EXCLUSION SETS IN THE $\Delta$-TYPE EIGENVALUE INCLUSION SET FOR TENSORS

YAOTANG LI* AND SUHUA LI
School of Mathematics and Statistics, Yunnan University
Kunming 650091, China

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Abstract. By excluding some sets which don’t include any eigenvalue of a given tensor from the $\Delta$-type eigenvalue inclusion set, two new $\Delta$-type eigenvalue inclusion sets of tensors are given. And two criteria for identifying nonsingular tensors are also provided by using the new $\Delta$-type eigenvalue inclusion sets.

1. Introduction. Let $C(R)$ be the set of all complex(real) numbers, and $C^n(R^n)$ be the set of all complex(real) vectors. An order $m$ dimension $n$ complex(real) tensor, denoted by $A = (a_{i_1i_2\cdots i_m}) \in C^{[m,n]}(R^{[m,n]})$, consists of $n^m$ entries:

$$a_{i_1i_2\cdots i_m} \in C(R), \forall i_k \in N = \{1, 2, \cdots, n\}, k = 1, 2, \cdots, m.$$  

Definition 1.1. [11] Let $A = (a_{i_1i_2\cdots i_m}) \in C^{[m,n]}$. If there are $\lambda \in C$ and $x \in C^n\{0\}$ such that

$$Ax^{m-1} = \lambda x^{[m-1]},$$

where $Ax^{m-1}$ and $x^{[m-1]}$ are dimension $n$ vectors with $i$th components

$$(Ax^{m-1})_i = \sum_{i_2, \cdots, i_m \in N} a_{i_1i_2\cdots i_m} x_{i_2} \cdots x_{i_m},$$

and

$$(x^{[m-1]})_i = x_i^{m-1},$$

then $\lambda$ is called an eigenvalue of $A$ and $x$ is a corresponding eigenvector. Particularly, if $\lambda \in R$ and $x \in R^n\{0\}$, then $x$ is called an H-eigenvector associated with the H-eigenvalue $\lambda$.

This definition respectively presented by Qi [11] and Lim [10] in 2005, where they assumed that $A \in R^{[m,n]}$. Although the eigenvalues of tensors have many applications in numerical multilinear algebra [12,15,16], the computation for which, like most tensor problems, is NP-hard [2]. Hence the localization for all eigenvalues of a given tensor has become increasingly important. In fact, Qi [11] proposed a Geršgorin set to locate the eigenvalues of real symmetric tensors, which was extended to complex tensors in [5,15].

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* Corresponding author: Yaotang Li.
Theorem 1.2. [5] Let $A = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$, and \( \sigma(A) = \{ \lambda : \lambda \) is an eigenvalue of \( A \} \). Then

\[
\sigma(A) \subseteq \Gamma(A) = \bigcup_{i \in N} \Gamma_i(A),
\]

where

\[
\Gamma_i(A) = \{ z \in \mathbb{C} : |z - a_{ii\cdots ii}| \leq r_i(A) \}, \quad r_i(A) = \sum_{i_2\cdots i_m \in N, \delta_{i_2\cdots i_m} = 0} |a_{ii\cdots ii}|,
\]

and

\[
\delta_{i_2\cdots i_m} = \begin{cases} 
1, & i = i_2 = \cdots = i_m, \\
0, & \text{otherwise}.
\end{cases}
\]

Furthermore, a Brauer-type eigenvalue inclusion set for tensors was given in [5]. And it was proved that the Brauer-type eigenvalue inclusion set is a subset of the Geršgorin set for tensors.

Theorem 1.3. [5] Let $A = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$, \( n \geq 2 \). Then

\[
\sigma(A) \subseteq \mathcal{K}(A) = \bigcup_{i,j \in N, j \neq i} \mathcal{K}_{ij}(A),
\]

where

\[
\mathcal{K}_{ij}(A) = \left\{ z \in \mathbb{C} : (|z - a_{ii\cdots i}| - r_i^j(A))|z - a_{jj\cdots j}| \leq |a_{ij\cdots j}|r_j(A) \right\},
\]

and

\[
r_i^j(A) = r_i(A) - |a_{ij\cdots j}|.
\]

Furthermore, \( \mathcal{K}(A) \subseteq \Gamma(A) \).

In 2016, Li and Li [6] gave a \( \Delta \)-type eigenvalue inclusion set of tensors, and proved that it is a subset of the Brauer-type eigenvalue inclusion set.

Theorem 1.4. [6] Let $A = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$, \( n \geq 2 \). Then

\[
\sigma(A) \subseteq \Theta(A) = \bigcup_{i,j \in N, j \neq i} \Theta_{ij}(A),
\]

where

\[
\Theta_{ij}(A) = \left\{ z \in \mathbb{C} : (|z - a_{ii\cdots i} - r_i^\Delta(A))|z - a_{jj\cdots j} - r_j(\Delta(A))r_j(A) \right\},
\]

\( \Delta_i = \{ (i_2, \cdots, i_m) : i_j = i \) for some \( j \in \{2, \cdots, m\} \), where \( i_2, \cdots, i_m \in N \} \),

\( \Delta_j = \{ (i_2, \cdots, i_m) : i_j \neq i \) for any \( j \in \{2, \cdots, m\} \), where \( i_2, \cdots, i_m \in N \} \),

\[
r_i^\Delta(A) = \sum_{(i_2, \cdots, i_m) \in \Delta_i} |a_{ii\cdots ii}|, \quad r_j(\Delta(A)) = \sum_{(i_2, \cdots, i_m) \in \Delta_j} |a_{ii\cdots ii}|,
\]

and

\[
r_i^{\Delta_j}(A) + r_j(\Delta(A)) = r_i(A).
\]

Furthermore, \( \Theta(A) \subseteq \mathcal{K}(A) \subseteq \Gamma(A) \).
Subsequently, several other eigenvalue inclusion sets for tensors were derived in [1, 4, 7–9, 13], and the relations of some of them were given.

In this paper, we continue to research the eigenvalue localization of tensors. By finding some sets which don’t contain any eigenvalue of tensor \( \mathbf{A} \) (called exclusion sets) in \( \Theta(\mathbf{A}) \) of Theorem 1.4 and excluding them from \( \Theta(\mathbf{A}) \), we get two new \( \Delta \)-type eigenvalue inclusion sets. It is proved that the proposed sets are subsets of \( \Theta(\mathbf{A}) \), hence they can be seen as modifications for \( \Theta(\mathbf{A}) \). As an application, two criteria for identifying nonsingular tensors \([11,14]\) are provided. In order to show the efficiency of the proposed results, numerical examples are given.

2. New \( \Delta \)-type eigenvalue inclusion sets. By removing some exclusion sets from \( \Theta(\mathbf{A}) \), we give two new \( \Delta \)-type eigenvalue inclusion sets in this section.

Theorem 2.1. Let \( \mathbf{A} = (|a_{ij||\cdots||im|}) \in \mathbb{C}^{[m,n]} \), \( n \geq 2 \). Then

\[
\sigma(\mathbf{A}) \subseteq C(\mathbf{A}) = \bigcup_{i,j \in \mathbb{N}, i \neq j} (\Theta_{ij}(\mathbf{A}) \setminus L_{ij}(\mathbf{A})),
\]

where

\[
L_{ij}(\mathbf{A}) = \{-z \in \mathbb{C} : \langle |z| - a_{ij}|z| + r_x^\Delta(\mathbf{A}) \rangle < \langle |z| - a_{ij}|z| - r_x^\Delta(\mathbf{A}) \rangle \}
\]

\[
r_x^\Delta(\mathbf{A}) = \sum_{(i_1\cdots,j_m) \in \Delta} |a_{i_1\cdots,j_m}|, r_x^\Delta(\mathbf{A}) = \sum_{(i_1\cdots,j_m) \in \Delta} |a_{i_1\cdots,j_m}|.
\]

Furthermore, \( C(\mathbf{A}) \subseteq \Theta(\mathbf{A}) \).

Proof. Suppose that \( x = (x_1, x_2, \cdots, x_n) \in \mathbb{C}^n \setminus \{0\} \) is an eigenvector corresponding to \( \lambda \in \sigma(\mathbf{A}) \), that is,

\[
\mathbf{A}x^{m-1} = \lambda x^{m-1}.
\]

Let

\[
|x_i| \geq |x_s| \geq \max\{|x_k| : k \neq s, k \neq t, k \in \mathbb{N}\},
\]

then \( |x_t| > 0 \). The \( th \) equation of (1) is

\[
(\lambda - a_{tt\cdots t})x_t^{m-1} = \sum_{(i_1\cdots,i_m) \in \Delta_t} a_{i_1\cdots i_m}x_i^{m-1} + \sum_{(i_1\cdots,i_m) \in \Delta_t} a_{i_1\cdots i_m}x_i^{m-1}.
\]

Taking modulus in both sides of (2) and using the triangle inequality give

\[
|\lambda - a_{tt\cdots t}| |x_t|^{m-1} \leq \sum_{(i_1\cdots,i_m) \in \Delta_t} |a_{i_1\cdots i_m}| |x_i|^{m-1} + \sum_{(i_1\cdots,i_m) \in \Delta_t} |a_{i_1\cdots i_m}| |x_i|^{m-1}\]

\[
\leq \sum_{(i_1\cdots,i_m) \in \Delta_t} |a_{i_1\cdots i_m}| |x_t|^{m-1} + \sum_{(i_1\cdots,i_m) \in \Delta_t} |a_{i_1\cdots i_m}| |x_s|^{m-1}\]

\[
= r_{\lambda}^\Delta(\mathbf{A}) |x_t|^{m-1} + r_{\lambda}^\Delta(\mathbf{A}) |x_s|^{m-1}.
\]

Hence

\[
(|\lambda - a_{tt\cdots t} - r_{\lambda}^\Delta(\mathbf{A})) |x_t|^{m-1} \leq r_{\lambda}^\Delta(\mathbf{A}) |x_s|^{m-1}.
\]
Meanwhile, for the $s$th equation of (1), we have
\[ |\lambda - a_{ss}x_s| |x_s|^{m-1} \leq \sum_{(i_2, \ldots, i_m) \in \Delta, \delta_{i_2}, \ldots, \delta_{i_m} = 0} |a_{si_2 \cdots i_m}| |x_i|^{m-1} = r_s(A)|x_t|^{m-1}. \] (4)

If $|x_s| > 0$, multiplying (3) with (4) yields
\[ (|\lambda - a_{tt}x_t| - r_t^{\Delta_t}(A))|\lambda - a_{ss}x_s| |x_s|^{m-1} |x_t|^{m-1} \leq r_t^{\Delta_t}(A)r_s(A)|x_t|^{m-1} |x_s|^{m-1}. \]

Hence
\[ (|\lambda - a_{tt}x_t| - r_t^{\Delta_t}(A))|\lambda - a_{ss}x_s| \leq r_t^{\Delta_t}(A)r_s(A), \]

which implies that
\[ \lambda \in \Theta_{ts}(A). \] (5)

If $|x_s| = 0$, we get $|\lambda - a_{tt}x_t| - r_t^{\Delta_t}(A) \leq 0$ from (3), then (5) holds.

Note that (2) can be rewritten as
\[ a_{ts}x_s^{m-1} = (\lambda - a_{tt}x_t)x_t^{m-1} - \sum_{(i_2, \ldots, i_m) \in \Delta, \delta_{i_2}, \ldots, \delta_{i_m} = 0} a_{ti_2 \cdots i_m} x_{i_2} \cdots x_{i_m}. \] (6)

Taking modulus in both sides of (6) and using the triangle inequality yield
\[ |a_{ts}x_s| |x_s|^{m-1} \leq |\lambda - a_{tt}x_t| |x_t|^{m-1} + \sum_{(i_2, \ldots, i_m) \in \Delta, \delta_{i_2}, \ldots, \delta_{i_m} = 0} |a_{ti_2 \cdots i_m}| |x_{i_2}| |x_i| \cdots |x_{i_m}| \]
\[ \leq |\lambda - a_{tt}x_t| |x_t|^{m-1} + \sum_{(i_2, \ldots, i_m) \in \Delta, \delta_{i_2}, \ldots, \delta_{i_m} = 0} |a_{ti_2 \cdots i_m}| |x_{i_2}| |x_i| \cdots |x_{i_m}| \]
\[ = (|\lambda - a_{tt}x_t| + r_t^{\Delta_t}(A))|x_t|^{m-1}, \]

that is
\[ |a_{ts}x_s| |x_s|^{m-1} \leq (|\lambda - a_{tt}x_t| + r_t^{\Delta_t}(A))|x_t|^{m-1}. \] (7)

Similarly, the $s$th equation of (1) can be rewritten as
\[ a_{st}x_t^{m-1} = (\lambda - a_{ss}x_s)x_t^{m-1} - \sum_{(i_2, \ldots, i_m) \in \Delta, \delta_{i_2}, \ldots, \delta_{i_m} = 0} a_{si_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \]
\[ - \sum_{(i_2, \ldots, i_m) \in \Delta, \delta_{i_2}, \ldots, \delta_{i_m} = 0} a_{si_2 \cdots i_m} x_{i_2} \cdots x_{i_m}. \] (8)

Taking modulus in both sides of (8) and using the triangle inequality yield
\[ |a_{st}x_t| |x_t|^{m-1} \leq |\lambda - a_{ss}x_s| |x_s|^{m-1} + \sum_{(i_2, \ldots, i_m) \in \Delta, \delta_{i_2}, \ldots, \delta_{i_m} = 0} |a_{si_2 \cdots i_m}| |x_{i_2}| |x_i| \cdots |x_{i_m}| \]
\[ + \sum_{(i_2, \ldots, i_m) \in \Delta, \delta_{i_2}, \ldots, \delta_{i_m} = 0} |a_{si_2 \cdots i_m}| |x_{i_2}| |x_i| \cdots |x_{i_m}| \]
\[ \leq |\lambda - a_{ss}x_s| |x_s|^{m-1} + \sum_{(i_2, \ldots, i_m) \in \Delta, \delta_{i_2}, \ldots, \delta_{i_m} = 0} |a_{si_2 \cdots i_m}| |x_i|^{m-1} \]
\[ \leq |\lambda - a_{ss}x_s| |x_s|^{m-1} + \sum_{(i_2, \ldots, i_m) \in \Delta, \delta_{i_2}, \ldots, \delta_{i_m} = 0} |a_{si_2 \cdots i_m}| |x_i|^{m-1}. \]
Let \( \lambda \neq \lambda / \)

Theorem 2.2. \( \lambda / \)

Because we don’t know which \( t \) and \( s \) are appropriate to each eigenvalue \( \lambda \), we only give

\[
\lambda \in \bigcup_{t, s \in N, t \neq s} (\Theta_{ts}(A) \setminus L_{ts}(A)).
\]

Then

\[
\sigma(A) \subseteq C(A).
\]

Additionally, since

\[
(\Theta_{ts}(A) \setminus L_{ts}(A)) \subseteq \Theta_{ts}(A), \forall t \in N, \forall s \in N, \text{ and } t \neq s,
\]

then

\[
C(A) \subseteq \Theta(A)
\]

can be easily obtained. \( \square \)

**Theorem 2.2.** Let \( A = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}, n \geq 2 \). Then

\[
\sigma(A) \subseteq V(A) = \bigcup_{t \in N} \left( \bigcup_{j \in N, j \neq t} \Theta_{ij}(A) \setminus \bigcup_{p \in N, p \neq i} H_{ip}(A) \right),
\]

where

\[
H_{ip}(A) = \{ z \in \mathbb{C} : |z - a_{ppp\cdots p}||z - a_{i1\cdots i}| + r_i(A) < (2|a_{p1\cdots i}|-r_p(A))|a_{ip}\cdots p| \}.
\]

Furthermore, \( V(A) \subseteq \Theta(A) \).

**Proof.** For each \( \lambda \in \sigma(A) \), assume that \( x = (x_1, x_2, \cdots, x_n) \in \mathbb{C}^n \setminus \{0\} \) is a corresponding eigenvector, then (1) holds. Let

\[
|x_i| \geq |x_s| \geq \max\{|x_k| : k \neq s, k \neq t, k \in N\},
\]

then \( |x_i| > 0 \). Analogy to the proof of Theorem 2.1,

\[
\lambda \in \Theta_{ts}(A)
\]
can be easily obtained. On the other hand, for any \( p \neq t \), the \( p \)th equation of (1) can be written as
\[
a_{pt \ldots t}x_{t}^{m-1} = (\lambda - a_{pp \ldots p})x_{p}^{m-1} - \sum_{i_2 \ldots \cdot i_m \in \mathbb{N}} a_{pi_2 \cdot \cdot i_m}x_{i_2} \cdot \cdot \cdot x_{i_m}. \tag{12}
\]
Taking modulus in both sides of (12) and using the triangle inequality yield
\[
|a_{pt \ldots t}|x_{t}^{m-1} \leq |\lambda - a_{pp \ldots p}|x_{p}^{m-1} + \sum_{i_2 \ldots \cdot i_m \in \mathbb{N}} |a_{pi_2 \cdot \cdot i_m}|x_{i_2} \cdot \cdot \cdot x_{i_m}.
\]
that is
\[
(2|a_{pt \ldots t}| - r_{p}(A))|x_{t}|^{m-1} \leq |\lambda - a_{pp \ldots p}|x_{p}^{m-1}. \tag{13}
\]
Similarly, the \( t \)th equation of (1) can be rewritten as
\[
a_{tp \ldots p}x_{p}^{m-1} = (\lambda - a_{tt \ldots t})x_{t}^{m-1} - \sum_{i_2 \ldots \cdot i_m \in \mathbb{N}} a_{ti_2 \cdot \cdot i_m}x_{i_2} \cdot \cdot \cdot x_{i_m}. \tag{14}
\]
Taking modulus in both sides of (14) and using the triangle inequality yield
\[
|a_{tp \ldots p}|x_{p}^{m-1} \leq |\lambda - a_{tt \ldots t}|x_{t}^{m-1} + \sum_{i_2 \ldots \cdot i_m \in \mathbb{N}} |a_{ti_2 \cdot \cdot i_m}|x_{i_2} \cdot \cdot \cdot x_{i_m}.
\]
Hence
\[
|a_{tp \ldots p}|x_{p}^{m-1} \leq (|\lambda - a_{tt \ldots t}| + r_{p}^{T}(A))|x_{t}|^{m-1}. \tag{15}
\]
If \( |x_{p}| > 0 \), multiplying (13) and (15) gives
\[
(2|a_{pt \ldots t}| - r_{p}(A))|a_{tp \ldots p}| \leq |\lambda - a_{pp \ldots p}|(|\lambda - a_{tt \ldots t}| + r_{p}^{T}(A)), \tag{16}
\]
that is
\[
\lambda \notin H_{tp}(A). \tag{17}
\]
If \( |x_{p}| = 0 \), from (13) we get
\[
2|a_{pt \ldots t}| - r_{p}(A) \leq 0,
\]
which also derives (17). Note that (17) holds for any \( p \neq t \), then
\[
\lambda \notin \bigcup_{p \in \mathbb{N}, \ p \neq t} H_{tp}(A),
\]
which together with (11) implies that
\[
\lambda \in \Theta_{ts}(A) \setminus \left( \bigcup_{p \in \mathbb{N}, \ p \neq t} H_{tp}(A) \right).
\]
Because it is unknown which \( t \) and \( s \) are appropriate to each eigenvalue \( \lambda \), we can only conclude that
\[ \lambda \in \bigcup_{t,s \in \mathbb{N}, t \neq s} \left( \Theta_{ts}(A) \setminus \left( \bigcup_{p \in \mathbb{N}, p \neq t} H_{tp}(A) \right) \right) = \bigcup_{t \in \mathbb{N}} \left( \left( \bigcup_{s \in \mathbb{N}, s \neq t} \Theta_{ts}(A) \right) \setminus \bigcup_{p \in \mathbb{N}, p \neq t} H_{tp}(A) \right). \]

Then

\[ \sigma(A) \subseteq V(A). \]

In addition, as

\[ \left( \Theta_{ts}(A) \setminus \left( \bigcup_{p \in \mathbb{N}, p \neq t} H_{tp}(A) \right) \right) \subseteq \Theta_{ts}(A) \text{ for all } t, s \in \mathbb{N} \text{ and } t \neq s, \]

therefore

\[ V(A) \subseteq \Theta(A) \]

can be easily derived.

**Remark 1.** There is no inclusion relation between the set \( C(A) \) in Theorem 2.1 and the set \( V(A) \) in Theorem 2.2, which can be illustrated by the following example.

**Example 1.** Consider tensor \( \mathcal{A}_0 = (a_{ijkl}) \in \mathbb{C}^{[3,4]} \) with

\[ a_{111} = 14, \ a_{222} = 1, \ a_{144} = 18 - 5i, \ a_{222} = 16, \ a_{233} = 0.6 + 10i, \]
\[ a_{311} = 1 + i, \ a_{322} = 4 - 0.1i, \ a_{333} = 0.1, \ a_{411} = 14 + i, \ a_{433} = 1 + i, \ a_{444} = 12, \]

and other \( a_{ijkl} = 0 \). The set \( C(\mathcal{A}_0) \) and \( V(\mathcal{A}_0) \) are drawn in Figure 1, and the exact eigenvalues of \( \mathcal{A}_0 \) are plotted with asterisks. It is not difficult to see that

\[ C(\mathcal{A}_0) \not\subseteq V(\mathcal{A}_0) \text{ and } C(\mathcal{A}_0) \not\supseteq V(\mathcal{A}_0). \]
The determinant of a tensor $A$, denoted by $\det(A)$, is equal to the product of all eigenvalues of $A$ [3,11]. In general, a tensor is called nonsingular [14] if $\det(A) \neq 0$. Therefore, the following two sufficient criteria for identifying nonsingular tensors can be obtained by Theorem 2.1 and Theorem 2.2.

**Corollary 1.** Let $A = (a_{i1i_2 \cdots i_m}) \in C^{[m,n]}$, $n \geq 2$. If for each $i \in N$,

$$|a_{ii-i}| - r_i^A(A)|a_{jj-j}| > r_i^A(A)r_j(A),$$

for any $j \neq i$ and $j \in N$,

or

$$|a_{ii-i}| + r_i^A(A)|a_{jj-j}| + r_j^A(A) < |a_{jj-j}|(2|a_{ii-i}| - r_j^A(A)),$$

for any $j \neq i$ and $j \in N$,

then $A$ is nonsingular.

**Corollary 2.** Let $A = (a_{i1i_2 \cdots i_m}) \in C^{[m,n]}$, $n \geq 2$. If for each $i \in N$,

$$|a_{ii-i}| - r_i^A(A)|a_{jj-j}| > r_i^A(A)r_j(A),$$

for any $j \neq i$ and $j \in N$,

or

$$|a_{pp-p}|(|a_{ii-i}| + r_p^A(A)) < (2|a_{ii-i}| - r_p(A))|a_{ip-p}|,$$ for some $p \neq i$ and $p \in N$,

then $A$ is nonsingular.

3. **Numerical examples.** To show the efficiency of Theorem 2.1 and Theorem 2.2, we give two numerical examples.

**Example 2.** Consider tensor $A_1 = (a_{i1i_2i_3}) \in C^{[3,4]}$, where

\[
\begin{align*}
a_{111} &= 14, \ a_{122} = i, \ a_{144} = 18 - 5i, \ a_{222} = 16, \ a_{233} = 0.1 - i, \\
a_{331} &= 1 + i, \ a_{332} = 4 - 0.1i, \ a_{333} = 0.1, \ a_{411} = 14 + i, \ a_{433} = 1 + i, \ a_{444} = 12, \\
\end{align*}
\]

and other $a_{i1i_2i_3} = 0$. The sets $\Theta(A_1)$ and $C(A_1)$ are drawn in Figure 2, and the exact eigenvalues of $A_1$ are plotted with asterisks. It is not difficult to see that $C(A_1)$ can locate the eigenvalues of $A_1$ more precisely than $\Theta(A_1)$.

**Example 3.** Consider tensor $A_2 = (a_{i1i_2i_3}) \in C^{[3,4]}$, where

\[
\begin{align*}
a_{111} &= 1, \ a_{122} = i, \ a_{144} = 18 - 2i, \ a_{222} = 16, \ a_{233} = 4 + 16i, \\
a_{331} &= 1 + i, \ a_{332} = 4 - 17i, \ a_{333} = 11, \ a_{411} = 14 - 12i, \ a_{433} = 1 + i, \ a_{444} = 10, \\
\end{align*}
\]

and other $a_{i1i_2i_3} = 0$. The sets $\Theta(A_2)$ and $V(A_2)$ are drawn in Figure 3, and the exact eigenvalues of $A_2$ are plotted with asterisks. It is not difficult to see that $V(A_2)$ can capture the eigenvalues of $A_2$ more precisely than $\Theta(A_2)$.

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E-mail address: liyaotang@ynu.edu.cn
E-mail address: suhuali66@126.com