^2}{2} W \bar{Z}_n Z_n + \frac{p^2}{2} \Phi'' Z_n \bar{Z}_n + (n + \frac{p_{\pm A}}{6})^2 Z_n \bar{Z}_n \right]. \tag{53}
\]

This is the same Hamiltonian as (21) when \( A = 0 \) with the only difference that the term \( n^2 ((X_n^1)^2 + (X_n^2)^2) \) is replaced by the term \( (n + \frac{p_{\pm A}}{6})^2 ((X_n^1)^2 + (X_n^2)^2) \). Also note that the momentum operator \( \hat{\Pi}_n^i \) that appears in appendix A changes to

\[
\hat{\Pi}_n^i = i(n + \frac{p_{\pm A}}{6}) \hat{X}_n^i \hat{P}_n^i - i(n - \frac{p_{\pm A}}{6}) \hat{X}_n^i \hat{P}_n^i. \tag{54}
\]

This momentum operator generates \( \sigma \) translations in \( \tilde{Z}, \bar{\tilde{Z}} \). This means that it generates covariant \( \sigma \) translations on \( Z, \bar{Z} \). The only difference is that \( \pm n \) has been replaced by \( (\pm n + \frac{p_{\pm A}}{6}) \). So the partition function becomes

\[
Z_A(\tau, \bar{\tau}) = C \int dp_+ dp_- e^{-2\pi p_+ p_-} \prod_{i=1}^{2} \prod_{n \in \mathbb{Z}} det^{-\frac{1}{2}} (D_i^{n + \frac{p_{\pm A}}{6}}). \tag{55}
\]
The product of determinants has been computed in appendix B so if we define

\[ \tilde{\omega}_{n+p+\frac{A}{6}}^{(1,2)} = \sqrt{(n + \frac{p+A}{6})^2 + p_+^2 S_-^2} \]  

we have

\[
\prod_{n \in \mathbb{Z}} \text{det}(D_{n+p+\frac{A}{6}}) = \prod_{n \in \mathbb{Z}} e^{2\pi \omega_{(n+p+\frac{A}{6})^2}} (1 - e^{-2\pi \omega_{(n+p+\frac{A}{6})^2}}) 
\cdot (1 - e^{-2\pi \omega_{(n+p+\frac{A}{6})^2}}) = e^{4\pi \tau_2 \Delta_{(n+p+\frac{A}{6})^2}} f_i \frac{p+A}{6}(\tau, \bar{\tau}) f_i \frac{p+A}{6}(\tau, \bar{\tau})
\]

where

\[
\Delta_{p+\frac{A}{6}}^{(1,2)} = \Delta_{p+\frac{A}{6}}(p_+ S_{(+, -)}) = \frac{1}{2} \sum_{n \in \mathbb{Z}} (n + \frac{p+A}{6})^2 + p_+^2 S_-^2 
\]

\[
= -\frac{1}{2\pi^2} \sum_{n=1}^{\infty} \int_0^\infty ds e^{-ns^2} \frac{p_+^2 S_-^2}{s} \cos(\frac{p+A}{3}n) 
\]

\[
f_i \frac{p+A}{6}(\tau, \bar{\tau}) = \prod_{n \in \mathbb{Z}} (1 - e^{-2\pi \omega_{(n+p+\frac{A}{6})^2}}) 
\]

In terms of the deformed theta functions of appendix D the partition function becomes

\[
Z_A(\tau, \bar{\tau}) = C \int dp_+ dp_- e^{-2\pi \tau_2 p_+ p_-} \frac{\hat{\Theta}_{p_+ S_+} \left[ \frac{1}{2} + \frac{p+\frac{A}{6}}{2} \right] (\tau, \bar{\tau}) \hat{\Theta}_{p_- S_-} \left[ \frac{1}{2} + \frac{p+\frac{A}{6}}{2} \right] (\tau, \bar{\tau})}{\hat{\eta}_{p_+ S_+} (\tau, \bar{\tau}) \hat{\eta}_{p_- S_-} (\tau, \bar{\tau})} \right]^{-\frac{1}{2}}
\]

This is modular invariant for fixed \( A \) since the \( p_- \) integration forces \( p_+ \) to be zero if we return to Minkowski space. Furthermore it is equal to the free field partition function upon regularization \[17\] \[18\]. Nevertheless, if we sum over all possible twists before we carry out the \( p_- \) integration, the partition function changes to

\[
Z(\tau, \bar{\tau}) = \int dC \frac{1}{\tau_2} \left( \frac{\hat{\Theta}_R \left[ \frac{1}{2} + \frac{C}{2} \right] (\tau, \bar{\tau})}{\hat{\eta}_R (\tau, \bar{\tau})} \right)^{-1} = \int dC \frac{1}{\tau_2} \left[ \frac{\eta(\tau)}{\Theta \left[ \frac{1}{2} + \frac{C}{2} \right] (0, \tau)} \right]^2.
\]

This is not modular invariant since we only have twists in the \( \sigma \)-direction. The reason for this is the choice of the light-cone gauge. Consider the action \( (7) \) in the conformal gauge. This can be written in the form

\[
S = \int \frac{1}{2\pi} d\sigma dt \left[ -\partial_t X^+ \partial_t X^- + \partial_\sigma X^+ \partial_\sigma X^- + (D_t X)_i (D_t X)_i - (D_\sigma X)_i (D_\sigma X)_i 
+ \bar{F} \partial_t X^+ \partial_t X^- - \bar{F} \partial_\sigma X^+ \partial_\sigma X^- \right]
\]

(61)
where

\[
(D_t X)_i = \partial_t X_i - \frac{1}{6} A_{ij}(X^+) \partial_\sigma X^+ X_j \\
(D_\sigma X)_i = \partial_\sigma X_i - \frac{1}{6} A_{ij}(X^+) \partial_t X^+ X_j \\
\tilde{F} = F + \frac{1}{36} A_{ij}(X^+) A_{il}(X^+) X_j X_l.
\] (62)

Note that (11) implies that \( \tilde{F} \) satisfies the equation

\[
\partial_T^2 \tilde{F} + 2 \Phi'' = 0.
\] (63)

The action (61) implies that we also have a covariant derivative in the \( t \)-direction. In the light-cone gauge \( (D_t X)_i = \partial_t X_i \) so the symmetry in the \( \sigma \)- and \( t \)-directions is broken. This explains why the partition function we obtained is not modular invariant. Upon the introduction of twists in the \( t \)-direction, similar to the twists in the \( \sigma \)-direction we get that

\[
Z(\tau, \bar{\tau}) = \int_0^1 \int_0^1 dCdD \frac{1}{\tau_2} \frac{|\eta(\tau)|^2}{\Theta \left[ \frac{1}{2} + C, \frac{1}{2} + D \right](0, \tau)}.
\] (64)

This is now explicitly modular invariant and it is the partition function when we integrate over axion fields of the form we have considered, as suggested by the covariant form of the action.

5 Conclusion

In this paper motivated by the fascinating structure of potentially exactly solvable plane wave backgrounds, we have investigated the modular properties of the one-loop amplitude for the closed bosonic string in four dimensions and in the presence of a metric, an axion and a dilaton field. We found that when an axion field is switched off \( (A = 0) \) the partition function is explicitly modular invariant. In the case the axion field is switched on \( (A \neq 0) \) the partition function is not modular invariant since we have a twist only in the \( \sigma \)-direction which breaks the equivalence of the \( \sigma \)- and \( t \)-directions. One way to recover modular invariance is to integrate over all the possible twists in both directions. However it seems necessary to develop a covariant quantization procedure to justify such an integration. Another way to obtain the modular invariant partition function is through equivalences that exist between the bosonic string in certain pp-wave backgrounds and Wess-Zumino-Witten models based on non-semisimple Lie groups of the type discussed in [16].

Appendix A

Let us consider the generators of \( t \) and \( \sigma \) translations on the \( n \)-mode part of the field operator \( \hat{X}^i(t, \sigma) \) expansion, \( \hat{X}_n^i(t) e^{i n \sigma} + \hat{X}_n^i(t) e^{-i n \sigma} \) for \( n > 0 \). In this appendix to simplify notation we omit the tilde from the \( \hat{X}_n^i \) and the \( \hat{P}_n^i \). The generator of \( t \) translations is just the harmonic oscillator Hamiltonian \( \hat{H}_n^i \) while the generator of \( \sigma \) translations is

\[
\hat{\Pi}_n^i = i n \left( \hat{X}_n^i \hat{P}_n^i - \hat{X}_n^i \hat{P}_n^i \right).
\] (A.1)
It satisfies
\[ [\hat{\Pi}_n^i, \hat{X}_m^j(t)e^{i\sigma} + \hat{\dot{X}}_{-m}^j(t)e^{-i\sigma}] = -i\delta^{ij}\delta_{nm} \frac{\partial}{\partial \sigma} \left( \hat{X}_n^i(t)e^{i\sigma} + \hat{\dot{X}}_n^j(t)e^{-i\sigma} \right). \]  
(A.2)

We want to compute the partition function of the oscillators on the torus. To do this we propagate oscillator states from \( \sigma, t - \pi \tau \) to \( \sigma + 2\pi \tau, t + \pi \tau \) along a path in the worldsheet which we discretize. Now we have
\[
< X_{\pm n}^{i,j+1}, \sigma_{j+1}, t_{j+1}| X_{\pm n}^{i,j}, \sigma_j, t_j > = \frac{1}{(2\pi)^2} \int dP_n^{i,j} dP_n^{i,j'} e^{iP_n^{i,j}(X_{\pm n}^{i,j+1} - X_{\pm n}^{i,j})} e^{iP_n^{i,j'}(X_{\pm n}^{i,j+1} - X_{\pm n}^{i,j})},
\]
(A.3)

Now observe that
\[
\delta(X_{n}^{i,j+1} - X_{n}^{i,j}) \delta(X_{n}^{i,j+1} - X_{n}^{-i,j}) = \frac{1}{2}\left[ e^{iP_n^{i,j}X_{\pm n}^{i,j+1}} e^{iP_n^{i,j}X_{\pm n}^{i,j}} \right],
\]
(A.4)

and similarly
\[
< X_{\pm n}^{i,j+1}| \hat{H}_n^i| X_{\pm n}^{i,j} > = \int dP_n^{i,j} dP_n^{i,j'} dP_n^{i,j} dP_n^{i,j'} < X_{\pm n}^{i,j+1}| P_{\pm n}^{i,j'} > < P_{\pm n}^{i,j'}| \hat{P}_n^i P_{\pm n}^{i,j'} > < P_{\pm n}^{i,j'}| X_{\pm n}^{i,j} > \]
\[
\left[ X^{i,j} < X_{\pm n}^{i,j+1}| P_{\pm n}^{i,j'} > < P_{\pm n}^{i,j'}| \hat{P}_n^i P_{\pm n}^{i,j'} > < P_{\pm n}^{i,j'}| X_{\pm n}^{i,j} > \right]
\]
\[
= \frac{1}{(2\pi)^2} \int dP_n^{i,j} dP_n^{i,j'} e^{iP_n^{i,j}(X_{\pm n}^{i,j+1} - X_{\pm n}^{i,j})} e^{iP_n^{i,j'}(X_{\pm n}^{i,j+1} - X_{\pm n}^{i,j})} \Pi_n^i(P_{\pm n}^{i,j}, X_{\pm n}^{i,j})
\]
(A.6)

Putting together (A.4), (A.6) and (A.7) we get
\[
< X_{\pm n}^{i,j+1}| e^{-i\hat{H}_n^i dt} e^{i\hat{\Pi}_n^i dt} | X_{\pm n}^{i,j} > = \frac{1}{(2\pi)^2} \int dP_n^{i,j} dP_n^{i,j'} e^{iP_n^{i,j}(X_{\pm n}^{i,j+1} - X_{\pm n}^{i,j})} e^{iP_n^{i,j'}(X_{\pm n}^{i,j+1} - X_{\pm n}^{i,j})} \left( 1 - i\delta t H_n^i(X_{\pm n}^{i,j}, X_{\pm n}^{i,j}) + i\delta \Pi_n^i(P_{\pm n}^{i,j}, X_{\pm n}^{i,j}) \right)
\]
\[
= \frac{1}{(2\pi)^2} \int dP_n^{i,j} dP_n^{i,j'} e^{iP_n^{i,j}(X_{\pm n}^{i,j+1} - X_{\pm n}^{i,j})} e^{iP_n^{i,j'}(X_{\pm n}^{i,j+1} - X_{\pm n}^{i,j})} e^{-i\delta t H_n^i(X_{\pm n}^{i,j}, X_{\pm n}^{i,j})} e^{i\delta \Pi_n^i(P_{\pm n}^{i,j}, X_{\pm n}^{i,j})} + \text{quad.}
\]
(A.8)

Now, since the momentum and the Hamiltonian commute, the partition function is independent of the path by which we join \( (\sigma, t - \pi \tau) \) to \( (\sigma + 2\pi \tau, t + \pi \tau) \) so we can take this path to be
a straight line. Hence we have that \(\delta \sigma = \frac{2}{\tau_2} \delta t\) and then
\[
Z_n^i \equiv < X_n^{i,final}, \sigma + 2\pi \tau_1, t + \pi \tau_2 | X_n^{i,initial}, \sigma, t - \pi \tau_2 > 
= C \int \prod_j dX_n^{i,j} dX_{-n}^{i,j} dP_n^{i,j} dP_{-n}^{i,j} e^{iP_n^{i,j}(X_n^{i,j+1} - X_n^{i,j})} e^{iP_{-n}^{i,j}(X_{-n}^{i,j+1} - X_{-n}^{i,j})} e^{-i\delta t H_n^i(p^{i,j}_\pm, X^{i,j}_\pm)}.
\]
(\text{A.9})

In the continuum limit this becomes
\[
Z_n^i = C \int DX_n^i DX_{-n}^i DP_n^i DP_{-n}^i e^{i \int (p_n^i \dot{X}_n^i + p_{-n}^i \dot{X}_{-n}^i - H_n^i(p^{i,j}_\pm, X^{i,j}_\pm) + \frac{\tau_1}{\tau_2} \Pi_n^i(p^{i,j}_\pm, X^{i,j}_\pm)) dt}.
\]
(\text{A.10})

Analytically continuing to imaginary time by letting \(t \to -it\), \(\tau_2 \to -i\tau_2\) and performing the \(P_{\pm}^i\) integrations we get
\[
Z_n^i = C \int DX_n^i e^{\int X_n^i \left( -\frac{d^2}{dt^2} + 2i n \frac{\tau_1}{\tau_2} \frac{d}{dt} + n^2 \frac{\tau_1^2}{\tau_2^2} + (\omega_n^i)^2 \right) dt} = C \det^{-\frac{1}{2}} \left( -\frac{d^2}{dt^2} + (\omega_n^i)^2 \right).
\]
(\text{A.11})

For the zero mode \(X_0^i\) formula (\text{A.11}) is no longer valid because we do not have a complex pair of modes, but instead we have a real mode. Doing a similar calculation we get
\[
Z_0^i = C \int DX_0^i e^{\int X_0^i \left( -\frac{d^2}{dt^2} + (\omega_0^i)^2 \right) dt} = C \det^{-\frac{1}{2}} \left( -\frac{d^2}{dt^2} + (\omega_0^i)^2 \right).
\]
(\text{A.12})

\section*{Appendix B}

Consider now the operator
\[
D_{(n+a)} = -\frac{d^2}{dt^2} + 2i(n + a) \frac{\tau_1}{\tau_2} \frac{d}{dt} + (n + a)^2 \frac{\tau_1^2}{\tau_2^2} + \omega^2(n+a)
\]
(\text{B.1})

acting on periodic functions on \((-\pi \tau_2, \pi \tau_2\)). The basis of eigenfunctions is
\[
f_m(t) = e^{im \frac{\pi}{\tau_2}}, \quad m \in \mathbb{Z}.
\]
(\text{B.2})

The corresponding eigenvalues are
\[
\lambda_{m}^{n+a} = \frac{(m - (n + a)\tau_1)^2}{\tau_2^2} + \omega^2(n+a).
\]
(\text{B.3})

In computing the determinant of \(D_n\) we are going to use the zeta function regularization. This means that
\[
\prod_{n=-\infty}^{\infty} \alpha = \alpha^{\zeta(0)+1} = 1, \quad \prod_{n=1}^{\infty} \alpha^n = e^{-\alpha \zeta'(0)} = (2\pi)^{\frac{\alpha}{2}}
\]
(\text{B.4})
and
\[ \prod_{n=-\infty}^{\infty} (n + \alpha) = \alpha \prod_{n=1}^{\infty} (-n^2) \left( 1 - \frac{\alpha^2}{n^2} \right) = 2i \sin \pi \alpha. \] (B.5)

Now
\begin{align*}
\det(D_{(n+a)}) &= \prod_{m \in \mathbb{Z}} \lambda_{m+n}^{\alpha} = \prod_{m \in \mathbb{Z}} \frac{\tau_2 \nu^2_{n+a} + (m - (n + a) \tau_1)^2}{\tau_2^2} \\
&= \prod_{m \in \mathbb{Z}} (m - (n + a) \tau_1 - i \nu_{n+a} \tau_2) (m - (n + a) \tau_1 + i \nu_{n+a} \tau_2) \\
&= -4 \sin[\pi((n + a) \tau_1 + i \nu_{n+a} \tau_2)] \sin[\pi((n + a) \tau_1 - i \nu_{n+a} \tau_2)] \\
&= -e^{2\pi \nu_{n+a} \tau_2} \left( 1 - e^{-2\pi \nu_{n+a} \tau_2 + 2\pi(n+a) \tau_1} \right) \left( 1 - e^{-2\pi \nu_{n+a} \tau_2 - 2\pi(n+a) \tau_1} \right). \tag{B.6}
\end{align*}

### Appendix C

Consider
\[ F(a) = \sum_{n \in \mathbb{Z}} ((n + a)^2 + c^2)^p. \tag{C.1} \]

It is possible to write \( F(a) \) as
\[ F(a) = \int_{-\infty}^{\infty} \delta(y - a) F(y) dy = \sum_{k \in \mathbb{Z}} e^{2i\pi ka} \int_{-\infty}^{\infty} e^{-2i\pi ky} (y^2 + c^2)^p dy. \tag{C.2} \]

Substituting the expression
\[ (y^2 + c^2)^p = \frac{1}{\Gamma(-p)} \int_{0}^{\infty} t^{-(1+p)} e^{-(y^2+c^2)t} dt \tag{C.3} \]

into (C.2) and performing the Gaussian integration we get
\[ F(a) = \frac{\sqrt{\pi}}{\Gamma(-p)} \sum_{k \in \mathbb{Z}} e^{2i\pi ka} \int_{0}^{\infty} t^{-(\frac{3}{2}+p)} e^{-(tc^2+c^2)\frac{t}{4}} dt \\
= \frac{\sqrt{\pi}}{\Gamma(-p)} \Gamma(-p - \frac{1}{2}) c^{1+2p} + \frac{\sqrt{\pi}}{\Gamma(-p)} \sum_{k=1}^{\infty} 2 \cos(2\pi ka) \int_{0}^{\infty} t^{-(\frac{3}{2}+p)} e^{-(tc^2+c^2)\frac{t}{4}} dt. \tag{C.4} \]

Substituting in formula (C.1) and expressing the result in terms of the modified Bessel function we have
\[ F(a) = \frac{\sqrt{\pi}}{\Gamma(-p)} c^{1+2p} \left( \Gamma(-p - \frac{1}{2}) + \sum_{k=1}^{\infty} 4(\pi kc)^{-p-\frac{1}{2}} \cos(2\pi kc) K_{-p-\frac{1}{2}}(2k\pi c) \right). \tag{C.5} \]

In the particular case \( a = 0 \) and using the symmetry \( n \to -n \) we get
\[ \sum_{n=1}^{\infty} (n^2 + c^2)^p = \frac{1}{2} \left[ \sum_{n \in \mathbb{Z}} (n^2 + c^2)^p - c^{2p} \right] = \frac{1}{2} \left[ F(0) - c^{2p} \right] \\
= -\frac{c^{2p}}{2} + \frac{\sqrt{\pi}}{2\Gamma(-p)} c^{1+2p} \left( \Gamma(-p - \frac{1}{2}) + \sum_{k=1}^{\infty} 4(\pi kc)^{-p-\frac{1}{2}} K_{-p-\frac{1}{2}}(2k\pi c) \right). \tag{C.6} \]
Appendix D

It is possible to define a deformed version of the product \( \eta(\tau) \eta(\bar{\tau}) \) where \( \eta(\tau) \) is the Jacobi \( \eta \)-function in such a way that it transforms simply under modular transformations. Define

\[
\hat{\eta}_c(\tau, \bar{\tau}) = e^{2\pi \tau_2 \Delta(c)} \prod_{n \in \mathbb{Z}} (1 - e^{-2\pi \tau_2 \sqrt{n^2 + c^2} + 2i\pi n \tau_1}) \tag{D.1}
\]

where

\[
\Delta(c) = -\frac{1}{2\pi^2} \sum_{n=1}^{\infty} \int_0^\infty ds e^{-n^2 s - \frac{c^2 s^2}{s}}. \tag{D.2}
\]

The modular properties satisfied by \( \hat{\eta}_c(\tau, \bar{\tau}) \) are given by

\[
\hat{\eta}_c(\tau + 1, \bar{\tau} + 1) = \hat{\eta}_c(\tau, \bar{\tau})
\]

\[
\hat{\eta}_c\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) = \hat{\eta}_c|_{\left|\tau\right|}(\tau, \bar{\tau}). \tag{D.3}
\]

Note that in particular \( \Delta(0) = -1/12 \). When \( c \to 0 \), \( \hat{\eta}_c(\tau, \bar{\tau}) \to 0 \), but it is possible to regularize it by dropping the factor that goes to zero. In this way we obtain

\[
\hat{\eta}_c^R(\tau, \bar{\tau}) \equiv \lim_{c \to 0} \frac{\hat{\eta}_c(\tau, \bar{\tau})}{1 - e^{-2\pi \tau_2}} = \eta(\tau)\eta(\bar{\tau}). \tag{D.4}
\]

It is also possible to define deformed \( \Theta \)-function bilinears. Define

\[
\hat{\Theta}_c \begin{bmatrix} a \\ b \end{bmatrix} (\tau, \bar{\tau}) = e^{2\pi \tau_2 \hat{\Delta}_a(\tau)} \prod_{n \in \mathbb{Z}} (1 - e^{-2\pi \tau_2 \sqrt{(n+a)^2 + c^2} + 2i\pi n \tau_1})
\]

\[
(1 - e^{-2\pi \tau_2 \sqrt{(n+a+1/2)^2 + c^2} + 2i\pi (n+a+1/2) \tau_1 + 2i\pi (b+1/2)})
\]

\[
(1 - e^{-2\pi \tau_2 \sqrt{(n+a+1/2)^2 + c^2} - 2i\pi (n+a+1/2) \tau_1 - 2i\pi (b+1/2)}) \tag{D.5}
\]

where

\[
\hat{\Delta}_a(c) = -\frac{1}{2\pi^2} \sum_{n=1}^{\infty} \int_0^\infty ds e^{-n^2 s - \frac{c^2 s^2}{s^2}} (1 + 2 \cos(2\pi an)). \tag{D.6}
\]

The modular transformations of the deformed \( \Theta \)-functions are given by

\[
\hat{\Theta}_c \begin{bmatrix} a \\ b \end{bmatrix} \left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) = \hat{\Theta}_c \begin{bmatrix} b \\ -a \end{bmatrix} (\tau, \bar{\tau})
\]

\[
\hat{\Theta}_c \begin{bmatrix} a \\ b \end{bmatrix} (\tau + 1, \bar{\tau} + 1) = \hat{\Theta}_c \begin{bmatrix} a \\ a + b + \frac{1}{2} \end{bmatrix} (\tau, \bar{\tau}). \tag{D.7}
\]

Again, when \( c \to 0 \), \( \hat{\Theta}_c \begin{bmatrix} a \\ b \end{bmatrix} (\tau, \bar{\tau}) \to 0 \), but it is possible to regularize it by dropping the factor that goes to zero. In this way we obtain

\[
\hat{\Theta}_c^R \begin{bmatrix} a \\ b \end{bmatrix} (\tau, \bar{\tau}) = \lim_{c \to 0} \frac{\hat{\Theta}_c \begin{bmatrix} a \\ b \end{bmatrix} (\tau, \bar{\tau})}{1 - e^{-2\pi \tau_2}}. \tag{D.8}
\]

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One can easily prove that this regularized deformed Θ-function is given by

\[ \hat{\Theta}^R_0 \left[ \begin{array}{c} a \\ b \end{array} \right] (\tau, \bar{\tau}) = \Theta \left[ \begin{array}{c} a \\ b \end{array} \right] (0, \tau) \Theta \left[ \begin{array}{c} a \\ b \end{array} \right] (0, \tau) \] (D.9)

where the Θ-functions that appear on the righthand side of (D.9) are the usual theta functions with characteristics. It is worth mentioning that in this context the factor \( \sqrt{-i\tau} \) that appears in the modular transformation properties of the usual Θ-functions can be attributed to the necessary regularization when \( c \to 0 \) since

\[ \hat{\Theta}^R_0 \left[ \begin{array}{c} a \\ b \end{array} \right] (-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}) = \lim_{c \to 0} \frac{1 - e^{-2\pi i c \tau}}{1 - e^{-2\pi i \bar{\tau} c}} \hat{\Theta}^R_0 \left[ \begin{array}{c} b \\ -a \end{array} \right] (\tau, \bar{\tau}) = |\tau| \hat{\Theta}^R_0 \left[ \begin{array}{c} b \\ -a \end{array} \right] (\tau, \bar{\tau}). \] (D.10)

Furthermore the deformed Θ-functions have the following symmetry properties

\[ \hat{\Theta}_c \left[ \begin{array}{c} a + 1 \\ b \end{array} \right] (\tau, \bar{\tau}) = \hat{\Theta}_c \left[ \begin{array}{c} a \\ b + 1 \end{array} \right] (\tau, \bar{\tau}) = \hat{\Theta}_c \left[ \begin{array}{c} a \\ b \end{array} \right] (\tau, \bar{\tau}) \] (D.11)

and

\[ \hat{\Theta}_c \left[ \begin{array}{c} -a \\ -b \end{array} \right] (\tau, \bar{\tau}) = \hat{\Theta}_c \left[ \begin{array}{c} a \\ b \end{array} \right] (\tau, \bar{\tau}). \] (D.12)

References

[1] A. Peres, “Some Gravitational Waves”, Phys. Rev. Lett. 3 (1959) 571.

[2] D. Amati and C. Klimcik, “Strings in a Shock Wave Background and Generation of Curved Geometry from Flat Space String Theory”, Phys. Lett. B210 (1988) 92; D. Amati and C. Klimcik, “Nonperturbative Computation of the Weyl Anomaly for a Class of Nontrivial Backgrounds”, Phys. Lett. B219 (1989) 443.

[3] C. Callan, D. Friedan, E. Martinec and M. Perry, “Strings in Background Fields”, Nucl. Phys. B262 (1985) 593.

[4] G. T. Horowitz and A. R. Steif, “Space-Time Singularities in String Theory” Phys. Rev. Lett. 64 (1990) 260; G. T. Horowitz and A. R. Steif, “Strings in Strong Gravitational Fields”, Phys. Rev. D42 (1990) 1950.

[5] C. R. Nappi and E. Witten, “A WZW Model Based on a Non-semisimple Group”, Phys. Rev. Lett. 71 (1993) 3751.

[6] T. Takayanagi, “Modular Invariance of Strings on PP-Waves with RR-Flux”, JHEP 0212 (2002) 022.

[7] M. Blau and M. O’Loughlin, “Homogeneous Plane Waves”, Nucl. Phys. B654 (2003) 135.

[8] G. Papadopoulos, J. G. Russo and A. A. Tseytlin, “Solvable Model of Strings in a Time-Dependent Plane-Wave Background”, Class. Quant. Grav. 20 (2003) 969.
[9] R. Brooks, “Plane Wave Gravitons, Curvature Singularities and string Physics”, Mod. Phys. Lett. A6 (1991) 841.

[10] H. J. de Vega, M. Ramon Medrano and N. Sanchez, “Clasical and Quantum Strings Near Space-Time Singularities: Gravitational Plane Waves with Arbitrary Polarization”, Class. Quant. Grav. 10 (1993) 2007; O. Jofre and C. Nunez, “Strings in Plane Wave Backgrounds Revisited”, Phys. Rev. D50 (1994) 5232.

[11] A. A. Tseytlin, “Exact Solutions of Closed String Theory”, Class. Quant. Grav. 12 (1995) 2365.

[12] R. R. Metsaev, “Type IIB Green-Schwarz Superstring in Plane Wave Ramond-Ramond Background”, Nucl. Phys. B625 (2002) 70.

[13] M. V. Cougo-Pinto, C. Farina and A. J. Segui-Santonja, “Schwinger’s Method for the Massive Casimir Effect”, Lett. Math. Phys. 31 (1994) 309.

[14] I. S. Gradshteyn and I. M. Ryzhik, “Table of Integrals, Series, and Products”, Academic Press, 6th ed (2000).

[15] O. Bergman, M. R. Gaberdiel and M. B. Green, “D-Brane Interactions in Type IIB Plane-Wave Background”, JHEP 0303 (2003) 002.

[16] E. Kiritsis and B. Pioline, “Strings in Homogeneous Gravitational Waves and Null Holography”, JHEP 0208 (2002) 048.

[17] J.G.Russo and A.A.Tseytlin ” Exactly Solvable String Models of Curved Space-Time Backgrounds”, Nucl. Phys. B449 (1995) 91.

[18] A. B. Hammou ” One Loop Partition Function in Plane Waves R-R Background”, JHEP 0211 (2002) 048.